Filling hypersurfaces by discs in almost complex manifolds of dimension 2

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Abstract. We study pseudoholomorphic discs with boundaries attached to a real hypersurface $E$ in an almost complex manifold of dimension 2. We prove that if $E$ contains no discs, then they fill a one-sided neighborhood of $E$.

MSC: 32H02, 53C15.
Key words: almost complex manifold, Bishop disc.

1 Introduction

The modern development of the analysis on almost complex manifolds began after the work of Gromov [3] who discovered its remarkable applications in symplectic geometry. One of the main tools of his approach involves the Bishop discs, which are pseudoholomorphic discs with boundaries attached to a prescribed real submanifold; they have been used later by many authors.

On the other hand, the method of Bishop discs is well known and powerful in complex analysis of several variables (see, e.g., [9, 11]). The main goal of this paper is to develop a systematic general approach to the theory of Bishop discs in almost complex manifolds. As an object of the study we choose a well known nontrivial result for the standard complex structure and generalize it to the almost complex category. Our main result is the following.

**Theorem 1.1** Let $E$ be a real hypersurface in an almost complex manifold $(M, J)$ of complex dimension 2. Suppose that $E$ contains no $J$-holomorphic discs. Then Bishop discs of $E$ fill a one-sided neighborhood of every point of $E$.

In the case of the standard complex structure, the corresponding result was obtained by Trépreau [8] for real hypersurfaces in $\mathbb{C}^n$ and by the second named author [10] for generic
submanifolds of arbitrary codimension in \( \mathbb{C}^n \). Although the restriction on the dimension is used in our paper, this is mainly due to technical reasons and we believe that after a suitable modification our approach can be generalized to higher dimension.

In fact, a stronger result is proved in [8, 10]: if Bishop discs don’t fill a one-sided neighborhood of \( p \in E \), then there exists a disc in \( E \) passing through \( p \). Thus we leave open an interesting question whether the same can be done in the almost complex case. We may want to address it elsewhere.

We now explain the organization of the paper and the main steps of our approach. Section 2 contains basic properties of almost complex manifolds, pseudoholomorphic discs and some special coordinate systems on almost complex manifolds used throughout the paper. In Section 3 we derive different versions of the Bishop equation for pseudoholomorphic discs. It can be viewed as a non-linear boundary value Riemann-Hilbert type problem of a quasilinear elliptic system in the unit disc. We prove that the Bishop discs attached to a real hypersurface at a given point form a Banach manifold and give an efficient local parametrization of this manifold. As an application in Section 4 we prove Theorem 1.1 under some additional restrictions (pseudoconvexity or finite type conditions). In particular, we prove Theorem 1.1 in the real analytic category. Our considerations there are primarily geometrical. They are based on non-isotropic dilations of suitable local coordinates which allow to represent the above mentioned Riemann-Hilbert type problem as a small perturbation of the corresponding problem for the standard complex structure in \( \mathbb{C}^n \).

Section 5 is dedicated to the analysis of the Bishop equation for pseudoholomorphic discs and is principal for our approach. We parametrize the tangent space to the manifold of Bishop discs by a space of holomorphic functions and solve the corresponding linearized boundary value problem by means of the generalized Schwarz integral. It turns out that the proof of Theorem 1.1 given in [10] for the standard complex structure does not go through in the almost complex category. The proof in [10] is based on the notion of the defect of a disc. For a Bishop disc through a fixed point of a real manifold \( E \), the infinitesimal perturbations of the direction of the disc at the fixed point and those of another boundary point of the disc are restricted in the same way, according to the defect of the disc. We were unable to find an analogue of this phenomenon in our case of complex dimension 2. In higher dimension no such analogue is possible due to an example by Ivashkovich and Rosay [4], in which all Bishop discs through a fixed point of a hypersurface \( E \) lie in \( E \) and cover all of \( E \). Baouendi, Rothschild, and Trépreau [1] interpret the defect of a disc in terms of its lifts attached to the conormal bundle of \( E \) in the cotangent bundle \( T^*\mathbb{C}^n \). Although an almost complex structure admits natural lifts to the cotangent bundle of an almost complex manifold (see, e.g., [13]), seemingly, they don’t give rise to a correct notion of the defect of Bishop discs.

In this paper we develop a new approach, in particular we give a different proof of the main result of [10] for a hypersurface in \( \mathbb{C}^2 \). Our key result is the following.

**Theorem 1.2** Let \( E \) be a real hypersurface in an almost complex manifold \( (M, J) \) of complex dimension 2 and let \( f_0 \) be a small enough embedded Bishop disc attached to \( E \) at a point \( p \in E \) (that is \( f_0(1) = p \)) and tangent to \( E \) at \( p \). Suppose that every Bishop disc \( f \) attached at \( p \) and
close enough to $f_0$ also is tangent to $E$ at $p$. Then the Levi form of $E$ vanishes identically along the boundary of $f_0$.

For the standard complex structure, this result follows from [10]. In the almost complex case the result is new. We show that the condition of tangency to the hypersurface $E$ of all Bishop discs attached to $E$ at a point $p$ is equivalent to the vanishing of some non-linear operator $\Phi$ defined on the Banach space of discs and valued in the space of smooth functions on the unit circle. We show that the Frechet derivative $\dot{\Phi}$ of $\Phi$ up to smoother terms is equal to the multiplication operator by the Levi determinant of $E$ and this allows to conclude the proof of Theorem 1.2. Now if in the hypothesis of Theorem 1.1 $E$ does not admit a transversal Bishop disc attached at a given point $p$ then Theorem 1.2 implies that $E$ is Levi flat along the boundaries of the Bishop discs through $p$ which allows to construct holomorphic discs contained in $E$ and leads to a contradiction. Hence there exists a transversal Bishop disc, whose perturbations fill a one sided neighborhood of $E$. Theorem 1.2 (and the method of its proof) gives a new powerful tool for constructing transversal Bishop discs, which may have further applications in the almost complex analysis and geometry.

This paper was written when the first named author visited the University of Illinois at Urbana-Champaign during the Spring semester 2005. He thanks this institution for hospitality and excellent conditions for work. In conclusion, the authors thank the referee for many useful remarks.

2 Preliminaries

In this section we briefly recall some basic properties of almost complex manifolds.

2.1 Almost complex manifolds

Let $(M, J)$ be a $C^\infty$-smooth almost complex manifold. Everywhere below we denote by $I \mathbb{D}$ the unit disc in $\mathbb{C}$ and by $J_{st}$ the standard complex structure in $\mathbb{C}^n$; the value of $n$ is usually clear from the context. Let $f$ be a smooth map from $I \mathbb{D}$ into $M$. We say that $f$ is $J$-holomorphic if $df \circ J_{st} = J \circ df$. We call such a map $f$ a $J$-holomorphic disc or a pseudoholomorphic disc.

The following frequently used statement shows that an almost complex manifold $(M, J)$ of complex dimension $n$ can be locally viewed as the unit ball $I \mathbb{B}$ in $\mathbb{C}^n$ equipped with a small almost complex deformation of $J_{st}$.

**Lemma 2.1** Let $(M, J)$ be an almost complex manifold of complex dimension $n$. Then for each $p \in M$, each $\delta_0 > 0$, and each $k \geq 0$ there exist a neighborhood $U$ of $p$ and a smooth coordinate chart $Z : U \rightarrow I \mathbb{B}$ such that $Z(p) = 0$, $dZ(p) \circ J(p) \circ dZ^{-1}(0) = J_{st}$, and the direct image $Z_*(J) := dZ \circ J \circ dZ^{-1}$ satisfies the inequality $||Z_*(J) - J_{st}||_{C^k(I \mathbb{B})} \leq \delta_0$. 

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Proof: There exists a diffeomorphism \( z \) from a neighborhood \( U' \) of \( p \in M \) onto \( \mathbb{B} \) such that \( Z(p) = 0 \) and \( dZ(p) \circ J(p) \circ dZ^{-1}(0) = J_{st} \). For \( \delta > 0 \) consider the isotropic dilation \( d_\delta: t \mapsto \delta^{-1}t \) in \( \mathbb{C}^n \) and the composite \( Z_\delta = d_\delta \circ Z \). Then \( \lim_{\delta \to 0} ||(Z_\delta)_*(J) - J_{st}||_{C^k(\mathbb{B})} = 0 \). Setting \( U = Z_\delta^{-1}(\mathbb{B}) \) for positive \( \delta \) small enough, we obtain the desired result.

Let \((M,J)\) be an almost complex manifold. Denote by \( TM \) the real tangent bundle of \( M \) and by \( T_\mathbb{C}M \) its complexification. Recall that \( T_\mathbb{C}M = T^{(1,0)}M \oplus T^{(0,1)}M \) where \( T^{(1,0)}M := \{ \xi \in T_\mathbb{C}M : J\xi = i\xi \} \) and \( T^{(0,1)}M := \{ \xi \in T_\mathbb{C}M : J\xi = -i\xi \} \). Let \( T^*M \) denote the cotangent bundle of \( M \). Identifying \( \mathbb{C} \otimes T^*M \) with \( T^*_\mathbb{C}M := \text{Hom}(T_\mathbb{C}M, \mathbb{C}) \) we consider the set of complex forms of type \((1,0)\) on \( M \): \( T_{(1,0)}M = \{ w \in T^*_\mathbb{C}M : w(X) = 0, \forall X \in T^{(1,0)}M \} \) and the set of complex forms of type \((0,1)\) on \( M \): \( T_{(0,1)}M = \{ w \in T^*_\mathbb{C}M : w(X) = 0, \forall X \in T^{(0,1)}M \} \). Then \( T^*_\mathbb{C}M = T_{(1,0)}M \oplus T_{(0,1)}M \).

So we define the operators \( \partial_J \) and \( \bar{\partial}_J \) on the space of smooth functions on \( M \): for a smooth complex function \( u \) on \( M \) we set \( \partial_J u = du_{(1,0)} \in T_{(1,0)}M \) and \( \bar{\partial}_J u = du_{(0,1)} \in T_{(0,1)}M \). As usual, differential forms of any bidegree \((p,q)\) on \((M,J)\) are defined by exterior multiplication.

As usual, an upper semicontinuous function \( u \) on \((M,J)\) is called \( J \)-plurisubharmonic on \( M \) if the composition \( u \circ f \) is subharmonic on \( \mathbb{D} \) for every \( f \in \mathcal{O}_J(\mathbb{D},M) \).

Let \( u \) be a \( C^2 \) function on \( M \), let \( p \in M \) and \( v \in T_pM \). The Levi form of \( u \) at \( p \) evaluated on \( v \) is defined by the equality \( L^J(u)(p)(v) := -d(J^*du)(v,Jv)(p) \).

The following result is well known (see, for instance, [4]).

Proposition 2.2 Let \( u \) be a \( C^2 \) real valued function on \( M \), let \( p \in M \) and \( v \in T_pM \). Then \( L^J(u)(p)(v) = \Delta(u \circ f)(0) \) where \( f \) is an arbitrary \( J \)-holomorphic disc in \( M \) such that \( f(0) = p \) and \( df(0)(\partial/\partial \text{Re } \zeta) = v \) (here \( \zeta \) is the standard complex coordinate variable in \( \mathbb{C} \)).

The Levi form is invariant with respect to \( J \)-biholomorphisms. More precisely, let \( u \) be a \( C^2 \) real valued function on \( M \), let \( p \in M \) and \( v \in T_pM \). If \( \Phi \) is a \((J,J')\)-biholomorphic diffeomorphism from \((M,J)\) into \((M',J')\), then \( L^J(u)(p)(v) = L^{J'}(u \circ \Phi^{-1})(\Phi(p))(d\Phi(p)(v)) \).

Finally, it follows from Proposition 2.2 that a \( C^2 \)-smooth real function \( u \) is \( J \)-pluri-

subharmonic on \( M \) if and only if \( L^J(u)(p)(v) \geq 0 \) for all \( p \in M \), \( v \in T_pM \). Thus, similarly to the case of the integrable structure one arrives in a natural way to the following definition: a \( C^2 \) real valued function \( u \) on \( M \) is strictly \( J \)-plurisubharmonic on \( M \) if \( L^J(u)(p)(v) \) is positive for every \( p \in M \), \( v \in T_pM \setminus \{0\} \).

It follows easily from Lemma 2.1 that for every point \( p \in M \) there exists a neighborhood \( U \) of \( p \) and a diffeomorphism \( Z : U \to \mathbb{B} \) with center at \( p \) (in the sense that \( Z(p) = 0 \)) such that the function \( |Z|^2 \) is \( J \)-plurisubharmonic on \( U \) and \( Z_*(J) = J_{st} + O(|Z|) \).

Let \( E \) be a real submanifold of codimension \( m \) in an almost complex manifold \((M,J)\) of complex dimension \( n \). For every \( p \) we denote by \( H^J_p(E) \) the maximal complex (with respect to \( J(p) \)) subspace of the tangent space \( T_p(E) \). Similarly to the integrable case, \( E \) is said to be a CR manifold if the complex dimension of \( H^J_p(E) \) is independent on \( p \); it is called the CR dimension of \( E \) and is denoted by \( \dim_{CR} E \). As usual, by a generic submanifold of a complex manifold one means a submanifold \( E \) such that at every point \( p \in E \) the complex linear span of \( T_p(E) \) coincides with the tangent space of the ambient manifold.
If $E$ is defined as the common zero set of real functions $\rho_1, \ldots, \rho_m$, then after the standard identification of $TM$ and $T^{(1,0)}M$, the space $H^J_p(E)$ can be defined as the zero subspace of the forms $\partial_J \rho_1, \ldots, \partial_J \rho_m$. In the present paper we only deal with the case $m = 1$ that is when $E$ is a real hypersurface.

Similarly to the integrable case, by the Levi form of a real hypersurface $E = \{ \rho = 0 \}$ at $p \in E$ we mean the conformal class of the Levi form $L^J_p(\rho)(p)$ of the defining function $\rho$ on the holomorphic tangent space $H^J_p(E)$. It is well-known that

$$L^J_p(\rho)(X_p) = J^* \, d\rho[X, JX]_p,$$

where a vector field $X$ is a smooth section of the $J$-holomorphic tangent bundle $H^J(E)$ of $E$ such that $X(p) = X_p$ for a given vector $X_p \in H^J_p(E)$.

2.2 Normal form of an almost complex structure along a pseudoholomorphic disc

Throughout the paper, we often use the standard notation $Z_\zeta := \frac{\partial Z}{\partial \zeta}$ and $Z_\overline{\zeta} := \frac{\partial Z}{\partial \overline{\zeta}}$. We don’t make a difference between $Z_\zeta$ and $Z_\overline{\zeta}$ as well as between $Z_\zeta$ and $Z_\overline{\zeta}$.

Let $J$ be a smooth almost complex structure on a neighborhood of the origin in $\mathbb{C}^2$ and $J(0) = J_{st}$. Denote by $Z = \left( \begin{array}{c} z \\ w \end{array} \right)$ the standard coordinates in $\mathbb{C}^2$. Then a map $Z : D \rightarrow \mathbb{C}^2$ is $J$-holomorphic if and only if it satisfies the following equation

$$Z_\zeta - A(Z)Z_\overline{\zeta} = 0,$$

where $A(Z)$ is the complex $2 \times 2$ matrix defined by

$$A(Z)v = (J_{st} + J(Z))^{-1}(J_{st} - J(Z))(\overline{v}).$$

It is easy to see that the right-hand side is $\mathbb{C}$-linear in $v \in \mathbb{C}^2$ with respect to the standard structure $J_{st}$, hence $A(Z)$ is well defined. Since $J(0) = J_{st}$, we have $A(0) = 0$. However, we will need a more precise choice of coordinates imposing additional restrictions on $A$. We first derive a rule of transformation of $A$ under diffeomorphisms.

**Lemma 2.3** Let $A(Z)$ be the matrix defined by (3). Let $Z' = Z'(Z)$ be a diffeomorphic change of coordinates such that $(Z_Z + Z_{\overline{Z}}A)^{-1}$ exists. Then in the new coordinates

$$A'(Z') = (Z'_Z A + Z'_{\overline{Z}})(Z_Z + Z_{\overline{Z}}A)^{-1}.$$

**Proof:** Consider a $J$-holomorphic disc $\zeta \mapsto Z(\zeta)$ and the disc $Z'(\zeta) = Z'(Z(\zeta))$. Then

$$Z_\zeta = A(Z)Z_\zeta$$

and

$$Z'_\zeta = A'(Z)Z'_\zeta.$$ 

Then we have

$$(Z'_Z A + Z'_{\overline{Z}})Z_\zeta = Z'_Z Z_\zeta + Z'_{\overline{Z}} Z_\zeta = Z'_\zeta = A'(Z)Z_\zeta = A'(Z_Z Z_\zeta + Z'_{\overline{Z}} Z_\zeta) = A'(Z_Z + Z_{\overline{Z}} A)Z_\zeta.$$

Since $Z_\zeta$ is arbitrary, the desired formula follows.
Lemma 2.4 Let $Z_0$ be a $J$-holomorphic disc close to the disc $\left( \begin{array}{c} 0 \\ \zeta \end{array} \right)$, $\zeta \in \mathbb{D}$. Then there exists a change coordinates in a neighborhood of $Z_0(\mathbb{D})$ such that in the new coordinates we have $Z_0 = \left( \begin{array}{c} 0 \\ \zeta \end{array} \right)$, $\zeta \in \mathbb{D}$. Moreover, $A(0, \zeta) = 0$, $A_Z(0, \zeta) = 0$ for $\zeta \in \mathbb{D}$.

Remark 1. If $A(p) = 0$, then the condition $A_Z(p) = 0$ means exactly that for all $J$-holomorphic maps $f : \mathbb{D} \rightarrow \mathbb{C}^2$ with $f(0) = p$, in addition to $f(0) = 0$ we also have $\Delta f(0) = 0$. On the other hand, the integrability condition for $J$ in terms of $A$ is equivalent to certain symmetry in the expression $A_{\overline{Z}} + A_ZA$, therefore for a non-integrable $J$, one generally cannot achieve $A = 0$, $A_{\overline{Z}} = 0$ by a change of coordinates, even at a point.

Remark 2. If $Z_0 \in C^k, 3 \leq k \leq \infty$, then we construct a change of coordinates of class $C^{k-2}$. We don’t think this smoothness is optimal.

Proof: After a local change of coordinates we have $Z_0 = \left( \begin{array}{c} 0 \\ \zeta \end{array} \right)$. The $J$-holomorphicity condition of $Z$ implies that in these coordinates we have

$$(A \circ Z)(\zeta) = \left( \begin{array}{c} \alpha(\zeta) \\ \beta(\zeta) \end{array} \right).$$

Consider a local change of coordinates of the form

$$z' = a_{10}z + a_{01} \overline{z} + a_{11}zz, \quad w' = w + b_{10}z + b_{01} \overline{z} + b_{11}zz,$$

where $a_{jk}$, $b_{jk}$ are smooth functions of $w$ with $|a_{10}| \neq |a_{01}|$. In the new coordinates the disc $Z_0$ does not change. We have

$$Z'_Z(0, w) = \left( \begin{array}{c} a_{10} \\ b_{10} \end{array} \right), \quad Z'_{\overline{Z}}(0, w) = \left( \begin{array}{c} a_{01} \\ b_{01} \end{array} \right).$$

By Lemma 2.3 we have

$$A'(Z') = (Z'_Z A + Z'_{\overline{Z}})(Z'_{\overline{Z}} + Z'_Z)^{-1}.$$ 

The condition $A'(0, w') = 0$ implies

$$\left( \begin{array}{c} a_{10} \\ b_{10} \end{array} \right) \left( \begin{array}{c} \alpha \\ \beta \end{array} \right) + \left( \begin{array}{c} a_{01} \\ b_{01} \end{array} \right) = 0$$

and therefore

$$\alpha a_{10} + a_{01} = 0, \quad \alpha b_{10} + \beta + b_{01} = 0.$$
Hence the condition $A'(0, w') = 0$ determines the functions $a_{01}$ and $b_{01}$ for given $\alpha$, $\beta$, $a_{10}$ and $b_{10}$.

Thus without loss of generality we assume that $A(0, w) = 0$ and $a_{01} = b_{01} = 0$. This also implies that $A_w(0, w) = A_z(0, w) = 0$. So we need to achieve $A'_z(0, w') = 0$.

We have $A'_z = A'_z z'_z + A'_z z'_w$. Since $a_{01} = b_{01} = 0$, then $Z'_z(0, w) = 0$. Then $A'_z(0, w) = a_{10}(w) A'_z(0, w')$. Hence $A'_z(0, w') = 0$ if and only if $A'(0, w) = 0$. Since $A(0, w) = 0$, then for $z = 0$ we have

$$A'_z(0, w) = (Z'_z A_z + Z'_w)(Z'_z + Z'_w A)^{-1}.$$ 

We also have

$$Z'_z(0, w) = \begin{pmatrix} z'_z & z'_w \\ w'_z & w'_w \end{pmatrix} = \begin{pmatrix} a_{11} & (a_{10})_w \\ b_{11} & (b_{10})_w \end{pmatrix}.$$ 

Put

$$A_z(0, w) = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}.$$ 

Then the condition $A'_z(0, w') = 0$ takes the form

$$\begin{pmatrix} a_{10} & 0 \\ b_{10} & 1 \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} + \begin{pmatrix} a_{11} & (a_{10})_w \\ b_{11} & (b_{10})_w \end{pmatrix} = 0,$$

so the coefficients $a_{11}$, $b_{11}$ are determined by $a_{10}$, $b_{10}$, $\alpha$, $\gamma$. The functions $a_{10}$ and $b_{10}$ are found as solutions of the following classical elliptic equations (see [12]):

$$(a_{10})_w + \beta a_{10} = 0,$$

$$(b_{10})_w + \beta b_{10} + \delta = 0.$$ 

This completes the proof of the lemma.

Let $E$ be a real hypersurface through the origin in $\mathbb{C}^2$ equipped with a smooth almost complex structure $J$. Even if $J(0) = J_{st}$, the Levi form of $E$ with respect to $J$ at the origin does not necessarily coincide with the Levi form with respect to $J_{st}$. However, if the coordinates are normalized according to the previous lemma, then the Levi forms with respect to $J$ and $J_{st}$ are the same.

**Lemma 2.5** Assume that $A(0) = A_z(0) = 0$. Then the Levi form of $E$ at the origin with respect to the structure $J$ coincides with the Levi form of $E$ at the origin with respect to the structure $J_{st}$. 

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Proof: We can assume that \( E \) is given by
\[
\rho(Z) = \text{Re} \ z + Q(Z) + H(Z) + o(|Z|^2) = 0,
\]
where \( Q(Z) \) is a complex quadratic form and \( H(Z) \) is a hermitian quadratic form. It is sufficient to consider the Levi form of \( \rho \) at the origin evaluated on the vector \( V = (0, 1) \). So we consider a \( J \)-holomorphic disc \( Z(\zeta) \) satisfying \( Z(\zeta) = \begin{pmatrix} z(\zeta) \\ w(\zeta) \end{pmatrix} = \begin{pmatrix} 0 \\ \zeta \end{pmatrix} + O(|\zeta|^2) \) (the existence of such a disc follows by the classical Nijenhuis-Woolf theorem, see for instance [7]). It follows from the assumption on the matrix \( A \) and the \( J \)-holomorphicity of \( Z \) that \( w_\zeta = a_\zeta + o(\zeta) \) so that \( w_\zeta(0) = 0 \). Therefore \( \Delta(\rho \circ Z)(0) = H(V) \) which proves the lemma.

3 Bishop discs and the Bishop equation

Let \((M,J)\) be a smooth almost complex manifold of real dimension \(2n\) and \( E \) a generating submanifold of \( M \) of real codimension \( m \). A \( J \)-holomorphic disc \( f: \mathbb{D} \to M \) continuous on \( \overline{\mathbb{D}} \) is called a Bishop disc if \( f(b\mathbb{D}) \subset E \), where \( b\mathbb{D} \) denotes the boundary of \( \mathbb{D} \). The existence and local parametrization of certain classes of Bishop discs attached to \( E \) are obtained in [5]. Here we give a more precise description of small Bishop discs which will be used in our constructions.

3.1 Bishop’s equation as the Riemann-Hilbert type problem for elliptic PDE systems

Let \( E \) be a smooth generic submanifold in a smooth (always supposed \( C^\infty \)) almost complex manifold \((M,J)\) defined as the zero set of an \( \mathbb{R}^m \)-valued function \( \rho = (\rho^1, ..., \rho^m) \) on \( M \). Then a smooth map
\[
f: \mathbb{D} \to M \quad f: \zeta \mapsto f(\zeta)
\]
continuous on \( \overline{\mathbb{D}} \) is a Bishop disc if and only if it satisfies the following non-linear boundary problem of the Riemann-Hilbert type for the quasi-linear operator \( \overline{\partial}_J \):
\[
(RH) : \begin{cases}
\overline{\partial}_J f(\zeta) = 0, & \zeta \in \mathbb{D} \\
\rho(f)(\zeta) = 0, & \zeta \in b\mathbb{D}
\end{cases}
\]

In order to obtain a local description of solutions of this problem we fix a chart \( U \subset M \) and a coordinate diffeomorphism \( Z: U \to \mathbb{B} \) where \( \mathbb{B} \) is the unit ball of \( \mathbb{C}^n \). Identifying \( M \) with \( \mathbb{B} \) we can assume that in these coordinates \( J = J_{st} + O(|Z|) \) and the norm \( \| J - J_{st} \|_{C^k(\mathbb{B})} \) is small enough for some positive real \( k \) in accordance with Lemma 2.1. (Here \( k > 1 \) can be chosen arbitrary; we assume it for convenience to be real positive non-integral and fix it throughout what follows.) More precisely, using the notation \( Z = (z,w) \), \( z = (z_1, ..., z_m) \),
z = x + iy, w = (w_1, ..., w_{n-m}) for the standard coordinates in \( \mathbb{C}^n \), we can also assume that \( E \cap U \) is described by the equations

\[
\rho(Z) = x - h(y, w) = 0
\]

with vector-valued \( C^\infty \)-function \( h : \mathbb{B} \to \mathbb{R}^m \) such that \( h(0) = 0 \) and \( \nabla h(0) = 0 \).

Similarly to the proof of Lemma 2.1 consider the isotropic dilations \( d_\delta : Z \mapsto \delta^{-1}Z \). In the new \( Z \)-variables (we drop the primes) the image \( E_\delta = d_\delta(E) \) is described by the equations

\[
\rho_\delta(Z) = \delta^{-1}\rho(\delta Z) = 0
\]

Since the function \( \rho_\delta \) approaches \( x \) as \( \delta \to 0 \), the manifolds \( E_\delta \) approach the flat manifold \( E_0 = \{x = 0\} \), which, of course, may be identified with the real tangent space to \( E \) at the origin. Furthermore, as seen in the proof of Lemma 2.1, the structures \( J_\delta : (d_\delta)_*(J) \) converge to \( J_{st} \) in the \( C^k \)-norm as \( \delta \to 0 \). This allows us to find explicitly the \( \partial J \)-operator in the \( Z \) variables.

Consider now a \( J_\delta \)-holomorphic disc

\[
Z : \mathbb{D} \to (\mathbb{B}, J_\delta)
\]

of class \( C^k(\overline{D}) \). The \( J_\delta \)-holomorphicity condition \( J_\delta(Z) \circ dZ = dZ \circ J_{st} \) can be written in the following form.

\[
Z_\zeta - A_{J,\delta}(Z)Z_\zeta = 0, \quad (5)
\]

where \( A_{J,\delta}(Z) \) is the complex \( n \times n \) matrix of an operator the composite of which with complex conjugation is equal to the endomorphism \((J_{st} + J_\delta(Z))^{-1}(J_{st} - J_\delta(Z))\) (which is an anti-linear operator with respect to the standard structure \( J_{st} \)). Hence the entries of the matrix \( A_{J,\delta}(Z) \) are smooth functions of \( \delta, Z \) vanishing identically in \( Z \) for \( \delta = 0 \).

Recall that the Cauchy-Green transform is defined by

\[
Tf(\zeta) = \frac{1}{2\pi i} \int_{\mathbb{D}} \int_{\overline{\mathbb{D}}} \frac{f(\tau)}{\tau - \zeta} d\tau \wedge d\overline{\tau}.
\]

It is well-known ([12], p. 56, Theorem 1.32) that \( T : C^k(\mathbb{D}) \to C^{k+1}(\overline{\mathbb{D}}) \) is a bounded operator when \( k \) is non-integer. Furthermore, \( (Tf)_\overline{\zeta} = f \). Therefore we can write \( \partial_{\overline{J}} \)-equation (5) as follows:

\[
[Z - T(A_{J,\delta}(Z)Z_\zeta)]_\overline{\zeta} = 0.
\]

The arising non-linear operator

\[
\Phi_{J,\delta} : Z \to \tilde{Z} = Z - T[A_{J,\delta}(Z)Z_\zeta]
\]

takes the space \( C^k(\overline{\mathbb{D}}) \) into itself. Thus, \( Z \) is \( J_\delta \)-holomorphic disc if and only if \( \Phi_{J,\delta}(Z) \) is a holomorphic disc (in the usual sense) on \( \mathbb{D} \). For sufficiently small positive \( \delta \) the map \( \Phi_{J,\delta} \)
is an invertible operator on a neighborhood of zero in $C^k(D)$ which establishes a one-to-one correspondence between the sets of $J_\delta$-holomorphic and holomorphic discs in $\mathbb{B}^n$.

These considerations allow us to replace the non-linear Riemann-Hilbert problem (RH) by the generalized Bishop equation

$$\rho_\delta(\Phi^{-1}_{J_\delta}(\tilde{Z}))(\zeta) = 0, \quad \zeta \in b\mathbb{D},$$

(7)

for an unknown holomorphic function $\tilde{Z}$ in $\mathbb{D}$.

If $\tilde{Z}$ is a solution of the boundary problem (7), then $Z = \Phi^{-1}_{J_\delta}(\tilde{Z})$ is a Bishop disc with boundary attached to $E_\delta$. Since the manifold $E_\delta$ is biholomorphic via isotropic dilations to the initial manifold $E$, the solutions of the equation (7) allow to describe the Bishop discs attached to $E$. Of course, this gives just the discs close enough in the $C^k$-norm to the trivial solution $Z \equiv 0$ of the problem (RH). We will call such discs small.

3.2 Schwarz-Green formula and Bishop’s equation

This is also useful to give a more explicit form of the Bishop equation as a non-linear system of singular integral equations. Let

$$P_0 f = \frac{1}{2\pi i} \int_{b\mathbb{D}} f(\tau) \frac{d\tau}{\tau}$$

denote the average value of $f$ on $b\mathbb{D}$ and let

$$S u(\zeta) = \frac{1}{2\pi i} \int_{b\mathbb{D}} \frac{\tau + \zeta}{\tau - \zeta} u(\tau) \frac{d\tau}{\tau}$$

denote the Schwarz integral. Consider also the Cauchy integral

$$K f(\zeta) = \frac{1}{2\pi i} \int_{b\mathbb{D}} \frac{f(\tau) d\tau}{\tau - \zeta},$$

(8)

We note

$$S = 2K - P_0.$$

We begin with a version of the Cauchy-Green formula replacing the Cauchy kernel by the Schwarz kernel; this is a variation of the classical results [12].

**Proposition 3.1** Let $f = u + iv$ a complex function of class $C^1$ on the unit disc $\mathbb{D}$ of $\mathbb{C}$. Then for every $\zeta \in \mathbb{D}$ we have

$$f(\zeta) = Su(\zeta) + iv_0 + T f_\zeta(\zeta) - T f_\zeta(1/\zeta),$$

where $v_0 = P_0 v$. 
Proof: Let 
\[ g(\zeta) = f(\zeta) - Su(\zeta) - iv_0 - Tf_{\zeta}(\zeta) + \overline{Tf_{\zeta}(1/\zeta)}. \]

Then the function \( g \) is holomorphic on \( \mathbb{D} \) because \( \frac{\partial}{\partial \zeta} Tf_{\zeta}(\zeta) = f_{\zeta}(\zeta) \) and the functions \( Su \) and \( Tf_{\zeta}(1/\zeta) \) are holomorphic. Furthermore, \( \text{Re} g|_{b\mathbb{D}} = 0 \) since \( \text{Re} Su|_{b\mathbb{D}} = u \) (in the sense of limiting boundary values from inside) and \( 1/\zeta = \zeta, Tf_{\zeta}(\zeta) = Tf_{\zeta}(1/\zeta) \) for \( \zeta \in b\mathbb{D} \). Hence \( g = ic \) where \( c \) is a real constant.

In order to prove that \( c = v_0 \), it suffices to show that \( P_0Tf_{\zeta} = 0 \). Note that for every bounded \( g \), in particular for \( g = f_{\zeta} \) the function \( Tg \) is holomorphic in \( \mathbb{C} \setminus \mathbb{D} \) and vanishes at the infinity. Hence \( P_0Tg = 0 \), and the proposition follows.

Let now \( E \) be a generic submanifold of codimension \( n - k \) in an almost complex manifold \((M,J)\) of complex dimension \( n \). Fixing local coordinates, we can assume that \( E \) is a submanifold of \( \mathbb{C}^n \) through the origin, \( J(0) = J_{st} \). Similarly to [11] we can also assume that \( E \) is given by the following parametric equations 

\[ \text{Re} Z = h(\text{Im} Z, t), \]

where \( Z \in \mathbb{C}^n \) are the standard complex coordinates and \( t \in \mathbb{R}^k \) is a parameter. Furthermore, 

\[ h(0,0) = 0, \quad \frac{\partial h}{\partial \text{Im} Z}(0,0) = 0, \quad \text{rank} \frac{\partial h}{\partial t}(0,0) = k. \]

Let \( Z : \zeta \mapsto Z(\zeta) \) be a \( J \)-holomorphic Bishop disc for \( E \) in a sufficiently small neighborhood of the origin. Then it satisfies the \( J \)-holomorphicity equations 

\[ Z_{\zeta} - A(Z(\zeta))\overline{Z}_{\zeta}(\zeta) = 0. \]

Since the disc \( Z \) takes its values in a neighborhood of the origin small enough, the norm of the matrix \( A \) also is supposed to be small. The boundary condition \( Z(b\mathbb{D}) \subset E \) means that 

\[ \text{Re} Z(\zeta) = h(\text{Im} Z(\zeta), t(\zeta)) \]

for some function \( t(\zeta) \) on \( b\mathbb{D} \). Set 

\[ Pf(\zeta) = Tf(\zeta) - \overline{Tf(1/\zeta)}. \]

Then we obtain that the above conditions are equivalent to the following Bishop equation for the map \( Z \):

\[ Z = Sh(\text{Im} Z, t) + ic_0 + P(A(Z)\overline{Z}_{\zeta}). \quad (9) \]

For a given non-integral \( \alpha > 1 \), sufficiently small function \( t(\zeta) \in (C^\alpha(b\mathbb{D}))^k \), and \( c_0 \in \mathbb{R}^n \) this equation has a unique solution of class \( C^\alpha(\overline{\mathbb{D}}) \) by the implicit function theorem.

In particular, if \( E \) is given by the equations
\[ x = \varphi(y, w), \]
\[ z = x + iy \in \mathbb{C}^{n-k}, \quad w = u + iv \in \mathbb{C}^k, \]
\[ \varphi(0, 0) = 0, \quad d\varphi(0, 0) = 0, \]
then
\[ \left( \begin{array}{c} x \\ u \end{array} \right) = h(y, v, t) = \left( \begin{array}{c} \varphi(y, t + iv) \\ t \end{array} \right) \]
can be used for Bishop’s equation (9).

We can modify Bishop’s equation to define
\[ Z(1) = Z_0 = X_0 + iY_0, \quad X_0 = h(Y_0, t_0). \]
Put
\[ S_1u = Su - Su(1), \quad P_1f = Pf - (Pf)(1). \]
Then we have the modified Bishop’s equation
\[ Z = S_1h(\text{Im} Z, t) + Z_0 + P_1(A\overline{Z}), \]
where \( t(1) = t_0 \). The solution satisfies \( Z(1) = Z_0 \).

We point out that it follows immediately from the equation (9) or (10) that Bishop’s discs depend smoothly on deformations of the almost complex structure \( J \).

### 3.3 Parametrization of discs

Let \( p \) be a point of a generic submanifold \( E \) in an almost complex manifold \((M, J)\). Fix local coordinates so that \( E \) is defined by (4) and \( p = 0 \). Our goal is to describe the solutions \( \tilde{Z} \) of the generalized Bishop equation (7) satisfying the condition \( \Phi^{-1}_J(\tilde{Z})(1) = 0 \). Our argument is similar to [5].

Let \( U \) be a neighborhood of the origin in \( \mathbb{R} \), \( X' \) a sufficiently small neighborhood of the origin in the Banach space \((C^k(\mathbb{D}) \cap O(\mathbb{D}))^m \) (with positive non-integral \( k > 1 \)), \( X'' \) a neighborhood of the origin in the Banach space \((C^k(b\mathbb{D}))^m \). If \( \tilde{z} \in X' \), \( \tilde{z} : \zeta \mapsto \tilde{z}(\zeta) \) and \( \tilde{\omega} \in X'' \), \( \tilde{\omega} : \zeta \mapsto \tilde{\omega}(\zeta) \) are holomorphic discs, then we denote by \( \tilde{Z} \) the holomorphic disc \( \tilde{Z} = (\tilde{z}, \tilde{\omega}) \).

Set \( X = X' \times X'' \). Given \( \tilde{Z} \in X \subset (C^k(\mathbb{D}) \cap O(\mathbb{D}))^n \), we put \( (z_\delta, w_\delta) = \Phi^{-1}_{J,\delta}(\tilde{Z}) \) and consider the map of Banach spaces \( R : X \times U \longrightarrow Y \times \mathbb{R}^m \times \mathbb{C}^{n-m} \) defined as follows:
\[ R : (\tilde{z}, \tilde{\omega}, \delta) \mapsto \left( \rho_\delta(\Phi^{-1}_{J,\delta}(\tilde{Z}))(\bullet), y_\delta(1), w_\delta(1) \right). \]

Let \( \phi \) be a \( C^{2k} \)-map between two domains in \( \mathbb{R}^n \) and \( \mathbb{R}^m \); it determines a map \( \omega_\phi \) acting by composition on \( C^k \)-smooth maps \( g \) into the source domain: \( \omega_\phi : g \mapsto \phi(g) \). The well-known fact is that \( \omega_\phi \) is a \( C^k \)-smooth map between the corresponding spaces of \( C^k \)-maps. In our case this means that the map \( R \) is of class \( C^k \).
Lemma 3.2 The tangent map

\[ DR := D_X R(0,0,0) : X \rightarrow Y \times \mathbb{R}^m \times \mathbb{C}^{n-m} \]

(the partial derivative with respect to the space \( X \)) is surjective.

Proof: This is easy to see that the map \( DR \) defined by the equality

\[ DR(\hat{Z}) = (\text{Re}\ \hat{z}_1, ..., \text{Re}\ \hat{z}_m, \hat{y}(1), \hat{w}(1)) \]

for any \( \hat{Z} = (\hat{z}, \hat{w}) \in X \). Recall that the Hilbert transform \( H \) can be defined as a singular integral operator

\[ Hu(\zeta) = \frac{1}{2\pi i} (p.v.) \int_0^{2\pi} \frac{e^{i\theta} + \zeta}{e^{i\theta} - \zeta} u(e^{i\theta}) d\theta. \]

The operator \( H \) is a bounded linear operator on \( C^k(b\mathbb{D}) \) for any non-integral positive \( k \). Set \( H_1u = Hu - (Hu)(1) \) so that \( (H_1u)(1) = 0 \). Now for given \( u \in (C^k(b\mathbb{D}))^m \), given \( c \in \mathbb{R}^m \) and \( q \in \mathbb{C}^{n-m} \) we set \( \hat{z} = u + iH_1u + ic \) and denote by \( \hat{w} \) a holomorphic function satisfying \( \hat{w}(1) = q \). Then \( DR(\hat{z}, \hat{w}) = (u, c, q) \) so the map \( DR \) is surjective. Moreover, its kernel is canonically isomorphic to the space \( (C^k(\mathbb{D}) \cap \mathcal{O}_1(\mathbb{D}))^{n-m} \) of holomorphic functions \( \hat{w} \) of class \( C^k(\mathbb{D}) \) satisfying \( \hat{w}(1) = 0 \).

Now given \( \hat{w} \) we look for a solution \( \hat{Z} \) of the equation (7). By the implicit function theorem there exist \( \delta_0 > 0 \), a neighborhood \( V_1 \) of the origin in \( X' \), a neighborhood \( V_2 \) of the origin in \( (C^k(\mathbb{D}) \cap \mathcal{O}_1(\mathbb{D}))^{n-m} \), and a \( C^k \) smooth map \( G : V_2 \times [0, \delta_0] \rightarrow V_1 \) such that for every \( (\hat{w}, \delta) \in V_2 \times [0, \delta_0] \) the holomorphic function \( \hat{Z} = G(\hat{w}, \delta)(\bullet) \) is the unique solution of generalized Bishop’s equation (7) belonging to \( V_1 \times V_2 \) and satisfying the condition \( \Phi_{J_{\delta}^1}(\hat{Z})(1) = 0 \).

Now, the pullback \( Z = \Phi_{J_{\delta}^1}(\hat{Z}) \) gives us a \( J_{\delta} \)-holomorphic disc attached to \( E_{\delta} \) and satisfying \( Z(1) = 0 \). Thus, the initial data consisting of \( \hat{w} \in (C^k(\mathbb{D}) \cap \mathcal{O}_1(\mathbb{D}))^{n-m} \) define for each small \( \delta \) a unique \( J_{\delta} \)-holomorphic disc \( Z_0 = Z(\hat{w})(\bullet) \) attached to \( E_{\delta} \) and satisfying \( Z_0(1) = 0 \). We will call this disc \( Z_0 \) the lift of \( \hat{w} \) attached at \( p = 0 \) (we write \( Z_p(\hat{w}) \) for an arbitrary \( p \)). Since the almost complex structures \( J \) and \( J_{\delta} \) are biholomorphic via isotropic dilations, we can give the following description of local solutions of Bishop’s equation (the problem (RH)).

Proposition 3.3 The set \( \mathcal{A}_p^J \) of \( J \)-holomorphic Bishop discs satisfying \( Z(1) = p \) is a Banach manifold and its tangent space at \( Z \equiv 0 \) is canonically isomorphic to \( (C^k(\mathbb{D}) \cap \mathcal{O}_1(\mathbb{D}))^{n-m} \). Moreover, the parametrization map \( (C^k(\mathbb{D}) \cap \mathcal{O}_1(\mathbb{D}))^{n-m} \times E \ni (\hat{w}, p) \mapsto Z_p[\hat{w}] \) is smooth.

The proof follows from the above analysis of the Bishop equation. One merely fixes some value of \( \delta \), \( 0 < \delta \leq \delta_0 \) and observes that the families of Bishop discs corresponding to distinct values of \( \delta \neq 0 \) are taken into one another by the corresponding dilations.
In what follows we say that a $J$-holomorphic Bishop disc $Z$ of $E$ satisfying $Z(1) = p$ is attached at $p \in E$.

The representation (10) of Bishop’s equation is convenient since the structure $J$ and defining functions of $E$ appear explicitly in the coefficients of integral equations. This implies immediately that solutions depends smoothly on deformations of the structure. In particular, if $E_1$ and $E_2$ are $C^{2k}$-close submanifolds of $\mathbb{C}^n$ defined by equations of the form (4) and $J_1$ and $J_2$ are $C^{2k}$ close almost complex structures, then there exists a (locally defined) diffeomorphism between the corresponding $C^k$ Banach submanifolds of Bishop discs $A_{01}^j$ and $A_{02}^j$ that depends smoothly on the pairs $E_j$, $J_j$, $j = 1, 2$, and is the identity in the case of equal pairs.

Moreover, if we apply the implicit function theorem to the equation (10) we obtain that every Bishop discs is uniquely determined by the initial data $Z_0$ and $t(\zeta)$. Thus, we obtain another parametrization of the manifold $A_j^p$.

4 Filling by pseudoholomorphic Bishop discs: pseudo-convex and finite type cases

In this section we consider the case where $E$ is a real hypersurface. The following simple statement will be crucially used.

**Proposition 4.1** Suppose that there exists $Z \in A_p^j$ such that the normal derivative vector $dZ(\partial_{\text{Re} \zeta}|_1)$ is not tangent to $E$ at $p$ (that is the Bishop disc $Z$ is attached to $E$ at $p$ transversally). Then Bishop discs from $\cup_{q \in E} A_q^j$ of $E$ fill a one-sided neighborhood of $p \in E$.

**Proof:** We can assume that $Z$ is the lift of $\hat{w} \in (C^k(\mathbb{D}) \cap O_1(\mathbb{D}))^{n-m}$ (defined in the proof of Proposition 3.3). It follows by Proposition 3.3 that for every point $q \in E$ in a neighborhood of $p$ there exists a disc $Z_q(\hat{w})$ (the lift of $\hat{w}$ attached to $E$ at $q$) which is a small deformation of $Z$. So the normals $Z_q([0, 1])$ of these discs at $1$ fill a one-sided neighborhood of $p$. This completes the proof.

**Remark.** If $Z$ is a Bishop disc attached to $E$ at $p = 0$, the tangent space $T_Z(A_p^j)$ of the manifold of Bishop discs attached at $p$ consists of maps $\hat{Z} : \mathbb{D} \rightarrow \mathbb{C}^n$ with $\hat{Z}(1) = 0$ satisfying a Riemann-Hilbert type system of the form

\[
\begin{cases}
\hat{Z} + a(\zeta)\overline{\hat{Z}} + b(\zeta)\hat{Z} + c(\zeta)\overline{\hat{Z}} = 0, \quad \zeta \in \mathbb{D}, \\
\text{Re}(\partial \rho(Z)\hat{Z}) = 0, \quad \zeta \in b\mathbb{D},
\end{cases}
\]

where the matrix valued coefficients $a$, $b$, $c$ involve $\overline{Z}$ and the values of the matrix $A$ and its derivatives $A_Z$ and $A_{\overline{Z}}$ on the disc $Z$. This system is obtained by varying the Bishop equation. The solutions $\hat{Z}$ of this problem are called the infinitesimal variations (or infinitesimal perturbations) of the disc $Z$. In general the study of the above boundary...
problem is quite complicated. Fortunately, in the present paper we deal with small Bishop discs only. This implies that the above Bishop disc $Z$ can be chosen small enough. Therefore, the above Riemann-Hilbert problem can be viewed as an arbitrarily small perturbation of the usual $\bar{\partial}$-equation on the unit disc with the boundary condition $\text{Re}(\partial \rho(0) \bar{Z}) = 0$. In particular the corresponding linear operator is surjective in suitable functional spaces and the above linearized equation indeed determines the tangent space to the manifold $A^J_p$. A neighborhood of the disc $Z$ in $A^J_p$ is in a one-to-one correspondence with a neighborhood of the origin in this tangent space by the implicit function theorem. The last proposition means that if the space of discs $\dot{Z}$ satisfying the above conditions contains a disc with the normal derivative vector at 1 transversal to $E$, then Bishop disc fill one-sided neighborhood of $E$. In this section we study the above linear Riemann-Hilbert problem by geometric tools: using suitable non-isotropic dilations of coordinates we represent it as a small deformation of the corresponding Riemann-Hilbert problem for the usual $\bar{\partial}$-operator. An analytic study of this problem is postponed to section 5.

We will need the following statement.

**Proposition 4.2** Let $E$ be a real hypersurface in an almost complex manifold $M$ with an integrable structure $J_0$. Assume that $E$ contains no complex hypersurfaces. Then for any almost complex structure $J$ close enough to $J_0$ in the $C^k$, $k > 2$, norm the $J$-holomorphic Bishop discs of $E$ fill a one-sided neighborhood of every point of $E$.

**Proof:** According to [10] the manifold $A^J_p$ contains a disc transversal to $E$ at $p \in E$. Since the manifolds $A^J_p$ depend smoothly on $J$, they also contain a transversal disc if $\| J - J_0 \|_{C^k}$ is small enough.

Of course, a similar statement remains true if $E$ is a generic submanifold in $(M, J_0)$ minimal in the sense of [10].

### 4.1 Pseudoconvex hypersurfaces

As a first consequence we obtain the following

**Proposition 4.3** Let $\rho$ be a $J$-plurisubharmonic function of class $C^2$ on an almost complex manifold $(M, J)$ of complex dimension 2 and the hypersurface $E = \rho^{-1}(0)$ contains no $J$-holomorphic discs (we assume $d\rho \neq 0$ on $E$). Then there exists a neighborhood $U$ of $E$ such that the Bishop discs of $E$ fill $U \cap \{\rho < 0\}$.

**Proof:** Consider a (sufficiently small) Bishop disc $Z \in A^J_p$. Then $\rho \circ Z \leq 0$ on $\mathbb{D}$ by the maximum principle. But then $dZ(\partial \text{Re} \zeta|_1)$ is transversal to $E$ by the Hopf lemma. So we apply Proposition 4.1.

The above result remains true for arbitrary dimension (see [1] for the case of $\mathbb{C}^n$).
Proposition 4.4  Let $E$ be a real hypersurface defined as the zero set of a $J$-plurisubharmonic function $\rho$ of class $C^2$ in an almost complex manifold $(M, J)$. Suppose that there exists a complex tangent vector field $X$ on $E$ such that for any $p \in E$ there are no (germs of) $J$-holomorphic discs in $E$ tangent to $X(p)$. Then there exists a neighborhood $U$ of $E$ such that Bishop discs of $E$ fill $U \cap \{ \rho < 0 \}$.

Proof:  If the Banach manifold $A^J_p$ contains no transversal discs, by the Hopf lemma all the discs from $A^J_p$ are contained in $E$. Similarly to the previous section, fix local coordinates centered at $p$ and consider isotropic dilations $\Lambda_\delta$. Fix a point $q \in E_0$ close enough to $p$. Then the tangent vectors at the centers of Bishop discs (with respect to $J_{st}$) attached to $E_0$ at 0 fill a real sphere in the holomorphic tangent space $H^{J_{st}}_q(E_0)$. By continuity, a similar property holds for $\delta$ small enough so there is a point in $E_\delta$ admitting a germ of a $J_\delta$-holomorphic disc in any complex tangent direction and contained in $E_\delta$. This contradiction shows that the manifold $A^J_p$ contains a transversal disc and we apply Proposition 4.1.

As a corollary we obtain the following global version of this statement.

Corollary 4.5  Let $\rho$ be a $C^2$ plurisubharmonic function on an almost complex manifold. Suppose that for some $c \in \mathbb{R}$ its level set $E := \rho^{-1}(c)$ is a compact hypersurface of class $C^2$ which contains no germs of holomorphic discs. Then there exists a neighborhood $U$ of $E$ such that Bishop discs of $E$ fill $U \cap \{ \rho < c \}$.

4.2 Finite type hypersurfaces

Consider a real hypersurface $E$ in an almost complex manifold $M$ of complex dimension 2. By its type we mean the supremum of tangency order of $E$ with regular $J$-holomorphic discs at $p \in E$ (the properties of finite type hypersurfaces in almost complex manifolds are studied in the recent work [2]).

Proposition 4.6  If $E$ is of finite type, then its Bishop discs fill a one-sided neighborhood of every point of $E$.

Let an integral $m > 1$ be the type of $E$ at the origin. There exists a regular disc $Z$ such that

$$(\rho \circ Z)(\zeta) = O(|\zeta|^m).$$

(11)

We can choose local coordinates so that $Z(\zeta) = (0, \zeta)$ and the $J$-holomorphicity conditions for a map $(w, z) : \mathbb{D} \to \mathbb{C}^2$ have the form

$$u\overline{\zeta} + a(w, z)\overline{\zeta} = 0,$$

$$z\overline{\zeta} + b(w, z)\overline{\zeta} = 0,$$
where

$$a(\bullet, 0) = 0, \quad b(0, 0) = 0.$$  \hfill (12)

Indeed, we can identify $M$ with $\mathbb{C}^2$ equipped with an almost complex structure $J$ such that $J(0) = J_{st}$. Using the classical result of Nijenhuis-Woolf (see, for instance, [7]) we construct a foliation of a neighborhood of the origin in $\mathbb{C}^2$ by a family of $J$-holomorphic discs containing $Z$. Similarly, consider also a transversal foliation. After a local diffeomorphism the discs of these foliations become translations of coordinate axis. In this system of coordinates the above equations hold and $Z$ has the above form. Moreover, the matrix $J$ becomes diagonal by blocks:

$$J = \begin{pmatrix} P & 0 \\ 0 & S \end{pmatrix},$$

where $P$ and $S$ are real $2 \times 2$-matrices satisfying $P(0) = S(0) = J_{st}.$

In view of condition (11) the Taylor expansion of $r$ in these coordinates has the form

$$\rho(w, z) = \Re z + \Re \sum_{j=1}^{m} q_{j}(w) z^{j} + p_{m}(w, \overline{w}) + o(|(w, z)|^{m}),$$

where $p_{m}$ is a non-zero homogeneous polynomial of degree $m$ in $w, \overline{w}$ and $q_{1}(0) = 0.$

**Lemma 4.7** The polynomial $p_{m}$ contains at least one non-zero non-harmonic term.

**Proof:** We need the following statement due to Ivashkovitch-Rosay [4].

**Proposition 4.8** Let $m \geq 1$ integer and $0 < \alpha < 1$. Let $J$ be a $C^{m+\alpha}$ almost complex structure defined in a neighborhood of $0$ in $\mathbb{R}^{2n}$. If $\phi : \mathbb{D} \to (\mathbb{R}^{2n}, J)$ is a smooth map such that $\overline{\partial} J \phi(\zeta) = o(|\zeta|^{m-1}),$ then there exists a $J$-holomorphic map $u$ from a neighborhood of $0$ in $\mathbb{C}$ into $\mathbb{R}^{2n}$ such that $|(u - \phi)(\zeta)| = o(|\zeta|^{m}).$

Suppose by contradiction that $p_{m}(w, \overline{w}) = (h_{m}w^{m} + \overline{h_{m}w^{m}})/2.$ Consider the map $\phi(\zeta) = (\zeta, -h_{m}\zeta^{m}).$ In view of (12) it satisfies the hypothesis of the last proposition so there exists a $J$-holomorphic disc $U$ of the form $u = (\zeta, -h_{m}\zeta^{m}) + o(|\zeta|^{m}).$ Then $\rho \circ u = o(|\zeta|^{m})$ which contradicts the condition that the type of $E$ at the origin is equal to $m.$ This proves the lemma.

Now we are able to finish the proof. For $\delta > 0$ consider the non-isotropic dilations $\Lambda_{\delta} : (w, z) \mapsto (\delta^{-1/m}w, \delta^{-1}z)$. Since in our system of coordinates $J$ is given by a real $(4 \times 4)$-matrix diagonal by $(2 \times 2)$-blocks, the structures $J_{\delta} := (\Lambda_{\delta})_{*}(J)$ tend to $J_{st}$ in $C^{\alpha}$-norm for any positive $\alpha$. The hypersurfaces $E_{\delta} := \Lambda_{\delta}(E)$ tend to the hypersurface $\tilde{E} = \{\Re z + p_{m}(w, \overline{w}) = 0\}$ which is of finite type in view of the last lemma. So we can apply Proposition 4.2.
We also point out that a Bishop disc transversally attached to $\tilde{E}$ easily can be given quite explicitly without using the general theorem of [10]. Consider a holomorphic function of the form $w(\zeta) = e^{i\theta}(\zeta - 1)$ where a value of the parameter $\theta \in [0, 2\pi]$ will be chosen later. After a standard biholomorphic change of coordinates we can assume that the polynomial $p_m$ contains no harmonic terms, so that $p_m = \sum_{k=1}^{m-1} q_k w^m w^{m-k}$ so that for $\zeta \in \partial \mathbb{D}$ we have

$$h(\zeta) := (p_m \circ w)(\zeta) = (\zeta - 1)^m \sum_{k=1}^{m-1} (-1)^{m-k} e^{i(2k-m)\theta} q_k \zeta^{k-m}.$$ 

Let $z$ be a holomorphic function on $\Delta$ such that $\text{Re } z|_{\partial \mathbb{D}} = h$ and $z(1) = 0$. The normal derivative $(\text{Re } z)_\nu(1)$ at 1 (that is the derivative with respect to $\text{Re } \zeta$) of $\text{Re } z$ is given by the derivation of the Poisson integral (since $h(1) = 0$):

$$(\text{Re } z)_\nu(1) = (p.v.) \frac{1}{\pi} \int_0^{2\pi} \frac{h(e^{i\tau})d\tau}{|e^{i\tau} - 1|^2} = (p.v.) \frac{i}{\pi} \int_{\zeta \in \partial \mathbb{D}} \frac{h(\zeta)}{(\zeta - 1)^2} d\zeta.$$ 

By the Cauchy residue theorem we obtain that the last integral is equal to $\sum_{k=1}^{m-1} q_k \alpha_k e^{i(2k-m)\theta}$ where $\alpha_k$ are some non-zero constants coinciding up to the signs with certain binomial coefficients. But this expression does not vanish after a suitable choice of $\theta$. So for any $\varepsilon > 0$ the map $\zeta \mapsto (\varepsilon w(\zeta), \varepsilon^m z(\zeta))$ is an (arbitrarily small) Bishop disc transversally attached to $\tilde{E}$ at the origin.

If $E$ is a real analytic hypersurface in a real analytic almost complex manifold of complex dimension 2, then $E$ is of finite type if and only if it contains no $J$-holomorphic discs (by Nagano’s theorem [6]).

This implies our main Theorem 1.1 in the real analytic category.

**Corollary 4.9** Let $(M, J)$ be a real analytic almost complex manifold of complex dimension 2 and $E$ be a real analytic hypersurface in $M$. Assume that $E$ contains no $J$-holomorphic discs. Then the Bishop discs of $E$ fill a one-sided neighborhood of every point of $E$.

A suitable modification of this method leads to a similar result in arbitrary dimension. We say that a real hypersurface $E$ in an almost complex manifold $(M, J)$ is of finite type $m$ at point $p \in M$ if there exists a complex tangent vector $X$ to $E$ at $p$ such that any regular $J$-holomorphic disc tangent to $X$ to $E$ has the order of tangency with $E$ less or equal to $m$.

**Proposition 4.10** If $E$ is a real finite type hypersurface in an almost complex manifold $(M, J)$ (of an arbitrary dimension), then its Bishop discs fill a one-sided neighborhood of every point of $E$.

We can assume that local coordinates are chosen so that the disc $Z = (\zeta, 0, \ldots, 0)$ is $J$-holomorphic and $J(0) = J_{st}$.

Then $J = J_{st} + R$ where $R$ is $(2n \times 2n)$-matrix formed by $2 \times 2$-blocks $R_{kl}$ and $R(0) = 0$. Moreover, it follows from the $J$-holomorphicity of $Z$ that

$$R_{k1}(\zeta, 0, \ldots, 0) \equiv 0. \quad (13)$$
Furthermore, we can assume that $\rho(w^1, ..., w^{n-1}, z) = \text{Re} z + o(|(w, z)|)$ and $(\rho \circ Z)(\zeta) = p_m(\zeta, \zeta) + o(|\zeta|^m)$. Here $p_m$ is a homogeneous polynomial of degree $m$. Using the above Ivashkovich-Rosay proposition and repeating the former argument we obtain that $p_m$ contains a non-zero harmonic term, because the condition (13) implies that $\overline{\partial}_J$ vanishes with order $m-1$ on the disc $\zeta \mapsto (\zeta, 0, ..., 0, a\zeta^m)$.

Finally, consider the dilations $\Lambda_\delta(z) = (\delta^{-1/m} w^1, \delta^{-1/m} w^2, ..., \delta^{-1} z)$. Again using condition (13), we obtain that the structures $J_\delta = (\Lambda_\delta)_* (J)$ converge to the standard structure $J_{st}$ in any $C^\alpha$ norm. The hypersurfaces $E_{\delta} = \Lambda_\delta (E)$ converge to the hypersurface $E_0 : \text{Re} z + P_m(w^1, \bar{w}^1) = 0$ in $\mathbb{C}^n$ which of finite type with respect to $J_{st}$. So we conclude as above.

5 Analysis of the Bishop equation

Our goal now is to prove Theorem 1.1 stated in introduction without any additional hypothesis of pseudoconvexity or real analyticity type. Everywhere we suppose that $E = \{\rho = 0\}$ is a real $C^\infty$-smooth hypersurface in an almost complex manifold $(M, J)$ of complex dimension 2. Since the statements of our main results are local, we work in local coordinates similarly to the previous sections. However, for technical reasons it is more convenient to consider Bishop discs attached to $E$ at the point $(0, 1)$. We also assume that $J(0, 1) = J_{st}$. Since in the present section matrix computations will play a substantial role, we everywhere write vectors $Z$ of $\mathbb{C}^2$ as vector-columns. We will use the following notation:

$$\overline{\zeta} := \frac{\partial}{\partial \zeta}, \partial := \frac{\partial}{\partial \zeta}.$$  

Recall that a map $\zeta \mapsto Z(\zeta)$ from the unit disc $\mathbb{D}$ to $\mathbb{C}^2$ is $J$-holomorphic if and only if it satisfies the following system of equations:

$$\overline{\partial} Z - A(Z) \overline{\partial Z} = 0,$$

where the matrix valued function $A$ is defined by (3). We consider here only maps valued in a small enough neighborhood of the point $(0, 1)$.

Consider small embedded $J$-holomorphic Bishop discs attached to $E$ at the point $(0, 1)$; their existence is proved in Section 3. Given such a disc $Z_0$ there exists a local diffeomorphism such that in the new coordinates we have $Z_0(\zeta) = \left( \begin{array}{c} 0 \\ \zeta \end{array} \right)$, $\zeta \in \mathbb{D}$ (see Lemma 2.4). We denote again by $J$ the representation of our almost complex structure in the new coordinates, $J(0, 1) = J_{st}$.

Our main goal is to establish the following

**Proposition 5.1** Suppose that for the Bishop disc $Z_0(\zeta) = \left( \begin{array}{c} 0 \\ \zeta \end{array} \right)$, $\zeta \in \mathbb{D}$, we have $A \circ Z_0 = 0$ and $A_Z \circ Z_0 = 0$ and

$$\rho_z(0, 1) = 1, \rho(0, 1) = \rho_w(0, 1) = 0.$$

(15)
Suppose that the derivatives $A_Z$ and the second derivatives of $\rho$ are small on $Z_0(\mathbb{D})$. Suppose further that for every Bishop disc $Z(\zeta) = \begin{pmatrix} z(\zeta) \\ w(\zeta) \end{pmatrix}$ with $Z(1) = Z_0(1) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ close enough to $Z_0$ we have $\partial z(1) = 0$. Then the Levi form of $E$ with respect to $J$ vanishes on the set $Z_0(b\mathbb{D})$.

Theorem 1.2 is an immediate corollary of this statement.

**Remark 1.** The assumptions of smallness of the derivatives of $A$ and $\rho$ are automatically satisfied because the original disc is small and the change of coordinates stretches it to the unit disc. The normalization condition (15) also can be achieved by a suitable change of coordinates which does not affect previous assumptions (Lemma 2.4).

**Remark 2.** We point out that Proposition 5.1 employs all $J$-holomorphic discs close to $Z_0$, not just the infinitesimal perturbations of $Z_0$. In some sense it uses the second variation of the condition of tangency. Given a disc $Z_0$ of class $C^k$ for sufficiently big $k$, we prove the desired vanishing of the Levi form under the hypothesis that there are no $C^{k-m}$ transversal disc close to $Z_0$, where $m$ is an unimportant constant. The “loss of smoothness” is due to the normalization of $A$ (Lemma 2.4) and changes of variables in the linearized Bishop equation involving $A$ and $\rho$. It is unrelated to the class $C^{1+\alpha}$, in which we consider the infinitesimal perturbations in Section 5.3.

We provide a proof in the next three subsections. In Section 5.1 we introduce and simplify the linearized Bishop equation. In Section 5.2 we describe its solutions for every disc $Z$ close to $Z_0$. We use infinitesimal perturbations $\dot{Z}$ of $Z$ to show that if no transversal perturbation exists, then $Z$ satisfies a certain nonlinear integral equation $\Phi(Z) = 0$ involving $Z$ and its derivatives. We only use the infinitesimal perturbations $\dot{Z}$ parametrized by polynomials, hence they are just as smooth as the coefficients of the linearized Bishop equation, so no loss of smoothness occurs here. Finally in Section 5.3 we use infinitesimal perturbations of $Z_0$ of class $C^{1+\alpha}$ to prove the desired conclusion about the Levi form.

### 5.1 Linearized Bishop equation

Suppose the hypotheses of Proposition 5.1 are fulfilled. For simplicity we assume that $A$ and $\rho$ are $C^\infty$. Consider a Bishop disc $Z(\zeta) = \begin{pmatrix} z(\zeta) \\ w(\zeta) \end{pmatrix}$, $\zeta \in \mathbb{D}$, attached to $E$ at the origin and close enough to $Z_0$. Then it satisfies the following boundary problem for the $\overline{\partial}_J$-operator:

\[
\overline{\partial} Z = A \overline{\partial} Z,
\]

\[
\rho \circ Z|_{b\mathbb{D}} = 0.
\]

Since we only deal with small Bishop discs, we can assume that for any fixed $k > 0$ the norm $\|A\|_{C^k}$ is small enough.
Consider now solutions $\dot{Z}$ of the corresponding linearized problem:

$$
\partial \dot{Z} = A \partial Z + A \overline{\dot{Z}},
\quad \text{Re}(\rho_Z(Z)\dot{Z})|_{\partial \Omega} = 0.
$$

Here and in the rest of the paper for a map $F$ defined on the space of smooth discs $Z : \Omega \to \Omega'$, we denote by $\dot{F}$ the Frechet derivative of $F$. In particular, if $F$ is defined by a smooth function on a set in $\Omega'$, which is the case for $A$, then

$$
\dot{F}(Z)(\dot{Z}) = F_Z(Z)\dot{Z} + F_{\overline{Z}}(Z)\overline{\dot{Z}}.
$$

We call the solutions $\dot{Z}$ of the linearized system the \textit{infinitesimal perturbations of the disc $Z$}.

In order to eliminate $\partial \dot{Z}$ in the right-hand side of the linearized equation, we perform the following change of (dependent) variables $Z_1 := \dot{Z} - A \overline{\dot{Z}}$. Then $\dot{Z} = I'(Z_1 + A \overline{Z}_1)$, where $I' = (I - A A)^{-1}$ and $I$ denotes the identity matrix.

We have

$$
\partial Z_1 = \partial (\dot{Z} - A \overline{\dot{Z}}) = (\partial \dot{Z} - A \partial \overline{\dot{Z}}) - \partial A \overline{\dot{Z}},
\quad
\overline{\partial} Z_1 = \overline{\partial \dot{Z}} - \overline{\partial A \overline{\dot{Z}}},
\quad
\overline{\partial} A = A \overline{\partial \dot{Z}} + A \overline{\partial A \overline{\dot{Z}}} = (A \overline{\partial \dot{Z}} + A \overline{\partial A \overline{\dot{Z}}}).
$$

Since $A Z$ and $A \overline{Z}$ are 3-index quantities, the above products are not all matrix products, but rather tensor products suitably contracted. We don’t specify the exact meaning of each product because it won’t matter for our analysis. We leave the details to the reader. The same applies to several lines below. We get

$$
\partial Z_1 = A_1 Z_1 + A_2 \overline{Z}_1,
\quad
A_1 = (A Z I' + A \overline{Z} \overline{T} A) \partial \overline{\dot{Z}} - (A Z A + A \overline{Z}) \overline{\partial \dot{Z} T} A,
\quad
A_2 = (A Z I' A + A \overline{Z} \overline{T}) \partial \overline{\dot{Z}} - (A Z A + A \overline{Z}) \overline{\partial A \overline{\dot{Z}} T}.
$$

Consider the Frechet derivatives $\dot{A}_1$ and $\dot{A}_2$ at $Z_0$. The condition $A = 0$ implies $I' = 0$. Obviously $\dot{A}_1$ and $\dot{A}_2$ are linear combinations of $\dot{Z}$, $\overline{Z}$ and $\overline{\partial \dot{Z}}$ with smooth coefficients depending on $\zeta$, which we write in the form

$$
\dot{A}_\nu(Z_0) = 0 \mod (\dot{Z}, \overline{Z}, \overline{\partial \dot{Z}}), \quad \nu = 1, 2.
$$

For the disc $Z = Z_0$ we have $\partial Z = \partial Z_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. The expression for $A_1$ implies

$$
\dot{A}_1(Z_0) = 0 \mod (\dot{Z}, \overline{Z}).
$$

We now express the boundary condition $\text{Re}(\rho_Z \dot{Z}) = 0$ in terms of $Z_1$. We obtain

$$
\rho_Z I'(Z_1 + A \overline{Z}_1) + \rho_Z T(Z_1 + A \overline{Z}_1) = 0,
$$

for the disc $Z = Z_0$.
We put
\[ \lambda = (\lambda_1, \lambda_2) = \rho Z I' + \rho \overline{Z} T A. \]
Then the boundary condition for \( Z_1 \) turns into
\[ \Re (\lambda Z_1)|_{b\mathbb{D}} = 0. \]

For simplicity of notation we put
\[ \partial_1 f = \partial f(1). \]

We note that the condition \( \partial_1 \dot{z} = 0 \) fulfilled for all the discs implies \( \partial_1 \dot{z} = 0 \). For the change of variables \( \dot{Z} \mapsto Z_1 = \begin{pmatrix} z_1 \\ w_1 \end{pmatrix} \), the condition \( A(0,1) = 0 \) (recall \( J(0,1) = J_{st} \) ) implies that \( \partial_1 z_1 = \partial_1 \dot{z} \). Thus for every \( Z_1 \) the conditions
\[ \overline{\partial} Z_1 = A_1 Z_1 + A_2 \overline{Z}_1, \]
\[ Z_1(1) = 0, \]
\[ \Re (\lambda Z_1)|_{b\mathbb{D}} = 0 \]
imply \( \partial_1 z_1 = 0 \). Consider the matrix
\[ \Lambda = \begin{pmatrix} \lambda_1 & \lambda_2 \\ 0 & 1 \end{pmatrix}, \]
where \( \lambda = (\lambda_1, \lambda_2) \) is defined above. We use that \( \Lambda \) is smoothly extended on the disc \( \mathbb{D} \), although only the values on \( b\mathbb{D} \) will matter.

In order to simplify the boundary conditions, we further change the variable to \( V = \Lambda Z_1 \) with \( V = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \). We put
\[ B_1 = (\overline{\partial} \Lambda) \Lambda^{-1} + \Lambda A_1 \Lambda^{-1}, \]
\[ B_2 = \Lambda A_2 \Lambda^{-1}. \]

It follows from the hypotheses of Proposition 5.1 that \( B_1 \) and \( B_2 \) are small on \( Z_0 \), which we will need in Section 5.2. The new unknown \( V \) satisfies
\[ \overline{\partial} V = B_1 V + B_2 V, \]
\[ V(1) = 0, \]
\[ \Re v_1|_{b\mathbb{D}} = 0. \]

For every solution \( V \) we have
\[ \partial_1 v_1 = 0. \]

The expressions for \( \dot{A}_1 \) and \( \dot{A}_2 \) on the disc \( Z_0 \) imply
\[ \dot{B}_1 = (\overline{\partial} \dot{\Lambda}) \Lambda^{-1} \mod (\dot{Z}, \overline{\partial} \overline{Z}), \] (16)
\[ \dot{B}_2 = 0 \mod (\dot{Z}, \overline{\partial} \overline{Z}). \] (17)

We will need this information about the derivatives \( \dot{B}_\nu \) in Section 5.3.
5.2 Solving the boundary value problem: the generalized Schwarz integral

We derive an integral formula for solving a linear boundary Riemann-Hilbert problem for elliptic systems of PDE in the unit disc. Following [12], we call it the generalized Schwarz integral. Consider the following boundary value problem in $\mathbb{D}$:

$$
\overline{\partial}V = B_1 V + B_2 \overline{V},
$$

$$
\text{Re} \, V \big|_{\partial \mathbb{D}} = \Gamma \ (\Gamma(1) = 0),
$$

$$
V(1) = 0.
$$

Recall the Cauchy integral $K$ and the Cauchy-Green transform $T$ are defined in the previous sections by (8) and (6). We also define the operators:

$$
K_1 u := K u - (K u)(1)
$$

and

$$
T_1 u := T u - (T u)(1).
$$

We also need the following operators:

$$
T^*_1 u = T^* u - (T^* u)(1),
$$

where

$$
T^*_1 u(\zeta) = \frac{1}{2\pi i} \int \int_{\mathbb{D}} \frac{\zeta u(\tau)d\tau \wedge d\overline{\tau}}{1 - \zeta \overline{\tau}}.
$$

The operator $T^*$ satisfies the following identities:

$$
T^* = -K T, \ T^*_1 = -K_1 T_1,
$$

which one can verify directly.

Then by the Schwarz-Green formula:

$$
V = 2K_1 \text{Re} \, V + T_1 \overline{\partial} V + T^*_1 \overline{\partial} V + V(1).
$$

Set $V_0 = 2K_1 \Gamma$. Then $V$ satisfies the following integral equation

$$
V = V_0 + T_1(B_1 V + B_2 \overline{V}) + T^*_1(B_1 \overline{V} + \overline{B_2} V).
$$

Consider the operators

$$
L_1 = T_1 \circ B_1 + T^*_1 \circ \overline{B_2},
$$

$$
L_2 = T_1 \circ B_2 + T^*_1 \circ \overline{B_1}.
$$
where $B_j$ are viewed as operators of left multiplication by the matrix functions $B_j$. Then $V$ satisfies the following equation:

$$V = V_0 + L_1V + L_2\overline{V}.$$  

We solve this equation using that $L_1$ and $L_2$ are small because so are $B_1$ and $B_2$. Then the solution has the form

$$V = V_0 + R_1V_0 + R_2\overline{V}_0,$$

where the resolvent operators $R_j$ are $\mathbb{C}$-linear. We now find an explicit expression for the resolvents $R_j$. We have

$$V_0 + R_1V_0 + R_2\overline{V}_0 = V_0 + L_1(V_0 + R_1V_0 + R_2\overline{V}_0) + L_2(\overline{V}_0 + R_1\overline{V}_0 + R_2\overline{V}_0).$$

Then (we temporarily drop the index 0 of $V_0$)

$$R_1V = L_1(V + R_1V) + L_2\overline{R}_2V,$$

$$R_2\overline{V} = L_1R_2\overline{V} + L_2(\overline{V} + \overline{R}_1\overline{V}),$$

so that

$$R_1 = L_1 + L_1R_1 + L_2\overline{R}_2,$$

$$R_2 = L_1R_2 + L_2(I + \overline{R}_1).$$

Substituting the equality

$$R_2 = (I - L_1)^{-1}L_2(I + \overline{R}_1)$$

into the expression for $R_1$, we obtain

$$R_1 = [I - L_1 - L_2(I - \overline{L}_1)^{-1}\overline{L}_2]^{-1}[L_2(I - \overline{L}_1)^{-1}\overline{L}_2 + L_1].$$

Since $(1 - x)^{-1}x = (1 - x)^{-1} - 1$, we can rewrite the above expression in the form

$$R_1 = [I - L_1 - L_2(I - \overline{L}_1)^{-1}\overline{L}_2]^{-1} - I.$$  

Hence $R_j$ are bounded operators $C^m \rightarrow C^{m+1}$ for any non-integral $m > 0$. We use the expression for $R_1$ to make some simple observations. Note that

$$R_1 = L_1 + O(2),$$

where $O(2)$ is a sum of products of 2 or more operators $L_1$, $L_2$, $\overline{L}_1$, $\overline{L}_2$. It will turn out that the $O(2)$ term is unimportant, but it will take considerable work. Furthermore, one can see that every term in $R_1$ starts with $L_1$ or $L_2$ and ends with $L_1$ or $\overline{L}_2$. Recalling the expressions for $L_1$ and $L_2$ we conclude that $R_1$ is a sum of terms of the form

$$P_0 \circ F_0 \circ P_1 \circ F_1 \circ ..., \circ P_n \circ F_n,$$

$24$
where each $P_j \in \{ T_1, T_1^*, T_1^\ast \}$ and $F_j \in \{ B_1, B_1, B_2, B_2 \}$. Moreover, $P_0 \in \{ T_1, T_1^* \}$ and $F_n \in \{ B_1, B_2 \}$.

We now interpret the condition $\partial_1 v_1 := \partial v_1(1) = 0$. The boundary conditions we derived in Section 5.1 imply $\Gamma = \begin{pmatrix} 0 \\ \gamma \end{pmatrix}$, where $\gamma$ is an arbitrary real function with $\gamma(1) = 0$.

Then $V_0 = \begin{pmatrix} 0 \\ \varphi \end{pmatrix}$, where $\varphi$ is an arbitrary holomorphic function $\varphi = 2K_1\gamma$. Then $v_1 = R_{12}^{12}\varphi + R_{12}^{12}\varphi$, where $R_{12}^{12}$ and $R_{12}^{12}$ are the (12) matrix entries of the matrix operators $R_1$ and $R_2$. The condition $\partial_1 v_1 = 0$ means

$$\partial_1 R_{12}^{12}\varphi + \partial_1 R_{12}^{12}\varphi = 0.$$ 

Since the first term is $\mathbb{C}$-linear in $\varphi$ and the second one is antilinear, we get

$$\partial_1 R_{12}^{12}\varphi = 0$$

for all $\varphi$ with $\varphi(1) = 0$.

Since every term in $R_1$ starts with $T_1$ or $T_1^*$, we need formulas for their derivatives at $\zeta = 1$. For every function $f$ of class $C^\alpha(I\bar{\mathbb{D}})$, $0 < \alpha < 1$, with $f(1) = 0$ we have

$$\partial_1 T_1 f = \partial_1 T f = \frac{1}{2\pi i} \int \int \frac{f(\tau)d\tau \wedge d\bar{\tau}}{(\tau - 1)^2},$$

$$\partial_1 T_1^* f = \partial_1 T^* f = \frac{1}{2\pi i} \int \int \frac{f(\tau)d\tau \wedge d\bar{\tau}}{\bar{(\tau - 1)}^2},$$

where the integrals converge in the usual sense.

We would like to interpret the condition $\partial_1 R_{12}^{12}\varphi = 0$ for all $\varphi$ as a moment condition and eliminate the arbitrary $\varphi$. In order to do so we need to reverse the order of integration in every term in $R_1$, which requires the following preparations. Let $P$ be a scalar integral operator with kernel $P(t, \tau)$, that is

$$Pf(\zeta) = \int \int \bar{\mathbb{D}} P(\zeta, \tau) f(\tau)d\tau \wedge d\bar{\tau}.$$ 

Define the operators $P^+$ and $P^-$ as the integral operators with the kernels

$$P^+(\zeta, \tau) = \frac{(\zeta - 1)^2}{(\tau - 1)^2} P(\tau, \zeta),$$

$$P^-(\zeta, \tau) = \frac{(\zeta - 1)^2}{\bar{(\tau - 1)}^2} P(\tau, \zeta).$$

We also put

$$\mu(\tau) = \frac{(\tau - 1)^2}{(\bar{\tau} - 1)^2}.$$ 

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Lemma 5.2 Let \( \varphi(\tau) = (\tau - 1)^2 \psi(\tau) \). We have
\[
\partial_1 TF_0 \circ P_1 \circ F_1 \circ \ldots \circ P_n \circ F_n \varphi = \frac{1}{2\pi i} \int \int_{D} P_n^+ (\ldots P_1^+ (F_0) F_1 \ldots) F_n \psi(\tau) d\tau \land d\bar{\tau},
\]
and the lemma follows.

\[
\partial_1 T^* F_0 \circ P_1 \circ F_1 \circ \ldots \circ P_n \circ F_n \varphi = \frac{1}{2\pi i} \int \int_{D} P_n^- (\ldots P_1^- (F_0) F_1 \ldots) F_n \psi(\tau) \mu(\tau) d\tau \land d\bar{\tau}.
\]

The proof follows by changing the order of integration.

Lemma 5.3 (Moment conditions.) Let \( F \in L^\infty(D) \). Then
\[
TF|_{\partial D} = 0
\]
if and only if for every holomorphic polynomial \( f \) we have
\[
\int \int_{D} F(\tau) \varphi(\tau) d\tau \land d\bar{\tau} = 0.
\]

Proof: The series
\[
\frac{1}{\tau - \zeta} = - \sum_{n=0}^{\infty} \frac{\tau^n}{\zeta^{n+1}}
\]
converges in \( L^1(D) \) for every fixed \( \zeta \in \partial D \). Integrating the series against \( F(\tau) \) yields
\[
TF(\zeta) = - \sum_{n=0}^{\infty} \frac{c_n}{\zeta^{n+1}}, \quad \zeta \in \partial D,
\]
where \( c_n = \frac{1}{2\pi i} \int \int_{D} F(\tau) \tau^n d\tau \land d\bar{\tau}, \)
and the lemma follows.

Lemma 5.4 Let \( f \in C^\gamma(D), 0 < \gamma < 1, f(1) = 0 \). Let \( |g| \leq C \) be bounded and
\[
|\partial g(\tau)| \leq \frac{C}{|\tau - 1|}, \quad |\bar{g} g(\tau)| \leq \frac{C}{|\tau - 1|}, \quad \tau \in D \setminus \{1\}.
\]
Then \( fg \in C^\gamma(D) \) and \( \| fg \|_{C^\gamma} \leq C' C \| f \|_{C^\gamma} \) for some constant \( C' \). For \( \gamma = 1 \) lemma holds in \( \operatorname{Lip}_1(D) \).

Proof: Put \( \Delta := (gf)(\zeta_1) - (gf)(\zeta_2) = (f(\zeta_1) - f(\zeta_2))g(\zeta_1) + f(\zeta_2)g(\zeta_1) - g(\zeta_2), \) \( \zeta_1 - \zeta_2 = \delta. \) Without loss of generality we can assume that \( |\zeta_2 - 1| \leq |\zeta_1 - 1| \). Then
\[
|\Delta| \leq C \| f \|_{C^\gamma} \delta \gamma + \| f \|_{C^\gamma} |\zeta_2 - 1| \gamma \min \left( 2C, \frac{\pi C \delta}{|\zeta_2 - 1|} \right) \leq \pi \delta,
\]
where \( \pi \) arises because we may have to connect \( \zeta_1 \) and \( \zeta_2 \) by going around 1.

Now consider the two cases separately: \( \delta \leq |\zeta_2 - 1| \) and \( \delta \geq |\zeta_2 - 1| \). In both cases we get \( |\Delta| \leq C' C \| f \|_{C^\gamma} \delta \gamma, \) which proves the lemma.

Lemma 5.5 If \( P \) is one of the operators \( T_1, T_1^*, T_1, T_1^* \), then \( P^+ \) and \( P^- \) are bounded operators \( C^\alpha(D) \to C^{1-\beta}(D) \) for every \( 0 < \alpha < 1, 0 < \beta < 1 \).
Proof: We directly obtain the following equalities:

\[ T^+_1 = -T_1, \quad (18) \]
\[ T^-_1 = -\mu^{-1}T_1\mu, \quad (19) \]
\[ T^{*+}_1 = -\mu T^{*-}_1. \quad (20) \]

By using the equality \( T^* = -K T \), we obtain

\[ T^{*+}_1 = \mu K_1 T_1, \quad (21) \]
\[ T^{*-}_1 = K_1 T_1 \mu. \quad (22) \]

We also need the following relation:

\[ T_1\mu(\zeta) = T\mu(\zeta) = \frac{\zeta - 1}{\zeta - 1}(1 - |\zeta|^2) \in \text{Lip}_1(I_D). \quad (23) \]

To obtain (23) we put \( \psi(\zeta) = -\frac{(\zeta - 1)^2}{\zeta - 1} \), then \( \partial_1 \psi = \mu \). By the Cauchy-Green formula we have

\[ T\mu = \psi - K\psi = \frac{\zeta - 1}{\zeta - 1}(1 - |\zeta|^2). \]

Since \((P)^+ = \overline{P^-}, (P)^- = \overline{P^+}\), it suffices to prove the lemma for \( P \in \{T_1, T^*_1\} \).

For \( T^+_1 \) by (18) we have \( T^+_1 : C^\alpha(\overline{D}) \rightarrow C^{1+\alpha}(\overline{D}) \). Consider \( T^-_1 \). Let \( u \in C^\alpha(\overline{D}) \). Then \( T_1(\mu u) = T_1(\mu u(1)) + T_1(\mu(u - u(1))) \). Then \( T_1(\mu u(1)) = u(1)T_1\mu \in \text{Lip}_1(\overline{D}) \) by (23). Then \( \mu(u - u(1)) \in C^{\alpha}(\overline{D}) \) by Lemma 5.4 and \( T_1(\mu(u - u(1))) \in C^{1+\alpha}(\overline{D}) \). Therefore \( T_1(\mu u) \in \text{Lip}_1(\overline{D}) \) and \( T^-_1 u = -\mu^{-1}T_1(\mu u) \in \text{Lip}_1(\overline{D}) \subset C^{1-\beta}(\overline{D}) \) by Lemma 5.4. The operators \( T^+_1 \) and \( T^-_1 \) are treated similarly by (21) and (22). This completes the proof of the lemma.

The property \( \partial_1 R^{12} \varphi = 0 \) for all holomorphic functions \( \varphi \) with \( \varphi(1) = 0 \), takes the form

\[ \partial_1[T_1 \circ B_1 + T^*_1 \circ \overline{B}_2 + \sum_{n \geq 1} P_0 \circ F_0 \circ \cdots \circ P_n \circ F_n]^{12} \varphi = 0, \]

where the summation includes all the terms in \( R_1 \), so \( P_0, \ldots, P_n \in \{T_1, T^*_1, \overline{T}_1, \overline{T}^*_1\} \), \( P_0 \in \{T_1, T^*_1\} \), \( F_n \in \{B_1, \overline{B}_2\} \) and (12) denotes the corresponding matrix entry. By Lemma 5.2 this condition is equivalent to

\[ \int \int_{\overline{D}} \left[ B_1 + \overline{B}_2 \mu(\tau) + \Sigma' + \Sigma'' \right]^{12} \psi(\tau) \, d\tau \wedge d\overline{\tau} = 0, \]

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Similarly, we define bounded operators
\[ \phi \in O \]
where \( \phi \) is an arbitrary holomorphic function. The terms in \( \Sigma' \) and \( \Sigma'' \) correspond to \( O(2) \) in \( R_1 \). They split into \( \Sigma' \) and \( \Sigma'' \) depending whether the term in \( R_1 \) starts with \( T_1 \) or \( T_1^* \). The series converges in \( C^{1-\beta}(\overline{D}) \).

By Lemma 5.3 we further obtain
\[
\Phi(Z) := T \left[ B_1 + \overline{D}_2 \mu(\tau) + \Sigma' + \Sigma'' \right]_{|bID} = 0. \tag{24}
\]
This property holds for every Bishop disc \( Z \) close enough to \( Z_0 \) with \( Z(1) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \) provided that for every polynomial \( \psi \) the corresponding infinitesimal perturbation \( \dot{Z} \) of \( Z \) is tangent to \( E \) at \( \zeta = 1 \).

### 5.3 Frechet derivative of \( \Phi \)

In this section we consider the map \( \Phi \) defined by (24). This map is defined in a neighborhood of the disc \( Z_0 \) in the Banach manifold \( \mathcal{A}_p^j(E) \) of \( J \)-holomorphic Bishop discs attached to \( E \) at \( p = (0,1) \). The condition (24) means that in fact the map \( \Phi \) vanishes identically, so its Frechet derivative \( \dot{\Phi} \) at \( Z_0 \) does. We will study the geometric consequences of the equality
\[
\dot{\Phi}(Z_0)(\dot{Z}) = 0, \quad \dot{Z} \in T_{Z_0} \mathcal{A}_p^j(E).
\]
According to the previous subsection, due to the special normalization of the matrix \( A \) along the disc \( Z_0 \), we have
\[
\dot{Z} = A^{-1}V
\]
where \( \varphi \) is an arbitrary holomorphic function in \( \mathbb{D} \) with \( \varphi(1) = 0 \).

This allows to consider \( \dot{Z} \) and therefore \( \Phi \) as \( \mathbb{R} \)-linear operators applied to a function \( \varphi \in \mathcal{O}(\mathbb{D}) \cap C^{1+\alpha}(\overline{\mathbb{D}}), \ 0 < \alpha < 1 \).

For the target spaces, given integral \( m \geq 0 \) and \( 0 < \alpha < 1 \) we introduce the spaces \( C^{m+\alpha}_1(\mathbb{D}) \) and \( C^{m+\alpha}_1(b\mathbb{D}) \) as spaces of functions which are \( C^m \) except at 1. More precisely we say that \( f \in C^{m+\alpha}_1(\mathbb{D}) \) if \( f \in L^\infty(\mathbb{D}) \) and for every \( \varepsilon > 0 \) we have \( \int_{\mathbb{D}\setminus B(1,\varepsilon)} f = C^{m+\alpha}(\mathbb{D}\setminus B(1,\varepsilon)) \), where \( B(1,\varepsilon) \) denotes the disc of radius \( \varepsilon \) centered at 1. We define \( C^{m+\alpha}_1(b\mathbb{D}) \) similarly. We do not introduce norms in these spaces, nevertheless, we say that \( P \) is a bounded linear operator \( C^{m+\alpha}_1(\mathbb{D}) \to C^{k+\beta}_1(\mathbb{D}) \) if for every \( \varepsilon > 0 \) there exists a constant \( C = C(\varepsilon) > 0 \) such that for every \( f \in C^{m+\alpha}_1(\mathbb{D}) \) we have
\[
\| Pf \|_{C^{k+\beta}(\mathbb{D}\setminus B(1,2\varepsilon))} + \| Pf \|_{L^\infty(\mathbb{D})} \leq C(\| f \|_{C^{m+\alpha}(\mathbb{D}\setminus B(1,\varepsilon))} + \| f \|_{L^\infty(\mathbb{D})}).
\]
Similarly, we define bounded operators \( C^{m+\alpha}_1(\mathbb{D}) \to C^{k+\beta}_1(\mathbb{D}) \) and when we have \( b\mathbb{D} \) in place of \( \mathbb{D} \).
Lemma 5.6 If \( P \in \{ T_1, T^*_1, T_1^*, T^*_1 \} \) then \( P^+ \) and \( P^- \) are bounded operators \( C^{m+\alpha}_1(\mathbb{D}) \to C^{m+\alpha}_1(\mathbb{D}) \) for all integral \( m \geq 0 \), \( 0 < \alpha < 1 \).

The lemma follows immediately by splitting integration

\[
\int \int_{\mathbb{D}} = \int \int_{\mathbb{D}\setminus B(1,\varepsilon)} + \int \int_{B(1,\varepsilon)}
\]

in the definition of each operator and the regularity of \( T \) and \( T^* \).

The Frechet derivative of \( \Phi \) has the form

\[
\dot{\Phi} = T \left[ \hat{B}_1 + \overline{B}_2 \mu(\tau) + \dot{\Sigma}' + \dot{\Sigma}'' \right]^{12} |_{\mathbb{D}} = 0.
\]

In the derivatives \( \dot{\Sigma}' \) and \( \dot{\Sigma}'' \) we distinguish the expressions

\[
\Psi' = \sum_{n \geq 1} P_n^+ (\ldots P_1^+(F_0)\ldots) F_n \quad \text{and} \quad \Psi'' = \sum_{n \geq 1} P_n^- (\ldots P_1^-(F_0)\ldots) F_n,
\]

in which \( F_n \) is not differentiated. Since \( F_n \in \{ B_1, B_2 \} \), then all other terms contain either \( \hat{B}_1 \) or \( \overline{B}_2 \), and we group them accordingly. Then

\[
\dot{\Phi} = T \left[ (I + a_1 + a_2 \mu) \hat{B}_1 + (b_1 + b_2 \mu) \overline{B}_2 + \Psi' + \mu \Psi'' \right]^{12} |_{\mathbb{D}} = 0.
\]

Here each matrix function \( a \in \{ a_1, a_2, b_1, b_2 \} \) has the form

\[
a = \sum P_n^\sigma (P_{n-1}^\sigma (\ldots P_1^\sigma (F_0)\ldots) F_{n-1}),
\]

where \( \sigma \) is + or −. (In each term in \( a \), the signs \( \sigma \) are the same, but different terms have different \( \sigma \), depending whether they come from \( \Sigma' \) or \( \Sigma'' \).) Then \( a \in C^{1-\beta}_1 \) and the norms of \( a_j, j = 1, 2 \), are small.

Furthermore, actually \( a \in C^{m+\alpha}_1(\mathbb{D}) \) for all \( m \geq 0 \), \( 0 < \alpha < 1 \). Indeed, since \( F_j \) are smooth, then each term in \( a \) is \( C^{m+\alpha}_1 \) for all \( m \) and \( \alpha \) by Lemma 5.6. To show that \( a \in C^{m+\alpha}_1 \) we split the terms containing \( m \) or more operators \( P_j^\sigma \) into finitely many groups of the form

\[
P_n^\sigma (P_{n-1}^\sigma (\ldots P_{n-m+1}^\sigma (\ldots) F_{n-m+1} \ldots) F_{n-1}).
\]

Each group consists of the terms with the same \( P_n^\sigma, \ldots, P_{n-m+1}^\sigma \) and \( F_{n-m+1}, \ldots, F_{n-1} \). Then the inside summation is of class \( C^{1-\beta}_1 \) by Lemma 5.5, hence each group is \( C^{m+1-\beta}_1 \) by Lemma 5.6. We say that \( a \in C^{\infty}_1(\mathbb{D}) \).

Since \( F_j \in \{ B_1, B_2, \hat{B}_1, \overline{B}_2 \} \) and \( \hat{Z} \in C^{1+\alpha} \), we have \( \dot{F}_j \in C^{\alpha} \) (because the expressions for \( \hat{B}_1, \hat{B}_2 \) include the term \( \dot{\Sigma} \)).

By Lemma 5.6 the terms \( \Psi' \) and \( \Psi'' \) define bounded operators \( C^{1+\alpha}_1(\mathbb{D}) \to C^{1+\alpha}_1(\mathbb{D}) \). Hence \( T \Psi' \) defines a bounded operator \( C^{1+\alpha}_1(\mathbb{D}) \to C^{2+\alpha}_1(\mathbb{D}) \). The same is true for \( T(\mu \Psi'') \) because the multiplication by \( \mu \) does not change the class \( C^{m+\alpha}_1(\mathbb{D}) \).

The next step is the following
Lemma 5.7  The term \( T[(b_1 + b_2\mu)\overline{B_2}] \) defines a bounded linear operator \( C^{1+\alpha}(\mathbb{D}) \rightarrow C^{2+\alpha}_1(b\mathbb{D}) \).

This simple result applies to all matrix elements, including (12)-entry. By (17)

\[
\overline{B_2} = 0 \mod (\dot{Z}, \overline{Z}, \partial \dot{Z}).
\]

The operator \( T \) takes the terms with \( \dot{Z} \) and \( \overline{Z} \) to \( C^{2+\alpha}_1(b\mathbb{D}) \). Now since \( \dot{Z} = \Lambda^{-1}V_0 + (S) \), where \( (S) \) denotes smoother terms, it suffices to consider \( T(b\partial\varphi) \), where \( b \in C^{\infty}_1 \). We have

\[
T(b\partial\varphi) = [T, b] \partial\varphi + bT(\partial\varphi),
\]

where \([T, b] := T \circ b - b \circ T\) denotes the commutator of two operators. In order to complete the proof of Lemma 5.7, we need the following two simple lemmas.

Lemma 5.8  Let \( b \in C^{\infty}_1(\mathbb{D}) \). Then \([T, b]\) defines a bounded operator \( C^m(\mathbb{D}) \rightarrow C^{m+2}_1(b\mathbb{D}) \) for any non-integral \( m > 0 \).

The proof is immediate because the kernel of the operator \([T, b]\) has the form

\[
b(\tau) - b(t) \overline{\tau}.\]

Lemma 5.9  Let \( \varphi \) be a holomorphic function. Then

\[
T\varphi(\tau) = \tau\varphi(\tau) - \frac{\varphi(\tau) - \varphi(0)}{\tau}.
\]

In particular if \(|\tau| = 1\), then \( T\varphi(\tau) = \frac{\varphi(0)}{\tau} \).

Proof:  Set \( f = \tau\varphi \). By the Cauchy-Green formula

\[
f = Kf + T\overline{\partial}f = Kf + T\varphi.
\]

Then \( Kf = K(\varphi/\tau) = \frac{\tau\varphi - \varphi(0)}{\tau} \) which proves the lemma.

We now conclude the proof of Lemma 5.7. It follows by Lemma 5.8 and 5.9 that \([T, b]\partial\varphi \in C^{2+\alpha}_1(\mathbb{D}) \) and \( T(\partial\varphi) \in C^{\infty}(b\mathbb{D}) \). This proves Lemma 5.7.

We will write \( P_1 \sim P_2 \) for two operators \( P_1 \) and \( P_2 \) if \( P_1 - P_2 \) is a bounded operator \( C^{1+\alpha}(\mathbb{D}) \cap \mathcal{O}(\mathbb{D}) \rightarrow C^{2+\alpha}_1(b\mathbb{D}) \). The result of the above analysis of \( \dot{\Phi} \) so far yields

\[
T((I + a_1 + a_2\mu)\dot{B}_1)^{12}|_{b\mathbb{D}} \sim 0.
\]

Put \( I_0 = I + a_1 + a_2\mu \). Using the commutator argument and Lemma 5.8, we obtain \((I_0T\dot{B}_1)^{12}|_{b\mathbb{D}} \sim 0\). Then

\[
0 \sim (I_0T\dot{B}_1)^{12}|_{b\mathbb{D}} = I_0^{11}T\dot{B}_1^{12} + I_0^{12}T\dot{B}_1^{22}.
\]

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Since
\[ \dot{\Lambda} = \begin{pmatrix} \dot{\lambda}_1 & \dot{\lambda}_2 \\ 0 & 0 \end{pmatrix}, \]
then \( T\dot{B}_1^{22}|_{b\mathbb{D}} \sim 0 \). Hence \( I_0^{11}(T\dot{B}_1)^{12}|_{b\mathbb{D}} \sim 0 \). Since \( I_0 \) is close to \( I \), then
\[ (T\dot{B}_1)^{12}|_{b\mathbb{D}} \sim 0. \]
Recall by (16)
\[ \dot{B}_1 = (\partial\dot{\Lambda})\Lambda^{-1} \mod (\dot{Z}, \overline{Z}). \]
On the disc \( Z_0 \) we have
\[ \Lambda = \begin{pmatrix} \rho_z & \rho_w \\ 0 & 1 \end{pmatrix}, \quad \Lambda^{-1} = \rho_z^{-1} \begin{pmatrix} 1 & -\rho_w \\ 0 & \rho_z \end{pmatrix}, \]
\[ [(\partial\dot{\Lambda})\Lambda^{-1}]^{12} = \rho_z^{-1}\partial\dot{\lambda} \begin{pmatrix} -\rho_w \\ \rho_z \end{pmatrix}. \]
By Lemma 5.8, \( T(\dot{B}_1)^{12}|_{b\mathbb{D}} \sim 0 \) implies
\[ T(\partial\dot{\lambda})|_{b\mathbb{D}} \begin{pmatrix} -\rho_w \\ \rho_z \end{pmatrix} \sim 0 \quad (25) \]
Since \( A_Z \circ Z_0 = 0 \), then
\[ \dot{\lambda} = (\rho_Z) + \rho_Z \overline{A}_Z = a\dot{Z} + b\overline{Z}, \]
where \( a = \rho_{ZZ} + \rho_Z \overline{A}_Z \) and \( b = \rho_{Z\overline{Z}} \). Recall that
\[ \dot{Z} = \begin{pmatrix} \dot{z} \\ \dot{w} \end{pmatrix} \sim \Lambda^{-1} \begin{pmatrix} 0 \\ \varphi \end{pmatrix} = \rho_z^{-1} \begin{pmatrix} -\rho_w \\ \rho_z \end{pmatrix} \varphi. \]
By the Cauchy-Green formula, \( T\overline{\partial}\varphi = 0 \) and \( T\overline{\partial}\varphi = \overline{\varphi} \). Then by Lemma 5.8,
\[ T\overline{\partial}\dot{\lambda}|_{b\mathbb{D}} \sim aT\overline{\partial}\dot{Z} + bT\overline{\partial}\overline{Z} \sim b\overline{Z} \sim (\overline{z}, \overline{w}) \begin{pmatrix} \rho_{z\overline{w}} & \rho_{w\overline{w}} \\ \rho_{z\overline{w}} & \rho_{w\overline{w}} \end{pmatrix}. \]
Then (25) turns into
\[ \det L(\rho)\varphi \sim 0, \]
where
\[ \det L(\rho) = (-\rho_{\overline{w}}, \rho_{\overline{z}}) \begin{pmatrix} \rho_z & \rho_w \\ \rho_{z\overline{w}} & \rho_{w\overline{w}} \end{pmatrix} \begin{pmatrix} -\rho_w \\ \rho_z \end{pmatrix} \]
is the Levi determinant of \( \rho \) with respect to \( J_{st} \). Thus \( \det L(\rho) \) becomes \( C^{2+\alpha}(b\mathbb{D}) \) after multiplication by any function \( \varphi \in C^{1+\alpha}(\mathbb{D}) \cap \mathcal{O}(\mathbb{D}) \). Hence \( \det L(\rho) \) vanishes identically on the boundary of the disc \( Z_0 \).
The conditions \( A \circ Z_0 = 0 \) and \( A_Z \circ Z_0 = 0 \) imply that the Levi form of \( E \) with respect to the structure \( J \) coincides with the Levi form of \( E \) with respect to the structure \( J_{st} \) at every point of the boundary of the disc \( Z_0 \) (Lemma 2.5). Thus, our proposition implies that the Levi form of \( E \) with respect to the structure \( J \) vanishes on the boundary of the disc \( Z_0 \), as desired.

This proves Proposition 5.1 and Theorem 1.2.

### 5.4 The case of degenerate rank

Consider the case in which the boundaries of the pseudoholomorphic discs attached to \( E \) through a fixed point do not cover an open set in \( E \).

**Proposition 5.10** Suppose that the boundaries of \( J \)-holomorphic discs \( \zeta \mapsto Z(\zeta) \) with \( Z(1) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \) attached to \( E \) and close to the disc \( Z_0(\zeta) = \begin{pmatrix} 0 \\ \zeta \end{pmatrix} \), \( \zeta \in \mathbb{D} \), do not cover an open set in \( E \). Then for every \( \zeta_0 \in \partial \mathbb{D}, \zeta_0 \neq 1 \) there exists a \( J \)-holomorphic disc attached to \( E \) at the point \( \begin{pmatrix} 0 \\ \zeta_0 \end{pmatrix} \) completely contained in \( E \).

Fix a point \( \zeta_0 \neq 1 \) in \( \partial \mathbb{D} \). For every Bishop disc \( Z(\zeta) \) close enough to \( Z_0(\zeta) \) consider the evaluation map

\[
\mathcal{F}_{\zeta_0} : Z \mapsto Z(\zeta_0).
\]

Then the tangent map \( \mathcal{F}_{\zeta_0}' \) of \( \mathcal{F} \) at \( Z_0 \) is the map

\[
\mathcal{F}_{\zeta_0}' : \dot{Z} \mapsto \dot{Z}(\zeta_0),
\]

where \( \dot{Z} \) is an infinitesimal perturbation of \( Z_0 \).

Since the boundaries of discs do not cover an open subset of \( E \), we have \( \text{rank} \mathcal{F}_{\zeta_0}' \leq 2 \) for all \( \zeta_0 \). The following statement implies that \( \text{rank} \mathcal{F}_{\zeta_0}' \geq 2 \).

**Lemma 5.11** For every \( w_0 \in \mathcal{C} \) there exists \( \dot{Z} = \begin{pmatrix} \dot{z} \\ \dot{w} \end{pmatrix} \) with \( \dot{w}(\zeta_0) = w_0 \).

**Proof:** We recall that \( A = 0 \) and \( A_Z = 0 \) on the discs \( Z_0 \). Then as above we have

\[
\dot{Z} = \Lambda^{-1} V,
\]

\[
V = \begin{pmatrix} 0 \\ \varphi \end{pmatrix} + R_1 \begin{pmatrix} 0 \\ \varphi \end{pmatrix} + R_2 \begin{pmatrix} 0 \\ \varphi \end{pmatrix},
\]

where \( \varphi \) is an arbitrary holomorphic function with \( \varphi(1) = 0 \). Then

\[
\dot{w} = \varphi + R_1^{22} \varphi + R_2^{22} \varphi,
\]

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where $R_{1}^{22}, R_{2}^{22}$ are $(22)$-matrix elements of $R_{1}$ and $R_{2}$. Plugging $\zeta = \zeta_{0}$, we get

$$\dot{w}(\zeta_{0}) = \varphi(\zeta_{0}) + \int_{\mathbb{D}} a_{1}(\zeta) \varphi(\zeta) d\zeta \wedge \overline{d\zeta} + \int_{\mathbb{D}} a_{2}(\zeta) \overline{\varphi(\zeta)} d\zeta \wedge d\zeta,$$

where $a_{1}$ and $a_{2}$ are integrable in $\mathbb{D}$. Assume that the rank of the map $\varphi \mapsto \dot{w}(\zeta_{0})$ is smaller than or equal to 1. Then there is $c \in \mathbb{C}\backslash\{0\}$ such that for every $\varphi$ we have $Re(c\dot{w}(\zeta_{0})) = 0$. Then for some $b_{1}, b_{2} \in L^{1}(\mathbb{D})$ we have

$$2Re(c\varphi(\zeta_{0})) + \int_{\mathbb{D}} b_{1}(\zeta) \varphi(\zeta) d\zeta \wedge \overline{d\zeta} + \int_{\mathbb{D}} b_{2}(\zeta) \overline{\varphi(\zeta)} d\zeta \wedge d\zeta = 0.$$

Splitting into linear and antilinear parts we obtain

$$c\varphi(\zeta_{0}) + \int_{\mathbb{D}} b_{1}(\zeta) \varphi(\zeta) d\zeta \wedge \overline{d\zeta} = 0.$$

This implies that $c = 0$. Indeed, take $\varphi = \psi^{n}$, where $\psi$ has a peak at $\zeta_{0}$, that is $\psi(0) = 0$, $\psi(\zeta_{0}) = 1$ and $|\psi(\zeta)| < 1$ for $\zeta \in \mathbb{D}\backslash\{\zeta_{0}\}$. Then passing to the limit as $n \rightarrow \infty$ we obtain that $c = 0$. This contradiction proves the lemma.

Since the rank of the map $\varphi \mapsto \dot{Z}(\zeta_{0})$ is equal to 2, then there are $k_{1}, k_{2} \in \mathbb{C}$ such that $\dot{z}(\zeta_{0}) = k_{1}\dot{w}(\zeta_{0}) + k_{2}\overline{\dot{w}(\zeta_{0})}$ for all $\varphi$. The equality $\dot{Z} = \Lambda^{-1}V$ implies

$$\dot{z} = -\rho_{z}^{-1}\rho_{w}\varphi + P_{1}\varphi + P_{2}\overline{\varphi},$$

where $P_{1}$ and $P_{2}$ are integral operators. Expressing $\dot{z}(\zeta_{0})$ and $\dot{w}(\zeta_{0})$ in terms of $\varphi$, we get

$$-\rho_{z}^{-1}\rho_{w}\varphi(\zeta_{0}) = k_{1}\varphi(\zeta_{0}) + k_{2}\overline{\varphi(\zeta_{0})} + \int_{\mathbb{D}} b_{1}(\zeta) \varphi(\zeta) d\zeta \wedge \overline{d\zeta} + \int_{\mathbb{D}} b_{2}(\zeta) \overline{\varphi(\zeta)} d\zeta \wedge d\zeta$$

for some $b_{1}, b_{2} \in L^{1}(\mathbb{D})$. As in lemma 5.11 we obtain $k_{1} = -\rho_{z}^{-1}\rho_{w}|_{\zeta = \zeta_{0}}$, $k_{2} = 0$. Since $\zeta_{0} \in b\mathbb{D}$ is arbitrary, we have $\rho_{z}\dot{z} + \rho_{w}\dot{w} = 0$ on $b\mathbb{D}$ that is

$$\dot{Z}(\zeta) \in H_{Z}^{1}(\zeta)E, \quad |\zeta| = 1.$$

By the hypothesis of proposition this is true for every disc close to $Z_{0}$. By the rank theorem, the image of the evaluation map $F_{\zeta_{0}}$ is a $J$-holomorphic curve in $E$. This completes the proof of the proposition.

### 5.5 Proof of Theorem 1.1

If $E$ admits a transversal Bishop disc attached at $p$, then the statement follows by Proposition 4.1. Suppose that there are no transversal Bishop discs. Then by Proposition 5.1 for every disc $Z$ attached to $E$ at $p$, the Levi form of $E$ with respect to $J$ vanishes on $Z(b\mathbb{D})$. If these discs fill an open subset $\Omega$ of $E$, then the Levi form of $E$ vanishes on $\Omega$ identically. Then it follows that $\Omega$ is foliated by $J$-holomorphic discs, which holds for Levi-flat hypersurfaces in any dimension [5]. In the case of complex dimension 2 the existence of the foliation follows immediately from the representation (1) for the Levi form and the Frobenius theorem. Finally, if the boundaries of Bishop discs do not cover an open piece of $E$, then there exist $J$-holomorphic discs in $E$ by Proposition 5.10. Thus, $E$ necessarily admits a transversal Bishop disc which proves the theorem.
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