Generalization and Another Proof of Two Conjectural Identities Posed by Sun

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Abstract. We study two conjectural identities involving roots of unity and determinants of Hermitian matrices which have been recently proved by using the famous eigenvector-eigenvalue identity for normal matrices. In this note, we extend these identities to a general form by considering the class of circulant matrices suggested by Petrov. Furthermore, we give another proof to Sun’s identities independent of the eigenvector-eigenvalue identity, where our strategy is built upon the similarity of an unnecessarily normal matrix to a particular one with integer eigenvalues, derived from the Fourier transform vectors suggested by Petrov.

1. Introduction

In 2018, Zhi-Wei Sun proposed a system of open problems in number theory and formalized numerous conjectural identities with meaningful arithmetic properties [7, 8, 9, 10, 11]. The following two identities about roots of unity and derangements have been conjectured by Sun in [10, 12]. Recently, Guo et al. proved (1) in [4] based on the famous eigenvector-eigenvalue identity for normal matrices [2]. With the same technique, Wang and Sun proved (2) in [12].

Let \( D(n) \) denote the set of all derangements \( \tau \) of indices \( j = 1, \ldots, n \) such that \( \tau(j) \neq j \) for all \( j = 1, \ldots, n \).

Theorem 1.1. (Trigonometric Identity-1) Let \( n > 1 \) be an odd number and \( \zeta \) a primitive \( n \)-th root of unity in the complex field \( \mathbb{C} \). Then

\[
\sum_{\tau \in D(n-1)} \text{sign}(\tau) \prod_{j=1}^{n-1} \frac{1}{1 - \zeta^{j-\tau(j)}} = (-1)^{\frac{n-1}{2}} \left( \frac{n-1}{2} \right)!^2. \tag{1}
\]

Theorem 1.2. (Trigonometric Identity-2) Let \( n > 1 \) be an odd number and \( \zeta \) a primitive \( n \)-th root of unity in the complex field \( \mathbb{C} \). Then

\[
\sum_{\tau \in D(n-1)} \text{sign}(\tau) \prod_{j=1}^{n-1} \frac{1 + \zeta^{j-\tau(j)}}{1 - \zeta^{j-\tau(j)}} = (-1)^{\frac{n-1}{2}} \left( (n-2)! \right)^2 \frac{2}{n}. \tag{2}
\]
In this note, we generalize Theorems 1.1 and 1.2 to a more general form based on circulant matrices. Furthermore, we give a different technique for proving the above identities without using the eigenvector-eigenvalue identity and solve for the integer eigenvalues of some related non-Hermitian (and unnecessarily normal) matrices.

2. Generalization by Circulant Matrices

Let $f_n : \mathbb{Z} \to \mathbb{C}$ be a function satisfying the following conditions:

(i) $f_n(i) = f_n(j)$ if $i \equiv j \pmod{n}$.

(ii) There exist an $s \in \mathbb{Z}$ and an $n$-th root of unity $\zeta \in \mathbb{C}$ such that

$$\sum_{k=0}^{n-1} f_n(k)\zeta^{-ks} = 0.$$ 

(iii) For any $i, j \in \{1, 2, \ldots, n\}$,

$$\sum_{k=1}^{n} f_n(i-k)\overline{f_n(j-k)} = \sum_{k=1}^{n} f_n(k-i)f_n(k-j),$$

where $\overline{a}$ denotes the complex conjugate of any $a \in \mathbb{C}$.

In other words, $f_n$ mapped from the rational integer domain is assumed to be a periodic function with period $n$ and has at least one coefficient of its discrete Fourier transform equal to zero. Furthermore, we require $f_n$ to have a symmetric property (condition (iii) above) which is clearly satisfied if $f_n(k) = \overline{f_n(-k)}$ for any integer $k$. Let $M = (m_{ij})_{1 \leq i,j \leq n}$ denote the $n \times n$ matrix with $m_{ij} = f(i-j)$, i.e., $M$ be a circulant matrix given by $f$. Then we have the following theorem about the determinant of the $(n-1) \times (n-1)$ minor $M_j$ of $M$.

**Theorem 2.1. (Determinants of Minors of Circulant Matrices)** Let $n > 1$ be an integer and $f_n$ a function satisfying conditions (i-iii) above. Then for any minor $M_j$ of $M = (f(i-j))_{1 \leq i,j \leq n}$ by deleting the $j$-th row and column, we have

$$\det M_j = \frac{1}{n} \prod_{i=1,i\neq s}^{n} \lambda_i(M),$$

where $(\lambda_i(M))_{1 \leq i \leq n}$ are the eigenvalues of $M$ given by

$$\lambda_i(M) = \sum_{k=0}^{n-1} f_n(k)\zeta^{-ki}.$$
Proof. Since $M$ is circulant (condition (i) above), we have that for any column vector
\[ v_i = \frac{1}{\sqrt{n}}(\zeta^i, \zeta^{2i}, \ldots, \zeta^{ni})' \]
and any $k \in \{1, 2, \ldots, n\}$ \[6\],
\[ \frac{1}{\sqrt{n}} \sum_{j=1}^{n} m_{kj} \zeta^{nj} = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} f_n(k-j) \zeta^{ji} = \frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} f_n(j) \zeta^{-ji} \zeta^{ki}. \] (5)
So $\sum_{j=0}^{n-1} f_n(j) \zeta^{-ji}$ is an eigenvalue of $M$ whose eigenvectors $(v_i)_{1 \leq i \leq n}$ forms an orthonormal basis of $\mathbb{C}^n$.

By condition (iii), $MM^* = M^*M$ and $M$ is normal. We can thus apply the following eigenvector-eigenvalue equality to $M$. Now the proof of (3) is completed by choosing $\lambda_i(M) = 0$ which exists by condition (ii).

**Theorem 2.2.** (Eigenvector-eigenvalue identity \[2\]) Let $\lambda_k(M_j)$ be the $k$th eigenvalue of the minor $M_j$ of a normal matrix $M$ by deleting the $j$-th row and column. Then we have
\[ |v_{ij}|^2 \prod_{k=1; k \neq i}^{n} (\lambda_i(M) - \lambda_k(M)) = \prod_{k=1}^{n-1} (\lambda_i(M) - \lambda_k(M_j)). \] (6)

It is not hard to observe that Theorems 1.1 and 1.2 are special cases of Theorem 2.1 with proper selections of $f_n$ satisfying conditions (i-iii). A more general example is given by
\[ f_n(k) = \begin{cases} \frac{a+bc^k}{1-\zeta^k}, & k \in \{1, \ldots, n-1\} \\ 0, & k = 0 \end{cases}, \] (7)
where $a, b, c \in \mathbb{Z}$ are chosen to satisfy condition (ii) and $\zeta \in \mathbb{C}$ is a primitive $n$-th root of unity. These values of $a, b, c$ can be easily found by (16) and (17) in Lemma 5.1.

3. Another Proof of Theorems 1.2 and 1.1

Now we proceed to give a proof of Theorems 1.1 and 1.2 that is independent of the eigenvector-eigenvalue equality (6).

Let $A$ denote the $(n-1) \times (n-1)$ Hermitian matrix with diagonal elements equal to zero and off-diagonal elements $(a_{ij})_{i \neq j}$ given by
\[ \frac{1}{1-x_{i-j}}, \quad 1 \leq i \neq j \leq n-1, \]
where
\[ x_k = \zeta^k, \quad \forall \ k. \]

Clearly the left-hand side of (1) is equal to \( \det(A) \). It is more convenient to multiply \( A \) from right with the following diagonal matrix \( B_s \) whose diagonal entries are given as
\[ 1 - x_{is}, \quad 1 \leq i \leq n - 1, \]
by fixing any \( s \in \{-\frac{n-1}{2}, \cdots, -1, 1, \cdots, \frac{n-1}{2}\} \). Define \( C_s \triangleq AB_s \). If \( n \) is prime or \( s = 1 \), we have
\[ \det(C_s) = \det(AB_s) = \det(A) \det(B_s) = n \det(A). \] (8)

The observation that \( \det(B_s) = n \) when \( n \) is prime or \( s = 1 \) can be easily proved by comparing the coefficient of the constant term in the following polynomial equation after cancelling 1 and the trivial term \( x \) from both sides:
\[ (1 - x)^n = 1. \] (9)

Note that if \( n \) is prime, then \( (x_{js})_{1 \leq j \leq n-1} \) is a permutation of \( (x_j)_{1 \leq j \leq n-1} \) for any fixed \( s \in \{-\frac{n-1}{2}, \cdots, -1, 1, \cdots, \frac{n-1}{2}\} \). In this case \( js_1 \not\equiv js_2 \pmod{n} \) if \( s_1 \not\equiv s_2 \pmod{n} \).

For general odd \( n > 1 \), Equation (1) is equivalent to the following identity:
\[ \det(C_1) = \det(AB_1) = \det(A) \det(B_1) = (-1)^{\frac{n-1}{2}} \left( \frac{n-1}{2}! \right)^2. \] (10)

To analyze (10), we use (16) in Lemma 5.1 to obtain the following identity for any fixed \( s \in \{0, 1, 2, \cdots, \frac{n-3}{2}, \frac{n-1}{2}, \cdots, n-1\} \) and \( k \in \{1, 2, \ldots, n-1\} \):
\[ \sum_{j=1, j \neq k}^{n-1} \frac{1 - x_{j(n-1)-s}}{1 - x_{k-j}} x_{js} = \sum_{j=1, j \neq k}^{n-1} \frac{x_{js} - x_j n^{-1}}{1 - x_{k-j}} = \left( \frac{n-1}{2} - s \right) x_{ks}. \] (11)

Note that the above equality also holds for \( s = \frac{n-1}{2} \), but we do not need this fact. It shows that \( s \) is an eigenvalue of \( C_s \) for each \( s \in \{-\frac{n-1}{2}, \cdots, -1, 1, \cdots, \frac{n-1}{2}\} \). If we can show that \( s \) is also an eigenvalue of \( C_1 \), then
\[ \det(C_1) = \prod_{-\frac{n-1}{2} \leq s < \frac{n-1}{2}, s \neq 0} \left( \frac{n-1}{2} \right)^2. \] (13)
To prove that $C_1$ has all the required integer eigenvalues, we construct a basis of the $(n-1)$-dimensional vector space on the complex field $\mathbb{C}$ by using the Fourier transform [5]. Denote by $u_j$ ($j = 0, 1, \ldots, n-1$) the column-vector with coordinates $(x_{ji})_{1 \leq i \leq n-1}$. Note that
\begin{equation}
 u_0 + u_1 + \ldots + u_{n-1} = 0. \tag{14}
\end{equation}
Furthermore, any $n-1$ vectors $u_j$'s form a $(n-1) \times (n-1)$ minor of the orthogonal basis for the classical Fourier vector transform in the $n$-dimensional complex space $\mathbb{C}^n$. So the $n-1$ vectors are linearly independent and form a basis of the $(n-1)$-dimensional complex space $\mathbb{C}^{n-1}$, for otherwise the original full $n \times n$ Fourier matrix would not have the full rank according to (14). Let us consider the specific basis $\{u_0, u_1, \ldots, u_{n-1}\} \setminus \{u_{(n-1)/2}\}$. According to (16) in Lemma 5.1, we have that
\begin{equation}
 C_1 u_s = \begin{cases} 
 \left(\frac{n-1}{2} - s\right) u_s - \left(\frac{n-1}{2} - s - 1\right) u_{s+1}, & s \in \{0, 1, \ldots, n-2\} \\
 -\frac{n-1}{2} u_{n-1} - \frac{n-1}{2} u_0, & s = n-1 
\end{cases} \tag{15}
\end{equation}
By ignoring the case of $s = \frac{n-1}{2}$ in the above, $C_1$ can be rewritten in the basis $\{u_0, u_1, \ldots, u_{n-1}\} \setminus \{u_{(n-1)/2}\}$ with the form
\[
\begin{pmatrix} X & Y \\ 0 & Z \end{pmatrix},
\]
where $X$ and $Z$ are lower-triangular. Therefore the eigenvalues of $C_1$ are just the diagonal elements of $X$ and $Z$, and these are exactly $-\frac{n-1}{2}, \ldots, -1, 1, \ldots, \frac{n-1}{2}$.

Now the proof of (2) follows a similar process as above by applying (17) in Lemma 5.1 to the corresponding matrix $C'_1 = A'B_1$ and considering the basis $\{u_1, \ldots, u_{n-1}\}$. We omit the details here.

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5. Appendix

The following lemma is well-known (see, e.g., [3, 1, 12]) and we give an induction proof here for easy connections to the proof in Sec. 3.
Lemma 5.1. For any integers $s \in \{0, 1, \ldots, n - 1\}$ and $k \in \{1, 2, \ldots, n - 1\}$, we have

\[
\sum_{j=1, j \neq k}^{n-1} \frac{x_{js}}{1 - x_{k-j}} = \left(\frac{n - 1}{2} - s\right) x_{ks} - \frac{1}{1 - x_k}, \tag{16}
\]

\[
\sum_{j=1, j \neq k}^{n-1} \frac{x_{js}}{1 - x_{j-k}} = \begin{cases} 
(s - \frac{n+1}{2}) x_{ks} - \frac{1}{1 - x_{k}}, & s > 0 \\
\frac{n-1}{2} - \frac{1}{1 - x_{k}}, & s = 0
\end{cases}. \tag{17}
\]

Proof.

\[
\sum_{j=1, j \neq k}^{n-1} \frac{x_{js}}{1 - x_{k-j}} = x_{ks} \sum_{j=1, j \neq k}^{n-1} \frac{x(j-k)s}{1 - x_{k-j}} \tag{18}
\]

\[
= x_{ks} \left[ \sum_{j=1}^{n-1} \frac{x_{js}}{1 - x_{j}} - \frac{x_{ks}}{1 - x_{k}} \right]. \tag{19}
\]

Note that

\[
\sum_{j=1}^{n-1} \frac{x_{js}}{1 - x_{j}} = \sum_{j=1}^{n-1} \frac{x_{j(s-1)} + x_{js}}{1 - x_{j}} \tag{20}
\]

\[
= -1 + \sum_{j=1}^{n-1} \frac{x_{j(s-1)}}{1 - x_{j}} \tag{21}
\]

\[
= \ldots \tag{22}
\]

\[
= -s + \sum_{j=1}^{n-1} \frac{1}{1 - x_{j}} \tag{23}
\]

\[
= \frac{n - 1}{2} - s. \tag{24}
\]

The last equality is obtained by comparing the coefficient of $x^{n-2}$ in the following polynomial equation after cancelling the trivial term $x^n$ and dividing by $n$ on both sides:

\[
\left(1 - \frac{1}{x}\right)^n = 1. \tag{25}
\]

The case of $s = 0$ is thus trivial in (17). For $s > 0$, we only need to observe that

\[
\sum_{j=1}^{n-1} \frac{x_{js}}{1 - x_j} = \sum_{j=1}^{n-1} \left[x_{j(s-1)} - x_{j(s-1)}\right] \tag{26}
\]
\[
1 + \sum_{j=1}^{n-1} \frac{x_j(s-1)}{1-x_j} = \cdots = s - 1 + \sum_{j=1}^{n-1} \left( \frac{1}{1-x_j} - 1 \right) = s - \frac{n + 1}{2}.
\]

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