Scaling and Asymptotic Scaling in the SU(2) Gauge Theory

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\textbf{ABSTRACT}

We determine the critical couplings for the deconfinement phase transition in SU(2) gauge theory on $N_\tau \times N_\sigma^3$ lattices with $N_\tau = 8$ and 16 and $N_\sigma$ varying between 16 and 48. A comparison with string tension data shows scaling of the ratio $T_c/\sqrt{\sigma}$ in the entire coupling regime $\beta = 2.30 - 2.75$, while the individual quantities still exhibit large scaling violations. We find $T_c/\sqrt{\sigma} = 0.69(2)$. We also discuss in detail the extrapolation of $T_c/\Lambda_{\text{MS}}$ and $\sqrt{\sigma}/\Lambda_{\text{MS}}$ to the continuum limit. Our result, which is consistent with the above ratio, is $T_c/\Lambda_{\text{MS}} = 1.23(11)$ and $\sqrt{\sigma}/\Lambda_{\text{MS}} = 1.79(12)$. We also comment upon corresponding results for SU(3) gauge theory and four flavour QCD.
1. Introduction

Ever since the pioneering work of M. Creutz [1] the approach to asymptotic scaling, and thus the continuum limit, was one of the central issues in studies of gauge theories on the lattice. Although the first results were promising, it soon became clear that simulations on large lattices are needed in order to establish asymptotic scaling for asymptotically free quantum field theories such as QCD. In fact, recent numerical studies of the $O(3)$ $\sigma$-model in two dimensions [2] suggest that the asymptotic scaling regime may not be reached even for quite large correlation length while at the same time ratios of physical observables show scaling behaviour.

The lack of asymptotic scaling as well as the scaling of certain ratios of physical observables has also been observed in $SU(N)$ gauge theories. In particular in the case of the $SU(3)$ gauge theory the deviations from asymptotic scaling are large and have been noticed early as a dip in the discrete $\beta$-function [3,4], which led to deviations from asymptotic scaling by more than 50% for certain values of the gauge couplings. Although the dip is not that pronounced for $SU(2)$ gauge theories, there are clear deviations from asymptotic scaling seen even for the largest values of the coupling, $\beta = 2N/g^2$ for colour group $SU(N)$, studied so far. In particular the analysis of the heavy quark potential and the string tension has been performed up to couplings as large as $\beta = 2.85$ [5]. Still there is no hint for asymptotic scaling at this large $\beta$-value, which already corresponds to lattice spacings as small as $\sim 0.05$ fm. Furthermore, an analysis of the short distance part of the heavy quark potential suggests that the approach to the continuum limit may even be as slow as in the two-dimensional $\sigma$-model [6].

One of the best studied quantities in $SU(2)$ gauge theory is the finite temperature deconfinement phase transition. For lattices of size $N_\tau \times N_\sigma^3$ with $N_\tau \leq 6$ the critical coupling has been determined with high accuracy [7,8] and an extrapolation to spatially infinite volume could be performed. Moreover, an analysis of the critical exponents at the transition point were in perfect agreement with those of the three-dimensional Ising model. Here it turned out that the Binder cumulant of the order parameter is an observable which is well suited to locate the critical coupling for given $N_\tau$ as finite spatial size corrections are only due to the presence of irrelevant operators.

So far the analysis of the scaling of the ratio $T_c/\sqrt{\sigma}$ was limited to a rather small coupling regime in the case of $SU(2)$ as the critical couplings for the decon-
finement transition have been determined only for \( N_\tau \leq 6 \). It is the purpose of this paper to further investigate the scaling properties of the \( SU(2) \) gauge theory. We will extend earlier studies of the deconfinement transition to lattices up to a temporal size \( N_\tau = 16 \). This will enable us to perform a quantitative test of scaling in \( SU(2) \). Furthermore, we can follow the apparent scaling violations over a large range of couplings, which allows us to analyze various extrapolation schemes to extract \( T_c/\Lambda_{\overline{\text{MS}}} \) in the continuum limit.

This paper is organized as follows: In the next section we discuss our strategy of calculating the critical couplings for the deconfinement transition on lattices with large temporal extent. In particular we discuss the finite-size scaling of cumulants of the Polyakov loop expectation value. In section 3 we present our numerical results. Section 4 is devoted to a discussion of scaling and asymptotic scaling and a detailed discussion of the extrapolation of these results to the continuum limit. Finally section 5 contains our conclusions.

2. Finite-Size Scaling and the Continuum Limit

Usually finite-size scaling (FSS) in the vicinity of a finite temperature phase transition is discussed for lattice \( SU(N) \) gauge models, without trying to make contact with the continuum limit, \( i.e., \) the scaling properties are studied on lattices of size \( N_\tau \times N_\sigma^d \) with fixed \( N_\sigma \) and varying \( N_\tau \), where \( d \) denotes the spatial dimension and the model is viewed as a \( d \)-dimensional spin system. In the continuum limit the FSS properties of these non-abelian models should, of course, be discussed in terms of the physical volume, \( V = L^d \), and the temperature, \( T \), in the vicinity of the deconfinement transition temperature \( T_c \). We will study here how the scaling behaviour of the continuum theory emerges from the lattice free energy on arbitrary lattices, \( i.e., \) when varying \( N_\tau \) and \( N_\sigma \).

For a continuum theory having a simple critical point and a characteristic length \( L = V^{1/d} \) the singular part of the free energy density,

\[
 f_s = \frac{F_s}{TV} = -\frac{\ln Z_s}{TV} ,
\]

is described by a universal finite-size scaling form \([9,10]\),

\[
 f_s(T, H; L) = L^{-d} Q_{f_s} \left( g_T L^{\frac{1}{\nu}}, g_H L^{\frac{\beta + \gamma}{\nu}} \right) .
\]

Here we assume that corrections to scaling from irrelevant scaling fields \( \xi, L^{\gamma_\nu} \), proportional to negative powers of \( L \), can be neglected; for \( N_\tau = 6 \) a value of
$y_1 = -0.9$ has been found for the $SU(2)$ gauge theory [8], showing that irrelevant contributions disappear rather fast with increasing $N_\sigma$.

On a lattice of size $N_\tau \times N^{d}_\sigma$ the length scale $L$ and the temperature $T$ are given in units of the lattice spacing

$$L = N_\sigma a ,$$

$$T^{-1} = N_\tau a .$$

In general the lattice spacing $a$ is a complicated function of the coupling $\beta = 2N/g^2$. The dependence of the lattice spacing on $\beta$ is known only in the continuum limit in the form of the renormalization group equation

$$a\Lambda_L = \left( \frac{\beta}{2Nb_0} \right)^{b_1/2b_0^2} \exp \left( -\frac{\beta}{4Nb_0} \right) .$$

Therefore it is advantageous to replace the length scale $L$ by the dimensionless combination,

$$L \cdot T = \frac{N_\sigma}{N_\tau} .$$

Using this ratio in the FSS relation for the singular part of the free energy density we get

$$f_s(t, h; N_\sigma, N_\tau) = \left( \frac{N_\sigma}{N_\tau} \right)^{-d} Q_{f_s} \left( g_t \left( \frac{N_\sigma}{N_\tau} \right)^{\frac{1}{\nu}}, g_h \left( \frac{N_\sigma}{N_\tau} \right)^{\frac{\beta+\gamma}{\nu}} \right) .$$

The scaling function $Q_{f_s}$ depends on the temperature $T$ and the external field strength $h$ through thermal and magnetic scaling fields,

$$g_t = c_t t(1 + b_t t) + O(t^3) ,$$

$$g_h = c_h h(1 + b_h h) + O(t^2 h^2, h^2) ,$$

with non-universal metric coefficients $c_t$, $c_h$, $b_t$ and $b_h$ still carrying a possible $N_\tau$ dependence. Here $t$ is the reduced temperature, $t = (T - T_c)/T_c$, which in the neighbourhood of the transition point can be approximated by

$$t = (\beta - \beta_{c,\infty}) \frac{1}{4Nb_0} \left[ 1 - \frac{2Nb_1}{b_0} \beta_{c,\infty}^{-1} \right] .$$

This approximation reproduces the correct reduced temperature in the continuum limit, which is easily verified by using Eq. (5). We note that $t$ has $O(\beta_{c,\infty}^{-1})$
corrections to the leading term, which, however, contribute less than 8\% in the relevant coupling regime, \( i.e., \beta > 2.0 \) for \( SU(2) \) and \( \beta > 5.5 \) for \( SU(3) \). This non-leading term introduces a logarithmic dependence of \( t \), and thus \( Q_{fs} \), on \( N_\tau \) through Eqs. (4) and (5). A priori we cannot exclude that this violation of the otherwise universal \( N_\sigma/N_\tau \) dependence is enhanced in the non-asymptotic scaling regime. However, as we will show later we do not find any hints for this in our Monte Carlo data.

A non-vanishing magnetic field strength \( h \) corresponds to adding a symmetry breaking term of the form \( hZ(a, N_\tau)N_\sigma^dP \) to the action. Here \( P \) denotes the Polyakov loop, defined as

\[
P = N_\sigma^{-d} \sum_{\vec{x}} \prod_{x_0=1}^{N_\tau} U(x_0, \vec{x}), 0
\]

and \( Z(a, N_\tau) \) is a renormalization factor necessary to remove divergent self-energy contributions to the Polyakov loop, which represents a static heavy quark source. A physical order parameter, \( \langle P_p \rangle \), not vanishing in the continuum limit, a susceptibility \( \chi_p \) and a normalized fourth cumulant \( g_4 \) may then be defined through derivatives of \( f_s \) with respect to the external magnetic field strength \( h \) at \( h = 0 \),

\[
\langle P_p \rangle = N_\tau^d Z(a, N_\tau) \langle P \rangle = -\frac{\partial f_s}{\partial h}\bigg|_{h=0},
\]

\[
\chi_p = N_\tau^d Z(a, N_\tau)^2 \chi = \frac{\partial^2 f_s}{\partial h^2}\bigg|_{h=0},
\]

\[
g_4 = \frac{\partial^4 f_s}{\partial h^4}\bigg|_{h=0} \left/ \left( \chi_p^2 \left( \frac{N_\sigma}{N_\tau} \right)^d \right) \right. \right.
\]

For the cumulant \( g_4 \) the renormalization factors cancel and we end up with the usual expression known for the Binder cumulant of the order parameter [11,12,13]. The general form of the scaling relations derived from Eq. (7) is

\[
\frac{\partial^n f_s}{\partial h^n}\bigg|_{h=0} = \left( \frac{N_\sigma}{N_\tau} \right)^{n \frac{\beta+\gamma}{\nu} - d} \cdot Q_n \left( g_t \left( \frac{N_\sigma}{N_\tau} \right)^{\frac{1}{\nu}} \right)
\]

where the function \( Q_n \) is defined as

\[
Q_n(x_1) = \frac{\partial^n}{\partial x_2^n} Q_{fs}(x_1, x_2)\bigg|_{x_2=0}.
\]
The finite-size scaling behaviour of higher order cumulants which are closely related to the derivatives defined in Eq. (15) is discussed in appendix A. We note that for $g_4$, as well as the higher cumulants, the prefactors of the scaling functions cancel. Thus they take on unique fixed point values at $g_t = 0$ even on finite lattices, if we ignore corrections from irrelevant operators. The cumulants are thus well suited to determine the critical coupling from simulations on finite lattices.

Let us first consider the case when $N_\tau$ is kept fixed. Then $N_\tau$ can be absorbed in the non-universal constants in $g_t$ and $g_h$ and we end up with the usual form of the finite-size scaling ansatz for $f_s$. The critical coupling can be determined from the fixed point of $g_4(\beta, N_\sigma)$ \[8,11,13,14\].

Next we consider $y = N_\sigma / N_\tau$ fixed, varying $N_\sigma$ and therefore $N_\tau$ accordingly as is needed to reach the continuum limit. Rescaling $N_\sigma$ and $N_\tau$ by a factor $b$ leads to a phenomenological renormalization $\tilde{\beta}(\beta; b; y)$ by the following identity for a scaling function $Q$

\[
Q(g_t(\beta, N_\tau) \left( \frac{N_\sigma}{N_\tau} \right)^{\frac{1}{2}}) = Q(g_t(\tilde{\beta}, bN_\tau) \left( \frac{bN_\sigma}{bN_\tau} \right)^{\frac{1}{2}}). \tag{17}
\]

Here we have kept explicit the dependence of $g_t$ on $N_\tau$ that comes in through the non-universal metric coefficients $c_t$ and $b_t$ in Eq. (8).

The property that the normalized fourth cumulant $g_4$ is directly a scaling function can be used to measure the discrete $\beta$-function, $\Delta \beta$, by using the above identity for $Q = g_4$. By writing Eq. (17) in terms of the scaling function $Q$ and not for the scaling field $g_t$ directly we do not have to determine the metric coefficients in Eq. (8). The discrete $\beta$-function is given by the shift in the coupling $\beta$, which is necessary to get the same value of $g_4$ for the two different lattice sizes

\[
g_4(\beta - \Delta \beta_y(\beta); N_\sigma; N_\tau) = g_4(\beta; bN_\sigma; bN_\tau). \tag{18}
\]

The function $\Delta \beta_y$ defined in Eq. (18) may depend on $y$ through contributions from irrelevant scaling fields $g_i(\beta, N_\tau)N_\sigma^{y_i}$ with negative exponents $y_i$. The discrete $\beta$-function can then be obtained by an extrapolation to $y = \infty$

\[
\Delta \beta(\beta) = \lim_{y \to \infty} \Delta \beta_y(\beta). \tag{19}
\]

The knowledge of $\beta_{c,\infty}(N_\tau)$ and $\Delta \beta$ allows to calculate the critical coupling for the rescaled lattice size

\[
\beta_{c,\infty}(bN_\tau) = \beta_{c,\infty}(N_\tau) + \Delta \beta. \tag{20}
\]
In the following we will use this approach to determine $\beta_{c,\infty}$ for $N_\tau = 8$ and 16. In particular we will check the $y$-independence for $N_\tau \leq 8$ and use this information to justify our calculations for $N_\tau = 16$ with moderate values for $y$.

3. Critical couplings and the Binder Cumulant for $N_\tau = 8$ and $N_\tau = 16$

As discussed in the previous section the Binder cumulants are well suited to determine the critical couplings for the deconfinement transition. Previously they have been used to determine the critical couplings for the $SU(2)$ gauge theory on lattices with temporal extent $N_\tau = 4$ and 6 [7,8]. We follow here the same strategy to determine the critical coupling for $N_\tau = 8$. We have performed simulations on lattices of size $8 \times N_\sigma^3$ with $N_\sigma = 16$, 24 and 32 at several values of $\beta$ in the vicinity of the estimated critical point. We have used an overrelaxed heat-bath algorithm with 14 overrelaxation steps between subsequent “incomplete” heat-bath updates [15]. We used the Kennedy-Pendleton [16] algorithm with one trial per link. The acceptance rate was always larger than 96%. Most runs were done on the massively parallel CM-2 and so the use of a complete heat-bath algorithm would have led to a considerable waste of resources. Measurements were taken after each heat-bath update. Details on our parameter choices and number of iterations are given in Table 1, where we also give results for the estimated integrated autocorrelation times for the expectation value of the Polyakov loop.

In Fig. 1 we show the Binder cumulant, which on a lattice is measured by

$$g_4 = \frac{\langle P^4 \rangle}{\langle P^2 \rangle^2} - 3,$$

(21)

with the Polyakov loop, $P$, given by Eq. (11). Also shown in this figure is an interpolation between results obtained at the various values of $\beta$, which is based on the density of states method (DSM) [17]. Our implementation of the DSM takes into account each measurement and does not require the usual division of the range of the expectation value of the plaquette operator $U_p = N^{-1} \text{Tr} U_1 U_2 U_3 U_4^\dagger$ in bins. This corresponds to an infinite number of bins and has the advantage to remove a free parameter, the number of bins, from the method. Because of the finite length of the Monte Carlo runs, the DSM provides reliable results only if histograms of $U_p$ belonging to adjacent couplings overlap. We convinced ourselves that at least 2.5% of the data of one sample is contained in each of the overlapping tails of the distribution.
Table 1: Run parameters.

| $N_\sigma$ | $N_\tau$ | $\beta$ | $N_{\text{meas}}$ | $\tau_{\text{int}}$ |
|------------|----------|--------|-----------------|-----------------|
| 16         | 8        | 2.500  | 25010           | 3.4             |
| 16         | 8        | 2.510  | 21055           | 3.7             |
| 16         | 8        | 2.520  | 21200           | 3.6             |
| 24         | 8        | 2.500  | 55500           | 10.6            |
| 24         | 8        | 2.505  | 29500           | 10.0            |
| 24         | 8        | 2.510  | 29500           | 13.5            |
| 24         | 8        | 2.515  | 39500           | 9.8             |
| 24         | 8        | 2.520  | 14500           | 14.9            |
| 32         | 8        | 2.500  | 30354           | 29.6            |
| 32         | 8        | 2.505  | 34750           | 17.1            |
| 32         | 8        | 2.510  | 38540           | 15.3            |
| 32         | 8        | 2.515  | 28750           | 18.0            |
| 32         | 8        | 2.520  | 18800           | 30.0            |
| 32         | 16       | 2.720  | 30910           | 10.9            |
| 32         | 16       | 2.740  | 40350           | 12.6            |
| 32         | 16       | 2.750  | 35000           | 12.0            |
| 48         | 16       | 2.740  | 43850           | 45.0            |

The DSM allows an accurate determination of the intersection points of $g_4(\beta; N_\sigma; N_\tau)$ and $g_4(\beta; bN_\sigma; N_\tau)$, where $N_\sigma$ is the spatial size of the smaller lattice and $b$ is given by the ratio $N_\sigma'/N_\sigma$. Taking also the largest irrelevant scaling field into account the coupling $\beta_c(N_\sigma, b)$ at which two cumulants intersect varies with $N_\sigma$ and $b$ as [8,11,13]

$$\beta_c(N_\sigma, b) = \beta_{c,\infty}(1 - ae)$$

$$\epsilon = N_\sigma^{y_1-1/\nu} \frac{1 - b^{y_1}}{b^{1/\nu} - 1}. \quad (22)$$

Using the value $\nu = 0.628$ calculated for the three-dimensional Ising model [14] and $y_1 = -1$ [8] we determine the critical coupling for $N_\tau = 8$ by an extrapolation to $\epsilon = 0$ with the result

$$\beta_{c,\infty}(N_\tau = 8) = 2.5115 \pm 0.0040. \quad (23)$$

The error has been obtained from a weighted linear fit to $\beta_c(N_\sigma, b)$, where the error for the intersection points has been calculated using the jackknife method. The extrapolation is shown in Fig. 2.
Figure 1. The cumulant $g_4$ for $N_\tau = 8$ and various values of the spatial lattice size as a function of the coupling $\beta$. Solid curves are interpolation curves based on the density of states method. The dashed lines indicate the error on the curves estimated by the jackknife method.

Taking into account the error band on the continuous curves obtained by the density of states method shown in Fig. 1 as well as the error on $\beta_{c,\infty}$ we obtain for the fixed point value of $g_4$,

$$\overline{g}_4(N_\tau) = \lim_{N_\sigma \to \infty} g_4(\beta_{c,\infty}; N_\sigma; N_\tau)$$

$$\overline{g}_4(N_\tau = 8) = -1.48 \pm 0.10 \quad (24)$$ \quad (25)

From the FSS theory as discussed in section 2 we expect, in fact, that $\overline{g}_4$ is independent of $N_\tau$, i.e., is universal also in the continuum limit. Indeed, this seems to be supported by our data. In Table 2 we give results for $\overline{g}_4(N_\tau)$ obtained from simulations with $N_\tau = 4$, 6 and 8.

All three values agree reasonably well within errors, although there seems to be a tendency to lower values of $\overline{g}_4$ for larger $N_\tau$. At least partially this may be related to still too-low statistics for lattices with large $N_\tau$ as $g_4$ is a non-self-
Figure 2. The coupling value at the intersection points of $g_4$ for $N_\sigma = 16, 24$ and 32 as a function of $\epsilon$ which is defined in Eq. (22). The critical coupling $\beta_{c,\infty}$ can be read from the figure as the section of the y axis at $\epsilon = 0$.

Table 2: Fixed point value of $g_4$ for different $N_\tau$.

| $N_\tau$ | $\bar{g}_4$       |
|----------|-----------------|
| 4        | $-1.38(3)$      |
| 6        | $-1.43(8)$      |
| 8        | $-1.48(10)$     |

As discussed in section 2, $g_4$ is a scaling function, which in the continuum limit
depends only on the reduced temperature, \( t = (T - T_c)/T_c \), and the dimensionless quantity \( y = LT \). In terms of lattice variables, \( y \) is given by
\[
y = \frac{N_{\sigma}}{N_{\tau}},
\]
and the reduced temperature in the asymptotic scaling regime is given by Eq. (10). For small values of \( t \) one expects,
\[
g_4(t, y) = g_4(ty^{1/\nu}) = \overline{g}_4 + g_{4,1}ty^{1/\nu}.
\]
This universal scaling behaviour indeed seems to be fulfilled quite well, as can be seen from Fig. 3, where we show \( g_4(t, y) \) for various values of \( N_{\tau} \) and \( N_{\sigma} \). The critical exponent \( \nu \) has been taken from the three-dimensional Ising model [14]. Otherwise the presentation in Fig. 3 is parameter-free. We note that indeed the slope \( g_{4,1} \) seems to be universal and within our accuracy does not show any further \( N_{\tau} \) or even \( N_{\sigma} \) dependence.

The universal scaling behaviour of \( g_4 \) in the vicinity of the critical point can be explored to determine the critical couplings for larger values of \( N_{\tau} \) without an explicit determination of the fixed point \( \overline{g}_4 \) from simulations on lattices of varying spatial size. We have calculated \( g_4 \) at three \( \beta \) values on a \( 16 \times 32^3 \) lattice and at one \( \beta \) value on a \( 16 \times 48^3 \) lattice. Details for these runs are also given in Table 1. In Fig. 4 we show the Binder cumulant as a function of \( \beta \). The critical coupling \( \beta_c(N_{\tau} = 16) \) can now be determined from the shift of \( \beta \) required to overlay the data of Fig. 4 with those for \( N_{\sigma} = 16 \) and 24 shown in Fig. 1. A matching procedure according to Eq. (18) gives the shift values \( \Delta \beta_{y=2} = 0.232(2) \) and \( \Delta \beta_{y=3} = 0.224(2) \). The two values show a noticeable difference, although the statistical error is relatively large. To investigate whether the observed \( y \) dependence is of significance, we checked if this is present also for \( N_{\tau} = 4, 6 \) and 8. We determined the shift \( \Delta \beta_y \) for \( y = 2, 3 \) and 4 using data from [7,8] and found that there is no significant \( y \) dependence of \( \Delta \beta_y \). Furthermore \( \Delta \beta_y \) agrees with the shift \( \Delta \beta = \beta_{c,\infty}(N_{\tau,1}) - \beta_{c,\infty}(N_{\tau,2}) \) calculated from the infinite volume critical coupling. For the largest lattice size \( N_{\tau} = 16 \) the critical slowing down is most severe, as is seen from \( \tau_{int} \) given in Table 1. As mentioned before \( g_4 \) is a non-self-averaging quantity and, as for the susceptibility, the expectation value will be too small, if one has not enough independent measurements [18]. This seems to be true for our dataset on the \( N_{\sigma} = 48, N_{\tau} = 16 \) lattice and we can not obtain the same precision as in the previous cases of \( N_{\tau} = 4, 6 \) and 8.
Figure 3. The Binder cumulant $g_4$ as a function of $t y^{1/\nu}$ for various lattice sizes as given in the figure. The critical exponent $\nu$ has been taken to be the one of the three-dimensional Ising model, $\nu = 0.628$. For $N_\tau = 16$ and $N_\sigma = 32$ and 48 we also mark the error in $x$ direction caused by the uncertainty in the critical coupling $\beta_{c, \infty}$.

Averaging over the two values of $\Delta \beta_y$ at $y = 2$ and 3 and assigning a large error, which includes both numbers, we get $\Delta \beta(N_\tau = 16) = 0.228(6)$, which is consistent with results obtained on lattices of size $32^4$ [20]. Using Eqs. (20) and (23) this corresponds to a critical coupling,

$$\beta_{c, \infty}(N_\tau = 16) = 2.7395 \pm 0.0100.$$  \hspace{1cm} (30)

We want to point out that the combination of Monte Carlo simulation and the DSM is especially suited to determine the intersection points of $g_4$ for small volumes. If the system becomes too large, the widths of the histograms of $U_p$ become very small and one needs a lot of simulations at different $\beta$ values to cover a given range of the coupling. This was the case for our results on lattices with $N_\tau = 16$ and for these we could only use the single histogram version of the DSM.
Figure 4. The cumulant $g_4$ for $N_\tau$ and $N_\sigma = 32$ (dots) and 48 (star) as a function of $\beta$. The straight line represents a linear fit to the data points for $N_\sigma = 32$. The four curves through the data points are obtained using the single histogram version of the DSM. The range of validity follows from the condition that at least 2.5% of the measurements in the data sample is contained in each tail of the distribution of $U_p$. This gives the range of validity in $U_P$, which is converted to a coupling range using the computed energy coupling relation. The dashed lines indicate the error on these curves.

4. Scaling and Asymptotic Scaling

The critical couplings determined in the previous section can be used to test scaling and asymptotic scaling in $SU(2)$ gauge theory over a much wider range of the coupling than was previously possible. Additionally we will compare the $SU(2)$ results to existing data for $SU(3)$ and four flavour QCD. After that we will try to point out common features of the data.

Recently the heavy quark potential has been studied in the pure $SU(2)$ gauge theory on quite large lattices and several values of the gauge coupling [5,6,21,22,23], which cover the range of couplings we have studied here. The string tension
extracted from the long distance part of the heavy quark potential can be used to test the scaling of dimensionless ratios of physical quantities. In particular we will discuss here the scaling of $T_c/\sqrt{\sigma}$. In Table 3 and Table 4 we have collected results from most recent calculations of the string tension in $SU(2)$ [5,6,21,22,23] as well as $SU(3)$ [24,25] gauge theory. For four flavour QCD we used the value $\sqrt{\sigma} a = 0.332(2)$ [26].

| $N_\sigma$ | $N_\tau$ | $\beta$ | $\sqrt{\sigma} a$ |
|---|---|---|---|
| 8  | 10  | 2.20 | 0.4690(100) |
| 10 | 10  | 2.30 | 0.3690( 30) |
| 16 | 16  | 2.40 | 0.2660( 20) |
| 32 | 32  | 2.50 | 0.1905(  8) |
| 20 | 20  | 2.60 | 0.1360( 40) |
| 32 | 32  | 2.70 | 0.1015( 10) |
| 48 | 56  | 2.85 | 0.0630( 30) |

**Table 3:** $SU(2)$ string tension.

| $N_\sigma$ | $N_\tau$ | $\beta$ | $\sqrt{\sigma} a$ |
|---|---|---|---|
| 16 | 32  | 5.60 | 0.5295(  9) |
| 16 | 24  | 5.70 | 0.4099(12) |
| 16 | 24  | 5.80 | 0.3302(15) |
| 16 | 24  | 5.90 | 0.2702(19) |
| 32 | 32  | 6.00 | 0.2269(62) |
| 24 | 32  | 6.20 | 0.1619(19) |
| 32 | 32  | 6.40 | 0.1214(12) |
| 32 | 32  | 6.80 | 0.0738(20) |

**Table 4:** $SU(3)$ string tension.

We have used a spline interpolation between these values to determine the string tension $\sqrt{\sigma} a(\beta)$ at the critical couplings of the deconfinement transition. These critical couplings are displayed in Table 5 and in Table 6 [27,28,29]. The resulting values for the ratio,

$$\frac{T_c}{\sqrt{\sigma}} = (\sqrt{\sigma} a(\beta_c^\infty(N_\tau)) N_\tau)^{-1},$$

(31)
are shown in Fig. 5 as a function of $aT_c \equiv 1/N_{\tau}$. Also shown in this figure is a result for four flavour QCD, which combines the measured critical coupling for $N_{\tau} = 8$ ($\beta_c = 5.15 \pm 0.05$) [30] and the result for the string tension measured at $\beta = 5.15$ on a $24 \times 16^3$ lattice [26].

It is rather remarkable that scaling of $T_c/\sqrt{\sigma}$ is valid in the case of $SU(2)$ to a high degree over the entire regime of lattice spacings from $aT_c \leq 0.25$ ($N_{\tau} \geq 4$) onwards. For $SU(3)$ this seems to hold for $aT_c \leq 0.125$ ($N_{\tau} \geq 8$), although for larger values of the lattice spacing deviations are also only on the level of a few percent. Averaging over the region $0.0625 \leq aT_c \leq 0.25$ ($4 \leq N_{\tau} \leq 16$) for the $SU(2)$ data and over $0.0714 \leq aT_c \leq 0.125$ ($8 \leq N_{\tau} \leq 14$) for $SU(3)$ we obtain

$$\frac{T_c}{\sqrt{\sigma}} = \begin{cases} 0.69 \pm 0.02 & , \text{SU}(2) \\ 0.56 \pm 0.03 & , \text{SU}(3) \\ 0.38 \pm 0.05 & , \text{QCD}, n_f = 4 \end{cases}$$

A strong decrease of this ratio with increasing number of degrees of freedom (par-

| $N_{\tau}$ | $\beta_c$ | $T_c/\Lambda_L$ | $T_c/\Lambda_{\text{MS}}$ | $T_c/\Lambda_{\text{MS}}^E$ |
|---|---|---|---|---|
| 2 | 1.8800(30) | 29.7(2) | 1.499(11) | 0.852(6) |
| 3 | 2.1768(30) | 41.4(3) | 2.089(16) | 1.213(11) |
| 4 | 2.2986(6) | 42.1(1) | 2.125(3) | 1.313(3) |
| 5 | 2.3726(45) | 40.6(5) | 2.047(23) | 1.360(21) |
| 6 | 2.4265(30) | 38.7(3) | 1.954(15) | 1.354(13) |
| 8 | 2.5115(40) | 36.0(4) | 1.815(18) | 1.325(16) |
| 16 | 2.7395(100) | 32.0(8) | 1.616(41) | 1.271(37) |

**Table 5:** Critical couplings for $SU(2)$.

| $N_{\sigma}$ | $N_{\tau}$ | $\beta_c$ | $T_c/\Lambda_L$ | $T_c/\Lambda_{\text{MS}}$ | $T_c/\Lambda_{\text{MS}}^E$ |
|---|---|---|---|---|---|
| 12 | 3 | 5.5500(100) | 85.70 ± 0.96 | 2.975 ± 0.033 | 1.217 ± 0.038 |
| $\infty$ | 4 | 5.6925(2) | 75.41 ± 0.02 | 2.618 ± 0.001 | 1.318 ± 0.012 |
| $\infty$ | 6 | 5.8941(5) | 63.05 ± 0.04 | 2.189 ± 0.001 | 1.299 ± 0.002 |
| 16 | 8 | 6.0010(250) | 53.34 ± 1.50 | 1.851 ± 0.052 | 1.152 ± 0.043 |
| 16 | 10 | 6.1600(70) | 51.05 ± 0.40 | 1.772 ± 0.014 | 1.155 ± 0.012 |
| 16 | 12 | 6.2680(120) | 48.05 ± 0.65 | 1.668 ± 0.023 | 1.110 ± 0.017 |
| 16 | 14 | 6.3830(100) | 46.90 ± 0.53 | 1.628 ± 0.018 | 1.111 ± 0.016 |
| 24 | 16 | 6.4500(500) | 46.27 ± 2.50 | 1.537 ± 0.087 | 1.064 ± 0.069 |

**Table 6:** Critical couplings for $SU(3)$.
Figure 5. The critical temperature in units of the square root of the string tension versus the $aT_c \equiv 1/N_T$ for the case of $SU(2)$ (circles) and $SU(3)$ (triangles) pure gauge theory as well as QCD with four flavours of dynamical fermions of mass $m_f = 0.01$ (cross). The straight lines correspond to one parameter fits to the data.

The critical energy density (in units of $T_c^2$) is obvious. Such a behaviour is expected, as a certain critical energy density can be reached already at a lower temperature when the number of partons is larger. Moreover, it is natural to assume that the string tension increases when the number of gluonic degrees of freedom increases. Both effects tend to lower the value of $T_c/\sqrt{\sigma}$ with increasing $N$ and/or $n_f$.

A more natural scale to compare the critical temperature in pure $SU(N)$ gauge theories is given by the glueball mass. The energy density in units of $T^4$ of a massive, free glueball gas is a function of $x = m_G/T$ only,

$$
\epsilon/T^4 = d \frac{x^3}{2\pi^2} \sum_{n=1}^{\infty} \left[ K_1(nx) + \frac{3}{nx} K_2(nx) \right],
$$

where $d$ is a degeneracy factor, $K_1$ and $K_2$ are modified Bessel functions. One thus might expect that the critical behaviour in the $SU(2)$ and $SU(3)$ theory is
controlled by the ratio $T_c/m_G$, with $m_G$ denoting the mass of the lightest glueball. In fact, using recent glueball data [21,31,32], we find that this ratio varies only slightly between $SU(2)$ and $SU(3)$,

$$\frac{T_c}{m_G} = \begin{cases} 
0.180 \pm 0.016 &, \ SU(2) \\
0.176 \pm 0.020 &, \ SU(3) 
\end{cases} \quad (34)$$

It is well known that asymptotic scaling does not hold in the coupling regime considered by us, despite the fact that scaling works remarkably well. This suggests that there are universal scaling violating terms, which are common to the different physical observables and thus may partially be absorbed in a redefinition of the coupling constant. The need for such a resummation has been noticed quite early and various schemes have been suggested [33,34,35,36,37]. In the case of the $SU(3)$ deconfinement transition a variant of the scheme originally introduced by Parisi [33] has been found to yield quite good results already for rather large lattice spacings [38]. Here the bare coupling, $\beta$, is replaced by an effective coupling, $\beta_E$, which is related to the plaquette expectation value and thus takes into account rapid fluctuations in the pure gauge action at intermediate values of the coupling. Further details on the definition of $\beta_E$ and a collection of plaquette expectation values needed to calculate $\beta_E$ are given in appendix B.

In Fig. 6 we show $T_c/\Lambda_{\overline{MS}}$, using a conversion factor $\Lambda_{\overline{MS}}/\Lambda_L$ [39] given by

$$\frac{\Lambda_{\overline{MS}}}{\Lambda_L} = 38.852704 \exp \left( -\frac{3\pi^2}{11N^2} \right) \quad (35)$$

In the case of full QCD with four flavours we used the value $\Lambda_{\overline{MS}}/\Lambda_L = 76.45$ for the conversion factor.

It is evident that the asymptotic scaling violations are much reduced when $\beta_E$ is used as a coupling constant. This is particularly true for the case of $SU(3)$ shown in Fig. 6b. The $O(g^7)$ term in the $SU(N)$ $\beta$-function would lead to violations of asymptotic scaling, which are of $O(1/\ln a)$. Our data for $T_c/\Lambda_{\overline{MS}}$ at finite values of the cut-off suggest, however, that the deviations from asymptotic scaling are well approximated by a correction term which is $O(a)$. This holds for both coupling schemes, with the bare coupling $\beta$ and with the effective coupling $\beta_E$. In the latter case, though, the coefficient of the correction term is much smaller. Performing a linear extrapolation of $T_c/\Lambda_{\overline{MS}}$ to $a = 0$ in the effective coupling scheme we find

$$\frac{T_c}{\Lambda_{\overline{MS}}} = \begin{cases} 
1.23 \pm 0.11 &, \ SU(2) \\
1.03 \pm 0.19 &, \ SU(3) 
\end{cases} \quad (36)$$
Figure 6. The critical temperature in units of $\Lambda_{\overline{\text{MS}}}$ for the SU(2) (a) and SU(3) (b) gauge theory. Shown are data obtained by using the bare coupling constant (triangles) and the effective coupling $\beta_E$ (circles) as input in the asymptotic renormalization group equation. The solid straight lines give linear extrapolations of these data sets to the continuum limit, $a = 0$, in the effective coupling scheme. The broken lines indicate a corresponding linear extrapolation using the bare coupling. The position of the filled symbols marks the result of the extrapolation. The dotted line in (a) marks how both coupling schemes approach in the continuum limit, where we assumed a linear form of $T_c/\Lambda_{\overline{\text{MS}}}$ as a function of $a$ given by the fit for the effective coupling scheme and used the third order expansion of the internal energy given in appendix B, Eq. (B.1).
where we used the region $0.0625 \leq aT_c \leq 0.1667$ ($6 \leq N_\tau \leq 16$) for the $SU(2)$ data and $0.0714 \leq aT_c \leq 0.1250$ ($8 \leq N_\tau \leq 14$) for $SU(3)$. We want to note that this result does not disagree with a linear extrapolation for the bare coupling scheme within errors.

It is remarkable that $T_c$ in units of $\Lambda_{\overline{MS}}$ differs only by less than 20% between $SU(2)$ and $SU(3)$ and also the corresponding value for QCD with four flavours, $T_c/\Lambda_{\overline{MS}} = 1.05(20)$ [30], is surprisingly close to the $SU(3)$ value.

A corresponding linear extrapolation of the string tension $\sqrt{\sigma}/\Lambda_{\overline{MS}}$ in the region $0.0625 \leq aT_c \leq 0.1667$ ($6 \leq N_\tau \leq 16$) for $SU(2)$ and $0.0714 \leq aT_c \leq 0.1250$ ($8 \leq N_\tau \leq 14$) for $SU(3)$ to the continuum limit $a = 0$ gives the result

$$\frac{\sqrt{\sigma}}{\Lambda_{\overline{MS}}} = \begin{cases} 1.79 \pm 0.12 & , \text{SU}(2) \\ 1.75 \pm 0.20 & , \text{SU}(3) \end{cases} .$$

(37)

The data from Table 4 together with the extrapolated values are shown in Fig. 7. The corresponding value for four flavour QCD is $\sqrt{\sigma}/\Lambda_{\overline{MS}} = 2.78 \pm 0.50$. This result is, of course, consistent with the scaling value of $T_c/\sqrt{\sigma}$ given in Eq. (32).

We note that our value of the $SU(2)$ string tension is only slightly larger than a recent estimate from the heavy quark potential [6], which yielded $\sqrt{\sigma}/\Lambda_{\overline{MS}} = 1.61(9)$. The agreement between these entirely different approaches gives additional support for our $a = 0$ extrapolation in the $\beta_E$-scheme.

5. Conclusions

We determined the critical coupling for the $SU(2)$ deconfinement transition on lattices with large temporal extent, $N_\tau = 8$ and 16, using a FSS analysis of the Binder cumulant $g_4$. We find

$$\beta_{c,\infty}(N_\tau = 8) = 2.5115(40) ,$$

$$\beta_{c,\infty}(N_\tau = 16) = 2.7395(100) .$$

A comparison with existing string tension data showed scaling in the entire coupling range $2.30 \leq \beta \leq 2.74$ under consideration. For the $SU(3)$ gauge group scaling sets in at a value of $\beta \simeq 6$. We observe a 20% increase in $T_c/\sqrt{\sigma}$ when going from $SU(3)$ to $SU(2)$, which can be understood in terms of the larger number of gluons in the $SU(3)$ gauge theory, which tend to decrease $T_c$ and increase $\sqrt{\sigma}$ at the same time.
Figure 7. The string tension $\sqrt{\sigma}$ in units of $\Lambda_{\overline{MS}}$ as a function of $a\Lambda_{\overline{MS}}$ obtained from the two loop renormalization group equation Eq. (5) for the $SU(2)$ (a) and $SU(3)$ (b) gauge theory. The convention for symbols and lines is the same as in Fig. 6.
In the bare coupling $\beta = 2N/g^2$ we observe no sign of asymptotic scaling up to $\beta = 2.74$. The alternative coupling scheme $\beta_E$, derived from the measured plaquette expectation value, shows much less deviation from asymptotic scaling. Our data indicate that these deviations are well described by $O(a)$ corrections and a linear extrapolation of $T_c/\Lambda_L$ and $\sqrt{\sigma}/\Lambda_L$ to $a = 0$ seems to be justified.

We note, however, that our analysis clearly is not sensitive to $O(1/\ln a)$ correction terms, which would be nearly constant in the rather small interval of couplings considered by us.

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We thank these computer centers for providing us with the necessary computational resources and the staff for their continuous support. In particular we thank Dr. Völpel, GMD, for making the CM-2 available to us during the Christmas break. Financial support from DFG under contract Pe 340/1-3 and from the Ministerium für Wissenschaft und Forschung NRW under contract IVA5-10600990 is gratefully acknowledged (JF and FK). The work of UMH was supported in part by the DOE under grant # DE-FG05-85ER250000. He also acknowledges partial support by the NSF under grant # INT-8922411 and the kind hospitality at the Fakultät für Physik at the University of Bielefeld while this research was begun.
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Appendix A: Cumulants of the order parameter

We define cumulants of the Polyakov loop expectation value in the following way:

\[ K_n = \left( \frac{N_{\sigma}}{N_{\tau}} \right)^{d(1-n)} \frac{\partial^n f_s}{\partial h^n} \bigg|_{h=0} \]  
(A.1)

On a finite lattice all odd cumulants are zero due to the \( Z(N) \) center symmetry.

\[ K_i = 0, \quad i \text{ odd} \]  
(A.2)

\[ K_2 = < P^2 > \]  
(A.3)

\[ K_4 = < P^4 > - 3 < P^2 >^2 \]  
(A.4)

\[ K_6 = < P^6 > - 15 < P^2 > < P^4 > + 30 < P^2 >^3 \]  
(A.5)

\[ K_8 = < P^8 > - 35 < P^4 >^2 - 28 < P^2 > < P^6 > + \]
\[ 420 < P^2 >^2 < P^4 > - 630 < P^2 >^3 \]  
(A.6)

Then moments of the order parameter can be expressed by cumulants:

\[ < P^2 > = K_2 \]  
(A.7)

\[ < P^4 > = K_4 + 3K_2^2 \]  
(A.8)

\[ < P^6 > = K_6 + 15K_2K_4 + 75K_2^3 \]  
(A.9)

\[ < P^8 > = K_8 - 28K_6K_2 - 35K_4^2 - 210K_4K_2^2 - 1785K_2^4 \]  
(A.10)

Now we consider three possible cumulant ratios given by:

\[ g_4 = \frac{K_4}{K_2^2} = \frac{< P^4 >}{< P^2 >^2} - 3 \]  
(A.11)

\[ g_6 = \frac{K_6}{K_2^3} = \frac{< P^6 >}{< P^2 >^3} - 15g_4 - 75 \]  
(A.12)

\[ g_8 = \frac{K_8}{K_2^4} = \frac{< P^8 >}{< P^2 >^4} + 28g_6 + 35g_4^2 - 210g_4 - 315 \]  
(A.13)

The general form of these cumulant ratios obtained from Eq. (A.1) is

\[ g_{nm} = \frac{K_{nm}}{K_n} = \left( \frac{N_{\tau}}{N_{\sigma}} \right)^{d(1-nm)} \frac{\partial^{nm} f_s}{\partial h^{nm}} \bigg|_{h=0} = \frac{Q_{nm}}{Q_n} \]  
(A.14)
The ratio $K_{nm}/K_m^n$ is directly a scaling function. Therefore it can be expressed as a function of $g_4$ since both are derived from $f_s$ and depend on the same argument $g_t (N_\sigma/N_\tau)^{1/\nu}$. Then we expect $g_{nm}$ to be constant for a fixed value of $g_4$.

In order to test the consistency of the FSS form for $f_s$, Eq. (7), we measured the normalized moments $M_6 = \langle P^6 \rangle / \langle P^2 \rangle^3$ and $M_8 = \langle P^8 \rangle / \langle P^2 \rangle^4$. From Eq. (A.12) and (A.13) we see that if $g_6$ and $g_8$ are constant then $M_6$ and $M_8$ should also be constant. The measured values of these moments together with the size of a typical error are given in Table 7. We see that $M_6$ and also $M_8$ are very stable for the different lattice sizes, thus supporting our FSS ansatz for the singular part of the free energy density.

| $N_\sigma$ | $N_\tau$ | $M_6$  | $M_8$  |
|-----------|-----------|--------|--------|
| 8         | 4         | 3.13   | 6.94   |
| 12        | 4         | 3.11   | 6.82   |
| 16        | 8         | 3.16   | 7.17   |
| 24        | 8         | 3.13   | 6.97   |
| 32        | 16        | 3.19(10)| 7.39(30)|

Table 7: Normalized moments at $g_4 = -1.4$.

Appendix B: The effective coupling scheme

In the $\beta_E$ scheme the bare coupling, $\beta$, is replaced by an effective coupling, $\beta_E$, which is related to the internal energy. The starting point is a perturbative weak coupling expansion of the internal energy, which is known up to $O(\beta^{-3})$ [40,41], *

\[
\langle U_p \rangle = 1 - c_1 \beta^{-1} - c_2 \beta^{-2} - c_3 \beta^{-3} + O(\beta^{-4})
\]

\[
c_1 = (N^2 - 1)(1/4)
\]

\[
c_2 = (N^2 - 1)N^2(0.0204277 - 1/(32N^2))
\]

\[
c_3 = (N^2 - 1)N^4(4/3)(0.0066599 - 0.020411/N^2 + 0.0343399/N^4)
\]

* We thank H. Panagopoulos for informing us about the result of his $O(\beta^{-3})$ calculation prior to publication.
An alternative coupling $\beta_E$ can now be defined from the first two terms of $\langle U_p \rangle$,

$$
\beta_E = \frac{c_1}{1 - \langle U_p \rangle},
$$

with $U_p$ denoting the plaquette operator as defined in the third section. The value of $\langle U_p \rangle$ has to be determined by numerical simulations. Our data for the average action are listed in Table 8. For $SU(3)$ the data of the average action were taken from large symmetric lattices [24,42,43,44]. They are summarized in Table 9.

| $\beta$  | $1 - \langle U_p \rangle$ |
|----------|---------------------------|
| 1.8800   | 0.52637(5)                |
| 2.1768   | 0.43158(3)                |
| 2.2986   | 0.39746(1)                |
| 2.3726   | 0.37661(2)                |
| 2.4265   | 0.36352(1)                |
| 2.5115   | 0.34564(1)                |
| 2.7395   | 0.30869(2)                |

**Table 8:** Expectation value of the average action for $SU(2)$.

| $\beta$  | $1 - \langle U_p \rangle$ |
|----------|---------------------------|
| 5.40     | 0.52823(30)               |
| 5.51     | 0.50120(100)              |
| 5.60     | 0.47520(2)                |
| 5.70     | 0.45100(80)               |
| 5.75     | 0.44105(9)                |
| 5.80     | 0.43236(5)                |
| 5.90     | 0.41825(6)                |
| 6.00     | 0.40626(2)                |
| 6.10     | 0.39592(3)                |
| 6.20     | 0.38635(1)                |
| 6.30     | 0.37788(1)                |
| 6.40     | 0.36935(1)                |
| 6.60     | 0.35438(4)                |
| 6.80     | 0.34078(1)                |
| 8.00     | 0.27935(2)                |

**Table 9:** Expectation value of the average action for $SU(3)$.  

26
The normalization in Eq. (B.2) is chosen such that the leading term in the weak coupling expansion of $\beta_E = 2N/g_E^2$ is identical to the bare coupling,

$$\beta_E = \beta - c_2/c_1 + O(\beta^{-1}) \quad . \quad \text{(B.3)}$$

Both couplings $g$ and $g_E$ agree in the limit when $g$ goes to zero

$$g_E^2 = g^2 + (c_2/c_1)g^4/(2N) + (c_3/c_1)g^6/(2N)^2 + O(g^8) \quad . \quad \text{(B.4)}$$

Additionally in both schemes the $\beta$-function $a dg/da$ starts with the two universal coefficients $b_0$ and $b_1$, which can be seen from an expansion in the bare coupling $g$

$$a \frac{dg_E}{da} = -b_0 g_3 - b_1 g_5 + (3/2)(c_2/c_1)b_0 g_5 + O(g^7)$$

$$= -b_0 g_E^3 - b_1 g_E^5 \quad . \quad \text{(B.5)}$$

The constant $c_2/c_1$, however, redefines the lattice $\Lambda$-parameter in the $\beta_E$-scheme relative to the $\beta$-scheme,

$$\frac{\Lambda_E}{\Lambda_L} = \exp \left( \frac{c_2/c_1}{4Nb_0} \right) \quad . \quad \text{(B.6)}$$

This yields a much larger $\Lambda$-parameter for the alternative coupling scheme,

$$\frac{\Lambda_E}{\Lambda_L} = \left\{ \begin{array}{ll} 1.7217 & , \text{SU(2)} \\ 2.0756 & , \text{SU(3)} \end{array} \right. \quad . \quad \text{(B.7)}$$

The $\beta$-function in the $\beta_E$ scheme can be separated into two terms

$$a \frac{dg_E}{da} = a \frac{dg}{da} \frac{dg_E}{dg} \quad \text{(B.8)}$$

$$\frac{dg_E}{dg} = \frac{1}{g_E g^3} \frac{16N^2}{N^2 - 1} \frac{dE}{d\beta} \quad . \quad \text{(B.9)}$$

The first term is the usual $\beta$-function while the second term $dE/d\beta$ is proportional to the “specific heat” $c_v$. * The entire function $dg_E/dg$ asymptotically approaches unity as the coupling $g$ goes to zero.

For the $SU(2)$ gauge theory a peak of the “specific heat” has been observed at $\beta \simeq 2.2$ which can be related to a nearby singularity in a generalized two parameter coupling space [46]. For $SU(3)$ the same structure was found in the form of a huge bump in the region $5.2 \leq \beta \leq 5.8$ [47].

As a consequence the dip in the discrete $\beta$-function of the bare coupling seems to be compensated by the peak of the “specific heat”, which motivates the choice of $g_E$ as a better behaved coupling.

* Here we consider the system as a $(d + 1)$ dimensional model at “inverse temperature” $\beta$, which is not to be confused with the physical temperature $T$. 

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