WELL-POSEDNESS AND SCATTERING FOR THE BOLTZMANN EQUATIONS: SOFT POTENTIAL WITH CUT-OFF

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Abstract. We prove the global existence of the unique mild solution for the Cauchy problem of the cut-off Boltzmann equation for soft potential model $\gamma = 2 - N$ with initial data small in $L^N_{x,v}$ where $N = 2, 3$ is the dimension. The proof relies on the existing inhomogeneous Strichartz estimates for the kinetic equation by Ovcharov [16] and convolution-like estimates for the gain term of the Boltzmann collision operator by Alonso, Carneiro and Gamba [1]. The global dynamics of the solution is also characterized by showing that the small global solution scatters with respect to the kinetic transport operator in $L^N_{x,v}$. Also the connection between function spaces and cut-off soft potential model $-N < \gamma < 2 - N$ is characterized in the local well-posedness result for the Cauchy problem with large initial data.

1. Introduction and Results

With the first appearance of Strichartz estimates for the kinetic equation in the note of Castella and Perthame [9], the Strichartz estimates have been applied to proving the existence of global weak solution with small initial data assumption for the kinetic equation, Bournaveas et al. [8] for a nonlinear kinetic system modeling chemotaxis and Arsénio [4] for the cut-off Boltzmann equation. We note that the result of Arsénio holds only for non-conventional collision kernel whose kinetic part is $L^p$ integrable for some $p$ depending on dimension and the weak solution is not unique.

We accomplish this approach to some extend for the case of the Boltzmann equation by proving the global existence of the unique mild solution for the Cauchy problem of the cut-off Boltzmann equation for soft potential model $\gamma = 2 - N$ with initial data small in $L^N_{x,v}$ where $N = 2, 3$ is the dimension. The proof relies on the existing inhomogeneous Strichartz estimates for the kinetic equation by Ovcharov [16] and convolution-like estimates for the gain term of the Boltzmann collision operator by Alonso, Carneiro and Gamba [1]. The global dynamics of the solution is also characterized by showing that small global solution scatters with respect to the kinetic transport operator in $L^N_{x,v}$. Also the connection between function spaces and cut-off soft potential model $-N < \gamma < 2 - N$ is characterized in the local well-posedness result for the Cauchy problem with large initial data.
To state the results precisely, we begin with the introduction of the necessary notations. We consider the Cauchy problem for the Boltzmann equation

\begin{equation}
\begin{aligned}
\partial_t f + v \cdot \nabla_x f &= Q(f, f) \\
f(0, x, v) &= f_0(x, v)
\end{aligned}
\end{equation}

in \((0, \infty) \times \mathbb{R}^N \times \mathbb{R}^N, N = 2, 3\), where the collision operator

\[Q(f, f)(v) = \int_{\mathbb{R}^N} \int_{\omega \in S^{N-1}} (f'(v') - f_*) B(v - v_*, \omega) d\Omega(\omega) dv_*\]

and \(d\Omega(\omega)\) is the solid element in the direction of unit vector \(\omega\). Here we have used the abbreviations \(f' = f(x, v', t)\), \(f'_* = f(x, v'_*, t)\), \(f_* = f(x, v_*, t)\), where the relation between the pre-collisional velocities of particles and after collision is given by

\[v' = v - [\omega \cdot (v - v_*))]\omega, \quad v'_* = v_* + [\omega \cdot (v - v_*)]\omega, \quad \omega \in S^{N-1}\]

The cut-off soft potential collision kernel takes the form

\begin{equation}
B(v - v_*, \omega) = |v - v_*|^\gamma b(\cos \theta), \quad 0 \leq \theta \leq \pi/2
\end{equation}

where

\[-N < \gamma < 0, \quad \cos \theta = \frac{(v - v_*) \cdot \omega}{|v - v_*|}\]

and the angular function \(b\) satisfies the Grad’s cut-off assumption

\begin{equation}
\int_{S^N} b(\cos \theta) d\Omega(\omega) < \infty.
\end{equation}

When \(\gamma = 0\), (1.2) is called the Maxwell molecules. For our purpose, we introduce the mixed Lebesgue norm

\[\|f(t, x, v)\|_{L_x^p L_v^q L_t^r} \]

where the notation \(L_x^p L_v^q L_t^r\) stands for the space \(L^p(\mathbb{R}; L^q(\mathbb{R}^N; L^r(\mathbb{R}^N)))\) and it is understood that we are using \(L_x^p(\mathbb{R}) = L_x^p([0, \infty))\) for the well-posedness problem which can be done by imposing support restriction to the inhomogeneous Strichartz estimates. We use \(L_{x,v}^a\) to denote \(L_x^a(\mathbb{R}^N; L_v^a(\mathbb{R}^N))\).

We also need to define the meaning of the solution scatters with respect to kinetic transport operator in our result. It seems strange to mention the notion of the scattering of the solution of the Boltzmann equation since it involves Boltzmann’s H-theorem (see for example [5] for more discussion). From the mathematical point of view, the solution scatters implies that the hyperbolic part of the equation dominates the solution all the time and the definition of scattering thus help us to understand the large time behavior of solution, see also remark 4 after Theorem 1.1 and Corollary 1.2. Here we say that a global solution \(f \in C([0, \infty), L_{x,v}^a)\) scatters in \(L_{x,v}^a\) as \(t \to \infty\) if there exits \(f_+ \in L_{x,v}^a\) such that

\begin{equation}
\|f(t) - U(t)f_+\|_{L_{x,v}^a} \to 0
\end{equation}

where \(U(t)f(x, v) = f(x - vt, v)\) is the solution map of the kinetic transport equation

\[\partial_t f + v \cdot \nabla_x f = 0.
\]

We note that the operator \(U(t)\) is time reversible and thus the scattering problem is still well-defined if we consider \(t \to -\infty\) or \(t\) goes from \(-\infty\) to \(\infty\). Since the results for these scattering problems are similar, we only present the case \(t \to \infty\).
The main result of this paper is the following.

**Theorem 1.1.** Let \( N = 2 \) or \( 3 \) and \( B \) defined in (1.2) satisfies (1.3) and \( \gamma = 2 - N \). The Cauchy problem (1.1) is globally wellposed in \( L^N_{x,v} \) when the initial data is small enough. More specifically, there exists \( R > 0 \) small enough such that for all \( f_0 \) in the ball \( B_R = \{ f_0 \in L^N_{x,v}(\mathbb{R}^N \times \mathbb{R}^N) : \| f_0 \|_{L^N_{x,v}} < R \} \) there exists a globally unique mild solution

\[
\begin{align*}
    f & \in C([0, \infty), L^N_{x,v}) \cap L^q([0, \infty], L^r_x L^p_v) \\
    \text{where the triple } (q, r, p) & \text{ lies in the set (1.5) }
\end{align*}
\]

\[
\{ (q, r, p) | \frac{1}{q} = \frac{N}{p} - 1, \frac{1}{r} = \frac{2}{N} - \frac{1}{p}, \frac{1}{p} < \frac{1}{N} < \frac{N+1}{N^2} \}.
\]

The solution map \( f_0 \in B_R \subset L^N_{x,v} \rightarrow f \in L^q_t L^r_x L^p_v \) is Lipschitz continuous and the solution \( f \) scatters with respect to the kinetic transport operator in \( L^N_{x,v} \).

Some comments about this result are given in the following.

1. The related earlier work using the the iterative scheme proposed by Kaniel and Shimbrot [13, 11, 6, 17, 2, 3] or fixed point argument [10] all require pointwise upper bound. One of advantages of the current approach is that it requires only the initial date is small in \( L^N_{x,v} \).

2. Arsénio [4] noted that \( L^3_x \) appeared in the well-posed result of the Boltzmann equation matches the critical space of Navier-Stokes equation [14]. It should be interesting to see how are they related. On the other hand, the generalized homogeneous Strichartz estimate [16] reads

\[
\| U(t) f_0 \|_{L^q_t L^r_x L^p_v} \leq C \| f_0 \|_{L^b_x L^c_v},
\]

where

\[
\frac{1}{q} + \frac{N}{r} = \frac{N}{b}, \quad HM(p, r) = HM(b, c) \overset{\text{def}}{=} a
\]

\[
p < b \leq a \leq c < r,
\]

with the definition of \( HM(p, r) \) given in Definition 2.1 below. This allows us to choose the initial data in \( L^b_x L^c_v \) space where \( b \) is less than \( N \) by paying the price of rising \( c \). It is not clear if this flexibility of choosing initial data in such spaces really reflects the difference between kinetic equation and hydrodynamic equation. Hence we retain the statement of the initial data in the current format.

3. Since the property of loss term is not fully utilized in our analysis, the exponent \( \gamma = 2 - N \) is the number where the dispersive effect from the kinetic transport part of the equation dominates the self-produced part from the collision operator when the small initial data is given. We expect that this mechanism should work for a more wide range of soft potential kernel if the loss term is properly used. One the other hand, the \( v \) variable estimates for the gain term of the Boltzmann collision operator with hard potential \( 0 < \gamma \leq 1 \), (see [12] and reference therein)

\[
\| Q^+(f, f)(v) \|_{H^{\gamma-\varepsilon}} \leq C \| f(v) \|_{L^1_v} \| f \|_{L^\gamma_v}
\]

suggests that the study of weighted Strichartz estimates is needed for applying such an approach to hard potential or hard sphere case.
4. The uniqueness of the small global solution implies that given a small enough scattering state \( f_+ \in L_{x,v}^N \), there exists a unique small enough initial data \( f_0 \in L_{x,v}^N \) whose corresponding global well-posed solution scatters to \( U(t)f_+ \) as \( t \to \infty \) (see the proof of the Theorem 1.1). In fact we can define the wave operator

(1.6) \[ \Omega_+: B_R \subset L_{x,v}^N \to B_R \subset L_{x,v}^N \]

by \( \Omega_+ f_+ = f_0 \) and have the following result.

**Corollary 1.2.** There exist \( R, \tilde{R} \) small enough such that the wave operator \( (1.6) \) is one-to-one and onto.

The next result is the local well-posedness for the large data Cauchy problem.

**Theorem 1.3.** Let \( N = 2 \) or \( 3 \) and \( B \) defined in (1.2) satisfies (1.3) and \( -N < \gamma < 2 - N \). The Cauchy problem (1.1) is locally wellposed in \( L_{x,v}^a, a = 2N/(\gamma + N) \). More specially, for any \( R > 0 \) there exists a \( T = T(r,p,R) \) such that for all \( f_0 \) in the ball \( B_R = \{ f_0 \in L_{x,v}^a(\mathbb{R}^N \times \mathbb{R}^N) : \|f_0\|_{L_{x,v}^a} < R \} \) there exist \( T \in (0,\infty] \) and a unique mild solution

\[ f \in C([0,T), L_{x,v}^a) \cap L^q([0,T], L_x^rL_v^p) \]

where the triple \( (q,r,p) \) lies in the set

(1.7) \[ \left( \frac{1}{q}, \frac{1}{r}, \frac{1}{p} \right) = \left( \frac{2(\alpha - 1)(\gamma + N)}{2} \right) = \left( \frac{1}{1 - \alpha}(\gamma + N), \frac{1}{\alpha}(\gamma + N) \right), \text{with } \frac{1}{2} < \alpha < \frac{N + 1}{2N}. \]

The solution map \( f_0 \in B_R \subset L_{x,v}^N \to f \in L^q([0,T]; L_x^rL_v^p) \) is Lipschitz continuous.

Some comments about this result are given in the following.

1. Heuristically, the solutions for large initial data exist during short time when the self-reproduced effect is not strong enough and the hyperbolic part of equation dominates the solution. This holds especially for soft collision where the non-local property of the collision operator is weaker. The result here indicates that the function spaces for the solutions depending on the exponent of kinetic part of the collision kernel at least for very beginning of evolution. It seems that this intuition and the fact \( a \to \infty \) when \( \gamma \to -3 \) suggests a local well-posed result in \( L_{x,v}^\infty \) for Landau equation.

2. Since the initial data lie in \( L_{x,v}^a \) space, it is suitable to discuss the propagation of singularity of solution in this setting though we are not pursuing it here.

2. **Proof of the Theorems**

In order to prove Theorem 1.1 and 1.3 we need to introduce the Strichartz estimates for the kinetic transport equation

(2.1) \[ \begin{cases} \partial_t u(t, x, v) + v \cdot \nabla_x u(t, x, v) = F(t, x, v), & (t, x, v) \in (0, \infty) \times \mathbb{R}^N \times \mathbb{R}^N, \\ u(0, x, v) = u_0(x, v). \end{cases} \]

To state the Strichartz estimates for the kinetic transport equation (2.1), we need the following definition.
Definition 2.1. We say that the exponent triplet \((q, r, p)\), for \(1 \leq p, q, r \leq \infty\) is KT-admissible if

\[
\frac{1}{q} = \frac{N}{2} \left( \frac{1}{p} - \frac{1}{r} \right)
\]

except in the case \(N = 1\), \((q, r, p) = (a, \infty, a/2)\). Here by \(a = \text{HM}(p, r)\) we have denoted the harmonic means of the exponents \(r\) and \(p\), i.e.,

\[
\frac{1}{a} = \frac{1}{2} \left( \frac{1}{p} + \frac{1}{r} \right)
\]

Furthermore, the exact lower bound \(p^*\) to \(p\) and the exact upper bound \(r^*\) to \(r\) are

\[
\begin{align*}
  p^*(a) &= \frac{Na}{N + 1}, & r^*(a) &= \frac{Na}{2 - a} & \text{if } 1 \leq a < \infty,
  \\
  p^*(a) &= 1, & r^*(a) &= \frac{a}{2 - a} & \text{if } 1 \leq a \leq \frac{N + 1}{N}.
\end{align*}
\]

The triplets of the form \((q, r, p) = (a, r^*(a), p^*(a))\) for \(\frac{N + 1}{N} \leq a < \infty\) are called endpoints. We note that the endpoint Strichartz estimate for the kinetic equation is false in all dimensions has been proved recently by Bennett, Bez, Gutiérrez and Lee [7].

The solution of (2.1) can be written as

\[
u = U(t)u_0 + W(t)F
\]

where

\[
U(t)u_0 = u_0(x - vt, v), \ W(t)F = \int_0^t U(t - s)F(s) ds.
\]

The estimates for the operator \(U(t)\) and \(W(t)\) respectively in the mixed Lebesgue norm \(\| \cdot \|_{L^q_t L^r_x L^p_v}\) are called homogeneous and inhomogeneous Strichartz estimates. These two estimates together are given in the following Proposition where we use \(p^*\) to denote the conjugate exponent of \(p\) and so on.

Proposition 2.2 ([15], [7]). Let \(u\) satisfies (2.1). The estimate

\[
\|u\|_{L^q_t L^r_x L^p_v} \leq C(q, r, p, N)\|u_0\|_{L^2_x} \ + \ \|F\|_{L^{q'}_t L^{r'}_x L^{p'}_v}
\]

holds for all \(u_0 \in L^2_x\) and all \(F \in L^{q'}_t L^{r'}_x L^{p'}_v\) if and only if \((q, r, p)\) and \((\tilde{q}, \tilde{r}, \tilde{p})\) are two KT-admissible exponents triplets and \(a = \text{HM}(p, r) = \text{HM}(\tilde{p}', \tilde{r}')\) with the exception of \((q, r, p)\) begin an endpoint triplet.

We also need the estimates for the Boltzmann collision operator. Recall that the collision operator can be split into gain and loss terms if the collision kernel satisfies Grad cut-off assumption [1, 2]. And it is convenient to introduce the bilinear gain term

\[
Q^+(f, g)(v) = \int_{v_* \in \mathbb{R}^n} f(v')g(v_*')B(v - v_*, \omega) d\Omega(\omega) dv_*
\]

and the bilinear loss term

\[
Q^-(f, g)(v) = \int_{v_* \in \mathbb{R}^n} f(v)g(v_*)B(v - v_*, \omega) d\Omega(\omega) dv_*
\]

The estimate we need for the gain term with the cut-off soft potential is due to Alonso, Carneiro and Gamba [1].
Proposition 2.3 ([1]). Let $1 < p_v, q_v, r_v < \infty$ with $-N < \gamma \leq 0$ and $1/p_v + 1/q_v = 1 + \gamma/N + 1/r_v$. Assume the kernel
\[
B(v - v_*, \omega) = |v - v_*|^\gamma b(\cos \theta)
\]
with $b(\cos \theta)$ satisfies [1.3]. The bilinear operator $Q^+$ extends to a bounded operator from $L^{p_v}(\mathbb{R}^N) \times L^{q_v}(\mathbb{R}^N) \to L^{r_v}(\mathbb{R}^N)$ via the estimate
\[
|Q^+(f, g)|_{L^{r_v}(\mathbb{R}^N)} \leq C\|f\|_{L^{p_v}(\mathbb{R}^N)}\|g\|_{L^{q_v}(\mathbb{R}^N)}.
\]

The case $\gamma = 0$ is due to H"older inequality. For $-\gamma < \gamma < 0$, we note that for the cut-off case $Q^-(f, g) = f(v)Lg(v)$ where
\[
Lg(v) = \int_{v_\ast \in \mathbb{R}^N} \int_{S^{N-1}} |v - v_*|^\gamma b(\cos \theta) g(v_\ast) d\Omega(\omega) dv_*
\]
is a convolution operator. Using H"older inequality, we have
\[
|Q^-(f, g)(v)|_{L^{r_v}(\mathbb{R}^N)} \leq \|f(v)\|_{L^{p_v}(\mathbb{R}^N)}\|Lg(v)\|_{L^2(\mathbb{R}^N)}, \quad \frac{1}{r_v} = \frac{1}{p_v} + \frac{1}{Z}.
\]
Since $-N < \gamma < 0$, we can invoke the Hardy-Littlewood-Sobolev inequality to have
\[
\|Lg(v)\|_{L^2(\mathbb{R}^N)} \leq C\|g\|_{L^{q_v}(\mathbb{R}^N)}
\]
where $-\frac{2}{n} = 1 - \left(\frac{1}{q_v} - \frac{1}{2}\right)$ and end the proof. \hfill \Box

Now we are ready to prove Theorem [1.1], Corollary [1.2] and Theorem [1.3]

Proof of Theorem [1.1]. We define the solution map by
\[
Sf(t, x, v) = f_0(x - vt, v) + \int_0^t Q(f, f)(s, x - (t - s)v, v) ds
\]
(2.4)
\[
= U(t)f_0 + \int_0^t U(t - s)Q(s) ds
\]
\[
= U(t)f_0 + W(t)Q(f, f)
\]
and wish to show that $S$ is a contraction mapping in the suitable Banach spaces. Applying the Strichartz estimates [2.3] to above, we have
\[
\|Sf(t, x, v)\|_{L^p_t L^q_x L^r_v} \leq C\left(\|f_0\|_{L^{p_v}_x} + \|Q(f, f)\|_{L^p_t L^q_x L^r_v}\right).
\]
The goal is to obtain the estimates of the form
\[
\|Sf(t, x, v)\|_X \leq C_1\|f_0(x, v)\|_Y + C_2\|f(t, x, v)\|_X^2
\]
(2.6)
where $X$ and $Y$ are suitable Banach spaces of the form $L^q_t L^r_x L^p_v$, $L^a_{x,v}$ respectively appearing in above estimates.

By Proposition 2.3, Lemma 2.4 and (2.6), we wish to have

\[ \frac{2}{p} = 1 + \frac{\gamma}{N} + \frac{1}{p'} \tag{2.7} \]

for the estimate of $v$ variables. For $x$ variables, we need

\[ 2r' = r, \quad r \geq 2 \tag{2.8} \]

for being able to apply the Hölder inequality. Furthermore the Strichartz inequality demands the relation of pairs $(p,r), (\tilde{p}', \tilde{r}')$,

\[ \frac{1}{p} + \frac{1}{r} = \frac{1}{p'} + \frac{1}{\tilde{r'}}. \tag{2.9} \]

In order to apply the Hölder inequality to $t$ variable, we wish to have

\[ \frac{2}{q} = \frac{1}{q'} < 1, \tag{2.10} \]

that is

\[ \frac{2}{q} + \frac{1}{\tilde{q}} = 1, \quad \frac{1}{q} < \frac{1}{2}. \tag{2.11} \]

Finally the KT-admissible conditions

\[ \frac{1}{q} = \frac{N}{2} \left( \frac{1}{p} - \frac{1}{r} \right) > 0, \tag{2.12} \]

\[ \frac{1}{\tilde{q}} = \frac{N}{2} \left( \frac{1}{\tilde{p}} - \frac{1}{\tilde{r}} \right) > 0 \tag{2.13} \]

must be fulfilled.

We note that once $\gamma, p, r$ are given, $q, \tilde{p}, \tilde{r}, \tilde{q}$ are determined. Rewrite these conditions as

\[ \frac{1}{p} + \frac{1}{r} = 1 + \frac{\gamma}{N} \tag{2.14a} \]

from (2.7) and (2.8), (2.10)

\[ \frac{1}{p} + \frac{1}{r} = \frac{2}{N} \tag{2.14b} \]

from (2.11) and (2.12), (2.9)

\[ 0 < \frac{1}{p} - \frac{1}{r} < \frac{1}{N} \tag{2.14c} \]

from $1/q < 1/2$ in (2.11) and (2.12)

\[ 0 < \frac{1}{\tilde{p}} - \frac{1}{\tilde{r}} < \frac{1}{2} \left( 1 + \frac{\gamma}{N} \right) \tag{2.14d} \]

from (2.13) and (2.7)

Therefore

\[ \gamma = 2 - N, \quad a = N. \]

Thus we have

\[ \frac{1}{N} < \frac{1}{p} < \frac{N+1}{N^2}, \quad \frac{N-1}{N^2} < \frac{1}{r} < \frac{1}{N}, \]

and conclude the set

\[ \{(p,r) | \frac{1}{N} < \frac{1}{p} < \frac{N+1}{N^2}, \frac{1}{r} = \frac{2}{N} - \frac{1}{p} \}. \]
Using the triplets \((q, r, p), (\tilde{q}, \tilde{r}, \tilde{p})\) satisfies above conditions, applying Proposition \(2.3\) and Lemma \(2.4\) to the right hand side of \(2.5\) by choosing \(r_v = \tilde{p}^\prime\), \(p_v = q_v = p\) and the H"older inequality to \(x, t\) variables, we conclude that
\[
\|Sf\|_{L^q_t L^r_x L^p_v} \leq C_1 \|f_0\|_{L^q_N L^r_x L^p_v} + C_2 \|f\|_{L^q_t L^r_x L^p_v}^2
\]
with \((q, r, p)\) being determined as above. With a similar argument one also obtains
\[
\|Sf_1 - Sf_2\|_{L^q_t L^r_x L^p_v} \leq C_2(\|f_1\|_{L^q_t L^r_x L^p_v} + \|f_2\|_{L^q_t L^r_x L^p_v})\|f_1 - f_2\|_{L^q_t L^r_x L^p_v}.
\]

Let \(\|f_0\|_{L^q_N L^r_x L^p_v} \leq R/2C_1\), \(\overline{B}_R = \{f \in L^q_t L^r_x L^p_v \mid \|f\|_{L^q_t L^r_x L^p_v} \leq R\}\) where \(R < 1\) is small enough so that
\[
2C_2R < 1,
\]
then from \((2.16), (2.17)\) and \((2.18)\) it follows that \(L : \overline{B}_R \to \overline{B}_R\) is a contraction mapping and there exists a unique fixed point \(f \in \overline{B}_R\) that is a solution to the integral equation \((2.4)\).

Now we show that \(f \in C([0,T], L^N_{x,v})\), \(T \in [0,\infty]\). It has been noted by Ovcharov \(15\) that \(U(t)f_0 \in C(\mathbb{R}; L^N_{x,v})\), hence it suffice to show that \(W(t)\) is also continuous. Let \(0 < t \in (0,\infty]\). Using inhomogeneous Strichartz with \(\tilde{q}^\prime, \tilde{r}^\prime, \tilde{p}^\prime\) as above, we see that
\[
\|W(t)Q(f, f)\|_{L^q_t([0,t]; L^N_{x,v})} = \int_0^t \|U(t-s)Q(f, f]\|_{L^N_{x,v}} ds
\]
is bounded. Since \(U(t)\) is continuous, we conclude that \(W(t)\) is continuous from above expression. The solution map \(f_0 \in B_R \subset L^N_{x,v} \to f \in L^q_t L^r_x L^p_v\) is Lipschitz continuous. For if \(f\) and \(g\) are two solutions with initial data \(f_0\) and \(g_0\) in \(B_R\), we have as above
\[
\|f - g\|_{L^q_t L^r_x L^p_v} \leq C_1 \|f_0 - g_0\|_{L^N_{x,v}} + C_2 \|f - g\|_{L^q_t L^r_x L^p_v}^2,
\]
and thus
\[
\|f - g\|_{L^q_t L^r_x L^p_v} \leq C_3 \|f_0 - g_0\|_{L^N_{x,v}}.
\]

Next we show that the global solution \(f\) scatters. We note that to show \(\|f(t) - U(t)f_+\|_{L^N_{x,v}} \to 0\) as \(t \to \infty\) is equivalent to show that \(\|U(-t)f(t) - f_+\|_{L^N_{x,v}} \to 0\) as \(t \to \infty\) since \(U(t)\) preserves the \(L^N_{x,v}\) norm. By the Duhamel formula, we have
\[
U(-t)f(t) = f_0 + \int_0^t U(-s)Q(f, f)(s)ds.
\]
Hence the scattering property is the consequence of the convergence of the integral
\[
\int_0^\infty U(-t)Q(f, f)(t)dt
\]
in \(L^N_{x,v}\) in which case \(f_+\) is given by
\[
(2.19) \quad f_+ = f_0 + \int_0^\infty U(-t)Q(f, f)(t)dt.
\]
Let \(U^*(t)\) be the adjoint operator of \(U(t)\), it is clearly that \(U^*(t) = U(-t)\). By duality, the homogeneous Strichartz estimate
\[
\|U(t)g\|_{L^q_t L^r_x L^p_v} \leq C|g|_g {\|L^N_{x,v}}
\]
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with \((q, r, p)\) admissible and \(1/p + 1/r = (N - 1)/N\) implies
\[
\| \int_0^\infty U^*(t)Q(f, f) \|_{L_x^N} \leq C \| Q(f, f) \|_{L_t^{q'} L_x^r L_v^p}
\]
with \(1/p' + 1/r' = N\). As the proof of existence of solution above, we see that the
right hand side of above inequality is bounded by \(\| f \|_{L_t^q L_x^r L_v^p}^2\) and thus bounded as
\(f \in \mathcal{B}_R\).

\[ \square \]

Proof of Corollary 1.2. First we exam the existence of \(\Omega_+ : f_+ \to f_0\). Using the
second formula of Duhamel representation (2.4) and the relation (2.19), we can
write
\[ f(t) = U(t)f_+ - \int_0^\infty U(t-s)Q(f, f)(s)ds. \]

Therefore the well-defined of \(\Omega_+\) is equivalent to being able to define (2.20) for
\(t = 0\) but this is just the reminiscence of global existence result above if
\(f_+\) is small enough in \(L_x^N\).

The map \(\Omega_+\) is one to one as a consequence of (2.19) and uniqueness of solution
\(f\). This mapping is also surjective as a consequence of the fact the the small global
solution scatters.

\[ \square \]

Proof of Theorem 1.3. Let \(\chi(r)\) be a smooth nonnegative bump even function sup-
ported on \(-2 \leq r \leq 2\) and satisfying \(\chi(r) = 1\) for \(-1 \leq r \leq 1\). Let \(T > 0\) be a
positive number which will be chosen later. We define the solution map by
\[ Sf(t, x, v) \]
\[ = \chi(t/T)f_0(x - vt, v) + \int_0^t Q(f, f)(s, x - (t-s)v, v)d\tau ds \]
\[ = \chi(t/T)U(t)f_0 + \chi(t/T)W(t)Q(f, f) \]
and wish to show that \(S\) is a contraction mapping in the suitable Banach spaces.
Applying the Strichartz estimates (2.3) to above, we have
\[ \|Sf(t, x, v)\|_{L^q_t L_x^r L_v^p} \leq C \left( \|f_0\|_{L_x^N} + \|Q(f, f)\|_{L_t^{q'} L_x^r L_v^p} \right) \]
The goal is to obtain the estimates of the from
\[ \|Sf(t, x, v)\|_X \leq C_1 \|f_0(x, v)\|_Y + C_2 T^\beta \|f(t, x, v)\|_X \]
with \(\beta > 0\) where \(X\) and \(Y\) are suitable Banach spaces of the form \(L_t^q L_x^r L_v^p, L_x^q\)
respectively appearing in above estimates.

The conditions posed on triplets \((q, r, p), (q', r', p')\) are similar. The only difference
is the exponents about \(t\) variables. For \(t\) variable, the condition \(\beta > 0\) is
equivalent to
\[ \frac{2}{q} < \frac{1}{q'} < 1, \]
that is
\[ \frac{2}{q} + \frac{1}{q'} < 1, \quad \frac{1}{q'} < 1. \]
Therefore we conclude a system of restrictions similar to that of Theorem 1.1

\begin{align}
(2.24a) \quad & \left\{ \frac{1}{p} + \frac{1}{r} = 1 + \frac{\gamma}{N} \right. \\
(2.24b) \quad & \left. \frac{1}{p} + \frac{1}{r} < \frac{2}{N} \right. \\
(2.24c) \quad & \left. 0 < \frac{1}{p} - \frac{1}{r} < \frac{1}{N} \right. \\
(2.24d) \quad & \left. 0 < \frac{1}{p} - \frac{1}{r} < \frac{1}{2}(1 + \frac{\gamma}{N}) \right. .
\end{align}

The conditions (2.24a) and (2.24b) imply that $-N < \gamma < -(N - 2)$. Thus (2.24c) holds by (2.24d). Since

$$\frac{1}{a} = \frac{1}{2} \cdot \frac{\gamma + N}{N} < \frac{1}{N} < \frac{N}{N + 1},$$

we also require

$$\frac{1}{2} \cdot \frac{\gamma + N}{N} < \frac{1}{p} < \frac{N + 1}{N} \cdot \frac{1}{2} \cdot \frac{\gamma + N}{N}$$

$$\frac{N - 1}{N} \cdot \frac{1}{2} \cdot \frac{\gamma + N}{N} < \frac{1}{r} < \frac{1}{2} \cdot \frac{\gamma + N}{N}$$

for the KT-admissible condition. Thus we conclude the set

\begin{align}
(2.25) \quad & \{(p, r) \mid \frac{1}{p} = \alpha \left(\frac{\gamma + N}{N}\right), \quad \frac{1}{r} = (1 - \alpha) \left(\frac{\gamma + N}{N}\right), \quad \text{with} \quad \frac{1}{2} < \alpha < \frac{N + 1}{2N}\}
\end{align}

satisfies all the conditions list above. Ans it is easy to check that $(q, r, p)$ and $(\tilde{q}, \tilde{r}, \tilde{p})$ are KT-admissible triplets when $(p, r)$ lies in set (2.25).

Using triplets $(q, r, p), (\tilde{q}, \tilde{r}, \tilde{p})$ above and the argument as the Theorem 1.1, we conclude

\begin{align}
(2.26) \quad & \|Sf(t, x, v)\|_{L^q([0, T]; L^r_x L^p_v)} \\
& \leq C_1\|f_0(x, v)\|_{L^q_{x,v}} + C_2 T^\beta \|f(t, x, v)\|^2_{L^q([0, T]; L^r_x L^p_v)}
\end{align}

where

$$\beta = \frac{(2 - N) - \gamma}{2} > 0.$$

With a similar argument one also obtains

\begin{align}
(2.27) \quad & \|L_1 f_1 - L_2 f_2\|_{L^q([0, T]; L^r_x L^p_v)} \\
& \leq C_4 T^\delta \left(\|f_1\|_{L^q([0, T]; L^r_x L^p_v)} + \|f_2\|_{L^q([0, T]; L^r_x L^p_v)}\right) \|f_1 - f_2\|_{L^q([0, T]; L^r_x L^p_v)}.
\end{align}

Let $R = 2C_1\|f_0\|_{L^q_{x,v}}$ be any positive number, $B_R = \{f \mid \|f\|_{L^q_{t,x} L^r_x L^p_v} \leq R\}$ and $T$ such that

$$C_4 T^\delta R < \frac{1}{2}.$$

then from (2.26), (2.27) and (2.28) it follows that $L : B_R \rightarrow B_R$ is a contraction mapping and there exists a unique fixed point $f \in B_R$ that is a solution to the integral equation (2.21).
Finally, we show that the uniqueness of solution. If \( f_1 \) and \( f_2 \) are two solutions, is easy to see that we have
\[
\|f_1 - f_2\|_{L^q([0,T];L^r_x L^p_v)} \leq C_2 t^\beta (\|f_1\|_{L^q([0,T];L^r_x L^p_v)} + \|f_2\|_{L^q([0,T];L^r_x L^p_v)})
\]
for \( 0 < t \leq T \). By choosing \( t \) small enough, we have
\[
\|f_1 - f_2\|_{L^q([0,T];L^r_x L^p_v)} \leq \frac{1}{2} \|f_1 - f_2\|_{L^q([0,T];L^r_x L^p_v)}
\]
and thus \( f_1 = f_2 \) on \([0,t]\). We can cover the interval \([0,T]\) by iterates this argument. The well-posedness of this case ends.

\[\Box\]

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