A SYSTEM WITH A RECURSION OPERATOR, 
BUT ONE HIGHER LOCAL SYMMETRY

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ABSTRACT

We construct a recursion operator for the system \((u_t, v_t) = (u^4 + v^2, \frac{1}{5} v_4)\), for which only one local symmetry is known and we show that the action of the recursion operator on \((u_t, v_t)\) is a local function.

1. Introduction.

We consider the system of evolution equations
\[\begin{align*}
  u_t &= u_m + v^2 \\
  v_t &= \lambda v_m, \quad \lambda \neq 0.
\end{align*}\] (1.1)

These systems are studied in [1] as examples of systems for which only one local symmetry is known. It is shown that for arbitrary \(\lambda\), they have classical and Lie-point symmetries \((\sigma, \eta) :\) \[(1, 0), \quad (2u, v), \quad (u_1, v_1), \quad (u_t, v_t),\] (1.2)

and it is shown that higher symmetries are of the form
\[ (au_n + \sum_{i=0}^{s} \beta_i v_i v_{n-m-i}, bv_n), \quad s = [(n - m)/2]. \] (1.3)

It is also shown that for \(m = 4\), and \(\lambda = \frac{1}{5}\) the system admits the 6th order symmetry
\[ (\frac{11}{25} u_6 + v_2 v + \frac{4}{5} v_1^2, \frac{1}{25} v_6) \] (1.4)

but no higher local symmetry of this equation could have been found up to order 53 [1].

We will show that the system (1.1) admits always a “formal” recursion operator \(R\), in the sense of being the solution of the operator equation
\[ R_t + [R, F'] = 0 \] (1.5)

where \(F'\) is the Frechet derivative of \(F = \left(\frac{u_m + v^2}{\lambda v_m}\right)\). Then, for \(m = 4\) and \(\text{Ord}(R) = 2\), we determine the coefficients of \(R\) explicitly and by appropriate choice of free parameters we show that \(R\) acting on the symmetry \((u_t, v_t)\) gives the 6th order symmetry (1.4).
2. Computation of the recursion operator.

It can be seen that the symmetries \((\sigma, \eta)\) satisfy the equation

\[
\sigma_t = D^m \sigma + 2v\eta, \\
\eta_t = \lambda D^m \eta.
\]

The recursion operator \(R\) is a matrix operator of the form

\[
R = \begin{pmatrix}
A & B \\
C & D
\end{pmatrix}.
\]

The operator equation (1.5) gives

\[
A_t + AD^m - D^m A = 0, \quad (2.2a)
\]
\[
B_t + 2Av + \lambda BD^m - D^m B - 2vE = 0, \quad (2.2b)
\]
\[
C_t + CD^m - \lambda D^m C = 0, \quad (2.2c)
\]
\[
E_t + 2Cv + \lambda ED^m - \lambda D^m E = 0. \quad (2.2d)
\]

We first show that \(C = 0\). If \(\text{Ord}(C) = k\), then the top term of (2.2c) is of order \(k + m\), with coefficient \((\lambda - 1)C_k\), which implies that \(C = 0\). Then it follows that \(A\) and \(E\) can be arbitrary constant coefficient operators and the problem is reduced to solving

\[
B_t + \lambda BD^m - D^m B + 2Av - 2vE = 0. \quad (2.3)
\]

If \(\text{Ord}(B) = n, \text{Ord}(A) = k\) and \(\text{Ord}(E) = l\), then it can be seen that provided that \(n + m = k\) or \(n + m = l\), the operator equation in (2.3) can always be solved for the coefficients of \(B\) recursively, because the top term involves the top term of \(B\) algebraically, with a nonzero coefficient. This situation is in contrast with the scalar case where the top term of the operator equation always involves the first derivative of the top coefficient to be determined at that stage, and the solvability of condition gives conserved densities that lead to a classification.

Thus the system (1.1) possesses a recursion operator for any \(m\) and any \(\lambda \neq 1\). We recall the existence of a “formal symmetry” is proposed as an integrability test in [2]. A formal symmetry is a pseudo-differential operator that satisfies the equation (1.5) up to a certain order, hence the existence of a recursion operator implies integrability in this sense. We note that in [2], the integrability test for systems of evolution equations involves a formal diagonalization procedure that first transforms the differential function \(F\) to a pseudo-differential operator, then the problem is reduced to finding formal symmetries for a set of scalar equations. Here we have used a direct computation of the recursion operator, because we were interested not only in proving the existence of a formal symmetry, but in finding the recursion operator explicitly.

In the following we will assume that

\[
A = aD^2, \quad E = bD^2. \quad (2.4)
\]
We note that the expansions of a pseudo-differential operator of the forms $B = \sum B_i D^{-i}$ or $B = \sum D^{-i} B_i$ are equivalent. Here we choose the second formulation. It can easily be seen that $B$ is of the form

$$B = \sum_{j=0}^{\infty} b_j D^{-j-2} v_j$$ \hspace{1cm} (2.5)

where the $b_j$'s are constants.

Substituting $B$ in (2.3) and using the commutation relation

$$D^{-1} \varphi = \varphi D^{-1} - D^{-1} \varphi_1 D^{-1}$$ \hspace{1cm} (2.6)

one can compute the coefficients of $D^2$, $D$, $1$, $D^{-1}$, ..., and obtain the following equations for the $b_i$'s.

$$(\lambda - 1)b_0 + 2a - 2b = 0,$$

$$(\lambda - 1)b_1 - 2(\lambda + 1)b_0 + 4a = 0,$$

$$(\lambda - 1)b_2 - (3\lambda + 1)b_1 + + (3\lambda - 1)b_0 + 2a = 0,$$

$$(\lambda - 1)b_3 - 4\lambda b_0 + 6\lambda b_1 - 4\lambda b_2 = 0,$$

and the recurrence relation

$$(\lambda - 1)b_n - 4\lambda b_{n-1} + 6\lambda b_{n-2} - 4\lambda b_{n-3} + 2\lambda b_{n-4} = 0.$$ \hspace{1cm} (2.8)

From this recurrence relation, it is easy to see that the series cannot terminate.

3. Existence of one local symmetry.

We will show that $B$ acting on $v_4$ results in a local function. For this we will need the following.

Lemma. Let $S = \sum_{n=4}^{\infty} b_n D^{n-2} v_n v_4$. Assume that

$$b_n = a_1 b_{n-1} + a_2 b_{n-2} + a_3 b_{n-3} + a_4 b_{n-4} \quad n \geq 8$$ \hspace{1cm} (3.1)

with $a_1 + a_2 + a_3 + a_4 = 0$, and

$$(2 + a_4)b_4 + (1 + a_3 + a_4)b_5 + (1 + a_2 + a_3 + a_4)b_6 + b_7 = 0.$$ \hspace{1cm} (3.2)

If the condition (3.2) is invariant under the transformation

$$\begin{pmatrix} b_4 \\ b_5 \\ b_6 \\ b_7 \end{pmatrix} \rightarrow \begin{pmatrix} a_4 & a_3 + a_4 & 1 + a_2 + a_3 + a_4 & 1 \\ a_4 & a_3 + a_4 & a_2 + a_3 + a_4 & 1 \\ a_4 & a_3 + a_4 & a_2 + a_3 + a_4 & 1 \\ 0 & a_4 & a_3 + a_4 & a_2 + a_3 + a_4 \end{pmatrix} \begin{pmatrix} b_4 \\ b_5 \\ b_6 \\ b_7 \end{pmatrix}$$ \hspace{1cm} (3.3)

then $S$ is zero.

Proof. Recall that $D^{-1} v_n v_k = v_{n-1} v_k - D^{-1} v_{n-1} v_{k+1}$. Hence by successive iterations of this formula, $D^{-n-2} v_n v_4$ will be a linear combination of $\{D^{-2n+4} v_4\}_{n=4} = \{D^{-6} v_4^2, D^{-8} v_5^2, \ldots \}$. We will show that the
coefficient of $D^{-2n+4}v_n^2$ vanishes for all $n \geq 4$. We rewrite $S$ by substituting the recurrence relations for $n \geq 8$.

$$S = \sum_{n=4}^{7} b_n D^{-n+2}v_n v_4 + \sum_{n=8}^{4} \sum_{k=1}^{\infty} a_k b_{n-k} D^{-n+2}v_n v_4.$$ 

Note that the series $S_4 = \sum_{n=8}^{\infty} b_{n-4} D^{-n+2}v_n v_4$ can be integrated to give

$$S_4 = \sum_{n=8}^{\infty} b_{n-4} D^{-n+1}v_{n-1} v_4 - \sum_{n=8}^{\infty} b_{n-4} D^{-n+2}v_{n-1} v_5$$

$$= b_4 D^{-9} v_4 + \sum_{n=9}^{\infty} b_{n-4} D^{-n+1}v_{n-1} v_4 - \sum_{n=8}^{\infty} b_{n-4} D^{-n+2}v_{n-1} v_5$$

$$= b_4 D^{-9} v_4 + \sum_{n=8}^{\infty} b_{n-3} D^{-n+2}v_n v_4 - \sum_{n=8}^{\infty} b_{n-4} D^{-n+2}v_{n-1} v_5.$$ 

By combining the first series above with the series $a_3 \sum_{n=8}^{\infty} b_{n-3} D^{-n+2}v_n v_4^2$, we obtain

$$S = \sum_{n=4}^{7} b_n D^{-n+2}v_n v_4 + a_4 b_4 D^{-9} v_4 - a_4 \sum_{n=8}^{\infty} b_{n-4} D^{-n+2}v_{n-1} v_5$$

$$+ a_1 \sum_{n=8}^{\infty} b_{n-1} D^{-n+2}v_n v_4 + a_2 \sum_{n=8}^{\infty} b_{n-2} D^{-n+2}v_n v_4$$

$$+ (a_3 + a_4) \sum_{n=8}^{\infty} b_{n-3} D^{-n+2}v_n v_4$$

Repeating the same procedure we obtain

$$S = \sum_{n=4}^{7} b_n D^{-n+2}v_n v_4 + a_4 b_4 D^{-9} v_4 + (a_3 + a_4) b_5 D^{-9} v_4$$

$$- a_4 \sum_{n=8}^{\infty} b_{n-4} D^{-n+2}v_{n-1} v_5 - (a_3 + a_4) \sum_{n=8}^{\infty} b_{n-3} D^{-n+2}v_{n-1} v_5$$

$$+ a_1 \sum_{n=8}^{\infty} b_{n-1} D^{-n+2}v_n v_4 + (a_2 + a_3 + a_4) \sum_{n=8}^{\infty} b_{n-2} D^{-n+2}v_n v_4$$

and finally

$$S = \sum_{n=4}^{7} b_n D^{-n+2}v_n v_4 + a_4 b_4 D^{-9} v_4 + (a_3 + a_4) b_5 D^{-9} v_4$$

$$+ (a_2 + a_3 + a_4) b_6 D^{-9} v_4 - a_4 \sum_{n=8}^{\infty} b_{n-4} D^{-n+2}v_{n-1} v_5$$

$$- (a_3 + a_4) \sum_{n=8}^{\infty} b_{n-3} D^{-n+2}v_{n-1} v_5 - (a_2 + a_3 + a_4) \sum_{n=8}^{\infty} b_{n-2} D^{-n+2}v_{n-1} v_5$$

$$+ (a_1 + a_2 + a_3 + a_4) \sum_{n=8}^{\infty} b_{n-1} D^{-n+2}v_n v_4$$

But the last series vanishes because the sum of the $a_i$’s is zero. By rearranging we obtain

$$S = b_4 D^{-6} v_4^2 + b_5 D^{-7} v_5 v_4 + b_6 D^{-8} v_6 v_4$$

$$+ [b_7 + a_4 b_4 + (a_3 + a_4) b_5 + (a_2 + a_3 + a_4) b_6] D^{-9} v_7 v_4$$

$$- \sum_{n=8}^{\infty} [a_4 b_{n-4} + (a_3 + a_4) b_{n-3} + (a_2 + a_3 + a_4) b_{n-2}] D^{-n+2}v_{n-1} v_5.$$
Using

\[ D^{-7}v_5v_4 = \frac{1}{2}D^{-6}v_4^2, \]
\[ D^{-8}v_6v_4 = \frac{1}{2}D^{-6}v_4^2 - D^{-8}v_5^2, \]
\[ D^{-9}v_7v_4 = \frac{1}{2}D^{-6}v_4^2 - D^{-8}v_5^2 - D^{-9}v_6v_5, \]

we get

\[
S = \left[ b_4 + \frac{1}{2}b_5 + \frac{1}{2}b_6 + \frac{1}{2}(b_7 + a_4b_4 + (a_3 + a_4)b_5 + (a_2 + a_3 + a_4)b_6) \right]D^{-6}v_4^2
\]
\[- \left[ b_6 + b_7 + a_4b_4 + (a_3 + a_4)b_5 + (a_2 + a_3 + a_4)b_6 \right]D^{-8}v_5^2
\[- \left[ b_7 + a_4b_4 + (a_3 + a_4)b_5 + (a_2 + a_3 + a_4)b_6 \right]D^{-9}v_6v_5
\-
\sum_{n=8}^{\infty} [a_4b_{n-4} + (a_3 + a_4)b_{n-3} + (a_2 + a_3 + a_4)b_{n-2}]D^{-n-2}v_{n-1}v_5
\]

The coefficient of \( D^{-6}v_4^2 \) is just the condition in (3.2), hence this term vanishes by assumption. Thus \( S \) is now of the form

\[ S = -\sum_{n=4}^{\infty} c_n D^{-n-4}v_{n+1}v_5 \]

where

\[ c_4 = b_6 + b_7 + a_4b_4 + (a_3 + a_4)b_5 + (a_2 + a_3 + a_4)b_6, \]
\[ c_4 = b_7 + a_4b_4 + (a_3 + a_4)b_5 + (a_2 + a_3 + a_4)b_6, \]

and

\[ c_n = a_4b_{n-2} + (a_3 + a_4)b_{n-1} + (a_2 + a_3 + a_4)b_n, \quad n \geq 6. \]

It is easy to see that the \( c_n \)'s satisfy the same recursion relation as the \( b_n \)'s, and they are related to them by the transformation formula (3.3). Hence the same procedure can be repeated to show that the coefficient of \( D^{-8}v_5^2 \) is zero. As all the arguments can be repeated by replacing \( v_4 \) with \( v_k \) it can be shown by induction that the coefficient of \( D^{-2(n-1)}v_n^2 \) in \( S \) is zero.

We will now show that the conditions of the Lemma hold for our system. In our case

\[ a_1 = \frac{4\lambda}{1-\lambda}, \quad a_2 = \frac{6\lambda}{1-\lambda}, \quad a_3 = -\frac{4\lambda}{1-\lambda}, \quad a_4 = \frac{2\lambda}{1-\lambda}, \quad (3.4) \]

Let \( A \) denote the matrix in (3.3). It can be seen that the minimal polynomial of \( A \) is

\[ A^2 + \frac{4\lambda}{1-\lambda}A + \frac{2\lambda}{1-\lambda} = 0. \quad (3.5) \]

The condition (3.2) can be interpreted as the orthogonality of the vector \( b = (b_4, b_5, b_6, b_7) \) with the fixed vector \( d = (2 + a_4, 1 + a_3 + a_4, 1 + a_2 + a_3 + a_4, 1) \). If \( b \) is chosen such that \( d^t b = 0 \) and \( d^t A b = 0 \), then as the minimal polynomial of \( A \) has order 2, the condition \( d^t A^n b = 0 \) will hold for all \( n \). This means that the initial condition (3.2) will hold for all iterations.
The conditions \( d^t b = 0 \) and \( d^t Ab = 0 \) determine \( b_7 \) and \( b_6 \) as

\[
\begin{align*}
    b_7 &= \frac{2}{(\lambda - 1)^2} [2(1 + 5\lambda)b_4 + (1 + 7\lambda)b_5] \\
    b_6 &= \frac{1}{\lambda - 1} [6b_4 + (3 + \lambda)b_5]
\end{align*}
\]  

(3.6)

On the other hand \( b_6 \) and \( b_7 \) are already given by recurrence relations. The compatibility of these expressions give a homogeneous system for \( a \) and \( b \). The determinant of this system is zero only for \( \lambda = \frac{1}{5} \). In this case \( a \) is determined as \( \frac{11}{5} b \). Substituting these values in (2.7), \( b_i \ i = 0, \ldots, 3 \) can be computed, and it can be seen that \( Bv_4 = b(3v_2 - 2v_1^2) \). In particular, the coefficient of \( D^{-4}v_2^2 \) vanishes without giving any further restriction. Then, the symmetry \( R(u, v) = (aD^2(u + v^2) + B(\frac{1}{5}v_4), bD^2(\frac{1}{5}v_4)) \) can be computed, and is equal to the expression in (1.4).

We present the first few \( b_i \)’s for \( b = \frac{1}{5} \) below.

| \( b_0 \) | \( b_1 \) | \( b_2 \) | \( b_3 \) | \( b_4 \) |
|---|---|---|---|---|
| 0.6 | 0.4 | 0 | 0 | -0.1 |
| 0.3 | -0.45 | 1 | -2.025 | 4.125 |
| -8.388 | 17.1 | -34.82 | 70.92 | -144.4 |
| 294.2 | -599.2 | 1220.0 | -2486.0 | 5062.0 |

4. Conclusion.

We have shown that the system (1.1) has always a recursion operator which is an infinite series in \( D^{-i} \) and we have calculated the coefficients for the case \( m = 4 \) given in (2.7) and (2.8). From the recurrence relation (2.8) it can be seen that the series cannot terminate, and the attempts to write the recursion operator in closed form were not successful. Nevertheless the series \( Bv_4 \) terminates and furthermore it is a local function. The action of \( B \) on higher symmetries in general do not give a series that terminates. The condition for the series to terminate is the orthogonality of the vectors \((b_n, b_{n+1}, b_{n+2}, b_{n+3})\) with the vectors \( d = (\frac{3}{2}, \frac{1}{2}, 1, \frac{1}{2}) \) and \( dA^t = (\frac{3}{2}, -2, 7, 4) \). These conditions do not follow from the recursion relations, and they are not satisfied for the next few symmetries.

The existence of a recursion operator, a formal symmetry or an eigenvalue problem are more or less equivalent methods that lead to a classification [2,3,4]. This example shows that the the existence of an infinite number of local symmetries is more restrictive.

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