New Dimensions for Wound Strings:
THE MODULAR TRANSFORMATION OF GEOMETRY TO TOPOLOGY

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We show, using a theorem of Milnor and Margulis, that string theory on compact negatively curved spaces grows new effective dimensions as the space shrinks, generalizing and contextualizing the results in \cite{1}. Milnor’s theorem relates negative sectional curvature on a compact Riemannian manifold to exponential growth of its fundamental group, which translates in string theory to a higher effective central charge arising from winding strings. This exponential density of winding modes is related by modular invariance to the infrared small perturbation spectrum. Using self-consistent approximations valid at large radius, we analyze this correspondence explicitly in a broad set of time-dependent solutions, finding precise agreement between the effective central charge and the corresponding infrared small perturbation spectrum. This indicates a basic relation between geometry, topology, and dimensionality in string theory.

December 2006
1. Introduction

Negatively curved spaces constitute a basic class of geometries, and hence a basic class of potential target spaces for string theory. At large curvature radius, one has an easily controlled description of the system via general relativity; in vacuum, for example, such spaces expand slowly with time in a way described by self-consistent solutions of the equations of motion.

At the same time, perturbative string theory probes spacetime geometry and topology differently from point particle theory. For example, on a space with nontrivial fundamental group the perturbative string spectrum contains winding string sectors; again this is a well-controlled statement at large radius.

A fundamental relation between the geometry and the topology of compact negatively curved spaces is described by Milnor’s theorem \[2\]. This says that negative sectional curvature and compactness imply exponential growth of the fundamental group \[2,3\]. Exponential growth of the fundamental group implies \[3\] that the number density \(\rho(l)\) of independent geodesics of length \(l\) is an exponentially growing function of \(l\):

\[
\rho(l) \sim l^\alpha e^{l/l_0}
\]

for some finite constants \(l_0\) and \(\alpha\).

In the context of perturbative string theory on such a space, in the large radius regime, this correspondence has a simple physical interpretation (introduced in a special case in \[1\]). The exponential growth in the fundamental group implies an exponential growth in the density of states of string winding modes (independent of oscillator modes), exhibiting compact negatively curved target spaces as supercritical backgrounds of string theory.

Moreover, the exponential growth in the density of states is related by a modular transformation to the IR behavior of the partition function \[4,5\]. The spectrum of IR perturbations is in turn sensitive to the curvature and compactness of the spatial slices.

In this work, we analyze in some detail the physical realization of this correspondence in a broad class of background solutions, exhibiting precise agreement with this prediction of modular invariance. Although we will not compute the full modular-invariant partition function, we will explicitly work out the leading effects in the deep IR and deep UV parts of the spectrum, which depend in a clear way on the compactness, curvature, and topology of our target space. In these limits, the semiclassical approximation controlling both the
infrared spectrum and the winding strings provides a straightforward means for consistently analyzing the one-loop partition function in a time-dependent background. One interesting outcome is that this check of modular invariance works timeslice by timeslice, i.e., it works before performing the path integration over the time variable.

It is interesting also to ask what the effective dimensionality becomes at small radius. In the case of genus \( h \) Riemann surfaces, numerous diagnostics – including a variant of Buscher’s worldsheet path integral argument for T-duality, D-brane systems, and algebro-geometric genericity arguments – all suggest that the system can naturally cross over to a supercritical theory on the \( 2h \)-dimensional Jacobian torus, with a rolling tachyon producing a transition from this to the large radius expanding space [9].

Altogether, we are being led to a generalization of T-duality and the Calabi-Yau/Landau-Ginzburg correspondence applicable to the generic case of curved compact target spaces. This provides a new mechanism by which spatial dimensions emerge, arising in a novel way from nontrivial topology.

The paper is organized as follows. We start by reviewing the modular invariance relation between the UV and IR limits of the partition function. Next we analyze these limits for solutions of the equations of motion with compact spatial slices of constant negative curvature, using their realization as freely acting orbifolds of hyperbolic \( n \)-dimensional space. We find precise agreement with the modular invariance prediction for the relation between the deep UV and deep IR contributions to the partition function. We consider a more subtle case of exponential growth, obtained from compact quotients of the Sol geometry, and a subtle case of power law growth, obtained from compact quotients of the Nil geometry. In all these homogeneous cases, modular invariance and the growth of effective dimensions checks out explicitly. We then discuss more general cases, analogies to other mechanisms for growing dimensions in string theory, and applications to mathematical and physical questions in the last two sections.

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1 See, e.g., [6,7,8] for previous works computing and interpreting a full partition function in a Lorentzian-signature target space. There remain significant subtleties with these models partly to do with their construction as a global orbifold of Minkowski space which introduces extra challenges in the very early time physics. In the present work we will evade these subtleties by working in a regime where IR quantum field theory and semiclassical cosmic strings are easily controlled, and by avoiding a global orbifold construction.

2 More generally, for higher dimensional examples the Albanese variety arises at small radius.
2. Structure of the One-loop Amplitude

Let us use the conventions of Polchinski’s book [10]. The one-loop vacuum amplitude in perturbative closed string theory takes the form

\[ \int \frac{d^2 \tau}{\tau_2} Z_1(\tau) = \text{Tr} \int \frac{d^2 \tau}{4\tau_2} (-1)^F q^{L_0} \bar{q}^{\bar{L}_0}, \]  

(2.1)

where \( \tau \) is the modular parameter of the worldsheet torus, \( F \) is the spacetime fermion number, \( q = e^{2\pi i \tau} \), and \( L_0 \pm \bar{L}_0 \) are the worldsheet Hamiltonian and momentum generators. Here \( Z_1(\tau) \) is defined so as to be modular invariant: \( Z_1(\tau) = Z_1(\tau + 1) = Z_1(-1/\tau) \).

The \( \tau_2 \to \infty \) region of the integral is dominated by the deep IR part of the spectrum; \( Z_1 \) can be expanded as

\[ Z_1/\tau_2 \to q^{L_0,\min} \bar{q}^{\bar{L}_0,\min}. \]  

(2.2)

Let us first review the IR and UV behavior of \( Z_1 \) in the 10-dimensional classical backgrounds of the superstring (including the tachyonic cases) and the 26-dimensional unstable background of the bosonic string. In terms of the spacetime masses and momenta, for level-matched contributions we have

\[ L_0 = (k^2 + m^2)\alpha'/4 = \bar{L}_0, \]  

(2.3)

so the IR limit of the partition function has an integrand going like

\[ Z_{\text{IR}} \equiv Z_1|_{\tau_2 \to \infty} \sim e^{-\pi \tau_2 \alpha' m_{\text{min}}^2}. \]  

(2.4)

The \( \tau_2 \to 0 \) region is dominated by the deep UV. We can compute directly the UV asymptotics of the torus amplitude using the density of states for large masses,

\[ \rho(m) \sim e^{m\sqrt{\alpha'} \pi \sqrt{2c_{\text{eff}}/3}}, \]  

(2.5)

where \( c_{\text{eff}} \) is the effective central charge. Summing over \( m \) with this measure by saddle point approximation gives

\[ Z_{\text{UV}} \equiv Z_1|_{\tau_2 \to 0} \sim \sum_m \rho(m)e^{-\pi \tau_2 \alpha' m^2} \sim e^{\pi c_{\text{eff}} \tau_2}. \]  

(2.6)

Here, as will be our custom, we ignore the subleading polynomial dependence of the asymptotic form. Equivalently, our statements that two functions are asymptotic should be interpreted as statements about the corresponding logarithms.
By modular invariance, the dominant contribution in the $\tau_2 \to 0$ region must go like

$$Z_{UV} \sim e^{-\pi \alpha' m_{\text{min}}^2 / \tau_2}. \quad (2.7)$$

So we have

$$c_{\text{eff}} = -6\alpha' m_{\text{min}}^2. \quad (2.8)$$

As a check, note that in the case of the type 0 theory, the lightest state is a tachyon $\alpha' m_{\text{min}}^2 = -2$. This corresponds to $c_{\text{eff}} = 12$, the correct value once one subtracts the 2 dimensions cancelled out by the ghosts. Similarly in the bosonic string, the lightest state is a tachyon with $\alpha' m_{\text{min}}^2 = -4$, corresponding to $c_{\text{eff}} = 24$.

For models in which the mass squared is nonnegative, such as spacetime-supersymmetric strings, there is no such exponential IR divergence in the partition function. Hence, by modular invariance, there is also no surviving contribution of order $e^{\text{const} / \tau_2}$. This requires a dramatic cancellation going well beyond the necessary condition that the effective central charges of the spacetime bosons and fermions agree, $c_{\text{eff}}^F = c_{\text{eff}}^B [5]$.

Theories which do have such IR divergences might naively be expected to universally suffer from dramatic instabilities on equal footing with those in the bosonic string theory. However, this is not the case; IR divergences can arise from modes, pseudotachyons, whose condensation does not cause a large back reaction (as occurs in the standard case of inflationary perturbation theory, as well as in more formal time-dependent string backgrounds [11,12]). This is the case we will encounter in our examples. In the next section we will study the UV and IR limits of the partition function, and the modular transformation between them, in the case of spacetimes with compact negatively curved spatial slices.

### 3. Exponential Growth: Examples

Any compact manifold of negative sectional curvature has a fundamental group of exponential growth [2], meaning that the number of independent elements grows exponentially with minimal word length in any basis of generators of the group. It follows [3] that the number of free homotopy classes also grows exponentially with geodesic length.

In this section, we reproduce this result in constant curvature examples using modular invariance in string theory. This analysis will provide precise results for the IR and UV limits of the partition function, and their modular equivalence, exhibiting these compact negatively curved target spaces as supercritical backgrounds of string theory as suggested in [1]. We will then consider more general examples and applications.
3.1. Orbifolds of Hyperboloids

Consider first $n$-dimensional manifolds of constant negative curvature. These have a universal cover $\mathbb{H}_n$, the $n$-dimensional noncompact hyperbolic space with metric

$$ds^2_{\mathbb{H}_n} = R_0^2 \left(dy^2 + \sinh^2 y \, d\Omega^2\right). \quad (3.1)$$

For some of this discussion, we will set $R_0 = 1$ for simplicity. For $n \leq 9$, a solution of the equations of motion of critical superstring theory is obtained by considering spacial slices (3.1) expanding in time, combined with a transverse $(9-n)$-dimensional Ricci flat space $X_\perp$ which we will take to be compact with metric $ds^2_\perp$:

$$ds^2 = -dt^2 + \left(t/t_0\right)^2 ds^2_{\mathbb{H}_n} + ds^2_\perp. \quad (3.2)$$

This, of course, is nothing but the inside of a light cone in flat spacetime $M_{n,1}$ (times $X_\perp$), with time coordinate chosen so that the $M_{n,1}$ component has spatial slices of constant negative curvature. The constant $t_0$ will often be set to $t_0 = R_0$.

Many other solutions are obtained by orbifolding the space $\mathbb{H}_n$ by a freely acting discrete group of isometries, $\Gamma$, such that the resulting space $\mathcal{M}_n \equiv \mathbb{H}_n/\Gamma$ is compact. This space has $\pi_1(\mathcal{M}_n) \cong \Gamma$ and metric

$$ds^2 = -dt^2 + \left(t/t_0\right)^2 ds^2_{\mathcal{M}_n = \mathbb{H}_n/\Gamma} + ds^2_\perp, \quad (3.3)$$

which is locally the same as flat spacetime (3.2). Globally, the spacetime (3.3) is very different from the noncompact flat covering spacetime (3.2); it is not Poincaré invariant, it is spatially compact, and it has a nontrivial (and exponentially growing) fundamental group. As we will see, the latter two features affect the IR and UV behavior of the one-loop partition function in a way consistent with modular invariance.

We will consider string theory on these target spaces and compare and contrast the structure of the one-loop partition function in the flat spacetime (3.2) to that in the expanding compact daughter space (3.3). Note that the model we analyze is not necessarily the same as any global orbifold of Minkowski space by $\Gamma$. We do find it useful to obtain our compact spatial slices by projection from the $\mathbb{H}_n$ spatial slices of (3.2), which sit within the future light cone of a point in Minkowski space. In any case we will restrict attention to the late-time large-radius system and require of the initial state only that it lead to a smooth large-radius geometry at late times.
Before proceeding, let us explain the framework we will use to analyze the perturbative string theory. In general, time-dependent target spaces of string theory are difficult to formulate explicitly. However, solving Einstein’s equations provides a semiclassical solution to the conditions for worldsheet conformal invariance, valid as the leading approximation in a self-consistent expansion at large curvature radius.

Formally, the worldsheet path integration then generates appropriate corrections to the semiclassical action obtained from this background solution; one integrates over the conformal factor in the metric, leading to a system whose total central charge is zero. However, a priori, the worldsheet path integral is not manifestly convergent. The path integral may be rendered convergent by a Wick rotation in some circumstances (such as in flat spacetime), but most backgrounds do not have a smooth Euclidean continuation. In quantum field theory on curved spacetime, one can work directly in Lorentzian signature, including a convergence factor in the calculation of amplitudes. A similar method may apply to the full worldsheet string path integral; indeed some works have succeeded in calculating the one-loop partition function of some simple time-dependent backgrounds using a Euclidean worldsheet and Lorentzian target space \cite{7}. For these reasons, we expect our backgrounds to provide consistent solutions, which at late times are accurately described by perturbative string theory and its general relativistic approximation. Our computations can be viewed as a test of this proposition, more specifically as a concrete test of modular invariance of the putative worldsheet string path integral, and of the existence of initial states consistent with our large-radius phase.

We will begin with a discussion of the bosonic contributions to the one-loop partition function in the IR and the UV. In §3.4 we will comment on the fermionic contributions, arguing both in the IR and the UV that they do not cancel out the effect.

3.2. IR Limit

Consider a massless scalar field $\eta(t, y, \Omega)$, a classical modulus of $X_\perp$, propagating on $M_{n,1}$, coordinatized as in (3.2)\footnote{A similar but more indexful analysis applies to perturbations of the metric.}. We will be interested in the spectrum of modes which propagate in the one-loop partition function.

The following feature of this spectrum will play an important role. In the parent theory (3.2), a complete set of normalizable eigenfunctions of the Laplacian $\nabla^2_{H_n}$ on the
spatial slices $\mathbb{H}_n$ is gapped \[3.14\], due to the rapid growth of the volume as a function of proper distance $y$ at large $y$; this can be shown as follows.

Consider the eigenvalue equation

$$\nabla^2_{\mathbb{H}_n} \tilde{f}_k(y, \Omega) = -k^2 \tilde{f}_k(y, \Omega), \quad (3.4)$$

where $\nabla^2_{\mathbb{H}_n}$ is the spatial Laplacian:

$$\nabla^2_{\mathbb{H}_n} = \frac{1}{\sinh^{n-1} y} \frac{\partial_y}{\partial y} (\sinh^{n-1} y \partial_y) + \frac{1}{\sinh^2 y} \nabla^2_\Omega. \quad (3.5)$$

Separate variables as $\tilde{f}_k(y, \Omega) = f_{k,L}(y) Y_L(\Omega)$ with $Y_L$ denoting spherical harmonics that satisfy $\nabla^2_\Omega Y_L = -L^2 Y_L$. Then (3.4) reduces to

$$\frac{1}{\sinh^{n-1} y} \frac{\partial_y}{\partial y} (\sinh^{n-1} y \partial_y f_{k,L}) - \frac{L^2}{\sinh^2 y} f_{k,L} = -k^2 f_{k,L}. \quad (3.6)$$

At large $y$ this becomes

$$e^{-(n-1)y} \partial_y \left(e^{(n-1)y} \partial_y f_{k,L} \right) = -k^2 f_{k,L}, \quad (3.7)$$

so that $f_{k,L}$ is in fact independent of $L$ at large $y$. We can solve this with the ansatz $f_{k,L}(y \to \infty) \sim f_0 e^{\lambda y}$ which, when plugged into (3.7), yields the two independent solutions

$$\lambda_{\pm} = \frac{1}{2} (- (n-1) \pm \sqrt{(n-1)^2 - 4k^2}).$$

Consider the range $k^2 < (n-1)^2/4$. In this range, the solution smooth in the interior is the solution with $\lambda = \lambda_+$, as noted in \[15\].

However, in this range $k^2 < (n-1)^2/4$ the solution is not normalizable:

$$\int_\infty dy \sqrt{-G} f_{k,L}^*(y) f_{k,L}(y) \sim \int_\infty dy e^{(n-1)y} |f_0|^2 e^{\left(-(n-1)+\sqrt{(n-1)^2 - 4k^2}\right)y} = \infty. \quad (3.8)$$

That is, there is a gap in the spectrum:

$$k^2 \geq k^2_{\text{gap}} = \frac{(n-1)^2}{4}. \quad (3.9)$$

The time dependence in (3.2) compensates for this, lowering the effective mass squared of each mode expanded about the time-dependent background, similarly to the situation

\[5\] This is straightforward to establish by rewriting the equation (3.6) as an equation for $\hat{f} = e^{(n-1)y/2} f$ and using it to constrain the relative signs of $\hat{f}$ and its derivatives. This enables one to rule out the possibility of a solution matching the $\lambda = \lambda_-$ behavior at infinity with the nonsingular (constant) solution at $y = 0$.\]
in the supercritical linear dilaton theory \[16\]. To see this, consider massless on-shell modes of $\eta$ in the background (3.2). The equation of motion for $\eta$ is

$$\nabla^2 \eta = 0,$$  \hspace{1cm} (3.10)

where $\nabla^2$ is the Laplacian acting on the full spacetime:

$$\nabla^2 = -\frac{1}{t^n} \partial_t (t^n \partial_t) + \frac{1}{t^2} \nabla_{H_a}^2.$$

(3.11)

We can decompose $\eta$ into modes as $\eta(t,y,\Omega) = u(t) f_{k,L}(y) Y_L(\Omega)$, with $f_{k,L}$ and $Y_L$ as above. Then the equation of motion (3.10) for $\eta(t,y,\Omega)$ reduces to an equation for $u(t)$ alone:

$$-\frac{1}{t^n} \partial_t (t^n \partial_t u(t)) = \frac{k^2}{t^2} u(t).$$

(3.12)

This has solutions

$$u(t) \propto t^\alpha, \quad \alpha = \frac{1}{2} \left( -(n - 1) \pm \sqrt{(n - 1)^2 - 4k^2} \right).$$

(3.13)

So we see that precisely for $k$ above the gap, satisfying (3.9), the massless solutions oscillate in time. Moreover, the frequency goes to zero as $k^2 \to (n - 1)^2 / 4$, so the modes behave like massless particles with zero momentum for $k = k_{\text{gap}}$. The effects of the spatial curvature and the effects of the time-dependent evolution precisely cancel.

This effect can be seen in the propagator constructed in the low energy field theory, which is closely related to the IR limit of the one-loop amplitude of interest in the perturbative string theory. This is obtained by solving

$$\nabla^2 D(x,x') = -\delta(x-x')/\sqrt{-g}$$

in terms of a complete basis of functions.

It proves convenient to consider the complete set of functions

$$\psi_{\omega,k,L}(t,y,\Omega) = u_\omega(t) f_{k,L}(y) Y_L(\Omega),$$

(3.15)

with $f_{k,L}(y)$ and $Y_L(\Omega)$ as described above and with $u_\omega(t)$ given by

$$u_\omega(t) = t^{\frac{1-n}{2}} e^{i\omega \ln t}.$$  \hspace{1cm} (3.16)

These satisfy

$$-\frac{1}{t^n} \partial_t (t^n \partial_t u_\omega) = \frac{1}{t^2} \left( \omega^2 + \frac{(n-1)^2}{4} \right) u_\omega.$$  \hspace{1cm} (3.17)
The full spacetime Laplacian (3.11) acting on $\psi_{\omega,k,L}$ will then give

$$\nabla^2 \psi_{\omega,k,L} = \frac{1}{t^2} \left( \omega^2 - k^2 + \frac{(n - 1)^2}{4} \right) \psi_{\omega,k,L}. \quad (3.18)$$

In (3.18) the gap is evident. At large $y$ the modes (3.15) take the form

$$\psi_{\omega,k,L}(t,y,\Omega) \sim t^{\frac{1-n}{2}} e^{i\omega \ln t} e^{(\frac{1-n}{2} \pm i \sqrt{k^2 - k_{gap}^2})y} Y_L(\Omega). \quad (3.19)$$

The Laplacian (3.11) is the low-energy approximation to the worldsheet Hamiltonian $H \equiv L_0 + \tilde{L}_0$. Note that we are not working in a basis of eigenfunctions of $H$, but instead are considering a basis of eigenfunctions of $t^2H$. The one-loop vacuum amplitude involves a trace (2.1), which can be evaluated in any basis. The basis (3.16) is slightly more convenient for our calculations than the basis of eigenfunctions of $H$ simply because the latter are more complicated Bessel functions.

The propagator can be expressed as

$$D(x,x') = (tt')^{\frac{1-n}{2}} \int \frac{d\omega}{2\pi} e^{i\omega(\ln t - \ln t')} \sum_L \int_{k^2 \geq (n-1)^2/4} \frac{f_{L,k}^* (y) f_{L,k} (y') Y_L^*(\Omega) Y_L(\Omega')}{-\omega^2 + k^2 - (n - 1)^2/4}. \quad (3.20)$$

It is straightforward to check that this provides a solution to the equation (3.14) for the two-point Green’s function.

In general, physical amplitudes depend on a choice of vacuum state, and as mentioned above we assume initial conditions consistent with a late-time phase with weakly curved spatial slices. In the Friedmann equation with negative spatial curvature, radiation, and matter, $(\dot{a}/a)^2 = 1/a^2 + g_s^2[(\rho_r/a^4) + (\rho_m/a^3)]$, curvature dominates at late times. The presence of a bath of particles, which is the typical relic of working in another vacuum (distinct from that killed by the annihilation operators corresponding to our modes (3.15)), will not significantly alter our results.

The one-loop vacuum amplitude in the point-particle approximation is

$$\text{Tr} \log(H) = \int d^d x \sqrt{-g} \Lambda(x)$$

$$= \int dt dy d\Omega t^n \sinh^{n-1} y \int \frac{d\tilde{\omega}}{2\pi} \sum_L \int_{k^2 \geq (n-1)^2/4} |f_{L,k}(y) Y_L(\Omega)|^2$$

$$\times \log \left( \frac{\tilde{\omega}^2 + k^2 - (n - 1)^2/4}{t^2} \right), \quad (3.21)$$
where we Wick rotated the $\omega$ integral via $\omega = i\tilde{\omega}$.

This can be expressed in a Schwinger formalism appropriate to the IR limit of the first-quantized string path integral as

$$
\int dt dy d\Omega t^n \sinh^{n-1} y \int \frac{d\tilde{\omega}}{2\pi} \sum_L \int_{k^2 \geq (n-1)^2/4} \left| f_{L,k}(y) Y_L(\Omega) \right|^2 \times \int \frac{d\tau_2}{\tau_2} \exp \left( -\pi\alpha'\tau_2 \left( \frac{\tilde{\omega}^2 + k^2 - (n-1)^2/4}{t^2} \right) \right).
$$

The net effect agrees with the standard flat space result. The fluctuations of $\eta$ are massless, and the one-loop partition function has no IR divergence as $\tau_2 \to \infty$, because normalizability requires $k^2 \geq (n-1)^2/4$.

In the compact daughter space $\mathcal{M}_n = \mathbb{H}_n/\Gamma$, in contrast, the constraint of normalizability on the spatial slices $\mathcal{M}_n = \mathbb{H}_n/\Gamma$ is much less restrictive: since the space is compact, the zero mode is normalizable, and there is no gap in the spectrum. Hence on $\mathcal{M}_n$, one sums over $k$ values including $k = 0$, leading to an IR-divergent term from the $\eta$ contribution to the one-loop amplitude:

$$Z_{\text{IR}} \sim e^{\pi\alpha'\tau_2(n-1)^2/(4t^2)}.\quad (3.23)$$

The time dependence still lowers the effective mass squared of the mode. The magnitude of the negative effective mass squared of the zero mode is given precisely by the gap in the spectrum on the parent space $\mathbb{H}_n$.

That this mode is only pseudotachyonic rather than being a catastrophic instability can be seen as follows. The action for the zero mode is of the form $S_0 = \int dt t^n (\dot{\eta})^2$, with solutions

$$\eta = \eta_0 + \eta_1/t^{n-1}.\quad (3.24)$$

The resulting energy density is of order $\dot{\eta}^2 \sim 1/t^{2n}$, which is much smaller than Hubble squared $H^2 \sim 1/t^2$ for our original vacuum solution. Hence the back reaction of the rolling $\eta$ mode on the spacetime geometry is negligible. Moreover, the solutions (3.24) show that the Hubble friction from the expanding space $\mathcal{M}_n$ prevents the modulus field $\eta$ from straying far from its initial value.

More generally, for massive or tachyonic particles in the parent theory (with nonzero mass squared $m^2$), we can carry out the above analysis working with a basis of mode solutions satisfying (c.f. (3.18))

$$(\nabla^2 - m^2) \psi_{\omega,k,L} = \left[ \frac{1}{t^2} \left( \omega^2 - k^2 + \frac{(n-1)^2}{4} \right) - m^2 \right] \psi_{\omega,k,L}.\quad (3.25)$$
In any perturbative string theory, the minimal mass squared obtained in the daughter theory is that in the parent theory \( m_{\text{min}}^{(0)} \), offset by the gap \( -(n-1)^2/(4t^2) \).

For example, we can formally apply our methods to the bosonic string theory, whose lowest lying mode in the flat spacetime (3.2) is a tachyon of mass squared \( m_T^2 = -4/\alpha' \). Once we project to the daughter space \( \mathcal{M}_n = \mathbb{H}_n/\Gamma \), the minimal mass squared in the bosonic theory drops to

\[
m_{\text{min}}^{(B)}(t) = -\frac{4}{\alpha'} - \frac{(n-1)^2}{4t^2}.
\]

(3.26)

### 3.3. Modular Transform and UV Limit

In the previous subsection, we found a gap in the spectrum of spatial modes at

\[
\frac{k_{\text{gap}}^2}{t^2} = \frac{(n-1)^2}{4t^2}.
\]

(3.27)

This led to a minimal mass squared

\[
m_{\text{min}}^2 = -\frac{(n-1)^2}{4t^2} + m_{\text{min}}^{(0)} \, ,
\]

(3.28)

where \( m_{\text{min}}^{(0)} \) is the minimal mass squared in Minkowski space (which is zero in the superstring, and equal to \( m_T^2 = -4/\alpha' \) in the bosonic string). We obtained this by evaluating the partition function in the deep IR limit, where low-energy effective field theory applies.

Now we would like to evaluate the partition function in the deep UV limit, as in (2.6), to check modular invariance.

Before implementing this, we must address one subtlety. In general, the string partition function in a consistent background is only guaranteed to be conformally invariant (and hence modular invariant) after performing the path integral over all the embedding coordinates, including \( t \); in the Hamiltonian formalism, equivalently, modular invariance only holds in general after taking the trace in (2.1). Using our complete basis of functions (3.15)(3.18), we will find that our test of modular invariance works before performing the \( t \) integral. A priori, this is not necessary, but it is a sufficient condition for passing this test of modular invariance.

\[\text{Footnote 6} \text{: The fact that it works for every } t \text{ may follow from a more extensive analysis of one-loop non-vacuum S-matrix amplitudes involving a set of different scattering processes each of which preserve modular invariance but which localize interactions at different times. It is somewhat reminiscent of “fiber-wise” string dualities.}\]
Let us now turn to the test of modular invariance. Plugging our result for \( m_{\text{min}} \) (3.28) into the modular relation (2.7) gives the small-\( \tau_2 \) behavior demanded by modular invariance:

\[
Z_{\text{UV}} \sim \exp \left( \frac{\pi \alpha'(n - 1)^2}{4t^2 \tau_2} + \frac{\pi c_{\text{eff}}^\text{crit}}{6\tau_2} \right).
\]

where \( c_{\text{eff}}^\text{crit} \) is the value of the effective central charge in the critical theory on Minkowski space (reviewed in §2 above). Comparing to (2.6) yields

\[
c_{\text{eff}} - c_{\text{eff}}^\text{crit} = \frac{3\alpha'(n - 1)^2}{2t^2}.
\]

This means that modular invariance can be satisfied in the late-time limit only if the compact negatively curved manifold \( \mathcal{M}_n \) provides a new contribution to the exponential density of states. In the superstring this further requires the new contribution to survive the sum over bosonic and fermionic contributions, an issue we will discuss below in §3.4.

The projection of \( \mathbb{H}_n \) by freely acting isometries \( \Gamma \) has several effects, which modify both the IR and UV limits of the spectrum. First, it compactifies the space, so that new modes become normalizable; this introduces a new state into the IR spectrum which causes a divergence in the one-loop partition function as we saw in §3.2. Second, it restricts unwound modes to invariant states. Third, it introduces new sectors of winding strings; we will see that these winding strings contribute a Hagedorn density of states to the UV limit of the spectrum in just the right way to provide the requisite effective central charge (3.30).\[7\]

The partition function includes a sum over winding sectors and a sum over states arising from transverse motion of the winding strings, as well as a sum over eigenstates of \( L_0, \tilde{L}_0 \) in each winding sector (which we will refer to as “oscillator modes”). The mass of each winding mode is

\[
m_{w}^2 = R^2/\alpha'^2
\]

when the string winds a cycle of length \( l \equiv 2\pi R \).

\[7\] In our case the winding strings which contribute to the exponential density are in one-to-one correspondence with free homotopy classes (conjugacy classes of the fundamental group). In more general examples, one might find additional contributions from metastable closed geodesics that are not supported by topology. This does not happen in cases where the sectional curvature is bounded above by a negative value \([17]\), in particular in our cases of constant curvature.
We are studying the one-loop partition function in the limit where it degenerates into a long thin tube. In the previous section, we studied this in the channel where worldsheet time ran around the long direction of the tube, which is dominated by IR modes. In order to study the UV density of states, including the contribution of long winding strings, we need to analyze the diagram in the opposite channel.

Consider a rectangular Euclidean worldsheet torus, with the worldsheet space direction $\sigma_1$ ranging from 0 to $2\pi \tilde{\tau}_2$, and the worldsheet time direction $\sigma_2$ ranging from 0 to $2\pi$. $\tilde{\tau}_2$ is related to the modular parameter $\tau_2$ of the previous section by a modular transform $\tilde{\tau}_2 = 1/\tau_2$. That is, to study the UV behavior $\tau_2 \to 0$, we are interested in the asymptotic behavior as $\tilde{\tau}_2 \to \infty$ of this diagram.  

Let us start with the case $n = 2$, where the simplest version of the Selberg trace formula computes for each value of $t$ the path integral we are interested in, in the degeneration limit $\tilde{\tau}_2 \to \infty$, as a sum over periodic orbits. Up to an overall constant, the sum over windings and their transverse motions contribute to $Z_{UV}$ as

$$
\int dl \rho(l) \frac{1}{\sinh(l/2t)} e^{-l^2/(4\pi\alpha'\tilde{\tau}_2)} \quad (n = 2),
$$

(3.32)

where $\rho(l)$ is the density of periodic geodesics. The latter is given \[3,18,19\] by

$$
\rho(l) = e^{l/t} \quad (n = 2)
$$

(3.33)

(up to powers of $l$). Here we are using the fact that $t$ is the curvature radius of $\mathcal{M}_n$ at time $t$.

Altogether, working out the saddle point approximation to the $l$ integral in (3.32) at large $l$ (large $\tilde{\tau}_2$) gives

$$
Z_{UV} \sim \exp \left( \frac{\pi \alpha'}{4t^2 \tilde{\tau}_2} + \frac{\pi c_{\text{eff}}^{\text{crit}}}{6 \tilde{\tau}_2} \right) \quad (n = 2),
$$

(3.34)

where $c_{\text{eff}}^{\text{crit}}$ arises from the usual sum over oscillator contributions. This result (3.34) reproduces directly in the winding channel the enhancement to the effective central charge needed to satisfy the modular invariance requirement (3.29).  

---

8 We can see the basic effect we need without keeping track of worldsheet angles determined by $\tau_1$, which we have set to zero for simplicity. As discussed further below, at least in familiar examples the exponential growth of the level-matched and non-level-matched spectra are the same.

9 Here we are computing the density of states for physical winding strings, whereas the trace appearing in the partition function (2.1) includes all states in the CFT, whether or not they are level matched. In the familiar case of flat space the density of states at high level scales the same for level-matched and non-level-matched states (though the subleading polynomial dependence differs). Our check will work for the level-matched states, which leads us to suspect that they scale the same as the non-level-matched states also in our system.
The form of (3.32) can be understood in a Hamiltonian description directly in the winding string channel. In particular, we interpret the factor of $1/\sinh(l/2t)$ in (3.32), which is novel relative to the simpler case of toroidal target spaces, as coming from the sum over transverse motions of the winding string. Unlike in the case of a toroidal target space, in our negatively curved target space the geodesics wrapped by the winding strings are shorter than any other curves in the same free homotopy class; moving the winding string in the normal direction, away from the minimal geodesic, will increase its length and hence its energy. So the transverse degrees of freedom will produce an approximate harmonic oscillator spectrum.\footnote{For strings wound on a flat cylinder, the analogous sum over transverse positions generates only a subexponential contribution to the UV limit of the partition function.}

In the winding channel of our diagram, the worldsheet space coordinate $\sigma_1$ runs from 0 to $2\pi\tilde{\tau}_2$, while the worldsheet Euclidean time direction $\sigma_2$ runs from 0 to $2\pi$. Integrating along the $\sigma_1$ (worldsheet space) direction, this leads to a Hamiltonian for the transverse motions of the form
\[
H = \frac{\alpha'}{2} \left( p^2_\perp + \frac{l[X_\perp]^2}{(2\pi\alpha')^2} \right) + \cdots, \tag{3.35}
\]
where the functional dependence $l[X_\perp]^2 \sim l^2(1 + X_\perp^2/t^2)$ incorporates the stretching of the string in the transverse directions in the constant curvature geometry, and (\cdots) indicates terms independent of the normal degrees of freedom $X_\perp, p_\perp$. The frequency of the resulting harmonic oscillator spectrum is $\omega = l/(2\pi t)$, and the Euclidean time direction in this channel is $\sigma_2 \in (0, 2\pi)$. So we have the following contribution from the transverse modes:\footnote{This interpretation owes its origin to some observations in \cite{7}, where a similar factor also arises from a harmonic oscillator spectrum (in that case from the time variable).}
\[
\sum_{N=0}^{\infty} e^{-(N+1/2)t/l} = \frac{1}{2\sinh(l/2t)}. \tag{3.36}
\]
The generalization to arbitrary $n$ is similar. The analogue of (3.32) is now
\[
\int dl\rho(l) \left( \frac{1}{\sinh(l/2t)} \right)^{n-1} e^{-l^2/(4\pi\alpha'\tilde{\tau}_2)}. \tag{3.37}
\]
The $n-1$ powers of $1/\sinh(l/2t)$ arise because there are $n-1$ transverse spatial directions $X_\perp$, each yielding a harmonic oscillator spectrum as in (3.36). The density of geodesics is, for general $n$,
\[
\rho(l) = e^{(n-1)t/l} \tag{3.38}
\]
The saddle point approximation now gives

$$Z_{UV} \sim \exp \left( \frac{\pi \alpha' (n - 1)^2}{4 l^2 \tau_2} + \frac{\pi c_{\text{crit}}^{\text{eff}}}{6 \tau_2} \right),$$

(3.39)

which again exactly reproduces the requirement (3.29).

Thus we see that the sum over the exponential density of winding modes provides a supercritical contribution to the effective central charge, related by modular invariance to the IR pseudotachyon. This test of modular invariance is satisfied by virtue of the mathematical relation between geometry (producing pseudotachyons) and topology (producing a Hagedorn spectrum of winding modes) \[2,3,20\]. It would be very interesting to work out the full partition function before doing the $t$ integral to understand the structure of the full string-theoretic amplitude. In any case the relation between topology and geometry is evident in the deep UV and deep IR limits.

### 3.4. Fermionic Contributions in the Superstring

Let us next address the contribution of fermionic states. These contribute with opposite sign from the bosons, and \textit{a priori} they may affect the net $c_{\text{eff}}$ obtained from the partition function. For the following reasons, we expect the contribution to $c_{\text{eff}}$ that we calculated above to survive in the full partition function, including the fermionic contributions.

First, let us discuss this question in the IR limit. In the parent flat spacetime theory, the fermion contribution cancels that of the bosons. In the daughter theory, we saw that the bosons develop new pseudotachyonic instabilities arising from the newly normalizable $k = 0$ modes. We do not expect the fermions to behave in the same way. The $k = 0$ mode is not generally an eigenstate of the Dirac operator which respects the spin structure and the reality of the action. In general, the hermiticity of the Dirac Hamiltonian $i \gamma^\mu \nabla_\mu$ would seem to preclude pseudotachyonic fermions.

Let us discuss this question next in the UV limit. Recall that in tachyon-free and pseudotachyon-free theories, the high energy density of states of bosons and of fermions must cancel exquisitely \[3\]. In the present case, the exponential density of winding modes is exactly that required for modular invariance provided it does not cancel out in the sum over bosonic and fermionic states. This non-cancellation is a generic expectation which can be motivated in our case as follows. Let us consider separately the sum over winding sectors and the sum over eigenstates of $L_0 + \tilde{L}_0$ in each winding sector (the latter we
will loosely refer to as the oscillator levels). In the unwound sector, the SUSY breaking is detected only by paths which propagate around a nontrivial cycle introduced by the projection to a compact space, since the parent theory was simply flat spacetime. These contributions are suppressed by a factor of \( e^{-R(t)m} \) for a cycle of length \( R(t) \), so summing over these high oscillator levels weighted by the Hagedorn density \( e^{m\sqrt{\alpha'}\sqrt{2c_{\text{crit}}/3}} \) does not yield a divergence in itself. The same might be expected for fixed winding number at high oscillator level \( m \gg R(t)/l_s^2 \), since the winding is a subleading contribution to the sum over random walks traced out by the string at high oscillator level (though to check this would require a more detailed accounting of the way in which the asymptotic SUSY cancellation of oscillator levels works). The sum over winding sectors at fixed oscillator level (say the zero mode) probes the supersymmetry breaking directly, and there is no reason to expect a similar exponential suppression.

It would be interesting to work out the fermion contributions in more detail. An important issue in that regard is the contribution of the worldsheet fermion determinants to the semiclassical path integral. Since the IR propagator for a given bosonic field is the same in the superstring as it is in the bosonic theory, this determinant must not change the leading \( l \) dependence in the expression (3.37). This seems plausible to us since the dependence on the winding string length \( l \) arises most directly in the boundary conditions on the bosonic embedding coordinates \( X^\mu \), and its role in the fermionic sector is more indirect. A full calculation of the fermionic contributions might make use of known variants of the Selberg trace formula involving superparticles. Such formulas would need to be used with care, however, to properly take into account the embedding of the spatial slices into the full spacetime.

The application of our correspondence to the superstring has the feature that the critical contribution to \( c_{\text{eff}} \) vanishes, so that the winding mode contribution is the only source for \( c_{\text{eff}} \). As we just discussed, we expect this contribution to survive the supertrace over bosons and fermions. Given this, the “ten dimensional” superstring theory on a (large) finite-volume compact hyperbolic target space is in fact a (slightly) supercritical background of string theory.

### 3.5. Generalization

Milnor’s theorem \(^2\) applies to arbitrary manifolds of negative sectional curvature. The proof uses a result of Günther that for any manifold \( \mathcal{M} \) of strictly negative sectional
curvature \(\leq -\alpha^2\), the volume \(V(y)\) of a ball of proper radius \(y\) about a point \(y_0\) in its universal cover \(\tilde{\mathcal{M}}\) satisfies

\[
V(y) \geq \frac{n\omega_n}{(n-1)(2\alpha)^n} e^{(n-1)\alpha y}
\] (3.40)

for \(y \to \infty\). Here \(\omega_n\) is the volume of a unit ball in \(\mathbb{R}^n\).

In some cases, compact manifolds may be obtained by quotienting by isometries in the covering space. It would be interesting to understand if the behavior (3.40) combined with the strictly negative sectional curvature always entails a gap in the spectrum of the right value to match the supercritical effective central charge arising from the winding modes, similarly to what we found for the constant curvature cases.

However, we should emphasize that a gap in the spectrum of the covering space is sufficient but not necessary to obtain pseudotachyons. In particular, there are examples which are not of strictly negative sectional curvature, and have exponential volume growth and hence exponential growth of the fundamental group, but no gap in the spectrum. We will turn to this case in the next subsection to see how the requisite pseudotachyons arise.

In any case, for consistent string backgrounds on more general manifolds of negative sectional curvature, Milnor’s theorem guarantees that extra effective dimensions emerge at finite volume. The fact that negative sectional curvature leads to pseudotachyons is intuitively plausible. Consider a scalar field \(\eta\) propagating on a compact manifold of negative sectional curvature. The negative curvature locally causes the space to stretch out rapidly enough that the friction on spatial modes of \(\eta\) can compete with the gradient energy in the modes. (If the gradient energy dominated at late times, then the modes would have positive effective mass squared; when the friction energy competes at late times this can lead to pseudotachyons \([12,17]\) in the way we saw above for the constant curvature case.)

It is clear moreover that the manifolds yielding pseudotachyons in the infrared include spaces without strictly negative sectional curvature. As long as there is a region with negative sectional curvatures, in which the modes of \(\eta\) have support, the friction will compete with the gradient energy and can produce a pseudotachyon. Turning this around, if modular invariance holds we can also conclude that any manifold with exponentially growing fundamental group must evolve with time in such a way as to contain regions expanding fast enough to produce a pseudotachyon.
3.6. Sol Geometry

An interesting class of examples with exponential growth is obtained from compact quotients of the Sol geometry by discrete isometries. In these examples, the fundamental group grows exponentially but the universal covering space, the Sol geometry

\[ ds^2 = dz^2 + \frac{1}{2}e^{-2z}dx^2 + \frac{1}{2}e^{2z}dy^2, \]  

(3.41)

has no gap in its spectrum. Nonetheless we will see that there are pseudotachyons in the spectrum, as needed for modular invariance.

A simple way to obtain solutions for the evolution of homogeneous but anisotropic spaces is to include many extra dimensions \( D \sim Q^2 \gg 1 \), in a timelike linear dilaton solution. The resulting friction renders the time evolution close to the RG flow of the corresponding sigma model. At large radius, the latter in turn reduces to Ricci flow, for which simple solutions have been worked out \cite{21}. Let us adopt this prescription for convenience, to illustrate the consistency of our methods in the case of the Sol geometry using the simplicity of the evolution under Ricci flow. In this setup, the effective central charge is large (\( \sim D \)) to begin with, and we will be interested in picking out the enhancement to it arising from the exponentially growing fundamental group in the Sol component of the target space.

For the Sol geometry, the late-time solution reviewed for example in \cite{21} translates into a string-frame metric

\[ ds_{S}^2 = -dt^2 + \frac{4t}{Q}dz^2 + \frac{1}{2}e^{-2z}dx^2 + \frac{1}{2}e^{2z}dy^2 + dx_{\perp}^2, \]  

(3.42)

where the RG scale is \( \mu e^{t/Q} \) and the extra dimensions are taken to be flat with coordinates \( x_{\perp} \). The string coupling decreases exponentially with \( t \): \( g_s \sim g_0 e^{-Qt} \).

Compactifying the \( x_{\perp} \) directions on a flat torus, and converting to Einstein frame in the remaining four dimensions, yields a metric

\[ ds_{4d,E}^2 = e^{2Qt} \left( -dt^2 + \frac{4t}{Q}dz^2 + \frac{1}{2}e^{-2z}dx^2 + \frac{1}{2}e^{2z}dy^2 \right). \]  

(3.43)

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\(^{12}\) We thank Michael Freedman for pointing out these features to us.

\(^{13}\) To match to the usual mathematical conventions for Ricci flow, we will set \( \alpha' = 2 \) in this subsection and in §4.2.
The Laplacian for a scalar field $\eta$ on this spacetime is
\[
\nabla^2 \eta = -\frac{1}{e^{4Qt}} \partial_t (\sqrt{t} e^{2Qt} \partial_t \eta) + e^{-2Qt} \left( \frac{Q}{4t} \partial_z^2 \eta + 2(e^{2z} \partial_z^2 \eta + e^{-2z} \partial_y^2 \eta) \right).
\]
(3.44)

If we look at the modes $\eta_{00}$ with $k_x, k_y = 0$, we get an equation of motion
\[
e^{2Qt} \nabla^2 \eta_{00} = -\ddot{\eta}_{00} - \left[ \frac{1}{2t} + 2Q \right] \dot{\eta}_{00} - \frac{Qk_z^2}{4t} \eta_{00} = 0.
\]
(3.45)

Relative to the timelike linear dilaton background, a new contribution to the Hubble friction (the first term in square brackets) is manifest in this equation. Moreover, this new contribution to the Hubble friction $H$ from the expanding $z$ direction competes with the gradient term for modes with nonzero $k_z$, just as in our spectrum (3.12) arising in the constant curvature case discussed in §3. In general, in an expanding space the effective mass squared for a canonically normalized scalar has a contribution proportional to $-H^2$ [12] (as well as a contribution of order $\dot{H}$, which in our example is subleading). If the magnitude of this is greater than the gradient energy at late times, then this enhances the IR divergence in the one-loop partition function. More prosaically, a differential equation of the form
\[
0 = \ddot{\eta} - 2H(t) \dot{\eta} + m^2_0(t) \eta
\]
(3.46)
may be rewritten (by substituting $\eta = a(t)y(t)$, with $H = \dot{a}/a$) as
\[
0 = \ddot{y} + m^2_{\text{eff}} y
\]
(3.47)
with
\[
m^2_{\text{eff}} \equiv m^2_0 - H^2 + \dot{H}.
\]
(3.48)

In the late-time Sol cosmology, $-H^2$ has a contribution going like $-Q/t$, which is bigger than the gradient term $Qk_z^2/t$ for sufficiently small $k_z^2$ (and is also bigger than the additional contribution $\dot{H} \sim -1/t^2$ to the effective mass). This provides the requisite new pseudotachyonic contribution to $m^2_{\text{min}}(t)$ in our background. The analysis in [12] applies directly to a field canonically normalized with respect to the proper time $e^{Qt}/Q$; it is straightforward to check that a similar scaling argument yields the same conclusion in that description.

There is no gap in the spectrum of the covering space, which is the Sol geometry (3.41). But there is still an important distinction between the covering space and the
closed manifold we obtain via projection. Namely, the modes with strictly zero momentum in the $x$ and $y$ directions, those with $k_x = k_y = 0$, do not exist as normalizable modes on the covering space. For any nonzero $k_x$ or $k_y$, however small, the additional friction term does not compete with the gradient term at late times, so on the covering space there are no pseudotachyons beyond those already present in the original timelike linear dilaton background. On the compact space obtained by orbifolding by isometries, the $k_x = k_y = 0$ mode is normalizable, and as just discussed this mode is pseudotachyonic.

4. Polynomial Growth

Next consider the case of critical string theory on target spaces with fundamental groups of polynomial growth. Since in this case there is not a Hagedorn density of winding strings, $c_{\text{eff}} = c_{\text{crit}}$, and modular invariance requires that the small perturbation spectrum contain no pseudotachyons.

4.1. Non-negative Curvature

In [2], Milnor proved that manifolds with non-negative sectional curvatures have polynomial growth of the fundamental group. In particular, the power law growth for an $n$-manifold of this type is $\rho(l) \leq l^n$. Thus such spaces do not have an enhancement of their effective central charge.

4.2. Nil Manifold

The Nil manifold $\mathcal{N}$ is an interesting case, with constant negative scalar curvature but without universally negative sectional curvatures. Although the growth of the fundamental group in this case is only power law, the number of words of length $s$ grows like a constant times $s^4$ (see e.g. [22] sections VI.6, VII.7, VII.22). This is a larger power than for $T^3$, or any 3-manifold of positive semidefinite curvature, whose fundamental group grows at most like $s^3$ [22]. This comparison suggests that even for critical $c_{\text{eff}}$, the number of degrees of freedom encoded in the subexponential contribution to the density of states increases with the potential energy (since negative scalar curvature corresponds to positive potential energy). This might provide a generalization of [23] to the closed string sector, with increasing potential energy corresponding to increasing numbers of degrees of freedom.
Following [21], we can consider the late-time Ricci flow in a very supercritical limit of our system, as in the above discussion of the Sol geometry in §3.5, which introduced the relevant framework and notation. In the Nil case, the string-frame metric is at large $t$

$$ds^2_S = -dt^2 + \frac{Q^{1/3}}{3t^{1/3}} \left( dx + \frac{1}{2} ydz - \frac{1}{2} zdy \right)^2 + \frac{t^{1/3}}{Q^{1/3}}(dy^2 + dz^2) + dx_1^2,$$

leading to a 4d Einstein frame metric

$$ds^2_{4d,E} = e^{2Qt} \left( -dt^2 + \frac{Q^{1/3}}{3t^{1/3}} \left( dx + \frac{1}{2} ydz - \frac{1}{2} zdy \right)^2 + \frac{t^{1/3}}{Q^{1/3}}(dy^2 + dz^2) \right)$$

and a scalar Laplacian

$$e^{2Qt} \nabla^2 \eta = -\partial_t^2 \eta - \left( \frac{1}{6t} + 2Q \right) \partial_t \eta + 3\frac{Q^{1/3}}{t^{1/3}} \partial_z^2 \eta + \frac{Q^{1/3}}{t^{1/3}} \left( \partial_y^2 \eta + \partial_z^2 \eta + (y\partial_z + z\partial_y)\partial_x \eta + \frac{1}{4}(y^2 + z^2)\partial_x^2 \eta \right).$$

We are interested in the possibility of an enhanced IR divergence in the compact quotient of the Nil geometry. Such new pseudotachyons would require normalizable modes for which Hubble friction dominates at late times. Since the $x$ direction shrinks, modes with momentum in the $k_x$ direction blueshift, with gradient energy dominating. Let us therefore consider $k_x = 0$ modes. For these, (4.3) boils down to

$$-\partial_t^2 \eta - \left[ \frac{1}{6t} + 2Q \right] \partial_t \eta + \frac{Q^{1/3}}{t^{1/3}} (\partial_z^2 \eta + \partial_y^2 \eta).$$

As in the Sol case, we can read off a new contribution to the Hubble friction arising from the Nil directions (the first term in square brackets). In contrast to the Sol case, here the new contribution to the Hubble friction squared is subdominant to the gradient energy in the $y, z$ directions. This means these modes should not gain a new pseudotachyonic contribution in addition to the usual one from the supercritical linear dilaton, which is consistent with the absence of exponential growth of the fundamental group. Also as in Sol, one obtains a similar result for the field canonically normalized with respect to the proper time $e^{Qt}/Q$. 

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5. Intermediate Growth?

There exist groups with intermediate growth, meaning the following. Express each group element minimally as a “word” formed from generators of the group. A group of intermediate growth has the property that the number of group elements $\rho(s)$ of word length $\leq s$ grows at large $s$ faster than any power of $s$, but slower than $e^{\lambda s}$ for any constant $\lambda$ [24]. However, all known examples are not finitely presentable. This has raised the question of whether or not there exist finitely presentable groups of intermediate growth [25, 24, 22].

Any finitely presentable group can be realized as the fundamental group of a smooth manifold of dimension $n \geq 4$ (see e.g. [26]). Those manifolds constituting good initial data for GR can be analyzed using the techniques developed here and in [1]. (The requirement of having good initial data is a very weak constraint, including as in §3.6 and §4.2 the rolling dilaton in string theory.)

Suppose, as a particular example of intermediate growth, that the density of winding modes on such a manifold $M_I$ grew like

$$\rho(l) \sim e^{\lambda l^\gamma}$$

for some real numbers $\lambda$ and $\gamma$, with $\gamma < 1$. Consider the perturbative string one-loop partition function on the spacetime obtained by time-evolving $M_I$. The UV limit of the sum over winding modes would then take the form

$$\int dl e^{\lambda l^\gamma} f(l) e^{-l^2/\tau_2},$$

where $f(l)$ is the analogue of the $1/\sinh(l/2)$ factors in (3.37).

Ignoring the $f(l)$ factor for a moment, and performing the $l$ integral by saddle point approximation about the stationary point $l_s^{2-\gamma} = \gamma \lambda / (2\tau_2)$, yields a winding contribution to $Z_{UV}$ of the form

$$\exp \left[ (2 - \gamma) \left( \frac{\lambda}{2} \right)^{2/(2-\gamma)} \left( \frac{\gamma}{\tau_2} \right)^{\gamma/(2-\gamma)} \right].$$

Assuming that smooth solutions of the long-distance equations of motion of string theory yield consistent, modular-invariant string backgrounds, the same one-loop partition function must have as an IR limit that goes like [13, 14].

\[\text{14 In another context, this question was addressed recently in [27], where the authors found modular invariance to arise nontrivially in a basic class of smooth string backgrounds.}\]

\[\text{15 In this section we have assumed } c_{\text{eff}}^{\text{crit}} = 0 \text{ for simplicity.}\]

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\[ Z_{\text{IR}} \sim \exp \left( -(\text{const}) \tau_2^{\gamma/(2-\gamma)} \right). \] (5.4)

For the putative theory with intermediate growth, \( \gamma < 1 \) and this expression (5.4) is not of the form (2.2). In order to recover an expression consistent with modular invariance, the factor \( f(l) \) in (5.2) would have to have the property that it completely cancelled out the leading \( e^{l^\gamma} \) behavior of the density of states. A similar cancellation would have to take place for any other intermediate growth function. While this seems unlikely in general, one could alternatively take it as a hint for how to attempt to construct manifolds that exhibit fundamental groups with intermediate growth.

6. Discussion: Lessons, Generalizations, Applications

So far we have seen in prototypical examples how the relation between geometry and topology fits neatly with perturbative string modular invariance. Let us now make some comments about mathematical and physical implications and generalizations.

6.1. Physical Lessons and Analogies

The emergence of new dimensions arising from the growth of the fundamental group of compact negatively curved spaces is novel, but in some ways reminiscent of previous phenomena in string theory. Let us compare and contrast our phenomenon to other familiar effects.

(i) In string/M theory duality, a new dimension arises at strong coupling in type IIA string theory, and in M theory on a \( T^2 \), the small volume limit builds up the 10-dimensional type IIB theory.

(ii) String theory on a very small \( T^n \) is equivalent to string theory on a large \( T^n \) via T-duality. Perhaps this is the closest existing analogue of the effect discussed here. At very small radius, there are significant indications [34] that the system naturally asymptotes to a large number of new dimensions which may be geometrical. However, in the torus case, the effective central charge is independent of the size; the effective number of dimensions in which strings can oscillate is fixed. In our case, \( c_{\text{eff}} \) grows as the space shrinks.

(iii) In open string theory, the tension of D-branes is related to the dimension of the open string Hilbert space [23]. Here, we have seen that positive potential energy arising
from negative scalar curvature correlates with extra degrees of freedom. In the case of negative sectional curvature, the new degrees of freedom build up an enhanced $c_{\text{eff}}$. In cases like the Nil geometry with negative scalar curvature but indefinite sectional curvature, there is no enhancement of $c_{\text{eff}}$, but we saw that the power law growth of the fundamental group exceeded that of any non-negatively curved target space.

6.2. Scope

Milnor’s proof [2] implies an exponential density of winding modes in any space of strictly negative sectional curvature. String theoretic modular invariance predicts a supercritical $c_{\text{eff}}$ for any consistent perturbative background with pseudotachyons in its spectrum of small fluctuations. We have seen that these two phenomena line up precisely in the examples discussed above. In more general circumstances, one or the other notion may apply independently.

It is interesting to ask whether the correspondence exhibited here extends to perturbatively metastabilized solutions, such as [28]. In particular, de Sitter and inflationary solutions have pseudotachyonic modes. To the extent that perturbative string theory applies, which in metastabilized solutions would require applying the Fischler-Susskind method for retaining conformal invariance [28], these modes translate via modular invariance to an enhanced $c_{\text{eff}}$. This interpretation is consistent with the present state of the art in moduli stabilization, which exhibits the following pattern. Low energy supersymmetric de Sitter models have crucial contributions from non-perturbative ingredients [30], so string perturbation theory (even at the level of Fischler-Susskind) does not apply and $c_{\text{eff}}$ is not well defined. Perturbative models with SUSY broken at or above the Kaluza-Klein scale in the geometry are explicitly supercritical [31] or involve a negatively curved compactification [28], which we now also understand to be supercritical.

6.3. Future Directions and Applications

There are many directions that suggest themselves. First, it is worth emphasizing that in the case we studied here, we analyzed only the deep UV and deep IR physics of the one-loop amplitude, focusing on the leading bosonic contributions. It would be very interesting to work out the fermionic contributions in detail, and to compute (if possible at finite time) the full one loop partition function including all the “oscillator” modes.

Mathematically, the relation between geometry and topology is a subject of current research, for example as discussed in the above section on intermediate growth. It arises
the classification of 3-manifolds (which in effect constitute \((3 + c_{\text{eff}})\)-dimensional spaces in string theory, suggesting perhaps an alternate classification scheme). For another example, the full characterization of groups that can appear as fundamental groups of Kähler manifolds is an open problem. It is interesting to consider whether a more detailed probe than the one-loop partition function could help determine the limitations on fundamental groups of Kähler manifolds \[32\].

We focused here on the compact case, for which Milnor’s theorem \[2\] is sufficient to derive the new effective dimensions from winding strings. Compactness is a sufficient, but not necessary, condition; we expect there to be examples of \textit{localized} pseudotachyons which correspond to an exponential density of winding modes arising in \textit{noncompact} manifolds with exponentially growing fundamental group. It might be interesting to study these cases in detail.

At large radius, it is interesting to ask whether group theoretic considerations can help determine features of the evolution and small perturbation spectrum of these spaces. For example, it is of interest to learn what cosmological phases arise from vacuum solutions obtained from compactification on generic curved manifolds. Initial conditions in such string cosmologies will be affected by the behavior at small radius, at very early times.

Physically, a key question is indeed the behavior of the system at very small radius. There are several independent indications that the system can naturally asymptote to a specific supercritical background of string theory with a rolling closed string tachyon; these are under investigation \[33\]. More specifically, for \(n = 2\) the Jacobian torus may arise naturally (and for higher even dimensions the analogous Albanese variety). This is the moduli space of a single wrapped D-brane probe; moreover for a spherical shell of wrapped D-branes this torus appears as its approximate moduli space, within a UV complete system providing a field theory dual. In the closed string sector alone, in even dimensions, the embedding of the compact manifold \(M_n\) into a torus of any dimension proceeds through the Jacobian/Albanese variety, which preserves the symmetries of the system by mapping one-cycles of \(M_n\) into those of the torus. Moreover, a preliminary analysis indicates that a variant of Buscher’s path integral derivation of T-duality produces in our case the Jacobian perturbed by a rolling tachyon. This substantiates the hint already available at large radius, that the high energy winding strings trace out a lattice random walk in

\[16\] See \[33\] for potentially related recent works.
2h dimensions for a Riemann surface of genus \( h \), suggesting that a continuum limit of 2h dimensions might arise [1].

Among homogeneous spaces, negative curvature is generic in dimensions \( n = 2 \) and 3. Three being the number of large spatial dimensions in the observed universe, there is strong motivation for getting a handle on these spaces in string theory. The winding string degrees of freedom (which go beyond general relativity) build up new effective dimensions, and these degrees of freedom become more important as the space shrinks.\[\] In more formal terms, it seems that a new type of string duality (perhaps best described as a “D(imensional) duality”) is emerging in the case of negatively curved compact target spaces.

Acknowledgements

We are grateful to A. Maloney for originally pointing out that the Selberg trace formula provides a precise method for calculating the enhancement to \( c_{\text{eff}} \), and to O. Aharony for many discussions on pseudotachyons. We are grateful to D. Green, A. Lawrence and D. Morrison for their many insights in the related work [3]. We thank A. Adams, M. Freedman, G. Horowitz, S. Kachru, N. Lambert, L. McAllister, J. Polchinski, and A. Tomasiello for useful discussions on this topic. E.S. and D.S. thank the KITP for hospitality during the completion of this work, and gratefully acknowledge useful discussions in the UCSB math/physics seminar on “Geometry, Topology, and Physics” run by D. Morrison. This research was supported in part by the National Science Foundation under Grant No. PHY99-07949. E.S. and D.S. are also supported in part by the DOE under contract DE-AC03-76SF00515, by the NSF under contract 9870115, and by an FQXI grant. The work of J.M. is supported in part by funds provided by the U.S. Department of Energy (D.O.E.) under cooperative research agreement DE-FG0205ER41360.

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It would be interesting to understand how this affects the question of summing over “three-manifolds” (which has interesting subtleties to do with the distribution of volumes at fixed curvature [34]).
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