Universality classes for the "ricepile" model with absorbing properties

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Abstract

The absorbing "ricepile" model with stochastic toppling rules has been numerically studied. Local limited, local unlimited, nonlocal limited and nonlocal unlimited versions of the absorbing model have been investigated. Transport properties and different dynamical regimes of all of the models have been analysed, from the point of view of self organized criticality (SOC). Phase transitions between different dynamical regimes were studied in detail. It was shown, that the absorbing models belong to two different universality classes.
1 Introduction

Self organized criticality (SOC) is widely studied phenomenon in the last ten years. Theory of SOC, proposed by Bak, Tang and Wiesenfeld [1], describes dynamical behaviour of many particle systems with local interactions. Paradigmatically, the description is based on the dynamics of a pile of sand. If the sandpile is randomly driven by slow addition of sandgrains, the slope of pile grows up and after some time local stability conditions are violated somewhere on the pile surface. The avalanche starts to slide down the slope. This type of dynamics is easily modeled by a cellular automaton. Such a model ”sandpile” is defined on a large n-dimensional lattice. The avalanche dynamics is, under the action of slow drive, governed by the local critical conditions (such as the local critical slope, for example), and local toppling rules. This dynamics leads to the steady state called self organized critical state, characterized by the critical scaling of the avalanche size distribution:

\[ p(s, L) = s^{-\tau} f \left( \frac{s}{L^D} \right) \]  

In (1), \( L \) is the system size and \( s \) exhibits the size of the avalanche. Critical exponents \( \tau \) and \( D \), depend on the particular model. Different models can be divided into the various universality classes, defined by a specific set of the critical exponents [10, 8].

A natural step from the sandpile model systems, leads to the investigation of real piles of granular material from the point of view of SOC dynamics. Several efforts have been made in this direction [2, 3, 4], but no clear evi-
dence of self organized critical dynamics (1) has been found. Finally, in 1996 an experiment has been done by the group of experimentalists and theorists in Oslo (2). In the Oslo experiment dynamical behaviour of the driven quasionedimensional pile of rice has been investigated. The avalanche sizes in the steady state were measured in terms of dissipated potential energy. Two type of grains were used (elongated and round ones), showing completely different dynamics. In the case of the ricepile, consisting of elongated grains, SOC state has been established, in which the avalanche size distribution has a power law character with critical exponent $\tau \approx 2.02$ [(2) ].

Soon after the experimental results were published, the model ”ricepile” cellular automata were suggested and numerically studied [(3)–(6) ]. The model ricepile is, in principle, cellular automaton defined on onedimensional lattice, with randomness incorporated into the toppling rules and with deterministic drive. Changes in toppling rules are often manifested by different dynamical behaviour of the model and different universality class into which the model belongs. [(3)–(6) ]

In this paper we study a model, which exhibits a modification of the two threshold ricepile model [(3)–(6)]. In [(3)–(6)] the gravity effects, grain friction and the local conditions on the pile are described by the parameter $p$. Two thresholds, namely the critical threshold and the gravity threshold are defined. Thresholds governs the movement of the grain on the pile surface.

We removed the second, gravity threshold of the ricepile model. This causes, that the model, depending on the parameter $p$ value, has absorbing
properties and interesting dynamical regimes [11]. One threshold model in
two dimensions has been studied by Tadić and Dhar [12].

Here, different versions of the absorbing model are defined using differ-
ent toppling rules. Dynamical behaviour of the local limited (LLIM), local
unlimited (LUNLIM), nonlocal limited (NLIM) and nonlocal unlimited
(NUNLIM) absorbing model is studied and two distinct universality classes
are recognized.

2 Ricepile models

The experimental results of the Oslo group motivated further theoretical
studies of the avalanches and dynamics of granular material [13, 14] . The
main question is, which physical properties of the pile granular material, are
important for SOC state to be established. What is, for example, the role
of friction, what is the role of the grain shape, the gravity and inertia of
rolling grains? The ricepile experiment reveals, that no SOC is possible if
the ricepile consists of round ricegrains. On the contrary, the dynamics of
pile consisting of elongated grains, is self organized critical [5] . Certainly,
the shape of grains, and the additional effects related to the shape, such
as better packing of grains due to the elongated shape, supressed rolling of
grains and thus supressed inertia effects, are of great importance [5, 14].

Ricepile models are cellular automata, in which friction and gravity effects
are taken into account in a simple way, through the parameter $p$. The value
of the parameter $p$ decides, whether the grain will stop on the site, or roll
further down the slope.

The two threshold ricepile model introduced in \cite{6, 7} is defined on a onedimensional lattice of size $L$, with a wall at the zero position and open boundary at the other end. At the open boundary, particles are free to flow out of the system.

As in the experimental set up, the system is driven by adding particles to the position one, at the closed end. Every time unit one particle is added. Two thresholds are defined in the model: the critical threshold $z_c$, which is the local condition for the onset of avalanche, and the gravity threshold $z_g$ ($z_c < z_g$). If the local slope

$$z_i = h_i - h_{i+1}, \quad (2)$$

(where $h_i$, $i = 1, 2, 3, ..., L$ is a height profile of pile) is less then $z_c$, it is too low and the friction stops the grain movement. The grain resides on the position $i$. If $z_i > z_g$, the local slope is too high and the grain moving downslope is not allowed to stop at the $i$-th position. But, in the case, that \(z_c < z_i < z_g\), the grain moves from $i$ to $i + 1$ with probability $p$. Through the parameter $p$, effective friction is introduced into the model. All supercritical slopes can topple, with probability $p$, but for local slopes, which are too big ($z_i > z_g$), the gravity becomes decisive and the site topples with probability $p = 1.0$.

The dynamics of the ricepile model \cite{6, 7} is as follows:

1. Each avalanche starts at $i = 1$. If $z_1 > z_c$, the site one is activated and topples a particle to the next nearest position (two), with the probability $p.$
If even $z_1 > z_g$; $p = 1.0$.

2. Every particle, sliding from the position $i$ to $i + 1$ activates three columns, namely $i - 1$, $i$ and $i + 1$. The position $i - 1$ is activated, because it possibly can become supercritical, when removing a grain from the $i$-th column. Columns $i$ and $i + 1$ are activated, because they are destabilized by sliding or stopping particles, respectively. In the next time step, all supercritical active sites topple a particle to the $i + 1$-st position with probability $p$ ($p = 1.0$, if $z_i > z_g$).

3. Step two is repeated, until there are no active sites in the system, that means, until the avalanche is not over.

Changing slightly the local toppling rules, different versions of the ricepile model are defined:

a) The number of particles toppled from the position $i$ is constant and independent on the supercritical local slope $z_i$ - the model is called **limited**.

b) The number of toppled particles is a function of the supercritical local slope $z_i$, the model is called **unlimited**.

c) If the particle (or more particles) topples from the site $i$ and moves only to the next nearest position $i + 1$, the model is defined as **local**.

d) The model is called **nonlocal**, if $n$ toppled particles moving from the $i$-th site are added subsequently to $n$ nearest downslope positions (one particle per site) $i + 1$, $i + 2$, ..., $i + n$.

Thus four different ricepile models are recognized:
1. local limited model (LLIM)
2. local unlimited model (LUNLIM)
3. nonlocal limited model (NLIM)
4. nonlocal unlimited model (NUNLIM)

Universality classes for the ricepile model were studied by Amaral and Lauritsen [8]. Their results show, that local models (LLIM, LUNLIM) belong to the wide universality class called local linear interface universality class (LLI class) [8, 13]. The authors also found, that the nonlocal toppling rules lead to two new universality classes, with a different set of critical exponents. But none of the universality classes is the one of the real ricepile.

3 Absorbing model

Our absorbing model [11] exhibits simplified, one threshold version of the ricepile model [6]. The gravity threshold is removed and all supercritical active sites are allowed to topple with probability $p < 1$. There, therefore, persists small, but nonzero probability, that also the extremely large local slopes are possible. Physically this seems to be quite plausible. It is not probable, that in real piles of granular material there exists a strict gravity threshold. There is rather a continuous transition to the local slopes, which are already so large that they, when activated, almost always topple.

We investigated numerically all four versions of the absorbing model: LLIM, LUNLIM, NLIM, NUNLIM. Several quantities were measured for all of the models:
1. Material transport as a ratio of the number of outgoing to ingoing particles

\[ J(p) = \frac{n_{\text{out}}(p)}{n_{\text{in}}(p)}, \]

and its dependence on the parameter \( p \).

2. Average material transport \( \langle J(p) \rangle \) as a function of \( p \).

3. Avalanche size distribution for different parameter values \( p \). Avalanche sizes are measured in terms of dissipated potential energy, in accordance with the experiment [5].

4. Changes in the pile profile, due to changes in the parameter \( p \) value.

   If the probability parameter \( p \) changes slowly in the interval \((0, 1)\), the model typically passes through different dynamical regimes:
   i) isolating, in which all particles are absorbed in the system and none of them reaches the open boundary;
   ii) partially conductive, in which the pile profile grows up as a bulk, because a certain fraction of the particles, depending on \( p \), is absorbed in the system (absorbing properties);
   iii) and totally conductive, when the number of ingoing and outgoing particles is balanced.

4 **Universality classes for the absorbing model**
4.1 LLIM

Dynamical properties of the local limited absorbing model are in details described in [11]. Here I only briefly list the main results.

Local limited toppling rules are defined as follows: active supercritical site topples one particle to the next nearest position with the probability $p$. Looking at the average material transport $\langle J(p) \rangle$, three dynamical regimes of the LLIM model are recognized (Fig.1a):

a) For $0 < p < p'$; $p' \approx 0.53865$, the system is completely isolating. The average transport $\langle J(p) \rangle$ is zero. For $p$ close to zero, almost all ingoing particles are absorbed. The avalanches die out soon, their size is exponentially bounded.

Close to the first phase transition point $p'$, the steepness of the pile is still high enough to say, that the local slopes are almost everywhere higher than the critical threshold $z_c$. This is the reason that the spreading of active sites in time is practically determined by the probability $p$; the same way as it is in the percolation process. In the space-time coordinate system, we have therefore a picture of directed percolation with three descendants and absorbing boundary [15]. $p'$ is thus simply the critical percolation threshold. Close to the percolation threshold $p'$, the average transport $J(p)$ scales with $p$ as

$$\langle J(p) \rangle - J' \propto (p - p')^{\delta'}$$  \hspace{1cm} (4)

$$\delta' = 0.9 \pm 0.01$$

where $J'$ is the current flowing due to the finite size of the system.
b) For the probability interval $p' < p < p_c$; $p_c \approx 0.7185$, the system is partially conductive, with constant average slope. That means, the height profile grows as a bulk with velocity $v(p)$. Fluctuations of transport $J(p)$ exhibit long range correlations.

Above the percolation threshold $p'$, the percolation picture breaks down. The subcritical, absorbing states are randomly distributed throughout the system and the avalanche can stop anywhere. As $p \to p_c$, the long range correlations in transport fluctuations are destroyed, and the region of small local slopes spans the whole system. The pile stops to grow and at $p = p_c$ it is pinned at the position $i = L$. Critical point $p_c$ is thus understood as the depinning transition point. Close to the depinning critical point, average transport scales as

$$1 - \langle J(p) \rangle \propto |p - p_c|^\delta$$

$$\delta = 0.9 \pm 0.01$$

c) In the interval $(p_c, 1)$ the system is completely conductive. Transport fluctuations are of white noise type and the average transport $J(p) = 1.0$. In the dynamical regimes b) and c) the system is in the SOC state, having power law distribution of avalanche sizes (I) with critical exponent $\tau = 1.57 \pm 0.05$.

4.2 LUNLIM

Local unlimited toppling rules are in the absorbing model defined as follows: In order to get a realistic profile of the pile, each supercritical active site
topples \( k, k = \text{int}(\frac{z_i}{2}) \), grains to the next nearest downslope position with probability \( p \). This way one gets smooth profile without cavities (Fig.2a).

Numerical investigations of \( \langle J(p) \rangle \) reveals, that only two dynamical regimes are clearly recognized (Fig.1b): the pile is either completely isolating, \( \langle J(p) \rangle = 0 \), or completely conductive \( \langle J(p) \rangle = 1.0 \). Partialy conductive dynamical regime is missing.

a) For \( 0 < p < p_c; p_c \approx 0.6995 \) (Fig.1b), the system is completely isolating. From the definition of local toppling rules it is clear, that absorbing states \( (z_i < z_c) \) are easily created even for very small values of the parameter \( p \). It means, that the percolation picture in the space - time coordinate system is not correct in the case of the LUNLIM model. Near transition point \( p_c \) the average transport \( J(p) \) scales with \( p \) as (Fig.3)

\[
J(p) \propto (p - p_c)^\delta
\]

\( \delta = 1.93 \pm 0.07 \)

b) In the second dynamical regime \( (p_c < p < 1.0) \) the system is completely conductive, with pile profile pinned at \( i = L \) (Fig.2a). \( J(p) \) as a function of time exhibits white noise features (Fig.4). Avalanche size distribution is critical \([1]\) with critical power law exponent \( \tau = 1.54 \pm 0.02 \) (Fig.5a).

### 4.3 NLIM

Nonlocal limited toppling rule means, that the supercritical active site topple, with probability \( p \), \( N \) particles to the \( N \) nearest downslope positions. The nonlocal limited toppling rules preserve three dynamical regimes, the same
way as it is in the LLIM case. In Fig.1c, isolating, partially conductive and totally conductive regimes are recognized.

a) For $0 < p < p'$, $p' \approx 0.267$ (Fig.1c), the pile is in the isolating regime. To understand the nature of the first phase transition point $p'$, the percolation picture in the time-space coordinate system is still useful. But now, the number of descendants is, in principle, greater than three. That is the reason for the fact, that the percolation threshold is shifted to the lower parameter values as one can see when comparing Fig.1a and Fig.1c, e.g. ($p'_{NLIM} < p'_{LLIM}$). In the model studied here, the number of particles toppling from the activated supercritical site is 4. Five sites are thus activated by every toppling from the position $i$, namely $i - 1$, $i$, $i + 1$, $i + 2$, $i + 3$. There are therefore five descendant sites in the directed percolation with absorbing boundary $\square$. In order to estimate $p'$ with greater accuracy, systematic studies of the dependence of percolation threshold on the number of descendant sites are necessary.

The average transport near the percolation threshold $p'$ scales as (Fig.6a):

$$J(p) \propto (p - p')^{\delta'}$$  \hspace{1cm} (7)

with the critical exponent

$$\delta' = 1.18 \pm 0.04$$

b) In the interval $p' < p < p_c$, the pile grows up with constant velocity $v(p)$, maintaining the global slope on a constant value for a constant probability
parameter $p$. Transport $J(p)$ as a function of time shows long range correlations, on the contrary to the totally conductive regime, where it has a character of white noise (Fig.7).

c) Depinning transition occurs at $p_c \approx 0.365$ (Fig.6b). Average transport scales with $p$ close to the critical point as:

$$1 - \langle J(p) \rangle \propto |p - p_c|^{\delta}$$

$$\delta = 1.12 \pm 0.06$$

For the probability interval $p_c < p < 1$, the profile of pile is pinned at $i = L$ (Fig.2b), and the average transport $\langle J(p) \rangle = 1.0$ (Fig.1c). Fig.5b demonstrates the avalanche size distribution in a case of partially conductive and conductive dynamical regimes. In both cases the dynamics of the pile is self organized critical, which is demonstrated by the critical, power law scaling (1). The critical exponent $\tau = 1.35 \pm 0.05$.

4.4 NUNLIM

The nonlocal unlimited toppling rule is defined as follows: $N(z_i)$ particles are released (with probability $p$) from the activated supercritical position and are added to $N(z_i)$ nearest downslope positions.

The nonlocal unlimited version of the absorbing model shows completely different behaviour. First, no distinct dynamical regimes are recognized. The pile is completely conductive already for $p$ close to zero as can be seen from
Fig. 8a. Pile profile is pinned at $i = L$ (Fig. 2c). Avalanche size distribution shows the critical scaling (1) with critical exponent $\tau = 1.51 \pm 0.05$ (Fig. 5e).

5 Discussion and conclusion

Probability density function (1) scales with the system size as

$$p(s, L) = L^{-\beta} g\left(\frac{s}{L^{D}}\right)$$

with $\beta = D\tau$. For the LLIM, LUNLIM and NUNLIM absorbing models the best data collapse has been found for $D = 2.24$, what indicats, that these models belong to the same universality class, called LLI universality class \[8, 13\].

On the contrary, for the NLIM absorbing model the best data collapse was found for $D = 1.55$. The critical exponents $\tau$ and $D$ are different from that of LLI class and defines a new universality class to which belongs also the NLIM version of the two threshold ricepile model \[8\].

The reason of lowering the $\tau$ exponent of the NLIM model in comparison with the LLIM model is as follows: The average slope of the NLIM and the LLIM pile is simillar. For example for $p = 0.8$, the average slope of the NLIM pile is $67.87^\circ$ and for the LLIM pile $60.53^\circ$. Due to the nonlocal toppling rules in the NLIM model, more columns are perturbed and thus the probability of greater avalanches is enhanced. Therefore the exponent $\tau$ is lowered.

The same argument could be used in the case of the NUNLIM and the LUNLIM model. But here the situation is different. The average slope changes significantly with changes in toppling rules from local to nonlocal.
For example if $p = 0.8$, the average slope of the LUNLIM model is $82.22^\circ$ and that of the NUNLIM model equals $53.6^\circ$. Because the number of toppled particles is proportional to the slope in the unlimited model, it seems, that the relatively small average slope of NUNLIM pile leads to relatively few particles released on average in one toppling. This fact should enhance the probability of small avalanches and the avalanche size distribution function should have $\tau$ exponent greater than $1.55$. This really happens for the NUNLIM two threshold ricepile model \cite{8}. In this model the probability of big local slope decreases exponentially with $z_i$. But looking at (Fig.2c) one can see, that it is not an exception to have a big local slopes in the NUNLIM absorbing pile. During a toppling event, the site with big local slope releases a number of particles (proportional to the local slope), which disturb a lot of downslope columns. This effect increases the probability of large avalanches. It seems, that in the case of the absorbing model the two described effects balance each other and thus the exponent $\tau$ remains untouched by changed toppling rules. This is different from the NUNLIM two threshold model \cite{8}. Here the first effect is decisive and the model belongs to a new universality class ($\tau = 1.63$).

Another question, which should be discussed, is the nonexistence of partially conductive regime of the LUNLIM model. First, the model is local. That means, in every toppling, only three columns are activated by each toppled particle. Therefore, the probability of avalanches, having a chance to reach the end of the system and thus to transport a material, doesn’t
increase due to more activated sites by every toppled particle. As it has been already told, in the LLIM model, there are no absorbing \((z_i < z_c)\) states in the system for the isolating dynamical regime. This is not the case of the LUNLIM model. Absorbing states, therefore, exhibits another obstacle for bigger avalanches to develop and transport the particles. The existence of absorbing states, even for a small parameter values, destroys the percolation picture of the spreading of active sites and this is also the reason for nonexisting critical percolation probability \(p'\) and only the transition to completely conductive dynamical regime is present.

Last, some words should be told about the pile slopes in all of the three dynamical regimes. In the isolating regime average transport \(\langle J(p) \rangle = 0\). All the added particles are absorbed in the system. Moreover, the avalanche sizes in this regime are exponentially bounded. That means, majority of the particles is absorbed on the first few columns of the pile. Therefore, if the driving time \(t\) tends to infinity, the average slope of the pile grows to infinity. That in consequence means, that also the local slopes become arbitrarily large.

In the partially conductive regime, constant amount of particles, depending on the parameter \(p\), is absorbed in the system. As \(t \to \infty\), average slope of the pile remains constant; pile grows up as a bulk. The local slopes are finite, except of \(z(L)\) (see (2)), which tends to infinity.

In the conductive regime \(\langle J(p) \rangle = 1.0\). The average slope of the pile is constant depending only on the parameter \(p\). All local slopes are finite in
this regime.

In conclusion, we have studied numerically the LLIM, LUNLIM, NLIM and the NUNLIM absorbing models. We have found, that the models belong to two different universality classes, characterized by different critical exponents. Both of the universality classes are different from the one of real pile of rice. We have studied the transport properties of all of the models and found phase transitions between different dynamical regimes. We state, that the dynamics of LLIM and NLIM model is directly mapped to the dynamics of directed percolation process at the absorbing boundary [15] for the defined interval of parameter $p$.

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6 Figure captions

Fig.1

Average transport ratio of particles through the system as a function of the parameter $p$.

(a) LLIM model: Three different dynamical regimes are recognized: isolating, $p \in (0, 0.53865)$; partially conductive, $p \in (0.53865, 0.7185)$ and conductive, $p \in (0.7185, 1.0)$.

(b) LUNLIM model: Only two different dynamical regimes are recognized: isolating, $p \in (0, 0.6995)$ and conductive, $p \in (0.6995, 1.0)$.

(c) NLIM model: Again, three different dynamical regimes are depicted: isolating, $p \in (0, 0.267)$; partially conductive, $p \in (0.267, 0.365)$ and totally conductive, $p \in (0.365, 1.0)$.

Fig.2

Pile profiles of the LUNLIM (Fig.2a), NLIM (Fig.2b) and NUNLIM (Fig.2c) absorbing models for different values of the probability parameter $p$. Notice, that in the totally conductive regime, the pile profile is pinned at $i = L$ (a), (c). The pile is growing as a bulk with velocity $v(p)$ in the case of the partially conductive regime and is pinned in the totally conductive regime (b).

Fig.3

Ln-ln plot of the average transport as a function of the distance from the critical point $p_c$ in the LUNLIM absorbing model, $\epsilon = (p - p_c)$. We find, that the best scaling is obtained for $p_c = 0.6995$. 

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Transport $J(p)$ as a function of time (in iterations) in the conductive regime of the LUNLIM model for two different values of $p$. For $p \geq p_c$, $p_c = 0.6995$, white noise is observed. For $p < p_c$ all particles are absorbed and therefore there are no fluctuations.

Ln -ln plots of the power law parts of the avalanche size distributions (unnormalized). The critical exponent $\tau$ of the LLIM, LUNLIM ($a$) and NUNLIM ($c$) models is $\tau = 1.55$. The NLIM model ($b$) belongs to the different universality class with $\tau = 1.35$. The system size in Figs. 5b, c is $L = 300$ and in Fig. 5a $L = 500$. In the NLIM model ($b$) the bump shift with system size has been numerically tested. With growing system size, the bump shifts to the higher values of $s$, indicating thus SOC.

(a) Ln-ln plot of the average transport as a function of the distance from the critical point $p'$ in the NLIM absorbing model, $\epsilon = (p - p_c)$. We find, that the best scaling is obtained for $p_c = 0.276$.

(b) Ln-ln plot of the average transport as a function of the distance from the critical point $p_c$ in the NLIM absorbing model, $\epsilon = (p_c - p)$. We find, that the best scaling is obtained for $p_c = 0.3441$.

Transport $J(p)$ as a function of time (in iterations) in the partially conductive regime ($p = 0.339$) and totally conductive regime ($p = 0.4$) of the
NLIM absorbing model. For $p \geq p_c$, $p_c = 0.365$, white noise is observed. For $p' < p < p_c$ the character of fluctuations is different. Long range correlations (reminiscent of Brownian motion) are observable in the time signal. For $p < p'$ all particles are absorbed and therefore there are no fluctuations.

Fig. 8

Transport $J(p)$ as a function of time (in iterations) in the NUNLIM model for three different values of the parameter $p$. The pile is totally conductive in wide range of the parameter $p$ and the signal has a character of white noise, with fluctuations depending on $p$. 
Fig. 1a, Markosova
Fig. 1b, Markosova
Fig. 1c, Markosova
Fig. 2a, Markosova

\[ h(i) \]

- \( p = 0.704 \) (solid line)
- \( p = 0.707 \) (dashed line)
- \( p = 0.710 \) (dotted line)
- \( p = 0.730 \) (dash-dotted line)
Fig. 2c, Markosova

- $p=0.1$
- $p=0.3$
- $p=0.6$
- $p=0.8$
Fig. 3, Markosova

ln <J>

Fig. 4, Markosova

p=0.80

p=0.73

0.6980

0.6990

0.6995

0.7000

0.7005

0.7010
Fig. 5a, Markosova

\[ \ln N(s) \]

\[ \ln s \]

(a)
Fig. 5b, Markosova

\[ \ln N(s) \]

- \( p = 0.3 \) +
- \( p = 0.6 \) ×
- \( p = 0.8 \) *

Fit: 

\[ \ln s \]
Fig. 6a, Markosova

Fig. 6b, Markosova

Fig. 7, Markosova

$p = 0.339$

$p = 0.400$
Fig. 8a, Markosova

(a)
Fig. 8c, Markosova

$p = 0.9$