THE RECURSION OPERATORS OF THE BKP HIERARCHY AND THE CKP HIERARCHY

MAOHUA LI\textsuperscript{1,2} JIPENG CHENG\textsuperscript{3} CHUANZHONG LI\textsuperscript{2} JINGSONG HE\textsuperscript{2*}

1. Department of Mathematics, USTC, Hefei, 230026 Anhui, P.R.China
2. Department of Mathematics, NBU, Ningbo, 315211 Zhejiang, P.R.China
3. Department of Mathematics, CUMT, Xuzhou, 221116 Jiangsu, P.R.China

ABSTRACT. In this paper, under the constraints of the BKP(CKP) hierarchy, a crucial observation is that the odd dynamical variable $u_{2k+1}$ can be explicitly expressed by the even dynamical variable $u_{2k}$ in the Lax operator $L$ through a new operator $B$. Using operator $B$, the essential differences between the BKP hierarchy and the CKP hierarchy are given by the flow equations and the recursion operators under the $(2n+1)$-reduction. The formal formulas of the recursion operators for the BKP and CKP hierarchy under $(2n+1)$-reduction are given. To illustrate this method, the two recursion operators are constructed explicitly for the 3-reduction of the BKP and CKP hierarchies. The $t_7$ flows of $u_3$ are generated from $t_1$ flows by the above recursion operators, which are consistent with the corresponding flows generated by the flow equations under 3-reduction.

Mathematics Subject Classifications(2000). 37K05, 37K10, 37K40.

Keywords: BKP hierarchy, CKP hierarchy, Flow equation, Recursion operator.

1. Introduction

The Kadomtsev-Petviashvili (KP) hierarchy [1] is an attractive research object in the mathematical physics since 1980s. It has two important sub-hierarchies, i. e. Kadomtsev-Petviashvili hierarchy of B-type (BKP hierarchy) and Kadomtsev-Petviashvili hierarchy of C-type (CKP hierarchy) [2] which are two interesting reductions of KP hierarchy associated with two infinite dimensional algebras $o(\infty)$ and $sp(\infty)$ respectively [3]. By the formulation of pseudo-differential operator, the Lax operator of KP hierarchy is $L = \sum_{l \geq 0} u_l \partial^{1-l} = \partial + u_2 \partial^{-1} + u_3 \partial^{-2} + \cdots$. These different algebraic structures also can be shown in some sense by the reduction conditions on the Lax operator, i. e. $L^* = -L$ for the CKP hierarchy and $L^* = -\partial L \partial^{-1}$ for the BKP hierarchy [3]. Here $L^*$ is the formally adjoint operator of $L$. Besides above two essential differences between the KP, BKP and CKP hierarchies from the view of algebra structure and Lax operator, there are also more interesting facts in the four aspects including the $\tau$ function [3], the gauge transformation [4]-[8], the additional symmetries and the ASvM formula [9]-[17], the freeze of the even flows and odd number dynamical variables $u_j, (j = 3, 5, 7, \cdots)$ [3]. Because of the importance of the flow equations and recursion operators, it is very natural to explore more differences from these two aspects among them.

* Corresponding author: hejingsong@nbu.edu.cn.
The recursion operator [18] for a given soliton equation is firstly introduced by using the KdV equation as example. The results of the recent thirty five years of the soliton theory show possessing a recursion operator is one of essential integrable properties, which is related to infinitely many conservation laws and symmetries, Hamiltonian structure, higher order flows, etc [2, 19, 20]. With the help of the recursion operator, the higher flows can be generated from the lower flows for an integrable hierarchy, which offers a natural way to construct the whole hierarchy from a single seed system [2, 20, 21, 22, 23, 24, 25, 26]. There are several ways to construct the recursion operator of a given integrable system, which is reviewed in reference [20]. The construction of recursion operator in 2+1 dimension was given in the papers by Fokas and Santini [24, 25, 27]. For the $n$-reduction KP hierarchy, it is very natural to extract recursion operator from the explicit flow equations, which has been done by W. Strampp and W. Oevel [19]. The advantage of this method is that the higher-order flows generated by recursion operator from lower-order ones are local even if the recursion operator has nonlocal term, because the higher-order flows are automatically identified with the local flows given by the Lax equations of the KP hierarchy.

To improve the understanding of more essential differences of the KP, BKP and CKP hierarchies, we shall study the explicit flows and the recursion operators for them. The main difficulties to apply Strampp and Oevel’s method [19] for the BKP hierarchy and the CKP hierarchy are due to the two constraints: the disappearance of the even flows and the odd dynamical variables $u_i, (i = 1, 3, 5, \ldots)$, which is originated from the reduction conditions on the Lax operator $L^* = -\partial L \partial^{-1}$ or $L^* = -L$. It is also crucial to know that there only exists $(2n+1)$-th reduction in the BKP hierarchy and the CKP hierarchy. The key step is to transmit the reduction conditions on the Lax operator to the flow equations, which can be realized by expressing the odd dynamical variable $u_j, (j = 1, 3, 5, \ldots)$ by the even ones.

This paper was organized as follows. The odd dynamical variable $u_j, (j = 1, 3, 5, \ldots)$ is expressed by the even ones in section 2 with the reduction condition on Lax operator $L^* = -L$ or $L^* = -\partial L \partial^{-1}$. An operator $B$ is introduced for this purpose. In section 3, the odd flow equations of the even dynamical variable $u_{2j}, (j = 1, 2, \ldots)$ are obtain from $L_{2m+1} = [L, (L^{2m+1})_\ldots]$. We also calculate the some odd flows of BKP(CKP) hierarchy as examples. In section 4 the recursion operator of BKP(CKP) hierarchy is discussed. This recursion operators are different from the recursion operator of KP hierarchy [19, 20, 26]. Section 5 is devoted on reduction of the $t_7$ flows of BKP(CKP) hierarchy by the recursion operator, and which are consistent with corresponding flows in section 2. Section 6 is a brief discussion on recursion operator and the future problem.

2. THE EVEN DYNAMICAL VARIABLES

In this section, we study the even dynamical variables and the odd ones of the Lax operator. The even dynamical variables will be expressed by the odd dynamical variables by a formula which be introduced below.
Firstly we given a pseudo-differential Lax operator
\[ L = \sum_{l \geq 0} u_l \partial^{1-l} \]
\[ = \partial + u_2 \partial^{-1} + u_3 \partial^{-2} + \cdots, \]  \hfill (2.1)
where we assume \( u_0 = 1, u_1 = 0, \) and \( u_2, u_3, \cdots \) are the functions of an infinite set of time variables \( t = (t_1 = x, t_3, \cdots) \) and \( \partial = \frac{\partial}{\partial x}, \) by imposing the following condition on \( L \)
\[ L^* = -\partial^k L \partial^{-k}. \]  \hfill (2.2)
And the formally adjoined operators are given by
\[ L^* = \sum_{l \geq 0} (-1)^{1-l} \partial^{1-l} u_l, \]  \hfill (2.3)
with \( k = 0, 1 \) corresponding to the Lax operator for CKP hierarchy and BKP hierarchy respectively. The operation of \( \partial^k \) with \( k \in \mathbb{Z} \) is defined by
\[ \partial^k u = \sum_{j \geq 0} C_k^j u^{(j)} \partial^{k-j}, \]  \hfill (2.4)
where \( u^{(j)} = \frac{\partial^j u}{\partial x^j}, \) with
\[ C_k^j = \frac{k(k-1)(k-2) \cdots (k-j+1)}{j!}, \]  \hfill (2.5)
The \( m \)-th power of \( L \) can be denote for
\[ L^m = (\partial + u_2 \partial^{-1} + u_3 \partial^{-2} + \cdots)^m \]
\[ = \sum_{j \leq m} p_j(m) \partial^j, \]  \hfill (2.6)
and \( (L^m)_+ = \sum_{j=0}^{m} p_j(m) \partial^j, \) i.e. \( (L^m)_+ \) is the non-negative projection of \( L^m, \) and \( (L^m)_- = L^m - (L^m)_+ \) is the negative projection of \( L^m. \) In particular,
\[ u_l = p_{l-1}(1), l = 2, 3, \cdots. \]  \hfill (2.7)
The dynamical equations of BKP hierarchy and CKP hierarchy are defined as follows,
\[ L_{t2m+1} = [(L^{2m+1})_+, L] = [L, (L^{2m+1})_-], \quad m = 0, 1, 2, \cdots \]  \hfill (2.8)
Remembering the corresponding constraints on the Lax operator, there are only the odd flows existed with the BKP hierarchy and CKP hierarchy \[. \]  \hfill (2.8)
And the \( (2n + 1) \)-th power of Lax operator \( L \) must be considered,
\[ L^{2n+1} = \sum_{j \leq 2n+1} p_j(2n + 1) \partial^j, \]  \hfill (2.9)
with \( n = 0, 1, 2, \cdots. \) From the constraints (2.2), we know
\[ \partial^{-k} L^{*2n+1} = -L^{2n+1} \partial^{-k}. \]  \hfill (2.10)
By considering the negative part of both sides
\[ (\partial^{-k} L^{2n+1})_- = -(L^{2n+1} \partial^{-k})_-, \] (2.11)
then
\[ \text{l.h.s of (2.11)} = \sum_{j \leq -1 + k} (-1)^j \partial^{-k} p_j(2n + 1) \]
\[ = \sum_{j \leq -1 + k} \sum_{l \geq 0} (-1)^j C^{l}_{j+k} p_j^{(l)}(2n + 1) \partial^{-k-l} \]
\[ = \sum_{j \leq -1 + k} \sum_{l \geq 0} (-1)^{j+l} C^{l}_{j+t-k} p_j^{(l)}(2n + 1) \partial^{-k}. \] (2.12)

On the other hand,
\[ \text{r.h.s of (2.11)} = - \sum_{j \leq -1 + k} p_j(2n + 1) \partial^{-k}. \]

So
\[ \sum_{j \leq -1 + k} \sum_{l \geq 0} (-1)^j C^{l}_{j+t-k} p_j^{(l)}(2n + 1) \partial^{-k} = - \sum_{j \leq -1 + k} p_j(2n + 1) \partial^{-k}. \] (2.12)

Comparing the coefficients of \( \partial^{-k} \) in above relation, we find
\[ \sum_{l \geq 0} (-1)^{j+l} C^{l}_{j+t-k} p_j^{(l)}(2n + 1) = -p_j(2n + 1), \quad j \leq -1 + k. \] (2.13)

Further,
\[ ((-1)^j + 1)p_j(2n + 1) = - \sum_{l \geq 1} (-1)^{j+l} C^{l}_{j+t-k} p_j^{(l)}(2n + 1), \quad j \leq -1 + k. \] (2.14)

Thus, one can find \( p_j(2n + 1) \) with \( j \) odd are independent. As for \( j \) being even number, we have
\[ p_j(2n + 1) = \frac{-1}{2} \sum_{l \geq 1} (-1)^{j+l} C^{l}_{j+t-k} p_j^{(l)}(2n + 1), \quad j \leq -1 + k \text{ and } j \text{ even.} \] (2.15)

In particular, \( p_0(2n + 1) = 0 \) for \( k = 1 \). Thus \( p_j(2n + 1) \) in (2.15) becomes
\[ p_j(2n + 1) = \frac{-1}{2} \sum_{l \geq 1} (-1)^l C^{l}_{j+t-k} p_j^{(l)}(2n + 1), \quad j = -2, -4, \ldots. \] (2.16)
By considering (2.7), we have \( l \) odd number dynamical variable

\[
\begin{align*}
\text{If } l &= 1, \\
\quad \tag{2.7}
\end{align*}
\]

\[
\begin{align*}
\quad &= -\frac{1}{2} \sum_{\mu \geq 1} (-1)^{1-l+\mu} C_{1-l+\mu-k}^{(\mu)} p_{1-l+\mu}(1) \\
\quad &= -\frac{1}{2} \sum_{\mu \geq 1} (-1)^{\mu} C_{1-l+\mu-k}^{(\mu)} u_{l-\mu}, \quad l = 3, 5, \ldots
\end{align*}
\]

That is,

\[
\begin{align*}
\text{If } l &= 1, \\
\quad \tag{2.17}
\end{align*}
\]

So we summarize above results for below. For BKP hierarchy,

\[
\begin{align*}
\quad &= -\frac{1}{2} \sum_{\mu \geq 1} (-1)^{\mu} C_{1-l+\mu-k}^{(\mu)} u_{l-\mu} = -\frac{1}{2} \sum_{\mu \geq 1} C_{l-2+k}^{(\mu)} u_{l-\mu}, \quad l = 3, 5, \ldots
\end{align*}
\]

For CKP hierarchy,

\[
\begin{align*}
\quad &= -\frac{1}{2} \sum_{\mu \geq 1} (-1)^{\mu} C_{1-l+\mu-k}^{(\mu)} u_{l-\mu}, \quad l = 3, 5, \ldots
\end{align*}
\]

But because the odd dynamical variable and even dynamical variable of it are not really separated by eq. (2.18, 2.20), we next want to separate the odd parts and even ones from above relation. Before doing this, let’s see a lemma first.

**Lemma 2.1.** If

\[
\begin{align*}
p_j(2n+1) &= -\frac{1}{2} \sum_{l \geq 1} (-1)^{l} C_{j+l}^{(l)} p_{j+l}(2n+1), \quad j = -2, -4, \ldots \\
\quad \tag{2.21}
\end{align*}
\]

where \( A_{j\mu} \) is an operator, then

\[
\begin{align*}
p_{-2l} &= -\frac{1}{2} \sum_{\mu \geq 1} (-1)^{\mu} C_{l+\mu-k}^{(\mu)} u_{l-\mu}, \quad l = 3, 5, \ldots
\end{align*}
\]

\[
\begin{align*}
p_{-2l} &= -\frac{1}{2} \sum_{\mu \geq 1} (-1)^{\mu} C_{l-2+k}^{(\mu)} u_{l-\mu}, \quad l = 3, 5, \ldots
\end{align*}
\]
Thus the lemma holds for $l$. Assume the lemma is correct for $B$. So the corresponding operators are, 

$$
B_{-2l,-2\mu+1} = \sum_{i_1,\ldots,i_\nu} (-1)^{\nu+1} C_{i_1}^{2i_1} C_{2i_1-k-1}^{2i_2} \cdots C_{2i_{\nu-1}}^{2i_{\nu}} \partial^{\nu+1} C_{2l-1+k-k_1-i_2-\cdots-i_{\nu-1}}^{2l+1+2l-1-i_2-\cdots-i_{\nu}} \partial^{2l-2\mu+1}.
$$

(2.24)

**Proof.** We prove the lemma by induction. Obviously the lemma is true for $l = 1$. We next assume the lemma is correct for $\leq l$, then for $l+1$ case. By (2.22),

$$
p_{-2l-2} = \sum_{\gamma \geq 1} A_{-2l-2,\gamma} p_{-2l-2+\gamma} = \sum_{\mu = 1}^{l+1} A_{-2l-2,2\mu-1} p_{-2l-2+2\mu-1} + \sum_{\mu = 1}^{l} A_{-2l-2,2\mu} p_{-2l-2+2\mu},
$$

$$
= \sum_{\mu = 1}^{l+1} A_{-2l-2,2\mu+1+2(l+1)} p_{-2\mu+1} + \sum_{\gamma = 1}^{l} A_{-2l-2,2\gamma} p_{-2(l-\gamma+1)}
$$

$$
+ \sum_{\mu = 1}^{l+1} \sum_{\gamma = 1}^{l} A_{-2l-2,2\gamma} A_{-2l-2+2\gamma, i_2} A_{-2l-2+2\gamma+i_2, i_3} \cdots A_{-2l-2+2\gamma+i_2+i_3+\cdots+i_\nu, i_\nu} A_{-2l+2\gamma+i_2+i_3+\cdots+i_\nu-2\mu+1+2(l+1)-2l-2\gamma-i_2-\cdots-i_\nu} p_{-2\mu+1}
$$

$$
= \sum_{\mu = 1}^{l+1} B_{-2l-2,2\mu+1} p_{-2\mu+1}.
$$

Thus the lemma holds for $l+1$ case.

We apply this lemma to BKP hierarchy and CKP hierarchy cases,

$$
A_{j\mu} = -\frac{1}{2} (-1)^{j+\mu} C_{j+k-\mu}^\mu \partial^\mu = -\frac{1}{2} C_{j+k-\mu}^\mu \partial^\mu = \begin{cases} 
-\frac{1}{2} C_{j-k}^\mu \partial^\mu, & k = 1, \text{BKP}, \\
-\frac{1}{2} C_{j-1}^\mu \partial^\mu, & k = 0, \text{CKP}.
\end{cases}
$$

(2.25)

So the corresponding $B$ operators are,

$$
B_{-2l,-2\mu+1}
$$

$$
= \sum_{i_1,\ldots,i_\nu} (-1)^{\nu+1} C_{i_1}^{2i_1} C_{2l-1+k-i_1}^{2i_2} \cdots C_{2i_{\nu-1}}^{2l-1+k-k_1-i_2-\cdots-i_{\nu-1}} C_{2l-1+k-k_1-1-i_2-\cdots-i_{\nu-1}}^{2l+1+2l-1-i_2-\cdots-i_{\nu}} \partial^{2l-2\mu+1},
$$

$$
= \begin{cases} 
-\frac{1}{2} C_{i_1}^{2i_1} C_{2l-1}^{2i_2} \cdots C_{2i_{\nu-1}}^{2l-1-i_2-\cdots-i_{\nu-1}} C_{2l-1-i_2-\cdots-i_{\nu-1}}^{2l+1+2l-1-i_2-\cdots-i_{\nu}} \partial^{2l-2\mu+1}, & \text{BKP}, \\
-\frac{1}{2} C_{i_1}^{2i_1} C_{2l-1-i_1}^{2i_2} \cdots C_{2i_{\nu-1}}^{2l-1-i_2-\cdots-i_{\nu-1}} C_{2l-1-i_1-i_2-\cdots-i_{\nu-1}}^{2l+1+2l-1-i_1-i_2-\cdots-i_{\nu}} \partial^{2l-2\mu+1}, & \text{CKP},
\end{cases}
$$

(2.26)
So according to Lemma 2.1, we get
\[ p_{-2l}(2n + 1) = \sum_{\mu=1}^{l} B_{-2l,-2\mu+1} p_{-2\mu+1}(2n + 1). \tag{2.27} \]

**Proposition 2.2.** All the odd dynamical variables \(u_{2l+1}\) of Lax operator \(L\) can be expressed by the even dynamical variables \(u_{2\mu}(\mu \leq l)\), that is,
\[ u_{2l+1} = \sum_{\mu=1}^{l} B_{-2l,-2\mu+1} u_{2\mu}, \tag{2.28} \]
where \(B_{-2l,-2\mu+1}\) is defined by (2.26).

*Proof.* From eq. (2.27), in particular, \(u_{2l+1} = p_{-2l}(1)\) for \(l = 1, 2, \ldots\). Then we find
\[ u_{2l+1} = p_{-2l}(1) = \sum_{\mu=1}^{l} B_{-2l,-2\mu+1} p_{-2\mu+1}(1) = \sum_{\mu=1}^{l} B_{-2l,-2\mu+1} u_{2\mu}. \]

\[ \square \]

The equation (2.28) is crucial to calculate the flow equation in section 3 and the recursion operator in section 4. From the above relation of \(u_j\), one can obtain that all the odd item \(u_{2l+1}\) can be expressed by the even item \(u_{2\mu}\), where \(\mu \leq l\).

Below let’s see some examples of (2.28). Firstly we will deal with the BKP hierarchy.
For \(l = 1\), we find \(\mu = 1, B_{-2l,-2\mu+1} = B_{-2,-1} = -\frac{1}{2}C_2^1 \partial = -\partial\), thus \(u_3 = -u_{2x}\).
For \(l = 2\), then \(\mu = 1, \mu = 2\), thus one only need \(B_{-4,-1}\) and \(B_{-4,-3}\), while
\[ B_{-4,-1} = -\frac{1}{2}C_4^3 \partial^3 + (-\frac{1}{2})^2 C_4^2 C_2^1 \partial^3 = \partial^3, \]
\[ B_{-4,-3} = -\frac{1}{2}C_4^1 \partial = -2\partial. \]

So
\[ u_5 = u_{2xxx} - 2u_{4x}. \]

For \(l = 3\), \(\mu = 1, 2, 3, B_{-6,-1}, B_{-6,-3}\) and \(B_{-6,-5}\) have the form
\[ B_{-6,-1} = \{(\frac{1}{2})C_6^5 + (\frac{1}{2})^2 C_6^2 C_4^3 + (\frac{1}{2})^2 C_6^4 C_4^1 + (\frac{1}{2})^3 C_6^2 C_4^1 C_2^1\} \partial^5 = -3\partial^5, \]
\[ B_{-6,-3} = \{(\frac{1}{2})C_6^3 + (\frac{1}{2})^2 C_6^2 C_4^1\} \partial^3 = 5\partial^3, \]
\[ B_{-6,-5} = (\frac{1}{2})C_6^1 \partial = -3\partial. \]

So
\[ u_7 = -3u_{2xxxxx} + 5u_{4xxx} - 3u_{6x}. \]
For \( l = 4 \) and \( \mu = 1, 2, 3, 4, B_{-8,-1}, B_{-8,-3}, B_{-8,-5} \) and \( B_{-8,-7} \) have the form

\[
B_{-8,-1} = \left\{ \left( -\frac{1}{2} \right) C_8^7 + \left( -\frac{1}{2} \right)^2 C_8^2 C_6^5 + \left( -\frac{1}{2} \right)^3 C_8^2 C_6^2 C_4^3 + \left( -\frac{1}{2} \right)^2 C_8^4 C_4^3 + \left( -\frac{1}{2} \right)^2 C_8^6 C_2^1 \right\} \partial^7 = \left( -\frac{1}{2} \right) C_8^1 \partial = -4 \partial.
\]

Then we consider the examples of CKP hierarchy.

When \( l = 1 \) and \( \mu = 1 \), \( B_{-2,-1} = -\frac{1}{2} C_1^1 \partial = -\frac{1}{2} \partial \). Thus \( u_3 = -\frac{1}{2} u_{2x} \).

For \( l = 2 \), then \( \mu = 1, 2 \), thus only need \( B_{-4,-1} \) and \( B_{-4,-3} \), while

\[
B_{-4,-1} = -\frac{1}{2} C_3^3 \partial^3 + \left( -\frac{1}{2} \right)^2 C_3^2 C_1^1 \partial^3 = \frac{1}{4} \partial^3,
\]

\[
B_{-4,-3} = -\frac{1}{2} C_3^1 \partial = -3 \partial.
\]

So

\[
u_5 = \frac{1}{4} u_{2xx} - \frac{3}{2} u_{4x}.
\]

For \( l = 3, \mu = 1, 2, 3 \), \( B_{-6,-1}, B_{-6,-3} \) and \( B_{-6,-5} \) have the form

\[
B_{-6,-1} = \left\{ \left( -\frac{1}{2} \right) C_5^5 + \left( -\frac{1}{2} \right)^2 C_5^2 C_3^3 + \left( -\frac{1}{2} \right)^3 C_5^2 C_3^2 C_1^1 \right\} \partial^5 = -\frac{1}{2} \partial^5,
\]

\[
B_{-6,-3} = \left\{ \left( -\frac{1}{2} \right) C_3^3 + \left( -\frac{1}{2} \right)^2 C_3^2 C_1^1 \right\} \partial^3 = \frac{5}{2} \partial^3,
\]

\[
B_{-6,-5} = -\frac{1}{2} C_1^1 \partial = -\frac{5}{2} \partial.
\]

So

\[
u_7 = -\frac{1}{2} u_{2xxxx} + \frac{5}{2} u_{4xx} - \frac{5}{2} u_{6x}.
\]
For \( l = 4, \mu = 1, 2, 3, 4, \) \( B_{-8,-1}, B_{-8,-3}, B_{-8,-5} \) and \( B_{-8,-7} \) have the form below

\[
B_{-8,-1} = \left\{ (-\frac{1}{2})^2C_7^2C_5^5 + (-\frac{1}{2})^3C_7^2C_5^2C_3^3 + (-\frac{1}{2})^2C_7^4C_3^3 + (-\frac{3}{2})^2C_7^6C_1^1, \right.
\]
\[
+ \left. (-\frac{1}{2})^3C_7^2C_3^3C_1^1 + (-\frac{1}{2})^3C_7^4C_3^1C_1^1 + (-\frac{1}{2})^4C_7^6C_3^1C_1^1 \right\} \partial^7 = \frac{17}{8} \partial^7,
\]

\[
B_{-8,-3} = \left\{ (-\frac{1}{2})C_7^5 + (-\frac{1}{2})^2C_7^2C_3^3 + (-\frac{1}{2})^2C_7^4C_3^1 + (-\frac{1}{2})^3C_7^6C_3^1 \right\} \partial^5 = -\frac{21}{2} \partial^5,
\]

\[
B_{-8,-5} = \left\{ (-\frac{1}{2})C_7^3 + (-\frac{1}{2})^2C_7^2C_3^1 \right\} \partial^3 = \frac{35}{4} \partial^3,
\]

\[
B_{-8,-7} = (-\frac{1}{2})C_7^1 \partial = -\frac{7}{2} \partial.
\]

So

\[
u_9 = \frac{17}{8} u_{2xxxxxx} - \frac{21}{2} u_{4xxxxx} + \frac{35}{4} u_{6xxx} - \frac{7}{2} u_{8xx}.
\]

We summarize above results below.

For BKP,

\[
\begin{align*}
\left\{
\begin{array}{l}
u_3 = -u_{2x}, \\
u_5 = u_{2xxx} - 2u_{4x}, \\
u_7 = -3u_{2xxxx} + 5u_{4xxx} - 3u_{6x}, \\
u_9 = 17u_{2xxxxxx} - 28u_{4xxxxx} + 14u_{6xxx} - 4u_{8x}, \\
\end{array}
\right. 
\end{align*}
\]

\[\cdots \] (2.29)

For CKP,

\[
\begin{align*}
\left\{
\begin{array}{l}
u_3 = -\frac{1}{2}u_{2x}, \\
u_5 = \frac{1}{4}u_{2xxx} - \frac{3}{2}u_{4x}, \\
u_7 = -\frac{1}{2}u_{2xxxx} + \frac{5}{2}u_{4xxx} - \frac{5}{2}u_{6x}, \\
u_9 = \frac{17}{8} u_{2xxxxxx} - \frac{21}{2} u_{4xxxxx} + \frac{35}{4} u_{6xxx} - \frac{7}{2} u_{8x}, \\
\end{array}
\right. 
\end{align*}
\]

\[\cdots \] (2.30)

**Remark:** From (2.28), one can know there are only the even dynamical variables of \( \{u_j, j \geq 1\} \) are independent, and the odd dynamical variables of \( \{u_j, j \geq 1\} \) can be expressed by the even ones of \( \{u_j, j \geq 1\} \). With this result, it is nature to discuss the odd flows of even dynamical variables in the next section.
3. Flow Equations

We next deal with the BKP hierarchy and CKP hierarchy in an unified way. First we derive the flow equations of the dynamical variables $u_{2j}$. Inserting (2.1) and (2.9) into (2.8), one finds

$$L(L^{2m+1})_- = (L^{2m+1})_- L$$

$$\begin{align*}
&= \sum_{r \geq 0} \sum_{h > 0} \left( u_r \partial^{1-r} p_{-h}(2m+1) \partial^{-h} - p_{-h}(2m+1) \partial^{-h} u_r \partial^{1-r} \right) \\
&= \sum_{r \geq 0} \sum_{\alpha \geq 0} \sum_{h > 0} \left( C_{1-r}^{\alpha} u_r p_{-h}(2m+1) - C_{-h}^{\alpha} p_{-h}(2m+1) u_r^{(\alpha)} \right) \partial^{1-r-h-\alpha} \\
&= \sum_{l \geq 0} \sum_{r \geq 0} \sum_{h > 0} \left( C_{l-r}^{l-r} u_r p_{-h}(2m+1) - C_{-h}^{l-r} p_{-h}(2m+1) u_r^{(l-r)} \right) \partial^{1-l-h} \\
&= \sum_{j \geq 1} \sum_{h \geq 1} \sum_{r \geq 0} \left( C_{j-r}^{j-r} u_r p_{-h}(2m+1) - C_{-h}^{j-r} p_{-h}(2m+1) u_r^{(j-r)} \right) \partial^{1-j}.
\end{align*}$$

Comparing with $L_{t_{2m+1}} = \sum_{j \geq 0} u_{j,t_{2m+1}} \partial^{1-j}$, we have

$$u_{0,t_{2m+1}} = 0, \quad u_{1,t_{2m+1}} = 0, \quad u_{j,t_{2m+1}} = \sum_{h=1}^{j} O_{j,h} p_{-h}(2m+1),$$

where

$$O_{j,h} = \sum_{r \geq 0} (C_{j-r}^{j-r} u_r \partial^{j-h-r} - C_{-h}^{j-r} u_r^{(j-r)}).$$

In particular, $O_{j,0} = 0, O_{j,j-1} = \partial$.

With (2.1) and (2.9), $p_j(2n+1)$ can be uniquely determined by $u_2, u_3, \cdots, u_{2n+1-j}$, i.e. it’s formula is

$$p_j(2n+1) = (2n+1) u_{2n+1-j} + f_{jn}(u_2, u_3, \cdots, u_{2n-j}), j \leq 2n+1$$

and $f_{jn}$ is a differential polynomials in $u_2, u_3, \cdots, u_{2n-j}$. With the help of Proposition 2.2, every dynamical variable can be expressed by the even ones. So $p_j(2n+1)$ can be expressed by $u_2, u_4, \cdots, u_{2n+1-j}$ which $j$ is odd. Now we consider the $(2n+1)$-reduction, i.e. for some fixed $2n+1, n \in \mathbb{Z}_+$,

$$L_{2n+1} = (L^{2n+1})_+.$$

This relation is equal to requiring the $p_j(2n+1) = 0$ for $j < 0$. Hence, one can recursively express all coordinates $u_j$ with $j \geq 2n+1$ in terms of $(u_2, u_3, \cdots, u_{j-1})$. But thanks for (2.28), all the odd dynamical variables $u_{2n+1}$ can be express by the even dynamical variables $u_{2\mu}$, where $\mu \leq l$. So only first $n$ coordinates $(u_2, u_4, \cdots, u_{2n})$ are independent for BKP(CKP) hierarchy in the case of $(2n+1)$-reduction.
On the other hand, according to formula \ref{eq:2.27} and the third formula of \ref{eq:3.1}, one has

\[
\begin{align*}
    u_{2j,t_{2m+1}} &= \sum_{h=1}^{j} O_{2j,2h-1} p_{-2h+1}(2m+1) + \sum_{h=1}^{j} O_{2j,2h} p_{-2h}(2m+1) \\
    &= \sum_{h=1}^{j} O_{2j,2h-1} p_{-2h+1}(2m+1) + \sum_{h=1}^{j} \sum_{\mu=1}^{h} O_{2j,2h} B_{-2h,-2\mu+1} p_{-2\mu+1}(2m+1) \\
    &= \sum_{h=1}^{j} O_{2j,2h-1} p_{-2h+1}(2m+1) + \sum_{\mu=1}^{j} \sum_{h=\mu}^{j} O_{2j,2\mu} B_{-2\mu,-2h+1} p_{-2h+1}(2m+1) \\
    &= \sum_{h=1}^{j} \left( O_{2j,2h-1} + \sum_{\mu=h}^{j} O_{2j,2\mu} B_{-2\mu,-2h+1} \right) p_{-2h+1}(2m+1).
\end{align*}
\]

Thus,

\[
    u_{2j,t_{2m+1}} = \sum_{h=1}^{j} Q_{jh} p_{-2h+1}(2m+1), \quad j \leq n,
\]  \hspace{1cm} (3.5)

which

\[
    Q_{jh} = O_{2j,2h-1} + \sum_{\mu=h}^{j} O_{2j,2\mu} B_{-2\mu,-2h+1}, \quad j \leq n.
\]  \hspace{1cm} (3.6)

Obviously, \( Q_{jj} = \partial \). So with the above formula \ref{eq:3.5} and the help of equations \ref{eq:2.29} \ref{eq:2.30}, all odd flow equations of the even dynamical coordinate \( u_{2j} \) can be obtained. This result implies that the flow equations are expressed by even dynamical variable \( u_2, u_4, \cdots, u_{2n} \). Due to the appearance of the operator \( B \) \ref{eq:2.26}, the flow equation of KP hierarchy, BKP hierarchy and CKP hierarchy are different.

We present some odd flow equations below calculated by Maple. The first several odd flow equations of BKP hierarchy are

\[
\begin{align*}
    u_{2,t_1} &= u_{2,x}, \\
    u_{2,t_3} &= 6u_{2} u_{2,x} + 3u_{4,x} - 2u_{2,xxx}, \\
    u_{2,t_5} &= 20u_{2} u_{4,x} + 20u_{4} u_{2,x} + 10u_{2} u_{2,xxx} + 5u_{6,x} + 60u_{2} u_{2,xx} - \frac{2}{3} u_{2,xxxx} + 30u_{2} u_{2,x}, \\
    u_{2,t_7} &= \frac{10}{3} u_{2,xxxxxx} + 406u_{2} u_{2,xx} + 210u_{2} u_{4} u_{2,xx} + 7u_{8,x} - 7u_{4,xxxxxx} \\
    &\quad + 14u_{6,xxx} + 112u_{2,xxx} + 42u_{6} u_{6,x} + 42u_{4} u_{4,xx} + 42u_{6} u_{2,xx} + 49u_{2} u_{4,xxx} \\
    &\quad + 98u_{4} u_{2,xxx} + 49u_{2} u_{2,xxxxxx} + 203u_{2} u_{4,xxx} + 252u_{4} u_{2,xx} + 294u_{2} u_{2,xxxx} \\
    &\quad + 609u_{2,xxx} u_{2,xxx} + 105u_{2} u_{4,x} + 91 u_{2} u_{2,xxx} + 140u_{2} u_{2,x}.
\end{align*}
\]  \hspace{1cm} (3.7)
The first several odd flow equations of CKP hierarchy are

\[
\begin{align*}
    u_{2,t_1} &= u_{2,x}, \\
    u_{2,t_3} &= 6u_2u_{2,x} + 3u_{4,x} - \frac{1}{3}u_{2,xxx}, \\
    u_{2,t_5} &= 20u_2u_{4,x} + 20u_4u_{2,x} + 5u_{6,x} + 95u_{2,x}u_{2,xx} + 30u_2u_{2,xxx} \\
        &\quad + 10u_{4,xxx} + 30u_2u_{2,x} - \frac{3}{2}u_{2,xxxxxx}, \\
    u_{2,t_7} &= 42u_2u_{6,x} + 42u_4u_{4,x} + 42u_6u_{2,x} + 49u_2u_{4,xxx} + 98u_{4,xxx} + 98u_{4,xxx} + 35u_2u_{2,xxxxxx} \\
        &\quad + 7u_{8,x} + \frac{385}{2}u_{2,x}u_{2,xx} + \frac{483}{2}u_{2,xxx}u_{4,x} + 154u_{2,x}u_{2,xxx} + 287u_{2,xxx}u_{2,xxx} \\
        &\quad + 105u_2^2u_{4,x} + 91u_2^2u_{2,xxx} + 140u_2^3u_{2,x} + 210u_2u_4u_{2,xx} + \frac{701}{2}u_2u_{2,xxx}u_{2,xxx} \\
        &\quad + 14u_{6,xxx} + \frac{427}{3}u_2^3 - 7u_{4,xxxxxx} + \frac{13}{6}u_{2,xxxxxxx}.
\end{align*}
\]

(3.8)

One can see the first flow equations (3.7, 3.8) of BKP(CKP) hierarchy are trivial equations. If consider the 3-reduction (3.10) when \(n = 1\), we can calculate the \(t_7\) flow from the \(t_1\) flow. The first three equations of 3-reduction of BKP hierarchy are

\[
\begin{align*}
    u_4 &= -u_2^2 + \frac{2}{3}u_{2,xx}, \\
    u_6 &= -2u_2u_4 - \frac{11}{3}u_2^2 - \frac{1}{3}u_2^3 - \frac{7}{3}u_2u_{2,xx} + \frac{2}{3}u_{4,xx} - \frac{1}{3}u_{2,xxx}, \\
    u_8 &= -2u_2u_6 - \frac{7}{3}u_2u_{4,xxx} + 9u_2u_{2,xxx} - \frac{1}{3}u_2^2u_{2,xxx} - \frac{32}{3}u_{2,xxx}u_{4,x} - 10u_{4,xxx,xxx} \\
        &\quad + \frac{7}{3}u_{2,xxx} + \frac{2}{3}u_{6,xxx} - \frac{1}{3}u_{2,xxxxxx} + \frac{1}{6}u_{2,xxxxxxx} - u_2^2 - u_2^2u_4.
\end{align*}
\]

(3.9)

If one substitute (3.9) into (3.7), then the \(t_7\) flow equation of \(u_2\) (3.7) of BKP hierarchy can be reduced for

\[
\begin{align*}
    u_{2,t_7} &= - \frac{7}{9}u_2u_{2,xxxxxx} - \frac{14}{9}u_{2,xxx}u_{2,xxx} - \frac{7}{3}u_{2,xxx}u_{2,xxxxxx} - \frac{14}{3}u_{2,xxx}u_2^2 \\
        &\quad - \frac{28}{3}u_2^2u_{2,x} + 14u_{2,x}u_{2,xxx} - \frac{7}{3}u_{2,x}^2 - \frac{1}{27}u_{2,xxxxxxx}.
\end{align*}
\]

(3.10)

The first three equations of 3-reduction of CKP hierarchy are

\[
\begin{align*}
    u_4 &= -u_2^2 + \frac{1}{6}u_{2,xxx}, \\
    u_6 &= -2u_2u_4 - \frac{17}{12}u_2^2 - \frac{1}{3}u_2^3 - \frac{4}{3}u_2u_{2,xx} + \frac{2}{3}u_{4,xx} - \frac{1}{3}u_{2,xxx}, \\
    u_8 &= -2u_2u_6 + \frac{7}{3}u_2u_{2,xxx} - \frac{49}{6}u_{2,xxx}u_{4,x} + \frac{37}{3}u_2u_{2,xxx} - u_2^2 - \frac{7}{3}u_2u_{2,xxx} - \frac{4}{3}u_2u_{2,xxx} \\
        &\quad - \frac{8}{3}u_{2,xxx} - 8u_{2,xxx} + \frac{17}{12}u_{2,xxx} + \frac{2}{3}u_{6,xxx} - \frac{1}{3}u_{2,xxxxxx} + \frac{1}{18}u_{2,xxxxxxx} - u_2^2u_4.
\end{align*}
\]

(3.11)

If one substitute (3.11) into (3.8), then the \(t_7\) flow equation of \(u_2\) (3.8) of CKP hierarchy can be reduced for

\[
\begin{align*}
    u_{2,t_7} &= - \frac{35}{6}u_2^3 - \frac{28}{3}u_2^3u_{2,x} - \frac{14}{3}u_2^2u_{2,xxx} - \frac{14}{3}u_{2,xxx}u_{2,xxx} - 21u_{2,x}u_{2,xxx} \\
        &\quad - \frac{49}{18}u_{2,xxx}u_{2,xxx} - \frac{1}{27}u_{2,xxxxxxx} - \frac{7}{9}u_{2,xxxxxxx}.
\end{align*}
\]

(3.12)

But it is not easy to find a relation between the more other higher order flow equations and the lower order flow equations. And we will find the recursion operator which can generate the higher order flow equations from the lower order flow equations in the next section.
For \((2n + 1)\)-reduction, it only has the odd reduction and even dynamical variable in the BKP(CKP) hierarchy. If we denote
\[
\hat{U}(2n) = (u_2, u_4, \cdots, u_{2n})^t,
\]
\[
\hat{P}(2n + 1, 2m + 1) = (p_{-1}(2m + 1), p_{-3}(2m + 1), \cdots, p_{-2n+1}(2m + 1))^t,
\]
\[
Q(n) = \begin{pmatrix}
Q_{11} & 0 & \cdots & 0 \\
Q_{21} & Q_{22} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
Q_{n1} & Q_{n2} & \cdots & Q_{nn}
\end{pmatrix},
\]
where the up index \(t\) denotes the transpose of the matrix, then \((3.5)\) can be rewritten for
\[
\hat{U}(2n)_{t_{2m+1}} = Q(n)\hat{P}(2n + 1, 2m + 1).
\]
(3.13)
It is trivial to know that all the flow equations in \(\hat{U}(2n)_{t_{2m+1}}\) are local. Next, we want to study the recursion relation between \(t_{2m+1}+2p(2n+1)\) flow and \(t_{2m+1}\) flow.

4. Recursion Operator

In this section, we will discuss the recursion operator of BKP hierarchy and CKP hierarchy starting from the recursion operator of KP hierarchy. To do this, we must find a recursion formula relation between \(\hat{U}(2n)_{t_{2m+1}}\) and \(\hat{U}(2n)_{t_{2m+4n+3}}\) under the \((2n+1)\)-reduction constraint. That is, we try to find an operator \(\tilde{\Phi}(2n + 1)\), s.t. \(\hat{U}(2n)_{t_{2m+4n+3}} = \tilde{\Phi}(2n + 1)\hat{U}(2n)_{t_{2m+1}}\). Recall the result of the recursion operator of KP hierarchy [19, 20] under \(n\)-reduction, we have
\[
P(n, m + n) = R(n)P(n, m),
\]
(4.1)
where

\[
P(n, m) = (p_{-1}(m), p_{-2}(m), \cdots, p_{-n+1}(m))^t,
\]
\[
R(n) = S(n) - T(n)M(n)^{-1}N(n),
\]
\[
S(n) = \begin{pmatrix}
C_{-1,0}(n) & C_{-1,1}(n) & \cdots & C_{-1,n-2}(n) \\
C_{-2,1}(n) & C_{-2,0}(n) & \cdots & C_{-2,n-3}(n) \\
\vdots & \vdots & \ddots & \vdots \\
C_{-n+1,2}(n) & C_{-n+1,3}(n) & \cdots & C_{-n+1,n}(n)
\end{pmatrix}_{(n-1)\times(n-1)}
\]
\[
T(n) = \begin{pmatrix}
C_{-1,1-n}(n) & C_{-1,n}(n) & 0 & \cdots & 0 \\
C_{-2,1-n}(n) & C_{-2,n}(n) & C_{-2,n-1}(n) & 0 & \cdots \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
C_{-n+1,2}(n) & C_{-n+1,3}(n) & \cdots & C_{-n+1,n}(n) & C_{-n+1,n-1}(n)
\end{pmatrix}_{(n-1)\times n}
If we set \( \Phi(n) = Q(n)R(n)Q^{-1}(n) \), then the recursion formula of KP hierarchy [19, 26] is

\[
U(n)_{t_{m+j}n} = \Phi^2(n)U(n)_{t_m},
\]

(4.2)

where \( U(n) = (u_2, u_3, u_4, \ldots, u_{n-1}, u_n)^t \). If we substitute \( 2m + 1 \) for \( m \), \( 2n + 1 \) for \( n \) and \( j = 2 \) in (4.2), we have

\[
U(2n + 1)_{t_{2m+1}2(2n+1)} = \Phi^2(2n + 1)U(2n + 1)_{t_{2m+1}}.
\]

(4.3)

We consider the even element of \( U(2n + 1) \), which are the dynamical variables of BKP(CKP) hierarchy. Then it is necessary to calculate the odd flow equations of the even dynamical variables \( u_{2k, t_{2m+1+2p(2n+1)}} \). And if let \( \Phi(n)_{i,j} \) denote the \((i, j)\)-th element of the matrix \( \Phi(n) \),
from \((4.3)\), the \((2k - 1)\)-th elements of \(U(2n + 1)_{t_{2m+1+2(2n+1)}}\) are

\[
u_{2k,t_{2m+1+2(2n+1)}} = \sum_{i=1}^{2n} (\Phi^2(2n + 1))_{2k-1,i} u_{i+1,t_{2m+1}}
\]

\[
= \sum_{i=1}^{n} (\Phi^2(2n + 1))_{2k-1,2i-1} u_{2i,t_{2m+1}} + \sum_{i=1}^{n} (\Phi^2(2n + 1))_{2k-1,2i} u_{2i+1,t_{2m+1}}
\]

\[
= \sum_{i=1}^{n} (\Phi^2(2n + 1))_{2k-1,2i-1} u_{2i,t_{2m+1}} + \sum_{i=1}^{n} \sum_{\mu=1}^{\mu} (\Phi^2(2n + 1))_{2k-1,2i} B_{2i,-2\mu+1} u_{2\mu,t_{2m+1}}
\]

\[
= \sum_{\mu=1}^{\mu} ((\Phi^2(2n + 1))_{2k-1,2\mu-1} + \sum_{i=1}^{n} (\Phi^2(2n + 1))_{2k-1,2i} B_{2i,-2\mu+1}) u_{2\mu,t_{2m+1}}, k \leq n. \tag{4.4}
\]

It is used the formula \((2.28)\) for the third equality. If denote \(\hat{\Phi}(2n + 1)_{k,\mu} = (\Phi^2(2n + 1))_{2k-1,2\mu-1} + \sum_{i=1}^{n} (\Phi^2(2n + 1))_{2k-1,2i} B_{2i,-2\mu+1}\), then \((4.4)\) become

\[
u_{2k,t_{2m+1+2(2n+1)}} = \sum_{\mu=1}^{\mu} \hat{\Phi}(2n + 1)_{k,\mu} u_{2\mu,t_{2m+1}}. \tag{4.5}
\]

Further we denote

\[
\hat{\Phi}(2n + 1) = (\hat{\Phi}(2n + 1)_{k,\mu})
\]

\[
= ((\Phi^2(2n + 1))_{2k-1,2\mu-1} + \sum_{i=1}^{n} (\Phi^2(2n + 1))_{2k-1,2i} B_{2i,-2\mu+1}), \tag{4.6}
\]

and \(\hat{\Phi}(2n + 1)\) is a \(n \times n\) matrix because \(1 \leq k \leq n\) and \(1 \leq \mu \leq n\). Then for \(\hat{U}(2n) = (u_2, u_4, \ldots, u_{2n})^t\), one has

\[
\hat{U}(2n)_{t_{2m+1+2(2n+1)}} = \hat{\Phi}(2n + 1) \hat{U}(2n)_{t_{2m+1}}. \tag{4.7}
\]

With the above prepared knowledge, we have a theorem below.

**Theorem 4.1.** The flow equations of BKP(CKP) hierarchy under the \((2n+1)\)-reduction possess a recursion operator \(\hat{\Phi}(2n + 1)\) such that

\[
\hat{U}(2n)_{t_{2m+1+2p(2n+1)}} = \hat{\Phi}^p(2n + 1) \hat{U}(2n)_{t_{2m+1}}, \tag{4.8}
\]

where \(\hat{\Phi}(2n + 1)\) is defined by \((4.6)\).
Proof. With (4.6) and (4.7), it is clear that we have
\[
\hat{U}(2n)_{t_{2m+1+2p(2n+1)}} = \hat{U}(2n)_{t_{2(m+(p-1)(2n+1))}+1+2(2n+1)}
\]
\[
= \hat{\Phi}(2n+1)\hat{U}(2n)_{t_{2(m+(p-1)(2n+1))}+1}
\]
\[
= \hat{\Phi}(2n+1)\hat{U}(2n)_{t_{2m+1+2(p-1)(2n+1)}}
\]
\[
\cdots
\]
\[
= \hat{\Phi}^n(2n+1)\hat{U}(2n)_{t_{2m+1}}.
\]

\[\square\]

**Remark:** Under the \((2n+1)-\)reduction, \(t_1, t_3, \cdots, t_{2n-1}, t_{2n+3}, t_{2n+5}, \cdots, t_{4n+1}\)-flows are independent, and only \(n\) coordinates \((u_2, u_4, \cdots, u_{2n})\) are independent. That is just the \(2n\) flows can generate the whole BKP(CKP) hierarchy under the action of the recursion operator \(\hat{\Phi}(2n+1)\) (4.6). Though the recursion operator \(\hat{\Phi}(2n+1)\) is nonlocal, it doesn’t generate the nonlocal higher flow equations. Because the flow equations (2.8) are local, and the recursion operator \(\hat{\Phi}(2n+1)\) is derived from these flow equations. In particular, the difference of the recursion operators in eq. (4.8) of the BKP hierarchy and the CKP hierarchy is reflected by the appearance of the operator \(B\).

5. Applications

In this section, we will give some examples for the applications of formula (4.8). Here we only consider 3-reduction of the BKP and CKP hierarchies. For an example, we generate the \(t_7\) flow equation from the \(t_1\) flow equation for 3-reduction.

For the BKP hierarchy, set \(n = 1, m = 0\) and \(p = 1\) in (4.8), one can calculate
\[
\Phi(3) = \begin{pmatrix}
\Phi_{11}(3) & \Phi_{12}(3) \\
\Phi_{21}(3) & \Phi_{22}(3)
\end{pmatrix},
\]
(5.1)
where
\[
\begin{align*}
\Phi_{11}(3) &= \frac{1}{9}\partial^3 + \frac{1}{3}a_1 \partial - \frac{1}{3}a_{1,x} - \frac{1}{3}a_{1,xx} \partial^{-1}, \\
\Phi_{12}(3) &= \frac{2}{9}\partial^2 + \frac{2}{3}a_1 + \frac{2}{3}a_{1,x} \partial^{-1}, \\
\Phi_{21}(3) &= -\frac{2}{9}\partial^4 - \frac{2}{3}a_1 \partial^2 - \frac{2}{3}a_{1,x} \partial - \frac{2}{3}a_{1,xx} + (\frac{1}{9}a_{1,xxx} - \frac{2}{3}a_{1,xx} \partial^{-1}, \\
\Phi_{22}(3) &= -\frac{1}{3}\partial^3 - \frac{1}{3}a_1 \partial - a_{1,x} - \frac{1}{3}a_{1,xx} \partial^{-1},
\end{align*}
\]
and \(a_1(3) = 3u_2, a_0(3) = 0\). Because \(B_{-2,-1} = -\partial\), then the recursion operator is
\[
\hat{\Phi}(3) = \Phi_{11}(3) + \Phi_{12}(3)\Phi_{21}(3) - (\Phi_{11}(3)\Phi_{12}(3) + \Phi_{12}(3)\Phi_{22}(3))\partial
\]
\[
= -\frac{1}{27}\partial^5 - \frac{2}{3}u_2 \partial^4 - u_{2,x} \partial^3 - (\frac{11}{9}u_{2,xx} + 3u_2^2) \partial^2 - (\frac{10}{9}u_{2,xxx} + 7u_2 u_{2,x}) \partial
\]
\[
= \frac{5}{9}u_{2,xxxx} + 2u_{2,xx}^2 + 4u_2^3 + \frac{16}{3}u_2 u_{2,xx} - u_{2,x} \partial^{-1}(\frac{2}{3}u_{2,xx} + u_2^2)
\]
\[
= \frac{1}{9}u_{2,xxxxx} + \frac{5}{9}u_{2,xxxx} + \frac{5}{9}u_{2,xx} u_{2,xx} + 5u_2 u_{2,x} \partial^{-1}. 
\]
(5.2)
With the recursion operator (4.16), we can generate \( t_7 \) flow from \( t_1 \) flow by
\[
  u_{2,t_7} = - \frac{7}{9} u_{2,xxxx} + \frac{14}{9} u_{2,xxx} - \frac{7}{3} u_{2,xx} u_{2,xxx} - \frac{14}{3} u_{2,xxx} u_2^3 - \frac{28}{3} u_{2,xx} - 14 u_{2,xx} u_{2,xx} - \frac{7}{3} u_{2,xx} - \frac{1}{27} u_{2,xxxxxx},
\]
(5.3)
which consistent with the flow eq. (3.10) of the BKP hierarchy under 3-reduction. With a scaling transformations for \( u_2 \to \frac{u_2}{3} \) and \( t_7 \to -27t \), the operator (5.2) consistent with the formula (B3) of Ref. [20], and (5.3) become the flow equation
\[
  u_t = 3 u_{xxxxxx} + 15 u u_{xxxxxx} + 189 u_{xxx} + 1134 uu u_{xxx} + 126 u_{xxx} + 756 u^3 u_x + 189 uu_{xxx} + 126 u^2 u_{xxx}.
\]
(5.4)
Set \( n = 1, m = 0 \) and \( p = 1 \) in eq.(1.8) for the CKP hierarchy, then
\[
  \Phi(3) = \begin{pmatrix}
    \Phi(3)_{11} & \Phi(3)_{12} \\
    \Phi(3)_{21} & \Phi(3)_{22}
  \end{pmatrix},
\]
(5.5)
where
\[
  \begin{align*}
  \Phi(3)_{11} &= \frac{1}{3} \partial^3 + \frac{1}{3} a_1 \partial - \frac{1}{3} a_{1,x} + a_0(3) + (\frac{2}{3} a_{0,x} - \frac{2}{3} a_{1,x}) \partial^{-1}, \\
  \Phi(3)_{12} &= \frac{2}{3} \partial^2 + \frac{2}{3} a_1 + \frac{1}{3} a_{1,x} \partial^{-1}, \\
  \Phi(3)_{21} &= -\frac{2}{3} \partial^4 - \frac{4}{3} a_{1,xx} \partial^3 - \frac{2}{3} a_{1,x} \partial - \frac{2}{3} a_{0,x} + (\frac{1}{3} a_{1,xxx} - \frac{2}{3} a_{1,xx} - \frac{1}{3} a_{0,xx}) \partial^{-1}, \\
  \Phi(3)_{22} &= -\frac{1}{3} \partial^3 - \frac{4}{3} a_1 \partial - a_{1,x} + a_0 + \frac{1}{3} (a_{0,x} - a_{1,xx}) \partial^{-1},
  \end{align*}
\]
and \( a_1(3) = 3 u_2, a_0(3) = \frac{3}{2} u_2, \). Because \( B_{-2,-1} = -\frac{1}{2} \partial \), then the recursion operator is
\[
  \tilde{\Phi}(3) = \Phi(3)_{11}
  = \frac{1}{3} \partial^3 + \frac{1}{3} a_1 \partial - \frac{1}{3} a_{1,x} + a_0(3) + (\frac{2}{3} a_{0,x} - \frac{2}{3} a_{1,x}) \partial^{-1} - \frac{1}{27} \partial^6 - \frac{2}{3} u_{2,xx} \partial^3 - (\frac{49}{18} u_{2,xxx} + 3 u_{2}^2) \partial^2 - (\frac{35}{18} u_{2,xxx} + 10 u_{2,xx}) \partial \\
  - (\frac{13}{18} u_{2,xxxx} + \frac{41}{6} u_{2,xxx} + \frac{23}{4} u_{2,xx} + 4 u_{2}^3) - \frac{1}{6} u_{2,xx} \partial^{-1} (u_{2,xx} + 6 u_{2}^2) \\
  - \frac{1}{9} u_{2,xxxxx} + \frac{5}{3} u_{2,xxxx} + \frac{25}{6} u_{2,xxx} u_{2,xx} + 5 u_{2,xx} \partial^{-1}.
\]
(5.6)
We can generate \( t_7 \) flow from \( t_1 \) flow by
\[
  u_{2,t_7} = - \frac{35}{6} u_{2,xx} - \frac{28}{3} u_{2,xxx} - \frac{14}{3} u_{2,xxx} + 14 u_{2,xxx} u_{2,xxx} - \frac{14}{3} u_{2,xxx} u_{2,xxx} - 21 u_{2,xxx} u_{2,xxx} \\
  - \frac{49}{18} u_{2,xxx} + \frac{1}{27} u_{2,xxxxxx} - \frac{7}{9} u_{2,xxxxxx},
\]
(5.7)
which consistent with flow eq. (3.12) for the CKP hierarchy. With a scaling transformations $u_2 \to \frac{2}{3} u$ and $t_7 \to -27 t$, (5.6) is nothing but the formula (30) of Ref. [20] and (5.7) become the flow of equation

$$u_t = u_{xxxxxx} + 14uu_{xxxx} + 49uu_{xxx} + 84u_xu_{xx} + 56u^2u_{xxx} + \frac{224}{3}u^3u_x + 256uu_xu_{xx} + 70u^3. \quad (5.8)$$

**Remark:** If let $n = 1, m = 1$ and $p = 1$ in (4.8), we can also obtain the second recursion relation by $\hat{\Phi}(3)$, i.e. the formula $\hat{U}(2)_{t_{11}} = \hat{\Phi}(3)\hat{U}(2)_{t_5}$. It is not difficult to generate $t_{11}$ flow equation of $u_2$ from $t_5$ flow equation of it. If we choose properly the number of coefficient $n, m, p$ in (4.8), then $\hat{U}(2)_{t_{13}} = \hat{\Phi}^2(3)\hat{U}(2)_{t_1}$ is obtained for $n = 1, m = 0$ and $p = 2$, and $\hat{U}(2)_{t_{17}} = \hat{\Phi}^2(3)\hat{U}(2)_{t_5}$ is obtained for $n = 1$ and $m = 2 = p$. And the highest order of $\partial$ in $\hat{\Phi}^2(3)$ is 12. Of course one can also use it to generate the higher order flows.

### 6. Conclusions and Discussions

In this paper, we found in Proposition 2.2 that the odd dynamical variable $u_{2k+1}$ of $\{u_j, j \geq 1\}$ can be expressed by the even dynamical variable $u_{2k}$ of $\{u_j, j \geq 1\}$ in Lax operator $L$ by considering the constraint of BKP(CKP) hierarchy. The flow equations and the recursion operators of the BKP and the CKP hierarchies are given in a unified approach, which also reflect the two essential differences between the two sub-hierarchies of the KP hierarchy because of the appearance of the operator $B$. Two examples of the recursion operator are given explicitly for the BKP and CKP hierarchy under the 3-reduction. The $t_7$ flows are generated by these recursion operators again, which are consistent with the flow equations in the Lax equation. So the validity of these recursion operators is confirmed.

This research depicts deeply the integrability of KP hierarchy and BKP(CKP) hierarchy. And it will also be helpful for studying the difference of Hamiltonian structure, Poisson bracket between the KP hierarchy and BKP(CKP) hierarchy, which will be studied later.

**Acknowledgments** This work is supported by the NSF of China under Grant No.10971109 and Science Fund in Ningbo University (No.xk1062, No.XYL11012). Jingsong He is also supported by Program for NCET under Grant No.NCET-08-0515.

**References**

[1] L. A. Dickey, Soliton Equations and Hamiltonian Systems, (2nd edn. World Scientific, Singapore, 2003).
[2] P. J. Olver, Applications of Lie groups to differential equations, (New York: Springer, 1993).
[3] M. Jimbo, and T. Miwa, Solitions and infinite dimensional lie algebras, Publ. RIMS, Kyoto Univ. 19(1983), 943-1001.
[4] L.-L. Chau, J.-C. Shaw and H.-C. Yen, Solving the KP hierarchy by Gauge Transformations, Commun. Math. Phys. 149(1992), 263-278.
[5] J. S. He, Y. Cheng and A. Rudolf Römer, Solving bi-directional soliton equations in the KP hierarchy by gauge transformation, JHEP 03(2006) 103.
[6] L.-L. Chau, J.-C. Shaw and M.-H. Tu, Solving the constrained KP hierarchy by gauge transformations, J. Math. Phys. 38(1997), 4128-4137.
[7] R. Willox, I. Loris, C. R. Gilson, Binary Darboux transformations for constrained KP hierarchies, Inverse Problems 13(1997), 849-865.
[8] J. S. He, Z. W. Wu and Y. Cheng, Gauge transformations for the constrained CKP and BKP hierarchies, J. Math. Phys. 48(2007), 113-519.
[9] A. Yu. Orlov and E.I. Schulman, Additional symmetries for integrable systems and conformal algebra representation, Lett. Math. Phys. 12 (1986), 171-179.
[10] L. A. Dickey, On additional symmetries of the KP hierarchy and Sato's Bäcklund transformation, Comm. Math. Phys. 167(1995), 227-233.
[11] M. Adler, T. Shiota and P. van Moerbeke, A Lax representation for the vertex operator and the central extension, Comm. Math. Phys. 171 (1995), 547-588.
[12] J. van de Leur, The Adler-Shiota-van Moerbeke formula for the BKP hierarchy, J. Math. Phys. 36 (1995), 4940-4951.
[13] M. H. Tu, On the BKP hierarchy: Additional symmetries, Fay identity and Adler-Shiota-van Moerbeke formula, Lett. Math. Phys. 81(2007), 93-105.
[14] J. S. He, K. L. Tian, A. Foerster and W. X. Ma, Additional Symmetries and String Equation of the CKP Hierarchy, Lett. Math. Phys. 81(2007), 119-134.
[15] J. P. Cheng, J. S. He and S. Hu, The "ghost" symmetry of the BKP hierarchy, J. Math. Phys. 51(2010), 053514.
[16] K. L. Tian, J. S. He, J.P. Cheng and Y. Cheng, Additional symmetries of constrained CKP and BKP hierarchies, Science China Mathematics 54(2011), 257-268.
[17] H. F. Shen and M.H. Tu, On the constrained B-type Kadomtsev-Petviashvili hierarchy: Hirota bilinear equations and Virasoro symmetry, J. Math. Phys. 52(2011), 032704.
[18] P. J. Olver, Evolution equations possessing infinitely many symmetries, J. Math. Phys. 18(1977), 1212-1215.
[19] W. Strampp and W. Oevel, Recursion operators and Hamiltonian structures in Satos theory, Lett. Math. Phys. 20(1990), 195-210.
[20] M. Gurses, A. Karasu and V. V. Sokolov, On construction of recursion operators from Lax representation, J. Math. Phys. 40(1999), 6473-6490.
[21] I. Loris, Recursion operator for a constraint BKP system, In:Boiti M, Martina L, etal ed. Proceedings of the Workshop on Nonlinearity, Integrability and All That Twenty years After NEEDS ‘79. Singapore: World Scientific, 1999. 325-330.
[22] C. Z. Li, K. L. Tian, J. S. He, etal, Recursion operator for a constrained CKP hierarchy, Acta Mathematica Scientia 31B(2011)，no.4, 1295-1302.
[23] M. Boiti, J.JP.Leon, L. Martina and F. Pempinelli, On the recursion operator for the KP hierarchy in two and three spatial dimensions, Phys. Lett. A 123(1987), 340-344.
[24] A. S. Fokas and P. M. Santini, The Recursion Operator of the Kadomtsev-Petviashvili Equation and the Squared Eigenfunction of the Schrödinger Operators, Stud. Appl. Math. 75(1986), 179-186.
[25] P. M. Santini and A. S. Fokas, Recursion operators and bi-Hamiltonian structures in multidimensions. I, Commun. Math. Phys. 115(1988), 375-419.
[26] J. P. Cheng, L. H. Wang and J. S. He, Resursion operators for KP, mKP and Harry-Dym hierarchies, J. Nonlinear Mathematical Physics 18(2011), 161-178.
[27] A. S. Fokas and P. M. Santini, Recursion operators and bi-Hamiltonian structures in multidimensions. II, Commun. Math. Phys. 116(1988), 449-474.
[28] J. A. Sanders and J. P. Wang, Integrable systems and their recursion operators, Nonlinear Analysis 47(2001), 5213-5240.