Scattering theory without large-distance asymptotics: scattering boundary condition

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Abstract: By large-distance asymptotics, in conventional scattering theory, at the cost of losing the information of the distance between target and observer, one arrives at an explicit expression for scattering wave functions represented by a scattering phase shift. In the present paper, together with a preceding paper (T. Liu, W.-D. Li, and W.-S. Dai, JHEP06(2014)087), we establish a rigorous scattering theory without imposing large-distance asymptotics. We show that even without large-distance asymptotics, one can also obtain an explicit scattering wave function represented also by a scattering phase shift, in which, of course, the information of the distance is preserved. Nevertheless, the scattering amplitude obtained in the preceding paper depends not only on the scattering angle but also on the distance between target and observer. In this paper, by constructing a scattering boundary condition without large-distance asymptotics, we introduce a scattering amplitude, like that in conventional scattering theory, depending only on the scattering angle and being independent of the distance. Such a scattering amplitude, when taking large-distance asymptotics, will recover the scattering amplitude in conventional scattering theory. The present paper, with the preceding paper, provides a complete scattering theory without large-distance asymptotics.

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1 Introduction

In our preceding work, ref. [1], we establish a rigorous scattering theory without imposing large-distance asymptotics. The information of the distance between target and observer, which is lost in conventional scattering theory due to large-distance asymptotics, is taken into account. The scattering amplitude without large-distance asymptotics introduced in ref. [1], however, depends both on scattering angle and distance between target and observer. As a comparison, recall that in conventional scattering theory, the scattering amplitude depends only on the scattering angle.

In this paper, we present a scattering boundary condition without large-distance asymptotics. Under this scattering boundary condition, the scattering amplitude, as same as that in conventional scattering theory, depends only on the scattering angle. This scattering boundary condition will reduce to the Sommerfeld radiation condition, the scattering boundary condition in conventional scattering theory, when taking large-distance asymptotics.

Scattering theory. In quantum mechanics, all is determined by the Schrödinger equation with a given boundary condition. For bound-state problems, the boundary condition is chosen to be $\psi(r)|_{r\in \text{boundary}} = 0$, i.e., the wave function vanishes on the boundary. For scattering problems, the boundary condition is chosen to be a given wave function at an asymptotic distance.
Concretely, for a spherical potential, a scattering problem is determined by the radial wave equation,

$$\frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{dR_l}{dr} \right) + \left[ k^2 - \frac{l(l+1)}{r^2} - V(r) \right] R_l = 0,$$

(1.1)

with the scattering boundary condition,

$$\psi = e^{ikr \cos \theta} + \psi^{sc},$$

(1.2)

where $e^{ikr \cos \theta}$ is the incident plane wave and $\psi^{sc}$ is a scattering wave function at an asymptotic distance.

To choose a scattering boundary condition is just to choose a $\psi^{sc}$. The scattering amplitude is defined by $\psi^{sc}$, the scattering part of the scattering boundary condition (1.2). Different choices of $\psi^{sc}$ define different scattering amplitudes.

The starting of a scattering theory is the following result: The radial wave equation (1.1) with $V(r) = 0$ can be exactly solved and the incident plane wave $e^{ikr \cos \theta}$ can be exactly expanded:

$$R_l(r) = C_l h_l^{(2)}(kr) + D_l h_l^{(1)}(kr),$$

(1.3)

$$e^{ikr \cos \theta} = \sum_{l=0}^{\infty} (2l+1) i^l j_l(kr) P_l(\cos \theta),$$

(1.4)

where $h_l^{(1)}(z)$ and $h_l^{(2)}(z)$ are the first and second kind spherical Hankel functions, $j_l(z)$ the spherical Bessel function, and $P_l(x)$ the Legendre polynomial [2].

Conventional scattering theory. In conventional scattering theory, the scattering boundary condition is chosen as the Sommerfeld radiation condition:

$$\psi = e^{ikr \cos \theta} + f(\theta) \frac{e^{ikr}}{r}, \quad r \to \infty.$$  

(1.5)

$\psi^{sc}$ here is chosen as $\psi^{sc} = f(\theta) e^{ikr}/r$; the scattering amplitude defined here is $f(\theta)$.

In coordination with the Sommerfeld radiation condition (1.5), the exact solution, eq. (1.3), and the exact expansion of the incident plane wave, eq. (1.4), are approximately replaced by their asymptotics [3]:

$$R_l(r) \underset{r \to \infty}{\sim} A_l \frac{\sin (kr - l\pi/2 + \delta_l)}{kr},$$

(1.6)

$$e^{ikr \cos \theta} \underset{r \to \infty}{\sim} \sum_{l=0}^{\infty} (2l+1) i^l \frac{\sin (kr - l\pi/2)}{kr} P_l(\cos \theta),$$

(1.7)

where $\delta_l$ is the scattering phase shift.

The reason why in the Sommerfeld radiation condition (1.5) $\psi^{sc}$ is chosen as being in proportion to $e^{ikr}/r$ is that the asymptotic solution of the radial wave equation (1.1) is $R_l \underset{r \to \infty}{\sim} e^{\pm ikr}/r$ and only the outgoing wave $R_l \underset{r \to \infty}{\sim} e^{ikr}/r$ remains in the scattering wave function when $r \to \infty$ [4].

The reason why the approximations (1.6) and (1.7) are used in conventional scattering theory is that only then can the scattering phase shift $\delta_l$ appear explicitly.
Scattering theory without large-distance asymptotics. In ref. [1], we show that, even without large-distance asymptotics, (1.6) and (1.7), the two approximations used in conventional scattering theory, one can still obtain a scattering theory in which the phase shift can also appear explicitly, and, of course, more information is preserved due to the fact that there is no approximation.

In ref. [1], the scattering boundary condition is taken to be

$$\psi = e^{ik\cos \theta} + f(r, \theta) \frac{e^{ikr}}{r}. \quad (1.8)$$

$\psi^{sc}$ here is chosen as $\psi^{sc} = f(r, \theta) e^{ikr}/r$; the scattering amplitude defined here is $f(r, \theta)$.

Moreover, the exact solution, eq. (1.3), and the expansion of the incident plane wave, eq. (1.4), without any approximation, are rewritten as [1]

$$R_l(r) = M_l \left( \frac{1}{ikr} \right) \frac{A_l}{kr} \sin \left[ kr - \frac{l\pi}{2} + \delta_l + \Delta_l \left( -\frac{1}{ikr} \right) \right], \quad (1.9)$$

$$e^{ikr\cos \theta} = \sum_{l=0}^{\infty} (2l+1) \frac{1}{4} M_l \left( \frac{1}{ikr} \right) \frac{1}{kr} \sin \left[ kr - \frac{l\pi}{2} + \Delta_l \left( -\frac{1}{ikr} \right) \right] P_l(\cos \theta), \quad (1.10)$$

where $M_l(x) = |y_l(x)|$ and $\Delta_l(x) = \text{arg} y_l(x)$ are the modulus and argument of the Bessel polynomial $y_l(x)$, respectively.

It can be directly seen that the phase shift $\delta_l$ appears explicitly in eq. (1.9), like that in conventional scattering theory, eq. (1.6).

Nevertheless, the scattering boundary condition (1.8) is a naive generalization of the Sommerfeld radiation condition. As a result, the scattering amplitude defined by the scattering boundary condition (1.8), $f(r, \theta)$, depends not only on the scattering angle $\theta$ but also on the distance $r$, rather than the scattering amplitude defined by the Sommerfeld radiation condition, $f(\theta)$. This implies that the scattering boundary condition (1.8) is not very suitable for the scattering theory without large-distance asymptotics, since the scattering part in eq. (1.8) is in proportion to $e^{ikr}/r$ which is not an exact solution of the radial wave equation (1.1) with $V(r) = 0$, unless $r \to \infty$.

The aim of the present paper is, by constructing a without-large-distance-asymptotics scattering boundary condition, to introduce a scattering amplitude depending only on the scattering angle $\theta$ but being independent of the distance $r$, like that in conventional scattering theory.

Without large-distance asymptotics, a new scattering boundary condition is constructed in Sec. 2. The expression of the scattering amplitude defined by the new scattering boundary condition is also given in this section. The relations among different scattering amplitudes, defined by different scattering boundary conditions, are given in Sec. 3. The scattering cross section is considered in Sec. 4. A discussion on the scattering phase shift is given in 5. Conclusions and outlook are given in Sec. 6.

2 Scattering boundary condition without large-distance asymptotics

In this section, 1) a scattering boundary condition without large-distance asymptotics is presented, 2) by which a scattering amplitude depending only on the scattering angle $\theta$ is
defined.

In a word, in the following, we construct a distance-independent scattering amplitude without the help of the large-distance asymptotic approximation. While, in ref. [1], we have to introduce a distance-dependent scattering amplitude for constructing a scattering theory without large-distance asymptotics.

2.1 Scattering boundary condition

The reason why in conventional scattering theory the information of the distance \( r \) is lost is that the exact outgoing solution, \( h_l^{(1)} (kr) \), is approximately replaced by its large-distance asymptotics: \( h_l^{(1)} (kr) \stackrel{r \to \infty}{\sim} e^{ikr}/r \). To retrieve the information of the distance, without large-distance asymptotics, we construct a scattering boundary condition by \( h_l^{(1)} (kr) \) rather than its asymptotics:

\[
\psi (r, \theta) = e^{ikr \cos \theta} + \sum_{l=0}^{\infty} a_l (\theta) h_l^{(1)} (kr) .
\]

(2.1)

\( \psi^{sc} \) here is chosen as \( \psi^{sc} = \sum_{l=0}^{\infty} a_l (\theta) h_l^{(1)} (kr) \); the scattering amplitude defined here is \( a_l (\theta) \) which is a partial wave scattering amplitude, rather than that in conventional scattering theory and in ref. [1].

It can be directly verified that the scattering boundary condition (2.1) reduces to the Sommerfeld radiation condition (1.5) when \( r \to \infty \):

\[
\sum_{l=0}^{\infty} a_l (\theta) h_l^{(1)} (kr) \stackrel{r \to \infty}{\sim} f (\theta) \frac{e^{ikr}}{r} .
\]

(2.2)

The relation between \( a_l (\theta) \) and \( f (\theta) \) will be given below.

2.2 Scattering amplitude \( a_l (\theta) \)

In this section, we calculate the scattering amplitude \( a_l (\theta) \) defined by the scattering boundary condition (2.1).

First, rewrite the scattering boundary condition (2.1) as

\[
\psi (r, \theta) = \sum_{l=0}^{\infty} (2l + 1) \frac{1}{2} \left[ h_l^{(1)} (kr) + h_l^{(2)} (kr) \right] P_l (\cos \theta) + \sum_{l=0}^{\infty} a_l (\theta) h_l^{(1)} (kr)
\]

\[
= \sum_{l=0}^{\infty} \left\{ (2l + 1) \frac{1}{2} P_l (\cos \theta) h_l^{(2)} (kr) + \left[ (2l + 1) \frac{1}{2} P_l (\cos \theta) + a_l (\theta) \right] h_l^{(1)} (kr) \right\},
\]

(2.3)

by use of the plane wave expansion (1.4) and \( j_l (x) = \frac{1}{2} \left[ h_l^{(1)} (x) + h_l^{(2)} (x) \right] \) [1].

Second, rewrite the radial wave function (1.3), the exact solution of the radial wave equation (1.1) with \( V (r) = 0 \), as

\[
R_l (r) = C_l h_l^{(2)} (kr) + D_l h_l^{(1)} (kr) = C_l \left[ h_l^{(2)} (kr) + e^{2i\delta_l} h_l^{(1)} (kr) \right],
\]

(2.4)
where $e^{2i\delta_l} = D_l/C_l$ defines the phase shift [1]. The wave function $\psi(r, \theta) = \sum_{l=0}^{\infty} R_l(r) P_l(\cos \theta)$, then, becomes

$$\psi(r, \theta) = \sum_{l=0}^{\infty} \left[ C_l P_l(\cos \theta) h_l^{(2)}(kr) + C_l e^{2i\delta_l} P_l(\cos \theta) h_l^{(1)}(kr) \right].$$

(2.5)

Finally, the scattering amplitude $a_l(\theta)$ can be achieved immediately by equating the coefficients of $h_l^{(1)}(kr)$ and $h_l^{(2)}(kr)$ in eqs. (2.3) and (2.5):

$$C_l = (2l + 1) i^{l+1/2},$$

(2.6)

$$C_l e^{2i\delta_l} P_l(\cos \theta) = (2l + 1) i^{l+1/2} P_l(\cos \theta) + a_l(\theta).$$

(2.7)

Substituting $C_l$ into eq. (2.7) gives

$$a_l(\theta) = (2l + 1) i^{l+1/2} \left( e^{2i\delta_l} - 1 \right) P_l(\cos \theta).$$

(2.8)

Notice that the scattering amplitude $a_l(\theta)$ is independent of the distance $r$.

### 3 Relations among scattering amplitudes: $a_l(\theta)$, $f(r, \theta)$, and $f(\theta)$

A scattering process is fully described by the scattering amplitude. In conventional scattering theory, with large-distance asymptotics, the scattering amplitude $f(\theta)$ is defined by the Sommerfeld radiation condition (1.5). In ref. [1], without large-distance asymptotics, the scattering amplitude $f(r, \theta)$, a function of the distance $r$, is defined by the scattering boundary condition (1.8). In the present paper, also without large-distance asymptotics, the scattering amplitude $a_l(\theta)$, which is independent of the distance $r$, is defined by the scattering boundary condition (2.1).

In the following, we reveal the relations among these three scattering amplitudes, $a_l(\theta)$, $f(r, \theta)$, and $f(\theta)$.

#### 3.0.1 Relation between $f(\theta)$ and $a_l(\theta)$

The relation between $a_l(\theta)$ and $f(\theta)$ can be achieved directly by performing large-distance asymptotics in the scattering part of the scattering boundary condition (2.1):

$$\sum_{l=0}^{\infty} a_l(\theta) h_l^{(1)}(kr) \bigg|_{r \to \infty} = \left[ \sum_{l=0}^{\infty} a_l(\theta) \frac{1}{i^{l+1} k} \right] \frac{e^{ikr}}{r} = f(\theta) \frac{e^{ikr}}{r},$$

(3.1)

where the asymptotics of the spherical Hankel function, $h_l^{(1)}(z) \underset{r \to \infty}{\sim} (-i)^{l+1} e^{iz}/z$ [1], is used.

Then we obtain the relation between $f(\theta)$ and $a_l(\theta)$:

$$f(\theta) = \sum_{l=0}^{\infty} a_l(\theta) \frac{1}{i^{l+1} k}.$$  

(3.2)

It can be seen that the information of the distance $r$ loses when taking large-distance asymptotics.
3.0.2 Relation between \( f(r, \theta) \) and \( a_l(\theta) \)

Without large-distance asymptotics, we have defined two scattering amplitudes: \( f(r, \theta) \) and \( a_l(\theta) \), where \( f(r, \theta) \) is defined by the scattering boundary condition (1.8) given in ref. [1] and \( a_l(\theta) \) is defined by the scattering boundary condition (2.1) given in the present paper. The relation between \( f(r, \theta) \) and \( a_l(\theta) \) can be achieved by comparing eqs. (1.8) and (2.1) directly:

\[
f(r, \theta) = e^{-ikr} \sum_{l=0}^{\infty} a_l(\theta) h_l^{(1)}(kr) \]

\[
= \frac{1}{k} \sum_{l=0}^{\infty} a_l(\theta) (-i)^{l+1} y_l \left( -\frac{1}{ikr} \right), \tag{3.3}
\]

where \( h_l^{(1)}(z) = (-i)^{l+1} (e^{iz}/z) y_l(i/z) \) [1] is used. From this result we can directly see how does \( f(r, \theta) \) depend on the distance \( r \).

The scattering amplitude \( f(r, \theta) \), introduced in ref. [1], then, by the relation (3.3), can be obtained directly:

\[
f(r, \theta) = \frac{1}{2ik} \sum_{l=0}^{\infty} (2l + 1) \left( e^{2i\delta_l} - 1 \right) P_l(\cos \theta) y_l \left( -\frac{1}{ikr} \right), \tag{3.4}
\]

which agrees with the result given in ref. [1].

4 Scattering cross section

In this section, we express the differential and total scattering cross sections by the scattering amplitude introduced in the present paper, \( a_l(\theta) \).

4.1 Differential scattering cross section

Without large-distance asymptotics, in ref. [1], we provide an exact expression of the differential scattering cross section. In this section, for simplicity, we consider the leading contribution of the differential scattering cross section,

\[
\frac{d\sigma}{d\Omega} = \frac{\mathbf{j}_s}{j_{\text{in}}} \cdot d\mathbf{S} = \frac{1}{k} \text{Im} \left( \psi_s^* \frac{\partial}{\partial r} \psi_s \right) r^2. \tag{4.1}
\]

Substituting \( \psi_s = \psi - \psi_{\text{in}} = \sum_{l=0}^{\infty} a_l(\theta) h_l^{(1)}(kr) \) (see eq. (2.1)) into eq. (4.1) and dropping the high-order contribution, we achieve a differential scattering cross section represented by the scattering amplitude \( a_l(\theta) \),

\[
\frac{d\sigma}{d\Omega} = r^2 \left| \sum_{l=0}^{\infty} a_l(\theta) h_l^{(1)}(kr) \right|^2. \tag{4.2}
\]
4.2 Total scattering cross section

The total scattering cross section can be achieved immediately by integrating the differential scattering cross section. For simplicity, we only take the leading contribution into account. We have

\[ \sigma (r) = 4\pi r^2 \sum_{l=0}^{\infty} (2l + 1) \sin^2 \delta_l |h^{(1)}_l (kr)|^2 \]
\[ = \frac{4\pi}{k^2} \sum_{l=0}^{\infty} (2l + 1) \sin^2 \delta_l \left| y_l \left( -\frac{1}{ikr} \right) \right|^2, \]  

which agrees with the result given in ref. [1].

5 A note on phase shift

In conventional scattering theory, with the help of large-distance asymptotics, it is proved that the phase shift is the only effect in an elastic scattering process, i.e., all information of an elastic scattering process is embedded in a scattering phase shift [3].

In this section, we show that such a statement also holds without large-distance asymptotics.

Without large-distance asymptotics, as shown in (2.3), the incident plane wave can be exactly expressed as

\[ \psi_{\text{in}} (r, \theta) = \sum_{l=0}^{\infty} \frac{1}{2} (2l + 1) i^l \left[ h^{(2)}_l (x) + h^{(1)}_l (x) \right] P_l (\cos \theta). \]  

After an elastic scattering, by eq. (2.5), the wave function becomes (for clarity and convenience, we rewrite eq. (2.5) here)

\[ \psi (r, \theta) = \sum_{l=0}^{\infty} \frac{1}{2} (2l + 1) i^l \left[ h^{(2)}_l (kr) + e^{2i\delta_l} h^{(1)}_l (kr) \right] P_l (\cos \theta). \]

Comparing the wave functions before and after the scattering, eqs. (5.1) and (5.2), we can see that, the incoming part, represented by \( h^{(2)}_l (kr) \), does not change anymore, while a phase factor \( e^{2i\delta_l} \) appears in the outgoing part, represented by \( h^{(1)}_l (kr) \). This shows that the only effect after an elastic scattering is a phase shift on the outgoing wave function, even without large-distance asymptotics.

6 Conclusions and outlook

A scattering theory without using large-distance-asymptotics approximation is established in our two papers, ref. [1] and the present paper: in ref. [1], we deal with the solution of the radial wave equation and the incident wave; in the present paper, we deal with the scattering boundary condition. Now, we have a complete scattering theory without large-distance asymptotics and, thus, without losing the information of distance.
Concretely, a scattering boundary condition without large-distance asymptotics is constructed in this paper. The scattering amplitude defined by such a scattering boundary condition, $a_l(\theta)$, rather than $f(r, \theta)$ (the scattering amplitude introduced in ref. [1]), depends only on the scattering angle $\theta$ and is independent of the distance $r$.

In further works, based on the scattering theory without large-distance asymptotics, we will systematically deal with a series of scattering related problems which are all treated under large-distance asymptotics in the frame of conventional scattering theory. Without large-distance asymptotics, we can also construct a complete treatment on the Lippmann-Schwinger equation. In conventional scattering theory, a very important problem is the analytic property of scattering amplitudes, which, of course, based on large-distance asymptotics [5–8]. Such a problem, now, can be discussed without large-distance asymptotics. Based on two important quantum field theory methods, scattering spectrum method [9–11] and heat kernel method [12–15], we establish a heat-kernel method for the calculation of the scattering phase shift through the relation between scattering spectrum method and heat kernel method given by Refs. [16, 17]; now, we can do this without large-distance asymptotics. The scattering theory given by ref. [1]) and the present paper is in fact a scalar scattering theory. Therefore, the result can be applied to any scalar scattering, such as acoustic scatterings. In an acoustic scattering, in comparison with the distance between target and observer, the wave length of an acoustic wave often cannot be ignored; in such cases, one can use our result to construct an acoustic scattering theory without imposing large-distance asymptotics, while in conventional acoustic scattering theory, large-distance asymptotics (far-field pattern) is imposed [18–20]. Moreover, the scalar scattering theory without large-distance asymptotics can be naturally generalized to vector and tensor scatterings. The scattering of electromagnetic waves is a vector scattering. We can establish a scattering theory of electromagnetic waves without large-distance asymptotics, especially for long wavelength cases. By generalizing our result to tensor wave scattering, we can consider the scattering theory of gravitational waves, which has been studied under large-distance asymptotics in literature, e.g., [21, 22]. Moreover, we can also consider the scattering of a wave scattered by a black hole; all discussions on this issue are based on large-distance asymptotics [23–27]. A relativistic scattering theory without large-distance asymptotics also can be established. Inverse scattering problems [28, 29] can also be systematically studied in the frame of the scattering theory without large-distance asymptotics. More topics on scatterings can be found in ref. [30]. Moreover, many scattering theories based on large-distance asymptotics can be treated without such an asymptotic approximation [31–33].

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