Symmetric Vertices for Symmetric Modules in Characteristic 2

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In memory of J. A. Green

Abstract

Let $G$ be a finite group and let $k$ be an algebraically closed field of characteristic 2. We develop the theory of $kG$-modules with symmetric $G$-forms using the notion of an involutary $G$-algebra. In particular we investigate orthogonal decompositions, induction from subgroups and relative projectivity by adapting as far as possible the theory of $G$-algebras.

Fong’s Lemma asserts that each nontrivial self-dual irreducible module has a symplectic geometry. We show that in addition to its vertex, such a module has a symplectic vertex.

1. Introduction

1.1. Main Results

Throughout the paper $G$ is a finite group, $k$ is an algebraically closed field of characteristic 2 and $M$ is a finite dimensional left $kG$-module. A symmetric $G$-form on $M$ is a non-degenerate symmetric bilinear form $B : M \times M \to k$ such that $B(gm_1, gm_2) = B(m_1, m_2)$, for all $m_1, m_2 \in M$ and $g \in G$. If $B$ exists we say that $M$ is of symmetric type and call $(M, B)$ a symmetric $kG$-module.

Each finite dimensional $k$-vector space has one isometry class of non-degenerate non-symplectic symmetric bilinear forms. We refer to these as diagonal forms. An even dimensional space has in addition one isometry class of non-degenerate symplectic bilinear forms. As $\text{char}(k) = 2$, each symplectic form is symmetric. We say that $(M, B)$ is of diagonal or symplectic type, depending on the type of $B$.

Recall that $M$ is said to be $H$-projective, for $H \leq G$, if $M$ is a component of an induced module $\text{Ind}_H^G L$, for some $kH$-module $L$. If $M$ is indecomposable, a vertex of $M$ is a 2-subgroup $V$ of $G$ which is minimal subject to $M$ being $V$-projective. By a result of J. A. Green \cite{6}, the vertices of $M$ are determined up to $G$-conjugacy.

There is a standard way of inducing a symmetric $kH$-module $(L, B_1)$ to a symmetric $kG$-module $\text{Ind}_H^G(L, B_1)$. We say that $B$ is $H$-projective if there is a $kG$-isometry $(M, B) \to \text{Ind}_H^G(L, B_1)$, for some symmetric $kH$-module $(L, B_1)$. We say $M$ is symmetrically $H$-projective if it has a symmetric $G$-form which is $H$-projective.

Now suppose that $M$ is indecomposable and of symmetric type. In \cite{2,3} we define a symmetric vertex of $M$ to be a subgroup $T$ of $G$ which is minimal subject to $M$ being symmetrically $T$-projective.

Theorem 1.1. Each symmetric vertex contains a vertex with index at most 2.
For any symmetric $G$-form $B$, we do not know whether the minimal subgroups $T \leq G$ for which $B$ is $T$-projective form a single $G$-orbit. However, $T$ is determined by $B$, if it happens to be a symmetric vertex:

**Theorem 1.2.** If $B$ is $T$-projective, where $T$ is a symmetric vertex of $M$, then $B$ is $H$-projective, for $H \leq G$, if and only if $T \leq_G H$.

The trivial module has a diagonal $G$-form, and P. Fong [2] noted that each non-trivial self-dual irreducible $kG$-module has a symplectic $G$-form, determined up to a non-zero scalar. So Theorem 1.2 implies:

**Theorem 1.3.** The symmetric vertices of a self-dual irreducible $kG$-module are determined up to $G$-conjugacy.

It is clear (from the proof) that this is true for any indecomposable $kG$-module which has only one isometry class of symmetric $G$-forms.

Let $e$ be a primitive idempotent in $Z(kG)$ with $e = e^o$. Then $ekG$ is an indecomposable $kG \times G$-module, and a real 2-block of $G$. In addition to defect groups, $ekG$ has extended defect groups, in the sense of [3].

**Theorem 1.4.** Let $E$ be an extended defect group of the real 2-block $ekG$. Then $\Delta E$ is a symmetric vertex of $ekG$, as $G \times G$-module.

In fact, let $B_1$ be the standard $G \times G$-invariant diagonal form on $kG$. Then its restriction $B_e$ to $ekG$ is non-degenerate and $\Delta E$-projective. We do not know whether $kGe$ can have a symmetric vertex which is not $G \times G$-conjugate to $\Delta E$.

We prove Theorems 1.2, 1.3 and 1.4 in 3.3.

As regards the rest of the paper, we begin in 1.2 by discussing the failure of unique factorization (the Krull-Schmidt Theorem) for symmetric $kG$-modules. So classical arguments about the vertices of indecomposable $kG$-modules cannot be directly applied to symmetric $kG$-modules.

Given a group representation $G \to \text{GL}(M)$, the endomorphism ring $E(M) = \text{End}_k(M)$ of $M$ is a $G$-algebra. We use $E_G(M)$ to denote the algebra of $kG$-endomorphisms in $E(M)$.

The adjoint of a $G$-form $B$ on $M$ is a $k$-involution $\sigma$ of $E(M)$. A key idea in this paper is to study $(M, B)$ via the involutory $G$-algebra $(E(M), \sigma)$. Note that $\sigma$ acts on the points, maximal ideals and multiplicity algebras of $E(M)$. The most general results that we obtain using this approach are Propositions 3.11, 3.13 and 3.14.

Lemma 2.1 is an idempotent lifting result for involutory $k$-algebras. This is a trivial, yet vital, generalization of [10], 1.4. In the rest of 2.1 we clarify the relationship between bilinear forms and their adjoints. Lemma 2.4 shows that each projective representation of $G$ lifts to a representation of $G$, in the presence of a $G$-equivariant involution.

We consider the action of $\sigma$ on $E_G(M)$ in 2.2 Lemma 2.6 gives a bijection between $E_G(M)$ and perfect $G$-pairings between pairs of submodules and Proposition 2.9 gives a bijection between non-degenerate submodules and self adjoint idempotents. We use these ideas to prove the main result of [5], in Lemma 2.12.

In 2.3 we explore the concept of $H$-projectivity for forms. The notion of ‘form induction’ appears in many places, for example [5], [13] and [14]. In particular [13] was a key inspiration for this paper. We prove a symmetric version of Higman’s Criterion in Lemma 2.15.
Many of the results in 3.1 on principal indecomposable modules appear in a weaker or disguised form in [4] or [13]. Corollary 3.7 gives a clean proof of a result of the author which is related to [15].

All our vector spaces and modules are finite dimensional and algebras and groups act on the left on their modules, unless stated otherwise.

1.2. Krull-Schmidt

Temporarily, let $k$ be a field of arbitrary characteristic. Each $kG$-module $M$ has a decomposition $M = M_1 + \ldots + M_k$ where the $M_i$ are indecomposable $kG$-modules. The Krull-Schmidt theorem is the assertion that the summands are uniquely determined up to isomorphism and ordering.

Now suppose that $B$ is a symmetric (alternating) $G$-form on $M$. Then $(M, B) = (M_1, B_1) \perp \ldots \perp (M_k, B_k)$ where each $(M_i, B_i)$ is orthogonally indecomposable. W. Willems proved in his Ph.D. Thesis [17] 3.11) that the analogue of Krull-Schmidt holds, if $\text{char}(k) \neq 2$; the $(M_i, B_i)$ are uniquely determined up to isometry and ordering. He also showed that this is false if $\text{char}(k) = 2$.

From now on $\text{char}(k) = 2$. The following examples show that in this case the obvious analogue of the Krull-Schmidt theorem does not hold for symmetric $kG$-modules.

**Example 1.** Let $M$ be a $kG$-module. There is a perfect $G$-pairing $P : M^* \times M \to k$ given by evaluation $P(f, m) := f(m)$, for all $f \in M^*$ and $m \in M$. This extends to a symmetric $G$-form, also denoted $P$, on $M^* \oplus M$, which is zero when restricted to $M^*$ or to $M$.

Now suppose that $M$ is indecomposable. Then $(M^* \oplus M, P)$ is orthogonally indecomposable. Suppose that $M$ has a symmetric $G$-form $B$. In particular $M \cong M^*$ as $kG$-modules. The ‘diagonal’ submodule of $(M, B) \perp (M, B) \perp (M, B)$ is isomorphic to $(M, B)$, and its orthogonal complement is isomorphic to $(M^* \oplus M, P)$. So we have incomparable orthogonal decompositions into indecomposables

$$(M, B) \perp (M, B) \perp (M, B) \cong (M, B) \perp (M^* \oplus M, P)$$

where even the dimensions of indecomposables do not match up.

In view of Lemma 2.1 and the proof of Lemma 2.12 this is a generic phenomenon originating in decompositions of symmetric $k$-spaces.

**Example 2.** [17] 3.13] Let $V_4 = \{1, r, s, t\}$ be the Klein 4-group, and let $M$ be the regular $kV_4$-module. The $V_4$-invariant symplectic bilinear forms on $M$ are $\{B_x \ | \ x \in kV_4, B_1(x, 1) = 0\}$, in the notation of 2.1 below. Moreover $B_x$ is non-degenerate if and only if $B_1(x, x) \neq 0$, and $(M, B_x) \cong (M, B_y)$ if and only if $y = \lambda x$, for some $\lambda \in k^\times$. Consider the symplectic $kV_4$-module $(M, B_r) \perp (M, B_s)$. Its non-degenerate submodules are parametrized by the nondiagonal 1-dimensional subspaces of $k^2$. So any two distinct orthogonal decompositions of $(M, B_r) \perp (M, B_s)$ give non-isomorphic indecomposable components, and the module has infinitely many such decompositions

$$(M, B_r) \perp (M, B_s) = (M, B_{ar + \beta s}) \perp (M, B_{\beta r + \alpha s}), \text{ if } \alpha \neq \beta.$$
Each \( \gamma \in \mathcal{M} \) of \( A \) is an idempotent \( \in A \). \( \hat{M} \), \( \hat{L} \) and \( (\gamma \eta) = \hat{M} \cdot \hat{L} \). \( M \rightarrow \hat{M} \) is a \( \mathcal{M} \)-isometry. If \( a \in \mathcal{M} \), \( \hat{M} \) is a \( \mathcal{M} \)-invariant symmetric bilinear form on \( M \) and \( \alpha_j : (M,\hat{M}) \rightarrow (L_j,\hat{B}_j) \) is an isometry.

### 2. General results on Forms and Adjoints

#### 2.1. Involutions, Forms and Adjoints

Let \( A \) be a \( k \)-algebra, with units group \( A^\times \). A point of \( A \) is a \( A^\times \)-conjugacy class \( \epsilon \) of primitive idempotents of \( A \). There is a unique maximal 2-sided ideal \( \mathcal{M}_\epsilon \) of \( A \) which does not contain \( \epsilon \). The multiplicity module of \( \epsilon \) is the irreducible \( A \)-module \( P_\epsilon \) whose annihilator is \( \mathcal{M}_\epsilon \), and the multiplicity algebra of \( \epsilon \) is \( E(P_\epsilon) = A/\mathcal{M}_\epsilon \). We use \( \pi_\epsilon \) to denote the projection \( A \rightarrow E(P_\epsilon) \).

An involution of \( A \) is a \( k \)-algebra anti-automorphism \( \tau \) on \( A \) whose square is the identity. We write \( a^\tau \) for the image of \( a \in A \) under \( \tau \). So \( \tau \) is a bijective \( k \)-linear map on \( A \) such that \( (a^\tau)^\tau = a \) and \( (ab)^\tau = b^\tau a^\tau \), for all \( a, b \in A \). We call \( (A,\tau) \) an involutary \( k \)-algebra.

Now \( \epsilon^\tau \) is a point of \( A \) and \( \mathcal{M}_\epsilon \cap \mathcal{M}_{\epsilon^\tau} \) is a \( \tau \)-invariant ideal of \( A \). So \( \tau \) defines an involution, also denoted by \( \tau \), on the \( k \)-algebra \( A/\mathcal{M}_\epsilon \cap \mathcal{M}_{\epsilon^\tau} \). We use \( \pi_{\epsilon,\epsilon^\tau} = \pi_\epsilon \times \pi_{\epsilon^\tau} \) to denote the projection \( A \rightarrow E(P_\epsilon) \times E(P_{\epsilon^\tau}) \).

If \( I \) is a 2-sided ideal of \( A \), set \( \overline{A} = A/I \) and \( \overline{\tau} = a + I \in \overline{A} \). This notation should be clear from context. We will make extensive use of an idempotent lifting result which is given in weaker form in [10] 1.4:

**Lemma 2.1.** Let \( (A,\tau) \) be an involutary \( k \)-algebra and let \( I \) be a \( \tau \)-invariant 2-sided ideal of \( A \). Suppose that \( \overline{\tau} \) is a \( \tau \)-invariant idempotent in \( \overline{A} \). Then there is a \( \tau \)-invariant idempotent \( e \in A \) such that \( \overline{\tau} = \overline{\pi}_\tau \) and \( e = f(aa^\tau) \) for some \( f \in \text{rk}[x] \). In particular \( e \in aAa^\tau \).

If \( \overline{\tau} \) is primitive in \( \overline{A} \) then \( e \) can be chosen to be primitive in \( A \).

**Proof.** Note that \( (\overline{A},\overline{\tau}) \) is an involutary \( k \)-algebra, via \( \overline{a^\tau} := \overline{a}^\tau \), for all \( a \in A \). We may assume that \( \overline{T} \) and \( \overline{\pi} \) are linearly independent in \( \overline{A} \). Set \( b = aa^\tau \). Then \( b^\tau = b \) and \( \overline{b} = \overline{\pi}_\tau \overline{\pi} = \overline{\pi} \). We apply idempotent lifting [8, (3.2)] to the \( k \)-algebra \( k[b] \) modulo its ideal \( k[b] \cap I \). So there is an idempotent \( e \in k[b] \) such that \( \overline{\pi} = \overline{b} \). Then \( e \) is \( \tau \)-invariant as \( b \) is \( \tau \)-invariant and \( k[b] \) is commutative. Write \( e = f(b) \), where \( f \in \text{rk}[x] \). Then \( \overline{\pi} = f(0)\overline{T} + (f(1) - f(0))\overline{\pi} \). So \( f(0) = 0 \).
Now suppose that $\pi$ is primitive in $\mathfrak{A}$. The proof of [8 (3.10)] shows that we may choose $e \in k[b]$ to be a primitive idempotent in $A$. \hfill \Box

Let $M$ be a $k$-vector space. The endomorphism ring $E(M)$ is isomorphic to a full matrix algebra over $k$. By a form on $M$ we mean a non-degenerate $k$-valued bilinear form. Let $B$ be a symmetric form on $M$. We call $(M, B)$ a symmetric $k$-space. The adjoint of $B$ is a $k$-algebra anti-automorphism $\sigma$ of $E(M)$: the adjoint $f^\sigma$ of $f \in E(M)$ is defined by $B(m_1, fm_2) = B(f^\sigma m_1, m_2)$, for all $m_1, m_2 \in M$. We note that $(E(M), \sigma)$ is an involutary $k$-algebra.

Let $M^* = \text{Hom}_k(M, k)$ be the dual space of $M$. So $M^* \otimes M^*$ is the space of bilinear forms on $M$. For $f \in E(M)$, define the bilinear form

$$B_f(m_1, m_2) := B(fm_1, m_2), \quad \text{for all } m_1, m_2 \in M. \quad (2.1)$$

So $f \rightarrow B_f$ is a $k$-isomorphism $E(M) \rightarrow M^* \otimes_k M^*$, $B_f$ is symmetric if and only if $f = f^\sigma$, and $B_f$ is non-degenerate if and only if $f \in E(M)$.

**Lemma 2.2.** (i) If $\sigma$ is an involution of $E(M)$, then up to scalars $M$ has a unique symmetric form with adjoint $\sigma$.

(ii) Let $M = M_1 + M_2$ and let $\sigma$ be an involution of $E(M_1) \times E(M_2)$ such that $1_{M_1} = 1_{M_2}$. Then $\sigma$ has a unique extension to an involution of $E(M)$. The associated form on $M$ is symplectic with $M_1^\perp = M_1$.

**Proof.** Let $B$ be a symmetric form on $M$ with adjoint $\tau$ on $E(M)$. In case (ii) we require in addition that $B$ is symplectic and $M_1, M_2$ are totally isotropic for $B$. In particular $1_{M_1} = 1_{M_2}$.

Assume (i). Then $\sigma \tau$ is an automorphism of $E(M)$. So by the Skolem-Noether theorem there is $g \in \text{GL}(M)$ such that $f^{\sigma \tau} = gf g^{-1}$, for all $f \in E(M)$. So $f^\sigma = g^{-\tau} f^\tau g^\tau$. Set $B_g(m_1, m_2) := B(gm_1, m_2)$, for all $m_1, m_2 \in M$. Then $B_g$ is a symmetric form whose adjoint is $\sigma$. Moreover $g$, and thus $B_g$, is determined up to a non-zero scalar.

Assume (ii). Then $1_{M_1}$ and $1_{M_2}$ are the projections onto $M_1$ and $M_2$ with kernels $M_2$ and $M_1$, respectively. We identify $1_{M_1} E(M_1) 1_{M_1} + 1_{M_2} E(M_1) 1_{M_2}$ with $E(M_1) \times E(M_2)$. Now $\sigma \tau$ maps each $E(M_i)$ onto itself and hence restricts to an automorphism on the semi-simple $k$-algebra $E(M_1) \times E(M_2)$. By the Skolem-Noether theorem there exists $g_i \in \text{GL}(M_1)$, and $g_2 \in \text{GL}(M_2)$, each determined up to a nonzero scalar, such that $(f_1 + f_2)^{\sigma \tau} = (g_1 f_1 g_1^{-1} + g_2 f_2 g_2^{-1})$, for all $f_1 \in E(M_1)$ and $f_2 \in E(M_2)$. Applying $\tau$ to both sides, we get $(f_1 + f_2)^\tau = g_2^{-\tau} f_2 g_2 + g_1^{-\tau} f_1 g_1$. Then

$$f_1 + f_2 = ((f_1 + f_2)^\sigma)^\tau = g_2^{-\tau} g_1 f_1 g_1^{-1} g_2 + g_1^{-\tau} g_2 f_2 g_2^{-1} g_1.$$

This holds for all $f_1 \in E(M_1)$. So there is $\lambda \in k^\times$ such that $g_2 = \lambda g_1$. Thus $g_1 = \lambda^{-1} g_2$. Now replace $g_2$ by $\lambda^{-1} g_2$. Then $g_2 = g_1$ and

$$(f_1 + f_2)^\sigma = g_1^{-1} f_1 g_1 + g_2^{-1} f_1 g_2, \quad \text{for all } f_1 \in E(M_1), f_2 \in E(M_2).$$

Note that $g_1 + g_2 \in \text{GL}(M)$. Then $B_{g_1 + g_2}$ is a symplectic form on $M$ whose adjoint is an extension of $\sigma$ to $E(M)$. Moreover this is the only involution on $E(M)$ which extends $\sigma$. \hfill \Box

An isometry is a $k$-linear map between symmetric spaces which preserves the forms. Note that an isometry is injective, but not necessarily surjective. Two symmetric spaces are isomorphic if there is a surjective isometry between them. Two symmetric forms are isometric if the corresponding symmetric spaces are isomorphic.

Set $n = \dim(M)$. As mentioned in the introduction, there are at most two isometry classes of symmetric forms on $M$:
- $B$ is a symplectic form if $B(m, m) = 0$, for all $m \in M$. Then $n$ is even and $M$ has a symplectic basis $\{m_i\}$ i.e. $B(m_i, m_j) = 1$ or 0, as $i \equiv j + n/2 \pmod{n}$ or not.
- $B$ is a diagonal form if it is symmetric but not symplectic. Then $M$ has an orthonormal basis with respect to $B$.

Now $\text{GL}(M, B) = \{g \in \text{GL}(M) \mid g^\sigma = g^{-1}\}$ is the group of isometries of $(M, B)$. It is not too hard to show that

$$\text{GL}(M, B) \cong \begin{cases} 
\text{Sp}(n, k), & \text{if } B \text{ is symplectic, } n \text{ even.} \\
\text{Sp}(n - 1, k), & \text{if } B \text{ is diagonal, } n \text{ odd.} \\
k^{n-1} : \text{Sp}(n-2, k), & \text{if } B \text{ is diagonal, } n \text{ even.}
\end{cases} \quad (2.2)$$

The following is well-known:

**Lemma 2.3.** Let $M$ be a $kG$-module which affords a diagonal $G$-form $B$. Then $M$ has a trivial submodule and a trivial quotient module.

**Proof.** Define $q(m) = \sqrt{B(m, m)}$ for all $m \in M$. Then $q : M \to kG$ is a non-zero $kG$-homomorphism. Let $\eta$ be the sum of the vectors in any orthonormal basis for $M$. Then $q(m) = B(\eta, m)$, for all $m \in M$. As a consequence $\eta$ spans a trivial submodule of $M$. \hfill $\Box$

The action of $\sigma$ on $E(M)$ maps $\text{GL}(M)$ onto itself and fixes the scalar matrices. So $\sigma$ acts as an isomorphism on $\text{PGL}(M) = \text{GL}(M)/k^\times 1_M$. We set $\text{PGL}(M, \sigma)$ as the centralizer of $\sigma$ in $\text{PGL}(M)$. The following result, which should be well-known, is true only in characteristic 2.

**Lemma 2.4.** Each projective representation $\theta : G \to \text{PGL}(M, \sigma)$ lifts to a group representation $G \to \text{GL}(M, B)$.

**Proof.** Let $\rho$ be the projection of $\text{GL}(M)$ onto $\text{PGL}(M)$ with kernel $k^\times 1_M$. So $\rho(a)f := gfg^{-1}$, for all $g \in \text{GL}(M)$ and $f \in E(M)$. We claim that $\rho$ restricts to an isomorphism $\text{GL}(M, B) \cong \text{PGL}(M, \sigma)$. As $\sigma$ inverts scalars, the identity is the only scalar in $\text{GL}(M, B)$. So $\rho$ is injective on $\text{GL}(M, B)$. Moreover, it is clear that $\rho$ maps $\text{GL}(M, B)$ into $\text{PGL}(M, \sigma)$.

Let $a \in \text{PGL}(M, \sigma)$. Choose $g \in \text{GL}(M)$ such that $a = \rho(g)$. Then

$$g^{-\sigma}fg^\sigma = (af\sigma)^\sigma = af = gfg^{-1}, \quad \text{for } f \in E(M).$$

So $g^\sigma = \lambda g^{-1}$ for some $\lambda \in k^\times$. Then $\sqrt{\lambda^{-1}}g \in \text{GL}(M, B)$ and $\rho(\sqrt{\lambda^{-1}}g) = a$. Our claim follows from this. \hfill $\Box$

We recall some results from [11] Appendix A. Assume the hypothesis and notation of Lemma 2.2 (ii) and let $B$ be a symplectic form on $M$ whose adjoint is $\sigma$. The stabilizer of $\{M_1, M_2\}$ in $\text{GL}(M)$ is the group $\text{GL}(M_1, M_2) = \text{GL}(M_1) \times \text{GL}(M_2) : \langle s \rangle$, where $s$ is an involution in $\text{GL}(M, B)$ which interchanges $M_1$ and $M_2$.

Let $\text{PGL}(M_1, M_2) \cong \text{PGL}(M_1) : \langle \sigma \rangle$ be the group of $k$-automorphisms of $E(M_1) \times E(M_2)$. Now $\text{GL}(M_1, M_2)$ acts by conjugation on $E(M_1) \times E(M_2)$ and the resulting map $\phi : \text{GL}(M_1, M_2) \to \text{PGL}(M_1, M_2)$ is surjective. Moreover $\ker(\phi) = \{a1_{M_1} + b1_{M_2} \mid a, b \in k^\times\} \cong k^\times \times k^\times$.

Set $\text{Sp}(M_1, M_2) = \text{GL}(M, B) \cap \text{GL}(M_1, M_2)$. If $\tau$ is transposition then

$$\text{Sp}(M_1, M_2) = \{(g, sg^{-1}s) \in \text{GL}(M_1) \times \text{GL}(M_2) \mid sgs = g^{-\tau}\} : \langle s \rangle$$
Now let $\text{PGL}(M_1, M_2, \sigma)$ be the centralizer of $\sigma$ in $\text{PGL}(M_1, M_2)$. Then $\phi$ restricts to a surjective map $\theta : \text{Sp}(M_1, M_2) \to \text{PGL}(M_1, M_2, \sigma)$. The kernel of $\theta$ is $K = \{(a, a^{-1}) \mid a \in k^*\}$. So $K \cong k^*$ and $s$ inverts each element of $K$.

**Lemma 2.5.** Each projective representation $\rho : G \to \text{PGL}(M_1, M_2, \sigma)$ is realised by a group representation $\chi : H \to \text{Sp}(M_1, M_2)$ which arises from a commutative diagram of finite groups with exact rows

$$
\begin{array}{ccccccccc}
1 & \longrightarrow & O & \xrightarrow{\text{inc}} & H & \xrightarrow{\nu} & G & \longrightarrow & 1 \\
& & \downarrow{\eta} & & \downarrow{x} & & \downarrow{\rho} & & \\
1 & \longrightarrow & K & \xrightarrow{\text{inc}} & \text{Sp}(M_1, M_2) & \xrightarrow{\theta} & \text{PGL}(M_1, M_2, \sigma) & \longrightarrow & 1
\end{array}
$$

Here $O$ is a cyclic group of odd order, $\eta$ is injective, $[H : C_H(O)] \leq 2$ and all elements of $H \setminus C_H(O)$ invert $O$.

**Proof.** The pull-back diagram associated with $\rho$ and $\theta$ is

$$
\begin{array}{ccccccccc}
1 & \longrightarrow & K & \xrightarrow{\text{inc}} & \hat{G} & \xrightarrow{\nu} & G & \longrightarrow & 1 \\
& & \downarrow{=} & & \downarrow{x} & & \downarrow{\rho} & & \\
1 & \longrightarrow & K & \xrightarrow{\text{inc}} & \text{Sp}(M_1, M_2) & \xrightarrow{\theta} & \text{PGL}(M_1, M_2, \sigma) & \longrightarrow & 1
\end{array}
$$

Every element of $G$ centralizes or inverts $K$. In this way $K \cong k^*$ is a (possibly non-trivial) $ZG$-module. Set $\gamma(\lambda) = \lambda|G|$, for all $\lambda \in K$. As $k$ is algebraically closed, $\gamma$ is a surjective endomorphism of $K$. We have a short exact sequence of abelian groups

$$
1 \longrightarrow O \xrightarrow{\eta} K \xrightarrow{\gamma} K \longrightarrow 1.
$$

Here $O$ is the set of roots of $x|G| - 1$ in $k$. So $O$ is a finite group. This induces a long exact sequence in cohomology, including

$$
\ldots \longrightarrow \text{H}^2(G, O) \xrightarrow{\eta} \text{H}^2(G, K) \xrightarrow{\gamma} \text{H}^2(G, K) \longrightarrow \ldots .
$$

Now $\gamma_*$ is the zero map, as multiplication by $|G|$ annihilates $\text{H}^2(G, K)$. Let $d \in \text{H}^2(G, K)$ be the factor set associated with

$$
1 \longrightarrow K \xrightarrow{\text{inc}} \hat{G} \xrightarrow{\nu} G \longrightarrow 1.
$$

Then there exists $c \in \text{H}^2(G, O)$ mapping onto $d$. This gives us the commutative diagram in the statement of the Lemma. }

We mention that Theorem A.5 in [11] wrongly claims (in the notation used here) that $H$ is a central extension of $G$. Now [11] 7.2 relies on Theorem A.5, but does not require $O \leq Z(H)$. So 7.2 is still correct.

2.2. *Forms and Modules*

Fix a symmetric $G$-form $B$ on $M$. If $B$ is non-degenerate on a submodule $M_1$ of $M$, then $M_1^+$ is a submodule of $M$ and $M = M_1 + M_1^+$; we call $M_1$ a $B$-direct summand of $M$. We say that $L$ is a $B$-component of $M$ if there is a $kG$-isometry $(L, B_1) \to (M, B)$, for some $G$-form $B_1$. In particular $L$ is of symmetric type.

Now in the notation 2.1 the $G$-invariant bilinear forms on $M$ are $\{B_\theta \mid \theta \in E_G(M)\}$. Clearly

- $B_\theta$ is non-degenerate if and only if $\theta$ is a unit in $E_G(M)$.
- $B_\theta$ is symmetric if and only if $\theta^* = \theta$.  

- $B_\theta$ is symplectic if and only if $\theta = \phi + \phi^\sigma$, for some $\phi \in E(M)$.

Suppose that $L_1$ and $L_2$ are $kG$-modules. Then $L_1 \cong L_2$ if and only if there is a perfect $G$-pairing $P$ between $L_1$ and $L_2$. So $P : L_1 \times L_2 \to k$ is a bilinear map such that $P(g\ell_1, g\ell_2) = P(\ell_1, \ell_2)$, for all $\ell_1 \in L_1$ and $g \in G$. Moreover the left and right radicals of $P$ are trivial.

Let $\theta \in E_G(M)$. Then $m_1 \in \ker(\theta)$ if and only if $B(m_1, \theta^\sigma m_2) = B(\theta m_1, m_2) = 0$, for all $m_2 \in M$. So $\ker(\theta) = (\theta^\sigma)^\perp$. There is a bilinear map $B_\theta : \theta M \times \theta^\sigma M \to k$ (well-)defined by
\[
B_\theta(\theta m_1, \theta^\sigma m_2) = B(\theta m_1, m_2) = B(m_1, \theta^\sigma m_2), \quad \text{for } m_1, m_2 \in M. \quad (2.3)
\]

**Lemma 2.6.** $\hat{B}_\theta$ is a perfect $G$-pairing, and thus $\theta^\sigma M \cong (\theta M)^*$. Conversely if $P$ is a perfect $G$-pairing between submodules $L_1$ and $L_2$ of $M$ there is $\psi \in E_G(M)$ such that $L_1 = \psi M, L_2 = \psi^\sigma M$ and $P = \hat{B}_\psi$.

**Proof.** Let $\theta^\sigma m_2$ be in the right radical of $\hat{B}_\theta$, where $m_2 \in M$. Then $\theta^\sigma m_2 = 0$, as $B(m_1, \theta^\sigma m_2) = 0$, for all $m_1 \in M$. Likewise the left radical of $\hat{B}_\theta$ is trivial.

Let $P : L_1 \times L_2 \to k$ be a perfect $G$-pairing. Then for each $m \in M$ there is $\psi m \in L_1$ such that $B(m, \ell_2) = P(\psi m, \ell_2)$ for all $\ell_2 \in L_2$. Check that $\psi \in E_G(M)$ and $\psi M = L_1$. Likewise there is $\psi^\sigma \in E_G(M)$ such that $B(\ell_1, m) = P(\ell_1, \psi^\sigma m)$ for all $\ell_1 \in L_1$. Now for all $m_1, m_2 \in M$
\[
B(m_1, \psi^\sigma m_2) = P(\psi m_1, \psi^\sigma m_2) = B(\psi m_1, m_2).
\]

It follows that $\psi^\sigma = \psi$ and $P = \hat{B}_\psi$. \hfill $\square$

Note that $\theta^\sigma = \theta$ if and only if $\theta M = \theta^\sigma M$ and $\hat{B}_\theta$ is symmetric.

**Corollary 2.7.** Let $\theta \in E_G(M)$. Then $B$ is non-degenerate on $\theta M$ if and only if $\theta^\sigma$ restricts to an isomorphism $\theta M \to \theta^\sigma M$.

**Proof.** We have $(\theta M)^\perp = \ker(\theta^\sigma)$. So $\theta M$ is non-degenerate if and only if $\theta^\sigma$ is injective on $\theta M$. Now $\dim(\theta M) = \dim(\theta^\sigma M)$. So the restriction $\theta^\sigma : \theta M \to \theta^\sigma M$ is injective if and only if it is surjective. \hfill $\square$

**Lemma 2.8.** Let $e \in E_G(M)$ be idempotent. Then each $G$-invariant bilinear form on $eM$ is the restriction of $B_\theta$, for a unique $\theta \in e^*E_G(M)e$.

**Proof.** Using Lemma 2.6, $e^*E(M)e = \text{Hom}(eM, e^\sigma M) \cong (eM)^* \otimes (eM)^*$. Also $B_\phi$ and $B_{e\phi}$ have the same restrictions to $eM$, for all $\phi \in E_G(M)$.

Let $\hat{B}$ be a $G$-invariant bilinear form on $eM$. Then $B(e\cdot e\cdot)$ defines a $G$-invariant bilinear form on $M$. So there exists $\theta \in e^*E_G(M)e$ with $\hat{B}(em_1, em_2) = B_\theta(m_1, m_2)$, for all $m_1, m_2 \in M$. \hfill $\square$

**Proposition 2.9** Orthogonal Projection. A $kG$-submodule $L$ of $M$ is a $B$-direct summand if and only if $L = eM$, for some $\sigma$-invariant idempotent $e \in E(M)$. If $e$ exists it is unique, $G$-invariant and $\ker(e) = L^\perp$.

**Proof.** Suppose that $e$ exists. Then $e^\sigma M = e^\sigma eM = eM$. So $B$ is non-degenerate on $L$, by Corollary 2.7. Moreover $\ker(e) = (e^\sigma M)^\perp = L^\perp$. This ensures that $e$ is unique, and this forces $e \in E_G(M)$. \hfill $\square$
Conversely, suppose that \( L \cap L^\perp = 0 \). Let \( e \) be projection onto \( L \) with kernel \( L^\perp \). Then \( \ker(e^\sigma) = (eM)^\perp = L^\perp \) and \( e^\sigma M = \ker(e)^\perp = L \). So \( e^\sigma \) is projection onto \( L \) with kernel \( L^\perp \). We deduce that \( e^\sigma = e \).

Let \( e, f \) be idempotents in \( E_G(M) \). By \( \text{[8]} \) \( eM \cong fM \) as \( kG \)-modules if and only if there exist \( x, y \in E_G(M) \) such that \( e = xy \) and \( f = yx \).

**Lemma 2.10.** Let \( e \) and \( f \) be \( \sigma \)-invariant idempotents in \( E_G(M) \). Then \( (eM, B) \cong (fM, B) \) if and only if
\[
e = h^\sigma h \quad \text{and} \quad f = hh^\sigma,
\]
for some \( h \in E_G(M) \).

**Proof.** Let \( h \in E_G(M) \) with \( e = h^\sigma h \) and \( f = hh^\sigma \). Then \( h : eM \rightarrow fM \) is a \( k \)-isomorphism, with inverse \( h^\sigma \). Now for all \( m_1, m_2 \in M \)
\[
B(hem_1, hem_2) = B(h^\sigma hem_1, em_2) = B(em_1, em_2).
\]
So \( h : (eM, B) \rightarrow (fM, B) \) is an isomorphism.

Conversely, let \( h : (eM, B) \rightarrow (fM, B) \) be an isomorphism. We may assume that \( h = fhe \in E_G(M) \). Then \( h^\sigma h = e \), as
\[
B(h^\sigma hem_1, em_2) = B(hem_1, hem_2) = B(em_1, em_2).
\]
Moreover \( h^\sigma : (fM, B) \rightarrow (eM, B) \) is an isometry as \( hM = fM \) and
\[
B(h^\sigma hm_1, h^\sigma hm_2) = B(em_1, em_2) = B(hm_1, hm_2).
\]
This implies that \( hh^\sigma = f \).

In our next result we may assume that \( k \) is an arbitrary field. The proof uses ideas from the proof of \( \text{[4]} \) Lemma 3.4].

**Lemma 2.11.** Let \( A \) be a semi-simple subalgebra of \( E(M) \) and let \( e \in A \) be an idempotent such that \( e^\sigma \in A \) and \( eM \) is a non-degenerate submodule of \( M \). Then orthogonal projection onto \( eM \) belongs to \( E_G(M) \cap A \).

**Proof.** Corollary \( \text{[2,4,1]} \) implies that \( e^\sigma eM = e^\sigma M \). So by the Artin-Wedderburn Theorem and the Jacobson Density Lemma \( e^\sigma = e^\sigma ea \) for some \( a \in A \). Set \( f = eae^\sigma \). Then
\[
f^\sigma f = (e^\sigma ea)^\sigma(eae^\sigma) = e^\sigma (e^\sigma ea)e^\sigma = e^\sigma (e^\sigma)^2 = f^\sigma.
\]
So \( f = (f^\sigma f)^\sigma = (f^\sigma f) \). Then \( f^2 = f^\sigma f = f \). Moreover \( f \in A \).

Now \( e^\sigma f = (e^\sigma ea)e^\sigma = e^\sigma \). So \( \text{rank}(f) \geq \text{rank}(e^\sigma) = \text{rank}(e) \). It follows that \( \dim(fM) \geq \dim(eM) \). But \( fM \subseteq eM \). So \( fM = eM \). Now \( f \in E_G(M) \), as \( f \) is projection onto \( eM \) and \( \ker(f) = (eM)^\perp \).

The next result is proved in \( \text{[5]} \) Proposition] and is part of the ‘folklore’ of the subject. Our proof anticipates the methods we use later.

**Lemma 2.12.** Suppose that \( M = M_1 + \ldots + M_t \) where each \( M_i \) is an indecomposable \( kG \)-module. Then for each \( i \)
\[(i) \ \text{B is non-degenerate on } M_i \text{ or}
(ii) \ B \text{ is non-degenerate on } M_i + M_j \text{ for some } j \neq i \text{ with } M_j \cong M_i^* .
\]
Proof. Write \(1_M = e_1 + \ldots + e_i\) where \(e_1, \ldots, e_i\) are pairwise orthogonal primitive idempotents in \(E_G(M)\) with \(e_jM = M_j\). Let \(\epsilon\) be the point of \(E_G(M)\) containing \(e_i\). We use \(\perp\) for images in \(E(P_e)\).

Suppose first that \(M_i \cong M^*_i\). Then \(\epsilon \sigma = \epsilon\), using Lemma 2.6. So \((E(P_e), \sigma)\) is an involutary \(k\)-algebra. Lemma 2.2 implies that \(P_e\) affords a symmetric form \(B_e\) with adjoint \(\sigma\). For all \(j\) with \(e_j \in \epsilon\), choose \(s_j \in \tau^{-1}P_e\) with \(s_j \neq 0\). Then the \(s_j\) form a basis of \(P_e\).

Say \(B_e(s_i, s_j) \neq 0\). Then \(B_e\) is non-degenerate on \(ks_i\). Let \(x_i + e_jE(P_e)\) be orthogonal projection onto \(ks_i\), as given by Proposition 2.9. By Lemma 2.1, there is a \(\sigma\)-invariant idempotent \(f_i \in \epsilon_iE_G(M)\) such that \(f_i = x_i\). Then \(f_iM = M_i\), as \(f_iM \subseteq M_i\) and \(M_i\) is indecomposable. So \(B\) is non-degenerate on \(M_i\), by Proposition 2.9.

Now say \(B_e(s_i, s_j) = 0\). As \(B_e\) is non-degenerate, we may choose \(j \neq i\) such that \(B_e(s_i, s_j) \neq 0\). So \(B_e\) is non-degenerate on \(ks_i + ks_j\). Let \(x_{ij} \in e_i + e_jE(P_e)\) be orthogonal projection onto \(ks_i + ks_j\), as given by Proposition 2.9. By Lemma 2.1, there is a \(\sigma\)-invariant idempotent \(f_{ij} \in (e_i + e_j)E_G(M)\) such that \(f_{ij} = x_{ij}\). Then \(f_{ij}M = M_i + M_j\), as \(f_{ij}M \subseteq M_i + M_j\) but \(f_{ij}\) is not primitive. So \(B\) is non-degenerate on \(M_i + M_j\), by Proposition 2.9.

Finally, suppose that \(M_i \not\cong M^*_i\). Then \(\epsilon \neq \epsilon\) and \((E(P_e), \epsilon, \epsilon)(P_e, \sigma)\) is an involutary \(k\)-algebra satisfying the hypothesis of Lemma 2.2(ii). Let \(B_{e, \sigma}\) be the corresponding symplectic form on \(P_e \oplus P_\sigma\). We proceed as above; there exists \(j\) such that \(e_j \in \epsilon\) and \(B_{e, \epsilon}\) is non-degenerate on \(\pi_{e, \epsilon}(e_i + e_j)(P_e \oplus P_\sigma)\). Then there is a \(\sigma\)-invariant idempotent \(f_{ij} \in (e_i + e_j)E_G(M)\) such that \(f_{ij}M = M_i + M_j\). So \(B\) is non-degenerate on \(M_i + M_j\), and \(M_i \cong M^*_i\). Q.E.D.

2.3. Form Induction

We continue to assume that \(M\) is a \(kG\)-module, \(B\) is a \(G\)-form on \(M\) and \(\sigma\) is the adjoint of \(B\) on \(E(M)\). For \(H \leq G\), let \(G/H\) be a left transversal to \(H\) in \(G\) and set \(E_H(M) = \text{End}_{kH}(M)\). The relative trace map \(\text{tr}_{H}^G : E_H(M) \to E_G(M)\) is the \(k\)-linear map \(\text{tr}_{H}^G(f) := \sum_{g \in G/H} gfg^{-1}\) for all \(f \in E_H(M)\). Its image is an ideal of \(E_G(M)\), which we denote by \(E_H(M)\). The \(\sigma\)-invariant elements in \(E_H(M)\) form a subspace, but not an ideal, of \(E_H(M)\).

Let \(L\) be a \(kH\)-module. We use \(L^G\) or \(\text{Ind}^G_H(L)\) for the induced \(kG\)-module, and \(M_H\) or \(\text{Res}^G_H(M)\) for the restricted \(kH\)-module. Set \(\#H = gHg^{-1}\), for \(g \in G\). So there is a conjugation isomorphism \(\#H \cong H\). Now \(\text{Ind}^G_H(L) = kG \otimes_{kH} L\) is a direct sum of the \(k\)-vector spaces \(\#L\), as \(g\) ranges over \(G/H\). Here \(\#L = g \otimes L\) is the \(\#H\)-module such that \(ghg^{-1}(g \otimes \ell) = g \otimes h\ell\), for all \(h \in H\) and \(\ell \in L\).

Let \(B_1\) be a symmetric \(H\)-form on \(L\). The induced symmetric \(kG\)-module \(\text{Ind}^G_H(L, B_1)\) is \(\text{Ind}^G_H(L)\) with the induced \(G\)-form \(B_1^G\), where

\[
B_1^G(g_1 \otimes \ell_1, g_2 \otimes \ell_2) = \begin{cases} 
B_1(g_1^{-1}g_1, \ell_1, \ell_2), & \text{if } g_1H = g_2H, \\
0, & \text{if } g_1H \neq g_2H.
\end{cases}
\]

Let \(\#B_1\) denote the restriction of \(B_1^G\) to \(\#L\). Then \(\text{Ind}^G_H(L, B_1)\) is the orthogonal direct sum of the symmetric \(k\)-spaces \((\#L, \#B_1)\). It is clear that \(B_1^G\) is symplectic if and only if \(B_1\) is symplectic.

There is a symmetric version of Mackey’s formula [12, 3.1.9]:

**Lemma 2.13.** Given \(K \leq G\), there is an isomorphism of symmetric \(kK\)-modules

\[
\text{Res}^G_K \text{Ind}^G_H(L, B_1) \cong \bigoplus_{g \in K \setminus G/H} \text{Ind}^K_{K \cap gH} \text{Res}^g_H(\#L, \#B_1).
\]
Proof. Let $g \in G$. The assignment $i \to ig$ maps $K/K \cap gH$ to a set of representatives for the left cosets of $H$ in $KgH$ and $\sum_{i \in K/K \cap gH} igL \cong \text{Ind}_{K/K \cap gH}^{K} gL$. This induces an isomorphism $\sum_{i \in K/K \cap gH} (gL, igB_1) \cong \text{Ind}_{K/K \cap gH}^{K} (gL, gB_1)$ of symmetric $kK$-modules. \hfill \square

Higman’s Criterion \cite[4.2.2]{12} is the definitive result on $H$-projectivity:

\begin{lem}
Let $L$ be a submodule of $\text{Res}_{H}^{G}(M)$. Then $M \mid \text{Ind}_{H}^{G}(L)$ if and only if there is $\phi \in E_{H}(M)$ such that $\text{tr}_{H}^{G}(\phi)$ is a unit in $E_{G}(M)$ and $\phi M$ is isomorphic to a submodule of $L$. So the following are equivalent:

(i) $M$ is a component of $\text{Ind}_{H}^{G} \text{Res}_{H}^{G} M$.

(ii) $M$ is a component of $\text{Ind}_{H}^{G} L$ for some $kH$-module $L$.

(iii) $\text{tr}_{H}^{G}(\phi)$ is a unit in $E_{G}(M)$, for some $\phi \in E_{H}(M)$.
\end{lem}

Now let $\theta$ be a $\sigma$-invariant unit in $E_{G}(M)$. Then we say that

- $\theta$ is $(H, \sigma)$-projective if $\theta = \text{tr}_{H}^{G}(\alpha)$ for some $\sigma$-invariant $\alpha \in E_{H}(M)$.

- $B_{\theta}$ is $H$-projective if $(M, B_{\theta})$ is an orthogonal direct summand of a symmetric $kG$-module induced from $H$.

- $M$ is symmetrically $H$-projective if it has a $H$-projective symmetric $G$-form.

We also say that

- $B_{\theta}$ is strongly $H$-projective if there is a $kG$-isometry

\[(M, B_{\theta}) \to \text{Ind}_{H}^{G} \text{Res}_{H}^{G}(M, B_{\theta}).\]

Note: if $B_{\theta}$ is $H$-projective and $H \leq K \leq G$ then $B_{\theta}$ is $K$-projective. However, if $B_{\theta}$ is strongly $H$-projective, we are not able to show that $B_{\theta}$ is strongly $K$-projective.

Recall the definition \cite[2.3]{12} of the perfect $H$-pairing $B_{\alpha}$, for $\alpha \in E_{H}(M)$.

\begin{lem}
$B_{\theta}$ is $H$-projective if and only if $\theta$ is $(H, \sigma)$-projective; given $\alpha \in E_{H}(M)$ such that $\alpha^{\sigma} = \alpha$ and $\text{tr}_{H}^{G}(\alpha) = \theta$, there is a $kG$-isometry $(M, B_{\theta}) \to \text{Ind}_{H}^{G}(\alpha M, B_{\alpha})$.
\end{lem}

Proof. Suppose first that $\theta$ is $(H, \sigma)$-projective and $\alpha$ is as given. Set $\phi = \text{tr}_{H}^{G}(1 \otimes \alpha) : M \to \text{Ind}_{H}^{G}(\alpha M)$. Then

\[B_{\theta}^{G}(\phi m_1, \phi m_2) = \sum_{g \in G/H} B_{\alpha}(g \alpha m_1, g \alpha m_2) = \sum_{g \in G/H} B(\alpha g m_1, g m_2) = B_{\theta}(m_1, m_2).\]

As $\phi : (M, B_{\theta}) \to \text{Ind}_{H}^{G}(\alpha M, B_{\alpha})$ is a $kG$-isometry, $B_{\theta}$ is $H$-projective.

Conversely, suppose that $B_{\theta}$ is $H$-projective. Then there is a $kG$-isometry $\phi : (M, B_{\theta}) \to \text{Ind}_{H}^{G}(L, B_{1})$ for some symmetric $kH$-module $(L, B_{1})$. Let $e \in E_{H}(L^{G})$ be orthogonal projection onto $1 \otimes L$. Now $(m_1, m_2) \to B_{1}(e \phi m_1, e \phi m_2)$ is a symmetric $H$-form on $M$. So there exists $\alpha \in E_{H}(M)$ such that $\alpha^{\sigma} = \alpha$ and

\[B_{1}(e \phi m_1, e \phi m_2) = B(\alpha m_1, m_2), \text{ for all } m_1, m_2 \in M.\]

As $\phi$ is an isometry, we have

\[B_{\theta}(m_1, m_2) = B_{1}(e \phi m_1, e \phi m_2) = \sum_{g \in G/H} B_{1}(e \phi g m_1, e \phi g m_2) = \sum_{g \in G/H} B(\alpha g m_1, g m_2) = B(\text{tr}_{H}^{G}(\alpha) m_1, m_2).\]
So $\theta = \text{tr}^G_H(\alpha)$ is $(H, \sigma)$-projective.

The proof of the next result is similar, and omitted.

**Lemma 2.16.** $B_\theta$ is strongly $H$-projective if and only if $\text{tr}^G_H(\alpha \theta \alpha^\sigma) = \theta$, for some $\alpha \in E_G(M)$.

We can now prove a symmetric analogue of Lemma 2.14.

**Proposition 2.17.** The following are equivalent:

(i) $M$ is symmetrically $H$-projective.

(ii) $B_\theta$ is $H$-projective for some $\sigma$-invariant unit $\theta \in E_G(M)$.

(iii) $\theta$ is $(H, \sigma)$-projective for some unit $\theta \in E_G(M)$.

**Proof.** (1) and (2) are equivalent, from the definitions. Lemma 2.15 shows that (2) and (3) are equivalent.

For indecomposable modules, we have:

**Lemma 2.18.** Suppose that $M$ is indecomposable and symmetrically $H$-projective. Then there is an indecomposable $kH$-module $L$ which has a symmetric $H$-form $B_1$ such that $M$ is a $B_1^G$-component of $L^G$.

**Proof.** Choose a symmetric $kH$-module $(L, B_1)$ with $\dim(L)$ minimal subject to the existence of a $kG$-isometry $\phi : (M, B_\theta) \to \text{Ind}_H^G(L, B_1)$, for some $\theta$. Using Lemma 2.15, $(L, B_1)$ is orthogonally indecomposable.

We claim that $L$ is indecomposable. Otherwise $L = L_1 \oplus L_2$, where $L_i \cong L_2$, byLemma 2.12. Let $e \in E_H(L^G)$ be orthogonal projection onto $1 \otimes L$ and for $i = 1, 2$, let $e_i = ee_ie$ be projection onto $1 \otimes L_i$ with kernel $1 \otimes L_{3-i}$. Now for $i, j = 1, 2$ there are $\alpha_{ij} \in E_H(M)$ such that

$$B_1(e_i \phi m_1, e_j \phi m_2) = B(\alpha_{ij} m_1, m_2),$$

for all $m_1, m_2 \in M$.

It is easy to check that $\alpha_{ij}^2 = \alpha_{ij}$.

As $e = e_1 + e_2$ and $\phi$ is an isometry, we have

$$\sum_{i,j=1,2} \text{tr}^G_H(\alpha_{ij}) = \theta.$$

But $\text{tr}^G_H(\alpha_{12}) + \text{tr}^G_H(\alpha_{21}) \in J(E_G(M))$, as $\text{tr}^G_H(\alpha_{12})^\sigma = \text{tr}^G_H(\alpha_{21})$ and $E_G(M)$ is a local ring. So $\theta_{ii} := \text{tr}^G_H(\alpha_{ii})$ is a unit, for some $i$. Lemma 2.15 gives an isometry $(M, B_{\theta_{ii}}) \to \text{Ind}_H^G(\alpha_{ii} M, B_{\theta_{ii}})$. But $\dim(\alpha_{ii} M) < \dim(L)$. This contradiction establishes our claim.

**Proof of Theorem 1.1** Let $T$ be a symmetric vertex of $M$ and let $V \leq T$ be a vertex of $M$. There is nothing to prove if $V = T$. So assume that $V \neq T$. By Lemma 2.14, $\text{tr}^G_V(\alpha) = 1_M$, for some $\alpha \in E_V(M)$, and by Proposition 2.17, $\theta = \text{tr}_T^G(\beta)$ for some $\sigma$-invariant $\beta \in E_T(M)$. Then

$$\theta = \text{tr}_V^G(\alpha) \text{tr}_T^G(\beta) \text{tr}_V^G(\alpha^\sigma) = \sum_{a, b \in G/V, c \in G/T} (^c a)(^c \beta)(^b \alpha^\sigma).$$
Now each $G$-orbit in $G/V \times G/T \times G/V$ contains a triple $(aV, T, bV)$. We say that this orbit is:
- diagonal if $aV = bV$,
- symmetric if $aV \neq bV$ but the orbit contains $(bV, T, aV)$,
- antisymmetric if the orbit does not contain $(bV, T, aV)$.

We denote the collections of such orbits by $O_d$, $O_s$, and $O_a$, respectively.

The stabilizer of $(aV, T, bV)$ is $^aV \cap {}^bV \cap T$. So the orbit sum is
\[ \text{tr}(a, b) := \text{tr}_{G/V \cap {}^bV \cap T}((^a \alpha \beta ^b \alpha^\sigma)) \in E_G(M). \]

Now $\text{tr}(a, b)^\sigma = \text{tr}(b, a)$. So $\theta$ is a sum, in $E_G(M)$, of $\sigma$-invariant terms
\[ \theta = \sum_{O_d} \text{tr}(a, a) + \sum_{O_s} \text{tr}(a, b) + \sum_{O_a} (\text{tr}(a, b) + \text{tr}(b, a)). \]

Write $\text{tr}(a, b) = \lambda_1 M + j$, with $\lambda \in k^\times$ and $j \in J(E_G(M))$. Then for each pair of anti-symmetric orbits $\text{tr}(a, b) + \text{tr}(b, a) = j + j^\sigma$ belongs to $J(E_G(M))$. Suppose that $\text{tr}(a, a)$ is a unit in $E_G(M)$, for some diagonal orbit. Then $B_{\text{tr}(a, a)}$ is a $(^aV \cap T)$-projective symmetric $G$-form on $M$. This is impossible, as $V \cap T \leq T$.

Now $\theta$ is a unit in the local ring $E_G(M)$. So we can choose a triple $(aV, T, bV)$ in a symmetric orbit such that $\text{tr}(a, b)$ is a unit. We then replace $V$ by a conjugate so that $a = 1$, to simplify the notation. Then $\text{tr}(1, b)$ is a unit and $(V, T, bV)$ is in a symmetric $G$-orbit.

As $(bV, T, V)$ is $G$-conjugate to $(V, T, bV)$ there is $t \in T$ with $tV = bV$ and $tbV = V$. So $t \in N_T(V \cap bV)$ and $t^2 \in V \cap bV$. Then $VbV = Vb^{-1}V$ is a self-dual double coset and $[(V \cap bV) \langle t \rangle : V \cap bV] = 2$.

Set $\gamma := \alpha \beta ^b \alpha^\sigma$. Then $\gamma^\sigma = ^b \alpha \beta \alpha^\sigma = \gamma t$. So $\gamma + t \gamma$ is fixed by both $\sigma$ and $(V \cap bV) \langle t \rangle \cap T$. Moreover $\text{tr}(1, b) = \text{tr}_{G/(V \cap bV) \langle t \rangle} (\gamma + t \gamma)$. As $T$ is a symplectic vertex of $M$, this forces $(V \cap bV) \langle t \rangle = T$. But $V \neq T$ and $t^2 \in V$. We deduce that $V = bV$, $T = V \langle t \rangle$ and $[T : V] = 2$. \qed

3. Symmetric Vertices

3.1. Projective Modules

We interpret some results from [4] and [11] on the symmetric forms of projective $kG$-modules. We denote the projective cover of a $kG$-module $M$ by $P(M)$.

The ring multiplication in $kG$ induces maps $\ell, r : kG \to E(kG)$, where
\[ \ell(x)(y) = r(y)(x) = xy, \quad \text{for all } x, y \in kG. \]

So $\ell$ is the structural $k$-homomorphism of the left regular $kG$-module and $r : kG^{op} \to E_kG(kG)$ is a $k$-algebra isomorphism.

The elements of $G$ form an orthonormal basis for a symmetric $G$-form $B_1$ on $kG$. Let $\sigma$ be the adjoint of $B_1$ on $E(kG)$ and let $o$ be the contragredient map on $kG$ i.e. $g^\sigma = g^{-1}$ for all $g \in G$. Then
\[ B_1(xy, z) = B_1(x, zy^o) = B_1(y, x^o z), \quad \text{for all } x, y, z \in kG. \]

Equivalently $\ell(x)^\sigma = \ell(x^o)$ and $r(y)^\sigma = r(y^o)$.

Our first result includes Fong’s Lemma:

**Lemma 3.1.** Each non-trivial selfdual irreducible $kG$-module $M$ affords a unique symplectic $G$-form, up to scalars. The form $B_1$ is degenerate on each direct summand of $kG$ that is isomorphic to $P(M)$. 
Proof. Let $\epsilon$ be the point of $E_{kG}(M)$ such that $P(M) \cong kG\epsilon$ for $\epsilon \in \epsilon$. Then the surjective $k$-algebra map $\pi_\epsilon : kG \to E(M)$ has kernel $M\epsilon$. As $P(M) \cong P(M)^*$, we have $M\epsilon = M\epsilon$. So $(E(M),^o)$ is an involutory $k$-algebra. Let $\hat{B}$ be a symmetric form on $M$ whose adjoint is $^o$, as given by Lemma 2.2. Then $\hat{B}$ is $G$-invariant as $\pi_\epsilon(g)^o = \pi_\epsilon(g)^{-1}$, for all $g \in G$, and symplectic, as $M$ has no trivial submodule.

Following Proposition 2.9, suppose that $\epsilon$ contains an $^o$-invariant idempotent $\epsilon$. So $B_1$ non-degenerate on $kG \cong P(M)$. Then $\pi_\epsilon(\epsilon)$ is an $^o$-invariant primitive idempotent in $E(M)$. So $\hat{B}$ is a diagonal form. As this is false, no such $\epsilon$ exists, proving the last statement.

Our next result includes a proof of [4 (1.6)].

Lemma 3.2. $B_1$ restricts to a diagonal $G$-form on each direct summand of $kG$ that is isomorphic to $P(kG)$. Also $\dim P(kG)/|G|_2$ is odd.

Proof. We may write $kG \cong \sum P(S)^{\dim S}$ where $S$ ranges over the irreducible $k$-modules. In particular $P(kG) \cong P(kG)^*$ occurs once in $kG$ and $B_1$ is symplectic on any direct summand not isomorphic to $P(kG)$. So $B_1$ is non-degenerate and diagonal on any direct summand isomorphic to $P(kG)$.

Now $|G|_2 = \sum (\dim P(S)/|G|_2) \dim S$, where each $\dim P(S)/|G|_2$ is an integer. If $S \cong S^*$ and $S \not\cong kG$ then $\dim S$ is even, by Fong’s Lemma. If $S \not\cong S^*$, then $S$ and $S^*$ contribute equally to the sum. We conclude that $1 = \dim P(kG)/|G|_2 \pmod{2}$.

Adapting the notation of Section 2.1 the $G$-invariant bilinear forms on $kG$ are $B_a$, for $a \in kG$. Here $B_a(x, y) = B_1(xa, y)$, for all $x, y \in kG$. Then $B_a$ is non-degenerate if and only if $a$ is a unit in $kG$, symmetric if and only if $a = a^o$ and symplectic if and only if $a = a^e$ and $B_1(a, 1) = 0$. In particular $B_t$ is a symplectic $G$-form, for each involution $t \in G$.

We use $B_1$ to identify $kG \otimes_k kG$ with $E(kG)$ as $k$-vector spaces: $x \otimes y \in kG \otimes kG$ gives the endomorphism

$$(x \otimes y)(z) = B_1(y, z)x, \quad \text{for all } z \in kG.$$ 

Then $(x \otimes y)^o = y \otimes x$ and $(x \otimes y) = gx \otimes gy$, for $g \in G$. Using this

$$\text{tr}_G(x \otimes y) = r(y^o x), \quad \text{for all } x, y \in kG.$$ 

It is useful to list the elements of $G$ as

$$1, t_1, \ldots, t_m, \quad \frac{g_1}{g_1^{-1}}, \ldots, \frac{g_n}{g_n^{-1}}$$

where each $t_i$ is an involution.

Lemma 3.3. A basis for the $\sigma$-invariant elements in $E_G(kG)$ is

$$r(1), r(t_1), \ldots, r(t_m), r(g_1 + g_1^{-1}), \ldots, r(g_n + g_n^{-1}).$$

Of these $r(1)$ and $r(g_i + g_i^{-1})$ are $(1, \sigma)$-projective, while $r(t_i)$ is $(H, \sigma)$-projective, for $H \leq G$, if and only if $^\sigma t_i \in H$, for some $g \in G$.

Proof. Clearly $\{r(g) \mid g \in G\}$ is a basis for $E_G(kG)$ and $r(g)^o = r(g^{-1})$ for all $g \in G$. The first statement follows from these facts.

Let $g \in G$. Then $1 \otimes 1$ and $g \otimes 1 + 1 \otimes g$ are $\sigma$-invariant and

$$\text{tr}_G^G(1 \otimes 1) = r(1), \quad \text{tr}_G^G(g \otimes 1 + 1 \otimes g) = r(g + g^{-1}).$$
So \( r(1) \) and \( r(g + g^{-1}) \), for \( g \neq g^{-1} \), are \((1, \sigma)\)-projective.

Let \( t = t_i \). Then \( t \otimes 1 + 1 \otimes t \in E_i(M) \) is \( \sigma \)-invariant and

\[
\text{tr}^G_i(t \otimes 1 + 1 \otimes t) = r(t).
\]

So \( r(t) \) is \((t, \sigma)\)-projective, if \( t \neq 1 \).

Let \( H \) be a subgroup of \( G \). Then the endomorphisms \( \text{tr}^H_1(a \otimes b) \) span \( E_H(M) \), as \( a, b \) range over all elements of \( G \). Now

\[
\text{tr}^H_1(a \otimes b)(g) = \begin{cases} gb^{-1}a, & \text{if } g \in Hb. \\ 0, & \text{if } g \in G \setminus Hb. \end{cases}
\]

So \( \text{tr}^H_1(a \otimes b) \) is \( \sigma \)-invariant if and only if \( Ha = Hb \) and \( b^{-1}a = a^{-1}b \) i.e. if and only if \( t := b^{-1}a \) is an involution such that \( b t \in H \). The last statement of the lemma follows from this. \( \square \)

**Lemma 3.4.** Let \( H \leq G \) and let \( a \) be a unit in \( kH \). Then

\[ \text{Ind}^G_H(kH, B_a) \cong (kG, B_a). \]

In particular \((kG, B_1) \cong \text{Ind}^G_1(k\langle t \rangle, B_1), \) for all \( t \in G \) with \( t^2 = 1 \).

**Proof.** Let \( r_H(a) \) be the endomorphism \( x \to xa \) of \( kH \). Then \( r_H(a) \) extends to a \( kH \)-endomorphism of \( kG \) (acting as 0 on \( k(G\setminus H) \)) and \( \text{tr}^H_G(r_H(a)) = r_G(a) \). The Lemma is a consequence of this fact. \( \square \)

**Lemma 3.5.** Two involutions \( s, t \in G \) are \( G \)-conjugate if and only if \( \langle k\langle s \rangle, B_s \rangle \) is a component of \( \text{Res}^G_{\langle s \rangle}(kG, B_t) \).

**Proof.** It is clear that there is an \( \langle s \rangle \)-isometry \( \langle k\langle s \rangle, B_s \rangle \to \text{Res}^G_{\langle s \rangle}(kG, B_t) \) if and only if \( B_t(x, sx) \neq 0 \), for some \( x \in kG \). If \( x = \sum_{g \in G} x_g g \), with \( x_g \in k \), then

\[
B_t(x, sx) = B_1(x_t, sx) = \sum_{g \in G} x_g x_{sgt} = \sum_{g \in G, g \neq sgt} x_g^2,
\]

using \( x_g x_{sgt} + x_{sgt} x_g = 0 \). So if \( B_t(x, sx) \neq 0 \) then \( g = sgt \), for some \( g \in G \). In that case \( s = gtg^{-1} \) is conjugate to \( t \).

Conversely, if \( s = gtg^{-1} \), then \( B_t(g, sg) = 1 \). \( \square \)

For the next two results we let \( e \) be a primitive idempotent in \( kG \).

**Lemma 3.6.** Let \( \hat{B} \) be a symplectic \( G \)-form on \( kGe \). Then there is an involution \( t \in G \) such that \( B_t \) is non-degenerate on \( kGe \) and \( \langle k\langle t \rangle, B_t \rangle \) is a component of \( \text{Res}^G_1(kG, \hat{B}) \).

**Proof.** By Lemma 2.8 there is \( a \in ekGe^G \) so that \( \hat{B}(xe, ye) = B_a(x, y) \), for all \( x, y \in kG \). Then \( a = a^\sigma \) and \( B_1(a, 1) = 0 \), as \( \hat{B} \) is symplectic. Write \( a = \sum_{i=1}^n \alpha_i t_i + \sum_{j=1}^n \beta_j (g_j + g_j^{-1}) \), with \( \alpha_i, \beta_j \in k \). Now \( E_G(kGe) \) is a local ring. So each \( \beta_j (g_j + g_j^{-1}) \) is degenerate on \( kGe \). It follows that \( B_{\alpha_i t_i} \) is non-degenerate on \( kGe \), for some \( i \). Set \( t = t_i \). Then \( B_t \) is non-degenerate on \( kGe \) and \( B_a(e, te) = \alpha_i \neq 0 \). So \( ke + kte \) is a \( B_a \)-direct summand of \( \text{Res}^G_1(kG) \) which is isomorphic to \( k\langle t \rangle \). We conclude that \( \langle k\langle t \rangle, B_t \rangle \) is a component of \( \text{Res}^G_1(kG, \hat{B}) \). \( \square \)

Our last result in this section strengthens [11, 6.5]:
COROLLARY 3.7. Suppose that $t \in G$, $t^2 = 1$ and $B_t$ is non-degenerate on $kGe$. Then there is a $kG$-isometry $(kGe, B_t) \to \text{Ind}^G_{(t)} \text{Res}^G_G(kGe, B_t)$.

Let $M = \text{hd}(kGe)$. Then $\text{Res}^G_{C_G(t)} M$ affords a diagonal $C_G(t)$-form. In particular $k_{C_G(t)}$ is a submodule and a quotient module of $\text{Res}^G_{C_G(t)} M$.

**Proof.** If $t = 1$, then $kGe \cong P(kGe)$, and Lemma 3.2 gives all conclusions. So assume that $t$ is an involution. By Lemma 3.6 there is an involution $s \in G$ and so an isometry $(k\langle s \rangle, B_s) \to \text{Res}^G_{(s)}(kGe, B_t)$. Then $s$ and $t$ are $G$-conjugate, according to Lemma 3.5. This proves the first assertion.

We may assume that $M \not\cong kG$. Recall the notation of Lemma 3.1. So there is a surjection $\pi : kG \to E(M)$ and $M$ has a symplectic $G$-form $\dot{B}$. Define the bilinear form $B_t(m_1, m_2) := B(tm_1, m_2)$, for all $m_1, m_2 \in M$. Then $B_t$ is a symmetric $C_G(t)$-form on $M$. Its adjoint $\sigma$ satisfies $\pi_\sigma(x) = \pi_\tau(tx^\sigma t)$, for all $x \in kG$.

Let $\tau(f)$ be the orthogonal projection onto $kGe$ with respect to $B_t$, as given by Proposition 3.4. So $kGe = kGf$ and $tf^\sigma t = f$ (as in [14], 3.1). It follows that $\pi_\sigma(f)$ is a non-zero $\tau$-invariant primitive idempotent in $E(M)$. As a consequence $B_t$ is a diagonal $C_G(t)$-form on $M$. The last assertion now follows from Lemma 2.8. \(\square\)

3.2. Indecomposable Modules

In this section $M$ is an indecomposable $kG$-module, with vertex $V$ and $V$-source $Z$. There is a point $\mu$ of $E_G(Z^G)$ such that $Z^G \cong M$, for all $a \in \mu$ and there is a point $\delta$ of $E_V(M)$ such that $dM \cong Z$, for all $d \in \delta$. The point $\mu$ induces an embedding $F : E(M) \to E(Z^G)$ of $G$-algebras. Set $\Delta$ as the point of $E_V(Z^G)$ containing $F(\delta)$.

Let $N_G(V, Z)$ be the stabilizer of $Z$ in $N_G(V)$. Set $N := N_G(V, Z)/V$. Then $E(P_\Delta)$ is an $N$-algebra. So $P_\Delta$ is a module for a twisted group algebra $k_\gamma N$ of $N$ over $k$. Likewise $E(P_\delta)$ is an $N$-algebra and $P_\delta$ is a module for a twisted group algebra $k_\gamma N$ of $N$ over $k$. According to [16], (26.1), $P_\delta$ is the regular $k_\gamma N$-module. Now $F$ induces an embedding of $N$-algebras $E(P_\delta) \to E(P_\Delta)$. This in turn induces a group isomorphism between the central extensions of $N$ corresponding to the cocycles $\gamma$ and $\gamma'$. In this way, $P_\delta$ can and will be identified with an indecomposable component of $P_\Delta$, as $k_\gamma N$-modules.

**Lemma 3.8.** Suppose that $Z \cong Z^*$ and either $Z$ or $M$ is of symmetric type. Then $k_\gamma N \cong kN$. So $P_\Delta$ is the regular $kN$-module.

**Proof.** Suppose first that $Z$ has a symmetric $V$-form $B_0$. Let $\sigma_0$ be the adjoint of $B_0^G$ on $E(Z^G)$. As $Z \cong Z^*$, Lemma 2.6 implies that $\Delta^{\sigma_0} = \Delta$. So $\mathfrak{M}_{\sigma_0} = \mathfrak{M}_{\Delta}$ and $\sigma_0$ is an involution on $E_V(Z^G)/\mathfrak{M}_{\Delta}$. In this way $(E(P_\Delta), \sigma_0)$ is a simple involutary $N$-algebra.

By Lemma 2.4 there is a symmetric form $B_{\sigma_0}$ on $P_\Delta$ such that the action of $N$ on $E(P_\Delta)$ lifts to a representation $N \to \text{GL}(P_\Delta, B_{\sigma_0})$. In particular $k_\gamma N \cong kN$ as twisted group algebras and $P_\Delta$ is the regular $kN$-module.

Conversely suppose that $M$ has a symmetric $G$-form $B$. Let $\sigma$ be the adjoint of $B$ on $E(M)$. Then $\Delta^{\sigma} = \delta$. So $\mathfrak{M}_{\sigma} = \mathfrak{M}_{\delta}$ and $\sigma$ induces an involution on $E_V(M)/\mathfrak{M}_{\delta}$. According to Lemma 2.4, the action of $N$ on $E(P_\delta)$ lifts to a representation $N \to \text{GL}(P_\delta, B_{\sigma})$, where $B_{\sigma}$ is a symmetric form on $P_\delta$. Thus $k_\gamma N \cong kN$ as twisted group algebras. But $k_\gamma N \cong k_\gamma N$. So as before $P_\Delta$ is the regular $kN$-module. \(\square\)

Set $N^* = N_G^G(V, Z)/V$, where $N_G^G(V, Z)$ is the stabilizer of $\{Z, Z^*\}$ in $N_G(V)$. So $[N^* : N] \leq 2$. The following is based on [16], (14.8)].
Lemma 3.9. Let $L$ be a component of $Z^G$ and let $e$ be a point of $E_V(L)$ contained in $\Delta$. For $V \leq H \leq G$ set $N_H = N_H(V,Z)/V$ and $N_H^* = N_H^*(V,Z)/V$. Then for all $f \in E_V(L)eE_V(L)$ we have:

(i) $\pi, tr_1^N f = tr_1^{N_H} \pi, tr_1^V f$ and $\pi, res_1^N f : E_H(L) \to E_{N_H}(P_e)$ is onto.

(ii) If $\sigma$ is a $G$-involution of $E(L)$, then $\pi, tr_1^N f = tr_1^{N_H} \pi, tr_1^V f$ and $\pi, res_1^N f : E_H(L) \to N_H^*(E(P_e) \times E(P_{e^*}))$ is onto.

Proof. (i) follows from Remark (19.9) in [16] as $E_H(L) = tr_1^H(E(V,L))$.

From the proof of [16] (14.7) we see that $\pi, tr_1^N f = tr_1^{N_H} \pi, tr_1^V f$. So $\pi, tr_1^N f$ maps $E_H(L)$ onto $tr_1^{N_H}(E(P_e) \times E(P_{e^*}))$. This is a 2-sided ideal of $N_H^*(E(P_e) \times E(P_{e^*}))$ which is contained in $\pi, tr_1^{N_H} E_H(L)$.

We now modify [16] (14.8). As $L \mid Z^G$, we have $1_L = tr_1^{H}(i)$ for some $i \in E_V(L)eE_V(L)$. So $1_{P_e + P_{e^*}} = tr_1^{N_H} (\pi, e) (i)$ and hence

$$tr_1^{N_H} (E(P_e) \times E(P_{e^*})) = N_H^*(E(P_e) \times E(P_{e^*})).$$

Now (ii) follows from this and the previous paragraph.

In our situation the Puig correspondence [16] (19.1) is a multiplicity preserving bijection between the indecomposable components of $Z^G$ with vertex $V$ and the indecomposable components of $P_\Delta$. More concretely, if $e$ is a primitive idempotent in $E_G(Z^G)$ such that $eZ^G$ has vertex $V$, then $\pi_\Delta(e)$ is a primitive idempotent in $E_N(P_\Delta)$, and $\pi_\Delta(e)P_\Delta$ is the Puig correspondent of $eZ^G$.

Proposition 3.10. Suppose that $Z$ has a symmetric $V$-form $B_0$. Then the Puig correspondent of $P(k_N)$ is the unique indecomposable $B_0^G$-component of $Z^G$ that has vertex $V$.

Proof. Lemma 3.8 applies, and we adopt its notation. Let $e \in \Delta$ be the orthogonal projection $Z^G \to 1 \otimes Z$. So $e, e = e$ and $tr_1^V(e) = 1_{Z^G}$. Set $\tau := \pi_\Delta(e)$. Then $1_{P_\Delta} = tr_1^N (\tau)$, using Lemma 3.9 (i). Moreover $\tau, V = \tau$. So $(P_\Delta, B_{n_\tau}) \mid \text{Ind}_1^N (\tau P_\Delta, B_\tau)$, according to Lemma 2.13. But $\text{dim}(\tau P_\Delta) = 1$. So $(\tau P_\Delta, B_\tau) \cong (k_1, B_1)$ and thus $\text{Ind}_1^N (\tau P_\Delta, B_\tau) \cong (kN, B_1)$. So we can and do identify $(P_\Delta, B_{n_\tau})$ with $(kN, B_1)$.

Write $Z^G = L_1 + L_2 + \ldots + L_n$, where the $L_i$ are indecomposable $kG$-modules and $L_1$ is the Puig correspondent of $P(k_N)$. Then $L_i \not= L_j^*$ for $i > 1$. So $B_0^G$ is non-degenerate on $L_1$, by Lemma 2.12.

Now suppose that $B_0^G$ is non-degenerate on $L_i$, where $L_i$ has vertex $V$. Then $L_i$ has $V$-source $Z$. Let $a \in E_G(Z^G)$ be orthogonal projection onto $L_i$. Then $\pi_\Delta(a)$ is a $\sigma_\psi$-invariant primitive idempotent in $E_N(P_\Delta)$. So $\pi_\Delta(a)kN \cong P(k_N)$, by Lemma 3.1. We deduce that $L_i = L_1$.

From now on we assume that $M$ is of symmetric type.

Proposition 3.11. The following are equivalent:

(i) $V$ is a symmetric vertex of $M$.

(ii) $Z$ has symmetric type and $M$ is the Puig correspondent of $P(k_N)$.

(iii) $Z$ has symmetric type and if $B_0$ is a symmetric $V$-form on $Z$ then $M$ is a $B_0^G$-component of $Z^G$.

(iv) $M$ has a symmetric $G$-form $B$ such that $Z$ is a $B$-component of $M_V$. 
Proof. Assume (i). Then by Lemma 2.4.18 there is a symmetric $kV$-module $(Y, B_t)$ such that $Y$ is indecomposable and $M$ is a $B_t$-component of $Y^G$. Then $Y$ is a $V$-source of $M$. But $Z = nY$, for some $n \in N_G(V)$. So (iii) holds. Moreover, (i) and (ii) are equivalent, by Proposition 3.10.

Lemma 3.8 applies if (ii), (iii) or (iv) hold. We adopt its notation.

Assume (iii). Then $M$ has a $V$-projective $G$-form. So (i) is true. Choose $a \in \mu$ such that $a^n = a$. But $B_0^G$ is non-degenerate on $aZ^G \cong M$. Set $\overline{a} = \pi_\Delta(a)$. Then $\pi_\Delta B_0 \cong P(kN)$. Lemma 3.2 implies that $B_1$ is non-degenerate on a 1-dimensional subspace of $\pi_\Delta kN$. So there is a $\sigma_\Delta$-invariant primitive idempotent $\overline{\pi_e} \in \pi E(kN)[\overline{\pi}]$, by Proposition 2.9.

Now $\pi_\Delta$ restricts to a surjective map $aE_V(Z^G)a \rightarrow \pi E(kN)[\overline{\pi}]$. So by Lemma 2.1 there is a primitive $\sigma_\Delta$-invariant idempotent $d \in aE_V(Z^G)a$ such that $\pi_\Delta(d) = \overline{d}$. In particular $d \in \Delta$ and $d = ad\bar{a}$. Then $B_0^G$ is non-degenerate on the $V$-component $dZ^G$ of $aZ^G \cong M$. So (iv) holds.

Assume (iv). By Proposition 2.9 this means that there is $d \in \delta$ with $d^\sigma = d$. As $\pi_\delta(d)$ is a $\sigma$-invariant primitive idempotent in $E(P_\delta)$, $B_\sigma$ is a diagonal $N$-form on $P_\delta$. So $P_\delta \cong P(kN)$, using Lemma 3.11. Now $P(kN)$ is the only self-dual principal indecomposable $kN$-module which has multiplicity 1 in $kN$. So $P_\delta \cong P(kN)$, when regarded as a component of $P_\Delta$. So (ii) holds.

The principal 2-block $b_0(G)$ is the block containing $kG$.

\textbf{Corollary 3.12.} If $V$ is a symmetric vertex of $M$, then $M$ is in $b_0(G)$.

Proof. From Proposition 3.11 $M$ is the Puig correspondent of $P(kN)$. Set $C = VC_G(V)$. Then $\text{Res}^M_C(V, P(kN)) \cong P(kC/V)^m$, for some $m \geq 1$. So $(V, P(kC/V))$ is a root of $M$, in the terminology of ([9]). Now $P(kC/V)$ is in $b_0(C)$, and Brauer's Third Main Theorem implies that $(V, b_0(C))$ is a $b_0(G)$-subpair. So $M$ belongs to $b_0(G)$, according to [9].

\textbf{Proposition 3.13.} Suppose that $Z \cong Z^\ast$. Let $B$ be a symmetric $G$-form on $M$ such that $Z$ is not a $B$-component of $M_V$. Then

(i) There is $V \leq T \leq N_G(V, Z)$ with $[T : V] = 2$ such that $Z^T$ is a $B$-component of $M_T$.

(ii) Let $B_0$ be a symmetric $T$-form on $Z^T$ such that $Z$ is a $B_0$-component of $(Z^T)_V$. Then $M$ is a $B_0$-component of $Z^G$.

(iii) Either $V$ or $T$ is a symmetric vertex of $M$.

Proof. Lemma 3.8 applies, and we adopt its notation.

As $B$ is degenerate on each submodule of $M_V$ isomorphic to $Z$, $\sigma$ does not fix any idempotent in $\delta$. So $\sigma$ does not fix any primitive idempotent in $E(P_\delta)$, in view of Lemma 2.4.1 This means that $(P_\delta, B_\sigma)$ is a symplectic $kN$-module. Lemma 3.6 gives an involution $t \in N$ such that $B_t$ is non-degenerate on $P_\delta$. Moreover $\text{Res}^G_{(t)}(P_\delta, B_\sigma)$ has a component $(k(t), B_t)$. So there is a $\sigma$-invariant primitive idempotent $\overline{\pi_e} \in E_{(t)}(P_\delta)$.

Let $T \leq N$ with $T/V = \langle t \rangle$. Then $\pi_\delta \text{Res}^T_V : E_T(M) \rightarrow E_{(t)}(P_\delta)$ is surjective, by Lemma 3.9 (i). Lemma 2.4.1 gives a primitive $\sigma$-invariant idempotent $y \in E_T(M)$ with $\pi_\delta(y) = \overline{\pi_e}$. So $Y = E_{(t)}(P_\delta)$ is a $B$-direct summand of $M_T$. Now $Y$ is $V$-projective, $|T/V| = 2$ and $Z$ is a component of $Y_V$. So $Y \cong Z^T$. The conclusion of (i) follows.

Assume the hypothesis of (ii). Note that $\text{Ind}_V^G(Y) \cong \text{Ind}_V^G(Z)$. Let $\sigma_0$ be the adjoint of $B_0^G$ on $E(Z^G)$. Now $(E(P_\Delta), \sigma_0)$ is an involutory $N$-algebra. So $k_N \cong kN$, $P_\Delta$ is the regular $kN$-module and $\sigma_0$ is the adjoint of a symmetric $N$-form $B_{\sigma_0}$ on $P_\Delta$. By hypothesis on $B_0$, the $N$-form $B_{\sigma_0}$ is symplectic.
Let $e \in E_V(Z^G)$ be orthogonal projection onto $1 \otimes Y$. Then $\text{tr}^G_T(e) = 1_{Z^G}$ and $e \in \text{tr}^G_T(E_V(Z^G) \Delta E_V(Z^G))$. Set $\pi = \pi_\Delta(e)$. Then

$$1_{E(P_\Delta)} = \pi \text{tr}^G_T(e) = \text{tr}^N_T(\pi),$$

using Lemma 3.9(i).

As $\pi^{\sigma_0} = \pi$, Lemma 2.15 gives a $kN$-isometry $(P_\Delta, B_{\sigma_0}) \rightarrow \text{Ind}_N^G(\pi P_\Delta, B_\pi)$. But $\dim(\pi P_\Delta) = 2$. So this isometry is surjective, as both sides have dimension $|N|$. Now $B_{\sigma_0}$ is symplectic, and $\langle t \rangle$ is cyclic of order 2. So $(\pi P_\Delta, B_\pi) \cong (k\langle t \rangle, B_t)$. We deduce that $(P_\Delta, B_{\sigma_0}) \cong (kN, B_t)$.

Now $P_\delta$ is a $B_1$-component of $P_\Delta$. So there is a primitive $\sigma_0$-invariant idempotent $\pi \in E_N(P_\Delta)$ with $\pi P_\Delta \cong P_\delta$. Since $\pi_\Delta \text{res}^{G^0}_G : E_G(Z^G) \rightarrow E_N(P_\Delta)$ is surjective, idempotent lifting gives a primitive $\sigma_0$-invariant idempotent $a \in E_G(Z^G)$ with $\pi_\Delta(a) = \pi$. So $aZ^G$ is a $B_0^G$-direct summand of $Z^G$. But $aZ^G \cong M$. The conclusion of (ii) follows.

(iii) holds as $M$ has a $T$-projective symmetric $G$-form. □

**Proposition 3.14.** Suppose that $Z \not\cong Z^*$ and $B$ is any symmetric $G$-form on $M$. Then

(i) There is $V \subseteq T \subseteq N_\sigma^0(V, Z)$ with $|T : V| = 2$ such that $Z^T$ is a $B$-component of $M_T$. Then $N^* = N : (T/V)$.

(ii) Let $B_0$ be any symmetric $T$-form on $Z^T$. Then $M$ is a $B_0^G$-component of $Z^G$.

(iii) $T$ is a symmetric vertex of $M$.

**Proof.** Set $\Delta^T$ as the point of $E_G(Z^G)$ corresponding to $Z^*$. Lemma 2.6 implies that $\delta^T \not= \delta$. So $(E(P_\delta) \times E(P_\sigma), \sigma)$ is an involutory $N^\sigma$-algebra satisfying the hypothesis of Lemma 2.2(ii). This algebra is embedded in the $N^\sigma$-algebra $E(P_\Delta) \times E(P_\sigma)$ as follows. According to Lemma 3.9(ii), the restriction $\pi_{\Delta, \Delta^T} : E_G(Z^G) \rightarrow N^\sigma(E(P_\Delta) \times E(P_\sigma))$ is surjective. Let $a \in \mu$. So $aZ^G \cong M$. Then $\pi : = \pi_{\Delta, \Delta^T}(a)$ is a primitive idempotent in $N^\sigma(E(P_\Delta) \times E(P_\sigma))$. We identify $E(P_\delta) \times E(P_\sigma)$ with $\pi E(P_\Delta) \times E(P_\sigma)$, and $P_\delta + P_\sigma$ with $\pi(P_\Delta + P_\sigma)$.

By Lemma 2.6 there is a commutative diagram

$$
\begin{array}{c}
1 \longrightarrow O \xrightarrow{\text{inc}} H \xrightarrow{\theta} N^* \longrightarrow 1 \\
\downarrow \eta \quad \downarrow \chi \quad \quad \downarrow \rho \\
1 \longrightarrow K \xrightarrow{\text{inc}} \text{Sp}(P_\delta, P_\sigma) \xrightarrow{\rho} \text{PGL}(P_\delta, P_\sigma, \sigma) \longrightarrow 1
\end{array}
$$

where $O$ is a finite cyclic group of odd order and $\theta(C_H(O)) = N$. Each element of $H(C_H(O))$ maps $P_\delta$ onto $P_\sigma$. Moreover $\sigma$ is the adjoint of a symplectic $H$-form $B_\sigma$ on $P_\delta + P_\sigma$, with

$$
(P_\delta)_{1} = P_\delta \quad \text{and} \quad (P_\sigma)_{1} = P_\sigma.
$$

Now $e_\eta = \frac{1}{|O|} \sum_{\lambda \in O} \eta(\lambda^{-1})\lambda$ is a central idempotent in $kH$ such that $P_\Delta + P_\Delta^\sigma \cong kHe_\eta$ as $kH$-modules. So $E_{N^\sigma}(P_\Delta + P_\Delta^\sigma) \cong e_\eta kHe_\eta$.

By Lemma 3.6 there is an involution $t \in H$ such $B_t$ is non-degenerate on $P_\delta + P_\sigma$ and $k(t)$ is a $B_\sigma$-component of $(P_\delta + P_\sigma)_{t}$. This means that there is $p \in P_\delta + P_\sigma$ such that $B_\sigma(p, tp) \neq 0$.

Write $p = p_1 + p_2$ where $p_1 \in P_\delta$ and $p_2 \in P_\sigma$.

We claim that $t \notin C_H(O)$. Otherwise, $tp_1 \in P_\delta$ and $tp_2 \in P_\sigma$. Then

$$B_\sigma(p, tp) = B_\sigma(p_1, tp_2) + B_\sigma(p_2, tp_1), \quad \text{by } 3.1
$$

$$= B_\sigma(p_1, tp_2) + B_\sigma(tp_1, p_2), \quad \text{as } B_\sigma \text{ is symmetric}
$$

$$= B_\sigma(p_1, p_2), \quad \text{as } t^\sigma = t.
$$

This contradicts our choice of $p$ and thus establishes our claim.

Now $tp_1 \in P_\sigma$ and $tp_2 \in P_\delta$. So $B_\sigma(p, tp) = B_\sigma(p_1, tp_1) + B_\sigma(p_2, tp_2)$. Replace $p$ by $p_1$ or $p_2$ so that $p \in P_\delta$ and $B_\sigma(p, tp) \neq 0$. Then replace $p$ by $\sqrt{B_\sigma(p, tp)}^{-1} p$, so that $B_\sigma(p, tp) = 1$. 


Define $\overline{\theta} \in E_{(t)}(P_3 + P_{3\sigma})$ by $\overline{\theta}(x) = B(x, tp)p + B(x, p)tp$, for all $x \in P_3 + P_{3\sigma}$. Then $\overline{\theta}$ is orthogonal projection onto $kp + ktp$. Moreover, $\overline{\theta}P_3 \subseteq P_3$ and $\overline{\theta}P_{3\sigma} \subseteq P_{3\sigma}$. So $\overline{\theta}$ is a $\sigma$-invariant primitive idempotent in $\langle t \rangle (E(P_3) \times E(P_{3\sigma}))$.

Let $T \geq V$ such that $T/V = (\theta(t))$. Then $N_G(V, Z) = N_G(V, Z)T$, $N_G(V, Z) \cap T = V$ and $T/V$ is a complement to $N$ in $N^*$. Now by Lemma 3.9(ii) the restriction $\pi_{\delta, \delta^*} : E_T(M) \to \langle t \rangle (E(P_3) \times E(P_{3\sigma}))$ is surjective. So by Lemma 2.1 there is a $\sigma$-invariant primitive idempotent $y \in E_T(M)$ such that $\pi_{\delta, \delta^*}(y) = \overline{\theta}$. Then $yM$ is a $B$-direct summand of $MT$ which lies over $Z$ and $Z^*$. But $|T/V| = 2$. So $yM \cong ZT$.

Let $B_0$ be any symplectic $T$-form on $Z^T$. Identifying $\text{Ind}_T^G(Z^T)$ with $\text{Ind}_V^G(Z)$, we regard $B_0^G$ as a symplectic $G$-form on $Z^G$. Let $\sigma_0$ be the adjoint of $B_0^G$ on $E(Z^G)$. Then $(E(P_\Delta) \times E(P_{\Delta^*})), \sigma_0)$ is an involutive $N^*$-algebra satisfying the hypothesis of Lemma 2.2(ii). So $\sigma_0$ is the adjoint of a symmetric $N^*$-form $B_{\delta_0}$ on $P_\Delta + P_{\Delta^*}$ such that $(P_{\Delta^*})^\perp = P_\Delta$ and $(P_{\Delta^*})^\perp = P_{E_G}$.

Let $e \in E_T(Z^G)$ be orthogonal projection onto $1 \otimes Z^T$. Then $\text{tr}_{E_G}^G(e) = 1_{Z^G}$ and $e \in \text{tr}_{E_T}^G(E_T(Z^G)\Delta E_T(Z^G))$. Set $\pi = \pi_{\Delta, \Delta^*}(e)$, a primitive idempotent in $\langle t \rangle (E(P_\Delta) \times E(P_{\Delta^*}))$. As $O$ acts trivially on $E(P_\Delta) \times E(P_{\Delta^*})$, we have

$$1_{P_\Delta + P_{\Delta^*}} = \pi_{\Delta, \Delta^*}\cdot \text{tr}_{E_G}^G(e) = \text{tr}_{\theta(t)}^G(\overline{\theta}), \text{ by Lemma 3.9(ii)}$$

$$= \text{tr}_1^G(\overline{\theta}), \text{ as } N^* = N : (\theta(t)).$$

$$= \text{tr}_C^G(O)) = \text{tr}_O^C(\overline{\theta}), \text{ as } |O| \text{ is odd.}$$

$$= \text{tr}_{E_T}^G(\overline{\theta}), \text{ as } H = C_H(O) : (t).$$

Now $(\overline{\pi}(P_\Delta + P_{\Delta^*}), B_{\sigma_0}) \cong (k(t), b_1)$ as symmetric $k(t)$-modules. It then follows from Proposition 2.17 that $(P_\Delta + P_{\Delta^*}, B_{\sigma_0}) \cong (kH_{\sigma_0}, b_1)$.

As $B_1$ is non-degenerate on $P_\Delta + P_{\Delta^*}$, we may choose $\overline{\pi}$ so that $\sigma_0$ is non-degenerate on $\overline{\pi}(P_\Delta + P_{\Delta^*})$. Since $\overline{\pi}$ belongs to the $\sigma_0$-invariant semi-simple subalgebra $E(P_\Delta) \times E(P_{\Delta^*})$ of $E(P_\Delta + P_{\Delta^*})$, it follows from Lemma 2.11 that the orthogonal projection $\pi$ onto $\overline{\pi}(P_\Delta + P_{\Delta^*})$ belongs to $N'(E(P_\Delta) \times E(P_{\Delta^*}))$.

Lemma 3.9(ii) states that $\pi_{\Delta, \Delta^*} : E_G(Z^G) \to N'(E(P_\Delta) \times E(P_{\Delta^*}))$ is surjective. So by Lemma 2.1 there is a $\sigma_0$-invariant primitive idempotent $b \in E_G(Z^G)$ such that $\pi_{\Delta, \Delta^*}(b) = \overline{\pi}$. Then $bZ^G$ is a $B_0^G$-direct summand of $Z^G$ isomorphic to $M$. This completes the proof of (i) and (ii).

(iii) holds as $M$ has a $T$-projective symmetric $G$-form, but $V$ is not a symmetric vertex of $M$.

3.3. Proofs of the main theorems

In this section $M$ is an indecomposable $kG$-module of symmetric type, with a symmetric vertex $T$, a vertex $V \leq T$ and a $V$-source $Z$.

Proof Proof of Theorem 12.1 The first statement and the ‘if’ implication holds by Proposition 2.17. The ‘only if’ holds if $T$ is a vertex of $M$. So we assume from now on that $T$ is not a vertex of $M$.

Suppose that there is a $kG$-isometry $(M, B_1) \to (L, B_1)$, where $H \leq G$ and $(L, B_1)$ is a symmetric $kH$-module. Let $(V, Z)$ be a vertex-source pair of $M$. Then according to Propositions 3.13 and 3.14 there is $V \leq S \leq N_G(V, Z)$ with $[S : V] = 2$, a $kS$-isometry $(Z^S, B_0) \to \text{Res}^G_S(M, B_1)$ and a $kG$-isometry $(M, B_2) \to \text{Ind}^G_S(Z^S, B_0)$, where $B_0$ is a symmetric $S$-form on $Z^S$ and $B_2$ is a symmetric $G$-form on $M$.

Composing the 3 isometries of the previous paragraph produces a $kG$-isometry $(M, B_2) \to \text{Ind}^G_S \text{Res}^G_S \text{Ind}^G_H(L, B_1)$. Now Lemma 2.12 gives

$$\text{Ind}^G_S \text{Res}^G_S \text{Ind}^G_H(L, B_1) \cong \bigoplus_{g \in S \cap G/H} \text{Ind}^G_{S \cap G/H} \text{Res}^H_{S \cap G/H}(gL, gB_1).$$
So by Lemma 1.15 there is a $kG$-isometry
\[(M, B_3) \to \text{Ind}_{S \cap H}^G \text{Res}_{S \cap H}^H (\varrho L, \varrho B_1),\]
for some $g \in G$ and some symmetric $G$-form $B_3$ on $M$. So $B_3$ is $S \cap H$-projective. But $S$ is a symmetric vertex of $M$. It follows that $S \leq \varrho H$.

Choosing $H = T$, the work above shows that $S = \varrho T$. Then taking $H$ to be any subgroup of $G$, we get $T \leq_G H$. \hfill \square

Here is a precise statement and proof of Theorem 1.3:

**Theorem 3.15.** If $M$ is self-dual and irreducible then its symmetric vertices are determined up to $G$-conjugacy.

Let $T$ be a symmetric vertex of $M$. Then $Z^T$ is a $B$-component of $M_T$ and for every symmetric $T$-form $B_0$ on $Z^T$, there is a $kG$-isometry $(M, B) \to \text{Ind}_T^G (Z^T, B_0)$.

**Proof.** Let $S$ be any symmetric vertex of $M$. As $B$ is the unique $G$-form on $M$, $B$ is both $T$ and $S$-projective. Then by Theorem 1.2, $T \leq_G S$ and $S \leq_G T$. So $T = G S$. This proves that there is one $G$-conjugacy class of symmetric vertices of $M$.

The other conclusions now follow from Propositions 3.11, 3.13 and 3.14. \hfill \square

Note that in case $T = V$ is a vertex of $M$, Proposition 3.11 implies that the defect multiplicity module $P_3$ of $E(M)$ is $P(k_N)$. But $P_3$ is an irreducible projective $kN$-module, by a well-known theorem of R. Knörr. This forces $P_3 = k_N$. So $V$ is a Sylow 2-subgroup of $N_G(V, Z)$.

We now turn to the blocks of $kG$. Recall that $kG$ is a left $kG \times G$-module via $(g_1, g_2)x := g_1 x g_2^{-1}$, for all $(g_1, g_2) \in G \times G$ and $x \in kG$. Clearly the elements of $G$ form a transitive $G \times G$-set under this action and the stabilizer of 1 is $\Delta G$. So $kG \cong \text{Ind}_{\Delta G}^{G \times G} (k_{\Delta G})$.

Let $C$ be a conjugacy class of $G$ and set $C^+ := \sum_{c \in C} c$ in $kG$. Then the $C^+$ form a basis for $Z(kG) \cong E_{G \times G}(kG)$; the corresponding $kG \times G$-endomorphism of $kG$ is $\ell(C^+) = r(C^+)$. It is easy to show that $r(C^+) = \text{tr}^{G \times G}_{\Delta G} (c \otimes 1)$ in $E(kG)$, for each $c \in C$. So $r(C^+) = \text{tr}^{G \times G}_{\Delta G} (c \otimes 1)$, where $Q$ is a Sylow 2-subgroup of $C_G(c)$. We call $Q$ a defect group of $c$; the $G$-conjugates of $Q$ are the defect groups of $C$. Set $C^\circ = \{c^{-1} | c \in C\}$ as the inverse conjugacy class of $C$.

Let $e$ be a primitive idempotent in $Z(kG)$, also called a block idempotent of $kG$. Then $ekG$ is an indecomposable $G \times G$-direct summand of $kG$, and a block of $kG$. Now $ekG$ has vertex $\Delta D$ as $kG \times G$-module, where $D \leq G$ is minimal subject to $e \in \text{tr}^{G \times G}_{\Delta D} (E_{\Delta D}(kG))$. J. A. Green [7] showed that each $D$ is a defect group of the block $ekG$, in the sense of [1]. Now $e = \sum_{g \in G} B_1(e, g) g$. If $B_1(e, g) \neq 0$, it is known that $g$ has 2-order and $D$ contains a conjugate of a defect group of $g$, and there is $c \in G$ such that $B_1(e, c) \neq 0$, and $D$ is a defect group of $c$.

Recall that $g \in G$ is said to be real (in $G$) if $g^2 = g^{-1}$, for some $t \in G$. Then $C_G^o (g) = C_G (g) \langle t \rangle$ is a subgroup of $G$, called the extended centralizer of $g$ in $G$. The Sylow 2-subgroups of $C_G^o (g)$ are called extended defect groups of $g$. So conjugacy classes have extended defect groups.

Now the block $ekG$ is said to be real if $e^o = e$. Then $B_1(e, g) = B_1(e, g^{-1})$, for all $g \in G$. R. Gow defined the extended defect groups of a real 2-block $ekG$ in [3]. This is a $G$-conjugacy class of 2-subgroups of $G$. If $ekG$ is the principal 2-block of $G$ then its extended defect groups are just its defect groups. For any other real block with a defect group $D$, there is an extended defect group $E \supseteq D$ with $[E : D] = 2$.

The author has shown that if $B_1(e, g) \neq 0$ and $g$ is real with extended defect group $R$, then there is $s \in G$ such that $R \leq \langle E \rangle$ and $R \cap D$ is a defect group of $g$. Moreover, there is $c \in G$ such that $B_1(e, c) \neq 0$, $c$ is real, $E$ is an extended defect group of $c$ and $D$ is a defect group of $c$. 
THEOREM 3.16. Let $e$ be a real 2-block of $kG$ and let $E$ be an extended defect group of $e$ in $G$. Set $B_e$ as the restriction of $B_1$ from $kG$ to $ekG$. Then $B_e$ is $\Delta E$-projective and $\Delta E$ is a symmetric vertex of $ekG$.

Proof. Write $e = \sum_{i=1}^n B_1(e, c_i)(C_i \cup C_i^\sigma)^+$, where the $C_i$ are distinct conjugacy classes of $G$, $c_i \in C_i$ and $B_1(e, c_i) \neq 0$, for all $i$. Let $D \leq E$ be a defect group of $ekG$ and let $E_i$ be an extended defect group of $c_i$. We choose the $C_i$ so that $D_i := D \cap E_i$ is a defect group of $c_i$, and also $E = DE_i$, if $c_i$ is real. As $c_i$ is 2-regular, $b_i^2 = c_i$ for $b_i \in \langle c_i \rangle$. Then $b_i$ has defect group $D_i$ and extended defect group $E_i$.

If $C_i \neq C_i^o$, set $\alpha_i = tr^{\Delta E}_{\Delta D_i}(b_i \otimes b_i^{-1} \otimes b_i)$. If $C_i = C_i^o$, set $\alpha_i = tr^{\Delta E}_{\Delta D_i}(b_i \otimes b_i^{-1})$. In both cases $\alpha_i = \alpha_i^o$ and $tr^{G \times G}_{\Delta \theta}(\alpha_i) = (C_i \cup C_i^\sigma)^+$. So

$$e = tr^{G \times G}_{\Delta \theta}(\sigma),$$

where $\theta = (\sum_{i=1}^n B_1(e, c_i)\alpha_i \otimes 1)$. Then $e = tr^{G \times G}_{\Delta \theta}(e \theta e)$ and $e \theta e$ is $\sigma$-invariant. As $E(ekG) = eE(kG)e$, we deduce that $B_e$ is $\Delta E$-projective.

If $ekG$ is the principal 2-block of $kG$, then $\Delta E = \Delta D$ is a vertex of $ekG$, as $kG \times G$-module. So $\Delta E$ is a symmetric vertex of $ekG$. If $ekG$ is not the principal 2-block of $kG$, then $ekG$ belongs to a non-principal 2-block of $G \times G$. So $\Delta D$ is not a symmetric vertex of $ekG$, by Corollary 3.12. As $|\Delta E : \Delta D| = 2$ and $ekG$ has a $\Delta E$-projective symmetric $G \times G$-form, it follows that $\Delta E$ is a symmetric vertex of $ekG$ in this case.

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