Some asymptotic formulae for Bessel process

Yuu Hariya*

Abstract

We recover in part a recent result of [4] on the asymptotic behaviors for tail probabilities of first hitting times of Bessel process. Our proof is based on a weak convergence argument. The same reasoning enables us to derive the asymptotic behaviors for the tail probability of the time at which the global infimum of Bessel process is attained, and for expected values relative to local infima. Some relevant results are also presented. In addition, we give another proof of the result of [4] with improvement of error estimates.

1 Introduction

For every \( \nu \in \mathbb{R} \) and \( a > 0 \), we denote by \( P_a^{(\nu)} \) the law on the space \( C([0, \infty); \mathbb{R}) \) of real-valued continuous paths over \([0, \infty)\), induced by Bessel process with index \( \nu \) (or dimension \( 2(1 + \nu) \)) starting from \( a \). For every \( b \geq 0 \), we denote by \( \tau_b \) the first hitting time to \( b \):

\[
\tau_b(\omega) := \inf\{t \geq 0; \omega(t) = b\}, \quad \omega \in C([0, \infty); \mathbb{R}).
\]

In Hamana-Matsumoto [4], they have shown the following asymptotic formulae for the tail probability of \( \tau_b \) in the case \( b < a \): for every \( \nu > 0 \),

\[
P_a^{(\nu)}(\infty > \tau_b > t) = \frac{1}{(2t)^\nu \Gamma(1 + \nu)} b^{2\nu} \left\{ 1 - \left( \frac{b}{a} \right)^{2\nu} \right\} + O(t^{-\nu-\varepsilon}), \quad (1.1)
\]

\[
P_a^{(-\nu)}(\tau_b > t) = \frac{1}{(2t)^\nu \Gamma(1 + \nu)} a^{2\nu} \left\{ 1 - \left( \frac{b}{a} \right)^{2\nu} \right\} + O(t^{-\nu-\varepsilon}) \quad (1.2)
\]

as \( t \to \infty \) for any \( \varepsilon \in (0, \nu/(1 + \nu)) \). Here \( \Gamma \) is the gamma function. Their proof uses computational estimates.

*Mathematical Institute, Tohoku University, Aoba-ku, Sendai 980-8578, Japan.
E-mail: hariya@math.tohoku.ac.jp

Key Words and Phrases. Bessel process; tail probability; weak convergence.

2010 Mathematical Subject Classification. Primary 60J60; Secondary 60B10.
One of the purposes of this paper is to give a different proof of these two formulae based on a weak convergence argument; while our proof does not give asymptotic estimates for remainder terms as in (1.1) and (1.2), we think that it is straightforward. The same reasoning also provides the asymptotic behaviors for the tail probability of the time at which the global infimum of Bessel process is attained, and for some expected values related to local infima.

We devote the latter half of the paper to another proof of (1.1) and (1.2). The proof is based on an identity for hitting distributions that is an immediate consequence of the strong Markov property of Bessel process. The identity differs from the one used in [3] and makes it possible to do more precise estimates; it will be seen that the remainder terms in fact decay faster than $t^{-\nu-\nu/(1+\nu)}$.

We organize this paper as follows: In Section 2 we first prove Theorem 2.1 which recovers principal terms in (1.1) and (1.2); we reduce the proof to showing that a given sequence of probability measures on $C([0,\infty);\mathbb{R})$ is weakly convergent. This argument also proves Proposition 2.1 which is then applied to derive in Theorem 2.2 the asymptotic behavior for the tail probability of the time Bessel process with positive index attains its global infimum, and those for expected values involving its local infima.

Some relevant results are presented in Remark 2.3 and Proposition 2.2. In Section 3 we prove Theorem 3.1, which improves (1.1) and (1.2); we do this by using an identity for hitting distributions given in Lemma 3.1. Finally in the appendix, we prove auxiliary facts that are referred to in Sections 2 and 3.

In the sequel we write $\Omega$ for $C([0,\infty);\mathbb{R})$. We equip $\Omega$ with the topology of compact uniform convergence. Unless otherwise stated, $R$ denotes the coordinate process on $\Omega$: $R_t(\omega) := \omega(t), \omega \in \Omega, t \geq 0$. We also set

$$\mathcal{F}_t := \sigma(R_s, s \leq t), \quad t \geq 0, \quad \text{and} \quad \mathcal{F} := \bigsqcup_{t \geq 0} \mathcal{F}_t.$$ 

For any $x, y \in \mathbb{R}$, we write $x \vee y = \max\{x, y\}$, $x \wedge y = \min\{x, y\}$. Other notation will be introduced as needed.

## 2 Main results and proofs

Throughout the paper we fix $\nu > 0$ and $a > 0$. One of the objectives of this section is to give a proof of

**Theorem 2.1.** It holds that for every $b \in [0, a)$,

(i) $\lim_{t \to \infty} t^{\nu} P_a^{(\nu)}(\infty > \tau_b > t) = \frac{1}{2^{\nu} \Gamma(1 + \nu)} b^{2\nu} \left\{ 1 - \left( \frac{b}{a} \right)^{2\nu} \right\};$

(ii) $\lim_{t \to \infty} t^{\nu} P_a^{(-\nu)}(\tau_b > t) = \frac{1}{2^{\nu} \Gamma(1 + \nu)} a^{2\nu} \left\{ 1 - \left( \frac{b}{a} \right)^{2\nu} \right\}.$
We start with the following lemma, which plays a key role throughout this section.

**Lemma 2.1.** For every $x > 0$, it holds that as $t \to \infty$,

$$ t^\nu E^{(\nu)}_x \left[ \left( \frac{1}{R_t} \right)^{2\nu} \right] \to \frac{1}{2^\nu \Gamma(1+\nu)}. \quad (2.1) $$

Here and below, $E^{(\nu)}_x$ denotes the expectation with respect to $P^{(\nu)}_x$.

**Proof.** By the scaling property of Bessel process, the left-hand side of (2.1) is equal to

$$ E^{(\nu)}_x \left[ \left( \frac{1}{R_t} \right)^{2\nu} \right] \to \frac{1}{2^\nu \Gamma(1+\nu)}; \quad (2.2) $$

where $I_\nu$ denotes the modified Bessel function of the first kind with index $\nu$. It is known that the function $(0, \infty) \ni z \mapsto z^{-\nu} I_\nu(z)$ is increasing and

$$ \lim_{z \to 0} z^{-\nu} I_\nu(z) = \frac{1}{2^\nu \Gamma(1+\nu)}; $$

moreover, $I_\nu(z)$ grows slower than $e^z$ as $z \to \infty$ (see, e.g., [7, Sections 5.7 and 5.16]). Therefore we may apply the dominated convergence theorem to, say, $t \geq 1$, to conclude that (2.2) converges to $(2^\nu \Gamma(1+\nu))^{-1}$ as $t \to \infty$. This ends the proof.

**Remark 2.1.** In fact, the expectation $E^{(\nu)}_x [(R_t)^{-2\nu}]$ admits a simple representation

$$ E^{(\nu)}_x \left[ \left( \frac{1}{R_t} \right)^{2\nu} \right] = \frac{1}{2^\nu \Gamma(\nu)} \int_t^\infty \frac{ds}{s^{1+\nu}} \exp \left( -\frac{x^2}{2s} \right); $$

see Remark A.1 in Appendix. One may easily derive (2.1) also from this.

For each $t \geq 0$, we set

$$ I_t \equiv I_t(R) := \inf_{0 \leq s \leq t} R_s. $$

We also write $I_\infty$ for $\inf_{t \geq 0} R_t$. Recall that for every $x > 0$ and $0 \leq y \leq x$,

$$ P^{(\nu)}_x (I_\infty > y) = 1 - \left( \frac{y}{x} \right)^{2\nu}. \quad (2.3) $$

The same as [4], we also use the following identity:

**Lemma 2.2.** For every $b \in [0, a)$ and $t > 0$, it holds that

$$ P^{(\nu)}_a (I_t > b) = 1 - \left( \frac{b}{a} \right)^{2\nu} + E^{(\nu)}_a \left[ \left( \frac{b}{R_t} \right)^{2\nu} ; I_t > b \right]. \quad (2.4) $$
Proof. By the Markov property of Bessel process and (2.3),

\[ P_a(I_{\infty} > b \mid \mathcal{F}_t) = P_{x}(y \wedge I_{\infty} > b \mid (R_t, I_t)) \]

\[ = 1_{(I_t > b)} \left\{ 1 - \left( \frac{b}{R_t} \right)^{2 \nu} \right\} \]

\[ P_a^{(\nu)} \text{-a.s. Taking the expectation on both sides leads to (2.4).} \]

Since \( P_a^{(\nu)}(\tau_b > t) = P_a^{(\nu)}(I_t > b) \) and \( P_a^{(\nu)}(\tau_b = \infty) = P_a^{(\nu)}(I_{\infty} > b) = 1 - (b/a)^{2 \nu} \) by (2.3), we have from (2.4)

\[ P_a^{(\nu)}(\infty > \tau_b > t) = E_a^{(\nu)} \left[ \left( \frac{b}{R_t} \right)^{2 \nu} \mid I_t > b \right]. \quad (2.5) \]

We are prepared to prove Theorem 2.1.

Proof of Theorem 2.1. (i) Fix arbitrarily a strictly increasing sequence \( \{t_n\}_{n \in \mathbb{N}} \subset (0, \infty) \) such that \( \lim_{n \to \infty} t_n = \infty \). For each \( n \), we define the probability measure \( \tilde{P}_n \) on \( \Omega \) by

\[ \tilde{P}_n(A) := \frac{E_a^{(\nu)}[(R_{t_n})^{-2 \nu} \mid (R_{t_n})^{-2 \nu} \in A]}{E_a^{(\nu)}[(R_{t_n})^{-2 \nu}]} \], \quad A \in \mathcal{F}, \]

where \( R_{t_n} := R_{t \wedge t_n}, \ t \geq 0 \).

First we show that \( \{ \tilde{P}_n \}_{n \in \mathbb{N}} \) is tight. Fix \( t > 0 \) and take \( A' \in \mathcal{F}_t \). If we let \( n \) be such that \( t_n \geq t \), then by the Markov property,

\[ \frac{E_a^{(\nu)}[(R_{t_n})^{-2 \nu} \mid A']}{{E_a^{(\nu)}[(R_{t_n})^{-2 \nu}]} = E_a^{(\nu)}[M_n \mid A'], \quad (2.6) \]

where we set

\[ M_n = \frac{E_a^{(\nu)}[(R_{t_n-t})^{-2 \nu}]}{E_a^{(\nu)}[(R_{t_n})^{-2 \nu}]}]. \]

By Lemma 2.1 \( M_n \to 1 \) a.s. as \( n \to \infty \). Moreover, \( M_n \geq 0 \) a.s. and \( \lim_{n \to \infty} E_a^{(\nu)}[M_n] = 1 \). Hence by Scheffé’s lemma, \( E_a^{(\nu)}[M_n - 1] \xrightarrow{n \to \infty} 0 \), from which it follows that (2.6) converges to \( P_a^{(\nu)}(A') \) as \( n \to \infty \). In particular, regarding each \( \tilde{P}_n \) as being defined on the path space \( \Omega_t = C([0, t]; \mathbb{R}) \) equipped with the uniform norm topology, we see that \( \{ \tilde{P}_n \}_{n \in \mathbb{N}} \) is tight as a family of probability measures on \( \Omega_t \), which is equivalent to

\[ \limsup_{\delta \downarrow 0} \sup_{n \in \mathbb{N}} \tilde{P}_n \left( \omega \in \Omega_t; \sup_{|u-v| \leq \delta} \sup_{0 \leq u, v \leq t} |\omega(u) - \omega(v)| > \varepsilon \right) = 0 \]
for any $\varepsilon > 0$ (see, e.g., [5, Theorem 2.4.10]). It is then clear that, with $\Omega_t$ replaced by $\Omega$, (2.7) holds for any $t > 0$ and $\varepsilon > 0$, and hence the tightness of $\{\tilde{P}_n\}_{n \in \mathbb{N}}$ follows.

As $t > 0$ is arbitrary, convergence of (2.6) also implies that $\{\tilde{P}_n\}_{n \in \mathbb{N}}$ converges weakly to $P^{(\nu)}_a$ in the sense of finite-dimensional distributions. Consequently, $\{\tilde{P}_n\}_{n \in \mathbb{N}}$ converges weakly to $P^{(\nu)}_a$. Since $P^{(\nu)}_a(I_{\infty} = b) = 0$ by (2.3), the weak convergence entails that

$$\tilde{P}_n(I_{\infty} > b) \xrightarrow{n \to \infty} P^{(\nu)}_a(I_{\infty} > b).$$

By the definition of $\tilde{P}_n$, the left-hand side of (2.8) is equal to

$$\frac{E^{(\nu)}_a[(R_t)^{-2\nu}; I_t > b]}{E^{(\nu)}_a[(R_t)^{-2\nu}]} E^{(\nu)}_a[(R_t)^{-2\nu}; I_t > b].$$

As the sequence $\{t_n\}_{n \in \mathbb{N}}$ is arbitrarily taken, we now conclude that

$$\lim_{t \to \infty} \frac{E^{(\nu)}_a[(R_t)^{-2\nu}; I_t > b]}{E^{(\nu)}_a[(R_t)^{-2\nu}]} = P^{(\nu)}_a(I_{\infty} > b),$$

which proves (i) by Lemma 2.1, (2.3) and (2.5).

(ii) Using the absolute continuity relationship between $P^{(\nu)}_a$ and $P^{(-\nu)}_a$ (see Proposition A.1), we have

$$P^{(-\nu)}_a(\tau_b > t) \equiv P^{(-\nu)}_a(I_t > b) = E^{(\nu)}_a\left[\left(\frac{a}{R_t}\right)^{2\nu}; I_t > b\right].$$

The assertion follows from this and (2.9).

As for the convergence of (2.6) to $P^{(\nu)}_a(A')$ for every $A' \in \mathcal{F}_t$, see also Proposition 2.2 below.

The same reasoning as the proof of Theorem 2.1 (i) also yields the

**Proposition 2.1.** For any continuous function $f : \mathbb{R} \to \mathbb{R}$, we have

$$\lim_{t \to \infty} t^\nu E^{(\nu)}_a[f(I_t)(R_t)^{-2\nu}] = \frac{2\nu}{2^\nu a^{2\nu} \Gamma(1 + \nu)} \int_0^a z^{2\nu - 1} f(z) \, dz.$$

**Proof.** We keep the notation in the proof of Theorem 2.1 (i). Note that the mapping

$$\Omega \ni \omega \mapsto \inf_{t \geq 0} \{a \wedge (\omega(t) \vee 0)\} =: I^{a,+}(\omega)$$

is bounded and continuous, and that $I_{\infty} = I^{a,+}(R) P^{(\nu)}_a$-a.s. As $\{\tilde{P}_n\}_{n \in \mathbb{N}}$ converges weakly to $P^{(\nu)}_a$, we have for any continuous function $f$ on $\mathbb{R}$,

$$\int_{\Omega} f(I^{a,+}(R)) \, d\tilde{P}_n \xrightarrow{n \to \infty} E^{(\nu)}_a[f(I_{\infty})].$$
By the definition of $\hat{P}_n$, the left-hand side is equal to

$$\frac{E_{\alpha}^{(\nu)}[f(I_n)(R_{t_n})^{-2\nu}]}{E_{\alpha}^{(\nu)}[(R_{t_n})^{-2\nu}]}.$$ 

The rest of the proof proceeds in the same way as the proof of Theorem 2.1(i). □

As an application of this proposition, we may prove further the following asymptotic formulae: We set

$$\rho_\infty := \inf\{t \geq 0; R_t = I_\infty\};$$

as we will see in Proposition A.2 below, $\rho_\infty$ is a.s. the unique time at which the global infimum $I_\infty$ is attained.

**Theorem 2.2.** (i) For any $0 \leq b \leq a$, it holds that

$$\lim_{t \to \infty} t^{\nu} P_{a}^{(\nu)}(I_t - I_\infty > b) = \frac{2\nu}{2^\nu a^{2\nu}\Gamma(1+\nu)} \int_b^a z^{2\nu - 1}(z - b)^{2\nu} dz. \quad (2.11)$$

In particular,

$$\lim_{t \to \infty} t^{\nu} P_{a}^{(\nu)}(\rho_\infty > t) = \frac{a^{2\nu}}{2^{1+\nu}\Gamma(1+\nu)}. \quad (2.12)$$

(ii) For any continuous function $g: \mathbb{R} \to \mathbb{R}$, it holds that

$$\lim_{t \to \infty} t^{\nu} \{ P_{a}^{(\nu)}[g(I_\infty)] - P_{a}^{(\nu)}[g(I_t)] \} = \frac{2\nu}{2^\nu a^{2\nu}\Gamma(1+\nu)} \int_0^a (a^{2\nu} - 2z^{2\nu})^{2\nu - 1} g(z) dz. \quad (2.13)$$

**Proof.** (i) By the Markov property of Bessel process and by (2.3),

$$P_{a}^{(\nu)}(I_t - I_\infty > b) = E_{a}^{(\nu)} \left[ P_{x}^{(\nu)}(y - y \wedge I_\infty > b) \right]_{(x,y) = (R_t, I_t)}$$

$$= E_{a}^{(\nu)} \left[ (I_t - b)^{2\nu} / (R_t)^{2\nu}; I_t > b \right]. \quad (2.13)$$

Taking $f(z) = \{(z - b) \lor 0\}^{2\nu}$ in Proposition 2.1 leads to (2.11). The latter equality (2.12) follows from taking $b = 0$ in (2.11); indeed, as seen in the proof of Proposition A.2, one has $P_{a}^{(\nu)}(\rho_\infty > t) = P_{a}^{(\nu)}(I_t > I_\infty)$.

(ii) Again by the Markov property,

$$E_{a}^{(\nu)} [g(I_\infty), F_t] = E_{x}^{(\nu)} [g(y \wedge I_\infty)]_{(x,y) = (R_t, I_t)} P_{a}^{(\nu)} \text{ - a.s.}$$

for every $t > 0$. By (2.3), the $P_{x}^{(\nu)}$-expectation on the right-hand side is calculated as

$$g(y) + \frac{h(y)}{x^{2\nu}}, \quad h(y) := 2\nu \int_0^y z^{2\nu - 1} g(z) dz - y^{2\nu} g(y).$$

Hence we have

$$E_{a}^{(\nu)} [g(I_\infty)] - E_{a}^{(\nu)} [g(I_t)] = E_{a}^{(\nu)} \left[ h(I_t) / (R_t)^{2\nu} \right].$$

Taking $f = h$ in Proposition 2.1 concludes the proof. □
We give a remark on Theorem 2.2 (ii).

Remark 2.2. (1) We may allow the function $g$ to have the set of discontinuity with Lebesgue measure 0; in particular, taking $g = 1_{(b, \infty)}$ recovers Theorem 2.1 (i).

(2) For the function $h$ defined in the proof, the process

$$g(I_t) + \frac{h(I_t)}{(R_t)^{2\nu}}, \quad t \geq 0,$$

is, by definition, a uniformly integrable $\{F_t\}$-martingale under $P_\nu^{(\nu)}$, which may be associated with the so-called Azéma-Yor martingales (see [1]); in fact, $\{(R_t)^{-2\nu}; t \geq 0\}$ is an $\{F_t\}$-local martingale and $\sup_{0 \leq s \leq t} (R_s)_{-2\nu} = (I_t)^{-2\nu}$.

We may also relate (2.12) to Theorem 2.1 (ii) in the following manner:

Proof of (2.12) via Theorem 2.1 (ii). Note that by taking $b = 0$ in (2.13),

$$P_a^{(\nu)}(\rho_\infty > t) = P_a^{(\nu)}(I_t > I_\infty)$$

$$= E_a^{(\nu)} \left( \frac{I_t}{R_t} \right)^{2\nu}.$$

By the absolute continuity relation Proposition A.1 and by Fubini’s theorem, this is rewritten as

$$E_a^{(-\nu)} \left[ \left( \frac{I_t}{a} \right)^{2\nu}; t < \tau_0 \right] = \frac{2\nu}{a^{2\nu}} \int_0^a z^{2\nu-1} P_a^{(-\nu)}(I_t > z) \, dz$$

$$= \frac{2\nu}{a^{2\nu}} \int_0^a z^{2\nu-1} P_a^{(-\nu)}(\tau_z > t) \, dz.$$

For every $z \in (0, a)$ we have by Theorem 2.1 (ii) and (2.10),

$$a^{2\nu} \geq \frac{P_a^{(-\nu)}(\tau_z > t)}{E_a^{(\nu)}[(R_t)^{-2\nu}]} \xrightarrow{t \to \infty} a^{2\nu} \left\{1 - \left( \frac{z}{a} \right)^{2\nu} \right\},$$

and hence the bounded convergence theorem yields

$$\lim_{t \to \infty} \frac{P_a^{(\nu)}(\rho_\infty > t)}{E_a^{(\nu)}[(R_t)^{-2\nu}]} \xrightarrow{t \to \infty} 2\nu \int_0^a z^{2\nu-1} \left\{1 - \left( \frac{z}{a} \right)^{2\nu} \right\} \, dz$$

$$= \frac{1}{2} a^{2\nu}.$$

This shows (2.12) by Lemma 2.1.

Remark 2.3. In the above proof, we have just observed the identity

$$P_a^{(\nu)}(\rho_\infty > t) = \frac{2\nu}{a^{2\nu}} \int_0^a z^{2\nu-1} P_a^{(-\nu)}(\tau_z > t) \, dz,$$

(2.14)
which may easily be extended, thanks to the Markov property, to
\[
P_a^{(-\nu)}(A \cap \{\rho_\infty > t\}) = \frac{2\nu}{a^{2\nu}} \int_0^a z^{2\nu-1} P_a^{(-\nu)}(A \cap \{\tau_z > t\}) \, dz
\]
for any \( A \in \mathcal{F}_t \). This relation shows that the process \( \{R_t; 0 \leq t \leq \rho_\infty\} \) under \( P_a^{(-\nu)} \) is identical in law with \( \{\xi_t; 0 \leq t \leq \tau_{Z(\xi)}\} \), where \( \xi \) is a Bessel process with index \(-\nu\) starting from \( a \) and \( Z \) is a random variable independent of \( \xi \) and distributed as \((2\nu/a^{2\nu})z^{2\nu-1} \, dz\), \( z \in (0, a) \). In the case \( \nu = 1/2 \), this partly recovers the path decomposition of 3-dimensional Bessel process due to D. Williams (e.g., [10, Theorem VI.3.11]).

We also send the reader to [3, Corollary 4.14] for identities as (2.14) in a general framework of diffusion processes.

We conclude this section by pointing out the following fact which, together with Proposition A.1, the proof of Theorem 2.1 (i) also indicates.

**Proposition 2.2.** Let \( s > 0 \). Then for any \( A \in \mathcal{F}_s \), we have
\[
\lim_{t \to \infty} P_a^{(-\nu)}(A \mid \tau_0 > t) = P_a^{(\nu)}(A).
\]

The proposition asserts that Bessel process with a negative index conditioned to stay positive is nothing but Bessel process with the opposite index. This seems to be a well-known fact and to have been rediscovered by several authors, see e.g., [12, Section 7]; we also refer to [2] for the case of drifted Brownian motions with nonsingular drift coefficients. The case \( \nu = 1/2 \) goes back to Knight [6, Theorem 3.1]. Roynette, Yor et al. extensively studied limit laws of Brownian motion normalized by various kinds of weight processes other than \( 1_{\{\tau_0 > t\}} \), referring to those studies as *penalisation problems*; see [11] and references therein, where usage of Scheffé’s lemma as in the proof of Theorem 2.1 (i) is also found. For related studies concerning quasi-stationary distributions (Yaglom limits), refer to [9].

**Proof of Proposition 2.2.** Take \( t > s \). Then by Proposition A.1
\[
P_a^{(-\nu)}(A \mid \tau_0 > t) = \frac{E_a^{(\nu)}[(R_t)^{-2\nu}; A]}{E_a^{(\nu)}[(R_t)^{-2\nu}]},
\]
to which the same argument as in the proof of Theorem 2.1 (i) applies.

\[\square\]

### 3 Asymptotic estimates for remainders

Independently of the argument used in the previous section, we prove in this section the following theorem, which improves (1.1) and (1.2):

Theorem 3.1. For every \( b \in [0, a) \), it holds that as \( t \to \infty \),

\[
\begin{align*}
(i) \quad P_a^{(\nu)}(\infty > \tau_b > t) &= \frac{1}{(2t)^\nu \Gamma(1 + \nu)} b^{2\nu} \left\{ 1 - \left( \frac{b}{a} \right)^{2\nu} \right\} + o(t^{-\nu - \frac{\nu}{1+\nu}}); \\
(ii) \quad P_a^{(-\nu)}(\tau_b > t) &= \frac{1}{(2t)^\nu \Gamma(1 + \nu)} a^{2\nu} \left\{ 1 - \left( \frac{b}{a} \right)^{2\nu} \right\} + o(t^{-\nu - \frac{\nu}{1+\nu}}).
\end{align*}
\]

Notice that by (2.5) and (2.10),

\[
P_a^{(\nu)}(\infty > \tau_b > t) = \left( \frac{b}{a} \right)^{2\nu} P_a^{(-\nu)}(\tau_b > t).
\]

Therefore we only need to prove the assertion (ii). The proof utilizes the following relation for hitting distributions:

Lemma 3.1. It holds that for every \( 0 \leq b \leq a \) and \( t > 0 \),

\[
P_a^{(-\nu)}(\tau_b > t) = \frac{1}{P_b^{(-\nu)}(\tau_0 \leq t)} \left\{ P_a^{(-\nu)}(\tau_0 > t) - P_b^{(-\nu)}(\tau_0 > t) - \int_{D_t} P_a^{(-\nu)}(\tau_b \in ds) P_b^{(-\nu)}(\tau_0 \in du) \right\},
\]

where

\[
D_t := \{(s, u) \in (0, \infty)^2; s + u > t, s \leq t, u \leq t\}.
\]

Proof. By the strong Markov property,

\[
P_a^{(-\nu)}(\tau_0 > t, \tau_b \leq t) = \int_0^t P_a^{(-\nu)}(\tau_0 > t) P_b^{(-\nu)}(\tau_0 \in du) ds.
\]

Rewriting this leads to the desired identity. \( \square \)

Prior to the proof of Theorem 3.1, we recover Theorem 2.1 (ii) using Lemma 3.1. Set

\[
I(t) := \frac{P_a^{(-\nu)}(\tau_0 > t) - P_b^{(-\nu)}(\tau_0 > t)}{P_b^{(-\nu)}(\tau_0 \leq t)}, \quad J(t) := \int_{D_t} P_a^{(-\nu)}(\tau_b \in ds) P_b^{(-\nu)}(\tau_0 \in du)
\]

so that

\[
P_a^{(-\nu)}(\tau_b > t) = I(t) - \frac{1}{P_b^{(-\nu)}(\tau_0 \leq t)} J(t)
\]

by Lemma 3.1. Since we have the expression

\[
P_b^{(-\nu)}(\tau_0 > t) = \frac{x^{2\nu}}{2^\nu \Gamma(\nu)} \int_t^\infty ds \exp \left( -\frac{x^2}{2s} \right), \quad t \geq 0,
\]

(3.3)
for every \( x \geq 0 \) (see Remark A.1), it is immediate that

\[
\lim_{t \to \infty} t^{\nu} I(t) = \frac{a^{2\nu} - b^{2\nu}}{2\nu \Gamma(1 + \nu)}.
\]

Therefore in order to recover Theorem 2.1(ii), it suffices to show that

\[
\lim_{t \to \infty} t^{\nu} J(t) = 0.
\]

To this end, take \( t, \lambda > 0 \) in such a way that \( t > \lambda \). Because of the inclusion

\[
D_t \subset \{(s, u) \in (0, \infty)^2; t + \lambda \geq s + u > t\}
\]

\[
\cup \{(s, u) \in (0, \infty)^2; s + u \geq t + \lambda, s \leq t, u \leq t\},
\]

we have

\[
J(t) \leq P^{(-\nu)}(t + \lambda \geq \tau_0 > t) + \int_t^t \lambda P^{(-\nu)}(\tau_0 \in du) P^{(-\nu)}(t \geq \tau_0 > t + \lambda - u)
\]

\[
=: J_1(t; \lambda) + J_2(t; \lambda),
\]

(3.4)

where the expression of \( J_1 \) is due to the strong Markov property. By (3.3) we have

\[
J_1(t; \lambda) \leq \frac{a^{2\nu}}{2\nu \Gamma(\nu)} \int_t^{t+\lambda} ds \frac{ds}{s^{1+\nu}} \left\{ \left( \frac{1}{t} \right)^{1+\nu} \right\}.
\]

Since there exists a positive constant \( C \) such that \( 1 - x^{\nu} \leq C(1 - x) \) for all \( 0 \leq x \leq 1 \), we obtain an estimate

\[
J_1(t; \lambda) \leq C_1 \frac{\lambda}{t^{1+\nu}}.
\]

(3.5)

Here and below every \( C_i \) denotes a positive constant dependent only on \( a, b \) and \( \nu \). As for \( J_2 \), we use (3.3) to rewrite

\[
J_2(t; \lambda) = \frac{b^{2\nu}}{2\nu \Gamma(\nu)} \int_t^t \lambda \frac{du}{(t + \lambda - u)^{1+\nu}} \exp \left\{ -\frac{b^2}{2(t + \lambda - u)} \right\} P^{(-\nu)}(t \geq \tau_b > u).
\]

(3.6)

Since \( \tau_b \leq \tau_0 \ P_a^{(-\nu)} \)-a.s.,

\[
P^{(-\nu)}(t \geq \tau_b > u) \leq P^{(-\nu)}(\tau_0 > u)
\]

\[
\leq \frac{a^{2\nu}}{2\nu \Gamma(\nu) u^{\nu}}
\]

by (3.3). We substitute this estimate into (3.6) to obtain a bound

\[
J_2(t; \lambda) \leq C_2 \int_t^t \lambda \frac{du}{u^{\nu}(t + \lambda - u)^{1+\nu}}.
\]

(3.7)
We now fix \( \varepsilon \in (0, 1) \) arbitrarily and let \( \lambda = \varepsilon t \). Then by (3.5),
\[
\limsup_{t \to \infty} t^{\nu} J_1(t; \varepsilon t) \leq C_1 \varepsilon.
\]
On the other hand, by (3.7),
\[
J_2(t; \varepsilon t) \leq \frac{C_2}{(\varepsilon t)^{1+\nu}} \int_{\varepsilon t}^t \frac{du}{u^{\nu}},
\]
whence
\[
\lim_{t \to \infty} t^{\nu} J_2(t; \varepsilon t) = 0.
\]
Combining these with (3.4), we have
\[
\limsup_{t \to \infty} t^{\nu} J(t) \leq C_1 \varepsilon.
\]
This shows Theorem 2.1 (ii) as \( \varepsilon \) is arbitrary.

We proceed to the proof of Theorem 3.1. We begin with the following lemma:

**Lemma 3.2.** One has for every \( x > 0 \) and \( t > 0 \),
\[
P_x^{(-\nu)}(\tau_0 > t) = \frac{x^{2 \nu}}{(2t)^\nu \Gamma(1 + \nu)} \exp \left( -\frac{x^2}{2t} \right) + P_x^{(-\nu-1)}(\tau_0 > t).
\]

**Proof.** By integration by parts,
\[
\int_1^\infty \frac{ds}{s^{1+\nu}} \exp \left( -\frac{x^2}{2s} \right) = \frac{1}{\nu t^{\nu}} \exp \left( -\frac{x^2}{2t} \right) + \frac{x^2}{2\nu} \int_t^\infty \frac{ds}{s^{2+\nu}} \exp \left( -\frac{x^2}{2s} \right).
\]
Plugging this expression into (3.3), we obtain the equality. \( \square \)

Using this lemma, we divide \( I(t) \) into three parts:
\[
I(t) = I_1(t) + I_2(t) + I_3(t),
\]
where we set
\[
I_1(t) = \frac{1}{(2t)^\nu \Gamma(1 + \nu)} \left\{ a^{2 \nu} \exp \left( -\frac{a^2}{2t} \right) - b^{2 \nu} \exp \left( -\frac{b^2}{2t} \right) \right\},
\]
\[
I_2(t) = \frac{\nu x^{2 \nu} \Gamma(2 + \nu)}{(2t)^{1+\nu} \Gamma(1 + \nu)} \left\{ a^{2 \nu} \exp \left( -\frac{a^2}{2t} \right) - b^{2 \nu} \exp \left( -\frac{b^2}{2t} \right) \right\},
\]
\[
I_3(t) = \frac{\nu \rho \Gamma(2 + \nu)}{(2t)^{2\nu} \Gamma(1 + \nu)} \left\{ a^{2 \nu} \exp \left( -\frac{a^2}{2t} \right) - b^{2 \nu} \exp \left( -\frac{b^2}{2t} \right) \right\}.
\]
Using the fact that \( (1 - e^{-x})/x \xrightarrow{x \to 0} 1 \) for \( I_1 \) and \( I_3 \) for \( I_2 \) and \( I_3 \), we see that
\[
I_1(t) = \frac{a^{2 \nu} - b^{2 \nu}}{(2t)^\nu \Gamma(1 + \nu)} - \frac{a^{2 \nu+2} - b^{2 \nu+2}}{(2t)^{1+\nu} \Gamma(1 + \nu)} + o(t^{-\nu-1}),
\]
\[
I_2(t) = \frac{a^{2 \nu+2} - b^{2 \nu+2}}{(2t)^{1+\nu} \Gamma(2 + \nu)} + o(t^{-\nu-1}),
\]
\[
I_3(t) = \frac{b^{2 \nu}(a^{2 \nu} - b^{2 \nu})}{(2t)^{2\nu} \Gamma(1 + \nu)^2} + o(t^{-2\nu}).
\]
We put together these asymptotics into a proposition.
Proposition 3.1. It holds that as $t \to \infty$,

$$I(t) = \frac{a^{2\nu} - b^{2\nu}}{(2t)^\nu \Gamma(1 + \nu)} + O(t^{-\kappa})$$

with $\kappa = (2\nu) \wedge (1 + \nu)$.

In view of (3.2), this proposition reduces the proof of Theorem 3.1 to that of Proposition 3.2.

Proposition 3.2. It holds that

$$\lim_{t \to \infty} t^{\nu + \alpha} J(t) = 0,$$

where $\alpha := 1/(1 + \nu)$.

Once Proposition 3.2 is shown, the proof of Theorem 3.1 is straightforward:

Proof of Theorem 3.1. Since $\nu + \alpha < (2\nu) \wedge (1 + \nu)$,

$$\lim_{t \to \infty} t^{\nu + \alpha} \left\{ I(t) - \frac{a^{2\nu} - b^{2\nu}}{(2t)^\nu \Gamma(1 + \nu)} \right\} = 0$$

by Proposition 3.1. Combining this with Proposition 3.2 and the identity (3.2) leads to (ii). The assertion (i) follows from (ii) and the relation (3.1). \qed

It remains to prove Proposition 3.2. Recall (3.4).

Proof of Proposition 3.2. Fix $\gamma > 0$ arbitrarily and set

$$C(\gamma) = \frac{a^{2\nu} - b^{2\nu}}{2\nu \Gamma(1 + \nu)} + \gamma.$$

Then by Theorem 2.1(ii), we may pick $\lambda$ so that

$$P_a\{\tau_b > u\} \leq C(\gamma) u^{-\nu} \text{ for all } u \geq \lambda.$$  \(\text{(3.8)}\)

By this estimate and (3.6), we bound $J_2(t; \lambda)$ from above in such a way that

$$J_2(t; \lambda) \leq \frac{b^{2\nu}}{2\nu \Gamma(\nu)} \left\{ C(\gamma) K_1(t; \lambda) - K_2(t; \lambda) P_a\{\tau_b > t\} \right\},$$  \(\text{(3.9)}\)

where we set

$$K_1(t; \lambda) = \int_{\lambda}^{t} \frac{du}{u^{\nu}(t + \lambda - u)^{\nu+1}}, \quad K_2(t; \lambda) = \int_{\lambda}^{t} \frac{du}{u^{1+\nu}} \exp \left( -\frac{b^2}{2u} \right).$$

By Lemma A.1 in Appendix,

$$K_1(t; \lambda) = \int_{\lambda}^{t} \frac{1}{\lambda^{\nu}(t + \lambda)^{2\nu}} \int_{\lambda}^{t} \frac{(u + \lambda)^{2\nu}}{u^{1+\nu}} \frac{du}{u^{\nu-1}} \leq \frac{1}{\lambda^{\nu} t^{2\nu}} \int_{\lambda}^{t} \left\{ u^{\nu-1} + C_3 \left( \lambda u^{\nu-2} + \frac{\lambda^{2\nu}}{u^{1+\nu}} \right) \right\} du.$$  \(\text{(3.10)}\)
As for $K_2$, 

$$K_2(t; \lambda) \geq \exp \left( -\frac{b^2}{2\lambda} \right) \cdot \frac{1}{\nu} \left( \frac{1}{\lambda^\nu} - \frac{1}{\nu} \right) \geq \frac{1}{\nu} \left( 1 - \frac{b^2}{2\lambda} \right) \left( \frac{1}{\lambda^\nu} - \frac{1}{\nu} \right). \tag{3.11}$$

Now we fix $\varepsilon > 0$ arbitrarily and take $\lambda = \varepsilon t^\alpha$ with $t$ sufficiently large so that (3.8) is valid. By (3.10), (3.11) and Theorem 2.1 (ii), we deduce readily from (3.9) that 

$$J_2(t; \varepsilon t^\alpha) \leq \frac{b^{2\nu}}{2\nu \Gamma(\nu) \nu \varepsilon^\nu} \left( C(\gamma) - \frac{a^{2\nu} - b^{2\nu}}{2\nu \Gamma(1 + \nu)} \right) \frac{1}{t^{\nu+\nu\alpha}} + o(t^{-\nu-\nu\alpha}),$$

and hence by the definition of $C(\gamma)$,

$$\limsup_{t \to \infty} t^{\nu+\nu\alpha} J_2(t; \varepsilon t^\alpha) \leq \frac{b^{2\nu}}{2\nu \Gamma(\nu) \nu \varepsilon^\nu \gamma}.$$

As $\gamma$ is arbitrary, we have 

$$\lim_{t \to \infty} t^{\nu+\nu\alpha} J_2(t; \varepsilon t^\alpha) = 0.$$

On the other hand, by (3.5),

$$\limsup_{t \to \infty} t^{1+\nu-\alpha} J_1(t; \varepsilon t^\alpha) \leq C_1 \varepsilon.$$

Combining these with (3.4) and noting $1 + \nu - \alpha = \nu + \nu\alpha$, we obtain

$$\limsup_{t \to \infty} t^{\nu+\nu\alpha} J(t) \leq C_1 \varepsilon \xrightarrow{\varepsilon \to 0} 0$$

as claimed.

We close this section with a remark on Theorem 3.1 and Proposition 3.2.

**Remark 3.1.** (1) Nonuse of Theorem 2.1 (ii) in the proof of Proposition 3.2 results in the weaker statement that the remainder term decays at the order of $t^{-\nu-\nu/(1+\nu)}$.

(2) Although we do not give details here, we may also prove that

$$\liminf_{t \to \infty} \theta_\nu(t) J(t) > 0,$$

where

$$\theta_\nu(t) = \begin{cases} 
\frac{t^2}{\log t} & (\nu = 1), \\
\frac{t^2}{1+\nu} & (\nu > 1).
\end{cases}$$

It seems plausible that

$$P_a^{(\nu)}(\tau_b > t) = \frac{a^{2\nu} - b^{2\nu}}{(2t)^\nu \Gamma(1 + \nu)} + O(1/\theta_\nu(t))$$

however, we do not have a proof at present.
Appendix

We append proofs of auxiliary facts referred to in preceding sections.

**Proposition A.1.** Let $\nu > 0$ and $a > 0$. Then it holds that for every $t > 0$,

$$P_a^{(\nu)}|_{\mathcal{F}_t} = \left( \frac{R_t}{a} \right)^{2\nu} P_a^{(-\nu)}|_{\mathcal{F}_t \cap \{ t < \tau_0 \}}. \quad (A.1)$$

This relationship seems well known; however, as far as we know, proofs are not found in any literature. For the reader’s convenience, we give a proof here. The absolute continuity relationship for Bessel processes with nonnegative indices is shown in [13].

**Proof of Proposition A.1.** Fix $a > 0$ and set $b = \log a$. Let $B = \{ B_t; t \geq 0 \}$ be a one-dimensional Brownian motion starting from $b$. For each $\mu \in \mathbb{R}$, we denote by $B^{(\mu)}_t$ the Brownian motion with drift $\mu$: $B^{(\mu)}_t = B_t + \mu t, t \geq 0$. Let $X$ denote the coordinate process on $\Omega = C([0, \infty); \mathbb{R})$. We define two functionals $A, \alpha$ of $X$ by

$$A_t(X) := \int_0^t e^{2X_s} ds, \quad \alpha_t(X) := \inf\{ s \geq 0; A_s(X) > t \}, \quad t \geq 0,$$

where we set $\inf\emptyset = \infty$. By Lamperti’s relation (see, e.g., [8, Section 3]), there exists a Bessel process $R^{(\mu)}$ with index $\mu$ starting from $a$ such that

$$\exp B^{(\mu)}_t = R^{(\mu)}_{A_t(B^{(\mu)})}, \quad t \geq 0, \quad (A.2)$$

and hence by the definition of $\alpha$,

$$\exp B^{(\mu)}_{\alpha_t(B^{(\mu)})} = R^{(\mu)}_t, \quad t < \tau_0(R^{(\mu)}). \quad (A.3)$$

Recall that $\tau_0(R^{(\mu)}) = \infty$ a.s. for $\mu \geq 0$ while $\tau_0(R^{(\mu)}) < \infty$ a.s. for $\mu < 0$; in fact,

$$\tau_0(R^{(\mu)}) = A_\infty(B^{(\mu)}) \quad (A.4)$$

by (A.2). We now fix $t > 0$ and take $\Gamma \in \mathcal{F}_t = \sigma(X_s, s \leq t)$. Let $\nu > 0$. Then by (A.3),

$$P \left( R^{(\nu)} \in \Gamma \right) = P \left( \exp B^{(\nu)}_{\alpha_t(B^{(\mu)})} \in \Gamma \right). \quad (A.5)$$

By definition, $\alpha_t(X)$ is a stopping time for the coordinate process $X$. Therefore the Cameron-Martin formula entails that (A.5) is equal to

$$e^{-2\nu b} \mathbb{E} \left[ \exp \left\{ 2\nu B_{\alpha_t(B^{(\mu)})}^{(\nu)} \right\}; \exp B^{(\nu)}_{\alpha_t(B^{(\mu)})} \in \Gamma, \alpha_t(B^{(\nu)}) < \infty \right]$$

$$= a^{-2\nu} \mathbb{E} \left[ \left( R^{(-\nu)}_t \right)^{2\nu}; R^{(-\nu)} \in \Gamma, \tau_0(R^{(-\nu)}) > t \right].$$

Here for the second line, we used (A.3), and the equivalence between $\alpha_t(B^{(-\nu)}) < \infty$ and $\tau_0(R^{(-\nu)}) > t$ that follows from (A.4). The proof is complete. \qed
Remark A.1. By (A.4) and Dufresne’s identity (see, e.g., [9, Section 2]), it holds that under $P_{a}^{(-\nu)}$,

$$
\tau_0(R) \overset{(d)}{=} \frac{a^2}{2\gamma_\nu}.
$$

Here $\gamma_\nu$ is a gamma random variable with parameter $\nu$. Therefore one may find that

$$
E_{a}^{(\nu)} \left( \left( \frac{a}{R_t} \right)^{2\nu} \right) = P_{a}^{(-\nu)}(\tau_0 > t)
$$

$$
= P(\gamma_\nu < a^2/(2t))
$$

$$
= \frac{a^{2\nu}}{2^{\nu} \Gamma(\nu)} \int_{t}^{\infty} ds \frac{1}{s^{1+\nu}} \exp\left( -\frac{a^2}{2s} \right),
$$

where the first equality follows from (A.1).

Proposition A.2. Let $\nu > 0$ and $a > 0$. Under $P_{a}^{(\nu)}$, the time at which the Bessel process $R$ attains its global infimum $I_\infty$ is a.s. unique.

Proof. Write $T \equiv T(R) = \{ s \geq 0; R_s = I_\infty \}$. Set $\rho_\infty = \inf T$ as in Section 2 and $\bar{\rho}_\infty = \sup T$. Note that $T$ is compact a.s. since $R$ is continuous and $\lim_{s \to \infty} R_s = \infty$ a.s. It then holds that

$$
\{ \rho_\infty > t \} = \{ R_s > I_\infty \text{ for all } s \in [0, t] \} \text{ a.s.,} \tag{A.6}
$$

namely the indicator functions of these two events are equal a.s. Indeed, it is obvious that the left-hand event is included in the right-hand event; for converse inclusion, since $t \notin T$ and $T$ is compact a.s., we have $t < \inf T = \rho_\infty$ a.s. By continuity, the right-hand side of (A.6) is written as $\{ \inf_{0 \leq s \leq t} R_s > I_\infty \}$, and hence we have

$$
\{ \rho_\infty > t \} = \{ I_t > \inf_{s \geq t} R_s \} \text{ a.s.}
$$

Therefore by the Markov property and (2.3),

$$
P_a^{(\nu)}(\rho_\infty > t) = E_a^{(\nu)} \left( (I_t/R_t)^{2\nu} \right).
$$

Similarly

$$
\{ \bar{\rho}_\infty < t \} = \{ I_t < \inf_{s \geq t} R_s \} \text{ a.s.,}
$$

from which it also follows that

$$
P_a^{(\nu)}(\bar{\rho}_\infty \geq t) = P_a^{(\nu)}(I_t \geq \inf_{s \geq t} R_s)
$$

$$
= E_a^{(\nu)} \left( (I_t/R_t)^{2\nu} \right).
$$
By the dominated convergence theorem, the mapping \([0, \infty) \ni t \mapsto E^{(\nu)}(I_t/R_t^{2\nu})\) is continuous. Combining these we see that \(\rho_\infty\) and \(\tilde{\rho}_\infty\) have the same distribution, which implies that \(\rho_\infty = \tilde{\rho}_\infty\) a.s. since \(\rho_\infty \leq \tilde{\rho}_\infty\). This ends the proof. \(\square\)

**Lemma A.1.** Let \(c > 0\) and \(\beta \in \mathbb{R}\). For all \(x \geq c\) one has

\[
\int_c^x \frac{dy}{y^\beta(x + c - y)^{\beta+1}} = \frac{1}{c^\beta(x + c)^{2\beta}} \int_c^x \frac{(y + c)^{2\beta}}{y^{\beta+1}} dy.
\]

**Proof.** We write \(f(x)\) for the left-hand side of the equality. Change of variables yields

\[
f(x) = \int_c^x \frac{dy}{(x + c - y)\beta y^{\beta+1}},
\]
and thus

\[
2f(x) = \int_c^x \frac{x + c}{y^{\beta+1}(x + c - y)^{\beta+1}} dy.
\]

We differentiate (A.7) with respect to \(x\) to see from (A.8) that \(f\) satisfies

\[
f'(x) = \frac{1}{c^\beta x^{\beta+1}} - \frac{2\beta f(x)}{x + c}
\]
with \(f(c) = 0\). The equality follows by solving this differential equation. \(\square\)

**References**

[1] J. Azéma, M. Yor, Une solution simple au problème de Skorokhod: In Séminaire de Probabilités, XIII, pp. 90–115, Lect. Notes in Math. 721, Springer, Berlin (1979).

[2] P. Collet, S. Martínez, J.S. Martín, Asymptotic laws for one-dimensional diffusions conditioned to nonabsorption, Ann. Probab. 23, 1300–1314 (1995).

[3] P.J. Fitzsimmons, Excursions above the minimum for diffusions, [arXiv:1308.5189v1 [math.PR]] (2013).

[4] Y. Hamana, H. Matsumoto, Asymptotics of the probability distributions of the first hitting times of Bessel processes, Electron. Commun. Probab. 19 (5), 1–5 (2014).

[5] I. Karatzas, S.E. Shreve, Brownian Motion and Stochastic Calculus, 2nd ed., Springer, New York (1991).

[6] F.B. Knight, Brownian local times and taboo processes, Trans. Amer. Math. Soc. 143, 173–185 (1969).

[7] N.N. Lebedev, Special Functions and Their Applications, Dover, New York (1972).
[8] H. Matsumoto, M. Yor, A relationship between Brownian motions with opposite drifts via certain enlargements of the Brownian filtration, Osaka J. Math. 38, 383–398 (2001).

[9] S. Méléard, D. Villemonais, Quasi-stationary distributions and population processes, Probab. Surv. 9, 340–410 (2012).

[10] D. Revuz, M. Yor, Continuous Martingales and Brownian Motion, 3rd ed., Springer, Berlin (1999).

[11] B. Roynette, M. Yor, Penalising Brownian Paths, Lect. Notes in Math. 1969, Springer, Berlin (2009).

[12] I. Shigekawa, One dimensional diffusions conditioned to be non-explosive, https://www.math.kyoto-u.ac.jp/~ichiro/kandai06_slide.pdf (2006).

[13] M. Yor, Loi de l’indice du lacet Brownien, et distribution de Hartman-Watson, Z. Wahrsch. Verw. Gebiete 53, 71–95 (1980).