GLOBAL EXISTENCE AND TIME-DECAY ESTIMATES OF SOLUTIONS TO THE COMPRESSIBLE NAVIER-STOKES-SMOLUCHOWSKI EQUATIONS

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Abstract. This paper is concerned with the Cauchy problem of the compressible Navier-Stokes-Smoluchowski equations in $\mathbb{R}^3$. Under the smallness assumption on both the external potential and the initial perturbation of the stationary solution in some Sobolev spaces, the existence theory of global solutions in $H^3$ to the stationary profile is established. Moreover, when the initial perturbation is bounded in $L^p$-norm with $1 \leq p < \frac{6}{5}$, we obtain the optimal convergence rates of the solution in $L^q$-norm with $2 \leq q \leq 6$ and its first order derivative in $L^2$-norm.

1. Introduction. In this paper, we consider the initial value problem of a fluid-particle interaction model called Navier-Stokes-Smoluchowski equations taking the form of

$$
\begin{cases}
\rho_t + \nabla \cdot (\rho u) = 0, \\
\rho (u_t + (u \cdot \nabla)u) + \nabla (p + \eta) - \mu \Delta u - (\mu + \mu') \nabla (\nabla \cdot u) = -(\eta + \beta \rho) \nabla \Phi, \\
\eta_t + \nabla \cdot [\eta (u - \nabla \Phi)] = \Delta \eta, \\
(\rho, u, \eta)(x, 0) = (\rho_0, u_0, \eta_0)(x) \to (\rho_\infty, 0, 0), \quad \text{as} \quad |x| \to \infty,
\end{cases}
$$

where $x = (x_1, x_2, x_3) \in \mathbb{R}^3$, $t > 0$, $\rho$ is the density of the fluid, $u := (u_1, u_2, u_3)$ is the fluid velocity field, and the density of the particles in the mixture $\eta \in \mathbb{R}$ is related to the probability distribution function $f(t, x, \xi)$ in the macroscopic description through the relation

$$
\eta(t, x) = \int_{\mathbb{R}^3} f(t, x, \xi) d\xi.
$$

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Here, the function $p_F = p_F(\rho)$ denotes the pressure of the fluid. Moreover, the time independent external potential $\Phi = \Phi(x) \in \mathbb{R}$ measures the effects of gravity and buoyancy, $\beta$ is a constant reflecting the differences in how the external force affects the fluid and the particles, $\mu$ and $\mu'$ are viscosity constants, satisfying $\mu > 0$, $2\mu + 3\mu' \geq 0$ which implies $\mu + \mu' > 0$. In addition, $(\rho_\infty, 0, 0)$ is the state of initial data at infinity, while $\rho_\infty > 0$ is a constant and $p_F(\rho)$ is smooth in a neighborhood of $\rho_\infty$ with $p_F(\rho_\infty) > 0$ and $p'_F(\rho_\infty) > 0$.

The fluid-particle interaction model plays an important role in sedimentation analysis of disperse suspensions of particles in fluids. Its applications in biotechnology, medicine, chemical engineering, mineral processes, etc. can be referred for instance to [3, 5, 19, 21, 22]. The system (1.1) consists in a Vlasov-Fokker-Planck equation to describe the microscopic motion of the particles coupled to the Navier-Stokes equations for a compressible fluid. Without the dynamic viscosity terms in (1.1), this system was derived formally by Carrillo and Goudon [6]. There are two different scaling limits for the coupling system between the kinetic and the fluid equations: the so-called bubbling and flowing regimes. They correspond to the diffusive approximation of the kinetic equation, the bubbling regime, written in (1.1), and the strong drag force and strong Brownian motion for the flowing regime (Refer to [6, 7] for more details). In [6], they considered the flowing regime and the bubbling regime under the two different scaling assumptions and investigated the stability and asymptotic limits finally. There have been some known results on the local and global well-posedness of the solutions of (1.1) in one dimension ([11, 18]). For the global existence of weakly dissipative solutions as well as their weak-strong uniqueness and low Mach number limits in high dimensions, please refer to Ballew-Trivisa’s work [4], Carrillo et. al’s work [7] and Ballew’s work [2], respectively. In particular, Carrillo et. al in their work [7] prove that the weak solutions exist globally in time and that the weak solutions converge to a stationary solution as time goes to $\infty$. More precisely, Carrillo et. al derived the following two theorems:

**Theorem A** (Carrillo-Karper-Trivisa: Global Existence). Assume that $(\Omega, \Phi)$ satisfy the “confinement hypotheses”. Then, the problem (1.1) supplemented with boundary conditions $u|_{\partial \Omega} = (\nabla_x \eta \cdot \nu + \eta \nabla_x \Phi \cdot \nu)|_{\partial \Omega} = 0$ and initial data $\rho(x, 0) = \rho_0 \in L_+^p(\Omega) \cap L_+^1(\Omega)$, $(\rho u)(x, 0) = m_0 \in L_+^p(\Omega) \cap L_+^1(\Omega)$, $\eta(x, 0) = \eta_0 \in L^2(\Omega) \cap L_+^1(\Omega)$ admits a weak solution $(\rho, u, \eta)$ on $\Omega \times (0, \infty)$.

**Theorem B** (Carrillo-Karper-Trivisa: Large-Time Asymptotics). Assume that $(\Omega, \Phi)$ satisfy the “confinement hypotheses”. Then, for any free energy solution $(\rho, u, \eta)$ of the problem (1.1), in the sense of “Definition 2.2”, there exist universal stationary states $\rho_s(x), \eta_s(x)$, such that

$$
\begin{align*}
\rho(t) &\to \rho_s \text{ strongly in } L^\gamma(\Omega), \\
\mathrm{ess}\sup_{\tau > t} \int_{\Omega} \rho(\tau)|u(\tau)|^2\,dx &\to 0, \\
\eta(t) &\to \eta_s \text{ strongly in } L^{p_2}(\Omega), \text{ for } p_2 > 1,
\end{align*}
$$

as $t \to \infty$. However, how fast does the solution converge to the stationary solutions as $t \to \infty$? It is still open. This motivates us to study the time-decay rates of the solutions.

Similar to the statement about the stationary solution of Navier-Stokes equations in [17], for system (1.1), there exists a stationary solution $(\rho_s, u_s, \eta_s)(x)$ in a small neighborhood of $(\rho_\infty, 0, 0)$ such that
To simplify the computations in the proof, we denote
\[ \int_{\rho_\infty}^{\rho_+(x)} \frac{p_F'(\zeta)}{\zeta} \, d\zeta + \beta \Phi(x) = 0, \quad u_*(x) = 0, \quad \eta_*(x) = 0, \quad (1.2) \]
and that
\[ \|\rho_* - \rho_\infty\|_{H^3} \leq C \|\Phi\|_{H^3} \quad (1.3) \]
for some positive constant \( C \). Here \( \|\Phi\|_{H^3} \leq \epsilon \) for some positive constant \( \epsilon \) as in (2.6).

2. Preliminary and main results. Before stating the main results, we would like to give a reformulation of (1.1). More precisely, denote
\[ \rho_1 = \frac{\mu}{\rho_\infty}, \quad \mu_2 = \frac{\mu + \mu'}{\rho_\infty}, \quad \gamma = \sqrt{\frac{p_F'\rho_\infty}{\rho_\infty}}, \]
\[ \tilde{\rho}(x, t) = \rho(x, t) - \rho_*(x), \quad \tilde{u}(x, t) = u(x, t), \quad \tilde{\eta}(x, t) = \eta(x, t), \]
and
\[ \tilde{\rho}(x) = \rho_*(x) - \rho_\infty. \]

Then the initial value problem (1.1) is reformulated as
\[ \begin{dcases}
\tilde{\rho}_t + \rho_\infty \nabla \cdot \tilde{u} = \tilde{S}_1, \\
\tilde{u}_t - \mu_1 \Delta \tilde{u} - \mu_2 \nabla \text{div} \tilde{u} + \frac{p_F'(\rho_\infty)}{\rho_\infty} \nabla \tilde{\rho} + \frac{1}{\rho_\infty} \nabla \tilde{\eta} = \tilde{S}_2, \\
\tilde{\eta}_t - \nabla \tilde{\eta} = \tilde{S}_3, \\
(\tilde{\rho}, \tilde{u}, \tilde{\eta})(x, t)|_{t=0} = (\rho_0 - \rho_*, u_0, \eta_0)(x) \to (0, 0, 0), \quad \text{as } |x| \to \infty,
\end{dcases} \quad (2.1) \]
where
\[ \tilde{S}_1 = \nabla \cdot \left[ (\tilde{\rho} + \tilde{\rho}) \tilde{u} \right], \]
\[ \tilde{S}_2 = -\tilde{u} \cdot \nabla \tilde{u} + \left( \frac{\mu}{\tilde{\rho} + \rho_*} - \frac{\mu}{\rho_\infty} \right) \Delta \tilde{u} + \left( \frac{\mu + \mu'}{\tilde{\rho} + \rho_*} - \frac{\mu + \mu'}{\rho_\infty} \right) \nabla \text{div} \tilde{u} + \frac{p_F'(\rho_\infty)}{\rho_\infty} \nabla \tilde{\rho} + \frac{1}{\rho_\infty} \nabla \tilde{\eta}, \]
\[ \tilde{S}_3 = -\nabla \cdot \left[ \tilde{\eta} (\tilde{u} - \nabla \Phi) \right]. \]

To simplify the computations in the proof, we denote
\[ \varrho(x, t) = \tilde{\rho}(x, t), \quad u(x, t) = \frac{\rho_\infty}{\sqrt{p_F'(\rho_\infty)}} \tilde{u}(x, t), \quad z(x, t) = \frac{1}{\sqrt{p_F'(\rho_\infty)}} \tilde{\eta}(x, t). \]

Then by (1.2), (2.1) can be rewritten as
\[ \begin{dcases}
\varrho_t + \gamma \nabla \cdot u = S_1, \\
u_t - \mu_1 \Delta u - \mu_2 \nabla \text{div} u + \gamma \nabla \varrho + \nabla z = S_2, \\
z_t - \Delta z = S_3, \\
(\varrho, u, z)(x, t)|_{t=0} = (\varrho_0, u_0, z_0)(x),
\end{dcases} \quad (2.2) \]
where
\[ S_1 = -\frac{\mu_1 \gamma}{\mu} \text{div} [\varrho (\varrho + \tilde{\rho}) u], \quad (2.3) \]
\[ S_2 = -\frac{\mu_1}{\mu} (u \cdot \nabla)u - \mu_1 h(\varrho, \bar{\rho}) \Delta u - \mu_2 h(\varrho, \bar{\rho}) \nabla \div u - \frac{\mu}{\mu_1} g_1(\varrho, \bar{\rho}) \nabla \bar{\rho} \]  
\[ S_3 = -\frac{\mu_1}{\mu} \nabla (z u) - \nabla \left( z \frac{p_1'(\rho_\ast)}{\beta \rho_\ast} \nabla \bar{\rho} \right), \]  
(2.4)

and

\[ h(\varrho, \bar{\rho}) := \frac{\varrho + \bar{\rho}}{\varrho + \bar{\rho} + \rho_\infty}, \quad \bar{h}(\varrho, \bar{\rho}) := \frac{p_1'(\bar{\rho} + \rho_\infty)}{(\bar{\rho} + \rho_\infty)(\varrho + \bar{\rho} + \rho_\infty)}, \]

\[ g_1(\varrho, \bar{\rho}) := \frac{p_1'(\varrho + \bar{\rho} + \rho_\infty)}{\varrho + \bar{\rho} + \rho_\infty} - \frac{p_1'(\bar{\rho} + \rho_\infty)}{\bar{\rho} + \rho_\infty}, \]

\[ g_2(\varrho, \bar{\rho}) := \frac{p_1'(\varrho + \bar{\rho} + \rho_\infty)}{\varrho + \bar{\rho} + \rho_\infty} - \frac{p_1'(\rho_\infty)}{\rho_\infty}. \]

The initial data consequently becomes

\[ (\varrho_0, u_0, z_0)(x) = \left( \varrho_0 - \rho_\ast, \frac{\rho_\infty}{\sqrt{p_1'(\rho_\infty)}} u_0, \frac{1}{\sqrt{p_1'(\rho_\infty)}} z_0 \right)(x) \to (0, 0, 0), \quad \text{as } |x| \to \infty. \]

Our main purpose in the paper is to study the global existence and time-decay rates of the solution \((\varrho, u, \eta)\) in a small perturbation of the stationary solution \((\rho_\ast, 0, 0)\), i.e., the existence and decay rates of the perturbed solution \((\varrho, u, z)\). In what follows, we state our main results. The first one is concerned with the global existence of the solution.

**Theorem 2.1** (Global existence). Let \((\varrho_0, u_0, z_0) \in H^3(\mathbb{R}^3)\) and \(\Phi \in H^4(\mathbb{R}^3)\). For given \(1 \leq p < \frac{6}{5}\), suppose that the potential function \(\Phi(x)\) and the initial perturbation satisfy

\[ \|\Phi\|_{H^4} + \|1 + |x|\|\nabla \Phi\|_{L^2 \cap L^3} \leq \epsilon, \]

\[ \|\varphi_0, u_0, z_0\|_{H^3} + \|z_0\|_{L^p} \leq \epsilon, \]

for some small constant \(\epsilon > 0\), then the Cauchy problem (2.2) admits a unique global solution \((\varrho, u, z)\).

**Remark 2.1.** In [7], the uniqueness of the weak solutions is still open. Here since the solutions have more regularity, the uniqueness holds.

**Remark 2.2.** When \(\eta \equiv 0\), the system (1.1) is reduced to the compressible Navier-Stokes equations. In this case, the global well posedness of the solution has been studied by Matsumura and Nishida in [17], where time-decay estimates were not necessary for proving the global well posedness. However, here the better time-decay rate of \(\|z(t)\|_{L^2}\) as in (3.40) plays a very important role in the proof of Theorem 2.1. For more details, please refer to (3.48). Thus this model has its own interest compared with compressible Navier-Stokes equations.

The second result is devoted to the time-decay rates of the solution.

**Theorem 2.2** (Decay estimates). Assume all hypothesis of Theorem 2.1 hold. In addition, for \(p \in \left[1, \frac{6}{5}\right]\), if \((\varrho_0, u_0)\) is bounded in \(L^p(\mathbb{R}^3)\), then there exists a constant \(\tilde{C}_0\) independent of \(t\), such that

\[ \|\nabla^k (\varrho, u, z)(t)\|_{L^2} \leq \tilde{C}_0 (1 + t)^{-\frac{k}{2} \left( \frac{1}{p} - \frac{5}{6} \right)} - \frac{k}{2}, \quad k = 0, 1, \]
and
\[ \|(\varrho, u, z)(t)\|_{L^q} \leq C_0(1 + t)^{-\frac{3}{2} \left(\frac{1}{p} - \frac{1}{q}\right)}, \quad \text{for } 2 \leq q \leq 6. \]

**Remark 2.3.** Compared with the stationary state in [7], we consider a special case of the stationary state of \(\eta\) for simplicity, i.e., \(\eta_*(x) = 0\). However, here we obtain the time-decay rate of the solution near the stationary state \((\rho_*, 0, 0)\).

**Remark 2.4.** In Theorem 2.2, we prove that the solution \((\rho, u, \eta)\) of (1.1) converges to the stationary solution \((\rho_*, 0, 0)\) at some time speed, and that the decay speeds in time for the zero-order spatially derivative and the first-order spatially derivative are different. For the study of the stability in time of more general stationary solutions, we leave it in the near future.

3. **Global existence.** Define the solution space of the initial value problem (2.1) with a norm as follows:

\[ X(0, T) = \{(\varrho, u, z) | \varrho \in C^0(0, T; H^3(\mathbb{R}^3)) \cap C^1(0, T; H^2(\mathbb{R}^3)), \]
\[ u, z \in C^0(0, T; H^3(\mathbb{R}^3)) \cap C^1(0, T; H^1(\mathbb{R}^3)), \]
\[ \nabla \varrho \in L^2(0, T; H^2(\mathbb{R}^3)), \nabla u, \nabla z \in L^2(0, T; H^3(\mathbb{R}^3))\}, \]

and

\[ N(0, T)^2 = \sup_{0 \leq t \leq T} \|(\varrho, u, z)(t)\|_{H^3}^2 + \int_0^T (\|\nabla \varrho(t)\|_{H^2}^2 + \|\nabla (u, z)(t)\|_{H^3}^2) dt, \]

for any \(0 \leq T \leq +\infty\). The following is the local existence of the solution to the problem (2.2).

**Proposition 3.1** (Local existence). Suppose that the initial data satisfy \((\varrho_0, u_0, z_0) \in H^3(\mathbb{R}^3)\) and (2.6). Then there exists a positive constant \(T_1 > 0\) depending on \(\varrho_0, u_0\) and \(z_0\), such that the initial value problem (2.2) has a unique solution \((\varrho, u, z) \in X(0, T_1)\) which satisfies \(N(0, T_1) \leq 2N(0, 0)\).

**Remark 3.1.** Proposition 3.1 is a special case of Theorem 2.1 in [9]. Thus, the proof is omitted here for simplicity. Refer also to [14, 17] for the ideas of the proof.

In what follows, we will establish some a priori estimates of the solutions globally in time. With the help of the local existence theory and those estimates, the global existence of solutions will be obtained by employing the standard continuity argument. To begin with, we make a priori assumption

\[ \sup_{0 \leq t \leq T} \|(\varrho, u, z)(t)\|_{H^3} \leq \delta, \quad (3.1) \]

for some \(T \in (0, T^*)\) where \(T^*\) represents the maximal time of existence of the solutions, and the constant \(\delta\) sufficiently small is chosen in (3.48). Using the Sobolev imbedding inequality, we are able to obtain that

\[ |\hat{h}(\varrho, \bar{\varrho})|, |\hat{g}_2(\varrho, \bar{\varrho})| \lesssim |\varrho| + |\bar{\varrho}|, \quad |\hat{g}_1(\varrho, \bar{\varrho})| \lesssim |\varrho| \quad \text{and} \quad |\hat{h}(\varrho, \bar{\varrho})| \leq C, \quad (3.2) \]

and

\[ |\partial_\alpha^\alpha \partial_\beta^\beta h(\varrho, \bar{\varrho})|, |\partial_\alpha^\alpha \partial_\beta^\beta g_2(\varrho, \bar{\varrho})|, |\partial_\alpha^\alpha \partial_\beta^\beta g_1(\varrho, \bar{\varrho})|, |\partial_\alpha^\alpha \partial_\beta^\beta \hat{h}(\varrho, \bar{\varrho})| \leq C \quad \text{with } |\alpha| + |\beta| \geq 1 \].

(3.3)

Here “\(\lesssim\)” represents that “\(\leq C \cdot\)” for some known constant \(C > 0\).

With the a priori assumption (3.1), we obtain the following estimates which can ensure the global existence of the solution. The first one is the \(L^2\) estimate of (\(\varrho, u\)).
Lemma 3.1. Under the assumptions of Proposition 3.1 and (3.1), it holds that
\[
\frac{d}{dt} \|(\varrho, u(t))\|_{L^2}^2 + D_1 \|\nabla u(t)\|_{L^2}^2 \leq C(\delta + \epsilon)(\|\nabla \varrho\|_{L^2}^2 + \|\nabla u\|_{H^1}^2 + \|\nabla z\|_{L^2}^2) + C\|z\|_{L^2}^2. 
\] (3.4)

Proof. Multiplying (2.2) by \(\varrho\) and \(u\), respectively, and then integrating by parts over \(\mathbb{R}^3\), we have from the sum of the resulting equalities that
\[
\frac{1}{2} \frac{d}{dt} \left(\|\varrho\|_{L^2}^2 + \|u\|_{L^2}^2\right) + \mu_1 \|\nabla u\|_{L^2}^2 + \mu_2 \|\text{div} u\|_{L^2}^2 \\
= - \langle u, \nabla z\rangle + \langle \varrho(t), S_1(t) \rangle + \langle u(t), S_2(t) \rangle. 
\] (3.5)

Using integration by parts and the Young inequality, we have
\[
- \langle u, \nabla z\rangle = \int_{\mathbb{R}^3} z \nabla \cdot u dx \leq \epsilon \|\nabla u\|_{L^2}^2 + \frac{C}{\epsilon} \|z\|_{L^2}^2. 
\] (3.6)

To estimate the last two terms on the right-hand side of (3.5), we notice that the source terms \(S_1\) and \(S_2\) have the following equivalent properties under the conditions of (2.6) and (3.1):
\[
S_1 \sim \nabla \varrho \cdot u + \varrho \nabla \cdot u + \nabla \rho \cdot u + \rho \nabla \cdot u, 
\] (3.7)
\[
S_2 \sim (u \cdot \nabla)u + \varrho \Delta u + \varrho \nabla \cdot u + \varrho \nabla \varrho + \rho \Delta u + \rho \nabla \varrho + \varrho \nabla z + \rho \nabla z + z \nabla \rho. 
\] (3.8)

With the help of (3.7), we use the Hölder inequality, Lemma 4.3, (1.2), (2.6) and the Young inequality, and then obtain
\[
\langle \varrho, S_1 \rangle \lesssim \|\varrho\|_{L^6} \|\nabla \varrho\|_{L^3} \|u\|_{L^3} + \|\varrho\|_{L^6} \|\varrho\|_{L^3} \|\nabla u\|_{L^2} \\
+ \|\varrho\|_{L^6} \|1 + |x|\| \nabla \varrho\|_{L^3} \left\| \frac{u}{1 + |x|} \right\|_{L^2} + \|\varrho\|_{L^6} \|\varrho\|_{L^3} \|\nabla u\|_{L^2} 
\lesssim (\delta + \epsilon) (\|\nabla \varrho\|_{L^2}^2 + \|\nabla u\|_{L^2}^2), 
\] (3.9)

where we also have used the following Hardy inequality
\[
\left\| \frac{u}{1 + |x|} \right\|_{L^2} \lesssim \|\nabla u\|_{L^2}. 
\]

With (3.8), similar to the proof of \(\langle \varrho, S_1 \rangle\), we get
\[
\langle u, S_2 \rangle \lesssim (\delta + \epsilon) (\|\nabla \varrho\|_{L^2}^2 + \|\nabla u\|_{H^1}^2 + \|\nabla z\|_{L^2}^2). 
\] (3.10)

Plugging (3.6)-(3.10) into (3.5) yields
\[
\frac{1}{2} \frac{d}{dt} \|(\varrho, u(t))\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 \lesssim (\delta + \epsilon) (\|\nabla \varrho\|_{L^2}^2 + \|\nabla u\|_{H^1}^2 + \|\nabla z\|_{L^2}^2) + \frac{C}{\epsilon} \|z\|_{L^2}^2. 
\]

Thus, we complete the proof of Lemma 3.1. \qed

Lemma 3.2. Under the assumptions of Proposition 3.1 and (3.1), it holds that
\[
\frac{d}{dt} \|\nabla^k (\varrho, u(t))\|_{L^2}^2 + D_2 \|\nabla^{k+1} u(t)\|_{L^2}^2 \\
\lesssim (\delta + \epsilon) (\|\nabla^k \varrho\|_{L^2}^2 + \|\nabla u\|_{H^2}^2 + \|\nabla z\|_{H^1}^2 + \|\nabla \varrho\|_{H^1}^2) + \|\nabla^{k+1} z\|_{L^2}^2, 
\] (3.11)

for \(k = 1, 2, 3\).
Proof. Multiplying $\nabla^k \varphi(2.2)_1$, $\nabla^k \varphi(2.2)_2$ by $\nabla^k \varphi$ and $\nabla^k u$, respectively, and then integrating by parts over $\mathbb{R}^3$, we have from the sum of the resultant equalities that
\begin{equation}
\frac{1}{2} \frac{d}{dt} \left( \|\nabla^k \varphi\|^2_{L^2} + \|\nabla^k u\|^2_{L^2} + \mu_1 \|\nabla^{k+1} u\|^2_{L^2} + \mu_2 \|\nabla^k \operatorname{div} u\|^2_{L^2} \right) = - \langle \nabla^k u, \nabla^{k+1} \varphi \rangle + \langle \nabla^k \varphi(t), \nabla^k S_1(t) \rangle + \langle \nabla^k u(t), \nabla^k S_2(t) \rangle.
\end{equation}

We are going to estimate the terms on the right hand side of the above equality. More precisely, for the first term on the right hand side of (3.12), we get
\begin{equation}
- \langle \nabla^k u, \nabla^{k+1} \varphi \rangle \leq \epsilon \|\nabla^k u\|^2_{L^2} + \frac{C}{\epsilon} \|\nabla^{k+1} \varphi\|^2_{L^2}.
\end{equation}

For the second term on the right hand side of (3.12), we obtain
\begin{align}
\langle \nabla^k \varphi(t), \nabla^k S_1(t) \rangle &= - \frac{\mu_1 \gamma}{\mu} \int_{\mathbb{R}^3} \nabla^k (\varphi \text{div} u) \nabla^k \varphi \, dx - \frac{\mu_1 \gamma}{\mu} \int_{\mathbb{R}^3} \nabla^k (\varphi \cdot u) \nabla^k \varphi \, dx \\
&\quad - \frac{\mu_1 \gamma}{\mu} \int_{\mathbb{R}^3} \nabla^k (\bar{\varphi} \text{div} u) \nabla^k \varphi \, dx - \frac{\mu_1 \gamma}{\mu} \int_{\mathbb{R}^3} \nabla^k (\bar{\varphi} \cdot u) \nabla^k \varphi \, dx \\
&:= I_1 + I_2 + I_3 + I_4.
\end{align}

For $I_1$, we have
\begin{align}
I_1 &\lesssim \|\nabla^k (\varphi \text{div} u)\|_{L^2} \|\nabla^k \varphi\|_{L^2} \\
&\lesssim \left( \|\nabla^k \varphi\|_{L^2} \|\text{div} u\|_{L^\infty} + \|\varphi\|_{L^\infty} \|\nabla^k \text{div} u\|_{L^2} \right) \|\nabla^k \varphi\|_{L^2} \\
&\lesssim \delta \left( \|\nabla^k \varphi\|^2_{L^2} + \|\nabla^k \text{div} u\|^2_{L^2} \right),
\end{align}
where we have used the Hölder inequality, (3.1) and (4.40).

For $I_2$, we have
\begin{align}
I_2 &= - \frac{\mu_1 \gamma}{\mu} \int_{\mathbb{R}^3} \nabla^k \varphi \nabla \cdot u \nabla^k \varphi \, dx \\
&\quad - \frac{\mu_1 \gamma}{\mu} \int_{\mathbb{R}^3} \nabla^k (\varphi \cdot u) - \nabla^k \varphi \cdot u \nabla^k \varphi \, dx \\
&\lesssim \|\nabla u\|_{L^\infty} \|\nabla^k \varphi\|^2_{L^2} \\
&\quad + \left( \|\nabla^k \varphi\|_{L^2} \|\nabla u\|_{L^\infty} + \|\varphi\|_{L^\infty} \|\nabla^k u\|_{L^2} \right) \|\nabla^k \varphi\|_{L^2} \\
&\lesssim \delta \left( \|\nabla^k \varphi\|^2_{L^2} + \|\nabla^{k+1} u\|^2_{L^2} \right),
\end{align}
where we have used the Hölder inequality, (3.1) and (4.41). Similar to the evaluations of $I_1$ and $I_2$, for $I_3$ and $I_4$, we get
\begin{align}
I_3 &= - \frac{\mu_1 \gamma}{\mu} \sum_{1 \leq i \leq k} C_k^i \int_{\mathbb{R}^3} \nabla^i \bar{\rho} \nabla^{k-i} \text{div} u \nabla^k \varphi \, dx \\
&\lesssim \sum_{1 \leq i \leq k-2} \|\nabla^i \bar{\rho}\|_{L^\infty} \|\nabla^{k-i+1} u\|_{L^2} \|\nabla^k \varphi\|_{L^2} \\
&\quad + \left( \|\nabla^{k-1} \bar{\rho}\|_{L^\infty} \|\nabla^2 u\|_{L^3} + \|\nabla^k \bar{\rho}\|_{L^2} \|\nabla u\|_{L^\infty} \right) \|\nabla^k \varphi\|_{L^2} \\
&\leq \epsilon \|\nabla^k \varphi\|^2_{L^2} + \frac{C}{\epsilon} \sum_{1 \leq i \leq k-2} \|\nabla^i \bar{\rho}\|^2_{L^2} + \|\nabla^{k-i+1} u\|^2_{L^2} + \epsilon \|\nabla^2 u\|^2_{H^1},
\end{align}
\begin{align}
&\leq \epsilon \|\nabla^k \varphi\|^2_{L^2} + C \epsilon \sum_{2 \leq i \leq k} \|\nabla^i u\|^2_{L^2},
\end{align}
and

\[
I_4 := - \frac{\mu_1 \gamma}{\mu} \left\{ \sum_{l=k} C_k \int_{\mathbb{R}^3} \nabla^l \nabla \tilde{\rho} \cdot \nabla^{k-l} u \nabla^k u \, dx \right. \\
\leq C \| \nabla \tilde{\rho} \|_{L^2}^2 + C \frac{L^4}{L^4 \epsilon} \sum_{l \leq k-1} \| \nabla^l \nabla \tilde{\rho} \cdot \nabla^{k-l} u \|_{L^2}^2 \\
\leq C \| \nabla \tilde{\rho} \|_{L^2}^2 + C \frac{L^4}{L^4 \epsilon} \sum_{l \leq k-1} \| \nabla^l \nabla \tilde{\rho} \|_{L^2}^2 \| \nabla^{k-l} u \|_{L^2}^2 \\
+ C \frac{L^4}{L^4 \epsilon} \| \nabla \tilde{\rho} \|_{L^6}^2 \| u \|_{L^3}^2, \\
\leq C \| \nabla \tilde{\rho} \|_{L^2}^2 + C \epsilon \sum_{l \leq k-1} \| \nabla^l u \|_{L^2}^2,
\]

where we have used (1.3) and (2.6).

For the third term on the right hand side of (3.12), we obtain

\[
\langle \nabla^k u(t), \nabla^k S_2(t) \rangle = - \frac{\mu_1 \gamma}{\mu} \int_{\mathbb{R}^3} \nabla \tilde{\rho} \cdot \nabla^k u \cdot \nabla^k u \, dx - \mu_1 \int_{\mathbb{R}^3} \nabla \tilde{\rho} \cdot \nabla^k (h(\varrho, \tilde{\rho}) \Delta u) \cdot \nabla^k u \, dx \\
- \mu_2 \int_{\mathbb{R}^3} \nabla \tilde{\rho} \cdot \nabla \left( \sum_{l=1}^7 J_l \right) \cdot \nabla^k u \, dx
\]

For \( J_1 \), we have

\[
J_1 := - \frac{\mu_1 \gamma}{\mu} \int_{\mathbb{R}^3} \nabla \tilde{\rho} \cdot \nabla^k \left( \sum_{l=1}^7 J_l \right) \cdot \nabla^k u \, dx
\]

where we have used the Hölder inequality, (4.40), (3.1), the Sobolev inequality and Gagliardo-Nirenberg inequality. With (1.3), (2.6), (3.2) and (4.42), similar to (3.20), we have,

\[
J_2 \approx \int_{\mathbb{R}^3} \nabla^{k-1} (h(\varrho, \tilde{\rho}) \Delta u) \cdot \nabla^{k+1} u \, dx
\]

\[
\lesssim \left( \| \nabla^{k-1} h(\varrho, \tilde{\rho}) \|_{L^6} \| \Delta u \|_{L^3} + \| h(\varrho, \tilde{\rho}) \|_{L^\infty} \| \nabla^{k+1} u \|_{L^2} \right) \| \nabla^{k+1} u \|_{L^2} \\
\lesssim (\delta + \epsilon) \left( \| \nabla \tilde{\rho} \|_{L^2}^2 + \| \nabla^2 u \|_{H^1}^2 + \| \nabla^{k+1} u \|_{L^2}^2 \right),
\]

and

\[
J_3 \lesssim (\delta + \epsilon) \left( \| \nabla \tilde{\rho} \|_{L^2}^2 + \| \nabla^2 u \|_{H^1}^2 + \| \nabla^{k+1} u \|_{L^2}^2 \right),
\]
and
\[ J_4 + J_5 + J_6 \lesssim (\delta + \epsilon) \left( \|\nabla^k \varrho\|_{L^2}^2 + \|\nabla^2 u\|_{H^1}^2 + \|\nabla^k z\|_{L^2}^2 \right) \]
\[ + (\delta + \epsilon) \left( \|\nabla^{k+1} u\|_{L^2}^2 + \|\nabla z\|_{H^1}^2 + \|\varrho\|_{H^1}^2 \right). \] (3.23)

For \( J_7 \), we have
\[ J_7 \approx \int_{\mathbb{R}^3} \nabla^{k-1} \left( \hat{h}(\varrho, \bar{\rho}) \nabla \right) \cdot \nabla^{k+1} u \, dx \]
\[ \lesssim \int_{\mathbb{R}^3} \left( \|\nabla^{k-1} (z, \nabla \rho)\|_{L^2} \|\hat{h}(\varrho, \bar{\rho})\|_{L^\infty} + \|z \nabla \rho\|_{L^3} \|\nabla^{k-1} \hat{h}(\varrho, \bar{\rho})\|_{L^5} \right) \|\nabla^{k+1} u\|_{L^2} \, dx \]
\[ \lesssim \int_{\mathbb{R}^3} \left( \|z\|_{L^6} \|\nabla^k \bar{\rho}\|_{L^3} + \|\nabla^{k-1} z\|_{L^6} \|\nabla \rho\|_{L^3} \right) \|\nabla^{k+1} u\|_{L^2} \, dx \]
\[ + \int_{\mathbb{R}^3} \|z\|_{L^\infty} \|\nabla \rho\|_{L^3} (\|\nabla^k \varrho\|_{L^2} + \|\nabla^k \bar{\rho}\|_{L^2}) \|\nabla^{k+1} u\|_{L^2} \, dx \]
\[ \lesssim (\delta + \epsilon) \left( \|\nabla^k \varrho\|_{L^2}^2 + \|\nabla^{k+1} u\|_{L^2}^2 + \|\nabla z\|_{H^1}^2 + \|\nabla^k z\|_{L^2}^2 \right). \] (3.24)

With those estimates above, we complete the proof of the lemma. \( \square \)

**Lemma 3.3.** Under the assumptions of Proposition 3.1 and (3.1), it holds that
\[ \frac{d}{dt} \|\nabla^k \varrho(t)\|_{H^1}^2 + C_2 \|\nabla^{k+1} \varrho(t)\|^2 \]
\[ \lesssim (\delta + \epsilon) \left( \|\nabla^k u\|_{H^1}^2 + \|\nabla^{k+1} u\|_{H^1}^2 + \|\nabla z\|_{H^1}^2 + \|\nabla u\|_{H^1}^2 + \|\varrho\|_{H^1}^2 \right) \] (3.25)
for \( k = 0, 1, 2 \).

**Proof.** Applying \( \nabla^k \) to (2.2) and then taking the \( L^2 \) inner product with \( \nabla \nabla^k \varrho \), we have
\[ \gamma \int_{\mathbb{R}^3} \|\nabla^k \varrho\|_{L^2}^2 \, dx \leq - \int_{\mathbb{R}^3} \nabla^k \partial_t u \cdot \nabla \nabla^k \varrho \, dx + C_2 \|\nabla^{k+2} u\|_{L^2} \|\nabla^{k+1} \varrho\|_{L^2} \]
\[ - \int_{\mathbb{R}^3} \nabla \nabla^k z \cdot \nabla \nabla^k \varrho \, dx + \|\nabla^k S_2\|_{L^2} \|\nabla^{k+1} \varrho\|_{L^2}. \] (3.26)

With (2.2), the first term on the right hand side of (3.26) is estimated as follows:
\[ - \int_{\mathbb{R}^3} \nabla^k \partial_t u \cdot \nabla \nabla^k \varrho \, dx = - \frac{d}{dt} \int_{\mathbb{R}^3} \nabla^k u \cdot \nabla \nabla^k \varrho \, dx - \int_{\mathbb{R}^3} \nabla^k \text{div} u \cdot \nabla \partial_t \varrho \, dx \]
\[ = - \frac{d}{dt} \int_{\mathbb{R}^3} \nabla^k u \cdot \nabla \nabla^k \varrho \, dx + \gamma \|\nabla^k \text{div} u\|_{L^2}^2 \] (3.27)
\[ + \frac{\mu_1 \gamma}{\mu} \int_{\mathbb{R}^3} \nabla^k \text{div} u \cdot \nabla \text{div} \left[ (\varrho + \bar{\rho}) u \right] \, dx. \]

Using the Hölder inequality, (1.3), (2.6), (3.1) and (4.40), we have
\[ \frac{\mu_1 \gamma}{\mu} \int_{\mathbb{R}^3} \nabla^k \text{div} u \cdot \nabla \text{div} \left[ (\varrho + \bar{\rho}) u \right] \, dx \]
\[ \lesssim \|\nabla^k \text{div} \left[ (\varrho + \bar{\rho}) u \right]\|_{L^2} \|\nabla^{k+1} u\|_{L^2} \]
\[ \lesssim (\|\nabla^{k+1} \varrho\|_{L^2} + \|\varrho\|_{L^\infty} \|\nabla^{k+1} u\|_{L^2}) \|\nabla^{k+1} u\|_{L^2} \]
\[ + (\|\nabla^{k+1} \bar{\rho}\|_{L^2} + \|\bar{\rho}\|_{L^\infty} \|\nabla^{k+1} u\|_{L^2}) \|\nabla^{k+1} u\|_{L^2} \]
\[ \lesssim (\delta + \epsilon) \left( \|\nabla^k \varrho\|_{L^2}^2 + \|\nabla^{k+1} u\|_{L^2}^2 + \|\nabla u\|_{H^1}^2 \right). \] (3.28)
The second term and the third term on the right hand side of (3.26) can be estimated as follows:

\[ \|\nabla^{k+2}u\|_{L^2}^2 + \|\nabla^{k+1}\varrho\|_{L^2}^2 \leq \epsilon \|\nabla^{k+1}\varrho\|_{L^2}^2 + \frac{C}{\epsilon} \|\nabla^{k+2}u\|_{L^2}^2 \]  

(3.29)

and

\[ -\int_{\mathbb{R}^3} \nabla\nabla^{k}z \cdot \nabla\nabla^{k}\varrho dx \leq \epsilon \|\nabla^{k+1}\varrho\|_{L^2}^2 + \frac{C}{\epsilon} \|\nabla\nabla^{k}z\|_{L^2}^2, \]

(3.30)

where the Cauchy inequality has been used twice. With (3.8), \( \|\nabla^{k}S_2\|_{L^2} \) on the fourth term of the right hand side of (3.26) can be estimated as follows.

\[ \|\nabla^{k}[(u \cdot \nabla)u]\|_{L^2} \lesssim \delta \|\nabla^{k+1}u\|_{L^2}, \]

(3.31)

and

\[ \|\nabla^{k}[h(\varrho, \bar{\rho})\Delta u]\|_{L^2} + \|\nabla^{k}[h(\varrho, \bar{\rho})\nabla \text{div} u]\|_{L^2} \lesssim (\delta + \epsilon) (\|\nabla^{k+1}\varrho\|_{L^2} + \|\nabla^2u\|_{H^1} + \|\nabla^{k+2}u\|_{L^2}), \]

(3.32)

and

\[ \|\nabla^{k}[h(\varrho, \bar{\rho})\nabla z]\|_{L^2} + \|\nabla^{k}[g_1(\varrho, \bar{\rho})\nabla \bar{\rho}]\|_{L^2} + \|\nabla^{k}[g_2(\varrho, \bar{\rho})\nabla \varrho]\|_{L^2} \lesssim (\delta + \epsilon) (\|\nabla z\|_{H^1} + \|\nabla \varrho\|_{H^1} + \|\nabla^{k+1}\varrho\|_{L^2} + \|\nabla^{k+1}z\|_{L^2}), \]

(3.33)

and

\[ \|\nabla^{k}[z \dot{h}(\varrho, \bar{\rho})\nabla \bar{\rho}]\|_{L^2} \lesssim (\delta + \epsilon) (\|\nabla^{k+1}\varrho\|_{L^2} + \|\nabla z\|_{H^1} + \|\nabla^{k+1}z\|_{L^2}). \]

(3.34)

Due to (3.26)-(3.34), it is easy to get (3.25).\(\square\)

**Lemma 3.4.** Under the assumptions of Proposition 3.1 and (3.1), it holds that

\[ \frac{d}{dt} \|\nabla^k z\|_{L^2}^2 + \|\nabla^{k+1}z\|_{L^2}^2 \lesssim (\epsilon + \delta) (\|\nabla^k \varrho\|_{L^2}^2 + \|\nabla^{k+1}u\|_{L^2}^2 + \|\nabla z\|_{H^1}^2), \]

(3.35)

for \( k = 0, 1, 2, 3. \)

**Proof.** Taking \( \nabla^k \) over (2.2)\( \_3 \), Multiplying the resulting equations by \( 2\nabla^k z \), and integrating by parts over \( \mathbb{R}^3 \), we have

\[ \frac{d}{dt} \|\nabla^k z\|_{L^2}^2 + \|\nabla^{k+1}z\|_{L^2}^2 \lesssim \langle \nabla^k S_3, \nabla^k z \rangle, \]

(3.36)

where

\[ \langle \nabla^k S_3, \nabla^k z \rangle \approx \int_{\mathbb{R}^3} \nabla^k (zu) \cdot \nabla^k \nabla z dx + \int_{\mathbb{R}^3} \nabla^k (z \dot{h}(\varrho, \bar{\rho})\nabla \bar{\rho}) \cdot \nabla^k \nabla z dx. \]

(3.37)

We are going to evaluate the terms on the right hand side of (3.37) term by term. More precisely, we have

\[ \int_{\mathbb{R}^3} \nabla^k (zu) \cdot \nabla^k \nabla z dx \leq \epsilon \|\nabla^{k+1}z\|_{L^2}^2 + \frac{C}{\epsilon} \|\nabla^k (zu)\|_{L^2}^2 \]

\[ \leq \epsilon \|\nabla^{k+1}z\|_{L^2}^2 + \frac{C}{\epsilon} \left( \|\nabla^k z\|_{L^2}^2 \|u\|_{L^3}^2 + \|z\|_{L^3}^2 \|\nabla^k u\|_{L^2}^2 \right) \]

\[ \leq \epsilon \|\nabla^{k+1}z\|_{L^2}^2 + \frac{C}{\epsilon} \left( \|\nabla^{k+1}z\|_{L^2}^2 \|u\|_{H^1}^2 + \|z\|_{H^1}^2 \|\nabla^{k+1}u\|_{L^2}^2 \right) \]

\[ \leq \epsilon \|\nabla^{k+1}z\|_{L^2}^2 + \frac{C}{\epsilon} \delta^2 \left( \|\nabla^{k+1}u\|_{L^2}^2 + \|\nabla^{k+1}z\|_{L^2}^2 \right) \]

(3.38)
Remark 3.2. The time-decay estimate of \( \|z\|_{L^2} \) in (3.41) will be used to get a time-independent bound of \( \int_0^t \|z(s)\|_{L^2}^2 \, ds \). It plays a very important role to close the a priori assumption (3.1). For more details, please refer to (3.48).

Proof. It follows from (2.2) and the classical theory of linear parabolic equations (refer for instance to [15, 23]) that \( z \) is given by

\[
z(t) = L(t)z_0 + \int_0^t L(t - \tau)S_\delta(\tau) \, d\tau, \quad t \geq 0,
\]

where \( L(t) : \phi \rightarrow v(\cdot, t) \) is the solution semigroup defined by \( L(t) = e^{-t\Delta} \), i.e.,

\[
v(x, t) = L(t)\phi = K(\cdot, t) * \phi(\cdot),
\]

where \( K := K(x, t) \) is the heat kernel

\[
K(x, t) = \frac{1}{(2\sqrt{\pi}t)^3} \exp \left\{ -\frac{|x|^2}{4t} \right\}.
\]

Notice that \( \|K(t)\|_{L^1} = 1 \) and

\[
\nabla^k L(t)\phi = \nabla^k K(t) * \phi,
\]

and that

\[
\|\nabla^k L(t)\phi\|_{L^p} \leq \|\nabla^k K(t)\|_{L^p} \|\phi\|_{L^p} \leq C(1 + t)^{-\frac{3}{2}(1 - \frac{1}{p}) - \frac{1}{2}} \|\phi\|_{L^p},
\]

where \( \frac{1}{p} = 1 + \frac{1}{2} - \frac{1}{2} \).

To prove the decay estimate of \( z \), we define

\[
\mathcal{Z}(t) = \sup_{0 \leq \tau \leq t} (1 + \tau)^{\frac{3}{2}(\frac{1}{p} - \frac{1}{2})} \|z(\tau)\|.
\]
Then, we have
\[
\|z(t)\|_{L^2} \lesssim \|L(t)z_0\|_{L^2} + \int_0^t \|L(t - \tau)S_\delta(\tau)\|_{L^2} d\tau
\]
\[
\lesssim (1 + t)^{-\frac{3}{2}(\frac{1}{6} - \frac{1}{2})}\|z_0\|_{L^p} + \int_0^t \|\nabla G(t - \tau) \ast (\frac{\mu_1 \gamma}{\mu} z u + z \frac{p'_{E}(\rho_\ast)}{\beta \rho_\ast} \nabla \bar{\rho})\|_{L^2} d\tau
\]
\[
\lesssim (1 + t)^{-\frac{3}{2}(\frac{1}{6} - \frac{1}{2})}\|z_0\|_{L^p} + \int_0^t (1 + t - \tau)^{-\frac{3}{2} - \frac{1}{2}} \|z\|_{L^2} \|u\|_{L^2} + \|\frac{p'_{E}(\rho_\ast)}{\beta \rho_\ast} \nabla \bar{\rho}\|_{L^2} d\tau
\]
\[
\lesssim (1 + t)^{-\frac{3}{2}(\frac{1}{6} - \frac{1}{2})}\|z_0\|_{L^p} + (\delta + \epsilon) \int_0^t (1 + t - \tau)^{-\frac{3}{2} - \frac{1}{2}} \|z\|_{L^2} d\tau
\]
\[
\lesssim (1 + t)^{-\frac{3}{2}(\frac{1}{6} - \frac{1}{2})}\|z_0\|_{L^p} + (\delta + \epsilon) \mathcal{Z}(t) \int_0^t (1 + t - \tau)^{-\frac{3}{2} - \frac{1}{2}} (1 + \tau)^{-\frac{3}{2}(\frac{1}{6} - \frac{1}{2})} d\tau
\]
\[
\lesssim (1 + t)^{-\frac{3}{2}(\frac{1}{6} - \frac{1}{2})}\|z_0\|_{L^p} + (\delta + \epsilon) \mathcal{Z}(t)(1 + t)^{-\frac{3}{2}(\frac{1}{6} - \frac{1}{2})},
\]
(3.43)
where we have used (1.3), (2.6) and (3.1). Multiplying (3.43) by \((1 + t)^{\frac{3}{2}(\frac{1}{6} - \frac{1}{2})}\), we have
\[
(1 + t)^{\frac{3}{2}(\frac{1}{6} - \frac{1}{2})}\|z(t)\|_{L^2} \leq \|z_0\|_{L^p} + (\delta + \epsilon) \mathcal{Z}(t).
\]
This combined with the smallness of \(\delta\) and \(\epsilon\) implies that
\[
\mathcal{Z}(t) \leq C \|z_0\|_{L^p}.
\]
The proof of Lemma 3.5 is complete.

Now we are in a position to close the \textit{a priori} assumption (3.1). From (3.4), (3.11), and (3.25), for a fixed small constant \(\epsilon_1 > 0\), we have
\[
\frac{d}{dt} \left\{ \sum_{0 \leq k \leq 3} \|\nabla^k(\rho, u)\|_{L^2}^2 + \epsilon_1 \sum_{0 \leq k \leq 2} \|\nabla^k \nabla \rho, \nabla^k u\|_{L^2}^2 \right\}
\]
\[
+ \sum_{0 \leq k \leq 3} \|\nabla^{k+1}u\|_{L^2}^2 + \epsilon_1 \sum_{0 \leq k \leq 2} \|\nabla^{k+1} \rho\|_{L^2}^2
\]
\[
\leq C(\delta + \epsilon) \left( \|\nabla \rho\|_{L^2}^2 + \|\nabla u\|_{H^1}^2 + \|\nabla z\|_{L^2}^2 \right) + C \|z\|_{L^2}^2
\]
\[
+ C \sum_{1 \leq k \leq 3} \left[ (\delta + \epsilon) \left( \|\nabla^k \rho\|_{L^2}^2 + \|\nabla^k u\|_{H^1}^2 + \|\nabla z\|_{L^2}^2 \right) + \|\nabla^k \rho\|_{H^1}^2 \right]
\]
\[
+ C \epsilon_1 \sum_{0 \leq k \leq 2} \left[ (\delta + \epsilon) \left( \|\nabla^k u\|_{L^2}^2 + \|\nabla^k \rho\|_{H^1}^2 + \|\nabla z\|_{L^2}^2 \right) + \|\nabla^k \rho\|_{H^1}^2 \right]
\]
\[
+ C \left( \|\nabla^{k+2}u\|_{L^2}^2 + \|\nabla^{k+1} z\|_{L^2}^2 \right),
\]
(3.44)
With (3.44) and the smallness of \(\epsilon\) and \(\delta\), we have
\[
\frac{d}{dt} A(t) + \epsilon_1 \|\nabla \rho\|_{H^2}^2 + \|\nabla u\|_{H^3}^2
\]
\[
\leq C \|z\|_{L^2}^2 + C \sum_{0 \leq k \leq 3} \left[ (\delta + \epsilon) \epsilon_1 \|\nabla z\|_{H^1}^2 + \epsilon_1 \|\nabla^{k+1} z\|_{L^2}^2 \right],
\]
(3.45)
The solution \( (\varrho, u, z)(t) = G(t) * (\varrho_0, u_0, z_0) \) of the linearized system (4.1) can be expressed as

\[
(q, u, z)^t(t) = G(t) + \varrho_0, \quad u|_{t=0} = u_0, \quad z|_{t=0} = z_0.
\]

Here \( G(t) := G(x, t) \) is the Green matrix of the system. Similar to the proof in [12], we have the following lemma.

**Lemma 4.1.** The Fourier transform \( \hat{G} \) of the Green’s matrix for the linearized system (4.1) is given by

\[
\hat{G}(\xi, t) = \begin{pmatrix}
\lambda_+e^{-\lambda_-t} - \lambda_+e^{-\lambda_-t} & -i\gamma \left( \frac{\lambda_+e^{-\lambda_-t} - \lambda_-e^{-\lambda_+t}}{\lambda_+ - \lambda_-} \right) \xi^t \\
-i\gamma \left( \frac{\lambda_+e^{-\lambda_-t} - \lambda_-e^{-\lambda_+t}}{\lambda_+ - \lambda_-} \right) & e^{-\mu_1|\xi|^2t} + \left( \frac{\lambda_+e^{-\lambda_-t} - \lambda_-e^{-\lambda_+t}}{\lambda_+ - \lambda_-} \right) \xi^t - e^{-\mu_1|\xi|^2t}
\end{pmatrix}
\]

The proof of Theorem 2.1 is completed.
where \( \tilde{\mu} = \mu_1 + \mu_2 \) and

\[
\lambda_{\pm} = -\frac{\tilde{\mu} |\xi|^2}{2} \pm \frac{1}{2} \sqrt{\tilde{\mu}^2 |\xi|^4 - 4\gamma^2 |\xi|^2},
\]

for \( |\xi| \neq 0 \), \( \frac{2\gamma}{\tilde{\mu}} \) when \( \tilde{\mu} > 1 \), and for \( |\xi| \neq 0 \), \( \frac{2\gamma}{\tilde{\mu}} \) when \( \tilde{\mu} \leq 1 \).

**Proof.** Taking Fourier transform with respect to the spatial variables on (4.1), we get

\[
\begin{align*}
\hat{\varrho}_t &= -i\gamma \xi \cdot \hat{u}, \\
\hat{u}_t &= (-\mu_1 |\xi|^2 I - \mu_2 \xi^t \xi) \hat{u} - i\gamma \xi \hat{\varrho} - i\xi \hat{z}, \\
\hat{z}_t &= -|\xi|^2 \hat{z}.
\end{align*}
\]

From (4.3)_3, we have

\[
\hat{z} = \hat{z}_0 e^{-|\xi|^2 t}. 
\]

Differentiating (4.3)_1 with respect to \( t \), substituting (4.3)_2 into the result, and using (4.4), we get

\[
\begin{align*}
\hat{\varrho}_t + \tilde{\mu} |\xi|^2 \hat{\varrho}_t + \gamma^2 |\xi|^2 \hat{\varrho} &= -\gamma |\xi|^2 \hat{z}_0 e^{-|\xi|^2 t}, \\
\hat{\varrho}(0,0) &= \hat{\varrho}_0(\xi), \quad \hat{\varrho}_0(0) = -i\gamma \xi \cdot \hat{u}_0(\xi).
\end{align*}
\]

By direct calculation, from (4.5), when the roots of the corresponding indicial equation \( \lambda_{\pm} := \lambda_{\pm}(\xi) \) is unequal, we obtain

\[
\hat{\varrho}(\xi, t) = \frac{\lambda_+ e^{\lambda_+ t} - \lambda_- e^{\lambda_- t}}{\lambda_+ - \lambda_-} \hat{\varrho}_0(\xi) - i\gamma \frac{e^{\lambda_+ t} - e^{\lambda_- t}}{\lambda_+ - \lambda_-} \xi \cdot \hat{u}_0(\xi)
\]

\[
- \gamma \frac{1}{(1 - \tilde{\mu}) |\xi|^2 + \gamma^2} \left( \frac{\lambda_+ e^{\lambda_+ t} - \lambda_- e^{\lambda_- t}}{2} + |\xi|^2 (e^{\lambda_+ t} - e^{\lambda_- t}) \right) \hat{z}_0(\xi).
\]

Now, we introduce components parallel to and orthogonal to \( \xi \). More precisely, we have

\[
\hat{u}(\xi, t) = a(\xi, t) \frac{\xi}{|\xi|} + b(\xi, t),
\]

where the vector \( b \) is perpendicular to \( \xi \), and where \( a \) is the scalar \( a = \hat{u} \cdot \xi / |\xi| \). We find that

\[
\begin{align*}
\begin{cases}
a_t &= -\mu_1 |\xi|^2 a - i\gamma |\xi| \hat{\varrho} - i|\xi| \hat{z}, \\
b_t &= -\mu_1 |\xi|^2 b.
\end{cases}
\end{align*}
\]

Thus, we obtain

\[
b(\xi, t) = e^{-\mu_1 |\xi|^2 t} \left( I - \frac{\xi \xi^t}{|\xi|^2} \right) \hat{u}_0,
\]

and

\[
a(\xi, t) = e^{-\tilde{\mu} |\xi|^2 t} \left[ a(\xi, 0) - i\gamma |\xi| \int_0^t e^{\tilde{\mu} |\xi|^2 s} \hat{\varrho}(\xi, s) ds - i|\xi| \int_0^t e^{\tilde{\mu} |\xi|^2 s} \hat{z}(\xi, s) ds \right].
\]
Then, we complete the proof of Lemma 4.1.

Let \( \hat{g}_{11}, \hat{g}_{12}, \hat{g}_{21}, \hat{g}_{22} \) denote the four nonzero components of \( \hat{G}_1 \), and \( \hat{g}_{13}, \hat{g}_{23}, \hat{g}_{33} \) denote the nonzero components of \( \hat{G}_2 \). Prior to estimates \( \hat{G} \), we let \( 0 < R < \min\{ \frac{2\gamma}{\beta}, \frac{2\gamma}{\sqrt{\beta}} \} \) be any fixed constant. While for \( |\xi| \leq R \), since \( |\lambda_+ - \lambda_-| = O(|\xi|) \), from the expression of \( \hat{G} \), we easily obtain that there exists a constant \( \eta' > 0 \) such that

\[
|\hat{g}_{11}(\xi, t), \hat{g}_{12}(\xi, t), \hat{g}_{21}(\xi, t), \hat{g}_{22}(\xi, t), \hat{g}_{13}(\xi, t), \hat{g}_{23}(\xi, t), \hat{g}_{33}(\xi, t)| \leq e^{-\eta'|\xi|^2 t}, \quad |\xi| \leq R.
\]

(4.12)

**Proposition 4.1.** Assume that \((\varrho, u, z)\) is the solution of the linearized Navier-Stokes-Smoluchowski system with the initial data \( \varrho_0 \in H^1 \), \( u_0 \in H^3 \) and \( z_0 \in H^3 \), then

\[
\int_{|\xi| \leq R} |\xi|^{2k} (|\varrho, u, z|)^2 d\xi \leq C (1 + t)^{-3(\frac{1}{2} - \frac{1}{k}) - k} \| (\varrho_0, u_0, z_0) \|^2_{L^p},
\]

(4.13)

where \( k = 0, 1, 2, 3 \).

**Proof.** Let \( R > 0 \) be a fixed constant as before. By using the Parseval theorem and the Young inequality, we obtain

\[
\int_{|\xi| \leq R} |\xi|^{2k}|\hat{\varrho}|^2 d\xi = \int_{|\xi| \leq R} |\xi|^{2k}|\hat{\varrho}_{11}(\xi, t)|^2 + |\hat{\varrho}_{12}(\xi, t)|^2 + |\hat{\varrho}_{13}(\xi, t)|^2 + |\hat{\varrho}_{23}(\xi, t)|^2 d\xi \\
\leq C \int_{|\xi| \leq R} e^{-\eta'|\xi|^2 t} |\xi|^{2k} \left( |\hat{\varrho}_0(\xi)|^2 + |\hat{\varrho}_{0}(\xi)|^2 + |\hat{z}_0(\xi)|^2 \right) d\xi \\
\leq C(1 + t)^{-3(\frac{1}{2} - \frac{1}{k}) - k} \| (\varrho_0, u_0, z_0) \|^2_{L^p}.
\]

(4.14)

Similarly, we have

\[
\int_{|\xi| \leq R} |\xi|^{2k} (|u|^2 + |z|^2) d\xi \leq C (1 + t)^{-3(\frac{1}{2} - \frac{1}{k}) - k} \| (\varrho_0, u_0, z_0) \|^2_{L^p}.
\]

(4.15)

Then, we complete the proof of Proposition 4.1. \( \square \)

4.2. Decay rates of nonlinear system. For the nonlinear system, we have the following decay estimates.

**Lemma 4.2.** Under the assumptions of Theorem 2.2, it holds that

\[
\left( \int_{|\xi| \leq R} |\xi|^{2k} (|\varrho, u, z|)^2 d\xi \right)^{\frac{1}{2}} \leq (1 + t)^{-\frac{3}{2} + \delta} \| (\varrho_0, u_0, z_0) \|_{L^p} \\
+ (\epsilon + \delta) \int_0^t (1 + t - \tau)^{-\frac{3}{2} - \frac{1}{2}} \| \nabla \varrho(u, z) \|_{L^d \tau} d\tau,
\]

(4.16)

where \( k = 0, 1, 2, 3 \).
Proof. From the Duhamel principle, we have
\[
(\rho, u, z) = G(x, t) \ast (\rho_0, u_0, z_0) + \int_0^t G(t - \tau) \ast (S_1, S_2, S_3)(\tau) d\tau. 
\]
Then, we get
\[
\left( \int_{|\xi| \leq R} |\xi|^{2k} |(\hat{\rho}, \hat{u}, \hat{z})|^2 d\xi \right)^{\frac{1}{2}} \leq (1 + t)^{-\frac{3}{2}(\frac{d}{2} - 1)} \|G(\rho_0, u_0, z_0)\|_{L^p}
\]
\[
+ \int_0^t (1 + t - \tau)^{-\frac{3}{2}} \|(S_1, S_2, S_3)\|_{L^1} d\tau. 
\]
For the terms including $\hat{\rho}$ on the r.h.s. of (4.18), by using the Hölder inequality, and the Hardy inequality, we get
\[
\|\nabla \hat{\rho} \cdot u\|_{L^1} \lesssim \|\nabla u\|_{L^2}, 
\]
\[
\|\hat{\rho} \nabla \cdot u\|_{L^1} \lesssim \|\hat{\rho}\|_{L^2} \|\nabla \cdot u\|_{L^2} \lesssim \|\nabla u\|_{L^2}. 
\]
Similarly, it holds that
\[
\|\hat{\rho} \Delta u\|_{L^1} \lesssim \|\nabla^2 u\|_{L^2}, 
\]
\[
\|\hat{\rho} \nabla \hat{\theta}\|_{L^1} \lesssim \|\nabla \hat{\theta}\|_{L^2}, 
\]
\[
\|\nabla \hat{\rho}\|_{L^1} \lesssim \|\hat{\rho}\|_{L^1} \lesssim \|\nabla \hat{\rho}\|_{L^1}. 
\]
It is easy to check that
\[
\|(S_1, S_2, S_3)\|_{L^1} \leq C(\epsilon + \delta) \|\nabla (\rho, u, z)\|_{L^2}. 
\]
Then, we complete the proof of Lemma 4.2. □

Now we are in a position to prove Theorem 2.2. From (3.11) and (3.25), for a fixed small constant $\epsilon_2 > 0$, we have
\[
\frac{d}{dt} \left\{ \sum_{1 \leq k \leq 3} \|\nabla^k (\rho, u)\|_{L^2}^2 + \epsilon_2 \sum_{1 \leq k \leq 2} \|\nabla^k \nabla \rho\|_{L^2} \right\} 
\]
\[
+ \sum_{1 \leq k \leq 3} \|\nabla^{k+1} u\|_{L^2}^2 + \epsilon_2 \sum_{1 \leq k \leq 2} \|\nabla^{k+1} \rho\|_{L^2}^2 
\]
\[
\lesssim \sum_{1 \leq k \leq 3} \left( (\delta + \epsilon) \left( \|\nabla^k \rho\|_{L^2}^2 + \|\nabla u\|_{H^2}^2 + \|\nabla z\|_{H^1}^2 + C \|\nabla^{k+1} z\|_{L^2}^2 \right) 
\]
\[
+ \epsilon_2 (\delta + \epsilon) \sum_{1 \leq k \leq 2} \left( \|\nabla^2 u\|_{H^1}^2 + \|\nabla^{k+1} u\|_{H^1}^2 + \|\nabla z\|_{H^1}^2 + \|\nabla u\|_{L^2}^2 + \|\nabla \rho\|_{H^1}^2 \right) 
\]
\[
+ C \epsilon_2 \sum_{1 \leq k \leq 2} \left( \|\nabla^k u\|_{L^2}^2 + \|\nabla^{k+1} z\|_{L^2}^2 \right). 
\]
Then, we have
\[
\frac{d}{dt} B(t) + \epsilon_2 \|\nabla^2 \rho\|_{H^1}^2 + \|\nabla^2 u\|_{H^2}^2 
\]
\[
\lesssim \sum_{1 \leq k \leq 3} \left[ (\delta + \epsilon) \|\nabla z\|_{H^1}^2 + C \|\nabla^{k+1} z\|_{L^2}^2 \right] + (\delta + \epsilon) \left( \|\nabla \rho\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 \right), 
\]
where 

\[ B(t) = \sum_{1 \leq k \leq 3} \| \nabla^k (\varrho, u) \|_{L^2}^2 + \epsilon_2 \sum_{1 \leq k \leq 2} \langle \nabla^k \varrho, \nabla^k u \rangle = O(\| \nabla (\varrho, u) \|_{H^3}^2). \]

From (3.35), we have

\[ \frac{d}{dt} \| \nabla \varrho \|_{H^2}^2 + \| \nabla^2 \varrho \|_{L^2}^2 \lesssim (\delta + \epsilon) \left( \| \nabla \varrho \|_{H^2}^2 + \| \nabla \varrho \|_{H^1}^2 \right). \]  
(4.27)

Putting (4.26) and (4.27) together, we get

\[ \frac{d}{dt} \left[ B(t) + \| \nabla \varrho \|_{H^2}^2 \right] + \epsilon_2 \| \nabla^2 \varrho \|_{H^1}^2 + \| \nabla^2 (u, z) \|_{H^2}^2 \lesssim (\epsilon + \delta) \left( \| \nabla \varrho \|_{H^2}^2 + \| \nabla \varrho \|_{H^1}^2 + \| \nabla u \|_{H^2}^2 \right). \]  
(4.28)

Using the Parseval Theorem, and splitting the integral into two parts, we have

\[ \| \nabla^2 \varrho \|_{L^2}^2 = \int_{|\xi| \leq R} |\xi|^4 |\hat{\varrho}|^2 d\xi + \int_{|\xi| \geq R} |\xi|^4 |\hat{\varrho}|^2 d\xi \geq C \int_{|\xi| \geq R} |\xi|^2 |\hat{\varrho}|^2 d\xi. \]

Similarly, we have

\[ \| \nabla^2 (u, z) \|_{L^2}^2 \geq C \int_{|\xi| \geq R} |\xi|^2 |(\hat{u}, \hat{z})|^2 d\xi. \]

Then, from (4.28), we have

\[ \frac{d}{dt} \left[ B(t) + \| \nabla \varrho \|_{H^2}^2 \right] + \epsilon_2 \| \nabla^2 \varrho \|_{H^1}^2 + \frac{1}{2} \| \nabla^2 (u, z) \|_{H^2}^2 \]

\[ \lesssim (\epsilon + \delta) \left( \int_{|\xi| \leq R} |\xi|^2 (|\hat{\varrho}|^2 + |\hat{u}|^2) d\xi + \int_{|\xi| \geq R} |\xi|^2 (|\hat{\varrho}|^2 + |\hat{u}|^2) d\xi \right). \]  
(4.29)

Due to (4.29) and the smallness of \( \epsilon \) and \( \delta \), there exists a positive constant \( D_1 \) such that

\[ \frac{d}{dt} \left[ B(t) + \| \nabla \varrho \|_{H^2}^2 \right] + D_2 \int_{|\xi| \leq R} |\xi|^2 |(\hat{\varrho}, \hat{u}, \hat{z})|^2 d\xi + \frac{\epsilon_2}{2} \| \nabla^2 \varrho \|_{H^1}^2 + \frac{1}{2} \| \nabla^2 (u, z) \|_{H^2}^2 \]

\[ \lesssim (\epsilon + \delta) \int_{|\xi| \leq R} |\xi|^2 (|\hat{\varrho}|^2 + |\hat{u}|^2) d\xi. \]  
(4.30)

Adding \( \int_{|\xi| \leq R} |\xi|^2 |(\hat{\varrho}, \hat{u}, \hat{z})|^2 d\xi \) on both sides of (4.30), we have

\[ \frac{d}{dt} \left[ B(t) + \| \nabla \varrho \|_{H^2}^2 \right] + D_2 \| \nabla (\varrho, u, z) \|_{L^2}^2 + \frac{\epsilon_2}{2} \| \nabla^2 \varrho \|_{H^1}^2 + \frac{1}{2} \| \nabla^2 (u, z) \|_{H^2}^2 \]

\[ \lesssim C \int_{|\xi| \leq R} |\xi|^2 |(\hat{\varrho}, \hat{u}, \hat{z})|^2 d\xi. \]  
(4.31)

Then, we have

\[ \frac{d}{dt} \left[ B(t) + \| \nabla \varrho \|_{H^2}^2 \right] + D_3 \| \nabla (\varrho, u, z) \|_{H^2}^2 \lesssim C \int_{|\xi| \leq R} |\xi|^2 |(\hat{\varrho}, \hat{u}, \hat{z})|^2 d\xi. \]  
(4.32)
To prove the decay estimates stated in Theorem 2.2, we define

$$M(t) = \sup_{0 \leq \tau \leq t} (1 + \tau)^{3(\frac{1}{p} - \frac{1}{2})+1} \|
abla(p, u, z)\|_{H^2}.$$

Notice that $M(t)$ is non-decreasing and

$$\|\nabla(p, u, z)(\tau)\|_{L^2} \lesssim (1 + \tau)^{-\frac{5}{6}} \sqrt{M(t)}, \quad 0 \leq \tau \leq t.$$ 

Then, we get

$$\left(\int_{|\xi| \leq R} |\xi|^2 |(\hat{\varphi}, \hat{\varphi}, \hat{z})|^2 \, d\xi\right)^{\frac{1}{2}} \lesssim (1 + t)^{-\frac{5}{6}} \|\nabla(p, u, z)\|_{L^p} + (\epsilon + \delta) \int_0^t (1 + t - \tau)^{-\frac{5}{6}} \|\nabla(p, u, z)\|_{L^2} \, d\tau$$

$$\lesssim (1 + t)^{-\frac{5}{6}} \|\nabla(p, u, z)\|_{L^p} + (\epsilon + \delta) \int_0^t (1 + t - \tau)^{-\frac{5}{6}} (1 + \tau)^{-\frac{1}{2}} \sqrt{M(t)} \, d\tau$$

$$\lesssim (1 + t)^{-\frac{5}{6}} \|\nabla(p, u, z)\|_{L^p} + (\epsilon + \delta) \left(\|\nabla(p, u, z)\|_{L^p} + (\epsilon + \delta) \sqrt{M(t)}\right).$$

(4.33)

From (4.32) and (4.33), we have

$$\|\nabla(p, u, z)\|_{H^2}^2 \lesssim e^{-D_3 t} \|\nabla(p, u, z)\|_{H^2}^2 + \int_0^t e^{-D_3 (t - \tau)} \left(\int_{|\xi| \leq R} |\xi|^2 |(p, u, z)|^2 \, d\xi\right) \, d\tau$$

$$\lesssim e^{-D_3 t} \|\nabla(p, u, z)\|_{H^2}^2$$

$$\lesssim e^{-D_3 t} \|\nabla(p, u, z)\|_{H^2}^2 + \int_0^t e^{-D_3 (t - \tau)} (1 + \tau)^{-3(\frac{1}{p} - \frac{1}{2})-1} \left[\|\nabla(p, u, z)\|_{L^p}^2 + (\epsilon + \delta) M(t)\right] \, d\tau$$

$$\lesssim (1 + t)^{-3(\frac{1}{p} - \frac{1}{2})-1} \left[\|\nabla(p, u, z)\|_{H^2}^2 + \|\nabla(p, u, z)\|_{L^p}^2 + (\epsilon + \delta) M(t)\right].$$

(4.34)

Using (4.34) and the smallness of $\epsilon$ and $\delta$, we have

$$M(t) \leq C \left(\|\nabla(p, u, z)\|_{H^2}^2 + \|\nabla(p, u, z)\|_{L^p}^2\right),$$

which implies that

$$\|\nabla(p, u, z)\|_{H^2} \lesssim (1 + t)^{-\frac{5}{6}} \left[\|\nabla(p, u, z)\|_{H^2} + \|\nabla(p, u, z)\|_{L^p}\right].$$

(4.35)

Similarly, due to (3.45) and (3.46), there exists a constant $D_4 > 0$ such that

$$\frac{d}{dt} \|\nabla(p, u, z)\|_{H^2}^2 + D_4 \|\nabla(p, u, z)\|_{H^2}^2 \lesssim \int_{|\xi| \leq R} |(\hat{\varphi}, \hat{\varphi}, \hat{z})|^2 \, d\xi.$$ 

(4.36)

From Lemma 4.2, we have

$$\left(\int_{|\xi| \leq R} |(\hat{\varphi}, \hat{\varphi}, \hat{z})|^2 \, d\xi\right)^{\frac{1}{2}} \lesssim (1 + t)^{-\frac{5}{6}} \|\nabla(p, u, z)\|_{L^p} + (\epsilon + \delta) \left(\|\nabla(p, u, z)\|_{H^2} + \|\nabla(p, u, z)\|_{L^p}\right).$$
\[ \int_0^t (1 + t - \tau)^{-\frac{3}{2}} (1 + \tau)^{-\frac{3}{2}(\frac{1}{p} - \frac{1}{2})} d\tau \]
\[ \lesssim (1 + t)^{-\frac{3}{2}(\frac{1}{p} - \frac{1}{2})} (\|\nabla (\varrho_0, u_0, z_0)\|_{H^2} + \| (\varrho_0, u_0, z_0) \|_{L^p}). \] (4.37)

By (4.36) and (4.37), we have
\[ \| (\varrho, u, z) \|_{H^2} \lesssim (1 + t)^{-\frac{3}{2}(\frac{1}{p} - \frac{1}{2})} (\|\nabla (\varrho_0, u_0, z_0)\|_{H^2} + \| (\varrho_0, u_0, z_0) \|_{L^p}). \] (4.38)

By the interpolation and Lemma 4.3, it holds that for any \( 0 \leq t \leq T \),
\[ \| (\varrho, u, z)(t) \|_{L^2} \leq \| (\varrho, u, z)(t) \|_{L^2}^{1-\theta} \| (\varrho, u, z)(t) \|_{L^\infty}^{\theta} \leq C(1 + t)^{-\frac{3}{2}(\frac{1}{p} - \frac{1}{2})}, \] for \( 2 \leq q \leq 6 \),
\[ \boxed{(4.39)} \]

where \( \theta = \frac{6-q}{2q} \).

With (4.35), (4.38) and (4.39), the proof of Theorem 2.2 is completed.

**Appendix.** We recall some known inequalities of Sobolev type. One can find them in [1, 8]

**Lemma 4.3. (Sobolev inequalities)** Let \( f \in H^2(\mathbb{R}^3) \), we have
\[ (1) \| f \|_{L^\infty} \leq C \| \nabla f \|_{H^1}^{\frac{1}{2}} \| \nabla f \|_{H^2}^{\frac{1}{2}} \leq C \| \nabla f \|_{H^1}; \]
\[ (2) \| f \|_{L^6} \leq C \| \nabla f \|_{H^1}; \]
\[ (3) \| f \|_{L^q} \leq C \| f \|_{H^1}, \quad 2 \leq q \leq 6. \]

**Lemma 4.4. (Gagliardo-Nirenberg inequality)** Let \( 0 \leq m, \alpha \leq l \), then we have
\[ \| \nabla^m f \|_{L^p} \lesssim \| \nabla^m f \|_{L^q}^{1-\theta} \| \nabla^l f \|_{L^r}^\theta, \]
where \( 0 \leq \theta \leq 1 \) and \( \alpha \) satisfies
\[ \frac{\alpha}{3} - 1 = \left( \frac{m}{3} - 1 \right) (1 - \theta) + \left( \frac{l}{3} - 1 \right) \theta. \]
Here when \( p = \infty \), we require that \( 0 < \theta < 1 \).

We recall the following estimates, cf. [13, 20]. Here, we give the proof of Lemma 5.4 for the convenience of readers.

**Lemma 4.5.** Let \( m \geq 1 \) be an integer and define the commutator
\[ [\nabla^m, f] g = \nabla^m (fg) - f \nabla^m g. \]
Then we have
\[ \| \nabla^m (fg) \|_{L^p} \lesssim \| \nabla^m f \|_{L^{p_1}} \| g \|_{L^{p_2}} + \| f \|_{L^{p_3}} \| \nabla^m g \|_{L^{p_4}}. \] (4.40)
and
\[ \| [\nabla^m, f] g \|_{L^p} \lesssim \| \nabla^m f \|_{L^{p_1}} \| \nabla^{m-1} g \|_{L^{p_2}} + \| \nabla^m f \|_{L^{p_3}} \| g \|_{L^{p_4}}, \] (4.41)
where \( p, p_2, p_3 \in (1, +\infty) \) and
\[ \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}. \]
Lemma 4.6. For $m \geq 1$ and $1 \leq p \leq +\infty$, we have
\[ \|\nabla^m (g(\varrho, \bar{\rho}))\|_{L^p} \leq C \|\nabla^m g\|_{L^p} + C \|\nabla^m \bar{\rho}\|_{L^p}. \] (4.42)

Proof.
\[ \nabla^m (g(\varrho, \bar{\rho})) = \sum_{1 \leq |\alpha| + |\beta| \leq m} \partial^\alpha_\varrho \partial^\beta_{\bar{\rho}} g(\varrho, \bar{\rho})(\nabla \varrho)^{\alpha_1} (\nabla^2 \varrho)^{\alpha_2} \cdots (\nabla^m \varrho)^{\alpha_m} \right \} \times (\nabla \bar{\rho})^{\beta_1} (\nabla^2 \bar{\rho})^{\beta_2} \cdots (\nabla^m \bar{\rho})^{\beta_m}, \] (4.43)
where
\[ \alpha_1 + \alpha_2 + \cdots + \alpha_m = |\alpha|, \quad \beta_1 + \beta_2 + \cdots + \beta_m = |\beta|, \]
\[ 1 \cdot \alpha_1 + 2 \cdot \alpha_2 + \cdots + m \cdot \alpha_m + 1 \cdot \beta_1 + 2 \cdot \beta_2 + \cdots + m \cdot \beta_m = m. \]

Using the Hölder inequality, we have
\[ \|\nabla^m (g(\varrho, \bar{\rho}))\|_{L^p} \lesssim \sum_{1 \leq |\alpha| + |\beta| \leq m} \|\nabla \varrho\|^{\alpha_1}_{L^{p_1}} \|\nabla^2 \varrho\|^{\alpha_2}_{L^{p_2}} \cdots \|\nabla^m \varrho\|^{\alpha_m}_{L^{p_m}} \times \|\nabla \bar{\rho}\|^{\beta_1}_{L^{q_1}} \|\nabla^2 \bar{\rho}\|^{\beta_2}_{L^{q_2}} \cdots \|\nabla^m \bar{\rho}\|^{\beta_m}_{L^{q_m}} \] (4.44)
where
\[ p_i = \frac{mp}{i \alpha_i}, \quad q_j = \frac{mp}{j \beta_j}. \]

Using the Nirenberg inequality, we have
\[ \|\nabla^i \varrho\|_{L^{\frac{mp}{i}}_{L^p}} \leq C \|\varrho\|^{1 - \frac{m}{i}}_{L^\infty} \|\nabla^m \varrho\|^{\frac{m}{i}}_{L^p} \leq C \|\nabla^m \varrho\|_{L^p}, \] (4.45)
\[ \|\nabla^j \bar{\rho}\|_{L^{\frac{mp}{j}}_{L^p}} \leq C \|\bar{\rho}\|^{1 - \frac{m}{j}}_{L^\infty} \|\nabla^m \bar{\rho}\|^{\frac{m}{j}}_{L^p} \leq C \|\nabla^m \bar{\rho}\|_{L^p}, \] (4.46)

Then, we have
\[ \|\nabla^m (g(\varrho, \bar{\rho}))\|_{L^p} \lesssim C \|\nabla^m \varrho\|_{L^p}^{\frac{m_1}{L^p}} \|\nabla^m \bar{\rho}\|_{L^p}^{\frac{m-m_1}{L^p}} \] (4.47)
where
\[ m_1 = 1 \cdot \alpha_1 + 2 \cdot \alpha_2 + \cdots + m \cdot \alpha_m. \]

By using the following inequality
\[ ab \leq \frac{1}{p} a^p + \frac{1}{q} b^q, \quad (a, b \geq 0, \frac{1}{p} + \frac{1}{q} = 1, \ 1 \leq p, q \leq \infty), \]
we have
\[ \|\nabla^m (g(\varrho, \bar{\rho}))\|_{L^p} \lesssim C \|\nabla^m \varrho\|_{L^p} + C \|\nabla^m \bar{\rho}\|_{L^p}. \]

Thus, we have completed the proof of Lemma 4.6.

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