Magnetic bottles on geometrically finite hyperbolic surfaces

Abderemane MORAME\textsuperscript{1} and Françoise TRUC\textsuperscript{2}

\textsuperscript{1} Université de Nantes, Faculté des Sciences, Dpt. Mathématiques, UMR 6629 du CNRS, B.P. 99208, 44322 Nantes Cedex 3, (FRANCE), E.Mail: morame@math.univ-nantes.fr

\textsuperscript{2} Université de Grenoble I, Institut Fourier, UMR 5582 CNRS-UJF, B.P. 74, 38402 St Martin d’Hères Cedex, (France), E.Mail: Francoise.Truc@ujf-grenoble.fr

Abstract

We consider a magnetic Laplacian $-\Delta_A = (id + A)^*(id + A)$ on a hyperbolic surface $M$, when the magnetic field $dA$ is infinite at the boundary at infinity. We prove that the counting function of the eigenvalues has a particular asymptotic behavior when $M$ has an infinite area.\footnote{\textit{Keywords} : spectral asymptotics, magnetic bottles, hyperbolic surface.}

1 Introduction

We consider a smooth, connected, complete and oriented Riemannian surface $(M,g)$ and a smooth, real one-form $A$ on $M$. We define the magnetic Laplacian

$$-\Delta_A = (id + A)^*(id + A),$$

$$(id + A)u = i du + uA, \ \forall u \in C^\infty_0(M; \mathbb{C}) .$$

The magnetic field is the exact two-form $\rho_B = dA$.

If $dm$ is the Riemannian measure on $M$, then

$$\rho_B = \tilde{b} \ dm, \ \text{with} \ \tilde{b} \in C^\infty(M; \mathbb{R}) .$$

The magnetic intensity is $b = |\tilde{b}|$.\footnote{\textit{Keywords} : spectral asymptotics, magnetic bottles, hyperbolic surface.}
It is well known, (see [Shu]), that $-\Delta_A$ has a unique self-adjoint extension on $L^2(M)$, containing in its domain $C^\infty_0(M; \mathbb{C})$, the space of smooth and compactly supported functions.

When $b$ is infinite at the infinity, (with some additional assumption), the spectrum of $-\Delta_A$ is discrete, and we denote by $(\lambda_j)_j$ the increasing sequence of eigenvalues of $-\Delta_A$, (each eigenvalue is repeated according to its multiplicity). Let

$$N(\lambda) = \sum_{\lambda_j < \lambda} 1. \quad (1.3)$$

We are interested by the hyperbolic surfaces $M$, when the curvature of $M$ is constant and negative.

In this case, when $M$ has finite area, the asymptotic behavior of $N(\lambda)$ seems to be the Weyl formula: $N(\lambda) \sim +\infty \frac{\lambda}{4\pi |M|}$.

S. Golénia and S. Moroianu in [Go-Mo] have such examples.

In the case of the Poincaré half-plane, $M = \mathbb{H}$, we prove in [Mo-Tr] that the Weyl formula is not valid: $\lim_{\lambda \to +\infty} \lambda^{-1} N(\lambda) = +\infty$.

For example when $b(z) = a_0^2(x/y)2m_0 + a_1^2y^{m_1} + a_2^2y^{-m_2}$, $a_j > 0$ and $m_j \in \mathbb{N}^*$, then

$$N(\lambda) \sim +\infty \lambda^{1+1/(2m_0)} \ln(\lambda) \alpha(m_0, m_1, m_2).$$

In this paper, we are interested by the hyperbolic surfaces with infinite area. When $M$ is a geometrically finite hyperbolic surface of infinite area and when the above example is arranged for this new situation, ($m_0$ is absent, $m_1$ appears in the cusps and $m_2$ in the funnels), we get

$$N(\lambda) \sim +\infty \lambda^{1+1/m_2} \alpha(m_2);$$

the cusps do not contribute to the leading part of $N(\lambda)$.

## 2 Main result

We assume that $(M, g)$ is a smooth connected Riemannian manifold of dimension two, which is a geometrically finite hyperbolic surface of infinite area; (see [Per] or [Bor] for the definition and the related references). More precisely

$$M = \left( \bigcup_{j=0}^{J_1} M_j \right) \bigcup \left( \bigcup_{k=1}^{J_2} F_k \right); \quad (2.1)$$
where the $M_j$ and the $F_k$ are open sets of $M$, such that the closure of $M_0$ is compact, and if $J_1 > 0$, the other $M_j$ are cuspidal ends of $M$, and the $F_k$ are funnel ends of $M$.

This means that, for any $j$, $1 \leq j \leq J_1$, there exist strictly positive constants $a_j$ and $L_j$ such that $M_j$ is isometric to $\mathbb{S} \times ]a_j^2, +\infty[$, equipped with the metric

$$ds_j^2 = y^{-2}(L_j^2 \, d\theta^2 + dy^2) ;$$

(2.2)

($\mathbb{S} = \mathbb{S}^1$ is the unit circle.)

In the same way, for any $k$, $1 \leq k \leq J_2$, there exist strictly positive constants $\alpha_k$ and $\tau_k$ such that $F_k$ is isometric to $\mathbb{S} \times [\alpha_k^2, +\infty[$, equipped with the metric

$$ds_k^2 = \tau_k^2 \cosh^2(t) d\theta^2 + dt^2 ;$$

(2.3)

moreover, for any two integers $j, k > 0$, we have $M_j \cap F_k = \emptyset$ if $j \neq k$.

Let us choose some $z_0 \in M_0$ and let us define

$$d : M \to \mathbb{R}_+ ; \quad d(z) = d_g(z, z_0) ;$$

(2.4)

$d_g(\ldots)$ denotes the distance with respect to the metric $g$.

We assume the smooth one-form $A$ to be given such that the magnetic field $\tilde{b}$ satisfies

$$\lim_{d(z) \to \infty} \tilde{b}(z) = +\infty .$$

(2.5)

If $J_1 > 0$, there exists a constant $C_1 > 0$ such

$$|X\tilde{b}(z)| \leq C_1(b(z) + 1)e^{d(z)}|X|_g ;$$

(2.6)

$\forall z \in M_j$, $\forall X \in T_z M$ and $\forall j = 1, \ldots J_1$.

There exists a constant $C_2 > 0$ such

$$|X\tilde{b}(z)| \leq C_2(b(z) + 1)|X|_g ;$$

(2.7)

$\forall z \in F_k$, $\forall X \in T_z M$ and $\forall k = 1, \ldots J_2$.

For any self-adjoint operator $P$, and for any real $\lambda$, we will denote by $E_\lambda(P)$ its spectral projection, and when its trace is finite we will denote it by

$$N(\lambda; P) = Tr(E_\lambda(P)) .$$

$N(\lambda; P)$ is the number of eigenvalues of $P$, (counted with their multiplicity), which are in $]-\infty, \lambda[$.

3
Theorem 2.1 Under the above assumptions, $-\Delta_A$ has a compact resolvent and for any $\delta \in \left[\frac{1}{3}, \frac{2}{5}\right]$, there exists a constant $C > 0$ such that

$$
\frac{1}{2\pi} \int_M (1 - \frac{C}{(b(m) + 1)(2 - 5\delta)/2}) \mathcal{N}(\lambda(1 - C\lambda^{-3\delta + 1}) - \frac{1}{4}, b(m)) \, dm
\leq N(\lambda, -\Delta_A) \leq \frac{1}{2\pi} \int_M (1 + \frac{C}{(b(m) + 1)(2 - 5\delta)/2}) \mathcal{N}(\lambda(1 + C\lambda^{-3\delta + 1}) - \frac{1}{4}, b(m)) \, dm
$$

(2.8)

where

$$
\mathcal{N}(\mu, b(m)) = b(m) \sum_{k=0}^{+\infty} [\mu - (2k + 1)b(m)]_+^0 \quad \text{if} \quad b(m) > 0,
$$

and

$$
\mathcal{N}(\mu, b(m)) = \mu/2 \quad \text{if} \quad b(m) = 0.
$$

$[\rho]_+^0$ is the Heaviside function:

$$
[\rho]_+^0 = \begin{cases} 1, & \text{if} \quad \rho > 0 \\ 0, & \text{if} \quad \rho \leq 0. \end{cases}
$$

The Theorem remains true if we replace $\int_M$ by $\sum_{k=1}^{J_2} \int_{F_k}$, due to the fact that the other parts are bounded by $C\lambda$.

Corollary 2.2 Under the assumptions of Theorem 2.1 and if the function

$$
\omega(\mu) = \int_M [\mu - b(m)]_+^0 \, dm
$$
satisfies, $\exists C_1 > 0 \text{ s.t. } \forall \mu > C_1, \forall \tau \in ]0, 1[,$

$$
\omega((1 + \tau)\mu) - \omega(\mu) \leq C_1 \tau \omega(\mu),
$$

(2.9)

then

$$
N(\lambda; -\Delta_A) \sim \frac{1}{2\pi} \int_M \mathcal{N}(\lambda - \frac{1}{4}, b(m)) \, dm.
$$

(2.10)
For example this allows us to consider magnetic fields of the following type:

\[\text{on } F_k, \quad b(\theta, t) = p_k(1/cosh(t)),\]
and on \( M_j, \quad j > 0, \quad b(\theta, y) = q_j(y),\)

where the \( p_k(s) \) and the \( q_j(s) \) are, for large \( s \), polynomial functions of order \( \geq 1 \). In this case, if \( d \) is the largest order of the \( p_k(s) \), then

\[N(\lambda; -\Delta_A) \sim \alpha \lambda^{1+1/d},\]

for some constant \( \alpha > 0 \), depending only on the funnels \( F_k \) where the order of \( p_k(s) \) is \( d \).

\section{Estimate for Dirichlet operators}

\subsection{The main propositions}

In this section, we consider some particular open set \( \Omega \) of \( \mathbb{M} \) with smooth boundary. To \( \Omega \) and \( -\Delta_A \), we associate the Dirichlet operator \( -\Delta_\Omega^A \), and we estimate \( N(\lambda; -\Delta_\Omega^A) \).

**Proposition 3.1** Let \( \Omega \) an open set of \( M_0 \) with smooth boundary. Then there exists a constant \( C_\Omega > 0 \) s.t.

\[\left| N(\lambda; -\Delta_\Omega^A) - \frac{\Omega}{4\pi} \lambda \right| \leq C_\Omega \sqrt{\lambda}; \quad \forall \lambda > 1.\]

As \( \overline{\Omega} \) is compact, the above estimate is well known. See for example Theorem 29.3.3 in [Hor].

**Proposition 3.2** Let \( j > 0 \) and \( \Omega \) an open set of the cusp \( M_j \), isometric to \( S \times [a^2, +\infty[, \), equipped with the metric

\[ds^2 = y^{-2} (L^2 \, d\theta^2 + dy^2); \quad (a \text{ and } L \text{ are strictly positive constants}).\]

Then \( -\Delta_\Omega^A \) has a compact resolvent and

\[N(\lambda; -\Delta_\Omega^A) \sim \frac{|\Omega|}{4\pi} \lambda; \quad \text{as } \lambda \to +\infty.\]
We will prove it in the next subsection.

**Proposition 3.3** Let $\Omega$ an open set of a funnel $F_k$, isometric to $\mathbb{S} \times [a^2, +\infty[$, equipped with the metric
\[
 ds^2 = L^2 \cosh^2(t) \, dt^2 + \, dt^2; \quad (a \text{ and } L \text{ are strictly positive constants}).
\]
Then $-\Delta_\Omega^A$ has a compact resolvent and for any $\delta \in ]\frac{1}{3}, \frac{2}{5}[$, there exists a constant $C > 0$ such that
\[
\frac{1}{2\pi} \int_{\Omega} (1 - \frac{C}{b(m) + 1} \frac{(2-5\delta)/2}{}) \mathcal{N}(\lambda(1 - C\lambda^{-3\delta+1}) - \frac{1}{4} \cdot b(m)) \, dm 
\leq \mathcal{N}(\lambda, -\Delta_\Omega^A) \leq 
\frac{1}{2\pi} \int_{\Omega} (1 + \frac{C}{b(m) + 1} \frac{(2-5\delta)/2}{}) \mathcal{N}(\lambda(1 + C\lambda^{-3\delta+1}) - \frac{1}{4} \cdot b(m)) \, dm.
\]
The proof comes easily following the ones in the Poincaré half-plane of $\text{[Mo-Tr]}$, using the method of $\text{[Col]}$, in the neighbourhood of the boundary at infinity. It corresponds to a context where the partitions of unity were fine, so they can be performed on $\mathbb{S} \times [a^2, +\infty[$, (instead of $\mathbb{R} \times [-\infty, 0[)$.

### 3.2 Proof of Proposition 3.2

For simplicity we change the unit circle $\mathbb{S} = S_1$ into the circle $S_L$, of radius $L$, so
\[
 \Omega = S_L \times [a^2, +\infty[; \quad ds^2 = y^{-2}(dx^2 + dy^2), \quad \text{and} \quad (3.1)
\]
\[
 -\Delta_\Omega^O u(z) = y^2[(D_x - A_1)^2 u(z) + (D_y - A_2)^2 u(z)];
\]
moreover $d(z, z') = \arg \cosh \frac{y^2 + y'^2 + d_{S_L}^2(x, x')}{2yy'}$.

We begin by proving the compactness of the resolvent of $-\Delta_\Omega^A$.

**Lemma 3.4** There exists $C_0 > 1$ such that
\[
 \int_{\Omega} (b(z) - C_0) |u(z)|^2 \, dm \leq \int_{\Omega} -\Delta_\Omega^O u(z) \overline{u(z)} \, dm; \quad \forall u \in C_0^\infty(\Omega).
\]
Proof. Let us denote the quadratic form

\[ q_\Omega^A(u) = \int_\Omega -\Delta_\Omega u(z)\overline{u(z)} \, dm \quad \forall u \in C_0^\infty(\Omega). \quad (3.2) \]

Then \( q_\Omega^A(u) = \int_\Omega \left[ (D_x - A_1)^2 u(z) + (D_y - A_2)^2 u(z) \right] \, dxdy, \)
and

\[ \left| \int_\Omega \overline{\tilde{b}(z)} u(z)^2 \, dm \right| = \left| \int_\Omega [(D_x - A_1)u(z)(D_y - A_2)u(z) - (D_y - A_2)u(z)(D_x - A_1)u(z)] \, dxdy \right|. \]

Therefore we get that \( \left| \int_\Omega \overline{\tilde{b}(z)} u(z)^2 \, dm \right| \leq q_\Omega^A(u). \)

As \( \tilde{b}(z) = \overline{b(z)} \to +\infty \) at the infinity, the Lemma comes easily.

The Lemma 3.4 and the assumption (2.5) prove that \(-\Delta_\Omega^A\) has compact resolvent.

Later on, we will need that the assumptions (2.5) and (2.6) ensure that there exists \( C > 1 \) such that

\[ b(z)/C \leq b(z') \leq C b(z), \quad \text{if} \quad |y - y'| \leq 1 \quad \text{and} \quad y > C. \quad (3.3) \]

This comes from the fact that \( d(z) \) is equivalent to \( \ln(y) \) for \( y(> 1) \) large enough, so the assumption (2.6) ensures that \( |\partial_x b(z)| + |\partial_y b(z)| \leq C(|b(z)| + 1). \)

Lemma 3.5 There exists a constant \( C_0 > 1 \) such that, for any \( \lambda > 1 \) and for any \( K \subset \Omega \) isometric to \( I_1 \times I_2 \), endowed with the metric in (3.1), with

\[ I_1 = [x_0 - \epsilon_1, x_0 + \epsilon_1], \quad I_2 = [y_0 - \epsilon_2, y_0 + \epsilon_2], \]
\[ \epsilon_1 \in ]C_0^{-1}, 1[, \quad \epsilon_2 = \sqrt{\frac{1}{y_0}}/\sqrt{b(z_0)}, \quad (y_0 > C_0); \]

the following estimates hold:

\[ [\lambda(1 - \frac{1}{\sqrt{y_0}}) - C_0] \frac{|K|}{4\pi} \leq N(\lambda; -\Delta_\Omega^A) \leq [\lambda(1 + \frac{1}{\sqrt{y_0}}) + C_0] \frac{|K|}{4\pi}. \quad (3.4) \]

Proof. If \( b(z_0) > C \lambda \), then, according to the estimate of Lemma 3.4 with \( K \) instead of \( \Omega \), \( N(\lambda; -\Delta_K^A) = 0 \).
So we can assume that \( b(z_0) \leq C \lambda \). 7
We use that the spectrum of $-\Delta^K_A$ is gauge-invariant, so we can suppose that in $K$
\[
A_2 = 0 \quad \text{and} \quad A_1(x, y) = -\int_{y_0}^y \frac{\tilde{b}(x, \rho)}{\rho^2} d\rho.
\]
Then $|A_1(x, y)| \leq C\epsilon_2 \frac{b(z_0)}{y_0^2}$.
From this estimate, we get that for any $\epsilon \in ]0, 1[$,
\[
-(1 - \epsilon)\Delta^K_0 - C\epsilon_2^2 \frac{b^2(z_0)}{y_0^2} \leq -\Delta^K_A \leq -(1 + \epsilon)\Delta^K_0 + C\epsilon_2^2 \frac{b^2(z_0)}{y_0^2}.
\]
We take $\epsilon = 1/\sqrt{y_0}$, to get
\[
-(1 - \frac{1}{\sqrt{y_0}})\Delta^K_0 - C \frac{b(z_0)}{\sqrt{y_0}} \leq -\Delta^K_A \leq -(1 + \frac{1}{\sqrt{y_0}})\Delta^K_0 + C \frac{b(z_0)}{\sqrt{y_0}}.
\]
As $b(z_0) \leq C\lambda$, the Lemma follows easily from the min-max principle and the well-known estimate for $N(\lambda; -\Delta^K_0)$.

**Proof of Proposition 3.2.**
It follows easily from Lemma 3.5 (for large $y$), using the same tricks as in [Mo-Tr].

4 Proof of the main Theorem 2.1
The proof comes easily from the three propositions 3.1 - - 3.3, following the method developped in [Mo-Tr].

5 Remark on the case of constant magnetic field
It is not always possible to have a constant magnetic field on $M$, (for topological reason), but for any $(b, \beta) \in \mathbb{R}^{J_1} \times \mathbb{R}^{J_2}$, there exists a one-form $A$, such that the corresponding magnetic field $dA$ satisfies

\[
dA = \tilde{b}(z) dm \quad \left\{
\begin{array}{l}
\tilde{b}(z) = b_j \quad \forall \ z \in M_j \\
\tilde{b}(z) = \beta_k \quad \forall \ z \in F_k
\end{array}
\right.
\]  
(5.1)
Theorem 5.1 Assume (2.1) and (5.1).

If $J_1 = 0$ and $J_2 > 0$, then the essential spectrum of $-\Delta_A$ is

$$\text{sp}_{\text{ess}}(-\Delta_A) = \left\{ \frac{1}{4} + \inf_k \beta_k^2, \quad +\infty \right\} \bigcup \bigcup_{k=1}^{J_2} S(\beta_k)$$  \hspace{1cm} (5.2)

with $S(\beta_k) = \emptyset$ when $|\beta_k| \leq 1/2$ and when $|\beta_k| > 1/2$.

$S(\beta_k) = \{(2j+1)|\beta_k| - j(j+1) ; \ j \in \mathbb{N}, \ j < |\beta_k| - 1/2\}$.

If $J_1$ and $J_2$ are $> 0$, then for any $j$, $1 \leq j \leq J_1$ and for any $z \in M_j$ there exists a unique closed curve through $z$, $C_{j,z}$ in $(M_j, g)$, not contractible and with zero $g$-curvature. The following limit exists and is finite:

$$[A]_{M_j} = \lim_{d(z) \to +\infty} \int_{C_{j,z}} A.$$  \hspace{1cm} (5.3)

If $J_1^A = \{ j \in \mathbb{N} \mid 1 \leq j \leq J_1 \text{ s.t. } [A]_{M_j} \in 2\pi \mathbb{Z} \}$, then

$$\text{sp}_{\text{ess}}(-\Delta_A) = \left\{ \frac{1}{4} + \min_{j \in J_1^A} \inf_{1 \leq k \leq J_2} \beta_k^2 \right\} \bigcup \bigcup_{k=1}^{J_2} S(\beta_k).$$  \hspace{1cm} (5.4)

If $J_2 = 0$ and $J_1^A = \emptyset$, then $\text{sp}_{\text{ess}}(-\Delta_A) = \emptyset$: $-\Delta_A$ has purely discrete spectrum, (its resolvent is compact).

Remark 5.2 In Theorem 5.1, one can change $C_{j,z}$ into $S_{j,z}$, the unique closed curve through $z$, not contractible and with minimal $g$-length. $S_{j,z}$ is not smooth at $z$, $S_{j,z}$ is part of two geodesics through $z$, so there is an out-going tangent and an incoming tangent at $z$. It is easy to see that $C_{j,z} \cap S_{j,z} = \{ z \}$, so by Stokes formula

$$\int_{S_{j,z}} (A - A^0) = \int_{C_{j,z}} (A - A^0),$$

where $A^0$ is a one-form on $M$, such that

$$dA = dA^0 \text{ on } M_j \text{ and } [A^0]_{M_j} = 0 ; \forall j.$$

The orientation in both cases $C_{j,z}$ and $S_{j,z}$ is chosen such that, if $u_z, v_z \in T_z M_j$, $g_z(u_z, v_z) = 0$, $dm(u_z, v_z) > 0$, and $u_z$ is tangent to the curve (in the positive direction), then $v_z$ points to boundary at infinity; (for $S_{j,z}$, one can take as $u_z$ the out-going tangent, or the incoming tangent).
Proof of Theorem 5.1. It is clear that

$$\text{sp}_{ess}(-\Delta_A) = \left( \bigcup_{j=1}^{J_1} \text{sp}_{ess}(-\Delta_{A_j}) \right) \bigcup \left( \bigcup_{k=1}^{J_2} \text{sp}_{ess}(-\Delta_{F_k}^A) \right); \quad (5.5)$$

so the proof will result on the two lemmas below.

Lemma 5.3

$$\text{sp}_{ess}(-\Delta_{F_k}^A) = \left[ \frac{1}{4} + \beta_k^2, +\infty \right] \cup S(\beta_k).$$

Proof. We have

$$-\Delta_{F_k}^A = \tau_k^{-2} \cosh^{-2}(t)(D_\theta - A_1)^2 + \cosh^{-1}(t)(D_t - A_2) [ \cosh(t)(D_t - A_2) ].$$

Since $\tilde{b} = \beta_k = \tau_k^{-1} \cosh^{-1}(t)(\partial_\theta A_2 - \partial_t A_1)$, there exists a function $\varphi$ such that $A - \tilde{A} = d\varphi$ if $\tilde{A} = (\xi - \beta_k \tau_k \sinh(t))d\theta$, (for some constant $\xi$).

So we can assume that $A = \tilde{A}$.

We change the density $dm = \tau_k \cosh(t)d\theta dt$ for $d\theta dt$, using the unitary operator $Uf = (\tau_k \cosh(t))^{1/2}f$, so

$$P = -U \Delta_{F_k}^A U^* = \tau_k^{-2} \cosh^{-2}(t)(D_\theta - A_1)^2 + D_t^2 + \frac{1}{4}(1 + \cosh^{-2}(t)).$$

We remind that $\lambda \in \text{sp}_{ess}(-\Delta_{F_k}^A)$ iff there exists a sequence $(u_j)_j \in \text{Dom}(-\Delta_{F_k}^A)$ converging weekly in $L^2(F_k)$ to zero, $\|u_j\|_{L^2(F_k)} = 1$ and such that the sequence $(-\Delta_{F_k}^A u_k - \lambda u_k)_k$ converges strongly to zero.

It is clear that $\text{sp}(-\Delta_{F_k}^A) = \text{sp}(\bigoplus_{\ell \in \mathbb{Z}} P_\ell)$.

$$P_\ell = D_t^2 + \tau_k^{-2} \cosh^{-2}(t)(\ell + \beta_k \tau_k \sinh(t) - \xi)^2 + \frac{1}{4}(1 + \cosh^{-2}(t)), $$

for the Dirichlet condition on $L^2(I; dt); \quad I = [\alpha_k^2, +\infty[$.

So $\text{sp}(-\Delta_{F_k}^A) = \bigcup_{\ell \in \mathbb{Z}} \text{sp}(P_\ell)$.

Writing that $P_\ell = D_t^2 + \left( \frac{\ell - \xi}{\tau_k \cosh(t)} + \beta_k \tanh(t) \right)^2 + \frac{1}{4}(1 + \cosh^{-2}(t))$, we get easily that $\text{sp}_{ess}(P_\ell) = \left[ \frac{1}{4} + \beta_k^2, +\infty \right]$, and that the number of eigenvalues $< \frac{1}{4} + \beta_k^2$ is finite for all $\ell < \xi$ and equal to zero for all $\ell \geq \xi$. Here
we assume $\beta_k > 0$. So \(\left[\frac{1}{4} + \beta_k^2, +\infty[\subset \text{sp}_{\text{ess}}(-\Delta^F_A)\) and the other part of \(\text{sp}_{\text{ess}}(-\Delta^F_A)\) is \(S_\infty = \{\lambda \mid \lambda = \lim_{j \to +\infty} \lambda_{\ell(j)}, \lambda_{\ell(j)} \in \text{sp}_d(P_{\ell(j)})\}\), where \((\ell(j))_j\) denotes any decreasing sequence of negative integers.

Now we use again the formula
\[
P_\ell = D_i^2 + \left(\frac{\ell - \xi}{\tau_k \cosh(t)} + \beta_k \tanh(t)\right)^2 + \frac{1}{4}(1 + \cosh^{-2}(t)).
\]
Assuming $\ell - \xi < 0$, we set $\rho = |\ell - \xi|/\tau_k$ and we introduce the new variable $y = 2\rho e^{-t}$. We get that $P_\ell$ is unitarily equivalent to $\tilde{P}_\rho$ defined as a Dirichlet type operator in $L^2(0, 2\rho e^{-\alpha^2t}; dy)$, (zero boundary condition is only required on the right boundary):

\[
\tilde{P}_\rho = D_y(y^2D_y) + W_\rho(y), \quad \text{with}
\]
\[
W_\rho(y) = \left(\beta_k \frac{(1 - y^2/(4\rho^2))}{1 + y^2/(4\rho^2)} - \frac{y}{1 + y^2/(4\rho^2)}\right)^2 + \left(\frac{y/(2\rho)}{1 + y^2/(4\rho^2)}\right)^2.
\]
So we have \(\lim_{\rho \to +\infty} W_\rho(y) = W_\infty(y) = (\beta_k - y)^2\), and the operator $\tilde{P}_\infty = D_y(y^2D_y) + W_\infty(y)$ on $L^2([0, +\infty[; dy)$ satisfies, (see [Mo-Tr]),

\[
\text{sp}(\tilde{P}_\infty) = \text{sp}_{\text{ess}}(\tilde{P}_\infty) \cup \text{sp}_d(\tilde{P}_\infty)
\]

\[
\text{sp}_{\text{ess}}(\tilde{P}_\infty) = \left[\frac{1}{4} + \beta_k^2, +\infty\right]; \quad \text{sp}_d(\tilde{P}_\infty) = S(\beta_k).
\]

We remind that the eigenfunctions associated to the eigenvalues in $S(\beta_k)$ of $\tilde{P}_\infty$ are exponentially decreasing, so if $\lambda_0(\rho) \leq \ldots \leq \lambda_j(\rho) \leq \lambda_{j+1}(\rho)\ldots$ are the eigenvalues of $\tilde{P}_\rho$ then for any $j$,

\[
\lim_{\rho \to +\infty} \lambda_j(\rho) = \lambda_j(\infty) = (2j + 1)\beta_k - j(j + 1), \text{ if } \beta_k > 1/2 \text{ and } j < \beta_k - 1/2,
\]

otherwise \(\lim_{\rho \to +\infty} \lambda_j(\rho) = \frac{1}{4} + \beta_k^2\).

Therefore we get that $S_\infty = S(\beta_k)$, or $S_\infty = S(\beta_k) \cup \left\{\frac{1}{4} + \beta_k^2\right\}$ : the formula of Lemma 5.3 follows.

**Lemma 5.4** If $1 \leq j \leq J_1$ and $j \notin J^A_1$, then

\[
\text{sp}_{\text{ess}}(-\Delta^F_A) = \emptyset.
\]

If $j \in J^A_1$, then

\[
\text{sp}_{\text{ess}}(-\Delta^F_A) = \left[\frac{1}{4} + b_j^2, +\infty[\right).
\]
Proof. Use the coordinate \( t = \ln y \) instead of \( y \), so

\[
M_j = S \times [\alpha_j^2, +\infty[ \quad \text{and} \quad ds_j^2 = L_j^2 e^{-2t} d\theta^2 + dt^2; \quad (\alpha_j = e^{\theta_j}).
\]

Then

\[
-\Delta_A^{M_j} = L_j^{-2} e^{2t} (D_\theta - A_1)^2 + e^t (D_t - A_2) (e^{-t} (D_t - A_2)),
\]

\[
\hat{b} = L_j^{-1} e^t (\partial_\theta A_2 - \partial_t A_1) \quad \text{and} \quad dm = L_j e^{-t} d\theta dt. \quad \text{As in Lemma 5.3, we have}
\]

\[
A - \tilde{A} = d\varphi \quad \text{if} \quad \tilde{A} = (\xi + L_j b_j e^{-t}) d\theta, \quad (\text{for some constant } \xi).
\]

So we can also assume that \( A = \tilde{A} \).

We replace the density \( dm \) by \( d\theta dt \), using the unitary operator

\[
Uf = \sqrt{L_j} e^{-t/2} f,
\]

so

\[
P = -U \Delta_A^{M_j} U^* = L_j^{-2} e^{2t} (D_\theta - A_1)^2 + D_t^2 + \frac{1}{4}.
\]

Then we get also that

\[
\text{sp}(\Delta_A^{M_j}) = \text{sp}(P) = \bigcup_{\ell \in \mathbb{Z}} \text{sp}(P_\ell); \quad P_\ell = D_t^2 + \frac{1}{4} + \left( e^t (\ell + \xi) + b_j \right)^2,
\]

for the Dirichlet condition on \( L^2(I; dt) ; \quad I = [\alpha_j^2, +\infty[ .

When \( \ell + \xi \neq 0 \), the spectrum of \( P_\ell \) is discrete. More precisely

\[
\text{sp}(P_\ell) = \text{sp}(P^\pm), \quad \text{where} \quad P^\pm = D_t^2 + \frac{1}{4} + (\pm e^t + b_j)^2
\]

for the Dirichlet condition on \( L^2(I_{j,\ell}; dt) ; \quad I_{j,\ell} = [\alpha_j^2 + \ln(|\ell + \xi|/L_j), +\infty[ .

and \( \pm = \frac{\ell + \xi}{|\ell + \xi|} \).

So \( \lim_{|\ell| \to \infty} \inf \text{sp}(P_\ell) = +\infty \), and then we get easily that the spectrum of \( -\Delta_A^{M_j} \) is discrete, when \( \xi = [A]_{M_j}/(2\pi) \notin \mathbb{Z} .

If \( \ell + \xi = 0 \), the spectrum of \( P_\ell \) is absolutely continuous:

\[
\text{sp}(P_{-\xi}) = \text{sp}_{\text{ess}}(P_{-\xi}) = \text{sp}_{\text{ac}}(P_{-\xi}) = [\frac{1}{4} + b_j^2, +\infty[ ;
\]

and then, when \( [A]_{M_j} \in 2\pi \mathbb{Z} \), \( \text{sp}_{\text{ess}}(-\Delta_A^{M_j}) = [\frac{1}{4} + b_j^2, +\infty[ .

This achieves the proof of Lemma 5.4.
References

[Bor] David Borthwick : Spectral Theory of Infinite-Area Hyperbolic Surfaces Birkhäuser, Boston, 2007.

[Col] Y. Colin de Verdière : L’asymptotique de Weyl pour les bouteilles magnétiques Comm. Math. Phys., 105, (1986), p. 327-335.

[Go-Mo] S. Golénia, S. Moroianu : Spectral Analysis of Magnetic Laplacians on Conformally Cusp Manifolds Ann. Henri Poincaré, 9, (2008), p. 131-179.

[Hor] Lars Hörmander : The Analysis of Linear P.D.O. IV Springer-verlag, Berlin, 1985.

[Mo-Tr] A. Morame, F. Truc : Magnetic bottles on the Poincaré half-plane: spectral asymptotics J. Math. Kyoto Univ. 48-3, (2008), p 597-616.

[Per] Peter Perry : The Spectral Geometry of Geometrically Finite Hyperbolic Manifolds Proc. Sympos. Pure Math., 76, Part1, AMS, (2007), p. 289-327.

[Shu] M. Shubin : The essential Self-adjointness for Semi-bounded Magnetic Schrödinger operators on Non-compact Manifolds J. Func. Anal., 186, (2001), p. 92-116.