A Semidefinite Program for Structured Blockmodels

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Abstract

Semidefinite programs have recently been developed for the problem of community detection, which may be viewed as a special case of the stochastic blockmodel. Here, we develop a semidefinite program that can be tailored to other instances of the blockmodel, such as non-assortative networks and overlapping communities. We establish label recovery in sparse settings, with conditions that are analogous to recent results for community detection. In settings where the data is not generated by a blockmodel, we give an oracle inequality that bounds excess risk relative to the best blockmodel approximation. Simulations are presented for community detection, for overlapping communities, and for latent space models.

1 Introduction

The stochastic blockmodel [19, 17] is a popular class of models for network data in which each node is assumed to belong to a latent class. Various sub-families of the blockmodel now exist, such as community structure [20], hierarchical community structure [23, 27], and overlapping blockmodels [29, 33, 32], as well as relatives such as latent space models [18], mixed membership [1], degree-corrected blockmodels [20], and time-varying blockmodels [24].

For all of these models, estimation of the latent nodal classes is an active area of research. For blockmodels, spectral methods are known to yield asymptotically consistent estimates, provided that the network is sufficiently large and dense [22, 25]. For the special case of community structure, it is additionally known that specialized methods can achieve weakly consistent estimates even when spectral methods fail completely due to sparsity [26, 21, 14, 2, 31]. Examples of such methods include semidefinite programming [3, 17, 25] and message passing [13, 32]. For other variants of the blockmodel and their relatives, estimation methods also exist but are less understood; in particular, theory analogous to that of community detection does not yet seem to exist for these cases.

To address this gap, we show in this paper that semidefinite programming can be applied not only to community detection, but also to other blockmodel
sub-families as well. Specifically, we propose a semidefinite program that can be tailored to any specific instance of the blockmodel. For this program, we prove estimation bounds that are analogous to those already known for community detection, including weak consistency in the bounded degree setting. When the data is not generated from a blockmodel, the semidefinite program can be used to construct a “de-noised” version of the data matrix, and we provide an oracle inequality bounding its error relative to the best blockmodel approximation.

The organization of the paper is as follows. Section 2 presents the semidefinite program. Section 3 presents a convergence analysis for sparse data. Section 4 discusses numerical optimization. Section 5 gives simulations results. Proofs are contained in the appendix.

2 Problem Formulation

In this section, we present the generative models that we will consider in this paper; derive a semidefinite relaxation for the combinatorial problem of estimating the latent nodal classes; and present estimators for blockmodel and non-blockmodel settings.

2.1 Preliminary Notation

Given a matrix $M \in \mathbb{R}^{nK \times nK}$, let $M^{(i)} = \mathbb{R}^{K \times K}$ for $i,j \in [n]$ denote its $(i,j)$th submatrix of size $K \times K$, and similarly for $x \in \mathbb{R}^{nK}$ let $x^{(i)} \in \mathbb{R}^{K}$ for $i = 1, \ldots, n$ denote its $i$th subvector of length $K$, so that

$$
M = \begin{bmatrix}
M^{(1,1)} & \cdots & M^{(1,n)} \\
\vdots & \ddots & \vdots \\
M^{(n,1)} & \cdots & M^{(n,n)}
\end{bmatrix}
\quad \text{and} \quad
x = \begin{bmatrix}
x^{(1)} \\
\vdots \\
x^{(n)}
\end{bmatrix}.
$$

We will use $M_{ab}^{(ij)}$ to denote the $(a,b)$th entry of the submatrix $M^{(ij)}$, and likewise use $x_a^{(i)}$ to denote the $a$th entry of the subvector $x^{(i)}$. This implies that $M_{ab}^{(ij)} = M_{a+(i-1)K,b+(j-1)K}$ and $x_a^{(i)} = x_{a+(i-1)K}$.

2.2 Generative Models

**Stochastic Blockmodel** Let $A \in \{0, 1\}^{n \times n}$ denote the symmetric adjacency matrix of a undirected network with $n$ nodes. In a stochastic blockmodel with $K$ classes, each node has a random latent class $z_i \in [K]$, and the upper triangular entries of $A$ are independent Bernoulli when conditioned on $z$:

$$ z_i \sim \text{Discrete}(\pi) \quad \text{and} \quad A_{ij} \sim \text{Bernoulli}(\theta_{z_i,z_j}), \quad i, j \in [n], \ i < j \quad (1) $$

$$ A_{ji} = A_{ij} \quad (2) $$
where \( \pi \) is a probability distribution over \([K]\) giving the expected class frequencies, and \( \theta \in [0, 1]^{K \times K} \) is a symmetric matrix that gives the connection probabilities between each class type.

**General Model** Under a more general model, \( A \in \{0, 1\}^{n \times n} \) is a random matrix generated by

\[
A_{ij} \sim \text{Bernoulli}(P_{ij}) \quad i < j
\]

\[
A_{ji} = A_{ij},
\]

where \( P \in [0, 1]^{n \times n} \) is symmetric and satisfies \( P_{ii} = 0 \) for \( i \in [n] \). It can be seen that the stochastic blockmodel is a special case of (3), where \( P_{ij} = \theta_{z_i, z_j} \).

### 2.3 Semidefinite program

We will assume that \( A \in \{0, 1\}^{n \times n} \) is observed, and the estimation task is to find \( z \in [K]^n \) maximizing the generic combinatorial problem

\[
\max_{z \in [K]^n} \sum_{i=1}^{n} \sum_{j=1}^{n} f_{ij}(z_i, z_j),
\]

under some choice of objective functions \( f_{ij} : [K]^2 \to \mathbb{R} \) for \( i, j \in [n] \). In this paper, we will let \( \{f_{ij}\} \) equal the likelihood function

\[
f_{ij}(z_i, z_j) = A_{ij} \log P_{z_i z_j} + (1 - A_{ij}) \log(1 - \hat{P}_{z_i z_j}),
\]

in which case (4) finds the maximum likelihood assignment of \( z \) under a specified parameter matrix \( \hat{P} \in [0, 1]^{K \times K} \). Note that \( \hat{P} \) may differ from the actual generative model for \( A \).

Optimizing (4) is not computationally tractable, so we will relax it into a semidefinite program. Let \( F \in \mathbb{R}^{nK \times nK} \) denote a matrix with submatrices \( F^{(ij)} \in \mathbb{R}^{K \times K} \) given by

\[
F^{(ij)}_{ab} = f_{ij}(a, b) \quad a, b \in [K],
\]

so that (4) can be expressed as

\[
\max_{z \in [K]^n} \sum_{i=1}^{n} \sum_{j=1}^{n} e_{z_i}^T F^{(ij)} e_{z_j},
\]

where \( e_1, \ldots, e_K \in \{0, 1\}^K \) denote the canonical basis in \( \mathbb{R}^K \). This can be further rewritten as

\[
\max_{z \in [K]^n, x \in \{0, 1\}^{nK}} \langle F, xx^T \rangle \quad \text{subject to} \quad x = \begin{bmatrix} e_{z_1} \\ \vdots \\ e_{z_n} \end{bmatrix}.
\]
This suggests the following semidefinite program relaxation, where \( xx^T \) is approximated by a positive semidefinite matrix \( X \):

\[
\begin{align*}
\max_{X \in \mathbb{R}^{nK \times nK}} & \langle F, X \rangle \\
\text{s.t.} & \quad X \succeq 0, X \geq 0 \\
& \quad \sum_{a=1}^{K} \sum_{b=1}^{K} X_{ab}^{(ij)} = 1 \quad \forall i, j \in [n],
\end{align*}
\]

where \( X \succeq 0 \) denotes that \( X \) positive semidefinite, and \( X \geq 0 \) denotes that \( X \) is elementwise non-negative. For any feasible \( X \in \mathbb{R}^{nK \times nK} \), each submatrix \( X^{(ij)} \in \mathbb{R}^{K \times K} \) is nonnegative and sums to one, and can be viewed as a relaxed version of the indicator matrix \( e_z e_z^T \) encoding the class pair \((z_i, z_j)\).

**2.4 Matrix Denoising and Blockmodel Estimation**

Let \( \hat{X} \in [0, 1]^{nK \times nK} \) denote the solution to the semidefinite program (8), and let \( A \) be generated from the general model (3), with generative matrix \( P \in [0, 1]^{n \times n} \). Let \( \hat{P} \in [0, 1]^{n \times n} \) denote a “MAP estimate” of \( P \), constructed by treating each submatrix \( \hat{X}^{(ij)} \) as a probability distribution over \([K]^2\):

\[
\hat{P}_{ij} = \arg \max_{\vartheta \in [0, 1]} \sum_{a=1}^{K} \sum_{b=1}^{K} \hat{X}_{ab}^{(ij)} \cdot 1\{\hat{\theta}_{ab} = \vartheta\}
\]

Alternatively, let \( \tilde{P} \) denote a randomized estimate of \( P \), where each dyad is an independent random variable with distribution

\[
\begin{align*}
\tilde{P}_{ij} &= \hat{\theta}_{ab} \text{ with probability } \hat{X}_{ab}^{(ij)} \quad \forall i < j \\
\tilde{P}_{ji} &= \tilde{P}_{ij}.
\end{align*}
\]

Let \( \hat{z} \in [K]^n \) denote the cluster labels found by spectral clustering of \( \hat{P} \) — i.e., applying \( K \)-means to the first \( K \) eigencoordinates of \( \hat{P} \). If \( A \) is generated by a blockmodel, then the generative \( P \) will be block structured, with blocks induced by \( \theta \) and \( z \). In this case, we will use \( \hat{z} \) to estimate \( z \), up to label-switching permutation.

To estimate \( \theta \) up to permutation, let \( \hat{\theta}_{\text{est}} \in [0, 1]^{K \times K} \) denote the matrix of between-block densities induced by \( \hat{z} \),

\[
\hat{\theta}_{\text{est}} = \frac{\sum_{i=1}^{n} \sum_{j=1}^{n} A_{ij} 1\{\hat{z}_i = a, \hat{z}_j = b\}}{\sum_{i=1}^{n} \sum_{j=1}^{n} 1\{\hat{z}_i = a, \hat{z}_j = b\}} \quad a, b \in [K].
\]

**2.5 Discussion**

**Related Work** Semidefinite programs have been used for community detection in [3 17 25], as well as [9] which allowed for outliers, and [11] which
allowed for degree heterogeneity. In each of these works, the network is required to exhibit assortative block structure. For the general model (3) without such restrictions, estimation has been considered in [10], but only for dense settings. To the best of our knowledge, networks that are both non-assortative and sparse (i.e., bounded degree), such as those presented in Sections 5.2 and 5.3, have not been considered in previous work.

Additionally, the semidefinite program presented here bears resemblance to one appearing in [34] for lower bounding the optimal objective value of the quadratic assignment problem (without finding a feasible solution) and also recent work on estimating pairwise alignments between objects [6].

3 Convergence Analysis

In this section, we analyze the solution of the semidefinite program (8) for both matrix denoising and label recovery. Analogous to existing results for community detection, our results will imply weak consistency (i.e., performance better than random guessing) in the regime where the average degree of $A$ is asymptotically bounded above some constant, and consistency of $\hat{z}$, as well as vanishing excess risk of $\hat{P}$, when the average degree $\to \infty$.

The organization of this section is the following: Section 3.1 defines basic notation; Section 3.2 states the required conditions; and Section 3.3 presents the convergence results, which are proven in Appendix A.

3.1 Preliminaries

The following notation will be used. Given $A \in \{0, 1\}^{n \times n}$ that is generated by (3) under some $P \in [0, 1]^{n \times n}$, let $\rho$ denote its expected density

$$\rho = \frac{1}{n(n-1)} \sum_{ij} P_{ij}.\hspace{1cm}(14)$$

Given $\hat{\theta} \in [0, 1]^{K \times K}$, let $F \in \mathbb{R}^{nK \times nK}$ denote the objective function of the semidefinite program (8), with submatrices $F^{(ij)} \in \mathbb{R}^{K \times K}$ given by

$$F^{(ij)}_{ab} = A_{ij} \log \hat{\theta}_{ab} + (1 - A_{ij}) \log(1 - \hat{\theta}_{ab}) \hspace{0.5cm} i, j \in [n], a, b \in [K].\hspace{1cm}(12)$$

Let $\hat{F} \in \mathbb{R}^{nK \times nK}$ denote an idealized version of $F$ in which $A$ is replaced by its expectation $P$, with submatrices $\hat{F}^{(ij)} \in \mathbb{R}^{K \times K}$ given by

$$\hat{F}^{(ij)}_{ab} = P_{ij} \log \hat{\theta}_{ab} + (1 - P_{ij}) \log(1 - \hat{\theta}_{ab}) \hspace{0.5cm} i, j \in [n], a, b \in [K].\hspace{1cm}(13)$$

Let $X \subset \mathbb{R}^{nK \times nK}$ denote the feasible region of the semidefinite program (8):

$$X = \left\{ X \in \mathbb{R}^{nK \times nK} : X \geq 0, X \geq 0, \text{ and } \sum_{a=1}^{K} \sum_{b=1}^{K} X^{(ij)}_{ab} = 1 \text{ for all } i, j \in [n] \right\}.\hspace{1cm}(14)$$

$^1$ which can be used as an intermediate step in a branch-and-bound algorithm.
Let \( \hat{X} \) denote any solution to the semidefinite program (8), which can be written as

\[
\text{maximize } \langle F, X \rangle \text{ over all } X \in \mathcal{X}.
\]

For any matrix \( B \in [0, 1]^{K \times K} \), let the function \( Q_B \) be given by

\[
Q_B(a, b) = \{(c, d) \in [K]^2 : B_{cd} = B_{ab}\} \quad a, b \in [K],
\]

so that \( Q_B \) identifies the subsets of \([K]^2\) that have equal values under \( B \).

### 3.2 Assumptions

Assumption 1 and 2 will apply when \( \hat{P} \) is used to estimate the general model (3). Assumption 1 requires \( P \) to have density \( \rho \) exceeding \( 1/n \). Assumption 2 bounds the entries of \( \hat{\theta} \) to differ from \( \rho \) by roughly at most a constant factor.

**Assumption 1.** Let \( A \in \{0, 1\}^{n \times n} \) be generated by (3), with \( P \in [0, 1]^{n \times n} \) (which evolves with \( n \)) satisfying \( \rho \geq 1/n \).

**Assumption 2.** The matrix \( \hat{\theta} \in [0, 1]^{K \times K} \) (which may evolve with \( n \)) satisfies

\[
\frac{\rho}{c} \leq \hat{\theta}_{ab} \leq c \rho \quad \text{for some } c > 0 \text{ and all } n \text{ and } a, b \in [K].
\]

Assumptions 3 and 4 will apply when \( A \) is generated by a stochastic blockmodel, and are sufficient to show that \( \hat{z} \) converges to the true \( z \), up to label switching. Assumption 3 describes a parametrization that is commonly used for sparse blockmodels. Assumption 4 places bounds on the misspecification between \( \hat{\theta} \) and \( \theta^* \) in the sparse blockmodel setting.

**Assumption 3.** For all \( n \), let \( A \in \{0, 1\}^{n \times n} \) and \( z \in [K]^n \) be generated by a stochastic blockmodel with parameters \((\pi, \theta^*)\). Let \( \pi \) be constant, and let \( \theta^* = \alpha B^* \), where \( \alpha \in \mathbb{R} \) satisfies \( \alpha \to 0 \) and \( \alpha > 1/n \), and \( B^* \in \mathbb{R}^{K \times K}_+ \) is constant, rank \( K \), and satisfies

\[
\sum_{a=1}^{K} \sum_{b=1}^{K} \pi_a \pi_b B^*_{ab} = 1. \quad (15)
\]

**Assumption 4.** Let \( \hat{\theta} = \hat{\alpha} \hat{B} \), where \( \hat{\alpha} = \frac{1}{n(n-1)} \sum_{i,j} A_{ij} \) and \( \hat{B} \in \mathbb{R}^{K \times K}_+ \) is a fixed matrix such that \( \hat{B} \) and \( B^* \) satisfy

\[
Q_{\hat{B}}(a, b) = Q_{B^*}(a, b) \quad \forall a, b \in [K] \quad (16)
\]

\[
B_{ab}^* \log \frac{B_{ab}}{B_{cd}} - (\hat{B}_{ab} - \hat{B}_{cd}) > 0 \quad \forall a, b \in [K] \text{ and } c,d \notin Q_{B^*}(a, b). \quad (17)
\]

Assumption 4 states that \( \hat{B} \) and \( B^* \) need not have identical values, but should have the same structure (as given by \( Q_{\hat{B}} \) and \( Q_{B^*} \)). Additionally, for all \( a, b \in [K] \), the entry \( \hat{B}_{ab} \) should be the closest element of \( \hat{B} \) to \( B_{ab}^* \), in terms of the Bregman divergence associated with the Poisson likelihood.
3.3 Results

Theorem 1 holds when $A$ is generated from the general model (3), including non-blockmodels. It gives an oracle inequality on the quality of the randomized estimate $\tilde{P}$ given by (10), relative to the best blockmodel approximation to the generative $P$.

**Theorem 1.** Let Assumptions 1 and 2 hold. Let $\tilde{P}$ denote the randomized estimate of $P$ given by (10). Then

$$\frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} KL(P_{ij}, \tilde{P}_{ij}) \leq \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} KL(P_{ij}, \hat{\theta}_{zi,zj}) + O_P\left(\frac{1}{\sqrt{n\rho}}\right),$$

with $O_P(\cdot)$ having constant terms depending only on $K$ and $c$ (which appears in Assumption 3).

Theorem 2 assumes the sparse blockmodel setting of Assumptions 3 and 4, and shows that both $\hat{P}$ and the randomized estimate $\tilde{P}$ asymptotically recover $P$, with vanishing fraction of incorrect values.

**Theorem 2.** Let Assumptions 3 and 4 hold. Let $\hat{P}$ denote the estimate of $P$ given by (9). Then

$$\frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} 1\{\hat{P}_{ij} \neq \hat{\theta}_{zi,zj}\} = O_P\left(\frac{1}{\sqrt{n\alpha}}\right),$$

with the same result if $\hat{P}$ is replaced by $\tilde{P}$ given by (10).

Corollary 1 follows from Theorem 2 and states that the eigencoordinates from $\hat{P}$, which are used to compute $\hat{z}$, converge (up to a unitary transformation) to those of $P$. As $P$ is block structured, this suggests that $\hat{z}$ will converge to $z$ up to a permutation of the labels. It is proven in Appendix C and is a direct application of the Davis-Kahan theorem in [30, Th. 4].

**Corollary 1.** Let Assumptions 3 and 4 hold, and let $V = (v_1, \ldots, v_K)$ denote the eigenvectors of $P$. Let $\hat{V} = (\hat{v}_1, \ldots, \hat{v}_K)$ denote the eigenvectors of the $K$ largest eigenvalues (in absolute value) of $\hat{P}$ or $\tilde{P}$. Let $O$ denote the set of $K \times K$ orthogonal matrices. It holds that

$$\min_{O \in O} \|\hat{V}O - V\|_F^2 \leq O_P\left(\frac{1}{\sqrt{n\alpha}}\right).$$

**Remark** Weak consistency in the bounded degree regime is implied by each of the results, in that the estimation error is bounded away from the performance of random guessing, provided that $\lim_{n \to \infty} \rho \geq c_0/n$ (for Theorem 1) or $\lim_{n \to \infty} \alpha \geq c_0/n$ (for Theorem 2), for some constant $c_0$. 

7
4 Numerical optimization

The semidefinite program \( \text{(8)} \) can be solved by the alternating direction method of multipliers (ADMM) \([8]\). While this semidefinite program is much larger than those previously introduced for community detection, speedups can often be achieved by exploiting problem structure, resulting in competitive computation times.

To solve \( \text{(8)} \) using ADMM, we introduce the decision variable \( X \in \mathbb{R}^{nK \times nK} \) and auxiliary variables \( Y, W, U, V \in \mathbb{R}^{nK \times nK} \), which are all initialized to zero and evolve according to the following rule:

\[
\begin{align*}
X_{t+1} &= \Pi_X \left( \frac{1}{2} \left( W^t - U^t + Y^t - V^t + \frac{1}{\rho} F \right) \right) \\
W_{t+1} &= \max(0, X_{t+1} + U^t) \\
Y_{t+1} &= \Pi_{S^+} (X_{t+1} + V^t) \\
U_{t+1} &= U^t + X_{t+1} - W_{t+1} \\
V_{t+1} &= V^t + X_{t+1} - Y_{t+1},
\end{align*}
\]

where \( F \in \mathbb{R}^{nK \times nK} \) is given by \( \text{(6)} \), \( \rho \in \mathbb{R} \) is a positive stepsize parameter, the operator \( \Pi_{S^+} \) denotes projection onto the set of positive semidefinite matrices, and the operator \( \Pi_X \) denotes projection onto the affine subspace of all matrices \( X \in \mathbb{R}^{nK \times nK} \) satisfying the linear constraints

\[
\sum_{a=1}^K \sum_{b=1}^K X_{ab}^{(ij)} = 1 \quad \forall i, j \in [n].
\]

The slowest step of the ADMM iterations is the computation of \( Y_{t+1} = \Pi_{S^+} (X_{t+1} + V^t) \), which requires the eigendecomposition of an \( nK \times nK \) matrix. In comparison, semidefinite programs for community detection require only the decomposition of an \( n \times n \) matrix in each ADMM iteration. However, in many settings of interest, \( X_{t+1} + V^t \) will be highly structured and have fast eigendecomposition methods.

In particular, let the \( K \times K \) submatrices \( [X_{t+1} + V^t]^{(ij)} \) for \( i, j \in [n] \) be symmetric and share a common set of orthonormal eigenvectors \( v_1, \ldots, v_K \). It then holds for some \( \ell \leq K \) and some partition \( S_0, \ldots, S_\ell \) of \([K]\) that

\[
[X_{t+1} + V^t]^{(ij)} = \sum_{l=0}^{\ell} \lambda_{ijl} \sum_{j \in S_l} v_j v_j^T \quad \forall i, j \in [n],
\]

where \( \lambda_{ijl} \) is the eigenvalue corresponding to the eigenvectors \( \{v_j : j \in S_l\} \). In this case, \( Y_{t+1} \) can be computed in the following manner:

1. Find the eigendecomposition for each submatrix \( [X_{t+1} + V^t]^{(ij)} \in \mathbb{R}^{K \times K} \), yielding eigenvectors \( v_1, \ldots, v_K \), sets \( S_0, \ldots, S_\ell \) partitioning \([K]\), and eigenvalues \( \{\lambda_{ijl}\} \) satisfying \( \text{(20)} \).

\( \setlength\parskip\emptytext \)
2. Let $\Lambda_0, \ldots, \Lambda_\ell \in \mathbb{R}^{n \times n}$ and $E_0, \ldots, E_\ell \in \mathbb{R}^{K \times K}$ be given by

$$
\Lambda_l(i,j) = \lambda_{ijl} \quad \text{and} \quad E_l = \sum_{j \in S_l} v_j v_j^T \quad l = 0, \ldots, \ell,
$$

so that

$$
X^{t+1} + V^t = \sum_{l=0}^\ell \Lambda_l \otimes E_l.
$$

3. Return

$$
\Pi_{S_+}(X^{t+1} + V^t) = \sum_{l=0}^\ell \Pi_{S_+}(\Lambda_l) \otimes E_l, \quad (21)
$$

which holds because the matrices $E_0, \ldots, E_\ell$ are orthogonal and positive semidefinite.

Since $\Lambda_0, \ldots, \Lambda_\ell \in \mathbb{R}^{n \times n}$, the above steps require $\ell + 1$ eigendecompositions of $n \times n$ matrices. As the computation time of eigendecomposition scales cubically in the matrix size, the resulting speedup can be quite significant in practice; for example, if $K = 10$ and $\ell = \log_2 K$, the speedup is a factor of roughly 200 (e.g., going from 3 hours to $\leq 1$ minute to solve the semidefinite program).

When will the decomposition (20) hold? A sufficient condition is that $\hat{\theta}$ is in the span of an association scheme, which is evidently a fundamental concept in combinatorics, and is defined as follows:

**Definition 1.** [15, Sec 2] A set of binary-valued and symmetric matrices $B_0, \ldots, B_\ell$ form an association scheme if the following properties hold:

1. $B_0 = I$

2. $\sum_{i=0}^\ell B_i = 11^T$

3. For all $i, j$, it holds that $B_iB_j \in \text{span}(B_0, \ldots, B_\ell)$

The key property of association schemes is the following result, which is Theorem 2.6 in [5], or follows from E1-E4 in [15]. The result states that $B_0, \ldots, B_\ell$ share a common set of eigenvectors.

**Lemma 1** ([15, 5]). Let $B_0, \ldots, B_\ell \in \mathbb{R}^{K \times K}$ be an association scheme. Then $B_0, \ldots, B_\ell$ have the same eigenvectors $v_1, \ldots, v_K$, and for some partition $S_0, \ldots, S_\ell$ of $[K]$ and scalars $\{\lambda_j^{(i)} \}_{i,j=0}^\ell$ it holds for that

$$
B_i = \sum_{j=0}^\ell \lambda_j^{(i)} \sum_{k \in S_j} v_k v_k^T, \quad i = 0, \ldots, \ell. \quad (22)
$$

Additionally, the vector $1$ will be one of the eigenvectors.
Theorem 3, which is proven in Appendix B, states that if the parameter matrix $\hat{\theta} \in \mathbb{R}^{K \times K}$ is in the span of an association scheme, then the decomposition \[ (20) \] will hold, with $\ell$ equal to the number of matrices in the association scheme, and with precomputable $\{S_i\}$ and $v_1, \ldots, v_K$:

**Theorem 3.** Let $F$ be defined as \[ (12) \], for any symmetric $A \in \{0, 1\}^{n \times n}$ and $\hat{\theta} \in \text{span}(B_0, \ldots, B_\ell)$, where $\{B_i\}_{i=0}^\ell$ form an association scheme. Let $X, W, Y, U, V \in \mathbb{R}^{nK \times nK}$ be initialized to zero and evolve by the ADMM equations \[ (19) \]. Then \[ (20) \] holds for all $t$, with $v_1, \ldots, v_K$ and $S_0, \ldots, S_\ell$ satisfying \[ (22) \].

In Section 5, we will give three examples of semidefinite programs in which $\hat{\theta}$ can be shown to belong to association schemes, with $\ell = 1, \ell = 2$, and $\ell \leq K$ respectively. We remark that association schemes were originally invented by statisticians for the study of experiment design \[ [4, 5] \], and that they have been used with semidefinite programming in \[ [12] \] to lower bound the optimal objective value of the traveling salesperson problem (without finding a feasible solution).

## 5 Simulation results

To illustrate the usage and performance of the semidefinite program, we show simulated results for three examples: community structure, overlapping communities, and latent space models.

### 5.1 Community Structure

In the best-understood blockmodel setting, $A$ is generated by $\theta \in [0, 1]^{K \times K}$ which has parameters $\gamma_0, \gamma_1 \in [0, 1]$, and equals \[ \theta = \gamma_0 I + \gamma_1 (11^T - I). \] \[ (23) \]

In this model, nodes connect with probability $\gamma_0$ if they are in the same class, and probability $\gamma_1$ if they are in different classes.

Under \[ (23) \], the parameter matrix $\theta$ can be written as \[ \theta = \gamma_0 B_0 + \gamma_1 B_1, \] for $B_0 = I$ and $B_1 = 11^T - I$. By manual verification, this can be seen to satisfy the requirements given in Definition 1 for an association scheme with $\ell = 1$ so that fast methods can be used to evaluate the ADMM iterations.

Figure 1 shows an adjacency matrix $A$ generated by \[ (23) \], the estimated class labels $\hat{z}$ found by spectral clustering of $\hat{P}$ (as discussed in Section 2.4), and the estimates found by direct spectral clustering of $A$. In this instance, the semidefinite program yields nearly perfect class estimates, while spectral clustering of $A$ fails due to its sparsity.

Figure 2 shows the average simulated performance, over a range of values for the network size $n$ and average degree $n\rho$. We see that for $n\rho \geq 10$, the
Figure 1: Community structured blockmodel (23) with \( n = 800 \) nodes, \( K = 4 \) classes, average degree = 18, and unbalanced class sizes. (a) adjacency matrix \( A \). (b) similarity matrix for \( \hat{z} \), which resulted in 4 miss-classified nodes. (c) shows the similarity matrix for class estimates found by spectral clustering of \( A \), which split the largest community, failed to find the smallest two communities, and resulted in 447 miss-classified nodes.

Figure 2: Misclassification rates for \( \hat{z} \) found by semidefinite programming (blue line, solid) vs. spectral clustering of \( A \) (red line, dotted), under the community structured blockmodel (23) with \( K = 4 \), \( n = \{100, 200, 400, 800, 1200\} \), and expected average degree \( n\rho = \{5, 10, 20, 30\} \). 100 simulations per trial, standard errors shown.

The semidefinite program has roughly constant error if the average degree is fixed, which is consistent with Theorem 2 (and other known results for community detection). In contrast, the misclassification rate for spectral clustering of \( A \) increases with the sparsity of the graph. This is exemplified by subplot (d), where spectral clustering of \( A \) performs well for small networks but degrades severely as \( n \) increases.

5.2 Two Overlapping Groups of Communities

Let parameters \( \gamma_0, \gamma_1 \in [0, 1] \) satisfy \( \gamma_0 > \gamma_1 \), and let \( K = k^2 \) for some integer \( k \), so that each \( a \in [K] \) has a \( k \)-ary representation \( (a_1, a_2) \in [k]^2 \). Let \( \theta \) equal

\[
\theta_{ab} = \begin{cases} 
\gamma_0 & \text{if } a_1 = b_1 \text{ or } a_2 = b_2 \\
\gamma_1 & \text{otherwise}
\end{cases} \quad a, b \in [K].
\]
In this model, there exist two types of community structure, each type comprised of \( k \) communities, where node belongs to one community of each type. Two nodes have a higher probability of connection if they share at least one community in common.

We observe that \( \theta = \gamma_0 B_0 + \gamma_0 B_1 + \gamma_1 B_2 \), where \( B_0, B_1, \text{ and } B_2 \in \{0, 1\}^{K \times K} \) are given by

\[
B_0 = I
\]
\[
B_1 = (I - 11^T) \otimes I + I \otimes (I - 11^T)
\]
\[
B_2 = (I - 11^T) \otimes (I - 11^T).
\]

By manual verification, \( \{B_0, B_1, B_2\} \) can be seen to satisfy the requirements given in Definition 1 for an association scheme with \( \ell = 2 \).

For this model, it may be of interest to not only compute \( \hat{z} \) and \( \hat{\theta}^{\text{est}} \), which estimate \( z \) and \( \theta \) up to label switching, but to also estimate the \( 2^k \) overlapping communities, which we will denote by \( C_1, \ldots, C_{2^k} \subset [n] \). It can be seen that \( C_1, \ldots, C_{2^k} \) are given by

\[
C_{l+(m-1)k} = \{i \in [n] : z_{im} = 1\} \quad l \in [k], m \in [2],
\]

where each \( z_i \in [K] \) has \( k \)-ary representation \((z_{i1}, z_{i2})\). Equivalently, \( C_1, \ldots, C_{2^k} \) may also be defined as follows: Let \( G \) denote a graph with \( K \) vertices, and with edges induced by thresholding \( \theta \) between \( \gamma_0 \) and \( \gamma_1 \):

\[
G_{ab} = \begin{cases} 
1 & \text{if } \theta_{ab} \geq (\gamma_0 + \gamma_1)/2 \\
0 & \text{otherwise.} 
\end{cases} \quad a, b \in [K].
\]

It can be seen that \( G \) has \( 2^k \) maximal cliques \( C_1, \ldots, C_{2^k} \), for which the communities \( C_1, \ldots, C_{2^k} \) satisfy

\[
C_{\ell} = \{i \in [n] : z_i \in C_{\pi(\ell)}\} \quad \ell = 1, \ldots, 2^k,
\]

for some permutation \( \pi \) of \([2^k]\).

Thus, to estimate \( C_1, \ldots, C_{2^k} \), we can construct \( \hat{\theta}^{\text{est}} \in [0, 1]^{K \times K} \) as given by (11), and estimate \( G \) by

\[
\hat{G}_{ab} = 1 \left\{ \hat{\theta}^{\text{est}}_{ab} \geq \frac{\gamma_0 + \gamma_1}{2} \right\} \quad a, b \in [K],
\]

which has maximal cliques \( \hat{C}_1, \ldots, \hat{C}_{m'} \subset [K] \) for some \( m' \). Using the maximal cliques of \( \hat{G} \), we can estimate overlapping communities \( \hat{C}_1, \ldots, \hat{C}_{m'} \subset [n] \) by

\[
\hat{C}_{\ell} = \{i \in [n] : \hat{z}_i \in \hat{C}_{\ell}\}, \quad \ell = 1, \ldots, m'.
\]

We remark that even in settings where the model (24) is not valid, by (26) and (27) the subsets \( \hat{C}_1, \ldots, \hat{C}_{m'} \) are still interpretable as overlapping subsets of densely connected nodes.
Figure 3: Overlapping community model (24), with \( n = 800 \), \( n\rho = 50 \), and 6 overlapping communities \( C_1, \ldots, C_6 \). (a) adjacency matrix \( A \). (b) similarity matrix for \( \hat{C}_1, \hat{C}_2, \hat{C}_3 \). (c) similarity matrix for \( \hat{C}_4, \hat{C}_5, \hat{C}_6 \). (d) similarity matrix for \( \hat{z} \), but assuming non-overlapping model (23). (e) similarity matrix using spectral clustering of \( A \). \( \hat{C}_1, \ldots, \hat{C}_6 \) had 32 errors, while \( \hat{z} \) using the non-overlapping model (23) had 397 errors, and spectral clustering of \( A \) had 187 errors.

Figure 3a shows an adjacency matrix \( A \) generated by (24), with \( k = 3 \) and \( K = 9 \). The pattern of \( \theta \) is clearly visible in \( A \). Figures 3b and 3c show the estimated communities \( \hat{C}_1, \ldots, \hat{C}_6 \) using (27). For comparison, Figure 3d shows \( \hat{z} \) when the semidefinite program assumes (23) instead of (24), and Figure 3e shows the estimate of \( z \) under spectral clustering of \( A \). In this instance, \( \hat{C}_1, \ldots, \hat{C}_6 \) accurately estimate the true communities \( C_1, \ldots, C_6 \), with a misclassification rate of 0.04. This is consistent with Theorem 2, which predicts that \( \hat{P} \) will be nearly block-structured, implying that the subsequent steps of estimating \( \hat{z}, \hat{\theta}^\text{est} \), and \( \hat{G} \) will succeed as well. In contrast, the two alternative methods give poor estimates for \( z \), with misclassification rates of 0.50 and 0.23, respectively.

Figure 4 shows the average misclassification rate for \( \hat{C}_1, \ldots, \hat{C}_6 \), in simulations over a range of values for the network size \( n \) and average degree \( n\rho \). For comparison, the misclassification rate for spectral clustering of \( A \) (which estimates \( z \) instead of \( C_1, \ldots, C_6 \) is shown as well. The misclassification rate of the semidefinite program is not quite constant in \( n \) for fixed \( n\rho \), suggesting that the asymptotic results of Theorem 2 may require larger \( n \) for this model compared to the results shown in Figure 2. However, as \( n\rho \) increases the semidefinite program estimates the overlapping communities (and hence \( z \) as well) with much better accuracy than spectral clustering of \( A \), which shows little improvement with increasing \( n\rho \).

### 5.3 Latent Space Models

We consider a latent space model, reminiscent of [18], in which each node is assigned a latent coordinate vector \( y_1, \ldots, y_n \in [0, 1]^d \). Conditional on \( \{y_i\}_{i=1}^n \), each dyad is independent Bernoulli with log odds given by

\[
\log \text{odds } A_{ij} = -\|y_i - y_j\|/\sigma, \tag{28}
\]

where \( \sigma \geq 0 \) is a bandwidth parameter. In general, (28) is not equivalent to a blockmodel.
Let $D \in \mathbb{R}^{n \times n}$ denote the matrix of squared distances between the latent coordinates, given by

$$D_{ij} = \|y_i - y_j\|^2.$$  

It is known that the first $d$ eigenvectors of $(I - 11^T/n)D(I - 11^T/n)$ will recover $y_1, \ldots, y_n$ up to a unitary transformation.

To estimate $D$, we will approximate (28) by a blockmodel with $K = (2k)^d$ classes, where each class represents a coordinate in $\mathbb{R}^d$, so that $\hat{\theta}$ equals

$$\text{logit } \hat{\theta}_{ab} = -\delta(\gamma_a, \gamma_b)/\sigma,$$

where $\gamma_1, \ldots, \gamma_K \in \mathbb{R}^d$, and $\delta$ is a distance metric. In order for $\hat{\theta}$ to belong to an association scheme, we choose $\gamma_1, \ldots, \gamma_K$ extending beyond $[0, 1]^d$ to form a grid in $[0, 2]^d$, and choose $\delta$ to be a toric distance. That is, given $a \in [K]$, let $(a_1, \ldots, a_d)$ denote its $(2k)$-ary representation, and let $\gamma_1, \ldots, \gamma_K$ be given by

$$\gamma_a = \left(\frac{a_1}{k}, \ldots, \frac{a_d}{k}\right), \quad a \in [K],$$

and let $\delta : [0, 2]^d \times [0, 2]^d \to \mathbb{R}$ equal the distance on a torus of circumference 2,

$$\delta(\gamma_a, \gamma_b) = \left[\sum_{j=1}^d \min(|\gamma_{a_j} - \gamma_{b_j}|, 2 - |\gamma_{a_j} - \gamma_{b_j}|)^2\right]^{1/2},$$

where $\gamma_{a_j}$ is the $j$th element of the vector $\gamma_a$. Since $\hat{\theta}_{ab}$ depends on $\gamma_a$ and $\gamma_b$ only through their element-wise differences $|\gamma_{a_j} - \gamma_{b_j}|$, for $j = 1, \ldots, d$, it follows that $\hat{\theta}$ can be written as a weighted sum of matrices

$$\hat{\theta} = \sum_{j_1=0}^{2k-1} \cdots \sum_{j_d=0}^{2k-1} \gamma_{j_1 \cdots j_d} C^{(j_1)} \otimes \cdots \otimes C^{(j_d)},$$
where \( C^{(j)} \in \{0,1\}^{2k \times 2k} \) for \( j = 0, \ldots, 2k - 1 \) is the circulant matrix given by
\[
C^{(j)}_{ab} = \begin{cases} 
1 & \text{if } \min(|a - b|, 2k - |a - b|) = j \\
0 & \text{otherwise}.
\end{cases}
\]

By manual inspection, it can be seen that \( C = \{C^{(0)}, \ldots, C^{(2k-1)}\} \) satisfies the requirements of an association scheme, and hence that also \( C \otimes \cdots \otimes C \) is an association scheme as well, with \( \ell \leq (2k)^d = K \). (See [5, Chapters 1-3] for a complete treatment.)

Let \( \hat{X} \) denote the solution to the semidefinite program [8] with \( \hat{\theta} \) given by (29). To estimate \( D \), let \( \hat{D} \in \mathbb{R}^{n \times n} \) equal the MAP-style estimate under \( \hat{X} \),
\[
\hat{D}_{ij} = \arg \max_{x \in \mathbb{R}^K} \sum_{a=1}^{K} \sum_{b=1}^{K} \hat{X}^{(ij)}_{ab} \{ \delta(\gamma_a, \gamma_b) \}^2 = x, \tag{30}
\]
and let \( \tilde{D} \in \mathbb{R}^{n \times n} \) denote the randomized estimate
\[
\tilde{D}_{ij} = \delta(\gamma_a, \gamma_b)^2 \text{ with probability } X_{ab}^{(ij)}. \tag{31}
\]

Given \( \hat{D} \), we will take the first \( d \) eigencoordinates of \( (I - 11^T/n)\hat{D}(I - 11^T/n) \) to estimate \( y_1, \ldots, y_n \) (or similarly using \( \tilde{D} \)).

Corollary 2 bounds the error between the randomized estimate \( \tilde{D} \) and the true distances \( D \), relative to rounding the coordinates \( \{y_i\}_{i=1}^n \) to their closest points on a grid over \([0,1]^d\):

**Corollary 2.** Let \( L : \mathbb{R} \times \mathbb{R} \mapsto \mathbb{R} \) denote the loss function
\[
L(x, x') = KL \left( \frac{e^{-x/\sigma}}{1 + e^{-x/\sigma}}, \frac{e^{-x'/\sigma}}{1 + e^{-x'/\sigma}} \right),
\]
and let \( \eta_1, \ldots, \eta_{kd} \in \mathbb{R} \) form a grid on \([0,1]^d\),
\[
\eta_a = \left( \frac{a_1}{k}, \ldots, \frac{a_d}{k} \right), \quad a \in [k^d],
\]
where \( (a_1, \ldots, a_d) \) denotes the \( k \)-ary representation of \( a \in [k^d] \). Then
\[
\frac{1}{n^2 \rho} \sum_{i=1}^{n} \sum_{j=1}^{n} L(\sqrt{D_{ij}}, \sqrt{\hat{D}_{ij}}) \leq \min_{z \in [k^d]^n} \frac{1}{n^2 \rho} \sum_{i=1}^{n} \sum_{j=1}^{n} L(\sqrt{D_{ij}}, \|\eta_{zi} - \eta_{zj}\|) + O_P \left( \frac{1}{\sqrt{n \rho}} \right)
\]

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Proof. It holds that

\[
\frac{1}{n^2 \rho} \sum_{i=1}^{n} \sum_{j=1}^{n} L \left( \sqrt{D_{ij}}, \sqrt{\tilde{D}_{ij}} \right) \leq \min_{z \in \mathbb{R}^n} \frac{1}{n^2 \rho} \sum_{i=1}^{n} \sum_{j=1}^{n} L \left( \sqrt{D_{ij}}, \delta(\gamma_{zi}, \gamma_{zj}) \right) + O_P \left( \frac{1}{\sqrt{n \rho}} \right)
\]

where the first inequality holds by Theorem 1 and definition of \( L \); the second inequality holds because the minimization is over a strictly smaller set than in the previous line; and the last inequality holds because \( \delta(x, x') = \|x - x'\| \) for all \((x, x') \in [0, 1]^d \times [0, 1]^d\).

Figure 5 shows latent coordinates \( y_1, \ldots, y_n \in \mathbb{R}^2 \) arranged in a circle, from which an adjacency matrix was generated by (28) with \( n = 500 \) and \( n \rho = 20 \). The figure also shows the estimated coordinates \( \hat{y} \) derived from the randomized \( \tilde{D} \) given by (31), from the MAP-style \( \hat{D} \) given by (30), and by applying the USVT method of \[10\] to the adjacency matrix \( A \), in which a spectral estimate of \( P \) is constructed from \( A \), and then inverted to form an estimate of \( D \). In this instance, both \( \tilde{D} \) and \( \hat{D} \) yield estimates \( \hat{y} \) that are similar, and are substantially more accurate than the USVT approach which failed due to the sparsity of \( A \).

Figure 6 shows a different configuration for \( y_1, \ldots, y_n \), with similar results.

Figure 7 shows the average simulated estimation accuracy using \( \hat{D} \) for a range of values for the network size \( n \) and average degree \( n \rho \). For comparison, the performance of the spectral USVT method is shown as well. We see that the estimation error for \( \hat{D} \) is near-constant when the average degree is fixed. In contrast, the estimation error for the spectral USVT method worsens with the sparsity of \( A \). This is exemplified by subplot (d), in which the USVT method performs well for small networks but degrades severely as \( n \) increases.

\section{A Proof of Theorems 1 and 2}

\subsection{A.1 Intermediate Results}

We first present intermediate results that will be used in the proof of Theorems 1 and 2. Let \( X^* \) denote any solution to the idealized problem

\[
\text{maximize } \langle \bar{F}, X \rangle \text{ over all } X \in X.
\]

Lemma 2 is the main technical result, and states that for the general matrix model \[3\], \( X \) nearly optimizes the desired objective function \( \bar{F} \), even though only the noisy proxy \( F \) is available. Its proof closely follows the approach of \[17\].
Figure 5: Latent space model (28), with $n = 500$, $n\rho = 20$, and latent coordinates $y_1, \ldots, y_n \in \mathbb{R}^2$ arranged in a circle. (a) Latent coordinates $y$. (b) Estimated $\hat{y}$ using randomized $\tilde{D}$ from (31). (c) Estimated $\hat{y}$ using MAP-style $\hat{D}$ from (30). (d) Estimated $\hat{y}$, using USVT spectral method [10] directly on $A$. RMS errors for (b), (c), and (d) were 0.08, 0.07, and 0.25 respectively.

Figure 6: Latent space model (28), with $n = 1000$, $n\rho = 140$, and latent coordinates $y_1, \ldots, y_n \in \mathbb{R}^2$ arranged in a cross. (a) Latent coordinates $y$. (b) Estimated $\hat{y}$ using randomized $\tilde{D}$ from (31). (c) Estimated $\hat{y}$ using MAP-style $\hat{D}$ from (30). (d) Estimated $\hat{y}$, using USVT spectral method [10] directly on $A$. RMS errors for (b), (c), and (d) were 0.044, 0.040, and 0.11 respectively.

Figure 7: Estimation error for $\hat{y}$ found by semidefinite programing (blue line, solid) vs. USVT method on $A$ (red line, dotted), under the latent space model (28) with $n = \{200, 300, 400, 600, 800, 1000, 1200\}$, expected average degree $n\rho = \{5, 15, 25, 35\}$, and latent coordinates $y_1, \ldots, y_n \in \mathbb{R}^2$ arranged in a circle. 100 simulations per trial, standard errors shown.
Lemma 2. Let Assumptions 1 and 2 hold. Then for some $C_1, C_2 > 0$ it holds for all $n$ that

$$
\mathbb{P} \left\{ \frac{1}{n^2 \rho} \langle \hat{F}, X^* - \hat{X} \rangle \geq \frac{C_1}{\sqrt{n \rho}} \right\} = e^{-C_2 n},
$$

where $C_1$ and $C_2$ depend only on $K$ and $c$.

Lemma 3 gives a condition under which $\hat{P}$ and $\tilde{P}$ will both be approximately block structured. It roughly states that if there exists $z \in [K]^n$ such that $(z_i, z_j) \in \text{arg max}_{ab} \hat{F}_{ab}^{(ij)}$ for all $i, j \in [n]$, then both $P$ and $\hat{P}$ will asymptotically have block structure corresponding to $z$.

Lemma 3. Let Assumptions 1 and 2 hold. If there exists $z \in [K]^n$ and $\Delta > 0$ such that $P, \hat{\theta}$ and $\bar{F}$ satisfy

$$
\hat{F}_{z_i, z_j}^{(ij)} \geq \bar{F}_{ab}^{(ij)} + \rho \Delta \quad \forall i, j \in [n] \text{ and } a, b \notin Q_{\hat{\theta}}(z_i, z_j),
$$

then it holds that

$$
\frac{1}{n(n-1)} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{a=1}^{K} \sum_{b=1}^{K} \hat{X}_{ab}^{(ij)} \cdot 1\{ (a, b) \notin Q_{\hat{\theta}}(z_i, z_j) \} \leq O_P \left( \frac{1}{\sqrt{n \rho}} \right),
$$

(33)

Lemma 4 states that the error between $P$ and the randomized estimate $\tilde{P}$ converges to its expectation.

Lemma 4. Let Assumptions 1 and 2 hold. Then

$$
\frac{1}{n(n-1)} \sum_{i=1}^{n} \sum_{j=1}^{n} KL \left( P_{ij}, \tilde{P}_{ij} \right) = \frac{1}{n(n-1)} \sum_{i=1}^{n} \sum_{j=1}^{n} \mathbb{E} \left[ KL \left( P_{ij}, \tilde{P}_{ij} \right) \right] + O_P \left( \frac{1}{\sqrt{n \rho}} \right).
$$

Bernstein’s inequality states that for independent $x_1, \ldots, x_n$ satisfying $|x_i| \leq b$, with variance $\sigma_i^2$ and expectation $\mathbb{E}x_i = 0$, it holds that

$$
\mathbb{P} \left( \frac{1}{n} \sum_{i=1}^{n} x_i \geq t \right) \leq \exp \left( -\frac{nt^2}{2 \sum_{i=1}^{n} \sigma_i^2 + 3bt/3} \right).
$$

Grothendieck’s inequality [17, Th. 3.1] states that there exists a universal constant $C_G$ such that for any matrix $M \in \mathbb{R}^{n \times n}$,

$$
\max_{X \in \mathcal{M}} |\langle M, X \rangle| \leq C_G \max_{s, t \in \{-1, 1\}^n} |s^T Mt|,
$$

(34)

where $\mathcal{M} = \{ U V^T : \text{ all rows of } U, V \in B_2^n \}$, and $B_2^n = \{ x \in \mathbb{R}^n : \|x\|_2 \leq 1 \}$ is the $n$-dimensional unit ball.

Lemmas 2, 3, and 4 are proven in Section A.3.
A.2 Proof of Theorems 1 and 2

Proof of Theorem 1. Given $z \in [K]^n$, let the vector $x(z) \in \{0,1\}^{nK}$ be given by

$$x(z) = \left[ e_{z_1}^T, \ldots, e_{z_n}^T \right]^T.$$  

Theorem 1 holds by the following steps:

$$\frac{1}{n(n-1)\rho} \sum_{i=1}^{n} \sum_{j=1}^{n} KL \left( P_{ij}, \hat{P}_{ij} \right)$$

$$= \frac{1}{n(n-1)\rho} \sum_{i=1}^{n} \sum_{j=1}^{n} E \left[ KL \left( P_{ij}, \hat{P}_{ij} \right) \right] + O_P \left( \frac{1}{\sqrt{n\rho}} \right) \quad (35)$$

$$= \frac{1}{n(n-1)\rho} \sum_{i=1}^{n} \sum_{j=1}^{n} H(P_{ij}) - \langle \vec{F}, \hat{X} \rangle + O_P \left( \frac{1}{\sqrt{n\rho}} \right) \quad (36)$$

$$\leq \frac{1}{n(n-1)\rho} \sum_{i=1}^{n} \sum_{j=1}^{n} H(P_{ij}) - \langle \vec{F}, X^* \rangle + O_P \left( \frac{1}{\sqrt{n\rho}} \right) \quad (37)$$

$$\leq \frac{1}{n(n-1)\rho} \sum_{i=1}^{n} \sum_{j=1}^{n} H(P_{ij}) - \max_{z \in [K]^n} \langle \vec{F}, x(z)x(z)^T \rangle + O_P \left( \frac{1}{\sqrt{n\rho}} \right) \quad (38)$$

$$\leq \min_{z \in [K]^n} \frac{1}{n(n-1)\rho} \sum_{i=1}^{n} \sum_{j=1}^{n} KL \left( P_{ij}, \hat{\theta}_{z_i,z_j} \right) + O_P \left( \frac{1}{\sqrt{n\rho}} \right) \quad (39)$$

where (35) holds by Lemma 4; (36) and (39) follow from the identity

$$KL(P_{ij}, \hat{\theta}_{ab}) = H(P_{ij}) - \hat{F}^{(ij)},$$

with (36) additionally using (10), the definition of $\hat{P}$; (37) holds by Lemma 2; and (38) holds because $x(z)x(z)^T \in \mathcal{X}$, implying that $\langle F, X^* - x(z)x(z)^T \rangle \geq 0$. \hfill \(\square\)

Proof of Theorem 2. Define $\hat{\pi}$ by

$$\hat{\pi}_a = \frac{1}{n} \sum_{i=1}^{n} 1\{z_i = a\}.$$  

By Bernstein’s inequality, it can be seen that $\hat{\alpha} = \rho(1 + o_P(1))$, and that $\rho = \frac{1}{\frac{1}{2}} \sum_{i<j} P_{ij}$ satisfies

$$\rho = \sum_{a=1}^{K} \sum_{b=1}^{K} \hat{\pi}_a \hat{\pi}_b \theta_{ab}^* (1 + o(1))$$

$$= \alpha(1 + o_P(1)).$$
so that $\hat{\alpha} \rightarrow \rho \rightarrow \alpha$. As a result, Assumptions [1] and [2] can be seen to hold (in probability). Let $\Delta'$ be defined by

$$\Delta' = \min_{a,b \in [K]^2} \min_{c,d \notin Q_B} B_{ab}^* \log \frac{\hat{B}_{ab}}{B_{cd}} - (\hat{B}_{ab} - \hat{B}_{cd}).$$

It can be seen that $\Delta' > 0$ by assumption. To bound $\tilde{F}_{z_i,z_j} - \tilde{F}^{(ij)}_{cd}$ so as to apply Lemma [3], observe for all $i,j \in [n]$ and $c,d \notin Q_B(z_i,z_j)$:

$$F_{z_i,z_j}^{(ij)} - \tilde{F}^{(ij)}_{cd} = P_{ij} \log \frac{\theta_{z_i,z_j}}{\theta_{cd}} + (1 - P_{ij}) \log \frac{1 - \theta_{z_i,z_j}}{1 - \theta_{cd}}$$

$$= \alpha B_{z_i,z_j}^* \log \frac{\hat{B}_{z_i,z_j}}{B_{cd}} + \log \frac{1 - \hat{\alpha} \hat{B}_{z_i,z_j}}{1 - \hat{\alpha} \hat{B}_{cd}} - \alpha B_{z_i,z_j}^* \log \frac{1 - \hat{\alpha} \hat{B}_{z_i,z_j}}{1 - \hat{\alpha} \hat{B}_{cd}}$$

$$= \rho B_{z_i,z_j}^* \log \frac{\hat{B}_{z_i,z_j}}{B_{cd}} - \rho (\hat{B}_{z_i,z_j} - \hat{B}_{cd}) + o_P(\alpha)$$

$$\geq \rho \Delta' + o_P(\alpha),$$

where the $o_P(\alpha)$ terms are bounded uniformly over all $(i,j) \in [n]^2$ and $(c,d) \in [K]^4$. This implies that for all $i,j \in [n]$ and all $c,d \notin Q_B(z_i,z_j)$, it holds that

$$\tilde{F}_{z_i,z_j}^{(ij)} - \tilde{F}^{(ij)}_{cd} \geq \rho (\Delta' + o_P(1)),$$

implying that the conditions of Lemma [3] will hold in probability for any $\Delta < \Delta'$. Lemma [3] thus implies

$$\frac{1}{n^2} \sum_{i=1}^n \sum_{j=1}^n \sum_{a=1}^K \sum_{b=1}^K \hat{X}_{ab}^{(ij)} \cdot 1\{(a,b) \notin Q_B(z_i,z_j)\} = O_P \left( \frac{1}{\sqrt{n\alpha}} \right).$$

Let $E_{ij}$ for $i,j \in [n]$ be given by

$$E_{ij} = \sum_{a=1}^K \sum_{b=1}^K \hat{X}_{ab}^{(ij)} \cdot 1\{(a,b) \notin Q_B(z_i,z_j)\}.$$

To show [18], we apply (40) as follows:

$$\frac{1}{n^2} \sum_{i=1}^n \sum_{j=1}^n \sum_{a=1}^n \sum_{b=1}^n 1\{\hat{P}_{ij} \neq \hat{\theta}_{z_i,z_j}\} \leq \frac{1}{n^2} \sum_{i=1}^n \sum_{j=1}^n \sum_{a=1}^n \sum_{b=1}^n 1\{E_{ij} \geq 1/2\}$$

$$\leq \frac{1}{n^2} \sum_{i=1}^n \sum_{j=1}^n \sum_{a=1}^n \sum_{b=1}^n 2E_{ij}$$

$$= O_P \left( \frac{1}{\sqrt{n\alpha}} \right),$$

20
where the last equality follows from \((40)\).

To show that \((18)\) holds with \(\tilde{P}\) in place of \(\hat{P}\), we observe the following bounds:

\[
E \left[ \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} 1\{ \hat{P}_{ij} \neq \hat{\theta}_{zi, zj} \} \right] = \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} \xi_{ij} = O_P \left( \frac{1}{\sqrt{n\alpha}} \right)
\]

and

\[
\text{Var} \left[ \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} 1\{ \hat{P}_{ij} \neq \hat{\theta}_{zi, zj} \} \right] = \frac{4}{n^4} \sum_{i < j} \text{Var} \left[ 1\{ \hat{P}_{ij} \neq \theta_{ij} \} \right] \\
\leq \frac{4}{n^4} \sum_{i < j} \mathbb{P} \left( \hat{P}_{ij} \neq \theta_{ij} \right) \\
= \frac{4}{n^4} \sum_{i < j} \xi_{ij} \\
= \frac{1}{n^2} O_P \left( \frac{1}{\sqrt{n\alpha}} \right).
\]

Applying Chebychev, which states that \(\mathbb{P} \left( |X - \mathbb{E}X| \geq k \sqrt{\text{Var} X} \right) \leq k^{-2}\), with \(k = n^{3/4}\alpha^{-1/4}\) yields that

\[
\frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} 1\{ \hat{P}_{ij} \neq \hat{\theta}_{zi, zj} \} = O_P \left( \frac{1}{\sqrt{n\alpha}} \right).
\]

\(\square\)

A.3 Proof of Lemmas 2, 3, and 4

Proof of Lemma 2. Let \(\tilde{F} \in \mathbb{R}^{nK \times nK}\) be a re-centered version of \(\hat{F}\), with submatrices \(\tilde{F}^{(ij)} \in \mathbb{R}^{K \times K}\) given by

\[
\tilde{F}^{(ij)} = \hat{F}^{(ij)} + (A_{ij} - P_{ij}) \log \rho \cdot 11^T.
\]

Algebraic manipulation yields that

\[
\langle \hat{F}, X^* \rangle - \langle \hat{F}, \hat{X} \rangle = \langle \tilde{F}, X^* \rangle + \langle \tilde{F} - \tilde{F}, X^* \rangle - \langle \tilde{F}, \hat{X} \rangle - \langle \tilde{F} - \tilde{F}, \hat{X} \rangle \\
= \langle \tilde{F}, X^* \rangle - \langle \tilde{F}, \hat{X} \rangle \\
\leq \langle \tilde{F}, X^* \rangle - \langle F, X^* \rangle + \langle F, \hat{X} \rangle - \langle \tilde{F}, \hat{X} \rangle \\
\leq 2 \max_{X \in \mathcal{X}} \left| \langle \tilde{F} - F, X \rangle \right| \\
\leq 2C_G \max_{s, t \in \{-1, 1\}^{nK}} s^T (F - \tilde{F}) t,
\]

(43)
By Bernstein’s inequality, it follows that $C \langle X \subset M \text{ inequality and because }$

combine (44) and (43) proves the lemma.

where (41) holds because $\tilde{X}$ maximizes $\langle F, X \rangle$, implying that $\langle F, X^* - \tilde{X} \rangle \leq 0$; and (43) follows by Grothendieck’s inequality and because $X \subset M$.

It remains to bound the right hand side of (43). By the definition of $\tilde{F}$, it can be seen that

$$F^{(i,j)} - \tilde{F}^{(i,j)} = (A_{i,j} - P_{i,j}) \log \frac{\hat{\theta}_{ab}/\rho}{1 - \theta_{ab}},$$

so that

$$s^T(F - \tilde{F}) = \sum_{i,j} \sum_{a,b} s^{(i)}_a t^{(j)}_b (A_{i,j} - P_{i,j}) \log \frac{\hat{\theta}_{ab}/\rho}{1 - \theta_{ab}}.$$ 

Given $s, t \in \{-1, 1\}^{nK}$ define $x_{ij} \equiv x_{ij}(s, t)$ by

$$x_{ij} = (A_{i,j} - P_{i,j}) \sum_{a,b} \left(s^{(i)}_a t^{(j)}_b + s^{(j)}_b t^{(i)}_a \right) \log \frac{\hat{\theta}_{ab}/\rho}{1 - \theta_{ab}}.$$ 

so that $s^T(F - \tilde{F}) = \sum_{i,j} x_{ij}$. Using $\rho / c \leq \hat{\theta}_{ab} / (1 - \theta_{ab}) \leq c \rho$ which holds by Assumption 2 and letting $C = 2 \log c$, it can be seen that

$$\text{Var}(x_{ij} + x_{ji}) \leq P_{i,j}(1 - P_{i,j}) K^4 C^2 \quad \text{and} \quad |x_{ij}| \leq K^2 C.$$ 

By Bernstein’s inequality, it follows that

$$P \left[ \frac{1}{(n/2)} \sum_{i,j} x_{ij} \geq \epsilon \right] \leq \exp \left( - \frac{(n/2)^2 \epsilon^2}{K^4 C^2 \rho + 2 K^2 C \epsilon} \right),$$

where we have used the fact that $\left(\frac{n}{2}\right)^{-1} \sum_{i<j} P_{i,j}(1 - P_{i,j}) \leq \rho$. Letting $\epsilon = C_1 \sqrt{\frac{\rho}{n}}$ for any value of $C_1$ implies

$$P \left[ \frac{1}{(n/2)} \sum_{i,j} x_{ij} \geq C_1 \sqrt{\frac{\rho}{n}} \right] \leq \exp \left( - \frac{\frac{1}{2} n C_1^2}{K^4 C^2 + 2 K^2 C} \right).$$

Applying a union bound over all $s, t \in \{-1, 1\}^{nK}$ implies

$$P \left[ \max_{s,t} \left( \frac{1}{(n/2)} \sum_{i,j} x_{ij} \right) \geq C_1 \sqrt{\frac{\rho}{n}} \right] \leq 2^{nK} \exp \left( - \frac{\frac{1}{2} n C_1^2}{K^4 C^2 + 2 K^2 C} \right),$$

which implies for all $C_1$ satisfying $\frac{1}{2} C_1^2 / (K^4 C^2 + 2 K^2 C C_1) > K \log 2$ that

$$P \left[ \max_{s,t} \left( \frac{1}{(n/2)} \sum_{i,j} x_{ij} \right) \geq C_1 \sqrt{\frac{\rho}{n}} \right] = e^{-C_2 n}, \quad (44)$$

where $C_2 = \theta C_1^2 / (K^4 C^2 + 2 K^2 C C_1) - K \log 2$. Since $s^T(F - \tilde{F}) = \sum_{i,j} x_{ij}$, combining (44) and (43) proves the lemma. 

□
Proof of Lemma 3. It holds that
\[
\langle \bar{F}, \hat{X} \rangle = \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{a=1}^{K} \sum_{b=1}^{K} \bar{F}_{ab}(\hat{X}^{(ij)}_{ab}) = \sum_{i=1}^{n} \sum_{j=1}^{n} \left[ \bar{F}_{z_{i},z_{j}}^{(ij)} + \sum_{a=1}^{K} \sum_{b=1}^{K} \left( \bar{F}_{ab}^{(ij)} - \bar{F}_{z_{i},z_{j}}^{(ij)} \right) \hat{X}_{ab}^{(ij)} \right] \leq \sum_{i=1}^{n} \sum_{j=1}^{n} \left[ \bar{F}_{z_{i},z_{j}}^{(ij)} + \rho \Delta \sum_{a=1}^{K} \sum_{b=1}^{K} \hat{X}_{ab}^{(ij)} \cdot 1\{ (a,b) \notin Q_{\hat{\theta}}(z_{i},z_{j}) \} \right].
\]
Rearranging and using \( \langle \bar{F}, X^{*} \rangle = \sum_{i,j} \bar{F}_{z_{i},z_{j}}^{(ij)} \) (as implied by (32)) yields
\[
\frac{\langle \tilde{F}, X^{*} \rangle - \langle \tilde{F}, \bar{X} \rangle}{\rho \Delta} \geq \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{a=1}^{K} \sum_{b=1}^{K} \hat{X}_{ab}^{(ij)} \cdot 1\{ (a,b) \notin Q_{\hat{\theta}}(z_{i},z_{j}) \},
\]
and dividing both sides by \( n^{2} \) and using Lemma 2 yields that with probability at least \( 1 - e^{-C_{2}n} \),
\[
\frac{C_{1} / \Delta}{\sqrt{n} \rho} \geq \frac{1}{n^{2}} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{a=1}^{K} \sum_{b=1}^{K} \hat{X}_{ab}^{(ij)} \cdot 1\{ (a,b) \notin Q_{\hat{\theta}}(z_{i},z_{j}) \},
\]
proving the lemma.

Proof of Lemma 4. Let \( x_{ij} \) for \( i,j \in [n] \) be given by
\[
x_{ij} = H(P_{ij}) - P_{ij} \log \rho - KL(P_{ij}, \hat{P}_{ij}).
\]
It can be seen that \( x_{ij} \) for \( i < j \) are independent random variables with distributions
\[
x_{ij} = P_{ij} \log \frac{\theta_{ab}/\rho}{1 - \theta_{ab}} + \log(1 - \theta_{ab}) \quad \text{with probability } \hat{X}_{ab}^{(ij)},
\]
with \( x_{ji} = x_{ij} \) for \( i > j \) and \( x_{ii} = 0 \). By definition of \( \{ x_{ij} \} \), it holds that
\[
\sum_{i=1}^{n} \sum_{j=1}^{n} \left( KL(P_{ij}, \hat{P}_{ij}) - E[kl(P_{ij}, \hat{P}_{ij})] \right) = \sum_{i=1}^{n} \sum_{j=1}^{n} (E[x_{ij} - x_{ij}]). \quad (45)
\]
To bound the right hand side of (45), observe that by Assumption 2
\[
|x_{ij}| \leq P_{ij} |\log c| + |\log(1 + cp)| \leq P_{ij} |\log c| + \frac{cp}{1 + cp}
\]
and hence it holds that
\[
\max_{i,j} |x_{ij} - \mathbb{E}x_{ij}| \leq O(1)
\]
\[
\frac{1}{n(n-1)} \sum_{i,j:i<j} \text{Var}(x_{ij} + x_{ji}) \leq O(\rho),
\]
Applying Bernstein’s inequality thus yields
\[
P\left( \frac{1}{(\binom{n}{2})} \sum_{i,j:i<j} (x_{ij} - \mathbb{E}x_{ij}) \geq \epsilon \right) \leq \exp \left( - \frac{(\epsilon^2)}{O(\rho) + O(1)\epsilon} \right),
\]
and letting \(\epsilon = (\rho/n)^{1/2}\) implies that
\[
\frac{1}{(\binom{n}{2})} \sum_{i,j} (x_{ij} - \mathbb{E}x_{ij}) = O_P((n\rho)^{-1/2}).
\]
Combining this bound with (45) proves the lemma.

\[\square\]

**B Proof of Theorem 3**

Lemma 1 can be found as E1-E4 in [15], and as Theorem 2.6 in [5].

**Proof of Theorem 3.** Let \(B = \text{span}(B_0, \ldots, B_\ell)\), and let \(A\) denote the set
\[
A = \{ X \in \mathbb{R}^{nK \times nK} : X^{(ij)} \in B \ \forall i,j \},
\]
and observe the following properties of \(A\):

1. If \(M \in A\), then its submatrices \(M^{(ij)}\) must be symmetric and have the same eigenvectors since they are in the span of an association scheme. Specifically, if \(X^{t+1} + V^t \in A\) then (20) holds.

2. \(A\) is a linear space, so if \(M_1, M_2 \in A\), then \(M_1 + M_2 \in A\) as well.

3. \(0 \in A\), so the initial values \(X^0, W^0, Y^0, U^0, V^0\) are in \(A\).

4. Let \(B = \sum_{i=0}^\ell \gamma_i B_i\) for some weights \(\gamma_0, \ldots, \gamma_\ell\). Because \(B_0, \ldots, B_\ell\) are binary with disjoint support, it holds that \(\max(0, B) = \sum_i \max(0, \gamma_i) B_i\) and \(\log(B) = \sum_i \log(\gamma_i) B_i\).

5. If \(\hat{\theta} \in B\) then \(\log \hat{\theta}\) and \(\log(1 - \hat{\theta})\) are in \(B\) as well, by property 4. This implies the matrix \(F\) given by (12) is in \(A\), since each submatrix \(F^{(ij)}\) is a linear combination of \(\log \hat{\theta}\) and \(\log(1 - \hat{\theta})\).

6. If \(M \in A\) then \(\max(0, M)\) is also in \(A\), by property 4.

7. If \(M \in A\), then the eigenvectors of each submatrix are orthogonal and include the vector 1, by Lemma [1]. As a result, the effect of the projection \(\Pi_A(M)\) is to change the eigenvalue associated with the vector 1 in each submatrix, implying that \(\Pi_A(M) \in A\).
8. If $X^{t+1} + V^t \in \mathcal{A}$, then (20) and hence (21) hold, which implies that
   \[ \Pi_{S_t}(X^{t+1} + V^t) \in \mathcal{A}. \]

Using these properties, we show by induction that $X^t, W^t, Y^t, U^t, V^t \in \mathcal{A}$ for all $t$. As for the base case, it holds that $X^0, W^0, Y^0, U^0, V^0 \in \mathcal{A}$ by property 3. Now suppose that $X^t, W^t, Y^t, U^t, V^t \in \mathcal{A}$ for any $t$. Then by (19) it follows that $X^{t+1} \in \mathcal{A}$ by properties 2, 5, and 7; that $W^{t+1} \in \mathcal{A}$ by properties 2 and 6; that $Y^{t+1} \in \mathcal{A}$ by properties 2 and 8; and that $U^{t+1}, V^{t+1} \in \mathcal{A}$ by property 2.

This completes the induction argument.

Since $X^t, W^t, Y^t, U^t, V^t \in \mathcal{A}$ for all $t$, it follows by property 2 that $X^{t+1} + V^t \in \mathcal{A}$ for all $t$. By property 1, this implies that (20) holds for all $t$, proving the theorem.

\[ \square \]

\section{C Proof of Corollary 1}

Here we prove Corollary 1, which states that the eigencoordinates of $\hat{P}$ or $\tilde{P}$ will approximate those of $P$ (up to a unitary transform) when Theorem 2 holds. This suggests that $\hat{z}$, which is computed by spectral clustering of $\hat{P}$, will converge to $z$ up to label permutation.

\subsection{C.1 Intermediate results}

Lemma 5 bounds the eigenvalues of $P$ under Assumptions 1 and 2.

\textbf{Lemma 5.} Let Assumptions 1 and 2 hold. Let $D \in [0,1]^{K \times K}$ denote the matrix $D = \text{diag}(\pi)$. Let $\lambda^*_1, \ldots, \lambda^*_K$ and $\lambda_1, \ldots, \lambda_K$ respectively denote the sorted eigenvalues of $D^{1/2}B^*D^{1/2}$ and $P$. It holds that

\[ \lambda_k = n\alpha(\lambda^*_k(1 + o_P(1))) \quad k = 1, \ldots, K. \]

We will use the following version of the Davis-Kahan theorem, taken from \cite[Th. 4]{30}:

\textbf{Theorem 3.} Let $P, \hat{P} \in \mathbb{R}^{n \times n}$ be symmetric, with singular values $\sigma_1 \geq \ldots \geq \sigma_n$ and $\hat{\sigma}_1 \geq \ldots \geq \hat{\sigma}_n$ respectively. Fix $1 \leq r \leq s \leq n$ and assume that $\min(\sigma^2_{r-1} - \sigma^2_r, \sigma^2_{s-1} - \sigma^2_s) > 0$, where $\sigma^2_0 = \infty$ and $\sigma^2_{n+1} = -\infty$. Let $d = s - r + 1$, and let $V = (v_r, v_{r+1}, \ldots, v_s) \in \mathbb{R}^{n \times d}$ and $\hat{V} = (\hat{v}_r, \hat{v}_{r+1}, \ldots, \hat{v}_s) \in \mathbb{R}^{n \times d}$ have orthonormal columns satisfying $Pv_j = \sigma_j u_j$ and $\hat{P}\hat{v}_j = \hat{\sigma}_j \hat{u}_j$ for $j = r, r+1, \ldots, s$. Then there exists orthogonal $\hat{O} \in \mathbb{R}^{d \times d}$ such that

\[ \|\hat{V}\hat{O} - V\|_F \leq \frac{2^{3/2}(2\sigma_1 + \|\hat{P} - P\|_{\text{op}})\|\hat{P} - P\|_F}{\min(\sigma^2_{r-1} - \sigma^2_r, \sigma^2_{s-1} - \sigma^2_s)} \]
C.2 Proof of Corollary 1 and Lemma 5

Proof of Corollary 1. Let $r = 1$ and $s = K = \text{rank}(P)$, so that $\sigma_s^2 - \sigma_{s+1}^2 = \lambda_K^2$. By Theorem 3, it holds that

$$\|\hat{V}\hat{O} - V\|_F^2 \leq \left(\frac{2^{3/2}(2\sigma_1 + \|\hat{P} - P\|_{\text{op}})\|\hat{P} - P\|_F}{\lambda_K^2}\right)^2.$$ 

It follows that

$$\|\hat{V}\hat{O} - V\|_F^2 \leq \left(\frac{2^{3/2}O(n\alpha)\|\hat{P} - P\|_F}{n\alpha\lambda_K^2(1 + o_P(1))}\right)^2,$$

where in the first inequality follows from $\lambda_K = n\alpha\lambda_K^* (1 + o_P(1))$ by Lemma 5, and also from

$$2\sigma_1 + \|\hat{P} - P\|_{\text{op}} \leq 3\|P\|_{\text{op}} + \|\hat{P}\|_{\text{op}}$$

$$\leq 3n \max_{ij} P_{ij} + n \max_{ij} \hat{P}_{ij}$$

$$\leq 3n \alpha \max_{ab} B_{ab}^* + n \alpha \max_{ab} \hat{B}_{ab}.$$ 

By Theorem 2 it holds that

$$\frac{1}{n^2} \|\hat{P} - P\|_F^2 = \frac{1}{n^2} \sum_{i=1}^n \sum_{j=1}^n (\hat{P}_{ij} - P_{ij})^2$$

$$\leq \frac{1}{n^2} \sum_{i=1}^n \sum_{j=1}^n 1\{\hat{P}_{ij} = P_{ij}\} \cdot \max_{i,j} (\hat{P}_{ij} - P_{ij})^2$$

$$\leq O_P \left(\frac{1}{\sqrt{n\alpha}}\right) \cdot O(\alpha^2).$$ (47)

Substituting (47) into (46) yields

$$\|\hat{V}\hat{O} - V\|_F^2 \leq \frac{O(1)}{\sqrt{n\alpha}},$$

completing the proof.

Proof of Lemma 5. Let $\hat{\pi} \in [0, 1]^K$ be given by

$$\hat{\pi}_a = \frac{1}{n} \sum_{i=1}^n 1\{z_i = a\},$$

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and let $\hat{D} = \text{diag}(\hat{\pi})$. Let $\hat{u} \in \mathbb{R}^K$ denote an eigenvector of $\hat{D}^{1/2}B^*\hat{D}^{1/2}$ with eigenvalue $\hat{\lambda}$, let $\hat{v} = \hat{D}^{-1/2}\hat{u}$, and let $v \in \mathbb{R}^n$ be given by $v_i = \hat{v}_{z_i}$ for $i \in [n]$.

It can be seen that

$$[Pv]_i = [n\alpha B^*\hat{D}\hat{v}]_{z_i}$$
$$= [n\alpha B^*\hat{D}^{1/2}\hat{u}]_{z_i}$$
$$= [n\alpha \hat{D}^{-1/2}\hat{D}^{1/2}B^*\hat{D}^{1/2}\hat{u}]_{z_i}$$
$$= [n\alpha \hat{D}^{-1/2}\hat{\lambda}\hat{u}]_{z_i}$$
$$= [n\alpha \hat{\lambda}\hat{v}]_{z_i}$$
$$= n\alpha \hat{\lambda}v_i,$$

showing that $v$ is an eigenvector of $P$ with eigenvalue $n\alpha \hat{\lambda}$. Since $\hat{D} \to D$, it follows that the eigenvalues of $\hat{D}^{1/2}B^*\hat{D}^{1/2}$ converge to those of $D^{1/2}B^*D^{1/2}$, completing the proof. 

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