Approximating Upper Degree-Constrained Partial Orientations*

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Abstract

In the Upper Degree-Constrained Partial Orientation problem we are given an undirected graph $G = (V, E)$, together with two degree constraint functions $d^-, d^+: V \to \mathbb{N}$. The goal is to orient as many edges as possible, in such a way that for each vertex $v \in V$ the number of arcs entering $v$ is at most $d^-(v)$, whereas the number of arcs leaving $v$ is at most $d^+(v)$. This problem was introduced by Gabow [SODA'06], who proved it to be MAXSNP-hard (and thus APX-hard). In the same paper Gabow presented an LP-based iterative rounding $4/3$-approximation algorithm.

Since the problem in question is a special case of the classic 3-Dimensional Matching, which in turn is a special case of the $k$-Set Packing problem, it is reasonable to ask whether recent improvements in approximation algorithms for the latter two problems [Cygan, FOCS’13; Sviridenko & Ward, ICALP’13] allow for an improved approximation for Upper Degree-Constrained Partial Orientation. We follow this line of reasoning and present a polynomial-time local search algorithm with approximation ratio $5/4 + \varepsilon$. Our algorithm uses a combination of two types of rules: improving sets of bounded pathwidth from the recent $4/3 + \varepsilon$-approximation algorithm for 3-Set Packing [Cygan, FOCS’13], and a simple rule tailor-made for the setting of partial orientations. In particular, we exploit the fact that one can check in polynomial time whether it is possible to orient all the edges of a given graph [Gyárfás & Frank, Combinatorics’76].

1 Introduction

During the last decades several graph orientation problems were studied (see Section 8.7 in [2] and Section 61.1 in [14]). One of the most recently introduced is the Upper Degree-Constrained Partial Orientation, abbreviated as UDPO. In the UDPO problem we are given an undirected graph $G = (V, E)$, together with two degree constraint functions $d^-, d^+: V \to \mathbb{N}$. The goal is to orient as many edges as possible, in such a way that for each vertex $v \in V$ the number of arcs entering $v$ is at most $d^-(v)$, whereas the number of arcs leaving $v$ is at most $d^+(v)$. This problem was introduced by Gabow [9], motivated by a variant of the maximum bipartite matching problem arising when planning a two-day event with several parallel sessions and each participant willing to attend one chosen session each day, but without a particular order on the two selected sessions (for the exact definition, see [9]).

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Upper Degree-Constrained Partial Orientation (UDPO)

**Input:** Undirected graph $G$, degree constraints $d^+, d^- : V(G) \rightarrow \mathbb{Z}_{\geq 0}$

**Find:** A subset $\mathcal{F} \subseteq E(G)$ which admits an orientation $F$ satisfying $\deg^+_F(v) \leq d^+(v)$ and $\deg^-_F(v) \leq d^-(v)$ for each $v \in V(G)$.

**Maximize:** $|\mathcal{F}|$

Gabow proved the problem to be MAXSNP-hard (thus also APX-hard), and showed an LP-based iterative rounding $4/3$-approximation algorithm. As already observed by Gabow, UDPO is a special case of the 3-DIMENSIONAL MATCHING problem, which in turn is a special case of $k$-SET PACKING. Both of these problems belong to the Karp’s list of 21 NP-complete problems, and until last year the best known polynomial-time approximation algorithm was due to Hurkens and Schrijver [13] with approximation ratio $(k+\varepsilon)/2$. However this was recently improved independently by Sviridenko and Ward [15] to $(k+2)/3$-approximation and by Cygan [7] to $(k+1+\varepsilon)/3$-approximation. The latter result involves colour coding and pathwidth, tools originating from the area called Fixed Parameter Tractability, in local search routines.

**k-set packing**

**Input:** A family $\mathcal{F}$ of subsets of a finite universe $U$, such that $|F| \leq k$ for every $F \in \mathcal{F}$

**Find:** A subfamily $\mathcal{F}_0 \subseteq \mathcal{F}$ of pairwise-disjoint subsets

**Maximize:** $|\mathcal{F}_0|$

### 1.1 Our results

Since $(k+1+\varepsilon)/3$-approximation for $k$-set packing implies a $(4+\varepsilon)/3$-approximation for UDPO, one can ask whether recent developments for the former may be used to obtain an improved algorithm for the latter. In this paper we follow this line of reasoning and present a local search $(5+\varepsilon)/4$-approximation algorithm, improving over the $4/3$-approximation ratio of Gabow [9]. In fact, our approximation ratio matches the $5/4$ lower bound on the integrality gap of the natural LP relaxation obtained by Gabow [9].

Our algorithm uses two types of rules trying to improve the current solution at hand. Firstly, we invoke the bounded pathwidth local search by Cygan [7] in a black-box manner, when treating the UDPO problem as an instance of 3-set packing. Secondly, we use a custom rule for UDPO, relying on the fact that using a polynomial-time algorithm of Gyárfás & Frank [10] one can check whether a given set of undirected edges admits a feasible orientation (satisfying the degree constraints).

In the analysis we focus on simple instances, where all the degree bounds are either zero or one, which means that each vertex can have only zero or one incoming and outgoing arcs. Interestingly, as shown in Section 4, for our local search routines simple instances are actually no easier than the arbitrary ones.

### 1.2 Organization of the paper

In the following subsection we discuss related work on the subject. Next, in Section 2.1 we recall the reduction from UDPO to 3-set packing, followed by Section 2.2 with a description of basic notation for the local search algorithm from previous work on $k$-set packing. Our algorithm is presented in Section 3. Its analysis on simple instances (with all degree bounds at most one) is
provided in Sections 5 and 6, preceded, in Section 4, by a reduction proving that the worst-case approximation ratio is already attained by simple instances.

1.3 Related work on \( k \)-set packing

Between the algorithms of Hurkens and Schrijver and the recent improvements for the \( k \)-set packing problem, quasipolynomial-time approximation algorithms were considered [11, 8].

There also is a line of research on the weighted variant of \( k \)-set packing, where we want to select a maximum-weight family of pairwise-disjoint sets from \( F \). Arkin and Hassin [1] gave a \((k-1+\varepsilon)\)-approximation algorithm, later Chandra and Hallårdsson [6] improved it to a \((2k+2+\varepsilon)/3\)-approximation. Currently, the best-known approximation ratio is \((k + 1 + \varepsilon)/2\) due to Berman [3]. All the mentioned results are based on local search.

For the standard (unweighted) \( k \)-set packing problem, Chan and Lau [5] also presented a strengthened LP relaxation with integrality gap \((k + 1)/2\).

On the other hand, Hazan et al. [12] proved that \( k \)-set packing is hard to approximate within a factor of \( O(k / \log k) \). Concerning small values of \( k \), Berman and Karpinski [4] obtained a 98/97 - \( \varepsilon \) hardness for 3-DIMENSIONAL MATCHING, which implies the same lower bound for 3-set packing.

2 Preliminaries

Let \( G \) be an undirected (multi)graph. We sometimes treat \( G \) as a directed graph, where each edge \( e \in E(G) \) is represented by a pair of oppositely directed arcs in \( A(G) \). For an arc \( e \in A(G) \) we denote by \( \overline{e} \) the corresponding edge in \( E(G) \), and by \( e^R \), the reverse arc. We also define \( \overline{A} = \{ \overline{e} : e \in A \} \) and \( A^R = \{ e^R : e \in A \} \) for an arbitrary subset \( A \subseteq A(G) \).

A partial orientation of \( G \) can be defined as a subset \( F \subseteq A(G) \) such that \( F^R \cap F = \emptyset \). It is called feasible (for degree constraints \( d = (d^+, d^-) \)), if \( \deg^+_F(v) \leq d^+(v) \) and \( \deg^-_F(v) \leq d^-(v) \) for each \( v \in V(G) \), that is, if the number of arcs leaving \( v \) and the number of arcs entering \( v \) do not violate the upper bounds. Now, UDPO can be reformulated as the problem of finding a maximum feasible partial orientation \( F \), rather than the corresponding set of undirected edges \( \overline{F} \).

For an undirected (multi)graph \( G \) and a set \( U \subseteq V(G) \) we also define \( N_G(U) \) as the set of vertices \( v \not\in U \) adjacent to some \( u \in U \); we also set \( N_G[U] = N_G(U) \cup U \).

2.1 Reduction to 3-set packing

The following reduction to 3-set packing was introduced by Gabow [9]. Let \( I = (G, d) \) be an instance of UDPO. We construct an equivalent instance of the 3-set packing problem, i.e., a set family \( \mathcal{F} \) over a universe \( U \).

The universe \( U \) is a disjoint union of three sets: \( V^+, V^- \) and \( E \). The set \( V^+ \) contains \( d^+(v) \) copies \( v^+_i \) of each \( v \in V(G) \), \( V^- \) contains \( d^-(v) \) copies \( v^-_i \) of each \( v \in V(G) \), and \( E \) is defined as \( E(G) \). The family \( \mathcal{F} \) consists of sets \( \{ u^+_i, v^-_j, e \} \) and \( \{ v^+_j, u^-_i, e \} \) for each edge \( e = \{ u, v \} \) and all possible indices \( i, j \).

Given a feasible partial orientation \( F \), the constraints clearly let us choose for each arc \( e = uv \) two copies \( u^+_i \) and \( v^-_j \), so that the choices are distinct across all arcs leaving \( u \) and entering \( v \), respectively. Consequently, the sets \( \{ u^+_i, v^-_j, \overline{e} \} \) form a disjoint subfamily of \( \mathcal{F} \). Similarly, given any disjoint set-family \( \mathcal{F}_0 \subseteq \mathcal{F} \) it is easy to see that orienting \( e \) from \( u \) to \( v \) for any \( \{ u^+_i, v^-_j, \overline{e} \} \in \mathcal{F}_0 \) gives a feasible partial orientation.
2.2 Local search for $k$-set packing

In this section we recall and reinterpret some of the results behind the recent $k + \frac{1}{3} \epsilon$-approximation algorithm by Cygan [7] for the $k$-set packing problem.

For an instance $(U, \mathcal{F})$ of the $k$-set packing problem, we build an undirected conflict graph $G = G(\mathcal{F})$ with $V(G) = \mathcal{F}$ and vertices $F, F'$ made adjacent if $F \cap F' \neq \emptyset$. Observe that solutions to this instance of $k$-set packing form independent sets in this graph.

The algorithm of [7] is based on the local-search principle. It maintains a solution $\mathcal{F}_0 \subseteq \mathcal{F}$ and tries to replace it with a larger, but similar solution. It tries to use a disjoint family $X \subseteq \mathcal{F} \setminus \mathcal{F}_0$ and replace $\mathcal{F}_0$ by $\mathcal{F}_0' = (\mathcal{F} \setminus N_G(X)) \cup X$, where $G = G(\mathcal{F})$ is the conflict graph. Note that $N_G(X) \cap \mathcal{F}_0$ consists exactly of those members of $\mathcal{F}_0$ which cannot be present together with $X$ in a single disjoint family. It is reasonable to perform this operation if the resulting family $\mathcal{F}_0'$ is larger than $\mathcal{F}_0$, or equivalently $|N_G(X) \cap \mathcal{F}_0| < |X|$. This leads to a notion of improving sets, defined for $\mathcal{F}_0 \subseteq \mathcal{F}$ as disjoint families $X \subseteq \mathcal{F} \setminus \mathcal{F}_0$ such that $|N_G(X) \cap \mathcal{F}_0| < |X|$.

The classic approach to the $k$-set packing problem is to search for improving sets of sufficiently large constant size, which leads to a $\frac{k+1}{2}$-approximation factor [13]. The novel idea of [7] was to consider larger improving sets satisfying structural properties, which let us efficiently find these sets. This is achieved using a structural parameter of a graph called pathwidth. In this paper we only use some results of [7] as a black-box, so we do not need to recall the relatively complex definition of pathwidth. Pathwidth of an undirected graph $G$, denoted as $\text{pw}(G)$, does not exceed the number of vertices of $G$. Pathwidth of an improving set $X$ is defined as $\text{pw}(G[N_G[X]])$ where $G = G(\mathcal{F})$ is the conflict graph and $G[N_G[X]]$ is the subgraph of $G$ induced by $N_G[X]$. The following theorem uses techniques of fixed-parameter tractability to find improving sets of logarithmic size and constant pathwidth in the conflict graph.

**Theorem 1 ([7], Theorem 3.6).** There is an algorithm, that given a $k$-set-packing instance $\mathcal{F}$, and a disjoint family $\mathcal{F}_0 \subseteq \mathcal{F}$, in $2^{O(r k)}|\mathcal{F}|^{O(\text{pw})}$ time determines whether there exists an improving set $X \subseteq \mathcal{F} \setminus \mathcal{F}_0$ of size at most $r$ and pathwidth at most $\text{pw}$, and if so, finds such an improving set.

Finally, let us make an easy observation, stating that the algorithm is monotone in a certain sense.

**Observation 2.** If no improving set can be found using Theorem 1 for $\mathcal{F}_0 \subseteq \mathcal{F}$, then one still cannot find an improving set if the instance $\mathcal{F}$ is restricted to any $\mathcal{F}'$ such that $\mathcal{F}_0 \subseteq \mathcal{F}' \subseteq \mathcal{F}$.

3 Algorithm for UDPO

Our algorithm for UDPO combines the local-search rule by Cygan [7] for 3-set packing, applied to an instance obtained through the reduction given in Section 2.1, with a new custom rule. This rule also tries to extend a feasible partial orientation $F$, but it works with partial orientations as sets of undirected edges rather than directed arcs. Given a partial orientation $F$ it tries to find a partial orientation $F'$ such that $|F'| > |F|$ and $\overline{F} \Delta \overline{F}'$, the symmetric difference between the underlying undirected versions of $F$ and $F'$, is of constant size. Polynomial time is sufficient to generate all possible choices of $\overline{F}'$, but it is not enough to check all orientations $F'$. To overcome this issue, for $(V, \overline{F})$ we apply a result of Gyárfás and Frank, who used maximum-flow techniques to find in polynomial time a (total) orientation satisfying degree constraints.

**Lemma 3 ([10]).** Given an undirected graph $G$ and upper-degree constraints $d$, one can in polynomial time decide whether there is a feasible partial orientation using all edges of $G$.
Corollary 4. There is an algorithm, that given a UDPO instance \((G,d)\) and a feasible partial orientation \(F\), in \(O(|E(G)|^r \text{poly}(|G|))\) time determines whether there exists a feasible partial orientation \(F'\) satisfying \(|F'| > |F|\) as well as \(|F' \Delta F| \leq r\), and if so, finds such a feasible partial orientation.

We conclude this section with a succinct description of the algorithm. Given an instance \(I = (G,d)\) of UDPO, it builds an equivalent instance \(I'\) of the 3-set packing problem using a reduction of Section 2.1. It maintains a feasible partial orientation \(F\) together with a corresponding disjoint subfamily \(F_0 \subseteq F\), while using the following two rules to improve \(F\):

1. apply Theorem 1 to find an improving set for \(F_0\) of size at most \(c_\varepsilon \log |U|\) with pathwidth at most \(c_\varepsilon\), where \(|U| = |E(G)| + \sum_{v \in V(G)} (d^+(v) + d^-(v))\) is the universe size of the underlying instance of 3-set packing.

2. apply Corollary 4 to find a partial orientation \(F'\) satisfying \(|F'| > |F|\) and \(|F' \Delta F| \leq c_\varepsilon\). The algorithm terminates if neither of the two rules is able to improve \(F\). Any such partial orientation \(F\) is called a local optimum. The remaining part of this paper is devoted to analyzing how big the local optimum can be compared to the global optimum. More precisely, we show that for every \(\varepsilon\) there is an appropriate choice of \(c_\varepsilon\) so that \(|F| \geq (\frac{1}{2} + \varepsilon)|OPT|\) for any local optimum \(F\) and global optimum \(OPT\).

4 Reduction to simple instances

An instance \(I = (G,d)\) of UDPO is called simple if \(d^+(v), d^-(v) \in \{0,1\}\) for every \(v \in V(G)\) and proper if \(\text{deg}_G(v) \geq \max(d^+(v), d^-(v)) > 0\) for every \(v \in V\). Clearly, any instance can be easily reduced to an equivalent proper instance. In this section we show that it suffices to analyze our local-search algorithm for simple instances.

Theorem 5. Fix a constant \(c_\varepsilon > 1\) for the algorithm of Section 3. Suppose that there exists an instance \(I\) of UDPO with a locally-optimum partial orientation \(F\) such that \(|F| = \alpha |OPT_1|\). Then there exists a simple instance \(I'\) of UDPO with a locally-optimum partial orientation \(F'\) satisfying \(|F'| = \alpha |OPT_1|\).

Let \(I = (G,d)\) be an arbitrary instance. For a pair of distinct non-adjacent vertices \(u,v \in V(G)\) we define the operation of joining \(u\) and \(v\) as follows: \(u\) and \(v\) are identified in \(G\) into a single vertex \(w\) and their degree constraints for \(w\) are obtain by summing the respective constraints for \(u\) and \(v\). Note that this operation preserves the set of edges. Observe that in terms of the instance of 3-set packing obtained through the reduction of Section 2.1, joining can be interpreted as introducing some sets to \(F\). Consequently, if a partial orientation is feasible in \(I\), it is also feasible in the resulting instance \(I'\), but the converse does not necessarily hold.

If \(I'\) is obtained from \(I\) by joining \(u\) and \(v\) into \(w\), we say that \(I\) can be obtained from \(I'\) by splitting \(w\). Splitting is said to preserve a partial orientation \(A\), if \(A\) is feasible in \(I'\) and remains feasible in \(I\).

Lemma 6. Let \(I = (G,d)\) be a proper instance with two feasible partial orientations \(A,B\). If \(\max(d^+(v), d^-(v)) \geq 2\) for some \(v \in V(G)\), then one can split \(v\) so that both \(A\) and \(B\) are preserved and the resulting instance \(I'\) is proper.
Proof. First, let us introduce an auxiliary vertex \( v' \) connected to \( v \) by \( d^+(v) + d^-(v) \) parallel edges. We extend \( d \) to \( v' \) setting the constraints large enough to accommodate all edges incident to \( v' \). Note that this operation has no effect on whether one can split \( v \).

Now, let us modify \( A \) to obtain \( A' \) by orienting \( d^+(v) - \deg_A^+(v) \) edges from \( v \) to \( v' \) and \( d^-(v) - \deg_A^-(v) \) edges from \( v' \) to \( v \). Note that \( A' \) is feasible in the extended graph and the degree constraints for \( v \) are tight. Analogously, we extend \( B \) to \( B' \). A larger partial orientation may only be harder to preserve, so it suffices to prove that one can split \( v \) preserving \( A' \) and \( B' \). Equivalently, the construction in this paragraph lets us assume that \( \deg_A^+(v) = \deg_B^+(v) = d^+(v) \) and \( \deg_A^-(v) = \deg_B^-(v) = d^-(v) \).

Both for \( A \) and \( B \) we classify edges of \( G \) incident to \( v \) into three types: oriented towards \( v \) (−), oriented towards the other endpoint (+) and not included in the orientation (0). In total, we get a partition of the set \( \delta(v) \), consisting of edges incident to \( v \), into nine sets \( E_{ab} \) with \( a, b \in \{+, -, 0\} \); here \( a \) corresponds to the orientation in \( A \) and \( b \) to the orientation in \( B \).

In some situations, one can clearly take a few edges incident to \( v \), and split \( v \) into two vertices, one new vertex \( v' \) incident to the selected edges, and the other, still denoted as \( v \), incident to the remaining edges. We refer to this operation as splitting out some edges. Note that in order to preserve both \( A \) and \( B \), we need to split out edges so that for \( v' \) the number incoming edges is the same in both orientations, similarly for the outgoing arcs. We shall make sure that this number is always 0 or 1, i.e., \((d^+(v'), d^-(v')) \in \{(0, 1), (1, 0), (1, 1)\}\). The constraints at \( v \) are decreased accordingly.

1. If \( E_{++} \neq \emptyset \), one can split out a single edge \( e \in E_{++} \) setting constraints (1, 0); symmetrically if \( E_{--} \neq \emptyset \) one sets \( (0, 1) \).

2. If \( E_{+-}, E_{-+} \neq \emptyset \), one can split out two edges – one of each type, setting constraints (1, 1).

3. If \( E_{0+}, E_{+0} \neq \emptyset \), one can split out two edges – one of each type, setting constraints (1, 0); symmetrically if \( E_{0-}, E_{-0} \neq \emptyset \) one sets \( (0, 1) \).

4. If \( E_{+-}, E_{0+}, E_{-0} \neq \emptyset \) one can split out three edges – one of each type, setting constraints (1, 1); symmetrically if \( E_{--}, E_{-0}, E_{0-} \neq \emptyset \) one also sets \( (1, 1) \).

We shall prove that one of these rules is always applicable. Note that the resulting instance is guaranteed to be proper as we have \( \max(d^+(v), d^-(v)) \geq 2 \), so it is impossible to leave \( v \) with both constraints equal to 0, which is forbidden in proper instances.

We proceed by contradiction, showing that if no rule is applicable, then \( d^+(v) = d^-(v) = 0 \), which is impossible because \( I \) is proper. Let \( n_{ab} = |E_{ab}| \). Recall that we have made an assumption that \( \deg_A^+(v) = \deg_B^+(v) = d^+(v) \) and \( \deg_A^-(v) = \deg_B^-(v) = d^-(v) \), which implies the following equalities:

\[
\begin{align*}
n_{0+} + n_{++} + n_{-+} &= d^+(v) = n_{+0} + n_{++} + n_{+-}, \\
n_{0-} + n_{+-} + n_{--} &= d^-(v) = n_{-0} + n_{++} + n_{--}.
\end{align*}
\]

If \( n_{++} > 0 \) or \( n_{--} > 0 \) we could apply rule 1. Therefore

\[
\begin{align*}
n_{0+} + n_{-+} &= d^+(v) = n_{+0} + n_{+-}, \\
n_{0-} + n_{+-} &= d^-(v) = n_{-0} + n_{--}.
\end{align*}
\]
If $n_{++} > 0$ and $n_{--} > 0$ we could apply rule 2; without loss of generality we assume $n_{+-} = 0$ and thus

$$n_{0+} + n_{-+} = d^+(v) = n_{+0},$$
$$n_{0-} = d^-(v) = n_{-0} + n_{-+}.$$  

Consequently, we have $n_{+0} \geq n_{0+}$ and $n_{0-} \geq n_{-0}$. Therefore, if $n_{0+} > 0$ or $n_{0-} > 0$, we could apply rule 3, which means that both these values are equal to 0 and

$$n_{0-} = n_{+0} = n_{-+} = d^+(v) = d^-(v).$$

However, if the common value of these variables was not equal to 0, we could apply rule 4. This way we get the announced contradiction.

**Corollary 7.** If $I$ is a proper instance with feasible partial orientations $A$ and $B$, then with a finite sequence of vertex splitting preserving both $A$ and $B$, one can obtain a simple proper instance $I'$.

**Proof.** It suffices to exhaustively apply Lemma 6. Observe that this process must terminate, as vertex splitting increases the number of vertices and changes neither $D^+ = \sum_{v \in V(G)} d^+(v)$ nor $D^- = \sum_{v \in V(G)} d^-(v)$, while $|V(G)| \leq D^+ + D^-$ for any proper instance.

For a proof of Theorem 5, it suffices to apply Corollary 7 for $A = F$ and $B = OPT_I$. Vertex splitting may only reduce the family of feasible partial orientations, so $OPT_I$ is still a global optimum. Also, this operation preserves $F$ as a local optimum with respect to rule 2. For rule 1 the analogous property follows from the fact that vertex splitting can be seen as removing sets in the underlying instance of 3-SET PACKING (without changing the size of the universe), and by Observation 2, the corresponding rule for 3-SET PACKING is monotone, i.e., removing sets from the universe does not make finding an improving set easier.

Therefore, Corollary 7 gives a simple instance $I'$ for which $F$ and $OPT_I$ are still a local and a global optimum, respectively.

## 5 Tools from $k$-set packing

In this section we recall and reinterpret several pieces of the analysis of the local search algorithms for $k$-SET PACKING, see [13, 7].

This analysis focuses on the subgraph of the conflict graph $G(F)$ induced by two solutions: a local and a global optimum. Sets belonging to both families can be ignored, which leads to a bipartite graph with degrees bounded by $k$. The following results are stated in the language of abstract bipartite graphs, so that we can also use them in a slightly different context.

**Definition 8.** Let $H = (A, B, E(H))$ be a bipartite graph. A set $X \subseteq B$ is called improving, if $|N_H(X)| < |X|$.

The following lemma is a part of the analysis of the classic $(k+\varepsilon)/2$-approximation local search, which goes back to Hurkens and Schrijver [13]. Our proof is based on the proof of Lemma 3.11 in [7]. Although that result uses larger class of improving sets to obtain a better bound on $|B|/|A|$, the overall line of reasoning remains the same.
Lemma 9. Fix a positive integer $k \geq 3$. For any $\varepsilon > 0$ there exists a constant $c_\varepsilon$ satisfying the following property. Let $H = (A, B, E(H))$ be a bipartite graph with degrees not exceeding $k$. If there is no improving set $X \subseteq B$ with $|X| \leq c_\varepsilon$, then $|B| \leq \frac{k + \varepsilon}{2}|A|$.

Proof. We are going to construct a sequence of at most $\frac{1}{2}$ induced subgraphs $H_i = H[A_i, B_i]$, with $A_i \subseteq A$ and $B_i \subseteq B$. These subgraphs shall satisfy the following two properties:

(a) in $H_i$ there is no subset $X \subseteq B_i$ such that $|X| \leq 2(k + 1)^{\frac{i}{k} - 1}$ and $|N_{H_i}(X)| < |X|$,

(b) $|A \setminus A_i| = |B \setminus B_i| \geq \varepsilon i |A|$.

We start with $H_0 = H$, which trivially satisfies (b). It suffices to take $c_\varepsilon = 2(k + 1)^{\frac{1}{k}}$ to make sure that (a) also holds.

Consider the graph $H_i$. Let us classify vertices of $B_i$ based on their degree in $H_i$: we define $B_i^d$ as the set of vertices of degree $d$, and $B_i^{d+}$ as the set of vertices of degree at least $d$. Note that (b) implies $i \leq \frac{k}{d}$, and thus $2(k + 1)^{\frac{i}{k} - 1} \geq 2$. Consequently, by (a), $B_i^0 = \emptyset$ and the vertices of $B_i^1$ have distinct neighbors (otherwise we would have an improving set of size one or two, respectively).

We consider two cases, depending on whether $|B_i^1| \leq \varepsilon |A|$. First, we suppose this inequality does not hold. Then we construct $H_{i+1}$ setting $B_{i+1} = B_i^{2+}$ and $A_{i+1} = A_i \setminus N_{H_i}[B_i^1]$. As we have noted, vertices in $B_i^1$ do not share neighbours, so $|A_i \setminus A_{i+1}| = |B_i^1| = |B_i \setminus B_{i+1}|$, and consequently $|B \setminus B_{i+1}| = |A \setminus A_{i+1}|$. Also, we clearly have $|B \setminus B_{i+1}| \geq \varepsilon i |A| + |B_i^1| \geq \varepsilon (i + 1) |A|$.

Therefore, it suffices to show that $H_{i+1}$ satisfies property (a). Take $X \subseteq B_{i+1}$ such that $|N_{H_{i+1}}(X)| < |X|$. We construct $X' \subseteq B_i$ with $|N_{H_i}(X')| < |X'|$ such that $|X'| \leq (k + 1) |X|$. Clearly, if $X$ then contradicts (a) for $H_{i+1}$, so does $X'$ for $H_i$. Recall that $H_i[B_i \setminus B_{i+1}, A_i \setminus A_{i+1}]$ is a perfect matching. We denote the unique neighbor of a vertex $v$ in this graph by $m(v)$. We simply define $X' = X \cup \{m(a) : a \in (A_i \setminus A_{i+1}) \cap N_{H_i}(X)\}$ (see also Figure 1). Then $N_{H_i}(X') = N_{H_i}(X) = N_{H_{i+1}}(X) \cup \{m(b) : b \in X' \setminus X\}$. Consequently, $|N_{H_i}(X')| = |N_{H_{i+1}}(X)| + |X' \setminus X| < |X| + |X' \setminus X| = |X'|$. Moreover, by the degree restriction in $H$, we have $|N_{H_i}(X)| \leq k |X|$, and thus $|X'| \leq |X| + |N_{H_i}(X)| \leq (k + 1) |X|$, as claimed.

![Figure 1: Lifting an improving set X in H_{i+1} to an improving set X' in H_i. Gray vertices belong to H_i but not to H_{i+1}.](image)

Therefore it suffices to consider the case when $|B_i^1| \leq \varepsilon |A|$. We count edges of $H_i$; clearly, $|E(H_i)| \leq k |A_i|$ since the degrees do not exceed $k$. On the other hand, $|E(H_i)| \geq |B_i^1| + 2|B_i^{2+}|$, and consequently $|B_i^1| + 2|B_i^{2+}| \leq k |A_i|$. Summing up, we get

$$2|B| = 2|B \setminus B_i| + 2|B_i| = 2|A \setminus A_i| + 2|B_i^1| + 2|B_i^{2+}| \leq 2|A \setminus A_i| + |B_i^1| + k |A_i| \leq (k + \varepsilon) |A|,$$
that is, $|B| \leq \frac{k+2}{2}|A|$, which concludes the proof.

The following lemma is, on the other hand, a slight generalization of Lemma 3.11 in [7], restricted to $k = 3$. Under the original assumptions it shows that the ratio $\frac{|B|}{|A|}$ is close to the worst-case $\frac{4}{3}$ only if (almost) all vertices in $A$ are of degree 3, and thus allows for a better bound if some fraction of vertices have degree at most 2.

**Lemma 10.** For any $\varepsilon > 0$ there exists a constant $c_\varepsilon$ satisfying the following property. Let $H = (A, B, E(H))$ be a bipartite graph with degrees not exceeding 3. If there is no improving set $X \subseteq B$ such that $|X| \leq c_\varepsilon \log |V(H)|$ and $pw(H[N_G[X]]) \leq c_\varepsilon$, then

$$|B| \leq (1 + \varepsilon)|A| + \frac{1}{3}|\{a \in A : \deg_H(a) \geq 3\}|.$$

**Proof.** We follow the notation and the main line of reasoning of the proof of Lemma 9, which for $k = 3$ has stronger requirements for $X$. We only alter the last step of the proof, i.e., the analysis when $|B_1^3| \leq \varepsilon|A|$. This requires the following reformulation of Claim 3.12 from [7], which is where we use the whole strength of the assumptions of Lemma 10.

**Claim 11** ([7]). For large enough $c_\varepsilon$ we have $|B_2^3| \leq (1 + \varepsilon)|A_i|$.

As before, we count edges $E(H_i)$. We clearly have $|E(H_i)| = |B_1^3| + 2|B_2^3| + 3|B_3^3|$. On the other hand, $|E(H_i)| \leq 2|A_i| + |A_3^i|$ where $A_3^i = \{a \in A_i : \deg_{H_i}(a) = 3\}$. Summing up, we obtain

$$3|B| = 3|B \setminus B_1| + 3|B_1^3| + 3|B_2^3| + 3|B_3^3| = 3|A \setminus A_i| + 2|B_1^3| + |B_2^3| + |E(H)| \leq 3|A \setminus A_i| + 2|A_i| + (1 + \varepsilon)|A_i| + 2|A_i| + |A_3^i| \leq 3(1 + \varepsilon)|A| + |\{a \in A : \deg_H(a) = 3\}|,$$

that is, $|B| \leq (1 + \varepsilon)|A| + |\{a \in A : \deg_H(a) = 3\}|$, which completes the proof.

## 6 Analysis

We start the analysis of the algorithm of Section 3 with a result which lets us construct the counterpart of the bipartite conflict graph with respect to two feasible solutions. Later we apply Theorem 5, which allows restricting to simple instances.

**Lemma 12.** Let $I$ be a simple instance of UDPO and let $A, B$ be a pair of feasible partial orientations. There exists a bipartite graph $H = (\overline{B} \setminus \overline{A}, \overline{A} \setminus \overline{B}, E(H))$ such that:

(a) degrees in $H$ do not exceed 4,

(b) for any $X \subseteq \overline{B} \setminus \overline{A}$ there is a feasible partial orientation $F$ with $F = (\overline{A} \setminus N_H(X)) \cup X$.

**Proof.** Let $A' = \overline{A} \setminus \overline{B}$, $B' = \overline{B} \setminus \overline{A}$ and $G' = (V(G), \overline{A} \cap B)$. For a connected component $C$ of $G'$ we define $\delta_G[C]$ as the set of edges $e \in E(G)$ incident to at least one vertex of $C$. We construct the graph $H$ as follows. We make $a \in A'$ adjacent in $H$ to $b \in B'$ if and only if both $a$ and $b$ belong to $\delta_G[C]$ for some connected component $C$.

Let us prove that $H$ satisfies the desired properties, starting with (a). Consider any connected component $C$ of $G'$. As $I$ is a simple instance, all the vertices in $G'$ are of degree at most two, which means that $C$ is either a path or a cycle. Consequently, in either case, again by the assumption that $I$ is simple, we have $|\delta_G[C] \cap A'| \leq 2$ and $|\delta_G[C] \cap B'| \leq 2$, because, both in $A$ and in $B$, at
most 2\(|C|\) arc endpoints can be incident to \(C\). Any edge is incident to at most two components of \(G\), for each of them we may have created at most two neighbors in \(H\), and thus the degrees in \(H\) are at most 4.

To prove (b) we take \(X \subseteq B'\) and consider a set \(\overline{F} = (\overline{A} \setminus N_H(X)) \cup X\). Note that for any component \(C\) of \(G'\) we have \(\overline{F} \cap \delta_G[C] \subseteq \overline{A} \cap \delta_G[C]\) (if \(X \cap \delta_G[C] = \emptyset\)) or \(\overline{F} \cap \delta_G[C] \subseteq \overline{B} \cap \delta_G[C]\) (otherwise). We can orient edges of \(\delta_G[C] \cap \overline{F}\) consistently with \(A\) in the former case and consistently with \(B\) in the latter. Note that if there is an edge \(e \in \overline{F}\) between two connected components of \(G'\), then \(e \notin A \cap B\), so both components are oriented consistently with \(A\) (if \(e \in \overline{A}\)) or \(B\) (if \(e \in \overline{B}\)), hence the proposed orientation is well-defined. It remains to argue that if we orient the edges in this manner, then all the capacity constraints are satisfied. Consider any vertex \(v\) of \(G'\). As it belongs to exactly one connected component of \(G\), its incident edges from \(\overline{F}\) are either oriented as in \(A\) or as in \(B\), in either case the degree constraints are obeyed.

Next, we apply the conflict graph and the technique similar to the standard analysis of the \((2 + \varepsilon)\)-local search approximation of 4-set packing. This lets us derive a bound with respect to rule 2.

**Lemma 13.** Fix \(\varepsilon > 0\). There exists a constant \(c_\varepsilon\) such that for any simple instance \(I\) of UDPO the following condition holds. Let \(F\) be a feasible partial orientation which cannot be improved using rule 2 and let \(OPT\) be an optimum partial orientation. Then \(|OPT \setminus F| \leq (2 + \varepsilon)|\overline{F} \setminus OPT|\).

*Proof.* We set \(c_\varepsilon\) as in Lemma 9 for \(k = 4\), and proceed with a proof by contradiction. Suppose that \(|OPT \setminus F| > (2 + \varepsilon)|\overline{F} \setminus OPT|\). We apply Lemma 12 to \(A = F\) and \(B = OPT\) to obtain a bipartite graph \(H\), which we plug to Lemma 9. This implies that there is a set \(X \subseteq V\) of size at most \(c_\varepsilon\|\) with \(|N_H(X)| < |X|\). By Lemma 12(b), replacing \(X\) with \(X'\), gives a feasible orientation, and rule 2 would actually be able to perform this improvement. This contradicts the assumption that \(F\) is a local optimum.

Finally, we combine the consequences of rule 2 (Lemma 13) with the strengthened analysis of rule 1 (Lemma 10) to derive the main result of this paper.

**Theorem 14.** Fix \(\varepsilon > 0\). There exists a constant \(c_\varepsilon\) such that for any instance of UDPO and any feasible partial orientation \(F\) which cannot be improved using rules 1 and 2, we have \(|OPT| \leq \left(\frac{5}{4} + \varepsilon\right)|F|\), where \(OPT\) is a maximum feasible partial orientation.

*Proof.* By Theorem 5, it suffices to prove the claim for simple instances only. Let \(C = OPT \cap F\). Note that \(F \setminus C\) and \(OPT \setminus C\) induce a bipartite subgraph \(H = (F \setminus C, OPT \setminus C, E(H))\) of the conflict graph in the underlying instance of 3-SET PACKING. Clearly, the degrees in \(H\) are bounded by 3. Moreover, by construction of the reduction, if \(\deg_H(e) = 3\) for some \(e \in F\), then \(eR \in OPT\), i.e., \(|\{e \in F : \deg_H(e) = 3\}| \leq |OPT \cap F|\).

We set \(c_\varepsilon\) large enough for Lemmas 10 and 13 to be applicable. The former lets us conclude that

\[
|OPT| = |C| + |OPT \setminus C| \leq |C| + (1 + \varepsilon)|F \setminus C| + \frac{1}{4}|\{e \in F \setminus C : \deg_H(e) = 3\}| \leq (1 + \varepsilon)|F| + \frac{1}{4}|OPT \cap F|.
\]

If \(|OPT \cap F| \leq \frac{1}{4}|F|\), this already concludes the proof. Otherwise \(|\overline{F} \setminus OPT| \leq \frac{1}{4}|F|\) and we apply Lemma 13 to get

\[
|OPT \setminus F| \leq (2 + \varepsilon)|\overline{F} \setminus OPT|,
\]

and we apply Lemma 12 to...
and consequently we obtain

\[ |OPT| = |OPT \setminus F| + |OPT \cap F| \leq (2 + \varepsilon)|F \setminus OPT| + |OPT \cap F| = (1 + \varepsilon)|F \setminus OPT| + |F| \leq \frac{5 + \varepsilon}{4}|F| \leq (\frac{5}{4} + \varepsilon)|F|, \]

which concludes the proof.

\[ \square \]

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