1. Introduction and summary

In this paper we focus on the following problem

(1) find colim-acyclic objects in $\text{Ab}^C$.

Here, $\text{Ab}$ denote the category of abelian groups and $\text{Ab}^C$ denote the (abelian) functor category for the small category $C$. The functor $\text{colim} : \text{Ab}^C \to \text{Ab}$ is the direct limit functor and $F \in \text{Ab}^C$ is colim-acyclic if $\text{colim}_i F = 0$ for $i \geq 1$ (see [16] and [5], and the classical books of Cartan and Eilenberg [2] and of MacLane [13]). It is clear that if $F$ is projective then it is colim-acyclic but, in the same way as not every flat module is projective (see, for example, [16, Section 3.2]), we may be missing colim-acyclic objects if we just consider projective ones.

We shall assume the hypothesis that the category $C$ is a graded partially ordered set (a graded poset for short). These are special posets in which we can assign an integer to each object (called the degree of the object) in such a way that preceding elements are assigned integers which differs in 1. Thus a graded poset can be divided into a set of “layers” (the objects of a fixed degree), and these layers are linearly ordered. Any simplicial complex (viewed as the poset of its simplices with the inclusions among them) and any subdivision category is a graded poset. Also, every $CW$-complex is (strong) homotopy equivalent to a simplicial complex, and thus to a graded poset.

To attack problem (1) we start giving a characterization of the projective objects in $\text{Ab}^C$. Recall that for any small category $C$ (not necessarily a poset), the projective objects in $\text{Ab}^C$ are well known to be, by the Yoneda Lemma, summands of direct sums of representable functors. Moreover, if $C$ is a poset with the descending chain condition (not necessarily graded) then [3, Corollary 3] the projective objects in $\text{Ab}^C$ are also direct sums of representable functors (see also [11, Proposition 7] and [4, Theorem 9] for related results). In case $C$ is a graded poset we characterize the projective functors in $\text{Ab}^C$ as those functors which satisfy two conditions:

**Theorem A** (Theorem 4.9). Let $C$ be a bounded below graded poset and let $F : C \to \text{Ab}$ be a functor. Then $F$ is projective if and only if:

(1) for any object $i_0$ of $C$ $\text{Coker}_F(i_0)$ is a free abelian group.
(2) $F$ is pseudo-projective.

Here, $\text{Coker}_F(i_0)$ is the quotient of $F(i_0)$ by the images of all the non-trivial morphisms arriving to $i_0$. For the actual definition of pseudo-projectiveness see Definition

\textit{Date:} February 5, 2008.
4.5. The boundedness condition in the theorem is related to the descending chain condition in the aforementioned result, and neither of these conditions can be dropped: consider the poset $C = \mathbb{Z}$ of the integers. This graded poset does not satisfy the descending chain condition, and thus neither is it bounded below. The constant functor of value $\mathbb{Z}$ over this poset is projective but it is not a sum of representable functors. The constant functor of value $\mathbb{Z}/n$ (for some $n \geq 1$) satisfies both conditions in Theorem A but it is not projective (see Remark 4.10).

Theorem A is the first step towards finding colim-acyclic objects in $\text{Ab}^C$. The reason is that the second of the conditions in the theorem, i.e., pseudo-projectiveness, implies colim-acyclicity:

**Theorem B** (Theorem 5.2). Let $F : C \to \text{Ab}$ be a pseudo-projective functor over a bounded below graded poset $C$. Then $F$ is colim-acyclic.

This is the main result of this work, and it gives a family of functors in $\text{Ab}^C$ which are colim-acyclic but not necessarily projective. To show that there exist functors in this situation consider the functor

$$
\mathbb{Z} \xrightarrow{x^2} \mathbb{Z} \xrightarrow{x^2} \mathbb{Z}
$$

This is a pseudo-projective functor which, by Theorem B, is acyclic. Moreover, it does not satisfy condition (1) in Theorem A and so it is not projective (see Examples 4.11 and 5.3). On the other hand, pseudo-projective functors do not cover all colim-acyclic functors: the functor

$$
0 \xrightarrow{0} \mathbb{Z} \xrightarrow{1} \mathbb{Z}
$$

is not pseudo-projective but, as a straightforward computation shows, it is acyclic. For vector spaces the notion of pseudo-projectiveness becomes identical to projectiveness as condition (1) in Theorem A is unnecessary in the context of functors to $k - \text{mod}$ (where $k$ is a field). Even in this favorable case the functor

$$
0 \xrightarrow{0} k \xrightarrow{1} k
$$

shows that there are acyclic functors which are not projective.

The main ingredient in the proof of Theorem B is a meticulous use of a spectral sequence built upon the grading of the partially ordered set $C$:

**Proposition C** (Proposition 3.2). For a (decreasing) graded poset $C$ and a functor $F : C \to \text{Ab}$:

- There exists a cohomological type spectral sequence $E_{p,*}^r$ with target $\text{colim}_* F$.
- There exists a homological type spectral sequence $(E^p)_{*,*}^r$ with target the column $E_{1,*}^{p,*}$ for each $p$.

Applications of these results to computation of integral cohomology of posets are given in [7]. The work also contains the dual version of the above, in which we consider injective objects in $\text{Ab}^C$, the right derived functors of the inverse limit functor $\lim : \text{Ab}^C \to \text{Ab}$ and the respective lim-acyclic objects.
The paper is structured as follows: in Section 2 we introduce preliminaries about graded partially ordered sets. In Section 3 we build some spectral sequences arising from the grading of a graded poset. Afterwards, in Section 4, we work out the characterization of projective objects in $\text{Ab}^C$. In Section 5 we prove that pseudo-projectiveness implies colim-acyclicity. We finish with Section 6, where the dual definitions and results for lim-acyclicity are stated without proof.

Acknowledgements: I would like to thank my Ph.D. supervisor Prof. A. Viruel for his support during the development of this work. Also, thanks to Prof. C.A. Weibel for all his fruitful suggestions and comments, in particular for a short proof of Lemma 4.2.

2. Graded posets

In this section we define a special kind of categories: graded partially ordered sets (graded posets for short). We shall think of a poset $\mathcal{P}$ as a category in which there is an arrow $p \rightarrow p'$ if and only if $p \leq p'$. The notion of graded poset is not new and it was already used in [6, pp. 29-33]. The definition there is weaker than the one given here, being the difference that here we ask for every morphism to factor through morphisms of degree 1 (some kind of “saturation” condition).

Definition 2.1. If $\mathcal{P}$ is a poset and $p < p'$ then $p$ precedes $p'$ if $p \leq p'' \leq p'$ implies that $p = p''$ or $p' = p''$.

Definition 2.2. Let $\mathcal{P}$ be a poset. $\mathcal{P}$ is called graded if there is a function $\text{deg} : \text{Ob}(\mathcal{P}) \rightarrow \mathbb{Z}$, called the degree function of $\mathcal{P}$, which is order preserving and that satisfies that if $p$ precedes $p'$ then $\text{deg}(p') = \text{deg}(p) + 1$. If $p$ is an object of $\mathcal{P}$ then $\text{deg}(p)$ is called the degree of $p$.

Notice that the degree function associated to a graded poset is not unique (consider the translations $\text{deg}' = \text{deg} + c_k$ for $k \in \mathbb{Z}$). According to the definition the degree function increases in the direction of the arrows: we say that this degree function is increasing. If the degree function is order reversing and satisfies the alternative condition that $p$ precedes $p'$ implies $\text{deg}(p') = \text{deg}(p) - 1$, i.e., $\text{deg}$ decreases in the direction of the arrows, then we say that $\text{deg}$ is a decreasing degree function. Clearly both definitions are equivalent (by taking $\text{deg}' = -\text{deg}$).

Example 2.3. The “pushout category” $b \leftarrow a \rightarrow c$, the “telescope category” $a \rightarrow b \rightarrow c \rightarrow \ldots$, and the opposite “telescope category” $\ldots \rightarrow c \rightarrow b \rightarrow a$ are graded posets. The integers $\mathbb{Z}$ is a graded poset. The rationals $\mathbb{Q}$ with the usual order is a poset but it is not a graded poset.

If $\mathcal{P}$ is a graded poset and $p < p'$ then it is straightforward that the number $\text{deg}(p') - \text{deg}(p)$ does not depend on the degree function $\text{deg}$. Thus, we can “extend” the degree function $\text{deg}$ to the morphisms set $\text{Hom}(\mathcal{P})$ by $\text{deg}(p \rightarrow p') = |\text{deg}(p') - \text{deg}(p)|$. Whenever $\mathcal{P}$ is a graded poset we denote by $\text{Ob}_n(\mathcal{P})$ the objects of degree $n$ and by $\text{Hom}_n(\mathcal{P})$ the arrows of degree $n$. 
2.1. **Boundedness on graded posets.** Often we will restrict to:

**Definition 2.4.** A graded poset \( P \) with increasing degree function \( \text{deg} \) is *bounded below* (*bounded above*) if the set \( \text{deg}(P) \subset \mathbb{Z} \) has a lower bound (an upper bound).

If the degree function \( \text{deg} \) of \( P \) is decreasing then \( P \) is *bounded below* (*bounded above*) if and only if \( \text{deg}(P) \subset \mathbb{Z} \) has an upper bound (a lower bound). If \( P \) is bounded below and over then \( N \overset{\text{def}}{=} \max(\text{deg}(P)) - \min(\text{deg}(P)) \) exists and it is finite, and it does not depend on the degree function \( \text{deg} \). We call it the dimension of \( P \), and we say that \( P \) is \( N \)-dimensional.

**Example 2.5.** The “pushout category” \( b \leftarrow a \rightarrow c \) is 1-dimensional, the “telescope category” \( a \rightarrow b \rightarrow c \rightarrow .. \) is bounded below but it is not bounded over. The opposite “telescope category” \( .. \rightarrow c \rightarrow b \rightarrow a \) is bounded over but it is not bounded below.

Notice that in a bounded above (below) graded poset there are maximal (minimal) elements, but that the existence of maximal (minimal) objects does not guarantee boundedness. Also it is clear that, in general, neither dcc posets are graded nor graded posets are dcc.

### 3. A spectral sequence

In this section we shall construct spectral sequences with targets \( \text{colim}_i F \) and \( \text{lim}^i F \) for \( F : C \to \text{Ab} \) with \( C \) a graded poset. Some conditions for (weak) convergence shall be given. We build the spectral sequences starting from filtered differential modules (see [15], where the notion of weak convergence we use is also given).

Recall that (see [8, Appendix II.3], [1, XII.5.5], [1, XI.6.2] or [9, p.409ff.]) there is, for any small category \( C \) and covariant functor \( F : C \to \text{Ab} \), a concrete chain (cochain) complex \( C_*(C, F) \) (\( C^*(C, F) \)) whose homology groups (cohomology groups) are precisely the left derived functors \( \text{colim}_i \) (right derived functors \( \text{lim}^i \)).

Let \( NC \) denote the nerve of the small category \( C \) whose \( n \)-simplices are chain of composable morphisms in \( C \): \( \sigma = \sigma_0 \xrightarrow{\alpha_1} \sigma_1 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{n-1}} \sigma_{n-1} \xrightarrow{\alpha_n} \sigma_n \). Then

\[
C_n(C, F) = \bigoplus_{\sigma \in NC_n} F_{\sigma},
\]

where \( F_{\sigma} = F(\sigma_0) \). Moreover, \( C_*(C, F) \) is a simplicial abelian group with face and degeneracy maps induced by those of the nerve \( NC \). The chain complex \( (C_*(C, F), d) \) with differential of degree \(-1\) \( d = \sum_{i=0}^{n} (-1)^i d_i \) satisfies

\[
\text{colim}_i F = H_i(C_*(C, F), d).
\]

For the inverse limit \( \text{lim} : \text{Ab}^C \to \text{Ab} \) there is a cosimplicial abelian group with simplices

\[
C^n(C, F) = \prod_{\sigma \in NC_n} F^\sigma,
\]
where $F^\sigma = F(\sigma_n)$. This cosimplicial object gives rise to a cochain complex $(C^*(\mathcal{C}, F), d)$ with differential of degree 1 $d = \sum_{i=0}^{n+1}(-1)^i d^i$. It is well known that
\begin{equation}
\lim^i F = H^i(C^*(\mathcal{C}, F), d).
\end{equation}

**Remark 3.1.** We can apply the Dold-Kan correspondence (see [16, 8.4]) to the simplicial and cosimplicial abelian groups constructed above. This means that we shall use the normalized chain (cochain) complex to compute the homology (cohomology) in Equation (2) (Equation (3)).

There is a decreasing filtration of the chain complex $(C^*(\mathcal{C}, F), d)$ given by
\[ L^p C_n(\mathcal{C}, F) = \bigoplus_{\sigma \in NC_n, \deg(\sigma) \geq p} F_\sigma. \]

It is straightforward that the triple $(C^*(\mathcal{C}, F), d, L^*)$ is a filtered differential graded $\mathbb{Z}$-module, so it yields a spectral sequence $(E^{'p, q}_r, d_r)$ of cohomological type whose differential $d_r$ has bidegree $(r, 1-r)$. The $E^{p, q}_1$ page is given by
\[ E^{p, q}_1 \cong H^{p+q}(L^p C/L^{p-1} C). \]

The differential graded $\mathbb{Z}$-module $L^p C/L^{p-1} C$ is in fact a simplicial abelian group because the face operators $d_i$ and the degeneracy operators $s_i$ respect the filtration $L^*$. The $n$-simplices are
\[ (L^p C/L^{p-1} C)_n = \bigoplus_{\sigma \in NC_n, \deg(\sigma) = p} F_\sigma. \]
Moreover, for each $p$, $L^p C/L^{p-1} C$ can be filtered again by the condition $\deg(\sigma_0) \leq p'$ to obtain a homological type spectral sequence. Then arguing as above we obtain:

**Proposition 3.2.** For a (decreasing) graded poset $\mathcal{C}$ and a functor $F : \mathcal{C} \to \text{Ab}$:
- There exists a cohomological type spectral sequence $E^{'p, q}_* \cong \text{colim}_* F$.
- There exists a homological type spectral sequence $(E^p)^{p', q}_* \cong \text{colim}_* F$ for each $p$.

Notice that the column $E^{p, q}_1$ is given by the cohomology of the simplicial abelian group formed by the simplices that end on objects of degree $p$, the column $(E^p)^{p', q}_1$ is given by the homology of the simplicial abelian group formed by the simplices that end on degree $p$ and begin on degree $p'$, and all the differentials in the spectral sequences above are induced by the completely described differential of $(C^*(\mathcal{C}, F), d)$.

As $\bigcup_p L^p C_n = C_n$ and $\bigcap_p L^p C_n = 0$ for each $n$ the spectral sequence $E^{'p, q}_*$ converges weakly to its target. In case the map $\deg$ has a bounded image, i.e., when $\mathcal{C}$ is $N$ dimensional, the filtration $L^*$ is bounded below and over, and so $E^{'p, q}_*$ collapses after a finite number of pages. The same assertions on weak converge and boundedness hold for the spectral sequences $(E^p)^{p', q}_*$. If we proceed in reverse order, i.e., filtrating first by the degree of the beginning object and later by the degree of the ending object, we obtain:

**Proposition 3.3.** For a (decreasing) graded poset $\mathcal{C}$ and a functor $F : \mathcal{C} \to \text{Ab}$:
Table 1. Filtrations and spectral sequences obtained

| Complex | Degree | First filtration | Second filtration | First ss. | Second ss. |
|---------|--------|------------------|-------------------|-----------|------------|
| $C_*(C, F)$ | decreasing | $deg(\sigma_n) \geq deg(\sigma_0)$ | $deg(\sigma_n) \geq deg(\sigma_0)$ | cohom. type | homol. type |
| $C_*(C, F)$ | decreasing | $deg(\sigma_0) \leq deg(\sigma_n)$ | $deg(\sigma_0) \leq deg(\sigma_n)$ | homol. type | cohom. type |
| $C_*(C, F)$ | increasing | $deg(\sigma_n) \leq deg(\sigma_0)$ | $deg(\sigma_n) \leq deg(\sigma_0)$ | homol. type | cohom. type |
| $C_*(C, F)$ | increasing | $deg(\sigma_0) \geq deg(\sigma_n)$ | $deg(\sigma_0) \geq deg(\sigma_n)$ | cohom. type | homol. type |
| $C^*(C, F)$ | decreasing | $deg(\sigma_0) \leq deg(\sigma_n)$ | $deg(\sigma_0) \leq deg(\sigma_n)$ | homol. type | cohom. type |
| $C^*(C, F)$ | decreasing | $deg(\sigma_n) \leq deg(\sigma_0)$ | $deg(\sigma_n) \leq deg(\sigma_0)$ | homol. type | cohom. type |
| $C^*(C, F)$ | increasing | $deg(\sigma_0) \geq deg(\sigma_n)$ | $deg(\sigma_0) \geq deg(\sigma_n)$ | cohom. type | homol. type |
| $C^*(C, F)$ | increasing | $deg(\sigma_n) \geq deg(\sigma_0)$ | $deg(\sigma_n) \geq deg(\sigma_0)$ | homol. type | cohom. type |

There exists a homological type spectral sequence $E^{*,*}$ with target colim$_*$ $F$.

There exists a cohomological type spectral sequence $(E^p)^{*,*}$ with target the column $E^{1}_{p,*}$ for each $p$.

If the degree function we take is increasing then the appropriate conditions for the filtrations are $deg(\sigma_n) \leq p$ and $deg(\sigma_0) \geq p'$, and the spectral sequences obtained in Propositions 3.2 and 3.3 are of homological (cohomological) type instead of cohomological (homological) type.

For the case of the cochain complex $(C^*(C, F), d)$ the choices for the filtrations are $deg(\sigma_n) \leq p$ and $deg(\sigma_0) \geq p'$ for a decreasing degree function and $deg(\sigma_n) \geq p$ and $deg(\sigma_0) \leq p'$ for an increasing one. Analogously we obtain spectral sequences with target $\lim^1 F$ which columns in the first page are computed by another spectral sequence.

Table 1 shows a summary of the types of the spectral sequences for all the cases. The statements on weak convergence and boundedness apply to any of the spectral sequences of the table.

Remark 3.4. It is straightforward that normalizing (see Remark 3.1) the simplicial (cosimplicial) abelian groups that computes the page 1 of the spectral sequences above has the same effect as considering the spectral sequences of the normalizations of $C_*(C, F)$ $(C^*(C, F))$.

4. Projective objects in $\text{Ab}^P$.

Consider the abelian category $\text{Ab}^P$ for some graded poset $\mathcal{P}$. In this section we shall determine the projective objects in $\text{Ab}^P$. Recall that in Ab the projective objects are the free abelian groups. Along the rest of the section $\mathcal{P}$ denotes a graded poset.

Suppose $F \in \text{Ab}^P$ is projective. How does $F$ look? Consider an object $i_0$ of $\mathcal{P}$. We show that the quotient of $F(i_0)$ by the images of the non-identity morphisms arriving to $i_0$ is free abelian. To prove it, write

$$\text{Im}_F(i_0) = \sum_{i \neq i_0, \alpha \neq 1} \text{Im}(\alpha) \quad \text{(or } \text{Im}_F(i_0) = 0 \text{ if the index set of the sum is empty)}$$

and

$$\text{Coker}_F(i_0) = F(i_0)/\text{Im}_F(i_0).$$

**Definition 4.1.**
It is straightforward that for a fixed object $i_0$ of $\mathcal{P}$ there is a functor
\[ \text{Coker}(i_0) : \text{Ab}^\mathcal{P} \to \text{Ab} \]
which maps $F$ to $\text{Coker}_F(i_0)$. This functor is left adjoint to the skyscraper functor $\text{Ab} \to \text{Ab}^\mathcal{P}$ which maps the abelian group $A_0$ to the functor $A : \mathcal{P} \to \text{Ab}$ with values
\[ A(i) = \begin{cases} A_0 & \text{for } i = i_0 \\ 0 & \text{for } i \neq i_0 \end{cases} \]
on objects, and values
\[ A(\alpha) = \begin{cases} 1_{A_0} & \text{for } \alpha = 1_{i_0} \\ 0 & \text{for } \alpha \neq 1_{i_0} \end{cases} \]
on morphisms. As the skyscraper functor is exact we obtain by [16, Proposition 2.3.10] that $\text{Coker}(i_0)$ preserves projective objects:

**Lemma 4.2.** Let $F : \mathcal{P} \to \text{Ab}$ be a projective functor over a graded poset $\mathcal{P}$. Then $\text{Coker}_F(i_0)$ is free abelian for every object $i_0$ of $\mathcal{P}$.

This means that we can write
\[ F(i_0) = \text{Im}_F(i_0) \oplus \text{Coker}_F(i_0) \]
with $\text{Coker}_F(i_0)$ free abelian for every object $i_0$ of $\mathcal{P}$, and also that

**Example 4.3.** For the category $\mathcal{P}$ with shape
\[ \cdot \rightarrow \cdot \]
the functor $F : \mathcal{P} \to \text{Ab}$ with values
\[ \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{Z} \]
is not projective as $\text{Coker}_F$ on the right object equals the non-free abelian group $\mathbb{Z}/n$.

Now that we know a little about the values that a projective functor $F : \mathcal{P} \to \text{Ab}$ takes on objects we can wonder about the values $F(\alpha)$ for $\alpha \in \text{Hom}(\mathcal{P})$. Do they have any special property? Recall that a feature of graded posets is that there is at most one arrow between any two objects, and also that

**Remark 4.4.** If $\mathcal{P}$ is graded then for any object $i_0$ of $\mathcal{P}$
\[ \text{Im}_F(i_0) = \sum_{i \rightarrow i_0, \deg(\alpha) = 1} \text{Im} F(\alpha) \]
because every morphism factors as composition of morphisms of degree 1.

We prove that the following property holds for $F$:

**Definition 4.5.** Let $F : \mathcal{P} \to \text{Ab}$ be a functor over a graded poset $\mathcal{P}$ with degree function $\deg$. Given $d \geq 0$ we say that $F$ is $d$-pseudo-projective if for any object
$i_0$ of $\mathcal{P}$ and $k$ different objects $i_j$ in $\mathcal{P}$, arrows $\alpha_j : i_j \rightarrow i_0$ with $deg(\alpha_j) = d$, and $x_j \in F(i_j)$ $j = 1, \ldots, k$ such that

$$\sum_{j=1, \ldots, k} F(\alpha_j)(x_j) = 0$$

we have that $x_j \in \text{Im}_F(i_j)$ $j = 1, \ldots, k$. If $F$ is $d$-pseudo-projective for each $d \geq 0$ we call $F$ pseudo-projective.

**Remark 4.6.** In case $k = 1$ and $\text{Im}_F(i_1) = 0$ the condition states that $F(\alpha_1)$ is a monomorphism. Notice that any functor is 0-pseudo-projective as the identity is a monomorphism.

Before proving that projective functors $F$ over a graded poset verify this property we define two functors $\text{Coker}_F$ and $\text{Coker}'_F$ and natural transformations $\sigma$ and $\pi$ that fit in the diagram

$$\begin{array}{ccc}
F & \downarrow \sigma \\
\text{Coker}'_F & \equiv & \text{Coker}_F \\
\downarrow & & \downarrow \\
0 & & \text{Coker}_F(i_0) \\
\end{array}$$

for any functor $F : \mathcal{P} \rightarrow \text{Ab}$ with $\mathcal{P}$ a graded poset. We begin defining $\text{Coker}_F$. Because for every $\alpha : i_1 \rightarrow i_0$ holds that $F(\alpha)(\text{Im}_F(i_1)) \leq \text{Im}_F(i_0)$ we can factor $F(\alpha)$ as in the diagram

$$\begin{array}{ccc}
F(i_1) & \overset{F(\alpha)}{\rightarrow} & F(i_0) \\
\downarrow & & \downarrow \\
\text{Coker}_F(i_1) & \overset{F(\alpha)}{\rightarrow} & \text{Coker}_F(i_0). \\
\end{array}$$

In fact, if $\alpha \neq 1_{i_1}$, then $F(\alpha) \equiv 0$ by definition. Because the identity $1_{i_0}$ cannot be factorized (by non-identity morphisms) in a graded poset then we have a functor $\text{Coker}_F$ with value $\text{Coker}_F(i)$ on the object $i$ of $\mathcal{P}$ and which maps the non-identity morphisms to zero. $\text{Coker}_F$ is a kind of “discrete” functor. Also it is clear that there exists a natural transformation $\sigma : F \Rightarrow \text{Coker}_F$ with $\sigma(i)$ the projection $F(i) \rightarrow \text{Coker}_F(i)$.

Now we define $\text{Coker}'_F$ from $\text{Coker}_F$ in a similar way as free diagrams are constructed. Let $\text{Coker}'_F$ be defined on objects by

$$\text{Coker}'_F(i_0) = \bigoplus_{\alpha : i \rightarrow i_0} \text{Coker}_F(i).$$

For $\beta \in \text{Hom}(\mathcal{P})$, $\beta : i_1 \rightarrow i_0$, $\text{Coker}'_F(\beta)$ is the only homomorphism which makes commute the diagram

$$\begin{array}{ccc}
\text{Coker}'_F(i_1) & \overset{\text{Coker}'_F(\beta)}{\rightarrow} & \text{Coker}'_F(i_0) \\
\downarrow & & \downarrow \\
\text{Coker}_F(i) & \overset{1}{\rightarrow} & \text{Coker}_F(i) \\
\end{array}$$
for each $\alpha : i \to i_1$. In the bottom row of the diagram, the direct summands $\text{Coker}_F(i)$ of $\text{Coker}'_F(i_1)$ and $\text{Coker}'_F(i_0)$ correspond to $\alpha : i \to i_1$ and to the composition $i \xrightarrow{\alpha} i_1 \xrightarrow{\beta} i_0$ respectively.

Then there exists a candidate to natural transformation $\pi : \text{Coker}'_F \Rightarrow \text{Coker}_F$ which value $\pi(i)$ is the projection $\pi(i) : \text{Coker}'_F(i) \to \text{Coker}_F(i)$ onto the direct summand corresponding to $\alpha : i \to i_1$. Thus, $\pi$ is a natural transformation if for every $\beta : i_1 \to i_0$ with $i_1 \neq i_0$ the following diagram is commutative

\[
\begin{array}{c}
\text{Coker}'_F(i_1) \\ \downarrow \pi(i_1) \\
\text{Coker}_F(i_1) \\
\end{array}
\begin{array}{c}
\text{Coker}'_F(i_0) \\ \downarrow \pi(i_0) \\
\text{Coker}_F(i_0) \\
\end{array}
\]

It is clear that this square commutes if the identity $1_{i_0}$ cannot be factorized (by non-identity morphisms), and this holds in a graded poset.

Now we have the commutative triangle

\[
\begin{array}{c}
\text{Coker}'_F \\ \downarrow \pi \\
\text{Coker}_F \\
\end{array}
\begin{array}{c}
\text{F} \\ \downarrow \sigma \\
0 \\
\end{array}
\]

where the natural transformation $\rho$ exists because $F$ is projective. To prove that $F$ is $d$-pseudo-projective for some $d \geq 0$ take an object $i_0$ of $\mathcal{P}$, $k$ objects $i_1, \ldots, i_k$, arrows $\alpha_j : i_j \to i_0$ with $\deg(\alpha_j) = d$ and elements $x_j \in F(i_j)$ for $j = 1, \ldots, k$ such that

\[
\sum_{j=1, \ldots, k} F(\alpha_j)(x_j) = 0.
\]

To visualize what is going on consider the diagram above near $i_0$ for $k = 2$

where $\pi$ is not drawn completely for clarity. Recall that we are supposing that $\{x_1, \ldots, x_k\}$ is such that $\sum_{j=1, \ldots, k} F(\alpha_j)(x_j) = 0$. Then

\[
0 = \rho(i_0)(0) = \sum_{j=1, \ldots, k} \rho(i_0)(F(\alpha_j)(x_j)) = \sum_{j=1, \ldots, k} \text{Coker}'_F(\alpha_j)(\rho(i_j)(x_j)).
\]
Now consider the projection $p_{j_0}$ for $j_0 \in \{1, \ldots, k \}$ from $\text{Coker}_{F}(i_{j_0})$ onto the direct summand $\text{Coker}_{F}(i_{j_0}) \hookrightarrow \text{Coker}_{F}(i_0)$ which corresponds to $\alpha_{j_0} : i_{j_0} \rightarrow i_0$

$$\text{Coker}_{F}(i_0) \twoheadrightarrow \text{Coker}_{F}(i_{j_0}).$$

Then

$$0 = p_{j_0}(0) = p_{j_0}(\rho(i_0)(0)) = \sum_{j=1, \ldots, k} p_{j_0}(\text{Coker}_{F}(\alpha_j)(\rho(i_j)(x_j))).$$

For any $y = \bigoplus_{\alpha : i \rightarrow i_j} y_{\alpha} \in \text{Coker}_{F}(y_j)$

$$p_{j_0}(\text{Coker}_{F}(\alpha_j)(y)) = \sum_{\alpha : i \rightarrow i_j, \alpha \circ \alpha = \alpha_{j_0}} y_{\alpha}.$$  

So if $y_j = \rho(i_j)(x_j) = \bigoplus_{\alpha : i \rightarrow i_j} y_{j, \alpha} \in \text{Coker}_{F}(i_j)$ then

$$p_{j_0}(\text{Coker}_{F}(\alpha_j)(\rho(i_j)(x_j))) = \sum_{\alpha : i \rightarrow i_j, \alpha \circ \alpha = \alpha_{j_0}} y_{j, \alpha}.$$  

This last sum runs over $\alpha : i_{j_0} \rightarrow i_j$ such that the following triangle commutes

$$\begin{array}{ccc}
 i_{j_0} & \stackrel{\alpha_{j_0}}{\longrightarrow} & i_0 \\
 \downarrow{\alpha} & & \downarrow{\alpha_j} \\
 i_j & \end{array}$$

Because we are in a graded poset and $\text{deg}(i_j) = d$ for each $j = 1, \ldots, k$ then the only chance is $i_j = i_{j_0}$ and $\alpha = 1_{i_{j_0}}$. Because the objects $i_1, \ldots, i_k$ are different this implies that $j = j_0$ too. Thus

$$p_{j_0}(\text{Coker}_{F}(\alpha_j)(\rho(i_j)(x_j))) = \begin{cases} y_{j_0,1_{i_{j_0}}} & \text{for } j = j_0 \\ 0 & \text{for } j \neq j_0 \end{cases}$$

and Equation (4) becomes

$$0 = p_{j_0}(0) = y_{j_0,1_{i_{j_0}}}.$$ 

Notice now that $y_{j_0,1_{i_{j_0}}}$ is the evaluation of $\pi(i_{j_0})$ on $y_{j_0} = \rho(i_{j_0})(x_{j_0})$ and then

$$0 = y_{j_0,1_{i_{j_0}}} = \pi(i_{j_0})(\rho(i_{j_0})(x_{j_0})) = \sigma_{i_{j_0}}(x_{j_0}).$$

This last equation means that $x_{j_0}$ goes to zero by the projection $F(i_{j_0}) \twoheadrightarrow \text{Coker}_{F}(i_{j_0}) = F(i_{j_0})/\text{Im}_{F}(i_{j_0})$, and then

$$x_{i_{j_0}} \in \text{Im}_{F}(i_{j_0}).$$

As $j_0$ was arbitrary this completes the proof of

**Lemma 4.7.** Let $F : \mathcal{P} \rightarrow \text{Ab}$ be a projective functor over a graded poset $\mathcal{P}$. Then $F$ is pseudo-projective.
Example 4.8. For the category $\mathcal{P}$ with shape
\[ \cdot \to \cdot \]
the functor $F : \mathcal{P} \to \text{Ab}$ with values
\[ \mathbb{Z} \xrightarrow{\text{red}_n} \mathbb{Z}/n \]
is not projective as $\text{red}_n$ is not injective, in spite of the $\text{Coker}_F$’s are $\mathbb{Z}$ and 0, which are free abelian.

Till now we have obtained (Lemmas 4.2 and 4.7) that projective functors $\mathcal{P} \to \text{Ab}$ are pseudo-projective and have $\text{Coker}_F(i_0)$ projective for any object $i_0$. In fact, as the next theorem shows, the restriction we did to graded posets is worthwhile:

**Theorem 4.9.** Let $F : \mathcal{P} \to \text{Ab}$ be a functor over a bounded below graded poset $\mathcal{P}$. Then $F$ is projective if and only if

1. for any object $i_0$ of $\mathcal{P}$ $\text{Coker}_F(i_0)$ is a free abelian group.
2. $F$ is pseudo-projective.

**Proof.** It remains to prove that a functor $F$ satisfying the conditions in the statement is projective. We can assume that the degree function $\text{deg}$ on $\mathcal{P}$ is increasing and takes values $\{0, 1, 2, 3, \ldots\}$, and that $\text{Ob}_0(\mathcal{P}) \neq \emptyset$.

To see that $F$ is projective in $\text{Ab}^{\mathcal{P}}$, given a diagram of functors with exact row as shown, we must find a natural transformation $\rho : F \Rightarrow A$ making the diagram commutative:

\[ \begin{array}{c}
F \\
\downarrow \rho \\
A \xrightarrow{\pi} B \xrightarrow{\pi} 0.
\end{array} \]

We define $\rho$ inductively, beginning on objects of degree 0 and successively on object of degrees 1, 2, 3, ...

So take $i_0 \in \text{Ob}_0(\mathcal{P})$ of degree 0, and restrict to the diagram in $\text{Ab}$ over $i_0$. By hypothesis (1) in the statement, as $\text{Im}_F(i_0) = 0$, $F(i_0) = \text{Coker}_F(i_0)$ is free abelian. So we can close the following triangle with a homomorphism $\rho(i_0)$

\[ \begin{array}{c}
F(i_0) \\
\downarrow \rho(i_0) \\
A(i_0) \xrightarrow{\pi(i_0)} B(i_0) \xrightarrow{\pi(i_0)} 0.
\end{array} \]

As there are no arrows between degree 0 objects we do not worry about $\rho$ being a natural transformation. Now suppose that we have defined $\rho$ on all objects of $\mathcal{P}$ of degree less than $n$ ($n \geq 1$), and that the restriction of $\rho$ to the full subcategory generated by these objects is a natural transformation and verifies $\pi \circ \rho = \sigma$.

The next step is to define $\rho$ on degree $n$ objects. So take $i_0 \in \text{Ob}_n(\mathcal{P})$ and consider the splitting

\[ F(i_0) = \text{Im}_F(i_0) \oplus \text{Coker}_F(i_0) \]
where
\[ \text{Im}_F(i_0) = \sum_{i \to i_0, \deg(\alpha) = 1} \text{Im} F(\alpha). \]

To define \( \rho(i_0) \) such that it makes commutative the diagram

\[
\begin{array}{ccc}
\text{Im}_F(i_0) \oplus \text{Coker}_F(i_0) & \xrightarrow{\rho(i_0)} & B(i_0) \\
A(i_0) & \xrightarrow{\pi(i_0)} & C(i_0) \end{array}
\]

we define it on \( \text{Im}_F(i_0) \) and \( \text{Coker}_F(i_0) \) separately. For \( \text{Coker}_F(i_0) \), as it is a free abelian group, we define it by any homomorphism that makes commutative the diagram above when restricted to \( \text{Coker}_F(i_0) \). For \( \text{Im}_F(i_0) \) take

\[ x = \sum_{j = 1, \ldots, k} \alpha_j(x_j) \]

where \( \{i_1, \ldots, i_k\} \) are \( k \) different objects, \( \alpha_j : i_j \to i_0, \deg(\alpha_j) = 1 \) and \( x_j \in F(i_j) \) for \( j = 1, \ldots, k \) (see Remark 4.4). Then define

\[ \rho(i_0)(x) = \sum_{j = 1, \ldots, k} (A(\alpha_j) \circ \rho(i_j))(x_j). \]

To check that \( \rho(i_0)(x) \) does not depend on the choice of the \( i_j \)'s, \( \alpha_j \)'s and \( x_j \)'s we have to prove that

\[ \sum_{j = 1, \ldots, k} F(\alpha_j)(x_j) = 0 \Rightarrow \sum_{j = 1, \ldots, k} (A(\alpha_j) \circ \rho(i_j))(x_j) = 0. \]

So suppose that

(5) \[ \sum_{j = 1, \ldots, k} F(\alpha_j)(x_j) = 0. \]

Then using that \( F \) is 1-pseudo-projective and Remark 4.4 we obtain objects \( i_{j,j'} \), arrows \( \alpha_{j,j'} \) of degree 1, and elements \( x_{j,j'} \) for \( j = 1, \ldots, k, j' = 1, \ldots, k_j \) such that

(6) \[ \sum_{j = 1, \ldots, k} F(\alpha_{j,j'})(x_{j,j'}) = x_j \]

for every \( j \in \{1, \ldots, k\} \). Notice that possibly not all the objects \( i_{j,j'} \) are different. Replacing Equation (6) in Equation (5) we obtain

(7) \[ \sum_{j = 1, \ldots, k} F(\alpha_j \circ \alpha_{j,j'})(x_{j,j'}) = 0. \]

Because in a graded poset there is at most one arrow between two objects, the condition \( i_{j,j'} = i_{j',j''} = i \) implies \( \alpha_j \circ \alpha_{j,j'} = \alpha_{j'} \circ \alpha_{j',j''} : i \to i_0 \). So, considering objects \( i \) in \( \mathcal{P} \), we can rewrite (7) as

(8) \[ \sum_{i \in \text{Ob}(\mathcal{P})} F(\alpha_j \circ \alpha_{j,j'})(\sum_{j,j' \mid i_{j,j'} = i} x_{j,j'}) = 0. \]
Call \( \{ i'_1, \ldots, i'_m \} = \{ i_{i,j'} \mid j = 1, \ldots, k, j' = 1, \ldots, k_j \} \) where these sets have \( m \) elements. Call \( \beta_l = \alpha_j \circ \alpha_{j,j'} \) if \( i'_l = i_{j,j'} \) and \( y_l = \sum_{j,j' \mid i_{j,j'} = i'_l} x_{j,j'} \) for \( l = 1, \ldots, m \). Notice that \( \deg(\beta_l) = 2 \) for each \( l \). Then Equation (8) becomes

\[
\sum_{l=1,\ldots,m} F(\beta_l)(y_l) = 0. \tag{9}
\]

Now we repeat the same argument: applying that \( F \) is 2-pseudo-projective and the Remark 4.4 to Equation (9) we obtain objects \( i''_{l,l'} \), arrows \( \beta_{l,l'} \) of degree 1, and elements \( y_{l,l'} \) for \( l = 1, \ldots, m, l' = 1, \ldots, k'_l \) such that

\[
\sum_{l=1,\ldots,m} F(\beta_{l,l'})(y_{l,l'}) = y_l \tag{10}
\]

for every \( l \in \{ 1, \ldots, m \} \). Substituting (10) in (9)

\[
\sum_{l=1,\ldots,m, l' = 1,\ldots,k'_l} F(\beta_l \circ \beta_{l,l'})(y_{l,l'}) = 0.
\]

Now proceed as before regrouping the terms in this last equation.

In a finite number of steps, after a regrouping of terms as above, we find objects \( i''''_s \), arrows \( \gamma_s \), and elements \( z_s \) of degree 0 for \( s = 1, \ldots, r \) which verify an equation

\[
\sum_{s=1,\ldots,r} F(\gamma_s)(z_s) = 0. \tag{11}
\]

Then pseudo-injectivity gives that \( z_s \in \text{Im}_F(i''''_s) \) for each \( s \). As \( \deg(i''''_s) = 0 \) then \( \text{Im}_F(i''''_s) = 0 \) and so \( z_s = 0 \) (notice that \( z_s = 0 \) for \( s = 1, \ldots, r \) does not imply \( x_j = 0 \) for any \( j \)).

Recall that we want to prove that

\[
\sum_{j=1,\ldots,k} (A(\alpha_j) \circ \rho(i_j))(x_j) = 0. \tag{12}
\]

Substituting (6) in \( \sum_{j=1,\ldots,k} (A(\alpha_j) \circ \rho(i_j))(x_j) \) we obtain

\[
\sum_{j=1,\ldots,k} (A(\alpha_j) \circ \rho(i_j))(x_j) = \sum_{j=1,\ldots,k} \sum_{j'=1,\ldots,k_j} (A(\alpha_j) \circ \rho(i_j) \circ F(\alpha_{j,j'}))(x_{j,j'})
\]

\[
= \sum_{j=1,\ldots,k} \sum_{j'=1,\ldots,k_j} (A(\alpha_j) \circ \alpha_{j,j'} \circ \rho(i_{j,j'}))(x_{j,j'})
\]

\[
= \sum_{j=1,\ldots,k} \sum_{j'=1,\ldots,k_j} (A(\alpha_j \circ \alpha_{j,j'} \circ \rho(i_{j,j'}))(x_{j,j'}),
\]

as \( \rho \) is natural up to degree less than \( n \). Then regrouping terms

\[
\sum_{j=1,\ldots,k} \sum_{j'=1,\ldots,k_j} (A(\alpha_j \circ \alpha_{j,j'} \circ \rho(i_{j,j'}))(x_{j,j'})) = \sum_{i \in \text{Ob}(\mathcal{P})} (A(\alpha_j \circ \alpha_{j,j'} \circ \rho(i_{j,j'}))(x_{j,j'}))
\]

\[
= \sum_{l=1,\ldots,m} (A(\beta_l \circ \rho(i'_l))(y_l)).
\]
Then, after a finite number of steps, we obtain
\[
\sum_{j=1,\ldots,k} (A(\alpha_j) \circ \rho(i_j))(x_j) = \sum_{s=1,\ldots,r} (A(\gamma_s) \circ \rho(i''_s))(z_s) = 0
\]
as \(z_s = 0\) for each \(z = 1,\ldots,r\).

So we have checked that \(\rho(i_0)(x)\) does not depend on the choice of \(i_j, \alpha_j\) and \(x_j\). It is straightforward that \(\rho(i_0)\) on \(\text{Im}_F(i_0)\) defined in this way is a homomorphism of abelian groups.

It remains to prove that \(\pi(i_0)\rho(i_0) = \sigma(i_0)\) when restricted to \(\text{Im}_F(i_0)\). So take 
\[
x = \sum_{j=1,\ldots,k} F(\alpha_j)(x_j) \quad \text{in} \quad \text{Im}_F(i_0).
\]
Then
\[
\pi(i_0)(\rho(i_0)(x)) = \sum_{j=1,\ldots,k} (\pi(i_0) \circ A(\alpha_j) \circ \rho(i_j))(x_j)
\]
\[
= \sum_{j=1,\ldots,k} (B(\alpha_j) \circ \pi(i_j) \circ \rho(i_j))(x_j), \quad \pi \text{ is a natural transformation}
\]
\[
= \sum_{j=1,\ldots,k} (B(\alpha_j) \circ \sigma(i_j))(x_j), \text{ by the inductive hypothesis}
\]
\[
= \sum_{j=1,\ldots,k} (\sigma(i_0) \circ F(\alpha_j))(x_j), \quad \sigma \text{ is a natural transformation}
\]
\[
= \sigma(i_0)(x)
\]

Defining \(\rho(i_0)\) in this way for every \(i_0 \in \text{Ob}_n(P)\) we have now \(\rho\) defined on all objects of \(P\) of degree less or equal than \(n\). Finally, to complete the inductive step we have to prove that \(\rho\) restricted to the full subcategory over these objects is a natural transformation. Take \(\alpha : i \to i_0\) in this full subcategory. If the degree of \(i_0\) is less than \(n\) then the commutativity of

\[
\begin{array}{ccc}
F(i) & \xrightarrow{F(\alpha)} & F(i_0) \\
\downarrow{\rho(i)} & & \downarrow{\rho(i_0)} \\
A(i) & \xrightarrow{A(\alpha)} & A(i_0)
\end{array}
\]

is granted by the inductive hypothesis. Suppose that the degree of \(i_0\) is \(n\). Take \(x' \in F(i)\). Because \(P\) is graded there exists \(\alpha_1 : i_1 \to i_0\) of degree 1 and \(\alpha' : i \to i_1\) such that \(\alpha = \alpha_1 \circ \alpha'\):

\[
\begin{array}{ccc}
& i & \\
\alpha' & \alpha_1 & i_0. \\
& \alpha & \downarrow{\alpha_1} & \downarrow{\alpha'} \\
& i_1 & \end{array}
\]
Write $x = F(\alpha)(x') = F(\alpha_1)(x_1)$ where $x_1 = F(\alpha')(x')$. Then, by definition of $\rho(i_0)$ on $\text{Im}_F(i_0)$,

$$\rho(i_0)(x) = (A(\alpha_1) \circ \rho(i_1))(x_1)$$

$$= (A(\alpha_1) \circ \rho(i_1))(F(\alpha')(x'))$$

$$= (A(\alpha_1) \circ \rho(i_1) \circ F(\alpha'))(x')$$

$$= (A(\alpha_1 \circ \alpha') \circ \rho(i))(x'), \ \rho \text{ is natural up to degree less than } n$$

$$= (A(\alpha) \circ \rho(i))(x')$$

and so the diagram commutes. \hfill \Box

**Remark 4.10.** As the following example shows the condition of lower boundedness of $P$ in Theorem 4.9 cannot be dropped:

Consider the inverse ‘telescope category’ $\mathcal{P}$ with shape

$$\cdots \to \cdot \to \cdot \to \cdot$$

It is a graded poset which is not bounded below. Consider the functor of constant value $\mathbb{Z}/p$, $c_{\mathbb{Z}/p} : \mathcal{P} \to \text{Ab}$:

$$\cdots \to \mathbb{Z}/p \to \mathbb{Z}/p \to \mathbb{Z}/p$$

It is straightforward that it satisfies the conditions in the theorem as all the cokernels are zero and all the arrows are injective. But it is not a projective object of $\text{Ab}^\mathcal{P}$ because, in that case, the adjoint pair $\text{colim} : \text{Ab}^\mathcal{P} \leftrightarrow \text{Ab} : \Delta$ would give that $\mathbb{Z}/p$ is projective in $\text{Ab}$ (see [5, 3.2, Ex7] or [16, Proposition 2.3.10]).

This theorem yields the following examples. The degree functions $\text{deg}$ for the bounded below graded posets appearing in the examples are indicated by subscripts $i_{\text{deg}(i)}$ on the objects $i$ of $\mathcal{P}$ and take values $\{0, 1, 2, 3, \ldots\}$.

**Example 4.11.** For the ‘pushout category’ $\mathcal{P}$ with shape

$$\begin{array}{c}
 a_0 \\
 f \\
 \downarrow g \\
 c_1
\end{array}$$

a functor $F : \mathcal{P} \to \text{Ab}$ is projective if and only if

- $F(a), F(b)/\text{Im } F(f)$ and $F(c)/\text{Im } F(g)$ are free abelian.
- $F(f)$ and $F(g)$ are monomorphisms.

For the ‘telescope category’ $\mathcal{P}$ with shape

$$\begin{array}{c}
 a_0 \\
 f_1 \\
 a_1 \\
 f_2 \\
 a_2 \\
 f_3 \\
 a_3 \\
 f_4 \\
 a_4 \\
 \vdots
\end{array}$$

a functor $F : \mathcal{P} \to \text{Ab}$ is projective if and only if

- $F(a_0)$ is free abelian.
• $F(a_i)/\text{Im} F(f_i)$ is free abelian, $F(f_i \circ f_{i-1} \circ \ldots \circ f_0)$ is a monomorphism and $	ext{Ker} F(f_i \circ f_{i-1} \circ \ldots \circ f_{i-d}) \subseteq \text{Im} F(f_{i-d})$ for $d = 0, 1, \ldots, i - 1$ for each $i = 1, 2, 3, 4, \ldots$.

5. Pseudo-projectivity

Consider a functor $F : \mathcal{P} \to \text{Ab}$ over a graded poset $\mathcal{P}$. In this section we find conditions on $F$ such that $\text{colim}_i F = 0$ for $i \geq 1$. We fix the following notation

**Definition 5.1.** Let $\mathcal{P}$ be a graded poset and $F : \mathcal{P} \to \text{Ab}$. We say that $F$ is colim-acyclic if $\text{colim}_i F = 0$ for $i \geq 1$.

Recall that for projective objects it holds that any left derived functor vanishes. Because being colim-acyclic is clearly weaker that being projective we can wonder if it is possible to weaken the hypothesis of projectiveness keeping the thesis of colim-acyclicity. It turns out that pseudo-projectiveness gives an appropriate weaker condition:

**Theorem 5.2.** Let $F : \mathcal{P} \to \text{Ab}$ be a pseudo-projective functor over a bounded below graded poset $\mathcal{P}$. Then $F$ is colim-acyclic.

**Proof.** We can suppose that the degree function $\text{deg}$ on $\mathcal{P}$ is increasing and takes values $\{0, 1, 2, 3, \ldots\}$, and that $\text{Ob}_0(\mathcal{P}) \neq \emptyset$. To compute $\text{colim}_i F$ we use the (normalized, Remark 3.4) spectral sequences corresponding to the third row of Table 1 in Chapter 3. That is, we first filter by the degree of the end object of each simplex to obtain a homological type spectral sequence $E_{\ast, \ast}^\ast$. To compute the column $E_{p, \ast}^1$, we filter by the degree of the initial object of each object to obtain cohomological type spectral sequences $(E_p^p)_{\ast, \ast}^p$. Fix $t \geq 1$. Notice that to prove that $\text{colim}_i F = 0$ is enough to show that $(E^1_{p, t-p})_{r, p-p-t} = 0$ if $r$ is big enough for each $p$ and $p' \leq p - t$. This implies that $\text{colim}_i F = 0$.

Consider the increasing filtration $L_p^n$ of $C_\ast(\mathcal{P}, F)$ that gives rise to the spectral sequence $E_{\ast, \ast}^\ast$. The $n$-simplices are

$$L_p^n = L_p^n C_\ast(\mathcal{P}, F) = \bigoplus_{\sigma \in N\mathcal{P}_n, \text{deg}(\sigma_n) \leq p} F_{\sigma}.$$

For each $p$ we have a decreasing filtration $M_{p'}^p$ of the quotient $L_p^n/L_{p-1}^n$ that gives rise to the spectral sequence $(E_{p})_{\ast, \ast}^{p}$ and which $n$-simplices are

$$(M_{p'}^p)_{\sigma} = \bigoplus_{\sigma \in N\mathcal{P}_n, \text{deg}(\sigma_n) \geq p', \text{deg}(\sigma_n) = p} F_{\sigma}.$$
For $p' \leq p - t$ the abelian group $(E_p)^{p',q'}_r$ at the $t = -(p' + q') + p$ simplices is given by
\[
(E_p)^{p',q'}_r = (M_p)^{p'}_t \cap d^{-1}(M_{p'+1})/(M_p)^{p'+1} \cap d^{-1}(M_{p'+1} + (M_p)^{p'}_t \cap d((M_p)^{p'-r+1}_t)
\]
where $d$ is the differential of the quotient $L_p/L_p^{-1}$ restricted to the subgroups of the filtration $(M_p)$. For $r > p - p' - (t - 1)$ there are not $(t - 1)$-simplices beginning in degree at least $p' + r > p - (t - 1)$ and ending in degree $p$, i.e., $(M_p)^{p'+r-1}_t = 0$. Because $P$ is bounded below for $r$ big enough $(M_p)^{p'-r+1}_t = (M_p)_t = (L_p/L_p^{-1})_{t+1}$, i.e., $(M_p)^{p'-r+1}_t$ equals all the $(t + 1)$-simplices that end on degree $p$. Thus there exists $r$ such that
\[
(E_p)^{p',q'}_r = (M_p)^{p'}_t \cap d^{-1}(0)/(M_p)^{p'+1} \cap d^{-1}(0) + (M_p)^{p'}_t \cap d((M_p)^{0}_t).
\]
Fix such an $r$ and take $[x] \in (E_p)^{p',q'}_r$ where
\[
x = \bigoplus_{\sigma \in NP_{t,deg(\alpha) \geq p',deg(\alpha)} = p} x_{\sigma}
\]
and $d(x) = 0$. Notice that by definition there is just a finite number of summands $x_{\sigma} \neq 0$ in the expression (14) for $x$. We prove that $[x] = 0$ in three steps:

**Step 1:** In this first step we find a representative $x'$ for $[x]$
\[
x' = \bigoplus_{\sigma \in NP_{t,deg(\alpha) \geq p',deg(\alpha)} = p} x'_{\sigma}
\]
such that $\text{deg}(\alpha_1) = 1$ for every $\sigma = \sigma_0 \xrightarrow{\alpha_1} \sigma_1 \xrightarrow{\alpha_2} \ldots \xrightarrow{\alpha_{t-1}} \sigma_{t-1} \xrightarrow{\alpha_t} \sigma_t$ with $x'_{\sigma} \neq 0$.

Take $\sigma$ such that $x_{\sigma} \neq 0$ and suppose that $\text{deg}(\alpha_1) > 1$, i.e., $\text{deg}(\sigma_0) < \text{deg}(\sigma_1) - 1$. Then, as in a graded poset every morphism factors as composition of degree 1 morphisms, there exists an object $\sigma_*$ of degree $\text{deg}(\sigma_0) < \text{deg}(\sigma_*) < \text{deg}(\sigma_1)$ and arrows $\beta_1 : \sigma_0 \rightarrow \sigma_*$ and $\beta_2 : \sigma_* \rightarrow \sigma_1$ with $\alpha_1 = \beta_2 \circ \beta_1$.

\[
\begin{array}{c}
\sigma_0 \\
\xrightarrow{\beta_1} \\
\sigma_* \\
\xrightarrow{\beta_2} \\
\sigma_1.
\end{array}
\]

Call $\tilde{\sigma}$ to the $(t + 1)$-simplex $\sigma = \sigma_0 \xrightarrow{\beta_1} \sigma_* \xrightarrow{\beta_1} \sigma_1 \xrightarrow{\alpha_2} \ldots \xrightarrow{\alpha_{t-1}} \sigma_{t-1} \xrightarrow{\alpha_t} \sigma_t$ and consider the $(t + 1)$-chain of $(M_p)^{0}_t y = i_{\tilde{\sigma}}(-x_{\sigma})$. Its differential in $L_p/L_p^{-1}$ equals
\[
d(y) = d_0(y) - d_1(y) + \sum_{i=2,\ldots,t} (-1)^i d_i(y) = d_0(y) + i_{\sigma}(x_{\sigma}) + \sum_{i=2,\ldots,t} (-1)^i d_i(y).
\]
Notice that the first morphisms appearing in the simplices $d_0(\tilde{\sigma})$ and $d_i(\tilde{\sigma})$ for $i = 2,\ldots,t$ have degree $\text{deg}(\beta_2)$ and $\text{deg}(\beta_1)$ respectively, which are strictly less than $\text{deg}(\alpha_1)$. Also notice that $d(y) \in (M_p)^{p'}_t \cap d((M_p)^{0}_{t+1})$ (which is zero in Equation (13)).
Taking the (finite) sum of the chains $y$ for each term $x_\sigma$ we find that $[x] = [x']$ where

$$x' = \bigoplus_{\sigma \in NP_t, \deg(\sigma_0) \geq p', \deg(\sigma_t) = p} x'_\sigma,$$

and the maximum of the degrees of the morphisms $\alpha_1$ of the simplices

$$\sigma = \sigma_0 \xrightarrow{\alpha_1} \sigma_1 \xrightarrow{\alpha_2} \ldots \xrightarrow{\alpha_{t-1}} \sigma_{t-1} \xrightarrow{\alpha_t} \sigma_t$$

with $x'_\sigma \neq 0$ is smaller than this maximum computed for $x$. So repeating this process a finite number of times we find a representative as wished. For simplicity we write also $x$ for this representative.

**Step 2:** By Step 1 we can suppose that $\deg(\alpha_1) = 1$ for every

$$\sigma = \sigma_0 \xrightarrow{\alpha_1} \sigma_1 \xrightarrow{\alpha_2} \ldots \xrightarrow{\alpha_{t-1}} \sigma_{t-1} \xrightarrow{\alpha_t} \sigma_t$$

with $x_\sigma \neq 0$. Now our objective is to find a representative $x'$ for $[x]$ that runs over simplices $\sigma$ with begin in degree $p'$. Begin writing $x$ as

$$x = \bigoplus_{i=p', \ldots, p-t} x_i$$

where

$$x_i = \bigoplus_{\sigma \in NP_t, \deg(\sigma_0) = i, \deg(\sigma_t) = p} x_\sigma.$$

Notice that the index $i$ just goes to $p-t$ (and not to $p$) because we are using normalized (Remark 3.4) spectral sequences. Now we prove

**Claim 5.2.1.** For each $i$ from $i = p - t$ to $i = p'$ there exists a representative $x'_i$ for $[x]$ such that

$$x'_i = \bigoplus_{\sigma \in NP_t, i \geq \deg(\sigma_0) \geq p', \deg(\sigma_t) = p} (x'_i)_\sigma$$

and $d(x'_i) = 0$ and $\deg(\sigma_0) < i$ imply $\deg(\alpha_1) = 1$.

Notice that taking $i = p'$ in the claim, the step 2 is finished. The case $i = p - t$ in the claim is fulfilled taking $x'_{p-t} = x$ (by step 1). Suppose the statement of the claim holds for $i$. Then we prove it for $i - 1$. We have $x'_i$ such that

$$x'_i = \bigoplus_{\sigma \in NP_t, i \geq \deg(\sigma_0) \geq p', \deg(\sigma_t) = p} (x'_i)_\sigma,$$

and

$$d(x'_i) = 0 \text{ and } [x] = [x'_i]. \text{ The differential } d \text{ on } L^p/L^{p-1} \text{ restricts to}$$

$$d : (M_p^p)_t \to (M_p^p)_{t-1}$$
and carries \( z \in F_\sigma \hookrightarrow \bigoplus_{\sigma \in NP_t, \deg(\sigma_0) \geq p', \deg(\sigma_t) = p} F_\sigma = (M_p)_t^{p'} \) to

\[
d(z) = \sum_{j=0,1,...,t-1} (-1)^j d_j(z)
\]

with \( d_j(z) \in F_{d_j(\sigma)} \hookrightarrow (M_p)_{t-1}^{p'} \). Notice that the initial object of \( d_j(\sigma) \) is \( \sigma_1 \) for \( j = 0 \) and \( \sigma_0 \) for \( j = 1, ..., t - 1 \). Also notice that the final object of \( d_j(\sigma) \) is \( \sigma_t \) for \( j = 0, ..., t - 1 \).

By hypothesis \( d(x'_i) = 0 \). So for every \( \epsilon \in NP_{t-1} \) with \( \deg(\epsilon_0) \geq p' \) and \( \deg(\epsilon_{t-1}) = p \) we can apply the projection

\[
\pi_\epsilon : (M_p)_{t-1}^{p'} \to F_\epsilon
\]

and obtain \( \pi_\epsilon(d(x'_i)) = 0 \). If \( \deg(\epsilon_0) > i \) then the remarks on the differential above and condition (15) imply that

\[
\pi_\epsilon(d(x'_i)) = \sum_{\sigma \in NP_t, \deg(\sigma_0) = i, \deg(\sigma_t) = \epsilon} F(\alpha_1)((x'_i)_\sigma)
\]

and thus

(16)

\[
0 = \sum_{\sigma \in NP_t, \deg(\sigma_0) = i, \deg(\sigma_t) = \epsilon} F(\alpha_1)((x'_i)_\sigma)
\]

for each \( \epsilon \in NP_{t-1} \) with \( \deg(\epsilon_0) > i \) and \( \deg(\epsilon_{t-1}) = p \). Notice that each summand \( (x'_i)_\sigma \) with \( \sigma \in NP_t, \deg(\sigma_0) = i \) and \( \deg(\sigma_t) = p \) appears in one and just one equation as (16) (take \( \epsilon = d_0(\sigma) \)).

Fix an \( \epsilon \in NP_{t-1} \) with \( \deg(\epsilon_0) > i \) and \( \deg(\epsilon_{t-1}) = p \) and consider the associated Equation (16). Then, as \( F \) is \((i - \deg(\epsilon_0))\)-pseudo-projective, \((x'_i)_\sigma \in \text{Im}_F(\epsilon_0)\) for every \( \sigma \in NP_t \) with \( \deg(\sigma_0) = i \) and \( \deg(\sigma_t) = \epsilon \). This means that for every such a \( \sigma \) there exists \( k_\sigma \) objects of degree \((i - 1)\), namely \( i^1_\sigma, ..., i^{k_\sigma}_\sigma \), arrows \( \beta^j_\sigma : i^j_\sigma \to \sigma_0 \) and elements \( x^j_\sigma \in F(i^j_\sigma) \) for \( j = 1, ..., k_\sigma \) such that

(17)

\[
(x'_i)_\sigma = \sum_{j=1,...,k_\sigma} F(\beta^j_\sigma)(x^j_\sigma).
\]

Consider the \((t + 1)\)-simplices for \( j = 1, ..., k_\sigma \)

\[
\sigma^j = i^j_\sigma \xrightarrow{\beta^j_\sigma} \sigma_0 \xrightarrow{\alpha_1} \sigma_1 \xrightarrow{\alpha_2} ... \xrightarrow{\alpha_{t-1}} \sigma_{t-1} \xrightarrow{\alpha_t} \sigma_t
\]

and the \((t + 1)\)-chain of \((M_p)_{t+1}^{i-1}\)

\[
y_\sigma = \oplus_{j=1,...,k_\sigma} i_{\sigma^j}(x^j_\sigma).
\]
The differential of $y_\sigma$ is

$$d(y_\sigma) = d_0(y_\sigma) + \sum_{j=1}^{t} (-1)^j d_j(y_\sigma)$$

$$= d_0(y_\sigma) + R_\sigma, \text{ where } R_\sigma = \sum_{j=1}^{t} (-1)^j d_j(y_\sigma)$$

$$= \sum_{j=1}^{k_\sigma} i_{d_0(\sigma_j)}(F(\beta^j_\sigma(x^j_\sigma))) + R_\sigma$$

$$= \sum_{j=1}^{k_\sigma} i_\sigma(F(\beta^j_\sigma(x^j_\sigma))) + R_\sigma$$

$$= i_\sigma(\sum_{j=1}^{k_\sigma} F(\beta^j_\sigma(x^j_\sigma))) + R_\sigma$$

$$= i_\sigma((x^j_\sigma) + R_\sigma$$

where the last equality is due to (17). Notice that $R_\sigma$ lives in the subgroup $\bigoplus_{\sigma \in N\mathcal{P}_t, \deg(\sigma_0) = i-1, \deg(\sigma_t) = p} F_\sigma \subseteq (M_p)_t^\prime$ of simplices beginning at degree $(i-1)$. Repeating the same construction for each $\sigma \in N\mathcal{P}_i$ with $\deg(\sigma_0) = i$ and $d_0(\sigma) = \epsilon$ we obtain $y_\epsilon = \sum_\sigma y_\sigma$ such that

$$d(y_\epsilon) = \bigoplus_{\sigma \in N\mathcal{P}_t, \deg(\sigma_0) = i, \deg(\sigma_t) = \epsilon} (x^j_\sigma) + R_\epsilon$$

where $R_\epsilon$ lives in the subgroup $\bigoplus_{\sigma \in N\mathcal{P}_t, \deg(\sigma_0) = i-1, \deg(\sigma_t) = p} F_\sigma \subseteq (M_p)_t^\prime$. Repeating the same argument for every $\epsilon \in N\mathcal{P}_{t-1}$ with $\deg(\epsilon_0) > i$ and $\deg(\epsilon_{t-1}) = p$ we obtain $y = \sum_\epsilon y_\epsilon$ such that

$$d(y) = \bigoplus_{\sigma \in N\mathcal{P}_t, \deg(\sigma_0) = i, \deg(\sigma_t) = p} (x^j_\sigma) + R$$

where $R$ lives in the subgroup $\bigoplus_{\sigma \in N\mathcal{P}_t, \deg(\sigma_0) = i-1, \deg(\sigma_t) = p} F_\sigma \subseteq (M_p)_t^\prime$. By construction $y \in (M_p)_{i+1}^{i-1} \subseteq (M_p)_t^0$ and $d(y) \in (M_p)_{i+1}^{i-1} \subseteq (M_p)_t^\prime$. Thus $d(y) \in (M_p)_{i+1}^{i-1} \cap d((M_p)_t^0_{i+1})$. Then, by (13), $[x^j_i] = [x^j_i - d(y)] = [x^j_{i-1}]$ where

$x^j_{i-1} = \bigoplus_{\sigma \in N\mathcal{P}_t, \deg(\sigma_0) \geq p', \deg(\sigma_t) = p} (x^j_\sigma) + R$

is a representative that lives in

$$\bigoplus_{\sigma \in N\mathcal{P}_t, \deg(\sigma_0) \geq p', \deg(\sigma_t) = p} F_\sigma \subseteq (M_p)_t^\prime$$

as wished. That condition (15) holds is clear from the definition of $x^j_{i-1}$.

**Step 3:** By Step 2 we can suppose that

$$x = \bigoplus_{\sigma \in N\mathcal{P}_t, \deg(\sigma_0) = p', \deg(\sigma_t) = p} x_\sigma.$$

Our objective now is to see that there exists $y \in (M_p)_t^0$ with $d(y) = x$. This implies that $[x] = 0$ and finishes the proof of the theorem. We need the
Claim 5.2.2. There exist chains \( x_i \in (M_p)_i^0 \) for \( i = p', \ldots, 0 \) and \( y_i \in (M_p)_{i+1}^0 \) for \( i = p', \ldots, 1 \) such that

\[
d(y_i) = x_i + x_{i-1}
\]

for \( i = p', \ldots, 1 \) with \( x_{p'} = x \) and \( x_0 = 0 \) such that

1. \( x_i \) lives on \( \bigoplus_{\alpha \in \Lambda \cap \tilde{F}} F_\alpha \subseteq (M_p)_i^0 \) for \( i = p', \ldots, 0 \).
2. \( d(x_i) = 0 \) for \( i = p', \ldots, 0 \).

Notice that the claim finishes Step 3: as \( x_0 = 0 \) then \( x_1 = d(y_1), x_2 = d(y_2) - x_1 = d(y_2 - y_1), x_3 = d(y_3) - x_2 = d(y_3 - y_2 + y_1), \ldots, x = x_{p'} = d(y_{p'}) - x_{p'-1} = d(y_{p'} - y_{p'-1} + \ldots + (-1)^{p'-1}y_1) \), and \( y_{p'} = 0 \).

Define \( x_{p'} \equiv x \). Then condition (1) and (2) are satisfied for \( i = p' \). We construct \( y_i \) and \( x_{i-1} \) from \( x_i \) recursively beginning on \( i = p' \). The arguments are similar to those used in step 2.

The differential \( d \) on \( L^p/L^{p-1} \) restricts to

\[
d : (M_p)_i^0 \rightarrow (M_p)_{i-1}^0.
\]

As \( d(x_{p'}) = d(x) = 0 \), for every \( \epsilon \in \Lambda \cap \tilde{F} \) with \( \deg(\epsilon) = p \) we can apply the projection

\[
\pi_\epsilon : (M_p)_{i-1}^0 \rightarrow F_\epsilon
\]

and obtain \( \pi_\epsilon(d(x)) = 0 \). If \( \deg(\epsilon_0) > p' \) then

\[
\pi_\epsilon(d(x)) = \sum_{\sigma \in \Lambda \cap \tilde{F}, \deg(\sigma) = \epsilon} F(\alpha_1)(x_\sigma)
\]

and thus

\[
0 = \sum_{\sigma \in \Lambda \cap \tilde{F}, \deg(\sigma) = \epsilon} F(\alpha_1)(x_\sigma)
\]

for each \( \epsilon \in \Lambda \cap \tilde{F} \) with \( \deg(\epsilon_0) > p' \) and \( \deg(\epsilon) = p \). Notice that each summand \( x_\sigma \) with \( \sigma \in \Lambda \cap \tilde{F}, \deg(\sigma_0) = p' \) and \( \deg(\sigma) = p \) appears in one and just one equation as (19) (take \( \epsilon = d_0(\sigma) \)). Using now pseudo-injectivity we build as before \( y_\epsilon, y_\epsilon = \sum_\sigma y_\sigma \) and \( y = \sum_\epsilon y_\epsilon \), where \( \epsilon \) runs over \( \epsilon \in \Lambda \cap \tilde{F} \) with \( \deg(\epsilon_0) > p' \) and \( \deg(\epsilon) = p \), such that

\[
d(y) = x + R
\]

with \( R \) living in \( \bigoplus_{\alpha \in \Lambda \cap \tilde{F}, \deg(\sigma_0) = p} F_\alpha \subseteq (M_p)_i^0 \). Call \( y_{p'} \equiv y \) and \( x_{p'-1} = R \). Then Equation (18) is satisfied. Condition (1) for \( i = p'-1 \) holds by the construction of \( R \) and condition (2) for \( i = p'-1 \) holds because \( d(x_{p'-1}) = d(R) = d(d(y) - x) = d^2(y) - d(x) = 0 - 0 = 0 \) as \( d \) is a differential and \( d(x) = 0 \) by hypothesis. The construction of \( y_i \) and \( x_{i-1} \) from \( x_i \) is totally analogous to the construction of \( y_{p'} \) and \( x_{p'-1} \) from \( x_{p'} \) that we have just made.

After we have built \( y_1 \) and \( x_0 \) if we try to build \( y = \sum_\epsilon y_\epsilon \) and \( R \) from \( x_0 \) we find that, because there are not objects of negative degree (thus if \( z \in \text{Im}(i') \) where \( \deg(i') = 0 \) then \( z = 0 \)), \( x_0 = 0 \).
The following examples come from Example 4.11. They show the weaker conditions that are needed for colim-acyclicity instead of projectiveness.

**Example 5.3.** For the “pushout category” \( \mathcal{P} \) with shape

\[
\begin{array}{ccc}
  a_0 & \xrightarrow{f} & b_1 \\
  \downarrow{g} & & \\
  c_1 & \xrightarrow{} & \\
\end{array}
\]

a functor \( F : \mathcal{P} \to \text{Ab} \) is colim-acyclic if \( F(f) \) and \( F(g) \) are monomorphisms.

For the “telescope category” \( \mathcal{P} \) with shape

\[
\begin{array}{ccc}
  a_0 & \xrightarrow{f_1} & a_1 \\
  & \xrightarrow{f_2} & a_2 \\
  & \xrightarrow{f_3} & a_3 \\
  & \xrightarrow{f_4} & a_4 \\
  \vdots \\
\end{array}
\]

a functor \( F : \mathcal{P} \to \text{Ab} \) is colim-acyclic if \( F(f_i \circ f_{i-1} \circ \ldots \circ f_1) \) is a monomorphism and \( \ker F(f_i \circ f_{i-1} \circ \ldots \circ f_{i-d+1}) \subseteq \text{Im} F(f_{i-d}) \) for \( d = 1, 2, 3, \ldots, i-1 \) for each \( i = 2, 3, 4, \ldots \)

Notice that for this it is enough that \( F(f_i) \) is a monomorphism for each \( i = 1, 2, 3, \ldots \).

**6. Dual results for injective objects in \( \text{Ab}^\mathcal{P} \).**

The appropriate notions to characterize the injective objects in the functor category \( \text{Ab}^\mathcal{P} \), where \( \mathcal{P} \) is a graded poset, are the following:

**Definition 6.1.** \( \ker F(i_0) = \bigcap_{\alpha} \ker F(\alpha) \) (or \( \ker F(i_0) = F(i_0) \) if the index set of the intersection is empty) and \( \operatorname{coim} F(i_0) = F(i_0)/\ker F(i_0) \).

**Definition 6.2.** Let \( F : \mathcal{P} \to \text{Ab} \) be a functor over a graded poset \( \mathcal{P} \) with degree function \( \deg \). Fix an integer \( d \geq 0 \). If for any object \( i_0 \) of \( \mathcal{P} \), different objects \( \{ i_j \}_{j \in J} \) of \( \mathcal{P} \), arrows \( \alpha_j : i_0 \to i_j \) with \( \deg(\alpha_j) = d \) and elements \( x_j \in \ker F(i_j) \) for each \( j \in J \), there is \( y \in F(i_0) \) with

\[ F(\alpha_j)(y) = x_j \]

for each \( j \in J \), we call \( F \) \( d \)-pseudo-injective. If \( F \) is \( d \)-pseudo-injective for each \( d \geq 0 \) we call \( F \) pseudo-injective.

Then we can prove the following

**Theorem 6.3.** Let \( \mathcal{P} \) be a bounded above graded poset and \( F : \mathcal{P} \to \text{Ab} \) be a functor. Then \( F \) is injective if and only if

1. for any object \( i_0 \) of \( \mathcal{P} \) \( \ker F(i_0) \) is injective in \( \text{Ab} \).
2. \( F \) is pseudo-injective.

Also in the dual case pseudo-injectiveness is enough for vanishing higher inverse limits:

**Definition 6.4.** Let \( \mathcal{P} \) be a graded poset and \( F : \mathcal{P} \to \text{Ab} \). We say \( F \) is lim-acyclic if \( \lim^i F = 0 \) for \( i \geq 1 \).

**Theorem 6.5.** Let \( F : \mathcal{P} \to \text{Ab} \) be a pseudo-injective functor over a bounded above graded poset \( \mathcal{P} \). Then \( F \) is lim-acyclic.
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