Autonomous quantum error correction and quantum computation

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In this work, we present a general theoretical framework for the study of autonomously corrected quantum devices. First, we identify a necessary and sufficient revised version of the Knill-Laflamme conditions for the existence of an engineered Lindbladian providing protection against at most \( c \) consecutive errors of natural dissipation, giving rise to an effective logical decoherence rate suppressed to order \( c \). Moreover, we demonstrate that such engineered dissipation can be combined with generalized realizations of error-transparent Hamiltonians (ETH) in order to perform a quantum computation in the logical space while maintaining the same degree of suppression of decoherence. Finally, we introduce a formalism predicting with precision the emergent dynamics in the logical code space resulting from the interplay of natural, engineered dissipations sources and the generalized ETH.

Introduction. Quantum error correction (QEC) plays a central role in the development of scalable quantum computers. While recent developments demonstrated the possibility to perform complex quantum information processing tasks with high fidelity in state-of-the-art architectures with 10-100 qubits, reliable large-scale quantum information processing still requires quantum error correction with efficient encoding and fault-tolerant implementation to suppress various practical imperfections. The standard procedure for QEC relies on classical adaptive control, which measures the error syndrome and performs feedback with a classical controller. Recent experimental demonstration of QEC of bosonic encoding used classical adaptive control to reach the break-even point by suppressing the natural errors due to excitation loss. However, the performance of QEC with classical controllers is often limited by the readout errors, decoherence from syndrome measurement, extra time delay and heating associated with classical feedback loop, as well as significant physical and computational resource overhead from the routine error recovery procedures.

Alternatively, we may implement QEC without classical adaptive control – an approach called autonomous quantum error correction (AutoQEC) which has attracted increasing attention recently. Instead of measuring the error syndrome with a classical device, AutoQEC coherently processes the error syndrome and extract the entropy via engineered dissipation – avoiding the measurement imperfection and overhead associated with classical feedback loop. AutoQEC has recently achieved efficient suppression of both cavity dephasing errors and single-photon loss using the bosonic cat encoding, which paves the way towards hardware-efficient QEC. Theoretical investigations have shown that AutoQEC can improve the decoherence rate to \( O(\kappa/R^c) \) at the logical level, where \( \kappa \) is the natural error rate and \( R \gg 1 \) is the ratio between the (good) engineered dissipation and (bad) natural error rates. Hence, it is desirable to use higher-order AutoQEC protocols, which can further improve logical decoherence rate to \( O(\kappa/R^c) \) with \( c > 1 \). In addition, from a scalability perspective (where small autonomously protected qubits would be used as basic elements of a larger architecture such as a topological code), it is also crucial to understand how logical gates can be applied while error correcting, and to assess the quality of the resulting logical qubits by quantifying systematically the post-AutoQEC residual logical error model.

In this work we perform a study of autonomously error corrected quantum devices and derive a fundamental result covering altogether AutoQEC and autonomous error-corrected approaches to quantum computation. Our formalism extends the Knill-Laflamme conditions to a continuous-time framework where natural and engineered dissipations are constantly acting on the system, and generalizes previous results to quantum codes of arbitrary order \( c \geq 1 \) and arbitrary sets of jump error operators \( F_k \). We prove that the Knill-Laflamme conditions are both necessary and sufficient conditions for AutoQEC. Under those same conditions, we show that autonomous error-corrected quantum computations with the same protection of order \( c \) can be performed by combining the engineered dissipation with generalized versions of the standard error-transparent Hamiltonian. In addition, we present a formalism allowing to assess with precision the post-AutoQEC dynamics in the restricted logical code space resulting from the interplay of natural and engineered dissipations, and the generalized ETH.

Problem setup. Let us have a Hilbert space \( \mathcal{H} \) with some code subspace \( \mathcal{C} \subset \mathcal{H} \), and some natural dissipation with jump operators \( F_k \) acting on \( \mathcal{H} \) with the characteristic rate \( \kappa \). One assumes that \( d_C = \dim(\mathcal{C}) \) and \( d_H = \dim(\mathcal{H}) \) are both finite and defines the code space projector \( P_C \) as well as its complementary projector \( Q_C = 1 - P_C \).

With appropriate choice of logical subspace, we expect that the jump error operators \( F_k \) cannot directly destroy the encoded quantum information. Instead, repeated action of the \( F_k \) should generates a hierarchy of submanifolds corresponding to errors of incrementing weight \( q \), as illustrated in Fig. 1 a). For a quantum code of order \( c \geq 1 \), i.e., a code robust
against at most $c$ errors, when the weight $q$ overcomes $c$ a logical error can however occur. Formally this structure should translate into Knill-Laflame conditions on an a certain error set, which is unknown at this stage and will be identified in a subsequent section. To obtain an effective suppression of decoherence of order $c$, a suitable set of additional jump operators $F_{E,j}$ needs to be engineered, such that correctable error states are brought back to the code space without their underlying logical content being affected. If engineered dissipation has a characteristic rate $R\kappa$ (with $R$ dimensionless), in the strong recovery regime ($R \gg 1$) the error submanifolds are expected to be strongly depleted, leading after kinetic equilibration between engineered and natural jumps to an occupancy of $q$-weighted error states scaling as $p_q \propto O(1/R^q)$ for $q \leq c$ (see Fig. 1a). Eventually logical errors occur when $c+1$ or more natural errors are left unrecovered. As a consequence one expects the scaling

$$
\kappa_{\text{eff}} \equiv \text{logical error rate} \sim \frac{\kappa}{R^c},
$$

for the resulting logical decoherence rate in the code space.

Preferably, this suppression of decoherence should hold not only for an idle quantum state, i.e., for the purpose of performing AutoQEC, but also while performing some computation in the logical basis. In addition to the engineered dissipation, the latter requirement will involve the introduction of an error-transparent Hamiltonian [15], or a generalization of an ETH. As sketched in Fig. 1b), in order to preserve the scaling Eq. (1), such Hamiltonian should not only operate in the error-free code space, but also couple correctable error states in a way which does not compromise the hierarchy of error submanifolds preexisting to the computation (e.g. by increasing the error weight), and such that the correct logical operation has still been performed upon recovery by the engineered jumps $F_{E,j}$. We formalize these concepts below.

**Definitions.** Let us consider the following Lindbladian time-continuous evolution within the Hilbert space $\mathcal{H}$:

$$
\frac{d\rho}{dt} = \mathcal{L}\rho = \kappa (\{-i\mathcal{H},\rho \} + R\mathcal{L}_{E}(\rho) + \mathcal{L}_{\alpha}(\rho))
$$

where $\mathcal{L}_{\alpha}(\rho) = \sum_{k=1}^{N} D[f_{E,k}](\rho)$ and $\mathcal{L}_{E}(\rho) = \sum_{j=1}^{N} D[f_{E,j}](\rho)$ are the Liouvillians associated with natural and engineered dissipation, and $D[A](\rho) = A\rho A^\dagger - \frac{1}{2}\{A^\dagger A,\rho\}$. $H$ is the control Hamiltonian for logical operations. Apart from $\kappa$ with a dimension of a frequency, all introduced quantities are dimensionless. In particular the parameters $R$ and $\kappa$ control respectively the relative speeds of engineered dissipation and Hamiltonian dynamics compared to natural dissipation. We introduce the useful notation $A \otimes \mathcal{B} \equiv \text{span}(|\mu\rangle\langle\nu|,|\mu\rangle \in A,|\nu\rangle \in \mathcal{B})$ and the superoperator projector $\mathcal{P}_{C}$ defined as $\mathcal{P}_{C}\rho = \mathcal{P}_{C}\rho\mathcal{P}_{C}$ for all $\rho \in \mathcal{H} \otimes \mathcal{H}$. Since $\mathcal{L}_{E}$ induces purely Lindbladian dynamics, the limit $\mathcal{P}_{E} \equiv \lim_{u \to \infty} e^{\mathcal{L}_{E} u}$ is well-defined [30, 31] and is a projector ($\mathcal{P}_{E}^2 = \mathcal{P}_{E}$).

First, let us consider the case $H = 0$ of a protected quantum memory with no logical operation [36]. Assuming an initial condition $\rho(0)$ in the code space, due to natural dissipation the population leaks from the code space and occupies various error spaces (see Fig. 1b) with a probability of order $O(1/R)$. In order to assess the amount of logical information in $\rho(t)$ which has not been irreversibly erased by natural dissipation, it is therefore necessary to perform an information-preserving quantum recovery onto $\rho(t)$ back into the code space before comparing it to the initial state $\rho(0)$. One stresses that such recovery does not have to be explicitly implemented, but is only a mathematical tool to assess deviations at a finite time $t$. We now point out that if the engineered dissipation $\mathcal{L}_{E}$ is indeed able to protect the code space and perform autonomous error correction up to order $c$, then the CPTP projector $\mathcal{P}_{E} = \lim_{u \to \infty} e^{\mathcal{L}_{E} u}$ should be a finite hypothetical final recovery to assess such deviation. This leads us to the following definition: the engineered dissipation $\mathcal{L}_{E}$ performs autonomous quantum error correction up to order $c$ with respect to the code space $\mathcal{C}$ and the natural dissipation $\mathcal{L}_{\alpha}$ iff there exists $M > 0$ such that for all $\rho(0) \in \mathcal{C} \otimes \mathcal{D}$, one has

$$
\|\mathcal{P}_{E}\rho(t) - \rho(0)\| \leq M\|\rho(0)\|\frac{\kappa t}{R^c}
$$

for all times $t \geq 0$ and engineered dissipation strength $R \geq 0$, where $\rho(t) = e^{\mathcal{L}_{E} t}\rho(0)$. We point out that since we work in a finite-dimensional space the choice of the norm $\|\rho\|$ for matrices in $\mathcal{H} \otimes \mathcal{H}$ only modifies the prefactor, which is not the focus of this investigation.

We now move to the generic case $H \neq 0$ where we want to perform a gate while preserving the same degree of accuracy. Consider some target dimensionless Hamiltonian $H_0 = \mathcal{P}_{C}H_0\mathcal{P}_{C}$ acting only on the logical space, we denote $U_{0}(s) = \exp[-iH_0 s]$ the unitary parametrized by the dimensionless time $s$. Accordingly, we say that the engineered dissipation $\mathcal{L}_{E}$ and the Hamiltonian $H$ perform an autonomous error-corrected quantum computation up to order $c$ with respect to the code space $\mathcal{C}$, the target logical Hamiltonian $H_0$ and natural dissipation $\mathcal{L}_{\alpha}$, iff there exists two positive constants $M, g_0 > 0$ such that for any initial condition

\[\begin{align*}
Q_{\rho(0)}(t) \equiv & \mathcal{P}_{E}\rho(t) - \mathcal{P}_{C}\rho(t) \\
\|Q_{\rho(0)}(t)\| & \leq M\|\rho(0)\|\frac{\kappa t}{R^c}
\end{align*}\]
\( \rho(0) \in \mathcal{C} \otimes \mathcal{D} \) one has

\[
\| \mathcal{P}_E \rho(t) - U_0(g \kappa t) \rho(0) U_0(g \kappa t)^\dagger \| \leq M \| \rho(0) \| \frac{\kappa t}{R}.
\]

(4)

for all \( t, R \geq 0 \), and all Hamiltonian coupling strengths \( g \) satisfying \( |g| \leq g_0 R \). We call any pair of \( \mathcal{L}_E \) and \( \mathcal{H} \) satisfying the latter condition respectively an error-correcting Liouvillian and a generalized error-transparent Hamiltonian of order \( c \).

**Error sets.** In connection to the usual QEC formalism [7] for discrete-time error channels with a Kraus representation, we expect an extension of Knill-Laflamme conditions to be satisfied on a certain error set (still to be identified) in order to be able to perform AutoQEC in the continuous-time case. To facilitate the discussion, we define the zeroth- and first-order error sets as: \( \mathcal{E}^{[0]} = \{1\} \), \( \mathcal{E}^{[1]} = \{F_1, \ldots, F_N\} \). At higher order we need to include errors from the no-jump evolution associated to the natural dissipation backaction Hamiltonian

\[
\mathcal{H}_{n,\text{BA}} = \sum_{k=1}^N F_k^\dagger F_k:
\]

for the second-order error set, this yields: \( \mathcal{E}^{[2]} = \{F_k F_l \mid k, l \in \{1, \ldots, N\}\} \cup \{\mathcal{H}_{n,\text{BA}}\} \). More generally, we construct higher-order error sets recursively as follows:

\[
\mathcal{E}^{[n]} = \{F_k E_i^{[n-1]} E_j^{[n-1]} \mid E_i^{[n-1]} \in \mathcal{E}^{[n-1]} \text{ and } k \in \{1, \ldots, N\}\} \cup \{\mathcal{H}_{n,\text{BA}} E_i^{[n-2]} E_j^{[n-2]} \in \mathcal{E}^{[n-2]}\}.\]

(5)

We employ some arbitrary labeling notation \( \mathcal{E}^{[n]} = \{E_j^{[n]} \mid 1 \leq j \leq |\mathcal{E}^{[n]}|\} \) for the errors \( E_j^{[n]} \) in \( \mathcal{E}^{[n]} \). Finally, we define the error set up to order \( n \) as:

\[
\mathcal{E}^{[-n]} = \bigcup_{k=0}^n \mathcal{E}^{[k]}.
\]

Below we formulate our main theorem on AutoQEC and autonomous error-corrected quantum computations, with its proof presented in [32].

**Theorem:** existence of engineered dissipation and generalized error-transparent Hamiltonian. Let us have some integers \( c \geq 0 \). The following statements are equivalent:

(1) Knill-Laflamme: The Knill-Laflamme condition is satisfied for the error set \( \mathcal{E}^{[-c]} \).

(2) AutoQEC: There exists a Liouvillian \( \mathcal{L}_E \) performing autonomous quantum error correction up to order \( c \) with respect to the code space \( \mathcal{C} \) and the natural dissipation \( \mathcal{L}_n \).

(3) Autonomous error-corrected quantum computations: There exists an engineered dissipation \( \mathcal{L}_E \) satisfying the following property: for all logical Hamiltonians \( H_0 = P_c H_0 P_c \), there exists a Hamiltonian \( \mathcal{H} \) such that \( \mathcal{L}_E \) and \( \mathcal{H} \) perform an autonomous error-corrected quantum computation up to order \( c \) with respect to the code space \( \mathcal{C} \), the target logical Hamiltonian \( H_0 \), and natural dissipation \( \mathcal{L}_n \).

Despite the resulting equivalence between all these logical assertions, hypothesis (3) appears significantly stronger than hypothesis (2) as it ensures the possibility of performing any logical operation while error correcting. The equivalence between hypotheses (1) and (2) on the other hand is a direct extension of known results of QEC [7] to the context of continuous feedback and dissipation. Yet, there are some peculiarities emerging in this specific context: first, the ability to correct against a set of quantum jump operations \( \{\alpha F_k + \beta F_j\} \) does not guarantee to be able to correct against errors associated with superposition of error jump operators \( \alpha F_k + \beta F_j \). For example, the modification of the backaction Hamiltonian \( \mathcal{H}_{n,\text{BA}} \) generates new terms of the type \( F_k^\dagger F_j \) which might not be included in the original error set, nor satisfy the Knill-Laflamme conditions. Second, although the backaction Hamiltonian \( \mathcal{H}_{n,\text{BA}} \) acts as a first-order term in the time evolution of the density matrix \( \rho(t) \), it only counts as a second-order error to be corrected (\( \mathcal{H}_{n,\text{BA}} \in \mathcal{E}^{[2]} \setminus \mathcal{E}^{[1]} \)). The proof of the theorem, heavily relies on the following lemma.

**Lemma:** Properties of generalized error-transparent Hamiltonians and engineered dissipation. Let us have some target Hamiltonian \( H_0 = P_2 H_0 P_2 \), some engineered dissipation \( \mathcal{L}_E \) and generic Hamiltonian \( H \). We denote \( H \) (resp. \( H_0 \)) as the superoperator associated with the Hamiltonian evolution \( \mathcal{H}(\rho) = [H, \rho] \) (resp. \( H_0(\rho) = [H_0, \rho] \)). The following statements are equivalent:

(1) \( \mathcal{L}_E \) and \( H \) perform an autonomous error-corrected quantum computation up to order \( c \) with respect to the code space \( \mathcal{C} \), the target logical Hamiltonian \( H_0 \) and natural dissipation \( \mathcal{L}_n \).

(2) for all sets of integers \( k_1, k_2, k_3 \in \mathbb{N} \times \mathbb{N} \times \{0, \ldots, c\} \) one has

\[
\mathcal{P}_E \cdot \mathcal{S}[\{\mathcal{H}, k_1\}, \{\mathcal{L}_E, k_2\}, \{\mathcal{L}_n, k_3\}] \mathcal{P}_C = \delta_{k_2,0} \delta_{k_3,0} \mathcal{H}_0^k \mathcal{P}_C.
\]

(6)
and thus any asymmetric contributions (which would be generated for time-dependent dynamics) are absent here and do not need to be corrected. While the complete proofs of the theorem and the Lemma can be found in [32], we detail below an explicit construction of engineered dissipation and Hamiltonian satisfying the Lemma conditions and thus performing AutoQEC and error-corrected autonomous computations.

Explicit construction of an engineered dissipation and Hamiltonian for error-corrected computations. We assume that the Knill-Laflamme condition is satisfied for the error set $\mathcal{E}^{(c)}$. Let us have $|\mu\rangle \in \mathcal{C}$. We construct an orthogonal basis of correctable error states as follows: $|\mu^\mathbf{E}\rangle \equiv (|\mu|)_{i=1}^{p_k} = (|\mu^1|, \ldots, |\mu|_c) = \text{G.S.}(|\mu|_1, E^1_1|\mu_1|, \ldots, E^1_c|\mu_1|, \ldots, E^c_c|\mu_c|).

The notation $\text{G.S.}(v_i)_{i \in I}$ stands for the uniquely defined orthonormal vector family generated by the Gram-Schmidt algorithm using as input the vector family $(v_i)_{i \in I}$ defined over an ordered ensemble $I$. For all $k \in \{0, \ldots, c\}$, the vectors $|\mu^{|\mathbf{E}}\rangle$ represent the orthonormal basis of the $k$-th order error states generated from the logical $|\mu\rangle \in \mathcal{C}$ which are orthogonal to all lower-order error states. The Knill-Laflamme conditions being satisfied, we can show for all $k \in \{0, \ldots, c\}$ that the number $p_k$ of generated states $\{\mu^{|\mathbf{E}}\}_1 \leq p_k$ is independent of $|\mu\rangle \in \mathcal{C}$, and that for two choices $|\mu\rangle \perp |\nu\rangle \in \mathcal{C}$, the vectors $|\mu^\mathbf{E}\rangle$ are orthogonal to the vectors of $|\nu^\mathbf{E}\rangle$. Finally, given a basis $\{|\mu_i\rangle, 1 \leq i \leq d_c\}$ of the code space, one defines the space $\mathcal{H}_{\text{res}}$ of residual states as

$$\mathcal{H}_{\text{res}} = \left(\text{span}\{\cup_{i=1}^{d_c} \mu^{|\mathbf{E}}\_i\}\right) \perp \text{span}\{|\phi_1\}, \ldots, |\phi_{q_{\max}}\rangle\rangle$$

where $q_{\max} = d_H - d_c$. An engineered dissipation $\mathcal{L}_{\mathcal{E}}$ and generalized error-transparent Hamiltonian $\mathcal{H}$ satisfying the condition (2) of the Lemma and thus performing an autonomous error-corrected quantum computation up to order $c$ are then constructed as follows:

$$\mathcal{L}_{\mathcal{E}} = \sum_{n=1}^c \sum_{i,n=1}^{p_n} D[|F_{E,i,n}\rangle] + \sum_{q=1}^{q_{\max}} D[|F_{E,q}\rangle], \quad (7)$$

$$\mathcal{H} = \sum_{j,k=1}^{d_c} \sum_{n=0}^{c_n} \sum_{i,n=1}^{p_n} (\langle \mu_j | H_0 | \mu_k\rangle |\mu^{|\mathbf{E}}\_j\rangle, |\mu^{|\mathbf{E}}\_k\rangle). \quad (8)$$

The jump operators associated with engineered dissipation are given by

$$F_{E,i,n} = \sum_{j=1}^{d_c} |\mu_j\rangle \langle \mu^{|\mathbf{E}}\_j|, \quad \text{for } 1 \leq n \leq c \text{ and } 1 \leq i, n \leq p_n,$$

$$F_{E,q} = |\Phi_q\rangle \langle \phi_q|, \quad \text{for } 1 \leq q \leq q_{\max}, \quad (9)$$

and $|\Phi_q\rangle$ is any (normalized) state in $\mathcal{C}$.

Generalized error-transparent Hamiltonians. Interestingly, in order to perform correctly an error-corrected computation of order $c$, an Hamiltonian does not necessarily need to be error-transparent in the standard sense, which has been originally formulated as a requirement of commutation with errors $\Box [2]$. While, the specific construction in Eq. (8) indeed accidentally satisfies both conditions $H E P_0 = E H P_0$ and $[H, E] P_0 = 0$ for all the error operators $E \in \mathcal{E}^{(c)}$ of our error set, from our Lemma, we see that a generalized ETH $H$ can be defined as any Hamiltonian satisfying the condition of Eq. (33). Such condition admits more general solutions: in [32] we outline in particular examples where a generalized ETH $H$ does not commute with errors, preserve the error syndrome, nor satisfy $H P_0 = H_0 P_0$ (in the latter case $H$ was found to not stabilize the code space).

Logical space dynamics. For practical purposes it is often useful to acquire a more precise knowledge of the resulting decoherence dynamics of the corrected logical qubit. Consider some engineered dissipation $\mathcal{L}_{\mathcal{E}}$ and Hamiltonian $\mathcal{H}$ satisfying Eq. (4). In particular, according to the Lemma, the recovery projector satisfies $P_{\mathcal{E}} P_{\mathcal{C}} = P_{\mathcal{C}}$. For simplicity, we will also assume that the steady-states of $\mathcal{L}_{\mathcal{E}}$ are all contained in $\mathcal{C}$, which translates as $P_{\mathcal{C} P_{\mathcal{E}}} = P_{\mathcal{C}}$. Note that our explicit construction for $\mathcal{L}_{\mathcal{E}}$ in Eq. (7) does satisfy this condition.

Under these conditions, in [32] we present the derivation of an effective master equation governing the dynamics of $P_{\mathcal{E}} \rho(t)$. Our formalism is based on Nakajima-Zwanzig projection operator techniques [33]; assuming an initial condition $\rho(0)$ in the code space, $P_{\mathcal{E}} \rho(t)$ follows the exact non-Markovian dynamics $\partial_t P_{\mathcal{E}} \rho = \mathcal{L}_{\mathcal{E}} P_{\mathcal{E}} \rho(t) + \int_0^t d\tau \Sigma(\tau) P_{\mathcal{E}} \rho(t-\tau)$, where $\Sigma(\tau) = P_{\mathcal{E}} \mathcal{L}_{\mathcal{E}} \rho_0 \mathcal{E} P_{\mathcal{E}} = Q_{\mathcal{E}} \mathcal{L}_{\mathcal{E}} \rho_0 \mathcal{E} P_{\mathcal{E}}$ is the Nakajima-Zwanzig memory kernel, and $Q_{\mathcal{E}} \equiv I - P_{\mathcal{E}}$ is the complementary projector to $P_{\mathcal{E}}$. After highlighting a strong time scale separation between the relaxation dynamics of the memory kernel $\Sigma(\tau)$ and of the projected density matrix $P_{\mathcal{E}} \rho(t)$ occurring respectively over $t \sim 1/(R \kappa)$ and $t \sim R^c / \kappa$ as soon as $R \gg 1$ [32], we derive the following time-local master equation $\partial_t P_{\mathcal{E}} \rho(t) = [-i g H_0 + \mathcal{L}_{\mathcal{E}}] P_{\mathcal{E}} \rho(t) + \{[\partial_t P_{\mathcal{E}} \rho(t)]_{\text{corr}},\}$ where

$$\mathcal{L}_{\mathcal{E}} = \int_0^\infty dt \left[ \Sigma(\tau) e^{i g H_0 \tau} \right] P_{\mathcal{C}}. \quad (10)$$

The mean density matrix satisfies $\rho(t) = \mathcal{T} P_{\mathcal{E}} \rho(t) + \delta(t)$, where $\mathcal{T} = I + \int_0^\infty d\tau e^{i \Sigma(\tau) P_{\mathcal{E}}} e^{i \Sigma(\tau) P_{\mathcal{E}}} e^{i \Sigma(\tau) P_{\mathcal{C}}}$. As desired, the effective Liouvillian superoperator $\mathcal{L}_{\mathcal{E}}$ only acts within the code space: $\mathcal{L}_{\mathcal{E}} = P_{\mathcal{C}} \mathcal{L}_{\mathcal{E}} = P_{\mathcal{C}} \mathcal{L}_{\mathcal{C}}$. The full density matrix $\rho(t)$ contains components in error spaces due to the application of $\mathcal{T}$ on the projected matrix $P_{\mathcal{E}} \rho(t)$. The corrections to these effective dynamics are shown to satisfy the following upper-bounds $[\{[\partial_t P_{\mathcal{E}} \rho(t)]_{\text{corr}}\} \leq \kappa |K/R^{2c+1} + K e^{-\beta Q R^c}/R^c|$, $||\delta(t)|| \leq A e^{-Q R^c}/R^c + C/R^c + 2$ for some $A, B, C, K > 0$. These bounds hold for all $\kappa, t, R^c, |g| \leq g_0 R$ for some constant $g_0 > 0$. We conclude that for large $R$ the derived estimate can become arbitrarily precise, and that the increase in precision occurs faster for error-correcting codes of larger order $c$.

The expressions of the integrals in $\mathcal{L}_{\mathcal{E}}$ and $\mathcal{T}$ are reported in the general case in the Supplemental Material [32]. We present here the corresponding results in the case $H = H_0 = \Box [2]$.
without computation in the logical basis:

\[
\mathcal{L}_{\text{eff}} = \kappa \sum_{k=c+1}^{+\infty} \left( \frac{-1}{R} \right)^{k-1} \mathcal{P}_E (\mathcal{L}_n \mathcal{L}^*_E)^k \mathcal{L}_n \mathcal{P}_C \tag{11}
\]

with correspondingly \( T = \sum_{n=0}^{+\infty} \left( \frac{-1}{R} \right)^n (\mathcal{L}^*_E \mathcal{L}_n)^k \mathcal{P}_C \). The quantity \( \mathcal{L}^*_E = -\int_0^{+\infty} da e^{ax} Q_E \) is well-defined as \( Q_E \) projects onto the relaxation modes of \( \mathcal{L}_E \), and is a pseudo inverse of \( \mathcal{L}_E \). Along with along \( \mathcal{P}_E \), \( \mathcal{L}^*_E \) possesses a relatively simple analytical expression within our explicit construction of Eqs. (78) (see [32]). As expected intuitively from the AutoQEC of order \( c \) hypothesis, the summation in Eq. (11) initiates with terms involving at least \( k = c+1 \) powers of natural dissipation, and the effective Liouvillian \( \mathcal{L}_{\text{eff}} \) is indeed suppressed as \( 1/R^c \) for a large \( R \). This scaling is demonstrated in [32] to persist in the presence of non-vanishing Hamiltonians \( H \) and \( H_0 \).

Concluding remarks and outlooks We have developed an extensive mathematical framework for autonomously error-corrected quantum devices. In particular, our study features a proof in the time-continuous context of the equivalence between Knill-Laflamme conditions and the possibility to perform both autonomous quantum error correction and error-corrected quantum computations to an arbitrary order \( c \geq 1 \). Our study encompasses all types of codes and Markovian error models. An important generalization of these results regards time-dependent and non-Markovian dissipation sources. Future work will also investigate approximated quantum error correction and noise-biased logical qubits in connection to the relevant applications to bosonic quantum computation.

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USEFUL IDENTITIES ON SYMMETRIZED PRODUCTS

Our Lemma in the main manuscript introduced symmetrized products of operators $S(\{A_1, k_1\}, \ldots, \{A_n, k_n\})$. The demonstrations of our results, detailed in the subsequent sections of this supplementary, heavily rely on the manipulations of such quantities. In order to facilitate the reader we present here a few basic relations involving symmetrized products.

- **Binomial expansion**
  \[
  (\sum_{i=1}^{n} A_i)^q = \sum_{k_1, \ldots, k_n \in \mathbb{N}} S(\{A_1, k_1\}, \ldots, \{A_n, k_n\})
  \]  
  This relation can be seen also as a definition for symmetrized products.

- **Invariance by permutation**
  \[
  S(\{A_1, k_1\}, \ldots, \{A_n, k_n\}) = S(\{A_{\sigma(1)}, k_{\sigma(1)}\}, \ldots, \{A_{\sigma(n)}, k_{\sigma(n)}\})
  \]  
  for all permutation $\sigma \in S_n$ of the symmetric group $S_n$.

- **Exponential power series expansion**
  \[
  \exp\left(\sum_{i=1}^{n} A_i\right) = \sum_{k_1, \ldots, k_n \in \mathbb{N}} S(\{A_1, k_1\}, \ldots, \{A_n, k_n\})/(k_1 + \ldots + k_n)!
  \]  

- **Recursivity**
  \[
  S(\{A_1, k_1\}, \ldots, \{A_n, k_n\}) = \sum_{i=1}^{n} A_i S(\{A_1, k_1\}, \ldots, \{A_i, k_i - 1\}, \ldots, \{A_n, k_n\})
  \]  
  In particular $S(\{A_1, k_1\}, \ldots, \{A_n, k_n = 0\}) = S(\{A_1, k_1\}, \ldots, \{A_{n-1}, k_{n-1}\})$.

PROOF THEOREM

In this supplementary section we present the proof of the theorem of the main manuscript, which is formulated as follows.

*Theorem: existence of engineered dissipation and generalized error-transparent Hamiltonian.*

Let us have some integer $c \geq 0$. The following statements are equivalent:

1. **Knill-Laflamme**: The Knill-Laflamme condition is satisfied for the error set $\mathcal{E}^{[-c]}$.

2. **AutoQEC**: There exists a Liouvillian $\mathcal{L}_E$ performing autonomous quantum error correction up to order $c$ with respect to the code space $\mathcal{C}$ and the natural dissipation $\mathcal{L}_n$.

3. **Autonomous error-corrected quantum computations**: There exists an engineered dissipation $\mathcal{L}_E$ satisfying the following property: for all logical Hamiltonians $H_0 = P_C H_0 P_C$, there exists a Hamiltonian $H$ such that $\mathcal{L}_E$ and $H$ perform an autonomous error-corrected quantum computation up to order $c$ with respect to the code space $\mathcal{C}$, the target logical Hamiltonian $H_0$, and natural dissipation $\mathcal{L}_n$.

More precisely we will prove the theorem assuming that the Lemma of the main manuscript has been proved, and postpone the proof of the Lemma to the subsequent supplementary section. First of all, the theorem hypothesis (3) trivially implies (2).
We now assume that the theorem hypothesis (1) is true and will prove the property (3). To do so we will show for any target Hamiltonian $H_0$ that our explicit construction for the engineered dissipation $L_E$ (which is independent of $H_0$) and the Hamiltonian $H$ in Eqs. (8) of the main manuscript satisfy the property (2) of the Lemma for all integers $k_1, k_2, k_3 \in \mathbb{N} \times \mathbb{N} \times \{0, ..., c\}$. As a preliminary result, it is helpful to note that recovery projector possesses an exact analytical expression

$$P_{E} \rho = P_C \rho P_C + \sum_{n=1}^{\infty} \sum_{m=0}^{q_{\max}} F_{n,m}^{[E]} \rho F_{n,m}^{[E]} | + \sum_{q=1}^{q_{\max}} F_{E,q}^{[res]} \rho F_{E,q}^{[res]} |$$

within our construction. For some integer $n \geq 0$, let us now define the set

$$\Pi[\sim n] \equiv \text{span}\{E_k^{[m]}|\mu\rangle \langle \nu| E_k^{[m]'}, E_k^{[m]} \in \mathcal{E}^{[m]}, E_k^{[m]'} \in \mathcal{E}^{[m]'}, m + m' \leq 2n, |\mu\rangle, |\nu\rangle \in \mathcal{C}\}.$$ 

First, $\Pi[\sim 0] = \mathcal{C} \otimes_d \mathcal{C}$, and it is simple to prove that given some $\rho \in \Pi[\sim n]$ one has

$$\Pi_{n}\rho \in \Pi[\sim n + 1], \quad \Pi_{E}\rho \in \Pi[\sim n], \quad \mathcal{H}\rho \in \Pi[\sim n].$$

Secondly, one can prove the two additional relations

$$P_{E}\mathcal{L}_E = 0, \quad P_{E}\mathcal{H} = \mathcal{H}_0 P_{E}(\mathcal{I} - P_{res}),$$

where $P_{res} \rho = P_{res} \rho P_{res}$ and $P_{res} = \sum_{q=1}^{q_{\max}} \phi_q \langle \phi_q|$ is the projector onto residual states. While the former identity is a simple consequence of the definition of the recovery projector, the latter is a consequence of the Knill-Laflamme conditions as well as of our explicit expression Eq. (16) for the recovery projector. Finally, for any integer $n \leq c$ and $\rho \in \Pi[\sim n]$ one has

$$(\mathcal{I} - P_{res}) \rho = \rho :$$

In fact, to have $P_{res} \rho \neq 0$, $\rho$ needs to contain errors of a weight superior to $c$ on both its sides and thus has to belong in $\mathcal{H} \otimes_d \mathcal{H}\backslash\Pi[\sim c]$. In the absence of natural dissipation events ($k_3 = 0$), identities Eq. (17-22) yield

$$P_{E} \mathcal{S}[\{\mathcal{H}, k_1\}, \{\mathcal{L}_E, k_2\} \{\mathcal{L}_n, 0\}] P_{C} = \delta_{k_2,0} \mathcal{H}_0^{k_1} P_{E} P_{C} = \delta_{k_2,0} \mathcal{H}_0^{k_1} P_{C},$$

which is the desired result. In the presence of natural dissipation events ($k_3 \neq 0$), the following recursive relation connects $k_3$-th order of natural dissipation error to $(k_3 - 1)$-th order for any $\rho \in \mathcal{C} \otimes_d \mathcal{C}$:

$$P_{E} \mathcal{S}[\{\mathcal{H}, k_1\}, \{\mathcal{L}_E, k_2\}, \{\mathcal{L}_n, k_3\}] \rho = \sum_{q=0}^{k_3} \mathcal{H}_0^{q} P_{C} \mathcal{L}_n \rho^{(q)},$$

where $\rho^{(q)} = \mathcal{S}[\{\mathcal{H}, k_1 - q\}, \{\mathcal{L}_E, k_2\}, \{\mathcal{L}_n, k_3 - 1\}] \rho \in \Pi[\sim k_3 - 1] \subset \Pi[\sim c - 1]$. Finally, exploiting the Knill-Laflamme conditions and our explicit construction one proves below that

$$P_{E} \mathcal{L}_n \rho = 0 \quad \forall \rho \in \Pi[\sim c - 1]$$

which combined to Eq. (24) leads to the property (2) of the Lemma.

We will now complete the proof by demonstrating the validity of the key identity Eq. (25). We will prove this for any matrix of the form $\rho = E_k^{[m]} |\mu\rangle \langle \nu| E_k^{[m]'}$ for any integers $m, m'$ such that $m + m' \leq 2(c - 1), E_k^{[m]}, E_k^{[m]'}$ respectively in $\mathcal{E}^{[m]}$ and $\mathcal{E}^{[m]'}$ and $\alpha, \beta \in \{1, ..., d_C\}$. The desired result will then proceed simply by linearity. Since, $\mathcal{L}_n(\rho) \in \Pi[\sim c]$, then
\( F_{E,q}^{[\text{res}]} (L_\rho) F_{E,q}^{[\text{res}] \dagger} = 0 \) for all \( q = 1, \ldots, q_{\text{max}} \), and one gets

\[
\mathcal{P}_E L_\rho = \mathcal{P}_C (L_\rho) \mathcal{P}_C + \sum_{n=1}^{c} \sum_{i,n} F_{E,i,n}^{[n]} (L_\rho) F_{E,i,n}^{[n] \dagger} = \sum_{a=1}^{d_c} \sum_{b=1}^{c} \sum_{l=1}^{p_n} \sum_{n=0}^{m} (|\mu_a^{[n]} \rangle \langle \mu_b^{[n]} | (L_\rho) |\mu_{a,i,n}^{[m]} \rangle)
\]

\[
= \sum_{a=1}^{d_c} \sum_{b=1}^{c} \sum_{l=1}^{p_n} \sum_{n=0}^{m} |\mu_a^{[n]} \rangle \langle \mu_b | \times \left\{ (|\mu_{a,i,n}^{[m]} | F_l E_k^{[m]} |\mu_a^{[m]} \rangle |\mu_{\beta,l}^{[m]} | F_l^{\dagger} |\mu_b^{[m]} \rangle + (|\mu_{a,i,n}^{[m]} | F_l E_k^{[m]} |\mu_a^{[m]} \rangle |\mu_{\beta,l}^{[m]} | F_l^{\dagger} |\mu_b^{[m]} \rangle) \right\} \}
\]

(26)

Moreover, as a consequence of the Knill-Laflamme conditions, one gets that the orthonormalized error states generated by the Gramm-Schmidt algorithm in the main manuscript satisfy

\[
|\mu_{a,i}^{[m]} \rangle = \sum_{E \in \mathcal{E}^{|m|}} \Gamma_{E,i,n}^{[m]} E |\mu_a \rangle
\]

(27)
as long as \( n \leq c \), where \( \Gamma_{E,i,n}^{[m]} \in \mathbb{C} \) is a constant independent of the code space basis state \( |\mu_a \rangle \) (\( a \in \{1, \ldots, d_c\} \)). From this one deduces further for any \( E \in \mathcal{E}^{c-c} \) and \( n \leq c \) that \( (|\mu_{a,i}^{[m]} | E |\mu_b \rangle = \Omega_{E,i,n}^{[m]} \delta_{a,b} \) with \( \Omega_{E,i,n}^{[m]} = (|\mu_{a,i}^{[m]} | E |\mu_b \rangle \) is independent of \( a \). We now assume that \( m < m' \) (the demonstration in the cases \( m = m' \) and \( m > m' \) is very similar) and thus in particular \( m \leq c - 2 \): thus \( E_k^{[m]} \), \( F_l E_k^{[m]} \) and \( F_l^{\dagger} E_k^{[m]} \) all belong in \( \mathcal{E}^{c-c} \). One can insert the latter result in Eq. (26), yielding:

\[
\mathcal{P}_E L_\rho = \sum_{a=1}^{d_c} \sum_{b=1}^{c} \sum_{l=1}^{p_n} \sum_{n=0}^{m} |\mu_a^{[n]} \rangle \langle \mu_b | \times \left\{ (|\mu_{a,i,n}^{[m]} | F_l E_k^{[m]} |\mu_a^{[m]} \rangle |\mu_{\beta,l}^{[m]} | F_l^{\dagger} |\mu_b^{[m]} \rangle + (|\mu_{a,i,n}^{[m]} | F_l E_k^{[m]} |\mu_a^{[m]} \rangle |\mu_{\beta,l}^{[m]} | F_l^{\dagger} |\mu_b^{[m]} \rangle) \right\}
\]

\[
= \sum_{a=1}^{d_c} \sum_{b=1}^{c} \sum_{l=1}^{p_n} \sum_{n=0}^{m} |\mu_a^{[n]} \rangle \langle \mu_b | \times \left\{ (|\mu_{a,i,n}^{[m]} | F_l E_k^{[m]} |\mu_a^{[m]} \rangle |\mu_{\beta,l}^{[m]} | F_l^{\dagger} |\mu_b^{[m]} \rangle + (|\mu_{a,i,n}^{[m]} | F_l E_k^{[m]} |\mu_a^{[m]} \rangle |\mu_{\beta,l}^{[m]} | F_l^{\dagger} |\mu_b^{[m]} \rangle) \right\}
\]

\[
= \sum_{a=1}^{d_c} \sum_{b=1}^{c} \sum_{l=1}^{p_n} \sum_{n=0}^{m} |\mu_a^{[n]} \rangle \langle \mu_b | \times \left\{ (|\mu_{a,i,n}^{[m]} | F_l E_k^{[m]} |\mu_a^{[m]} \rangle |\mu_{\beta,l}^{[m]} | F_l^{\dagger} |\mu_b^{[m]} \rangle + (|\mu_{a,i,n}^{[m]} | F_l E_k^{[m]} |\mu_a^{[m]} \rangle |\mu_{\beta,l}^{[m]} | F_l^{\dagger} |\mu_b^{[m]} \rangle) \right\}
\]

\[
= \sum_{a=1}^{d_c} \sum_{b=1}^{c} \sum_{l=1}^{p_n} |\mu_a^{[n]} \rangle \langle \mu_b | \times \left\{ (|\mu_{a,i,n}^{[m]} | F_l E_k^{[m]} |\mu_a^{[m]} \rangle |\mu_{\beta,l}^{[m]} | F_l^{\dagger} |\mu_b^{[m]} \rangle + (|\mu_{a,i,n}^{[m]} | F_l E_k^{[m]} |\mu_a^{[m]} \rangle |\mu_{\beta,l}^{[m]} | F_l^{\dagger} |\mu_b^{[m]} \rangle) \right\}
\]

\[
= 0
\]

(28)

where we used the fact that \( \sum_{n=0}^{c} \sum_{i=0}^{p_n} |\mu_{a,i}^{[m]} \rangle \langle \mu_{b,i}^{[m]} | E |\mu_b \rangle = E |\mu_b \rangle \) for any \( E \in \mathcal{E}^{c-c} \).

\( (2) \Rightarrow (1) \)

We now assume (2) and will prove that Knill-Laflamme conditions are satisfied for the error set \( \mathcal{E}^{[c]} \). Thus there exists an engineered dissipation \( L_\mathbb{E} \) performing AutoQEC up to order \( c \). In particular, the statement (2) of the Lemma is satisfied for \( H_0 = H = 0 \), which for \( k_1 = k_2 = 0 \) yields \( \mathcal{P}_E L_\mathbb{E}^{[k]} \mathcal{P}_C = \delta_{E_0,0} \mathcal{P}_C \) for all \( k \in \{1, \ldots, c\} \). We will show from this that \( \mathcal{P}_E \) is a good quantum recovery in the traditional sense of non-autonomous QEC for the error set \( \mathcal{E}^{[c]} \), a result known to be enforce the Knill-Laflamme conditions to be satisfied onto that same error set. First \( L_\mathbb{E} \) being a Lindbladian, the recovery projector \( \mathcal{P}_E = \lim_{t \to +\infty} [L_\mathbb{E}, t] \) is a CPTP map and it admits a Kraus representation: \( \mathcal{P}_E = \sum_{l=1}^{L} R_l \cdot R_l^\dagger \) with \( \sum_{l=1}^{L} R_l^\dagger R_l = 1 \). We then prove inductively over \( k \in \{0, \ldots, c\} \) the following statement:
For all $E \in \mathcal{E}(^{\sim k})$ and $l \in \{1, L\}$, one has $R_l E P_C = \Gamma_{l,E} P_C$ for some constant $\Gamma_{l,E}$.

First, one has $P_C \bullet P_C = P_E P_C = \sum_{l=1}^L R_l P_C \bullet P_C R_l^\dagger$. By unicity of the Kraus operators of quantum channels up to unitaries, we deduce that $\forall l \in \{1, L\}$, exists $\Gamma_{l,E}$ such that $R_l P_C = \Gamma_{l,E} P_C$. Since the identity is the only error in $\mathcal{E}(^{\sim k=0})$ one gets that the desired statement is true for $k = 0$. One now assume that the statement is true up to $k - 1 \leq c - 1$. Expanding natural dissipation to order $k$ yields:

$$
\mathcal{L}_n^k = \sum_{k'=0}^k \left(\frac{1}{2}\right)^{k-k'} \sum_{(d_i) \in \{1,\ldots,N\}^{k'}} \sum_{\sum_n m_i = k-k'} B_{(n_i),(d_i)}
$$

where $B_{(n_i),(d_i)} = \prod_{i=1}^{k'+1} [C^{n_i+m_i}_{n_i} E_{(d_i),(n_i)} E_{(d_i),(m_i)}]^d_i$, $C^n_k$ is the binomial coefficient and $E_{(d_i),(n_i)} \equiv H_{n_{BA}}^{k+k'} \prod_{i=1}^{k'} (F_{d_i} H_{n_{BA}}^{m_i})$. Let us have $|\mu\rangle \in \mathcal{C}$, and $|\nu\rangle \in \mathcal{H}$ such that $|\nu\rangle \perp |\mu\rangle$. Considering the identity $0 = \langle \nu | (P_E \mathcal{L}_n^k (|\mu\rangle |\mu\rangle)) |\nu\rangle$, and applying the inductive hypothesis for $k - 1$ one gets:

$$
0 = \sum_{l=1}^L \sum_{k' = 0}^k \frac{1}{2^{k-k'}} \sum_{(d_i) \subseteq \{1,\ldots,N\}^{k'}} \sum_{\sum_n m_i = k-k'} \left[ \prod_{i=1}^{k'+1} C^{n_i}_{n_i+m_i} \right] \langle \nu | R_l E_{(d_i),(n_i)} P_C |\mu\rangle \langle P_C^\dagger E_{(d_i),(m_i)} R_l^\dagger |\nu\rangle
$$

Note that the scalar products in Eq. (30) only involves $E_{(d_i),(n_i)}$ and $E_{(d_i),(m_i)}$ of identical weight $k$, where the backaction Hamiltonian $H_{n_{BA}}$ has been applied the same amount of times $(k - k')/2$ on both left and right sides, and $k'$ jump operators $F_{d_i}$ have been applied simultaneously on both sides: indeed since $|\mu\rangle \in \mathcal{C}$ and $|\nu\rangle \perp |\mu\rangle$ one can show via the inductive hypothesis that asymmetric terms with unequal weight must vanish (as either $\langle \nu | R_l E_{(d_i),(n_i)} P_C |\mu\rangle$ or $\langle P_C E_{(d_i),(n_i)} R_l^\dagger |\nu\rangle$ necessarily involve an error of weight inferior or equal to $k - 1$). Eq. (30) can be reformulated in the more compact form:

$$
0 = \sum_{l=1}^L \sum_{k' = 0}^k \left(\frac{1}{2}\right)^{k-k'} \sum_{(d_i) \subseteq \{1,\ldots,N\}^{k'}} \langle X_{(d_i),(n_i)} | \mathcal{M}_k | X_{(d_i),(n_i)} \rangle
$$

where $\mathcal{M}_k = \bigotimes_{k' = 0}^{k'+1} \mathcal{M}^{(k')}_{k'}$, $\mathcal{M}^{(k)}_{k'} = \sum_{(d_i) \subseteq \{1,\ldots,N\}^{k'}} (k-k')/2 C^{n_i}_{n_i+m_i} |n_i\rangle \langle m_i|$ is the symmetric Pascal matrix defined over some abstract quantum system $\mathcal{H}_k = \text{span}[|n_i\rangle, 0 \leq n \leq k - k'/2]$ of dimension $(k - k')/2 + 1$ and

$$
|X_{(d_i),(n_i)}\rangle = \sum_{(n_i) \subseteq \{1,\ldots,N\}^{k'+1}} \langle \nu | R_l E_{(d_i),(n_i)} P_C |\mu\rangle \bigotimes_{i=1}^{k'+1} |n_i\rangle
$$

is a vector representing a state in a tensor product of $k' + 1$ copies of such system. One notices that $\mathcal{M}^{(k)}_{k'} = \exp(A)^{\dagger} \exp(A)$, where $A = \sum_{n=1}^{(k-k')/2} n |n-1\rangle \langle n|$, and thus the Pascal matrix is Hermitian positive definite. From this, one concludes that $\langle \nu | R_l E_{(d_i),(n_i)} P_C |\mu\rangle = 0$ for all $l \in \{1,\ldots,L\}$ and $E \in \mathcal{E}^{(k')}$ (indeed all the errors in $\mathcal{E}^{(k)}$ can be written as $E_{(d_i),(n_i)}$ for some sequences $(d_i)_{1 \leq i \leq k'}$ and $(n_i)_{1 \leq i \leq k'+1}$). The previous result being true for all $|\nu\rangle \in \mathcal{H}$ such that $|\mu\rangle \perp |\nu\rangle$, one deduces that

$$
R_l E |\mu\rangle = \gamma_{l,E} |\mu\rangle.
$$

Since Eq. (32) is true for all $|\mu\rangle \in \mathcal{C}$ the proportionality constant $\gamma_{l,E}$ needs to be independent of the specific code state $|\mu\rangle$. One concludes that there exists $\Gamma_{l,E}$ such that $R_l E P_C = \Gamma_{l,E} P_C$. Thus one has proved that the desired inductive statement is also true for $k$, and thus is true up to $k = c$. As a conclusion one deduces for all $E, E' \in \mathcal{E}(^{\sim c})$ that $P_C E^t P_C = P_C E'^t (\sum_{l=1}^L R_l^\dagger R_l) E = \Gamma_{E',\mathcal{E}} P_C$, where $\Gamma_{E',\mathcal{E}} = \sum_{l} \Gamma_{l,E} \Gamma_{l,E}$ which is precisely the Knill-Laflamme condition. \hfill $\square$

**PROOF LEMMA**

In this supplementary section we present the proof of the Lemma of the main manuscript, which is formulated as follows.

**Lemma:** Properties of generalized error-transparent Hamiltonians and engineered dissipation. Let us have some target Hamiltonian $H_0 = P_C H_0 P_C$, some engineered dissipation $\mathcal{L}_E$ and generic Hamiltonian $H$. We denote $\mathcal{H}$ (resp. $\mathcal{H}_0$) as the superoperator associated with the Hamiltonian evolution $H(\rho) = [H, \rho]$ (resp. $H_0(\rho) = [H_0, \rho]$). The following statements are equivalent:

1. For all $E \in \mathcal{E}(^{\sim k})$ and $l \in \{1, L\}$, one has $R_l E P_C = \Gamma_{l,E} P_C$ for some constant $\Gamma_{l,E}$.
2. $R_l E = \Gamma_{l,E} P_C$ for all $E \in \mathcal{E}(^{\sim k})$.
3. $R_l E P_C = \Gamma_{l,E} P_C$ for all $E \in \mathcal{E}(^{\sim k})$.
4. $R_l E = \Gamma_{l,E} P_C$ for all $E \in \mathcal{E}(^{\sim k})$.
(1) \( \mathcal{L}_E \) and \( H \) perform an autonomous error-corrected quantum computation up to order \( c \) with respect to the code space \( \mathcal{C} \), the target logical Hamiltonian \( H_0 \) and natural dissipation \( \mathcal{L}_n \).

(2) for all sets of integers \( k_1, k_2, k_3 \in \mathbb{N} \times \mathbb{N} \times \{0, ..., c\} \) one has
\[
\mathcal{P}_E \mathcal{S}([\mathcal{H}, k_1] \cup \mathcal{L}_E, k_2, \mathcal{L}_n, k_3) \mathcal{P}_C = \delta_{k_2,0} \delta_{k_3,0} H_0^{k_1} \mathcal{P}_C
\]
\[
(1) \Rightarrow (2)
\]

We now assume that (1) is true and thus \( \mathcal{L}_E \) and \( H \) perform an autonomous error-corrected quantum computation up to order \( c \) wrt the Hamiltonian \( H_0 \). Let us have \( M > 0 \) and \( g_0 > 0 \) the positive constants involved in the definition of the latter property (see main manuscript). Let us have some dimensionless numbers \( u, \tilde{g} > 0 \) such that \( |\tilde{g}| < g_0 \). For all \( R > 0 \), we will consider the time \( t = u/ (R \kappa) \) and the Hamiltonian coupling strength \( g = R \tilde{g} \). For all \( \rho(0) \) in the code space and for all \( R > 0 \) one has
\[
\| \mathcal{P}_E \exp([\mathcal{L}_E + \mathcal{L}_n/R - i\tilde{g}H]u) \rho(0) - \exp[-i\tilde{g}H_0u] \rho(0) \| = \| \mathcal{P}_E \rho(t) - \exp[-igH_0t] \rho(0) \| \leq M \frac{u}{R^{c+1}} \| \rho(0) \|
\]
which in the limit \( R \to +\infty \) gives:
\[
\| \mathcal{P}_E \exp([\mathcal{L}_E - i\tilde{g}H]u) \rho(0) - \exp[-i\tilde{g}H_0u] \rho(0) \| \leq 0
\]
and thus \( \mathcal{P}_E \exp([\mathcal{L}_E - i\tilde{g}H]u) \rho(0) = \exp[-i\tilde{g}H_0u] \rho(0) \). This property holds for all initial conditions in the code space, as well as all dimensionless \( u \) and \( \tilde{g} \) such that \( |\tilde{g}| \leq g_0 \) enabling thus an identification between the left and right handsides of all terms with identical powers in \( g \) and \( \kappa \), which according to the relations Eqs. (14-15) on symmetrized products yields
\[
\mathcal{P}_E \mathcal{S}([\mathcal{H}, k_1] \cup \mathcal{L}_E, k_2, \mathcal{L}_n, k_3 = 0) \mathcal{P}_C = \mathcal{P}_E \mathcal{S}([\mathcal{H}, k_1] \cup \mathcal{L}_E, k_2) \mathcal{P}_C = \delta_{k_2,0} H_0^{k_1} \mathcal{P}_C
\]
for all \( k_1, k_2 \in \mathbb{N} \times \mathbb{N} \) which is the desired result for \( k_3 = 0 \). Reasoning by reductio ad absurdum, we now will make the hypothesis that condition (2) of the Lemma is not true and will reach ultimately a logical contradiction. Therefore, there exists \( k_1, k_2, k_3 \in \mathbb{N} \times \mathbb{N} \times \{0, ..., c\} \) such that
\[
\mathcal{P}_E \mathcal{S}([\mathcal{H}, k_1] \cup \mathcal{L}_E, k_2, \mathcal{L}_n, k_3) \mathcal{P}_C \neq \delta_{k_2,0} \delta_{k_3,0} H_0^{k_1} \mathcal{P}_C
\]
Due to Eq. (36), this implies \( k_3 > 0 \). One defines \( k_3^M \leq c \) as the minimum of the non-empty set:
\[
\mathcal{A} = \{ k_3 \in \{1, ..., c\}, \exists (k_1, k_2) \in \mathbb{N}^2 : \mathcal{S}([\mathcal{H}, k_1] \cup \mathcal{L}_E, k_2, \mathcal{L}_n, k_3) \mathcal{P}_C \neq 0 \}.
\]
Based on Eq. (36), one gets by expanding \( \exp([\mathcal{L}_E + \mathcal{L}_n/R - i\tilde{g}H]u) \) in symmetrized products
\[
\mathcal{P}_E \rho(t) = \exp[-i\tilde{g}H_0 t] \rho(0) + \left( \frac{u}{R} \right)^{k_3^M} \sum_{k_1, k_2} \mathcal{P}_E \mathcal{S}([\mathcal{H}, k_1] \cup \mathcal{L}_E, k_2, \mathcal{L}_n, k_3^M) \rho(0) (-i\tilde{g})^{k_1} u_{k_1+k_2} \]
\[
\left( \frac{u}{R} \right)^{k_3^M+1} \sum_{k_1, k_2, k_3} \mathcal{P}_E \mathcal{S}([\mathcal{H}, k_1] \cup \mathcal{L}_E, k_2, \mathcal{L}_n, k_3^M+1 + k_3) \rho(0) (-i\tilde{g})^{k_1} u_{k_1+k_2+k_3} \left( \frac{1}{R} \right)^{k_3}
\]
Given that \( k_3^M \in \mathcal{A} \) there exists \( u_1, \tilde{g}_1 > 0 \) with \( \tilde{g}_1 < g_0 \) such that \( \Delta = \sum_{k_1, k_2} \mathcal{P}_E \mathcal{S}([\mathcal{H}, k_1] \cup \mathcal{L}_E, k_2, \mathcal{L}_n, k_3^M) \rho(0) (-i\tilde{g}_1)^{k_1} u_{k_1+k_2} \neq 0 \). Thus, \( \forall R > 0 \), defining \( t = u_1/(R \kappa) \) and \( g = R \tilde{g}_1 < g_0 R \), one has:
\[
\mathcal{P}_E \rho(t) - \exp[-i\tilde{g}H_0 t] \rho(0) = \left( \frac{u_1}{R} \right)^{k_3^M} \Delta + \left( \frac{u_1}{R} \right)^{k_3^M+1} \Delta(R)
\]
\[
= \frac{\kappa t}{R^{k_3^M-1}} u_1^{k_3^M-1} \Delta + \frac{nt}{R^{k_3^M+2}} u_1^{k_3^M} \Delta(R)
\]
with
\[
\Delta = \sum_{k_1, k_2} \mathcal{P}_E \mathcal{S}([\mathcal{H}, k_1] \cup \mathcal{L}_E, k_2, \mathcal{L}_n, k_3^M) \rho(0) (-i\tilde{g}_1)^{k_1} u_{k_1+k_2} \neq 0
\]
\[
\Delta(R) = \sum_{k_1, k_2, k_3} \mathcal{P}_E \mathcal{S}([\mathcal{H}, k_1] \cup \mathcal{L}_E, k_2, \mathcal{L}_n, k_3^M+1 + k_3) \rho(0) (-i\tilde{g}_1)^{k_1} u_{k_1+k_2+k_3} \left( \frac{1}{R} \right)^{k_3}
\]
Let us have some $R_0 > 0$. A careful analysis shows that $\tilde{\Delta}(R)$ is necessarily bounded in the domain $[R_0, +\infty]$. One thus finds that the projected density matrix verifies the asymptotic behavior:

$$\mathcal{P}_E \rho(t = u_1/(R\kappa)) - \rho(0) \sim \frac{\kappa t}{Rk_1^M - 1} u_1^{k-1} \Delta$$

(44)

Since $k_1^M - 1 < c$, one concludes that for all $M > 0$ there exists a specific choice of time $t = u_1/(R\kappa)$ and couplings $g = R\tilde{g}_1$, $R \geq 0$ satisfying $\|\mathcal{P}_E \rho(t) - \exp[-i\mathcal{H}_0 t] \rho(0)\| > M\kappa t/R^c$, which is in logical contradiction with our initial hypothesis. One concludes that $\mathcal{S}(\{\mathcal{H}, k_1\}, \{\mathcal{L}_E, k_2\}, \{\mathcal{L}_n, k_3\}) \mathcal{P}_C = \delta_{k_2,0} \delta_{k_1,0} \mathcal{H}_0^0 \mathcal{P}_C$.

(2) \Rightarrow (1)

We will make use of the following identity

$$e^{(A+B)t} = e^{At} + \int_0^t ds e^{(A+B)(t-s)} Be^{As}. \quad (45)$$

Given some integer $N > 1$ integer, this can be recursively generalized as:

$$e^{(A+B)t} = e^{At} + \sum_{k=1}^{N-1} \int \left( \prod_{i=1}^k \frac{d\tau_i}{\tau_i \leq t} \right) e^{A(t-\sum_i \tau_i)} \prod_{i=1}^k \left( B e^{A \tau_i} \right) + \int \left( \prod_{i=1}^k \frac{d\tau_i}{\tau_i \leq t} \right) e^{[A+B](t-\sum_i \tau_i)} \prod_{i=1}^k \left( B e^{A \tau_i} \right). \quad (46)$$

We denote $\mathcal{G}(t) = e^{\kappa(R\mathcal{L}_E + \mathcal{L}_n - ig\mathcal{H})t}$, $\mathcal{G}_1(t) = e^{\kappa(R\mathcal{L}_E - ig\mathcal{H})t}$ and $\mathcal{G}_0(t) = e^{\kappa\mathcal{L}_E t}$. Using identity Eq. (46) we get:

$$\mathcal{P}_E \rho(t) = \mathcal{P}_E \mathcal{G}(t) \rho(0)$$

$$= \mathcal{P}_E \mathcal{G}_1(t) \rho(0) + \sum_{k=1}^c \kappa^k \int \left( \prod_{i=1}^k \frac{d\tau_i}{\tau_i \leq t} \right) \mathcal{P}_E \mathcal{G}_1 \left( t - \sum_i \tau_i \right) \prod_{i=1}^k \left( \mathcal{L}_n \mathcal{G}_1(\tau_i) \right) \rho(0) + \kappa^{c+1} \int \left( \prod_{i=1}^{c+1} \frac{d\tau_i}{\tau_i \leq t} \right) \mathcal{P}_E \mathcal{G} \left( t - \sum_i \tau_i \right) \prod_{i=1}^{c+1} \left( \mathcal{L}_n \mathcal{G}_1(\tau_i) \right) \rho(0). \quad (47)$$

Having assumed the hypothesis (2) of the lemma to be satisfied, from Eq. (14) we deduce $\mathcal{P}_E \mathcal{G}_1(t) \mathcal{P}_C = e^{-i\mathcal{H}_0 t} \mathcal{P}_C$. Then, expanding $\mathcal{G}_1(t)$ in powers of the Hamiltonian superoperator $\mathcal{H}$ as following

$$\mathcal{G}_1(t) = \mathcal{G}_0(t) + \sum_{q=1}^{+\infty} \frac{(-ik)^q}{q!} \int \left( \prod_{i=1}^q \frac{d\tau_i}{\tau_i \leq t} \right) \mathcal{G}_0 \left( t - \sum_i \tau_i \right) \prod_{i=1}^q \left( \mathcal{H} \mathcal{G}_0(\tau_i) \right) \quad (48)$$

we get

$$\mathcal{P}_E \rho(t) - e^{-i\mathcal{H}_0 t} \rho(0) = \sum_{k=1}^c \sum_{q=0}^{+\infty} \kappa^k \frac{(-ik)^q}{q!} \int \left( \prod_{i=1}^k \frac{d\tau_i}{\tau_i \leq t} \right) \mathcal{P}_E \mathcal{G}_0 \left( t - \sum_i \tau_i \right) \prod_{i=1}^k \left( \mathcal{A}_i \mathcal{G}_0(\tau_i) \right) \rho(0)$$

$$+ \kappa^{c+1} \sum_{q=0}^{+\infty} \frac{(-ik)^q}{q!} \int \left( \prod_{i=1}^{c+1} \frac{d\tau_i}{\tau_i \leq t} \right) \mathcal{P}_E \mathcal{G} \left( t - \sum_i \tau_i \right) \mathcal{L}_n \mathcal{G}_0(\tau_{c+1}) \prod_{i=1}^{c+1} \left( \mathcal{A}_i \mathcal{G}_0(\tau_i) \right) \rho(0), \quad (49)$$

where we defined the set

$$\mathcal{O}_{k,q} = \{(A_j)_{1 \leq j \leq k+q} | \text{with } A_j \in \{\mathcal{H}, \mathcal{L}_n\} \text{ and } A_j = \mathcal{L}_n \text{ for exactly } k \text{ values of } j \in \{1, ..., k+q\}\}.$$  \quad (50)
Moreover, one remarks that
\[
\sum_{q=0}^{+\infty} (-i\kappa g)^q \sum_{(\mathcal{A}_i) \in \mathcal{O}_{k,q}} \int \left( \prod_{i=1}^{k+q} d\tau_i \right) G_0 \left( t - \sum_i \tau_i \right) \prod_{i=1}^{k+q} (A_i G_0(\tau_i)) = \\
\sum_{q=0}^{+\infty} (-i\kappa g)^q \sum_{(\mathcal{A}_i) \in \mathcal{O}_{k-1,q}} \int \left( \prod_{i=1}^{k+q} d\tau_i \right) G_0 \left( t - \sum_i \tau_i \right) \prod_{i=1}^{k+q-1} (A_i G_0(\tau_{i+1})) L_n G_1(\tau_1).
\] (51)

This is obtained by resummating all the Hamiltonian terms in the left hand site of the previous identity (starting from the first term on the right hand side of the product) for each \((\mathcal{A}_i)\) until the first index \(i_\mathcal{A}\) such that \(A_{i_\mathcal{A}} = L_n\). This leads to the following expansion for the full projected propagator:
\[
\mathcal{P}_E \rho(t) - e^{-i\mathcal{H} \rho(t)} = \sum_{k=1}^{c} \kappa^k \sum_{q=0}^{+\infty} (-i\kappa g)^q \mathcal{O}_{k-1,q} \left( \mathcal{P}_E - i\kappa g \right) \left( I - \mathcal{P}_E \right) \mathcal{E}_n G_n(\tau_{c+q+1}) \mathcal{L}_n G_0(\tau_1) \rho(0),
\]
(52)

The first line in the right hand side of this expansion is actually vanishing. Indeed, by identification between the expansion of \(G(t)\) in powers of \(\mathcal{L}_n\) and \(\mathcal{H}\) in both the Schrödinger and interaction pictures we can show that
\[
\sum_{k=1}^{c} \kappa^k \sum_{q=0}^{+\infty} (-i\kappa g)^q \mathcal{O}_{k-1,q} \left( \mathcal{P}_E - i\kappa g \right) \left( I - \mathcal{P}_E \right) \mathcal{E}_n G_n(\tau_{c+q+1}) \mathcal{L}_n G_0(\tau_1) \rho(0) = 0.
\] (54)

for all integers \(k, q\). Combining this identification with Lemma hypothesis (2) and the identity Eq. (51) one gets
\[
\sum_{q=0}^{+\infty} (-i\kappa g)^q \sum_{(\mathcal{A}_i) \in \mathcal{O}_{k-1,q}} \int \left( \prod_{i=1}^{k+q} d\tau_i \right) G_0 \left( t - \sum_i \tau_i \right) \prod_{i=1}^{k+q} (A_i G_0(\tau_i)) = 0.
\] (56)

The relation of Eq. (55) can be obtained recursively using Eq. (54) by inserting the identity \(I = \mathcal{P}_E + \mathcal{Q}_E\) between each \(\mathcal{L}_n\) or \(\mathcal{A}_i\) and each \(G_0(\tau_i)\) in the left hand side of Eq. (55). This leads to
\[
\mathcal{P}_E \rho(t) - e^{-i\mathcal{H} \rho(t)} = \kappa^{c+1} \sum_{q=0}^{+\infty} (-i\kappa g)^q \sum_{(\mathcal{A}_i) \in \mathcal{O}_{k,q}} \int \left( \prod_{i=1}^{k+q} d\tau_i \right) G_0 \left( t - \sum_i \tau_i \right) \prod_{i=1}^{k+q} (A_i G_0(\tau_i)) L_n G_1(\tau_1) \rho(0).
\] (57)
By resummation of the previous expansion in powers of the projected Hamiltonian superoperator $Q_E H Q_E$ the last identity can be rewritten as:

$$P_E \rho(t) - e^{-iH_{\text{act}}t} \rho(0) = \kappa^{c+1} \int \left( \prod_{i=1}^{c+1} dt_i \right) P_E G \left( t - \sum \tau_i \right) L_n G_1^{Q_E}(\tau_{c+1}) Q_E \prod_{i=2}^{c} \left( L_n G_1^{Q_E}(\tau_i) Q_E \right) L_n G_1(\tau_1) \rho(0). \tag{58}$$

with $G_1^{Q_E}(\tau) = \exp[(\kappa R_L - i\kappa gQ_E H Q_E)t]$ (we used the fact that $L_E = Q_E L_E = L_E Q_E$). To prove the desired bound, we will use the following property:

Exist $g_0, N, \lambda_0 > 0$, such that for all $R > 0$ and for all $\kappa g \in \mathbb{R}$ such that $|\kappa g| < g_0 R \kappa$ one has

$$\left\| G_1^{Q_E}(t) Q_E \right\| \leq M e^{-R \kappa \lambda_0 t} \tag{59}$$

The bound Eq. (59) can be proved by expanding this $G_1^{Q_E}(t) Q_E$ in powers of $Q_E H Q_E$:

$$G_1^{Q_E}(t) Q_E = G_0(t) Q_E + \sum_{k=1}^{+\infty} \int \left( \prod_{i=1}^{k} dt_i \right) G_0(t - \sum \tau_i) Q_E \prod_{i=1}^{k} \left( \kappa g Q_E H Q_E G_0(\tau_i) \right) Q_E. \tag{60}$$

Since $Q_E$ projects on relaxation eigenmodes of $L_E$, we deduce that there exists $\tilde{M} > 0$ such that $G_0(\tau_i) Q_E \leq \tilde{M} e^{-R \kappa \lambda_0 t}$ for all $R, \kappa, t > 0$, $0 < \alpha \leq 1$ and $\lambda_0 = -\max[\text{Re}(\text{Sp}(L_E)) \setminus \{0\}] > 0$ is the real part of the eigenvalue corresponding to the slowest relaxation eigenmode of $L_E$. From this we get:

$$\left\| G_1^{Q_E}(t) Q_E \right\| \leq \tilde{M} e^{-R \kappa \lambda_0 t} + \sum_{k=0}^{+\infty} \int \left( \prod_{i=1}^{k} dt_i \right) \tilde{M}^{k+1} e^{-R \kappa \lambda_0 t} \left( |\kappa g||Q_E H Q_E| \right)^k \tag{61}$$

$$= \tilde{M} e^{-R \kappa \lambda_0 t} \sum_{k=0}^{+\infty} \frac{(\tilde{M}|\kappa g||Q_E H Q_E|t)^k}{k!} \tag{62}$$

$$= \tilde{M} e^{-\left(R \kappa \lambda_0 - \tilde{M}|\kappa g||Q_E H Q_E|t\right)} \tag{63}$$

where we used the identity

$$\int \left( \prod_{i=1}^{k} dt_i \right) 1 = t^k/k!$$

for the integral over regular triangular domain of dimension $k$. We see that for $|\kappa g| \leq g_0 R \kappa$ with $g_0 = \alpha \lambda_0/(2 \tilde{M}|Q_E H Q_E|)$ we get the expected bound with $M = \tilde{M}$ and $\lambda_0 = \alpha \lambda_0/2$.

Inserting the bound Eq. (59) in the identity Eq. (58) and using the fact that the first propagator $G_1(\tau_1)$ in the expansion of Eq. (58) is a CPTP map and thus is bounded by a time-independent constant we find that there exists $M > 0$ such that the following bound is verified for all $\rho(0) \in \mathbb{C}$, for all $\kappa, R, t \geq 0$ and $g \in \mathbb{R}$ such that $|\kappa g| < g_0 R \kappa$:

$$\left\| P_E \rho(t) - e^{-i\kappa g H_{\text{act}}t} \rho(0) \right\| \leq \frac{N \kappa t}{R} \left\| \rho(0) \right\| \tag{64}$$

which is the desired result.

**SOME GENERALIZED ERROR-TRANSPARENT HAMILTONIANS**

In Eq. (8) of the main manuscript we see that our specific explicit construction for the Hamiltonian $H$ satisfies both conditions $H E P_C = E H_0 P_C$ and $[H, E] P_C = 0$, for all the error operators $E \in \mathcal{E}^{\text{act}}$ of our error set, coinciding thus exactly with the most standard definitions of the error-transparent Hamiltonian [1, 2]. However it is possible to show that the condition (2) of the Lemma admits more solutions and that not all possible generalized Hamiltonians can be cast in this form. We outline here two constructions (without going through the derivation) going beyond the framework of the standard error-transparent Hamiltonian.
A generalized ETH which does not preserve the error syndrome or commute with the errors. Standard error-transparent gates have the property that a natural error happening before the gate operation produces the same final state as the error happening immediately after the gate. Intuitively one expects however a gate generated by an Hamiltonian \( H \) producing various final states (depending on when the natural dissipation error happened) with the same logical content but differing errors syndromes to be still functional as long as the various generated errors have the same weight: under those conditions the robustness of the code is not hindered by the Hamiltonian and an autonomously error-corrected computation of order \( c \) should still be achievable.

In that perspective, a simple example of viable modification to \( H \) would be to add to the Hamiltonian in Eq. (8) of the main manuscript an extra contribution of the form \( \Delta H = \sum_{n=1}^{c} \sum_{i=1}^{c} E_{E,n,j,i}^{n} |\hat{\mu}_{k,i,n}^{n}\rangle\langle \hat{\mu}_{k,i,n}^{n}| + h.c. \) coupling states corresponding to different error syndromes, but with identical error weight and underlying logical word, via a coupling strength \( E_{E,n,j,i}^{n} \) independent of the code word. With respect to our explicit construction \( \mathcal{L}_{E} \) does need to be modified. Since \( \Delta H \) does not discriminate between the various code states and their associated correctable error-states \( (E_{E,n,j,i}^{n}) \) does not depend on the code word, such correction to \( H \) does not impact at all the logical computation. Moreover since \( H \) does not affect the error weight, it does not fragilize further the code against natural dissipation error, and the protection up to order \( c \) is preserved.

The fact that the modified Hamiltornian \( H \) does not preserves the error syndrome given by the indices \( n, i, n \) of the states \( |\hat{\mu}_{k,i,n}^{n}\rangle \) is connected to a non-commutativity with the error model: to understand this, let us consider a toy model composed of a 2-dimensional code embedded in a 6-dimensional Hilbert space. A basis of this Hilbert space is given by two code space states \( |\mu_{1}\rangle, |\nu_{1}\rangle \), and 4 error states \( |\mu_{2}\rangle, |\nu_{2}\rangle \). We consider the natural dissipation model corresponding to the set of jumps \( \{ F_{a}, a = 1, 2 \} \) with \( F_{a} = |\mu_{a}\rangle\langle \mu_{a}| + |\nu_{a}\rangle\langle \nu_{a}| + |\mu_{a}\rangle\langle \nu_{a}| + |\nu_{a}\rangle\langle \mu_{a}| \). It is simple to show that the set \( \mathcal{E} \{ 1 \} = \{ 1, F_{1}, F_{2} \} \) satisfies the Knill-Laflame conditions and thus this model is suitable for 1st-order AutoQEC.

Moreover, as it is introduced, the above Hilbert space basis is already a naturally Gram-Schmidt orthonomalized error state basis which can be used for the construction of our engineered dissipation and generalized ETH in Eqs. (7-8) of the main manuscript. Let us now consider the following error-syndrome modifying additive contribution \( \Delta H = \mu_{1}\rangle\langle \mu_{2}| + |\nu_{1}\rangle\langle \nu_{2}| + h.c \) to the generalized ETH \( H \) in Eq. (8) of the main manuscript. One finds that it does not satisfy the commutation relation of a standard ETH with the error set \( \Delta H F_{1}, F_{2} = F_{1}\Delta H P_{C} \neq F_{1}F_{2}P_{C} \). This is directly related to an exchange of the error syndromas \( a = 1, 2 \) upon application of \( \Delta H \).

A generalized ETH which does not preserve the code space: Interestingly, \( H \) does not even have to preserve the code space. Indeed, let us hypothetically extend the Hilbert space by adding a copy \( \tilde{\mathcal{H}} \) of \( \mathcal{H} \). The copy space \( \tilde{\mathcal{H}} \) consists of a copy \( \tilde{\mathcal{C}} = \text{span}\{ |\tilde{\mu}_{i}\rangle \} \) of the code space \( \mathcal{C} \), in direct sum with a copy of the space of correctable states (\( \text{span}\{ |\tilde{\mu}_{k,i,n}\rangle \} \) and a copy of the space of residual states (\( \text{span}\{ |\tilde{\phi}_{q}\rangle \} \)). We assume that \( \tilde{\mathcal{H}} \) is free of any natural dissipation and that \( \mathcal{H} \) and \( \tilde{\mathcal{H}} \) do not couple via natural dissipation. It is then possible to show that the following engineered dissipation and generalized ETH are suitable for error-corrected quantum computations of order \( c \):

\[
\mathcal{L}_{E} = \sum_{n=1}^{c} \sum_{i=1}^{c} D[F_{E,i,n}^{n}] + \sum_{q=1}^{q_{\text{max}}} D[F_{E,q}^{\text{res}}] + \sum_{n=1}^{c} \sum_{i=1}^{c} D[F_{E,q}^{\text{res}}] + \sum_{q=1}^{q_{\text{max}}} D[F_{E,q}^{\text{res}}] \tag{65}
\]

\[
H = \sum_{j,k=1}^{d_{C}} \sum_{n=0}^{c} \sum_{i=1}^{c} \langle \mu_{j}| H_{0} | \mu_{k}\rangle |\mu_{j,i,n}^{n}\rangle |\mu_{k,i,n}^{n}\rangle + \sum_{j,k=1}^{d_{C}} \sum_{n=0}^{c} \sum_{i=1}^{c} \langle \mu_{j}| H_{0} | \mu_{k}\rangle |\mu_{j,i,n}^{n}\rangle |\tilde{\mu}_{j,i,n}^{n}\rangle |\tilde{\mu}_{k,i,n}^{n}\rangle + \sum_{c=0}^{d_{C}} \sum_{k=0}^{c} \sum_{n=0}^{c} \sum_{i=1}^{c} \langle \tilde{\phi}_{q}| H_{0} | \tilde{\mu}_{k,i,n}^{n}\rangle |\tilde{\mu}_{k,i,n}^{n}\rangle + h.c + \sum_{q=1}^{q_{\text{max}}} \langle \tilde{\phi}_{q}| \tilde{\phi}_{q}\rangle + h.c \tag{66}
\]

From the second line in Eq. (66) we see that those contributions to \( H \) completely swap the Hilbert spaces \( \mathcal{H} \) and \( \tilde{\mathcal{H}} \): in particular the code space \( \tilde{\mathcal{C}} \) is not stable under application of \( H \) as it couples to \( \tilde{\mathcal{C}} \) and one finds \( H P_{C} \neq H_{0} P_{C} \). The second contribution in the first line Eq. (66) ensures that the computation is still ‘error-transparent’ in the copy space \( \tilde{\mathcal{H}} \). To ensure that the probabilibty converges back to the orginal Hilbert space and eventually the code space, the engineered dissipation is completed with respect to our original construction by an extra series of jumps of the form \( \tilde{F}_{E,i,n}^{n} = \sum_{j=1}^{d_{C}} |\mu_{j,i,n}^{n}\rangle |\tilde{\mu}_{j,i,n}^{n}\rangle \) and \( F_{E,q}^{\text{res}} = |\tilde{\phi}_{q}\rangle |\tilde{\phi}_{q}\rangle \).

Physically speaking, a copy of such Hilbert space and the associated swapping induced by the Hamiltonian \( H \) is straightforwardly obtained by adding a two-level and an extra ancilla, which Rabi oscillates between its fundamental and excited states via a \( c \) contribution in \( H \). Such generalized ETH associated with the additional ancilla is related to subsystem
code, with the extra ancilla plays the role of gauge degrees of freedom [34]. While this would require further verification, our intuition is that a single extra engineered jump operator of the form \( \tilde{\mathcal{E}}_{E} = \sigma_{-} \) for the extra ancilla might as well work instead of having an extra jump \( \tilde{F}_{E,i}^{[n]} \) and \( \tilde{F}_{E,q}^{[\text{res}]} \) for each error syndrome and residual state.

**EFFECTIVE LOGICAL DECOHERENCE DYNAMICS**

Let us have some engineered dissipation \( L_{E} \) and an Hamiltonian \( H \) performing an error-corrected autonomous quantum computation up to order \( c \) wrt the code space \( \mathcal{C} \) and the natural dissipation \( \mathcal{L}_{n} \). In particular, due to the Lemma, the recovery projector satisfies \( \mathcal{P}_{E}\mathcal{P}_{C} = \mathcal{P}_{C} \). As stated in the main manuscript we also assume \( \mathcal{P}_{C}\mathcal{P}_{E} = \mathcal{P}_{E} \).

**Projected density matrix**

\( \mathcal{P}_{E} \) being a projector one can apply Nakajima-Zwanzig projection operator techniques [33]: assuming an initial condition \( \rho(0) \) in the code space (i.e. satisfying \( \mathcal{P}_{E}\rho(0) = \mathcal{P}_{E} \)), the recovered density matrix \( \mathcal{P}_{E}\rho(t) \) follows the exact non-Markovian dynamics

\[
\partial_{t}\mathcal{P}_{E}\rho = \mathcal{P}_{E}\mathcal{L}_{E}\mathcal{P}_{E} + \int_{0}^{t} d\tau \Sigma(\tau)\mathcal{P}_{E}\rho(t - \tau). \tag{67}
\]

where the memory kernel is defined as

\[
\Sigma(\tau) \equiv \mathcal{P}_{E}\mathcal{L}_{E}\mathcal{Q}_{E}\exp[\mathcal{Q}_{E}\mathcal{L}_{E}\mathcal{Q}_{E}\tau]\mathcal{L}_{E}\mathcal{P}_{E} \tag{68}
\]

and represents the sum of all processes leaving the projective space associated to \( \mathcal{P}_{E} \) (in our case the code space) evolving for a time \( \tau \) in the complementary space associated to the projector \( \mathcal{Q}_{E} \) (here the relaxation modes of engineered dissipation) and coming back finally in the projective space. Importantly as a consequence of our hypothesis \( \mathcal{P}_{C}\mathcal{P}_{E} = \mathcal{P}_{E} \), the memory kernel can be rewritten as

\[
\Sigma(\tau) \equiv \mathcal{P}_{E}\mathcal{L}_{E}\mathcal{Q}_{E}\exp[\mathcal{Q}_{E}\mathcal{L}_{E}\mathcal{Q}_{E}\tau]\mathcal{L}_{E}\mathcal{P}_{C}\mathcal{P}_{E}. \tag{69}
\]

The insertion of a code space projector \( \mathcal{P}_{C} \) on the right hand side plays a key role in our derivation, as it can then be shown that expanding \( \Sigma(\tau) \) in function of the various processes \( \mathcal{L}_{n}, \mathcal{L}_{E} \) and \( \mathcal{H} \) will lead to the cancellation of all \( c + 1 \) lowest-order contributions in powers of \( \mathcal{L}_{n} \) via the property (2) of the Lemma. Remarking that \( \mathcal{P}_{E}\mathcal{L}_{E}\mathcal{P}_{E} = \mathcal{P}_{E}\mathcal{L}_{C}\mathcal{P}_{E} = -i g_{0}H_{0}\mathcal{P}_{E} \), the master equation Eq. (67) can be rewritten as:

\[
\partial_{t}\mathcal{P}_{E}\rho = -i g_{0}\mathcal{H}_{0} + \mathcal{L}_{\text{eff}}\mathcal{P}_{E}\rho(t) + [\partial_{t}\mathcal{P}_{E}\rho]_{\text{nloc}} + [\partial_{t}\mathcal{P}_{E}\rho]_{\text{kink}} \tag{70}
\]

where the effective Liouvillian is given by

\[
\mathcal{L}_{\text{eff}} = \int_{0}^{+\infty} d\tau \left[ \Sigma(\tau)e^{i g_{0}\mathcal{H}_{0}\tau} \right] \mathcal{P}_{C}. \tag{71}
\]

The corrections

\[
[\partial_{t}\mathcal{P}_{E}\rho]_{\text{nloc}} = \int_{0}^{t} d\tau \Sigma(\tau)\left[ \mathcal{P}_{E}\rho(t - \tau) - e^{i g_{0}\mathcal{H}_{0}\tau}\mathcal{P}_{E}\rho(t) \right] \tag{72}
\]

\[
[\partial_{t}\mathcal{P}_{E}\rho]_{\text{kink}} = \int_{t}^{+\infty} d\tau \Sigma(\tau)e^{i g_{0}\mathcal{H}_{0}\tau}\mathcal{P}_{E}\rho(t). \tag{73}
\]

correspond to memory-related non-local effects and the initial kink of the dynamics.

Let us now have \( \tilde{g}_{0} > 0 \) so that Eq. (59) in the supplementary is satisfied for all \( R > 0 \) and \( |g| \leq \tilde{g}_{0}R \). On one hand, the memory kernel \( \Sigma(\tau) \) is expected to decay exponentially to zero over the short time scale \( \sim 1/(R\kappa) \) for a large enough engineered dissipation strength \( R \) and a controlled Hamiltonian coupling \( |g| \leq \tilde{g}_{0}R \). This is because \( \Sigma(\tau) \) contains as only time-dependent factor the term \( \exp[\mathcal{Q}_{E}\mathcal{L}_{E}\mathcal{Q}_{E}\tau]\mathcal{L}_{E} \); after expanding such term in powers of \( \mathcal{L}_{n} \), it is easy to prove this result by using the fact that \( \mathcal{G}_{1}^{(E)}(\tau)\mathcal{Q}_{E} = \exp[(\kappa R\mathcal{C}_{E} - i\kappa g\mathcal{Q}_{E}\mathcal{H}\mathcal{Q}_{E})\tau]\mathcal{Q}_{E} \) also decays over the time scale \( \sim 1/(R\kappa) \) as a consequence of Eq. (59).

On the other hand, since \( \mathcal{H} \) and \( \mathcal{L}_{E} \) perform autonomous error-corrected quantum computation up to order \( c \), one expects that beyond a simple time-dependent and deterministic unitary rotation \( U_{0}(t) = \exp[-i g_{0}H_{0}t] \) that the relaxation dynamics of...
$P_E \rho(t)$ should be very slow (occurring over the time scale $t \sim R^c / \kappa$ as soon as $R \gg 1$), and thus the memory kernel should be suppressed for large $R$. More precisely, using the additional hypothesis $P_C P_E = P_E$ and the fact that $L_E$ and $H$ satisfies the lemma condition (2), one proves that all $c + 1$ leading orders in the expansion of the memory kernel in powers of natural dissipation cancel out exactly.

Ultimately, similarly to was done in Sec. of the supplementary, the kernel can be bounded as $\| \Sigma(t) \| \leq A \kappa^2 \exp(-R \lambda_0 \kappa \tau) / R^{c-1}$ for all $\tau, R \geq 0$ and all $g \leq \tilde{g}_0 R$ for some dimensionless constants $A, \lambda_0 > 0$. Combined with Eq. (67), this bound implies that the relaxation dynamics of $P_E \rho(t)$ must be slow, which leads the following bound

$$\left\| P_E \rho(t) - \hat{U}_0(gk(t-t')) (P_E \rho(t')) \hat{U}_0^\dagger(gk(t-t')) \right\| \leq \tilde{M} \kappa |t-t'| / R^c$$

(74)

for some $\tilde{M} > 0$, for all $R, t, t' \geq 0$ and $g$ such that $|g| \leq \tilde{g}_0 R$, which is a generalization of Eqs. (3-4) of the main manuscript.

As a consequence of this very slow decoherence and the fast relaxation of $\Sigma(t)$, one expects thus that non time-local features of the Eq. (67) quantified by the correction Eq. (72) should be negligible. Similarly, the kink term in Eq. (73) should only have some limited impact restrained to the very early stage of the dynamics due to the fast relaxation of the memory kernel. A precise analysis yields the exact upper bounds:

$$[\partial_t P_E \rho]_{\text{nloc}} \leq \tilde{N} \frac{\kappa}{R^c+1} \| \rho(0) \|$$

(75)

$$[\partial_t P_E \rho]_{\text{kink}} \leq \beta R t / R^c \| \rho(0) \|$$

(76)

for some positive dimensionless numbers $\tilde{M}, \tilde{N}, \beta > 0$ for all $\rho(0) \in \mathbb{C} \otimes \mathbb{C}$ and $R, \kappa, t \geq 0$, $g$ satisfying $|g| \leq \tilde{g}_0 R$.

Considering that $\partial_t P_E \rho \sim \kappa / R^c$, one finds thus that the non-local contribution $[\partial_t P_E \rho]_{\text{nloc}}$ can be completely neglected in the limit $R \to +\infty$. The kink contribution $[\partial_t P_E \rho]_{\text{kink}}$ is relevant at very short times $t \approx 1 / (\kappa R)$, but since it decays immediately afterwards, it leads to overall negligible deviations $O(1 / R^{c+1})$ in the evolution of the density matrix.

**Full density matrix**

Our formalism enables us to retrieve information about the full density matrix $\rho(t)$ from our knowledge of the dynamics of the recovered matrix $P_E \rho(t)$: if $\rho(0) \in \mathbb{C} \otimes \mathbb{C}$ then at later times ones has exactly:

$$Q_E \rho(t) = \int_0^t d\tau e^{Q_E \rho(t - \tau)} P_E \rho(t - \tau).$$

(77)

Proceeding similarly to the previous section one can show that $Q_E \rho(t) = [Q_E \rho(t)]_{\infty} + [Q_E \rho(t)]_{\text{kink}} + [Q_E \rho(t)]_{\text{nloc}}$, where the dominant contribution is given by

$$[Q_E \rho(t)]_{\infty} = \int_0^{+\infty} d\tau e^{Q_E \rho(t - \tau)} Q_E L e^{igk \mathcal{H} \tau} P_E \rho(t).$$

(78)

Likewise, as a consequence of the slow logical space decoherence dynamics associated to AutoQEC the remaining corrections are bounded as following

$$\| [Q_E \rho(t)]_{\text{kink}} \| \leq \frac{A}{R} e^{-R \kappa B t}$$

(79)

$$\| [Q_E \rho(t)]_{\text{nloc}} \| \leq \frac{C}{R^{c+2}},$$

(80)

for all $R, t, t' \geq 0$ and $g$ such that $|g| \leq \tilde{g}_0 R$, where $A, B, C$ are some positive constants. Adding together the complementary projected parts of the density matrix $\rho(t) = P_E \rho(t) + Q_E \rho(t)$, based on the previously obtained bounds we find at all times the following estimate:

$$\rho(t) = T P_E \rho(t) + \delta \rho(t)$$

(81)

where

$$T = I + \int_0^{+\infty} d\tau e^{Q_E L Q_E T} Q_E L e^{igk \mathcal{H} \tau} P_E$$

(82)

$$\| \delta \rho(t) \| \leq \frac{A}{R} e^{-R \kappa B t} + \frac{C}{R^{c+2}},$$

(83)
Final expressions for the effective dynamics

Here we analyse further the expression of the superoperators $\mathcal{L}_{\text{eff}}$ and $T$ governing the dynamics of $\mathcal{P}_E \rho(t)$ and $\rho(t)$.

General case

Let us have a complete set $\{ | \mu_v^{H_0} \rangle \in \mathcal{C} \}$ of code space eigenstates of the logical Hamiltonian $H_0$, and we denote $E_v$ the corresponding eigenenergies. We introduce the extra notations $| \mu_v^{H_0}, \mu_w^{H_0} \rangle = | \mu_v^{H_0} \rangle | \mu_w^{H_0} \rangle$ and $E_{v,w} = E_v - E_w$. Similarly to what was done in Sec. one can expand the memory kernel $\Sigma(\tau)$ in powers of natural dissipation and rule out the $c - 1$ lowest-order contributions since property (2) if the Lemma is satisfied. After the expansion, one can compute the required integral and one finds the following expressions:

$$
\langle \langle \mu_v^{H_0}, \mu_w^{H_0} | \mathcal{L}_{\text{eff}}^0 | \mu_v^{H_0}, \mu_w^{H_0} \rangle \rangle = \kappa \sum_{k=0}^{\infty} \left( \frac{1}{R^k} \right) \langle \langle \mu_v^{H_0}, \mu_w^{H_0} | \mathcal{P}_E (\mathcal{L}_{\text{inv}}^{v,w})^k \mathcal{L}_u | \mu_v^{H_0}, \mu_w^{H_0} \rangle \rangle \tag{84}
$$

$$
\langle \langle \mu_v^{H_0}, \mu_w^{H_0} | \mathcal{L}_{\text{eff}}^0 | \mu_v^{H_0}, \mu_w^{H_0} \rangle \rangle = -ig \kappa \sum_{k=0}^{\infty} \left( \frac{1}{R^k} \right) \langle \langle \mu_v^{H_0}, \mu_w^{H_0} | \mathcal{P}_E \mathcal{H} \mathcal{F}_{\text{inv}}^{v,w} (\mathcal{L}_{\text{inv}}^{v,w})^k \mathcal{L}_u | \mu_v^{H_0}, \mu_w^{H_0} \rangle \rangle \tag{85}
$$

$$
\langle \langle \mu_v^{H_0}, \mu_w^{H_0} | \mathcal{L}_{\text{eff}}^1 | \mu_v^{H_0}, \mu_w^{H_0} \rangle \rangle = -ig \kappa \sum_{k=0}^{\infty} \left( \frac{1}{R^k} \right) \langle \langle \mu_v^{H_0}, \mu_w^{H_0} | \mathcal{P}_E (\mathcal{L}_{\text{inv}}^{v,w})^k \mathcal{H} | \mu_v^{H_0}, \mu_w^{H_0} \rangle \rangle \tag{86}
$$

$$
\langle \langle \mu_v^{H_0}, \mu_w^{H_0} | \mathcal{L}_{\text{eff}}^0 | \mu_v^{H_0}, \mu_w^{H_0} \rangle \rangle = -g^2 \kappa \sum_{k=0}^{\infty} \left( \frac{1}{R^k} \right) \langle \langle \mu_v^{H_0}, \mu_w^{H_0} | \mathcal{P}_E \mathcal{H} \mathcal{F}_{\text{inv}}^{v,w} (\mathcal{L}_{\text{inv}}^{v,w})^k \mathcal{H} | \mu_v^{H_0}, \mu_w^{H_0} \rangle \rangle \tag{87}
$$

and

$$
\mathcal{T} | \mu_v^{H_0}, \mu_w^{H_0} \rangle \rangle = \mathcal{I} + \sum_{k=1}^{\infty} \left( \frac{1}{R^k} \right) (\mathcal{F}_{\text{inv}}^{v,w} \mathcal{L}_n)^k (\mathcal{F}_{\text{inv}}^{v,w} \mathcal{L}_n - ig \mathcal{H}) | \mu_v^{H_0}, \mu_w^{H_0} \rangle \rangle \tag{88}
$$

The superoperator $\mathcal{F}_{\text{inv}}^{v,w} = -\int_0^{\infty} dt \exp \left( \int_0^t d\tau \mathcal{L}_{E,H}^{v,w} \right) Q_E$, with

$$
\mathcal{L}_{E,H}^{v,w} = \mathcal{L}_E - ig/R (Q_E \mathcal{H} Q_E - E_{v,w} Q_E) \tag{89}
$$

is well defined for an Hamiltonian coupling $g$ satisfying $|g| \leq \bar{g}_0 R$, as $\bar{g}_0$ was chosen (and proven to exist) so to satisfy the bound Eq. (58) in the supplementary material. Since engineered dissipation and the Hamiltonian superoperators are all assumed to perform an autonomous error-corrected quantum computation of order $c$ with respect to natural dissipation, the summations in the expressions of Eqs. (84)-(87) initiate only $k = c$ because all lower-order contributions cancelled exactly as a consequence of the applicability of the Lemma.

We notice that $\mathcal{L}_{E,H}^{v,w} Q_E = Q_E \mathcal{L}_{E,H}^{v,w} = \mathcal{L}_{E,H}^{v,w} Q_E = \mathcal{P}_E \mathcal{L}_{E,H}^{v,w} = 0$, thus $\mathcal{L}_{E,H}^{v,w}$ is an operator which can be restricted to the relaxation subspace of projected states $\hat{\rho} = Q_E \hat{\rho} Q_E$ of the complementary projector $Q_E$. Moreover we can also show that $\mathcal{L}_{E,H}^{v,w}$ is also necessarily invertible with its eigenvalues having strictly negative real parts in the relaxation subspace since $g$ is chosen so that the bound Eq. (58) is satisfied: to prove so we remark that the component $i(g/R) E_{v,w} Q_E$ in $\mathcal{L}_{E,H}^{v,w}$ is just an imaginary shift proportional to the identity in that subspace and thus does not affect the (strictly negative) real parts of the eigenvalues of $\mathcal{L}_E - ig/R \mathcal{H} Q_E$, $\mathcal{L}_E - ig/R \mathcal{H}$ $Q_E$. As a consequence, the above defined superoperator $\mathcal{F}_{\text{inv}}^{v,w}$ is indeed well-defined, and $\mathcal{F}_{\text{inv}}^{v,w}$ can be seen as the exact inverse of $\mathcal{L}_{E,H}^{v,w}$ in the relaxation subspace. Coming back to the full Hilbert space, $\mathcal{F}_{\text{inv}}^{v,w}$ is a pseudo-inverse of $\mathcal{L}_{E,H}^{v,w}$, i.e., it satisfies $\mathcal{F}_{\text{inv}}^{v,w} \mathcal{L}_{E,H}^{v,w} \mathcal{F}_{\text{inv}}^{v,w} = \mathcal{F}_{\text{inv}}^{v,w}$ and $\mathcal{L}_{E,H}^{v,w} \mathcal{F}_{\text{inv}}^{v,w} \mathcal{L}_{E,H}^{v,w} = \mathcal{L}_{E,H}^{v,w}$.

Finally, one remarks that there exists some constants $A, R_0 > 0$ such that for any $R \geq R_0$ and $|g| \leq \bar{g}_0 R$ the power series in the expressions of $\mathcal{L}_{\text{eff}}^{ab}$ in Eq. (87) are all convergent and that the contributions $\mathcal{L}_{\text{eff}}^{ab}$ ($a,b = 0, 1$) satisfy $\| \mathcal{L}_{\text{eff}}^{ab} \| \leq A/R^c$. Substituting the logical Hamiltonian component, the suppression up to order $c$ of the effective code space Liouvillian $\| \mathcal{L}_{\text{eff}} - i g \mathcal{H} \| < 4A/R^c$ confirms the picture of an autonomous error-corrected computation of order $c$.

Special case: $H = H_0 = 0$

In absence of an Hamiltonian the expression derived in the previous section are drastically simplified, and one obtains
\[ \mathcal{L}_{\text{eff}} = \kappa \sum_{k=c+1}^{\infty} \left( \frac{-1}{R} \right)^{k-1} \mathcal{P}_E (\mathcal{L}_n \mathcal{L}_E^*)^k \mathcal{L}_n \mathcal{P}_C \] (90)

\[ \mathcal{T} = \sum_{n=0}^{\infty} \left( \frac{-1}{R^k} \right)^k (\mathcal{L}_E^* \mathcal{L}_n)^k \mathcal{P}_C, \] (91)

The quantity \( \mathcal{L}_E = -\int_0^{\infty} du [e^{\mathcal{L}_E u} \mathcal{Q}_E] \) is well-defined (as the projector \( \mathcal{Q}_E \) restricts the dynamics to relaxation eigenmodes of \( \mathcal{L}_E \)), and is a pseudo inverse of \( \mathcal{L}_E \). As previously, effective dissipation in the logical space is suppressed as \( 1/R_c \).

We remark in the specific case were we chose the engineered dissipation \( \mathcal{L}_E \) according to the explicit construction introduced in Eq. (7) of the main manuscript, the pseudo-inverse and the various projectors in Eqs. (90-91) have some explicit analytical expressions:

\[ \mathcal{P}_E \rho = \mathcal{P}_C \rho \mathcal{P}_C + \sum_{n=1}^{c} \sum_{i=1}^{p_n} \mathcal{F}_{E,i_n} \rho \mathcal{F}_{E,i_n}^\dagger + \sum_{q=1}^{q_{\text{max}}} \mathcal{F}_{E,q} \rho \mathcal{F}_{E,q}^\dagger \] (92)

\[ \mathcal{Q}_E \rho = \rho - \mathcal{P}_E \rho \] (93)

\[ \mathcal{L}_E^* \rho = -2(\mathcal{Q}_C \rho \mathcal{P}_C + \mathcal{P}_C \rho \mathcal{Q}_C) - \mathcal{Q}_C \rho \mathcal{Q}_C + \sum_{n=1}^{c} \sum_{i=1}^{p_n} \mathcal{F}_{E,i_n} \rho \mathcal{F}_{E,i_n}^\dagger + \sum_{q=1}^{q_{\text{max}}} \mathcal{F}_{E,q} \rho \mathcal{F}_{E,q}^\dagger \] (94)