A note on the automorphism group of Schubert varieties

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Abstract

In [1], the authors determined that the automorphisms of a Schubert divisor are those automorphisms which fix a particular subspace. In this work we extend those results to all Schubert varieties. We study the Schubert conditions which define a Schubert variety and the action upon these conditions by the automorphism group of the Grassmannian variety. We conclude that the automorphisms of the Grassmannian which map a Schubert variety to itself if and only if it fixes the subspaces which do not give redundant conditions used to define the Schubert variety.

1 Introduction

In this article we let $V = \mathbb{F}_q^m$ with its usual $\mathbb{F}_q$–linear vector space structure. We denote by $[a] := \{0,1,2,\ldots,a\}$. We also consider a subset $\alpha \subseteq [m]$ as an ordered tuple. That is $\alpha = (a_1 < a_2 < \ldots < a_\ell)$.

Definition 1. A flag of $V$ is a sequence of nested subspaces

$$\mathcal{A} := A_1 \subsetneq A_2 \subsetneq A_3 \subsetneq \cdots \subsetneq A_\ell \subseteq V,$$

For $\alpha = (a_1, a_2, \ldots, a_\ell)$ be a subset of $[m]$ we denote the flag

$$\mathcal{A} := A_1 \subsetneq A_2 \subsetneq A_3 \subsetneq \cdots \subsetneq A_\ell \subseteq V$$

as an $\alpha$–flag if $\dim A_i = a_i$.

Note that in the case of an $\alpha$–flag, there exists a basis $a_1, a_2, \ldots, a_m$ such that $A_i$ is spanned by $\{a_j \mid 1 \leq j \leq a_i\}$.

Definition 2. The $\ell$–Grassmannian of $\mathbb{F}_q^m$ is the set of all subspaces of $\mathbb{F}_q^m$ whose dimension is $\ell$ that is:

$$G_{\ell,m} := \{W \leq \mathbb{F}_q^m \mid \dim W = \ell\}.$$

Lemma 3. A matrix $M \in GL_m(\mathbb{F}_q)$ acts on a row vector $x \in \mathbb{F}_q^m$ by mapping $x$ to the vector $xM$. This action is extended to a subspace $W \leq \mathbb{F}_q^m$ as follows: If $W = \langle w_1, w_2, \ldots, w_r \rangle$, then $M$ maps $W$ to $M(W) := \langle w_1 M, w_2 M, \ldots, w_r M \rangle$. 
Lemma 4. Let $\theta$ be a field automorphism of $\mathbb{F}_q$. This automorphism $\theta$ acts on the space $\mathbb{F}_q^m$ by mapping the vector $(x_1, x_2, \ldots, x_m) = x \in \mathbb{F}_q^m$ to the vector $x^{\theta} := (\theta(x_1), \theta(x_2), \ldots, \theta(x_m)) \in \mathbb{F}_q^m$. This is extended to a subspace $W \leq \mathbb{F}_q^m$ as follows: If $W = \langle w_1, w_2, \ldots, w_r \rangle$, then the automorphism $\theta$ maps $W$ to $W^{\theta} := \langle w_1^{\theta}, w_2^{\theta}, \ldots, w_r^{\theta} \rangle$.

Definition 5. For $\alpha \subseteq [m]$, we denote the set $m - \alpha := \{m - a_i \mid a_i \in \alpha\}$.

W.L. Chow proved the following:

Proposition 6. [3, Chow]

Let $1 < \ell < m - 1$. The permutations of $G_{\ell,m}$ which map lines to lines is given by the group $\Gamma L(F_q)$. That is, these permutations are given by compositions of the following permutations:

- The permutation $\sigma_M$ where $\sigma_M(W) = W.M$ for $M \in GL_m(F_q)$.
- The permutation $\sigma_\theta$ where $\sigma_\theta(W) = W^{\theta}$ for $\theta$ a field automorphism of $F_q$.
- If $\ell = m - \ell$, the permutation $\sigma_\perp$ where $\sigma_\perp(W) = W^\perp$ where $W^\perp$ is the orthogonal complement of $W$.

With an orthogonal basis for $V$, the permutation $\sigma_\perp$ is also given by the Hodge star operator on $\bigwedge^\ell V$. Although the relation between $\bigwedge^\ell V$ and $G_{\ell,m}$ is well known, for our purposes, we need only to consider the permutations of $G_{\ell,m}$ onto itself given by the elements of $\Gamma L(F_q)$. Note that $\sigma_\perp$ maps $G_{\ell,m}$ onto $G_{m-\ell,m}$. As such we will also consider $\Gamma L(F_q)$ acting on $\bigcup_{i=1}^{m-1} G_{i,m}$. This action is extended to flags as follows.

Definition 7. Let $M \in GL_m(F_q)$ and $\theta$ be a field automorphism. Suppose $\alpha = (a_1, a_2, \ldots, a_\ell)$. Let $A := A_1 \subseteq A_2 \subseteq A_3 \subseteq \cdots \subseteq A_\ell$ be an $\alpha$–flag. Then we define:

- The linear transformation $M$ maps the $\alpha$–flag $A$ to the $\alpha$–flag:
  $$M(A) := M(A_1) \subseteq M(A_2) \subseteq \cdots \subseteq M(A_\ell).$$

- The field automorphism $\theta$ maps the $\alpha$–flag $A$ to the $\alpha$–flag
  $$A^\theta := A_1^\theta \subseteq A_2^\theta \subseteq \cdots \subseteq A_\ell^\theta.$$

- The orthogonal complement, $\perp$ maps the $\alpha$–flag $A$ to the $m - \alpha$–flag
  $$A^\perp := A_\ell^\perp \subseteq \cdots \subseteq A_1^\perp.$$
2 Schubert Varieties

Schubert varieties are special subvarieties of $G_{\ell,m}$. By considering Schubert subvarieties, one can answer many geometrical questions about projective spaces in general and study the Grassmannian as well. The classical reference to Schubert varieties is [2].

**Definition 8.** Let 
\[ \alpha = (a_1 < a_2 < \cdots < a_\ell) \subseteq [m]. \]

Let $\mathcal{A}$ be an $\alpha$–flag. The Schubert variety is defined as
\[ \Omega^\mathcal{A}_\alpha := \{ W \in G_{\ell,m} \mid \dim(W \cap A_i) \geq i \}. \]

We have included the $\alpha$–flag $\mathcal{A}$ in the notation for the Schubert variety $\Omega^\mathcal{A}_\alpha$ because we shall consider what happens to the Schubert varieties when the flag is changed. For any two $\alpha$–flags, $\mathcal{A}$ and $\mathcal{B}$, the varieties $\Omega^\mathcal{A}_\alpha$ and $\Omega^\mathcal{B}_\alpha$ are isomorphic. However, the choice of flag may change the Schubert variety.

Some of the Schubert conditions $\dim(W \cap A_i) \geq i$ may be redundant. Suppose $\alpha \subseteq [m]$ has two consecutive elements, say $a_i = a_{i-1} + 1$. Each $W \in \Omega^\mathcal{A}_\alpha$ satisfies $\dim(W \cap A_i) \geq i$. As $\dim A_i = a_i$ and $\dim A_{i-1} = \dim A_{i-1}$, the inequality $\dim(W \cap A_i) \geq i$ implies $\dim(W \cap A_{i-1}) \geq i - 1$. Therefore the condition $\dim(W \cap A_{i-1}) \geq i - 1$ is redundant. This motivates the following definition.

**Definition 9.** Let $\alpha \subseteq [m]$. We define the nonconsecutive subset of $\alpha$ as
\[ \alpha_{nc} := \{ a_i \mid a_i + 1 \notin \alpha \}. \]

The previous discussion implies the following.

**Lemma 10.** Let $\alpha = (a_1, a_2, \ldots, a_\ell) \subseteq [m]$. Suppose
\[ \mathcal{A} := A_1 \subseteq A_2 \subseteq A_3 \subseteq \cdots \subseteq A_\ell \subseteq V \]
and
\[ \mathcal{B} := B_1 \subseteq B_2 \subseteq B_3 \subseteq \cdots \subseteq B_\ell \subseteq V \]
are two $\alpha$–flags.

If $A_i = B_i \ \forall i \in \alpha_{nc}$, then
\[ \Omega^\mathcal{A}_\alpha = \Omega^\mathcal{B}_\alpha. \]

**Proof.** As we have discussed, the conditions given by $A_i$ and $B_i$ where $\dim A_i = \dim B_i \in \alpha_{nc}$ imply the remaining conditions. By hypothesis, $A_i = B_i$ whenever $\dim A_i = \dim B_i \in \alpha_{nc}$. Equality follows. \qed

Laksov and Kleiman [2] proved that two Schubert varieties are isomorphic if and only if they have the same dimension sequence. Therefore we have stated that proposition as follows.
Theorem 12. Let $\alpha = (a_1, a_2, \ldots, a_\ell) \subseteq [m]$. Let $\mathcal{A}$ and $\mathcal{B}$ be two $\alpha$–flags. Then

$$\Omega^A_\alpha = \Omega^B_\beta \text{ if and only if } A_i = B_i, \forall a_i \in \alpha_{nc}.\]$$

Proof. From the previous discussion, the veracity of the only if direction is clear.

Let $a_s$ be the largest element in $\alpha_{nc}$ such that $A_s \neq B_s$. Let $a_{r_1}$ be the next smallest index in $\alpha_{nc}$ and let $A_{r_2}$ be the next largest index in $\alpha_{nc}$. The choice of $s$ implies $A_r = B_r$ for any index in $\alpha_{nc}$ greater than $s$.

As $\mathcal{A}$ and $\mathcal{B}$ are $\alpha$–flags, there exists $a_1, a_2, \ldots, a_m$ such that $A_i$ is spanned by $\{a_j \mid 1 \leq j \leq a_i\}$, and there exists $b_1, b_2, \ldots, b_m$ such that $B_i$ is spanned by $\{b_j \mid 1 \leq j \leq a_i\}$.

If $a_s = a_\ell$ is the largest element, there exists $x \in A_\ell \setminus B_\ell$. The vector space $W$ spanned by $a_1, a_2, \ldots, a_{\ell-1}$ and $x$ is in $\Omega^A_\alpha$ but not in $\Omega^B_\beta$. Thus $\Omega^A_\alpha \neq \Omega^B_\beta$.

If $a_s$ is not the largest element in $\alpha_{nc}$. Note that $A_s \neq B_s$ but

$$A_{r_1} = B_{r_1} \subseteq A_s, B_s \subseteq A_{r_2} = B_{r_2}.\]$$

Let $x \in A_s \setminus B_s$. In this case consider the vector space $W$ spanned by the set $\{a_u \mid 1 \leq u \leq \ell, u \neq s\} \cup \{x\}$. In this case $\dim W \cap A_u = u$ for each $u \in \alpha_{nc}$, but $\dim W \cap B_s = s - 1$. Therefore $W \in \Omega^A_\alpha$ but not in $\Omega^B_\beta$. \qed

Now we aim find the automorphism group of $\Omega^A_\alpha$.

Lemma 13. Let $\alpha = (a_1, a_2, \ldots, a_\ell) \subseteq [m]$. Suppose

$$A := A_1 \subsetneq A_2 \subsetneq A_3 \subsetneq \cdots \subsetneq A_\ell \subsetneq V$$

is an $\alpha$–flag. Let $\tau \in \text{Aut}(\mathcal{G}_{\ell,m}))$. Suppose $\tau$ preserves the dimension of any linear subspace of $V$. Then $\tau(\Omega^A_\alpha) = \Omega^\tau(\mathcal{A})$.

Proof. The Schubert variety $\Omega^A_\alpha$ is defined by

$$\{W \in \mathcal{G}_{\ell,m} \mid \dim W \cap A_i \geq i\}.\]$$

The automorphism $\tau \in \text{Aut}(\mathcal{G}_{\ell,m})$ maps $\Omega^A_\alpha$ to

$$\tau(\Omega^A_\alpha) = \{\tau(W) \in \mathcal{G}_{\ell,m} \mid \dim \tau(W \cap A_i) \geq i\}.\]$$

In this case, $\tau(W \cap A_i) = \tau(W) \cap \tau(A_i)$. As $\tau$ is a permutation of the Grassmannian, we change the indexing variable to $\tau(W) = U$. Now the Schubert variety has the form:

$$\tau(\Omega^A_\alpha) = \{U \in \mathcal{G}_{\ell,m} \mid \dim U \cap \tau(A_i) \geq i\}.\]$$

The right hand side is clearly $\Omega^{\tau(\mathcal{A})}_\alpha$ and equality follows. \qed
Theorem 14. Let $\alpha = (a_1, a_2, \ldots, a_{\ell}) \subseteq [m]$. Suppose
$$A := A_1 \subsetneq A_2 \subsetneq A_3 \subsetneq \cdots \subsetneq A_{\ell} \subsetneq V$$
is an $\alpha$–flag. Let $\tau \in \text{Aut}(G_{\ell,m})$. Suppose $\tau$ preserves the dimension of any linear subspace of $V$. Then $\tau \in \text{Aut}(\Omega^A_\alpha)$ if and only if $\tau(A_i) = A_i \forall a_i \in \alpha_{nc}$.

Proof. Lemma 13 implies $\tau \in \text{Aut}(G_{\ell,m})$ maps $\Omega^A_\alpha$ to $\Omega^{\tau(A)}_\alpha$. Theorem 12 implies $\Omega^A_\alpha = \Omega^{\tau(A)}_\alpha$ if and only if $\tau(A_i) = A_i \forall a_i \in \alpha_{nc}$. \qed

When $\ell \neq m - \ell$ the only line preserving bijections are those which preserve the dimension. On the remainder of the article, we shall assume $\ell = m - \ell$. Now we shall determine what happens when $\tau \in \text{Aut}(G_{\ell,m})$ is a contravariant mapping. That is when $\dim \tau$ change. Now we study how $\alpha$ maps the $C$ subspace of each dimension. That is a complete flag is a sequence of nested $C$ linear subspace of $V$. Then $\tau \in \text{Aut}(\Omega^A_\alpha)$ if and only if $\tau(A_i) = A_i \forall a_i \in \alpha_{nc}$.

Theorem 14. Let $\alpha = (a_1, a_2, \ldots, a_{\ell}) \subseteq [m]$. Suppose $\tau \in \text{Aut}(G_{\ell,m})$. Suppose $\tau$ preserves the dimension of any linear subspace of $V$. Then $\tau \in \text{Aut}(\Omega^A_\alpha)$ if and only if $\tau(A_i) = A_i \forall a_i \in \alpha_{nc}$.

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Definition 15. A complete flag is a $[m]$–flag. That is, it is a flag which contains a subspace of each dimension. That is a complete flag is a sequence of nested subspaces $C = C_0 = \{0\} \subsetneq C_1 \subsetneq C_2 \subsetneq \cdots \subsetneq C_m = F^m_q$ where $C_i = i$.

If a complete flag $C$ contains the subspaces $A_i$ where
$$A := A_1 \subsetneq A_2 \subsetneq A_3 \subsetneq \cdots \subsetneq A_{\ell} \subsetneq V$$
then $A$ is known as a subflag of $C$.

Lemma 16. Let $\tau \in \text{Aut}(G_{\ell,m})$ be a contravariant mapping. Let $\tau(\Omega^A_\alpha)$ be the image of $\Omega^A_\alpha$. Then $\tau(\Omega^A_\alpha) = \Omega^{\tau(A)}_{\beta}$ where $\beta = \{m + 1 - j | j \notin \alpha\}$.

Proof. Let $A = A_1 \subsetneq A_2 \subsetneq A_3 \subsetneq \cdots \subsetneq A_{\ell} \subsetneq V$ be an $\alpha$–flag. Suppose $C = C_0 = \{0\} \subsetneq C_1 \subsetneq C_2 \subsetneq \cdots \subsetneq C_m$ is a complete flag with $A$ as a subflag.

The Schubert conditions $\dim A_i \cap W \geq i$ can be extended to the subspaces of $C$ Simply note that for $A_i \subseteq C_s \subseteq A_{i+1}$ the condition $\dim C_s \cap W \geq i$ holds. Now we have the following Schubert conditions on the complete flag $C$.
$$\dim C_s \cap W \geq i, \text{ for } a_i \leq s < a_{i+1}.$$

Thus given $\alpha$, an $\alpha$–flag $A$ and a complete flag $C$ containing $A$ we may rewrite the conditions as follows: Let $w_0, w_1, w_2, \ldots, w_m$ be a sequence of integers such that $w_s = i$ for $a_i \leq s < a_{i+1}$. Then
$$\dim C_s \cap W \geq w_s.$$

Note that $w_i$ increases by 1 only on the positions corresponding to $\alpha$. That is
$$\alpha = \{s | n_s = n_{s-1} - 1\}.$$

Now we shall apply $\tau$ to $\dim C_s \cap W \geq w_s$. The Schubert conditions become
$$\dim \tau(C_s \cap W) \leq m - w_s.$$
As \( \tau(\mathcal{C}_s \cap W) \) is the vector space spanned by \( \tau(\mathcal{C}_s) \) and \( \tau(W) \), we have the conditions
\[
\dim \tau(\mathcal{C}_s) + \tau(W) \leq m - w_s.
\]

This is equivalent to
\[
\dim \tau(\mathcal{C}_s) + \dim \tau(W) - \dim \tau(\mathcal{C}_s) \cap \tau(W) \leq m - w_s.
\]

In order to simplify our notation we shall set \( r = m+1-s, D_r = \tau(C_{m+1-s}), \tau(W) = U \), and \( u_r = m-w_s \). Note that \( \tau \) maps \( \mathcal{G}_{\ell,m} \) to itself so \( U \) also represents any element of the Grassmannian. The Schubert conditions become
\[
\dim D_r + \dim U - \dim D_r \cap U \leq u_r.
\]

As \( \dim D_r = r, \dim U = \ell \) we rearrange the terms and obtain:
\[
\dim D_r \cap U \geq r + \ell - u_r.
\]

Let \( n_r = r + \ell - u_r \). Now we determine \( \beta = \{ j \in [m] \mid n_{j+1} = n_j + 1 \} \).

Recall that these are the entries where \( \dim D_j \cap U > \dim D_{j-1} \cap U \). In this case there are some stringent conditions on \( \beta \) from the equality \( \tau(\Omega^A_\alpha) = \Omega^\tau(\mathcal{A})_\beta \) and Proposition [11]

Suppose \( n_r = n_{r+1} \). In this case \( r + \ell - u_r = r + 1 + \ell - u_{r+1} \). From the definition of \( r \) and \( u_r \) we have that \( m+1-s+\ell = (m-w_{m+1-s}) = m-s+\ell-(m-w_{m-s}) \). Therefore increases in \( \dim D_r \cap U \) do not occur for \( w_{m-s} + 1 = w_{m-s+1} \). Likewise \( \dim D_r \cap U \) increases when \( w_{m-s} = w_{m-s+1} \). Therefore the set \( \{ j \in [m] \mid n_{j+1} = n_j + 1 \} = \{ m+1-i \mid i \not\in \alpha \} \). Thus \( \tau(\Omega^A_\alpha) = \Omega^\tau(\mathcal{A})_\beta \).

For a contravariant mapping \( \tau \in \text{Aut}(\mathcal{G}_{\ell,m}) \) we find when \( \tau(\Omega^A_\alpha) = \Omega^A_\alpha \).

**Lemma 17.** Let \( \tau \in \text{Aut}(\mathcal{G}_{\ell,m}) \) be a contravariant mapping, \( \alpha = (a_1, a_2, \ldots, a_t) \subseteq [m] \) and \( \mathcal{A} \) an \( \alpha \)-flag. Then \( \tau(\Omega^A_\alpha) = \Omega^\tau(\mathcal{A})_\alpha \) and only if \( \alpha = \{ m+1-j \mid j \not\in \alpha \} \) and the sets \( \{ \tau(A_i) \mid a_i \in \alpha_{nc} \} = \{ A_i \mid a_i \in \alpha_{nc} \} \) are equal.

**Proof.** Lemma [10] and Proposition [11] imply that \( \tau(\Omega^A_\alpha) = \Omega^\tau(\mathcal{A})_\alpha \). Theorem [12] states that \( \Omega^\tau(\mathcal{A})_\alpha = \Omega^A_\alpha \) if and only if they have the same subspaces in for the nonconsecutive indices. \( \square \)

**Theorem 18.** An automorphism of \( \mathcal{G}_{\ell,m} \) is an automorphism of \( \Omega^A_\alpha \) if and only if it maps the set \( \{ A_i \mid a_i \in \alpha_{nc} \} \) to itself.

**Proof.** It follows from Theorem [14] and Lemma [17] \( \square \)
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