MODULAS AT BOUNDARY POINTS, FIBERWISE BERGMAN KERNELS, AND LOG-SUBHARMONICITY

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Abstract. In this article, we consider Bergman kernels with respect to modules at boundary points, and obtain a log-subharmonicity property of the Bergman kernels, which deduces a concavity property related to the Bergman kernels. As applications, we reprove the sharp effectiveness result related to a conjecture posed by Jonsson-Mustata and the effectiveness result of strong openness property of the modules at boundary points.

1. Introduction

The strong openness property of multiplier ideal sheaves (i.e. $\mathcal{I}(\psi) = \mathcal{I}_+(\psi) := \bigcup_{\epsilon > 0} \mathcal{I}(1 + \epsilon \psi)$) is an important feature and has been widely used in the study of several complex variables, complex algebraic geometry and complex differential geometry (see e.g. [39, 47, 10, 22, 60, 42, 61, 62, 23, 48, 12]), where $\psi$ is a plurisubharmonic function on a complex manifold $M$ (see [13]) and multiplier ideal sheaf $\mathcal{I}(\psi)$ is the sheaf of germs of holomorphic functions $f$ such that $|f|^2 e^{-\psi}$ is locally integrable (see e.g. [58, 50, 53, 16, 17, 15, 18, 49, 54, 55, 14, 43]).

The strong openness property was conjectured by Demailly [15] and proved by Guan-Zhou [39] (the 2-dimensional case was proved by Jonsson-Mustata [45]). Recall that in order to prove the strong openness property, Jonsson and Mustata (see [46], see also [15]) posed the following conjecture, and proved the 2-dimensional case [45]:

Conjecture J-M: If $c^F_\alpha(\psi) < +\infty$, $\frac{1}{r^2} \mu(\{c^F_\alpha(\psi) \psi - \log |F| < \log r \})$ has a uniform positive lower bound independent of $r \in (0, 1)$, where $\mu$ is the Lebesgue measure on $\mathbb{C}^n$, and $c^F_\alpha(\psi) := \sup\{c \geq 0 : |F|^2 e^{-2c\psi} \text{ is locally } L^1 \text{ near } o\}$. Using the strong openness property, Guan-Zhou [41] proved Conjecture J-M.

Independent of the strong openness property, Bao-Guan-Yuan [3] considered minimal $L^2$ integrals with respect to a module at a boundary point of the sublevel sets, and established a concavity property of the minimal $L^2$ integrals, which deduced a sharp effectiveness result related to Conjecture J-M, and completed the approach from Conjecture J-M to the strong openness property.

As a generalization of Berndtsson’s log-plurisubharmonicity result of fiberwise Bergman kernels (see [4]), in [1] (see also [2]), we obtained the log-plurisubharmonicity of fiberwise Bergman kernels with respect to functionals over the space of holomorphic germs by using the optimal $L^2$ extension theorem (see [40]) and Guan-Zhou method (see [51]). As applications, we gave new approaches to the effectiveness results of strong openness property (11) and $L^p$ strong openness property (2).
As continuity work of [11] and [12], in this article, we consider Bergman kernels with respect to the modules at boundary points, and obtain a log-subharmonicity property of the Bergman kernels (as a generalization of Berndtsson’s log-subharmonicity result of fiberwise Bergman kernels in [11]), which deduces a new approach from Conjecture J-M to the strong openness property. We also give a reproof for the effectiveness result of the strong openness property related to the modules at boundary points.

1.1. Main result.

Let $D$ be a pseudoconvex domain in $\mathbb{C}^n$, and the origin $o \in D$. Let $F \neq 0$ be a holomorphic function on $D$, and $\psi$ be a negative plurisubharmonic function on $D$. Let $\varphi_0$ be a plurisubharmonic function on $D$. Denote that

$$\Psi := \min\{\psi - 2 \log |F|, 0\}.$$  

If $F(w) = 0$ for $w \in D$, set $\Psi(w) = 0$.

We recall some notations in [3]. Denote that

$$J\Psi := \{f \in \mathcal{O}(\{\Psi < -t\} \cap V) : t \in \mathbb{R}, V \text{ is a neighborhood of } o\},$$

and

$$J\Psi := J\Psi/\sim,$$

where the equivalence relation $\sim$ is as follows:

$$f \sim g \iff f = g \text{ on } \{\Psi < -t\} \cap V, \text{ where } t \gg 1, V \text{ is a neighborhood of } o.$$  

For any $f \in J\Psi$, denote the equivalence class of $f$ in $J\Psi$ by $f_o$. And for any $f_o, g_o \in J\Psi$, and $(h, o) \in \mathcal{O}_o$, define

$$f_o + g_o := (f + g)_o, \quad (h, o) \cdot f_o := (hf)_o.$$  

It is clear that $J\Psi$ is an $\mathcal{O}_o$-module. For any $a \geq 0$, denote that

$$I(\alpha \Psi + \varphi_0)_o := \{f_o \in J\Psi : \exists t \gg 1, V \text{ is a neighborhood of } o, \text{ s.t. } \int_{\{\Psi < -t\} \cap V} |f|^2 e^{-a \Psi - \varphi_0} < +\infty\}.$$  

Then it is clear that $I(\alpha \Psi + \varphi_0)$ is an $\mathcal{O}_o$-submodule of $J\Psi$. Especially, we denote that $I(\varphi_0)_o := I(0 \Psi + \varphi_0)_o$, $I(\Psi)_o := I(\Psi + 0)_o$, and $I_o := I(0 \Psi + 0)_o$.

Then $I(\alpha \Psi + \varphi_0)$ is an $\mathcal{O}_o$-submodule of $I(\varphi_0)_o$ for any $a > 0$.

For any $t \in [0, +\infty)$ and $\lambda > 0$, denote that

$$\Psi_{\lambda, t} := \lambda \max\{\Psi + t, 0\},$$

and for any $f \in A^2(\{\Psi < 0\}, e^{-\varphi_0})$ and $\lambda > 0$, denote that

$$\|f\|_{\lambda, t} := \left(\int_{\{\Psi < 0\}} |f|^2 e^{-\varphi_0 - \Psi_{\lambda, t}}\right)^{1/2},$$

where $A^2(\{\Psi < 0\}, e^{-\varphi_0}) := \{f \in \mathcal{O}(\{\Psi < 0\}) : \int_{\{\Psi < 0\}} |f|^2 e^{-\varphi_0} < +\infty\}$ (if $\varphi_0 \equiv 0$, we may denote $A^2(\{\Psi < 0\}) := A^2(\{\Psi < 0\}, e^0)\}$). It is clear that $e^{-\lambda / 2} \|f\|_{\lambda, 0} \leq \|f\|_{\lambda, t} \leq \|f\|_{\lambda, 0} < +\infty$ for any $t \geq 0$.

For any $\xi \in A^2(\{\Psi < 0\}, e^{-\varphi_0})^*$ (the dual space of $A^2(\{\Psi < 0\}, e^{-\varphi_0})$), denote that the Bergman kernel related to $\xi$ is

$$K_{\xi, \Psi, \lambda}^E(t) := \sup_{f \in A^2(\{\Psi < 0\}, e^{-\varphi_0})} \frac{|\xi \cdot f|^2}{\|f\|_{\lambda, t}^2},$$

for any $t \in [0, +\infty)$. 
Denote $E := \{ w \in \mathbb{C} : \text{Re } w \geq 0 \} \subset \mathbb{C}$. We obtain the following log-subharmonicity property of the Bergman kernel $K^{\varphi_0}_{\xi, \psi, \lambda}$.

**Theorem 1.1.** Assume that $K^{\varphi_0}_{\xi, \psi, \lambda}(0) \in (0, +\infty)$. Then $\log K^{\varphi_0}_{\xi, \psi, \lambda}(\text{Re } w)$ is subharmonic with respect to $w \in E$.

Let $J$ be an $O_o$-submodule of $I(\varphi_0)_o$. Denote that

$$A^2(\{ \Psi < 0 \}, e^{-\varphi_0}) \cap J := \{ f \in A^2(\{ \Psi < 0 \}, e^{-\varphi_0}) : f_o \in J \}.$$

Assume that $A^2(\{ \Psi < 0 \}, e^{-\varphi_0}) \cap J$ is proper subspace of $A^2(\{ \Psi < 0 \}, e^{-\varphi_0})$ (and we will state that it is a closed subspace).

Using Theorem 1.1, we obtain the following concavity and monotonicity property related to $K^{\varphi_0}_{\xi, \psi, \lambda}$.

**Theorem 1.2.** Assume that $J \supset I(\Psi + \varphi_0)_o$, and assume that $\xi \in A^2(\{ \Psi < 0 \}, e^{-\varphi_0})^*$ such that $\xi|_{A^2(\{ \Psi < 0 \}, e^{-\varphi_0}) \cap J}$ $\equiv 0$ and $K^{\varphi_0}_{\xi, \psi, \lambda}(0) \in (0, +\infty)$. Then $- \log K^{\varphi_0}_{\xi, \psi, \lambda}(t) + t$ is concave and increasing with respect to $t \in [0, +\infty)$.

**Remark 1.3.** Let $\xi \in A^2(\{ \Psi < 0 \}, e^{-\varphi_0})^*$. According to Theorem 1.1, if there exist $k > 0$ and $T > 0$, such that $e^{-kt}K^{\varphi_0}_{\xi, \psi, \lambda}(t)$ is increasing and not a constant function on $[0, T]$, then $e^{-kt}K^{\varphi_0}_{\xi, \psi, \lambda}(t)$ is strictly increasing on $[T, +\infty)$.

1.2. Applications.

As applications of Theorem 1.1 and Theorem 1.2, we give new proofs of some results in [31].

Let $D$ be a pseudoconvex domain in $\mathbb{C}^n$, and the origin $o \in D$. Let $F \not\equiv 0$ be a holomorphic function on $D$, and $\psi$ be a negative plurisubharmonic function on $D$. Denote that

$$\Psi := \min\{ \psi - 2 \log |F|, 0 \}.$$

If $F(w) = 0$ for $w \in D$, set $\Psi(w) = 0$.

Let $f$ be a holomorphic function on $D$. Recall the definition of the minimal $L^2$ integral related to $J$ ([31])

$$G(t; \Psi, J, f) := \inf \left\{ \int_{\{ \Psi < -t \}} |\tilde{f}|^2 : \tilde{f} \in O(\{ \Psi < -t \}) \& (\tilde{f} - f)_o \in J \right\}$$

for any $O_o$-submodule $J$ of $I_o$ and $t \in [0, +\infty)$. Denote that

$$\Psi_1 := \min\{ 2c_o^{\xi F}(\psi) \psi - 2 \log |F|, 0 \},$$

and

$$I_+(\Psi)_o := \bigcup_{a > 1} I(a \Psi)_o,$$

where $c_o^{\xi F}(\psi) := \sup\{ c \geq 0 : |fF|^2 e^{-2c\psi} \text{ is locally } L^1 \text{ near } o \}$.

Theorem 1.2 deduces a reproof of the following lower bound of $L^2$ integrals.

**Corollary 1.4 ([31]).** If $f \in A^2(\{ \Psi_1 < 0 \})$, and $c_o^{\xi F}(\psi) \not\equiv +\infty$, then for any $r \in (0, 1]$,

$$\frac{1}{r^2} \int_{c_o^{\xi F}(\psi) \psi - \log |F| < \log r} |f|^2 \geq G(0; \Psi_1, I_+(\Psi)_o, f) > 0.$$

**Remark 1.5.** The proof of the inequality $G(0; \Psi_1, I_+(\Psi)_o, f) > 0$ can be referred to [31].
When $f \equiv 1$, Corollary 1.6 deduces a reproof of the sharp effectiveness result related to a conjecture posed by Jonsson-Mustata.

**Corollary 1.6** (3). If $f \in A^2(\{\Psi < 0\})$ and $c^F_\psi(\psi) < +\infty$, then for any $r \in (0, 1]$, 

$$\frac{1}{\pi r^2} \mu(\{c^F_\psi(\psi) - \log |F| < \log r\}) \geq G(0; \Psi_1, I_+(\Psi_1)_o, 1) > 0,$$

where $\Psi_1 := \min\{2c^F_\psi(\psi) - 2 \log |F|, 0\}$, and $c^F_\psi(\psi) := \sup\{c \geq 0 : |F|^2 e^{-2c\psi} \text{ is locally } L^1 \text{ near } o\}$.

Let $\varphi_0$ be a plurisubharmonic function on $D$, and let $f$ be a holomorphic function on $\{\Psi < 0\}$. Denote that $a^f_\psi(\Psi; \varphi_0) := \sup\{a \geq 0 : f_\circ \in I(2a\Psi + \varphi_0)_o\}$,

$$I_+(a\Psi_1 + \varphi_0)_o := \bigcup_{a > a^f_\psi(\Psi; \varphi_0)_o} I(a\Psi_1 + \varphi_0)_o$$

for any $a \geq 0$, and

$$C(\Psi, \varphi_0, J, f) := \inf \left\{ \int_{\{\Psi < 0\}} |\tilde{f}|^2 e^{-\varphi_0} : (\tilde{f} - f)_\circ \in J \& \tilde{f} \in \mathcal{O}(\{\Psi < 0\}) \right\}$$

for any $\mathcal{O}_o$-submodule $J$ of $I(\varphi_0)_o$. The following effectiveness result of strong openness property of the module $I(o\Psi + \varphi_0)_o$ can be reproved by Theorem 1.2.

**Corollary 1.7** (31). Let $C_1$ and $C_2$ be two positive constants. If

1. $\int_{\{\Psi < 0\}} |f|^2 e^{-\varphi_0 - \Psi} \leq C_1$;
2. $C(\Psi, \varphi_0, I_+(2a^f_\psi(\Psi; \varphi_0)\Psi + \varphi_0)_o, f) \geq C_2$,

then for any $q > 1$ satisfying

$$\theta(q) := \frac{C_1}{C_2},$$

we have $f_\circ \in I(q\Psi + \varphi_0)_o$, where $\theta(q) = \frac{C_1}{C_2}$.

2. Preparations

2.1. $L^2$ methods.

We recall the optimal $L^2$ extension theorem.

Let $\Omega$ be a pseudoconvex domain in $\mathbb{C}^{n+1}$ with coordinate $(z, t)$, where $z \in \mathbb{C}^n$, $t \in \mathbb{C}$. Let $p$ be the natural projections $p(z, t) = t$ on $\Omega$. Denote that $\omega := p(\Omega)$ and $\Omega_t := p^{-1}(t)$ for any $t \in \omega$. Let $\varphi$ be a plurisubharmonic function on $\Omega$.

**Lemma 2.1** (Optimal $L^2$ extension theorem (38, see 37, 10, 38)). Let $\omega = \Delta_{t_0, r}$ be the disc in the complex plane centered at $t_0$ with radius $r$. Then for any $f$ in $A^2(\Omega_{t_0}, e^{-\varphi})$, there exists a holomorphic function $\tilde{f}$ on $\Omega$, such that $\tilde{f}|_{\Omega_{t_0}} = f$, and

$$\frac{1}{\pi r^2} \int_{\Omega} |\tilde{f}|^2 e^{-\varphi} \leq \int_{\Omega_{t_0}} |f|^2 e^{-\varphi}.$$

The following $L^2$ method will be used to prove Theorem 1.2.

Let $D$ be a pseudoconvex domain in $\mathbb{C}^n$, and the origin $o \in D$. Let $F \neq 0$ be a holomorphic function on $D$, and $\psi$ be a negative plurisubharmonic function on $D$. Let $\varphi_0$ be a plurisubharmonic function on $D$. Denote that

$$\varphi := \varphi_0 + 2 \max\{\psi, 2 \log |F|\},$$

and

$$\Psi := \min\{\psi - 2 \log |F|, 0\}.$$
If $F(w) = 0$ for $w \in D$, set $\Psi(w) = 0$.

Lemma 2.2 (see [30] [31] [3]). Let $t_0 \in (0, +\infty)$ be arbitrary given. Let $f$ be a holomorphic function on $\{\Psi < -t_0\}$ such that

$$\int_{\{\Psi < -t_0\} \cap K} |f|^2 e^{-\Psi} < +\infty$$

for any compact subset $K \subset D$. Then there exists a holomorphic function $\tilde{F}$ on $D$ such that

$$\int_D |\tilde{F} - (1 - b_{t_0}^t(\Psi))fF^2|e^{\Psi - \Psi_0} \leq C \int_D I_{(-t_0, -t_0)} |f|^2 e^{-\Psi},$$

where $b_{t_0}(t) = \int_{-\infty}^t \int_{-t_0}^{s-t_0} ds$ and $C$ is a positive constant.

2.2. Some lemmas about submodules of $I(\varphi_0)_o$.

Recall that $D$ is a pseudoconvex domain in $\mathbb{C}^n$, and the origin $o \in D$. Let $F \neq 0$ be a holomorphic function on $D$, and $\psi$ be a negative plurisubharmonic function on $D$. Let $\varphi_0$ be a plurisubharmonic function on $D$. Denote that

$$\Psi := \min\{\psi - 2 \log |F|, 0\}.$$

If $F(w) = 0$ for $w \in D$, set $\Psi(w) = 0$. We recall the following lemma.

Lemma 2.3 ([31]). Let $J_o$ be an $O_{\mathbb{C}^n,o}$-submodule of $I(\varphi_0)_o$ such that $I(\Psi + \varphi_0)_o \subset J_o$. Assume that $f_o \in J(\Psi)_o$. Let $U_0$ be a Stein open neighborhood of $o$. Let $\{f_j\}_{j \geq 1}$ be a sequence of holomorphic functions on $U_0 \cap \{\Psi < -t_j\}$ for any $j \geq 1$, where $t_j \in (T, +\infty)$. Assume that $t_0 = \lim_{j \to +\infty} t_j \in [T, +\infty)$,

$$\limsup_{j \to +\infty} \int_{U_0 \cap \{\Psi < -t_j\}} |f_j|^2 e^{-\varphi_0} \leq C < +\infty,$$

and $(f_j)_o \in J_o$. Then there exists a subsequence of $\{f_j\}_{j \geq 1}$ compactly convergent to a holomorphic function $f_0$ on $\{\Psi < -t_0\} \cap U_0$ which satisfies

$$\int_{U_0 \cap \{\Psi < -t_0\}} |f_0|^2 e^{-\varphi_0} \leq C,$$

and $(f_0 - f)_o \in J_o$.

It is well-known that $A^2(\{\Psi < 0\}, e^{-\varphi_0})$ is a Hilbert space. Let $J$ be an $O_o$-submodule of $I(\varphi_0)_o$. We state that $A^2(\{\Psi < 0\}, e^{-\varphi_0}) \cap J := \{f \in A^2(\{\Psi < 0\}, e^{-\varphi_0}) : f_o \in J\}$ is a closed subspace of $A^2(\{\Psi < 0\}, e^{-\varphi_0})$ if $J \supset I(\Psi + \varphi_0)_o$.

Lemma 2.4. If $J \supset I(\Psi + \varphi_0)_o$, then $A^2(\{\Psi < 0\}, e^{-\varphi_0}) \cap J$ is closed in $A^2(\{\Psi < 0\}, e^{-\varphi_0})$.

Proof. Let $\{f_j\}$ be a sequence of holomorphic functions in $A^2(\{\Psi < 0\}, e^{-\varphi_0}) \cap J$, such that $\lim_{j \to +\infty} f_j = f_0$ under the topology of $A^2(\{\Psi < 0\}, e^{-\varphi_0})$. Then $\{f_j\}$ compactly converges to $f_0$ on $\{\Psi < 0\}$, and $(f_j)_o \in J$ for any $j$. According to Lemma 2.3, we can get that $(f_0 - f)_o \in J$, which means that $(f_0)_o \in J$, i.e. $f_0 \in A^2(\{\Psi < 0\}, e^{-\varphi_0}) \cap J$. The we know that $A^2(\{\Psi < 0\}, e^{-\varphi_0}) \cap J$ is closed in $A^2(\{\Psi < 0\}, e^{-\varphi_0})$. 

The following two lemmas can be referred to [3] or [31].
Lemma 2.5 ([31]). For any \( a \geq 0 \), there exists \( a' > a \) such that \( I(a'\Psi + \varphi_0) = I_+(a\Psi + \varphi_0) \).

Let \( f \) be a holomorphic function on \( D \). Denote that
\[
\Psi_1 := \min\{2c_o^f(\psi)\psi - 2\log |F|, 0\},
\]
where \( c_o^f(\psi) := \sup\{c \geq 0 : |F|^2e^{-2c\psi} \text{ is locally } L^1 \text{ near } o\} \).

Lemma 2.6 ([3], see also [31]). \( f_o \notin I_+(\Psi_1)_o \).

2.3. Some lemmas about functionals on \( A^2((\Psi < 0), e^{-\varphi_0}) \).

The following two lemmas will be used in the proof of Theorem 1.1. For the convenience of readers, we recall the proofs.

Lemma 2.7. Let \( D \) be a domain in \( \mathbb{C}^n \), and let \( \varphi_0 \) be a plurisubharmonic function on \( D \). Let \( \{f_j\} \) be a sequence in \( A^2(D, e^{-\varphi_0}) \), such that \( \int_D |f_j|^2 e^{-\varphi_0} \) is uniformly bounded for any \( j \in \mathbb{N}_+ \). Assume that \( f_j \) compactly converges to \( f_0 \in A^2(D, e^{-\varphi_0}) \). Then for any \( \xi \in A^2(D, e^{-\varphi_0})^* \),
\[
\lim_{j \to +\infty} \xi \cdot f_j = \xi \cdot f_0.
\]

Proof. For any \( f \in A^2(D, e^{-\varphi_0}) \), denote that \( \|f\|^2 := \int_D |f|^2 e^{-\varphi_0} \). Let \( \{f_{k_j}\} \) be any subsequence of \( \{f_j\} \). Since \( A^2(D, e^{-\varphi_0}) \) is a Hilbert space, and \( \|f_{k_j}\|^2 \) is uniformly bounded, there exists a subsequence of \( \{f_{k_j}\} \) (denoted by \( \{f_{k_{j_l}}\} \)) weakly convergent to some \( \tilde{f} \in A^2(D, e^{-\varphi_0}) \). Note that for any \( z \in D \), the functional \( e_z \in A^2(D, e^{-\varphi_0})^* \), where
\[
e_z : A^2(D, e^{-\varphi_0}) \to \mathbb{C}
\]
\[
f \mapsto e_z(f).
\]
Then we have
\[
f_0(z) = \lim_{j \to +\infty} e_z \cdot f_j = \lim_{j \to +\infty} e_z \cdot f_{k_{j_l}} = e_z \cdot \tilde{f} = \tilde{f}(z), \ \forall z \in D,
\]
thus \( f_0 = \tilde{f} \). It means that \( \{f_{k_{j_l}}\} \) has a subsequence weakly convergent to \( f_0 \). Since \( \{f_{k_{j_l}}\} \) is an arbitrary subsequence of \( \{f_j\} \), we get that \( \{f_j\} \) weakly converges to \( f_0 \).

In other words, for any \( \xi \in A^2(D, e^{-\varphi_0})^* \),
\[
\lim_{j \to +\infty} \xi \cdot f_j = \xi \cdot f_0.
\]

Let \( \Omega := D \times \omega \subset \mathbb{C}^{n+1} \), where \( D \) is a domain in \( \mathbb{C}^n \), \( \omega \) is a domain in \( \mathbb{C} \). Denote the coordinate on \( \Omega \) by \( (z, \tau) \), where \( z \in D \), \( \tau \in \omega \). Let \( \varphi_0 \) be a plurisubharmonic function on \( D \). Let \( f \) be a holomorphic function on \( \Omega \), such that
\[
\int_{\Omega} |f(z, \tau)|^2 e^{-\varphi_0(z)} < +\infty.
\]
Denote \( f_{\tau} := f|_{D \times \{\tau\}} \).

Lemma 2.8. For any \( \xi \in A^2(D, e^{-\varphi_0})^* \), \( \xi \cdot f_{\tau} \) is holomorphic with respect to \( \tau \in \omega \).
Proof. We only need to prove that $h(\tau) := \xi \cdot f_\tau$ is holomorphic near any $\tau_0 \in \omega$. Since $\tau_0 \in \omega$, there exists $r > 0$ such that $\Delta(\tau_0, 2r) \subset \subset \omega$. Then for any $\tau \in \Delta(\tau_0, r)$, according to sub-mean value inequality of subharmonic functions, we have

$$\int_D |f_\tau(z)|^2 e^{-\varphi_0(z)} \leq \frac{1}{\pi r^2} \int_{D \times \Delta(t, r)} |f(z, \tau)|^2 e^{-\varphi_0(z)} \leq \frac{1}{\pi r^2} \int_\Omega |f|^2 e^{-\varphi_0} < +\infty,$$

which implies that $f_\tau \in A^2(D, e^{-\varphi_0})$ and there exists $M > 0$ such that $\int_D |f_\tau|^2 e^{-\varphi_0} \leq M$ for any $\tau \in \Delta(\tau_0, r)$.

Fix $z_0 \in D$. According to Lemma 2.9 in Appendix, we can find a sequence $\{\xi_k\} \subset \ell_0 \subset A^2(D, e^{-\varphi_0})^*$, such that

$$\lim_{k \to +\infty} \|\xi_k - \xi\|_{A^2(D, e^{-\varphi_0})^*} = 0,$$

where

$$\ell_0 := \{ \eta = (\eta_\alpha)_{\alpha \in \mathbb{N}^+} : \exists k \in \mathbb{N}, \text{ such that } \eta_\alpha = 0, \forall |\alpha| \geq k \}.$$

Here for any $\eta = (\eta_\alpha)_{\alpha \in \mathbb{N}^+} \in \ell_0$ and $f \in A^2(D, e^{-\varphi_0})$, define that

$$\eta \cdot f := \sum_{\alpha \in \mathbb{N}^+} \eta_\alpha \frac{f^{(\alpha)}(z_0)}{\alpha!}.$$

Note that for any $\xi_k \in \ell_0$, $\xi_k \cdot f_\tau$ can be written as

$$\xi_k \cdot f_\tau = \sum_{\alpha \in \mathbb{N}^+, |\alpha| \leq l_k} c_{\alpha, k} \frac{\partial^\alpha f(z, \tau)}{\partial z^\alpha}(z_0, \tau),$$

where $l_k$ is a finite integer, and $c_{\alpha, k} \in \mathbb{C}$ are constants. It is clear that $h_k(\tau) := \xi_k \cdot f_\tau$ is holomorphic with respect to $\tau \in \omega$ for any $k \in \mathbb{N}^+$, since any $\frac{\partial^\alpha f(z, \tau)}{\partial z^\alpha}(z_0, \tau)$ is holomorphic with respect to $\tau$ for any $\alpha \in \mathbb{N}^+$. Note that for any $\tau \in \Delta(\tau_0, r)$, we have

$$|h_k(\tau) - h(\tau)|^2 = |(\xi_k - \xi) \cdot f_\tau|^2 \leq 2 \|\xi_k - \xi\|^2_{A^2(D, e^{-\varphi_0})^*} \int_D |f_\tau|^2 e^{-\varphi_0} \leq M \|\xi_k - \xi\|^2_{A^2(D, e^{-\varphi_0})^*},$$

which means that $h_k$ uniformly converges to $h$ on $\Delta(\tau_0, r)$. According to Weierstrass theorem, we know that $h$ is holomorphic on $\Delta(\tau_0, r)$, i.e. near $\tau_0$. Then we get that $\xi \cdot f_\tau$ is holomorphic with respect to $\tau \in \omega$. \qed

2.4. Some properties of $K_{\xi, \Psi, \lambda}(t)$.

In this section, we prove some properties of the Bergman kernel $K_{\xi, \Psi, \lambda}(t)$.

Let $\xi \in A^2(\{\Psi < 0\}, e^{-\varphi_0})^* \setminus \{0\}$.

Lemma 2.9. For any $t \in [0, +\infty)$, if $K_{\xi, \Psi, \lambda}(t) \in (0, +\infty)$, then there exists $\tilde{f} \in A^2(\{\Psi < 0\}, e^{-\varphi_0})$, such that

$$K_{\xi, \Psi, \lambda}(t) = \frac{|\xi \cdot \tilde{f}|^2}{\|\tilde{f}\|^2_{\lambda, t}}.$$
Lemma 2.10. Then the following lemma holds. We get that $K_{\xi,\Psi,\lambda}(t)$, such that $\|f\|_{\lambda,t} = 1$, and $\lim_{t \to +\infty} |\xi \cdot f|^2 = K_{\xi,\Psi,\lambda}(t)$. Then $\int_{\Psi < 0} |f_j|^2 e^{-\varphi_0}$ is uniformly bounded. Following from Montel's theorem, we can get a subsequence of $\{f_j\}$ compactly convergent to a holomorphic function $\tilde{f}$ on $\{\Psi < 0\}$. According to Fatou's lemma, we have $\|\tilde{f}\|_{\lambda,t} \leq 1$, and according to Lemma 2.7, we have $|\xi \cdot \tilde{f}|^2 = K_{\xi,\Psi,\lambda}(t)$, thus $K_{\xi,\Psi,\lambda}(t) \leq \frac{|\xi \cdot \tilde{f}|^2}{\|\tilde{f}\|_{\lambda,t}^2}$. Note that $\|\tilde{f}\|_{\lambda,t} \leq 1$ implies $\tilde{f} \in A^2(\{\Psi < 0\}, e^{-\varphi_0})$, which means $K_{\xi,\Psi,\lambda}(t) \geq \frac{|\xi \cdot \tilde{f}|^2}{\|\tilde{f}\|_{\lambda,t}^2}$.

We get that $K_{\xi,\Psi,\lambda}(t) = \frac{|\xi \cdot \tilde{f}|^2}{\|\tilde{f}\|_{\lambda,t}^2}$.

Let $J$ be an $\mathcal{O}_\alpha$-submodule of $I(\varphi_0)_\alpha$ such that $J \supset I(\Psi + \varphi_0)_\alpha$, and let $f \in A^2(\{\Psi < 0\}, e^{-\varphi_0})$, such that $f_0 \notin J$. Recall the minimal $L^2$ integral (\cite{[3]} $\text{[31]}$)

$$C(\Psi, \varphi_0, J, f) := \inf \left\{ \int_{\{\Psi < 0\}} |\tilde{f}|^2 e^{-\varphi_0} : (\tilde{f} - f)_0 \in J & \tilde{f} \in \mathcal{O}(\{\Psi < 0\}) \right\}.$$

Then the following lemma holds.

Lemma 2.10. Assume that $C(\Psi, \varphi_0, J, f) \in (0, +\infty)$, then

$$C(\Psi, \varphi_0, J, f) = \sup_{\xi \in A^2(\{\Psi < 0\}, e^{-\varphi_0}) \setminus \{0\} : \xi \perp A^2(\{\Psi < 0\}, e^{-\varphi_0}) \cap J = 0} \frac{|\xi \cdot f|^2}{K_{\xi,\Psi,\lambda}(0)}. \quad (2.1)$$

Proof. Note that $\xi \cdot \tilde{f} = \xi \cdot f$ for any $\tilde{f} \in A^2(\{\Psi < 0\}, e^{-\varphi_0})$ with $(\tilde{f} - f)_0 \in J$ and $\xi \in A^2(\{\Psi < 0\}, e^{-\varphi_0})^*$ satisfying $\xi \perp A^2(\{\Psi < 0\}, e^{-\varphi_0}) \cap J = 0$. Then we have

$$K_{\xi,\Psi,\lambda}(0) = \sup_{h \in A^2(\{\Psi < 0\}, e^{-\varphi_0})} \frac{|\xi \cdot h|^2}{\int_{\{\Psi < 0\}} |h|^2 e^{-\varphi_0}} \geq \sup_{\tilde{f} \in A^2(\{\Psi < 0\}, e^{-\varphi_0}) : (\tilde{f} - f)_0 \in J} \frac{|\xi \cdot \tilde{f}|^2}{\int_{\{\Psi < 0\}} |\tilde{f}|^2 e^{-\varphi_0}} = \sup_{\tilde{f} \in A^2(\{\Psi < 0\}, e^{-\varphi_0}) : (\tilde{f} - f)_0 \in J} \frac{|\xi \cdot \tilde{f}|^2}{\int_{\{\Psi < 0\}} |\tilde{f}|^2 e^{-\varphi_0}}.$$

Thus we get that

$$C(\Psi, \varphi_0, J, f) = \sup_{\xi \in A^2(\{\Psi < 0\}, e^{-\varphi_0}) \setminus \{0\} : \xi \perp A^2(\{\Psi < 0\}, e^{-\varphi_0}) \cap J = 0} \frac{|\xi \cdot f|^2}{K_{\xi,\Psi,\lambda}(0)} \leq \inf_{\tilde{f} \in A^2(\{\Psi < 0\}, e^{-\varphi_0}) : (\tilde{f} - f)_0 \in J} \int_{\{\Psi < 0\}} |\tilde{f}|^2 e^{-\varphi_0} = C(\Psi, \varphi_0, J, f).$$

Since $A^2(\{\Psi < 0\}, e^{-\varphi_0})$ is a Hilbert space, and $A^2(\{\Psi < 0\}, e^{-\varphi_0}) \cap J$ is a closed proper subspace of $A^2(\{\Psi < 0\}, e^{-\varphi_0})$ (using Lemma 2.4), there exists a closed subspace $H$ of $A^2(\{\Psi < 0\}, e^{-\varphi_0})$ such that $H = (A^2(\{\Psi < 0\}, e^{-\varphi_0}) \cap J)^\perp \neq \{0\}$. Then for $f \in A^2(\{\Psi < 0\}, e^{-\varphi_0})$, we can make the decomposition $f = f_J + f_H$. Theorem 2.11
such that \( f_j \in A^2(\{ \Psi < 0 \}, e^{-\varphi_0}) \cap J \), and \( f_H \in H \). Note that the linear functional \( \xi_f \) defined as follows:

\[
\xi_f \cdot g := \int_{\{ \Psi < 0 \}} g \overline{f} e^{-\varphi_0}, \quad \forall g \in A^2(\{ \Psi < 0 \}, e^{-\varphi_0}),
\]

satisfies that \( \xi_f \in A^2(\{ \Psi < 0 \}, e^{-\varphi_0})^\ast \setminus \{ 0 \} \) and \( \xi_f|_{A^2(\{ \Psi < 0 \}, e^{-\varphi_0}) \cap J} \equiv 0 \). Then we have

\[
\sup_{\xi \in A^2(\{ \Psi < 0 \}, e^{-\varphi_0})^\ast \setminus \{ 0 \}} \frac{|\xi \cdot f|^2}{K_{\xi,\Psi,\lambda}(0)} \geq \frac{|\xi_f \cdot f|^2}{K_{\xi,\Psi,\lambda}(0)}.
\]

Besides, we can know that

\[
K_{\xi,\Psi,\lambda}(0) = \sup_{h \in A^2(\{ \Psi < 0 \}, e^{-\varphi_0})} \frac{\int_{\{ \Psi < 0 \}} |h \overline{f} e^{-\varphi_0}|^2}{\int_{\{ \Psi < 0 \}} |h|^2 e^{-\varphi_0}} \leq \int_{\{ \Psi < 0 \}} |f_H|^2 e^{-\varphi_0},
\]

and

\[
\xi_f \cdot f = \xi_f \cdot (f_j + f_H) = \xi_f \cdot f_H = \int_{\{ \Psi < 0 \}} |f_H|^2 e^{-\varphi_0}.
\]

Then we have

\[
\frac{|\xi_f \cdot f|^2}{K_{\xi,\Psi,\lambda}(0)} \geq \int_{\{ \Psi < 0 \}} |f_H|^2 e^{-\varphi_0} \geq C(\Psi, \varphi_0, J, f),
\]

which implies that

\[
\sup_{\xi \in A^2(\{ \Psi < 0 \}, e^{-\varphi_0})^\ast \setminus \{ 0 \}} \frac{|\xi \cdot f|^2}{K_{\xi,\Psi,\lambda}(0)} \geq C(\Psi, \varphi_0, J, f).
\]

Lemma 2.10 is proved. \( \square \)

### 3. Proof of Theorem 1.1

We prove Theorem 1.1 by using Lemma 2.1 (optimal \( L^2 \) extension theorem).

**Proof of Theorem 1.1.** Denote that \( \Omega := \{ \Psi < 0 \} \times E = \{ \Psi < 0 \} \times \{ w \in \mathbb{C} : \text{Re} \; w \geq 0 \} \), and the coordinate on \( \Omega \) is \( (z, w) \), where \( z \in \{ \Psi < 0 \} \subset \mathbb{C}^n \) and \( w \in E = \{ w \in \mathbb{C} : \text{Re} \; w \geq 0 \} \). Note that \( D \setminus \{ F = 0 \} \) is a pseudoconvex domain in \( \mathbb{C}^n \), and \( \{ \Psi < 0 \} = \{ \psi + 2 \log |1/F| < 0 \} \) on \( D \setminus \{ F = 0 \} \). Then \( \{ \Psi < 0 \} \) is a pseudoconvex domain in \( \mathbb{C}^n \), and \( \Psi \) is a plurisubharmonic function on \( \{ \Psi < 0 \} \). We get that \( \Omega \) is a pseudoconvex domain in \( \mathbb{C}^{n+1} \). For any \( (z, w) \in \Omega \), let

\[
\tilde{\Psi}(z, w) := \varphi_0(z) + \Psi_{\lambda, \text{Re} \; w} = \varphi_0(z) + \lambda \max \{ \Psi(z) + \text{Re} \; w, 0 \}.
\]

Then \( \tilde{\Psi} \) is a plurisubharmonic function on \( \Omega \).

Denote that

\[
K(w) := K_{\xi,\Psi,\lambda}(\text{Re} \; w)
\]

for any \( w \in E \). We prove that \( \log K(w) \) is a subharmonic function with respect to \( w \in E \).

Firstly we prove that \( \log K(w) \) is upper semicontinuous. Let \( w_j \in E \) such that \( \lim_{j \to +\infty} w_j = w_0 \in E \). We assume that \( \{ w_{k_j} \} \) is the subsequence of \( \{ w_j \} \) such that

\[
\lim_{j \to +\infty} K(w_{k_j}) = \lim \sup_{j \to +\infty} K(w_j).
\]
By Lemma 2.9, there exists a sequence of holomorphic functions \( \{f_j\} \) on \( \{\Psi < 0\} \) such that \( f_j \in A^2(\{\Psi < 0\}, e^{-\varphi_0}) \), \( \|f_j\|_{\lambda, \text{Re } w_j} = 1 \), and \( |\xi \cdot f_j|^2 = K(w_j) \), for any \( j \in \mathbb{N}_+ \). Since \( \{w_j\} \) is bounded in \( \mathbb{C} \), there exists some \( s_0 > 0 \), such that \( \text{Re } w_j < s_0 \) for any \( j \), which implies that

\[
\int_{\{\Psi < 0\}} |f_j|^2 e^{-\varphi_0} \leq e^{K_{s_0}} \|f_j\|_{\lambda, \text{Re } w_j}^2 = e^{K_{s_0}}, \quad \forall j \in \mathbb{N}_+.
\]

Then following from Montel’s theorem, we can get a subsequence of \( \{f_{k_j}\} \) (denoted by \( \{f_k\} \) itself) compactly convergent to a holomorphic function \( f_0 \) on \( \{\Psi < 0\} \). According to Fatou’s lemma, we have

\[
\|f_0\|_{\lambda, \text{Re } w_0} = \int_{\{\Psi < 0\}} |f_0(z)|^2 e^{-\varphi_0(z)} e^{-\varphi_0(z) - \lambda \max\{\Psi(z) + \text{Re } w_0, 0\}}
\]

\[
= \int_{\{\Psi < 0\}} \lim_{j \to +\infty} |f_{k_j}(z)|^2 e^{-\varphi_0(z) - \lambda \max\{\Psi(z) + \text{Re } w_{k_j}, 0\}}
\]

\[
\leq \liminf_{j \to +\infty} \int_{\{\Psi < 0\}} |f_{k_j}(z)|^2 e^{-\varphi_0(z) - \lambda \max\{\Psi(z) + \text{Re } w_{k_j}, 0\}}
\]

\[
= \liminf_{j \to +\infty} \|f_{k_j}\|_{\lambda, \text{Re } w_j} = 1.
\]

Then \( \int_{\{\Psi < 0\}} |f_0|^2 e^{-\varphi_0} \leq e^{K_{s_0}} \|f_0\|^2_{\lambda, \text{Re } w_0} \leq e^{K_{s_0}} < +\infty \), which implies that \( f_0 \in A^2(\{\Psi < 0\}, e^{-\varphi_0}) \). Lemma 2.7 shows that \( |\xi \cdot f_0|^2 = \lim_{j \to +\infty} |\xi \cdot f_{k_j}|^2 = \limsup_{j \to +\infty} K(w_j) \). Thus

\[
K(w_0) \geq \frac{|\xi \cdot f_0|^2}{\|f_0\|^2_{\lambda, \text{Re } w_0}} \geq \limsup_{j \to +\infty} K(w_j),
\]

which means

\[
\log K(w_0) \geq \limsup_{j \to +\infty} \log K(w_j).
\]

Then we get that \( \log K(w) \) is upper semicontinuous with respect to \( w \in E \).

Secondly we prove that \( \log K(w) \) satisfies the sub-mean value inequality.

Let \( \Delta(w_0, r) \subset E \) be the disc centered at \( w_0 \) with radius \( r \), and let \( \Omega' := \{\Psi < 0\} \times \Delta(w_0, r) \subset \mathbb{C}^{n+1} \). Let \( f_0 \in A^2(\{\Psi < 0\}, e^{-\varphi_0}) \) such that

\[
K(w_0) = \frac{|\xi \cdot f_0|^2}{\|f_0\|^2_{\lambda, \text{Re } w_0}}.
\]

by Lemma 2.9.

Note that \( \Omega' \) is a pseudoconvex domain in \( \mathbb{C}^{n+1} \), and \( \tilde{\Psi}(z, w) = \varphi_0(z) + \Psi_{\lambda, \text{Re } w} = \varphi_0(z) + \lambda \max\{\Psi(z) + \text{Re } w, 0\} \) is a plurisubharmonic function on \( \Omega' \). Using Lemma 2.1 (optimal \( L^2 \) extension theorem), we can get a holomorphic function \( \tilde{f} \) on \( \Omega' \) such that \( \tilde{f}(z, w_0) = f_0(z) \) for any \( z \in \{\Psi < 0\} \), and

\[
\frac{1}{\pi r^2} \int_{\Omega'} |\tilde{f}(z, w)|^2 e^{-\tilde{\Psi}(z, w)} \leq \int_{\{\Psi < 0\}} |f_0(z)|^2 e^{-\Psi(z, w_0)}.
\]
Denote that $\tilde{f}_w(z) = \tilde{f}(z, w) = \tilde{f}|_{\{\Psi < 0\} \times \{w\}}$. Since the function $y = \log x$ is concave, according to Jensen’s inequality and inequality (3.1), we have
\[
\log \|f_0\|_{\lambda, \text{Re } w_0}^2 = \log \left( \int_{\{\Psi < 0\}} |f_0(z)|^2 e^{-\Psi(z, w_0)} \right) \\
\geq \log \left( \frac{1}{\pi R^2} \int_{\Omega r} |\tilde{f}(z, w)|^2 e^{-\Psi(z, w)} \right) \\
= \log \left( \frac{1}{\pi R^2} \int_{\Delta(w_0, r)} \int_{\{\Psi < 0\} \times \{w\}} |\tilde{f}_w(z)|^2 e^{-\tilde{\Psi}(z, w)} \right) \\
\geq \frac{1}{\pi R^2} \int_{\Delta(w_0, r)} \log \left( \|\tilde{f}_w\|_{\lambda, \text{Re } w}^2 \right) \\
\geq \frac{1}{\pi R^2} \int_{\Delta(w_0, r)} \left( \log |\xi \cdot \tilde{f}_w|^2 - \log K(w) \right).
\]
(3.2)

It follows from Lemma 2.8 that $\xi \cdot \tilde{f}_w$ is holomorphic with respect to $w$, which implies that $\log |\xi \cdot \tilde{f}_w|^2$ is subharmonic with respect to $w$. Combining with $\tilde{f}_w = f_0$, we have
\[
\log |\xi \cdot f_0|^2 \leq \frac{1}{\pi R^2} \int_{\Delta(w_0, r)} \log |\xi \cdot \tilde{f}_w|^2.
\]
Combining with inequality (3.2), we get
\[
\log \|f_0\|_{\lambda, \text{Re } w_0}^2 \geq \log |\xi \cdot f_0|^2 - \frac{1}{\pi R^2} \int_{\Delta(w_0, r)} \log K(w),
\]
which means
\[
\log K(w_0) \leq \frac{1}{\pi R^2} \int_{\Delta(w_0, r)} \log K(w).
\]

Since $\log K(w)$ is upper semicontinuous and satisfies the sub-mean value inequality, we know that $\log K(w)$ is a subharmonic function on the interior of $E$. In addition, since $\log K(w)$ is upper semicontinuous near $\{0\} + \sqrt{-1}R$, and $\log K(w)$ is only dependent on the real part of $w$, we know that $\log K(w)$ is a subharmonic function on $E$. \hfill \Box

4. Proofs of Theorem 1.2 and Remark 1.3

In this section, we give the proofs of Theorem 1.2 and Remark 1.3. We need the following lemma.

Lemma 4.1 (see [13]). Let $D = I + \sqrt{-1}R := \{ z = x + \sqrt{-1}y \in \mathbb{C} : x \in I, y \in \mathbb{R} \}$ be a subset of $\mathbb{C}$, where $I$ is an interval in $\mathbb{R}$. Let $\phi(z)$ be a subharmonic function on $D$ which is only dependent on $x = \text{Re } z$. Then $\phi(x) := \phi(x + \sqrt{-1}R)$ is a convex function with respect to $x \in I$.

Proof of Theorem 1.2. It follows from Theorem 1.1 that $K_{(\varphi_0)}^{\xi, \Psi, \lambda}(\text{Re } w)$ is subharmonic with respect to $w \in [0, +\infty) + \sqrt{-1}R$. Note that $\log K_{(\varphi_0)}^{\xi, \Psi, \lambda}(\text{Re } w)$ is only dependent on $\text{Re } w$, then following from Lemma 4.1 we get that $\log K_{(\varphi_0)}^{\xi, \Psi, \lambda}(t) = \log K_{(\varphi_0)}^{\xi, \Psi, \lambda}(t + \sqrt{-1}R)$ is convex with respect to $t \in [0, +\infty)$, which implies that $-\log K_{(\varphi_0)}^{\xi, \Psi, \lambda}(\text{Re } w) + t$ is concave with respect to $t \in [0, +\infty)$. Then for any $\xi \in A^2(\{\Psi < 0\}, e^{-\varphi_0})^* \cap \partial_{\varphi_0}(\{\Psi < 0\}, e^{-\varphi_0}) \equiv 0$, to prove that $\log -K_{(\varphi_0)}^{\xi, \Psi, \lambda}(t) + t$
Note that according to inequality (4.4), we can get that \( \tilde{\xi} \) is increasing, we only need to prove that \( \log -K_{\tilde{\xi},\Psi,J}(t) + t \) has a lower bound on \([0, +\infty)\).

Using Lemma 2.9 we obtain that there exists \( f_t \in A^2(\{ \Psi < 0 \}, e^{-\varphi_0}) \) for any \( t \in [0, +\infty) \), such that \( \xi \cdot f_t = 1 \) and

\[
K_{\tilde{\xi},\Psi,J}(t) = \frac{1}{\|f_t\|_{\lambda,t}^2}. \tag{4.1}
\]

In addition, according to Lemma 2.2 there exists a holomorphic function \( \tilde{F} \) on \( D \) such that

\[
\int_D |\tilde{F} - (1 - b_t(\Psi))f_t F^2|e^{-\varphi_0+\psi_t(\Psi)-\Psi} \leq C \int_D \int_{\{t-1 < \Psi < -t\}} |f_t|^2 e^{-\varphi_0-\Psi}, \tag{4.2}
\]

where

\[
\varphi = \varphi_0 + 2 \max\{\psi, 2 \log |F|\},
\]

and \( C \) is a positive constant. Then it follows from inequality (4.2) that

\[
\int_{\{\Psi < 0\}} |\tilde{F} - (1 - b_t(\Psi))f_t F^2|e^{-\varphi_0+\psi_t(\Psi)-\Psi} \leq \int_D |\tilde{F} - (1 - b_t(\Psi))f_t F^2|e^{-\varphi_0+\psi_t(\Psi)-\Psi} \leq C \int_D \|\Psi\|_{(-t-1, \Psi < -t)} |f_t|^2 e^{-\varphi_0-\Psi} \leq Ce^{t+1} \int_{\{\Psi < -t\}} |f_t|^2 e^{-\varphi_0}. \tag{4.3}
\]

Denote that \( \tilde{F}_t := \tilde{F}/F^2 \) on \( \{ \Psi < 0 \} \), then \( \tilde{F}_t \) is a holomorphic function on \( \{ \Psi < 0 \} \). Note that \( |F|^4 e^{-\varphi} = e^{-\varphi_0} \) on \( \{ \Psi < 0 \} \). Then inequality (4.3) implies that

\[
\int_{\{\Psi < 0\}} |\tilde{F}_t - (1 - b_t(\Psi))f_t|^2 e^{-\psi_0+\psi_t(\Psi)-\Psi} \leq Ce^{t+1} \int_{\{\Psi < -t\}} |f_t|^2 e^{-\varphi_0} < +\infty. \tag{4.4}
\]

According to inequality (4.4), we can get that \( (\tilde{F}_t - f_t)_0 \in I(\Psi + \varphi_0) \subset J \), which means that \( \xi \cdot \tilde{F}_t = \xi \cdot f_t = 1 \). Besides, since \( v_t(\Psi) \geq \Psi \), we have

\[
\left( \int_{\{\Psi < 0\}} |\tilde{F}_t - (1 - b_t(\Psi))f_t|^2 e^{-\psi_0+\psi_t(\Psi)-\Psi} \right)^{1/2} \geq \left( \int_{\{\Psi < 0\}} |\tilde{F}_t - (1 - b_t(\Psi))f_t|^2 e^{-\varphi_0} \right)^{1/2} \geq \left( \int_{\{\Psi < 0\}} |\tilde{F}_t|^2 e^{-\varphi_0} \right)^{1/2} - \left( \int_{\{\Psi < 0\}} |(1 - b_t(\Psi))f_t|^2 e^{-\varphi_0} \right)^{1/2} \geq \left( \int_{\{\Psi < 0\}} |\tilde{F}_t|^2 e^{-\varphi_0} \right)^{1/2} - \left( \int_{\{\Psi < -t\}} |f_t|^2 e^{-\varphi_0} \right)^{1/2}. \]
Combining with inequality (4.4), we have
\[ \int_{\{\Psi<0\}} |\tilde{F}_t|^2 e^{-\varphi_0} \leq 2 \int_{\{\Psi<0\}} |\tilde{F}_t - (1 - b_t(\Psi))f_t|^2 e^{-\varphi_0 + v_t(\Psi) - \Psi} + 2 \int_{\{\Psi<-t\}} |f_t|^2 e^{-\varphi_0} \]
\[ \leq 2(Ce^{t+1} + 1) \int_{\{\Psi<-t\}} |f_t|^2 e^{-\varphi_0}. \]

Note that
\[ \|f_t\|_{\lambda,t}^2 = \int_{\{\Psi<0\}} |f_t|^2 e^{-\varphi_0 - \Psi - \lambda t} = \int_{\{\Psi<-t\}} |f_t|^2 e^{-\varphi_0} + \int_{\{0<\Psi<-t\}} |f_t|^2 e^{-\varphi_0 - \lambda(\Psi + t)} \]
\[ \geq \int_{\{\Psi<-t\}} |f_t|^2 e^{-\varphi_0}. \]

Then we have
\[ \int_{\{\Psi<0\}} |\tilde{F}_t|^2 e^{-\varphi_0} \leq 2(Ce^{t+1} + 1)\|f_t\|_{\lambda,t}^2 = C_1 \frac{e^t}{K_{\xi,\Psi,\lambda}(t)}, \]
where \( C_1 := 2(\epsilon C + 1) \) is a positive constant. In addition, \( \xi \cdot \tilde{F}_t = 1 \) implies that
\[ \int_{\{\Psi<0\}} |\tilde{F}_t|^2 e^{-\varphi_0} = \|\tilde{F}_t\|_{\lambda,0}^2 \geq (K_{\xi,\Psi,\lambda}(0))^{-1}. \]

Then we get that
\[ -\log K_{\xi,\Psi,\lambda}(t) + t \geq C_2, \forall t \in [0, +\infty), \]
where \( C_2 := \log(C_1^{-1} K_{\xi,\Psi,\lambda}(0)) \) is a finite constant. Since \( -\log K_{\xi,\Psi,\lambda}(t) + t \) is concave, we get that \( -\log K_{\xi,\Psi,\lambda}(t) + t \) is increasing with respect to \( t \in [0, +\infty) \).

In the following we give the proof of Remark 1.3.

Proof of Remark 1.3. Denote that \( K(t) := K_{\xi,\Psi,\lambda}(t) \) for any \( t \in [0, +\infty) \). According to Theorem 1.1 and Lemma 4.1 we can know that \( \log K(t) - kt \) is convex on \([0, +\infty)\). Combining with that \( e^{-kt} K(t) \) is increasing and not a constant function on \([0, T]\), which implies that \( \log K(t) - kt = \log(e^{-kt} K(t)) \) is increasing and not a constant function on \([0, T]\), we have that \( \log K(t) - kt \) is strictly increasing on \([T, +\infty)\). Then \( e^{-kt} K_{\xi,\Psi,\lambda}(t) = \exp(\log K(t) - kt) \) is strictly increasing on \([T, +\infty)\). \( \square \)

5. Proof of Corollary 1.4

In this section, we give the proof of Corollary 1.4

Proof of Corollary 1.4. For any \( p \in (1, 2), \lambda > 0, \) let \( \xi \in A^2(\{\Psi_1 < 0\}) \setminus \{0\} \), such that \( \xi|_{A^2(\{\Psi_1<0\}) \cap J_p} \equiv 0 \), where \( J_p := I(p\Psi_1) \). Denote that
\[ K_{\xi,p,\lambda}(t) := \sup_{f \in A^2(\{\Psi_1<0\})} \|\xi \cdot \tilde{f}\|_{\lambda,t}^2 / \|\tilde{f}\|_{p,\lambda,t}^2. \]
where
\[ \|f\|_{p,\lambda,t} := \left( \int_{\{\psi_1 < 0\}} |f|^2 e^{-\lambda \max\{p\psi_1 + t, 0\}} \right)^{1/2}, \]
and \( t \in [0, +\infty) \). Note that
\[ p\psi_1 = \min\{(2pc^2F(\psi)\psi + (4 - 2p) \log |F|) - 2 \log |F|^2, 0\} \]
and Lemma 2.6 shows \( f_o \notin J_p \), which implies that \( A^2(\{\psi_1 < 0\}) \cap J_p \) is a proper subspace of \( A^2(\{\psi_1 < 0\}) \), and \( K_{\xi,\lambda}(0) \in (0, +\infty) \). Theorem 1.2 tells us that
\[ -\log K_{\xi,\lambda}(t) + t \text{ is increasing with respect to } t \in [0, +\infty), \]
which implies that
\[ -\log K_{\xi,\lambda}(t) + t \geq -\log K_{\xi,\lambda}(0), \forall t \in [0, +\infty). \tag{5.1} \]
Since \( f \in A^2(\{\psi_1 < 0\}) \), following from inequality \( 5.1 \), we get that
\[ \|f\|_{p,\lambda,t}^2 \geq \frac{|\xi \cdot f|^2}{K_{\xi,\lambda}(t)} \geq e^{-t}\frac{|\xi \cdot f|^2}{K_{\xi,\lambda}(0)}, \forall t \in [0, +\infty). \]
In addition, since \( f_o \notin J_p \), according to Lemma 2.10 we have
\[ \|f\|_{p,\lambda,t}^2 \geq \sup_{\xi \in A^2(\{\psi_1 < 0\}) \setminus \{0\} \atop \xi I_{\lambda^2(\{\psi_1 < 0\})}) \cap J_p \equiv 0} e^{-t}\frac{|\xi \cdot f|^2}{K_{\xi,\lambda}(0)} \tag{5.2} \]
\[ = e^{-t}C(p\psi_1, 0, J_p, f), \forall t \in [0, +\infty). \]
Note that for any \( t \in [0, +\infty) \),
\[ \|f\|_{p,\lambda,t}^2 = \int_{\{\psi_1 < -t\}} |f|^2 + \int_{\{\psi_1 \geq -t\}} |f|^2 e^{-\lambda(p\psi_1 + t)}. \tag{5.3} \]
Since for any \( \lambda > 0 \),
\[ \int_{\{\psi_1 \geq -t\}} |f|^2 e^{-\lambda(p\psi_1 + t)} \leq \int_{\{\psi_1 \geq -t\}} |f|^2 < +\infty, \]
and \( \lim_{\lambda \to +\infty} e^{-\lambda(p\psi_1 + t)} = 0 \) on \( \{0 > p\psi_1 \geq -t\} \), according to Lebesgue’s dominated convergence theorem, we have
\[ \lim_{\lambda \to +\infty} \int_{\{\psi_1 \geq -t\}} |f|^2 e^{-\lambda(p\psi_1 + t)} = 0. \]
Then equality \( 5.3 \) implies
\[ \lim_{\lambda \to +\infty} \|f\|_{p,\lambda,t}^2 = \int_{\{\psi_1 < -t\}} |f|^2, \forall t \in [0, +\infty). \tag{5.4} \]
Letting \( \lambda \to +\infty \) in inequality \( 5.2 \), we get that for any \( t \in [0, +\infty) \),
\[ \int_{\{\psi_1 < -t\}} |f|^2 \geq e^{-t}C(p\psi_1, 0, J_p, f). \tag{5.5} \]
Note that \( \{p\psi_1 < 0\} = \{\psi_1 < 0\} \) and \( J_p \subset I_+(\psi_1)_o \) for any \( p \in (1, 2) \). Then we have
\[ C(p\psi_1, 0, J_p, f) \geq C(\psi_1, 0, I_+(\psi_1)_o, f), \forall p \in (1, 2). \]
Since \( f_{\{ q, \varphi < 0 \}} \) \( |f|^2 < +\infty \), it follows from Lebesgue’s dominated convergence theorem and inequality (5.5) that
\[
\int_{\{ \Psi < r \}} |f|^2 = \lim_{p \to r + 0} \int_{\{ \Psi < r \}} |f|^2 \\
\geq \lim_{p \to r + 0} e^{-t} C(p\Psi_1, 0, J_p, f) \\
\geq e^{-t} C(\Psi_1, 0, I_+(\Psi_1)_0, f), \quad \forall t \in [0, +\infty).
\]

Let \( r = e^{-t/2} \), and we get that
\[
\frac{1}{r^2} \int_{\{ \Psi < 2 \log r \}} |f|^2 \geq C(\Psi_1, 0, I_+(\Psi_1)_0, f), \quad \forall r \in (0, 1].
\] (5.7)

Note that \( C(\Psi_1, 0, I_+(\Psi_1)_0, f) = G(0; \Psi_1, I_+(\Psi_1)_0, f) > 0 \), thus Corollary 1.7 holds.

6. Proof of Corollary 1.7

In this section, we give the proof of Corollary 1.7.

Proof of Corollary 1.7 Let \( \Psi_q := q\Psi \) for any \( q > 2a^+_f(\Psi; \varphi_0) \geq 1 \). Note that
\[
q\Psi = \min\{2q\Psi + (2[q] - 2q) \log |F| - 2 \log |F[\Psi]|, 0\},
\]
where \([q] = \min\{m \in \mathbb{Z} : m \geq q\} \). By the definition of \( a^+_f(\Psi; \varphi_0) \), we have \( f_o \notin I(2q\Psi + \varphi_0)_a \) for any \( q > 2a^+_f(\Psi; \varphi_0) \). For any fixed \( q > 2a^+_f(\Psi; \varphi_0) \), \( \lambda > 0 \), let \( \xi \in A^2(\{ \Psi < 0 \}, e^{-\varphi_0}) \setminus \{0\} \), such that \( \xi|_{A^2(\{ \Psi < 0 \}, e^{-\varphi_0}) \cap J_q} \equiv 0 \), where \( J_q := I(q\Psi + \varphi_0)_a \). Denote that
\[
K_{\xi,q,\lambda}(t) := \sup_{f \in A^2(\{ \Psi < 0 \}, e^{-\varphi_0})} \frac{\langle \xi, f \rangle^2}{\| f \|^2_{q,\lambda,t}},
\]
where
\[
\| f \|^2_{q,\lambda,t} := \left( \int_{\{ \Psi < 0 \}} |\hat{f}|^2 e^{-\varphi_0 - \lambda \max(q\Psi + t, 0)} \right)^{1/2},
\]
and \( t \in [0, +\infty) \). Theorem 1.2 tells us that \( -\log K_{\xi,q,\lambda}(t) + t \) is increasing with respect to \( t \in [0, +\infty) \), which implies that
\[
-\log K_{\xi,q,\lambda}(t) + t \geq -\log K_{\xi,q,\lambda}(0), \quad \forall t \in [0, +\infty). \] (6.1)

Since \( \int_{\{ \Psi < 0 \}} |f|^2 e^{-\varphi_0} \leq \int_{\{ \Psi < 0 \}} |f|^2 e^{-\varphi_0 - \Psi} < +\infty \), following from inequality (5.4), we get that
\[
\| f \|^2_{q,\lambda,t} \geq \frac{|\xi, f|^2}{K_{\xi,q,\lambda}(t)} \geq e^{-t} \frac{|\xi, f|^2}{K_{\xi,q,\lambda}(0)}, \quad \forall t \in [0, +\infty).
\]
According to Lemma 2.11, we have
\[
\| f \|^2_{q,\lambda,t} \geq \sup_{\xi \in A^2(\{ \Psi < 0 \}, e^{-\varphi_0}) \setminus \{0\} \atop \xi|_{A^2(\{ \Psi < 0 \}, e^{-\varphi_0}) \cap J_q} \equiv 0} e^{-t} \frac{|\xi, f|^2}{K_{\xi,q,\lambda}(0)} \]
\[= e^{-t} C(q\Psi, \varphi_0, J_q, f), \quad \forall t \in [0, +\infty). \] (6.2)
Note that for any $t \in [0, +\infty)$,
\[
\|f\|^2_{q, \lambda, t} = \int_{\{q\Psi < -t\}} |f|^2 e^{-\varphi_0} + \int_{\{0 > q\Psi \geq -t\}} |f|^2 e^{-\varphi_0 - \lambda(q\Psi + t)}. \tag{6.3}
\]
Since for any $\lambda > 0$,
\[
\int_{\{0 > q\Psi \geq -t\}} |f|^2 e^{-\varphi_0 - \lambda(q\Psi + t)} \leq \int_{\{0 > q\Psi \geq -t\}} |f|^2 e^{-\varphi_0} < +\infty,
\]
and $\lim_{\lambda \to +\infty} e^{-\lambda(q\Psi + t)} = 0$ on $\{0 > q\Psi \geq -t\}$, according to Lebesgue’s dominated convergence theorem, we have
\[
\lim_{\lambda \to +\infty} \int_{\{0 > q\Psi \geq -t\}} |f|^2 e^{-\varphi_0 - \lambda(q\Psi + t)} = 0.
\]
Then equality (6.3) implies
\[
\lim_{\lambda \to +\infty} \|f\|^2_{q, \lambda, t} = \int_{\{q\Psi < -t\}} |f|^2 e^{-\varphi_0}, \forall t \in [0, +\infty). \tag{6.4}
\]
Thus letting $\lambda \to +\infty$ in inequality (6.2), we get that for any $t \in [0, +\infty)$,
\[
\int_{\{q\Psi < -t\}} |f|^2 e^{-\varphi_0} \geq e^{-t} C(q\Psi, \varphi_0, J_q, f) = e^{-t} C(\Psi, \varphi_0, J_q, f). \tag{6.5}
\]
Note that $J_q \subset I_+ (2a_o^f(\Psi; \varphi_0) \Psi + \varphi_0)_o$ for any $q > 2a_o^f(\Psi; \varphi_0)$. Then we have
\[
C(\Psi, \varphi_0, J_q, f) \geq C(\Psi, \varphi_0, I_+ (2a_o^f(\Psi; \varphi_0) \Psi + \varphi_0)_o, f), \forall q > 2a_o^f(\Psi; \varphi_0).
\]
Then it follows from inequality (6.5) that
\[
\int_{\{q\Psi < -t\}} |f|^2 e^{-\varphi_0} \geq e^{-t} C(\Psi, \varphi_0, I_+ (2a_o^f(\Psi; \varphi_0) \Psi + \varphi_0)_o, f) \tag{6.6}
\]
for any $q > 2a_o^f(\Psi; \varphi_0)$ and $t \in [0, +\infty)$.

According to Fubini’s theorem, we have
\[
\int_{\{q\Psi < -t\}} |f|^2 e^{-\varphi_0} \Psi \\
= \int_{\{q\Psi < -t\}} \left( |f|^2 e^{-\varphi_0} \int_0^{e^{-\Psi}} ds \right) \\
= \int_0^{+\infty} \left( \int_{\{q\Psi < -t\}} \right) |f|^2 e^{-\varphi_0} ds \\
= \int_0^{+\infty} \left( \int_{\{q\Psi < -q\Psi \cap \{q\Psi < 0\}} |f|^2 e^{-\varphi_0} \right) e^t dt.
\]
Inequality (6.6) implies that for any $q > 2a_o^f(\Psi; \varphi_0)$,
\[
\int_0^{+\infty} \left( \int_{\{q\Psi < -q\Psi \cap \{q\Psi < 0\}} |f|^2 e^{-\varphi_0} \right) e^t dt \\
\geq \int_0^{+\infty} e^{-qt} C(\Psi, \varphi_0, I_+ (2a_o^f(\Psi; \varphi_0) \Psi + \varphi_0)_o, f) \cdot e^t dt \\
= \frac{1}{q - 1} C(\Psi, \varphi_0, I_+ (2a_o^f(\Psi; \varphi_0) \Psi + \varphi_0)_o, f),
\]
Remark 7.2. It follows from Lemma 7.1 that under the topology of $A$, we have $q < q$ for any $\alpha \eta$. Thus, if $q > 1$ satisfying

$$\int_{\{\psi < 0\}} |f|^q e^{-\varphi_0 - \Psi} \geq \frac{q}{q-1} C(\psi, \varphi_0, I+(2a_0^f(\psi; \varphi_0)\Psi + \varphi_0)_{\alpha}, f).$$

(6.7)

for any $q > 2a_0^f(\Psi; \varphi_0)$. Let $q \rightarrow 2a_0^f(\Psi; \varphi_0) + 0$, then inequality (6.7) also holds for $q \geq 2a_0^f(\Psi; \varphi_0)$. Thus, if $q > 1$ satisfying

$$\int_{\{\psi < 0\}} |f|^q e^{-\varphi_0 - \Psi} < \frac{q}{q-1} C(\psi, \varphi_0, I+(2a_0^f(\psi; \varphi_0)\Psi + \varphi_0)_{\alpha}, f),$$

(6.8)

we have $q < 2a_0^f(\Psi; \varphi_0)$, which means that $f_0 \in I(q\psi + \varphi_0)$. Proof of Corollary 1.7 is done. $\square$

7. APPENDIX

Let $D$ be a domain in $\mathbb{C}^n$, and $\varphi$ be a plurisubharmonic function on $D$. Denote that

$$\ell_0 := \{\eta = (\eta_\alpha)_{\alpha \in \mathbb{N}^n} : \exists k \in \mathbb{N}, \text{ such that } \eta_\alpha = 0, \forall |\alpha| \geq k\},$$

where for any $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, $|\alpha| := \alpha_1 + \cdots + \alpha_n$. Let $z_0 \in D$, and $\eta = (\eta_\alpha) \in \ell_0$. For any $f \in \mathcal{O}(D)$, denote that

$$\eta \cdot f := \sum_{\alpha \in \mathbb{N}^n} \eta_\alpha \frac{f^{(\alpha)}(z_0)}{\alpha!}.$$

(7.1)

It can be shown that for any $\eta \in \ell_0$, there is a finite constant $C_\eta > 0$, such that

$$|\eta \cdot f|^2 \leq C_\eta \int_D |f|^2 e^{-\varphi},$$

for any $f \in A^2(D, e^{-\varphi})$ (see [12]). Then any $\eta \in \ell_0$ can be seen as an element in $A^2(D, e^{-\varphi})^*$ by equality (7.1). Note that $A^2(D, e^{-\varphi})$ is a Hilbert space. By Riesz representation theorem, there exists $g_{\eta} \in A^2(D, e^{-\varphi})$, such that

$$\eta \cdot f = \int_D f g_{\eta} e^{-\varphi}, \forall f \in A^2(D, e^{-\varphi}),$$

which induces a map from $\ell_0$ to $A^2(D, e^{-\varphi})$. We denote the map by $T_{\varphi, z_0}$:

$$T_{\varphi, z_0} : \ell_0 \rightarrow A^2(D, e^{-\varphi})$$

$$\eta \mapsto g_{\eta}.$$  

We state the following lemma.

**Lemma 7.1.** The image of $T_{\varphi, z_0}$ is dense in $A^2(D, e^{-\varphi})$, i.e., $\overline{T_{\varphi, z_0}(\ell_0)} = A^2(D, e^{-\varphi})$, under the topology of $A^2(D, e^{-\varphi})$.

**Remark 7.2.** It follows from Lemma 7.1 that $\ell_0$ is dense in $A^2(D, e^{-\varphi})^*$, under the strong topology of $A^2(D, e^{-\varphi})^*$. 

$$\int_{-\infty}^{0} \left( \int_{\{\psi < -qt \} \cap \{\psi < 0\}} |f|^2 e^{-\varphi_0} \right) e^t dt \geq \int_{-\infty}^{0} C(\Psi, \varphi_0, I+(2a_0^f(\psi; \varphi_0)\Psi + \varphi_0)_{\alpha}, f) \cdot e^t dt = C(\Psi, \varphi_0, I+(2a_0^f(\psi; \varphi_0)\Psi + \varphi_0)_{\alpha}, f).$$
Proof of Lemma 7.1. We introduce some notations before the proof.

For \( \alpha = (\alpha_1, \cdots, \alpha_n), \beta = (\beta_1, \cdots, \beta_n) \in \mathbb{N}^n \), denote that \( \alpha < \beta \), if \( |\alpha| < |\beta| \), or \( |\alpha| = |\beta| \) but there exists \( k \) with \( 1 \leq k \leq n \), such that \( \alpha_1 = \beta_1, \cdots, \alpha_{k-1} = \beta_{k-1}, \alpha_k < \beta_k \).

We may assume that \( z_0 = o \in D \) is the origin in \( \mathbb{C}^n \), and denote \( T_{\varphi, z_0} \) by \( T \). We will choose a countable sequence \( \{\eta[\alpha]\} \) of elements in \( \ell_0 \), such that

\[
\text{span}\{T(\eta[\alpha])\} = A^2(D, e^{-\varphi}),
\]

which can imply Lemma 7.1. For any \( \alpha \in \mathbb{N}^n \), we set \( \eta[\alpha] \in \ell_0 \), with \( \eta[\alpha] \gamma = \gamma! \cdot b^\alpha_\gamma \in \mathbb{C} \) (which will be determined in the following discussions) for any \( \gamma < \alpha \), \( \eta[\alpha]_\alpha = \alpha! \), and \( \eta[\alpha]_\gamma = 0 \) for any \( \gamma > \alpha \). Denote that \( g_\alpha := g_{\eta_\alpha} = T(\eta[\alpha]) \in A^2(D, e^{-\varphi}) \). We will choose \( b^\alpha_\gamma \) such that

\[
\int_D g_\alpha \overline{g_\beta} e^{-\varphi} = 0, \quad \forall \alpha \neq \beta;
\]

\[
g_\beta^{(\gamma)}(o) = 0, \quad \forall \gamma < \beta; \tag{7.2}
\]

\[
g_\alpha^{(\alpha)}(o) = \int_D |g_\alpha|^2 e^{-\varphi}, \quad \forall \alpha.
\]

And for any \( \alpha \in \mathbb{N}^n \), denote that \( \alpha \in S_2 \) if \( g_\alpha \equiv 0 \). Otherwise we denote that \( \alpha \in S_1 \).

Firstly, for \( \alpha = (0, \cdots, 0) \), we set

\[
\eta((0, \cdots, 0)) = (1, 0, \cdots, 0, \cdots) \in \ell_0.
\]

Denote that

\[
T((1, 0, \cdots, 0, \cdots)) = g_{(0, \cdots, 0)} \in A^2(D, e^{-\varphi}),
\]

then for any \( f \in A^2(D, e^{-\varphi_0}) \),

\[
f(o) = \int_D f g_{(0, \cdots, 0)} e^{-\varphi}. \tag{7.3}
\]

Let \( f = g_{(0, \cdots, 0)} \) in equality (7.3), we get

\[
g_{(0, \cdots, 0)}(o) = \int_D |g_{(0, \cdots, 0)}|^2 e^{-\varphi}.
\]

For some \( \alpha \in \mathbb{N}^n \), we assume that for any \( \beta < \alpha \), \( \eta(\beta) \in \ell_0 \) (i.e. the complex number sequence \( \{b^\alpha_\gamma\} \)) has been chosen to satisfy

\[
\int_D g_\beta \overline{g_\beta} e^{-\varphi} = 0, \quad \forall \beta_1 \neq \beta_1, \beta_1, \beta_2 < \alpha;
\]

\[
g_\beta^{(\gamma)}(o) = 0, \quad \forall \gamma < \beta; \tag{7.4}
\]

\[
g_\beta^{(\beta)}(o) = \int_D |g_\beta|^2 e^{-\varphi}, \quad \forall \beta < \alpha.
\]

By the choice of \( \eta[\alpha] \), for any \( f \in A^2(D, e^{-\varphi}) \),

\[
\sum_{\gamma < \alpha} b^\alpha_\gamma f^{(\gamma)}(o) + f^{(\alpha)}(o) = \int_D f g_\alpha e^{-\varphi}. \tag{7.5}
\]

Since we want

\[
\int_D g_\beta \overline{g_\beta} e^{-\varphi} = 0, \quad \forall \beta < \alpha,
\]
then there must be
\[ \sum_{\gamma < \alpha} b_\beta^\gamma g_\beta(\gamma)(o) + g_\beta(\alpha)(o) = 0, \ \forall \beta < \alpha, \]
which is equivalent to
\[ b_\beta^\beta g_\beta(\beta)(o) + \sum_{\beta < \gamma < \alpha} b_\beta^\gamma g_\beta(\gamma)(o) = -g_\beta(\alpha)(o), \ \forall \beta < \alpha, \]
(7.6)
by the choice of \( \{g_\beta\} \). Equality (7.6) can be seen as a linear equations system for \( (b_\beta^\gamma)_{\gamma < \alpha} \). Note that for \( \beta \in S_1 \), \( g_\beta(\beta) = 0 \). We set \( b_\beta^\beta = 0 \) for any \( \beta < \alpha \) with \( \beta \in S_1 \).

And we also note that
\[ \prod_{\beta < \alpha, \beta \in S_2} g_\beta(\beta)(o) = \prod_{\beta < \alpha, \beta \in S_2} \int_D |g_\beta|^2 e^{-\varphi} > 0. \]
It means that there exists \( (b_\beta^\alpha)_{\gamma < \alpha} \) satisfying equality (7.6), where \( b_\beta^\beta = 0 \) for any \( \beta < \alpha \) with \( \beta \in S_1 \). Now suppose that \( (b_\beta^\alpha)_{\gamma < \alpha} \) is the solution as we described above. Then \( g_\alpha \) satisfies
\[ \int_D g_\beta g_\alpha e^{-\varphi} = 0, \ \forall \beta < \alpha. \]
Note that in the process of induction, for any \( \beta < \alpha \), we have
\[ \sum_{\gamma < \beta} b_\beta^\gamma f(\gamma)(o) + f(\beta)(o) = \int_D f g_\beta e^{-\varphi}. \]
Let \( f = g_\alpha \), then we get
\[ \sum_{\gamma < \beta} b_\beta^\gamma g_\beta(\gamma)(o) + g_\alpha(\beta)(o) = \int_D g_\beta g_\alpha e^{-\varphi}, \ \forall \beta < \alpha. \]
(7.7)
In equality (7.7), by induction, we can know that for any \( \beta < \alpha \),
\[ g_\alpha(\beta)(o) = \int_D g_\alpha g_\beta e^{-\varphi} = \int_D g_\beta g_\alpha e^{-\varphi} = 0. \]
(7.8)
In addition, in equality (7.6), Letting \( f = g_\alpha \), we have
\[ \sum_{\gamma < \alpha} b_\beta^\gamma g_\beta(\gamma)(o) + g_\alpha(\alpha)(o) = \int_D |g_\alpha|^2 e^{-\varphi}. \]
Then it follows from equality (7.8) that
\[ g_\alpha(\alpha)(o) = \int_D |g_\alpha|^2 e^{-\varphi}. \]
Now, by induction, we can choose out \( \eta[\alpha] \in \ell_0 \) for any \( \alpha \in \mathbb{N}^n \) satisfying what we described before equality (7.2), and \( \{g_\alpha\}_{\alpha \in \mathbb{N}^n} \subset A^2(D, e^{-\varphi}) \) satisfies equality (7.2). In the following we prove that
\[ \text{span}\{g_\alpha : \alpha \in S_2\} = A^2(D, e^{-\varphi}). \]
(7.9)
For any \( f \in A^2(D, e^{-\varphi}) \), let \( \{a_\alpha\}_{\alpha \in \mathbb{N}^n} \) be a sequence of complex numbers (which will be determined in the following). Denote
\[ f_\alpha(z) = \sum_{\beta \leq \alpha} a_\beta g_\beta(z). \]
(7.10)
We choose \(a_\beta\) for \(\beta \leq \alpha\), such that
\[
f_\alpha^{(\beta)}(o) = f^{(\beta)}(o). \tag{7.11}
\]
Firstly, for \(\beta = (0, \ldots, 0)\), if \((0, \ldots, 0) \in S_2\), we can see that
\[
a_{(0, \ldots, 0)} = \frac{f(o)}{g_{(0, \ldots, 0)}(o)}
\]
satisfy equality (7.11), and if \((0, \ldots, 0) \in S_1\), we have \(f(o) = 0\) according to equality (7.3), which implies \(a_{(0, \ldots, 0)} = 0\) satisfies inequality (7.11).

Secondly, assume that for some \(\gamma \leq \alpha\), all \(\beta < \gamma\) have been choosen to satisfy equality (7.11). According to equality (7.10) and equality (7.2), we have
\[
f^{(\gamma)}(o) = \sum_{\beta < \gamma} a_\beta g_\beta^{(\gamma)}(o) + a_\gamma g_\gamma^{(\gamma)}(o). \tag{7.12}
\]
Then
\[
f^{(\gamma)}(o) = f^{(\gamma)}(o) \iff \sum_{\beta < \gamma} a_\beta g_\beta^{(\gamma)}(o) + a_\gamma g_\gamma^{(\gamma)}(o). \tag{7.13}
\]
Note that for \(\gamma \in S_2\), \(g_\gamma^{(\gamma)}(o) = \int_D |g_\gamma|^2 e^{-\varphi} > 0\), then we can choose
\[
a_\gamma = (g_\gamma^{(\gamma)}(o))^{-1} \left( f^{(\gamma)}(o) - \sum_{\beta < \gamma} a_\beta g_\beta^{(\gamma)}(o) \right)
\]
to satisfy equality (7.11). If \(\gamma \in S_1\), following from equality (7.1), we have
\[
f^{(\gamma)}(o) = -\sum_{\beta < \gamma} b_\beta f^{(\beta)}(o) = -\sum_{\beta < \gamma} b_\beta f^{(\beta)}_{\alpha}(o)
\]
\[
= -\sum_{\beta < \gamma} b_\beta \left( \sum_{\beta' \leq \beta} a_{\beta'} g_{\beta'}^{(\beta)}(o) \right)
\]
\[
= -\sum_{\beta' < \gamma} a_{\beta'} \left( \sum_{\beta' \leq \beta < \gamma} b_\beta g_{\beta'}^{(\beta)}(o) \right)
\]
\[
= \sum_{\beta' < \gamma} a_{\beta'} g_{\beta'}^{(\gamma)}(o).
\]
Then we can choose \(a_\gamma = 0\) according to equation (7.13).

Finally, by induction, we can know that the sequence \(\{a_\beta\}\) can be choosen to satisfy equality (7.11). In addition, we have \(a_\beta = 0\) for \(\beta \in S_1\).

Now we have \((f - f_{\alpha})^{(\beta)}(o) = 0\) for any \(\beta \leq \alpha\). Then it follows from equality (7.3) that
\[
\int_D (f - f_{\alpha}) \overline{g_{\alpha}} e^{-\varphi} = 0,
\]
which means that
\[
\int_D f \overline{g_{\alpha}} e^{-\varphi} = \int_D f_{\alpha} \overline{g_{\alpha}} e^{-\varphi} = a_{\alpha}.
\]
Then combining with equality (7.2), we get that
\[
\sum_{\alpha} |a_{\alpha}|^2 \int_D |g_{\alpha}|^2 e^{-\varphi} \leq \int_D |f|^2 e^{-\varphi}.
\]
Denote
\[ h(z) := \sum_{\alpha \in \mathbb{N}^n} a_\alpha g_\alpha(z), \]
then \( h \in A^2(\Omega, e^{-\phi}) \). In addition, we have
\[ h(z) = \lim_{|\alpha| \to +\infty} f_\alpha \Rightarrow h^{(\beta)}(o) = \lim_{|\alpha| \to +\infty} f^{(\beta)}_\alpha(o) \]
for any \( \beta \in \mathbb{N}^n \), inducing that \( f \equiv h = \sum_{\alpha \in \mathbb{N}^n} a_\alpha g_\alpha(z) \). Note that \( a_\alpha = 0 \) for \( \alpha \in S_1 \).

By the arbitrariness of \( f \in A^2(D, e^{-\phi}) \), we get
\[ \text{span}\{ g_\alpha : \alpha \in S_2 \} = A^2(D, e^{-\phi}), \quad (7.14) \]
which implies
\[ \text{span}\{ T(\eta(\alpha)) \} = A^2(D, e^{-\phi}). \]
Then we know that Lemma 7.1 holds. \( \square \)

Remark 7.4. It follows from Lemma 7.3 that \( \text{span}\{ e_z : z \in D \} \) is dense in \( A^2(D, e^{-\phi})^* \), under the strong topology. And in Lemma 2.8, for any \( \eta \in \text{span}\{ e_z : z \in D \} \) such that
\[ \eta = \sum_{k=1}^{N} c_k e_{z_k}, \]
where \( N \) is a finite positive integer, \( c_k \in \mathbb{C} \), and \( z_k \in D \) for any \( k \), we have that
\[ \eta \cdot f = \sum_{k=1}^{N} c_k e_{z_k} \cdot f = \sum_{k=1}^{N} c_k f(\tau, z_k) \]
is holomorphic with respect to \( \tau \). Then with a similar discussion in the proof of Lemma 2.8, we can know that Lemma 2.8 can also be induced by Lemma 7.3.

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