Quantum LDPC codes with positive rate and minimum distance proportional to $n^{1/2}$

Jean-Pierre Tillich* Gilles Zémor†

January 15, 2009

Abstract

The current best asymptotic lower bound on the minimum distance of quantum LDPC codes with fixed non-zero rate is logarithmic in the block-length. We propose a construction of quantum LDPC codes with fixed non-zero rate and prove that the minimum distance grows proportionally to the square root of the blocklength.

1 Introduction

LDPC codes [11] and their variants are one of the most satisfying answers to the problem of devising codes guaranteed by Shannon’s theorem. They display outstanding performance for a large class of error models with a fast decoding algorithm. Generalizing these codes to the quantum setting seems a promising way to devise powerful quantum error correcting codes for protecting, for instance, the very fragile superpositions manipulated in a quantum computer. It should be emphasized that a fast decoding algorithm could be even more crucial in the quantum setting than in the classical one. In the classical case, when error correction codes are used for communication over a noisy channel, the decoding time translates directly into communication delays. This has been the driving motivation to devise decoding schemes of low complexity, and is likely to be important in the quantum setting as well. However, there is an important additional motivation for efficient decoding in the quantum setting. Quantum computation is likely to require active stabilization. The decoding time thus translates into computation delays, and most importantly in error suppression delays. If errors accumulate faster than they can be identified, quantum computation may well become infeasible: fast decoding is an essential ingredient to fault-tolerant computation.

Quantum generalizations of LDPC codes have indeed been proposed in [18]. However, it has turned out that the design of high performance quantum LDPC
codes is much more complicated than in the classical setting. This is due to several reasons, the most obvious of which being that the parity-check matrix of quantum LDPC codes must satisfy certain orthogonality constraints. This complicates significantly the construction of such codes. In particular, the plain random constructions that work so well in the classical setting are pointless here. There have been a number of attempts at overcoming this difficulty and a variety of methods for constructing quantum LDPC codes have been proposed [19, 16, 18, 5, 6, 17, 13, 8, 21, 12, 14, 8, 21, 1, 2, 14, 23]. However, all of these constructions suffer from disappointingly small minimum distances, namely whenever they have non-vanishing rate and parity-check matrices with bounded row-weight, their minimum distance is either proved to be bounded, or unknown and with little hope for unboundedness. The point has been made several times that minimum distance is not everything, because there are complex decoding issues involved, whose behaviour depends only in part on the minimum distance, and also because a poor asymptotic behaviour may be acceptable when one limits oneself to practical lengths. Nevertheless, the minimum distance has been the most studied parameter of error-correcting codes and given that asymptotically good (dimension and minimum distance both linear in the blocklength) quantum LDPC codes are expected to exist, it is of great theoretical interest, and possibly also practical, to devise quantum LDPC codes with large, growing, minimum distance. This is the problem that we address in the present paper, leaving aside decoding issues for discussion elsewhere.

Besides the above constructions, we must mention the design of quantum LDPC codes based on tessellations of surfaces [16, 3, 7], among which the most prominent example is the toric code of [16]. Toric codes have minimum distances which grow like the square root of the blocklength and parity-check equations of weight 4 but unfortunately have fixed dimension which is 2, and hence zero rate asymptotically. It turns out that by taking appropriate surfaces of large genus, quantum LDPC codes of non vanishing rate can be constructed with minimum distance logarithmic in the blocklength, this has actually been achieved in [10, Th. 12.4], see also [24], [15]. To the best of our knowledge, this is until now the only known family of quantum LDPC codes of non-vanishing rate that yields a (slowly) growing minimum distance.

We improve here on these surface codes in several ways, by providing a flexible construction of quantum LDPC codes from any pair \((H_1, H_2)\) of parity-check matrices of binary LDPC codes \(C_1\) and \(C_2\). Although the constructed quantum code belongs to the CSS class [4, 22], there is no restriction on \(C_1\) and \(C_2\). For instance, they do not need to be mutually orthogonal spaces as in the CSS construction. In particular we can choose \(C_1 = C_2\), in which case our main result reads:

**Theorem 1** Let \(H\) be a full-rank \((n - k) \times n\) parity-check matrix of a classical LDPC code \(C\) of parameters \([n, k, d]\). There is a construction of a quantum LDPC code with \(H\) as building block, of length \(N = n^2 + (n - k)^2\), dimension \(k^2\), and quantum minimum distance \(d\). The quantum code has a parity-check matrix with row weights of the form \(i + j\), where \(i\) and \(j\) are respectively row and column weights of the original parity check matrix \(H\).
In particular, any family of classical asymptotically good LDPC codes of fixed rate yields a family of quantum LDPC codes of fixed rate and minimum distance proportional to a square root of the block length.

2 Overview of the construction

The quantum codes we will consider are CSS codes. A CSS code of length \( n \) is determined by two binary parity-check matrices \( H_X \) and \( H_Z \) of two classical codes of length \( n \), \( C_X \) and \( C_Z \) respectively, with the property that every row of \( H_X \) is orthogonal to every row of \( H_Z \), in other words the row-spaces \( C_X^\perp \) and \( C_Z^\perp \) of \( H_X \) and \( H_Z \) are orthogonal subspaces of \( \mathbb{F}_2^n \). The parameters of the associated quantum code are \([n, k, d]\), where \( n \) is the blocklength, \( k \) is its dimension and is given by \( n - \dim C_X^\perp - \dim C_Z^\perp \), and the minimum distance \( d \) is given by the minimum weight of the non-zero vectors that are either in \( C_X \) but not in \( C_Z^\perp \) or in \( C_Z \) but not in \( C_X^\perp \).

For details on why the above defines the CSS construction and why it is relevant to quantum error correction see for example [18]. We are interested in families of CSS codes that have sparse matrices \( H_X \) and \( H_Z \), i.e. whose row weight is bounded by a constant, in which case we shall say that we have a quantum LDPC (CSS) code.

Our construction borrows both from classical LDPCs and Kitaev’s toric quantum code. To get a clear picture of the construction it is desirable to take a close look at the toric code and explain how we shall generalize it.

The toric code is based on the graph \( \mathcal{G} \) represented on figure 1 which is a tiling of the 2-dimensional torus. The vertex set of the graph is \( V = \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z} \) and there is an edge between every vertex \((x, y)\) and the four vertices \((x \pm 1, y), (x, y \pm 1)\). Now number the edges from 1 to \( m^2 = n \) so as to identify the edge set with \([1, n]\). The ambient space \( \mathbb{F}_2^n \) is therefore identified with subsets of edges. The matrix \( H_X = (h_{ij}) \) is the vertex-edge incident matrix, rows are indexed by vertices of the graph \( \mathcal{G} \), and \( h_{ij} = 1 \) iff vertex \( i \) is incident to edge \( j \). The associated code \( C_X \) is the cycle code of \( \mathcal{G} \), a cycle being by definition a set of edges \( Z \) such that every vertex is incident to an even number of edges of \( \mathbb{Z} \). Elements of the row-space \( C_X^\perp \) are called cocycles, rows of \( H_X \) are called elementary cocycles, and the row-space itself \( C_X^\perp \) is also known as the cocycle code of \( \mathcal{G} \).

The second matrix \( H_Z \) of the quantum code is defined as the face-edge incidence matrix. The faces are defined as the 4-cycles \((x, y), (x + 1, y), (x + 1, y + 1), (x, y + 1)\).

The rowspace \( C_Z^\perp \) of \( H_Z \) is therefore a subspace of the cycle code \( C_X \), and the quotient \( C_X/C_Z^\perp \) is readily seen to have dimension 2, coset leaders of the quotient being given by cycles of the form \((a, 0), (a, 1), \ldots, (a, m - 1)\) and \((0, a), (1, a), \ldots, (m - 1, a)\), as represented by the thick lines on figure 1. The dimension of the quantum code is therefore equal to 2 and the minimum weight of a vector of \( C_X \) not in \( C_Z^\perp \) is therefore equal to \( m \).

To conclude that the minimum distance of the quantum code is actually \( m \), it remains to determine the minimum weight of a vector of \( C_Z \) that is not in
C⊥ X, i.e. that is not a cocycle. This particular graph G has the nice property of being a tiling of a surface (the torus). This means that it has a dual graph. The (Poincaré) dual graph G′ has vertex set equal to the faces of G, and there is an edge between two vertices of G′ if the corresponding faces of G have a common edge in G. Furthermore the dual graph G′ of G is isomorphic to G itself, and given that the edges of G define the edges of G′, the ambient space $\mathbb{F}_2^n$ can be identified with the edge set of the dual graph G′. With this identification, the elementary cocycles of G become the faces of G′ and the faces of G become the elementary cocycles of G′. Hence the minimum weight of a vector of $C_Z$ that is not in $C⊥ X$ is exactly the same as the minimum weight of a vector of $C_X$ not in $C⊥ Z$ and the minimum distance of the quantum code is exactly m.

This duality argument is quite powerful, and for this reason a number of quantum codes that arise by replacing the graph G by different tilings of different surfaces have been investigated (surface codes). Here we shall consider a different generalization that does not destroy the graph duality but generalizes it.

Our first remark is that the graph G is a product graph: it is the product of two graphs each equal to an elementary cycle of length m. The product $G_1 \cdot G_2$ of two graphs $G_1$ and $G_2$ has vertex set made up of couples $(x, y)$, where x is a vertex of $G_1$ and y of $G_2$. The edges of the product graph connect two vertices $(x, y)$ and $(x', y')$ if either $x = x'$ and $\{y, y'\}$ is an edge of $G_2$ or $y = y'$ and $\{x, x'\}$ is an edge of $G_1$. Note that any two edges $\{a, b\}$ and $\{x, y\}$ of $G_1$ and $G_2$ define the 4-cycle of $G_1 \cdot G_2$:

$$
\begin{align*}
(a, y) & \rightarrow (b, y) \\
\downarrow & \downarrow \\
(a, x) & \rightarrow (b, x)
\end{align*}
$$

Now, we are tempted to define a quantum code by, as before, declaring $H_X$ to be the vertex-edge incident matrix of a product graph $G = G_1 \cdot G_2$ of two arbitrary graphs, and by declaring $H_Z$ to be the matrix whose rows are the characteristic vectors of all faces, i.e. the 4-cycles of the form $(\star)$. This is
a quantum code which generalizes the toric code, since the latter corresponds to the case when $S_1$ and $S_2$ are two cycles of length $m$. This construction loses graph duality however, and our objective was to preserve it. But a closer look shows us that graph duality has not completely gone: the dual has simply become a hypergraph, whose vertex set is a the set of faces of $S$ and where the hyperedges are the subsets of those faces of $S$ that meet in a common edge of $S$. This observation shows us that we really should consider products of hypergraphs rather than graph products to start with.

Our construction will proceed as follows. We will consider a product $\mathcal{H} = \mathcal{H}_1 \cdot \mathcal{H}_2$ of two hypergraphs. The matrix $H_X$ of the quantum code will be defined as before, by the vertex-hyperedge incidence matrix of $\mathcal{H}$. The 4-cycles of the form $(\star)$ will be replaced by similar structures that we will call chambers (to be defined precisely below). The matrix $H_Z$ of the quantum code will be the chamber-hyperedge incidence matrix. There will again be a duality notion for product hypergraphs, such that the chambers of the dual hypergraph are the elementary (hyper)cocycles of the original graph, and the chambers of the original hypergraph are the elementary (hyper)cocycles of the dual graph.

What about the parameters of the quantum code? The two hypergraphs $\mathcal{H}_1$ and $\mathcal{H}_2$ can be identified with their vertex-edge incidence matrices $H_1$ and $H_2$. The parameters of the quantum code will be directly related to the dimensions and minimum distances of the codes $C_1$ and $C_2$ for which $H_1$ and $H_2$ are parity-check matrices. They will also depend on the dimensions and minimum distances of two associated codes $C_1^T$ and $C_2^T$ that we will call the transpose codes and that have the transpose matrices $H_1^T$ and $H_2^T$ for parity-check matrices. Some of these codes may be trivial and equal $\{0\}$.

The dimension of the quantum code is computed by standard linear algebra arguments. The computation of the minimum distance involves two ideas, one is the duality for product hypergraphs sketched above, and the other is a dimension argument. If a (hyper)cycle of the product hypergraph has a weight which is too small, then it has to be included in a sub-product-hypergraph whose associated quantum dimension is shown to be zero, which means the cycle has to belong to the chamber code.

We now move on to precise definitions, statements and proofs.

3 Hypergraphs

Let $H$ be a sparse parity-check matrix of some binary linear code $C$. We will view $H$ as a hypergraph $\mathcal{H}$, which is just a set of vertices $\mathcal{V}$ together with a collection $\mathcal{E}$ of subsets of $\mathcal{V}$ called hyperedges (henceforth edges). Given $H$, the vertices of $\mathcal{H}$ are its rows and the edges of $\mathcal{H}$ are the columns of $H$, and vertex $i$ belongs to edge $j$ if the corresponding entry of $H$ equals 1. In particular if $N = |\mathcal{E}|$ is the number of edges of $\mathcal{H}$, we shall identify the Hamming space $\{0,1\}^N$ with $\{0,1\}^\mathcal{E}$, i.e. coordinates are labeled with the edges of $\mathcal{H}$ and vectors of $\{0,1\}^N$ are identified with subsets of edges of $\mathcal{H}$.

We use the language of hypergraphs, even though it is not always familiar to coding theorists, because it serves to highlight the connection with topological
quantum codes and because it is better suited to our proof techniques. In particular the quantum code that we will define has an underlying hypergraph structure that generalizes the grid structure behind Kitaev’s toric code. The reader may nevertheless translate hypergraphs $\mathcal{H}$ into parity-check matrices $H$ to be on more familiar ground.

An elementary cocycle of $\mathcal{H}$ is the subset of edges incident to a given vertex $v \in V$. A cocycle is a sum of elementary cocycles. The set of cocycles is a linear code (the cocycle code) that is the dual code $Z(\mathcal{H})^\perp$ of $Z(\mathcal{H})$.

If $\mathcal{H}$ is a hypergraph with vertex set $V$ and edge set $E$, we define the transpose hypergraph $\mathcal{H}^T$ with vertex set $V^T = E$ and edge set $E^T = V$, to be the hypergraph whose vertex-edge incidence matrix is the transpose matrix $H^T$ of the vertex-edge incidence matrix $H$ of $\mathcal{H}$. We will denote an edge of $\mathcal{H}^T$ simply by $x \in V$. Note that we have

$$\dim Z(\mathcal{H}^T) = |V| - \dim Z(\mathcal{H})^\perp. \tag{1}$$

The hypergraph $\mathcal{H}^T$ is sometimes called the dual hypergraph by graph theorists. We avoid this terminology here because we have too many notions of duality to deal with.

Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two hypergraphs with respective vertex sets $V_1, V_2$ and edge sets $E_1, E_2$.

The product $\mathcal{H} = \mathcal{H}_1 \times \mathcal{H}_2$ of the two hypergraphs is defined as the hypergraph having vertex set $V = V_1 \times V_2$ and edge set $E = E_L \cup E_R$ where

- $E_L$ is the set of edges $\{(a, y_1), (a, y_2), \ldots, (a, y_v)\}$ for $a$ a vertex of $V_1$ and $\{y_1, y_2, \ldots, y_v\}$ an edge of $E_2$
- $E_R$ is the set of edges $\{(x_1b), (x_2b), \ldots, (x_ub)\}$ for $b$ a vertex of $V_2$ and $\{x_1, x_2, \ldots, x_u\}$ an edge of $E_1$.

To lighten notation we shall write vertices of $V$ as $ab$ instead of $(a, b)$ and similarly, if $\alpha = \{z_1, z_2, \ldots, z_t\}$ is an edge of $E_1$ (respectively $E_2$) and $x$ is a vertex of $V_2$ (respectively $V_1$) we shall write $\alpha x$ (respectively $x\alpha$) to mean the edge $\{z_1x, z_2x, \ldots, z_tx\}$ (respectively $\{xz_1, xz_2, \ldots, xz_t\}$) of $E$.

Note that the edge set $E$ is indexed by the set $V_1 \times E_2 \cup V_2 \times E_1$ so that we have

$$|E| = |V_1||E_2| + |V_2||E_1|.$$ 

Let $\alpha = \{x_1, x_2, \ldots, x_u\}$ and $\beta = \{y_1, y_2, \ldots, y_v\}$ be two edges of $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively. We now define the chamber $C_{\alpha \beta}$ of $\mathcal{H}$ as the set of edges of $E$

$$C_{\alpha \beta} = \{\alpha y_i, i = 1 \ldots v\} \cup \{x_i\beta, i = 1 \ldots u\}.$$ 

The number of edges belonging to $C_{\alpha \beta}$ is therefore $|C_{\alpha \beta}| = u + v$. 

6
Remark. When the hypergraphs $\mathcal{H}_1$ and $\mathcal{H}_2$ are graphs (edges are incident to exactly two vertices) then chambers are 4-cycles of type ($\star$).

We now define the Poincaré dual hypergraph $\mathcal{G}$ of $\mathcal{H} = \mathcal{H}_1 \cdot \mathcal{H}_2$ by $\mathcal{G} = \mathcal{H}^T_1 \cdot \mathcal{H}^T_2$. We have:

**Proposition 2** $\mathcal{G}$ has vertex set $E_1 \times E_2$ and it edges are indexed by $V_1 \times E_2 \cup V_2 \times E_1$ so that we may identify its edge set with the edge set $\mathcal{E}$ of $\mathcal{H}$. With this identification, the elementary cocycles of $\mathcal{H}$ are the chambers of $\mathcal{G}$ and the chambers of $\mathcal{H}$ are the elementary cocycles of $\mathcal{G}$.

**Proof:** Let $v = xy$ be a vertex of $\mathcal{V}$. The elementary cocycle associated to $v$ is the union

$$\{\alpha y, x \in \alpha\} \cup \{x\beta, y \in \beta\}.$$

It follows from the definition of a chamber and of the transpose hypergraphs $\mathcal{H}^T_1$ and $\mathcal{H}^T_2$ that this set is exactly the chamber $C_{xy}$ of $\mathcal{G} = \mathcal{H}^T_1 \cdot \mathcal{H}^T_2$. 

### 4 The quantum code $Q_{\mathcal{H}}$ associated with the product hypergraph $\mathcal{H} = \mathcal{H}_1 \cdot \mathcal{H}_2$

We now define the quantum code $Q_{\mathcal{H}}$ associated to the product hypergraph $\mathcal{H}$. The code $Q_{\mathcal{H}}$ is a CSS code of length $N$ where $N = |\mathcal{E}|$ is the number of edges of $\mathcal{H}$. As announced before, $Q_{\mathcal{H}}$ is defined by the two matrices $H_X$ and $H_Z$ where $H_X$ is the vertex-edge incidence matrix of $\mathcal{H}$ and $H_Z$ is the chamber-edge incidence matrix. Equivalently, the quantum code $Q_{\mathcal{H}}$ is defined by the two linear codes $C_X$ and $C_Z$ of respective parity-check matrices $H_X$ and $H_Z$.

**Proposition 2** implies that the quantum code $Q_{\mathcal{H}}$ associated to the hypergraph $\mathcal{H}$ is the same as the quantum code $Q_{\mathcal{G}}$ associated to the dual hypergraph $\mathcal{G}$. Note also that $C_X$ is equal to the cycle code $Z(\mathcal{H})$ and that $C_Z$ is equal to the cycle code $Z(\mathcal{G})$ of the dual hypergraph $\mathcal{G}$. The dual codes $C^\perp_X$ and $C^\perp_Z$ are respectively the cocycle code of $\mathcal{H}$ and the code generated by the chambers of $\mathcal{H}$ that we call the chamber code of $\mathcal{H}$. By **Proposition 2** the chamber code of $\mathcal{H}$ is the cocycle code of $\mathcal{G}$ and the cocycle code of $\mathcal{G}$ is the chamber code of $\mathcal{H}$.

**Row and column weights of $H_X$ and $H_Z$.** We see from the definitions that any column weight of $H_X$ is the number of vertices of either an edge of $\mathcal{H}_1$ or of an edge of $\mathcal{H}_2$. A row weight of $H_X$ is the sum of the degree of a vertex of $\mathcal{H}_1$ and the degree of a vertex of $\mathcal{H}_2$, where the degree of a vertex is the number of edges incident to it. A column weight of $H_Z$ is equal to either the degree of a vertex of $\mathcal{H}_1$ or of a vertex of $\mathcal{H}_2$. Finally, a row weight of $H_Z$ is equal to the sum of the cardinality of an edge of $\mathcal{H}_1$ and the cardinality of an edge of $\mathcal{H}_2$.

We say that a hypergraph is $t$-uniform if every edge is incident to $t$ vertices and it said to be regular of degree $\Delta$, or $\Delta$-regular, if every vertex is incident to $\Delta$ edges. Suppose that the hypergraphs $\mathcal{H}_1$ and $\mathcal{H}_2$ are $t_1$- and $t_2$-uniform and $\Delta_1$- and $\Delta_2$-regular respectively. In other words their vertex-edge incidence matrices have column weights $t_1$ and $t_2$ and row weights $\Delta_1$ and $\Delta_2$ respectively.
Then the matrices $H_X$ and $H_Z$ each have two column weights, equal to $t_1$ and $t_2$ for $H_X$ and $\Delta_1$ and $\Delta_2$ for $H_Z$. The matrices $H_X$ and $H_Z$ each have constant row weight equal to $\Delta_1 + \Delta_2$ and $t_1 + t_2$ respectively.

It should be clear that we have

**Proposition 3** $C_X \perp Z(H) = C_X$

so that $C_X^\perp$ and $C_Z^\perp$ are mutually orthogonal subspaces which justifies the definition of the quantum code $Q_{\mathcal{H}}$.

The following proposition states that there are redundancies between the rows of the generators of $H_Z$.

**Proposition 4** Let $\mathcal{Z}_1$ be a cycle of $\mathcal{H}_1$ and let $\mathcal{Z}_2$ be a cycle of $\mathcal{H}_2$. Then

$$\sum_{(\alpha, \beta) \in \mathcal{Z}_1 \times \mathcal{Z}_2} C_{\alpha \beta} = 0.$$  \hspace{1cm} (2)

**Proof**: Let $E$ be an edge of $\mathcal{H}$ of the form $E = \alpha y$ for $\alpha \in \mathcal{Z}_1$ and $y \in \mathcal{V}_2$. By definition of a cycle, the number of edges $\beta$ of $\mathcal{Z}_2$ such that $y \in \beta$ is even. Therefore the edge $E = \alpha y$ belongs to an even number of chambers $C_{\alpha \beta}$ for any $y$ belonging to an edge of the cycle $\mathcal{Z}_2$. Proceed similarly for edges of the form $E = x \beta$, $x \in \mathcal{V}_1$, $\beta \in \mathcal{Z}_2$. \hfill \blacksquare

**Proposition 5** Let $k = \dim Z(\mathcal{H}_1)$ and $h = \dim Z(\mathcal{H}_2)$. Then the dimension of the chamber code $C_Z^\perp$ equals

$$\dim C_Z^\perp = |\mathcal{E}_1||\mathcal{E}_2| - kh.$$  \hspace{1cm} (3)

**Proof**: Let $\mathcal{F} \subset \mathcal{E}_1 \times \mathcal{E}_2$ be such that

$$\sum_{(\alpha, \beta) \in \mathcal{F}} C_{\alpha \beta} = 0,$$

which means that every edge appearing in this sum appears an even number of times. Note that every edge of the form $x \beta \in \mathcal{E}$ for a fixed $\beta$ only appears in chambers of the form $C_{\alpha \beta}$. Consider the subset $\mathcal{F}_{(\cdot, \beta)}$ of $\mathcal{E}_1$ defined by

$$\mathcal{F}_{(\cdot, \beta)} = \{\alpha \in \mathcal{E}_1, \ (\alpha, \beta) \in \mathcal{F}\}.$$  \hspace{1cm} (3)

Now the number of times that an edge $x \beta$ appears in the sum (3) equals the number of edges $\alpha$ of $\mathcal{F}_{\beta}$ that $x$ belongs to. Since this number is even for every $x \in \mathcal{V}_1$ we have that $\mathcal{F}_{(\cdot, \beta)}$ is a cycle of $\mathcal{H}_1$ for every edge $\beta$ of $\mathcal{E}_2$. Similarly, the subset $\mathcal{F}_{(\alpha, \cdot)}$ of $\mathcal{E}_2$ defined by

$$\mathcal{F}_{(\alpha, \cdot)} = \{\beta \in \mathcal{E}_2, \ (\alpha, \beta) \in \mathcal{F}\}$$

is a cycle of $\mathcal{H}_2$ for every edge $\alpha$ of $\mathcal{E}_1$.

Therefore the set of linear combinations (3) is a vector space isomorphic to the product code $Z(\mathcal{H}_1) \otimes Z(\mathcal{H}_2)$, and its dimension is therefore $kh$. The result follows. \hfill \blacksquare

Now by duality (Proposition 2), Proposition 5 becomes:
Proposition 6 Let \( r = \dim Z(\mathcal{H}_1^T) \) and \( s = \dim Z(\mathcal{H}_2^T) \). Then the dimension of the cocycle code \( C_x^\perp \) equals
\[
\dim C_x^\perp = |V_1||V_2| - rs.
\]

From Propositions 5 and 6 we obtain

Theorem 7 The dimension of the quantum code \( Q \mathcal{H} \) is equal to:
\[
\dim Q \mathcal{H} = 2rs + r(|E_2| - |V_2|) + s(|E_1| - |V_1|) = 2kh + k(|V_2| - |E_2|) + h(|V_1| - |E_1|).
\]

Proof: We have
\[
\dim Q \mathcal{H} = |E| - \dim C_x^T - \dim C_z^T = |E_1||V_2| + |V_2||E_1| - \dim C_x^T - \dim C_z^T = |E_1||V_2| + |V_2||E_1| - (|V_1||V_2| - rs) - (|E_1||E_2| - kh)
\]
by Propositions 5 and 6. From (1) we have
\[
k = |E_1| - |V_1| + r, \quad h = |E_2| - |V_2| + s
\]
and the result follows after rearranging.

Corollary 8 If
- either \( \dim Z(\mathcal{H}_1^T) = \dim Z(\mathcal{H}_2^T) = 0 \),
- or \( \dim Z(\mathcal{H}_1) = \dim Z(\mathcal{H}_2) = 0 \),
then \( \dim Q \mathcal{H} = 0 \), equivalently, \( C_x^T \) and \( C_z^T \) are dual to each other.

Proof: This follows directly from Theorem 7 and from Poincaré duality, since \( Q \mathcal{G} = Q \mathcal{H} \).

5 Minimum distance

Let us adopt the convention that the minimum distance of a code reduced to the all-zero codeword is \( \infty \). Let \( d_1 \) and \( d_2 \) be the minimum distances of the cycle codes \( Z(\mathcal{H}_1) \) and \( Z(\mathcal{H}_2) \). Let \( d_1^T \) and \( d_2^T \) be the minimum distances of the cycle codes of the transpose hypergraphs \( Z(\mathcal{H}_1^T) \) and \( Z(\mathcal{H}_2^T) \). Let \( D \) be the minimum distance of the quantum code \( Q \mathcal{H} \). We have:

Theorem 9 The minimum distance \( D \) of the quantum code \( Q \mathcal{H} \) satisfies:
\[
D \geq \min(d_1, d_2, d_1^T, d_2^T).
\]
Proof: Consider first a vector of $C_X$, i.e. a cycle $Z$ of $\mathcal{H}$, that is not a sum of chambers. Suppose furthermore that its weight $|Z|$ (number of edges) is minimum, and that this minimum is strictly smaller than $\min(d_1, d_2)$.

Now let $\mathcal{H}_1(Z)$ be the subhypergraph of $\mathcal{H}_1$ with vertex set $V_1$ and edge set made up of all those edges $\alpha$ such that $\alpha y$ is an edge of $Z$ for some $y \in V_2$. Similarly, let $\mathcal{H}_2(Z)$ be the subhypergraph of $\mathcal{H}_2$ with vertex set $V_2$ and edge set made up of all those edges $\beta$ such that $x\beta$ is an edge of $Z$ for some $x \in V_1$.

Notice that $\mathcal{H}_1(Z)$ is a subhypergraph of $\mathcal{H}_1$, i.e. every edge of $\mathcal{H}_1(Z)$ is an edge of $\mathcal{H}_1$. Similarly, $\mathcal{H}_2(Z)$ is a subhypergraph of $\mathcal{H}_2$. Notice also that the product graph $\mathcal{H}(Z) = \mathcal{H}_1(Z) \cdot \mathcal{H}_2(Z)$ is a subhypergraph of $\mathcal{H}$, therefore chambers of $\mathcal{H}(Z)$ are also chambers of $\mathcal{H}$. Furthermore, all edges of $Z$ are edges of $\mathcal{H}(Z)$ so that, since $Z$ is a cycle of $\mathcal{H}$, $Z$ is also a cycle of $\mathcal{H}(Z)$.

Now since $\mathcal{H}_1(Z)$ is a subhypergraph of $\mathcal{H}_1$, all cycles of $\mathcal{H}_1(Z)$ are also cycles of $\mathcal{H}_1$. But cycles of $\mathcal{H}_1$ have at least $d_1$ edges and the number of edges $\mathcal{H}_1(Z)$ is at most $|Z|$ and we have supposed $|Z| < d_1$. Therefore the only cycle in $\mathcal{H}_1(Z)$ is the empty cycle and $\dim Z(\mathcal{H}_1(Z)) = 0$. Similarly, $|Z| < d_2$ implies $\dim Z(\mathcal{H}_2(Z)) = 0$. Now Corollary 8 implies that all cycles of $\mathcal{H}(Z)$ are chambers of $\mathcal{H}(Z)$. Hence $Z$ is a chamber of $\mathcal{H}(Z)$ and of $\mathcal{H}$, a contradiction.

By duality we obtain that vectors of $C_Z$ (cycles of $\mathcal{G} = \mathcal{H}_1^T \cdot \mathcal{H}_2^T$) that are not cocycles of $\mathcal{H}$ (or chambers of $\mathcal{G}$) have weight at least $\min(d_1^T, d_2^T)$. This proves that $D \geq \min(d_1, d_2, d_1^T, d_2^T)$. 

The following lemma shows that the above bound is exact, except for some degenerate cases. With the notation of Theorem 9 we have:

**Lemma 10** Suppose $d_1 < \infty$ and $d_2^T < \infty$. Then $D \leq d_1$. Similarly, if $d_2 < \infty$ and $d_1^T < \infty$. Then $D \leq d_2$.

Proof: Suppose $d_1 < \infty$ and $d_2^T < \infty$. Let $Z_1 \subset E_1$ be a cycle of $\mathcal{H}_1$ of minimum weight $d_1$. Now let $y \in V_2$ be a vertex of $\mathcal{H}_2$. Recall that $y$ is also an edge of the transpose hypergraph $\mathcal{H}_2^T$. Now we claim that the subset $\{y\}$ cannot be a cocycle of $\mathcal{H}_2^T$ for every $y \in V_2$. Otherwise the cocycle code of $\mathcal{H}_2^T$ is the whole space $\{0, 1\}^{V_2}$, but we have supposed $d_2^T < \infty$, meaning that the cycle code of $\mathcal{H}_2^T$ is not $\{0\}$, so the cocycle code of $\mathcal{H}_2^T$ cannot be the whole space.

Let therefore $y \in V_2$ be some vertex of $\mathcal{H}_2$, such that $\{y\}$ is not a cocycle of $\mathcal{H}_2^T$. Now let $Z$ be the set of edges of $\mathcal{H} = \mathcal{H}_1 \cdot \mathcal{H}_2$ consisting of all edges $\alpha y$ for which $\alpha \in Z_1$. We have therefore $|Z| = |Z_1| = d_1$. It is easy to check that $Z$ is a cycle of $\mathcal{H}$. Suppose now that $Z$ belongs to the chamber code, so that we have

$$Z = \sum_{(\alpha, \beta) \in \mathcal{F}} C_{\alpha\beta}$$ \hspace{1cm} (4)

for some set $\mathcal{F} \subset E_1 \times E_2$. Now let $\alpha$ be some fixed edge of $Z_1$. Now (4) implies :

$$\{\alpha y\} = \sum_{\beta, (\alpha, \beta) \in \mathcal{F}} \sum_{y' \in \beta} \alpha y'$$
but this implies in turn
\[ \{ y \} = \sum_{\beta, (\alpha, \beta) \in \mathcal{F}} \beta. \]

But this means that \( \{ y \} \) belongs to cocycle code of \( \mathcal{H}_T \), contrary to our assumption. Therefore \( Z \) is a cycle of \( \mathcal{H} \) of weight \( d_1 \) that does not belong to the chamber code. This proves the first claim of the Lemma. The second is obtained analogously. 

6 The quantum code associated to a classical LDPC code

Let \( C \) be a classical LDPC code of parameters \([n, k, d]\) associated to a \((n-k) \times n\) parity-check matrix \( \mathbf{H} \). Suppose that \( \mathbf{H} \) is full-rank, i.e. \( n-k \).

Let \( \mathcal{H}_1 = (\mathcal{V}_1, \mathcal{E}_1) \) be the hypergraph whose vertex-edge incidence matrix is given by \( \mathbf{H} \). It is tempting to consider the quantum code associated the product hypergraph \( \mathcal{H}_1 \cdot \mathcal{H}_1 \): but a full-rank parity-check matrix \( \mathbf{H} \) means that \( \dim Z(\mathcal{H}_1^T) = 0 \) and unfortunately Theorem 7 will give zero dimension for this quantum code. If we want the associated quantum code to be non-trivial we must start with a parity-check matrix \( \mathbf{H} \) with redundant rows, which is possible but not straightforward. However, we obtain an interesting non-zero quantum code by considering the product hypergraph \( \mathcal{H} = \mathcal{H}_1 \cdot \mathcal{H}_2 \), where \( \mathcal{H}_2 = \mathcal{H}_1^T \) is the transpose hypergraph of \( \mathcal{H}_1 \).

With the notation of Theorems 7 and 9 we have:
\[
\begin{align*}
\dim Z(\mathcal{H}_1) &= k \\
\dim Z(\mathcal{H}_2) &= h = 0 \\
\dim Z(\mathcal{H}_1^T) &= r = 0 \\
\dim Z(\mathcal{H}_2^T) &= s = k \\
d_1 &= d \\
d_2 &= \infty \\
d_1^T &= \infty \\
d_2^T &= d
\end{align*}
\]

and the quantum code \( Q_{2\mathcal{H}} \) is a code of parameters \([N, K, D]\) where
\[
N = |\mathcal{V}_1||\mathcal{E}_2| + |\mathcal{V}_2||\mathcal{E}_1| = (n-k)^2 + n^2
\]
and
\[
K = k^2
\]
by Theorem 7. We have \( D \geq d \) by Theorem 9, and \( D \leq d \) by Lemma 10 so that \( D = d \).

This proves Theorem 11.

Remark. If the original classical code \( C \) is a regular LDPC code, i.e. if its parity-check matrix has constant row weight \( \Delta \) and constant column weight \( t \), then the discussion at the beginning of section 4 shows that \( \mathbf{H}_X \) and \( \mathbf{H}_Z \) both have constant row weight equal to \( t + \Delta \) and both have two column weights equal to \( t \) and to \( \Delta \).
References

[1] S. A. Aly. A class of quantum LDPC codes derived from Latin squares and combinatorial objects. Technical report, Department of Computer Science, Texas A&M University, April 2007.

[2] S. A. Aly. A class of quantum LDPC codes constructed from finite geometries. In Proc. of IEEE GLOBECOM, pages 1–5, December 2008.

[3] H. Bombin and M. A. Martin-Delgado. Homological error correction: classical and quantum codes. J. Math. Phys., 48, 052105 (2007).

[4] A. R. Calderbank and P. W. Shor. Good quantum error-correcting codes exist. Phys. Rev. A, 54:1098–1105, 1996.

[5] T. Camara, H. Ollivier, and J.-P. Tillich. Constructions and performance of classes of quantum LDPC codes, 2005. http://arxiv.org/abs/quant-ph/0502086v2

[6] T. Camara, H. Ollivier, and J.-P. Tillich. A class of quantum LDPC codes: construction and performances under iterative decoding. In Proc. of ISIT, pages 811–815, Nice, June 2007.

[7] C. D. de Albuquerque, R. Palazzo, and E. B. da Silva. Construction of topological quantum codes on compact surfaces. In Proc. of ITW, pages 391–395, Porto, May 2008.

[8] I. B. Djordjevic. Quantum LDPC codes from incomplete block designs. IEEE Communication Letters, 12(5):389–391, May 2008.

[9] D. Poulin and Y. Chung. On the iterative decoding of sparse quantum codes. Quantum Information and Computation, 8:987, 2008.

[10] M. H. Freedman, D. A. Meyer, and F. Luo. Z2-systolic freedom and quantum codes. In Mathematics of quantum computation, Chapman & Hall/CRC, pages 287–320, Boca Raton, FL, 2002.

[11] R. G. Gallager. Low Density Parity Check Codes. M.I.T. Press, Cambridge, Massachusetts, 1963.

[12] J. Garcia-Frias and K. Liu. Design of near-optimum quantum error-correcting codes based on generator and parity-check matrices of LDGM codes. In Proc. of CISS, pages 562–567, Princeton, March 2008.

[13] M. Hagiwara and H. Imai. Quantum quasi-cyclic LDPC codes. In Proc. ISIT’07, pages 806–811, Nice, June 2007.

[14] M-H. Hsieh, T. A. Brun, and I. Devetak. Quantum quasi-cyclic low-density parity check codes, March 2008. http://arxiv.org/abs/0803.0100v1

[15] I. H. Kim. Quantum codes on Hurwitz surfaces, S.B. thesis, MIT, 2007. http://dspace.mit.edu/handle/1721.1/40917
[16] A. Y. Kitaev. Fault-tolerant quantum computation by anyons. *Ann. Phys.*, 303:2, 2003.

[17] H. Lou and J. Garcia-Frias. On the application of error-correcting codes with low-density generator matrix over different quantum channels. In *Proc. of Turbo-coding*, Munich, April 2006.

[18] D. J. C. MacKay, G. Mitchison, and P. L. McFadden. Sparse graph codes for quantum error-correction. *IEEE Trans. Info. Theory*, 50(10):2315–2330, 2004.

[19] M. S. Postol. A proposed quantum low density parity check code, 2001. http://arxiv.org/abs/quant-ph/0108131v1

[20] D. Poulin, J.-P. Tillich, and H. Ollivier. Quantum serial turbo-codes, 2007. http://arxiv.org/abs/0712.2888v1

[21] K. P. Sarvepalli, M. Rötteler, and A. Klappenecker. Asymmetric quantum LDPC codes. In *Proc. of ISIT*, pages 305–309, Toronto, July 2008.

[22] A. M. Steane. Multiple particle interference and quantum error correction. *Proc. R. Soc. Lond. A*, 452:2551–2577, 1996.

[23] P. Tan and J. Li. New classes of LDPC stabilizer codes using ideas from matrix scrambling. In *Proc. of ICC*, pages 1166–1170. May 2008.

[24] G. Zémor. On Cayley graphs, surface codes and the limits of homological coding for quantum error correction, December 2008. preprint. http://www.math.u-bordeaux.fr/~zemor/surface.pdf