Quasi-invariants and quantum integrals of the deformed Calogero–Mosser systems

M. Feigin $^1$ and A. P. Veselov $^{2,3}$

1 Chair of Mathematics and Financial Applications, Financial Academy, Leningradsky prospect, 49, Moscow, 125468, Russia
2 Department of Mathematical Sciences, Loughborough University, Loughborough, LE11 3TU, UK
3 Landau Institute for Theoretical Physics, Kosygina 2, Moscow, 117940, Russia
E-mail addresses: mfeigin@dnttm.ru, A.P.Veselov@lboro.ac.uk

Abstract

The rings of quantum integrals of the generalized Calogero-Moser systems related to the deformed root systems $A_n(m)$ and $C_n(m,l)$ with integer multiplicities and corresponding algebras of quasi-invariants are investigated. In particular, it is shown that these algebras are finitely generated and free as the modules over certain polynomial subalgebras (Cohen-Macaulay property). The proof follows the scheme proposed by Etingof and Ginzburg in the Coxeter case. For two-dimensional systems the corresponding Poincare series and the deformed $\mathfrak{m}$-harmonic polynomials are explicitly computed.

1 Introduction

Quantum Calogero-Moser (CM) problem [1] with the Hamiltonian

$$L = \Delta - \sum_{i<j}^n \frac{2m(m+1)}{(x_i - x_j)^2}$$

was generalized by Olshanetsky and Perelomov [2] for any Coxeter group as

$$L = \Delta - \sum_{\alpha \in \mathcal{A}} \frac{m_\alpha(m_\alpha + 1)(\alpha,\alpha)}{(\alpha,x)^2}$$

where $\mathcal{A} = \mathcal{R}_+$ is a positive part of the corresponding root system. In 1996 O. Chalykh and the authors [3] showed that there are non-Coxeter integrable generalizations as well (so-called deformed Calogero-Moser problems). Recently A.N. Sergeev suggested an explanation
of these deformations in relation with Lie superalgebras \[4\]. A systematic approach to the deformed quantum CM problems from this point of view has been developed in \[5\].

In this paper, which can be considered as a sequel to \[6, 7\], we investigate the algebras of quantum integrals for two series $A_n(m)$ and $C_{n+1}(m, l)$ of the deformed quantum Calogero-Moser systems discovered in \[8, 9\]. According to \[5\] these series are the only non-Coxeter cases among the deformed CM systems when all the parameters are integer.

The corresponding operators have the forms \[8, 9\]

$$L = \Delta - \sum_{i<j}^n \frac{2m(m+1)}{(x_i - x_j)^2} - \sum_{i=1}^n \frac{2(m+1)}{x_i - \sqrt{mx_{n+1}}},$$

(3)

and

$$L = \Delta - \sum_{i<j}^n \frac{4\kappa(\kappa+1)(x_i^2 + x_j^2)}{(x_i^2 - x_j^2)^2} - \sum_{i=1}^n \frac{m(m+1)}{x_i^2} -$$

$$- \frac{l(l+1)}{x_{n+1}^2} - \sum_{i=1}^n \frac{4(\kappa+1)(x_i^2 + \kappa x_{n+1}^2)}{(x_i^2 - \kappa x_{n+1}^2)^2},$$

(4)

where the parameters $m, \kappa, l$ satisfy the relation $\kappa = \frac{2m+1}{2l+1}$ and $\Delta$ stands for the standard Laplace operator in $n + 1$-dimensional Euclidean space. When the parameter $m = 1$ the first operator becomes a special case of the Calogero operator \[10\]. The second operator is the deformation of the generalized CM operator \[2\] related to the root system $C_{n+1}$, which corresponds to the case $m = l$.

In this paper we consider the case when all the parameters (multiplicities) in these operators are integer. The importance of the last condition was first demonstrated by Chalykh and one of the authors \[10\], who discovered that in such a case the ring of integrals of quantum CM system is much bigger than for the generic parameters.

More precisely, for any configuration $A$ which is a finite set of vectors $\alpha$ in the Euclidean space $V$ with prescribed multiplicities $m_\alpha \in \mathbb{Z}_+$ one can introduce the following algebra of quasi-invariants $Q = Q^A$. It consists of all polynomials $q$ on $V$ with the following property:

$$q(s_\alpha(x)) = q(x) + o((\alpha, x)^{2m_\alpha})$$

(5)

near the hyperplane $(\alpha, x) = 0$ for any $\alpha \in A$, where $s_\alpha$ denotes reflection with respect to this hyperplane. For the special configurations, in particular for the Coxeter systems and configurations $A_n(m)$ and $C_{n+1}(m, l)$, there exists a homomorphism from the algebra of quasi-invariants into the ring of quantum integrals of the corresponding generalized CM operator, which under some assumptions can be shown to be an isomorphism.

For the Coxeter configurations the algebraic structure of the rings of quasi-invariants $Q^A$ was investigated in \[6, 7, 11\]. In particular, Etingof and Ginzburg \[11\] proved that this
ring is a free module over the subring of invariant polynomials (Cohen-Macaulay property) confirming some of the conjectures from [6].

The main result of this paper is the proof of a similar fact for the rings of quasi-invariants related to the deformed configurations \( A_n(m) \) and \( C_{n+1}(m, l) \). We introduce certain polynomial subalgebra \( P^A \subset Q^A \) and show that \( Q^A \) is free as a module over \( P^A \). The proof follows the Etingof-Ginzburg scheme from [11]. In two-dimensional case we find the explicit formulas for the Poincare series of the corresponding rings of quasi-invariants and show that these rings are Gorenstein. We introduce also the deformed \( m \)-harmonic polynomials and compute them explicitly in \( A_2(m) \)-case.

2 Deformed quantum Calogero-Moser systems

The quantum systems we are going to discuss are related to the following configurations introduced in [3, 8, 9].

The first configuration \( A_n(m) \) consists of the vectors \( e_i - e_j \) with multiplicity \( m \), where \( 1 \leq i < j \leq n \), and the vectors \( e_i - \sqrt{m}e_{n+1} \) with multiplicity 1. When the parameter \( m = 1 \) this is the classical root system of type \( A_n \).

The second configuration \( A = C_{n+1}(m, l) \) consists of the following vectors

\[
C_{n+1}(m, l) = \begin{cases} 
  e_i \pm e_j & \text{with multiplicity } \kappa \\
  2e_i & \text{with multiplicity } m \\
  2\sqrt{\kappa}e_{n+1} & \text{with multiplicity } l \\
  e_i \pm \sqrt{\kappa}e_{n+1} & \text{with multiplicity } 1 
\end{cases}
\]

where \( m, \kappa \) and \( l \) are parameters with the relation \( \kappa = \frac{2m+1}{2l+1} \) (so only two of them, say \( m, l \) are independent) and \( 1 \leq i < j \leq n \). In the case of \( C_2(m, l) \) system there are no vectors of \( e_i \pm e_j \) type, and the parameters \( m, l \) can be arbitrary. In the case \( l = m \) the system \( C_{n+1}(m, l) \) coincides with the classical root system \( C_{n+1} \) (or \( D_{n+1} \) for \( l = m = 0 \)).

Although in the rest of the paper only the case of integer multiplicities will be considered at the beginning we will not assume this and consider general values of parameters. Corresponding deformed quantum CM problems are given by the general formula (2) or more explicitly by the formulas (3), (4) respectively.

We will actually be using these operators in a different ("radial") gauge and consider the operators \( \mathcal{L} = gLg^{-1} \) with \( g = \prod_{\alpha \in A}(\alpha, x)^{m_\alpha} \), which have the following forms:

\[
\mathcal{L}_{A_n(m)} = \Delta - \sum_{i<j}^{n} \frac{2m}{x_i - x_j}(\partial_i - \partial_j) - \sum_{i=1}^{n} \frac{2}{x_i - \sqrt{m}x_{n+1}}(\partial_i - \sqrt{m}\partial_{n+1}).
\] (6)
and

\[ \mathcal{L}_{C_{n+1}(m,l)} = \Delta - \sum_{i<j}^n \frac{4\kappa(x_i \partial_i - x_j \partial_j)}{x_i^2 - x_j^2} - \sum_{i=1}^n \frac{2m \partial_i}{x_i} - \frac{2l \partial_{n+1}}{x_{n+1}} - \sum_{i=1}^n \frac{4(x_i \partial_i - \kappa x_{n+1} \partial_{n+1})}{x_i^2 - \kappa x_{n+1}^2} \]  

(7)

The existence of such forms for \( A_n(m) \) and \( C_{n+1}(m, l) \) is one of the remarkable properties of these systems and is due to the following identity valid for both systems:

\[ \sum_{\alpha \neq \beta} m_\alpha m_\beta (\alpha, \beta)(\alpha, x)(\beta, x) \equiv 0, \]  

(8)

which is equivalent to the set of relations

\[ \sum_{\beta \neq \alpha} m_\beta (\alpha, \beta)(\beta, x) \equiv 0 \]  

at \((\alpha, x) = 0\) (9)

for any \( \alpha \in A \).

As it was shown in [3, 9] the quantum systems related to the deformations \( A_n(m) \) and \( C_{n+1}(m, l) \) are integrable. More precisely, consider the following two series of polynomials

\[ p_s = x_1^s + x_2^s + \ldots + x_n^s + m \frac{s-1}{2} x_{n+1}^s, \]  

(10)

and

\[ q_s = x_1^{2s} + x_2^{2s} + \ldots + x_n^{2s} + \kappa^{s-1} x_{n+1}^{2s}, \]  

(11)

where \( s = 1, 2, \ldots \)

**Theorem 1** [2] For any \( s \in \mathbb{N} \) there exists differential operator \( \mathcal{L}_s \) with the highest term \( p_s(\partial) \) given by (10) such that \([\mathcal{L}_s, \mathcal{L}_t] = 0\), \( t \in \mathbb{N} \). The operator \( \mathcal{L}_2 \) coincides with the Calogero–Moser operator (6) related to the system \( A_n(m) \). The same is true for the polynomials \( q_s \) and the operator (7) related to \( C_{n+1}(m, l) \).

Thus for generic parameters we have a commutative algebra of quantum integrals of the deformed CM problems generated by \( \mathcal{L}_s \), which is isomorphic to the subalgebra generated by the polynomials \( p_s \) in \( A_n(m) \) case and by \( q_s \) in \( C_{n+1}(m, l) \) case. One can show that these subalgebras are finitely generated for generic values of the parameters (see [4]).

However we will be using the smaller subalgebras \( P^{A_n(m)} = C[p_1, p_2, \ldots, p_{n+1}] \) and \( P^{C_{n+1}(m, l)} = C[q_1, q_2, \ldots, q_{n+1}] \), which are freely generated by the first \( n + 1 \) polynomials \( p_s \) and \( q_s \) respectively. Since when \( m = 1 \) and \( \kappa = 1 \) these algebras coincide with the corresponding algebras of invariants for the Coxeter groups of type \( A_{n+1} \) and \( C_{n+1} \) we will call them the *algebras of deformed invariants.*
Proposition 1  If parameter \( m \notin \{0, -1, -\frac{1}{2}, -\frac{1}{3}, \ldots, -\frac{1}{n}\} \) then

1. The dimension of the quotient

\[
\mathbb{C}[x_1, \ldots, x_{n+1}]/I,
\]

where \( I \) is ideal generated by \( p_1, \ldots, p_{n+1} \), is finite and equals to the number \( \mu^{A_n(m)} = (n+1)! \) of different complex solutions to the algebraic system \( p_i(x) = c_i, i = 1, \ldots, n+1 \) for generic \( c_i \in \mathbb{C} \).

2. The ring \( \mathbb{C}[x_1, \ldots, x_{n+1}] \) is a free module over \( \mathbb{C}[p_1, \ldots, p_{n+1}] \) of rank \( \mu^{A_n(m)} \).

The same is true for the system \( C_{n+1}(m, l) \) if we replace \( m \) by \( \kappa \), \( p_s \) by \( q_s \) and \( \mu^{A_n(m)} \) by \( \mu^{C_{n+1}(m,l)} = 2^{n+1}(n+1)! \).

For the proof we need the following

Lemma 1  The system of equations

\[ p_s(x) = 0, \quad s = 1, \ldots, n+1 \]

has unique solution \( x = 0 \) if \( p_s \) are given by \( \text{(10)} \) where \( m \notin \{0, -1, -\frac{1}{2}, -\frac{1}{3}, \ldots, -\frac{1}{n}\} \).

Proof  Let us scale the variable \( x_{n+1} \) introducing \( y = m^{1/2}x_{n+1} \). Then the system of equations takes the form

\[
\begin{align*}
  x_1 + \ldots + x_n + m'y &= 0 \\
  x_1^2 + \ldots + x_n^2 + m'y^2 &= 0 \\
  \vdots &
\end{align*}
\]

(12)

where \( m' = \frac{1}{m} \). We have to show that \( \text{(12)} \) implies that \( x_1 = \ldots = x_n = y = 0 \). One can see that this is not the case when \( m' = 0, -1, -2, \ldots, -n \). Indeed, if \( m' = 0 \) then there exists nonzero solution \( x_i = 0, i = 1, \ldots, n \), and \( y \in \mathbb{C} \) is arbitrary. If \( m' = -s \) then the nonzero solution will be \( x_1 = x_2 = \ldots = x_s = y \in \mathbb{C} \) and \( x_{s+1} = \ldots = x_n = 0 \). We will show that there are no nonzero solutions for all other values of \( m' \).

Denote by \( \tau_i \) the power sum

\[ \tau_i = x_1^i + \ldots + x_n^i. \]

Then \( \tau_1, \ldots, \tau_n \) form a basis in the ring of symmetric functions of \( n \) variables \( x_1, \ldots, x_n \), in particular, \( \tau_{n+1} = P(\tau_1, \ldots, \tau_n) \) for some polynomial \( P(a_1, \ldots, a_n) \). If we assign degree \( i \) to the variable \( a_i \) then \( P \) is homogeneous polynomial of degree \( n+1 \). The first \( n \) equations of \( \text{(12)} \) can be written as \( \tau_i = -m'y^i \). Then the last equation of the system can be written as

\[
P(-m'y, -m'y^2, \ldots, -m'y^n) + m'y^{n+1} = 0.
\]
Now due to weighted homogeneity of $P$,
\[ P(-m'y, -m'y^2, \ldots, -m'y^n) = y^{n+1}P(-m', -m', \ldots, -m'), \]
and usual degree $\deg P \leq n + 1$. It is obvious from \[12\] that if nonzero solution exists then $y \neq 0$. Therefore $m'$ must satisfy
\[ P(-m', -m', \ldots, -m') + m' = 0 \]
with $\deg P \leq n + 1$. Hence either $m'$ is arbitrary or there are not more than $n + 1$ possible values for $m'$ to have nonzero solution to \[12\]. Since for $m' = 1$ there are no nonzero solutions, and since we know that for $m' = 0, -1, \ldots, -n$ there are nonzero solutions, we conclude that if $m' \neq 0, -1, \ldots, -n$ then there are no nonzero solutions to system \[12\]. Lemma is proven.

Now the part 1 of Proposition 1 for the system $A_n(m)$ follows from the Lemma and the standard results about isolated zeros of analytic maps \[12\]. The part 2 is a consequence of part 1 and the Cohen–Macaulay property of the polynomial ring. For $C_{n+1}(m, l)$-system the proof is similar.

Let now $\mathcal{A}$ denote one of the systems $A_n(m)$ or $C_{n+1}(m, l)$ with the parameters $m$ and $\kappa$ which do not belong to the exceptional set $\{0, -1, -\frac{1}{2}, -\frac{1}{3}, \ldots, -\frac{1}{n}\}$ and let $\gamma_1(x) = 1, \gamma_2(x), \ldots, \gamma_\mu(x)$ be some homogeneous basis in $\mathbb{C}[x_1, \ldots, x_{n+1}]$ as a module over the corresponding algebra of the deformed invariants $P^A$.

Consider the system of the differential equations
\[ L_i f = \lambda_i f, \quad i = 1, \ldots, n + 1, \]
where $L_i = L_i^A$ are the commuting quantum integrals defined in Theorem \[11\]. Let $F(x)$ be the vector function
\[ F(x) = (f(x), \gamma_2(\partial)f(x), \ldots, \gamma_\mu(\partial)f(x)). \]

**Proposition 2** The system of equations \[13\] is equivalent to the first order system
\[ \frac{\partial F}{\partial x_i} = A_i(x, \lambda)F, \quad i = 1, \ldots, n + 1, \]
where $A_i(x)$ are matrix valued functions analytic if $\prod_{\alpha \in \mathcal{A}}(\alpha, x) \neq 0$ and such that
\[ [\partial_i - A_i, \partial_j - A_j] = 0. \]
The space of local analytic solutions to \[13\] has the dimension $\mu^A$.

For the root systems such a statement (in a trigonometric version) was proven in \[14\]. The proof from \[14\] can be easily adapted for our case if we use Proposition 1 instead of the Chevalley theorem about the invariants of the Weyl group.

In the case when all the multiplicities are integer we can actually claim that all the solutions of the system \[13\] are analytic everywhere but for this we will need some results from \[9\] which we discuss in the next section.
3 Baker-Akhiezer function and quantum integrals

Let us assume now that all the parameters of the systems are integer: for the system $A_n(m)$ this means that $m \in \mathbb{N}$, and for the system $C_{n+1}(m,l)$ all $m,l$ and $\kappa = \frac{2m+1}{2l+1}$ must be positive integers.

According to [9] in that case there exists so-called *Baker–Akhiezer function* related to such a configuration. This function $\psi^A(k,x)$ has the form

$$\psi^A(k,x) = (P^A_N + \ldots + P^A_0) e^{(k,x)},$$

where $P^A_i = P^A_i(k,x)$ are some polynomials in $k,x$ of degree $i$ in $k$ and in $x$ with the highest term of the form

$$P^A_N(k,x) = \prod_{\alpha \in A} (\alpha,k)^{m_\alpha}(\alpha,x)^{m_\alpha}.$$

The following *quasi-invariance condition* determines the function $\psi^A$ uniquely (see [9]):

$$\psi^A(s_\alpha(k),x) - \psi^A(k,x) = o((\alpha,k)^{2m_\alpha}) \text{ near } (\alpha,k) = 0, \ \alpha \in A,$$

where $s_\alpha$ is the reflection with respect to the hyperplane $(\alpha,k) = 0$.

Such a function $\psi^A$ exists only for very special class of configurations including the Coxeter systems with invariant integer multiplicities and the configurations $A_n(m)$, $C_{n+1}(m,l)$ (see [13] for the latest results in this direction).

For any such configuration $A \subset V$ one can introduce the following important notion. We will call a polynomial $q$ on $V$ *quasi-invariant for the system $A$* if for all $\alpha \in A$ it is invariant up to order $2m_\alpha$ with respect to the reflections $s_\alpha$:

$$q(s_\alpha(k)) = q(k) + o((\alpha,k)^{2m_\alpha})$$

near the hyperplane $(\alpha,k) = 0$ for any $\alpha \in A$. Equivalently we can say that for each $\alpha \in A$ the odd normal derivatives $\partial^s_\alpha q = (\alpha, \frac{\partial}{\partial \alpha})^s q$ vanish on the hyperplane $(\alpha,k) = 0$ for $s = 1, 3, 5, \ldots, 2m_\alpha - 1$.

We denote the corresponding ring of quasi-invariants by $Q^A$. We should mention that the terminology is slightly abusing. The group generated by the reflections $s_\alpha$ is not a finite group any more, and generally there are no polynomials except those of $q(k) = k^2$ which are invariant with respect to all the reflections $s_\alpha$. Nevertheless we will see that the rings $Q^A$ have some nice properties similar to the Coxeter case.

The following general result explains the relation of this ring to the generalized quantum CM problems. Let $D_A$ denote the ring of differential operators which are regular outside the hyperplanes $(\alpha,x) = 0, \alpha \in A$.

**Theorem 2** [10] *If a system $A$ admits the Baker–Akhiezer function $\psi^A(k,x)$, then there is a homomorphism $\chi^A : Q^A \to D_A$ mapping a quasi-invariant $q(k)$ to the differential operator $L_q(x, \frac{\partial}{\partial x})$ such that*

$$L_q \psi^A(k,x) = q(k) \psi^A(k,x).$$
Under this homomorphism the quasi-invariant \(k^2\) is mapped to the (gauged) generalized Calogero–Moser operator

\[
L = \Delta - \sum_{\alpha \in \mathcal{A}} \frac{2m_\alpha}{(\alpha, x)} \partial_\alpha.
\]

In fact we can actually claim that the ring of quasi-invariants \(Q^\mathcal{A}\) for our configurations \(\mathcal{A}_n(m)\) and \(\mathcal{C}_{n+1}(m, l)\) is isomorphic to the ring of all quantum integrals for the operator \(L_\mathcal{A}\) in the following sense. Denote by \(D^\mathcal{A}\) the maximal commutative ring of differential operators with rational coefficients which contains the operators

\[
L_i = \chi^{\mathcal{A}_n(m)}(p_i) \quad \text{for } \mathcal{A} = \mathcal{A}_n(m),
\]

and

\[
L_i = \chi^{\mathcal{C}_{n+1}(m, l)}(q_i) \quad \text{for } \mathcal{A} = \mathcal{C}_{n+1}(m, l),
\]

\(i = 1, \ldots, n+1\). The following result can be proved similarly to the Coxeter case \([7]\).

**Theorem 3** The map \(\chi^\mathcal{A}\) is an isomorphism between the ring \(Q^\mathcal{A}\) and the ring of quantum integrals \(D^\mathcal{A}\) of the corresponding deformed CM systems \([8], [7]\).

Another relation which will be important for us is the invariance of the quasi-invariants under the action of all the quantum integrals (cf. \([7]\)).

**Proposition 3** For the systems \(\mathcal{A} = \mathcal{A}_n(m), \mathcal{C}_{n+1}(m, l)\) the space \(Q^\mathcal{A}\) of quasi-invariants is invariant under the action of the operators \(L_q, q \in Q^\mathcal{A}\).

**Proof.** We will use the result from \([9]\) which says that the operator \(L_\mathcal{A}\) preserves the space \(\Phi\) of meromorphic functions \(\phi(x)\) such that

\[
\phi(x) \prod_{\alpha \in \mathcal{A}} (\alpha, x)^{m_\alpha} \text{ is holomorphic in } \mathbb{C}^{n+1},
\]

and

\[
\partial_\alpha^{2s-1}(\phi(x)(\alpha, x)^{m_\alpha})|_{(\alpha, x)=0} = 0
\]

for \(s = 1, \ldots, m_\alpha\), and \(\alpha \in \mathcal{A}\) (see Lemma in the proof of Theorem 3.1 in \([9]\)). Since \(L_\mathcal{A} = L_{k^2} = g_\mathcal{A}L_\mathcal{A}g_\mathcal{A}^{-1}\) for \(g = \prod_{\alpha \in \mathcal{A}} (\alpha, x)^{m_\alpha}\), the operator \(L_\mathcal{A}\) preserves the space \(\Phi\) consisting of the functions \(f(x)\) which are holomorphic in \(\mathbb{C}^{n+1}\) and satisfy the relations

\[
\partial_\alpha^{2s-1}(f(x) \prod_{\beta \neq \alpha} (\beta, x)^{-m_\beta})|_{(\alpha, x)=0} = 0,
\]

\(s = 1, \ldots, m_\alpha, \alpha \in \mathcal{A}\).
Now we use the identities (9) to claim that
\[
\partial_{s}^{2s-1}(\prod_{\beta \neq \alpha} (\beta, x)^{-m_{\beta}})|_{(\alpha, x)=0} = 0,
\]
(17)
s = 1, \ldots, m_{\alpha}, \alpha \in A. Indeed, when \( s = m_{\alpha} = 1 \) these relations are equivalent to (9). If \( \alpha \) is such that \( m_{\alpha} > 1 \) then \( \prod_{\beta \neq \alpha} (\beta, x)^{-m_{\beta}} \) is symmetric with respect to reflection \( s_{\alpha} \) and thus (17) obviously holds.

Due to (17) the relations (16) can be rewritten as
\[
\partial_{s}^{2s-1}f(x)|_{(\alpha, x)=0} = 0
\]
which means that \( f(x) \) is quasi-invariant. Thus \( \mathcal{L}_{A} \) preserves the space of quasi-invariant functions. Due to its form, \( \mathcal{L}_{A} \) also preserves the subspace of quasi-invariant polynomials. Now the Proposition follows from Berest’s formula [15]: \( \mathcal{L}_{q} = \text{const} \ ad^{\deg q} \mathcal{L}_{A} q \).

In particular we have

**Corollary 1** For any \( q \in Q^{A} \) one has \( \mathcal{L}_{q} 1 = 0 \).

### 4 Structure of the rings of quasi-invariants

In this section we are going to investigate the rings \( Q^{A} \) of quasi-invariants related to the configurations \( A_{n}(m) \), \( C_{n+1}(m, l) \) in more detail.

First of all we would like to mention that all the deformed Newton sums \( p_{s} \) and \( q_{s} \) given by (10), (11) are the quasi-invariants for the systems \( A_{n}(m) \) and \( C_{n+1}(m, l) \) respectively. Since they are symmetric with respect to the first \( n \) coordinates the quasi-invariance must be checked only for the hyperplanes with multiplicity one, which is an easy calculation.

In particular the corresponding algebras of deformed invariants \( P^{A} \) generated by the first \( n + 1 \) deformed Newton sums are the subalgebras in \( Q^{A} \).

**Theorem 4** The algebras of quasi-invariants \( Q^{A} \) are finitely generated.

Indeed by Proposition 1 the algebra of all polynomials is a finitely generated free module over \( P^{A} \) since positive integers never belong to the exceptional set. To conclude the theorem one can use a standard result from commutative algebra (see e.g. Proposition 7.8 from Atiyah-Macdonald [16]).

We are going to prove that the algebras \( Q^{A} \) are freely generated as the modules over \( P^{A} \). This means that the rings of quasi-invariants \( Q^{A} \) have the Cohen-Macaulay property. We will follow the scheme from Etingof-Ginzburg paper [11] where a similar result for the Coxeter configurations was proved. A very essential fact for the considerations in [11] is the lemma claiming that the value of the Baker–Akhiezer function at zero is nonzero. The proof was based on Opdam’s results from [17] and valid only in the Coxeter case.

We present now a different proof of this statement which works both in Coxeter and in deformed cases.
Proposition 4 The values of the Baker–Akhiezer functions $\psi^A$ related to the systems $A = A_n(m), C_{n+1}(m,l)$ at zero are not zero:

$$\psi^A(0,0) \neq 0.$$

Proof. Consider the system of differential equations (13):

$$L_i \varphi = \lambda_i \varphi, \quad i = 1, \ldots, n+1,$$

where $L_i = \chi^A_n(m)(p_i)$ or $\chi^C_{n+1}(m,l)(q_i)$. By Proposition 2 we can fix a solution $\varphi = \varphi(\lambda,x; x_0,a)$ locally at generic point $x_0$ by requirements $\gamma_i(\partial)\varphi(x_0) = a_i, i = 1, \ldots, \mu$ and consider its analytic continuation in $\mathbb{C}^{n+1} \times \mathbb{C}^{n+1}$. The corresponding function $\varphi$ is analytic at $x \notin \Sigma : \{\cup_{\alpha \in A}(\alpha,x) = 0\}$ and at arbitrary $\lambda \in \mathbb{C}^{n+1}$.

On the other hand for generic $\lambda \in \mathbb{C}^{n+1}$ there exist $\mu = \mu^A$ linearly independent solutions $\psi_i(\lambda,x) = \psi^A(k^{(i)},x)$ to the system (18) corresponding to different solutions $k^{(1)}, \ldots, k^{(\mu)}$ of the systems of the algebraic equations $p_i(k) = \lambda_i, \quad i = 1, \ldots, n+1$ (and respectively $q_i(k) = \lambda_i, \quad i = 1, \ldots, n+1$). Since by Proposition 2 the solution space to (18) has dimension $\mu$, for generic $\lambda$ we have representation

$$\varphi(\lambda,x; x_0,a) = \sum_{i=1}^{\mu} c_i(\lambda)\psi_i(\lambda,x), \quad (19)$$

and in particular $\varphi(\lambda,x; x_0,a)$ is holomorphic everywhere in $x$. By Hartogs theorem it follows that $\varphi(\lambda,x; x_0,a)$ is holomorphic in $\mathbb{C}^{n+1} \times \mathbb{C}^{n+1}$.

Let us consider the value $\varphi(\lambda,0; x_0,a)$. Suppose that

$$\psi^A(0,0) = \psi^A(k,0) = \psi^A(0,x) = 0.$$

Then from representation (19) we conclude that $\varphi(\lambda,0; x_0,a) = 0$ for generic and therefore for all $\lambda$. Now we choose $a = (1,0,\ldots,0)$. By Proposition 2 the solution $\varphi = \varphi(0,x; x_0,a)$ is uniquely defined by the properties $\varphi(x_0) = 1, \gamma_i(\partial)\varphi(x_0) = 0, i = 2, \ldots, \mu$. By Corollary 1 we have that $\varphi(0,x; x_0,a) \equiv 1$, which is a contradiction with $\varphi(0,0; x_0,a) = 0$. This proves the Proposition.

Notice that as a by-product we have proved the following

Proposition 5 If all the multiplicities are integer all locally analytic solutions of the system (13) are analytic everywhere in $\mathbb{C}^{n+1}$.

Now everything is prepared for the next step, which is to define the bilinear form on $Q^A$. For any $p, q \in Q^A$ we define

$$(p,q)^A = L_pq|_{x=0}, \quad (20)$$

where the operator $L_p = \chi^A(p)$ is the operator defined in Theorem 2. As in the Coxeter case [7], [11] this form defines a scalar product on $Q^A$.

Let $c_A$ be the value of the Baker-Akhiezer function at zero: $\psi^A(0,0) = c_A \neq 0$ by Proposition 4.
Theorem 5. For the systems $A = A_n(m), C_{n+1}(m, l)$ bilinear form (20) is symmetric and non-degenerate. It can be written as

$$(p, q)^A = c_A^{-1} L_p^{(x)} L_q^{(k)} \psi^A(k, x)|_{k=x=0}. \quad (21)$$

With respect to this form the differential operator $L_q$ is adjoint to the operator of multiplication by $q \in QA$.

Proof. We closely follow [11] here. Denote by $g_i(k)$ any homogeneous basis in the space $QA$ of quasi-invariants. Consider the Baker–Akhiezer function $\psi^A(k, x)$. Since it satisfies the quasi-invariant conditions, we can write its Taylor series in the following form

$$\psi^A(k, x) = \sum_{i=0}^{\infty} g_i(k) g^i(x), \quad (22)$$

where $g^i(x)$ are some polynomials. Indeed, we may think of (22) as of decomposition of $\psi^A$ via the basis $g_i(k)$. Then coefficients $g^i(x)$ of this decomposition turn out to be the quasi-invariant polynomials as well. Indeed, as it is established in [9] the function $\psi^A(k, x)$ is symmetric:

$$\psi^A(k, x) = \psi^A(x, k). \quad (23)$$

Hence it satisfies the quasi-invariant conditions in $x$ variables as well. Since $g_i(k)$ is a basis, each polynomial $g^i(x)$ must satisfy quasi-invariant conditions. Now for a given quasi-invariant $q(k)$ consider the following equations

$$q(k) \psi^A(k, x) = L_q \psi^A(k, x) = \sum_i g_i(k) L_q g^i(x).$$

We put $x = 0$ and use the fact that $\psi^A(k, 0) = \psi^A(0, 0) = c_A$ to obtain the relation

$$c_A q(k) = \sum_i g_i(k) L_q g^i\big|_{x=0}. \quad (24)$$

Since $g_i(k)$ is a basis in $QA$, taking $q(k) = g_j(k)$ we get

$$L_{g_j} g^i\big|_{x=0} = c_A \delta^i_j. \quad (25)$$

This proves that the bilinear form (20) is non-degenerate.

Now the formula (21) simply follows: for any $p, q \in QA$ we have

$$L_p^{(x)} L_q^{(k)} \psi^A(k, x)|_{k=x=0} = L_p^{(x)} \left(q(x) \psi^A(k, x)\right)|_{k=x=0} = c_A L_p^{(x)} \left(q(x)\right)|_{x=0} = c_A(p, q)^A.$$
For arbitrary three quasi-invariants \( p, q, r \in Q^A \) we have
\[
(qr, p)^A = L_{qr}p|_{x=0} = L_rL_qp|_{x=0} = (r, L_qp)^A,
\]
so the operator of multiplication by \( q \) is adjoint to the operator \( L_q \). This completes the proof of the theorem.

As in the Coxeter case \([11]\) the defined form can be extended to the pairing between \( Q^A \) and the formal series of quasi-invariants \( \hat{Q}^A \) still being non-degenerate.

Consider now the space \( H^A(\lambda) \) which consists of formal series solutions to the corresponding system (13).

**Lemma 2** For any \( \lambda \in \mathbb{C}^{n+1} \) the following inclusions hold
\[
H^{A_n(m)}(\lambda) \subset \hat{Q}^{A_n(m)}, \quad H^{C_{n+1}(m,l)}(\lambda) \subset \hat{Q}^{C_{n+1}(m,l)}.
\]

**Proof** We actually will prove a stronger statement. Let some power series \( f(x) \) satisfy the equation
\[
\mathcal{L}_A f = Ef, \quad E \in \mathbb{C}
\]
where \( \mathcal{L}_A \) is the deformed Calogero–Moser operator. We claim that \( f(x) \) satisfies the quasi-invariance conditions (c.f. [9]).

Let \( L_A = g^{-1}\mathcal{L}_Ag, \quad g = \prod_{\alpha \in A}(\alpha, x)^{m_\alpha} \) be the potential form of the operator \( \mathcal{L}_A \), then we have the equation
\[
L_A (g^{-1}f) = Eg^{-1}f. \tag{24}
\]
Consider the expansions at \( (\alpha, x) = 0, \alpha \in A \), where \( \alpha \) is normalized to have unit length:
\[
g^{-1}f = (\alpha, x)^{-m_\alpha} \left( f_0 + f_1(\alpha, x) + f_2(\alpha, x)^2 + \ldots \right), \tag{25}
\]
\[
L_A = \partial_\alpha^2 + \tilde{\Delta} - \frac{m_\alpha(m_\alpha + 1)}{(\alpha, x)^2} - u_0 - u_1(\alpha, x) - u_2(\alpha, x)^2 - \ldots, \tag{26}
\]
where \( \tilde{\Delta} \) is the Laplacian in the hyperplane \( (\alpha, x) = 0 \), and all the coefficients \( f_i, u_i \) are functions on the hyperplane. One can check by direct calculation that for the systems \( A_n(m), C_{n+1}(m,l) \)
\[
u_1 = u_3 = \ldots = u_{2m_\alpha - 1} = 0.
\]
Substitution of the expansions (25), (26) into equation (24) shows that
\[
f_1 = f_3 = \ldots = f_{2m_\alpha - 1} = 0,
\]
so that
\[
f = \prod_{\beta \neq \alpha} (\beta, x)^{m_\beta} \left( f_0 + f_2(\alpha, x)^2 + \ldots \right). \tag{27}
\]
We have to show that
\[ \partial^s f = 0 \quad \text{at} \quad (\alpha, x) = 0 \quad (28) \]
for \( s = 1, 3, \ldots, 2m_\alpha - 1 \). Since the systems \( \mathcal{A}_n(m), \mathcal{C}_{n+1}(m, l) \) are symmetric with respect to each hyperplane with multiplicity \( m_\alpha > 1 \) for such hyperplane these conditions are obviously satisfied. If the multiplicity \( m_\alpha = 1 \) then the condition (28) follows from the form (27) and the relation
\[ \partial_\alpha \left( \prod_{\beta \neq \alpha} (\beta, x)^{m_\beta} \right) = 0 \quad \text{if} \quad (\alpha, x) = 0, \]
which is equivalent to identity (9). Lemma is proven.

Let us introduce now the following ideal \( I^A(\lambda) \subset \hat{Q}^A \) generated by the polynomials \( p_s - \lambda_s, \quad s = 1, \ldots, n+1 \) in the case \( \mathcal{A}_n(m) \) and by the polynomials \( q_s - \lambda_s, \quad s = 1, \ldots, n+1 \) in \( \mathcal{C}_{n+1}(m, l) \) case.

**Proposition 6** The following two dimensions are equal
\[ \dim Q^A/I^A(\lambda) = \dim H^A(\lambda), \quad (29) \]
where \( \mathcal{A} = \mathcal{A}_n(m), \mathcal{C}_{n+1}(m, l) \).

The proof follows from the following two lemmas.

**Lemma 3** The orthogonal complement to the ideal \( I^A(\lambda) \) in the completion \( \hat{Q}^A \) is the solution space \( H^A(\lambda) \).

**Lemma 4** The space \( H^A(\lambda) \) is isomorphic to the dual space \( (Q^A/I^A(\lambda))^* \). The element \( h \in H^A(\lambda) \) defines a functional on the factor space by the formula
\[ q + I^A(\lambda) \to (q + I^A(\lambda), h) \]
where \( q + I^A(\lambda) \) is arbitrary element from \( Q^A/I^A(\lambda) \).

The proof is based on the non-degeneracy of the scalar product (20).

The next claim is that the dimension of \( Q^A/I^A(\lambda) \) is the largest when \( \lambda = 0 \).

**Lemma 5** For any \( \lambda \in \mathbb{C}^{n+1} \) the inequality
\[ \dim Q^A/I^A(0) \geq \dim Q^A/I^A(\lambda) \]
holds. If \( r_1, \ldots, r_N \) is a homogeneous basis in the complement to the ideal \( I^A(0) \) then the classes \( \bar{r}_i = r_i + I^A(\lambda) \) generate \( Q^A/I^A(\lambda) \).
**Proof** Consider a homogeneous basis $r_1, \ldots, r_N$ in the complement to the ideal $I^A(0)$. For arbitrary $q \in Q^A$ we have

$$ q = \sum_{i=1}^{N} \mu_i r_i + q', $$

where $q' \in I^A(0)$ and $\mu_i$ are some constants. Since $q' \in I^A(0)$, for $q'$ there is a representation

$$ q' = \sum_{i=1}^{n+1} t_i p_i $$

when $A = A_n(m)$, and one should write polynomials $q_i$ instead of $p_i$ for the system $A = C_{n+1}(m, l)$. The polynomials $t_i \in Q^A$ have the degrees less than $\deg q$, and

$$ q = \sum_{i=1}^{N} \mu_i r_i + \sum_{i=1}^{n+1} t_i p_i. \quad (30) $$

For $t_i$ we have a similar representation

$$ t_i = \sum_{j=1}^{N} \mu_{ij} r_j + f_i \quad (31) $$

with $f_i \in I^A(0)$. Next we should write $f_i$ as a combination of $p_1, \ldots, p_{n+1}$ and substitute (31) back to (30). Continuing the process we arrive to the presentation

$$ q = \sum_{i=1}^{N} \tau_i r_i, \quad (32) $$

where $\tau_i$ are polynomials in $p_1, \ldots, p_{n+1}$. Now, consider relation (32) modulo ideal $I^A(\lambda) = (p_i - \lambda_i)$. We get

$$ \bar{q} = \sum c_i \bar{r}_i, $$

where the constants $c_i = \tau_i(\lambda)$. Thus we see that $\bar{r}_i$ generate the whole space $Q^A/I^A(\lambda)$, and hence $\dim Q^A/I^A(\lambda) \leq N$. The lemma is proven.

**Proposition 7** For any $\lambda \in \mathbb{C}^{n+1}$ the dimensions

$$ \dim Q^A/I^A(\lambda) = \dim Q^A/I^A(0) < \infty. $$

**Proof.** We will use Proposition 6. Recall that the spaces $H^A(\lambda)$ consist of formal solutions to the system (13). But by Proposition 5 any locally analytic solution of (13) is actually holomorphic everywhere in $\mathbb{C}^{n+1}$. This implies that

$$ \dim H^A(\lambda) \geq \mu^A, $$

14
where $\mu^A$ denotes the dimension of the space of (locally) analytic solutions.

Consider now the value $\lambda = 0$. We claim that all the formal series solutions $F$ to systems \[ (13) \] with $\lambda = 0$ are actually polynomial. Indeed, if $F \in \mathbb{C}[[x]]$ is a formal solution then every homogeneous component of $F$ is also a solution to the system due to homogeneity of system \[ (13) \] if $\lambda = 0$. So if a formal solution exists then there are infinitely many analytic solutions. But the space of analytic solutions has finite dimension $\mu^A$. Therefore $\dim H^A(0)$ is also equal to $\mu^A$.

Thus using Lemma 5 and Proposition 6 we have

$$\mu^A = \dim Q^A/I^A(0) \geq \dim Q^A/I^A(\lambda) \geq \mu^A,$$

which proves the statement.

Finally we have the main result of this section.

**Theorem 6** The algebra of quasi-invariants $Q^A$ is a free module of rank $\mu^A$ over its polynomial subalgebra $P^A$.

**Proof** Consider a homogeneous basis $r_1, \ldots, r_\mu$ in the complement to the ideal $I^A(0)$. As we established in the proof of Lemma 5 every element $q \in Q^A$ has a representation

$$q = \sum_{i=1}^\mu \tau_i r_i,$$

where $\tau_i$ are some polynomials in $p_1, \ldots, p_{n+1}$, and the classes $\bar{r}_i = r_i + I^A(\lambda)$ generate the space $Q^A/I^A(\lambda)$. From Proposition 7 it follows that the elements $\bar{r}_i$ are linearly independent. Now, if the polynomials $r_1, \ldots, r_\mu$ were dependent over the ring generated by $p_1, \ldots, p_{n+1}$ we would have a relation

$$\sum_{i=1}^\mu s_i r_i = 0$$

where $s_i$ are some polynomials in $p_1, \ldots, p_{n+1}$. Hence, taking this relation modulo $I^A(\lambda)$ we get

$$\sum_{i=1}^\mu c_i \bar{r}_i = 0$$

where $c_i = s_i(\lambda)$. Since generically all $c_i$ are nonzero, this contradicts to linear independence of $\bar{r}_i$. The theorem is proven.

### 5 Poincare series for quasi-invariants

In this section we calculate the Poincare series for two-dimensional deformations $A_2(m)$ and $C_2(m, l)$. 


Let \( P^A(t) \) be the Poincare polynomial for the complement to the ideal \( I^A(0) \) in \( Q^A \). If \( p^A(t) \) is the Poincare series for the quasi-invariants then from Theorem 6 it follows that

\[
p^A_2(m)(t) = \frac{P^A_2(m)(t)}{(1 - t)(1 - t^2)(1 - t^3)}, \tag{33}
\]

\[
p^C_2(m,l)(t) = \frac{P^C_2(m,l)(t)}{(1 - t^2)(1 - t^4)}. \tag{34}
\]

We are going to compute the polynomials \( P^A_2(m)(t) \) and \( P^C_2(m,l)(t) \).

**Theorem 7** The Poincare polynomials for \( \mathcal{A}_2(m) \) and \( \mathcal{C}_2(m,l) \) have the form:

\[
P^A_2(m)(t) = 1 + t^4 + t^5 + t^{2m+2} + t^{2m+3} + t^{2m+7}, \tag{35}
\]

\[
P^C_2(m,l)(t) = 1 + t^6 + t^{2m+3} + t^{2m+5} + t^{2l+3} + t^{2l+5} + t^{2(m+l+1)} + t^{2(m+l+4)}. \tag{36}
\]

Since these Poincare polynomials are palindromic according to the general Stanley result \[18\] we have the following result which we believe to be true in arbitrary dimension \( n \).

**Corollary 2** The rings \( Q^A_2(m) \) and \( Q^C_2(m,l) \) are Gorenstein.

**Proof of the Theorem.** We first consider \( \mathcal{C}_2(m,l) \) case. We will actually compute the Poincare series for \( Q^C_2(m,l) \) by direct computation.

Consider an arbitrary polynomial of degree \( n \) in two variables:

\[
q(x, y) = \sum_{i=0}^{n} a_i x^i y^{n-i}.
\]

The quasi-invariance conditions on the lines \( x = 0, y = 0 \) have the following simple form:

\[
a_1 = a_3 = \ldots = a_{2m-1} = 0, \tag{37}
\]

and

\[
a_{n-1} = a_{n-3} = \ldots = a_{n-(2l-1)} = 0. \tag{38}
\]

We have two more quasi-invariance conditions on the lines \( \Pi_{\pm} : \xi x \pm y = 0 \), where

\[
\xi = \sqrt{\frac{2l+1}{2m+1}}. \tag{39}
\]
Let us write these conditions

\[ (\xi \partial_x \pm \partial_y)q(x, y)|_{\Pi_{\pm}} = \sum_{i=0}^{n} (\xi i a_i x^{n-i} y_i \pm (n-i) a_i x^{n-i} y_{n-i})|_{\Pi_{\pm}} = \]

\[ \sum_{i=0}^{n} (\xi i a_i (\mp \xi)^{n-i} \pm (n-i) a_i (\mp \xi)^{n-i} x^{n-i})|_{\Pi_{\pm}} = \]

\[ \sum_{i=0}^{n} (\pm i a_i (\mp \xi)^{n-i+1} \pm (n-i) a_i (\mp \xi)^{n-i} x^{n-i-1})|_{\Pi_{\pm}} = 0. \]

Thus we obtain

\[ \sum_{i=0}^{n} a_i (\pm \xi)^{n-i-1} ((n-i) - \xi^2 i) = 0. \]

These equations are equivalent to the following

\[ \sum_{i=0}^{n} a_i \xi^{n-i-1} ((n-i) - \xi^2 i) = 0, \quad (40) \]

\[ \sum_{i=0}^{n} a_i \xi^{n-i-1} ((n-i) - \xi^2 i) = 0, \quad (41) \]

Now the question is: how many linear independent conditions on the coefficients \( a_0, \ldots, a_n \) are among \( (37), (38), (40), (41) \)?

Consider first the case when \( n \) is odd. The equations on the coefficients \( a_i \) split into the equations for the coefficients with odd and even indices. Due to the equations \( (37), (38) \) we have \( \frac{n+1}{2} - l \) nontrivial even coefficients, and \( \frac{n+1}{2} - m \) nontrivial odd coefficients. Both equations \( (40), (41) \) are nontrivial, each of them is a linear equation for the nontrivial coefficients \( a_i \) with even, and, correspondingly, odd, indices \( i \). Summarizing, for a given odd \( n \) we have \( \frac{n+1}{2} - l - 1 \) - dimensional space of the coefficients with even indices, and \( \frac{n+1}{2} - m - 1 \) - dimensional space of the coefficients with odd indices. Therefore the Poincare series for the quasi-invariants with odd degrees has the form

\[ p_{\text{odd}} = \sum_{n \geq 2l+1} \left( \frac{n + 1}{2} - l - 1 \right) t^n + \sum_{n \geq 2m+1} \left( \frac{n + 1}{2} - m - 1 \right) t^n = \]

\[ t^{2l+3}(1 + 2t^2 + 3t^4 + 4t^6 + \ldots) + t^{2m+3}(1 + 2t^2 + 3t^4 + 4t^6 + \ldots) = \]

\[ \frac{t^{2l+3} + t^{2m+3}}{(1-t^2)^2}. \quad (42) \]
Now let us consider the even part of the Poincare series. When \( n \) is even conditions (37), (38) are both the conditions for the coefficients \( a_i \) with odd indices \( i \). Therefore for the coefficients \( a_i \) with even \( i \) the only restriction is given by (40), and it is nontrivial unless \( n = 0 \). Thus the space of possible even coefficients has the dimension \( \frac{n^2}{2} + 1 - \frac{n}{2} = \frac{n^2}{2} \) if \( n \neq 0 \).

If \( n = 0 \) we still have one-dimensional space of quasi-invariants given by the constants \( a_0 \).

The conditions on the odd coefficients \( a_i \) are independent from the conditions on the even coefficients. Therefore the Poincare series \( p_{even} \) of even degree quasi-invariants has the form

\[
p_{even} = p_{odd} + p_{even}^{even}.
\]

As we just analyzed,

\[
p_{even} = 1 + \sum_{n=2k, n>0} \frac{n}{2} t^n = 1 + t^2(1 + 2t^2 + 3t^4 + \ldots) = 1 + \frac{t^2}{(1 - t^2)^2}.
\]

We are left to calculate the Poincare subseries \( p_{odd} \) corresponding to quasi-invariants which have even degree and which are odd in \( x \) and \( y \). We have \( \frac{n}{2} \) odd coefficients \( a_1, a_3, \ldots, a_{n-1} \) and conditions (37), (38), (41). If \( m + l \geq \frac{n}{2} \) then by conditions (37), (38) all the coefficients must be zero and there are no quasi-invariants of the form required. If \( m + l = \frac{n}{2} - 1 \) then we have \( m + l \) vanishing conditions (37), (38), and equation (41) for the only nontrivial coefficient \( a_{2m+1} \). But

\[
(n - i) - \xi^2 i = 0
\]

if \( i = 2m + 1 \) and \( m + l = \frac{n}{2} - 1 \) (see the form (39)). Hence equation (41) gives no additional restrictions, and there exists one dimensional space of quasi-invariants when \( n \) satisfies \( m + l = \frac{n}{2} - 1 \), i.e. when \( n = 2(m + l + 1) \). If \( n > 2(m + l + 1) \) then the equation (41) is already nontrivial and we have the dimension of quasi-invariants being equal to \( \frac{n}{2} - m - l - 1 \).

Thus we have

\[
p_{even}^{odd} = t^{2(m+l+1)} + \sum_{k=0}^{\infty} kt^{2(m+l+1)+2k} =
\]

\[
t^{2(m+l+1)} + t^{2(m+l+1)+2} \left( 1 + 2t^2 + 3t^4 + \ldots \right) = t^{2(m+l+1)} + \frac{t^{2(m+l+1)+2}}{(1 - t^2)^2}
\]

Altogether collecting (42), (43) and (44) we get the following expression for the Poincare series

\[
p = \frac{t^{2l+3} + t^{2m+3}}{(1 - t^2)^2} + 1 + \frac{t^2}{(1 - t^2)^2} + t^{2(m+l+1)} + \frac{t^{2(m+l+1)+2}}{(1 - t^2)^2} =
\]

\[
1 + t^6 + \frac{t^{2l+3} + t^{2m+3} + t^{2l+5} + t^{2m+5} + t^{2(m+l+1)} + t^{2(m+l+4)}}{(1 - t^2)(1 - t^4)}.
\]
This proves the formula (36).

The case $A_2(m)$ can be reduced to the previous one. The observation is that the system $A_2(m)$ considered as a system in $C^2 \simeq \{ x \in C^3 | x_1 + x_2 + \frac{1}{\sqrt{m}} x_3 = 0 \}$ is identical to $C_2(m, l)$ with multiplicity $l = 0$. Since the Poincare series for the quasi-invariants in $C^2$ and $C^3$ satisfy obvious relation

$$\frac{p_{A_2(m) \subset C^2}}{1-t} = p_{A_2(m) \subset C^3}$$

we have

$$p_{A_2(m) \subset C^2} = (1-t)p_{A_2(m) \subset C^3} = \frac{P_{A_2(m)}}{(1-t^2)(1-t^3)} = \frac{P_{C_2(m,0)}}{(1-t^2)(1-t^4)}.$$

Thus we have

$$P_{C_2(m,0)}(1-t^3) = P_{A_2(m)}(1-t^4),$$

which together with (36) implies the formula (35).

**Remark.** Note that the "beginnings" of the Poincare polynomials (35), (36) do not depend on the multiplicities. We conjecture that the corresponding "stable" parts of Poincare polynomials for the general $n$ have the form

$$P_{A_n(m)}^{\text{stable}} = 1 + t^{n+2} + t^{n+3} + \ldots + t^{2n+1}$$

and

$$P_{C_{n+1}(m,l)}^{\text{stable}} = 1 + t^{2(n+2)} + t^{2(n+3)} + \ldots + t^{2(2n+1)}.$$

This form is motivated by the formula (24) from [5] for the Poincare series for the subalgebra of quasi-invariants generated by all deformed Newton sums for generic values of deformation parameter. It is natural also to suggest that for large multiplicities $m$ the corresponding $n$ deformed Newton sums $p_s$ with $s = n+2, \ldots, 2n+1$ can be chosen as the first $n$ free generators for the module $Q_{A_n(m)}$ over $P_{A_n(m)}$.

### 6 Deformed $m$-harmonic polynomials

In the paper [7] we introduced the notion of $m$-harmonic polynomials related to any Coxeter configuration. Their deformed versions related to configurations $A_n(m)$ and $C_{n+1}(m, l)$ are defined in a similar way as the solutions of the system

$$\mathcal{L}_i f = 0, \quad i = 1, \ldots, n+1,$$

where $\mathcal{L}_i$ are the operators from the Theorem [4].

**Theorem 8** All the solutions of the system (45) are polynomial and moreover are quasi-invariants. The degrees of a basis of homogeneous solutions to (45) coincide with the degrees of homogeneous generators of $Q^A$ as a module over $P^A$.
The proof follows from the results of section 4. Indeed, the first part follows from Lemma 2, Propositions 5 and the homogeneity of the system \(45\). To prove the second part note that according to Lemma 4 the space of the deformed \(m\)-harmonic polynomials \(H^A(0)\) is dual to the quotient \(Q^A/I^A(0)\) and use the homogeneity of the bilinear form \(20\).

In the Coxeter case we have conjectured in \([6]\) that these polynomials can be chosen as free generators for the corresponding algebra of quasi-invariants over invariants (like in the classical case due to Chevalley and Steinberg \([19, 20]\)). Although this turned out to be false in general (Etingof and Ginzburg \([11]\) found a counterexample related to \(B_6\) system) the question how often this fails is not clear yet (see \([7]\) for the discussion of this). It is interesting therefore to look what happens in the deformed case. In this section we consider only the simplest case \(A_2(m)\).

Let us consider the system \(A_2(m)\) as a system in \(C^2\) consisting of the vectors \(e_1\) with multiplicity \(m\) and \(e_1 \pm \sqrt{2m+1}e_2\) with multiplicity 1. In this embedding the first deformed Newton sums have the form

\[
p_1 = x^2 + y^2, \quad p_2 = (2m - 1)y^3 - 3x^2y.
\]

The corresponding quantum integrals are

\[
L_1 = \partial_x^2 + \partial_y^2 + \frac{4(1 + 2m)y}{x^2 - (2m + 1)y^2} \partial_y + \frac{2m(1 + 2m)y^2 - 2(2 + m)x^2}{x(x^2 - (2m + 1)y^2)} \partial_x,
\]

and

\[
L_2 = (2m - 1)\partial_y^2 - 3\partial_x^2\partial_y - \frac{6(4m^2 - 1)y}{(2m + 1)y^2 - x^2} \partial_y^2 + \frac{6(2m + 1)y}{(2m + 1)y^2 - x^2} \partial_x^2
\]

\[
+ \frac{6((m + 2)x^2 - m(2m + 1)y^2)}{x(x^2 - (2m + 1)y^2)} \partial_x \partial_y
\]

\[
- \frac{12(2m + 1)y((m - 2)x^2 + m(1 + 2m)y^2)}{x(x^2 - (2m + 1)y^2)^2} \partial_x - \frac{12(2m + 1)(x^2 + (1 - 4m^2)y^2)}{(x^2 - (2m + 1)y^2)^2} \partial_y. \tag{46}
\]

The deformed \(m\)-harmonic polynomials are solutions to the system

\[
\begin{aligned}
L_1 f &= 0 \\
L_2 f &= 0.
\end{aligned} \tag{47}
\]

**Proposition 8** The space of the deformed \(m\)-harmonic polynomials for the configuration
\( A_2(m) \) is generated by the following 6 quasi-invariants:

\[
q_1 = 1, \quad q_2 = x^4 + 2(2m + 1)x^2 y^2 + (1 - 4m^2)y^4, \quad q_3 = x^4 y - \frac{1}{5}(2m - 3)(2m + 1)y^5, \\
q_4 = x^{2m+1}y, \quad q_5 = x^{2m+1}(x^2 + (2m + 3)y^2), \\
q_6 = 5x^{7+2m} + x^{5+2m}(-35y^2 - 10my^2) + x^{3+2m}(35y^4 + 80my^4 + 20m^2y^4) + \\
x^{1+2m}(-21y^6 - 62my^6 - 44m^2y^6 - 8m^2y^6).
\]

Analysis of these formulas shows that the deformed \( m \)-harmonics fail to give a basis for the quasi-invariants module already for the first non-trivial case \( m = 2 \). Indeed, in that case the polynomial \( q_3 \) has the form

\[
q_3 = y(x^2 - y^2)(x^2 + y^2) = -\frac{1}{3}p_1p_2
\]

and thus belongs to the ideal generated by \( p_1 \) and \( p_2 \).

Note that the degrees of \( q_i \) are

\[
0, 4, 5, 2m + 2, 2m + 3, 2m + 7
\]

which is in agreement with Theorem 7. When \( m = 1 \) (i.e. in the Coxeter case) these degrees are known for all \( n \) (see [21]).

7 Acknowledgements

We are grateful to Yu. Berest, O. Chalykh, P. Etingof and A.N. Sergeev for useful discussions. The second author (A.P.V.) is grateful to IHES (Bures-sur-Yvette, France) for the hospitality in February 2003 when the final version of this paper was prepared.

This work was partially supported by EPSRC (grant GR/M69548).

References

[1] F. Calogero Solution of the one-dimensional \( n \)-body problem with quadratic and/or inversely quadratic pair potential // J. Math. Phys. 1971. V. 12. P. 419–436.

[2] M.A. Olshanetsky, A.M. Perelomov Quantum completely integrable systems connected with semi-simple Lie algebras. // Lett. Math. Phys. 2 1977 P. 7–13.

[3] Veselov A.P., Feigin M.V., Chalykh O.A. New integrable deformations of quantum Calogero–Moser problem. // Russian Math Surveys 1996. V. 51, N.3. P. 185–186.
[4] Sergeev A.N. *Superanalogs of the Calogero operators and Jack polynomials.* // J. Nonlin. Math. Phys. V. 8, 2001, no. 1, 59–64.

[5] Sergeev A.N., Veselov A.P. *Deformed quantum Calogero-Moser problems and Lie superalgebras.* Preprint IHES /M/03/14, February 2003.

[6] Feigin M., Veselov A.P. *Quasi-invariants of Coxeter groups and m-harmonic polynomials.* // Preprint 2001, math-ph/0105014

[7] Feigin M., Veselov A.P. *Quasi-invariants of Coxeter groups and m-harmonic polynomials.* // Intern. Math. Res. Notices 2002, N. 10, P. 521–545.

[8] Chalykh O.A., Feigin M.V., Veselov A.P. *New integrable generalizations of Calogero-Moser quantum problem* // J. Math. Phys. 1998. V. 39, No. 2. P. 695–703.

[9] Chalykh O.A., Feigin M.V., Veselov A.P. *Multidimensional Baker–Akhiezer Functions and Huygens’ Principle* // Commun. Math. Phys. 1999. V. 206. P. 533–566.

[10] Chalykh O.A., Veselov A.P. *Commutative rings of partial differential operators and Lie algebras* // Commun. Math. Phys. 1990. V. 126. P. 597–611.

[11] Etingof P., Ginzburg V. *Om m-quasi-invariants of Coxeter groups* // Preprint 2001, math.QA/0106175, to appear in Math. Moscow Journal

[12] Arnold V.I., Varchenko A. N., Gussein-Zade S.M. *Singularities of differentiable maps* // Moscow, ”Nauka”, 1982.

[13] O.A. Chalykh, A.P.Veselov *Locus configurations and ∨-systems.* // Phys. Lett. A, V. 285, 2001, 339-349.

[14] G.J. Heckman, E.M. Opdam *Root systems and hypergeometric functions I* // Comp. Math. 1988.

[15] Berest Yu. *Huygens’ principle and the bispectral problem* // CRM Proceedings and Lecture Notes, 1998, V. 14, P. 11–30.

[16] M. Atiyah, I.G. Macdonald *Introduction to Commutative Algebra.* Addison-Wesley, 1969.

[17] Opdam E.M. *Dunkl operators, Bessel functions and the discriminant of a finite Coxeter group* // Comp. Math., V. 85, 1993, 333-373.

[18] Stanley R. *Hilbert functions for graded algebras.* // Adv.Math., V. 28, 1978, 57–83.

[19] Chevalley C. *Invariants of finite groups generated by reflections.* // Amer. J. Math, 1955, V. 77, 778-782.

[20] Steinberg R. *Differential equations invariant under finite reflection groups* //Trans. Amer. Math. Soc.,V.112 (1964), 392-400.
[21] Felder G., Veselov A.P. *Action of Coxeter groups on m-harmonic polynomials and KZ equations.* // Preprint 2001, math.QA/0108012 (to appear in Moscow Math. J.).