Quantum discord plays no distinguished role in characterization of complete positivity: Robustness of the traditional scheme

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The traditional scheme for realizing open-system quantum dynamics takes the initial state of the system-bath composite as a simple product. Currently, however, the issue of system-bath initial correlations possibly affecting the reduced dynamics of the system has been attracting considerable interest. The influential work of Shabani and Lidar [PRL 102, 100402 (2009)] famously related this issue to quantum discord, a concept which has in recent years occupied the centre-stage of quantum information theory and has led to several fundamental results. They suggested that reduced dynamics is completely positive if and only if the initial system-bath correlations have vanishing quantum discord. Here we show that there is, within the Shabani-Lidar framework, no scope for any distinguished role for quantum discord in respect of complete positivity of reduced dynamics. Since most applications of quantum theory to real systems rests on the traditional scheme, its robustness demonstrated here could be of far-reaching significance.

A specific, carefully detailed, and precise formulation of the issue of initial system-bath correlations possibly influencing the reduced dynamics was presented not long ago by Shabani and Lidar [8] (SL hereafter [11]). In this formulation, the distinguished bath state $\rho_B^{\text{fid}}$ is replaced by a collection of (possibly correlated) system-bath initial states $\Omega^{SB} \in \mathcal{B}(\mathcal{H}_S \otimes \mathcal{H}_B)$, where $\mathcal{H}_S, \mathcal{H}_B$ are the Hilbert spaces of the system and bath, the dimensions being $d_S, d_B$ respectively. The dynamics gets defined through a joint unitary $U_{SB}(t)$:

$$
\rho_{SB}(0) \rightarrow \rho_{SB}(t) = U_{SB}(t) \rho_{SB}(0) U_{SB}(t)^\dagger, \\
\forall \rho_{SB}(0) \in \Omega^{SB}.
$$

This composite dynamics induces on the system the QDP

$$
\rho_S(0) \rightarrow \rho_S(t),
$$

with $\rho_S(0)$ and $\rho_S(t)$ defined through this natural imaging from $\Omega^{SB}$ to the system state space $\mathcal{L}_S$:

$$
\rho_S(0) = \text{Tr}_B \rho_{SB}(0), \quad \rho_S(t) = \text{Tr}_B \rho_{SB}(t).
$$

It is evident that the folklore scheme obtains as the special case $\Omega^{SB} = \{ \rho_S \otimes \rho_B^{\text{fid}} | \rho_B^{\text{fid}} = \text{fixed} \}$. This generalized formulation of QDP allows SL to transcribe the fundamental issue to this question: What are the necessary and sufficient conditions on the collection $\Omega^{SB}$ so that the induced QDP $\rho_S(0) \rightarrow \rho_S(t)$ in Eq. (3) is guaranteed to be CP for all joint unitaries $U_{SB}(t)$? Motivated by the work of Rodriguez-Rosario et al. [5], and indeed highlighting it as ‘a recent breakthrough’, SL advance the following resolution to this issue:

**Theorem 1** (SL): The QDP in Eq. (4) is CP for all joint unitaries $U_{SB}(t)$ if and only if the quantum discord vanishes for all $\rho_{SB} \in \Omega^{SB}$, i.e., if and only if the initial system-bath correlations are purely classical.
The SL theorem has come to be counted among the more important recent results of quantum information theory, and has influenced an enormous number of authors.

In order that the QDP in Eq. (3) be well defined in the first place, the set \( \Omega^{SB} \) should necessarily satisfy the following two properties; since our entire analysis rests critically on these properties, we begin by motivating them.

**Property 1:** No state \( \rho_S(0) \) can have two (or more) pre-images in \( \Omega^{SB} \). To see this fact unfold, assume to the contrary that

\[
\text{Tr}_B \rho_{SB}(0) = \text{Tr}_B \rho'_{SB}(0), \quad \rho_{SB}(0) \neq \rho'_{SB}(0),
\]

for two states \( \rho_{SB}(0), \rho_{SB}'(0) \in \Omega^{SB} \).

It is clear that the difference \( \Delta \rho_{SB}(0) = \rho_{SB}(0) - \rho'_{SB}(0) \) should necessarily meet the property \( \text{Tr}_B \Delta \rho_{SB}(0) = 0 \). Let \( \{ \lambda_u \}_{u=1}^{d_S^2-1} \) be a set of orthonormal hermitian traceless matrices \( d_S \times d_S \) matrices so that together with the unit matrix \( \lambda_0 = \mathbb{1}_{d_S d_S} \) these matrices form a basis for \( \mathcal{B}(\mathcal{H}_S) \), the set of all \( d_S \times d_S \) (complex) matrices. Let \( \{ \gamma_v \}_{v=1}^{d_B^2-1}, \gamma_0 = \mathbb{1}_{d_B d_B} \) be a similar basis for \( \mathcal{B}(\mathcal{H}_B) \). Then the \( (d_S d_B)^2 \) tensor products \( \{ \lambda_u \otimes \gamma_v \} \) form a basis for \( \mathcal{B}(\mathcal{H}_S \otimes \mathcal{H}_B) \), and \( \Delta \rho_{SB}(0) \) can be written in the form

\[
\Delta \rho_{SB}(0) = \sum_{u=0}^{d_S^2-1} \sum_{v=0}^{d_B^2-1} C_{uv} \lambda_u \otimes \gamma_v, \quad C_{uv} \text{ real}.
\]

Now, the property \( \text{Tr}_B \Delta \rho_{SB}(0) = 0 \) is strictly equivalent to the demand that the expansion coefficient \( C_{00} = 0 \), for all \( u = 0, 1, \cdots, d_S^2 - 1 \). Since the \( [(d_S d_B)^2 - 1] \)-parameter unitary group \( SU(d_S d_B) \) acts irreducibly on the \( [(d_S d_B)^2 - 1] \)-dimensional subspace of \( \mathcal{B}(\mathcal{H}_S \otimes \mathcal{H}_B) \) consisting of all traceless \( d_S d_B \)-dimensional matrices [this is the adjoint representation of \( SU(d_S d_B) \)], there exists an \( U_{SB}(t) \in SU(d_S d_B) \) which takes \( \Delta \rho_{SB}(0) \neq 0 \) into a matrix whose expansion coefficient \( C_{00} \neq 0 \) for some \( u \). That is, if the initial \( \Delta \rho_{SB}(0) \neq 0 \) then one and the same system state \( \rho_S(0) \) will evolve into two distinct

\[
\rho_S(t) = \text{Tr}_B \left[ U_{SB}(t) \rho_{SB}(0) U_{SB}(t)^\dagger \right],
\]

\[
\rho'_S(t) = \text{Tr}_B \left[ U_{SB}(t) \rho'_{SB}(0) U_{SB}(t)^\dagger \right]
\]

for some \( U_{SB}(t) \), rendering the QDP in Eq. (3) one-to-many, and hence ill-defined.

**Property 2:** While every system state \( \rho_S(0) \) need not have a pre-image actually enumerated in \( \Omega^{SB} \), the set of \( \rho_S(0) \)'s having pre-image should be sufficiently large. Indeed, Rodriguez-Rosario et al.\[5\] have rightly emphasised that it should be ‘a large enough set of states such that the QDP in Eq. (3) can be extended by linearity to all states of the system’. It is easy to see that if \( \Omega^{SB} \) fails this property, then the very issue of CP would make no sense. For, in carrying out verification of CP property, the QDP would be required to act, as is well known\[12\], on \( \{|j⟩⟨k|\} \) for \( j, k = 1, 2, \cdots, d_S \); i.e., on generic complex \( d_S \)-dimensional square matrices, and not just on positive or hermitian matrices alone. Since the basic issue on hand is to check if the QDP as a map on \( \mathcal{B}(\mathcal{H}_S) \) is CP or not, it is essential that it be well defined (at least by linear extension) on the entire complex linear space \( \mathcal{B}(\mathcal{H}_S) \).

With the two properties of \( \Omega^{SB} \) thus motivated, we proceed to prove our main result. We ‘assume’, for the time being, that every pure state \( |ψ⟩ \) of the system has a pre-image in \( \Omega^{SB} \). This assumption may appear, at first sight, to be a drastic one. But we show later that it entails indeed no loss of generality.

It is evident that, for every pure state \( |ψ⟩ \), the pre-image in \( \Omega^{SB} \) has to necessarily assume the (uncorrelated) product form \( |ψ⟩⟨ψ| \otimes \rho_B \), \( \rho_B \) being a state of the bath which could possibly depend on the system state \( |ψ⟩ \). Now, let \( \{|ψ_k⟩⟩\}_{k=1}^{d_S} \) be an orthonormal basis in \( \mathcal{H}_S \) and \( \{|ϕ_α⟩⟩\}_{α=1}^{d_s} \) be another orthonormal basis related to the former through a complex Hadamard unitary matrix \( U \). Recall that a unitary \( U \) is Hadamard if \( |U_{kα}| = 1/\sqrt{d_S} \), independent of \( k, α \). For instance, the characters of the cyclic group of order \( d_S \) written out as a \( d_S \times d_S \) matrix is Hadamard. The fact that the \( \{|ψ_k⟩⟩\} \) basis and the \( \{|ϕ_α⟩⟩\} \) basis are related by a Hadamard means that the magnitude of the inner product \( ⟨ψ_k|ϕ_α⟩ \) is independent of both \( k \) and \( α \), and hence equals \( 1/\sqrt{d_S} \) uniformly. We may refer to such a pair as relatively unbiased bases.

Let \( |ψ_k⟩⟩ \otimes O_k \) be the pre-image of \( |ψ_k⟩⟨ψ_k| \) and \( |ϕ_α⟩⟩ \otimes \tilde{O}_α \) be that of \( |ϕ_α⟩⟨ϕ_α| \), \( k, α = 1, 2, \cdots, d_S \). Possible dependence of the bath states \( O_k \) on \( |ψ_k⟩⟩ \) and \( \tilde{O}_α \) on \( |ϕ_α⟩⟩ \) has not been ruled out as yet. Since the maximally mixed system state can be expressed in two equivalent ways as \( d_S^{-1} \sum_k |ψ_k⟩⟨ψ_k| = d_S^{-1} \sum_α |ϕ_α⟩⟨ϕ_α| \), uniqueness of its pre-image in \( \Omega^{SB} \) demands (Property 1)

\[
\sum_{k=1}^{d_S} |ψ_k⟩⟨ψ_k| \otimes O_k = \sum_{α=1}^{d_s} |ϕ_α⟩⟨ϕ_α| \otimes \tilde{O}_α.
\]

Taking projection of both sides on \( |ψ_j⟩⟨ψ_j⟩ \), and using \( |ϕ_j⟩⟨ϕ_j| = d_S^{-1} \), we have

\[
O_j = \frac{1}{d_S} \sum_{α=1}^{d_s} \tilde{O}_α, \quad j = 1, 2, \cdots, d_S,
\]

while projection on \( |ϕ_β⟩⟨ϕ_β| \) leads to

\[
\tilde{O}_β = \frac{1}{d_S} \sum_{k=1}^{d_S} O_k, \quad β = 1, 2, \cdots, d_S.
\]

These \( 2d_S \) constraints together imply that \( O_j = \tilde{O}_β \) uniformly for all \( j, β \). Thus the pre-image of \( |ψ_k⟩⟩ \otimes \rho_B \) and that of \( |ϕ_α⟩⟩ \otimes \rho_B \), for all \( k, α \), \( \rho_B \), for some fixed bath state \( \rho_B \). And, perhaps more importantly, the pre-image of the maximally mixed state \( d_S^{-1} \mathbb{1} \) necessarily equals the product \( d_S^{-1} \mathbb{1} \otimes \rho_B \).

Taking another pair of relatively unbiased bases \( \{|ψ_k⟩⟩\}, \{|ϕ_α⟩⟩\} \) one similarly concludes that the pure
states $|\psi_1\rangle^d, |\phi\rangle^d$ too have pre-images $|\psi_1\rangle^d \otimes \rho_B, |\phi\rangle^d \otimes \rho_B$ respectively, with the same fixed fiducial bath state $\rho_B^d$. This is so, since the maximally mixed state is common to both sets.

Considering in this manner enough number of pure states or projections $|\psi\rangle\langle\psi|$ sufficient to span—by linearity—the entire system state space $\Lambda_S$, and hence $B(H_S)$, one readily concludes that every element of $\Omega^{SB}$ necessarily needs to be of the product form $\rho_S(0) \otimes \rho_B^d$, for some fixed bath state $\rho_B$. But this is exactly the folklore realization of non-unitary dissipative dynamics given in Eq. (1), to surpass which was the primary goal of the SL scheme. We have thus proved our principal result:

**Theorem 2:** No initial correlations—even classical ones—are permissible within the SL scheme.

As we have noted, if at all a pure state $\rho_S(0) = |\psi\rangle\langle\psi|$ has a pre-image in $\Omega^{SB}$ it would necessarily be of the product form $|\psi\rangle\langle\psi| \otimes \rho_B$, for some (possibly $|\psi\rangle$-dependent) bath state $\rho_B$. While this is self-evident and is independent of SL, it is instructive to view it as a consequence of the necessary condition part of SL theorem. Then our principal conclusion above can be rephrased to say that validity of SL theorem for pure states of the system readily leads to the folklore product-scheme as the only solution within the SL framework. This interesting aspect comes through in an even more striking manner in our proof below that our earlier ‘assumption’ is one without loss of generality.

Our assumption entails no loss of generality: Let us focus, to begin with, on the convex hull $\Omega^{SB}$ of $\Omega^{SB}$ rather than the full (complex) linear span of $\Omega^{SB}$ to which we are entitled. Let us further assume that the image of $\Omega^{SB}$ under the convexity-preserving linear imaging (projection) map $\rho_{SB}(0) \rightarrow \text{Tr}_{B|\rho_{SB}(0)}$ fills not the entire (convex) state space—the $(d_S^2 - 1)$-dimensional generalized Bloch sphere $\Lambda_S$—of the system, but only a portion thereof, possibly a very small part. Even so, in order that our QDP in Eq. (3) be well-defined, this portion would occupy a non-zero volume of the $(d_S^2 - 1)$-dimensional state space $\Lambda_S$ of the system (Property 2).

Let us consider one set of all mutually commuting elements of $\Lambda_S$. If the full state space $\Lambda_S$ were available under the imaging $\rho_{SB}(0) \rightarrow \text{Tr}_{B|\rho_{SB}(0)}$ of $\Omega^{SB}$, then the resulting mutually commuting images would have filled the entire $(d_S - 1)$-simplex, the classical state space of a $d_S$-level system, this being the tetrahedron when $d_S = 2$ [13]. Since the full state space is assumed to be not available as image of $\Omega^{SB}$, these commuting elements fill a, possibly very small but of nontrivial measure, proper convex subset of the $(d_S - 1)$-simplex, depicted in Fig. 1 as region $R$ for the case $d_S = 3$ (qutrit).

Elements of these simultaneously diagonal density matrices of the system can be expressed as convex sums of pure states or one-dimensional projections. For a generic element in this region, the spectrum is non-degenerate, and hence the projections are unique and commuting, being the eigenstates of $\rho_S(0)$, and correspond to the $d_S$ vertices of the $(d_S - 1)$-simplex. In the case of qutrit, it is pictorially seen in Fig. 1 that only the points on the bisectors (the three dotted lines) correspond to doubly degenerate density matrices and the centre alone is triply degenerate, rendering transparent the fact that being nondegenerate is a generic attribute of region $R$.

Now consider the pre-image $\rho_{SB}(0)$ in $\Omega^{SB}$ of such a non-degenerate $\rho_S(0) \in R$. Application of the SL requirement of vanishing discord (again, only the necessity part of the SL theorem) to this $\rho_{SB}(0)$ implies that this pre-image has the form $\rho_{SB}(0) = \sum_{j=1}^{d_S} p_j |j\rangle\langle j| \otimes \rho_{Bj}(0)$, (4)

where the probabilities $p_j$ and the pure states $|j\rangle\langle j|$ are uniquely determined (in view of nondegeneracy) by the spectral resolution

$$\rho_S(0) = \text{Tr}_{B} \rho_{SB}(0) = \sum_{j=1}^{d_S} p_j |j\rangle\langle j|.$$ 

And $\rho_{Bj}(0)$'s are bath states, possibly dependent on $|j\rangle\langle j|$ as indicated by the label $j$ in $\rho_{Bj}(0)$. These considerations hold for every nondegenerate element of region $R$ of probabilities $\{p_j\}$. In view of generic nondegeneracy, the requirement (4) implies that each of the $d_S$ pure states $|j\rangle\langle j|$ has pre-image of the form $|j\rangle\langle j| \otimes \rho_{Bj}(0)$ in the linear span of the pre-image of $R$—at least as seen by the evolution (2). That is $\rho_{Bj}(0)$’s can’t be dependent on the probabilities $\{p_j\}$.

Since every pure state of the system constitutes one of the vertices of some $(d_S - 1)$-simplex comprising one of all mutually commuting density operators $\rho_S(0)$, the conclusion that a pure state effectively has in the

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**FIG. 1:** Depicting, for the case $d_S = 3$ (qutrit), the image of $\Omega^{SB}$ under $\text{Tr}_{B}(\cdot)$ in the plane spanned by the commuting (diagonal) $\lambda$-matrices ($\lambda_3, \lambda_8$).
linear span of $\Omega^{SB}$ a pre-image, and one necessarily of the product form, applies to every pure state, showing that the ‘assumption’ in our earlier analysis indeed entails no loss of generality.

To summarize, it is clear that the dynamics described by (2) and (3) would ‘see’ only the full (complex) linear span of $\Omega^{SB}$, and not so much the actual enumeration of $\Omega^{SB}$ as such. This is notwithstanding the fact that, as indicated by the projection map $\rho_{SB}(0) \rightarrow \rho_S(0) = \text{Tr}_B \rho_{SB}(0)$, the only elements of this linear span which are immediately relevant for the QDP are those which are hermitian, positive semidefinite, and have unit trace.

As no system state can have two or more pre-images (Property 1), in order that the QDP in (4) be well defined these relevant elements are forced to constitute a faithful linear embedding of (a nontrivial convex subset of) the system’s state space $\Lambda_S$ in $\mathcal{B}(\mathcal{H}_S \otimes \mathcal{H}_B)$. In the SL scheme of things, this leaves us with just the folklore embedding $\rho_S(0) \rightarrow \rho_{SB}(0) = \rho_S(0) \otimes \rho_B^{\text{fid}}$. This is the principal conclusion that emerges.

Let us view this from a slightly different position. Since there is no conceivable manner in which a linear map acting on elements of $\Omega^{SB}$ could be prevented from acting on convex sums (indeed, the linear span) of such elements, we may assume—without loss of generality—$\Omega^{SB}$ to be convex and ask, consistent with the SL theorem: What are the possible choices for the set $\Omega^{SB}$ to be convex and at the same time consist entirely of states of vanishing quantum discord. One possibility comprises elements of the form $\rho_{SB}(0) = \rho_S(0) \otimes \rho_B^{\text{fid}}$, for a fixed bath state $\rho_B^{\text{fid}}$ and arbitrary system state $\rho_S(0)$. This is recognized to be simply the folklore case. The second one consists of elements of the form $\rho_{SB}(0) = \sum_j p_j |j\rangle\langle j| \otimes \rho_{Bj}(0)$, for a fixed (complete) set of orthonormal pure states $\{|j\rangle\}$. a case restricted to mutually commuting density operators of the system. This seems to be the case studied by Rodriguez-Rosario et al. [5], but the very notion of CP is unlikely to make much sense in this non-quantum case of classical state space (of dimension $d_S - 1$ rather than $d_S^2 - 1$), the honorific ‘a recent breakthrough’ notwithstanding.

The stated goal of SL was to give a complete characterization of possible initial correlations that lead to CP maps. It is possibly in view of the belief that there was a large class of permissible initial correlations out there within the SL framework, and that that class now stands fully characterized by the SL theorem [6], that a large number of recent papers tend to list complete characterization of CP maps among the principal achievements of quantum discord [15]. Our result implies, with no irreverence whatsoever to quantum discord, that characterization of CP maps may not yet be rightfully paraded as one of the achievements of quantum discord.

The SL theorem has influenced an enormous number of authors, and it is possible that those results of these authors which make essential use of the sufficiency part of the SL theorem need recalibration in the light of our result.

There are other, potentially much deeper, implications of our finding. Our analysis—strictly within the SL framework—has shown that this framework brings one exactly back to the folklore scheme itself, as if it were a fixed point. This is not at all a negative result for two reasons. First, it shows that quantum discord is no ‘cheaper’ (to accommodate) than entanglement as far as complete positivity of QDP is concerned. Second, and more importantly, the fact that the folklore product-scheme survives attack under this well-defined and fairly general SL framework demonstrates its, perhaps unsuspected, robustness. In view of the fact that this scheme has been at the heart of most applications of quantum theory to real situations, virtually in every area of physical science, and even beyond, its robustness the SL framework helps to demonstrate is likely to prove to be of far-reaching significance.
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