Model-free two-step design for improving transient learning performance in nonlinear optimal regulator problems

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Abstract—Reinforcement learning (RL) provides a model-free approach to designing an optimal controller for nonlinear dynamical systems. However, the learning process requires a considerable number of trial-and-error experiments using the poorly controlled system, and accumulates wear and tear on the plant. Thus, it is desirable to maintain some degree of control performance during the learning process. In this paper, we propose a model-free two-step design approach to improve the transient learning performance of RL in an optimal regulator design problem for unknown nonlinear systems. Specifically, a linear control law pre-designed in a model-free manner is used in parallel with online RL to ensure a certain level of performance at the early stage of learning. Numerical simulations show that the proposed method improves the transient learning performance and efficiency in hyperparameter tuning of RL.

I. INTRODUCTION

Reinforcement learning (RL) [1], [2] is a class of machine learning methods that learn the optimal policy through trial and error. This method has been attracting attention because it can be applied to obtain the optimal policy, i.e., desirable control law, even when the model of the plant is unknown [3]. There have already been many RL methods that design nonlinear control laws for nonlinear systems with unknown models [3]–[7]. However, these methods were not developed in consideration of the performance or the number of trials during the learning process. Since RL requires many trial and error experiments by operating the poorly controlled system, the learning process would cause wear on the plant. Therefore, it is desirable to develop a method that maintains the performance of the control law to some extent during the trial and error process, and to perform learning in as few trials as possible.

To overcome this problem, existing works proposed the use of model-based linear controllers in addition to an RL controller to improve the transient learning performance [8], [9]. Also, literature [10] proposed a method that combines RL and model predictive control (MPC) to tackle the safety and stability issues in the exploration phase. However, these studies assumed that some form of plant model is available, which is often unlikely to be the case in practical situations. This motivates us to further develop a model-free approach for assisting the learning process of RL.

In this paper, we propose a completely model-free approach to design an optimal control law, especially a nonlinear optimal regulator, for nonlinear systems while improving the transient learning performance and efficiency of RL. The proposed approach consists of two steps. In the first step, we design a linear control law by an offline model-free method, which achieves a certain degree of performance and assists the learning process of RL. Then, in the second step, we use an online RL method to design a nonlinear control law that is connected, in parallel, to the pre-designed linear controller. As a result of this two-step design approach, the designed control law achieves a performance that cannot be realized by the linear control law alone. The proposed approach has two main advantages for its practical use: 1) it can be applied even if the models of the plants are unavailable beforehand, 2) it can prevent wear on actual plants because the performance during the learning process is ensured by improving transient learning performance, and the efficiency in hyperparameter tuning is improved.

The organization of this paper is as follows. In Section II, we provide the problem formulation. In Section III, we describe the proposed approach in detail. Then, in Section IV, the effectiveness of the proposed method is verified by numerical simulations with an inverted pendulum as the plant. Section V illustrates the advantage of the proposed method that hyperparameters can be efficiently tuned. Finally, in Section VI, we give concluding remarks of this paper.

II. PROBLEM FORMULATION

In this section, we describe the problem formulation. We consider a continuous-time nonlinear system described by

\[ \dot{x}(t) = f(x(t), u(t)), \]

where \( x(t) \in \mathbb{R}^n \) and \( u(t) \in \mathbb{R}^m \) are the state and input at time \( t \), and \( f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n \) is a smooth nonlinear function. The equilibrium point of interest is at the origin, and the state \( x(t) \) can be measured directly.

We define \( A \in \mathbb{R}^{n \times n} \) and \( B \in \mathbb{R}^{n \times m} \) by

\[ A = \frac{\partial f}{\partial x} \bigg|_{x=0,u=0}, \quad B = \frac{\partial f}{\partial u} \bigg|_{x=0,u=0}. \]  

Throughout this paper, we make the following assumptions.

Assumption 1 A linear control law \( K_0 \in \mathbb{R}^{m \times n} \) such that \( A + BK_0 \) is Hurwitz is given.

Assumption 2 The nonlinear system (1) is in operation near the origin with \( K_0 \).
The proposed approach

**Step 1:** Design the linear control law $K \in \mathbb{R}^{m \times n}$ using an offline model-free method for the linear quadratic regulator (LQR) problem.

**Step 2:** Design the nonlinear control law $\mu$ (adjust the parameter $W \in \mathcal{W}$) by an online RL method where the closed-loop system consisting of the linear control law $K$ and the plant is regarded as a single environment (plant for RL).

The linear control law designed in Step 1 is quasi-optimal since the plant is nonlinear, but it contributes to reducing wear on the plant because of the following reasons: (i) it improves the transient learning performance of the online RL since a certain level of performance, if not optimal, is already attained at the early stage of learning, (ii) it makes the hyperparameter tuning for online RL easier since the system controlled with the quasi-optimal control law does not require large inputs from RL. It should be noted that, in Step 1, the linear control law can be obtained without operating the actual plant, because $K$ is updated offline after data collection.

In Step 2, the control input $u(t)$ is given by

$$u(t) = u^L(t) + u^{RL}(t),$$

where $u^L(t) = Kx(t)$, and $u^{RL}(t)$ is the control input obtained by an online RL method based on $\mu$. In the following subsections, we provide specific instances of the linear and nonlinear control law design methods.

### B. Step 1: Offline RL for design of linear control law

In this study, we use the algorithm proposed in the prior study [11] that solves an LQR problem for an unknown linear plant with collected input-output data. This algorithm is shown in Algorithm 1, where the quasi-optimal control law is obtained by recursively solving eq. (5) for $K_i+1$ without using the model of the plant. In eq. (5), the number of data sets $l$ is an integer that satisfies the condition $l \geq n(n+1)/2 + mn$ so that the equation is overdetermined. When the plant is linear, the solution of eq. (5) is guaranteed to converge to the LQR solution. For completeness, we summarize the results given in literature [11] as Theorem 1.

**Theorem 1** Suppose Assumption 1 holds, and $K_{i+1}$ in eq. (5) has a unique solution for all $i = 0, 1, \ldots$. Then, the solution of the recursive equation (5) asymptotically converges to $K^*$ as $i \to \infty$, where $K^*$ is the LQR control law for the linear system $\dot{x} = Ax + Bu$.

**Proof:** Differentiating both sides of eq. (5) by $T_{dx}$,

$$\frac{dS_i}{dt} = -x^T(Q + K_i^TRK_i)x - 2u^T(K_ix + u) + 2R_iK_{i+1}x,$$

is obtained, where $S_i = x^TP_ix$. On the other hand, we have

$$\frac{dS_i}{dt} = x^T(A_i^TP_i + P_iA_i)x + 2(-K_ix + u)^TB^TP_ix,$$

1. In creating this pseudocode representation, we used [12] as a reference.
Algorithm 1 Model-Free Policy Iteration Algorithm for State Feedback LQR [11]

Data Collection. Apply $u(t) = K_0x(t) + v(t)$ to the system to collect data for $t \in [t_0, t_0 + T_{dc})$, where $T_{dc}$ is the interval length for data collection and $v(t) \in \mathbb{R}^m$ is the exploration signal.

Policy Evaluation and Improvement. Based on the collected data, perform the following iterations for $i = 0, 1, \ldots$

(I) Find the solution $P_i \in \mathbb{R}^{n \times n}$ and $K_{i+1} \in \mathbb{R}^{m \times n}$ of the following learning equations:

$$x^T(t)P_i x(t) - x^T(t - T_{dc})P_i x(t - T_{dc}) = - \int_{t - T_{dc}}^t x^T(\tau)(Q + K_i^T R K_i) x(\tau) d\tau$$

$$- 2 \int_{t - T_{dc}}^t (u(\tau) - K_i x(\tau))^T R K_{i+1} x(\tau) d\tau,$$

for $t = t_0 + T_{dc}, \ldots, t_0 + iT_{dc}$. (5)

(II) Go to (I) with $i \leftarrow i + 1$ until $\|P_i - P_{i-1}\| < \epsilon$, for some very small positive constant $\epsilon$ and $i \geq 1$.

Equations (8) and (9) are the well-known recursive algebraic Riccati equations. The convergence of $K_{i+1}$ to $K^*$ was shown in literature [13].

C. Step 2: Online RL for design of nonlinear control law

The algorithm for online RL in Step 2 is given in Algorithm 2, which is an Actor-Critic method with eligibility traces combined with the linear control law [11, 8]. Specifically, we set the estimated value function $V(x)$ for the state $x$ to be

$$V(x; \theta) = \theta^T \phi(x),$$

where $\phi(x) \equiv [\phi_1(x), \ldots, \phi_N(x)]^T$ is the vector representation of basis functions $\{\phi_j\}_{j=1}^N$, $\theta \in \mathbb{R}^N$ is the state value weight, and $N$ is the number of basis functions. Also, the policy function $\pi$ is represented by the Gaussian probability density function of the following form:

$$\pi(u_{RL}|x; W) = \frac{1}{(2\pi)^{\frac{m}{2}}|\Sigma|^\frac{1}{2}} \exp \left( - \frac{1}{2} (u_{RL} - \mu(x; W))^T \Sigma^{-1} (u_{RL} - \mu(x; W)) \right),$$

where $u_{RL} = u_{RL} - \mu(x; W)$. The mean $\mu \in \mathbb{R}^m$ with the adjustable policy parameter $W \in \mathbb{R}^{N \times m}$ is given by

$$\mu(x; W) = W^T \phi(x),$$

and $\Sigma \in \mathbb{R}^{m \times m}$ is the covariance matrix. During the learning process, $u_{RL}$ is selected stochastically according to eq. (11), and when the learning is done, $u_{NL} = \mu(x; W)$ is used deterministically as the nonlinear control law. The weights of both the state value function and the policy function are updated according to Algorithm 2, where $z^\theta \in \mathbb{R}^N$ and $Z^W \in \mathbb{R}^{N \times m}$ represent the eligibility traces of $\theta$ and $W$, respectively, and $\lambda^\theta \in [0, 1]$ and $\lambda^W \in [0, 1]$ are their trace-decay parameters. In addition, $\delta \in \mathbb{R}$ represents the temporal difference (TD) error, $\gamma \in [0, 1]$ represents the discount rate, and $\alpha \in (0, 1)$ and $\beta \in (0, 1)$ represent learning rates. Because this algorithm is used to maximize a cumulative reward, the instantaneous reward $r_k$ is defined by

$$r_k = -(x_{k+1}^T Q x_{k+1} + u_k^T R u_k),$$

where $Q$ and $R$ are set to the same values as in Step 1. Furthermore, the inputs $\theta_0 \in \mathbb{R}^N$ and $W_0 \in \mathbb{R}^{N \times m}$ in this algorithm are set arbitrarily in the first episode; otherwise, they are set with the outputs $\theta$ and $W$ obtained in the last episode.

IV. DEMONSTRATION EXAMPLE

In this section, we illustrate the validity of the proposed method through numerical simulations using an inverted pendulum as a plant.

A. Plant and overall flow of the simulation

The nonlinear state equation of the inverted pendulum is given by

$$\dot{x}(t) = \left[ \frac{\mu}{T} \sin \psi(t) - \frac{\xi(t)}{MT^2} \xi(t) + \frac{1}{MT^2} u(t) \right],$$

where $x = [\psi, \xi]^T$ is the state of the inverted pendulum, with the angle and angular velocity of the pendulum denoted by $\psi \in [-\pi, \pi]$ and $\xi \in (-\infty, \infty)$, respectively, and $u \in \mathbb{R}$ is
the torque input from the motor attached to the fulcrum of the pendulum. The definitions and values of the other variables in eq. (14) are shown in Table I.

In this simulation, we first regard the nonlinear system (14) as a plant and obtain the linear control law $K$ through Algorithm 1, assuming that the model of the plant is unknown (Step 1). Then, with this $K$, we design the nonlinear control law $\mu$ by repeating Algorithm 2 for $T_{epi} = 3$ s as one episode (Step 2). We compare the results of the proposed method with those of the online RL controller alone.

**B. Step 1: design of linear control law $K$**

In Step 1, we set each parameter as listed in Table I, and run the iterations in Algorithm 1. The initial value of the state for data collection is set to $x(0) = 0$, and the following exploration signal $\nu(t)$ is added to $K_0 x(t)$:

$$\nu(t) = 0.5 \sum_{i=1}^{100} \sin(\omega_i t),$$

where $\omega_i \ (i = 1, 2, \cdots, 100)$ are selected randomly from $[-500, 500]$.

The quasi-optimal linear control law $K$ is obtained as $K = [\begin{array}{c} -10.7560 \\ -1.2997 \end{array}]$. We also calculate the optimal control law $K^* = [\begin{array}{c} -10.7620 \\ -1.2952 \end{array}]$ for the linearized model of the nonlinear plant (14) around the equilibrium point $[\begin{array}{c} \psi \\ \xi \end{array}] = 0$. We now compare the performance of the three control laws, $K$, $K^*$, and $K_0$. The trajectories of the pendulum angle when each of the three control laws is used as a feedback gain for the plant (14) are shown in Fig. 2. In this simulation, the initial value of the state is set to $x(0) = [0.4, 0]^T$. It can be seen in Fig. 2 that the performance of the control law $K$ is better than that of the initially given control law $K_0$ because the result using $K$ shows better state transient characteristics and faster convergence. Also, compared with $K_0$, $K$ has almost the same performance as $K^*$.

**C. Step 2: design of nonlinear control law $\mu$**

In Step 2, we set the variance $\Sigma = \sigma_E^2$ in episode $E$ to

$$\sigma_E^2 = \sigma^2 \times \left(10^{-4}\right)^{T_{epi}},$$

so that the variance decreases as the number of episodes increases, where $\sigma^2$ denotes the initial variance and $N_{epi}$ denotes the total number of simulation episodes. The details of simulation parameters used in Step 2 are listed in Table II. Also, in the case of $|\psi_k| \geq 0.5$ rad, we give the online RL controller a penalty of $r_k = -100$, and terminate the current episode with the state value set to 0. We discretize eq. (14) using the fourth-order Runge-Kutta method under a sufficiently small sampling period $T_s = 0.03$ s.

The comparison of the control costs in one episode $J = \sum_{k=0}^{T_{epi}/T_s} (x_{k+1}^\top Q x_{k+1} + R u_k^2)$ is shown in Fig. 3. It can be seen from Fig. 3 that the cost of using the online RL controller alone (RL alone) is significantly high in the early stage of learning. This indicates that the inverted pendulum falls over quickly when the controller is inexperienced, resulting in significantly low transient learning performance. On the other hand, the cost of using the proposed method (K+RL) is lower than that of using the online RL controller alone, even in the early stage of learning. This indicates that the proposed method can maintain the control performance to some extent even in inexperienced states. Furthermore, it can be observed that the cost of using the proposed method is lower than that of using the quasi-optimal linear control law alone ($K$ alone) in the final episode. This result means that the performance of the control law leaned by the proposed method is better than that of the linear control law alone.
TABLE III: Cost associated with each combination of control laws obtained in different phases

| Control law (phase) | Cost J  |
|---------------------|---------|
| K₀ alone (initial)  | 249.3   |
| K alone (after step 1) | 72.8    |
| K+RL (after step 2)  | 66.2    |
| RL alone (reference)| 66.9    |

Table III shows the control costs associated with each controller: the initially given control law alone (K₀ alone), the linear control law alone (K alone), the proposed method (K+RL), and the online RL controller alone (RL alone). It is shown in Table III that the cost of using K alone is lower than that of K₀ alone. This indicates that the performance is improved through Algorithm 1 in Step 1. In addition, the cost of using the proposed method is lower than that of using K alone, which indicates that the nonlinear control law learned by the online RL in Step 2 achieves a performance that cannot be realized by the linear control law alone. Furthermore, we can see that the performance of the proposed method is better than that of the online RL controller alone.

V. ROBUSTNESS AGAINST HYPERPARAMETER SETTING

In RL, inputs are selected stochastically, so learning is not always successful; in the case of the inverted pendulum, for example, the resulting control law may fail to stabilize the pendulum, or it may deteriorate the performance even if the pendulum is stabilized. The rate of successful learning depends on hyperparameters. In particular, the learning rate β and the initial variance of the policy function σ² are two major factors that directly affect the rate. In this section, we show, through numerical simulations, that the stability and the performance of the proposed system are more robust against hyperparameter settings when compared to the system with the online RL controller alone (RL alone) and the RL controller with the initially given control law K₀ (K₀+RL).

A. Evaluation method

We vary the learning rate β and the initial variance of the policy function σ² and perform Nₜₙₑₙ = 100 sets of simulations for the system with each control law under each hyperparameter setting. Then, the percentage of successful learning is calculated by q/Nₜₙₑₙ × 100, where q is the number of times that learned control laws successfully stabilized the pendulum. Furthermore, the percentage of improvement in performance is calculated by p/Nₜₙₑₙ × 100, where p is the number of times that the control costs of using learned control laws are reduced compared to that of using the linear control law alone.

B. Evaluation results

We first present the results when the initial variance of the policy function (11) is fixed to σ² = 2.5 × 10⁻², and the learning rate β for the policy parameter is varied in three ways. The percentages of successful learning and improvement in performance are shown in Fig. 4 and Fig. 5, respectively. Firstly, we observe that both percentages are higher with the proposed method (K+RL) than the online RL controller alone (RL alone). According to Fig. 4 and Fig. 5, the method that uses K₀ and the online RL controller in parallel (K₀+RL) is less prone to falling over when the learning rate β (the step size to update the policy parameter) is small, and the learning is more likely to succeed, but the performance is not improved because the inputs are restricted to small values, resulting in a conservative solution. On the contrary, when the value of β is large, the percentage of improvement in performance increases, but the percentage of successful learning decreases. This implies that there is a trade-off between improvement in performance and successful learning, and the method of using K₀ and an online RL controller in parallel requires an appropriate adjustment of the learning rate β.

On the other hand, it can be seen from Fig. 4 and Fig. 5 that, for the proposed method (K+RL), the percentage of improvement in performance and the percentage of successful learning are both 100% under every β. In other words, the proposed method can achieve high rates of improvement in performance and successful learning, regardless of the learning rate. Furthermore, because the performance is ensured to some extent by the linear control law designed in Step 1, we have the prior knowledge that the proposed method does not require large inputs from the online RL controller. Therefore, it can be inferred that the learning rate β can be set close to zero and no fine-tuning is required.

We observe the same trends for different values of the hyperparameter σ² as illustrated in Fig. 6 and Fig. 7, where the percentage of successful learning and improvement in performance are shown respectively. These results imply that the proposed method is more robust in achieving improvement in the performance of the control system with respect
to hyperparameter settings than the other two methods. The above results show that the proposed method is relatively easy to adjust the hyperparameters compared to the other two methods, making it highly useful in practical applications.

VI. CONCLUSION

In this paper, we have proposed a model-free two-step approach to design the optimal control law for unknown nonlinear plants to improve the transient learning performance and increase the efficiency of learning. Specifically, a linear control law that achieves a certain degree of performance is first designed using an offline method, and then a nonlinear control law is designed by using an online RL method and the linear controller in parallel to achieve performance that cannot be realized with the linear control law alone. Using an inverted pendulum as an example, we have shown that the proposed method improves the transient learning performance of online RL and robustly achieves improvement in the performance of the control system against hyperparameter tuning.

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