Continuous vs. discrete models for the quantum harmonic oscillator and the hydrogen atom

Miguel Lorente

Departamento de Física, Universidad de Oviedo, 33007 Oviedo, Spain

Abstract

The Kravchuk and Meixner polynomials of discrete variable are introduced for the discrete models of the harmonic oscillator and hydrogen atom. Starting from Rodrigues formula we construct raising and lowering operators, commutation and anticommutation relations. The physical properties of discrete models are figured out through the equivalence with the continuous models obtained by limit process.

PACS: 02.20.+b, 03.65.Bz, 03.65.Fd

Key words: orthogonal polynomials; difference equation; raising and lowering operators; quantum oscillator; hydrogen atom.

1 Introduction

The method of finite difference is becoming more powerful in physics for different reasons: difference equations are more suitable to computational physics and numerical calculations; lattice gauge theories explore the physical properties of the discrete models before the limit is taken; some modern theories have proposed physical models with discrete space and time [1].

In recent papers [2], [3] we have presented the mathematical properties of hypergeometric functions of continuous and discrete variable. We have worked out general formulas for the differential/difference equation, recurrence relations, raising and lowering operators, commutation and anticommutation relations. The starting point is the general properties of classical orthogonal

1 E-mail: mlp@pinon.ccu.uniovi.es
polynomials of continuous and discrete variable [4] of hypergeometric type, in particular the Rodrigues formula from which the raising and lowering operators are derived. Similar results were obtained with more sophisticated method using the factorization of the Hamiltonian [6] [7]. In those papers the term “oscillator” is used for the Hamiltonian that has equally spaced eigenvalues.

In this paper we study the physical properties of two simple examples, the harmonic oscillator and the hydrogen atom on the lattice. We make the Ansatz that the discrete model has the same elements (Hamiltonian, eigenvalues, eigenvectors, expectation values, dispersion relations) as the continuous one, provided the difference equation, the raising and lowering operator the commutation and anticommutation relations become in the limit the equivalent elements in the continuous case. To this scheme we can add the evolution of the fields with discrete time, using some difference equation that replace Heisenberg equation. [8]

Some applications of discrete models have been presented elsewhere. Bijker et al. [9] have apply the SO(3) algebra of the Wigner functions to the one-dimensional anharmonic (Morse) oscillator. Bank and Ismail have apply Laguerre functions to the attractive Coulomb potential [10].

2 The quantum harmonic oscillator of discrete variable

We start from the orthogonal polynomials of a discrete variable, the Kravchuk polynomials \(K_n^{(p)}(x)\) and the corresponding normalized Kravchuk functions [1]

\[
K_n^{(p)}(x) = d_n^{-1} \sqrt{\rho(x)} k_n^{(p)}(x),
\]

where \(d_n^2 = \frac{N!}{n!(N-n)!} (pq)^n\) is a normalization constant, \(\rho(x) = \frac{N! p^x q^{N-x}}{x!(N-x)!} (pq)^n\) is the weight function, with \(p > 0, \quad q > 0, \quad p + q = 1, \quad x = 0, 1, \ldots, N + 1\).

The Kravchuk functions satisfy the orthonormality condition

\[
\sum_{x=0}^{N} K_n^{(p)}(x) K_{n'}^{(p)}(x) = \delta_{nn'},
\]

and the following difference and recurrence equations [2]:

\[
\sqrt{pq(N-x)(x+1)} K_n^{(p)}(x+1) + \sqrt{pq(N-x+1)x} K_n^{(p)}(x-1) + [x(p-q) - Np + n] K_n^{(p)}(x) = 0,
\]

where \(d_n^2 = \frac{N!}{n!(N-n)!} (pq)^n\) is a normalization constant, \(\rho(x) = \frac{N! p^x q^{N-x}}{x!(N-x)!} (pq)^n\) is the weight function, with \(p > 0, \quad q > 0, \quad p + q = 1, \quad x = 0, 1, \ldots, N + 1\).
\[
\sqrt{pq(N-n)(x+1)}K_{n+1}^{(p)}(x) + \sqrt{pq(N-n+1)nK_{n-1}^{(p)}(x) + [n(q-p) + Np - x]}K_{n}^{(p)}(x) = 0, \quad (4)
\]

From the properties of the Kravchuk polynomials we can construct raising and lowering operators for the Kravchuk functions [2]

\[
L^+(x, n)K_{n}^{(p)}(x) = pq(x + n - N)K_{n}^{(p)}(x) + \sqrt{pq(N-x+1)xK_{n}^{(p)}(x-1)} = \sqrt{pq(N-n)(n+1)K_{n+1}^{(p)}(x)}, \quad (5)
\]

\[
L^-(x, n)K_{n}^{(p)}(x) = pq(x + n - N)K_{n}^{(p)}(x) + \sqrt{pq(N-x)(x+1)K_{n}^{(p)}(x+1)} = \sqrt{pq(N-n+1)nK_{n-1}^{(p)}(x)}. \quad (6)
\]

The raising operator satisfies

\[
K_{n}^{(p)}(x) = \sqrt{(pq)^n(N-n)! \over N!n!} \prod_{k=0}^{n-1} L^+(x, n-1-k)K_{0}^{(p)}(x),
\]

where \(K_{0}^{(p)}(x) = \sqrt{Np'q^{N-x} \over x!(N-x)!} \) is the solution of the difference equation

\[
L^-(x, 0)K_{0}^{(p)}(x) = 0.
\]

It can be proved that the raising and lowering operators \(L^+(x, n)\) and \(L^-(x, n)\) are mutually adjoint with respect to the scalar product (2).

If we define the difference equation (3) as the operator equation \(H(x, n)K_{n}^{(p)}(x) = 0\), we can factorize this equation as follows

\[
L^+(x, n-1)L^-(x, n) = pq(N-n+1)n + pq(x + n - 1 - N)H(x, n), \quad (7)
\]
\[
L^-(x, n+1)L^+(x, n) = pq(N-n)(n+1) + pq(x + n + 1 - N)H(x, n). \quad (8)
\]

Now we make connection between the Kravchuk function and the Wigner functions that appear in the generalized spherical functions [4]

\[
(-1)^{m-m'}d_{m'm}^{j} = K_{n}^{(p)}(x), \quad (9)
\]

where \(j = N/2\), \(m = j - n\), \(m' = j - x\), \(p = \sin^2(\beta/2)\), \(q = \cos^2(\beta/2)\).

Then formulas (3) to (8) can be written down in terms of the Wigner functions, namely,
\[
\frac{1}{2} \sin \beta \sqrt{(j + m')(j - m' + 1)} d_{m,m'}^{-1}(\beta) \\
+ \frac{1}{2} \sin \beta \sqrt{(j - m')(j + m' + 1)} d_{m,m'}^{j}(\beta) + (m' - m \cos \beta) d_{m,m'}^{j}(\beta) = 0,
\]

(3a)

\[
\frac{1}{2} \sin \beta \sqrt{(j + m)(j - m + 1)} d_{m',m}^{j}(\beta) \\
+ \frac{1}{2} \sin \beta \sqrt{(j - m)(j + m + 1)} d_{m',m}^{j}(\beta) - (m' - m \cos \beta) d_{m,m'}^{j}(\beta) = 0,
\]

(4a)

\[
L^+(m', m) d_{m,m'}^{j}(\beta) = \sin^2 \frac{\beta}{2} (m + m') d_{m,m'}^{j}(\beta) \\
+ \frac{1}{2} \sin \beta \sqrt{(j - m')(j + m' + 1)} d_{m,m'}^{j}(\beta) \\
= \frac{1}{2} \sin \beta \sqrt{(j + m)(j - m + 1)} d_{m',m}^{j}(\beta),
\]

(5a)

\[
L^-(m', m) d_{m,m'}^{j}(\beta) = \sin^2 \frac{\beta}{2} (m + m') d_{m,m'}^{j}(\beta) \\
+ \frac{1}{2} \sin \beta \sqrt{(j + m')(j - m' + 1)} d_{m,m'}^{j}(\beta) \\
= \frac{1}{2} \sin \beta \sqrt{(j - m)(j + m + 1)} d_{m+1,m'}^{j}(\beta).
\]

(6a)

Notice that (3a) and (4a) are equivalent if we interchange \( m \leftrightarrow m' \) and take into account the general property of Wigner functions

\[
d_{m,m'}^{j}(\beta) = (-1)^{m-m'} d_{m',m}^{j}(\beta).
\]

(10)

The same property is satisfied between (5a) and (6a).

Expressions (5a) and (6a) can be obtained directly from the properties of Wigner functions. In fact, it is known [4, formula 5.1.19] that

\[
\frac{d}{d\beta} d_{m,m'}^{j}(\beta) + \frac{m' - m \cos \beta}{\sin \beta} d_{m',m}^{j}(\beta) = \sqrt{(j - m)(j + m + 1)} d_{m+1,m'}^{j}(\beta),
\]

(11)

\[
-\frac{d}{d\beta} d_{m,m'}^{j}(\beta) + \frac{m' - m \cos \beta}{\sin \beta} d_{m',m}^{j}(\beta) = \sqrt{(j + m)(j - m + 1)} d_{m-1,m'}^{j}(\beta),
\]

(12)
The last equation (11), after interchanging \( m \leftrightarrow m' \) and using (9), can be transformed into
\[
\frac{d}{d\beta} d^{j}_{m,m'}(\beta) - \frac{m - m' \cos \beta}{\sin \beta} d^{j}_{m,m'}(\beta) = \sqrt{(j + m')(j - m' + 1)d^{j}_{m,m'-1}(\beta)}.
\]
(12a)

Combining (11) and (12a) we obtain (6a) and by similar method we obtain (5a) In order to give a physical interpretation of the difference equation and raising and lowering operators for the Kravchuk functions we take the limit when \( N \) goes to infinity and the discrete variable \( x \) becomes continuous \( s \).

First of all, we take the limit of Kravchuk functions. We write
\[
K_{n}^{(p)}(x) = \left\{ \frac{n!(N-n)!}{N!(pq)^{n}} \frac{(Npq)^{n}}{2^{n}(n!)^{2}} \right\}^{1/2} \left\{ \frac{N!p^{x}q^{N-x}}{x!(N-x)!} \right\}^{1/2} \left\{ \frac{2}{Npq} \right\}^{n/2} n!K_{n}^{(p)}(x)
\]
\[
\overset{n \to \infty}{\longrightarrow} \left\{ \frac{1}{2^{n}n!} \right\}^{1/2} \left\{ \frac{1}{\sqrt{2\pi Npq}} e^{-s^{2}} \right\}^{1/2} H_{n}(s) = \psi_{n}(s),
\]
(13)

where the last braket becomes the weight for the Hermite functions and the functions \( \psi_{n}(s) \) are the solution of the continuous harmonic oscillator (up to the constant \( (2Npq)^{-1/4} \)).

In order to get the continuous limit of (4) we multiply it by \( 2/\sqrt{2Npq} \) and substitute \( x = Np + \sqrt{2Npq} s \); after simplification we get
\[
\sqrt{\left(1 - \frac{n}{N}\right)} 2(n+1) K_{n+1}^{(p)}(x) + \sqrt{\left(1 - \frac{n-1}{N}\right)} 2n K_{n-1}^{(p)}(x) - 2 \left( s + \frac{(2p-1)n}{\sqrt{2Npq}} \right) K_{n}^{(p)}(x) = 0.
\]

In the limit \( N \to \infty \) this equation becomes the familiar recurrence relation for the normalized Hermite functions
\[
\sqrt{2(n+1)} \psi_{n+1}(s) + \sqrt{2n} \psi_{n-1}(s) = 2s \psi_{n}(s)
\]
(14)

Before we take the limit of (5) we redefine the raising operator, subtracting from it one half equation (3), namely,
\[ L^+(x, n) K_n^{(p)}(x) = \frac{1}{2} \left\{ [(x - Np) + n(p - q)] K_n^{(p)}(x) \right. \\
- \sqrt{pq(N - x)(x + 1)} K_n^{(p)}(x - 1) + \sqrt{pq(N - x + 1)x} K_n^{(p)}(x - 1) \left\} \right. \\
= \sqrt{pq(N - n)(n + 1) K_{n+1}^{(p)}(x)} \] (15)

We divide this expression by \( h \sqrt{2Npq} \sqrt{Npq} \) where \( h \sqrt{2Npq} = 1 \). After simplification we get

\[ \frac{1}{\sqrt{Npq}} L^+(x, n) K_n^{(p)}(x) = \frac{1}{\sqrt{2}} \left\{ s + \frac{n(p - q)}{\sqrt{2Npq}} \right\} K_n^{(p)}(x) \right. \\
- \frac{1}{2h} \left[ \sqrt{1 - \sqrt{\frac{2p}{Nq}}} \left( 1 + \sqrt{\frac{2q}{Np}} s + \frac{1}{Np} \right) K_n(x + 1) \right. \\
- \left[ 1 - \sqrt{\frac{2p}{Nq}} s + \frac{1}{Nq} \right] \left( 1 + \sqrt{\frac{2q}{Np}} s \right) K_n(x - 1) \left\} \right. \\
= \sqrt{1 - \frac{n}{N}} (n + 1) K_{n+1}^{(p)}(x) \]

\[ \rightarrow_{N \rightarrow \infty} \frac{1}{\sqrt{2}} \left\{ s - \frac{1}{2h} [\psi_n(s + h) - \psi_n(s - h)] \right\} \]

\[ \rightarrow_{h \rightarrow 0} \frac{1}{\sqrt{2}} \left\{ s - \frac{d}{ds} \right\} \psi_n(s) = \sqrt{n + 1} \psi_n(s), \] (16)

Similarly from (6) we get

\[ \frac{1}{\sqrt{Npq}} L^-(x, n) K_n^{(p)}(x) \rightarrow_{N \rightarrow \infty} \frac{1}{\sqrt{2}} \left\{ s + \frac{d}{ds} \right\} \psi_n(s) = \sqrt{n} \psi_n(s). \] (17)

Therefore the raising and lowering operators for the Kravchuk functions become, in the limit, creation and annihilation operators for the normalized Hermite functions.

We still have an other connection between the raising and lowering operators of Wigner functions with the generators of the SO(3) algebra.

From (5a) and (6a) we define

\[ A^+ d^j_{mm'}(\beta) \equiv \frac{1}{\sqrt{Npq}} L^+ d^j_{mm'}(\beta) = \sqrt{(j + m)(j - m + 1)} d^j_{m-1,m'}(\beta), \] (18)
\[ A^- d_{mm'}^j(\beta) \equiv \frac{1}{\sqrt{Npq}} L^- d_{mm'}^j(\beta) = \sqrt{\frac{(j-m)(j+m+1)}{2j}} d_{m+1,m'}^j(\beta). \quad (19) \]

Multiplying both expressions by the spherical harmonics \( Y_{jm'} \), adding for \( m' \) and using the property of Wigner functions
\[ Y_{jm} = \sum_{m'} d_{m,m'}^j Y_{jm'}, \quad (20) \]
we obtain
\[ A^+ Y_{jm} = \sqrt{\frac{(j+m)(j-m+1)}{2j}} Y_{j,m-1} = \frac{1}{\sqrt{2j}} J_- Y_{jm} \quad (21) \]
\[ A^- Y_{jm} = \sqrt{\frac{(j-m)(j+m+1)}{2j}} Y_{j,m+1} = \frac{1}{\sqrt{2j}} J_+ Y_{jm} \quad (22) \]
where \( J_+, J_- \) are the generators of SO(3) algebra.

For the commutation relations of these operators we have
\[ (AA^+ - A^+ A) Y_{jm} = \frac{m}{j} Y_{jm} = \frac{1}{2j} 2J_z Y_{jm} = \left( 1 - \frac{n}{j} \right) Y_{jm}, \quad (23) \]

we substitute (20) in this expression and then take the limit:
\[ [AA^+] d_{mm'}^j(\beta) = \left( 1 - \frac{n}{j} \right) d_{mm'}^j(\beta) \xrightarrow{j \to \infty} [a, a^+] Y_{jm}, \]

For the anticommutation relation we have from (7) and (8):
\[ (AA^+ + A^+ A) Y_{jm} = \frac{1}{j} \left( (j(j+1) - m^2) Y_{jm} = \frac{1}{j} \left( J_z^2 - J_z^2 \right) Y_{jm}. \quad (24) \]

Again substituting (20) and taking the limit
\[ \left( AA^+ + A^+ A \right) d_{mm'}^j(\beta) = \left\{ \left( 2n + 1 \right) - \frac{n^2}{j} \right\} d_{mm'}^j(\beta) \xrightarrow{j \to \infty} (aa^+ + a^+ a) \psi_n(s) = (2n + 1) \psi_n(s). \]

This correspondence suggests that the operator algebra for the quantum harmonic oscillator on the lattice is expanded by the generators of the SO(3)
groups. The commutation relations \([J_+, J_-] = 2J_z\) play the role of the Heisenberg algebra, and the anticommutation relation multiply by \(\hbar \omega / 2\) play the role of the Hamiltonian. In order to complete the picture we define the position and momentum operators on the lattice as follows:

\[
X : i \left( \frac{\hbar}{2M\omega} \right)^{1/2} (A + A^+) d^j_{mm'} (\beta) = i \left( \frac{\hbar}{2M\omega} \right)^{1/2} m' - m \cos \beta \frac{\sqrt{J}}{\sin J} d^j_{mm'} (\beta)
\]

\[
P : \left( \frac{M\hbar\omega}{2} \right)^{1/2} (A - A^+) d^j_{mm'} (\beta) = \left( \frac{M\hbar\omega}{2} \right)^{1/2} \times \sqrt{\frac{(j + m')(j - m' + 1)}{2j}} d^j_{m,m' - 1} (\beta) - \sqrt{\frac{(j - m')(j + m' + 1)}{2j}} d^j_{m,m' + 1} (\beta)
\]

with dispersion with respect to the state \(K_n^{(p)}(x)\)

\[
(\Delta X)^2_n = \langle X^2 \rangle_n = \frac{\hbar}{2M\omega} \langle AA^+ + A^+ A \rangle_n = \frac{\hbar}{2M\omega} \left( 2n + 1 - \frac{n^2}{j^2} \right)
\]

\[
(\Delta P)^2_n = \langle P^2 \rangle_n = \frac{M\hbar\omega}{2} \langle AA^+ + A^+ A \rangle_n = \frac{M\hbar\omega}{2} \left( 2n + 1 - \frac{n^2}{j^2} \right)
\]

from which the uncertainty relation follows:

\[
(\Delta X)_n (\Delta P)_n = \frac{\hbar}{2} \left( 2n + 1 - \frac{n^2}{j^2} \right)
\]

The eigenvalues of the Hamilton operator on the lattice are connected with the index \(m = j - n\) of the eigenvectors \(d^j_{mm'} (\beta)\). These eigenvalues are equally separated by \(\hbar \omega\) but finite \((m = -j, \ldots + j)\). The eigenvalues of the position operator on the lattice are connected with the index \(m' = j - x\) of \(d^j_{mm'} (\beta)\). These eigenvalues are equally separated by \(\sqrt{\frac{\hbar}{M\omega}}\) but finite \((m' = -j, \ldots + j)\). Therefore the Planck constant \(\hbar\) plays a role with respect to the discrete space coordinate similar to the discrete energy eigenvalues.

### 3 Wave equation for the hydrogen atom with discrete variables

Our model is based on the properties of generalized Laguerre polynomials as continuous limit of the Meixner polynomials of discrete variable.

We start from the generalized Laguerre functions

\[
\psi_n^\alpha (s) = d_n^{-1} \sqrt{\rho_n(s)} L_n^\alpha(s)
\]
with \( \alpha > -1 \), \( d_n^2 = \Gamma(n + \alpha + 1)/n! \), \( \rho_1(s) = s^{\alpha+1}e^{-s} \), that satisfy the orthonormality condition
\[
\int_0^\infty \psi_n^\alpha(s) \psi_{n'}^\alpha(s) s^{-1} ds = \rho_{nn'} \tag{26}
\]
from the differential equation and Rodrigues formula for the Laguerre polynomials [4] we deduce the following properties for the Laguerre functions (25)

i) differential equation
\[
\psi_n^\alpha(s) + \left[ \frac{\lambda}{s} - \frac{1}{4} - \frac{\alpha^2 - 1}{s^2} \right] \psi_n^\alpha(s) = 0, \quad \lambda = n + \frac{1}{2}(\alpha + 1). \tag{27}
\]

ii) Recurrence relations
\[
-\sqrt{(n + \alpha + 1)(n + 1)} \psi_{n+1}^\alpha(s) - \sqrt{(n + \alpha)} n \psi_{n-1}^\alpha(s) + (2n + \alpha + 1 - s) \psi_n^\alpha(s) = 0. \tag{28}
\]

iii) Raising operator
\[
L^+(s, n) \psi_n^\alpha(s) = -\frac{1}{2} (2n + \alpha + 1 - s) \psi_n^\alpha(s) - s \frac{d}{ds} \psi_n^\alpha(s) = -\sqrt{(n + 1)(n + \alpha + 1)} \psi_{n+1}^\alpha(s) \tag{29}
\]

iv) Lowering operator
\[
L^-(s, n) \psi_n^\alpha(s) = -\frac{1}{2} (2n + \alpha + 1 - s) \psi_n^\alpha(s) + s \frac{d}{ds} \psi_n^\alpha(s) = -\sqrt{n(n + \alpha)} \psi_{n-1}^\alpha(s) \tag{30}
\]
from (29) and (30) we get the factorization of (27)
\[
L^-(s, n+1)L^+(s, n) = (n+1)(n+\alpha+1) - s^2 \left\{ \frac{d^2}{ds^2} + \frac{\lambda}{s} - \frac{1}{4} - \frac{\alpha^2 - 1}{s^2} \right\} \tag{31}
\]
\[
L^+(s, n-1)L^-(s, n) = n(n+\alpha) - s^2 \left\{ \frac{d^2}{ds^2} + \frac{\lambda}{s} - \frac{1}{4} - \frac{\alpha^2 - 1}{s^2} \right\} \tag{32}
\]
Notice that (27) corresponds to some self adjoint operator of Sturm-Liouville type. Also (29) and (30) are mutually adjoint operators with respect to the scalar product (26).

Similarly, in the discrete case, we defined the normalized Meixner functions
\[
M_n^{(\gamma,\mu)}(x) \equiv d_n^{-1} \sqrt{\rho_1(x)} m_n^{(\gamma,\mu)}(x) \tag{33}
\]
where \( m_n^{(\gamma, \mu)}(x) \) are the Meixner polynomials,

\[
d_n^2 = \frac{n! \Gamma(n+\gamma)}{\mu^n(1-\mu)\gamma \Gamma(\gamma)}, \quad \rho_1(x) = \frac{\mu^x \Gamma(x+\gamma+1)}{\Gamma(x+1) \Gamma(\gamma)},
\]

and \( \gamma, \mu \) are some constants \( 0 < \mu < 1, \quad \gamma > 0 \). The functions (33) satisfy the orthonormality condition

\[
\sum_{x=0}^{\infty} M_n^{(\gamma, \mu)}(x) M_{n'}^{(\gamma, \mu)}(x) \frac{1}{\mu(x+\gamma)} = \rho_{nn'} \tag{34}
\]

and the following properties:

i) Difference equation

\[
\sqrt{\frac{\mu(x+\gamma)(x+1)(x+\gamma)}{x+\gamma+1}} M_n(x+1) + \sqrt{\mu(x+\gamma)x} M_n(x-1) - [\mu(x+\gamma) + x - n(1-\mu)] M_n(x) = 0 \tag{35}
\]

ii) Recurrence relation

\[
- \sqrt{\mu(n+\gamma)(n+1)} M_{n+1}(x) - \sqrt{\mu(n+\gamma-1)n} M_{n-1}(x) + (\mu x + \mu n + \mu \gamma + n - x) M_n(x) = 0 \tag{36}
\]

iii) Raising operator

\[
L^+(x, n) M_n(x) = -\mu(x+\gamma+n) M_n(x) + \sqrt{\mu(x+\gamma)x} M_n(x-1) = \sqrt{\mu(n+\gamma)(n+1)} M_{n+1}(x) \tag{37}
\]

iii) Lowering operator

\[
L^-(x, n) M_n(x) = -\mu(x+\gamma+n) M_n(x) + \sqrt{\frac{\mu(x+\gamma)(x+1)(x+\gamma)}{x+\gamma+1}} M_n(x+1) = -\sqrt{\mu(n+\gamma-1)n} M_{n-1}(x) \tag{38}
\]

A redefinition of (37) and (38) can be obtained subtracting one half the difference equation (35) from both:

\[
10
\]
\[ L^+(x, n)M_n(x) = -\frac{1}{2} (2n\mu + \mu \gamma + n(1 - \mu) + (\mu - 1)x) M_n(x) \]
\[ + \frac{1}{2} \left\{ \sqrt{\mu(x + \gamma)x}M_n(x - 1) - \sqrt{\frac{\mu(x + \gamma)(x + 1)(x + \gamma)}{x + \gamma + 1}} M_n(x + 1) \right\} \] (37a)

\[ L^-(x, n)M_n(x) = -\frac{1}{2} (2n\mu + \mu \gamma + n(1 - \mu) + (\mu - 1)x) M_n(x) \]
\[ + \frac{1}{2} \left\{ \sqrt{\frac{\mu(x + \gamma)(x + 1)(x + \gamma)}{x + \gamma + 1}} M_n(x + 1) - \sqrt{\mu(x + \gamma)x}M_n(x - 1) \right\} \] (38a)

It can be proved that the difference equation (35) corresponds to some self-adjoint operator of Sturm-Liouville type. Also the raising and lowering operators (37) and (38) are mutually adjoint with respect to the scalar product (34).

The anticommutation relations for the raising and lowering operators (37a) and (38a) are

\[ \frac{1}{2} \left\{ L^+(x, n - 1)L^-(x, n) + L^-(x, n + 1) L^+(x, n) \right\} \]
\[ = \frac{1}{4} (\mu \gamma + (\mu + 1)n + (\mu - 1)x)^2 \]
\[ - \frac{\mu}{4} \left\{ \left( \frac{(x + \gamma)^2(x + 1)}{x + \gamma + 2} \right)(E^+)^2 - (x + \gamma - 1)x \right\} \]
\[ - (x + \gamma)(x + 1) + \sqrt{(x + \gamma)x(x + \gamma - 1)(x - 1)} (E^-)^2 \]
\[ + \frac{1}{2} (\mu + 1) \left\{ \sqrt{\frac{\mu(x + \gamma)(x + 1)(x + \gamma)}{x + \gamma + 1}} M_n(x + 1) \right\} \]
\[ - \sqrt{\mu(x + \gamma)x}M_n(x - 1) \] (39)

The commutation relations read

\[ L^+(x, n - 1) L^-(x, n) - L^-(x, n + 1) L^+(x, n) = -\mu(2n + \gamma) \] (40)

In order to make connection between the Meixner functions of discrete variable (33) and Laguerre functions of continuous variable (25) we substitute \( \gamma = \alpha + 1, \mu = 1 - h, x = \frac{s}{h} \) and then take the limit \( h \to 0 \).
The explicit expression of (33) in terms of \( \alpha, h \), and \( s \) reads

\[
M_n^{(\gamma, \mu)}(x) = \sqrt{\frac{\mu^{n+1} n!}{\Gamma(n + \gamma)}} \sqrt{\frac{\mu^x \Gamma(x + \gamma + 1)}{\Gamma(x + 1)}} \frac{m_n^{(\gamma, \mu)}(x)}{n!} \\
= \sqrt{\frac{(1 - h)^{n+1} n!}{\Gamma(n + \alpha + 1)}} \exp\left(\frac{s}{n} \ln(1 - h)\right) h^{\alpha+1} \left(x^{\alpha+1} + O\left(\frac{1}{x^2}\right)\right) M_n^{\alpha+1}\left(\frac{s}{n}\right)
\]

where we have used the asymptotic expansion

\[
\frac{\Gamma(x + a)}{\Gamma(x + b)} = x^{a-b} \left(1 + O\left(\frac{1}{x^2}\right)\right), \quad x \to \infty
\]

In the limit we obtain (25), namely

\[
M_n^{(\gamma, \mu)}(x) \xrightarrow{\substack{h \to 0 \\ x \to \infty}} \sqrt{\frac{n!}{\Gamma(n + \alpha + 1)}} e^{-s/s^{\alpha+1}} L_n^\alpha(s) = \psi_n^\alpha(s)
\]

Also

- (36) \( \xrightarrow{\substack{h \to 0}} \) (28)
- (37a) \( \xrightarrow{\substack{h \to 0}} \) (29)
- (38a) \( \xrightarrow{\substack{h \to 0}} \) (30)
- (39) \( \xrightarrow{\substack{h \to 0}} \frac{1}{2} \left[(30) + (31)\right]

In order to make application to the hydrogen atom we take the reduced radial equation [5]

\[
\frac{d^2 u}{d\rho^2} + \left[\frac{\nu}{\rho} - \frac{1}{4} - \frac{l(l + 1)}{\rho^2}\right] u(\rho) = 0
\]

where

\[
\rho \equiv \sqrt{\frac{8M |E|}{\hbar}} r, \quad \nu = \frac{Ze^2}{\hbar} \sqrt{\frac{\mu}{2 |E|}}
\]

If we compare (41) with (27) we must have \( \rho \equiv s, \ l(l+1) = (\alpha^2 - 1)/4 \), hence, \( \alpha = 2l + 1 \), and \( \nu = n + \frac{(\alpha+1)}{2} = n + l + 1 \); since \( l \) is integer, \( \alpha \) and \( \nu \) should be also integer. The energy eigenvalues are given by

\[
E_{\nu l} = -\frac{1}{2} \frac{M (Ze^2)^2}{\hbar^2} \frac{1}{\nu^2}, \quad \nu = 1, 2, \ldots
\]

For fixed \( \nu \) we still have degeneracy for \( l = 0, 1, \ldots \nu - 1 \)
The corresponding eigenvectors are given by (25), which after the substitution
\( \alpha = 2l + 1 \), and \( n = \nu - l - 1 \), become

\[
\psi_{\nu l}(\rho) = \left\{ \frac{(\nu - l - 1)!}{(\nu + l)!} \right\}^{\frac{1}{2}} \rho^{l+1} e^{\frac{\rho}{2}} L_{\nu-l-1}^{2l+1}(\rho)
\]  

(43)

From the connection between the Meixner and Laguerre functions given above,
we can make the ansatz of a discrete model for the hydrogen atom where
the reduced radial equation is substituted by the difference equation (35) with
\( \gamma = \alpha + 1 = 2l + 2 \), \( \nu = \nu - l - 1 \), \( x = 0, 1, \ldots \)

The raising and lowering operators (29), (30), respect to the radial quantum
number \( n \), are substituted by the raising and lowering operators (37) and (38)
with respect to \( n \) or \( \nu = n + l + 1 \).

The anticommutation relations (32) + (31), which is proportional to hamiltonian
of the hydrogen atom (for the radial part), is substituted by the anticommutation
relations (39). The commutation relations for the raising and lowering operators
defining the Lie algebra of the SU(1,1) group, are substituted by equation (40).

The expectation value of the discrete variable \( x \) with respect to the Meixner
functions \( M_n^{(\gamma,\mu)}(x) \) is

\[
\langle x \rangle_{n\gamma} = \sum M_n^{(\gamma,\mu)}(x) x M_n^{(\gamma,\mu)}(x),
\]

that can be calculated with the help of the recurrence relation (36).

Finally the term \( l(l+1) \) in the Hamiltonian for the continuous and discrete
case can be interpreted as the eigenvalues of the Casimir operator for the
SO(3) group,

\[
\vec{L}^2 = \frac{1}{2} \left( L^+ L^- + L^- L^+ \right) + L_3^2
\]

where the operators \( L^+ \), \( L^- \) y \( L_3 \) are acting on discrete space (see formulas
(5) (6)).

Acknowledgements

This work has been partially supported by D.G.I.C.Y.T. under contract PB-96-0538. The autor wants to express his gratitude to Profs. A. Ronveaux and
Y. Smirnov for valuable conversations.
References

[1] M. Lorente, “Quantum Mechanics on discrete space” en M. Ferrero, A van der Merwe (ed.) New Developments on Fundamental Problems in Quantum Physics. Kluwer Academic, Dordrecht 1997, p. 213-224.

[2] M. Lorente, “Creation and annihilation operators for orthogonal polynomials of continuous and discrete variable”, Electron. Trans. Num. Anal. (E.T.N.A.) 9, 102-111 (1999).

[3] M. Lorente, “Raising and lowering operators, factorization and differential/difference operators of hypergeometric type”. J. Phys. A: Math. Gen. 34 (2001) 569-588.

[4] A.F. Nikiforov, S.K. Suslov, V.B. Uvarov, Classical Orthogonal polynomials of a discrete variable, Springer, Berlin 1991.

[5] A. Galindo, P. Pascual, Mecánica Cuántica I. Eudema, Madrid 1989, p. 300.

[6] N.M. Atakishiyev, E. I. Jafarov, S.M. Nagiyev, K.B. Wolf, “Meixner oscillators” Rev. Mex. Fis. 44, 235-244 (1998).

[7] N. M. Atakishiyev, S. K. Suslov, “Difference Analog of the harmonic oscillator” Theor. Math. Phys. 85, 1055-1062 (1991).

[8] M. Lorente, “On some integrable one-dimensional quantum mechanical systems” Phys. Lett. B. 232, 345-350 (1989).

[9] A. Frank, R. Lemus, R. Bijker, F. Prez Bernal, J.M. Arias, “A general algebraic model for molecular vibrational spectroscopy” Annals of Phys. 252, 211 (1996).

[10] E. Bank, M.E.H. Ismail, “The Attractive Coulomb Potential polynomials”, Constr. Approx. 1, 103-119 (1985).