Longitudinal waves dispersion control through the affine deformation of the interlaced wire medium

Denis Sakhno,1 Eugene Koreshin,1 and Pavel A. Belov1

1ITMO University, Kronverksky pr. 49, 197101 St. Petersburg, Russia

We have studied dispersion properties of interlaced wire media with geometry deformed by affine transformation. It is revealed that the metamaterials support eigenmodes with longitudinal polarization at low frequency range for all deformations. The isofrequency surfaces of the metamaterial are centered at the points at the edges of the Brillouin zone (not at Γ point) and have shape of ellipsoids. The refraction indexes corresponding to the ellipsoids are analysed both analytically and numerically.

I. INTRODUCTION

Metamaterials are artificially engineered media with electromagnetic properties not observed in natural materials. The metamaterials have been studied for the last decades and so far this topic has not exhausted itself. Metamaterials made of wires also known as wire media1 are the simplest (in terms of geometry) representatives of class while they feature extremely complicated electromagnetic properties due to nonlocal response. One-dimensional array of parallel wires, the so-called simple wire medium,2,3, allows one to reach subwavelength resolution of near-field imaging4, and is also used in magnetic resonance imaging (MRI) in order to improve the quality of the images5. The non-connected double wire medium2,3 exhibits hyperbolic behavior7, while the triple one has the interesting property of having four optical axes8, which once again emphasizes the dominant role of the non-local behavior of the materials under discussion.

An interesting example is a three-dimensional metal mesh6,9 that exhibits plasma-like behavior, i.e. which prohibits the propagation of electromagnetic waves below a certain plasma frequency, and also supports longitudinal modes above it. On the basis of the artificial plasma, a large subclass of complex interlaced wire metamaterials10 has been proposed, which are consisting of several such independent lattices inserted one into the other. Unexpectedly, such materials have $N - 1$ low-frequency modes, where $N$ is the number of independent meshes.

So far, double interlaced wire metamaterial dispersion properties have been studied11–14. Thus, the longitudinal nature of the only low-frequency mode was established12,14, and it was demonstrated that isofrequency contours are born at the edges of the Brillouin zone14 in one of the material configurations (symmetrical), where two identical lattices are used.

The subject of interest of this article will also be the double interlaced wire medium, the deformation will not affect the longitudinal polarization of the low-frequency mode and its splitting, which confirms the scalar nature of the mode predicted in10.

II. OBJECT OF STUDY

The object of research of this paper is the interlaced wire metamaterial1,10–12 which consists of two metal lattices inserted one into the other. In the original configuration of the material, each of the identical lattices is cubic with period $a$, and the metamaterial itself has body-centered cubic symmetry, the primitive unit cell of which is a rhombohedron (this is discussed in more detail in our previous work14).

![Figure 1](https://example.com/figure1.png)

Figure 1. (a) Cube skeleton from a children’s magnetic constructor. (b) Trigonal trapezohedron skeleton resulting from stretching of the cubic one.

Our work is intended to answer the question of what will happen if the original lattices are deformed. For substantiation and more clear description of the deformation used in our work, we will resort to a simple visualization.

Firstly, let’s assemble a cube skeleton from the popular children’s magnetic constructor (Fig. 1(a)), this cube will represent an elementary block of one of the identi-
cal sublattices (in the original configuration of the material). The second step is to try to pull (or squeeze) it by two opposite ball-vertices (along one of the main diagonals), the skeleton will easily change its shape (Fig. 1(b)). What is important about such deformation is that the lengths of the frame ribs remain unchanged, i.e. each face of the frame will be a rhombus. Regardless of the degree of stretching (or compression) the frame itself will save a trigonal trapezohedron form, since the main diagonal (along which stretching (or compression) occurs) will stay the 3-fold rotation axis for the skeleton and all ribs are equal.

Here it is important to note the fact that the lengths of the ribs of the skeleton (Fig. 1) remain unchanged during our deformation. When the transformation will be applied to the metamaterial, length of wires will be unchanged, i.e. the wire length $a$ is a deformation invariant (Fig. 2).

Next, we describe the proposed deformation analytically by introducing the deformation operator. This operator will be applied to the interlaced wire metamaterial, i.e. to its elementary cell (unit cell of the original metamaterial shown in Fig. 2(a) and discussed in more details), which will retain the shape of a trigonal trapezohedron (faces are equal rhombuses) like the skeleton assembled from the constructor in Fig. 1. Thus, any rhombohedron with wires (of length $a$) in the equal faces diagonals will be an elementary cell of the metamaterial belonging to the family. The simplest (in form) unit cell to imagine is the cubic cell shown in Fig. 2(b). This is where we start our description.

As were described above the proposed deformation keeps the length of wires $a$ (i.e. periods of lattices) and preserves the 3-fold rotation axis $A_0B_0$ (see Fig. 3) which coincides with the axis of stretching (or compression).

Any trigonal trapezohedron and, accordingly, the modified metamaterial unit cell can be restored by only two parameters (see Fig. 3(b)): by a face angle $\angle A_3B_0A_1 = \theta$ ($\angle A_3B_0A_1 = \angle A_3B_0A_2 = \angle A_1B_0A_2 = \angle B_1A_0B_2 = \angle B_3A_0B_2 = \angle B_3A_0B_2 = \theta$), which value is limited by the range $\theta \in (0, \frac{2\pi}{3})$ (the upper limit is $2\pi/3$, when the cell becomes flat), and by a length of a rib $b_0$ since all ribs are of the same length. The rib length can also be found via the face angle and a face diagonal size. In the case of the deformation under consideration, it is more convenient to use the length of the diagonal, which is a bisector of the angle $\theta$, where the wire lies. The size of that diagonal (wire) is the deformation invariant and equal to $a$.

For example, the unit cell of the original interlaced wire metamaterial (meshes are cubic) in Fig. 2(a) is describing by the deformation angle equal to $acos(-1/3)$ (further we will use $\theta_\perp$ for that angle value), and the cubic unit cell in Fig. 3(b), obviously – by $\pi/2$.

Thus $\theta$ is the only necessary parameter to restore the structure geometry and will be define the so-called degree of the deformation (further we will use the term deformation angle for $\theta$ and the $\theta$-material (or -cell) for the corresponding metamaterial configuration).

For more comfort of the reader, we will analytically describe the deformation using a cubic cell as an initial (Fig. 3(b)). In some places of the Letter, we will use the description of the deformation, in which the original cell (Fig. 3(c)) is taken as the initial one, but we will indicate it separately.

![Image](https://via.placeholder.com/150.png?text=Figure+2.+Double+interlaced+wire+metamaterial+consisting+of+two+metal+non-connected+networks+(dark-blue+and+pink):+(a)+the+original+configuration+of+the+double+interlaced+wire+metamaterial+with+a+rhombohedral+unit+cell+(composed+of+networks+with+a+cubic+cell,+Fig.+1(a));+(b)+a+stretched+along+x-axis+metamaterial+with+a+cubic+unit+cell+(composed+of+networks+with+a+rhombohedral+cell,+Fig.+1(b)).+Unit+cells+are+shown.)
Figure 3. Trigonal trapezohedron unit cells of metamaterials from the family which geometry is defined by the deformation angle (a) $\theta = 5\pi/12$; (b) $\theta = \pi/2$ (the cubic unit cell); (c) $\theta = \theta_\perp = \arccos(-1/3) > \pi/2$ (the original interlaced wire metamaterial); (d) $\theta = 2\pi/3 - \pi/90 > \theta_\perp$. The length of all wires (blue or red lines) is the same in all cases ($a = A_0A_1 = A_0A_2 = A_0A_3 = B_0B_1 = B_0B_2 = B_0B_3$). Axes: $OX || B_0A_0$ and $OY || B_1B_2 || A_1A_2$.

Compose a matrix $V_\theta$ of coordinates (in axes: $OX || B_0A_0$ and $OY || B_1B_2 || A_1A_2$, see Fig. 3) of $\theta$-cell direct lattice vectors $(B_0A_j(\theta)$, where Fig. 3$)$ for the arbitrary deformation angle $\theta$:

$$V_\theta = \sum_j B_0A_j(\theta) \otimes \hat{n}_j$$

where $\hat{n}_j$ – axes unit vectors $(j = 1, 2, 3)$ It allows us to represent our deformation as:

$$V_\theta = D_\theta \cdot V_{\pi/2},$$

where $D_\theta$ is an operator that transform the matrix

$$V_{\pi/2} = \begin{pmatrix} \sqrt{2} & \sqrt{2} & \sqrt{2} \\ -\sqrt{3} & \sqrt{3} & 0 \\ -1 & -1 & 2 \end{pmatrix} \frac{a}{2\sqrt{3}}$$

for the cubic-cell metamaterial (Fig. 3(b)) to the matrix $V_\theta$ for the $\theta$-material.

It is possible to show that $D_\theta$ is a diagonal matrix (deformation is an affine transformation) in the chosen coordinate system and depends on $\theta$ as:

$$D_\theta = \text{diag} \left( \frac{\sqrt{2}\cos\theta + 1}{\sqrt{2}\cos(\theta/2)}, \tan(\theta/2), \tan(\theta/2) \right).$$

It is clear that it is possible to express $V_\theta$ in Eq. (2) via the another initial matrix $V_\theta$ (in general case specified by an arbitrary angle $\beta$) in the right hand side of the equation. The main purpose here is to choose the material with dispersion properties is known to derive and to apply to generalize properties of the whole family. The convenient candidate is $V_{\theta_\perp} = D_{\theta_\perp} \cdot V_{\pi/2}$:

$$V_\theta = D_\theta V_{\pi/2} = D_\theta \left( D_{\theta_\perp}^{-1} V_{\theta_\perp} \right) = I_\theta V_{\theta_\perp},$$

which implies the unit cell of the original metamaterial (the one with dispersion properties described in Refs. 12,14, Fig. 2(a)) as the initial cell for the deformation. In that case the deformation matrix $I_\theta$ in Eq. (5) is defined as:

$$I_\theta = \text{diag} \left( \frac{\sqrt{2}\cos\theta + 1}{\cos(\theta/2)}, \frac{\tan(\theta/2)}{\sqrt{2}}, \frac{\tan(\theta/2)}{\sqrt{2}} \right)$$

and gives the matrix of lattice vectors $V_\theta$ of the $\theta$-metamaterial after applying it to the matrix $V_{\theta_\perp}$ for the original material.

In addition to the deformation angle the following parameter which determines the degree of the deformation of the original metamaterial can be introduced:

$$p_{\text{orig}}(\theta) = \frac{A_0B_0(\theta)}{A_0B_0(\theta_\perp)} = I_{1,1}^{(1,1)} = \frac{\sqrt{2}\cos\theta + 1}{\cos(\theta/2)},$$

where $A_0B_0(\theta)$ is of the $\theta$-defined unit cell diagonal $A_0B_0$ (Fig. 3). So, for the original geometry (Fig. 3(c), $\theta_\perp$) this parameter is equal to 1, for the metamaterial with the cubic unit cell (Fig. 3(b), $\theta = \pi/2$) to $\sqrt{2}$.

**IV. LOW FREQUENCY POINTS**

Every material from the deformation family consists of two non-connected identical wire meshes, therefore the low-frequency mode which is supported by the original interlaced wire metamaterial should exist for each family member. It can be shown via the quasistatic limit approach as it was discussed in Refs. 14,15.

In order to determine the reciprocal space points coordinates where zero-frequency mode appears, we appeal to the unit cell geometry. In the quasistatic limit each mesh has its own potential $(\varphi_A$ and $\varphi_B$) since there is no any electric contact between them. The Bloch’s theorem can be applied to a potential difference of networks in each cell face, thus linking the differences on the opposite faces via the corresponding exponent (for example $(\varphi_B - \varphi_A)$ in $B_0A_2B_1A_3$ and $(\varphi_A - \varphi_B)$ in $BA_1B_3A_0B_2$ are linked by $\exp[i(k_0 \cdot B_0A_1(\theta))]$, see Fig. 3). Thereby a following system of equations can be composed:

$$\begin{cases}
(\varphi_B - \varphi_A) = (\varphi_A - \varphi_B)e^{i(k_0 \cdot B_0A_1(\theta))} \\
(\varphi_B - \varphi_A) = (\varphi_A - \varphi_B)e^{i(k_0 \cdot B_0A_2(\theta))} \\
(\varphi_B - \varphi_A) = (\varphi_A - \varphi_B)e^{i(k_0 \cdot B_0A_3(\theta))}
\end{cases}$$
Thus, the wave vectors $k_\theta = (k_x, k_y, k_z)^T(\theta)$ – coordinates of low-frequency points for $\theta$-material, – that satisfy Eq. (8) can be defined via following exhaustive condition:

$$V_\theta^T k_\theta = \left( \mathcal{D}_\theta V_{\pi/2} \right)^T k_\theta = \begin{pmatrix} 2n_1 + 1 \\ 2n_2 + 1 \\ 2n_3 + 1 \end{pmatrix} \frac{\pi}{a},$$

(9)

where the superscript $T$ implies the transpose matrix.

Taking into account the diagonal form of the matrix $\mathcal{D}_\theta$ (Eq. (4)) the solution of Eq. (9) can be expressed as:

$$k_\theta = \mathcal{D}_\theta^{-1} k_{\pi/2},$$

(10)

where $k_{\pi/2}$ are coordinates of low-frequency points in $k$-space for the metamaterial with the cubic unit cell, i.e. the solution of Eq. (9) with substitution of the unit matrix $\mathcal{D}_{\pi/2}$ and the matrix $V_{\pi/2}$.

We can also express $k_\theta$ in terms of the $k_{\perp}$ coordinates for the original interlaced wire metamaterial ($\theta_{\perp}$-material, Fig. 3(c)) and rewrite Eq. (10) as:

$$k_\theta = \mathcal{D}_{\theta_{\perp}}^{-1} k_{\perp} = \mathcal{D}_{\theta_{\perp}}^{-1} \left( \mathcal{D}_{\theta_{\perp}} k_{\perp} \right) = \mathcal{K}_\theta k_{\perp}.$$ 

(11)

This representation will allow us to further appeal to the shape of isofrequency surfaces, since for the original metamaterial isofrequency surfaces have spherical shapes at low frequencies\textsuperscript{14}.

V. LOW-FREQUENCY POINTS COORDINATES

Investigation of a relative position of the Brillouin zone of the deformed metamaterial and its low-frequency points becomes more demonstrative after a visualization in $k$-space (see Fig. 4). The absence of the $\Gamma$-born surfaces for the considered metamaterials family confirmed not only by Fig. 4 but also by the computed (using thin PEC wires in COMSOL model) dispersion diagrams presented in Fig. 5.

In the case of cubic unit cell metamaterial ($\theta_2 = \pi/2$, Fig. 3(b)) the Brillouin zone is a cube (Simple Cubic Lattice) and isofrequency surfaces are born in its eight corner points (see Fig. 4(b)): two in Q-points (lie on the $k_x$ axis, symmetrically with respect to the point $\Gamma$) and six at R-points (outside the $k_x$ axis).

The slightest change in the deformation angle $\theta$ leads to a change in the relative position of the $k$-spheroids and the Wigner-Seitz cell in the reciprocal space (blue polyhedrons in Fig. 4).

For $\theta < \pi/2$ (Fig. 3(a)): Q-points lie on the surface of the Brillouin zone of the $\text{RHL1 type}$ ($\text{Rhombohedral Lattice type 1}$, Fig. 4(a)), and R-ellipsoids gradually sitting at its corners when the deformation angle $\theta$ approaches to $\pi/2$.

For angles $\theta > \pi/2$ the opposite is true: R-ellipsoids are located in the corners of the $\text{RHL2 cell}$ (in the case of $\theta = \theta_{\perp}$ the Bravais lattice is body-centered cubic; the Brillouin zone has the $\text{BCC type}$, Fig. 4(c)), and Q-ellipsoids moving away from its vertices with increasing of deformation angle (Fig. 4(c,d)).

In the chosen coordinate system, the motion of the Q-ellipsoids birth points upon deformation of the cubic cell will occur exclusively along the $k_x$ axis and can be described by the first non-zero element of the diagonal
Figure 5. Dispersion QΓR-path (see Fig. 4) diagrams for the metamaterial defined by the deformation angle: (a) \( \theta_1 = 5\pi/12 \); (b) \( \theta_2 = \pi/2 \); (c) \( \theta_3 = \theta_\perp = \arccos(-1/3) \); (d) \( \theta_4 = 2\pi/3 - \pi/90 \).

matrix, inverse to the matrix \( \mathcal{D}_\theta \) according to Eq. (10):

\[
\Gamma Q^\pm (\theta) = \mathcal{D}_\theta^{-1} \Gamma Q^\pm (\pi/2) = \begin{pmatrix}
\pm \sqrt{3} \cos(\theta/2) \\
\sqrt{2} \cos \theta + 1 \\
0 \\
\end{pmatrix} \frac{2\pi}{a} .
\]

The movement description of R-points in accordance with the same Eq. (10) will be a bit more complicated, since the coordinates of these points in the \( k \)-space are not so trivially defined:

\[
\Gamma R_j(\theta) = \mathcal{D}_\theta^{-1} \Gamma R_j(\pi/2) = \mathcal{D}_\theta^{-1} (\Gamma Q^\pm (\pi/2) \mp QR_j(\pi/2)) ,
\]

where \( j = 1, 2, 3 \) and reciprocal vectors \( QR_j(\pi/2) \) coordinates are (see Fig. 4(b))

\[
QR_1(\pi/2) = \frac{1}{\sqrt{3}} \begin{pmatrix}
\sqrt{2} \\
\sqrt{3} \\
-1
\end{pmatrix} \frac{2\pi}{a} ,
\]

\[
QR_2(\pi/2) = \frac{1}{\sqrt{3}} \begin{pmatrix}
\sqrt{2} \\
0 \\
2
\end{pmatrix} \frac{2\pi}{a} ,
\]

\[
QR_3(\pi/2) = \frac{1}{\sqrt{3}} \begin{pmatrix}
\sqrt{2} \\
-\sqrt{3} \\
-1
\end{pmatrix} \frac{2\pi}{a} .
\]

Dependencies of Q and R low frequency points distances from Γ point (\( |\Gamma Q^\pm(\theta)| \) and \( |\Gamma R_j(\theta)| \) from Eqs. (12) and (13)) versus the deformation angle \( \theta \) are plotted in Fig. 6 by solid red and dashed blue lines respectively.

Let’s introduce a parameter in the reciprocal space, which is characterise the ratio of the distance from Γ-point to the nearest low-frequency point located along \( x \)-axis, Q-point, to the distance to the nearest out-of-axis isofrequency surface center – one of \( R_j \) points (where \( j = 1, 2, 3 \)). This parameter will indirectly define the Brillouin zone geometry:

\[
g_{BZ}(\theta) = \frac{|\Gamma Q(\theta)|}{|\Gamma R_j(\theta)|} ,
\]

for example, the cubic Brillouin zone (Fig. 4(b)) corresponds to the metamaterial with the cubic unit cell (Fig. 3(b)) and, obviously, \( g_{BZ}(\pi/2) = 1 \). The parameter becomes less than zero after any stretching of the cubic-cell metamaterial, i.e. for any metamaterial from the family with \( \theta < \pi/2 \). Greater than zero – for materials with \( \theta > \pi/2 \) (see Table 1).

The ability to analytically predict the location of low-frequency points during the deformation of the unit cell

| \( \theta \) | \( p_{orig} \) | \( g_{BZ} \) | Brillouin zone type |
|---|---|---|---|
| \( \theta_1 \) | 1.232 | 0.72 | RHL1 |
| \( \theta_2 = \pi/2 \) | \( \sqrt{2} \) | 1 | CUB |
| \( \theta_3 = \theta_\perp \) | 1 | 1.732 | BCC |
| \( \theta_4 \) | 0.247 | 2.6 | RHL2 |

Table I. Table of \( p_{orig} \) (Eq. 7) and \( g_{BZ} \) (Eq. 15) parameters for different \( \theta \)-materials.
allows us to consider a zone in the $k$-space bounded by a hexahedron whose vertices are two Q-points and six R-points. This polyhedron is highlighted in yellow in Fig. 4. Its shape is easier (by Eqs. (12) and (13)) than the volume of the Wigner-Seitz cell in reciprocal space. For the arbitrary deformed configuration, we obtain a spheroid (oblate or prolate depending on $\theta$) with $k_x$ as the symmetry axis (Fig. 4). Thus, we can artificially determine the longitudinal $n_\parallel$ and transverse $n_\perp$ refractive indices at low frequencies from the shape of the isofrequency surfaces:

$$|k_\parallel| = |k_x| = k_0 n_\parallel, \quad |k_\perp| = \sqrt{k_0^2 + k^2} = k_0 n_\perp,$$

where $|k_\parallel|$ is the distance from centre to pole along the symmetry axis, $|k_\perp|$ is equatorial radius of the spheroid, $k_0$ — vacuum wavenumber.

We performed numerical simulations in COMSOL 5.5 of the structure with thin wires (of length $a = 1$ cm from PEC, in which the deformation operator $D_\theta$ acted exclusively on the geometry of the unit cell. Thus it is possible to determine a dependence of the index $n_\parallel$ and $n_\perp$ values on $\theta$ and plot the graph in Fig. 8 (black and white circle markers). The physical properties of the medium (vacuum) where the metamaterial is located did not change during these calculations and the environment remained isotropic, i.e. the transformation optics theory was not applied here. Hence, there appears a noticeable discrepancy ($15 - 20\%$) between the simulation results and the theoretical dependences of the corresponding matrix $K_\theta$ terms (red dash-dotted and blue solid lines in Fig. 8).

The $K_\theta$ matrix will accurately predict the shape of the isofrequency surfaces upon deformation of the original

VI. ISO FREQUENCY SURFACES AND POLARIZATION

A spherical form of isofrequency surfaces confirmed for the original interlaced wire metamaterial ($\theta = \theta_\perp$) at low frequencies, allows us to predict their shape in $k$-space for the arbitrary deformed configuration.

The deformation matrix $K_\theta$ from Eq. (11) has the form:

$$K_\theta = \text{diag} \left( \frac{\cos (\theta/2)}{\sqrt{2 \cos \theta + 1}}, \sqrt{2} \cot (\theta/2), \sqrt{2} \cot (\theta/2) \right)$$

and is the inverse of the matrix $\mathcal{I}_\theta$ from Eq. (5). Applying it to a sphere in $k$-space, we obtain a spheroid (oblate or prolate depending on $\theta$) with $k_x$ as the symmetry axis (Fig. 4). Thus, we can artificially determine the longitudinal $n_\parallel$ and transverse $n_\perp$ refractive indices at low frequencies from the shape of the isofrequency surfaces:

$$|k_\parallel| = |k_x| = k_0 n_\parallel, \quad |k_\perp| = \sqrt{k_0^2 + k^2} = k_0 n_\perp,$$

where $|k_\parallel|$ is the distance from centre to pole along the symmetry axis, $|k_\perp|$ is equatorial radius of the spheroid, $k_0$ — vacuum wavenumber.

We performed numerical simulations in COMSOL 5.5 of the structure with thin wires (of length $a = 1$ cm from PEC, in which the deformation operator $D_\theta$ acted exclusively on the geometry of the unit cell. Thus it is possible to determine a dependence of the index $n_\parallel$ and $n_\perp$ values on $\theta$ and plot the graph in Fig. 8 (black and white circle markers). The physical properties of the medium (vacuum) where the metamaterial is located did not change during these calculations and the environment remained isotropic, i.e. the transformation optics theory was not applied here. Hence, there appears a noticeable discrepancy ($15 - 20\%$) between the simulation results and the theoretical dependences of the corresponding matrix $K_\theta$ terms (red dash-dotted and blue solid lines in Fig. 8).

The $K_\theta$ matrix will accurately predict the shape of the isofrequency surfaces upon deformation of the original
Figure 8. Refractive indexes versus $\theta$ plot: (1) $n_\parallel$ (red dash-dotted line) and $n_\perp$ (solid blue line) obtained from analytics when $\mathcal{I}_\theta$ from Eq. (6) applied to unit cell geometry and background medium (using transformation optics theory, Eq. (18)); (2) $n_\parallel$ (white circle markers) and $n_\perp$ (black circle markers) obtained from the modeling when deformation applied to unit cell geometry only (vacuum is undeformed).

cell (in this case – the original interlaced wire metamaterial cell) only in the case when the host medium parameters $\hat{\eta}_{host}$ (tensor $\bar{\varepsilon}_{host}$ or $\bar{\mu}_{host}$) will change according to the transformation optics theory\textsuperscript{16,17}:

$$\bar{\varepsilon}_{host}(\theta) = \bar{\mu}_{host}(\theta) = \frac{\mathcal{I}_\theta \mathcal{I}_\theta^T}{\det \mathcal{I}_\theta}, \quad (18)$$

where $\mathbf{I}$ is a unit tensor which is standing for $\bar{\varepsilon}_{vac}$ (or $\bar{\mu}_{vac}$) – permittivity (or permeability) tensors of the vacuum. If such vacuum deformation is taken into account in simulations, similar dependencies will coincide with the theoretical estimate.

The last point of the Letter is a field polarization verification for the low-frequency mode supported by the deformed material. As were mentioned in introduction, the corresponding results for the original interlaced wire metamaterial, which confirms the longitudinal polarization of the mode, were presented in the work\textsuperscript{14} as well as a methodology for such calculation.

Isofrequency contours for different deformed configurations in $\Gamma Q^+ R_2$ plane of the reciprocal space (in the vicinity of $Q^+$ and $R_2$ points, see Fig. 4) are shown in Fig. 7. We can see that the original isofrequency circle (Fig. 7 (c)) doesn’t splits in two curves during the deformation of the metamaterial, because of its longitudinal polarization.

Contours shape (Fig. 7) remain ellipses that shrinks (or stretches) along the $k_x$-axis during the structure deformation and the longitudinal polarisation of the mode preserves (which indirectly confirms its scalar nature\textsuperscript{10}). It is assumed that a similar result can be obtained for an anisotropic host medium, however a detailed study will be elsewhere.

\section*{VII. DISCUSSION}

Spherical isofrequency surfaces are characteristic of isotropic materials. Although in the case of original wire metamaterial, surfaces are not born at $\Gamma$-point, as was shown above (Eq. (10), Fig. 4), we will draw some analogy between the original configuration of the material and the artificial environment of metal spheres located at the nodes of the body-centered cubic lattice.

Since the polarizabilities for a sphere are the same in all directions, the bcc metamaterial from the spheres will be isotropic (the mode with a spherical isofrequency surface is born at $\Gamma$-point), and the mode itself will be degenerate in polarization due to the identity of the magnetic and electric polarizabilities (see ch. 12 in\textsuperscript{18}).

Under the the deformation (Eq. 6) the BCC symmetry of the lattice will be violated and prolate/oblate spheroids will be located at the nodes of the deformed lattice. For the spherical, the statement about the equivalence of directions becomes incorrect, since the polarizability along stretching/compression axis will differ from the polarizabilities in other directions\textsuperscript{18}. Also magnetic and electrical polarizabilities cease to be equal to each other. Thus, the mode ceases to be degenerate and splits: the material acquires a preferred direction (optical axis) along the deformation axis.

\section*{VIII. CONCLUSION}

In this work, we considered the deformations of a metamaterial consisting of two independent metal networks. It was shown that the described type of deformations leads to a theoretically predictable shift of the points where isofrequency surfaces are born. Isofrequency surfaces have an analytically predictable shape of ellipsoids, the dimensions of the semi-axes of which change monotonically with a change in the deformation angle $\theta$ and become equal to each other for the original interlaced wire metamaterial. At the same time the low-frequency mode polarisation remain unchanged during the deformation and stays longitudinal\textsuperscript{12,14}.
C. R. Simovski, P. A. Belov, A. V. Atrashchenko, and Y. S. Kivshar, “Wire metamaterials: physics and applications,” *Advanced Materials*, vol. 24, no. 31, pp. 4229–4248, 2012.

P. A. Belov, R. Marques, S. I. Maslovski, I. S. Nefedov, M. Silveirinha, C. R. Simovski, and S. A. Tretyakov, “Strong spatial dispersion in wire media in the very large wavelength limit,” *Physical Review B*, vol. 67, no. 11, p. 113103, 2003.

C. Simovski and P. Belov, “Low-frequency spatial dispersion in wire media,” *Physical Review E*, vol. 70, no. 4, p. 046616, 2004.

P. A. Belov, Y. Zhao, S. Tse, P. Ikonen, M. G. Silveirinha, C. R. Simovski, S. Tretyakov, Y. Hao, and C. Parini, “Transmission of images with subwavelength resolution to distances of several wavelengths in the microwave range,” *Physical Review B*, vol. 77, no. 19, p. 193108, 2008.

A. P. Slobozhanyuk, A. N. Poddubny, A. J. E. Raaijmakers, C. A. T. van den Berg, A. V. Kozachenko, I. A. Dubrovin, I. V. Melchakova, Y. S. Kivshar, and P. A. Belov, “Enhancement of magnetic resonance imaging with metasurfaces,” *Advanced Materials*, vol. 28, no. 9, pp. 1832–1838, 2016.

M. G. Silveirinha and C. A. Fernandes, “Homogenization of 3-d-connected and nonconnected wire metamaterials,” *IEEE transactions on microwave theory and techniques*, vol. 53, no. 4, pp. 1418–1430, 2005.

T. A. Morgado, J. S. Marcos, J. T. Costa, J. R. Costa, C. A. Fernandes, and M. G. Silveirinha, “Reversed rainbow with a nonlocal metamaterial,” *Applied Physics Letters*, vol. 105, no. 26, p. 264101, 2014.

D. Sakhno, E. Koreshin, and P. Belov, “Quadraxial metamaterial,” *Opt. Lett.*, vol. 47, pp. 4451–4454, Sep 2022.

M. G. Silveirinha, “Artificial plasma formed by connected metallic wires at infrared frequencies,” *Physical Review B*, vol. 79, no. 3, p. 035118, 2009.

J. Shin, J.-T. Shen, and S. Fan, “Three-dimensional electromagnetic metamaterials that homogenize to uniform non-maxwellian media,” *Phys. Rev. B*, vol. 76, p. 113101, Sep 2007.

J. Shin, J.-T. Shen, and S. Fan, “Transmission through a scalar wave three-dimensional electromagnetic metamaterial and the implication for polarization control,” *Journal of nanoscience and nanotechnology*, vol. 10, no. 3, pp. 1737–1740, 2010.

H. Latifi and M. G. Silveirinha, “Light tunneling anomaly in interlaced metallic wire meshes,” *Phys. Rev. B*, vol. 96, p. 195132, Nov 2017.

A. W. Powell, R. C. Mitchell-Thomas, S. Zhang, D. A. Cadman, A. P. Hibbins, and J. R. Sambles, “Dark mode excitation in three-dimensional interlaced metallic meshes,” *ACS photonics*, vol. 8, no. 3, pp. 841–846, 2021.

D. Sakhno, E. Koreshin, and P. A. Belov, “Longitudinal electromagnetic waves with extremely short wavelength,” *Phys. Rev. B*, vol. 104, p. L100304, Sep 2021.

W.-J. Chen, B. Hou, Z.-Q. Zhang, J. B. Pendry, and C.-T. Chan, “Metamaterials with index ellipsoids at arbitrary k-points,” *Nature communications*, vol. 9, no. 1, pp. 1–10, 2018.

J. B. Pendry, D. Schurig, and D. R. Smith, “Controlling electromagnetic fields,” *science*, vol. 312, no. 5781, pp. 1780–1782, 2006.

F. Sun, B. Zheng, H. Chen, W. Jiang, S. Guo, Y. Liu, Y. Ma, and S. He, “Transformation optics: From classic theory and applications to its new branches,” *Laser & Photonics Reviews*, vol. 11, no. 6, p. 1700034, 2017.

R. E. Collin, *Field theory of guided waves*, vol. 5. John Wiley & Sons, 1990.