Uncertainty relations for the relativistic Jackiw-Nair anyon: A first-principles derivation

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Abstract – In this paper we have explicitly computed the position-position and position-momentum (Heisenberg) uncertainty relations for the relativistic particles with arbitrary spin, proposed by Jackiw and Nair (JACKIW R. and NAIR V. P., Phys. Rev. D, 43 (1991) 1933) as a model for anyon, in a purely quantum mechanical framework. This supports (via Schwarz inequality) the conjecture that anyons live in a 2-dimensional non-commutative space. We have computed the non-trivial uncertainty relation between anyon coordinates, \( \sqrt{\Delta x^2 \Delta y^2} = \hbar \Theta_{xy} \), using the recently constructed anyon wave function (MAJHI J. et al., Phys. Rev. Lett., 123 (2019) 164801), in the framework of BIALYNICKI-BIRULA I. and BIALYNICKA-BIRULA Z., New J. Phys., 21 (2019) 07306. We also compute the Heisenberg (position-momentum) uncertainty relation for anyons. Lastly we show that the identical formalism when applied to electrons, yield a trivial position uncertainty relation, consistent with their living in a 3-dimensional commutative space.

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Introduction. – Consider \( x, y \) to be the spatial coordinates for excitations in a (2+1)-dimensional field theoretic model, proposed by Jackiw and Nair [1] (see also [2]), that describes relativistic particles of arbitrary spin, purportedly referred to as anyons [3–5] (although its arbitrary statistics was not shown). In this paper we explicitly show, in a quantum mechanical framework, that the position uncertainty relation \( \sqrt{\Delta x^2 \Delta y^2} = \Theta_{xy} \), for these anyons has a non-zero lower bound for \( \Theta_{xy} \), where \( \sqrt{\Delta A^2} \) is the uncertainty (dispersion or fluctuation from the mean value, i.e., standard deviation) of a generic Hermitian operator \( A \). Our result agrees with the commonly used assumption that anyons live in Non-Commutative (NC) space. The Schwarz inequality

\[
\sqrt{\Delta A^2 \Delta B^2} \geq \frac{1}{2} \langle [A, B] \rangle,
\]

if considered for \( x, y \), indicates \( \langle [x, y] \rangle \neq 0 \). An interesting observation is that \( \Theta_{xy} \) is the minimum effective spatial area occupied by an anyon. We tentatively claim the (length)^2-dimensional NC-parameter \( \Theta_{xy} \) to be a new physical constant. We also compute the Heisenberg Uncertainty Relation (HUR) for anyon \( \sqrt{\Delta x^2 \Delta p^2} = \hbar \) where \( r = \sqrt{x^2 + y^2}, \quad p = \sqrt{p_x^2 + p_y^2}. \) These results for anyons are new.

In deriving the above, we have used a technique, recently formulated by Bialynicki-Birula and Bialynicka-Birula [6], who compute HUR for free electrons, using spinor wave functions (see also [7–9]). We follow the same procedure to derive the above uncertainty relations for anyons, making use of the recently constructed anyon wave function from a work involving the present authors [10].

Since our demonstration of the counterintuitive non-zero result \( \Theta_{xy} \) for anyons is new, we need to show that the result is not a spurious artifact of our formalism. Thus, we use the same formalism [6] to calculate spatial uncertainty relation for conventional (3+1)-dimensional electrons; this turns out to be zero (as expected) thus establishing robustness of the formalism.

It is to be noted that non-commutativity of \( x, y \) and non-zero \( \Theta_{xy} \) are two completely different issues and here we are directly concerned only with the latter. We stress that the possible non-commutative nature of the \((x, y)\)-plane of anyon does not play any role in our study. In fact our result of non-zero \( \Theta_{xy} \) is (possibly a necessary but) not a sufficient condition condition for non-commutative \((x, y)\)-plane; proving non-commutativity is beyond the scope of our work. It needs to be duly emphasized that the present analysis is done purely in a quantum...
mechanical framework using anyon wave functions in calculating expectation values \(\langle A\rangle\) because so far, in modelling anyons, an NC algebra (or non-canonical algebra) between anyon dynamical variables was simply posited in a semi-classical formalism. In some variants a generalized (spinning) point particle model was constructed which had constraints that yielded Dirac brackets, to be identified with the NC algebra (see for example [11–17]). The NC algebra was geared to generate the arbitrary anyon spin. A non-relativistic limit of the Jackiw-Nair anyon, as suggested in [18], also yields a non-commutative spatial algebra, as proved in [19]. In the present work we have provided a totally quantum mechanical treatment for the anyon, based on the Jackiw-Nair anyon model.

The impact of anyons in theoretical and applied physics is easily established from its ubiquitous role in High Energy (via Chern-Simons theory [20,21]) to Condensed Matter (quantum Hall effect [22], high \(T_c\) superconductivity [23]) to the exciting arena of non-Abelian anyons (in fault tolerant quantum computation [24,25]). The possibility of non-commutative space and its effect on modern physics can be seen in [26–28]. Thus, a thorough understanding of anyon theory and its living in a noncommutative space is necessary.

The quantum mechanical Jackiw-Nair model of [1] was extended in [10] to the full construction of anyon wave function. The scheme of [6] is well-suited for our purpose since the Jackiw-Nair equation for an anyon is structurally similar to the Dirac equation for electron, both being first order in derivatives. Crucial difference in solutions is that anyon wave function is an infinite component one although constraints are imposed to reduce it to a single polarizability [1,10]. Apart from the anyonic position \(UR\) we also show in the framework of [6] that \(\Delta x^2\Delta p^2 = 0\) for electron in (3+1)-dimensions which is indeed reassuring.

**Outline of the formalism.** – The probability densities in position and momentum space, for a generic system, are defined as \(\rho_x(r) = |\psi_x(r)|^2\) and \(\rho_p(p) = |\tilde{\psi}(p)|^2\), where \(\psi(x)\) is the Fourier transform of \(\psi(r)\) and \(\psi_x\) and \(\psi_p\) are the components of the column vector \(\psi\). Now \(\psi_x(r, t)\) can be expressed as

\[
\psi_x(r, t) = \int \frac{dp}{2\pi} \tilde{F}_x(p) \psi_x(p)e^{i(p \cdot r - E_xt)},
\]

where \(F_x(p)\) is the orthonormalized free particle solution and \(\psi_x(p)\) is an arbitrary function with \(\beta\) referring to components of the column vector.

The standard expressions for the dispersions \(\Delta r^2\) and \(\Delta p^2\) in terms of the probability densities in position and momentum space

\[
\Delta r^2 = \frac{1}{N^2} \int d^3r |\tilde{r} - (r)|^2 \rho_x(r), \tag{3}
\]

\[
\Delta p^2 = \frac{1}{N^2} \int d^3p |\tilde{p} - \langle p \rangle|^2 \rho_p(p), \tag{4}
\]

are calculated, where \(N^2\) is the normalization constant,

\[
N^2 = \int d^3r \rho_x(r) = \int d^3p \rho_p(p). \tag{5}
\]

Uncertainty relations are always formulated at a fixed time. Without loss of generality, \(\langle r \rangle\), \(\langle p \rangle\) can be dropped from (3), (4) [6]. In terms of the wave functions, we find

\[
\Delta r^2 = \int d^3p |v(p)|^2, \tag{6}
\]

where the function \(v(p)\) of momentum variable represents the independent degrees of freedom of an anyon moving in free space.

\[
\Delta p^2 = \frac{1}{N^2} \int d^3p |v(p)|^2, \tag{7}
\]

whereas \(\Delta r^2\) can be found by using the following identity [6]:

\[
r\psi_x(r) = \int \frac{d^3p}{(2\pi)^{3/2}} \nabla_p(F_xv(p)e^{ip \cdot r}), \tag{8}
\]

using (2) with \(v(p)\) being an arbitrary weight function as mentioned above.

**Anyon wave function.** – Let us start with the anyon wave function for an anyon of arbitrary spin \(s = 1 - \lambda\). The dynamical equation for a spin-one particle in (2+1)-dimensions in co-ordinate and momentum space (\(i\partial_a = p_a\)) is given by [1]

\[
\partial_a e^{abc} F_{c} \pm m F^b = 0, \quad (p \cdot j)_a^b F^b + ms F^a = 0 \tag{9}
\]

with \(j\) being the angular momentum operator and \(m\) is the mass term. If \(J\) is the total spin contribution to the Lorentz generators \(K\), then \(J = K + j\).

The solution of the three-vector \(F^a\), \(a = 1, 2, 3\) in the Minkowski metric \(\eta_{\mu\nu} = \text{diag}(1,-1,-1)\) is given by

\[
F^a(p) = \frac{m}{\sqrt{2E}} \begin{bmatrix} 0 \\ 1 \\ \frac{p^x + ip^y}{m(E+m)} \end{bmatrix}, \tag{10}
\]

This construction has been extended in an elegant way to the Jackiw-Nair anyon equation [1] to describe an anyon of arbitrary spin \(s = 1 - \lambda\), whose momentum space dynamics is given by

\[
P \cdot (J + j)_{\alpha a'} f_{\alpha a'}^{\lambda, \lambda'} + ms f_{\alpha}^{\lambda} = 0, \quad (D_a f_{\alpha}^{\lambda, \alpha'})_{\alpha} = 0, \tag{11}
\]

\[
(D_{\alpha a'})_{\alpha a'} = e^{ba} P^b K_{\alpha a'}^{ba} - e^{ba} P^b K^{ba}_{\alpha a'}
\]

where the second equation is the subsidiary (constraint) relation and \(n\) runs from 0 to \(-\infty\).

For \(\lambda = 0\) the anyon reduces to the spin-one model discussed earlier. The notation and other details can be found in [10] (see also [1]). \(K^a\) are the Lorentz generators with action defined as

\[
K^0 f_{\alpha}^{\lambda +, \lambda +} = (\lambda + n) f_{\alpha}^{\lambda +, \lambda +},
\]

\[
K^a f_{\alpha}^{\lambda +, \lambda +} = \sqrt{(2\lambda + n)(n + 1)} f_{n+1}^{\lambda +, \lambda +}.
\]

\[
K^0 f_{\alpha}^{\lambda +, \lambda -} = (\lambda - n) f_{\alpha}^{\lambda +, \lambda -},
\]

\[
K^a f_{\alpha}^{\lambda +, \lambda -} = \sqrt{(2\lambda - n)(n - 1)} f_{n-1}^{\lambda +, \lambda -}.
\]
where “+” superscript denotes representations bounded below. There is an analogous bounded above representation and $K^x = K^z \mp iK^y$ [1]. Thus, the explicit form of the free anyon solution, $(\infty \geq n \geq 0)$ is given by [10]

$$f^{\lambda,n}_{n,n} = \left( \frac{m}{E + m} \right)^{\lambda} \sqrt{\frac{\Gamma(2\lambda + n)}{n!\Gamma(2\lambda)}} \left( p^2 + i\beta \right)^n E^{\lambda}(p)e^{-ipx}, \quad (13)$$

where $F^a(p)$ is the same as the spin-1 case defined in (10).

Before proceeding further let us ensure: i) the probability distribution thus generated is conserved; ii) the Lorentz generators defined earlier are self-adjoint. Complications can stem from the sum over index $n$ running from zero to infinity and possible appearance of null or negative norm states. These issues are addressed in [29] in detail. Furthermore, in [29] an alternative and more compact anyon model was proposed, where the spin-one base used here [1, 10], was replaced by a Majorana-Dirac spin-1/2 base.

i) Anyon current conservation: We will explicitly derive the conservation law for probability current $\partial^\mu j^{\lambda}_{\mu} = 0$, where $j^{\lambda}_{\mu}$ denotes the probability density (computational steps are provided in the Supplemental Material of [10]). Using the explicit form of anyon wave function (13) a long calculation yields [10]

$$\sum_{n=0}^{\infty} \left[ \left( f^{\mu}_{n,n} f^{\mu}_{n,n} + f^{\mu}_{n,n} f^{\mu}_{n,n} + f^{\mu}_{n,n} f^{\mu}_{n,n} \right) -i \left( f^{\mu}_{n,n} f^{\mu}_{n,n} f^{\mu}_{n,n} - f^{\mu}_{n,n} f^{\mu}_{n,n} f^{\mu}_{n,n} \right) \right] = 0,$$

where the sum over $n$ has been carried out in [10].

ii) Self-adjointness of Lorentz generators $K^x$: We have to check whether the following matrix elements are equal for arbitrary $f$ for a specific value of $\lambda$:

$$\langle f^{\lambda}(p_{n}), K^{a} f^{\lambda}(p_{n}) \rangle = \langle K^{a} f^{\lambda}(p_{n}), f^{\lambda}(p_{n}) \rangle.$$  (15)

Since $K^{a}$ acts only on the index $n$, the non-trivial part of the matrix element is written following the convention:

$$K^{n}_{n,n'} = \langle \lambda, n | K^{\lambda} | \lambda, n' \rangle = (\lambda + n)\delta_{nn'},$$

$$K^{0}_{n,n'} = \langle \lambda, n | K^{\lambda} | \lambda, n' \rangle = (\lambda + n)\sqrt{\frac{2\lambda + n'}{2\lambda + n}} \sqrt{(2\lambda + n') \delta_{nn'}} + 1,$$

$$K^{-}_{n,n'} = (\lambda, n | K^{-} | \lambda, n' \rangle = \sqrt{\frac{2\lambda + n - 1}{2\lambda + n}} \delta_{nn'}, \quad (16)$$

We recover

$$K^{x} = \frac{1}{2} (K^{+} + K^{-}) \quad K^{y} = i \frac{1}{2} (K^{+} - K^{-})$$

and obtain in a straightforward way:

$$\langle K_{n,n'}^{\lambda} = (\lambda + n)\delta_{nn'}, \quad (17a)$$

$$K^{0}_{n,n'} = \frac{1}{2} \left[ \sqrt{(2\lambda + n - 1)\delta_{nn'} - \frac{1}{2}} + \sqrt{(2\lambda + n)\delta_{nn'} - \frac{1}{2}} \right], \quad (17b)$$

$$K^{y}_{n,n'} = \frac{i}{2} \left[ -\sqrt{(2\lambda + n - 1)\delta_{nn'} - \frac{1}{2}} - \sqrt{(2\lambda + n)\delta_{nn'} - \frac{1}{2}} \right]. \quad (17c)$$

The volume integral in the matrix element introduces $\delta(p_{n} - p_{n'})$. To calculate $\langle f^{\lambda}(p_{n}), K^{a} f^{\lambda}(p_{n}) \rangle$, once again we invoke the $n$-summation identities [10] and get, for $K^{0}$:

$$\sum_{n,n'=0}^{\infty} (f^{0}_{n}) \int K^{0}_{n,n'} f^{0}_{n'} = \sum_{n=0}^{\infty} (f^{0}_{n}) \int E_{\lambda} f^{0}_{n} = \frac{E_{\lambda}}{m}, \quad (18)$$

and for $K^{x}$:

$$\sum_{n=0}^{\infty} (f^{0}_{n}) \int K^{x} f^{0}_{n} = \sum_{n=0}^{\infty} \left[ (f^{0}_{n}) \int \frac{1}{2} \sqrt{(2\lambda + n - 1)n} f^{0}_{n} + \sqrt{(2\lambda + n)(n + 1)} f^{0}_{n} \right] = \frac{p_{x}\lambda}{m}, \quad (19)$$

and for $K^{y}$:

$$\sum_{n=0}^{\infty} (f^{0}_{n}) \int K^{y} f^{0}_{n} = \sum_{n=0}^{\infty} \left[ (f^{0}_{n}) \int \frac{1}{2} \sqrt{(2\lambda + n - 1)n} f^{0}_{n} - \sqrt{(2\lambda + n)(n + 1)} f^{0}_{n} \right] = \frac{p_{y}\lambda}{m}. \quad (20)$$

In an exactly similar manner we calculate $\langle K^{a} f^{\lambda}(p_{n}), f^{\lambda}(p_{n}) \rangle$:

$$\sum_{n,n'=0}^{\infty} (f^{0}_{n}) \int K^{0}_{n,n'} f^{0}_{n'} = \sum_{n=0}^{\infty} (f^{0}_{n}) \int \frac{E_{\lambda}}{m} = \frac{E_{\lambda}}{m}, \quad (21)$$

$$\langle K^{x}_{n,n'} f^{0}_{n'} \rangle = \int \frac{1}{2} \sqrt{(2\lambda + n' - 1)n'} f^{0}_{n'}. \quad (22)$$
We arrive at identical results as the ones derived above in (18)–(20). This proves the self-adjointness of the Lorentz generators $K^a$.

**Spatial uncertainty relation for anyon.**– To settle this critical issue, let us consider $\Delta x^2 \Delta y^2 = h^2 \Theta_{xy}^2$. Computations with anyon wave functions (13) yield (in polar coordinates)

$$
\Delta x^2 = \frac{1}{N^2} \int_0^\infty dp \int_0^{2\pi} d\theta \left[ \left| \cos \theta \partial_p v(p, \theta) - \frac{1}{p} \sin \theta \partial_\theta v(p, \theta) \right|^2 + \frac{1}{2} \left( \frac{m^2 + \lambda E^2}{E^4} + \frac{\sin^2 \theta}{p^2} (3 + 4\lambda (\lambda - 2)) \right) + \frac{(\lambda - 1)}{(m/E) + \frac{\lambda (1 + 2\lambda) p^2}{m^2}} \sin \theta \times |v(p, \theta)|^2 \right. \\
\times \left. v^*(p, \theta) \left( \cos \theta \partial_p v(p, \theta) - \frac{1}{p} \sin \theta \partial_\theta v(p, \theta) \right) \right].
$$

(24)

It is straightforward to check that similar results are obtained for $\Delta y^2$. Following [6] we simplify the system by invoking spherical symmetry $v(p, \theta) = v(p)$, to obtain

$$
\Delta x^2 = \frac{\pi}{N^2} \int_0^\infty dp \left[ \frac{1}{p^2} \left| \partial_p v(p) \right|^2 + \frac{1}{2} \left( \frac{m^2 + \lambda E^2}{E^4} + \frac{\sin^2 \theta}{p^2} (3 + 4\lambda (\lambda - 2)) + \frac{(\lambda - 1) \lambda E}{m} \right) \left| v(p) \right|^2 \right].
$$

(25)

One can check that $\Delta y^2$ is also given by the above relation, a consequence of spherical symmetry. Note that we are interested only in the minimum value for $\Theta_{xy}$. Hence it is justified to restrict to $v(p)$ only [6] since angular contributions can only increase $\Theta_{xy}$.

Our aim is to compute $\Delta x^2 \Delta y^2$ and show that it has a non-zero minimum. It needs to be emphasized that only as far as numerical numbers are involved, spherical symmetry dictates that $\Delta x^2 = \Delta y^2$ whereas in reality $\Delta x^2$ and $\Delta y^2$ are two distinct and independent entities, each pertaining to two coordinate directions that are mutually orthogonal. Hence we define $\Delta x^2 \Delta y^2 = h^2 \Theta_{xy}^2$ keeping in mind that there are two independent parameters involved and the naive equality between $\Delta x^2$ and $\Theta_{xy}$ is not to be considered. We obtain a Schrödinger-like-like variational equation for $v(p)$ whose minimum eigenvalue will provide the cherished value of $\Theta_{xy}^2$ [6],

$$
-\partial_q^2 - \frac{1}{q} \partial_q + U^{(d,\lambda)}(q) v_{(d,\lambda)}(q) = \frac{\Theta_{xy}}{m} v_{(d,\lambda)}(q).
$$

(26)

where, at this stage we are allowed to use the numerical equality $\Theta_{xy}/(1 + \Delta x^2/2\Delta y^2) = \Theta_{xy}/2 = \Theta_{xy}$ and the potential is

$$
2U^{(d,\lambda)}(q) = \frac{(1 + \lambda (1 + d^2 q^2))}{(1 + d^2 q^2)} + \frac{1}{q} \left[ 3 + 4\lambda (\lambda - 2) + 4(\lambda - 1) \left( 1 - \frac{1}{1 + d^2 q^2} \right) \right] + \lambda (1 + 2\lambda) d^2 q^2.
$$

(27)

Rescaled variables used are $q = \frac{p}{mc^2}$, $d = \frac{h^2}{mc^2} (\Delta x^2/\Delta y^2)^{1/4}$ [30]. Following [6] one can consider a non-relativistic limit $m \to \infty$ or $c \to \infty$ leading to $d \to 0$ (keeping $\Theta_{xy}$ fixed) when there is a surprising cancellation of $\lambda$ terms yielding $U^{(s,0)}(q) = -1/(2q^2)$,

$$
-\partial_q^2 - \frac{1}{q} \partial_q - \frac{1}{2q^2} v_0(q) = \Theta_{xy(0)} v_0(q).
$$

(28)

We are only interested in finding the lowest positive eigenvalue $\Theta_{xy(0)}$. The dependence of the potential profile of $d$ and $\lambda$ are shown in fig. 1. Few of the lowermost eigenvalues of $\Theta_{xy}$ are shown in each of the panels. The numerically computed eigenvalues for $\Theta_{xy}$ are revealed in the $\Theta_{xy}$-d-$\lambda$ diagram in the lowest positive eigenvalues on the parameters $d$ and $\lambda$ which is shown in fig. 2. Notice that the minimum value $\Theta_{xy} \approx 0.002$ (time/mass), the dimension comes from the definition $\Delta x^2 \Delta y^2 = h^2 \Theta_{xy}^2$ and also agrees with the dimension coming from (28) and above, $dimension[1/q^2] = dimension \Theta_{xy}$. As explained above, this corresponds to $d = 0$ which can be contrasted to the commutative space for electron discussion (see below (43)). This is our most significant result showing that, thanks to the Schwarz inequality, the claim that anyons live in a non-commutative space is mathematically consistent.

**Heisenberg uncertainty relation for anyon.**– As before, we exploit the spherical symmetry by introducing polar coordinates,

$$
N^2 = \int_0^\infty dp \int_0^{2\pi} d\theta \left| v(p, \theta) \right|^2 d\theta,
$$

(29)

$$
\Delta p^2 = \frac{1}{N^2} \int_0^\infty dp \int_0^{2\pi} d\theta p^2 \left| v(p, \theta) \right|^2,
$$

(30)

20002-p4
Following [6] we define $\Theta_{xp} = \sqrt{\Delta r^2 \Delta \rho^2} / \hbar$ and try to find the minimum value of $\Theta_{xp}$ by varying $v^*$ and equating it to zero. For the same reasons mentioned earlier we invoke spherical symmetry since angular contributions can only increase the eigenvalue [6]. The variational equation for $\Delta r^2 \Delta \rho^2$ with respect to $v^*$ reduces to

$$\frac{\Delta \rho^2}{\Delta r^2} \left[ -\frac{\partial^2}{\partial \rho^2} - \frac{1}{\rho} \frac{\partial}{\partial \rho} - \frac{\rho^2}{E^2} - \frac{4(\lambda - 1)(E - m)(E \lambda + m)}{m Ep} \right] \left[ \frac{1}{E^2} + \frac{2 \lambda + 1}{m^2} \right] + \frac{p^2 \Delta r^2 - 2 \Theta_{xp}^2}{\Delta \rho^2} \right] \right] \frac{\partial}{\partial \theta} v(p, \theta) = 0. \quad (32)$$

Shifting to dimensionless variables [6] $q = \frac{p}{\hbar \Delta \rho}$, $d = \frac{1}{mc \sqrt{\Delta \rho^2}}$ we recover a Schrödinger-like equation

$$\frac{1}{2} \left[ -\frac{\partial^2}{\partial q^2} - \frac{1}{q} \frac{\partial}{\partial q} + \frac{2}{q^2} \frac{(\lambda - 1)(E - m)(E \lambda + m)}{m Ep} \right] \left[ \frac{1}{E^2} + \frac{2 \lambda + 1}{m^2} \right] + \frac{p^2 \Delta r^2 - 2 \Theta_{xp}^2}{\Delta \rho^2} \right] \right] \frac{\partial}{\partial \theta} v(p, \theta) = 0. \quad (32)$$

for $v_{(d, \lambda)}(q)$.

$$\frac{1}{2} \left[ -\frac{\partial^2}{\partial q^2} - \frac{1}{q} \frac{\partial}{\partial q} + \frac{2}{q^2} \frac{(\lambda - 1)(E - m)(E \lambda + m)}{m Ep} \right] \left[ \frac{1}{E^2} + \frac{2 \lambda + 1}{m^2} \right] + \frac{p^2 \Delta r^2 - 2 \Theta_{xp}^2}{\Delta \rho^2} \right] \right] \frac{\partial}{\partial \theta} v(p, \theta) = 0. \quad (32)$$

where the potential is

$$2U_{(d, \lambda)}(q) = -\frac{q^2 d^4}{(1 + q^2 d^2)^2} + \lambda d^2 \left( \frac{1}{q^2 d^2 + 1} + 2 \lambda + 1 \right)$$

$$- \frac{4(\lambda - 1)(\sqrt{q^2 d^2 + 1} - \lambda \sqrt{q^2 d^2 + 1} + 1) + q^2}{q^2 \sqrt{q^2 d^2 + 1}}. \quad (34)$$

Notice that $d \to 0$ yields the non-relativistic limit, for which (33) becomes independent of $\lambda$.

$$\frac{1}{2} \left[ -\frac{\partial^2}{\partial q^2} - \frac{1}{q} \frac{\partial}{\partial q} + \frac{2}{q^2} \right] v_{0}(q) = \Theta_{xp(0)} v_{0}(q). \quad (35)$$

Incidentally (35) is identical to the corresponding equation for electron in [6] apart from a factor of 2 in the $\partial^2 / \partial q^2$ term for space dimensional mismatch. The solution of the above equation is a Gaussian $v_{0}(q) = \exp(-q^2/2)$ with the lowest eigenvalue $\Theta_{xp(0)} = 1$. We recover the HUR for anyon as $\sqrt{\Delta r^2 \Delta \rho^2} = \hbar$, due to the two (spatial) dimensional nature of the system. This is one of our major results.

It is interesting to note that the relativistic limit $d \to \infty$ exists only for $\lambda = 0$ (i.e., for spin $s = 1 - \lambda = 1$, otherwise some $\lambda$-dependent terms diverge) leading to

$$\frac{1}{2} \left[ -\frac{\partial^2}{\partial q^2} - \frac{1}{q} \frac{\partial}{\partial q} + \frac{3}{q^2} \right] v_{(\infty, 0)}(q) = \Theta_{xp(\infty, 0)} v_{(\infty, 0)}(q). \quad (36)$$

The solution of the above differential equation is $v_{(\infty, 0)}(q) = q \sqrt{\pi} e^{-q^2/2}$ and $\Theta_{xp(\infty, 0)} = 1 + \sqrt{3}$. A few representative results are given in table 1.

In fig. 3 a three-dimensional plot for $U_{(d, \lambda)}(q)$-$q$-$d$ for different $\lambda$ shows how the potential separates into sheets.
leading to
\[
P \frac{\nu kT}{\Theta} = 1 + a_2(\nu \Theta_{xy}) + a_3(\nu \Theta_{xy})^2 + \ldots . \quad (39)
\]
(For a detailed discussion see [31].) The above is a heuristic presentation of a possible significance of \(\Theta_{xy}\), the anyonic spatial uncertainty parameter.

**Commutative space for electrons.** – Let us check the consistency of our scheme by recovering the commutative space for electrons, in the framework of [6]. Using free spinor solutions \(F_\nu^e\) for electron, the dispersion for \(x\) and \(y\) is straightforward to obtain. The expression simplifies considerably by invoking spherical symmetry \((f(p, \theta, \phi) = f(p))\),
\[
\Delta x^2 = \frac{4\pi}{3N^2} \int_0^\infty dp \sum_s \left[ p^2 | \partial_p f(p) |^2 + \left( 1 + \frac{m}{E} + \frac{m^2 p^2}{4E^4} \right) | f(p) |^2 \right]. \quad (40)
\]
Let us define
\[
\Delta x^2 \Delta y^2 = h^2 \Theta_{xy(e)}, \quad (41)
\]
where \(\Theta_{xy(e)}\) is the NC parameter of dimension \((\text{length})^2\) (if it turns out to be non-zero) and \(e\) stands for electron. Integrating over \(\theta\) and \(\phi\) and considering the variation with respect to \(f^\ast\) we have
\[
\left[ \Delta x^2 \left( -\partial^2_p - \frac{2}{q} \partial_q + \left( \frac{1}{q^2} - \frac{m}{E}q^2 + \frac{m^2}{4E^2} \right) \right) \right] f(p) = \Theta^2_{xy(e)} f(p), \quad (42)
\]
where \(\Theta_{xy(e)} = \frac{\Theta_{xy} d}{2}.\) Introducing the dimensionless variable \(q = \frac{q}{mc}\) and parameter \(d = \frac{1}{mc} \left( \frac{h^2 A^2}{\Delta x^2} \right)^{1/4}\), the above equation reduces to
\[
\left[ -\partial^2_q - \frac{2}{q} \partial_q + V_d^{(x,y)}(q) \right] f_d(q) = \Theta_{xy(e)} f_d(q), \quad (43)
\]
\[
V_d^{(x,y)}(q) = \frac{1}{q^2} - \frac{1}{q^2 \sqrt{1 + d^2 q^2}} + \frac{d^2}{4(1 + q^2 d^2)}. \quad (44)
\]
In fig. 5, the profile of \(q\) vs. \(V_d^{(x,y)}(q)\) for different values of \(d\) is shown. For \(\infty \geq d \geq 0\), the potential approaches zero for large \(q\). In the two limits of \(d\), eq. (43) reduces to
\[
\left[ -\partial^2_q - \frac{2}{q} \partial_q \right] f_0(q) = \Theta_{xy(e,0)} f_0(q), \quad d \to 0, \quad (45)
\]
\[
\left[ -\partial^2_q - \frac{2}{q} \partial_q + \frac{1}{q^2} \right] f_\infty(q) = \Theta_{xy(e,\infty)} f_\infty(q), \quad d \to \infty. \quad (46)
\]
We are interested in the smallest eigenvalue and clearly \(\Theta_{xy(e,0)} < \Theta_{xy(e,\infty)}\) (due to the positive contribution from \(1/q^2\) in (46)) and the lowest possible value for \(\sigma_0\) is zero consistent with free particle solution \(f_0 \sim \exp(-i \sqrt{\Theta_{xy(e,0)} q})\) for large \(q\). Comparison with electron
\(\Delta r^2 \Delta p^2\) HUR results in [6] reveals that the presence of \(q^2\) (harmonic oscillator) term in the overall potential that induced a non-zero minimum eigenvalue is absent here in \(\Delta x^2 \Delta y^2\) HUR for electron. Thus, \((3+1)\)-dimensional electrons live in a commutative space. Indeed, this is not a new result but rederived in the present framework, to be compared with the anyon result, derived earlier.

**Conclusion and future prospects.** – To summarise, we have computed minimum values of products of dispersions between coordinates \(x, y\) and between coordinate and momentum \(r, p\) for \((2+1)\)-dimensional arbitrary spin particles, referred to as anyons, utilising the explicit form of anyon wave function [10], in the framework of [6]. Non-zero value for the former yields the spatial uncertainty relation for anyon and strongly suggests that anyons live in non-commutative space. Incidentally we also show, in the same formalism, that Dirac electrons live in commutative space, which is reassuring. We have briefly indicated how \(\Theta_{xy}\) might appear in anyon equation of state.

Indeed more research is required but still, it is tempting to interpret the non-commutativity parameter \(\Theta_{xy}\) as a new and independent constant for planar quantum physics, similar to \(\hbar\) in phase space. It might be interesting to consider Zitterbewegung effect for anyons and also introduce Foldy-Wouthuysen or Newton-Wigner coordinates in the study of anyon.

More and more applications of anyons in modern physics (especially in the area of quantum computing) make it necessary to grasp the underpinnings of anyon theory at the microscopic level. We plan to extend our work to model an action principle for anyon that can be generalized to non-Abelian anyons, the ultimate goal of this project.

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**Data availability statement:** All data that support the findings of this study are included within the article (and any supplementary files).

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