Finite Unified Theories

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Abstract. All-loop Finite Unified Theories (FUTs) are very interesting N=1 GUTs in which a complete reduction of couplings has been achieved. FUTs realize an old field theoretical dream and have remarkable predictive power. Reduction of dimensionless couplings in N=1 GUTs is achieved by searching for renormalization group invariant (RGI) relations among them holding beyond the unification scale. Finiteness results from the fact that there exists RGI relations among dimensionless couplings that guarantee the vanishing of the β-functions in certain N=1 supersymmetric GUTS even to all orders. Furthermore, developments in the soft supersymmetry breaking sector of N=1 GUTs and FUTs lead to exact RGI relations also in this dimensionful sector of the theories. Of particular interest for the construction of realistic theories is a RGI sum rule for the soft scalar masses holding to all orders.

1. Introduction
The success of the Standard Model (SM) of Elementary Particle Physics is seriously limited by the presence of a plethora of free parameters. An even more disturbing fact is that the best bet for Physics beyond the SM namely the minimal supersymmetric extension of the SM (MSSM), which is expected to bring us one step further towards a more fundamental understanding of Nature, introduces around a hundred additional free parameters.

In our studies [1–7] we have developed a strategy in searching for a more fundamental theory possibly at the Planck scale, whose basic ingredients are GUTs and supersymmetry, but its consequences certainly go beyond the known ones.

Our method consists of hunting for renormalization group invariant (RGI) relations holding below the Planck scale, which in turn are preserved down to the GUT scale. This programme, called Gauge–Yukawa unification scheme, applied in the dimensionless couplings of supersymmetric GUTs, such as gauge and Yukawa couplings, had already noticeable successes by predicting correctly, among others, the top quark mass in the finite and in the minimal N=1 supersymmetric SU(5) GUTs. An impressive aspect of the RGI relations is that one can guarantee their validity to all-orders in perturbation theory by studying the uniqueness of the resulting relations at one-loop, as was proven in the early days of the programme of reduction of couplings [8–11]. Even more remarkable is the fact that it is possible to find RGI relations among couplings that guarantee finiteness to all-orders in perturbation theory [12–14].

Although supersymmetry seems to be an essential feature for a successful realization of the above programme, its breaking has to be understood too, since it has the ambition to supply the SM with predictions for several of its free parameters. Indeed, the search for
RGI relations has been extended to the soft supersymmetry breaking sector (SSB) of these theories, [6, 15] which involves parameters of dimension one and two. Interesting progress was made in [16–23] concerning the renormalization properties of the SSB parameters, based conceptually and technically on the work of Yamada [24]. In here, [24] the powerful supergraph method [25–28] for studying supersymmetric theories has been applied to the softly broken ones by using the “spurion” external space-time independent superfields [29–32]. In the latter method a softly broken supersymmetric gauge theory is considered as a supersymmetric one in which the various parameters such as couplings and masses have been promoted to external superfields that acquire “vacuum expectation values”. Based on this method the relations among the soft term renormalization and that of an unbroken supersymmetric theory have been derived. In particular the $\beta$-functions of the parameters of the softly broken theory are expressed in terms of partial differential operators involving the dimensionless parameters of the unbroken theory. The key point in the strategy [21–23] in solving the set of coupled differential equations so as to be able to express all parameters in a RGI way, was to transform the partial differential operators involved to total derivative operators. This is indeed possible to be done on the RGI surface which is defined by the solution of the reduction equations.

On the phenomenological side there exist some serious developments too. Previously an appealing “universal” set of soft scalar masses was asumed in the SSB sector of supersymmetric theories, given that apart from economy and simplicity, (1) they are part of the constraints that preserve finiteness up to two-loops [33, 34], (2) they are RGI up to two-loops in more general supersymmetric gauge theories, subject to the condition known as $P = 1/3 \frac{Q}{3}$ [15], and (3) they appear in the attractive dilaton dominated supersymmetry breaking superstring scenarios [35–37]. However, further studies have exhibited a number of problems all due to the restrictive nature of the “universality” assumption for the soft scalar masses. For instance, (a) in finite unified theories the universality predicts that the lightest supersymmetric particle is a charged particle, namely the superpartner of the $\tau$ lepton $\tilde{\tau}$, (b) the MSSM with universal soft scalar masses is inconsistent with the attractive radiative electroweak symmetry breaking, [37] and (c) which is the worst of all, the universal soft scalar masses lead to charge and/or colour breaking minima deeper than the standard vacuum [38]. Therefore, there have been attempts to relax this constraint without loosing its attractive features. First an interesting observation was made that in $N = 1$ Gauge–Yukawa unified theories there exists a RGI sum rule for the soft scalar masses at lower orders; at one-loop for the non-finite case [39] and at two-loops for the finite case [40]. The sum rule manages to overcome the above unpleasant phenomenological consequences. Moreover it was proven [23] that the sum rule for the soft scalar massses is RGI to all-orders for both the general as well as for the finite case. Finally the exact $\beta$-function for the soft scalar masses in the Novikov-Shifman-Vainstein-Zakharov (NSVZ) scheme [41–43] for the softly broken supersymmetric QCD has been obtained [23]. Armed with the above tools and results we are in a position to study the spectrum of finite supersymmetric models.

2. The Reduction of Couplings Method

Let us outline briefly the idea of reduction of couplings. Any RGI relation among couplings (which does not depend on the renormalization scale $\mu$ explicitly) can be expressed, in the implicit form $\Phi(g_1, \cdots, g_A) = \text{const.}$, which has to satisfy the partial differential equation (PDE)

$$\mu \frac{d\Phi}{d\mu} = \nabla \cdot \vec{\beta} = \sum_{a=1}^{A} \beta_a \frac{\partial \Phi}{\partial g_a} = 0 , \quad (1)$$
where $\beta_a$ is the $\beta$-function of $g_a$. This PDE is equivalent to a set of ordinary differential equations, the so-called reduction equations (REs), [8–11]

$$\beta_g \frac{dg_a}{dg} = \beta_a , \ a = 1, \cdots, A ,$$  \hspace{1cm} \text{(2)}

where $g$ and $\beta_g$ are the primary coupling and its $\beta$-function, and the counting on $a$ does not include $g$. Since maximally $(A - 1)$ independent RGI “constraints” in the $A$-dimensional space of couplings can be imposed by the $\Phi_a$’s, one could in principle express all the couplings in terms of a single coupling $g$. The strongest requirement is to demand power series solutions to the REs,

$$g_a = \sum_{n=0}^{\infty} \rho_a^{(n)} g^{2n+1} ,$$  \hspace{1cm} \text{(3)}

which formally preserve perturbative renormalizability. Remarkably, the uniqueness of such power series solutions can be decided already at the one-loop level [8–11]. To illustrate this, let us assume that the $\rho_a^{(n)}$’s with $n \leq r$ have been uniquely determined. To obtain the $\rho_a^{(r+1)}$’s, we insert the power series (3) into the REs (2) and collect terms of $O(g^{2r+3})$ and find

$$\sum_{d \neq g} M(r)^d_a \rho_d^{(r+1)} = \text{lower order quantities} ,$$

where the r.h.s. is known by assumption, and

$$M(r)^d_a = 3 \sum_{b,c \neq g} \beta_a^{(1)bcd} \rho_b^{(1)} \rho_c^{(1)} \rho_d^{(1)} + \sum_{b \neq g} \beta_a^{(1)b} g_g^2 \rho_b^{(1)} + \cdots ,$$  \hspace{1cm} \text{(5)}

$$0 = \sum_{b,c,d \neq g} \beta_a^{(1)bcd} \rho_b^{(1)} \rho_c^{(1)} \rho_d^{(1)} + \sum_{d \neq g} \beta_a^{(1)d} \rho_d^{(1)} - \beta_g^{(1)} \rho_g^{(1)} ,$$  \hspace{1cm} \text{(6)}

Therefore, the $\rho_a^{(n)}$’s for all $n > 1$ for a given set of $\rho_a^{(1)}$’s can be uniquely determined if $\det M(n)^d_a \neq 0$ for all $n \geq 0$.

Our experience examining specific examples has taught us that the various couplings in supersymmetric theories have easily the same asymptotic behaviour. Therefore searching for a power series solution of the form (3) to the REs (2) is justified and moreover, one can rely that in applications keeping only the first terms a good approximation is obtained. This is not the case in non-supersymmetric theories.

3. Reduction of Couplings and Finiteness in $N = 1$ SUSY Gauge Theories

Consider a chiral, anomaly free, $N = 1$ globally supersymmetric gauge theory based on a group $G$ with gauge coupling constant $g$. The superpotential of the theory is given by

$$W = \frac{1}{2} m_{ij} \phi_i \phi_j + \frac{1}{6} C_{ijk} \phi_i \phi_j \phi_k ,$$  \hspace{1cm} \text{(7)}
where $m_{ij}$ and $C_{ijk}$ are gauge invariant tensors and the matter field $\phi_i$ transforms according to the irreducible representation $R_i$ of the gauge group $G$. The renormalization constants associated with the superpotential (7), assuming that supersymmetry is preserved, are

$$\phi_i^0 = (Z_i^j)^{(1/2)} \phi_j,$$  

$$m_{ij}^0 = Z_i^{j'} m_{j'i'},$$  

$$C_{ijk}^0 = Z_i^{j'k'} C_{j'k'}. \tag{10}$$

The $N = 1$ non-renormalization theorem [27, 44–46] ensures that there are no mass and cubic-interaction-term infinities and therefore

$$Z^{i'j'k'} Z_i^{1/2 j_1} Z_j^{1/2 j''_1} Z_k^{1/2 k''_1} = \delta_{i}^{(i)} \delta_{j}^{(j')} \delta_{k}^{(k')},$$  

$$Z_{ij}^{j_1} Z_i^{1/2 j_1'} Z_{j}^{1/2 j''_1} = \delta_{i}^{(i')} \delta_{j}^{(j')}. \tag{11}$$

As a result the only surviving possible infinities are the wave-function renormalization constants $Z_i^j$, i.e., one infinity for each field. The one-loop $\beta$-function of the gauge coupling $g$ is given by [47–53]

$$\beta_g^{(1)} = \frac{dg}{dt} = \frac{g^3}{16\pi^2} \left[ \sum_i l(R_i) - 3 C_2(G) \right], \tag{12}$$

where $l(R_i)$ is the Dynkin index of $R_i$ and $C_2(G)$ is the quadratic Casimir of the adjoint representation of the gauge group $G$. The $\beta$-functions of $C_{ijk}$, by virtue of the non-renormalization theorem, are related to the anomalous dimension matrix $\gamma_{ij}$ of the matter fields $\phi_i$ as:

$$\beta_{ijk} = \frac{dC_{ijk}}{dt} = C_{ijl} \gamma_l^i + C_{ikl} \gamma_l^j + C_{jkl} \gamma_l^i. \tag{13}$$

The conditions for finiteness for $N = 1$ field theories with $SU(N)$ gauge symmetry are discussed in [54], and the analysis of the anomaly-freedom and no-charge renormalization requirements for these theories can be found in [55]. At one-loop level $\gamma_{ij}$ is [47–54]

$$\gamma_{ij}^{(1)} = \frac{1}{32\pi^2} \left[ C_{ijkl} - 2 g^2 C_2(R_i) \delta_{ij} \right], \tag{14}$$

where $C_2(R_i)$ is the quadratic Casimir of the representation $R_i$, and $C_{ijk} = C_{iji}^*.$

As we already mentioned, the non-renormalization theorem ensures there are no extra mass and cubic-interaction-term renormalizations, implying that the $\beta$-functions of $C_{ijk}$ can be expressed as linear combinations of the anomalous dimensions $\gamma_{ij}$ of $\phi_i$. Therefore, all the one-loop $\beta$-functions of the theory vanish if $\beta_{ij}^{(1)}$ and $\gamma_{ij}^{(1)}$, given in Eqs. (12) and (14) respectively, vanish, i.e.

$$\sum_i l(R_i) = 3C_2(G),$$  

$$C_{ijkl} = 2 \delta_{ij} g^2 C_2(R_i), \tag{16}$$

A very interesting result is that the conditions (15,16) are necessary and sufficient for finiteness at the two-loop level [47–53].

In case supersymmetry is broken by soft terms, one-loop finiteness of the soft sector imposes further constraints on it [33]. In addition, the same set of conditions that are sufficient for one-loop finiteness of the soft breaking terms render the soft sector of the theory two-loop finite [34].
The one- and two-loop finiteness conditions (15,16) restrict considerably the possible choices of the irreps, \( R_i \) for a given group \( G \) as well as the Yukawa couplings in the superpotential (7). Note in particular that the finiteness conditions cannot be applied to the supersymmetric standard model (SSM), since the presence of a \( U(1) \) gauge group is incompatible with the condition (15), due to \( C_2[U(1)] = 0 \). This naturally leads to the expectation that finiteness should be attained at the grand unified level only, the SSM being just the corresponding, low-energy, effective theory.

Another important consequence of one- and two-loop finiteness is that supersymmetry (most probably) can only be broken by soft breaking terms. Indeed, due to the unacceptability of gauge singlets, \( F \)-type spontaneous symmetry breaking \([56]\) terms are incompatible with finiteness, as well as \( D \)-type \([57]\) spontaneous breaking which requires the existence of a \( U(1) \) gauge group.

A natural question to ask is what happens at higher loop orders. The answer is contained in a theorem \([12–14]\) which states the necessary and sufficient conditions to achieve finiteness at all orders. Before we discuss the theorem let us make some introductory remarks. The finiteness conditions impose relations between gauge and Yukawa couplings. To require such relations which render the couplings mutually dependent at a given renormalization point is trivial. What is not trivial is to guarantee that relations leading to a reduction of the couplings hold at any renormalization point. As we have seen, the necessary, but also sufficient, condition for this to happen is to require that such relations are solutions to the REs (2) and hold at all orders. As we have seen, remarkably the existence of all-order solutions to (2) can be decided at the one-loop level.

3.1. The all-loop finiteness theorem
Let us now turn to the all-order finiteness theorem, \([12–14]\) which states when a \( N = 1 \) supersymmetric gauge theory can become finite to all orders in the sense of vanishing \( \beta \)-functions, that is of physical scale invariance. It is based on (a) the structure of the supercurrent in \( N = 1 \) SYM, \([13,58,59]\) and on (b) the non-renormalization properties of \( N = 1 \) chiral anomalies \([12–14]\). Details on the proof can be found in refs. \([12–14,60]\). Here, following mostly Piguet, \([60]\) we present a comprehensible sketch of the proof.

Consider a \( N = 1 \) supersymmetric gauge theory, with simple Lie group \( G \). The content of this theory is given at the classical level by the matter supermultiplets \( S_i \), which contain a scalar field \( \phi_i \) and a Weyl spinor \( \psi_{ia} \), and the gauge fields \( V_a \), which contain a gauge vector field \( A^a_\mu \) and a gaugino Weyl spinor \( \lambda^a_\alpha \).

Let us first recall certain facts about the theory:
(1) A massless \( N = 1 \) supersymmetric theory is invariant under a \( U(1) \) chiral transformation \( R \) under which the various fields transform as follows
\[
A'_\mu = A_\mu, \quad \lambda'_\alpha = \exp(-i\theta)\lambda_\alpha, \quad \phi'_l = \exp(-i\frac{2}{3}\theta)\phi, \quad \psi'_{\alpha a} = \exp(-i\frac{1}{3}\theta)\psi_{\alpha a}, \quad \cdots
\] (17)

The corresponding axial Noether current \( J^\mu_R(x) \) is
\[
J^\mu_R(x) = \bar{\lambda} \gamma^\mu \gamma^5 \lambda + \cdots
\] (18)
is conserved classically, while in the quantum case is violated by the axial anomaly
\[
\partial_\mu J^\mu_R = r(e^{\mu\nu\rho}\epsilon_{\mu\nu\rho} F_{\nu\rho} + \cdots).
\] (19)

From its known topological origin in ordinary gauge theories, \([61–63]\) one would expect that the axial vector current \( J^\mu_R \) to satisfy the Adler-Bardeen theorem \([64]\) and receive corrections
only at the one-loop level. Indeed it has been shown that the same non-renormalization theorem holds also in supersymmetric theories [13]. Therefore

\[ r = \hbar \beta^1 g. \] (20)

(2) The massless theory we consider is scale invariant at the classical level and, in general, there is a scale anomaly due to radiative corrections. The scale anomaly appears in the trace of the energy momentum tensor \( T_{\mu\nu} \), which is traceless classically. It has the form

\[ T_{\mu}^{\mu} = \beta^g F^{\mu\nu} F_{\mu\nu} + \cdots \] (21)

(3) Massless, \( N = 1 \) supersymmetric gauge theories are classically invariant under the supersymmetric extension of the conformal group – the superconformal group. Examining the superconformal algebra, it can be seen that the subset of superconformal transformations consisting of translations, supersymmetry transformations, and axial R-transformations is closed under supersymmetry, i.e. these transformations form a representation of supersymmetry. It follows that the conserved currents corresponding to these transformations make up a supermultiplet represented by an axial vector superfield called supercurrent \( J \), [58]

\[ J = \{ J^R_{\mu}, Q^\alpha_{\mu}, T^\mu_{\nu}, \ldots \}, \] (22)

where \( J^R_{\mu} \) is the current associated to R-invariance, \( Q^\alpha_{\mu} \) is the one associated to supersymmetry invariance, and \( T^\mu_{\nu} \) the one associated to translational invariance (energy-momentum tensor).

The anomalies of the R-current \( J^R_{\mu} \), the trace anomalies of the supersymmetry current, and the energy-momentum tensor, form also a second supermultiplet, called the supertrace anomaly

\[ S = \{ \text{Re} S, \text{Im} S, S_{\alpha} \} = \{ T_{\mu}^{\mu}, \partial_{\mu} J^R_{\mu}, \sigma^\mu_{\alpha\beta} \bar{Q}^\beta_{\mu} + \cdots \}, \] (23)

where \( T_{\mu}^{\mu} \) is defined in Eq.(21) and

\[ \partial_{\mu} J^R_{\mu} = \beta^g \epsilon^{\mu\nu\sigma\rho} F_{\mu\nu} F_{\sigma\rho} + \cdots \] (24)

\[ \sigma^\mu_{\alpha\beta} \bar{Q}^\beta_{\mu} = \beta^3 \delta_{\alpha\beta} \sigma^\mu_{\alpha\beta} F_{\mu\nu} + \cdots \] (25)

(4) It is very important to note that the Noether current defined in (18) is not the same as the current associated to R-invariance that appears in the supermultiplet \( J \) in (22), but they coincide in the tree approximation. So starting from a unique classical Noether current \( J^R_{\mu(\text{class})} \), the Noether current \( J^R_{\mu} \) is defined as the quantum extension of \( J^R_{\mu(\text{class})} \) which allows for the validity of the non-renormalization theorem. On the other hand \( J^R_{\mu} \), is defined to belong to the supercurrent \( J \), together with the energy-momentum tensor. The two requirements cannot be fulfilled by a single current operator at the same time.

Although the Noether current \( J^R_{\mu} \) which obeys (19) and the current \( J^R_{\mu} \) belonging to the supercurrent multiplet \( J \) are not the same, there is a relation [12–14] between quantities associated with them

\[ r = \beta^g (1 + x^g) + \beta_{ijk} x^{ijk} - \gamma_A r^A, \] (26)

where \( r \) was given in Eq. (20). The \( r^A \) are the non-renormalized coefficients of the anomalies of the Noether currents associated to the chiral invariances of the superpotential, and –like \( r \) – are strictly one-loop quantities. The \( \gamma_A \)’s are linear combinations of the anomalous dimensions of the matter fields, and \( x^g \), and \( x^{ijk} \) are radiative correction quantities. The structure of equality (26) is independent of the renormalization scheme.
One-loop finiteness, i.e. vanishing of the $\beta$-functions at one-loop, implies that the Yukawa couplings $\lambda_{ijk}$ must be functions of the gauge coupling $g$. To find a similar condition to all orders it is necessary and sufficient for the Yukawa couplings to be a formal power series in $g$, which is solution of the REs (2).

We can now state the theorem for all-order vanishing $\beta$-functions.

*Theorem:* Consider an $N = 1$ supersymmetric Yang-Mills theory, with simple gauge group. If the following conditions are satisfied

(i) There is no gauge anomaly.
(ii) The gauge $\beta$-function vanishes at one-loop

$$\beta_g^{(1)} = 0 = \sum_i l(R_i) - 3C_2(G). \quad (27)$$

(iii) There exist solutions of the form

$$\lambda_{ijk} = \rho_{ijk}g, \quad \rho_{ijk} \in \mathbb{C} \quad (28)$$

to the conditions of vanishing one-loop matter fields anomalous dimensions

$$\gamma_j^{(1)} = 0 = \frac{1}{32\pi^2} \left[ C^{ijkl} C_{jkl} - 2g^2 C_2(R_i) \delta_{ij} \right]. \quad (29)$$

(iv) these solutions are isolated and non-degenerate when considered as solutions of vanishing one-loop Yukawa $\beta$-functions:

$$\beta_{ijk} = 0. \quad (30)$$

Then, each of the solutions (28) can be uniquely extended to a formal power series in $g$, and the associated super Yang-Mills models depend on the single coupling constant $g$ with a $\beta$ function which vanishes at all-orders.

It is important to note a few things: The requirement of isolated and non-degenerate solutions guarantees the existence of a formal power series solution to the reduction equations. The vanishing of the gauge $\beta$-function at one-loop, $\beta_g^{(1)}$, is equivalent to the vanishing of the R current anomaly (19). The vanishing of the anomalous dimensions at one-loop implies the vanishing of the Yukawa couplings $\beta$-functions at that order. It also implies the vanishing of the chiral anomaly coefficients $r^A$. This last property is a necessary condition for having $\beta$ functions vanishing at all orders.

*Proof:* Insert $\beta_{ijk}$ as given by the REs into the relationship (26) between the axial anomalies coefficients and the $\beta$-functions. Since these chiral anomalies vanish, we get for $\beta_g$ an homogeneous equation of the form

$$0 = \beta_g(1 + O(h)). \quad (31)$$

The solution of this equation in the sense of a formal power series in $h$ is $\beta_g = 0$, order by order. Therefore, due to the REs (2), $\beta_{ijk} = 0$ too.
3.2. Soft symmetry breaking sector, sum rule and all-loop results

The method of reducing the dimensionless couplings has been extended [6] to the soft supersymmetry breaking (SSB) dimensionful parameters of $N = 1$ supersymmetric theories. In addition it was found [39] that RGI SSB scalar masses in Gauge-Yukawa unified models satisfy a universal sum rule. Here we will describe first how the use of the available two-loop RG functions and the requirement of finiteness of the SSB parameters up to this order leads to the soft scalar-mass sum rule [40].

Consider the superpotential given by (7) along with the Lagrangian for SSB terms

$$-L_{\text{SB}} = \frac{1}{6} h^{ijk} \phi_i \phi_j \phi_k + \frac{1}{2} b^{ij} \phi_i \phi_j + \frac{1}{2} (m^2)_i^j \phi^* i \phi_j + \frac{1}{2} M \lambda \lambda + \text{H.c.},$$

where the $\phi_i$ are the scalar parts of the chiral superfields $\Phi_i$, $\lambda$ are the gauginos and $M$ their unified mass. Since we would like to consider only finite theories here, we assume that the gauge group is a simple group and the one-loop $\beta$ function of the gauge coupling $g$ vanishes. We also assume that the reduction equations admit power series solutions of the form

$$C^{ijk} = g \sum_{n=0} \rho^{ijk}_{(n)} g^n.$$  \hspace{1cm} (33)

According to the finiteness theorem, [12–14] the theory is then finite to all orders in perturbation theory, if, among others, the one-loop anomalous dimensions $\gamma_i^{(1)}$ vanish. The one- and two-loop finiteness for $h^{ijk}$ can be achieved by

$$h^{ijk} = - M C^{ijk} + \cdots = - M \rho^{ijk}_{(0)} g + O(g^5).$$  \hspace{1cm} (34)

Now, to obtain the two-loop sum rule for soft scalar masses, we assume that the lowest order coefficients $\rho^{ijk}_{(0)}$ and also $(m^2)_i^j$ satisfy the diagonality relations

$$\rho^{ijk}_{(0)} \rho^{jpq}_{(0)} \propto \delta_i^q \text{ for all } p \text{ and } q \text{ and } (m^2)_i^j = m^2_j \delta_i^j,$$  \hspace{1cm} (35)

respectively. Then we find the following soft scalar-mass sum rule

$$(m_i^2 + m_j^2 + m_k^2)/(MM^\dagger) = 1 + \frac{g^2}{16\pi^2} \Delta^{(1)} + O(g^4)$$  \hspace{1cm} (36)

for $i, j, k$ with $\rho^{ijk}_{(0)} \neq 0$, where $\Delta^{(1)}$ is the two-loop correction

$$\Delta^{(1)} = - 2 \sum_i [(m_i^2/MM^\dagger) - (1/3)] T(R_i),$$  \hspace{1cm} (37)

which vanishes for the universal choice in accordance with previous findings [34].

If we know higher-loop $\beta$-functions explicitly, we can follow the same procedure and find higher-loop RGI relations among SSB terms. However, the $\beta$-functions of the soft scalar masses are explicitly known only up to two loops. In order to obtain higher-loop results, we need something else instead of knowledge of explicit $\beta$-functions, e.g. some relations among $\beta$-functions.
By use of the spurion technique [25–32] it is possible to find the following all-loop relations among SSB $\beta$-functions, [16–22]

$$
\beta_M = 2\mathcal{O} \left( \frac{\beta_i}{g} \right),
$$

(38)

$$
\beta_{h}^{ijk} = \gamma_i^j h^{ljk} + \gamma_j^i h^{ikl} + \gamma_k^i h^{ijl} - 2\gamma_1^i C^{ljk} - 2\gamma_1^j C^{ikl} - 2\gamma_1^k C^{ijl},
$$

(39)

$$
(\beta_{m^2})^i_j = \left[ \Delta + X \frac{\partial}{\partial g} \right] \gamma^i_j,
$$

(40)

$$
\mathcal{O} = \left( M g^2 \frac{\partial}{\partial g^2} - h^{lmn} \frac{\partial}{\partial C^{lmn}} \right),
$$

(41)

$$
\Delta = 2\mathcal{O} \mathcal{O}^* + 2|M|^2 g^2 \frac{\partial}{\partial g^2} + \tilde{C}^{lmn} \frac{\partial}{\partial C^{lmn}} + \tilde{C}^{lmn} \frac{\partial}{\partial C^{lmn}},
$$

(42)

where $(\gamma_1)^i_j = \mathcal{O} \gamma^i_j$, $C_{lmn} = (C^{lmn})^*$, and

$$
\tilde{C}^{ijk} = (m^2)^i_j C^{ljk} + (m^2)^j_i C^{ilk} + (m^2)^k_i C^{ijl}.
$$

(43)

It was also found [22] that the relation

$$
h^{ijk} = -M(C^{ijk})' \equiv -M \frac{dC^{ijk}(g)}{d\ln g},
$$

(44)

among couplings is all-loop RGI. Furthermore, using the all-loop gauge $\beta$-function of Novikov et al. [41–43] given by

$$
\beta_{NSVZ}^{\gamma_i} = \frac{g^3}{16\pi^2} \left[ \sum_l T(R_l) \left( 1 - \gamma_i / 2 \right) - 3C(G) \right] / \left[ 1 - g^2 C(G) / 8\pi^2 \right],
$$

(45)

it was found the all-loop RGI sum rule, [23]

$$
m_i^2 + m_j^2 + m_k^2 = |M|^2 \left\{ \frac{1}{1 - g^2 C(G) / 8\pi^2} \frac{d\ln C^{ijk}}{d\ln g} + \frac{1}{2} \frac{d^2 \ln C^{ijk}}{d(\ln g)^2} \right\}
$$

$$
+ \sum_l \frac{m_l^2 T(R_l)}{C(G) - 8\pi^2 / g^2} \frac{d\ln C^{ijk}}{d\ln g}.
$$

(46)

In addition the exact-$\beta$ function for $m^2$ in the NSVZ scheme has been obtained [23] for the first time and is given by

$$
\beta_{m^2}^{NSVZ} = \left[ |M|^2 \left\{ \frac{1}{1 - g^2 C(G) / 8\pi^2} \frac{d\ln C^{ijk}}{d\ln g} + \frac{1}{2} \frac{d^2 \ln C^{ijk}}{d(\ln g)^2} \right\}
$$

$$
+ \sum_l \frac{m_l^2 T(R_l)}{C(G) - 8\pi^2 / g^2} \frac{d\ln C^{ijk}}{d\ln g} \right] \gamma_i^{NSVZ}.
$$

(47)

Surprisingly enough, the all-loop result (46) coincides with the superstring result for the finite case in a certain class of orbifold models [40] if $d\ln C^{ijk} / d\ln g = 1$. 


4. Conclusions
In the present contribution we have presented a summary of the theoretical developments that permit us to construct all-loop Finite Grand Unified Theories beyond the unification scale. These theories, after the spontaneous symmetry breaking at the Grand Unification scale, are expected to lead to finiteness-constrained versions of the MSSM with huge predictive power. A finiteness-constrained MSSM has first to be confronted with the existing experimental data, while the remaining predictions are expected to be tested in the future experiments starting with LHC. Examples of such theories are presented in another contribution of the present proceedings.

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