Stability of interconnected impulsive systems with and without time-delays using Lyapunov methods

Sergey Dashkovskiy\textsuperscript{a}, Michael Kosmykov\textsuperscript{b}, Andrii Mironchenko\textsuperscript{b}, Lars Naujok\textsuperscript{b,*}

\textsuperscript{a}University of Applied Sciences Erfurt, Department of Civil Engineering, Erfurt, Germany
\textsuperscript{b}University of Bremen, Centre of Industrial Mathematics, P.O.Box 330440, 28334 Bremen, Germany

\begin{abstract}
In this paper, we consider input-to-state stability (ISS) of impulsive control systems with and without time-delays. We prove that if the time-delay system possesses an exponential Lyapunov-Razumikhin function or an exponential Lyapunov-Krasovskii functional, then the system is uniformly ISS provided that the average dwell-time condition is satisfied. Then, we consider large-scale networks of impulsive systems with and without time-delays and prove that the whole network is uniformly ISS under the small-gain and the average dwell-time condition. Moreover, these theorems provide us with tools to construct a Lyapunov function (for time-delay systems - a Lyapunov-Krasovskii functional or a Lyapunov-Razumikhin function) and the corresponding gains of the whole system, using the Lyapunov functions of the subsystems and the internal gains, which are linear and satisfy the small-gain condition. We illustrate the application of the main results on
\end{abstract}

\begin{flushright}
Preprint submitted to Nonlinear Analysis: Hybrid Systems February 23, 2012
\end{flushright}
1. Introduction

Impulsive systems combine continuous and discontinuous behavior of a dynamical system [1]. The continuous dynamics is typically described by differential equations and the discontinuous behavior are instantaneous state jumps that occur at given time instants, also referred to as impulses. Impulsive systems are closely related to hybrid systems [1] and switched systems [2] and have wide range of applications.

In this paper we study the input-to-state stability (ISS) property of impulsive systems. ISS was first introduced for continuous systems in [3]. Lyapunov functions provide a useful tool to verify the ISS property (see [4]) as well as for other variants of ISS, namely input-to-state dynamical stability [5], local ISS [6, 7] and integral ISS (iISS) [8]. Investigation of ISS for hybrid systems can be found in [9]. For time-delay systems the ISS property can be verified by Lyapunov-Razumikhin functions [10] or Lyapunov-Krasovskii functionals [11].

For impulsive systems the ISS and iISS properties were studied in [12] for the delay-free case and in [13] for non-autonomous time-delay systems. Sufficient conditions, which assure ISS and iISS of an impulsive system, were derived using locally Lipschitz continuous exponential ISS-Lyapunov(-Razumikhin) functions. In [12] the average dwell-time condition was used, whereas in [13] a fixed dwell-time condition was utilized. The average dwell-time condition was introduced in [14] for switched systems.

In this paper we provide a Lyapunov-Krasovskii type and a Lyapunov-Razumikhin type ISS theorem for single impulsive time-delay systems using the average dwell-time condition. The proofs use the idea of the proof of [12]. For the Razumikhin type ISS theorem we require as an additional condition that the Lyapunov gain fulfills a small-gain condition. To prove this theorem we show the equivalence of ISS and the conjunction of two properties, one of them was also used in ISS definition from [10], namely uniform global stability and a uniform convergence property. In contrast to the Razumikhin-type theorem from [13] we consider autonomous time-delay systems and the average dwell-time condition. This condition considers the

examples.

*Keywords:* Impulsive systems, Large-scale systems, Lyapunov methods, Input-to-state stability, Time-delays
average of impulses over an interval, whereas the fixed dwell-time condition considers the (minimal or maximal) interval between two impulses. Our theorem allows to verify the ISS property for larger classes of impulse time sequences, however, we have used an additional technical condition on the Lyapunov-gain in our proofs.

Considering large-scale networks of impulsive systems our main goal is to find sufficient conditions which assure ISS of interconnections of impulsive systems with and without time-delays. To this end, we use the approach used for networks of continuous systems. The first results about the ISS property for the delay-free case were given for two coupled continuous systems in [15] and for an arbitrarily large number \((n \in \mathbb{N})\) of coupled continuous systems in [16], using a small-gain argument which is a condition on the interconnection structure of the system. Lyapunov versions of the ISS small-gain theorems were proved in [17] (two systems) and [18] (\(n\) systems), where ISS-Lyapunov functions for the overall system are constructed. There are also known results for the ISS property of hybrid systems, see [19] (two systems) and [20] (\(n\) systems), as well as for the infinite-dimensional systems [21].

In [22] Lyapunov-Razumikhin functions and Lyapunov-Krasovskii functionals are used to verify the ISS property of large-scale time-delay systems, where a small-gain condition is used. An approach with vector Lyapunov functions can be found in [23].

We prove that under a small-gain condition with linear gains and a dwell-time condition according to [12] a large-scale network of impulsive systems has the ISS property and construct the exponential ISS-Lyapunov, Lyapunov-Razumikhin and Lyapunov-Krasovskii function(al) and the corresponding gains of the whole system.

The paper is organized as follows: In Section 2 we note some basic definitions. Single impulsive systems are studied in Section 3, where the Lyapunov-Krasovskii and the Lyapunov-Razumikhin methodologies for impulsive time-delay systems are introduced, including the first main results of this paper. In Section 4, large-scale networks of impulsive systems with and without time-delays are considered and three different Lyapunov-type theorems are proved. An illustrative example of the application of the main results for networks can be found in Section 5. Finally, Section 6 concludes this paper with a short summary.
2. Preliminaries

By $x^T$ we denote the transposition of a vector $x \in \mathbb{R}^N$, $N \in \mathbb{N}$, furthermore $\mathbb{R}_+ := [0, \infty)$ and $\mathbb{R}_+^N$ denotes the positive orthant $\{x \in \mathbb{R}^N : x \geq 0\}$ where we use the partial order for $x, y \in \mathbb{R}^N$ given by

\[
x \geq y \iff x_i \geq y_i, \quad i = 1, \ldots, N \quad \text{and} \quad x \nless y \iff \exists i : x_i < y_i,\]

\[
x > y \iff x_i > y_i, \quad i = 1, \ldots, N.\]

We denote the Euclidean norm by $|\cdot|$. For a piecewise continuous function $x : I \to \mathbb{R}^N$ we define $\|x\|_I := \sup_{t \in I} |x(t)|$. $\nabla V$ denotes the gradient of a function $V$. The upper right-hand side derivative of a locally Lipschitz continuous function $V : \mathbb{R}^N \to \mathbb{R}_+$ along trajectory $x(\cdot)$ is defined by

\[
D^+V(x(t)) = \limsup_{h \to 0^+} \frac{V(x(t + h)) - V(x(t))}{h}.
\]

The function $x^t : [-\theta, 0] \to \mathbb{R}^N$ is given by $x^t(\tau) := x(t + \tau), \quad \tau \in [-\theta, 0]$, where $\theta \in \mathbb{R}_+$ is the maximum involved delay and we denote the norm $\|x^t\| := \max_{-\theta \leq s \leq t} |x(s)|$. For $a, b \in \mathbb{R}, a < b$, let $PC([a, b]; \mathbb{R}^N)$ denote the Banach space of piecewise right-continuous functions defined on $[a, b]$ equipped with the norm $\|\cdot\|_{[a, b]}$ and take values in $\mathbb{R}^N$.

To define the stability notion we use the following classes.

**Definition 2.1.** Classes of comparison functions are:

- $\mathcal{K} := \{\gamma : \mathbb{R}_+ \to \mathbb{R}_+ | \gamma \text{ is continuous, } \gamma(0) = 0 \text{ and strictly increasing}\}$,
- $\mathcal{K}_\infty := \{\gamma \in \mathcal{K} | \gamma \text{ is unbounded}\}$,
- $\mathcal{L} := \{\gamma : \mathbb{R}_+ \to \mathbb{R}_+ | \gamma \text{ is continuous and decreasing with } \lim_{t \to \infty} \gamma(t) = 0\}$,
- $\mathcal{KL} := \{\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+ | \beta(\cdot, t) \in \mathcal{K}, \beta(r, \cdot) \in \mathcal{L}, \forall t, r \geq 0\}$.

Note that for $\gamma \in \mathcal{K}_\infty$ the inverse function $\gamma^{-1}$ always exists and $\gamma^{-1} \in \mathcal{K}_\infty$.

3. Single impulsive systems

We consider single impulsive systems without time-delays of the form

\[
\dot{x}(t) = f(x(t), u(t)), \quad t \neq t_k, \quad k \in \mathbb{N},
\]

\[
x(t) = g(x^-(t), u^-(t)), \quad t = t_k, \quad k \in \mathbb{N},
\]
where $t \in \mathbb{R}_+$, $x \in \mathbb{R}^N$ is absolutely continuous between impulses, $u \in \mathbb{R}^M$ is a locally bounded, Lebesgue-measurable input and $\{t_1, t_2, t_3, \ldots \}$ is a strictly increasing sequence of impulse times in $(t_0, \infty)$ for some initial time $t_0 < t_1$. The set of impulse times is assumed to be either finite or infinite and unbounded and impulse times $t_k$ have no finite accumulation point. Given a sequence $\{t_k\}$ and a pair of times $s, t$ satisfying $t_0 \leq s < t$, $N(t, s)$ denotes the number of impulse times $t_k$ in the semi-open interval $(s, t)$.

Furthermore, $f : \mathbb{R}^N \times \mathbb{R}^M \to \mathbb{R}^N$, $g : \mathbb{R}^N \times \mathbb{R}^M \to \mathbb{R}^N$, where we assume that $f$ is locally Lipschitz. All signals ($x$ and inputs $u$) are assumed to be right-continuous and to have left limits at all times and we denote $x^− := \lim_{s \uparrow t} x(s)$, $u^−(t) := \lim_{s \uparrow t} u(s)$.

We are interested in the stability of systems of the form (1), where we use the following stability property, introduced in [3] and adapted to impulsive systems in [12] as follows:

**Definition 3.1.** Assume that a sequence $\{t_k\}$ is given. We call system (1) input-to-state stable (ISS) if there exist functions $\beta \in \mathcal{KL}$, $\gamma \in \mathcal{K}_\infty$, such that for every initial condition $x(t_0)$ and every input $u$ the corresponding solution to (1) exists globally and satisfies

$$|x(t)| \leq \max\{\beta(|x(t_0)|, t - t_0), \gamma(\|u\|_{[t_0, t]})\}, \quad \forall t \geq t_0. \quad (2)$$

The impulsive system (1) is uniformly ISS over a given class $\mathcal{S}$ of admissible sequences of impulse times if (2) holds for every sequence in $\mathcal{S}$, with functions $\beta$ and $\gamma$ that are independent of the choice of the sequence.

For the stability analysis of impulsive systems we use exponential ISS-Lyapunov functions, see [12]. Here, we assume that these functions are locally Lipschitz continuous, which are differentiable for almost all (f.a.a.) $x$ by Rademacher’s Theorem (see e.g., [24] Theorem 5.8.6). For the stability analysis of interconnected systems it is sufficient to consider locally Lipschitz continuous functions instead of smooth functions and they were also used for example in [18].

**Definition 3.2.** We say that a function $V : \mathbb{R}^N \to \mathbb{R}_+$ is an exponential ISS-Lyapunov function for (1) with rate coefficients $c, d \in \mathbb{R}$ if $V$ is locally Lipschitz, positive definite, radially unbounded, and whenever $V(x) \geq \gamma(\|u\|)$ holds it follows

$$\nabla V(x) \cdot f(x, u) \leq -cV(x) \text{ f.a.a. } x, \text{ all } u \text{ and}$$

$$V(g(x, u)) \leq e^{-d}V(x) \forall x, u. \quad (3)$$

$$5$$
where $\gamma$ is some function from $\mathcal{K}_\infty$.

Without loss of generality we use the same function $\gamma$ in (3) and (4). Choosing $\gamma_c \in \mathcal{K}_\infty$ in (3) and $\gamma_t \in \mathcal{K}_\infty$ in (4) and taking the maximum of these two functions, we get $\gamma$.

Note that conditions (3) and (4) are in implication form. This is equivalent to the usage of the dissipative form in [12], which was proved in [9], Proposition 2.6., where the coefficients $c, d$ are different in general.

In [12] the following theorem was proved which establishes stability of a single impulsive system.

**Theorem 3.3 (Lyapunov-type theorem).** Let $V$ be an exponential ISS-Lyapunov function for (1) with rate coefficients $c,d \in \mathbb{R}$ with $d \neq 0$. For arbitrary constants $\mu, \lambda > 0$, let $\mathcal{S}[\mu, \lambda]$ denote the class of impulse time sequences $\{t_k\}$ satisfying

$$-dN(t,s) - (c - \lambda)(t - s) \leq \mu, \quad \forall t \geq s \geq t_0. \quad (5)$$

Then the system (1) is uniformly ISS over $\mathcal{S}[\mu, \lambda]$.

**Remark 3.4.** Note that in [12] this theorem was proved, using exponential ISS-Lyapunov functions in dissipative form. However, the same statement holds also if the exponential ISS-Lyapunov function is given in the implication form (3) and (4). The proof is similar to the proof of Theorem 3.7.

If $d = 0$, then the jumps do not destabilize the system, and the whole system will be ISS, if the corresponding continuous dynamics is ISS. This case was investigated in more detail in [12], Section 6.

Note, that condition (5) guarantees stability of the impulsive system even if the continuous or discontinuous behavior is unstable. For example, if the continuous behavior is unstable, which means $c < 0$, then this condition assumes that the discontinuous behavior has to stabilize the system ($d > 0$) and the jumps have to occur often enough. Conversely, if the discontinuous behavior is unstable ($d < 0$) and the continuous behavior is stable ($c > 0$) then the jumps have to occur rarely, which stabilizes the system.

### 3.1. Systems with time-delays

We consider single impulsive system with time-delays of the form

$$\dot{x}(t) = f(x^t, u(t)), \quad t \neq t_k, \quad k \in \mathbb{N},$$

$$x(t) = g((x^t)^-, u^-(t)), \quad t = t_k, \quad k \in \mathbb{N}, \quad (6)$$
where we make the same assumptions as in the delay-free case, where \( f : PC([-\theta, 0]; \mathbb{R}^N) \times \mathbb{R}^M \rightarrow \mathbb{R}^N \) is locally Lipschitz and \( g : PC([-\theta, 0]; \mathbb{R}^N) \times \mathbb{R}^M \rightarrow \mathbb{R}^N \). We denote \((x^t)^- := \lim_{s \uparrow t} x_s\).

We assume that the regularity conditions (see e.g., [25]) for the existence and uniqueness of a solution of system (6) are satisfied. We denote the solution of (6) corresponding to the given input \( u \) by \( x(t, t_0, \xi, u) \) or \( x(t) \) for short, for any \( \xi \in PC([-\theta, 0], \mathbb{R}^N) \) that exists in a maximal interval \([-\theta, b)\), \( 0 < b \leq +\infty \), satisfying the initial condition \( x(t_0) = \xi \).

In the following subsections we present tools, namely Lyapunov-Krasovskii functionals and Lyapunov-Razumikhin functions, to check, if a time-delay system has the ISS property.

3.1.1. The Lyapunov-Krasovskii methodology

In this subsection we adapt the Lyapunov-Krasovskii methodology, introduced in [11], to impulsive time-delay systems. As one of the results of this paper we prove that from the existence of an exponential ISS-Lyapunov-Krasovskii functional the ISS property follows, provided that the dwell-time condition (5) is satisfied.

We consider another type of impulsive systems with time-delays of the form

\[
\dot{x}(t) = f(x^t, u(t)), \ t \neq t_k, \ k \in \mathbb{N},
\]

\[
x^t = g((x^t)^-, u^-(t)), \ t = t_k, \ k \in \mathbb{N},
\]

where we make the same assumptions as before and the functional \( g \) is now a map from \( PC([-\theta, 0]; \mathbb{R}^N) \times \mathbb{R}^M \) into \( PC([-\theta, 0]; \mathbb{R}^N) \).

According to [26], Section 2, the initial state and the input together determine the evolution of the system according to the right-hand side of the differential equation. Therefore, for time-delay systems we denote the state by the function \( x^t \in PC([-\theta, 0], \mathbb{R}^N) \) and we change the discontinuous behavior in (7): In contrary to the system (6) at an impulse time \( t_k \) not only the point \( x(t_k) \) “jumps”, but all the states \( x^t \) in the interval \( (t_k - \theta, t_k) \). Due to this change the Lyapunov-Razumikhin approach cannot be applied. In this case we propose to use Lyapunov-Krasovskii functionals for the stability analysis of systems of the form (7).

Another approach using Lyapunov functionals can be found in [27]. There, Lyapunov functionals for systems of the form (6) with zero input are used for stabilization of impulsive systems, where the definition of such a functional is different to the approach presented here according to impulse times.
The ISS property is redefined with respect to time-delays, see [13]:

**Definition 3.5.** Suppose that a sequence \( \{t_k\} \) is given. We call system (6) (or (7)) input-to-state stable (ISS) if there exist functions \( \beta \in KL, \gamma_u \in K_\infty \), such that for every initial condition \( \xi \in PC([-\theta,0],\mathbb{R}^N) \) and every input \( u \) the corresponding solution to (6) (or (7)) exists globally and satisfies

\[
|x(t)| \leq \max\{\beta(\|\xi\|_{[-\theta,0]}, t - t_0), \gamma_u(\|u\|_{[t_0,t]})\}, \ \forall t \geq t_0. \tag{8}
\]

The impulsive system (6) (or (7)) is uniformly ISS over a given class \( S \) of admissible sequences of impulse times if (8) holds for every sequence in \( S \), with functions \( \beta \) and \( \gamma_u \), which are independent on the choice of the sequence.

Given a locally Lipschitz continuous functional \( V : PC([-\theta,0];\mathbb{R}^N) \to \mathbb{R}_+ \), the upper right-hand derivative \( D^+ V \) of the functional \( V \) along the solution \( x(t,t_0,\xi,u) \) is defined according to [28], Chapter 5.2:

\[
D^+ V(\phi,u) := \limsup_{h \to 0^+} \frac{1}{h} (V(x^{t+h}) - V(\phi)), \tag{9}
\]

where \( x^{t+h} \in PC([-\theta,0];\mathbb{R}^N) \) is generated by the solution \( x(t,t_0,\phi,u) \) of the system (7) with \( x^{t_0} := \phi \in PC([-\theta,0];\mathbb{R}^N) \).

With the symbol \( |\cdot|_a \) we indicate any norm in \( PC([-\theta,0];\mathbb{R}^N) \) such that for some positive reals \( b, \tilde{c} \) the following inequalities hold

\[
b|\phi(0)| \leq |\phi|_a \leq \tilde{c}\|\phi\|_{[-\theta,\infty)}, \ \forall \phi \in PC([-\theta,0];\mathbb{R}^N). \]

**Definition 3.6.** A functional \( V : PC([-\theta,0];\mathbb{R}^N) \to \mathbb{R}_+ \) is called an exponential ISS-Lyapunov-Krasovskii functional with rate coefficients \( c,d \in \mathbb{R} \) for system (7), if \( V \) is locally Lipschitz continuous, there exist \( \psi_1, \psi_2 \in K_\infty \) such that

\[
\psi_1(|\phi(0)|) \leq V(\phi) \leq \psi_2(|\phi|_a), \ \forall \phi \in PC([-\theta,0];\mathbb{R}^N) \tag{10}
\]

and there exists a function \( \gamma \in K \) such that whenever \( V(\phi) \geq \gamma(|u|) \) holds it follows

\[
D^+ V(\phi,u) \leq -cV(\phi) \text{ and } \tag{11}
V(g(\phi,u)) \leq e^{-d}V(\phi), \tag{12}
\]

for all \( \phi \in PC([-\theta,0];\mathbb{R}^N) \) and \( u \in \mathbb{R}^M \).
Now, we present as a result a counterpart of Theorem 1 in [12] and Theorems 1 and 2 in [13] for impulsive systems with time-delays using the Lyapunov-Krasovskii approach:

**Theorem 3.7 (Lyapunov-Krasovskii-type theorem).** Let $V$ be an exponential ISS-Lyapunov-Krasovskii functional for system (7) with $c, d \in \mathbb{R}$, $d \neq 0$. For arbitrary constants $\mu, \lambda \in \mathbb{R}_+$, let $S[\mu, \lambda]$ denote the class of impulse time sequences $\{t_k\}$ satisfying the dwell-time condition (5). Then the system (7) is uniformly ISS over $S[\mu, \lambda]$.

**Proof.** From (11) we have for any two consecutive impulses $t_{k-1}, t_k$, $\forall t \in (t_{k-1}, t_k)$
\[
V(x^t) \geq \gamma(|u(t)|) \Rightarrow \text{D}^+V(x^t, u(t)) \leq -cV(x^t)
\]
and similarly with (12) for every impulse time $t_k$
\[
V(x^{t_k}) \geq \gamma(|u^-(t_k)|) \Rightarrow V(g((x^{t_k})^-, u^-(t_k))) \leq e^{-d}V(x^{t_k}).
\]
Because of the right-continuity of $x$ and $u$ there exists a sequence of times $t_0 := \tilde{t}_0 < \tilde{t}_1 < \tilde{t}_1 < \tilde{t}_2 < \ldots$ such that we have
\[
V(x^t) \geq \gamma(||u||_{[t_0, t]}) \quad \forall t \in [\tilde{t}_i, \tilde{t}_{i+1}), \ i = 0, 1, \ldots,
\]
\[
V(x^t) \leq \gamma(||u||_{[t_0, t]}) \quad \forall t \in [\tilde{t}_i, \tilde{t}_i), \ i = 1, 2, \ldots,
\]
where this sequence breaks the interval $[t_0, \infty)$ into a disjoint union of subintervals. Suppose $t_0 < \tilde{t}_1$, so that $[t_0, \tilde{t}_1)$ is nonempty. Otherwise skip forward to the line below (17). Between any two consecutive impulses $t_{k-1}, t_k \in (t_0, \tilde{t}_1]$ with (15) and (13) we have $\text{D}^+V(x^t, u(t)) \leq -cV(x^t) \forall t \in (t_{k-1}, t_k)$ and therefore
\[
V((x^{t_k})^-) \leq e^{-c(t_k-t_{k-1})}V(x^{t_{k-1}}).
\]
From (14) and (15) we have $V(x^{t_k}) \leq e^{-d}V((x^{t_k})^-)$. Combining this it follows
\[
V(x^{t_k}) \leq e^{-d}e^{-c(t_k-t_{k-1})}V(x^{t_{k-1}})
\]
and by the iteration over the $N(t, t_0)$ impulses on $(t_0, t]$ we obtain the bound
\[
V(x^t) \leq e^{-dN(t,t_0)-c(t-t_0)}V(\xi), \ \forall t \in (t_0, \tilde{t}_1].
\]
Using the dwell-time condition (5) we get
\[ V(x^t) \leq e^{\mu - \lambda (t-t_0)} V(\xi), \quad \forall t \in (t_0, \tilde{t}_1]. \] (17)

Now, on any subinterval of the form \([\tilde{t}_i, \tilde{t}_i]\) we already have (16) as a bound. If \(\tilde{t}_i\) is not an impulse time, then (16) is a bound for \(t = \tilde{t}_i\). If \(\tilde{t}_i\) is an impulse time, then we have
\[ V(x^{\tilde{t}_i}) \leq e^{-\lambda (\tilde{t}_i - t_0)} V(\xi), \quad \forall t \in (\tilde{t}_i, \tilde{t}_i]. \] (17)

and in either case
\[ V(x^t) \leq e^{\mu - \lambda (t-t_0)} V(\xi), \quad \forall t \in (\tilde{t}_i, \tilde{t}_i], \quad i \geq 1, \] (18)

where this bound holds for all \(t \geq \tilde{t}_i\), if \(\tilde{t}_i = \infty\). Now consider any subinterval of the form \([\tilde{t}_i, \tilde{t}_{i+1}]\), \(i \geq 1\). Repeating the argument used to establish (17) with \(\tilde{t}_i\) in place of \(t_0\) and using (18) with \(t = \tilde{t}_i\), we get
\[ V(x^t) \leq e^{\mu - \lambda (\tilde{t}_i - t_0)} V(x^{\tilde{t}_i}) \leq e^{\mu + |d| \gamma} (\|u\|_{[t_0, \tilde{t}_i]}), \forall t \in (\tilde{t}_i, \tilde{t}_{i+1}], \quad i \geq 1. \] Combining this with (17) and (18) we obtain the global bound \(\forall t \geq t_0\)
\[ V(x^t) \leq \max \{ e^{\mu - \lambda (t-t_0)} V(\xi), e^{\mu + |d| \gamma} (\|u\|_{[t_0, t]}) \}. \]

The uniformly ISS property follows then from (10) by definition of \(\beta(\xi, t - t_0) := \psi_1^{-1}(e^{-\lambda (t-t_0)} \psi_2(c\|\xi\|_{[-\theta, 0]}))\) and \(\gamma_u(r) := \psi_1^{-1}(e^{\mu + |d| \gamma} (r))\), where the global existence of solutions follows from the boundedness of \(x\). Note that \(\beta\) and \(\gamma_u\) do not depend on the particular choice of the time sequence and therefore uniformity is clear. \(\square\)

3.1.2. The Lyapunov-Razumikhin methodology

To study the ISS property of impulsive systems with time-delays of the form (6) one can use ISS-Lyapunov-Razumikhin functions.

Definition 3.8. A function \(V : \mathbb{R}^N \to \mathbb{R}_+\) is called an exponential ISS-Lyapunov-Razumikhin function for system (6) with rate coefficients \(c, d \in \mathbb{R}\), if \(V\) is locally Lipschitz continuous and there exist functions \(\psi_1, \psi_2, \gamma_l, \gamma_u \in \mathcal{K}_\infty\) such that
\[ \psi_1(|\phi(0)|) \leq V(\phi(0)) \leq \psi_2(|\phi(0)|) \]
and whenever 
\[ V(\phi(0)) \geq \max\{\gamma_t(\|V'(\phi)\|), \gamma_u(\|u\|)\} \]
holds it follows
\[ D^+ V(\phi(0)) \leq -c V(\phi(0)) \quad \text{and} \]
\[ V(g(\phi, u)) \leq e^{-d} V(\phi(0)), \quad (20) \]
for all \( \phi \in PC([-\theta, 0] ; \mathbb{R}^N) \) and \( u \in \mathbb{R}^M \), where \( V^t : PC([-\theta, 0] ; \mathbb{R}^N) \to PC([-\theta, 0] ; \mathbb{R}^N) \) is defined by \( V^t(x)(\tau) := V(x(t + \tau)) \), \( \tau \in [-\theta, 0] \).

For the main result of this section, we need the following:

**Definition 3.9.** Assume that a sequence \( \{t_k\} \) is given. We call system (6) (or (7)) globally stable (GS) if there exist functions \( \varphi, \gamma \in K_{\infty} \), such that for every initial condition \( \xi \in PC([-\theta, 0], \mathbb{R}^N) \) and every input \( u \) it holds
\[ |x(t)| \leq \max\{\varphi(\|\xi\|_{[-\theta, 0]}), \gamma(\|u\|_{[t_0, t]})\}, \quad \forall t \geq t_0. \quad (21) \]
The impulsive system (6) (or (7)) is uniformly GS over a given class \( S \) of admissible sequences of impulse times if (21) holds for every sequence in \( S \), with functions \( \varphi \) and \( \gamma_u \), which are independent on the choice of the sequence.

To prove the Razumikhin-type theorem for impulsive time-delay systems, we need the following characterization of the uniform ISS property:

**Lemma 3.10.** The system (6) (or (7)) is uniformly ISS over \( S \) if and only if it is
- uniformly globally stable over \( S \) and
- \( \exists \gamma \in K \) such that for each \( \epsilon > 0 \), \( \eta_x \in \mathbb{R}_+ \), \( \eta_u \in \mathbb{R}_+ \) there exists \( T \geq 0 \) (which does not depend on an impulse time sequence from \( S \)) such that
\[ \|\xi\|_{[-\theta, 0]} \leq \eta_x \text{ and } \|u\|_{\infty} \leq \eta_u \text{ imply } |x(t)| \leq \max\{\epsilon, \gamma(\|u\|_{[t_0, t]})\}, \quad \forall t \geq T + t_0. \]

**Proof.** We start with necessity. Let (6) (or (7)) be uniformly ISS over \( S \). Then it is uniformly GS over \( S \) with \( \varphi(\cdot) := \beta(\cdot, 0), \gamma_u \equiv \gamma \).

Take arbitrary \( \epsilon > 0 \), \( \eta_x \in \mathbb{R}_+ \). For all \( \|\xi\|_{[-\theta, 0]} \leq \eta_x \), all \( u \) and all impulse time sequences from \( S \) it holds
\[ |x(t)| \leq \max\{\beta(\eta_x, t - t_0), \gamma_u(\|u\|_{[t_0, t]})\}, \quad \forall t \geq t_0. \]
If $\varepsilon > \beta(\eta_x,0)$, then we choose $T$ in Lemma 3.10 as $T := 0$. Otherwise take $T$ as solution (which is unique) of the equation $\beta(\eta_x, T - t_0) = \varepsilon$. Clearly, $T$ does not depend on the choice of the sequence from $\mathcal{S}$. Thus, the second property in the statement of the lemma is verified with $\gamma_u \equiv \gamma$.

Now let us prove sufficiency. Without loss of generality we take $\kappa := \eta_x = \eta_u$ and fix it. From uniform global stability it follows that for all $\|\xi\|_{[-\theta,0]} \leq \kappa$, for all $u \in L_\infty(\mathbb{R}^+, \mathbb{R}^m)$ and all impulse time sequences from $\mathcal{S}$ it holds

$$|x(t)| \leq \max\{|\varphi(\kappa)|, \gamma(\|u\|_{[t_0,t]})\}, \forall t \geq t_0.$$ 

Define $\varepsilon_n := \frac{1}{2^n}\varphi(\kappa)$. The second assumption of the lemma implies the existence of a sequence of times $T_n := T(\varepsilon_n, \kappa)$, which without loss of generality we assume to be strictly increasing such that for all $\xi : \|\xi\|_{[-\theta,0]} \leq \kappa$, for all $u : \|u\|_\infty \leq \kappa$ and for all $t \geq T_n + t_0$ it holds

$$|x(t)| \leq \max\{\varepsilon_n, \gamma(\|u\|_{[t_0,t]})\}.$$ 

Define $\omega(\kappa, T_n) := \varepsilon_n-1, n \in \mathbb{N}, n \neq 0$. Extend this function for $t \in \mathbb{R}^+ \backslash \{T_n, n \in \mathbb{N}\}$ such that $\omega(\kappa, \cdot) \in \mathcal{L}$. For all $t \in (T_n + t_0, T_{n+1} + t_0)$ it holds

$$|x(t)| \leq \max\{\varepsilon_n, \gamma(\|u\|_{[t_0,t]})\} \leq \max\{\omega(\kappa, t - t_0), \gamma(\|u\|_{[t_0,t]})\}.$$ 

Doing this for all $\kappa \in \mathbb{R}^+$ we obtain the function $\omega$, defined on $\mathbb{R}^+ \times \mathbb{R}^+$. Now define $\beta(r,t) = \sup_{0 \leq s \leq r} \omega(s,t) \geq \omega(r,t)$. It follows that $\beta$ is continuous, $\beta(\cdot,t) \in \mathcal{K}$ (if it is not true, we can always find $\widetilde{\beta}(\cdot,t) \in \mathcal{K}$ without losing continuity in the second argument such that $\beta(r,t) \leq \widetilde{\beta}(r,t)$ for all $r > 0)$ and $\beta(r, \cdot) \in \mathcal{L}$, since $\omega(r, \cdot) \in \mathcal{L}$. Thus, $\beta \in \mathcal{KL}$ and we obtain

$$|x(t)| \leq \max\{\beta(\max\{\|\xi\|_{[-\theta,0]}, \|u\|_{[t_0,t]}\}, t - t_0), \gamma(\|u\|_{[t_0,t]})\}$$

$$= \max\{\beta(\|\xi\|_{[-\theta,0]}, t - t_0), \beta(\|u\|_{[t_0,t]}, t - t_0), \gamma(\|u\|_{[t_0,t]})\}$$

$$\leq \max\{\beta(\|\xi\|_{[-\theta,0]}, t - t_0), \gamma_u(\|u\|_{[t_0,t]}), \gamma(\|u\|_{[t_0,t]})\},$$

where $\gamma_u(r) := \max\{\beta(r,0), \gamma(r)\}$. This proves uniform ISS over $\mathcal{S}$. \qed

**Remark 3.11.** Note that in [10] for time-delay systems without impulse times a different definition of the ISS property has been used. However, with the help of Lemma 3.10 one can show that from the Assumptions of Theorem 1 in [10] it follows not only ISS in the sense of [10], but also ISS in the sense of Definition 3.5. Until now, it is an open problem whether the definition of ISS in [10] is equivalent to the ISS property in Definition 3.5 for time-delay systems without impulse times, see [20, 30].
We need one more technical result for the main result of this section. For a given sequence of impulse times, we denote the number of jumps within the time-span \([s, t]\) by \(N^*(t, s)\). The set of impulse time sequences, for which (5) holds with \(N^*(t, s)\) instead of \(N(t, s)\), is denoted by \(S^*[\mu, \lambda]\). The next lemma shows that \(S^*[\mu, \lambda]\) is equal to \(S[\mu, \lambda]\).

**Lemma 3.12.** Let \(c, d \in \mathbb{R}\), \(d \neq 0\) be given. Then, \(S[\mu, \lambda] = S^*[\mu, \lambda]\) for all \(\mu, \lambda > 0\).

**Proof.** Firstly, consider the case, when \(d < 0\). Let \(T \in S^*[\mu, \lambda]\). Since \(N(t) \leq N^*(t) \leq \frac{1}{d}((c - \lambda)(t - s) + \mu)\), it is clear that \(S[\mu, \lambda] \supset S^*[\mu, \lambda]\).

Let \(T \in S[\mu, \lambda]\). Then, since \([s, t] \subset (s - \varepsilon, t]\) for arbitrary \(\varepsilon > 0\) it holds

\[N^*(t, s) \leq N(t, s - \varepsilon) \leq \frac{1}{d}((c - \lambda)(t - (s - \varepsilon)) + \mu).\]

Tending \(\varepsilon \to 0\), we obtain that \(S[\mu, \lambda] \subset S^*[\mu, \lambda]\).

Now let \(d > 0\). Take \(T \in S^*[\mu, \lambda]\). Then, \(N^*(t, s) \geq N(t, s)\) implies \(S[\mu, \lambda] \subset S^*[\mu, \lambda]\). On the other side, since for arbitrary \(\varepsilon > 0\) it holds

\[N(t, s) \geq N^*(t, s + \varepsilon) \geq \frac{1}{d}((c - \lambda)(t - (s + \varepsilon)) + \mu),\]

for \(T \in S^*[\mu, \lambda]\), tending \(\varepsilon \to 0\) we have \(S[\mu, \lambda] \supset S^*[\mu, \lambda]\). \(\square\)

Now, we prove the main result of this section.

**Theorem 3.13 (Lyapunov-Razumikhin-type theorem).** Let \(V\) be an exponential ISS-Lyapunov-Razumikhin function for system (6) with \(c, d \in \mathbb{R}\), \(d \neq 0\). For arbitrary constants \(\mu, \lambda > 0\), let \(S[\mu, \lambda]\) denote the class of impulse time sequences \(\{t_k\}\) satisfying the dwell-time condition (5). If \(\gamma_t\) satisfies \(\gamma_t(r) < e^{-\mu r}, r > 0\), then the system (6) is uniformly ISS over \(S[\mu, \lambda]\).

**Proof.** From (19) we have for any two consecutive impulses \(t_{k-1}, t_k, \forall t \in (t_{k-1}, t_k)\)

\[V(x(t)) \geq \max\{\gamma_t(\|V'(x^t)\|), \gamma_u(|u(t)|)\} \Rightarrow D^+V(x(t)) \leq -cV(x(t)) \quad (22)\]

and similarly with (20) for every impulse time \(t_k\)

\[V(x(t_k)) \geq \max\{\gamma_t(\|V'((x^{t_k})^-)\|), \gamma_u(|u^-(t_k)|)\}\]

\[\Rightarrow V(g((x^{t_k})^-, u^-(t_k))) \leq e^{-d}V(x(t_k)). \quad (23)\]
Because of the right-continuity of $x$ and $u$ there exists a sequence of times $t_0 := \bar{t}_0 < \bar{t}_1 < \bar{t}_2 < \bar{t}_2 < \ldots$ such that for $i = 0, 1, \ldots$ we have

$$V(x(t)) \geq \max\{\gamma_t(\sup_{r \in [t_0,t]} \|V^r(x^r)\|), \gamma_u(\|u\|_{[t_0,t]}), \forall t \in [\bar{t}_i, \bar{t}_{i+1}) \tag{24}$$

and for all $i = 1, 2, \ldots$ it holds

$$V(x(t)) \leq \max\{\gamma_t(\sup_{r \in [t_0,t]} \|V^r(x^r)\|), \gamma_u(\|u\|_{[t_0,t]}), \forall t \in [\bar{t}_i, \bar{t}_i), \tag{25}$$

where this sequence breaks the interval $[t_0, \infty)$ into a disjoint union of subintervals. Suppose $t_0 < \bar{t}_1$, so that $[t_0, \bar{t}_1)$ is nonempty. Otherwise skip forward to the line below (27). Between any two consecutive impulses $t_{k-1}, t_k \in [t_0, \bar{t}_1]$ from (24) and (22) we have $D^+V(x(t))) \leq -cV(x(t)), \forall t \in (t_{k-1}, t_k)$ and therefore

$$V(x^{-(t_k)}) \leq e^{-c(t_k-t_{k-1})}V(x(t_{k-1})).$$

From (23) and (24) we have $V(x(t_k)) \leq e^{-dV(x^{-(t_k)})}$. Combining this it follows

$$V(x(t_k)) \leq e^{-d-c(t_k-t_{k-1})}V(x(t_{k-1}))$$

and by the iteration over the $N(t, t_0)$ impulses on $[t_0, t]$ we obtain the bound

$$V(x(t)) \leq e^{-dN(t,t_0)-c(t-t_0)}V(x(t_0)), \forall t \in [t_0, \bar{t}_1]. \tag{26}$$

Using the dwell-time condition (5) we get

$$V(x(t)) \leq e^{\mu-\lambda(t-t_0)}V(x(t_0)), \forall t \in [t_0, \bar{t}_1]. \tag{27}$$

For any subinterval of the form $[\bar{t}_i, \bar{t}_i), i = 1, 2, \ldots$ we have (25) as a bound for $V(x(t))$. Now consider two cases.

Let $\bar{t}_i$ be not an impulse time, then (25) is a bound for $t = \bar{t}_i$. Consider the subinterval $[\bar{t}_i, \bar{t}_{i+1})$. Repeating the argument used to establish (27), with $\bar{t}_i$ in place of $t_0$ and using (25) with $t = \bar{t}_i$ we get $\forall t \in (\bar{t}_i, \bar{t}_{i+1})$

$$V(x(t)) \leq e^{\mu-\lambda(t-\bar{t}_i)}V(x(\bar{t}_i)) \leq e^\mu \max\{\gamma_t(\sup_{r \in [t_0,\bar{t}_i]} \|V^r(x^r)\|), \gamma_u(\|u\|_{[t_0,\bar{t}_i]}), \forall t \in [t_0, \bar{t}_{i+1}) \}.$$
Now let $\tilde{t}_i$ be an impulse time. Then we have

$$V(x(\tilde{t}_i)) \leq e^{-d} \max \{ \gamma(t) \sup_{r \in [t_0, \tilde{t}_i]} \| V^t(x^r) \|, \gamma_u(\|u\|_{[t_0, \tilde{t}_i]}) \}. \quad (28)$$

Consider for one more time the subinterval of the form $[\tilde{t}_i, \tilde{t}_{i+1})$. According to the estimate (26) we obtain

$$V(x(t)) \leq e^{-dN(t, \tilde{t}_i) - c(t-\tilde{t}_i)} V(x(\tilde{t}_i)) = e^{-d(N(t, \tilde{t}_i) + 1) - c(t-\tilde{t}_i)} V(x(\tilde{t}_i)) = e^{d} e^{-dN(t, \tilde{t}_i) - c(t-\tilde{t}_i)} V(x(\tilde{t}_i)).$$

From Lemma 3.12 we know that $\mathcal{S}[\mu, \lambda] = \mathcal{S}^*[\mu, \lambda]$. Thus, we can continue:

$$V(x(t)) \leq e^{d} e^{-\lambda(t-\tilde{t}_i)} V(x(\tilde{t}_i)) \leq e^{\mu} \max \{ \gamma(t) \sup_{r \in [t_0, \tilde{t}_i]} \| V^t(x^r) \|, \gamma_u(\|u\|_{[t_0, \tilde{t}_i]}) \}. \quad (29)$$

Overall, we obtain $\forall t \geq t_0$

$$V(x(t)) \leq \max \left\{ e^{\mu - \lambda(t-t_0)} V(x(t_0)), e^{\mu} \gamma(t) \sup_{r \in [t_0, \tilde{t}]} \| V^t(x^r) \|, e^{\mu} \gamma_u(\|u\|_{[t_0, \tilde{t}]}), e^{\mu} \right\}. \quad (30)$$

and it holds

$$\sup_{t \geq t_0} \| V^t(x^*) \| \leq \max \left\{ \| V^t(x^0) \|, \sup_{t \geq t_0} V(s) \right\}. \quad (30)$$

We take the supremum over $[t_0, t]$ in (29) and insert it into (30). Then, using $\gamma(t) < e^{-\mu t}$ and the fact that for all $a, b > 0$ from $a \leq \max \{ b, e^{\mu} \gamma(a) \}$ it follows $a \leq b$, we obtain

$$\sup_{t \geq t_0} \| V^t(x^*) \| \leq \max \left\{ e^{\mu} \psi_2(\|x\|_{[-\theta, 0]}), e^{\mu} \gamma_u(\|u\|_{[t_0, \tilde{t}]}), e^{\mu} \right\}$$

and therefore

$$|x(t)| \leq \max \left\{ \psi^{-1}_1(e^{\mu} \psi_2(\|x\|_{[-\theta, 0]})), \psi^{-1}_1(e^{\mu} \gamma_u(\|u\|_{[t_0, \tilde{t}]})) \right\}, \quad \forall t \geq t_0,$$

which means, that the system (6) is uniformly GS over $\mathcal{S}[\mu, \lambda]$. Note that $\bar{\varphi}(: \psi^{-1}_1(e^{\mu} \psi_2(\cdot))$ is a $\mathcal{K}_\infty$-function. Now, for given $\epsilon, \eta_x, \eta_u > 0$ such that $\|x\|_{[-\theta, 0]} \leq \eta_x, \|u\|_\infty \leq \eta_u$ let $\kappa := \max \{ e^{\mu} \psi_2(\eta_x), e^{\mu} \gamma_u(\eta_u) \}$. It holds
sup_{t \geq s \geq t_0} \| V^t(x^s) \| \leq \kappa. \text{ Let } \rho_2 > 0 \text{ be such that } e^{-\lambda \rho_2} \kappa \leq \psi_1(\epsilon) \text{ and let } \rho_1 > \theta. \text{ Then, by the estimate (29) we have}

\sup_{t \geq s \geq t_0 + \rho_2} \| V^t(x^s) \| \leq \sup_{t \geq s \geq t_0} V(x(s))

\leq \max \left\{ \psi_1(\epsilon), e^{\mu \gamma_t} \left( \sup_{s \in [t_0,t]} \| V^t(x^s) \| \right), e^{\mu \gamma_u} \left( \| u \|_{[t_0,t]} \right) \right\}.

In the previous inequality we put \( t_0 + \rho_1 + \rho_2 \) instead of \( t_0 \) and obtain

\sup_{t \geq s \geq t_0 + 2(\rho_1 + \rho_2)} \| V^t(x^s) \|

\leq \max \left\{ \psi_1(\epsilon), e^{\mu \gamma_t} \left( \sup_{s \in [t_0+\rho_1+\rho_2,t]} \| V^t(x^s) \| \right), e^{\mu \gamma_u} \left( \| u \|_{[t_0+\rho_1+\rho_2,t]} \right) \right\}

\leq \max \left\{ \psi_1(\epsilon), (e^{\mu \gamma_t})^2 \left( \sup_{s \in [t_0,t]} \| V^t(x^s) \| \right), e^{\mu \gamma_u} \left( \| u \|_{[t_0,t]} \right) \right\}.

Since \( e^{\mu \gamma_t} < 1 \), there exists a number \( \tilde{n} \in \mathbb{N} \), which depends on \( \kappa \) and \( \epsilon \) such that

\( (e^{\mu \gamma_t})^{\tilde{n}}(\kappa) := (e^{\mu \gamma_t}) \circ \ldots \circ (e^{\mu \gamma_t})(\kappa) \leq \max \left\{ \psi_1(\epsilon), e^{\mu \gamma_u} \left( \| u \|_{[t_0,t]} \right) \right\} \).

By induction we conclude that

\sup_{t \geq s \geq t_0 + \tilde{n}(\rho_1 + \rho_2)} \| V^t(x^s) \| \leq \max \left\{ \psi_1(\epsilon), e^{\mu \gamma_u} \left( \| u \|_{[t_0,t]} \right) \right\},

and finally we obtain

\| x(t) \| \leq \max \left\{ \epsilon, \psi_1^{-1}(e^{\mu \gamma_u} \left( \| u \|_{[t_0,t]} \right)) \right\}, \forall t \geq t_0 + \tilde{n}(\rho_1 + \rho_2). \quad (31)

Thus, the system (6) satisfies the second property from the Lemma 3.10, which implies that (6) is uniformly ISS over \( S[\mu, \lambda] \). \quad \square

**Remark 3.14.** Another Razumikhin-type theorem (for non-autonomous systems) has been proposed in [13]. In that paper it was used so-called fixed dwell-time condition to characterize the class of impulse time sequences, over which the system is uniformly ISS. In contrast to Theorems 1,2 from [13], we...
prove the Razumikhin-type theorem over the class of sequences, which satisfy the average dwell-time condition, which is larger than the class of sequences, which satisfy the fixed dwell-time condition. However, the small-gain condition, that we have used in this paper, \( \gamma_i(r) < e^{-\mu r}, \) is stronger, than that from \([13]\).

In the next section we investigate general networks of impulsive systems in view of stability and establish a dwell-time condition for such interconnections.

4. Large-scale networks of impulsive systems

We consider an interconnection of \( n \) impulsive subsystems with inputs of the form

\[
\dot{x}_i(t) = f_i(x_1(t), \ldots, x_n(t), u_i(t)), \quad t \neq t_k, \\
x_i(t) = g_i(x^-_1(t), \ldots, x^-_n(t), u^-_i(t)), \quad t = t_k,
\]

\( k \in \mathbb{N}, \ i = 1, \ldots, n, \) where the state \( x_i(t) \in \mathbb{R}^{N_i} \) of the \( i \)th subsystem is absolutely continuous between impulses; \( u_i(t) \in \mathbb{R}^{M_i} \) is a locally bounded, Lebesgue-measurable input and \( x_j(t) \in \mathbb{R}^{N_j}, \ j \neq i \) can be interpreted as internal inputs of the \( i \)th subsystem. Note that the impulse sequences for all subsystems are assumed to be equal.

Furthermore, \( f_i : \mathbb{R}^{N_1} \times \ldots \times \mathbb{R}^{N_n} \times \mathbb{R}^{M_i} \to \mathbb{R}^{N_i} \) and \( g_i : \mathbb{R}^{N_1} \times \ldots \times \mathbb{R}^{N_n} \times \mathbb{R}^{M_i} \to \mathbb{R}^{N_i}, \) where we assume that the \( f_i \) are locally Lipschitz for all \( i = 1, \ldots, n. \) All signals \((x_i \text{ and inputs } u_i, \ i = 1, \ldots, n)\) are assumed to be right-continuous and to have left limits at all times and we denote \( x_\downarrow_i(t) := \lim_{s \nearrow t} x_i(s), \ u_\downarrow_i(t) := \lim_{s \nearrow t} u_i(s). \)

We define \( N := N_1 + \ldots + N_n, \ M := M_1 + \ldots + M_n, \ x := (x_1^T, \ldots, x_n^T)^T, \ u := (u_1^T, \ldots, u_n^T)^T, \ f := (f_1^T, \ldots, f_n^T)^T \) and \( g := (g_1^T, \ldots, g_n^T)^T \) such that the interconnected system \((32)\) is of the form \((1)\). We investigate under which conditions the whole system has the ISS property. In case of a system with several inputs the definition of ISS reads as follows:

Assume that a sequence \( \{t_k\} \) is given. The \( i \)th subsystem of \((1)\) is ISS if there exist \( \beta_i \in \mathcal{KL}, \gamma_{ij}, \gamma_i \in \mathcal{K}_\infty \cup \{0\} \) such that for every initial condition \( x_i(t_0) \) and every input \( u_i \) the corresponding solution to \((32)\) exists globally and satisfies for all \( t \geq t_0 \)

\[
|x_i(t)| \leq \max\{|\beta_i(x_i(t_0), t-t_0)|, \max_{j,j \neq i} \gamma_{ij}(\|x_j\|_{[t_0,t]}), \gamma_i(\|u\|_{[t_0,t]})\}. \tag{33}
\]
Functions $\gamma_{ij}$ are called gains. The impulsive system (32) is \textit{uniformly ISS} over a given class $\mathcal{S}$ of admissible sequences of impulse times if (33) holds for every sequence in $\mathcal{S}$, with functions $\beta_i$ and $\gamma_i, \gamma_{ij}$ that are independent of the choice of the sequence.

Similarly, the Lyapunov functions for a system with several inputs are as follows:

Assume that for each subsystem of the interconnected system (32) there is a given function $V_i : \mathbb{R}^{N_i} \to \mathbb{R}_+$, which is continuous, proper, positive definite and locally Lipschitz continuous on $\mathbb{R}^{N_i}\setminus\{0\}$. For $i = 1, \ldots, n$ the function $V_i$ is called an \textit{exponential ISS-Lyapunov function} for the $i$th subsystem of (32) with rate coefficients $c_i, d_i \in \mathbb{R}$ if whenever $V_i(x_i) \geq \max\{\max_{j,j \neq i} \gamma_{ij}(V_j(x_j)), \gamma_i(|u_i|)\}$ holds it follows

$$\nabla V_i(x_i) \cdot f_i(x, u_i) \leq -c_i V_i(x_i) \text{ f.a.a. } x, \text{ all } u_i$$

and for all $x$ and $u_i$ it holds

$$V_i(g_i(x, u_i)) \leq \max\{e^{-d_i}V_i(x_i), \max_{j,j \neq i} \gamma_{ij}(V_j(x_j)), \gamma_i(|u_i|)\},$$

where $\gamma_{ij}, \gamma_i$ are some functions from $\mathcal{K}_\infty$. A different formulation can be obtained by replacing (35) by

$$V_i(x_i) \geq \max_{j,j \neq i} \tilde{\gamma}_{ij}(V_j(x_j)), \tilde{\gamma}_i(|u_i|) \Rightarrow V_i(g_i(x, u_i)) \leq e^{-d_i}V_i(x_i),$$

where $\tilde{\gamma}_{ij}, \tilde{\gamma}_i \in \mathcal{K}_\infty$.

In general, even if all subsystems of (32) are ISS, the whole system (1) may be not ISS. Thus, we need conditions that guarantee ISS of (1). In this paper we are concerned with an interconnection of impulsive systems that have exponential ISS-Lyapunov functions $V_i$ with linear gains $\gamma_{ij}$.

By slight abuse of notation we denote throughout the paper $\gamma_{ij}(r) = \gamma_{ij}^r, \gamma_{ij}, r \geq 0$. To derive sufficient stability conditions for such an interconnection we collect the linear gains $\gamma_{ij}$ of the subsystems in a matrix $\Gamma = (\gamma_{ij})_{n \times n}, i, j = 1, \ldots, n$ denoting $\gamma_{ii} := 0, i = 1, \ldots, n$ for completeness, see [31, 16, 32]. Note that this matrix describes in particular the interconnection topology of the whole network, moreover it contains the information about the mutual influence between the subsystems. We also introduce the gain operator $\Gamma : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ defined by

$$\Gamma(s) := \left(\max_j \gamma_{1j} s_j, \ldots, \max_j \gamma_{nj} s_j\right)^T, s \in \mathbb{R}^n_+. $$

(36)
For the stability analysis of interconnected systems, we need the following:

**Definition 4.1.** We say that \( \Gamma \) satisfies the small-gain condition, if it satisfies
\[
\Gamma(s) \not\geq s, \quad \forall \; s \in \mathbb{R}^n_+ \setminus \{0\}.
\] (37)

As the gains in the matrix \( \Gamma \) are linear, condition (37) is equivalent to
\[
\rho(\Gamma) < 1,
\] (38)
where \( \rho \) is the spectral radius of \( \Gamma \), see [31, 32]. Note also that \( \rho(\Gamma) < 1 \) implies that there exists a vector \( s \in \mathbb{R}^n, \; s > 0 \) such that
\[
\Gamma(s) < s.
\] (39)

Now, we can formulate one of the main results that is an ISS small-gain theorem for impulsive systems without time-delays. This theorem allows to construct an exponential ISS-Lyapunov function for the whole interconnection.

**Theorem 4.2.** Assume that each subsystem of (32) has an exponential ISS-Lyapunov function \( V_i \) with corresponding linear ISS-Lyapunov gains \( \gamma_{ij} \) and rate coefficients \( c_i, \; d_i, \; d_i \neq 0 \). If \( \Gamma = (\gamma_{ij})_{n \times n} \) satisfies the small-gain condition (37), then the exponential ISS-Lyapunov function for the whole system (32) can be chosen as
\[
V(x) := \max_i \left\{ \frac{1}{s_i} V_i(x_i) \right\},
\] (40)
where \( s = (s_1, \ldots, s_n)^T \) is from (39). Its gain \( \gamma \) is given by
\[
\gamma(r) := \max\{e^d, 1\} \max_i \frac{1}{s_i} \gamma_i(r)
\]
and the rate coefficients are
\[
c := \min_i c_i \quad \text{and} \quad d := \min_{i,j, j \neq i} \{d_i, -\ln(\frac{s_j}{s_i} \gamma_{ij})\}.
\]
In particular, for all \( \mu, \lambda > 0 \) (32) is uniformly ISS over \( S[\mu, \lambda] \).

The small-gain condition is used in [31, 16, 18], for example, to verify the ISS property of interconnected systems. Note that \( \gamma_i \) are allowed to be nonlinear.

**Proof.** As the small gain condition (37) is satisfied, it follows from (38) that there exists \( s \in \mathbb{R}^n, \; s > 0 \) satisfying (39), see [31]. Let us define \( V \) as in (40) and show that this function is an exponential ISS-Lyapunov function for
the system (1). It can be easily checked that this function is locally Lipschitz, positive definite and radially unbounded.

For any $i \in \{1, \ldots, n\}$ consider open domains $M_i \in \mathbb{R}^N \setminus \{0\}$ defined by

$$M_i := \{(x_1^T, \ldots, x_n^T)^T \in \mathbb{R}^N \setminus \{0\} : \frac{1}{s_i} V_i(x_i) > \max_{j \neq i} \frac{1}{s_j} V_j(x_j)\}. \quad (41)$$

Take arbitrary $\hat{x} = (\hat{x}_1^T, \ldots, \hat{x}_n^T)^T \in \mathbb{R}^N \setminus \{0\}$ for which there exist $i \in \{1, \ldots, n\}$, such that $\hat{x} \in M_i$. It follows that there is a neighborhood $U$ of $\hat{x}$ such that $V(x) = \frac{1}{s_i} V_i(x_i)$ is differentiable for almost all $x \in U$.

We define $\gamma(r) := \max \frac{1}{s_i} \gamma_i(r), \ r \geq 0$ and assume $V(x) \geq \gamma(|u|)$. It follows from (39) that

$$V_i(x_i) = s_i V(x) \geq \max \{\max_j \gamma_{ij} s_j V(x), s_i \gamma(|u|)\}$$

$$\geq \max \{\max_j \gamma_{ij} V_j(x_j), \gamma_i(|u_i|)\}. \quad (42)$$

Then, from (34) we obtain for almost all $x$

$$\dot{V}(x) = \frac{1}{s_i} \nabla V_i(x_i) \cdot f_i(x, u_i) \leq -\frac{1}{s_i} c_i V_i(x_i) = -c_i V(x).$$

We have shown that for $c = \min_i c_i$ the function $V$ satisfies (3) with $\bar{\gamma}$ for all $\hat{x} \in \bigcup_{i=1}^n M_i$. To treat the points $\hat{x} \in \bigcup_{i=1}^n M_i$ one can use the technique from [31].

With $d := \min_{i, j \neq i} \{d_i, -\ln(\frac{d_i}{s_i} \gamma_{ij})\}$ and using (35) it holds

$$V(g(x, u)) = \max_i \{\frac{1}{s_i} V_i(g(x_1, \ldots, x_n, u_i))\}$$

$$\leq \max_i \{\frac{1}{s_i} \max_j \{e^{-d_i} V_i(x_i), \max_{j \neq i} \gamma_{ij} V_j(x_j), \gamma_i(|u_i|)\}\}$$

$$\leq \max_i \{\frac{1}{s_i} e^{-d_i} V(x) + \frac{1}{s_i} \gamma_{ij} s_j V(x), \frac{1}{s_i} \gamma_i(|u_i|)\}$$

$$\leq \max_e \{e^{-d} V(x), \hat{\gamma}(|u|)\}. \quad (43)$$

Define $\hat{\gamma}(r) := e^d \hat{\gamma}(r)$. If it holds $V(x) \geq \hat{\gamma}(|u|)$, it follows

$$V(g(x, u)) \leq \max_e \{e^{-d} V(x), \hat{\gamma}(|u|)\} = \max_e \{e^{-d} V(x), e^{-d} \hat{\gamma}(|u|)\} \leq e^{-d} V(x) \quad (44)$$

for all $x$ and $u$ and $V$ satisfies (4) with $\hat{\gamma}$. Define $\gamma(r) := \max \{e^d, 1\} \max_i \frac{1}{s_i} \gamma_i(r)$, then we conclude that $V$ is an exponential ISS-Lyapunov function with rate coefficients $c, d$ and gain $\gamma$.

All conditions of Definition 3.2 are satisfied and thus $V$ is the exponential ISS-Lyapunov function of the system (1). We can apply Theorem 3.3 and the overall system is uniformly ISS over $S[\mu, \lambda], \mu, \lambda > 0$. \hfill $\square$
Remark 4.3. Using \( c = \min_i c_i, \ d = \min_{i,j, \ j \neq i} \{d_i, -\ln\left(\frac{2}{s_i} \gamma_{ij}\right)\} \) in the dwell-time condition some kind of conservativeness may occur, which means that the ISS property for an interconnected impulsive system cannot be verified by the application of Theorem 4.2, although the system possesses the ISS property.

4.1. Example with nonlinear gains

We have formulated Theorem 4.2 for the case of linear gains. Note that not for all interconnections with nonlinear gains one can construct an exponential Lyapunov function for the whole system, even if the small-gain condition is satisfied. However, if the small-gain condition holds, then we can construct non-exponential Lyapunov functions for the whole system as in [31], [18]. In this case the dwell-time condition from [12] cannot be applied and one has to develop more general conditions that guarantee ISS of the system.

Nevertheless, in some special cases the treatment of the interconnections with nonlinear gains is also possible. For example, if the continuous and discrete dynamics of all subsystems are stabilizing, i.e., with \( c_i, d_i > 0 \), and the small-gain condition holds, then we can construct an ISS-Lyapunov function (non-exponential in general) for the whole system using the methodology from [18]. Then, according to Theorem 2 in [12] the interconnection will be ISS. Note that in this case the dwell-time condition is not needed anymore.

For the case that the discrete or continuous dynamics is unstable we consider the following example with a construction of an ISS-Lyapunov function.

Let \( T = \{t_k\} \) be a sequence of impulse times. Consider two interconnected nonlinear systems

\[
\begin{align*}
\dot{x}_1(t) &= -x_1(t) + x_2^2(t), \quad t \notin T, \\
x_1(t) &= e^{x_1}(t), \quad t \in T
\end{align*}
\]

and

\[
\begin{align*}
\dot{x}_2(t) &= -x_2(t) + \frac{1}{2} \sqrt{|x_1(t)|}, \quad t \notin T, \\
x_2(t) &= e^{x_2}(t), \quad t \in T.
\end{align*}
\]

The exponential ISS-Lyapunov functions and Lyapunov gains for these subsystems are given by

\[
\begin{align*}
V_1(x_1) &= |x_1|, \quad \gamma_{12}(r) = \frac{1}{1-\varepsilon_1} r^2, \\
V_2(x_2) &= |x_2|, \quad \gamma_{21}(r) = \frac{1}{2(1-\varepsilon_2)} \sqrt{r},
\end{align*}
\]
where \( \varepsilon_1, \varepsilon_2 \in (0, 1) \). We have the estimates

\[
|x_1| \geq \gamma_{12}(|x_2|) \Rightarrow \dot{V}_1(x_1) \leq -\varepsilon_1 V_1(x_1),
\]
\[
|x_2| \geq \gamma_{21}(|x_1|) \Rightarrow \dot{V}_2(x_2) \leq -\varepsilon_2 V_2(x_2).
\]

The small-gain condition

\[
\gamma_{12} \circ \gamma_{21}(r) = \frac{1}{4(1 - \varepsilon_1)(1 - \varepsilon_2)^2} r < r, \quad \forall r > 0
\]  

is satisfied, if

\[
h(\varepsilon_1, \varepsilon_2) := (1 - \varepsilon_1)(1 - \varepsilon_2)^2 > \frac{1}{4}
\]

holds. In this case, an ISS-Lyapunov function for the interconnection is given by

\[
V(x) = \max\{|x_1|, \frac{1}{a^2}x_2^2\}, \quad \text{where} \; a \in \left(\frac{1}{2(1 - \varepsilon_2)}, \sqrt{1 - \varepsilon_1}\right)
\]

and we have the estimate

\[
V(g(x)) = V(e \cdot x) \leq e^2 V(x).
\]  

Thus, \( d = -2 \) for the interconnection. The Lyapunov estimates of the continuous dynamics for \( V \) are as follows:

\[
\frac{d}{dt}(|x_1|) \leq -\varepsilon_1 |x_1|,
\]
\[
\frac{d}{dt} \left( \frac{1}{a^2}x_2^2 \right) = \frac{d}{dt} \left( \frac{1}{a^2} V_2(x_2)^2 \right) \leq -2\varepsilon_2 \frac{1}{a^2} (V_2(x_2))^2.
\]

Using estimates as in the proof of Theorem 4.2, we obtain

\[
\frac{d}{dt} V(x) \leq - \min\{\varepsilon_1, 2\varepsilon_2\} V(x).
\]  

Function \( h \), defined by (43), is increasing on both arguments (remember, that \( \varepsilon_i \in (0, 1) \)), which implies that to maximize \( \varepsilon := \min\{\varepsilon_1, 2\varepsilon_2\} \), we have to choose \( \varepsilon_1 = 2\varepsilon_2 \). Then, from (42) we obtain the inequality

\[
(1 - 2\varepsilon_2)(1 - \varepsilon_2)^2 > \frac{1}{4}.
\]
Thus, the best choice for $\varepsilon_2$ is $\varepsilon_2 \approx 0.267$. From the inequalities (44) and (45) we see that if the dwell-time condition (5) with $d = -2$ and $c = 2 \cdot 0.267$ is satisfied, then the system under investigation is stable according to Theorem 1 in [12].

In this example we were able to construct an exponential ISS Lyapunov function for the whole system due the special type of nonlinearities in internal gains. For general nonlinear internal gains this construction is not always possible.

4.2. Systems with time-delays

Now we consider $n$ interconnected impulsive systems with time-delays of the form
\[
\dot{x}_i(t) = f_i(x_1^t, \ldots, x_n^t, u_i(t)), \quad t \neq t_k,
\]
\[
x_i(t) = g_i((x_1^t)^-, \ldots, (x_n^t)^-, u_i^-(t)), \quad t = t_k,
\]
where the same assumptions on the system as in the delay-free case are considered with the following differences: We denote $x_i^t(\tau) := x_i(t + \tau), \tau \in [-\theta, 0]$, where $\theta$ is the maximum involved delay and $(x_i^t)^-(\tau) := \lim_{\tau \searrow 0} x_i(s + \tau), \tau \in [-\theta, 0]$. Furthermore, $f_i : PC([-\theta, 0], \mathbb{R}^N_1) \times \ldots \times PC([-\theta, 0], \mathbb{R}^N_n) \times \mathbb{R}^M_i \to \mathbb{R}^N_i$, and $g_i : PC([0, \theta], \mathbb{R}^N_1) \times \ldots \times PC([0, \theta], \mathbb{R}^N_n) \times \mathbb{R}^M_i \to \mathbb{R}^N_i$, where we assume that $f_i$ are locally Lipschitz, $i = 1, \ldots, n$.

If we define $N$, $M$, $x$, $u$, $f$ and $g$ as in the delay-free case, then (46) becomes the system of the form (4). The ISS property for systems with several inputs and time-delays is as follows:

Suppose that a sequence $\{t_k\}$ is given. The $i$th subsystem of (46) is ISS, if there exist $\beta_i \in K\mathcal{L}, \gamma_{ij}, \gamma_i^u \in K_{\infty} \cup \{0\}$ such that for every initial condition $\xi_i$ and every input $u_i$ the corresponding solution to the $i$th subsystem of (46) exists globally and satisfies
\[
|x_i(t)| \leq \max_{j,j \neq i} \{\beta_i(\|\xi_i\|_{[-\theta,0]}, t-t_0), \gamma_{ij}(\|x_j\|_{[t_0-\theta,t_0]}, \gamma_i^u(\|u\|_{[t_0,t]})}\}
\]
for all $t \geq t_0$. The $i$th subsystem of (46) is uniformly ISS over a given class $\mathcal{S}$ of admissible sequences of impulse times if (47) holds for every sequence in $\mathcal{S}$, with functions $\beta_i$, $\gamma_{ij}$ and $\gamma_i^u$ that are independent of the choice of the sequence.

In the following subsections, we present useful tools to analyze a system of the form (46) in view of stability: ISS-Lyapunov-Razumikhin functions and ISS-Lyapunov-Krasovskii functionals for the subsystems.


4.2.1. Lyapunov-Razumikhin functions

Assume that for each subsystem of the interconnected system there is a given function $V_i : \mathbb{R}^{N_i} \rightarrow \mathbb{R}_+$, which is continuous, proper, positive definite and locally Lipschitz continuous on $\mathbb{R}^{N_i} \setminus \{0\}$. For $i = 1, \ldots, n$ the function $V_i$ is called an exponential ISS-Lyapunov-Razumikhin function of the $i$th subsystem of (46), if there exist $\gamma_i^u \in \mathcal{K} \cup \{0\}$, $\gamma_{ij} \in \mathcal{K}_\infty \cup \{0\}$, $j = 1, \ldots, n$ and scalars $c_i, d_i \in \mathbb{R}$, such that whenever $V_i(\phi_i(0)) \geq \max\{\max_j \gamma_{ij}(\|V_j^i(\phi_j)\|), \gamma_i^u(|u_i|)\}$ holds it follows

$$D^+V_i(\phi_i(0)) \leq -c_i V_i(\phi_i(0))$$

for all $\phi = (\phi_1^T, \ldots, \phi_n^T)^T \in PC([-\theta, 0], \mathbb{R}^N)$ and $u_i \in \mathbb{R}^{M_i}$, where $V_j^i : PC\left([-\theta, 0] ; \mathbb{R}^{N_j}\right) \rightarrow PC\left([-\theta, 0] ; \mathbb{R}^{N_j}\right)$ with $V_j^i(x_j(t)) := V_j(x_j(t))$, $\tau \in [-\theta, 0]$ and it holds

$$V_i(g_i(\phi_1, \ldots, \phi_n, u_i)) \leq \max_j e^{-d_j} V_i(\phi_i(0)), \max_j \gamma_{ij}(\|V_j^i(\phi_j)\|), \gamma_i^u(|u_i|),$$

(49)

for all $\phi \in PC([-\theta, 0], \mathbb{R}^N)$ and $u_i \in \mathbb{R}^{M_i}$. Another formulation can be obtained by replacing \(49\) by

$$V_i(\phi_i(0)) \geq \max_j \tilde{\gamma}_{ij}(\|V_j^i(\phi_j)\|), \tilde{\gamma}_i^u(|u_i|) \quad \Rightarrow \quad V_i(g_i(\phi, u)) \leq e^{-d_i} V_i(\phi_i(0)),$$

where $\tilde{\gamma}_{ij}, \tilde{\gamma}_i^u \in \mathcal{K}_\infty$.

We consider linear gains $\gamma_{ij}$ and define the gain-matrix $\Gamma := (\gamma_{ij})_{n \times n}$ and the map $\Gamma : \mathbb{R}^n_+ \rightarrow \mathbb{R}^n_+$ by $\Gamma(s) := (\max_j \gamma_{1j}s_j, \ldots, \max_j \gamma_{nj}s_j)^T$, $s \in \mathbb{R}^n_+$.

Now, we state one of our main results: the ISS small-gain theorem for interconnected impulsive systems with time-delays and linear gains $\gamma_{ij}$. We construct the Lyapunov-Razumikhin function and the gain of the overall system under a small-gain condition, which is here of the form

$$\Gamma(s) \not\geq \min\{e^{-\mu}, e^{-d-\mu}\} s, \forall \ s \in \mathbb{R}^n_+ \setminus \{0\}$$

$$\Leftrightarrow \exists s \in \mathbb{R}^n_+ \setminus \{0\} : \Gamma(s) < \min\{e^{-\mu}, e^{-d-\mu}\} s,$$

(50)

where $\mu$ is from the dwell-time condition (5) and $d := \min_i d_i$. The dwell-time condition on the size of the time intervals between impulses is the same as in the delay-free case.
Theorem 4.4. Assume that each subsystem of (46) has an exponential ISS-Lyapunov-Razumikhin function with \( c_i, d_i \in \mathbb{R}, d_i \neq 0 \) and gains \( \gamma_i^u, \gamma_{ij} \), where \( \gamma_{ij} \) are linear. Define \( c := \min_i c_i \) and \( d := \min_i d_i \). If for some \( \mu > 0 \) the operator \( \Gamma \) satisfies the small-gain condition (50), then for all \( \lambda > 0 \) the whole system (3) is uniformly ISS over \( \mathcal{S}[\mu, \lambda] \) and the exponential ISS-Lyapunov-Razumikhin function is given by \( V(x) := \max_i \{ \frac{1}{s_i} V_i(x_i) \} \), where \( s = (s_1, \ldots, s_n)^T \) is from (50). The gains are given by \( \gamma_i(r) := \max \{ e^d, 1 \} \max_{k,j} \frac{1}{s_k} \gamma_{kj} s_j r, \gamma_u(r) := \max \{ e^d, 1 \} \max_i \frac{1}{s_i} \gamma_i^u(r) \).

The proof goes along the lines of the proof of Theorem 4.2 with corresponding changes due to time-delay systems and the additional gain \( \gamma_i(r) \).

Proof.

We define \( V(x) \) as in (40) and show that \( V \) is the exponential ISS-Lyapunov-Razumikhin function for the overall system. Note that \( V \) is locally Lipschitz continuous, positive definite and radially unbounded. For any \( i \in \{1, \ldots, n\} \) consider open domains \( M_i \in \mathbb{R}^N \setminus \{0\} \) defined as in (41).

Take arbitrary \( \hat{x} = (\hat{x}_1^T, \ldots, \hat{x}_n^T)^T \in \mathbb{R}^N \setminus \{0\} \) for which there exist \( i \in \{1, \ldots, n\} \) such that \( \hat{x} \in M_i \). It follows that there is a neighborhood \( U \) of \( \hat{x} \) such that \( V(x) = \frac{1}{s_i} V_i(x_i) \) and \( D^+ V(x) \) exists.

We define the gains \( \gamma_i(r) := \max_{k,j} \frac{1}{s_k} \gamma_{kj} s_j r, \gamma_u(r) := \max_i \frac{1}{s_i} \gamma_i^u(r), r > 0 \) and assume \( V(x(t)) \geq \max \{ \tilde{\gamma}_i(||x_i(t)||), \tilde{\gamma}_u(||u(t)||) \} \). It follows

\[
V_i(x_i(t)) \geq s_i \max \{ \max_{k,j} \frac{1}{s_k} \gamma_{kj} s_j \|V^i(x^i(t))\|, \max_i \frac{1}{s_i} \gamma_i^u(||u(t)||) \}
\geq \max \{ \max_j \gamma_{ij} \|V^j(x^j(t))\|, \gamma_i^u(||u_i(t)||) \}.
\]

Then, from (48) we obtain

\[
D^+ V(x(t)) = D^+ \frac{1}{s_i} V_i(x_i(t)) \leq -\frac{1}{s_i} c_i V_i(x_i(t)) = -c_i V(x(t)).
\]

We have shown that for \( c = \min_i c_i \) the function \( V \) satisfies (19) with \( \gamma_i, \gamma_u \) for all \( \hat{x} \in \bigcup_{i=1}^n M_i \). To treat the points \( \hat{x} \in \mathbb{R}^N \setminus \bigcup_{i=1}^n M_i \) one can use the technique from [31] or [21].

25
With \( d := \min_i d_i \) and using (49) it holds
\[
V(g(x^t, u(t))) = \max_i \left\{ \frac{1}{s_i} V_i(g_i(x_1^t, \ldots, x_n^t, u_i(t))) \right\}
\leq \max_i \left\{ \frac{1}{s_i} \max_j \{ e^{-d_i} V_j(x_i(t)), \max \gamma_{ij} \| V_j'(x_j) \|, \bar{\gamma}_i^n(|u_i(t)|) \} \right\}
\leq \max_{i,j} \left\{ \frac{1}{s_i} e^{-d_i} s_i V(x(t)), \frac{1}{s_i} \gamma_{ij} s_j \| V'(x^t) \|, \frac{1}{s_i} \bar{\gamma}_i^n(|u_i(t)|) \right\}
\leq \max \{ e^{-d} V(x(t)), \bar{\gamma}_i V'(x^t) \}, \bar{\gamma}_u(|u(t)|) \}.
\]
Define \( \bar{\gamma}_t(r) := e^d \bar{\gamma}_u(r) \) and \( \bar{\gamma}_u(r) := e^d \bar{\gamma}_u(r) \).
If \( V(x(t)) \geq \max \{ \bar{\gamma}_t(||V'(x^t)||), \bar{\gamma}_u(|u(t)|) \} \) holds, it follows
\[
V(g(x^t, u(t))) \leq \max \{ e^{-d} V(x(t)), \bar{\gamma}_t(||V'(x^t)||), \bar{\gamma}_u(|u(t)|) \}
= \max \{ e^{-d} V(x(t)), e^{-d} \bar{\gamma}_t(||V'(x^t)||), e^{-d} \bar{\gamma}_u(|u(t)|) \}
\leq e^{-d} V(x(t)),
\]
i.e., \( V \) satisfies the condition (20) with \( \bar{\gamma}_t, \bar{\gamma}_u \). Now, define \( \gamma_t(r) := \max \{ \bar{\gamma}_t(r), \bar{\gamma}_t(r) \} \)
and \( \gamma_u(r) := \max \{ \bar{\gamma}_u(r), \bar{\gamma}_u(r) \} \). Then, \( V \) satisfies (19) and (20) with \( \gamma_t, \gamma_u \).

By (50) it holds
\[
\gamma_t(r) = \max \{ \bar{\gamma}_t(r), e^d \bar{\gamma}_t(r) \} < \max \{ \min \{ e^{-\mu}, e^{-d-\mu} \}, \min \{ e^{-d-\mu}, e^{-\mu} \} \} r = e^{-\mu r}.
\]
All conditions of Definition 3.8 are satisfied and \( V \) is the exponential ISS-Lyapunov-Razumikhin function of the whole system of the form (6). We can apply Theorem 3.13 and the whole system is uniformly ISS over \( \delta[\mu, \lambda] \). \( \square \)

4.2.2. Lyapunov-Krasovskii functionals

Let us consider \( n \) interconnected impulsive subsystems of the form
\[
\begin{align*}
\dot{x}_i(t) &= f_i(x_1^t, \ldots, x_n^t, u_i(t)), \quad t \neq t_k, \\
x_i^t &= g_i((x_1^t)^-, \ldots, (x_n^t)^-, u_i^t(t)), \quad t = t_k,
\end{align*}
\]
\( k \in \mathbb{N}, \ i = 1, \ldots, n, \) where we make the same assumptions as in the previous subsections and the functionals \( g_i \) are now maps from \( PC([-\theta, 0]; \mathbb{R}^N) \times \ldots \times PC([-\theta, 0]; \mathbb{R}^N) \times \mathbb{R}^M \) into \( PC([-\theta, 0]; \mathbb{R}^N) \).

Note that the ISS property for systems of the form (51) is the same as in (47).
If we define \( N, M, x, u, f \) and \( g \) as in the previous subsection, then (51) becomes the system of the form (7). ISS-Lyapunov-Krasovskii functionals of systems with several inputs and time-delays are as follows:
Assume that for each subsystem of the interconnected system (51) there is a given functional \( V_i : PC \left( [−\theta, 0]; \mathbb{R}^{N_i} \right) \rightarrow \mathbb{R}_+ \), which is locally Lipschitz continuous, positive definite and radially unbounded. For \( i = 1, \ldots, n \) the functional \( V_i \) is called an exponential ISS-Lyapunov-Krasovskii functional of the \( i \)th subsystem of (51), if there exist \( \gamma_i \in K \cup \{0\}, \gamma_{ij} \in K_\infty \cup \{0\}, \gamma_{ii} \equiv 0, i, j = 1, \ldots, n \) and scalars \( c_i, d_i \in \mathbb{R} \) such that whenever \( V_i(\phi) \geq \max \{ \max_{j \neq i} \gamma_{ij}(V_j(\phi_j)), \gamma_i(|u_i|) \} \) holds it follows

\[
D^+ V_i(\phi_i, u_i) \leq -c_i V_i(\phi_i)
\]

and

\[
V_i(g_i(\phi, u_i)) \leq \max \{ e^{-d_i} V_i(\phi_i), \max_{j \neq i} \gamma_{ij}(V_j(\phi_j)), \gamma_i(|u_i|) \}
\]

for all \( \phi \in PC \left( [−\theta, 0]; \mathbb{R}^{N_i} \right), u_i \in \mathbb{R}^{M_i} \). A different formulation can be obtained by replacing (53) by

\[
V_i(\phi_i) \geq \max_{j \neq i} \gamma_{ij}(V_j(\phi_j)), \gamma_i(|u_i|) \Rightarrow V_i(g_i(\phi, u_i)) \leq e^{-d_i} V_i(\phi_i),
\]

where \( \tilde{\gamma}_{ij}, \tilde{\gamma}_i \in K_\infty \).

The next result is an ISS small-gain theorem for impulsive systems with time-delays using the Lyapunov-Krasovskii methodology. This theorem allows to construct an exponential ISS-Lyapunov-Krasovskii functional and the corresponding gain for the whole interconnection under a dwell-time and a small-gain condition.

**Theorem 4.5.** Assume that each subsystem of (51) has an exponential ISS-Lyapunov-Krasovskii functional \( V_i \) with corresponding gains \( \gamma_i, \gamma_{ij} \), where \( \gamma_{ij} \) are linear, and rate coefficients \( c_i, d_i, d_i \neq 0 \). Define \( c := \min_i c_i \) and \( d := \min_{i,j, j \neq i} \{ d_i, -\ln(\gamma_i) \} \). If \( \Gamma = (\gamma_{ij})_{n \times n}, \gamma_{ii} \equiv 0 \) satisfies the small-gain condition (37), then the impulsive system (7) is uniformly ISS over \( S[\mu, \lambda], \mu, \lambda > 0 \) and the exponential ISS-Lyapunov-Krasovskii functional is given by

\[
V(\phi) := \max_i \{ \frac{1}{s_i} V_i(\phi_i) \},
\]

where \( s = (s_1, \ldots, s_n)^T \) is from (39), \( \phi \in PC \left( [−\theta, 0]; \mathbb{R}^{N} \right) \). The gain is given by \( \gamma(r) := \max \{ e^d, 1 \} \max_i \frac{1}{s_i} \tilde{\gamma}_i(r) \).
Proof. Let $0 \neq x^t = ((x_1^t)^T, \ldots, (x_n^t)^T)^T$ and $V$ be defined by $V(x^t) := \max_i \{ \frac{1}{s_i} V_i(x_i^t) \}$. For any $i \in \{1, \ldots, n\}$ consider open domains $M_i \in \mathbb{R}^N \backslash \{0\}$ defined by

$$M_i := \{((x_1^t)^T, \ldots, (x_n^t)^T)^T \in PC \left([-\theta, 0]; \mathbb{R}^N\right) : \frac{1}{s_i} V_i(x_i^t) > \max_j \frac{1}{s_j} V_j(x_j^t)\}.$$

Take arbitrary $\hat{x}^t = ((\hat{x}_1^t)^T, \ldots, (\hat{x}_n^t)^T)^T \in PC \left([-\theta, 0]; \mathbb{R}^N\right)$ for which there exist $i \in \{1, \ldots, n\}$, such that $\hat{x}^t \in M_i$. It follows that there is a neighborhood $U$ of $\hat{x}$ such that $V(x) = \frac{1}{s_i} V_i(x_i)$ and $D^+ V(x)$ exists.

By similar calculations as in the proof of Theorem 4.2, $V$ is the exponential ISS-Lyapunov-Krasovskii functional for the overall system of the form (7). We can apply Theorem 3.7 and the whole system is uniformly ISS over $S[\mu, \lambda]$. □

5. Example: networked control systems with time-delays

We consider a class of networked control systems given by an interconnection of linear systems with time-delays [12], [33], [34] and [35]. The $i$th subsystem is described as follows

$$\dot{x}_i = -a_i x_i + \sum_{j, j \neq i} a_{ij} x_j(t - \tau_{ij}) + b_i \nu_i, \quad a_i > 0, \quad y_i = x_i + \mu_i, \quad i = 1, \ldots, n. \tag{55}$$

Here $\tau_{ij} \in [0, \theta]$ is a time-delay of the input from other subsystems with maximum involved delay $\theta > 0$, $\nu_i$ is an input disturbance, $\mu_i$ a measurement/quantization noise. The sequence $\{t_1, t_2, \ldots\}$ is a sequence of time instances at which measurements of $x_i$ are sent. It is allowed to send only one measurement per each time instant. Between the sending of new measurements the estimate $\hat{x}_i$ of $x_i$ is given by

$$\dot{\hat{x}}_i(t) = -a_i \hat{x}_i(t) + \sum_{j, j \neq i} a_{ij} \hat{x}_j(t - \tau_{ij}), \quad t \notin \{t_1, t_2, \ldots\}. \tag{56}$$

At time $t_k$ the node $i_k$ gets access to the measurement $y_{i_k}$ of $x_{i_k}$ and all other nodes stay unchanged:

$$\hat{x}_i(t_k) = \begin{cases} y_{i_k}(t_k), & i = i_k, \\ \hat{x}_i(t_k), & i \neq i_k. \end{cases}$$
An estimation error is defined by $e_i := \hat{x}_i - x_i$. The dynamics of $e_i$ can be then given by the following impulsive system:

$$
\dot{e}_i(t) = -a_i e_i(t) + \sum_{j,j \neq i} a_{ij} e_j(t - \tau_{ij}) - b_i \nu_i, \ t \neq t_k, \ k \in \mathbb{N}, \tag{57}
$$

$$
e_i(t_k) = \begin{cases}
\mu_k^{-}(t_k), & i = i_k,
\mu_k^{+}(t_k), & i \neq i_k.
\end{cases} \tag{58}
$$

The decision to which node a measurement will be sent is performed using some protocol, for examples see [34].

Let us show that the error of the whole interconnected system (57), (58) is uniformly ISS using Lyapunov-Razumikhin approach. Firstly, we will find an ISS-Lyapunov-Razumikhin function candidate for each subsystem.

Consider the function $V_i(e_i) := |e_i|$. If $t = t_k$, then $V_i(g_i(e_i)) \leq \max\{|e_i|, |\mu_i|\} = \max\{e^{-d_i V_i(e_i)}, |\mu_i|\}$ with $d_i = 0$. Consider now the case $t \neq t_k$. If $|e_i| \geq \max\{\max_j n_{\frac{|a_{ij}|}{a_i - e_i}} \max_{t - \theta \leq s \leq t} V_j(e_j(s)), \frac{|b_i \nu_i|}{a_i - e_i}\}, e_i \in [0, a_i]$, then

$$
D^+ V_i(e_i) = (-a_i e_i + \sum_{j,j \neq i} a_{ij} e_j(t - \tau_{ij}) - b_i \nu_i) \cdot \text{sign} e_i \\
\leq -a_i |e_i| + \sum_{j,j \neq i} |a_{ij}| |e_j(t - \tau_{ij})| + |b_i \nu_i| \\
\leq -a_i |e_i| + (a_i - \epsilon_i)|e_i| \\
= -\epsilon_i |e_i| = -\epsilon_i V_i(e_i) =: -c_i V_i(e_i)
$$

Thus, the function $V_i(e_i) = |e_i|$ is an exponential ISS-Lyapunov-Razumikhin function for the $i$th subsystem with $c_i = \epsilon_i$, $d_i = 0$, $\gamma_i(r) = n_{\frac{|b_i|}{a_i - e_i}} r$ and $\gamma_i(|(\mu_i, \nu_i)|) = \max\{1, n_{\frac{|b_i|}{a_i - e_i}} |(\mu_i, \nu_i)|\}$.

To prove ISS of the whole error system we need to check the dwell-time condition (5) and the small-gain condition (50), see Theorem 4.4. Let us check condition (5). We have $d = \min_i d_i = 0$, $c = \min_i c_i = \min_i \epsilon_i > 0$. Taking $0 < \lambda \leq c$ and any $\mu > 0$ the dwell-time condition is satisfied for any $t \geq s \geq 0$ and time sequence $\{t_k\}$:

$$
-dN(t, s) - (c - \lambda)(t - s) = -(c - \lambda)(t - s) \leq 0 < \mu.
$$

The fulfillment of the small gain condition (50) can be checked by slightly modifying the cycle condition [32]: for all $(k_1, ..., k_n) \in \{1, ..., n\}^n$, where
$k_1 = k_p$, it holds

$$
\gamma_{k_1k_2} \circ \gamma_{k_2k_3} \circ \ldots \circ \gamma_{k_{p-1}k_p} < e^{-\mu} \text{Id}, 
$$

where in this example we can choose $\mu$ arbitrarily small. If the cycle condition is fulfilled, then the the small gain condition is satisfied. Let us check this condition for the following parameters: $n = 3$, $b_i = 1$, $\nu_i = 2$, $\epsilon_i = 0.1$, $i = 1, 2, 3$; $\tau_{ij} = \theta = 0.03$, $i, j = 1, 2, 3$, $i \neq j$; $e(s) = (0.9; 0.3; 0.6)^T$, $s \in [-\theta, 0]$; $a_1 = 1$, $a_2 = 2$, $a_3 = 0.5$,

$$
A := (a_{ij})_{3x3} = \begin{pmatrix}
0 & 0.25 & 0.25 \\
0.7 & 0 & 0.65 \\
0.15 & 0.1 & 0
\end{pmatrix}.
$$

The system uses TOD-like protocol \[34\]. The protocol sends measurements at $t_k = 0.1k$, $k \in \mathbb{N}$.

The gain matrix $\Gamma$ is then given by

$$
\Gamma := (\gamma_{ij})_{3x3} = \begin{pmatrix}
0 & 0.8333 & 0.8333 \\
1.1053 & 0 & 1.0263 \\
1.1250 & 0.7500 & 0
\end{pmatrix}.
$$

It is easy to check that all the cycles are less than the identity function multiplied by $e^\mu$, because $\mu$ can be chosen arbitrarily small. Thus, the cycle condition is satisfied and by application of Theorem 4.4 the error system (57), (58) is uniformly ISS. The trajectory of the Euclidean norm of the error is given in Figure 1.

6. Conclusions

In this paper, we have established several theorems: At first we have introduced the Lyapunov-Krasovskii methodology and the Lyapunov-Razumikhin approach for establishing ISS of single impulsive systems with time-delays. Then, we have considered networks of impulsive subsystems without time-delays. As one of the results, we have proved an ISS-Lyapunov small-gain theorem, which guarantees that the whole network has the ISS property under a small-gain condition with linear gains and a dwell-time condition. To prove this, we have constructed the ISS-Lyapunov function and the gain of the whole system. Under consideration of time-delays in such networks,
we have proved two theorems to show that a network has the ISS property provided that a small-gain condition with linear gains and a dwell-time condition is satisfied. On the one hand, we have used ISS-Lyapunov-Razumikhin functions and on the other hand we have used ISS-Lyapunov-Krasovskii functionals. An application was illustrated for networked control systems with time delays.

An open question to be investigated in the future is the usage of general Lyapunov functions instead of exponential Lyapunov functions for single systems, where a different dwell-time condition has to be formulated. Then, for the interconnection one can also use general Lyapunov functions and general gains instead of only linear gains. Furthermore, for interconnected systems the case could be investigated the case, when the impulse sequences of sub-systems are different.

References

[1] W. M. Haddad, V. S. Chellaboina, S. G. Nersesov, Impulsive and hybrid dynamical systems, Princeton Series in Applied Mathematics, Princeton University Press, Princeton, NJ, 2006.

[2] R. Shorten, F. Wirth, O. Mason, K. Wulff, C. King, Stability criteria for switched and hybrid systems, SIAM Rev. 49 (4) (2007) 545–592.
[3] E. D. Sontag, Smooth stabilization implies coprime factorization, IEEE Trans. Automat. Control 34 (4) (1989) 435–443.

[4] E. D. Sontag, Y. Wang, On characterizations of the input-to-state stability property, Systems Control Lett. 24 (5) (1995) 351–359.

[5] L. Grüne, Input-to-state dynamical stability and its Lyapunov function characterization, IEEE Trans. Automat. Control 47 (9) (2002) 1499–1504.

[6] E. D. Sontag, Y. Wang, New characterizations of input-to-state stability, IEEE Trans. Automat. Control 41 (9) (1996) 1283–1294.

[7] S. N. Dashkovskiy, B. S. Rüffer, Local ISS of large-scale interconnections and estimates for stability regions, Systems & Control Letters 59 (3-4) (2010) 241 – 247.

[8] E. D. Sontag, Comments on integral variants of ISS, Systems Control Lett. 34 (1-2) (1998) 93–100.

[9] C. Cai, A. Teel, Characterizations of input-to-state stability for hybrid systems, Systems & Control Letters 58 (1) (2009) 47–53.

[10] A. R. Teel, Connections between Razumikhin-type theorems and the ISS nonlinear small gain theorem, IEEE Trans. Automat. Control 43 (7) (1998) 960–964.

[11] P. Pepe, Z.-P. Jiang, A Lyapunov-Krasovskii methodology for ISS and iISS of time-delay systems, Systems Control Lett. 55 (12) (2006) 1006–1014.

[12] J. P. Hespanha, D. Liberzon, A. R. Teel, Lyapunov conditions for input-to-state stability of impulsive systems, Automatica J. IFAC 44 (11) (2008) 2735–2744.

[13] W.-H. Chen, W. X. Zheng, Brief paper: Input-to-state stability and integral input-to-state stability of nonlinear impulsive systems with delays, Automatica (Journal of IFAC) 45 (6) (2009) 1481–1488.

[14] J. P. Hespanha, A. S. Morse, Stabilization of nonholonomic integrators via logic-based switching, Automatica J. IFAC 35 (3) (1999) 385–393.

32
[15] Z.-P. Jiang, A. R. Teel, L. Praly, Small-gain theorem for ISS systems and applications, Math. Control Signals Systems 7 (2) (1994) 95–120.

[16] S. Dashkovskiy, B. S. Rüffer, F. R. Wirth, An ISS small gain theorem for general networks, Math. Control Signals Systems 19 (2) (2007) 93–122.

[17] Z.-P. Jiang, I. M. Y. Mareels, Y. Wang, A Lyapunov formulation of the nonlinear small-gain theorem for interconnected ISS systems, Automatica J. IFAC 32 (8) (1996) 1211–1215.

[18] S. N. Dashkovskiy, B. S. Rüffer, F. R. Wirth, Small Gain Theorems for Large Scale Systems and Construction of ISS Lyapunov Functions, SIAM Journal on Control and Optimization 48 (6) (2010) 4089–4118.

[19] D. Nesic, A. Teel, A Lyapunov-based small-gain theorem for hybrid ISS systems, in: Proceedings of the 47th IEEE Conference on Decision and Control, Cancun, Mexico, Dec. 9-11, 2008, pp. 3380–3385.

[20] S. Dashkovskiy, M. Kosmykov, Stability of networks of hybrid ISS systems, in: Proceedings of the 48th IEEE Conference on Decision and Control, Shanghai, China, Dec. 16-18, 2009, pp. 3870–3875.

[21] S. Dashkovskiy, A. Mironchenko, Local ISS of Reaction-Diffusion Systems, in: Proceedings of the 18th IFAC World Congress, 28.08.-02.09., Milan, Italy, 2011, pp. 11018–11023.

[22] S. Dashkovskiy, L. Naujok, Lyapunov-Razumikhin and Lyapunov-Krasovskii theorems for interconnected ISS time-delay systems, in: Proceedings of the 19th MTNS, Budapest, Hungary, July 5-9, 2010, pp. 1179–1184.

[23] I. Karafyllis, Z.-P. Jiang, A vector small-gain theorem for general nonlinear control systems, IMA Journal of Mathematical Control and Information 28 (2011) 309–344.

[24] L. C. Evans, Partial Differential Equations, Vol. 19 of Graduate Studies in Mathematics, American Mathematical Society, 1998.

[25] G. Ballinger, X. Liu, Existence and uniqueness results for impulsive delay differential equations, Dynam. Contin. Discrete Impuls. Systems 5 (1-4) (1999) 579–591, differential equations and dynamical systems (Waterloo, ON, 1997).
[26] D. Hinrichsen, A. J. Pritchard, Mathematical systems theory. I, Vol. 48 of Texts in Applied Mathematics, Springer-Verlag, Berlin, 2005, modelling, state space analysis, stability and robustness.

[27] J. Shen, Z. Luo, X. Liu, Impulsive Stabilization of Functional Differential Equations via Liapunov Functionals, Journal of Mathematical Analysis and Applications 240 (1-5) (1999) 1–15.

[28] J. K. Hale, S. M. Verdun Lunel, Introduction to functional-differential equations, Vol. 99 of Applied Mathematical Sciences, Springer-Verlag, New York, 1993.

[29] S. Tiwari, Y. Wang, Z.-P. Jiang, A Nonlinear Small-Gain Theorem for Large-Scale Time Delay Systems, in: Proceedings of the 48th IEEE CDC, Shanghai, China, Dec. 16-18, 2009, pp. 7204–7209.

[30] S. Tiwari, Y. Wang, Razumikhin-type Small-Gain Theorems for Large-Scale Systems with Delay, in: Proceedings of the 49th IEEE CDC, Atlanta, GA, USA, Dec. 15-17, 2010, pp. 7407–7412.

[31] S. Dashkovskiy, B. S. Rüffer, F. R. Wirth, An ISS Lyapunov function for networks of ISS systems, in: Proceedings of the 17th International Symposium on Mathematical Theory of Networks and Systems (MTNS), Kyoto, Japan, July 24-28, 2006, pp. 77–82.

[32] B. Rüffer, Monotone dynamical systems, graphs, and stability of large-scale interconnected systems, Ph.D. thesis, Fachbereich 3 (Mathematik & Informatik) der Universität Bremen (2007).

[33] G. C. Walsh, H. Ye, L. G. Bushnell, J. Nilsson, B. Bernhardsson, Stability analysis of networked control systems, Automatica 34 (1999) 57–64.

[34] D. Nesic, A. Teel, Input-to-state stability of networked control systems, Automatica 40 (12) (2004) 2121 – 2128.

[35] J. Hespanha, P. Naghshtabrizi, Y. Xu, A survey of recent results in networked control systems, Proceedings of the IEEE 95 (1) (2007) 138–162.