On the intersection of homoclinic classes in intransitive sectional-Anosov flows

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Abstract

We show that if $X$ is a Venice mask (i.e. nontransitive sectional-Anosov flow with dense periodic orbits, [9], [25], [24], [18]) supported on a compact 3-manifold, then the omega-limit set of every non-recurrent point in the unstable manifold of some singularity is a closed orbit. In addition, we prove that the intersection of two different homoclinic classes in the maximal invariant set of a sectional-Anosov flow can be decomposed as the disjoint union of, singular points, a non-singular hyperbolic set, and regular points whose alpha-limit set and omega-limit set is formed by singular points or hyperbolic sets.

1 Introduction

The dynamical systems theory is interested to describes the behavior as time goes to infinity for the majority of orbits in a determinated system. An important tool for hyperbolic sets is the known connecting lemma [15], [2], [10]. Specifically, the lemma says that if $X$ is an Anosov flow on a compact manifold $M$ and $p, q \in M$ satisfy that for all $\varepsilon > 0$ there is a trajectory from a point $\varepsilon$-close to $p$ to a point $\varepsilon$-close to $q$, then there is a point $x \in M$ such that $\alpha_X(x) = \alpha_X(p)$ and $\omega_X(x) = \omega_X(q)$.

In [7] was proved a similar result for sectional-Anosov flows, which is known as sectional-connecting lemma. Recall, the sectional hyperbolic sets and sectional Anosov flows were introduced in [21] and [19] respectively as a generalization of the hyperbolic sets and Anosov flows to include important examples such as the saddle-type hyperbolic attracting sets, the geometric and multidimensional Lorenz attractors [1], [11], [14] and certain robustly transitive sets. A fundamental hypothesis in the sectional-hyperbolic case consists in the alpha-limit set of $p \in$
M(X) to be non-singular. As the unstable manifold of every singularity σ of a sectional-Anosov X is contained in the maximal invariant set M(X), would be interesting to know what is the omega-limit set of a point in W^u_X(σ). In fact, it can be seen as an extension of the sectional-connecting lemma.

On the other hand, the class of Venice masks (i.e. intransitive sectional-Anosov flows with dense periodic orbits) has a particular interest since its existence shows that the spectral decomposition theorem \[29\] is not valid in the sectional-hyperbolic case. Its study has been collected by different authors during the last years. The examples exhibited in \[9, 18, 25\] are characterized because the maximal invariant set can be decomposed as the disjoint finite union of homoclinic classes. In addition, the intersection between two different homoclinic classes is contained in the closure of the unstable manifold of the singularities. Specifically, this intersection can be decomposed as the disjoint union of, a singularity σ, a closed orbit C, and regular points such that its alpha-limit set is σ and the omega-limit set is C. Particularly, was proved in \[25, 24\] that every Venice mask with a unique singularity has these properties.

In search of properties which allow to characterized the dynamic of Venice masks, will be studied the behavior of homoclinic classes and its relation with the unstable manifolds of the singularities.

Let us state our results in a more precise way.

Consider a Riemannian compact manifold M of dimension n (a compact n-manifold for short). M is endowed with a Riemannian metric \(\langle \cdot, \cdot \rangle\) and an induced norm \(\| \cdot \|\). We denote by \(\partial M\) the boundary of M. Let \(\mathcal{X}^1(M)\) be the space of \(C^1\) vector fields in M endowed with the \(C^1\) topology. Fix \(X \in \mathcal{X}^1(M)\), inwardly transverse to the boundary \(\partial M\) and denotes by \(X_t\) the flow of X, \(t \in \mathbb{R}\).

The omega-limit set of \(p \in M\) is the set \(\omega_X(p)\) formed by those \(q \in M\) such that \(q = \lim_{n \to \infty} X_{t_n}(p)\) for some sequence \(t_n \to \infty\). The alpha-limit set of \(p \in M\) is the set \(\alpha_X(p)\) formed by those \(q \in M\) such that \(q = \lim_{n \to -\infty} X_{t_n}(p)\) for some sequence \(t_n \to -\infty\). The non-wandering set of X is the set \(\Omega(X)\) of points \(p \in M\) such that for every neighborhood \(U\) of \(p\) and every \(T > 0\) there is \(t > T\) such that \(X_t(U) \cap U \neq \emptyset\). Given \(\Lambda \subseteq M\) compact, we say that \(\Lambda\) is invariant if \(X_t(\Lambda) = \Lambda\) for all \(t \in \mathbb{R}\). We also say that \(\Lambda\) is transitive if \(\Lambda = \omega_X(p)\) for some \(p \in \Lambda\); singular if it contains a singularity and attracting if \(\Lambda = \cap_{t > 0} X_t(U)\) for some compact neighborhood \(U\) of it. This neighborhood is often called isolating block. It is well known that the isolating block \(U\) can be chosen to be positively invariant, i.e., \(X_t(U) \subset U\) for all \(t > 0\). An attractor is a transitive attracting set. An attractor is nontrivial if it is not a closed orbit.

The maximal invariant set of X is defined by \(M(X) = \cap_{t \geq 0} X_t(M)\).
**Definition 1.1.** A compact invariant set \( \Lambda \) of \( X \) is hyperbolic if there are a continuous tangent bundle invariant decomposition \( T_\Lambda M = E^s \oplus E^X \oplus E^u \) and positive constants \( C, \lambda \) such that

- \( E^X \) is the vector field’s direction over \( \Lambda \).
- \( E^s \) is contracting, i.e., \( \|DX_t(x)|_{E^s_x}\| \leq Ce^{-\lambda t}, \) for all \( x \in \Lambda \) and \( t > 0 \).
- \( E^u \) is expanding, i.e., \( \|DX_t(x)|_{E^u_x}\| \leq Ce^{-\lambda t}, \) for all \( x \in \Lambda \) and \( t > 0 \).

A compact invariant set \( \Lambda \) has a dominated splitting with respect to the tangent flow if there are an invariant splitting \( T_\Lambda M = E \oplus F \) and positive numbers \( K, \lambda \) such that

\[
\|DX_t(x)e_x\| \cdot \|f_x\| \leq Ke^{-\lambda t}\|DX_t(x)f_x\| \cdot \|e_x\|, \quad \forall x \in \Lambda, t \geq 0, (e_x, f_x) \in E_x \times F_x.
\]

Notice that this definition allows every compact invariant set \( \Lambda \) to have a dominated splitting with respect to the tangent flow (See [8]): Just take \( E^x_x = T^x_xM \) and \( F^x_x = 0 \), for every \( x \in \Lambda \) (or \( E^x_x = 0 \) and \( F^x_x = T^x_xM \) for every \( x \in \Lambda \)).

A compact invariant set \( \Lambda \) is partially hyperbolic if it has a partially hyperbolic splitting, i.e., a dominated splitting \( T_\Lambda M = E \oplus F \) with respect to the tangent flow whose dominated subbundle \( E \) is contracting in the sense of Definition 1.1.

The Riemannian metric \( \langle \cdot, \cdot \rangle \) of \( M \) induces a 2-Riemannian metric [27],

\[
\langle u, v/w \rangle_p = \langle u, v \rangle_p \cdot \langle w, w \rangle_p - \langle u, w \rangle_p \cdot \langle v, w \rangle_p, \quad \forall p \in M, \forall u, v, w \in T_pM.
\]

This in turns induces a 2-norm [13] (or areal metric [17]) defined by

\[
\|u, v\| = \sqrt{\langle u, u/v \rangle_p} \quad \forall p \in M, \forall u, v \in T_pM.
\]

Geometrically, \( \|u, v\| \) represents the area of the paralellogram generated by \( u \) and \( v \) in \( T_pM \).

If a compact invariant set \( \Lambda \) has a dominated splitting \( T_\Lambda M = E \oplus F \) with respect to the tangent flow, then we say that its central subbundle \( F \) is sectionally expanding if

\[
\|DX_t(x)u, DX_t(x)v\| \geq K^{-1}e^{\lambda t}\|u, v\|, \quad \forall x \in \Lambda, u, v \in F_x, t \geq 0.
\]

By a sectional-hyperbolic splitting for \( X \) over \( \Lambda \) we mean a partially hyperbolic splitting \( T_\Lambda M = E \oplus F \) whose central subbundle \( F \) is sectionally expanding.

**Definition 1.2.** A compact invariant set \( \Lambda \) is sectional-hyperbolic for \( X \) if its singularities are hyperbolic and if there is a sectional-hyperbolic splitting for \( X \) over \( \Lambda \).
Definition 1.3. We say that $X$ is a sectional-Anosov flow if $M(X)$ is a sectional-hyperbolic set.

The Invariant Manifold Theorem [3] asserts that if $x$ belongs to a hyperbolic set $H$ of $X$, then the sets

$$W^s_X(p) = \{ x \in M : d(X_t(x), X_t(p)) \to 0, t \to \infty \}$$

and

$$W^u_X(p) = \{ x \in M : d(X_t(x), X_t(p)) \to 0, t \to -\infty \},$$

are $C^1$ immersed submanifolds of $M$ which are tangent at $p$ to the subspaces $E^s_p$ and $E^u_p$ of $T_pM$ respectively.

$$W^s_X(p) = \bigcup_{t \in \mathbb{R}} W^s(X_t(p)) \quad \quad W^u_X(p) = \bigcup_{t \in \mathbb{R}} W^u(X_t(p))$$

are also $C^1$ immersed submanifolds tangent to $E^s_p \oplus E^X_p$ and $E^u_p \oplus E^u_p$ at $p$ respectively.

Recall that a singularity of a vector field is hyperbolic if the eigenvalues of its linear part have non zero real part.

Definition 1.4. We say that a singularity $\sigma$ of a sectional-Anosov flow $X$ is Lorenz-like if it has three real eigenvalues $\lambda^{ss}, \lambda^s, \lambda^u$ with $\lambda^{ss} < \lambda^s < 0 < -\lambda^s < \lambda^u$. such that the real part of the remainder eigenvalues are outside the compact interval $[\lambda^s, \lambda^u]$. $W^s_X(\sigma)$ is the manifold associated to the eigenvalues with negative real part. The strong stable foliation associated to $\sigma$ and denoted by $\mathcal{F}^{ss}_X(\sigma)$, is the foliation contained in $W^s_X(\sigma)$ which is tangent to space generated by the eigenvalues with real part less than $\lambda^s$.

Definition 1.5. A periodic orbit of $X$ is the orbit of some $p$ for which there is a minimal $t > 0$ (called the period) such that $X_t(p) = p$. An orbit is called closed if it is a periodic orbit or a singularity.

A homoclinic orbit of a hyperbolic periodic orbit $O$ is an orbit $\gamma \subset W^s(O) \cap W^u(O)$. If additionally $T_qM = T_qW^s(O) + T_qW^u(O)$ for some (and hence all) point $q \in \gamma$, then we say that $\gamma$ is a transverse homoclinic orbit of $O$. The homoclinic class $H(O)$ of a hyperbolic periodic orbit $O$ is the closure of the union of the transverse homoclinic orbits of $O$. We say that a set $\Lambda$ is a homoclinic class if $\Lambda = H(O)$ for some hyperbolic periodic orbit $O$.

Definition 1.6. A Venice mask is a sectional-Anosov flow with dense periodic orbits which is not transitive.

If $A$ is a compact invariant set of $X$ we denote $\text{Sing}_X(A)$ the set of singularities of $X$ in $A$, and $\text{Sing}(X) = \text{Sing}_X(M(X))$. The closure of $B \subset M$ is denoted by $\text{Cl}(B)$. With these definitions we can state our main results.
2 Main statements

We show that if $X$ is a Venice mask supported on a compact 3-manifold, then the omega-limit set of every non-recurrent point in the unstable manifold of some singularity is a closed orbit. In addition, we prove that the intersection of two different homoclinic classes in the maximal invariant set of a sectional-Anosov flow can be decomposed as the disjoint union of, singular points, a non-singular hyperbolic set, and regular points whose alpha-limit set and omega-limit set is formed by singular points or hyperbolic sets.

Specifically, we have the following statements.

**Theorem A.** If $X$ is a three-dimensional Venice mask and $\sigma$ is a singularity of $X$, then for every $q \in W^u_X(\sigma)$ such that $q$ is non-recurrent we have the following dichotomy:

- $\omega_X(q) \in \text{Sing}(X)$.
- $\omega_X(q) = O$, where $O$ is a hyperbolic periodic orbit.

**Theorem B.** The intersection of two different homoclinic classes $H_1, H_2$ in the maximal invariant set of a sectional-Anosov flow $X$ is the disjoint union of a set $S$ (possibly empty) of singularities, a non-singular hyperbolic set $H$ (possibly empty), and a set $R$ (possibly empty) of regular points such that if $q \in R$ then $\alpha_X(q) \subset H \cup S$ and $\omega_X(q) \subset H \cup S$.

3 Preliminaries

We mention the following results which are essentials to proving the theorems.

**Theorem 3.1** ([26]). Let $\Lambda$ be a sectional-hyperbolic set with dense periodic orbits. Then, every $\sigma \in \text{Sing}_X(\Lambda)$ is Lorenz-like and satisfies $\Lambda \cap F^s_X(\sigma) = \{\sigma\}$.

We observe that $W^s_X(\sigma) \setminus F^s_X(\sigma)$ is decomposed by two connected components $W^s_{X^+}(\sigma)$ and $W^s_{X^-}(\sigma)$ (see figure 3). Hence for a Venice mask, a regular point in $M(X)$ contained in the stable manifold of some singularity $\sigma$, necessarily is contained either $W^s_{X^+}(\sigma)$ or $W^s_{X^-}(\sigma)$.

**Lemma 3.2** (Hyperbolic lemma [26]). A compact invariant set without singularities of a sectional-hyperbolic set is hyperbolic saddle-type.

**Remark 3.3.** Theorem 3.1 and the Hyperbolic Lemma imply that every Venice mask has singularities, and these are Lorenz-like.
**Definition 3.4.** We say that a $C^1$ vector field $X$ with hyperbolic closed orbits has the Property $(P)$ if for every periodic orbit $O$ there is a singularity $\sigma$ such that

$$W^u_X(O) \cap W^s_X(\sigma) \neq \emptyset.$$ 

(1)

The above definition is useful by the interesting fact below.

**Lemma 3.5.** Every point in the closure of the periodic orbits of a vector field with the Property $(P)$ is accumulated by points for which the omega-limit set is a singularity.

Moreover, we have an important property.

**Lemma 3.6 ([25]).** Every sectional-Anosov flow with singularities and dense periodic orbits on a compact 3-manifold has the Property $(P)$.

**Remark 3.7.** By Lemma 3.5 and Lemma 3.6 we can assert that every Venice mask $X$ has the Property $(P)$ and $W^s(Sing(X)) \cap M(X)$ is dense in $M(X)$.

**Definition 3.8.** Given $\Sigma \subset M$ we say that $q \in M$ satisfies Property $(P)_\Sigma$ if $Cl(O^+(q)) \cap \Sigma = \emptyset$ and there is open arc $I$ in $M$ with $q \in \partial I$ such that $O^+(x) \cap \Sigma \neq \emptyset$ for every $x \in I$.

We finish to exhibit the preliminar statements with the following characterization.

**Theorem 3.9 ([6]).** Let $X$ be a $C^1$ vector field in a compact 3-manifold $M$. If $q \in M$ has sectional-hyperbolic omega-limit set $\omega(q)$, then the following properties are equivalent:
• $\omega(q)$ is a closed orbit.

• $q$ satisfies $(P)_{\Sigma}$ for some closed subset $\Sigma$.

In Figure 2 is exhibited the case when the omega-limit set $\omega(q)$ of the point $q$ is a hyperbolic singularity of saddle-type.

4 Characterizing the omega-limit set

In this section we will prove the Theorem A. The idea is to consider a sequence of points satisfying the Property $(P)_{\Sigma}$, which approximates a point $q$ in the unstable manifold of a fixed singularity. We show that $q$ satisfies the Property $(P)_{\Sigma}$ too. Hereafter in this section, we assume that every regular point $q \in W^u(Sing(X))$ is non-recurrent.

First, we mention some facts of topology. Given a compact metric space $(Y, d)$, define a distance function between any point $x$ of $Y$ and any non-empty set $B$ of $Y$ by:

$$d(x, B) = \inf\{d(x, y) | y \in B\}.$$  

Now, consider the collection $\mathcal{C}(Y) = \{C \in Y : C \text{ is a non-empty compact subset of } (Y, d)\}$. For $\mathcal{C}(Y)$, take the Hausdorff
metric $d_H$ defined as the distance function between any two non-empty sets $A$ and $B$ of $Y$ by:

$$d_H(A, B) = \sup\{d(x, B) | x \in A\}.$$  

**Lemma 4.1.** Let $\{A_n : n \in \mathbb{N}\}$ be a sequence of closed sets contained in a compact metric space $(Y, d)$, such that $A_n \to A$ in the Hausdorff metric induced by $d$. Then $\partial A_n \to \partial A$.

For now and on this section, let $M$ be a Riemannian compact 3-manifold, and let $X$ be a Venice mask on $M$. So, for a hyperbolic point $p$ of $X$, $W^s_X(p)$ is just denoted by $W^s(p)$. The same interchanging $s$ by $u$.

### 4.1 Existence of singular partitions

We introduce the following definition which can also be found in [4] and [5], and extends the notion given in [23].

A cross section of $X$ is a codimension one submanifold $S$ transverse to $X$. We denote the interior and the boundary (in topological sense) of $S$ by $\text{Int}(S)$ and $\partial S$ respectively. If $\mathcal{R} = \{S_1, \cdots, S_k\}$ is a collection of cross sections we still denote by $\mathcal{R}$ the union of its elements. Moreover

$$\partial \mathcal{R} := \bigcup_{i=1}^k \partial S_i \quad \text{and} \quad \text{Int}(\mathcal{R}) := \bigcup_{i=1}^k \text{Int}(S_i)$$

The size of $\mathcal{R}$ will be the sum of the diameters of its elements.

**Definition 4.2.** A singular partition of an invariant set $H$ of a vector field $X$ is a finite disjoint collection $\mathcal{R}$ of cross sections of $X$ such that $H \cap \partial \mathcal{R} = \emptyset$ and

$$H \cap \text{Sing}(X) = \{y \in H : X_t(y) \notin \mathcal{R}, \forall t \in \mathbb{R}\}.$$  

For a Lorenz-like singularity $\sigma$, the center unstable manifold $W^c_X(\sigma)$ associated is divided by $W^u(\sigma)$ and $W^s(\sigma) \cap W^c(\sigma)$ in the four sectors $s_{11}$, $s_{12}$, $s_{21}$, $s_{22}$. $\pi : V_{\sigma} \to W^c(\sigma)$ is the projection defined in a neighborhood $V_{\sigma}$ of $\sigma$. Figure 3 exhibits the case when $\pi(M(X) \cap V_{\sigma})$ intersects $s_{11}$ and $s_{12}$.

**Lemma 4.3.** Consider $\sigma$ a Lorenz-like singularity of a Venice mask $X$, and $O$ a hyperbolic periodic orbit satisfying $\text{Cl}(W^u(O)) \cap W^{s,+}(\sigma) \neq \emptyset$ and $\text{Cl}(W^u(O)) \cap W^{s,-}(\sigma) \neq \emptyset$. Moreover, $\pi(\text{Cl}(W^u(O))) \cap s_{1i} \neq \emptyset$ and $\pi(\text{Cl}(W^u(O))) \cap s_{2i} \neq \emptyset$ for some $i \in \{1, 2\}$. If $q$ is a regular point in $W^u(\sigma) \cap \text{Cl}(s_{1i}) \cap \text{Cl}(s_{2i})$, then $O = \omega_X(q)$. 

8
Figure 3: Center unstable manifold of $\sigma$.

Proof. We take $q \in W^u(\sigma)$ a regular point close to $\sigma$. We assert that $q \in W^s(O)$. Indeed, if we suppose that is not the case, we will get a contradiction.

So, we assume $q \in W^u(\sigma) \setminus W^s(O)$. Then, there is a sequence $p_n^- \to q$ such that $p_n^- \in W^u(O)$ for all $n$. In addition, $\{O_X(p_n^+) : n \in \mathbb{N}\}$ accumulates some regular point $p^-$ in $W^{s,-}(\sigma)$ or in $W^{s,+}(\sigma)$. We can suppose the accumulation in some point of $W^{s,-}(\sigma)$. Also, we can take $\{p_n^+ : n \in \mathbb{N}\} \subset W^u(O)$ be a sequence such that $p_n^+ \to q$. Moreover, $\{O_X(p_n^+) : n \in \mathbb{N}\}$ accumulates $\sigma$ and some point $p^+$ in $W^{s,+}(\sigma)$. We have $p_n^+,p_n^- \notin W^u(\sigma)$ for all $n$. On the other hand, $q \in Cl(W^u(O))$ and the invariance of $W^u(\sigma)$ imply $O_X(q) \subset Cl(W^u(O))$. But $Cl(W^u(O))$ is a closed set, therefore $Cl(O_X(q)) \subset Cl(W^u(O))$. Applying the compactness of $Cl(W^u(O))$ and Tubular Flow Box Theorem [28] in a neighborhood of $O^+(q)$ we obtain that $\{O^+(p_n^+) : n \in \mathbb{N}\}$ and $\{O^+(p_n^-) : n \in \mathbb{N}\}$ accumulate all point in $W^u(\sigma)$ close to $\omega_X(q)$.

As $O$ and $\omega_X(q)$ are invariant closed sets, then they are disjoints and $d(x,\omega_X(q)) > 0$ for all $x \in O$. This implies that there exists $\varepsilon > 0$ such that every point $y$ close to $\omega_X(q)$ satisfies $d(y,O) > \varepsilon$. Moreover $y \notin O_X(q)$ and, $\{O^+(p_n^+) : n \in \mathbb{N}\}$, $\{O^+(p_n^-) : n \in \mathbb{N}\}$ accumulate $y$. The positive orbits of $p_n^+$ and $p_n^-$ cannot intersect $\omega_X(q)$. So, we have two possibilities, either any orbit intersects $O_X(q)$, or no orbit does it. The first case means that there is a point
\[ w \in W^u(\sigma) \cap W^u(O) \] which is absurd. So, neither orbit intersects \( O_X(q) \). Now, \( q \) is a non-recurrent point. Then, \( \{O_X(p_n^+): n \in \mathbb{N}\} \) does not accumulate on \( W^{s,+}(\sigma) \). But this contradicts the choice of the sequences. Therefore \( q \in W^s(O) \). So, we conclude \( O = \omega_X(q) \).

From Lemma 4.3 we obtain the following corollary.

**Corollary 4.4.** Consider \( \sigma \) a Lorenz-like singularity of a Venice mask \( X \), and \( O \) a hyperbolic periodic orbit satisfying \( W^u(O) \cap W^{s,+}(\sigma) \neq \emptyset \) and \( W^u(O) \cap W^{s,-}(\sigma) \neq \emptyset \). Let \( q \) be a regular point in \( W^u(\sigma) \cap Cl(W^u(O)) \) and let \( \{p_n : n \in \mathbb{N}\} \subset Cl(W^u(O)) \cap W^s(O) \) be a sequence such that \( p_n \rightarrow q \). Then \( p_n \in O_X(q) \) for all \( n \) large.

**Proof.** For this is sufficient to observe that \( O_X(q) \) is contained in \( W^s(O) \).

**Remark 4.5.** Corollary 4.4 says that for \( i \in \{1,2\} \) and for every hyperbolic periodic orbit \( O \) of \( X \), it is not possible \( H(O) \cap s_{1i} \neq \emptyset \) and \( H(O) \cap s_{2i} \neq \emptyset \) simultaneously.

**Lemma 4.6.** Let \( \sigma \) be a singularity of a Venice mask \( X \), and let \( O \) be a hyperbolic periodic orbit such that \( W^u(O) \cap W^s(\sigma) \neq \emptyset \). Then for \( q \in W^u(\sigma) \setminus \{\sigma\} \), \( \omega_X(q) \) has singular partitions of arbitrarily small size.

**Proof.** We adapt the proof of Theorem 17 given in [5]. Observe that \( \omega_X(q) \) is sectional-hyperbolic. Therefore, if \( \omega_X(q) \) is a closed orbit, then Theorem 3.9 implies that \( q \) satisfies the property \( (P)_\Sigma \) for some closed subset \( \Sigma \). Moreover, we can apply Theorem 16 in [5] to conclude that \( \omega_X(q) \) has singular partitions of arbitrarily small size.

Hereafter, we assume \( \omega_X(q) \) is not a closed orbit. By Proposition 3 in [5] is sufficient to prove that for all \( z \in \omega_X(q) \) there is cross section \( \Sigma_z \) close to \( z \) such that \( z \in Int(\Sigma_z) \) and \( \omega_X(q) \cap \partial \Sigma_z = \emptyset \).

We assert that \( \omega_X(q) \) cannot contain any local strong stable manifold. Indeed, we first assume that \( \omega_X(q) \) has no singularities. By Hyperbolic lemma, it is hyperbolic saddle-type. Suppose \( \omega_X(q) \) containing a local strong stable manifold. Then, by Lemma 11 in [5], \( q \) would be a recurrent point. Therefore using Lemma 5.6 in [2], there is \( x^* \in Per(X) \cap \omega_X(q) \) such that \( q \in W^s_X(x^*) \). This means that \( \omega_X(q) \) is a periodic orbit which contradicts our assumption. Now, if \( \omega_X(q) \) is a sectional-hyperbolic set with singularities, applying Main
Theorem in [20], $\omega_X(q)$ cannot contain any local strong stable manifold.

We can fix a foliated rectangle of small diameter $R^0_z$ such that $z \in \text{Int}(R^0_z)$ and $\omega_X(q) \cap \partial^h R^0_0 = \emptyset$. By Theorem 3.1, the intersection of $W^u(O)$ with $W^s(\sigma)$ occurs in some connected component $W^{s,+}(\sigma)$ or $W^{s,-}(\sigma)$ (or both). We initially assume the intersection in $W^{s,+}(\sigma)$.

Since $z \in \omega_X(q)$ and the omega-limit set is not a closed orbit, we have that the positive orbit of $q$ intersects either only one or the two connected components of $R^0_0 \setminus F^s(z, R^0_z)$.

Assume the intersection is occurring in just one component only, we shall consider the following cases:

- $W^{s,-}(\sigma) \cap M(X) = \emptyset$.

  Using this and linear coordinates around $\sigma$, we can construct an open interval $I^+ = I^+_q \subset W^u(O)$, contained in a suitable cross section through $q \in W^u(\sigma) \setminus \{\sigma\}$ and $q \in \partial I^+$. As $W^u(O) \cap W^{s,+}(\sigma)$ is dense in $W^u(O)$ we have $I^+ \cap W^{s,+}(\sigma)$ is dense in $I^+$.

  It is possible to assume $I^+$ is contained in that component of $R^0_0 \setminus F^s(z, R^0_z)$. It is because of the positive orbit of $q$ carries the positive orbit of $I^+$ into such a component. Furthermore, the stable manifolds through $I^+$ form a subrectangle $R^+_I$ in there. So, $W^{s,+}(\sigma) \cap R^+_I$ is dense in $R^+_I$.

Now, as in Theorem 17 of [5], we suppose $\omega_X(q) \cap \text{Int}(R^+_I) \neq \emptyset$ to obtain a contradiction. By hypothesis, the omega-limit set of $q$ is not a periodic orbit. Then Lemma 5.6 in [22] implies that the positive orbit of $q$ cannot intersects $F^s(q, R^0_0)$ infinitely many times. Now, if it intersects $R^+_I$, then by the density of $W^{s,+}(\sigma) \cap R^+_I$ in $R^+_I$, we can assert that the positive orbit of a point $p$ in $W^{s,-}(\sigma)$ would intersect $R^+_I$. Therefore $p \in \text{Cl}(W^u(O)) \subset M(X)$ which we get a contradiction. So $\omega_X(q) \cap \text{Int}(R^+_I) = \emptyset$.

To continue, we choose a point $z' \in \text{Int}(R^+_I)$ and a point $z''$ in the connected component $R^0_0 \setminus F^s(z, R^0_0)$ not intersected by the positive orbit of $q$. The desired rectangle $\Sigma_z$ is a subrectangle of $R^0_0$ bounded by $F^s(z', R^0_0)$ and $F^s(z'', R^0_0)$.

- $W^s(\sigma) \cap W^u(O) \subset W^{s,+}(\sigma)$ and $W^s(\sigma) \cap W^u(O') \subset W^{s,-}(\sigma)$ for some hyperbolic periodic orbit $O' \neq O$. 

11
In this way, we have the hypotheses of Theorem 17 in [5]. Therefore there exists an interval $I^- \subset W^u(O')$ contained in that component of $R^0 \setminus F^s(z, R^0)$, such that $q \in \partial I^-$ and $I^- \cap W^{s,-}(\sigma)$ is dense in $I^-$. The stable manifolds through $I = I^+ \cup \{q\} \cup I^-$ form a subrectangle $R_I$ in there, with $\text{Int}(R_I) \cap \omega_X(q) = \emptyset$. So, the existence of $\Sigma_2$ is guaranteed such as last item.

- $W^{s,+}(\sigma) \cap W^u(O) \neq \emptyset$ and $W^{s,-}(\sigma) \cap W^u(O) \neq \emptyset$.

We assert that there are $O_1, O_2$ hyperbolic periodic orbits such that, $W^{s}(\sigma) \cap W^u(O_1) \subset W^{s,+}(\sigma)$ and $W^{s}(\sigma) \cap W^u(O_2) \subset W^{s,-}(\sigma)$. Indeed, we take $q_1 \in W^{s,+}(\sigma) \cap W^u(O)$ and $q_2 \in W^{s,-}(\sigma) \cap W^u(O)$. As $M(X)$ is union of homoclinic classes and $W^u(O) \subset M(X)$, there are hyperbolic periodic orbits $O_1, O_2$ satisfying $q, q_1 \in H(O_1)$ and $q, q_2 \in H(O_2)$. Therefore $O_X(q_1) \subset H(O_1)$ and $O_X(q_2) \subset H(O_2)$. Moreover, since the homoclinic classes are closed set we have that $\sigma$ and $O$ are in $H(O_1) \cap H(O_2)$. From Remark 4.5 follows $H(O_1) \cap W^s(\sigma) \subset W^{s,+}(\sigma)$ and $H(O_2) \cap W^s(\sigma) \subset W^{s,-}(\sigma)$. On the other hand, let $W^+(O)$ be the connected component of $W^u(O) \setminus O$ containing $q_1$, then $W^+(O) \subset H(O_1)$. Analogously, for $W^-(O)$, the connected component of $W^u(O) \setminus O$ containing $q_2$, we have $W^-(O) \subset H(O_2)$. Therefore $W^u(O_1) \cap W^s(\sigma) \subset W^{s,+}(\sigma)$ and $W^u(O_2) \cap W^s(\sigma) \subset W^{s,-}(\sigma)$. Again we have the hypotheses of Theorem 17 in [5].

- $W^{s,+}(\sigma) \cap W^u(O) \neq \emptyset$ and $W^{s,-}(\sigma) \cap H(O) \neq \emptyset$.

It is not possible by Corollary 4.4.

- $W^{s,+}(\sigma) \cap W^u(O) \neq \emptyset$, $W^{s,-}(\sigma) \cap \text{Cl}(W^u(O')) \neq \emptyset$ and $q \in \text{Cl}(W^u(O'))$, where $O'$ is a hyperbolic periodic orbit of $X$.

From last item $O' \not\in H(O)$. As $X$ satisfies the Property $(P)$, there is $\sigma' \in \text{Sing}(X)$ such that $W^u(O') \cap W^s(\sigma') \neq \emptyset$. If $\sigma' = \sigma$ then $W^u(O')$ intersects $W^{s,+}(\sigma)$ or $W^{s,-}(\sigma)$. Observe that those alternatives were already analyzed. If $\sigma' \neq \sigma$, then we can obtain an interval $J^-$ such that $J^- \subset W^u(O')$ and $J^- \cap W^s(\sigma')$ is dense in $J^-$. Moreover we can assume $W^s(\sigma) \cap W^u(O) \subset W^{s,+}(\sigma)$ to obtain an interval $I^+$ such that $I^+ \subset W^u(O) \cap I^+ \cap W^{s,+}(\sigma)$ is dense in $I^+$. Since $O' \not\in H(O)$, follows that $W^u(O') \not\subset H(O)$. Therefore $W^u(O')$ cannot intersect $W^{s,+}(\sigma)$. In this way, there is an open arc $I^- \subset \bigcup_{t \geq 0} X_t(J^-)$ such that $q \in \partial I^-$. $I^-$ works such as in second item. The stable manifolds through $I = I^+ \cup \{q\} \cup I^-$ generates a subrectangle $R_I$. This acts such as Theorem 17 in [5].
Now assume the positive orbit intersects both components of \( R_0^y \setminus \mathcal{F}^s(z, R_0^y) \).
Therefore we take I (or \( I^+ \) to first case) with the positive orbit as before to obtain two subrectangles \( R^l_1 \) and \( R^r_1 \), like \( R_I \) (or \( R^+_I \) to first case), in each component. Then we select two points \( z' \in \text{Int}(R^l_1) \) and \( z'' \in \text{Int}(R^r_1) \) and define \( \Sigma_z \) as the rectangle in \( R_0^x \) bounded by \( \mathcal{F}^s(z', R_0^y) \) and \( \mathcal{F}^s(z'', R_0^y) \).

From Proposition 4.8 in [5] we conclude the result.

\[ \square \]

We remember the concept of singular cross section that appears in [24]. For a disjoint collection of rectangles \( S = \{ S_1, \ldots, S_l \} \) we denote \( S^o = S \setminus \partial S \) and \( \partial^* S = \bigcup_{s \in S} \partial^* S \) for \( * = h, v, o \).

**Definition 4.7.** A singular cross section of \( X \) is a finite disjoint collection \( S \) of foliated rectangles with \( \partial^h S = \emptyset \) such that for every \( S \in S \) there is a leaf \( l_S \) of \( \mathcal{F}^s \) in \( S^o \) such that the return time \( t_S(x) \) for \( x \in S \cap \text{Dom}(\Pi_S) \) goes uniformly to infinity as \( x \) approaches \( l_S \).

We define the singular curve of \( S \) as the union,

\[
l_S = \bigcup_{s \in S} l_S.
\]

**Proposition 4.8.** Let \( q \) be a regular point in \( W^u(\sigma) \), with \( \sigma \) a singularity of a Venice mask \( X \), and let \( O \) be a hyperbolic periodic orbit such that \( W^u(O) \cap W^s(\sigma) = \emptyset \). Then \( \omega_X(q) \) is a closed orbit.

**Proof.** If \( \omega_X(q) \) is a singularity, then it is done. Hereafter, we assume that \( \omega_X(q) \) is not a singularity. From Lemma 4.6 follows that \( \omega_X(q) \) has singular partitions of arbitrarily small size. On the other hand, let \( T_U: M = \tilde{F}^s_U \oplus \tilde{F}^c_U \) be a continuous extension of the sectional-hyperbolic splitting \( T_{\omega_X(q)} M = F^s_{\omega_X(q)} \oplus F^c_{\omega_X(q)} \) of \( \omega_X(q) \) to a neighborhood \( U \) of \( \omega_X(q) \). Let \( I \) be an arc tangent to \( \tilde{F}^c_U \), transverse to \( X \), with \( q \) as boundary point. Theorem 18 in [5] guarantees for every singular partition \( \mathcal{R} = \{ S_1, \ldots, S_k \} \) of \( \omega_X(q) \), the existence of \( S \in \mathcal{R} \), \( \delta > 0 \), a sequence \( q'_1, q'_2, \ldots \in S \) in the positive orbit of \( q \), and a sequence of intervals \( J'_1, J'_2, \ldots \subset S \) in the positive orbit of \( I \) with \( q'_j \) as a boundary point of \( J'_j \) for all such that \( \text{length}(J'_j) \geq \delta \), for all \( j = 1, 2, 3, \ldots \).

We can assume \( I = J'_1 \). As \( q, q'_j \in M(X) \) and \( X \) is a Venice mask, we can use the Lemma 3.5 to obtain a sequence \( \{ q_n : n \in \mathbb{N} \} \subset M \) such that \( q_n \to q \) and \( \omega(q_n) \) is a singularity for any \( n \). As \( X \) has just a finite singular points, we can take \( \omega(q_n) = \{ \sigma' \} \) for all \( n \), and some \( \sigma' \in \text{Sing}(X) \). If \( q_n \in W^u(\sigma) \) for all \( n \), then \( \omega(q) = \{ \sigma' \} \) which contradicts our assumption. Therefore \( q_n \notin W^u(\sigma) \) for any \( n \). We can take \( q_n \) such that \( q_n \in S \) for all \( n \).
On the other hand, for $\sigma'$ are possible the following two alternatives, either $\sigma' \in \omega_X(q)$, or $\sigma' \notin \omega_X(q)$. We begin to consider $\sigma' \in \omega_X(q)$. Lemma 14 in [5] asserts $O^+(q) \cap R = \{\hat{q}_1, \hat{q}_2, \cdots\}$ an infinite sequence ordered in a way that $\Pi(\hat{q}_n) = \hat{q}_{n+1}$, and the existence of a curve $c_n \subset W^s(Sing(X) \cap \omega_X(q)) \cap B_\delta(\hat{q}_n)$ such that

$$ B_\delta(\hat{q}_n) \subset Dom(\Pi) \quad and \quad \Pi|_{B_\delta(\hat{q}_n)} \text{ is } C^1, $$

where $B_\delta(\hat{q}_n)$ denotes the connected component of $B_\delta(\hat{q}_n)$ containing $\hat{q}_n$. In particular, we can reduce $\delta$ to obtain $\Pi_S = \Pi|_S$ such that

$$ (\Pi_S)|_{B_\delta(q)} \text{ is } C^1. $$

However $W^s(\sigma')$ accumulates $q$ on $S$, so we obtain a contradiction. Therefore the first alternative cannot occur. We conclude $\sigma' \notin \omega_X(q)$.

Hartman-Grobman’s Theorem implies the existence of a neighborhood $V_{\sigma'}$ of $\sigma'$, where the flow is $C^0$-conjugated to its linear part. Let $\eta > 0$ be such that $V_{\sigma'} \subset B_\eta(\sigma')$ and $O^+(q) \cap V_{\sigma'} = \emptyset$. From Lemma 2.2 in [24] there are singular cross sections $\Sigma^+, \Sigma^- \subset V_{\sigma'}$ such that every orbit of $M(X)$ passing close to some point in $W^s(\Sigma^+) (\text{respectively } W^s(\Sigma^-))$ intersects $\Sigma^+(\text{respectively } \Sigma^-)$. Moreover Lemma 2.3 in [3] guarantees the existence of two disks $\Lambda^+, \Lambda^- \subset V_{\sigma'}$ transverse to $X$ such that for $B_\epsilon(\sigma') \subset V_{\sigma'}$, and for any point $x \in B_\epsilon(\sigma')$, there are two numbers $t_- < 0 < t_+$ with $X_{t_-}(x) \in \Sigma^+ \cup \Sigma^-$ and $X_{t_+}(x) \in \Lambda^+ \cup \Lambda^-$. In addition, $X_t(x) \in V_{\sigma'}$ for all $t \in (t_-, t_+)$. See Figure 2.

As $q_n \to q$, we can take a sequence of open arcs $I_1, I_2, \cdots$ with $q_n$ as a boundary point of $I_n$ such that $Cl(I_n)$ converges to $Cl(I)$. In particular, we can assume $\delta \leq \text{length}(I_n) < \epsilon$ for all $n = 1, 2, 3, \cdots$ and $\text{diam}(S) = \epsilon$. In addition, we can take $I_n \subset S$ for all $n$. On the other hand, $q_n \in W^s(\sigma')$ implies that $O^+(q_n) \cap \Sigma^+ \cup \Sigma^-$. Assume that the intersection occurs in $\Sigma^+$ for all $n$. As we can choose the singular partition of arbitrarily small size and $q$ is non-recurrent, there is $\epsilon' > 0$ such that $\text{diam}(R) = \epsilon'$ and $O^+(s_n) \cap \Sigma^+ \neq \emptyset$ for all $s_n \in I_n$.

We assert that $q$ satisfies the property $(P)_\Sigma$, where $\Sigma = \Sigma^+$. Indeed, from $O^+(q) \cap V_{\sigma'} = \emptyset$ follows $O^+(q) \cap \Sigma^+ = \emptyset$. Now, for $x \in I$ there are $\beta_1, \beta_2 > 0$ such that $B_{\beta_1}(x) \cap \partial I = \emptyset, B_{\beta_2}(x) \cap \{q_l\} = \emptyset$ and $B_{\beta_2}(x) \cap I \neq \emptyset$ for all $I$ large. We define $\beta = \min\{\beta_1, \beta_2\}$. Let $\{x_t\}_t$ be a sequence with $x_t \in I_t \cap B_{\beta}(x)$ such that $x_t \to x$. As in [3], we define the holonomy map $\Pi_{S, \Sigma^+}$ from $S$ to $\Sigma^+$ by

$$ \text{Dom}(\Pi_{S, \Sigma^+}) = \{y \in S : X_t(y) \in \Sigma^+ \text{ for some } t > 0\} $$

Figure 4: Proof Proposition 4.8

and

$$\Pi_{S, \Sigma^+}(y) = X_{t_{S, \Sigma^+}(y)}(y),$$

where $t_{S, \Sigma^+}(y) = \inf\{t > 0 : X_t(y) \in \Sigma^+\}.$

Therefore $x_l \in \text{Dom}(\Pi_{S, \Sigma^+})$ for all $n.$ From Lemma 19 and Theorem 22 in [5] follows that $x \in \text{Dom}(\Pi_{S, \Sigma^+}).$

Finally, Theorem 3.9 implies that $\omega_X(q)$ is a closed orbit. As we assume $\omega_X(q)$ not being a singularity, then we conclude that the omega-limit set of $q$ is a periodic orbit.

4.2 Property $(P_{\sigma'})_q^+$

Definition 4.9. Let $\sigma, \sigma' \in \text{Sing}(X)$ and $q$ be a regular point in $W^u(\sigma).$ We say that an open arc $I \subset M$ satisfies the Property $(P_{\sigma'})_q^+$ if $q \in \partial I$ and $I \cap W^{s,+}(\sigma')$ is dense in $I.$ In a similar way, an open arc $J \subset M$ satisfies the Property $(P_{\sigma'})_q^-$ if $q \in \partial J$ and $J \cap W^{s,-}(\sigma')$ is dense in $J.$
Proposition 4.10. Let $O$ be a hyperbolic periodic orbit of a Venice mask $X$. Assume $\sigma' \in \text{Sing}(X)$ satisfying $\emptyset \neq W^u(O) \cap W^s(\sigma') \subset W^{s,+}(\sigma')$. Let $q$ be a regular point with $q \in W^u(\sigma) \cap \text{Cl}(W^u(O))$, for some $\sigma \in \text{Sing}(X)$. Then there is an open arc satisfying the Property $(P_{\sigma'})_q^+$. The same interchanging $+$ by $-$.

Proof. Let $p \in W^u(\sigma')$ be a regular point. We assert that there is an open interval $J$ satisfying the Property $(P_{\sigma'})_p^+$. Indeed, $\sigma'$ and $p$ are contained in $\text{Cl}(W^u(O))$. As $W^u(O)$ intersects $W^{s,+}(\sigma')$, then $W^u(O) \cap W^s(\sigma)$ is dense in $W^{s,+}(\sigma')$. Consider an open arc $J \subset W^u(O)$ with $p \in \partial J$. So, the density of $W^u(O) \cap W^{s,+}(\sigma)$ in $W^u(O)$ implies that $J \cap W^{s,+}(\sigma')$ is dense in $J$.

If $\sigma = \sigma'$, then we obtain the desired result. Now, we consider $\sigma \neq \sigma'$. From Lemma 4.8 follows that the omega-limit set of every point in $W^u(\sigma')$ is a closed orbit. Now, take two point $p_1, p_2$, one on each branch of $W^u(\sigma') \setminus \{\sigma'\}$. We analize the following cases which are illustrated in Figure 5.

- $\omega_X(p_1)$ is a singularity. Let $\sigma_1$ be a singularity with $\omega_X(p_1) = \{\sigma_1\}$. If $\omega_X(p_1) = \{\sigma'\}$, then $\omega_X(p_2) \neq \{\sigma'\}$. Indeed, $\omega_X(p_1) = \{\sigma'\} = \omega_X(p_2)$ implies either $W^u(O) \cap W^s(\sigma) \neq \emptyset$ or $\text{Cl}(W^u(O)) \cap W^s(\sigma) \neq \emptyset$. But $W^u(O) \cap W^s(\sigma) = \emptyset$ by hypothesis. Moreover $\sigma \in \text{Cl}(W^u(O))$. So, $\sigma_1 \neq \sigma'$.

Let $w \in W^u(\sigma') \cap W^s(\sigma_1)$ be a point in $O^+_X(p_1)$ close to $\sigma_1$. Using it and linear coordinates around $\sigma_1$, we can construct an open interval
$J_1 \subset \bigcup_{t \geq 0} X_t(J) \subset W^u(O)$ contained in a suitable cross section through $w$, such that $w \in \partial J_1$. From Inclination lemma [28], follows that $W^u(O)$ accumulates points in some branch of $W^u(\sigma_1)$. Therefore, for $q_1 \in (W^u(\sigma_1) \cap Cl(W^u(O))) \setminus \{\sigma_1\}$ there is an open arc $I_1$ such that $I_1 \subset \bigcup_{t \geq 0} X_t(J_1)$ and $q_1 \in \partial I_1$. The density of $W^{s,+}(\sigma') \cap W^u(O)$ in $W^u(O)$ implies the density of $W^{s,+}(\sigma') \cap I_1$ in $I_1$. Then $I_1$ satisfies $(P_{\sigma'})^+_q$.

- When the omega-limit set of $p_1$ and $p_2$ are respectively hyperbolic periodic orbits $O_1, O_2$, we have that $W^u(O_i)$ intersects the stable manifold of some singularity $\sigma_i$ of $X$, $i = 1, 2$. We first assume $\sigma_1 = \sigma_2 = \sigma'$. That intersection cannot just only occurs in $W^s(\sigma')$ because of this would imply $\sigma \notin Cl(W^u(O_1) \cup W^u(O_2))$ and $Cl(W^u(O)) \subset Cl(W^u(O_1) \cup W^u(O_2))$. But $\sigma \in Cl(W^u(O))$ which produces a contradiction. Therefore we can assume that $W^u(O_1) \cap W^s(\sigma_1) \neq \emptyset$ with $\sigma_1 \neq \sigma'$.

Applying Inclination lemma, $Cl(W^u(O))$ and $\bigcup_{t \geq 0} X_t(J)$ intersect $W^s(\sigma_1)$ transversally. Again, let $w \in W^s(O) \cap W^s(\sigma)$ be a point in $\bigcup_{t \geq 0} X_t(J)$ close to $\sigma_1$. Using it and linear coordinates around $\sigma_1$, we can construct an open interval $J_1 \subset W^u(O)$ contained in a suitable cross section through $w$. $J_1 \setminus \{w\}$ is formed by two open arcs $J^+_1, J^-_1 \subset W^u(O)$. Therefore, for $q_1 \in W^u(\sigma_1) \setminus \{\sigma_1\}$ there is an open arc $I_1$ such that and $q_1 \in \partial I_1$ and, $I_1 \subset \bigcup_{t \geq 0} X_t(J^+_1)$, or $I_1 \subset \bigcup_{t \geq 0} X_t(J^-_1)$. The density of $W^{s,+}(\sigma') \cap W^u(O)$ in $W^{s,+}(\sigma')$ implies the density of $W^{s,+}(\sigma) \cap I_1$ in $I_1$. Then $I_1$ satisfies $(P_{\sigma'})^+_q$.

If $\sigma_1 = \sigma$, then the result is obtained. Otherwise, we apply a similar process to $\sigma_1$ to get $\sigma_3 \in Sing(X)$ with $\sigma_3 \notin \{\sigma', \sigma_1\}$, and an open arc $I_3 \subset Cl(W^u(O))$ such that $I_3$ satisfies the Property $(P_{\sigma'})^+_q$.

As $\sigma \in Cl(W^u(O))$ and $X$ just has finitely many singularities, we conclude the existence of some open arc satisfying the Property $(P_{\sigma'})^+_q$ for $q \in W^u(\sigma) \cap Cl(W^u(O))$.

\[\square\]

### 4.3 Proof of Theorem [A]

It is sufficient to prove the existence of singular partitions of arbitrarily small size.

Let $q$ be a regular point in $W^u(\sigma)$, where $\sigma \in Sing(X)$.

As $M(X)$ is union of homoclinic classes, there is a hyperbolic periodic orbit $O$ such that $\sigma$ and $q$ are contained in the homoclinic class associated to $O$, denoted by $H(O)$. In addition $H(O)$ intersects only one or the two connected
components \(W^{s, +}(\sigma), W^{s, -}(\sigma)\) of \(W^s(\sigma) \setminus F_X^{ss}(\sigma)\). We begin to analyze the intersection in \(W^{s, +}(\sigma)\). On the other hand, \(\bar{X}\) satisfies the Property \((P)\). This implies that there is a singularity \(\sigma' \in \text{Sing}(X)\) with \(W^u(O) \cap W^s(\sigma') \neq \emptyset\). By Theorem \ref{thm:intersection}, the intersection of \(W^u(O)\) with \(W^s(\sigma')\) is either only one or the two connected components \(W^{s, +}(\sigma'), W^{s, -}(\sigma')\) of \(W^s(\sigma') \setminus F_X^{ss}(\sigma')\). If \(\sigma = \sigma'\) then from Lemma \ref{lem:existence} follows the existence of singular partitions of arbitrarily small size. Hereafter, we assume \(\sigma \neq \sigma'\) and \(W^{s, +}(\sigma') \cap W^u(O) \neq \emptyset\).

If \(\text{Cl}(W^u(O)) \cap W^{s, -}(\sigma') \neq \emptyset\), then Lemma \ref{lem:existence} and Proposition \ref{prop:intersection} imply that for some \(p \in W^u(\sigma') \cap \text{Cl}(W^u(O))\), \(O = \omega_X(p)\) and \(H(O) \subset \text{Cl}(W^u(\sigma'))\). But \(q \notin W^u(\sigma')\). This contradicts \(q \in H(O)\). So, \(\text{Cl}(W^u(O)) \cap W^{s, -}(\sigma') = \emptyset\). Proposition \ref{prop:existence} guarantees the existence of an open arc \(I^+ \subset M\) satisfying the Property \((P_{\sigma'})\).

We suppose \(\omega_X(q)\) is not a periodic orbit. Let \(z\) be a point in \(\omega_X(q)\). In a similar way as Lemma \ref{lem:existence}, we fix a foliated rectangle of small diameter \(R_z^0\) such that \(z \in \text{Int}(R_z^0)\) and \(\omega_X(q) \cap \partial^h R_z^0 = \emptyset\). The positive orbit of \(q\) intersects either only one or the two connected components of \(R_z^0 \setminus F^s(z, R_z^0)\).

Assume the intersection is occurring in just one component only. Now, analyze the following cases:

- \(q \notin H(O')\) for all hyperbolic periodic orbit \(O'\) of \(X\) such that \(H(O') \cap W^{s, -}(\sigma) = \emptyset\).

The existence of the singular partitions of arbitrarily small size is obtained such as the first case in Lemma \ref{lem:existence}.

- There is a sequence \(\{p_n\}_n \subset W^u(O)\) such that \(p_n \to p \in W^{s, -}(\sigma)\), and there is a sequence \(\{q_n\}\) such that \(q_n \in O_X(p_n)\) and \(q_n \to q\).

From Lemma \ref{lem:existence} follows that \(\omega_X(q) = O\). But this contradicts our assumption that the omega-limit set is not a periodic orbit.

- For some periodic orbit \(O' \neq O\), there is a sequence \(\{p_n : n \in \mathbb{N}\} \subset W^u(O')\) such that \(p_n \to p \in W^{s, -}(\sigma)\), and there is a sequence \(\{q_n : n \in \mathbb{N}\}\) satisfying \(q_n \in O_X(p_n)\) and \(q_n \to q\).

Again, Lemma \ref{lem:existence} implies that \(W^u(O')\) does not intersect the open arc \(I^+\). From Property \((P)\), there is \(\sigma'' \in \text{Sing}(X)\) such that \(W^u(O') \cap W^s(\sigma'') = \emptyset\). Then for some \(r \in W^u(\sigma'')\) there is an interval \(J^- \subset W^u(O')\), such that \(r \in \partial J^+\) and \(J^- \cap W^s(\sigma'')\) is dense in \(J^-\). Also there is an open arc \(I^- \subset \bigcup_{t \geq 0} X_t(J^-)\) satisfying \(q \in \partial I^-\). Therefore \(I^- \subset W^u(O')\) and \(I^- \cap W^s(\sigma'')\) is dense in \(I^-\). In addition, \(W^{s, +}(\sigma) \cap I^- = \emptyset\). The stable manifolds throught
\[ I = I^+ \cup \{q\} \cup I^- \] generates a subrectangle \( R_I \). This rectangle acts such as Lemma 17 in [5].

The existence of the singular partition of arbitrarily small size is obtain such as Lemma 4.6.

If the intersection of \( O^+_z(q) \) with \( R^u_0 \) occurs in both connected components of \( R^u_0 \setminus F^s(z, R^u_0) \), then we proceed such as Lemma 4.6 to get a cross section \( \Sigma_z \) with \( z \in \Sigma_z \) and \( \partial \Sigma_z \cap \omega_X(q) = \emptyset \).

In this way, Proposition 3 in [5] implies the existence of the singular partition of arbitrarily small size for \( \omega_X(q) \).

Finally, we follow the proof of Proposition 4.8 to conclude that \( \omega_X(q) \) is a closed orbit.

5 Intersection of homoclinic classes

In this section we are interested in the study of the intersection of homoclinic classes in a sectional-Anosov flow. We follow some ideas developed in [8] to obtain Theorem B. More specifically, we prove that in this context, this intersection can be decomposed in three specific sets. a non-singular hyperbolic set, finitely many singularities and regular orbits joining them. Recall that an invariant set is nontrivial if it does not reduces to a single orbit. The conclusion of Theorem B is obvious when \( H_1 \) or \( H_2 \) are trivial invariant sets. Hereafter, \( H_1 \) and \( H_2 \) are two non trivial different homoclinic classes in \( M(X) \).

Let \( \Lambda \) be the intersection between \( H_1 \) and \( H_2 \). We start with the following lemma.

**Lemma 5.1.** Assume that there is a singularity \( \sigma \in \Lambda \), then for \( \delta > 0 \) small, every sequence \( \{x_n : n \in \mathbb{N}\} \subset \Lambda \cap B_\delta(\sigma) \) such that \( x_n \to \sigma \) is contained in \( W^s(\sigma) \cup W^u(\sigma) \).

**Proof.** We suppose by contradiction that there is a sequence \( \{x_n : n \in \mathbb{N}\} \subset \Lambda \cap B_\delta(\sigma) \) such that \( x_n \to \sigma \) and \( x_n \notin W^s(\sigma) \cup W^u(\sigma) \) for all \( n \).

So, we obtain two sequences \( x^s_n, x^u_n \), in the orbit of \( x_n \) such that \( x^s_n \to y^s \) and \( x^u_n \to y^u \) for some \( y^s \in W^s(\sigma) \setminus \{\sigma\} \) and \( y^u \in W^u(\sigma) \setminus \{\sigma\} \) close to \( \sigma \). Let \( O_1, O_2 \) be two orbits such that \( H(O_1) = H_1 \) and \( H(O_2) = H_2 \). Then there exist sequences \( \{p_n : n \in \mathbb{N}\} \subset (W^u(O_1) \cap W^s(O_2)) \) and \( \{q_n : n \in \mathbb{N}\} \subset (W^u(O_2) \cap W^s(O_1)) \) satisfying \( p_n \to x^s_n \) and \( q_n \to x^s_n \). We can assume \( p_n \notin H_2 \) for all \( n \). This means that \( p_n \to x^s \) and \( q_n \to x^s \) too. The behavior of the orbits of \( x_n, p_n \) and \( q_n \) nearby \( \sigma \), are as described in Figure 6.

Since homoclinic classes have density of periodic points [16], for each \( n \) we have that \( p_n \) and \( q_n \) are approximated respectively by a sequence of periodic
orbits \( \{O_{1m}^n : m \in \mathbb{N}\} \) and \( \{O_{2m}^n : m \in \mathbb{N}\} \). Define the map \( \pi : B_\delta(\sigma) \to W^u(\sigma) \) such as in Subsection 4.1. Observe that \( \{\pi(W^u(O_{1m}^n)) : m \in \mathbb{N}\} \) and \( \{\pi(W^u(O_{2m}^n)) : m \in \mathbb{N}\} \) accumulate \( y^s \) in the same sector \( s_{ij} \) of \( W^u(\sigma) \). Follows from Lemma 3.1 in [12] that these sequences can be chosen in a way that, for \( i = 1, 2 \) and for all \( n,m \), \( W^s(O_{i,m}^{n,m}) \) is uniformly bounded away from zero. This implies that for \( m_1,m_2,n_1,n_2 \) large, \( W^u(O_{1,m_1}^{n_1}) \cap W^s(O_{2,m_2}^{n_2}) \neq \emptyset \). Consider \( x \in W^u(O_{1,m_1}^{n_1}) \cap W^s(O_{2,m_2}^{n_2}) \). As \( O_{1,m_1}^{n_1} \subset (H_1 \setminus H_2) \) and \( O_{2,m_2}^{n_2} \subset H_2 \), then there is \( x^* \in O_X(x) \) such that \( x^* \in \Lambda \). But \( \Lambda \) is an invariant closed set, then \( O_{1,m_1}^{n_1} \subset Cl(O_X(x^*)) = Cl(O_X(x^*)) \subset \Lambda \). However \( O_{1,m_1}^{n_1} \not\subset H_2 \) and \( \Lambda \subset H_2 \), which is a contradiction.

We conclude \( x_n \in W^s(\sigma) \cup W^u(\sigma) \) for all \( n \in \mathbb{N} \).

\[ \Box \]

5.1 Proof theorem \( \mathbb{B} \)

Theorem \( \mathbb{B} \) gives a description about the set \( \Lambda \).

Proof. The idea of the proof is the same given in Lemma 3.3 by [8]. Follows to Lemma 5.1 that there is \( \delta > 0 \) such that \( \Lambda \cap B_\delta(\sigma) \subset W^s(\sigma) \cup W^u(\sigma) \), and the balls \( B_\delta(\sigma) \) are pairwise disjoint for every \( \sigma \in \Lambda \cap Sing(X) = S \). Define

\[ H = \bigcap_{(t,\sigma) \in \mathbb{R} \times S} X_t(\Lambda \setminus B_\delta(\sigma)). \]

By construction, \( H \) is a non-singular, compact invariant sectional-hyperbolic set. So, applying Lemma 3.2 we have that \( H \) is hyperbolic. Now define \( R = \)
For $x \in R$ there is $(t, \sigma) \in \mathbb{R} \times S$ with $X_t(x) \in B_\delta(\sigma)$, and by Lemma 5.1 $X_t(x) \in W^s(\sigma) \cup W^u(\sigma)$. If $x \in W^u(\sigma)$ we obtain $\alpha(x) \subset H \cup S$. Assume $X_s(x) \notin \bigcup_{\rho \in S} B_\delta(\rho)$ for all $s \geq 0$, then $\omega(x) \subset H$. Now, if there is $(s, \rho) \in \mathbb{R} \times S$ such that $X_s(x) \in B_\delta(\rho)$ then $x \in W^s(\rho)$. So $\omega(x) \subset H \cup S$.

With a similar argument we have $\alpha(x) \subset H \cup S$ and $\omega(x) \subset H \cup S$ for $x \in W^*(\sigma)$. So, we conclude the result.

\[ \square \]

6 Some conjectures

Because of the study developed in this work, different questions have appeared. All known examples of Venice mask are characterized because the maximal invariant set is the finite union of homoclinic classes and the intersection between two different homoclinic classes $H_1$ and $H_2$ is contained in $\text{Cl}(W^u(\text{Sing}(X)))$.

Moreover, every regular point $q \in W^u(\text{Sing}(X)) \cap H_1 \cap H_2$ is non-recurrent.

Consider a Venice mask $X$ supported on a compact 3-manifold $M$. Let $H_1$ and $H_2$ be two different homoclinic classes in $M(X)$ and let $\Lambda$ be the intersection between $H_1$ and $H_2$. Assume the decomposition of $\Lambda$ given in Theorem B, it is $\Lambda = S \cup H \cup R$.

We announce the following conjecture.

**Conjecture 6.1.** Every regular point $q \in R$ is non-recurrent.

From Lemma 5.1 we have that for $\delta > 0$ small, $x \in B_\delta(\sigma)$ implies $x \in W^s(\sigma) \cup W^u(\sigma)$ for some $\sigma \in S$. If $x \in W^u(\sigma)$ then $\alpha(x) = \{\sigma\}$. Now we take $x \in W^s(\sigma) \setminus W^u(\sigma)$. Therefore we shall consider two cases, either $\alpha(x) = \{\rho\}$ for some $\rho \in S$ or $\alpha(x) \subset H$. In the first case, we obtain the desired result. If we prove that the second case cannot occur, then the following conjecture would be true.

**Conjecture 6.2.** $\Lambda \subset \text{Cl}(W^u(\text{Sing}(X)))$.

Let us state direct consequence of the hyperbolic Lemma 3.2 that appears in [5].

**Corollary 6.3.** Every periodic orbit of a sectional-Anosov flow on a compact manifold is hyperbolic. In particular, all such flows have countably many closed orbits.

This implies that the maximal invariant set of every Venice mask is union of countably many homoclinic classes. So, if Conjecture 6.1 and Conjecture 6.2 are true, then would be possible to realize the following statement.
Conjecture 6.4. The maximal invariant set of every Venice mask is finite union of homoclinic classes.

Proof. Let $X$ be a Venice mask supported on a compact 3-manifold $M$. Then $X$ has finite many singularities, we say $n$. Let $H_1, H_2$ be two different homoclinic classes associated to $M(X)$. From Conjectures 6.1 and 6.2 is possible to apply Theorem [A] to conclude that for each singularity $\sigma$ of $X$, $Cl(W^u(\sigma)) = \{\sigma\} \cup W^u(\sigma) \cup C_\sigma$, it is a disjoint union and $C_\sigma$ is a closed orbit. On the other hand, the branches of $W^u(\sigma)$ are uni-dimensional. Therefore Theorem 6.2 implies $H_1 \cap H_2$ has just only a finite number of possibilities to occur. Moreover, at most three homoclinic classes can contain the branch of the unstable manifold of some singularity.

This finishes the proof. \qed

References

[1] Afraimovich, V. S., Bykov, V. V., and Shilnikov, L. P. On structurally unstable attracting limit sets of lorenz attractor type. Trudy Moskov. Mat. Obshch. 44, 2 (1982), 150–212.

[2] Araújo, V., and Pacífico, M. J. Three-dimensional flows, vol. 53. Springer Science & Business Media, 2010.

[3] Arroyo, A., and Pujals, E. Dynamical properties of singular-hyperbolic attractors. Discrete Contin. Dyn. Syst. 19, 1 (2007), 67–87.

[4] Bautista, S. Sobre conjuntos singulares-hiperbólicos. Tese de Doutorado, UFRJ (2005).

[5] Bautista, S., and Morales, C. A. Lectures on sectional-anosov flows. http://preprint.impa.br/Shadows/SERIE_D/2011/86.html.

[6] Bautista, S., and Morales, C. A. Characterizing omega-limit sets which are closed orbits. J. Differential Equations 245, 3 (2008), 637–652.

[7] Bautista, S., and Morales, C. A. A sectional-anosov connecting lemma. Ergodic Theory Dynam. Systems 30, 2 (2010), 339–359.

[8] Bautista, S., and Morales, C. A. On the intersection of sectional-hyperbolic sets. J. of Modern Dynamics 10, 1 (2016), 1–16.
[9] Bautista, S., Morales, C. A., and Pacifico, M. J. On the intersection of homoclinic classes on singular-hyperbolic sets. *Discrete and continuous Dynamical Systems* 19, 4 (2007), 761–775.

[10] Bonatti, C., Diaz, L., and Viana, M. *Dynamics beyond uniform hyperbolicity. A global geometric and probabilistic perspective.*, Encyclopaedia of Mathematical Sciences, 102. Mathematical Physics, III. Springer-Verlag, Berlin, 2005.

[11] Bonatti, C., Pumariño, A., and Viana, M. Lorenz attractors with arbitrary expanding dimension. *C. R. Acad. Sci. Paris Sér. I Math.* 325, 8 (1997), 883–888.

[12] Carballo, C. M., and Morales, C. A. Omega-limit sets close to singular-hyperbolic attractors. *Illinois Journal of Mathematics* 48, 2 (2004), 645–663.

[13] Gähler, S. Lineare 2-normierte räume. *Mathematische Nachrichten* 28, 1-2 (1964), 1–43.

[14] Guckenheimer, J., and Williams, R. F. Structural stability of lorenz attractors. *Publications Mathématiques de l’IHÉS* 50, 1 (1979), 59–72.

[15] Hayashi, S. Connecting invariant manifolds and the solution of the $c^1$-stability and $\omega$-stability conjectures for flows. *Annals of Math.* 145 (1997), 81–137.

[16] Katok, A., and Hasselblatt, B. *Introduction to the Modern Theory of Dynamical Systems*, vol. 54. Cambridge University Press, 1997.

[17] Kawaguchi, A., and Tandai, K. On areal spaces i. *Tensor NS* 1 (1950), 14–45.

[18] López Barragan, A. M., and Sánchez, H. M. S. Sectional anosov flows: Existence of venice masks with two singularities. *Bulletin of the Brazilian Mathematical Society, New Series* 48, 1 (2017), 1–18.

[19] Metzger, R., and Morales, C. A. Sectional-hyperbolic systems. *Ergodic Theory and Dynamical Systems* 28, 05 (2008), 1587–1597.

[20] Morales, C. A. Strong stable manifolds for sectional-hyperbolic sets. *Discrete and Continuous Dynamical Systems* 17, 3 (2007), 553–560.

[21] Morales, C. A. Sectional-anosov flows. *Monatshefte für Mathematik* 159, 3 (2010), 253–260.
[22] Morales, C. A., and Pacífico, M. J. Mixing attractors for 3-flows. *Nonlinearity* 14, 2 (2001), 359–378.

[23] Morales, C. A., and Pacífico, M. J. A dichotomy for three-dimensional vector fields. *Ergodic Theory Dynam. Systems* 23, 5 (2003), 1575–1600.

[24] Morales, C. A., and Pacífico, M. J. Sufficient conditions for robustness of attractors. *Pacific journal of mathematics* 216, 2 (2004), 327–342.

[25] Morales, C. A., and Pacífico, M. J. A spectral decomposition for singular-hyperbolic sets. *Pacific Journal of Mathematics* 229, 1 (2007), 223–232.

[26] Morales, C. A., Pacífico, M. J., and Pujals, E. R. Singular hyperbolic systems. *Proceedings of the American Mathematical Society* 127, 11 (1999), 3393–3401.

[27] Morales, C. A., and Vilches, M. On 2-riemannian manifolds. *SUT J. Math.* 46, 1 (2010), 119–153.

[28] Palis, J., and De Melo, W. *Geometric theory of dynamical systems*. Springer, 1982.

[29] Smale, S. Differentiable dynamical systems. *Bulletin of the American mathematical Society* 73, 6 (1967), 747–817.

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