Euler sums of generalized hyperharmonic numbers

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Abstract

In this paper, we mainly show that Euler sums of generalized hyperharmonic numbers can be expressed in terms of linear combinations of the classical Euler sums.

Keywords: generalized hyperharmonic numbers, Euler sums, Faulhaber’s formula, Bernoulli numbers

1 Introduction

The main investigating object of the this paper is the so called Euler sums of generalized hyperharmonic numbers

$$\zeta_{H(p,r)}(m) := \sum_{n=1}^{\infty} \frac{H_n^{(p,r)}}{n^m} \quad (p, r, m \in \mathbb{N} := \{1, 2, 3, \cdots \})$$,

where

$$H_n^{(p,r)} := \sum_{j=1}^{n} H_j^{(p,r-1)} \quad (n, p, r \in \mathbb{N})$$

are the generalized hyperharmonic numbers (see [4, 10]). Furthermore, $H_n^{(p,1)} = H_n^{(p)} = \sum_{j=1}^{n} 1/n^p$ are the generalized harmonic numbers and $H_n^{(1,r)} = h_n^{(r)}$ are the classical hyperharmonic numbers. In particular $H_n^{(1,1)} = H_n$ are the classical harmonic numbers.

Many researchers have been studying Euler sums of harmonic and hyperharmonic numbers (see [4, 6, 7, 9] and references therein), since they play
important roles in combinatorics, number theory, analysis of algorithms and many other areas (see e.g. [8]). It is interesting that the Riemann zeta function \( \zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s} \) often appears in such expressions. A well-known result [6] that can be traced back to the time of Euler is as the following:

\[
2 \sum_{n=1}^{\infty} \frac{H_n}{n^m} = (m + 2)\zeta(m + 1) - \sum_{n=1}^{m-2} \zeta(m - n)\zeta(n + 1), \quad m = 2, 3, \cdots.
\]

From that time on, various similar types of infinite series had been investigated. For instance, Flajolet and Salvy [6] developed the contour integral representation approach to the evaluation of Euler sums involving the classical harmonic numbers. Following Flajolet-Salvy’s paper [6], we write the classical linear Euler sums as

\[
S_{p,q}^{+,+} := \sum_{n=1}^{\infty} \frac{H_n^{(p)}}{n^q}.
\]

Mező and Dil [9] considered the infinite sum

\[
\sum_{n=1}^{\infty} \frac{h_{n}^{(r)}}{n^m}
\]

for an integer \( m \geq r + 1 \), and showed that it could be expressed in terms of infinite sums of the Hurwitz zeta function values, where the well-known Hurwitz zeta function is defined as

\[
\zeta(s, a) = \sum_{n=0}^{\infty} \frac{1}{(n + a)^s} \quad (s \in \mathbb{C}, \Re(s) > 1, a > 0).
\]

Note that \( \mathbb{C} \) denotes the set of complex numbers and \( \Re(s) \) denotes the real part of the complex number \( s \).

Later Dil and Boyadzhiev [4] extended this result to infinite sums involving multiple sums of the Hurwitz zeta function values.

If we regard \( \sum_{n=1}^{\infty} \frac{h_{n}^{(r)}}{n^s} \) as a complex function in variable \( s \), there are some more progresses toward this direction. For instance, Kamano [7] expressed the complex variable function \( \sum_{n=1}^{\infty} \frac{h_{n}^{(r)}}{n^s} \) in terms of the Riemann zeta function, and showed that it could be meromorphically continued to the whole complex plane. In addition, the residue at each pole was also given.
As a natural generalization, Dil, Mező and Cenkci [5] considered Euler sums of generalized hyperharmonic numbers

\[
\zeta_{H(p,r)}(m) := \sum_{n=1}^{\infty} \frac{H_n^{(p,r)}}{n^m}.
\]

They proved that for positive integers \( p, r \) and a positive integer \( m \) with \( m > r \), \( \zeta_{H(p,r)}(m) \) could be expressed in terms of series of multiple sums of the Hurwitz zeta function values. For \( r = 1, 2, 3 \), \( \zeta_{H(p,r)}(m) \) were also given explicit expressions as linear combinations of the multiple zeta values. On the contrary, Ömür and Koparal defined two \( n \times n \) matrices \( A_n \) and \( B_n \) with \( a_{i,j} = H_i^{(j,r)} \) and \( b_{i,j} = H_i^{(p,j)} \), respectively, and gave some interesting factorizations and determinants of the matrices \( A_n \) and \( B_n \).

The motivation of this paper arises from Dil, Mező and Cenkci’s result (see [5]). Although they reduced \( \zeta_{H(p,r)}(m) \) to zeta values for small \( p, r, m \), they didn’t find a general formula. Our main aim is to establish a general formula to express \( \zeta_{H(p,r)}(m) \) in terms of linear combinations of the classical Euler sums. Since a big family of the classical Euler sums can be reduced to zeta values (see [6] and references therein), we can reduce \( \zeta_{H(p,r)}(m) \) to zeta values for appropriate values of \( p, r, m \). In addition, we also present several conjectures on these coefficients.

\section{Main theorem}

We are now going to prove our main theorem of this section. Before going further, we introduce some notations and lemmata.

It is well-known that the sum of powers of consecutive integers \( 1^k + 2^k + \cdots + n^k \) can be explicitly expressed in terms of Bernoulli numbers or Bernoulli polynomials. Faulhaber’s formula can be written as

\[
\sum_{\ell=1}^{n} \ell^k = \frac{1}{k+1} \sum_{j=0}^{k} \binom{k+1}{j} B_j^+ n^{k+1-j}
\]

(1)

\[
= \frac{1}{k+1} \left( B_{k+1}(n+1) - B_{k+1}(1) \right) \quad \text{(2)},
\]

where Bernoulli numbers \( B_n^+ \) are determined by the recurrence formula

\[
\sum_{j=0}^{k} \binom{k+1}{j} B_j^+ = k + 1 \quad (k \geq 0)
\]
or by the generating function
\[
\frac{t}{1 - e^{-t}} = \sum_{n=0}^{\infty} B_n^+ t^n / n!,
\]
and Bernoulli polynomials \(B_n(x)\) are defined by the following generating function
\[
\frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) t^n / n!.
\]
It is known that the Conway-Guy formula [1, 3] for the hyperharmonic numbers has a simple expression
\[
H^{(1,r)}_n = h^{(r)}_n = \frac{(n + r - 1)}{r - 1} (H_{n+r-1} - H_{r-1}). \tag{3}
\]
It seems difficult to find a similar formula for the generalized hyperharmonic numbers \(H^{(p,r)}_n\), because it looks hard to reduce the \((p,r,n)\) parameter to one parameter as in (3), where \(r\) was reduced to \(r = 1\). We make further investigations in this direction, since it is important in the proof of the main theorem. We try to reduce the \((p,r,n)\) parameter to the \((p,n)\) parameter.

**Lemma 1.** For \(r, n, t \in \mathbb{N}\), defining
\[
T(r, n, t) := \sum_{k_1=t}^{n} \sum_{k_2=t}^{k_1} \cdots \sum_{k_{r-1}=t}^{k_{r-2}} 1, \tag{4}
\]
then we have
\[
T(r, n, t) = \sum_{m=0}^{r-1} B(r, t, m)n^m, \tag{5}
\]
where \(B(r, t, m)\) satisfy the following recurrence relations
\[
B(r + 1, t, \ell) = \sum_{m=\ell-1}^{r-1} \frac{B(r, t, m)}{m + 1} \binom{m + 1}{m - \ell + 1} B_{m-\ell+1}^+ \quad (1 \leq \ell \leq r), \tag{6}
\]
\[
B(r + 1, t, 0) = - \sum_{j=0}^{r-1} (t - 1)^{1+j} \sum_{m=j}^{r-1} \frac{B(r, t, m)}{m + 1} \binom{m + 1}{m - j} B_{m-j}^+. \tag{7}
\]
with boundary value \(B(1, t, 0) = 1\). In addition, \(B(r, t, m)\) denote polynomials in variable \(t\) of at most \(r - m - 1\) degree.
Proof. From the definition of $T(r, n, t)$, we have

$$T(r+1, n, t) = \sum_{m=0}^{r-1} B(r, t, m) \sum_{k_1=t}^{n} k_1^m$$

$$= \sum_{m=0}^{r-1} B(r, t, m) \frac{1}{m+1} \sum_{j=0}^{m} \binom{m+1}{j} B_j^+(n^{m+1-j} - (t-1)^{m+1-j})$$

$$= \sum_{j=0}^{r-1} n^{1+j} \sum_{m=j}^{r-1} B(r, t, m) \frac{1}{m+1} \binom{m+1}{m-j} B_{m-j}^+$$

$$- \sum_{j=0}^{r-1} (t-1)^{1+j} \sum_{m=j}^{r-1} B(r, t, m) \frac{1}{m+1} \binom{m+1}{m-j} B_{m-j}^+. $$

Since $T(r+1, n, t) = \sum_{m=0}^{r} B(r+1, t, m)n^m$, comparing the coefficients of $n^m$ gives the recurrence relations.

We prove the claim on the degree of $B(r, t, m)$ by induction on $r$. For $r = 1$, from the definition of $T(r, n, t)$, we have $B(1, t, 0) = 1$. The claim is true for $r = 1$. Assume the claim is true for $r$, thus we show that the claim is true for $r + 1$. Since $B(r, t, m)$ is of at most $r - m - 1$ degree, with the help of formula (6), we obtain that for $B(r+1, t, \ell)$ is of at most $r - \ell$ degree. From formulas (6) and (7), we have

$$B(r+1, t, 0) = -\sum_{j=0}^{r-1} (t-1)^{1+j} B(r+1, t, j+1).$$

Since each term is of at most $r$ degree, we get that $B(r+1, t, 0)$ is of at most $r$ degree. 

Since $B(r, t, m)$ is of at most $r - m - 1$ degree, we can write $B(r, t, m)$ more precisely. For $r \in \mathbb{N}$ and $0 \leq m \leq r - 1$, let

$$B(r, t, m) = \sum_{j=0}^{r-1-m} b(r, m, j)t^j.$$

Since $B(1, t, 0) = 1$, we have $b(1, 0, 0) = 1$. We now give recurrence relations for $b(r, m, j)$ more precisely.
Lemma 2. For \( r \in \mathbb{N} \), one has

\[
b(r + 1, \ell, j) = \sum_{m=\ell-1}^{r-1-j} \frac{b(r, m, j)}{m+1} \binom{m+1}{m-\ell+1} B_{m-\ell+1}^+ \quad (1 \leq \ell \leq r, 0 \leq j \leq r - \ell),
\]

\[
b(r + 1, 0, p) = -\sum_{j=0}^{r-1} \sum_{\ell = \max\{0, p+1+j-r\}}^{\min\{1+j,p\}} C(r, p, j, \ell) \quad (0 \leq p \leq r),
\]

where

\[
C(r, p, j, \ell) = \binom{1+j}{\ell} (-1)^{1+j-\ell} \sum_{m=j}^{r-1-p+\ell} \frac{b(r, m, p-\ell)}{m+1} \binom{m+1}{m-j} B_{m-j}^+.
\]

Proof. From the formula (6), we have

\[
B(r + 1, t, \ell) = \sum_{m=\ell-1}^{r-1} \frac{B(r, t, m)}{m+1} \binom{m+1}{m-\ell+1} B_{m-\ell+1}^+
\]

\[
= \sum_{m=\ell-1}^{r-1} \sum_{j=0}^{r-1-m} b(r, m, j) t^j \frac{1}{m+1} \binom{m+1}{m-\ell+1} B_{m-\ell+1}^+
\]

\[
= \sum_{j=0}^{r-\ell} t^j \sum_{m=\ell-1}^{r-1-j} b(r, m, j) \frac{1}{m+1} \binom{m+1}{m-\ell+1} B_{m-\ell+1}^+.
\]

Since \( B(r + 1, t, \ell) = \sum_{j=0}^{r-\ell} b(r + 1, \ell, j) t^j \), comparing the coefficients of \( t^j \) gives the first recurrence relation.

From the formula (7), we have

\[
B(r + 1, t, 0)
\]

\[
= -\sum_{j=0}^{r-1} (t-1)^{1+j} \sum_{m=j}^{r-1} \frac{b(r, m, j)}{m+1} \binom{m+1}{m-j} B_{m-j}^+
\]

\[
= -\sum_{j=0}^{r-1} (t-1)^{1+j} \sum_{m=j}^{r-1-m} \sum_{\ell=0}^{r-1-\ell} b(r, m, \ell) t^\ell \frac{1}{m+1} \binom{m+1}{m-j} B_{m-j}^+
\]

\[
= -\sum_{j=0}^{r-1} \sum_{k=0}^{1+j} \binom{1+j}{k} (-1)^{1+j-k} t^k \sum_{\ell=0}^{r-1-\ell} \sum_{m=j}^{r-1-\ell} b(r, m, \ell) \frac{1}{m+1} \binom{m+1}{m-j} B_{m-j}^+.
\]
\[
= - \sum_{p=0}^{r} t^p \sum_{j=0}^{r-1} \sum_{\ell=\max\{0, p+1+j-r\}}^{\min\{1+j,p\}} \left( \frac{1+j}{\ell} \right) (-1)^{1+j-\ell} \\
\times \sum_{m=j}^{r-1-p+\ell} \frac{b(r,m,p-\ell)}{m+1} \left( \frac{m+1}{m-j} \right) B_{m-j}^{+}.
\]

Since \( B(r+1,t,0) = \sum_{p=0}^{r} b(r+1,0,p) t^p \), comparing the coefficients of \( t^p \) gives the second recurrence relation.

If we write \( a(r,m,j) := b(r,j,m) \), then we have

\[
T(r,n,t) = \sum_{m=0}^{r-1} \sum_{j=0}^{r-1-m} a(r,m,j) n^j t^m. \tag{8}
\]

We give a simple expression for \( H_{n}^{(p,r)} \):

\[
H_{n}^{(p,r)} = \sum_{t=1}^{n} \frac{1}{t^p} \sum_{k_1=t}^{n} \sum_{k_2=t}^{k_1} \cdots \sum_{k_{r-1}=t}^{k_{r-2}} 1 \\
= \sum_{t=1}^{n} \frac{1}{t^p} T(r,n,t) \\
= \sum_{m=0}^{r-1} \sum_{j=0}^{r-1-m} a(r,m,j) n^j H_{n}^{(p-m)}. \tag{9}
\]

Note that, when \( \ell \geq 0 \), \( H_{n}^{(-\ell)} \) is understood to be the sum \( \sum_{x=1}^{n} x^\ell \).

Now we are able to prove our main theorem.

**Theorem 1.** Let \( r, p, m \in \mathbb{N} \) with \( m \geq r + 1 \), we have,

\[
\zeta_{H^{(p,r)}}(m) = \sum_{\ell=0}^{r-1} \sum_{j=0}^{r-1-\ell} a(r,\ell,j) S_{p-\ell,m-j}^{+}. \tag{10}
\]

Therefore \( \zeta_{H^{(p,r)}}(m) \) can be expressed in terms of linear combinations of the classical Euler sums. Moreover, it can be expressed as linear combinations of multiple zeta sums.
Proof. Using formula (9), we can write

\[
\zeta_{H(p,r)}(m) = \sum_{n=1}^{\infty} \frac{1}{n^m} \sum_{\ell=0}^{r-1} \sum_{j=0}^{r-1-\ell} a(r, \ell, j) n^j H_n^{(p-\ell)}
\]

\[
= \sum_{\ell=0}^{r-1} \sum_{j=0}^{r-1-\ell} a(r, \ell, j) \sum_{n=1}^{\infty} \frac{H_n^{(p-\ell)}}{n^{m-j}}.
\]

Note that

\[
\sum_{n=1}^{\infty} \frac{H_n^{(p-\ell,1)}}{n^{m-j}} = \sum_{n=1}^{\infty} \frac{H_n^{(p-\ell,1)}}{n^{m-j}} + \zeta(m - j + p - \ell),
\]

and \(\sum_{n=1}^{\infty} \frac{H_n^{(p-\ell,1)}}{n^{m-j}}\) is a specialization of a multiple zeta function (see [5] and references therein), we get the desired result. \(\square\)

3 More results on \(a(r, m, \ell)\)

Lemma 2 gives a recurrence relation for \(b(r, m, \ell)\). In this section, We will establish another recurrence relation for \(a(r, m, \ell)\).

**Theorem 2.** For \(r \in \mathbb{N}\), one has

\[
a(r + 1, r, 0) = - \sum_{m=0}^{r-1} a(r, m, r - m - 1) \frac{1}{r - m},
\]

\[
a(r + 1, m, \ell) = \sum_{j=\max\{0, m-\ell\}}^{r-1-m} \frac{a(r, m, j)}{j + 1} \binom{j + 1}{j - \ell + 1} B_{j+\ell+1}^+ (0 \leq m \leq r - 1, 1 \leq \ell \leq r - m),
\]

\[
a(r + 1, m, 0) = - \sum_{y=0}^{m} \sum_{j=\max\{0, m-y\}}^{r-1-y} a(r, y, j) D(r, m, j, y) (0 \leq m \leq r - 1),
\]

where

\[
D(r, m, j, y) = \sum_{\ell=\max\{0, m-y\}}^{j} \frac{1}{j + 1} \binom{j + 1}{j - \ell} B_{j+\ell}^+ \binom{\ell + 1}{m - y} (-1)^{1+\ell-m+y}.
\]

**Proof.** From the formula (8), we have

\[
T(r + 1, n, t)
\]
\[ \begin{align*}
&= \sum_{m=0}^{r-1} \sum_{j=0}^{r-1-m} a(r, m, j)t^m \sum_{k_1=t}^n k_1^j \\
&= \sum_{m=0}^{r-1} \sum_{j=0}^{r-1-m} a(r, m, j)t^m \frac{1}{j+1} \sum_{\ell=0}^j \binom{j+1}{\ell} B_{\ell}^+ (n^{i+1-\ell} - (t-1)^{j+1-\ell}) \\
&= \sum_{m=0}^{r-1} \sum_{j=0}^{r-1-m} a(r, m, j)t^m \frac{1}{j+1} \sum_{\ell=0}^j \binom{j+1}{\ell} B_{j-\ell}^+ \sum_{x=0}^{1+\ell} \binom{\ell+1}{x} t^x (-1)^{1+\ell-x} \\
&= \sum_{m=0}^{r-1} \sum_{j=0}^{r-1-m} a(r, m, j)t^m \frac{1}{j+1} \sum_{\ell=0}^j \binom{j+1}{\ell} B_{j-\ell}^+ n^{1+\ell} \\
&= \sum_{m=0}^{r-1} \sum_{j=0}^{r-1-m} \frac{a(r, m, j)}{j+1} \sum_{x=0}^{1+\ell} \sum_{\ell=\max\{0, x-1\}}^j \binom{\ell}{j-\ell} B_{j-\ell}^+ \left(\binom{\ell+1}{x} t^x (-1)^{1+\ell-x}\right) \\
&= \sum_{m=0}^{r-1} \sum_{j=0}^{r-1-m} a(r, m, j) \frac{1}{j+1} \sum_{\ell=0}^j \binom{j+1}{\ell} B_{j-\ell}^+ n^{1+\ell} \\
&= \sum_{m=0}^{r-1} \sum_{y=0}^{\min\{r-1, m\}} \sum_{j=\max\{0, m-y-1\}}^{r-1-y} a(r, y, j) D(r, m, j, y).
\end{align*} \]

On the other hand,

\[ T(r+1, n, t) = \sum_{m=0}^r \sum_{j=0}^{r-m} a(r+1, m, j)n^j t^m, \]

comparing the coefficients gives the desired result. \( \square \)

Next we give explicit values of \( a(r, m, j) \) for small \( r \).

**Case** \( r = 1 \):

\[ a(1, 0, 0) = 1. \]

**Case** \( r = 2 \):

\[ a(2, 1, 0) = -1, \]

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\[ a(2, 0, 0) = 1, \quad a(2, 0, 1) = 1. \]

Case \( r = 3 \):

\[ a(3, 2, 0) = \frac{1}{2}, \quad a(3, 1, 0) = -\frac{3}{2}, \quad a(3, 1, 1) = -1, \]
\[ a(3, 0, 0) = 1, \quad a(3, 0, 1) = \frac{3}{2}, \quad a(3, 0, 2) = \frac{1}{2}. \]

Case \( r = 4 \):

\[ a(4, 3, 0) = -\frac{1}{6}, \quad a(4, 2, 0) = 1, \quad a(4, 2, 1) = \frac{1}{2}, \]
\[ a(4, 1, 0) = -\frac{11}{6}, \quad a(4, 1, 1) = -2, \quad a(4, 1, 2) = -\frac{1}{2}, \]
\[ a(4, 0, 0) = 1, \quad a(4, 0, 1) = \frac{11}{6}, \quad a(4, 0, 2) = 1, \quad a(4, 0, 3) = \frac{1}{6}. \]

Case \( r = 5 \):

\[ a(5, 4, 0) = \frac{1}{24}, \quad a(5, 3, 0) = -\frac{5}{12}, \quad a(5, 3, 1) = -\frac{1}{6}, \]
\[ a(5, 2, 0) = \frac{35}{24}, \quad a(5, 2, 1) = \frac{5}{4}, \quad a(5, 2, 2) = \frac{1}{4}, \]
\[ a(5, 1, 0) = -\frac{25}{12}, \quad a(5, 1, 1) = \frac{35}{12}, \quad a(5, 1, 2) = -\frac{5}{4}, \quad a(5, 1, 3) = -\frac{1}{6}, \]
\[ a(5, 0, 0) = 1, \quad a(5, 0, 1) = \frac{25}{12}, \quad a(5, 0, 2) = \frac{35}{24}, \quad a(5, 0, 3) = \frac{5}{12}, \quad a(5, 0, 4) = \frac{1}{24}. \]

Observing the above facts, we present the following four conjectures.

**Conjecture 1.** For \( r, m, \ell \in \mathbb{N} \) with \( 0 \leq m \leq r - 1 \) and \( 0 \leq \ell \leq r - 1 - m \), we conjecture that

\[ a(r, m, \ell) = (-1)^{m+\ell} a(r, \ell, m). \]
Conjecture 2. For $r \in \mathbb{N}$ with $0 \leq n \leq r - 1$, we conjecture that
\[
\sum_{\ell=0}^{n} a(r, n - \ell, \ell) = \delta_{n0},
\]
where $\delta_{nm}$ is the Kronecker delta, that is, $\delta_{nn} = 1$, $\delta_{nm} = 0$ for $n \neq m$.

Conjecture 3. For $r \in \mathbb{N}$, we conjecture that
\[
\sum_{\ell=0}^{r-1} a(r, 0, \ell) = r \quad (r \geq 1),
\]
and
\[
\sum_{\ell=0}^{r-1} a(r, \ell, 0) = 0 \quad (r \geq 2).
\]

Conjecture 4. For $r, m \in \mathbb{N}$ with $0 \leq m \leq r - 1$, we conjecture that
\[
\text{sgn}(a(r, m, \ell)) = (-1)^m \quad (0 \leq \ell \leq r - 1 - m),
\]
where $\text{sgn}(x)$ is the signum function defined by
\[
\text{sgn}(x) = \begin{cases} 
1 & x > 0, \\
0 & x = 0, \\
-1 & x < 0.
\end{cases}
\]
In particular
\[
a(r, m, \ell) \neq 0 \quad (0 \leq \ell \leq r - 1 - m).
\]

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