Differential Algebras in Non-Commutative Geometry

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Abstract

We discuss the differential algebras used in Connes’ approach to Yang-Mills theories with spontaneous symmetry breaking. These differential algebras generated by algebras of the form functions $\otimes$ matrix are shown to be skew tensor products of differential forms with a specific matrix algebra. For that we derive a general formula for differential algebras based on tensor products of algebras. The result is used to characterize differential algebras which appear in models with one symmetry breaking scale.
1 Introduction

In spite of the great experimental success of the Standard Model of electroweak interaction there is a general feeling that the theoretical understanding of this interaction is far from being complete. Not only that the regularities, like the appearance of the elementary particles in families, and the irregularities, like the mass-matrix, are a complete mystery. Also the concept of spontaneous symmetry breaking seems to be arbitrarily introduced by hand in order to turn Yang-Mills theories into experimentally relevant models.

However, there are recent new and promising attempts to solve at least the problem related to the Higgs-mechanism and spontaneous symmetry breaking. They are more or less based on or inspired by Connes’ non-commutative geometry [1]. There is one line of approach initiated by Connes himself [2] which lateron was generalized to a Grand Unification Model [3, 4].

There is another line of approach followed in [5, 6]. It is based on superconnections in the sense of Matthai, Quillen [12]. The key-idea in these models is to extend the usual exterior differential by a matrix derivation. The physical motivation for this was given in [4]. The connection is then taken to be an element of a graded Lie-algebra $SU(m|n)$ which has been extended to a module over differential forms. This line of approach seems to be related to the one of Connes. However, until now it was not known to what extent they are similar and what the precise differences are. A precise comparison between the models [2, 3] and [5, 6, 7] became possible only after the present construction. The results of this comparison will appear in another publication [10].

In this article we want to investigate Connes’ approach to Yang-Mills theory with spontaneous symmetry breaking. More precisely, we will discuss the differential algebra $\Omega_{\mathcal{D}}\mathcal{A}$ in Connes construction. This algebra is a derived object, obtained from an associative algebra $\mathcal{A}$ via the universal differential enveloping algebra and a k-cycle. Therefore $\Omega_{\mathcal{D}}\mathcal{A}$ is not known in general and has to be computed for each specific example, we shall give a quite detailed characterization of this object for general situations. The fact that all physical quantities like connections or curvatures are objects in the differential algebra $\Omega_{\mathcal{D}}\mathcal{A}$ underlines its importance. Some attempts for the construction of the algebra $\Omega_{\mathcal{D}}\mathcal{A}$ were done by [1]. A.Connes gives some special examples in [1].

We show that this algebra is in fact a skew tensor product of a specific differential matrix algebra with differential forms, i.e. matrix valued differential forms.

In the case of Yang-Mills theories with spontaneous symmetry breaking the algebra $\mathcal{A}$ is given as a tensor product of $\mathcal{F}$, the algebra of smooth functions, and $\mathcal{A}_M$, a matrix algebra. The differential algebra $\Omega_{\mathcal{D}}\mathcal{F}$ for the algebra of functions is the usual de Rham
algebra \([\mathbb{I}]\). \(\Omega_\delta A_m\) for the matrix algebra will be easy to compute, as we shall see in sec. 3. Therefore we want to make use of this fact and derive in sec. 4 a general formula which relates \(\Omega_\delta (A_\infty \otimes A_\epsilon)\) of a product algebra to the differential algebras \(\Omega_\delta A_\infty\) and \(\Omega_\delta A_\epsilon\) of the factor algebras. In our case, where the tensor product of an algebra of functions and a matrix algebra is taken, the general relation becomes much simpler. Thus it is straightforward to write down the complete algebra \(\Omega_\delta (\mathcal{F} \otimes A_m)\) which we will do for the 2-point case in sec. 5. This article ends with conclusions drawn in sec. 6. However, we shall first give a brief introduction to the general subject in the next section.

### 2 The Universal Differential Envelope and k-Cycles

We start with a brief review of the basic concepts of non-commutative geometry needed to describe Yang-Mills theories with spontaneous symmetry breaking. This will allow us to fix the notation and to introduce some useful definitions. For a more comprehensive presentation of this subject we refer to [1, 11, 13].

Let \(\mathcal{A}\) be an associative unital algebra. We can construct a bigger algebra \(\Omega A\) by associating to each element \(A \in \mathcal{A}\) a symbol \(\delta A\). \(\Omega A\) is the free algebra generated by the symbols \(A, \delta A, A \in \mathcal{A}\) modulo the relation

\[
\delta(AB) = \delta A B + A \delta B.
\]

With the definition

\[
\delta(A_0 \delta A_1 \cdots \delta A_k) := \delta A_0 \delta A_1 \cdots \delta A_k
\]

\(\Omega A\) becomes a \(\mathbb{N}\)-graded differential algebra with the odd differential \(\delta\), \(\delta^2 = 0\). \(\Omega A\) is called the universal differential envelope of \(A\).

The next element in this formalism is a k-cycle \((\mathcal{H}, \mathcal{D})\) over \(\mathcal{A}\), where \(\mathcal{H}\) is a Hilbert space such that there is an algebra homomorphism \(\pi: \mathcal{A} \rightarrow B(\mathcal{H})\).

\(B(\mathcal{H})\) denotes the algebra of bounded operators acting on \(\mathcal{H}\). \(\mathcal{D}\) is a Dirac operator such that \([D, \pi(A)]\) is bounded for all \(A \in \mathcal{A}\). We can use this operator to extend \(\pi\) to an algebra homomorphism of \(\Omega A\) by defining

\[
\pi(A_0 \delta A_1 \cdots \delta A_k) := \pi(A_0)[D, \pi(A_1)] \cdots [D, \pi(A_k)]
\]

However, in general \(\pi(\Omega A)\) fails to be a differential algebra. In order to repair this, one has to divide out the two sided \(\mathbb{N}\)-graded differential ideal \(J\) given by

\[
J := \bigoplus J^\parallel, \quad J^\parallel := K^\parallel + \delta K^\parallel \cdot \infty, \quad K^\parallel := \ker \pi \cap \otimes A^\parallel.
\]
Now we are ready to define our basic object of interest, $\Omega_D$, as

$$\Omega_D := \bigoplus_{k \in \mathbb{N}} \pi(\Omega^k) / \pi(J^k).$$

$\Omega_D$ is an $\mathbb{N}$-graded differential algebra, where the differential $d$ is defined by

$$d[\pi(\omega)] := [\pi(\delta \omega)], \quad \omega \in \Omega.\]$$

If we take, for example, $A = \mathcal{F}$, the algebra of smooth functions on a compact spin-manifold, the space of square-integrable spin-sections as $\mathcal{H}$ and $D = i\partial$ then $\Omega_D$ is the usual de Rham-algebra $[1]$.\]

Since we are dealing with tensor products of algebras and want to make use of this fact we also have to extend k-cycles over factor algebras to a k-cycle of product algebras. One possibility is provided by the notion of product k-cycles. Suppose we have two k-cycles, $(D_1, \mathcal{H}_\infty)$ over $A_\infty$, $(D_2, \mathcal{H}_\in)$ over $A_\in$ and suppose there is a $\mathbb{Z}_2$-grading on $\mathcal{H}_\infty$ given by $\Gamma_1$. The product of k-cycles $(D_{12}, \mathcal{H}_{\infty\in})$ over $A = A_\infty \otimes A_\in$ is given as

$$\mathcal{H}_{\infty\in} := \mathcal{H}_\infty \otimes \mathcal{H}_\in$$

$$D_{12} := D_1 \otimes 1 + \Gamma_1 \otimes D_2. \quad (2.3)$$

3 Differential Forms of Associative Matrix Algebras

In this section we shall derive some general properties of the differential algebra generated by an associative matrix algebra $A$. The k-cycle $(\mathcal{H}, D)$ over $A$ is specified as

$$\mathcal{H} = \mathfrak{g}^N; \quad D := [\mathcal{M}, \cdot], \quad \mathcal{M} \in \mathfrak{g}^{N \otimes N}. \quad (3.1)$$

Without any loss of generality we may assume that for the algebra homomorphism

$$\pi: A \rightarrow \mathfrak{g}^{N \otimes N},$$

we have

$$\ker (\pi(A)) = \{0\}.$$

Obviously in this case $J^\infty$ is given by

$$J^\infty = \ker (\pi(\otimes A)) \cap \otimes^\infty A.$$

Therefore we conclude that the first non-trivial contributions of the differential operator can only appear at degree $k \geq 2$ of $J$.

For quite general cases the next lemma shows that $J$ is generated by $J^\in$ and as a consequence the differential on $\Omega_D$ is given by a supercommutator.
Lemma 1 Let $\mathcal{A}$ be an associative algebra and $(\mathcal{H}, D)$ a K-cycle over $\mathcal{A}$ as in (3.1) and $\pi$ the corresponding algebra homomorphism with $\ker(\pi(\mathcal{A})) = \{0\}$. If
\[
[\mathcal{M}^e, \pi(\mathcal{A})] \subset \pi(\mathcal{J}^e)
\]
then

i) $\pi(\mathcal{J})$ is generated by $\pi(\mathcal{J}^e)$, i.e.
\[
\pi(\mathcal{J}^\|) = \sum_{\| \neq \varepsilon} \pi(\otimes \mathcal{A} \mathcal{J}^e \otimes \mathcal{J}^{\|} - \varepsilon \mathcal{A}) \ , \ \| \geq \varepsilon ;
\]

ii) the differential $d$ on $\Omega^\| \mathcal{A}$ is given by the graded supercommutator
\[
d[\omega^k] = \left[ [\mathcal{M}, \pi(\omega^{\|})]_\varepsilon \right] = \left[ \mathcal{M} \pi(\omega^{\|}) - (-\infty)^\| \pi(\omega^{\|}) \mathcal{M} \right]
\]
with $[\omega^k] \in \Omega^k \mathcal{A}$ and $\omega^k \in \Omega^k \mathcal{A}$.

Proof: Let us consider $\pi(\omega^k) = [\mathcal{M}, -\infty] \cdots [\mathcal{M}, -\|]$ with $a_1[\mathcal{M}, -\varepsilon] \cdots [\mathcal{M}, -\|] = 1$, $a_i \in \pi(\mathcal{A})$, i.e. $\omega^k \in \mathcal{J}^\|$. We have
\[
\pi(\omega^k) = [\mathcal{M}, -\infty] \cdots [\mathcal{M}, -\|] = -a_1[\mathcal{M}, -\varepsilon] \cdots [\mathcal{M}, -\|] + a_1[\mathcal{M}, -\varepsilon] \mathcal{M} \cdots [\mathcal{M}, -\|]
\]
\[
\vdots
\]
\[
= -\sum_{j=2}^{k} (-1)^j a_1[\mathcal{M}, -\varepsilon] \cdots [\mathcal{M}^{\varepsilon, -\|}] \cdots [\mathcal{M}, -\|] + (-\infty)^\| + \infty [\mathcal{M}, -\varepsilon] \cdots [\mathcal{M}, -\|] \mathcal{M} .
\]
The last term in the last equation vanishes by assumption and the sum is of the form as in eq. (3.3) which proves i).

The second part of the lemma is proved by a similar calculation. We now take $\pi(\omega^k) = a_0[\mathcal{M}, -\infty] \cdots [\mathcal{M}, -\|]$ as a representative of $[\omega^k] \in \Omega^\| \mathcal{A}$. We have to show that the differential
\[
d[\omega^k] = \left[ \mathcal{M} \pi(\omega^{\|}) - (-\infty)^\| \pi(\omega^{\|}) \mathcal{M} \right]
\]
coincides with the one induced by the differential on the universal differential envelope of $\mathcal{A}$. The representative of the first term of the right hand side of eq. (3.4) can be rewritten as
\[
\mathcal{M} \pi(\omega^{\|}) = \mathcal{M}^{-1} [\mathcal{M}, -\infty] \cdots [\mathcal{M}, -\|]
\]
\[
= [\mathcal{M}, -\|] [\mathcal{M}, -\infty] \cdots [\mathcal{M}, -\|] + \cdots + [\mathcal{M}, -\infty] [\mathcal{M}, -\infty] \cdots [\mathcal{M}, -\|]
\]
\[
\vdots
\]
\[
= [\mathcal{M}, -\|] [\mathcal{M}, -\infty] \cdots [\mathcal{M}, -\|] - \sum_{i=\infty}^{\|} (-\infty)^{\| - i \|} [\mathcal{M}, -\varepsilon] \cdots [\mathcal{M}^{\varepsilon, -\|}] \cdots [\mathcal{M}, -\|]
\]
\[
+ (-1)^k a_1[\mathcal{M}, -\varepsilon] \cdots [\mathcal{M}, -\|] \mathcal{M} .
\]
Thus we see that
\[ \mathcal{M}\pi(\omega^\parallel) = [\pi(\omega^\parallel)] + (-\infty)^{\parallel} [\pi(\omega^\parallel) \mathcal{M}] . \]

Inserting this in eq.(3.4) we obtain
\[ d[\omega^k] = [\pi(\delta\omega^k)] . \]

Since \( \mathcal{J} \) is a differential ideal the result is independent of the choice of a representative
and we have proved the lemma.

Our next task is to find algebras for which the condition (3.2) is fulfilled. The next
lemma shows that the matrix algebras which are building blocks in models discussed
in [2] for the two point case and in [3] for the n-point case meet condition (3.2). They
all have in common that they are direct sums of algebras \( \mathcal{A} = \mathcal{A}_\infty \oplus \cdots \oplus \mathcal{A}_1 \) such that
the algebra homomorphism maps them into a block diagonal matrix of the form
\[
\pi(\mathcal{A}) = \begin{pmatrix}
\pi_1(\mathcal{A}_\infty) & \cdots \\
& \ddots \\
& & \pi_n(\mathcal{A}_1)
\end{pmatrix}
\]

where \( \pi_j \) denotes the restriction of \( \pi \) to \( \mathcal{A}_j \). The Dirac operator for those algebras is
off-diagonal which will be made more precise in the following lemma.

**Lemma 2** Let \( \mathcal{A} \) be an associative algebra which can be decomposed into a direct sum
\( \mathcal{A} = \mathcal{A}_\infty \oplus \cdots \oplus \mathcal{A}_1 \) of unital algebras and \( (\mathcal{H},\mathcal{D}) \) a K-cycle over \( \mathcal{A} \) as in (3.1). \( P_i, \)
\( i = 1, \ldots, n \) are projection operators on \( \mathcal{H} \) with
\[
P_i\pi(\mathcal{A}_j) = \pi(\mathcal{A}_j) = \pi(\mathcal{A}_j)P_j \ ; \ P_jP_i = \delta_j^i \ ; \ \sum_j P_j = \infty .
\]

If
\[
P_i\mathcal{M}P_j = \mathbf{1} , \ \ j = 1, \ldots, n,
\]

then
\[
[\mathcal{M}^\varepsilon, \pi(\mathcal{A})] \subset \pi(\mathcal{J}^\varepsilon) .
\]

**Proof:** We introduce the following notation
\[
\mathcal{M}_j := P_j\mathcal{M}P_j , \quad [\cdot] := P_j\pi(\mathcal{A}) , \quad [\cdot] := P_j\pi(\mathcal{B}) , \quad \mathcal{A}, \mathcal{B} \in \mathcal{A}.
\]

Elements \( \pi(\omega^2) , \omega^2 \in \mathcal{J}^\varepsilon \) can be written as
\[
\pi(\omega^2) = \pi(\mathcal{A})[\mathcal{M}^\varepsilon, \pi(\mathcal{B})] .
\]
where $A, B$ obey the condition

$$\pi(A)[M, \pi(B)] = i \cdot$$ \hspace{1cm} (3.8)

In the notation introduced in (3.6) eqs. (3.7, 3.8) take the form

$$P_i \pi(\omega^2) P_j = \sum_{k=1}^{n} a_i M_{jk} \parallel M \parallel | - \sum_{\parallel M \parallel = \infty} \pi[|M|] \parallel M \parallel$$

$$0 = a_i M_{jk} | - \pi[|M|] \parallel M \parallel$$.

For any $A'_i \in A_i$ one can choose

$$A_1 = A'_i \quad B_1 = 1_i$$

$$A_2 = -1_i \quad B_2 = A'_i \cdot$$ \hspace{1cm} (3.9)

Since $M_i = \iota$ it is straightforward to verify that

$$\sum_{r \in \{1, 2\}} \pi(A_r)[M, \pi(B_i)] = \iota$$

and

$$\sum_{r \in \{1, 2\}} \pi(A_r)[\mathcal{E}, \pi(B_i)] = [\mathcal{E}, \pi(A')]$$.

This shows that

$$[\mathcal{E}, \pi(A)] \subset \pi(J^\mathcal{E})$$.

Since elements $A_r, B_r \in A$ as in eq. (3.9) exist for any $i \in \{1, \ldots, n\}$ the lemma is proved.

We now want to apply these results to a two point case, i.e. to a matrix algebra which is given as the direct sum of the algebras $A_\infty = \mathbb{C}_{\times \times 1}$ and $A_\in = \mathbb{C}_{\times \times \times \times m}$ of complex $n \times n$ resp. $m \times m$ matrices. The representation of the algebra and the Dirac operator take the form

$$\pi(A) = \begin{pmatrix} A_\infty & 0 \\ 0 & A_\in \end{pmatrix}, \quad \mathcal{M} = \begin{pmatrix} 0 & \mu^* \\ \mu & 0 \end{pmatrix}$$

where $\mu$ denotes an arbitrary (non-zero) complex $n \times m$ matrix. Let us first consider the algebra generated by

$$A_\infty[\mu^*, A_\infty]A_\infty \cdot$$ \hspace{1cm} (3.10)

There are two possibilities. Either $\mu^* \mu \sim 1_{m \times m}$, then the commutator in (3.10) is zero and no non-trivial algebra can be generated, or $\mu^* \mu \not\sim 1_{m \times m}$, then the whole algebra $A_\infty$ is generated. There is the same situation for

$$A_\in[\mu \mu^*, A_\in]A_\in$$

and therefore we may distinguish three cases:
i. $\mu^*\mu \sim 1_{m \times m}$ and $\mu\mu^* \sim 1_{n \times n}$ which is possible only for $m = n$, i.e. $A_\infty = A_\varepsilon$. In this case we have $\mathcal{J} = \{t\}$ and
\[
\Omega_{DA}^{2n} A = \begin{pmatrix} A_\infty & 0 \\ 0 & A_\infty \end{pmatrix}, \quad \otimes_{DA}^{\varepsilon, \ldots, \infty} A = \begin{pmatrix} 0 & A_\infty \\ A_\infty & 0 \end{pmatrix}, \quad \varepsilon \in \mathbb{N}.
\]
The multiplication is just the ordinary matrix multiplication of $2m \times 2m$ matrices.

ii. $\mu^*\mu \not\sim 1_{m \times m}$ and $\mu\mu^* \not\sim 1_{n \times n}$. Here only $\Omega_{DA}^1 A$ survives since $\pi(\Omega^2 A) = \pi(A) = \mathcal{J}^\varepsilon$ and therefore $\pi(\Omega^k A) = \mathcal{J}^\parallel \| \varepsilon \geq \varepsilon$. In this case one may view
\[
\Omega_{DA}^1 A = \{A \in \mathbb{C}^{\times \times } \} \oplus \{B \in \mathbb{C}^{\times \times } \}
\]
as a module over $A$. There is no non-trivial multiplication of elements in $\Omega_{DA}^1 A$, i.e. for $\nu, \omega \in \Omega_{DA}^1 A$ we have $\nu \cdot \omega = 0$.

iii. $m \leq n$, $\mu^*\mu \sim 1_{m \times m}$ and $\mu\mu^* \not\sim 1_{n \times n}$. Again we have
\[
\Omega_{DA}^1 A = \{A \in \mathbb{C}^{\times \times } \} \oplus \{B \in \mathbb{C}^{\times \times } \}
\]
as a module over $A$. However, in this case $\Omega_{DA}^2 A$ is non-trivial since $J^2 = \begin{pmatrix} 0 & 0 \\ 0 & A_\varepsilon \end{pmatrix}$ implies $\pi(\Omega^k A) = \mathcal{J}^\parallel \| \geq \varepsilon$. and therefore
\[
\Omega_{DA}^2 A = \begin{pmatrix} A_\infty & 0 \\ 0 & 0 \end{pmatrix}
\]
and all higher degrees of $\Omega_{DA} A$ are trivial. The multiplication $\circ$ of two elements $(A, B), (A', B') \in \Omega_{DA}^1 A$ is given by
\[
(A, B) \circ (A', B') = \begin{pmatrix} A \cdot B' & 0 \\ 0 & 0 \end{pmatrix} \in \Omega_{DA}^2 A
\]
where $\cdot$ denotes the usual matrix multiplication. A representation for the matrix algebra is now on the zeroth and first degree given by
\[
\Omega_{DA}^0 A = \begin{pmatrix} A_\infty & 0 \\ 0 & A_\varepsilon \end{pmatrix}, \quad \otimes_{DA}^{\infty} A = \begin{pmatrix} 0 & \eta \mathbb{C}^{m \times n} \\ \eta' \mathbb{C}^{n \times m} & 0 \end{pmatrix}.
\]
The relations for the formal elements $\eta, \eta'$ are
\[
\eta \eta' \neq 0, \quad \eta' \eta = 0.
\]
Although these relations seem a little awkward it is not difficult to find a representation for them. E.g.

\[
\eta = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \eta' = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}
\]

is such an representation.

4 \( \Omega_D \) of the Tensor Product of Algebras

In this section we are going to establish a relation allowing the computation of \( \Omega_D \) for a tensor product of two algebras \( A = A_\infty \otimes A_\varepsilon \). The idea is to use the knowledge of \( \Omega_D(A_i) \) in order to calculate \( \Omega_D(A) \). This cannot be done by simply taking tensor products of \( \Omega_D A_\infty \) and \( \Omega_D A_\varepsilon \). We observe that already at the level of universal differential envelopes we have in general

\[
\Omega(A_\infty \otimes A_\varepsilon) \neq \Omega A_\infty \otimes \Omega A_\varepsilon.
\]

Let us denote by \( \delta_{12} \) the differential on \( \Omega(A_\infty \otimes A_\varepsilon) \). The differential on \( \Omega A_\infty \otimes \Omega A_\varepsilon \) is \( \delta_S := \delta_1 \otimes 1 + \chi \otimes \delta_2 \), where \( \delta_i \) denotes the differentials on \( \Omega A_i \) and \( \chi \) is the \( \mathbb{Z}_2 \) grading on \( \Omega A_\infty \). We apply \( \delta_S \) on some arbitrary element \((a \otimes b) \in A_\infty \otimes A_\varepsilon\)

\[
\delta_S(a \otimes b) = \delta_1 a \otimes b + a \otimes \delta_2 b
\]

\[
= (1 \otimes b)(\delta_1 a \otimes 1) + (a \otimes 1)(1 \otimes \delta_2 b)
\]

\[
= (1 \otimes b)\delta_S(a \otimes 1) + (a \otimes 1)\delta_S(1 \otimes b).
\]

For \( \delta_{12} \) we can only use relation (2.1) to write

\[
\delta_{12}(a \otimes b) = \delta_{12}(a \otimes 1)(1 \otimes b) + (a \otimes 1)\delta_{12}(1 \otimes b),
\]

there is no relation which tells us that

\[
\delta_{12}(a \otimes 1)(1 \otimes b) - (1 \otimes b)\delta_{12}(a \otimes 1) = 0.
\]

Thus we see that \( \Omega(A_\infty \otimes A_\varepsilon) \) and \( \Omega A_\infty \otimes \Omega A_\varepsilon \) are not isomorphic.

However, we can prove the following lemma.

Lemma 3 Let \((\omega_1, \delta_1)\) and \((\omega_2, \delta_2)\) be two graded differential algebras which are both generated by the same algebra \( A \) as zeroth grading and their respective differentials. Let \( B \) be an algebra and let

\[
\pi_1 : \omega_1 \rightarrow B
\]

\[
\pi_2 : \omega_2 \rightarrow B
\]

\(^1\)The hat on the tensor product denotes the \( \mathbb{Z}_2 \) graded tensor product.
be algebra homomorphisms. $J_{π_i}$ are the corresponding differential ideals as defined in (2.2) and $Ω_{π_i} = \bigoplus_{k\in\mathbb{N}} \omega_k^k / J_{π_i}^k$ the induced differential algebras.

If
\[ π_1(a) = π_2(a) \] and \[ π_1(δ_1a) = π_2(δ_2a) \] for all $a \in A$

then
\[ Ω_{π_1} = Ω_{π_2} . \]

Proof: As $ω_1$ and $ω_2$ are generated by elements $a_0δ_1a_1 . . . δ_i a_k$ $i = 1, 2$, we obviously have
\[ π_1(ω_1) = π_2(ω_2) \]

since $π_i$ are algebra homomorphisms. Therefore the relation $a_0δ_1a_1 . . . δ_i a_k \in \ker_{π_1}^k$ implies $a_0δ_2a_1 . . . δ_2 a_k \in \ker_{π_2}^k$ and we get
\[ π_1(J_{π_1}) = π_2(J_{π_2}) . \]

This establishes the identity of subalgebras of $B$.

An example for this situation is $Ω(A^∞ \otimes A)$ and $Ω(A^∞) \hat{⊗} Ω(A)$. The lemma allows to use the second algebra for the calculation of $Ω(Δ(A^∞ \otimes A))$ because of our choice of the product k-cycle. Thus we now have to analyze
\[ \frac{π((Ω(A^∞) \otimes A))^\parallel)}{π(δ\ker^\parallel−∞ + \ker^\parallelπ)} . \]

This can be done by splitting the mapping
\[ π = π_1 \otimes π_2 : ΩA^∞ \hat{⊗} ΩA \longrightarrow B(H^∞ \otimes H) \]

into $π = π_Σ \circ π_{⊕}$ with $π_{⊕}$ defined by:
\[ π_{⊕} : \begin{align*}
(Ω A^∞ \otimes A) & \longrightarrow \bigoplus_{i+j=k} π_1(Ω A^i) \otimes π_2(Ω A^j) \\
a_0δ_1a_1 . . . δ_i a_i \otimes b_0δ_2b_1 . . . δ_j b_j & \mapsto π_1(a_0δ_1a_1 . . . δ_i a_i) \otimes π_2(b_0δ_2b_1 . . . δ_j b_j)
\end{align*} \]

Accordingly $π_Σ$ is then given by the summation:
\[ π_Σ : \bigoplus_{i+j=k} π_1(Ω A^i) \otimes π_2(Ω A^j) \longrightarrow B(H^∞ \otimes H) \]
\[ \sum_{i+j=k} (v_i \otimes ω_j) \mapsto \sum_{i+j=k} v_i \otimes ω_j \]

I.e. $π_{⊕}$ maps each term of the sum
\[ (Ω A^∞ \otimes A) = \bigoplus_{i+j=|\parallel|} A^i \otimes A^j^{|\parallel|} \]
The reason for this splitting will become clearer by calculating the quotient
\[ \frac{\pi(\Omega(A_{\infty} \otimes A_{\epsilon})^p)}{\pi(\delta \ker \pi^{\infty} + \ker \pi_{\oplus})} \]

We fix the notation:
\[ J^k_{\oplus} = \delta \ker \pi^{k-1}_{\oplus} + \ker \pi^k_{\oplus} \]

With the following result from multilinear algebra
\[ \ker \pi^{k}_{\oplus} = \bigoplus_{i+j=k} \ker \pi^i_1 \otimes \Omega A^{j}_\epsilon + \Omega A^{i}_\infty \otimes \ker \pi^j_2 , \]

this gives
\[ \pi(\delta \ker \pi^{i-1}_i) = \pi(J^{i}_{\pi_2}) \]

There is now:
\[ \frac{\pi(\Omega(A_{\infty} \otimes A_{\epsilon})^p)}{\pi(J^p_{\oplus})} = \bigoplus_{i+j=k} \pi_1(J^i_{\pi_1}) \otimes \pi_2(\Omega A^{j}_\epsilon) + \pi_1(\Omega A^{j}_\infty) \otimes \pi_2(J^j_{\pi_2}) \]

The multiplication is here given by the skewsymmetric multiplication
\[ \nu_1 \otimes \omega_1 \cdot \nu_2 \otimes \omega_2 = (-1)^{\partial_{\nu_1} \partial_{\nu_2}} \nu_1 \nu_2 \otimes \omega_1 \omega_2 \]

and the differential is:
\[ d(\nu \otimes \omega) = d_1 \nu \otimes \omega + (-1)^{\partial_{\nu}} \nu \otimes d_2 \omega \]

Thus we see that the differential algebra related to \( \pi_{\oplus} \) is just the tensor product
\[ \Omega_{D} A_{\infty} \otimes_{D} A_{\epsilon} . \]

Suppose the \( \Omega_{D} A_{\epsilon} \) are known. The non-trivial step in the calculation of \( \Omega_{D}(A_{\infty} \otimes A_{\epsilon}) \) is then related to the map \( \pi_{\Sigma} \). Before we formulate the proposition relating the tensor product (4.2) to \( \Omega_{D}(A_{\infty} \otimes A_{\epsilon}) \) via \( \pi_{\Sigma} \) we need one further definition:
Definition 1 If $\beta : \omega \to \beta(\omega)$ is an algebra homomorphism and $j$ an ideal in $\omega$ then let $\beta_q$ be the homomorphism
$$
\beta_q : \frac{\omega}{j} \to \frac{\beta(\omega)}{\beta(j)}.
$$

Proposition 1 Let $(\omega, d), (\omega_1, d_1), (\omega_2, d_2)$ be $N$-graded differential algebras, $\alpha$ and $\beta$ differential algebra homomorphisms
$$
\omega \xrightarrow{\alpha} \omega_1 \xrightarrow{\beta} \omega_2
$$
and let $\pi = \beta \circ \alpha$ be the composition of these maps. Then we have
$$
\Omega_\pi = \bigoplus_{k \in N} \frac{\beta_q(\Omega^k)}{\beta_q(J^k_{\beta_q})}
$$
where $\Omega_\pi, \Omega_\alpha$ are given as in lemma 4. $\beta_q$ is given by definition 3 and
$$
J^k_{\beta_q} = d_{\Omega_\alpha} \ker \beta_q^{k-1} + \ker \beta_q^k.
$$

Proof: We shall go through the following proof in two steps. First the vector space isomorphism is established at the level of the $k$th grading. Then we show that multiplication and differential are respected thus giving an isomorphism of differential algebras. Now $\ker \alpha \subset \ker \pi$ implies $J_\alpha \subset J_\pi$, hence we get
$$
\Omega^k_\pi = \frac{\pi(\omega)}{\pi(J^k_\pi)} = \frac{\pi(\omega)}{\pi(J^k_\alpha)} = \frac{\beta_q(\alpha(\omega))}{\beta_q(\alpha(J^k_\alpha))} = \frac{\beta_q(\alpha(J^k_\alpha))}{\beta_q(J^k_{\beta_q})}
$$
by definition of $\beta_q$. Next we show that
$$
\alpha(J^k_\pi) = d_{\Omega_\alpha} \ker \beta_q^{k-1} + \ker \beta_q^k
$$
and therefore we first prove the identity
$$
\ker \beta_q^k = \frac{\alpha(\ker \pi^k + J^k_\alpha)}{\alpha(J^k_\alpha)}.
$$
Clearly ”$\supseteq$" in equation (4.4) is given. On the other hand an element of $\ker \beta_q^k$ is represented by $\nu \in \alpha(\omega^k)$ with $\beta(\nu) \in \pi(J^k_\alpha)$. This means $\beta(\nu) = \beta \circ \alpha(j)$ for an element $j \in J^k_\alpha$ and $\nu = \alpha(\rho)$ for some $\rho \in \omega$. Now $\nu = \alpha(\rho - j) + \alpha(j)$ with $\pi(\rho - j) = 0$. Therefore $\nu \in \alpha(\ker \pi^k + J^k_\alpha)$. This gives equation (4.6). Equation (4.5) is derived by:
$$
d_{\Omega_\alpha} \ker \beta_q^{k-1} + \ker \beta_q^k = d_{\Omega_\alpha} \left( \frac{\alpha(\ker \pi^{k-1} + J^{k-1}_\alpha)}{\alpha(J^{k-1}_\alpha)} + \frac{\alpha(\ker \pi^k + J^k_\alpha)}{\alpha(J^k_\alpha)} \right) = \frac{\alpha(d \ker \pi^{k-1})}{\alpha(J^k_\alpha)} + \frac{\alpha(\ker \pi^k + d \ker^{k-1})}{\alpha(J^k_\alpha)} = \frac{\alpha(\ker \pi^{k-1} + \ker \pi^k)}{\alpha(J^k_\alpha)} = \frac{\alpha(J^k_\pi)}{\alpha(J^k_\alpha)}.
$$
On the other hand

\[ \Omega^k_\alpha = \frac{\alpha(\omega^k)}{\alpha(J^k_\alpha)} , \]

such that equation \[4.4\] now reads

\[ \Omega^k_\pi = \frac{\beta_q(\Omega^k_\alpha)}{\beta_q(J^k_\beta q)} . \]

This proves the vector space identity. Going through the proof so far it becomes clear that the isomorphism \(i\) given by equation \[4.3\] is the identity map on representatives in \(\pi(\omega)\) followed by different quotient building mechanisms. The quotient in \(\Omega_\pi\) is split up into a double quotient. Using the definition of \(d_{\Omega_\pi}\) on \(\Omega_\pi\) and \(d_{RHS}\) given on

\[ \bigoplus_{k \in N} \frac{\beta_q(\Omega^k_\alpha)}{\beta_q(J^k_\beta q)} \]

by

\[ d_{RHS}[\nu]_{J_a} \beta_q(J_{\beta q}) = [d\Omega_\alpha][\nu]_{J_a} \beta_q(J_{\beta q}) = [(\alpha(d\nu))[J_a] \beta_q(J_{\beta q}) \]

we have

\[ i \circ d_{\Omega_\pi} = d_{RHS} \circ i . \]

The corresponding relation holds for the multiplication defined on representatives in a similar fashion:

\[ [\nu_1]_{\Omega_\pi} \circ [\nu_2]_{\Omega_\pi} = [\nu_1 \nu_2]_{\Omega_\pi} \]

\[ [(\nu_1)[J_a] \beta_q(J_{\beta q}) \cdot [(\nu_2)[J_a] \beta_q(J_{\beta q}) = [(\nu_1 \nu_2)[J_a] \beta_q(J_{\beta q}) \]

such that

\[ i([\nu_1][\nu_2]) = i([\nu_1]) \cdot i([\nu_2]) . \]

By these rules for multiplication and differential we thus have established an isomorphism of differential algebras.

We now apply proposition \[\ref{proposition}\] to our situation.

**Theorem 1**

\[
\Omega_p(A_\infty \otimes A_\epsilon) = \bigoplus_{\parallel \in \mathcal{N}} \frac{\pi_{\pm 1} \left( \left( \otimes_p A_\infty \otimes \otimes_p A_\epsilon \right) \right)}{\pi_{\pm 1} \left( \left( ker \pi_{\pm 1} \right) + ker \pi_{\pm 1} \right)} \tag{4.7}
\]

**Proof:** In proposition \[\ref{proposition}\] we set

\[ \omega = \Omega A_\infty \otimes \Omega A_\epsilon \]
which is justified by lemma 3. For the maps we insert
\[ \alpha = \pi_\oplus \quad \beta = \pi_\Sigma. \]
Using the result 4.1 it follows immediately that \( \Omega_\pi \) is isomorphic to \( \Omega_D(\mathcal{A}_\infty \otimes \mathcal{A}_\epsilon) \).

For all examples of interest for particle physics the algebra has the following form
\[ \mathcal{A} = \mathcal{F} \otimes \mathcal{A}_M, \]
that is \( \mathcal{A}_\infty = \mathcal{F} \) is the algebra of smooth functions on a manifold and \( \mathcal{A}_\epsilon = \mathcal{A}_M \) is a matrix algebra. For this special case the next lemma shows, that the denominator in (4.7) vanishes. It is not necessary to go as far as theorem 1 since it suffices to use:
\[ \Omega_D(\mathcal{A}_\infty \otimes \mathcal{A}_\epsilon)^\parallel = \frac{\pi(\otimes(\mathcal{A}_\infty \otimes \mathcal{A}_\epsilon)^\parallel)}{\pi(\mathcal{J}_\parallel)} \]

If \( \pi_1 : \mathcal{F} \rightarrow \mathcal{L}(\mathcal{H}_\infty) \) is a representation of smooth functions on the square-integrable spinors as described in [1], \( \pi_2 : \mathcal{A}_\epsilon \rightarrow M_n(C) \) an injective representation of a matrix algebra \( \mathcal{A}_\epsilon \) on \( \mathfrak{C}^\infty \) we can construct a representation of \( \Omega_\mathcal{F} \otimes \mathcal{A}_M \) on \( \mathcal{H}_\infty \otimes \mathfrak{C} \) using the derivation
\[ \mathcal{D} = [\mathcal{D}, \cdot] + [\gamma^\nabla \otimes \mathcal{M}, \cdot] \]
with the Dirac operator \( D = i\partial_\mu \gamma^\mu \) and an \( n \times n \)-matrix \( \mathcal{M} \). \( \mathcal{A}_\epsilon \) and \( \mathcal{M} \) should be chosen in such a way that the condition (3.5) is satisfied.

**Lemma 4** *With the above preliminaries*
\[ \frac{\pi(J^k_\parallel)}{\pi(\mathcal{J}_\parallel)} = \{0\} \]

**Proof:** During this proof we shall use the following shorthands:
\[ \pi_1(\Omega_\mathcal{F}^\parallel) = \omega^\parallel_\infty \quad \pi_\epsilon(\otimes \mathcal{A}_M^\parallel) = \omega^\parallel_\epsilon \quad \pi_\parallel(\mathcal{J}_\parallel^\parallel) = \| \]

The important property of \( \Omega(\mathcal{F}) \) is for \( k \geq 2 \):
\[ j^k_1 = \omega^{k-2}_1 \subseteq \omega^k_1 \]
Thus we have
\[ \pi(J^k_\parallel) = \sum_{i+j=k} j^i_1 \otimes \omega^j_2 + \omega^i_1 \otimes j^j_2 \]
\[ = \sum_{i+j=k} \omega^i_1 \otimes j^j_2. \]
We shall now choose a representative $\alpha \in \pi(J^h_\pi)$ and show that $\alpha \in \pi(J^h_\oplus)$. $\alpha$ can be written as $\alpha = \pi(\delta k)$ with

$$ k = \bigoplus_{i+j=k-1} k_i^j \otimes k_2^j \in \ker \pi^{k-1} \quad (4.10) $$

and

$$ k_i^j = f_0^j \delta_1 f^i_1 \cdots \delta_1 f^i_1, \quad k_2^j = A^j_0 \delta_2 A^j_1 \cdots \delta_2 A^j_2. $$

where $f$ are functions and $A \in A_M$. In equation (4.10) we have suppressed a further summation due to the tensor product in order to simplify notation. Then

$$ \alpha = \pi(\delta k) = \sum_{i+j=k-1} \pi_1(\delta_1 k^j_1)(\gamma^j) \otimes \pi_2(\delta_2 k^j_2) + (-1)^i \pi_1(k^j_1)(\gamma^j) \otimes \pi_2(\delta_2 k^j_2) $$

$$ = \sum_{i+j=k-1} d_d f_0^i d_d f^i_1 \cdots d_d f^i_1(\gamma^j) \otimes A^j_0 d_M A^j_1 \cdots d_M A^j_2 $$

$$ + \sum_{i+j=k-1} (-1)^i f_0^i d_d f^i_1 \cdots d_d f^i_1(\gamma^j) \otimes d_M A^j_0 d_M A^j_1 \cdots d_M A^j_2 \quad (4.11) $$

with

$$ d_d f = [D, f] = \pi_1(\delta_1 f) \quad , \quad d_M A = [M, A] = \pi_\epsilon(\delta_\epsilon A). $$

Since $k \in \ker \pi^{k-1}$ we also have

$$ 0 = \sum_{i+j=k-1} f_0^i d_d f^i_1 \cdots d_d f^i_1(\gamma^j) \otimes A^j_0 d_M A^j_1 \cdots d_M A^j_2 \quad (4.12) $$

We shall first deal with the second term of the r.h.s. of (4.11). We know from section 2 that $d_M$ can be written as a supercommutator up to elements generated by $[M^\epsilon, \cdot]$ which are in the ideal (4.9). Thus we can rewrite this term as

$$ \left[ \gamma^5 \otimes M, \sum_{i+j=k-1} f_0^i d_d f^i_1 \cdots d_d f^i_1(\gamma^j) \otimes A^j_0 d_M A^j_1 \cdots d_M A^j_2 \right] + \text{Terms in } [M^\epsilon, \cdot] $$

and therefore it is contained in $\pi(J^h_\oplus)$. We now turn to the first term of the r.h.s. of (4.11). By using $d_d f = [D, f]$ and equation (4.12) we obtain

$$ \text{first term of (4.11)} = - \sum_{i+j=k-1} f_0^i D(Df^i_1) \cdots (Df^i_1)(\gamma^j) \otimes A^j_0 d_M A^j_1 \cdots d_M A^j_2. $$

At least two of the $\gamma$-matrices appearing in this expression are identical, therefore this term is contained in

$$ \sum_{i+j=k-2} \omega^i \otimes \omega^j $$

and therefore in $\pi(J^h_\oplus)$ according to (4.9). This gives $\alpha \in \pi(J^h_\oplus)$, thus completing the proof.
The results of this section can be summarized as follows:

For all algebras $\mathcal{A} = \mathcal{F} \otimes \mathcal{A}_M$ fulfilling the preliminaries of lemma 4, that is for all algebras relevant for particle physics, $\Omega_D$ is given by (using the conventions 4.8)

$$\Omega_D(\mathcal{F} \otimes \mathcal{A}_M) = \frac{\omega_{\parallel} \otimes \omega_{\parallel} + \ldots + \omega'_{\parallel} \otimes \omega'_{\parallel}}{\omega_{\parallel} \otimes \omega_{\parallel} + \ldots + \omega'_{\parallel} \otimes \omega'_{\parallel} + \omega_{\parallel} \otimes \omega_{\parallel} + \ldots + \omega'_{\parallel} \otimes \omega'_{\parallel}}$$ (4.13)

This can easily be calculated for any specific example once one has determined the relevant terms for $\mathcal{F}$ and $\mathcal{A}_M$.

5 The Two Point Case

In this section we want to apply the general results, developed in the previous section, to the two point case. The algebra $\mathcal{A}$ is

$$\mathcal{A} = \mathcal{F} \otimes (\mathcal{A}_\infty \oplus \mathcal{A}_\epsilon) = \mathcal{F} \otimes \mathcal{A}_M$$

where $\mathcal{A}_\infty$, $\mathcal{A}_\epsilon$ denote $\mathbb{C} ^{m \times m}$ resp. $\mathbb{C} ^{n \times n}$ matrix algebras. The Dirac operator for $\Omega(\mathcal{A}_\infty \oplus \mathcal{A}_\epsilon)$ is off-diagonal as in (3.5). Thus $\Omega_D(\mathcal{A}_\infty \oplus \mathcal{A}_\epsilon)$ is known and we can distinguish three different cases as shown at the end of sec. 3. $\Omega_D \mathcal{F}$ is the de-Rham complex and the Dirac operator for the product k-cycle is

$$\mathcal{D} = i \partial \otimes 1 + [\gamma^5 \otimes \mathcal{M}, \cdot]$$

A general result is

$$\Omega_D \mathcal{A}' = \mathcal{A}, \quad \otimes_D \mathcal{A}^\infty = (\otimes_D \mathcal{F} \otimes_D (\mathcal{A}_\infty \oplus \mathcal{A}_\epsilon))^\infty$$

as one immediately infers from equation (4.13).

We now analyze the higher degrees of $\Omega_D \mathcal{A}$ for the three different $\Omega_D(\mathcal{A}_\infty \oplus \mathcal{A}_\epsilon)$ as in sec. 3.

i. $\mu^* \mu \sim 1_{m \times m}$ and $\mu \mu^* \sim 1_{n \times n}$, i.e. $\mathcal{A}_\infty = \mathcal{A}_\epsilon$. Because of the isomorphism (3.11) we have:

$$\pi_1(\Omega \mathcal{F}^\parallel) \otimes \pi_\epsilon(\otimes \mathcal{A}_M^\parallel) = \pi_\parallel(\otimes ^\parallel \mathcal{F}^\parallel) \otimes \pi_\epsilon(\otimes \mathcal{A}_M^\parallel)$$

Inserting this in equation (4.13) and using the fact that

$$\pi_1(\mathcal{J} \mathcal{F}^k) = \pi_1(\Omega \mathcal{F}^\parallel - \epsilon)$$

and

$$\pi_2(\mathcal{J} \mathcal{A}_M) = \{t\}$$

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one obtains for $\kappa = 2$

$$\Omega^\kappa_{D,A} = \frac{\Omega^1(\Omega^\infty_F) \otimes \pi_\infty(\otimes \mathcal{A}'_M) + \pi_\infty(\otimes \mathcal{A}^\infty_M)}{\pi_1(\Omega^1_\mathcal{F}')} \otimes \pi_\infty(\otimes \mathcal{A}'_M)}$$

$$= \Lambda^2 \otimes \mathcal{A}_M + \ast^\infty \otimes \mathcal{M} \mathcal{A}_M$$

and for $\kappa > 2$:

$$\Omega^\kappa_{D,A} = \frac{\Omega^1(\Omega^\kappa_\mathcal{F}) \otimes \pi_\infty(\otimes \mathcal{A}'_M) + \pi_\infty(\otimes \mathcal{A}^\infty_M)}{\pi_1(\Omega^\kappa_\mathcal{F}')} \otimes \pi_\infty(\otimes \mathcal{A}'_M) + \pi_\infty(\otimes \mathcal{A}^\infty_M)}$$

$$= \Lambda^\kappa \otimes \mathcal{A}_M + \ast^\kappa \otimes \mathcal{M} \mathcal{A}_M$$

In the quotient we suppressed further terms which trivially cancel out. $\Lambda^k$ denotes the space of differential forms of degree $k$. The degree of an element $\alpha \in \Omega^\kappa_{D,A}$ is the sum of the form degree and the degree of the matrix algebra. We see that although all degrees $\kappa \in \mathbb{N}$ of the matrix algebra $\Omega^\kappa_{D,A}$ are non-trivial, in the algebra $\Omega^1_{D,A}$ only the zeroth and first matrix degrees appear. However, this situation can change if we include a ”generation-space”, i.e. if we allow for a bigger representation space for the algebra $\mathcal{A}_M$. The homomorphism

$$\pi_2 : \mathcal{A}_M \longrightarrow \mathfrak{g} \times \mathfrak{g}$$

is extended to

$$\pi'_2 : \mathcal{A}_M \longrightarrow \mathfrak{g} \times \mathfrak{g} \otimes \mathfrak{g}$$

where $\mathfrak{g}$ is the ”generation-space”. The new homomorphism is given as

$$\pi'_2 = \pi_2 \otimes 1$$

This by itself would not yield higher matrix degrees in $\Omega^\kappa_{D,A}$. In order to get that we have to use the larger freedom in the choice of the matrix $\mathcal{M}$. We now take

$$\mathcal{M}' = \begin{pmatrix} 0 & \mu^* \otimes G^* \\ \mu \otimes G & 0 \end{pmatrix}.$$ 

Here $G$ denotes an arbitrary $\mathfrak{g} \otimes \mathfrak{g}$-matrix. The effect of this extension is that we now can distinguish between elements $\alpha \in \pi_1(\Omega^\kappa_\mathcal{F} \otimes \otimes \mathcal{A}'_M)$ and $\beta \in \pi_1(\Omega^\kappa_\mathcal{F} \otimes \otimes \mathcal{A}^\infty_M)$ for $p \neq q$ as long as the powers of $G^*G$ resp. $GG^*$ are linearly independent for $p$ and $q$. Therefore they cannot be cancelled by the denominator of equation (1.13). However there is an integer $p_0 \leq g$ for which the powers of the matrices become linearly dependent. In this case any element $\alpha_{p_0} \in \pi(\Omega^\kappa_\mathcal{F} \otimes \otimes \mathcal{A}'_M)$ can be written as a linear combination of elements with smaller matrix degree:

$$\alpha_{p_0} = \sum_{q=1}^{q \leq p_0/2} \alpha_{p_0-2q}$$, \quad $\alpha_{p_0-2q} \in \pi(\Omega^\kappa_\mathcal{F} \otimes \otimes \mathcal{A}'_M^{\kappa-\infty})$
As a consequence all terms with matrix degree \( p \geq p_0 \) in \( \pi(\Omega A) \) are cancelled by the denominator of (4.13).

These results can be summarized by defining the following representation for \( \Omega_D A \). The matrix part \( \mathcal{A}'_M \) of the algebra is generated by the zeroth order

\[
\mathcal{A}'_M = \begin{pmatrix} A_\infty & 0 \\ 0 & A_\infty \end{pmatrix}
\]

and the first order

\[
\mathcal{A}'_M = \begin{pmatrix} 0 & \eta A_\infty \\ \eta A_\infty & 0 \end{pmatrix}.
\]

Here we have introduced a formal element \( \eta \) which has the property that

\[
\eta^{p_0} = 0 ; \quad \eta^p \neq 0, \quad p < p_0
\]

and it commutes with all other elements of the algebra. Thus \( \mathcal{A}'_M \) is a graded algebra with highest degree \( p_0 - 1 \) and an induced \( \mathbb{Z}_2 \) grading. The full algebra \( \Omega_D A \) is obtained by taking the graded tensor product of \( \mathcal{A}'_M \) and the de Rham algebra \( \Lambda \)

\[
\Omega_D A = \ast \otimes \mathcal{A}'_M.
\]

The degree of elements in \( \Omega_D A \) is the sum of form degree and matrix degree. The derivation on an element \( \alpha \in \Omega_D A^\bullet \) is given as

\[
d\alpha = d_C \alpha + \left[ \begin{pmatrix} 0 & \eta \\ \eta & 0 \end{pmatrix}, \alpha \right]
\]

where \( d_C \) denotes the usual exterior derivative and the commutator is the graded commutator.

One now might wonder what has happened to the ”generation” space? In fact, it is not needed at the level of the algebra since it was introduced to separate the matrix degrees. This task has been taken over by the element \( \eta \) which also has the nilpotency property \( \eta^{p_0} = 0 \). However, the ”generation”-matrix \( M \) may have a physical interpretation as a mass-matrix for fermions and one might wish to keep it in the algebra. This is of course possible and does not change any algebraic properties.

ii. \( \mu^* \mu \not\sim 1_{m \times m} \) and \( \mu \mu^* \not\sim 1_{n \times n} \). In this case \( \Omega_D A_M \) is the highest matrix degree in \( \Omega_D \mathcal{F} \otimes \mathcal{D} A_M \) and therefore no further cancellations appear in equation (4.13), i.e.,

\[
\ker \pi_{\Sigma_y} = \{0\}.
\]

Thus we infer that

\[
\Omega_D A = \otimes \mathcal{F} \otimes \mathcal{D} A_M = \ast \otimes \mathcal{D} A_M
\]

Note that the introduction of a ”generation”-space would not change the situation.
iii. \( m \leq n, \mu^*\mu \sim 1_{m \times m} \) and \( \mu^*\mu \not\sim 1_{n \times n} \). In this case the highest matrix degree is 2 but
\[
\pi(\Omega^\text{F} \otimes \otimes A^\epsilon_M) \subset \pi(\otimes \otimes F \otimes \otimes A'_M) .
\]
Therefore the highest matrix degree in \( \Omega_D A \) is 1 and we can represent this algebra in the same way as in ii. The extension by a "generation"-space as in i. can be used to make \( \pi(\Omega^\text{F} \otimes \otimes A^\epsilon_M) \) and \( \pi_2(\Omega^\text{F} \otimes \otimes A'_M) \) distinguishable. In this case we again have
\[
\ker\pi_{\Sigma_q} = \{0\}
\]
and therefore
\[
\Omega_D A = \otimes_D F \otimes_D A_M = \ast \otimes_D A_M .
\]

6 Conclusions

We derived a general formula which relates the differential algebra \( \Omega_D (A_\infty \otimes A_\epsilon) \) of a product algebra to the differential algebras \( \Omega_D A_\infty \) and \( \Omega_D A_\epsilon \) of the factor algebras. This considerably simplifies the calculation of \( \Omega_D (A_\infty \otimes A_\epsilon) \) once the differential algebras of the factor algebras are known.

However, in the context of Yang-Mills theories with spontaneous symmetry breaking, all relevant algebras are of the form \( A = F \otimes A_M \) with \( F \), the algebra of smooth functions on space-time and \( A_M \), a matrix-algebra. In this case the differential algebras of each factor algebra is known. For the algebra of functions it is the usual de Rham-algebra \([1]\) and the differential algebras for matrix-algebras are described in sec. 3. With this information it is possible to compute the full differential algebra \( \Omega_D (F \otimes A_M) \) as we showed for the two-point case in some detail. The generalization to three or more points, necessary to handle models with several symmetry breaking scales, is obvious although it would require a new calculation along the lines described in sec. 4.

Since physical models, at least the bosonic part, are constructed out of objects in \( \Omega_D A \), which is an \( A \)-module, the explicit knowledge of the differential algebra for a given algebra \( A \) allows for a very economical derivation of physical quantities like connection and curvature. This can be done in the usual way by taking an antihermitian one form as connection form and the curvature as the square of the connection. However, the construction of physically relevant models requires a more careful discussion, e.g. the imbedding of charge and iso-spin enforces a certain structure on the Higgs-sector. We shall come back to this point in a future publication.

It is now also possible to use the explicit knowledge of \( \Omega_D A \) to discuss the precise relation of Connes’ approach to Yang-Mills theory with spontaneous symmetry breaking and the model presented in \([3, 5, 7]\). This latter model is based on superconnection a
la Matthai, Quillen \([12]\). Here, the usual exterior differential is extended by a matrix differential, connections are elements of odd degree in a graded \(SU(n|m)\) algebra extended to a module over differential forms. There are several features in this approach similar to the differential algebras derived in sec. 5, namely the general settings in matrix valued differential forms, the matrix derivation and Cartan's derivation giving the building principles for connection and curvature. However, we also note an important difference: the quotient building described in sec. 5 does not occur there and therefore the model is not based on a differential algebra, but an algebra with derivation.

So far, we have only discussed the bosonic sector of physical models. For the derivation of \(\Omega_D A\) one has to introduce a Dirac operator in order to represent the differential envelope. It is considered as one nice property of Connes' approach, that this Dirac operator can be used to write down the fermionic Lagrangian. However, if one starts with \(\Omega_D A\) for the construction of physical models, then there is no Dirac operator given automatically. Of course, such a Dirac operator can be derived by requiring the usual physical properties.

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