Tri-Sasaki 7-metrics over the QK limit of the Plebanski-Demianski metrics

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Abstract

We consider a family of conical hyperkahler 8-metrics and we find the corresponding tri-Sasaki 7-metrics. We find in particular, a 7-dimensional tri-Sasaki fibration over a quaternion Kahler limit of the Plebanski-Demianski (or AdS-Kerr-Newmann-Taub-Nut) metrics, and we consider several limits of its parameters. We also find an squashed version for these metrics, which are of weak $G_2$ holonomy. Construction of supergravity backgrounds is briefly discussed, in particular examples which do not possess $AdS_4$ near horizon limit.

1. An sketch of the idea

The holographic principle [2] renewed the interest in constructing 5 and 7-dimensional Einstein manifolds and in particular those admitting at least one conformal Killing spinor. The number of such spinors will be related to the number of supersymmetries of the dual conformal field theory. This leads to consider weak $G_2$ holonomy spaces, Einstein-Sasaki spaces and tri-Sasaki ones. This spaces corresponds to $N = 1, 2, 3$ supersymmetries respectively.

Several of such spaces have been constructed, for instance, in [12]-[21]. One of the recent achievements is the construction of Einstein-Sasaki spaces fibered over orthotoric Kahler-Einstein spaces [19]-[20]. It was indeed noticed in [20] that this spaces can be obtained by taking certain scaling limit of the euclidean Plebanski-Demianski metrics. In particular, there were found toric Einstein-Sasaki 5-dimensional metrics defined over $S^2 \times S^3$.

The present work deals also with Einstein-Sasaki fibrations over the euclidean Plebanski-Demianski metric. But we will obtain a different type of metrics. Our metrics are 7-dimensional instead, and not only Einstein-Sasaki but tri-Sasaki. The idea behind our construction is indeed very simple. Our starting point are the Swann metrics [27], which are hyperkahler fibrations over quaternion Kahler metrics. Their generic form is

$$g_s = |q|^2 g_4 + |dq + \omega q|^2,$$

being $q$ certain quaternion coordinate and $\omega$ an imaginary quaternion valued 1-form associated to the quaternion Kahler metric $g_4$. The metric $g_4$ does not depend on $q$, thus under the transformation $q \rightarrow \lambda q$ these metrics are scaled by a factor $g_s \rightarrow \lambda^2 g_s$. It means that they are conical, therefore they define family tri-Sasaki 7-metrics. We will consider as our base metrics $g_4$ the euclidean Plebanski-Demianski (or the AdS-Kerr-Newman-Taub-Nut) metrics which, in some limiting case of the parameters, are quaternion Kahler [28]. We will extend them to Swann metrics and finally, we will find the 7-metric over which $g_s$ is a cone.

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This will be a tri-Sasaki metric fibered over the euclidean Plebanski-Demianski 4-metric, which is what we were looking for.

We also present an squashed version of these metrics, which are of weak $G_2$ holonomy. In the last section we construct several 11-dimensional supergravity backgrounds, some of them with $AdS_4 \times X_7$ near horizon limit and some others without this property. We achieve this by finding a set of harmonic functions over the internal Ricci-flat space (the Swann space) which depends not only on the radius, but also on other angular coordinates of the cone.

The key ingredient in order to construct these functions is to identify the coordinate system for which the Swann metric takes the generalized Gibbons-Hawking form [14].

2. Tri-Sasaki fibrations over quaternion Kahler spaces

2.1 Quaternion Kahler spaces in brief

A quaternion Kähler space $M$ is an euclidean $4n$ dimensional space with holonomy group $\Gamma$ included into the Lie group $Sp(n) \times Sp(1) \subset SO(4n)$ [6]-[9]. This affirmation is non trivial if $D > 4$, but in $D = 4$ there is the well known isomorphism $Sp(1) \times Sp(1) \simeq SU(2)_L \times SU(2)_R \simeq SO(4)$ and so to state that $\Gamma \subseteq Sp(1) \times Sp(1)$ is equivalent to state that $\Gamma \subseteq SO(4)$. The last condition is trivially satisfied for any oriented space and gives almost no restrictions, therefore the definition of quaternion Kähler spaces should be modified in $d = 4$.

Here we do a brief description of these spaces, more details can be found in the appendix and in the references therein. For any quaternion exists three automorphism $J^i$ ($i = 1, 2, 3$) of the tangent space $TM_x$ at a given point $x$ with multiplication rule $J^i \cdot J^j = -\delta_{ij} + \epsilon_{ijk} J^k$. The metric $g_q$ is quaternion hermitian with respect to this automorphism, that is

$$g_q(X, Y) = g(J^i X, J^i Y),$$

being $X$ and $Y$ arbitrary vector fields. The reduction of the holonomy to $Sp(n) \times Sp(1)$ implies that the $J^i$ satisfy the fundamental relation

$$\nabla_X J^i = \epsilon_{ijk} J^j \omega^k_-, \tag{2.2}$$

being $\nabla_X$ the Levi-Civita connection of $M$ and $\omega^\pm$ its $Sp(1)$ part. As a consequence of hermiticity of $g$, the tensor $\mathcal{T}_{ab} = (J^i)^c_a g_{cb}$ is antisymmetric, and the associated 2-form

$$\mathcal{T}^i = \mathcal{T}_{ab} e^a \wedge e^b$$

satisfies

$$d\mathcal{T}^i = \epsilon_{ijk} \mathcal{T}^j \wedge \omega^k_-, \tag{2.3}$$

being $d$ the usual exterior derivative. Corresponding to the $Sp(1)$ connection we can define the 2-form

$$F^i = d\omega^i_- + \epsilon_{ijk} \omega^j_- \wedge \omega^k_-.$$

For any quaternion Kähler manifold it follows that

$$R^i_- = 2n \kappa \mathcal{T}^i, \tag{2.4}$$

$$F^i = \kappa \mathcal{T}^i, \tag{2.5}$$
being a certain constant and \( \kappa \) the scalar curvature. The tensor \( R^a \) is the \( Sp(1) \) part of the curvature. The last two conditions implies that \( g \) is Einstein with non zero cosmological constant, i.e., \( R_{ij} = 3\kappa(g_q)_{ij} \) being \( R_{ij} \) the Ricci tensor constructed from \( g_q \). The \((0, 4)\) and \((2, 2)\) tensors
\[
\Theta = J^1 \wedge J^1 + J^2 \wedge J^2 + J^3 \wedge J^3,
\]
\[
\Xi = J^1 \otimes J^1 + J^2 \otimes J^2 + J^3 \otimes J^3
\]
are globally defined and covariantly constant with respect to the usual Levi Civita connection for any of these spaces. This implies in particular that any quaternion Kähler space is orientable.

In four dimensions the Kähler triplet \( J_2 \) and the one forms \( \omega^a \) are
\[
\omega^a = \omega^a_0 - \epsilon_{abc}\omega^b_c, \quad J_1 = e^1 \wedge e^2 - e^3 \wedge e^4,
\]
\[
J_2 = e^1 \wedge e^3 - e^4 \wedge e^2 \quad J_3 = e^1 \wedge e^4 - e^2 \wedge e^3.
\]
In this dimension quaternion Kähler spaces are defined by the conditions \((2.5)\) and \((2.4)\). This definition is equivalent to state that quaternion Kähler spaces are Einstein and with self-dual Weyl tensor.

In the Ricci-flat limit \( \kappa \to 0 \) the holonomy of a quaternion Kahler space is reduced to a subgroup of \( Sp(n) \) and the resulting spaces are hyperkahler. It follows from \((2.5)\) and \((2.2)\) that the almost complex structures \( J_i \) are covariantly constant in this case. Also, there exist a frame for which \( \omega^a_\pm \) goes to zero. In four dimensions this implies that the spin connection of this frame is self-dual.

### 2.2 A general tri-Sasaki family

As is well known, any hyperkahler conical metric define a tri-Sasaki metric by means of \( g_8 = dr^2 + r^2 g_7 \). A well known family of conical hyperkahler metrics are the Swann metrics [27], these are 4n dimensional metrics but we will focus only in the case \( d = 8 \). The metrics reads
\[
g_8 = |u|^2 g_q + |du + u\omega_-|^2, \quad (2.6)
\]
being \( g_q \) any 4-dimensional quaternion Kahler metric. In the expression for the metric we have defined the quaternions
\[
u = u_0 + u_1I + u_2J + u_3K, \quad \overline{\nu} = u_0 - u_1I - u_2J - u_3K,
\]
and the quaternion one form
\[
\omega_- = \omega_-^1I + \omega_-^2J + \omega_-^3K,
\]
constructed with the anti-self-dual spin connection. The multiplication rule for the quaternions \( I, J \) and \( K \) is deduced from
\[
I^2 = J^2 = K^2 = -1, \quad IJ = K, \quad JI = -K
\]
The metric \( g_q \) is assumed to be independent on the coordinates \( u_i \). We easily see that if we scale \( u_0, u_1, u_2, u_3 \) by \( t > 0 \) this scales the metric by a homothety \( t \), which means that the
metrics (2.6) are conical. Therefore they define a family of tri-Sasaki metrics, which we will find now. We first obtain, by defining \( \tilde{u}_i = u_i/u \) that

\[
|du + u\omega_\perp|^2 = (du_0 + u_0\omega_\perp)^2 + (du_i + u_0\omega_\perp + \frac{\epsilon_{ijk}}{2}u_k\omega_j)^2
\]

\[
= (\tilde{u}_0du + ud\tilde{u}_0 - u\tilde{u}_i\omega_\perp)^2 + (\tilde{u}_i du + ud\tilde{u}_i + u\tilde{u}_0\omega_\perp + u\frac{\epsilon_{ijk}}{2}\tilde{u}_j\omega_k)^2
\]

\[
= du^2 + u^2(\tilde{u}_0 - \tilde{u}_i\omega_\perp)^2 + u^2(\tilde{u}_i + \tilde{u}_0\omega_\perp + \frac{\epsilon_{ijk}}{2}\tilde{u}_j\omega_k)^2
\]

\[
+ 2u\tilde{u}_0du(\tilde{u}_0 - \tilde{u}_i\omega_\perp) + 2u\tilde{u}_i du(\tilde{u}_i + \tilde{u}_0\omega_\perp + \frac{\epsilon_{ijk}}{2}\tilde{u}_j\omega_k).
\]

It is not difficult to see that the last two terms are equal to

\[
2u\tilde{u}_0du(\tilde{u}_0 - \tilde{u}_i\omega_\perp) + 2u\tilde{u}_i du(\tilde{u}_i + \tilde{u}_0\omega_\perp + \frac{\epsilon_{ijk}}{2}\tilde{u}_j\omega_k) = \frac{d(\tilde{u}_i^2)}{2} + \frac{\epsilon_{ijk}}{2}\tilde{u}_i\tilde{u}_j\omega_k.
\]

But the second term is product of a antisymmetric pseudotensor with a symmetric expression, thus is zero, and the first term is zero due to the constraint \( \tilde{u}_i^2 = 1 \). Therefore this calculation shows that

\[
|du + u\omega_\perp|^2 = du^2 + u^2(\tilde{u}_0 - \tilde{u}_i\omega_\perp)^2 + u^2(\tilde{u}_i + \tilde{u}_0\omega_\perp + \frac{\epsilon_{ijk}}{2}\tilde{u}_j\omega_k)^2.
\]

(2.7)

By introducing (2.7) into (2.6) we find that \( g_8 \) is a cone over the following metric

\[
g_7 = g_q + (\tilde{u}_0 - \tilde{u}_i\omega_\perp)^2 + (\tilde{u}_i + \tilde{u}_0\omega_\perp + \frac{\epsilon_{ijk}}{2}\tilde{u}_j\omega_k)^2.
\]

(2.8)

This is the tri-Sasaki metric we were looking for. By expanding the squares appearing in (2.8) we find that

\[
g_7 = g_q + (\tilde{u}_0)^2 + (\omega_\perp)^2 + 2\omega_\perp(\tilde{u}_0d\tilde{u}_1 - \tilde{u}_1d\tilde{u}_0 + \tilde{u}_2d\tilde{u}_3 - \tilde{u}_3d\tilde{u}_2)
\]

\[
+ 2\omega_\perp^2(\tilde{u}_0d\tilde{u}_2 - \tilde{u}_2d\tilde{u}_0 + \tilde{u}_2d\tilde{u}_1 - \tilde{u}_1d\tilde{u}_3) + 2\omega_\perp^3(\tilde{u}_0d\tilde{u}_3 - \tilde{u}_3d\tilde{u}_0 + \tilde{u}_1d\tilde{u}_2 - \tilde{u}_2d\tilde{u}_1).
\]

(2.9)

But the expression in parenthesis are a representation of the \( SU(2) \) Maurer-Cartan 1-forms, which are defined by

\[
\sigma_1 = -(\tilde{u}_0d\tilde{u}_1 - \tilde{u}_1d\tilde{u}_0 + \tilde{u}_2d\tilde{u}_3 - \tilde{u}_3d\tilde{u}_2)
\]

\[
\sigma_2 = -(\tilde{u}_0d\tilde{u}_2 - \tilde{u}_2d\tilde{u}_0 + \tilde{u}_2d\tilde{u}_1 - \tilde{u}_1d\tilde{u}_3)
\]

\[
\sigma_3 = -(\tilde{u}_0d\tilde{u}_3 - \tilde{u}_3d\tilde{u}_0 + \tilde{u}_1d\tilde{u}_2 - \tilde{u}_2d\tilde{u}_1).
\]

Therefore the metric (2.8) can be reexpressed in simple fashion as

\[
g_7 = g_q + (\sigma_i - \omega_\perp)^2.
\]

(2.10)

This is one of the expressions that we will use along this work.

Let us recall that there exist a coordinate system for which the Maurer-Cartan forms are expressed as

\[
\sigma_1 = \cos \varphi d\theta + \sin \varphi \sin \theta d\tau,
\]

\[
\sigma_2 = -\sin \varphi d\theta + \cos \varphi \sin \theta d\tau,
\]

\[
\sigma_3 = d\varphi + \cos \theta d\tau.
\]

(2.11)
With the help of this coordinates we will write (2.10) in more customary form for tri-Sasaki spaces, namely

\[ g_7 = (d\tau + H)^2 + g_6, \]  

(2.12)
as in (??). Here \( g_6 \) a Kahler-Einstein metric with Kahler form \( \mathcal{J} \) and \( H \) a 1-form such that

\[ dH = 2\mathcal{J}. \]

A lengthy algebraic calculation shows that the fiber metric is

\[
(\sigma_i - \omega_i^\perp)^2 = (d\tau + \cos \theta d\varphi - \sin \theta \sin \varphi \omega_i^\perp - \cos \theta \sin \varphi \omega_i^\perp - \cos \theta \omega_i^\perp)^2
+ (\sin \theta d\varphi - \cos \theta \sin \varphi \omega_i^\perp - \cos \theta \cos \varphi \omega_i^\perp + \sin \theta \omega_i^\perp)^2 + (d\theta - \sin \varphi \omega_i^\perp \cos \varphi \omega_i^\perp)^2,
\]

from where we read that

\[ H = \cos \theta d\varphi - \sin \theta \sin \varphi \omega_i^\perp - \cos \theta \sin \varphi \omega_i^\perp - \cos \theta \omega_i^\perp. \]  

(2.13)
The Kahler-Einstein six dimensional metric should be

\[ g_6 = (\sin \theta d\varphi - \cos \theta \sin \varphi \omega_i^\perp - \cos \theta \cos \varphi \omega_i^\perp + \sin \theta \omega_i^\perp)^2 + (d\theta - \sin \varphi \omega_i^\perp \cos \varphi \omega_i^\perp)^2 + g_q. \]  

(2.14)
Although we have given the formulas needed for our work, we consider instructive to give an alternative deduction. This is our next task.

**An alternative deduction of the tri-Sasaki metric (2.10)**

For any quaternion Kahler space \( M \), a linear combination of the almost complex structures of the form \( J = \tilde{v}_i J_i \) will be also an almost complex structure on \( M \). Here \( \tilde{v}_i \) denote three "scalar fields" \( \tilde{v}^i = v^i/v \) being \( v = \sqrt{v_i v^i} \). This fields are supposed to be constant over \( M \) and are obviously constrained by \( \tilde{v}_i \tilde{v}^i = 1 \). This means that the bundle of almost complex structures over \( M \) is parameterized by points on the two sphere \( S^2 \). This bundle is known as the twistor space \( Z \) of \( M \). The space \( Z \) is endowed with the metric

\[ g_6 = \theta_i \theta_i + g_q, \]  

(2.15)
where \( \theta_i = d(\tilde{v}^i) + \epsilon^{ijk} \omega_j \tilde{v}^k \). The constraint \( \tilde{v}_i \tilde{v}^i = 1 \) implies that the metric (2.15) is six dimensional. It have been shown that this metric together with the sympletic two form [3], [5]

\[ \mathcal{J} = -\tilde{v}_i \mathcal{J}_i + \frac{i \epsilon^{ijk} \tilde{v}_j \theta_j \wedge \theta_k}{2}, \]  

(2.16)
constitute a Kähler structure. The calculation of the Ricci tensor of \( g_6 \) shows that it is also Einstein, therefore the space \( Z \) is Kähler-Einstein. The expressions given below are written for a quaternion Kahler metric normalized such that \( \kappa = 1 \), for other normalizations certain coefficients must be included in (2.16). By parameterizing the coordinates \( \tilde{v}_i \) in the spherical form

\[ \tilde{v}_1 = \sin \theta \sin \varphi, \quad \tilde{v}_2 = \sin \theta \cos \varphi, \quad \tilde{v}_3 = \cos \theta, \]  

(2.17)
we find that (2.15) is the same as (2.14). The isometry group of the Kahler-Einstein metrics is in general \( SO(3) \times G \), being \( G \) the isometry group of the quaternion Kahler basis which also preserve the forms \( \omega_i^\perp \). The \( SO(3) \) part is the one which preserve the condition \( \tilde{v}_i \tilde{v}_i = 1 \). Globally the isometry group could be larger.

From the definition of Einstein-Sassakian geometry, it follows directly that the seven dimensional metric

\[ g_7 = (d\tau + H)^2 + g_6 = (d\tau + H)^2 + \theta_i \theta_i + g_q, \]  

(2.18)
will be Einstein-Sassaki if \( dH = 2\vec{J} \), and we need to find an explicit expression for such \( H \). Our aim is to show that this form is indeed (2.13). The expression (2.16) needs to be simplified as follows. We have that 
\[
\theta_i = d(\bar{\nu}^i) + \epsilon^{ijk} \bar{\nu}_j \bar{\omega}^k.
\]
Also by using the condition \( \bar{\nu}_i \bar{\nu}_i = 1 \) it is found that
\[
\bar{\nu}_i \theta_i = \bar{\nu}_i d\bar{\nu}_i + \epsilon^{ijk} \bar{\nu}_j \bar{\omega}^k - \bar{\nu}_k \bar{\omega}_j = 0.
\]
From the last equality it follows the orthogonality condition \( \bar{\nu}_i \theta_i = 0 \) which is equivalent to
\[
\theta_3 = -\frac{(\bar{\nu}_1 \theta_1 + \bar{\nu}_2 \theta_2)}{\bar{\nu}_3}.
\]
The last relation implies that
\[
\frac{\epsilon^{ijk} \bar{\nu}_j \theta_k}{2} = \frac{\theta_1 \wedge \theta_2}{\bar{\nu}_3} = \frac{d\bar{\nu}_1 \wedge d\bar{\nu}_2}{\bar{\nu}_3} - d\bar{\nu}_i \wedge \omega^i + \frac{\epsilon^{ijk} \bar{\nu}_j \omega^k}{2}.
\]
By another side in a quaternion Kahler manifold with \( \kappa = 1 \) we always have
\[
\bar{J}_i = d\omega^i + \frac{\epsilon^{ijk} \omega^j}{2} \wedge \omega^k.
\]
Inserting the last two expressions into (2.16) gives a remarkably simple expression for \( \vec{J} \), namely
\[
\vec{J} = -d(\bar{\nu}_i \omega^i) + \frac{d\bar{\nu}_1 \wedge d\bar{\nu}_2}{\bar{\nu}_3}.
\]  
(2.19)
By using (2.17) it is obtained that
\[
\frac{d\bar{\nu}_1 \wedge d\bar{\nu}_2}{\bar{\nu}_3} = -d\varphi \wedge d \cos \theta.
\]
With the help of the last expression we find that (2.19) can be rewritten as
\[
\vec{J} = -d(\bar{\nu}_i \omega^i) - d\varphi \wedge d \cos \theta,
\]
from where it is obtained directly that the form \( H \) such that \( dH = \vec{J} \) is [26]
\[
H = -\bar{\nu}_i \omega^i + \cos \theta d\varphi,
\]  
(2.20)
up to a total differential term. By introducing (2.17) into (2.20) we find that \( H \) is the same than (2.13), as we wanted to show.

It will be of importance for the purposes of the present work to state these results in a concise proposition.

**Proposition** Let \( g_q \) be a four dimensional Einstein space with self-dual Weyl tensor, i.e., a quaternion Kahler space. We assume the normalization \( \kappa = 1 \) for \( g_q \). Then the metrics
\[
g_6 = (\sin \theta d\varphi - \cos \theta \sin \varphi \omega^1 - \cos \theta \cos \varphi \omega^2 + \sin \theta \omega^3)^2 + (d\theta - \sin \varphi \omega^2 + \cos \varphi \omega^1)^2 + g_q,
\]
are Kahler-Einstein whilst
\[
g_7 = (\sigma_i - \omega^i_\perp)^2 + g_q, \quad g_8 = dr^2 + r^2 g_7
\]
are tri-Sassakian and hyperkahler respectively. Here $\omega^i_-$ is the $\text{Sp}(1)$ part of the spin connection and $\sigma_i$ are the usual Maurer-Cartan one forms over $\text{SO}(3)$. Moreover the “squashed” family
\[ g_7 = (\sigma_i - \omega^i_-)^2 + 5g_q, \]
is of weak $G_2$ holonomy.

We will consider the last sentence of this proposition in the next section. In order to finish this section we would like to describe a little more the Swann bundles. Under the transformation $u \to Gu$ with $G : M \to \text{SU}(2)$ the $\text{SU}(2)$ instanton $\omega_-$ is gauge transformed as $\omega_- \to G\omega_-G^{-1} + GdG^{-1}$. Therefore the form $du + \omega_- u$ is transformed as
\[ du + \omega_- \to d(Gu) + (G\omega_-G^{-1} + GdG^{-1})Gu = Gdu + (dG + G\omega_- - dG)u = G(du + \omega_-), \]
and it is seen that $du + \omega_- u$ is a well defined quaternion-valued one form over the chiral bundle. The metric (2.6) is also well defined over this bundle. Associated to the metric (2.6) there is a quaternion valued two form
\[ \tilde{J} = uJ + (du + u\omega_-) \wedge (du + u\omega_-), \tag{2.21} \]
and it can be checked that the metric (2.22) is hermitian with respect to any of the components of (2.21). Also
\[ d\tilde{J} = du \wedge (J + d\omega_- - \omega_- \wedge \omega_-)u + u \wedge (\tilde{J} + d\omega_- - \omega_- \wedge \omega_-)du \]
\[ + u(d\tilde{J} + \omega_- \wedge d\omega_- - d\omega_- \wedge \omega_-)u. \]
The first two terms of the last expression are zero due to (2.5). Also by introducing (2.5) into the relation (2.3) it is seen that
\[ d\tilde{J} + \omega_- \wedge d\omega_- - d\omega_- \wedge \omega_- = 0, \]
and therefore the third term is also zero. This means that the metric (2.6) is hyperkahler with respect to the triplet $\tilde{J}$. The Swann metrics have been considered in several context in physics, as for instance in [37]-[40]. It is an important tool also in differential geometry because the quaternion Kahler quotient construction correspond to hyperkahler quotients on the Swann fibrations.

### 2.3 The squashed version of the tri-Sasaki family

In [4] there were probably constructed the first examples of $\text{Spin}(7)$ holonomy metrics. These examples are fibered over four dimensional quaternion Kahler metrics defined over manifold $M$. This resembles the Swann metrics that we have presented in (2.6), although the Bryant-Salamon were found first. The anzatz for the $\text{Spin}(7)$ is
\[ g_8 = g|u|^2g + f|du + u\omega_-|^2, \tag{2.22} \]
where $f$ and $g$ are two unknown functions $f(r^2)$ and $g(r^2)$ which will be determined by the requirement that the holonomy is in $\text{Spin}(7)$, i.e, the closure of the associated 4-form $\Phi_4$. The analogy between the anzatz (2.22) and (2.6) is clear, in fact, if $f = g = 1$ the holonomy will be reduced to $\text{Sp}(2)$. A convenient (but not unique) choice for $\Phi_4$ is the following
\[ \Phi = 3fg[\alpha \wedge \overline{\alpha} \wedge e \wedge e + e \wedge e \wedge \alpha \wedge \overline{\alpha}] + g^2 e^2 \wedge e \wedge e \wedge e + f^2 \alpha \wedge \overline{\alpha} \wedge \alpha \wedge \overline{\alpha} \tag{2.23} \]
where \( \alpha = du + u\omega \). After imposing the condition \( d\Phi_4 = 0 \) to (2.22) it is obtained a system of differential equations for \( f \) and \( g \) with solution

\[
\begin{align*}
  f & = \frac{1}{(2\kappa r^2 + c)^{2/5}}, \\
  g & = (2\kappa r^2 + c)^{3/5},
\end{align*}
\]

and the corresponding metric

\[
g_s = (2\kappa r^2 + c)^{3/5}g + \frac{1}{(2\kappa r^2 + c)^{2/5}|\alpha|^2}.
\]

Spaces defined by (2.24) are the Bryant-Salamon Spin(7) ones. The metrics (2.24) are non compact (because \( |u| \) is not bounded), and asymptotically conical. They will be exactly conical only if \( c = 0 \). This is better seen by introducing spherical coordinates for \( u \)

\[
\begin{align*}
  u_1 & = |u| \sin \theta \cos \varphi \cos \tau, \\
  u_2 & = |u| \sin \theta \cos \varphi \sin \tau, \\
  u_3 & = |u| \sin \theta \sin \varphi, \\
  u_4 & = |u| \cos \theta,
\end{align*}
\]

and defining the radial variable

\[
r^2 = \frac{9}{20}(2\kappa|u|^2 + c)^{3/5},
\]

from which we obtain the spherical form of the metric

\[
g = \frac{dr^2}{\kappa(1 - c/r^{10/3})} + \frac{9}{100\kappa}r^2 \left( 1 - \frac{c}{r^{10/3}} \right) \left( \sigma^i - \omega_i \right)^2 + \frac{9}{20}r^2 g
\]

being \( \sigma^i \) the left-invariant one-forms on \( SU(2) \)

\[
\begin{align*}
  \sigma_1 & = \cos \varphi d\theta + \sin \varphi \sin \theta d\tau \\
  \sigma_2 & = -\sin \varphi d\theta + \cos \varphi \sin \theta d\tau \\
  \sigma_3 & = d\varphi + \cos \theta d\tau.
\end{align*}
\]

In this case it is clearly seen that (2.26) are of cohomogeneity one and thus, by the results presented [10]-[11], they define a weak \( G_2 \) holonomy metric.

Let us fix the normalization \( \kappa = 1 \), as before. Then in the limit \( r >> c \) it is found the behavior

\[
g \approx dr^2 + r^2\Omega,
\]

being \( \Omega \) a seven dimensional metric asymptotically independent of the coordinate \( r \), namely

\[
\Omega = (\sigma^i - \omega_i)^2 + 5g_q
\]

In particular the subfamilies of (2.26) with \( c = 0 \) are exactly conical and their angular part is (2.28). This seven dimensional metric is of weak \( G_2 \) holonomy and possesses an \( SO(3) \) isometry action associated with the \( \sigma^i \). If also the four dimensional quaternion
Kahler metric has an isometry group $G$ that preserve the $\omega^i$, then the group is enlarged to $SO(3) \times G$.

This metric can also be obtained by introducing four new coordinates $(r, \theta, \phi, \varphi)$ and using an ansatz of the form
\[
g = \alpha^2 dr^2 + \beta^2 (\sigma_i - A_i)^2 + \gamma^2 g, \tag{2.29}
\]
being $\sigma_i$ the $SU(2)$ left-invariant one forms. The unknown functions $\alpha$, $\beta$ and $\gamma$ are supposed only functions of $r$. By requiring $d\tilde{\Phi} = 0$ for the four-form constructed with
\[
e^0 = \alpha dr, \quad e^i = \beta (\sigma_i - A_i), \quad e^a = \gamma e^a, \tag{2.30}
\]
the metrics (2.26) will be obtained again.

### 2.4 A test of the formulas

It is extremely important to compare the weak $G_2$ holonomy metrics (2.28) and the tri-Sasaki metrics (2.10). The only difference between the two metrics is a factor 5 in front of $g_q$ in (2.28). Both metrics possess the same isometry group. At first sight it sounds possible to absorb this factor 5 by a simple rescale of the coordinates and therefore to conclude that both metrics are the same. But this is not true. We are fixing the normalization $\kappa = 1$ in both cases, thus this factor should be absorbed only by an rescaling on the coordinates of the fiber. There is no such rescaling. Therefore, due to this factor 5, both metrics are different. This is what one expected, since they are metrics of different type.

We can give an instructive example to understand why this is so. With this purpose in mind let us consider the Fubini-Study metric on CP(2). There exists a coordinate system for which the metric take the form
\[
g_f = 2d\mu^2 + \frac{1}{2} \sin^2 \mu \sigma_3^2 + \frac{1}{2} \sin^2 \mu \cos^2 \mu (\bar{\sigma}_1^2 + \bar{\sigma}_2^2). \tag{2.31}
\]
We have denoted the Maurer-Cartan one-forms of this expression as $\tilde{\sigma}_i$ in order to not confuse them with the $\sigma_i$ appearing in (2.28) and (2.10). The anti-self-dual part of the spin connection is
\[
\omega_+ = -\cos \mu \tilde{\sigma}_1, \quad \omega_- = \cos \mu \tilde{\sigma}_2, \quad \omega_3 = -\frac{1}{2} (1 + \cos \mu) \tilde{\sigma}_3. \tag{2.32}
\]
The two metrics that we obtain by use of (2.28) and (2.10) are
\[
g_7 = 2b d\mu^2 + \frac{1}{2} \sin^2 \mu \sigma_3^2 + b \frac{1}{2} \sin^2 \mu \cos^2 \mu (\sigma_1^2 + b \sigma_2^2) \tag{2.33}
\]
\[
+ (\sigma_1 + \cos \mu \tilde{\sigma}_1)^2 + (\sigma_2 - \cos \mu \tilde{\sigma}_2)^2 + (\sigma_3 + \frac{1}{2} (1 + \cos \mu) \tilde{\sigma}_3)^2.
\]
If (2.28) and (2.10) are correct, then $b = 1$ corresponds to a tri-Sasaki metric and $b = 5$ to a weak $G_2$ holonomy one. This is true. Locally this metrics are the same that $N(1, 1)_I$ and $N(1, 1)_II$ given in [42], which are known to be tri-Sasaki and weak $G_2$. We see therefore that this number five in front of the quaternion Kahler metric is relevant and change topological properties of the metric (such as the number of conformal Killing spinors). In other words, we will construct in this work an infinite doublet of 7-dimensional metrics, one with one conformal Killing spinor and other with three.
3. Explicit tri-Sasaki metrics over manifolds and orbifolds

3.1 Quaternion Kahler limit of AdS-Kerr-Newman-Taub-Nut

In this subsection we will describe certain toric quaternion Kahler orbifolds which are obtained by a Wick rotation of the Plebanski and Demianski solution \cite{28}. In fact, these spaces has been discussed in detail in \cite{29}-\cite{34}. The distance element is

\[ g_q = \frac{x^2 - y^2}{P} dx^2 + \frac{x^2 - y^2}{Q} dy^2 + \frac{P}{x^2 - y^2} (d\alpha + y^2 d\beta)^2 + \frac{Q}{x^2 - y^2} (d\alpha + x^2 d\beta)^2 \]  

(3.34)

being \( P \) and \( Q \) polynomials of the form

\[ P(x) = q - 2sx - tx^2 - \kappa x^4, \quad Q(y) = -P(y), \]

being \((q, s, t, \kappa)\) four parameters. These expressions can be rewritten as

\[ P(x) = -\kappa(x - r_1)(x - r_2)(x - r_3)(x - r_4), \quad Q(y) = -P(y), \]

\[ r_1 + r_2 + r_3 + r_4 = 0, \]

the last condition comes from the fact that \( P(x) \) contains no cubic powers of \( x \). The two commuting Killing vectors are \( \partial_\alpha \) and \( \partial_\beta \).

The metric (3.34) is invariant under the transformation \( x \leftrightarrow y \). The transformations \( x \to -x, \ y \to -y, \ r_i \to -r_i \) are also a symmetry of the metric. In addition the symmetry \( (x, y, \alpha, \beta) \to (\lambda x, \lambda y, \frac{\alpha}{\lambda}, \frac{\beta}{\lambda}) \), \( r_i \to \lambda r_i \) can be used in order to put one parameter equal to one, so there are only three effective parameters here. The domains of definition are determined by

\[ (x^2 - y^2)P(x) \geq 0, \quad (x^2 - y^2)Q(y) \geq 0. \]

The anti-self-dual part of the spin connection is

\[ \omega^1_+ = \sqrt{\frac{PQ}{y - x}} d\beta, \quad \omega^3_+ = \frac{1}{x - y} \left( \sqrt{\frac{Q}{P}} dx + \sqrt{\frac{P}{Q}} dy \right), \]

\[ \omega^2_+ = -\kappa(x - y) d\alpha + \frac{1}{x - y} \left( q - s(x + y) - tx y - \kappa x^2 y^2 \right) d\beta, \]  

(3.35)

(see for instance \cite{32}). We will need (3.35) in the following. These solutions are the self-dual limit of the AdS-Kerr-Newmann-Taub-Nut solutions, the last one corresponds to the polynomials

\[ P(x) = q - 2sx - tx^2 - \kappa x^4, \quad Q(y) = -q + 2s'x + tx^2 + \kappa x^4, \]

and are always Einstein, but self-dual if and only if \( s' = s \). We will concerned with this limit in the following, because is the one which is quaternion Kahler. If we define the new coordinates

\[ y = \bar{r}, \quad x = a \cos \bar{\theta} + N, \]

\[ \alpha = t + \left( \frac{N^2}{a} + a \right) \bar{\phi} \Xi, \quad \beta = -\frac{\bar{\phi}}{a \Xi}, \]

where we have introduced the parameters

\[ \Xi = 1 - \kappa a^2, \quad q = -a^2 + N^2(1 - 3\kappa a^2 + 3\kappa N^2), \]
then the functions \( P \) and \( Q \) are expressed as
\[
P = -a^2 \sin^2 \tilde{\theta} [1 - \kappa (4aN \cos \tilde{\theta} + a^2 \cos^2 \tilde{\theta})],
\]
\[
Q = -\tilde{r}^2 - N^2 + 2s' \tilde{r} + a^2 + \kappa (\tilde{r}^4 - a^2 \tilde{r}^2 - 6\tilde{r}^2 N^2 + 3a^2 N^2 - 3N^4),
\]
and the metric take the AdS-Kerr-Newman-Taub-Nut form
\[
g_q = \frac{R^2}{1 - \kappa (a^2 \cos^2 \tilde{\theta} + 4aN \cos \tilde{\theta})} d\tilde{\theta}^2 + \frac{R^2}{\lambda^2} d\tilde{r}^2 + \frac{\lambda^2}{R^2} (d\tilde{t} + \frac{(a \sin^2 \tilde{\theta} - 2N \cos \tilde{\theta})}{\Xi} d\tilde{\phi})^2 + \frac{\sin^2 \tilde{\theta} [1 - \kappa (a^2 \cos^2 \tilde{\theta} + 4aN \cos \tilde{\theta})]}{R^2} (d\tilde{t} - \tilde{r}^2 - a^2 - N^2 \frac{R}{d\tilde{\phi}})^2,
\]
being \( R \) and \( \lambda \) defined by
\[
R = \tilde{r}^2 - \left(a \cos \tilde{\theta} + N\right)^2,
\]
\[
\lambda = \tilde{r}^2 + N^2 - 2s' \tilde{r} - a^2 - \kappa (\tilde{r}^4 - a^2 \tilde{r}^2 - 6\tilde{r}^2 N^2 + 3a^2 N^2 - 3N^4).
\]
Notice that the self-dual limit corresponds to the choice \( s' = N(1 - \kappa a^2 + 4aN^2) \) in all the expressions. The parameter \( \kappa \) is the scalar curvature of the metric and we fix \( \kappa = 1 \), as we did previously.

These metrics have interesting limits. For \( a = 0 \) and \( N \) different from zero becomes the AdS Taub-Nut solution with local metric
\[
g_q = V(\tilde{r}) (d\tilde{t} - 2N \cos \tilde{\theta} d\tilde{\phi})^2 + \frac{d\tilde{r}^2}{V(\tilde{r})} + (\tilde{r}^2 - N^2)(d\tilde{\theta}^2 + \sin^2 \tilde{\theta} d\tilde{\phi}^2),
\]
being \( V(\tilde{r}) \) given by
\[
V(\tilde{r}) = \frac{\lambda}{R^2} = \frac{1}{\tilde{r}^2 - N^2} \left( \tilde{r}^2 + N^2 - (\tilde{r}^4 - 6N^2 \tilde{r}^2 - 3N^4) - 2s' \tilde{r} \right).
\]
This metric has been considered in different context \([29]-[34]\). The parameter \( s' \) is a mass parameter and \( N \) is a nut charge. Both parameters are not independent in the quaternion Kahler limit, in fact the self-duality condition \( s' = s \) relates them as \( s' = N(1+4N^2) \). If the mass were arbitrary then the metric will possess a "bolt", but in this case the metric will possess a "nut", that is, a zero dimensional regular fixed point set. The isometry group of this metric is enhanced from \( U(1) \times U(1) \) to \( SU(2) \times U(1) \) in this limit. The anti self-dual part of the spin connection reads
\[
\omega_1 = -\sqrt{(\tilde{r} + N)V(\tilde{r})} \sin \tilde{\theta} d\tilde{\phi}, \quad \omega_3 = \sqrt{(\tilde{r} + N)V(\tilde{r})} d\tilde{\theta},
\]
\[
\omega_2 = (\tilde{r} - N)d\tilde{t} + g(\tilde{r}) \cos \tilde{\theta} d\tilde{\phi},
\]
being \( g(\tilde{r}) \) defined by
\[
g(\tilde{r}) = \left( \frac{N^2(\tilde{r} - N) + N(1 + 4N^2) + (1 + 6N^2)\tilde{r} - 2N\tilde{r}^2}{\tilde{r} - N} \right).
\]
By taking the further limit \( N = 0 \), that is, but switching off the mass and the charge, we obtain after introducing the new radius \( \tilde{r} = \sin \tilde{\rho} \) the following distance element
\[
g_q = \cos^2 \tilde{\rho} dt^2 + d\tilde{\rho}^2 + \sin^2 \tilde{\rho} (d\tilde{\theta}^2 + \sin^2 \tilde{\theta} d\tilde{\phi}^2).
\]
The anti-self-dual spin connection takes the simple form
\[ \omega_1 = \cos \tilde{\rho} \sin \tilde{\theta} d\tilde{\phi}, \quad \omega_2 = \sin \tilde{\rho} d\tilde{t} + \cos \tilde{\theta} d\tilde{\phi}, \quad \omega_3 = \cos \tilde{\rho} d\tilde{\theta}, \]
and it follows that we have obtained the metric of the sphere \( S^4 = SO(5)/SO(4) \). If we would choose negative scalar curvature instead, this limit would correspond to the non compact space \( SO(4,1)/SO(4) \). Both cases are maximally symmetric and for this reason this is called the \( AdS_4 \) limit of the AdS-Taub-Nut solution. The only known quaternion kahler manifolds in 4-dimensions are the \( S^4 \) and \( \mathbb{C}P^2 \). The \( \mathbb{C}P^2 \) manifold limit is obtained by defining the new coordinates \( \tilde{r} = N(\bar{r} - N) \) and \( \tilde{t} = 2N\xi \) and taking the limit \( N \to \infty \). The result, after defining \( \tilde{\rho} = \tilde{r}^2/4(1 + \tilde{r}^2) \), is the metric
\[ g_q = \frac{\tilde{r}^2}{2(1 + \tilde{r}^2)^2} \left( d\xi - \cos \tilde{\theta} d\tilde{\phi} \right)^2 + \frac{2d\tilde{r}^2}{(1 + \tilde{r}^2)^2} + \frac{\tilde{r}^2}{2(1 + \tilde{r}^2)^2} \left( d\tilde{\theta}^2 + \sin^2 \tilde{\theta} d\tilde{\phi}^2 \right). \] (3.39)
By noticing that \( \sigma_3 = d\xi - \cos \tilde{\theta} d\tilde{\phi} \) and that \( \sigma_1^2 + \sigma_2^3 = d\tilde{\theta}^2 + \sin^2 \tilde{\theta} d\tilde{\phi}^2 \), we recognize from (3.39) the Bianchi IX form for the Fubbini-Study metric on \( \mathbb{C}P^2=SU(3)/SU(2) \). Instead, if we put \( N = 0 \) in (3.40), the result will be the AdS-Kerr euclidean solution in Boyer-Linquidst coordinates, namely
\[ g_q = \frac{\tilde{r}^2 - a^2 \cos \tilde{\theta}^2}{1 - a^2 \cos^2 \tilde{\theta}} d\tilde{\theta}^2 + \frac{\tilde{r}^2 - a^2 \cos \tilde{\theta}^2}{(\tilde{r}^2 - a^2)(1 - \tilde{r}^2)} d\tilde{r}^2 + \frac{(\tilde{r}^2 - a^2)(1 - \tilde{r}^2)}{\tilde{r}^2 - a^2 \cos \tilde{\theta}^2} \left( d\tilde{t} + \frac{a \sin^2 \tilde{\theta}}{\Xi} d\tilde{\phi} \right)^2 \] (3.40)
\[ + \frac{\sin^2 \tilde{\theta}(1 - a^2 \cos^2 \tilde{\theta})}{\tilde{r}^2 - a^2 \cos^2 \tilde{\theta}} ( \tilde{a} d\tilde{t} - \tilde{r}^2 - a^2 d\tilde{\phi} )^2. \]
The anti-self-dual connection \( \omega^\perp \) is in this case
\[ \omega_1 = -\frac{1}{\tilde{r} - a \cos \tilde{\theta}} \sqrt{(1 - a^2 \cos \tilde{\theta}^2)(\tilde{r}^2 - a^2)(1 - \tilde{r}^2)} \frac{\sin \tilde{\theta}}{\Xi} d\tilde{\phi}, \]
\[ \omega_2 = (\tilde{r} - a \cos \tilde{\theta}) d\tilde{t} + \frac{1}{\tilde{r} - a \cos \tilde{\theta}} \frac{W(\tilde{r},\tilde{\theta})}{\Xi} d\tilde{\phi}, \]
\[ \omega_3 = \frac{1}{\tilde{r} - a \cos \tilde{\theta}} \left( \frac{(\tilde{r}^2 - a^2)(1 - \tilde{r}^2)}{1 - a^2 \cos^2 \tilde{\theta}} d\tilde{\theta} - \sqrt{\frac{1 - a^2 \cos^2 \tilde{\theta}}{(\tilde{r}^2 - a^2)(1 - \tilde{r}^2)} a \sin \tilde{\theta} d\tilde{r}} \right), \] (3.41)
where we have defined the function
\[ W(\tilde{r},\tilde{\theta}) = [(\tilde{r} - a \cos \tilde{\theta})^2 - a + (1 + a^2)\tilde{r} \cos \tilde{\theta} - a\tilde{r}^2 \cos^2 \tilde{\theta}]. \] (3.42)
The parameter \( a \) is usually called rotational parameter, although we have no the notion of rotational black hole in euclidean signature. The mass parameter \( s \) and the nut charge are zero in this case.

### 3.2 Tri-Sassaki and weak \( G_2 \) over AdS-Kerr and AdS-Taub-Nut

We are now in position to find new compact tri-Sasaki and weak \( G_2 \) holonomy metrics. The main ingredient in this construction is the proposition 1, applied to limiting cases of the euclidean Plebanski-Demianski solution (3.40). Let us turn our attention to this task.
The AdS-Taub-Nut case

It is direct, by using proposition 1 and the lifting formula (2.28), to work out tri-Sassaki and weak $G_2$ holonomy metrics fibered over the AdS-Taub-Nut metrics (3.37), the result is

\[
g_\tau = (\sqrt{\bar{\tau} + N})V(\bar{\tau})\sin \bar{\theta}d\bar{\phi} + \sigma_1)^2 + \left( (\bar{\tau} - N)d\bar{\tau} + g(\bar{\tau})\cos \bar{\theta}d\bar{\phi} - \sigma_2 \right)^2
\]

\[
+ (\sqrt{\bar{\tau} + N})V(\bar{\tau})d\bar{\theta} - \sigma_3)^2 + b \left( V(\bar{\tau})(d\bar{\tau} - 2N\cos \bar{\theta}d\bar{\phi})^2 + \frac{d\bar{r}^2}{V(\bar{\tau})} + (\bar{r}^2 - N^2)(d\bar{\theta}^2 + \sin^2 \bar{\theta}d\bar{\phi}^2) \right).
\]

Although the base quaternion Kahler space possess $SU(2) \times U(1)$ isometry, this group does not preserve the fibers, so the isometry group is $SU(2) \times U(1)^2$, being the $SU(2)$ group related to the Maurer-Cartan forms of the fiber metric and $U(1)^2$ generated by $\partial_\tau$ and $\partial_\phi$. Let us notice that we have a third commuting Killing vector, which is the Reeb vector $\partial_\tau$, which is present in the expression for the Maurer-Cartan forms $\sigma_i$. Therefore we have a $T^3$ subgroup of isometries. By taking into account the explicit form of the $\sigma_i$s given in (2.11) we obtain the following metric components

\[
g_{\bar{\tau}\bar{\tau}} = (\bar{\tau} - N)^2 + bV(\bar{\tau}), \quad g_{\bar{\phi}\bar{\phi}} = 4bN^2V(\bar{\tau})\cos^2 \bar{\theta} + (\bar{r}^2 - N^2)\sin^2 \bar{\theta}
\]

\[
g_{\bar{\theta}\bar{\theta}} = b(\bar{r}^2 - N^2) + (\bar{\tau} + N)V(\bar{\tau}), \quad g_{\bar{r}\bar{r}} = \frac{b}{V(\bar{\tau})}, \quad g_{\bar{\tau}\bar{\phi}} = g_{\bar{\phi}\bar{\phi}} = g_{\theta\theta} = 1
\]

\[
g_{\bar{\tau}\bar{\phi}} = -2NbV(\bar{\tau})\cos \bar{\theta} + (\bar{\tau} - N)g(\bar{\tau})\cos \bar{\theta}, \quad g_{\bar{r}\bar{\phi}} = -bV(\bar{\tau})\cos \bar{\theta} \sin \phi
\]

\[
g_{\bar{\theta}\bar{\tau}} = \sqrt{(\bar{\tau} + N)V(\bar{\tau})}\sin \bar{\theta} \sin \bar{\theta} \sin \phi + g(\bar{\tau})\cos \bar{\theta} \sin \phi \cos \phi
\]

\[
g_{\bar{\theta}\bar{\phi}} = \sqrt{(\bar{\tau} + N)V(\bar{\tau})}\cos \bar{\phi} \cos \bar{\phi} + g(\bar{\tau})\cos \bar{\theta} \sin \phi \cos \phi
\]

the remaining components are all zero. The parameter $b$ take the values 1 or 5, $b = 1$ corresponds to an Einstein-Sasaki metric, while $b = 5$ corresponds to a weak $G_2$ holonomy metric.

The AdS-Kerr-Newmann case

For the rotating case, that is, for the AdS-Kerr-Newman metrics (3.40) we obtain the metrics

\[
g_\phi = \left( \frac{f(\bar{\tau}c(\bar{\tau})d(\bar{\tau})}{e(\bar{\tau}, \theta)} \sin \bar{\theta}d\bar{\phi} - \sigma_1 \right)^2 + \left( \frac{e(\bar{\tau}, \bar{\phi})d\bar{\tau} + W(\bar{\tau}, \theta)}{\Xi e(\bar{\tau}, \theta)} d\bar{\phi} - \sigma_2 \right)^2
\]

\[
+ \left( \frac{c(\bar{\tau})d(\bar{\tau})}{f(\bar{\tau})} \frac{d\bar{\theta}}{e(\bar{\tau}, \theta)} - \sqrt{\frac{f(\bar{\tau})}{c(\bar{\tau})d(\bar{\tau})}} \frac{a \sin \bar{\theta}}{e(\bar{\tau}, \theta)} d\bar{\tau} - \sigma_3 \right)^2 + \frac{b f(\bar{\tau})\sin^2 \bar{\theta}}{\Xi^2 - a^2 \cos^2 \theta} \left( a d\bar{\tau} - \frac{c(\bar{\tau})d(\bar{\tau})}{\Xi} d\bar{\phi} \right)^2
\]

\[
+ \frac{b c(\bar{\tau})d(\bar{\tau})}{\Xi^2 - a^2 \cos^2 \theta} \left( d\tau + \frac{a \sin^2 \bar{\theta}}{\Xi} d\bar{\phi} \right)^2 + \frac{\bar{r}^2 - a^2 \cos \bar{\theta}^2}{c(\bar{\tau})d(\bar{\tau})} \frac{d\bar{\tau}^2}{b^2 d\tau^2} + \frac{\bar{r}^2 - a^2 \cos \bar{\theta}^2}{c(\bar{\tau})d(\bar{\tau})} \frac{d\bar{\phi}^2}{b^2 d\tau^2},
\]
where we have introduced the functions

\[ f(\tilde{\vartheta}) = 1 - a^2 \cos^2 \tilde{\vartheta}, \quad c(\tilde{r}) = \tilde{r}^2 - a^2, \quad d(\tilde{r}) = 1 - \tilde{r}^2 \quad e(\tilde{r}, \tilde{\vartheta}) = \tilde{r} - a \cos \tilde{\vartheta}. \]

The local isometry is $SU(2) \times U(1)^2$ and as before, the vectors $\partial_{\tilde{r}}, \partial_{\tilde{\varphi}}$, and $\partial_{\tilde{\tau}}$ generate a $T^3$ isometry subgroup. From expression (3.44) we read the following components

\[
\begin{align*}
g_{\tilde{\vartheta}\tilde{\vartheta}} &= \frac{b c(\tilde{r}) d(\tilde{r})}{\tilde{r}^2 - a^2 \cos \theta^2} + \frac{a^2 b f(\tilde{\vartheta}) \sin^2 \tilde{\vartheta}}{\tilde{r}^2 - a^2 \cos^2 \theta} + e^2(\tilde{r}, \tilde{\vartheta}) \\
g_{\tilde{\varphi}\tilde{\varphi}} &= \frac{b c(\tilde{r}) d(\tilde{r})}{\tilde{r}^2 - a^2 \cos \theta^2} \frac{a^2 \sin^4 \tilde{\vartheta}}{\tilde{r}^2 - a^2 \cos^2 \theta} + \frac{b f(\tilde{\vartheta}) \sin^2 \tilde{\vartheta} \cdot c(\tilde{r})}{\tilde{r}^2 - a^2 \cos^2 \theta} \frac{W^2(\tilde{r}, \tilde{\vartheta})}{\tilde{r}^2 e^2(\tilde{r}, \tilde{\vartheta})} + \frac{2 f(\tilde{\vartheta}) c(\tilde{r}) d(\tilde{r})}{e^2(\tilde{r}, \tilde{\vartheta})} \frac{\sin^2 \tilde{\vartheta}}{\tilde{r}^2 e^2(\tilde{r}, \tilde{\vartheta})} \\
g_{\tilde{\omega}\tilde{\omega}} &= \frac{\tilde{r}^2 - a^2 \cos \tilde{\vartheta}^2}{f(\tilde{\vartheta})} b + \frac{1}{f(\tilde{\vartheta})} c(\tilde{r}) d(\tilde{r}), \quad g_{\tilde{\varphi}\tilde{\varphi}} = \frac{\tilde{r}^2 - a^2 \cos \tilde{\vartheta}^2}{f(\tilde{\vartheta})} b + \frac{f(\tilde{\vartheta}) a^2 \sin^2 \tilde{\vartheta}}{c(\tilde{r}) d(\tilde{r})} e^2(\tilde{r}, \tilde{\vartheta}), \\
g_{\tilde{\varphi}\tilde{\varphi}} &= \sqrt{\frac{f(\tilde{\vartheta})}{c(\tilde{r}) d(\tilde{r})}} e(\tilde{r}, \tilde{\vartheta}), \quad g_{\tilde{\varphi}\tilde{\varphi}} = \sqrt{\frac{f(\tilde{\vartheta}) \cos \theta a \sin \tilde{\tau} \cos \vartheta}{c(\tilde{r}) d(\tilde{r})}} e(\tilde{r}, \tilde{\vartheta}), \\
g_{\tilde{\vartheta}\tilde{\vartheta}} &= \frac{b c(\tilde{r}) d(\tilde{r})}{\tilde{r}^2 - a^2 \cos \theta^2} \frac{a^2 \sin^2 \tilde{\vartheta}}{\tilde{r}^2 - a^2 \cos^2 \theta} + \frac{a f(\tilde{\vartheta}) \sin^2 \tilde{\vartheta} \cdot c(\tilde{r})}{\tilde{r}^2 - a^2 \cos^2 \theta} \frac{W(\tilde{r}, \tilde{\vartheta})}{\tilde{r}^2 e(\tilde{r}, \tilde{\vartheta})}, \\
g_{\tilde{\varphi}\tilde{\varphi}} &= \frac{\sqrt{f(\tilde{\vartheta}) c(\tilde{r}) d(\tilde{r})}}{e(\tilde{r}, \tilde{\vartheta})} \cos \varphi, \quad W(\tilde{r}, \tilde{\vartheta}) \sin \theta \cos \varphi \quad (3.45) \\
g_{\tilde{\vartheta}\tilde{\vartheta}} &= \frac{W(\tilde{r}, \tilde{\vartheta})}{e(\tilde{r}, \tilde{\vartheta})} \sin \varphi + \frac{\sqrt{f(\tilde{\vartheta}) c(\tilde{r}) d(\tilde{r})}}{e(\tilde{r}, \tilde{\vartheta})} \cos \varphi \\
g_{\tilde{\omega}\tilde{\omega}} &= \frac{\sqrt{c(\tilde{r}) d(\tilde{r})}}{f(\tilde{\vartheta})} \cos \theta, \quad g_{\tilde{\omega}\tilde{\omega}} = \frac{\sqrt{c(\tilde{r}) d(\tilde{r})}}{f(\tilde{\vartheta})} \frac{1}{e(\tilde{r}, \tilde{\vartheta})}, \\
g_{\tilde{\omega}\tilde{\omega}} &= -e(\tilde{r}, \tilde{\vartheta}) \sin \varphi, \quad g_{\tilde{\varphi}\tilde{\varphi}} = \cos \theta
\end{align*}
\]

and the other components are zero.

The manifold limit

We saw that the $S^4$ and CP(2) metrics are limits of (3.37). By taking these limits, a doublet of tri-Sasaki and weak $G_2$ metric fibered over these manifolds arise from (3.43). For $S^4$ the result is the following metric

\[
g_7 = (\sin \tilde{\rho} \tilde{d} + \cos \tilde{\vartheta} \tilde{d} \tilde{\varphi} + \sin \varphi \tilde{d} \theta - \cos \varphi \sin \theta \tilde{d} \tau)^2 + (\cos \tilde{\rho} \tilde{d} \tilde{\omega} - \tilde{d} \varphi - \cos \theta \tilde{d} \tau)^2
+ (\cos \tilde{\rho} \sin \tilde{\omega} \tilde{d} \tilde{\varphi} - \cos \varphi \tilde{d} \theta - \sin \varphi \sin \theta \tilde{d} \tau)^2 + b \cos^2 \tilde{\rho} \tilde{d} \tilde{\varphi}^2 + bd \rho^2 + b \sin^2 \tilde{\rho} (\tilde{d} \tilde{\omega}^2 + \sin^2 \tilde{\omega} \tilde{d} \tilde{\varphi}^2). \quad (3.46)
\]

The expression for the tri-Sasaki 7-metrics fibered over CP(2) was already given in (2.33), it is well known and have been considered already, so we will not discuss it again.
### 3.3 AdS and non AdS backgrounds from harmonic functions

In the context of the $AdS/CFT$ correspondence [1]-[2] there is of interest to construct eleven dimensional supergravity backgrounds of the form

$$g_{11} = H^{-2/3}(-dt^2 + dx^2 + dy^2) + H^{1/3}(dr^2 + r^2 g_7),$$

$$F = \pm dx \wedge dy \wedge dt \wedge dH^{-1}$$

where the conical metric $g_8 = dr^2 + r^2 g_7$ is Ricci flat and $H$ is an harmonic function over the space $M_8$ where $g_8$ is defined. Usually one consider radial harmonic functions given by

$$H(r) = 1 + \frac{2^{5/2} \pi^2 N l^6}{r^6}. $$

This solution describe $N$ M2 branes. The near horizon limit of this geometry is obtained taking the 11 dimensional Planck length $l_p \to 0$ and keeping fixed $U = r^2/l_p^3$. The resulting background is $AdS_4 \times X_7$, being $X_7$ is an Einstein manifold with cosmological constant $\Lambda = 5$, and the radius of $AdS_4$ is $2R_{AdS} = l_p (2^{5/2} \pi^2 N)^{1/6}$. Such solutions have the generic form

$$g_{11} = g_{AdS} + g_7, \quad F_4 \sim \omega_{AdS},$$

being $\omega_{AdS}$ the volume form of $AdS_4$. This backgrounds are in general associated to three dimensional conformal field theories arising as the infrared limit of the world volume theory of $N$ coincident M2 branes located a the singularity of $M_3 \times X_8$. Also in this case, the number of supersymmetries of the field theory is determined by the holonomy of $X_8$. In the case of $Spin(7)$, $SU(4)$ or $Sp(2)$ holonomies we have $N = 1, 2, 3$ supersymmetries, respectively. This implies that the 7-dimensional cone will be of weak $G_2$ holonomy (if the eight dimensional metric is of cohomogeneity one, see below), tri-Sasaki or a Sasaki-Einstein, respectively. If $g_8$ is flat, then we have the maximal number of supersymmetries, namely eight.

Non $AdS_4$ backgrounds are also of interest because they give rise to non conformal field theory duals. Therefore it is of interest to find harmonic functions which are functions not only of the radius $r$, but also of other coordinates of the internal space. We will now give here a simple way to construct non trivial harmonic functions over the Swann hyperkahler metrics that we have presented.

First let us notice that all the 4-dimensional quaternion Kahler orbifolds we have considered possess two commuting Killing vectors, namely $\partial_\phi$ and $\partial_t$, which also preserve the one forms $\omega_\rho$ also. Thus these vectors also preserve the Kahler triplet $d\overrightarrow{J} = \omega_- + \omega_- \wedge \omega_-$. Consequently they preserve the hyperkahler triplet (2.21) for the corresponding Swann fibration. Such vectors are therefore Killing and tri-holomorphic (thus tri-hamiltonian) vectors of $g_8$. For any eight dimensional hyperkahler metric with two commuting tri-holomorphic Killing vectors there exist a coordinate system in which it takes the form [14]

$$g_8 = U_{ij} dx^i \cdot dx^j + U^{ij} (dt_i + A_i)(dt_j + A_j),$$

being $(U_{ij}, A_i)$ solutions of the generalized monopole equation

$$F_{x^i x^j} = \epsilon_{\mu \nu} \nabla_{x^\mu} U_j,$$

$$\nabla_{x^\rho} U_j = \nabla_{x^\rho} U_i,$$

(3.50)
The hyperkahler form corresponding to \(3.49\) is \([14]\)

\[
\mathcal{F}_k = (dt_i + A_i) dx^i_k - U_{ij}(dx^i \wedge dx^j)_k,
\]

From the last expression it follows that the coordinates \(dx^i_k\) are defined by means of the relation

\[
dx^i_k = i_{\partial_k} \mathcal{F}_k, \quad dx^2_k = i_{\partial_2} \mathcal{F}_k. \tag{3.51}\]

The coordinates \((x^1, x^2)\) defined by (3.51) are known as the momentum maps of the isometries \(\partial_1\) and \(\partial_2\). Thus, the generic Gibbons-Hawking form is obtained by going to the momentum map system.

In the momentum map system the 11-dimensional supergravity solution reads

\[
g_{11} = H^{-2/3} g_{2,1} + H^{1/3}[U_{ij} dx^i \cdot dx^j + U^{ij}(dt_i + A_i)(dt_j + A_j)], \tag{3.52}\]

\[
F = \pm \omega(E^{2,1}) \wedge dH^{-1}, \tag{3.53}\]

and the harmonic condition on \(H\) is expressed as

\[
U^{ij} \partial_i \cdot \partial_j H = 0.
\]

Let us recall that, as a consequence of (3.50), we have that \(\partial_i \cdot \partial_j U_{ij} = 0\), which implies that \(U^{ij} \partial_i \cdot \partial_j U_{ij} = 0\). This means that any entry \(U_{ij}\) is an harmonic function over the hyperkahler cone. The matrix \(U^{ij}\) is determined by the relation \(U^{ij} = g_8(\partial_i, \partial_j)\), and the inverse matrix \(U_{ij}\) will give us three independent non trivial harmonic functions for the internal space in consideration. This give a way to find harmonic functions which are not only radial, but depends on other coordinates of the cone.

As an example we can consider the cone \(g_8 = du^2 + u^2 g_7\) being \(g_7\) the tri-Sasaki metric corresponding to the AdS-Kerr-Newmann solution (3.43). For this cone we have that

\[
U^\Pi = u^2(\tilde{r} - N)^2 + u^2 V(\tilde{r}), \quad U^{\tilde{\phi}} = 4N^2 u^2 V(\tilde{r}) \cos^2 \tilde{\theta} + (\tilde{r}^2 - N^2) u^2 \sin^2 \tilde{\theta}
\]

\[
U^{i\tilde{\phi}} = -2Nu^2 V(\tilde{r}) \cos \tilde{\theta} + u^2(\tilde{r} - N)g(\tilde{r}) \cos \tilde{\theta}. \tag{3.54}\]

By defining \(\Delta = U^\Pi U^{\tilde{\phi}} - (U^{i\tilde{\phi}})^2\) we obtain the following harmonic functions

\[
U^\Pi = \frac{U^\tilde{\phi}}{\Delta}, \quad U^{i\tilde{\phi}} = -\frac{U_i}{\Delta}, \quad U_{\tilde{\phi}} = \frac{U^{i\tilde{\phi}}}{\Delta}. \tag{3.55}\]

For the AdS-Kerr-Newman case (3.45) we have

\[
U^\Pi = u^2 \frac{c(\tilde{r})d(\tilde{r})}{\tilde{r}^2 - a^2 \cos^2 \tilde{\theta}^2} + a^2 u^2 \frac{f(\tilde{\theta}) \sin^2 \tilde{\theta}}{\tilde{r}^2 - a^2 \cos^2 \tilde{\theta}^2} + u^2 e^2(\tilde{r}, \tilde{\theta})
\]

\[
U^{\tilde{\phi}} = u^2 \frac{c(\tilde{r})d(\tilde{r})}{\tilde{r}^2 - a^2 \cos^2 \tilde{\theta}^2} \frac{a^2 \sin^4 \tilde{\theta}}{\Xi^2} + u^2 \frac{f(\tilde{\theta}) \sin^2 \tilde{\theta}}{\tilde{r}^2 - a^2 \cos^2 \tilde{\theta}^2} \frac{c^2(\tilde{r})}{\Xi^2} + u^2 \frac{W^2(\tilde{r}, \tilde{\theta})}{\Xi^2 e^2(\tilde{r}, \tilde{\theta})} + u^2 \frac{f(\tilde{\theta}) c(\tilde{r}) d(\tilde{r}) \sin^2 \tilde{\theta}}{c^2(\tilde{r}, \tilde{\theta}) \Xi^2} \tag{3.56}\]

\[
U^{i\tilde{\phi}} = u^2 \frac{c(\tilde{r}) d(\tilde{r})}{\tilde{r}^2 - a^2 \cos^2 \tilde{\theta}^2} \frac{a \sin^2 \tilde{\theta}}{\Xi} + u^2 \frac{f(\tilde{\theta}) \sin^2 \tilde{\theta}}{\tilde{r}^2 - a^2 \cos^2 \tilde{\theta}^2} \frac{c(\tilde{r})}{\Xi} + u^2 \frac{W(\tilde{r}, \tilde{\theta})}{\Xi} \tag{3.57}\]
and again, the three functions $U^{ij}/\Delta$ are harmonic functions over the internal hyperkahler space. Notice that $\Delta \sim r^4$ and therefore all these harmonic functions depends on $r$ as $1/r^2$.

In the $S^4$ manifold limit we obtain

$$U^{\phi \phi} = -\frac{1}{u^2} \left( \frac{1}{\sin^2 \theta + \sin^2 \tilde{\rho} \cos^2 \tilde{\theta} - \sin^4 \tilde{\rho} \cos^2 \theta} \right),$$

$$U^{\phi t} = \frac{1}{u^2} \left( \frac{\sin^2 \tilde{\theta} + \sin^2 \tilde{\rho} \cos^2 \tilde{\theta}}{\sin^2 \theta + \sin^2 \tilde{\rho} \cos^2 \theta - \sin^4 \tilde{\rho} \cos^2 \theta} \right)$$

$$U^{\phi i} = \frac{1}{u^2} \left( \frac{\sin^2 \tilde{\rho} \cos \tilde{\theta}}{\sin^2 \theta + \sin^2 \tilde{\rho} \cos^2 \theta - \sin^4 \tilde{\rho} \cos^2 \theta} \right)$$

where the soldering forms $e^i$ for AdS-Taub-Nut metric are

$$e^1 = \frac{d\tilde{r}}{\sqrt{V(\tilde{r})}}, \quad e^2 = -\sqrt{\tilde{r}^2 - N^2} d\tilde{\theta},$$

$$e^3 = \sqrt{V(\tilde{r})}(d\tilde{t} - 2N \cos \tilde{\theta} d\tilde{\phi}), \quad e^4 = -\sqrt{\tilde{r}^2 - N^2} \sin \tilde{\theta} d\tilde{\phi}.$$

We need to find the solution of the system (3.51) being $t_1 = \tilde{t}$ and $t_2 = \tilde{\phi}$. For an arbitrary vector field $X$ we have that

$$i_X \tilde{\mathcal{J}} = u(\mathcal{J}(X) + \omega_-(X) \omega_- - \omega_- \omega_-(X)) \tilde{\mu} + u \omega_-(X) d\tilde{\mu} + du \omega_-(X) \tilde{\mu}. \quad (3.58)$$

By applying the contraction formula (3.58) with $X = \partial_t$ or $X = \partial_{\tilde{\phi}}$ and taking into account the expressions (3.38) and (3.57) we will obtain a complicated expression for $dx^\tilde{t}$ and $dx^\tilde{\phi}$.

After a lengthy calculation it is found a compact expression for the momentum maps, which in quaternion form reads

$$x^1 = u \left[ (\tilde{r} - N) J \right] \tilde{\mu}, \quad x^2 = u \left[ g(\tilde{r}) \cos \tilde{\theta} + \sqrt{(\tilde{r} + N)V(\tilde{r})} \sin \tilde{\theta} K \right] J \tilde{\mu}. \quad (3.59)$$
We can express (3.59) in pure angular fashion by parameterizing \( u \) as in (2.25).

The next step is to find the one forms \( A^i \), which are determined by

\[
A^i = U_{ij} \ g_8 \left( \frac{\partial}{\partial \bar{t}^j} \cdot \right).
\]

(3.60)

The only quantity to calculate in (3.60) is \( g_8 \left( \frac{\partial}{\partial \bar{t}}, \cdot \right) \), the matrix \( U_{ij} \) is defined by (3.55) and (3.54). The result is finally

\[
g_8 \left( \frac{\partial}{\partial \bar{t}}, \cdot \right) = -4 \ N \ u^2 \ V(\bar{r}) \ \cos \bar{\theta} d\bar{\phi} + 2 \ u^2 \ (\bar{r} - N) \left( g(\bar{r}) \cos \bar{\theta} d\bar{\phi} - \sigma_2 \right)
\]

\[
+ 2 \ u^2 \ g(\bar{r}) \ \cos \bar{\theta} \left( (\bar{r} - N) \ d\bar{t} - \sigma_2 \right).
\]

We have calculated all the relevant quantities and the procedure is finished.

4. Discussion

We have found a family of tri-Sasaki metrics fibered over the an euclidean quaternion Kahler version of the Plebanski-Demianski metrics. There exist in the literature examples of 5-dimensional Einstein-Sasaki spaces related to the Plebanski-Demianski metrics [19]-[22], but the metrics that we are presenting are different, they are indeed 7-dimensional. We have also presented an squashed version with weak \( G_2 \) holonomy. We have considered different limits of the quaternion kahler basis, which in the general case are orbifolds, but in certain limiting cases reduce to CP(2) or \( S^4 \), the two compact quaternion Kahler spaces which are manifolds. We have also extended our solutions to certain 11-dimensional supergravity backgrounds, some with \( AdS_4 \) near horizon limit and others that not.

We would like to remark that the examples that we have presented here are compact with \( T^3 \) isometry and therefore are suitable for applications related to marginal deformations of field theories [41]. This will be part of a forthcoming paper.

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