Revisiting the screening mechanism in $f(R)$ gravity

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We reexamine the screening mechanism in $f(R)$ gravity using N-body simulations. By explicitly examining the relation between the extra scalar field $\delta f_R$ and the gravitational potential $\phi$ in the perturbed Universe, we find that the relation between these two fields plays an important role in understanding the screening mechanism. We show that the screening mechanism in $f(R)$ gravity depends mainly on the depth of the potential well, and find a useful condition for identifying unscreened halos in simulations. We also discuss the potential application of our results to real galaxy surveys.

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I. INTRODUCTION

Compelling cosmological observations [1,3] have shown that our Universe is undergoing a phase of accelerated expansion. The leading theoretical explanation to this cosmic acceleration is a cosmological constant within the context of General Relativity (GR). Despite its notable success in explaining current cosmological data sets, the standard paradigm suffers from several serious problems: the measured value of the cosmological constant is far smaller than that predicted by quantum field theory and there is also a coincidence problem as to why the energy densities of matter and vacuum energy are of the same order today (see Ref. [4] for review).

On the other hand, general relativity might not be accurate on the cosmological scales, and modified gravity theories are proposed as alternatives to explain the observed cosmic acceleration. One of the simplest attempts is the so-called $f(R)$ gravity, in which the scalar curvature $R$ in the Einstein-Hilbert action of general relativity is replaced by an arbitrary function of $R$ [5-16]. $f(R)$ gravity introduces a new scalar field degree of freedom that has profound impacts on cosmology. At the background level, the freedom of this scalar field enables the theory to produce any cosmic expansion history with desired effective dark energy equation of state $w(a)$. At the perturbed level, the local scalar curvature $R$ does not necessarily follow the matter density field and thus high density might not imply high curvature in $f(R)$ cosmology. If the curvature is significantly lower than the prediction from general relativity for the same density field, the local spacetime will be altered and the model would fail to pass the local tests of gravity. Therefore, for viable $f(R)$ models the standard local space-time should be recovered in high-density regions. To this end, a screening mechanism [17] is essential and plays an important role in $f(R)$ gravity.

The aim of this paper is to further investigate this important issue. Instead of studying the screening mechanism based on individual isolated galactic halos [18-20], we will examine the relation between the scalar field, $\delta f_R$, and the gravitational potential, $\phi$, in $f(R)$ cosmologies, using N-body simulations.

We will show that this relation plays an important role in understanding the screening mechanism in $f(R)$ gravity. In order to strengthen our argument, we will investigate two different $f(R)$ models: one that can exactly reproduce the $\Lambda$CDM background expansion [21] and the Hu-Sawicki model (H-S hereafter) [18].

This paper is organized as follows: In Sec. II we will introduce the details of the $f(R)$ models investigated in this work. In Sec. III we will briefly review the technique details of N-body simulations. In Sec. IV we will discuss the distribution of the scalar curvature $R$ in the void regions and the screening mechanism in the high-density regions. In Sec. V we will discuss the screening mechanism in the dark halos. In Sec. VI we will summarize and conclude this work.

II. F(R) MODEL

We work with the 4-dimensional modified Einstein-Hilbert action

$$S = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} [R + f(R)] + \int d^4x \mathcal{L}^{(m)}, \quad (1)$$

where $\kappa^2 = 8\pi G$ with $G$ being Newton’s constant. $g$ is the determinant of the metric $g_{\mu\nu}$, $\mathcal{L}^{(m)}$ is the Lagrangian density for matter and $f(R)$ is an arbitrary function of the Ricci scalar curvature $R$ [5-16] (see Refs. [22, 23] for reviews). In this work, we will consider two different $f(R)$ models.

Firstly, we consider a model proposed by one of us, which can exactly reproduce the $\Lambda$CDM background expansion history [21]. We call this ‘our model’, and it is specified by

$$f(R) = -6\Omega_0^0 H_0^2 - \frac{3D\Omega_0^0 H_0^2}{\rho_+ - 1} \left( \frac{3\Omega_0^0 H_0^2}{R - 12\Omega_0^0 H_0^2} \right)^{p_+ - 1} \times 2F_1 \left[ q_+, p_+ - 1; r_+; - \frac{3\Omega_0^0 H_0^2}{R - 12\Omega_0^0 H_0^2} \right]. \quad (2)$$

The indices in the above expression are given by

$$q_+ = \frac{1 + \sqrt{73}}{12}, \quad r_+ = 1 + \sqrt{73}, \quad p_+ = \frac{5 + \sqrt{73}}{12}.$$

$2F_1 [a, b; c; z]$ is the hypergeometric function. When $c > b >
0, the hypergeometric function has the integral representation
\[ 2F_1[a, b; c; z] = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1} (1-t)^{c-b-1} (1-zt)^{-a} dt \]
where \( \Gamma(x) \) is the Euler gamma function. \( 2F_1[a, b; c; z] \) is a real function that is well defined in the range \( -\infty < z < 1 \) in this case. \( H_0 \) is the Hubble constant today. \( \Omega_m^0 \) is the matter density today and \( \Omega_d^0 = 1 - \Omega_m^0 \). \( D \) is an additional parameter that characterizes the \( f(R) \) model. For the instability issue as discussed in Ref. [23], \( D \) must be constrained as \( D < 0 \). Our model predicts a lower bound for the scalar curvature \( \Omega \) across the Universe
\[ R \in (4\Lambda, +\infty) \]
where
\[ \Lambda = 3\Omega_d^0 H_0^2 \]
Secondly, we also consider the Hu-Sawicki model [18], for which
\[ f(R) = -\Omega_m^0 H_0^2 \frac{c_1 \left( \frac{R}{\Omega_m^0 H_0^2} \right)^n}{c_2 \left( \frac{R}{\Omega_m^0 H_0^2} \right)^n + 1} \]
This model is designed to explain the late-time cosmic acceleration without a cosmological constant. In the high-curvature limit, where
\[ \frac{R}{\Omega_m^0 H_0^2} \gg 1 \]
however, \( f(R) \) actually does reduce to a phenomenological cosmological constant \( 2\Omega_d^0 \Omega_m^0 H_0^2 \sim 4\Lambda \) [18]. In the opposite limit, it satisfies \( f(R = 0) = 0 \). If one chooses \( |f_{R0}| \ll 1 \), the background expansion of the H-S model can not be practically distinguished from the \( \Lambda CDM \) model. For simplicity, we will focus on \( n = 1 \) for the H-S model throughout this work.

III. N-BODY SIMULATIONS

In this section, we will briefly summarize the basic equations that are used in \( f(R) \) cosmological simulations and also present the technical details of our simulations.

A. Non-linear perturbation equations

The formation of large-scale structure in \( f(R) \) gravity is governed by the modified Poisson equation
\[ \nabla^2 \phi = \frac{16\pi G}{3} \delta \rho - \frac{\delta R}{6} \]
and the equation of motion for the scalar field \( f_R \). If \( |f_R| \ll 1 \) the equation for \( f_R \) approximately becomes
\[ \nabla^2 f_R = \frac{1}{3c^2} \delta R - 8\pi G \delta \rho \]
where \( \phi \) denotes the gravitational potential, \( \delta f_R = f_R(R) - f_R(R_0) \), \( \delta R = R - R_0 \), and \( \delta \rho = \rho - \rho_0 \). The overbar denotes the background quantities, and \( \nabla \) is the derivative with respect to the physical coordinates. Equations (8) and (9) are derived in linear perturbation theory under the quasi-static approximation, but can also be used in the non-linear regime [18].

In order to incorporate nonlinear effects into \( f(R) \) simulations, we need to express \( R \) in terms of \( f_R \). In practice, however, it is difficult to do this by inverting the exact expression Eq. (2) for our model. Instead, we use a fitting formula
\[ f(R) \sim -6\Omega_m^0 H_0^2 - \frac{3D\Omega_m^0 H_0^2 R}{p+1} \left( \frac{3\Omega_m^0 H_0^2}{R - 12\alpha \Omega_m^0 H_0^2} \right)^{p+1} \]
and
\[ f(R) \sim D \left( \frac{3\Omega_m^0 H_0^2}{R - 12\alpha \Omega_m^0 H_0^2} \right)^{p+1} \]
where \( \alpha \) is a fitting parameter depending on \( \Omega_m^0 \). Taking the derivative of the above equation, we obtain
\[ \frac{\Delta f_R}{f_R} = \left| \frac{f_{R_{app}} - f_{R_{exact}}}{f_{R_{exact}}} \right| \]
By fitting \( \alpha \), Eq. (11) is found to be relatively a good approximation to the exact derivative of Eq. (2). In Fig. 1, we show the relative error of our fitting formula with respect to the exact expression, where
\[ \frac{\Delta f_R}{f_R} = \left| \frac{f_{R_{app}} - f_{R_{exact}}}{f_{R_{exact}}} \right| \]
In Fig. 1, we set \( \alpha = 0.9436 \) for \( \Omega_m^0 = 0.316 \). The relative error between Eq. (11) and the exact derivative of Eq. (2) is less than 5.5% for \( R > R_0 \) where \( R_0 \) is the Ricci curvature today. When \( R > 3.3 R_0 \), the error drops rapidly down to 1%. At \( R \sim R_0 \), the error is around 1.5%, and it only goes up to 10% when \( R \) approaches \( 4\Lambda \) where \( \gamma = 1.0338 \). However, as we shall show later, \( 4\Lambda \) is the minimal value of \( R \) that can be found in our simulations, which is actually a rare case.

Using this fitting formula, we can express \( R \) in terms of \( f_R \) as
\[ R = 12\alpha \Omega_m^0 H_0^2 + 3\Omega_m^0 H_0^2 \left( \frac{f}{f_R} \right)^{\frac{1}{p+1}} \]
for our model. In the mean time, for the H-S \( f(R) \) model, if \( R \gg H_0^2 \Omega_m^0 \), the scalar field \( f_R \) can be approximated as [18]
\[ f_R(R) \sim -\frac{c_1}{c_2} \left( \frac{\Omega_m^0 H_0^2}{R} \right)^{n+1} \]
Fig. 1 also shows the accuracy of this approximation, and we can see that it is less accurate when \( R \sim R_0 \), where the error goes up to 7%. We can similarly invert this equation to get \( R \) as a function of \( f_R \) for the H-S model.

Our cosmological simulations are performed using the ECOSMOG code [20] which is itself based on the N-body code RAMSES [27]. The code uses the supercomoving coordinates
\[ \tilde{x} = \frac{x}{aB}, \quad \rho = \frac{\rho a^3}{\rho_0 \Omega_m}, \quad \tilde{v} = \frac{av}{BH_0}, \quad \tilde{\phi} = \frac{a^2 \phi}{(BH_0)^2}, \quad \tilde{\delta} = \frac{\delta}{BH_0}, \]
where \( B \) is the Hubble parameter today.
FIG. 1. The error of the approximation for \( f_R(R) \) relative to the exact expressions. When the curvature is high, the error in our model drops very quickly. When \( R > 3.3 R_\odot \), the error is below 1%. When the curvature is low, e.g., \( R \approx R_\odot \), the error is around 1.5%. The error goes up to 10\% when \( R \) is around \( 4\gamma \Lambda \), where \( \gamma = 1.0338 \). However, \( 4\pi \Lambda \) is the minimal value of \( R \) in our simulations, which actually is a rare case. The results show that the overall accuracy of the approximate expression of \( f_R(R) \) for our model is better than that for the Hu-Sawicki model with \( n = 1 \).

where \( x \) is the comoving coordinate, \( \rho_c \) is the critical density today, \( c \) is the speed of light and \( D \) is the size of the simulation box in units of \( h^{-1}\text{Mpc} \). For our \( f(R) \) model, in code units Eq. (8) and Eq. (9) can be written as,

\[
\nabla^2 \hat{\phi} = 2a\Omega_m^0 (\dot{\rho} - 1) + \frac{\alpha}{2} \Omega_m^0 - \frac{a^4 \Omega_m^0}{2} \left( \frac{Da^2}{f_R(\rho)} \right)^{\frac{1}{\gamma}} + 2a^4 (1 - \alpha) \Omega_d^0, \\
\nabla^2 \hat{f}_R = -\frac{a^4 \Omega_m^0}{c^2} (\dot{\rho} - 1) + \frac{a^4 \Omega_m^0}{c^2} \left( \frac{Da^2}{f_R} \right)^{\frac{1}{\gamma}} - \frac{4a^4 (1 - \alpha) \Omega_d^0}{c^2} - \frac{a^4 \Omega_m^0}{c^2},
\]

FIG. 2. Upper Panel: The numerical solution of the Gaussian field on the 256\(^3\) domain grids, as well as the first and second refinements. The solid line is the analytical solution. We take \( |f_{R0}| \approx 10^{-5} \) in the tests, and the size of the simulation box is 150\(h^{-1}\text{Mpc}\). Lower Panel: The errors of the numerical results relative to the exact solution on the domain grid and each refinement.

\[
\delta(x) = \frac{(x - \frac{1}{2})^2}{W^2} - \frac{1}{2} 4\beta a^2 \tilde{f}_R(a) \exp \left( \frac{(x - \frac{1}{2})^2}{W^2} \right) \\
+ a^3 \left[ \frac{D}{\tilde{f}_R(a)} \left( 1 - \beta \exp \left( - \frac{(x - \frac{1}{2})^2}{W^2} \right) \right) \right]^{\frac{1}{\gamma}} \\
- 4a^3 (1 - \alpha) \frac{\Omega_d^0}{\Omega_m^0} - 1, \\
\]

which admits the following solution to the field \( \tilde{f}_{R'} \):

\[
\tilde{f}_{R'}(x) = a^2 \tilde{f}_R(a) \left[ 1 - \beta \exp \left( - \frac{(x - \frac{1}{2})^2}{W^2} \right) \right],
\]

where \( W \) and \( \beta \) are constants. We use \( W = 0.1, \beta = 0.999999 \) in the test. In Fig. 2, we show the numerical results on domain grids, as well as the first and second refinements. The numerical results are in good agreement with the analytical solutions. In addition to the Gaussian field test, we have also tested the code with both sine and homogenous fields, and found the numerical results to be in good agreement with the analytical solutions. However, for simplicity, we do not present these results here.

Since these equations are different from those in the default ECOSMOG code, we need to test the accuracy of our modified code. Following \( \text{26,29,31} \), we take the density \( \delta \) as a one dimensional (in the \( x \) direction without loss of generality) Gaussian field

\[
\delta(x) = \frac{(x - \frac{1}{2})^2}{W^2} - \frac{1}{2} 4\beta a^2 \tilde{f}_R(a) \exp \left( \frac{(x - \frac{1}{2})^2}{W^2} \right) \\
+ a^3 \left[ \frac{D}{\tilde{f}_R(a)} \left( 1 - \beta \exp \left( - \frac{(x - \frac{1}{2})^2}{W^2} \right) \right) \right]^{\frac{1}{\gamma}} \\
- 4a^3 (1 - \alpha) \frac{\Omega_d^0}{\Omega_m^0} - 1, \\
\]

which admits the following solution to the field \( \tilde{f}_{R'} \):

\[
\tilde{f}_{R'}(x) = a^2 \tilde{f}_R(a) \left[ 1 - \beta \exp \left( - \frac{(x - \frac{1}{2})^2}{W^2} \right) \right],
\]

where \( W \) and \( \beta \) are constants. We use \( W = 0.1, \beta = 0.999999 \) in the test. In Fig. 2, we show the numerical results on domain grids, as well as the first and second refinements. The numerical results are in good agreement with the analytical solutions. In addition to the Gaussian field test, we have also tested the code with both sine and homogenous fields, and found the numerical results to be in good agreement with the analytical solutions. However, for simplicity, we do not present these results here.

The perturbation equations in code units for the H-S model have been presented in Refs. \( \text{26,29,31} \), interested readers are referred to these papers for further details, and we will not repeat them here.
The cosmological parameters used in our simulations are $\Omega_0^b = 0.049, \Omega_0^c = 0.267, \Omega_0^d = 0.684, h = 0.671, n_s = 0.962$, and $\sigma_8 = 0.834$, which are the Planck best-fit values for the standard ΛCDM model. We use the MGGRAF package [28] to generate initial conditions at $z = 49$. The number of particles in our simulations is $N = 256^3$ and the box size is $L_{\text{box}} = 150h^{-1}\text{Mpc}$. We run four realisations for each model. In each realisation, the different models share the same initial conditions. In Fig. 3, we show the ratio of the power spectra

$$\Delta P/P = P_{f(R)}/P_{\Lambda m} - 1$$

at $z = 0$, measured using the POWMES [35] code. The power spectra are averaged over the four realisations. The parameter $f_{R0}$ is taken as $f_{R0} = -10^{-4}, -10^{-5}, -10^{-6}$ for both our model and the H-S model. Compared with our previous work [35], we have significantly improved the accuracy of the background field $f_R$ in the regime $R \approx R_0$ by introducing the parameter $\alpha$ in the fitting formula Eq. (10). When $\alpha = 0$, the perturbation equations Eq. (15) and Eq. (16) reduce to the equations used in Ref. [35].

IV. COSMOLOGICAL INEQUALITIES

In this section we will lay out the theoretical framework for the screening mechanism in $f(R)$ gravity. We will begin by discussing the importance of the homogenous field solution in the $f(R)$ simulations and then introduce two inequalities found in $f(R)$ cosmology. Using these inequalities, we will explain how the screening mechanism works in $f(R)$ gravity. In the next section, we will apply the theory presented here to dark matter halos.

A. homogeneous fields

We begin by discussing the solutions of Eq. (8) and Eq. (9) for a homogenous density field ($\delta \rho = 0$). From Eq. (9), the vanishing of $\delta f_R$ gives

$$f_R = \bar{f}(R) = D \left( \frac{3\Omega_m^0 H_0^2}{R - 12\alpha \Omega_d^0 H_0^2} \right)^{p_+} ,$$

where

$$\bar{R}(a) = \left[ 3\Omega_m^0 a^{-3} + 12\Omega_d^0 H_0^2 \right] .$$

The error of the field $f_R$ obtained from Eq. (19) relative to the exact expression of the derivative of the background field Eq. (2) is shown in Fig. 1. The maximal deviation is about 5.5% in the range $R_0 < R < 3R_0$ and when $R > 3R_0$, the error rapidly drops down to 1%. For the modified Poisson equation Eq. (8), $\delta \rho = 0$ gives the homogenous solution of the field $\phi = 0$, namely the zero point of the potential, which, as we shall show later, plays an important role in understanding the screening mechanism in $f(R)$ cosmology.

On the other hand, roughly speaking, when the local density in the simulations is above the background density ($\rho > \bar{\rho}$), the potential $\phi$ is negative ($\phi < 0$) and $\delta f_R$ is positive ($\delta f_R > 0$). When the local density is below the background density ($\rho < \bar{\rho}$), the potential $\phi$ is positive ($\phi > 0$) and $\delta f_R$ is negative ($\delta f_R < 0$). However, as we shall show later, the ratio $-\frac{\delta \phi}{\delta f_R}$ is usually positive $-\frac{\delta \phi}{\delta f_R} > 0$ because $\phi$ and $\delta f_R$ will change their signs simultaneously as $\phi$ crosses the zero point.

B. voids

In this subsection, we will discuss solutions of the fields in void regions, where $\rho \sim 0$. In $f(R)$ cosmology, voids are permeated with the scalar field $f_R$. The solutions of Eq. (9) in these regions are usually quite complicated – they depend not only on the size of the void but also on the environment surrounding it [34]. However, if we consider an extreme case where, for a large enough void, the distribution of the cosmic field $f_R$ near the void centre is nearly homogeneous ($\delta f_R \sim 0$), we have $\nabla^2 \delta f_R \sim 0$. Equation (9) thus yields

$$R \sim 4\Lambda ,$$

where we have used the expression for the background Ricci curvature $R$

$$\bar{R} = 8\pi G\bar{\rho} + 4\Lambda ,$$

and the assumption that at the void centre $\rho \sim 0$ so that $\delta \rho \sim -\bar{\rho}$.

Equation (21) implies that in the perturbed Universe, even at the center of the void, the local curvature $\bar{R}$ in $f(R)$ cosmology has a nonzero lower bound $4\Lambda$. In the above analysis, we did not assume any specific functional form of $f(R)$ but just required that the background expansion is practically
indistinguishable from that of the ΛCDM model. This conclusion therefore is general. In order to check it explicitly, we generate a two-dimensional map from our simulations by finding the minimal value of the curvature $R$ along the $z$ direction through the simulation box and then project them onto the $x$-$y$ plane. As shown in Fig. 4 in the cases with $|f_{R0}| = 10^{-6}$, the minimal values of $R$ are very close to $4\Lambda$, and we can see clearly that $R > 4\Lambda$ for both $f(R)$ models. In the cases with $|f_{R0}| = 10^{-4}$, the minimal values of $R$ are very close to $R_0$ and the distribution of $\min|R|$ is nearly homogeneous. These numerical checks thus confirm that

$$R > 4\Lambda .$$  \hspace{1cm} (23)

From this inequality, we know that the approximate formulae for the background fields $f_R$ (e.g., Eq. (11) and Eq. (14)) only need to be accurate in the range $R > 4\Lambda$. Furthermore, $f(R = 0) = 0$ is not a necessary condition for $f(R)$ models, given the fact that the point $R = 0$ will never be arrived at in the Universe since $R > 4\Lambda$ if the background expansion of the $f(R)$ model is practically indistinguishable from the ΛCDM model. Nevertheless, our model explicitly predicts $R > 4\Lambda$ and is therefore naturally consistent with this inequality.

C. high-density regions

In this subsection, we discuss the solutions of Eq. (8) and Eq. (9) in high-density regions. There are two types of solutions. If $\delta R \sim 8\pi G\delta \rho$, the solution is called the high-curvature solution. Correspondingly, the solution with $\delta R \ll 8\pi G\delta \rho$ is called the low-curvature solution. Note that high density does not necessarily imply high curvature in $f(R)$ gravity.

The low-curvature solution is usually arrived at when the amplitude of the background field, $|f_R|$, is large compared to the local potential: $c^2|f_R| \gg |\phi|$ (29). The terms associated with the perturbation of the curvature, $\delta R(f_R) = \frac{\partial f_R}{\partial f_0} \delta f_R \ll 8\pi G\delta \rho$, in Eqs. (8) and Eq. (9) have a minor effect and can be neglected. These equations can therefore be linearized and reduced to

$$\nabla^2 \phi \sim \frac{16\pi G}{3} \delta \rho$$

$$\nabla^2 \delta f_R \sim \frac{8\pi G}{3c^2} \delta \rho .$$

Equation (24) and Eq. (25) indicate that, given the density field $\delta \rho$ and under the same (e.g., periodic) boundary conditions, their solutions satisfy the relation $c^2 \delta f_R \sim -\frac{2\phi}{\phi}$, where $\delta f_R = f_R - \bar{f}_R$. In this extreme case, the scalar field $|\delta f_R|$ and the local potential $|\phi|$ attain their maximum values as $|\phi| = \frac{2G\rho}{3}$ and $|\delta f_R| = \frac{4\rho}{3c^2}$, respectively, where $\phi_N$ is the standard Newtonian potential for a given density field $\delta \rho$. Combining Eq. (8) and Eq. (9), we obtain

$$\nabla^2 \left( \phi + \frac{c^2 \delta f_R}{2} \right) = 4\pi G\delta \rho .$$

The standard Newtonian potential $\phi_N$ is related to the potential $\phi$ and the scalar field $c^2 \delta f_R$ as

$$\phi_N = \phi + \frac{c^2 \delta f_R}{2} .$$

In general, if the background field $|f_R|$ is not large enough, we have

$$c^2 |\delta f_R| \leq \frac{2\phi}{3} ,$$

which is a known result in the literature (12). Furthermore, in high-density regions, we usually have $\phi_N < 0$, $\phi < 0$ and $\delta f_R > 0$, inserting Eq. (27) into Eq. (28) we can obtain

$$c^2 |\delta f_R| \leq \frac{\phi}{2} ,$$

which only involves the quantities $\delta f_R$ and $\phi$ ($\phi_N$ is not a physical quantity in $f(R)$ gravity). In high-density regions, applying Eq. (27) and Eq. (29), and using $\phi_N < 0$, $\phi < 0$, $\delta f_R > 0$, we obtain

$$|\phi_N| \leq |\phi| \leq \frac{4}{3} \phi ,$$

where the left and right limits correspond to the extreme cases of high-curvature and low-curvature solutions, respectively. It is evident that Eq. (30) is also equivalent to the well-known result that $G \leq G_{\text{eff}} \leq \frac{4}{3} G$ in $f(R)$ gravity, where $G_{\text{eff}}$ is the effective Newtonian constant which is defined by

$$G_{\text{eff}} = \frac{4}{3} - \frac{\delta R}{3c^2\delta \rho} .$$

From Equation (30), we can find that Eq. (29) imposes a tighter constraint on the scalar field perturbation $c^2 |\delta f_R|$ than Eq. (28) does. We therefore will focus on Eq. (29) throughout this work, and take it as the starting point of our analysis in the next few sections. We will first examine its validity and show how well it holds in numerical simulations for $f(R)$ models. Then we will attempt to quantitatively understand the screening mechanism in $f(R)$ gravity based on this inequality.

In order to check Eq. (29) using our simulations, we statistically compare the values of $-c^2 \delta f_R$ and $\phi$. We divide the potential $\phi$ into 100 equal bins from the minimal value to the maximal value. For convenience, $\phi$ is in code units. We then count the number of occurrences of $-c^2 \delta f_R$ and calculate its arithmetic average in each bin. The results are shown in the upper panels in each plot of Fig. 5. Included in Fig. 5 are the results at $z = 0$ for our $f(R)$ model (red) and the H-S model (black), each with different parameters $f_{R0} = -10^{-4}, -10^{-5}, -10^{-6}$. We can clearly see that $-c^2 \delta f_R / \phi$ is a positive and rather smooth function with respect to the potential $\phi$, except in the vicinity of $\phi = 0$, where the discontinuities are due to numerical errors. We find that the maximal value of $-c^2 \delta f_R / \phi$ is 1/2, which only happens in the $f_{R0} = -10^{-4}$ case. In the other two cases
Recall that from which we have
\begin{equation}
\phi < -\frac{2\pi G \rho}{\Lambda} = -4\Lambda f_R \phi < -\frac{2\pi G \rho}{\Lambda} = -4\Lambda f_R
\end{equation}
for every \( f_R \) smaller than the value of the background field, \( |f_R| < |\bar{f}_R| \) (see Fig. 6), implying that \( \delta f_R > 0 \). Equation (29) in this case can be rewritten as
\begin{equation}
-\frac{\phi}{2} \geq c^2 (f_R - \bar{f}_R) = c^2 \delta f_R > 0
\end{equation}
from which we have
\begin{equation}
c^2 f_R \leq -\frac{\phi}{2} + c^2 \bar{f}_R
\end{equation}
Recall that \( f_R \) must satisfy the physical constraint \( f_R < 0 \) due to the stability considerations of the perturbation evolution in the high curvature regime \( H-S \). It can be shown that if the right hand side of Eq. (33) is less than zero or, equally, \( c^2 \bar{f}_R < \frac{\phi}{2} \), the absolute value of \( f_R \) will have a nonzero lower bound \(-c^2|f_R| \geq -\frac{\phi}{2} + c^2 f_R| > 0 \). If the background field \( |f_R| \) is large \( (c^2|f_R| \gg |\frac{\phi}{2}|) \), this lower bound will be very high as well \((-\frac{\phi}{2} + c^2 |f_R| \gg 0 \), which means that \( |f_R| \) cannot be adequately suppressed in high-density regions, leading to a strong fifth force. This physical picture can also be viewed in a different way: the existence of the lower bound for \( |f_R| \), for both \( f(R) \) models studied in this work, conversely, means that there is an upper bound on the curvature: \( R_{\text{max}} = R(f_R = -| -\frac{\phi}{2} + f_R| ) \) in high-density regions. If \( R_{\text{max}} \ll 8\pi G \rho \), the solution to the curvature is far below the GR prediction, so that the model does not have a high-curvature solution in high-density regions and would be ruled out. Therefore, \( c^2|f_R| \gg | -\frac{\phi}{2} \) is a sufficient condition for the model to admit the low-curvature solution.

On the other hand, if \(-\frac{\phi}{2} + c^2 f_R \sim 0 \), the magnitude of the scalar field \( f_R \) can be sufficiently suppressed: \( |f_R| \to 0 \) and \( R_{\text{max}} \) can be close enough to its GR solution, \( R_{\text{max}} \sim 8\pi G \rho \).
FIG. 5. The statistics of $-\Sigma^2 \delta f_R$ and $G_{\text{eff}}/G$ for our model and the H-S model at $z = 0$. The horizontal axis is the potential $\phi$ in code units. The condition $|\phi| > 2c^2 |f_R|$ is equivalent to $|\phi| > |\phi_c|$ where $\phi_c = 2c^2 f_R$ is the critical potential and is indicated by red (black) solid vertical lines for our (the H-S) model. We can see clearly that when $|\phi| > |\phi_c|$, the screening mechanism starts to work.

so that the model could admit the high-curvature solution. Moreover, if the local scalar field $\phi$ satisfies $|\phi| > 2c^2 |f_R|$, there will be no constraint on the maximal value of the local scalar curvature ($R_{\text{max}} = +\infty$), and the high-curvature solution can possibly be arrived at too. $|\phi| \gtrsim 2c^2 |f_R|$ is therefore the necessary condition for the high-curvature solution. However, this is not a sufficient condition: as we shall show later, to guarantee a high-curvature solution ($G_{\text{eff}} \sim G$), the potential well $\phi$ need to be deep enough relative to the background field $2c^2 |f_R|$.

In order to test the above conclusions, we perform a similar statistical analysis, to that of $-\Sigma^2 \delta f_R$, for the effective Newtonian constant $G_{\text{eff}}$, which is defined by Eq. (31). Recall that $G_{\text{eff}} \sim G$ indicates the high-curvature solution ($\delta R \sim \kappa^2 \delta \rho$) and $G_{\text{eff}} \sim \frac{G}{2}$ implies the low-curvature solution ($\delta R \ll \kappa^2 \delta \rho$). The numerical results on the statistics of $G_{\text{eff}}/G$ are shown in the lower panels in each plot of Fig. 5 and Fig. 6.

We define a critical potential as $\phi_c = 2c^2 f_R$, and in Fig. 5 and Fig. 6 $\phi_c$ (in code units) is indicated by vertical lines. As we have expected, when the magnitude of the local potential $|\phi|$ is higher than the critical potential $|\phi_c|$, the screening mechanism starts to work, as can be seen clearly in Fig. 5 for both $f(R)$ models studied, and for different values of the parameter $f_{R0}$. For completeness, we also check this conclusion at higher redshifts ($z = 0.5, 1, 1.5, 2$). We take $f_{R0} = -10^{-4}$ for illustration purposes. Fig. 6 shows that $|\phi_c|$ lies accurately at the point above which the screening mechanism starts to work. These numerical results are in good agreement with our above analysis. From Fig. 5 and Fig. 6, we can also see that high-curvature solutions with an effective Newtonian constant close to the standard gravity, $G_{\text{eff}} \sim G$, usually happen in regimes where the potential well $\phi$ is noticeably deeper than $\phi_c$. 

\begin{align*}
\end{align*}
Before leaving this section, we briefly summarize the main results obtained from the above analysis:

- $2c^2|\bar{f}_R| \gg | - \phi|$ is a sufficient condition for the low-curvature solution. Combining the constraint $R > 4 \Lambda$ obtained above, the curvature scalar $R$ is bounded locally as

$$4 \Lambda < R < R(f_R = -| - \phi | 2c^2 + \bar{f}_R)$$

for the low-curvature solution. If this occurs in the Solar system, the model is ruled out. Using $| - \phi | 2c^2 \phi N \geq | - \phi |$, it can be shown that $2c^2|\bar{f}_R| \gg | - \phi |$ is sufficient for the low-curvature solution and is indeed stronger than the condition $2c^2|\bar{f}_R| \gg | - \phi |$ because, logically, we have

$$2c^2|\bar{f}_R| \gg | - \frac{4}{3} \phi N | \Rightarrow 2c^2|\bar{f}_R| \gg | - \phi |.$$  

- $| - \phi | \geq 2c^2|\bar{f}_R|$ is a necessary but not sufficient condition for the high-curvature solution. Combining $| - \frac{4}{3} \phi N | \geq | - \phi |$, we can show that $| - \frac{4}{3} \phi N | \geq 2c^2|\bar{f}_R|$ is also a necessary condition for the high-curvature solutions. However, it is much weaker than that of $| - \frac{4}{3} \phi N | \geq 2c^2|\bar{f}_R|$ because, logically, we have

$$| - \phi | \geq 2c^2|\bar{f}_R| \Rightarrow | - \frac{4}{3} \phi N | \geq 2c^2|\bar{f}_R|.$$  

In addition to the above results, we also find that the critical potential $\phi_c = 2c^2\bar{f}_R$ is a good indicator which tells us
when the screening mechanism starts to work. Such a universal criterion applies to both $f(R)$ models studied here, with different parameters ($f_{R0} = -10^{-6}, -10^{-5}, -10^{-4}$) at different redshifts (see Fig. 6). Regions where the local potential $\phi$ is below $|\phi_c|$ are usually completely unscreened.

A potential application of the result obtained above is that the condition $|\frac{1}{2}\phi_N| < |\phi_c|$ can be used to identify unscreened galaxies and make screening maps in galaxy surveys [37]. Such maps play an important role in astrophysical constraints on $f(R)$ gravity [36], which can place much tighter constraint than that can be obtained from cosmological observations.

In the next section, we will apply our results derived in this section to dark matter halos.

V. DARK MATTER HALOS

From the previous analysis, we know that the screening in $f(R)$ gravity depends mainly on the depth of the gravitational potential. From the condition $| - \phi | \gtrsim 2 \epsilon^2 |f_R|$, we can infer that there are two possible ways for a dark matter halo to be screened. Firstly, the halo itself is so massive that it can generate a deep enough potential well that satisfies $| - \phi | \gg | - \phi_c |$: this case is called self-screening [37, 39–41]. Secondly, for a halo too small to be self-screened but lying in a very deep potential well, if the magnitude of the total local potential satisfies $| - \phi | \gg | - \phi_c |$, then the halo can still become screened: this case is called environmental-screening [37, 39–41]. In the following, we will discuss these two different screening scenarios in detail.

We identify halos in our simulations using a modified version of the AHF code [42]. We follow the standard procedure in the AHF code to locate density peaks as the positions of the dark matter halos, but remove the unbound particles in halos by taking into account the modification to gravity. We use the effective density $\delta_{\text{eff}} = \frac{\delta\rho_{\text{eff}}}{\rho}$ instead of $\delta\rho$ to calculate the gravitational potential. In order to characterise screened and unscreened dark matter halos, we follow [41] by defining the lensing mass $M_L$ and dynamical mass $M_D$ for a dark matter halo.

The lensing mass is the bare mass of the dark matter halos, which is defined by

$$M_L = \int \delta\rho(x) \, dV \quad . \tag{35}$$

The dynamical mass, on the other hand, is defined by

$$M_D = \int \delta\rho_{\text{eff}}(x) \, dV \quad , \tag{36}$$

which includes the effect of the scalar field. For a totally unscreened halo, the ratio between the two masses is $\frac{M_D}{M_L} \sim \frac{4}{3}$, while for a completely screened halo we have $\frac{M_D}{M_L} \sim 1$. In general, however, the value of $\frac{M_D}{M_L}$ is somewhere in between.

We now present our results for several representative models. We show in Figs. 7–9 the numerical results for the two $f(R)$ models with $f_{R0} = -10^{-4}$ at $z = 1$ (In Fig. 7 note that we do not show the $z = 0$ results for $f_{R0} = -10^{-6}$, because all halos in this case are simply unscreened) and the models with $f_{R0} = -10^{-5}, f_{R0} = -10^{-6}$ at $z = 0$, respectively. In these figures, each point represents a dark matter halo and the color of the point represents the ratio between the dynamic mass and the lensing mass. We find the maximal value of the gravitational potential $\bar{\phi}$ inside a dark matter halo and show $\text{Max}[\bar{\phi}]$ with respect to the lensing mass of that halo. For convenience, the potential $\bar{\phi}$ is in code units, and $\bar{\phi}_c = 2^{1/2} \bar{R}$ is the critical potential as discussed in the previous section. From these figures, we can see that if $| - \phi | > 0$, the completely screened dark matter halos ($\frac{M_D}{M_L} \sim 1$) only appear in potentials much deeper than the critical potential $\bar{\phi}_c$. It is also obvious that below this critical potential, almost all the halos are completely unscreened ($\frac{M_D}{M_L} \sim \frac{4}{3}$). These observations apply for both $f(R)$ models and for different values of $f_{R0}$.

Next, we look at the two different ways of screening halos as mentioned before. The efficiency of the screening depends on the depth of the potential well. In the $f_{R0} = -10^{-4}$ case, as shown in Fig. 7, the dark matter halos, even the largest ones, cannot generate a deep enough potential well for self-screening, and most of them are completely unscreened. However, we can see that there are several small halos that are well screened. In these cases, the screened halos are environmentally screened, because they reside in deep potential wells generated by nearby structures. In order to confirm this point, in Fig. 10 we show the minimal value of the gravitational potential $\phi$ found inside dark matter halos with respect to the lensing mass of the halos. Compared with Fig. 7 for the large halos, we find that although the maximal depth of the potential well $\text{Max}[\bar{\phi}]$ inside the halos is far above the critical potential, the minimal depth $\text{Min}[\bar{\phi}]$ can be below it: the large halos are therefore only partially screened, leading to $\frac{M_D}{M_L} > 1$. On the other hand, for the well-screened small halos, from Fig. 10 we can see that even the minimal depths of the potential inside the halos are far above the critical potential (see the blue points in Fig. 10): since the small halos themselves cannot produce such deep potentials, the latter must have been generated by their environments.

If the background field $|f_R|$ is small (e.g., $f_{R0} = -10^{-6}$), most halos can generate relatively deeper potential wells than the small critical potential $\phi_c$ and thus easily be self-screened. From Fig. 9 we can see that all halos more massive than about $10^{13}$ solar mass are well screened. However, not all the small halos less massive than about $10^{13}$ solar mass are unscreened. As explained previously, there are a substantial fraction of the small halos which can be environmentally screened: as the critical potential $|\phi_c|$ is smaller for $f_{R0} = -10^{-6}$, there will be more regions in which nearby structures can create a potential well deeper than $|\phi_c|$.

So far, our analysis of the screening mechanism is based on comparing the local gravitational potential $-\phi$, to the value of the background field $c^2 |f_R|$. The condition $| - \phi | \lesssim 2 \epsilon^2 |f_R|$ is useful for identifying unscreened halos theoretically. However, in practice a global map of potential $-\phi$ may not be easily constructed in real galaxy surveys, and we need to use the standard Newtonian potential $\phi_N$, namely the lensing potential, which is related to $\phi$ by Eq. (27). There are two reasons for this:
FIG. 7. Scatter plot for the maximal value of the gravitational potential \( \text{Max}[\phi] \) inside a dark matter halo with respect to the lensing mass of the halo for \( f(R) \) models with \( f_{R0} = -10^{-4} \) at \( z = 1 \). Each point represents a dark matter halo and its color encodes the ratio between the dynamical mass and the lensing mass (see the colorbar on the right hand side). \( |\phi_c| = 2c^2|f_R| \) is the critical value in code units, above which the screening mechanism starts to work. Halos with the maximal depth of the potential well \(|-\phi|\) below the threshold \(|\phi_c|\) are completely unscreened in this case. On the right panel, some small halos are well screened due to environmental screening. However, in this case large halos cannot generate deep enough potential wells for self-screening and therefore are only partially screened.

FIG. 8. Scatter plot for the maximal value of the gravitational potential \( \text{Max}[\phi] \) inside a dark matter halo with respect to the lensing mass of the halo for \( f(R) \) models with \( f_{R0} = -10^{-5} \) at \( z = 0 \). It is clear that below the horizontal line, which represents the critical potential \(|\phi_c| = 2c^2|f_R|\), the halos are completely unscreened. It is also clear that most of the well-screened halos lie in very deep potential wells.

- First, a global map of \( \phi_N \) can be easily constructed in real galaxy surveys if a group catalog [43] is available, because \( \phi_N \) satisfies the linear equation Eq. (26). \( \phi \), on the other hand, can only be constructed by solving the more complicated nonlinear scalar field equation.
- Second, galaxy shear measurements also have the potential to reconstruct the 3-dimensional map of the lensing potential \( \phi_N \) using the weak lensing tomography technique [44].

As we have discussed in the previous section, for identifying the unscreened halos, the condition \(|-\frac{4}{3}\phi_N| \lesssim 2c^2|f_R|\) is stronger than \(|-\phi| \lesssim 2c^2|f_R|\). Let us now examine the power of the condition \(|-\frac{4}{3}\phi_N| \lesssim 2c^2|f_R|\) for identifying unscreened halos. In Fig. 11, we show the maximal value of the Newtonian potential \( \phi_N \) inside a halo (\( \text{Max}[\phi_N] \)) with respect to the lensing mass of the halo for \( f_{R0} = -10^{-4} \) models at \( z = 1 \) (top panel) and \( f_{R0} = -10^{-5} \) (middle panel) and \( f_{R0} = -10^{-6} \) (bottom panel) models at \( z = 0 \). The horizontal lines indicate the critical potentials for the Newtonian
FIG. 9. Scatter plot for the maximal value of the gravitational potential $\text{Max}[-\phi]$ inside a dark halo with respect to the lensing mass of the halo for $f(R)$ models with $f_{R0} = -10^{-6}$ at $z = 0$. In this case, most of the massive halos (e.g. $M_{\text{vir}} > 10^{13} M_\odot$) can generate deep enough potential well and get self-screened. It is also obvious that a substantial fraction of the small halos are also well screened due to the environment-screening. Below the horizontal line, which represents the critical potential $|\tilde{\phi}| = 2\tilde{c}^2|f_R|$, most of the halos are completely unscreened.

FIG. 10. Scatter plot for the minimal value of the gravitational potential $\text{Min}[-\phi]$ inside a dark halo with respect to the lensing mass of the halo for $f(R)$ models with $f_{R0} = -10^{-4}$ at $z = 1$. The small halos indicated by the blue points are embedded in potential wells significantly deeper than the threshold $|\tilde{\phi}_c|$, and are therefore well-screened. However, the minimal depth of the potential well $\text{Min}[-\phi]$ for the massive halos are not far above the threshold of the potential $|\tilde{\phi}_c|$. These massive halos are only partially screened (e.g. $M_D/M_L \sim 1.20$).

potential $\phi_N$, which is defined by

$$\phi_{Nc} = \frac{3}{2} \tilde{c}^2 f_R.$$  

We can see that $|\phi_N| > |\phi_{Nc}|$ is not very useful for identifying screened halos. However, the opposite case $|\phi_N| < |\phi_{Nc}|$ is very accurate for identifying completely unscreened halos in $f_{R0} = -10^{-4}$ and $f_{R0} = -10^{-5}$ cases. For $f_{R0} = -10^{-6}$ cases, as shown in Fig. [11], not all halos with $\text{Max}[-\phi_N] < |\phi_{Nc}|$ are completely unscreened and several halos (mainly the more massive ones) are only partially unscreened. However, the condition $|\phi_N| < |\phi_{Nc}|$ in this case does distinguish unscreened halos (including partially unscreened ones) from well-screened halos (dark blue points in Fig. [11]). In order to show this point, in Fig. [12] we present a histogram for the distribution of the well-screened dark halos ($\frac{M_D}{M_L} - 1 < 0.01$) with respect to the maximal potential $-\phi_N$ inside the halo. It is clear that below the threshold $|\phi_{Nc}|$, the number counts of well-screened halos are fairly low.
FIG. 11. Scatter plot for the maximal value of the standard Newtonian potential $\text{Max}[-\phi_N]$ inside a dark halo with respect to the lensing mass of the halo for $f(R)$ models with $f_{R0} = -10^{-4}$ at $z = 1$ (top row), and $f_{R0} = -10^{-5}$ (middle row), $-10^{-6}$ (bottom row) at $z = 0$, respectively. The values of the Newtonian potential are evaluated by $\phi_N = \phi + \frac{c^2}{2} f_R$. From the plots, we can see clearly that $|\tilde{\phi}_N| > |\tilde{\phi}_{Nc}|$ is not accurate enough for identifying the screened halos. However, $|\phi_N| < |\phi_{Nc}|$ is accurate for identifying the unscreened halos in the $f_{R0} = -10^{-4}$ and $f_{R0} = -10^{-5}$ cases. In the $f_{R0} = -10^{-6}$ case, below the horizontal line, most of the halos are completely unscreened, though several of them are only partially unscreened.
VI. SUMMARY AND DISCUSSION

The chameleon screening plays an important role in the viability of \( f(R) \) gravity. In this paper, we have reexamined the screening in \( f(R) \) cosmology using a suite of N-body simulations and found a number of useful results, which are summarized as follows.

- In low-density regions, we find that the local curvature \( R \) has a non-zero lower bound given by
  \[
  R > 4\Lambda .
  \] (38)

  This conclusion applies to a large family of \( f(R) \) models that can closely mimic the ΛCDM background expansion regardless the functional form of \( f(R) \). A practical application of this result is that the approximation for the scalar field \( f_R \) no longer needs to be accurate in the range \( R > 4\Lambda \).

- In high-density regions, we have found an inequality
  \[
  c^2 |\delta f_R| \leq \left| \frac{\Delta \phi}{2} \right| ,
  \] (39)

  that has important implications on the screening mechanism in \( f(R) \) gravity. We have shown that screening only happens if the depth of the local potential well \( (-\phi) \) is close to or above the value of the background field, i.e., \( | -\phi | \gtrsim 2c^2 |f_R| \). However, this condition is not sufficient for all halos to be well screened. On the other hand, we find that the opposite case, \( | -\phi | \leq 2c^2 |f_R| \), is very powerful for identifying completely unscreened halos in the simulations.

In order to make our results applicable to real galaxy surveys, we have also expressed the condition in terms of the standard Newtonian potential \( \phi_N \), or the lensing potential. We have shown that
  \[
  -\frac{4}{3} \phi_N \leq 2c^2 |f_R| ,
  \] (40)

  is stronger and more conservative for identifying unscreened halos. It works very well in the \( f_R = -10^{-4} \) and \( f_R = -10^{-5} \) cases, where below the threshold potential \( |\phi_{Nc} = \frac{3}{2}c^2 f_R| \) all the dark matter halos are completely unscreened. In the \( f_R = -10^{-6} \) case, although the condition Eq. (40) cannot guarantee all the selected halos are completely unscreened \( f_R \sim \frac{4}{3} \), it does cleanly separate unscreened halos from the well-screened ones. The contamination of the unscreened samples is very low.

Further, we would like to remark here that the efficiency of screening depends on the absolute depth of the potential well. This is due to the non-linear nature of the scalar field equation Eq. (9). The reference of the depth of the potential well \( \delta f_R \) cannot be chosen arbitrarily because \( \delta f_R \) should vanish for the homogenous density field, which actually defines the zero point of \( \delta f_R \). The solution of the Newtonian potential \( \phi_N \) should vanish for the homogenous density field as well. To apply our results to real galaxies surveys, we need to carefully take into account this point.

Comparing the properties of galaxies in screened and unscreened regions can potentially provide one of the most robust tests of \( f(R) \) gravity \([36-37,39]\), because the formation and evolution of galaxies in these regions should differ significantly due to the 1/3 enhancement of the gravitational force. However, caution must be taken when performing and interpreting these tests, due to the difficulty of correctly modelling the nonlinear environmental effects. Detailed simulations and analysis of galaxy formation in \( f(R) \) gravity are needed before drawing any quantitative conclusions.

In real galaxy surveys, the first step for this study is to build a screening map \([37]\). The unscreened samples are of particular interest because a real galaxy might not be completely unscreened even if its host dark matter halo is completely unscreened. Massive components in the galaxy (e.g., stars) can still be self-screened if they can generate deep enough local potential wells \( |\phi_N| \gg |\phi_{Nc}| \), where the thresholds potential \( \phi_{Nc}/c^2 \) for models with different values of \( f_{R0} \) at \( z \approx 0 \) are listed in table I. The Sun typically has the potential as \( \phi_{Nc}/c^2 \sim 10^{-9} \) and, therefore, the main sequence stars similar to or more massive than the Sun could be self-screened for \( f(R) \) models with \( |f_{R0}| \leq 10^{-6} \). Only low density components like the gaseous disk and low-mass stars, in unscreened halos, are unscreened. This picture of partially-screened galaxy opens a novel opportunity to test \( f(R) \) gravity by examining the different dynamics between their screened and unscreened components \([38]\). However, as pointed out in this work, to accurately separate unscreened galaxies from screened ones in real galaxy surveys, we need to estimate the Newtonian potential \( \phi_N \), considering the galaxies as tracers for the underlying dark matter field. A group

![FIG. 12. Histogram for the well-screened dark halos (\( |\frac{MD}{ML} - 1| < 0.01 \)) with respect to the maximal potential \( -\phi_N \) inside them for the \( f_{R0} = -10^{-6} \) cases. It is clear that almost all the well-screened dark halos lie above the critical potential \( | -\phi_{Nc} | \), and below the threshold \( | -\phi_{Nc} | \) the number counts are very small.](image-url)
Furthermore, the galaxy shear measurements may have the potential of determining the Newtonian potential $\phi_N$, namely the lensing potential, with greatly improved precisions. Upcoming surveys such as Euclid \cite{45} will be able to reconstruct the three-dimensional lensing potential using the weak lensing tomography technique \cite{44}. With these, the unscreened samples in the galaxy surveys can be selected reliably, based on the method presented in this paper. Combining galaxy shear measurements, galaxy surveys and additional observations on the galaxy properties may yield one of the most determinative tests on $f(R)$ gravity in the near future.

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| $f_{R0}$ | $\phi_{N}/c^2 = \phi_{N}/c^2 = \frac{3}{2} f_{R0}$ |
|--------|---------------------------------|
| $-10^{-4}$ | $-1.5 \times 10^{-4}$ |
| $-10^{-5}$ | $-1.5 \times 10^{-5}$ |
| $-10^{-6}$ | $-1.5 \times 10^{-6}$ |

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