An Involution on Semistandard Skyline Fillings

NEIL J.Y. FAN, PETER L. GUO, NICOLAS Y. LIU

Abstract. Non-attacking skyline fillings were used by Haglund, Haiman and Loehr to establish a combinatorial formula for nonsymmetric Macdonald polynomials. Semistandard skyline fillings are non-attacking skyline fillings with both major index and coinversion number equal to zero, which serve as a combinatorial model for key polynomials. In this paper, we construct an involution on semistandard skyline fillings. This involution can be viewed as a vast generalization of the classical Bender–Knuth involution. As an application, we obtain that semistandard skyline fillings are compatible with the Demazure operators, offering a new combinatorial proof that nonsymmetric Macdonald polynomials specialize to key polynomials.

1 Introduction

The key polynomials \( \kappa_\alpha(x) \) associated to compositions \( \alpha \in \mathbb{Z}_{\geq 0}^n \), also called Demazure characters, are characters of the Demazure modules for the general linear groups \([8,9]\). These polynomials are defined based on the Demazure operator \( \pi_i = \partial_i x_i \). Here, \( \partial_i \) is the divided difference operator sending a polynomial \( f(x) \in \mathbb{Z}[x_1, x_2, \ldots, x_n] \) to

\[
\partial_i(f(x)) = \frac{f(x) - s_if(x)}{x_i - x_{i+1}},
\]

where \( s_if(x) \) is obtained from \( f(x) \) by interchanging \( x_i \) and \( x_{i+1} \). Precisely, if \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \) is a partition (i.e., \( \alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n \)), then set \( \kappa_\alpha(x) = x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_n^{\alpha_n} \). Otherwise, choose an index \( i \) such that \( \alpha_i < \alpha_{i+1} \), and let \( \alpha' \) be obtained from \( \alpha \) by interchanging \( \alpha_i \) and \( \alpha_{i+1} \). Set

\[
\kappa_\alpha(x) = \pi_i(\kappa_{\alpha'}(x)) = \partial_i(x_i \kappa_{\alpha'}(x)).
\] (1.1)

The above definition is independent of the choice of \( i \) since the Demazure operators satisfy the Coxeter relations: \( \pi_i \pi_j = \pi_j \pi_i \) for \( |i - j| > 1 \), and \( \pi_i \pi_{i+1} \pi_i = \pi_{i+1} \pi_i \pi_{i+1} \).

Combinatorial constructions of key polynomials have received extensive attention. Lascoux and Schützenberger [14] initiated a combinatorial treatment for key polynomials, showing that \( \kappa_\alpha(x) \) can be expressed as a sum of the weights of semistandard Young tableaux with right key bounded by a certain key, see also Reiner and Shimozono [23]. Kohnert [13] provided a diagram interpretation, now called Kohnert diagrams, for \( \kappa_\alpha(x) \) by giving a bijection between Kohnert diagrams and semistandard Young tableaux with bounded right key, see also Assaf and Quijada [3]. Assaf and Searles [4] defined the notion of Kohnert tableaux, and established a bijection between Kohnert tableaux and Kohnert diagrams. Assaf [2] introduced semistandard key tableaux to give another combinatorial construction of key polynomials. Comparing the definitions of Kohnert tableaux [4] and semistandard key tableaux [2], it can be checked that Kohnert tableaux are in bijection with semistandard key tableaux [22].
On the other hand, Ion [11] proved that $\kappa_\alpha(x)$ can be realized as a specialization of the nonsymmetric Macdonald polynomial $E_\alpha(x; q, t)$ at $q = t = 0$, namely,

$$\kappa_\alpha(x) = E_\alpha(x; q = 0, t = 0). \tag{1.2}$$

Nonsymmetric Macdonald polynomials were developed by Opdam [21], Macdonald [17] and Cherednik [7]. Haglund, Haiman and Loehr [10] established a combinatorial formula for $E_\alpha(x; q, t)$ in terms of non-attacking skyline fillings for $\alpha$ with $x$ counting the weights, $q$ counting the coinversion numbers, and $t$ counting the major indices. This, combined with (1.2), implies that $\kappa_\alpha(x)$ is a weighted counting of non-attacking skyline fillings for $\alpha$ with both major index and coinversion number equal to zero. Such non-attacking fillings are called semistandard skyline fillings, see for example Monical [19] or Monical, Pechenik and Searles [20].

As pointed out by Assaf [2, Proposition 3.1], semistandard skyline fillings are exactly semistandard key tableaux. Assaf [2] also gave an alternative proof of the relation (1.2) based on an expansion of key polynomials into fundamental slide polynomials established in [1, Corollary 3.16].

In Figure 2.2, we summarize the relationships among the above mentioned combinatorial constructions of key polynomials.

Figure 1.1: Relations among some combinatorial models of key polynomials.

In this paper, we construct an involution on semistandard skyline fillings. This involution significantly generalizes the classical Bender–Knuth involution from semistandard Young tableaux to semistandard skyline fillings. To be specific, when the parts of $\alpha$ are weakly increasing, the key polynomial $\kappa_\alpha(x)$ becomes a Schur polynomial, and our involution specifies to the Bender–Knuth involution [5]. As an application, it immediately follows that semistandard skyline fillings are compatible with the Demazure operators. This provides a new simple proof of the relation (1.2) between nonsymmetric Macdonald polynomials and key polynomials.

We end this section with some remarks, which are also parts of the motivation of this paper. The first two remarks concern combinatorial realizations of the Demazure operators in two formulas for key polynomials. As comparison, our treatment avoids using the properties of the “plactic congruence” or the RSK insertion algorithm.
Remark 1.1. As aforementioned, Lascoux and Schützenberger [14] established the following formula of $\kappa_\alpha(x)$:

$$\kappa_\alpha(x) = \sum_{T \in \mathcal{T}(\alpha)} x^T,$$

where

$$\mathcal{T}(\alpha) = \{T : K_+(T) \leq \text{key}(\alpha)\}$$

is the set of semistandard Young tableaux $T$ of shape $\lambda(\alpha)$ (namely, the partition by rearranging the parts of $\alpha$) such that the right key $K_+(T)$ of $T$ is entry-wisely less than or equal to $\text{key}(\alpha)$. Here, $\text{key}(\alpha)$ is the semistandard Young tableau of shape $\lambda(\alpha)$ whose first $\alpha_j$ columns contain the letter $j$ for all $j$. The definition of the right key $K_+(T)$ of $T$ is quite subtle, and we refer the reader to [23] for a detailed description. Lascoux and Schützenberger [14] proved (1.3) by introducing a combinatorial version of the Demazure operator on semistandard Young tableaux, see also [15, Section 4] or [16, Section 2]. The proof relies heavily on properties of “plactic congruence”.

Remark 1.2. Another combinatorial description of $\kappa_\alpha(x)$ also due to Lascoux and Schützenberger [14] is as follows:

$$\kappa_\alpha(x) = \sum_{u \in \mathcal{W}(\alpha)} x^u,$$

where $\mathcal{W}(\alpha)$ is the set of certain flagged row-frank words, see [23] for the precise definition. Reiner and Shimozono [23] gave a proof of (1.4) by showing combinatorially that the set $\mathcal{W}(\alpha)$ is compatible with the Demazure operators, namely,

$$\pi_r \left( \sum_{v \in \mathcal{W}(\alpha')} x^v \right) = \sum_{u \in \mathcal{W}(\alpha)} x^u,$$

where $\alpha'$ is a composition obtained from $\alpha$ by interchanging $\alpha_i$ and $\alpha_{i+1}$ with $\alpha_i < \alpha_{i+1}$. The proof of (1.5) uses implicit properties of the “r-paring” operation and the RSK insertion algorithm.

The third remark discusses whether the involution could be extended to give a combinatorial proof of a formula for Lascoux polynomials.

Remark 1.3. The Lascoux polynomial $L_\alpha(x)$ is the $K$-theory analog of the key polynomial $\kappa_\alpha(x)$. In fact, $\kappa_\alpha(x)$ can be obtained from $L_\alpha(x)$ by extracting the lowest degree component. Monical [19] gave a conjectural formula for $L_\alpha(x)$ in terms of semistandard set-valued skyline fillings for $\alpha$, see also [20]. This conjecture was recently confirmed by Buciumas, Scrimshaw and Weber [6] using the colored five-vertex model. We do not know if our involution could be extended to a set-valued version to give a combinatorial proof of Monical’s formula for Lascoux polynomials. We point out that Miller [18] showed that the classical reduced pipe dreams (or, RC-graphs) for the constructions of Schubert polynomials are compatible with the divided difference operators by using the mitosis algorithm [12]. Grothendieck polynomials are $K$-theory analogs of Schubert polynomials. The mitosis algorithm has been extended by Tyurin [25] to non-reduced pipe dreams for the construction of Grothendieck polynomials.
This paper is structured as follows. In Section 2, we construct the promised involution on simistandard skyline fillings. The key point in the construction is a classification of two consecutive entries appearing in a simistandard skyline filling. In Section 3, we apply this involution to give a combinatorial explanation that semistandard skyline fillings are compatible with the Demazure operators.

2 The involution on simistandard skyline fillings

In this section, we aim to construct an involution on semistandard skyline fillings. Let us start with an overview of semistandard skyline fillings.

2.1 Semistandard skyline fillings

Let \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{Z}_{\geq 0}^n \) be a composition. The skyline diagram \( D(\alpha) \) of \( \alpha \) is a left-justified array with \( \alpha_i \) boxes in row \( i \). Here, the row indices increase from top to bottom, and the column indices increase from left to right. We use \((i, j)\) to denote the box in row \( i \) and column \( j \). For example, Figure 2.2 is an illustration of the skyline diagram of \((1, 3, 0, 2)\).

![Figure 2.2: The skyline diagram for \( \alpha = (1, 3, 0, 2) \).](image)

A skyline filling for \( \alpha \) is an assignment \( F \) of positive integers into the boxes of \( D(\alpha) \). Write \( F(i, j) \) for the entry filled in the box \((i, j)\). A skyline filling \( F \) is non-attacking if the entries are distinct in each column, and \( F(i, j) \neq F(i', j + 1) \) for two boxes in consecutive columns with \( i > i' \). Non-attacking skyline fillings were used by Haglund, Haiman and Loehr \cite{10} to give a combinatorial formula for the nonsymmetric Macdonald polynomial \( E_\alpha(x; q, t) \). A semistandard skyline filling is a non-attacking skyline filling with both major index and coinversion number equal to zero.

As mentioned in Introduction, Assaf \cite[Proposition 3.1]{2} showed that semistandard skyline fillings are exactly semistandard key tableaux. In this paper, we adopt the equivalent definition due to Assaf, which is relatively easier to describe. To be specific, a semistandard skyline filling is a skyline filling such that

(i) the entries are weakly decreasing from left to right in each row;

(ii) each entry cannot exceed its row index (flag condition);

(iii) the entries are distinct in each column;
(iv) if some entry $a$ is below and in the same column as an entry $b$ with $a < b$, then there is an entry to the immediately right of $b$, say $c$, such that $a < c$. See Figure 2.3 for an illustration.

\[
\begin{array}{c|c}
  b & c \\
  \vdots & \vdots \\
  a & \\
\end{array}
\]

if $a < b$, then there exits $c$ such that $a < c$.

Figure 2.3: An illustration of the condition (iv).

Let $SSF(\alpha)$ denote the set of semistandard skyline fillings for $\alpha$. For example, there are 13 semistandard skyline fillings for $\alpha = (1, 3, 0, 2)$, as illustrated in Figure 2.4.

\[
\begin{array}{cccccccccc}
  1 & & & & & & & & & \\
  2 & 2 & 2 & & & & & & & \\
  4 & 4 & & & & & & & & \\

  1 & & & & & & & & & \\
  2 & 2 & 1 & & & & & & & \\
  4 & 4 & & & & & & & & \\

  1 & & & & & & & & & \\
  2 & 1 & 1 & & & & & & & \\
  4 & 4 & & & & & & & & \\

  1 & & & & & & & & & \\
  2 & 2 & 2 & & & & & & & \\
  4 & 3 & & & & & & & & \\

  1 & & & & & & & & & \\
  2 & 2 & 1 & & & & & & & \\
  4 & 3 & & & & & & & & \\

  1 & & & & & & & & & \\
  2 & 1 & 1 & & & & & & & \\
  3 & 3 & & & & & & & & \\

  3 & 3 & & & & & & & & \\
  4 & 2 & & & & & & & & \\

  4 & 1 & & & & & & & & \\
\end{array}
\]

Figure 2.4: Semistandard skyline fillings for $(1, 3, 0, 2)$.

For $F \in SSF(\alpha)$, write $x^F$ for the monomial generated by $F$, namely,

\[x^F = \prod_{(i, j) \in D(\alpha)} x_{F(i, j)}.\]

Using the combinatorial formula for the nonsymmetric Macdonald polynomial $E_\alpha(x; q, t)$ [10] together with the relation (1.2), it follows that

\[\kappa(x) = \sum_{F \in SSF(\alpha)} x^F,\]

see Assaf [2] for an alternative proof of (2.1).

The following lemma will be used later, which is in fact the non-attacking condition satisfied by a semistandard skyline filling. Since we adopt the definition of semistandard skyline fillings due to Assaf [2], we give a simple proof here.

**Lemma 2.1.** For $F \in SSF(\alpha)$, if $F(i, j) = F(i', j + 1)$, then we have $i \leq i'$.

**Proof.** Suppose otherwise that $i > i'$. Let $a = F(i', j)$. Since the entries in each row of $F$ are weakly decreasing and the entries in each column of $F$ are distinct, we obtain that
a > F(i, j). However, by item (iv) in the definition of a semistandard skyline filling, $F(j', j + 1)$ would be strictly larger than $t$, leading to a contradiction.

In the rest of this section, we always fix a composition $\alpha \in \mathbb{Z}_{\geq 0}^n$, a row index $r$ and a positive integer $t$. Our involution, denoted $\Phi_{r,t}$, is parameterized by $r$ and $t$. The construction of $\Phi_{r,t}$ is based on two operators: the lowering operator $L_{r,t}$ and the raising operator $R_{r,t}$. To define these two operators, we classify the entries $t$ and $t+1$ appearing in $F \in \text{SSF}(\alpha)$. This classification is the main ingredient in the construction of $\Phi_{r,t}$.

### 2.2 Classification of two consecutive entries

Let $F \in \text{SSF}(\alpha)$. Recall that the entries in every column of $F$ are distinct. So every column of $F$ contains at most one $t$ and at most one $t+1$. An entry $t$ (respectively, $t+1$) in $F$ is called paired if there is also a $t+1$ (respectively, $t$) in the same column, otherwise it is called unpaired. The unpaired entries $t$ or $t+1$ in $F$ will be further classified into pseudo-free or free.

The pseudo-free entries appear in pairs (but not in the same column). Precisely, two unpaired entries $t$ and $t+1$ in $F$ are called pseudo-free if

1. the entry $t+1$ appears to the upper right of the entry $t$; and
2. each column of $F$ between this pair of $t$ and $t+1$ contains both $t$ and $t+1$ (By (1) and Lemma 2.1 in each such column, the entry $t+1$ must be above the entry $t$).

An entry $t$ or $t+1$ is called free if it is neither paired nor pseudo-free.

For example, Figure 2.5 depicts a semistandard skyline filling. Choose $t = 2$. We use boldface to signify the entries $t$ and $t+1$, where free entries are underlined and pseudo-free entries are circled.

![Figure 2.5: Classification of the entries $t$ and $t + 1$.](image)

The following lemma gives an alternative characterization of pseudo-free entries.

**Lemma 2.2.** Let $F \in \text{SSF}(\alpha)$, and $F(i, j) = t$ be an unpaired entry. Assume that $i'$ is a row index such that $i' < i$. Then the following are equivalent:

1. $t+1$ appears to the upper right of $t$;
2. each column of $F$ between $t$ and $t+1$ contains both $t$ and $t+1$. (By (1) and Lemma 2.1 in each such column, the entry $t+1$ must be above the entry $t$).

An entry $t$ or $t+1$ is called free if it is neither paired nor pseudo-free.
(1) \( t \) is pseudo-free and its associated pseudo-free entry \( t + 1 \) is in row \( i' \);

(2) \( F(i', j + 1) = t + 1 \).

**Proof.** We first show that (2) implies (1). If the entry \( t + 1 \) in the box \((i', j + 1)\) of \( F \) is unpaired, then it is the pseudo-free entry associated with \( t \). Otherwise, column \( j + 1 \) contains an entry \( t \), we assert that
\[ F(i', j + 2) = t + 1. \]  
(2.2)

By Lemma 2.1, the entry \( t \) in column \( j + 1 \) lies in row \( i \) or in a row below row \( i \). It follows that in column \( j + 1 \), the entry \( t + 1 \) is above the entry \( t \). By item (iv) in the definition of a semistandard skyline filling, we have \( F(i', j + 2) > t \), which, together with the constraint that the entries in row \( i' \) are weakly decreasing, forces that \( F(i', j + 2) = t + 1 \). This verifies (2.2).

If column \( j + 2 \) does not contain \( t \), then the entry \( t + 1 \) in column \( j + 2 \) is the pseudo-free entry associated with \( t \). Otherwise, we use the same arguments as above to conclude that \( F(i', j + 3) = t + 1 \). Continuing this procedure, we can eventually locate a column index, say \( j' \), such that \( F(i', j) = t + 1 \) and column \( j' \) of \( F \) does not contain \( t \), as desired.

We next verify that (1) implies (2). Assume that the pseudo-free entry \( t + 1 \) associated to \( t \) is in column \( j' \). If \( j' = j + 1 \), then we are done. We next consider the case \( j' > j + 1 \). By (2) in the definition of pseudo-free entries, in column \( j' - 1 \), the entry \( t + 1 \) is above the entry \( t \). The proof for (2.2) implies that \( F(i', j' - 1) \) must equal \( t + 1 \). Repeating this procedure will lead to \( F(i', j + 1) = t + 1 \). So the proof is complete.

In view of the proof of Lemma 2.2, a typical local configuration of a pseudo-free entry \( t \) and its associated pseudo-free entry \( t + 1 \) is illustrated in Figure 2.6, where the pseudo-free entries are signified in boldface.

Figure 2.6: An illustration of pseudo-free entries.

### 2.3 The lowering operator

Based on the classification of the entries \( t \) and \( t + 1 \), we can now define the lowering operator \( L_{r,t} \) on semistandard skyline fillings. Note that, according to the flag condition \( \text{iii} \), we naturally have \( r \geq t \).

Let \( F \in \text{SSF}(\alpha) \), and denote \( F' = L_{r,t}(F) \). Consider the free entries \( t + 1 \) in row \( r \) of \( F \). If row \( r \) of \( F \) does not contain any free entry \( t + 1 \), set \( F' = F \). Otherwise, locate the rightmost free entry \( t + 1 \) in row \( r \). Assume that such a \( t + 1 \) lies in column \( j \). Locate the
smallest column index $j' < j$, such that for any $j' \leq k < j$, $F(r, k) = t + 1$, and there is an entry $t$ in column $k$ lying below row $r$. Let $F'$ be obtained from $F$ by replacing the entry $t + 1$ in column $j$ by $t$, and then exchanging the entries $t$ and $t + 1$ in column $k$ for $j' \leq k < j$. If such column index $j'$ does not exist, then $F'$ is obtained from $F$ by just replacing the entry $t + 1$ in column $j$ by $t$. This operation is illustrated in Figure 2.7.

![Figure 2.7: An illustration of the lowering operation.](image)

**Proposition 2.3.** For $F \in \text{SSF}(\alpha)$, the skyline filling $L_{r,t}(F)$ also belongs to $\text{SSF}(\alpha)$.

Clearly, $L_{r,t}(F)$ satisfies items (i) and (iii) in the definition of a semistandard skyline filling. To conclude Proposition 2.3, we need to show that $L_{r,t}(F)$ satisfies items (iv) and (v), which will be verified in Lemma 2.4 and Lemma 2.5 respectively.

**Lemma 2.4.** For $F \in \text{SSF}(\alpha)$, the skyline filling $L_{r,t}(F)$ satisfies item (iv).

**Proof.** Let $F' = L_{r,t}(F)$, and let $j, j'$ be the column indices as used in definition of $L_{r,t}$. For $j' \leq k < j$, assume that the entry $t$ in column $k$ of $F$ lies in row $r_k$. We need to show that the entries in row $r$ as well as in row $r_k$ of $F'$ are weakly decreasing.

Let us first consider row $r$ of $F'$. It suffices to verify that if $(r, j + 1)$ is a box of $D(\alpha)$, then $F(r, j + 1) \neq t + 1$. Suppose otherwise that $F(r, j + 1) = t + 1$. Since $F(r, j) = t + 1$ and this $t + 1$ is free, by definition, the entry $t + 1$ in the box $(r, j + 1)$ is not pseudo-free. Recalling that the entry $t + 1$ in the box $(r, j)$ is the rightmost free entry in row $r$, it follows that the entry $t + 1$ in $(r, j + 1)$ is paired. Assume that the entry $t$ in column $j + 1$ of $F$ is in row $p$. There are two cases.

Case 1. $p < r$. Since row $p$ of $F$ is weakly decreasing and column $j$ of $F$ does not contain $t$, we have $F(p, j) > t$. Since column $j$ of $F$ has distinct entries, we obtain that $F(p, j) > t + 1$, which, in view of item (iv), leads to $F(p, j + 1) > t + 1$, contrary to the assumption that $F(p, j + 1) = t$.

Case 2. $p > r$. By the arguments in first two paragraphs in the proof of Lemma 2.2, we can find a column index $j'' > j$ such that $F(r, j'') = r + 1$, column $j''$ of $F$ does not contain $t$, and for each $j < k < j''$, column $k$ of $F$ contains both $t$ and $t + 1$. Clearly, the unpaired entry $t + 1$ in column $j''$ is not pseudo-free, so it must be free. This contradicts the assumption that in row $r$ of $F$, the entry $t + 1$ in column $j$ is the rightmost free entry.

By the above arguments, the assumption that $F(r, j + 1) = t + 1$ is false, and hence row $r$ of $F'$ is weakly decreasing.

We next consider row $r_k$ of $F'$ for $j' \leq k < j$. By Lemma 2.1, we see that
\[
r_{j'} \leq r_{j'+1} \leq \cdots \leq r_{j-1}.\]
So we can find column indices \( j' = a_1 < a_2 < \cdots < a_m = j \) such that for \( 1 \leq h \leq m - 1 \),
\[
    r_{ah} = \cdots = r_{ah+1-1}.
\]
Now we need to show that for each \( 1 \leq h \leq m - 1 \), row \( r_{ah} \) of \( F' \) is weakly decreasing. For \( 1 < h < m - 1 \), since column \( a_h - 1 \) of \( F \) contains both \( t \) and \( t+1 \) lying strictly above row \( r_{ah} \), we see that \( F(r_{ah}, a_h - 1) > t + 1 \), and so row \( r_{ah} \) of \( F' \) is weakly decreasing.

It remains to check that row \( r_{j'} = r_{a_1} \) of \( F' \) is weakly decreasing. For simplicity, write \( p = r_{j'} \). We need to verify that \( F(p, j' - 1) > t \). Suppose to the contrary that \( F(p, j' - 1) = t \). The discussion is divided into two cases.

Case 1. Column \( j' - 1 \) of \( F \) does not contain \( t + 1 \). In this case, the entry \( t \) in column \( j' - 1 \) and the entry \( t + 1 \) in column \( j \) would be pseudo-free, leading to a contradiction.

Case 2. Column \( j' - 1 \) of \( F \) contains \( t + 1 \). Let \( q \) be the row index such that \( F(q, j' - 1) = t + 1 \). Since \( F(r, j') = t + 1 \), by Lemma 2.1, we see that \( q \leq r \). By the choice of the column index \( j' \), it follows that \( q \neq r \). So we have \( q < r \). In light of the proof of (2.2), we obtain that \( F(q, j') = t + 1 \), contrary to the fact \( F(r, j') = t + 1 \).

By the above arguments, the assumption that \( F(p, j' - 1) = t \) is false. This completes the proof.

**Lemma 2.5.** For \( F \in \text{SSF}(\alpha) \), the skyline filling \( L_{r,t}(F) \) satisfies item (iv).

**Proof.** Still, let \( F' = L_{r,t}(F) \), and \( j, j' \) be the column indices used in the definition of \( L_{r,t} \). Let \( a' < b' \) be two entries in the same column of \( F' \), say column \( m \), such that \( a' \) is below \( b' \). We need to check that there is an entry \( c' \) in \( F' \) to the immediately right of \( b' \) satisfying \( a' < c' \). This is clearly true for \( m < j' - 1 \) or \( m > j \), since the columns, which are strictly to the right of column \( j \) or strictly to the left of column \( j' \), coincide in \( F' \) and \( F \). We next verify the case when \( j' - 1 \leq m \leq j \).

Assume that the entries \( a' \) and \( b' \) in \( F' \) are filled in the boxes \( A \) and \( B \), respectively. Denote by \( a \) and \( b \) the entries in \( F \) that are filled in \( A \) and \( B \), respectively.

Let us first consider the case \( m = j \). Keep in mind that column \( j \) of \( F' \) is obtained from column \( j \) of \( F \) by replacing \( t + 1 \) by \( t \). Since \( b' > a' \) in \( F' \), it is easily seen that \( b > a \) in \( F \). So there is an entry \( c \) in \( F \) immediately to the right of \( b \) such that \( a < c \). Note that column \( j + 1 \) in \( F \) and \( F' \) coincide. So the entry in \( F' \) immediately to the right of \( b' \) is also \( c \), which is larger than \( a' \) by noticing that \( a > a' \).

We next consider the case \( m = j - 1 \). In this case, column \( j - 1 \) of \( F' \) is obtained from column \( j - 1 \) of \( F \) by interchanging \( t + 1 \) and \( t \). It is readily checked that the only possible pair that might violate item (iv) would be \( a' = t \) and \( b' = t + 1 \). However, this cannot occur since \( t \) lies above \( t + 1 \) in column \( j - 1 \) of \( F' \). Using similar arguments, one can verify the cases for \( k = j - 2, j - 3, \ldots, j' - 1 \). This completes the proof.

### 2.4 The raising operator

The raising operator \( R_{r,t} \) is the reverse procedure of the lowering operator. When implementing the raising operator, we shall always assume that \( r \geq t + 1 \), since otherwise the resulting skyline filling would violate the flag condition (ii).
Let \( F \in \text{SSF}(\alpha) \). Define \( F' = R_{r,t}(F) \) as follows. Consider the free entries \( t \) in row \( r \) of \( F \). If row \( r \) of \( F \) does not contain any free entry \( t \), then set \( F' = F \). Otherwise, locate the leftmost free entry \( t \) in row \( r \), and assume that such a \( t \) lies in column \( j \). Locate the smallest column index \( j' < j \), such that for any \( j' \leq k < j \), \( F(r,k) = t \), and there is an entry \( t + 1 \) in column \( k \) and below row \( r \). Let \( F' \) be obtained from \( F \) by replacing the entry \( t \) in column \( j \) by \( t + 1 \) and then exchanging the entries \( t \) and \( t + 1 \) in each column \( k \) for \( j' \leq k < j \). If such \( j' \) does not exist, then \( F' \) is obtained from \( F \) by just replacing the entry \( t \) in column \( j \) by \( t + 1 \). This raising operator is illustrated in Figure 2.8.

![Figure 2.8: An illustration of the raising operation.](image)

Using nearly the same arguments as for Proposition 2.3, we can obtain that the raising operator is a map on semistandard skyline fillings.

**Proposition 2.6.** For \( F \in \text{SSF}(\alpha) \), the skyline filling \( R_{r,t}(F) \) belongs to \( \text{SSF}(\alpha) \).

By the constructions of \( L_{r,t} \) and \( R_{r,t} \), we have the following property.

**Corollary 2.7.** Let \( F \) be a semistandard skyline filling in \( \text{SSF}(\alpha) \).

1. If row \( r \) of \( F \) has a free entry \( t + 1 \), then \( R_{r,t}(L_{r,t}(F)) = F \);
2. If row \( r \) of \( F \) has a free entry \( t \), then \( L_{r,t}(R_{r,t}(F)) = F \).

**Proof.** We only give a proof of (1). Let \( F \in \text{SSF}(\alpha) \), and \( F' = L_{r,t}(F) \). Assume that the rightmost free entry \( t + 1 \) in row \( r \) of \( F \) is in column \( j \). Then \( L_{r,t} \) replaces this \( t + 1 \) by \( t \). By the constructions of \( L_{r,t} \) and \( R_{r,t} \), to show that \( R_{r,t}(F') = F \), we need to check that the entry \( t \) in the box \((r, j)\) of \( F' \) is still free. Equivalently, we need to check that the entry \( t \) in the box \((r, j)\) of \( F' \) is not pseudo-free. Suppose to the contrary that it is pseudo-free. By Lemma 2.2, there is a row index \( r' < r \) such that \( F'(r', j+1) = t + 1 \). So \( F(r', j+1) = F'(r', j+1) = t + 1 \), which contradicts Lemma 2.1 since \( F(r, j) = t + 1 \). This completes the proof.

### 2.5 The involution \( \Phi_{r,t} \) on \( \text{SSF}(\alpha) \)

We can now describe the involution \( \Phi_{r,t} \), where \( r \geq t + 1 \), on semistandard skyline fillings. Let \( F \in \text{SSF}(\alpha) \). Assume that in row \( r \) of \( F \), the number of free entries \( t + 1 \) is \( n_1 \), and the number of free entries \( t \) is \( n_2 \). There are three cases.

1. \( n_1 = n_2 \). In this case, set \( \Phi_{r,t}(F) = F \).
(II) $n_1 > n_2$. Write $m = n_1 - n_2$. Define $\Phi_{r,t}(F) = L_{r,t}^m(F)$ to be the skyline filling obtained by applying the lowering operator $m$ times to $F$.

(III) $n_1 < n_2$. Write $m' = n_2 - n_1$. Define $\Phi_{r,t}(F) = R_{r,t}^{m'}(F)$ to be the skyline filling obtained by applying the raising operator $m'$ times to $F$.

For example, Figure 2.9 illustrates the involution $\Phi_{3,1}$, where $r = 3$ and $t = 1$. For the leftmost skyline filling, we have $n_1 = 2, n_2 = 0$, thus $\Phi_{3,1} = L_3^2$.

By Corollary 2.7, we have the following conclusion.

**Corollary 2.8.** The map $\Phi_{r,t}$ is an involution on $SSF(\alpha)$. Particularly, $\Phi_{r,t}$ exchanges the numbers of free entries $t$ and $t+1$ in row $r$ of any skyline filling in $SSF(\alpha)$.

It can be readily seen that for two distinct row indices $r$ and $r'$, the columns interfered by $\Phi_{r,t}$ differ from the columns interfered by $\Phi_{r',t}$, and thus they are commutative, namely,

$$\Phi_{r',t} \circ \Phi_{r,t} = \Phi_{r,t} \circ \Phi_{r',t}. \quad (2.3)$$

**Remark 2.9.** Consider a composition $\alpha = (\alpha_1, \ldots, \alpha_n)$ whose parts are weakly increasing. This means its reverse $\alpha^\text{rev} = (\alpha_n, \ldots, \alpha_1)$ is a partition. Let $F \in SSF(\alpha)$. Using item (iv), it is easy to check that the entries in each column of $F$ are strictly increasing. Let $\overline{F}$ be obtained from $F$ by applying a reflection about the horizontal line. Then each row of $\overline{F}$ is weakly decreasing, and each column of $\overline{F}$ is strictly decreasing. So $\overline{F}$ is a reverse semistandard Young tableau (also called a column-strict plane partition) of shape $\alpha^\text{rev}$ [24, Chapter 7.10]. In this case, the involution $\Phi_{r,t}$ specifies to the classical Bender–Knuth involution [5], which has the form as illustrated in Figure 2.10. Note that in such a case, each entry $t$ or $t+1$ is either paired or free, there are no pseudo-free entries.

![Figure 2.10: The Bender–Knuth involution.](image-url)
3 Application to key polynomials

In this section, we always assume that $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{Z}^n$ is a composition, $r$ is the first ascent of $\alpha$ (namely, $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_r$, and $\alpha_r < \alpha_{r+1}$), and $\alpha'$ is obtained from $\alpha$ by interchanging $\alpha_r$ and $\alpha_{r+1}$. We aim to apply the involution $\Phi_{r,t}$ to give a combinatorial interpretation of the relation

\[
\pi_r \left( \sum_{F' \in \text{SSF}(\alpha')} x^{F'} \right) = \sum_{F \in \text{SSF}(\alpha)} x^F.
\] (3.1)

Using the involution $\Phi_{r,t}$, we define a new involution $\Phi_r$ on $\text{SSF}(\alpha')$. For $F' \in \text{SSF}(\alpha')$, let $\Phi_r(F')$ be obtained from $F'$ by applying $\Phi_{i,r}$ for each $i \geq r + 1$. In view of (2.3), this operation does not depend on the order of the involutions $\Phi_{i,r}$.

By Corollary 2.8, we have the following consequence.

**Corollary 3.1.** For $F' \in \text{SSF}(\alpha')$, the number of free entries $r$ (respectively, $r+1$) below row $r$ of $F'$ is the same as the number of free entries $r+1$ (respectively, $r$) below row $r$ of $\Phi_r(F')$.

To prove (3.1), we need three simple lemmas.

**Lemma 3.2.** Let $F' \in \text{SSF}(\alpha')$. Assume that row $r$ of $F'$ has $m$ free entries $r$. Then

\[
x^{F'} + x^{\Phi_r(F')} = x_r^m f(x),
\] (3.2)

where $f(x)$ is a polynomial symmetric in $x_r$ and $x_{r+1}$.

**Proof.** Note that $F'$ and $\Phi_r(F')$ have the same entries other than $r$ and $r+1$. These entries contribute a common factor to $x^{F'}$ and $x^{\Phi_r(F')}$, which does not contain the variables $x_r$ and $x_{r+1}$. Note also that the lowering operator or the raising operator keeps paired entries and pseudo-free entries unchanged. So the paired entries and the pseudo-free entries together contribute a common factor of the form $(x_r x_{r+1})^h$ to $x^{F'}$ and $x^{\Phi_r(F')}$. Now we consider the free entries in $F'$ and $\Phi_r(F')$. First, the entries $r$ and $r+1$ in $F'$ or $\Phi_r(F')$ cannot occur above row $r$. Second, there are no entries $r+1$ in row $r$ of $F'$ or $\Phi_r(F')$, and so the free entries in row $r$ of $F'$ and $\Phi_r(F')$ contribute a common factor $x_r^m$. Finally, by Corollary 3.1, the total free entries below row $r$ of $F'$ and $\Phi_r(F')$ together generate a term symmetric in $x_r$ and $x_{r+1}$. The above analysis allows us to conclude (3.2).

Let $F' \in \text{SSF}(\alpha')$. Suppose that row $r$ of $F'$ has $m$ free entries $r$. Define $m+1$ semistandard skyline fillings $F_0, F_1, \ldots, F_m$ belonging to $\text{SSF}(\alpha)$ as follows. Let $F_0$ be obtained from $F'$ by moving the last $\alpha_{r+1} - \alpha_r$ boxes, together with the filled entries, down to row $r + 1$. It can be checked that $F_0 \in \text{SSF}(\alpha)$ by noticing the following two observations:

(i) Let $F \in \text{SSF}(\alpha)$. For $1 \leq i \leq r$, each box in row $i$ of $T$ is filled with $i$. The first $\alpha_r$ boxes in row $r + 1$ of $F$ are filled with $r + 1$. 


(ii) Let \( F' \in \text{SSF}(\alpha') \). For \( 1 \leq i < r \) or \( i = r + 1 \), each box in row \( i \) of \( F' \) is filled with \( i \). The first \( \alpha_r \) boxes in row \( r \) of \( F' \) are filled with \( r \).

For \( 1 \leq k \leq m \), let
\[
F_i = R_{r+1,r}^k(F_0)
\]
be the skyline filling obtained by applying the raising operator \( R_{r+1,r} \) \( k \) times to \( F_0 \). We call \( F_0, F_1, \ldots, F_m \) the derived fillings of \( F' \), and we denote
\[
\text{DF}(F') = \{ F_0, F_1, \ldots, F_m \}.
\]

For example, Figure 3.11 displays the construction of \( \text{DF}(F') \) for a skyline filling \( F' \in \text{SSF}(\alpha') \), where \( \alpha = (4, 2, 6, 0, 4) \), \( r = 2 \), and \( m = 2 \).

![Figure 3.11: An illustration of the construction of \( \text{DF}(F') \).](image)

**Lemma 3.3.** For \( F' \in \text{SSF}(\alpha') \), we have
\[
\pi_r \left( x^{F'} + x^{\Phi_r(F')} \right) = \sum_{F \in \text{DF}(F')} x^F + \sum_{F \in \text{DF}(\Phi_r(F'))} x^F. \tag{3.3}
\]

**Proof.** For a polynomial \( f_1(x) \) symmetric in \( x_r \) and \( x_{r+1} \), it is easy to check that for any polynomial \( f_2(x) \),
\[
\partial_r(f_1(x) f_2(x)) = f_1(x) \partial_r(f_2(x)).
\]
So, by Lemma 3.2,
\[
\pi_r \left( x^{F'} + x^{\Phi_r(F')} \right) = \partial_r(x^{m+1} f(x)) = \partial_r(x^{m+1}) f(x)
\]
\[
= (x_r^m + x_r^{m-1} x_{r+1} + \cdots + x_{r+1}^m) f(x). \tag{3.4}
\]
In view of the construction of the derived fillings of \( F' \), it is easily seen that (3.4) coincides with the right-hand side of (3.3).

**Lemma 3.4.** The set \( \text{SSF}(\alpha) \) is a disjoint union of \( \text{DF}(F') \), where \( F' \) runs over semistandard skyline fillings in \( \text{SSF}(\alpha') \). That is,
\[
\text{SSF}(\alpha) = \bigcup_{F' \in \text{SSF}(\alpha')} \text{DF}(F').
\]
Proof. It is easy to see that the sets DF(F') are disjoint. On the other hand, given a skyline filling \( F \in \text{SSF}(\alpha) \), the corresponding skyline filling \( F' \in \text{SSF}(\alpha') \) such that \( F \in \text{DF}(F') \) can be constructed as follows. Suppose that there are \( k \) free entries \( r + 1 \) in row \( r \) of \( F \). Let \( \overline{F} \) be obtained from \( F \) by applying the lowering operator \( L_{r+1,r} \) \( k \) times. Let \( F' \) be obtained from \( \overline{F} \) by moving the last \( \alpha_{r+1} - \alpha_r \) boxes, together with the filled entries, up to row \( r \). It is routine to check that \( F' \) belongs to \( \text{SSF}(\alpha') \). Moreover, it is easy to see that \( F \) belongs to \( \text{DF}(F') \). This completes the proof.

Lemma 3.3 provides an algorithm to generate inductively the semistandard skyline fillings for key polynomials. By Lemma 3.3 we obtain that

\[
\pi_r \left( \sum_{F' \in \text{SSF}(\alpha')} x^{F'} \right) = \sum_{F' \in \text{SSF}(\alpha')} \sum_{F \in \text{DF}(F')} x^{F}. \tag{3.5}
\]

By Lemma 3.4 the right-hand side of (3.5) is the same as the right-hand side of (3.1). This establishes relation (3.1).

Acknowledgments. We are grateful to Oliver Pechenik and Dominic Searles for explaining the connections among several combinatorial structures for the construction of key polynomials. This work was supported by the National Science Foundation of China (Grant No. 11971250, 12071320) and the Sichuan Science and Technology Program (Grant No. 2020YJ0006).

References

[1] S. Assaf, Weak dual equivalence for polynomials, arXiv:1702.04051.

[2] S. Assaf, Nonsymmetric Macdonald polynomials and a refinement of Kostka–Foulkes polynomials, Trans. Amer. Math. Soc. 370 (2018), 8777–8796.

[3] S. Assaf and D. Quijada, A Pieri rule for Demazure characters of the general linear group, arXiv:1908.08502v1.

[4] S. Assaf and D. Searles, Kohnert tableaux and a lifting of quasi-Schur functions, J. Combin. Theory Ser. A 156 (2018), 85–118.

[5] E. Bender and D.E. Knuth, Enumeration of plane partitions, J. Combin. Theory Ser. A 13 (1972), 40–54.

[6] V. Buciumas, T. Scrimshaw and K. Weber, Colored five-vertex models and Lascoux polynomials and atoms, J. Lond. Math. Soc. 2 (2020), 1–20.

[7] I. Cherednik, Nonsymmetric Macdonald polynomials, Internat. Math. Res. Notices (1995), 483–515.

[8] M. Demazure, Désingularisation des variétés de Schubert généralisées, Ann. Sci. École Norm. Sup. 7 (1974), 53–88.
[9] M. Demazure, Une nouvelle formule des caractères, Bull. Sci. Math. (2) 98 (1974), 163–172.

[10] J. Haglund, M. Haiman, and N. Loehr, A combinatorial formula for nonsymmetric Macdonald polynomials, Amer. J. Math. 130 (2008), 359–383.

[11] B. Ion, Nonsymmetric Macdonald polynomials and Demazure characters, Duke Math. J. 116 (2003), 299–318.

[12] A. Knutson and E. Miller, Gröbner geometry of Schubert polynomials, Ann. Math. 161 (2005), 1245–1318.

[13] A. Kohnert, Weintrauben, Polynome, Tableaux, Dissertation, Universität Bayreuth, Bayreuth, 1990. Bayreuth. Math. Schr. No. 38 (1991), 1–97.

[14] A. Lascoux and M.-P. Schützenberger, Keys & standard bases, Invariant Theory and Tableaux (Minneapolis, MN, 1988), 125–144, IMA Vol. Math. Appl., 19, Springer, New York, 1990.

[15] C. Lenart, A unified approach to combinatorial formulas for Schubert polynomials, J. Algebraic Combin. 20 (2004), 263–299.

[16] S. Mason, An explicit construction of type A Demazure atoms, J. Algebraic Combin. 29 (2009), 295–313.

[17] I.G. Macdonald, Affine Hecke algebras and orthogonal polynomials, Astérisque 237 (1996), 189–207, Séminaire Bourbaki 1994/95, Exp. No. 797.

[18] E. Miller, Mitosis recursion for coefficients of Schubert polynomials, J. Combin. Theory Ser. A 103 (2003), 223–235.

[19] C. Monical, Set-valued skyline fillings, Sém. Lothar. Combin. 78B (2017), Art. 35, 12 pp.

[20] C. Monical, O. Pechenik and D. Searles, Polynomials from combinatorial K-theory, Canad. J. Math., to appear.

[21] E.M. Opdam, Harmonic analysis for certain representations of graded Hecke algebras, Acta Math. 175 (1995), 75–121.

[22] O. Pechenik and D. Searles, Private communication.

[23] V. Reiner and M. Shimozono, Key polynomials and a flagged Littlewood-Richardson rule, J. Combin. Theory Ser. A 70 (1995), 107–143.

[24] R.P. Stanley, Enumerative Combinatorics, Vol. 2, Cambridge Studies in Advanced Mathematics, 62, Cambridge University Press, Cambridge, 1999.

[25] D.N. Tyurin, Mitosis algorithm for Grothendieck polynomials, J. Combin. Theory Ser. A 154 (2018), 32–48.
NEIL J.Y. FAN, DEPARTMENT OF MATHEMATICS, SICHUAN UNIVERSITY, CHENGDU 610064, P.R. CHINA. Email address: fan@scu.edu.cn

PETER L. GUO, CENTER FOR COMBINATORICS, NANKAI UNIVERSITY, TIANJIN 300071, P.R. CHINA. Email address: lguo@nankai.edu.cn

NICOLAS Y. LIU, CENTER FOR COMBINATORICS, NANKAI UNIVERSITY, TIANJIN 300071, P.R. CHINA. Email address: yiliu@mail.nankai.edu.cn