DEL PEZZO SURFACES OF PICARD NUMBER ONE ADMITTING A TORUS ACTION

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Abstract. We present efficient classification algorithms for log del Pezzo surfaces with torus action of Picard number one and given Gorenstein index. Explicit results are obtained up to Gorenstein index 200.

1. Introduction

This paper contributes to the classification of del Pezzo surfaces. Recall that a del Pezzo surface is a normal algebraic surface $X$ over an algebraically closed field $K$ of characteristic zero that admits an ample anticanonical divisor $-K_X$. The smooth del Pezzo surfaces are well known: the product $\mathbb{P}_1 \times \mathbb{P}_1$ of the projective line with itself, the projective plane $\mathbb{P}_2$ and the blowing-ups of $\mathbb{P}_2$ in up to eight points in general position. For the singular del Pezzo surfaces, we need to impose suitable conditions on the singularities in order to end up with any kind of finiteness features allowing a classification comparable to the smooth case.

In the singular case, it is common to restrict to log del Pezzo surfaces $X$, which means that all discrepancies of some resolution of singularities $X' \to X$ are greater than $-1$. Log del Pezzo surfaces are necessarily rational [22, Prop. 3.6] but still form a huge class without suitable boundedness features. A common strategy is to filter by the Gorenstein index, that means the smallest positive integer $\iota_X$ such that $\iota_X K_X$ is a Cartier divisor. The simplest case, $\iota_X = 1$ gives the Gorenstein del Pezzo surfaces $X$, which have been classified by Hidaka/Watanabe [17, Thm. 2.2]. Moreover, Alexeev/Nikulin [1] and Nakayama [22] succeeded in classifying the log del Pezzo surfaces of Gorenstein index two. Nakayama’s approach was extended by Fujita/Yasutake [11] to treat the case of Gorenstein index three and also to provide a strategy to investigate higher Gorenstein indices.

The situation becomes much more accessible and explicit if one considers del Pezzo surfaces $X$ coming with a (non-trivial) torus action. In this setting, the most symmetric ones are the toric del Pezzo surfaces, that means those with an effective action of the two-dimensional torus $K^* \times K^*$; these are automatically log del Pezzo. Kasprzyk, Kreuzer and Nill [20] provide us with a classification of the toric del Pezzo surfaces up to Gorenstein index 16. The other possible case is given by the non-toric del Pezzo $K^*$-surfaces, that means those allowing only an effective action of a one-dimensional torus $K^*$. In this case, complete classifications exist up to Gorenstein index 3; where Huggenberger [18] treated the Gorenstein case, in the Gorenstein indices 2 and 3, the case of Picard number one has been settled by Süß [23] and Hättig [13] was successful with all the higher Picard numbers.

In the present paper, we focus on log del Pezzo surfaces of Picard number one coming with a torus action. In this setting, the toric ones are precisely the fake weighted projective planes. These and their higher dimensional analogues, the fake weighted projective spaces have been studied by several authors [4, 7, 19]. In our classification, we benefit from a close connection to decompositions of $1/\iota$, where $\iota$
stands for the Gorenstein index, into a sum of three unit fractions, which finally leads to our Classification Algorithm 3.14. Up to isomorphy, the algorithm delivers 117,065 toric log del Pezzo surfaces of Picard number one. In the non-toric case, the $\mathbb{K}^*$-surfaces with at most cyclic quotient singularities form the richest case. Here we use again a connection to unit fractions in the corresponding Classification Algorithm 6.5. The cases admitting more serious singularities turn out to be less productive and can be directly addressed via the bounds provided in Proposition 6.6. We obtain 154,138 families of on-toric log del Pezzo $\mathbb{K}^*$-surfaces.

Let us give a summarizing impression of our classification results. We refer to Propositions 7.1 and 7.2 for more details. Moreover, the resulting data will be made available at [24].

**Theorem 1.1.** There are 271,203 families of log del Pezzo surfaces with torus action of Picard number one and Gorenstein index at most 200. The numbers of families for given Gorenstein index develop as follows:

| Gorenstein Index | Number of Surfaces |
|------------------|--------------------|
| 0                | 1,000              |
| 20               | 1,500              |
| 40               | 2,000              |
| 60               | 2,500              |
| 80               | 3,000              |
| 100              | 3,500              |
| 120              | 4,000              |
| 140              | 4,500              |

The computational treatment of the log del Pezzo surfaces $X$ with torus action is made possible by an encoding of the surfaces in terms of certain integral matrices $P$. If $X$ is acted on effectively by a two-dimensional torus, then it is a toric surface, and $P$ is the $2 \times 3$ matrix with the primitive generators of the describing fan of $X$ as its columns. If $X$ only allows an effective action a one-dimensional torus, then it can be realized in a very specific way as a subvariety of a toric variety $Z$ of higher dimension and the matrix $P$ encoding $X \subseteq Z$ has the primitive generators of the describing fan of $Z$ as its columns. These approaches rely on the theories of toric varieties [9, 10] and rational varieties with a torus action of complexity one initiated in [14, 16]. Restricting to Picard number one means that ambient toric varieties $Z$ showing up are all fake weighted projective spaces. Section 2 introduces to these varieties in terms of basic algebraic geometry with clear interfaces to toric geometry when using methods from there. In Section 3 we present and prove the classification algorithm for the log del Pezzo surfaces of Picard number one admitting an effective action of a two-dimensional torus. Sections 4 and 5 serve to develop the necessary parts of the theory on rational $\mathbb{K}^*$-surfaces. Adapting to the case of Picard number one allows us to stay in terms of basic algebraic geometry, giving clear interfaces the general theory when necessary. In Section 6, we present our classification procedure for the log del Pezzo $\mathbb{K}^*$-surfaces of Picard number one. and Section 7 provides tables on the classification results.
2. Basics on fake weighted projective spaces

The projective space $\mathbb{P}_n$ is the set of all lines through the origin in the affine $(n+1)$-plane. We may regard $\mathbb{P}_n$ as well as the quotient of the pointed affine $(n+1)$-plane by the one-dimensional torus $\mathbb{K}^*$ acting via scalar multiplication. Allowing more generally one-dimensional quasitorus actions, this point of view brings us to the fake weighted projective spaces. We give a basic introduction, define all necessary notions, present two possible constructions of fake weighted projective spaces in detail and show how to turn fake weighted projective spaces into toric varieties.

A torus is an algebraic group isomorphic to some standard $n$-torus $\mathbb{T}^n = (\mathbb{K}^*)^n$. We denote by $C(m) \subseteq \mathbb{K}^*$ the group of $m$-th roots of unity. A quasitorus is an algebraic group isomorphic to a standard $n$-quasitorus, that means a direct product

$$\mathbb{T}^n \times C, \quad C = C(m_1) \times \ldots \times C(m_k).$$

A character of a quasitorus $H$ is a homomorphism $\chi : H \to \mathbb{K}^*$ of algebraic groups. Explicitly, given a standard quasitorus $\mathbb{T}^n \times C$, set $\Gamma := \mathbb{Z}/m_1 \mathbb{Z} \times \ldots \times \mathbb{Z}/m_k \mathbb{Z}$. Then every $\omega = (w, \eta) \in \mathbb{Z}^n \times \Gamma$ defines a character

$$\chi^\omega : \mathbb{T}^n \times C \to \mathbb{K}^*, \quad (s, \zeta) \mapsto s^w \zeta^\eta, \quad s^w := s_1^{w_1} \cdots s_n^{w_n}, \quad \zeta^\eta := \zeta_1^{\eta_1} \cdots \zeta_k^{\eta_k},$$

and this assignment sets up an isomorphism $\mathbb{Z}^n \times \Gamma \to \mathbb{K}^{(\mathbb{T}^n \times C)}$ onto the character group of $\mathbb{T}^n \times C$. We are ready to present our first concrete construction of fake weighted projective spaces.

**Construction 2.1** (Fake weighted projective spaces). Consider a one-dimensional quasitorus, given as a direct product

$$H = \mathbb{K}^* \times C, \quad C = C(m_1) \times \ldots \times C(m_k)$$

and characters $\chi^{w_0}, \ldots, \chi^{w_n}$, where $\omega_i = (w_i, \zeta_i) \in \Gamma$ with $w_0, \ldots, w_n > 0$. Then we have an action

$$H \times \mathbb{K}^{n+1} \to \mathbb{K}^{n+1}, \quad h \cdot z := (\chi^{w_0}(h)z_0, \ldots, \chi^{w_n}(h)z_n)$$

having $0 \in \mathbb{K}^{n+1}$ as an attractive fixed point. The fake weighted projective space associated with $\omega_0, \ldots, \omega_n$ is the orbit space

$$\mathbb{P}(\omega_0, \ldots, \omega_n) := (\mathbb{K}^{n+1} \setminus \{0\})/H.$$ 

**Proposition 2.2.** Consider a fake weighted projective space $\mathbb{P}(\omega_0, \ldots, \omega_n)$ resulting from Construction 2.1 and the canonical map

$$\pi : \mathbb{K}^{n+1} \setminus \{0\} \to \mathbb{P}(\omega_0, \ldots, \omega_n), \quad z \mapsto H \cdot z.$$ 

Then $\mathbb{P}(\omega_0, \ldots, \omega_n)$ is an irreducible, normal, projective variety of dimension $n$ once we installed the quotient topology w.r.t. $\pi$ and the structure sheaf

$$\mathcal{O}(U) := \{f : U \to \mathbb{K}; \ f \circ \pi \in \mathcal{O}(\pi^{-1}(U))\} = \mathcal{O}(\pi^{-1}(U))^H.$$
Moreover, \( \pi: \mathbb{P}^{n+1} \setminus \{ 0 \} \to \mathbb{P}(\omega_0, \ldots, \omega_n) \) is an affine \( H \)-invariant morphism and every \( H \)-invariant morphism \( \varphi: \mathbb{P}^{n+1} \setminus \{ 0 \} \to X \) uniquely factors as

\[
\begin{array}{ccc}
\mathbb{P}(\omega_0, \ldots, \omega_n) & \xrightarrow{\varphi} & X \\
\downarrow \pi & & \downarrow \\
\mathbb{P}(\omega_0, \ldots, \omega_n) & \xrightarrow{\varphi} & \mathbb{P}(\omega_0, \ldots, \omega_n)
\end{array}
\]

Proof. All this is a consequence of the fact that \( \mathbb{P}(\omega_0, \ldots, \omega_n) \) is a GIT-quotient of the action of \( H \) on \( \mathbb{P}^{n+1} \) in the sense of [21]. More explicitly, the \( H \)-action on \( \mathbb{P}^{n+1} \) stems from the \( (\mathbb{Z} \times \Gamma) \)-grading of \( \mathbb{K}[T_0, \ldots, T_n] \) given by \( \deg(T_i) = \omega_i \). Thus, projectivity of \( \mathbb{P}(\omega_0, \ldots, \omega_n) \) and the universal property of \( \pi \) follow from [3 Prop. 3.1.2.2]. Finally, using normality of \( \mathbb{K}^{n+1} \) and the universal property of \( \pi \), we obtain normality of \( \mathbb{P}(\omega_0, \ldots, \omega_n) \).

Note that for the special case \( H = \mathbb{K}^* \), Construction 2.1 delivers precisely the weighted projective spaces. Moreover, it shows that every fake weighted projective space is covered by a weighted one.

Remark 2.3. Every fake weighted projective space is a quotient of a weighted projective space by a finite abelian group: there is a commutative diagram

\[
\begin{array}{ccc}
\mathbb{P}(w_0, \ldots, w_n) & \xrightarrow{\mathbb{K}^*} & \mathbb{P}(\omega_0, \ldots, \omega_n) \\
\downarrow /H & & \downarrow /C \\
\mathbb{P}(\omega_0, \ldots, \omega_n) & \xrightarrow{\mathbb{K}^*} & \mathbb{P}(\omega_0, \ldots, \omega_n)
\end{array}
\]

where \( H = \mathbb{K}^* \times C \) and \( \omega_i = (w_i, \eta_i) \) are as in Construction 2.1 and we have the induced action of \( C \) on the weighted projective space \( \mathbb{P}(w_0, \ldots, w_n) \).

Example 2.4. Consider the one-dimensional quasitorus \( H = \mathbb{K}^* \times C(4) \) and the action of \( H \) on \( \mathbb{K}^3 \) given by

\[
(s, \zeta) \cdot (z_0, z_1, z_2) = (s^2 \zeta z_0, s \zeta^2 z_1, s \zeta z_2).
\]

Then we have \( \Gamma := \mathbb{Z}/4\mathbb{Z} \) and, denoting the elements of \( \Gamma \) by \( \bar{0}, \bar{1}, \bar{2}, \bar{3} \), the \( \omega_i = (w_i, \eta_i) \) are explicitly given as

\[
\omega_0 = (2, \bar{1}), \quad \omega_1 = (1, \bar{2}), \quad \omega_2 = (1, \bar{1}).
\]

We arrive at a fake weighted projective plane \( \mathbb{P}(\omega_0, \omega_1, \omega_2) = (\mathbb{K}^3 \setminus \{ 0 \})/H \), which by Remark 2.3 comes with a 4:1 cover by the weighted projective plane \( \mathbb{P}(2, 1, 1) \).

Many of the well known concepts around the projective space directly generalize to the fake weighted projective spaces. For instance, we can introduce analogues of homogeneous coordinates, coordinate hyperplanes and the standard affine charts as follows.

Definition 2.5. Consider a fake weighted projective space \( \mathbb{P}(\omega_0, \ldots, \omega_n) \) arising from Construction 2.1. Given \( z = (z_0, \ldots, z_r) \) in \( \mathbb{K}^{n+1} \setminus \{ 0 \} \), set

\[
[z] = [z_0, \ldots, z_n] := H \cdot z = \mathbb{P}(\omega_0, \ldots, \omega_n).
\]

Then we call \([z]\) a presentation of \( H \cdot z \) in homogeneous coordinates. For any two points \( z, z' \in \mathbb{K}^{n+1} \setminus \{ 0 \} \), we have

\[
[z] = [z'] \iff z' = h \cdot z \text{ for some } h \in H.
\]

Moreover, for \( k = 0, \ldots, n \) we define the \( k \)-th coordinate divisor to be the closed, irreducible \( (n-1) \)-dimensional subvariety

\[
D_k := \{ [z]; \ z \in \mathbb{K}^{n+1} \setminus \{ 0 \}, \ z_k = 0 \} \subseteq \mathbb{P}(\omega_0, \ldots, \omega_n).
\]
Finally, for \( k = 0, \ldots, n \), the \( k \)-th affine chart is the open affine subvariety obtained by removing the \( k \)-th coordinate divisor: 
\[
Z_k := \mathbb{P}(\omega_0, \ldots, \omega_n) \setminus D_k.
\]

**Remark 2.6.** Every coordinate divisor of a fake weighted projective space is itself a fake weighted projective space. Moreover, intersecting coordinate divisors, we obtain coordinate subspaces, which again are fake weighted projective spaces.

An important feature of fake weighted projective spaces is that they are examples of toric varieties; recall that a toric variety is a normal variety \( X \) together with an effective action of a torus \( T \) having an open orbit \( T \cdot x_0 \subseteq X \). The torus \( T \) is called the acting torus of \( X \).

**Remark 2.7.** Consider a fake weighted projective space \( \mathbb{P}(\omega_0, \ldots, \omega_n) \) produced by Construction 2.4. Then the torus \( \mathbb{T}^{n+1} = (\mathbb{K}^*)^{n+1} \) acts on \( \mathbb{K}^{n+1} \setminus \{0\} \) via 
\[
t \cdot z = (t_0 z_0, \ldots, t_n z_n).
\]
This action commutes with the \( H \)-action and induces an effective almost transitive action of the torus \( \mathbb{T}^n \cong \mathbb{T}^{n+1}/H \) on \( \mathbb{P}(\omega_0, \ldots, \omega_n) \), turning it into a toric variety.

In order to benefit from the rich theory of toric varieties [9, 10, 12], we provide another way to construct fake weighted projective spaces, starting with a certain integral matrix as input data. Besides strengthening the connection to toric varieties, this approach also yields an appropriate encoding for our subsequent computational considerations.

**Construction 2.8** (Fake weighted projective spaces via integral matrices). Consider an integral \( n \times (n+1) \) matrix 
\[
P = [v_0, \ldots, v_n],
\]
the columns \( v_0, \ldots, v_n \) of which are primitive vectors in \( \mathbb{Z}^n \) generating \( \mathbb{Q}^n \) as a convex cone. The matrix \( P = (p_{ij}) \) defines a homomorphism of tori 
\[
p: \mathbb{T}^{n+1} \to \mathbb{T}^n, \quad t \mapsto (t_0^{p_{0\star}}, \ldots, t_n^{p_{n\star}}),
\]
where \( t^{p_{i\star}} = t_0^{p_{i0}} \cdots t_n^{p_{in}} \) is the monomial in \( t_0, \ldots, t_n \) having the \( i \)-th row of \( P \) as its exponent vector. This in turn gives us a one-dimensional quasitorus 
\[
H := \ker(p) \subseteq \mathbb{T}^{n+1}.
\]
Denote by \( \chi_{\omega_i} \in \mathbb{X}(H) \) the character obtained by restricting the \( i \)-th coordinate function \( \mathbb{T}^{n+1} \to \mathbb{K}^* \). Then the subgroup \( H \subseteq \mathbb{T}^{n+1} \) acts on \( \mathbb{K}^{n+1} \) via 
\[
h \cdot z = (\chi_{\omega_0}(h) z_0, \ldots, \chi_{\omega_n}(h) z_n).
\]
Suitably splitting \( H = \mathbb{K}^* \times C \) with \( C = C(1) \times \cdots \times C(k) \), we can write \( \omega_i = (w_i, \eta_i) \) as in Construction 2.4 and arrive at a fake weighted projective space 
\[
Z(P) := (\mathbb{K}^{n+1} \setminus \{0\})/H = \mathbb{P}(\omega_0, \ldots, \omega_n).
\]

**Example 2.9.** Let us see how to obtain the fake weighted projective plane from Example 2.4 by means of Construction 2.8. Consider the matrix 
\[
P = \begin{bmatrix} 1 & 1 & -3 \\ 0 & 4 & -4 \end{bmatrix}.
\]
Note that the columns of \( P \) are primitive and generate \( \mathbb{Q}^2 \) as a cone. The homomorphism of tori associated with \( P \) is given as 
\[
p: \mathbb{T}^3 \to \mathbb{T}^2, \quad (t_0, t_1, t_2) \mapsto \left( \frac{t_0 t_1}{t_2}, \frac{t_1^4}{t_2^3} \right).
\]
One directly computes
\[ H := \ker(p) = \{(s^2, s^2, s) : s \in \mathbb{K}^*, \zeta \in C(4)\} \subseteq \mathbb{T}^1. \]
Moreover, \((s, \zeta) \mapsto (s^2, s^2, s)\) yields a splitting \(\mathbb{K}^* \times C(4) \cong H\). Thus, we indeed arrive at Example 2.4 again:
\[ Z(P) = (\mathbb{K}^1 \setminus \{0\})/H = \mathbb{P}(\omega_0, \omega_1, \omega_2), \quad \omega_0 = (2, 1), \ \omega_1 = (1, 2), \ \omega_2 = (1, 1). \]

Observe that Construction 2.8 is more special than Construction 2.1 in the sense that it will not produce all character lists that will appear within Construction 2.1 as for instance \(\langle \omega_0, \omega_1 \rangle = (2, 2)\). More precisely, we can say the following.

**Remark 2.10.** In Construction 2.8 the matrix \(P\) has pairwise distinct primitive columns. This merely means that any \(n\) members of the resulting weight list \((\omega_0, \ldots, \omega_n)\) generate \(Z \times \Gamma\) as a group; use [3, Lemma 2.1.4.1]. Thus, Construction 2.8 delivers precisely the well-formed weight lists \((\omega_0, \ldots, \omega_n)\).

Remark 2.7 tells us that any fake weighted projective space is a toric variety. For the \(Z(P)\) arising from Construction 2.8 we will use the homomorphism \(p : \mathbb{T}^{n+1} \to \mathbb{T}^n\) given by the matrix \(P\) in order to understand the torus action more concretely.

**Remark 2.11.** Consider a fake weighted projective space \(Z(P)\) arising from Construction 2.8 with the action of the torus \(\mathbb{T}^{n+1}/H\) as provided by Remark 2.7. Then we obtain a commutative diagram
\[
\begin{array}{ccc}
\mathbb{T}^{n+1} & \subseteq & \mathbb{K}^{n+1} \setminus \{0\} \\
p & & \pi \\
\mathbb{T}^n & \longrightarrow & Z(P),
\end{array}
\]
where the lower arrow is an isomorphism from the torus \(\mathbb{T}^n\) onto the torus \(\mathbb{T}^{n+1}/H\) which in turn equals the open set \(\pi(\mathbb{T}^{n+1}) \subseteq Z(P)\). This allows us to regard \(\mathbb{T}^n\) as the acting torus of the toric variety \(Z(P)\).

**Proposition 2.12.** Consider a fake weighted projective space \(Z(P)\) provided by Construction 2.8. Then the quasitorus \(H\) acts freely on the open set
\[
\bigcup_{0 \leq j < k \leq n} \mathbb{K}^{n+1} \setminus V(T_j, T_k) \subseteq \mathbb{K}^{n+1}.
\]
The isotropy group of \(\mathbb{T}^n\) at any point \([z] \in D_k\) with \(z_j \neq 0\) for \(j \neq k\) is given in terms of the \(k\)-th column \(v_k = (v_{k1}, \ldots, v_{kn})\) as
\[
\mathbb{T}^n_{[z]} = \{(s^{v_{k1}}, \ldots, s^{v_{kn}}) : s \in \mathbb{K}^*\} \subseteq \mathbb{T}^n.
\]
The fixed points of the \(\mathbb{T}^n\)-action on \(Z(P)\) are precisely the points \(z(k) \in Z\) having all homogeneous coordinates except the \(k\)-th one equal to zero:
\[
z(0) = [1, 0, \ldots, 0], \quad z(1) = [0, 1, 0, \ldots, 0], \quad \ldots, \quad z(n) = [0, \ldots, 0, 1].
\]
The affine chart \(Z_k\) is the minimal \(\mathbb{T}^n\)-invariant open set containing the point \(z(k)\). Moreover, identifying \(\mathbb{T}^n\) with its open orbit in \(Z\), we have
\[
Z_0 \cap \ldots \cap Z_n = \mathbb{T}^n = Z \setminus (D_0 \cup \ldots \cup D_n).
\]

**Proof.** Let us see why \(H\) acts freely on the set of points \(z = (z_0, \ldots, z_n)\) with at most one coordinate equal to zero. For instance, consider \(z = (0, z_1, \ldots, z_n)\). We have to show that the isotropy group \(H_z\) is trivial. For any \(h \in H\), we have
\[
h \cdot z = z \iff \chi^{\omega_1}(h) = \ldots = \chi^{\omega_n}(h) = 1.
\]
Remark 2.11 tells us that \(\omega_1, \ldots, \omega_n\) generate \(\mathbb{Z} \times \Gamma\) as a group. Thus, \(\chi^{\omega_1}, \ldots, \chi^{\omega_n}\) generate the character group \(X(H)\), hence \(\mathcal{O}(H)\) and thus separate the points of \(H\). Consequently, \(h \cdot z = z\) only happens for \(h = 1\).

We determine the isotropy group \(\mathbb{T}_e\), where \(z\) has exactly one coordinate equal to zero. Again we discuss exemplarily \(z = (0, z_1, \ldots, z_n)\). In terms of the homomorphism \(p: \mathbb{T}^{n+1} \to \mathbb{T}^n\) defined by the matrix \(P\) and \(H = \ker(p)\) we have
\[
\mathbb{T}_e^n = \{p(t); t \in \mathbb{T}^{n+1}, t \cdot z \in H \cdot z\} = p(\mathbb{T}_e^{n+1}).
\]
The isotropy group \(\mathbb{T}_e^{n+1}\) consists precisely of the elements \((s, 1, \ldots, 1) \in \mathbb{T}^{n+1}\), where \(s \in \mathbb{K}^*\). These are mapped via \(p: \mathbb{T}^{n+1} \to \mathbb{T}^n\) precisely to the elements \((s^{v_1}, \ldots, s^{v_n}) \in \mathbb{P}^n\), where \(s \in \mathbb{K}^*\). \(\square\)

The \(n\)-dimensional toric varieties are in correspondence with fans in \(\mathbb{Z}^n\), that means finite sets \(\Sigma\) of pointed, polyhedral, convex cones in \(\mathbb{Q}^n\) such that for any cone in \(\Sigma\) all its faces belong to \(\Sigma\) as well and any two cones of \(\Sigma\) intersect in a common face. Let us see how to detect the defining fan of a fake weighted projective space \(Z\) given by Construction 2.8 and, as a direct consequence, its divisor class group \(\text{Cl}(Z)\) and Cox ring
\[
\mathcal{R}(Z) = \bigoplus_{\text{Cl}(Z)} \Gamma(Z, \mathcal{O}(D)).
\]

A reference on Cox rings of toric varieties is [8]; see also [3] Sections 2.1.3 and 2.1.4 and [9] Chap. 5 for more details on Cox rings and Cox’s quotient construction. However, a deeper understanding of the theory around Cox rings is not needed throughout this text.

Proposition 2.13. Let \(P = [v_0, \ldots, v_n]\) be as in Construction 2.8. Then \(Z = Z(P)\) is the toric variety associated with the fan \(\Sigma\) in \(\mathbb{Z}^n\) given by
\[
\Sigma = \{\text{cone}(v_{i_1}, \ldots, v_{i_m}); 0 \leq i_1 < \ldots < i_m \leq n, m \leq n\}.
\]

With \(K := \mathbb{Z}^{n+1}/P\mathbb{Z}^n\), we obtain the character group of the quasitorus \(H\) and the divisor class group of the fake weighted projective space \(Z\) as
\[
X(H) \cong K \cong \text{Cl}(Z).
\]

With the canonical projection \(Q: \mathbb{Z}^{n+1} \to K\), these isomorphisms allow to relate the characters \(\chi^{\omega_i}\) and the coordinate divisors \(D_i\) to each other via
\[
\omega_i = Q(e_i) = [D_i], \quad i = 0, \ldots, n.
\]

Moreover, the Cox ring of the toric variety \(Z = Z(P)\) equals its \(K\)-homogeneous coordinate ring and thus is given as the \(K\)-graded polynomial ring
\[
\mathcal{R}(Z) = \mathbb{K}[T_0, \ldots, T_n], \quad \deg(T_i) = Q(e_i) \in K.
\]

Finally Cox’s quotient presentation of the toric variety \(Z = Z(P)\) is precisely the quotient map showing up in its construction:
\[
\pi: \mathbb{K}^{n+1} \setminus \{0\} \to Z = (\mathbb{K}^{n+1} \setminus \{0\})/H, \quad z \mapsto [z] := H \cdot z.
\]

Proof. In the set \(\Sigma\), the cones \(\sigma_i := \text{cone}(v_k; k \neq i)\) are maximal with respect to inclusion. Since \(v_0, \ldots, v_n\) generate \(\mathbb{Q}^n\) as a cone, each \(\sigma_i\) is \(n\)-dimensional, hence pointed, and any two of them intersect in a common facet. Thus, \(\Sigma\) is a fan and there is an associated toric variety \(Z(\Sigma)\). Cox’s quotient presentation [9, Sec. 5.1] and [3] Sec. 2.1.3 reproduces \(Z(\Sigma)\) as the quotient of \(\mathbb{K}^{n+1} \setminus \{0\}\) by \(H = \ker(p) \subseteq \mathbb{T}^{n+1}\), acting exactly as in Construction 2.8. We conclude \(Z(\Sigma) = Z(P)\). For the remaining statements, we refer to [3] Constr. 2.1.3.1. \(\square\)

Proposition 2.14. Let \(Z\) be a fake weighted projective space. Then \(Z \cong Z(P)\) holds with \(Z(P)\) arising from Construction 2.8.
Proof. By Remark 2.7 any fake weighted projective space $Z$ is a toric variety and thus $Z \cong Z(\Sigma)$ with a toric variety $Z(\Sigma)$ given by a fan $\Sigma$ in $\mathbb{Z}^n$. Applying Cox’s quotient presentation [9, Sec. 5.1] and [3, Sec. 2.1.3] to $Z(\Sigma)$, we see that the matrix $P$ having the primitive generators of the fan $\Sigma$ as its columns satisfies $Z(\Sigma) \cong Z(P)$. □

Proposition 2.15. We have $Z(P) \cong Z(P')$ if and only if $P' = A \cdot P \cdot S$ holds with a unimodular matrix $A$ and a permutation matrix $S$.

Proof. First we note that $Z(P)$ and $Z(P')$ are isomorphic as algebraic varieties if and only if they are isomorphic as toric varieties; the reason for this is that toric varieties have a linear algebraic automorphism group and thus any two maximal tori of the automorphism group are conjugate. Next, we remark that $Z(P)$ and $Z(P')$ are isomorphic as toric varieties if and only if there is a unimodular matrix $A$ mapping the fan of $Z(P)$ cone-wise to the fan of $Z(P')$, which in turn means $P' = A \cdot P \cdot S$ with the matrix $A$ and a permutation matrix $S$. □

3. Classifying fake weighted projective planes

We present our procedure for efficiently classifying fake weighted projective planes of given Gorenstein index. Recall that a normal variety $X$ is $\mathbb{Q}$-Gorenstein if some positive multiple of its canonical divisor $K_X$ is Cartier. The Gorenstein index of a $\mathbb{Q}$-Gorenstein variety $X$ is the smallest positive integer $\iota_X$ such that $\iota_X K_X$ is Cartier. Here is the main result of the section.

Theorem 3.1. There are 117,065 isomorphy classes of toric del Pezzo surfaces of Picard number one and Gorenstein index at most 200. The numbers of isomorphy classes for given Gorenstein index develop as follows:

![Graph showing the number of isomorphy classes for given Gorenstein index]

The result is obtained by applying the classification algorithm [4.14] developed throughout this section. We first provide the necessary facts on the geometry of fake weighted projective spaces $Z(P)$, using their structure as toric varieties; see [9] Prop. 4.1.2, Thm. 4.1.3 and [3] Prop. 2.1.2.7 for the details.

Proposition 3.2. Consider $Z = Z(P)$ given by Construction 2.8 and a character $\chi^u \in \chi(\mathbb{T}^n)$. Then $\chi^u$ is a rational function on $Z$ with divisor

$$\text{div}(\chi^u) = \langle u, v_0 \rangle D_0 + \cdots + \langle u, v_n \rangle D_n.$$ 

Proposition 3.3. Every fake weighted projective space $Z$ is $\mathbb{Q}$-factorial that means that for any Weil divisor on $Z$ some positive multiple is a Cartier divisor.

Proof. By Proposition 2.14 we may assume $Z = Z(P)$. Since $v_0, \ldots, v_n$ generate $\mathbb{Q}^n$ as a cone, each of the cones of the fan $\Sigma$ is generated by a linearly independent collection of the $v_j$. Now [9] Prop. 4.2.7] tells us that $Z$ is $\mathbb{Q}$-factorial. □
**Proposition 3.4.** For a fake weighted projective space $Z = Z(P)$ and any Weil divisor $D = a_0D_0 + \ldots + a_mD_m$ on $Z$, the following statements are equivalent.

(i) The divisor $D$ is Cartier on a neighbourhood of $z(i) \in Z$.

(ii) The divisor $D$ is Cartier on the affine open subvariety $Z_i \subseteq Z$.

(iii) There is a linear form $u \in \mathbb{Z}^n$ such that $D = \text{div}(\chi^u)$ holds on $Z_i \subseteq Z$.

(iv) There is a linear form $u \in \mathbb{Z}^n$ such that $\langle u, v_j \rangle = a_j$ for all $j \neq i$.

(v) The class $a_0\omega_0 + \ldots + a_m\omega_m \in K$ is a multiple of $\omega_i \in K$.

**Proof.** Assume that (i) holds. Then $D$ is Cartier on an open neighborhood $U \subseteq Z$ of $z(i) \in Z$. Since $D$ is $\mathbb{T}^n$-invariant, $D$ is also Cartier on $\mathbb{T}^n \cdot U$. Since $Z_i \subseteq Z$ is the smallest $\mathbb{T}^n$-invariant open neighbourhood containing $z(i)$, we have $Z_i \subseteq \mathbb{T}^n \cdot U$ and conclude that $D$ is Cartier on $Z_i$. Now assume that (ii) holds. Then [9, Prop. 4.2.2] tells us that the restriction of $D$ to $Z_i$ is the divisor of a character function $\chi^u$. Assertions (iii) and (iv) are equivalent due to Proposition 3.2. If (iii) holds, then we have $D - \text{div}(\chi^u) = bD_i$ with $b \in \mathbb{Z}$. Finally, if (v) holds, then the class $a_0\omega_0 + \ldots + a_m\omega_m$ equals $b\omega_i$ with some $b \in \mathbb{Z}$. Consequently, $D - bD_i$ is principal on $Z_i$ and hence $D$ is principal on the neighbourhood $Z_i$ of $z(i)$.

For a normal variety $X$ and a point $x \in X$, one defines the **local class group** $\text{Cl}(X, x)$ to be the factor group of the Weil divisor group $\text{WDiv}(X)$ by the subgroup $\text{PDiv}(X, x)$ of all Weil divisors being principal on a neighbourhood of $x$. Proposition 3.4 allows us to determine these groups for the toric fixed points of a fake weighted projective space.

**Corollary 3.5.** Consider a fake weighted projective space $Z = Z(P)$, the points $z(i) \in Z$ and the divisor classes $\omega_i = [D_i] \in K = \text{Cl}(Z)$. Then we have

$$\mathbb{Z}\omega_i = \{ [D]; D \in \text{PDiv}[Z, z(i)] \} \subseteq K = \text{Cl}(Z).$$

In particular, the local class groups $\text{Cl}(Z, z(i))$ of points $z(i) \in Z$ and the respective group orders are given as

$$\text{Cl}(Z, z(i)) = K/\mathbb{Z}\omega_i, \quad |\text{Cl}(Z, z(i))| = |\text{det}(v_0, \ldots, v_n)|.$$

Finally, the Picard group of $Z$, being the intersection over all the local class groups in $\text{Cl}(Z)$, can be presented as

$$\text{Pic}(Z) = \mathbb{Z}\omega_0 \cap \ldots \cap \mathbb{Z}\omega_n \subseteq \text{Cl}(Z).$$

**Proof.** For computing the order of $\text{Cl}(Z, z(i))$, we use [3, Lemma 2.1.4.1]. All other assertions are direct consequences of Proposition 3.4.

The **local Gorenstein index** of a point $x \in X$ is the smallest positive integer $\iota(x)$ such that $\iota(x)K_X$ is Cartier on a neighbourhood of $x \in X$.

**Remark 3.6.** For every $\mathbb{Q}$-Gorenstein variety, the Gorenstein index $\iota_X$ equals the least common multiple of the local Gorenstein indices $\iota(x)$, where $x \in X$.

**Proposition 3.7.** Consider a fake weighted projective space $Z = Z(P)$.

(i) The variety $Z$ is $\mathbb{Q}$-factorial and thus $\mathbb{Q}$-Gorenstein. An anticanonical divisor of $Z$ is explicitly given by $-K_Z = D_0 + \ldots + D_n$.

(ii) For each $k = 0, \ldots, n$, there is a unique linear form $u_k \in \mathbb{Q}^n$ such that for all $j \neq k$ we have $\langle u_k, v_j \rangle = 1$.

(iii) The local Gorenstein index of $z(k) \in Z$ is the minimal positive integer $\iota_k$ such that $\iota_ku_k$ is a primitive vector in $\mathbb{Z}^n$.

(iv) The Gorenstein index of $Z$ is the least common multiple of the local Gorenstein indices $\iota_0, \ldots, \iota_n$. 


Proof. Proposition 3.3 ensures that $Z$ is $\mathbb{Q}$-factorial. The statement on the anticanonical divisor is a special case of [9, Thm. 8.2.3]. This proves (i). Assertion (ii) is clear because for each $k = 0, \ldots, n$ the family $(v_j; j \neq k)$ is a basis for $\mathbb{Q}^n$. We show (iii). Let $\nu_k$ be a positive integer such that the anticanonical divisor $-\nu_k K_Z$ from (ii) is Cartier on the open set $Z_k \subseteq Z$. Then Proposition 3.9 tells us that $-\nu_k K_Z$ equals $\text{div}(\chi_{\nu_k a_k})$ on $Z_k$ and $\nu_k a_k \in \mathbb{Z}^n$. The claim follows. Assertion (iv) is clear.

□

Definition 3.8. Given an $n$-dimensional fake weighted projective space $Z = Z(P)$, we call $u_0, \ldots, u_n$ from Proposition 3.7 (iii) the Gorenstein forms of $Z$.

The following elementary observation helps to identify Gorenstein forms and local Gorenstein indices in the case of fake weighted projective planes.

Lemma 3.9. Let $v_1 = (a, c)$ and $v_2 = (b, d)$ be non-collinear primitive vectors in $\mathbb{Z}^2$. Moreover, consider

$$u := \begin{bmatrix} d - c & a - b \\ ad - bc & ad - bc \end{bmatrix}, \quad \iota := \frac{|ad - bc|}{\gcd(d - c, a - b)}.$$

Then $\langle u, v_i \rangle = 1$ holds for $i = 1, 2$ and $\iota$ is the unique positive integer such that $\iota \cdot u$ is a primitive vector in $\mathbb{Z}^2$.

We provide the necessary statements for our classification algorithm for fake weighted projective planes with given Gorenstein index.

Lemma 3.10. Fix a positive integer $\iota$ and consider the following matrix, depending on three other positive integers $a_0, a_1, a_2$:

$$G := G(\iota, a_0, a_1, a_2) := \begin{bmatrix} \iota - a_0 & \iota & \iota \\ \iota & \iota - a_1 & \iota \\ \iota & \iota & \iota - a_2 \end{bmatrix}.$$ 

Then $G$ is of rank greater or equal to two and it is of rank exactly two if and only if we have the following identity of unit fractions:

$$\frac{1}{\iota} = \frac{1}{a_0} + \frac{1}{a_1} + \frac{1}{a_2}.$$

If a triple of positive integers $(a_0, a_1, a_2)$ satisfies this identity, then the kernel of $G$ is generated by the primitive vector

$$w(a_0, a_1, a_2) := \frac{1}{\gcd(a_1 a_2, a_0 a_2, a_0 a_1)} \begin{bmatrix} a_1 a_2 \\ a_0 a_2 \\ a_0 a_1 \end{bmatrix} \in \mathbb{Z}^3.$$

Proof. All statements made in the Lemma can be directly verified via elementary computations.

□

Now we are going to produce systematically fake weighted projective planes $Z(P)$ of given Gorenstein index $\iota$ with local Gorenstein indices $\iota_0, \iota_1, \iota_2$.

Proposition 3.11. Fix positive integers $\iota, \iota_0, \iota_1, \iota_2$ with $\iota = \text{lcm}(\iota_0, \iota_1, \iota_2)$, assume that $a_0, a_1, a_2 \in \mathbb{Z}_{\geq 1}$ satisfy

$$\frac{1}{\iota} = \frac{1}{a_0} + \frac{1}{a_1} + \frac{1}{a_2}$$

and let $w := w(a_0, a_1, a_2) = (w_0, w_1, w_2)$ be the primitive generator of the kernel of the matrix $G(\iota, a_0, a_1, a_2)$. Consider the $2 \times 3$ matrices of the form

$$P := \begin{bmatrix} 1 & ax + 1 & -\frac{w_0 + w_1 + w_2 x_1}{x_2} \\ 0 & x_2 & \frac{-w_0 - w_1 + w_2 x_1}{x_2} \end{bmatrix}, \quad x \in \mathbb{Z}_{\geq 1}, x w_1 | (w_0 + w_1 + w_2) \iota_1, \frac{1}{a_0} \leq a < \frac{1}{a_2}.$$
If the columns of $P$ are primitive vectors in $\mathbb{Z}^2$, then it defines a fake weighted projective plane $Z = Z(P)$. Moreover, if each of 

$$
t_0 \cdot \left[ \begin{array}{c} -\frac{w_0 + w_2}{w_0 + w_1 + w_2 + a x (w_1 + w_2)} \\ \frac{w_0 + w_1 + w_2}{w_0 + w_1 + w_2 + a x (w_1 + w_2)} \\ x x_2 w_0 \end{array} \right], \quad t_1 \cdot \left[ \begin{array}{c} 1 \\ -\frac{w_0 + w_1 + w_2 + a x w_1}{x x_2 w_0} \\ x x_2 w_0 \end{array} \right], \quad t_2 \cdot \left[ \begin{array}{c} 1 \\ -\frac{a}{x x_2} \\ x x_2 w_0 \end{array} \right]$$

is a primitive vector in $\mathbb{Z}^2$, then $Z$ is of Gorenstein index $\iota$ with the local Gorenstein indices $\iota_k$ of the toric fixed points $z(k)$, where $k = 0, 1, 2$.

Proof. We show that the columns of $P$ generate $\mathbb{Q}^2$ as a cone. Given $v \in \mathbb{Q}^2$, we can clearly write it as a linear combination over the columns of $P$, that is $v = P \cdot u$ for some $u \in \mathbb{Q}^3$. Now observe $P \cdot w = 0$. Thus, since all entries of $w$ are strictly positive, we obtain a positive combination $v = P \cdot (u + \xi w)$ by choosing $\xi \in \mathbb{Q}_0$ big enough. Thus, if the columns of $P$ are integral vectors in $\mathbb{Z}^2$, then it defines a fake weighted projective plane. If also the three displayed vectors are integral and primitive, then, using Lemma 3.9, one directly checks that these are the $\iota_k$-fold local Gorenstein form $w_k$ of the toric fixed points $z(k)$. $\square$

Finally, we show that the above example yields indeed all fake weighted projective planes of given Gorenstein index.

**Proposition 3.12.** Consider a fake weighted projective plane $Z$ of Gorenstein index $\iota$. Then $Z \cong Z(P)$ with a matrix $P$ as provided by Proposition 3.11.

Proof. We know that $Z = Z(P)$ holds with a $2 \times 3$ matrix $P$. The divisor class group of $Z$ is given as

$$\text{Cl}(Z) \cong \mathbb{Z}^3/P^*\mathbb{Z}^2 \cong \mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}.$$ 

In particular, $\text{Cl}(Z)$ has a cyclic torsion part and thus, the associated degree matrix is of the form

$$Q = \left[ \begin{array}{ccc} w_0 & w_1 & w_2 \\ \eta_0 & \eta_1 & \eta_2 \end{array} \right], \quad w_0, w_1, w_2 \in \mathbb{Z}_{\geq 1}, \quad \eta_0, \eta_1, \eta_2 \in \mathbb{Z}/n\mathbb{Z}.$$ 

As $Z$ is of Gorenstein index $\iota$, the divisor class $-\iota K_Z$ is Cartier and thus Proposition 3.14 provides us with positive integers $a_0, a_1, a_2$ such that

$$\left[ \begin{array}{ccc} t - a_0 & t & t \\ t & t - a_1 & t \\ t & t & t - a_2 \end{array} \right] \cdot \left[ \begin{array}{c} w_0 \\ w_1 \\ w_2 \end{array} \right] = 0.$$ 

In particular, Lemma 3.10 tells us $(w_0, w_1, w_2) = w(a_0, a_1, a_2)$. Let us see that the matrix $P$ can be chosen as in Proposition 3.11. First, we may assume

$$P = \left[ \begin{array}{ccc} 1 & \alpha & \beta \\ 0 & \gamma & \delta \end{array} \right], \quad \alpha, \beta, \gamma, \delta \in \mathbb{Z}, \quad 0 \leq \alpha < \gamma.$$ 

By Lemma 3.9 the local Gorenstein index $\iota_2$ divides $\gamma = \det(w_0, w_1)$. Thus, $\gamma = \iota_2 x$ with an integer $x \geq 1$. Moreover, the Gorenstein form $w_2 \in \mathbb{Q}^2$ is given as

$$w_2 = [1, q] = \left[ \begin{array}{c} 1 \\ \frac{1 - \alpha}{\gamma} \end{array} \right].$$

Thus, $\iota_2 q$ is an integer. Using $\gamma = \iota_2 x$, we see $\alpha = 1 - \iota_2 q x$. Thus, with $a := -\iota_2 q$, the first two columns are as wanted. Resolving $P \cdot w = 0$ for $\beta$ and $\gamma$, we arrive at

$$P = \left[ \begin{array}{ccc} 1 & 1 + ax & -w_0 - (1 + ax) w_1 \\ 0 & \iota_2 x & -\frac{w_0 + (1 + ax) w_1}{w_2} \\ 0 & \iota_2 x & -\frac{w_0 + (1 + ax) w_1}{w_2} \end{array} \right].$$

The condition $x w_1 | \iota_1(w_0 + w_1 + w_2)$ holds because the $\iota_1$-fold of the Gorenstein form $u_1$ is integral and $-1/x \leq \alpha < \iota_2 - 1/x$ reflects $0 \leq \alpha < \gamma$. $\square$
We come to the algorithmic part. The key to efficient classification algorithms is the computability of all presentations of any given rational number $0 < q < 1$ as a sum of a fixed number $n + 1$ of unit fractions:

$$q = \left[ a_0, \ldots, a_n \right] := \frac{1}{a_0} + \ldots + \frac{1}{a_n}, \quad a_0, \ldots, a_n \in \mathbb{Z}_{\geq 1}.$$  

We refer to $[a_0, \ldots, a_n]$ as a UFP of length $n + 1$ of $q$. The list of all UFP of length 2 of a given $q$ is quickly determined: for each $1/q < a_0 \leq 2/q$ check $a_1 := a_0/a_0(q - 1) \in \mathbb{Z}$ and if so, add $[a_0, a_1]$ to the list.

Our first algorithm computes in particular all weight vectors $(w_0, w_1, w_2)$ of fake weighted projective planes $Z(P)$ of given Gorenstein index $\iota$ arising from Construction 2.8. Recall that the vector $(w_0, w_1, w_2)$ lists the weights of the covering weighted projective plane and that it is well formed in the sense that any two of its entries are coprime.

**Algorithm 3.13** (Computation of well formed weight vectors). *Input:* A positive integer $\iota$, the prospective Gorenstein index. *Algorithm:*

- open an empty list $W$ for the weight vectors;
- for $a_0$ from $\iota + 1$ to $3\iota$ do
  - determine $q := \frac{1}{a_0}$;
  - compute the list $Q$ of all UFP $[a_1, a_2]$ of $q$;
  - for every UFP $[a_1, a_2]$ from the list $Q$ do
    - determine $w := \frac{1}{\gcd(a_1, a_2, a_0a_1)}(a_1a_2, a_0a_2, a_0a_1)$;
    - if the entries of $w$ are pairwise coprime, then add $w$ to $W$;
  - end do;
- end do;

*Output:* The list $W$. Then $W$ contains in particular the weight vectors $(w_0, w_1, w_2)$ of all fake weighted projective planes $Z(P)$ of Gorenstein index $\iota$ arising from Construction 2.8.

**Proof.** Obviously, the algorithm terminates. The fact that the output list $W$ contains the weight vectors of all fake weighted projective planes $Z(P)$ of Gorenstein index $\iota$ arising from Construction 2.8 follows from Propositions 3.11 and 3.12. □

**Algorithm 3.14** (Classification of fake weighted projective planes). *Input:* A positive integer $\iota$, the prospective Gorenstein index. *Algorithm:*

- use Algorithm 3.13 to compute the list $W$ of weight vectors for $\iota$;
- open an empty list $L$ for the defining matrices $P$;
- for each triple $\iota_0 \leq \iota_1 \leq \iota_2$ of positive integers with $\lcm(\iota_0, \iota_1, \iota_2) = \iota$ do
  - for every weight vector $(w_0, w_1, w_2)$ from the list $W$ do
    - for every pair of integers $(x, a) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}_{\geq -1}$ satisfying the conditions
      $$xw_1[\iota_0w_0 + w_1 + w_2], \quad -\frac{1}{2} \leq a \leq 2, \quad \gcd(a, \iota_2) = 1,$$
      plug the actual values of the variables $\iota_0, \iota_1, \iota_2, w_0, w_1, w_2, x$ and $a$ into
      $$P := \begin{bmatrix} 1 & a_1x + 1 & -\frac{w_0 + w_1 + w_2}{x_12} \\ 0 & x_12 & -\frac{2}{x_12} \\ -\frac{w_0 + w_1 + w_2}{x_12} & \frac{w_0}{x_12} & \frac{x_12}{x_12} \end{bmatrix},$$
      $$u_0 := \iota_0 \begin{bmatrix} 1 \\ 0 \\ -\frac{w_0 + w_1 + w_2}{x_12} \end{bmatrix},$$
      $$u_1 := \iota_1 \begin{bmatrix} 1 \\ -\frac{w_0 + w_1 + w_2}{x_12} \\ \frac{x_12}{x_12} \end{bmatrix},$$
• if the columns of $P$ and $u_0, u_1$ are primitive integral and $P \neq P'$ for all $P'$ of $L$, then add $P$ to $L$;
• end do;
• end do;

Proof. Obviously, the algorithm terminates. The statements on $L$ are due to Propositions 2.13, 3.11 and 3.12.

□

4. Constructing $K^*$-surfaces of Picard number one

We adapt the more general framework provided in [14]-[15] for constructing systematically varieties with torus action to our setting. The result is a construction delivering rational projective $K^*$-surfaces $X$ of Picard number one embedded into a fake weighted projective space $Z$, where they are given as the closure of a linear subvariety of the acting torus. Moreover, the $K^*$-action of $X \subseteq Z$ is given by a one-dimensional subtorus of the ambient acting torus $T^0$ of $Z$. We begin by looking at an example.

Example 4.1. The weighted projective space $Z = \mathbb{P}_{1,3,2,3}$ is obtained via Construction 2.3 as the variety $Z = Z(P)$ given by the defining matrix

$$P = \begin{bmatrix}
-3 & -1 & 3 & 0 \\
-3 & -1 & 0 & 2 \\
-2 & -1 & 1 & 1
\end{bmatrix}.$$

Consider the homomorphism $p: \mathbb{T}^4 \to \mathbb{T}^3$ of tori associated with $P$ and the quotient map $\pi: \mathbb{K}^4 \setminus \{0\} \to Z$ from 2.8. Remark 2.11 provides us with the identification

$$\mathbb{T}^3 = p(\mathbb{T}^4) = \pi(\mathbb{T}^4) \subseteq Z,$$

so that we can regard $\mathbb{T}^3$ as the acting torus of $Z$. In terms of the coordinates $s_1, s_2, s_3$ of $\mathbb{T}^3$, set $h := 1 + s_1 + s_2$, look at its zero set $V(h) \subseteq \mathbb{T}^3$ and the closure

$$X := X(P) := V(h) \subseteq Z = Z(P).$$

Then $X$ is an irreducible, rational surface in $Z$. We install a $K^*$-action on $X$. First, take the action of $K^*$ on $\mathbb{T}^3$ given in terms of the coordinates $s_1, s_2, s_3$ as

$$t \cdot s := (s_1, s_2, ts_3).$$

Since $h$ doesn’t depend on $s_3$, its zero set $V(h) \subseteq \mathbb{T}^3$ stays invariant. Now, $\mathbb{T}^3 \subseteq Z$ is the acting torus and thus the $K^*$-action extends to $Z$, leaving $X$ invariant.

In fact, this introductory example turns out to be the well-known $E_6$-singular cubic surface. The following general construction distinguishes the types (ee) and (ep): as we will see, the first one admits only isolated $K^*$-fixed points, whereas the second one always comes with a curve of $K^*$-fixed points.

Construction 4.2 (Rational projective $K^*$-surfaces of Picard number one). Consider defining $(r + 1) \times (r + 2)$ matrices $P$ of the following types:

Type (ee)

$$P = \begin{bmatrix}
v_0, v_3, v_1, \ldots, v_r
\end{bmatrix} = \begin{bmatrix}
-l_{01} & -l_{02} & l_1 & \cdots & \vdots & \cdots & \cdots \\
\vdots & \vdots & \ddots & \ddots & \vdots & \ddots & \vdots \\
-l_{01} & -l_{02} & l_r & \cdots & \vdots & \ddots & \vdots \\
-d_{01} & d_{02} & d_1 & \ldots & d_r
\end{bmatrix}, \quad l_1, \ldots, l_r \geq 2, \quad \frac{d_{01}}{v_{01}} = \frac{d_{02}}{v_{02}}.$$
Type \((ep)\)

\[
P = [v_0, v_1, \ldots, v_r, v^-] = \begin{bmatrix}
-l_0 & l_1 & 0 \\
\vdots & \ddots & \vdots \\
-l_0 & l_r & 0 \\
d_0 & d_1 & \ldots & d_r & -1
\end{bmatrix}, \quad l_0, l_1, \ldots, l_r \geq 2, \\
1 < d_0 \geq \ldots \geq d_r > 0.
\]

Let \(Z(P)\) be the associated fake weighted projective space and \(\mathbb{T}^{r+1} \subseteq Z(P)\) the acting torus. Fix pairwise different \(1 = \lambda_2, \lambda_3, \ldots, \lambda_r \in \mathbb{K}^*\) and set

\[
h_i(s_1, \ldots, s_{r+1}) := \lambda_i + s_i, \quad i = 2, \ldots, r,
\]

where \(s_1, \ldots, s_{r+1}\) are the coordinate functions on \(\mathbb{T}^{r+1}\). Passing to the closure of the set of common zeroes of \(h_2, \ldots, h_r\) in \(\mathbb{T}^{r+1}\), we obtain a projective surface

\[
X(P) := \overline{V(h_2, \ldots, h_r)} \subseteq Z(P).
\]

As the functions \(h_2, \ldots, h_r\) don’t depend on the last coordinate \(s_{r+1}\) of \(\mathbb{T}^{r+1}\), we see that \(\mathbb{K}^*\) acts effectively as a subtorus of \(\mathbb{T}^{r+1}\) on \(X(P) \subseteq Z(P)\) via

\[
t \cdot x = (1, \ldots, 1, t) \cdot x.
\]

We also need a description of \(X = X(P)\) in terms of homogeneous coordinates of its ambient projective space \(Z = Z(P)\). For this, we suitably homogenize the defining equations \(h_2, \ldots, h_r\) for \(X \cap \mathbb{T}^3\), which then provides us with defining equations \(g_2, \ldots, g_r\) in homogeneous coordinates. Let us first take another look at the introductory example.

**Example 4.3.** Consider \(X = X(P)\) in \(Z = Z(P)\) from Example 4.1. We relate the homogeneous coordinates on \(Z\) to the coordinates of \(\mathbb{T}^3 \subseteq Z\) via the homomorphism of tori \(p: \mathbb{T}^4 \to \mathbb{T}^3\) associated with \(P\), concretely given as

\[
p: \mathbb{T}^4 \to \mathbb{T}^3, \quad (t_{01}, t_{02}, t_1, t_2) \mapsto \left(\frac{t_1^3}{t_{01}^2 t_{02}}, \frac{t_2^3}{t_{01}^2 t_{02}}, \frac{t_1 t_2}{t_{01}^2 t_{02}}\right),
\]

where the particular numbering of coordinates on \(\mathbb{T}^4\) indicates that \(t_{01}\) and \(t_{02}\) form the denominator monomial of the first two components. Pulling back the defining equation \(h = 1 + s_1 + s_2\) gives the \(H\)-invariant Laurent polynomial

\[
p^* h(t_{01}, t_{02}, t_1, t_2) = 1 + \frac{t_1^3}{t_{01}^2 t_{02}} + \frac{t_2^3}{t_{01}^2 t_{02}}.
\]

By definition, the hypersurface \(V(p^* h) \subseteq \mathbb{T}^4\) maps onto \(V(h) \subseteq \mathbb{T}^3\) via \(p\). We are interested in the closure \(\overline{X} \subseteq \mathbb{K}^4\) of \(V(p^* h) \subseteq \mathbb{T}^4\). This is the hypersurface \(\overline{X} = \overline{V(g)}\) defined by the \(H\)-homogeneous polynomial

\[
g(t_{01}, t_{02}, t_1, t_2) = \frac{t_{01}^3 t_{02} p^* h}{t_{01}^2 t_{02}} = \frac{t_{01}^3 t_{02} + t_1^3 + t_2^3}{t_{01}^2 t_{02}}.
\]

**Construction 4.4.** Consider \(X = X(P)\) in \(Z = Z(P)\) and set \(K = \mathbb{Z}^{r+2} / P^* \mathbb{Z}^{r+1}\). According to the type of \(P\), the \(K\)-homogeneous coordinate ring of \(Z(P)\) is

\[
(\text{ee}) \quad \mathbb{K}[T_{01}, T_{02}, T_1, \ldots, T_r], \quad \deg(T_{01}) = Q(e_0), \quad \deg(T_1) = Q(e_1),
\]

\[
(\text{ep}) \quad \mathbb{K}[T_0, T_1, \ldots, T_r], \quad \deg(T_0) = Q(e_i), \quad \deg(T^-) = Q(e^-),
\]

where \(Q: \mathbb{Z}^{r+2} \to K\) denotes the projection. According to the type of \(P\), we define for \(i = 2, \ldots, r\) the \(K\)-homogenization \(g_i\) of the affine form \(h_i\) as

\[
(\text{ee}) \quad g_i := T_{01}^{l_0} T_{02}^{l_1} p^* h_i = \frac{\lambda_i T_{01}^{l_0} T_{02}^{l_1} + T_{01}^{l_1} + T_{02}^{l_2}}{T_1^{l_1} + T_{01}^{l_1} + T_{02}^{l_2} \in \mathbb{K}[T_{01}, T_{02}, T_1, \ldots, T_r]},
\]

\[
(\text{ep}) \quad g_i := T_{01}^{l_0} p^* h_i = \frac{\lambda_i T_{01}^{l_0} + T_1^{l_1} + T_{01}^{l_1}}{T_{01}^{l_1} + T_{01}^{l_1} + T_{01}^{l_1} \in \mathbb{K}[T_0, T_1, \ldots, T_r, T^-]}.
\]
Note that in the case of $\text{Construction 4.2}$, we can describe $X \subseteq Z$ in homogeneous coordinates as

$$X = V(g_2, \ldots, g_r) := \pi(X) \subseteq Z.$$ 

Remark 4.6. Consider a defining matrix $P$ as in Construction 4.2. According to the type of $P$, the admissible operations are:

- (ee) adding integral multiples of the upper $r$ rows to the last one, multiplying the last row by $-1$, interchanging the data sets $(l_1, d_1), \ldots, (l_r, d_r)$ of the columns $v_1, \ldots, v_r$.
- (ep) adding integral multiples of the upper $r$ rows to the last one, interchanging the data sets $(l_0, d_0), \ldots, (l_r, d_r)$ of the columns $v_0, \ldots, v_r$ and adapting the $\lambda_i$ accordingly.

Then, for any two defining matrices $P$ and $P'$ the resulting $\mathbb{K}^*$-surfaces are isomorphic to each other if and only if $P$ can be transformed into $P'$ via admissible operations.

From the corresponding results from the general theory we infer that our Construction 4.2 delivers basically all non-toric, normal, rational, projective $\mathbb{K}^*$-surface of Picard number one. More precisely, we have the following.

**Theorem 4.7.** Every non-toric, normal, rational, projective $\mathbb{K}^*$-surface of Picard number one is isomorphic to a $\mathbb{K}^*$-surface $X(P)$ arising from Construction 4.2.
Proposition 4.8. Consider $X = X(P)$ in $Z = Z(P)$ of type (ce) as provided by Construction 4.2 and the coordinate divisors on $Z$, given as

$$D_z^j = V(T_{0j}) \subseteq Z, \quad j = 1, 2, \quad D_z^j = V(T_i) \subseteq Z, \quad i = 1, \ldots, r.$$  

Cutting down the coordinate divisors to the subvariety $X \subseteq Z$ yields $\mathbb{K}^*$-invariant prime divisors on $X$, namely

$$D_X^j := X \cap D_z^j \subseteq X, \quad j = 1, 2, \quad D_X^i := X \cap D_z^i \subseteq X, \quad i = 1, \ldots, r.$$  

Each of the $D_X^j$ and $D_X^i$ is the closure of a non-trivial $\mathbb{K}^*$-orbit having $C(l_0j)$ and $C(l_i)$ as its isotropy group, respectively. The $D_X^j$ and $D_X^i$ intersect as follows

The points $x^+ = z(02)$ and $x^- = z(01)$ together with the unique intersection point $x_h \in D_X^{01} \cap D_X^{02}$ form the fixed point set of the $\mathbb{K}^*$-surface $X$.

Proof. We indicate why the $D_X^j$ and $D_X^i$ are indeed prime divisors. Consider $\mathcal{X} = V(g_2, \ldots, g_r)$ in $\mathbb{K}^{r+2}$. Then we obtain $H$-invariant hypersurfaces $\mathcal{X} \cap V(T_{0j})$ and $\mathcal{X} \cap V(T_k)$ in $\mathcal{X}$. It turns out that $H$ transitively permutes the components of each of these hypersurfaces and thus, their images $D_X^j$ and $D_X^i$ are prime divisors; see [14] for a detailed arguing.

We determine the isotropy groups of the non-trivial $\mathbb{K}^*$-orbits of the $D_X^j$ and $D_X^i$. Consider exemplarily a point $x \in D_X^{0j}$ from the non-trivial $\mathbb{K}^*$-orbit. Then $x = [z]$, where $z_{01} = 0$ and $z_{02}, z_1, \ldots, z_r$ are all non-zero. Then we have

$$\mathbb{K}_x^* = T_x^{r+1} \cap \mathbb{K}^* = \{(s^{-l_01}, \ldots, s^{-l_01}, s^{l_01}); \ s \in \mathbb{K}^*\} \cap \{1, \ldots, 1, t\}; \ t \in \mathbb{K}^*\},$$  

where we allow ourselves to write just $\mathbb{K}_x^*$ for the subgroup $\{1\} \times \ldots \times \{1\} \times \mathbb{K}^*$ of the acting torus $\mathbb{T}_x^{r+1}$ of $Z(P)$ and Proposition 2.12 provides us with the description of the isotropy group $T_x^{r+1}$.

Proposition 4.9. Consider $X = X(P)$ in $Z = Z(P)$ of type (ep) as provided by Construction 4.2 and the coordinate divisors on $Z$, given as

$$D_z^i = V(T_i) \subseteq Z, \quad i = 0, \ldots, r, \quad D_z^i = V(T^-) \subseteq Z.$$  

Cutting down the coordinate divisors to the subvariety $X \subseteq Z$ yields $\mathbb{K}^*$-invariant prime divisors on $X$, namely

$$D_X^i := X \cap D_z^i \subseteq X, \quad i = 0, \ldots, r, \quad D_X^i := X \cap D_z^i \subseteq X.$$  

Each of the $D_X^i$ is the closure of a non-trivial $\mathbb{K}^*$-orbit having $C(l_i)$ as its isotropy group and $D_X^i$ consists of fixed points. The $D_X^i$ and $D_X^i$ intersect as follows
The fixed point set of the $\mathbb{K}^*$-surface $X$ consists of the point $x^+ = z(01)$ and all the points of the curve $D^{-}_X$.

Proof. Follow the lines of the proof of Proposition 4.8. \hfill \Box

Let us shed some light on the meaning of the notations (ee), (ep), $x^+$, $x^-$, $D^{-}_X$ and the arrows. Whenever $\mathbb{K}^*$ acts morphically on a normal projective variety $X$, each point $x \in X$ gives rise to a morphism

$$\mu_x : \mathbb{P}^1 \rightarrow X,$$

extending the orbit map $\mathbb{K}^* \rightarrow X$, $s \mapsto s \cdot x$. This allows us to define the limit points of the $\mathbb{K}^*$-orbit through $x$ as

$$x_0 := \lim_{s \rightarrow 0} s \cdot x := \mu_x(0), \quad x_{\infty} := \lim_{s \rightarrow \infty} s \cdot x := \mu_x(\infty).$$

Moreover, there are precisely one source $F^+ \subseteq X$ and precisely one sink $F^- \subseteq X$; these are components of the fixed point set such that

$$X^+ := \{ x \in X; \ x_0 \in F^+ \}, \quad X^- := \{ x \in X; \ x_{\infty} \in F^- \}$$

are open subsets of $X$. If $X$ is a surface, then there are exactly three types of fixed points $x \in X$, namely, $x$ is called

- (e) elliptic, if $\{x\}$ is the source or the sink,
- (p) parabolic, if $x$ belongs to a curve consisting of fixed points,
- (h) hyperbolic, if $x$ lies in the closure of precisely two non-trivial $\mathbb{K}^*$-orbits.

Remark 4.10. Consider a $\mathbb{K}^*$ surface $X = X(P)$. Then, according to the type of $X$, the following holds

- (ee) $X$ has $\{x^+\}$ as its source and $\{x^-\}$ as its sink. Moreover, we have
  $$X^+ = X \setminus D^0_X, \quad X^- = X \setminus D^1_X.$$
- (ep) $X$ has $\{x^+\}$ as its source and $D^{-}_X$ as its sink. Moreover, we have
  $$X^+ = X \setminus D^{-}_X, \quad X^- = X \setminus \{x^+\}.$$

The key observation is that Construction 4.2 of the $\mathbb{K}^*$-surface $X = X(P)$ delivers us for free the divisor class group $\text{Cl}(X)$ and the Cox ring

$$R(X) = \bigoplus_{\text{Cl}(X)} \Gamma(X, O(D)).$$

We refer to [3, Sec. 1.1.4] for the precise definition of the Cox ring and to [3, Sec. 3.4.3] for the details of the following.

Construction 4.11. Consider $X = X(P)$ in $Z = Z(P)$ as provided by Construction 4.2. Then we obtain homomorphisms of abelian groups

$$\mathbb{Z}^{r+2} \rightarrow \text{WDiv}(Z), \quad a \mapsto D_Z(a), \quad \mathbb{Z}^{r+2} \rightarrow \text{WDiv}(X), \quad a \mapsto D_X(a),$$
by prescribing their values on the canonical basis vectors of $\mathbb{Z}^{r+2}$ according to the type of $P$ as follows:

\[(ee) \quad \mathbb{Z}^{r+2} \to \text{WDiv}(Z), \quad e_{0j} \mapsto D^0_Z, \quad e_i \mapsto D^i_Z, \]
\[(ep) \quad \mathbb{Z}^{r+2} \to \text{WDiv}(X), \quad e_{0j} \mapsto D^0_X, \quad e_i \mapsto D^i_X, \]
\[(ii) \quad \mathbb{Z}^{r+2} \to \text{WDiv}(Z), \quad e_1 \mapsto D^1_Z, \quad e^- \mapsto D^-_Z, \]
\[(ii) \quad \mathbb{Z}^{r+2} \to \text{WDiv}(X), \quad e_1 \mapsto D^1_X, \quad e^- \mapsto D^-_X. \]

**Remark 4.13.** Given $X = X(P)$ in $Z = Z(P)$ with $K = \mathbb{Z}^{r+2}/P^{*}\mathbb{Z}^{r+1}$ and the projection $Q: \mathbb{Z}^{r+2} \to K$. Then we have a commutative diagram

\[
\begin{array}{ccc}
\text{WDiv}(Z) & \xrightarrow{D \to [D]} & \mathbb{Z}^{r+2} \xrightarrow{\alpha \mapsto D_X(a)} \text{WDiv}(X) \\
\text{Cl}(Z) & \cong & K & \cong & \text{Cl}(X) \\
\end{array}
\]

where $\alpha^*$ extends the pullback of $T^n$-invariant Cartier divisors. In particular, for every character function $\chi^n \in \mathbb{K}(Z)$, we have

$$\text{div}(\chi^n) = D_Z(P^*u) = D_X(P^*u) = \text{div}(\chi^n|_X).$$

**Proposition 4.14.** Let $X = X(P)$ in $Z = Z(P)$ arise from Construction 4.2 and let $x \in X$. Then, for any $a \in \mathbb{Z}^{r+2}$, the following statements are equivalent.

1. The divisor $D_X(a)$ in $\text{WDiv}(X)$ is Cartier near $x \in X$.
2. The divisor $D_Z(a)$ in $\text{WDiv}(Z)$ is Cartier near $x \in Z$.

In particular, $X$ inherits $\mathbb{Q}$-factoriality from $Z$, is of Picard number $\rho(X) = 1$ and for any $x \in X$, the local class group $\text{Cl}(X,x)$ equals $\text{Cl}(Z,x)$.

We turn to the Cox ring. The more general [14] Thm. 10.5 provides us with the Cox rings of all normal, rational, projective varieties with a torus action of complexity one. If we pick out those describing surfaces of Picard number one, we arrive at the following.

**Proposition 4.15.** Consider $X = X(P)$ in $Z = Z(P)$ with $K = \mathbb{Z}^{r+2}/P^{*}\mathbb{Z}^{r+1}$ and the projection $Q: \mathbb{Z}^{r+2} \to K$. Then the Cox ring of $X$ is given as

$$\mathcal{R}(X) = \mathcal{R}(Z)/\langle g_2, \ldots, g_r \rangle,$$

where the variables $T_{0j}$, $T_i$ and $T^-$ are of degree $Q(e_{0j})$, $Q(e_i)$ and $Q(e^-)$ in $\text{Cl}(X) = K = \text{Cl}(Z)$, according to the type of $P$.

**Corollary 4.16.** Every $\mathbb{K}^*$-surface arising from Construction 4.2 is non-toric.

**Proof.** On the one side, the Cox ring of any projective toric variety is a polynomial ring [8]. On the other side, the Cox ring $\mathcal{R}(X)$ of any $\mathbb{K}^*$-surface $X = X(P)$, has as its spectrum $\tilde{X} = V(g_2, \ldots, g_r) \subseteq \mathbb{K}^{r+2}$, coming with a singularity at the origin. Thus $\mathcal{R}(X)$ is not a polynomial ring. \qed
5. Singularities of \(K^*-\)surfaces of Picard number one

We discuss the possible singularities of a \(K^*-\)surface \(X = X(P)\). First note that by normality of \(X\), all its singularities are isolated and thus must be \(K^*\)-fixed points. We call \(x \in X\) quasi-smooth if \(x = \pi(\hat{x})\) holds with a smooth point \(\hat{x} \in \hat{X}\). It turns out that the quasi-smooth points of \(X\) are precisely those which are at most cyclic quotient singularities.

**Proposition 5.1.** Consider a \(K^*-\)surface \(X = X(P)\) with \(P\) of type (ee). Then all points of \(X\) different from \(x^+\), \(x^-, x_h\) are smooth. Moreover:

(i) \(x^+\) is quasi-smooth if and only if \(r = 2\) and \(l_{01} = 1\),

(ii) \(x^+\) is smooth if and only if \(r = 2\) and \(l_{01} = 1\) and \(\det(v_{01}, v_1, v_2) = \pm 1\),

(iii) \(x^-\) is quasi-smooth if and only if \(r = 2\) and \(l_{02} = 1\),

(iv) \(x^-\) is smooth if and only if \(r = 2\) and \(l_{02} = 1\) and \(\det(v_{02}, v_1, v_2) = \pm 1\),

(v) \(x_h\) is quasi-smooth,

(vi) \(x_h\) is smooth if and only if \(l_{01}d_0 - l_{02}d_1 = 1\).

**Proof.** We show (i). In order to see that \(x^+ = z(02) = [0, 1, 0, \ldots, 0]\) is quasi-smooth, we take a look at the gradients of the homogenized defining relations:

\[
\text{grad}(g_j) = (l_{01}T_{01}^{l_{01}-1}T_{02}^{l_{02}-1}, l_{02}T_{01}^{l_{01}-1}l_1T_1^{l_1-1}, 0, \ldots, 0, l_jT_j^{l_j-1}, \ldots, 0).
\]

Due to \(l_j \geq 2\), evaluating at \(x^+\) gives \((l_{01}T_{01}^{l_{01}-1}, 0, \ldots, 0)\) in all cases. Thus, the Jacobian of \(g_2, \ldots, g_r\) is of full rank at \(x^+\) if and only if \(r = 2\) and \(l_{01} = 1\).

For Assertion (ii), recall that \(x^+\) is the (only) point in the \(\mathbb{T}^n\)-orbit of \(Z = Z(P)\) corresponding to the cone\((v_{01}, v_1, \ldots, v_r)\). Thus, [3, Cor. 3.3.1.12] tells us that \(x^+\) is smooth in \(\hat{X}\) if and only if it is quasi-smooth in \(X\) and smooth in \(Z\). The latter two translate to \(r = 2\), \(l_{01} = 1\) and \(\det(v_{01}, v_1, \ldots, v_r) = \pm 1\). Assertions (iii) and (iv) are settled analogously. For (v), note that \(x_h = [0, 0, z_1, \ldots, z_l]\) with all \(z_j\) non-zero and thus the Jacobian of \(g_2, \ldots, g_r\) always is of full rank at \(x^+\). For (vi), note that \(x_h\) lies in the \(\mathbb{T}^n\)-orbit of \(Z\) corresponding to cone\((v_0, v_{02})\). Again, [3, Cor. 3.3.1.12] yields the claim. \(\square\)

**Proposition 5.2.** Consider a \(K^*-\)surface \(X = X(P)\) with \(P\) of type (ep). Then all points of \(X\) different from \(x^+\) and \(x^-\) are smooth. Moreover:

(i) \(x^+\) is not quasi-smooth,

(ii) each of \(x^-_0, \ldots, x^-_r\) is quasi-smooth but not smooth.

**Proof.** As in the preceding proof, we look at the Jacobian of \(g_2, \ldots, g_r\), and see that quasi-smoothness of \(x^+\) would allow \(l_i \neq 1\) at most twice, which is not possible. This proves (i). Concerning the points of \(D^-\), observe that each point of \(\hat{X} \cap V(T^-)\) is smooth in \(\hat{X}\). Thus, all points of \(D^-\) are quasi-smooth. Any \(x \in D^-\) different from \(x^-_0, \ldots, x^-_r\) lies in the \(\mathbb{T}^n\)-orbit of \(Z(P)\) corresponding to cone\((v_i)\) which consists of smooth points of \(Z(P)\). Moreover, \(x^-_i\) lies in the \(\mathbb{T}^n\)-orbit of \(Z(P)\) corresponding to cone\((u, v^-)\), which consists of singular points of \(Z(P)\) due to \(l_i \geq 2\). Thus, [3, Cor. 3.3.1.12] yields the assertions. \(\square\)

For checking the del Pezzo property and computing Gorenstein indices, we need an explicit description of an (anti-)canonical divisor. As our surfaces have a complete intersection Cox ring, we can apply [3, Prop. 3.3.3.2] to obtain the following.

**Proposition 5.3.** Let \(X = X(P)\) in \(Z = Z(P)\) arise from Construction. Then, according to the type of \(X\), we obtain anticanonical divisors on \(X\) by

**ee** \(-K_X^e = D^0_X + D^1_X + D^2_X + \ldots + D^r_X - (r - 1)(l_{01}D^0_X + l_{02}D^0_X),

\(-K_X^e = D^0_X + D^1_X + D^2_X + \ldots + D^r_X - (r - 1)l_iD^i_X,

**ep** \(-K_X^e = D^0_X + \ldots + D^r_X + D^r_X - (r - 1)l_iD^i_X).\)
We express the local Gorenstein indices of the fixed points of a $\mathbb{K}^*$-surface $X(P)$. For this, we explicitly determine the defining linear forms representing an anticanonical divisor near the fixed points.

**Lemma 5.4.** Let integers $l_0, \ldots, l_r > 0$ and $d_0, \ldots, d_r$ be given and define rational numbers

$$m_i := \frac{d_i}{l_i}, \quad m := m_0 + \ldots + m_r.$$

Assume that $m \not= 0$ holds and consider the $r + 1$ by $r + 1$ matrix $B$ and $u_B \in \mathbb{Q}^{r+1}$ given by

$$B := \begin{bmatrix} -l_0 & l_1 & \cdots & \cdots & l_r \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ -l_0 & d_1 & \ldots & d_r \\ \end{bmatrix}, \quad u_B := \frac{1}{m}(u_1, \ldots, u_r, \ell), \quad u_i := (r-1)m_i + \sum_{j \not= i} \frac{m_j}{l_j} - \frac{m_i}{l_i}, \quad \ell := \frac{1}{m} + \ldots + \frac{1}{l_r} - r + 1.$$

Then the linear form $u_B$ evaluates on the columns $v_0, v_1, \ldots, v_r$ of the matrix $B$ as follows:

$$\langle u_B, v_0 \rangle = 1 - (r-1)l_0, \quad \langle u_B, v_i \rangle = 1, \quad i = 1, \ldots, r.$$

**Proof.** One explicitly computes the evaluation of $u_B$ on each of the columns of the matrix $B$. □

**Proposition 5.5.** Let $X = X(P)$ be of type (ee) with $P = [v_0, v_1, v_2, \ldots, v_r]$. Consider the matrices

$$B^+ = [v_0, v_1, \ldots, v_r], \quad B^- = [v_0, v_1, \ldots, v_r],$$

the associated linear forms $u^+ = u_B^+$ and $u^- = u_B^-$ according to Lemma 5.4 and the linear form

$$u_h = \frac{1}{l_0d_0 - l_0l_2}(d_0 - d_0 - d_2, 0, \ldots, 0, l_0 - d_0) \in \mathbb{Q}^{r+1}.$$

Then the local Gorenstein indices of $x^+, x^-, x_h \in X$ are the unique positive integers $i^+, i^-, i_h$ such that $i^+u^+, i^-u^-, i_hu_h$ are primitive integral vectors.

**Proof.** By definition, the local Gorenstein index of $x \in X$ is the minimal positive integer $i(x)$ such that the $i(x)$-fold of some anticanonical divisor of $X$ is Cartier near $x$. For $x^+, x^- \in X$, consider $-\mathcal{K}_X^0$ from Proposition 5.3 and 4.4 ensure that any integral multiple $-t^\pm\mathcal{K}_X^0$ is Cartier near $x^\pm$ if and only if it equals $\text{div}(\chi_{x^\pm}^+) = \text{div}(\chi_{x^\pm}^-)$ near $x^\pm$ with an integral linear form $i^\pm u^\pm$. This proves the assertion for $x^\pm$. For $x_h$ we argue analogously, using that $-i_h\mathcal{K}_X^1$ from Proposition 5.3 equals $\text{div}(\chi_{x_h}^+) \cap X_h$, provided that $i_hu_h$ is integral. □

**Proposition 5.6.** Let $X = X(P)$ be of type (ep) with $P = [v_0, v_1, \ldots, v_r, e_{r+1}]$. Consider the matrix

$$B^- = [v_0, \ldots, v_r],$$

the associated linear form $u^- = u_B^-$ as in Lemma 5.4 and, for $i = 0, 1, \ldots, r$, the linear forms

$$u_0 = \frac{1}{l_0} (1 - d_0, 0, \ldots, 0, -l_0), \quad u_i = \frac{1}{l_i} (0, \ldots, 0, d_i - 1, 0, \ldots, 0, -l_i).$$

Then the local Gorenstein indices of $x^-, x_0, \ldots, x_r \in X$ are the unique positive integers $i^-, i_0, \ldots, i_r$ such that $i^-u^-, i_0u_0, \ldots, i_ru_r$ are primitive integral vectors.

**Proof.** Follow the lines of the proof of Proposition 5.5. □
We characterize log terminality of rational $K^*$-surfaces. Recall that for any normal surface $X$ with a $\mathbb{Q}$-Cartier canonical divisor $K_X$ one considers a resolution of singularities $\pi: X' \to X$ and the associated ramification formula
\[ K'_X = \pi^* K_X + \sum a(E) E, \]
where $E$ runs through the exceptional prime divisors and the $a(E) \in \mathbb{Q}$ are the discrepancies of $\pi: X' \to X$. Then a point $x \in X$ is called log terminal if we have $a_E > -1$ for all exceptional divisors $E$ contracting to $x$.

**Proposition 5.7.** Let $X = X(P)$ in $Z = Z(P)$ arise from Construction 4.2. Then, according to the type of $P$, we have the following:

- (ee) $x^+ \in X$ is log terminal if and only if $\frac{1}{l_{01}} + \frac{1}{l_{1}} + \ldots + \frac{1}{l_{r}} > r - 1$
- (ii) $x^- \in X$ is log terminal if and only if $\frac{1}{l_{02}} + \frac{1}{l_{1}} + \ldots + \frac{1}{l_{r}} > r - 1$
- (ep) $x^- \in X$ is log terminal if and only if $\frac{1}{l_{03}} + \ldots + \frac{1}{l_{r}} > r - 1$.

**Proof.** The statements are special cases of the more general results [6, Cor. 4.6] and [2 Thm. 3.1].

**Remark 5.8.** A tuple $(q_0, \ldots, q_r)$ of positive integers is called platonic if the following inequality is satisfied:
\[ q_0^{-1} + \ldots + q_r^{-1} > r - 1. \]
If $q_0 \geq \ldots \geq q_r$ holds, then platonicity of the tuple is equivalent to $q_3 = \ldots = q_r = 1$ and $(q_0, q_1, q_2)$ being one of
\[ (q_0, q_1, 1), \quad (q_0, 2, 2), \quad (5, 3, 2), \quad (4, 3, 2), \quad (3, 3, 2). \]

**Proposition 5.9.** If a $K^*$-surface $X = X(P)$ is log terminal, then the possible tuples $(l_{01}, l_{02}, l_{1}, \ldots, l_{r})$ for type (ep) are the following:

- (eAeA): $(1, 1, x_1, x_2)$,
- (eAeD): $(1, y, 2, 2)$,
- (eAeE): $(1, z, 3, 2)$,
- (eDeD): $(2, 2, y, 2)$,
- (eDeE): $(2, 2, z, 2)$,
- (eEp): $(z, 3, 2)$,

where $2 \leq x_1, x_2, y, y_1, y_2$, we have $3 \leq z, z_1, z_2 \leq 5$ and the notation “eA, eD, eE” refers to log terminal $x^\pm \in X$ with index one cover of type $A, D, E$.

**Proof.** Recall that for type (ee) we have $l_1, \ldots, l_r \geq 2$ and for type (ep) we have $l_0, \ldots, l_r \geq 2$. Thus for a log terminal $X = X(P)$, Proposition 5.7 and Remark 5.8 force $r \leq 3$ in the case (ee) and $r = 2$ in the case (ep). Now the possible configurations result from the possible choices of platonic triples, presented in Remark 5.8.

**Corollary 5.10.** Every non-toric, rational, log terminal, projective $K^*$-surface $X$ of Picard number one is del Pezzo.

**Proof.** We have to show that $X$ has an ample anticanonical divisor. As $X$ is of Picard number one, it suffices to find an effective anticanonical divisor. For $X$ of type (ee) Proposition 5.3 provides us with the anticanonical divisor
\[ -K_X^\prime = D_X^{l_{01}} + D_X^{l_{02}} + D_X^1 + \ldots + D_X^r - (r - 1)(l_{01}D_X^{l_{01}} - l_{02}D_X^{l_{02}}). \]
In order to verify effectivity of $-K_X^0$, we have to show that the corresponding class in $\mathbb{Q} \otimes \mathbb{Z} \text{Cl}(X) = \mathbb{Q}$ is positive. This is done by going through the cases of Proposition 5.9. For instance, consider $(z_1, z_2, 3, 2)$ from (eEeE). There we have
\[
z_1\omega_0^1 + z_2\omega_0^2 = 3\omega_1^X, \quad z_1\omega_0^1 + z_2\omega_0^2 = 2\omega_2^X,
\]
reflecting the relations among the $\omega_0^j$ and $\omega_i^X$ in $\text{Cl}(X)$ given by the first two rows of the defining matrix $P$. Together with $3 \leq z_1, z_2 \leq 5$ this allows us to estimate the anticanonical class of $X$ as follows:
\[
[-K^0_X] = \omega_0^1 + \omega_1^X + \omega_2^X - z_1\omega_0^1 - z_2\omega_0^2
\]
\[
= \left(\frac{z_1}{3} + \frac{z_2}{2} + (1 - z_1)\right)\omega_0^1 + \left(\frac{z_2}{3} + \frac{z_2}{2} + (1 - z_2)\right)\omega_0^2
\]
\[
= \left(\frac{5}{6} + \frac{1}{z_1} - 1\right)z_1\omega_0^1 + \left(\frac{5}{6} + \frac{1}{z_2} - 1\right)z_2\omega_0^2
\]
\[
> 0.
\]
The other cases of type (ee) are settled by analogous computations. For the type (ep), we exemplarily discuss $(y, 2, 2)$. Using the anticanonical divisor $-K^0_X$ from Proposition 5.3, we obtain
\[
[-K^0_X] = \omega_0^0 + \omega_1^X + \omega_2^X - y\omega_0^0
\]
\[
= \omega_0^0 + \frac{y}{2}\omega_1^X + \frac{y}{2}\omega_2^X + \omega_0^X - y\omega_0^0
\]
\[
= \omega_0^0 + \omega_0^X
\]
\[
> 0,
\]
where we made use of the relations $y\omega_0^0 = 2\omega_1^X$ and $y\omega_0^0 = 2\omega_2^X$ in $\text{Cl}(X)$ given by the first two rows of the defining matrix $P$. The remaining case in type (ep) is settled similarly.

Note that any toric del Pezzo surface is log terminal and, moreover, any Gorenstein rational del Pezzo surface is log terminal as well [17]. Concerning higher Gorenstein indices, we have the following.

Remark 5.11. Corollary 5.10 has no converse in the sense that any del Pezzo $K^*$-surface $X$ of Picard number one should be log terminal. For instance, $P_l := \begin{bmatrix} -6 & -1 & l & 0 \\ -6 & -1 & 0 & 2 \\ -7 & -1 & \frac{l+1}{2} & 1 \end{bmatrix}$ defines a del Pezzo $K^*$-surface $X_l = X(P_l)$ of Picard number one and Gorenstein index two for every $l$ in the range given above, but none of the $X_l$ is log terminal.

6. Classifying $K^*$-surfaces of Picard number one

We present our classification procedure for non-toric, log terminal, rational, projective $K^*$-surfaces $X$ of Picard number one and given Gorenstein index $\iota$. Recall from Corollary 6.10 that all these surfaces are in fact del Pezzo. We provide explicit results for $1 \leq \iota \leq 200$, summarized as follows.

Theorem 6.1. There are 154,138 families of non-toric, log terminal, rational, projective $K^*$-surfaces of Picard number one and Gorenstein index at most 200. The numbers of families for given Gorenstein index develop as follows:
The classification uses the presentation $X = X(P)$ of the surfaces in question and is performed entirely in terms of the defining matrices $P$. The task is to derive suitably efficient bounds on $P$ from the geometric properties of the surface $X(P)$. A first step was done in Proposition 5.9, where we studied the effect of log terminality on the possible choices of the entries $l_0$ and $l$ of $P$.

Basically, we go through the list given in Proposition 5.9 and provide bounds on $P$ in each case. The case $(eAeA)$, comprising precisely the quasi-smooth surfaces $X(P)$, turns out to be by far the richest one. In our treatment of this case, unit fractions will play again a central role and pop up in a similar way but more visibly as in [5], where the Gorenstein threefold case is considered.

**Proposition 6.2.** Let $a, e^+, e^-, l_1, l_2, d_2 \in \mathbb{Z}_{>0}$ and $b \in \mathbb{Z}_{<0}$ such that $l_1, l_2 \geq 2$ and $0 < d_2 < l_2$ hold. Assume that the columns of the matrix

$$
P := \begin{bmatrix}
-1 & -1 & l_1 & 0 \\
-1 & 0 & l_2 \\
0 & a^+ - b^- & l_1 & 0 \\
\end{bmatrix}
$$

are pairwise distinct, integral and primitive. Then $P$ defines a quasi-smooth log del Pezzo $\mathbb{K}^*$-surface $X = X(P)$ with $\rho(X) = 1$. Assume in addition that

$$
\gamma^+ := \left[ \frac{a^+ - b^- - l_1^2 - l_2^2}{al_2}, \frac{a^+ - l_1 - l_2}{al_2}, l_1 + l_2 \right],
$$

$$
\gamma^- := \left[ \frac{a^+ + l_1 + l_2 - b^- - l_1^2 - l_2^2}{bl_2}, \frac{b^- - l_1 - l_2}{bl_2}, l_1 + l_2 \right]
$$

are primitive vectors in $\mathbb{Z}^3$. Then the local Gorenstein indices of the elliptic fixed points $x^+, x^- \in X$ and the Gorenstein index of $X$ are given by

$$
e^+ = \iota(x^+), \quad e^- = \iota(x^-), \quad \iota_X = \text{lcm}(e^+, e^-).
$$

Finally, if $P$ and $\gamma^+, \gamma^-$ are as above, then there exist positive integers $a_1 \leq a_2$ such that we have

$$
\frac{1}{\iota_X} = \frac{1}{a_1 l_1} + \frac{1}{a_2 l_2} + \frac{1}{a_1 l_2} + \frac{1}{a_2 l_1}.
$$

**Proof.** From the conditions of Construction 4.2 on $P$, we only have to check that its columns $v_0, v_0, v_1, v_2$ generate $\mathcal{O}^3$ as cone. Since $b < 0 < a$, this follows from

$$
l_1l_2v_0 + l_2v_1 + l_1v_2 = ax^+e_3, \quad l_1l_2v_1 + l_2v_1 + l_1v_2 = bx^-e_3.
$$

Now, $X = X(P)$ is of Picard number one, quasi-smooth and log terminal due to Propositions 4.14, 5.1 and 5.7. With $u^+$ and $u^-$ from Proposition 4.3, we have

$$
\gamma^+ = \iota^+u^+, \quad \gamma^- = \iota^-u^-.
$$
Consequently, $x^\pm$ is the local Gorenstein index of $x^\pm \in X$. Moreover, Proposition \[\text{5.2} \] shows $\iota(x_h) = 1$. Proposition \[\text{5.3} \] provides us with the anticanonical divisor $-K_X^* = D_X^1 + D_X^2$.

Write $\iota := \iota_X$ for the Gorenstein index of $X$. Then $-\iota K_X^*$ is Cartier near $x^-$, $x^+$ and the classes $\omega_{0j} = (w_{0j}, \eta_{0j})$ and $\omega_i = (w_i, \eta_i)$ of the $D_{X}^{0j}$ and $D_X^i$ satisfy

\[
a_1 w_{01} = \iota w_1 + \iota w_2, \quad a_2 w_{02} = \iota w_1 + \iota w_2,
\]

see Propositions \[\text{5.2} \] and \[\text{4.14} \]. Moreover, the first two rows of $P$ encode a relation among the $\omega_{0j}$ and $\omega_i$ such that altogether we obtain

\[
\begin{pmatrix}
-1 & -1 & l_1 & 0 \\
-1 & -1 & 0 & l_2 \\
a_1 & 0 & \iota & \iota \\
0 & -a_2 & \iota & \iota
\end{pmatrix}
\begin{pmatrix}
w_{01} \\
w_{02} \\
w_1 \\
w_2
\end{pmatrix}
= 0.
\]

This identity implies that the above matrix has vanishing determinant. A simple computation shows that the latter is equivalent to

\[
\frac{1}{\iota} = \frac{1}{a_1 l_1} + \frac{1}{a_2 l_2} + \frac{1}{a_1 l_2} + \frac{1}{a_2 l_1}.
\]

Proposition 6.3. Let $X$ be a quasi-smooth, rational, projective $\mathbb{K}^*$-surface of Picard number one. Then $X \cong X(P)$ with $P$ given by $a, b, \iota^+, \iota^-, l_1, l_2, d_2$ as in Proposition 6.3 satisfying all assumptions made there.

Proof. According to Theorem 4.7 we may assume that $X = X(P)$ holds. As $X$ is quasismooth, Proposition 6.1 tells us that $P$ must be of type (ee) with $r = 2$ and $l_01 = l_02 = 1$. Thus,

\[
P = \begin{bmatrix}
-1 & -1 & l_1 & 0 \\
-1 & -1 & 0 & l_2 \\
0 & d_{02} & d_1 & d_2
\end{bmatrix}.
\]

The local Gorenstein index $\iota^+ := \iota(x^+)$ equals the order of the subgroup generated by $K_X$ in the local class group of $x^+$. Thus, Lagrange’s Theorem, Corollary 5.5 and Proposition 6.1 provide us with an $a \in \mathbb{Z}_{\geq 1}$ such that

\[
a \iota^+ = |\text{Cl}(X, x^+)| = |\text{Cl}(Z, z(02))| = \det(v_{01}, v_1, v_2) = l_1 d_2 + l_2 d_1.
\]

Resolving for $d_{22}$, gives the desired entry at the third place of the column $v_1$. Using the same argument for $\iota^- := \iota(x^-)$, we gain the desired entry at the third place of the column $v_{02}$ from

\[
b \iota^- = \det(v_{02}, v_1, v_2) = d_{02} l_1 l_2 + a \iota^+.
\]

Now one directly checks that the vectors $\gamma^\pm$ from Proposition 6.2 represent the Cartier divisor $-\iota \pm K_X$, where $-K_X = D_X^{01} + D_X^{02}$, near $x^\pm$ and thus are integral and primitive.

Remark 6.4. For given $\iota$, the presentation of $1/\iota$ as a sum of four unit fractions provided by Proposition 6.2 allows only finitely many choices of the pairs $(l_1, l_2)$. Moreover, there we have $0 \leq d_2 < l_2$ and all entries of $P$ and $\gamma^+, \gamma^-$ are integers. Thus, for fixed Gorenstein index $\iota = \text{lcm}(\iota^+, \iota^-)$, all involved numbers $a, b, \iota^+, \iota^-, l_1, l_2, d_2$ are effectively bounded.

Algorithm 6.5 (Classification algorithm for non-toric, quasi-smooth, rational, projective $\mathbb{K}^*$-surfaces of Picard number one). Input: A positive integer $\iota$, the prospective Gorenstein index. Algorithm:

- open an empty list $S$ for defining matrices $P$;
Proposition 6.6. Let $X$ be a non-toric, non-quasi-smooth, log terminal, rational, projective $\mathbb{K}^*$-surface of Picard number one. Then $X \cong X(P)$ with a defining matrix $P$ taken from the following list:

**Type (eAeD):**

$$
\begin{bmatrix}
-1 & 2 & 0 \\
-1 & 0 & 2 \\
0 & 1 \\
\end{bmatrix}
$$

$z = 3.4, a = 1.2, b = 2.4, c = 3.5, d = 4.6, e = 5.6, f = 6.7,$

$z = 3.4, a = 1.2, b = 2.4, c = 3.5, d = 4.6, e = 5.6, f = 6.7,$

$z = 3.4, a = 1.2, b = 2.4, c = 3.5, d = 4.6, e = 5.6, f = 6.7,$

$z = 3.4, a = 1.2, b = 2.4, c = 3.5, d = 4.6, e = 5.6, f = 6.7,$

**Type (eAeE):**

$$
\begin{bmatrix}
-1 & -2 & z \\
0 & 1 \\
\end{bmatrix}
$$

$z = 3.4, a = 1.2, b = 2.4, c = 3.5, d = 4.6, e = 5.6, f = 6.7,$

**Type (eDeD):**

$$
\begin{bmatrix}
-2 & -2 & a \\
-2 & 0 & 2 \\
0 & 1 \\
\end{bmatrix}
$$

$z = 3.4, a = 1.2, b = 2.4, c = 3.5, d = 4.6, e = 5.6, f = 6.7,$

$z = 3.4, a = 1.2, b = 2.4, c = 3.5, d = 4.6, e = 5.6, f = 6.7,$

$z = 3.4, a = 1.2, b = 2.4, c = 3.5, d = 4.6, e = 5.6, f = 6.7,$

$z = 3.4, a = 1.2, b = 2.4, c = 3.5, d = 4.6, e = 5.6, f = 6.7,$

**Type (eDeE):**

$$
\begin{bmatrix}
-2 & -3 & z \\
-2 & 0 & 2 \\
0 & 1 \\
\end{bmatrix}
$$

$z = 3.4, a = 1.2, b = 2.4, c = 3.5, d = 4.6, e = 5.6, f = 6.7,$

$z = 3.4, a = 1.2, b = 2.4, c = 3.5, d = 4.6, e = 5.6, f = 6.7,$

$z = 3.4, a = 1.2, b = 2.4, c = 3.5, d = 4.6, e = 5.6, f = 6.7,$

$z = 3.4, a = 1.2, b = 2.4, c = 3.5, d = 4.6, e = 5.6, f = 6.7,
Type \((eEeE)\):

\[
\begin{bmatrix}
-2 & -2 & z & 0 \\
-2 & -2 & 0 & 3 \\
-1 & -c-1 & a^n + 2d + x & d
\end{bmatrix}
\]

Moreover, for fixed \(i \in \mathbb{Z}_{>0}\), the defining matrices \(P\) from the above list having integral primitive \(i\)-Gorenstein forms give us all \(\mathbb{K}^*\)-surfaces \(X(P)\) of Gorenstein index \(i\).

**Proof.** Let \(X\) be a non-toric, non-quasi-smooth, rational, projective \(\mathbb{K}^*\)-surface of Picard number one. By Theorem 4.7 we may assume \(X = X(P)\) and Proposition 5.5 gives us the possible configurations of \((l_0, l_0, l_1, \ldots, l_r)\) for type \((ee)\) and \((l_0, \ldots, l_r)\) for type \((ep)\). We exemplarily discuss the case \((1, 1, y, 2, 2)\) from \((eDeD)\).

There, after suitable admissible operations, we can assume

\[
P = \begin{bmatrix}
-1 & -1 & y & 0 & 0 \\
-1 & -1 & 0 & 2 & 0 \\
-1 & -1 & 0 & 0 & 2 \\
0 & d_0 & d_1 & 1 & 1
\end{bmatrix}
\]

Let \(v^+, v^-, v_h\) be the local Gorenstein indices of the fixed points \(x^+, x^-, x_h\), respectively. Proposition 5.5 shows \(i(X_h) = 1\) and applying Corollary 3.5 and Proposition 5.5 to \(x^+\) yields an integer \(a > 0\) such that

\[a v^+ = l^+ m^+ = \det(v_{01}, v_{11}, v_{22}, v_{33}) = 4y + 4d_1.\]

Thus, we can replace the entry \(d_1\) with \((a v^+ - 4y)/4\). As before, applying Corollary 3.5 and Proposition 4.13 to \(x^-\) yields an integer \(b < 0\) with

\[b v^- = l^- m^- = \det(v_{01}, v_{11}, v_{22}, v_{33}) = -a v^+ - 4yd_0.\]

This allows us to replace the entry \(d_0\) with \(-(a v^+ + b v^-)/4\). Thereafter, the matrix \(P\) looks as in the assertion. The remaining task is to bound \(a, b\) and \(y\). We compute

\[v^+ u^+ = \begin{bmatrix} \frac{4}{a} \end{bmatrix}, \quad v^- u^- = \begin{bmatrix} \frac{a v^+ + b v^- - 4y}{by} \end{bmatrix},\]

according to Proposition 5.5. As \(v^+ u^+\) and \(v^- u^-\) are in particular integral vectors, we arrive at the bounding conditions

\[a \mid 4, \quad b \mid 4, \quad 4y \mid (a v^+ + b v^-)\]

Since \(x^\pm\) both are of type \(D\), their canonical multiplicity \(\zeta^\pm\) is one or two; see [2]. By definition, \(\zeta^\pm\) equals the last component of \(v^\pm u^\pm\), which excludes \(a = 1\) and \(b = -1\).  

\[\square\]
### Proposition 7.1

For the 117,065 sporadic isomorphism classes of log del Pezzo surfaces of Picard number one and Gorenstein index at most 200 that admit an effective action of a two-dimensional torus, the numbers \( \mu_i \) of isomorphism classes of Gorenstein index \( i \) are the following:

| \( i \) | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| \( \mu_i \) | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 |
| 89 | 91 | 92 | 93 | 94 | 95 | 96 |
| 145 | 166 | 187 | 208 | 229 | 250 | 271 | 292 | 313 | 334 | 355 | 376 |
| 49 | 50 | 51 | 52 | 53 | 54 | 55 | 56 | 57 | 58 | 59 | 60 |
| 315 | 326 | 337 | 348 | 359 | 370 | 381 | 392 | 403 | 414 | 425 | 436 |
| 61 | 62 | 63 | 64 | 65 | 66 | 67 | 68 | 69 | 70 | 71 | 72 |
| 216 | 227 | 238 | 249 | 260 | 271 | 282 | 293 | 304 | 315 | 326 | 337 |
| 73 | 74 | 75 | 76 | 77 | 78 | 79 | 80 | 81 | 82 | 83 | 84 |
| 249 | 283 | 704 | 338 | 552 | 377 | 497 | 311 | 531 | 351 | 571 | 391 |
| 85 | 86 | 87 | 88 | 89 | 90 | 91 | 92 | 93 | 94 | 95 | 96 |
| 764 | 807 | 910 | 1013 | 1116 | 1219 | 1322 | 1425 | 1528 | 1631 | 1734 | 1837 |
| 97 | 98 | 99 | 100 | 101 | 102 | 103 | 104 | 105 | 106 | 107 | 108 |
| 341 | 551 | 899 | 549 | 352 | 583 | 385 | 539 | 1377 | 383 | 409 | 536 |
| 109 | 110 | 111 | 112 | 113 | 114 | 115 | 116 | 117 | 118 | 119 | 120 |
| 377 | 840 | 756 | 589 | 377 | 642 | 1058 | 512 | 1010 | 462 | 1191 | 807 |
| 121 | 122 | 123 | 124 | 125 | 126 | 127 | 128 | 129 | 130 | 131 | 132 |
| 702 | 402 | 811 | 478 | 888 | 876 | 416 | 406 | 569 | 946 | 480 | 868 |
| 133 | 134 | 135 | 136 | 137 | 138 | 139 | 140 | 141 | 142 | 143 | 144 |
| 1202 | 483 | 1321 | 680 | 456 | 772 | 505 | 1172 | 931 | 522 | 1395 | 707 |
| 145 | 146 | 147 | 148 | 149 | 150 | 151 | 152 | 153 | 154 | 155 | 156 |
| 1204 | 482 | 1319 | 540 | 518 | 997 | 499 | 745 | 1261 | 1204 | 1205 | 965 |
| 157 | 158 | 159 | 160 | 161 | 162 | 163 | 164 | 165 | 166 | 167 | 168 |
| 493 | 543 | 1088 | 919 | 1477 | 748 | 517 | 670 | 2128 | 590 | 631 | 1160 |
| 169 | 170 | 171 | 172 | 173 | 174 | 175 | 176 | 177 | 178 | 179 | 180 |
| 895 | 1211 | 1395 | 613 | 562 | 962 | 2017 | 907 | 1156 | 646 | 689 | 1285 |
| 181 | 182 | 183 | 184 | 185 | 186 | 187 | 188 | 189 | 190 | 191 | 192 |
| 554 | 1338 | 1119 | 864 | 1442 | 963 | 1710 | 762 | 1864 | 1307 | 655 | 865 |
| 193 | 194 | 195 | 196 | 197 | 198 | 199 | 200 |
| 579 | 661 | 2507 | 1025 | 647 | 1319 | 651 | 1169 |

### Proposition 7.2

For the 154,138 families of non-toric log del Pezzo surfaces of Picard number one and Gorenstein index at most 200 that admit an effective action of a one-dimensional torus, any two members of a family share the same Gorenstein index, any two members stemming from different families are not isomorphic to each other and the numbers \( \nu_i \) of families of Gorenstein index \( i \) are the following:

| \( i \) | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| \( \nu_i \) | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 |
| 108 | 109 | 110 | 111 | 112 | 113 | 114 | 115 | 116 | 117 | 118 | 119 |
| 158 | 159 | 160 | 161 | 162 | 163 | 164 | 165 | 166 | 167 | 168 | 169 |
| 554 | 1338 | 1119 | 864 | 1442 | 963 | 1710 | 762 | 1864 | 1307 | 655 | 865 |
| 193 | 194 | 195 | 196 | 197 | 198 | 199 | 200 |
| 579 | 661 | 2507 | 1025 | 647 | 1319 | 651 | 1169 |
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