Short Paper

Synthesis and Generalization of Parallel Algorithm for Matrix-vector Multiplication

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Abstract: Recently, there have been more chances to calculate matrix-vector multiplication due to the growing use of the neural network. We have proposed the method to automatically synthesize the optimum parallel algorithm for the given environment and synthesized an algorithm for matrix-vector multiplication of a specific size matrix with 4 nodes connected in a one-way ring. This paper proposes a method to generalize the synthesized algorithm to deal with any size matrix. We generalized the synthesized algorithm for the $32 \times 32$ matrix to calculate $N \times N$ matrix-vector multiplication.

Keywords: partial synthesis, program synthesis, automatic parallelization

1. Introduction

Matrix-vector multiplication (MVM) is a time-consuming calculation in a neural network. It requires many Multiply and Accumulate (MAC) operations. As each multiplication is independent, there has been much effort to parallelize MVM\cite{1,2,3}.

Some research has been conducted to parallelize designs automatically\cite{4,5,6,7}. They partitioned a dataflow into several blocks and distribute them among nodes considering data dependency and communication. We expect that we can derive a more efficient parallel design by transforming the dataflow according to the computing environment before partitioning it.

We have proposed the method to automatically synthesize the optimum algorithm in the given parallel environment from the specified input-output relation\cite{8}. We have applied our automatic synthesis method to MVM to derive the algorithm executable with nodes (processing units) connected in one-way-ring-topology. We were able to synthesize the optimum algorithm for the $32 \times 32$ matrix with 4 nodes by analyzing the algorithms synthesized for smaller sizes of matrices.

This paper proposes a method to generalize the synthesized algorithm to obtain the algorithm for an arbitrary size matrix.

This paper is organized as follows: Section 2 explains partial synthesis, which is fundamental to our work. Section 3 describes our previous work on the synthesis of the optimum algorithm. Section 4 explains our method to generalize the algorithm. Section 5 discusses the effectiveness of the proposed method, and Section 6 concludes the paper.

2. Partial Synthesis

Partial synthesis generates a new design equivalent to the specification by filling the blanks in the given template. The specification is a correct design or a set of correct input-output patterns. The template is an incomplete design, which has several blanks inside where each blank has candidates to fill itself.

We can formulate the synthesis problem as a 2-level Quantified Boolean Formula (2QBF) shown in Eq. (1). The formula is satisfied when there exists the assignment of candidates for all blanks ($y$) satisfying the following condition: the output pattern of the specification ($SPEC$) is equal to that of Template ($TMPL$) for every input pattern ($x$). This 2QBF can be solved by iteratively solving SAT problems\cite{9}. The synthesis is successful if Eq. (1) is satisfied.

\[
\exists y. \forall x. SPEC(x) = TMPL(x, y)
\] (1)

3. Synthesis of Parallel Algorithm

We modeled parallel environments as Fig. 1. Each node has several registers and a processor with a fixed number of inputs. Each register stores one variable. All nodes are synchronized and any operation in a processor takes one cycle. Nodes are connected...
by oneway connections. The communication of one variable by a connection takes one cycle overlapped with operation. The modeled environment can be realized as a Coarse Grained Reconfigurable Array (CGRA), which works at 100–400 MHz when implemented on FPGA [10].

Our previous method creates templates as Fig. 2 based on the model of a parallel environment. We denote a blank by \( \{} \) with its candidates inside. A function beginning from ‘f’ is a blank function, which is an arbitrary function to be determined by the synthesis.

We first try synthesis with the template which takes just one cycle. If it fails, we try again with a new template with the number of cycles increased by one until the synthesis succeeds. The synthesized program takes the minimum possible number of cycles because SAT solver implicitly proves that there is no correct program with that number of cycles when the synthesis fails.

We performed the synthesis of MVM for \( N \times N \) matrix with \( M \) nodes connected in oneway-ring-topology. We were able to synthesize a program for \( N = M = 2 \) but not \( N > 2 \) and \( M \geq 2 \) with a timeout of 1 day. To synthesize programs for larger matrices, we devised additional constraints as follows:

- Each processor performs only MAC operation
- \( N \) is divisible by \( M \)
- Nodes communicate only input-vector-elements
- The movement of data in registers is symmetric among nodes
- The movement of data is repeated every \( N \) cycles
- The movement of data is repeated every \( M \) cycles except the \( i \times N \)-th cycle where \( i \) is an integer

The synthesized program takes the minimum possible number of cycles according to \( N \) and \( M \) and one for \( N \times M \) matrix with \( N \)\( \times \)M multiplications with \( M \) nodes doing MAC operation.

The candidates of blank \( s \) and \( c \). The candidates of blank \( s \) are

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We synthesized a program for \( N = 16, M = 4 \) in 22.5 seconds and one for \( N = 32, M = 4 \) in 3 hours using the same additional constraints. The synthesized programs finish in the minimum possible number of cycles, \( N^2/M \), where we must do \( N^2 \) multiplications with \( M \) nodes doing MAC operation.

### 4. Generalization of Algorithm

The synthesis method explained in Section 3 seems impossible to synthesize a program for a larger matrix such as \( N = 1000, M = 4 \) in a feasible time. The synthesis time increases exponentially according to \( N \) as we can see from the previous results. It is reasonable because the synthesis problem is originally one of PSPACE problems and the size of problem increases according to \( N \).

A way to compute MVM for \( N = 1000, M = 4 \) is to divide the matrix into 15625 sub-matrices of \( 8 \times 8 \) and execute the program synthesized for \( N = 8 \) and \( M = 4 \) for each sub-matrix. However, it is not obvious whether the communication between each execution of the program can be completely hidden or not. If not, the entire calculation will take extra time for communication.

We propose a method to generalize the synthesized program. We use the generalization-template in Fig. 3. According to the constraints used in the synthesis, it has four loops where the number of iterations is \( N/M \) for two loops and \( M \) for the other loops. A matrix-element used in node \( p \) at cycle \( k \) is defined by \( X \) and \( Y \): \( X \) represents the row of the element, and \( Y \) represents the column of the element. They are defined with blanks \( s \) and \( c \). The candidates of blank \( s \) are \( N, M, 1, 0, -N, -M \) and \( -1 \). The candidates of blank \( c \) are \( N, M, \) and INF which is such a large int that \( x \times INF \) is \( x \) for any int \( x \). The number of lines for \( X \) is the same as that for \( Y \). It is initially 1 and is increased by one when the generalization fails.

We synthesized \( X \) and \( Y \) with assigning 32 to \( N \) and 4 to \( M \) such that the generalization-template is equivalent to the program synthesized in our previous work for \( N = 32, M = 4 \). \( X \) and \( Y \) were synthesized as Eq. (2) in 1 second. The generalized algorithm takes \( N^2/M \) cycles, the minimum possible number of cycles.

\[
X = -M \ast i + p + N - M \quad \quad Y = -M \ast j + N - M + (-k + p + N) \ast N/M
\]

\[(2)\]

for ( int \( i = 0; i < N/M; i++ \) ) {
  for ( int \( j = 0; j < N/M; j++ \) ) {
    for ( int \( k = 0; k < M; k++ \) ) {
      for ( int \( p = 0; p < M; p++ \) ) {
        // \{s\} = \{N, M, 1, 0, -N, -M, -1\}
        // \{c\} = \{N, M, INF\}
        X = \{s\}*i + \{s\}*j + \{s\}*k + \{s\}*p + \{s\} \% \{c\}
        + \{s\}*i + \{s\}*j + \{s\}*k + \{s\}*p + \{s\} \% \{c\}
        Y = \{s\}*i + \{s\}*j + \{s\}*k + \{s\}*p + \{s\} \% \{c\}
        + \{s\}*i + \{s\}*j + \{s\}*k + \{s\}*p + \{s\} \% \{c\}
      }
      // Out vec \{X\} = matrix \{X\}[Y] \ast vec \{Y\}
    }
  }
}

Fig. 2 The template for a modeled parallel environment.

Fig. 3 The generalization-template.
As of now, we do not have a method to prove the generalized algorithm is still correct when we change \( N \) and \( M \) to any number. When we use the generalized program, we should verify it after assigning specific values to \( N \) and \( M \).

The verification consists of 2 parts: whether the outputs are correct and whether the communication is valid in the environment. We verify the first part by checking that each node uses all matrix-weights involved in the output-vector-elements the node outputs. For the second part, we adopted the assumption that a node must use the received input-vector-element in the next MAC operation whenever it receives. Then, we can verify the second part by checking that the input-vector-element used in a node was the most recently used in its adjacent nodes. The verification of Eq. (2) took 0.1 seconds for the first part and 5 seconds for the second part when we assign \( N = 1000, M = 4 \).

This template of generalization is different from a tiling method, although they look similar. A tiling method exploits the cache reuse. Our method generalizes the synthesized MVM program based on the constraints imposed in the synthesis.

## 5. Effectiveness

The synthesized algorithm takes the minimum number of cycles, which is proved through iterative synthesis. It must be faster or no less faster than what is generated by existing approaches.

We can see the effectiveness of the synthesized algorithm by comparing it with the straightforward algorithm shown in Fig. 4 when \( N = M = 4 \). It is assumed that loading from external storage takes much longer time than communication among nodes. The straightforward algorithm requires 3 more registers for each node and takes 7 cycles in total including 3 cycles to final storage takes much longer time than communication among nodes.

6. Conclusion

We proposed the method to generalize the result of the synthesis method for the parallel MVM algorithm. We assumed that the algorithm can be represented by several loops with their number of iterations defined by the size of matrix and the number of nodes. We were able to generalize the program with those loops. Its correctness can be verified after the size of matrix and the number of nodes is defined.

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