Causal Implication and the Origin of Time Dilation

George Jaroszkiewicz
School of Mathematical Sciences, University of Nottingham, University Park, Nottingham NG7 2RD, UK

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Abstract

We discuss the emergence of time dilation as a normal feature expected of any system where a central processor may have to wait one or more clock cycles before concluding a local calculation. We show how the process of causal implication in a typical Newtonian cellular automaton leads naturally to Lorentz transformations and invariant causal structure.

1 Introduction

In this paper we develop the idea that time dilation of the sort encountered in special relativity occurs naturally in any universe which behaves like a cellular automaton. This theme was discussed by Minsky nearly two decades ago [1], but at that time the “wrong” dilation factors emerged and the idea seems not to have been taken further. We obtain our results by focusing on the frame of references employed, rather than on wave packets as was the case of the Minsky paper, and this makes the conclusions more generic.

Our discussion is based on an approach to causality introduced by us [2], in which time emerges as an index of causal implication as carried out by a Theorist standing outside of space-time. By this we mean the following. Between times $n$ and $n+1$ of the Theorist’s internal discrete clock time (referred to as physiotime in [2]), the Theorist works out the $n + 1^{th}$ causal implication, which is the maximal set of future data values logically inferable from all the information available to the Theorist at time $n$, using the given laws of physics. This process is repeated, with $n$ replaced by $n + 1$, \textit{ad infinitum}. This gives a natural dating process to events.

From this point of view the Theorist behaves effectively as a Turing computing machine, the input of which involves the dynamical data in some of the events or cells of space-time, including conventional past and future events. If at time $n$ a given cell is blank (i.e. unwritten) and the local rules of the cellular automaton allow it, then the Theorist can make a causal implication at that cell, writes in the value determined by the dynamics, and dates the cell $n + 1$. Otherwise the cell is left alone until some possibly future physiotime. This model reconciles the notion of a static block universe with the process view of time, where the present is a real, distinguishable moment.
Our approach utilises only the local discrete topology of cellular automata (the relationship between adjacent cells), which makes it independent of the details of the dynamical rules relating state values in adjacent cells. Therefore, the results are generic and should find their analogues in a wide variety of cellular automata. Our conclusion is that the notion of time dilation is not specific to Minkowski space-time but occurs also as a generic feature of systems such as modern computers where information is processed by a central processing unit governed by an internal clock. On occasions, the processor may have to wait various clock cycles before certain local computations can be completed, simply because necessary information is not yet available, and this is equivalent to time dilation.

2 Discrete space-time

Cellular automata are of interest to physicists for a number of reasons [3]. First, they deal with discrete space-time, which is currently a fashionable notion. Second, they generate in a completely natural way intrinsic structures analogous to the light-cone structure of special relativity. Third, they can provide good approximations to the differential equations of conventional physics. From our point of view, however, we prefer to put it the other way around and regard conventional physics as a good approximation, in an appropriate continuum limit, to some discrete space-time cellular automaton.

In conventional cellular automata, such as discussed by Minsky [1], there is an external absolute time \( n \) which regulates the dynamical evolution in a strict way. At each click of this time, each cell (or event in the terminology of [2]) \( C^m \) carries a dynamical variable (the state of the event in the terminology of [2]) which has value \( \psi^m_n \) at that time. For such automata, there will be some deterministic law or rule such that the value \( \psi^m_{n+1} \) of the state in \( C^m \) at time \( n + 1 \) is completely determined (or causally resolved in the terminology of [2]) by knowledge of some or all of the values \( \{ \psi^m_n : -\infty < m < \infty \} \) at an earlier time \( n \).

In our approach we have such a structure, but it is modified in an important way as follows. First, there will in general also be rules (called links in the terminology of [2]) which relate cell values at different values of \( n \) and \( m \). Examples are

\[
\psi^{m+1}_n + \psi^{m+1}_n + \psi^{m}_n + \psi^{m-1}_n + \psi^{m}_n - 1, \quad -\infty < m < \infty, \quad \psi^i \in C
\]

(1)

and

\[
g^{m+1}_n g^{m+1}_n g^m_n g^{m-1}_n g^m_n - 1, \quad g^i \in SU(2)
\]

(2)

where \( e \) is the identity element of the group. These two quite different links have the same local discrete topology, in the sense that the pattern of \( n \)'s and \( m \)'s is the same. These links will generate identical causal structures, even though they represent quite different dynamical systems. In the following we shall discuss cellular automata with two indices, \( m \) and \( n \) as above, but our conclusions are general.

The difference between our approach and conventional cellular automata is that we allow the external Theorist (the central processor) to work with cells which may be scattered over the \( m - n \) domain in a more general way than just lined up at equal values of \( n \), say, corresponding to conventional “planes of simultaneity”. We shall show how standard Lorentz time dilation and Fitzgerald length contraction emerges using these ideas, based on the local discrete topology implied by (1) and (2). This topology was not chosen by accident, as it emerges naturally in simple discretisations of the Klein-Gordon equation as shown below.
3 Causal implication

Given the continuous time equation

\[ \frac{\partial^2}{\partial t^2} \psi(t, x) - \frac{\partial^2}{\partial x^2} \psi(t, x) + \mu^2 \psi(t, x) = 0, \]  

in 1 + 1 space-time dimensions, with the speed of light and Planck’s constant chosen to be unity, a simple discretisation of space-time defined by

\[ t_m \equiv nT, \quad x_m \equiv mL, \]
\[ \psi_m \equiv \psi(t_m, x_m) \]

where \( T \) and \( L \) are scale constants satisfying \( T/L = 1 \), gives the link equations

\[ \psi_{m+1} + \psi_{m-1} - 2\psi_m + \mu^2 T^2 \psi_m + O(T^4) = 0. \]

Neglecting the higher order terms, we arrive at an equation of the type

\[ \psi_{m+1} = \mathcal{F}(\psi_m, \psi_{m+1}, \psi_{m-1}, \psi_{m-1}), \]

which has the same local discrete topology as (1) and (2).

In our approach, such an equation can only be resolved (or worked out) if at Clock Time (\( \equiv \) physiotime) \( p \), the Theorist (\( \equiv \) central processor) actually knows what the state values \( \psi_m, \psi_{m+1}, \psi_{m-1} \) and \( \psi_{m-1} \) in the cells \( C_m, C_{m+1}, C_{m-1}, C_{m-1} \) are. If these are not available, the Theorist cannot complete this calculation and must wait.

If these values had in fact been computed at previous Clock Times \( a, b, c, d \) respectively then the value \( \psi_{n-1} \) on the left hand side of (3) could be evaluated or resolved, but no earlier than at Clock Time \( p = \max(a, b, c, d) \). In such a case the calculation is completed and dated, and forms part of the \((p + 1)th\) implication. This provides a temporal ordering or dating over space-time.

3.1 The algorithm

Specifically, the algorithm for causal implication in such a model is as follows.

1. At Clock Time \( p \), the Theorist inspects every cell \( C_i^j \) in space-time and marks each one down for possible implication, or not, as follows:

   (a) if a particular cell \( C_i^j \) already has a date, this means that the cell value (\( \equiv \) event state) \( \psi_i^j \) has already been evaluated at that cell at some previous Clock Time, and the cell is left alone and the cell value never changes;

   (b) if a particular cell \( C_i^j \) has no date, then the Theorist must look at cells \( C_{i-2}^j, C_i^{j+1}, C_i^{j-1}, C_{i-1}^j \). If one or more of these cells has no date then \( C_i^j \) cannot be resolved at Clock Time \( p \). If however, each of the cells \( C_{i-2}^j, C_i^{j+1}, C_i^{j-1}, C_{i-1}^j \) already has a date (which necessarily must be less or equal to \( p \)) then \( C_i^j \) is marked down for resolution.

2. The Theorist now resolves (\( \equiv \) evaluates) each cell which has been marked down for resolution in the first step of this algorithm and dates it with time \( p + 1 \).
We note the following:

1. If the $m - n$ plane is infinite the Theorist may take an infinite amount of physiotime to inspect each cell. This is not regarded as a problem. In practice, physicists only deal with finite regions of space-time, and we may idealise this to the infinite extent situation.

2. The actual duration of a Clock Time interval is not significant here. Ticks of Clock Time only occur when a given process of causal implication has been completed by the Theorist.

3. To paraphrase the words of Omar Al-Khayyam, who was a mathematician as well as a poet, once the moving finger of the Theorist has written in a cell, it moves on and never rewrites that cell.

4. A cell which is being resolved cannot be used in the process of causal resolution for any other cell during that particular implication process. The role of cells is entirely classical here, in that a given cell can only play one role at any given Clock Time. Either it is being resolved, does nothing, or else is being used in the resolution of other cells.

5. Conventional spreadsheets are very convenient for analysing cellular automata along the above lines.

**Example:** Suppose our initial data set $\sigma_0$ consists of cells with given values $\psi_{m-1}^m$ and $\psi_0^m$ for $-\infty < m < \infty$. Using (6) we can work out the first implication $\sigma_1$, given by

$$\sigma_1 = \left\{ \psi_1^m : \psi_1^m = \mathcal{F}(\psi_0^m, \psi_0^{m+1}, \psi_0^{m-1}, \psi_{-1}^m), -\infty < m < \infty \right\}.\quad (7)$$

Here we are using events and event states interchangeably in our notation, and it should be clear what is meant.

Once we have completed this task, we are in a position to repeat the process of implication, but with the difference that the data set used for the second implication $\sigma_2$ now involves the state values $\psi_1^m$ and $\psi_0^m$ for $-\infty < m < \infty$.

Clearly this process can be continued indefinitely. There is no difference in this example between our approach and that taken in conventional cellular automata, but this is not true in general.

### 4 Inertial frames

The initial data set $\sigma_0$ in the above example would be naturally identified as relating to the fundamental rest frame $\mathcal{F}_0$ of the cellular automaton space-time lattice, in which the dynamical rules are specified. If we wish to simulate inertial frames moving with respect to $\mathcal{F}_0$ we need to start with different initial data sets. We are free to choose a number of subsets of the lattice for this, but some choices are more natural than others. We shall work with a reasonable choice as follows. First, we choose a frame velocity $v \equiv r/s$, where $r$ and $s$ are integers and $|r| < s$. In this approach, therefore, all velocities are rational fractions of the speed of light (chosen to be unity).
We define our initial data set $\sigma_0(v)$ as shown in Fig. 1. In this diagram $n$ runs left to right and $m$ runs upwards. Cells belonging to $\sigma_0(v)$ are labelled by zeros and shaded, and contain given starting state values. State values are not shown in such a diagram.

By the process of causal implication, we may fill in cells on either side of the zig-zag initial data set. We shall only discuss implications running to the right, as this represents the forwards direction of time in our model.

The pattern of implication settles down after several clock cycles and assumes a regularity in which a relativistic pattern may be discerned. In Fig. 1 we show lines which connect cells which carry dates suggestive of times and position coordinates in a frame $F_0$ moving with velocity $v = 3/5$ relative to the rest frame $F_0$ of the original lattice.

These patterns are rather general for this particular cellular automaton and for analogous

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**Figure 1:** The frame velocity is chosen such that $r = 3$, $s = 5$. Cell numbers indicate implication dates and solid skew lines represent lines of constant $t'$ and $x'$. Numbers in bold indicate some of the cells which have integer coordinates in the primed frame and the dotted lines show a typical forwards lightcone.
initial data sets, so we now give a more general discussion, which helps to illuminate features of Fig. 1.

5 The general case

We may generalise the above example and consider transforming to a frame of reference $F'$ moving with velocity $v \equiv r/s$ relative to $F_0$, where $r$ and $s$ are integers with the proviso $|v| < 1$. Now the initial data set consists of a double layer of zeros zig-zagging over the $n - m$ plane in a manner identical to that shown in Fig. 1, except now instead of five steps upwards followed by three steps to the right, and so on, we have $s$ steps upwards followed by $r$ steps to the right, and so on ad infinitum.

With such an initial data set, experience soon confirms the following pattern of implication, analogous to that shown in Fig. 1. During the initial phases of causal implication, there is something like a boundary layer, starting with the first implication, during which the final patterns begin to get established. Care should be taken to ensure that any discussion concerning frames of reference does not involve cells inside this boundary layer. This is particularly the case if such a model is evaluated using a spreadsheet. A spreadsheet is an excellent tool to use for calculating such causality diagrams, as single changes to any cell or cells in the initial data set are immediately translated throughout the rest of the space-time, and the causal structure of light-cones can be very readily discerned.

An important point concerns the initial data set. It is possible that such a set is a “Garden of Eden” set [3], which means that it cannot be derived from the rules of the particular cellular automaton being discussed. In Fig. 1 it will be observed that some of the initial data set cells appear to be inconsistent with the rules. The five cells shown with a bold border in Fig. 1. are an example where the right most cell cannot be an implication of the others, since the lowest one is dated two implications later. However, a strict reading of the algorithm shows that the whole process is in fact consistent.

We note that such a scenario occurs in the causal boundary layer. Once the process of implication has cleared this layer, no apparent clashes occur. This will be important in cases such as the discretised Klein-Gordon equations where the causal implication is reversible, i.e., causal implication can run in two or more directions depending on the boundary conditions.

The depth of the causal boundary layer will depend on the values of $r$ and $s$. It is remarkable that quite without any further input, the final pattern beyond the boundary layer emerges as a result of some sort of self-organisation within the boundary layer. What is clear from our results is that complex behaviour can emerge from simple dynamical rules, but may depend critically on starting conditions.

Beyond the causal boundary layer, the flow of time as indicated by the implication dates assumes a regular pattern of behaviour, much like Minsky’s wave packet “machines” [4]. Again, the periods discernible in this pattern depend on $r$ and $s$. One immediate observation is that the date of an implication in a cell does not in general increase strictly uniformly with “coordinate time” $n$, but over and above any regular deviation, proceeds proportionately with $n$ and $m$. From now on we shall assume we are beyond the causal boundary layer and that normal space-time has stabilised.

By inspection of a number of examples, we are led to the following. First, denote points in the $m - n$ plane by vectors

$$\mathbf{x}_{nm} \equiv (n, m).$$
Then
\[ x^0_p \equiv (p, 0) \quad p \text{ an integer} \] (9)
represents a time-like vector whereas
\[ x^0_q \equiv (0, q) \quad q \text{ an integer} \] (10)
represents a space-like vector.

Let
\[ x^m_n \equiv (t^m_n, x^m_n) . \] (11)
Then
\[ t^m_n = n, \quad x^m_n = m \] (12)
are time and space coordinates in our absolute rest frame \( F_0 \), in which the rules of the cellular automaton are defined.

The objective now is to transform to some new frame \( F^r_s \) with coordinates
\[ t^m_n', x^m_n' , \] (13)
where now
\[ t^m_n' = \text{date}(x^m_n) , \] (14)
where \text{date} returns the implication date based on the given initial data set \( \sigma (r, s) \). The interpretation of \( x^m_n' \) remains to be determined.

By inspection of a “light-line”, representing the maximal flow of causality along the diagonal \( m = n \) in \( F_0 \), we found in all examples studied that the implication date \( t^m_n' \) along such a line satisfies the rule
\[ t^m_n' = t^m_n , \] (15)
assuming the zero of counting has been reset to occur inside normal space-time. This simply amounts to subtracting off a suitable positive constant from all dates in space-time.

However, since we also expect something like a Lorentz transformation to take us from \( F_0 \) coordinates to \( F^r_s \) we write
\[ t^m_n' = \Gamma^r_s \left( t^m_n - \frac{r}{s} x^m_n \right) , \]
\[ x^m_n' = \Gamma^r_s \left( x^m_n - \frac{r}{s} t^m_n \right) , \] (16)
where \( \Gamma^r_s \) is some scale factor to be determined.

Along the above light-line, we may use (15) and the relation
\[ t^m_n = x^m_n \] (17)
to find
\[ \Gamma^r_s = \frac{s}{s-r} = \frac{1}{1-v} , \] (18)
which is always positive.

A problem now emerges. All cells in \( F_0 \) are labelled by integers \( n \) and \( m \). In fact, it is true the other way around. Given any pair of integers \((a, b)\) then there exists a unique cell \( C^b_a \) in \( F_0 \) corresponding to this pair. However, this is not true of the frame \( F^r_s \). The coordinates \( t^m_n' \) and \( x^m_n' \) given by the rule (16) are not in general integers for arbitrary choice
of integers $m$ and $n$. The reason for this can be understood in more than one way. First, the transformation (16) is a pseudo-rotation of a cubic lattice, and it is obvious that such a transformation will not in general rotate one cubic lattice exactly into another cubic lattice. This problem is one frequently encountered in lattice gauge theories, and is addressed by an appeal to the continuum limit, i.e., it is argued that loss of rotational invariance is recovered in the limit of zero lattice spacing.

This argument does not help us here, and indeed, we would not wish to use it. We are not interested in the continuum limit at this stage, and it is essential for us to retain discreteness. Therefore, we have to get around this problem in a different way.

A second way of looking at this issue is to recognise that in such a model, only the fundamental rest frame $F_0$ “exists” in an meaningful way on the microscopic level, where the rules of the cellular automaton are defined. Other frames of reference such as $F_s$ are convenient fictions, which play a significant role on much larger scales, when the continuum versions of relativity begin to hold.

In fact, only some cells in the fictitious lattice $F_s$ transform under (16) from some cells in $F_0$. Other cells in $F_s$ are convenient interpolations and need not have any physical significance.

To determine which cells should be used in the transformation, consider the vectors in $F_0$ given by

$$m \equiv (r, s), \quad n \equiv (s, r).$$

(19)

Then consider those cells in $F_0$ at the sites

$$x \equiv (t, x) = pn + qm,$$

(20)

where $p$ and $q$ are integers. Then the coordinates $t$ and $x$ of these cells are integers given by

$$t = ps + qr,$$
$$x = pr + qs.$$

(21)

This defines a sub-lattice of $F_0$. With the transformation rule (16) and dropping the sub- and super-scripts, we find

$$x' \equiv (t', x') = (s + r)(p, q),$$

(22)

i.e.

$$t' = p(s + r),$$
$$x' = q(s + r),$$

(23)

which are integers. Essentially, we have a transformation between a sub-lattice of $F_0$ to a sub-lattice of $F_s$.

The inverse transformation exists and is given by

$$t = \Gamma_{-s}^r \left( t' + \frac{r}{s} x' \right),$$
$$x = \Gamma_{-s}^r \left( x' + \frac{r}{s} t' \right),$$

(24)

and we find that provided $t'$ and $x'$ satisfy (23), then we recover integer valued $t$ and $x$. 
The light-cone structure of the cellular automaton is invariant to these transformations. We find

\[ t'^2 - x'^2 = \left( \frac{s + r}{s - r} \right) (t^2 - x^2) \]
\[ = \left( \frac{1 + v}{1 - v} \right) (t^2 - x^2), \quad (25) \]

so that space-like, time-like, and light-like intervals in \( F_0 \) appear as space-like, time-like and light-like intervals respectively in \( F^r_s \). Causality therefore is invariant to these transformations.

6 Interpretation and discussion

We note that we started off without any concept of Minkowski metric. Indeed, the discrete topology of our cellular automaton rules is similar to those found in cellular automata such as Conway’s “Game of Life”, which is quite Newtonian in its operation. The essential difference is the role of causal implication and the occasional need for the Central processor (the Theorist) to delay causal resolution in a cell until information becomes available.

It is remarkable, therefore, that a special relativistic structure emerges from such a Newtonian basis. In this approach, there is an absolute time underlying the running of the universe, but it is the Clock Time of the Theorist, and not that of the space-time diagram per se. Proper time and relativistic time dilation can be now understood as signals of this relationship. It is also very clear from the model why time dilation occurs, and not time contraction. Computers can always lose time, but cannot go faster than their natural clock rate.

Several points need to be addressed. First, what about Minsky’s apparently wrong time dilation factor? We note that the dilation factor \( \Gamma_s \) that we found is not the conventional factor

\[ \gamma(v) \equiv \frac{1}{\sqrt{1 - v^2}} \quad (26) \]

either. However, looking at the transformation equations (16) and the causality invariance property (25), we see that the problem is one merely of an overall choice of scale of units. It is only by choosing to apply the principle of special relativity, i.e., that all inertial frames of reference should be formally equivalent, that we would be led to rescale our definitions of \( t' \) and \( x' \) by the rule

\[ t' \rightarrow \tilde{t} \equiv \sqrt{\frac{1 - v}{1 + v}} t' \]
\[ x' \rightarrow \tilde{x} \equiv \sqrt{\frac{1 - v}{1 + v}} x'. \quad (27) \]

This is something invoked for conventional macroscopic scale physics, but on a microscopic level, there is no reason to do this. Therefore, we suggest that Minsky was entirely correct in his original message.

Another problem is the fact that the transformations relate integer labelled cells only for sub-lattices of \( F_0 \) and \( F^r_s \). This is related to the need also to consider only rational fractions of the speed of light. There are two answers which come to mind. Either we allow inertial
frames moving or spatially rotated with respect to the fundamental frame to be distorted cubic lattices, or we appeal to physical scales. We may imagine that the scales $T$ and $L$ discussed earlier relate to something like Planck scales, i.e., we may suppose $T$ is of the order $10^{-44}$ seconds and $L$ is of the order $10^{-35}$ metres. This would mean that there is considerable scope for taking relatively large multiples of $T$ or $L$ (such as $s, r \sim 10^{10}$) and still leave such vast number of points per second or metre in the lattice $F_s$ for a continuum to provide a good approximation.

There may be regimes accessible to current physics which might permit investigation of this last point, but we will leave investigation of this to a future paper. In other words, there may be empirical consequences to these ideas.

Four more points need to be discussed.

1. In the model chosen and with the initial data set used in Fig. 1, causal implication would give resolution moving to the left as well as to the right. This is because the dynamics is reversible. This is not a problem, as such behaviour occurs in conventional mechanics. The Theorist needs to decide which is the correct physical direction of time flow and ignore other disconnected pieces of the causal implication.

2. There is a fundamental frame, $F_0$ in this model, in which the lattice $(n, m)$ makes sense. Lest this be regarded as contrary to the principles of relativity, two things should be considered: $(i)$ the Cosmic Background Radiation Field does provide a physically sound marker for identifying a special local inertial frame at each point in the Universe, and $(ii)$ if indeed relativity emerges as a continuum approximation to a discrete theory such as the above, then relativistic principles cannot be used to criticise the discrete theory.

3. The use of a Theorist standing outside space-time may appear ad-hoc and without explanation, but if the universe is indeed a vast cellular automaton, then it is quite feasible for complex structures to emerge within it which behave for all intents and purposes as a Theorist. This by no means invalidates Penrose’s notion that non-computational physics must be involved somewhere in the origin of consciousness [4]. Our model utilises only the end result, which is to suppose the existence of a Theorist who behaves effectively like a Turing machine. This is a curious reversal of the usual idea which advocates of artificial intelligence might discuss, namely the question whether a Turing machine might ever behave to all intents and purposes in the fashion of a human.

4. The role of quantum mechanics in such a scheme remains to be investigated and we will report on this presently.

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References

[1] Marvin Minsky, *Cellular Vacuum*, Int. J. Theor. Phys., 21(6/8), pp537-551 (1982)

[2] George Jaroszkiewicz, *Discrete Spacetime: Classical Causality, Prediction, Retrodiction and the Mathematical Arrow of Time*, in *First International Interdisciplinary Workshop: Studies on the Structure of Time: from Physics to Psycho(Patho)Logy*, edited by V. DiGesu, R. Buccheri and M.Saniga, 23-24 November 1999, CNR-Area della Ricerca di Palermo, Sicily, published by Kluwer, Dordrecht. Also available at [gr-qc/0004026](http://gr-qc/0004026).

[3] Stephen Wolfram, *Theory and Applications of Cellular Automata*, World Scientific, 1986

[4] Roger Penrose, *Shadows of the Mind*, Oxford University press, 1994