Complexity and Non-Commutativity of Learning Operations on Graphs

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Abstract

We present results from numerical studies of supervised learning operations in recurrent networks considered as graphs, leading from a given set of input conditions to predetermined outputs. Graphs that have optimized their output for particular inputs with respect to predetermined outputs are asymptotically stable and can be characterized by attractors which form a representation space for an associative multiplicative structure of input operations. As the mapping from a series of inputs onto a series of such attractors generally depends on the sequence of inputs, this structure is generally non-commutative. Moreover, the size of the set of attractors, indicating the complexity of learning, is found to behave non-monotonically as learning proceeds. A tentative relation between this complexity and the notion of pragmatic information is indicated.
1 Introduction

Graph theory has recently received increasing attraction for applications to complex systems in various disciplines (Gernert 1997, Paton 2002a,b, Bornholdt and Schuster 2003). The characterization of systems (with interrelated constituents) by graphs (with linked vertices) is comparably general as their characterization in terms of categories (with elements related by morphisms). Despite its generality, graph theory has turned out to be a powerful tool for gaining very specific insight into structural and dynamical properties of complex systems (see Jost and Joy 2002, Atmanspacher et al. 2005 for examples).

An area of particularly intense interest, in which complex systems abound, is biological information processing. This ranges from evolutionary biology over genetics to the study of neural systems. Theoretical and computational neuroscience have become rapidly growing fields (Hertz et al. 1991, Haykin 1999, Dayan and Abbott 2001) in which graph theoretical methods have gained considerable significance (cf. Sejnowski 2001).

Two basic classes of biological networks are feedforward and recurrent networks. In networks with purely feedforward (directed) connectivities, neuronal input is mapped onto neuronal output through a feedforward synaptic weight matrix. In recurrent networks, there are additional (directed or bi-directed) connectivities between outputs and other network elements, giving rise to a recurrent synaptic weight matrix. Much recurrent modeling incorporates the theory of nonlinear and complex dynamical systems (cf. Smolensky 1988, see also beim Graben 2004 for discussion).

Hopfield networks are an example of a fully recurrent network in which all connectivities are bidirectional and the output is a deterministic function of the input. Their stochastic generalizations are known as Boltzmann machines. Another important distinction with respect to the implementation of neural networks refers to the way in which the neuronal states are characterized: the two main options are firing rates and action potentials (for more details see Haykin 1999).

A key topic of information processing in complex biological networks is learning, for which three basically different scenarios are distinguished in the literature (see Dayan and Abbott 2001, Chap. III): unsupervised, supervised and reinforcement learning. In unsupervised (also self-supervised) learning a network responds to inputs solely on the basis of its intrinsic structure and dynamics. A network learns by evolving into a state that is constrained by its own properties and the given inputs, an important modelling strategy for implicit learning processes.

In contrast, supervised learning presupposes the definition of desired input-output relations, so the learned state of the network is additionally constrained by its outputs. Usually, the learning process in this case develops by minimizing the difference between the actual output and the desired output. The corresponding optimization procedure is not intrinsic to the evolution of the system itself, but has to be externally arranged, hence the learning is called supervised. If the supervision is in some sense “naturalized” by coupling a network to an environment, which provides
evaluative feedback, one speaks of reinforcement learning.

In this contribution we are interested in supervised learning (see Duda et al. 2000 for a review) on small, fully recurrent networks implemented on graphs (cf. Jordan 1998). We start with a general formal characterization in terms of dynamical systems (Sec. 2.1), describe how they are implemented on graphs (Sec. 2.2), and show how it reaches asymptotically stable states (attractors) when the learning process is terminated, i.e. is optimized for given inputs and (random) initial conditions with respect to predetermined outputs (Sec. 2.3).

We shall characterize the learning operations by a multiplicative structure characterizing successively presented inputs in Sec. 3.1. In this context we confirm and specify earlier conjectures (e.g., Gernert 1997) about the non-commutativity of learning operations for a concrete model. In Sec. 3.2, we study how the size of the set of attractors representing the derived structure changes during the process for perfectly and imperfectly optimized networks. The number of attractors is proposed to indicate the complexity of learning, and in Sec. 4 this is tentatively related to pragmatic information as a particular measure of meaning.

2 Supervised Learning in Recurrent Networks

2.1 General Notation

Let \( M \) be a set, and let \( M = X \cup B \), with \( X \cap B = \emptyset \), be a partition of \( M \) into two disjoint subsets. If \( M \) is some closed subset of \( \mathbb{R}^n \), \( B \) may be the boundary of \( M \). (Later we will specify \( M \) as the vertices of a graph, \( B \) as a set of “external” or “boundary” vertices, and \( X \) as a set of “internal” vertices.)

We consider the dynamics of fields \( u(x,y,t) \in U \), where \( x \in X \), \( y \in B \), \( t \) represents time as parametrized discretely or continuously, and \( U \) is the space of admissible state values for the fields. The dynamics of \( u \) can be described by an equation

\[
F[u(x,y,t)] = 0. \tag{1}
\]

For a continuous time variable and \( M \subset \mathbb{R}^n \), a typical example is the diffusion equation

\[
F[u(x,t)] = \frac{\partial u(x,t)}{\partial t} - \lambda \Delta u(x,t) \tag{2}
\]

where \( \Delta \) is the Laplace operator and \( \lambda \) the diffusion constant. The only constraint on Eq. 2 is that a state \( u(x,y,0) \) at time \( t = 0 \) determines uniquely the solution for any time \( t > 0 \).

We now define a set of external conditions \( \{ b_i : B \to U \} \) specifying field values \( b_i \) on \( B \) which will be kept fixed during the time evolution of the fields on \( M \). This is to say that the dynamics of fields is effectively restricted to \( X \):

\[
F[u(x,b_i,t)] = 0. \tag{3}
\]
Since the state of the system at time $t = 0$ uniquely determines the states for all $t > 0$, we can define a mapping $\Phi_t$, the so-called time evolution operator, acting on the set of field states. For an initial state $u$ at $t = 0$, $\Phi_t[u]$ yields the state of the system at $t > 0$. Taking into account that different external conditions initiate different evolutions, we have to specify the time evolution operator as a mapping $\Phi_{t,b_i} : F_X \to F_X$, where $F_X$ is the set of states $u(x) : X \to U$, by the following construction: Let $u(x,t = 0)$ be the initial condition for Eq. (3), then

$$\Phi_{t,b_i}[u](x) = u(x,b_i,t)$$

is the state of the corresponding solution at time $t$ under the external condition $b_i$.

In principle, the state space of $\Phi_{t,b_i}$ can be the entire set of states $F_X$. However, for reasons which will become clear below, we are interested in dissipative systems evolving into attractors $a_i$ in the limit of large $t$. If one of the states belonging to an attractor is chosen as an initial condition, the image of $\Phi_{t,b_i}$ will again be one of the attractor states. This allows us to reduce the number of possible states on which the mappings $\Phi_{t,b_i}$ close.

Denoting the flow operator $B_i \equiv \Phi_{t,b_i}$ as the input under the external condition $b_i$, we now consider the set of states $A \in F_X$ belonging to attractors after time $t$. Then all mappings $B_i$, applied to an attractor $a$, lead to images in $A$:

$$B_i[a] \in A \quad \text{for } a \in A.$$ (4)

In general, the set of all attractor states $A$ does not contain a proper subset which is mapped onto itself by the set of mappings $\{B_i\}$; otherwise $A$ can be reduced to such a subset. Each single mapping $B_i$ may not be surjective, but the union of the images of all $\{B_i\}$ equals $A$.

Due to condition (4), we can define a composition of mappings $B_i$. In this way, the external conditions $\{b_i\}$ give rise to an associative multiplicative structure $\{B_i\}$. This structure is represented on the set of attractors $\{a_i\}$.

### 2.2 Implementation on Graphs

We now implement the general notions developed so far on graphs (see Wilson 1985 for an introduction to graph theory) and specify the set $M$ as the set of vertices $V$ of a graph. For simplicity we consider directed graphs with single connections for each direction between any two vertices and without self-loops. Such a graph gives rise to non-reflexive relations on $V$ and can be represented by an adjacency matrix $Ad$. For two vertices $x_1$ and $x_2$ we have:

$$Ad(x_1,x_2) = \begin{cases} 
1 & \text{if there exists a directed line from } x_2 \text{ to } x_1 \\
0 & \text{otherwise} 
\end{cases} \quad (5)$$

If $Ad$ is symmetric, the graph is undirected.
The set of vertices $V$ is decomposed into a set of external vertices $V_{\text{ext}}$ and a set of internal vertices $V_{\text{int}}$. If $N$ is the total number of vertices, $N_{\text{ext}}$ the number of external vertices and $N_{\text{int}}$ the number of internal vertices, we have $N = N_{\text{ext}} + N_{\text{int}}$.

Next we consider fields $u(z, t)$ on a graph with vertices $z \in V$ evolving in discrete time steps $t \in \mathbb{N}$ according to:

$$u(z, t + 1) = f \left( \sum_{y \to z} u(y, t) \right) = f \left( \sum_{y} Ad(z, y) u(y, t) \right).$$

The value of the field $u$ at vertex $z$ and time $t + 1$ depends only on the sum of the field values at neighboring vertices $y$ at time $t$.

The fields $u(z, t)$ assume integer values $\{0, 1, \ldots, I_{\text{max}}\}$, and the function $f$ is defined as:

$$f(x) = \begin{cases} 
\text{int}(I_{\text{max}} \cdot (x/n_0)) & \text{for } x < n_0 \\
\text{int}(I_{\text{max}} \cdot (n_1 - x)/(n_1 - n_0)) & \text{for } n_0 \leq x < n_1 \\
0 & \text{for } x \geq n_1
\end{cases}$$

where $\text{int}(x)$ denotes the nearest integer-rounded $x$. The function $f(x)$ is shown in Fig. 1.

Figure 1: The function $f(x)$ according to Eq. (7). The values $I_{\text{max}} = 10$, $n_0 = 10$, and $n_1 = 30$ are used in the simulations.

The restriction of $u(z, t)$ to integer values implies that there is only a finite number of states. Starting from an arbitrary initial state in $F_X$, the system runs into an attractor after a few time steps. In many cases, this attractor is a fixed point, i.e. one single state that is asymptotically stable. Sometimes the attractor is a limit cycle, i.e. a periodic succession of several (usually few) states. Strange attractors do not occur since the number of states is finite.

The external conditions $\{b_i\}$ are defined as fixed states on the external vertices, i.e., the state values on the external vertices remain unaffected by the dynamics. Of course, the external conditions are supposed to influence the dynamics of the internal vertices.

The graphs used in our investigations consist of a total of $N = 24$ vertices with $N_{\text{ext}} = 16$ external vertices and $N_{\text{int}} = 8$ internal vertices. The maximal value of
Figure 2: The 11 input patterns $b_i$ on the 16 external vertices, represented as three basic types of $4 \times 4$-matrices: $\circ$ indicates field value 0, $\bullet$ indicates field value $I_{\text{max}}$.

$u(z,t)$ is defined to be $I_{\text{max}} = 10$. We consider 11 different input patterns $b_i$ which are shown in Fig. 2.

In order to obtain a minimal set $A$ of attractor states under which the multiplication of the evolution operators $B_i$ is closed, we first determine the attractor state $a_1$ (or states $a_i$, $i = 1, \ldots, m$ for a limit cycle of period $m$) corresponding to input $B_1$, starting from a random initial distribution of states in $F_X$. Subsequently, $B_2$ is applied to $a_1$, and so on until $B_{11}$ provides the final attractor state(s).

Next we apply all evolution operations $B_1, \ldots, B_{11}$ to the obtained set of attractors until no new attracting states are generated. The resulting set $A$ can be represented in terms of a mapping diagram. An example for such a mapping dia-

| inputs $B_i$ | attractor states $a_i$ |
|-------------|------------------------|
|             | 1 2 3 4 5 6 7 8 9 10 |
| 1           | 1 1 1 1 1 1 1 1 1 1   |
| 2           | 2 2 2 2 2 2 2 2 2 2   |
| 3           | 3 3 3 3 3 3 3 3 3 3   |
| 4           | 2 2 2 2 2 2 2 2 2 2   |
| 5           | 3 3 3 3 3 3 3 3 3 3   |
| 6           | 4 4 4 4 4 4 4 4 4 4   |
| 7           | 4 4 4 4 4 4 4 4 4 4   |
| 8           | 5 5 5 5 5 5 5 5 5 5   |
| 9           | 6 2 2 7 5 6 7 7 10 10 |
| 10          | 8 8 8 5 5 8 5 8 5 5   |
| 11          | 4 9 9 4 4 9 4 9 9 9   |

Table 1: Example of a mapping diagram for 11 inputs $B_i$ and a system with 10 different attractors $a_i$. The entries show the number $i$ of the attractor state which is obtained by applying $B_i$ (plotted vertically) to $a_i$ (plotted horizontally).
gram with a relatively small number of 10 attractor states, which are all fixed-point attractors, is shown in Tab. 1. The corresponding field values on the eight internal vertices are listed in Tab. 2, and the corresponding adjacency matrix of the graph is given in Tab. 3.

| attractor states $a_i$ | field values on internal vertices |
|------------------------|----------------------------------|
|                        | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| 1                      | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 2                      | 0 | 10| 0 | 0 | 0 | 10| 0 | 0 |
| 3                      | 0 | 10| 0 | 0 | 0 | 0 | 0 | 10|
| 4                      | 10| 0 | 0 | 10| 0 | 0 | 0 | 0 |
| 5                      | 10| 0 | 0 | 0 | 0 | 0 | 10| 0 |
| 6                      | 5 | 5 | 0 | 0 | 0 | 5 | 5 | 0 |
| 7                      | 9 | 1 | 0 | 0 | 0 | 1 | 9 | 0 |
| 8                      | 9 | 1 | 0 | 0 | 0 | 0 | 9 | 1 |
| 9                      | 9 | 1 | 0 | 9 | 0 | 0 | 0 | 1 |
| 10                     | 8| 2 | 0 | 0 | 0 | 2 | 8 | 0 |

Table 2: Configuration of field values on internal vertices for the 10 attractors $a_i$ of Tab. 1.

$$Ad(x, y) = 0 \text{ for } x \leq 16$$

Table 3: The adjacency matrix $Ad$ for the mapping diagram in Tab. 1 with $17 \leq x \leq 24$ plotted vertically and $1 \leq y \leq 24$ plotted horizontally. Since there are no directed lines from internal vertices to external vertices and no lines between external vertices, $Ad(x, y) = 0$ for $x \leq 16$, only rows $x > 16$ are shown. As explained in Sec. 2.3 there are no direct connections from the 16 external vertices to the first two internal vertices serving as outputs.

From the mapping diagram one can deduce the multiplicative structure of the operations $B_i$. A simple indicator for the complexity of this structure is the minimal number of attractor states necessary for the structure to close. For the structure corresponding to mapping diagram in Tab. 1 one can see that the first eight inputs...
Table 4: Optimal output states for all inputs.

| inputs $B_i$ | output states 1 2 |
|--------------|-------------------|
| 1            | 0 0               |
| 2            | 0 $I_{\text{max}}$ |
| 3            | 0 $I_{\text{max}}$ |
| 4            | 0 $I_{\text{max}}$ |
| 5            | 0 $I_{\text{max}}$ |
| 6            | $I_{\text{max}}$ 0 |
| 7            | $I_{\text{max}}$ 0 |
| 8            | $I_{\text{max}}$ 0 |
| 9            | $I_{\text{max}}$ 0 |
| 10           | $I_{\text{max}}$ 0 |
| 11           | $I_{\text{max}}$ 0 |

give rise to very simple relations:

$$B_i B_j = B_i \quad \text{for } i \leq 8 \text{ and } j \text{ arbitrary},$$

representing projection operators. Furthermore, some of these elements are identical:

$$B_4 = B_2 \quad , \quad B_5 = B_3 \quad , \quad B_6 = B_7 .$$

The remaining three elements generate new elements of the multiplicative structure. Simple products of these three elements yield four relations,

$$B_9^2 = B_9 \quad , \quad B_{10}^2 = B_{10} \quad , \quad B_{11}^2 = B_{11} \quad , \quad B_{10} B_{11} = B_8 ,$$

leaving us with five new elements. The total multiplicative structure contains more than 20 elements.

### 2.3 Learning on Graphs

In a very elementary way, the described graphs can be used to simulate simple supervised learning processes. This can be achieved by considering the inputs as stimuli to which the rest of the graph reacts in order to produce an optimal output. In order to define such an optimal output, two of the internal vertices (vertex 1 and 2 in Tab. 2) are defined as output vertices, on which particular field values as given in Tab. 4 are defined as optimal. As we want to investigate how input from the 16 external vertices is processed onto the two output vertices by the remaining six internal vertices, direct connections from external vertices to output vertices are excluded.
The learning process is intended to produce field states on the six internal vertices which map external vertices $B_i$ onto output states as close as possible to those given in Table 4. The internal structure of the graph is, thus, optimized in such a way that its links (connectivities) and vertices (field values) finally give rise to optimal output states.

The following measure of variance serves to quantify the distance of actual output states $u(z_i)$ from optimal output states $u(z_{i,\text{opt}})$ ($i = 1, 2$):

$$v = \sum_{\text{ext. cond.}} \sum_{t=10}^{30} \sum_{\text{output states}} (u(z_i, t) - u(z_{i,\text{opt}}))^2.$$  

The sum extends over all 11 external conditions, over 20 time steps (beginning after the first 10 transient time steps), and over the two output vertices. A variance $v = 0$ implies optimal learning, i.e. an optimized structure of the six internal vertices has been reached. For a random graph, $v$ is of the order of $15-20 \times 10^4$. (Note that the fitness of the graph is related to the inverse of its variance $v$.)

In order to find an optimized graph (an "optimal learner") with respect to a predefined input-output pattern, a random graph is used as an initial condition and randomly selected single-link changes (insertion or deletion of a directed link) are offered successively, implying changes of the state values on internal and output vertices according to Eqs. (6) and (7). (Note that this strategy differs from optimization based on changing the strength of links, e.g. by Hebb's rule.) The initial random graph contains only undirected links, and there are no connections of input-input, input-output, and output-output vertices.

If the variance of a graph after a link change decreases, it is accepted, otherwise rejected. In this way, a sequence of graphs is generated with improving output behavior. In many cases the sequence terminates with a variance much larger than 0 (between $10^2$ and $10^4$). In such cases the evolution of the graph ends in a local minimum far away from optimal behavior. In other cases the sequence ends with an optimal learner, $v = 0$.

### 3 Non-Commutativity of Inputs and Non-Monotonic Complexity of Learning

#### 3.1 Output Dependence on the Sequence of Inputs

The inputs $B_i$ for the learning process are always presented in the same sequence from $i = 1$ up to $i = 11$. Each input is presented for 30 time steps, after which the next input follows. Except the random initialization of the fields on the internal and output vertices at the beginning of the learning run, there is no randomization when inputs are changed. In this case, the field values start with the attractor state of the previous input.
Table 5: The mapping diagram for a perfect learner with 11 inputs $B_i$ and 11 attractors $a_i$. The entries show the number $i$ of the attractor state which is obtained by applying $B_i$ (plotted vertically) to $a_i$ (plotted horizontally).

| input | attractor state $a_i$ |
|-------|-----------------------|
|       | 1 2 3 4 5 6 7 8 9 10 11 |
| $B_i$ | 1 1 1 1 1 1 1 1 1 1 1 |
| 1     | 2 2 2 2 2 2 2 2 2 2 2 |
| 2     | 3 3 3 3 3 3 3 3 3 3 3 |
| 3     | 4 4 4 4 4 4 4 4 4 4 4 |
| 4     | 5 5 5 5 5 5 5 5 5 5 5 |
| 5     | 6 6 6 6 6 6 6 6 6 6 6 |
| 6     | 7 7 7 7 7 7 7 7 7 7 7 |
| 7     | 8 8 8 8 8 8 8 8 8 8 8 |
| 8     | 9 9 9 9 9 9 9 9 9 9 9 |
| 9     | 10 10 10 10 10 10 10 10 10 10 |
| 10    | 11 11 11 11 11 11 11 11 11 11 |
| 11    | 11 11 11 11 11 11 11 11 11 11 |

It turns out that the graphs not only learn to provide optimal outputs for individual inputs, but they learn to do so for particular sequences of inputs. In most cases, input $i + 1$ is recognized correctly (in the sense that the fields on the output vertices assume the optimal values) only if the previous input $i$ was recognized correctly and the starting configuration of the fields for input $i + 1$ corresponds to the attractor for input $i$.

The multiplicative structure introduced above expresses how sensibly the reaction of graphs to the presentation of an input depends on previous inputs. For “perfect learners”, optimally recognizing each input independently of the previous configuration, the multiplicative structure of the inputs is quite trivial: for any initial state, each input operation $B_i$ simply projects the system onto its corresponding attractor. This gives rise to a mapping diagram as in Tab. 5.

The multiplicative structure associated with Tab. 5 consists of the 11 elements $B_i$ which are idempotent,

$$B_i^2 = B_i \text{ for all } i,$$

and satisfy the relation

$$B_i B_j = B_i \text{ for all } i, j,$$

hence they are non-commutative, though associative:

$$B_i(B_j B_k) = (B_i B_j)B_k = B_i \text{ for all } i, j, k.$$

Since the optimal reaction of a graph to an input is not uniquely related to that input, the attractor providing an optimal output can be identical for different inputs. Therefore, the multiplicative structure of input operations can be even simpler in
the sense that some of the attractors are identical. Table 1 shows a corresponding example with less than 11 attractors.

Deviations from Eq. (10) indicate a more complicated structure of learning operations. If the elements in the same row (i.e. for the same input) of the mapping diagram differ from each other, the reaction of the graph with respect to an input depends on the previous input. This means that the result of a learning process depends on the sequence in which successive learning steps are carried out. This implies that the multiplicative structure of input operations deviates from Eq. (10). Since the $B_i$ are mappings, associativity is valid trivially. However, the structure will generally be non-commutative,

$$B_iB_j \neq B_jB_i,$$ (12)

although it may happen that particular inputs commute, for instance when they project onto the same attractor, such as $B_2$ and $B_4$, or $B_3$ and $B_5$, or $B_6$ and $B_7$ in Tab. 1.

We can now understand how an optimal learner differs from a perfect learner, which recognizes inputs independently of the sequence of their presentation. Comparing Tabs. 2 and 4 shows that attractor $a_1$ leads to the optimal output (field values on the first two vertices) for input $B_1$, attractors $a_2$ and $a_3$ yield the optimal output for inputs $B_2 - B_5$, and attractors $a_4$ and $a_5$ yield the optimal output for inputs $B_6 - B_{11}$. In these cases, optimal learning coincides with perfect learning.

From Tab. 1 we see that inputs $B_1 - B_8$ are recognized independently of previous inputs. By contrast, inputs $B_9$, $B_{10}$ and $B_{11}$ are recognized correctly only if the previous input is $B_8$, $B_9$ and $B_{10}$, respectively. Table 2 shows that attractors $a_7 - a_{10}$ lead to an “almost” correct output for inputs $B_9 - B_{11}$, and the output of $a_6$ differs considerably from any optimal output. Although these situations represent optimal learning, they are different or even far from perfect learning.

If the attractor for a particular input does not consist of one single state (fixed point), but of a periodic sequence of states (limit cycle), idempotency 9 does no longer hold. (Strictly speaking, this is only correct if the number of time steps $t$ in the mapping $B_i = \Phi_{t,b_i}$ and the length of the cycle have no common denominator. Otherwise, the attractor may consist of more states than can be detected by the mapping diagram or the set of inputs $B_i$.)

Note that the structure of learning operations derived here is more general than an algebra (as conjectured by Gernert 2000). There is no identity element, there is no neutral element, and no addition of the elements $B_i$ is defined.

### 3.2 Number of Attractors Versus Variance

In order to investigate the evolution of the set of attractors during the learning process, we focus on the number $N$ of attractor states as a function of learning steps for the entire sequence of graphs starting from a random graph until a graph with optimal learning is reached. Since a large number of attractors intuitively relates
to quite complex structures of the graph during the learning process, we propose to refer to the size of the set of attractors as a possible measure for the complexity of learning. However, it should be emphasized that a rigorous definition of complexity (cf. Wackerbauer et al. 1994) is not yet associated with this notion.

Initially, the graphs are (almost) random and exhibit large variances of the order of $2 \times 10^4$. For these graphs the number of attractor states with respect to the inputs varies over a large range; typical are numbers between 30 and 50. As learning begins, the variance decreases, but the number of attractor states increases, sometimes up to a few hundred. A further decrease in variance, below a value of 6000, causes the number of attractor states to decrease again. For optimal learners (graphs with vanishing variance) the number of attractor states terminates at around $N = 10$. A typical example is shown in Fig. 3.

![Figure 3: Number $N$ of attractor states (vertical axis) as a function of variance $v$ (horizontal axis), starting from a random graph with $v \approx 18000$ and terminating at a graph with optimal response and $v = 0$. The points refer to those graphs which were accepted during the learning process, so that decreasing variance indicates progressive learning. The non-monotonic behavior of the complexity of learning is clearly visible.](image)

We now select a sample of 116 learning sequences starting from random graphs and terminating as (almost) optimal learners. For this sample we count the number of attractor states, i.e. the complexity of learning, for those graphs which were accepted during the process, i.e., for which the variance was always smaller than for any previous graph in the sequence. Their behavior can be seen in Fig. 4 where $N$ is plotted as a function of $v$. It confirms the impression from Fig. 3 that, as learning proceeds, its complexity evolves non-monotonically.

In about 50% of the cases the sequence started with less than $N = 50$ attractor states. The final $N$ was much smaller, and for intermediate stages of learning $N$
Figure 4: Number $N$ of attractor states (vertical axis) as a function of variance $v$ (horizontal axis) for 116 learning sequences starting from random graphs and terminating as (almost) optimal learners with a variance of below 10. The plot shows only those graphs which were accepted during learning. The non-monotonic behavior of the complexity of learning for optimal learners is clearly visible.

Figure 5: Number $N$ of attractor states (vertical axis) as a function of variance $v$ (horizontal axis) for 98 learning sequences starting from random graphs and terminating as non-optimal learners with a variance of above 9700. The plot shows only those graphs which were accepted during learning. The non-monotonic behavior of the complexity of learning is visible for non-optimal learners as well.
reached a maximum during the learning process. In about 85% of all cases the final number of attractor configurations was smaller than 20. The largest final number of attractor states for an optimal learner was 56.

Exceptions from this behavior occur if the initial (random) graph has a number of attractor states that is extremely large, exceeding any other number of attractor states in the sequence. For this case we find a total number of 15 sequences. In 12 of these sequences the initial number of attractors is larger than 100 (with a maximum of 747).

Figure 5 shows a plot of number of attractors as a function of variance for 98 non-optimal learners whose final variance is $\nu > 9700$. Keeping in mind that decreasing variance corresponds to progressive learning, the general trend of Figs. 3 and 4 reappears: the size of the set of attractors, i.e. the complexity of learning, evolves non-monotonically as learning proceeds.

As the main observation of the present subsection, we can state that the number $N$ of attractors required to optimally map a given input onto a predetermined output evolves non-monotonically during the process of learning. While $N$ increases during the initial phase of learning, it decreases again until the learning process is terminated. We interpret this behavior as a non-monotonic complexity of the learning process.

### 4 Is the Complexity of Learning Related to Meaning?

Non-monotonic as opposed to monotonic measures of complexity have been developed and investigated for about two decades; for a comparative overview see Wackerbauer et al. (1994). The property of monotonicity is usually understood as a function of (some measure of) randomness of the pattern or process considered. Monotonic complexity essentially increases as randomness increases: most random features are also most complex. Non-monotonic complexity shows convex behavior as a function of increasing randomness: highest complexity is assigned to features with a mixture of random and non-random elements, while both very low and very high randomness yield minimal complexity.

There is an interesting relationship between the two classes of complexity measures and measures of information; for more details see Atmanspacher (1994) or Atmanspacher (2005). It turns out that monotonic complexity usually corresponds to syntactic information, whereas non-monotonic (convex) complexity corresponds to semantic information or other measures of meaning (see Fig. 6).

As a particularly interesting approach, pragmatic information has been proposed (Weizsäcker 1972) as an operationalized measure of meaning. Its essence is that purely random messages keep providing complete novelty (or primordiality) as they are delivered, while purely non-random messages keep providing complete confirmation (after initial transients). Pragmatic information refers to meaning in
terms of a mixture of confirmation and novelty. Extracting meaning from a message depends on the capability to transform novel elements into knowledge using confirming elements.

It has been speculated (Atmanspacher 1994) that systems having this capacity are able to reorganize themselves in order to flexibly modify their complexity relative to the task that they are supposed to solve. A learning process, in which insight is gained and meaning is understood, may start at low complexity (high randomness, much novelty) and terminate at low complexity (high regularity, much confirmation), but it passes through an intermediate stage of maximal complexity.

The notion of pragmatic information was earlier utilized in this sense for non-equilibrium phase transitions in multimode lasers (Atmanspacher and Scheingraber 1990). It could be shown that a particular well-defined type of pragmatic information, adapted to that case, behaves precisely as indicated above. Pragmatic information is maximal at the unstable stage of the phase transition, and it is low in the preceding and successive stages. However, lasers are physical systems, and it is problematic to ascribe something like an “understanding of meaning” to their behavior.

Figure 6: Schematic illustration of two different classes of complexity measures, corresponding to different information measures and distinguished by their functional dependence on randomness.
Biological networks such as studied in this paper are more realistic systems for a concrete demonstration of the basic idea. The non-monotonic complexity of learning processes as indicated in Sec. 3.2 starts with random graphs and ends with graphs of minimized variance (maximized fitness), which are as non-random as possible under the given conditions. In this sense, a scenario has been established in which the complexity of learning on graphs qualitatively satisfies the conditions required for relating it to a measure of pragmatic information. Within this scenario, our approach suggests that the actual “release of meaning” during learning does not occur when the output is optimized but rather when the complexity is maximized. It is a long-standing desideratum to identify meaning-related physiological features in the brain (Freeman 2003). Since learning is a key paradigm in which the emergence of meaning can be studied, we hope that our approach may offer a useful perspective for progress concerning this problem.

5 Summary

In this contribution an example of supervised learning in recurrent networks of small size implemented on graphs is studied numerically. The elements of the network are treated as vertices of graphs and the connections among the elements are treated as links of graphs. Eleven inputs and two outputs are predefined, and the learning process within the remaining six internal vertices is carried out such as to minimize the difference between the actual output and the predetermined output. Optimization of outputs is achieved by stable configurations at the internal vertices that can be characterized as attractors.

Two particular features of the learning behavior of the network are investigated in detail. First, it is shown that, in general, the mapping from inputs to outputs depends on the sequence of inputs. Thus, the associative multiplicative structure of input operations represented by sets of attractors is, in general, non-commutative. Second, the size of the set of attractors changes as the learning process evolves. With increasing optimization (fitness), the number of attractors increases up to a maximum and then decreases down to a usually small final set for optimal network performance.

Assuming that the size of the set of attractors indicates the complexity of learning, its non-monotonic behavior is of special interest. Since non-monotonic measures of complexity can be related to pragmatic information as a measure of meaning, it is tempting to consider the maximum of complexity as reflecting the release of meaning in learning processes. Further work will be necessary to substantiate this speculation.
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