THE COLORED JONES POLYNOMIAL AND KONTSEVICH-ZAGIER SERIES FOR DOUBLE TWIST KNOTS, II

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Abstract. Let \( K_{(m,p)} \) denote the family of double twist knots where \( 2m - 1 \) and \( 2p \) are non-zero integers denoting the number of half-twists in each region. Using a result of Takata, we prove a formula for the colored Jones polynomial of \( K_{(-m,-p)} \) and \( K_{(-m,p)} \). The latter case leads to new families of \( q \)-hypergeometric series generalizing the Kontsevich-Zagier series. These generalized Kontsevich-Zagier series are conjectured to be quantum modular forms. We also use Bailey pairs and formulas of Walsh to find Habiro-type expansions for the colored Jones polynomials of \( K_{(m,p)} \) and \( K_{(m,-p)} \).

1. Introduction

Let \( K \) be a knot and \( J_N(K; q) \) be the usual \( N \)th colored Jones polynomial, normalized to be 1 for the unknot. Formulas for \( J_N(K; q) \) in terms of \( q \)-hypergeometric series have been proved for several families of knots \([14, 16, 17, 22, 26, 32]\); these have played a prominent role in numerous studies in quantum topology and modular forms \([6, 13, 18, 19, 20, 21, 35]\). In recent work \([24]\), the authors used a theorem of Takata \([30]\) to find \( q \)-hypergeometric expressions for the colored Jones polynomial of double twist knots where each of the two regions consisted of an even number of half-twists. This led to a doubly infinite family of \( q \)-series generalizing the famous Kontsevich-Zagier series \([34, 35]\),

\[
F(q) = \sum_{n \geq 0} (1 - q)(1 - q^2) \cdots (1 - q^n). \tag{1.1}
\]

These generalized Kontsevich-Zagier series are conjectured to be new families of quantum modular forms. Comparing with previously known expressions for the colored Jones polynomials of double twist knots due to Lauridsen \([23]\) led to generalizations of a \( q \)-series “identity” involving \( F(q) \) due to Bryson, Ono, Pitman, and Rhoades \([7]\) – namely, for any root of unity \( q \) one has

\[
F(q^{-1}) = \sum_{n \geq 0} q^{n+1}(1 - q)^2 \cdots (1 - q^n)^2. \tag{1.2}
\]

For a complete description of these results, see \([24]\).

Here we turn our attention to double twist knots where one region has an odd number of half-twists. Recall the standard \( q \)-hypergeometric notation

\[
(a)_n = (a; q)_n := \prod_{k=0}^{n-1} (1 - aq^k)
\]

Date: January 13, 2020.
2010 Mathematics Subject Classification. 33D15, 57M27.
Key words and phrases. double twist knots, colored Jones polynomial.
and the usual \( q \)-binomial coefficient

\[
\left[ \begin{array}{c} n \\ k \end{array} \right] = \left[ \begin{array}{c} n \\ k \end{array} \right] := \frac{(q)_n}{(q)_{n-k}(q)_k}.
\] (1.3)

Consider the family of double twist knots \( K_{(m,p)} \) where \( 2m - 1 \) and \( 2p \) are non-zero integers denoting the number of half-twists in each respective region of Figure 1. Positive integers correspond to right-handed half-twists and negative integers correspond to left-handed half-twists.

![Figure 1. Double twist knots](image)

To state the case \( K_{(-m,-p)} \), we define the functions \( \epsilon_{i,j,m} \) and \( \gamma_{i,m} \) by

\[
\epsilon_{i,j,m} = \begin{cases} 
1, & \text{if } j \equiv -i \text{ or } -i - 1 \pmod{2m + 1}, \\
-1, & \text{if } j \equiv i \text{ or } i - 1 \pmod{2m + 1}, \\
0, & \text{otherwise}
\end{cases}
\] (1.4)

where \( 1 \leq i < j \leq (2m + 1)p - 1 \) with \( (2m + 1) \nmid i \) and \( j \not\equiv m \pmod{2m + 1} \) and

\[
\gamma_{i,m} = \begin{cases} 
1, & \text{if } i \equiv 1, \ldots, m - 1 \pmod{2m + 1}, \\
-1 & \text{if } i \equiv 0, m + 1, \ldots, 2m \pmod{2m + 1}, \\
0 & \text{if } i \equiv m \pmod{2m + 1}
\end{cases}
\] (1.5)

where \( 1 \leq i \leq (2m + 1)p - 2 \). Our first main result is the following.

**Theorem 1.1.** For positive integers \( m \) and \( p \), we have

\[
J_N(K_{(-m,-p)}; q) = q^{(p-1)(N-1)} \sum_{N-1 \geq n_{(2m+1)p-1} \geq \cdots \geq n_1 \geq 0} (q^{1-N})_{n_{(2m+1)p-1}} (-1)^{n_{(2m+1)p-1}} q^{\left(\binom{n_{(2m+1)p-1}}{2} - 1\right)}
\times \prod_{1 \leq i < j \leq (2m+1)p-1 \atop j \equiv m \pmod{2m+1}} q^{\epsilon_{i,j,m}n_i n_j} \prod_{i \equiv m, 2m+1 \pmod{2m+1}} (-1)^{n_i} q^{N n_i + \binom{n_i+1}{2}}
\]
For an example of Theorem 1.1, take $m = 3$ and $p = 1$. We then have

$$J_N(K_{(-3,-1)}; q) = \sum_{N-1 \geq n_6 \geq n_5 \geq n_4 \geq n_3 \geq n_2 \geq n_1 \geq 0} (q^{1-N})_{n_6} (-1)^{n_3 + n_6} q^{N n_3 + (n_3^2 + 1) - (n_6^2 + 1)} \times q^{n_1(n_5+n_6)+n_2(n_4+n_5)-n_1n_2-n_2n_3-n_4n_5-n_5n_6} \times q^{n_1+n_2-n_4-n_5} \left[ \begin{array}{c} n_6 \\ n_5 \\ n_4 \\ n_3 \\ n_2 \\ n_1 \end{array} \right].$$

For the case of $K_{(-m,p)}$, define the functions $\Delta_{i,j,m}$ and $\beta_{i,m}$ by

$$\Delta_{i,j,m} = \begin{cases} 1, & \text{if } j \equiv -i \text{ or } -i+1 \pmod{2m+1}, \\ -1, & \text{if } j \equiv i \text{ or } i+1 \pmod{2m+1}, \\ 0, & \text{otherwise} \end{cases} \quad (1.7)$$

where $1 \leq i < j \leq (2m+1)p$ with $(2m+1) \nmid i$ and $j \not\equiv m+1 \pmod{2m+1}$ and

$$\beta_{i,m} = \begin{cases} 1, & \text{if } i \equiv 1, \ldots, m \pmod{2m+1}, \\ -1, & \text{if } i \equiv m+1, \ldots, 2m \pmod{2m+1}, \\ 0, & \text{if } i \equiv 0 \pmod{2m+1} \end{cases} \quad (1.8)$$

where $1 \leq i \leq (2m+1)p - 1$. For convenience, we define $\beta_{i,0} = 0$ for $1 \leq i \leq p - 1$. Our second main result is the following.

**Theorem 1.2.** For a nonnegative integer $m$ and positive integer $p$, we have

$$J_N(K_{(-m,p)}; q) = q^{p(1-N)} \sum_{N-1 \geq n_1 \geq \cdots \geq n_1 \geq 0} (q^{1-N})_{n_1} (-1)^{n_1} q^{-(n_1^2 + 1)} \times \prod_{1 \leq i < j \leq (2m+1)p \atop j \not\equiv m+1 \pmod{2m+1}} q^{\Delta_{i,j,m} n_i n_j} \prod_{i=m+2 \pmod{2m+1} = 1}^{(2m+1)p-1} (-1)^{n_i} q^{-N n_i + \binom{n_i+1}{2}} \times \prod_{i=1}^{(2m+1)p-1} q^{\beta_{i,m} n_i} \left[ \begin{array}{c} n_i+1 \\ n_i \end{array} \right]. \quad (1.9)$$

The case $m = 0$ of Theorem 1.2 was proved by Hikami [16]. Here $K_{(0,p)} = T_{(2,2p+1)}$, the family of right-handed torus knots. Thus, one recovers $J_N(T_{(2,2p+1)}; q)$ by taking $m = 0$ in (1.9). To see this, we first rewrite (1.9) as

$$J_N(K_{(-m,p)}; q)$$
\[ J_N(T_{(2,2p+1)}; q) = q^{p(1-N)} \sum_{N-1 \geq n_p \geq \cdots \geq n_1 \geq 0} (q^{1-N})_{n_p} q^{-Nn_p} \prod_{i=1}^{p-1} q^{n_i+1-N(n_i+1-2N)} \left[ \frac{n_i+1}{n_i} \right]. \] (1.11)

For another example of Theorem 1.2, consider \( m = p = 2 \). We then have

\[ J_N(K_{(-2,2)}; q) = q^{3(1-N)} \sum_{N-1 \geq n_2 \geq n_1 \geq 0} (q^{1-N})_{n_2} q^{-n_2(n_1+1-2N)} q^{-N(n_2+n_2+n_1)}. \]

Recall that

\[ J_N(K; q^{-1}) = J_N(K^*; q), \] (1.12)

where \( K^* \) denotes the mirror image of the knot \( K \). Thus, since \( K_{(-m,-p)} \) is the mirror image of \( K_{(m+1,p)} \) and \( K_{(0,-p)} \) is the mirror image of \( K_{(0,p)} \), equations (1.6) and (1.9) cover all of the double twist knots in this family, up to a substitution of \( q \) by \( q^{-1} \). Combined with Theorems 1.1 and 1.2 in [24], we have \( q \)-hypergeometric series expressions of this type for all double twist knots.

Another type of \( q \)-hypergeometric formula for the colored Jones polynomial can be deduced from formulas of Walsh [32] together with the theory of Bailey pairs. These formulas are our third main result.

**Theorem 1.3.** For positive integers \( m \) and \( p \), we have

\[ J_N(K_{(m,p)}; q) \]
\[ F = \frac{(q^{1+N})_{n}(q^{1-N})_{n}}{(q)_{n_1}} q^{\sum_{i=1}^{m-1} q^{n_i^2+n_i + n_i+1}} \prod_{j=1}^{p-1} q^{s_j+1} \] 

(1.13)

and

\[ J_N(K_{(m,-p)}; q) = q^{-p(1-N^2)} \sum_{n \geq 0} \frac{(q^{1+N})_{n}(q^{1-N})_{n}}{(q)_{n_1}} (-1)^{n} q^{-(n+1)} \]

\[ \times \prod_{i=1}^{m-1} q^{n_i^2+n_i + n_i+1} \prod_{j=1}^{p-1} q^{-s_j-s_j+1} \] 

(1.14)

In view of (1.9) and (1.14), we define the q-series \( F_{m,p}(q) \) for \( m \geq 0 \) and \( p \geq 1 \) and \( U_{m,p}(x; q) \) for \( m, p \geq 1 \) by

\[ F_{m,p}(q) = q^{(2m+1)p-1} \sum_{n \geq 0} (q^{2m+1})_{m} (-1)^{n} q^{-(n+1)} \prod_{j=1}^{(2m+1)p-1} q^{-xq} \] 

(1.15)

and

\[ U_{m,p}(x; q) = q^{-p} \sum_{n \geq 0} \frac{(-xq)_{n}}{(q)_{n_1}} (-1)^{n} q^{-(n+1)} \]

\[ \times \prod_{i=1}^{m-1} q^{n_i^2+n_i + n_i+1} \prod_{j=1}^{p-1} q^{-s_j-s_j+1} \] 

(1.16)

Note that neither \( F_{m,p}(q) \) nor \( U_{m,p}(x; q) \) is defined anywhere except at roots of unity. In this case, we have

\[ F_{m,p}(\zeta_N) = J_N(K_{(-m,p)}; \zeta_N) \] 

(1.17)
and
\[ U_{m,p}(-1; \zeta_N) = J_N(K_{(-m,-p)}; \zeta_N) \] (1.18)
for any \( N \)th root of unity \( \zeta_N \). By (1.12), (1.17) and (1.18) and since the mirror image of \( K_{(-m,p)} \) is \( K_{(m+1,-p)} \), we immediately have the following.

**Corollary 1.4.** If \( \zeta_N \) is any root \( N \)th root of unity, then we have
\[ F_{m,p}(\zeta_N) = U_{m+1,p}(-1; \zeta_N^{-1}). \] (1.19)

Similar “dualities” involving \( q \)-hypergeometric series at roots of unity can be found in [7, 9, 10, 20, 24]. As the case \( F_{0,1}(q) \) is equal to \( q \) times the Kontsevich-Zagier series (1.1) we refer to the \( q \)-series \( F_{m,p}(q) \) as the Kontsevich-Zagier series for odd double twist knots.

Similarly, motivated by (1.6) and (1.13), we define the \( q \)-series \( \mathcal{F}_{m,p}(q) \) and \( \Omega_{m,p}(x; q) \) for \( m, p \geq 1 \) by
\[
\mathcal{F}_{m,p}(q) = q^{1-p} \sum_{n(2m+1)p-1 \geq \cdots \geq n_1 \geq 0} (q)_{n(2m+1)p-1} (-1)^n q^{-\left(\begin{array}{c} n(2m+1)p-1+1 \\ 2 \end{array}\right)} \prod_{1 \leq i < j \leq (2m+1)p-1 \atop j \not\equiv m \pmod{2m+1}} q^{i,j,m,n_i n_j} \times \prod_{i=1 \atop i \equiv m, 2m+1 \pmod{2m+1}} (2m+1)p-2 \left(\begin{array}{c} n \\ 2 \end{array}\right) \prod_{i=1}^{(2m+1)p-2} \left(\begin{array}{c} n_i+1 \\ 2 \end{array}\right) \prod_{i=1}^{(2m+1)p-2} q^{-n_i n_i+1+\gamma_i, m, n_i} \left[\begin{array}{c} n_i+1 \\ n_i \end{array}\right] \right) \] (1.20)

and
\[
\Omega_{m,p}(x; q) = q^p \sum_{n \geq 0} \frac{(-x q)_{n} (-x^{-1} q)_{n}}{(q)_{n_1}} \prod_{i=1}^{m-1} q^{n_i^2+n_i} \left[\begin{array}{c} n_i+1 \\ n_i \end{array}\right] \prod_{j=1}^{p-1} q^{s_j^2+s_j} \left[\begin{array}{c} s_j+1 \\ s_j \end{array}\right]. \] (1.21)

Here, \( \Omega_{m,p}(x; q) \) is well-defined for \( |q| < 1 \) and for \( q \) a root of unity when \( x = -1 \) while \( \mathcal{F}_{m,p}(q) \) is only defined at roots of unity. Then
\[ \mathcal{F}_{m,p}(\zeta_N) = J_N(K_{(-m,-p)}; \zeta_N) \] (1.22)
and
\[ \Omega_{m,p}(-1; \zeta_N) = J_N(K_{(m,p)}; \zeta_N) \] (1.23)
for any \( N \)th root of unity \( \zeta_N \), giving the following.

**Corollary 1.5.** If \( \zeta_N \) is any root \( N \)th root of unity, then we have
\[ \mathcal{F}_{m,p}(\zeta_N) = \Omega_{m+1,p}(-1; \zeta_N^{-1}) \] (1.24)

The rest of this paper is organized as follows. In Section 2, we recall Takata’s main theorem and provide some preliminaries. In Sections 3 and 4, we prove Theorems 1.1 and 1.2. In Section 5, we prove Theorem 1.3. In Section 6, we conclude with some remarks.
2. Preliminaries

We begin by recalling the setup from [30]. Let \( l \) and \( t \) be coprime odd integers with \( l > t \geq 1 \) and \( p' := \frac{l-1}{2} \). For \( 1 \leq j \leq p' \), define integers \( r(j) \) such that \( r(j) \equiv (2j - 1)t \mod{2l} \) and \(-l < r(j) < l\). We put \( \sigma_j := (-1)^{\frac{(2j-1)t}{2}} \), \( r'(j) := \frac{|r(j)|+1}{2} \) and \( i_{r'(j)} = j \) (and thus \( i_k = j \) if and only if \( r'(j) = k \)). For an integer \( i \), \( \text{sgn}(i) \) denotes the sign of \( i \). Let \( n = (n_1, \ldots, n_{p'}) \) and \( n_s = 0 \) for \( s \leq 0 \). Finally, define

\[
\kappa(p') = \begin{cases} 
-Nn_{p'} & \text{if } \sigma_{p'} = -1, \\
0 & \text{if } \sigma_{p'} = 1
\end{cases} \tag{2.1}
\]

and

\[
\tau(j) = \begin{cases} 
(-1)^{n_j - n_j - 1} & \text{if } \sigma_j = -1, \\
\frac{q^{n_j - n_j - 1} + 1}{2} & \text{if } \sigma_j = 1.
\end{cases} \tag{2.2}
\]

Consider the family of 2-bridge knots \( b(l, t) \) (see [8] or [25]). The main result in [30] is an explicit formula for the colored Jones polynomial of \( b(l, t)^* \).

**Theorem 2.1.** We have

\[
J_N(b(l, t)^*; q) = \sum_{N-1 \geq n' \geq \cdots \geq n_1 \geq 0} q^{a(n) N + b_1(n) + b_2(n)} X(n) \tag{2.3}
\]

where\(^1\)

\[
a(n) = -\frac{1}{2} \sum_{j=1}^{p'} \left( \sum_{k=r'(j)}^{j} (\sigma_{i_k} + \sigma_{i_{{p'+1-k}}}) (n_j - n_{j-1}) - \frac{1}{2} \sum_{j=1}^{p'-1} (\sigma_{j+1} + \sigma_{i_{p'+1-j}}) n_j \right) \\
- \frac{1}{2} (\sigma_{p'} + 1) n_{p'} - \sum_{j=1}^{p'} \sigma_j,
\]

\[
b_1(n) = -a(n) + \sum_{k=1}^{l-t} \frac{1 - \sigma_{i_k}}{2} n_{i_k-1} - \sum_{k=\frac{l-t}{2}+1}^{p'} n_{i_k-1} + \sum_{k=\frac{l-t}{2}+1}^{p'} \frac{1 + \sigma_{i_k}}{2} n_{i_k} - (1 + \sigma_{p'}) n_{p'} \\
+ \frac{1}{2} \sum_{j=1}^{p'-1} (\sigma_{j+1} - \sigma_j) n_j \\
- \frac{1}{2} \sum_{k=1}^{p'-1} \sum_{k'=k+1}^{p'} \frac{1 + \text{sgn}(i_k - i_{k'})}{2} (\sigma_{i_k} - \sigma_{i_{p'}}) (n_{i_k} - n_{i_k-1}) (n_{i_{k'}} - n_{i_{k'}-1}) \\
+ \sum_{j=1}^{p'} \sigma_j \left( \sum_{k=1}^{r'(j)} (n_{i_k} - n_{i_k-1}) \right) n_{j-1},
\]

\(^1\)Note that there is a misprint in the definition of \( X(n) \) in [30]. Each \( \overline{q} \) in the prefactor should be \( q \).
Lemma 2.2. For $l = 4mp + 2p - 1$ and $t = 4mp - 1$, we have

(i) $\sigma_j = \begin{cases} 1 & \text{if } j \equiv 1, 2, \ldots, m \pmod{2m+1}, \\ -1 & \text{if } j \equiv 0, m+1, \ldots, 2m \pmod{2m+1}. \end{cases}$

(ii) To compute $i_k$, apply the following algorithm. Divide the integers from 1 to $p'$ into $2m$ intervals, each of length $p$, and a final interval of length $p - 1$. The value of $i_k$ is $(2m+1)(k-1) + m + 1$ in the first interval and $(2m+1)(2p-k) + m$ in the second. If $j > 1$ is odd, then to obtain the value of $i_k$ in the $j$th interval, subtract $2(2m+1)p - 1$ from the formula for $i_k$ in the $(j-2)$th interval. If $j > 2$ is even, then to obtain the value of $i_k$ in the $j$th interval, add $2(2m+1)p - 1$ to the formula for $i_k$ in the $(j-2)$th interval.

(iii) To compute $\sigma_{ik}$, apply the following algorithm. Divide the integers from 1 to $p'$ into $2m$ intervals, each of length $p$, and a final interval of length $p - 1$. The value of $\sigma_{ik}$ alternates between $-1$ and $1$ starting with $-1$ in the first interval.

Lemma 2.3. Let $l = 4mp + 2p - 1$ and $t = 4mp - 1$. Then for $1 \leq k \leq p'$ and $1 \leq j \leq p' - 1$ we have

(i) $\sigma_{ik} + \sigma_{p' + 1 - k} = \begin{cases} 2 & \text{if } ip + 1 \leq k \leq (i+1)p - 1 \text{ for } i = 1, 3, \ldots, 2m - 1, \\ -2 & \text{if } ip + 1 \leq k \leq (i+1)p - 1 \text{ for } i = 0, 2, \ldots, 2m, \\ 0 & \text{if } k = ip \text{ for } i = 1, 2, \ldots, 2m. \end{cases}$

(ii) $\sigma_{j+1} + \sigma_{p' + 1 - j} = \begin{cases} -2 & \text{if } j \equiv m \pmod{2m+1}, \\ 0 & \text{otherwise}. \end{cases}$

Lemma 2.4. For $l = 4mp + 2p + 1$ and $t = 4mp + 1$, we have

(i) $\sigma_j = \begin{cases} 1 & \text{if } j \equiv 1, 2, \ldots, m+1 \pmod{2m+1}, \\ -1 & \text{if } j \equiv 0, m+2, \ldots, 2m \pmod{2m+1}. \end{cases}$
Applying (2.4), (2.5), (2.7), (2.8) and reindexing yields that a

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2.5 to obtain we have

We now illustrate the computation of

(ii) To compute $i_k$, apply the following algorithm. Divide the integers from 1 to $p'$ into $2m+1$ intervals, each of length $p$. The value of $i_k$ is $(2m+1)(k-1) + m + 1$ in the first interval and $(2m+1)(2p - k) + m + 2$ in the second. If $j > 1$ is odd, then to obtain the value of $i_k$ in the $j$th interval, subtract $2(2m+1)p + 1$ from the formula for $i_k$ in the $(j-2)$th interval. If $j > 2$ is even, then to obtain the value of $i_k$ in the $j$th interval, add $2(2m+1)p + 1$ to the formula for $i_k$ in the $(j-2)$th interval.

(iii) To compute $\sigma_{ik}$, apply the following algorithm. Divide the integers from 1 to $p'$ into $2m+1$ intervals, each of length $p$. The value of $\sigma_{ik}$ alternates between 1 and $-1$ starting with 1 in the first interval.

Lemma 2.5. Let $l = 4mp + 2p + 1$ and $t = 4mp + 1$. Then for $1 \leq k \leq p'$ and $1 \leq j \leq p' - 1$ we have

\[
(i) \quad \sigma_{ik} + \sigma_{ip' + 1 - k} = \begin{cases} 
2 & \text{if } ip + 1 \leq k \leq (i + 1)p \text{ for } i = 0, 2, \ldots, 2m, \\
-2 & \text{if } ip + 1 \leq k \leq (i + 1)p \text{ for } i = 1, 3, \ldots, 2m - 1.
\end{cases}
\]

\[
(ii) \quad \sigma_{j+1} + \sigma_{p' + 1 - j} = \begin{cases} 
2 & \text{if } j \equiv 0 \pmod{2m+1}, \\
0 & \text{otherwise}.
\end{cases}
\]

We now illustrate the computation of $a(n)$ and $b_1(n) + b_2(n)$ for $l = 10p + 1$ and $t = 8p + 1$. The routine evaluation of $X(n)$ is left to the reader. First, we take $m = 2$ in Lemmas 2.4 and 2.5 to obtain

\[
\sigma_j = \begin{cases} 
1 & \text{if } j \equiv 1, 2, 3 \pmod{5}, \\
-1 & \text{if } j \equiv 0, 4 \pmod{5}
\end{cases} \quad (2.4)
\]

\[
i_k = \begin{cases} 
5(k-1) + 3 & \text{if } 1 \leq k \leq p, \\
5(2p - k) + 4 & \text{if } p + 1 \leq k \leq 2p, \\
5k - 10p - 3 & \text{if } 2p + 1 \leq k \leq 3p, \\
20p - 5k + 5 & \text{if } 3p + 1 \leq k \leq 4p, \\
5k - 20p - 4 & \text{if } 4p + 1 \leq k \leq 5p
\end{cases} \quad (2.5)
\]

\[
\sigma_{ik} = \begin{cases} 
1 & \text{if } 1 \leq k \leq p, \\
-1 & \text{if } p + 1 \leq k \leq 2p, \\
1 & \text{if } 2p + 1 \leq k \leq 3p, \\
-1 & \text{if } 3p + 1 \leq k \leq 4p, \\
1 & \text{if } 4p + 1 \leq k \leq 5p
\end{cases} \quad (2.6)
\]

\[
\sigma_{ik} + \sigma_{ip' + 1 - k} = \begin{cases} 
2 & \text{if } 1 \leq k \leq p, \\
-2 & \text{if } p + 1 \leq k \leq 2p, \\
2 & \text{if } 2p + 1 \leq k \leq 3p, \\
-2 & \text{if } 3p + 1 \leq k \leq 4p, \\
2 & \text{if } 4p + 1 \leq k \leq 5p
\end{cases} \quad (2.7)
\]

and

\[
\sigma_{j+1} + \sigma_{p' + 1 - j} = \begin{cases} 
2 & \text{if } j \equiv 0 \pmod{5}, \\
0 & \text{otherwise}.
\end{cases} \quad (2.8)
\]

Applying (2.4), (2.5), (2.7), (2.8) and reindexing yields that $a(n)$ equals
By (2.4)–(2.13), the sum of \( b \) and \( b \) is

\[
-\frac{1}{2} \sum_{j=1}^{5p} \left( \sum_{k=r(j)}^{5p} \left( \sigma_{ik} + \sigma_{i_{5p+1-k}} \right) \right) (n_j - n_{j-1}) = \sum_{j=1}^{p-1} n_{5j} - p
\]

\[
= -\frac{1}{2} \sum_{j=1}^{p} \left( \sum_{k=4p-j+1}^{5p} \left( \sigma_{ik} + \sigma_{i_{5p+1-k}} \right) \right) (n_{5j} - n_{5j-1})
\]

\[
+ \sum_{j=1}^{p} \left( \sum_{k=4p+j}^{5p} \left( \sigma_{ik} + \sigma_{i_{5p+1-k}} \right) \right) (n_{5j-4} - n_{5j-5}) + \sum_{j=1}^{p} \left( \sum_{k=2p+j}^{5p} \left( \sigma_{ik} + \sigma_{i_{5p+1-k}} \right) \right) (n_{5j-3} - n_{5j-4})
\]

\[
+ \sum_{j=1}^{p} \left( \sum_{k=3p+j}^{5p} \left( \sigma_{ik} + \sigma_{i_{5p+1-k}} \right) \right) (n_{5j-2} - n_{5j-3}) + \sum_{j=1}^{p} \left( \sum_{k=2p-j+1}^{5p} \left( \sigma_{ik} + \sigma_{i_{5p+1-k}} \right) \right) (n_{5j-1} - n_{5j-2})
\]

\[
- \sum_{j=1}^{p-1} n_{5j} - p
\]

\[= \sum_{j=1}^{p} n_{5j-2} - \sum_{j=1}^{p-1} n_{5j} - p. \tag{2.9}\]

By (2.4) and (2.6), the second and fifth sums in \( b_1(n) \) are zero. We then use (2.4)–(2.6) and reindex to obtain

\[ - \sum_{k=p+1}^{5p} n_{ik-1} = - \left( \sum_{k=p+1}^{2p} n_{ik-1} + \sum_{k=2p+1}^{3p} n_{ik-1} + \sum_{k=3p+1}^{4p} n_{ik-1} + \sum_{k=4p+1}^{5p} n_{ik-1} \right) \]

\[= - \left( \sum_{j=1}^{p} (n_{5j-2} + n_{5j-4} + n_{5j-1} + n_{5j-5}) \right), \tag{2.10}\]

\[\sum_{k=p+1}^{5p} \frac{1+\sigma_{ik}}{2} n_{ik} = \sum_{k=2p+1}^{3p} n_{ik} + \sum_{k=3p+1}^{4p} n_{ik} = \sum_{j=1}^{p} (n_{5j-3} + n_{5j-4}), \tag{2.11}\]

\[\frac{1}{2} \sum_{j=1}^{p-1} (\sigma_{j+1} - \sigma_{j}) n_j = \sum_{j=1}^{p} n_{5j} - 2 \sum_{j=1}^{p-1} n_{5j-2} \tag{2.12}\]

and

\[b_2(n) = \sum_{k=p+1}^{2p+1} \frac{1+\sigma_{ik}}{2} n_{ik-1} = \sum_{k=2p+1}^{3p} n_{ik-1} = \sum_{j=1}^{p} n_{5j-4}. \tag{2.13}\]

By (2.9)–(2.13), the sum of \( b_2(n) \) and the first six terms in \( b_1(n) \) equals

\[p + \sum_{j=1}^{p-1} n_{5j} + \sum_{j=1}^{p} (n_{5j-4} + n_{5j-3} - n_{5j-2} - n_{5j-1}). \tag{2.14}\]
To compute the seventh term in \( b_1(n) \), we use (2.5) and (2.6) to observe that \( k < k' \) and \( \sigma_{ik} \neq \sigma_{ik'} \) if and only if either \( 1 \leq k \leq p \) and \( p + 1 \leq k' \leq 2p \) or \( 1 \leq k \leq p \) and \( 3p + 1 \leq k' \leq 4p \) or \( p + 1 \leq k \leq 2p \) and \( 2p + 1 \leq k' \leq 3p \) or \( p + 1 \leq k \leq 2p \) and \( 4p + 1 \leq k' \leq 5p \) or \( 2p + 1 \leq k \leq 3p \) and \( 3p + 1 \leq k' \leq 4p \) or \( 3p + 1 \leq k \leq 4p \) and \( 4p + 1 \leq k' \leq 5p \). Also, \( \text{sgn}(i_k - i_{k'}) = 1 \) if and only if \( i_k > i_{k'} \) and either \( i_k = 5k - 2 \) for \( 1 \leq k \leq p \) and \( i_{k'} = 10p - 5k' + 4 \) for \( p + 1 \leq k' \leq 2p \) or \( i_k = 5k - 2 \) for \( 1 \leq k \leq p \) and \( i_{k'} = 20p - 5k + 5 \) for \( 3p + 1 \leq k \leq 4p \) or \( i_k = 20p - 5k + 4 \) for \( p + 1 \leq k \leq 2p \) or \( i_{k'} = 5k' - 10p - 3 \) for \( 2p + 1 \leq k' \leq 3p \) or \( i_k = 20p - 5k + 4 \) for \( p + 1 \leq k \leq 2p \) and \( i_{k'} = 5k' - 20p - 4 \) for \( 4p + 1 \leq k \leq 5p \) or \( i_k = 5k - 10p - 3 \) for \( 2p + 1 \leq k \leq 3p \) and \( i_{k'} = 20p - 5k' + 5 \) for \( 3p + 1 \leq k \leq 4p \) or \( i_k = 20p - 5k + 5 \) for \( 3p + 1 \leq k \leq 4p \) and \( i_{k'} = 5k' - 20p - 4 \) for \( 4p + 1 \leq k \leq 5p \). Taking these cases into account and reindexing, we have

\[
-\frac{1}{2} \sum_{k=1}^{5p-1} \sum_{k'=k+1}^{5p} \frac{1 + \text{sgn}(i_k - i_{k'})}{2} (\sigma_{ik} - \sigma_{ik'}) (n_{ik} - n_{ik-1})(n_{ik'} - n_{ik'-1})
\]

\[
= -\sum_{k=1}^{p} \sum_{k'=2p-k+1}^{2p} (n_{ik} - n_{ik-1})(n_{ik'} - n_{ik'-1}) - \sum_{k=1}^{p} \sum_{k'=4p-k+2}^{4p} (n_{ik} - n_{ik-1})(n_{ik'} - n_{ik'-1})
\]

\[
+ \sum_{k=p+1}^{3p} \sum_{k'=2p-k+1}^{4p} (n_{ik} - n_{ik-1})(n_{ik'} - n_{ik'-1}) + \sum_{k=p+1}^{4p} \sum_{k'=6p-k+2}^{8p-1} (n_{ik} - n_{ik-1})(n_{ik'} - n_{ik'-1})
\]

\[
= -\sum_{j=1}^{j'=1} \sum_{j=1}^{p} (n_{5j-2} - n_{5j-3})(n_{5j'-6} - n_{5j'-7}) - \sum_{j=1}^{j'=1} \sum_{j=1}^{p} (n_{5j-2} - n_{5j-3})(n_{5j'-5} - n_{5j'-6})
\]

\[
+ \sum_{j=1}^{j'=1} \sum_{j=1}^{p} (n_{5j-1} - n_{5j-2})(n_{5j'-4} - n_{5j'-5}) + \sum_{j=1}^{j'=1} \sum_{j=1}^{p} (n_{5j-1} - n_{5j-2})(n_{5j'-4} - n_{5j'-5})
\]

\[
- \sum_{j=1}^{j'=1} \sum_{j=1}^{p} (n_{5j-3} - n_{5j-4})(n_{5j'-6} - n_{5j'-7}) + \sum_{j=1}^{j'=1} \sum_{j=1}^{p} (n_{5j-3} - n_{5j-4})(n_{5j'-4} - n_{5j'-5}).
\]
Thus, combining (2.9)–(2.16) implies that $b_1(n) + b_2(n)$ equals
\[p + \sum_{j=1}^{p-1} n_{5j} + \sum_{j=1}^{p} (n_{5j-4} + n_{5j-3} - n_{5j-2} - n_{5j-1})\]

\[- \sum_{j=1}^{p} \sum_{j'=1}^{j} (n_{5j-2} - n_{5j-3})(n_{5j'-6} - n_{5j'-7}) - \sum_{j=1}^{p} \sum_{j'=1}^{j} (n_{5j-2} - n_{5j-3})(n_{5j'-5} - n_{5j'-6})\]

\[+ \sum_{j=1}^{p} \sum_{j'=1}^{j} (n_{5j-1} - n_{5j-2})(n_{5j'-3} - n_{5j'-4}) + \sum_{j=1}^{p} \sum_{j'=1}^{j} (n_{5j-1} - n_{5j-2})(n_{5j'-4} - n_{5j'-5})\]

\[+ \sum_{j=1}^{p} (\sum_{j'=j+1}^{j} (n_{5j'-4} - n_{5j'-5}) + n_{5j}) n_{5j-5}\]

\[+ \sum_{j=1}^{p} (\sum_{j'=j+1}^{j} (n_{5j'-3} - n_{5j'-4}) + \sum_{j'=j+1}^{j} (n_{5j'-1} - n_{5j'-3}) n_{5j-4}\]

\[+ \sum_{j=1}^{p} (\sum_{j'=j+1}^{j} (n_{5j'-2} - n_{5j'-3}) n_{5j-3}\]

\[- \sum_{j=1}^{p} (\sum_{j'=j+1}^{j} (n_{5j'-1} - n_{5j'-4}) + \sum_{j'=j}^{j} (n_{5j'} - n_{5j'-4}) n_{5j-1}\]

\[- \sum_{j=1}^{p} (\sum_{j'=j+1}^{j} (n_{5j'-2} - n_{5j'-3}) + \sum_{j'=j}^{j} (n_{5j'-1} - n_{5j'-3}) n_{5j-2}.\]

3. Proof of Theorem 1.1

Proof of Theorem 1.1. Using Lemmas 2.2 and 2.3, one can check that for \(l = 4mp + 2p - 1\) and \(t = 4mp - 1\)

\[a(n) = \sum_{j=1}^{p-1} n_{(2m+1)j} + \sum_{j=1}^{p} n_{(2m+1)j-(m+1)} + p - 1 \quad (3.1)\]

and \(b_1(n) + b_2(n)\) equals

\[1 - p + n_{(2m+1)p-1} + \sum_{j=1}^{p} \left( \sum_{i=1}^{m-1} n_{(2m+1)j-2m+i-1} - \sum_{i=m+1}^{2m} n_{(2m+1)j-2m+i-1} \right)\]

\[+ \sum_{j=1}^{p} \sum_{j'=1}^{j} \sum_{k=1}^{m} \sum_{k'=1}^{k} \left( n_{(2m+1)j-k} - n_{(2m+1)j-k-1} \right) \left( n_{(2m+1)j'+k-2m-k'-1} - n_{(2m+1)j'+k-2m-k'-1} \right)\]
Also, by (2.1) and (2.2), $X(n) = \sum_{j=1}^{p} (n(2m+1)j - n(2m+1)j + n(2m+1)j' - 2m - k')$ equals

$$- \sum_{j=1}^{p} \sum_{j'=1}^{m} \sum_{k'=1}^{m-k+1} (n(2m+1)j - m - k - n(2m+1)j - m - k - 1)(n(2m+1)j' - 2m - k' - n(2m+1)j' - 2m - k' - 1)$$

$$+ \sum_{j=1}^{p} \sum_{j'=1}^{m} \sum_{k'=1}^{m-k+1} (n(2m+1)j' - s - n(2m+1)j' - 2m + s - 1)$$

$$+ \sum_{j=1}^{p} \sum_{j'=1}^{m} \sum_{k'=1}^{m-k+1} (n(2m+1)j' - s - n(2m+1)j' - 2m + s - 2) + n(2m+1)j - 2m - 1 - n(2m+1)j' - 2m + 1$$

$$- \sum_{j=1}^{p} \sum_{j'=1}^{m} \sum_{k'=1}^{m-k+1} (n(2m+1)j' - m + s + 1 - n(2m+1)j' - m - s)$$

$$+ \sum_{j=1}^{p} \sum_{j'=1}^{m} \sum_{k'=1}^{m-k+1} (n(2m+1)j' - m + s + 2 - n(2m+1)j' - m - s)$$

$$+ \sum_{j=1}^{p} \sum_{j'=1}^{m} \sum_{k'=1}^{m-k+1} (n(2m+1)j' - m + s + 2 - n(2m+1)j' - m + s + 1)$$

Also, by (2.1) and (2.2), $X(n) = \sum_{j=1}^{p} \sum_{j'=1}^{m} \sum_{k'=1}^{m-k+1} (n(2m+1)j - m - k - n(2m+1)j - m - k - 1)(n(2m+1)j' - 2m - k' - n(2m+1)j' - 2m - k' - 1)$
\[\begin{align*}
- \sum_{j=1}^{p-1} \left( \sum_{j'=j+1}^{p} (n_{3j'-1} - n_{3j'-3}) + n_{3j} \right) n_{3j-1} & - \sum_{j=1}^{p} \sum_{j'=1}^{j} (n_{3j'-1} - n_{3j'-2}) n_{3j-2} \\
- \sum_{j=1}^{p} n_{3j-2} n_{3j-3} & \end{align*}\]

equals

\[\sum_{1 \leq i < j \leq 3p-1} \sum_{i \equiv j \pmod{3}} n_{i} n_{j} - \sum_{i=1}^{3p-2} n_{i} n_{i+1}\]

(3.6)

where \(\epsilon_{i,j,m}\) is given by (1.4). Here, we have used the fact that

\[\begin{align*}
- \sum_{j=1}^{p-1} n_{3j-1} = \sum_{i=1}^{3p-2} \gamma_{i,1} n_{i} + \sum_{i=1}^{p-1} n_{3i},
\end{align*}\]

(3.7)

where \(\gamma_{i,m}\) is given by (1.5), together with the identities

\[\frac{1}{2} \sum_{j=1}^{p} S(1, j, 1) = \sum_{j=1}^{p} \binom{n_{3j-3}}{2} + \sum_{j=1}^{p} \binom{n_{3j-2} + 1}{2} - n_{3j-2} n_{3j-3}\]

(3.8)

where \(S(m, j, s)\) is given by (3.4) and

\[\begin{align*}
\sum_{j=1}^{p} \binom{n_{3j-3}}{2} + \sum_{i=1}^{p-1} n_{3i} = \sum_{i=1}^{p-1} \binom{n_{3i} + 1}{2}.
\end{align*}\]

(3.9)

We now explain how to proceed from (3.5) to (3.6). After taking out the \(j' = j\) term from the fourth sum in the third line of (3.5) and simplifying, we obtain

\[\begin{align*}
\sum_{j=1}^{p} \sum_{j'=1}^{j} n_{3j'-1} n_{3j'-2} + \sum_{j=1}^{p} \sum_{j'=1}^{j-1} n_{3j'-1} n_{3j-3} - \sum_{j=1}^{p} \sum_{j'=1}^{j-1} n_{3j'-2} n_{3j-3} - \sum_{j=1}^{p-1} \sum_{j'=j+1}^{p} n_{3j'-1} n_{3j-1} \\
- \sum_{j=1}^{p-1} n_{3j} n_{3j-1} - \sum_{j=1}^{p} n_{3j-2} n_{3j-3} - \sum_{j=1}^{p} n_{3j-1} n_{3j-2}.\end{align*}\]

(3.10)

The first line of (3.10) corresponds to the first sum in (3.6); namely, the first two sums correspond to \((i, j) \equiv (i, -i) \pmod{3}\) and \((i, j) \equiv (i, -i - 1) \pmod{3}\), respectively, while the second two sums correspond to \((i, j) \equiv (i, i - 1) \pmod{3}\) and \((i, j) \equiv (i, i) \pmod{3}\), respectively. The three sums in the second line of (3.10) match the second sum of (3.6). Thus, we have proven that (3.5) equals (3.6).

We now turn to the general case \(m \geq 2\). Upon comparing (2.3) and (3.1)–(3.4) with (1.6) and then simplifying, it suffices to prove that
\[
\sum_{j=1}^{p} \sum_{j'=1}^{j} \sum_{k=1}^{m} \sum_{k'=1}^{j} \left( n_{(2m+1)j-k} - n_{(2m+1)j-k-1} \right) \left( n_{(2m+1)j'-k} - n_{(2m+1)j'-k-1} \right) \\
- \sum_{j=1}^{p} \sum_{j'=1}^{j} \sum_{m=1}^{m-1} \sum_{k'=1}^{j} \left( n_{(2m+1)j-m-k} - n_{(2m+1)j-m-k-1} \right) \left( n_{(2m+1)j'-m-k} - n_{(2m+1)j'-m-k-1} \right) \\
+ \sum_{s=1}^{m} \sum_{j=1}^{p} \left( \sum_{j'=1}^{j} \left( n_{(2m+1)j'-s} - n_{(2m+1)j'-2m-s-1} \right) \right) n_{(2m+1)j-2m+s-2} \\
- \sum_{j=1}^{p} \left( \sum_{j'=j+1}^{p} \left( n_{(2m+1)j'-1} - n_{(2m+1)j'-(2m+1)} + n_{(2m+1)j} \right) \right) n_{(2m+1)j-1} \\
- \sum_{s=1}^{m} \sum_{j=1}^{p} \left( \sum_{j'=1}^{j} \left( n_{(2m+1)j'-m+s-1} - n_{(2m+1)j'-m-s} \right) \right) n_{(2m+1)j-m+s-2} \\
+ \sum_{j=1}^{p} \left[ \left( \frac{n_{(2m+1)j-2m+1}}{2} \right) - n_{(2m+1)j-2m} n_{(2m+1)j-2m-1} + \left( \frac{n_{(2m+1)j-m-2}}{2} \right) \\
- n_{(2m+1)j-m-1} n_{(2m+1)j-m-2} + \frac{1}{2} \sum_{s=2}^{m-1} S(m, j, s) \right] \\
\text{equals} \\
\sum_{1 \leq i < j \leq (2m+1)p-1}^{(2m+1)p-2} \epsilon_{i,j,m} n_i n_j - \sum_{i=1}^{((2m+1)p-1)} n_i n_{i+1}. 
\] 

(3.12)

Here, we have used the fact that

\[
\sum_{j=1}^{p} \left( \sum_{i=1}^{m-1} n_{(2m+1)j-2m+i-1} - \sum_{i=m+1}^{2m-1} n_{(2m+1)j-2m+i-1} \right) - \sum_{j=1}^{p-1} n_{(2m+1)j-1} \\
= \sum_{i=1}^{(2m+1)p-2} \gamma_{i,m} n_i + \sum_{i=1}^{p-1} n_{(2m+1)i},
\]

(3.13)

together with the identities
\[ \frac{1}{2} \sum_{j=1}^{p} \sum_{s=1}^{m} S(m, j, s) = \sum_{j=1}^{p} \left( \binom{n(2m+1)j-2m-1}{2} + \binom{n(2m+1)j-m-1 + 1}{2} \right) \]

\[ + \sum_{j=1}^{p} \left[ \binom{n(2m+1)j-2m+1}{2} - n(2m+1)j-2m n(2m+1)j-2m-1 + \binom{n(2m+1)j-m-2}{2} \right. \]

\[ \left. - n(2m+1)j-m-1 n(2m+1)j-m-2 + \frac{1}{2} \sum_{s=2}^{m-1} S(m, j, s) \right] \] (3.14)

and

\[ \sum_{j=1}^{p} \left( \binom{n(2m+1)j-2m-1}{2} \right) + \sum_{i=1}^{p-1} n(2m+1)i = \sum_{i=1}^{p-1} \binom{n(2m+1)i + 1}{2}. \] (3.15)

We now sketch how to proceed from (3.11) to (3.12). For 1 \( \leq i \leq 9 \), let \( L_i \) denote the \( i \)th line of (3.11). First note that

\[ L_8 + L_9 = \sum_{j=1}^{p} \sum_{i=1}^{p-1} n^2_{(2m+1)j+i} - \sum_{j=1}^{p-1} \sum_{i=1}^{m} n(2m+1)j+i n(2m+1)j+i-1. \] (3.16)

Next, the sum over \( k' \) in both \( L_1 \) and \( L_2 \) telescopes, and we obtain

\[ L_1 = \sum_{k=1}^{m} \sum_{j=1}^{p} \sum_{j'=1}^{j} (n(2m+1)j-k - n(2m+1)j-k-1)n(2m+1)j'+k-2m-1 \] (3.17)

\[ - \sum_{k=1}^{m} \sum_{j=1}^{p} \sum_{j'=1}^{j} (n(2m+1)j-k - n(2m+1)j-k-1)n(2m+1)j'-2m-1 \]

and

\[ L_2 = - \sum_{k=1}^{m} \sum_{j=1}^{p} \sum_{j'=1}^{j} (n(2m+1)j-m-k - n(2m+1)j-m-k-1)n(2m+1)j'-2m-1 \] (3.18)

\[ + \sum_{k=1}^{m} \sum_{j=1}^{p} \sum_{j'=1}^{j} (n(2m+1)j-m-k - n(2m+1)j-m-k-1)n(2m+1)j'-3m+k-2. \]

Now the sum over \( k \) in the second line of (3.17) and the first line of (3.18) both telescope and so

\[ L_1 = \sum_{k=1}^{m} \sum_{j=1}^{p} \sum_{j'=1}^{j} n(2m+1)j-k n(2m+1)j'+k-2m-1 - \sum_{k=1}^{m} \sum_{j=1}^{p} \sum_{j'=1}^{j} n(2m+1)j-k-1 n(2m+1)j'+k-2m-1 \]

\[ + \sum_{j=1}^{p} \sum_{j'=1}^{j} n(2m+1)j-m-1 n(2m+1)j'+2m-1 - \sum_{j=1}^{p} \sum_{j'=1}^{j} n(2m+1)j-1 n(2m+1)j'-2m-1 \] (3.19)
and

\[
L_2 = \sum_{k=1}^{m} \sum_{j=1}^{p} \sum_{j'=1}^{j} n(2m+1)j - n(2m+1)j' - 3m + k - 2
\]

\[- \sum_{k=1}^{m} \sum_{j=1}^{p} \sum_{j'=1}^{j} n(2m+1)j - m - k - 1n(2m+1)j' - 3m + k - 2 + \sum_{j=1}^{p} \sum_{j'=1}^{j} n(2m+1)j - 2m - 1n(2m+1)j' - 2m - 1
\]

\[- \sum_{j=1}^{p} \sum_{j'=1}^{j} n(2m+1)j - m - 1n(2m+1)j' - 2m - 1.
\]

Observe that the third sum in (3.19) and the fourth sum in (3.20) cancel. Moreover, if we take \( s = 1 \) in the triple sum in \( L_4 \),

\[
\sum_{s=1}^{m} \sum_{j=1}^{p} \sum_{j'=1}^{j} (n(2m+1)j' - s - n(2m+1)j' - 2m + s - 2)n(2m+1)j - 2m + s - 2;
\]

and exchange \( j \) and \( j' \) we see that this cancels with the fourth sum in (3.19) and the third sum in (3.20). Putting this and (3.16) together and expanding all of the sums we find that (3.11) equals

\[
\sum_{k=1}^{m} \sum_{j=1}^{p} \sum_{j'=1}^{j} n(2m+1)j - k\sum_{j'=1}^{k} n(2m+1)j' + k - 2m - 1
\]

\[+ \sum_{k=1}^{m} \sum_{j=1}^{p} \sum_{j'=1}^{j} n(2m+1)j - m - k\sum_{j'=1}^{j} n(2m+1)j' - 3m + k - 2 \]

\[+ \sum_{j=1}^{p} \sum_{j'=1}^{j} n(2m+1)j' - s\sum_{j'=1}^{j} n(2m+1)j' - 2m + s - 2 - \sum_{j=1}^{j} \sum_{j'=1}^{j} n(2m+1)j' - 2m + s - 1n(2m+1)j - 2m + s - 2
\]

\[- \sum_{j=1}^{p} \sum_{j'=1}^{j} \sum_{j'=1}^{j} n(2m+1)j' - 2m + s - 2n(2m+1)j - 2m + s - 2
\]

\[- \sum_{j=1}^{p} \sum_{j'=1}^{j} n(2m+1)j' - 1n(2m+1)j - 1 + \sum_{j=1}^{p} \sum_{j'=1}^{j} n(2m+1)j' - (2m+1)n(2m+1)j - 1
\]

\[- \sum_{j=1}^{p} n(2m+1)j - (2m+1)j - 1
\]

\[- \sum_{j=1}^{p} \sum_{j'=1}^{j} n(2m+1)j' - m + s - 1n(2m+1)j - m + s - 2 + \sum_{j=1}^{p} \sum_{j'=1}^{j} n(2m+1)j' - m - s\sum_{j'=1}^{j} n(2m+1)j - m + s - 2
\]
We then take out the term to cancel with the first sum on the last line of (3.22). In the second line of (3.22), perform the

$$\sum_{j=0}^{p-1} m_i n_i (2m+1)_{j+i} - \sum_{j=0}^{p-1} m_i n_i (2m+1)_{j+i-1}.$$  \hspace{1cm} (3.22)

In the second sum on the fourth line of (3.22), we exchange $j$ and $j'$ and reindex to obtain

$$- \sum_{s=2}^{m-1} n_i (2m+1)_{j-2m+s-2} (2m+1)_{j'-2m+s-2}.$$  \hspace{1cm} (3.23)

We then take out the term $j' = j$ and shift the indices in this term by $j \rightarrow j+1$ and $s \rightarrow s+1$ to cancel with the first sum on the last line of (3.22). In the second line of (3.22), perform the shift $j' \rightarrow j'+1$ and start the sum at $j' = 1$ (as $j' = 0$ gives 0) to obtain

$$- \sum_{s=2}^{m-1} \sum_{j=1}^{p-j} n_i (2m+1)_{j-m-k} n_i (2m+1)_{j'-m+k-1}.$$  \hspace{1cm} (3.23)

Now, in the second sum of the penultimate line of (3.22), we exchange $j$ and $j'$ and reindex, shift by $s \rightarrow s+1$, then remove the $s = 0$ term. Note that what remains cancels with the second sum in (3.23) after removing the $k = m$ term. In total, this yields that (3.11) equals
the last line. Thus, (3.11) equals
\[ i \]

with the second sum in the second line. Finally, the sum in the sixth line is the
\[ s \]

We now simplify further. The \( s = 1 \) term of the first sum in the penultimate line cancels with
the second sum in the same line. Remove the \( j' = j \) term from the first sum in the seventh
line and write it in the last line. The \( s = 1 \) term of the remaining triple sum cancels with the
\( k = 1 \) term of the first sum on the second line. The first sum on the fourth line cancels with
the second sum of the first line once we remove the \( k = m \) term. This \( k = m \) term then cancels
with the \( s = 1 \) term of the second sum of the seventh line. The first sum in the fifth line is the
\( s = m + 1 \) term of the first sum in the penultimate line. The second sum in the fifth line cancels
with the second sum in the second line. Finally, the sum in the sixth line is the \( i = 0 \) term in
the last line. Thus, (3.11) equals
\[
\sum_{s=1}^{m} \sum_{j=1}^{p} \sum_{j'=1}^{p} n(2m+1)j' - m + s - 2n(2m+1)j - m + s - 2 + \sum_{j=1}^{p} \sum_{j'=1}^{p-1} n(2m+1)j - m - 1n(2m+1)j' - m - 1
\]
\[
- \sum_{j=0}^{p-1} m \sum_{i=1}^{m} n(2m+1)j + in(2m+1)j + i - 1. \tag{3.24}
\]

Now we see that this is equal to (3.12) as follows. The first five lines of (3.25) correspond to
the first term in (3.12); namely, the first line of (3.25) corresponds to \( (i, j) \equiv (i, -i) \) \((\text{mod } 2m + 1)\)
while the second line corresponds to \( (i, j) \equiv (i, -i - 1) \) \((\text{mod } 2m + 1)\). The first sums in the
third and fifth lines correspond to \( (i, j) \equiv (i, i - 1) \) \((\text{mod } 2m + 1)\) while the sum in the fourth
line and the second sum in the fifth line correspond to \( (i, j) \equiv (i, i) \) \((\text{mod } 2m + 1)\). Finally, the
sixth line of (3.25) matches the second sum of (3.12). Thus, we have proven that (3.11) equals
(3.12).

\[ \square \]
4. Proof of Theorem 1.2

Proof of Theorem 1.2. As (1.9) reduces to (1.11) when \( m = 0 \) and this case was proven in [16], we assume that \( m \geq 1 \). Using Lemmas 2.4 and 2.5, one can check that for \( t = 4mp + 1 \) and 
\[
\begin{align*}
a(n) &= - \sum_{j=1}^{p-1} n(2m+1)j - \sum_{j=1}^{p} n(2m+1)j - (m) - p \\
\end{align*}
\]
and \( b_1(n) + b_2(n) \) equals
\[
\begin{align*}
p + \sum_{j=1}^{p-1} n(2m+1)j &+ \sum_{j=1}^{p} \left( \sum_{i=1}^{m} n(2m+1)j - 2m - i - 1 \right) - \sum_{i=m+1}^{2m} n(2m+1) - 2m - i - 1 \right) \\
+ \sum_{j=1}^{p} \sum_{j'=1}^{m} \sum_{m-k}^{m-k+1} \sum_{k'=1}^{k} (n(2m+1)j - k) - n(2m+1)j - k (n(2m+1)j' + k - 2m - k' - 1) \right) \\
- \sum_{j=1}^{p} \sum_{j'=1}^{m} \sum_{m-k}^{m-k+1} \sum_{k'=1}^{k} (n(2m+1)j - m - k) - n(2m+1)j - m (n(2m+1)j' + m - k' - 1) \right) \\
+ \sum_{s=1}^{m} \sum_{j=1}^{p} \sum_{j'=1}^{k} (n(2m+1)j' - s) - n(2m+1)j' - 2m - s - 1) \right) \\
+ \sum_{j=1}^{p} \left( \sum_{j'=1}^{p} (n(2m+1)j' - s) - n(2m+1)j' - 2m - s - 1) \right) n(2m+1)j - 2m - s - 1 \right) \\
+ \sum_{j=1}^{p} \left( \sum_{j'=1}^{p} (n(2m+1)j' - n(2m+1)j' - 2m) + n(2m+1)j \right) n(2m+1)j - (2m+1) \right) \\
- \sum_{s=1}^{m} \sum_{j=1}^{p} \left( \sum_{j'=1}^{k-1} (n(2m+1)j' - s) - n(2m+1)j' - 2m - s - 1) \right) \\
+ \sum_{j'=1}^{p} \left( (n(2m+1)j' - s) - n(2m+1)j' - 2m - s - 1) \right) n(2m+1)j - s \right). \tag{4.2}
\end{align*}
\]

Also, by (2.1) and (2.2), \( X(n) \) equals
\[
\begin{align*}
(-1)^{n(2m+1)p} q^{-N n(2m+1)p} \frac{(q)^{N-1}(q)^{p(2m+1)p}}{(q)^{N-n(2m+1)p-1}} \prod_{j=1}^{m+1} \frac{(-1)^{n(2m+1)-n(2m+1)-m}}{q^{\frac{m+1}{2}} \sum_{s=1}^{2m+1} S(m,j,s)} \tag{4.3}
\end{align*}
\]
where \( S(m, j, s) \) is given by (3.4).
Upon comparing (2.3) and (4.1)–(4.3) with (1.9) and then simplifying, it suffices to prove that for \( m \geq 1 \)

\[
\sum_{j=1}^{p} \sum_{j'=1}^{j} \sum_{k=1}^{m} \sum_{k'=1}^{m} (n(2m+1)j-k+1 - n(2m+1)j-k)(n(2m+1)j'+k-2m-k' - n(2m+1)j'+k-2m-k'-1)
\]

\[
- \sum_{j=1}^{p} \sum_{j'=1}^{j} \sum_{m=1}^{m} (n(2m+1)j-m-k+1 - n(2m+1)j-m-k)(n(2m+1)j'+k-2m-k' - n(2m+1)j'-2m-k'-1)
\]

\[
+ \sum_{s=1}^{m} \sum_{j=1}^{p} (\sum_{j'=1}^{j} (n(2m+1)j'-s - n(2m+1)j'-2m+s-1))
\]

\[
+ \sum_{j'=1}^{j} \sum_{j=1}^{p} (n(2m+1)j'-s - n(2m+1)j'-2m+s) \sum_{s=1}^{n(2m+1)j-s} n(2m+1)j-(2m+1)
\]

\[
- \sum_{s=1}^{m} \sum_{j=1}^{p} (\sum_{j'=1}^{j-1} (n(2m+1)j'-s - n(2m+1)j'-2m+s-1))
\]

\[
+ \sum_{j'=1}^{j} \sum_{j=1}^{p} (n(2m+1)j'-s+1 - n(2m+1)j'-2m+s-1) \sum_{s=1}^{n(2m+1)j-s} n(2m+1)j-s
\]

\[
+ \sum_{j=1}^{p} \left[ \binom{n(2m+1)j-2m+1}{2} - n(2m+1)j-2m n(2m+1)j-2m-1 + \binom{n(2m+1)j-m-1}{2}
\right]
\]

\[
- n(2m+1)j-m \sum_{s=2}^{m} S(m, j, s) + \frac{1}{2} \sum_{s=2}^{m} S(m, j, s)
\]

equals

\[
\sum_{1 \leq i < j \leq (2m+1)p} \Delta_{i,j,m} n_i n_j
\]

(4.5)

where \( \Delta_{i,j,m} \) is given by (1.7). Here, we have used (3.15),

\[
\frac{1}{2} \sum_{j=1}^{p} \sum_{s=1}^{m+1} S(m, j, s) = \sum_{j=1}^{p} \left[ \binom{n(2m+1)j-2m-1}{2} + \binom{n(2m+1)j-m+1}{2}
\right]
\]

\[
+ \sum_{j=1}^{p} \left[ \binom{n(2m+1)j-2m+1}{2} - n(2m+1)j-2m n(2m+1)j-2m-1 + \binom{n(2m+1)j-m-1}{2}
\right]
\]
\[ -n\left(2m+1\right)j-mn\left(2m+1\right)j-m-1 + \frac{1}{2} \sum_{s=2}^{m} S(m, j, s) \]

(4.6)

and the fact that

\[ \sum_{j=1}^{p-1} \left( \sum_{i=1}^{m} n\left(2m+1\right)j-2m+i-1 - \sum_{i=m+1}^{2m} n\left(2m+1\right)j-2m+i-1 \right) = \sum_{i=1}^{(2m+1)p-1} \beta_{i,m} n_i \]

(4.7)

where \( \beta_{i,m} \) is given by (1.8). We now sketch how to proceed from (4.4) to (4.5). For \( 1 \leq i \leq 9 \), let \( \hat{L}_i \) denote the \( i \)th line of (4.4). First, note that

\[ \hat{L}_8 + \hat{L}_9 = \sum_{j=0}^{p-1} \sum_{i=1}^{m} n\left(2m+1\right)j+i - \sum_{j=0}^{p-1} \sum_{i=1}^{m} n\left(2m+1\right)j+i-1. \]

(4.8)

Next, the sum over \( k' \) in both \( \hat{L}_1 \) and \( \hat{L}_2 \) telescope and we obtain

\[ \hat{L}_1 = \sum_{k=1}^{m} \sum_{j=1}^{p} \sum_{j'=1}^{j} \left( n\left(2m+1\right)j-k+1 - n\left(2m+1\right)j-k \right) n\left(2m+1\right)j'+k-2m-1 \]

(4.9)

\[ - \sum_{k=1}^{m} \sum_{j=1}^{p} \sum_{j'=1}^{j} \left( n\left(2m+1\right)j-k+1 - n\left(2m+1\right)j-k \right) n\left(2m+1\right)j'-2m-1 \]

and

\[ \hat{L}_2 = - \sum_{k=1}^{m} \sum_{j=1}^{p} \sum_{j'=1}^{j} \left( n\left(2m+1\right)j-m-k+1 - n\left(2m+1\right)j-m-k \right) n\left(2m+1\right)j'-2m-1 \]

(4.10)

\[ + \sum_{k=1}^{m} \sum_{j=1}^{p} \sum_{j'=1}^{j} \left( n\left(2m+1\right)j-m-k+1 - n\left(2m+1\right)j-m-k \right) n\left(2m+1\right)j'-3m+k-2. \]

Now the sum over \( k \) in the second line of (4.9) and the first line of (4.10) both telescope and so

\[ \hat{L}_1 = \sum_{k=1}^{m} \sum_{j=1}^{p} \sum_{j'=1}^{j} n\left(2m+1\right)j-k+1 n\left(2m+1\right)j'+k-2m-1 - \sum_{k=1}^{m} \sum_{j=1}^{p} \sum_{j'=1}^{j} n\left(2m+1\right)j-k n\left(2m+1\right)j'+k-2m-1 \]

(4.11)

\[ + \sum_{j=1}^{p} \sum_{j'=1}^{j} n\left(2m+1\right)j-m n\left(2m+1\right)j'-2m-1 - \sum_{j=1}^{p} \sum_{j'=1}^{j} n\left(2m+1\right)j n\left(2m+1\right)j'-2m-1 \]
and

\[
\hat{L}_2 = \sum_{k=1}^{m} \sum_{j=1}^{p} \sum_{j' = 1}^{j} n_{2(m+1)j'' - m - k + 1} n_{2(m+1)j'' - 3m + k - 2} - \sum_{k=1}^{m} \sum_{j=1}^{p} \sum_{j' = 1}^{j} n_{2(m+1)j'' - m - k} n_{2(m+1)j'' - 3m + k - 2}
\]

\[
+ \sum_{j=1}^{p} \sum_{j' = 1}^{j} n_{2(m+1)j'' - 2m} n_{2(m+1)j'' - 2m - 1} - \sum_{j=1}^{p} \sum_{j' = 1}^{j} n_{2(m+1)j'' - m} n_{2(m+1)j'' - 2m - 1}.
\]

\tag{4.12}

Observe that the first sum in the second line of (4.11) cancels with the second sum in the third line of (4.12). Combine the remaining double sums, then remove the \( j' = j \) term to obtain cancellation with the double sum in \( \hat{L}_5 \). The second sum in this \( j' = j \) term then cancels with the remaining sum in \( \hat{L}_5 \). Next, the \( i = 1 \) term of the second sum of (4.8) cancels with the first sum in this \( j' = j \) term. Putting this together and expanding sums, we now have that (4.4) equals

\[
\sum_{k=1}^{m} \sum_{j=1}^{p} \sum_{j' = 1}^{j} n_{2(m+1)j'' - k + 1} n_{2(m+1)j'' + k - 2m - 1} - \sum_{k=1}^{m} \sum_{j=1}^{p} \sum_{j' = 1}^{j} n_{2(m+1)j'' - k} n_{2(m+1)j'' + k - 2m - 1}
\]

\[
+ \sum_{j=1}^{p} \sum_{j' = 1}^{j} n_{2(m+1)j'' - m - k + 1} n_{2(m+1)j'' - 3m + k - 2} - \sum_{j=1}^{p} \sum_{j' = 1}^{j} n_{2(m+1)j'' - m - k} n_{2(m+1)j'' - 3m + k - 2}
\]

\[
+ \sum_{s=1}^{m} \sum_{j=1}^{p} \sum_{j' = 1}^{j} n_{2(m+1)j'' - s} n_{2(m+1)j'' - 2m + s - 1} - \sum_{s=1}^{m} \sum_{j=1}^{p} \sum_{j' = 1}^{j} n_{2(m+1)j'' - 2m + s - 1} n_{2(m+1)j'' - 2m + s - 1}
\]

\[
+ \sum_{s=1}^{m} \sum_{j=1}^{p} \sum_{j' = j+1}^{j} n_{2(m+1)j'' - s} n_{2(m+1)j'' - 2m + s - 1} - \sum_{s=1}^{m} \sum_{j=1}^{p} \sum_{j' = j+1}^{j} n_{2(m+1)j'' - 2m + s - 1} n_{2(m+1)j'' - 2m + s - 1}
\]

\[
- \sum_{s=1}^{m} \sum_{j=1}^{p} \sum_{j' = 1}^{j} n_{2(m+1)j'' - s} n_{2(m+1)j'' - s} + \sum_{s=1}^{m} \sum_{j=1}^{p} \sum_{j' = 1}^{j} n_{2(m+1)j'' - 2m - s - 1} n_{2(m+1)j'' - s}
\]

\[
- \sum_{s=1}^{m} \sum_{j=1}^{p} \sum_{j' = j}^{j} n_{2(m+1)j'' - s} n_{2(m+1)j'' - s} + \sum_{s=1}^{m} \sum_{j=1}^{p} \sum_{j' = j}^{j} n_{2(m+1)j'' - 2m - s - 1} n_{2(m+1)j'' - s}
\]

\[
+ \sum_{j=0}^{p-1} \sum_{i=1}^{m+1} n_{2(m+1)j'' - i} + \sum_{j=0}^{p-1} \sum_{i=2}^{m+1} n_{2(m+1)j'' - i} - \sum_{j=0}^{p-1} \sum_{i=2}^{m+1} n_{2(m+1)j'' - i}.
\]

\tag{4.13}

We combine the \( j' = j \) term from the first sum on the third line in (4.13) with the first sum in the fourth line and then cancel with the second sum in the first line. Next, the \( j' = j \) term in the second sum of the third line cancels with the first sum in the last line. Thus, (4.13) equals
\[
\sum_{k=1}^{m} \sum_{j=1}^{p} \sum_{j'=1}^{j} n(2m+1)j-k+1 n(2m+1)j'+k-2m-1
\]
\[
+ \sum_{k=1}^{m} \sum_{j=1}^{p} \sum_{j'=1}^{j} n(2m+1)j-m-k+1 n(2m+1)j'-3m+k+2
\]
\[
- \sum_{k=1}^{m} \sum_{j=1}^{p} \sum_{j'=1}^{j} n(2m+1)j-m-k n(2m+1)j'-3m+k-2
\]
\[
+ \sum_{s=1}^{m} \sum_{j=1}^{p} \sum_{j'=1}^{j} n(2m+1)j'-s n(2m+1)j-2m+s-1
\]
\[
- \sum_{s=1}^{m} \sum_{j=1}^{p} \sum_{j'=1}^{j} n(2m+1)j'-2m+s-1 n(2m+1)j-2m+s-1
\]
\[
- \sum_{s=1}^{m} \sum_{j=1}^{p} \sum_{j'=1}^{j} n(2m+1)j'-2m+s n(2m+1)j-2m+s-1
\]
\[
- \sum_{s=1}^{m} \sum_{j=1}^{p} \sum_{j'=1}^{j} n(2m+1)j'-s n(2m+1)j-2m+s-1
\]
\[
+ \sum_{s=1}^{m} \sum_{j=1}^{p} \sum_{j'=1}^{j} n(2m+1)j'-2m+s-1 n(2m+1)j-2m+s-1
\]
\[
- \sum_{s=1}^{m} \sum_{j=1}^{p} \sum_{j'=1}^{j} n(2m+1)j'-2m+s n(2m+1)j-2m+s-1
\]
\[
- \sum_{s=1}^{m} \sum_{j=1}^{p} \sum_{j'=1}^{j} n(2m+1)j'-s n(2m+1)j-2m+s-1
\]
\[
+ \sum_{s=1}^{m} \sum_{j=1}^{p} \sum_{j'=1}^{j} n(2m+1)j'-2m+s-1 n(2m+1)j-2m+s-1
\]
\[
- \sum_{s=1}^{m} \sum_{j=1}^{p} \sum_{j'=1}^{j} n(2m+1)j'-2m+s n(2m+1)j-2m+s-1
\]
\[
- \sum_{s=1}^{m} \sum_{j=1}^{p} \sum_{j'=1}^{j} n(2m+1)j'-s n(2m+1)j-2m+s-1
\]
\[
+ \sum_{s=1}^{m} \sum_{j=1}^{p} \sum_{j'=1}^{j} n(2m+1)j'-2m+s-1 n(2m+1)j-2m+s-1
\]
\[
- \sum_{s=1}^{m} \sum_{j=1}^{p} \sum_{j'=1}^{j} n(2m+1)j'-2m+s n(2m+1)j-2m+s-1
\]
\[
- \sum_{s=1}^{m} \sum_{j=1}^{p} \sum_{j'=1}^{j} n(2m+1)j'-s n(2m+1)j-2m+s-1
\]
\[
+ \sum_{s=1}^{m} \sum_{j=1}^{p} \sum_{j'=1}^{j} n(2m+1)j'-2m+s-1 n(2m+1)j-2m+s-1
\]
\[
- \sum_{s=1}^{m} \sum_{j=1}^{p} \sum_{j'=1}^{j} n(2m+1)j'-2m+s n(2m+1)j-2m+s-1
\]
\[
- \sum_{s=1}^{m} \sum_{j=1}^{p} \sum_{j'=1}^{j} n(2m+1)j'-s n(2m+1)j-2m+s-1
\]
\[
+ \sum_{s=1}^{m} \sum_{j=1}^{p} \sum_{j'=1}^{j} n(2m+1)j'-2m+s-1 n(2m+1)j-2m+s-1
\]
\[
- \sum_{s=1}^{m} \sum_{j=1}^{p} \sum_{j'=1}^{j} n(2m+1)j'-2m+s n(2m+1)j-2m+s-1
\]
\[
- \sum_{s=1}^{m} \sum_{j=1}^{p} \sum_{j'=1}^{j} n(2m+1)j'-s n(2m+1)j-2m+s-1
\]
\[
+ \sum_{s=1}^{m} \sum_{j=1}^{p} \sum_{j'=1}^{j} n(2m+1)j'-2m+s-1 n(2m+1)j-2m+s-1
\]
\[
- \sum_{s=1}^{m} \sum_{j=1}^{p} \sum_{j'=1}^{j} n(2m+1)j'-2m+s n(2m+1)j-2m+s-1
\]
\[
- \sum_{s=1}^{m} \sum_{j=1}^{p} \sum_{j'=1}^{j} n(2m+1)j'-s n(2m+1)j-2m+s-1
\]
\[
+ \sum_{s=1}^{m} \sum_{j=1}^{p} \sum_{j'=1}^{j} n(2m+1)j'-2m+s-1 n(2m+1)j-2m+s-1
\]
\[
- \sum_{s=1}^{m} \sum_{j=1}^{p} \sum_{j'=1}^{j} n(2m+1)j'-2m+s n(2m+1)j-2m+s-1
\]
\[
- \sum_{s=1}^{m} \sum_{j=1}^{p} \sum_{j'=1}^{j} n(2m+1)j'-s n(2m+1)j-2m+s-1\]
5. Proof of Theorem 1.3

Before proving Theorem 1.3, we briefly review the theory of Bailey pairs [3, 4]. Two sequences $(\alpha_n, \beta_n)$ are said to form a Bailey pair relative to $a$ if

\[
\beta_n = \sum_{k=0}^{n} \frac{\alpha_k}{(q)_{n-k} (aq)_{n+k}}. \tag{5.1}
\]

The Bailey lemma says that if $(\alpha_n, \beta_n)$ form a Bailey pair relative to $a$, then so do $(\alpha_n', \beta_n')$, where

\[
\alpha_n' = \frac{(\rho_1)_n (\rho_2)_n (aq/\rho_1 \rho_2)^n}{(aq/\rho_1)_n (aq/\rho_2)_n} \alpha_n \tag{5.2}
\]

and

\[
\beta_n' = \sum_{k=0}^{n} \frac{(\rho_1)_k (\rho_2)_k (aq/\rho_1 \rho_2)_{n-k} (aq/\rho_1 \rho_2)^k}{(aq/\rho_1)_n (aq/\rho_2)_n (q)_{n-k}} \beta_k. \tag{5.3}
\]

Iterating (5.2) and (5.3) gives what is called the Bailey chain.

We shall not require the full power of the Bailey chain, but only two special cases. First, take the Bailey pair relative to $q$ [27, p.468, B(3)],

\[
\alpha_n = \frac{(-1)^n q^{n(n+1)/2} (1 - q^{2n+1})}{1 - q}. \tag{5.4}
\]

and

\[
\beta_n = \frac{1}{(q)_n}. \tag{5.5}
\]

Iterating (5.4) and (5.5) using (5.2) and (5.3) with $\rho_1, \rho_2 \to \infty$ at each step, we find that $(\alpha_n^{(p)}, \beta_n^{(p)})$ is a Bailey pair relative to $q$, where

\[
\alpha_n^{(p)} = \frac{(-1)^n q^{n(n-1)/2 + p(n^2+n)} (1 - q^{2n+1})}{1 - q}. \tag{5.6}
\]

and

\[
\beta_n^{(p)} = \frac{1}{(q)_n} \sum_{n_p \geq n_{p-1} \geq \ldots \geq n_1 \geq 0} \prod_{j=1}^{p-1} q^{n_j^2 + n_j} \left[ \begin{array}{c} n_{j+1} \\ n_j \end{array} \right]. \tag{5.7}
\]

Next take the Bailey pair relative to $q$ [33, Eq. (4.12)],

\[
\alpha_n = \frac{q^{n^2} (1 - q^{2n+1})}{1 - q}. \tag{5.8}
\]

and

\[
\beta_n = \frac{1}{(q)_n^2}. \tag{5.9}
\]
Performing the same iteration as above to (5.8) and (5.9), we find that \((\alpha_n^{(p)}, \beta_n^{(p)})\) is a Bailey pair relative to \(q\), where

\[
\alpha_n^{(p)} = \frac{q^{m^2+(p-1)n(1-q^{2n+1})}}{1-q}
\]  

(5.10)

and

\[
\beta_n^{(p)} = \frac{1}{(q)_n} \sum_{n=n_p \geq n_{p-1} \geq \cdots \geq n_1 \geq 0} \frac{1}{(q)_{n_1}} \prod_{j=1}^{p-1} q^{n_j^2+n_j} \left[ n_{j+1} \atop n_j \right].
\]  

(5.11)

We are now ready to prove Theorem 1.3.

**Proof of Theorem 1.3.** We began by recalling a formula of Walsh [32, Cor 4.2.4, corrected]. Namely, for \(m \geq 1\) and \(p \neq 0\), we have

\[
J_N(K_{(m,p)}; a^2) = \frac{a^{2p(1-N^2)}}{[N]} \sum_{n=0}^{N-1} \frac{[N+n]!}{[N-n-1]![2n+1]!} \left( -1 \right)^n \frac{(n+1)!}{1!(a-a^{-1})^{2n}} \frac{[2k+1]}{[n+k+1]![n-k]!} \frac{\mu_{2k}^p}{[n]!},
\]  

(5.12)

where

\[
\mu_i = a^{i+2i}, \quad \{n\} = a^n - a^{-n}, \quad [n] = \frac{a^n - a^{-n}}{a - a^{-1}},
\]

\[
\{n\}! = \{n\} \{n-1\} \cdots \{1\}, \quad [n]! = [n][n-1] \cdots [1]
\]

and

\[
c'_{n,p} = \frac{1}{(a-a^{-1})^n} \sum_{k=0}^{n} (-1)^k \mu_{2k}^p \frac{[2k+1]}{[n+k+1]![n-k]!}.
\]

(5.13)

We note that the prefactor \(a^{2p(1-N^2)}\) and the normalization factor \(\frac{1}{[N]}\) are both missing in [32].2

Some routine (but tedious) simplification shows that (5.12) can be written as

\[
J_N(K_{(m,p)}; q) = q^{p(1-N^2)} \sum_{n=0}^{\infty} q^n (q^{1+N})_n (q^{1-N})_n c_{p,n}(q) d_{m,n}(q),
\]

(5.14)

where

\[
c_{p,n}(q) = (q)_n \sum_{k=0}^{n} \frac{(-1)^k q^{(k)}}{(q)_{n-k}(q)_{n+k+1}} q^{mk^2+(p+1)(1-q^{2k+1})}
\]

(5.15)

and

\[
d_{m,n}(q) = (q)_n \sum_{k=0}^{n} \frac{q^{mk^2+(m-1)k(1-q^{2k+1})}}{(q)_{n-k}(q)_{n+k+1}}.
\]

(5.16)

Here, we have used that \(a^2 = q\). Now, recalling (5.1) and comparing (5.14) to (5.6) and (5.7), we have that for \(p > 0\),

\[
c_{p,n}(q) = \sum_{n=n_p \geq n_{p-1} \geq \cdots \geq n_1 \geq 0} \prod_{j=1}^{p-1} q^{n_j^2+n_j} \left[ n_{j+1} \atop n_j \right].
\]

We thank Katherine Walsh Hall for providing us with the corrected version.

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2We thank Katherine Walsh Hall for providing us with the corrected version.
Similarly, comparing (5.15) to (5.10) and (5.11), we have that for \( m > 0 \),
\[
d_{m,n}(q) = \sum_{n=n_m \geq n_{m-1} \geq \cdots \geq n_0 \geq 0} \frac{1}{(q)_{n_1}} \prod_{j=1}^{m-1} q^{n_j^2 + n_j} \left[ \begin{array}{c} n_j+1 \\ n_j \end{array} \right].
\] (5.17)

Inserting (5.16) and (5.17) in (5.13) gives (1.13).

For the case \( p < 0 \), a calculation using the fact that \((1/q; 1/q)_n = (q^{-1})_n\) shows that
\[
c_{p,n}(1/q) = (-1)^n q^{n(n+3)/2} c_{-p,n}(q).
\] (5.18)

Using (5.18) together with the fact that
\[
\left[ \begin{array}{c} n \\ k \end{array} \right]_{1/q} = q^{k^2 - nk} \left[ \begin{array}{c} n \\ k \end{array} \right]_q
\]
gives that for \( p > 0 \),
\[
c_{-p,n}(q) = (-1)^n q^{-n(n+3)/2} \sum_{n=n_p \geq n_{p-1} \geq \cdots \geq n_0 \geq 0} \prod_{j=1}^{p-1} q^{-n_j - n_j + n_j} \left[ \begin{array}{c} n_j+1 \\ n_j \end{array} \right].
\] (5.19)

Inserting (5.19) and (5.15) in (5.13) gives (1.14), which completes the proof. □

6. Concluding Remarks

Recall that Habiro [15] showed that for a knot \( K \), the colored Jones polynomial has a cyclotomic expansion of the form
\[
J_N(K; q) = \sum_{n \geq 0} (q^{1+N})_n (q^{1-N})_n C_n(K; q),
\] (6.1)
where the cyclotomic coefficients \( C_n(K; q) \) are Laurent polynomials independent of \( N \). The formulas in (1.13) and (1.14) for \( J_N(K_{(m,p)}; q) \) and \( J_N(K_{(m,-p)}; q) \) closely resemble the expansion in (6.1), but the coefficients are neither polynomials nor independent of \( N \). It would be highly desirable to find the correct cyclotomic expansions for these knots. We note that this has already been done by Hikami and the first author in the case of the left-handed torus knots \( K_{(1,-p)} \); where we have [20, Prop. 3.2]
\[
C_n(K_{(1,-p)}; q) = q^{n+1-p} \sum_{n+1=k_0 \geq k_{p-1} \geq \cdots \geq k_1 \geq 1} \prod_{i=1}^{t-1} q^{k_i^2} \left[ \begin{array}{c} k_{i+1} + k_i - i + 2 \sum_{j=1}^{i-1} k_j \\ k_{i+1} - k_i \end{array} \right].
\] (6.2)

Another topic for future study would be to generalize facts about the Kontsevich-Zagier series (1.1) to the generalized Kontsevich-Zagier series \( F_{m,p}(q) \) (and/or for \( \tilde{F}_{m,p}(q) \) ). First, given the relation to the colored Jones polynomial in (1.17), we conjecture that the \( F_{m,p}(q) \) are quantum modular forms. Second, as the coefficients of \( F(1-q) \) enjoy a wide variety of combinatorial interpretations (see A022493 in [28]) and interesting congruence properties [1, 2, 5, 11, 12, 29], it would be of great interest to determine if the same is true for \( F_{m,p}(1-q) \).

Finally, can one prove Theorems 1.1 and 1.2 using difference equations? This approach was used in [16, 17] to compute (1.11).
Acknowledgements

The authors would like to thank Paul Beirne and Katherine Walsh Hall for their helpful comments and suggestions. The second author would like to thank the Max-Planck-Institut für Mathematik for their support during the completion of this paper.

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