Computing Skeletons for Rectilinearly Convex Obstacles in the Rectilinear Plane

Marcus Volz\textsuperscript{1} · Marcus Brazil\textsuperscript{1} · Charl Ras\textsuperscript{2} · Doreen Thomas\textsuperscript{1}

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Abstract
We introduce the concept of an obstacle skeleton, which is a set of line segments inside a polygonal obstacle $\omega$ that can be used in place of $\omega$ when performing intersection tests for obstacle-avoiding network problems in the plane. A skeleton can have significantly fewer line segments compared to the number of line segments in the boundary of the original obstacle, and therefore performing intersection tests on a skeleton (rather than the original obstacle) can significantly reduce the CPU time required by algorithms for computing solutions to obstacle-avoidance problems. A minimum skeleton is a skeleton with the smallest possible number of line segments. We provide an exact $O(n^2)$ algorithm for computing minimum skeletons for rectilinear obstacles in the rectilinear plane that are rectilinearly convex.

Keywords  Skeletons · Obstacle avoidance · Rectilinear · Steiner trees

Mathematics Subject Classification  90B10 · 52B05 · 68U05

1 Introduction
Obstacle-avoiding shortest network problems arise in many applications in industry. In these problems, it may be necessary to perform intersection tests to determine whether a part of the network intersects one or more obstacles, since a network that intersects an obstacle is not a feasible solution. For polygonal obstacles, the number of edges can significantly contribute to the CPU time required by algorithms for computing shortest obstacle-avoiding networks. It is therefore desirable to reduce the number of
Fig. 1 A shortest obstacle-avoiding rectilinear network interconnecting points $t_1$, $t_2$, $t_3$ and $t_4$. The network has two straight edges ($t_1s_1$ and $t_2s_1$) and four bent edges ($s_1v_1$, $v_1t_3$, $t_3v_2$ and $v_2t_4$)

edges that need to be considered for a given obstacle, since this will reduce the number of obstacle intersection tests that need to be performed.

Throughout this paper, an obstacle (denoted by $\omega$) is a simple polygon, i.e. a closed and bounded polygonal region that does not have holes and whose boundary does not intersect itself. In obstacle-avoiding network problems, the network is not permitted to intersect the interior of $\omega$. In this paper, we introduce the concept of a skeleton, which is a representation of an obstacle that consists of a set of line segments inside the obstacle. A skeleton can have a significantly smaller number of line segments than the number of line segments in the boundary of the original obstacle.

Our aim in this paper is to develop an algorithm for computing obstacle skeletons with a minimum number of edges. We do this in the context of shortest networks with respect to the rectilinear metric (Fig. 1). Obstacle-avoiding rectilinear network problems and shortest rectilinear path problems have been well studied (see [1–13]) and have a range of applications including VLSI design and motion planning. Such a network can be represented as an embedding in the Euclidean plane, where each edge is a rectilinear shortest path between its endpoints; that is, a shortest path composed only of horizontal and vertical line segments.

In this paper, we focus on computing minimum skeletons for obstacles that are themselves rectilinear, i.e. simple polygons for which each edge of the polygon is either horizontal or vertical. In particular, we focus on rectilinear obstacles that are rectilinearly convex, meaning that any two points in $\omega$ can be joined by a shortest rectilinear path that is inside $\omega$. This is a reasonable restriction of the problem to study initially, as it provides insight into the structure of minimum skeletons, and the intuition gained through studying minimum skeletons for rectilinearly convex obstacles can then be extended to other contexts. Moreover, rectilinearly convex obstacles are often found in VLSI problem instances (see for instance the many examples in the SteinLib database [14]). For the remainder of this paper, the term rectilinearly convex will be used to refer to an obstacle that is both rectilinear and rectilinearly convex.
2 Preliminaries

Shortest rectilinear networks can be constructed from two types of edges (which represent geodesics in the rectilinear metric): a straight edge is a single line segment that is either horizontal or vertical; and a bent edge consists of a series of horizontal and vertical line segments, where each adjacent pair of orthogonal line segments meet at a corner point. Any bent edge connecting a given pair of points can be embedded with a single corner point and has exactly two such embeddings. In many problem contexts, it is sufficient to consider shortest networks for which each edge has at most one corner point (see for example the discussion below on obstacle-avoiding rectilinear Steiner trees).

On this basis, we provide a formal definition of skeletons for obstacles in the context of rectilinear shortest networks as follows (note that a set of embedded edges $S$ is considered to be inside $\omega$ if $\bigcup_{s \in S} s \subseteq \omega$):

**Definition 2.1** Let $S$ be a set of closed line segments inside a polygonal obstacle $\omega$. Then, $S$ is said to be a skeleton for $\omega$ if, for any given pair of points outside the interior of $\omega$, if every rectilinear shortest path with at most one corner point between the pair of points meets the interior of $\omega$, then each such path intersects an element of $S$. A minimum skeleton $S^*$ is a skeleton with the smallest possible number of line segments.

An example of a minimum skeleton for an obstacle $\omega$ is given in Fig. 2. Note that shortest rectilinear paths between points outside the interior of $\omega$ with more than one corner point (i.e. zigzag paths) that pass through $\omega$ need not intersect a skeleton. For example in Fig. 2, a shortest rectilinear path between $p$ and $q$ with multiple corner points (shown as a red line) intersects $\omega$ but does not intersect the skeleton. The skeleton is legitimate, however, because the two shortest rectilinear paths with one corner point (i.e. $pc_1q$ and $pc_2q$) intersect the skeleton.
The requirement in Definition 2.1 that a skeleton edge be inside \( \omega \) ensures that all shortest rectilinear paths with at most one corner point that intersect the interior of a skeleton also intersect \( \omega \). This requirement makes it necessary to treat obstacles individually when determining their skeletons (since any one skeleton constructed for multiple disjoint obstacles would in part lie outside the interior of the obstacles). The related question, of whether a rectilinear path that intersects a skeleton edge (say, at an endpoint of the skeleton edge) actually enters the interior of the obstacle or simply runs along part of the obstacle boundary, is addressed in Sect. 2.

An important potential application of skeletons is to the obstacle-avoiding rectilinear Steiner tree problem, which involves finding a minimum length obstacle-avoiding rectilinear network interconnecting a given set of points. The problem is applicable to VLSI physical network design (or microchip design); see Section 3.6 of [12]. An exact (exponential-time) algorithm [11] exists for solving this problem, based on the more general GeoSteiner algorithm [15]. A major bottleneck in this algorithm is the construction of a set of candidate trees on subsets of the given points that guarantee a minimum solution can be found by taking a union of some elements of this set. It is desirable to reduce the vast number of obstacle intersection tests that need to be performed in this phase, which is something a minimum skeleton can help achieve.

The most effective GeoSteiner algorithms assume that each edge of the minimum network contains at most one corner point. This makes sense in VLSI applications as any change in direction in an edge incurs a cost, since it usually involves moving from one layer of the microchip to another and is also a reasonable assumption in many other applications. In all of these cases, any rectilinear edge embedded with at most one corner point that does not intersect any skeleton edges has an embedding that is obstacle avoiding.

In order to apply skeletons to these network design problems, we need to specify the conditions under which a rectilinear edge (with at most one corner point) that intersects a skeleton edge also intersects the interior of the obstacle. Let \( s \) be a closed line segment inside a rectilinearly convex obstacle \( \omega \) with both endpoints on the boundary of \( \omega \). We can classify each point on \( s \) as either a boundary point if it lies on the boundary of \( \omega \), or a non-boundary point if it lies in the interior of \( \omega \). Let \( P \) be a rectilinear path connecting two points outside the interior of \( \omega \), and containing at most one corner point. We first define what it means for \( s \) to block \( P \).

**Definition 2.2** With \( s \), \( P \) and \( \omega \) defined as above, we say that \( s \) blocks \( P \) if:

1. \( P \) intersects \( s \) at a non-boundary point; or,
2. \( P \) meets a boundary point of \( s \) not at a corner point of \( P \), and the direction of \( P \) at this meeting point is not equal to the direction of any edge of the boundary of \( \omega \) containing this boundary point of \( s \); or,
3. \( s \) is neither horizontal nor vertical, and \( P \) meets an endpoint of \( s \) at a corner point of \( P \), and \( P \) includes part or all of the interior of a rectilinear bounding box edge of \( s \) that intersects the interior of \( \omega \).

It is straightforward to see that this definition is precisely what is required to get the following theorem.
Theorem 2.1 Let $S$ be a skeleton for a rectilinearly convex obstacle $\omega$. Let $P$ be a rectilinear path connecting two points outside the interior of $\omega$, and containing at most one corner point. Then, an edge of $S$ blocks $P$ if and only if every embedding of $P$ with at most one corner point intersects the interior of $\omega$.

Once a skeleton has been constructed for a given obstacle $\omega$, information about how each skeleton edge interacts with the boundary of $\omega$ can be recorded. This means that it is simple to check whether a skeleton edge blocks a given rectilinear path, using only this supplementary information, and without needing to refer back to $\omega$.

3 Classification and Properties of Rectilinearly Convex Obstacles

Throughout this paper, we will use the terms “edges of $\omega$” and “vertices of $\omega$” as shorthand to refer to the edges and vertices (that is, corner points) of the boundary of $\omega$ (since $\omega$ is a region). The extreme edges of $\omega$ are the edges of $\omega$ that intersect its bounding box, where, as usual, the bounding box is the smallest closed axis-oriented rectangle that encloses $\omega$. The term bounding box will be similarly applied more generally to any set of line segments, such as skeletons. The function that returns the bounding box for a given set of line segments will be denoted by $B(\cdot)$. An extreme corner of $\omega$ is a vertex of $\omega$ at the intersection of two extreme edges. A rectilinearly convex obstacle has exactly four extreme edges and up to four extreme corners.

A staircase walk of $\omega$ is a shortest rectilinear path on the boundary of $\omega$ between adjacent extreme edges (including the extreme edges themselves). A pair of extreme corners (edges) will be called adjacent if they coincide with adjacent vertices (edges) of the bounding box of $\omega$, or opposite otherwise. An extreme corner $c$ and an extreme edge $e$ are opposite if $e$ lies on an edge of the bounding box of $\omega$ that does not intersect $c$. For example, if $c$ lies at the bottom-left corner of $\omega$, then the right and top extreme edges of $\omega$ are opposite to $c$.

Rectilinearly convex obstacles can be categorised into six types based on the number and relative locations of extreme corners (see Fig. 3):

- A rectangle has exactly four extreme corners.
- An L-obstacle has exactly three extreme corners.
- A T-obstacle has exactly two extreme corners, which are adjacent.
- A staircase obstacle has exactly two extreme corners, which are opposite.
- A partial staircase has exactly one extreme corner.
- A general obstacle has no extreme corners.

Fig. 3 Classification of rectilinearly convex obstacles based on the relative locations of extreme corners. Extreme edges are shown as red lines, and extreme corners as red dots.
General obstacles can be further sub-classified. Suppose $\omega$ is a general obstacle. A pair of parallel extreme edges of $\omega$ will be said to overlap if the orthogonal projection of one edge onto the other is not empty, and a pair of non-overlapping extreme edges will be called positively (negatively) sloped if the gradient of the line segment between the midpoints of the two edges is positive (negative). Up to symmetry, $\omega$ can be sub-classified into four types as shown in Fig. 4:

- Type (a): Both pairs of parallel extreme edges overlap.
- Type (b): Exactly one pair of parallel extreme edges overlaps.
- Type (c): Neither pair of parallel extreme edges overlap and the slopes of the two pairs of parallel extreme edges have different signs.
- Type (d): Neither pair of parallel extreme edges overlap and the slopes of the two pairs of parallel extreme edges have the same sign.

### 3.1 Obstacle Ends

Let $e_i$ and $e_j$ be adjacent extreme edges of a rectilinearly convex obstacle $\omega$ and assume that $e_i$ is horizontal and $e_j$ is vertical. Then, $\{e_i, e_j\}$ is called an end of $\omega$ if:

1. $e_i$ and $e_j$ do not overlap with their corresponding opposite parallel extreme edges,
2. there exists a corner $c$ of $B(\omega)$ such that $e_i$ is the closest horizontal extreme edge to $c$ in the horizontal direction and $e_j$ is the closest vertical extreme edge to $c$ in the vertical direction.

A staircase or partial staircase obstacle with overlapping parallel extreme edges has no ends; otherwise, a staircase obstacle has two ends, one corresponding to each extreme corner, while a partial staircase has one end corresponding to its extreme corner $c$ and the other end corresponding to the two extreme edges that are not incident to $c$. Rectangles, L-obstacles and T-obstacles do not have ends, since their pairs of parallel extreme edges overlap. Type (a), (b) and (c) general obstacles do not have ends, while Type (d) general obstacles have two ends. For example, the obstacle in Fig. 4d has two ends: $\{e_1, e_2\}$ and $\{e_3, e_4\}$ corresponding to $c_1$ and $c_3$. 

![Fig. 4 Sub-classification of general obstacles based on the relative positions of parallel extreme edges](image)
3.2 Point and Edge Visibility

Two points, \( p \) and \( q \), inside a rectilinearly convex obstacle \( \omega \) will be called \textit{mutually visible} if the line segment between \( p \) and \( q \) is inside \( \omega \). Two line segments \( s_1 \) and \( s_2 \) in \( \omega \) are mutually visible if there exists a point \( p_1 \) on \( s_1 \) and a point \( p_2 \) on \( s_2 \) such that \( p_1 \) and \( p_2 \) are mutually visible. A point \( p \) and a line segment \( s \) in \( \omega \) are mutually visible if there exists a point \( q \) on \( s \) such that \( p \) and \( q \) are mutually visible.

A \textit{visibility edge} \( e \) is any line segment that is inside \( \omega \). An \textit{auxiliary point} is an endpoint of a visibility edge that lies in the interior of an edge in the boundary of \( \omega \).

We define the following special types of visibility edge configurations, which may or may not exist for a given obstacle (refer to Fig. 5):

- A \textit{diagonal} of \( \omega \) is a line segment inside \( \omega \) between a pair of opposite extreme corners.
- An \textit{opposite (adjacent) extreme visibility edge} is a line segment inside \( \omega \) that has endpoints on parallel (orthogonal) extreme edges.
- A \textit{maximum length adjacent extreme visibility edge} is an adjacent extreme visibility edge with the maximum length among all adjacent extreme visibility edges, for a given pair of extreme edges (for example, if \( e_1 \) and \( e_2 \) are the left and bottom extreme edges, respectively, and then the maximum length extreme visibility edge connects the top endpoint of \( e_1 \) and the right endpoint of \( e_2 \)).
- A \textit{cross} is a pair \( s_H, s_V \) of opposite extreme visibility edges, where \( s_H \) (\( s_V \)) has an endpoint on each horizontal (vertical) extreme edge.
- A \textit{perpendicular extreme visibility edge} is a visibility edge that is perpendicular to its corresponding extreme edge.

4 Connectivity Properties of Skeletons

In this section, we present a necessary and sufficient condition for a set of edges to constitute a skeleton. Consider a visibility edge \( s \) inside a rectilinearly convex obstacle. Then, \( s \) will be called a \textit{maximum length visibility edge} if it extends as far as possible in both directions to the boundary of \( \omega \), subject to \( s \) remaining inside \( \omega \). Clearly, for any obstacle, a minimum skeleton exists such that all edges of the skeleton are maximum length visibility edges. Therefore, for the remainder of this paper we will assume that all skeleton edges and visibility edges have this property.
Note that a skeleton must meet every extreme edge of the obstacle. We therefore define an extreme skeleton edge to be a skeleton edge with at least one endpoint on an extreme edge. It is possible to show that, if a set $S$ of maximum length visibility edges of $\omega$ is connected and meets all extreme edges, then $S$ is a skeleton of $\omega$ (see Lemma A.1 of “Appendix”, or see the supplementary document [16]). However, the converse of this is not true (see Fig. 6).

To find a characterisation of skeletons, we therefore define a different kind of connectivity, which we will refer to as weak connectivity.

**Definition 4.1** Let $\omega$ be a rectilinearly convex obstacle, and let $S$ be a set of line segments inside $\omega$. Let $\{S_i\}$ be the set of maximal connected components of $S$.

- Two components $S_j$ and $S_k$ will be called weakly connected if $S_j \cap B_k \neq \emptyset$ and $S_k \cap B_j \neq \emptyset$, where $B_j$ and $B_k$ are the respective (closed) bounding boxes of $S_j$ and $S_k$.
- A skeleton is called weakly connected if, when each connected component is treated as a single vertex and an edge inserted between each pair of weakly connected components, the resulting graph is connected.

**Theorem 4.1** Let $\omega$ be a rectilinearly convex obstacle and let $S$ be a set of line segments inside $\omega$. Then, $S$ is a skeleton for $\omega$ if and only if (1) $S$ intersects the four extreme edges of $\omega$ and (2) $S$ is weakly connected.

**Proof** ($\rightarrow$) Refer to Fig. 7 and assume that $S$ is a skeleton. Recall that $S$ necessarily intersects the four extreme edges of $\omega$. Now, assume contrary to the theorem that $S$ is not weakly connected. Then, $S$ can be partitioned into two disjoint subsets $S_j$ and $S_k$ such that $S_j \cap B_k = \emptyset$. Since $S$ is a skeleton, the projection of $S_j \cup S_k$ must cover the projection of $\omega$ (for otherwise it would be possible to pass an axis-aligned line through $\omega$ without intersecting $S$) and therefore $B_j \cap B_k \neq \emptyset$. Hence, there exists a shortest rectilinear path with one corner point between $p$ and $q$ (where the path follows the boundary of $B_j$ at some small distance outside $B_j$) that is not intersected by $S$ (as illustrated in Fig. 7). Therefore, $S$ is not a skeleton, giving a contradiction.
Now, assume that $S$ intersects the four extreme edges of $\omega$ and that $S$ is weakly connected and assume contrary to the theorem that $S$ is not a skeleton for $\omega$. Then, there exist points $p$ and $q$ outside the interior of $\omega$ such that each shortest rectilinear path between $p$ and $q$ with at most one corner point intersects $\omega$, but at least one of these paths, say $pcq$, does not intersect $S$. Then, $pcq$ partitions $S$ into two sets, and since each set necessarily intersects at least one extreme edge, part of the skeleton must lie strictly on each side of $pcq$. This contradicts the assumption that $S$ is weakly connected.

5 Minimum Skeletons by Obstacle Type

In this section, we present methods for constructing minimum skeletons by obstacle type. We begin with some simple cases.

Let $|S|$ denote the cardinality of $S$ as a set of line segments.

Lemma 5.1 Let $\omega$ be a rectilinearly convex obstacle and let $S^*$ be a minimum skeleton for $\omega$. Then, $|S^*| = 1$ if and only if $\omega$ has a diagonal.

Proof $(\rightarrow)$ If $S^* = 1$, then the single edge $s$ in $S^*$ must intersect the four extreme edges of $\omega$. This is only possible if the endpoints of $s$ are mutually visible opposite extreme corners of $\omega$. $(\leftarrow)$ A diagonal $\{d\}$ is a connected set of edges inside $\omega$ that intersects the four extreme edges of $\omega$. Therefore, $\{d\}$ is a skeleton for $\omega$ by Lemma A.1 in “Appendix”. It is also a minimum skeleton since a skeleton clearly must have at least one edge.

Corollary 5.1 Let $\omega$ be either a rectangle, $L$-obstacle or $T$-obstacle. Let $S$ be a minimum skeleton for $\omega$. 

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1. If $\omega$ is a rectangle, then $S$ is either of the two diagonals of $\omega$.
2. If $\omega$ is an L-obstacle, then $S$ is a diagonal of $\omega$, if one exists, or else a cross of $\omega$.
3. If $\omega$ is a T-obstacle, then $S$ is a cross of $\omega$.

Corollary 5.1(1) is known in the literature, as discussed in Section 4.2.3 of [12] (see Theorem 4.18). Parts (2) and (3) of the corollary both follow from Lemma 5.1, and the observations that every L-obstacle and T-obstacle contains a cross, and if $S$ is a cross then $|S| = 2$.

Note that minimum skeletons for L-obstacles without diagonals and T-obstacles are not unique; there are infinitely many configurations of crosses that constitute minimum skeletons.

5.1 Minimum Skeletons for Staircase Obstacles

While minimum skeletons for rectangles, L-obstacles and T-obstacles have either one or two edges, there is no fixed upper bound (independent of the number of obstacle vertices) on the number of edges in minimum skeletons for the remaining three types in the classification, and constructing minimum skeletons for these types is in general nontrivial. In this section, we develop an iterative procedure for constructing minimum skeletons for staircase obstacles. We start by introducing the concept of a maximal extreme visibility edge.

5.1.1 Maximal Extreme Visibility Edges

Let $\omega$ be a rectilinearly convex obstacle and let $S$ be a skeleton for some connected subregion of $\omega$. Then, the vertical (horizontal) projection of $S$ is a single vertical (horizontal) line segment. Adding a new edge $s$ to $S$ potentially extends this line segment in one direction or the other, or both. The vertical (horizontal) advancement of $s$ is the length of the vertical (horizontal) projection of $s$ that is not covered by the corresponding projection of $S$.

**Definition 5.1** A maximal extreme visibility edge $s^*$ is an extreme visibility edge for an extreme edge $e$, such that:

- If $e$ is horizontal, then $s^*$ maximises its vertical advancement and, subject to this, maximises its horizontal advancement towards the other horizontal extreme edge.
- If $e$ is vertical, then $s^*$ maximises its horizontal advancement and, subject to this, maximises its vertical advancement towards the other vertical extreme edge.

The importance of maximising the secondary advancement towards the opposite extreme edge is highlighted in the example of Fig. 8 (a partial staircase). It can be shown in this case that $\{s_1, s_2, s_3\}$ is a minimum skeleton for the obstacle shown. If the first edge added to the skeleton is a maximal extreme visibility edge associated with $e_1$, then $s_1$, rather than $s_1'$, should be chosen, since $s_1$ has greater horizontal advancement towards $e_3$, even though $s_1'$ has a larger horizontal projection (note that if $s_2$ is added first, then $s_1'$ has no horizontal advancement). Any set of line segments containing $s_1'$ will not be a minimum skeleton.
Fig. 8 A minimum skeleton does not contain $s_1'$ even though the horizontal projection of $s_1'$ is greater than the horizontal projection of $s_1$, since $s_1$ maximises its horizontal advancement towards $e_3$.

Fig. 9 Proof of Lemmas 5.2 and 5.3

If a maximal extreme visibility edge has an endpoint at an extreme corner, we refer to it as a maximal extreme corner visibility edge. In the following lemma, we show that for each extreme corner of a staircase obstacle, there exists an associated maximal extreme corner visibility edge.

**Lemma 5.2** Let $\omega$ be a staircase obstacle with an extreme corner $c$ at the intersection of extreme edges $e_1$ and $e_2$, and let $s_1$ and $s_2$ be maximal extreme visibility edges with endpoints on $e_1$ and $e_2$, respectively. Then, $c$ is an endpoint of at least one of $s_1$ and $s_2$.

**Proof** We can assume, without loss of generality, that $e_1$ is vertical and $e_2$ is horizontal; let $v_1$ and $v_2$ denote the respective endpoints of $s_1$ and $s_2$ that are opposite $e_1$ and $e_2$. Then, (with reference to Fig. 9a) $v_1$ must be on the bottom-right staircase walk and $v_2$ on the top-left staircase walk (if $v_1$ is on the top-left staircase walk, then it must lie on a horizontal edge of $\omega$, and moving it to the right will increase its...
horizontal advancement). Therefore, the two line segments must intersect at some point \(I\). Suppose that neither \(s_1\) nor \(s_2\) have an endpoint at \(c\). Let \(c'\) denote the point at the intersection of the vertical line through \(v_1\) and the horizontal line through \(v_2\). Since \(\omega\) is a staircase obstacle, the polygon with vertices \(I, v_1, c', v_2\) is inside \(\omega\). Let \(c'\) denote the point at the intersection of the vertical line through \(v_1\) and the horizontal line through \(v_2\).

Since \(\omega\) is a staircase obstacle, the polygon with vertices \(I, v_1, c', v_2\) is inside \(\omega\). Let \(s_3\) denote the line segment between \(c\) and \(v_3\), where \(v_3\) is the point on the ray from \(c\) through \(I\) that intersects the rectilinear path \(v_1c'v_2\). If \(v_3\) is on the line segment \(v_1c'\), then \(s_3\) has the same horizontal advancement as \(s_1\) and has additional vertical advancement, and therefore \(s_1\) is not maximal, leading to a contradiction. Otherwise, \(v_3\) is on the line segment \(v_2c'\), and in this case \(s_3\) has the same vertical advancement as \(s_2\) and has additional horizontal advancement. Therefore, \(s_2\) is not maximal, again leading to a contradiction.

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\]

In the following lemma, we show that a maximal extreme corner visibility edge is unique for a given extreme corner of a staircase obstacle.

**Lemma 5.3** Let \(\omega\) be a staircase obstacle and let \(s\) be a maximal extreme corner visibility edge in \(\omega\). Then, \(s\) maximises both its vertical and horizontal advancement.

**Proof** Without loss of generality, assume that \(c\) is an extreme corner at the bottom-left corner of \(\omega\) (see Fig. 9b). Let \(s_1\) and \(s_2\) be maximal extreme visibility edges which maximise their horizontal and vertical advancements of \(\omega\), respectively, and, subject to this, maximise their respective vertical and horizontal advancements. Let \(v_1\) and \(v_2\) denote the endpoints of \(s_1\) and \(s_2\) that are opposite to \(c\), and suppose \(v_1\) and \(v_2\) are distinct points. Let \(v_3\) denote the point at the intersection of the vertical line through \(v_1\) and the horizontal line through \(v_2\). Since \(\omega\) is a staircase obstacle, the polygon with vertices \(c, v_1, v_3, v_2\) is contained in \(\omega\). Hence, the edge \(s_3\) between \(c\) and \(v_3\) lies in \(\omega\), and this edge has the same horizontal and vertical advancement as \(s_1\) and \(s_2\) (respectively), but it has greater advancement in the respective orthogonal directions. Hence, \(s_1\) and \(s_2\) are not maximal, giving a contradiction.

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We now show that for any staircase obstacle there exists a minimum skeleton that contains maximal extreme corner visibility edges.

**Lemma 5.4** Let \(\omega\) be a staircase obstacle with extreme corners \(c_1\) and \(c_2\). Then, there exists a minimum skeleton for \(\omega\) containing maximal extreme corner visibility edges at \(c_1\) and \(c_2\).

**Proof** Let \(e_1\) and \(e_2\) denote the extreme edges that intersect at \(c_1\) (Fig. 10). Now, both \(e_1\) and \(e_2\) must be intersected by any minimum skeleton for \(\omega\). Let \(s_1\) and \(s_2\) denote maximal extreme skeleton edges with respective endpoints on \(e_1\) and \(e_2\). By Lemma 5.2, at least one of \(s_1\) and \(s_2\) has an endpoint at \(c_1\). Denote this edge by \(s'_1\). By Lemma 5.3, \(s'_1\) is the unique edge that maximises both its horizontal and vertical advancement. Clearly, \(s'_1\) is a skeleton for the part of \(\omega\) that intersects the bounding box of \(s'_1\), since \(s'_1\) is connected and intersects all four extreme edges of this part of the obstacle.

Let \(s''_1\) be a skeleton edge with an endpoint at \(c_1\) that is not maximal. Let \(\omega'\) (\(\omega''\)) denote the part of \(\omega\) that remains when the bounding box of \(s'_1\) (\(s''_1\)) is subtracted from \(\omega\). Let \(S'\) (\(S''\)) denote the minimum sets of edges required to construct skeletons for
\( \omega \) given the inclusion of \( s' (s'') \) in the skeleton, respectively. Then, \(|S'| \leq |S''|\) (since \( \omega' \subset \omega'' \)). Therefore, \( s' \) requires the minimum possible number of additional edges to construct a skeleton for \( \omega \), and hence a minimum skeleton for \( \omega \) exists that contains \( s' \).

\[ \Box \]

### 5.1.2 Frontiers and Maximal Frontier Visibility Edges

Let \( S \) be a set of line segments inside a rectilinearly convex obstacle \( \omega \) such that \( S \) is a skeleton for \( \omega \cap B \), where \( B \) is the bounding box of \( S \). A frontier is the closure of a maximal connected component of the intersection of the boundary of \( B \) and the interior of \( \omega \). Examples of frontiers are shown as dashed red lines in Fig. 10. The function that returns the set of frontiers, for a given set of line segments, will be denoted by \( f (\cdot) \).

A frontier can be a single (horizontal or vertical) line segment, or it can be \( L \)-shaped. An \( L \)-shaped frontier, which we will treat as a single frontier, can only occur if the intersection of the interior of the frontier with \( S \) is empty (provided that all elements of \( S \) are maximum length visibility edges). A set \( S \) can have multiple associated frontiers. Frontiers provide a mechanism for constructing skeletons in an iterative fashion. The following lemma provides a useful general property of frontiers which is applicable to any rectilinearly convex obstacle.

**Lemma 5.5** Let \( \omega \) be a rectilinearly convex obstacle and let \( S^* \) be a minimum skeleton for \( \omega \). Let \( S \subset S^* \) be a connected or weakly connected set of line segments in \( S^* \) and let \( F \) be the set of frontiers associated with \( S \). Then, \( f \cap S^* \setminus S \neq \emptyset \) for all \( f \in F \).

**Proof** Suppose that there exists a frontier \( f \in F \) that is not intersected by \( S^* \setminus S \). Since \( \omega \) is convex and, given the assumption that all elements of \( S \) are maximum length visibility edges, it follows that \( f \) divides \( \omega \) into two regions, one containing \( S \) and the other containing a set \( S' \subseteq S^* \setminus S \) that lies on the opposite side of \( f \) to \( S \). Clearly, \( S' \cap B(S) = \emptyset \), and therefore \( S \) and \( S' \) are not weakly connected. This contradicts the assumption that \( S^* \) is a skeleton. \[ \Box \]
A **frontier visibility edge** is a visibility edge that intersects a frontier $f$. A **maximal frontier visibility edge** $s$ is a frontier visibility edge with the following properties:

- If $f$ is a vertical line segment, then $s$ maximises its horizontal advancement and, subject to this, maximises its vertical advancement towards the opposite (vertical) extreme edge that is parallel to $f$.
- If $f$ is a horizontal line segment, then $s$ maximises its vertical advancement and, subject to this, maximises its horizontal advancement towards the opposite (horizontal) extreme edge that is parallel to $f$.
- If $f$ is L-shaped, then $s$ maximises either its horizontal or vertical advancement and, subject to this, maximises its vertical or horizontal advancement (respectively) towards the opposite vertical or horizontal extreme edge.

**Lemma 5.6** Let $\omega$ be a staircase obstacle or a partial staircase obstacle such that $\omega$ has a top-right extreme corner and let $f$ be a frontier associated with a connected or weakly connected set line segments inside $\omega$ that intersects the left and bottom extreme edges of $\omega$. Let $s$ be a maximal frontier visibility edge associated with $f$. Then, $s$ maximises both its vertical and horizontal advancement.

**Proof** The proof is similar to the proofs of Lemmas 5.2 and 5.3. Assume, without loss of generality, that the direction of advancement is towards the top-right, as in Fig. 11. Let $S$ be a set of skeleton edges for $\omega$ that intersects the bottom and left extreme edges of $\omega$, and let $B$ be the bounding box of $S$. We initially assume that the frontier $f$ corresponding to $B$ is vertical, as in Fig. 11a. Suppose contrary to the lemma that $s_1$ and $s_2$ are maximal frontier visibility edges which maximise their horizontal and vertical advancements, respectively, and, subject to this, maximise their respective vertical and horizontal advancements. (Note that in this case the part of $\omega$ below $f_h$, the horizontal line through the topmost point of $f$, has already been covered in the vertical direction.) Let $v_1$ and $v_2$ denote the endpoints of $s_1$ and $s_2$ not in $B$. Let $v_3$ denote the point at the intersection of the vertical line through $v_1$ and the horizontal line through $v_2$. Let $I$ denote the intersection of $s_1$ and $s_2$. Since $s_1$ and $s_2$ are visibility edges (and are therefore in $\omega$), it follows that $I$ must be inside $\omega$, and therefore the polygon with vertices $I$, $v_1$, $v_3$, $v_2$ is inside $\omega$. Let $s_3$ denote the line segment between $v_3$ and the point where the ray from $v_3$ through $I$ intersects the boundary of $\omega$. Hence, the edge $s_3$ is inside $\omega$, and this edge has the same horizontal and vertical advancement as $s_1$ and $s_2$ (respectively), and greater advancement in the respective orthogonal directions. Hence, $s_1$ and $s_2$ are not maximal, giving a contradiction. Similar arguments apply if the frontier $f$ is horizontal or L-shaped (an example of the latter case is shown in Fig. 11b). \[\square\]

**Lemma 5.7** Let $\omega$ be a staircase obstacle and let $S$ be a set of line segments inside $\omega$ such that $S$ intersects an extreme corner $s_c^*$ of $\omega$ and $S$ is a minimum skeleton for $\omega \cap B(S)$. Let $f$ denote the frontier for $S$, and let $s_f^*$ be the (unique) maximal frontier visibility edge associated with $f$. Then, there exists a minimum skeleton $S^*$ for $\omega$ such that $S^*$ contains $S \cup s_f^*$.
Fig. 11 Proof of Lemma 5.6

Proof By Lemma 5.6, \( s_f^* \) maximises both its horizontal and vertical advancement. Therefore, \( \omega - B(S \cup s_f^*) \) requires the fewest possible number of edges to complete the skeleton. By Theorem 4.1, \( S \cup s_f^* \) is a skeleton for \( B(S \cup s_f^*) \) since \( S \) and \( s_f^* \) are weakly connected.

\[ \square \]

5.1.3 Iterative Algorithm for Computing Minimum Skeletons for Staircase Obstacles

The preceding results allow us to construct an iterative procedure for computing a minimum skeleton for a staircase obstacle \( \omega \) (Algorithm 1). Note that the algorithm is stated here in a general form, in which the inputs include an initial frontier and a termination frontier. This is because the algorithm will be used later as a sub-routine for computing minimum skeletons for partial staircases and general obstacles. For staircase obstacles, the initial frontier and termination frontiers are considered to be points, namely the extreme corners of the obstacles.

Algorithm 1: STAIRCASESKELETON

| Input: | (1) A staircase obstacle \( \omega \). (2) An initial frontier \( f_0 \). (3) A termination frontier \( f_t \). |
|--------|---------------------------------------------------------------|
| Output:| A minimum skeleton \( S^* \) for \( \omega \). |
| 1 \( S^* := \emptyset \) (initialise the skeleton). |
| 2 \( f := f_0 \) (initialise the current frontier). |
| 3 \( \text{terminate} := \text{FALSE} \) (initialise the termination variable). |
| 4 while \( \neg\text{terminate} \) do |
| 5 \( s^* := \text{a maximal frontier visibility edge for } f \). |
| 6 \( S^* := S^* \cup \{s^*\} \). |
| 7 if \( s^* \cap f_t \neq \emptyset \) then \( \text{terminate} := \text{TRUE} \). else \( f := \text{the new frontier for } S^* \). |
| 9 return \( S^* \) |

Note that Algorithm 1 could alternatively be implemented as a recursive algorithm. We demonstrate the application of Algorithm 1 to the example provided in Fig. 12. At the first iteration (Fig. 12a), the (unique) maximal extreme corner visibility edge
Constructing a minimum skeleton using Algorithm 1

s\textsuperscript{*} is added to \(S\) and the frontier \(f_1\) associated with \(s\textsuperscript{*}_1\) is computed. In the second and third iterations, the maximal frontier visibility edges \(s\textsuperscript{*}_2\) and then \(s\textsuperscript{*}_3\) are added to \(S\), and in each case the corresponding frontier is computed. Finally, in the fourth iteration, the maximal frontier visibility edge \(s\textsuperscript{*}_4\) is added to \(S\). In this case, there are a number of candidate frontier skeleton edges that complete the skeleton; our current implementation selects the longest edge from these candidates. The algorithm now terminates since \(S\) intersects the termination frontier \(c_2\).

**Theorem 5.1** \textsc{StaircaseSkeleton}(\(\omega, c_1, c_2\)) computes a minimum skeleton \(S\) for a given staircase obstacle \(\omega\), where \(c_1\) and \(c_2\) are the extreme corners of \(\omega\).

**Proof** The correctness of the algorithm follows from a straightforward inductive argument. By Lemma 5.2, a minimum skeleton \(S\) for \(\omega\) exists such that \(S\) contains a maximal extreme corner skeleton edge \(s\textsuperscript{*}_1\), and by Lemma 5.3, \(s\textsuperscript{*}_1\) is unique. The inductive step makes use of Lemma 5.7. At each iteration, the algorithm adds a new maximal frontier visibility edge to the existing set of skeleton edges such that the resulting edge set, \(S\), is a minimum skeleton for \(\omega \cap B(S)\). \(\square\)

### 5.2 Minimum Skeletons for Partial Staircase Obstacles

A partial staircase has exactly one extreme corner, which will be denoted by \(c\). Denote the two extreme edges that do not have an endpoint at \(c\) by \(e_1\) and \(e_2\). Now, both \(e_1\) and \(e_2\) must be intersected by \(S\). The example in Fig. 13 demonstrates that a minimum skeleton for a partial staircase does not necessarily intersect \(c\). In this case, any edge that has at endpoint at \(c\) will be redundant, since \(c\) is not visible to \(e_1\) or \(e_2\), and therefore two additional edges are required with endpoints on \(e_1\) and \(e_2\) to complete the skeleton (since \(e_1\) and \(e_2\) are not mutually visible).

The following lemma provides a method for constructing skeleton edges at the end of \(\omega\) corresponding to \(e_1\) and \(e_2\).

**Lemma 5.8** Let \(\omega\) be a partial staircase obstacle with extreme edges \(e_1\) and \(e_2\) that are opposite to the extreme corner of \(\omega\), and assume that \(\omega\) does not admit a cross. Let \(s\textsuperscript{*}_1\) and \(s\textsuperscript{*}_2\) be maximal extreme visibility edges with endpoints on \(e_1\) and \(e_2\), respectively. If \(\omega\) has a maximum length adjacent extreme visibility edge \(s\textsuperscript{*}_{12}\) between \(e_1\) and \(e_2\), then there exists a minimum skeleton \(S\) such that \(s\textsuperscript{*}_{12} \in S\). Otherwise, there exists a minimum skeleton \(S\) such that \(s\textsuperscript{*}_1 \in S\) and \(s\textsuperscript{*}_2 \in S\).
Fig. 13 A minimum skeleton for a partial staircase \( \omega \) does not necessarily intersect the extreme corner of \( \omega \)

Proof Suppose that \( \omega \) has a maximum length adjacent extreme visibility edge \( s_{12}^* \) between \( e_1 \) and \( e_2 \) (Fig. 14a). Let \( f \) denote the frontier associated with \( s_{12}^* \) (\( f \) is either L-shaped as in Fig. 14a, or it is a horizontal or vertical line). Let \( s_3^* \) denote the maximal frontier visibility edge for \( f \) (where the uniqueness of \( s_3^* \) follows from Lemma 5.6). Assume, without loss of generality, that the extreme corner of \( \omega \) is at the top right of \( \omega \), and hence that \( s_{12}^* \) is negatively sloped (as in Fig. 14a). Then, \( s_3^* \) can be assumed to be positively sloped, since any line segment intersecting \( f \) with a negative slope can be reflected about the vertical line through its midpoint and at least one of the endpoints of the resulting line segment can be extended to increase its horizontal and/or vertical advancement towards \( c \). Any positively sloped line segment inside \( \omega \) (whose endpoints lie on the boundary of \( \omega \)) that intersects \( f \) must have an endpoint on the bottom-left staircase walk of \( \omega \). By the convexity of \( \omega \), and since \( s_{12}^* \) is a maximum length adjacent extreme visibility edge, we can assume that \( s_{12}^* \) is the diagonal of the bounding box for \( s_{12}^* \cup e_1 \cup e_2 \), and therefore, \( s_{12}^* \cup s_3^* \) is connected. It is also clear that both the horizontal and vertical advancement of \( s_3^* \) is at least as great as that of each of \( s_1^* \) and \( s_2^* \). It follows that \( s_{12}^* \cup s_3^* \) is a minimum skeleton for \( \omega \cap B(s_{12}^* \cup s_3^*) \), and the region \( \omega - B(s_{12}^* \cup s_3^*) \) requires the fewest number of edges to complete the skeleton.

Now, suppose that \( e_1 \) and \( e_2 \) are not mutually visible (Fig. 14b). Using an argument similar to the argument used in Lemma 5.6, it can be shown that \( s_1^* \) and \( s_2^* \) are unique, and since \( s_1^* \) and \( s_2^* \) are maximal, the region \( \omega - B(s_1^* \cup s_2^*) \) requires the fewest number of edges to complete the skeleton.

Consider a partial staircase obstacle that has a maximum length adjacent extreme visibility edge \( s_{12}^* \), and let \( B(s_{12}^*) \) denote the bounding box of \( s_{12}^* \). When \( B(s_{12}^*) \) is subtracted from \( \omega \), the remaining obstacle \( \omega' \) has two possible forms (see Fig. 15): (a) \( \omega' \) is a sub-staircase with a frontier that is either horizontal or vertical. (b) \( \omega' \) can be treated as a sub-staircase with a ‘notch’ removed from it and an \( L \)-shaped frontier. Using the same argument used in the proof of Lemma 5.8, it can be seen that in both cases a maximal frontier visibility edge associated with \( f \) necessarily passes through \( f \) and intersects \( s_{12}^* \), thereby forming a connected network. We now show that if the minimum skeleton for \( \omega \) is not a cross, then there exists a minimum skeleton that intersects the extreme corner of \( \omega \).
**Lemma 5.9** Let $\omega$ be a partial staircase obstacle. If $\omega$ has a cross $X$, then $X$ is a minimum skeleton for $\omega$; otherwise, there exists a minimum skeleton $S^*$ for $\omega$ such that $S^*$ intersects the extreme corner $c$ of $\omega$.

**Proof** A partial staircase does not contain a diagonal, since by definition it has exactly one extreme corner. Therefore, $|S^*| \geq 2$ by Lemma 5.1. If $\omega$ has a cross $X$, then $X$ is a skeleton for $\omega$ and $|X| = 2$; hence, $X$ is a minimum skeleton for $\omega$. Now, suppose that $\omega$ does not contain a cross. When the bounding box of the extreme visibility edge or extreme visibility edges that are incident to $e_1$ and $e_2$ (i.e. either $s_{12}^*$ or $s_1^* \cup s_2^*$ using the notation of Lemma 5.8) is subtracted from $\omega$, then a sub-staircase $\omega'$ is obtained. The argument used in the proof of Lemma 5.2 can be applied to $\omega'$ to show that there exists a minimum skeleton $S^*$ for $\omega$ such that $S^*$ intersects the extreme corner $c$ of $\omega$. \qed
Algorithm 2 provides a procedure for computing a minimum skeleton for a partial staircase obstacle.

**Algorithm 2**: PARTIALSTAIRCASESKELETON

**Input**: A partial staircase obstacle \( \omega \).

**Output**: A minimum skeleton \( S^* \) for \( \omega \).

1. if \( \omega \) has a cross \( X \) then return \( X \) else let \( e_1, e_2, e_3, e_4 \) denote the extreme edges of \( \omega \), such that the extreme corner \( c \) of \( \omega \) is at the intersection of \( e_3 \) and \( e_4 \). if \( e_1 \) and \( e_2 \) are mutually visible then

   construct the maximum length adjacent extreme visibility edge \( s^*_{12} \) and let \( S^* = s^*_{12} \). else

   construct maximal extreme visibility edges \( s^*_1 \) and \( s^*_2 \) for \( e_1 \) and \( e_2 \), respectively, and let

   \[ S^* = s^*_1 \cup s^*_2. \]

2. \( \omega = \omega - B(S^*) \).

3. \( f = f(S^*) \).

4. return \( S^* \cup \text{STAIRCASESKELETON}(\omega, f, c) \).

**Theorem 5.2** PARTIALSTAIRCASESKELETON(\( \omega \)) computes a minimum skeleton \( S^* \) for a given partial staircase obstacle \( \omega \).

**Proof** There are three possibilities for \( \omega \) that need to be considered (note that by definition a partial staircase obstacle does not have a diagonal):

1. \( \omega \) has a cross: The existence, or otherwise, of a cross can be determined by checking for the existence of opposite extreme visibility edges.

2. \( \omega \) does not have a cross and has a maximum length adjacent extreme visibility edge \( s^*_{12} \): By Lemma 5.8, there exists a minimum skeleton \( S^* \) such that \( s^*_{12} \in S^* \). After constructing the frontier \( f \) associated with \( s^*_{12} \), the remainder of the skeleton can be constructed using STAIRCASESKELETON with initial frontier \( f \) and termination frontier \( c \).

3. \( \omega \) does not have a cross or a maximum length adjacent extreme visibility edge: By Lemma 5.8, there exists a minimum skeleton \( S^* \) such that \( s^*_1 \in S^* \) and \( s^*_2 \in S^* \). After constructing the frontier \( f \) associated with \( s^*_1 \cup s^*_2 \), the remainder of the skeleton can be constructed using STAIRCASESKELETON with initial frontier \( f \) and termination frontier \( c \).

\[ \square \]

### 5.3 Minimum Skeletons for General Obstacles

Finally, we examine general obstacles (obstacles with no extreme corners), starting with the special cases where \(|S^*| = 2\) and \(|S^*| = 3\).

**Lemma 5.10** Let \( \omega \) be a general obstacle and let \( S^* \) be a minimum skeleton for \( \omega \). Then, \(|S^*| = 2\) if and only if \( \omega \) has a cross \( X \).

**Proof** \((\rightarrow)\) If \( S^* = 2 \), then each extreme edge of \( \omega \) contains an endpoint of one of the two edges of \( S^* \), say \( s^*_1 \) and \( s^*_2 \). Therefore, \( s^*_1 \) and \( s^*_2 \) are either both opposite extreme visibility edges, or they are both maximum length adjacent extreme visibility edges.
In the former case, \( s_1^* \) and \( s_2^* \) form a cross. In the latter case, recall that a necessary condition for \( S^* \) to be a skeleton is that the projections of \( s_1^* \) and \( s_2^* \) cover the projections of \( \omega \). It follows that the bounding boxes of \( s_1^* \) and \( s_2^* \) must intersect, and a cross \( s_h \cup s_v \) can be constructed between the endpoints of \( s_1^* \) and \( s_2^* \) (see Fig. 16a). \((-\rightarrow)\) A cross \( X \) is a connected set of edges inside \( \omega \) that intersects the four extreme edges of \( \omega \). Therefore, \( X \) is a skeleton for \( \omega \) by Lemma A.1. It is also a minimum skeleton, since \( \omega \) has no extreme corners and hence does not have a diagonal, and therefore \( |S^*| > 1 \).\(\blacksquare\)

**Corollary 5.2** Let \( S^* \) be a minimum skeleton for a general obstacle that does not admit a cross. Then, \( |S^*| \geq 3 \).

The following lemma deals with the case when \( \omega \) has exactly one opposite extreme visibility edge.

**Lemma 5.11** Let \( \omega \) be a general obstacle that does not admit a cross. If \( \omega \) has an opposite extreme visibility edge \( s_{ik}^* \) between extreme edges \( e_i \) and \( e_k \), then \( s_{ik}^* \cup s_j^\perp \cup s_l^\perp \) is a minimum skeleton for \( \omega \), where \( s_j^\perp \) and \( s_l^\perp \) are perpendicular extreme visibility edges for \( e_j \) and \( e_l \), respectively.

**Proof** Let \( e_j \) and \( e_l \) be the two extreme edges of \( \omega \) corresponding to \( s_j^* \) and \( s_l^* \), respectively (Fig. 16b). When the bounding box of \( s_{ij}^* \) is subtracted from \( \omega \), two rectilinearly convex pieces remain, one containing \( e_j \) and the other containing \( e_l \). From the convexity of \( \omega \), the perpendicular extreme edges \( s_j^\perp \) and \( s_l^\perp \) clearly intersect \( s_{ik}^* \). The three edges together form a connected set of line segments that intersect all four extreme edges of \( \omega \) with the least possible number of line segments. \(\blacksquare\)

The following lemma addresses the case where \( \omega \) has an adjacent extreme visibility edge.

**Lemma 5.12** Let \( \omega \) be a general obstacle with extreme edges \( e_i, e_j, e_k, e_l \) such that the ends of \( \omega \) (if they exist) are \( \{e_i, e_j\} \) and \( \{e_k, e_l\} \), and such that \( \omega \) does not admit a cross...
or an opposite extreme visibility edge. If \( \omega \) has an adjacent extreme visibility edge \( s_{i_k}^* \), then \( s_{i_k}^* \cup s_{i_j}^\perp \cup s_{i_l}^\perp \) is a minimum skeleton for \( \omega \), where \( s_{i_j}^\perp \) and \( s_{i_l}^\perp \) are perpendicular extreme visibility edges for \( e_j \) and \( e_l \), respectively.

**Proof** When the bounding box of \( s_{i_j}^* \) is subtracted from \( \omega \), two rectilinearly convex pieces remain, one piece containing \( e_k \) and the other piece containing \( e_l \). The perpendicular extreme visibility edges \( s_{i_k}^\perp \) and \( s_{i_l}^\perp \) necessarily intersect \( s_{i_j}^* \) (from the convexity of \( \omega \)), and the three edges together form a connected set of line segments that intersect all four extreme edges of \( \omega \) with the least possible number of line segments. \( \square \)

### 5.3.1 Type (c) General Obstacles

The following result is required to construct minimum skeletons for Type (c) general obstacles.

**Lemma 5.13** Let \( \omega \) be a Type (c) general obstacle such that no pair of extreme edges of \( \omega \) are mutually visible, and let \( s_{i_i}^\perp \) denote a perpendicular extreme visibility edge for extreme edge \( e_i \). Then, \( s_{i_1}^\perp \cup s_{i_2}^\perp \cup s_{i_3}^\perp \cup s_{i_4}^\perp \) is a minimum skeleton for \( \omega \).

**Proof** The proof is similar to the proof of Lemma 5.11. Suppose that \( e_1, e_2, e_3 \) and \( e_4 \) correspond to the left, bottom, right and top extreme edges, respectively, and assume that \( e_1 \) is vertically higher than \( e_3 \) (and therefore \( e_2 \) is horizontally to the left of \( e_4 \)). From the convexity of \( \omega \), \( e_{i_1}^\perp \) extends from \( e_1 \) to a point that is at least as far to the right as the right-most endpoint of \( e_4 \), and \( e_{i_3}^\perp \) extends from \( e_3 \) to a point that is at least as far to the left as the left-most endpoint of \( e_2 \). Therefore, the horizontal projection of \( s_{i_1}^\perp \cup s_{i_3}^\perp \) covers the horizontal projection of \( \omega \). When the bounding box of \( s_{i_1}^\perp \cup s_{i_3}^\perp \) is subtracted from \( \omega \), two rectilinearly convex pieces remain, one containing \( e_2 \) and the other containing \( e_4 \). From the convexity of \( \omega \), the perpendicular extreme visibility edges \( s_{i_2}^\perp \) and \( s_{i_4}^\perp \) clearly intersect \( e_{i_1}^\perp \cup e_{i_3}^\perp \). The four edges together form a connected set of line segments that intersect all four extreme edges of \( \omega \) with the least possible number of line segments. \( \square \)

### 5.3.2 Type (d) General Obstacles

The following lemma provides a result for constructing skeleton edges at the ends of Type (d) general obstacles.

**Lemma 5.14** Let \( \omega \) be a general obstacle with ends \( \{e_1, e_2\} \) and \( \{e_3, e_4\} \) such that \( \omega \) does not have a cross, an opposite extreme visibility edge or an adjacent extreme visibility edge between extreme edges at opposite ends of \( \omega \). If \( \omega \) has an adjacent extreme visibility edge \( s_{i_{12}}^* \) for the end \( \{e_1, e_2\} \), then there exists a minimum skeleton \( S^* \) such that \( s_{i_{12}}^* \in S^* \). Otherwise, there exists a minimum skeleton \( S^* \) such that \( s_{i_1}^* \in S^* \) and \( s_{i_2}^* \in S^* \), where \( s_{i_1}^* \) and \( s_{i_2}^* \) are maximal extreme visibility edges with endpoints on \( e_1 \) and \( e_2 \), respectively.
Proof \ Refer to Fig. 17. Without loss of generality, let $e_1$ and $e_2$ denote the left and bottom extreme edges of $\omega$, respectively. Suppose initially that $e_1$ and $e_2$ are mutually visible and let $s_{12}^*$ denote an adjacent extreme visibility edge of $\omega$. Let $f$ denote the frontier associated with $s_{12}^*$. Let $s_1^*, s_2^*, s_3^*, s_4^*$ denote the maximal extreme visibility edges associated with $e_1, e_2, e_3, e_4$, respectively. Let $s_v^*$ and $s_h^*$ denote the two candidates for the maximal frontier visibility edge associated with $f$ (depending on the orientation of $f$), where $s_v^*$ is the line segment intersecting $f$ that maximises its vertical advancement and subject to this then maximises its horizontal advancement, while $s_h^*$ is the line segment intersecting $f$ that maximises its horizontal advancement and, subject to this, maximises its vertical advancement. Now, we have two cases (illustrated in Fig. 17):

- Case (a): Suppose that $s_v^* = s_h^*$ (this is equivalent to saying that the endpoint of $s_v^*$ opposite $e_1$ and $e_2$ does not meet the top-right staircase walk of $\omega$). Then, $s_{12}^* \cup s_v^*$ has at least as much horizontal and vertical advancement as $s_1^* \cup s_2^*$, and therefore, $\omega - B(s_{12}^* \cup s_v^*)$ requires the fewest possible number of edges to complete the skeleton.

- Case (b): Now, suppose that $s_v^* \neq s_h^*$ and assume initially that $s_v^*$ has an endpoint on the top-right staircase walk of $\omega$. If $e_3$ and $e_4$ are mutually visible (and have a corresponding adjacent extreme visibility edge $s_{34}^*$), then $s_v^*$ necessarily intersects $s_{34}^*$, and therefore, $s_{12}^* \cup s_v^* \cup s_{34}^*$ is a minimum skeleton for $\omega$. If $e_3$ and $e_4$ are not mutually visible, then $s_v^*$ necessarily intersects both of the extreme visibility edges $s_3^*$ and $s_4^*$. In this case, $s_{12}^* \cup s_v^* \cup s_3^* \cup s_4^*$ is a minimum skeleton for $\omega$. The preceding arguments are also applicable if $s_h^*$ has an endpoint on the top-right staircase walk of $\omega$. If both $s_v^*$ and $s_h^*$ have an endpoint on the top-right staircase walk of $\omega$, then either of the two edges can be selected for inclusion in $S^*$.

Now, suppose that $e_1$ and $e_2$ are not mutually visible. Again, there are two cases (shown in Fig. 17):

- Case (c) If $\omega - B(s_1^* \cup s_2^*)$ is a single region $\omega'$, then $\omega'$ requires the fewest possible number of edges to complete the skeleton.

- Case (d) Suppose $\omega - B(s_1^* \cup s_2^*)$ is comprised of two disjoint regions. If $s_{34}^*$ exists, then $s_1^* \cup s_2^* \cup s_{34}^*$ is a minimum skeleton for $\omega$; otherwise, $s_1^* \cup s_2^* \cup s_3^* \cup s_4^*$ is a minimum skeleton for $\omega$. \hfill \Box
Algorithm 3 provides a procedure for computing a minimum skeleton for a general obstacle.

**Algorithm 3: GENERALSKELETON**

**Input:** A general obstacle \( \omega \).

**Output:** A minimum skeleton \( S^* \) for \( \omega \).

1. Find the extreme edges of \( \omega \).
2. For each extreme edge \( e_i \), let \( s_i^* \) be the corresponding maximal extreme visibility edge and let \( s_i^\perp \) be any perpendicular extreme visibility edge.
3. **if** \( \omega \) has a cross \( X \) **then** return \( X \)
   **else if** \( \omega \) has an opposite extreme visibility edge \( s_{ik}^* \) **then** return \( s_{ik}^* \cup s_j^\perp \cup s_l^\perp \)
   **else if** \( \omega \) has an adjacent extreme visibility edge \( s_{ik}^* \) and \( \omega \) is not a Type (d) obstacle **then** return \( s_{ik}^* \cup s_j^\perp \cup s_l^\perp \)
   **else if** \( \omega \) is a Type (c) obstacle **then** return \( s_i^\perp \cup s_j^\perp \cup s_k^\perp \cup s_l^\perp \)
4. **else** determine the two ends \( \{e_1, e_2\} \) and \( \{e_3, e_4\} \) of \( \omega \) (as per Fig. 4d).
5. **if** there exists an adjacent extreme visibility edge \( s_{14}^* \) **then** return \( s_{14}^* \cup s_2^\perp \cup s_3^\perp \)
   **else if** there exists an adjacent extreme visibility edge \( s_{12}^* \) for \( e_1 \) and \( e_2 \) then \( S_{12}^* = s_{12}^* \)
   **else** \( S_{34}^* = s_3^* \cup s_4^* \)
6. **if** \( S_{12}^* \) and \( S_{34}^* \) are weakly connected **then** return \( S^* \)
   **else** \( \omega = \omega - B(S_{12}^*) - B(S_{34}^*) \)
7. \( S^* = S^* \cup \) STAIRCASESKELETON(\( \omega \), \( f(S_{12}^*) \), \( f(S_{34}^*) \)).

**Theorem 5.3** GENERALSKELETON(\( \omega \)) computes a minimum skeleton \( S^* \) for a given general obstacle \( \omega \).

**Proof** If \( \omega \) has a cross, an opposite extreme visibility edge, an adjacent visibility edge or if \( \omega \) is a Type (c) obstacle, then \( S^* \) can be constructed directly as in Lines 3–6 of the algorithm, by Lemmas 5.10, 5.11, 5.12 and 5.13. Otherwise, there exists a minimum skeleton \( S^* \) such that \( S_{12}^* \) and \( S_{34}^* \) are elements of \( S^* \) by Lemma 5.14. If \( S_{12}^* \) and \( S_{34}^* \) are weakly connected, then \( S_{12}^* \cup S_{34}^* \) is a minimum skeleton for \( \omega \) since \( |S^*| > 2 \), by Lemma 5.14. Otherwise, the remainder of the skeleton can be constructed using STAIRCASESKELETON with initial frontier \( f(S_{12}^*) \) and termination frontier \( f(S_{34}^*) \).

6 Exact Algorithm for Computing Minimum Skeletons

Algorithm 4 provides an exact algorithm for computing minimum skeletons for rectilinearly convex obstacles.

**Theorem 6.1** Algorithm 4 computes a minimum skeleton \( S^* \) for a given rectilinearly convex obstacle \( \omega \).

**Proof** Recall that \( \omega \) belongs to one of six possible classifications, based on the number and adjacency of extreme corners. Once the classification of the given obstacle has been identified, a minimum skeleton for the obstacle is obtained as follows:
Algorithm 4: \textsc{ComputeSkeleton}

\textbf{Input:} A rectilinearly-convex obstacle $\omega$.

\textbf{Output:} A minimum skeleton $S^*$ for $\omega$.

1. Find the extreme edges and extreme corners of $\omega$, and hence classify $\omega$.
2. \textbf{if} $\omega$ is a rectangle, L-obstacle or T-obstacle \textbf{then}
   3. \quad \textbf{if} $\omega$ has a diagonal $d$ \textbf{then} $S^* = \{d\}$ \textbf{else} $S^* = X$, where $X$ is a cross of $\omega$.
4. \textbf{else if} $\omega$ is a staircase obstacle \textbf{then} $S^* = \textsc{StaircaseSkeleton}(\omega, c_1, c_2)$, where $c_1$ and $c_2$ are the extreme corners of $\omega$.
5. \textbf{else if} $\omega$ is a partial staircase obstacle \textbf{then} $S^* = \textsc{PartialStaircaseSkeleton}(\omega)$.
6. \textbf{else} $S^* = \textsc{GeneralSkeleton}(\omega)$.
7. \textbf{return} $S^*$

– If $\omega$ is a rectangle, L-obstacle or T-obstacle, then by Corollary 5.1 if $\omega$ has a diagonal $d$ then $S^* = \{d\}$; otherwise, $S^*$ is a cross.
– The correctness of the final three cases follows by Theorems 5.1, 5.2 and 5.3, respectively.

Regarding the complexity of Algorithm 4, we note first that the algorithm is exact and finite. However, there are numerous computational steps, such as the construction of maximal visibility edges, that need to be implemented in an efficient way that will scale to obstacles with a large number of vertices. We defer discussion of most of the technical implementation issues to “Appendix” and the supplementary document (see [16]).

The running time of Algorithm 4 is governed by the construction of the set $G$ of candidate visibility edges, where each edge in $G$ is computed by the rotational plane sweep procedure described in “Appendix”.

Let $\omega$ be a rectilinearly convex obstacle with $n$ vertices, and let $v$ be a point on the boundary of $\omega$, where $v$ could be a non-convex vertex, an auxiliary point or an endpoint of an extreme edge. The rotational plane sweep procedure requires computing the gradients of the line segments between $v_i$ and every other non-convex vertex (or endpoint of an extreme edge) to the right of $v_i$. The vertices are then sorted by increasing gradient (this sorting can be undertaken in $O(n \log n)$ time using, for instance, the well-known Heapsort algorithm [17]), and a series of constant-time checks are performed on each vertex in the sorted list (see Fig. 19). In the worst case, a total of $\frac{n}{2}$ checks are required (i.e. one series of checks for each non-convex vertex of $\omega$).

To construct $G$, the rotational plane sweep procedure is applied to each of the $\frac{n}{2}$ non-convex vertices of $\omega$. In addition, the sweeping procedure is applied for each auxiliary point constructed during the running of Algorithm 4. As shown Lemma A.3 in “Appendix”, a minimum skeleton has at most $\frac{n}{2}$ edges, and therefore at most $\frac{n}{2}$ auxiliary points. As a consequence, an additional $2 \times \frac{n}{2}$ sweeps are required for auxiliary edges (since at most two sweeps are required for each auxiliary edge). From the preceding discussion, we conclude that the overall running time for Algorithm 4 is $O(n^2)$. 
7 Conclusions

We have introduced the concept of an obstacle skeleton, which is a set of line segments inside an obstacle \( \omega \) that can be used, in place of \( \omega \), when performing intersection tests for obstacle-avoiding shortest network problems in the plane. A skeleton can have significantly fewer line segments compared to the number of line segments on the boundary of the original obstacle, and therefore performing intersection tests on a skeleton (rather than the original obstacle) can significantly reduce the CPU time required by algorithms for computing shortest obstacle-avoiding networks. We have provided an exact \( O(n^2) \) algorithm for computing minimum skeletons for obstacles in the rectilinear plane that are rectilinearly convex (obstacles whose edges are either horizontal or vertical, and for which any two points in the obstacle have a shortest rectilinear path that is entirely inside the obstacle), in the context of the obstacle-avoiding rectilinear Steiner tree problem.

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Appendix

A.1 A Sufficient Condition for Skeletons

Lemma A.1 Let \( \omega \) be a rectilinearly convex obstacle, and let \( S \) be a set of line segments inside \( \omega \) such that (1) \( S \) intersects the four extreme edges of \( \omega \) and (2) \( S \) is connected. Then, \( S \) is a skeleton for \( \omega \).

Proof Let \( S \) be a connected set of line segments inside \( \omega \) that intersects the four extreme edges of \( \omega \), and suppose that \( S \) is not a skeleton for \( \omega \). Then, there exist points \( p \) and \( q \) outside the interior of \( \omega \) such that all shortest rectilinear paths between \( p \) and \( q \) with at most one corner point intersect the interior of \( \omega \), but at least one such path does not intersect \( S \).

If \( p \) and \( q \) lie on a horizontal or vertical line, then there is a unique shortest rectilinear path between \( p \) and \( q \) (i.e. the line segment \( pq \)) which enters and exits \( \omega \) at distinct points on the boundary of \( \omega \). The obstacle \( \omega \) can be partitioned into two regions, one on each side of \( pq \), and each region contains exactly one extreme edge of \( \omega \) that is parallel to \( pq \), each of which is intersected by \( S \). If \( pq \) does not intersect \( S \), then \( S \) must be disconnected, giving a contradiction.

Now, suppose that \( p \) and \( q \) do not lie on a horizontal or vertical line. Then, there are two shortest rectilinear paths with at most one corner point between \( p \) and \( q \), which we denote by \( pc_1q \) and \( pc_2q \). Since both \( pc_1q \) and \( pc_2q \) intersect \( \omega \), it follows that \( pc_1q \) cannot enter and exit \( \omega \) on the same staircase walk; otherwise, \( pc_2q \) would be outside the interior of \( \omega \) (due to the convexity of \( \omega \)).

Suppose that \( pc_1q \) does not intersect \( S \). If \( c_1 \) lies outside the interior of \( \omega \), then at least one of the line segments \( pc_1 \) and \( c_1q \) enters and exits \( \omega \) at distinct points on the boundary of \( \omega \), and the argument above (for the case where \( p \) and \( q \) are on a vertical or horizontal line) can be applied to arrive at the same contradiction. Otherwise, \( c_1 \) is
in the interior of $\omega$, and $pc_1q$ enters and exits $\omega$ at two locations on distinct staircase walks. Therefore, $pc_1q$ partitions $\omega$ into two regions, one on each side of $pc_1q$, and each region contains at least one extreme edge of $\omega$ (due to the convexity of $\omega$), each of which is intersected by $S$. If $pc_1q$ does not intersect $S$, it follows that $S$ must be disconnected, giving a contradiction. □

A.2 Implementation Details: Pre-computing the Candidate Set of Skeleton Edges (Excluding Auxiliary Edges)

We define an auxiliary edge to be a maximal frontier visibility edge with an endpoint that lies on its corresponding frontier. Let $S^*$ be a minimum skeleton that has been computed by Algorithm COMPUTESKELETON. Then, $S^*$ is composed of the following types of edges: (1) maximal extreme visibility edges (including opposite, adjacent and extreme-corner visibility edges and diagonals); (2) maximal frontier visibility edges that are not auxiliary edges; and (3) auxiliary edges. Let $G$ denote the set of all candidate skeleton edges that are not auxiliary edges. Then, $G$ can be constructed as a pre-processing step to the algorithm, while auxiliary edges are constructed in the course of the algorithm.

In order to construct $G$, we require the following property of maximal frontier visibility edges. We will refer to a staircase obstacle as being positively sloped if its extreme corners are located at the bottom-left and top-right corners of the obstacle (denoted by $c_1$ and $c_3$, respectively). The top-left and bottom-right staircase walks of the obstacle will be referred to simply as the top and bottom staircase walks, respectively.

**Lemma A.2** Let $\omega$ be a staircase obstacle that is positively sloped and let $e$ be a maximal frontier visibility edge in $\omega$ for some frontier $f$ of $\omega$, such that $e$ is not an auxiliary edge. Then, $e$ satisfies the following properties:

1. $e$ passes through two vertices of $\omega$, say $v_i = (x_i, y_i)$ and $v_j = (x_j, y_j)$, such that $x_j > x_i$ and $y_j > y_i$, and $v_i$ and $v_j$ are on different staircase walks; and
2. If $v_j \neq c_3$ and $v'_j$ is the endpoint of $e$ that is closest to $v_j$, then $v'_j$ is on the same staircase walk as $v_i$.

**Proof** Without loss of generality, assume that $v_i$ is on the bottom staircase walk. Then, there are four cases, three of which do not satisfy the property of the lemma (Fig. 18):

- (a) If $v_j$ and $v'_j$ are on the top and bottom staircase walks, respectively, then the conditions of the lemma are satisfied, and there is no continuous transformation of $e$ that increases its advancement.
- (b) If $v_j$ and $v'_j$ are both on the top staircase walk, then the advancement of $e$ can be increased by rotating $e$ clockwise about $v_i$.
- (c) If $v_j$ and $v'_j$ are on the bottom and top staircase walks, respectively, then the advancement of $e$ can be increased by rotating $e$ clockwise about $v_j$.
- (d) If $v_j$ and $v'_j$ are both on the bottom staircase walk, then the advancement of $e$ can be increased by transposing $e$ vertically upwards.

In the three cases (a)–(c) in which the candidate edge does not satisfy the property of the lemma, the specified transformation (translation or rotation) increases the hor-
Fig. 18  a A valid skeleton edge has points $v_i$, $v_j$ and $v'_j$ that alternate between opposite staircase walks.  

b–e Examples of edges that are not maximal.

izontal and/or vertical advancement of $e$. Since $e$ intersects the interior of $f$, it will continue to do so under any sufficiently small transformation, and therefore, $e$ is not a maximal frontier visibility edge, giving a contradiction. □

We now develop an efficient rotational plane sweep method for computing candidate skeleton edges. We begin by looking at maximal frontier visibility edges in staircase obstacles using Lemma A.2.

Let $\omega$ be a staircase obstacle with vertices $V = \{v_1, \ldots, v_n\}$. Assume without loss of generality that $\omega$ is positively sloped, and that its vertices are labelled in counterclockwise order around its boundary, starting with $v_1$ at the bottom-left extreme corner and denote the top-right extreme corner by $v_k$. Let $V_b$ and $V_t$ denote the non-convex vertices of $\omega$ on the bottom and top staircase walks (excluding the extreme corners), where a non-convex vertex is a vertex whose interior angle to the obstacle is 270 degrees. The candidate set of skeleton edges for $\omega$ (excluding auxiliary edges and extreme skeleton edges) can be efficiently generated as follows: For each non-convex vertex, $v_i \in V_b$ do the following (refer to Fig. 19).

1. Construct a half line $\rho$ starting at $v_i$ that initially points in the positive $x$-direction.
2. Rotate $\rho$ counterclockwise around $v_i$ until it intersects a non-convex vertex $v_j = (x_j, y_j)$ of $\omega$. There are six possible cases that can occur, each of which is illustrated in Fig. 19:
Fig. 19 Construction of maximal frontier visibility edges

(a) If $v_j \in V_b$ and there exists at least one vertex in $V_b \cup v_k$ that has not already been encountered in the rotational plane sweep by $\rho$, then continue to rotate $\rho$ counterclockwise.

(b) If $v_j \in V_b$ and all other vertices in $V_b \cup v_k$ have already been encountered in the rotational plane sweep by $\rho$, then there are no feasible visibility edges from $v_i$, since the requirements of Lemma A.2 cannot be satisfied.

(c) If $v_j \in V_t \cup v_k$ and (1) there exists at least one vertex in $V_b$ that has not been encountered in the rotational plane sweep and whose $y$-coordinate is in $[y_i, y_j]$, or (2), there exists at least one other vertex in $V_t$ that has been previously encountered in the rotational plane sweep and whose $y$-coordinate is in $[y_i, y_j]$, then either continue to rotate $\rho$ counterclockwise, or return no visibility edge if all vertices with $y$-coordinate in $[y_i, y_j]$ have now been encountered in the rotational plane sweep. If continued sweeping is possible,
all vertices in $V_t$ whose $y$-coordinate is greater than $y_j$ can be disregarded from the sweep.

(d) If $v_j = v_k$ and all vertices in $V_b$ whose $y$-coordinate is in $[y_i, y_j]$ have already been encountered in the rotational plane sweep, then stop the procedure and do not return a visibility edge (since any visibility edge with an endpoint at $v_k$ will be computed by a separate sweeping procedure applied to the extreme corner).

(e) If $v_j \in V_t$ and all vertices in $V_b$ whose $y$-coordinate is in $[y_i, y_j]$ have already been encountered in the rotational plane sweep, then compute the point $v'_j$ obtained when the line segment between $v_j$ is extended along the line through $v_i$ and $v_j$ to the boundary of $\omega$. If $v'_j$ lies on a horizontal edge of $\omega$, then there are no feasible visibility edges from $v_i$ satisfying the alternating property of Lemma A.2.

(f) Otherwise, $v_j \in V_t$ and all vertices in $V_b$ whose $y$-coordinate is in $[y_i, y_j]$ have already been encountered in the rotational plane sweep, and $v'_j$ lies on a vertical edge of $\omega$, in which case there is a feasible visibility edge that passes through $v_i$ and $v_j$.

In Case (f), the line segment between $v_i$ and $v_j$ is extended so that its endpoints lie on the boundary of $\omega$. This can be done without intersection computations as follows. To compute $v'_j$, take the set $V'_j$ of non-convex vertices of $\omega$ whose $y$-coordinate is greater than $y_j$ and sort the vertices by increasing $y$-coordinate. Search through the vertices by increasing $y$-coordinate until a vertex $v_k$ on the bottom staircase walk is encountered such that $g(v_iv_k) > g(v_iv_j)$, where $g(\cdot)$ denotes the gradient of a line segment. Then, $x_j' = x_k$ and $y_j' = y_i + g(v_iv_j)(x_j' - x_i)$. The extension $v'_i$ of $v_i$ is similarly computed, except that non-convex vertices must be checked on the top and bottom staircase walks, since $v'_i$ can be on either side.

A modified version of the above process is also applied to non-convex vertices on the top staircase walk. In this case, $\rho$ initially points in the positive $y$-direction and is rotated in the clockwise direction.

This process is also used to construct other types of skeleton edges, the first three of which can also be constructed during the pre-processing stage:

- Maximal extreme corner edges (at the ends of staircase obstacles): The rotational sweep is executed in the clockwise direction with $\rho$ starting at the extreme corner $c_1$. If a feasible edge is not found, then the sweep is executed in the clockwise direction with $\rho$ starting in the positive $y$-direction. To compute the maximal extreme corner visibility edge for $c_3$, the staircase obstacle is reflected about the $x$-axis and the $y$-axis, and the rotational sweep/s performed from $c_3$ (after the reflection $c_3$ is located at the bottom-left corner of $\omega$).
- Maximal extreme visibility edges for partial staircase and general obstacles are also computed from one or two rotational plane sweeps (after appropriate reflections have been made to $\omega$).
- Adjacent extreme visibility edges are constructed by starting with $v_i$ at an appropriate endpoint of either of the two extreme edges, and performing the rotational plane sweep in the relevant direction.
An auxiliary edge $e_f$ (an edge with an endpoint coinciding with an endpoint $v_f$ of a frontier during the construction of staircase skeletons) is also constructed by applying either a clockwise or a counterclockwise plane sweep with $v_f$ located at $v_f$ (if $e_f$ does not terminate at an extreme corner, then it either passes through a vertex on the top staircase walk and terminates on a vertical edge on the bottom staircase walk, in which case a counterclockwise sweep is required, or it passes through a vertex on the bottom staircase walk and terminates on a horizontal edge on the top staircase walk, in which case a clockwise sweep is required). These edges are constructed during the running of Algorithm STAIRCASE_SKELETON, as each new frontier is established.

A.3 Upper Bound on the Number of Edges in a Minimum Skeleton

Although there is no upper bound (independent of the number of obstacle vertices) on the number of edges in a minimum skeleton for staircase obstacles, partial staircase obstacles and general obstacles, we can nevertheless bound the number of skeleton edges based on the number of vertices of $\omega$ as follows.

Lemma A.3 Let $\omega$ be a rectilinearly convex obstacle with vertex set $V$ and edge set $E$, and let $S^*$ be a minimum skeleton for $\omega$. Then, $|S^*| \leq \frac{|V|}{2} = \frac{|E|}{2}$.

Proof For any polygonal obstacle, it is clear that the number of vertices is the same as the number of edges. If $\omega$ is a rectangle, then $|V| = 4$ and $|S^*| = 1$. If $\omega$ is an L-obstacle, then $|V| \geq 6$ and $|S^*| \leq 2$. If $\omega$ is a T-obstacle, then $|V| \geq 8$ and $|S^*| = 2$.

Now, suppose that $\omega$ is a staircase obstacle and assume without loss of generality that $\omega$ is positively sloped (Fig. 20a). Let $W_u$ and $W_l$ denote the sets of edges in the top-left and bottom-right staircase walks of $\omega$, respectively, and without loss of generality assume that $|W_u| \leq |W_l|$. Then, $W_u$ is a skeleton for $\omega$ since it is a connected set of line segments that intersects the four extreme edges of $\omega$, and $|W_u|$ has at most $\frac{|V|}{2}$ edges since $|W_u| + |W_l| = |E| = |V|$.

Now, suppose that $\omega$ is a partial staircase obstacle and without loss of generality assume that $e_1$ and $e_2$ are the (disjoint) left and bottom extreme edges and $c$ is an extreme corner at the top right of $\omega$ (Fig. 20b). Let $W_u$ and $W_l$ denote the top-left and bottom-right staircase walks of $\omega$ excluding $e_1$ and $e_2$, respectively, and assume that $|W_u| \leq |W_l|$. Then, $W_u \cup s^*_2$ is a skeleton for $\omega$ (where $s^*_2$ is the maximal extreme visibility edge for $e_2$) with $|W_u| + 1$ edges and since $\omega$ has at most $2|W_u| + 4$ edges, we have that $|S| \leq \frac{|V|}{2} - 1$. If $e_1$ and $e_2$ are mutually visible, then the same argument applies when $s^*_2$ is replaced by the adjacent extreme visibility edge $s^*_1$.

Similar arguments apply to the case where $\omega$ is a general obstacle (Fig. 20c). In this case, $W_u \cup s^*_2 \cup s^*_3$ is a skeleton for $\omega$ with at most $\frac{|V|}{2} - 2$ edges.

The bound stated in the lemma is tight and can be achieved by constructing a ‘skinny’ staircase obstacle for which $|W_u| \leq |W_l|$ and $W_u$ and $W_l$ are closely aligned, in which case $W_u$ is a minimum skeleton.
Fig. 20 Upper bound on the number of edges in a minimum skeleton. a Staircase obstacle. b Partial staircase obstacle. c General obstacle

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