Generalized covariant gyrokinetic dynamics of magnetoplasmas

C. Cremaschini\textsuperscript{a}, M. Tessarotto\textsuperscript{b,c}, P. Nicolini\textsuperscript{b} and A. Beklemishev\textsuperscript{d}

\textsuperscript{a}Department of Astronomy, University of Trieste, Trieste, Italy, \textsuperscript{b}Department of Mathematics and Informatics, University of Trieste, Trieste, Italy, \textsuperscript{c}Consortium of Magneto-fluid-dynamics, University of Trieste, Trieste, Italy, \textsuperscript{d}Budker Institute of Nuclear Physics, Novosibirsk, Russia

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Abstract

A basic prerequisite for the investigation of relativistic astrophysical magnetoplasmas, occurring typically in the vicinity of massive stellar objects (black holes, neutron stars, active galactic nuclei, etc.), is the accurate description of single-particle covariant dynamics, based on gyrokinetic theory (Beklemishev et al., 1999-2005). Provided radiation-reaction effects are negligible, this is usually based on the assumption that both the space-time metric and the EM fields (in particular the magnetic field) are suitably prescribed and are considered independent of single-particle dynamics, while allowing for the possible presence of gravitational/EM perturbations driven by plasma collective interactions which may naturally arise in such systems. The purpose of this work is the formulation of a generalized gyrokinetic theory based on the synchronous variational principle recently pointed out (Tessarotto et al., 2007) which permits to satisfy exactly the physical realizability condition for the four-velocity. The theory here developed includes the treatment of nonlinear perturbations (gravitational and/or EM) characterized locally, i.e., in the rest frame of a test particle, by short wavelength and high frequency. Basic feature of the approach is to ensure the validity of the theory both for large and vanishing parallel electric field. It is shown that the correct treatment of EM perturbations occurring in the presence of an intense background magnetic field generally implies the appearance of appropriate four-velocity corrections, which are essential for the description of single-particle gyrokinetic dynamics.

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I. INTRODUCTION

From the mathematical viewpoint, it is well known that the Hamiltonian dynamics of charged single particles can be expressed, in principle, in arbitrary hybrid (i.e., non-canonical) state variables. The search of phase-space transformations yielding “simpler” equations of motion has motivated in the past theoretical research in physics and mathematical physics. Among such transformations, a particularly significant case is provided by the gyrokinetic transformation, yielding generally hybrid variables defined in such a way that one of them, the so-called gyrophase angle, results ignorable \([1, 2]\). This transformation, which is typically based on a perturbative expansion (the so-called Larmor radius expansion), can only be introduced when particles move in a sufficiently strong magnetic field. The gyrokinetic description of particle dynamics is useful because in these variables the equations of motion for single charged particles generally exhibit a significantly lower computational complexity (with respect to the case of non-gyrokinetic variables). This is due to the fact that the dependence on the fast gyro-angle formally disappears in this representation. The same approach, nevertheless, can be adopted in principle also for the formulation of the kinetic and fluid descriptions of relativistic plasmas, provided the treatment of these plasmas can be based on single-particle dynamics, i.e., radiation effects (such as, in particular, radiation-reaction effects) are negligible. For a certain class of problems in plasma-physics and astrophysics, involving the study of relativistic plasma flows in strong gravitational fields, the adoption of the gyrokinetic formulation results of basic importance. This involves the description of plasmas in which the strength of the magnetic field becomes very large, which are observed or assumed to exist in accretion disks and related plasma jets around neutron stars, black holes, and active galactic nuclei \([3, 4]\). The conventional non-relativistic gyrokinetic theory (GKT) has been previously generalized to include only some relativistic effects, which are essential for description of confinement of fusion products in thermonuclear reactors, namely, the particle itself is considered relativistic, while its drift velocity is not \([6]\). Other approaches have instead investigated the formulation of covariant relativistic theories applicable in astrophysical problems. A first attempt at a formulation of this type was the Ph.D. thesis of Boghosian \([5]\). In line with previous non-covariant treatments, this approach assumed however a flat space-time, a vanishing parallel component of the electric field, ignoring non-linear EM perturbations and invoking the usual gyrokinetic
ordering for admissible wave forms. This implies that, the Maxwell’s equations in his approach were actually modified to maintain the ordering properties invoked for the linear EM perturbations. These deficiencies were corrected by Beklemishev et al. (1999-2005, [7, 8, 9]) who developed a fully covariant theory (derivation of the gyrokinetic transformation) holding in principle for arbitrary curved space-times. This theory, in particular, is applicable in the presence of non-uniform and non-linear electromagnetic (EM) and gravitational fields, as well as arbitrary wave-fields, i.e., non-linear perturbations of the EM and gravitational fields. Basic features of their approach, consistent with the allowance of relativistic drifts typical of many astrophysical plasmas, were the assumption of non-vanishing parallel electric field and the description of particle dynamics in the field-related basis tetrad, allowing a simpler formulation of the relativistic equations of motion. In addition, the perturbative theory was employed to derive the relativistic Vlasov-Maxwell equations expressed in gyrokinetic variables accurate through second order in $\varepsilon$ (which is the ratio of the gyro-radius/gyro-time to the equilibrium gradient length/time), while the wave fields are taken into account in the second order in the amplitude. Another remarkable feature of the formulation was to extend the customary applicability limits of GKT to include wave fields which can be fast oscillating in time ($\omega \rho_L/c \sim 1$) as well as in space ($k \rho_L \sim 1$), without involving the customary assumption $k_\parallel \rho_L \ll 1$ ($k_\parallel$ denoting the parallel wave-vector of the perturbation with respect to the local direction of the magnetic field). Nevertheless, the approach of Beklemishev et al. contained a limitation linked to the case of a small or vanishing background electric field. If $E$ and $B$ denote the invariant eigenvalues of the electromagnetic (EM) field tensor $F_{\mu\nu}$, this happens if there results locally $E/B \ll 1$ (denoted as weak electric field assumption (WEA); see also below) or even it occurs $E = 0$. For a certain class of problems in plasma-physics and astrophysics this limitation may be serious. In particular, this may occur (in relativistic plasmas) either if:

1. The plasma is characterized by a strong turbulence. Turbulence can manifest itself by the possible simultaneous presence of perturbations characterizing both the EM field and the plasma (to be described either in terms of kinetic or fluid descriptions). In such cases, due to strong perturbations, the electric field may locally vanish or become much smaller than the magnetic field.

2. However, difficulties with the perturbative gyrokinetic treatment adopted in [4, 8,
may occur for slowly varying (or "quasi-equilibrium") plasmas and even in the absence of wave fields if WEA occurs. This may happen when the standard gyrokinetic perturbative theory is carried out at higher orders in the Larmor-radius expansion.

3. The magnetic field is produced primarily by strong diamagnetic currents. In fact, even in the absence of an electric field (i.e., letting $E = 0$), it is well known that electric currents are produced in a magnetized plasma by gradients of the relevant fluid fields (in particular, fluid velocity, number density, temperature, magnetic field intensity, etc.).

The physical motivations of the present investigation are related to the occurrence of strongly magnetized relativistic plasmas in curved space-time, which may be strongly characterized by the presence of EM and kinetic turbulence. In the astrophysical context they are typically related to the existence of accretion disks, plasma inflows and outflows and relativistic jets, all occurring typically close to massive objects, such as neutron stars, black holes (BH) and active galactic nuclei (Mohanty [3]). These plasmas are believed to be dominated by collisionless, fully relativistic dynamics in which the charged particles of the plasma interact mutually primarily by means for the very EM mean field which they produce. This limitation was overcome by the GKT approach developed by Tessarotto et al. (Ref.1 [10]) extending the earlier formulation [7, 8, 9]. The new approach is based on key features: 1) the adoption of a synchronous hybrid Hamilton variational principle to describe single-particle covariant dynamics; 2) the introduction of an extended gyrokinetic transformation, which includes now also a suitable 4-velocity transformation and is taken of the form

$$\left( r^\alpha, u^\beta \right) \leftrightarrow y^i \equiv (r'^\alpha, u'^\alpha), \quad (1)$$

where $r^\alpha$ and $u^\beta$ coincide with the countervariant components of the 4-vectors $r$ (4-position) and $u$ (4-velocity), $r^\alpha$ and $u^\alpha$ denote the gyrokinetic variables; 3) the 4-velocity transformation is defined in such a way to warrant the fulfillment of the physical realizability condition for the 4-velocity, $u_\beta u^\beta = 1$, also by the transformed 4-velocity, namely $u'_\beta u'^\beta = 1$; 4) the gyrokinetic transformation (17) is defined in terms of a power series expansions of the form

$$r^\mu = r'^\mu + \sum_{s=1}^{\infty} \varepsilon^s r^\mu_s \equiv r'^\mu + \varepsilon\tilde{r}^\mu_1(y', \varepsilon), \quad (2)$$

$$u^\mu = u'^\mu \oplus \sum_{s=1}^{\infty} \varepsilon^s u^\mu_s (y) \equiv u'^\mu \oplus \varepsilon\tilde{v}^\mu_1(y', \varepsilon) \quad (3)$$
(extended gyrokinetic transformation), where \( r^\mu_i (y^i) \) and \( v^\alpha_i (y^i) \) (\( i = 1, \ldots \)) denote suitable perturbations and the operator \( \oplus \) is defined by the 4-velocity addition law \[6\] which permits to satisfy the physical realizability conditions for all 4-vectors involved (\( u^\mu, u'^\mu \) and \( \epsilon \tilde{\gamma}^\mu (y', \epsilon) \)). This requires that, if \( a_\mu \) and \( b_\mu \) denote two arbitrary 4-velocities, i.e., two unit time-like 4-vectors, such that \( a_\mu a^\mu = 1 \) and \( b_\mu b^\mu = 1 \), the \( \oplus \) operator is defined so that

\[
a_\mu \oplus b_\mu = \frac{a_\mu + b_\mu}{\sqrt{2(1 + a_\mu b^\mu)}} \tag{4}
\]
defines a 4-velocity too. In addition, here the perturbations \( r^\mu_i (y^i) \) and \( v^\alpha_i (y^i) \) are defined in such a way that both have - by assumption - a pure oscillatory with respect to the gyrophase \( \phi' \). In Ref.1 this approach was adopted to treat finite Larmor-radius effects on particle dynamics in the presence of non-uniform electromagnetic (EM) and gravitational fields. Nevertheless, an important issue concerns the extension of the theory (developed in Ref.1) to take into account also the possible presence of wave-fields, in principle both EM and gravitational. In this paper we intend to extend the applicability range of the GKT of Ref.1 to such a case. On the other hand since the theory relies on a perturbative expansion to determine relevant equations of motion, critical issues are its asymptotic convergence and accuracy. In this paper we intend to develop a perturbative solution scheme for the covariant gyrokinetic equations of motion for a particle immersed in a strong background magnetic field, which exhibits the correct asymptotic convergence properties even in the presence of EM perturbations. In detail, the main goals of the investigation are as follows:

1. generalization of the approach developed in Ref.1 to include the treatment of wave-fields (EM and gravitational) perturbations based on the introduction of a suitable gyrokinetic 4-velocity transformation;

2. explicit construction of the relevant extended gyrokinetic transformation appropriate for the gyrokinetic treatment of linear wave perturbations. In particular, we intend to show that, for prescribed wave-fields, the 4-velocity transformation can be reduced to the construction of a unique sequence of Lorentz transformations to be constructed in such a way to assure a) the fulfillment of the physical realizability condition \( u_\mu u^\mu = 1 \) for the 4-velocity; b) the proper convergence of the perturbative expansion also in the case of validity of the weak electric field assumption.
These features appear particularly relevant for the treatment of turbulent plasmas in the astrophysical context, thus providing an effective tool for use in plasma-physics-related problems in astrophysics.

II. FORMULATION OF COVARIANT GKT

Let us briefly summarize the basic steps of the covariant GKT approach based on the extended gyrokinetic transformation (2),(3). For this purpose, we adopt here a direct perturbative construction method rather than a non-canonical Lie-transform method [11]. As a starting point, this involves the introduction of a suitable perturbative expansion also for the EM 4-potential $A_\mu$ and the metric tensor $g_{\mu\nu}$. In particular let us assume that they are of the form

$$A_\mu = \frac{1}{\epsilon} \tilde{A}_\mu(r^\alpha, \epsilon) + \lambda a^\mu(r^\alpha/\lambda),$$

$$g_{\mu\nu} = G_{\mu\nu}(r^\alpha, \epsilon) + \lambda g^1_{\mu\nu}(r^\alpha/\lambda),$$

where $\epsilon$ and $\lambda$ are suitable dimensionless real parameters, both to be assumed infinitesimal and with $\epsilon \ll \lambda$. Here the notation is standard (see Ref.1). Thus, $A_\mu$ and $g_{\mu\nu}$ are respectively the counter-variant and covariant components of the EM 4-potential of the metric tensor, while $\frac{1}{\epsilon} \tilde{A}_\mu(r^\alpha, \epsilon)$, $G_{\mu\nu}(r^\alpha, \epsilon)$ and $\lambda a^\mu(r^\alpha/\lambda), \lambda g^1_{\mu\nu}(r^\alpha/\lambda)$ denote respectively the equilibrium terms and the so-called wave-fields, to be identified with suitable, rapidly-varying perturbations. All functions here considered are generally assumed to be represented by smooth, i.e., $C^{(\infty)}$ in functions. In particular, invoking the gyrokinetic transformation (2) for the 4-vector $r^\alpha$, it follows that “equilibrium” fields can be expanded in Taylor series near the guiding-center 4-position $r'^\alpha$. This implies that, neglecting corrections of order $o(\epsilon)$, $\frac{1}{\epsilon} \tilde{A}_\mu(r^\alpha)$ and $G_{\mu\nu}(r^\alpha, \epsilon)$ can be approximated respectively as

$$\frac{\tilde{A}_\mu}{\epsilon} = A^\mu_0 [1 + o(\epsilon)]$$

and

$$G_{\mu\nu} = G'_{\mu\nu} + \epsilon G^1_{\mu\nu} [1 + o(\epsilon)].$$

Here the primed quantities are all evaluated, as usual, at the guiding-center position ($A'^\mu = A^\mu(r'^\alpha)$, $G'_{\mu\nu} = g_{\mu\nu}(r'^\alpha)$, $A^\mu_0 = r'^\alpha \partial_{\alpha} A'^\mu$ and $G^1_{\mu\nu} \equiv r'^\alpha \partial_{\alpha} G'_{\mu\nu}$), while the perturbations (of relevant fields) depend generally also on the gyrophase angle $\phi'$ through the perturbation $\epsilon r'^\alpha(y^i)$. The perturbative scheme converges, at least in an asymptotic sense, only if one assumes the validity of a suitable ordering conditions (Larmor radius ordering), implying that EM contributions must dominate, in a suitable sense, over inertial ones in the variational functional. For-
mally this corresponds to replace the electron electric charge $e$ by $e/\varepsilon$, leaving the rest mass $m_0$ unchanged and considering the 4-velocity of order $o(e^0)$. To obtain the GKT for particle dynamics the relevant variational functional $S(y)$ must be expressed in terms of suitable gyrokinetic variables, $y^i$, which include in particular the gyrophase $\phi$ describing the fast gyration motion of particles around magnetic flux lines. As shown in Ref.1 a convenient formulation can be achieved by identifying the variational functional with the action $S(y) \equiv \int_1^2 \gamma(y) + dF$, where $y$ denotes the set of variables $y \equiv (r^\mu, u^\mu, \lambda)$ and $\gamma$ is the differential 1-form $\gamma(y) = g_{\mu\nu} [qA^\nu + u^\nu] dr^\mu + \xi [g_{\mu\nu}u^\mu u^\nu - 1]$, with $\xi$ a Lagrange multiplier and $q = Ze/m_0$ the normalized electric charge. It is immediate to show that phase-space trajectory of the point particle with 4-position $r^\nu$ and 4-velocity $u^\nu$ results, by construction, an extremal curve determined by the synchronous variational principle $\delta S(y) = 0$, where the synchronous variations $\delta r^\mu, \delta u^\mu, \delta \xi$ are all considered independent [10]. Indeed, the physical realizability condition $u_\nu u^\nu = 1$ is satisfied only by the extremal curve. To construct the transformations (2)-(3) corresponding to (5)-(6) the perturbations $r^\mu_i(y^i)$ and $u^\alpha_i(y^i)$ must be suitably determined. As usual, $u^\alpha$ can be expressed in terms of suitable independent (generally hybrid) gyrokinetic variables, one of which includes by definition the gyrophase $\phi'$. This is obtained by projecting $u^\alpha$ along the four independent directions defining the so-called fundamental tetrad unit 4-vectors $e^\mu_a$ (with $a = 0, ..., 3$), hereon also denoted $(l^\mu, l'^\mu, l^\mu, \tau^\mu)$ [or in symbolic notation $(l, l', l', \tau)$], where $l^\mu, l'^\mu, l^\mu$ and $\tau^\mu$ are respectively space-like and time-like orthogonal unit vectors. In particular, the latter can always be identified with the EM fundamental tetrad, i.e., the basis vectors of the EM field tensor, in this case to be evaluated at the gyrocenter position, $F'_{\mu\nu} \equiv F_{\mu\nu}(x^{\alpha\sigma})$, with eigenvalues $B$ and $E$. In particular, the space-like vectors $e^\mu_2, e^\mu_3$ are assumed to satisfy the eigenvalue equations $F'_{\mu\nu}e^\nu_2 = Be^\mu_3$, $F'_{\mu\nu}e^\nu_3 = -Be^\mu_2$. Here $e_{\alpha\mu} = g_{\mu\nu}e^\nu_a$ are the covariant components of $e^\mu_a$ and the labels $a$ assume a tensor meaning in the tangent space [so that in terms of the Minkowskian tensor $\eta = diag (1, -1, -1, -1)$ there results $e^a_\mu = \eta^{ab}e_b\mu$].

III. CONSTRUCTION OF THE EXTENDED GYROKINETIC TRANSFORMATION

To illustrate the key new features of the present approach let us point out the role of the extended gyrokinetic transformations (2) and (3). GKT is a perturbative theory which can
in principle be carried out at any order - both in \( \varepsilon \), and \( \lambda \) - provided the perturbations (i.e., \( \varepsilon \hat{r}_1^\mu \) and \( \varepsilon \hat{v}_1^\mu \)) are suitably small, i.e., in particular there results uniformly, in an appropriate subset of phase-space,

\[
\varepsilon \hat{r}_1^\mu \sim o(\varepsilon). \tag{7}
\]

This requirement is clearly necessary for the convergence of the gyrokinetic perturbative expansion. Hence it’s must be viewed as a consistency condition for the validity of GKT. To elucidate the point, it is sufficient to consider the perturbative formulation of GKT based on an expansion in \( \lambda \) which is accurate only to leading order in the dimensionless parameter \( \lambda \). As indicated above, \( \lambda \) represents the magnitude of the wave fields defined by Eqs.(5) and (6). Hence in this case a linear approximation in the wave-fields is actually considered for the perturbative expansion. If the velocity perturbations in Eq.(3) are neglected, one can prove that to assure that the differential 1-form \( \gamma \) results gyro-phase independent, the covariant formulation of GKT requires that to leading order in \( \lambda \) the following two constraints equations must be satisfied \[7\]:

\[
\begin{align*}
\left( u'_\mu + \lambda q a'_\mu - q r_1^\nu F'_{\mu\nu} \right) & \sim 0 \\
\left\{ \left( u'_\mu + \lambda q a'_\mu - \frac{1}{2} q r_1^\nu F'_{\mu\nu} \right) \frac{\partial r_1^\mu}{\partial \phi'} \right\} & \sim \frac{\partial R}{\partial \phi'},
\end{align*}
\tag{8,9}
\]

where \( R \) represents in principle an arbitrary gauge function (which can always be set \( R \equiv 0 \)). Assuming that \( EB \neq 0 \) and ignoring possible 4-velocity corrections [see Eq.(3)], Eq.(8) delivers a formal solution for the perturbation \( r_1^\nu \) (Larmor-radius 4-vector) which to leading-order in \( \lambda \) yields

\[
r_1^\nu = \frac{\omega'}{q} D^{\nu\mu} \left[ l_\mu \cos \phi' + l''_\mu \sin \phi' \right] + \lambda D^{\nu\mu} \hat{a}_\mu.
\]

Here \( D^{\nu\mu} \) is the inverse Faraday tensor which reads

\[
D^{\mu\nu} \equiv F^{\nu-1}_{\mu\nu} = \frac{1}{E} \left( l^{\nu} \tau^\mu - l^{\mu} \tau^\nu \right) - \frac{1}{E} \left( l''^{\nu} l^\mu - l''^{\mu} l^{\nu} \right).
\]

This expression manifestly diverges in the limit \( E \to 0 \). In particular, for \( |E| \) sufficiently small it is obvious that the second term on the r.h.s. can become arbitrarily large thus violating the ordering condition \[7\] (\textit{weak electric field assumption}). To assure the proper convergence of GKT also in this case it is manifest that an extended formulation of GKT is required. In particular, as indicated in Ref.1, it is actually necessary to introduce a perturbative expansion also for the 4-velocity. This can be conveniently represented in terms of the 4-velocity perturbative expansion \[8\] and by imposing the 4-velocity addition law \[1\]. In such a case the constraints \[8\] and \[9\] are replaced by the three equations:

\[
\left( u'_\mu \oplus \lambda v_1^\mu \right) \sim + \lambda q a'_\mu - q r_1^\nu F'_{\mu\nu} = 0, \tag{10}
\]

8
\[
\left\{ \left( u'_\mu \oplus \lambda v'_\mu \right) + \lambda q a'_\mu - \frac{1}{2} q r'_\nu F'_{\mu\nu} \right\} \frac{\partial r'_\nu}{\partial \phi'} \sim = \frac{\partial R}{\partial \phi'}, \tag{11} \right.
\]
\[
\left\{ [u'_0 \oplus \lambda u'_{01}]^2 - [u'_1 \oplus \lambda u'_{11}]^2 - [w' \oplus \lambda w'_1]^2 - 1 \right\} \sim = 0. \tag{12} \]

It is immediate to prove that - to leading-order in \( \lambda \) - these equations can be satisfied identically for arbitrary values of \( E \), by suitable definition of the 4-velocity perturbation \( \lambda v_{1\mu} \), defined in such a way to realize appropriate velocity cancellations for the wave-field perturbations \( \lambda q a_{\mu} \) occurring in the previous equations. In particular it can be proven that this result can be achieved by means of the following two Lorentz transformations:

- **First transformation (Lorentz boost, space-time rotation):** Let us first introduce a space-time 4-rotation acting along the directions \( (\tau, l^0) \) and let us impose that the spatial component of the transformed velocity cancels identically the corresponding component of the wave perturbation \( \wedge \). The 4-rotation takes the form:

\[
\begin{cases}
\wedge u'_{||} = (u'_{||} \oplus \lambda v'_{||}) \cosh \alpha + (u'_{0} \oplus \lambda v'_{0}) \sinh \alpha \\
\wedge u'_{0} = (u'_{0} \oplus \lambda v'_{0}) \sinh \alpha + (u'_{0} \oplus \lambda v'_{10}) \cosh \alpha
\end{cases} \tag{13}
\]

where \( \wedge u'_{||} \) and \( \wedge u'_{0} \) represent the two surviving components of the 4-velocity after the transformation.

- **Second transformation (Lorentz-boost, space-time rotation):** Let us introduce now a second space-time rotation acting along the two remaining directions \( (l^1, l^2, \tau) \), and which leave unchanged the parallel component \( \wedge u'_{||} \) (along the unit 4-vector \( \tau \)) already determined by the first Lorentz boost. We therefore define the second 4-rotation:

\[
\begin{cases}
\wedge w' = (w' \oplus \lambda w'_1(y)) \cosh \beta + u'_0 \sinh \beta \\
\wedge u'_0 = u'_0 \cosh \beta + (w' \oplus \lambda w'_1(y)) \sinh \beta
\end{cases} \tag{14}
\]

After the second transformation the 4-velocity takes the (desired) final form

\[
u_{\mu} = \wedge w' \left[ l'_\mu \cos(\phi') + l''_\mu \sin(\phi') \right] + \wedge u'_{||}_\mu + \wedge u'_0 \tau_{\mu}. \tag{15}\]

As a result one finds that the correct definition of the Larmor 4-vector is

\[
r''_{\mu} = \frac{1}{q} D'_{\mu\nu} \left( u'_\nu \right)^{\sim} + \lambda D'_{\mu\nu} \wedge a'_{\mu} =
\]

\[
= \frac{\wedge w'}{q} D'_{\mu\nu} \left[ (l'_\mu \cos \phi' + l''_\mu \sin \phi') \right] + \lambda D'_{\mu\nu} \left( a'_{2l'_\mu} + a'_{3l''_\mu} \right), \tag{16}\]
while the gyrokinetic differential 1-form (which is by construction gyro-phase independent) reads

$$\gamma'(\mathbf{y}') = \left( \frac{q}{\varepsilon} A_\mu' + u_\mu' \oplus \lambda q \mathbf{a}' \mu \right) dr'^\mu + \frac{1}{2} \left< \left( u_\mu' \oplus \lambda q \mathbf{a}' \mu \right) \frac{\partial r'^\mu}{\partial \phi'} \right> d\phi' +$$

$$+ \xi \left[ \left< u_\mu u^\mu \right> - 1 \right] ds$$

This expression is formally similar to that determined in Ref.1, except for the explicit appearance of terms containing the 4-velocity addition law (14) on the r.h.s. of Eq. (17), which have been introduced to preserve their proper physical interpretation (as 4-velocities) in the corresponding Euler-Lagrange equations. This implies that non-linear corrections in the wave-fields are, actually, taken into account in the gyrokinetic differential 1-form. This result shows the (formal) intrinsic simplicity of the perturbative approach to GKT here developed.

IV. CONCLUSIONS

In this paper we have formulated a covariant gyrokinetic approach for single-particle dynamics utilizing the variational approach developed in Ref.1. We have shown that:

- the theory goes beyond the domain of validity of the treatment previously considered [7, 8, 9] and includes the case in which the electric field can locally vanish or result much smaller than the magnetic field;

- the treatment includes the effect produced by the presence of wave-fields (produced by EM and gravitational perturbations). In particular, we have shown that this generally requires the introduction of an extended gyrokinetic transformation based on the introduction of an appropriate 4-velocity perturbative expansion;

- a relativistic addition law has been invoked for the 4-velocity perturbative expansion;

- non-linear corrections in the wave fields have been take into account in the perturbative theory to assure that the condition of physical realizability is satisfied by the transformed (gyrokinetic) 4-velocity.

The simplicity of the approach and its applicability to a wide range of possible physical situations make the present theory particularly suitable for applications to the investigation of relativistic plasmas, potentially both in astrophysics and laboratory plasmas.
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[11] We notice, in fact, that due to the assumed form of the 4-velocity transformation the adoption of Lie transform methods would result cumbersome in the present case. Indeed, Lie transformations acting on the 4-velocity do not generally satisfy the relativistic 4-velocity addition law.