EIGENFUNCTIONS OF MACDONALD’S $q$-DIFFERENCE OPERATOR FOR THE ROOT SYSTEM OF TYPE $C_n$

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Abstract. We construct an integral representation of eigenfunctions for Macdonald’s $q$-difference operator associated with the root system of type $C_n$. It is given in terms of a restriction of a $q$-Jordan-Pochhammer integral. Choosing a suitable cycle of the integral, we obtain an integral representation of a special case of the Macdonald polynomial for the root system of type $C_n$.

1. Introduction

Macdonald introduced the $q$-difference operators \( [5] \) to define his orthogonal polynomials associated with root systems. In the case of a root system of type $C_n$, his $q$-difference operator is given by

\[
E = \sum_{a_1, \ldots, a_n = \pm 1} \prod_{1 \leq i < j \leq n} \frac{1 - ty_i y_j}{1 - y_i y_j} \prod_{1 \leq i \leq n} \frac{1 - ty_i^{2i}}{1 - y_i^{2i}} T_{yi}^{a_i} \frac{1}{a_i},
\]

where \((T_y f)(y_1, \ldots, y_n) = f(y_1, \ldots, qy_i, \ldots, y_n)\).

The present paper is devoted to study the eigenvalue problem associated with this operator \( E \). In particular, we construct an integral representation, which is given by a restriction of a $q$-Jordan-Pochhammer integral, of eigenfunctions in some special cases. It turns out that, taking a suitable cycle, such an integral expresses the Macdonald polynomial of $C_n$ type parametrized by the partition \((\lambda, 0, \ldots, 0)\). This representation leads to a more explicit expression.

Here we recall the definition of the Macdonald polynomial \( P_\mu(y|q,t) \) associated with the root system of type $C_n$. It is the eigenfunction of $E$ with respect to the eigenvalue

\[
c_\mu = q^{-\frac{1}{2}(\mu_1 + \cdots + \mu_n)} \prod_{i=1}^n (1 + q^{\mu_i} t^{n-i+1})
\]

of the form

\[
P_\mu(y|q,t) = m_\mu + \sum_{\nu < \mu} a_\mu \nu \nu \nu,
\]

where \( \mu = (\mu_1, \ldots, \mu_n) \) is a partition, a sequence of non-negative integers in decreasing order, \( m_\mu = \sum_{\nu \in W(C_n) \mu} e^\nu \) with \( W(C_n) \) the Weyl group of type $C_n$ and \( \nu < \mu \) is defined to be \( \mu - \nu \in Q^+ \) with \( Q^+ \) the positive cone of the root lattice.

Besides the $A_{n-1}$ case, the solution of the eigenvalue problem for the Macdonald operator is not well studied (See \([9, 10, 11]\) and \([7]\)). We expect that this paper

1991 Mathematics Subject Classification. Primary 33D45; Secondary 33D55, 33D70, 33D80.

Key words and phrases. Macdonald’s $q$-difference operator, Macdonald polynomials, $q$-Jordan-Pochhammer integral.
For related works on references therein. It is noteworthy that even in the classical \((q=1)\) case was not previously known that such an integral gives spherical functions associated with the root system \(BC_n\). For related works on \(BC_n\) type spherical functions, we refer the reader to [3] and references therein.

Throughout this paper, \(q\) is regarded as a real number satisfying \(0 < q < 1\), and \(t = q^k\) where \(k \in \mathbb{Z}_{\geq 1}\).

2. A RESTRICTION OF A \(q\)-JORDAN-Pochhammer INTEGRAL

Let us introduce a 1-form
\[
\Phi = x^\lambda \prod_{1 \leq j \leq n} \frac{(ty_j/x; q)_\infty (ty_j^{-1}/x; q)_\infty}{(y_j/x; q)_\infty (y_j^{-1}/x; q)_\infty} \frac{dx}{x},
\]
where \(\lambda \in \mathbb{Z}_{\geq 0}, (a; q)_\infty = \prod_{i \geq 0} (1 - aq^i)\) and \((a; q)_m = (a; q)_\infty / (q^m a; q)_\infty\). This can be regarded as a 1-form corresponding to a restriction of a \(q\)-Jordan-Pochhammer integral
\[
x^\lambda \prod_{1 \leq j \leq 2n} \frac{(ty_j/x; q)_\infty}{(y_j/x; q)_\infty} \frac{dx}{x},
\]
which is studied in [3] and [11].

Our first result is the following:

**Theorem 1.** For any cycle \(C\), the function \(\int_C \Phi\) satisfies the equation
\[
E \int_C \Phi = c(\lambda, 0, \ldots, 0) \int_C \Phi.
\]

This implies that linearly independent solutions are obtained by choosing several cycles. Indeed, if we put \(C_i^{(+)}\) (or \(C_i^{(-)}\)) for each \(i = 1, \ldots, n\) to be a path with the counterclockwise direction so that the poles at \(w = y_i, y_i q, \ldots, y_i q^{k-1}\) (or \(w = y_i^{-1}, y_i^{-1} q, \ldots, y_i^{-1} q^{k-1}\), respectively) are inside the path and other poles from \(\Phi\) are outside, we have the following rational solutions:

\[
\frac{1}{2\pi \sqrt{-1}} \int_{C_i^{(+)}} \Phi = y_i^\lambda \frac{1}{(q; q)_k \prod_{1 \leq j \leq n} (y_j/y_i; q)_k} \prod_{1 \leq j \leq n} (y_j^{-1} y_i^{-1}; q)_k \times \sum_{l=0}^{k-1} \prod_{j=1}^n \frac{(t^{-1} q y_j/y_i; q)_l (t^{-1} q y_i y_j; q)_l (t^{2n} q^\lambda)_l}{(q y_j/y_i; q)_l (q y_i y_j; q)_l (t^{2n} q^\lambda)_l} (2.2)
\]

and

\[
\frac{1}{2\pi \sqrt{-1}} \int_{C_i^{(-)}} \Phi = y_i^{-\lambda} \frac{1}{(q; q)_k \prod_{1 \leq j \leq n} (y_i/y_j; q)_k} \prod_{1 \leq j \leq n} (y_j y_i; q)_k \times \sum_{l=0}^{k-1} \prod_{j=1}^n \frac{(t^{-1} q y_j/y_i; q)_l (t^{-1} q y_i y_j; q)_l (t^{2n} q^\lambda)_l}{(q y_j/y_i; q)_l (q y_i y_j; q)_l (t^{2n} q^\lambda)_l} (2.3)
\]
The calculation is carried out by means of the residue calculus.

Since \( \lambda \) is a non-negative integer, the sum of the pathes \( \sum_{i=1}^{n} C_i^+ + \sum_{i=1}^{n} C_i^- \) is homologous to a path \( C \) which circles the origin in the positive sense so that all poles from \( \Phi \) are inside the path. The integral on this cycle \( C \) gives the Macdonald polynomial \( P_{(\lambda,0,...,0)}(y|q,t) \).

**Theorem 2.** If the cycle \( C \) is that above, we have

\[
\frac{1}{2\pi i} \oint_C \Phi = \frac{(t;q)_\lambda}{(q;q)_\lambda} P_{(\lambda,0,...,0)}(y|q,t). \tag{2.4}
\]

Moreover, applying the \( q \)-binomial theorem

\[
\sum_{i \geq 0} \frac{(a;q)_i}{(q;q)_i} z^i = \frac{(az;q)_\infty}{(z;q)_\infty} (|z| < 1)
\]

with the residue calculus to our integral, we obtain an exact expression of \( P_{(\lambda,0,...,0)}(y|q,t) \).

**Corollary**.

\[
P_{(\lambda,0,...,0)}(y|q,t) = \frac{(q;q)_\lambda}{(t;q)_\lambda} \sum_{\sum_{i \in I} a_i \pm 1} \prod_{1 \leq i < j \leq n} \frac{1 - t y_i^{a_i} y_j^{a_j}}{1 - y_i^{a_i} y_j^{a_j}} \prod_{1 \leq i \leq n} \frac{1 - t y_i^{2a_i}}{1 - y_i^{2a_i}} = \prod_{i=1}^{n} (1 + t^i). \tag{2.2}
\]

From the formula by Macdonald [4] about the Poincaré series of Coxeter systems, we have

\[
\sum_{w \in W(C_n)} w \left\{ \prod_{1 \leq i < j \leq n} \frac{1 - t y_i y_j}{1 - y_i y_j} \prod_{1 \leq i \leq n} \frac{1 - t y_i}{1 - y_i} \right\} = \prod_{i=1}^{n} \frac{(1 - t^{2i})}{(1 - t)^n} \tag{3.1}
\]

and

\[
\sum_{w \in W(A_{n-1})} w \left\{ \prod_{1 \leq i < j \leq n} \frac{1 - t y_i/y_j}{1 - y_i/y_j} \right\} = \prod_{i=1}^{n} \frac{(1 - t^i)}{(1 - t)^n}. \tag{3.2}
\]

Here \( W(C_n) \) or \( W(A_{n-1}) \) denotes the Weyl group of the root system of type \( C_n \) or \( A_{n-1} \), respectively. By applying the formula (3.2) to (3.1), we obtain

\[
\sum_{w \in W(C_n)} w \left\{ \prod_{1 \leq i < j \leq n} \frac{1 - t y_i y_j}{1 - y_i y_j} \prod_{1 \leq i \leq n} \frac{1 - t y_i}{1 - y_i} \right\} = \prod_{i=1}^{n} \frac{(1 - t^i)}{(1 - t)^n} \sum_{a_1,...,a_n = \pm 1} \prod_{1 \leq i \leq n} \frac{1 - t y_i^{a_i} y_j^{a_j}}{1 - y_i^{a_i} y_j^{a_j}} \prod_{1 \leq i \leq n} \frac{1 - t y_i^{2a_i}}{1 - y_i^{2a_i}}.
\]

Hence we derive the desired relation. \qed
Firstly, let us take the residue of the left-hand side of (3.3) at $x = t^i$:

$$\prod_{a_1, \ldots, a_n = \pm 1} 1 - ty_i^{a_i} y_j^{a_j} \prod_{1 \leq i \leq n} \frac{1 - ty_i^{2a_i} y_i^{-1}/x}{1 - y_i^{a_i} y_i^{-a_i}/x}$$

$$= \prod_{i=1}^{n-1} (1 + t^i) \left\{ 1 + t^n \prod_{i=1}^{n} \frac{(1 - ty_i/x)(1 - y_i^{-1}/x)}{(1 - ty_i/x)(1 - y_i^{-1}/x)} \right\}.$$

**Proof.** We prove the desired equality by means of partial fraction decompositions. Firstly, let us take the residue of the left-hand side of (3.3) at $x = ty_1$:

$$\text{Res}_{x=ty_1} \left\{ \sum_{a_1, \ldots, a_n = \pm 1} \prod_{1 \leq i < j \leq n} \frac{1 - ty_i^{a_i} y_j^{a_j}}{1 - y_i^{a_i} y_j^{a_j}} \prod_{1 \leq i \leq n} \frac{(1 - ty_i^{2a_i})(1 - ty_i^{2a_i}/x)}{(1 - y_i^{a_i} y_i^{-a_i})(1 - ty_i^{a_i}/x)} \right\} dx$$

$$= \frac{(1 - t^{-1})(1 - ty_1^2)}{1 - ty_1} \prod_{2 \leq i < j \leq n} \frac{1 - ty_1 y_j^{a_j}}{1 - y_j^{a_j}} \prod_{2 \leq i \leq n} \frac{(1 - ty_1^{2a_i})(1 - ty_1^{2a_i}/y_1)}{(1 - y_1^{a_i} y_1^{-a_i})(1 - ty_1^{a_i}/y_1)}$$

$$= t^{1-n} \left(1 - t^{-1})(1 - ty_1^2) \right) \prod_{2 \leq j \leq n} \frac{1 - ty_1 y_j^{a_j}}{1 - y_j^{a_j}} \prod_{2 \leq i \leq n} \frac{1 - ty_1^{2a_i}}{1 - y_1^{a_i}}.$$

This is equal to

$$t^{1-n} \left(1 - t^{-1})(1 - ty_1^2) \right) \prod_{i=1}^{n-1} (1 + t^i) \prod_{2 \leq j \leq n} \frac{(1 - ty_1 y_j)(1 - y_1 y_j)}{(1 - y_1 y_j)(1 - y_1 y_j)}.$$

from Lemma 1.

Secondly, by noticing that

$$\text{Res}_{x=ty_1} \prod_{1 \leq i \leq n} \frac{(1 - y_i/x)(1 - y_i^{-1}/x)}{(1 - ty_i/x)(1 - ty_i^{-1}/x)} dx$$

$$= t^{1-2n} \left(1 - t^{-1})(1 - ty_1^2) \right) \prod_{2 \leq j \leq n} \frac{(1 - ty_1 y_j)(1 - y_1 y_j)}{(1 - y_1 y_j)(1 - y_1 y_j)},$$

we know that the residue of the right-hand side of (3.3) at $x = ty_1$ is equal to (3.4). Hence, the symmetry of (3.3) with respect to the variables $y_1^{\pm 1}, \ldots, y_n^{\pm 1}$ leads to the fact that the residues of both sides of (3.3) at each $x = ty_i$ ($i = 1, \ldots, n$) or $x = ty_i^{-1}$ ($i = 1, \ldots, n$) are equal.

On the other hand, if $x$ goes to $\infty$, the left-hand side of (3.3) tends to

$$\sum_{a_1, \ldots, a_n = \pm 1} \prod_{1 \leq i < j \leq n} \frac{1 - ty_i^{a_i} y_j^{a_j}}{1 - y_i^{a_i} y_j^{a_j}} \prod_{1 \leq i \leq n} \frac{1 - ty_i^{2a_i}}{1 - y_i^{a_i} y_i^{-a_i}},$$

which is equal to $\prod_{i=1}^{n} (1 + t^i)$ by Lemma 1, and the right-hand side of (3.3) tends also to $\prod_{i=1}^{n} (1 + t^i)$. This completes the proof of Lemma 2.

Let us proceed to prove our Theorem 1.
Here, to derive the third equality, we have used the relation
\[ \prod_{1 \leq i \leq n} T_{y_i}^\frac{\lambda}{x} \int \Phi = \int_C x^\lambda \prod_{i=1}^n \left( \frac{q^{\frac{\lambda}{2} t y_i}}{x} \right)_\infty \left( \frac{q^{-\frac{\lambda}{2} t y_i^{-1}}}{x} \right)_\infty \frac{dx}{x} \]
\[ = q^{-\frac{\lambda}{2}} \int_C x^\lambda \prod_{i=1}^n \left( \frac{q^{\frac{\lambda}{2} (1-a_i) t y_i}}{x} \right)_\infty \left( \frac{q^{-\frac{\lambda}{2} (1-a_i)} y_i^{-1}}{x} \right)_\infty \frac{dx}{x} \]
\[ = q^{-\frac{\lambda}{2}} \int_C \prod_{i=1}^n \frac{1 - y_i^{n_i}/x}{1 - t y_i^{n_i}/x} \Phi, \]
where the second equality is given by the change of integration variable such that \( x \mapsto q^{-\frac{\lambda}{2}} x \).

Therefore, by using Lemma 2, we obtain
\[ E \int_C \Phi = q^{-\frac{\lambda}{2}} \int_C \left\{ \sum_{a_1, \ldots, a_n = 1} \prod_{1 \leq i < j \leq n} \frac{1 - t y_i^{a_i} y_j^{a_j}}{1 - y_i^{a_i} y_j^{a_j}} \prod_{1 \leq i \leq n} \frac{(1 - t y_i^{n_i})(1 - y_i^{a_i}/x)}{(1 - t y_i^{n_i})(1 - y_i^{n_i}/x)} \right\} \Phi \]
\[ = q^{-\frac{\lambda}{2}} \prod_{i=1}^{n-1} (1 + t^i) \int_C \left\{ 1 + t^n \prod_{i=1}^n \frac{(1 - y_i/x)(1 - y_i^{-1}/x)}{(1 - t y_i/x)(1 - t y_i^{-1}/x)} \right\} \Phi \]
\[ = (q^{-\frac{\lambda}{2}} + q^{\frac{\lambda}{2}} t^n) \prod_{i=1}^{n-1} (1 + t^i) \int_C \Phi. \]

Here, to derive the third equality, we have used the relation
\[ \int_C \Phi = q^{-\lambda} \int_C \prod_{i=1}^n \frac{(1 - y_i/x)(1 - y_i^{-1}/x)}{(1 - t y_i/x)(1 - t y_i^{-1}/x)} \Phi, \]
which is given by the change of integration variable such that \( x \mapsto q^{-1} x \).

This completes the proof of Theorem 1.

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