MINORS AND DIMENSION

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Abstract. Streib and Trotter proved in 2012 that posets with bounded height and with planar cover graphs have bounded dimension. Recently, Joret et al. proved that the dimension is bounded for posets with bounded height whose cover graphs have bounded tree-width. In this paper, it is proved that posets of bounded height whose cover graphs exclude a fixed (topological) minor have bounded dimension. This generalizes both the aforementioned results and verifies a conjecture of Joret et al. The proof relies on the Robertson-Seymour and Grohe-Marx structural decomposition theorems.

1. Introduction

In this paper, we are concerned with finite partially ordered sets, which we simply call posets. The dimension of a poset $P$ is the minimum number of linear orders that form a realizer of $P$, that is, their intersection gives rise to $P$. This notion was introduced in 1941 by Dushnik and Miller [3] and since then has been one of the most extensively studied parameters in the combinatorics of posets. Great part of this research was focused on understanding when and why the dimension is bounded, and this is also the focus of the current paper. See the monograph [20] for a comprehensive introduction to poset dimension theory.

To some extent, the dimension for posets behaves like the chromatic number for graphs. There is a natural construction of a poset with dimension $d$, the so-called standard example $S_d$ (see Figure 1). A standard example plays a similar role to a clique in the graph setting. Every poset that contains $S_d$ as a subposet must have dimension at least $d$. On the other hand, there are posets of large dimension not containing $S_3$ as a subposet, just like there are triangle-free graphs with large chromatic number. Moreover, it is NP-complete to decide whether a poset has dimension at most $d$ for any $d \geq 3$ [23].

This analogy between posets and graphs breaks when planarity comes in. A poset $P$ is planar if its cover graph (the graph of those comparabilities which cannot be inferred from other ones using transitivity) can be drawn in the plane in such a way that the ordering of points according to the vertical coordinate agrees with the order of $P$ and there are no edge crossings. Kelly [13] constructed planar posets which contain arbitrarily large standard examples as subposets and thus have arbitrarily large dimension (see Figure 2). This contrasts with the fact that $K_5$ is non-planar and with the famous four-color theorem.

Kelly’s construction brings the following natural question: what structural restrictions on the cover graph guarantee that the dimension is bounded? Noting that Kelly’s planar poset of dimension $d$ has height $d + 1$, Felsner, Li and Trotter [6] conjectured that posets of bounded height with planar cover graphs have
bounded dimension. This has been verified by Streib and Trotter [19]. Recently, Joret et al. [11] proved that the dimension is bounded for the posets of bounded height whose cover graphs have bounded tree-width, establishing a connection between poset dimension theory and structural graph theory. The restriction that the height is bounded cannot be dropped, as the posets in Kelly’s construction have tree-width 3. However, Biró, Keller and Young [2] proved that the posets with cover graphs of path-width 2 (and with no restriction on the height) have bounded dimension. This was generalized by Joret et al. [12] to posets with cover graphs of tree-width 2. Many other results bounding the dimension of posets whose cover graphs have some specific structure are discussed in the introductory sections of [11].

Joret et al. [11] also conjectured that posets of bounded height whose cover graphs exclude a fixed graph as a minor have bounded dimension. This is verified (and further generalized to excluded topological minors) in the present paper.

**Theorem 1.** The posets of bounded height whose cover graphs exclude a fixed graph as a topological minor have bounded dimension.

It follows from Theorem 1 that the posets whose comparability graphs exclude a fixed graph as a topological minor have bounded dimension. Indeed, if the comparability graph of a poset $P$ excludes a graph $H$ as a topological minor, then so does the cover graph of $P$ and additionally the height of $P$ is less than the number of vertices of $H$. This generalizes an old result which asserts that the posets with comparability graphs of bounded maximum degree have bounded dimension [7, 18].

The proof of Theorem 1 relies on structural decomposition theorems of graphs excluding a fixed (topological) minor due to Robertson and Seymour [16] and Grohe and Marx [9]. For a fixed bound on the height and a fixed excluded topological minor, it gives a polynomial-time algorithm constructing a family of linear extensions witnessing the bound on the dimension. The function that bounds the dimension in terms of the height and the excluded topological minor is enormous, and no effort has been made to compute its precise order of magnitude.

Bounding the height of a poset $P$ is the same as excluding a long chain as a subposet of $P$. One can ask what other posets can be excluded instead of bounding the height in the statement of Theorem 1 so that it remains valid. An obvious candidate is the standard example $S_5$. Gutowski and Krawczyk [10] suggested another candidate—the poset made of two incomparable chains of length $k$, denoted...
by \(k+k\). This is motivated by recent results on on-line algorithms, that some on-line problems “hard” for general posets become “tractable” for \((k+k)\)-free posets. For instance, it has been known that there is any algorithm trying to build a realizer on-line for a poset of width \(w\) can be forced to use arbitrarily many linear extensions even when \(w = 3\) \cite{14}, although the posets of width \(w\) have dimension at most \(w\). On the other hand, for \((k+k)\)-free posets of width \(w\), Felsner, Krawczyk and Trotter \cite{5} devised an on-line algorithm that builds a realizer of size bounded in terms of \(k\) and \(w\). Whether excluding \(S_d\) or \(k+k\) instead of bounding the height in Theorem \[1\] or its predecessors keeps the dimension bounded remains a challenging open problem.

2. Preliminaries

2.1. Graph terminology, notation, and basic properties. We denote by \(V(G)\) the set of vertices and by \(E(G)\) the set of edges of a graph \(G\). For \(X \subset V(G)\), we denote by \(G[X]\) the subgraph of \(G\) induced on \(X\).

A graph \(H\) is a minor of a graph \(G\) if \(H\) can be obtained from \(G\) by deleting some vertices, deleting some edges, and contracting some edges, where contracting an edge \(uv\) means replacing \(u\) and \(v\) by a single vertex that becomes connected to all neighbors of \(u\) or \(v\). A graph \(H\) is a topological minor of a graph \(G\) if \(H\) can be obtained from \(G\) by deleting some vertices, deleting some edges, and contracting some edges with at least one endpoint of degree 2. A class of graphs \(\mathcal{G}\) is

- minor-closed if every minor of every graph in \(\mathcal{G}\) belongs to \(\mathcal{G}\),
- topologically closed if every topological minor of every graph in \(\mathcal{G}\) belongs to \(\mathcal{G}\),
- monotone if every subgraph of every graph in \(\mathcal{G}\) belongs to \(\mathcal{G}\),
- proper if \(\mathcal{G}\) does not contain all graphs.

Every minor-closed class is topologically closed, and every topologically closed class is monotone. For every graph \(H\), the class of graphs excluding \(H\) as a minor or a topological minor is proper minor-closed or proper topologically closed, respectively.

A tree decomposition of a graph \(G\) is a tree \(T\) such that

- the vertices of \(T\) are subsets of \(V(G)\), called the bags of \(T\),
- every edge of \(G\) is contained in \(G[X]\) for at least one bag \(X\),
- for every \(v \in V(G)\), the set of bags containing \(v\) forms a subtree of \(T\).

An adhesion set of a bag \(X\) of \(T\) is a set of the form \(X \cap Y\) with \(XY \in E(T)\). The adhesion of \(T\) is the maximum size of an adhesion set of a bag of \(T\). The torso of a bag \(X\) of \(T\) is the graph obtained from \(G[X]\) by adding all edges in every adhesion set of \(X\).

The tree-width of a graph \(G\) is the minimum number \(k\) such that \(G\) has a tree decomposition with every bag of size at most \(k+1\). The radius of a graph \(G\) is the minimum number \(r\) such that \(G\) has a vertex whose distance from every vertex of \(G\) is at most \(r\) if \(G\) is connected, or it is \(\infty\) if \(G\) is disconnected. The local tree-width of a graph \(G\) is the function \(f: \mathbb{N} \to \mathbb{N}\) such that \(f(r)\) is the maximum tree-width of a subgraph of \(G\) with radius at most \(r\). Here and further on \(\mathbb{N}\) denotes the set of positive integers.

For a function \(f: \mathbb{N} \to \mathbb{N}\), let \(\mathcal{L}_f\) denote the class of graphs whose all minors have local tree-width bounded by \(f\). For \(d \in \mathbb{N}\), let \(\mathcal{D}_d\) denote the class of graphs with maximum degree bounded by \(d\). For a class of graphs \(\mathcal{G}\) and for \(t \in \mathbb{N}\), let
\( \mathcal{A}_t(\mathcal{G}) \) denote the class of graphs \( G \) such that there is \( A \subset V(G) \) with \( |A| \leq t \) and \( G \setminus A \in \mathcal{G} \). We call a vertex an apex to indicate that it can be connected to arbitrary other vertices of the graph. Hence \( \mathcal{A}_t(\mathcal{G}) \) is the class of graphs obtained from graphs in \( \mathcal{G} \) by adding at most \( t \) apices. For a class of graphs \( \mathcal{G} \) and for \( s \in \mathbb{N} \), let \( \mathcal{T}_s(\mathcal{G}) \) denote the class of graphs \( G \) that have a tree decomposition \( T \) of adhesion at most \( s \) such that every torso of \( T \) belongs to \( \mathcal{G} \). The following facts about classes of graphs are straightforward consequences of these definitions:

- for every \( f : \mathbb{N} \to \mathbb{N} \), the class \( \mathcal{L}_f \) is minor-closed;
- for every \( d \in \mathbb{N} \), the class \( \mathcal{D}_d \) is topologically closed;
- for every \( t \in \mathbb{N} \), if \( \mathcal{G} \) is monotone, topologically closed, or minor-closed, then \( \mathcal{A}_t(\mathcal{G}) \) is monotone, topologically closed, or minor-closed, respectively;
- for every \( s \in \mathbb{N} \), if \( \mathcal{G} \) is monotone, topologically closed, or minor-closed, then \( \mathcal{T}_s(\mathcal{G}) \) is monotone, topologically closed, or minor-closed, respectively.

### 2.2. Graph structure theorems

The classical results of Kuratowski [15] and Wagner [22] assert that the class of planar graphs is characterized by excluding \( K_5 \) and \( K_{3,3} \) as (topological) minors. For \( g \geq 1 \), the class of graphs with genus at most \( g \) is minor-closed as well, but its complete list of excluded minors is unknown. Robertson and Seymour, in a monumental series of papers culminating in [17], proved that the list of minimal excluded minors is finite for every minor-closed class of graphs. One of the most important results of this series [16] is a structural decomposition theorem of proper minor-closed classes of graphs \( \mathcal{G} \): every graph in \( \mathcal{G} \) admits a tree decomposition where every torso is almost embeddable\(^1\) on a surface of bounded genus with the exception of a bounded number of apices. This is an approximate structural characterization of proper minor-closed classes—the class of graphs satisfying the conclusion of the decomposition theorem is also proper minor-closed, although usually much broader than \( \mathcal{G} \).

Grohe [8] proved that graphs almost embeddable in a surface of bounded genus and all their minors have bounded local tree-width, thus generalizing earlier results of Baker [1] for planar graphs and of Eppstein [4] for graphs of bounded genus. This yields an approximate characterization of proper minor-closed classes that does not involve any topology.

**Theorem 2** (Robertson, Seymour [16], Grohe [8]). For every proper minor-closed class of graphs \( \mathcal{G} \), there are \( s, t \in \mathbb{N} \) and \( f : \mathbb{N} \to \mathbb{N} \) such that \( \mathcal{G} \subset \mathcal{T}_s(\mathcal{A}_t(\mathcal{L}_f)) \).

Topologically closed classes of graphs can be substantially richer than minor-closed ones. In particular, for \( d \geq 3 \), the class \( \mathcal{D}_d \) of graphs with maximum degree at most \( d \) is topologically closed, but every graph is a minor of some graph in \( \mathcal{D}_d \). Grohe and Marx [9] showed that incorporating graphs with bounded maximum degree to the structural description considered before is enough for an approximate characterization of all topologically closed classes of graphs. Specifically, for any proper topologically closed class \( \mathcal{G} \), they proved that every graph in \( \mathcal{G} \) admits a tree decomposition whose every torso (a) belongs to a fixed proper minor-closed class, or (b) has bounded maximum degree except for a bounded number of apices. This altogether yields the following approximate characterization of proper topologically closed classes.

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\(^{1}\)The precise definition of almost embeddable is unimportant for this paper and thus omitted.
Theorem 3 (Robertson, Seymour [16], Grohe [8], Grohe, Marx [9]). For every proper topologically closed class of graphs \( \mathcal{G} \), there are \( s, t, d \in \mathbb{N} \) and \( f : \mathbb{N} \to \mathbb{N} \) such that \( \mathcal{G} \subset \mathcal{T}_s(\mathcal{A}_t(\mathcal{L}_f \cup \mathcal{D}_d)) \).

It is important for algorithmic applications of Theorems 2 and 3 that for a fixed class \( \mathcal{G} \) and a graph \( G \in \mathcal{G} \), a tree decomposition of \( G \) and sets of apices in its torsos witnessing \( G \in \mathcal{T}_s(\mathcal{A}_t(\mathcal{L}_f)) \), as claimed in Theorem 2 or \( G \in \mathcal{T}_s(\mathcal{A}_t(\mathcal{L}_f \cup \mathcal{D}_d)) \), as claimed in Theorem 3 can be computed in time polynomial in the size of \( G \) [8, 9].

2.3. Poset terminology, notation, and basic properties. We denote by \( \leq_P \) and \(<_P\) the weak and strong order relations of a poset \( P \). The comparability graph of \( P \) is the graph built on the ground set of \( P \) whose edges correspond to the comparabilities of \( P \). The height of a poset \( P \) is the maximum size of a set of pairwise comparable elements of \( P \), which is the maximum size of a clique in the comparability graph of \( P \). For \( x, y \in P \), we say that \( y \) covers \( x \) if \( x <_P y \) there is no \( z \in P \) with \( x <_P z <_P y \). The cover graph of \( P \) is the graph built on the ground set of \( P \) whose edges correspond to the cover relations of \( P \). We will identify \( P \) with the set of vertices of the cover graph of \( P \) and simply call the elements of \( P \) vertices. We let \( \uparrow_P v = \{ x \in P : x \geq_P v \} \) and \( \downarrow_P v = \{ x \in P : x \leq_P v \} \). For \( X \subset P \), we denote by \( P[X] \) the subposet of \( P \) induced on \( X \).

The dimension of a poset \( P \) is the minimum number \( d \) of linear extensions \( L_1, \ldots, L_d \) of \( P \) such that \( x \leq_P y \) holds if and only if \( x \leq_{L_i} y \) holds for every \( i \in \{1, \ldots, d\} \). An incomparable pair in a poset \( P \) is an ordered pair of vertices of \( P \) that are incomparable. We denote by \( \text{Inc}(P) \) the set of all incomparable pairs of \( P \). A bad cycle is a tuple of incomparable pairs \( (x_1, y_1), \ldots, (x_k, y_k) \in \text{Inc}(P) \) such that \( y_i \leq_P x_{i+1} \) for every \( i \in \{1, \ldots, k\} \), where the indices go cyclically over \( \{1, \ldots, k\} \).

The following elementary lemma relates the dimension of \( P \) to bad cycles in \( \text{Inc}(P) \).

Lemma 4 (Trotter, Moore [21]). When \( I \subset \text{Inc}(P) \), then there is a linear extension \( L \) of \( P \) such that \( x <_L y \) for every \( (x, y) \in I \) if and only if \( I \) contains no bad cycle.

We call a coloring of a set \( I \subset \text{Inc}(P) \) good if it contains no bad cycle of one color. As a corollary to Lemma 4, the dimension of \( P \) is the minimum number of colors in a good coloring of \( \text{Inc}(P) \), and this characterization of the dimension will be used further in the paper.

2.4. Preliminary bounds on the dimension. First of all, we will use the following result already mentioned in the introduction.

Theorem 5 (Joret et al. [11]). The posets of bounded height with cover graphs of bounded tree-width have bounded dimension.

The above and the next lemma allow us to conclude that the dimension is also bounded for posets of bounded height whose cover graphs and all their minors have bounded local tree-width. The next lemma in a slightly more restricted setting appears implicitly in the work of Streib and Trotter [19]. We present its full proof in the appendix for the reader’s convenience.

Lemma 6 (Streib, Trotter [19]). Let \( \mathcal{G} \) be a minor-closed class of graphs, \( h \in \mathbb{N} \), and \( \mathcal{G}_{2h-2} \) be the class of graphs in \( \mathcal{G} \) with radius at most \( 2h - 2 \). If the posets
of height at most $h$ with cover graphs in $G_{2h-2}$ have bounded dimension, then the posets of height at most $h$ with cover graphs in $G$ have bounded dimension.

**Corollary 7.** For every function $f : \mathbb{N} \to \mathbb{N}$, the posets of bounded height with cover graphs in $L_f$ have bounded dimension.

**Proof.** Let $f : \mathbb{N} \to \mathbb{N}$ and $h \in \mathbb{N}$. The graphs in $L_f$ of radius at most $2h - 2$ have bounded tree-width, so by Theorem 5, the posets of height at most $h$ that have these graphs as cover graphs have bounded dimension. The class $L_f$ is minor-closed, so the conclusion follows from Lemma 6. \hfill \square

Finally, we will use the fact the posets of bounded height with cover graphs of bounded maximum degree have bounded dimension, which is a consequence of the following old result.

**Theorem 8** (Rödl, Trotter [18]; Füredi, Kahn [7]). The posets with comparability graphs of bounded maximum degree have bounded dimension.

**Corollary 9.** For every $d \in \mathbb{N}$, the posets of bounded height with cover graphs in $D_d$ have bounded dimension.

**Proof.** Whenever $x$ and $y$ are comparable in a poset $P$, there is a path between $u$ and $v$ of length at most $h - 1$ in the cover graph of $P$, where $h$ denotes the height of $P$. Consequently, if the cover graph of $P$ has maximum degree $d$, then the comparability graph of $P$ has maximum degree at most $d^{h-1}$. The conclusion now follows from Theorem 8. \hfill \square

3. **Proof of Theorem 1**

3.1. **Overview.** The starting point of the proof are Corollaries 7 and 9 that the posets of bounded height with cover graphs in $G = L_f \cup D_d$ for any $f : \mathbb{N} \to \mathbb{N}$ and $d \in \mathbb{N}$. We extend the class of graphs $G$ that allows us to bound the dimension of posets with bounded height and with cover graphs in $G$ first by adding a bounded number of apices, and then by going through tree decomposition.

Adding apices is dealt with in the next section. Although removing $t$ apices from a poset can decrease its dimension by at most $t$, it can change the cover graph dramatically—many new cover relations can arise by transitivity through the removed apices, and this is the main issue when dealing with apices.

Going through tree decomposition is the main technical content of this paper. We assume we are given a poset $P$, a tree decomposition of the cover graph of $P$ with bounded adhesion, and good colorings of the incomparable pairs of the subposets of $P$ induced on the bags with a bounded number of colors, and we construct a good coloring of $\text{Inc}(P)$ with a bounded number of colors. Several ideas are borrowed from the proof of Theorem 5 in [11]. However, that proof heavily relies on that the bags of the tree decomposition have bounded size so that the tree decomposition very accurately describes the entire structure of the cover graph. Here, the structure of each bag can be very complex, and all we can make use of are the good colorings of the incomparable pairs in the bags.

Here is the rough idea of the argument. We will define a *signature* $\Sigma(x, y)$ for every $(x, y) \in \text{Inc}(P)$ so that the number of distinct signatures is bounded and the
incomparable pairs with a common signature contain no bad cycle. These signatures will be the colors in a requested good coloring of Inc($P$).

The signatures will somehow encode good colorings of the subposets of $P$ induced on the bags. We will also define mappings of vertices of $P$ so that if there is a bad cycle in incomparable pairs of $P$ with a common signature, then one of the mappings turns it into a monochromatic bad cycle in incomparable pairs of the subposet of $P$ induced on some bag. Since the latter contradicts the assumption that the colorings of the incomparable pairs in the subposets of $P$ induced on the bags are good, it will follow that there is no bad cycle in incomparable pairs of $P$ with a common signature.

In order to keep the structure of a bad cycle under the above-mentioned mappings, we cannot map directly into subposets of $P$; instead, we will map into some extensions of these subposets, called gadget extensions. For every vertex $v$ that needs to be mapped into a gadget extension of $P[X]$, there will be a vertex $v'$ in the gadget extension whose comparabilities to the vertices of $P[X]$ are the same as those of $v$, and $v$ will be mapped into $v'$.

In section 3.3, we will prove that gadget extensions of the subposets of $P$ induced on the bags have bounded dimension. This will give us good colorings of incomparable pairs of the gadget extensions with a bounded number of colors.

3.2. Apices. This short section is devoted to the proof of the following.

**Lemma 10.** Let $G$ be a monotone class of graphs and $h, t \in \mathbb{N}$. If the posets of height at most $h$ with cover graphs in $G$ have bounded dimension, then the posets of height at most $h$ with cover graphs in $A_t(G)$ have bounded dimension.

*Proof.* It is clear that $A_t(G) = A_t^{|t\{\}}(G)$, so it is enough to consider the case of $t = 1$.

Let $P$ be a poset of height at most $h$ with cover graph $G \in A_1(G)$. Hence there is a vertex $v \in P$ such that $G \setminus \{v\} \in G$. The cover graphs of $P \setminus \uparrow_{pv}$ and $P \setminus \downarrow_{pv}$ are $G \setminus \uparrow_{pv}$ and $G \setminus \downarrow_{pv}$, respectively. Both $P \setminus \uparrow_{pv}$ and $P \setminus \downarrow_{pv}$ are subposets of $P$, so they have height at most $h$. Both $G \setminus \uparrow_{pv}$ and $G \setminus \downarrow_{pv}$ are subgraphs of $G \setminus \{v\}$, so they belong to $G$. It follows that $P \setminus \uparrow_{pv}$ and $P \setminus \downarrow_{pv}$ have bounded dimension. Therefore, there is a good coloring of Inc($P \setminus \uparrow_{pv}$) and Inc($P \setminus \downarrow_{pv}$) with a bounded number of colors. Since every element of $P \setminus \uparrow_{pv}$ is comparable to every element of $P \setminus \downarrow_{pv}$, the above coloring covers all incomparable pairs of $P \setminus \{v\}$. Two more colors are enough for the incomparable pairs of $P$ involving $v$—one for those having $v$ as the first member, and the other for those having $v$ as the second member. \qed

3.3. Gadget extensions. An $s$-gadget extension of a graph $G$ is a graph obtained from $G$ by adding, for some distinct cliques $K_1, \ldots, K_n$ in $G$ of size at most $s$, new vertex sets $X_1, \ldots, X_n$ of size at most $2^s + 1$ and new edges so that each new edge connects vertices in $K_i \cup X_i$ for some $i$. For a class of graphs $G$ and for $s \in \mathbb{N}$, let $E_s(G)$ denote the class of $s$-gadget extensions of graphs in $G$. The following are direct consequences of the definition:

- for every $s \in \mathbb{N}$, if $G$ is monotone, then $E_s(G)$ is monotone;
- for any $s, t \in \mathbb{N}$, we have $E_s(A_t(G)) \subset A_t(E_s(G))$.

The following properties are quite straightforward as well, and their proofs are omitted.
Lemma 11. If $s \in \mathbb{N}$ and $\mathcal{G}$ is a class of graphs with bounded local tree-width, then $\mathcal{E}_s(\mathcal{G})$ has bounded local tree-width.

Proof. Let $G' \in \mathcal{E}_s(\mathcal{G})$. It follows that $G'$ has an induced subgraph $G \in \mathcal{G}$ and there are distinct cliques $K_1, \ldots, K_n$ in $G$ and a partition of $V(G') \setminus V(G)$ into sets $X_1, \ldots, X_n$ such that

- $|K_i| \leq s$ and $|X_i| \leq 2^{s+1}$ for every $i$,
- every edge in $E(G') \setminus E(G)$ connects a pair of vertices in some $K_i \cup X_i$.

Let $H$ be a subgraph of $G'$ with radius at most $r$. Hence $G \cap H$ has radius at most $r$, so it has tree-width at most $f(r)$, where $f$ is the bound on the local tree-width of the graphs in $\mathcal{G}$. Let $T$ be a tree-decomposition of $G \cap H$ of width at most $f(r)$. Since every $K_i \cap V(H)$ is a clique, there is a bag $Y_i$ of $T$ such that $K_i \cap V(H) \subset Y_i$. We add every $(K_i \cup X_i) \cap V(H)$ to $T$ as a new bag that becomes connected to $Y_i$ by an edge of $T$. We thus obtain a tree decomposition of $H$ of width at most $\max\{f(r), 2^{s+1} + s - 1\}$. This shows that $G'$ has local tree-width bounded by the function $\max\{f, 2^{s+1} + s - 1\}$. □

Lemma 12. If $s \in \mathbb{N}$ and $\mathcal{G}$ is a class of graphs with bounded maximum degree, then $\mathcal{E}_s(\mathcal{G})$ has bounded maximum degree.

Proof. Let $G' \in \mathcal{E}_s(\mathcal{G})$. It follows that $G'$ has an induced subgraph $G \in \mathcal{G}$ and there are distinct cliques $K_1, \ldots, K_n$ in $G$ and a partition of $V(G') \setminus V(G)$ into sets $X_1, \ldots, X_n$ such that

- $|K_i| \leq s$ and $|X_i| \leq 2^{s+1}$ for every $i$,
- every edge in $E(G') \setminus E(G)$ connects a pair of vertices in some $K_i \cup X_i$.

Let $d$ be the bound on the maximum degree of the graphs in $\mathcal{G}$. It follows that every clique in $G$ has size at most $d + 1$. Let $v$ be a vertex of $G'$. If $v \in X_i$, then its neighborhood is contained in $(K_i \cap X_i) - \{v\}$, so the degree of $v$ is at most $2^{s+1} + d$. Now, suppose $v \in V(G)$. Every clique in $G$ containing $v$ is a subset of the closed neighborhood of $v$ in $G$. Therefore, the number of cliques $K_i$ with $v \in K_i$ is at most $2^{d}$. Each such clique $K_i$ gives at most $2^{s+1}$ additional neighbors of $v$ in $X_i$. Therefore, the total degree of $v$ in $G'$ is at most $2^{d+s+1} + d$. This shows that $G'$ has maximum degree bounded by $2^{d+s+1} + d$. □

Corollary 13. For $h, s, t, d \in \mathbb{N}$ and $f : \mathbb{N} \to \mathbb{N}$, the posets of height at most $h$ with cover graphs in $\mathcal{E}_s(A_t(\mathcal{L}_f \cup \mathcal{D}_d))$ have bounded dimension.

Proof. By Corollary [8], Corollary [9] Lemma [11] Lemma [12] and Lemma [13] the posets of height at most $h$ with cover graphs in $A_t(\mathcal{E}_s(\mathcal{L}_f \cup \mathcal{D}_d))$ have bounded dimension. The latter class contains $\mathcal{E}_s(A_t(\mathcal{L}_f \cup \mathcal{D}_d))$. □

If $P$ is a poset with cover graph $G$, then an $s$-gadget extension of $P$ is a poset whose cover graph is an $s$-gadget extension of $G$.

3.4. Tree decomposition. This entire section is devoted to the proof of the following.

Lemma 14. Let $\mathcal{G}$ be a monotone class of graphs and $h, s \in \mathbb{N}$. If the posets of height at most $h$ with cover graphs in $\mathcal{E}_s(\mathcal{G})$ have bounded dimension, then the posets of height at most $h$ with cover graphs in $T_s(\mathcal{G})$ have bounded dimension.
For this entire section, we assume the setting of Lemma 14. \(G\) is a monotone class of graphs and \(h, s \in \mathbb{N}\). Let \(P\) be a poset of height at most \(h\) with cover graph \(G \in \mathcal{T}_s(G)\). Let \(T\) be a tree decomposition of \(G\) with adhesion \(s\) such that every torso of \(T\) belongs to \(G\). Assume that the posets of height at most \(h\) with cover graphs in \(\mathcal{E}_s(G)\) have bounded dimension.

Let \(H\) be the graph obtained from \(G\) by adding all edges in every adhesion set of \(T\). Clearly, \(T\) is also a tree decomposition of \(H\).

Fix an arbitrary bag as the root of \(T\). We will envision \(T\) as a planted tree—a tree drawn in the plane so that the root lies at the bottom and the tree grows upwards. Let \(\preceq\) denote the bottom-to-top order of \(T\). That is, we have \(X \preceq Y\) whenever a bag \(X\) lies on a path in \(T\) from a bag \(Y\) down to the root. Note that \(\preceq\) is an order on the bags and should not be confused with the order \(\preceq_P\) of the poset \(P\).

For every \(x \in P\), let \(L(x)\) denote the unique lowest bag of \(T\) containing \(x\). For bags \(X\) and \(Y\), let \(X \triangleright_k Y\) denote that there is a sequence of bags \(Z_0, \ldots, Z_\ell\) such that \(0 \leq \ell \leq k\), \(Z_0 = X\), \(Z_\ell = Y\), and there is a sequence of vertices \(z_1, \ldots, z_\ell \in P\) such that \(z_i \in Z_{i-1} \cap Z_i\) and \(L(z_i) = Z_i\) for \(1 \leq i \leq \ell\).

**Lemma 15** (c.f. Lemma 5.2 (1)–(6) in [11]). The relation \(\triangleright_k\) has the following properties:

1. if \(X \triangleright_k Y\) and \(k \leq \ell\), then \(X \triangleright_\ell Y\),
2. \(X \triangleright_0 Y\) if and only if \(X = Y\),
3. \(X \triangleright_{k+\ell} Z\) if and only if there exists a bag \(Y\) such that \(X \triangleright_k Y\) and \(Y \triangleright_\ell Z\),
4. if \(X \triangleright_k Y\), then \(X \preceq Y\),
5. if \(X \triangleright_k Z\) and \(X \preceq Y \preceq Z\), then \(Y \triangleright_k Z\),
6. if \(X \triangleright_k Y\) and \(X \triangleright_k Z\), then \(Y \triangleright_k Z\) or \(Z \triangleright_k Y\).

**Proof.** Properties (1)–(4) follow directly from the definition of \(\triangleright_k\). For the proof of (5) let \(Z_0, \ldots, Z_k\) be a sequence of bags and \(z_1, \ldots, z_k\) be a sequence of vertices such that \(Z_0 = X\), \(Z_k = Z\), and \(z_i \in Z_{i-1} \cap Z_i\) and \(L(z_i) = Z_i\) for \(1 \leq i \leq k\). Since \(X \preceq Y \preceq Z\), there is an index \(i\) such that \(Z_{i-1} \preceq Y \preceq Z_i\). It follows that \(z_i \in Z\), so the sequences \(Y, Z_1, \ldots, Z_k\) and \(z_1, \ldots, z_k\) witness \(Y \triangleright_k Z\). To see (6) observe that \(X \triangleright_k Y\) and \(X \triangleright_k Z\) imply \(X \preceq Y \preceq Z\) or \(X \preceq Z \preceq Y\), and the conclusion follows from (5). \(\square\)

We will mostly use the properties listed in Lemma 15 implicitly, without a reference.

For every bag \(X\), let \(B_k(X)\) denote the family of bags \(Y\) such that \(X \triangleright_k Y\). Thus \(B_k(X)\) lies entirely on the path in \(T\) from \(X\) down to the root.

**Lemma 16** (c.f. Lemma 5.2 (7) in [11]). For every bag \(X\), we have \(|B_k(X)| \leq 1 + s + \ldots + s^k\).

**Proof.** If \(X\) is the root of \(T\), then \(B_k(X) = \{X\}\) and thus \(|B_k(X)| = 1\) for every \(k\). Thus suppose \(X\) is not the root of \(T\). We have \(B_0(X) = \{X\}\), hence \(|B_0(X)| = 1\).

We have \(B_1(X) \setminus \{X\} = \{L(z) : z \in X\) and \(L(z) \neq X\}\). Let \(Y\) be the bag directly following \(X\) on the path in \(T\) down to the root. It follows that if \(z \in X\) and \(L(z) \neq X\), then \(z \in X \cap Y\). Since \(T\) has adhesion at most \(s\), we have \(|X \cap Y| \leq s\). Therefore, we have \(|B_1(X) \setminus \{X\}| \leq s\) for every bag \(X\) of \(T\). This implies that \(|B_k(X) \setminus B_{k-1}(X)| \leq s^k\) for \(k \geq 1\), by a straightforward induction. Hence the lemma follows. \(\square\)
For any \( x, y \in P \) such that \( \mathcal{B}_h(L(x)) \cap \mathcal{B}_h(L(y)) \neq \emptyset \), let \( A(x, y) \) denote the highest bag in \( \mathcal{B}_h(L(x)) \cap \mathcal{B}_h(L(y)) \).

**Lemma 17** (c.f. Lemma 5.3 in [11]). If \( x \geq_P y \), then \( \mathcal{B}_{h-1}(L(x)) \cap \mathcal{B}_{h-1}(L(y)) \neq \emptyset \) and there is \( z \in A(x, y) \) such that \( x \geq_P z \geq_P y \).

**Proof.** Suppose \( x \geq_P y \). Let \( z_0, \ldots, z_k \) be a shortest sequence of vertices of \( P \) with the following properties:
- \( x = z_0 \geq_P \ldots \geq_P z_k = y \),
- \( z_{i-1} z_i \in E(H) \) for \( 1 \leq i \leq k \).

Since the height of \( P \) is at most \( h \), we have \( k \leq h - 1 \). Suppose there are indices \( i \) and \( j \) with \( 0 < i < j < k \) such that \( L(z_{i-1}) < L(z_i) = \ldots = L(z_j) > L(z_{j+1}) \).

Let \( X = L(z_i) = \ldots = L(z_j) \), and let \( Y \) be the bag directly following \( X \) on the path in \( T \) down to the root. Since \( z_{i-1} z_i, z_j z_{j+1} \in E(H) \), we have \( z_{i-1}, z_{j+1} \in X \cap Y \). Hence \( z_{i-1} z_{j+1} \in E(H) \), as \( X \cap Y \) is an adhesion set. This contradicts the choice of the sequence \( z_0, \ldots, z_k \). Therefore, there is an index \( j \) such that \( L(z_0) \geq \ldots \geq L(z_j) \leq \ldots \leq L(z_k) \). It follows that \( L(z_j) \in \mathcal{B}_{h-1}(L(x)) \cap \mathcal{B}_{h-1}(L(y)) \).

Hence \( L(x) \geq A(x, y) \geq L(z_j) \), so there is an index \( i \) with \( 0 \leq i \leq j \) such that \( z_i \in A(x, y) \). \( \square \)

Now, we are going to color incomparable pairs that are in a sense “distant” in \( T \). Let \( \ell \) denote the depth-first search labeling of \( T \) which visits the successors of every node in the left-to-right order, and let \( r \) denote the depth-first search labeling of \( T \) which visits the successors of every node in the right-to-left order. Define the following four families of incomparable pairs:

\[
\begin{align*}
I_1 &= \{ (x, y) \in \text{Inc}(P) : \ell(L(x)) < r(Y) \text{ for every } Y \in \mathcal{B}_{h-1}(L(y)) \}, \\
I_2 &= \{ (x, y) \in \text{Inc}(P) : r(L(x)) < r(Y) \text{ for every } Y \in \mathcal{B}_{h-1}(L(y)) \}, \\
I_3 &= \{ (x, y) \in \text{Inc}(P) : \ell(L(y)) < \ell(X) \text{ for every } X \in \mathcal{B}_{h-1}(L(x)) \}, \\
I_4 &= \{ (x, y) \in \text{Inc}(P) : r(L(y)) < r(X) \text{ for every } X \in \mathcal{B}_{h-1}(L(x)) \}.
\end{align*}
\]

**Lemma 18.** None of \( I_1, I_2, I_3, I_4 \) contains a bad cycle.

**Proof.** Suppose there is a bad cycle \( (x_1, y_1), \ldots, (x_k, y_k) \in I_1 \). It follows that \( \ell(L(x_i)) < \ell(A(y_i, x_{i+1})) \leq \ell(L(x_{i+1})) \) for every \( i \). This is a contradiction, as the indices go cyclically over \( \{1, \ldots, k\} \). Similar arguments show that there are no bad cycles in \( I_2, I_3 \) and \( I_4 \). \( \square \)

Let \( I = \text{Inc}(P) \setminus (I_1 \cup I_2 \cup I_3 \cup I_4) \).

**Lemma 19.** For every \( (x, y) \in I \), we have \( \mathcal{B}_h(L(x)) \cap \mathcal{B}_h(L(y)) \neq \emptyset \).

**Proof.** Let \( (x, y) \in I \). Since \( (x, y) \notin I_1 \cup I_2 \), there are \( Y_1, Y_2 \in \mathcal{B}_{h-1}(L(y)) \) such that \( \ell(L(x)) \geq \ell(Y_1) \) and \( r(L(x)) \geq r(Y_2) \). We have \( Y_1 \leq Y_2 \) or \( Y_1 \geq Y_2 \). Let \( Y = \min\{Y_1, Y_2\} \in \mathcal{B}_{h-1}(L(y)) \). It follows that \( \ell(L(x)) \geq \ell(Y) \) and \( r(L(x)) \geq r(Y) \).

This implies \( L(x) \geq Y \). Similarly, there is \( X \in \mathcal{B}_{h-1}(L(x)) \) such that \( L(y) \geq X \).

We have \( X \leq Y \) or \( Y \leq X \). If \( X \geq Y \), then the highest bag \( X' \in \mathcal{B}_{h-1}(L(y)) \) with \( X \geq X' \) satisfies \( X \triangleright X' \), hence \( L(x) \triangleright_h X' \). Similarly, if \( Y \geq X \), then the highest bag \( Y' \in \mathcal{B}_{h-1}(L(x)) \) with \( Y \geq Y' \) satisfies \( L(y) \triangleright_h Y' \). In both cases we conclude that \( \mathcal{B}_h(L(x)) \cap \mathcal{B}_h(L(y)) \neq \emptyset \). \( \square \)
It follows from Lemma 19 that $A(x, y)$ is defined for every $(x, y) \in I$. Now, it
remains to construct a good coloring of $I$.

Lemma 19 implies that there is a coloring $\phi$ of the bags of $T$ with at most
$1 + s + \ldots + s^{2h}$ colors such that no two bags in the relation $\triangleright_{2h}$ have the same
color. We can construct such a coloring greedily by processing the bags bottom-up
and assigning to each bag $X$ a color that is not used on the bags in $B_{2h}(X) \setminus \{X\}$.
From now on, the color of a bag will always refer to the coloring $\phi$. The following
asserts that the color assigned by $\phi$ uniquely determines a member of $B_{2h}(X)$.

Lemma 20 (c.f. Lemma 5.4 in [11]). If $X \triangleright_{2h} Y$, $X \triangleright_{2h} Z$, and $\phi(Y) = \phi(Z)$,
then $Y = Z$.

Proof. Suppose $Y \neq Z$. Since $X \triangleright_{2h} Y$ and $X \triangleright_{2h} Z$, we have $Y \triangleright_{2h} Z$ or $Z \triangleright_{2h} Y$.
Whichever of these holds, the definition of $\phi$ yields $\phi(Y) \neq \phi(Z)$.

Let $\tau(X)$ denote the sequence of the colors of the members of $B_h(X)$ listed in the
decreasing (top-to-bottom) order of $T$. In particular, $\phi(X)$ occurs first in $\tau(X)$.
For two such sequences $\tau_1$ and $\tau_2$, let $\tau_1 \oplus \tau_2$ denote the sequence obtained from $\tau_1$
and $\tau_2$ by the following procedure:
- remove from $\tau_1$ all colors that do not occur in $\tau_2$, thus obtaining $\tau'_1$;
- remove from $\tau_2$ all colors that do not occur in $\tau_1$, thus obtaining $\tau'_2$;
- let $\tau_1 \oplus \tau_2$ be the longest common suffix of $\tau'_1$ and $\tau'_2$.

Lemma 21. For any two bags $X$ and $Y$, the sequence of the colors of the members
of $B_h(X) \cap B_h(Y)$ listed in the decreasing (top-to-bottom) order of $T$ is a suffix of
$\tau(X) \oplus \tau(Y)$.

Proof. Let $Z \in B_h(X) \cap B_h(Y)$. The color $\phi(Z)$ occurs in both $\tau(X)$ and $\tau(Y)$.
Let $Z'$ be a bag in $B_h(X)$ or $B_h(Y)$ such that $Z > Z'$. It follows that $Z \triangleright_h Z'$, so
$Z' \in B_{2h}(X) \cap B_{2h}(Y)$. By Lemma 20, no member of $B_{2h}(X) \cup B_{2h}(Y)$ other than
$Z'$ can have color $\phi(Z')$. Therefore,
- if $Z' \in B_h(X) \cap B_h(Y)$, then $\phi(Z')$ occurs after $\phi(Z)$ in both $\tau(X)$ and $\tau(Y)$;
- if $Z' \in B_h(X) \setminus B_h(Y)$, then $\phi(Z')$ does not occur in $\tau(Y)$;
- if $Z' \in B_h(Y) \setminus B_h(X)$, then $\phi(Z')$ does not occur in $\tau(X)$.
This shows that the colors of the bags in $B_h(X) \cap B_h(Y)$ listed in the decreasing
order of $T$ indeed form a suffix of $\tau(X) \oplus \tau(Y)$.

Now, we describe a construction of $s$-gadget extensions $P_Z$ and $P'_Z$ of the induced
subposet $P[Z]$ for every bag $Z$ of $T$. Fix a bag $Z$. For every bag $X$ with $X > Z$,
let $N_Z(X)$ denote the lowest bag $X'$ such that $X \triangleright X' > Z$. For every clique $K$ in
$H[Z]$, let $X_{Z,K} = \{X : X > Z$ and $Z \cap N_Z(X) = K\}$. If $X_{Z,K} \neq \emptyset$, then $K$ is an
adhesion set of $Z$, so $|K| \leq s$. For the common vertex set of $P_Z$ or $P'_Z$, we take $Z$
and add, for every clique $K$ in $H[Z]$ such that $X_{Z,K} \neq \emptyset$, a gadget that consists of
- a vertex $x_{Z,K,S}$ for every set $S$ of the form $S = Z \cap \downarrow_{P} x$ for some $x \in P$ with
$L(x) \in X_{Z,K}$,
- a vertex $y_{Z,K,S}$ for every set $S$ of the form $S = Z \cap \uparrow_{P} x$ for some $x \in P$ with
$L(x) \in X_{Z,K}$.
Since $|K| \leq s$ and the comparabilities of a vertex $x$ such that $L(x) \in X_{Z,K}$ to the
vertices in $Z$ are uniquely determined by the comparabilities of $x$ to the vertices in $K$,
every gadget has at most $2^{s+1}$ vertices. The order $\preceq_P$ is defined so that
The order \( \leq_{P'_Z} \) is defined like \( \leq_{P_Z} \) except that the last condition above is replaced by the following:

- \( x_{Z,K,S} \geq_{P'_Z} y_{Z,K',S} \) whenever \( K = K' \) or \( S \cap S' \neq \emptyset \).

It is straightforward to verify that \( \leq_{P_Z} \) and \( \leq_{P'_Z} \) are indeed partial orders. It follows that every \( x_{Z,K,S} \) is a maximal vertex and every \( y_{Z,K,S} \) is a minimal vertex of both \( P_Z \) and \( P'_Z \).

It is straightforward to verify that the cover graphs of \( P_Z \) and \( P'_Z \) are \( s \)-gadget extensions of the torso \( H[Z] \). Therefore, by the assumption of Lemma 14, \( P_Z \) and \( P'_Z \) have bounded dimension. For any incomparable pair \((x, y)\) of \( P_Z \) or \( P'_Z \), let \( \sigma_Z(x, y) \) or \( \sigma'_Z(x, y) \), respectively, denote the color of \((x, y)\) in a good coloring of Inc\((P_Z)\) or Inc\((P'_Z)\), respectively, witnessing the bound on the dimension.

Next, we define mappings \( \mu_Z \) and \( \nu_Z \) of the vertices \( x \in P \) such that \( L(x) \geq Z \) into the common vertex set of \( P_Z \) and \( P'_Z \) as follows:

\[
\mu_Z(x) = \begin{cases} 
   x_{Z,K,S} & \text{if } x \notin Z, \text{ where } K = Z \cap N_Z(L(x)) \text{ and } S = Z \cap \downarrow P x, \\
   x & \text{if } x \in Z, 
\end{cases}
\]

\[
\nu_Z(x) = \begin{cases} 
   y_{Z,K,S} & \text{if } x \notin Z, \text{ where } K = Z \cap N_Z(L(x)) \text{ and } S = Z \cap \uparrow P x, \\
   x & \text{if } x \in Z. 
\end{cases}
\]

For any \( x, y \in P \) and every bag \( Z \in B_h(L(x)) \cap B_h(L(y)) \), let \( \Pi_Z(x, y) \) denote the predicate that is true if and only if \( x, y \notin Z \) and \( Z \cap N_Z(L(x)) = Z \cap N_Z(L(y)) \). Therefore, \( \Pi_Z(x, y) \) indicates whether \( x \) and \( y \) are mapped into the same gadget by \( \mu_Z \) and \( \nu_Z \). Clearly, if \( A(x, y) > Z \), then \( \Pi_Z(x, y) \).

**Lemma 22.**

1. If \((x, y) \in I\) and \( Z \in B_h(L(x)) \cap B_h(L(y)) \), then \( (\mu_Z(x), \nu_Z(y)) \in \text{Inc}(P_Z) \).
2. If \((x, y) \in I\), \( Z \in B_h(L(x)) \cap B_h(L(y)) \), and \( \neg \Pi_Z(x, y) \), then \( (\mu_Z(x), \nu_Z(y)) \in \text{Inc}(P'_Z) \).
3. If \( x \geq_{P} y \), \( Z \in B_h(L(x)) \cap B_h(L(y)) \), and there is \( z \in Z \) such that \( x \geq_{P} z \geq_{P} y \), then \( \mu_Z(x) \geq_{P_Z} \nu_Z(y) \).
4. If \( x \geq_{P} y \) and \( Z \in B_h(L(x)) \cap B_h(L(y)) \), then \( \mu_Z(x) \geq_{P'_Z} \nu_Z(y) \).

**Proof.** For the proof of (1) suppose \( \mu_Z(x) \geq_{P_Z} \nu_Z(y) \). It follows from the definition \( \mu_Z \) and \( \nu_Z \) that there is \( z \in Z \) such that \( x \geq_{P} z \geq_{P} y \), a contradiction. We cannot have \( \mu_Z(x) \geq_{P_Z} \nu_Z(y) \) either, so \( (\mu_Z(x), \nu_Z(y)) \in \text{Inc}(P_Z) \). The statement (2) follows directly from (1) and the definition of \( P_Z \). For the proof of (3) note that the existence of such \( z \) and the definition of \( \mu_Z \) and \( \nu_Z \) yield \( \mu_Z(x) \geq_{P_Z} \nu_Z(y) \). It remains to prove (4). If \( \Pi_Z(x, y) \), then \( \mu_Z(x) \geq_{P'_Z} \nu_Z(y) \) follows from the definition of \( P'_Z \). Now, suppose \( \neg \Pi_Z(x, y) \). It follows that \( A(x, y) = Z \). By Lemma 17, there is \( z \in Z \) such that \( x \geq_{P} z \geq_{P} y \). This and (3) imply \( \mu_Z(x) \geq_{P'_Z} \nu_Z(y) \). \( \square \)

Assign unique identifiers \( \text{id}(X) \in \mathbb{N} \) to all bags \( X \) of \( T \). For any bags \( X, Y, Z \) with \( Z \in B_h(X) \cap B_h(Y) \), let \( p_Z(X, Y) \) denote the color preceding \( \phi(Z) \) in \( \tau(X) \oplus \tau(Y) \).
or a special value $\perp$ if $\phi(Z)$ is the first color in $\tau(X) \oplus \tau(Y)$, and define

$$t_Z(X, Y) = \begin{cases} 1 & \text{if } p_Z(X, Y) \neq \perp \text{ and } \id(X') < \id(Y'), \text{ where } X \geq X' > Z, \\ Y > Y' > Z, \text{ and } \phi(X') = \phi(Y') = p_Z(X, Y), \\ 2 & \text{if } p_Z(X, Y) \neq \perp \text{ and } \id(X') > \id(Y'), \text{ where } X \geq X' > Z, \\ Y > Y' > Z, \text{ and } \phi(X') = \phi(Y') = p_Z(X, Y), \\ 3 & \text{otherwise.} \end{cases}$$

For every bag $Z$ of $T$ and every clique $K \subseteq Z$ such that $X_{Z,K} \neq \emptyset$, assign unique identifiers $\id_{Z,K}(x_{Z,K,S}) \in \{1, \ldots, 2^s\}$ to all vertices $x_{Z,K,S}$. For every $(x, y) \in I$, define a signature $\Sigma(x, y)$ which encodes the following information:

- $\tau(L(x))$ and $\tau(L(y))$,
- a function that maps the color $\phi(Z)$ of every bag $Z \in B_h(L(x)) \cap B_h(L(y))$ to $\Pi_Z(x, y)$, $\sigma_Z(\mu_Z(x), \nu_Z(y))$, $\sigma'_Z(\mu_Z(x), \nu_Z(y))$, and $t_Z(L(x), L(y))$,
- a function that maps the color $\phi(Z)$ of every bag $Z \in B_h(L(x)) \cap B_h(L(y)) \setminus \{L(z)\}$ to $\id_{Z,K}(\mu_Z(x))$, where $K = Z \cap N_Z(L(x))$.

It is clear that the number of distinct signatures is bounded. We use $\Sigma(x, y)$ as the color of $(x, y)$ in the requested coloring of $I$ with a bounded number of colors. To complete the proof of Lemma 13 it remains to show the following.

**Lemma 23.** No set of incomparable pairs in $I$ with the same signature contains a bad cycle.

**Proof.** For this entire proof, once we fix $k$, all the indices go cyclically over $\{1, \ldots, k\}$. Suppose for a contradiction that there is a bad cycle $(x_1, y_1), \ldots, (x_k, y_k) \in I$ such that $\Sigma(x_i, y_i) = \Sigma$ for every $i$. Assume without loss of generality that it is a shortest cycle with this property. By Lemmas 17 and 19, $A(x_i, y_i)$ and $A(y_i, x_{i+1})$ are defined for every $i$. Let $Z$ be the highest bag such that $Z \leq A(x_i, y_i)$ and $Z \leq A(y_i, x_{i+1})$ for every $i$.

Suppose $Z < A(x_i, y_i)$ and $Z < A(y_i, x_{i+1})$ for every $i$. It follows that the bags $L(x_i)$ and $L(y_i)$ altogether span at least two subtrees of $T$ rooted at immediate successors of $Z$. However, for every $i$, the bags $L(x_i)$ and $L(y_i)$ belong to the same of these subtrees, as $Z < A(x_i, y_i)$, and so do the bags $L(y_i)$ and $L(x_{i+1})$, as $Z < A(y_i, x_{i+1})$. This is a clear contradiction. Therefore, we have $Z = A(x_i, y_i)$ or $Z = A(y_i, x_{i+1})$ for some $i$. In particular, we have $L(x_i) \triangleright_h Z$ for some $i$ and $L(y_i) \triangleright_h Z$ for some $i$. Since $\tau(L(x_i))$ and $\tau(L(y_i))$ are encoded in $\Sigma$, the color $\phi(Z)$ occurs in $\tau(L(x_i))$ and $\tau(L(y_i))$ for every $i$.

Suppose $L(x_i) \triangleright_h Z$. We have $L(y_i) \triangleright_h A(x_i, y_i)$ and $L(x_i) \triangleright A(x_i, y_i) \triangleright Z$. By Lemma 15, we have $A(x_i, y_i) \triangleright_h Z$, hence $L(y_i) \triangleright_{2h} Z$. Since $\phi(Z)$ occurs in $\tau(L(y_i))$, there is a bag $Z'$ with $L(y_i) \triangleright_h Z'$ and $\phi(Z') = \phi(Z)$. By Lemma 20, we have $Z' = Z$, so $L(y_i) \triangleright_h Z$. Now, suppose only $L(y_i) \triangleright_h Z$. We have $L(x_{i+1}) \triangleright_h A(y_i, x_{i+1})$ and $L(y_i) \triangleright A(y_i, x_{i+1}) \triangleright Z$. The same argument as above shows $L(x_{i+1}) \triangleright_h Z$. Repeated application of this argument gives $L(x_i) \triangleright_h Z$ and $L(y_i) \triangleright_h Z$ for all $i$.

Since $\Pi_Z(x_i, y_i)$ is encoded in $\Sigma$, we have $\Pi_Z(x_i, y_i)$ for every $i$ or $-\Pi_Z(x_i, y_i)$ for every $i$.

**Case 1.** $\Pi_Z(x_i, y_i)$ for every $i$. Suppose that for some $i$, there is $z \in Z$ with $y_{i-1} \leq z \leq x_i$, but there is no $z' \in Z$ with $y_i \leq z' \leq x_{i+1}$. It follows from Lemma 17 that $A(y_i, x_{i+1}) > Z$, which implies $\Pi_Z(y_i, x_{i+1})$. This and $\Pi_Z(x_i, y_i)$
implies $\Pi_Z(x_i, x_{i+1})$. Hence $\mu_Z(x_i) = x_{Z,K,S}$ and $\mu_Z(x_{i+1}) = x_{Z,K,S'}$, where $K = Z \cap N_Z(L(x_i)) = Z \cap N_Z(L(x_{i+1}))$, $S = Z \cap \downarrow p x_i$, and $S' = Z \cap \downarrow p x_{i+1}$. Since $id_{Z,K}(\mu_Z(x_i))$ is encoded in $\Sigma$, we have $id_{Z,K}(x_{Z,K,S}) = id_{Z,K}(x_{Z,K,S'})$, so $S = S'$. Since $z \leq x_i$, we have $z \in S = S'$, so $y_{i-1} \leq p z \leq p x_{i+1}$. This implies that $k > 2$ and $(x_1, y_1), . . . , (x_{i-1}, y_{i-1}), (x_{i+1}, y_{i+1}), . . . , (x_k, y_k)$ is a shorter bad cycle in $I$, a contradiction. Therefore, there is $z \in Z$ with $y_i \leq p z \leq p x_{i+1}$ either for every $i$ or for no $i$.

Subcase 1.1. For every $i$, there is $z \in Z$ with $y_i \leq p z \leq p x_{i+1}$. By Lemma 22 [1], we have $(\mu_Z(x_i), \nu_Z(y_i)) \in \text{Inc}(P_Z)$ for every $i$. By Lemma 22 [3], we have $\nu_Z(y_i) \leq p z \mu_Z(x_{i+1})$ for every $i$. Hence $(\mu_Z(x_1), \nu_Z(y_1)), . . . , (\mu_Z(x_k), \nu_Z(y_k))$ is a bad cycle in $\text{Inc}(P_Z)$. Since $\sigma_Z(\mu_Z(x_i), \nu_Z(y_i))$ is encoded in $\Sigma$, it is the same for every $i$. This contradicts the assumption that $\sigma_Z$ is a good coloring of $\text{Inc}(P_Z)$.

Subcase 1.2. For every $i$, there is no $z \in Z$ with $y_i \leq p z \leq p x_{i+1}$. By Lemma 17, we have $A(y_i, x_{i+1}) > Z$ for every $i$. For every $i$, let $Z_i'$ be the lowest bag in $B_0(\Lambda(y_i)) \cap B_0(L(x_{i+1}))$ with $Z_i' > Z$. Hence we have $A(y_i, x_{i+1}) \geq Z_i'$, and by Lemma 21, $\phi(Z_i') = p_Z(L(y_i), L(x_{i+1}))$. The color $p_Z(L(y_i), L(x_{i+1}))$ is the same for all $i$, as it is determined by $\tau(L(y_i))$ and $\tau(L(x_{i+1}))$, which are encoded in $\Sigma$. Since $A(y_i, x_{i+1}) > Z$ for every $i$, we have $A(x_i, y_i) = Z$ for some $i$. This and $p_Z(L(x_i), L(y_i)) \neq \perp$ imply $t_Z(L(x_i), L(y_i)) \in \{1, 2\}$ for some $i$. Therefore, since $t_Z(L(x_i), L(y_i))$ is encoded in $\Sigma$, we have $id(Z_i') < id(Z_i'_{i+1})$ for every $i$ or $id(Z_i') > id(Z_i'_{i+1})$ for every $i$. This is a contradiction, as the indices go cyclically over $\{1, . . . , k\}$.

Case 2. $\neg \Pi_Z(x_i, y_i)$ for every $i$. By Lemma 22 [2], we have $(\mu_Z(x_i), \nu_Z(y_i)) \in \text{Inc}(P_Z)$ for every $i$. By Lemma 22 [4], we have $\nu_Z(y_i) \leq p_z \mu_Z(x_{i+1})$ for every $i$. It follows that $(\mu_Z(x_1), \nu_Z(y_1)), . . . , (\mu_Z(x_k), \nu_Z(y_k))$ is a bad cycle in $\text{Inc}(P_Z')$. Since $\sigma_Z'(\mu_Z(x_i), \nu_Z(y_i))$ is encoded in $\Sigma$, it is the same for every $i$. This contradicts the assumption that $\sigma_Z'$ is a good coloring of $\text{Inc}(P_Z')$.

\section{3.5. Summary.} Let $G$ be a proper topologically closed class of graphs, and let $h \in \mathbb{N}$. By Theorem 3, we have $G \subset \mathcal{T}_k(A_c(L_f \cup D_d))$, where $s$, $t$, $f$ and $d$ depend only on $G$. By Corollary 13, the posets of height at most $h$ with cover graphs in $E_s(A_c(L_f \cup D_d))$ have bounded dimension. Therefore, by Lemma 14, the posets of height $h$ with cover graphs in $\mathcal{T}_k(A_c(L_f \cup D_d))$ have bounded dimension. This completes the proof of Theorem 1.

Theorem 5 plays an important role in the proof of Theorem 1 above. However, we can use the machinery established in this paper to reprove Theorem 5 as well. Fix $k \in \mathbb{N}$. If $S_{k+1}$ denotes the class of graphs with at most $k + 1$ vertices, then $\mathcal{T}_k(S_{k+1})$ is the class of graphs of tree-width at most $k$. The graphs in $E_k(S_{k+1})$ have bounded size, so the posets with cover graphs in $E_k(S_{k+1})$ have bounded dimension. By Lemma 13, for any $h, k \in \mathbb{N}$, the posets of height at most $h$ with cover graphs in $\mathcal{T}_k(S_{k+1})$ have bounded dimension, which is the statement of Theorem 5.

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Appendix

Proof of Lemma 6. Let $P$ be a poset of height at most $h$ with cover graph $G \in \mathcal{G}$. We can assume without loss of generality that $G$ is connected, because the dimension of a disjoint union of posets is bounded in terms of the maximum of the dimensions of the individual posets. Let $v$ be an arbitrary minimal vertex in $P$. Let $A_0, \ldots, A_n$ be the levels of breadth-first search of the comparability graph of $P$ starting from $v$. That is, $A_0 = \{v\}$, $A_i = \{x \in P \setminus (A_0 \cup \ldots \cup A_{i-1}) :$ there is $y \in A_{i-1}$ comparable to $x \}$ for $i \geq 1$. Since $G$ is connected, there is an edge in $G$ from $A_i$ to $A_{i+1}$ for each $i$. Since $G$ is planar, the levels $A_0, \ldots, A_n$ are planar. Therefore, the dimension of $P$ is at most $n + 1$. □
to $x$ in $P$} for $i \geq 1$, and $n$ is greatest such that $A_n \neq \emptyset$. The following properties are straightforward:

1. If $x \in A_i$, $y \in A_{i-1}$, $x$ and $y$ are comparable, and $i$ is odd, then $x \geq_P y$.
2. If $x \in A_i$, $y \in A_{i-1}$, $x$ and $y$ are comparable, and $i$ is even, then $x \leq_P y$.
3. If $x \in A_i$, $y \in A_{i-1}$, and $|i - j| \geq 2$, then $x$ and $y$ are incomparable.
4. If $x \in A_i$ and $i$ is odd, then there is $y \in A_{i-1}$ such that $x \geq_P y$.
5. If $x \in A_i$ and $i$ is even, then $x = v$ or there is $y \in A_{i-1}$ such that $x \leq_P y$.

Let $P_i = P[A_0 \cup A_i]$. It follows that the cover graph $G_i$ of $P_i$ is $G[A_0 \cup A_i]$ and thus $G_i \in \mathcal{G}$. For $2 \leq i \leq n$, let $P_i$ be the poset obtained from $P[A_0 \cup \ldots \cup A_i]$ by contracting $A_0 \cup \ldots \cup A_{i-2}$ to a single vertex. That is, the whole set $A_0 \cup \ldots \cup A_{i-2}$ is replaced by a single vertex $v'$ such that

- $v' \leq_{P_i} x$ for every $x \in A_{i-1}$ if $i$ is even,
- $v' \geq_{P_i} x$ for every $x \in A_{i-1}$ if $i$ is odd,
- $v'$ is incomparable with every vertex in $A_i$.

It follows that the cover graph $G_i$ of $P_i$ is obtained from $G[A_0 \cup \ldots \cup A_i]$ by contracting $A_0 \cup \ldots \cup A_{i-2}$ to the vertex $v'$. Since $G[A_0 \cup \ldots \cup A_{i-2}]$ is connected, $G_i$ is a minor of $G$, so $G_i \in \mathcal{G}$. For $1 \leq i \leq n$, the height of $P_i$ is at most $h$, hence every comparability in $P_i$ is witnessed by a path in $G_i$ of length at most $h - 1$. Therefore, the distance of every vertex from $v'$ (or $v$ if $i = 1$) in $G_i$ is at most $2h - 2$, so $G_i \in \mathcal{G}_{2h-2}$. This and the assumption of the lemma imply that $P_i$ has bounded dimension.

Let $d$ be a common bound on the dimension of all the $P_i$. We are ready to construct a good coloring of $\text{Inc}(P)$ with a bounded number of colors. First, observe that by the property (3) above, neither of the following two sets of incomparable pairs contains a bad cycle:

\[ I_1 = \{(x, y) \in \text{Inc}(P) : x \in A_i \text{ and } y \in A_j \text{ for some } i \text{ and } j \text{ with } i - j \geq 2\}, \]

\[ I_2 = \{(x, y) \in \text{Inc}(P) : x \in A_i \text{ and } y \in A_j \text{ for some } i \text{ and } j \text{ with } j - i \geq 2\}. \]

Hence we can color $I_1 \cup I_2$ with just two colors. Every remaining incomparable pair $(x, y) \in \text{Inc}(P)$ satisfies $x, y \in A_{i-1} \cup A_i$ for some $i$ and therefore it is also in $\text{Inc}(P_i)$ for some $i$. Fix a good coloring of every $\text{Inc}(P_i)$ with colors $1, \ldots, d$ if $i$ is even or $d + 1, \ldots, 2d$ if $i$ is odd. Every incomparable pair $(x, y) \in \text{Inc}(P)$ such that $x, y \in A_{i-1} \cup A_i$ and at least one of $x, y$ is in $A_i$ is assigned its color in $\text{Inc}(P_i)$. Suppose that these incomparable pairs contain a monochromatic bad cycle $(x_1, y_1), \ldots, (x_k, y_k)$. For every $j$, we have $(x_j, y_j) \in A_{i-1} \cup A_i$ for some $i$, but the parity of $i$ is determined by the color of the bad cycle and thus common for all $j$. This and the properties (1)--(3) imply that the $i$ is common for all $j$, which contradicts the assumption that we have chosen a good coloring of $\text{Inc}(P_i)$. This shows that the coloring of $\text{Inc}(P)$ that we have constructed is good. \hfill $\square$

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