Chaotic Dynamics of Comet 1P/Halley; Lyapunov Exponent and Survival Time Expectancy

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Abstract

The orbital elements of comet Halley are known to a very high precision, suggesting that the calculation of its future dynamical evolution is straightforward. In this paper we seek to characterize the chaotic nature of the present day orbit of comet Halley and to quantify the timescale over which its motion can be predicted confidently. In addition, we attempt to determine the timescale over which its present day orbit will remain stable. Numerical simulations of the dynamics of test particles in orbits similar to that of comet Halley are carried out with the Mercury 6.2 code. On the basis of these we construct survival time maps to assess the absolute stability of Halley’s orbit, frequency analysis maps, to study the variability of the orbit and we calculate the Lyapunov exponent for the orbit for variations in initial conditions at the level of the present day uncertainties in our knowledge of its orbital parameters. On the basis of our calculations of the Lyapunov exponent for comet Halley, the chaotic nature of its motion is demonstrated. The e-folding timescale for the divergence of initially very similar orbits is approximately 70 years. The sensitivity of the dynamics on initial conditions is also evident in the self-similarity character of the survival time and frequency analysis maps in the vicinity of Halley’s orbit, which indicates that, on average, it is unstable on a timescale of hundreds of thousands of years. The chaotic nature of Halley’s present day orbit implies that a precise determination of its motion, at the level of the present day observational uncertainty, is difficult to predict on a timescale of approximately 100 years. Furthermore, we also find that the ejection of Halley from the solar system or its collision with another body could occur on a timescale as short as 10,000 years.

Key words: Solar System: dynamics – Chaos: theory – Comets: general.

1 INTRODUCTION

Halley’s comet is probably one of the most studied and therefore best known minor bodies in the Solar System to date. Historical records of comet Halley start in the year 240 B.C. (Kiang 1972), but it is until its last perihelion passage, in February 1986, when it became visible to modern telescopes and even physically accessible to spacecrafts, that the amount of available data has hugely grown. In particular, the parameters of its retrograde orbit, semimajor axis, \(a\), and eccentricity, \(e\), are since then determined with a precision of the order of \(10^{-6}\) (\(\sigma_a = 7.71 \times 10^{-9}\) and \(\sigma_e = 9.1 \times 10^{-8}\) respectively, where \(\sigma\) is standard deviation and \(q\) the perihelion distance, according to Landgraf 1986).

The origin of Halley’s comet has been a matter of discussion for decades. One of the likely sources of Halley-type comets (i.e. short-period comets with Tisserand parameters \(T < 2\) with respect to Jupiter, periods \(20 < P < 200\) years and semimajor axes less than 40 AU), seems to be the Oort cloud (Fernandez 1980). Indeed, since giant planet perturbations of trans-Neptunian objects in the vicinity of the Kuiper Belt, will not generally drive comets to the observed inclinations on Halley-type comets, so the origin of their orbits must be different. In their computations, Fernandez (1980), and also Duncan et al. (1988) and Quinn et al. (1990), show that the dynamical evolution of comets from the Oort cloud toward the inner solar system, is a probable origin of the random inclination of Halley-type comets, since they tend to preserve their random orbital inclinations. On the other hand, Levison et al. (2001), noticed that the inclination distribution of Halley-type comets is not isotropic, meaning that they could not be readily explained as origi-
nating from a rather isotropic source as the Oort cloud, but from a flattened component or inner disc-like portion of the Oort cloud. Earlier, Duncan & Levison (1997) had investigated a viable origin of Jupiter family comets coming from the already known flattened scattered disc. Lu et al. (1997). They found that approximately 1% of their scattered disc objects survived the full 4 Gyr simulation, where some of these reach semimajor axis of thousands of AU. In a more recent paper, Levison et al. (2006) show that these objects, once they reach a semimajor axis of the order of 10$^4$ AU are rapidly reduced in their perihelia due to galactic tides. If just 0.01% of these comets then evolve, due to giant planet interactions, onto Halley-type orbits, the resultant statistical orbital distribution is consistent with observations.

The physical properties of comet Halley are also known as never before (for a review see Mumma & Charnley 2011). A'Hearn et al. (1995) have shown that Halley-type comets differ from Jupiter Family comets on their average coma carbon abundances, suggesting a different origin for both families. Comet Halley seems to lose mass at an approximate rate of 0.5% every perihelion passage (Whipple 1951; Kresak & Kresakova 1987). At this rate, the comet might be severely diminished or even vanished in about 15,000 years. Halley’s comet has been spreading particles that settle down on the known Orionid stream for thousands of years (Sekhar & Asher 2013).

From the dynamical point of view the evolution of this comet has also been profusely studied through numerical integrations (Yeomans & Kiang 1981; Dvorak & Kribbel 1990; Levison & Duncan 1994; Bailey & Emel’Yanenko 1996; van der Helm & Jeffers 2012; Sekhar & Asher 2013). Detailed calculations show for example that Halley’s comet has been trapped in the past by resonances with Jupiter and secular perturbations, such as the Kozai resonance and other secular resonances (Quinn et al. 1990; Bailey & Emel’Yanenko 1996; Thomas & Morbidelli 1996) affecting considerably its long term dynamical evolution (Sekhar & Asher 2013). From numerical simulations under various approximations, Halley’s comet and other Halley-type comets seems to be intrinsically chaotic (Petrosky 1986; Froeschle & Gomezi 1988; Chirikov & Vecheslavov 1989).

In the pioneering work of Chirikov & Vecheslavov (1989) they used a simple analytical model based on the results of integrations by Yeomans & Kiang (1981) which resembles the perihelion passage over the past ~3000 years, in order to obtain a measure of the chaotic behavior of the Halley’s comet. They found that Jupiter plays the major role in driving the local instabilities of the motion while Saturn contributes an order of magnitude less in the random changes suffered by the comet. Apart from these early work, not many attempts have been made to obtain a reliable quantification of the chaotic behavior of Halley’s comet with modern numerical integration tools. This means that a standard Lyapunov exponent or Lyapunov time has not been established with confidence for Halley by means of direct numerical integrations.

In this work we explore numerically the evolution of test particles in the surrounding phase space of Halley’s comet in order to determine the chaoticity of this region quantified through frequency analysis maps and a simple auxiliary visual tool that we have called “survival time maps”. Additionally we compute, for the first time directly from numerical integration, the Lyapunov exponent for Halley’s comet in its current orbit considering its observational uncertainty. By finding a positive exponent we have determined that current known orbit is actually chaotic. We also estimated the e-folding time scale for the separation of neighbouring orbits. Finally we provide an estimation of sojourn time in the solar system for the comet in a qualitative similar manner to that used by Chirikov & Vecheslavov (1989) finding similar results.

The outline of this paper is as follows: in Section 2 we describe the three different analysis techniques to determine the chaotic nature of the dynamics of Halley’s comet and the corresponding numerical simulations done for this purpose. In Section 3 we show results of the numerical simulations while discussing these results in Section 4. Finally we give our conclusions in Section 5.

2 METHODS AND SIMULATIONS

We have used 3 different analysis methods to determine the chaotic nature of Halley’s comet dynamics. We describe these in the next subsections.

2.1 Survival time maps

Survival time maps (STM hereafter) are an auxiliary tool to visualize the absolute stability, against ejection from the system or collision with other bodies in orbits of a given phase space region, corresponding in our case to the plane of semimajor axis, $a$, vs eccentricity, $e$, surrounding Halley’s current position in this plane. In the STM a color code indicates the total survival time in the simulation according to the particle’s initial condition given by its position in the phase space plane. In order to explore this we have used the MERCURY integrator package developed by Chambers (1999) to integrate the orbits of clones of comet Halley as test particles in a Solar System N-body simulation. The clones were generated in such a way that their initial conditions cover a region on the $a- e$ plane with different zoom levels.

We performed 3 different simulations to construct STMs corresponding to different ranges of $a$ and $e$. In the first simulation a) semimajor axis, $a$, ranges from 15 to 20 AU and $e$ ranges from 0 to 1 with both intervals divided uniformly in 100 values each, giving a total of $10^4$ particles. In the second simulation b) $a$ ranges from 17.385 to 18.385 AU divided uniformly in 100 values and $e$ ranges from 0 to 1 for a total of 3300 particles (33 values in $e$). In the third simulation c) the domain of $a- e$ space covers the observational error according to Landgraf (1986) both in a ($a_{\pm} \sim 10^{-6}$ AU) and $e$ ($e_{\pm} \sim 10^{-5}$) sampled with $10^5$ total particles. In all cases the angular elements, argument of pericentre, $\omega$, longitude of the ascending node, $\Omega$, mean anomaly, $M$ and inclination, $i$, were set the same as that of Halley’s comet as given by the Horizons web site, at the beginning date of the simulations, Oct. 1, 2012 in heliocentric coordinates. A final simulation d) was performed varying semimajor axis and inclination and letting the 4 remaining orbital parameters, including eccentricity, be the same as for comet Halley

1 http://ssd.jpl.nasa.gov/horizons.cgi
on the same date, and covering the observational error in $a - i$ phase space (with $\sigma_i \sim 10^{-5}$ deg, according to Landgraf 1986). We divide both ranges in 100 uniformly spaced values for a total of $10^4$ particles.

For a) and b) simulations the RADAU integrator from Everhart (1985) as implemented in the MERCURY package was selected for the integrations with a tolerated accuracy parameter of $10^{-10}$ and an initial time step of 20 days. Simulations c) and d) were carried on using the optimized Bulirsch-St"oer integrator implemented in the MERCURY package (BS2) with a tolerated accuracy parameter of $10^{-12}$ and an initial time step of 1 day. All simulations spanned over $10^6$ years and include as N-bodies the planets except Mercury in order to avoid incorporating relativistic effects. We also include 5 Dwarf Planets (all but Sedna which orbits far beyond the 100 AU sojourn limit imposed in the simulation) and 5 of the greatest minor bodies (Orcus, Quaoar, Varuna, Ixion and 2002 AW197) not yet classified as dwarf planets but likely to be in the near future.

According to the sojourn criterion for STM, the particle is still part of the simulation at the end of the $1 \times 10^6$ years of it, regardless the changes in orbital parameters of each particle. So a particle is lost when its $a$ becomes greater than 100 AU, collides with the Sun or with another planet.

2.2 Frequency analysis map

We have used the frequency analysis introduced by Laskar (1993, 1990) to quantify the chaotic behavior of particles around current position of Halley’s comet. According to Correia et al. (2005), Morbidelli (2002), Laskar (1993), the difference ($D$) in the value of the fundamental frequency of the motion ($n$) of a particle under consideration, obtained over two consecutive and equal time intervals ($T$), is a measure of the secular stability of its trajectory and a reliable indicator of chaos. In this context we have explored the $D$ parameter, or diffusion parameter, for clone particles covering a grid of $a$ vs $e$ phase space. For each particle in the map we calculate using a lomb-scargle analysis (Scargle & Black 1980) the mean frequencies of the mean longitude of the particle, $\lambda(t) = M(t) + \omega(t) + \Omega(t)$, in the adjacent time intervals from 0 to $2.5 \times 10^5$ yr and from here to $5 \times 10^5$ yr. The mean motion $n$ in each interval is defined as the amplitude of the mean frequency of the series over $2\pi$ in that interval and then, according to the definition the $D$ parameter is calculated as:

$$D = \frac{|n - n'|}{T}$$

If $D$ is close to zero, this is if $n \approx n'$, then it means that the orbit is stable. Otherwise high values of $D$ reflect important changes in the motion of the particle related to unstable or even chaotic dynamics.

Figure 1. Survival Time Map for orbits with semimajor axis up to a few AU from comet Halley’s orbit, all with the same inclination. The color scale indicates the survival time as a function of initial semimajor axis and eccentricity of the orbit. Halley’s present day orbit is indicated by the black circle. The thick black and white lines correspond to particles crossing the orbit of Saturn and Jupiter, respectively. Also shown with vertical yellow lines are the strongest mean motion resonances in the region, 1:2 with Saturn and 1:1 with Uranus (Gallardo 2006).
We then performed one $5 \times 10^5$ yr simulation using the RADAU integrator from the MERCURY package with a toleration accuracy parameter of $10^{-10}$ and an initial time step of 20 days. We cover a semimajor axis span of just 1 AU from 17.385 to 18.385 AU, this is $\pm 0.5$ AU from the current semimajor axis of comet Halley, and from 0 to 1 in $e$. We have used 100 uniformly spaced particles in $a$ and 33 particles to uniformly cover $e$, in order to save computational time as we are bounded to record an extensive output cadence of parameters evolution.

2.3 Lyapunov Exponent

We have calculated the Lyapunov exponent for 6 neighbouring orbits around the current orbit for comet Halley separated by the present day observational uncertainty. The 6 orbits are the result of varying initial position by $\pm 10^{-6}$ AU in each cartesian axis X,Y,Z in a heliocentric reference frame, while maintaining the velocity at the same value of the fiducial orbit. This means that each one of the 6 orbits differs initially from the fiducial one by a distance of $10^{-6}$ AU distance, $\delta(0)$.

According to Morbidelli (2002) in a numerical experiment one can define a small initial separation distance between an arbitrary orbit and the fiducial one; the modified orbit is characterized by the vectors $\delta q(0)$ and $\delta p(0)$, where $q$ and $p$ are generalized coordinates in the problem. One can compute their evolution, $\delta q(t)$, $\delta p(t)$ over some constant time interval $T$ after which one can measure the separation distance of the modified orbit from the fiducial one and define the quantity:

$$s_j = ||\delta q(T), \delta p(T)||/||\delta q(0), \delta p(0)||$$

(2)

which we use to determine a new initial separation for the modified orbit in the next step of the integration as $\delta q(0) = \delta q(T)/s_j$ and $\delta p(0) = \delta p(T)/s_j$. In this manner, as was demonstrated by Benettin et al. (1980), the Lyapunov exponent, $\mathcal{L}$, can be calculated in an iterative process according to:

$$\mathcal{L} = \lim_{t \to +\infty} \frac{1}{T} \sum_{i=1}^{T} \ln s_j$$

(3)

where $\mathcal{L}$ is independent of the choice of $T$. In our case, as we are using only an initial separation in distance, i.e. in $q$, it follows that $\delta q(0) = \delta(0) = 10^{-6}$ AU for all our $i$ steps.

In this context we have used the Bulirsch-Stoer integrator from the MERCURY package in order to perform a set of 30 simulations, extending over 100 years each in succession, of Halley’s fiducial orbit plus 6 orbits around it. The total simulated time is 3000 years, starting from Oct. 1st, 2012, in order to avoid the first close encounter of the fiducial Halley with Jupiter. These close encounters produce random variations in the orbit that are capable of modifying the separation of the orbits by several AU on a very short timescale (see figure 2 for details). The initial time step was chosen to be 3 days with an accuracy parameter of $10^{-12}$ in order to obtain high enough precision to calculate a real separation of the orbits, resulting from dynamical effects, and not a computational artifact. In this manner we have calculated the Lyapunov exponent for each one of the 6 orbits previously defined using the iterative process just as described, and a maximum Lyapunov exponent for the full set of orbits.

3 RESULTS

In order to assess the stability of the orbit of comet Halley (and possibly other Halley-type comets), and to characterize its dynamics, we have carried out numerical simulations as previously described. The results of these studies are presented in this section.

3.1 Survival Time Maps

Figure 1 shows the survival time map (STM) for orbits with initial semimajor axis extending from 15 to 20 AU and eccentricity less than 1. The test particles sampling these orbits all have the same inclination, corresponding to the present day inclination of comet Halley, 162.18042 deg according to the Horizons website for the beginning date of our simulations. For comparison, we also show in the figure the nominal orbit of comet Halley (black dot).

Evident in Figure 1 is the fact that a wide range of orbits crossing that of Jupiter are strongly unstable, with a typical lifetime, for such orbits, of $5.8 \times 10^7$ years. Note that this lifetime is a lower limit for the average lifetime of small bodies in this region of $a-e$ phase space, as the survival time of stable orbits may be much greater than the 1 Myr timescale considered in our simulations. This result is also seen in Figure 2 which shows the STM zoom-in on the semimajor axis to a range extending within 0.5 AU of Halley’s orbit. In this case the typical survival time is $5.6 \times 10^9$ years which is consistent with the result from the previous map.

An important feature streaming from the comparison of the sequence of increasing zoom Figures 1, 2, and 3 is the last one corresponding to a semimajor axis and eccentricity range covered by the observational uncertainty for comet Halley (Landgraf 1986) is the fact that the “structure” of the stable/unstable zones is apparently self-similar likely fractal in nature. The determination of the fractal dimension for these structures is beyond the scope of the present study.
This result is already suggestive of strongly chaotic dynamics for small bodies in the region, as orbits within a very small neighborhood in $a-e$ space can have extremely different behaviour regarding their stability properties.

A similar result is found in the structure of the stable/unstable zones, where we plot them as a function of orbit inclination of the test particles. Figure 4 shows the STM for orbits with the present day eccentricity of comet Halley, 0.967623, again according to the Horizons website, but different inclinations spanning over the $10^{-5}$ deg observational uncertainty in this parameter (Landgraf 1986).

In both $a-e$ and $a-i$ phase space stability maps extending over the observational uncertainty (figures 3 and 4) the mean survival time for particles in the region is found to be $3 \times 10^5$ yr. This value is then a reliable estimate of the stability time for the current known orbit of comet Halley in the Solar System.

3.2 Frequency analysis maps

A more detailed analysis of the stability of orbits within 0.5 AU of Halley’s orbit (with its nominal inclination) is shown in Figure 5, where contours of the rate of change in the mean motion of particles initially on a given orbit, over the extent of a $5 \times 10^5$ year simulation, are plotted. Red and orange coloured areas in this figure denote regions of phase space where orbital parameters, particularly the mean motion, change by more than 5% over this timescale. While these orbits are not unstable in the sense of the results presented in the STMs of section 3.1 they do change significantly and may lead to a different dynamical evolution of particles on these zones.

As in the STM corresponding to the same zoom level (Figure 2), a region of strong orbital variation is shown in Figure 5 for orbits with perihelia that cross the orbit of Jupiter, indicated by the (nearly) horizontal, dashed white line. A large proportion of orbits above this line exhibit significant variation over the length of the simulation. In addition, a pair of families of unstable orbits are also found, corresponding to those particles that cross the orbit of Saturn and do not cross that of Jupiter, and to those particles with aphelia large enough to cross the orbit of Uranus but not to cross Saturn’s orbit. These features are not found on the basis of the STMs previously presented, as they do not correspond to strictly unstable orbits. A mean value for the $D$ parameter in each region is found to be as follows:
Figure 5. Frequency analysis map for orbits with comet Halley’s inclination but differing in semimajor axis (by up to 0.5 AU) and eccentricity. Red colored regions correspond to the most unstable orbits. Horizontal lines indicate orbits with perihelia crossing Jupiter’s orbit (white dashed), Saturn’s orbit (black dashed) and with aphelia crossing the orbit of Uranus (yellow dashed). The black vertical line indicates the position of the 2:5 MMR with Saturn, the stronger one in the region (Gallardo 2006).

$D \sim 10^{-7.8}$ for particles crossing Jupiter’s orbit, $D \sim 10^{-9.2}$ for particles crossing Saturn’s orbit but not Jupiter’s, and $D \sim 10^{-10.1}$ for particles that reach the orbit of Uranus but not Saturn’s.

A simple numerical estimation of the time scale for orbits to evolve significantly according to the $D$ parameter, $\tau_D$, can be found from the definition of diffusion parameter given in equation 1, from which it is clear that $\tau_D \sim n/D \sim 1/\Pi$ years, where $\Pi$ is the orbital period. As $P \sim 75$ years across this whole region of phase space, the corresponding time scale in the three regions mentioned above is $\sim 8 \times 10^5$, $\sim 2 \times 10^7$ and $\sim 2 \times 10^8$ years, respectively. The timescale $\tau_D$ can be understood as the time required for the particle to suffer major changes in its orbit. The instability time obtained from STM and Laskar frequency analysis is roughly consistent for particles crossing Jupiter’s orbit. In addition, these estimations are also consistent with the fact that particles not crossing Jupiter’s orbit can survive the whole 1 Myr simulation, as found from the STMs.

Islands of stability/instability (blue/red zones) exist throughout the region mapped in Figure 5, where no relation with major resonances or other bodies in the solar system are easily identified. Although we have not carried out simulations to accomplish the Laskar frequency analysis at different zoom levels, the rapidly changing structure of the stable/unstable regions of phase space suggests a strong dependence on the initial conditions for the dynamical evolution of small bodies in the region.

3.3 Lyapunov Exponent

The strong sensitivity on initial conditions depicted by our results in connection to the STMs and the Laskar stability analysis, is further illustrated in Figure 6 which shows the separation distance between 2 orbits and the nominal solution for comet Halley. One of the orbits considered (red line in Figure 6) is for a particle starting from initial coordinates with the $X$ component of its position greater than the nominal solution by an amount $\delta_0 = 10^{-6}$ AU, the reported observational error for the ephemerides of comet Halley. The other orbit differs initially from Halley’s nominal solution in the same amount but in the $Y$ coordinate (blue line in Figure 6). Both orbits rapidly diverge from the nominal solution.

As illustrated in Figure 6 there are 2 processes leading to an increase in the separation of neighbouring orbits. A gradual increase, as that occurring up to 3,400 yr during which the separation increases by approximately 5 orders of magnitude due to the effect of distant encounters with other bodies in the system. And a phase of abrupt change in orbital parameters due to a close encounter of the particle with Jupiter. During this phase, the dynamical evolution is particularly sensitive to the initial conditions, as can be seen from the widely different separation for the orbits considered following the close encounters. These events, during which the particle approaches the planet to within 3 Hill radii ($R_H = (M_p/3M_\odot)^{1/3}$), are marked by black vertical
The formal definition of the term chaos is matter of debate. In this paper we follow Strogatz et al. (1994) who proposes a working definition of chaotic behaviour as “aperiodic long-term behaviour in a deterministic system that exhibits sensitive dependence on initial conditions.” Our system is clearly deterministic and the sensitive dependence on initial conditions is usually measured in terms of the Lyapunov exponents which quantify the exponential rates at which neighbouring orbits diverge (or converge) as the system evolves in time. If such system exhibits at least one positive Lyapunov exponent, then it is said that the dynamics of the system is chaotic.

To formally assess the chaotic behaviour of comet Halley we have computed the maximum Lyapunov exponent for particles on Halley’s nominal orbit, with respect to variations in the orbital parameters at the level of their current observational uncertainty. The resulting Lyapunov exponent is shown in Figure 7. The corresponding Lyapunov timescale is approximately 70 yr. It is important to point out that this timescale corresponds to the time scale for orbits to diverge by 1 part in $10^6$, which is approximately the observational uncertainty.

4 DISCUSSION

In this section we discuss in more detail some of the implications of our results and analyse the effect of some of the assumptions and methodology used in our study.

4.1 Predictability of Halley’s orbit

One of the important consequences of the chaotic nature of Halley’s orbit is the possibility of making long term predictions of its motion. On one hand, we can interpret our results of section 3.3 in terms of the so-called Lyapunov timescale, $\tau_L$, defined as the inverse of $L$, corresponding to the e-folding timescale of the separation of initially neighbouring orbits. The fact that $\tau_L \approx 70$ yr, is an indication that over such timescale two initially separated in semimajor axis, eccentricity or inclination by 1 part in $10^6$, diverge from one another by a factor of $e$ in separation. This precludes the possibility of making high precision predictions, at a level similar to that of the present-day observational uncertainty in its orbital parameters, the “short” term, $\sim 70$ yr, evolution of the orbit of comet Halley, i.e. from one perihelion passage to the next.

On the other hand, the fact that the contours in the survival time maps, which reflect the structure of regions of dynamical stability in a $e-i$ phase space, have an apparently self-similar or fractal nature (commonly found in chaotic dynamical systems) as can be seen from comparing Figures 1 and 2 corresponding to STMs at different “zoom” levels around Halley’s orbit, also suggests that the survival time of the comet can not be accurately determined. As evident in Figure 2, closely neighbouring orbits (within the observational uncertainty) can have widely different stability properties. The time over which Halley remains stable can range from $10^3$ to $10^6$ yr (or more), with the average value in a domain ranging over the observational uncertainty of $a$ and $e$ being approximately $3 \times 10^5$ yr, at the zoom level shown in Figure 5. Furthermore, this result implies that an orbit, such as Halley’s nominal solution, apparently located in a stability island on a given “zoom” level, may turn out to be much more unstable as we zoom-in around it. A similar conclusion is reached from the distribution of values for the Laskar $D$ index which measures the change in the orbit and has also an apparent “fractal” structure as shown in Figure 6.

Another illustration of the strong dependence of Halley’s motion on its initial position and velocity, and of the difficulty in predicting its future motion, is given in Figure 5 which shows the evolution of the orbital parameters $a$, $e$, $i$ and the Tisserand parameter with respect to Jupiter, $T_J$, for 7 particles located at extreme values of these parameters, within the box defined by the observational uncertainty for Halley’s orbit. Over a timescale of $10^5$ years or more, even orbits starting out with a difference in orbital parameters of less than the observational uncertainty, can have widely different outcomes.
4.2 On the fate of Halley’s orbit

Halley-type comets as a class (HTCs hereafter), are characterized by orbital periods less than 200 yr and Tisserand parameter with respect to Jupiter smaller than 2. In the present study we have assumed that comet Halley is already on its present day orbit and followed its subsequent evolution, not much can be said about the origin of its orbit.

Nevertheless, the results presented in Section 3.1 indicate that the survival time for objects in Halley-type orbits ranges from $10^4$ to $10^6$ yr, implying that a body in a Halley-like orbit can remain on it, or on one very similar to it, for no more than approximately 1 Myr. This, together with the fact that Halley is still an active comet, which suggests that it has probably been in its present, short period and small perihelion orbit, less than $10^4$ yr, implies that Halley will probably remain on it for at least a similar timescale.

In order to estimate the probability that comet Halley remains in a stable orbit similar to the one it has in the present day, we have computed the fraction of particles remaining in the simulations as a function of time. Figure 9 shows the number of particles colliding with the Sun or ejected from the solar system, for a collection of 10,000 particles started from initial conditions within a domain in $a - e$ phase space defined by the observational uncertainties in these parameters, as we did in the construction of the STM of Figure 3. We find that $10^3$ yr after the start of the simulation, 2% of all particles have collided with the Sun and approximately 10% have been ejected from the system. Hence, we could estimate that comet Halley has approximately a 90% chance of surviving at least $10^5$ yr in the solar system. However, in a similar manner we can estimate that by $10^6$ yr, there is a 28% chance that it will collide with the Sun and a 50% chance that it will be ejected from the system.

One final piece of interesting information to be gathered from our results is that the orbits consistent with comet Halley (within the observational uncertainty) that are not lost from the system due to ejection or collision with the Sun, have a tendency to evolve conserving a Tisserand parameter with respect to Jupiter, as seen in Figure 10 which shows the evolution of $a$, $e$, $i$ and $T_J$ for 200 randomly chosen particles in our simulations. Again, the initial orbit of all these particles is consistent with Halley’s present day orbit, and most of the particles in the ensemble are seen to evolve as follows:

- A rapid increase, on a timescale of a few thousand years, in the spread of $a$, $e$ and $i$ representing the diversity of possible orbits to be followed by Halley if it survives.
- A large fraction of orbits evolve into periods greater than 200 years, i.e. they are no longer short period comets.
- The average eccentricity of the ensemble increases slightly on a timescale of $10^5$ yr.
- The inclination of most possible orbits consistent with Halley’s current orbit tends to evolve into lower inclination, almost polar orbits.

Although there is a wide spread in possible outcomes for comet Halley as we have discussed in previous sections, it seems likely that, if it survives for more than $10^5$ yr, its orbit will evolve into a more eccentric and less inclined orbit.

4.3 Origin of chaos

It is generally believed that chaotic dynamics of small bodies in the solar system, such as asteroids in the Main Belt or comets in the Kuiper Belt, results from the overlapping of mean-motion and secular resonances with the major bodies in the solar system (for a review of the topic see Malhotra 1998, an references therein).

Comet Halley however, is not trapped in any of the known strong resonances with the outer planets, and this is not expected due to the comet’s high eccentricity and inclination. So the origin of the chaotic character of its orbit is not straightforward to identify. Furthermore, it appears that the chaotic motion we have identified is not directly related to the close encounters of the comet with the giant planets, particularly with Jupiter. This is a well known source of strong chaos in the system which however, develops on
a greater timescale of many orbital periods. This is illustrated in Figure 6, where the first close encounters of the comets with Jupiter are shown to occur after approximately 50 orbital periods.

The weak chaotic behaviour we have characterized by the Lyapunov exponent analysis of section 3.3 is not directly related to these close encounters, as it develops long before the first close encounter occurs. We intend to analyse this issue in more detail in a future work.

4.4 Neglected dynamical effects

In the present study we have neglected the effect of non-gravitational forces known to affect the dynamics of active comets near perihelion (Marsden 1968). The effect on comet Halley is particularly well known, but along with this is the fact that the estimated time in which the gaseous jet forces are actively present in comets, is just a few hundred perihelion passages, which corresponds to a few thousand years of the dynamical life-time of Halley-type comets. As we are interested in the long-term dynamical evolution of Halley’s comet, we decided to neglect this effect in the simulations presented in this work.

To test the previous assumption, we carried out a simulation of the long-term evolution of Halley’s comet including non-gravitational forces as prescribed in the Horizons website, to construct a STM for the conditions studied in Figure 1. In doing that we are highly overestimating the importance of this effect because we implicitly assume that the comet is active for the whole 1 Myr simulation. Nevertheless, even in this extreme scenario, the results as far as survival expectancies are statistically equivalent to the one without non-gravitational forces presented in the previous sections.

Finally, it is worth mentioning that the results we have obtained are strictly related with the gravitational interaction of test particles with the major solar system bodies, except for Mercury. The mass of Mercury could be added to that of the sun, but this adds just $\sim 1.6 \times 10^{-7} M_\odot$ to the total mass of the Sun, therefore, in the long-term, there is not a measurable difference for Halley’s comet fate. To correctly account for the influence of Mercury in a simulation of the solar system, it would require a relativistic treatment of the gravitational interaction, which is not possible with the short time-scale of the simulations, compared to the solar system life-time, we considered this a second order effect in our analysis to be considered in future studies.

5 CONCLUSIONS

We have carried out a series of numerical simulations aimed to assessing the dynamical stability of Halley’s comet. Three types of analysis are carried out to demonstrate and characterize the chaotic behaviour of the Halley’s orbit on the basis of Survival Time Maps, Frequency Analysis Maps and a direct calculation of the Lyapunov exponent.
From our analysis of Survival Time Maps we conclude that it is common for Halley-like orbits to be unstable up to the point of being ejected from the system, or colliding with another solar system body. We also find that the long term evolution of comet Halley, even with the high precision of its observed orbital parameters, is difficult to predict on account of the strong dependence on the initial conditions. The timescale for particles in orbits differing in less than today’s observational uncertainties in $a$ or $e$, can range by at least 2 orders of magnitude, from $10^4$ to $10^6$ yr approximately. The average timescale for the survival of particles in a range of $a$ and $e$ within the observational constraints is $3 \times 10^5$ yr.

Based on the Laskar stability analysis in a neighborhood of Halley’s comet, we conclude that even stable orbits, in which the particles are not ejected or collide with a solar system body, can change significantly on a timescale of millions of years. Again, the precise determination of what will be the fate of comet Halley is hindered by the strong dependence on the initial conditions, even within today’s observational uncertainties at the level of 1 part in $10^6$. On timescales of more than $10^5$ yrs, it seems likely that the Halley, if it does not collide with the Sun or is ejected from the solar system, will evolve into a higher eccentricity, lower inclination orbit.

Finally, we have computed the Lyapunov exponent for the present day Halley’s orbit and a series of other orbits differing in $a$ and $e$ by an amount equivalent to the observational uncertainty. We have found that $\mathcal{L}$ is greater than zero with a value of approximately $10^{-2}$, indicating that the orbit is indeed chaotic. The corresponding timescale for the prediction of Halley’s orbit to within present day observational constraints is less than 100 years, suggesting that the orbit of Halley’s comet can not be accurately predicted for timescales much greater than this. An important finding in our work is that the chaotic behaviour is not related to close encounters of Halley with any of the planets in the solar system, nor to the overlap of any known system of resonances. The origin of the chaos in such eccentric orbits is a subject to be explored in more detail in future studies.

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