A Survey on Satisfiability Checking for the \( \mu \)-Calculus through Tree Automata\(^*\)

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Abstract. Algorithms for model checking and satisfiability of the modal \( \mu \)-calculus start by converting formulas to alternating parity tree automata. Thus, model checking is reduced to checking acceptance by tree automata and satisfiability to checking their emptiness. The first reduces directly to the solution of parity games but the second is more complicated.

We review the non-emptiness checking of alternating tree automata by a reduction to solving parity games of a certain structure, so-called emptiness games. Since the emptiness problem for alternating tree automata is ExpTime-complete, the size of these games is exponential in the number of states of the input automaton. We show how the construction of the emptiness games combines a (fixed) structural part with (history-)determinization of parity word automata. For tree automata with certain syntactic structures, simpler methods may be used to handle the treatment of the word automata, which then may be asymptotically smaller than in the general case.

These results have direct consequences in satisfiability and validity checking for (various fragments of) the modal \( \mu \)-calculus.

1 Introduction

The modal \( \mu \)-calculus extends modal logic with least and greatest fixpoint operators [15]. The \( \mu \)-calculus is expressive enough to express many temporal logics, in particular it can capture CTL\(^*\) and its fragments LTL and CTL [6]. At the same time, the \( \mu \)-calculus has interesting algorithmic and algebraic properties. For example, the \( \mu \)-calculus model-checking problem is equivalent to the solution of parity games, a well known (still) open problem attracting much research. This combination led to high interest in the \( \mu \)-calculus, studying many aspects of the logic.

Here, we are interested in the question of satisfiability of the \( \mu \)-calculus. The problem is ExpTime-complete and the first algorithms of this complexity were automata based [3]. Much like for other temporal logics, initial treatment of the logic was done through nondeterministic automata [25]. However, later, the richer structure of alternating automata enabled translations that are more natural and direct [19]. This translation defers the complicated handling of the satisfiability problem to standard automata constructions.

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With this approach, the satisfiability problem for the modal $\mu$-calculus reduces to the non-emptiness problem of alternating parity tree automata. The latter problem is solved either by constructing equivalent nondeterministic parity tree automata or by a direct reduction to two-player perfect information parity games.

We revisit the reduction and present it in a way that separates the tree acceptance and the parity acceptance aspects of a parity tree automaton $A$. The method creates an arena $G_A$ (strategy arena), which captures all the decisions made at the same location by $A$ and a nondeterministic parity word automaton $T_A$ (tracking automaton) that “accepts” bad branches in run-trees of the original automaton. The original automaton then is non-empty if and only if the combination of $G_A$ with $T_A$ as losing condition is won by the existential player. By using a history deterministic (or fully deterministic) word automaton $H_A$ that accepts the same language as $T_A$, we construct a parity game $G^*_A$.

This approach reduces the algorithmic content of non-emptiness checking for alternating automata (and for satisfiability checking in the modal $\mu$-calculus) to a fixed construction of a game arena and (history-)determinization of word automata that depend on the exact structure of the original automaton ($\mu$-calculus formula).

We then show that as the structure of $T_A$ strongly depends on the structure of $A$, specialized history determinization and determinization constructions lead to complexity results that match bespoke algorithms for different fragments of the $\mu$-calculus. These results are summarized in the following table, where LL stands for limit linear, LD for limit deterministic, N for nondeterministic, HD for history deterministic, D for deterministic, W for weak, B for Büchi, C for co-Büchi, and P for parity. For example, LL-CW is a limit linear co-Büchi word automaton and LD-WT is a limit deterministic weak tree automaton.

| | type of $T_A$ | method | type of $H_A$ | size of $H_A$ |
|---|---|---|---|---|
| det. | LL-CW | circle method | DCW | $n^2 \cdot 2^n$ |
| det. | NCW | Miyano-Hayashi | DCW | $3^n$ |
| history-det. | LD-CW | focus method | HD-CW | $n \cdot 2^n$ |
| det. | LD-BW | permutation method | DPW | $O(n!)$ |
| history-det. | NBW | Safra-Piterman | DPW | $O((n!)^2)$ |
| | NBW | Henzinger-Piterman | HD-PW | $O(3^n)$ |
Going back to the $\mu$-calculus, the following table depicts relations between various syntactic properties of $\mu$-calculus formulas (formally defined later) and automata. Thus, the separated treatment of the arena and the acceptance, and the different constructions for word automata summarize in one framework complexity results relating to various different fragments of the $\mu$-calculus.

| property of $\varphi$         | type of $A$ | type of $T_A$ |
|-------------------------------|-------------|---------------|
| limit-linear                  | LL-WT       | LL-CW         |
| alternation-free              | AWT         | NCW           |
| aconjunctive alternation-free | LD-WT       | LD-CW         |
| aconjunctive                  | LD-PT       | LD-BW         |
| unrestricted                  | APT         | NBW           |

2 The Modal $\mu$-Calculus

We are concerned with satisfiability checking for different syntactic fragments of the branching-time $\mu$-calculus, introduced by Kozen [15].

Syntax. Formulas of the $\mu$-calculus are generated by the following grammar, where $\text{At}$ and $\text{Var}$ are countable sets of atoms and fixpoint variables, respectively:

$$\varphi, \psi : = p \mid \neg p \mid \varphi \land \psi \mid \varphi \lor \psi \mid \Diamond \varphi \mid \Box \varphi \mid X \mid \xi X.\varphi \quad p \in \text{At}, X \in \text{Var}, \xi \in \{\mu, \nu\}$$

Fixpoint operators $\mu X.$ and $\nu X.$ bind their variable $X$, giving rise to standard notions of bound and free variables; for $\mu X.\psi$, free occurrences of $X$ in $\psi$ are least fixpoint variables and for $\nu X.\psi$, free occurrences of $X$ in $\psi$ are greatest fixpoint variables.

The Fischer-Ladner closure [15] $\text{FL}(\varphi)$ (or just closure) of a closed formula $\varphi$ is the least set of formulas that contains $\varphi$ and is closed under taking subformulas for non-fixpoint operators and under unfolding for fixpoint operators; e.g. $\mu X.\psi \in \text{FL}(\varphi)$ implies $\psi[X \mapsto \mu X.\psi] \in \text{FL}(\varphi)$, where $\psi[X \mapsto \mu X.\psi]$ is the formula that is obtained from $\psi$ by replacing every free occurrence of the variable $X$ in $\psi$ by the formula $\mu X.\psi$. We have $\text{FL}(\varphi) \leq |\varphi|$ where $|\varphi|$ is the number of operators that are required to write $\varphi$ (that is, the number of nodes in the syntax tree of $\varphi$).

A formula $\varphi$ is clean, if all fixpoint variables are bound at most once in it. Then we denote the fixpoint formula $\xi X.\psi$ that binds a variable $X$ in $\varphi$ by $\theta_\varphi(X)$. While transforming an arbitrary formula to a clean formula (by renaming bound variables accordingly) can increase the closure size, a translation to tree automata that does not rely on cleanness has recently been given in [16]. For brevity of presentation we assume throughout that target formulas are clean but remark that this does not affect the stated complexity results since the more involved translation from [16] can be used to obtain tree automata (of suitable size and rank) from arbitrary formulas.

Remark 2.1. Another common constraint on the syntactic structure of formulas is guardedness, which requires that there is always at least one modal operator...
between a fixpoint operator and occurrences of the fixpoint variable that it binds. It is currently an open question whether there is a guardedness-transformation with polynomial blow-up of the closure size, and it has been shown that such a polynomial transformation would yield a polynomial algorithm for parity game solving [16]. Throughout this work, we assume that formulas are guarded.

**Alternation-depth.** Given a clean formula $\varphi$, and two subformulas $\xi X.\psi$ and $\xi' Y.\chi$ of $\varphi$, $\xi' Y.\chi$ depends on $\xi X.\psi$ if $X$ has a free occurrence in $\chi$. We define the dependent nesting order $\succeq_{\varphi}$ to be the partial order on fixpoint subformulas of $\varphi$ obtained by taking the reflexive-transitive closure of the dependency ordering. The alternation-depth $\text{ad}(\varphi)$ of $\varphi$ then is defined to be the maximal length of an alternating $\succ_{\varphi}$-path, where a $\succ_{\varphi}$-path is alternating if all its transitions switch the fixpoint type. Given a fixpoint subformula $\chi = \eta X.\psi$ of $\varphi$, let $d$ be the maximal length of an alternating $\succeq_{\varphi}$-path that starts at $\chi$. We define the alternation-level $\text{al}(\chi)$ of $\chi$ to be $2\lceil d/2 \rceil - 1$ if $\eta = \mu$ and $2\lceil d/2 \rceil$ if $\eta = \nu$. Hence least fixpoint formulas have odd alternation-level while greatest fixpoint formulas have even alternation-level; furthermore, we always have $\text{al}(\chi) \leq \text{ad}(\varphi)$.

**Semantics.** Formulas of the $\mu$-calculus are evaluated over pointed Kripke structures. A pointed Kripke structure is $K = (W, w_0, R, L)$, where $W$ is a set of worlds, $w_0 \in W$ is an initial world, $R \subseteq W \times W$ is a transition relation, and $L : W \to \mathcal{P}(\text{At})$ is a labeling function. We denote by $R(w) = \{w' \in W \mid (w, w') \in R\}$ the set of worlds connected by $R$ to $w$. We restrict attention to structures where for every $w \in W$ we have $R(w) \neq \emptyset$.

Given a $\mu$-calculus formula $\varphi$ and a pointed Kripke structure $K$, the semantics of the formula is defined based on a valuation function $\eta$ assigning each variable appearing in $\varphi$ to a set of worlds of $K$. Given such a function $\eta$ we denote by $\eta[X \leftarrow S]$ the function $\eta'$ where $\eta'(X) = S$ and $\eta'(Y) = \eta(Y)$ for every $Y \neq X$. The semantics of the $\mu$-calculus is included in Figure 1. It is simple to see that in the case that a formula $\varphi$ is closed its semantics does not depend on the initial valuation $\eta$. Thus, for a closed formula we write $\llbracket \varphi \rrbracket$. A formula $\varphi$ is satisfiable if there exists a structure $K = (W, w_0, R, L)$ such that $w_0 \in \llbracket \varphi \rrbracket$. 

\[
\begin{align*}
\llbracket p \rrbracket\eta &= \{ w \mid p \in L(w) \} \\
\llbracket \neg p \rrbracket\eta &= \{ w \mid p \notin L(w) \} \\
\llbracket \varphi \lor \psi \rrbracket\eta &= \llbracket \varphi \rrbracket\eta \cup \llbracket \psi \rrbracket\eta \\
\llbracket \varphi \land \psi \rrbracket\eta &= \llbracket \varphi \rrbracket\eta \cap \llbracket \psi \rrbracket\eta \\
\llbracket \diamond \varphi \rrbracket\eta &= \{ w \mid R(w) \cap \llbracket \varphi \rrbracket\eta \neq \emptyset \} \\
\llbracket \Box \varphi \rrbracket\eta &= \{ w \mid R(w) \subseteq \llbracket \varphi \rrbracket\eta \} \\
\llbracket \mu X.\varphi \rrbracket\eta &= \bigcap \{ T \subseteq W \mid \llbracket \varphi \rrbracket_{\eta[X \leftarrow T]} \subseteq T \} \\
\llbracket \nu X.\varphi \rrbracket\eta &= \bigcup \{ T \subseteq W \mid T \subseteq \llbracket \varphi \rrbracket_{\eta[X \leftarrow T]} \}
\end{align*}
\]
Theorem 2.2 ([5]). Given a $\mu$-calculus formula $\varphi$, deciding whether $\varphi$ is satisfiable is \textit{ExpTime}-complete.

Fragments of the $\mu$-calculus. We consider the following fragments of the $\mu$-calculus:

- The limit-linear fragment of the $\mu$-calculus consists of all formulas $\varphi$ such that for all subformulas $\mu X.\psi$ of $\varphi$, $X$ has exactly one occurrence in $\psi$, and this occurrence is not in the scope of a fixpoint subformula of $\psi$. Computation tree logic (CTL) is a fragment of the limit-linear $\mu$-calculus.

- The alternation-free fragment of the $\mu$-calculus consists of all formulas $\varphi$ such that $\text{ad}(\varphi) \leq 1$. It has been shown that satisfiability checking for a guarded alternation-free formula of size $n$ can be done by solving a Büchi game of size $3^n$ [9].

- The aconjunctive fragment of the $\mu$-calculus consists of all formulas $\varphi$ such that for all conjunctions $\psi \land \chi$ that occur as a subformula in $\varphi$, at most one of the conjuncts $\psi$ or $\chi$ contains a free least fixpoint variable. Satisfiability checking for a (weakly) aconjunctive formula of size $n$ and with $k$ priorities can be done by solving a parity game of size $e \cdot (nk)!$ and with $2nk$ priorities [11].

- The alternation-free aconjunctive fragment of the $\mu$-calculus is the intersection of the alternation-free fragment and the aconjunctive fragment. In particular, every limit-linear formula is alternation-free and aconjunctive.

3 Two-Player Games and Alternating Parity Tree Automata

We give background on two-player games and tree automata. Our notations are based on those developed by Wilke [26].

Definition 3.1. A game is $G = (V, V^\Diamond, V^\Box, E, \alpha)$, where $V$ is a set of nodes, $V^\Diamond$ and $V^\Box$ form a partition of $V$ to player $\Diamond$ and player $\Box$ nodes, $E \subseteq V \times V$ is a set of edges, and $\alpha \subseteq V^\omega$ is a winning condition. A play is a sequence $\pi = v_0, v_1, \ldots$ such that for every $i$ we have $(v_i, v_{i+1}) \in E$. A play $\pi$ is winning for player $\Diamond$ if $\pi \in \alpha$. A strategy for player $\Diamond$ is $\sigma : V^* \cdot V^\Diamond \to V$ such that $(v, \sigma(vw)) \in E$ for $wv \in V^* \cdot V^\Diamond$. A play $\pi$ is compatible with $\sigma$ if whenever $v_i \in V^\Diamond$ we have $v_{i+1} = \sigma(v_0, \ldots, v_i)$. A strategy for player $\Diamond$ is winning from node $v$ if all the plays starting in $v$ that are compatible with $\sigma$ are winning for her. Strategies for player $\Box$ are defined similarly.

In a parity game, there exists a priority function $\Omega : V \to \mathbb{N}$ and a play $\pi$ is winning for player $\Diamond$ if the maximal priority occurring infinitely often in $\pi$ is even. A Büchi game is a parity game with just the priorities 1 and 2. Given an infinite sequence $\pi \in V^\omega$ let $\text{inf}(\pi)$ denote the set of nodes occurring in $\pi$ infinitely often and put $\text{inf}_{\Omega}(\pi) = \{\Omega(v) \mid v \in \text{inf}(\pi)\}$. Then the parity winning condition induced by $\Omega$ is $\alpha = \{\pi \in V^\omega \mid \max(\text{inf}_{\Omega}(\pi)) \text{ is even}\}$. The
complexity of analyzing parity games is a hot area of research\[13]\). Here we denote by Parity\((n,k)\)-Time the time complexity of solving parity games with \(n\) nodes and \(k\) priorities. We do not refer to space complexity, however, a similar general dependency on space can be stated. Our results produce parity games of different parameters depending on the exact shape of a \(\mu\)-calculus formula. We hence use this parametric form to give complexity results.

**Theorem 3.2 (Parity and Büchi games\[12]\).** Parity games with \(n\) nodes and \(k\) priorities can be solved in time quasipolynomial in \(n\) and \(k\), more specifically\[1\] in time \(n^{2\log(k/\log n)+\mathcal{O}(1)}\), and in polynomial time if \(k < \log n\). Büchi games with \(n\) nodes can be solved in time \(\mathcal{O}(n^2)\).

We sometimes consider games where edges are labeled. In such a case, there exists a set of labels \(D\) and we have \(E \subseteq V \times D \times V\). Then, \(\alpha\) can be a subset of \(V \cdot (D \cdot V)^{\omega}\).

When the winning condition is not important, we call \((V,V_0,V_{\Box},E)\) an arena.

**Definition 3.3.** An alternating parity tree automaton \(A = (\Sigma,Q,q_0,\delta,\Omega)\) consists of a finite alphabet \(\Sigma\), a finite set of states \(Q\), an initial state \(q_0 \in Q\), a transition function \(\delta : Q \times \Sigma \rightarrow P(Q)\), and a priority function \(\Omega : Q \rightarrow \mathbb{N}\); furthermore, each state \(q \in Q\) is marked as either local-existential, local-universal, modal-existential or modal-universal (denoted by \(q \in Q_v\), \(q \in Q_{\lambda}\), \(q \in Q_{\diamond}\), and \(q \in Q_{\Box}\), respectively). We also denote \(Q_l = Q_v \cup Q_{\lambda}\), where \(l\) stands for local, and \(Q_m = Q_{\diamond} \cup Q_{\Box}\), where \(m\) stands for modal. Without loss of generality, we assume that modal-existential and modal-universal states have exactly one successor, formally: \(|\delta(q)| = 1\) if \(q \in Q_{\diamond}\) or \(q \in Q_{\Box}\). We overload \(\delta(q)\) to denote \(q'\) if \(\delta(q) = \{q'\}\). The **rank** of \(A\) is the number \(|\Omega(Q)|\) of priorities appearing in its priority function. We assume that \(A\) does not have local loops. That is, for every letter \(\sigma\) and for every sequence of states \(q_1, \ldots, q_l \in Q_+\) such that for every \(i \geq 1\) we have \(q_{i+1} \in \delta(q_i, \sigma)\) we have \(q_l \neq q_1\).

A tree automaton is **weak** if for all its strongly connected components \(C\), either all states in \(C\) have priority 0 or all states in \(C\) have priority 1. A weak tree automaton is **limit-linear** if for each \(q \in Q\) such that \(\Omega(q) = 1\), there is exactly one path from \(q\) to \(q\). That is, within rejecting strongly connected components the looping behavior is deterministic. A tree automaton is **limit-deterministic** if for each odd priority \(p\) and each state \(q\) such that \(\Omega(q) = p\), we have that for all \(a \in \Sigma\) and all states \(q' \in Q_{\lambda}\) that are reachable from \(q\) by visiting nodes with priority at most \(p\), \(|\delta(q', a) \cap Q_{\leq p}| \leq 1\), where \(Q_{\leq p} = \{q \in Q \mid \Omega(q) \leq p\}\). In particular, every limit-linear automaton is limit-deterministic.

Alternating tree automata read pointed Kripke structures. An alternating tree automaton \(A\) accepts a Kripke structure \(K = (W,w_0,R,E)\) if player \(\hat{\diamond}\) wins the node \((w_0,q_0)\) in the acceptance game \(G_{A,R}\). Formally, \(G_{A,R} = (V,V_0,V_{\Box},E,\alpha)\), where \(V = W \times Q\), \(V_0 = W \times (Q_v \cup Q_{\diamond})\), \(V_{\Box} = W \times (Q_{\lambda} \cup Q_{\Box})\), \(\alpha\) is induced by the priority function \(\Omega'(w,q) = \Omega(q)\), and \(E\) is defined as follows.

\[
E = \{(w,q),(w,q') \mid q \in Q_l \text{ and } q' \in \delta(q,L(w))\} \cup \\
\{(w,q),(w',q') \mid q \in Q_m, q' \in \delta(q,L(w)), \text{ and } w' \in R(w)\}
\]

1 This improved bound has been shown in \[13\]
An automaton $A$ is non-empty if there exists a Kripke structure that it accepts.

We now state (the well known result) that given a $\mu$-calculus formula, we can construct an alternating tree automaton accepting exactly the models of the formula.

**Definition 3.4 (Formula automaton).** Given a closed and clean $\mu$-calculus formula $\varphi$ that mentions atoms $A \subseteq \text{At}$, we define an alternating parity tree automaton $A(\varphi) = (\Sigma, Q, q_0, \delta, \Omega)$ by putting $\Sigma = \mathcal{P}(A)$, $Q = \mathcal{FL}(\varphi) \cup \{\top, \bot\}$, and $q_0 = \varphi$. We define a partial priority function $\Omega' : Q \rightarrow \{0, \ldots, \text{ad}(\varphi)\}$ by putting $\Omega'(\eta X. \psi) = \text{al}(\theta(X))$ (recalling that $\theta(X)$ is the subformula of $\varphi$ that binds $X$), $\Omega'(\bot) = 1$ and $\Omega'(\top) = 0$; then $\Omega'$ assigns a priority to at least one state on each cycle in $A(\varphi)$. The total priority function $\Omega : V \rightarrow \{0, \ldots, \text{ad}(\varphi)\}$ is obtained by putting, for each state $q \in Q$ such that $\Omega'(q)$ is undefined, $\Omega(q) = p$ where $p$ is the minimum priority such that all paths from $q$ to $q$ visit priority at most $p$; states that do not belong to a strongly connected component obtain priority $0$. Furthermore, we put

$$
\delta(q, P) = \begin{cases} 
\{\psi_0, \psi_1\} & \text{if } q = \psi_0 \land \psi_1 \text{ or } q = \psi_0 \lor \psi_1 \\
\{\psi\} & \text{if } q = \Box \psi \text{ or } q = \Box \psi \\
\{\psi[X \mapsto \eta X. \psi]\} & \text{if } q = \eta X. \psi \\
\{\top\} & \text{if } q = p \text{ and } p \in P \text{ or } q = \neg p \text{ and } p \notin P \\
\{\bot\} & \text{if } q = p \text{ and } p \notin P \text{ or } q = \neg p \text{ and } p \in P \\
\{q\} & \text{if } q = \top \text{ or } q = \bot
\end{cases}
$$

for $q \in Q$, $P \in \Sigma$. Finally, we put

$$
Q_2 = \{\psi_0 \lor \psi_1, \eta X. \psi \in \mathcal{FL}(\varphi)\} \cup \{\bot\} \quad Q_0 = \{\Box \psi \in \mathcal{FL}(\varphi)\} \\
Q_\Box = \{\psi_0 \land \psi_1, p, \neg p \in \mathcal{FL}(\varphi)\} \cup \{\top\} \quad Q_\Box = \{\Box \psi \in \mathcal{FL}(\varphi)\}.
$$

**Theorem 3.5 (2616).** We have $L(A(\varphi)) = \{(W, w_0, R, L) \mid w_0 \in \llbracket \varphi \rrbracket\}$. Furthermore, $|Q| \leq |\mathcal{FL}(\varphi)| + 2$ and $A(\varphi)$ has rank $\text{ad}(\varphi) + 1$.

**Corollary 3.6.** Deciding if a Kripke structure with set of worlds $W$ satisfies a $\mu$-calculus formula $\varphi$ is in parity($|W| \cdot (|\mathcal{FL}(\varphi)| + 2), \text{ad}(\varphi) + 1$)-Time.

It follows from Theorem 3.5 that by checking whether the language of $A(\varphi)$ is empty we can decide whether $\varphi$ is satisfiable. In the next section, we proceed to show how to determine whether the language of an automaton is empty.

Before proceeding, we show that in case the $\mu$-calculus formula has a special structure, as defined in Section 2, the automaton resulting from the translation above has also a special structure.

**Lemma 3.7.** – If $\varphi$ is alternation-free, then $A(\varphi)$ is a weak tree automaton.
– If $\varphi$ is limit linear, then $A(\varphi)$ is limit linear.
– If $\varphi$ is aconjunctive, then $A(\varphi)$ is limit deterministic.
Proof. – Let \( \varphi \) be alternation-free so that \( \text{ad}(\varphi) \leq 1 \) and \( \text{al}(\psi) \leq 1 \) for all \( \psi \in \text{FL}(\varphi) \). Hence \( A(\varphi) \) uses just the priorities \( \{0, 1\} \). Furthermore, every strongly connected component in \( A(\varphi) \) belongs to either a greatest or a least fixpoint and hence consists only of states with priority 0 or only of states with priority 1, as claimed.

– Let \( \varphi \) be limit linear so that for all subformulas \( \mu X.\psi \) of \( \varphi \), \( X \) has exactly one occurrence in \( \psi \). Then \( \varphi \) is alternation-free so that \( A(\varphi) \) is weak by the previous item. Furthermore, all states in the strongly connected component of \( \theta(X) \) belong to \( \mu X.\psi \) and hence have priority 1. Since \( \varphi \) is limit-linear, there is exactly one circular path in the strongly connected component of \( \theta(X) \). Hence \( A(\varphi) \) is limit linear.

– Let \( \varphi \) be a conjunctive, let \( p \) be an odd number, let \( q \in Q \) such that \( \Omega(q) = p \) and let \( q' \in Q_\varphi \) be a state that is reachable from \( q \) by visiting states with priority at most \( p \). It remains to show that for all \( a \in \Sigma \), we have \( |\delta(q', \Sigma) \cap Q_{\leq p}| \leq 1 \), where \( Q_{\leq p} = \{ q \in Q \mid \Omega(q) \leq p \} \). Since \( q \in Q_\Lambda \), we have \( q = \psi_1 \land \psi_2 \) for some \( \psi_1, \psi_2 \in \text{FL}(\varphi) \). As \( \varphi \) is aconjunctive, there is at most one \( i \) such that \( \psi_i \) contains a free least fixpoint variable, and such that an odd priority is reachable from \( \psi_i \) without first passing a priority greater than \( p \). Hence \( \Omega(\psi_1) > p \) or \( \Omega(\psi_2) > p \), showing that \( |\delta(q', \Sigma) \cap Q_{\leq p}| \leq 1 \). \( \square \)

4 Emptiness of Alternating Tree Automata

We now show how the decision whether the language of an alternating automaton is non-empty can be reduced to deciding the winner in a two-player game. We start by constructing a game with labeled edges and an acceptance condition that is defined by a nondeterministic word automaton. We show that player \( \Diamond \) wins in this game if the language of the alternating automaton is not empty. Then, by manipulating the word automaton, we construct a parity game with the same quality: player \( \Diamond \) wins if the language of the alternating automaton is not empty. This is interesting because it unifies many results about fragments of the \( \mu \)-calculus to results about word automata.

4.1 Nondeterministic Parity Word Automata

Before proceeding we introduce nondeterministic and history deterministic word automata.

**Definition 4.1 (Nondeterministic Parity Word Automata).** A nondeterministic parity word automaton is \( N = (\Sigma, Q, q_0, \delta, \Omega) \), where \( \Sigma \) is a finite alphabet, \( Q \) a finite set of states, \( q_0 \in Q \) an initial state, and \( \delta : Q \times \Sigma \rightarrow \mathcal{P}(Q) \) a transition function. The priority function \( \Omega : Q \rightarrow \mathbb{N} \) assigns priorities to states. Given an automaton \( N \), the rank of \( N \) is its maximal priority, that is \( \max \{ \Omega(q) \mid q \in Q \} \). Given an infinite word \( w = a_0a_1 \ldots \in \Sigma^\omega \), a run of \( N \) on \( w \) is an infinite sequence \( \tau = q_0, q_1, \ldots \) of states such that \( q_{i+1} \in \delta(q_i, a_i) \) for
all $i \geq 0$. A run $\tau = q_0, q_1, \ldots$ is accepting if the highest priority that occurs infinitely often in $\tau$ is even. Formally, reusing the notation $\inf_{\Omega}$ introduced for parity games, run $\tau$ is accepting if and only if $\max\{\inf_{\Omega}(\tau)\}$ is an even number. The language accepted by $N$ is

$$L(N) = \{w \in \Sigma^\omega \mid \text{there is an accepting run of } N \text{ on } w\}.$$ 

**Definition 4.2 (History-deterministic Word Automata [12]).** Given a nondeterministic word automaton $N$, a resolver for $N$ is a function $\sigma : \Sigma^* \to Q$ such that $\sigma(\epsilon) = q_0$ and for all sequences $wa \in \Sigma^+$, we have $\sigma(wa) \in \delta(\sigma(w), a)$. Given a word $w = a_0a_1 \cdots \in \Sigma^*$, the outcome of $\sigma$ on $w$, denoted $\sigma(w)$, is the run $r = q_0, q_1, \ldots$ such that for all $i \geq 0$ we have $q_i = \sigma(a_0 \ldots a_{i-1})$. We say that $N$ is history-deterministic if there is a resolver $\sigma$ such that for every word $w$ we have that $w \in L(N)$ if and only if $\sigma(w)$ is an accepting run of $N$.

A word automaton is deterministic if for every state $q \in Q$ and every letter $a \in \Sigma$ we have $|\delta(q, a)| \leq 1$. In particular, every deterministic automaton is history deterministic.

**Theorem 4.3 ([22][21][12]).** Given a nondeterministic parity word automaton $N$, there exist a history deterministic parity automaton $H$ and a deterministic parity automaton $D$ such that $L(N) = L(H) = L(D)$.

In Section 5 we mention several determinization and history determinization constructions that take nondeterministic word automata and construct equivalent (history) deterministic automata.

### 4.2 The Emptiness Games

Using these definitions we are ready to proceed with the construction of the games capturing emptiness of an alternating parity tree automaton.

**Definition 4.4 (Strategy Arena).** Given an alternating parity tree automaton $A = (\Sigma, Q, q_0, \delta, \Omega)$ we define the strategy arena $G_A = (V, V_\diamond, V_\Box, E)$, where the components of $G_A$ are as follows. We label the edges in $E$ as we explain below.

- $V = (\mathcal{P}(Q) \times \Sigma) \cup \mathcal{P}(Q)$
- $V_\diamond = \mathcal{P}(Q) \cup \{(s, \sigma) \mid s \cap Q_l \neq \emptyset\}$
- $V_\Box = \{(s, \sigma) \mid s \cap Q_l = \emptyset\}$

That is, nodes correspond to either sets of states of $A$ with a letter from $\Sigma$ or just a set of states of $A$. A node is in $V_\diamond$ if either it is a plain subset of states of $A$ or if it contains local states of $A$. A node is in $V_\Box$ if it does not contain local states of $A$. 
A choice function for \( a \in \Sigma \) is \( d : Q_\lor \to Q \) such that for every \( q \in Q_\lor \) we have \( d(q) \in \delta(q,a) \). We denote by \( D_a \) all the choice functions for \( a \) and by \( D \) all the choice functions for all letters \( a \in \Sigma \).

Let \( D_A = D \cup Q_\lor \cup \Sigma \) be the set of labels.

Intuitively, an edge \( e \) from a node (set of states) \( s \) to set of states \( s' \) corresponds to one of three cases.

- Either \( e \) corresponds to a set of transitions taken by local states of \( A \), in which case \( e \) is labeled by the choice function associating each existential state in \( s \) to the successor chosen for it.
- Edge \( e \) corresponds to a set of transitions taken by modal states of \( A \). In this case the edge corresponds to the transitions of exactly one existential modal state and potentially many universal modal states. In this case \( e \) is labeled by the existential modal state whose transition was taken.
- Or \( e \) corresponds to a choice of a letter in \( \Sigma \).

Given a set of states \( s \subseteq Q_\lor \), \( q \in s \cap Q_\lor \), a letter \( \sigma \), and a choice \( d \in D_\sigma \) we define the update of \( s \) as follows:

\[
\text{update}_l(s,\sigma,d) = (s \setminus Q_\lor) \cup \{q' \mid \exists q \in s \cap Q_\lor \text{ and } q' \in \delta(q,\sigma) \text{ or } q' \in d(q)\}
\]

\[
\text{update}_m(s,\sigma,q) = \{q' \mid q' \in \delta(q,\sigma) \text{ or } \exists q'' \in s \cap Q_\square \text{ and } q' \in \delta(q'',\sigma)\}
\]

That is, a local update consists of the set of all the successors of all the local universal states in \( s \) and all the chosen successors of all the local existential states in \( s \). A modal update consists of the successors of the (modal existential) state \( q \in s \) and all the successors of all the modal universal states in \( s \).

The set of edges is:

\[
E = \{(s,\sigma,d,(s',\sigma)) \mid (s,\sigma) \in V_\lor, d \in D_\sigma, \text{ and } s' = \text{update}_l(s,\sigma,d)\} \cup
\{(s,\sigma,q,(s',\sigma)) \mid (s,\sigma) \in V_\square, q \in s \cap Q_\lor, \text{ and } s' = \text{update}_m(s,\sigma,q)\} \cup
\{(s,\sigma,(s,\sigma)) \mid \sigma \in \Sigma\}
\]

That is, a node \((s,\sigma)\), where \( s \) contains local states of \( A \), has successors that correspond to taking a transition from all the local states. For existential local states only one successor is taken (according to the choice labeling the edge) and for universal local states all successors are taken. A node \((s,\sigma)\), where \( s \) contains no local states, has successors that correspond to taking a transition from one existential modal state in \( s \) (according to the state labeling the edge) and taking the transitions of all the universal modal states in \( s \). A node \( s \subseteq Q_\lor \) has successors that correspond to choosing a letter \( \sigma \in \Sigma \) (labeling the edge) and moving to \((s,\sigma)\).

We now define the winning condition associated with the strategy arena. For a labeled arena, the winning condition is a subset of \( V \cdot (D_A \cdot V)^\omega \). We construct a word automaton to define the winning condition.
Definition 4.5 (Tracking Automaton). Given an alternating parity tree automaton \( A = (\Sigma, Q, q_0, \delta, \Omega) \), and the arena \( G_A \), we define the tracking automaton \( T_A = (\Sigma_A, Q, q_0, \Gamma, \Omega) \) to be a nondeterministic parity word automaton. The alphabet of \( T_A \) is \( \Sigma_A = (\Sigma \times (D \cup Q^\diamond)) \cup \Sigma \). That is \( T_A \) reads either a letter in \( \Sigma \) and either a choice function or a modal existential state or simply a letter in \( \Sigma \). The transition function of \( T_A \) is defined by putting

\[
\Gamma(q,(\sigma,d)) = \begin{cases} 
  d(q) & \text{if } q \in Q_v \text{ and } d \in D \\
  \emptyset & \text{if } q \in Q_v \text{ and } d \in Q^\diamond \\
  \delta(q,\sigma) & \text{if } q \in Q^\land \\
  \delta(q,\sigma) & \text{if } q \in Q^\diamond \text{ and } d = q \\
  \emptyset & \text{if } q \in Q^\diamond \text{ and } d \in Q \setminus \{q\} \\
  q & \text{if } q \in Q^\diamond \text{ and } d \notin Q \\
  \delta(q,\sigma) & \text{if } q \in Q^\square \text{ and } d \in Q \\
  q & \text{if } q \in Q^\square \text{ and } d \notin Q 
\end{cases}
\]

for \( q \in Q \), \( \sigma \in \Sigma \), and \( d \in D \cup Q^\diamond \).

For \( q \in Q \) and \( \sigma \in \Sigma \) we put \( \Gamma(q,\sigma) = \{q\} \).

Notice that the only transitions of \( T_A \) that lead to sets with more than one element are when \( q \in Q^\land \). Indeed, when \( q \in Q_m \), by assumption, we have \(|\delta(q,\sigma)| = 1 \). Finally, \( \Omega \) is obtained from \( \Omega \) by setting \( \Omega(q) = \Omega(q) + 1 \).

We are now ready to define the acceptance condition \( \alpha \). Recall, that a labeled play \( \pi \) in \( G_A \) is \( \pi \in V \cdot (D_A \cdot V)^\omega \). Given a pair \((v,d)\), their projection onto \( \Sigma_A \), denoted \([v,d]\), is \(([s,\sigma],d) = (\sigma,d) \) and \([s,\sigma] = \sigma \). Given an infinite sequence \( \pi = v_0d_0v_1d_1 \cdots \in V \cdot (D_A \cdot V)^\omega \), we denote by \([\pi]\) the sequence \([v_0,d_0][v_1,d_1] \cdots \). We define \( \alpha_A \) as follows.

\[
\alpha_A = \{\pi \mid [\pi] \notin L(T_A)\}
\]

Let \( G_A^+ \) be the combination of the arena \( G_A \) with \( \alpha_A \). We sum up the relation between \( A \) and \( G_A^+ \) as follows.

Theorem 4.6 (Simulation). Let \( A \) be an alternating tree automaton \( A \). Then \( A \) is non-empty if and only if player \( \diamond \) wins \( G_A^+ \) from \( \{q_0\} \).

Proof. Let \( A = (\Sigma, Q, q_0, \delta, \Omega) \) be non-empty so that there is some Kripke structure \( K = (W,w_0,R,L) \) such that player \( \diamond \) wins \((w_0,q_0)\) in \( G_{A,K} \). Let \( \kappa_{G_A,K} \) be a positional strategy for player \( \diamond \) in \( G_{A,K} \) witnessing this. We inductively construct a history-dependent strategy \( \kappa_{G_A^+} \) for player \( \diamond \) in \( G_A^+ \) as follows. Let \( \tau = v_0d_0v_1d_1 \cdots v_n \in V \cdot (D \cdot V)^* \) be a finite labeled play of \( G_A^+ \) according to the strategy that has been constructed so far. We write \( v_j = s_j \) if \( v_j \in \mathcal{P}(Q) \) and \( v_j = (s_j,\sigma_j) \) if \( v_j \in \mathcal{P}(Q) \times \Sigma \). We will use the following invariant in our construction:

There is a fixed state \( w_o \in W \) such that for all \( q \in s_o \), player \( \diamond \) wins \((w_o,q)\) in \( G_{A,K} \).
Initially, the invariant holds trivially since player \( \diamond \) wins \((w_0, q_0)\) in \(G_{A,K}\) by assumption.

- If \(v_o \in \mathcal{P}(Q)\), then we put \(\kappa_{G_A^+}(\tau) = (s_o, L(w_o))\) and label this transition with \(L(w_o)\). The invariant trivially holds for \(\tau \cdot L(w_o) \cdot (s_o, L(w_o))\).
- If \(v_o \in \mathcal{P}(Q) \times \Sigma\) and \(s_o \cap Q_\ell \neq \emptyset\) then we put \(\kappa_{G_A^+}(\tau) = (s', \sigma_o)\) where 
  \[s' = update_l(s_o, \sigma_o, d)\]
  and where \(d\) is obtained by putting \(d(q) = q'\) for 
  \[q \in s_o \cap Q_\ell,\]
  where \(q'\) is the state such that \(\kappa_{G_A,K}(w, q) = (w, q')\). Label this transition with \(d\). Since \(\kappa_{G_A,K}\) is a winning strategy and since player \(\diamond\) wins \((w_o, q)\) in \(G_{A,K}\) for each \(q \in s_o\) by the inductive invariant, player \(\diamond\) also wins \((w_o, q')\) in \(G_{A,K}\) for each \(q' \in s'\), showing that the invariant holds for \(\tau \cdot d \cdot \langle s', \sigma_o\rangle\).
- If \(v_o \in \mathcal{P}(Q) \times \Sigma\) and \(s_o \cap Q_\ell = \emptyset\), then we have to show that the invariant holds for all successors of \(v_o\).
  Consider \(s'\) and \(q \in s_o \cap Q_\ell\) such that we have \(\langle(s_o, \sigma_o), q, s'\rangle \in E\). Recall that player \(\diamond\) wins \((w_o, q')\) for all \(q' \in s_o\) by the inductive invariant. In particular, player \(\diamond\) wins \((q, w_o)\) in \(G_{A,K}\) so that there is \(w_{o+1} \in R(w)\) such that player \(\diamond\) wins \((\delta(q, \sigma_o), w_{o+1})\) in \(G_{A,K}\). For all \(q'' \in s_o \cap Q_\ell\) and 
  \[q' \in \delta(q'', \sigma_o),\]
  player \(\diamond\) wins \((q', w')\) for all \(w' \in R(w)\) since player \(\diamond\) wins \((q'', w_o)\) in \(G_{A,K}\); in particular, player \(\diamond\) wins \((q', w_{o+1})\). Hence the invariant holds for \(\tau \cdot t \cdot \langle s', \sigma_o\rangle\).

The function \(\kappa_{G_A^+}\) is a strategy for player \(\diamond\) in \(G_A^+\) by construction. To see that \(\kappa_{G_A^+}\) is a winning strategy, let \(\pi = v_0d_0v_1\ldots\) be a labeled play of \(G_A^+\) that follows \(\kappa_{G_A^+}\). It remains to show that \(|\pi| \notin L(T_A)\). In the case that \(\pi\) is finite, clearly \(|\pi| \notin L(T_A)\). Consider the case that \(\pi\) is infinite. In this case, \(\pi\) corresponds to an infinite play in \(G_{A,K}\) with potentially some finite stuttering. Let \(\pi = v_0d_0v_1\ldots\) and let \(v_0w_1\ldots\) be the worlds of \(K\) in the construction of the invariant above.

Any run of \(T_A\) on \(|\pi|\) is a sequence \(q_0\ldots\) maintaining \(q_o \in s_o\) for all \(o \geq 0\), where \(v_o\) is either of the form \((s_o, \sigma_o)\) or of the form \(s_o\). Furthermore, transitions where \(T_A\) does not change its state due to our construction stutter a finite number of times. It follows that \(q_0q_1\ldots\) is winning for player \(\diamond\) in \(G_{A,K}\) and thus cannot satisfy the parity condition of \(T_A\), which is \(\bar{\Omega}\).

For the converse direction, let player \(\diamond\) win \(G_A^+\) from \(\{q_0\}\). Let \(\kappa_{G_A^+}\) be a strategy for player \(\diamond\) in \(G_A^+\) that witnesses this. We inductively construct a Kripke structure \(K = (W, w_0, R, L)\) and a strategy \(\kappa_{G_A,K}\). To this end, we proceed by induction over the length of prefixes \((w_0, q_0)\ldots(w_o, q_o)\) of plays in \(G_{A,K}\). Let \(o\) be a number such that \(K\) and \(\kappa_{G_A,K}\) have been constructed for all plays \((w_0, q_0)\ldots(w_o, q_o)\) of \(G_{A,K}\). We use the following inductive invariant.

For all plays \(\pi = (w_0, q_0)\ldots(w_o, q_o)\) of \(G_{A,K}\), there exists a play \(v_0, \ldots, v_l\) of \(G_A^+\) compatible with \(\kappa_{G_A^+}\) and \(v_l = (s_l, \sigma_l, i_l)\) where \(L(q_o) = \sigma_l\) and \(q_o \in s_l\).

Initially, \(G_A^+\) starts from \(\{q_0\}\) and \(\kappa_{G_A^+}\) chooses a successor \((\{q_0\}, \sigma, 1)\) for some \(\sigma \in \Sigma\). We add to \(K\) the initial world \(w_0\) such that \(L(w_0) = \sigma\) and
consider the prefix \((w_0,q_0)\) in \(G_{A,K}\). We have \((w_0,q_0)\) associated with the prefix \(\{q_0\},\{\{q_0\},\sigma,1\}\) compatible with \(\kappa_{G_{A,K}^+}\).

In the inductive step, we extend \(K\) and \(\kappa_{G_{A,K}}\) as follows. Consider a prefix \((w_0,q_0)\cdots(w_o,q_o)\) and the associated play \(v_0,\ldots,v_l\) of \(G_{A,K}^+\). Let \(v_l = (s_l,\sigma_l)\).

We start by extending \(K\). We build temporarily a function \(f_{w_o}\) that associates a newly created successor world \(w_q\) of \(w_o\) with combinations of states \(q \in Q_0\) and plays \(v_0,\ldots,v_l,v_{l+1},\ldots,v_{l'}\) that are compatible with \(\kappa_{G_{A,K}^+}\). By construction either \(v_l \in V_0\) or \(\kappa_{G_{A,K}^+}\) goes through a finite sequence \(v_{l+1},\ldots,v_{l+p}\) of nodes in \(V_0\) until it gets to a node \(v_{l+p+1}\) in \(V_0\). This sequence is finite due to the automaton having no loops of local states. For each state \(q \in s_{l+p+1} \cap Q_0\), there exists a successor \(v_q\) such that \((v_{l+p+1},q,v_q)\) is an edge in \(G_{A,K}^+\). It follows that \(v_q\) is of the form \(s_q\) and that \(q \in s_q\). Furthermore, \(\kappa_{G_{A,K}^+}\) chooses a successor \(v_q' = (s_q,\sigma_q)\) of \(v_q\). We extend \(K\) by adding to it a new world \(w_q\) that is a successor of \(w_o\) such that \(L(w_q) = \sigma_q\). The function \(f_{w_o}\) associates \(w_q\) with \(q\) and the play \(v_0,\ldots,v_l,v_{l+1},\ldots,v_{l+p},v_{l+p+1},v_{q'}\).

In order to extend \(\kappa_{G_{A,K}}\) we consider the following cases:

- \(q_o \in Q_\Box\) - by assumption \(q_0 \in s_l\). As \(q_0 \in s_l\) we know that \(v_l\) is in \(V_0\). Hence, \(\kappa_{G_{A,K}^+}\) singles out a successor \(v_{l+1}\) of \(v_l\). Let \(d\) be the label of the edge \((v_l,d,v_{l+1})\). Let \(q_{o+1} = d(q_o)\). By definition \(q_{o+1} \in \delta(q_o,\sigma_l) = \delta(q_o,L(w_o))\).

- \(q_o \in Q_\land\) - by assumption \(q_0 \in s_l\). We continue as above except that for every \(q' \in \delta(q_o,L(w_o))\) we add a successor \((q',w_o)\) as compatible with \(\kappa_{G_{A,K}}\). The successor \((q',w_o)\) is associated with the same extended play in \(G_{A,K}^+\) as above.

- \(q_o \in Q_\lor\) - by assumption \(q_0 \in s_l\). Let \(q_{o+1}\) be the unique state such that \(q_{o+1} \in \delta(q_o,L(w_o))\).

  - If \(v_l \in V_\Box\) then \(v_l\) has been used in the construction of \(K\) to identify the successor \(w_{q_o}\) of \(w_o\) that is associated with \(q_o\) in the construction of \(K\). It follows that \(q_{o+1} \in s_{q_o}\). We extend the play compatible with \(\kappa_{G_{A,K}}\) by adding to it the pair \((w_{q_o},q_{o+1})\) and associate with it the play pre-prepared for \(w_{q_o}\) in the construction of \(K\).

- \(q_o \in Q_\Box\) - as before, let \(q_{o+1}\) be the unique state such that \(q_{o+1} \in \delta(q_o,L(w_o))\).

We now proceed similar to the case of \(q_o \in Q_0\).

As before, we find that \(q_o\) is in the node \(v'\) used to construct all the successors of \(w_o\). For every successor \(w_q\) of \(w_o\) we create a play extended by \((w_q,q_{o+1})\) and associate it with the pre-prepared play that ends in \(w_q\). The construction of the arena \(G_{A,K}^+\) ensures that \(q_{o+1} \in s_q\).

Let \(K^\infty\) and \(\kappa_{G_{A,K}^\infty}\) denote the Kripke structure and strategy, respectively, that are obtained in this way. It remains to show that \(\kappa_{G_{A,K}^\infty}\) is a winning strategy for player \(\Diamond\) in \(G_{A,K}^\infty\). Consider an infinite play \(\pi = (w_0,q_0)\ldots\) that is an outcome of \(\kappa_{G_{A,K}^\infty}\). During the construction of \(K^\infty\) and \(\kappa_{G_{A,K}^\infty}\) each
prefix \((w_0,q_0),\ldots,(w_i,q_i)\) of \(\pi\) was associated with a prefix \(v_0,\ldots,v_i\) of a play in \(G_A^+\) compatible with \(\kappa_{G_A^+}\). It follows that the limit of all these prefixes is an infinite play compatible with \(\kappa_{G_A^+}\) and hence winning in \(G_A^+\). Let \(d_0,\ldots\) be the sequence of labels such that \((v_i,d_i,v_{i+1})\) are the edges in \(G_A^+\) taken in this play. It follows that the sequence of state \(q_0,\ldots\) corresponds to a run of \(T_A\) reading \([v_0,d_0][v_1,d_1]\cdots\) with (potentially) additional stuttering resulting from entering a modal state \(q_i\) while the matching \(v_i\) is of the form \((s,L(w_i))\) where \(s \cap Q_l \neq \emptyset\). As the stuttering is finite, it follows that \(\inf(q_0,\ldots)\) is the same as the run of \(T_A\). As the run of \(T_A\) is rejecting, the sequence \(q_0,\ldots\) is winning for player \(\lozenge\). \(\square\)

Consider the automaton \(T_A\). By Theorem 4.3 there exists an equivalent history deterministic automaton \(H_A\). Let \(H_A = (\Sigma_A, T, t_0, \rho, I')\). By using \(H_A\) we can turn the game \(G_A\) to a parity game \(G_A^+\) capturing the non-emptiness of \(A\).

**Definition 4.7** \((G_A^+)\). Consider the strategy arena \(G_A = (V, V_\lozenge, V_\boxtimes, E)\) and the automaton \(H_A\). We construct the parity game \(G_A^+ = (V', V_\lozenge', V_\boxtimes', E', \Omega')\), where the components of \(G_A^+\) are as follows.

- \(V' = (V \times T) \cup (\Sigma_A \times V \times T)\)
- \(V_\lozenge' = V_\lozenge \times T\)
- \(V_\boxtimes' = (V_\boxtimes \times T) \cup (\Sigma_A \times V \times T)\)
- The set of edges is:

\[E' = \{ \{(s,\sigma), t\}, ((\sigma,d),v',t) \mid ((s,\sigma),d,v') \in E\} \cup \{
\{(s,t), (\sigma,v',t)\} \mid (s,\sigma,v') \in E\} \cup \{
\{((\sigma,d),v',t), (v',t')\} \mid t' \in \rho(t, (\sigma,d))\} \cup \{
\{(\sigma,v',t), (v',t')\} \mid t' \in \rho(t, \sigma)\}\}

- The priority function \(\Omega''\) is obtained from \(\Omega'\) by setting \(\Omega''(v,t) = \Omega'(t) + 1\) and \(\Omega''(a,v,t) = \Omega'(t) + 1\).

**Theorem 4.8** (Game Translation). Player \(\lozenge\) wins in \(G_A^+\) from some state \((s,\sigma)\) if and only if player \(\lozenge\) wins in \(G_A^+\) from \(((s,\sigma),t_0)\).

**Proof.** We can show that player \(\square\) wins in \(G_A^+\) if and only if she wins in \(G_A^+\). The proof follows the proof that history deterministic automata can be used in combination with games as in [12].

**Remark 4.9.** An alternative way to view the construction of \(T_A\) and \(H_A\) is to think about the dual of \(T_A\) as a universal automaton recognizing plays that are winning for player \(\lozenge\). Then, the dual of \(H_A\) would be a history-deterministic universal parity automaton recognizing the same language. The resolution of the transition function of a history-deterministic universal automaton is delegated to player \(\square\) just like it is in the construction of \(G_A^+\). In particular, every history-determinization for nondeterministic automata is, in fact, also a...
history-determinization for universal automata. This implies that the history-determinization construction of Henzinger and Piterman [12] can be also used for under-approximating the losing region in an LTL game, which was left as an open question in their paper.

The following is a direct implication of Theorems 4.6 and 4.8.

**Corollary 4.10 (Emptiness).** Let $A$ be an alternating tree automaton. Then $A$ is non-empty if and only if player ♦ wins $G_A^*$ from the node $(\{q_0\}, t_0)$.

We now consider the complexity of the decision problem. Let $A$ be an alternating tree automaton reading an alphabet of size $m$ with $n$ states and rank $k$. Then $G_A$ has $(m + 1) \cdot 2^n$ vertices and $T_A$ has $n$ states and rank $k$ as well. Let $S_{HD}(n, k)$ denote the number of states and $R_{HD}(n, k)$ denote the rank of a history deterministic automaton obtained from a nondeterministic word automaton with $n$ states and rank $k$.

**Corollary 4.11 (Complexity).** Let $A$ be an alternating tree automaton reading an alphabet of size $m$, with $n$ states and rank $k$. The complexity of emptiness of $A$ is \(\text{parity}((m + 1) \cdot 2^n \cdot S_{HD}(n, k), R_{HD}(n, k))\)-Time.

**Remark 4.12.** In case that $H_A$ is a deterministic automaton, a winning strategy for player ♦ in $G_A^*$ directly induces a Kripke structure $K$ with set of worlds $W = V_\Box \times T$ such that $A$ accepts $K$. Hence non-empty alternating parity tree automata accept some structure of size at most $(m + 1) \cdot 2^n \cdot ((nk)!)^2$ by Lemma 5.1 and Lemma 5.11 below.

5 **Transformations of Word Automata**

We specialize parity acceptance conditions to the special cases of Büchi and Co-Büchi conditions. In a Büchi condition the priority function uses only the priorities 1, 2. In a Co-Büchi condition the priority function uses only the priorities 0, 1. For Büchi automata, we put $F = \{ q \in Q \mid \Omega(q) = 2 \}$; for Co-Büchi automata, we put $F = \{ q \in Q \mid \Omega(q) = 0 \}$; in both cases, we put $\overline{F} = Q \setminus F$.

The Büchi acceptance requires accepting runs to contain infinitely many accepting states, while Co-Büchi acceptance requires accepting runs to contain only finitely many non-accepting states. A Co-Büchi automaton is weak if for all its strongly connected components $C$, we have $C \subseteq F$ or $C \subseteq \overline{F}$. For deterministic automata, we extend $\delta$ from letters to finite words in the obvious way.

**Lemma 5.1 ([14]).** Let $A = (\Sigma, Q, q_0, \delta, \Omega)$ be a nondeterministic parity word automaton of rank $k$. Then there is a nondeterministic Büchi word automaton $A' = (\Sigma, Q', q_0, \delta', F)$ such that $L(A) = L(A')$ and $|Q'| \leq \left(\left\lceil \frac{k}{2} \right\rceil + 1 \right) \cdot |Q|$. 

*Proof.* We just recall the construction of $A'$ and refer to [14] for the proof of $L(A) = L(A')$. Intuitively, the automaton $A'$ nondeterministically guesses a position and an even priority $p$ such that there is a run of $A$ on the input word
such that from the guessed position on, no state with priority greater than $p$ is visited and some state with priority $p$ is visited infinitely often. Formally, we put $Q'' = Q \times \{ i \in \mathbb{N} \mid 0 \leq i \leq k \text{ and } i \text{ is even} \}$ and

$$Q' = Q \cup Q'' \quad F = \{(q, i) \in Q'' \mid \Omega(q) = i\}$$

so that the claimed bound on the size of $A'$ follows immediately. The transition function $\delta'$ is defined, for $q \in Q$, even $i$ such that $0 \leq i \leq k$ and $a \in \Sigma$, by putting

$$\delta'(q, a) = \delta(q, a) \cup \{(q', i) \in Q'' \mid q' \in \delta(q, a) \text{ and } \Omega(q') = i\}$$

and

$$\delta'((q, i), a) = \{(q', i) \in \delta(q, a) \times \{i\} \mid \Omega(q') \leq i\}.$$

\[\square\]

**Definition 5.2 (Limit-linear Co-Büchi Automata).** A Co-Büchi automaton $A = (\Sigma, Q, q_0, \delta, \Omega)$ is **limit-linear** if for all $q \in F$, there is exactly one $\delta$-path that stays in $F$ and leads from $q$ to $q$.

**Definition 5.3 (Limit-deterministic Word Automata).** Fix a parity word automaton $A = (\Sigma, Q, q_0, \delta, \Omega)$. Given a state $q \in Q$, the compartment $C_q$ of $q$ consists of all states that are reachable from $q$ by a path that visits states with priority at most $\Omega(q)$. We say that $A$ is **limit-deterministic (LD)** if for all states $q$ such that $\Omega(q)$ is even, $C_q$ is internally deterministic, that is, $|\delta(q', a) \cap C_q| \leq 1$ for all $q' \in C_q$ and $a \in \Sigma$.

Thus a Büchi automaton is limit-deterministic if all its states that are reachable from an accepting state are deterministic. A Co-Büchi automaton is limit-deterministic if all its accepting states are deterministic. In particular, every limit-linear Co-Büchi automaton is limit-deterministic.

**Lemma 5.4.** The construction in Lemma 5.1 preserves limit determinism.

*Proof.* Let $A$ be a limit-deterministic parity automaton. Then we claim that $A'$ as constructed in Lemma 5.1 is limit-deterministic. Since $A'$ is a Büchi automaton and since all states that are reachable from some state in $F$ are contained in $Q \times \{p\}$ for some even $p$, it suffices to show that for all even $p$, all states $(q, p) \in Q \times \{p\}$ and all $a \in \Sigma$, we have $|\delta'((q, p), a)| \leq 1$. So let $(q, p) \in Q \times \{p\}$. Then, by construction of $A'$, $(q, p)$ is contained in the compartment $C$ of some state with priority $p$ in $A$. By definition of $\delta'$, we have $\delta'((q, p), a) \subseteq C$ since $C$ is a compartment. Since $A$ is limit-deterministic, $C$ is internally deterministic which shows $|\delta'((q, p), a)| \leq 1$, as required. \[\square\]
5.1 Determinizing Word Automata

We give specialized determinization constructions for limit-linear Co-Büchi automata, nondeterministic Co-Büchi automata, limit-deterministic Büchi automata, and finally for general Büchi automata and parity automata.

Lemma 5.5 (Circle method). Let $A$ be a limit-linear Co-Büchi automaton with $n$ states. Then there is a deterministic Co-Büchi automaton $A'$ with $n'$ states such that $L(A) = L(A')$ and $n' \leq n^2 \cdot 2^n$.

Proof. Let $A = (\Sigma, Q, q_0, \delta, F)$. If $F = Q$, then $A$ is deterministic and we put $A' = A$. Otherwise, we have $|F| < n$ and proceed with the following construction, which is similar to the powerset construction, but additionally annotates macro-states with a single state and a counter. The states of accepting components are arranged in a cycle since $A$ is limit-linear. Intuitively, the single state component of macro-states identifies exactly one state in exactly one accepting cycle that has a token. The determinized automaton then checks whether it is possible to stay within this cycle forever, moving the token according to the letters that are read. If this is not possible, the automaton reduces the counter by one and moves the token to the next state in the current cycle and again checks whether is possible to stay in the cycle forever when moving the token according to the read word. When this fails so often that the counter reaches 0, the automaton picks a state from another accepting cycle, moves the token to this state and resets the counter. It is crucial that the moving of tokens between accepting cycles is done in a fair way, so that if the token changes cycles infinitely often, the token visits every accepting cycle infinitely often. Then the token eventually stays forever within one accepting cycle if and only if there is an accepting run.

Formally, we proceed as follows. For moving the token between accepting cycles, we assume a function $\text{next} : Q \to Q$ such that for $q \in Q$, $\text{next}(q)$ is some arbitrary but fixed state from an accepting cycle of $A$ such that iterative application of $\text{next}$ cycles through all accepting cycles of $A$ in a fair manner. We also assume a function $\text{step} : F \to F$ that cycles through the states of a single accepting cycle of $A$ in a fair manner; formally, we put $\text{step}(q) = q'$ where $q'$ is the state such that there is some $a \in \Sigma$ such that $\delta(q, a) \cap F = \{q'\}$. We define the deterministic Co-Büchi automaton $A' = (\Sigma, Q', u_0, \delta', F')$ by putting

$$Q' = 2^Q \times Q \times \{0, \ldots, |F|\}$$

$$F' = \{(U, q, c) \in Q' \mid c \neq 0\}$$

and $u_0 = (\{q_0\}, q_0, 0)$. The claimed bound on the size of $A'$ follows immediately since $|F| < n$ so that $\{|0, \ldots, |F|\}| \leq n$. Finally, the transition relation $\delta'$ is defined by putting, for $(U, q, c) \in Q'$ and $a \in \Sigma$,

$$\delta'((U, q, c), a) = (\delta(U, a), q', c)$$

if $c \neq 0$, $\delta(q, a) \cap F = \{q'\}$ and $q \in U$; this moves the token within the current accepting cycle according to the input letter, if possible. Otherwise, the run...
represented by the token does not stay in the current accepting cycle and we move the token to another state. This is achieved by putting

\[ \delta'(\langle U, q, c \rangle, a) = \begin{cases} \langle \delta(U, a), \text{next}(q), |F| \rangle & \text{if } c = 0 \\ \langle \delta(U, a), \text{step}(\text{step}(q)), c - 1 \rangle & \text{if } c > 0 \end{cases} \]

If \( c = 0 \), then the token is moved to the next accepting cycle and the counter is reset to \( |F| \); if \( c > 0 \), then the token is moved to the next state in the current accepting cycle (to also incorporate the \( a \)-transition that takes place, we apply \text{step} twice) and the counter is reduced by 1.

To see \( L(A) \subseteq L(A') \), let \( \tau = q_0, q_1, \ldots \) be an accepting run of \( A \) on some word \( w \). Then \( \tau \) stays within one accepting component \( C \) of \( A \) from some point on; let \( i \) be a position from which on this the case. Since \( A \) is limit-linear, \( C \) forms a cycle. Let \( \pi \) be the run of \( A' \) on \( w = a_0, a_1, \ldots \). We have to show that there is a position \( j \) such that all states in \( \pi \) are of the shape \( \langle U_j, q_j', c_j \rangle \) from position \( j \) on. If there is some position \( j \) and some accepting component \( C' \neq C \) such that the states in \( \pi \) are of the shape \( \langle U_j, q_j', c_j \rangle \) such that \( q_j', c_j \in C' \) for all \( j' \geq j \), then we are done. Otherwise, it suffices to show that there is a position \( j \geq i \) such that the \( j \)-th state in \( \pi \) is of the shape \( \langle U_j, q_j, c_j \rangle \), since we then have \( \delta(q_j', a_j) \cap F = \{ q_{j+1}' \} \) and \( q_j' \in U_j \) for all \( j' \geq j \), since \( A \) is limit-linear. Since next cycles through the strongly connected components of \( A \) in a fair manner and since there is no position \( j \) such that there is an accepting component \( C' \neq C \) such that the states in \( \pi \) are of the shape \( \langle U_j, q_j', c_j \rangle \) such that \( q_j', c_j \in C' \) for all \( j' \geq j \), there is some \( j'' \) such that the \( j'' \)-th state in \( \pi \) is of the shape \( \langle U_j, q_{j''}, c_{j''} \rangle \), where \( q_{j''}, c_{j''} \in C \). We proceed by induction of the length \( m(q_{j''}, q_{j''}) \) of the path from \( q_{j''} \) to \( q_{j''} \). If \( m(q_{j''}, q_{j''}) = 0 \), then \( q_{j''} = q_{j''} \) so that we are done. If \( m(q_{j''}, q_{j''}) > 1 \) then we distinguish cases. If \( \delta(q_j', a_j) \cap F \neq \emptyset \) and \( q_{j''}, c_{j''} \in U_{j''} \) for all \( j'' \geq j \), then we are done. Otherwise, we eventually reach a state \( \langle U_o, q_o, c_o \rangle \) such that \( \delta(q_o, a_o) \cap F = \emptyset \) or \( q_o \in U_o \). For the next state \( \langle U_{o+1}, q_{o+1}, c_{o+1} \rangle \) we then have \( c_{o+1} = c_o - 1 \) and \( q_{o+1}' = \delta(\text{step}(q_o), a) \). We have \( m(q_{o+1}', q_o) < m(q_o, q_o) \) so that the inductive hypothesis finishes the case.

For the converse direction, we have to show \( L(A) \subseteq L(A') \). Let \( w = a_0, a_1, \ldots \) be a word such that \( w \in L(A') \) and let \( \pi \) be the accepting run of \( A' \) on \( w \). Since \( \pi \) is accepting, there is a position \( i \) such that the states in \( \pi \) are, for all \( i' \geq i \), of the shape \( \langle U_{i'}, q_{i'}, c \rangle \) for some \( c \) and we have \( \delta(q_i', a_i) \cap F = \{ q_{i'-1}' \} \) and \( q_{i'} \in U_{i'} \). In particular, \( q_{i'} \in F \) for all \( i' \geq i \). Let \( \tau_i \) be a run of \( A' \) on \( a_0, \ldots, a_i \) that ends in \( q_i \) and put \( \tau = \tau_1; q_i+1, q_{i+2}, \ldots \) so that \( \tau \) is an accepting run of \( A \) on \( w \).

**Example 5.6.** Consider the limit-linear Co-Büchi automaton \( A \) depicted below, and the equivalent deterministic Co-Büchi automaton \( A' \) obtained by using the construction from Lemma 5.5 to be able show a complete example, \( A \) is picked to be a very simple automaton (accepting just the word \( (ab)\omega \)). For brevity, we depict only the reachable part of \( A' \) and collapse all macro-states of the shape \( (\emptyset, q, c) \) to a single non-accepting sink state \( \bot \). Any macro-state in \( A' \) that has a nonzero counter value is accepting. We have \( \text{step}(y) = u \) and \( \text{step}(u) = y \). Since
there is just one accepting strongly connected component in A, we assume that \( \text{next}(x) = \text{next}(y) = \text{next}(z) = y \), and \( \text{next}(u) = u \).

The automaton \( A' \) starts with the token at \( x \) and with counter value 0. When reading \( a \), the token is moved to \( \text{next}(x) = y \) and the counter is reset to 2. Afterwards, there are two cases: If the automaton reads \( bb \), it is not possible in \( A \) to move the token accordingly from \( y \) and stay in the accepting cycle between \( y \) and \( u \). Thus \( A' \) transitions to \( \{ \{ z \}, u, 1 \} \), intuitively moving the token to the next accepting cycle, which in this example moves the token to \( u \). This state however is not contained in the powerset component \( \{ z \} \) so that the automaton rejects the word, which is reflected by the fact that \( \{ \{ z \}, u, 1 \} \) accepts the empty language. The other option to proceed from \( \{ y, z \}, y, 2 \) is by reading sequences \( (ba)^* \), which results in repeatedly moving the token from \( y \) to \( u \) and back to \( y \); if this continues forever, the word is accepted by \( A' \). Otherwise, a sequence \( aa \) or \( bb \) is read eventually and the automaton transitions to the sink state and rejects the word.

**Lemma 5.7 (Miyano-Hayashi [15]).** Given a nondeterministic Co-Büchi automaton \( A = (\Sigma, Q, q_0, \delta, F) \), there is a deterministic Co-Büchi automaton \( A' = (\Sigma, Q', u_0, \delta', F') \) such that \( L(A) = L(A') \) and \( |Q'| \leq 3^{|Q|} \).

**Proof.** We just show the construction of \( A' \); for the proof of \( L(A) = L(A') \) we refer to [15]. The construction is similar to the powerset construction but additionally tracks subsets \( V \) of the accepting states \( U \cap F \) of macro-states \( U \subseteq Q \). Intuitively, there is, for each state in \( V \), a run of \( A \) that has not left \( F \) recently. Whenever this set is the empty set, it is reset to all accepting states of the current macro-state. A run of \( A' \) then is accepting if such resetting steps happen only finitely often, ensuring the existence of a run of \( A \) that from some point on stays within \( F \) forever. Formally, we put

\[
Q' = \{(U,V) \mid U \subseteq Q, V \subseteq U \cap F\} \quad F' = \{(U,V) \in Q' \mid V \neq \emptyset\}
\]

and \( u_0 = (\{q_0\}, \emptyset) \). The claimed bound on the size of \( A' \) follows since macro-states \( (U,V) \in Q' \) can be coded by functions \( f : Q \rightarrow \{0,1,2\} \) where \( f(q) = 0 \) if \( q \notin U \), \( f(q) = 1 \) if \( q \in U \) but \( q \notin V \) and \( f(q) = 2 \) if \( q \in V \); the number of such functions is bounded by \( 3^{|Q|} \). We define \( \delta' \) by putting

\[
\delta'((U,V), a) = \begin{cases} 
\delta(U,a), \delta(V,a) \cap F & \text{if } V \neq \emptyset \\
\delta(U,a), \delta(U,a) \cap F & \text{if } V = \emptyset
\end{cases}
\]

for \( (U,V) \in Q' \) and \( a \in \Sigma \). \( \centerdot \)
Example 5.8. Consider the nondeterministic Co-Büchi automaton $A$ depicted below, and the equivalent deterministic automaton $A'$ obtained by using the construction from Lemma 5.7; both automata accept exactly the infinite words over $\Sigma = \{a, b\}$ that contain $a$ finitely often. For brevity, we depict only the reachable part of $A'$ and label macro-states $(U, V)$ with $U \setminus V, V$.

The $a$-transition from the accepting macro-state $(\{y\}, \{z\})$ in $A'$ leads to the non-accepting macro-state $(\{y, z\}, \emptyset)$ and not to $(\{y\}, \{z\})$: the tracked set of accepting states is then reset to $\{z\}$ after a further $a$- or $b$-transition. This reflects the fact that no run of $A$ can stay in the accepting state $z$ by reading the letter $a$ so that all words that contain $a$ infinitely often are rejected by both $A$ and $A'$.

Lemma 5.9 (Permutation method [7,11]). Let $A$ be a limit-deterministic Büchi automaton with $n$ states. Then there is a deterministic parity automaton $A'$ with $n'$ states and $2n$ priorities such that $L(A) = L(A')$ and $n' \leq e(n + 1)!$.

Proof. We sketch just the construction of $A'$ and refer to [7,11] for the proof of equivalence of $A$ and $A'$. Intuitively, $A'$ is similar to the powerset automaton of $A$, but additionally keeps a permutation on the deterministic states in macro-states, indicating the order in which runs leading to the respective states have last seen an accepting state. Additionally, states in $A'$ contain a third component which indicates the leftmost position in the permutation that is active or ending by the transitions leading to the current state in $A'$. Here, a position is active in an $a$-transition, if the state at this position in the current permutation is accepting; a position is said to be ending if all runs of $A$ that are represented by the state at this position end when reading the letter $a$ or lead to a state at an older position. A parity condition then uses this information to detect a position in the permutation components that is active infinitely often but, from some point on, never ends. This ensures the existence of a continuous run of $A$ that visits some accepting state infinitely often.

Formally, we proceed as follows. Given a limit-deterministic Büchi automaton $A = (\Sigma, Q, q_0, \delta, F)$ with sets $Q_D, Q_N \subseteq Q$ of deterministic and nondeterministic states, respectively, we have that every state reachable from $F$ is contained in $Q_D$. We assume without loss of generality that $q_0 \in Q_N$. We let $\text{perm}(Q_D)$ denote the set of partial permutations over $Q_D$, that is, $\text{perm}(Q_D)$ consists of all partial functions $f : Q_D \to |Q_D|$ such that $f(q) \neq f(q')$ for all $q, q' \in \text{dom}(f)$ such that $q \neq q'$. We denote the empty permutation by $[]$ ($\text{dom}([]) = \emptyset$). Then
we define the deterministic parity automaton $A' = (\Sigma, Q', u_0, \delta', \Omega)$ by putting

$$Q' = 2^{Q_N} \times \text{perm}(Q_D) \times \{1, \ldots, 2|Q_D| + 1\}$$

and $u_0 = ([\emptyset], \emptyset, 1)$. The claimed bounds on the size and number of priorities of $A'$ follows. The transition function $\delta'$ is defined by putting, for $(U, f, p) \in Q'$ and $a \in \Sigma$,

$$\delta'((U, f, p), a) = (\delta(U, a) \cap Q_N, f', p'),$$

where $f'$ denotes the partial permutation that is obtained by applying $a$-transitions from $\delta$ to the partial permutation $f$, keeping the ordering intact but removing elements that do not have an outgoing $a$-transition; here it is crucial that all states in $f$ are deterministic so that it is never the case that additional elements are inserted between any two elements of the permutation. Furthermore, we add all states from $\delta(U, a) \cap Q_D$ that do not already occur in this new permutation to the end of it (the order of these elements is irrelevant). Let $i \geq 1$ be the leftmost position in $f'$ such that $f(i)$ is defined and $\delta(f(i), a) \neq f'(i)$ (including the case that $f'(i)$ is undefined), or we have $f'(i) \in F$. Thus $i$ identifies the leftmost position in the partial permutation that is active or ending (possibly both). If no such $i$ exists, put $p' = 1$. Otherwise, if $\delta(f(i), a) \neq f'(i)$, then put $p' = 2(|Q_D| - i + 3)$; if $\delta(f(i), a) = f'(i) \in F$, then put $p' = 2(|Q_D| - i) + 2$. □

Example 5.10. For the limit-deterministic Büchi automaton $A$ with $Q_N = \{x, z\}$ and $Q_D = \{y, u\}$ depicted below, we obtain the equivalent deterministic parity automaton $A'$ using the construction from Lemma 5.9. For brevity, we depict $A'$ with edge priorities, thus moving the priority component $p$ of macro-states $(U, f, p)$ to the edges.

![Diagram of automata A and A']

Let $\delta$ be the transition relation of $A$. In $A'$ there is an $a$-transition with priority 1 from the initial state $((\{x\}, \emptyset))$ to $((x, z), [y])$. This is the case since $\delta(\{x\}, a) = \{x, y, z\} \cap Q_N = \{x, z\}$. Since $y \in \delta(\{x\}, a) \cap Q_D$, we add it to the end of the permutation component which thereby changes from $\emptyset$ to $[y]$. We have $y \notin F$ so that there is no position in the permutation that ends or is active. Thus the priority of this transition is 1. There is an $a$-loop with priority 3 at $((x, z), [y, u])$. This is the case since $\delta(\{x, z\}, a) = \{x, y, z, u\}$ so that $\delta(\{x, z\}, a) \cap Q_N = \{x, z\}$. Also we update the permutation component $[y, u]$ according to reading the letter $a$: We have $\delta(y, a) = y$ and $\delta(u, a) = y$ and hence obtain a temporary permutation $[y]$. Now $\delta(\{x, z\}, a) \cap Q_D$ contains the state...
that is appended to the permutation, resulting in \([y, u]\) as new permutation component. As \(y \notin F\) and \(\delta(y, a) = y\), the leftmost position in the permutation component is neither active nor ending. We also have \(u \in F\) and \(\delta(u, a) \neq u\) so that position 2 is both ending and active. Hence the priority of this transition is \(2(|Q_D| - i) + 3 = 2(2 - 2) + 3 = 3\). This reflects the fact that even though an accepting state can be reached from \(\{x, y, z, u\}\) by an \(a\)-transition in \(A\) (as \(u \in \delta(z, a)\)), all runs that have visited an accepting state at least once before are residing in the state \(y\) after reading \(a\). Thus it is not possible to construct a continuous run that only reads the letter \(a\) and still visits \(u\) more than once. Intuitively, reading the letter \(a\) merges all runs leading to \(y\) or \(u\), so that both positions in the permutation component \([y, u]\) are merged into the new first position containing just \(y\); the second position thus is ending. The deterministic state \(u\) to which there is no \(a\)-transition from \(y\) or \(u\) then is appended as new (accepting) position to the permutation.

**Lemma 5.11 (Safra-Piterman [22,21])**. Let \(A = (\Sigma, Q, q_0, \delta, F)\) be a B"uchi automaton. Then there is a deterministic parity automaton \(A' = (\Sigma, Q', u_0, \delta', \Omega)\) such that \(L(A) = L(A')\), \(|Q'| \in \mathcal{O}(|Q|!^2)\) and \(A'\) has at most \(2|Q|\) priorities.

**Lemma 5.12 (Parity Determinization [24])**. Let \(A = (\Sigma, Q, q_0, \delta, \Omega)\) be a parity automaton of rank \(k\). Then there is a deterministic parity automaton \(A' = (\Sigma, Q', u_0, \delta', \Omega')\) such that \(L(A) = L(A')\), \(|Q'| \in \mathcal{O}(|Q|!^{2k})\) and \(A'\) has at most \(2|Q|k\) priorities.

### 5.2 History-determinizing Word Automata

Next we give specialized history determination constructions for limit-deterministic Co-B"uchi automata and for general B"uchi automata.

**Lemma 5.13 (History-determinizing by focusing)**. Let \(A = (\Sigma, Q, q_0, \delta, F)\) be a limit-deterministic Co-B"uchi word automaton. Then there is a history-deterministic Co-B"uchi word automaton \(A' = (\Sigma, Q', u_0, \delta', F')\) such that \(L(A) = L(A')\) and \(|Q'| \leq (|F| + 1) \cdot 2^{|Q|}\).

**Proof.** Intuitively, the determination procedure is similar to the powerset construction but uses the limited nondeterminism that is allowed in history-deterministic automata to guess a run that eventually stays in \(F\) forever. Information about the guessed runs is kept by annotating macro-states with a focus, that is, the state in which the run currently resides. If the guess turns out to be wrong and the run leaves \(F\), a new guess is taken (a refocusing step takes place). The resulting automaton then can be shown to be history-deterministic by using a resolver function that refocuses in a fair manner, guaranteeing that no run is overlooked. Formally, we put \(Q'' = \{ (U, q) \in 2^Q \times F \mid q \in U \}\) and

\[
Q' = 2^Q \cup Q'' \\
u_0 = \{ q_0 \} \\
F' = Q''
\]
from which the claimed bound on the size of $A'$ follows since

$$|Q'| = |2^Q \cup Q''| = 2^{|Q|} + (2^{|Q|} \cdot |F|) = (|F| + 1) \cdot 2^{|Q|}. $$

The transition relation $\delta'$ is defined by putting, for $a \in \Sigma$ and $U \subseteq Q$,

$$\delta'(U, a) = \{ \delta(U, a) \} \cup \{ (\delta(U, a), q') \mid q' \in \delta(U, a) \cap F \}$$

and, for $a \in \Sigma$ and $(U, q) \in Q''$,

$$\delta'((U, q), a) = \begin{cases} \{ (\delta(U, a), q') \} & \text{if } \delta(q, a) \cap F = \{ q' \} \\ \{ \delta(U, a) \} & \text{if } \delta(q, a) \cap F = \emptyset, \end{cases}$$

noting that since $A$ is limit-deterministic, we have $|\delta(q, a) \cap F| \leq 1$ if $q \in F$, so that the case distinction above is exhaustive. Given $(U, q) \in Q''$, we refer to $q$ as the focus and for $U \in Q'$ we say that the focus is finished at $U$. Outgoing transitions from $U \subseteq Q$ to $(U', q) \in Q''$ are refocusing transitions.

It remains to show that $L(A) = L(A')$ and that $A'$ is history-deterministic. For the first item, let $w = a_0a_1 \ldots \in L(A)$ and let $\tau = q_0, q_1, \ldots$ be an accepting run of $A$ on $w$. Let $i$ be a position such that $q_j \in F$ for all $j \geq i$; such $i$ exists since $\tau$ is accepting. Construct a run $\pi$ of $A'$ as follows: Let $\pi_1 = \{ q_0 \}, T_1, \ldots, T_i$ (where $T_{i'} \in Q'$ for $1 \leq i' \leq i$) be a run of $A'$ on the first $i$ letters of $w$. Continue the run $\pi_1$ deterministically, and let $j$ be the first position such that $T_j \subseteq Q$. If no such $j$ exists, then the run $\pi$ that is obtained by deterministically continuing $\pi_1$ along $w$ is an accepting run ($\pi_1$ can be continued deterministically since all states $T_j \in Q''$ are deterministic states and no states $T_j \notin Q''$ is reached by assumption). Otherwise, let $\pi_2 = \{ q_0 \}, T_1, \ldots, T_j$ be the deterministic continuation of $\pi_1$ up to the position $j$. We have $q_{j+1} \in T_{j+1} = \delta(T_j, a_j)$. We extend $\pi_2$ with the transition

$$T_j \xrightarrow{a_j} (\delta(T_j), a_j), q_{j+1})$$

which is a transition in $\delta'$. Since the focus $q_{j+1}$ is never finished by assumption, the nondeterministic continuation of $\pi_2$ is an accepting run.

For the converse direction, let $w = a_0a_1 \ldots \in L(A')$ and let

$$\tau = \{ q_0 \}, T_1, T_2 \ldots$$

be an accepting run of $A'$ on $w$. We construct an accepting run $\pi$ of $A$ as follows. Let $i$ be the first position such that $T_j = (U_j, q_j) \in Q''$ for all $j \geq i$; such $j$ exists since $\tau$ is accepting. Then there is a run $\pi_1$ of $A$ on the first $i$ letters of $w$ that ends in $q_i$. Continue $\pi_1$ with the sequence $q_{j+1}, q_{j+2}, \ldots$. This results in a run of $A$ on $w$ since $q_{j+1} \in \delta(q_j, a_j)$ for all $j \geq i$. Furthermore, the resulting run $\pi$ is accepting since $q_i \in F$ and for all $j \geq i$.

To see that $A'$ is history-deterministic, we have to define a suitable resolver. Let $w = a_0, a_1, \ldots \in L(A')$. For any partial run $\{ q_0 \}, T_1, \ldots, T_m$ of $L(A')$ such that $T_m \subseteq Q$, define $\sigma(\{ q_0 \}, T_1, \ldots, T_m, a_m) = (\delta(T_m, a_m), q_{m+1})$, where
\( q_{m+1} \in \delta(T_m, a_m) \cap F \) is a state from \( \delta(T_m, a_m) \cap F \) with maximal age. Here, the age of a state \( q \in \delta(T_m, a_m) \) is the least position \( j \), such that \( T_j = U_j \) or \( T_j = (U_j, q_j) \) and \( U_j \) contains some state \( q' \) such that \( q \in \delta(q', a_{j+1}, ..., a_m) \).

If \( \delta(T_m, a_m) \cap F = \emptyset \), then put \( \sigma(\{q_0\}, T_1, ..., T_m, a_m) = \delta(T_m, a_m) \). Hence the resulting function \( \sigma \) picks a run \( \sigma(w) \) for each word \( w \). If \( w \in L(A') \), then \( \sigma(w) \) is an accepting run since then there is a run in \( A \) that eventually stays in \( F \) forever and since \( \sigma(w) \) refocuses in a fair manner (preferring older traces over younger traces), it is guaranteed to eventually pick a focus that is never finished.

We note that the automaton \( A' \) in the above construction is not a weak automaton, even if \( A \) is a weak automaton.

**Example 5.14.** Consider the limit-deterministic Co-Büchi automaton \( A \) depicted below, and the equivalent history-deterministic Co-Büchi automaton \( A' \) obtained by using the construction from Lemma 5.13; both automata accept exactly the infinite words over \( \Sigma = \{a, b\} \) that contain either \( a \) or \( b \) finitely often. For brevity, we depict only the reachable part of \( A' \) and label macro-states \((U, q)\) with the elements of \( U \) with focus \( q \) underlined. Every macro-state in \( A' \) that has a focus is accepting.

Let \( \delta \) be the transition relation of \( A \). Then we have \( a \)-transitions from \( x \) to \( x, y, z \), \( x, y, z \), and to \( x, y, z \). This is the case since \( \delta(x, a) = \{x, y, z\} \). Since both \( y \) and \( z \) are accepting states in \( A \), \( A' \) has, when reading \( a \) at the state \( x \), the history-deterministic choice to focus either \( y \) or \( z \) (or none of the two).

On the other hand, we have e.g. an \( a \)-transition from \( x, y \) to the non-accepting macro-state \( x, y, z \) since \( \delta(y, a) = x \) is not an accepting state. Hence the \( a \)-transition from \( x, y \) finishes the focus and another refocusing step is necessary in order for a run to be accepting. Thus \( A' \) accepts for instance the word \((aba)b^\omega \) (also accepted by \( A \)) by staying unfocused when reading \( abab \), leading to the partial run \( x \xrightarrow{a} x, y, z \xrightarrow{b} x, y \xrightarrow{a} x, y, z \xrightarrow{b} x, y \); then the automaton can focus on \( y \), continuing the run with \( x, y \xrightarrow{b} x, y \xrightarrow{b} x, y \ldots \), resulting in an overall accepting run. For the word \( a^\omega \) however, there is a non-accepting run \( x \xrightarrow{a} x, y, z \xrightarrow{a} x, y, z \xrightarrow{a} x, y, z \) in which \( y \) is focused infinitely often, but also finished infinitely often. This shows that fair order of focusing is crucial in resolving the history-determinism: Every run in which the automaton eventually focuses the state \( z \) is accepting.
Lemma 5.15 (Henzinger-Piterman [12]). Let $A$ be a nondeterministic Büchi word automaton with $n$ states. Then there is a history-deterministic parity word automaton $A'$ with $n'$ states such that $L(A) = L(A')$ and $n' \in \mathcal{O}(3^n)$. Notice that the size of a history-deterministic automaton, in the general case, is larger than the size of the deterministic automaton. The potential advantage of using history determinization would be to have a simpler structure of the resulting automaton.

5.3 Application to Emptiness Checking and $\mu$-Calculus Satisfiability

To conclude this section, we state the connection between the structure of the alternating parity tree automaton $A$ and the structure of the tracking automaton $T_A$.

Lemma 5.16. – If $A$ is an alternating weak tree automaton, then the tracking automaton $T_A$ is a weak word automaton. – If $A$ is limit deterministic, then the tracking automaton $T_A$ is limit deterministic.

Proof. – Let $A$ be a weak automaton. Then all strongly connected components in $T_A$ either contain only states with priority 2 or only states with priority 1. Regarding states with priority 1 as non-accepting and states with priority 2 as accepting, $T_A$ can be seen as a weak automaton (and as a Co-Büchi automaton).

– Recall that the transition relation of $T_A$ is $\Gamma$ and the priority function of $T_A$ is $\Omega$. Let $A$ be limit deterministic, let $q \in Q$ such that $\Omega(q) = p$ is even, let $q' \in C_q$ and let $a \in \Sigma$. We have to show that $|\Gamma(q', a) \cap C_q| \leq 1$. Since $q' \in C_q$, $q'$ is reachable from $q$ by a path that visits states with priority at most $p$. The only case where we have $|\Gamma(q', a)| > 1$ is when $q' \in Q_A$. Also $\Omega(q) = \Omega(q) - 1 = p - 1$ is odd and $q'$ is reachable in $A$ by a path that visits nodes with priority at most $p - 1$. Since $A$ is limit-deterministic, we have $|\delta(q', a) \cap Q_{\leq p-1}| \leq 1$ which implies $|\Gamma(q', a) \cap C_q| \leq 1$ since $C_q \subseteq Q_{\leq p-1}$ by definition of compartments and since $\Gamma(q', a) \subseteq \delta(q', a)$ by definition of $\Gamma$.

Remark 5.17. If $A$ is limit linear, then $A$ is weak. By the above lemma, $T_A$ is a weak automaton with $F$ being the states with priority 2. Given $q \in F$ so that $\Omega(q) = \Omega(q) - 1 = 1$, there is exactly one path from $q$ to $q$ in $A$, since $A$ is limit linear. Except for the self-loops introduced by non-manipulating transitions in $T_A$, there is exactly one path from $q$ to $q$ in $T_A$ that stays in $F$. We note that the concept of limit-linear word automata can be slightly extended to accommodate self-loops, using a notion of synchronizing transitions in such a way that the method from Lemma 5.5 can be employed, obtaining the same complexity result. For brevity, we omit the technical details here and refer to [10] instead.

By using the bespoke determinization and history-determinization constructions stated above we achieve below better complexity bounds.
Corollary 5.18. Let $A$ be an alternating parity tree automaton reading an alphabet of size $m$, with $n$ states and rank $k$. Depending on the structure of $A$, the complexity of emptiness checking for $A$ is as follows (where $s = (nm + 1) \cdot 2^n$).

- If $A$ is limit-linear: \(\text{parity}(s \cdot n^2 \cdot 2^n, 2^n)\)-Time
- If $A$ is limit-deterministic and weak: \(\text{parity}(s \cdot n \cdot 2^n, 2^n)\)-Time
- If $A$ is weak: \(\text{parity}(s \cdot e \cdot (nk)!, 2nk)\)-Time
- In any case: \(\text{parity}(s \cdot \mathcal{O}((n)^{2k}), 2nk)\)-Time

Remark 5.19. Given a formula $\varphi$, the automaton $A(\varphi)$ makes very limited use of the alphabet $\Sigma$ (in fact, it is only used to check for satisfaction of propositional atoms). For satisfiability checking, the guessing and memorizing of letters in the emptiness game $G_A^*$ can hence be avoided by letting player $\square$ immediately win all nodes whose state component $s \subseteq Q$ in the strategy arena contains $\{p, \neg p\}$ for some atom $p$. Furthermore, the state component $s \subseteq Q$ of nodes in the strategy arena is always contained in the label of states of the (history) deterministic variant of the tracking automaton $H_A$. Hence $G_A^*$ can be slightly adapted to obtain the following complexity bounds, matching previously known results for guarded formulas.

Corollary 5.20. Let $\varphi$ be a $\mu$-calculus formula and let $n = |\text{FL}(\varphi)|$, $k = \text{ad}(\varphi)$. Then the time complexity of deciding satisfiability of $\varphi$ is as follows.

- If $\varphi$ is limit-linear: \(\text{parity}(n^2 \cdot 2^n, 2^n)\)-Time
- If $\varphi$ is aconjunctive and alternation-free: \(\text{parity}(n \cdot 2^n, 2^n)\)-Time
- If $\varphi$ is alternation-free: \(\text{parity}(3^n, 2^n)\)-Time
- If $\varphi$ is aconjunctive: \(\text{parity}(e \cdot (nk)!, 2nk)\)-Time
- In any case: \(\text{parity}(\mathcal{O}((n)^{2k}), 2nk)\)-Time

Let $\varphi$ be satisfiable. Then $\varphi$ has a model of size $2^{\mathcal{O}(nk) \log n}$, and of size $3^n$ if $\varphi$ is alternation-free.

Remark 5.21. In [8], the authors present a tableaux-based satisfiability algorithm for unguarded formulas; unguardedness is handled by an auxiliary tableau rule and by extending the tracking automaton with an additional priority to detect inactive traces. Using this approach however, the tracking automaton for unguarded alternation-free formulas is (in contrast to our framework) not a Co-Büchi automaton and Co-Büchi methods for (history)-determinization can not be used to obtain Büchi games that characterize satisfiability.

Our treatment of aconjunctive and alternation-free formulas employs a focusing method [Lemma 5.13] to history-determinize limit-deterministic Co-Büchi automata. This generalizes the focus games for CTL [17] to the aconjunctive alternation-free $\mu$-calculus and sheds light on the automata theoretic background of focus games.
We surveyed the approach to deciding the satisfiability of the modal $\mu$-calculus through a reduction to alternating parity tree automata emptiness. We present the solution to the emptiness of alternating parity tree automata as a combination of a structural game construction with word automata for defining the winning condition. Interestingly the structural game construction remains fixed regardless of the exact structure of the automaton. The exact structure, however, greatly affects the properties of the word automata for the winning condition. This, in turn, can be exploited to give improved complexity results for various fragments of the $\mu$-calculus by concentrating on bespoke word automata conversion constructions.

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