Longitudinal Fracture Analysis of a Two-Dimensional Functionally Graded Beam

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Abstract. Longitudinal fracture in a two-dimensional functionally graded beam is analyzed. The modulus of elasticity varies continuously in the beam cross-section. The beam is clamped in its right-hand end. The external loading consists of one longitudinal force applied at the free end of the lower crack arm. The longitudinal crack is located in the beam mid-plane. The fracture is studied in terms of the strain energy release rate. The solution derived is used to elucidate the effects of material gradients along the height as well as along the width of the beam cross-section on the fracture behaviour. The results obtained indicate that the fracture in two-dimensional functionally graded beams can be regulated efficiently by employing appropriate material gradients.

1. Introduction
The application of the functionally graded materials has been constantly increasing for the last thirty years due mainly to the fact that by gradually changing the composition of the constituent materials during manufacturing, one can optimize the performance of structural members to the external loading [1-4]. Since the material properties of the functionally graded materials are distributed in one or more spatial directions, the study of fracture behaviour is especially significant [5-11].

The present study is motivated by the fact that the functionally graded materials can be built up layer by layer [12], which is a premise for appearance of longitudinal cracks between layers. The basic purpose of the present paper is to obtain a solution to the strain energy release rate for a longitudinal crack in a two-dimensional functionally graded beam. The solution is applied to elucidate the effects of material gradients on the fracture behaviour.

2. Analysis of the strain energy release rate
The functionally graded beam configuration analyzed in the present paper is shown in figure 1. A longitudinal crack of length, \(a\), is located in the beam mid-plane. The beam length is \(l\). The beam is loaded by a longitudinal force, \(F\), applied at the free end of the lower crack arm. It is obvious that the upper crack arm is free of stresses. The right-hand end of the beam is clamped. The beam cross-section is a rectangle of width, \(b\), and height, \(2h\). The material is two-dimensional functionally graded in the beam cross-section. The modulus of elasticity, \(E\), varies continuously in the beam cross-section according to the following law:

\[
E(y_3, z_3) = E_D + \frac{E_H - E_D}{b} \left( \frac{b}{2} + y_3 \right) + \frac{E_K - E_D}{4h^2} (z_3 + h)^2
\]  

(1)
where $E_H$, $E_I$ and $E_K$ are material properties ($E_H$ and $E_K$ govern the material gradient along the width and the height of the beam cross-section, respectively). The centroidal axes, $y_3$ and $z_3$, are shown in figure 1.

The strain energy release rate, $G$, is obtained by differentiating the beam strain energy, $U$, with respect to the crack area, $A$ [13].

$$G = \frac{dU}{dA}$$

where

$$dA = bda$$

In (3), $da$ is an elementary increase of the crack length.

![Figure 1. A two-dimensional functionally graded beam with a longitudinal crack.](image)

The beam strain energy is derived by integrating the strain energy density in the lower crack arm (the upper crack arm is free of stresses) and in the un-cracked beam portion ($a < x < l$)

$$U = a \left[ \int_{\frac{h}{2}}^{\frac{b}{2}} u_L dy_1 \right] dz_1 + \int_{-h}^{h} \left[ (l-a) \int_{\frac{b}{2}}^{\frac{h}{2}} u_U dy_2 \right] dz_2$$

where $u_L$ and $u_U$ are the strain energy densities in the lower crack arm and the un-cracked beam portion, respectively. The centroidal axes of the lower crack arm cross-section are marked by $y_1$ and $z_1$. In (4), $y_2$ and $z_2$ are the centroidal axes of the cross-section of the un-cracked beam portion.

By substituting of (3) and (4) in (2), one arrives at

$$G = \frac{1}{b} \int_{\frac{h}{2}}^{\frac{b}{2}} u_L dy_1 \right] dz_1 - \frac{1}{h} \int_{-h}^{h} \left[ \int_{\frac{b}{2}}^{\frac{h}{2}} u_U dy_2 \right] dz_2$$

The strain energy density in the lower crack arm is written as
\[ u_L = \frac{1}{2} E \varepsilon^2 \quad (6) \]

where \( \varepsilon \) is the distribution of the longitudinal strains.

The strains are analyzed by applying the Bernoulli’s hypothesis for plane sections, since the span to height ratio of the beam under consideration is large. Besides, since the beam (Fig. 1) is loaded in eccentric tension, the only non-zero strains are the longitudinal strains. Therefore, according to the small strains compatibility equations, \( \varepsilon \) is distributed linearly in the beam cross-section. Thus, \( \varepsilon \) in the lower crack arm is written as

\[ \varepsilon = \varepsilon_{C_1} + \kappa_{y_1} y_1 + \kappa_{z_1} z_1 \quad (7) \]

where \( \varepsilon_{C_1} \) is the longitudinal strain in the lower crack arm cross-section centre, \( \kappa_{y_1} \) and \( \kappa_{z_1} \) are the curvatures of lower crack arm in the \( x_1 y_1 \) and \( x_1 z_1 \) planes, respectively.

By using (1), the distribution of \( E \) in the lower crack arm cross-section is written as

\[ E(y_1, z_1) = E_D + \frac{E_B - E_D}{b} \left( \frac{b}{2} + y_1 \right) + \frac{E_K - E_B}{4h^2} \left( z_1 + r \right)^2 \quad (8) \]

where \( r = 3h/2 \).

By substituting of (7) and (8) in (6), one obtains

\[ u_L = \frac{1}{2} \int \left[ E_D + \frac{E_B - E_D}{b} \left( \frac{b}{2} + y_1 \right) + \frac{E_K - E_B}{4h^2} \left( z_1 + r \right)^2 \right] \left( \varepsilon_{C_1} + \kappa_{y_1} y_1 + \kappa_{z_1} z_1 \right)^2 \quad (9) \]

The following equations for equilibrium of the cross-section of the lower crack arm are used to determine \( \varepsilon_{C_1} \), \( \kappa_{y_1} \) and \( \kappa_{z_1} \):

\[ N_{y_1} = \int_{\frac{h}{2}}^{\frac{h}{2}} \int_{\frac{b}{2}}^{\frac{b}{2}} E \varepsilon_{y_1} dy_1 \] \[ M_{y_1} = \int_{\frac{h}{2}}^{\frac{h}{2}} \int_{\frac{b}{2}}^{\frac{b}{2}} E \varepsilon_{y_1} dy_1 \] \[ M_{z_1} = \int_{\frac{h}{2}}^{\frac{h}{2}} \int_{\frac{b}{2}}^{\frac{b}{2}} E \varepsilon_{z_1} dy_1 \]

where \( N_{y_1} \) is the axial force, \( M_{y_1} \) and \( M_{z_1} \) are the bending moments about \( y_1 \) and \( z_1 \), respectively.

It is obvious that \( N_1 = F \), \( M_{y_1} = 0 \) and \( M_{z_1} = 0 \) (figure 1). After substituting of (7) and (8) in (10), (11) and (12) the equations obtained should be solved with respect to \( \varepsilon_{C_1} \), \( \kappa_{y_1} \) and \( \kappa_{z_1} \) by using the MatLab computer program.

The strain energy density in the un-cracked beam portion, \( u_{y_1} \), can be obtained by formula (9). For this purpose, \( y_1 \), \( z_1 \), \( r \), \( \varepsilon_{C_1} \), \( \kappa_{y_1} \) and \( \kappa_{z_1} \) should be replaced with \( y_2 \), \( z_2 \), \( h \), \( \varepsilon_{C_2} \), \( \kappa_{y_2} \) and \( \kappa_{z_2} \), respectively (\( \varepsilon_{C_2} \) is the longitudinal strain in the centre of the cross-section of the un-cracked beam
portion, $\kappa_y$ and $\kappa_z$, are the curvatures of the un-cracked beam portion in the $x_2y_2$ and $x_2z_2$ planes, respectively). The same replacements should be done in equations (10), (11) and (12) in order to determine $\varepsilon_C$, $\kappa_y$ and $\kappa_z$. Besides, $M_{y_2}$ should be substituted equal to $Fh/2$ in (11).

After substituting of $u_L$ and $u_U$ in (5), the integration should be performed by using the MatLab computer program.

The solution to the strain energy release rate is verified by analyzing the longitudinal crack with the help of the $J$-integral [14]. The solution of the $J$-integral is obtained by using an integration contour, $\Gamma$, which coincides with the beam contour (figure 1). Thus, the $J$-integral has non-zero values only in the segments, $\Gamma_1$ and $\Gamma_2$, of the integration contour ($\Gamma_1$ coincides with the free end of the lower crack arm, $\Gamma_2$ coincides with the clamping). Therefore, the $J$-integral solution is written as $J = J_{\Gamma_1} + J_{\Gamma_2}$, where $J_{\Gamma_1}$ and $J_{\Gamma_2}$ are the $J$-integral values in segments $\Gamma_1$ and $\Gamma_2$, respectively. The integration of the $J$-integral is carried-out by using the MatLab computer program. The result obtained matches exactly the solution to the strain energy release rate. This fact is a verification of the fracture analysis performed in the present paper.

\[ \frac{G_{y_2}}{E_D} \times 10^9 \]

\[ \frac{E_K}{E_D} = 0.5 \]

\[ \frac{E_K}{E_D} = 1 \]

\[ \frac{E_K}{E_D} = 2 \]

\[ \frac{E_H}{E_D} \]

Figure 2. The strain energy release rate in non-dimensional form plotted against $E_H / E_D$ ratio at three $E_K / E_D$ ratios.

The solution to strain energy release rate obtained in the present paper is applied to elucidate the effects of material gradients on the fracture behaviour of the two-dimensional functionally graded beam configuration shown in figure 1. It is assumed that $b = 0.02$ m, $h = 0.006$ m and $F = 40$ N. The material gradients along the width and height of the beam are characterized by $E_H / E_D$ and
$E_K/E_D$ ratios, respectively. It should be noted that $E_D$ is kept constant in the calculations. Thus, $E_H$ and $E_K$ are varied in order to generate various $E_H/E_D$ and $E_K/E_D$ ratios. The strain energy release rate is presented in non-dimensional form by using the formula $G_N = G/(E_D b)$. The strain energy release rate in non-dimensional form is plotted against $E_H/E_D$ ratio in figure 2 at three $E_K/E_D$ ratios. The curves in figure 2 indicate that the strain energy release rate decreases with increasing of $E_H/E_D$ and $E_K/E_D$ ratios. This finding is attributed to the increase of the beam stiffness.

3. Conclusions
The longitudinal fracture behaviour of a two-dimensional functionally graded cantilever beam configuration is studied in terms of the strain energy release rate. The beam is loaded by one longitudinal force applied at the free end of the lower crack arm. The modulus of elasticity varies non-linearly in the beam cross-section. In order to verify the solution obtained, the fracture is analyzed also by the $J$-integral approach. The solution obtained is applied to elucidate the effects of material gradients along the width as well as along the height of the beam on the fracture. The analysis reveals that the strain energy release rate decreases with increasing of $E_H/E_D$ and $E_K/E_D$ ratios. It can be summarized that the fracture behaviour of two-dimensional functionally graded beams can be efficiently regulated by employing appropriate material gradients in the design stage of the beams.

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