RESEARCH ARTICLE

Logarithmic bounds on Fujita’s conjecture

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Abstract
Let $X$ be a smooth complex projective variety of dimension $n$. We prove bounds on Fujita’s basepoint freeness conjecture that grow as $n \log \log(n)$, where $\log$ is the logarithm with natural base.

INTRODUCTION

The purpose of this paper is to prove the following result:

**Theorem 1.1.** Let $X$ be a smooth projective variety of dimension $n \geq 2$ defined over an algebraically closed field of characteristic zero and let $A$ be an ample Cartier divisor on $X$. Then $|K_X + mA|$ is basepoint free for any positive integer $m \geq n(\log \log(n) + 2.34)$.

A conjecture of Fujita [4] states that, in the hypothesis of Theorem 1.1, $|K_X + mA|$ is basepoint free for all $m \geq n + 1$. Since maps to projective space are one of the main tools used in the study of projective varieties, Fujita’s conjecture has received considerable attention. In the case of curves, the statement is an immediate consequence of the Riemann–Roch theorem. Reider proved the conjecture for surfaces in [15] shortly after its formulation by using Bogomolov’s instability.

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theorem for rank two vector bundles. Ein and Lazarsfeld proved it for 3-folds in [3] by introducing techniques from the Minimal Model Program. Later, Kawamata proved the conjecture for 4-folds in [9] and Ye and Zhu recently proved it for 5-folds in [17] and [18].

For 6-folds, we prove the following:

**Theorem 1.2.** Let $X$ be a smooth projective variety of dimension 6 defined over an algebraically closed field of characteristic zero and let $A$ be an ample Cartier divisor on $X$. Then $|K_X + mA|$ is basepoint free for any positive integer $m \geq 8$.

The method of proof of Theorem 1.1 recovers the above cases of Fujita’s conjecture up to dimension 4, but not the case of dimension 5 (see Theorem 5.2 and the values of Table 1 in Section 4). For 6-folds, it only gives basepoint freeness of $|K_X + mA|$ for $m \geq 9$, so the proof of Theorem 1.2 requires a more refined analysis.

While these sporadic cases may be considered as evidence for the conjecture to hold true in general, in higher dimensions much less is known. By the cone theorem for smooth projective varieties [11, Theorem 1.24] we have that $|K_X + mA|$ is nef for $m \geq n + 1$. The first general result on basepoint freeness is due to Angehrn and Siu, who used techniques of analytic algebraic geometry to prove that $|K_X + mA|$ is basepoint free for all $m \geq (n^2 + n + 2)/2$ in [1]. Kollár adapted their proof to the algebraic setting in [10]. By using a different idea, later Helmke [7] also established a general method that essentially leads to a quadratic bound. Heier [6] combined Angehrn–Siu’s approach and Helmke’s approach to give a bound that is $O(n^{4/3})$.

Once one knows that the linear series $|K_X + mA|$ gives a morphism to projective space, some questions naturally arise. For instance, it is interesting to know if the morphism is birational. More in general, we say that $|K_X + mA|$ separates $r$ points if the restriction morphism $H^0(X, \mathcal{O}_X(K_X + mA)) \to H^0(T, \mathcal{O}_X(K_X + mA)|_T)$ is surjective for any reduced 0-dimensional subscheme $T$ of length $r$. Naturally, $|K_X + mA|$ separates two points if and only if the morphism defined by $|K_X + mA|$ is birational. Angehrn and Siu [1] (see also [10]) showed that $|K_X + mA|$ separates $r$ points for all $m \geq (n^2 + 2rn - n + 2)/2$. In this direction we prove:

**Theorem 1.3.** Let $X$ be a smooth projective variety of dimension $n \geq 2$ defined over an algebraically closed field of characteristic zero and let $A$ be an ample Cartier divisor. Then $|K_X + mA|$ separates $r$ points for any positive integer $m \geq r + n - 1 + \sqrt{r} n (\log \log n + 2.34)$.

Let us now briefly explain the ideas behind the proofs of Theorems 1.1–1.3. Let $x$ be a point in a smooth projective variety $X$ and let $A$ be an ample Cartier divisor. Suppose that we wish to find a section of $H^0(X, \mathcal{O}_X(K_X + A))$ that does not vanish at $x$. If $A^n$ is large enough, by Lemma 2.3 we may find a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor $D \in |A|_\mathbb{Q}$ that has large order of vanishing at $x$. The pair $(X, D)$ has a non-Kawamata log terminal center $Z$ containing $x$, and therefore one tries to use vanishing theorems to lift sections of $H^0(Z, \mathcal{O}_Z(K_X + A)|_Z)$. If $Z$ is 0-dimensional this is easily done, but $Z$ may well be higher dimensional. Thus, one needs to cut down $Z$ in dimension. At this stage, two approaches are possible. Helmke’s approach is to insist in finding a divisor $D' \in |A|_{Z, \mathbb{Q}}$ with large order of vanishing at $x$ and then lifting $D'$ to $X$. Finding such $D'$ is now harder than it was finding $D$, as $Z$ may be singular at $x$. However, Helmke proved that if $Z$ is a log canonical center of dimension $d$, then $\text{mult}_x Z \leq \binom{n-1}{d}$ holds. By cutting down one step at a time, one eventually gets a 0-dimensional log canonical center. Angehrn and Siu’s method is instead to
find a divisor $D' \in |A|_Z|_Q$ that is highly singular at a smooth point $y$ near $x$, and then take the limit as $y$ approaches $x$. As we mentioned earlier, both methods give a quadratic bound on $m$.

We will follow Helmke’s approach. The crucial new ingredient, however, is to consider all steps simultaneously rather than one at a time. By doing so, we rephrase the problem of bounding $m$ into an optimization problem of a linear function on a polytope. This approach allows us to estimate very efficiently the maximum of the linear function, as it suffices to evaluate it at the vertices of the polytope. We illustrate this idea in the simple case of dimension 2 (refer to Section 2 for the relevant notions). Let $X$ be a smooth projective surface, let $A$ be an ample Cartier divisor and fix a point $x \in X$. By Lemma 2.3, for any $\varepsilon > 0$ we may find a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor $D \in |A|_Q$ such that $\text{ord}_x D \geq 1 - \varepsilon$. Let us assume here for simplicity that in fact $\text{ord}_x D \geq 1$ holds. Since the error term $\varepsilon$ may be made arbitrarily small, this assumption does not affect the final estimate. Consider the log canonical threshold

$$t_1 = \sup \{c | (X, cD) \text{ is log canonical at } x \}.$$

Clearly, we have

$$0 \leq t_1 \leq 2. \tag{1}$$

Let $D_1 = t_1 D$ and let $Z$ be the minimal log canonical center of $(X, D_1)$ at $x$. If $Z$ is 0-dimensional, then we are done. If not, then $Z$ is a curve, which is smooth at $x$ [9, Theorem 1.6]. At this point we introduce the following important definition, due independently to Helmke and Ein (see [7, (3.1.1)] and [2, p. 212]).

**Definition 1.4.** Let $(X, \Delta)$ be a log pair. Let $n$ be the dimension of $X$ and let $x$ be a smooth point in $X$. Let $\pi : Y \to X$ be the blowing up of $X$ at $x$ with exceptional divisor $E$. The local discrepancy $b_x(X, \Delta)$ of $(X, \Delta)$ over $x$ is:

$$\inf \{b \mid \text{there is a non-klt center of } (Y, \pi^* \Delta - (n - 1 - b)E) \text{ in } E \}.$$

In our example, let $b_1 = b_x(X, D_1)$. The discrepancy of $E$ with respect to $(X, D_1)$ is $1 - t_1 \cdot \text{ord}_x D \leq 1 - t_1$. Therefore,

$$0 \leq b_1 \leq 2 - t_1. \tag{2}$$

Also, since $Z$ is 1-dimensional, by inversion of adjunction we have

$$0 \leq b_1 \leq 1. \tag{3}$$

Next, we want to cut down $Z$. Since $Z$ is smooth at $x$, we may pick a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor $D' \in |A|_Z|_Q$ such that $\text{ord}_x D' \geq 1$. Let $D''$ be a general lifting of $D'$ to $X$. Finally, let

$$t_2 = \sup \{c | (X, D_1 + cD'') \text{ is log canonical at } x \}$$

and set $D_2 = D_1 + t_2 D''$. By inversion of adjunction, it is again easy to see

$$0 \leq t_2 \leq b_1. \tag{4}$$
Notice that $x$ is a log canonical center of $(X, D_2)$ and that $D_2 \sim_{\mathbb{Q}} (t_1 + t_2)A$. Fix any positive integer $m$ such that $m > t_1 + t_2$ holds, so that $mA - D_2$ is ample. By Nadel’s vanishing theorem we may then deduce that $\mathcal{O}_X(K_X + mA)$ has a section that does not vanish at $x$, so now the problem is to bound $t_1 + t_2$. Consider the set $C \subseteq \mathbb{R}^3$ consisting of points $(t_1, t_2, b_1)$ satisfying conditions (1) – (4) above. Then $C$ is the convex hull of the points $(0,0,0), (0,0,1), (2,0,0), (1,0,1), (1,1,1), (0,1,1)$. Therefore, for any point of $C$ we have $t_1 + t_2 \leq 2$, and thus $|K_X + mA|$ is basepoint free for $m \geq 3$.

This same idea applies to higher dimensions. However, the situation is considerably more complicated due to the presence of singularities of $Z$ and due to the fact that the geometry of the polytope $C$ becomes increasingly complex. In order to deal with the problem more efficiently, in higher dimensions we do not compute all the vertices of $C$, but only those for which $\sum_i t_i$ is large. In the above example, this amounts to noticing that one may rewrite (2) as

$$t_1 \leq 2 - b_1.$$ 

By combining this with (4), we get

$$t_1 + t_2 \leq (2 - b_1) + b_1 \leq 2.$$ 

The generalization of the expression $(2 - b_1) + b_1$ to higher dimensions is the function $f(b_1, d, n, 1)$ of Section 4. Much of the work of the paper is devoted to carefully estimating $f$ in terms of $n$ only, which then leads to the result. We would like to give here an idea on how this is done at least in the case when $X$ is a 3-fold. First, define $D_1$, $b_1$, and $t_1$ as above and suppose we have $\text{LLC}(X, D_1, x) = \{Z_1\}$, where $Z_1$ is an irreducible surface. By Theorem 3.4 we have

$$m_1 = \text{mult}_x Z_1 \leq 3 - \lceil b_1 \rceil.$$ 

For the next step, choose a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor $D' \in |A|_{Z_1, \mathbb{Q}}$ with large order of vanishing at $x$. Notice that, differently than above, the best bound we may hope for is $\text{ord}_x D' \geq \frac{1}{\sqrt{m_1}}$, due to the fact that $Z_1$ is possibly singular (see Definition 2.2 for the notion of order of vanishing in this context). Let $D''$ be a general lifting of $D'$ to $X$, let

$$t_2 = \sup \{ c \mid (X, D_1 + cD'') \text{ is log canonical at } x \}$$

and set $D_2 = D_1 + t_2D''$. If we define $b_2 = b_x(X, D_2)$, then by an argument due to Helmke and Ein (see also Proposition 3.1), we have

$$b_2 \leq b_1 - t_2 \cdot \text{ord}_x D' \leq b_1 - \frac{t_2}{\sqrt{m_1}}.$$ 

Now suppose that the minimal log canonical center of $(X, D_2)$ at $x$ has dimension 1. After perturbing the coefficients of $D_2$, we may assume $\text{LLC}(X, D_2, x) = \{Z_2\}$, where $Z_2$ is an irreducible curve. Then we have that $Z_2$ is smooth near $x$ by inversion of adjunction. Let $D' \in |A|_{Z_2, \mathbb{Q}}$ be a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor such that $\text{ord}_x D' \geq 1$ and let $D''$ be a general lifting of $D'$ to $X$. Finally, let

$$t_3 = \sup \{ c \mid (X, D_2 + cD'') \text{ is log canonical at } x \}$$
and set $D_3 = D_2 + t_3D''$. We have $t_3 \leq b_2$ and that $x$ is a log canonical center of $(X, D_3)$. Putting everything together, we get

$$t_1 + t_2 + t_3 \leq 3 - b_1 + (b_1 - b_2)\sqrt{3 - \lfloor b_1 \rfloor} + b_2. \quad (5)$$

Notice that $b_1 \leq \dim(Z_1) = 2$ holds. Therefore, $\lfloor b_1 \rfloor \neq 3$ holds and expression (5) does not decrease if we decrease $b_2$. In particular, we may assume $b_2 = 0$. But then we get

$$t_1 + t_2 + t_3 \leq 3 - b_1 + b_1\sqrt{3 - \lfloor b_1 \rfloor} \leq 2 + \sqrt{2} < 4.$$

This proves Fujita’s basepoint freeness conjecture in dimension 3, at least in the case when the log canonical centers constructed inductively have dimension 2 and 1, respectively (the other cases being similar but simpler). As $n$ grows larger, however, bounding $f(b, d, n, 1)$ becomes increasingly difficult. For example, if $X$ is a 4-fold and if there are four steps in the inductive process, the upper bound on $\sum_i t_i$ is

$$4 - b_1 + (b_1 - b_2)\sqrt{4 - \lfloor b_1 \rfloor} + (b_2 - b_3)\sqrt{\frac{4 - \lfloor b_2 \rfloor}{2}} + b_3.$$

We refer to Section 4 for the details of the estimates on $f(b, d, n, r)$ and we refer to the Appendix for a proof of their optimality. These estimates involve Lambert’s productlog function $W$ and the number 2.34 appearing in Theorem 1.1 and Theorem 1.2 is simply an upper bound of the value $1/W(1) - \log W(1)$.

## 2 | PRELIMINARIES

### 2.1 | Notation

We work over an algebraically closed field $k$ of characteristic zero. Most of the following notation is standard. The set $\mathbb{N}$ is the set of natural numbers, 0 included. We denote the logarithmic function with natural base as $\log : \mathbb{R}^+ \to \mathbb{R}$. We denote by $W : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ the principal branch of Lambert’s productlog function (see Definition 4.6). A $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor $D$ on a normal variety $X$ is nef if $D \cdot C \geq 0$ for any curve $C \subset X$. We use the symbol $\sim_{\mathbb{Q}}$ to indicate $\mathbb{Q}$-linear equivalence and the symbol $\equiv$ to indicate numerical equivalence. We denote by $|D|_{\mathbb{Q}}$ the $\mathbb{Q}$-linear series of a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor $D$. A pair $(X, \Delta)$ consists of a normal variety $X$ and a Weil $\mathbb{Q}$-divisor $\Delta$ such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier. If $\Delta \geq 0$, we say $(X, \Delta)$ is a log pair. If $f : Y \to X$ is a birational morphism, we may write $K_Y + f^{-1}_*\Delta = f^*(K_X + \Delta) + \sum_i a_i E_i$ with $E_i$ $f$-exceptional divisors. A log pair $(X, \Delta)$ is called log canonical (or lc) if $a_i \geq -1$ for every $i$ and for every $f$, and it is called Kawamata log terminal (or klt) if $a_i > -1$ for every $i$ and $f$, and furthermore $|\Delta| = 0$ holds. The rational numbers $a_i$ are called the discrepancies of $E_i$ with respect to $(X, \Delta)$ and do not depend on $f$. We say that a subvariety $V \subset X$ is a non-klt center if it is the image of a divisor of discrepancy at most $-1$. A non-klt center $V$ is a log canonical center if $(X, \Delta)$ is log canonical at a general point of $V$. A non-klt place (respectively, log canonical place) is a valuation corresponding to a divisor of
discrepancy at most (respectively, equal to) $-1$. By a point $x$ we mean a closed point, unless differently specified. The set of all log canonical centers passing through $x \in X$ is denoted by $\text{LLC}(X, \Delta, x)$, and the union of all the non-klt centers through $x$ is denoted by $\text{Nklt}(X, \Delta, x)$. Finally, if $K_X$ and $\Delta$ are $\mathbb{Q}$-Cartier, then the log canonical threshold of $(X, \Delta)$ at a point $x$ is $\text{lct}(X, \Delta, x) = \sup \{ c \geq 0 | (X, c\Delta) \text{ is lc at } x \}$.

2.2 Log canonical centers

We recall here some standard definitions and results in birational geometry for the convenience of the reader.

**Definition 2.1.** Let $X$ be an irreducible projective variety of dimension $n$ and let $D$ be a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor. Let $m$ be a positive integer such that $mD$ is Cartier. The volume of $D$ is

$$\text{vol}(X, D) = \limsup_{k \to \infty} \frac{n! h^0(X, \mathcal{O}_X(kmD))}{(km)^n}.$$ 

**Definition 2.2.** Let $X$ be an irreducible projective variety of dimension $n$, let $x$ be a point of $X$ and let $D$ be an effective $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on $X$. Let $m$ be a positive integer such that $mD$ is Cartier and let $f \in \mathcal{O}_{X,x}$ be a defining equation. Then we define the order of vanishing of $D$ at $x$ as

$$\text{ord}_x D = \frac{1}{m} \max \{ s \in \mathbb{N} | f \in \mathfrak{m}_x^s \}$$

We define the multiplicity of $X$ at $x$ as

$$\text{mult}_x X = \lim_{k \to \infty} \frac{n! l(\mathcal{O}_{X,x}/\mathfrak{m}_x^k)}{k^n},$$

where $l(\mathcal{O}_{X,x}/\mathfrak{m}_x^k)$ is the length as an $\mathcal{O}_{X,x}$-module. See [13] for a reference.

**Lemma 2.3.** Let $X$ be an irreducible projective variety of dimension $n$ and let $D$ be a big $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on $X$. Let $T$ be a finite set of points of $X$ and fix any positive real number $\varepsilon > 0$ such that $\varepsilon \leq \frac{\text{vol}(X, D)}{|T| \cdot \text{mult}_x X}$ for all $x \in T$. Then there is a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor $D' \in |D|_\mathbb{Q}$ such that

$$\text{ord}_x D' \geq \left( \frac{\text{vol}(X, D)}{|T| \cdot \text{mult}_x X} - \varepsilon \right)^{1/n}$$

for all $x \in T$.

**Proof.** For large $k_x \in \mathbb{N}$, we have

$$l(\mathcal{O}_{X,x}/\mathfrak{m}_x^{k_x}) = \frac{k^n}{n!} \text{mult}_x X + p_s(k_x)$$
for all $x$ in $T$, where $p_x(k_x)$ is a polynomial of degree $n - 1$. Fix a positive rational number $\epsilon' > 0$. By the definition of volume, we may pick $k'$ large such that

$$\frac{n! h^0(X, O_X(k'mD))}{(k'm)^n} > \text{vol}(X, D) - \epsilon'$$

holds. Therefore, we have

$$h^0(X, O_X(k'mD)) > \sum_{x \in T} l(O_{X,x}/m_{X,x}^{k_x})$$

for all $x$ in $T$ if

$$\frac{(\text{vol}(X, D) - \epsilon') \cdot (k'm)^n}{n!} \geq \sum_{x \in T} \left( \frac{k_x^n}{n!} \cdot \text{mult}_x X + p_x(k_x) \right)$$

(*)

holds. Define

$$k_x = k'm \left( \frac{\text{vol}(X, D)}{|T| \cdot \text{mult}_x X - \epsilon} \right)^{1/n},$$

where $k'$ is large and divisible enough. With this choice, condition (*) becomes

$$\text{vol}(X, D) - \epsilon' \geq \sum_{x \in T} \left( \text{vol}(X, D)/|T| - \epsilon \cdot \text{mult}_x X + n! \frac{p_x(k_x)}{(k'm)^n} \right).$$

This inequality is satisfied provided that $k'$ is sufficiently large and $\epsilon'$ is sufficiently small. Thus, there is a divisor $H \in |k'mD|$ such that

$$\text{ord}_x H \geq \left( \frac{\text{vol}(X, D)}{|T| \cdot \text{mult}_x X - \epsilon} \right)^{1/n} \cdot k'm$$

holds for all points $x$ in $T$. We conclude by taking $D' = H/(k'm)$.

\[\square\]

**Definition 2.4.** Let $(X, \Delta)$ be a log pair with $X$ smooth, and let $\mu : Y \to X$ be a log resolution of $(X, \Delta)$. We define the multiplier ideal sheaf of the pair $(X, \Delta)$ to be

$$I(X, \Delta) = \mu_* \mathcal{O}_Y(K_Y - [\mu^* \Delta]) \subseteq \mathcal{O}_X.$$

We have that $(X, \Delta)$ is klt if and only if $I(X, \Delta) = \mathcal{O}_X$, and it is lc if and only if $I(X, (1 - \epsilon)\Delta) = \mathcal{O}_X$ for any $0 < \epsilon \ll 1$. Therefore, $\text{Nklt}(X, \Delta) = \text{Supp}(\mathcal{O}_X/I(X, \Delta))$ holds as closed sets.

**Theorem 2.5** (Nadel vanishing theorem). Let $X$ be a smooth projective variety and $\Delta \geq 0$ a Weil $\mathbb{Q}$-divisor on $X$. Let $D$ be any integral divisor such that $D - \Delta$ is big and nef. Then, $H^i(X, O_X(K_X + D) \otimes I(X, \Delta)) = 0$ for $i > 0$.

**Proof.** See [12, Section 9.4.B].

\[\square\]
The next two results are [12, Proposition 9.3.2] and [9, Proposition 1.5], respectively. Note that $\text{ord}_x \Delta$ is denoted by $\text{mult}_x \Delta$ in [12, Proposition 9.3.2].

**Proposition 2.6.** Let $X$ be an irreducible variety of dimension $n$ and let $\Delta$ be an effective $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on $X$. If $\text{ord}_x \Delta \geq n$ at some smooth point $x \in X$, then $I(X, \Delta)_x \subseteq m_x$, where $m_x$ is the maximal ideal of $x$.

**Lemma 2.7.** Let $\mathcal{O}_X$ be a log pair such that $\Delta$ is $\mathbb{Q}$-Cartier. Assume that $X$ is klt and $\mathcal{O}_X, \mathcal{O}_\Delta$ is lc. If $W_1$ and $W_2$ are log canonical centers of $(X, \Delta)$ and $W$ is an irreducible component of $W_1 \cap W_2$, then $W$ also is a log canonical center of $(X, \Delta)$. In particular, if $(X, \Delta)$ is not klt at $x \in X$, then there exists the unique minimal element of $\text{LLC}(X, \Delta, x)$.

We will refer to the following result as ‘tie breaking’.

**Lemma 2.8.** Let $(X, \Delta)$ be a log pair such that $X$ is klt and $\Delta$ is $\mathbb{Q}$-Cartier. Let $S$ be a finite set of points of $X$. Suppose that there is a point $x \in S$ such that $\{x\} \in \text{LLC}(X, \Delta, x)$ and that for each point $y \in S \setminus \{x\}$ there is a non-klt center of $(X, \Delta)$ containing $y$ but not $x$. Let $D$ be an ample $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor. Then, there exists a positive rational number $a > 0$ such that for any $0 < \epsilon \ll 1$ there exists a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor $E \in |aD_\mathbb{Q}|$ such that

1. $(X, (1-\epsilon)\Delta + \epsilon E)$ is not klt at any point of $S$,
2. $(X, (1-\epsilon)\Delta + \epsilon E)$ is lc at $x$,
3. $\text{LLC}(X, (1-\epsilon)\Delta + \epsilon E, x) = \{x\}$ holds.

**Proof.** This is an analog of [7, Proposition 6.2]. For each $y \in S \setminus \{x\}$ let $v_y$ be a non-klt place of $(X, \Delta)$ whose center $Z_y$ contains $y$ but not $x$. Let $b$ be a positive rational number such that there exists $E_1 \in |bD_\mathbb{Q}|$ with $v_y(E_1) > v_y(\Delta)$ for all $y \in S \setminus \{x\}$. After possibly taking a larger $b$ we may assume that the common support of all such $E_1$ is exactly the union of the $Z_y$. Similarly, let $v_x$ be a log canonical place of $(X, \Delta)$ with center $\{x\}$ and let $c$ be a positive rational number such that there exists $E_2 \in |cD_\mathbb{Q}|$ with $v_x(E_2) > v_x(\Delta)$. Again, after possibly taking a larger $c$ we may assume that the common support of all such $E_2$ is exactly $x$. Set $a = b + c$. For general choices of $E_1$ and $E_2$, the pair $(X, (1-\epsilon)\Delta + \epsilon E_1 + \epsilon E_2)$ is not lc at any point of $S$ for any small $\epsilon$. Let

$$t = \sup\{d| (X, (1-\epsilon)\Delta + \epsilon E_1 + d\epsilon E_2) \text{ is lc at } x\}.$$ 

Clearly $t < 1$ holds. Finally, take $E_3 \in |(1-t)cD_\mathbb{Q}|$ general enough. We have

$$\text{LLC}(X, (1-\epsilon)\Delta + \epsilon (E_1 + tE_2 + E_3), x) = \{x\}.$$ 

Furthermore, $S$ is contained in $\text{Nklt}(X, (1-\epsilon)\Delta + \epsilon (E_1 + tE_2 + E_3))$ and $E_1 + tE_2 + E_3 \in |aD_\mathbb{Q}|$. Therefore, we may take $E = E_1 + tE_2 + E_3$.

### 3 | THE INDUCTIVE METHOD

In this section, we describe an inductive method for cutting down the dimension of non-klt centers. This essentially due to Helmke (see [7, Proposition 3.2], [7, Proposition 6.3]) and Ein (see [2,
Lemma 4.6). Since we will need these results in a slightly different form, we go over the proofs and make the appropriate changes.

**Proposition 3.1.** Let \((X, \Delta + \Gamma)\) be a log pair, where \(X\) is a smooth projective variety of dimension \(n\). Suppose that \((X, \Delta + \Gamma)\) is log canonical at a point \(x\) in \(X\) and that the minimal log canonical center \(Z\) of \((X, \Delta)\) at \(x\) has dimension at least one. Then we have

\[
b_x(X, \Delta + \Gamma) \leq b_x(X, \Delta) - \text{ord}_x(\Gamma|_Z).
\]

**Proof.** By slightly perturbing the coefficients of \(\Delta\), we may assume \(\text{LLC}(X, \Delta, x) = \{Z\}\). Let \(\pi : \tilde{X} \to X\) be the blowing up of \(x\) with exceptional divisor \(E\) and let \(g : Y \to \tilde{X}\) be a log resolution of \((\tilde{X}, \pi^*(\Delta + \Gamma))\). Denote by \(f = \pi \circ g : Y \to X\) the composition. By definition of local discrepancy, there is an exceptional divisor \(E_1\) with coefficient \(-1\) in

\[
\Theta = K_Y - f^*(K_X + \Delta) - b_x(X, \Delta) \cdot g^*E.
\]

Let \(E_0\) be a log canonical place in \(Y\) for \((X, \Delta)\) over \(Z\). Notice that \(g(E_0)\) is not contained in \(E\) since \(\dim Z \neq 0\) by assumption. Therefore, the divisor \(E_0\) also appears with coefficient \(-1\) in \(\Theta\). By connectedness of the non-klt locus ([11, Theorem 5.48], [7, Lemma 2.2]), after possibly replacing \(E_0\), we may assume that \(E_0\) and \(E_1\) intersect. Since \(f\) is a log resolution, we have

\[
\text{ord}_{E_1} f^*\Gamma = \text{ord}_{E_0 \cap E_1} (f^*\Gamma|_{E_0}) = \text{ord}_{E_0 \cap E_1} (f^*(\Gamma|_Z))
\]

\[
= \text{ord}_{E_0 \cap E_1} (g^*\pi^*(\Gamma|_Z))) \geq \text{ord}_{E_0 \cap E_1} (\text{ord}_x(\Gamma|_Z) \cdot (g|_{E_0})^* E)
\]

\[
= \text{ord}_x(\Gamma|_Z) \cdot \text{ord}_{E_1} g^*E.
\]

By the definition of local discrepancy, we have that the coefficient of \(E_1\) in

\[
K_Y - f^*(K_X + \Delta + \Gamma) - b_x(X, \Delta + \Gamma) \cdot g^*E
\]

is at least \(-1\). Therefore,

\[
\text{ord}_{E_1} (b_x(X, \Delta) \cdot g^*E) \geq \text{ord}_{E_1} (f^*\Gamma + b_x(X, \Delta + \Gamma) \cdot g^*E).
\]

In conclusion,

\[
b_x(X, \Delta) \cdot \text{ord}_{E_1} g^*E \geq (\text{ord}_x(\Gamma|_Z) + b_x(X, \Delta + \Gamma)) \cdot \text{ord}_{E_1} g^*E.
\]

Since \(\text{ord}_{E_1} g^*E \neq 0\), we are done. \(\square\)

**Proposition 3.2.** Let \((X, \Delta)\) be a log pair, where \(X\) is a smooth projective variety of dimension \(n\). Let \(S\) be a finite set of points contained in \(\text{Nklt}(X, \Delta)\) and let \(r\) be the cardinality of \(S\). Let \(T\) be a nonempty subset of \(S\) such that

1. \((X, \Delta)\) is log canonical at all points of \(T\),
2. there is a log canonical center \(Z\) of \((X, \Delta)\) that is minimal at all points of \(T\),
(3) every point in $S \setminus T$ is contained in a non-klt center of $(X, \Delta)$ that does not contain any point of $T$.

Fix a positive real number $\varepsilon > 0$. Consider $d = \dim Z$ and let $D$ be an ample $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor. If $d > 0$, then there exist a nonempty subset $T'$ of $T$, a rational number $t$ such that

$$0 \leqslant t \leqslant b_x(X, \Delta) \left( \frac{r \mult_x Z}{D^d \cdot Z} \right)^{1/d} + \varepsilon$$

for every point $x \in T$, and a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor $D' \in |D|_\mathbb{Q}$ such that

(1) $(X, \Delta + tD')$ is log canonical at all points of $T'$,
(2) there is a log canonical center $Z'$ of $(X, \Delta + tD')$ strictly contained in $Z$ that is minimal at all points of $T'$,
(3) every point in $S \setminus T'$ is contained in a non-klt center of $(X, \Delta + tD')$ that does not contain any point of $T'$.

Furthermore,

$$b_x(X, \Delta + tD') \leqslant b_x(X, \Delta) - t \cdot \left( \frac{D^d \cdot Z}{r \mult_x Z} \right)^{1/d} + \varepsilon$$

holds for all points $x \in T'$.

Proof. Notice $D^d \cdot Z = \operatorname{vol}(Z, D|_Z)$. Fix a positive real number $\varepsilon' > 0$. Then by Lemma 2.3, there exists a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor $D'' \in |D|_\mathbb{Q}$ such that

$$\operatorname{ord}_x D'' \geqslant \left( \frac{D^d \cdot Z}{|T| \cdot \mult_x Z} - \varepsilon' \right)^{1/d} \geqslant \left( \frac{D^d \cdot Z}{r \mult_x Z} - \varepsilon' \right)^{1/d}$$

holds for every point $x$ in $T$. By Serre’s vanishing theorem, we may take $D' \in |D|_\mathbb{Q}$ to be a general lifting of $D''$ to $X$. Consider

$$t = \sup \{ c \mid (X, \Delta + cD') \text{ is log canonical at some point of } T \}.$$

By Proposition 3.1, we have

$$0 \leqslant t \leqslant b_x(X, \Delta) \left( \frac{r \mult_x Z}{D^d \cdot Z} \right)^{1/d} + \varepsilon$$

for all points $x$ in $T$, if $\varepsilon'$ is sufficiently small. Let $T_1$ be the set of points of $T$ where $(X, \Delta + tD')$ is log canonical. For each $x$ in $T_1$ let $Z_x$ be the minimal log canonical center of $(X, \Delta + tD')$ at $x$. By construction, for any $x \in T_1$ we have that $Z_x$ is strictly contained in $Z$. Choose a maximal element $Z'$ in the set $\{ Z_x \mid x \in T_1 \}$ ordered by inclusion and define $T' = \{ x \in T_1 \mid Z_x = Z' \}$. Now, if $x \in S \setminus T$ holds, then there is a non-klt center of $(X, \Delta + tD')$ containing $x$ but none of the points of $T'$ by hypothesis. If $x \in T \setminus T'$ either $(X, \Delta + tD')$ is not log canonical at $x$, or $(X, \Delta + tD')$ is log canonical at $x$ but the minimal log canonical center $Z_x$ does not contain $Z'$ by the maximality
assumption. In either case, if \( x \in S \setminus T \) holds there is a non-klt center, which does not contain any of the points of \( T' \).

The final statement follows from Proposition 3.1.

Remark 3.3. If \( \Delta = 0 \) in Proposition 3.2, then we take \( Z = X \).

Proposition 3.2 shows that it is crucial to have control over the singularities of log canonical centers. In this direction, we have the following theorem of Helmke [7, Theorem 4.3]; see also [16, Proposition 3.9] for a different proof.

**Theorem 3.4.** Let \( X \) be a smooth projective variety and \((X, \Delta)\) be log canonical at \( x \in X \). Let \( Z_d \) be the union of the elements of LLC\((X, \Delta, x)\) of dimension \( d \). Then,

\[
\text{mult}_x Z_d \leq \left( n - \left\lfloor b_x (X, \Delta) \right\rfloor \right) \frac{n - d}{n - d}.
\]

4 | OPTIMIZATION

Let \( s < n \) be nonnegative integers and let \( r \) be any positive integer. Consider the set \( R_{s,n} \subseteq \mathbb{R}_{\geq 0}^{s+2} \times \mathbb{N}_{\geq 0}^{s+2} \) consisting of elements

\[
(b, d) = (b_0, b_1, \ldots, b_s, b_{s+1}, d_0, d_1, \ldots, d_s, d_{s+1})
\]

satisfying the following conditions:

\[
0 = b_{s+1} < b_s < \ldots < b_1 < b_0 = n,
\]

\[
0 = d_{s+1} < d_s < \ldots < d_1 < d_0 = n,
\]

and \( b_i \leq d_i \) for all \( 1 \leq i \leq s \). This section is devoted to the study of the functions

\[
f(b, d, n, r) = \sum_{i=0}^{s} (b_i - b_{i+1}) \left[ r \left( n - \left\lfloor b_i \right\rfloor \right) \right]^{1/d_i}
\]

and

\[
F(n, r) = \max\{f(b, d, n, r) \mid (b, d) \in \cup_{s=0}^{n-1} R_{s,n}\}.
\]

In particular, we aim to prove the following upper bounds.

**Theorem 4.1.** Let \( n \geq 2 \) and \( r \) be positive integers. Then we have

1. \( F(n, 1) < n(\log \log(n) + 2.34) \),
2. \( F(n, r) < r + n - 1 + \sqrt{r} n(\log \log(n) + 2.34) \).

We start by pointing out that in order to maximize \( f \), it is sufficient to consider integral values of \( b_i \).
TABLE 1 Values of $\lfloor F(n, r) \rfloor$ for $2 \leq n \leq 17$ and $r = 1, 2$.

| $r$ | 2   | 3   | 4   | 5   | 6   | 7   | 8   | 9   | 10  | 11  | 12  | 13  | 14  | 15  | 16  | 17  |
|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| 1   | 2   | 3   | 4   | 6   | 8   | 9   | 11  | 13  | 15  | 17  | 19  | 21  | 24  | 26  | 28  | 30  |
| 2   | 3   | 4   | 6   | 8   | 10  | 12  | 14  | 16  | 18  | 20  | 22  | 24  | 26  | 29  | 31  | 33  |

Lemma 4.2. Fix $(b, d) \in R_{s,n}$. Then there are an integer $s' \leq s$ and an element $(b', d') \in R_{s', n}$ such that the vector $b'$ consists of integers and $f(b, d, n, r) \leq f(b', d', n, r)$.

Proof. Consider the set $B_s \subseteq \mathbb{R}_{\geq 0}^{s+2}$ consisting of elements

$$b' = (b'_0, \ldots, b'_{s+1})$$

satisfying the conditions

$$0 = b'_{s+1} \leq b'_s \leq \cdots \leq b'_1 \leq b'_0 = n,$$

$$\lceil b_i \rceil - 1 \leq b'_i \leq \lceil b_i \rceil.$$

Consider the linear function

$$L(b') = \sum_{i=0}^{s} (b'_i - b'_{i+1}) \left[ r \left( \frac{n - \lceil b_i \rceil}{n - d_i} \right) \right]^{1/d_i}.$$

Notice that if $b'$ is in the interior of $B_s$, then we have $f(b', d, n, r) = L(b')$. Also, $f(b, d, n, r) = L(b)$ holds. The set $B_s$ is a polytope in $\mathbb{R}_{\geq 0}^{s+2}$, therefore $L$ achieves its maximum value at a vertex $b'' \in B_s$. By construction, all the vertices of $B_s$ have integral coordinates. We have

$$f(b, d, n, r) = L(b) \leq L(b'') \leq f(b'', d, n, r).$$

If there is an index $i$ for which $b'' = b''_{i+1}$, then erase $b''$ and $d_i$ from $b''$ and $d$. Note that $f(b'', d, n, r)$ is nondecreasing under such an operation. After possibly repeating the above procedure sufficiently many times, we may assume that $(b'', d'')$ belongs to $R_{s', n}$ for some $s' \leq s$.

If $n$ and $r$ are fixed, then Lemma 4.2 reduces the computation of $F(n, r)$ to finitely many steps. If $n$ is small enough, this computation may be carried out by a computer (the interested reader may consult the authors’ personal websites for a possible implementation). We list in Table 1 the first few values of $\lfloor F(n, r) \rfloor$.

Next, by dropping the condition that $d_{j+1}$ is strictly less than $d_j$ for all $j$, we show that we may reduce to the case in which $b$ is the sequence of natural numbers $(0, 1, \ldots, n)$ ranging from 0 to $n$.

Lemma 4.3. Fix $(b, d) \in R_{n,n}$. Then, there is a function $d : \{1, \ldots, n\} \to \mathbb{N}$ with $b \leq d(b) \leq n$ for which we have

$$f(b, d, n, r) \leq \sum_{b=1}^{n} \left[ r \left( \frac{n - b}{n - d(b)} \right) \right]^{1/d(b)}.$$
Proof. By Lemma 4.2, we may assume that all the $b_i$ are integers. Now if $x \leq n$ is a positive real number, we define $i(x)$ by the property $b_{i(x)+1} < x \leq b_{i(x)}$ and set $d(x) = d_{i(x)}$. Therefore, we have

$$ f(b, d, n, r) = \sum_{i=0}^{s} (b_i - b_{i+1}) \left[ r \left( \frac{n - b_i}{n - d_i} \right) \right]^{1/d_i} $$

$$ = \sum_{i=0}^{s} \sum_{b=b_{i+1}+1}^{b_i} \left[ r \left( \frac{n - b_i}{n - d_i} \right) \right]^{1/d_i} $$

$$ = \sum_{i=0}^{s} \sum_{b=b_{i+1}+1}^{b_i} \left[ r \left( \frac{n - b_i}{n - d(b)} \right) \right]^{1/d(b)} $$

$$ \leq \sum_{i=0}^{s} \sum_{b=b_{i+1}+1}^{b_i} \left[ r \left( \frac{n - b}{n - d(b)} \right) \right]^{1/d(b)} $$

$$ = \sum_{b=1}^{n} \left[ r \left( \frac{n - b}{n - d(b)} \right) \right]^{1/d(b)} $$

□

We now need to measure the contribution of each term in Lemma 4.3. We start with the following elementary estimate, which is not optimal, but already implies an upper bound on $F(n, r)$ that is quadratic in $n$ and essentially linear in $r$.

**Lemma 4.4.** Let $b \leq d \leq n$ and $r$ be positive integers. Then, we have

$$ \left[ r \left( \frac{n - b}{n - d} \right) \right]^{1/d} \leq \sqrt[2]{r + n - b}. $$

Proof. First, by $\left( \frac{n - b}{n - d} \right) \leq n^{d-b}$, we have

$$ r^{1/d} \left( \frac{n - b}{n - d} \right)^{1/d} = \left( \frac{b}{\sqrt{r}} \right)^{b/d} \left( \frac{n - b}{n - d} \right)^{1/d} $$

$$ \leq \left( \frac{b}{\sqrt{r}} \right)^{b/d} n^{1-b/d}. $$

Then, using Young’s inequality $A^{\lambda}B^{1-\lambda} \leq \lambda A + (1 - \lambda)B$ for $\lambda \in [0, 1]$, we get

$$ \left( \frac{b}{\sqrt{r}} \right)^{b/d} n^{1-b/d} \leq \frac{b}{d} \sqrt{r} + \left( 1 - \frac{b}{d} \right) n $$

$$ \leq \frac{b}{b} \sqrt{r} + \left( 1 - \frac{b}{n} \right) n $$

$$ = \sqrt{r} + n - b. $$

□
Corollary 4.5. For all positive integers \( n \) and \( r \), we have
\[
F(n, r) \leq \frac{n(n - 1)}{2} + \sum_{b=1}^{n} \sqrt{b} \sqrt{r}.
\]

Proof. This follows from Lemma 4.3 and Lemma 4.4. \( \square \)

4.1 Optimization with Lambert’s \( W \) function

In order to give sharper estimates on \( F(n, r) \) for large values of \( n \), it is convenient to introduce the Lambert function, also known as \textit{productlog}.

Definition 4.6. Consider the bijective function \( u : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0} \) defined by \( u(x) = xe^x \). We define the Lambert function \( W : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0} \) as the inverse of \( u \). In other words, \( w = W(u) \) is the solution to the equation \( we^w = u \).

Note that since \( u(x) = xe^x \) is a nonnegative differentiable function with positive derivative, the function \( W \) is nonnegative, differentiable, and strictly increasing.

Lemma 4.7. Let \( b \leq d \leq n \) and \( r \) be positive integers. Then, we have
\[
\left[ r \left( \frac{n - b}{n - d} \right) \right]^{1/d} \leq \sqrt[d]{r \exp \left( W \left( \frac{n}{b} \sqrt{r} \right) \right)}.
\]

Proof. If \( b = d \) holds, then there is nothing to prove, because the binomial on the left-hand side reduces to 1, while the exponential on the right-hand side is \( \geq 1 \) because \( W \geq 0 \).

If \( b < d \) holds, we start with the crude estimate
\[
\left( \frac{n - b}{n - d} \right) = \frac{(n - b) \cdot (n - b - 1) \cdots (n - d + 1)}{(d - b)!} \leq \frac{n^{d-b}}{(d - b)!}.
\]

By the basic version of Stirling’s inequality \( A! \geq (A/e)^A \), we have
\[
\frac{n^{d-b}}{(d - b)!} \leq \left( \frac{en}{d - b} \right)^{d-b}.
\]

Therefore, we have
\[
\left[ r \left( \frac{n - b}{n - d} \right) \right]^{1/d} \leq r^{1/d} \left( \frac{en}{d - b} \right)^{d-b/d}. \tag{1}
\]

Let \( \delta = d - b \) and write the right-hand side of (1) as
\[
\psi(\delta) = r^{1/b+\delta} \left( \frac{en}{\delta} \right)^{\delta/b+\delta}.
\]

We are now going to treat the expression \( \psi(\delta) \) as a function of a continuous positive real variable \( \delta \), leaving \( b, n, \) and \( r \) as fixed constant positive integer parameters. By taking the logarithmic
derivative in $\delta$, we see that $\psi(\delta)$ is maximized when

$$\log(r) + \delta = b(\log(n) - \log(\delta))$$

holds. This equation admits a unique solution $\delta \in (0, \infty)$, because the left-hand side is a linear function of $\delta$ with positive slope, while the right-hand side is a continuous function, which is strictly decreasing and surjective onto $(-\infty, \infty)$.

If we divide through by $b$ and take the exponential on both sides, then we get

$$\frac{\sqrt[r]{e^\delta}}{b} = \frac{n}{\delta}.$$  

Let $w = \delta/b$. The last displayed equation becomes

$$\sqrt[r]{e^w} = \frac{n}{bw},$$  

which may be rewritten as

$$we^w = \frac{n}{b \sqrt[r]{r}}.$$  

By Definition 4.6, the solution of this equation is given by $w = w_0$, where

$$w_0 = W\left(\frac{n}{b \sqrt[r]{r}}\right)$$  

with $W$ being the Lambert productlog function.

To summarize, the inequality

$$\psi(\delta) \leq \psi(bw_0)$$

holds for every given positive parameters $b, n, r$, and for every $\delta > 0$, where $w_0$ is given by (3). By Equation (2), we have

$$\psi(bw_0) = \sqrt[r]{e^{w_0}} \left(e^{\sqrt[r]{r}w_0}\right)^{\frac{w_0}{w_0+1}} = \sqrt[r]{e^{w_0}}.$$  

Concatenating our estimates, we finally get

$$\left[\frac{r}{n-d}\right]^{1/d} \leq \psi(d-b) \leq \psi(bw_0) = \sqrt[r]{e^{w_0}}.$$  

Remark 4.8. By Equation (2) in the previous proof, the statement of Lemma 4.7 may also be written in the following way:

$$\left[\frac{r}{n-d}\right]^{1/d} \leq \psi(d-b) \leq \psi(bw_0) = \sqrt[r]{e^{w_0}}.$$  

We are ready to prove Theorem 4.1. We start with its first part.
**Theorem 4.9.** Let $n$ be a positive integer. Then,

$$F(n, 1) < \max\{n + 1, n(\log \log(n) + 2.34)\}.$$  

**Proof.** By Lemma 4.3 and Lemma 4.7 with $r = 1$, we get

$$f(b, d, n, 1) \leqslant \sum_{b=1}^{n} e^{W(n/b)}.$$  

Since $W$ is an increasing function, the function $x \mapsto W(n/x)$ is instead decreasing for $x > 0$. Therefore, from the use of Riemann sums, we have

$$\sum_{b=1}^{n} e^{W(n/b)} \leqslant e^{W(n)} + \int_{1}^{n} e^{W(n/x)} dx.$$  

By the change of variable $t = n/x$, we get

$$\int_{1}^{n} e^{W(n/x)} dx = n \int_{1}^{n} e^{W(t)} t^2 dt.$$  

The integrand $e^{W(t)} / t^2$ admits a nice closed-form primitive: $\log W(t) - 1 / W(t)$. To see this, we denote by $W'(t)$ the derivative of $W(t)$, and we reason as follows. First, differentiating the identity $W(t)e^{W(t)} = t$, we get

$$W'(t)(W(t) + 1)e^{W(t)} = 1.$$  

Then, the derivative of $\log W(t) - 1/W(t)$ for $t > 0$ is

$$\frac{W'(t)}{W(t)} + \frac{W'(t)}{W(t)^2} = \frac{W'(t)(W(t) + 1)}{W(t)^2} = \frac{W(t)'(W(t) + 1)e^{W(t)} \cdot e^{W(t)}}{(W(t)e^{W(t)})^2} = \frac{e^{W(t)}}{t^2}.$$  

Therefore, we have

$$f(b, d, n, 1) \leqslant e^{W(n)} + n \left[ \log W(t) - \frac{1}{W(t)} \right]_1^n.$$  

Using the identity $e^{W(n)} = n/W(n)$ for the first term of the right-hand side of the last inequality, we obtain

$$f(b, d, n, 1) \leqslant n \left( \frac{1}{W(n)} + \log W(n) - \frac{1}{W(n)} - \log W(1) + \frac{1}{W(1)} \right) = n \left( \log W(n) - \log W(1) + W(1)^{-1} \right).$$
Now, for $n \geq 3$, we have $W(n) \leq \log n$ since $W(n)e^{W(n)} = n \leq \log n \cdot e^{\log n}$. Thus, if $n \geq 3$ we have

$$f(b, d, n, 1) \leq n \left( \log \log n - \log W(1) + W(1)^{-1} \right).$$

The number $W(1)$ is sometimes known as the Omega constant. It is approximately equal to $0.56714329 \ [14]$ and so, by a direct numerical computation, the following estimate to the second digit holds

$$-\log W(1) + 1/W(1) < 2.34.$$ 

For $n = 2$, we may use Table 1 instead.

Similarly, we prove now the second part of Theorem 4.1.

**Theorem 4.10.** Let $n, r \geq 2$ be positive integers. Then we have

$$F(n, r) < r + n - 1 + \sqrt{rn \log \log(n) + 2.34}.$$ 

**Proof.** As in the case $r = 1$, we start with Lemma 4.3, which gives

$$f(b, d, n, r) \leq \sum_{b=1}^{n} \left[ r \left( \frac{n-b}{n-d(b)} \right) \right]^{1/d(b)}$$

for some $d: \{1, \ldots, n\} \rightarrow \mathbb{N}$ with $b \leq d(b) \leq n$. This time, however, we estimate with Lemma 4.7 only the terms of the sum with $b \geq 2$. For the first term, instead, we use Lemma 4.4. We get

$$f(b, d, n, r) \leq r + n - 1 + \sum_{b=2}^{n} \frac{\sqrt{n}}{\sqrt{r}} e^{W(n/(b \sqrt{r}))}.$$ 

Since $W$ is monotonic increasing, we have $W(n/(b \sqrt{r})) \leq W(n/b)$ for each $b \geq 2$. Moreover, $\sqrt{r} \leq \sqrt{r}$ also holds for $b \geq 2$. Therefore, we have

$$f(b, d, n, r) \leq r + n - 1 + \sum_{b=2}^{n} \sqrt{r} e^{W(n/b)}.$$ 

Using again the fact that $W$ is monotonic increasing, we may continue as in the proof of Theorem 4.9:

$$f(b, d, n, r) \leq r + n - 1 + \sqrt{r} \int_{1}^{n} e^{W(n/x)} dx$$

$$\leq r + n - 1 + \sqrt{r} n (\log W(n) - W(n)^{-1} + 2.34)$$

$$\leq r + n - 1 + \sqrt{r} n (\log \log n + 2.34).$$
Theorem 5.1. Let $X$ be a smooth projective variety of dimension $n$ and let $D$ be an ample $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor. Let $S$ be any finite set of points of $X$ of cardinality $r$. Suppose

$$D^d \cdot Z \geq 1$$

for all irreducible $d$-dimensional subvarieties $Z$ containing at least one point of $S$. Fix any positive rational number $0 < \varepsilon \ll 1$. Then there exists a point $x \in S$ and a $\mathbb{Q}$-divisor $\Delta \in |tD|_\mathbb{Q}$ such that the following hold:

1. $t < F(n, r) + \varepsilon$,
2. $(X, \Delta)$ is log canonical but not Kawamata log terminal at $x$,
3. $\text{LLC}(X, \Delta, x) = \{x\}$,
4. $S$ is contained in $\text{Nklt}(X, \Delta)$.

Proof. We define inductively a sequence of $\mathbb{Q}$-divisors, subvarieties, finite sets of points, positive rational numbers, and positive integers $(D_i, Z_i, T_i, t_i, d_i)$ for $0 \leq i \leq s+1$ as follows. Set $D_0 = 0$, $Z_0 = X, T_0 = S, t_0 = 0$, and $d_0 = n$. Now suppose that we are given $(D_i, Z_i, T_i, t_i, d_i)$. If $i > 0$ holds, suppose the following:

1. $(X, D_i)$ is log canonical at all points of $T_i$,
2. there is a log canonical center $Z_i$ of $(X, D_i)$ that is minimal at all points of $T_i$,
3. every point in $S \setminus T_i$ is contained in a non-klt center of $(X, D_i)$ that does not contain any point of $T_i$,
4. $d_i = \text{dim}(Z_i)$.

If $d_i = 0$, we stop and we set $s = i - 1$. If not, we construct $(D_{i+1}, Z_{i+1}, T_{i+1}, t_{i+1}, d_{i+1})$ as follows. By Proposition 3.2 (see also Remark 3.3 for the case $i = 0$), for any $\varepsilon' > 0$, there exist a nonempty subset $T'$ of $T_i$, a rational number $t$, and a $\mathbb{Q}$-divisor $D' \in |D|_\mathbb{Q}$ such that

1. $(X, D_i + tD')$ is log canonical at all points of $T'$;
2. there is a log canonical center $Z'$ of $(X, D_i + tD')$, which is strictly contained in $Z_i$ and is minimal at all points of $T'$;
3. every point in $S \setminus T'$ is contained in a non-klt center of $(X, D_i + tD')$ that does not contain any point of $T'$;
4. for all points $x \in T'$, we have

$$b_x(X, D_i + tD') \leq b_x(X, D_i) - \frac{t}{(r \text{ mult}_x Z_i)^{1/d_i}} + \varepsilon'.$$

We set $D_{i+1} = D_i + tD', Z_{i+1} = Z', T_{i+1} = T', t_{i+1} = t$, and $d_{i+1} = \text{dim}(Z')$. By construction, $Z_{s+1}$ is 0-dimensional and nonempty. Let $x$ be any point contained in $Z_{s+1}$. We define a sequence of positive rational numbers and positive integers $(b_i, m_i)$ for $0 \leq i \leq s + 1$ as follows. We set $b_0 = n$ and $m_0 = 1$. For any $i > 0$, we set $b_i = b_x(X, D_i)$ and $m_i = \text{mult}_x Z_i$. By (4) above for all $0 \leq i \leq s$ we have

$$t_{i+1} \leq (b_i - b_{i+1} + \varepsilon') \cdot (r m_i)^{1/d_i} + \varepsilon'.$$

By Theorem 3.4 we have that

$$m_i \leq \left(\frac{n - \lfloor b_i \rfloor}{n - d_i}\right).$$
Therefore by (*), if $\epsilon'$ is sufficiently small, we have

$$\sum_{i=0}^{s} t_{i+1} \leq f(b, d, n, r) + \epsilon/2 \leq F(n, r) + \epsilon/2.$$  

We tie break with Lemma 2.8 using $D$ as the ample $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor. For any $\epsilon'' > 0$, we get a divisor $\Delta$ in $|tD|_\mathbb{Q}$ for

$$t \leq (1 - \epsilon'')(F(n, r) + \epsilon/2) + a\epsilon'',$$

where $a$ is the fixed value coming from Lemma 2.8. We may now conclude by taking $\epsilon''$ sufficiently small. 

**Theorem 5.2.** Let $X$ be a smooth projective variety of dimension $n$, let $A$ be an ample divisor, and let $r$ be a positive integer. Then, $|K_X + mA|$ separates $r$ points for any positive integer $m > F(n, r)$.

**Proof.** Since $A$ is Cartier we have that $A^d \cdot Z \geq 1$ for every irreducible $d$-dimensional subvariety $Z$. Let $m > F(n, r)$ be any positive integer. We prove that $|K_X + mA|$ separates $r$ points by induction on $r$. The first step is to show that $|K_X + mA|$ is basepoint free. Let $x$ be any point of $X$ and let $t$ and $\Delta \in |tA|_\mathbb{Q}$ be as in Theorem 5.1 with $S = \{x\}$. Then $t < F(n, 1) + \epsilon \leq F(n, r) + \epsilon$ for a small positive rational number $\epsilon$. In particular, $mA - \Delta$ is ample. Consider the short exact sequence

$$0 \to \mathcal{O}_X(K_X + mA) \otimes I(X, \Delta) \to \mathcal{O}_X(K_X + mA) \to 0.$$  

Since $\text{LLC}(X, \Delta, x) = \{x\}$, we have that $\mathcal{O}_x$ is a direct summand of

$$\frac{\mathcal{O}_X(K_X + mA)}{I(X, \Delta) \otimes \mathcal{O}_X(K_X + mA)}.$$  

Therefore, by taking the associated long exact sequence and by using Theorem 2.5, we get a surjection

$$H^0(X, \mathcal{O}_X(K_X + mA)) \to k,$$

which is what we wanted. Now suppose that $|K_X + mA|$ separates all $r - 1$ points. Fix any set $S$ of $r$ points of $X$. Again, let $t$ and $\Delta \in |tA|_\mathbb{Q}$ be as in Theorem 5.1 with this choice of $S$ and let $x$ be a point of $S$ such that $\text{LLC}(X, \Delta, x) = \{x\}$. Consider once more the short exact sequence (1). We have a splitting

$$\frac{\mathcal{O}_X(K_X + mA)}{I(X, \Delta) \otimes \mathcal{O}_X(K_X + mA)} = \mathcal{O}_x \oplus \frac{\mathcal{O}_X(K_X + mA)}{I' \otimes \mathcal{O}_X(K_X + mA)}$$

for some ideal sheaf $I'$. By lifting a section that is 0 on the second factor and 1 on the first factor, we get a section $s \in H^0(X, \mathcal{O}_X(K_X + mA))$ that vanishes on $S \setminus \{x\}$ and that does not vanish on $x$. Since $|K_X + mA|$ separates the points of $S \setminus \{x\}$ by induction, we have that $|K_X + mA|$ separates the points of $S$. Since $S$ is arbitrary, the statement follows. □
Proof (of Theorem 1.1 and Theorem 1.3). It is immediate from Theorem 5.2 and Theorem 4.1. □

We also record here the following result.

**Corollary 5.3.** Let $X$ be a smooth projective 3-fold and let $A$ be an ample divisor. Then $|K_X + 5A|$ defines a birational morphism.

**Proof.** This is a consequence of Theorem 5.2 and the fact that $F(3, 2) < 5$ holds by Table 1. □

## 5.1 6-folds

Table 1 in Section 4 shows that for $n \leq 4$ the values of the function $F(n, 1)$ are enough to prove Fujita’s basepoint freeness conjecture. For larger values of $n$, however, the geometry of the problem is not fully reflected in the combinatorics of $F$. In fact, it is possible to carry out a slightly finer study by sharpening the inequalities appearing in the proof of Theorem 5.1 in certain geometric situations. This was, for instance, done in [17] and [18] to prove Fujita’s freeness conjecture for $n = 5$. This kind of study does not change the asymptotic behavior in $n$, so we only carry it out here for $n = 6$ as an example.

Let $s < n$ be two positive integers. Consider the set $U_{s, n} \subseteq \mathbb{R}_{\geq 0}^{s+2} \times \mathbb{N}^{s+2} \times \mathbb{N}^{s+2}$ consisting of elements

$$(b, d, m) = (b_0, \ldots, b_{s+1}, d_0, \ldots, d_{s+1}, m_0, \ldots, m_{s+1})$$

satisfying the following conditions:

1. $0 = b_{s+1} < b_s < \cdots < b_1 < b_0 = n$,
2. $0 = d_{s+1} < d_s < \cdots < d_1 < d_0 = n$,
3. $b_i \leq d_i$ for all $1 \leq i \leq s$,
4. $m_i \leq \binom{n-[b_i]}{n-d_i}$ for all $1 \leq i \leq s$,
5. $b_i \leq 2/m_i$ if $d_i = 2$,
6. $d_1 \neq n - 1$.

Consider now the functions

$$g(b, d, m, n) = \sum_{i=0}^{s} (b_i - b_{i+1}) \sqrt{m_i}$$

and

$$G(n) = \max\{g(b, d, m, n) | (b, d, m) \in \bigcup_{s=0}^{n-1} U_{s, n}\}.$$

**Lemma 5.4.** We have $G(6) < 8$.

**Proof.** Fix $s, d, m$, and $n = 6$. Then $U_{s, n} \cap (\mathbb{R}_{\geq 0}^{s+2} \times \{d\} \times \{m\})$ is a polytope and therefore it is the convex hull of its vertices. By linearity of $g$ in the $b_i$ entries, $g$ is maximal at one such vertex $(b, d, m)$. The result is then checked simply by running a computer program. For ease of implementation, we distinguish two cases. The first case is when $b$ consists of integral entries, which presents no difficulties. The second case is when $b$ does not consist of integral entries, which
implies $2 \in \{d_i\}$ holds. Let $i_0$ be the unique index such that $d_{i_0} = 2$. By possibly erasing all entries $(b_i, d_i)$ such that $b_i = b_{i_0}$ for $i < i_0$, we may assume that $i_0$ is the smallest index for which we have that $b_{i_0}$ is not integral. Note that $b_{i_0} = 2/m_{i_0}$ and $m_{i_0} \geq 3$. By $\sum_{i=i_0}^{s} b_i - b_{i+1} \leq b_{i_0} = \frac{2}{m_{i_0}}$ and $m_i^{1/d_i} \leq m_{i_0}^{1/2}$ for $i \geq i_0$, we have

$$\sum_{i=i_0}^{s} (b_i - b_{i+1}) m_i^{1/d_i} \leq \frac{2}{m_{i_0}^{1/2}}.$$ 

Therefore, $g(b, d, m, n)$ is bounded above by the expression

$$\sum_{i=0}^{i_0-2} (b_i - b_{i+1}) \left( \frac{n - \lfloor b_i \rfloor}{n - d_i} \right)^{1/d_i} + (b_{i_0-1} - 2/m_{i_0}) \left( \frac{n - \lfloor b_{i_0-1} \rfloor}{n - d_{i_0-1}} \right)^{1/d_{i_0-1}} + \frac{2}{m_{i_0}^{1/2}}.$$ 

Since the entries $b_i$ are all integral for $i < i_0$, we have once again an easily tractable case. The interested reader may find an implementation of the above algorithm on the authors’ personal website. □

Theorem 5.1 may be slightly sharpened by the following.

**Theorem 5.5.** Let $X$ be a smooth projective variety of dimension $n$ and let $D$ be an ample $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor. Let $x$ be a point of $X$. Suppose

$$D^d \cdot Z \geq 1$$

for all irreducible $d$-dimensional subvarieties $Z$ containing $x$. Fix any positive rational number $0 < \varepsilon \ll 1$. Then there exists a $\mathbb{Q}$-divisor $\Delta \in |tD|_\mathbb{Q}$ such that

1. $t < G(n) + \varepsilon$,
2. $(X, \Delta)$ is log canonical but not Kawamata log terminal at $x$,
3. LLC$(X, \Delta, x) = \{x\}$.

In particular, if $D$ is Cartier and $n = 6$, then $|K_X + mD|$ is basepoint free for all $m \geq 8$.

**Proof.** Consider the sequence $(D_1, Z_1, T_1, t_1, d_1, b_1, m_1)$ obtained by applying the proof of Theorem 5.1. Suppose first that $d_1 \neq n - 1$. Then $(b_1, d_1, m_1)$ belongs to $U_{s,n}$. In fact, conditions (1) – (4) are clear and (5) follows, for example, from [18, Theorem B.1]. Therefore, in this case we are done by relation $(\ast)$ of the proof of Theorem 5.1 and by Theorem 5.2.

On the other hand, if $d_1 = n - 1$, then by the proof of [8, Theorem 4.4] applied to $(n + \varepsilon)D$, we have

$$t_1 \leq n - b_1 \cdot \frac{n-1}{m_1}.$$ 

If one uses this estimate for $t_1$ and relation $(\ast)$ for $t_i$ with $i \geq 2$, we see that $b_1$ simplifies from the expression. Therefore, we may always assume that $(b_1, d_1, m_1)$ belongs to $U_{s,n}$.

Finally, if $n = 6$, then the result follows from Lemma 5.4. □
APPENDIX

In this appendix, we complement Section 4 by showing that the asymptotic bound of Theorem 4.1(1) is optimal. We also show some alternative ways of bounding $F(n, r)$.

A.1 Optimality

Theorem 4.1(1) asserts that $F(n, 1) = O(n \log \log(n))$. Here we show the following:

**Theorem A.1.** For all $n$, we have

$$F(n, 1) \geq \frac{1}{4e} n \log \log(n).$$

We start with the following.

**Lemma A.2.** Let $b \leq d \leq n$ be positive integers such that

$$bW(n/b) \leq d - b \leq 2bW(n/b)$$

and $b \leq n/10$ hold. Then, we have

$$\left(\frac{n - b}{n - d}\right)^{1/d} \geq \frac{1}{4e} \frac{n}{bW(n/b)}.$$

**Proof.** First, note that the identity $W(n/b)e^{W(n/b)} = n/b$ implies $bW(n/b) = ne^{-W(n/b)}$. Since $W$ is monotonic increasing, we conclude that the function $b \mapsto bW(n/b)$ is increasing as well. In particular, from $b \leq n/10$, we get

$$d \leq b + 2bW\left(\frac{n}{b}\right) \leq \frac{n}{10}(1 + 2W(10)) < \frac{n}{2}.$$

We have $W(10) < 2$, because of the inequality $2e^2 > 10$. Therefore, we have

$$d \leq b + 2bW\left(\frac{n}{b}\right) \leq \frac{n}{10}(1 + 2W(10)) < \frac{n}{2}.$$

Then, we may estimate $\left(\frac{n-b}{n-d}\right)$ as

$$\left(\frac{n - b}{n - d}\right) = \frac{(n-b) \cdot (n-b-1) \cdots (n-d+1)}{(d-b)!} \geq \frac{(n/2)^{d-b}}{(d-b)^{d-b}}.$$

Let $w = W(n/b)$. Since $d \geq b + bw$, we have

$$\frac{d - b}{d} \geq \frac{bw}{b + bw} = \frac{w}{w + 1}.$$
by monotonicity with respect to $d$ of the expression on the left-hand side. Since $n \geq 2d \geq 2(d - b)$, we have $\frac{n}{2(d-b)} \geq 1$ and so we have

$$\left( \frac{n - b}{n - d} \right)^{1/d} \geq \left( \frac{n}{2(d - b)} \right)^{(d-b)/d} \geq \left( \frac{n}{2(d - b)} \right)^{w/(1+w)}.$$ 

From $0 \leq w/(1 + w) \leq 1$ and from $d - b \leq 2bw$, we get

$$\left( \frac{n}{2(d - b)} \right)^{w/(1+w)} \geq \left( \frac{n}{4bw} \right)^{w/(1+w)} = \left( \frac{1}{4e} \right)^{w/(w+1)} \left( \frac{en}{bw} \right)^{w/(w+1)} \geq \frac{1}{4e} \left( \frac{en}{bw} \right)^{w/(w+1)}.$$

To conclude, we note that the equality

$$\left( \frac{en}{bw} \right)^{w/(w+1)} = \frac{n}{bw}$$

holds because both sides are equal to $e^w$. □

Lemma A.2 allows us to state sufficient conditions on $(b, d)$ for which $f(b, d, n, 1)$ grows asymptotically as $n \log \log n$.

**Lemma A.3.** Let $n \geq 10$ and let

$$(b_{s+1}, \ldots, b_1) = (0, 1, \ldots, \lfloor n/10 \rfloor)$$

be the whole reverse sequence of integers up to $n/10$. Now suppose $(b, d) \in R_{s,n}$ satisfies the condition

$$b_j W(n/b_j) \leq d_j - b_j \leq 2b_j W(n/b_j)$$

for all $1 \leq j \leq s$. Then, we have

$$f(b, d, n, 1) \geq \frac{1}{4e} n \log \log n.$$

**Proof.** By Lemma A.2, we have

$$f(b, d, n, 1) = n - \lfloor n/10 \rfloor + \sum_{j=1}^{s} \left( \frac{n - b_j}{n - d_j} \right)^{1/d_j}$$
$$\geq \frac{9}{10}n + \frac{1}{4e} \sum_{b=1}^{\lfloor n/10 \rfloor} \frac{n}{bW(n/b)}$$

$$\geq \frac{9}{10}n + \frac{1}{4e} \int_1^{n/10} \frac{n}{bW(n/b)} db.$$  

Then by the substitution $z = n/b$, we get

$$\int_1^{n/10} \frac{n}{bW(n/b)} db = n \int_{10}^{n} \frac{1}{zW(z)} dz.$$  

This last integral admits a closed form as in the proof of Theorem 4.9, because $1/(zW(z)) = e^{W(z)/z^2}$. Therefore, we have

$$n \int_{10}^{n} \frac{1}{zW(z)} dz = n \int_{10}^{n} \frac{e^{W(z)}}{z^2} dz = n \left[ \log W(z) - \frac{1}{W(z)} \right]_{10}^{n}.$$  

Note that $-1/W(n) + 1/W(10) \geq 0$ and $\log W(10) < \log 2$. Thus, we get the following estimate:

$$f(b, d, n, 1) \geq \frac{n}{4e} \log W(n) + \frac{9}{10}n - \frac{\log 2}{4e} n.$$  

Since the inequality

$$W(n) \geq \frac{1}{2} \log n$$  

holds for all $n$, and since $\log 2/(4e) < 0.4$, we are done. □

We are now ready to prove Theorem A.1.

Proof (of Theorem A.1). For $n < 110$, we use the simple inequality $F(n, 1) \geq n$. Indeed, since $\log \log 110 < 4e$, the required estimate follows immediately in this case. For larger $n$, it suffices to show that there exist $(b, d) \in R_{s,n}$ as in Lemma A.3. To this aim, let $n \geq 110$, let $(b_{s+1}, \ldots, b_1) = (0, 1, \ldots, \lfloor n/10 \rfloor)$ and let

$$d_j = b_j + \lceil b_j W(n/b_j) \rceil.$$  

Note that $d_j \leq b_j + 2b_j W(n/b_j)$ for all $1 \leq j \leq s$ because

$$b_j W(n/b_j) \geq W(10) > 1.$$  

It now suffices to show

$$0 = d_{s+1} < d_3 < \cdots < d_1 < d_0 = n.$$
The inequality $d_1 < n$ follows from the following computation:

$$d_1 \leq \frac{n}{10} + \frac{n}{10}W(11) + 1$$

$$< \frac{3n}{10} + 1 \leq n,$$

where we tacitly used inside $W$ the estimate $\lfloor n/10 \rfloor \geq n/11$, valid for all $n \geq 110$. Finally, note that the function

$$\delta(b) = bW(n/b)$$

has derivative given by the formula

$$\delta'(b) = \frac{W(n/b)^2}{W(n/b) + 1}.$$  

If $b \leq n/10$, then $W(n/b) \geq W(10) > 1.7$ and so $\delta'(b) > 1$. This implies that for all $1 \leq j \leq s - 1$, we have

$$d_j - d_{j+1} \geq 2$$

and so in particular $d_{j+1} < d_j$. $\square$

Remark A.4. It is well known that it is possible to get better estimates on Fujita’s conjecture if in the inductive process the dimension of the log canonical centers decreases only by one (that is, if $d_{j+1} = d_j - 1$ for some $j$). See, for example, [5, 8, 9, 17, 18] and see also condition (5) in the definition of $U_s$ in Section 5.1. In the proof of Theorem A.1, however, we have constructed an element $(b, d)$ for which $f(b, d, n, 1)$ grows as $n \log \log n$ and such that

$$d_{j+1} \leq d_j - 2$$

for all $1 \leq j \leq s$. In particular, this shows that a study like the one carried out in Section 5.1 does not change the asymptotic behavior of $F$.

A.2 | Other estimates

We record the following elementary refinement of Lemma 4.4, which is enough to give logarithmic bounds on the problem of separation of $r$ points.

Lemma A.5. Let $b \leq d \leq n$ and $r$ be positive integers. Then,

$$\left[ r \left( \frac{n - b}{n - d} \right) \right]^{1/d} \leq \frac{\tilde{b}r}{e} + e \frac{n}{b} - e.$$  

Proof. By the basic version of Stirling’s inequality $A! \geq (A/e)^A$ applied to $(d - b)!$, we have that

$$\left( \frac{n - b}{n - d} \right)^{1/(d-b)} \leq \frac{e(n - b)}{d - b}.$$
Then, using Young’s inequality \( A^{\lambda} B^{1-\lambda} \leq \lambda A + (1 - \lambda)B \), we get

\[
\frac{r^{1/d}}{n} \binom{n-b}{n-d}^{1/d} = \left( \sqrt[2d]{r} \right)^{b/d} \cdot \left( \frac{n-b}{n-d} \right)^{1/(d-b)}^{1-b/d}
\]

\[
\leq \frac{b}{d} \sqrt[d]{r} + \left( \frac{1-b}{d} \right) \left( \frac{n-b}{n-d} \right)^{1/(d-b)}
\]

\[
\leq \frac{b}{d} \sqrt[d]{r} + \frac{e}{d}(n-b)
\]

\[
\leq \frac{b}{b} \sqrt[d]{r} + \frac{e}{b}(n-b)
\]

\[
= \sqrt[d]{r} + e \frac{n}{b} - e.
\]

\[
\square
\]

**Corollary A.6.** Let \( n, r \) be positive integers. Then

\[
F(n, r) \leq en \log n + \sum_{b=1}^{n} \sqrt[r]{r}.
\]

**Proof.** Let \((b, d) \in R_{s,n}\). By Lemma 4.3, we have

\[
f(b, d, n, r) \leq \sum_{b=1}^{n} \left[ r \left( \frac{n-b}{n-d} \right) \right]^{1/d(b)}
\]

for some function \( d : \{1, \ldots, n\} \to \mathbb{N} \) with \( b \leq d(b) \leq n \). We estimate every term in the sum with Lemma A.5, so we get

\[
f(b, d, n, r) \leq \sum_{b=1}^{n} \left( \sqrt[r]{r} + e \frac{n}{b} - e \right)
\]

\[
= en \left( -1 + \sum_{b=1}^{n} \frac{1}{b} \right) + \sum_{b=1}^{n} \sqrt[r]{r}
\]

\[
\leq en \log n + \sum_{b=1}^{n} \sqrt[r]{r}.
\]

\[
\square
\]

We conclude by including the following simple estimates from below.

**Proposition A.7.** Let \( n, r \) be positive integers. Then,

\[
F(n, r) \geq \max \left\{ \frac{r^{1/n}}{4e} n \log n, \sum_{b=1}^{n} \sqrt[r]{r} \right\}.
\]

**Proof.** The first lower bound follows from Theorem A.1 and the simple observation that

\[
F(n, r) \geq r^{1/n} F(n, 1).
\]
To prove the second lower bound, let
\[ b = d = (n, n - 1, \ldots, 1, 0). \]
Then,
\[ F(n, r) \geq f(b, d, n, r) = \sum_{b=1}^{n} \sqrt[b]{r}. \]

If one fixes \( n \) and let \( r \) be large enough, it is possible to compute \( F(n, r) \) exactly.

**Proposition A.8.** Fix a positive integer \( n \). Then for each large enough positive integer \( r \), we have
\[ F(n, r) = \sum_{b=1}^{n} \sqrt[b]{r}. \]

**Proof.** By Proposition A.7, it is sufficient to prove the inequality \( F(n, r) \leq \sum_{b=1}^{n} \sqrt[b]{r} \). Let \((b, d) \in R_{n, n}\). By Lemma 4.3, we have
\[ f(b, d, n, r) \leq \sum_{b=1}^{n} r \left( \frac{n - b}{n - d(b)} \right)^{1/d(b)} \]
for some \( b \leq d(b) \leq n \). If \( r \) is large enough, then each term of the sum is dominated, respectively, by a \( b \)-th root of \( r \):
\[ r^{1/d(b)} \left( \frac{n - b}{n - d(b)} \right)^{1/d(b)} \leq r^{1/b}. \]

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**REFERENCES**

1. U. Angehrn and Y. T. Siu, *Effective freeness and point separation for adjoint bundles*, Invent. Math. 122 (1995), no. 2, 291–308.
2. L. Ein, *Multiplier ideals, vanishing theorems and applications*, Algebraic geometry—Santa Cruz 1995, vol. 62, Part 1 of Proc. Sympos. Pure Math., Amer. Math. Soc., Providence, RI, 1997, pp. 203–219.
3. L. Ein and R. Lazarsfeld, *Global generation of pluricanonical and adjoint linear series on smooth projective threefolds*, J. Amer. Math. Soc. 6 (1993), no. 4, 875–903.
4. T. Fujita, *On polarized manifolds whose adjoint bundles are not semipositive*, Algebraic geometry, Sendai, 1985, Adv. Stud. Pure Math., vol. 10, North-Holland, Amsterdam, 1987, pp. 167–178.
5. T. Fujita, *Remarks on Ein-Lazarsfeld criterion of spannedness of adjoint bundles of polarized threefolds*, arXiv preprint math/9311013, 1993.
6. G. Heier, *Effective freeness of adjoint line bundles*, Doc. Math. **7** (2002), 31–42.
7. S. Helmke, *On Fujita’s conjecture*, Duke Math. J. **88** (1997), no. 2, 201–216.
8. S. Helmke, *On global generation of adjoint linear systems*, Math. Ann. **313** (1999), no. 4, 635–652.
9. Y. Kawamata, *On Fujita’s freeness conjecture for 3-folds and 4-folds*, Math. Ann. **308** (1997), no. 3, 491–505.
10. J. Kollár, *Singularities of pairs*, Algebraic geometry—Santa Cruz 1995, Proc. Sympos. Pure Math., vol. 62, Amer. Math. Soc., Providence, RI, 1997, pp. 221–287.
11. J. Kollár and S. Mori, *Birational geometry of algebraic varieties*, Cambridge Tracts in Mathematics, vol. 134, Cambridge University Press, Cambridge, 1998. With the collaboration of C. H. Clemens and A. Corti, Translated from the 1998 Japanese original.
12. R. Lazarsfeld, *Positivity in algebraic geometry. II*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], vol. 49, Springer-Verlag, Berlin, 2004. Positivity for vector bundles, and multiplier ideals.
13. H. Matsumura, *Commutative ring theory*, Cambridge Studies in Advanced Mathematics, vol. 8, 2nd ed., Cambridge University Press, Cambridge, 1989. Translated from the Japanese by M. Reid.
14. OEIS Foundation Inc. *Decimal expansion of LambertW(1): the solution to x*exp(x) = 1*, Entry A030178 in the On-Line Encyclopedia of Integer Sequences, 2023. Published electronically at https://oeis.org/A030178.
15. I. Reider, *Vector bundles of rank 2 and linear systems on algebraic surfaces*, Ann. of Math. (2) **127** (1988), no. 2, 309–316.
16. K. Shibata, *Multiplicity and invariants in birational geometry*, J. Algebra **476** (2017), 161–185.
17. F. Ye and Z. Zhu, *Global generation of adjoint line bundles on projective 5-folds*, Manuscripta Math. **153** (2017), no. 3-4, 545–562.
18. F. Ye and Z. Zhu, *On Fujita’s freeness conjecture in dimension 5*, Adv. Math. **371** (2020), 107210, 56. With an appendix by Jun Lu.