Thouless time analysis of Anderson and many-body localization transitions

Piotr Sierant,1 Dominique Delande,2,3 and Jakub Zakrzewski1,4
1Instytut Fizyki im. Mariana Smoluchowskiego, Uniwersytet Jagielloński, Lojasiewicza 11, 30-348 Kraków, Poland
2Laboratoire Kastler Brossel, UPMC-Sorbonne Universités, CNRS, ENS-PSL Research University
3Collège de France; 4 Place Jussieu, 75005 Paris, France
4Mark Kac Complex Systems Research Center, Uniwersytet Jagielloński, Kraków, Poland.

(Dated: November 15, 2019)

Spectral statistics of disordered systems encode Thouless and Heisenberg time scales whose ratio determines whether the system is chaotic or localized. Identifying similarities between system size and disorder strength scaling of Thouless time for disordered quantum many-body systems with results for 3D and 5D Anderson models, we argue that the two-parameter scaling breaks down in the vicinity of the transition to the localized phase signalling subdiffusive dynamics.

*Instytut Fizyki im. Mariana Smoluchowskiego, Uniwersytet Jagielloński, Lojasiewicza 11, 30-348 Kraków, Poland
1, 4Mark Kac Complex Systems Research Center, Uniwersytet Jagielloński, Kraków, Poland.

Introduction. Phenomenon of many-body localization (MBL) [1, 2], the most robust mechanism of ergodicity breaking in quantum world [3–5] has received a lot of attention over the last decade. Investigations of MBL in lattice models, pioneered in spin systems [6–8], were extended to bosonic models [9, 10] and to systems of spinful fermions [11–14]. Remarkably, MBL, usually thought of as Anderson localization [15] in presence of interactions, was shown to occur in systems with completely delocalized single particle states either due to random interactions [16–18] or in a quasiperiodic Fibonacci chain [19]. MBL was also found in disorder-free systems as a result of gauge invariance [20, 21] or due to Wannier-Stark localization of single particle states [22, 23]. Further examples include systems with power-law interactions [24–27], with interactions of infinite range [28] or driven Floquet MBL systems [29]. Existence of local integrals of motion (LIOMs) [30–34] provides a common framework to understand features of MBL such as area-law entanglement entropy of eigenstates [35, 36], logarithmic growth of bipartite entanglement entropy in quench from a separable state [37, 38] or Poisson statistics (PS) of energy levels.

Level statistics of ergodic systems are described by an appropriate ensemble of random matrices [39, 40], e. g. time reversal invariant systems follow predictions of Gaussian Orthogonal Ensemble (GOE) of random matrices. The ensuing crossover between GOE level statistics of an ergodic system and PS of MBL phase seems to be well understood [41–45]. However, a recent analysis [46] of the Spectral Form Factor (SFF), $K(\tau)$, in the wide regime of slow thermalization on the ergodic side of the crossover [47–50] questions the very existence of the MBL phase in the thermodynamic limit predicting a two-parameter scaling of Thouless time

$$t_{Th} = t_0 e^{W/\Omega} L^2,$$

where $L$ is system size, $W$ is disorder strength, $t_0$ and $\Omega$ are constants.

The Thouless time $t_{Th}$ is defined as the time scale beyond which the SFF follows the universal GOE form. Another important time scale, the Heisenberg time $t_H = 2\pi/\Delta$ is defined by the average level spacing $\Delta$ which scales exponentially with size $L$ of a many-body system $\Delta \propto e^{cL}$. The Heisenberg time $t_H$ is an upper bound beyond which the discrete nature of the energy spectrum manifests itself and where specific, system dependent quantum effects are unavoidable. In the thermodynamic limit, (1) implies $t_{Th}/t_H \to 0$. Hence, Ref. [46] arrives at the surprising conclusion that the considered disordered quantum spin chain has spectral properties following the random matrix predictions regardless of the disorder strength $W$ and that MBL is merely a finite-size effect. Thus, developing a better understanding of long-range spectral correlations the ergodic-MBL crossover [51] is a central issue in studies of non-equilibrium dynamics of quantum many-body systems.

In this letter we analyse the SFF in the delocalized phase and its modifications when approaching the transition to the localized phase. We show that the Thouless time scales like $L^2$, in agreement with (1), in the deep delocalized phase, but that the scaling with $L$ evolves to a larger power at the critical point, a phenomenon that we correlate with the diffusive and subdiffusive transport properties respectively in the delocalized phase and at the metal-insulator transition. Results obtained for 3D and 5D Anderson models with known localization properties put the conclusions of [46] about the scaling of Thouless time $t_{Th}$ in disordered many-body systems in a considerable doubt, indicating presence of MBL phase at sufficiently strong disorder when finite-size effects are properly taken into account.

Thouless time. In a non-interacting system, the Thouless time was originally introduced as a time to diffuse through the system and reach its boundary [52]. It determines the energy scale below which the level statistics are well described by GOE [53], whereas its ratio with the Heisenberg time fixes the dimensionless conductance of the system [54] and enters the scaling theory of Anderson localization transition [55]. The Thouless time $t_{Th}$ in disordered many-body systems can be probed by examining the behavior of the SFF [56–59] defined as

$$K(\tau) = \frac{1}{Z} \left\langle \left| \sum_{j=1}^{N} g(\epsilon_j) e^{-i\epsilon_j \tau} \right|^2 \right\rangle,$$

where $Z$ is a normalizing factor. This letter provides a framework for understanding the Thouless time in both the disordered and the ergodic limits.
where $\epsilon_i$ are eigenvalues of the system after the so called unfolding [60] which sets their density to unity, $g(\epsilon)$ is a Gaussian function that reduces influence of the edges of the spectrum, the average is taken over disorder realizations and $N$ is the dimension of the Hilbert space. For a GOE matrix, the SFF is given as

$$K_{\text{GOE}}(\tau) = \begin{cases} 2\tau - \tau \log(1 + 2\tau), & \tau \leq 1 \\ 2 - \tau \log\left(\frac{2\tau+1}{2\tau-1}\right), & \tau > 1. \end{cases}$$  \hspace{1cm} (3)$$

The linear ramp $K_{\text{GOE}}(\tau) \approx 2\tau$ of SFF starting at $\tau = 0$ reflects correlations between all pairs of eigenvalues in a GOE matrix. In contrast, SFF $K(\tau)$ calculated for a physical system follows the GOE predictions $K(\tau) = K_{\text{GOE}}(\tau)$ only for $\tau > \tau_{TH}$ defining $\tau_{TH}$, which, in turn, is proportional to the Thouless time $t_{TH} = \tau_{TH} t_H$. The proportionality factor comes from the fact that unfolded eigenvalues $\epsilon_i$ enter the definition of $K(\tau)$ and is equal to the Heisenberg time $t_H$, determined by the inverse level spacing in the middle of spectrum.

In the diffusive regime, this definition of $t_{TH}$ coincides - up to a constant multiplicative factor - with the original definition of Thouless. In the deeply localized regime where the localization length is much smaller than the system size, a particle never explores the full system size, so that the original Thouless time eventually diverges and becomes larger than the Heisenberg time. In contrast, Poisson statistics are characteristic for the localized regime where the SFF is independent of time; the Thouless time deduced from the SFF is thus equal to the Heisenberg time.

FIG. 1. Thouless time as function of disorder strength $W$ extracted from the spectral form factor for 3D (upper plot) and 5D Anderson model (lower plot) for system size $L$. The black solid lines denote the scaling, grey vertical lines denote the critical disorder strength $W_{C}^{3D} = 16.54$ ($W_{C}^{5D} = 57.3$) in 3D (5D) Anderson model. Dashed lines denote the Heisenberg time $t_H$. The insets show Thouless time rescaled by $L^3$ ($L^5$) in the 3D (5D) case, the lines for different system sizes $L$ clearly cross respectively at $W_{C}^{3D}$ and $W_{C}^{5D}$.

3D Anderson model is characterized by subdiffusion [66] and multifractal wave functions [67, 68]. Properties of the Anderson model in $D > 3$ dimensions are not so extensively explored. However, accurate numerical studies of transport properties of 4D and 5D Anderson model [69] find a localization transition, consistently with studies of level statistics [70] giving the critical disorder in 5D at $W_{C}^{5D} = 57.3$.

Level statistics in 3D Anderson model were studied in [53, 71–74]. Most of the attention was devoted by short-range eigenvalue correlations reflected by the level spacing distribution. Therefore, Thouless times presented in Fig. 1 unveil a new aspect of level statistics in Anderson model associated with long-range correlations of eigenvalues. Examples of SFF of Anderson model and details on the calculation of Thouless time are given in [75].

At small disorder strengths $W$, the Thouless times depicted in Fig. 1 depend quadratically on system size $L$, following precisely the scaling (1) which simply means that the dynamics is diffusive, in accordance with Thouless original work. As $W$ increases, the situation is different for the 3D and 5D cases. For 3D Anderson model, the $t_{TH}/L^2 \propto e^{W/\Omega}$ behavior persists up to $W \approx 12$. For bigger disorder strength, the quadratic scaling with the
system size is no longer valid. Directly at the transition, $W = W_C^{3D}$, the Thouless time should scale as the Heisenberg time i.e. $t_{Th} \propto L^3$. This is indeed the case as the inset in the upper plot in Fig. 1 demonstrates. Further increase of disorder strength leads to a slow increase of the Thouless time $t_{Th}$ with eventual saturation to the Heisenberg time $t_H$.

In the deep delocalized phase where $t_{Th}$ scales with $L^2$, the ratio $t_{Th}/L^2$ is nothing but - up to a constant multiplicative factor - the inverse of the diffusion coefficient $D(W)$, in accordance with the original definition of the Thouless time. The dependence of $D(W)$ with $W$ is not known analytically, but it is known that it decreases quickly with $W$, vanishing at the critical point and scaling like $(W - W_c)^s$ below it, with the critical exponent $s \approx 1.574$. In any case, it is definitely not $e^{-W/\Omega}$ as in (1). It may be that, in a limited range of $W$ values, $D(W)$ can be approximately fitted by an exponential decrease, but other forms could do the job as well.

The 5D case is essentially identical, except that the Thouless time scales like $L^5$ instead of $L^3$ at the critical point. The growth of the Hilbert space size as the fifth power of system size prevents reaching system sizes $L \geq 10$ with exact diagonalization. Nevertheless, the obtained Thouless times $t_{Th}$, when rescaled by $L^5$ as suggested by the relation $t_{Th} \sim t_H$ valid at the transition, lead to a clear crossing of the $t_{Th}/L^5$ curves at $W_C^{5D}$. The $W$ dependence of $t_{Th}/L^2$ in the deeply delocalized regime is again approximately reproduced by an exponential, although it certainly fails near the critical point.

The bigger the system size $L$, the higher the value of disorder strength $\tilde{W}$ for which deviations from (1) are observed. This could lead to the conclusion that there is no transition to the localized phase in the 5D Anderson model, contrary to earlier results [69, 70].

**Diffusion and subdiffusion in Anderson models.** To demonstrate that the obtained behaviors of the Thouless time $t_{Th}$ are related to time dynamics in 3D and 5D Anderson systems, we consider the initial state $|\psi_0\rangle$ with a particle located at a given lattice site with periodic boundary conditions. The time evolved state $|\psi(t)\rangle = e^{-iHt}|\psi_0\rangle$ is obtained employing the standard Chebyshev technique [76] that allows us to get results for system sizes up to $L = 120$ and $L = 30$ for the 3D and 5D cases, respectively. The mean square displacement

$$\langle r^2(t) \rangle = \langle |\psi(t)| \sum_{i=1}^D (\hat{r}_i - \tau_i)^2 |\psi(t)\rangle, \quad (5)$$

where $r_i$ is $i$-th component of the position operator $\hat{r}$ and $\tau_i = \langle \psi(t)|\hat{r}_i|\psi(t)\rangle$, allows us to distinguish diffusive $\langle r^2(t) \rangle \propto Dt$, subdiffusive $\langle r^2(t) \rangle \propto t^\alpha$ and localized behavior which occurs when $\langle r^2(t) \rangle$ saturates after the initial expansion of the wave packet. Time dependence of the mean square displacement is well reflected by the function $\alpha(t) = d \log\langle r^2(t) \rangle/d \log t$. In the case of diffusion $\alpha(t) = 1$, for subdiffusion $0 < \alpha(t) = \alpha < 1$ and in the localized case $\alpha(t) \to 0$. We indeed observe the expected diffusive, subdiffusive and localized regimes as shown in Fig. 2. On the delocalized side of the transition in 3D and 5D Anderson models, respectively for $W < W_C^{3D}$ and $W < W_C^{5D}$ we observe that $\alpha(t)$ initially increases over time. This increase continues to longer times and eventually leads to a larger maximal $\alpha(t)$ value for larger system size. It seems natural to assume that the limit $\alpha(t) \to 1$ is reached in the large system size limit. Subsequent decrease of $\alpha(t)$ occurs when the wave packet ceases to spread as its size approaches the system size. The situation is qualitatively different at the transition, when, regardless of the system size $\alpha(t)$ approaches constant value $\alpha_{3D} = 2/3$ in the 3D case [66] or $\alpha_{5D} = 2/5$ in the 5D case. This is a smoking gun evidence for subdiffusion at the Anderson transition. The eventual decrease of $\alpha(t)$ for the 3D Anderson model is again due to finite system size. Finally, in the localized case, we observe that $\alpha(t)$ decreases with time and its value is nearly independent of the system size – a clear sign of localization in the system.

The observed diffusion and subdiffusion for both 3D and 5D Anderson models are in line with results obtained for the Thouless time $t_{Th}$ from the SFF. In diffusive system $\langle r(t)^2 \rangle = Dt$ which means that the time of reaching of the boundary of the system by initially localized particle, $t^B_{Th} \propto L^2$. In the case of subdiffusion, $\langle r(t)^2 \rangle \propto t^\alpha$ implies that $t^B_{Th} \propto L^{2/\alpha}$. Given the values for $\alpha_{3D}$ and $\alpha_{5D}$ we see that the obtained scalings of $t^B_{Th}$ on the delocalized side of the transition and at the transition agree with the scalings $t^B_{Th} \propto L^2$ and $t^B_{Th} \propto L^3$ (or $t^B_{Th} \propto L^5$ in the 5D case) obtained from the SFF.

**Thouless time in disordered many-body systems.** Consider 1D disordered spin-1/2 chains with Hamiltonian:
\[ H = J_1 \sum_{i=1}^{L} (S_i^x S_{i+1}^x + S_i^y S_{i+1}^y + \Delta S_i^z S_{i+1}^z) + \sum_{i=1}^{L} h_i S_i^z + J_2 \sum_{i=1}^{L} (S_i^x S_{i+2}^x + S_i^y S_{i+2}^y + \Delta S_i^z S_{i+2}^z), \]

where \( S_i \) are spin-1/2 matrices, \( J_1 = 1 \) is fixed as the energy unit, periodic boundary conditions are assumed and \( h_i \in [-W, W] \) are independent, uniformly distributed random variables.

Setting \( J_2 = 0 \) and \( \Delta = 1 \) we arrive at a disordered Heisenberg spin chain, widely studied in the MBL context \([77–84]\). An average gap ratio \( \tau \) \([7, 85]\) that reflects short-range correlations of energy levels has been used in \([78]\), alongside other probes, to find the critical value of disorder strength \( W_C = 3.72(6) \) for transition to MBL phase. However, impact of finite-size effects on that result is notable, as a similar analysis in system with open boundary conditions yields \( W_C = 3.29(9) \) \([86]\).

The choice \( J_1 = J_2 \) and \( \Delta = 0.55 \) in (6) leads to the \( J_1 - J_2 \) model studied in \([46]\). We shall present results for this model side-by-side with results for the disordered Heisenberg chain for a clear comparison between ergodic-MBL crossovers in both models. Average gap ratio data \( \tau \) in \([46]\), if rescaled as in \([78]\) would yield the critical value \( W_C \approx 9 \) for transition to MBL phase in the \( J_1 - J_2 \) mode, we give this value only to roughly describe the disorder strength in the \( J_1 - J_2 \) model.

To extract values of Thouless times in the many-body systems we perform exact diagonalization for \( L \leq 18 \) and use a shift-and-invert method \([87]\) for \( L = 20 \). The results are averaged over more than 1000 (500) disorder realizations for \( L \leq 16 \) (\( L = 18, 20 \)) and shown in Fig. 3.

In the case of \( J_1 - J_2 \) model, Thouless times obtained for available system sizes seem to follow the scaling (1) as for increasing system size \( L \), the point \( W(L) \) of deviating of \( t_{Th}/L^2(W) \) from the \( e^{W/\Omega} \) behavior shifts to larger and larger disorder strength. An interpretation of this behavior along the lines of \([46]\) is that one assumes that system size dependence of \( W(L) \) continues indefinitely, so that the scaling (1) holds in the thermodynamic limit which implies that there is no transition to MBL phase. However, the Thouless time scaling obtained for available system sizes in 5D Anderson model, exhibited in Fig. 1, is very similar with larger system sizes deviating from the (1) at increasing disorder strength. Taking into account that such behavior of Thouless time in 5D Anderson model occurs despite the localization transition taking place at \( W_{5D}^C \), we may give a second possible interpretation of the result: the scaling (1) is not broken in such a way that \( t_{Th}/L^2 \) exceeds the \( e^{W/\Omega} \) line because of strong finite size effects. It is still possible to devise the location of critical point \( W_{5D}^C \) provided one knows the correct value of exponent \( \alpha \) governing the subdiffusion at the Anderson transition in 5D, it is not clear how to rescale the Thouless times \( t_{Th} \) obtain the transition point in the many-body case since the transport properties at the delocalized side of MBL transition are not fully understood, with various approaches suggesting a subdiffusive behavior with exponent \( \alpha \) vanishing in vicinity of the transition \([88]\). Presumably, a sensible criterion for the transition in the many-body case would be \( t_{Th} \propto t_H \propto e^{cL} \).

In any case, the main observation in \([46]\) is that \( t_{Th}/L^2 \) is approximately equal to \( e^{W/\Omega} \) in the deeply localized regime. This implies in turn that the diffusion coefficient \( D(W) \) decreases like \( e^{-W/\Omega} \), exactly like in the 3D and 5D Anderson models. Concluding that is never vanishes, as done in \([46]\), is a grossly wrong extrapolation, as demonstrated by the Anderson models. In other words, while \( D(W) \) is clearly well fitted by a decreasing exponential in some \( W \) range, one cannot deduce that it never vanishes.

The finite size effects in the \( J_1 - J_2 \) model are necessarily enhanced by the next-to-nearest neighbor coupling term, thus we may expect weaker finite size effects for disordered Heisenberg chain. The scaling of Thouless time for this model is presented in the lower panel of Fig. 3 and it follows (1) only for disorder strengths \( W \in [1, 2] \).
We observe two important differences with the results for $J_1 - J_2$. Firstly, at small disorder strengths $W$, the exponential dependence of the Thouless time $t_{Th}$ on $W$ is weaker than in the interval $W \in [1, 2]$. This does not seem to be a finite size effect, we believe that such behavior is due to the proximity of the integrable point at $W = 0$ [89, 90] with Poisson level statistics and $t_{Th} = t_H$.

Secondly and more importantly, we see a breakdown of the scaling (1) for $W \geq 2$ where the data for $L = 22$ and $L = 24$ exceed the $t_0 e^{W/\Omega}$ line even though the Thouless time is still an order of magnitude smaller than the Heisenberg time $t_H$. Once more, it indicates that the exponential scaling with $W$ is only a heuristic numerical observation in a limited range, without any generality. The data for $L = 22$ and $L = 24$ are available only for $W \geq 2$ and $W > 2.2$ due to a limitation of total number of consecutive eigenvalues in a single disorder realization to 100 and 50 [91]. Nevertheless, the breakdown of the scaling (1) by data for $L = 22, 24$ at $W \approx 2.2$ is apparent, suggesting that the $L^2$ scaling of Thouless time breaks down, which, in turn reflects the slow-down on transport and approaching MBL transition when $t_{Th} \propto t_H \propto e^{-L}$.

Conclusions. Our results show that the Thouless time, defined by the behavior of the spectral form factor reflects the transport properties in disordered non-interacting models as we have shown on the examples of 3D and 5D Anderson models. In particular, the scaling of the Thouless time $t_{Th}$ directly at the transition encodes the subdiffusive behavior of the mean square displacement $\langle r^2(t) \rangle \sim t^\alpha$ with the exponent $\alpha_{3D} = 2/3$ and $\alpha_{5D} = 2/5$ leading to scaling $t_{Th} \sim L^{5/\alpha}$ with system size at the transition.

The scaling of Thouless time for $J_1 - J_2$ spin chain seems to follow $t_{Th} \sim t_0 L^2 e^{W/\Omega}$, however, the behavior of $t_{Th}$ is directly analogous to the case of 5D Anderson model. The 5D Anderson model undergoes a transition to a localized phase and the Thouless time do not exceed the $t_0 L^2 e^{W/\Omega}$ curve only because of strong finite size effects at available system sizes. It is plausible that the situation is the same in the $J_1 - J_2$ model, raising doubts about the lack of transition to the MBL phase as claimed in [46]. Our results for disordered Heisenberg model demonstrate that the $L^2$ scaling of Thouless time $t_{Th}$, valid deep in delocalized phase is evidently broken at $W \approx 2.2$, signaling transition to MBL phase at sufficiently strong disorder.

Finally, let us mention alternative definitions of Thouless time as extracted from properties of off-diagonal matrix elements of a local observable [92, 93] or directly from time dynamics [94–98]. Comparison of these different approaches is in progress. While finalizing this manuscript, we became aware of the related work [99].

Acknowledgements. We are most grateful to Fabien Alet for kindly sharing with us the eigenvalues for $L = 22, 24$ Heisenberg spin chain as well as discussions on subjects related to this work. The computations have been performed in PL-Grid Infrastructure. We acknowledge the support of National Science Centre (Poland) under projects 2015/19/B/ST2/01028 (P.S. and J.Z.), 2018/28/T/ST2/00401 (doctoral scholarship – P.S.) as well as Polish-French bilateral grant Polonium.

* piotr.sierant@uj.edu.pl

[1] I. V. Gornyi, A. D. Mirlin, and D. G. Polyakov, Phys. Rev. Lett. 95, 206603 (2005).
[2] D. Basko, I. Aleiner, and B. Altshuler, Ann. Phys. (NY) 321, 1126 (2006).
[3] R. Nandkishore and D. A. Huse, Ann. Rev. Cond. Mat. Phys. 6, 15 (2015).
[4] F. Alet and N. Laflorencie, Comptes Rendus Physique 19, 498 (2018).
[5] D. A. Abanin, E. Altman, I. Bloch, and M. Serbyn, Rev. Mod. Phys. 91, 021001 (2019).
[6] L. F. Santos, G. Rigolin, and C. O. Escobar, Phys. Rev. A 69, 042304 (2004).
[7] V. Oganesyan and D. A. Huse, Phys. Rev. B 75, 155111 (2007).
[8] A. Pal and D. A. Huse, Phys. Rev. B 82, 174411 (2010).
[9] P. Sierant and J. Zakrzewski, New Journal of Physics 20, 043032 (2018).
[10] T. Orell, A. A. Michailidis, M. Serbyn, and M. Silveri, Phys. Rev. B 100, 134504 (2019).
[11] R. Mondaini and M. Rigol, Phys. Rev. A 92, 041601 (2015).
[12] P. Prelovšek, O. S. Barišić, and M. Žnidarič, Phys. Rev. B 94, 241104 (2016).
[13] J. Zakrzewski and D. Delande, Phys. Rev. B 98, 041203 (2018).
[14] M. Kozarzewski, P. Prelovšek, and M. Mierzejewski, Phys. Rev. Lett. 120, 246602 (2018).
[15] P. W. Anderson, Phys. Rev. 109, 1492 (1958).
[16] P. Sierant, D. Delande, and J. Zakrzewski, Phys. Rev. A 95, 021601 (2017).
[17] Y. Bar Lev, D. R. Reichman, and Y. Sagi, Phys. Rev. B 94, 201116 (2016).
[18] X. Li, D.-L. Deng, Y.-L. Wu, and S. Das Sarma, Phys. Rev. B 95, 020201 (2017).
[19] N. Macê, N. Laflorencie, and F. Alet, SciPost Phys. 6, 50 (2019).
[20] A. Smith, J. Knolle, R. Moessner, and D. L. Kovrizhin, Phys. Rev. Lett. 119, 176601 (2017).
[21] M. Brenes, M. Dalmonte, M. Heyl, and A. Scardicchio, Phys. Rev. Lett. 120, 030601 (2018).
[22] M. Schulz, C. A. Hooley, R. Moessner, and F. Pollmann, Phys. Rev. Lett. 122, 040606 (2019).
[23] E. van Nieuwenburg, Y. Baum, and G. Refael, Proceedings of the National Academy of Sciences 116, 9269 (2019).
[24] G. De Tomasi, Phys. Rev. B 99, 054204 (2019).
[25] A. Safavi-Naini, M. L. Wall, O. L. Acevedo, A. M. Rey, and R. M. Nandkishore, Phys. Rev. A 99, 033610 (2019).
[26] T. Botzung, D. Vodola, P. Naldesi, M. Müller, E. Ercolessi, and G. Pupillo, Phys. Rev. B 100, 155136 (2019).
[27] A. O. Maksymov and A. L. Burin, (2019), arXiv:1905.02286.
[28] P. Sierant, K. Biedroń, G. Morigi, and J. Zakrzewski,
SUPPLEMENTARY: EXTRACTING THOULESS TIME FROM SPECTRAL FORM FACTOR

The spectral form factor $K(t/t_H)$ obtained with use of expression (2) for 3D Anderson model is shown in Fig. 4. SFF for Anderson model behaves similarly to spectral form factors of interacting many-body systems shown in [46]. To calculate the SFF, we use a gaussian function $g(\epsilon) \propto \exp(- (\epsilon - \bar{\epsilon})^2 / 2\eta \sigma^2)$, where $\bar{\epsilon}$ denotes the average of the unfolded eigenvalues for given disorder realization $\epsilon_i$, $\sigma$ is the standard deviation of $\{\epsilon_i\}$ and $\eta = 0.3$. The unfolding of eigenvalues that sets the mean level density to unity is performed by fitting of 10th order polynomial to level staircase for systems for which an exact diagonalization is performed and the full eigenvalue spectrum is obtained (those include 3D, 5D Anderson models, Heisenberg spin chain and $J_1 - J_2$ model of size $L \leq 18$). For larger system, for which only a fraction of eigenvalues obtained from shift-and-invert method is available, we perform unfolding with a 3rd oder polynomial. Importantly, whenever only a fraction of spectrum is available, we are able to obtain the SFF $K(t/t_H)$ only for times bounded from below by certain $t_M$ dependent on the number of available eigenvalues and system size.

To extract Thouless time from the SFF, we follow [46] and consider a function

$$\Delta K(t/t_H) = \left| \log \left( \frac{K(t/t_H)}{K_{GOE}(\tau = t/t_H)} \right) \right|. \quad (S.1)$$

The Thouless time $t_{Th}$ is the smallest time bigger than 0 for which $\Delta K(t/t_H) < 0.05$. The resulting Thouless times are denoted in Fig. 4 by dots and correspond to moment at which SFF for given disorder strength deviates from the $K_{GOE}(\tau = t/t_H)$ function.