A conceptual problem for non-commutative inflation and the new approach for non-relativistic inflationary equation of state

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In a previous paper, we connected the phenomenological non-commutative inflation of Alexander, Brandenberger and Magueijo (2003) and Koh S and Brandenberger (2007) with the formal representation theory of groups and algebras and analyzed minimal conditions that the deformed dispersion relation should satisfy in order to lead to a successful inflation. In that paper, we showed that elementary tools of algebra allow a group like procedure in which even Hopf algebras (roughly the symmetries of non-commutative spaces) could lead to the equation of state of inflationary radiation. In this paper, we show that there exists a conceptual problem with the kind of representation that leads to the fundamental equations of the model. The problem comes from an incompatibility between one of the minimal conditions for successful inflation (the momentum of individual photons is bounded from above) and the group structure of the representation which leads to the fundamental inflationary equations of state. We show that such a group structure, although mathematically allowed, would lead to problems with the overall consistency of physics, like in scattering theory, for example. Therefore, it follows that the procedure to obtain those equations should be modified according to one of two possible proposals that we consider here. One of them relates to the general theory of Hopf algebras while the other is based on a representation theorem of Von Neumann algebras, a proposal already suggested by us to take into account interactions in the inflationary equation of state. This reopens the problem of finding inflationary deformed dispersion relations and all developments which followed the first paper of Non-commutative Inflation. In particular, this analysis shows us that we really need to consider the Hopf algebra language (here explained at an elementary level) in order to adequately formulate non-commutative inflation, a point only heuristically suggested in the original papers.

I. INTRODUCTION

Non-commutative inflation, as originally conceived [1], is a phenomenological model of inflation inspired by a varying speed of light cosmology [2 and 3]. It was proposed as a possible mechanism connecting non-commutative spaces with inflation. In this model, radiation, rather than a scalar field, drives inflation. Non-commutative inflation is an application of a conjecture that the non-relativistic character of non-commutative spaces [4 and 5], modeled by a non-relativistic dispersion relation, was enough to determine the thermodynamics of radiation.

Despite the original constraint with non-commutative spaces, related to many interesting Cosmological applications like in [6 and 7], non-commutative inflation is actually a bridge connecting general violations of Lorentz symmetry with inflation, an idea that can be related with a wider literature in high energy physics [8, 9, 10, 11, 12 and 13]. Instead of the usual Lagrangian approach of physics, in which non-relativistic effects are modeled by non-relativistic terms that can affect the cosmological perturbations, such as in the so called trans-Planckian problem of inflation [14 and 15], the connection between non-commutative spaces and the fundamental equations of non-commutative inflation can be more directly related with the Wigner approach for QFT and the idea of Hopf algebra deformations of the Poincare Lie algebra due to non-commutative spaces [16], rather than the star product approach [17]. Although a general procedure to deform a Lie algebra as a function of non-commutative space information is not available in the literature, postulating the existence of a generalization of the analysis [16], it is possible to define the quantum theory of matter, not by a quantization of a deformed action assumed to be invariant by Hopf algebra, as previously done, or the star product approach to non-commutative field theories [17], but as a deformation of the Wigner procedure to define quantum theory in terms of representations of the Poincaré group as quantum symmetries [18]. This procedure is assured, by the general techniques of algebra, like the GNS construction (see [19] and [20]), to have much in common with a Lie group representation, in particular, the fundamental role of the dispersion relation in the scheme.

The Hopf algebra concept, more than an effort to bridge non-commutative spaces with non-commutative inflation, is actually a key concept in a more rigorous formulation of non-commutative inflation, being a useful concept in group theory in terms of which one can ask what is the group (algebraic) structure related to inflation. Something similar to ask what is the algebraic structure related to a fundamental invariant length scale
and a fundamental speed, leading to the seminal paper of the doubly special relativity \cite{22} (see a generalization of this idea to include an invariant expansion rate \cite{21}). More than that, the algebra is the key connecting a phenomenological deformed dispersion relation with the effective energy-momentum tensor of inflationary matter.

Our main conclusion in this paper is that the Fock space structure, the basis underlying non-commutative inflation, is incompatible with minimum duration of inflation. Also, we propose alternatives to this structure that should be applied in order to obtain new fundamental equations.

The paper is organized as follows:

In section (II) we sketch a Wigner like argument which could be (at first sight) considered as a more formal proof that the fundamental equations of non-commutative inflation are actually a consistent way to connect deformed Poincaré symmetries with a particular kind of equation of state for radiation in the early universe.

In section (III) we introduce a useful concept for our analysis, the Hopf Algebra concept. This is a generalization of the concept of Lie Algebra which is the algebraic structure underlying the Poincaré group. Previous works in the literature suggest that Hopf algebras play an important role in defining the meaning of symmetry in non-commutative spaces.

In section (IV), we introduce elementary tools of algebra which allow a connection between Hopf algebras and the familiar structure of physical quantum theories in Hilbert space. This, at first sight, suggests that there should not be any modification in the argument sketched in section (II). We then state an algebraic prescription to define Non-commutative inflation that is in harmony with the original equations. That is essentially a Fock space like structure written in group theory terms.

In section (V), we show that if we want to consider a more general symmetry than Poincaré, minimal physical requirements will constrain the way in which we are allowed to do that. These are related to things like (among others):

- A system which is in a stable state according to one inertial frame should be in a stable state in all other inertial frames.

- If an observer observes a scattering experiment preserving energy and momentum in one reference frame, all other observers will agree that energy and momentum are conserved by scattering.

In section (VI), we recall a minimal condition for successful inflation which is a result of our previous paper. That is that the condition that zero mass one-particle states should have an upper bound to momentum. We then show that if we construct the more general representation of a symmetry group satisfying the conditions of the previous section, we find that the momentum of the theory itself must have an upper bound, including the classical macroscopic world. However, this excludes the prescription outlined in section (IV). This also implies that we are no longer dealing with Lie Algebras, but general Hopf algebras, and therefore strengthens the bond between non-commutative spaces and non-commutative inflation. We argue that what happens here is similar to what happens in relativistic quantum theory in which representations with no lower bound to energy are mathematically possible but not physically allowed. It follows that the Fock space representation is mathematically allowed but not physically allowed.

In section (VII), we use concepts and methods of previous sections to propose two alternative ways (which are consistent with the requirements of section (V)) to use the one-particle information to construct the multiparticle theory. Neither of them is a Fock space like structure. One of those can in principle be applied to an interacting case and will assure a well defined finite and unitary quantum theory with a low energy relativistic limit if we already have the relativistic counterpart. The equivalence of the two prescriptions or not is still not established. These prescriptions still require closed form for the equation of state of the Non-commutative radiation. This is postponed for a future work. We then present our conclusions in the last section.

\section{A Group Theory Argument for Non-commutative Inflation}

The basic idea explored in \cite{22} and \cite{1} is that the effect of generic models of non-commutative space-time can be codified in the modification of the energy-momentum relation. This is indicated in most of the implementations of the non-commutative principle to field theory \cite{1}, \cite{22}. This would affect the calculation of the canonical partition function for radiation that in turn affects the early phases of the universe in thermal equilibrium.

This idea is formalized in the Wigner approach to relativistic quantum theory, as already discussed in \cite{18}, in which the basic problem is to construct representations of the Poincaré group as quantum symmetries, not the quantization of a particular classical field. That is, finding the correspondence \(\pi\) among Lorentz transformations

\[
x \rightarrow \Lambda x + a^\mu,\text{ denoted by } (\Lambda, a^\mu),\text{ and the transformation }\pi[(\Lambda, a^\mu)]; \mathcal{H}_P \rightarrow \mathcal{H}_P,\text{ that satisfies:}
\]

\[
\pi[(\Lambda_1, a_1^\mu)] \cdot \pi[(\Lambda_2, a_2^\mu)] = \pi[(\Lambda_1 a_1^\mu) \cdot (\Lambda_2, a_2^\mu)],
\]

\[
\pi[(\Lambda_1, a_1^\mu)]^{-1} = \pi[(\Lambda_1, a_1^\mu)^{-1}],
\]

(2)

(3)

where \(\mathcal{H}_P\) is the projective Hilbert space, that is, the space of rays, \(\cdot\) denotes the composition and \((-)^{-1}\) the inverse. The symmetry is characterized by the invariance of transition amplitudes:

\[
|\langle T\Phi, T\Psi \rangle| = |\langle \Phi, \Psi \rangle|.
\]

(4)

According to the Wigner theorem, a quantum symmetry can be extended from rays in \(\mathcal{H}_P\) describing quantum
states to the entire Hilbert space $\mathcal{H}$ as a linear unitary or a antilinear antiunitary transformation. After such an extension, we might be dealing with projective representations, i.e. eq. $\mathbf{1}$ might be valid except for a phase $e^{i\theta}$.

It is implicit in the Wigner prescription to quantum symmetries that all transformations act in the same physical space, which is left invariant by them, all transformations have an inverse and the composition is also an allowed transformation, they therefore form a group. That is, despite all of the possible complex algebraic structures that one may consider, we are always dealing with groups.

The Poincaré group $P$ is a very particular group, a Lie group, that is, a group with topological structure. In addition, a Lie group which possesses a connected subgroup, the proper and orthochronous part $P^+$ (for which the Lorentz subgroup satisfies $\Lambda^0_0 > 0$ and $\det \Lambda = 1$). The algebraic structure of the group, codified in the multiplication rule, does not constrain its topology and we could consider a particularly useful representative of this group, the so called universal enveloping group, in which every element is part of a one-parameter subgroup and thus is the square of some other element. In addition, every element of a connected Lie group can be written as a product of elements in an arbitrary small neighborhood of the identity. The square of an antilinear antiunitary operator is linear and unitary.

For connected Lie groups, the symmetry representation problem reduces to construct unitary representations, since, at least in the neighborhood of the identity described by canonical coordinates, every group element is part of a one-parameter subgroup and thus is the square of some other element. In addition, every element of a connected Lie group can be written as a product of elements in an arbitrarily small neighborhood of the identity. The square of an antilinear antiunitary operator is linear and unitary. The continuous one parameter family of unitary (non-projective) transformations affords a further simplification due to the Stone theorem, that can be put in the block diagonal form. If this process cannot be continued, every block is an example of the simplest kind of representation, the so called irreducible representation. The unitary representation of a group summarizes into constructing the irreducible pieces by which every other representation can be constructed by. These basic parts are identified with the Hilbert space of one particle states. The dispersion relation $C(p)$ is the fundamental information in this process because it is a (self-adjoint) function of space and time translation generators which commutes with all other generators of the symmetry group (Casimir of the Lie algebra) and defines a bounded operator $\hat{Z} = (e^{iC(p)})$ which commutes with every element of the group. By an infinite dimensional version of Schur's lemma, a unitary representation of a group is irreducible if, and only if, every bounded operator which commutes with every element of the group is a multiple of the identity. Indeed, for the Poincaré group, the algebraically independent set of Casimir operators is:

$$P^2 = H^2 - \beta^2,$$

$$W = -\psi^2; \quad \text{com} w_\rho = \frac{1}{2}\epsilon_{\alpha\mu\nu\rho}P^\alpha M^{\mu\nu},$$

$$\text{sign}(P^0) = \theta(P^2)e(P^0),$$

Given two irreducible representations of $SL(2, C)$, $U_1$ and $U_2$ in the Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively (with possible additional symmetries such as parity in the electrodynamics case), we can construct another representation in the tensor product Hilbert space $\mathcal{H}_1 \otimes \mathcal{H}_2$ which describes two non-interacting relativistic particles:

$$U_1 \otimes U_2(\sum_{ij} c_{ij} \Psi_i \otimes \Phi_j) = \sum_{ij} c_{ij}(U_1 \Psi_i) \otimes (U_2 \Phi_j).$$

The generators of the Lie algebra are given by:

$$X_i^\otimes = X_i^{(1)} \otimes I + I \otimes X_i^{(2)},$$

$X_i^{(i)}$ generators of the Lie algebra of the representation $U_i$.

In particular, we can make $U_1 = U_2$ and take into account indistinguishable particles by introducing a Hilbert space basis in the symmetrized form:

$$S_N(\psi_1 \otimes \cdots \otimes \psi_N) = \frac{1}{N!} \sum_\sigma \psi_{\sigma(1)} \otimes \cdots \otimes \psi_{\sigma(N)}, \quad \psi_i \in \mathcal{H}$$

where $\sigma$ is an index permutation, or in the antisymmetrized for:

$$A_N(\psi_1 \otimes \cdots \otimes \psi_N) = \frac{1}{N!} \sum \text{sign}(\sigma)\psi_{\sigma(1)} \otimes \cdots \otimes \psi_{\sigma(N)},$$

$A$ an operator $A$ is bounded if $||A\Psi|| \leq C||\Psi||$, with $C$ independent of $\Psi$.

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[1] i.e. $\langle \Phi | U(\lambda) | \Psi \rangle$ continuous for every $\Phi$ in the Hilbert space $\mathcal{H}$.
where \( \text{sign}(\sigma) \) is 1 for an even permutation and \(-1\) for an odd permutation. This defines two kinds of Hilbert spaces:

\[
\mathcal{H}_s^N = \{ \psi \in \mathcal{H}^N; S_N \psi = \psi \} \tag{13}
\]

and

\[
\mathcal{H}_A^N = \{ \psi \in \mathcal{H}^N; A_N \psi = \psi \}. \tag{14}
\]

\( \mathcal{H}^N \) denoting the tensor product of \( N \) Hilbert spaces.

We then define the Fock space of radiation:

\[
U_F = \sum_{N=0}^{\infty} \oplus (U_{\lambda}^{\otimes N})_S (\mathcal{P}) \tag{15}
\]

where \( U_{\lambda}^{\otimes N} \) is the tensor product of \( N \) photon irreducible representations characterized by the set of eigenvalues \( \lambda \), in particular \( H^2 - P^2 = 0 \). \( U_F \) is defined in the Hilbert space:

\[
\mathcal{H} = \sum_{N=0}^{\infty} \oplus (\mathcal{H}_s^N \otimes \mathcal{H}_A^N) \tag{16}
\]

\( \mathcal{H}_s^N \otimes \mathcal{H}_A^N \) being the symmetrized tensor product of the Hilbert spaces where the photon irreducible representations acts. \( N = 0 \) corresponds to the trivial vacuum representation \( \pi([\Lambda_1, a_j^\dagger]) = I \).

The associated Hamiltonian is:

\[
H_F = \sum_{N=1}^{\infty} (H \otimes I \cdots \otimes I + I \otimes H \otimes \cdots \otimes I + \cdots + I \otimes I \cdots \otimes H)_N, \tag{17}
\]

In the irreducible representation, \( H \) can be put a form in which the momentum \( P^j \) is diagonal. Being an irreducible representation, all we need to specify is the dimension \( d \) of the eigenspace associated with the eigenvalue \( p^j \) that can be proved to be the same for all \( p^j \) in irreducible representations:

\[
H = \text{diag}(E(p), E(p), \cdots E(p)), \tag{18}
\]

denotes a \( d \times d \) diagonal matrix, \( E(p) \) being the photon dispersion relation. The dimension \( d \) is determined by the dimension of the corresponding little group, that is, the group subspace that leaves the four-momentum of the irreducible representation invariant. For \( P^2 = 0 \) and \( P^j \not= 0 \), we can choose the four-momentum \( k = (1,0,0,1) \) and using the correspondence between the Lorentz group action and \( SL(2, \mathbb{C}) \) action on \( 2 \times 2 \) hermitian matrix, in such a way that \( k \) corresponds to the matrix \( \tilde{p} \) given by:

\[
\tilde{p} = \begin{pmatrix}
1 & 0 \\
0 & 0
\end{pmatrix}, \tag{19}
\]

that is left invariant by the action of the elements \( A \in SL(2, \mathbb{C}) \) given by \( (\tilde{p} = A \tilde{p} A^*) \):

\[
\gamma_\phi = \begin{pmatrix}
e^{i\phi} & 0 \\
0 & e^{-i\phi}
\end{pmatrix}, \quad \gamma_\eta = \begin{pmatrix}
1 & \eta \\
0 & 1
\end{pmatrix}, \tag{20}
\]

\( \gamma_\phi, \gamma_\eta \) real (mod \( 2\pi \)), is a representation of the rotation group in two dimensions \( SO(2) \), while \( \gamma_\eta, \eta \) is complex, the translation group. The associated little group is therefore \( ISO(2) \) of the rotations and translations in \( \mathbb{R}^2 \); this subgroup is associated with translations and only admits infinite dimensional representations (except the trivial one) associated with the continuous spin representations. Since we do not have knowledge of infinite internal degrees of freedom for particles, we postulate that this subgroup is only mapped into the trivial representation (all elements go into the identity). The \( SO(2) \) subgroup has one dimensional irreducible representations, but since the electrodynamics is invariant by parity, that connects opposite helicities, the dimension of the little group is two.

The canonical partition function is then given by:

\[
Z(\beta, V) = Tr (e^{-\beta H_F}) \tag{21}
\]

In order to determine the dependence with volume, we impose periodic boundary conditions on the one parameter subgroups of space translations in the \( x^j \) direction:

\[
U_x(i) = U_x(L). \tag{22}
\]

It follows that:

\[
U(x_i) = \sum_n e^{-\frac{2\pi n x_i}{L}} \int_0^L 1 \sum_n e^{\frac{2\pi n y}{L}} U(y) dy = \sum_n e^{-\frac{2\pi n x_i}{L}} E_n, \tag{23}
\]

where \( E_n \) are mutually orthogonal projections and:

\[
\lim_{n \to \infty} \sum_n E_n \Psi = \Psi. \tag{24}
\]

This implies momentum quantization, since \( U(t) = e^{-i \frac{P \cdot x}{\hbar}} \), where

\[
P = \sum_n n E_n \tag{25}
\]

The allowed values of momentum are:

\[
p_j = \frac{2\pi n j}{L}, \tag{26}
\]

where \( n_j \) an integer.

The idea of non-commutative inflation [23] is to introduce the phenomenological modification

\[
E^2 = P^2 \to E^2 = P^2 f^2(E) \tag{27}
\]

straight into [18] and [14], modifying the calculation of [21] and obtaining, from the usual thermodynamic relations:

\[
\rho(E,T) = \frac{1}{\pi^2} \frac{E^3}{\exp E/T - 1} \frac{1}{f^3} \left| 1 - \frac{E f}{T} \right| \tag{28}
\]
\[ p = \frac{1}{\beta} \frac{\partial}{\partial V} \ln (Z(\beta, V)) = \frac{1}{3} \int \frac{\rho(E, T)}{1 - E^3} dE \]  
(29)

\[ \rho = -\frac{1}{V} \frac{\partial}{\partial \beta} \ln (Z(\beta, V)) = \int \rho(E, T) dE. \]  
(30)

We consider here \( c = k_B = \hbar = 1 \). To find an inflationary behavior, [1] used the ansatz:

\[ f = 1 + (\lambda E)\alpha, \]  
(31)

We extended this to a set of inequalities on \( f(E) \) related to minimal requirements for inflation [24]. We use this extension as a starting point of our argument.

### III. THE HOPF ALGEBRA CONCEPT

A question addressed by us in [15] is whether or not the above analysis follows unaffected under the non-commutative hypothesis, since this model is supposed to be a mechanism for inflation applicable to non-commutative spaces. Nonetheless, what is meant by a non-commutative hypothesis, since this model is supposed to codify a metric.

Although the procedure (described in [16]) to take into account the information that defines the non-commutative space, the \( C^* \)-algebra, and deform the Lie algebra structure is not defined for general non-commutative spaces, if we assume that such a procedure must exist, we are able to define a formal deformation of the relativistic quantum theory in terms of algebraic methods [15]. That is very similar to what is done in quantum field theory in curved space-time. By doing this, we are following a different approach of the very authors of the Hopf algebra deformation procedure, in which, once possessing the associated Hopf algebra, they define the physics (only for real free scalar field) by a Lagrangian action to be quantized. A Lagrangian that is required to be invariant under Hopf Algebra action:

\[ S[\phi(x)] = \int \phi(x)(\Box \lambda - M^2)\phi(x), \]  
(32)

where \( \Box \lambda \) is a non-relativistic differential operator.

The Hopf algebra concept, instead of a formal need to a possible connection with non-commutative spaces, is actually a fundamental ingredient of non-commutative inflation.

There are many ways to define a Hopf algebra, a possible way which suggests its connections with non-commutative spaces is saying that the Hopf algebra is a set of transformations which acts not only on a vector space (like the Hilbert space), but on a algebra, like the \( C^* \)-algebra, and acts in a way dependent on the algebra product, that codifies non-commutativity.

A Hopf algebra \( H \) acts in a algebra \( A \) as a set of transformations not necessarily invertible, \( H : A \rightarrow A \), this action being denoted by \( h \triangleright f \), \( h \in H \) and \( f \in \mathcal{F} \). \( H \) is an algebra, that is, one can multiply elements, and construct linear combinations with complex coefficients:

\[ (h_1 \cdot h_2) \triangleright f = h_1 \triangleright (h_2 \triangleright f) \]  
(33)

\[ (\alpha h_1 + \beta h_2) \triangleright f = \alpha (h_1 \triangleright f) + \beta (h_2 \triangleright f). \]  
(34)

In this way, \( H \) depends on the product of the elements of \( \mathcal{F} \) by a generalization of the Leibniz rule: \( h \triangleright (f \cdot g) = \sum_i (h_{i(1)} \triangleright f) \cdot (h_{i(2)} \triangleright g), \) or, by using a short notation:

\[ h \triangleright (f \cdot g) = (h_{(1)} \triangleright f) \cdot (h_{(2)} \triangleright g), \]  
(35)

where the rule

\[ \Delta : H \rightarrow H \otimes H \]  
(36)

given by \( \Delta : h \rightarrow \sum h_{i(1)} \otimes h_{i(2)} \) is called a coproduct.

It leads us to an alternative way to define a Hopf algebra, a way which is particularly important for the purposes of this paper. A Hopf algebra \( H \) is an algebra which has a rule to construct tensor products that furnish representations of the same algebra.
Consider for example a Lie Algebra realized in the Hilbert space \( \mathcal{H} \). Its generators satisfy:

\[
[X_i, X_j] = C_{ij}^k X_k \tag{37}
\]

But we can define new representations of the very same Lie algebra \( \{ \mathcal{H}, \mathcal{H}_1 \} \) in the tensor product of Hilbert spaces \( \mathcal{H}_1 \otimes \mathcal{H}_1 \) by:

\[
\Delta X_i = X_i \otimes I + I \otimes X_i \tag{38}
\]

we observe that this is exactly what is done in (37).

We can actually construct arbitrary tensor products, because \( \Delta \) has a property called \textit{coassociativity}:

\[
(id \otimes \Delta) \Delta h = (\Delta \otimes id) \Delta h \tag{39}
\]

Here \( id \) is the identity among the operators \( \mathcal{H} \to \mathcal{H} \). This follows from \( H \triangleright (abc) \), where we apply the rule (39) first on the pair \( a \cdot (bc) \), but it must yield the same result as applied first on the pair \( (ab) \cdot c \).

A generalization of rule (39) for the product of \( N \) elements of \( \mathcal{A} \) follows from coassociativity:

\[
H \triangleright (\prod_i f_i) = \prod_i (h \triangleright f_i) \tag{40}
\]

In addition, we can define a generalization of the rule (36) \( \Delta^N : H \to H^{\otimes N+1} \):

\[
\Delta^N h = (\Delta \otimes id \cdots \otimes id) \cdots (\Delta \otimes id \otimes id)(\Delta \otimes id) \Delta h \tag{41}
\]

which would yield, by coassociativity, the same result, even if in each term \( (\Delta \otimes id \cdots \otimes id) \) we changed the positions of the \( \Delta \). In this sense, we can identify the \( \Delta \) itself with \( (\Delta \otimes id \cdots \otimes id) \) while acting in \( H^{\otimes N} \).

If we have a rule \( \Delta : H^{\otimes N} \to H^{\otimes N+1} \), we must have a rule to do the inverse. This rule is obtained when one of the multiplying elements of \( \mathcal{A} \), in the product rule (35), is the identity of \( \mathcal{A} \):

\[
h \triangleright (1 \cdot g) = (h \triangleright 1) \cdot (h \triangleright g) = \epsilon(h_1) \cdot (h \triangleright g) = h \triangleright g;
\]

\[
h \triangleright (g \cdot 1) = (h \triangleright g) \cdot (h \triangleright 1) = (h \triangleright g) \cdot \epsilon(h_2) = h \triangleright g;
\]

where the rule \( \epsilon : H \to \mathbb{C} \) is called the \textit{counity}. The counity undoes the \( \Delta \) action, in the following sense:

\[
\Delta h = h \otimes (1 \otimes h) \to h \otimes 1 = \epsilon(h_2) \otimes h_1 = \epsilon(h_2) = h
\]

We therefore set \( \epsilon : H^{\otimes N+1} \to H^{\otimes N} \), where \( \epsilon \) is identified with \( (\epsilon \otimes id \cdots \otimes id) \), independent of the position of \( \epsilon \).

The Hopf algebra \( H \) acts in \( \mathcal{A} \) as a set of transformations not necessarily invertible. In spite of that, there is a generalized notion of inverse called antipode, \( S : H \to H \).

The antipode is such that if \( H \) has an inverse, it will satisfy \( S(h) = h^{-1} \). The antipode is (uniquely) defined by the properties \( S(h_1) \cdot h_2 = h_1 \cdot S(h_2) = \epsilon(h)1_H \) where \( 1_H \) is the identity of \( H \). In a short notation:

\[
\epsilon(S \otimes \Delta) \Delta h = (\epsilon \otimes \epsilon) \Delta h = \epsilon(h)1_H \tag{42}
\]

Given \( h_1 \otimes \cdots \otimes h_{N+1} = \Delta^N h \), we apply \( (\epsilon \otimes \epsilon) \otimes \cdots \otimes \epsilon \), possibly changing the position of the term \( 1_H \), which yields \( \Delta^{N-2} h \).

\section{IV. Connecting Hopf Algebras with the Hilbert Space}

As already observed, in [16], the authors suggest that, once possessing a Hopf algebra which replaces the Poincaré Lie algebra, they define the physics (only for a real free scalar field) by an action functional to be quantized. An action functional that is required to be invariant under Hopf Algebra action:

\[
\mathcal{S}[\omega = \omega(x)] = \mathcal{S}[\phi(x)], \tag{43}
\]

where \( \phi(x) \) is an element of the non-commutative \( C^* \)-algebra which defines the non-commutative space. In our work, [18], we suggest an alternative formulation. This formulation consists in applying the Wigner prescription for relativistic quantum theory by assigning irreducible representations of the Hopf algebra to Hilbert spaces of one particle systems. A useful concept here is the GNS construction of the mathematical \( C^* \) algebra theory. The GNS construction is of fundamental importance in physics and mathematics. It is the basis of the proof of the Von Neumann theorem, which states that all irreducible representations of the Heisenberg algebra are unitarily equivalent. The GNS construction is what assures us that the representation problem of \( C^* \) algebras always has a solution, provided that all algebraic conditions of its definition are satisfied. It creates a correspondence between the algebra and a set of Hilbert space operators.

The fundamental idea is that for all state \( \omega : \mathcal{A} \to \mathbb{C} \), i.e. positive definite linear functional, defined by
\( \omega(A^*A) \geq 0 \), corresponds a representation \( \pi_\omega : \mathcal{A} \rightarrow \mathcal{O}(\mathcal{H}_\omega), \mathcal{O}(\mathcal{H}_\omega) \) being operators in the Hilbert space \( \mathcal{H}_\omega \). It is done in such a way that \( \mathcal{H}_\omega \) has a cyclic vector \( \Psi_\omega \), that is, the application, in \( \Psi_\omega \), of the operators of the representation, generates the entire Hilbert space (more exactly, a dense subspace), in such a way that \( \omega(A) = (\Psi_\omega, \pi_\omega(A)\Psi_\omega) \). Furthermore, any other representation \( \pi \) in Hilbert space \( \mathcal{H}_\pi \) with a cyclic vector \( \Psi \) which satisfies \( \omega(A) = (\Psi, \pi(A)\Psi) \) is unitarily equivalent to \( \pi_\omega \), that is, there exists an unitary transformation \( U : \mathcal{H}_\pi \rightarrow \mathcal{H}_\omega \) such that:

\[
U \pi(A) U^{-1} = \pi_\omega(A), \quad U \Psi = \Psi_\omega.
\] (44)

The GNS construction is based on the fact that the \( C^* \) algebra is already an Hilbert space, except for a scalar product to be defined. This can be taken as \( (A,B) = \omega(A^*B) \), where \( \omega \) is a state. The inner product is positive definite by positivity of the state, but \( (A,A) = 0 \) does not imply in general \( A = 0 \), an inner product property. We define then the set:

\[
J = \{ A \in \mathcal{A}, \omega(B^*A) = 0, \forall B \in \mathcal{A} \} \quad (45)
\]

that is such that \( A \cdot J \subseteq J \), property which makes \( J \) a left ideal. The states of the Hilbert space are then equivalence classes defined in terms of \( J \), that is, any two elements that differ by a \( J \) element are equivalent: \( [A] = A + B, B \in J \), which defines the so called quotient space \( \mathcal{A}/J \). The Hilbert space operators are then given by:

\[
\pi_\omega(A)[B] = [AB].
\] (46)

and such that:

\[
\omega(A) = \langle \Psi_\omega | A | \Psi_\omega \rangle \quad (47)
\]

The existence of representations is therefore assured by the existence of the states, positive definite linear functionals, but it is part of the \( C^* \) algebra theory to prove that such states do exist, provided that the algebraic properties of the definition are satisfied.

Furthermore, different kinds of states are related to different kinds of representations. In particular, the irreducible representations are related to normalized states (i.e. \( \omega(1) = 1 \)) that cannot be decomposed as convex linear combinations of any other two normalized states, i.e., \( \omega(A) \neq \lambda \omega_1(A) + (1 - \lambda) \omega_2(A) \), \( A \in [0,1] \) and \( \omega_i \) an state, \( \omega_i \neq \omega \). In particular, Schur’s lemma is valid, and a state related to irreducible representation satisfies \( \omega(P(C)) = P(\omega(C)) \), where \( C \) is a Casimir of the algebra and \( P(C) \) in a polynomial in \( C \).

We can do the same thing with Hopf algebras. There exists many ways in which a Hopf algebra can act on other algebraic structures and many ways to realize its action. The particular case given by Eq. (39) is called a left corregular action and it is the particular way explored in (10) to define the Poincaré Lie algebra deformation under non-commutative hypothesis. In group theory, for example, the left multiplication of an element \( g \) by another element \( g' \) furnishes a representation of the group: \( \pi_\omega(gg') = g'g \), but it is not the case in Hopf algebras. To do that we must consider the linear functionals on it, the so called dual Hopf algebra, \( H^* : H \rightarrow \mathbb{C} \), denoted by \( \langle \phi, h \rangle \) → \( \mathbb{C} \), \( \phi \in H^* \), \( h \in H \). We can transform \( H^* \) into a Hopf algebra through the structure of \( H^* : \langle \phi, h \rangle = \langle \phi \otimes \psi, \Delta h \rangle = \langle \Delta \psi, g \circ h \rangle = \langle \psi, g \cdot h \rangle, \langle 1, h \rangle = \langle \epsilon(h) \rangle \) and \( \epsilon(\psi) = \langle \psi, 1 \rangle \).

We then define the action of \( H \) into \( H^* \) according to (39), which transforms \( H^* \) into the so called \( H \)-module algebra, yielding the so called left corregular representation \( R^* : H^* \rightarrow H^* \):

\[
\langle R^*_g(\phi), h \rangle = \langle \phi, hg \rangle = \langle \phi(1), h \rangle \cdot \langle \phi(2), g \rangle . \quad (48)
\]

To proceed in the GNS construction, we need to define a involution \( * : H \rightarrow H \), the adjoint operation of \( C^* \) algebras, characterized by the properties \( (A+B)^* = A^* + B^* \), \( (\lambda A)^* = \overline{\lambda} A^* \) such that \( (AB)^* = B^* A^* \) and \( (A^*)^* = A \). We need actually to assure that the self adjoint property \( X = X^* \) is compatible with the algebra in such a way that we can transform it into generators of one parameter subgroups. Furthermore, we need compatibility with the Hopf algebra structure in the following sense: \( \Delta h^* = (\Delta h)^{\circ *}, \epsilon(h^*) = \epsilon(h) \) and \( (S \circ *)^2 = id \). It assures things like, if \( H \) is a self-adjoint operator, \( \Delta H \) will be a self-adjoint operator too.

As in the \( C^* \) algebra case, \( H^* \) is a Hilbert space, except for a inner product to be defined. This product must be such that \( \langle \phi, h \triangleright \psi \rangle = \langle h \triangleright \phi, \psi \rangle \phi, \psi \in H^* \). The Hopf algebra theory yields a natural choice of inner product, called the right integral \( f : H^* \rightarrow \mathbb{C} \) defined by the property:

\[
(\int \otimes id) \circ \Delta = 1_{H^*} \cdot \int . \quad (49)
\]

Analogously we can define the left integral, both being unique, except by a multiplicative factor.

The importance of the right integral is how it is related to the left corregular action:

\[
\int (\phi \triangleright g)^* f = \int g^*(\phi \triangleright f) \quad (50)
\]

We can define the ideal \( J_{H^*} : \)

\[
J_{H^*} = \{ f \in H^* | \int g^* f = 0, \forall g \in H^* \} \quad (51)
\]

that defines the Hilbert space \( H^*/J_{H^*} \). It follows that we transform the Hilbert space into a non-commutative algebra in which the Hopf algebra acts following the rule (39), the same rule as in it acts on the non-commutative \( C^* \)-algebra.

We could, however, interpret the Hopf algebra simply as an algebra which can be realized as a \( C^* \)-algebra, by functionals that act on \( H \) itself, not in \( H^* \), that is, ignoring the rule (39), and worrying about the coproduct only if we have to deal with tensor products. This is going to be the point of view of the rest of this work.
The prescription to define the physics of the radiation that would lead to the same equations of the noncommutative inflation, \( \mathcal{P}_N \), would be:

**Prescription 1** (\( \pi_0 \)). Be \( \mathcal{P}_N \) a Hopf algebra with generators \( X_i \) which deforms Poincaré Lie algebra \( \mathbb{L} \) with a set of Casimir elements \( C_0, C_1, \ldots \) and be \( \pi_0 : \mathcal{P}_N \to O(D) \) the irreducible representation \( \mathbb{L} \) with Casimir eigenvalues \( \lambda' = (\lambda_0, \lambda_1) \). Be \( \mathcal{P}_N \) the group generated by elements of the form \( e^{-itX_\mu} \) and \( \mathcal{U}^{\mathcal{P}_N}_\pi \) an irreducible representation of it given by \( \pi_\mathcal{P}_N(e^{-itX_\mu}) = e^{-it\pi_0(X_\mu)}, t \in \mathbb{R} \). The physics of the radiation is defined by a representation of \( \mathcal{P}_N \) obtained by (33) by the replacement:

\[
\mathcal{U}_\lambda \to \mathcal{U}^{\mathcal{P}_N}_\pi,
\]

for some set \( \lambda' \). This prescription leads us to the following representation of \( \mathcal{P}_N \):

\[
\mathcal{U}^{\mathcal{P}_N}_\pi = \sum_{N=0}^{\infty} \otimes (\mathcal{U}_N^{\mathcal{P}_N} \otimes \mathbb{N})_S
\]

Observe that it is a Fock space structure, the difference is the mathematical structure of one-particle states. Nonetheless, if there are some things in common between the new one-particle structure and the usual relativistic one, equations (25) and (29) will be the same.

Also, observe that the role of the GNS construction, if not directly applied to realize the algebra, is going to be telling us that, once we define a representation of it to describe the radiation and drive the inflation. This deformation, (28) and (29), would be:

\[
\pi_0 \to \pi_\mathcal{P}_N.
\]

The first one is that \( \mathcal{P}_N \) must form a nontrivial algebra, that is, its generators must satisfy a set of algebraic equations not trivially satisfied by every set of Hilbert space operators. It may appear a trivial condition, since a thing that we usually take for granted is that, no matter what is the group we are talking about, the generators will always for an algebra. Nevertheless, that is not mathematically true. Physics cannot be formulated without an underlying algebraic structure for symmetry generators though. It is in terms of those nontrivial algebraic equations that the physical requirements are expressed. On the other hand, such a specification tell us what is the particular \( \mathcal{P}_N \) that we are considering.

Furthermore, the generators of this algebra and the algebraic constraints must be compatible with the self adjoint condition \( X = X^* \), for an involution satisfying \((A+B)^* = A^* + B^*, (\lambda A)^* = \overline{\lambda} A^* \) and such that \((AB)^* = B^* A^*\).

A physical requirement related to the possibility of measure Energy and Momentum at same time, or the very concept of dispersion relation will not make sense, is that \( \mathcal{P}_N \) must have a commutative subalgebra associated to space-time translation generators. We will associate this algebra with the energy \( E \) and momentum \( P \). Besides, the algebra must have an Casimir \( C(E,P) \) obtained as a nontrivial algebraic function of \( E, P \) that is compatible with the positive energy condition.

Such compatibility is in the following sense: the solutions of the equation \( C(E,P) = \lambda \) define surfaces in the \((E,P)\) variables, that we call a **mass-shell deformation**, denoted by \( \mathbb{C}_\lambda \), such that for some subset \( M_{\mathcal{P}_N} \) of \( \lambda \), \( \mathbb{C}_\lambda \) has a connected component satisfying \( E \geq 0 \) and \( E = 0 \iff P^\mu = 0 \).

We are going to call the set of the \((E,P)\) points such that \( C(E,P) \in M_{\mathcal{P}_N} \) and \( E \geq 0 \) the **deformed forward light cone** \( V_{\mathcal{P}_N}^+ \). This property is part of the conditions that will assure that there are irreducible representations of \( \mathcal{P}_N \) with positive-definite energy.

The algebra generated by energy and momentum, denoted by \( t \), must be an ideal in the Lie algebra. That is, for any generator \( X_j \), \([t^\mu, X_j] = G^\mu(\nu)\), where \( G^\mu(\nu) \) is an algebraic function of energy and momentum. This condition assures that energy eigenstates, related to stable states, are mapped into energy eigenstates by the action of \( \mathcal{P}_N \). That is, if the system is observed to be in a stable state in one reference frame, it will be observed to be in a stable state after the application of \( \mathcal{P}_N \). The

[7] Suppose \( \Psi \) is an energy eigenstate, since \([E,P] = 0\), it is also a momentum eigenstate. We can then write \( P^\mu \Psi = p^\mu \Psi \), or, equivalently, \( e^{ip^\mu} \Psi = e^{ip^\mu} \Psi \). A referential change by \( \mathcal{P}_N \) implies \( e^{ip^\mu} \Upsilon \to U e^{ip^\mu} \Upsilon^{-1} = e^{ic^\mu} e^{ip^\mu} \), what implies that \( e^{ic^\mu} = U e^{ip^\mu} U^{-1} e^{-ip^\mu} \). We can write \( U = e^{iX_\mu} \), and apply the integral form of the Baker-Hausdorff formula:

\[
e^{ic^\mu} = e^{A^\mu} e^{B^\mu} = B + \int_0^1 g(e^{iatA} e^{ibtB}) \text{Ad}t; \quad g = \ln(z)/(z - 1),\]
fact that the subalgebra \( t \) is an ideal essentially assures that:
\[
U(g_1)P^\mu U^{-1}(g_1) = G_{g_1}(P^\mu),
\]
where \( U(g_1) \) is a Hilbert Space representation of an element \( g_1 \in \mathcal{P}_N \) and \( G_{g_1} \) is a realization of the same element as transformation in the energy-momentum space.

From this, we formulate the additional condition that assures positivity of energy for at least some irreducible representations of \( \mathcal{P}_N \) as:
\[
[P^\mu, X^\nu] = e^{(\nu \mu)}_0 + e^{(\nu \mu)}_\alpha P^\alpha + e^{(\nu \mu)}_{\alpha \beta} P^\alpha P^\beta + \cdots,
\]
where \( e^{(\nu \mu)}_0 = 0 \)

(55)

e^{(\nu \mu)} \) being complex coefficients.

The next requirement is that \( \mathfrak{p}_N \) has the Poincaré Lie algebra as the low energy-momentum limit. A way to assure this is as follows: Let \( C_{ij}^{(0)} \) be the structure constants of the Poincaré Lie algebra (5)-(7), and \( C_{ij}^{(E, P)} \) algebraic functions of the energy and momentum operators such that \( C_{ij}^{(0)}(0,0) = C_{ij}^{(0)} \) (Since \( [E, P] = 0 \), we can put the Hilbert space in a representation in which \( E \) and \( P \) are multiplicative operators). We can define \( \mathfrak{p}_N \) as:
\[
[X_i, X_j] = iX^kC_{ij}^{(k)}(H, P).
\]

(56)

It is possible to prove that in a irreducible representation satisfying (54) the states can be written as:
\[
\Psi = \sum_\sigma \int d^3 \mu (p) \phi(p^\mu, \sigma) \Psi_{p^\mu, \sigma},
\]

(57)

(The conclusion that follows from (54) is that the dimension of eigenspace of \( P^\mu \) with eigenvalue \( p^\mu \) is independent of \( p^\mu \))

We write then a state in the form (57) as:
\[
\Psi_{p^\mu, \sigma} = \int d\mu(p) \phi(p, \sigma) \Psi_{p^\mu, \sigma},
\]

(58)

\[
\text{supp} \{ \phi(p, \sigma) \} \in \{ E < E_{\text{max}}, P^i < p^i_{\text{max}} \}
\]

where \( adA \) is the adjoint action given by \( adA(B) = [A, B] \). We conclude that \( C \in \mathfrak{t} \) and, furthermore, \( e^{iC} e^{iP^\mu} = e^{iP^\mu} \) where \( P^\mu \) is an analytic function of \( P^\mu \) that defines a representation.

[8] The measurement in such representations yields expected values for four-momentum \( \omega(P^\mu) = \langle \Psi \mid P^\mu \mid \Psi \rangle \) belonging to the linear convex closure of the set \( \mathcal{C}_\lambda \), that we denote \( \mathcal{C}_\lambda \), and is the set of all linear combinations of the type \( \lambda \rho^\mu_{\alpha} + (1 - \lambda) \rho^\mu_{\beta} \), \( \rho^\mu_{\alpha, \beta} \in \mathcal{C}_\lambda, \lambda \in [0,1] \). Being a regular representation, that is, a representation in which \( e^{-iX^j} \mid \Psi \rangle \) is continuous for every state \( \mid \Psi \rangle \), \( \omega_i(P^\mu) = \langle \Psi \mid e^{iX^j} P^\mu e^{-iX^j} \mid \Psi \rangle \) is a continuous path in \( \mathcal{C}_\lambda \) that only achieves the region with \( E \leq 0 \) by the point \( p^\mu = 0 \), or, equivalently by the state \( \omega_{i0}(P^\mu) = 0 \), but \( \frac{\partial}{\partial \mu} \omega_i(P^\mu) \mid _{\mu = 0} = \omega_i [X_{t0}, P^\mu] = \omega_i (H(P^\mu)) = 0 \), where \( X_{t0} = e^{iX_{t0}} e^{-iX_{t0}} \), where we have made use of (55).

The action of generators of \( \mathfrak{p}_N \) on those states satisfy:
\[
[X^i, X^j] \triangleright \Psi_{p^\mu, \sigma} = iX^kC_{ij}^{(k)}(H, P) \triangleright \Psi_{p^\mu, \sigma}
\]
\[
\rightarrow iX^kC_{ij}^{(0)}(H, P) \triangleright \Psi_{p^\mu, \sigma}
\]

(59)

when \( p^\mu_{\text{max}} \rightarrow 0 \) and \( E_{\text{max}} \rightarrow 0 \). \( \triangleright \) denotes the action (as an operator) in the one particle states, while \( \rightarrow \) denotes the strong convergence (i.e. \( ||X^kC_{ij}^{(0)}(H, P) \triangleright \Psi - X^kC_{ij}^{(0)} \triangleright \Psi|| \rightarrow 0 \)). In other words, the typical eigenvalues of the generators effectively change the commutation relations. The property (56) further assures that the algebra generated by \( E, P \) is an ideal.

Given the property (54), it makes sense to talk about the little group, the subgroup \( \mathcal{W}_{p^\mu} \) of \( \mathcal{P}_N \), in representation (54), that leaves a particular \( p^\mu \) invariant. The little group of any \( p^\mu \) in the same \( C_k \) is the same (isomorphic). Indeed, in an irreducible representation, any \( p^\mu \) can be sent into any other by the action of some \( G_{g_1} \). Therefore, given \( w_1 \in \mathcal{W}_{p^\mu} \) we can obtain \( w_2 \in \mathcal{W}_{p^\mu} \) by \( w_2 = G_{g_1} w_1 G_{g_1}^{-1} \), where \( G_{g_1} |p^\mu_1) = |p^\mu_2) \). It follows that, the little group that acts in low energy and momentum should be the same as that at high energies. What leads us to the requirement that every little group of \( \mathcal{P}_N \), \( \mathcal{W}_{p^\mu} \), for \( \lambda \in M_{\mathcal{P}_N} \) must be isomorphic to some little group \( \mathcal{W}(\Lambda, p^\mu) \) of the Poincaré group associated to some positive-definite energy irreducible representation.

What additionally assures that the number of internal degrees of freedom for particles is the same as that of relativistic particles, which affects the calculation of the partition function, as in (10).

A further condition is that \( \mathfrak{p}_N \) must preserve the \( SO(3) \) Lie algebra, the rotation group, and \( [H, M_i] = 0 \), where \( M_i \) is the generator of the rotation subgroup along the \( i \) direction. Indeed, it is related to the thermodynamic description of a perfect fluid that is used in non-commutative inflation. That description is no longer valid if the thermodynamic state is no longer rotationally invariant, since the perfect fluid is the one that is isotropic in the reference frame that follows the fluid. In the thermodynamic description, the matter is described by the non-pure state \( \langle \Psi_T | A | \Psi_T \rangle = Tr(e^{-\beta H} A) / Tr(e^{-\beta H}), \forall A \leftrightarrow [M_i, H] = 0 \)

(60)

[9] For selfadjoint operators, or \( ||X^kC_{ij}^{(0)}(H, P) \triangleright \Psi - X^kC_{ij}^{(0)} \triangleright \Psi|| \rightarrow 0 \), or it is not a convergent sequence
B. Conditions on the quantum representation

Other thing we usually take for granted is that if some representation of a group has a well defined algebraic structure for generators, all other representations will have too; besides that, they will have the same one. Nonetheless, it is not generally valid. All of these are valid for Lie groups, but if we are no longer dealing with Lie groups, it will be no longer generally valid, the algebraic structure of the generators will no longer be a concept attached to the group itself but to specific representations.

It leads us to the last condition, which is related to the application of prescription 1. \[17\] must be a generator of the \(p_{N'}\) if \(H \in p_{N'}\). Although this is a subtle point, it is the main point of this paper. \[18\] is actually a valid representation of the group \(P_{N'}\), since it satisfies \[19\], but, at least that it is a Lie Group, the generators of the one-parameter subgroups are not assured to form the same algebra as \(p_{N'}\). If it is the case, it will mean that we do not have a uniquely defined notion of a dispersion relation that replaces the relativistic one. In particular, the generators of these subgroups may not form an algebra at all, not satisfying the above discussed requirements. If the direct product of two one-particle representations do form an algebra satisfying all the above requirements, but do not form the same algebra than that of the one particle subspace, then an \(S\) matrix connecting a one-particle with the two particles would not be compatible with energy conservation in all reference frames, since the energy of in and out states would not transform as the same function as in Eq. \[20\].

This leads us to consider two kinds of representations of the group \(P_{N'}\), the one whose generators of one-parameter subgroups do not form the same algebra than \(p_{N'}\), that we will call the \(\pi_p\) representation, and that one whose generators of the one-parameter subgroups do form the same algebra as \(p_{N'}\), that we will call the \(\pi_A\) representation.

VI. THE NON-EXISTENCE OF A LIE GROUP THAT REALIZES NON-COMMUTATIVE INFLATION

In our previous work \[15\], we obtained a set of conditions to be satisfied by a dispersion relation that assures minimal desirable conditions for inflation, like the minimum duration of it, related to problems like the flatness of the universe (\[34\] and \[44\]). These conditions were given by a set of inequalities that contains the particular ansatz applied by the original authors of non-commutative inflation \[3\]. It was shown that, given the original equations of the model, \[28\]-\[40\], the minimum duration of inflation is related to a limitation of the photon’s momentum (a consequence of condition 3 of the theorem 2 in \[18\]).

Since Casimir is an algebraic entity of \(p_{N'}\), those conditions should be included in the set of algebraic conditions for inflation discussed in the last section. Let us show here that the limitation of the photon’s momentum is precisely what implies that \[14\] cannot be the time translation generator of a representation of \(p_{N'}\), provided that all \(C_{\lambda}\) (the mass shell deformation) with positive definite energy is connected and allows a zero momentum state, that is, \(C_{\lambda}\) contains the point \((E, P) = (E^0, 0, 0)\).

Consider the deformed forward light cone \(V^+_{p_{N'}}\) defined in the previous section. Suppose that \(C(E, P) = \lambda P \in M_{p_{N'}}\) (\(M_{p_{N'}}\) defined in the previous section) for the irreducible representation that describes radiation. \(C_{\lambda}\) should contain the point \((E, P) = (0, 0, 0)\) since, in the low momentum and energy limit, the deformed dispersion relation should be the usual photon’s dispersion relation. Consider additionally that for any other \(\lambda \in M_{p_{N'}}\), \(C_{\lambda}\) contains the point \((E, P) = (E^0, m^2, 0)\), i.e. we are assuming a relativistic low energy-momentum limit for every particle’s deformed dispersion relation associated to \(p_{N'}\). Since \(C_{\lambda}\) is rotationally invariant by hypothesis (a condition related to the perfect fluid description), let us consider the graph relating the momentum’s magnitude to the energy in a particular irreducible representation with the dispersion relation Casimir \(\lambda\): \(||P|| = P_{\lambda}(E)\).

It happens that \(P_{\lambda}(E) \geq 0\) and there is no \(E\) such that \(P_{\lambda_1}(E) = P_{\lambda_2}(E)\) for \(\lambda_1 \neq \lambda_2\), since \((C(E), P)\) is an analytic function. Therefore \(P_{\lambda}(E^0) = 0 < P_{\lambda_2}(E^0)\), which implies that \(P_{\lambda}(E) < P_{\lambda_2}(E)\) for all \(E\). Indeed, if there is an \(E\) such that \(P_{\lambda}(E) = P_{\lambda_2}(E)\), it follows by the continuity of \(P_{\lambda}(E)\) (by analyticity of the Casimir \(C(E, P)\)) that there must be an \(E\) such that \(P_{\lambda}(E) = P_{\lambda_2}(E)\). But such a point cannot exist. The conclusion is that if each \(C_{\lambda}\) with positive definite energy is connected (to use the continuity argument above) and has relativistic energy in the low energy-momentum limit, a limitation of photon momentum implies a limitation of momentum for all positive definite energy irreducible representations.

The reasoning above is only valid for irreducible representations. Let us now argue about the general case. To do that, let us consider an important result in the representations of Von Neumann algebra\[35\], that additionally is going to be the basis of a possible new procedure to obtain the correct non-relativistic inflationary equation of state. This result comes from a powerful form of the spectral theorem. Indeed, in quantum mechanics, we know that, given an irreducible representation of the Heisenberg algebra, there exists a unitary transform \(U\) such that we can put the position or momentum opera-

\[10\] A Von Neumann algebra \(\mathcal{A}\) is a subalgebra of the operators in the Hilbert space \(\mathcal{B}(\mathcal{H})\) that is an *-algebra, that is, it contains the adjoint of each element, and is equal to its bicommutant, that is, \(\mathcal{A} = \mathcal{A}'\). The bicommutant of \(\mathcal{A}\) is the set of all operators in the Hilbert space that commutes with the commutant \(\mathcal{A}'\), where the commutant is the set of all operators that commute with \(\mathcal{A}\).
A self-adjoint operator \( A \) in some domain \( \mathcal{D} \) in the Hilbert space \( L^2(\mathbb{R}) \) of the square (Lebesgue) integrable functions, that is: \( \hat{x} |\Psi\rangle = x|\Psi(x)\rangle, \) \( \Psi(x) \in L^2(\mathbb{R}) \). This diagonal form, however, is not general. It is only valid for the so called cyclic operators[11] with spectrum equal to \( \mathbb{R} \). The general form of this decomposition is the so called direct integral representation (see [35] for a proof).

Given a self adjoint operator \( A \) in some domain \( \mathcal{D} \) in a separable (i.e., with countable basis) Hilbert space \( \mathcal{H} \) for each \( \lambda \). This is the Hilbert space whose vectors are written as

\[
|\Psi\rangle = \int \cdots |\Psi(\lambda)\rangle d\mu(\lambda), \quad \Psi(\lambda) \in \mathcal{H}(\lambda) \quad (62)
\]

and the inner product is:

\[
\langle \Phi, \Psi \rangle = \int \langle \Phi(\lambda), \Psi(\lambda) \rangle \lambda d\mu(\lambda), \quad (63)
\]

\( \langle \Phi(\lambda), \Psi(\lambda) \rangle_\lambda \) being the inner product of the Hilbert space \( \mathcal{H}(\lambda) \), \( \mu(\lambda) \) a suitable integration measure[12]. Furthermore, we have:

\[
UAU^{-1} \int \cdots |\Psi(\lambda)\rangle d\mu(\lambda) = \int \cdots |\lambda \Psi(\lambda)\rangle d\mu(\lambda). \quad (64)
\]

The domain of the operator \( UAU^{-1}, U(\mathcal{D}) \), is the set:

\[
\int ||\lambda \Psi(\lambda)||_2^2 d\mu(\lambda) < \infty, \quad (65)
\]

\( ||\lambda|| \) being the norm of the Hilbert space \( \mathcal{H}(\lambda) \).

We can do the same thing with a set of commuting self-adjoint operators \( A_i \) with a common domain \( \mathcal{D} \)[13], defining the unitary transform (this is actually the Von Neumann’s theorem for Abelian Von Neumann algebras):

\[
U_D : \mathcal{H} \rightarrow \mathfrak{h} = \int \cdots \mathcal{H}(\lambda_1, \lambda_2, \cdots) d\mu(\lambda_1, \lambda_2, \cdots), \quad (66)
\]

such that \( U_i AU_i^{-1} \int \cdots |\psi(\lambda_1, \cdots \lambda_N)\rangle d\mu(\lambda_1, \cdots \lambda_N) = \int \cdots \lambda_i \psi(\lambda_1, \cdots \lambda_N) d\mu(\lambda_1, \cdots \lambda_N) \).

We can define the operator \( X(\lambda) \) in some domain \( \mathcal{D} \) in each \( \mathcal{H}(\lambda) \):

\[
X \Psi = \int \cdots X(\lambda) \Psi(\lambda) d\mu(\lambda). \quad (67)
\]

We denote:

\[
X = \int \cdots X(\lambda) d\mu(\lambda) \quad (68)
\]

as the operator whose domain is the set \( \int \cdots |\Psi(\lambda)\rangle d\mu(\lambda), \Psi(\lambda) \in \mathcal{D} \) and \( \int ||X(\lambda) \Psi(\lambda)||_2^2 d\mu(\lambda) < \infty \). It is possible to prove that the algebraic function \( f[X] \) of \( X \) can be written as:

\[
f[X] = \int \cdots f[X(\lambda)] d\mu(\lambda), \quad (69)
\]

which shows the algebraic independence of the operators defined in each \( \mathcal{H}(\lambda) \).

Now suppose a representation of \( \mathfrak{p}_\mathcal{N} \) in the Hilbert space \( \mathcal{H} \) and \( C_1, C_2, \cdots, C_N \) is the set of Casimir operators of the representation. By the uniform transform we put \( H \) in a direct integral form such that all Casimir operators are multiplicative:

\[
\mathfrak{h} = \int \cdots \mathcal{H}(\lambda) d\mu(\lambda), \quad (70)
\]

where \( \lambda \) is the entire set of Casimir eigenvalues. But in each \( \mathcal{H}(\lambda) \), the Casimir is a multiple of the identity, therefore, by Schur’s lemma, there must exist an irreducible representation of \( \mathfrak{p}_\mathcal{N} \) on each \( \mathcal{H}(\lambda) \) whose generators are written in the form [38].

We observe that the set of Casimir operators can include operators that are not predicted by the algebra of \( \mathfrak{p}_\mathcal{N} \), that is, other than [38], in the case of Poincaré Lie algebra. Indeed, suppose \( \pi : \mathfrak{p}_\mathcal{N} \rightarrow \mathcal{O}(\mathcal{D}) \) (\( \mathcal{O}(\mathcal{D}) \) stands for operators in the domain \( \mathcal{D} \)) is an irreducible representation, and the algebra predicts a particular Casimir set \( C_1, C_2, \cdots \), written as algebraic functions of the generators of \( \mathfrak{p}_\mathcal{N} \). We can write another representation in terms of the irreducible one: \( \pi' = \pi + \pi' \). This is a reducible representation, but there is no other Casimir in the image of the representation \( \pi' \) other than \( \pi'(C_1) \). However, \( C'' = (a\pi(C_1)) \oplus (b\pi(C_1)) \), \( a, b \in \mathbb{R}, a \neq b \) is a Casimir operator that is not a multiple of the identity and it is not in the image of the original representation. This is because the representation of an algebra like the Lie algebra (actually its enveloping algebra) is not a Von Neumann algebra (see footnote [10]) such that \( A = A'' \), that is, that contains all its Casimir elements. It is contained in some Von Neumann algebra. Those additional Casimirs are related to the degenerance of the representation, i.e., the number of times, finite or infinite, that it appears in the decomposition.

We then represent this decomposition as:

\[
\mathfrak{h} = \int \cdots \mathcal{H}(\lambda, \sigma) d\mu(\lambda, \sigma), \quad (71)
\]
where $\lambda$ is the algebraic Casimir eigenvalues, and $\sigma$ is related to the degeneracy.

Now suppose that in the decomposition of the representation of $pX$ there appears irreducible representations $\pi_{\lambda}$ with dispersion relation eigenvalues $\lambda_{1} \notin M_{pX}$ for some set $B$ of $\lambda$ with non-zero measure. Suppose also there is a measurable function $f(\lambda)$ such that $\int d\mu(\lambda)||f(\lambda)||^{2} < \infty$. If $B$ has a finite measure, this function can be chosen as the characteristic function of $B$, $\chi_{B}$, which is equal to unity in $B$ and zero otherwise. We can choose a normalized vector $\chi_{\lambda}^{|E} f(\lambda)\psi_{\Lambda}$ belongs to the Hilbert space $\mathcal{H}$ and has negative energy. The conclusion is that for all direct integral decompositions of $pX$ with positive definite energy, only irreducible representations with $\lambda_{0} \in M_{pX}$ do appear (except by zero measure sets).

Furthermore, we can write $P_{\mu}$ as a multiplicative operator if we apply a further unitary transform in $\mathfrak{h}$ that represents each $\mathcal{H}(\lambda, \sigma)$ in (71) as the Hilbert space of $N$ component functions $\mathcal{H}_{\mu}$ defined in the place of the points $(E, P) \in \mathbb{R}^{4}$ in the energy-momentum space that satisfies $C(E, P) = \lambda_{1}$ with measure $\nu_{\lambda_{1}}$ (i.e. $(E, P) \in \mathcal{C}_{\lambda_{1}}$, recalling that $\lambda = (\lambda_{1}, \lambda_{2}, \cdots)$). The combination of the measures in $\mathcal{C}_{\Lambda_{1}}$ and $\lambda = (\lambda_{1}, \lambda_{2}, \cdots)$, we define a new measure $\mu'$ in four-momentum space and write:

\[
\mathfrak{h} = \int \mathcal{H}(p^{\mu}, \lambda') \mu'(d\mu')\nu(\lambda', \sigma),
\]

where $\lambda'$ denotes the remaining algebraic Casimir elements $\lambda' = (\lambda_{2}, \cdots)$. The support of the measure $\mu'$, i.e., the region giving contributions to the integral, is the set of points belonging to $\mathcal{C}_{\lambda_{1}}$ for each $\lambda$ belonging to the direct integral decomposition. For simplicity of notation, let us denote

\[
\mathfrak{h} = \int \mathcal{H}(p^{\mu}, \sigma) d\chi,
\]

where $\sigma = (\lambda', \sigma)$. We have the operator $P_{\mu}$ defined in $\mathfrak{h}$ by the operator $P_{\mu}(p^{\mu}, \sigma)$ in each $\mathcal{H}(p^{\mu}, \sigma)$ such that, for all $\Psi(p^{\mu}, \sigma) \in \mathcal{H}(p^{\mu}, \sigma)$,

\[
P_{\mu}(p^{\mu}, \sigma) \Psi(p^{\mu}, \sigma) = p^{\mu} \Psi(p^{\mu}, \sigma), \quad p^{\mu} \in V_{pX}^{+} \subseteq \mathbb{R}^{4}.
\]

Then, for all normalized $\Psi \in \mathfrak{h}$, $\langle \Psi | P^{\mu} | \Psi \rangle \in V_{pX}^{+}$, where $\overline{V_{pX}^{+}}^{c}$ denotes the convex linear closure $\mathbb{R}^{+}$.

Indeed, we can partition the set $V_{pX}^{+}$ in regions $U_{n} = V_{pX}^{+} \cap R_{ijkl}$, where $n$ is a set of indexes $n = (i, j, k, n)$ and $R_{ijkl}$ are the rectangles:

\[
R_{ijkl} = [p_{i}^{0}, p_{i}^{0} + 1] \times [p_{j}^{1}, p_{j}^{1} + 1] \times [p_{k}^{2}, p_{k}^{2} + 1] \times [p_{n}^{3}, p_{n}^{3} + 1],
\]

obtained from partitions $p_{i}^{n}$, $i = 1, 2, \cdots, N$, $\mu = 0, 1, 2, 3$, such that $\cup_{n} R_{n} \supset \mathbb{R}$.

It follows that $\langle \Psi | P^{\mu} | \Psi \rangle = \int d\chi ||p^{\mu}||^{2} \chi$, which implies the inequalities:

\[
\sum_{n} \inf_{U_{n}} \left( \int_{U_{n}} ||p^{\mu}||^{2} d\chi \right) \leq \langle \Psi | P^{\mu} | \Psi \rangle
\]

\[
\langle \Psi | P^{\mu} | \Psi \rangle \leq \sum_{n} \sup_{U_{n}} \left( \int_{U_{n}} ||p^{\mu}||^{2} d\chi \right)
\]

but $||p^{\mu}||^{2}$ is positive definite and $\int_{\overline{V_{pX}^{+}}^{c}} d\chi ||p^{\mu}||^{2} = 1$, in such a way that the right side of the first inequality and the left side of the last one can be written as:

\[
\sum_{n} p_{n}^{\mu} c_{n}, \text{ where } \sum_{n} c_{n} = 1, \quad p_{n}^{\mu} \in V_{pX}^{+} \text{ and } c_{n} > 0.
\]

Suppose that $\sum_{n=0}^{N} p_{n}^{\mu} c_{n} \in V_{pX}^{+}$ for $\sum_{n=0}^{N} c_{n} = 1$, then $\sum_{n=0}^{N} p_{n}^{\mu} c_{n} \in V_{pX}^{+}$ for $\sum_{n=0}^{N} c_{n} = 1$. Indeed, we can write

\[
\sum_{n=0}^{N+1} p_{n}^{\mu} c_{n} = \sum_{n=0}^{N} p_{n}^{\mu} c_{n} + p_{N+1}(1 - \sum_{n=0}^{N} c_{n}) + \sum_{n=0}^{N} p_{n}^{\mu} c_{n} + p_{N+1}(1 - \sum_{n=0}^{N} c_{n})
\]

But for $N = 1$ we have $p_{0}^{\mu} c_{0} + p_{1}^{\mu}(1 - c_{0}) \in V_{pX}^{+}$.

Therefore, if $p^{\mu} \in V_{pX}$ for every $(p^{\mu}, \sigma)$, $||p^{\mu}|| < p_{\text{max}}$, it follows that for every representation of $pX$ with positive definite energy the moment is limited by photon’s momentum:

\[
\langle \Psi | P^{\mu} | \Psi \rangle \leq p_{\text{max}}^{2}.
\]

since every convex linear combination with limited momentum has the same bound to the momentum.

But the generators associated to representation (55) are of the form:

\[
X_{r} = \sum_{N=1}^{\infty} \otimes (X \otimes I \cdots \otimes I + I \otimes X \cdots \otimes I + \cdots + I \otimes I \cdots \otimes X)_{N}
\]

Therefore, if $p^{\mu} \in V_{pX}$ for every $(p^{\mu}, \sigma)$, $||p^{\mu}|| < p_{\text{max}}$, it follows that for every representation of $pX$ with positive definite energy the moment is limited by photon’s momentum:

\[
\langle \Psi | P^{\mu} | \Psi \rangle \leq p_{\text{max}}^{2}.
\]

since every convex linear combination with limited momentum has the same bound to the momentum.
In particular, the energy of this representation is positive definite and the moment is unbounded (it assumes every integer multiple of the irreducible representation momentum). Therefore, the generators of \( p \) do not realize \( \mathcal{P}_N \). Besides, since every representation of a Lie group induces a representation of the same Lie algebra and \( p \) is a representation of the group \( \mathcal{P}_N \), there is no Lie group that realizes non-commutative inflation.

VII. THE ALTERNATIVE PRESCRIPTION FOR THE NON-RELATIVISTIC INFLATIONARY EQUATION OF STATE

The conclusion of the previous section is that, given the condition for minimum duration of inflation in non-commutative inflation \([18]\), which implies that the momentum of an individual photon is limited, and assuming that all \( C_\lambda \) (the mass shell deformation) with positive definite energy is connected and allows a zero momentum state, the non-commutative inflation is then necessarily associated with a non-trivial Hopf algebra structure with a non-trivial coproduct structure. Therefore Eq. \([53]\), which leads to the fundamental equations of non-commutative inflation, cannot be the appropriate coproduct, counity and antipode) that satisfies all the requirements discussed in section \( IV \) and leads to a particular photon phenomelogical dispersion relation \([54]\). We can use an alternative prescription only based on algebra properties. This can be applied to interacting field theory as well as to free fields and assuring that we are constructing a physics that is based on representations of a group \( \mathcal{P}_N \) that realizes the same algebra of generators \( p \). That is, based on \( \pi_A \) representations.

The basic idea is exploring the fact that every relativistic quantum field theory, associated to some Lagrangian density, has associated with it a representation of the Poincaré Lie group in the positive definite Hilbert space (i.e. such that the norm of states is positive). This representation enters in the relativistic covariance law of the fields:

\[
A^i_j([\Lambda, a])|\phi_i(\Lambda x + a) = U([\Lambda, a])|\phi_j(x)U^{-1}([\Lambda, a]) \quad (85)
\]

where \( A^i_j([\Lambda, a]) \) is a finite dimensional representation of the Poincaré Lie group. It happens that \( \mathcal{P}_N \) may not be satisfied in the positive definite Hilbert space, as in the case of electrodynamics, but we can restrict the action of \( U([\Lambda, a]) \) to the equivalence class of physical states (something similar to what we have done in the GNS construction to construct equivalence classes that describe the same state).

This representation can be recovered from the Wightman functions by the so called Wightman reconstruction theorem, that is an application of the GNS construction. Alternatively, we can obtain it by other classes

\[\phi(x^\mu) \rightarrow \phi(\Lambda x^\mu)\]

They are contravariant vectors, or first order differential operators. Consider the diffeomorphism \( \Phi : (E, p) \rightarrow (\tilde{E}, \tilde{p}) \) given by \( \tilde{E} = E : \tilde{p} = p/f(E) \), the \( f(E) \) given in \([42]\). Define the new algebra as multiplication operators \( E, p \) and \( X_i = \Phi \circ X_i \), the pushforward operator \( \Phi \circ X_i \circ f = X_i \circ f \circ \Phi \). This new algebra satisfy \( [E, \tilde{p}] = 0; [X_i, \tilde{X}_j] = C_{ij}^k \tilde{X}_k, C_{ij}^k \) the same structure constants of the original Lorentz group, but \( [X_i, E] = (\tilde{X}_i \circ E) = F(E, p) \) and \( [X_i, p] = (\tilde{X}_i \circ p) = G(E, p) \). As required \( C(E, p), E] = [C(E, p), p] = [C(E, p), X_i] = 0 \), where \( C(E, p) = f^2(E)p^2 - E^2 \). All the conditions of the section \( IV \) are satisfied

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\[\phi(x^\mu) \rightarrow \phi(\Lambda x^\mu)\]
of functions like the time-ordered Green’s functions or the Schwinger functions (see some original papers on this subject \[32, 37\] and \[38\]). In the case of free fields, we already have that representation, it is simply \[15\] for a suitable choice of irreducible representation of Poincaré group.

We then put the associated representation in the direct integral form:

$$\mathcal{U}^p = \mathcal{U}_0 \oplus \int_{\mathbb{R}} d\mu(\lambda, \sigma) \mathcal{U}_{\lambda, \sigma}^p,$$  

(86)

that is defined in:

$$\mathcal{H}^p = c\mathcal{U}_0 \oplus \int_{\mathbb{R}} d\mu(\lambda, \sigma) \mathcal{H}_{\lambda, \sigma}^p,$$  

(87)

where \(\mathcal{U}_0\) is the trivial vacuum representation, that leaves the vacuum subspace \(c\mathcal{U}_0\) invariant, while \(\lambda\) denotes the eigenvalues of those Casimir elements predicted by the Poincaré Lie algebra \[16\] and \(\sigma\) denotes the Casimir eigenvalues related to degeneracy. That is, the irreducible representation \(\mathcal{U}_{\lambda, \sigma}^p\) is the same for all \(\sigma\) and we have a \(\sigma\) value for each time that the same representation appears in the decomposition.

We then postulate:

**Prescription 2** (\(\pi_\lambda\)). *Be \(p_\lambda\) an algebra (or Hopf algebra) with generators \(X_i\) and \(p_\lambda\) the group generated by one parameter subgroups of the form \(e^{-i\lambda t_1}, t_1 \in \mathbb{R}\). Be \(\pi_\lambda : p_\lambda \rightarrow O(D)\) the irreducible representation of \(p_\lambda\) in Hilbert space (obtained by GNS construction with states in \(p_\lambda\) or \(p_\lambda^*\)) associated to algebraic Casimir eigenvalues \(\lambda = (\lambda_1, \lambda_2, \ldots)\). Be \(\mathcal{U}_{\lambda, \sigma}^{p_{\lambda}}\) the representation of \(\mathcal{P}_\lambda\) generated by \(e^{-i\lambda t_1}\). The representation of \(\mathcal{P}_\lambda\) that deforms a relativistic quantum field theory is obtained by the associated Poincaré Lie group representation (restricted to positive definite Hilbert space of physical states) put in the direct integral form:

$$\mathcal{U}^p = \mathcal{U}_0 \oplus \int_{\mathbb{R}} d\mu(\lambda, \sigma) \mathcal{U}_{\lambda, \sigma}^p,$$  

(88)

by the replacement:

$$\mathcal{U}_\lambda \rightarrow \mathcal{U}_{\lambda, \sigma}^{p_{\lambda}},$$  

(89)

for the set of Casimir eigenvalues \(\lambda'\) satisfying the condition:

$$\left( u(\Psi_{\lambda}^{p_{\lambda}}), e^{-i\lambda t_1}[X_{\mu}^{p_{\lambda}}], u(\Phi_{\lambda}^{p_{\lambda}}) \right) \rightarrow \left( \Psi_{\lambda}^{p_{\lambda}}, e^{-i\lambda t_1}[X_{\mu}], \Phi_{\lambda}^{p_{\lambda}} \right),$$  

(90)

when \((E_{\max}, p_{\max}) \rightarrow (m, 0)\) for normalized states \(\Psi_{\lambda}^{p_{\lambda}}\) and \(\Phi_{\lambda}^{p_{\lambda}}\) written in the form \[35\].

Here, \(u : \mathcal{H}_1 \rightarrow \mathcal{H}_2\) defines an unitary transform connecting the Hilbert spaces where the representations of the Poincaré group and \(\mathcal{P}_\lambda\) are defined. As before, \(X^{p_{\lambda}}\) is the corresponding operator in \(p_\lambda\) to the Poincaré Lie algebra element \(X^p\), \(m\) is the relativistic energy at zero momentum that, by hypothesis, is the same for the corresponding \(\mathcal{P}_\lambda\) representation.

This prescription defines the new time translation generator in the direct integral form:

$$\mathcal{H}_{\lambda, \sigma}^{p_{\lambda}} = 0 \oplus \int_{\mathbb{R}} d\mu(\lambda, \sigma) \mathcal{H}_{\lambda, \sigma}^{p_{\lambda}},$$  

(91)

where \(\mathcal{H}_{\lambda, \sigma}^{p_{\lambda}}\) is the Hamiltonian of the irreducible representation of \(p_\lambda\) associated to the set of eigenvalues \(\lambda'\) (that is diagonal in the energy-momentum representation and related to the momentum by \(C(H_{\lambda, \sigma}^{p_{\lambda}}, P) = \lambda_1\)). That is, not only the photon dispersion relation enters in the Hamiltonian of the theory but the dispersion relations of all of the species of particles.

The condition \[34\] is actually stronger than \[35\] and allows us to assure the convergence of the \(\mathcal{P}_\lambda\) representation to the original relativistic representation in the weak sense (i.e., in the sense of matrix elements: \(\langle u(\Psi), \mathcal{U}^{p_{\lambda}} u(\Phi) \rangle \rightarrow \langle \Psi, \mathcal{U}^{p_{\lambda}} \Phi \rangle\), where \(u\) is a unitary map that connects the Hilbert spaces where the Poincaré representation is defined to the one where the \(\mathcal{P}_\lambda\) representation is defined) if all intermediate states of direct integral decomposition converge as \[38\]. This corresponds to states of sufficiently low energy and momentum \[29\].

**VIII. CONCLUSION**

In a previous paper \[13\], we showed that the fundamental equations of non-commutative inflation, first obtained in \[22\] and applied in a model of inflation in \[1\], can actually be related to a representation of some group that deforms Poincaré Lie group, a group that can be constructed by starting from a Hopf algebra (allowing a possible connection with the original motivation of non-commutative spaces, along the lines of \[16\]). We showed that the condition of minimum duration of inflation is actually related to a limitation of the momentum of an individual photon. In the current paper, we showed that a group that replaces Poincaré must satisfy some important physical constraints already satisfied by the Poincaré Lie group. These constraints do not depend on cosmological requirements and affect not only the group’s abstract algebraic properties, but the representations themselves.

[20] Lebesgue’s theorem says that if \(g_N(x) \rightarrow g(x)\) for all \(x\) and \(|g_N(x)| \leq f(x)\) for some integrable \(f(x)\), then \(g(x)\) is Lebesgue integrable and \(\int g_N(x) \rightarrow \int g(x)\). Since we know that, for the original QFT, there exists Hilbert space vectors \(\Psi_{\lambda}^{p_{\lambda}}\) in each \(\mathcal{H}_{\lambda, \sigma}\) such that the inner product integral converges, and the unitary transformation satisfies \(\|u(\Psi_{\lambda, \sigma})\| = \|\mathcal{U}_{\lambda, \sigma}^{p_{\lambda}} u(\Psi_{\lambda, \sigma})\| \leq \|\Psi_{\lambda, \sigma}\| \leq \|\Phi_{\lambda, \sigma}\|^2 + \|\Psi_{\lambda, \sigma}\|^2\) which is a measurable function with finite integral, we have the fulfillment of conditions of the Lebesgue theorem.
Considering these constraints, the limitation of the photon’s momentum, together with the minimal additional conditions on the deformed mass shell, such as being connected at least for those solutions of positive definite energy and the same solutions allowing a zero momentum state, implies that the notion of Fock space is not suitable and a Hamiltonian like \( 17 \) cannot be the starting point of the fundamental equations of the non-commutative inflation.

Everything starts by the fact that we are replacing the Poincaré group \( \mathcal{P} \) by a group \( \mathcal{P}_N \) that has connected one-parameter subgroups that allow us, by Wigner and Stone theorems, to define generators of the representations of these one-parameter subgroups as self-adjoint operators that we assume that form an algebra \( \mathfrak{p}_N \). A thing that we usually take for granted is that, not matter what is the group we are talking about, the generators will always for an algebra. Nevertheless, that is not mathematically true. However, as we explained in section \( \nabla \) Physics cannot be formulated without an underlying algebraic structure for symmetry generators. Other thing that we usually take for granted is that if some representation of a group has a well defined algebraic structure for generators, all other representations will have too; besides that, they will have the same one. It is not generally valid too. All of these are valid for Lie groups and, once we are no longer dealing with Lie groups, it is no longer generally valid. Once we leave the domain of Lie groups, the algebraic structure of the generators is no longer a concept attached to the group itself but to specific representations. We show, however, that if we don’t have the very same algebra of generators for all symmetry representations that we apply in physics, physics will not be consistent.

Essentially, we show that, although a Fock space representation (53) is a valid mathematical representation for \( \mathcal{P}_N \), in the sense that it preserves the group structure given by Eqs. (10), for the kind of deformed dispersion relation that drives inflation with minimum duration, this representation does not satisfy the same algebra of generators used in the irreducible representations. As a consequence, non-commutative inflation is not related to some Lie group, since every representation of a Lie group induces a representation of the very same Lie algebra of generators.

As we argue, the use of this Fock like representation is a problem for a physical theory, since it might imply that a stable state in a particular reference frame is an unstable state in another one, or a possible S matrix connecting a one particle state with a \( n \)-particle state does not conserve energy in all reference frames. In particular, we do not have a unique well defined notion of a dispersion relation that replaces the usual relativistic one.

It does not mean that we cannot use a group like \( \mathcal{P}_N \) in physics. It means that we must restrict the class of representations of \( \mathcal{P}_N \) that is used to construct physics, something similar to restricting the representations of the Poincaré group that appear in QFT to those with positive definite energy. But now, besides that, we must restrict the representations of the \( \mathcal{P}_N \) group to those with positive definite energy and that induces a representation of the same algebra of generators \( \mathfrak{p}_N \). We here call these representations \( \pi_A \).

An alternative way to say that (53) is not a suitable representation of \( \mathcal{P}_N \) is to say that (17) is not a time-translation generator for \( \mathfrak{p}_N \) if \( H \) is. Since, in the Hopf algebra language, (17) is the coproduct rule for Linear Lie algebras, this suggests to us that non-commutative inflation is actually related to some non-trivial Hopf algebra, by understanding that a Hopf algebra is an algebra with a rule to construct tensor products that satisfy the same algebra.

The Hopf algebra concept is a possible link between the non-commutative space concept and non-commutative inflation, since an alternative interpretation for Hopf algebras is that it is a set of transformations that acts on other algebraic structures in a way dependent on the product rule. The product rule is the essential information of non-commutative spaces (see (20)). We can therefore use the coproduct rule of Hopf algebra to define the suitable non-commutative inflationary Hamiltonian that defines the canonical partition function that describes the radiation thermodynamics as in (53). This is our first possible prescription to define the non-commutative inflation. This prescription, however, needs the full knowledge of the Hopf algebraic structure associated to some phenomenological dispersion relation, a problem by itself, and can only be applied to free fields.

We can consider an alternative prescription that is based only on algebraic aspects, that is, it does not depend on the coalgebraic aspects, which means that it does not depend on the information that differs the Hopf algebra from a simple algebra, as the coproduct. This prescription is based on the infinite dimensional generalization of the concept of direct sum, the direct integral. It was originally proposed by us in (18), as an alternative to recover the interacting case and give a qualitative account of the behavior of the interacting non-commutative inflation. This prescription is essentially an application of an important Von Neumann theorem about the general representation of Von Neumann algebras and allows us to define a new Hamiltonian for non-commutative inflation that is indeed a time-translation generator of the very same algebra of generators that defines the dispersion relation Casimir. That is, allow us to deal only with \( \pi_A \) representations. Besides that, it assures to recover the corresponding Poincaré representation at low energy and momentum. The new Hamiltonian (91) is a function not only of the photon’s dispersion relation but a function of the dispersion relation of all other particles of the theory.

To obtain the new Hamiltonian for non-commutative inflation, we need the direct integral decomposition of the representation (18), that is a solved problem in the literature since the early days of QFT (for example, (39), (40)). We will address the problem of obtaining the new equations of state in a future paper. The main consequence of
our results is that all the work previously done on non-commutative inflation must be redone, that is, finding a new equation of state, finding inflationary dispersion relations that assure successful inflation, etc. In particular, using the same class of dispersion relations of the original equations of the model will imply the limitation of the momentum of the theory, even the macroscopic one. We must, however, use the same rule to construct the Hamiltonian even if our theory does not have that limitation of momentum, otherwise we would fall on the very same problem discussed in this paper.

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