EXISTENCE OF MINIMIZERS FOR A QUASILINEAR ELLIPTIC SYSTEM OF GRADIENT TYPE

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Abstract. The aim of this paper is to investigate the existence of weak solutions for the coupled quasilinear elliptic system of gradient type

\[
\begin{cases}
-\text{div}(a(x, u, \nabla u)) + A_t(x, u, \nabla u) = g_1(x, u, v) & \text{in } \Omega, \\
-\text{div}(B(x, v, \nabla v)) + B_t(x, v, \nabla v) = g_2(x, u, v) & \text{in } \Omega, \\
u = v = 0 & \text{on } \partial \Omega,
\end{cases}
\]

where \(\Omega \subset \mathbb{R}^N\) is an open bounded domain, \(N \geq 2\) and \(A(x, t, \xi)\) and \(B(x, t, \xi)\) are \(C^1\)-Carathéodory functions on \(\Omega \times \mathbb{R} \times \mathbb{R}^N\) with partial derivatives \(A_t = \frac{\partial A}{\partial t}\), \(a = \nabla_\xi A\), respectively \(B_t = \frac{\partial B}{\partial t}\), \(b = \nabla_\xi B\), while \(g_1(x, t, s)\) and \(g_2(x, t, s)\) are given Carathéodory maps defined on \(\Omega \times \mathbb{R} \times \mathbb{R}\) which are partial derivatives with respect to \(t\) and \(s\) of a function \(G(x, t, s)\).

We prove that, even if the general form of the terms \(A\) and \(B\) makes the variational approach more difficult, under suitable hypotheses, the functional related to the problem is bounded from below and attains its minimum in a “right” Banach space \(X\).

The proof, which exploits the interaction between two different norms, is based on a weak version of the Cerami–Palais–Smale condition and a suitable generalization of the Weierstrass Theorem.

1. Introduction. This paper aims at investigating the existence of solutions for the following quasilinear elliptic system with homogeneous Dirichlet boundary conditions

\[
\begin{cases}
-\text{div}(a(x, u, \nabla u)) + A_t(x, u, \nabla u) = g_1(x, u, v) & \text{in } \Omega, \\
-\text{div}(B(x, v, \nabla v)) + B_t(x, v, \nabla v) = g_2(x, u, v) & \text{in } \Omega, \\
u = v = 0 & \text{on } \partial \Omega,
\end{cases}
\]

2020 Mathematics Subject Classification. Primary: 35J20, 35J92; Secondary: 35J25, 58E05.

Key words and phrases. Quasilinear elliptic system, weak Cerami–Palais–Smale condition, Minimum Principle, “sublinear” growth.

The research that led to the present paper was partially supported by MIUR-PRIN project “Qualitative and quantitative aspects of non linear PDEs” (2017 JPCAPN-005), Fondi di Ricerca di Ateneo 2017/18 “Problemi differenziali non lineari”.

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For example, a family of model problems is given by:

\[
A_t(x,t,\xi) = \frac{\partial A}{\partial t}(x,t,\xi), \quad a(x,t,\xi) = \left( \frac{\partial A}{\partial \xi_1}(x,t,\xi), \ldots, \frac{\partial A}{\partial \xi_N}(x,t,\xi) \right),
\]

\[
B_t(x,t,\xi) = \frac{\partial B}{\partial t}(x,t,\xi), \quad b(x,t,\xi) = \left( \frac{\partial B}{\partial \xi_1}(x,t,\xi), \ldots, \frac{\partial B}{\partial \xi_N}(x,t,\xi) \right),
\]

and a function \( G : \Omega \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R} \) exists such that

\[
\frac{\partial G}{\partial t}(x,t,s) = g_1(x,t,s), \quad \text{for a.e. } x \in \Omega, \text{ for all } (t,s) \in \mathbb{R}^2.
\]

We note that the dependence of the functions \( A \) and \( B \) on solutions and on their gradient makes the variational approach more difficult.

In fact, also investigating the existence of solution for just an equation

\[
\left\{ \begin{aligned}
-\text{div}(a(x,u,\nabla u)) + A_t(x,u,\nabla u) &= g(x,u) & \text{in } \Omega, \\
u &= 0 & \text{on } \partial\Omega,
\end{aligned} \right.
\]

requires suitable approaches such as nonsmooth techniques or null Gâteaux derivative only along “good” directions or a suitable variational setting \([1, 4, 5, 9, 12, 14, 16, 17]\). For example, a family of model problems is given by:

\[
A(x,t,\xi) = \left( \sum_{i,j=1}^N a_{i,j}(x,t)\xi_i\xi_j \right)^{\frac{p_1}{2}} , \quad B(x,t,\xi) = \left( \sum_{i,j=1}^N b_{i,j}(x,t)\xi_i\xi_j \right)^{\frac{p_2}{2}},
\]

where \( p_1 > 1, p_2 > 1 \) and \((a_{i,j}(x,t))_{1 \leq i,j \leq N}, (b_{i,j}(x,t))_{1 \leq i,j \leq N}\) are two elliptic matrices.

In particular, if \( A, B : \Omega \times \mathbb{R} \rightarrow \mathbb{R} \) are given functions such that

\[
a_{i,j}(x,t) = \left( \frac{1}{p_1} A(x,t) \right)^{\frac{p_1}{2}} \delta_{i}^{j}, \quad b_{i,j}(x,t) = \left( \frac{1}{p_2} B(x,t) \right)^{\frac{p_2}{2}} \delta_{i}^{j},
\]

then

\[
A(x,t,\xi) = \frac{1}{p_1} A(x,t)\xi^p_1, \quad B(x,t,\xi) = \frac{1}{p_2} B(x,t)\xi^p_2,
\]

and the problem \((1.1)\) reduces to

\[
\left\{ \begin{aligned}
-\text{div}(A(x,u)|\nabla u|^{p_1-2}\nabla u) + \frac{1}{p_1} A_t(x,u)|\nabla u|^{p_1} &= g_1(x,u,v) & \text{in } \Omega, \\
-\text{div}(B(x,v)|\nabla v|^{p_2-2}\nabla v) + \frac{1}{p_2} B_t(x,v)|\nabla v|^{p_2} &= g_2(x,u,v) & \text{in } \Omega, \\
u = v &= 0 & \text{on } \partial\Omega,
\end{aligned} \right.
\]

which solutions are the critical points of the corresponding functional

\[
\mathcal{J}(u,v) = \frac{1}{p_1} \int_{\Omega} A(x,u)|\nabla u|^{p_1} dx + \frac{1}{p_2} \int_{\Omega} B(x,v)|\nabla v|^{p_2} dx - \int_{\Omega} G(x,u,v) dx,
\]

under suitable growth condition of nonlinear term \( G \) (see \([10]\)).
In the particular case $A(x,u) = A^*(x)$ and $B(x,v) = B^*(x)$, problem (1.4) becomes
\[
\begin{aligned}
-\text{div}(A^*(x)|\nabla u|^{p_1-2}\nabla u) &= g_1(x,u,v) \quad \text{in } \Omega, \\
-\text{div}(B^*(x)|\nabla v|^{p_2-2}\nabla v) &= g_2(x,u,v) \quad \text{in } \Omega, \\
u = v = 0 &\quad \text{on } \partial \Omega,
\end{aligned}
\]
with $A^*, B^* : \Omega \to \mathbb{R}$. If $A^*(x), B^*(x)$ are measurable, bounded and strictly positive functions, then (1.5) generalizes the classical $(p_1,p_2)$–Laplacian system
\[
\begin{aligned}
-\Delta_{p_1} u &= g_1(x,u,v) \quad \text{in } \Omega, \\
-\Delta_{p_2} v &= g_2(x,u,v) \quad \text{in } \Omega, \\
u = v = 0 &\quad \text{on } \partial \Omega.
\end{aligned}
\]
(1.6)

The interest in $(p_1,p_2)$–Laplacian systems, or their generalization such as (1.1), arises from the fact that they allow to model various physical phenomena. For example, they describe problems related to the equilibrium of anisotropic media which possibly are somewhere “perfect” insulators or “perfect” conductors. So that the couple $(p_1,p_2)$ represents the characteristic of the medium which involves a pseudoplastic fluid if $p_i < 2$, a dilatant fluid if $p_i > 2$ or a Newtonian fluid if $p_i = 2$. Moreover, problem (1.1) arises in the theory of quasiregular and quasiconformal mappings and is useful for investigating population dynamics or the spread of microorganisms.

As a consequence, many authors have studied problem (1.6) obtaining existence results under hypotheses of sublinear, superlinear or resonant type of the functions $g_1(x,u,v)$ and $g_2(x,u,v)$.

In particular, in [2] the authors found a solution of (1.6) as critical point of the related action functional
\[
\mathcal{I}(u,v) = \frac{1}{p_1} \int_{\Omega} |\nabla u|^{p_1} dx + \frac{1}{p_2} \int_{\Omega} |\nabla v|^{p_2} dx - \int_{\Omega} G(x,u,v) dx.
\]
We note that $\mathcal{I}$ is well defined on the space $W^{1,p_1}_0(\Omega) \times W^{1,p_2}_0(\Omega)$ if $G$ is a continuous function on $\mathbb{R} \times \mathbb{R}$ such that:
\[
|G(x,u,v)| \leq c(1 + |u|^{p_1^*} + |v|^{p_2^*}),
\]
where $p_1^*, p_2^*$ denote the critical exponents for the Sobolev Embeddings and $p_i \leq p_i^*$ for $i = 1,2$. Moreover, if more restrictive subcritical assumptions for $g_1$ and $g_2$ hold, functional $\mathcal{I}$ is of class $C^1$ and its critical points are weak solutions of (1.6).

On the other hand, the presence of coefficients $A(x,u)$ and $B(x,v)$ makes the variational approach more difficult. Indeed, even in the simplest case in which $G(x,u,v) \equiv 0$ and $A(x,u), B(x,v)$ are smooth, bounded and far away from zero, the corresponding functional for the problem (1.4)
\[
\frac{1}{p_1} \int_{\Omega} A(x,u)|\nabla u|^{p_1} dx + \frac{1}{p_2} \int_{\Omega} B(x,u)|\nabla v|^{p_2} dx
\]
is well defined in $W^{1,p_1}_0(\Omega) \times W^{1,p_2}_0(\Omega)$. But instead, if $\frac{\partial A}{\partial u}(x,t) \neq 0, \frac{\partial B}{\partial v} \neq 0$, it is Gâteaux differentiable only along directions of the Banach space $(W^{1,p_1}_0(\Omega) \cap L^{\infty}(\Omega)) \times (W^{1,p_2}_0(\Omega) \cap L^{\infty}(\Omega))$ (see [13] for the case of a single equation).
In this paper, we want to extend to our quasilinear system (1.1) the existence result stated in [2] for \((p_1, p_2)\)-Laplacian system (1.6) with a “sublinear like” growth for the terms \(g_1\) and \(g_2\).

To this aim, following the approach introduced in [4, 5] for quasilinear equations and extended in [10] to quasilinear systems, we look for solution of (1.1) as critical points of the functional:

\[
J(u, v) = \int_{\Omega} A(x, u, \nabla u) dx + \int_{\Omega} B(x, v, \nabla v) dx - \int_{\Omega} G(x, u, v) dx \quad (1.7)
\]

in the Banach space \(X_1 \times X_2\), where \(X_i = W^{1,p_i}_0(\Omega) \cap L^\infty(\Omega)\), for \(i = 1, 2\).

We note that functional \(J\) does not satisfy the Palais–Smale condition or its Cerami’s variant since, even in the case of a single equation, a Palais–Smale sequence, converging in the \(W^{1,p}_0(\Omega)\) norm but unbounded in \(L^\infty(\Omega)\), can be found (see, for example, [7, Example 4.3]). Therefore, by exploiting the interaction between two different norms on \(X\), we introduce the weak Cerami–Palais–Smale condition (see Definition 2.7) and apply a suitable generalized version of Weierstrass Theorem in order to prove the existence of at least one solution (see Theorems 2.3 and 2.5).

2. Main theorem and variational principle. From now on, let \(\Omega \subset \mathbb{R}^N\) be an open bounded domain, \(N \geq 2\) and consider \(A, B : \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}\) two functions with partial derivatives \(A_i(x, t, \xi), a(x, t, \xi), B_i(x, t, \xi), b(x, t, \xi)\), according to the notation (1.2) and a function \(G : \Omega \times \mathbb{R} \to \mathbb{R}\) with partial derivatives \(g_1(x, t, s), g_2(x, t, s)\) as in (1.3).

Assume that two real numbers \(p_1, p_2 > 1\) and a radius \(R \geq 1\) exist so that the following assumptions are held:

\((H_0)\) \(A\) and \(B\) are \(C^1\)-Carathéodory functions, i.e.,

\[
\begin{align*}
A(\cdot, t, \xi) : x \in \Omega &\mapsto A(x, t, \xi) \in \mathbb{R} \text{ is measurable for all } (t, \xi) \in \mathbb{R} \times \mathbb{R}^N, \\
A(x, \cdot, \cdot) : (t, \xi) \in \mathbb{R} \times \mathbb{R}^N &\mapsto A(x, t, \xi) \in C^1 \text{ for a.e. } x \in \Omega; \\
B(\cdot, t, \xi) : x \in \Omega &\mapsto B(x, t, \xi) \in \mathbb{R} \text{ is measurable for all } (t, \xi) \in \mathbb{R} \times \mathbb{R}^N, \\
B(x, \cdot, \cdot) : (t, \xi) \in \mathbb{R} \times \mathbb{R}^N &\mapsto B(x, t, \xi) \in C^1 \text{ for a.e. } x \in \Omega;
\end{align*}
\]

\((H_1)\) some positive continuous functions \(\Phi_i, \phi_i : \mathbb{R} \to \mathbb{R}, i \in \{1, 2\}\) exist such that:

\[
\begin{align*}
|A_i(x, t, \xi)| &\leq \Phi_1(t) + \phi_1(t)|\xi|^{p_1} \quad \text{a.e. } x \in \Omega, \text{ for all } (t, \xi) \in \mathbb{R} \times \mathbb{R}^N, \\
|a(x, t, \xi)| &\leq \Phi_2(t) + \phi_2(t)|\xi|^{p_1-1} \quad \text{a.e. } x \in \Omega, \text{ for all } (t, \xi) \in \mathbb{R} \times \mathbb{R}^N, \\
|B_i(x, t, \xi)| &\leq \Phi_1(t) + \phi_1(t)|\xi|^{p_2} \quad \text{a.e. } x \in \Omega, \text{ for all } (t, \xi) \in \mathbb{R} \times \mathbb{R}^N, \\
|b(x, t, \xi)| &\leq \Phi_2(t) + \phi_2(t)|\xi|^{p_2-1} \quad \text{a.e. } x \in \Omega, \text{ for all } (t, \xi) \in \mathbb{R} \times \mathbb{R}^N;
\end{align*}
\]

\((H_2)\) some constants \(\mu_0\) and \(\sigma_1, \sigma_2 \geq 0\) exist such that

\[
\begin{align*}
A(x, t, \xi) &\geq \mu_0 (1 + |t|^{p_1\sigma_1}) |\xi|^{p_1} \quad \text{a.e. } x \in \Omega, \text{ for all } |(t, \xi)| \geq R, \\
B(x, t, \xi) &\geq \mu_0 (1 + |s|^{p_2\sigma_2}) |\xi|^{p_2} \quad \text{a.e. } x \in \Omega, \text{ for all } |(t, \xi)| \geq R;
\end{align*}
\]

\((H_3)\) there exists \(\eta_1 > 0\) such that

\[
\begin{align*}
|A(x, t, \xi)| &\leq \eta_1 \quad \text{a.e. } x \in \Omega, \text{ if } |(t, \xi)| \leq R, \\
|B(x, t, \xi)| &\leq \eta_1 \quad \text{a.e. } x \in \Omega, \text{ if } |(t, \xi)| \leq R;
\end{align*}
\]
(H₁) there exists $\eta_2 > 0$ such that
\[
A(x, t, \xi) \leq \eta_2 a(x, t, \xi) \cdot \xi \quad \text{a.e. } x \in \Omega, \text{ if } |(t, \xi)| \geq R,
\]
\[
B(x, t, \xi) \leq \eta_2 b(x, t, \xi) \cdot \xi \quad \text{a.e. } x \in \Omega, \text{ if } |(t, \xi)| \geq R;
\]

(\textit{H₅}) there exists $\mu_1 > 0$ such that
\[
a(x, t, \xi) \cdot \xi + A_1(x, t, \xi) t \geq \mu_1 a(x, t, \xi) \cdot \xi \quad \text{a.e. } x \in \Omega, \text{ if } |(t, \xi)| \geq R,
\]
\[
b(x, t, \xi) \cdot \xi + B_1(x, t, \xi) t \geq \mu_1 b(x, t, \xi) \cdot \xi \quad \text{a.e. } x \in \Omega, \text{ if } |(t, \xi)| \geq R;
\]

(\textit{H₆}) there exists $\mu_2 > 0$ such that
\[
a(x, t, \xi) \cdot \xi \geq \mu_2 \xi^1 \quad \text{a.e. } x \in \Omega, \text{ if } |(t, \xi)| \leq R,
\]
\[
b(x, t, \xi) \cdot \xi \geq \mu_2 \xi^2 \quad \text{a.e. } x \in \Omega, \text{ if } |(t, \xi)| \leq R;
\]

(\textit{H₇}) for all $\xi, \xi^* \in \mathbb{R}^N, \xi \neq \xi^*$, it is
\[
\begin{align*}
[a(x, t, \xi) - a(x, t, \xi^*)] : [\xi - \xi^*] > 0 & \quad \text{a.e. } x \in \Omega, \text{ for all } t \in \mathbb{R},
[b(x, t, \xi) - b(x, t, \xi^*)] : [\xi - \xi^*] > 0 & \quad \text{a.e. } x \in \Omega, \text{ for all } t \in \mathbb{R};
\end{align*}
\]

(\textit{G₀}) $G$ is a $C^1$-Carathéodory function such that $G(\cdot, 0, 0) \in L^\infty(\Omega)$;

(\textit{G₁}) some real numbers $q_i \geq 1$, $s_i \geq 0$ if $i \in \{1, 2\}$, and $c > 0$ exist such that
\[
\begin{align*}
g_1(x, u, v) & \leq c (1 + |u|^{q_1} + |v|^{s_1}) \quad \text{a.e. } x \in \Omega, \text{ for all } (u, v) \in \mathbb{R}^2, \\
g_2(x, u, v) & \leq c (1 + |u|^{s_2} + |v|^{q_2}) \quad \text{a.e. } x \in \Omega.
\end{align*}
\]

(\textit{G₂})
\[
g_1(x, 0, 0) = g_2(x, 0, 0) = 0 \quad \text{a.e. } x \in \Omega.
\]

**Remark 2.1.** From conditions (\textit{H₂}) and (\textit{H₃}) a constant $\eta_3 > 0$ exists such that
\[
A(x, t, \xi) \geq \mu_0 (1 + |\xi|^p)^{p_1} \xi^1 - \eta_3
\]
\[
B(x, t, \xi) \geq \mu_0 (1 + |\xi|^p)^{p_2} \xi^2 - \eta_3
\]
a.e. $x \in \Omega$, for all $(t, \xi) \in \mathbb{R} \times \mathbb{R}^N$.

On the other hand, from (\textit{H₁}), (\textit{H₃}) and (\textit{H₄}) direct computations imply
\[
A(x, t, \xi) \leq \eta_1 + \eta_2 \Phi_2(t) + \eta_2 (\Phi_2(t) + \phi_2(t)) |\xi|^{p_1}
\]
\[
B(x, t, \xi) \leq \eta_1 + \eta_2 \Phi_2(t) + \eta_2 (\Phi_2(t) + \phi_2(t)) |\xi|^{p_2}
\]
a.e. $x \in \Omega$, for all $(t, \xi) \in \mathbb{R} \times \mathbb{R}^N$.

**Remark 2.2.** From conditions (\textit{G₀}) - (\textit{G₁}), the Main Value Theorem and standard computations we have that:
\[
|G(x, u, v)| \leq c_1 (1 + |u| + |u|^{q_1} + |u||v|^{s_1} + |v| + |u|^{s_2} |v| + |v|^{q_2}) \quad (2.2)
\]
a.e. $x \in \Omega$, for all $(u, v) \in \mathbb{R}^2$.

In fact, from (\textit{G₀}) - (\textit{G₁}) and the Main Value Theorem, for a.e. $x \in \Omega$ and for any $(u, v) \in \mathbb{R}^2$ there exists $\theta \in [0, 1]$ such that
\[
|G(x, u, v)| \leq |G(x, u, v) - G(x, 0, 0)| + |G(x, 0, 0)|
\]
\[
= |g_1(x, \theta u, \theta v) u + g_2(x, \theta u, \theta v) v| + |G(x, 0, 0)|
\]
\[
\leq c (1 + |u|^{q_1-1} + |v|^{s_1}) |u| + c (1 + |u|^{s_2} + |v|^{q_2-1}) |v| + |G(\cdot, 0, 0)|_{\infty}
\]
and so (2.2) follows with $c_1 = \max \{c, |G(\cdot, 0, 0)|_{\infty}\}$.

The following existence result holds:
Theorem 2.3. Assume that \((H_0) - (H_7), (G_0) - (G_1)\) hold. If
\[
1 < q_1 < p_1(1 + \sigma_1), \quad 1 < q_2 < p_2(1 + \sigma_2),
\]
\[
0 \leq s_1 < \min \left\{ p_2(1 + \sigma_2) \left( 1 - \frac{1}{p_1(1 + \sigma_1)} \right), \frac{p_1p_2^*}{N} \right\} \left( 1 - \frac{1}{p_1(1 + \sigma_1)} \right),
\]
\[
0 \leq s_2 < \min \left\{ p_1(1 + \sigma_1) \left( 1 - \frac{1}{p_2(1 + \sigma_2)} \right), \frac{p_1p_2^*}{N} \right\} \left( 1 - \frac{1}{p_2(1 + \sigma_2)} \right),
\]
then system (1.1) has at least a weak bounded solution \((u, v)\).

Remark 2.4. From direct computations the hypothesis (2.4) becomes:
\[
0 \leq s_1 < p_2(1 + \sigma_2) \left( 1 - \frac{1}{p_1(1 + \sigma_1)} \right),
\]
\[
0 \leq s_2 < p_1(1 + \sigma_1) \left( 1 - \frac{1}{p_2(1 + \sigma_2)} \right),
\]
if and only if \(N \leq p_1 + p_2\).

In particular, if moreover it is \(\sigma_1 = \sigma_2 = \sigma\), than the condition (2.4) reduces to:
\[
0 \leq s_1 < \left( 1 + \sigma - \frac{1}{p_1} \right) p_2, \quad 0 \leq s_2 < \left( 1 + \sigma - \frac{1}{p_2} \right) p_1.
\]

We note that if the additional assumption \((G_2)\) holds, then \((u, v) \equiv (0, 0)\) is a trivial solution of (1.1).

The next result states the existence of a non-trivial solution.

Theorem 2.5. Assume that \((H_0) - (H_2), (H_4) - (H_7), (G_0) - (G_2)\) and (2.3), (2.4) hold. Moreover, suppose that a constant \(\eta_4\) exists such that
\(\eta_4 > 0\) such that
\[
|A(x, t, \xi)| \leq \eta_4|\xi|^{p_1},
\]
\[
|B(x, t, \xi)| \leq \eta_4|\xi|^{p_2},
\]

a.e. \(x \in \Omega\), for all \((t, \xi) \in \mathbb{R} \times \mathbb{R}^N\) with \(|t| \leq \delta\).

\(\eta_4 = \max \lambda_{i, 1}\) uniformly a.e. in \(\Omega\),

where for \(i \in \{1, 2\}\) \(\lambda_{i, 1}\) is the first eigenvalue of \(-\Delta_{p_i}\) in \(W^{1, p_i}_0(\Omega)\).

Then, system (1.1) has at least a weak bounded non-trivial solution.

Remark 2.6. It is known that for \(i \in \{1, 2\}\) the first eigenvalues \(\lambda_{i, 1}\) of \(-\Delta_{p_i}\) in \(W^{1, p_i}_0(\Omega)\), is characterized as
\[
\lambda_{i, 1} = \inf_{\xi \in \mathbb{C}^1} \frac{\int_\Omega |\nabla \xi|^{p_i} dx}{\int_\Omega |\xi|^{p_i} dx},
\]
and is strictly positive, simple, isolated and has a unique eigenfunction \(\varphi_{i, 1}\) such that
\[
\varphi_{i, 1} > 0 \text{ a.e. } \Omega, \quad \varphi_{i, 1} \in L^\infty(\Omega) \quad \text{and} \quad |\varphi_{i, 1}|_{p_i} = 1. \quad (2.5)
\]

The proofs of Theorems 2.3 and 2.5 are based on a suitable generalization of the Weierstrass Theorem.

We denote by \((X, \| \cdot \|_X)\) a Banach space with dual \((X', \| \cdot \|_{X'})\) and also by \((W, \| \cdot \|_W)\) another Banach space such that \(X \hookrightarrow W\) continuously and finally by \(J\) a given \(C^1\) functional on \(X\).
Then, we introduce a suitable weaker version of the Cerami’s variant of Palais–Smale condition.

Taking $\beta \in \mathbb{R}$, we say that a sequence $(\xi_n)_n \subset X$ is a Cerami–Palais–Smale sequence at level $\beta$, briefly (CPS)$_\beta$ sequence, if

$$\lim_{n \to +\infty} J(\xi_n) = \beta \quad \text{and} \quad \lim_{n \to +\infty} dJ(\xi_n)_{X'}(1 + \|\xi_n\|_X) = 0.$$ 

Moreover, $\beta$ is a Cerami–Palais–Smale level, briefly (CPS)–level, if there exists a (CPS)$_\beta$–sequence.

As (CPS)$_\beta$–sequences may exist which are unbounded in $\|\cdot\|_X$ but converge with respect to $\|\cdot\|_W$, we have to weaken the classical Cerami–Palais–Smale condition in a suitable way according to the ideas already developed in previous papers (see, e.g., [4, 5, 6]).

**Definition 2.7.** The functional $J$ satisfies the weak Cerami–Palais–Smale condition at level $\beta$ ($\beta \in \mathbb{R}$), briefly (wCPS)$_\beta$ condition, if for every (wCPS)$_\beta$ sequence $(\xi_n)_n \in X$ a point $\xi \in X$ exists, such that

(i) $\lim_{n \to +\infty} \|\xi_n - \xi\|_W = 0$ (up to subsequences),

(ii) $J(\xi) = \beta$, $dJ(\xi) = 0$.

If $J$ satisfies the (wCPS)$_\beta$ condition at each level $\beta \in I$, $I$ real interval, we say that $J$ satisfies the (wCPS) condition in $I$.

Condition (wCPS)$_\beta$ implies that the set of critical point of $J$ at level $\beta$ is compact with respect to $\|\cdot\|_W$, hence a Deformation Lemma and some abstract critical point Theorems can be stated (see [6]). In particular, the following Minimum Principle holds (for the proof, see [6, Theorem 1.6]).

**Theorem 2.8.** (Minimum Principle) If $J \in C^1(X, \mathbb{R})$ is bounded from below in $X$ and (wCPS)$_\beta$ holds at level $\beta = \inf_X J \in \mathbb{R}$, then $J$ attains its infimum, i.e. $\xi_0 \in X$ exists such that $J(\xi_0) = \beta$.

3. **Variational setting and first properties.** From now on, we denote by:

- $\text{meas}(D)$ the usual Lebesgue measure of a measurable set $D$ in $\mathbb{R}^N$;
- $L^r(\Omega)$ the Lebesgue space with norm $\|\xi\|_r = (\int_\Omega |\xi|^r \, dx)^{1/r}$ if $1 \leq r < +\infty$;
- $L^\infty(\Omega)$ the space of Lebesgue–measurable and essentially bounded functions $\xi : \Omega \to \mathbb{R}$ with norm $\|\xi\|_\infty = \text{ess sup}_\Omega |\xi|$;
- $W^{1,p}_0(\Omega)$ the classical Sobolev Space with norm $\|\xi\|_{W^{1,p}_0} = |\nabla \xi|_p$, where $1 \leq p < +\infty$.

For simplicity, here and in the following we denote by $|\cdot|$ the standard norm on any Euclidean space, as the dimension of the considered vector is clear and no ambiguity occurs, and by $C$ any strictly positive constant which arises by computation.

In order to look for weak solutions of the nonlinear problem (1.1), consider $p_1$, $p_2 > 1$ and, for $i \in \{1, 2\}$, the related Sobolev space

$$W_i = W^{1,p_i}_0(\Omega) \quad \text{with norm} \quad \|\cdot\|_{W_i} = \|\cdot\|_{W^{1,p_i}_0}.$$ 

From the Sobolev Embedding Theorem, for any $r \in [1, p_i^*]$ with $p_i^* = \frac{N p_i}{N - p_i}$ if $N > p_i$, or any $r \in [1, +\infty]$ if $p_i \geq N$, $W_i$ is continuously embedded in $L^r(\Omega)$, i.e. a positive constant $\tau_{i,r}$ exists such that

$$|\xi|_r \leq \tau_{i,r} \|\xi\|_{W_i}, \quad \text{for all} \ \xi \in W_i.$$
For simplicity, we put
\[ p_i^* = +\infty \quad \text{and} \quad \frac{1}{p_i^*} = 0 \quad \text{if} \ p_i \geq N. \]

Here, the notation introduced for the abstract setting in Section 2 is referred to our problem with
\[ W = W_1 \times W_2 \]
while the Banach space \((X, \| \cdot \|_X)\) is defined as
\[ X = X_1 \times X_2, \]
where
\[ X_1 := W_1 \cap L^\infty(\Omega), \quad X_2 := W_2 \cap L^\infty(\Omega) \]
with the norms
\[ \|u\|_{X_1} = \|u\|_{W_1} + |u|_\infty \quad \text{if} \ u \in X_1 \quad \text{and} \quad \|v\|_{X_2} = \|v\|_{W_2} + |v|_\infty \quad \text{if} \ v \in X_2. \]

Since \((W_i, \| \cdot \|_{W_i})\) is a reflexive Banach space for both \(i = 1\) and \(i = 2\), so it is \((W_i, \| \cdot \|_{W_i})\), where for \((u, v) \in W\) it is
\[ \|(u, v)\|_W = \|u\|_{W_1} + \|v\|_{W_2}. \]

Setting
\[ L := L^\infty(\Omega) \times L^\infty(\Omega) \quad \text{with} \quad \|(u, v)\|_L = |u|_\infty + |v|_\infty, \]
we have that \(X\) in (3.1) can also be written as
\[ X = W \cap L \]
and can be equipped with the norm
\[ \|(u, v)\|_X = \|(u, v)\|_W + \|(u, v)\|_L = \|u\|_{X_1} + \|v\|_{X_2}. \]

By definition, for \(i \in \{1, 2\}\) we have \(X_i \hookrightarrow W_i\) and \(X_i \hookrightarrow L^\infty(\Omega)\) with continuous embeddings.

**Remark 3.1.** If \(i \in \{1, 2\}\) is such that \(p_i > N\), then \(X_i = W_i\), as \(W_i \hookrightarrow L^\infty(\Omega)\). Hence, if both \(p_1 > N\) and \(p_2 > N\), then \(X = W_1 \times W_2\) and all the arguments and the proofs can be simplified and the classical Weierstrass Theorem (see, e.g. [3]) can be used, if required. Hence, from now, we assume \(1 < p_1 \leq N\) and \(1 < p_2 \leq N\).

Now, we note that if conditions \((H_0), \ (H_1), \ (G_0), \ (G_1)\) hold, the functional \(J(u, v)\) in (1.7) is well defined for all \((u, v) \in X\). Moreover, taking any \((u, v), (w, z) \in X\), the Gâteaux differential of functional \(J\) in \((u, v)\) along the direction \((w, z)\) is
\[ dJ(u, v)[(w, z)] = \int_\Omega a(x, u, \nabla u) \cdot \nabla w \ dx + \int_\Omega A_t(x, u, \nabla u) w dx \]
\[ + \int_\Omega B(x, v, \nabla v) \cdot \nabla z \ dx + \int_\Omega B_t(x, v, \nabla v) z dx \]
\[ - \int_\Omega g_1(x, u, v) w \ dx - \int_\Omega g_2(x, u, v) z \ dx. \quad (3.2) \]

For simplicity, we put
\[ \frac{\partial J}{\partial u}(u, v) : w \in X_1 \mapsto \frac{\partial J}{\partial u}(u, v)[w] = dJ(u, v)[(w, 0)] \in \mathbb{R}, \]
\[ \frac{\partial J}{\partial v}(u, v) : z \in X_2 \mapsto \frac{\partial J}{\partial v}(u, v)[z] = dJ(u, v)[(0, z)] \in \mathbb{R}. \]
So, from (3.2) it follows that
\[
\frac{\partial J}{\partial u}(u, v)[w] = \int_{\Omega} a(x, u, \nabla u) \cdot \nabla w \, dx + \int_{\Omega} A_t(x, u, \nabla u)wdx - \int_{\Omega} g_1(x, u, v)w \, dx,
\]
\[
\frac{\partial J}{\partial v}(u, v)[z] = \int_{\Omega} b(x, v, \nabla v) \cdot \nabla z dx + \int_{\Omega} B_t(x, v, \nabla v)zdx - \int_{\Omega} g_2(x, u, v)zdx.
\]
(3.3)

Remark 3.2. Taking \((u, v) \in X\), since \(dJ(u, v) \in X'\), then
\[
\frac{\partial J}{\partial u}(u, v) \in X_1', \quad \frac{\partial J}{\partial v}(u, v) \in X_2'.
\]
and
\[
dJ(u, v)(w, z) = \frac{\partial J}{\partial u}(u, v)[w] + \frac{\partial J}{\partial v}(u, v)[z] \quad \forall (w, z) \in X.
\]
Moreover, direct computations imply that
\[
\left\| \frac{\partial J}{\partial u}(u, v) \right\|_{X_1'} \leq \left\| dJ(u, v) \right\|_{X'}, \quad \left\| \frac{\partial J}{\partial v}(u, v) \right\|_{X_2'} \leq \left\| dJ(u, v) \right\|_{X'},
\]
and
\[
\left\| dJ(u, v) \right\|_{X'} \leq \left\| \frac{\partial J}{\partial u}(u, v) \right\|_{X_1'} + \left\| \frac{\partial J}{\partial v}(u, v) \right\|_{X_2'}.
\]
Clearly, we have
\[
dJ(u, v) = 0 \text{ in } X \iff \frac{\partial J}{\partial u}(u, v) = 0 \text{ in } X_1 \text{ and } \frac{\partial J}{\partial v}(u, v) = 0 \text{ in } X_2.
\]

The following regularity result holds:

**Proposition 3.3.** Assume that conditions \((H_0), (H_1), (G_0), (G_1)\) hold.
Let \(((u_n, v_n))_n \subset X\) and \((u, v) \in X\) be such that
\[
u_n \to u \text{ in } W^1_1, \quad u_n \to u \text{ a.e. in } \Omega \text{ if } n \to \infty,
\]
\[
v_n \to v \text{ in } W^2_2, \quad v_n \to v \text{ a.e. in } \Omega \text{ if } n \to \infty,
\]
and, moreover, \(M > 0\) exists such that
\[
|u_n|_\infty \leq M \quad \text{and} \quad |v_n|_\infty \leq M \quad \forall n \in \mathbb{N}.
\]
(3.6)
Then,
\[
J(u_n, v_n) \to J(u, v) \quad \text{and} \quad \|dJ(u_n, v_n) - dJ(u, v)\|_{X'} \to 0 \quad \text{as } n \to \infty.
\]
Hence, \(J\) is a \(C^1\) functional on \(X\), with Fréchet differential defined as in (3.2).

**Proof.** First of all, consider the functionals \(J_1, J_2\) defined as:
\[
J_1(u) = \int_{\Omega} A(x, u, \nabla u)dx, \quad u \in X_1
\]
\[
J_2(v) = \int_{\Omega} B(x, v, \nabla v)dx, \quad v \in X_2.
\]
From assumptions \((H_0), (H_1)\) and Remark 2.1, arguing as in the Proof of [5, Proposition 3.1] it follows that \(J_1\) and \(J_2\) are of class \(C^1\) with Fréchet differential defined as
\[
d J_1(u)[w] = \int_\Omega a(x, u, \nabla u) \cdot \nabla w \, dx + \int_\Omega A_t(x, u, \nabla u) \, dx, \quad u, w \in X_1
\]
and
\[
d J_2(v)[z] = \int_\Omega b(x, v, \nabla v) \cdot \nabla z \, dx + \int_\Omega B_t(x, v, \nabla v) \, dx, \quad v, z \in X_2.
\]

On the other hand, reasoning as in [10, Proposition 3.7], \((G_0), (G_1), (3.4) – (3.6)\) imply that the functional
\[
J_3(u, v) = \int_\Omega G(x, u, v) \, dx, \quad (u, v) \in X
\]
is of class \(C^1\) with Fréchet differential defined as
\[
d J_3(u, v)[(w, z)] = \int_\Omega g_1(x, u, v) \, dx + \int_\Omega g_2(x, u, v) \, dx, \quad (w, z) \in X.
\]
Hence, we conclude that \(J = J_1 + J_2 + J_3\) is \(C^1\) on \(X\) with Fréchet differential as in (3.2).

We point out that up to now no sub-critical growth is required for \(g_1(x, u, v)\) and \(g_2(x, u, v)\) since in our setting it is \(X_i \hookrightarrow L^r(\Omega) \quad \forall r \geq 1 \text{ and } i \in \{1, 2\}\).

Anyway, subcritical growth assumptions need for proving in general the weak Cerami–Palais–Smale at any level, while further growth assumptions on \(A, B\) and \(G\) allows us to prove that \(J\) is bounded from below. To this aim, we will use the following result.

**Proposition 3.4.** Assume that condition \((G_0), (G_1)\) hold. Then, there exist \(\overline{q}_1, \overline{q}_2\) such that
\[
|G(x, u, v)| \leq c(1 + |u|^{\overline{q}_1} + |v|^{\overline{q}_2}). \tag{3.7}
\]
Moreover, if we assume \((2.3)\) and
\[
0 \leq s_1 < p_2(1 + \sigma_2) \left(1 - \frac{1}{p_1(1 + \sigma_1)}\right),
\]
\[
0 \leq s_2 < p_1(1 + \sigma_1) \left(1 - \frac{1}{p_2(1 + \sigma_2)}\right), \tag{3.8}
\]
it follows that
\[
1 < \overline{q}_1 < p_1(1 + \sigma_1), \quad 1 < \overline{q}_2 < p_2(1 + \sigma_2). \tag{3.9}
\]

**Proof.** Firstly we recall that \(G\) verifies (2.2). Now, the Young inequality implies that for any \(s_3 > 1\) it is
\[
|u||v|^{s_3} \leq \frac{1}{s_3} |u|^{s_3} + \left(1 - \frac{1}{s_3}\right) |v|^{s_1 + \frac{s_3}{s_3 - s_1}}. \tag{3.10}
\]
We note that the constant \(s_3\) can be choosen such that
\[
1 < s_3 < p_1(1 + \sigma_1), \quad 0 \leq s_4 = s_1 - \frac{s_3}{s_3 - 1} < p_2(1 + \sigma_2) \tag{3.11}
\]
or equivalently
\[
\frac{p_2(1 + \sigma_2)}{p_2(1 + \sigma_2) - s_1} < s_3 < p_1(1 + \sigma_1)
\]
if and only if $0 \leq s_1 < p_2(1 + \sigma_2) \left(1 - \frac{1}{p_1(1 + \sigma_1)}\right)$.

Arguing in a similar way, we have that

$$|u|^{s_2} \leq \left(1 - \frac{1}{s_5}\right)|u|^{s_2} + \frac{1}{s_5} |v|^{s_5}$$

(3.12)

where we can choose the constant $s_5$ such that:

$$0 \leq s_6 = s_2 \frac{s_5}{s_5-1} < p_1(1 + \sigma_1), \quad 1 < s_5 < p_2(1 + \sigma_2),$$

or equivalently

$$0 \leq \frac{p_1(1 + \sigma_1)}{p_1(1 + \sigma_1) - s_2} < s_5 < p_2(1 + \sigma_2)$$

if and only if $0 \leq s_2 < p_1(1 + \sigma_1) \left(1 - \frac{1}{p_2(1 + \sigma_2)}\right)$.

Summing up, from (2.2), (3.10) and (3.12), estimate (3.7) follows with $\overline{q}_1$, $\overline{q}_2$ given by

$$\overline{q}_1 = \max \left\{ q_1, s_3, s_2 \frac{s_5}{s_5-1} \right\} \quad \text{and} \quad \overline{q}_2 = \max \left\{ q_2, s_5, s_1 \frac{s_3}{s_3-1} \right\}.$$  (3.14)

Moreover, if (2.3) and (3.8) hold, then (3.11) and (3.13) are verified, hence $\overline{q}_1, \overline{q}_2$ verify (3.9).

**Remark 3.5.** We point out that $(G_0)$, $(G_1)$ imply only that $G$ has a polynomial growth, as in (3.7). Under the additional assumptions (2.3) and (3.8), (3.9) holds, i.e. $G$ has a “little more” than “subquadratic growth”. In particular, if we take $\sigma_1 = \sigma_2 = 0$ in $(H_2)$, then $G$ has a “subquadratic like growth”, assuming that

$$\left\{ \begin{array}{l} 1 < q_1 < p_1, \quad 1 < q_2 < p_2, \\ 0 \leq s_1 < p_2 \left(1 - \frac{1}{p_1}\right), \quad 0 \leq s_2 < p_1 \left(1 - \frac{1}{p_2}\right). \end{array} \right.$$  (4.1)

**Remark 3.6.** If $p_1 = p_2 = p$, condition (2.3) and (3.8) reduce to:

$$1 < q_1 < p(1 + \sigma_1), \quad 1 < q_2 < p(1 + \sigma_2),$$

$$0 \leq s_1 < p(1 + \sigma_2) - \frac{1 + \sigma_2}{1 + \sigma_1}, \quad 0 \leq s_2 < p(1 + \sigma_1) - \frac{1 + \sigma_1}{1 + \sigma_2}.$$  

In particular, if $p_1 = p_2 = 2$ and $\sigma_1 = \sigma_2 = 0$, the classical subquadratic growth for $G$ is obtained.

4. **Proof of the main theorem.** In this section we will apply Theorem 2.8 (Minimum Principle) to our functional $\mathcal{F} : X \to \mathbb{R}$ defined as in (1.7). To this aim, we will use the following results.

**Remark 4.1.** Taking $(u, v) \in X$, it results

$$|u|^{q_1} u, |v|^{q_2} v \in W_0^{1, p_1}(\Omega) \times W_0^{1, p_2}(\Omega)$$

for a.e. $x \in \Omega$. Indeed, it results

$$|\nabla (|u|^{p_1} u)|^{p_1} = (1 + \sigma_1)^{p_1} |u|^{p_1} \sigma_1 |\nabla u|^{p_1},$$

$$|\nabla (|v|^{p_2} v)|^{p_2} = (1 + \sigma_2)^{p_2} |u|^{p_2} \sigma_2 |\nabla v|^{p_2}.$$  (4.1)
Lemma 4.2. Assume that conditions \((H_0) - (H_5), (G_0), (G_1), (2.3)\) and \((3.8)\) hold.

Then, some positive constants \(c_i\) exist such that
\[
J(u, v) \geq \mu_0 ||u||_{W_1^{\sigma_1}}^2 + \frac{\mu_0}{(1 + \sigma_1)^{p_i}} |||u||_{W_1^{\sigma_1}}^2 - c_1 ||u||_{W_1^{\sigma_1}}^2 v_{\frac{\tau_i}{p_i}} \\
+ \mu_0 ||v||_{W_2^{\sigma_2}}^2 + \frac{\mu_0}{(1 + \sigma_2)^{p_2}} ||v||_{W_2^{\sigma_2}}^2 - c_2 ||v||_{W_2^{\sigma_2}}^2 - c_3
\]  

(4.2)

Hence, it follows that
\[
\inf_X J(u, v) > -\infty.
\]  

(4.3)

Proof. Fixing any \((u, v) \in X\), from Remark 2.1, (3.7) and (4.1) it follows that
\[
J(u, v) \geq \mu_0 \int_\Omega |\nabla u|^{p_i} dx + \frac{\mu_0}{(1 + \sigma_1)^{p_i}} \int_\Omega |\nabla(|u|^\sigma_1 u)|^{p_i} dx \\
- c \int_\Omega ||u||_{W_1^{\sigma_1}}^2 v_{\frac{\tau_i}{p_i}} dx + \mu_0 \int_\Omega |\nabla v|^{p_2} dx \\
+ \frac{\mu_0}{(1 + \sigma_2)^{p_2}} \int_\Omega |\nabla(|v|^\sigma_2 v)|^{p_2} dx - c \int_\Omega ||v||_{W_1^{\sigma_2}}^2 v_{\frac{\tau_2}{p_2}} dx - 2n_\text{meas}(\Omega),
\]

then (4.2) follows from (2.3), (3.8) and the Sobolev Embedding Theorem.

Clearly, (4.2) implies
\[
J(u, v) \geq \frac{\mu_0}{(1 + \sigma_1)^{p_i}} ||u||_{W_1^{\sigma_1}}^2 - c_1 ||u||_{W_1^{\sigma_1}}^2 v_{\frac{\tau_i}{p_i}} \\
+ \frac{\mu_0}{(1 + \sigma_2)^{p_2}} ||v||_{W_2^{\sigma_2}}^2 - c_2 ||v||_{W_2^{\sigma_2}}^2 - c_3.
\]

Then (4.3) follows from (3.9). \(\square\)

Remark 4.3. From (4.2), (2.3), (3.8) and direct computations, we obtain
\[
J(u, v) \geq \mu_0 ||u||_{W_1^{\sigma_1}}^2 + c_4 ||u||_{W_1^{\sigma_1}}^{p_i} + \mu_0 ||v||_{W_2^{\sigma_2}}^2 + c_5 ||v||_{W_2^{\sigma_2}}^{p_2} - c_6
\]

for suitable constants \(c_4, c_5, c_6 > 0\).

In order to prove that functional \(J\) satisfies the \((wCPS)\) condition in \(\mathbb{R}\), we need the following results.

Lemma 4.4. Fix \(p > 1\) and \(\sigma \geq 0\) and let \((\xi_n)_n \subset W_0^{1,p}(\Omega) \cap L^\infty(\Omega)\) be a sequence such that
\[
\left( \int_\Omega (1 + |\xi_n|^{p^*}) |\nabla \xi_n| dx \right)_n \text{ is bounded.}
\]

Then, \(\xi \in W_0^{1,p}(\Omega)\) exists such that \(|\xi|^{\sigma} \xi \in W_0^{1,p}(\Omega)\), too, and up to subsequences, if \(n \to \infty\), we have
\[
\xi_n \to \xi \text{ weakly in } W_0^{1,p}(\Omega), \quad |\xi_n|^{\sigma} \xi_n \to |\xi|^{\sigma} \xi \text{ in } W_0^{1,p}(\Omega), \\
\xi_n \to \xi \text{ a.e. in } \Omega, \\
\xi_n \to \xi \text{ strongly in } L^r(\Omega) \text{ for each } r \in [1, p^*(1 + \sigma)].
\]

Proof. For the proof, see [8, Lemma 3.8] \(\square\)
Lemma 4.5. Let $\Omega$ be an open bounded subset of $\mathbb{R}^N$ and consider $\xi \in W_0^{1,p}(\Omega)$ with $p \leq N$. Suppose that $\gamma > 0$ and $k_0 \in \mathbb{N}$ exist such that
\[
\int_{\Omega_k^+} |\nabla \xi|^p \, dx \leq \gamma \left( \int_{\Omega_k^+} (\xi - k)^r \, dx \right)^{\frac{p}{r}} + \gamma \sum_{j=1}^{m} k^\alpha_j \{\text{meas}(\Omega_k^+)^1\}^{-\frac{1}{p} + \varepsilon_j} \quad \text{for all } k \geq k_0,
\]
with $\Omega_k^+ := \{x \in \Omega : \xi(x) > k\}$ and $r, m, \alpha_j, \varepsilon_j$ are positive constants such that
\[
1 \leq r < p^*, \quad \varepsilon_j > 0, \quad p \leq \alpha_j < \varepsilon_j p^* + p.
\]
Then $\text{ess sup}_\Omega \xi$ is bounded from above by a positive constant which can be chosen so that it depends only on $\text{meas}(\Omega), p, \gamma, k_0, r, m, \varepsilon_j, \alpha_j, |\xi|_{p^*}$ (eventually, $|\xi|$ for some $i > r$ if $p^* = +\infty$).

Proof. For the proof, see [15, Theorem II 5.1]. \qed

Proposition 4.6. Assume that hypotheses $(H_0)$–$(H_7)$, $(G_0)$–$(G_1)$ and (2.3)–(2.4) hold. Then, functional $\mathcal{J}$ satisfies condition (wCPS) in $\mathbb{R}$.

Proof. Let $\beta \in \mathbb{R}$ be fixed and consider $((u_n, v_n))_n \subset X$ a sequence such that, if $n \to +\infty$, then
\[
\mathcal{J}(u_n, v_n) \to \beta \quad \text{and} \quad \|d\mathcal{J}(u_n, v_n)\|_{X'}(1 + \|(u_n, v_n)\|_X) \to 0. \quad (4.4)
\]
Our proof is divided in several steps:

1. $((u_n, v_n))_n$ is bounded in $W$.
   Hence, there exists $(u, v) \in W$ such that $(|u|^\sigma_1 u, |v|^\sigma_2 v) \in W$ and up to subsequences, we have that
   \[
   (u_n, v_n) \to (u, v) \text{ weakly in } W; \quad (4.5)
   \]
   \[
   (|u_n|^{\sigma_2} u_n, |v_n|^{\sigma_2} v_n) \to (|u|^{\sigma_2} u, |v|^{\sigma_2} v) \text{ weakly in } W; \quad (4.6)
   \]
   \[
   (u_n, v_n) \to (u, v) \text{ a.e. in } \Omega; \quad (4.7)
   \]
   \[
   (u_n, v_n) \to (u, v) \text{ in } L^{r_1}(\Omega) \times L^{r_2}(\Omega). \quad (4.8)
   \]
   for any $(r_1, r_2) \in [1, p^*_1(1 + \sigma_1)] \times [1, p^*_2(1 + \sigma_2)]$;

2. $(u, v) \in L$;

3. for any $k > 0$, defining $T_k : \mathbb{R} \to \mathbb{R}$ such that
   \[
   T_k t := \begin{cases} t & \text{if } |t| \leq k \\ k \frac{t}{|t|} & \text{if } |t| > k \end{cases}
   \]
   and
   \[
   T_k : (t_1, t_2) \in \mathbb{R}^2 \mapsto T_k(t_1, t_2) = (T_k t_1, T_k t_2) \in \mathbb{R}^2,
   \]
   then, if $k \geq \max\{|(u, v)|_L, R\} + 1$ (with $R \geq 1$ as in our set of hypotheses), we have
   \[
   \|d\mathcal{J}(T_k(u_n, v_n))\|_{X'} \to 0
   \]
   and
   \[
   \mathcal{J}(T_k(u_n, v_n)) \to \beta;
   \]

4. $\|(T_k u - u, T_k v - v)\|_W \to 0$ and then $\|(u_n - u, v_n - v)\|_W \to 0$ (up to subsequences);

5. $\mathcal{J}(u, v) = \beta, d\mathcal{J}(u, v) = 0$. 

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Step 1. Firstly, we note that (2.4) implies (3.8). Then from \((H_0)-(H_3), (G_0),(G_1),(2.3)\) and (2.4), Remark 4.3 holds. Hence, (4.1) and (4.4) imply that the sequences:

\[
\left( \int_{\Omega} (1 + |u_n|^{p_1 \sigma_1}) |\nabla u_n|^{p_1} dx \right)_n \quad \text{and} \quad \left( \int_{\Omega} (1 + |v_n|^{p_2 \sigma_2}) |\nabla v_n|^{p_2} dx \right)_n
\]

are bounded and (4.5) – (4.8) follow from Lemma 4.4.

Step 2. Arguing by contradiction, we assume that either \(u \notin L^\infty(\Omega)\) or \(v \notin L^\infty(\Omega)\). If \(u \notin L^\infty(\Omega)\), either

\[
\text{ess sup}_{\Omega} u = +\infty \quad (4.9)
\]

or

\[
\text{ess sup}_{\Omega} (-u) = +\infty. \quad (4.10)
\]

For example, suppose that (4.9) holds. Then, for any fixed \(k \in \mathbb{N}, k > R\), we have

\[
\text{meas}(\Omega_k^+) > 0, \quad (4.11)
\]

with \(\Omega_k^+ := \{x \in \Omega \mid u(x) > k\}\).

Now, for any \(\tilde{k} > 0\), we consider the new function \(R_{k}^{\tilde{k}} : \mathbb{R} \to \mathbb{R}\) such that

\[
R_{k}^{\tilde{k}} t = \begin{cases} 0 & \text{if } t \leq \tilde{k} \\ t - \tilde{k} & \text{if } t > \tilde{k} \end{cases}
\]

Taking \(\tilde{k} = k^{\sigma_1 + 1}\), from (4.5) it follows that

\[
R_{k^{\sigma_1 + 1}}^{k^{\sigma_1 + 1}}(|u_n|^{\sigma_1} u_n) \rightharpoonup R_{k^{\sigma_1 + 1}}^{k^{\sigma_1 + 1}}(|u|^{\sigma_1} u) \quad \text{in } W_1.
\]

Thus, by the sequentially weakly lower semicontinuity of \(\| \cdot \|_{W^1}\), it follows that

\[
\int_{\Omega} |\nabla R_{k^{\sigma_1 + 1}}^{k^{\sigma_1 + 1}}(|u|^{\sigma_1} u)|^{p_1} dx \leq \liminf_{n \to +\infty} \int_{\Omega} |\nabla R_{k^{\sigma_1 + 1}}^{k^{\sigma_1 + 1}}(|u_n|^{\sigma_1} u_n)|^{p_1} dx,
\]

i.e.,

\[
\int_{\Omega_k^+} |\nabla (u)^{\sigma_1 + 1}|^{p_1} dx \leq \liminf_{n \to +\infty} \int_{\Omega_{n,k}^{+}} |\nabla (u_n)^{\sigma_1 + 1}|^{p_1} dx, \quad (4.12)
\]

as \(|t|^{\sigma_1 + 1} t > k^{\sigma_1 + 1} \iff t > k\) with \(\Omega_{n,k}^{+} := \{x \in \Omega \mid u_n(x) > k\}\).

On the other hand, from \(\|R_{k}^{k_n} u_n\|_{X_1} \leq \|u_n\|_{X_1} (4.4)\) and (4.11) it follows that \(n_k \in \mathbb{N}\) exists so that:

\[
dJ(u_n, v_n)[R_{k}^{k_n} u_n] < \text{meas}(\Omega_k^+) \quad \text{for all } n \geq n_k. \quad (4.13)
\]
Taking any $k > R$ and $n \in \mathbb{N}$, from (3.3), (H5) with $\mu_1 < 1$, (H4), (H2) and (4.1), it follows that:

$$
\frac{\partial J}{\partial u}(u_n, v_n) [R^+_k u_n] = \int_{\Omega^+_k} \left( 1 - \frac{k}{u_n} \right) (a(x, u_n, \nabla u_n) \cdot \nabla u_n + A_k(x, u_n, \nabla u_n) u_n) \, dx \\
+ \int_{\Omega^+_k} \frac{k}{u_n} a(x, u_n, \nabla u_n) \cdot \nabla u_n \, dx - \int_{\Omega} g_1(x, u_n, v_n) R^+_k u_n \, dx \\
\geq \mu_1 \int_{\Omega^+_k} a(x, u_n, \nabla u_n) \cdot \nabla u_n \, dx - \int_{\Omega} g_1(x, u_n, v_n) R^+_k u_n \, dx \\
\geq \frac{\mu_1 \mu_1}{\eta_2} \int_{\Omega^+_k} u_n^{\mu_1 \sigma_1} |\nabla u_n|^p_1 \, dx - \int_{\Omega} g_1(x, u_n, v_n) R^+_k u_n \, dx = \\
= \frac{\mu_0 \mu_1}{\eta_2 (\sigma_1 + 1)^{p_1}} \int_{\Omega^+_k} (\nabla(u_n)^{\sigma_1 + 1}) \, dx - \int_{\Omega} g_1(x, u_n, v_n) R^+_k u_n \, dx
$$

which, together with (4.13), implies that

$$
\int_{\Omega^+_k} |\nabla(u_n)^{\sigma_1 + 1}| \, dx \leq \frac{\eta_2 (\sigma_1 + 1)^{p_1}}{\mu_0 \mu_1} \text{meas}(\Omega^+_k) + \frac{\eta_2 (\sigma_1 + 1)^{p_1}}{\mu_0 \mu_1} \int_{\Omega} g_1(x, u_n, v_n) R^+_k u_n \, dx
$$

for all $n \geq n_k$.

We note that, from (4.7) and (G0), we have

$$
g_1(x, u_n, v_n) R^+_k u_n \rightarrow g_1(x, u, v) R^+_k u \quad \text{a.e. in } \Omega.
$$

Since $|R^+_k u_n(x)| \leq |u_n(x)|$ a.e. $x \in \Omega$, from (G1) and (3.10), we have

$$
|g_1(x, u_n, v_n) R^+_k u_n| \leq c(|u_n| + |u_n|^q + |v_n|^q)
$$

$$
\leq c(|u_n| + |u_n|^q + \frac{|u_n|^{q_3}}{s_3} + \frac{s_3 - 1}{s_3} |v_n|^{q_4} + s_4).
$$

Thus, from (3.11) and (4.8), we have that a function $h \in L^1(\Omega)$ exists such that

$$
|g_1(x, u_n, v_n) R^+_k u_n| \leq C(|u_n| + |u_n|^q + |u_n|^{s_3} + |v_n|^{s_4}) \leq h(x) \quad \text{for a.e. } x \in \Omega,
$$

up to subsequences (see, e.g., [3, Theorem 4.9]).

So, by using the Dominated Convergence Theorem, we have

$$
\lim_{n \to +\infty} \int_{\Omega} g_1(x, u_n, v_n) R^+_k u_n \, dx = \int_{\Omega} g_1(x, u, v) R^+_k u \, dx.
$$

Hence, summing up, from (4.12), (4.14) and (4.15), passing to the limit, we obtain

$$
\int_{\Omega^+_k} |\nabla(u)^{\sigma_1 + 1}| \, dx \leq C \left( \text{meas}(\Omega^+_k) + \int_{\Omega^+_k} g_1(x, u, v) R^+_k u \, dx \right),
$$

which implies, by using again (2.1), that

$$
\int_{\Omega^+_k} |\nabla(u)^{\sigma_1 + 1}| \, dx \leq C \left( \text{meas}(\Omega^+_k) + \int_{\Omega^+_k} |u| \, dx \right)
$$

$$
+ C \left( \int_{\Omega^+_k} |u|^{q_1} \, dx + \int_{\Omega^+_k} |v|^{q_4} \, dx \right).
$$
Now, from (3.10), Hölder inequality with conjugate exponents $\frac{p_s^{2(1+\sigma_2)}}{s_3} > 1$ and $\frac{p_s^{2(1+\sigma_2)}}{p_s^{2(1+\sigma_2)}-s_3}$ and direct computations, it follows that

$$
\int_{\Omega_k^+} |u| |v|^{p_1} dx \leq \frac{1}{s_3} \int_{\Omega_k^+} |u|^{p_3} dx \\
+ \frac{s_3 - 1}{s_3} |v|^{s_3} \left[ v \right]^{s_3}_{p_s^{2(1+\sigma_2)}} \left[ \text{meas}(\Omega_k^+) \right]^{1-\frac{s_3}{p_s^{2(1+\sigma_2)}}}.
$$

(4.17)

Taking $q_1$ as in (3.14) so from (4.16) and (4.17) it results

$$
\int_{\Omega_k^+} |\nabla (u)^{p_1+1} |p_1 dx \leq C \left( \int_{\Omega_k^+} |u|^{q_1} dx + \text{meas}(\Omega_k^+) \right) \\
+ C \left( [\text{meas}(\Omega_k^+)]^{1-\frac{s_3}{p_s^{2(1+\sigma_2)}}} \right).
$$

(4.18)

with $C = C(\|v\|_{W_2}) > 0$.

Now, if we set $\pi = |u|^{p_1} u$, as $\pi \in W^{1,p_1}_0(\Omega)$ and $\Omega_k^+ := \{ x \in \Omega | \pi(x) > k^{\sigma_1+1} \}$, (in particular, $\pi = u^{\sigma_1+1}$ in $\Omega_k^+$), from (4.18) we obtain

$$
\int_{\Omega_k^+} |\nabla \pi|^{p_1} dx \leq C \left( \int_{\Omega_k^+} |\pi|^{q_1} dx + \text{meas}(\Omega_k^+) \right) + \left[ \text{meas}(\Omega_k^+) \right]^{1-\frac{s_3}{p_s^{2(1+\sigma_2)}}}.
$$

At last, we note that

$$
\int_{\Omega_k^+} |\pi|^{q_1} dx = \int_{\Omega_k^+} |\pi - k + k|^{q_1} dx \\
\leq 2^{q_1} \left( \int_{\Omega_k^+} |\pi - k|^{q_1} dx + \int_{\Omega_k^+} k^{q_1} dx \right) \\
= 2^{q_1} \left( \int_{\Omega_k^+} |\pi - k|^{q_1} dx \right) \left( \int_{\Omega_k^+} k^{q_1} dx \right) \\
+ 2^{q_1} k^{q_1} \text{meas}(\Omega_k^+) \\
\leq 2^{q_1} \left( \int_{\Omega_k^+} k^{q_1} dx \right) \left( \int_{\Omega_k^+} \left| \pi - k \right|^{q_1} dx \right) \\
+ 2^{q_1} k^{q_1} \text{meas}(\Omega_k^+).
$$

Thus, from (3.9) we obtain

$$
\int_{\Omega_k^+} |\nabla \pi|^{p_1} dx \leq C \left( \int_{\Omega_k^+} |\pi - k|^{q_1} dx \right) \left( \int_{\Omega_k^+} k^{q_1} dx \right) \\
+ C k^{p_1} \text{meas}(\Omega_k^+) + C \text{meas}(\Omega_k^+) \left( \int_{\Omega_k^+} k^{q_1} dx \right),
$$

with $C = C(\|u\|_{W_1}, \|v\|_{W_2}) > 0$.

Using the notations of Lemma (4.5), we note that, from (2.3) it is

$$
1 \leq r = \frac{q_1}{\sigma_1 + 1} < p_1^*,
$$

$$
k^{p_1} \text{meas}(\Omega_k^+) = k^{p_1} \left( \text{meas}(\Omega_k^+) \right)^{1-\frac{s_3}{p_s^{2(1+\sigma_2)}}}.
$$
with
\[ \epsilon_1 = \frac{p_1}{N} > 0, \quad p_1 = \alpha_1 < \epsilon_1 p_1^* + p_1. \]

At last, it results:
\[ (\text{meas}(\Omega_k^+))^{1 - \frac{s_4}{p_2(1 + \sigma_2)}} = (\text{meas}(\Omega_k^-))^{1 - \frac{p_1}{N} + \epsilon_2}, \]

where
\[ \epsilon_2 = \frac{p_1}{N} - \frac{s_4}{p_2(1 + \sigma_2)} > 0 \]
i.e.
\[ s_4 < \frac{p_1 p_2^*}{N}(1 + \sigma_2), \]
as (2.4) holds.

Indeed, since
\[ s_4 = s_1 \frac{s_3}{s_3 - 1}, \]
we can choose \( s_3 \) such that
\[ \begin{cases} 
1 < s_3 < p_1(1 + \sigma_1) \\
0 \leq s_4 < \frac{p_1 p_2^*}{N}(1 + \sigma_2) 
\end{cases} \tag{4.19} \]
if and only if it results
\[ s_1 < \frac{p_1 p_2^*}{N}(1 + \sigma_2) \left( 1 - \frac{1}{p_1(1 + \sigma_1)} \right), \]
which is true from (2.4).

So, as \( k \geq 1 \) implies \( 1 \leq k^{p_1} \), Lemma (4.5) applies and yields a contradiction to (4.9).

Now, suppose that (4.10) holds which implies that, fixing any \( k \in \mathbb{N}, k \geq R \), it is
\[ \text{meas}(\Omega_k^+) > 0 \quad \text{with} \quad \Omega_k^- = \{ x \in \Omega : u(x) < -k \}. \]
In this case, by replacing function \( R_k^+ \) with \( R_k^- : \mathbb{R} \to \mathbb{R} \) such that
\[ R_k^- t := \begin{cases} 
0 & \text{if } t \geq -k \\
t + k & \text{if } t < -k 
\end{cases}, \]
we can reason as above so to apply again Lemma 4.5 which yields a contradiction to (4.10). Then, it has to be \( u \in L^\infty(\Omega) \).

Similar arguments, but considering \( \frac{\partial J}{\partial v}(u_n, v_n) \) and \( R_k^+ v_n \), respectively \( R_k^- v_n \), and the related sets, allow us to prove that it has to be also \( v \in L^\infty(\Omega) \).

Step 3. The proof can be obtained by arguing as in the proof of [11, Proposition 4.9] (see also [10, Theorem 4.9]) but when the system has only two equations and replacing assumptions (3.5) and (3.6) in [11] with (2.3) and (2.4) and using conditions (4.19) and the fact that we can choose \( s_5 \) such that
\[ \begin{cases} 
1 < s_5 < p_2(1 + \sigma_2) \\
0 \leq s_6 < \frac{p_2 p_1^*}{N}(1 + \sigma_1), 
\end{cases} \]
instead of (3.24) and (3.25) in [11].

Steps 4. and 5. The proofs are as in the corresponding steps of [10, Proposition 4.9] (see also [7, Proposition 4.6]). \( \square \)

**Proof of Theorem 2.3.** The proof follows from the Minimum Principle (Theorem 2.8), since we have proved that the functional \( J \) is bounded from below in \( X \) (Lemma 4.2) and satisfies condition \((u \, wCPS)\) in \( \mathbb{R} \) (Proposition 4.6). Thus, \( J \) admits a minimum point in \( X \), which is a weak bounded solution of the system (1.1). \( \square \)
Proof of Theorem 2.5. From now on, without loss of generality, we assume that
\[ \int_{\Omega} G(x, 0, 0)dx = 0, \]  
\hspace{1cm} (4.20)
otherwise we can replace \( \mathcal{J}(u, v) \) with \( \mathcal{J}(u, v) + \int_{\Omega} G(x, 0, 0)dx \) which has the same differential \( d\mathcal{J}(u, v) \) in (3.2).
Hence, from (\( H_3 \)) and (4.20) it is
\[ \mathcal{J}(0, 0) = 0. \]  
\hspace{1cm} (4.21)
On the other hand, as (\( G_3 \)) holds, we can consider \( \lambda \in \mathbb{R} \) such that
\[ \liminf_{(u,v) \to (0,0)} \frac{G(x,u,v)}{|u|^{p_1} + |v|^{p_2}} > \lambda > \eta_4 \max \{\lambda_{1,1},\lambda_{2,1}\}. \]
Then, fixing any \( \epsilon > 0, \epsilon \) small enough, a constant \( \delta_\epsilon > 0 \) exists such that if we fix \( (u,v) \) such that \(|u|^2 + |v|^2 < \delta_\epsilon^2\), then it is
\[ \frac{G(x,u,v)}{|u|^{p_1} + |v|^{p_2}} > (\lambda - \epsilon) > \eta_4 \max \{\lambda_{1,1},\lambda_{2,1}\}. \]  
\hspace{1cm} (4.22)
Thus, taking \(|t|\) small enough (see Remark 2.6), it follows that
\[ G(x,t\varphi_{1,1},t\varphi_{2,1}) > (\lambda - \epsilon) (|t|^{p_1} |\varphi_{1,1}|^{p_1} + |t|^{p_2} |\varphi_{2,1}|^{p_2}). \]  
\hspace{1cm} (4.23)
Then, from (\( H_3 \)), (4.23) and (2.5) for \( t \) small enough it results
\[ \mathcal{J}(t\varphi_{1,1},t\varphi_{2,1}) \leq \eta_4 |t|^{p_1} \lambda_{1,1} |\varphi_{1,1}|^{p_1} + \eta_4 |t|^{p_2} \lambda_{2,1} |\varphi_{2,1}|^{p_2} 
- (\lambda - \epsilon) \left( |t|^{p_1} |\varphi_{1,1}|^{p_1} + |t|^{p_2} |\varphi_{2,1}|^{p_2} \right) 
= (\eta_4 \lambda_{1,1} - \lambda + \epsilon) |t|^{p_1} + (\eta_4 \lambda_{2,1} - \lambda + \epsilon) |t|^{p_2}. \]
Thus, (4.22) and (4.21) imply that
\[ \mathcal{J}(t\varphi_{1,1},t\varphi_{2,1}) < 0 = \mathcal{J}(0,0). \]
Hence, the minimum point of \( \mathcal{J} \) in \( X \) (see Theorem 2.3) is a non trivial solution of system (1.1). \( \square \)

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Received for publication August 2021; early access January 2022.

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