Algebraic structure of the space of homotopy classes of cycles and singular homology

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Abstract

Described the algebraic structure on the space of homotopy classes of cycles with marked topological flags of disks. This space is a non-commutative monoid, with an Abelian quotient corresponding to the group of singular homologies $H_k(M)$. For the marked flag contracted to a point the multiplication becomes commutative and the subgroup of spherical cycles corresponds to the usual homotopy group $\pi_k(M)$.

1 Basic example

Take $M = \mathbb{R}^2 - \{0\}$. Then a non-contractible cycle $c$ in $M$ corresponding to a map of $S^1$ is a submanifold, homeomorphic to $S^1$. For a flag $D$ of two disks $pt = D^0 \subset D^1$ with $D^0 \subset \partial D^1$ we have a subset $[c]_D$ of cycles with $D \subset c$. This subset is a disjoint union of two components $[c]_D = [c]_D^+ \cup [c]_D^-$ which differ by the isotopy class of $D \hookrightarrow [c]$ (see Fig.1).

Choosing for each homotopy class $[c]_D$ of flagged cycles $D \hookrightarrow S^1 \subset \mathbb{R}^2 - \{0\}$ a positive (and complementary negative) component we get a composition law (see Fig.1)

$$[c_1] \circ [c_2] := [c_1]^+ \# [c_2]^-$$

Proposition 1.1 This composition law defines on the set of homotopy classes of $S^1$-cycles the structure of the group, isomorphic to $\pi_1(\mathbb{R}^2 - \{0\})$.

2 General construction

2.1 Composition law on homotopy classes of cycles

Fix a flag $D$ of disks $D^0 \subset D^1 \subset \cdots \subset D^k \subset M^n$, where $D^i \subset \partial D^{i+1}$. Denote by $D'$ the subflag $D^0 \subset D^1 \subset \cdots \subset D^{k-1}$ of codimension 1. By a cycle in $M$ will be called a map $K \to M$ of a closed complex $K$ modulo automorphisms of the domain of the map. Then the homotopy of two cycles is weaker then the

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Figure 1:

Figure 2:

\[ \{0\} \times \{0\} := \quad \sim \quad \{0\} \]
homotopy of maps (see Section 1). Each homotopy class $[c]$ of $k$-cycles on $M$ has a subset of representatives $[c]_{D'} = \{c \in [c] \mid D^i \subset c\}$, i.e. cycles containing $D^i$. In particular $[c]_{D} \subset [c]_{D'} \subset \cdots \subset [c]_{D^{(k)}} \subset [c]$.

Denote by $\Pi_k(M; X)$ the space of homotopy classes of $k$-cycles with a fixed subset $X$.

Fix an orientation $o$ on $D \subset M$. Any two representatives of $[c]_{D'}$ are connected by a $D'$-fixed homotopy, so $[c]_{D'}$ is an element (a class) in $\Pi_k(M; D')$. Take a representative $c \in [c]_{D'}$. Then for a pair of representatives $c_1, c_2 \in [c]_{D'}$, we have a $D'$-fixed homotopy $c_1 \sim_{D'} c_2 = c_1 \sim_{D'} c \sim_{D'} c_2$. This induces an orientation on $D_c \subset c$ (where $D_c$ is $D$ as a submanifold in $c$) for all $c \in [c]_D$. But if for some ("positive") cycle $c^+$ we choose an orientation on $D_{c^+}$ to be equal to $o$ then there is a free (with no fixed set) homotopy of $c^+$ to a cycle $c^- \in [c]_D$ with the homotopy-induced orientation of $D_{c^-}$ opposite to $o$. So the space $[c]_D$ has two components $[c]_D = [c]_D^+ \cup [c]_D^-$, where $[c]_D^+$ is the set of cycles with the above homotopy-induced orientation on $D_{c^+}$ equal to $o$, and $[c]_D^-$ those with the opposite to $o$. Elements of each component $[c]_D^\sigma$ are connected by homotopies with fixed $D$ (i.e. belong to the same class of $\Pi_k(M; D)$), while homotopies between elements from opposite components belong to a wider class of $D'$-fixed homotopy. Then

**Proposition 2.1** There is a well defined double-covering $s : \Pi_k(M; D) \to \Pi_k(M; D')$, with $s^{-1}(c) = \{[c]_D^+, [c]_D^-\} \subset \Pi_k(M; D')$.

Choosing a "positive" representative as above fix a splitting $[c]_D = [c]_D^+ \cup [c]_D^-$ for each class $[c]_D \supset [c]_D$, which may be formulated as choosing a "positive section" of the above covering $\Pi_k(M; D) \to \Pi_k(M; D')$.

For each choice of a pair of representatives $a^\sigma \in [a]_D^+, b^\tau \in [b]_D^-$ we may take their connected sum along the common $D$, denoted by $a^\sigma \#^\tau b$. After making the pairing of representatives $a^\sigma \#^\tau b^\tau$, we take a class of the result with respect to homotopies with $D'$ fixed. The $D'$-fixed homotopy class of the result does not depend on the choice of representatives incide $[a]_D^+$ and $[b]_D^-$, while $[a]_D^+, [b]_D^-$ are homotopy invariants of classes $[a]$ and $[b]$.

**Proposition 2.2** $\#^\tau$, for $\sigma, \tau = +, -, $ is a well-defined pairing on the space $\Pi_k(M; D')$ of classes of homotopy of $k$-cycles with fixed $D'$.

Then we have 4 different well-defined ways of pairing for homotopy classes of cycles $[a]$ and $[b]$, corresponding to different choices of $\sigma, \tau$

\[\begin{align*}
[a]^+ \#^+ [b], & \ [a]^+ \#^-[b], & \ [a]^-[+][b], & \ [a]^-[\#][b]
\end{align*}\]

The results of these pairings are in general 4 different elements of $\Pi_k(M; D')$.

**Example 2.1** Take $M$ to be $\mathbb{R}^3$ with two deleted unlinked circles, $D : D^0 \subset D^1 \subset D^2$, $D' : D^0 \subset D^1$, $a := T_1$, $b := T_2$ being two tori. Choose decompositions $[T_1]_D = [T_1]_D^+ \cup [T_1]_D^-$, $[T_2]_D = [T_2]_D^+ \cup [T_2]_D^-$ in a way that the pairing $T_1^+ \#^+ T_2$ is $D'$-homotopic to that shown on Fig.2.1.

Then the pairing $T_1^+ \#^- T_2$ is $D'$-homotopic to that shown on Fig.2.4 (here some $D'$-homotopy has sort of "180-degrees twisted" $D^2$ around horizontal axis while $D^1$ being fixed), which has no $D'$-fixed homotopy to $T_1^+ \#^- T_2$.

So $[T_1]^+ \#^- [T_2]$ and $[T_1]^+ \#^+ [T_2]$ are different elements of $\Pi_2(M; D')$. 


Figure 5:

Having \([a] \#^\tau [b] \in \Pi_k(M; D')\) we can pair it in 4 different ways with any other class \([c] \in \Pi_k(M; D')\). These combinations of pairings may be represented as a composition of "signed" binary trees Fig.2.1.

Note, that \(a^\sigma \#^\tau b \equiv b^\tau \#^\sigma a\). Then each of the four pairings has two equivalent presentations in terms of words of letters \(a^+, a^-, b^+, b^-\) as follows:

\[
\begin{align*}
\{a^+ \#^- b \equiv b^- \#^+ a\} & : \{ab \sim ba^- \} \\
\{a^- \#^+ b \equiv b^+ \#^- a\} & : \{a^- b^- \sim ba \} \\
\{a^+ \#^+ b \equiv b^+ \#^+ a\} & : \{ab^- \sim ba^- \} \\
\{a^- \#^- b \equiv b^- \#^- a\} & : \{a^- b^- \sim ba \} 
\end{align*}
\]

Take an ordered set \(\{[c_1], \ldots, [c_N]\}\) of generators of \(\Pi_k(M; D')\). Let us choose for each pairing of \([c_i], [c_j]\) one presentation (of the two above) in terms of words of letters \(c_\sigma^\pm\) which will be called "canonical" presentation. For instance, for \(i < j\) let \(c_i c_j, c_j c_i, c_i c_i^-, c_j c_i\) be canonical presentations of the corresponding product classes (connected sums) of \([c_i]\) and \([c_j]\). For a canonical presentation \(c = c_{i_1}^\sigma \ldots c_{i_n}^\sigma\) the complement ("anti-canonical") presentation \(c_{i_n}^- \ldots c_{i_1}^-\) will be denoted by \(c^-\). If we have the canonical presentations \(a\) and \(b\) for classes \([a], [b]\), then one (of two) presentation for \([c] = [a] \#^\tau [b]\) is computed as \(a^\sigma b^\tau\).

**Example 2.2**

Having computed a presentation for the product class we set either it or its complement to be canonical. Then we may use these "signed" presentations of \([c]\) to compute presentations of pairing of \([c]\) with other classes according to the rule above and so on.

Note, that the ambiguity of choosing one of the two presentations of \([c]\) as the canonical corresponds exactly to the ambiguity while fixing signs of components in the decomposition \([c]_D = [c]_D^+ \cup [c]_D^-\).

**Proposition 2.3** The above procedure gives an isomorphism between the space of signed binary trees of pairings of \(\{[c_1], \ldots, [c_N]\}\) and the free monoid \(< c_1^\sigma, \ldots, c_N^\sigma >\) generated by letters \(c_\sigma^\pm\), \(\sigma = +, -\).

Since the space of signed trees covers \(\Pi_k(M; D)\) and the composition there corresponds to connected sums of cycles in \(\Pi_k(M; D)\) then we have an epimorphism \(p : < c_1^\sigma, \ldots, c_N^\sigma > \to \Pi_k(M; D)\), so
Proposition 2.4 $\Pi_k(M; D)$ is a quotient of the free monoid $< c_1^+, \ldots, c_N^+ >$.

To prove the associativity holds we note that the equivalence of presentations $(ab)c = a(bc)$ corresponds to the equality $([a]^{\#}([b])^{\#}[c]) = [a]^{\#}([b]^{\#}[-c])$ which may be checked directly to be true for connected sums.

Note, that $cc^-[c]^{\#}$ except for the case of fundamental group $\pi_1$ in general does not belong to the kernel of the above epimorphism, so $\Pi_k(M; D)$ is really a monoid (see Example 2.3).

2.2 Morphism to homology group $\Pi_k(M; D) \to H_k(M)$

Homotopic cycles are homologous, so we have a well-defined map of sets $h: \Pi_k(M; D) \to H_k(M,\mathbb{Z})$.

Proposition 2.5 For any class $[c] \in \Pi_k(M; D)$ the product $[c]^{\#}+[c]$ is a class of cycles with $h$-image homologous to $0 \in H_k(M)$.

Proof: Since $c$ is connected, then $\partial D$ is contractible in $c - D$ to a subcycle $c' \subset c$ of codimension 1. Then $c^{\#}+[c]$ is contractible in $M$ to the boundary of a tubular neighbourhood of $c'$. □

Example 2.3 Take $M = \mathbb{R}^3 - S^1$ and $c = T^2$ to be a torus, having this deleted $S^1$ as its axial circle.

Then the product $T^{+}c^{\#}+T$ is contractible onto a 1-dimensional cycle and thus is homologous to $0 \in H_2(M)$.

According to Proposition 2.3, we may use elements of the monoid $< c_1^+\ldots,c_N^+ >$ generated by letters $c_i^\sigma$, $\sigma = +, -$ to denote elements of $\Pi_k(M; D)$. We have a map of $< c_i^\pm >$ to the abelian group $(c_1,\ldots,c_N), c_i^{\sigma_1}\cdots c_i^{\sigma_n} \mapsto n_1 c_1^{\sigma_1} + \cdots + n_N c_N$ where $n_i$ is the difference of the numbers of $c_i^+$ and $c_i^-$ entering $c_i^{\sigma_1}\cdots c_i^{\sigma_n}$.

Proposition 2.6 The diagram

\[
\begin{array}{ccc}
< c_1^+, \ldots, c_N^+ > & \longrightarrow & (c_1, \ldots, c_N) \\
p \downarrow & & \downarrow \\
\Pi_k(M; D) & \xrightarrow{h} & H_k(M)
\end{array}
\]

is commutative. So $h: \Pi_k(M; D) \to H_k(M)$ is an epimorphism of monoids.

3 Discussion

1. Since $[c]_{D(\nu)} \subset [c]_{D(\nu+1)}$ then the structure of the monoid $\Pi_k(M; D'')$ of homotopies with the fixed subflag of codimension 2 will be some quotient of the structure of $\Pi_k(M; D')$ and we have a sequence

$\Pi_k(M; D') \to \Pi_k(M; D'') \to \cdots \to \Pi_k(M; D^0) \leftrightarrow \pi_k(M)$

2. In paper 11 it was developed a calculus of differential forms with coefficients in (matrix) algebra, with the integration defined over the elements of the group of "flagged spheres". In the current framework the integration of a closed form with coefficients in algebra $\mathfrak{g}$ (higher analogue of flat connection forms) over elements of $\Pi_k(M)$ must give a representation of the homotopy monoid $\Pi_k(M) \to \exp \mathfrak{g}$ in the corresponding Lie group.
References

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