THE HIRZEBRUCH GENERA OF COMPLETE INTERSECTIONS

JIANBO WANG, ZHIWANG YU, YUYU WANG

Abstract. Following Brooks’s calculation of the \( \hat{A} \)-genus of complete intersections, a new and more computable formula about the \( \hat{A} \)-genus and \( \alpha \)-invariant will be described as polynomials of multi-degree and dimension. We also give an iterated formula of \( \hat{A} \)-genus and the necessary and sufficient conditions for the vanishing of \( \hat{A} \)-genus of complex even dimensional spin complete intersections. Finally, we obtain a general formula about the Hirzebruch genus of complete intersections, and calculate some classical Hirzebruch genera as examples.

1. Introduction

A genus of a multiplicative sequence is a ring homomorphism, from the ring of smooth compact manifolds (up to suitable cobordism) to another ring. A multiplicative sequence is completely determined by its characteristic power series \( Q(x) \). Moreover, every power series \( Q(x) \) determines a multiplicative sequence. Given a power series \( Q(x) \), there is an associated Hirzebruch genus, which is also denoted as a \( \varphi_Q \)-genus. For more details, please see Section 2. The focus of this paper is mainly on the description of the Hirzebruch genus of complete intersections with multi-degree and dimension.

A complex \( n \)-dimensional complete intersection \( X_n(d_1, \ldots, d_r) \subset \mathbb{C}P^{n+r} \) is a smooth complex \( n \)-dimensional manifold given by a transversal intersection of \( r \) nonsingular hypersurfaces in complex projective space. The unordered \( r \)-tuple \( d := (d_1, \ldots, d_r) \) is called the multi-degree, which denotes the degrees of the \( r \) nonsingular hypersurfaces, and the product \( d := d_1d_2 \cdots d_r \) is called the total degree. It is well-known that the diffeomorphism type of the real \( 2n \)-dimensional manifold \( X_n(d_1, \ldots, d_r) \) depends only on the multi-degree and dimension. For the abbreviation of notation, set \( X_n(d) := X_n(d_1, \ldots, d_r) \). By the Lefschetz hyperplane section theorem, the inclusion \( X_n(d) \subset \mathbb{C}P^{n+r} \) is \( n \)-connected. Let \( x \in H^2(X_n(d); \mathbb{Z}) \) be the pullback of the first Chern class of the Hopf line bundle \( H \) (the dual bundle of canonical line bundle) over \( X_n(d) \).
\(\mathbb{C}P^{n+r}\), the total Chern class and total Pontrjagin class of \(X_n(d)\) ([16]) are given by

\[
c(X_n(d)) = (1 + x)^{n+r+1} \prod_{i=1}^{r} (1 + d_i \cdot x)^{-1},
\]

\[
p(X_n(d)) = (1 + x^2)^{n+r+1} \prod_{i=1}^{r} (1 + d_i^2 \cdot x^2)^{-1}.
\]

In particular, the first Chern class is

\[
c_1(X_n(d)) = \left(n + r + 1 - \sum_{i=1}^{r} d_i\right) \cdot x.
\]

In addition, one can show that the evaluation of \(x^n\) on the fundamental class of \(X_n(d)\) is the total degree \(d = d_1 \cdots d_r\):

\[
\int_{X_n(d)} x^n = d.
\]

Note that, for a complete intersection \(X_n(d)\), the \(i\)-th Pontrjagin class \(p_i\) must be an integral multiple of \(x^{2i}\), where \(x \in H^2(X_n(d); \mathbb{Z}) \cong \mathbb{Z}\) is a generator \((n \neq 2)\). This multiple is independent of the choice of the generator \(x\) since \(p_i \in H^{4i}(X_n(d); \mathbb{Z})\), so we can compare the Pontrjagin classes of complete intersections with the same dimension and different multi-degrees. Related to the diffeomorphism classification of complete intersections, there is the following conjecture, often called the Sullivan conjecture after Dennis Sullivan.

**Sullivan conjecture.** When \(n \neq 2\), two different complete intersections \(X_n(d)\) and \(X_n(d')\) are diffeomorphic if and only if they have the same total degree, Pontrjagin classes and Euler characteristics.

The partial known results on the Sullivan conjecture for complete intersections can be found in [6, 8, 13, 20]. The latest progress by Crowley and Nagy proves the Sullivan conjecture for \(n = 4\) in [5].

Brooks [3] calculated the \(\mathring{A}\)-genus of complex hypersurfaces \(X_{2n}(d_1)\) and complete intersections \(X_{2n}(\underline{d})\) with \(\underline{d} = (d_1, \ldots, d_r)\):

\[
\mathring{A}(X_{2n}(d_1)) = \frac{2}{(2n + 1)!} \prod_{j=-n}^{n} \left(\frac{d_1}{2} - j\right),
\]

\[
\mathring{A}(X_{2n}(\underline{d})) = \frac{1}{2\pi \sqrt{-1}} \oint_{\Gamma(0)} (1 + z)^{n-\frac{1}{2} - \sum_{i=1}^{r} \frac{d_i}{2}} \cdot \prod_{i=1}^{r} \left(\frac{(1 + z)^{d_i} - 1}{z^{2n+r+1}}\right) dz,
\]

where \(\Gamma(0)\) denotes a small circle around 0, and \(\frac{1}{2\pi \sqrt{-1}} \oint_{\Gamma(0)} f(z) dz\) means the residue of \(f(z)\) at \(z = 0\). Note that the \(\mathring{A}\)-genus of complex odd dimensional complete intersection is zero. (1.4) can also be found in [15, page 298]. Evidently, (1.4) is better relative to (1.5). One starting point for writing this paper is to give an explicit description of
\[ \hat{A}(X_{2n}(d)) \], which might be simpler and more computable than (1.5). After all, (1.5) looks more cumbersome. The first main theorem in this paper is as follows:

**Theorem 1.1.** The \( \hat{A} \)-genus of complete intersection \( X_{2n}(d) = X_{2n}(d_1, \ldots, d_r) \) is

\[
\hat{A}(X_{2n}(d)) = \sum_{j=0}^{r} (-1)^{r-j} \sum_{1 \leq k_1 < \cdots < k_j \leq r} \left( \frac{1}{2} c_1 - 1 + d_{k_1} + \cdots + d_{k_j} \right) \frac{n + r}{2n + r},
\]

where \( c_1 = 2n + r + 1 - \sum_{i=1}^{r} d_i \), i.e. \( c_1 \cdot x \) is the first Chern class \( c_1(X_{2n}(d)) \) as in (1.3),

\[
\binom{a}{k} := \frac{a(a - 1)(a - 2) \cdots (a - k + 1)}{k!},
\]

and \( d_{k_1} + \cdots + d_{k_j} \) vanishes if \( j = 0 \).

Combine the above theorem, we can give the \( \alpha \)-invariant of complete intersections \( X_n(d_1, \ldots, d_r) \) with \( c_1 = n + r + 1 - \sum_{i=1}^{r} d_i \) even.

**Theorem 1.2.** Let \( d = (d_1, \ldots, d_r) \). Then for a complete intersection \( X_n(d) \) with \( c_1 = n + r + 1 - \sum_{i=1}^{r} d_i \) even, the \( \alpha \)-invariant \( \alpha(X_n(d)) \) is equal to

1. \( \hat{A}(X_n(d)) \), if \( n \equiv 0 \mod 4 \);
2. \( \sum_{1 \leq k_1 < \cdots < k_j \leq r-1} \left( \frac{1}{2} c_1 - 1 + d_{k_1} + \cdots + d_{k_j} \right) \frac{n + r}{2n + r} \) (mod 2), if \( n \equiv 1 \mod 4 \);
3. \( \frac{1}{2} \hat{A}(X_n(d)) \), if \( n \equiv 2 \mod 4 \);
4. 0, if \( n \equiv 3 \mod 4 \).

Note that for \( n > 1 \), the complete intersection \( X_n(d) \) is spin if and only if \( c_1 \) is even. The complex one dimensional complete intersection \( X_1(d) \) is a closed orientable surface with the second Stiefel-Whitney class \( \omega_2 = 0 \), and hence is spin.

In [9], Hirzebruch defined the virtual index and virtual generalized Todd genus for a virtual submanifold \((v_1, \ldots, v_r)\) of \( M \), where \( v_1, \ldots, v_r \in H^2(M; \mathbb{Z}) \), and \( M \) is a compact oriented manifold or a compact almost complex manifold. Similarly, we define the virtual Hirzebruch genus of \((v_1, \ldots, v_r)\), which is equal to the Hirzebruch genus of a submanifold \( V \) with codimension \( 2r \) of \( M \). As an application, an iterated formula about \( \hat{A} \)-genus of complete intersections is given as follows:

**Theorem 1.3.** For any positive integers \( d_1, \ldots, d_r \), assume that \( d_{r-1} \geq d_r \geq 2 \), then we have

\[
\hat{A}(X_{2n}(d_1, \ldots, d_{r-1}, d_r)) = \sum_{k=0}^{d_r-1} \hat{A}(X_{2n}(d_1, \ldots, d_{r-2}, d_{r-1} + d_r - 1 - 2k)).
\]
Brooks shows that, for a spin complete intersections \( X_{2n}(d_1, \ldots, d_r) \), its \( \hat{A} \)-genus vanishes if and only if \( c_1 > 0 \) ([3]), where \( c_1 = 2n + r + 1 - \sum_{i=1}^{r} d_i \). We can use Theorem 1.3 to give a new proof for the vanishing of \( \hat{A}(X_{2n}(d_1, \ldots, d_r)) \).

As another application of virtual Hirzebruch genus, we give a general description of any Hirzebruch genus of complete intersections:

**Theorem 1.4.** For the given power series \( Q(x) = x/R(x) \), the Hirzebruch genus \( \varphi_Q \) on the complete intersection \( X_n(d) \) with \( d = (d_1, \ldots, d_r) \) satisfies that:

\[
\varphi_Q(X_n(d)) = \frac{1}{2\pi \sqrt{-1}} \int_{\Gamma(0)} \frac{\prod_{i=1}^{r} R(d_iz)}{(R(z))^{n+r+1}} \, dz.
\]

When this paper is near completion, we notice Baraglia’s paper [2]. Baraglia also give the similar description of the \( \hat{A} \)-genus and alpha invariant of complete intersections by the different methods. Ours results in Theorem 1.1 and 1.2 are equivalent to the results in [2, Theorem 1.3 & 1.2].

This paper is organized as follows. In Section 2, we introduce some necessary concepts and properties on Hirzebruch genus. In Section 3, we prove Theorem 1.1 and Theorem 1.2. In Section 4, the virtual Hirzebruch genus is defined. As an application of virtual genus, we prove Theorem 1.3 and determine when the \( \hat{A} \)-genus of complex even dimensional spin complete intersections vanishes. In Section 5, Theorem 1.4 is proved by using the virtual Hirzebruch genus of complete intersections, and we calculate some classical Hirzebruch genera of complete intersections as examples.

## 2. Hirzebruch genus

For references about the Hirzebruch genus, [4, Appendix E], [9] and [10, §1.6-1.8] are recommended. Unless otherwise stated, all the manifolds mentioned in this paper are smooth, oriented, closed, and connected. A general real \( n \)-dimensional manifold is denoted by \( M^n \). We omit the real dimension \( n \) when there is no danger of ambiguity. If \( M \) admits an almost complex structure, it must be even dimensional. For an almost complex manifold \( M^{2n} \), let \( TM \) be the complex tangent \( n \)-vector bundle, and \( (TM)_\mathbb{R} \) be the underlying real tangent \( 2n \)-vector bundle of \( M \). For a differentiable manifold \( M^n \), \( TM \) is the tangent \( n \)-vector bundle of \( M \).

Every homomorphism \( \varphi : \Omega^U \to \Lambda \) from the complex bordism ring to a commutative ring \( \Lambda \) with unit can be regarded as a multiplicative characteristic of manifolds which is an invariant of bordism classes. Such a homomorphism is called a (complex) \( \Lambda \)-genus. The term multiplicative genus is also used, to emphasize that such a genus is a ring homomorphism.

Assume that \( \Lambda \) does not have additive torsion, then every \( \Lambda \)-genus is fully determined by the corresponding homomorphism \( \Omega^U \otimes \mathbb{Q} \to \Lambda \otimes \mathbb{Q} \), which is also denoted by \( \varphi \). Every homomorphism \( \varphi : \Omega^U \otimes \mathbb{Q} \to \Lambda \otimes \mathbb{Q} \) can be interpreted as a sequence of homogeneous polynomials \( \{ K_i(c_1, \ldots, c_i), i \geq 0 \} \), \( \deg K_i = 2i \), with certain imposed
conditions. The conditions may be described as follows: an identity
\[ 1 + c_1 + c_2 + \cdots = (1 + c'_1 + c'_2 + \cdots) \cdot (1 + c''_1 + c''_2 + \cdots) \]
implies that
\[ \sum_{n \geq 0} K_n(c_1, \ldots, c_n) = \sum_{i \geq 0} K_i(c'_1, \ldots, c'_i) \cdot \sum_{j \geq 0} K_j(c''_1, \ldots, c''_j). \quad (2.1) \]
A sequence of homogeneous polynomials \( K = \{K_i(c_1, \ldots, c_i), i \geq 0\} \) with \( K_0 = 1 \) is called **multiplicative Hirzebruch sequence** if \( K \) satisfies the \((2.1)\). Write the following abbreviated notation:
\[ K(c_1, c_2, \ldots) := 1 + K_1(c_1) + K_2(c_1, c_2) + \cdots, \]
then \((2.1)\) is
\[ K(c_1, c_2, \ldots) = K(c'_1, c'_2, \ldots) \cdot K(c''_1, c''_2, \ldots). \]
Moreover, a multiplicative sequence \( K \) is one-to-one correspondence with a power series
\[ Q(x) = 1 + q_1 x + q_2 x^2 + \cdots \in \Lambda \otimes \mathbb{Q}[[x]], \]
where the notation \( \Lambda \otimes \mathbb{Q}[[x]] \) means the ring of power series in \( x \) over the ring \( \Lambda \otimes \mathbb{Q} \), \( x = c_1 \), and \( q_i = K_i(1,0,\ldots,0) \) (see [4, Appendix E] or [9, §1]), i.e.,
\[ Q(x) = 1 + K_1(x,0,\ldots,0) + K_2(x,0,\ldots,0) + \cdots. \]

By \((2.1)\), an equality
\[ 1 + c_1 + \cdots + c_n = (1 + x_1) \cdots (1 + x_n) \]
implies the equality
\[ Q(x_1) \cdots Q(x_n) = 1 + K_1(c_1) + \cdots + K_n(c_1, \ldots, c_n) + K_{n+1}(c_1, \ldots, c_n, 0) + \cdots \]
\[ = K(c_1, \ldots, c_n). \]
It follows that the \( n \)-th term \( K_n(c_1, \ldots, c_n) \) in the multiplicative Hirzebruch sequence \( K = \{K_i(c_1, \ldots, c_i)\} \) corresponding to a genus \( \varphi : \Omega^U \otimes \mathbb{Q} \rightarrow \Lambda \otimes \mathbb{Q} \) is the degree \( 2n \) part of the series \( \prod_{i=1}^n Q(x_i) \in \Lambda \otimes \mathbb{Q}[[c_1, \ldots, c_n]] \). Follow the statement in [4, Appendix E], the series \( \prod_{i=1}^n Q(x_i) \) can be regarded as a universal characteristic class of a complex \( n \)-vector bundles. For example, for a complex \( n \)-vector bundle \( \xi \) with the total Chern class \( c(\xi) = 1 + c_1 + \cdots + c_n = (1 + x_1) \cdots (1 + x_n) \), where \( x_1, \ldots, x_n \) are the formal roots of \( c(\xi) \), we can define the \( \varphi_Q \)-class of \( \xi \) by
\[ \varphi_Q(\xi) = K(c_1, \ldots, c_n) = \prod_{i=1}^n Q(x_i). \quad (2.2) \]

**Definition 2.1.** Let \( Q(x) \) be a power series with constant term being one. For an almost complex manifold \( M^{2n} \), let \( x_1, \ldots, x_n \) be the formal roots of the total Chern class of \( TM \), i.e., \( c(TM) = (1 + x_1) \cdots (1 + x_n) \). The **Hirzebruch genus** \( \varphi_Q \) corresponding to \( Q(x) \) is defined by
\[ \varphi_Q(M) = \int_M \varphi_Q(TM) = \int_M \prod_{i=1}^n Q(x_i), \]
where \( \int_M \varphi_Q(TM) = \langle \varphi_Q(TM), [M] \rangle \) is the evaluation of the degree 2n part of \( \varphi_Q(TM) \) on the fundamental class \([M]\) of \(M\).

A parallel theory of genera exists for oriented manifolds. These genera are homomorphisms \(\Omega^\text{SO} \to \Lambda\) from the oriented bordism ring to \(\Lambda\). Once again we assume that the ring \(\Lambda\) does not have additive torsion. Then every oriented \(\Lambda\)-genus \(\varphi\) is determined by the corresponding homomorphism \(\Omega^\text{SO} \otimes \mathbb{Q} \to \Lambda \otimes \mathbb{Q}\), which we also denote by \(\varphi\). Such genera are in one-to-one correspondence with even power series \(Q(x) = 1 + q_2 x^2 + q_4 x^4 + \cdots \in \Lambda \otimes \mathbb{Q}[[x]]\).

Similarly as the complex case above, for a real \(n\)-vector bundle \(\xi\) with the total Pontrjagin class \(p(\xi) = 1 + p_1 + \cdots + p_{\lfloor \frac{n}{4} \rfloor} = (1 + x_1^2)(1 + x_2^2) \cdots (1 + x_{\lfloor \frac{n}{4} \rfloor}^2)\), we can define the \(\varphi_Q\)-class of \(\xi\) by

\[
\varphi_Q(\xi) = K(p_1, \ldots, p_{\lfloor \frac{n}{4} \rfloor}) = \prod_{i=1}^{\lfloor \frac{n}{4} \rfloor} Q(x_i).
\] (2.3)

**Definition 2.2.** For a compact, oriented, differentiable manifold \(M\) of dimension \(n\), the Hirzebruch genus \(\varphi_Q\) corresponding to an even power series \(Q(x)\) is defined by

\[
\varphi_Q(M) = \int_M \varphi_Q(TM) = \int_M \prod_{i=1}^{\lfloor \frac{n}{4} \rfloor} Q(x_i),
\]

where the \(x_1^2, \ldots, x_{\lfloor \frac{n}{4} \rfloor}^2\) are the formal roots of the total Pontrjagin class of \(TM\), i.e., \(p(TM) = (1 + x_1^2) \cdots (1 + x_{\lfloor \frac{n}{4} \rfloor}^2)\).

An oriented genus \(\varphi : \Omega^\text{SO} \to \Lambda\) defines a complex genus given by the composition \(\Omega^U \to \Omega^\text{SO} \xrightarrow{\varphi} \Lambda\) with the homomorphism \(\Omega^U \to \Omega^\text{SO}\) forgetting the stably complex structure. Since \(\Omega^U \to \Omega^\text{SO}\) is onto modulo torsion, and \(\Lambda\) is assumed to be torsion-free, one loses no information by passing from an oriented genus to a complex one. On the other hand, a complex genus \(\varphi : \Omega^U \to \Lambda\) factors through an oriented genus \(\Omega^\text{SO} \to \Lambda\) whenever its corresponding power series \(Q(x)\) is even.

### 3. The \(\hat{A}\)-Genus and \(\alpha\)-Invariant of Complete Intersections

For a compact oriented differentiable manifold \(M^{4k}\), Hirzebruch ([9, Page 197]) defined the \(\hat{A}\)-sequence of \(M^{4k}\) as a certain polynomials in the Pontrjagin classes of \(M^{4k}\). More concretely, the power series

\[
Q(x) = \frac{\frac{1}{2}x}{\sinh \left( \frac{1}{2}x \right)} = 1 - \frac{x^2}{24} + \frac{7x^4}{5760} - \frac{31x^6}{967680} + \frac{127x^8}{154828800} + \cdots
\]
defines a multiplicative sequence \( \{ \hat{A}(p_1, \ldots, p_j) \} \). For example, the first four terms of the sequence are as follows (for abbreviation, \( p_j := p_j(M) = p_j(TM) \)):

\[
\begin{align*}
\hat{A}_1 &= -\frac{p_1}{24}, \\
\hat{A}_2 &= \frac{7p_1^2 - 4p_2}{5760}, \\
\hat{A}_3 &= -\frac{31p_1^3 + 44p_1p_2 - 16p_3}{967680}, \\
\hat{A}_4 &= \frac{381p_1^3 - 904p_1^2p_2 + 208p_2^2 + 512p_1p_3 - 192p_4}{464486400}.
\end{align*}
\]

The \( \hat{A} \)-genus \( \hat{A}(M) \) is the \( \varphi_Q \)-genus with the power series \( Q(x) \) as above, and

\[
\hat{A}(M) = \int_M \hat{A}(p_1, \ldots, p_k),
\]

where \( \hat{A}(p_1, \ldots, p_k) = \varphi_Q(TM) \) is the \( \hat{A} \)-class defined as in Section 2.

It is a result of Lichnerowicz that the \( \hat{A} \)-genus must vanish for a compact spin manifold which admits a Riemannian metric with positive scalar curvature (see [17]). In [11], Hitchin generalized this result to \( \alpha(M) = 0 \) if \( M \) has a metric of positive scalar curvature, where \( \alpha : \Omega^{2\text{pin}}_+ \to KO^{-n}(pt) \) is a homomorphism, and the \( \alpha \)-invariant \( \alpha(M) \in KO^{-n}(pt) \) is a \( KO \)-characteristic number for a spin manifold \( M \) of dimension \( n \). This is a strict generalization of the result of Lichnerowicz since \( \alpha(M) = \hat{A}(M) \) if \( n = 0 \) (mod 8) and \( \alpha(M) = \frac{1}{2} \hat{A}(M) \) if \( n = 4 \) (mod 8). Furthermore, Stolz showed that a spin manifold \( M \) of dimension \( n \geq 5 \) admits a positive scalar curvature metric if and only if \( \alpha(M) \) vanishes (see [18]), which proved the Gromov-Lawson conjecture. By virtue of the \( \alpha \)-invariants of complex \( 4k+1 \)-dimensional complete intersection \( X_{4k+1}(d) \) (\( k \geq 1 \)) and Seiberg-Witten theory in complex dimension 2, Fang and Shao in [7] gave a classification list of complete intersections admitting Riemannian metrics with positive scalar curvature.

The \( \hat{A} \)-genus is a special case of elliptic genera. There are many results concerning the \( \hat{A} \)-genus which are related with group actions (see [12]). From the Atiyah-Singer index theorem, the \( \hat{A}(M) \) can be interpreted as the index of the Dirac operator acting on spin-bundles over \( M \). A profound development of the classical result by Atiyah-Hirzebruch (see [1]): If \( M \) is connected spin and admits a nontrivial smooth \( S^1 \)-action, then the \( \hat{A} \)-genus \( \hat{A}(M) \) vanishes.

Based on the Brooks’s calculation (1.5), we can obtain a more computable formula about the \( \hat{A} \)-genus of complex even dimensional complete intersection.

**Theorem 3.1.** The \( \hat{A} \)-genus of complete intersection \( X_{2n}(d) = X_{2n}(d_1, \ldots, d_r) \) is

\[
\hat{A}(X_{2n}(d)) = \sum_{j=0}^{r} (-1)^{r-j} \sum_{1 \leq k_1 < \cdots < k_j \leq r} \left( \frac{1}{2} c_1 - 1 + d_{k_1} + \cdots + d_{k_j} \right),
\]

where \( c_1 \) is the first Chern class of the tangent bundle of \( X_{2n}(d) \).
where \( c_1 = 2n + r + 1 - \sum_{i=1}^r d_i \), i.e. \( c_1 \cdot x \) is the first Chern class \( c_1(X_{2n}(d)) \) as in (1.3),
\[
\binom{a}{k} := \frac{a(a-1)(a-2) \cdots (a-k+1)}{k!}, \quad \text{and } d_{k_1} + \cdots + d_{k_j} \text{ vanishes if } j = 0.
\]

**Proof.** Let \( \Gamma(0) \) denote a small circle around 0. By (1.5)
\[
\hat{A}(X_{2n}(d)) = \frac{1}{2\pi \sqrt{-1}} \oint_{\Gamma(0)} (1 + z)^{n-\frac{1}{2} - \sum_{i=1}^r \frac{d_i}{2}} \cdot \prod_{i=1}^r \left( (1 + z)^{d_i - 1} - 1 \right) \cdot \frac{\prod_{i=1}^r (1 + z)^{d_i - 1}}{2n+r+1} \, dz.
\]

Thus, by the residue theorem, the \( \hat{A} \)-genus of \( X_{2n}(d) \) is the coefficient of \( z^{2n+r} \) in the double summation
\[
\sum_{j=0}^r (-1)^{r-j} \sum_{1 \leq k_1 < \cdots < k_j \leq r} (1 + z)^{\frac{1}{2}c_1 - 1 + d_{k_1} + \cdots + d_{k_j}} \cdot \frac{\prod_{i=1}^r (1 + z)^{d_i - 1}}{2n+r+1} \, dz.
\]

We can give another proof of Theorem 3.1 by calculating the \( \hat{A} \)-genus directly.

**2nd Proof of Theorem 3.1.** Firstly, for any almost complex manifold \( M \), the \( \hat{A} \)-class of the underlying real tangent bundle \( (TM)_R \) is
\[
\hat{A}((TM)_R) = \sum_{k=0}^{\infty} \hat{A}_k(p_1, \ldots, p_k).
\]
By (1.2) and (2.3), the \( \hat{A} \)-class of \((T(X_n(d)))_\mathbb{R}\) is

\[
\hat{A}((T(X_n(d)))_\mathbb{R}) = \prod_{i=1}^{r} \frac{\sinh \left( \frac{d_i x}{2} \right)}{\frac{d_i x}{2}} \cdot \left( \frac{\frac{1}{2} x}{\sinh \left( \frac{1}{2} x \right)} \right)^{n+r+1} \\
= \prod_{i=1}^{r} \frac{\exp \left( \frac{d_i x}{2} \right) - \exp \left( -\frac{d_i x}{2} \right)}{d_i x} \cdot \left( \frac{x}{\exp \left( \frac{1}{2} x \right) - \exp \left( -\frac{1}{2} x \right)} \right)^{n+r+1} \\
= \frac{1}{d^r} x^{n+1} \cdot \prod_{i=1}^{r} \left( \frac{\exp \left( \frac{d_i x}{2} \right) - \exp \left( -\frac{d_i x}{2} \right)}{\exp \left( \frac{1}{2} x \right) - \exp \left( -\frac{1}{2} x \right)} \right)^{n+r+1} \\
= \frac{1}{d^r} x^{n+1} \cdot \exp \left( \frac{1}{2} (n + r + 1)x - \frac{1}{2} \sum_{i=1}^{r} d_i x \right) \cdot \prod_{i=1}^{r} \frac{\exp (d_i x) - 1}{(\exp (x) - 1)^{n+r+1}}, \quad (3.2)
\]

where \( d = d_1 \cdots d_r \) is the total degree of \( X_n(d) \) and \( c_1 = n + r + 1 - \sum_{i=1}^{r} d_i \).

By (3.2), we have

\[
\hat{A}((T(X_{2n}(d)))_\mathbb{R}) = \frac{1}{d^r} x^{2n+1} \cdot \exp \left( \frac{1}{2} c_1 x \right) \cdot \prod_{i=1}^{r} \frac{\exp (d_i x) - 1}{(\exp (x) - 1)^{2n+r+1}}.
\]

Then

\[
\hat{A}(X_{2n}(d)) = \int_{X_{2n}(d)} \hat{A}((T(X_{2n}(d)))_\mathbb{R}) \\
= \int_{X_{2n}(d)} \frac{1}{d} x^{2n+1} \cdot \exp \left( \frac{1}{2} c_1 x \right) \cdot \frac{\exp (d_i x)}{(\exp (x) - 1)^{2n+r+1}} \prod_{i=1}^{r} \left( \exp (d_i x) - 1 \right).
\]

Since what we are interested in is some coefficient in a polynomial, we use residue theorem to calculate this integral. Notice that \( \int_{X_{2n}(d)} x^{2n} = d \) is the total degree of the complete intersection \( X_{2n}(d) \), then we have

\[
\hat{A}(X_{2n}(d)) = \frac{1}{2\pi \sqrt{-1}} \oint_{\Gamma(0)} \frac{\exp \left( \frac{1}{2} c_1 z \right)}{(\exp (z) - 1)^{2n+r+1}} \prod_{i=1}^{r} \left( \exp (d_i z) - 1 \right) dz \\
= \frac{1}{2\pi \sqrt{-1}} \oint_{\Gamma(0)} \frac{\exp \left( (\frac{1}{2} c_1 - 1) z \right)}{(\exp (z) - 1)^{2n+r+1}} \prod_{i=1}^{r} \left( \exp (d_i z) - 1 \right) d \exp (z).
\]
By means of variable substitution: \( \omega = \exp(z) - 1 \),

\[
\hat{A}(X_{2n}(d)) = \frac{1}{2\pi\sqrt{-1}} \int_{\Gamma(0)} \frac{(1 + \omega)^{\frac{1}{2}c_1 - 1}}{\omega^{2n+r+1}} \prod_{i=1}^{r} ((1 + \omega)^{d_i} - 1) d\omega
\]

\[
= \sum_{j=0}^{r} (-1)^{r-j} \sum_{1 \leq k_1 < \cdots < k_j \leq r} \left( \frac{1}{2}c_1 - 1 + d_{k_1} + \cdots + d_{k_j} \right)
\]

Again, Theorem 3.1 follows. \( \square \)

The alpha invariant \( \alpha(M) \in KO^{-n}(pt) \) is a \( KO \)-characteristic number for a spin \( n \) dimensional manifold \( M \), and \( KO^{-n}(pt) \) is known by the Bott periodicity theorem:

| \( n \pmod{8} \) | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|-----------------|---|---|---|---|---|---|---|---|
| \( KO^{-n}(pt) \) | \( \mathbb{Z} \) | \( \mathbb{Z}/2 \) | \( \mathbb{Z}/2 \) | 0 | \( \mathbb{Z} \) | 0 | 0 | 0 |

For a complex \( n \)-dimensional complete intersection \( X_n(d) \), \( \alpha(X_n(d)) \in KO^{-2n}(pt) \). Then \( \alpha(X_n(d)) \) vanishes if \( n = 3 \pmod{4} \). Combine Theorem 3.1 and the results in [7], we can obtain the \( \alpha \)-invariant of complete intersections \( X_n(d_1, \ldots, d_r) \) with \( c_1 = n + r + 1 - \sum_{i=1}^{r} d_i \) even.

**Theorem 3.2.** Let \( d = (d_1, \ldots, d_r) \). Then for a complete intersection \( X_n(d) \) with \( c_1 = n + r + 1 - \sum_{i=1}^{r} d_i \) even, the \( \alpha \)-invariant \( \alpha(X_n(d)) \) is equal to

1. \( \hat{A}(X_n(d)), \) if \( n = 0 \pmod{4}, \) and is equal to \( \frac{1}{2} \hat{A}(X_n(d)), \) if \( n = 2 \pmod{4}, \)

2. \( \sum_{0 \leq j \leq r-1} \left( \frac{1}{2}c_1 - 1 + d_{k_1} + \cdots + d_{k_j} \right) \) (mod 2), if \( n = 1 \pmod{4}, \) where \( d_{k_1} + \cdots + d_{k_j} \) vanishes if \( j = 0 \).

3. 0, if \( n = 3 \pmod{4}. \)

**Proof.** (1) follows directly from [11, Proposition 4.2], and \( \hat{A}(X_n(d)) \) is as in Theorem 3.1. (3) is trivial.

For (2), when \( n = 1 \pmod{4}, \) by [7, (8)], the \( \alpha \)-invariant of \( X_n(d) \) is as follows:

\[
\alpha(X_n(d)) = \sum \left( \frac{1}{2} \left( n + r - 1 \pm d_1 \pm \cdots \pm d_{r-1} + d_r \right) \right) \pmod{2}, \tag{3.3}
\]

where the summation sums over all the possibilities \( \pm d_1 \pm \cdots \pm d_{r-1} + d_r \). By [7, Remark 2], the choice of \( d_r \) in (3.3) is not important to the result, since

\[
\left( \frac{n-1}{2} + k \right) = \left( \frac{n-1}{2} - k \right) \pmod{2}.
\]
Therefore we can make \((3.3)\) stepped forward:
\[
\alpha(X_n(d)) = \sum \left( \frac{1}{2} \left( n + r - 1 \pm d_1 \pm \cdots \pm d_{r-1} + d_r \right) \right) \pmod{2}
\]
\[
= \sum \left( \frac{1}{2} \left( n + r + 1 \pm d_1 \pm \cdots \pm d_{r-1} - d_r \right) - 1 \right) \pmod{2}
\]
\[
= \sum \left( \frac{1}{2}(n + r + 1 - \sum_{i=1}^{r} d_i \pm d_1 \pm d_2 \pm \cdots \pm d_{r-1} \pm d_r) - 1 \right)
\]
\[
= \sum \left( \frac{1}{2}c_1 + \frac{1}{2}(d_1 \pm d_2 \pm \cdots \pm d_{r-1} \pm d_r - 1) \right)
\]
\[
= \sum_{\varepsilon_1, \ldots, \varepsilon_{r-1} \in \{0, 1\}} \left( \frac{1}{2}c_1 - 1 + \varepsilon_1 d_1 + \varepsilon_2 d_2 + \cdots + \varepsilon_{r-1} d_{r-1} \right) \pmod{2}
\]
\[
= \sum_{j=0}^{r-1} \sum_{j=\varepsilon_1 + \cdots + \varepsilon_{r-1} \in \{0, 1\}} \left( \frac{1}{2}c_1 - 1 + \varepsilon_1 d_1 + \varepsilon_2 d_2 + \cdots + \varepsilon_{r-1} d_{r-1} \right) \pmod{2}
\]
\[
= \sum_{j=0}^{r-1} \sum \left( \frac{1}{2}c_1 - 1 + d_{k_1} + \cdots + d_{k_j} \right) \pmod{2}
\]
Thus,
\[
\alpha(X_n(d)) = \sum_{0 \leq j \leq r-1} \left( \frac{1}{2}c_1 - 1 + d_{k_1} + \cdots + d_{k_j} \right) \pmod{2}. \quad \Box
\]

In fact, the key formula \((3.3)\) we used in the above proof is deduced from the following formula \((7, (5))\):
\[
\alpha(X_n(d_1, \ldots, d_{r-1}, d_r)) = \int_{X_{n+1}(d_1, \ldots, d_{r-1})} \hat{A}((T(X_{n+1}(d_1, \ldots, d_{r-1})))) \cdot \exp \left( \frac{d_r x}{2} \right) \pmod{2}.
\]
Begin with this formula, we can directly prove Theorem 3.2 \((2)\).

2nd Proof of Theorem 3.2 \((2)\).
\[
\alpha(X_n(d_1, \ldots, d_{r-1}, d_r))
\]
\[
= \int_{X_{n+1}(d_1, \ldots, d_{r-1})} \hat{A}((T(X_{n+1}(d_1, \ldots, d_{r-1})))) \cdot \exp \left( \frac{d_r x}{2} \right) \pmod{2}
\]
\[
= \int_{X_{n+1}(d_1, \ldots, d_{r-1})} \hat{A}((T(X_{n+1}(d_1, \ldots, d_{r-1})))) \cdot \exp \left( -\frac{d_r x}{2} \right) \pmod{2}.
\]
By (3.2), we have
\[
\alpha(X_n(d_1, \ldots, d_{r-1}, d_r)) = \int_{X_{n+1}(d_1, \ldots, d_{r-1})} \frac{x^{n+2}}{\prod_{i=1}^{r-1} d_i} \cdot \prod_{i=1}^{r-1} \left( \exp \left( \frac{d_i}{2} x \right) - \exp \left( -\frac{d_i}{2} x \right) \right) \cdot \exp \left( -\frac{d_r}{2} x \right).
\]

By residue theorem,
\[
\alpha(X_n(d_1, \ldots, d_{r-1}, d_r)) = \frac{1}{2\pi \sqrt{-1}} \int_{\Gamma(0)} \prod_{i=1}^{r-1} \left( \exp \left( \frac{d_i}{2} z \right) - \exp \left( -\frac{d_i}{2} z \right) \right) \cdot \exp \left( -\frac{d_r}{2} z \right) \exp (z - 1) dz
\]
\[
= \frac{1}{2\pi \sqrt{-1}} \int_{\Gamma(0)} \left( \frac{1}{2} \sum_{i=1}^{r} d_i \right) \cdot \prod_{i=1}^{r-1} \left( \exp \left( \frac{d_i}{2} z \right) - \exp \left( -\frac{d_i}{2} z \right) \right) \cdot \exp \left( -\frac{d_r}{2} z \right) \exp (z - 1) dz
\]
\[
= \frac{1}{2\pi \sqrt{-1}} \int_{\Gamma(0)} \left( \frac{1}{2} \sum_{i=1}^{r} d_i \right) \cdot \prod_{i=1}^{r-1} \left( \exp \left( \frac{d_i}{2} z \right) - \exp \left( -\frac{d_i}{2} z \right) \right) \cdot \exp \left( -\frac{d_r}{2} z \right) \exp (z - 1) dz
\]
\[
= \frac{1}{2\pi \sqrt{-1}} \int_{\Gamma(0)} \left( \frac{1}{2} \sum_{i=1}^{r} d_i z \right) \cdot \prod_{i=1}^{r-1} \left( \exp \left( \frac{d_i}{2} z \right) - \exp \left( -\frac{d_i}{2} z \right) \right) \cdot \exp \left( -\frac{d_r}{2} z \right) \exp (z - 1) dz
\]
\[
= \frac{1}{2\pi \sqrt{-1}} \int_{\Gamma(0)} \left( \frac{1}{2} \sum_{i=1}^{r} d_i z \right) \cdot \prod_{i=1}^{r-1} \left( \exp \left( \frac{d_i}{2} z \right) - \exp \left( -\frac{d_i}{2} z \right) \right) \cdot \exp \left( -\frac{d_r}{2} z \right) \exp (z - 1) dz
\]

By means of variable substitution: \( \omega = \exp(z) - 1 \),
\[
\alpha(X_n(d_1, \ldots, d_{r-1}, d_r)) = \frac{1}{2\pi \sqrt{-1}} \int_{\Gamma(0)} (1 + \omega)^{\frac{1}{2} c_1} \cdot \prod_{i=1}^{r-1} \left( (1 + \omega)^{d_i} - 1 \right) \omega^{n+r+1} \cdot (1 + \omega)^{-1} \cdot \exp \left( -\frac{d_r}{2} \omega \right) \exp (\omega - 1)
\]
\[
= \frac{1}{2\pi \sqrt{-1}} \int_{\Gamma(0)} (1 + \omega)^{\frac{1}{2} c_1} \cdot \prod_{i=1}^{r-1} \left( (1 + \omega)^{d_i} - 1 \right) \omega^{n+r+1} \cdot \exp \left( -\frac{d_r}{2} \omega \right) \exp (\omega - 1)
\]
\[
= \sum_{0 \leq j_{k_1}, \ldots, j_{k_j} \leq r-1} \left( \frac{1}{2} c_1 - 1 + d_{k_1} + \cdots + d_{k_j} \right) \mod 2.
\]

\[\square\]

4. The virtual Hirzebruch genus of complete intersections

Inspired by the virtual index and virtual generalized Todd genus in [9, §9, §11], we define the virtual Hirzebruch genus. First of all, let’s introduce the concept of virtual submanifold. The following part about the virtual submanifold in [10, §3.1]

According to Thom [19], every 2-dimensional integral cohomology class \( v \in H^2(M^n; \mathbb{Z}) \) of a compact oriented differentiable manifold \( M^n \) can be represented by a submanifold \( V^{n-2} \), i.e., the cohomology class \( v \) is the Poincaré duality of the fundamental class of the oriented submanifold \( V^{n-2} \). Therefore \( v \in H^2(M; \mathbb{Z}) \) is called a virtual submanifold. Assume that \( v_1, v_2, \ldots, v_r \in H^2(M; \mathbb{Z}) \), and \( v_1 \) is a virtual submanifold represented by a submanifold \( V^{n-2} \) of the manifold \( M \), that the restriction of \( v_2 \) to \( V^{n-2} \) represents a submanifold \( V^{n-4} \) of \( V^{n-2} \), ..., and finally that the restriction of \( v_r \) to \( V^{n-2(r-1)} \).
represents a submanifold $V^{n-2r}$ of $V^{n-2(r-1)}$. The manifold $V^{n-2r}$ is then a submanifold of $M$ of codimension $2r$, which represents the virtual submanifold $(v_1, \ldots, v_r)$. An alternative construction is the following: If $v_1, \ldots, v_r$ can be represented by codimension 2 submanifolds $V'_1, \ldots, V'_r$ of $M$, which are transversal to one another, then their transversal intersection represents the virtual submanifold $(v_1, \ldots, v_r)$.

Let’s set up the following convention:

1. If $M$ is a compact almost complex manifold, $Q(x) = x/R(x)$ is taken as any power series with constant term being one.
2. If $M$ is a compact oriented differentiable manifold, $Q(x) = x/R(x)$ is taken as an even power series with constant term being one.

**Definition 4.1.** For the integral cohomology classes $v_1, v_2, \ldots, v_r$ of $H^2(M; \mathbb{Z})$, the **virtual Hirzebruch genus** $\varphi_Q$ of the virtual submanifold $(v_1, v_2, \ldots, v_r)$ is defined as follows:

$$
\varphi_Q(v_1, \ldots, v_r)_M = \int_M \prod_{j=1}^r R(v_j) \cdot \varphi_Q(TM),
$$

where $\varphi_Q(TM)$ is the $\varphi_Q$-class defined as in Section 2.

In fact, the virtual Hirzebruch genus of the virtual submanifold $(v_1, v_2, \ldots, v_r)$ is related to the Hirzebruch genus of the submanifold $V^{n-2r}$ of $M^n$. Similar to the argument in [9, §9, §11], we have the following result.

**Proposition 4.2.** As the notations in Definition 4.1. Assume that the virtual submanifold $(v_1, \ldots, v_r)$ is represented by a submanifold $V^{n-2r}$ of the manifold $M^n$, then

$$
\varphi_Q(v_1, \ldots, v_r)_M = \varphi_Q(V^{n-2r}).
$$

**Proof.** We only give the proof when $M$ is a compact oriented differentiable $n$-manifold. This proof is adapted to the case of almost complex manifold (see [9, §11]).

Let $v \in H^2(M^n; \mathbb{Z})$ be a virtual manifold represented by a submanifold $V^{n-2}$, and $j : V^{n-2} \to M^n$ be the embedding of the oriented submanifold $V^{n-2}$ of $M^n$. Let $TV^{n-2}$ and $TM^n$ be tangle bundles of $V^{n-2}$, $M^n$ respectively and $\mathcal{N}$ is the normal bundle of $V^{n-2}$ in $M^n$, then

$$
j^*(TM^n) = TV^{n-2} \oplus \mathcal{N}.
$$

Let $p(V^{n-2})$ and $p(M^n)$ be the total Pontrjagin classes of $V^{n-2}$ and $M^n$ respectively. Then

$$
j^*p(M^n) = p(V^{n-2})p(\mathcal{N}) \mod \text{torsion}.
$$

Since $p(\mathcal{N}) = j^*(1 + v^2)$, then

$$
p(V^{n-2}) = j^*((1 + v^2)^{-1}p(M^n)).
$$

Note that $(1 + v^2)^{-1}$ is unique in the cohomology class of $M^n$. Thus, for every Hirzebruch genus $\varphi_Q$ associated to an even power series $Q(x)$, we have

$$
\varphi_Q(TM^{n-2}) = j^* \left( \frac{R(v)}{v} \varphi_Q(TM^n) \right).
$$
For a cohomology class $u \in H^{n-2}(M^n; \mathbb{Z})$, by Poincaré duality, we have

$$\int_{V^{n-2}} j^*(u) = \int_{M^n} vu.$$

Therefore,

$$\varphi_Q(V^{n-2}) = \int_{V^{n-2}} \varphi_Q(TV^{n-2}) = \int_{V^{n-2}} j^* \left( \frac{R(v)}{v} \varphi_Q(TM^n) \right)$$

$$= \int_{M^n} R(v) \varphi_Q(TM^n) = \varphi_Q(v)_M.$$  

By finite induction, it implies that $\varphi_Q(v_1, \ldots, v_r)_M = \varphi_Q(V^{n-2r})$. \qed

Now we consider the complete intersection $X_n(d_1, \ldots, d_r)$ embedded in $\mathbb{C}P^{n+r}$. Let $x \in H^2(\mathbb{C}P^{n+r}; \mathbb{Z})$ be the first Chern class of the Hopf line bundle $H$ over $\mathbb{C}P^{n+r}$, then the virtual submanifold $(d_1x, \ldots, d_rx)$ is represented by the complete intersection $X_n(d_1, \ldots, d_r)$. So by Proposition 4.2, we have

**Corollary 4.3.** The $\varphi_Q$-genus of $X_n(d_1, \ldots, d_r)$ is the virtual $\varphi_Q$-genus of the virtual submanifold $(d_1x, \ldots, d_rx)$, i.e.,

$$\varphi_Q(d_1x, \ldots, d_rx)_{\mathbb{C}P^{n+r}} = \varphi_Q(X_n(d_1, \ldots, d_r)).$$

As a special case of Definition 4.1, we can also define the virtual $\hat{A}$-genus.

**Definition 4.4.** Let $M$ be a compact oriented differentiable manifold. For $v_1, \ldots, v_r \in H^2(M; \mathbb{Z})$, the virtual $\hat{A}$-genus of $(v_1, \ldots, v_r)$ is defined as follows:

$$\hat{A}(v_1, \ldots, v_r)_M = \int_M \prod_{j=1}^r R(v_j) \cdot \hat{A}(TM),$$

where $Q(x) = \frac{x}{R(x)} = \frac{\frac{1}{2}x}{\sinh \left( \frac{1}{2}x \right)}$.

**Proposition 4.5.** Let $M$ be a compact oriented differentiable manifold. For any $v_1, \ldots, v_r, w \in H^2(M; \mathbb{Z})$, the virtual $\hat{A}$-genus satisfies the following equality:

$$\hat{A}(v_1, \ldots, v_{r-2}, v_{r-1} + w, v_r + w)_M$$

$$= \hat{A}(v_1, \ldots, v_{r-2}, v_{r-1})_M + \hat{A}(v_1, \ldots, v_{r-2}, v_{r-1} + v_r + w, w)_M.$$

**Proof.** The $\hat{A}$-genus is corresponding to the power series $Q(x) = \frac{x}{R(x)} = \frac{\frac{1}{2}x}{\sinh \left( \frac{1}{2}x \right)}$, and $R(x) = 2 \sinh \left( \frac{1}{2}x \right)$ satisfies the following identity

$$R(u + w)R(v + w) = R(u)R(v) + R(u + v + w)R(w).$$
By Definition 4.4, for any \( w \in H^2(M; \mathbb{Z}) \), it induces the following equality
\[
\hat{A}(v_1, \ldots, v_{r-2}, v_{r-1} + w, v_r + w)_M \\
= \int_M \prod_{j=1}^{r-2} R(v_j) \cdot R(v_{r-1} + w) \cdot R(v_r + w) \hat{A}(TM) \\
= \int_M \prod_{j=1}^{r} R(v_j) \cdot \hat{A}(TM) + \int_M \prod_{j=1}^{r-2} R(v_j) \cdot R(v_{r-1} + v_r + w) \cdot R(w) \hat{A}(TM) \\
= \hat{A}(v_1, \ldots, v_{r-2}, v_{r-1}, v_r)_M + \hat{A}(v_1, \ldots, v_{r-2}, v_{r-1} + v_r + w, w)_M. \]
\( \Box \)

By Corollary 4.3 and Proposition 4.5, we have

**Corollary 4.6.** For any positive integers \( d_1, \ldots, d_r, e \), we have the following equality on the \( \hat{A} \)-genus of complete intersections
\[
\hat{A}(X_{2n}(d_1, \ldots, d_{r-2}, d_{r-1} + e, d_r + e)) \\
= \hat{A}(X_{2n}(d_1, \ldots, d_{r-1}, d_r)) + \hat{A}(X_{2n}(d_1, \ldots, d_{r-2}, d_{r-1} + d_r + e, e)).
\]

**Theorem 4.7.** For any positive integers \( d_1, \ldots, d_r \), assume that \( d_{r-1} \geq d_r \geq 2 \), then we have
\[
\hat{A}(X_{2n}(d_1, \ldots, d_{r-1}, d_r)) = \sum_{k=0}^{d_r-1} \hat{A}(X_{2n}(d_1, \ldots, d_{r-2}, d_{r-1} + d_r - 1 - 2k)). \quad (4.1)
\]

Note that, if \( d_r = 1 \), \( \hat{A}(X_{2n}(d_1, \ldots, d_{r-1}, 1)) = \hat{A}(X_{2n}(d_1, \ldots, d_{r-1})) \).

**Proof.** Applying Corollary 4.6, then
\[
\hat{A}(X_{2n}(d_1, \ldots, d_{r-2}, d_{r-1}, d_r)) \\
= \hat{A}(X_{2n}(d_1, \ldots, d_{r-2}, d_{r-1} - 1, d_r - 1)) + \hat{A}(X_{2n}(d_1, \ldots, d_{r-2}, d_{r-1} + d_r - 1, 1)) \\
= \hat{A}(X_{2n}(d_1, \ldots, d_{r-2}, d_{r-1} - 1, d_r - 1)) + \hat{A}(X_{2n}(d_1, \ldots, d_{r-2}, d_{r-1} + d_r - 1)).
\]

Let’s iterate the above process, the proof is finished. \( \Box \)

Consider complex even dimensional spin complete intersection \( X_{2n}(d) \) with multi-degree \( d = (d_1, \ldots, d_r) \). Note that \( X_{2n}(d) \) is spin if and only if \( c_1 = 2n + r + 1 - \sum_{i=1}^{r} d_i = 0 \) (mod 2). In [3, Theorem 2], Brooks can determine when \( \hat{A}(X_{2n}(d)) = 0 \) for the spin complete intersections \( X_{2n}(d) \). Now, we give a new proof for the necessary and sufficient conditions related to the vanishing of \( \hat{A}(X_{2n}(d)) \).

**Theorem 4.8.** For each complex even dimensional spin complete intersection \( X_{2n}(d) \) with \( d = (d_1, \ldots, d_r) \), the \( \hat{A} \)-genus of \( X_{2n}(d) \) has the following properties:

1. \( \hat{A}(X_{2n}(d)) = 0 \), if \( c_1 > 0 \);
2. \( \hat{A}(X_{2n}(d)) > 0 \), if \( c_1 \leq 0 \),
where \( c_1 = 2n + r + 1 - \sum_{i=1}^{r} d_i \).

**Proof.** Since the diffeomorphism type of \( X_{2n}(d) \) is independent of the order of the degrees \( d_1, \ldots, d_{r-1}, d_r \), without losing generality, we assume that \( d_{r-1} \geq d_r \).

We prove the theorem by induction on \( r \).

For \( r = 1 \), \( \hat{A}(X_{2n}(d_1)) = \frac{2}{(2n+1)!} \prod_{j=-n}^{n} \left( \frac{d}{2} - j \right) \), \( c_1 = 2n + 2 - d_1 \), and this case is trivial.

For \( r > 1 \), if \( c_1 > 0 \), then
\[
2n + r + 1 - \sum_{i=1}^{r} d_i = 2n + (r - 1) + 1 - \left( \sum_{i=1}^{r-2} d_i + (d_{r-1} + d_r - 1) \right) > 0,
\]
\[
2n + (r - 1) + 1 - \left( \sum_{i=1}^{r-2} d_i + (d_{r-1} + d_r - 1 - 2k) \right) > 0, 0 \leq k \leq d_r - 1.
\]

Thus, by induction hypothesis,
\[
\hat{A}(X_{2n}(d_1, \ldots, d_{r-2}, d_{r-1} + d_r - 1 - 2k)) = 0, 0 \leq k \leq d_r - 1,
\]
i.e. the right side of (4.1) is zero, then \( \hat{A}(X_{2n}(d_1, \ldots, d_r)) = 0 \);

Conversely, if \( c_1 \leq 0 \), then
\[
2n + r + 1 - \sum_{i=1}^{r} d_i = 2n + (r - 1) + 1 - \left( \sum_{i=1}^{r-2} d_i + (d_{r-1} + d_r - 1) \right) \leq 0.
\]

Thus, by induction hypothesis,
\[
\hat{A}(X_{2n}(d_1, \ldots, d_{r-2}, d_{r-1} + d_r - 1)) > 0,
\]
and all other summation terms in (4.1) are nonnegative, so \( \hat{A}(X_{2n}(d_1, \ldots, d_r)) > 0 \). \( \square \)

**5. Calculation of classical Hirzebruch genera of complete intersections**

In this section, for the power series \( Q(x) = x/R(x) \), we discuss the associated Hirzebruch genus \( \varphi_Q \) of complete intersections \( X_n(d) \) with multi-degree \( d = (d_1, \ldots, d_r) \).

Then for certain given power series \( Q(x) \), we calculate the associated classical Hirzebruch genera of complete intersections as examples.

**Theorem 5.1.** The Hirzebruch genus \( \varphi_Q \) on the complete intersection \( X_n(d) \) satisfies that:
\[
\varphi_Q(X_n(d)) = \frac{1}{2\pi \sqrt{-1}} \oint_{\Gamma(0)} \frac{\prod_{i=1}^{r} R(d_i z)}{(R(z))^{n+r+1}} dz.
\]
Proof. Let \( x \in H^2(\mathbb{C}P^n, \mathbb{Z}) \) be the first Chern class of the Hopf line bundle \( H \) over \( \mathbb{C}P^n \). By Corollary 4.3 and Definition 4.1,

\[
\varphi_Q(X_n(d)) = \varphi_Q(d_1 x, \ldots, d_r x)_{\mathbb{C}P^n} \\
= \int_{\mathbb{C}P^n} \prod_{i=1}^r R(d_i x) \cdot \varphi_Q(T(\mathbb{C}P^n)).
\]

The total Chern class of \( \mathbb{C}P^n \) is \( c(\mathbb{C}P^n) = (1 + x)^{n+r+1} \), so

\[
\varphi_Q(T(\mathbb{C}P^n)) = (Q(x))^{n+r+1}.
\]

Thus

\[
\varphi_Q(X_n(d)) = \int_{\mathbb{C}P^n} \prod_{i=1}^r R(d_i x) \cdot (Q(x))^{n+r+1} \\
= \int_{\mathbb{C}P^n} x^{n+r+1} \prod_{i=1}^r R(d_i x) \cdot (R(x))^{n+r+1}.
\]

By residue theorem,

\[
\varphi_Q(X_n(d)) = \frac{1}{2\pi i} \prod_{j=1}^r \frac{R(d_j z)}{(R(z))^{n+r+1}} dz.
\]

Example 5.2. For \( T_y \)-genus (or generalized Todd genus) in [9, page 93], the associated power series is \( Q(y; x) = \frac{x}{R(y; x)} \), where

\[
R(y; x) = \frac{\exp(x(y + 1)) - 1}{\exp(x(y + 1)) + y}.
\]

Hence by Theorem 5.1 we have

\[
T_y(X_n(d)) = \frac{1}{2\pi i} \prod_{j=1}^r \frac{\exp(d_j z(y + 1)) - 1}{\exp(d_j z(y + 1)) + y} \cdot \left( \frac{\exp(z(y + 1)) + y}{\exp(z(y + 1)) - 1} \right)^{n+r+1} dz.
\]
Example 5.3. In Example 5.2, when \( y = 0 \), \( R(y; x) \) is \( R(0; x) = 1 - \exp(-x) \). The \( T_0 \)-genus is the Todd genus, which has the following form:

\[
\text{Td}(X_n(d)) = \frac{1}{2\pi \sqrt{-1}} \int_{\Gamma(0)} \frac{\prod_{j=1}^{r} (1 - \exp(-d_j z))}{(1 - \exp(-z))^{n+r+1}} dz
\]

\[
= \frac{1}{2\pi \sqrt{-1}} \int_{\Gamma(0)} \frac{\prod_{j=1}^{r} (1 - \exp(-d_j z))}{(1 - \exp(-z))^{n+r+1}} \cdot \exp(z) d(1 - \exp(-z))
\]

\[
= \frac{1}{2\pi \sqrt{-1}} \int_{\Gamma(0)} \frac{\prod_{j=1}^{r} (1 - (1 - \omega)^{d_j})}{\omega^{n+r+1}} \cdot (1 - \omega)^{-1} d\omega
\]

\[
= \frac{1}{2\pi \sqrt{-1}} \int_{\Gamma(0)} \frac{1}{\omega^{n+r+1}} \sum_{j=0}^{r} (-1)^j \sum_{1 \leq k_1 < \cdots < k_j \leq r} (1 - \omega)^{-1+d_{k_1}+\cdots+d_{k_j}} d\omega
\]

\[
= \sum_{j=0}^{r} (-1)^j \sum_{1 \leq k_1 < \cdots < k_j \leq r} (1 + d_{k_1} + \cdots + d_{k_j})\frac{(-1)^{n+r} + d_{k_1} + \cdots + d_{k_j}}{n + r}.
\]

(5.2)

Executing another process, we have the following result:

\[
\text{Td}(X_n(d)) = \frac{1}{2\pi \sqrt{-1}} \int_{\Gamma(0)} \frac{\prod_{j=1}^{r} (1 - \exp(-d_j z))}{(1 - \exp(-z))^{n+r+1}} dz
\]

\[
= \frac{1}{2\pi \sqrt{-1}} \int_{\Gamma(0)} \exp(c_1 z) \prod_{j=1}^{r} \frac{(\exp(d_j z) - 1)}{(\exp(z) - 1)^{n+r+1}} dz
\]

\[
= \frac{1}{2\pi \sqrt{-1}} \int_{\Gamma(0)} \exp(c_1 z) \prod_{j=1}^{r} \frac{(\exp(d_j z) - 1)}{(\exp(z) - 1)^{n+r+1}} \cdot \exp(-z) d(\exp(z) - 1)
\]

\[
= \frac{1}{2\pi \sqrt{-1}} \int_{\Gamma(0)} \prod_{j=1}^{r} \frac{((1 + \omega)^{d_j} - 1)}{\omega^{n+r+1}} \cdot (1 + \omega)^{c_1-1} d\omega
\]

\[
= \sum_{j=0}^{r} (-1)^{r-j} \sum_{1 \leq k_1 < \cdots < k_j \leq r} \left( c_1 - 1 + d_{k_1} + \cdots + d_{k_j} \right)\frac{n + r}{n + r},
\]

(5.3)

where \( c_1 = n + r + 1 - \sum_{i=1}^{r} d_i \).

In fact, (5.2) and (5.3) are equivalent. Since \( \binom{a}{k} \) is a generalized binomial coefficients coventioned as in Theorem 3.1, it is easy to know \((-1)^k \binom{a}{k} = \binom{-a + k - 1}{k}\), then we
have

\[
(-1)^{n+r} \cdot \frac{(-1 + d_{k_1} + \cdots + d_{k_j})}{n + r} \\
= \frac{(n + r - 1 + 1 - (d_{k_1} + \cdots + d_{k_j}))}{n + r} \\
= \frac{(n + r + 1 - \sum_{i=1}^r d_i) - 1 + \sum_{i=1}^r d_i - (d_{k_1} + \cdots + d_{k_j})}{n + r} \\
= \left( c_1 - 1 + \frac{\left( \sum_{i=1}^r d_i - (d_{k_1} + \cdots + d_{k_j}) \right)}{n + r} \right).
\]

So (5.2) implies that

\[
\text{Td}(X_n(d)) = \sum_{j=0}^r (-1)^j \sum_{1 \leq k_1 < \cdots < k_j \leq r} (-1)^{n+r} \frac{(-1 + d_{k_1} + \cdots + d_{k_j})}{n + r} \\
= \sum_{j=0}^r (-1)^j \sum_{1 \leq k_1 < \cdots < k_j \leq r} \left( c_1 - 1 + \frac{\left( \sum_{i=1}^r d_i - (d_{k_1} + \cdots + d_{k_j}) \right)}{n + r} \right) \\
= \sum_{j=0}^r (-1)^{r-j} \sum_{1 \leq k_1 < \cdots < k_j \leq r} \left( c_1 - 1 + \frac{d_{k_1} + \cdots + d_{k_j}}{n + r} \right).
\]

Thus, the two forms (5.2) and (5.3) about Todd genus of \(X_n(d)\) are equivalent.

**Example 5.4.** In Example 5.2, when \(y = -1\), \(R(y; x)\) is

\[
R(-1; x) = \lim_{y \to -1} \frac{\exp(x(y + 1)) - 1}{\exp(x(y + 1)) + y} \\
= \lim_{y \to -1} \frac{x \exp(x(y + 1))}{x \exp(x(y + 1)) + 1} \\
= \frac{x}{1 + x},
\]

and (5.1) is the Euler characteristic of complete intersection \(X_n(d)\):

\[
\chi(X_n(d)) = \frac{1}{2\pi \sqrt{-1}} \oint_{R(0)} \frac{\prod_{j=1}^r \frac{d_j z}{1 + d_j z}}{(z + 1)^{n+r+1}} dz.
\]

Let \(\omega = \frac{z}{1 + z}\), then \(z = \frac{\omega}{1 - \omega}\). It implies that

\[
\frac{d_j z}{1 + d_j z} = \frac{d_j \omega}{1 + (d_j - 1) \omega},
\]

\[
d\omega = (1 - \omega)^2 dz \quad \text{(the differential of } \omega).\]
Hence,

\[
\chi(X_n(d)) = \frac{1}{2\pi \sqrt{-1}} \oint_{\Gamma(0)} \prod_{j=1}^{r} \frac{d_j \omega}{1 + (d_j - 1) \omega} \cdot \frac{1}{\omega^{n+r+1}} \cdot \frac{1}{(1-\omega)^2} d\omega
\]

\[
= \prod_{j=1}^{r} d_j \cdot \frac{1}{2\pi \sqrt{-1}} \oint_{\Gamma(0)} \frac{1}{1 + (d_j - 1) \omega} \cdot \frac{1}{\omega^{n+1}} d\omega
\]

\[
= \prod_{j=1}^{r} d_j \cdot h_n(1, 1, 1 - d_1, \ldots, 1 - d_r),
\]

where \( h_n(a_1, \ldots, a_k) = \sum_{1 \leq i_1 < \cdots < i_n \leq k} a_{i_1} a_{i_2} \cdots a_{i_n} \) is the coefficient of \( z^n \) in \( \prod_{j=1}^{k} \frac{1}{1 - a_j z} \), and \( h_n(a_1, \ldots, a_k) \) is called a complete homogeneous symmetric polynomial of degree \( n \) on indeterminates \( a_1, \ldots, a_k \).

**Example 5.5.** In Example 5.2, when \( y = 1 \), \( R(y; x) \) is \( R(1; x) = \tanh x \) and (5.1) is the corresponding signature of complete intersection \( X_n(d) \).

\[
\tau(X_n(d)) = \frac{1}{2\pi \sqrt{-1}} \oint_{\Gamma(0)} \prod_{j=1}^{r} \frac{\tanh(d_j z)}{(\tanh z)^{n+r+1}} dz
\]

\[
= \frac{1}{2\pi \sqrt{-1}} \oint_{\Gamma(0)} \frac{\prod_{j=1}^{r} \exp(2d_j z) - 1}{\exp(2d_j z) + 1} \cdot \left( \frac{\exp(2z) + 1}{\exp(2z) - 1} \right)^{n+r+1} dz
\]

\[
= \frac{1}{2\pi \sqrt{-1}} \oint_{\Gamma(0)} \frac{\prod_{j=1}^{r} \exp(2d_j z) - 1}{\exp(2d_j z) + 1} \cdot \left( \frac{\exp(2z) + 1}{\exp(2z) - 1} \right)^{n+r+1} \cdot \frac{\exp(-2z)}{2} d(\exp(2z) - 1)
\]

\[
= \frac{1}{2\pi \sqrt{-1}} \oint_{\Gamma(0)} \prod_{j=1}^{r} \frac{(1 + \omega)^{d_j} - 1}{(1 + \omega)^{d_j} + 1} \cdot \left( \frac{2 + \omega}{\omega} \right)^{n+r+1} \cdot \frac{1}{2(1 + \omega)} d\omega.
\]

**Example 5.6.** For Witten genus [10, page 82], the power series is that

\[
Q(x) = \frac{x}{R(L; x)} = \frac{x}{\sigma_L(x)},
\]

where \( R(L; x) = \sigma_L(x) \) is the Weierstrass \( \sigma \)-function for Lattice \( L \). Hence, by Theorem 5.1, the Witten genus of \( X_n(d) \) is

\[
W(X_n(d)) = \frac{1}{2\pi \sqrt{-1}} \oint_{\Gamma(0)} \frac{\prod_{j=1}^{r} \sigma_L(d_j z)}{(\sigma_L(z))^{n+r+1}} dz.
\]

**Example 5.7.** For \( \hat{A} \)-genus, the power series corresponding to \( \hat{A} \)-genus is

\[
Q(x) = \frac{x}{R(x)} = \frac{1}{2} x \cdot \frac{1}{\sinh \left( \frac{1}{2} x \right)} \cdot \frac{1}{\sinh \left( \frac{1}{2} x \right)}, \quad \text{where} \quad R(x) = 2 \sinh \left( \frac{1}{2} x \right).
\]
Then by Theorem 5.1,
\[
A(X_{2n}(d_1, \ldots, d_r)) = \frac{1}{2\pi \sqrt{1}} \int_{\mathbb{R}(0)} \prod_{j=1}^{r} \left(2 \sinh \left(\frac{1}{2}d_jz\right)\right) \left(\begin{array}{c} 2n + r + 1 - \sum_{i=1}^{r} d_i \end{array}\right) \left(\begin{array}{c} \prod_{j=1}^{r} (\exp (d_jz) - 1) \\ (\exp (z) - 1)^{2n + r + 1} \end{array}\right) dz
\]
\[
= \frac{1}{2\pi \sqrt{1}} \int_{\mathbb{R}(0)} \prod_{j=1}^{r} \left(\frac{1}{2} \left(2n + r + 1 - \sum_{i=1}^{r} d_i\right)z\right) \left(\begin{array}{c} \prod_{j=1}^{r} (\exp (d_jz) - 1) \\ (\exp (z) - 1)^{2n + r + 1} \end{array}\right) dz
\]
\[
= \frac{1}{2\pi \sqrt{1}} \int_{\mathbb{R}(0)} \exp \left(\frac{1}{2} \left(2n + r + 1 - \sum_{i=1}^{r} d_i\right)z\right) \left(\begin{array}{c} \prod_{j=1}^{r} (\exp (d_jz) - 1) \\ (\exp (z) - 1)^{2n + r + 1} \end{array}\right) d\omega
\]
\[
= \sum_{j=0}^{r} (-1)^{r-j} \sum_{1 \leq k_1 < \cdots < k_j \leq r} \left(\frac{1}{2} c_1 - 1 + d_{k_1} + \cdots + d_{k_j}\right),
\]
where \(c_1 = 2n + r + 1 - \sum_{i=1}^{r} d_i\).

**Example 5.8.** In [14], Kričever proved that the values of the Hirzebruch genera \(A_k\), \(k = 2, 3, \ldots\), are obstructions to the existence of nontrivial \(S^1\)-actions on a unitary manifold whose first Chern class is divisible by \(k\). For Hirzebruch genera \(A_k\) in [14, §2], the associated power series is
\[
Q(x) = \frac{x}{R(x)} = \frac{kx \exp (x)}{\exp (kx) - 1}, \quad k = 2, 3, \ldots,
\]
where \(R(x) = \frac{\exp (kx) - 1}{k \exp (x)}\).

Then by Theorem 5.1,
\[
A_k(X_n(d)) = \frac{1}{2\pi \sqrt{1}} \int_{\mathbb{R}(0)} \prod_{i=1}^{r} \frac{\exp (kd_iz) - 1}{k \exp (z)} \left(\frac{k \exp \left(\frac{1}{k}z\right)}{\exp (z) - 1}\right)^{n+r+1} dz
\]
\[
= \frac{1}{k} \cdot \frac{1}{2\pi \sqrt{1}} \int_{\mathbb{R}(0)} \prod_{i=1}^{r} \frac{\exp (d_iz) - 1}{k \exp \left(\frac{1}{k}d_iz\right)} \left(\frac{k \exp \left(\frac{1}{k}z\right)}{\exp (z) - 1}\right)^{n+r+1} dz
\]
\[
= \frac{1}{k} \cdot \frac{1}{2\pi \sqrt{1}} \int_{\mathbb{R}(0)} \prod_{i=1}^{r} \frac{\exp (d_iz) - 1}{k \exp \left(\frac{1}{k}d_iz\right)} \left(\frac{k \exp \left(\frac{1}{k}z\right)}{\exp (z) - 1}\right)^{n+r+1} \exp (-z) d(\exp (z) - 1)
\]
\[
\frac{1}{k} \cdot \frac{1}{2\pi \sqrt{-1}} \oint_{\Gamma(0)} \left( \prod_{i=1}^{r} \frac{(1+\omega)^{d_i} - 1}{k(1+\omega)^{\frac{1}{k}} \omega} \right)^{n+r+1} \cdot (1 + \omega)^{-1} \, d\omega
\]

\[
= k^n \cdot \frac{1}{2\pi \sqrt{-1}} \oint_{\Gamma(0)} \frac{\prod_{i=1}^{r} \left( (1+\omega)^{d_i} - 1 \right)}{\omega^{n+r+1}} \cdot (1 + \omega)^{\frac{n+r+1}{2} - \sum d_i} \, d\omega
\]

\[
= k^n \cdot \frac{1}{2\pi \sqrt{-1}} \oint_{\Gamma(0)} \frac{\prod_{i=1}^{r} \left( (1+\omega)^{d_i} - 1 \right)}{\omega^{n+r+1}} \cdot (1 + \omega)^{\frac{c_1}{2} - 1} \, d\omega
\]

\[
= k^n \sum_{j=0}^{r} (-1)^{r-j} \sum_{1 \leq k_1 < \cdots < k_j \leq r} \left( \frac{\frac{1}{k}c_1 - 1 + d_{k_1} + \cdots + d_{k_j}}{n + r} \right),
\]

where \( c_1 = n + r + 1 - \sum_{i=1}^{r} d_i \). Note that \( A_1(X_n(d)) \) is the Todd genus \( Td(X_n(d)) \), and \( A_2(X_n(d)) \) coincides with the \( \hat{A} \)-genus \( \hat{A}(X_n(d)) \) up to a factor \( 2^n \).

References

[1] M. Atiyah and F. Hirzebruch, *Spin-manifolds and group actions*, Essays on topology and related topics (Mémoires dédiés à Georges de Rham), Springer, New York, 1970, 18-28.

[2] D. Baraglia, *The alpha invariant of complete intersections*, arXiv: 2002.06750v1, https://arxiv.org/abs/2002.06750v1

[3] R. Brooks, *The A-genus of complex hypersurfaces and complete intersections*, Proc. Amer. Math. Soc. 87 (1983) 528-532.

[4] V.M. Buchstaber and T.E. Panov, *Toric topology*, Mathematical Surveys and Monographs 204, American Mathematical Society, 2015.

[5] D. Crowley and C. Nagy, *The smooth classification of 4-dimensional complete intersections*, arXiv: 2003.09216, https://arxiv.org/abs/2003.09216

[6] F. Fang and S. Klaus, *Topological classification of 4-dimensional complete intersections*, Manuscripta Math. 90 (1996) 139–147.

[7] F. Fang and P. Shao, *Complete intersections with metrics of positive scalar curvature*, C. R. Acad. Sci. Paris, Ser. I 347 (2009) 797-800.

[8] F. Fang and J. Wang, *Homeomorphism classification of complex projective complete intersections of dimension 5, 6 and 7*, Math. Z. 266 (2010) 719–746.

[9] F. Hirzebruch, *Topological methods in algebraic geometry*, Reprint of the 1978 edition. Classics in Mathematics. Springer-Verlag, Berlin, 1995.

[10] F. Hirzebruch, T. Berger and R. Jung, *Manifolds and modular forms*, Aspects of Mathematics, E20, Friedr. Vieweg and Sohn, Braunschweig, 1992.

[11] N. Hitchin, *Harmonic spinors*, Adv. Math. 14 (1974) 1-55.

[12] K. Kawakubo, *The theory of transformation groups*, The Clarendon Press, Oxford University Press, New York, 1991.

[13] M. Kreck, *Surgery and duality*, Ann. of Math. (2) 149 (1999) 707-754.

[14] I. M. Kríčever, *Obstructions to the existence of S1-actions. Bordism of ramified coverings*, (Russian) Izv. Akad. Nauk SSSR Ser. Mat. 40 (1976), no. 4, 828844; Math. USSR-Izv. 10 (1976) 783-797.

[15] H. B. Lawson and M. L. Michelsohn, *Spin geometry*, Princeton Mathematical Series, 38. Princeton University Press, Princeton, New Jersey, 1989.
[16] A. S. Libgober and J. W. Wood, *Differentiable structures on complete intersections I*, Topology 21 (1982) 469-482.

[17] A. Lichnerowicz, *Spineurs harmoniques*, C. R. Acad. Sci. Paris 257 (1963) 7-9.

[18] S. Stolz, *Simply connected manifolds of positive scalar curvature*, Ann. of Math. (2) 136 (1992) 511-540.

[19] R. Thom, *Quelques propriétés globales des variétés différentiables*, Comment. Math. Helv. 28 (1954) 17-86.

[20] C. Traving, *Klassification vollständiger Durchschnitte*, Diplomarbeit, University of Mainz, 1985.

(Jianbo Wang) School of Mathematics, Tianjin University, Tianjin 300350, China
*E-mail address: wjianbo@tju.edu.cn*

(Zhiwang Yu) School of Mathematics, Tianjin University, Tianjin 300350, China
*E-mail address: yzhwang@tju.edu.cn*

(Yuyu Wang) College of Mathematical Science, Tianjin Normal University, Tianjin 300387, China
*E-mail address: wdoubleyu@aliyun.com*