Effects of Homogeneous Plasma on Strong Gravitational Lensing of Kerr Black Holes

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Abstract

Considering a Kerr black hole surrounded by a homogenous unmagnetised plasma medium, we study the strong gravitational lensing on the equatorial plane of the Kerr black hole. We find that the presence of the uniform plasma can increase the photon-sphere radius $r_{ps}$, the coefficient $\bar{a}$, $\bar{b}$, the angular position of the relativistic images $\theta_{\infty}$, the deflection angle $\alpha(\theta)$ and the angular separation $s$. However the relative magnitudes $r_m$ decreases in the presence of the uniform plasma medium. It is also shown that the impact of the uniform plasma on the effect of strong gravitational become smaller as the spin of the Kerr black increace in prograde orbit ($a > 0$). Especially, for the extreme black hole ($a=0.5$), the effect of strong gravitational lensing in homogenous plasma medium is the same as the case in vacuum for the prograde orbit.

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I. INTRODUCTION

It is fact that a plasma exists around a supermassive black hole at the centre of the Galaxy\cite{1}. Consider a plasma cloud surrounding a black hole, the light’s propagation deviates from lightlike geodesics in a way that depends on the frequency of the plasma. Thus, compared with the general relativistic vacuum propagation effects, the astrophysical plasma as refractive and dispersive medium can influence the relativistic images due to the gravitational lensing and the shape of accretion disk effect ranging from pulsars and X-ray binaries to active galactic nucleus\cite{2–5}. The study of astrophysical processes in plasma medium surrounding black hole becomes very interesting and important.

A general theory of geometrical optics in a curved space-time, in an isotropic dispersive medium was proposed in the textbook of Synge\cite{6}. It is based on the elegant abstract Hamiltonian theory of rays and waves. In the book of Perlick\cite{7} the method was developed for consideration of a deflection of the light rays in presence of the gravity and plasma. The general formulae for the exact light deflection angle in the Schwarzschild and Kerr metric, in presence of plasma with spherically symmetric distribution of concentration, are obtained in the form of integrals. The generally covariant equations describing the propagation of waves with an arbitrary dispersion relation in a nonuniform, unmagnetised plasma medium was derived in paper\cite{8}. The geometric optics approximation through a magnetised plasma in the vicinity of a compact object was presented in paper\cite{2}. On the basis of his general approach of the geometric optics, the gravitational lensing in inhomogeneous and homogeneous plasma around black holes has been recently studied in \cite{9–16} as extension of vacuum studies. However, their research work restricted to the static spacetime and slowly rotating compacted object in plasma medium.

In this paper, we will study the strong gravitational lensing by Kerr black hole in a unmagnetised homogeneous plasma medium. It is well known that the gravitational lensing is regarded as a powerful indicator of the physical nature of the central celestial objects. It is shown that the relativistic images due to the gravitational lensing effect carry some essential signatures about the central celestial objects and could provide the profound verification of alternative theories of gravity in their strong field regime\cite{17–32}. The main purpose of this paper is to study the strong gravitational lensing by the Kerr Black hole in the unmagnetised homogeneous plasma medium and extend the results of the paper of Bisnovatyi-Kogan and Tsupko\cite{10} to the case of rotating gravitational lens, and to see the impact of Homogeneous Plasma on photon sphere radius, the deflection angle, the coefficients and the observable quantities of strong gravitational lensing. Moreover,
we will explore how it differs from the Kerr black hole lensing in vacuum.

The paper is organized as follows: In Sec. II, we will derive the expression for the deflection angle of light in Kerr black hole in the presence of homogeneous plasma. In Sec. III, we study the physical properties of the strong gravitational lensing by Kerr black hole and probe the effects of homogeneous plasma on the deflection angle, the coefficients and the observable quantities for gravitational lensing in the strong field limit. We end the paper with a summary.

**II. ROTATING KERR SPACETIME AND RADIUS OF PHOTON SPHERE**

Considering a rotating Kerr black hole surrounded by plasma. The Kerr metric in the standard Boyer-Lindquist coordinates can be expressed as

\[
ds^2 = -\left(1 - \frac{2Mr}{\Sigma}\right) dt^2 - \left(\frac{4Mar \sin^2 \theta}{\Sigma}\right) dt d\phi + \frac{\Sigma}{\Delta} dr^2 + \Sigma d\theta^2 + \left(r^2 + a^2 + \frac{2Ma^2r \sin^2 \theta}{\Sigma}\right) \sin^2 \theta d\phi^2, \tag{1}\]

with

\[
\Delta \equiv r^2 - 2Mr + a^2, \quad \Sigma \equiv r^2 + a^2 \cos^2 \theta. \tag{2}\]

We assume that the spacetime is filled with a non magnetized cold plasma whose electron plasma frequency \(\omega_p\) is a function of the radius coordinate only,

\[
\omega_p(r)^2 = \frac{4\pi e^2}{m} N(r). \tag{3}\]

Here \(e\) is the charge of the electron, \(m\) is the electron mass, and \(N(r)\) is the number density of the electrons in the plasma. When \(\omega_p\) is constant, plasma is homogeneous. In this paper we only consider homogeneous plasma. The refraction index \(n\) of this plasma depends on the plasma frequency \(\omega_p\) and on the frequency \(\omega\) of the photon as it is measured by a static observer,

\[
n^2 = 1 - \frac{\omega_p^2}{\omega^2}. \tag{4}\]

Let us now study the strong gravitational lensing of the rotating Kerr black hole surrounded by plasma. As in refs. [21,31], we just consider that both the observer and the source lie in the equatorial plane of the black hole and the whole trajectory of the photon is limited on the same plane. Using the condition \(\theta = \pi/2\) and taking \(2M = 1\), the metric (1) is reduced to

\[
ds^2 = -A(r) dt^2 + B(r) dr^2 + C(r) d\phi^2 - 2D(r) dt d\phi, \tag{5}\]
with

\[ A(r) = 1 - \frac{1}{r}, \quad (6) \]
\[ B(r) = \frac{r^2}{a^2 - r + r^2}, \quad (7) \]
\[ C(r) = a^2 + \frac{a^2}{r} + r^2, \quad (8) \]
\[ D(r) = \frac{a}{r}. \quad (9) \]

The general relativistic geometrical optics on the background of the curved space-time, in a refractive and dispersive plasma medium, was developed by Synge [6]. Based on the Hamiltonian approach for the description of the geometrical optics. The Hamiltonian for the photon around the Kerr black hole surrounded by plasma has the following form [8]

\[ H(x^i, p_i) = \frac{1}{2} \left[ g^{ik} p_i p_k + \hbar^2 \omega_p^2 \right] = 0. \quad (10) \]

It is interested to notice that the expression of the Hamiltonian for the photon around the Kerr black hole surrounded by homogeneous plasma is similar to the Hamiltonian of the massive particle in vacuum. Using the Hamiltonian for the photon around the Kerr black hole, we can get the Hamiltonian differential equations.

\[ \frac{dx^i}{d\lambda} = \frac{\partial H}{\partial p_i}, \quad \frac{dp_i}{d\lambda} = -\frac{\partial H}{\partial x^i}, \quad (11) \]

then we get two constants of motions are the energy and the angular momentum of the particle

\[ E = -p_t = \hbar \omega, \quad L = P_\phi. \quad (12) \]

Let us consider a homogeneous plasma with \( \omega_p = \text{const} \). We introduce notations of \( \hat{E} \) and \( \hat{L} \)

\[ \frac{-p_t}{\hbar \omega_p} = \frac{\omega}{\omega_p} = \hat{E} > 1, \quad \frac{L}{\hbar \omega_p} = \hat{L} > 0. \quad (13) \]

From this equations, we find an expression for the \( i, r, \phi \) in terms of \( \hat{E} \) and \( \hat{L} \)

\[ \frac{dt}{d\lambda} = \frac{\hbar \omega_p (\hat{E} C(r) - \hat{L} D(r))}{D(r)^2 + A(r) C(r)}, \quad (14) \]
\[ \frac{d\phi}{d\lambda} = \frac{\hbar \omega_p (\hat{E} D(r) + \hat{L} A(r))}{D(r)^2 + A(r) C(r)}, \quad (15) \]
\[ \left( \frac{dr}{d\lambda} \right)^2 = \frac{\hbar^2 \omega_p^2 (\hat{E}^2 C(r) - (A(r) C(r) + D(r)^2 - 2\hat{E} \hat{L} D(r) - \hat{L}^2 A(r)))}{B(r)[D(r)^2 + A(r) C(r)]}. \quad (16) \]
Where $\lambda$ is an affine parameter along the geodesics. With the condition $\frac{dr}{d\lambda}|_{r=r_0} = 0$, we can obtain the angular momentum $\hat{L}(r_0)$

$$\hat{L}(r_0) = \frac{-\hat{E}D(r_0) + \sqrt{A(r_0)PC(r_0) + \hat{E}^2D^2(r_0)}}{A(r_0)},$$

(17)

$$PC(r_0) = \hat{E}^2C(r_0) - (A(r_0)C(r_0) + D(r_0)^2).$$

(18)

For the photon moving in the plasma have the effective mass $m_{eff} = \hbar\omega_p$, the impact parameter can be expressed as

$$u = \frac{L}{\sqrt{E^2 - m_{eff}^2}} = \frac{\hat{L}}{\sqrt{E^2 - 1}}.$$  

(19)

Moreover, the photon sphere is a time-like hyper-surface ($r = r_{ps}$) on which the deflect angle of the light becomes unboundedly large as $r_0$ tends to $r_{ps}$. In this spacetime, the equation for the photon sphere reads

$$A(r)PC'(r) - A'(r)PC(r) + 2\hat{L}\hat{E}[A'(r)D(r) - A(r)D'(r)] = 0.$$  

(20)

The biggest real root external to the horizon of this equation is defined as the radius of the photon sphere $r_{ps}$. In the case of the Kerr black hole surrounded by the plasma, the analytical expression of marginally circular photon orbits takes a form

$$a^4 + 2a^2(2 - a^2)r + (4 + 8a^2 + a^4 - 4\hat{E}^2a^2 - 4\hat{E}^4a^4 + 2a^2(4 - 3\hat{E}^2) - 4a^2\hat{E}^2(\hat{E}^2 - 1))r^2$$

$$+ (4 + (4 - 3\hat{E}^2)(6\hat{E}^2 - 4(\hat{E}^2 - 1)(8 + a^2 - 2a^2\hat{E}^2 - 3\hat{E}^2))r^5$$

$$+ \frac{3\hat{E}^2 - 4 + \hat{E}\sqrt{-8 + 9\hat{E}^2}}{4(\hat{E}^2 - 1)}.$$  

(21)

Obviously, circular photon orbits is depend on the plasma frequency. Especially, for the photon radius in a static Schwarzschild black hole surrounded by homogeneous plasma have the analytical expression from the equation (20)

$$r_{ps} = \frac{3\hat{E}^2 - 4 + \hat{E}\sqrt{-8 + 9\hat{E}^2}}{4(\hat{E}^2 - 1)}.$$  

(22)

This just the result obtained in reference [10]. As $\hat{E} \to \infty$, the photon radius $r_{ps} \to \frac{3}{2}$ corresponds to the photons of the static Schwarzschild black hole in the vacuum. In Fig. (1), we present the variation of the photon-sphere radius $r_{ps}$ with the rotational parameter $a$ in different plasma medium and vacuum. as
expected, compared with the case in the vacuum, the presence of plasma can increase the photon sphere radius $r_{ps}$. It also shown that the growth of the photon-sphere radius in prograde orbit ($a > 0$) is less than the case in retrograde orbit ($a < 0$). Especially, for the extreme black hole ($a=0.5$), the photon sphere radius in plasma medium is the same as in vacuum for the prograde orbit.

\[FIG. 1: \text{Variation of the radius of the photon sphere } r_{ps} \text{ with the parameter } a \text{ in different plasma medium and vacuum of the Kerr black hole. parameter } \hat{E} = \frac{\omega}{\omega_p} \text{ describe the ratio photon frequency to the plasma frequency.}\]

III. STRONG GRAVITATIONAL LENSING OF ROTATING KERR BLACK HOLE IN PLASMA MEDIUM

In this section we will study the gravitational lensing of the rotating Kerr black hole in plasma medium which has a photon sphere and then probe the impact of plasma on the coefficients and the observable quantities of the strong gravitational lensing.
A. Coefficients of strong gravitational lensing

The deflection angle for the photon coming from infinite in a stationary, Kerr black hole in plasma medium can be given as follow:

\[ \alpha(r_0) = I(r_0) - \pi, \quad (23) \]

with

\[ I(r_0) = 2 \int_{r_0}^{\infty} \frac{\sqrt{B(r)|A(r_0)|[\hat{E}D(r) + \hat{L}A(r)] dr}}{\sqrt{D^2(r) + A(r)C(r)} \sqrt{\text{sgn}(A(r_0))[A(r_0)PC(r) - A(r)PC(r_0) + 2\hat{E}\hat{L}[A(r)D(r_0) - A(r_0)D(r)]]}}. \quad (24) \]

where \( \text{sgn}(X) \) gives the sign of \( X \).

It is obvious that the deflection angle increases as the parameter \( r_0 \) decreases. For a certain value of \( r_0 \) the deflection angle becomes \( 2\pi \), so that the light ray makes a complete loop around the lens before reaching the observer. If \( r_0 \) is equal to the radius of the photon sphere \( r_{ps} \), we can find that the deflection angle diverges and the photon is captured by the compact object.

In order to find the behavior of the deflection angle when the photon is close to the photon sphere, we use the evaluation method proposed by Bozza \[21\]. The divergent integral in Eq. (24) is first split into the divergent part \( I_D(r_0) \) and the regular one \( I_R(r_0) \), and then both of them are expanded around \( r_0 = r_{ps} \) with sufficient accuracy. This technique has been widely used in the study of the strong gravitational lensing for various black holes \[21\,31\]. Let us now to define a variable

\[ z = 1 - \frac{r_0}{r}, \quad (25) \]

and rewrite the Eq. (24) as

\[ I(r_0) = \int_{0}^{1} R(z, r_0)f(z, r_0)dz, \quad (26) \]

with

\[ R(z, r_0) = \frac{2r_0}{\sqrt{PC(z)(1-z)^2}} \frac{\sqrt{B(z)|A(r_0)|[\hat{E}D(z) + \hat{L}A(z)]}}{\sqrt{D^2(z) + A(z)C(z)}}, \quad (27) \]

\[ f(z, r_0) = \frac{1}{\sqrt{\text{sgn}(A(r_0))[A(r_0) - A(z)\frac{PC(r_0)}{PC(z)} + \frac{2\hat{E}\hat{L}[A(z)D(r_0) - A(r_0)D(z)]]}}}. \quad (28) \]

Obviously, the function \( R(z, r_0) \) is regular for all values of \( z \) and \( r_0 \). However, the function \( f(z, r_0) \) diverges as \( z \) tends to zero, i.e., as the photon approaches the photon sphere. Thus, the integral (26) can be separated.
FIG. 2: Variation of the coefficients $\bar{a}, \bar{b}$ for the strong gravitational lensing with the parameter $a$ in different plasma medium and vacuum of the Kerr black hole. Parameter $\hat{E} = \omega_{\omega_{p}}$ describe the ratio of photon frequency to the plasma frequency.

into two parts $I_{D}(r_{0})$ and $I_{R}(r_{0})$

$$I_{D}(r_{0}) = \int_{0}^{1} R(0, r_{ps})f_{0}(z, r_{0})dz,$$
$$I_{R}(r_{0}) = \int_{0}^{1} [R(z, r_{0})f(z, r_{0}) - R(0, r_{0})f_{0}(z, r_{0})]dz. \quad (29)$$

Expanding the argument of the square root in $f(z, r_{0})$ to the second order in $z$, we have

$$f_{0}(z, r_{0}) = \frac{1}{\sqrt{p(r_{0})z + q(r_{0})z^2}}. \quad (30)$$

where

$$p(r_{0}) = \frac{r_{0}}{PC(r_{0})}\left\{A(r_{0})PC''(r_{0}) - A'(r_{0})PC(r_{0}) + 2\hat{E}\hat{L}[A'(r_{0})D(r_{0}) - A(r_{0})D'(r_{0})]\right\},$$
$$q(r_{0}) = \frac{r_{0}}{2PC^{2}(r_{0})}\left\{2\left(PC(r_{0}) - r_{0}PC'(r_{0})\right)\left[A(r_{0})PC'(r_{0}) - A'(r_{0})PC(r_{0})\right] + 2\hat{E}\hat{L}[A'(r_{0})D(r_{0}) - A(r_{0})D'(r_{0})]\right\} + r_{0}PC(r_{0})\left[A(r_{0})PC''(r_{0}) - A''(r_{0})PC(r_{0})\right] + 2\hat{E}\hat{L}[A''(r_{0})D(r_{0}) - A(r_{0})D''(r_{0})]\right\}. \quad (31)$$

From Eq. (31), we can find that if $r_{0}$ approaches the radius of photon sphere $r_{ps}$ the coefficient $p(r_{0})$ vanishes and the leading term of the divergence in $f_{0}(z, r_{0})$ is $z^{-1}$, which implies that the integral (26) diverges.
FIG. 3: Variation of deflection angles $\alpha(\theta)$ evaluated at $u = u_{ps} + 0.00326$ with the parameter $a$ in different plasma medium and vacuum of the Kerr black hole. Parameter $\hat{E} = \frac{\omega}{\omega_p}$ describe the ratio of photon frequency to the plasma frequency.

logarithmically. The coefficient $q(r_0)$ takes the form

$$q(r_{ps}) = \frac{\text{sgn}(A(r_{ps}))r_{ps}^2}{2PC(r_{ps})}\left\{A(r_{ps})PC''(r_{ps}) - A''(r_{ps})PC(r_{ps}) + 2\hat{E}\hat{L}[A''(r_{ps})D(r_{ps}) - A(r_{ps})D''(r_{ps})]\right\},$$

(32)

Therefore, the deflection angle in the strong field region can be expressed as

$$\alpha(\theta) = -\tilde{a} \log \left(\frac{\theta D_{OL}}{u_{ps}} - 1\right) + \tilde{b} + O(u - u_{ps}),$$

(33)
with
\[
\bar{a} = \frac{R(0, r_{ps})}{2\sqrt{q(r_{ps})}},
\]
\[
\bar{b} = -\pi + b_R + \bar{a} \log\left\{ \frac{2q(r_{ps})PC(r_{ps})}{u_{ps}\sqrt{E^2 - 1}A(r_{ps})[\hat{E}D(r_{ps}) + \hat{L}_{ps}A(r_{ps})]} \right\},
\]
\[
b_R = I_R(r_{ps}),
\]
\[
u_{ps} = \frac{-\hat{E}D(r_{ps}) + \sqrt{A(r_{ps})PC(r_{ps}) + \hat{E}^2D^2(r_{ps})}}{A(r_{ps})\sqrt{E^2 - 1}},
\]
(34)

where the quantity \(D_{OL}\) is the distance between observer and gravitational lens. Making use of Eqs. (33) and (34), we can study the properties of strong gravitational lensing in the rotating Kerr black hole in presence of homogenous plasma. In Fig. (2), we plot the changes of the coefficients \(\bar{a}\) and \(\bar{b}\) with \(a\) for a different ratio of photon frequency to plasma frequency \(\omega_p\). It is shown that the coefficients (\(\bar{a}\) and \(\bar{b}\)) in the strong field limit are functions of the parameters \(a\) and \(\hat{E}\). Compared with the vacuum the case, the presence of plasma can increase the coefficients \(\bar{a}\) and \(\bar{b}\). It is also shown that the growth of the coefficients \(\bar{a}\) and \(\bar{b}\) in prograde orbit\((a > 0)\) is less than the case in retrograde orbit\((a < 0)\). Especially, for the extreme black hole\((a=0.5)\), the coefficients \(\bar{a}\) and \(\bar{b}\) in plasma medium is the same as in vacuum for the prograde orbit. With the help of the coefficients \(\bar{a}\) and \(\bar{b}\), we plot the change of the deflection angles evaluated at \(u = u_{ps} + 0.00326\) with the rotational parameter \(a\) for a different ratio of photon frequency to plasma frequency \(\omega_p\) in Fig. (3). It is shown that in the strong field limit the deflection angles \(\alpha(\theta)\) have the similar properties of the coefficient \(\bar{a}\).

### B. Observable quantities of strong gravitational lensing

Let us now to study the effect of the homogeneous plasma medium on the observable quantities of strong gravitational lensing. Here we consider only the case in which the source, lens and observer are highly aligned so that the lens equation in strong gravitational lensing can be approximated well as

\[
\gamma = \frac{D_{OL} + D_{LS}}{D_{LS}}\theta - \alpha(\theta) \mod 2\pi,
\]
(35)

where \(D_{LS}\) is the lens-source distance and \(D_{OL}\) is the observer-lens distance, \(\gamma\) is the angle between the direction of the source and the optical axis, \(\theta = u/D_{OL}\) is the angular separation between the lens and the image. Following ref. [22], we can find that the angular separation between the lens and the \(n\)-th relativistic image is

\[
\theta_n \simeq \theta_n^0\left( 1 - \frac{\nu_{ps}e_n(D_{OL} + D_{LS})}{\bar{a}D_{OL}D_{LS}} \right),
\]
(36)
FIG. 4: Variation of the innermost relativistic image $\theta_\infty$, the relative magnitudes $r_m$ and the angular separation $s$ with the parameter $a$ in different plasma medium and vacuum of the Kerr black hole. Parameter $\tilde{E} = \frac{\omega}{\omega_p}$ describe the ratio of photon frequency to the plasma frequency.

with

$$\theta_n^0 = \frac{u_p s}{D_{OL}}(1 + e_n), \quad e_n = e^{-\frac{\delta + \gamma - 2\pi n}{\alpha}}, \quad (37)$$

where the quantity $\theta_n^0$ is the image positions corresponding to $\alpha = 2n\pi$, and $n$ is an integer. According to the past oriented light ray which starts from the observer and finishes at the source the resulting images stand
on the eastern side of the black hole for direct photons \((a > 0)\) and are described by positive \(\gamma\). Retrograde photons \((a < 0)\) have images on the western side of the compact object and are described by negative values of \(\gamma\). In the limit \(n \to \infty\), we can find that \(e_n \to 0\), which means that the relation between the minimum impact parameter \(u_{ps}\) and the asymptotic position of a set of images \(\theta_{\infty}\) can be simplified further as

\[
u_{ps} = D_{OL}\theta_{\infty}. \tag{38}\]

In order to obtain the coefficients \(\bar{a}\) and \(\bar{b}\), we need to separate at least the outermost image from all the others. As in refs. [21, 22], we consider here the simplest case in which only the outermost image \(\theta_1\) is resolved as a single image and all the remaining ones are packed together at \(\theta_{\infty}\). Thus the angular separation between the first image and other ones can be expressed as

\[s = \theta_1 - \theta_{\infty} = \theta_{\infty}e^{\frac{r_{ps}}{\bar{a}}}. \tag{39}\]

By measuring \(s\) and \(\theta_{\infty}\), we can obtain the strong deflection limit coefficients \(\bar{a}\), \(\bar{b}\) and the minimum impact parameter \(u_{ps}\). Comparing their values with those predicted by the theoretical models, we can obtain information of Kerr Black hole.

The mass of the central object of our Galaxy is estimated recently to be \(4.4 \times 10^6 M_\odot\) and its distance is around 8.5kpc, so that the ratio of the mass to the distance \(M/D_{OL} \approx 2.4734 \times 10^{-11}\). Making use of Eqs. (34), (38) and (39) we can estimate the values of the coefficients and observable quantities for gravitational lens in the strong field limit. The numerical value for the angular position of the relativistic images \(\theta_{\infty}\), the angular separation \(s\) and the relative magnitudes \(r_m\) are plotted in Fig. 1. we find that the variation of the angular position of the relativistic images \(\theta_{\infty}\) with the rotational parameter \(a\) in different plasma medium and vacuum is similar to that of the photon-sphere radius \(r_{ps}\). However, the variation of the relative magnitudes \(r_m\) is contrary to the case of the photon-sphere radius \(r_{ps}\). We also find that the variation of the angular separation \(s\) with the rotational parameter \(a\) in different plasma medium and vacuum is similar to that of the deflection angle \(\alpha(\theta)\).

IV. SUMMARY

In this paper, we have investigated the strong gravitational lensing of Kerr black hole surrounded by homegeneous plasma. We derived the expression for the deflection angle of light in Kerr black hole in the presence of homogeneous plasma and numerically calculated the coefficient of the deflection angle. It is shown that the presence of the uniform plasma increases the photon-sphere radius \(r_{ps}\), the coefficient \(\bar{a}\), \(\bar{b}\), the angular
position of the relativistic images $\theta_{\infty}$, the deflection angle $\alpha(\theta)$ and the angular separation $s$. However the relative magnitudes $r_m$ decrease in presence of the uniform plasma medium. It is also shown that the impact of the uniform plasma on the effect of strong gravitational become smaller as the spin of Kerr black increase in prograde orbit ($a > 0$). Especially, for the extreme black hole ($a=0.5$), the effect of strong gravitational lensing in homogenous plasma medium is the same as the case in vacuum for the prograde orbit. In reality the plasma in the neighborhood of the compact objects can be significantly non-homogeneous. Such cases can be calculated numerically and much more complicated, in the later research work we will consider the influence of the non-homogeneous plasma medium on the strong gravitational lensing of Kerr black hole.

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