Involutions on 3-Manifolds and Self-dual, Binary Codes

Matthias Kreck and Volker Puppe

February 1, 2008

Key words: Involutions, 3-manifolds, codes, cohomology
Subject classification: 57M60, 94B05, 57M50, 57R91

Abstract We study a correspondence between orientation reversing involutions on compact 3-manifolds with only isolated fixed points and binary, self-dual codes. We show in particular that every such code can be obtained from such an involution. We further relate doubly even codes to Pin$^-$-structures and Spin-manifolds.

1 Introduction

Binary, self-dual codes play an important role in coding theory and have been studied extensively, see [RS] for a comprehensive survey and literature. A connection to involutions on 3-manifolds was made in [Pu]. It is shown there that an involution $\tau : M \rightarrow M$ on a 3-dimensional, closed manifold $M$ with "maximal number" of isolated, fixed points (i.e. with only isolated fixed points such that the number of fixed points $k := |M^\tau|$ equals $\dim_{\mathbb{Z}/2}(\oplus_i H^i(M;\mathbb{Z}/2))$ determines a binary self-dual code of length $k$. In turn this code determines the cohomology algebra $H^*(M;\mathbb{Z}/2)$ and the equivariant cohomology $H^*_G(M;\mathbb{Z}/2)$, where the action of $G := \mathbb{Z}/2$ is given by the involution. In fact, the code corresponds to the inclusion $H^*_G(M;\mathbb{Z}/2) \longrightarrow H^*_G(M^G;\mathbb{Z}/2) \cong (\mathbb{Z}/2[t])^k$, and an equivalence of codes given by permuting the coordinates, corresponds to an equivalence of inclusions, given by an automorphism of the algebra $(\mathbb{Z}/2[t])^k$.

In Section 2 we generalize these results by considering involutions on 3-manifolds which have a finite number of fixed points which need not be maximal. The code corresponding to such an involution is described in two ways, firstly using equivariant cohomology as in [Pu], secondly using the ordinary homology (with $\mathbb{Z}/2$ coefficients) of the complement (of a neighbourhood) of the fixed point in the orbit space.
In Section 3 we show that every binary, self-dual code can be obtained from an involution on a 3-manifold with a finite number of fixed points, in fact, using surgery we get that the manifold can be chosen so that the number of fixed points is maximal.

In Section 4 we relate doubly even codes to Spin-manifolds. We define the concept of a Spin-involution and show that Spin-involutions give doubly even codes. Finally we show that each doubly even code comes from a 3-manifold with Spin-involution.

## 2 Self-dual codes from involutions on 3-manifolds

Let \( \tau : M \to M \) be an involution on a closed 3-manifold \( M \) with finitely many fixed points \( x_1, \ldots, x_k \). By Smith theory \( k \leq \text{dim}_{\mathbb{Z}/2}(\oplus_i H^i(M; \mathbb{Z}/2)) \). The famous localization Theorem for equivariant cohomology gives in this context that the map

\[
\tilde{G} \rightarrow \tilde{H}_G^*(M) \cong (\mathbb{Z}/2[t])^k
\]

(here and in the following we always take coefficients in \( \mathbb{Z}/2 \)), induced by the inclusion \( M^G \xrightarrow{i} M \), becomes an isomorphism after inverting the powers of \( t \in \mathbb{Z}/2[[t]] = H^*(BG) \).

The kernel of \( i^*_G \) is the torsion submodule \( T \subset H^*_G(M) \); in particular \( i^*_G \) is injective if and only if \( H^*_G(M) \) is a free \( \mathbb{Z}/2[t]- \)module. As in the "maximal number of fixed points" case, the inclusion \( H^*_G(M)/T \xrightarrow{i^*_G} (\mathbb{Z}/2[t])^k \) determines a self-dual code \( C \), and in turn \( i^*_G \) is determined by this code. In more detail \( C \) is the image of \( \bigoplus_{i=0}^1 H^i_G(M) \) in \( \mathbb{Z}/2^k \) under the evaluation map, putting \( t = 1 \) (cf. [Pu]). This is the same as the image of \( \tilde{H}_G^1(M) \) in \( H^*_G(M^G) \cong \mathbb{Z}/2 \). The code \( C \) is self-dual. This is shown in [Pu] for the "maximal case" (i.e. \( T = 0 \)), and can be seen generally as follows (cf. [AP], Exercise (1.15)):

Let \( \rho : H^*_G(M) \to H^*(M) \) be the restriction to the fibre in the Borel construction \( M \to M \times_G EG \to BG \). The map \( \rho \) fits into the long exact Gysin sequence of the covering \( M \cong M \times EG \to M \times G EG \), i.e.

\[
\cdots \to H^{i-1}_G(M) \xrightarrow{\cup t} H^i_G(M) \xrightarrow{\rho} H^i(M) \xrightarrow{\rho} H^{i+1}_G(M) \xrightarrow{\cup t} H^{i+1}_G(M) \to \cdots
\]

is exact and hence so is

\[
0 \to H^*_G(M) \otimes_{\mathbb{Z}/2[t]} \mathbb{Z}/2 \to H^*(M) \to Tor_{\mathbb{Z}/2[t]}(H^*_G(M), \mathbb{Z}/2) \to 0.
\]

Splitting \( H^*_G(M) \cong T \oplus F \) into a direct sum of the torsion submodule \( T \) and a free complement \( F \), one sees that \( \text{dim}_{\mathbb{Z}/2}(T \otimes_{\mathbb{Z}/2[t]} \mathbb{Z}/2) = \text{dim}_{\mathbb{Z}/2} Tor_{\mathbb{Z}/2[t]}(H^*_G(M), \mathbb{Z}/2) \). We claim that with respect to the Poincaré duality pairing in \( H^*(M) \), the orthogonal complement of \( \rho(H^*_G(M)) \) is \( \rho(T) \).

Because of the dimension equality above it suffices to show that \( \langle x, y \rangle = \sigma(x \cup y) = 0 \) for \( x \in \rho(H^*_G(M)) \) and \( y \in \rho(T) \), where \( \sigma : H^*(M) \to \mathbb{Z}/2 \) is the orientation of \( M \). If \( \bar{x} \)
and \( \tilde{\gamma} \) are liftings of \( x \) and \( y \) with respect to \( \rho \), then \( \tilde{x} \cup \tilde{y} \in T \). Therefore \( \tilde{x} \cup \tilde{y} \) is mapped to zero under the equivariant orientation \( \tilde{\sigma} : H^*_G(M) \to \mathbb{Z}/2[t] \) (cf. e.g. [AP], Chap. 1). Hence \( \tilde{\sigma}(\tilde{x} \cup \tilde{y}) = 0 \) and thus \( \sigma(x \cup y) = 0 \).

We get that the graded algebra \( H^*_G(M)/T \) is a subquotient of \( H^*(M) \), which fulfills Poincaré duality, and \( H^*_G(M)/T \) as a \( \mathbb{Z}/2[t] \)-module is isomorphic to \( (\rho(H^*_G(M))/\rho(T)) \otimes \mathbb{Z}/2 \mathbb{Z}/2[t] \). We therefore have, analogous to the case \( T = 0 \), that

\[
H^*_G(M)/T \xrightarrow{\tilde{\sigma}_G} (\mathbb{Z}/2[t])^k
\]
determines a binary, self-dual code and, in turn, is determined by this code. Note, though, that we only get the quotient algebra \( H^*_G(M)/T \) (and \( \rho(H^*_G(M))/\rho(T) \cong (H^*_G(M)/T) \otimes \mathbb{Z}/2[t] \) \( \mathbb{Z}/2 \)) from the code which means that in case \( T \neq 0 \) the algebras \( H^*_G(M) \) and \( H^*(M) \) are not completely determined by the code.

In view of the construction below, we describe the code coming from an involution \( \tau : M \to M \) on a closed 3-manifold \( M \) with isolated fixed points \( x_1, \ldots, x_k \) in a second way.

Let \( W := (M \setminus +_k D^3)/\tau \), where \( D^3 \) are equivariant discs around \( x_i \). We consider the Mayer-Vietoris sequence in equivariant cohomology for \( M = (M \setminus +_k D^3) \cup (+_k D^3) : \)

\[
\cdots \to H^0_G(+_k S^2_1) \to H^1_G(M) \to H^1_G(M \setminus +_k \circ D^3) \oplus H^1_G(+_k D^3) \to H^1_G(+_k S^2) \to \cdots
\]
since the equivariant cohomology of a free \( G \)-space is the non-equivariant cohomology of the orbit space, one gets the exact sequence

\[
\cdots \to H^0(+_k \mathbb{R}P^2) \to H^1_G(M) \to H^1(W) \oplus (H^0_G(+_k D^3) \to H^1(+_k \mathbb{R}P^2) \to \cdots
\]

It is easy to see that the map \( (H^1_G(+_k D^3) \to H^1(+_k \mathbb{R}P^2) \) is an isomorphism. (For one disk \( D^3 \) one has that \( S^2 \xrightarrow{(id,i)} S^2 \times S^\infty \xrightarrow{(j,id)} D^3 \times S^\infty \xrightarrow{p_3} S^\infty \) is the canonical inclusion, and hence \( H^1_G(D^3) \cong H^1(\mathbb{R}P^\infty) \to H^1(\mathbb{R}P^2) = H^1_G(S^2) \). It therefore follows from the Mayer-Vietoris sequence that \( Im(H^1_G(M) \to H^1_G(M^G)) = Im(H^1(W) \to H^1(+_k \mathbb{R}P^2)) \). Dually to taking \( Im(H^1(W) \to H^1(+_k \mathbb{R}P^2)) \), we can take \( Ker(H^1(+_k \mathbb{R}P^2) \to H^1(W)) \). Since the kernel of the map on the middle homology of the boundary of a compact manifold to the inner is a self annihilating subspace (with respect to the intersection form) of half rank and the intersection form on \( H_1(+_k \mathbb{R}P^2) = (\mathbb{Z}/2)^k \) is the standard ”Euclidean” form, one gets in another way that the code is self-dual.

We summarize the above considerations somewhat vaguely as

**Theorem 1.** Every involution with only isolated fixed points on a compact 3-manifold determines a binary, self-dual code.
3 All self-dual codes come from 3-manifolds

Proposition 2. Every binary, self-dual code can be obtained from an involution on an orientable 3-manifold.

Proof: Let \( k = 2r \). Let \( C \subset \mathbb{Z}/2^k \) be a self-dual code. We choose a map \( f : +_k\mathbb{R}\mathbb{P}^2 \to (\mathbb{R}\mathbb{P}^\infty)^r \) such that the sequence (with \( \mathbb{Z}/2 \) coefficients)

\[
0 \to C \to H_1(_k\mathbb{R}\mathbb{P}^2) \xrightarrow{f_*} H_1((\mathbb{R}\mathbb{P}^\infty)^r) \to 0
\]

is exact.

Next we note that the first Stiefel-Whitney class \( w_1(_k\mathbb{R}\mathbb{P}^2) \) is in the image \( f^* \). The reason is that \( \Delta \) is in the code (\( \Delta \) is dual to \( w_1(_k\mathbb{R}\mathbb{P}^2) \)) and so \( <w_1(_k\mathbb{R}\mathbb{P}^2), x> = <\Delta, x> = 0 \) for all \( x \in C \). This implies that there is a real line bundle \( L \) over \((\mathbb{R}\mathbb{P}^\infty)^r\) pulling back to the non-trivial line bundle over each copy of \( \mathbb{R}\mathbb{P}^2 \). Thus \((+_k\mathbb{R}\mathbb{P}^2, f)\) is an orientable singular manifold, where the orientation is twisted by the line bundle \( L \). This means that the bundle \( \nu(+_k\mathbb{R}\mathbb{P}^2) - f^*(L) \) is orientable.

After choosing an orientation the pair \((+_k\mathbb{R}\mathbb{P}^2, f)\) represents an element in the bordism group \( (\Omega_2((\mathbb{R}\mathbb{P}^\infty)^r; L)) \) of singular manifold with orientation (twisted by \( L \)). We claim that this element is trivial.

The Atiyah-Hirzebruch spectral sequence implies that

\[
\Omega_2((\mathbb{R}\mathbb{P}^\infty)^r; L) \xrightarrow{\approx} H_2((\mathbb{R}\mathbb{P}^\infty)^r; \mathbb{Z}_t).
\]

\[
[F, h] \mapsto h^*([F])
\]

Here, \( \mathbb{Z}_t \) stands for twisted homology, where the coefficient system is given by the representation

\[
\pi_1((\mathbb{R}\mathbb{P}^\infty)^r) \to \pi_1((\mathbb{R}\mathbb{P}^\infty) \xrightarrow{\approx} (\pm 1) = Aut(\mathbb{Z})
\]

and the map is induced by the classifying map of \( L \). We note that

\[
H_2((\mathbb{R}\mathbb{P}^\infty)^r; \mathbb{Z}_t) \to H_2((\mathbb{R}\mathbb{P}^\infty)^r; \mathbb{Z}/2)
\]

is injective. The reason is that \( H_2((\mathbb{R}\mathbb{P}^\infty)^r; \mathbb{Z}_t) \) consists only of elements of order 2 (and 0).

Thus it is enough to control the image of the fundamental class in \( H_2((\mathbb{R}\mathbb{P}^\infty)^r; \mathbb{Z}/2) \). The vanishing is equivalent to

\[
h^*x \cup h^*y = 0
\]

for all \( x, y \in H^1((\mathbb{R}\mathbb{P}^\infty)^r; \mathbb{Z}/2) \). This follows since by construction the intersection form vanishes on \( Ker(H_1(+_k\mathbb{R}\mathbb{P}^2; \mathbb{Z}/2) \to H_1((\mathbb{R}\mathbb{P}^\infty)^r; \mathbb{Z}/2) \), which under the isometry between \( H_1(+_k\mathbb{R}\mathbb{P}^2; \mathbb{Z}/2) \cong H^1(+_k\mathbb{R}\mathbb{P}^2; \mathbb{Z}/2) \) corresponds to the image of \( H^1((\mathbb{R}\mathbb{P}^\infty)^r; \mathbb{Z}/2) \to
Summarizing the information so far, we have shown that 

\[ [+_k \mathbb{RP}^2; f] = 0 \in \Omega_2((\mathbb{RP}^\infty)^r; L). \]

Let \( h : W \to (\mathbb{RP}^\infty)^r \) be a zero bordism. We claim that the kernel of the map induced by the inclusion 

\[ H_1(+_k \mathbb{RP}^2; \mathbb{Z}/2) \to H_1(W; \mathbb{Z}/2) \]

is our code \( C \). Since \( h\big|_{+_k \mathbb{RP}^2} = f \), we conclude that \( \text{Ker}(H_1(+_k \mathbb{RP}^2; \mathbb{Z}/2) \to H_1(W; \mathbb{Z}/2)) \) is contained in \( C \). But this kernel has dimension \( r = \dim C \) implying the statement.

Finally we consider the classifying map \( g \) of \( L \) and the composition 

\[ gh : W \to \mathbb{RP}^\infty, \]

to construct the induced 2-fold covering \( \tilde{W} \) over \( W \). Since \( W \) is oriented (twisted by \( L \)), \( \tilde{W} \) is an orientable manifold. The boundary of \( W \) is \( +_k S^2 \) and the restriction of the deck transformation to the boundary is \(-id\) on each summand. Thus we obtain an involution \( \tau \) on 

\[ M := \tilde{W} \cup +_k D^3 \]

which on \( \tilde{W} \) is the deck transformation and on each \( D^3 \) is \(-id\). By construction the code associated to this 3-manifold \( M \) and \( \tau \) is the given code finishing the argument.

q.e.d.

The above construction depends on the choice of the zero cobordism \( h : W \to (\mathbb{RP}^\infty)^r \), and it is not clear whether one obtains a manifold \( M \) with involution, which has maximal number of isolated fixed points. We will show that one can change \( W \) by surgery to reduce the cohomology of \( M \), and obtain a pair \((M, \tau)\) with maximal number of isolated fixed points. By Smith theory the maximality condition is equivalent to the injectivity of 

\[ H^*_G(M; \mathbb{Z}_2) \to H^*_G(M^G; \mathbb{Z}_2) \cong (\mathbb{Z}_2[t])^k, \]

resp. the surjectivity of 

\[ H^*_G(M^G; \mathbb{Z}_2) \to H^*_G(M; \mathbb{Z}_2). \]

In our case \( M = \tilde{W} \cup (+_k D^3) \). The equivariant Mayer-Vietoris sequence (with \( \mathbb{Z}_2 \) coefficients) gives:

\[ \cdots \to H^*_G(+_k S^2) \to H^*_G(\tilde{W} \cup (+_k D^3)) \to H^*_G(M) \to H^*_{-1}(+_k S^2) \to \cdots \]

One has 

\[ H^*_G(+_k S^2) \cong (H_*(+_k \mathbb{RP}^2) \text{ and } H^*_G(\tilde{W}) \cong H_*(\tilde{W}) \text{ since the actions on } +_k S^2 \text{ and } \tilde{W} \text{ are free; and } H^*_G(+_k D^3) \cong H^*_G(M^G). \]

The inclusion \( S^2 \subset D^3 \) induces the inclusion 

\[ H_*(\mathbb{RP}^2) \to H_*(\mathbb{RP}^\infty). \]

Hence the map \( H^*_G(M^G) \to H^*_G(M) \) is surjective if and only if 

\[ H_i(+_k \mathbb{RP}^2) = H_i(\partial W) \to H_i(W) \]

is surjective for \( i = 1, 2 \). But the long exact sequence of the Poincaré pair \((W, \partial W)\) gives that surjectivity for \( i = 1 \) already implies surjectivity for
To verify the maximality condition it therefore suffices to show that $H_1(\partial W) \to H_1(W)$ is surjective. We want to arrive at this condition by applying surgery to $W$ (if necessary). Assume that $H_1(\partial W) \to H_1(W)$ is not surjective. We consider the following diagram:

$$
\begin{array}{cccc}
\to & H_1(\partial W) & \xrightarrow{i_1} & H_1(W) & \to & H_1(W, \partial W) \\
\downarrow f_1 & & \uparrow h_1 & & & \\
& H_1((\mathbb{R}P^\infty)^r) & & & & \\
\end{array}
$$

We already know that $i_1$ and $f_1$ have the same kernel, namely the code $C$, and $f_1$ is surjective by construction. Therefore $i_1$ is surjective if and only if $h_1$ is injective. Assume that there exists an $a \in H_1(W), a \neq 0$, with $h_1(a) = 0$. Since $W$ is orientable (twisted by $L$), the normal bundle of an embedded circle representing $a$ is trivial. Performing surgery with respect to $\alpha$ kills the class $a$ and its dual with respect to the intersection pairing. The map $h_1 : H_1(W) \to H_1((\mathbb{R}P^\infty)^r)$ factors through the quotient $H_1(W)/<a>$. Hence we can find a map $h' : W' \to (\mathbb{R}P^\infty)^r$ of our new manifold $W'$, which restricts to $f$ on the boundary $\partial W' = \partial W = +_k \mathbb{R}P^2$. Iterating the process (if necessary) gives the following result.

**Theorem 3.** Every binary, self-dual code can be obtained from an involution on an orientable 3-manifold with maximal number of isolated fixed points.

### 4 Spin-structures and doubly even codes

Let $M$ be an oriented closed 3-manifold with involution $\tau$ having exactly $k$ isolated fixed points. We will construct from these data a 4-manifold by starting with $M \times S^1$ and dividing by the involution, which on $M$ is $\tau$ and on $S^1$ is complex conjugation. This is a manifold with $2k$ isolated singularities, where $k$ is the number of fixed points of $\tau$. All fixed points singularities are cones over $\mathbb{R}P^3$, which are the links of the singularities. Since the involution on $M \times S^1$ is orientation preserving, the orientation on $M \times S^1$ induces an orientation on the quotient (after removing the fixed points), which in turn gives an orientation on each link $\mathbb{R}P^3$. Now we remove open cones around the singularities and replace them by the disc bundle of the complex line bundle over $\mathbb{C}P^1$ with Chern class $-2$. The reason for choosing this sign of the Chern class (and not $+2$) is that the induced orientation on $\mathbb{R}P^3$ above is the opposite of this orientation (we will discuss this in more detail in the proof of the following result). This implies that the orientations fit together and so the result is an oriented 4-manifold denoted $N(M, \tau)$. We say that $\tau$ is a Spin-involution if $N(M, \tau)$ admits a Spin structure compatible with the given orientation.

The construction of $N(M, \tau)$ is well known in the case of the 3-torus $T^3$ with $\tau$ complex conjugation. Then $N(T^3, \tau)$ is the $K3$-surface, which has a Spin structure.
Theorem 4. Let $M$ be a closed oriented 3-manifold with involution $\tau$ with finitely many fixed points. If $\tau$ is a Spin-involution, then the code $C(M, \tau)$ is doubly even.

Proof: We assume that the reader is familiar with $Pin^-$-structures [KT]. We recall that a $Pin^-$-structure on a smooth manifold $M$ is a Spin-structure on $TM \oplus \text{Det}(TM)$. Here we note that $TM \oplus \text{Det}(TM)$ has a natural orientation, which we assume to be compatible with the Spin-structure. Thus the $Pin^-$-structures are classified by $H^1(M; \mathbb{Z}/2)$.

A $Pin^-$-structure on a surface $F$ determines a quadratic refinement $q : H^1(F; \mathbb{Z}/2) \to \mathbb{Z}/4$, such that $q(x + y) = q(x) + q(y) + 2 < x, y >$, where $< x, y >$ is the intersection form. The two $Pin^-$-structures on $\mathbb{RP}^2$ are distinguished by the quadratic form, which can take the values $\pm 1$. For all this see [KT]. If $W$ is a 3-dimensional $Pin^-$-manifold with boundary $F$, then on the image of $H^1(W; \mathbb{Z}/2) \to H^1(\partial W; \mathbb{Z}/2)$ the intersection form and the quadratic refinement vanish. This follows from [KT] as explained in [T].

Now suppose that the disjoint union of $k$ copies of $\mathbb{RP}^2$ is the boundary of a $Pin^-$-manifold $W$ and the induced $Pin^-$-structure is equal on all components of the boundary, then if $x \in H^1(W; \mathbb{Z}/2)$, and $y = i^*(x) = (y_1, \ldots, y_k)$ we conclude that $\sum_i y_i = 0 \text{ mod } 4$. Thus we are finished if the condition that $\tau$ is a Spin-involution implies that $(M \setminus +kB^3)/\tau$ has a $Pin^-$-structure, which on all boundary components is the same. Here $B^3$, is an open ball around the $i$-th fixed point.

To see this we first note that a $Pin^-$-structure on $\mathbb{RP}^2$ is the same as a Spin-structure on the total space of $\text{TRP}^2 \oplus \text{Det TRP}^2$. Since $\text{Det TRP}^2$ is the normal bundle of $\mathbb{RP}^2$ in $\mathbb{RP}^3$, we can via a tubular neighbourhood identify $\text{TRP}^2 \oplus \text{Det TRP}^2$ with an open subset of $\mathbb{RP}^3$, which is homotopy equivalent to $\mathbb{RP}^3 - pt$. Thus a $Pin^-$-structure on $\mathbb{RP}^2$ determines a Spin-structure on $\mathbb{RP}^3$ and vice versa. In particular this means that the $Pin^-$-structure on $\mathbb{RP}^2$ determines an orientation on $\mathbb{RP}^3$. We note that $\mathbb{RP}^3$ is the total space of the complex line bundle over $\mathbb{CP}^1$ with first Chern class 2. Using the complex orientation on $\mathbb{CP}^1$ and on the complex line bundle we obtain an orientation on $\mathbb{RP}^3$. It is not difficult to show that this orientation agrees with the orientation coming from the $Pin^-$-structure on $\mathbb{RP}^2$ (one only has to compare the orientations at one point). Thus, if this is the orientation on a component of the boundary of some 4-manifold $V$, then we obtain an oriented manifold by gluing the disk bundle of the complex line bundle over $\mathbb{CP}^1$ with Chern class $-2$ (this induces the negative orientation compared to the orientation above, and so the orientations fit together). The key observation for our proof is, that since this disc bundle is simply connected there is a unique Spin-structure on it.

Now we consider $M \times S^1$ with the involution given by $\tau$ and complex conjugation $c$. Each fixed point of $M$ and each fixed point of $S^1$ gives a fixed point of $M \times S^1$ and for each fixed point the link of the corresponding singularity in $M \times S^1/(\tau \times c)$ is $\mathbb{RP}^3$ containing the link of the corresponding singularity in $M/\tau$. Thus a $Pin^-$-structure on each link in $M/\tau$ determines a Spin-structure of the two (for each fixed point of $S^1$ corresponding links in $M \times S^1/(\tau \times c)$) and vice versa. If $N(M, \tau)$ has a Spin-structure, this is the
same on each disk bundle of the complex line bundle with Chern class $-2$ over $\mathbb{C}P^1$, since there is a unique $Spin$-structure with the given orientation. Thus the restriction of the $Pin^-$-structure to each link $\mathbb{R}P^2$ is the same. As explained above this implies the theorem. q.e.d.

Next we prove that for each doubly even self-dual code $C$ there is a 3-manifold $M$ with $Spin$-involution $\tau$ such that the corresponding code is $C$.

**Proposition 5.** Let $C$ be a doubly even self-dual code. Then there is a 3-manifold $M$ with $Spin$-involution $\tau$ whose code is $C$.

**Proof:** We proceed as in the proof of Theorem 3 and use the notation from there. Now we consider $+k\mathbb{R}P^2$ as a $Pin^-$-manifold, where all copies are equipped with the same $Pin^-$-structure, which, if we pass to the corresponding $Spin$-structure on $\mathbb{R}P^3$ can be extended to the disc bundle of the complex line bundle with first Chern class $-2$ over $\mathbb{C}P^1$. Together with the map $f$ we obtain an element of $\Omega_2^{Pin^-}(\mathbb{R}P^\infty)^r$. We compute this bordism group with the Atiyah-Hirzebruch spectral sequence. We use from [KT] that $\Omega_0^{Pin^-} = \mathbb{Z}$, $\Omega_1^{Pin^-} = \mathbb{Z}/2$ and $\Omega_2^{Pin^-} = \mathbb{Z}/8$, where the latter group is generated by $\mathbb{R}P^2$ with any $Pin^-$-structure.

The Atiyah-Hirzebruch spectral sequence computing $\Omega_2^{Pin^-}((\mathbb{R}P^\infty)^r)$ has the entries:

$$\Omega_2^{Pin^-},$$

$$H_1((\mathbb{R}P^\infty)^r; \mathbb{Z}/2),$$

$$H_2((\mathbb{R}P^\infty)^r; \mathbb{Z}).$$

The component in the first entry is given by the $k$-fold sum of $\mathbb{R}P^2$ with the given $Pin^-$-structure, which is zero if and only if $k = 0 mod 8$. But this is the case for doubly even self-dual codes.

The last entry is as in the case of oriented bordism (twisted by $L$) detected by the image of the fundamental class with coefficients in $\mathbb{Z}/2$, which - as shown before - vanishes if the code is self dual.

The second entry is a bit delicate. We only have to detect the corresponding entry for $\Omega_2^{Pin^-}(\mathbb{R}P^\infty)$, since we can project to the different components. Then the corresponding entry is in $\mathbb{Z}/2$. By the fact that the bordism group is a module over $\Omega_*^{Pin^-}$ we see that the non-trivial element is represented by $(S^1 \times \eta, i\eta_1)$, where $\eta$ is $S^1$ with the non-trivial $Pin^-$-structure (which for 1-manifolds is the same as a $Spin$-structure) and $i$ is the inclusion $S^1 \to \mathbb{R}P^\infty$. We are free to choose a $Pin^-$-structure on the first factor. If we choose the $Spin$-structure again to be the non-trivial one, we see that the induced 2-fold cover is $\eta \times \eta$, which is the non-trivial element in $\Omega_2^{Spin^-}$. We note that whatever $Pin^-$-structure we choose on the first factor, we can change it, if necessary, to the non-trivial one, by modifying it with the non-trivial element in the image of $H^1(\mathbb{R}P^\infty; \mathbb{Z}/2)$. The upshot of these
considerations is that we can detect the second term in the Atiyah-Hirzebruch spectral sequence of an element \([N, g] \in \Omega^2_{Pin^-} (\mathbb{R}P^\infty)\) whose underlying \(Pin^-\)-bordism class is zero and whose fundamental class maps to zero by the following criterion: It is zero, if and only if for all modifications of the \(Pin^-\)-structure by elements in \(H^1(\mathbb{R}P^\infty; \mathbb{Z}/2)\) the induced 2-fold covering is zero bordant. Applying this to the case, where \(N = +_8\mathbb{R}P^2\) we note that the induced covering is an \(S^2\)'s over each summand which maps non-trivial to \(\mathbb{R}P^\infty\) (which is zero bordant) and that it is \(\mathbb{R}P^2 + \mathbb{R}P^2\) for each summand which maps trivially.

But since \(\mathbb{R}P^2\) is a generator of \(\Omega^2_{Pin^-} \cong \mathbb{Z}/8\), this implies that if the number of summands which are mapped trivial is \(0 \mod 4\), then the bordism class is trivial. Returning to the situation given by our code we see that if the code is doubly even, this criterion applies.

Thus we have shown that for doubly even codes the bordism class vanishes in \(\Omega^2((\mathbb{R}P^\infty)^r)\), and as in the proof of Theorem 2 we construct a 3-manifold \(M\) with involution \(\tau\) giving the code. Since the \(Pin^-\)-structure is the same for all copies of \(\mathbb{R}P^2\), we obtain a \(Spin\)-involution. Namely the 4-manifold we construct is the blow up of a \(Spin\)-manifold obtained by replacing the open cones over the individual \(\mathbb{R}P^3\)'s by the disc bundle of the complex line bundle with Chern class \(-2\) over \(\mathbb{C}P^1\). After perhaps changing the orientation before the gluing, the resulting manifold is oriented. Since the \(Spin\)-structure on all \(\mathbb{R}P^3\)'s extend to this disc bundle, the manifold is a \(Spin\)-manifold.

q.e.d.

As before one can apply surgery, this time taking into account the \(Pin^-\)-structure to get the following result.

**Theorem 6.** Every binary, doubly even, self-dual code can be obtained from a \(Spin\)-involution with maximal number of isolated fixed points on an orientable 3-manifold.

The first author was partially supported by a DFG grant. He also would like to thank the Mathematics Department of Columbia University and the Courant Institute for support and a stimulating atmosphere while part of the project was carried out.

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