A FLOW ON $S^2$ PRESENTING THE BALL AS ITS MINIMAL SET

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Abstract. The main goal of this paper is to present the existence of a vector field tangent to the unit sphere $S^2$ such that $S^2$ itself is a minimal set. This is reached using a piecewise smooth (discontinuous) vector field and following the Filippov’s convention on the switching manifold. As a consequence, none regularization process applied to the initial model can be topologically equivalent to it and we obtain a vector field tangent to $S^2$ without equilibria.

1. Introduction. Let $M \subset \mathbb{R}^n$ be a manifold and consider the flow $\phi : \mathbb{R} \times M \to M$. Associated with $\phi$ there exists a vector field $X$ given by $X = \frac{d}{dt} \phi$ which carries a point $p \in M$ to a vector in $T_p M$ the tangent space to $M$ at $p$.

A set $A \subset M$ is invariant for $\phi$ (or $X$) if for each $p \in A$ and all trajectory $\Gamma_\phi(t,p)$ passing through $p$ it holds $\Gamma_\phi(t,p) \subset A$. Moreover, a set $A \subset M$ is minimal for $\phi$ (or $X$) if:

(a) $A \neq \emptyset$;
(b) $A$ is compact;
(c) $A$ is invariant for $\phi$ (or $X$);
(d) $A$ does not contain proper subset satisfying (a)-(c).

Questions concerning the structure and characterization of minimal sets are of great relevance in the theory of dynamical systems. When $M$ is a two dimensional manifold, a trivial minimal set $\Lambda$ of $\phi$ (or $X$) is either an equilibrium point or a closed trajectory or else the whole manifold $M$, provide that $\Lambda = M$ is the torus and $\phi$ is (topologically equivalent to) an irrational flow. The following result characterizes minimal sets, according to the differentiability of the two-dimensional flow.

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Smoothing Theorem: (see [15]) Let \( \phi : \mathbb{R} \times M \to M \) be a continuous flow on a compact \( C^\infty \) two dimensional manifold \( M \). Then there exists a \( C^1 \) flow \( \psi \) on \( M \) which is topologically equivalent to \( \phi \). Furthermore, the following conditions are equivalent:

(a) any minimal set of \( \phi \) is trivial;
(b) \( \phi \) is topologically equivalent to a \( C^2 \) flow;
(c) \( \phi \) is topologically equivalent to a \( C^\infty \) flow.

The assertion that (b) implies (a) is the celebrated Denjoy-Schwartz Theorem (see [10, 27]), that generalizes to closed two-dimensional manifolds the planar, and even more celebrated, Poincaré-Bendixson Theorem (see [23]). We stress that the existence of \( C^1 \) flows which are not topologically equivalent to \( C^2 \) flows is given in [10] (see also [28]).

It is well known that a large range of problems in applied science are modeled using Ordinary Differential Equations (ODEs for short). However, in recent years researches around the world are increasingly convinced that the quantity of phenomena modeled using smooth ODEs is considerably smaller than those modeled using discontinuous ODEs. Inside the universe of discontinuous systems of ODEs (or, discontinuous vector fields governed by ODEs) we highlight the Filippov’s vector field. These systems appear when the laws that model the system must be abruptly changed. For example, when an on-off key is triggered. In this case, the system of ODEs before the key is activated is completely distinct from that one after the key activation. The exact instant when we change the laws of the system is characterized by a switching manifold on the state space where a new vector field is defined as an average between both vector fields. For instance, see [26] for applications in control theory, [12, 19] in mechanics models, [6, 17] in electrical circuits, [11, 16] in relay systems, [7, 18, 24, 25] in biological models, among others.

At the main result of this paper, we state that there exists a Filippov’s vector field tangent to \( S^2 \) (a compact two dimensional \( C^\infty \) manifold) such that the minimal set is \( S^2 \) itself. So, we have the following main result.

**Theorem 1.1.** There exists a piecewise smooth vector field \( Z \) tangent to the unit sphere \( S^2 \) such that \( S^2 \) itself is a non-trivial minimal set.

The regularization method established by Sotomayor and Teixeira in [30] allow us to use the well established theory of smooth vector fields to provide properties of discontinuous vector fields. The method consists of approximately a piecewise smooth vector field by a one parameter family of smooth vector fields which agree with the original discontinuous system outside a strip around the switching manifold. See [29] for details. Some papers (see, for instance, [1, 9, 20, 21]) take a Filippov’s vector field and regularize it, through appropriate assumptions, giving rise to a smooth vector field. However, this approach can fail in some cases. In fact, combining Theorem 1.1 and the Smoothing Theorem above, we are able to show (immediately) that:

**Corollary 1.** None regularization process can establish a topological equivalence between the piecewise smooth vector field of Theorem 1.1 and a smooth vector field (both tangent to \( S^2 \)).

In fact, if the vector field of Theorem 1.1 could be regularized giving rise to a smooth vector field \( \tilde{X} \) topologically equivalent to it, then \( S^2 \) is a non-trivial minimal set of \( \tilde{X} \). This produces a contradiction, according to the Smoothing Theorem.
Another issue that can be addressed using Theorem 1.1 is the attainment of a vector field tangent to $S^2$ without equilibrium points. From the Hairy Ball Theorem (see [2, 13, 22]), it is well known that it is not possible the existence of a continuous vector field tangent to $S^2$ without equilibrium points. However, by Theorem 1.1 we get a discontinuous vector field tangent to $S^2$ without equilibria (note that an equilibrium point is a minimal set). This problem was addressed in [5].

The paper is organized as follows. In Section 2 we present the essential theory about discontinuous vector fields. We also present the stereographic and central projections that will be very useful in this work. In Section 3 we consider two planar vector fields that will generate a piecewise smooth vector field in $S^2$ through the stereographic projection. In Section 4 we use the central projection in the trajectories defined in $S^2$ to prove some of its properties. Lastly, in Section 5, we show the existence of non-trivial minimal sets and we prove the main result.

2. Preliminaries. A piecewise smooth vector field on $\mathbb{R}^n$ is a pair of $C^r$-vector fields $X$ and $Y$, where $X$ and $Y$ are restricted to regions of $\mathbb{R}^n$ separated by a smooth codimension one manifold $\Sigma$ of $\mathbb{R}^n$ given by $\Sigma = f^{-1}(0)$, where $f: \mathbb{R}^n \to \mathbb{R}$ is a smooth function having $0 \in \mathbb{R}$ as a regular value. The switching manifold $\Sigma$ is the separating boundary of the regions $\Sigma^+ = \{q \in \mathbb{R}^n \mid f(q) > 0\}$ and $\Sigma^- = \{q \in \mathbb{R}^n \mid f(q) < 0\}$. Thus, a piecewise smooth vector field is given by:

$$Z(q) = \begin{cases} X(q), & \text{if } f(q) > 0, \\ Y(q), & \text{if } f(q) \leq 0, \end{cases}$$

(1)

Usually, a PSVF is denoted by $Z = (X, Y, \Sigma)$ or simply by $Z = (X, Y)$ when the switching manifold $\Sigma$ is well understood. We notice that $Z$ is considered multivalued at points of $\Sigma$. Call $\chi^r$ the space of piecewise smooth vector fields. We endow $\chi^r$ with the product topology.

To establish a definition for the trajectories of $Z$ and investigate its behavior, we need a criterion for the transition of the orbits across $\Sigma$. The contact between the smooth vector field $X$ (or $Y$) and the switching manifold $\Sigma = f^{-1}(0)$ is characterized by the expression $Xf(p) = \langle \nabla f(p), X(p) \rangle$ where $\langle\cdot,\cdot\rangle$ is the usual inner product in $\mathbb{R}^n$. Following the terminology of Filippov (see [14]) we can divide the switching manifold in the following sets:

- Crossing set: $\Sigma^c = \{p \in \Sigma : Xf(p) \cdot Yf(p) > 0\}$.
- Sliding set: $\Sigma^s = \{p \in \Sigma : Xf(p) < 0, Yf(p) > 0\}$.
- Escaping set: $\Sigma^e = \{p \in \Sigma : Xf(p) > 0, Yf(p) < 0\}$.

Moreover, let us consider the subsets $\Sigma^{c+} = \{p \in \Sigma : Xf(p) > 0, Yf(p) > 0\}$ and $\Sigma^{c-} = \{p \in \Sigma : Xf(p) < 0, Yf(p) < 0\}$ of $\Sigma^c$. Note that the escaping $\Sigma^e$ or sliding $\Sigma^s$ sets are respectively defined on points of $\Sigma$ where both vector fields $X$ and $Y$ simultaneously point outwards or inwards from $\Sigma$ while the interior of its complement in $\Sigma$ defines the crossing region $\Sigma^c$. The complementary of the union of those regions is the set formed by the tangency points between $X$ or $Y$ with $\Sigma$.

**Definition 2.1.** A point $p \in \Sigma$ is called a tangency point of $X$ (resp. $Y$) if it satisfies $Xf(p) = 0$ (resp. $Yf(p) = 0$). A tangency point is called a fold point of $X$ if $X^2 f(p) \neq 0$. Moreover, $p \in \Sigma$ is a visible (resp. invisible) fold point of $X$ if $X^2 f(p) > 0$ (resp. $X^2 f(p) < 0$).
In addition, a tangential singularity $q$ is singular if $q$ is an invisible tangency for both $X$ and $Y$. On the other hand, a tangential singularity $q$ is regular if it is not singular.

To define a trajectory of a PSVF passing through a crossing point, it is sufficient to concatenate the trajectories of $X$ and $Y$ by that point. However, in the sliding and escaping sets we need to define an auxiliary vector field.

**Definition 2.2.** Given a point $p \in \Sigma^+ \cup \Sigma^- \subset \Sigma$, we define the sliding vector field at $p$ as the vector field $Z^T(p) = m - p$ with $m$ being the point of the segment joining $p + X(p)$ and $p + Y(p)$ such that $m - p$ is tangent to $\Sigma$. (see Figure 1).

The sliding vector field is given by the expression

$$Z^T(p) = \frac{Yf(p)X(p) - Xf(p)Y(p)}{Yf(p) - Xf(p)}, \quad (2)$$

Moreover, the sliding vector field can be extended to $\Sigma^+ \cup \Sigma^-$. Any point $p \in \Sigma^+ \cup \Sigma^-$ such that $Z^T(p) = 0$ is called a pseudo equilibrium of $Z$.

In the next definitions we establish the concepts of local and global trajectories of a PSVF.

**Definition 2.3.** The local trajectory (orbit) $\phi_Z(t, p)$ of a PSVF given by (1) through $p \in V$ is defined as follows:

(i) For $p \in \Sigma^+$ and $p \in \Sigma^-$ the trajectory is given by $\phi_Z(t, p) = \phi_X(t, p)$ and $\phi_Z(t, p) = \phi_Y(t, p)$ respectively, where $t \in I$.

(ii) For $p \in \Sigma^+$ and taking the origin of time at $p$, the trajectory is defined as $\phi_Z(t, p) = \phi_Y(t, p)$ for $t \in I \cap \{t \leq 0\}$ and $\phi_Z(t, p) = \phi_X(t, p)$ for $t \in I \cap \{t \geq 0\}$. For the case $p \in \Sigma^-$ the definition is the same reversing time.

(iii) For $p \in \Sigma^-$ and taking the origin of time at $p$, the trajectory is defined as $\phi_Z(t, p) = \phi_Z(t, p)$ for $t \in I \cap \{t \leq 0\}$ and $\phi_Z(t, p)$ is either $\phi_X(t, p)$ or $\phi_Y(t, p)$ or $\phi_Z(t, p)$ for $t \in I \cap \{t \geq 0\}$. For $p \in \Sigma^+$ the definition is the same reversing time.

(iv) For $p$ a regular tangency point and taking the origin of time at $p$, the trajectory is defined as $\phi_Z(t, p) = \phi_1(t, p)$ for $t \in I \cap \{t \leq 0\}$ and $\phi_Z(t, p) = \phi_2(t, p)$ for $t \in I \cap \{t \geq 0\}$, where each $\phi_1, \phi_2$ is either $\phi_X$ or $\phi_Y$ or $\phi_Z$.

(v) For $p$ a singular tangency point, $\phi_Z(t, p) = p$ for all $t \in \mathbb{R}$.

**Definition 2.4.** A global trajectory (orbit) $\Gamma_Z(t, p_0)$ of $Z$ passing through $p_0$ is a union

$$\Gamma_Z(t, p_0) = \bigcup_{i \in \mathbb{Z}} \sigma_i(t, p_i) : t_i \leq t \leq t_{i+1}$$

of preserving-orientation local trajectories $\sigma_i(t, p_i)$ satisfying

$$\sigma_i(t_{i+1}, p_i) = \sigma_{i+1}(t_{i+1}, p_{i+1}) = p_{i+1}.$$

2.1. Limit sets in PSVF. Now we introduce some concepts about $\alpha$-limit and $\omega$-limit of a global trajectory of a PSVF. To define it we need to consider each trajectory passing through a given point since we do not have the uniqueness of solutions.

**Definition 2.5.** Given $\Gamma_Z(t, p_0)$ a global trajectory passing through $p_0$, the set

$$\omega(\Gamma_Z(t, p_0)) = \{q \in V : \exists (t_n) \text{ satisfying } \Gamma_Z(t_n, p_0) \to q \text{ when } t \to +\infty\}$$
is called \( \omega \)-limit set of \( \Gamma_Z(t, p_0) \). In a similar way, we can define the \( \alpha \)-limit set of \( \Gamma_Z(t, p_0) \) as the set
\[
\alpha(\Gamma_Z(t, p_0)) = \{ q \in V : \exists (t_n) \text{ satisfying } \Gamma_Z(t_n, p_0) \to q \text{ when } t \to -\infty \}.
\]
The \( \omega \)-limit (respectively, \( \alpha \)-limit) set of a point \( p \) is the union of the \( \omega \)-limit (respectively, \( \alpha \)-limit) sets of all global trajectories passing through \( p \).

We can notice that for single-valued trajectories the definition above coincides with the classical one. In what follows we bring some results concerning understand the role that trajectories play in non-smooth vector fields with sliding motion.

**Remark 1.** In general, the \( \omega \)-limit set is not connected. In fact, since the global orbit through a given point may not be unique, the positive global trajectory can follow distinct paths. In Figure 2 the trajectory through \( p \) can follow three paths, namely, \( \Gamma_1, \Gamma_2, \) and \( \Gamma_3 \). Notice that the \( \omega \)-limit of \( \Gamma_i \), for \( i = 1, 2, 3 \) is, respectively, a focus, a pseudo equilibrium and a limit cycle. So, the \( \omega \)-limit set of \( p \) is not connected. Nonetheless, in this example, the \( \alpha \)-limit set of \( p \) is a connected set composed by a pseudo equilibrium.

In opposite direction from Remark 1, the next proposition holds for piecewise smooth vector fields as well as for smooth vector fields, with the same proof (which we include for sake of completeness).

**Proposition 1.** The \( \alpha \) and \( \omega \)-limit sets of a trajectory \( \Gamma \) of a PSVF are closed subsets of \( \mathbb{R}^n \). In particular, if \( \Gamma \) is contained in a compact subset of \( \mathbb{R}^n \) then the \( \alpha \) and \( \omega \)-limit sets are non-empty compact subsets of \( \mathbb{R}^n \).

**Proof.** In fact, let \( p_n \) be a sequence of points in \( \omega(\Gamma) \) with \( p_n \to p \). Given \( x_0 \in \Gamma \) since \( p_n \in \omega(\Gamma) \) for each \( n = 1, 2, \ldots \) there is a sequence \( t_k^{(n)} \to \infty \) as \( k \to \infty \) such that \( \lim_{k \to \infty} \Gamma(t_k^{(n)}, x_0) = p_n \). We may assume \( t_k^{(n+1)} > t_k^{(n)} \) since otherwise we can choose a subsequence of \( t_k^{(n)} \). So, for all \( n \geq 2 \) there is a sequence of integers \( K(n) > K(n+1) \) such that for \( k > K(n) \),
\[
|\Gamma(t_k^{(n)}, x_0) - p_n| < \frac{1}{n}.
\]
Let \( t_n = t_k^{(n)}(n) \). Then \( t_n \to \infty \) and
\[
|\Gamma(t_k^{(n)}, x_0) - p| \leq |\Gamma(t_k^{(n)}, x_0) - p_n| + |p_n - p| \leq \frac{1}{n} + |p_n - p| \to 0
\]
as \( n \to \infty \). Thus, \( p \in \omega(\Gamma) \). If \( \Gamma \subset K \), a compact subset of \( \mathbb{R}^n \) and \( \Gamma(t_n, x_0) \to p \in \omega(\Gamma) \) then \( p \in K \). Thus, \( \omega(\Gamma) \subset K \) and therefore \( \omega(\Gamma) \) is compact since it is a closed subset of a compact set. Furthermore, \( \omega(\Gamma) \neq \emptyset \) since the sequence of points \( \Gamma(t_n, x_0) \) contains a convergent subsequence which converges to a point in \( \omega(\Gamma) \subset K \).

**2.2. Stereographic and central projections.** Consider the unit sphere \( S^2 = \{(x, y, z) : x^2 + y^2 + z^2 = 1 \} \). In order to establish a correspondence between a trajectory of a vector field in the plane and a trajectory of a vector field in the unit sphere \( S^2 \) we are going to use stereographic and central projections. Given a point \( p = (x, y, 0) \) in the plane \( z = 0 \) we can consider the straight line between \( p \) and the north pole \( N = (0, 0, 1) \) which is given by \( \lambda(t) = tp + (1-t)N \). This line will intersect
S^2 if, and only if, \( t = 2/(x^2 + y^2 + 1) \). So, we can define \( \pi_N : \mathbb{R}^2 \rightarrow S^2 \setminus \{N\} \) given by

\[
\pi_N(x, y) = \left( \frac{2x}{x^2 + y^2 + 1}, \frac{2y}{x^2 + y^2 + 1}, \frac{x^2 + y^2 - 1}{x^2 + y^2 + 1} \right)
\]  

(3)

We can notice that the inverse function of \( \pi_N \) is

\[
\pi_N^{-1} : S^2 \setminus \{N\} \rightarrow \mathbb{R}^2
\]

\[
(x, y, z) \mapsto \pi_N^{-1}(x, y, z) = \left( \frac{x}{1-z}, \frac{y}{1-z} \right)
\]

(4)

Similarly, we may find the stereographic projection related to the south pole \( S = (0, 0, -1) \). Consequently, we can determine \( \pi_S : \mathbb{R}^2 \rightarrow S^2 \setminus \{S\} \) given by

\[
\pi_S(x, y) = \left( \frac{2x}{x^2 + y^2 + 1}, \frac{2y}{x^2 + y^2 + 1}, \frac{1 - x^2 - y^2}{x^2 + y^2 + 1} \right)
\]

(5)

whose inverse function is

\[
\pi_S^{-1} : S^2 \setminus \{S\} \rightarrow \mathbb{R}^2
\]

\[
(x, y, z) \mapsto \pi_S^{-1}(x, y, z) = \left( \frac{x}{1 + z}, \frac{y}{1 + z} \right)
\]

(6)

Another projection that will be very helpful in this work is the central projection. In order to obtain an expression for the central projection let us consider the plane \( P = \{(x, y, z) : y = 1\} \) and the straight line \( \alpha \) between the origin \( O \) and \( q \in S^2 \setminus H_1 \), where \( H_1 = \{(x, y, z) \in S^2 : y > 0\} \). We have that the line \( \alpha(t) = (tx, ty, tz) \) belongs to \( P \) if \( t = 1/y \). So, the central projection \( \pi_c : S^2 \cap H_1 \rightarrow \mathbb{R}^2 \) is given by

\[
\pi_c(x, y, z) = \left( \frac{x}{y}, \frac{z}{y} \right)
\]

(7)

The inverse function of the central projection is

\[
\pi_c^{-1} : \mathbb{R}^2 \rightarrow S^2 \cap H_1
\]

\[
(u, v) \mapsto \pi_c^{-1}(u, v) = \left( \frac{u}{\sqrt{u^2 + v^2 + 1}}, \frac{1}{\sqrt{u^2 + v^2 + 1}}, \frac{v}{\sqrt{u^2 + v^2 + 1}} \right)
\]

(8)

An analogous of the central projection can be defined in the rear hemisphere \( H_2 = \{(x, y, z) \in S^2 : y < 0\} \) of \( S^2 \). We will call it rear projection and it is given by

\[
\pi_r(x, y, z) = \left( \frac{-x}{y}, \frac{-z}{y} \right)
\]

(9)

3. **Stereographic projection of a trajectory into \( S^2 \).** This section is devoted to describe some important features of the planar vector fields that we will consider. Particularly, we analyze the behavior of their trajectories and their projections on the unit sphere to create a flow in \( S^2 \) with a non-trivial minimal set. Let us consider the system

\[
\begin{align*}
\dot{x} &= 2; \\
\dot{y} &= y(x^3 - 3x)
\end{align*}
\]

(10)

where the overhead dot means the derivative with respect to time \( t \) and the associated vector field is \( X(x, y) = (2, y(x^3 - 3x)) \). Observe that the vector field \( X \) is tangent to \( S^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\} \) if, and only if, the scalar product between \( X(x, y) \) and \( (x, y) \) is zero, where \((x, y) \in S^1 \). Therefore, for positive values of \( y \), the vector field \( X \) is tangent to \( S^1 \) at the points \( p_0 = (0, 1), p_1 = (\sqrt{2 - \sqrt{3}}, \sqrt{-1 + \sqrt{3}}) \) and \( p_2 = (-\sqrt{2 - \sqrt{3}}, \sqrt{-1 + \sqrt{3}}) \). Since \( \dot{x} = dx/dt = 2 \)
we have that \( dy/dx = y(x^3 - 3x)/2 \). Therefore, every integral curve of (10) can be described by \( y_1(x) \) given by

\[
y_1(x) = y_0 e^{x^4 - \frac{3x^2}{2}}. \]

We may also observe that \( y_1(x) = y_1(-x) \) which means the function \( y_1 \) is symmetric with respect to axis \( y \). In what follows, we project the trajectories of system (10) in the south hemisphere \( V = \{(x, y, z) \in S^2 : z \leq 0\} \) of \( S^2 \). Consider \( W = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\} \). Given a point \((x, y_1(x)) \in W\) we have that

\[
\pi_N(x, y_1(x)) = \left( \frac{2x}{x^2 + y_1^2(x) + 1}, \frac{2y_1(x)}{x^2 + y_1^2(x) + 1}, \frac{x^2 + y_1^2(x) - 1}{x^2 + y_1^2(x) + 1} \right) \tag{11}
\]

is a point of \( V \). See Figure 3(a). We can notice that, as the projection \( \pi_N \) is the identity in \( S^1 \), we have that the points \( p_0, p_1 \) and \( p_2 \) are still the tangency points between \( \pi_N \) and \( S^1 \subset S^2 \).

In a similar way, we can project another vector field in the north hemisphere \( U = \{(x, y, z) \in S^2 : z \geq 0\} \) of \( S^2 \) using the south stereographic projection \( \pi_S \). Indeed, consider the following system

\[
\begin{align*}
\dot{x} &= -1; \\
\dot{y} &= x \left( \frac{x}{2} - y \right). \tag{12}
\end{align*}
\]

The vector field \( Y(x, y) = (-1, x(x/2 - y)) \) is tangent to \( S^1 \) if, and only if, \( x = 0 \). So, \( Y \) is tangent to \( S^1 \) at the points \( p_3 = (0, 1) \) and \( p_4 = (0, -1) \). Since \( dx = -dt \) we have

\[
\frac{dy}{dx} = -\frac{x^2}{2} + xy
\]

and therefore

\[
y_2(x) = \frac{1}{4} \left( 2x + 4e^{\frac{x^2}{2}} c_0 - e^{\frac{x^2}{2}} \sqrt{2\pi} \text{ Erf} \left( \frac{x}{\sqrt{2}} \right) \right),
\]

with \( y_2(0) = c_0 \), describe the trajectories of (12). Here, \( \text{Erf} (\cdot) \) is the Gauss error function (see [31]). By the south stereographic projection we get

\[
\pi_S(x, y_2(x)) = \left( \frac{2x}{x^2 + y_2^2(x) + 1}, \frac{2y_2(x)}{x^2 + y_2^2(x) + 1}, \frac{1 - x^2 - y_2^2(x)}{x^2 + y_2^2(x) + 1} \right) \tag{13}
\]

where \((x, y_2(x)) \in W\) and, consequently, \( \pi_S(x, y_2(x)) \in U \). See Figure 3(b). Attaching the flows given by Equations (11) and (13) we have piecewise smooth trajectories in \( S^2 \) as we see illustrated in Figure 4. Associated with this trajectories there exists a three dimensional piecewise smooth vector field \( Z \) whose switching manifold is \( S^1 \).

We can notice that these trajectories will not configure periodic orbits since \( Y \) is asymmetric. In fact, let us consider the system \( Y \) given by (12). Considering an initial condition \( \left( x_0, \sqrt{1 - x_0^2} \right) \in S^1 \), where \( x_0 \in (0, 1) \), and solving the initial value problem given by

\[
\begin{align*}
\dot{x} &= -1; \\
\dot{y} &= x \left( \frac{x}{2} - y \right); \\
x(0) &= x_0; \\
y(0) &= \sqrt{1 - x_0^2}, \tag{14}
\end{align*}
\]

...
we can find an explicit solution \( y(t) = y(-x + x_0) \) whose expression is complicated and will be omitted. Let us consider the displacement function \( P(x_0) = y(x_0) - y(-x_0) \). We can observe that the displacement function \( P \) allow us to analyze the \( y \)-coordinate of the trajectory passing through \( x_0 \) at the transversal section \( \{ x = -x_0 \} \). See Figure 5. By a straightforward calculation we have that \( P(x_0) \) is given by

\[
P(x_0) = x_0 - e\frac{\sqrt{2}}{2} \sqrt{\frac{\pi}{2}} \text{Erf} \left( \frac{x_0}{\sqrt{2}} \right).
\]

Since \( P(0) = 0 \) and the derivative of \( P \) is strictly negative for \( x_0 \in (0, 1) \) we can conclude that \( P \) does not have real roots for \( x_0 \in (0, 1) \). Therefore, we have that \( y(x_0) < y(-x_0) \) for all \( x_0 \in (0, 1) \). Since the stereographic projection \( \pi_S \) is the identity in \( S^1 \) we will not have symmetry in the projected trajectories of \( Y \) in \( S^2 \).

4. Central projection of the trajectories of \( S^2 \). In order to show some properties of the vector field \( Z \) associated with the trajectories in \( S^2 \) we project its orbits in a plane using the central projection. Since we have a bijection between a hemisphere of the sphere and \( W \subset \mathbb{R}^2 \), the properties of these two systems will be related. In this way, let us consider the central projection given by Equation (7). Applying the central projection in the hemisphere \( H_1 = \{(x, y, z) \in S^2 : y > 0 \} \) of \( S^2 \) we get the trajectories in \( \mathbb{R}^2 \) are modeled by \( \pi_c(\pi_N(x, y_1(x))) \) and \( \pi_c(\pi_S(x, y_2(x))) \) where

\[
\pi_c(\pi_N(x, y_1(x))) = \left( \frac{x}{y_1(x)}, \frac{x^2 + y_1^2(x) - 1}{2y_1(x)} \right) \tag{15}
\]

and

\[
\pi_c(\pi_S(x, y_2(x))) = \left( \frac{x}{y_2(x)}, \frac{1 - x^2 - y_2^2(x)}{2y_2(x)} \right). \tag{16}
\]

We observe that \( \pi_c \circ \pi_N \) and \( \pi_c \circ \pi_S \) provide a change of variables in \( \mathbb{R}^2 \). Indeed, by \( \pi^{-1}_N(\pi^{-1}_c(u, v)) = (x, y_1(x)) \) we obtain the change of variables

\[
\begin{cases}
x &= \frac{u}{\sqrt{u^2 + v^2 + 1} - 1}, \\
y_1(x) &= \frac{1}{\sqrt{u^2 + v^2 + 1} - 1}.
\end{cases} \tag{17}
\]

Similarly, by \( \pi^{-1}_S(\pi^{-1}_c(u, v)) = (x, y_2(x)) \) we have the change of variables

\[
\begin{cases}
x &= \frac{u}{\sqrt{u^2 + v^2 + 1} + 1}, \\
y_2(x) &= \frac{1}{\sqrt{u^2 + v^2 + 1} + 1}.
\end{cases} \tag{18}
\]

Due to these changes of variables, we can find the piecewise smooth vector field associated with these trajectories. Indeed, by Equation (17) we have

\[
\frac{du}{dt} = (\sqrt{u^2 + v^2 + 1} - 1 - v) \left( 2 - \frac{u^4}{(\sqrt{u^2 + v^2 + 1} - v)^4} + \frac{3u^2}{(\sqrt{u^2 + v^2 + 1} - v)^2} \right) =: G_1(u, v)
\]

and

\[
\frac{dv}{dt} = \frac{u(-1 + 2u^4 + v^2 + 4v^2\sqrt{u^2 + v^2 + 1} - 4v^2\sqrt{u^2 + v^2 + 1} + u^2(2 + 8v^2 - 6v\sqrt{u^2 + v^2 + 1}))}{(v - \sqrt{u^2 + v^2 + 1})^4} =: G_2(u, v)
\]
Therefore, the system in the plane $uv$, for $v < 0$, associated with the trajectories of the vector field $X$ is given by

\[
\begin{cases}
\dot{u} = G_1(u, v) \\
\dot{v} = G_2(u, v)
\end{cases}
\]  \hspace{1cm} (19)

In the same way, by Equation (18) we have

\[
\frac{du}{dt} = -\frac{2 + u^3 + 4v^2 + 4v\sqrt{1 + u^2 + v^2}}{2(v + \sqrt{1 + u^2 + v^2})} =: F_1(u, v)
\]

and

\[
\frac{dv}{dt} = \frac{u(4 + 2u^2 + 6v^2 + 6v\sqrt{1 + u^2 + v^2} - u(1 + v^2 + v\sqrt{1 + u^2 + v^2}))}{2(v + \sqrt{1 + u^2 + v^2})^2} =: F_2(u, v)
\]

Then, the system in the plane $uv$, for $v > 0$, associated with the trajectories of the vector field $Y$ is given by

\[
\begin{cases}
\dot{u} = F_1(u, v) \\
\dot{v} = F_2(u, v)
\end{cases}
\]  \hspace{1cm} (20)

Therefore, we obtain the piecewise smooth vector field

\[
Z_1(u, v) = \begin{cases}
F(u, v) = (F_1(u, v), F_2(u, v)) & \text{if } v \geq 0, \\
G(u, v) = (G_1(u, v), G_2(u, v)) & \text{if } v \leq 0.
\end{cases}
\]  \hspace{1cm} (21)

whose switching manifold is given by $\Sigma = f^{-1}(0)$ with $f(u, v) = v$. See Figure 6.

The next result will bring some properties of the vector field $Z_1$.

**Proposition 2.** Let $Z_1$ be given by (21). Then the fold points of the vector field $G$ are $q_0 = (0, 0)$, $q_1 = \left(\frac{-2 - \sqrt{3}}{\sqrt{-1 + \sqrt{3}}}, 0\right)$ and $q_2 = \left(\frac{2 - \sqrt{3}}{\sqrt{-1 + \sqrt{3}}}, 0\right)$. Moreover, $q_0$ is a visible fold point while $q_1$ and $q_2$ are invisible fold points.

**Proof.** In fact, by a straightforward calculation we get $Gf(u, v) = G_2(u, v)$. Then $Gf(u, 0) = 0$ if, and only if,

\[G_2(u, 0) = \frac{u(-1 + 2u^2 + 2u^4)}{(1 + u^2)^2} = 0\]

which nulls at zero, $\frac{2 - \sqrt{3}}{\sqrt{-1 + \sqrt{3}}}$ and $-\frac{2 - \sqrt{3}}{\sqrt{-1 + \sqrt{3}}}$. Also, we have $G^2f(q_0) = -2 < 0$ and $G^2f(q_1) = G^2f(q_2) = 8\sqrt{6}/(1 + \sqrt{3})^2 > 0$. This concludes the proof. \hfill \Box

We may notice that the fold points of the vector field $G$ are images of the points $p_0$, $p_1$ and $p_2$ by the function composition $\pi_c \circ \pi_N$.

**Proposition 3.** Let $Z_1$ be given by (21). Then the unique fold point of the vector field $F$ is $q_0 = (0, 0)$ which is an invisible fold point.

**Proof.** Indeed, we have $Ff(u, v) = \langle F(u, v), \nabla f(u, v) \rangle = F_2(u, v)$. Since

\[F_2(u, 0) = \frac{u(4 - u + 2u^2)}{2(1 + u^2)}\]

then the equation $Ff(u, 0) = F_2(u, 0) = 0$ has $u = 0$ as unique real root. Furthermore, $F^2f(0, 0) = -2 < 0$ and therefore $q_0$ is an invisible fold point. \hfill \Box
Note that the vector field $Z_1$ given by (21) has a sliding region $\Sigma^1_s$ given by the interval $(-\sqrt{2} - \sqrt{3}/\sqrt{-1 + \sqrt{3}}, 0)$ and it has an escaping region $\Sigma^1_e$ given by the interval $(0, \sqrt{2} - \sqrt{3}/\sqrt{-1 + \sqrt{3}})$ where we can define the sliding vector field given by Definition 2.2.

**Proposition 4.** The sliding vector field associated with the vector field $Z_1$ does not have equilibria.

**Proof.** Let $Z^T_1$ be the sliding vector field associated with $Z_1$. By Equation (2), given $(u, v) \in \Sigma^1_s \cup \Sigma^1_e \subset \Sigma = \{v = 0\}$, we have that

$$Z^T_1(u, v) = \frac{Gf(u, v)F(u, v) - Ff(u, v)G(u, v)}{Gf(u, v) - Ff(u, v)}.$$

Note that, by Definition 2.2, the sliding vector field $Z^T_1$ is defined just in the escaping and sliding regions of the switching manifold $\Sigma = \{v = 0\}$, which is a straight line in this case. So, $Z^T_1$ is one-dimensional. Then, by a straightforward calculation, we get

$$Z^T_1(u) = Z^T_1(u, 0) = \frac{2(3 - u + 18u^2 - 4u^3 + 17u^4 - u^5 + 4u^6 + u^7)}{\sqrt{1 + u^2}(-6 + u - 2u^2 + u^3 + 2u^4)}.$$  \hfill (22)

So, it is easy to see that $Z^T_1$ does not have equilibria in its domain $\Sigma^1_s \cup \Sigma^1_e$. Moreover, since the sliding vector field can be extended to its closure we need to verify that $\lim_{u \to a} Z^T_1(u, 0)$ is non-zero for all $a \in \Sigma^1_s \cup \Sigma^1_e$. Since $\lim_{u \to 0} Z^T_1(u, 0) = 1$ and

$$\lim_{u \to a} Z^T_1(u, 0) = \lim_{u \to a} Z^T_1(u, 0) = \frac{2}{(-1 + \sqrt{3})^2},$$

we can conclude that $Z^T_1$ does not have equilibria. \hfill $\square$

The analogous of the central projection can be applied to the hemisphere $H_2 = \{(x, y, z) \in S^2 : y < 0\}$ of $S^2$ in order to obtain a planar piecewise smooth vector field $Z_2$ with similar properties of system $Z_1$ given by (21). Indeed, let us consider the projection $\pi_r$ given by Equation (9). Applying $\pi_r$ in the hemisphere $H_0$ of $S^2$ we get the trajectories in $\mathbb{R}^2$ are modeled by $\pi_r(\pi_N(x, y_1(x)))$ and $\pi_r(\pi_S(x, y_2(x)))$ where

$$\pi_r(\pi_N(x, y_1(x))) = \left(\frac{-x}{y_1(x)}, \frac{1 - x^2 - y_1^2(x)}{2y_1(x)}\right)$$ \hfill (23)

and

$$\pi_r(\pi_S(x, y_2(x))) = \left(\frac{-x}{y_2(x)}, \frac{x^2 + y_2^2(x) - 1}{2y_2(x)}\right).$$ \hfill (24)

Proceeding in the same way used to find the vector field $Z_1$, applying the changes of variables provided by $\pi^{-1}_S(\pi^{-1}_N(u, v)) = (x, y_2(x))$ and $\pi^{-1}_N(\pi^{-1}_S(u, v)) = (x, y_1(x))$ we can obtain the piecewise smooth vector field

$$Z_2(u, v) = \begin{cases} 
Q(u, v) = (Q_1(u, v), Q_2(u, v)) & \text{if } v \geq 0, \\
G(u, v) = (G_1(u, v), G_2(u, v)) & \text{if } v \leq 0.
\end{cases}$$ \hfill (25)

where the switching manifold is given by $\Sigma = f^{-1}(0)$ with $f(u, v) = v$, $Q_1(u, v) = \frac{-2 + w^3 - 4v^2 - 4v\sqrt{1 + u^2 + v^2}}{2(v + \sqrt{1 + u^2 + v^2})}$.
and
\[ Q_2(u, v) = \frac{u (4 + 2u^2 + 6v^2 + 6v\sqrt{1 + u^2 + v^2} + u (1 + v^2 + v\sqrt{1 + u^2 + v^2}))}{2 (v + \sqrt{1 + u^2 + v^2})^2}. \]

See Figure 7.

**Proposition 5.** Let \( Z_2 \) be given by (25). Then the unique fold point of the vector field \( Q \) is \( q_0 = (0, 0) \) which is an invisible fold point.

**Proof.** Indeed, we have \( Qf(u, v) = \langle Q(u, v), \nabla f(u, v) \rangle = Q_2(u, v) \). Since
\[ Q_2(u, 0) = \frac{u(4 + u + 2u^2)}{2(1 + u^2)} \]
then the equation \( Qf(u, 0) = Q_2(u, 0) = 0 \) has \( u = 0 \) as unique real root. Furthermore, \( Q^2f(0, 0) = -2 < 0 \) and therefore \( q_0 \) is a invisible fold point.

The vector field \( Z_2 \) given by (25) has a sliding region given by the interval \( \Sigma_s^2 = (-\sqrt{1 - \frac{1}{\sqrt{3}}}/\sqrt{1 + \frac{1}{\sqrt{3}}, 0}) \) and it has an escaping region given by \( \Sigma_e^2 = \left(0, \sqrt{2 - \frac{1}{\sqrt{3}}}/\sqrt{1 + \frac{1}{\sqrt{3}}} \right) \) where we can define the sliding vector field associated with the vector field \( Z_2 \) given by definition 2.2.

**Proposition 6.** The sliding vector field associated to the vector field \( Z_2 \) does not have equilibria.

**Proof.** Let \( Z_2^T \) be the sliding vector field associated with \( Z_2 \). By Equation (2) we have that
\[ Z_2^T(u, v) = \frac{Gf(u, v)Q(u, v) - Qf(u, v)G(u, v)}{Gf(u, v) - Qf(u, v)}. \]
Note that, by Definition 2.2, the sliding vector field \( Z_2^T \) is one-dimensional since the switching manifold \( \Sigma = \{v = 0\} \) is a straight line. Then we get
\[ Z_2^T(u) = Z_2^T(u, 0) = \frac{2\sqrt{1 + u^2} (3 + u + 18u^2 + 4u^3 + 17u^4 + u^5 + 4u^6 - u^7)}{6 + u + 8u^2 + 2u^3 + u^5 + 2u^6}. \] (26)
So, it is easy to see that \( Z_2^T \) does not have equilibria in its domain \( \Sigma_s^2 \cup \Sigma_e^2 \). Moreover, since the sliding vector field can be extended to its closure we need to verify that
\[ \lim_{u \to a} Z_2^T(u, 0) \text{ is non-zero for all } a \in \Sigma_s^2 \cup \Sigma_e^2. \]
Since \( \lim_{u \to 0} Z_2^T(u, 0) = 1 \) and
\[ \lim_{u \to q_1} Z_2^T(u, 0) = \lim_{u \to q_2} Z_2^T(u, 0) = \frac{2}{(-1 + \sqrt{3})^{3/2}}, \]
we can conclude that \( Z_2^T \) does not have equilibria.

5. **Obtainment of non-trivial minimal sets.** To show the existence of non-trivial minimal sets associated with the trajectories in \( S^2 \) we are going to use the planar piecewise smooth vector fields found in Section 4. The next definitions (written for PSVF’s) are presented in [3, 4]. We stress that these definitions are analogous to those for smooth vector fields stated at the beginning of this paper.

**Definition 5.1.** A set \( M \subset \mathbb{R}^n \) is invariant for \( Z \in \Omega^r \) if for each \( p \in M \) and all global trajectory \( \Gamma_Z(t, p) \) passing through \( p \) it holds \( \Gamma_Z(t, p) \subset M \).

**Definition 5.2.** Consider \( Z \in \Omega^r \). A set \( M \subset \mathbb{R}^n \) is minimal for \( Z \) if:

(a) \( M \neq \emptyset \);
(b) $M$ is compact;
(c) $M$ is invariant for $Z$;
(d) $M$ does not contain proper subset satisfying (a)-(c).

Since the orbit passing through sliding or escaping region on the switching manifold can depart from it when the time goes to future or past, we can introduce some definitions concerning positive and negative invariance for global trajectories.

**Definition 5.3.** A set $M \subset \mathbb{R}^n$ is positively invariant (respectively, negatively invariant) for $Z = \Omega^+$ if for each $p \in M$ and all positive global trajectory $\Gamma_Z^+(t, p)$ (respectively, negative global trajectory $\Gamma_Z^-(t, p)$) passing through $p$ it holds $\Gamma_Z^+(t, p) \subset M$ (respectively, $\Gamma_Z^-(t, p) \subset M$).

**Remark 2.** We may notice that a set is invariant if, and only if, it is positively invariant and negatively invariant.

Let us consider the vector field $Z_1$ given by (21) and the particular trajectory $\varphi_G(q_0)$ of the vector field $G$ passing through $q_0$. Let $a$ be the intersection point between $\varphi_G(q_0)$ and $\Sigma \cap \{u > 0\}$ and consider $\varphi_F(a)$. Furthermore, let us consider $c = \varphi_F(a) \cap (\Sigma \cap \{u < 0\})$. These particular curves and the segment $I_1 = (a, c)$ contained in $\Sigma$, delimit a bounded region of the plane that we call $K_1$. See Figure 8.

**Lemma 5.4.** For all $p \in K_1$ we have that $q_0$ belongs to an orbit passing through $p$, where $q_0 = (0, 0)$ is a visible fold point for $G$ and an invisible fold point for $F$.

**Proof.** In fact, the positive global trajectory of $p \in K_1$ reaches the sliding region between $q_2$ and $q_0$ and slides to $q_0$ after some time $t_p$. \hfill $\square$

**Proposition 7.** Consider the vector field $Z_1$ given by (21). The set $K_1$ is positively invariant for $Z_1$.

**Proof.** Indeed, by Lemma 5.4 a positive global trajectory of any point $p$ in $K_1$ meets $q_0$ for some time $t_p$. Since $q_0$ is a visible tangency point for $G$ and $q_0 \in \partial \Sigma_e \cap \partial \Sigma_s$ according to the fourth item of Definition 2.3 any trajectory passing through $q_0$ remains in $K_1$. Therefore, $K_1$ is $Z_1$-positively invariant. \hfill $\square$

**Remark 3.** We may observe that $K_1$ is not $Z_1$-negatively invariant.

Let us consider the vector field $Z_2$ given by (25) and the particular trajectory $\varphi_G(q_0)$ of the vector field $G$ passing through $q_0$. Let $-a$ be the intersection point between $\varphi_G(q_0)$ and $\Sigma \cap \{u < 0\}$ and consider $\varphi_F(-a)$. Furthermore, let us consider $b = \varphi_F(-a) \cap (\Sigma \cap \{u > 0\})$. These particular curves and the segment $I_2 = (b, a)$ contained in $\Sigma$, delimit a bounded region of the plane that we call $K_2$. See Figure 9.

**Lemma 5.5.** For all $p \in K_2$ we have that $q_0$ belongs to an orbit passing through $p$, where $q_0 = (0, 0)$ is a visible fold point for $G$ and an invisible fold point for $Q$.

**Proof.** In fact, the negative global trajectory of $p \in K_2$ reaches the escaping region between $q_0$ and $q_1$ and slides to $q_0$ after some time $t_p$. \hfill $\square$

The next result follows in a similar way from Proposition 7.

**Proposition 8.** Consider the vector field $Z_2$ given by (25). The set $K_2$ is negatively invariant for $Z_2$. Moreover, $K_2$ is not positively invariant.
Since we have a bijection between the trajectories of the planar piecewise smooth vector field $Z_1$ in $W$ and the trajectories defined in the hemisphere $H_1$ of $S^2$ we have a respective behavior in the trajectories given by Equations (11) and (13) defined in the sphere $S^2$. Similarly, there is a bijection between the trajectories of the planar piecewise smooth vector field $Z_2$ and the trajectories defined in the hemisphere $H_2$ of $S^2$. We can notice that the trajectories defined in the sphere $S^2$ by Equations (11) and (13) generate a piecewise smooth vector field $Z$ tangent to $S^2$ whose switching manifold is $S^1$.

**Proposition 9.** The $\omega$-limit of every point of $S^2$ is contained in $\pi_c^{-1}(K_1)$.

**Proof.** In fact, since $\pi_c^{-1}(K_1)$ is $Z$-positively invariant, given $p \in \pi_c^{-1}(K_1)$ we have that $\omega(p) \in \pi_c^{-1}(K_1)$. Now, let us suppose $p = (-1, 0, 0)$. Since the vector field $X$ is symmetric we have that the orbit passing through $p$ goes to the point $q = (1, 0, 0)$.

Solving the initial value problem given by (14) with $x_0 = 1$ we can conclude that the orbit passing through the point $(1, 0)$ in the $xy$-plane intersects $S^1$ in a point $(x, y)$ with $y > 0$. So, the trajectory in $S^2$ passing through $p$ will reach $\pi_c^{-1}(\Sigma)$ and goes to $\pi_c^{-1}(K_1)$ by Proposition 7. See Figure 10. For $p \in H_1$ we have that $\pi_c^{-1}(K_1)$ is an attractor. Since $\pi_c^{-1}(K_2)$ is a repeller the orbit passing through $p \in H_2$ will reach $H_1$ and therefore goes to $\pi_c^{-1}(K_1)$.

The next result follows in a similar way from Proposition 9.

**Proposition 10.** The $\alpha$-limit of every point of $S^2$ is contained in $\pi_c^{-1}(K_2)$.

In this context, we can prove the following result.

**Proposition 11.** $S^2$ is minimal for the vector field $Z$ generated by the trajectories described by Equations (11) and (13).

**Proof.** Indeed, it is easy to see that $S^2$ is a non-empty compact set. Moreover, $S^2$ is $Z$-invariant. Now, let $M$ be a $Z$-invariant proper subset of $S^2$. We have that the $\omega$-limit of every point of $S^2$ is contained in $\pi_c^{-1}(K_1)$ and the $\alpha$-limit is contained in $\pi_c^{-1}(K_2)$. So, by Lemma 5.4, given $p \in M$ we have that $q_0$ belongs to a positive trajectory of $p$. Since $M$ is $Z$-invariant we conclude that $q_0 \in M$. Let us consider $q \notin M$. Similarly we can conclude that $q_0$ belongs to a positive trajectory of $q$. Therefore, since $q_0 \in M$ we have that $q \in M$. This is a contradiction.

Theorem 1.1 is an immediate consequence of Proposition 11. Therefore, we assure the existence of a piecewise smooth vector field tangent to $S^2$ where $S^2$ itself is a non-trivial minimal set.

5.1. **Final conclusions.** At this point, we can introduce a discussion about the density of orbits in minimal sets for PSVF and about (non)wandering points for PSVF.

As defined in [8],

a point $p$ is a nonwandering point for a PSVF $Z$ if each one of its neighborhoods meets arbitrarily large iterations of itself, for some global trajectory.

When a point is not nonwandering then it is called wandering. This definition coincides with the classical one when the PSVF is a smooth vector field (taking $X = Y$ in System (1)).

Consider the sets $\pi_c^{-1}(K_1)$ and $\pi_c^{-1}(K_2)$ described above. Given a point $q \in S^2 \setminus (\pi_c^{-1}(K_1) \cup \pi_c^{-1}(K_2))$, every global trajectory passing through $q$ converges to $\pi_c^{-1}(K_1)$ (see Proposition 9). Moreover, given a small neighborhood $V_q \subset S^2$ of $q$,
none global trajectory passing through a point \( \tilde{q} \in V_q \) returns to \( V_q \) since it enters the positively invariant set \( \pi^{-1}_c(K_1) \) (see Proposition 7). As a consequence, \( q \) is a wandering point for \( \bar{Z} \) and there is neither a dense nor periodic orbit passing through \( q \). However, it is easy to see that given a point \( \tilde{q} \in \{ \pi^{-1}_c(K_1) \cup \pi^{-1}_r(K_2) \} \subset S^2 \), then \( \tilde{q} \) is a nonwandering point for \( \bar{Z} \) and there is not a dense orbit passing through \( \tilde{q} \). According to the choice on \( \tilde{q} \), it is possible the occurrence of a periodic orbit passing through \( \tilde{q} \).

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![Figure 1. Sliding Vector Field.](image)

![Figure 2. The \( \omega \)-limit of \( p \) is disconnected.](image)

**REFERENCES**

[1] D. C. Braga, A. F. da Fonseca and L. F. Mello, Study of limit cycles in piecewise smooth perturbations of Hamiltonian centers via regularization method, *Electronic Journal of Qualitative Theory of Differential Equations*, 79 (2017), 1–13.

[2] L. E. J. Brouwer, On continuous vector distributions on surfaces, in *Proceedings of the Royal Netherlands Academy of Arts and Sciences (KNAW)*, 11 (1909), 850–858, [https://www.dwc.knaw.nl/DL/publications/PU00013599.pdf](https://www.dwc.knaw.nl/DL/publications/PU00013599.pdf).
Figure 3. Item (a) shows the projection of the trajectories of $X$ on $S^2$ by $\pi_N$. In (b) we have the projection of the trajectories of $Y$ by $\pi_S$.

Figure 4. Trajectories in $S^2$.

Figure 5. Displacement function.

[3] C. A. Buzzi, T. de Carvalho and R. D. Euzébio, Chaotic planar piecewise smooth vector fields with non-trivial minimal sets, *Ergodic Theory and Dynamical Systems*, 36 (2016), 458–469.
Figure 6. Trajectories of the vector field $Z_1$.

Figure 7. Trajectories of the vector field $Z_2$.

Figure 8. Piecewise smooth vector field $Z_1$ and region $K_1$.

Figure 9. Piecewise smooth vector field $Z_2$ and region $K_2$. 

Figure 10. Trajectory in $S^2$ passing through $p = (-1, 0, 0)$.

[4] C. A. Buzzi, T. Carvalho and R. D. Euzébio, On Poincaré-Bendixson theorem and non-trivial minimal sets in planar nonsmooth vector fields, *Publicacions Matemàtiques*, 62 (2018), 113–131.

[5] T. Carvalho and L. F. Gonçalves, Combing the hairy ball using a vector field without equilibria, *Journal of Dynamical and Control Systems*, 26 (2020), 233–242.

[6] R. Cristiano, T. Carvalho, D. J. Tonon and D. J. Pagano, Hopf and Homoclinic bifurcations on the sliding vector field of switching systems in $\mathbb{R}^3$: A case study in power electronics, *Physica D: Nonlinear Phenomena*, 347 (2017), 12–20.

[7] T. Carvalho, D. D. Novaes and L. F. Gonçalves, Sliding Shilnikov connection in Filippov-type predator-prey model, *Nonlinear Dynamics*, 100 (2020), 2973–2987.

[8] T. de Carvalho, On the closing lemma for planar piecewise smooth vector fields, *Journal de Mathématiques Pures et Appliquées*, 106 (2016), 1174–1185.

[9] T. de Carvalho and D. J. Tonon, Generic bifurcations of planar Filippov systems via geometric singular perturbations, *Bull. Belg. Math. Soc. Simon Stevin*, 18 (2011), 861–881.

[10] A. Denjoy, Sur les courbes définies par les équations différentielles à la surface du tore, *Journal de Mathématiques Pures et Appliquées*, 11 (1932), 333–376, http://eudml.org/doc/234887.

[11] M. di Bernardo, K. H. Johansson and F. Vasca, Self-oscillations and sliding in relay feedback systems: Symmetry and bifurcations, *International Journal of Bifurcation and Chaos*, 11 (2001), 1121–1140.

[12] D. D. Dixon, Piecewise deterministic dynamics from the application of noise to singular equations of motion, *Journal of Physics A: Mathematical and General*, 28 (1995), 5539–5551.

[13] N. M. Drissa, *Fixed Point, Game and Selection Theory: From the Hairy Ball Theorem to A Non Hair-Pulling Conversation*, PhD thesis, Université Paris 1 Panthéon-Sorbonne, 2016, http://hdl.handle.net/10579/8840.

[14] A. F. Filippov, *Differential Equations with Discontinuous Righthand Sides*, Mathematics and its Applications, 1st edition, Springer Netherlands, 1988.

[15] C. Gutiérrez, Smoothing continuous flows on two-manifolds and recurrences, *Ergodic Theory and Dynamical Systems*, 6 (1986), 17–44.

[16] A. Jacquemard and D. J. Tonon, Coupled systems of non-smooth differential equations, *Bulletin des Sciences Mathématiques*, 136 (2012), 239–255.

[17] T. Kousaka, T. Kido, T. Ueta, H. Kawakami and M. Abe, Analysis of border-collision bifurcation in a simple circuit, in *2000 IEEE International Symposium on Circuits and Systems. Emerging Technologies for the 21st Century. Proceedings (IEEE Cat No.00CH36353)*, 2 (2000), 481–484.

[18] V. Kriván, On the gause predator-prey model with a refuge: A fresh look at the history, *Journal of Theoretical Biology*, 274 (2011), 67–73.
[19] R. Leine and H. Nijmeijer, *Dynamics and Bifurcations of Non-Smooth Mechanical Systems*, Lecture Notes in Applied and Computational Mechanics, 1st edition, Springer-Verlag Berlin Heidelberg, 2004.

[20] J. Llibre, P. R. Silva and M. A. Teixeira, Regularization of discontinuous vector fields on $\mathbb{R}^3$ via singular perturbation, *Journal of Dynamics and Differential Equations*, 19 (2007), 309–331.

[21] J. Llibre and M. A. Teixeira, Regularization of discontinuous vector fields in dimension three, *Discrete & Continuous Dynamical Systems - A*, 3 (1997), 235–241.

[22] J. Milnor, Analytic proofs of the “hairy ball theorem” and the brouwer fixed point theorem, *The American Mathematical Monthly*, 85 (1978), 521–524.

[23] L. Perko, *Differential Equations and Dynamical Systems*, Texts in Applied Mathematics, 3rd edition, Springer-Verlag New York, 2001.

[24] S. H. Piltz, M. A. Porter and P. K. Maini, Prey switching with a linear preference trade-off, *SIAM Journal on Applied Dynamical Systems*, 13 (2014), 658–682.

[25] D. S. Rodrigues, P. F. A. Mancera, T. Carvalho and L. F. Gonçalves, Sliding mode control in a mathematical model to chemoimmunotherapy: The occurrence of typical singularities, *Applied Mathematics and Computation*, 387 (2020), 124782.

[26] F. D. Rossa and F. Dercole, Generic and generalized boundary operating points in piecewise-linear (discontinuous) control systems, in *2012 IEEE 51st IEEE Conference on Decision and Control (CDC)*, (2012), 7714–7719.

[27] A. J. Schwartz, A generalization of a Poincaré-Bendixson Theorem to closed two-dimensional manifolds, *American Journal of Mathematics*, 85 (1963), 453–458.

[28] P. A. Schweitzer, Counterexamples to the Seifert Conjecture and opening closed leaves of foliations, *Annals of Mathematics*, 100 (1974), 386–400.

[29] J. Sotomayor and A. L. F. Machado, Structurally stable discontinuous vector fields in the plane, *Qualitative Theory of Dynamical Systems*, 3 (2002), 227–250.

[30] J. Sotomayor and M. A. Teixeira, Regularization of discontinuous vector fields, in *International Conference on Dynamical Equations, Lisboa, 1995*, World Scientific Publishing, (1998), 207–223.

[31] E. T. Whittaker and G. Robinson, *The Calculus of Observations: A Treatise on Numerical Mathematics*, 4th edition, Blackie & Son limited, 1954.

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