THE RIEMANN HYPOTHESIS FOR CERTAIN INTEGRALS OF EISENSTEIN SERIES

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Abstract. This paper studies the non-holomorphic Eisenstein series $E(z, s)$ for the modular surface $PSL(2, \mathbb{Z}) \backslash \mathbb{H}$, and shows that integration with respect to certain non-negative measures $\mu(z)$ gives meromorphic functions $F_\mu(s)$ that have all their zeros on the line $\Re(s) = \frac{1}{2}$. For the constant term $a_0(y, s)$ of the Eisenstein series the Riemann hypothesis holds for all values $y \geq 1$, with at most two exceptional real zeros, which occur exactly for those $y > 4\pi e^{-\gamma} = 7.0555+$. The Riemann hypothesis holds for all truncation integrals with truncation parameter $T \geq 1$. At the value $T = 1$ this proves the Riemann hypothesis for a zeta function $Z_{2, Q}(s)$ recently introduced by Lin Weng, associated to rank 2 semistable lattices over $\mathbb{Q}$.

1. Introduction

Basic objects in the theory of automorphic forms are Eisenstein series, whose Fourier coefficients, particularly their constant terms, give information about $L$-functions. We consider the (completed) non-holomorphic Eisenstein series $E^*(z, s)$ for the modular group $PSL(2, \mathbb{Z})$, which is given for $z = x + iy \in \mathbb{H}$ with $y > 0$ and $\Re(s) > 1$ by

$$E^*(z, s) := \pi^{-s} \Gamma(s) \left( \frac{1}{2} \sum_{(m,n) \neq (0,0)} \frac{y^s}{|mz + n|^2s} \right) = \pi^{-s} \Gamma(s) E(z, s)$$

or, equivalently,

$$E^*(z, s) = \pi^{-s} \Gamma(s) \zeta(2s) \left( \frac{1}{2} \sum_{(c,d) = 1} \frac{y^s}{|cz + d|^2s} \right).$$

It is well-known that for fixed $z$, $E^*(z, s)$ meromorphically continues to the $s$-plane, and satisfies the functional equation

$$E^*(z, s) = E^*(z, 1 - s),$$

and its only singularities are simple poles at $s = 0$ and $s = 1$ with residues $-\frac{1}{2}$ and $\frac{1}{2}$, respectively. In addition $E^*(z, s)$ behaves like a modular form of weight 0, on $\overline{PSL(2, \mathbb{Z}) \backslash \mathbb{H}}$, satisfying

$$E^* \left( \frac{az + b}{cz + d}, s \right) = E^*(z, s) \text{ for } \left[ \begin{array}{cc} a & b \\ c & d \end{array} \right] \in PSL(2, \mathbb{Z}).$$
In particular $E^*(z + 1, s) = E^*(z, s)$ so it has a Fourier expansion

$$E^*(z, s) = \sum_{n=-\infty}^{\infty} a_n(y, s)e^{2\pi inx},$$

with

$$a_n(y, s) = \int_0^1 E^*(x + iy, s)e^{2\pi inx} dx.$$  

The non-constant Fourier coefficients ($n \neq 0$) are given by

$$a_n(y, s) = 2|n|^{s-\frac{1}{2}}\sigma_1(n)\sqrt{y}K_{s-\frac{1}{2}}(2\pi|n|y),$$

in which

$$\sigma_s(n) = \sum_{d|n} d^s = \prod_{p|n} \frac{1 - p^{(e+1)s}}{1 - p^s},$$

and, for positive real $y$, the $K$-Bessel function is

$$K_s(y) = \int_0^\infty e^{-\frac{u}{2}(e^x + e^{-x})} \frac{1}{2}(e^{us} + e^{-us}) du$$

$$= \frac{1}{2} \int_{-\infty}^\infty e^{-\frac{u}{2}(x^2 + x^{-2})} \frac{dx}{x}.$$  

In particular, $a_n(y, s) = a_{-n}(y, s)$, and for fixed $y$ these are entire functions of $s$. The constant term ($n = 0$) is given by the more complicated expression

$$a_0(y, s) = \zeta^*(2s)y^s + \zeta^*(2-2s)y^{1-s},$$

in which $\zeta^*(s)$ is the completed zeta function

$$\zeta^*(s) := \pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s).$$

The constant term $a_0(y, s)$ is a meromorphic function of $s$, and has simple poles at $s = 0$ and $s = 1$ with residues $-\frac{1}{2}$ and $\frac{1}{2}$ independent of $y$; these account for the poles of $E^*(z, s)$.

The object of this paper is to study certain such integrals, related to the constant term of the Eisenstein series, and show conditions under which they satisfy the Riemann hypothesis: All zeros of $F_\mu(s)$ lie on $\Re(s) = \frac{1}{2}$. The measures we consider are nonnegative real-valued measures of finite mass.
The first integral we consider gives a special case of zeta functions recently introduced by Lin Weng ([25], [26, Sec. B.4], [28]) whose general definition uses an integral representation motivated in part by Arakelov geometry. The rank $n$ zeta function $Z_{n,K}(s)$ of a number field $K$ is given in [28, Sec. II] as a Mellin-type integral over a moduli space of rank $n$ semi-stable lattices; this can be reduced to an integral of an Eisenstein series associated to the group $PSL(n,\mathbb{Z})$ over a certain subset of the allowable lattices. Weng [28, Main Theorem A] shows that this zeta function meromorphically continues to $\mathbb{C}$, with singularities being simple poles at $s = 0$ and $s = 1$, and satisfies the functional equation $Z_{n,K}(s) = Z_{n,K}(1 - s)$. For the case of rank 1 lattices over $\mathbb{Q}$, one obtains $Z_{1,\mathbb{Q}}(s) = \zeta^*(s)$, recovering the usual completed Riemann zeta function. For the case of rank 2 lattices over $\mathbb{Q}$, the resulting definition can be simplified to the following integral of an Eisenstein series:

$$Z_{2,\mathbb{Q}}(s) := \int_{D_{ss}} E^*(z, s) d\mu_H(z) = \int \int_{D_{ss}} E^*(x + iy, s) \frac{dx dy}{y^2},$$

(13)
as we explain in the appendix to this paper. Here the set

$$D_{ss} := \{ z = x + iy \mid -\frac{1}{2} \leq x \leq \frac{1}{2}, \ 0 < y \leq 1, \ \text{and} \ x^2 + y^2 \geq 1 \};$$

(14)
represents the set of two-dimensional semi-stable lattices. This integral can be proved to converge absolutely for $0 < \Re(s) < 1$. Using it one can deduce that $Z_{2,\mathbb{Q}}(s)$ extends to a meromorphic function on $\mathbb{C}$ satisfying the functional equation $Z_{2,\mathbb{Q}}(s) = Z_{2,\mathbb{Q}}(1 - s)$, whose only singularities are simple poles at $s = 0$ and $s = 1$, with residue at $s = 1$ (resp. $s = 0$) given by $c_1 = \frac{1}{2}(\pi^2 - 1)$ (resp. $-c_1$). We show that this function satisfies the Riemann hypothesis.

**Theorem 1.** The meromorphic function

$$Z_{2,\mathbb{Q}}(s) := \zeta^*(2s) \frac{1}{s - 1} - \zeta^*(2 - 2s) \frac{1}{s}$$

(15)
has all its zeros on the critical line $\Re(s) = \frac{1}{2}$.

A further result in §4 shows that all these zeros are simple zeros. These results supply evidence that the zeta functions associated to semi-stable lattices introduced by Weng [28] are of interest and deserve further investigation. Note that for $n \geq 2$ the rank $n$ zeta function does not have an Euler product.

We next consider functions given by integration of the Eisenstein series against hyperbolic measure over the truncation region

$$D^T := \{ z = x + iy \mid -\frac{1}{2} \leq x \leq \frac{1}{2} \text{ and } y \geq T \}$$

(16)
in the upper half-plane $\mathbb{H}$. These regions have finite hyperbolic measure, and the associated integral is

$$I(T, s) := \int \int_{D^T} E^*(x + iy, s) \frac{dx dy}{y^2}.$$

(17)
It can be shown that the integral $I(T, s)$ converges absolutely for all $T > 0$, when $0 < \Re(s) < 1$. The integral (17) over the $x$ variable removes all Fourier terms but the constant term, yielding the alternate formula

$$I(T, s) = \int_0^\infty a_0(y, s) \frac{dy}{y^2} = -\zeta^*(2s)\frac{T^{s-1}}{s-1} + \zeta^*(2 - 2s)\frac{T^{-s}}{s},$$

valid for all $T > 0$. For each $T$ this function has at most simple poles at $s = 0$ and $s = 1$ and satisfies the functional equation $I(T, s) = I(T, 1 - s)$; its residues at these points depend on $T$. At $s = 1$ the residue is $-\frac{1}{2}(\pi - \frac{1}{2})$ and $I(T, s)$ is entire at the point $T = \frac{3}{\pi} = 0.9548$. The integration region $D_T$ is contained in the standard fundamental domain

$$\mathcal{F} := \{z | |z| > 1, -\frac{1}{2} < \Re(z) < \frac{1}{2}\}$$

of the modular surface, when $T \geq 1$. We obtain the following result.

**Theorem 2.** For each fixed $T \geq 1$, the meromorphic function

$$I(T, s) = -\zeta^*(2s)\frac{T^{s-1}}{s-1} + \zeta^*(2 - 2s)\frac{T^{-s}}{s}$$

has all its zeros in the critical line $\Re(s) = \frac{1}{2}$.

The hypothesis $T \geq 1$ cannot be relaxed. It can be shown that the Riemann hypothesis fails to hold for all values $0 < T < 1$ by an argument principle method as in Hejhal[12, p. 89]. In §4 we establish for all $T \geq 1$ that all zeros of $I(T, s)$ are simple, using results of [16] (see Theorem 5).

Theorem 2 in the special case $T = 1$ yields Theorem 1, since we have

$$Z_{2, Q}(s) = -\int \int_{D^1} E^*(z, s)d\mu_H(z) = -I_1(s).$$

Zagier [30, Example 1] observes the Eisenstein series integral identity

$$\int \int_{\mathcal{F}} E^*(z, s)d\mu_H(z) \equiv 0,$$

for $0 < \Re(s) < 1$. The identity (21) follows by combining this with the fact that $\mathcal{F}$ is partitioned into the union of $D_{ss}$ and $D^1$ (up to a hyperbolic measure zero set).

The second set of integrals we consider are those giving the constant term $a_0(y, s)$ for fixed $y$, i.e. (6) for $n = 0$. Study of these integrals is motivated by an observation of Dan Bump which is stated in [5, p. 6]: for each $n \neq 0$ and each $y > 0$, the Fourier coefficient $a_n(y, s)$ satisfies the Riemann hypothesis in the $s$-variable. (This fact was noted earlier by D. Hejhal [12, p. 85]. Bump’s point in [5] is to make an analogy with metaplectic Eisenstein series, see the discussion in §5.) The observation follows because the finite Dirichlet series $\sigma_s(n)$ is easily shown from its Euler product to have all zeros on the imaginary axis $\Re(s) = 0$. In addition, for fixed $y > 0$ the $K$-Bessel function $K_{\lambda}(y)$, which is an entire function in the $s$ variable, is known to have all its zeros on the imaginary axis, a result first shown by Polya [17, p. 308]. (Pólya’s result
is stated in terms of $J$-Bessel functions, but is identifiable with a $K$-Bessel function using well-known identities [1, (9.6.2), (9.6.3)]. Therefore one can ask: does a similar property hold for the constant term $a_0(y, s)$, which is now a meromorphic function of $s$? There is an interesting answer for $y \geq 1$. To state it, recall that the modified Riemann hypothesis for a function asserts that all its zeros are either on the line $\Re(s) = \frac{1}{2}$ or on the real axis in the interval $0 < x < 1$. In 1990 D. Hejhal [12, Prop. 5.3 (f)], established that the modified Riemann hypothesis holds for $a_0(y, s)$ for $y \geq 1$ using the Maass-Selberg relations. Here we extend this result by determining the occurrence of real zeros.

**Theorem 3.** For each $y \geq 1$ the constant term of the Eisenstein series

$$a_0(y, s) := \zeta^*(2s)y^s + \zeta^*(2 - 2s)y^{1-s}$$

is a meromorphic function that satisfies the modified Riemann hypothesis. There is a critical value

$$y^* := 4\pi e^{-\gamma} = 7.055507 +$$

such that the following hold:

1. All zeros of $a_0(y, s)$ lie on the critical line for $1 \leq y \leq y^*$.
2. For $y > y^*$ there are exactly two zeros off the critical line. These are real simple zeros $\rho_y, 1 - \rho_y$ with $\frac{1}{2} < \rho_y < 1$. The zero $\rho_y$ is a nondecreasing function of $y$, and $\rho_y \to 1$ as $y \to \infty$.

Hejhal [12, p. 89] noted that for $0 < y < 1$ the function $a_0(y, s)$ has complex zeros off the critical line, with arbitrarily large real part. Recently Haseo Ki [13, Corollary 1] obtained a generalization and strengthening of Hejhal’s results, remarked on further below. In a preprint Ki [14] shows that all non-real zeros of $a_0(y, s)$ are simple zeros on the critical line.

The two real zeros off the critical line given in Theorem 3 seem of some interest because they behave like Landau-Siegel “exceptional zeros” as $y \to \infty$. The occurrence of real zeros for certain Epstein zeta functions was observed in the 1960’s by Bateman and Grosswald [3], and a precise description of the “exceptional zero” phenomenon in this case was given by Stark [18]. Epstein zeta functions are identical (up to a nonzero factor) with $E(z, s)$ for fixed $z$. However the Riemann hypothesis fails in general for Epstein zeta functions; they generically have zeros off the critical line, and often have zeros with $\Re(s) > 1$; see Arenstorf and Brewer [2] for some numerical examples. We do not address the issue of proving simplicity of zeros of $a_0(y, s)$, but Ki’s result [14] together with Theorem 3 gives that for each $y \geq 1$ all zeros of $a_0(y, s)$ are simple, except for a multiple zero at $s = \frac{1}{2}$ occurring for $y = y^*$. As discussed in §4, the simple zeros result is potentially provable along the lines used for $I(T, s)$, but would require additional analysis.

We briefly discuss the proofs. The proof of Theorem 2 rests on inequalities proved on a zero-by-zero basis in a Hadamard product expansion, for a linear combination of two shifted functions. One may trace methods of this kind back to Pólya [17], see also
Here we make a slight change of hypothesis, considering only functions that have both a functional equation $F(s) = \pm F(1 - s)$ and are real on the real axis, so have the reflection symmetry $F(\overline{s}) = \overline{F(s)}$. In the case of automorphic forms, this condition corresponds to self-duality. We formalize the argument in Theorem 4 in §2.

The proof of Theorem 3 follows Hejhal[12, Prop. 5.3 (f)] in deducing the modified Riemann hypothesis using the Maass-Selberg relations. The real zeros are determined by inequality estimates. An alternate proof of the modified Riemann hypothesis here can be given using an extension of the approach of Theorem 1, which is less elegant and relies on numerical calculations. This alternate method generalizes to give information on zeros of the functions

$$H(y, s) := p(s)\zeta^*(s)y^s + p(1 - s)\zeta^*(2 - 2s)y^{1-s}$$

for $y \geq 1$, provided $p(s)$ is a polynomial with real coefficients. It shows that all but finitely many of the zeros of $H(y, s)$ lie on the critical line, that the zeros off the line are confined to a compact set independent of $y \geq 1$ and their number is uniformly bounded for all $y \geq 1$. We hope to treat this method elsewhere.

We now review related work. In the early 1940’s P. R. Taylor, a student of E. C. Titchmarsh, proved a result similar in form to Theorem 3 for $y = 1$. His work was published posthumously [21]. He showed that $\zeta^*(s + \frac{1}{2}) - \zeta^*(s - \frac{1}{2})$ has all its zeros on the critical line. Making the change of variable $s = 2\tilde{s} - \frac{1}{2}$, which maps the critical line to itself, this asserts $F(\tilde{s}) = \zeta^*(2\tilde{s}) - \zeta^*(2\tilde{s} - 1)$, satisfies the RH; the functional equation yields $\zeta^*(2\tilde{s} - 1) = \zeta^*(2 - 2\tilde{s})$. In a different direction, the functions $I(T, s)$ were considered by A. I. Vinogradov and L. Taktajan [24] in 1980, who used them in an interesting heuristic argument in support of the Riemann hypothesis for $\zeta(s)$. In 1981 D. Zagier [29] considered integrals of Eisenstein series against certain nonnegative measures supported either at collections of (special) points or on collections of (special) closed geodesics. He showed that the zero sets of the resulting functions $F_{\mu}(s)$ contained the zeros of the Riemann zeta function, and used them to construct a vector space of functions carrying a representation of $SL(2, \mathbb{R})$ including principal series representations supported at the zeros of the Riemann zeta function. Recently Haseo Ki [13] proved general results strengthening and extending those of Hejhal [12], showing that finite truncations of the fourier expansion of Eisenstein series (summed from $-N$ to $N$) with $z \in \mathbb{Q}(i)$ and $\Im(z)$ sufficiently large have all but finitely many of their zeros simple and lying on the critical line. In a preprint Ki [14] proves that the constant term $a_0(y, s)$ has all its non-real zeros simple and on the critical line; his method easily adapts to determine real zeros, and could be used to give independent proofs of Theorems 1, 2 and 3 above.

The contents of the paper are as follows. In §2 we give a theorem allowing one to deduce that all zeros are on a line, and then use it to prove Theorem 2. In §3 we deduce Theorem 3. In §4 we establish some results on the density of zeros on the critical line of $I(T, s)$ and $a_0(y, s)$, and give some numerical data. We show that for $T \geq 1$ the zeros of $I(T, s)$ are simple, and those with $\Im(\rho) > 0$ have their imaginary parts decrease monotonically as $T$ increases. In §5 we make concluding remarks and
raise further topics for investigation. In an appendix we describe the interpretation of L. Weng’s zeta functions using semistable lattices.

H. Ki (private communication) observes that there are now three distinct approaches to proving Theorem 3. The first is an extension of the proof of Theorem 2 above, which is based on a variant of the Pólya approach, and requires some numerical calculations. The second uses the Maass-Selberg relations as in Lemma 1 below, and is the one given here. The third is the one taken in [14], which uses the Hermite-Biehler theorem, and also requires some numerical calculations; this last approach establishes simplicity of the zeros.

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2. Proof of Theorem 2.

The general mechanism used in the proof of Theorem 2 is formalized in the next result. This result is similar in flavor to a lemma \(^1\) in Titchmarsh[23, p. 280], which traces back to a result of Pólya [17, Hilfssatz II]. Titchmarsh’s lemma assumes (in our notation) that the function \(F(s)\) below has all its zeros on the critical line (and is real there) which we relax to (ii) by assuming extra symmetries in (i).

**Theorem 4.** Let \(F(s)\) be an entire function of genus zero or one, that has the following properties.

(i) \(F(s)\) is real on the real axis, and satisfies a functional equation of form

\[
F(s) = \pm F(1 - s),
\]

for some choice of sign.

(ii) There exists \(a > 0\) such that all zeros of \(F(s)\) lie in the vertical strip

\[
\left| \Re(s) - \frac{1}{2} \right| < a.
\]

Then for any real \(c \geq a\),

\[
\left| \frac{F(s + c)}{F(s - c)} \right| > 1 \quad \text{if} \quad \Re(s) > \frac{1}{2},
\]

\(^1\)Both Titchmarsh and Pólya state their results in terms a linear change of variable \(H(w) = F(\frac{1}{2} + iw)\), asserting that \(G(w) = H(w + ic) + H(w - ic)\) has real zeros if \(H(w)\) is real on the real axis and has only real zeros.
and
\[
\left| \frac{F(s + c)}{F(s - c)} \right| < 1 \quad \text{if } \Re(s) < \frac{1}{2}. \quad (28)
\]

In particular, for any \(0 \leq \theta < 2\pi\) the function
\[
G_{\theta}(s) := F(s + c) + e^{i\theta} F(s - c)
\]
has all its zeros falling on the line \(\Re(s) = \frac{1}{2}\).

**Proof.** The genus one assumption is equivalent to the assertion that the Hadamard product factorization
\[
F(s) = e^{A + Bs} s^R \prod_{\rho} \left( 1 - \frac{s}{\rho} \right) e^{\frac{s}{\rho}} \quad (30)
\]
converges absolutely and uniformly on compact subsets of \(\mathbb{C}\), see Titchmarsh [22, Sec. 8.23, 8.24]. This assumption is also equivalent to the bound
\[
\sum_{\rho} \frac{1}{|\rho|^2} < \infty.
\]
Hypothesis (i) implies symmetries of the zeros around both the real axis and the line \(\Re(s) = \frac{1}{2}\); i.e. under both \(\rho \mapsto 1 - \rho\) and \(\rho \mapsto \bar{\rho}\). It follows that set of zeros \(\rho = \beta + i\gamma\), counted with multiplicity, can be partitioned into blocks \(B(\rho)\) comprising \(\{\rho, 1 - \rho, \bar{\rho}, 1 - \bar{\rho}\}\) if \(\beta \neq \frac{1}{2}\); \(\{\rho, 1 - \rho\}\) if \(\beta = \frac{1}{2}\) and \(\gamma \neq 0\); and \(\{\rho\}\) if \(\rho = \frac{1}{2}\). Each block is labelled with the unique zero in it having \(\beta \leq \frac{1}{2}\) and \(\gamma \geq 0\). We next assert, using hypothesis (ii), that the modified Hadamard product obtained by grouping over blocks, can have the convergence factors \(e^{\frac{s}{\rho}}\) removed, i.e.
\[
F(s) = e^{A + B's} s^R \prod_{B(\rho)} \left( \prod_{\rho \in B(\rho)} \left( 1 - \frac{s}{\rho} \right) \right), \quad (31)
\]
where the outer product on the right side converges absolutely and uniformly on compact subsets of \(\mathbb{C}\). This assertion holds because the block convergence factors \(e^{c(B(\rho))s}\) are given by
\[
c(B(\rho)) = \begin{cases} 
\frac{\beta}{|\rho|^2} + \frac{1 - \beta}{|1 - \rho|^2} & \text{if } \beta \neq \frac{1}{2}; \\
\frac{1}{|\rho|^2} & \text{if } \beta = \frac{1}{2}, \gamma \neq 0; \\
\frac{1}{|\rho|^2} & \text{if } \rho = \frac{1}{2}.
\end{cases}
\]
Hypothesis (ii) gives \(-a < \beta - \frac{1}{2} < a\) hence
\[
\sum_{B(\rho)} |c(B(\rho))| \leq (1 + 2a) \left( \sum_{\rho} \frac{1}{|\rho|^2} \right) < \infty.
\]
Thus the convergence factors can be pulled out of the product, yielding (31), with
\[
B' = B + \sum_{B(\rho)} c(B(\rho)).
\]
Using the functional equation (25) we can further infer that the constant $B' = 0$ in (31), so that
\[ F(s) = e^A \prod_{B(\rho)} \left( \prod_{\rho \in B(\rho)} \frac{1 - \frac{s}{\rho}}{1 - \frac{s + c}{\rho}} \right). \] (32)

Indeed the change of variable $s \mapsto 1 - s$ permutes the factors in each block $B(\rho)$, with a possible sign change for $\rho = \frac{1}{2}$, so it must be that $e^{A + B's} = \pm e^{A + B'(1-s)}$, which forces $B' = 0$.

To establish (27) and (28) we can now proceed block by block in (32), using the factorization
\[ \left| \prod_{\rho \in B(\rho)} \frac{1 - \frac{s + c}{\rho}}{1 - \frac{s - c}{\rho}} \right|. \] (33)

In a single block we can clear denominators to obtain
\[ \prod_{\rho \in B(\rho)} \left| \frac{1 - \frac{s + c}{\rho}}{1 - \frac{s - c}{\rho}} \right| = \prod_{\rho \in B(\rho)} \frac{\left| \frac{s + c - \rho}{s - c - \rho} \right|}{\left| \frac{s + c - \rho}{s - c - \rho} \right|}. \]

The main point is now to compare the term in the numerator with $\rho$ against the term in the denominator with $\rho' := \frac{1}{1 - \rho} = 1 - \bar{\rho}$. We assert that
\[ \left| \frac{s + c - \rho}{s - c - (1 - \rho)} \right|^2 > 1 \text{ if } \Re(s) > \frac{1}{2}, \] (34)
and
\[ \left| \frac{s + c - \rho}{s - c - (1 - \rho)} \right|^2 < 1 \text{ if } \Re(s) < \frac{1}{2}. \] (35)

If (34) is shown, then we may conclude for $\Re(s) > \frac{1}{2}$ that the absolute value of the product over terms in each block on the right in (33) exceeds 1, and (27) follows. Similarly (35) implies that for $\Re(s) < \frac{1}{2}$ the product of terms over each block is smaller than 1, and (28) follows.

It remains to show (34) and (35). Writing $s = \sigma + it$, we have
\[ \left| \frac{s + c - \rho}{s - c - (1 - \rho)} \right|^2 = \frac{(\sigma + c - \beta)^2 + (t - \gamma)^2}{(\sigma - c - 1 + \beta)^2 + (t - \gamma)^2}. \]

Now (34) reduces to the assertion that
\[ (\sigma + c - \beta)^2 > (\sigma - c - 1 + \beta)^2 \text{ when } \Re(s) > \frac{1}{2}, \] (36)

To show this we note that $\Re(s) > \frac{1}{2}$ gives
\[ \sigma + c - \beta > \frac{1}{2} + a - \beta > 0, \]
whence (36) makes the two assertions

\[ \sigma + c - \beta > \sigma - c - 1 + \beta, \]
\[ \sigma + c - \beta > -(\sigma - c - 1 + \beta). \]

The second of these asserts that \( \sigma > \frac{1}{2} \), while the first asserts that \( 2c > 2(\beta - \frac{1}{2}) \), which holds since \( c \geq a > \beta - \frac{1}{2} \). Thus (36) holds, whence (34) holds.

A similar argument is used to establish (35). It reduces to the assertion that

\[ (\sigma + c - \beta)^2 < (\sigma - c - 1 + \beta)^2 \text{ when } \Re(s) < \frac{1}{2}. \]

We have

\[ -(\sigma - c - 1 + \beta) \geq \frac{1}{2} + a - \beta > 0, \]

so that (37) is equivalent to the two assertions

\[ -\sigma + c + 1 - \beta > -(\sigma + c - \beta), \]
\[ -\sigma + c + 1 - \beta > \sigma + c - \beta. \]

The second of these is equivalent to \( \sigma < 1/2 \) and the first to \( c + 1/2 - \beta > 0 \), which holds.

The conclusion that \( G_\theta(s) \) has all its zeros on the line \( \Re(s) = \frac{1}{2} \) follows, because the two terms on the right side of (29) have differing absolute values off this line. \( \square \)

**Remarks.** (1) In the special case when \( e^{i\theta} = \pm 1 \), the function \( G_\theta(s) \) has a functional equation

\[ G_\theta(s) = \pm G_\theta(1 - s) \]

inherited from the functional equation of \( F(s) \). This need not hold for other values of \( \theta \).

(2) The proof of Theorem 4 has two main steps: first, establishing the existence of a modified Hadamard product factorization grouping zeros into finite blocks, with the convergence factors dropped, and second, a local inequality argument that applies to each block separately. The local inequality argument used in the proof paired zeros \( \rho \) and \( 1 - \rho \), a condition that requires only that \( F(s) \) have constant modulus on the critical line, and does not require symmetry around the real axis. This condition is still met when hypothesis (i) is relaxed to:

\[ (i') \text{ } F(s) \text{ satisfies a functional equation of form } \]
\[ F(s) = e^{i\alpha}F(1 - s) \]

for some \( 0 \leq \alpha < 2\pi \).

The hypothesis (i') allows general \( L \)-functions. In the argument above hypothesis (i) was used to get the modified Hadamard product factorization. It is possible to prove an alternative version of Theorem 4 assuming (i') and (ii), provided some extra
hypothesis (iii) is added restricting the locations of zeros, that guarantees the existence of a modified Hadamard product (31) that converges absolutely and uniformly on all compact subsets of the plane. When (i') holds the zeros no longer need to be symmetric about the real axis, so blocks must be chosen differently to get convergence; such blocks remain invariant under the map $\rho \mapsto 1 - \rho$. The functional equation (i') then implies only that $\Re(B') = 0$, but this is sufficient to obtain the result.

It can be shown that automorphic $L$-functions (principal $L$-functions for $GL(n)$) do possess such modified Hadamard product expansions, using asymptotic formulas for zeros given in [15, Theorem 2.1 (4)].

**Proof of Theorem 2.** Let $\xi(s)$ be the Riemann $\xi$-function

$$\xi(s) := \frac{1}{2}s(s-1)\zeta^*(s),$$

which is an entire function. Theorem 4 applies to $F(s) = \xi(2s - \frac{1}{2})$, which satisfies the functional equation $F(s) = F(1-s)$, is real on the real axis, and the zeros of $F(s)$ are confined to $\frac{1}{4} < \Re(s) < \frac{3}{4}$, so we can take $a = \frac{1}{4}$.

Now we apply Theorem 4 with $c = \frac{1}{4}$, which for the function

$$G(s) := F(s + \frac{1}{4}) + F(s - \frac{1}{4}) = \xi(2s) + \xi(2s - 1)$$

yields, for $\Re(s) > \frac{1}{2}$, that

$$\left| \frac{F(s + \frac{1}{4})}{F(s - \frac{1}{4})} \right| = \left| \frac{\xi(2s)}{\xi(2s-1)} \right| > 1 \text{ for } \Re(s) > \frac{1}{2}. \quad (41)$$

Recall now that

$$I(T, s) := -\frac{\zeta^*(2s)}{s-1}T^{s-1} + \frac{\zeta^*(2-2s)}{s}T^{-s} = -\frac{\zeta^*(2s)}{s-1}T^{s-1} + \frac{\zeta^*(2s-1)}{s}T^{-s},$$

using the functional equation $\xi(s) = \xi(1-s)$. For fixed real $T > 0$ the function $I(T, s)$ has simple poles at $s = 0, 1$ and is analytic elsewhere, and satisfies the functional equation $I(T, s) = I(T, 1-s)$. At $s = 1$ it has residue $c_1(T) = -\frac{1}{2}(\frac{\pi}{2} - \frac{\pi}{3})$, and at $s = 0$ residue $-c_1(T)$; both terms in the expression (20) for $I(T, s)$ contribute to these residues. To prove the theorem it suffices to study zeros of the entire function

$$H(T, s) := \frac{1}{4}(2s)(2s-1)(2s-2)I(T, s) = -\xi(2s)T^{s-1} + \xi(2s-1)T^{-s}.$$

This function has a zero at $s = \frac{1}{2}$ and satisfies the functional equation

$$H(T, s) = -H(T, 1-s).$$

Applying the result (41), we have for $\Re(s) > \frac{1}{2}$ and $T \geq 1$ that

$$\left| \frac{-\xi(2s)T^{s-1}}{\xi(2s-1)T^{-s}} \right| = \left| \frac{\xi(2s)}{\xi(2s-1)} \right| T^{2s-1} \geq \left| \frac{\xi(2s)}{\xi(2s-1)} \right| > 1.$$
We conclude for $T \geq 1$ that $H(T, s) \neq 0$ when $\Re(s) > \frac{1}{2}$, and the functional equation (43) then yields $H(T, s) \neq 0$ when $\Re(s) < \frac{1}{2}$. Thus for $T \geq 1$ all zeros of $H(T, s)$ must have $\Re(s) = \frac{1}{2}$. □

3. PROOF OF THEOREM 3.

The simplest entire function associated to $a_0(y, s)$ is

$$G(y, s) := (2s)(2s - 2)a_0(y, s), \quad (44)$$

which behaves similarly to the Riemann $\zeta$-function, satisfying the functional equation $G(y, s) = G(y, 1 - s)$, being real on the real axis and on the line $\Re(s) = \frac{1}{2}$. It also has $G(y, \frac{1}{2}) = (\log 4\pi - \gamma - \log y)\sqrt{y}$, where $\gamma$ is Euler’s constant. However to establish Theorem 3 it proves useful to study instead the entire function

$$H(y, s) := \frac{1}{2}(s - \frac{1}{2})G(y, s) = (s - 1)\xi(2s)y^s + s\xi(2s - 1)y^{1-s}. \quad (45)$$

which adds an extra zero at $s = \frac{1}{2}$. The function $H(y, s)$ satisfies the functional equation $H(y, s) = -H(y, 1 - s)$, but has the advantage that both terms on the right side of (45) are entire functions.

First we will show using the Maass-Selberg relations ( Hejhal [12, Prop. 5.3(f)]) that all zeros of $a_0(y, s)$ lie on the critical line for any $y \geq 1$ except for real zeros. Second we will determine the location of real zeros of $a_0(y, s)$.

Lemma 1 (Hejhal). For any $y \geq 1$, all zeros of $a_0(y, s)$ lie on the critical line except for real zeros.

Proof. Let

$$E^*_T(z, s) := \begin{cases} E^*(z, s) - a_0(y, s) & \text{if } z \in D^T, \\ E^*(z, s) & \text{if } z \in D - D^T \end{cases} \quad (46)$$

where $T \geq 1$. The Maass-Selberg relation is stated as (cf.[11, pp.154–155])

$$(s - \bar{s})(1 - s - \bar{s})\int_D |E^*_T(z, s)|^2 \frac{dxdy}{y^2} = a_0(T, s)(s\zeta^*(2s)T^{s-1} + (1 - \bar{s})\zeta^*(2\bar{s} - 1)T^{-\bar{s}}) - a_0(T, \bar{s})(s\zeta^*(2s)T^{s-1} + (1 - s)\zeta^*(2s - 1)T^{-s}). \quad (47)$$

If $a_0(T, s) = 0$ then, by the reflection principle, $a_0(T, s) = 0$ as well, so that the right side of (47) is zero. But the left side of (47) is non-zero whenever $\Re(s) \neq 0$ and $\Re(s) \neq 1/2$ both hold. □

Lemma 1 shows that we need only to determine the locations of the zeros of $H(y, s) = s(s - 1)(2s - 1)a_0(y, s)$ on the real axis.
Lemma 2. For any fixed $y > 0$, the constant term $a_0(y, \sigma)$ has no zero outside of the open interval $(0, 1)$ as a function of $\sigma$ on $\mathbb{R}$.

Proof. For real $\sigma > 1$ we have $E^*(z, \sigma) > 0$ from the series representation (1). If $a_0(y, \sigma_0) = 0$ for some $\sigma_0 > 1$, then the integral $\int_0^1 E^*(x + iy, \sigma_0)dx$ is equal to 0. This is a contradiction. Thus $a_0(y, \sigma_0) \neq 0$ for $\sigma > 1$ and the same holds for $\sigma < 0$ using the functional equation. \qed

Using Lemma 2 and the functional equation it suffices to deal with $H(y, s)$ for real $s = \sigma$ in the interval $(\frac{1}{2}, 1)$. To prove the theorem it suffices to show that $H(y, \sigma) \neq 0$ on $(\frac{1}{2}, 1)$ for any fixed $1 < y \leq y^* = 4\pi e^{-\gamma}$ and that there is only one zero of $H(y, \sigma)$ in $(\frac{1}{2}, 1)$ for any fixed $y > y^*$.

We can rewrite the condition $H(y, \sigma) = 0$ as

$$\frac{1 - \sigma}{\sigma} \frac{\xi(2\sigma)}{\xi(2\sigma - 1)} y^{2\sigma - 1} = 1. \quad (48)$$

Here we consider the function

$$F(y, \sigma) := y^{2\sigma - 1} f(\sigma) \quad \text{with} \quad f(\sigma) := \frac{1 - \sigma}{\sigma} \frac{\xi(2\sigma)}{\xi(2\sigma - 1)}. \quad (49)$$

From the product formula (31) we find that $f(\sigma) > 0$ for $0 < \sigma < 1$. In fact we see that

$$f(\sigma) = \frac{1 - \sigma}{\sigma} \prod_{B(\rho)} \left( \prod_{\rho \in B(\rho), \gamma > 0} \frac{(2\sigma - \beta)^2 + \gamma^2}{(2\sigma - 2 + \beta)^2 + \gamma^2} \right) > 0. \quad (50)$$

Now let

$$F'(y, \sigma) := \frac{d}{d\sigma} F(y, \sigma). \quad (51)$$

Then we have

$$F'(y, \sigma) = 2y^{2\sigma - 1} f(\sigma) \left( \log y + \frac{1}{2} \frac{f'(\sigma)}{f(\sigma)} \right). \quad (52)$$

Now (50) implies that for $y > 0$ the function $F'(y, \frac{1}{2})$ is a strictly increasing function of $y$. Here we establish the following lemma.

Lemma 3.

(a) For $y > 0$, $F'(y, \frac{1}{2}) = 0$ at the unique value $y^* = 4\pi e^{-\gamma} = 7.055507^+$. 
(b) The function $-f'(\sigma)/f(\sigma)$ is a strictly increasing function of $\sigma$ on $(\frac{1}{2}, 1)$. 
Proof. First we prove (a). Because \( f(\sigma) > 0 \) for \( 0 < \sigma < 1 \), \( F'(y, 1/2) = 0 \) implies that
\[ 2 \log y = -f'(1/2)/f(1/2). \]
We have
\[ -\frac{f'(1/2)}{f(1/2)} = 4 - 2\frac{\xi'(1)}{\xi(1)} + 2\frac{\xi'(0)}{\xi(0)} = 4(1 + \frac{\xi'(0)}{\xi(0)}), \]
using \( \xi(0) = \xi(1) \) and \( \xi'(0) = -\xi'(1) \). Therefore the unique value \( y^* \) where \( F'(y, 1/2) = 0 \) is given by
\[ \log y^* = 2(1 + \frac{\xi'(0)}{\xi(0)}). \]
We recall the fact that
\[ \frac{\xi'(0)}{\xi(0)} = \frac{1}{2} \gamma - 1 + \frac{1}{2} \log 4\pi = -0.0230957+, \]
where \( \gamma = 0.57721+ \) is Euler’s constant, see Davenport [8, pp. 80–82]. Hence \( y^* = 4\pi e^\gamma \).

Next we prove (b). We have
\[ \left( -\frac{f'(\sigma)}{f(\sigma)} \right)' = \frac{2\sigma - 1}{\sigma^2(1 - \sigma)^2} + 2 \left( -\frac{\xi'(2\sigma)}{\xi(2\sigma)} + \frac{\xi'(2\sigma - 1)}{\xi(2\sigma - 1)} \right). \]
The first term in the right hand side in (54) is positive for \( \frac{1}{2} < \sigma < 1 \), so it suffices to show that the second term in the right hand side in (54) is also positive for \( \frac{1}{2} < \sigma < 1 \).
By using the product formula (31) we obtain
\[ -\frac{\xi'(2\sigma)}{\xi(2\sigma)} + \frac{\xi'(2\sigma - 1)}{\xi(2\sigma - 1)} = \sum_{B(\rho) \rho \in B(\rho)} \left( -\frac{1}{2\sigma - \rho} + \frac{1}{2\sigma - 1 - (1 - \rho')} \right) \]
\[ = \sum_{B(\rho) \rho \in B(\rho)} \left( -\frac{2(2\sigma - \beta)}{(2\sigma - \beta)^2 + \gamma^2} + \frac{2(2\sigma - 2 + \beta)}{(2\sigma - 2 + \beta)^2 + \gamma^2} \right), \]
where \( \rho = \beta + i\gamma \).
We show that the derivatives of each term in the second line of (55) are positive for \( \frac{1}{2} < \sigma < 1 \). Take
\[ g(\sigma) := \frac{d}{d\sigma} \left( -\frac{2(2\sigma - \beta)}{(2\sigma - \beta)^2 + \gamma^2} + \frac{2(2\sigma - 2 + \beta)}{(2\sigma - 2 + \beta)^2 + \gamma^2} \right). \]
Then we have
\[ g(\sigma) = \frac{-4\gamma^2 - 4(2\sigma - \beta)^2}{((2\sigma - \beta)^2 + \gamma^2)^2} + \frac{4\gamma^2 + 4(2\sigma - 2 + \beta)^2}{((2\sigma - 2 + \beta)^2 + \gamma^2)^2} \]
\[ = \frac{4(2\sigma - 1)(1 - \beta)|4\gamma^2\{(2\sigma - \beta)^2 + (2\sigma - 2 + \beta)^2\} + 12\gamma^4 - 4(2\sigma - \beta)^2(2\sigma - 2 + \beta)^2|}{((2\sigma - \beta)^2 + \gamma^2)^2} \]
\[ \cdot \frac{(2\sigma - 2 + \beta)^2 + \gamma^2)^2}{(2\sigma - 2 + \beta)^2 + \gamma^2)^2}. \]
Because \( 0 < \beta < 1 \)
\[ 4(2\sigma - \beta)^2(2\sigma - 2 + \beta)^2 < 16 \]
for $\frac{1}{2} < \sigma < 1$ and we know that $\gamma > 14$. Hence $g(\sigma) > 0$ for $\frac{1}{2} < \sigma < 1$. This implies $(-f'(\sigma)/f(\sigma))' > 0$ for $\frac{1}{2} < \sigma < 1$, which proves the lemma. \square

To complete the proof of Theorem 3, from Lemma 3 (b), we have $F'(y, \sigma) = 0$ for at most one $\sigma \in (\frac{1}{2}, 1)$ for any fixed $y \geq 1$. Now suppose $1 \leq y \leq y*$. We then have

$$\log y \leq \log y* = -\frac{1}{2} \frac{f'(\frac{1}{2})}{f(\frac{1}{2})} < -\frac{1}{2} \frac{f'(\sigma)}{f(\sigma)}$$

(57)

for $\frac{1}{2} < \sigma < 1$. Therefore

$$\log y + \frac{1}{2} \frac{f'(\sigma)}{f(\sigma)} < 0$$

(58)

so that $F'(y, \sigma) < 0$ on $(\frac{1}{2}, 1)$. Since $F(y, \frac{1}{2}) = 1$, $F(y, \sigma) \neq 1$ for any $\frac{1}{2} < \sigma < 1$, and this implies that $H(y, \sigma) \neq 0$ on $(\frac{1}{2}, 1)$.

Next suppose $y > y*$. Then there is a unique $\sigma_0$ with $\frac{1}{2} < \sigma_0 < 1$ such that

$$\log y* < \log y = -\frac{1}{2} \frac{f'(\sigma_0)}{f(\sigma_0)}.$$  

(59)

because $-f'(\sigma)/f(\sigma) \to +\infty$ monotonically as $\sigma \to 1$. Further

$$-\frac{1}{2} \frac{f'(\sigma_1)}{f(\sigma_1)} < -\frac{1}{2} \frac{f'(\sigma_0)}{f(\sigma_0)} < -\frac{1}{2} \frac{f'(\sigma_2)}{f(\sigma_2)}$$

for $\frac{1}{2} \leq \sigma_1 < \sigma_0 < \sigma_2 < 1$,

hence $F'(y, \sigma_1) > 0$ for $\frac{1}{2} \leq \sigma_1 < \sigma_0$ and $F'(y, \sigma_2) < 0$ for $\sigma_1 < \sigma_2 < 1$. Since $F(y, \frac{1}{2}) = 1$ and $F(y, 1) = 0$, these imply there is a unique value $\sigma_y$ in $(\frac{1}{2}, 1)$ such that $F(y, \sigma_y) = 1$, and this value has $\sigma_0 < \sigma_y < 1$. This is exactly the condition for $H(y, \sigma) = 0$ so we conclude that the unique value where this occurs in $(\frac{1}{2}, 1)$ is $\sigma = \sigma_0$.

We also find that $\sigma_0 \to 1$ as $y \to +\infty$, which implies that $\sigma_y \to 1$ as $y \to +\infty$. This completes the proof of Theorem 3. \square

4. Distribution of Zeros

We have shown in §2 that the Riemann hypothesis holds for $I(T, s)$ for each fixed $T \geq 1$ and in §3 that the modified Riemann hypothesis holds for $a_0(y, s)$ for each $y \geq 1$. Here we consider how the zeros behave as the parameter $T$ or $y$ is varied, and also consider the issue of simplicity of zeros. Note that the zeros of these functions vary continuously in $T$ as the parameter $T$ is varied, and vary analytically in $T$ as long as they are simple zeros.

In what follows we let $N(f, U)$ count the number of zeros of the function $f(s)$ having $|\Im(s)| \leq U$.

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\[2\] Asymptotically we see that $\sigma_0 = 1 + O(\frac{1}{\log y})$. This suggests that $\sigma_y = 1 + O(\frac{1}{\log y})$. 

**Theorem 5.** (1) For each fixed $T \geq 1$ the function $I(T, s)$ has simple zeros. The number of zeros of $I(T, s)$ with $|\Im(s)| \leq U$ satisfies

$$N(I(T, s); U) = N(\xi(2s); U) + \frac{2}{\pi}(\log T)U + O(\log U).$$

(2) As $T \geq 1$ increases, each zero $\rho$ of $I(T, s)$ with $\Im(\rho) > 0$ has imaginary part that is a strictly decreasing function of $T$.

**Remark.** Standard estimates from the theory of the Riemann zeta function show that

$$N(\xi(2s), U) = \frac{2}{\pi}U \log U - \frac{2}{\pi}(\log \pi + 1)U + O(\log U).$$

see for example [8, p. 98].

**Proof.** We study the variation in argument of the entire function

$$H(T, s) := \frac{1}{4}(2s)(2s - 1)(2s - 2)I(T, s) = -\xi(2s)T^{s-1} + \xi(2 - 2s)T^{-s},$$

on the critical line $s = \frac{1}{2} + it$. The function $H(T, s)$ has the same zeros as $I(T, s)$, with multiplicity, except for an extra zero at $s = \frac{1}{2}$ (where $I(T, \frac{1}{2}) \neq 0$) and by Theorem 2 all its zeros lie on $\Re(s) = \frac{1}{2}$. It is pure imaginary-valued on the critical line, with

$$H(T, \frac{1}{2} + it) = -\xi(1 + 2it)T^{-\frac{1}{2} + it} + \xi(1 - 2it)T^{-\frac{1}{2} - it}.$$ 

Now set $\xi(1 + it) = R(t)e^{i\theta(t)}$ with $R(t) = |\xi(1 + it)|$, where the argument $\theta(t)$ is measured continuously from $\theta(0) = 0$ on the real axis. Then we have

$$H(T, \frac{1}{2} + it) = -\frac{2i}{\sqrt{T}}R(2t)\sin(\theta(2t) + t\log T),$$

We detect zeros exactly when

$$f(T, t) := \theta(2t) + t\log T \equiv 0 \pmod{\pi},$$

but a priori have no information on their multiplicity.

(1) To show the zeros detected by (62) are simple zeros of $H(T, s)$, we use results from Lagarias [16]. We treat the cases $T = 1$ and $T > 1$ separately. For $T = 1$ we use a special case of Theorem 2.1 of [16], which asserts that

$$\xi_{1/2, \pi/2}(s) := -\frac{1}{2i}\left(\xi(s + \frac{1}{2}) - \xi(s - \frac{1}{2})\right)$$

has simple zeros which all lie on the critical line $\Re(s) = \frac{1}{2}$. Using the functional equation we have

$$\xi_{1/2, \pi/2}(2s - \frac{1}{2}) = -\frac{1}{2i}\left(\xi(2s) - \xi(2s - 1)\right) = -\frac{1}{2i}\left(\xi(2s) - \xi(2 - 2s)\right) = -\frac{1}{2i}H(1, s)$$

It follows that all the zeros of $H(1, s)$ are simple and lie on the critical line.

For $T > 1$ it is sufficient to show that $\frac{d}{dt}f(t, T) > 0$ for all $t \in \mathbb{R}$. We have $\frac{d}{dt}f(t, T) = \frac{d}{dt}f(t, 1) + \log T$, and since $\log T > 0$ it suffices to establish $\frac{d}{dt}f(t, 1) \geq 0$ for all $t$. We now apply Lemma 2.2 of [16] to $E(s) = \xi(s + \frac{1}{2})$, whose hypotheses are
met by Lemma 2.1 of that paper. Now equation (2.10) of that paper yields \( \frac{d}{dt} \theta(t) \geq 0 \) for all \( T \), as required.

It follows that \( H(T, s) \) has simple zeros for each \( T \geq 1 \), hence so does \( I(T, s) \). For fixed \( T \) we now may number the zeros of \( H(T, s) \) with \( \Im(\rho) > 0 \) in order of increasing imaginary part as \( 0 < \gamma_1(T) < \gamma_2(T) < \ldots \); and those with \( \Im(\rho) < 0 \) are numbered by \( \gamma_{-n}(T) = -\gamma_n(T) \); here \( \gamma_0(T) = 0 \) is the zero of \( H(T, s) \) at \( s = \frac{1}{2} \).

To count zeros, we have from (62) that

\[
N(I(T, s); U) = \frac{2}{\pi} f(T, U) + O(1) = \frac{2}{\pi} f(1, U) + \frac{2}{\pi} (\log T) U + O(1).
\] (63)

The function \( f(1, U) = \theta(2U) \) measures the change in argument of \( \xi(s) \), from \( s = 1 \) to \( s = 1 + 2iU \). Following Davenport [8, Chap. 15], this argument change differs from that of \( \xi(\frac{1}{2} + 2iU) \) by \( O(\log U) \), and the argument change to this point is \( \frac{1}{2}(2\pi) N(\xi(2s), U) + O(1) \), since the value \( (2\pi)N(\xi(2s), U) \) is a contribution of four terms of this type. We conclude that \( \frac{2}{\pi} f(1, U) = N(\xi(2s), U) + O(\log U) \). Substituting this in (63) yields (60).

(2) This fact follows directly from (62). That is, \( f(T, t) \) is strictly increasing in both \( t \) and \( T \), so increasing \( T \) increases the rate of turning of the function, so the \( n \)-th time the value \( 0(\mod \pi) \) is reached is a strictly decreasing function of \( T \). \( \square \)

The first few zeros of \( I(T, s) \) for \( T = 1 \) and \( T = y^* = 4\pi e^{-\gamma} = 7.055507+ \) are given in Table 1 below. (The value \( T = y^* \) was chosen for comparison with the zeros of \( a_0(y, s) \) zeros at this point.)

| \( T = 1 \) | \( T = y^* \) |
|---|---|
| 1 | 7.769080112 | 1.570199673 |
| 2 | 11.019004020 | 3.136172650 |
| 3 | 13.110798330 | 4.688303082 |
| 4 | 15.580525820 | 6.172131737 |
| 5 | 17.073670930 | 7.073883755 |
| 6 | 19.215398180 | 7.990000858 |
| 7 | 20.836592680 | 9.408931700 |
| 8 | 22.247541620 | 10.42818054 |
| 9 | 24.269734590 | 11.18590514 |
| 10 | 25.396497160 | 12.29085328 |
| 11 | 26.956100300 | 12.94293137 |
| 12 | 28.595714660 | 14.21495758 |
| 13 | 29.931196390 | 15.15516152 |
| 14 | 31.035610850 | 15.87628765 |
| 15 | 32.837371700 | 16.56289948 |

Table 1. Zeros of constant term \( I(T, s) \) on critical line
For the constant term \( a_0(y, s) \) similar properties of its zeros should hold, with minor modifications. One can show by similar arguments that

\[
N(a_0(y, s), U) = N(\xi(2s), U) + \frac{4}{\pi} \log yU + O(1).
\]

The motion of the zeros as \( y \geq 1 \) is slightly more complicated than for \( I(y, s) \), since a multiple zero occurs at \( y = y^* \), and two zeros eventually migrate to the real axis. It appears that, aside from these two zeros, all zeros are simple and their ordinates monotonically decrease as the parameter \( y \) increases. To prove this rigorously one have to consider instead of the argument of \( \xi(2s)y^{s-1} \) the argument of \( (2s - 2)\xi(2s)y^s \) on the critical line, and the argument of \(-1 + 2it\) turns in the wrong direction. Numerical and analytic estimates would be needed to control this effect. Presumably the contribution of \(-1 + 2it\) and its conjugate to the argument accounts for the escape of the two zeros to the real axis. The first few complex zeros of \( a_0(y, s) \) are given the values \( y = 1 \) and \( y = y^* \) in Table 2 below. Note that the two zeros at \( s = \frac{1}{2} \) of \( a_0(y, s) \) for \( y = y^* \) are omitted from this table. If the zeros were renumbered to include them as the first two zeros, it appears that each zero of \( a_0(y^*, s) \) will be closer to the real axis than the corresponding zero of \( I(y^*, s) \).

| \( y = 1 \)          | \( y = y^* \)          |
|----------------------|------------------------|
| 1  6.9746831333      | 2.244794235            |
| 2 10.40228756        | 3.851296383            |
| 3 12.42264167        | 5.404657031            |
| 4 15.08382464        | 6.732441081            |
| 5 16.40560282        | 7.383718196            |
| 6 18.68201963        | 8.670185248            |
| 7 20.34995710        | 10.02271471            |
| 8 21.60499108        | 10.69728308            |
| 9 23.85087057        | 11.78575276            |
| 10 24.8364580        | 12.56610869            |
| 11 26.40277087       | 13.53142535            |
| 12 28.11180718       | 14.79167003            |
| 13 29.54150449       | 15.42550847            |
| 14 30.39424164       | 16.28902291            |
| 15 32.41487455       | 16.93621484            |

Table 2. Zeros of constant term \( a_0(y, s) \) on critical line

5. Concluding Remarks

The observation of Bump ([5, p. 6]) mentioned in the introduction, concerning the Riemann hypothesis property holding for the Fourier coefficients \( a_n(y, s) \) \((n \neq 0)\), consisted of a direct verification, and gave no conceptual explanation why the Riemann
hypothesis holds in these cases. Bump observed, more strikingly, that the truth of the Riemann hypothesis for certain Fourier coefficients of metaplectic Eisenstein series would imply the Riemann hypothesis for the Riemann zeta function and for various Dirichlet $L$-functions. The function field analogue for metaplectic Eisenstein series was unconditionally proved by D. Cardon ([6],[7]) in some cases. Cardon’s proofs also are direct verifications, making use of the truth of the Riemann hypothesis for curves. Here our result Theorem 3 is again a direct verification without providing a mechanism. A challenging open question is to find a conceptual explanation (if there is one) for the Riemann hypothesis property for Fourier coefficients of such Eisenstein series.

There also remains the problem of conceptually explaining the difference in behavior of the constant term compared to the other Fourier coefficients; why must the extra condition $y \geq 1$ be imposed, and what is the meaning of “exceptional zeros”? Theorem 2 suggests the possibility that the “truncated” Fourier coefficients be considered, for there the Riemann hypothesis does hold for the “truncated” zero-th Fourier coefficient for $y \geq 1$. This encourages further study of related integrals.

One may ask whether the integrated versions of the Fourier coefficients $a_n(y, s)$ over $\mathcal{D}_T$ satisfy the Riemann hypothesis. The integrals in question are

$$J_n(T, s) := \int \int_{\mathcal{D}_T} a_n(y, s) \frac{dxdy}{y^2}. \tag{64}$$

for integer $n$. In effect we are integrating $E(z, s)$ against the twisted hyperbolic measure $e^{-2\pi inx} \frac{dxdy}{y^2}$. This question is related to determining the location of the zeros of the special function

$$\tilde{K}_s(y) =: \int_y^\infty u^{-\frac{3}{2}} K_s(u)du \tag{65}$$

for fixed $y > 0$. If for all $y > 0$ the zeros of $\tilde{K}_s(y)$ lie on the imaginary axis, then the Riemann hypothesis will hold for all $J_n(T, s)$ for all $T > 0$.

Another interesting integral of Eisenstein series is the integrated version of the Fourier coefficients $a_n(y, s)$ over the entire modular surface $\mathcal{F}$, i.e.

$$K_n(s) := \int \int_{\mathcal{F}} a_n(y, s) \frac{dxdy}{y^2}.$$ 

The observation of Zagier states that $K_0(s) \equiv 0$, because $K_0(s)$ is an eigenfunction of the non-Euclidean Laplacian. However this is not the case for the other $K_n(s)$, which are potentially interesting functions.

It might also be of interest to study properties of the vertical distribution of the zeros of these functions, particularly as $y$ varies. Should some analogue of the GUE property hold for these zeros? The results of [16] on a related problem suggest that one should expect the zeros of these functions to be very smoothly spaced, with no GUE behavior.

The existence of zeros on the real axis for individual Epstein zeta functions was noted long ago. In 1964 Bateman and Grosswald [3] gave a criterion for individual Epstein zeta functions to have a real zero, whose main term was $y^* = 4\pi e^{-\gamma}$ with a very small
error term depending on a parameter $k$. In terms of this paper, their parameter $k = y$, and one of their results can be rephrased as saying that each function $E(z, s)$ with $z = x + iy$ always has a real zero when $y \geq 7.0556$ and never has a real zero when $k \leq 7.0554$. For the constant term $a_0(y, s)$ obtained the exact cutoff value $y^* = 4\pi e^{-\gamma}$ in Theorem 3, and we note that the constant term $a_0(y, s)$ is obtained by averaging over $x$ of $E(z, s)$, for fixed $y$.

6. Appendix: Stable and semi-stable lattices

The rank $n$ zeta function $A_{N,K}(s)$ of a number field $K$ is given in [28] as a Mellin-type integral over a moduli space of rank $n$ semi-stable lattices; this can be reduced to an integral of an Eisenstein series associated to the group $PSL(n, \mathbb{Z})$ over a certain subset of the allowable lattices.

The notion of semistability for lattices was introduced by Stuhler [19], in analogy with a notion of semistability for vector bundles on curves, as given in Harder and Narasimhan [10]. The definition was given more generally for $\mathcal{O}_K$-lattices, which are lattices having a ring of endomorphisms including $\mathcal{O}_K$, the ring of algebraic integers of a number field $K$. For simplicity we treat here the case $K = \mathbb{Q}$, where $\mathcal{O}_\mathbb{Q} = \mathbb{Z}$, and a $\mathbb{Z}$-lattice is just a lattice. We associate to a rank $r$ lattice $L$ embedded in $\mathbb{R}^n$ (with $r \leq n$) having basis $V_L = \mathbb{Z}[v_1, ..., v_r]$ of row vectors $v_j = (v_{j,1}, ..., v_{j,n})$, so that $V_L$ is an $n \times r$ matrix, a covolume

$$\text{Vol}(L) := |\det(V_L v_L^T)|^{\frac{1}{2}}.$$  \hspace{1cm} (66)

The slope $s(L)$ of $L$ to be

$$s(L) = \frac{\log(\text{Vol}(L))}{\dim(L)} = \frac{1}{r} \log(\text{Vol}(L)).$$ \hspace{1cm} (67)

Now suppose that $L$ is an $n$-dimensional lattice $L$ embedded in $\mathbb{R}^n$. For $1 \leq r \leq n$ we define the invariants

$$\kappa_r(L) := \min\{\text{Vol}(L') : L' \text{ an rank } r \text{ sublattice of } L\}.$$ \hspace{1cm} (68)

The minimal slope of rank $r$ sublattices of $L$ is given by

$$s_r(L) := \frac{1}{r} \log \kappa_r(L), \hspace{1cm} 1 \leq r \leq n.$$ \hspace{1cm} (69)

In particular $s_n(L) = \frac{1}{n} \log \text{Vol}(L)$. We also artificially define $s_0(L) = 0$.

Next we plot the points $(0, 0)$ and $\{(r, s_r(L)) : 1 \leq r \leq n\}$ as points in the plane, and form and their convex hull, which either forms a polygon, or in degenerate cases a line segment. The relevance of convexity is the inequality (Stuhler [19, Prop.2])

$$\text{Vol}(L_1 \cap L_2) \text{Vol}(L_1 + L_2) \leq \text{Vol}(L_1) \text{Vol}(L_2).$$ \hspace{1cm} (70)

valid for two sublattices $L_1, L_2$ of a lattice $L$, cf. Grayson [9, Theorem 1.12]).

Now consider the lower envelope of this polygon (the lowest points in it when intersected with vertical lines), which forms a finite union of line segments with different slopes, increasing from left to right, with vertices occurring at ranks $0 = r_0 < r_1 < \ldots$
\[
\cdots < r_k = n. \text{ Grayson [9, p. 608] terms this envelope the \textit{canonical polygon} of } L, \text{ and the endpoints of these segments } \{(r, s_r(L))\} \text{ comprise the \textit{canonical vertices}. The canonical vertices always include } (0,0) \text{ and } (n, s(L)). \text{ The key fact about them, due to Stuhler (cf. Grayson [9, Theorem 1.18]), is that there is a unique sublattice } L_{r_j} \text{ in } L \text{ of rank } r_j \text{ having slope } s_{r_j}(L), \text{ and the set of such lattices are totally ordered by inclusion. This chain of lattices is analogous to of the Harder-Narasimhan canonical filtration for vector bundles on curves. }
\]

The definition of Stuhler [19] is the following. A \( \mathbb{Z} \)-lattice \( L \) is \textit{semistable} if the canonical polygon is a line segment joining \((0,0)\) to \((n, s(L))\), and it is \textit{stable} if there are no other points \((r, s_r(L))\) on this line segment. It is \textit{unstable} otherwise, i.e. if there are at least two line segments of different slopes in the canonical polygon. For example, the lattice \( \mathbb{Z}^n \) for \( n \geq 2 \) is semistable but not stable, because it has \( \kappa_j(\mathbb{Z}^n) = 1 \) for \( 1 \leq j \leq n \) whence the points determining the polygon are \( \{(r,0) : 0 \leq r \leq n\} \).

For the general case of integers \( \mathcal{O}_K \) of an algebraic number field \( K \) the notion of \( \mathcal{O}_K \)-semistability is defined considering only the slopes of those rank \( r \) sublattices that the action of \( \mathcal{O}_K \) takes into themselves; see Stuhler [19], [20] or Grayson [9] for details.

Returning to the case of \( \mathbb{Z} \)-lattices, the space of \( n \)-dimensional \( \mathbb{Z} \)-lattices can be identified with \( GL(n, \mathbb{Z})\backslash GL(n, \mathbb{R}) \), in which \( GL(n, \mathbb{R}) \) represents the set of all bases of the lattice, and the action of \( GL(n, \mathbb{Z}) \) determines equivalent bases. Each point of this space can be assigned the property: semistable or unstable. The property of being semistable is invariant under rotations of \( \mathbb{R}^n \), which is an operation that leaves the canonical polygon unchanged. The property of semistability is also preserved under homotheties \( L \mapsto \lambda L \) for \( \lambda > 0 \), although this does change the shape of the canonical polygon to \((0,0)\cup\{(r, s_r(L) + \log \lambda) : 1 \leq r \leq n\}\). These properties allow a notion of semistability (or unstability) to be unambiguously assigned to points of \( SL(n, \mathbb{Z})\backslash SL(n, \mathbb{R})/O(n, \mathbb{R}) \).

Now we consider the special case \( n = 2 \). The space \( SL(2, \mathbb{Z})\backslash SL(2, \mathbb{R})/O(2, \mathbb{R}) \) can be identified with the upper half-plane \( \mathbb{H} := \{z = x + iy : y > 0\} \equiv SL(2, \mathbb{R})/O(2, \mathbb{R}) \) under the action of the modular group \( SL(2, \mathbb{Z}) \) acting by linear fractional transformations. The point \( z \in \mathbb{H} \) corresponds to the lattice \( L_z = \mathbb{Z}[1, z] = \mathbb{Z}[(1,0), (x, y)] \), which has \( \det(L_z) = y \). Now restrict to the standard fundamental domain

\[
\mathcal{F} = \{z = x + iy : -\frac{1}{2} \leq x \leq \frac{1}{2}, |z|^2 \geq 1\}
\]

of \( SL(2, \mathbb{Z})\backslash \mathbb{H} \). The shortest vector in the lattice \( L_z \) then has length 1, which gives \( \kappa_1(L_z) = 1 \). The canonical polygon is therefore generated by the points \((0,0),(1,0)\) and \((2, \frac{1}{2}\log(y^2))\). The condition for semistability for \( z = x + iy \in \mathcal{F} \) is that \( y \leq 1 \), cf. Grayson [9, Example 1.25]. Thus the points in \( \mathcal{D}_{ss} \) given by (13) in §1 represent the moduli space of rank 2 semistable lattices over \( \mathbb{Q} \), quotiented by the action of homothety and rotations of space.
The rank \( r \) vector bundle \( L \)-function \( \zeta_{Q,r}(s) \) associated to the rational field \( Q \) that was introduced by Lin Weng can be expressed ([27, p. 8]) as

\[
\zeta_{Q,r}(s) = \frac{r}{2} \pi^{-\frac{r^2}{2}} \Gamma\left(\frac{rs}{2}\right) \int_{\mathcal{M}_{Q,r}[1]} \left( \sum_{x \in \Lambda \setminus 0} \frac{||x||^{-rs}}{||x||} \right) d\mu_1(\Lambda)
\]

The measure \( \mu_1(\Lambda) \) is the usual measure on lattices induced from a (suitably normalized) Haar measure on \( GL(n, \mathbb{R}) \), whose column vectors represent a basis of \( \Lambda \); equivalence of bases corresponds to a \( GL(n, \mathbb{Z}) \) action. The subset \( \mathcal{M}_{Q,r}[1] \) corresponds to the set of semistable lattices of determinant one, choosing a basis of positive determinant. The inner sum in the integral above is an Epstein zeta function of the positive definite quadratic form in \( r \) variables giving the squared norm of vectors in the lattice \( \Lambda \), which is \( Q(x) = x^T B^T B x \). In the case \( r = 2 \) this can be identified with the non-holomorphic Eisenstein series in the paper, and the integral above is simplifiable to the integral (13).

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