Tight Approximation Bounds for Vertex Cover on Dense $k$-Partite Hypergraphs

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Abstract

We establish almost tight upper and lower approximation bounds for the Vertex Cover problem on dense $k$-partite hypergraphs.

1 Introduction

A hypergraph $H = (V, E)$ consists of a vertex set $V$ and a collection of hyperedges $E$ where a hyperedge is a subset of $V$. $H$ is called $k$-uniform if every edge in $E$ contains exactly $k$ vertices. A subset $C$ of $V$ is a vertex cover of $H$ if every edge $e \in E$ contains at least a vertex of $C$.

The Vertex Cover problem in a $k$-uniform hypergraph $H$ is the problem of computing a minimum cardinality vertex cover in $H$. It is well known that the problem is $NP$-hard even for $k = 2$ (cf. [13]). On the other hand, the simple greedy heuristic which chooses a maximal set of nonintersecting edges, and then outputs all vertices in those edges, gives a $k$-approximation algorithm for the Vertex Cover problem restricted to $k$-uniform hypergraphs. The best known approximation algorithm achieves a slightly better approximation ratio of $(1 - o(1))k$ and is due to Halperin [11].

On the intractability side, Trevisan [22] provided one of the first inapproximability results for the $k$-uniform vertex cover problem and obtained an inapproximability factor of $k^\Omega$ assuming $P \neq NP$. In 2002, Holmerin [11] improved the factor to $k^{1-\epsilon}$. Dinur et al. [7, 8] gave consecutively two lower bounds, first $(k - 3 - \epsilon)$ and later on $(k - 1 - \epsilon)$. Moreover, assuming Khot’s Unique Games Conjecture (UGC) [17], Khot and Regev [18] proved an inapproximability factor of $k - \epsilon$ for the Vertex Cover problem on $k$-uniform hypergraphs. Therefore, it implies that the currently achieved ratios are the best possible.

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The Vertex Cover problem restricted to \(k\)-partite \(k\)-uniform hypergraphs, when the underlying partition is given, was studied by Lovász [20] who achieved a \(\frac{\epsilon}{2}\) approximation. This approximation upper bound is obtained by rounding the natural LP relaxation of the problem. The above bound on the integrality gap was shown to be tight in [11]. As for the lower bounds, Guruswami and Saket [10] proved that it is NP-hard to approximate the Vertex Cover problem on \(k\)-partite \(k\)-uniform hypergraphs to within a factor of \(\frac{k}{2} - \epsilon\) for \(k \geq 5\). Assuming the Unique Games Conjecture, they also provided an inapproximability factor of \(\frac{k}{2} - \epsilon\) for \(k \geq 3\). More recently, Sachdeva and Saket [21] claimed a nearly optimal NP-hardness factor.

To gain better insights on lower bounds, dense instances of many optimization problems has been intensively studied [2, 15, 16, 14]. The Vertex Cover problem has been investigated in the case of dense graphs, where the number of edges is within a constant factor of \(n^2\), by Karpinski and Zelikovsky [16], Eremeev [9], Clementi and Trevisan [6], later by Bar-Yehuda and Kehat [4] as well as Imamura and Iwama [12].

The Vertex Cover problem restricted to dense balanced \(k\)-partite \(k\)-uniform hypergraphs was introduced and studied in [5], where it was proved that this restricted version of the problem admits an approximation ratio better than \(\frac{k}{2}\) if the given hypergraph is dense enough.

In this paper, we give a new approximation algorithm for the Vertex Cover problem restricted to dense \(k\)-partite \(k\)-uniform hypergraphs and prove that the achieved approximation ratio is almost tight assuming the Unique Games Conjecture.

2 Definitions and Notations

Given a natural number \(i \in \mathbb{N}\), we introduce for notational simplicity the set \([i] = \{1, ..., i\}\) and set \([0] = \emptyset\). Let \(S\) be a finite set with cardinality \(s\) and \(k \in [s]\). We will use the abbreviation \(\binom{S}{k} = \{S' \subseteq S \mid |S'| = k\}\).

A \(k\)-uniform hypergraph \(H = (V(H), E(H))\) consists of a set of vertices \(V\) and a collection \(E \subseteq \binom{V}{k}\) of edges. For a \(k\)-uniform hypergraph \(H\) and a vertex \(v \in V(H)\), we define the neighborhood \(N_H(v)\) of \(v\) by \(\left(\bigcup_{e \in E | v \in e} e\right) \setminus \{v\}\) and the degree \(d_H(v)\) of \(v\) to be \(|\{e \in E \mid v \in e\}|\). We extend this notion to subsets of \(V(H)\), where \(S \subseteq V(H)\) obtains the degree \(d_H(S)\) by \(|\{e \in E \mid S \subseteq e\}|\).

A \(k\)-partite \(k\)-uniform hypergraph \(H = (V_1, ..., V_k, E(H))\) is a \(k\)-uniform hypergraph such that \(V\) is a disjoint union of \(V_1, ..., V_k\) with \(|V_i \cap e| = 1\) for every \(e \in E\) and \(i \in [k]\). In the remainder, we assume that \(|V_i| \geq |V_{i+1}|\) for all \(i \in [k-1]\) and \(k = O(1)\).

A balanced \(k\)-partite \(k\)-uniform hypergraph \(H = (V_1, ..., V_k, E(H))\) is a \(k\)-partite \(k\)-uniform hypergraph with \(|V_i| = \frac{|V|}{k}\) for all \(i \in [k]\). We set \(n = |V|\) and \(m = |E|\) as usual.

For a \(k\)-partite \(k\)-uniform hypergraph \(H = (V_1, ..., V_k, E(H))\) and \(v \in V_k\), we introduce the \(v\)-induced hypergraph \(H(v)\), where the edge set of \(H(v)\) is defined by
\{ e \setminus \{ v \} \mid v \in e \in E(H) \} and the vertex set of \( H(v) \) is partitioned into \( V_i \cap N_H(v) \) with \( i \in [k - 1] \).

A vertex cover of a \( k \)-uniform hypergraph \( H = (V(H), E(H)) \) is a subset \( C \) of \( V(H) \) with the property that \( e \cap C \neq \emptyset \) holds for all \( e \in E(H) \). The Vertex Cover problem consists of finding a vertex cover of minimum size in a given \( k \)-uniform hypergraph. The Vertex Cover problem in \( k \)-partite \( k \)-uniform hypergraphs is the restricted problem, where a \( k \)-partite \( k \)-uniform hypergraph and its vertex partition is given as a part of the input.

We define a \( k \)-partite \( k \)-uniform hypergraph \( H = (V_1, ..., V_k, E(H)) \) as \( \epsilon \)-dense for an \( \epsilon \in [0, 1] \) if the following condition holds:

\[
|E(H)| \geq \epsilon \prod_{i \in [k]} |V_i|
\]

For \( \ell \in [k - 1] \), we introduce the notion of \( \ell \)-wise \( \epsilon \)-dense \( k \)-partite \( k \)-uniform hypergraphs. Given a \( k \)-partite \( k \)-uniform hypergraph \( H \), if there exists an \( I \in \binom{[k]}{\ell} \) and an \( \epsilon \in [0, 1] \) such that for all \( S \) with the property \( |V_i \cap S| = 1 \) for all \( i \in I \) the condition

\[
d_H(S) \geq \epsilon \prod_{i \in [k] \setminus I} |V_i|
\]

holds, we define \( H \) to be \( \ell \)-wise \( \epsilon \)-dense.

## 3 Our Results

In this paper, we give an improved approximation upper bound for the Vertex Cover problem restricted to \( \epsilon \)-dense \( k \)-partite \( k \)-uniform hypergraphs. The approximation algorithm in [5] yields an approximation ratio of

\[
\frac{k}{k - (k - 2)(1 - \epsilon) e^{-\ell}}
\]

for \( \ell \)-wise \( \epsilon \)-dense balanced \( k \)-partite \( k \)-uniform hypergraphs. Here, we design an algorithm with an approximation factor of

\[
\frac{k}{2 + (k - 2)\epsilon}
\]

for the \( \epsilon \)-dense case which also improves on the \( \ell \)-wise \( \epsilon \)-dense balanced case for all \( \ell \in [k - 2] \) and matches their bound when \( \ell = k - 1 \). A further advantage of this algorithm is that it applies to a larger class of hypergraphs since the considered hypergraph is not necessarily required to be balanced.

As a byproduct, we obtain a constructive proof that a vertex cover of an \( \epsilon \)-dense \( k \)-partite \( k \)-uniform hypergraph \( H = (V_1, ..., V_k, E(H)) \) is bounded from below by \( \epsilon |V_k| \), which is shown to be sharp by constructing a family of tight examples.

On the other hand, we provide inapproximability results for the Vertex Cover problem restricted to \( \ell \)-wise \( \epsilon \)-dense balanced \( k \)-partite \( k \)-uniform hypergraphs under the Unique Games Conjecture. We also prove that this reduction yields a
matching lower bound if we use a conjecture on the Unique Games hardness of the Vertex Cover problem restricted to balanced $k$-partite $k$-uniform hypergraphs. This means that further restrictions such as $\ell$-wise density cannot lead to improved approximation ratios and our proposed approximation algorithm is best possible assuming this conjecture. In addition, we are able to prove an inapproximability factor under $P \neq NP$.

4 Approximation Algorithm

In this section, we give a polynomial time approximation algorithm with improved approximation factor for the Vertex Cover problem restricted to $\epsilon$-dense $k$-partite $k$-uniform hypergraphs.

We state now our main result.

**Theorem 1.** There exists a polynomial time approximation algorithm with approximation ratio

$$\frac{k}{2 + (k - 2)\epsilon}$$

for the Vertex Cover problem in $\epsilon$-dense $k$-partite $k$-uniform hypergraphs.

A crucial ingredient of the proof of Theorem 1 is Lemma 1 in which we show that we can extract efficiently a large part of an optimal vertex cover of a given $\epsilon$-dense $k$-partite $k$-uniform hypergraph $H = (V_1, \ldots, V_k, E(H))$. More precisely, we obtain in this way a constructive proof that the size of a vertex cover of $H$ is bounded from below by $\epsilon|V_k|$. The procedure for the extraction of a part of an optimal vertex cover is given in Figure 1.

We now formulate Lemma 1:

**Lemma 1.** Let $H = (V_1, \ldots, V_k, E(H))$ be an $\epsilon$-dense $k$-partite $k$-uniform hypergraph with $k \geq 1$. Then, the procedure $\text{Extract}(\cdot)$ computes in polynomial time a collection $R$ of subsets of $V_k$ such that the size of $R$ is polynomial in $|V_k|$ and $R$ contains a set $S$, which is a subset of an optimal vertex cover of $H$ and its cardinality is at least $\epsilon|V_k|$.

As a consequence, we obtain directly:

**Corollary 1.** Given an $\epsilon$-dense $k$-partite $k$-uniform hypergraph $H = (V_1, \ldots, V_k, E(H))$ with $k \geq 1$, the cardinality of an optimal vertex cover of $H$ is bounded from below by $\epsilon|V_k|$.

Before we prove Lemma 1 we describe the main idea of the proof. Let $\text{OPT}$ denote an optimal vertex cover of $H$. The procedure $\text{Extract}(\cdot)$ tests for the set $R = \{v_1, \ldots, v_p\}$ of the $p$ heaviest vertices of $V_k$, if $\{v_1, \ldots, v_{u-1}\} \subseteq \text{OPT}$ and $v_u \notin \text{OPT}$ for every $u \in [p]$. Clearly, either $R \subseteq \text{OPT}$ or there exists a $v_u$ such that $v_u \notin \text{OPT}$. If the procedure already possesses a part of $\text{OPT}$ denoted by $R_u$, then,
**Procedure Extract(·)**

Input: $\epsilon$-dense $k$-partite $k$-uniform hypergraph $H = (V_1, ..., V_k, E)$ with $k \geq 1$

1. IF $k = 1$ THEN
   (a) RETURN $\{\bigcup_{e \in E} e\}$

2. ELSE:
   (a) Let $(v_1, ..., v_p)$ be the vector consisting of the first $p = \left\lceil \frac{|E|}{\prod_{l \in [k-1]} |V_l|} \right\rceil$ heaviest vertices of $V_k$ with $d_H(v_i) \geq d_H(v_{i+1})$
   (b) $R = \{\{v_1, ..., v_p\}\}$
   (c) FOR $i = 1, ..., p$ DO:
      i. $R_i = \{v_k \mid k \in [i-1]\}$
      ii. Invoke $\text{Extract}(H(v_i))$ with output $O$
      iii. $R = R \cup \{R_i \cup S \mid S \in O\}$

3. RETURN $R$

---

**Figure 1: Procedure Extract**

$\text{Extract}(\cdot)$ tries to obtain a large part of an optimal vertex cover of the $v_u$-induced hypergraph $H(v_u)$. Hence, we have to show that $H(v_u)$ must still be dense enough. We now give the proof of Lemma 1.

**Proof.** The proof of Lemma 1 will be split in several parts. In particular, we show that given an $\epsilon$-dense $k$-partite $k$-uniform hypergraph $H = (V_1, ..., V_k, E(H))$, the procedure $\text{Extract}(\cdot)$ and its output $R$ possess the following properties:

1. $\text{Extract}(\cdot)$ constructs $R$ in polynomial time and the cardinality of $R$ is $O(n^k)$.

2. There is a $S \in R$ such that $S$ is a subset of an optimal vertex cover of $H$.

3. For every $S \in R$, the cardinality of $S$ is at least $|S| \geq \epsilon |V_k|$.

(1.) Clearly, $R$ is upper bounded by $|V_1|^k = O(n^k)$ and therefore, the running time of $\text{Extract}(\cdot)$ is $O(n^k)$.

(2.) and (3.) We prove the remaining properties by induction. If we have $k = 1$, the set $\bigcup_{e \in E(H)} e$ is by definition an optimal vertex cover of $H = (V_1, E(H))$. Since $H$ is $\epsilon$-dense, the cardinality of $|E(H)|$ is lower bounded by $\epsilon |V_1|$. We assume that $k > 1$. Let $H = (V_1, ..., V_k, E(H))$ be an $\epsilon$-dense $k$-partite $k$-uniform hypergraph and $OPT \subseteq V(H)$ an optimal vertex cover of $H$. Let $(v_1, ..., v_p)$ be the vector consisting of the first $p = \left\lceil \frac{|E(H)|}{\prod_{l \in [k-1]} |V_l|} \right\rceil$ heaviest vertices of $V_k$ with $d_H(v_i) \geq d_H(v_{i+1})$. If $\{v_1, ..., v_p\}$ is contained in $OPT$, we have constructed a subset of an
optimal vertex cover with cardinality

\[ p = \left\lceil \frac{|E(H)|}{\prod_{i \in [k-1]} |V_i|} \right\rceil \geq \frac{\epsilon \prod_{i \in [k]} |V_i|}{|V_k|} \geq \epsilon |V_k|. \]

Otherwise, there is an \( u \in [p] \) such that \( R_u \subseteq OPT \) and \( v_u \notin OPT \). But this means that an optimal vertex cover of \( H \) contains an optimal vertex cover of the \( v_u \)-induced \((k - 1)\)-partite \((k - 1)\)-uniform hypergraph \( H(v_u) \) in order to cover the edges \( e \in \{e \in E \mid v_u \in e\} \). The situation is depicted in Figure 2.

**Figure 2**: The \( v_u \)-induced \((k - 1)\)-partite \((k - 1)\)-uniform hypergraph \( H(v_u) \)

By our induction hypothesis, \( \text{Extract}(H(v_u)) \) contains a set \( S_u \) which is a subset of a minimum vertex cover of \( H(v_u) \) and of \( OPT \). The only claim, which remains to be proven, is that the cardinality of \( S_u \) is large enough. More precisely, we show that \( |S_u| \) can be lower bounded by \( \epsilon |V_k| - |R_u| \). Therefore, we need to analyze the density of the \( v_u \)-induced hypergraph \( H(v_u) \). The edge set of \( H(v_u) \) is given by \( \{e \setminus \{v_u\} \mid v_u \in e \in E\} \). Thus, we have to obtain a lower bound on the degree of \( v_u \). Since \( \{|e \in E \mid e \cap R_u \neq \emptyset\}| \) is upper bounded by \( |R_u| \prod_{i \in [k-1]} |V_i| \), the vertices in \( V_k \setminus R_u \) possess the average degree of at least

\[
\frac{\sum_{v \in V_k \setminus R_u} \text{deg}_H(v)}{|V_k \setminus R_u|} \geq \frac{\epsilon \prod_{i \in [k]} |V_i| - |\{e \in E \mid e \cap R_u \neq \emptyset\}|}{|V_k \setminus R_u|} \quad (1)
\]

\[
\geq \frac{\epsilon \prod_{i \in [k]} |V_i| - |R_u| \prod_{i \in [k-1]} |V_i|}{|V_k \setminus R_u|} \quad (2)
\]

\[
\geq \frac{(\epsilon |V_k| - |R_u|) \prod_{i \in [k-1]} |V_i|}{|V_k \setminus R_u|} \quad (3)
\]
Since the heaviest vertex in \( V_k \setminus R_u \) must have a degree of at least 
\[
\frac{(\epsilon|V_k| - |R_u|) \prod_{i \in [k-1]} |V_i|}{|V_k \setminus R_u|},
\]
we deduce that the edge set of \( H(v_u) \) denoted by \( E_u \) can be lower bounded by
\[
|E_u| \geq \frac{(\epsilon|V_k| - |R_u|) \prod_{i \in [k-1]} |V_i|}{|V_k \setminus R_u|} |V_{k-1}|
\]
Let \( H(v_u) \) be defined by \((V_i^u, \ldots, V_{k-1}^u, E_u)\) with \(|V_i^u| \leq |V_i|\) for all \( i \in [k-1] \). By our induction hypothesis, the size of every set contained in \( \text{Extract}(\cdot) \) is at least
\[
\frac{|E_u|}{\prod_{i \in [k-1]} |V_i^u|} |V_{k-1}| \geq \frac{(\epsilon|V_k| - |R_u|) \prod_{i \in [k-1]} |V_i|}{|V_k \setminus R_u|} |V_{k-1}|
\]
\[
\geq \frac{(\epsilon|V_k| - |R_u|) \prod_{i \in [k-1]} |V_i|}{|V_k \setminus R_u|} |V_k| = \epsilon|V_k| - |R_u|
\]
In (4), we used the fact that \(|V_i^u| \leq |V_i|\) for all \( i \in [k-1] \). Whereas in (5), we used our assumption \(|V_k| \geq |V_{k-1}|\). All in all, we obtain
\[
|R_u \cup S_u| \geq |R_u| + (\epsilon|V_k| - |R_u|) = \epsilon|V_k|.
\]
Clearly, this argumentation on the size of \( R_u \cup S_u \) holds for every \( u \in [p] \) and the proof of Lemma 4 follows.

Before we state our approximation algorithm and prove Theorem 1 we show that the bound in Lemma 4 is tight. In particular, we define a family of \( \epsilon \)-dense \( k \)-partite \( k \)-uniform hypergraphs \( H(k, l, \epsilon) = (V_1, \ldots, V_k, E(H)) \) with \(|V_i| = \frac{|V_k|}{l}\) for all \( i \in [k], k \geq 1, \epsilon \in \left\{ \frac{\ell}{l} \mid u \in [l] \right\} \) and \( l \geq 1 \) such that \( \text{Extract}(\cdot) \) returns a subset of an optimal vertex cover with cardinality of exactly \( \epsilon|V_k| \).

**Lemma 2.** The bound of Lemma 4 is tight.

**Proof.** Let us define \( H(k, p, \epsilon) = (V_1, \ldots, V_k, E) \). For a fixed \( p \geq 1 \) and \( k \geq 1 \), every partition \( V_i \) with \( i \in [k] \) consists of a set of \( l \) vertices. Let us fix a \( \epsilon = \frac{u}{l} \) with \( u \in [l] \). Then, \( H(k, l, \epsilon) \) contains the set \( V_i^u \subseteq V_k \) of \( u \) vertices such that \( E = \{\{v_1, v_2, \ldots, v_k\} \mid v_1 \in V_i^u, v_2 \in V_2, \ldots, v_k \in V_k\} \). An example of such a hypergraph is depicted in Figure 3.

Notice that \( H(k, l, \epsilon) = (V_1, \ldots, V_k, E) \) is \( \epsilon \)-dense, since
\[
\frac{|E|}{\prod_{j \in [k]} |V_j|} = \frac{|V_i^u|}{|V_k|} = \frac{u}{l} = \epsilon.
\]
The procedure $\text{Extract}(\cdot)$ returns a set $R$, in which $V_k^u$ is contained, since $V_k^u$ is the set of the $p$ heaviest vertices of $V_k$. Hence, we obtain $|V_k^u| = \frac{|V_k^u|}{|V_k|}|V_k| = \epsilon|V_k|$. On the other hand, the remaining hypergraph $H’ = (V_1, \ldots, V_k \setminus V_k^u, E(H’))$ with edge set $E(H’) = \{e \in E \mid e \cap V_k^u = \emptyset\}$ is already covered, since $E(H’)$ is by definition of $H(k, p, \epsilon)$ the empty set. Therefore, $V_k^u$ is a vertex cover of $H(k, p, \epsilon)$ and since, according to Corollary 1, every vertex cover is bounded from below by $\epsilon|V_k|$, $V_k^u$ must be an optimal vertex cover.

Next, we state our approximation algorithm for the Vertex Cover problem in $\epsilon$-dense $k$-partite $k$-uniform hypergraphs defined in Figure 4. The approximation algorithm combines the procedure $\text{Extract}(\cdot)$ to generate a large enough subset of an optimal vertex cover together with the $\frac{k}{2}$-approximation algorithm due to Lovász [20] applied to the remaining instance.

Algorithm $\text{Approx}(\cdot)$

Input: $\epsilon$-dense $k$-partite $k$-uniform hypergraph $H = (V_1, \ldots, V_k, E)$ with $k \geq 3$

1. $T = \{V_k\}$
2. invoke procedure $\text{Extract}(H)$ with output $R$
3. for all $S \in R$ do :
   (a) $H_S = (V(H) \setminus S, \{e \in E(H) \mid e \cap S = \emptyset\})$
   (b) obtain a $(\frac{k}{2})$-approximate solution $S_k$ for $H_S$
   (c) $T = T \cup \{S_k \cup S\}$
4. Return the smallest set in $T$
We now prove Theorem 1.

**Proof.** Let \( H = (V_1, \ldots, V_k, E) \) be an \( \epsilon \)-dense \( k \)-partite \( k \)-uniform hypergraph. From Lemma 1, we know that the procedure \( \text{Extract}(\cdot) \) returns in polynomial time a collection \( C \) of subsets of \( V(H) \) such that there is a set \( S \) in \( C \), which is contained in an optimal vertex cover of \( H \). Moreover, we know that the size of \( S \) is lower bounded by \( \epsilon |V_k| \).

Next, we analyze the approximation ratio of our approximation algorithm \( \text{Approx}(\cdot) \). Clearly, the size of an optimal vertex cover of \( H \) is upper bounded by \( |V_k| \). Let us denote by \( \text{OPT}' \) the size of an optimal vertex cover of the remaining hypergraph \( H' \) defined by removing all edges \( e \) of \( H \) with \( e \cap S \neq \emptyset \). Furthermore, let \( S' \) be the solution of the \( \frac{k}{2} \)-approximation algorithm applied to \( H' \). The approximation ratio of \( \text{Approx}(\cdot) \) is bounded by

\[
\frac{|S| + |S'|}{|S'| + |\text{OPT}'|} \leq \frac{k}{2 + (k - 2)\epsilon |V_k|} \quad (9)
\]

In (11), we used the fact that the size of the output of \( \text{Approx}(\cdot) \) is upper bounded by \( |V_k| \). Therefore, we have \( |S| + \frac{k}{2} |\text{OPT}'| \leq |V_k| \). In (12), we know from Lemma 1 that \( |S| \geq \epsilon |V_k| \).

\[ \square \]

### 5 Inapproximability Results

In this section, we prove hardness results for the Vertex Cover problem restricted to \( \ell \)-wise \( \epsilon \)-dense balanced \( k \)-uniform \( k \)-partite hypergraphs under the Unique Games Conjecture [17] as well as under the assumption \( P \neq NP \).

#### 5.1 UGC-Hardness

The Unique Games-hardness result of [10] was obtained by applying the result of Kumar et al. [19], with a modification to the LP integrality gap due to Ahorani et al. [1]. More precisely, they proved the following inapproximability result:
Theorem 2. [10] For every \( \delta > 0 \) and \( k \geq 3 \), there exist a \( n_\delta \) such that given \( H = (V_1, \ldots, V_k, E(H)) \) as an instance of the Vertex Cover problem in balanced \( k \)-partite \( k \)-uniform hypergraphs with \( |V(H)| \geq n_\delta \), the following is UGC-hard to decide:

- The size of a vertex cover of \( H \) is at least \( |V| \left( \frac{1}{2(k-1)} - \delta \right) \).
- The size of an optimal vertex cover of \( H \) is at most \( |V| \left( \frac{1}{k(k-1)} + \delta \right) \).

As the starting point of our reduction, we use Theorem 2 and prove the following:

Theorem 3. For every \( \delta > 0 \), \( \epsilon \in (0, 1) \), \( \ell \in [k-1] \), and \( k \geq 3 \), there exists no polynomial time approximation algorithm with an approximation ratio

\[
\frac{k}{2 + \frac{2(k-1)(k-2)}{k} \epsilon} - \delta
\]

for the Vertex Cover problem in \( \ell \)-wise \( \epsilon \)-dense \( k \)-partite \( k \)-uniform hypergraphs assuming the Unique Games Conjecture.

Proof. First, we concentrate on the \( \epsilon \)-dense case and afterwards, we extend the range of \( \ell \). As a starting point of the reduction, we use the \( k \)-partite \( k \)-uniform hypergraph \( H = (V_1, \ldots, V_k, E(H)) \) from Theorem 2 and construct an \( \epsilon \)-dense \( k \)-partite \( k \)-uniform hypergraph \( H' = (V'_1, \ldots, V'_k, E') \).

Let us start with the description of \( H' \). First, we join the set \( C_i \) of \( \frac{\epsilon}{1-\epsilon} \frac{n}{k} \) vertices to \( V_i \) for every \( i \in [k] \) and add all possible edges \( e \) of \( H' \) to \( E' \) with the restriction \( C_i \cap e \neq \emptyset \). Thus, we obtain \( |V'_i| = \frac{n}{k} + \frac{\epsilon}{1-\epsilon} \frac{n}{k} \) for all \( i \in [k] \).

Now, let us analyze how the size of the optimal solution of \( H' \) transforms. We denote by \( OPT' \) an optimal vertex cover of \( H' \). The UGC-hard decision question from Theorem 2 transforms into the following:

\[
n \left( \frac{1}{2(k-1)} - \delta \right) + \frac{\epsilon}{1-\epsilon} \frac{n}{k} \leq |OPT'| \quad \text{or} \quad |OPT'| \leq n \left( \frac{1}{k(k-1)} + \delta \right) + \frac{\epsilon}{1-\epsilon} \frac{n}{k}
\]

Assuming the UGC, this implies the hardness of approximating the Vertex Cover problem in \( \epsilon \)-dense hypergraphs for every \( \delta' > 0 \) to within:

\[
\frac{n \left( \frac{1}{2(k-1)} - \delta \right) + \frac{\epsilon}{1-\epsilon} \frac{n}{k}}{n \left( \frac{1}{k(k-1)} + \delta \right) + \frac{\epsilon}{1-\epsilon} \frac{n}{k}} = \frac{1-\epsilon}{2(k-1)} - \delta(1-\epsilon) + \frac{\epsilon}{k} = \frac{1-\epsilon}{k(k-1)} + \delta(1-\epsilon) + \frac{\epsilon}{k}
\]

\[
= \frac{(1-\epsilon)k}{2(k-1)k} + \frac{2\epsilon(k-1)}{2k(k-1)} - \delta'
\]

(15)
Finally, we have to verify that the constructed hypergraph $H'$ is indeed $\epsilon$-dense. Notice that $H'$ can have at most $(\left|V_1\right|)^k = \left(\frac{n}{k} + \frac{\epsilon}{1-\epsilon} \frac{n}{k}\right)^k$ edges. Therefore, we obtain the following:

\[
\left(\frac{\epsilon}{1-\epsilon} \frac{n}{k}\right) \left(\frac{n}{k} + \frac{\epsilon}{1-\epsilon} \frac{n}{k}\right)^{k-1} = \frac{n}{k} \left(1 + \frac{\epsilon}{1-\epsilon}\right) = \frac{n}{k} \left(1 + \frac{\epsilon}{1-\epsilon}\right) = \epsilon
\]

Notice that the constructed hypergraph is also $\ell$-wise $\epsilon$-dense balanced. Hence, we obtain the same inapproximability factor in this case as well.

Next, we combine the former construction with a conjecture about Unique Games hardness of the Vertex Cover problem in balanced $k$-partite $k$-uniform hypergraphs. In particular, we postulate the following:

**Conjecture 1.** Given a balanced $k$-partite $k$-uniform hypergraph $H = (V_1, \ldots, V_k, E(H))$ with $k \geq 3$, let $OPT$ denote an optimal vertex cover of $H$. For every $\delta > 0$, the following is UGC-hard to decide:

\[
|V| \left(\frac{1}{k} - \delta\right) \leq |OPT| \quad \text{or} \quad |OPT| \leq |V| \left(\frac{2}{k^2} + \delta\right)
\]

Combining Conjecture 1 with the construction in Theorem 3, it yields the following inapproximability result which matches precisely the approximation upper bound achieved by our approximation algorithm described in Section 4:

**Theorem 4.** For every $\delta > 0$, $\epsilon \in (0, 1)$, $\ell \in [k-1]$, and $k \geq 3$, there exists no polynomial time approximation algorithm with an approximation ratio

\[
\frac{k}{2 + (k-2)\epsilon} - \delta
\]
for the Vertex Cover problem in ℓ-wise ϵ-dense k-partite k-uniform hypergraphs assuming Conjecture [1].

Proof. The UGC-hard decision question from Conjecture [1] transforms into the following:

\[
n \left( \frac{1}{k} - \delta \right) + \frac{\epsilon}{1 - \epsilon} \frac{n}{k} \leq |OPT| \quad \text{or} \quad |OPT| \leq n \left( \frac{2}{k^2} + \delta \right) + \frac{\epsilon}{1 - \epsilon} \frac{n}{k}
\]

Assuming the UGC, this implies the hardness of approximating the Vertex Cover problem in ϵ-dense k-partite k-uniform hypergraphs for every δ’ > 0 to within:

\[
\frac{n \left( \frac{1}{k} - \delta \right) + \epsilon}{n \left( \frac{2}{k^2} + \delta \right) + \frac{\epsilon}{1 - \epsilon} \frac{n}{k}} = \frac{n \left( \frac{1}{k} - \delta \right) (1 - \epsilon) + \frac{\epsilon n}{k}}{n \left( \frac{2}{k^2} + \delta \right) (1 - \epsilon) + \frac{\epsilon n}{k}} \quad \text{(23)}
\]

\[
= \frac{n}{n \left( \frac{2}{k^2} \right) (1 - \epsilon) + \frac{k\epsilon n}{k^2}} - \delta' \quad \text{(24)}
\]

\[
= \frac{k}{2(1 - \epsilon) + k\epsilon} - \delta' \quad \text{(25)}
\]

\[
= \frac{k}{2 + (k - 2)\epsilon} - \delta' \quad \text{(26)}
\]

5.2 NP-Hardness

Recently, Sachdeva and Saket proved in [21] a nearly optimal NP-hardness of the Vertex Cover problem on balanced k-uniform k-partite hypergraphs. More precisely, they obtained the following inapproximability result:

Theorem 5. [21] Given a balanced k-partite k-uniform hypergraph \( H = (V, E) \) with \( k \geq 4 \), let OPT denote an optimal vertex cover of \( H \). For every \( \delta > 0 \), the following is NP-hard to decide:

\[
|V| \left( \frac{k}{2(k+1)(2(k+1)+1)} - \delta \right) \leq |OPT|
\]

or

\[
|V| \left( \frac{1}{k(2(k+1)+1)} + \delta \right) \geq |OPT|
\]

Combining our reduction from Theorem [2] with Theorem [5] we prove the following inapproximability result under the assumption \( P \neq NP \):

Theorem 6. For every \( \delta > 0 \), \( \epsilon \in (0, 1) \), \( \ell \in [k-1] \), and \( k \geq 4 \), there is no polynomial time approximation algorithm with an approximation ratio:

\[
\frac{k^2(1 - \epsilon) + \epsilon 2(k+1)(2(k+1)+1)}{2(k+1)[1 - \epsilon + \epsilon(2(k+1)+1)]} - \delta
\]
for the Vertex Cover problem in \( \ell \)-wise \( \epsilon \)-dense \( k \)-partite \( k \)-uniform hypergraphs assuming \( P \neq NP \).

**Proof.** The NP-hard decision question from Theorem 5 transforms into the following:

\[
\begin{align*}
\text{or}
\frac{k}{2(k+1)(2(k+1)+1)} - \delta & + \frac{\epsilon}{1-\epsilon k} n \leq |OPT'|

\frac{1}{k(2(k+1)+1)} + \delta & + \frac{\epsilon}{1-\epsilon k} n \geq |OPT'|
\end{align*}
\]

Assuming \( NP \neq P \), this implies the hardness of approximating the Vertex Cover problem in \( \epsilon \)-dense hypergraphs for every \( \delta' > 0 \) to within:

\[
\begin{align*}
\frac{k(1-\epsilon)}{2(k+1)(2(k+1)+1)} + \frac{\epsilon}{k} & - \delta' = \frac{k^2(1-\epsilon)+2(k+1)(2(k+1)+1)+1}{k(2(k+1)+1)} - \delta' \\
& = \frac{k^2(1-\epsilon)+\epsilon(2(k+1)+1)(2(k+1)+1)}{2(k+1)[1-\epsilon+\epsilon(2(k+1)+1)]} - \delta' \tag{27}
\end{align*}
\]

\[
\begin{align*}
\frac{k^2(1-\epsilon)+\epsilon(2(k+1)+1)(2(k+1)+1)}{2(k+1)[1-\epsilon+\epsilon(2(k+1)+1)]} - \delta' \tag{28}
\end{align*}
\]

\section{Further Research}

An interesting question remains about even tighter lower approximation bounds for our problem, perhaps connecting it more closely to the integrality gap issue of the LP of Lovász \cite{20}.

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