Time-dependent transitions with time–space noncommutativity and its implications in quantum optics

Nitin Chandra

Centre for High Energy Physics, Indian Institute of Science, Bangalore 560 012, India

E-mail: nitin@cts.iisc.ernet.in

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Abstract

We study the time-dependent transitions of a quantum-forced harmonic oscillator in noncommutative $\mathbb{R}_{1,1}^\theta$ perturbatively to linear order in the noncommutativity $\theta$. We show that the Poisson distribution gets modified, and that the vacuum state evolves into a ‘squeezed’ state rather than a coherent state. The time evolutions of uncertainties in position and momentum in vacuum are also studied and imply interesting consequences for modeling nonlinear phenomena in quantum optics.

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(Some figures may appear in colour only in the online journal)

1. Introduction

There seems to be a growing consensus among physicists that our classical notion of spacetime has to be drastically revised in order to find a consistent formulation of quantum mechanics and gravity [1–3]. One possible generalization that has attracted much interest is that of noncommutative Moyal spacetime [4–14]. In situations where the time coordinate remains commutative, i.e. only the spatial coordinates do not commute with each other, the quantum theory is conceptually straightforward (but nonetheless may display novel phenomena) [15–21]. In this paper, we will concentrate on understanding some implications of quantum mechanics with time–space noncommutativity; specifically, we will work with the Moyal plane $\mathbb{R}_{1,1}^\theta$. We will use the formalism of unitary quantum mechanics on this space as developed by Balachandran et al. [22] (see also [23]).

When time and space do not commute with each other, it is not unreasonable to expect that the dynamics of the time-dependent processes get altered. We will verify this explicitly in the context of a simple model of the forced harmonic oscillator (FHO) with the forcing term switched on only for a finite duration of time. In the commutative case, this is a much
studied model. We will compute deviations from the commutative case to leading order in $\theta$. These deviations suggest that time–space noncommutativity can capture certain nonlinear effects seen in quantum optics. This paper is organized as follows. In section 2, we will briefly review the formulation of unitary quantum mechanics on $\mathbb{R}^{1,1}_\theta$ [22]. In section 3, we will solve the problem of the FHO perturbatively in $\theta$ and compute corrections to the transition probabilities between simple harmonic oscillator (SHO) states. These corrections suggest the noncoherent nature of the time-evolved vacuum state and are reminiscent of those seen in nonlinear quantum optics [24]. To flesh out this analogy better, we study the time evolution of uncertainties in position and momentum in section 4. Encouraged by these results we, in section 5, suggest a correspondence between the nonlinearity in quantum optics and the quantum mechanics on $\mathbb{R}^{1,1}_\theta$. We conclude with a summary of our results in section 6.

2. Unitary quantum mechanics on $\mathbb{R}^{1,1}_\theta$

The noncommutative space $\mathbb{R}^{1,1}_\theta$ is described by the coordinates $\hat{x}_\mu$ satisfying

$$[\hat{x}_\mu, \hat{x}_\nu] = i\theta\epsilon_{\mu\nu}, \quad \text{with} \quad \epsilon_{\mu\nu} = -\epsilon_{\nu\mu} \text{and} \epsilon_{01} = 1,$$

(1)

where $\mu$ and $\nu$ can take values 0 and 1. Without loss of generality, we can take $\theta > 0$, as its sign can always be flipped by changing $\hat{x}_1$ to $-\hat{x}_1$. Let $A_\theta(\mathbb{R}^{1,1})$ be the unital algebra generated by $\hat{x}_0$ and $\hat{x}_1$. We associate with each $\hat{a} \in A_\theta(\mathbb{R}^{1,1})$, its left and right representations $\hat{a}^L$ and $\hat{a}^R$:

$$\hat{a}^L \hat{b} = \hat{a} \hat{b}, \quad \hat{a}^R \hat{b} = \hat{b} \hat{a}, \quad \hat{b} \in A_\theta(\mathbb{R}^{1,1}).$$

(2)

Unless stated, we work with the left representation.

For a quantum theory, what we need are (1) a suitable inner product on $A_\theta(\mathbb{R}^{1,1})$, (2) a Schrödinger constraint and time evolution. The operators $\hat{P}_0$ and $\hat{P}_1$, given by

$$\frac{i}{\theta} \frac{\partial}{\partial x_0} \equiv \hat{P}_0 = -\frac{1}{\theta} \text{ad} \hat{x}_1, \quad -i \frac{\partial}{\partial x_1} \equiv \hat{P}_1 = -\frac{1}{\theta} \text{ad} \hat{x}_0,$$

(5)

generate time and space translations, respectively. The Hamiltonian $\hat{H}$, in general, may depend on $\hat{x}_0^R$, $\hat{x}_1^R$ and $\hat{P}_1$. The possible dependence of $\hat{x}_0^R$ and $\hat{x}_1^R$ can be bypassed by

$$\hat{x}_0^R = \theta \hat{P}_0 + \hat{x}_1^R, \quad \hat{x}_1^R = -\theta \hat{P}_1 + \hat{x}_0^R.$$

(6)

Also, there is no dependence on $\hat{P}_0$ assumed in the line of the commutative case where there is never such dependence of $\hat{H}$ on $i\partial_{x_0}$ for $\theta = 0$. Now, note that the inner product (4) has an explicit dependence on the parameter $t$ and hence there exist more than one null vectors with respect to this inner product (actually any vector which vanishes at $\hat{x}_0 = t$ is a null vector). But
First we assume our system to be in an eigenstate

\[ (\hat{P}_0 - \hat{H}) \hat{\psi} = 0. \]  

(7)

It is easy to see that now there are no non-trivial null vectors. The Hamiltonian \( \hat{H} \) depends on \( \hat{x}_0^R, \hat{\dot{x}}_0^R \) and \( \hat{P}_1 \). Since \( \hat{x}_0^R \) commutes with \( \hat{x}_1^R \) and \( \hat{P}_1 \), we will choose \( \hat{x}_0^R \) as ‘time’.

It is easy to write down the formal solution of the Schrödinger constraint and find the time evolution. The time evolution is given by \( \hat{x}_0 \to \hat{x}_0 + \tau \) (or equivalently by \( \hat{x}_0^R \to \hat{x}_0^R + \tau \)). Thus, the amount of time translation is always commutative, though the time operator itself is noncommutative. The time-evolved wavefunctions satisfying the Schrödinger constraint are of the form \( \hat{\psi}(\hat{x}_0, \hat{x}_1) = \hat{U}(\hat{x}_0^R, \tau_1) \hat{\chi}(\hat{x}_1) \), where

\[ \hat{U}(\hat{x}_0^R, \tau_1) = \left( T \exp \left[ -i \left( \int_{\tau_1}^{\infty} \mathrm{d} \tau \hat{H}(\tau, \hat{x}_1^R, \hat{P}_1) \right) \right] \right) \bigg|_{\tau_1=\tau_2}. \]  

(8)

3. QFHO in \( \mathbb{R}^{1,1}_0 \) and their transition probabilities

Let us recall the dynamics of a quantum forced harmonic oscillator (QFHO) in ordinary spacetime. The Hamiltonian of this system is given by

\[ H(t) = \frac{\hat{p}^2}{2m_0} + \frac{1}{2} m_0 \omega^2 \hat{x}^2 + f(t) \hat{x} + g(t) \hat{p}, \]  

where \( m_0 \) is the mass of the particle and \( \omega \) is the angular frequency of the oscillator. We are interested in real functions obeying

\[ f(t), g(t) = 0 \quad \text{for } t \to \pm \infty. \]  

(10)

At \( t \to -\infty \), the Hamiltonian is simple harmonic, and we assume the system to be in one of the eigenstates of this SHO Hamiltonian. At \( t \to \infty \), the Hamiltonian again becomes simple harmonic and we try to find the probability (the transition probability) for the system to be in any arbitrary eigenstate of the SHO Hamiltonian subject to the fact that the system was in some already given eigenstate at \( t \to -\infty \). For this what we do is the following.

- First we assume our system to be in an eigenstate \( \phi_n(x) \) at \( t = t_i \to -\infty \).
- The state \( \phi_n(x) \) evolves under the SHO Hamiltonian from \( t = t_i \to -\infty \) to \( t = T_1 \).
- At \( t = T_1 \) the interaction gets switched on.
- The system then evolves under the full Hamiltonian (9) from \( t = T_1 \) to \( t = T_2 \).
- At \( t = T_2 \) the interaction gets switched off.
- The system again evolves under the SHO Hamiltonian from \( t = T_2 \) to \( t = t_f \to \infty \).
- We find the inner product of the final state with the eigenstate \( \phi_m(x) \) and obtain the \( t = t_f \to \infty \) limit. This gives the transition amplitude \( A_{mn} \) while its absolute square gives the transition probability \( P_{mn} \).
The generalization of the above Hamiltonian in \( \mathbb{R}^{1,1}_g \) is
\[
\hat{H} = \frac{\hat{\mathbf{p}}^2}{2m_0} + \frac{1}{2} m_0 \omega^2 \hat{x}^2 + \frac{1}{2} [ f(\hat{x}_0) \hat{x}_1 + \hat{x}_1 f(\hat{x}_0)] + g(\hat{x}_0) \hat{\mathbf{p}}_1 = \hat{H}_0 + \hat{H}_1, \tag{11}
\]
with
\[
\hat{H}_0 = \frac{\hat{\mathbf{p}}^2}{2m_0} + \frac{1}{2} m_0 \omega^2 \hat{x}^2, \quad \hat{H}_1 = \frac{1}{2} [ f(\hat{x}_0) \hat{x}_1 + \hat{x}_1 f(\hat{x}_0)] + g(\hat{x}_0) \hat{\mathbf{p}}_1. \tag{12}
\]
As \( \hat{x}_0 \) and \( \hat{p}_1 \) commute with each other, the ordering does not matter in the last term.

To define the transitions for the above Hamiltonian consider the time evolution by an amount \( \tau \). The functions \( f(\hat{x}_0) \) and \( g(\hat{x}_0) \) have the properties of vanishing in the far past and the far future, i.e.
\[
f, g(\hat{x}_0 + \tau) \to 0 \quad \text{as} \quad \tau \to \pm \infty. \tag{13}
\]

We shall find the transition probabilities \( \langle P_{m,n} \rangle \) for an SHO state ‘\( n \)’ at initial time \( \tau \to -\infty \) to go to some other SHO state ‘\( m \)’ at final time \( \tau \to +\infty \) after evolving under the Hamiltonian (11). The spectral map tells us that the energy spectrum of the SHO Hamiltonian in \( \mathbb{R}^{1,1}_g \) is the same as that of the commutative one, i.e.
\[
E_n = \hbar \omega \left( n + \frac{1}{2} \right), \quad \psi_n(\hat{x}_0, \hat{x}_1) = \phi_n(\hat{x}_1) e^{-i \omega (n + \frac{1}{2}) \hat{x}_0}, \tag{14}
\]
where \( \phi_n(\hat{x}_1) \) is the eigenfunctions of the commutative SHO Hamiltonian. The orthonormality (apart from a phase factor which comes because of the time evolution) of the eigenfunctions \( \psi_n(\hat{x}_0, \hat{x}_1) \) with respect to the inner product defined in section 2 can easily be checked.

The transition probabilities for our problem can be found by computing the same for the commutative Hamiltonian obtained after replacing
\[
\hat{x}_0 \to -\frac{\theta}{\hbar} p + t, \quad \hat{x}_1 \to x, \quad \hat{p}_1 \to p \tag{15}
\]
in the Hamiltonian (11). Here, \( t \) has come in place of the ‘time’ \( \hat{x}_0^R \) which commutes with \( \hat{x}_1 \) and \( \hat{p}_1 \). To linear order in \( \theta \) we obtain the following commutative Hamiltonian:
\[
H(t) = H_0 + H_1(t) = H_0 + H_0(t) + \theta H_1(t), \tag{16}
\]
with
\[
\begin{align*}
H_0 &= \frac{\hat{\mathbf{p}}^2}{2m_0} + \frac{1}{2} m_0 \omega^2 \hat{x}^2 = \hbar^2 \left( a^\dagger a + \frac{1}{2} \right), \\
H_0(t) &= f(t) x + g(t) p = z^*(t) a + z(t) a^\dagger, \\
H_1(t) &= \frac{1}{\hbar} \left( -g(t) p^2 - \frac{1}{2} f'(t)(xp + px) \right) = i \sqrt{\frac{m_0 \hbar \omega}{2}} \left( z^*(t) a^2 - z(t) a^\dagger a^\dagger + i \sqrt{\frac{m_0 \hbar \omega}{2}} g(t)(2a^\dagger a + 1) \right). \tag{17}
\end{align*}
\]

The function \( z(t) \) is related to \( f(t) \) and \( g(t) \) as
\[
z(t) = \sqrt{\frac{\hbar}{2m_0 \omega}} \left( f(t) + im_0 \omega g(t) \right). \tag{18}
\]

Also, \( a \) and \( a^\dagger \) are the annihilation and creation operators, respectively, defined as
\[
a = \sqrt{\frac{m_0 \omega}{2\hbar}} \left( x + \frac{p}{m_0 \omega} \right), \quad a^\dagger = \sqrt{\frac{m_0 \omega}{2\hbar}} \left( x - \frac{p}{m_0 \omega} \right), \quad x = \sqrt{\frac{\hbar}{2m_0 \omega}} (a^\dagger + a) \Rightarrow \]
\[
a = \sqrt{-\frac{m_0 \omega}{2\hbar}} \left( x + i \frac{p}{m_0 \omega} \right), \quad p = i \sqrt{-\frac{m_0 \hbar \omega}{2}} (a^\dagger - a).
\]
The nonlinearity in the Hamiltonian (16) is purely due to the noncommutativity. This provokes us to model certain types of nonlinear phenomena in quantum optics by the noncommutativity between time and space coordinates. This analogy will be further studied in section 5. Let us now continue with calculating the transition amplitude which is given by

$$A_{m,n}(t_f; T_1; t_i, t_i) = \langle \phi_{m}\rangle U_0^*(t_f, t_i) U_0(t_f, T_2) U(T_2, T_1) U_0(T_1, t_i)|\phi_n\rangle,$$

where $U_0(t', t)$ and $U(t', t)$ are the time evolution operators from time $t$ to time $t'$ for the Hamiltonians $H_0$ and $H(t)$, respectively, i.e.

$$U_0(t', t) = e^{-\frac{i}{\hbar} H_0(t'-t)}, \quad U(t', t) = T\left[e^{-\frac{i}{\hbar} \int_0^t dt' H(t')}\right],$$

the latter one being the time-ordered exponential. This gives

$$A_{m,n}(t_f, T_2; T_1, t_i, t_i) = e^{\frac{i}{\hbar} [E_m(T_2) - E_n(T_1)]} \langle \phi_{m}\rangle U(T_2, T_1) |\phi_n\rangle.$$

The state $|\psi(t)\rangle = U(t, T_1)|\phi_n\rangle$ evolves according to the Schrödinger equation for the Hamiltonian (16)

$$\left(i\hbar \frac{d}{dt} - H_0\right) |\psi(t)\rangle = H_f(t) |\psi(t)\rangle,$$

with the initial condition $|\psi(t = T_1)\rangle = |\phi_n\rangle$. If we define Green’s operator function $G(t, t_0)$ as

$$G(t, t_0) = \delta(t, t_0),$$

then the solution of the Schrödinger equation (22) will be

$$|\psi(t)\rangle = |\phi(t)\rangle + \int_{-\infty}^{+\infty} dt_0 G(t, t_0) H_f(t_0) |\psi(t_0)\rangle,$$

which in turn gives the Born series

$$|\psi(t)\rangle = |\phi(t)\rangle + \int_{-\infty}^{+\infty} dt_0 G(t, t_0) H_f(t_0) |\phi(t_0)\rangle + \int_{-\infty}^{+\infty} dt_0 \int_{-\infty}^{+\infty} dt_1 G(t, t_0) H_f(t_0) G(t_0, t_1) H_f(t_1) |\phi(t_1)\rangle + \cdots.$$  \hfill (24)

Here, $|\phi(t)\rangle$ is the solution of the homogeneous equation $\left(i\hbar \frac{d}{dt} - H_0\right) |\phi(t)\rangle = 0$, which is nothing but the Schrödinger equation for the SHO. $G$ has been found in the appendix A (see (A.1)). Note that the $\Theta$-function in the expression of the $G$ restricts the integration over $t_j$ in (24) within the limit of $-\infty$ to $t_{j-1}$ ($t_{j-1} = t$). Thus, at $t = T_1$ the integrations are only in the intervals when the interaction was switched off, i.e. $H_f = 0$. Hence, we obtain $|\psi(t = T_1)\rangle = |\phi(t = T_1)\rangle = |\phi_n\rangle$. The solution of the homogeneous part $|\phi(t)\rangle$ with this initial condition is

$$|\phi(t)\rangle = e^{-\frac{i}{\hbar} E_m(T_1-t_f)} |\phi_n\rangle.$$  \hfill (25)

Now, putting (24) with $t = T_2$ for $U(T_2, T_1)|\phi_n\rangle$ in (21), we obtain

$$A_{m,n}(t_f, T_2; T_1, t_i, t_i) = \sum_{j=0}^{\infty} B_j(t_f, T_2; T_1, t_i),$$

with

$$B_0(t_f, T_2; T_1, t_i) = \delta_{m,n},$$

$$B_j(t_f, T_2; T_1, t_i) = \int_{-\infty}^{+\infty} dt_0 \int_{-\infty}^{+\infty} dt_1 \cdots \int_{-\infty}^{+\infty} dt_{j-1} F_{m,n}^j(t_f, T_2; t_0, t_1, \ldots, t_{j-1}; T_1, t_i).$$  \hfill (27)
for \( j = 1, 2, \ldots \) Here,

\[
F_{m,n}^j(t_0, t_{i-1}, t_i, \ldots, t_{i-j}; T, t_i) = \left(-\frac{i}{\hbar}\right)^j \Theta(T - t_0) \Theta(t_0 - t_i) \cdots \Theta(t_0 - t_{i-j})
\]

\[
\times e^{\frac{i}{\hbar}(E_n - E_m)\theta}\langle \phi_m|H_{I1}^{int}(t_0)H_{I1}^{int}(t_1) \cdots H_{I1}^{int}(t_{i-j})|\phi_n\rangle.
\]

(28)

The \( H_{I1}^{int} \)s are defined as

\[
H_{I1}^{int}(t) = e^{\frac{i}{\hbar}\theta H_{12} H_{12}^{-1}}(t) e^{-\frac{i}{\hbar}H_{12}^d}.
\]

(29)

Separating the \( \theta \)-dependent and independent parts we obtain

\[
A_{m,n}(t_0, t_{i-1}, t_i, \ldots, t_{i-j}; T, t_i) = e^{\frac{i}{\hbar}(E_n - E_m)\theta}\langle \phi_m|A^{(0)}(T, t_i) + \phi A^{(1)}(T, t_i)|\phi_n\rangle,
\]

(30)

with

\[
A^{(0)}(T, t_i) = 1 + \int_{-\infty}^{t_{i-1}} d\tau_0 \Theta(T - t_0)H_{I1}^{int}(t_0)
\]

\[
\times \left( -\frac{i}{\hbar} \right)^2 \int_{-\infty}^{t_0} d\tau_1 \Theta(T - t_0)\Theta(t_0 - t_1)
\]

\[
\times \left[ H_{I1}^{int}(t_0)H_{I1}^{int}(t_1) + H_{I1}^{Dm}(t_0)H_{I1}^{Dm}(t_1) \right]
\]

\[
\times \left( -\frac{i}{\hbar} \right)^3 \int_{-\infty}^{t_1} d\tau_2 \Theta(T - t_0)\Theta(t_0 - t_1)\Theta(t_1 - t_2)
\]

\[
\times \left[ H_{I1}^{int}(t_0)H_{I1}^{int}(t_1)H_{I1}^{int}(t_2) + H_{I1}^{Dm}(t_0)H_{I1}^{Dm}(t_1)H_{I1}^{Dm}(t_2) + H_{I1}^{Dm}(t_0)H_{I1}^{Dm}(t_1)H_{I1}^{Dm}(t_2) \right]
\]

\[+ \cdots.
\]

(31)

The above expression for \( A^{(1)}(T, t_i) \) can be simplified to (see appendix B)

\[
A^{(1)}(T, t_i) = -\frac{i}{\hbar} \int_{-\infty}^{t_i} d\tau_0 [A^{(0)}(t_0, t_i)]^{-1} H_{I1}^{int}(t_0) A^{(0)}(t_0, T).
\]

(33)

\( A^{(0)}(t, t') \) with arbitrary arguments is defined in (B.3). Putting this in equation (30) we obtain

\[
A_{m,n}(t_0, t_{i-1}, t_i, \ldots, t_{i-j}; T, t_i) = e^{\frac{i}{\hbar}(E_n - E_m)\theta}\langle \phi_m|A^{(0)}
\]

\[
\times \left[ 1 - \frac{i}{\hbar} \int_{-\infty}^{t_i} d\tau_0 [A^{(0)}(t_0, t_i)]^{-1} H_{I1}^{int}(t_0) A^{(0)}(t_0, T) \right]|\phi_n\rangle.
\]

(34)

A straightforward use of the identity

\[
e^{A^d B} e^{-\lambda A} = B + \lambda \frac{\hbar}{1!} [A, B] + \frac{\lambda^2}{2!} [A, [A, B]] + \frac{\lambda^3}{3!} [A, [A, [A, B]]] + \cdots
\]

(35)

gives all the \( H_{I1}^{int} \)s. Also, to get rid of the time-ordered exponentials we follow the discussions given in [25, pp 338–40]. This finally gives the expression of the transition amplitude as

\[
A_{m,n}(t_0, t_{i-1}, t_i, \ldots, t_{i-j}; T, t_i) = e^{\frac{\theta}{\hbar} (E_n - E_m)\theta} \left[ D_{m,n}(\xi) - \frac{\theta}{\hbar} (\beta_1 D_{m,n}(\xi) + \beta_2 \sqrt{n} D_{m,n-1}(\xi))
\]

\[
+ \beta_3 \sqrt{n+1} D_{m,n+1}(\xi) + \beta_5 \sqrt{n(n-1)} D_{m,n-2}(\xi)
\]

\[+ \beta_7 \sqrt{(n+1)(n+2)} D_{m,n+2}(\xi) \right].
\]

(36)
with
\[ \xi = -i \hbar \int_{-\infty}^{\infty} d\tau \ e^{i\omega \tau} z(\tau) \] (37)
\[ \beta = \frac{i}{2\hbar} \int_{-\infty}^{\infty} d\tau \int_{-\infty}^{\infty} d\tau' \ [z^* (\tau_1) z (\tau_2) e^{-i\sigma (\tau_1 - \tau_2)} - z (\tau_1) z^* (\tau_2) e^{i\sigma (\tau_1 - \tau_2)}] = \text{real} \] (38)
\[ \beta_1 = -m_0 \omega \int_{-\infty}^{+\infty} d\tau \ [g' (\tau) \xi (\tau)]^2 \] (39)
\[ \beta_2 = -m_0 \omega \int_{-\infty}^{+\infty} d\tau \ [g' (\tau) \xi^* (\tau)] + 2i \hbar \sqrt{\frac{m_0 \hbar \omega}{2}} \int_{-\infty}^{+\infty} d\tau' \ [z (\tau') \xi (\tau') e^{-2i\omega t} - z' (\tau') \xi^* (\tau') e^{2i\omega t}] \] (40)
\[ \beta_3 = \frac{i}{\hbar} \sqrt{\frac{m_0 \hbar \omega}{2}} \int_{-\infty}^{+\infty} d\tau \ [z (\tau) \xi (\tau) e^{-2i\omega t}]. \] (41)

Here, the function \( \xi (\tau) \) is given as
\[ \xi (\tau) = -i \hbar \int_{-\infty}^{\tau} d\tau' \ e^{i\omega \tau} z(\tau). \] (42)

\( D_{m,n}(\xi) \) are the matrix elements of the displacement operator \( D(\xi) = e^{-\xi^a a^+ \xi^a} \) given by [26]
\[ D_{m,n}(\xi) = \sqrt{\frac{n!}{m!}} e^{-\frac{1}{2} |\xi|^2} \xi^{m-n} L_n^{m-n}(|\xi|^2), \] (43)

\( L_n^m(x) \) are the associated Laguerre polynomials. Also, the limits of the integrations have been extended to \(-\infty\) and \(\infty\) as the integrands are zero in the extended region. The transition probability is given by
\[ P_{m,n} = |A_{m,n}(t_f, T_2; T_1, t_i)|^2 \] (44)
as usual. The arguments have been omitted as the transition probability does not depend on the times \( t_f, T_2; T_1, t_i \).

3.1. \( n = 0 \)

For the initial state \(|\phi_0\rangle\), the transition amplitude is
\[ A_{m,0}(t_f, T_2; T_1, t_i) = e^{i\beta_1} e^{\frac{iE_0 - E_m}{\hbar} t_f} \frac{\xi^m}{\sqrt{m!}} \left[ 1 - \frac{i}{\hbar} \theta \left( \beta_1 - \beta_2^* \xi^* + \beta_3^* \xi^2 \right) \right. \] (45)
\[ + m \frac{1}{|\xi|^2} (\beta_2^* \xi^* - 2 \beta_3^* \xi^2) + m(m-1) \frac{1}{|\xi|^4} \beta_3^* \xi^2 \left. \right] \]
and the transition probability becomes (up to linear order in \( \theta \))
\[ P_{m,0} = |A_{m,0}(t_f, T_2; T_1, t_i)|^2 = e^{-|\xi|^2} \frac{2m}{m!} \left[ 1 + \frac{2}{\hbar} \theta [A_1 + mA_2 + m(m-1)A_3] \right]. \] (46)
with
\[ A_1 = \text{Im} (\beta_2 \xi) - \text{Im} (\beta_3 \xi^2), \quad A_2 = \frac{1}{|\xi|^2} (2 \text{Im} (\beta_3 \xi^2) - \text{Im} (\beta_2 \xi)), \]
\[ A_3 = -\frac{1}{|\xi|^4} \text{Im} (\beta_3 \xi^2). \] (47)
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Note that as $m \to \infty$, the $\theta$-correction starts dominating, and in this case the expansion up to linear order in $\theta$ is no longer meaningful. Hence, the above result is valid only for those $m$ values which are far smaller than $1/\sqrt{m_{\text{top}}}$ (in the unit $\hbar = 1$). For $\theta \to 0$, the transition probability becomes the well-known Poisson distribution as expected.

As a specific example, let us work with the functions $f(t)$ and $g(t)$ of the form (see figure 1)

\[
f(t) = f_0 [\Theta(t + T) - \Theta(t - T)], \quad g(t) = g_0 [\Theta(t + T) - \Theta(t - T)], \quad T > 0.
\]

For these functions, we obtain

\[
A_1 = \frac{2f_0}{m_0\omega^3} \left( f_0^2 + m_0^2\omega^2 g_0^2 \right) \sin^2 \omega T \cos 2\omega T,
\]

\[
A_2 = m_0\omega g_0 \sin 2\omega T - f_0 \cos 2\omega T,
\]

\[
A_3 = -\frac{m_0^2\omega^2 g_0 \cot \omega T}{f_0^2 + m_0^2\omega^2 g_0^2}.
\]

Now, the choices $m_0 = 1$, $\omega = 1$, $f_0 = \sqrt{5}$, $g_0 = \sqrt{5}$ and $T = \frac{\pi}{2}$ in the commutative case ($\theta = 0$) give the following Poisson distribution: $P_{m,0} = e^{-\frac{20\omega^2}{m}}$, while for nonzero $\theta$ the probability distribution modifies to $P_{m,0} = e^{-\frac{20\omega^2}{m}} [1 + 2\theta\sqrt{5}(m - 20)]$. The $\theta$-correction becomes of the order of the $\theta$-independent part when $m$ approaches the value $\tilde{m}(\theta) = \left(20 + \frac{1}{2\sqrt{5}}\right)$.

Hence, our result is valid only in the region where $m < \tilde{m}(\theta)$. Note that $\tilde{m}(\theta) \sim \frac{1}{\theta}$ rather than $\frac{1}{\sqrt{\theta}}$ because $A_3$ is identically zero for the choices taken. We choose $\theta = 0.01$ ($m(\theta = 0.01) \approx 42$) and obtain

\[
P_{m,0} = e^{-\frac{20\omega^2}{m}} [1 + 0.02\sqrt{5}(m - 20)].
\]

This deformed distribution along with the Poisson distribution is shown in figure 2. Such deformation of the Poisson distribution suggests that the vacuum no longer evolves to be a coherent state. To explore this further let us look at the time evolution of position and momentum uncertainties.

4. The time evolution of $\Delta_x$ and $\Delta_p$

The expectation value of any operator $\hat{O}$ in a state $\hat{\psi}(\hat{x}_0, \hat{p}_1)$ at any time $t$ is defined to be

\[
\langle \hat{O} \rangle_t = \langle \hat{\psi}, \hat{O} \hat{\psi} \rangle_t.
\]

Figure 1. The behavior of functions $f(t)$ and $g(t)$ with $t$. 
Figure 2. The modified distribution \((\theta = 0.01)\) along with the Poisson distribution \((\theta = 0)\) for \(m_0 = 1, \omega = 1, f_0 = \sqrt{3}, g_0 = \sqrt{3}\) and \(T = \frac{\pi}{2}\).

Also,

\[\langle \hat{\mathcal{O}} \rangle_{t+\tau} = (\hat{\psi} \hat{\mathcal{O}} \hat{\psi})_{t+\tau} = (\hat{\psi}(\hat{x}_0 + \tau, \hat{x}_1), \hat{\mathcal{O}} \hat{\psi}(\hat{x}_0 + \tau, \hat{x}_1))_t.\]

Thus, the time evolution of the expectation value of an operator is given by that of the state in which it is being calculated. For the QFHO in \(\mathbb{R}^{1,1}\), the time evolution of any operator \(\hat{\mathcal{O}}\) will be given by

\[
\frac{d}{dt} \langle \hat{\mathcal{O}} \rangle = \frac{\partial}{\partial t} \langle \hat{\mathcal{O}} \rangle + \frac{i}{\hbar} \langle [H(t), \hat{\mathcal{O}}] \rangle, \tag{54}\]

where \(H(t)\) is the Hamiltonian (16). The uncertainty in any observable \(\hat{\mathcal{O}}\) is given by

\[
\Delta_\mathcal{O} = \sqrt{\langle \hat{\mathcal{O}}^2 \rangle - \langle \hat{\mathcal{O}} \rangle^2}. \tag{55}\]

Thus, the evolution of \(\Delta_x^2\) and \(\Delta_p^2\) is

\[
\frac{d}{dt} \Delta_x^2 = 2 \left( \frac{1}{m_0} - \frac{\theta}{\hbar} g'(t) \right) \frac{1}{2} (\langle xp \rangle + \langle px \rangle) - \frac{2\theta}{\hbar} f'(t) \Delta_x^2 \right) \right], \tag{56}\]

Defining

\[
\Delta_{xp} = \frac{1}{2} \langle xp \rangle + \langle px \rangle, \tag{57}\]

we find the following first-order coupled equations

\[
\frac{d}{dt} \Delta_x^2 = 2 \left( \frac{1}{m_0} - \frac{\theta}{\hbar} g'(t) \right) \Delta_{xp} - \frac{2\theta}{\hbar} f'(t) \Delta_x^2 \right) \right], \tag{58}\]

As the initial state is the vacuum, the initial conditions for the above are

\[
\Delta_x^2(t \to -\infty) = \frac{\hbar}{2m_0\omega}, \quad \Delta_p^2(t \to -\infty) = \frac{m_0\hbar\omega}{2}, \quad \Delta_{xp}(t \to -\infty) = 0. \tag{59}\]
Our strategy for solving these equations is simple. We do so perturbatively in \( \theta \). A straightforward computation gives

\[
\Delta_x(t) = \sqrt{\frac{\hbar}{2m_0\omega}} - \theta \sqrt{\frac{m_0}{2\hbar\omega}} \left[ \frac{2}{m_0} f(t) + \frac{4\omega}{m_0} \int_{-\infty}^{t} \mathrm{d} \tau \sin[2\omega(\tau - t)] f(\tau) \right] \\
+ 2\omega^2 \int_{-\infty}^{t} \mathrm{d} \tau \cos[2\omega(\tau - t)] g(\tau)
\]

\[
\Delta_y(t) = \sqrt{\frac{m_0\hbar\omega}{2}} + \theta \frac{m_0}{2} \sqrt{\frac{m_0\hbar}{2\hbar}} \left[ \frac{2}{m_0} f(t) + \frac{4\omega}{m_0} \int_{-\infty}^{t} \mathrm{d} \tau \sin[2\omega(\tau - t)] f(\tau) \right] \\
+ 2\omega^2 \int_{-\infty}^{t} \mathrm{d} \tau \cos[2\omega(\tau - t)] g(\tau)
\]

\[
\Delta_{xy}(t) = \theta \left[ -\frac{m_0\omega}{2} g(t) + 2\omega \int_{-\infty}^{t} \mathrm{d} \tau \cos[2\omega(\tau - t)] f(\tau) \right. \\
- \frac{m_0\omega^2}{2} \int_{-\infty}^{t} \mathrm{d} \tau \sin[2\omega(\tau - t)] g(\tau)
\]

The fundamental uncertainty product (to linear order in \( \theta \)) is

\[
\Delta_x(t) \Delta_y(t) = \frac{\hbar}{2}.
\]

Thus, the vacuum state evolves to a ‘squeezed state’ rather than a coherent state as in the commutative case [27]. The uncertainties in the commutative case depend only on the product \( m_0\omega \). But their \( \theta \)-corrections change with \( \omega \) even if \( m_0\omega \) is kept constant. Also, the squeezing effect is oscillatory in time as is obvious from the \( \theta \)-dependent terms in (60). For the specific forms of \( f(t) \) and \( g(t) \) of (48), we obtain

\[
\Delta_x(t) \begin{cases} 
\frac{\hbar}{2m_0\omega}; & t < -T \\
\frac{\hbar}{2m_0\omega} - \theta \sqrt{\frac{1}{2m_0\omega}} \left[ 2f_0 \cos[2\omega(t + T)] + m_0\omega \sin[2\omega(t + T)] \right]; & -T < t < T \\
\frac{\hbar}{2m_0\omega} - \theta \sqrt{\frac{1}{2m_0\omega}} \left[ 2f_0 \cos[2\omega(t + T)] - \cos[2\omega(t - T)] \right. \\
+ \frac{m_0}{2m_0\omega} \sin[2\omega(t + T)] - \sin[2\omega(t - T)] \right]; & t > T,
\end{cases}
\]

\[
\Delta_y(t) \begin{cases} 
\frac{m_0\hbar\omega}{2}; & t < -T \\
\frac{m_0\hbar\omega}{2} + \theta \sqrt{\frac{m_0\hbar}{2\hbar}} \left[ 2f_0 \cos[2\omega(t + T)] + m_0\omega \sin[2\omega(t + T)] \right]; & -T < t < T \\
\frac{m_0\hbar\omega}{2} + \theta \sqrt{\frac{m_0\hbar}{2\hbar}} \left[ 2f_0 \cos[2\omega(t + T)] - \cos[2\omega(t - T)] \right. \\
+ \frac{m_0}{2m_0\omega} \sin[2\omega(t + T)] - \sin[2\omega(t - T)] \right]; & t > T
\end{cases}
\]
\[ \Delta_{sp}(t) = \begin{cases} 0; & t < -T \\ \frac{\theta}{2} [2f_0 \sin(2\omega(t + T)) - m_0 \omega g_0 \cos(2\omega(t + T))]; & -T < t < T \\ \frac{\theta}{2} [2f_0 \sin(2\omega(t + T)) - \sin(2\omega(t - T)) - m_0 \omega g_0 \cos(2\omega(t + T)) - \cos(2\omega(t - T))]; & t > T. \end{cases} \] (64)

For the same choice of parameters as in the last section we obtain

\[ \Delta_x(t) = \begin{cases} \frac{1}{\sqrt{2}}; & t < \frac{-\pi}{2} \\ 1 \frac{1}{\sqrt{2}} + 0.01 \sqrt{\frac{5}{2}} \left( \cos 2t + \frac{1}{2} \sin 2t \right); & \frac{-\pi}{2} < t < \frac{\pi}{2} \\ 1 \frac{1}{\sqrt{2}} + 0.01 \sqrt{\frac{5}{2}} \sin 2t; & t > \frac{\pi}{2}. \end{cases} \] (65)

\[ \Delta_p(t) = \begin{cases} \frac{1}{\sqrt{2}}; & t < \frac{-\pi}{2} \\ 1 \frac{1}{\sqrt{2}} - 0.01 \sqrt{\frac{5}{2}} \left( \cos 2t + \frac{1}{2} \sin 2t \right); & \frac{-\pi}{2} < t < \frac{\pi}{2} \\ 1 \frac{1}{\sqrt{2}} - 0.01 \sqrt{\frac{5}{2}} \sin 2t; & t > \frac{\pi}{2}. \end{cases} \] (66)

\[ \Delta_{xp}(t) = \begin{cases} 0; & t < \frac{-\pi}{2} \\ 0.01 \sqrt{5} \left( \frac{1}{2} \cos 2t - \sin 2t \right); & \frac{-\pi}{2} < t < \frac{\pi}{2} \\ -0.02 \sqrt{5} \sin 2t; & t > \frac{\pi}{2}. \end{cases} \] (67)

Figures 3 and 4 show the time dependence of the different uncertainties. The discontinuities at \( t = \pm \frac{\pi}{2} \) are simply the manifestation of the fact that the functions \( f(t) \) and \( g(t) \) themselves are discontinuous at these times. Before the interaction was switched on, the uncertainties had values equal to those for the vacuum state. During the time of nonvanishing interaction (and even after the interaction gets switched off!), they oscillate with a frequency equal to twice that of the oscillator.

5. Implications in quantum optics

In quantum optics a monochromatic (single-mode) coherent light field is usually described by the harmonic oscillator coherent states [28]. It has also been shown that a coherent state (in particular the vacuum state) remains coherent under the FHO Hamiltonian [29]. The annihilation and creation operators for photons are related to the field quadratures \( X_1 \) and \( X_2 \) by

\[ a = X_1 + iX_2, \quad a^\dagger = X_1 - iX_2, \] (68)

with \( X_1 \) and \( X_2 \) being Hermitian. The commutation \([a, a^\dagger]\) = 1 translates to \([X_1, X_2]\) = \( \frac{1}{2} \). The coherent state has different uncertainties as \( \Delta_{x_1} = \frac{1}{2}, \Delta_{x_2} = \frac{1}{2} \) and \( \Delta_{x_1 x_2} = 0 \Rightarrow \Delta_{x_1} \Delta_{x_2} = \frac{1}{4} \) which is the minimum. Also, the photon count (probability of having a certain number of
photons) in the coherent state is given by the transition probabilities of the corresponding number eigenstate and the profile is Poissonian.

The FHO Hamiltonian

\[ H(t) = \hbar \omega (X_1^2 + X_2^2) + \sqrt{\frac{2\hbar}{m_0\omega}} f(t) X_1 + \sqrt{2\hbar m_0 \omega g(t)} X_2 \]

\[ = \hbar \omega \left( a^+ a + \frac{1}{2} \right) + z^*(t) a + z(t) a^+ \]

Figure 3. The time-dependences of the uncertainties \( \Delta_x \) and \( \Delta_p \) for \( m_0 = 1, \omega = 1, f_0 = \sqrt{5}, g_0 = \sqrt{5}, T = \frac{\pi}{2} \) and \( \theta = 0.01 \).

Figure 4. The time dependence of \( \Delta_{xp} \) for the same choice of values.
\[
[t, X_1] = i \frac{m_0 \omega}{2 \hbar} \tag{70}
\]
will allow us to use the calculation of the previous sections. The photon count will be given by (46), while the uncertainties in the field quadratures will get modified as
\[
\begin{align*}
\Delta_X(t) &= \frac{1}{2} + \frac{\theta m_0}{4 \hbar} \int_{-\infty}^{t} d\tau \sin(2 \omega (\tau - t)) \left[ \text{Im} \xi'(\tau) - \frac{1}{m_0 \omega} f''(\tau) \right] \\
\Delta_\chi(t) &= \frac{1}{2} - \frac{\theta m_0}{4 \hbar} \int_{-\infty}^{t} d\tau \sin(2 \omega (\tau - t)) \left[ \text{Im} \xi'(\tau) - \frac{1}{m_0 \omega} f''(\tau) \right] \\
\Delta_X\chi(t) &= \frac{\theta}{2 \hbar} \left( \frac{1}{2 \omega} f'(t) - \frac{m_0}{2} \int_{-\infty}^{t} d\tau \cos(2 \omega (\tau - t)) \left[ \text{Im} \xi'(\tau) + \frac{1}{m_0 \omega} f''(\tau) \right] \right). \tag{71}
\end{align*}
\]
We further study the correlation among the photons. The time-evolved vacuum state
\[
|i(t \to \infty)\rangle = \sum_{m=0}^{\infty} A_{m,0} |m\rangle \tag{72}
\]
will give
\[
\bar{N} = \langle i(t \to \infty) | a^\dagger a | i(t \to \infty) \rangle = \sum_{m=0}^{\infty} m P_{m,0} = |\xi|^2 - \frac{2\theta}{\hbar^2} \text{Im}(\beta_2 \xi), \tag{73}
\]
with \(\bar{N}\) being the average number of photons in state \(|i(t \to \infty)\rangle\). Also,
\[
\langle i(t \to \infty) | a^\dagger a^\dagger a a | i(t \to \infty) \rangle = \sum_{m=2}^{\infty} m (m-1) P_{m,0}
= |\xi|^4 - \frac{4\theta}{\hbar^2} (\text{Im}(\beta_3 \xi^2) + |\xi|^2 \text{Im}(\beta_2 \xi)). \tag{74}
\]
This, to linear order in \(\theta\), gives the second-order correlation among photons with zero time delay to be equal to (see appendix C)
\[
g^{(2)}(0) = 1 - \frac{4\theta}{\hbar^2} \text{Im}(\beta_3 \xi^2) \left( |\xi|^4 - \frac{2\theta}{\hbar} |\xi|^2 \text{Im}(\beta_2 \xi) \right) = 1 - \frac{4\theta}{N^2} \text{Im}(\beta_3 \xi^2) \tag{75}
\]
For the case \(\text{Im}(\beta_3 \xi^2) < 0 \Rightarrow g^{(2)}(0) > 1\), the photons try to bunch together, while for \(\text{Im}(\beta_3 \xi^2) > 0 \Rightarrow g^{(2)}(0) < 1\), they anti-bunch [24]. For the functions (48), we obtain
\[
\text{Im}(\beta_3 \xi^2) = \frac{2(\tilde{j}_0^2 + m_0^2 \omega^2 \delta_0^2)}{\omega^2} \frac{\sin^2 \omega T}{g_0} g_0 \sin 2 \omega T, \tag{76}
\]
which implies that the bunching or anti-bunching will depend only on the sign of the factor \(g_0 \sin 2 \omega T\). For the choices taken in figures 2, 3 and 4, \(\omega T = \frac{\pi}{2}\) and hence no bunching or anti-bunching occurs.

6. Conclusions

In this paper, we developed a formalism to compute the transitions between states of a quantum mechanical system with noncommutative time. We found that for a free Hamiltonian in \(\mathbb{R}_1^{1,1}\) which is independent of time, the transitions are equal to the same for a different Hamiltonian in \(\mathbb{R}_1^{1,1}\) found after the replacements (15). The time evolution of an operator and its expectation value (and hence also its uncertainty) can also be found in a similar manner. Specifically, for the
FHO the transition probabilities get modified and are given by (36) and (44). The Poissonian distribution for the ‘vacuum to any state transition’ also gets modified and is given by (46). The study of uncertainties in position and momentum reveals that the time-evolved state is no longer coherent. It gets some squeezing effect due to the noncommutativity, keeping the product of the uncertainties minimum. These uncertainties are explicitly found and are given in (60). The leading-order corrections in these uncertainties are oscillatory in time and they depend independently on the mass of the particle $m_0$ and the frequency of the oscillator $\omega$ (note that the commutative uncertainties depend only on the product $m_0 \omega$). These results suggest a possible modeling of the noncommutativity for the nonlinear phenomena in quantum optics. The noncommutativity results in the following nonlinear effects.

1. The photon count gets modified from the usual Poisson distribution.
2. The uncertainties in the field quadratures change keeping the product minimum (the squeezing effect).
3. The second-order correlation function $g^{(2)}(0)$ gets modified producing new effects like bunching or anti-bunching of photons depending on the value of $\text{Im}(\beta_3 \xi^2)$.

All these observations suggest that the noncommutativity produces incoherence in the otherwise coherent field.

As a future work, one can try to formulate the scattering process in higher dimensions and study its implications in quantum optics. The correspondence found in this paper between noncommutativity and quantum optics also encourages one to study such possibilities in other forms of time–space noncommutativity. As an example one can start with assuming the spacetime-dependent noncommutative parameter $\theta$ [31–33].

### Appendix A. Green’s operator function

Expanding $G(t, t_0)$ as a Fourier integral we obtain

$$G(t, t_0) = G(t - t_0) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} d\omega' g(\omega') e^{-i\omega'(t-t_0)},$$

where $g(\omega')$ is given by $g(\omega') = (\hbar \omega' - H_0)^{-1}$. Introducing the simple harmonic eigenstate basis and using the completeness relation we obtain

$$G(t, t_0) = \sum_{j=0}^{\infty} \left( \lim_{\epsilon \to 0} \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-i\omega'(t-t_0)} \frac{1}{\hbar (\omega' - \frac{E_n}{\hbar} + i\epsilon)} \right) |\phi_j\rangle \langle \phi_j|. $$

Here, $i\epsilon$ has been introduced to avoid the pole on the contour (real axis). After finding the integral inside the summation using the complex analysis we get Green’s function to be

$$G(t, t_0) = -\frac{i}{\hbar} \Theta(t - t_0) e^{-\frac{E_n}{2\hbar}(t-t_0)},$$

where $\Theta(x)$ is the Heaviside step function.

### Appendix B. Simplifying $A^{(1)}(T_2, T_1)$

To simplify (32) what we do is find the differential equation for $A^{(1)}(t, T_1)$ with an independent variable $t$ and solve it with proper initial conditions. The differential equation has been found to be

$$\left( i\hbar \frac{\partial}{\partial t} - H_{11}^{\text{int}}(t) \right) A^{(1)}(t, T_1) = H_{11}^{\text{int}}(t) A^{(0)}(t, T_1),$$

where

$$A^{(0)}(t, T_1) = \begin{cases} 1 & t < T_1 \\ 0 & t > T_1 \end{cases}$$

and

$$H_{11}^{\text{int}}(t) = \begin{cases} 0 & t < T_1 \\ H_{11} & t > T_1 \end{cases}$$
which is a first-order equation and hence \( A^{(1)}(t, T_1) \) is unique if an initial condition is given. The initial condition comes from the fact that \( A^{(0)}(t, T_1) \) and \( A^{(1)}(t, T_1) \) must become identity and zero respectively for \( t = T_1 \), i.e. no interaction. Thus, \( A^{(1)}(T_1, T_1) = 0 \). To solve the equation, we define Green’s operator function \( G_{\text{int}}(t, t_0) \) as

\[
\left( \frac{i\hbar}{\partial t} - H_{\text{int}}^{\text{m}}(t) \right) G_{\text{int}}(t, t_0) = \delta(t - t_0).
\]  

(B.2)

Now, generalizing solution (A.1) for the time-dependent case we obtain

\[
G_{\text{int}}(t, t_0) = -\frac{i}{\hbar} \Theta(t - t_0)T[e^{-\frac{i}{\hbar} \int_{t_0}^{t} dt' H_{\text{int}}^0(t')} - \frac{i}{\hbar} \Theta(t - t_0)A^{(0)}(t, t_0).
\]  

(B.3)

It can be easily checked that the above expression for the \( G_{\text{int}}(t, t_0) \) satisfies the corresponding differential equation. The solution for \( A^{(1)}(t, T_1) \) is then given by

\[
A^{(1)}(t, T_1) = A^{(1)}_{\text{hom}}(t, T_1) + \int_{-\infty}^{+\infty} dt_0 \, G_{\text{int}}(t, t_0)H_{\text{int}}^{\text{m}}(t_0)A^{(0)}(t_0, T_1),
\]  

(B.4)

where \( A^{(1)}_{\text{hom}}(t, T_1) \) is the solution of the homogeneous equation \( (i\hbar \frac{\partial}{\partial t} - H_{\text{hom}}^{\text{m}}(t))A^{(1)}_{\text{hom}}(t, T_1) = 0 \). The \( \Theta \)-function in the expression of \( G_{\text{int}}(t, t_0) \) and the fact that interaction was off before \( T_1 \), with the initial condition for \( A^{(1)}(T_1, T_1) = 0 \), give the initial condition for \( A^{(1)}_{\text{hom}}(t, T_1) \), i.e.

\[
A^{(1)}_{\text{hom}}(T_1, T_1) = 0.
\]

The only solution of the homogeneous equation with this initial condition is \( A^{(1)}_{\text{hom}}(t, T_1) = 0 \). Thus, we obtain

\[
A^{(1)}(t, T_1) = \int_{-\infty}^{+\infty} dt_0 \, G_{\text{int}}(t, t_0)H_{\text{int}}^{\text{m}}(t_0)A^{(0)}(t_0, T_1).
\]  

(B.5)

Now, putting the expression of \( G_{\text{int}}(t, t_0) \) above and introducing \( A^{(0)}(t_0, T_1)[A^{(0)}(t_0, T_1)]^{-1} \), we obtain \((33)\). Here, we have also used the following property of \( A^{(0)}(t, T_1) \):

\[
A^{(0)}(t_0)A^{(0)}(t_0, T_1) = A^{(0)}(t, T_1).
\]

Appendix C. The correlation function

An operator corresponding to the detection of a photon by a detector should be proportional to the annihilation operator \( a \) (say \( ka \)) [24, 30]. Hence if \( |i\rangle \) is the initial state of the radiation field, the state after the detection of one photon is \( ka|i\rangle \). The amplitude for going to the final state \( |f\rangle \) is given by \( k\langle f|a|i\rangle \). The corresponding probability is \( |k|^2 |\langle f|a|i\rangle|^2 \). Thus, the probability of the detection of one photon in the state \( |i\rangle \),

\[
P_1 = \sum_f |k|^2 |\langle f|a|i\rangle|^2 = |k|^2 \sum_f |\langle f|a^\dagger|i\rangle|^2 = |k|^2 |\langle i|a^\dagger a|i\rangle|^2.
\]  

(C.1)

Similarly, the probability of the detection of two photons with a time delay of \( \tau \) is

\[
P_2 = \sum_{f} |k|^4 |\langle f|a(t + \tau)a(t)|i\rangle|^2 = |k|^4 \sum_{f} |\langle f|a(t)^a(t + \tau)|i\rangle|^2 = |k|^4 |\langle i|a(t)^a(t + \tau)a(t + \tau)a(t)|i\rangle|^2.
\]  

(C.2)

The second-order correlation function with a time delay \( \tau \) is defined as

\[
g^{(2)}(\tau) = \frac{P_2(\tau)}{P_1^2} = \frac{\langle i|a(t)^a(t + \tau)a(t + \tau)a(t)|i\rangle \langle i|a(t)^a(t + \tau)a(t + \tau)a(t)|i\rangle}{\langle i|a(t)^a(t)|i\rangle \langle i|a(t)^a(t)|i\rangle}.
\]  

(C.3)

For \( \tau = 0 \),

\[
g^{(2)}(0) = \frac{\langle i|a(t)^a(t)a(t)|i\rangle \langle i|a(t)a(t)|i\rangle}{\langle i|a(t)^a(t)|i\rangle \langle i|a(t)^a(t)|i\rangle} = \frac{\langle i\rangle \langle i|a(t)^a(t)a(t)|i\rangle}{\langle i\rangle \langle i|a(t)^a(t)a(t)|i\rangle}.
\]  

(C.4)

For a coherent state it can be calculated to be equal to 1.
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