Nonlinear Controllability Assessment of Aerial Manipulator Systems using Lagrangian Reduction

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Abstract: This paper analyzes the nonlinear Small-Time Local Controllability (STLC) of a class of underactuated aerial manipulator robots. We apply methods of Lagrangian reduction to obtain their lowest dimensional equations of motion (EOM). The symmetry-breaking potential energy terms are resolved using advected parameters, allowing full SE(3) reduction at the cost of additional advection equations. The reduced EOM highlights the shifting center of gravity due to manipulation and is readily in control-affine form, simplifying the nonlinear controllability analysis. Using Sussmann’s sufficient condition, we conclude that the aerial manipulator robots are STLC near equilibrium condition, requiring Lie bracket motions up to degree three.

Keywords: Controllability; Geometric Mechanics; Lagrangian Reduction; Flying robots; UAVs.

1. INTRODUCTION

Aerial manipulators, which combine a multi-rotor aerial platform with a multi-jointed robot arm, have the mobility of multi-rotors and can interact with the environment utilizing its manipulator. A variety of aerial manipulators have been introduced, ranging from a fully-actuated aerial base with a three-link manipulator (Sanchez-Cuevas et al. (2020)) to a helicopter with a seven degree-of-freedom manipulator (Kondak et al. (2014)). As summarized by Ruggiero et al. (2018), no matter the complexity of the fully-actuated appended arm, the underactuation of the aerial base poses limitations on the overall system ability. Despite many successful designs, there is no formal guideline for designing an aerial manipulator based on task objectives, nor has there been any analytical assessment of the limitation inherent in an underactuated base. To the best of the authors’ knowledge, there has not been any theoretical work studying the underactuated aerial system’s controllability properties. Controllability, which describes a system’s ability to drive its states to arbitrary values by choices of feasible inputs, is widely used in the study of stabilization, feedback controller design, and state-space reduction. Nonlinear controllability analysis is more challenging than the linear case, but it must be considered for aerial manipulators to ensure sufficient manipulability.

To tackle the algebraic complexity of nonlinear controllability analysis, we establish a lower-dimensional EOM by utilizing Lagrangian reduction. This reduction technique yields a set of the first-order EOM on a lower-dimensional phase space that excludes the symmetry group, which is algebraically simpler in the subsequent controllability assessment. Methods of Lagrangian reduction have successfully been applied to evaluating controllability of complex undulatory robots (Ostrowski (1996)) and the modeling and control of spherical robotic vehicles (Burkhardt (2018)). We consult our earlier work (Burkhardt and Burdick (2016)) to tackle the symmetry-breaking potential terms of aerial manipulators via advected parameters. We analyze nonlinear controllability under small-time, local, and near-equilibrium conditions, using the sufficient conditions given by Sussmann (1987), which applies to systems with a control-affine form, a non-zero drift, and bilateral (both positive and negative) control inputs. However, aerial manipulators have unilateral control inputs (non-negative thrust), making them ill-suited for a direct application. Based on the rotor dynamics, we select bilateral rotor RPM rate as the replacement control inputs. Our controllability result can be viewed as a certification of an aerial manipulator’s fundamental capability to control the multirotor platform during aerial manipulation. The set of reduced EOM is mathematically compact and reveals geometric properties that could serve in the motion planning and controller synthesis process.

Paper Outline. Section 2 defines the class of aerial manipulator system studied. Section 3 reviews the basics of reduction and reconstruction under broken symmetries, as well as controllability of nonlinear systems. The EOM for a general class of aerial manipulator system is developed in Section 4. Section 5 presents the main contribution: proofs that our class of aerial manipulators are Small-Time Locally Accessible and Small-Time Locally Controllable. We conclude in Section 6 with future applications.

2. SYSTEMS DESCRIPTION

We consider the following class of Aerial Manipulators:
Definition: A multirotor aerial platform with the following characteristics is an Aerial Manipulator (AM):

- The multi-rotor includes n-pairs of identical rotors where n ≥ 2. Each rotor pair consists of one clockwise (CW) and one counterclockwise (CCW) rotating rotor, distributed in a cross configuration.
- All rotor thrusts point in the positive direction of the z_0 axis, and the distances from their rotating axis to the z_0 axis are equal, see Fig. 1.
- The 2-link manipulator is attached to the multi-rotor’s geometric center and operates in x_0, z_0 plane. Each link is approximated by a uniform cylinder.
- All system components are rigid and complex fluid structure interactions are ignored.

The manipulator is limited to planar operation only for simplicity: it is sufficiently complicated to demonstrate the effectiveness of the reduction process and the dynamic coupling between the multi-rotor and arm dynamics.

Our model is derived using the following reference frames:

- The earth-fixed inertial frame \( E = \{ O^e, x_e, y_e, z_e \} \).
- The aerial-base body frame \( B = \{ O^b, x_b, y_b, z_b \} \).
- Manipulator \( i \)-th link frame \( L_i = \{ O^{L_i}, x_{L_i}, y_{L_i}, z_{L_i} \} \).
- The \( j \)-th rotor frame \( T_j = \{ O^{T_j}, x_{T_j}, y_{T_j}, z_{T_j} \} \).

Hereafter, \( s_{ab} \in \mathbb{R}^3 \) and \( R_{sb} \in SO(3) \) denote the position and orientation of the origin of frame \( B \) relative to the origin of frame \( A \), respectively. \( \eta_1, \eta_2 \in \mathbb{S} \) be the relative joint angles. The aerial-base linear velocity in \( B \) frame is defined to as \( \dot{s}_b = R^T_{eb}\dot{s}_{eb} \). Using standard roll, pitch, and yaw (Euler) angles \( \Theta = [\phi, \theta, \psi]^T \), the angular velocity of the aerial-base in \( B \) frames, \( \omega_b \), is the following:

\[
\omega_b = \begin{bmatrix} 0 & -\sin(\theta) & \cos(\theta) \\
0 & \cos(\phi) \cos(\theta) & -\sin(\phi) \cos(\theta) \\
0 & \sin(\phi) \cos(\theta) & \cos(\phi) \cos(\theta) \end{bmatrix} \begin{bmatrix} \dot{\phi} \\
\dot{\theta} \\
\dot{\psi} \end{bmatrix} = T(\Theta) \dot{\Theta}. \tag{1}
\]

Lastly, \( R_{eb} = R_{eb}S(\omega_b) \) where \( S(\cdot) \) is the \( 3 \times 3 \) skew-symmetric matrix such that \( S(\omega_b)x = \omega_b \times x \).

3. PRELIMINARIES
A mechanical system is characterized by the tuple \( \Sigma = (Q, L, T) \), where \( Q \) is its finite-dimensional configuration space. \( TQ \) denotes the tangent bundle of \( Q \), and \( T_qQ \) is the tangent space to \( Q \) at \( q \). \( L : TQ \to \mathbb{R} \) is the system Lagrangian. \( T(q, \dot{q}) \in T^*Q \) represents the external forces field acting on the system, where \( T^*Q \) is the dual of \( TQ \).

Broadly speaking, a Lagrangian possesses a symmetry if there is an action on its arguments that renders the Lagrangian invariant. This symmetry allows the reduction of the dynamical system to a lower dimensional phase space. This paper is concerned with Lie group symmetries, which naturally arise in rigid body systems. For the rest of the paper, we will assume that \( Q \) is a trivial principle bundle, i.e., \( Q = G \times M \), where the fiber-space \( G \) is a Lie group. Smooth manifold \( M \) denotes the internal shape space. Consequently, coordinates on \( q \in Q \) can be partitioned as \( q = (g, r) \), where \( g \in G \) and \( r \in M \). Associated with a Lie group is its Lie algebra, \( g \), a vector space isomorphic to the tangent space at the group identity, i.e. \( g \cong T_eG \).

In the context of reduction, we are interested in the left action of a Lie group \( X \) on a smooth manifold \( P \), which is defined as the map \( \Phi_g : P \times X \to P : p \to pg \) for any \( p \in P \). The Lie group of an aerial manipulator robot’s configuration space is a semidirect product space, defined as \( G = H \ltimes V \) where \( H \) is a Lie subgroup and \( V \) is a vector space. We denote the Lie algebra of \( H \) as \( h \), and the group \( H \) acts on the space \( V \) from the left, and on \( Q \) via a left-action. The Lie algebra of the semidirect-product group can be written as \( g = h \oplus TV \cong h \oplus V \) with elements \((\xi_h, \xi_v) \in g\). In local/body coordinates, \( \xi_h = h^{-1}\dot{h} \in h \) and \( \xi_v = h^{-1}\dot{v} \in TV \) where \((\dot{h}, \dot{v}) \in TG \) is an arbitrary tangent vector. The lifted action results from the left-action \( \Phi_g(q) \) on tangent vectors is defined as the map \( T\Phi_g : TQ \to TQ : (q, v) \to (\Phi_g(q), T_q\Phi_g(v)) \). For \( g = (h, v), T_q\Phi_g(I, \dot{p}, \dot{r}) = (h\dot{h}, h\dot{v}, \dot{r}) \).

A mechanical system possess a symmetry with respect to Lie group \( G \) if its Lagrangian \( L : TQ \to \mathbb{R} \) and external forces \( F(q, \dot{q}) \) are Lie Group Invariant. See Schwarz (1975). As the appended arm moves during its manipulation tasks, the system’s overall center of mass displaces, breaking symmetry in the Lagrangian’s potential. Thus, we consider the mechanics under a broken symmetry condition, which leads to a modified Euler-Poincaré set of equations as will be discussed next.

3.1 Lagrangian Reduction and reconstruction
The reduction technique based on advected parameter, introduced by Holm et al. (1998), is employed to reformulate the symmetry breaking potential contributions. In the context of mechanical system, an advected parameter, \( \gamma(t) \), is a vector expressed in a body-fixed reference frame satisfying the following differential equation:

\[
\frac{d}{dt} + g^{-1}(t)\dot{g}(t) \gamma(t) = 0. \tag{2}
\]

where \( g \in G \). For AM, we choose an advected parameter, \( \gamma(t) \triangleq R_{eb}^{Tq}c_e \in V \), to address the symmetry breaking gravity. Another symmetry breaking term of the AM is its dependency on the aerial base position. Let \( \zeta \triangleq R_{eb}^{Tq}s_{eb}, \) together with \( \gamma \), the AM potential energy can be expressed as \( V(r, \gamma, \zeta) \), where \( r \) denotes the shape variables.

Define the Augmented Lagrangian \( \mathcal{T} : TG \times V \to \mathbb{R} \) by augmenting the state with advected parameter \( \gamma \) and base position \( \zeta \). If the Augmented Lagrangian is \( \Phi_z \)-invariant, then it can be reduced to \( h \times \nu \times TM \) (Burkhardt (2018)). Applying the Lagrange-d’Alembert principle to the Augmented Lagrangian yields a modified form of the Euler-Poincaré equations. See Schneider (2002) for details.
One important takeaway from Schneider’s result is the following reduced variational principle:

$$\delta \left( \int_{t_0}^{t_f} L(q, \dot{q}, r, \dot{r}) dt - \int_{t_0}^{t_f} V(r, \gamma, \zeta) dt \right) = 0. \quad (3)$$

Merging (3) with the Lagrange-d’Alembert principle yields:

$$\delta \left( \int_{t_0}^{t_f} L(q, \dot{q}, r, \dot{r}) dt - \int_{t_0}^{t_f} \sum_{i=1}^{m} N_i(r, \dot{r}) \cdot \delta q_i dt \right) = \delta \left( \int_{t_0}^{t_f} dV(r, \gamma, \zeta) \cdot \delta q dt \right), \quad (4)$$

where $-dV(r, \gamma, \zeta)$ represents conservative forces and moments arise from the potential energy.

Based on Bullo and Lewis (2004), we can conclude the dynamical equivalence between the mechanical systems $\Sigma = (Q, L, T)$ and the reduced system $\Sigma_r = (Q, K, -dV + T)$. At the expanse of two additional equations for $\dot{\gamma}$ and $\dot{\zeta}$, reduction by $G$ (SE(3) for AMs) is made possible on an augmented configuration space.

**Theorem 1.** (Burkhardt (2018)) Let a mechanical system with a Lagrangian $L$ and generalized momenta $q_i$ arising from the potential energy.

The generalized momenta $q_i$ are defined as $q_i(\xi) = \{b_i, \xi\}$ along the symmetry directions. To recover the spatial motion of the system, we employ a reconstruction equation or connection, using trajectories in the reduced phase space. It defines a horizontal space of $T\Sigma Q$ as $H_\Sigma Q \cong \text{Ker}(A(q))$, where $A$ is a principle connection form and describes motion along the fiber as the flow of a left-invariant vector field. See Bloch et al. (1996) and Ostrowski (1996). The general form of connection is:

$$\zeta_k = h^{-1} h = -A(r, \gamma) \dot{r} + B^{-1}(r, \gamma) \varphi,$$  

where $A(r, \gamma)$ and $B(r, \gamma)$ are mass and inertia tensors.

**3.2 Nonlinear Control Theory Review**

Consider steering problems in control-affine form:

$$\Sigma : \dot{q} = f(q) + \sum_{i=1}^{m} g_i(q) u_i \in U \subset \mathbb{R}^n,$$  

from an initial state $q_0 \in Q$ to a final state $q_f \in Q$ by controls $u : [0, T] \to U$. $f$ is the drift vector field, $g_i, i \in \{1, \ldots, m\}$, are actuation vector fields which assumed to be real-valued smooth functions. Define a distribution $\Delta$ of the system $\Sigma : \Delta \triangleq \text{span}\{g_1, \ldots, g_m\}$. The followings are fundamentals for analyzing $\Sigma$’s controllability:

**Definition:** The reachable set, denoted as $R^V(q_0, T)$, is the set of all points $q_f$ such that there exists an input $u(t) \in U, 0 \leq t \leq T$, that steers the system (10) from $q(0) = q_0$ to $q(T) = q_f$, where the trajectories $q(t), 0 \leq t \leq T$, remain inside a neighborhood $V$ of $q_0 \in Q$.

**Definition:** $\Sigma$ is Small-Time Locally Accessible (STLA) from $q_0$ if the reachable set contains a full n-dimensional subset of $Q \forall$ neighborhoods $V$ and all $T > 0$.

**Definition:** $\Sigma$ is Small-Time Locally Controllable (STLC) from $q_0$ if the reachable set contains a neighborhood of $q_0$ neighborhoods $V$ and all $T > 0$.

The Lie bracket (product) between vector fields $g_1(q)$ and $g_2(q)$ is $[g_1(q), g_2(q)] \triangleq \partial_{q} g_2(q) - \partial_{q} g_1(q)$, which quantifies how the derivative of vector field $g_2(q)$ varies along the flow of $g_1(q)$. The Lie bracket $[g_1(q), g_2(q)]$ (and higher order brackets) may enable infinitesimal movements locally in the system tangent space along directions that are not in $\Delta$. In summary, STLC is to access the Lie brackets’ ability to generate independent actuation vector fields. STLC is to further examined the brackets’ ability to overcome the drift $f$, leading to the following definition.

**Definition:** A Lie product is considered to be a bad bracket if the drift term $f$ appears an odd number of times in the product and each control vector field $g_i, i \in \{1, \ldots, m\}$, appears an even number of times (including zero). If a Lie product is not bad, it is a good bracket.

**Theorem 2.** (Hermann (1963)) The system (10) is STLC from $q$ if it satisfies the Lie algebra rank condition (LARC): $\text{Lie}(f, g_1, \ldots, g_m)(q) = T\Sigma Q$, where $\text{Lie}(\cdot)$ is the closure of Lie algebra over the span of all input vector fields and their iterated Lie brackets, $\{f(q), g_1(q), \ldots, g_m(q)\}$.

For driftless systems, i.e. $f(q) = 0, \forall q \in Q$, STLA and STLC properties are equivalent, and LARC alone
where we can conveniently express using the manipulator rotor w.r.t. frame \( j \). To properly capture the yaw dynamics and control, we separately model the rotors' dynamics. Explicitly, the translational kinetic energy of the \( j \)th rotor \( K_{t,j} = \frac{1}{2} m_{j} \omega_{j}^{2} \), where \( m_{j} \) denotes the mass of an individual rotor, and its rotational kinetic energy 

\[
K_{r,j} = \frac{1}{2} \left( J_{\omega,j}^{2} + \omega_{j}^{T} I_{\omega,wb} + 2 \Omega_{j} \omega_{j}^{T} \right),
\]

where \( I_{*,\omega} = diag(a,a,J) \) is the moment of inertia of the \( j \)th rotor w.r.t. frame \( T_{e} \), and rotors are modeled as cylinders.

Suppose the mass, length, and inertia tuple of the link 1 and 2 w.r.t. frame \( L_{1} \) and \( L_{2} \) are \( \{m_{1},d_{1},I_{1}\} \) and \( \{m_{2},d_{2},I_{2}\} \), respectively. The 2-link manipulator dynamics can be conveniently expressed using the manipulator Jacobian matrix:

\[
\begin{bmatrix}
\dot{\theta}_{b1} \\
\dot{\theta}_{b2} \\
\dot{\omega}_{b} \\
\dot{\Omega}_{b}
\end{bmatrix}
= \begin{bmatrix}
J_{\theta,b1}^{T}(\eta_{1}) & J_{\theta,b2}^{T}(\eta_{1},\eta_{2}) & J_{\omega,b}^{T}(\eta_{1}) & J_{\Omega,b}^{T}(\eta_{1},\eta_{2})
\end{bmatrix}
\begin{bmatrix}
\dot{\theta}_{1} \\
\dot{\theta}_{2} \\
\dot{\omega}_{1} \\
\dot{\omega}_{2}
\end{bmatrix},
\]

where \( J_{\theta,b}^{T}(\eta_{1}) \in \mathbb{R}^{6 \times 1} \) is the manipulator spatial Jacobian matrix defined in Murray et al. (1994). Similar to the multi-rotor case, the kinetic and potential energy of each link can be calculated by summing the translational and rotational contributions. Details will be omitted.

4.1 Overall System Dynamics

Let \( \dot{q} = [\dot{\theta}_{b}^{T}, \omega_{b}^{T}, \Omega_{b}, \dot{\eta}_{1}, \dot{\eta}_{2}]^{T} \in \mathbb{R}^{8+2n} \). The kinetic energy of the AM system can be rewritten as the following:

\[
K_{AM} = \frac{1}{2} \dot{q}^{T} \mathcal{M} \dot{q}
\]

where \( \mathcal{M} \in \mathbb{R}^{8+2n \times 8+2n} \) is the overall system mass, \( \mathcal{P} \in \mathbb{R}^{3 \times 3} \), \( \mathcal{M}_{c} \in \mathbb{R}^{2n \times 2n} \), and \( \mathcal{M}_{t} \in \mathbb{R}^{2 \times 2} \) are symmetric mass and inertia matrices of the multirotor structure, the rotors, and the manipulator with respect to \( B \) frame. Matrices \( \mathcal{M}_{p} \in \mathbb{R}^{3 \times 3} \), \( \mathcal{M}_{d} \in \mathbb{R}^{3 \times 2} \), \( \mathcal{M}_{c} \in \mathbb{R}^{2n \times 2n} \), and \( \mathcal{M}_{t} \in \mathbb{R}^{2 \times 2} \) describe coupling effects. The total potential energy of the AM system with respect to inertial frame is 

\[
V_{AM} = g \mathcal{E}^{T}(\text{ } m_{0} + 2 m_{i} s_{eb} + m_{1} s_{eb} + m_{2} s_{eb} \text{ })
\]

Notice \( V_{AM} \) is a function of the direction of gravity, the position of the vehicle, and manipulator joint angles, thus making the AM systems suitable to apply theorem 1.

4.2 Gravitational Potential Forces and Moments

The previously defined advected parameter \( \gamma \) satisfies the following advection equations:

\[
\begin{bmatrix}
\dot{\gamma} \\
\dot{\zeta}
\end{bmatrix} = -S(\omega_{c}) \gamma + \dot{\zeta} S(\omega_{c}) + s_{k}
\]

The Augmented Lagrangian, parametrized by \( \gamma \) and \( \zeta \), is \( SE(3) \)-invariant by Theorem 1. The Augmented Lagrangian \( \mathcal{L}(q, \gamma, \zeta) \) for the system is:

\[
\mathcal{L} = \frac{1}{2} \dot{q}^{T} \mathcal{M}(q) \dot{q} - g^{T} (m_{1} \gamma - m_{1} s_{eb} + m_{2} s_{eb})
\]

where the total vehicle mass is lumped as \( m_{1} = m_{0} + m_{1} + m_{2} + 2 m_{i} \). The conservative forces and moments due to gravity can be derived from (4):

\[
-dV = -\frac{\partial \mathcal{L}}{\partial \dot{q}} dt
\]

4.3 Non-Conservative Forces and Moments

This section describes the non-conservative forces arising from aerodynamic effects and manipulator actuation, and we only address aerodynamic effects near hovering conditions. The thrust force produced by the \( j \)th rotor is modeled as the product of a lumped thrust coefficient \( c_{T} \) with the rotating velocity squared. Letting \( \Omega = [\Omega_{1}, \cdots, \Omega_{2n}]^{T} \), be the rotor RPMs, and the total thrust is \( \tau_{r} = c_{T} \Omega^{T} \Omega \).

4.4 Reduced Dynamics

In the standard basis for \( se(3) \), the generalized rotation momentum and translational momentum in \( B \) frame are defined as \( I \frac{d\Omega}{dt} \in T_{q} SO(3) \) and \( \mathbf{p} \frac{d\mathbf{p}}{dt} \in T_{q} \mathbb{R}^{3} \).
\[ \begin{bmatrix} p \\ \eta \end{bmatrix} = \begin{bmatrix} M_{p} & M_{\omega p} \\ M_{\omega p}^T & M_{\omega} \end{bmatrix} \begin{bmatrix} \delta \eta \\ \omega \end{bmatrix} + \begin{bmatrix} M_{s \omega} & M_{s \eta} \end{bmatrix} \begin{bmatrix} \text{diag}(\Omega) \Omega \\ \eta \end{bmatrix}. \]

where \( M_{s \omega} \) and \( M_{s \eta} \) are non-zero because of the unbalance in rotor and tilt moments, and the momentum equation including conservative forces \(-dV\) and nonconservative forces and moments \( \tau_c \) becomes:

\[ \dot{\eta} = M^{-1}_{\eta} \left( -f_\eta + g_\eta \Omega + \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix}^T \right), \]

where \( M_\eta = M_0 + M_\omega \delta M_\omega \). Further, \( f_\eta \) and \( g_\eta \) are:

\[ f_\eta = M_\eta \dot{\eta} + \frac{d}{dt} \left( M_\omega (M_\omega^{-1}) \right) \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \\ \dot{\eta} \\ \dot{\theta} \\ \dot{\eta} \end{bmatrix}, \]

\[ g_\eta = -M_\omega \delta M_\omega \Omega. \]

In summary, the reduced EOM for AM system are the momenta equations (23), the shape dynamics (24), and the advection equations (14). The Connection (21) can be incorporated to reconstruct the system’s motion in \( SE(3) \).

### 5. Nonlinear Controllability Assessment

To prepare for nonlinear controllability analysis, the EOM need to be organized into control affine form. We defined the overall system state variable to be:

\[ q = [p^T, \phi, \theta, \psi, \eta_1, \eta_2, \eta_3, \Omega_1, \Omega_2, \Omega_3] \in \mathbb{R}^{2n+13} \]

where we use the Euler angle instead of the advected parameter \( \gamma \) to track the the multiriotor orientation. Further, the control affine form of the full AM system is:

\[ \dot{q} = f(q) + G(q)u, \]

where the control input of the system is

\[ u = [\eta_1, \eta_2, \Omega_1, \Omega_2, \Omega_3, \Omega_2 n] \in \mathbb{R}^{2n+2}. \]

Further, the drift term and actuation vector fields are:

\[ f(q) = \begin{bmatrix} p \times \omega_0 - m_\omega \dot{\gamma} + c_{\omega} \Omega \Omega^T \\ p \times \delta_\eta + \frac{1}{2} \omega_0 + \eta + W_{\text{diag}}(\Omega) \Omega \\ T(\Omega)^{-1} \omega_0 \\ -M_\omega^{-1} f_\eta \\ 0_{(2n) \times (2n+2)} \end{bmatrix}, \]

\[ G(q) = \begin{bmatrix} g_1(q) \\ \ldots \\ g_{2n} + (q) \end{bmatrix}, \]

respectively, where unknown states \( \omega_0 \) and \( \delta_\eta \) can be expressed in terms of \( q \) using the connection (21) and \( \gamma \) is a function of \( \gamma \) and \( \eta \) given in (18) - (19).

#### 5.1 Equilibrium Condition

Nonlinear controllability can only be assessed in the neighborhood of equilibrium conditions. Setting \( q = 0 \), one can verify the following relationship must be true:

\[ \begin{bmatrix} \dot{p} \\ \dot{\eta} \end{bmatrix} = \begin{bmatrix} p^T \tilde{p}^* \\ \eta^T \tilde{\eta}^* \end{bmatrix}, \]

\[ \begin{bmatrix} \tilde{p}^* \\ \tilde{\eta}^* \end{bmatrix} = J(\Omega^*)^T \Omega^*, \]

\[ \text{and } T(\Omega^*)^{T} \Omega^* = m_{\omega} g. \]

#### 5.2 Small-Time Locally Accessible and Controllable

Theorem 4. All AMs are Small-Time Locally Accessible (STLA) evaluated at the equilibrium conditions (27) - (29), except at manipulator singularity \( \eta = [\pi/2, 0]^T \).

Proof. By Theorem 2, we must check that system (25) satisfied the LARC when evaluated at the equilibrium conditions, equivalently \( Lie(f_1, g_1, \ldots, g_m)(q^*) = T_{q^*}Q \).

First note that \( g_1(q^*) \) and \( g_2(q^*) \) are functions of the free state \( \eta_2 \) and \( \forall \eta_2, \text{rank}([g_1(q^*), g_2(q^*)]) = 2 \), spanning the states corresponds to \( \eta_1 \) and \( \eta_2 \). Despite coupling with the \( \eta_1 \) and \( \eta_2 \), the remaining first degree actuation fields have \( \text{rank}(g_1, \ldots, g_{2n+2}) = 2n \), annihilating any nonzero \( \Omega \) with appropriate combinations of \( u \). Since \( g_1 \) and \( g_2 \) correspond to manipulator maneuvers in the \( x_3 \) and \( x_2 \) plane, the Lie brackets between \( g_1, g_2 \) with the drift \( f \) are:

\[ [f, g_i]q^* = [0_{1 \times 3}, 0_{2 \times 1}]^T, \]

which are two independent actuation fields that primarily influences \( \eta_1 \) and \( \eta_2 \) and also perturbs the angular velocities and rotational momentum because of the shifting in center of gravity. The remaining \( 2n \) actuation vector fields together span \( \mathbb{R}^{2n} \) and \( \dot{p} \).

For \( i \in \{3, 2n + 2\} \), we have

\[ [f, g_i]q^* = [0_{1 \times 2}^T, 0_{1 \times 2}, 0_{1 \times 2}^T, 0_{1 \times 2}^T]^T, \]

where \( f_{1 \times 3}^2 \) is non-zero because of the unbalance in rotor thrust leads to a non-zero resistive torque resulting in nonzero body rates. The manipulator joints are also coupled with the rotor thrust rate via \( g_\eta \) in (24). Therefore, any perturbation in rotor thrust will leads to a non-zero rotation. Aided by the asymmetry, all degree two brackets together add six more independent control distributions. Looking at the third degree brackets and for \( i \in \{1, 2\}, \)

\[ [(f, g_i), f]q^* = [0_{1 \times 2}^T, 0_{1 \times 2}, 0_{1 \times 2}, 0_{1 \times 2}, 0_{1 \times 2}^T, 0_{1 \times 2}^T], \]

which provides two more independent bases for \( T_{q^*}Q \). Physically speaking, the third degree brackets with \( g_1 \) and \( g_2 \) are similar to the second degree brackets where shifting the vehicle center of gravity perturbs the hovering motion. It is important to note that the perturbation will come in the form of forces which leads to nonzero translational
accelerations. Further, the remaining third order brackets provided 3 more independent bases. For \( i \in \{3, 2n + 2\}, \)
\[
\left[ [f, g], f \right]_{q^*} = \left[ f^\epsilon(p_1, q_1) f^\epsilon(p_2, q_2) f^\epsilon(p_3, q_3) f^\epsilon(p_4, q_4) 0^{n+2} \right]^T.
\]
Together, there are 13 + 2n good brackets that can be used in spanning the tangent space \( T_{\eta}Q \) except at the manipulator singularity (\( \eta = [\pi/2, 0] \)). Therefore, by theorem 2, we conclude AM are STLA evaluated at the equilibrium condition (27)-(29) except at system singularity. ■

Theorem 5. All AMs are Small-Time Locally Controllable (STLC) evaluated at the equilibrium conditions (27) - (29), except at manipulator singularity \( \eta = [\pi/2, 0] \). ■

Proof. By definition of good and bad brackets, all Lie brackets used to establish STLA are good brackets, and the highest degree bracket used in spanning \( T_{\eta}Q \) is three. To apply theorem 3, we need to show all bad brackets of degree \( j \leq 3 \) can be expressed as a linear combination of good brackets of degree less than \( j \). By its definition, there will be no bad brackets associated with an even degree. The third degree bad brackets can be expressed generally as \( \left[ [f, g], [g, h] \right] \). More specifically, for \( i \in \{1, 2\} \) the bad brackets evaluated at the equilibrium condition takes the following form:
\[
\left[ [f, g], g_i \right]_{q^*} = \left[ 0^{n+1} h_i \right]_{q^*} = \left[ 0^{n+1} h_i \right]_{q^*} = \left[ 0^{n+1} h_i \right]_{q^*} = \left[ 0^{n+1} h_i \right]_{q^*},
\]
which can be annihilated with good, degree one bracket \( g_1 \) and \( g_2 \). The remaining 2n third degree bad brackets take the following form where \( i \in \{3, \ldots, 2n\} \):
\[
\left[ [f, g], g_i \right]_{q^*} = \left[ 0^{n+1} h_i \right]_{q^*} = \left[ 0^{n+1} h_i \right]_{q^*} = \left[ 0^{n+1} h_i \right]_{q^*} = \left[ 0^{n+1} h_i \right]_{q^*} = \left[ 0^{n+1} h_i \right]_{q^*},
\]
which can be annihilated by good degree two brackets \([f, g]_i\) and degree one brackets \( g_1 \) and \( g_2 \). The final bad bracket is the drift term evaluated at equilibrium conditions. Since by the definition of equilibrium condition, we have \( \dot{q}^* = f(q^*) + G(q^*)u^* = 0 \). Thus, all bad brackets of degree \( j \leq 3 \) can be written as combinations of good Lie brackets of degree \( j \leq k \), establishing STLC. ■

It is important to note that when interpreting these results, the net motions generated from higher degree Lie brackets (\( d > 1 \)) are “slower” than motions driven directly by actuation vector fields \( g_1, \ldots, g_m \). In fact, the net motions are \( O(\epsilon^d) \) for time \( O(\epsilon) \), where \( \epsilon \ll 1 \). See Choset et al. (2005). Aided by net external motions resulting from the coupling of internal shape changes, the AM system can achieve local accessibility and controllability. However, we suggest the more valuable takeaway from our SLTA and SLTC analysis should be the compositions of good and bad lie brackets since they contain information about how well the system can maneuver.

In summary, Lagrangian reduction process, directly produces the minimum set of first-order dynamical equations, substantially simplifies AM systems’ controllability analysis. Using the connection, the “reduced” EOM can be reformulated into the control-affine form effortlessly. In comparison, to assess controllability without the reduced dynamics, using the EOM given by Kim et al. (2013) requires a costly symbolic inversion of a \((12 + 2n) \times (12 + 2n)\) sized mass matrix to reformulate. Further, the reduction and reconstruction process can be generalized to analyze other flying vehicles that include conventional quadcopter, tilt-rotors, multi-rotors with asymmetric rotor placements, tethers, and pendulum appendages. Concurrently, we are extending the geometric reduction and controllability analysis to more general flying vehicles.

6. CONCLUSION

This paper analyzed two aspects of a class of aerial-manipulator robots. First, aerial manipulator robots’ EOM was developed using Lagrangian reduction and reconstruction despite having broken symmetry. Second, the AM system properties, STLA and STLC, are formally analyzed. We have concluded that underactuated multi-rotor with planar two-link manipulators are STLA and STLC near equilibrium condition. A physical interpretation of this result is if the AM is hovering, there exists control actions to maintain hover while the manipulator joints track a non-singular and smooth trajectory. However, our SLTC result should be applied with caution since a controllability certificate does not assess the relative control effort needed to realize the trajectory or hovering state.

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