Conditional Expectation in an Uncertainty Space

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Abstract. In the classical (probability) setting, the Radon-Nikodym Theorem is used to prove the existence of conditional expectations. This paper states the Radon-Nikodym Theorem for uncertain measures which were introduced by Liu (2007) to model belief degrees. This version of the theorem is adapted from Graf (1980) who established the theorem for non-additive measures called capacities. This paper then uses the theorem to define the conditional expectation of an uncertain variable with respect to a σ-algebra. Properties of the conditional expectation are established.

1. Introduction
In 2007, Liu [6] developed uncertainty theory in order to rationally deal with belief degrees. The most fundamental object of this theory is the uncertain measure, a type of set function satisfying some of the axioms that apply to measures, except that uncertain measures are only subadditive. Liu then introduced the uncertain variable which is a measurable function from an uncertainty space to the set of real numbers. Subsequently, further research led to applications in mathematical finance, including option pricing. This direction opens an alternative approach to the theory of option pricing which, in the classical setting, is mostly based on a probability measure.

In [7], Liu gave a definition of the European option price that is based on the price formula in the classical setting. Another approach that can be pursued, instead of presenting the price formula as a definition, is to derive one by following the framework in probability theory. This approach starts by setting up a riskless portfolio in an arbitrage free market. This requires martingales, which in turn requires conditional expectations. This paper focuses on the latter requirement. Specifically, the conditional expectation of an uncertain variable with respect to a σ-algebra is presented and some of its properties established.

This paper defines the expectation of a nonnegative uncertain variable as a Choquet integral with respect to a capacity, which is a non-additive measure. Further, it adapts the Radon-Nikodym theorem for capacities [5]. We point out below that an uncertain measure in a continuous uncertainty space is a capacity.

This paper is organized as follows: In Section 2, basic definitions and theorems related to uncertainty theory and Choquet integrals are stated. Section 3 presents the Radon-Nikodym theorem for uncertain measures. Section 4 gives the main results on the conditional expectation, its properties, and a brief introduction to martingales.
2. Preliminaries
In this section, basic concepts in uncertainty theory and of Choquet integrals needed in the discussion are presented.

2.1. Uncertainty Theory
These definitions and results are adapted from Liu [8].

Definition 2.1. [8] Let $\Gamma$ be a nonempty set, and let $\mathcal{F}$ be a $\sigma$-algebra over $\Gamma$. The pair $(\Gamma, \mathcal{F})$ is referred to as a measurable space. Each element $A \in \mathcal{F}$ is called an event. A set function $\mu: \mathcal{F} \rightarrow [0, 1]$ is said to be an uncertain measure if it satisfies the following conditions:

(i) (Normality) $\mu(\Gamma) = 1$;
(ii) (Duality) For any event $A$, $\mu(A) + \mu(A^c) = 1$;
(iii) (Countable Subadditivity) For every countable sequence of events $\{A_n\}$,

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} \mu(A_n).$$

The triple $(\Gamma, \mathcal{F}, \mu)$ is called an uncertainty space.

Theorem 2.2. [8] (Monotonicity) An uncertain measure $\mu$ is a monotone increasing set function. That is, for every event $A_1$ and $A_2$ with $A_1 \subset A_2$,

$$\mu(A_1) \leq \mu(A_2).$$

Example 2.3. Suppose $\mathcal{F}$ is the power set of $\mathbb{N}$ and the set function $\mu: \mathcal{F} \rightarrow [0, 1]$ is defined by

$$\mu(A) = \begin{cases} 
0 & \text{if } A = \emptyset \\
0.3 & \text{if } 1 \in A, A \neq \mathbb{N} \\
0.7 & \text{if } 1 \notin A, A \neq \emptyset \\
1 & \text{if } A = \mathbb{N}.
\end{cases}$$

It can be verified that $\mu$ is an uncertain measure.

The following definition of a continuous uncertainty space is a consequence of the characterization found in Gao [3].

Definition 2.4. [3] An uncertainty space $(\Gamma, \mathcal{F}, \mu)$ is said to be continuous if for any sequence of events $\{A_n\}$ with $A_1 \subset A_2 \subset \cdots$, then

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \lim_{n \to \infty} \mu(A_n).$$

In this case, we say that $\mu$ is continuous.

Definition 2.5. If $\mathcal{F}$ is a $\sigma$-algebra on a nonempty set $\Gamma$, then a function $f: \Gamma \rightarrow \mathbb{R}$ is said to be $\mathcal{F}$-measurable, or simply measurable, if for any Borel set $B$ of real numbers, the set $\{\gamma \in \Gamma: f(\gamma) \in B\}$ is in $\mathcal{F}$.

Definition 2.6. [8] Given an uncertainty space $(\Gamma, \mathcal{F}, \mu)$, an uncertain variable is an $\mathcal{F}$-measurable function from $\Gamma$ to the set of real numbers.

As in classical probability, certain statements involving uncertain variables should be understood in the almost sure sense. Thus, for instance, when we say $\xi_1 \leq \xi_2$ where $\xi_1$ and $\xi_2$ are uncertain variables, we mean $\xi_1 \leq \xi_2 \mu$-almost surely (or $\mu$-a.s., or simply a.s.), i.e., $\mu(\gamma \in \Gamma: \xi_1(\gamma) \leq \xi_2(\gamma)) = 1$.

In the following, the notation $\{\xi \geq r\}$ is shorthand for the set $\{\gamma \in \Gamma: \xi(\gamma) \geq r\}$. The meaning of $\{\xi \leq r\}$ is analogous.
Definition 2.7. [6] Let $\xi$ be an uncertain variable. Then the expected value of $\xi$ is defined by

$$E[\xi] = \int_{0}^{\infty} \mu(\{\xi \geq r\}) \, dr - \int_{-\infty}^{0} \mu(\{\xi < r\}) \, dr,$$

provided that at least one of the two integrals is finite.

2.2. Choquet Integrals

We now define the Choquet integral with respect to an uncertain measure and present its properties. Choquet [1] originally introduced the integral with respect to a nonadditive set function called a capacity. The definition of a capacity, as well as its relationship with uncertain measures, is given at the end of this section.

Definition 2.8. Let $(\Gamma, F)$ be a measurable space and $f : \Gamma \to \mathbb{R}$ a measurable function on $(\Gamma, F)$. The Choquet integral of $f$ with respect to an uncertain measure $\mu$ on $\Gamma$ is defined by

$$(C) \int f \, d\mu = \int_{0}^{\infty} \mu(\{f \geq r\}) \, dr - \int_{-\infty}^{0} \mu(\{f < r\}) \, dr,$$

where the integrals on the right-hand side are taken in the sense of Lebesgue.

If $-\infty < (C) \int f \, d\mu < \infty$, we say that $f$ is Choquet integrable. For $A \in F$, we will write $(C) \int_A f \, d\mu$ for $(C) \int 1_A f \, d\mu$. Note that when the measurable function $f$ is nonnegative,

$$(C) \int_A f \, d\mu = \int_{0}^{\infty} \mu(A \cap \{f \geq r\}) \, dr.$$

Remark 2.9. The expected value of an uncertain variable $\xi$ is given by

$$E[\xi] = (C) \int \xi \, d\mu.$$

The following are the basic properties of the Choquet integral adapted from Denneberg [2] while Property (5) is found in Grabisch [4]. The properties there, stated in terms of capacities (defined below in Definition 2.11), apply as well for uncertain measures since the proofs would be very similar. For completeness, we also state these properties in terms of integrals over $A \in F$.

Proposition 2.10. Let $(\Gamma, F, \mu)$ be an uncertainty space, $f, g : \Gamma \to \mathbb{R} = [-\infty, +\infty]$ measurable functions on $(\Gamma, F, \mu)$, $A \in F$, and $k \in \mathbb{R}$. Then the following properties hold:

1. $(C) \int 1_A \, d\mu = \mu(A)$;
2. (Constant preserving) $(C) \int k \, d\mu = k$ and $(C) \int_A k \, d\mu = k\mu(A)$;
3. (Positive homogeneity) $(C) \int kf \, d\mu = k \cdot (C) \int f \, d\mu$ and $(C) \int_A kf \, d\mu = k \cdot (C) \int_A f \, d\mu$, where $k \geq 0$;
4. (Monotonicity) If $f \leq g$, then $(C) \int f \, d\mu \leq (C) \int g \, d\mu$ and $(C) \int_A f \, d\mu \leq (C) \int_A g \, d\mu$;
5. $(C) \int (f + k1_A) \, d\mu = (C) \int f \, d\mu + k\mu(A)$.
(6) (Translation invariance) \((C)\int (f + k) \, d\mu = (C)\int f \, d\mu + k\mu (A)\).

Each of the statements above can be written in terms of expectations, which we do not state explicitly anymore.

**Definition 2.11.** [5] Let \(\Omega\) be a nonempty set, and \(\mathcal{F}\) a \(\sigma\)-algebra over \(\Omega\). A set function \(\psi: \mathcal{F} \rightarrow \mathbb{R}_+\) is called a capacity if it satisfies the following conditions:

(i) \(\psi (\emptyset) = 0\);

(ii) (Subadditivity) For all events \(A\) and \(B\), \(\psi (A \cup B) \leq \psi (A) + \psi (B)\);

(iii) (Monotonicity) For all events \(A\) and \(B\), if \(A \subset B\), then \(\psi (A) \leq \psi (B)\);

(iv) (Continuity) For every increasing sequence of events \(A_1, A_2, \ldots\), we have

\[
\psi \left( \bigcup_{n=1}^{\infty} A_n \right) = \lim_{n \to \infty} \psi (A_n).
\]

Conditions (ii) and (iv) imply countable subadditivity. From the definition above, it also follows that an uncertain measure in a continuous space is a capacity.

### 3. Radon-Nikodym Theorem for Uncertain Measures

This section presents the Radon-Nikodym theorem for uncertain measures adapted from Graf [5] where the theorem is established for capacities. Since a continuous uncertain measure is a capacity, the main results in [5] immediately follow.

**Definition 3.1.** Let \((\Gamma, \mathcal{F}, \mu)\) be an uncertainty space. An uncertain measure \(\nu\) on \((\Gamma, \mathcal{F})\) is absolutely continuous relative to \(\mu\), denoted by \(\nu \ll \mu\), if for any event \(A\),

\[\mu (A) = 0 \implies \nu (A) = 0\.

**Definition 3.2.** [5] Let \((\Gamma, \mathcal{F})\) be a measurable space, \(\mu\) an uncertain measure, and \(\nu\) a capacity. The pair \((\nu, \mu)\) is said to possess the Strong Decomposition Property if, for every \(\alpha > 0\), there exists \(A_\alpha \in \mathcal{F}\) such that the following conditions hold:

(i) For all events \(A\) and \(B\), \(B \subset A \subset A_\alpha\) implies \(\alpha (\mu (A) - \mu (B)) \leq \nu (A) - \nu (B)\);

(ii) For any event \(A\), \(\alpha (\mu (A) - \mu (A \cap A_\alpha)) \geq \nu (A) - \nu (A \cap A_\alpha)\).

**Theorem 3.3.** [5] (Radon-Nikodym Theorem for Uncertain Measures) Let \((\Gamma, \mathcal{F})\) be a measurable space, \(\mu\) a continuous uncertain measure and \(\nu\) a capacity. Then there exists a measurable function \(f: \Gamma \rightarrow \mathbb{R}_+\) such that

\[\nu (A) = (C)\int_{A} f \, d\mu\]

for all \(A \in \mathcal{F}\) if \(\nu\) is absolutely continuous relative to \(\mu\) and \((\nu, \mu)\) has the Strong Decomposition Property.

Note that only the absolute continuity is required to establish the Radon-Nikodym theorem for probability measures. For a proof, see Graf [5].

### 4. Conditional Expectation for Uncertain Measures

In this section, given a measurable space \((\Gamma, \mathcal{F})\) we define a conditional expectation of a nonnegative uncertain variable given a sub-\(\sigma\)-algebra of \(\mathcal{F}\) and determine some of its basic properties.
4.1. Definition, Existence and Uniqueness

**Theorem 4.1.** Let $(\Gamma, \mathcal{F}, \mu)$ be a continuous uncertainty space. For an uncertain variable $\xi \geq 0$, define

$$\nu(A) = (c) \int_A \xi \, d\mu$$

for all $A \in \mathcal{F}$. Then $\nu$ is a capacity on $\mathcal{F}$. Moreover, $\nu$ is absolutely continuous relative to $\mu$ and $(\nu, \mu)$ has the strong decomposition property on $\mathcal{F}$.

**Proof:**

The fact that $\nu$ is a capacity is shown in Graf [5].

Next, we show that $\nu$ is absolutely continuous relative to $\mu$. Suppose $A \in \mathcal{F}$ with $\mu(A) = 0$. We need to show that $\nu(A) = 0$ as well. Indeed,

$$\nu(A) = (c) \int_A \xi \, d\mu = \int_0^\infty \mu(A \cap \{\xi \geq r\}) \, dr \leq \int_0^\infty \mu(A) \, dr = \int_0^\infty 0 \, dr = 0.$$

Since $\nu(A) \geq 0$, then $\nu(A) = 0$.

Finally, we show that $(\nu, \mu)$ has the S.D.P. That is, for every $\alpha > 0$, there exists some event $A_\alpha$ such that the following conditions hold:

(i) For all events $A$ and $B$, $B \subset A \subset A_\alpha$ implies $\alpha(\mu(A) - \mu(B)) \leq \nu(A) - \nu(B)$;

(ii) For any event $A$, $\alpha(\mu(A) - \mu(A \cap A_\alpha)) \geq \nu(A) - \nu(A \cap A_\alpha)$.

We take $A_\alpha = \{\xi \geq \alpha\}$. Since $\xi$ is an uncertain variable, it follows that $A_\alpha \in \mathcal{F}$.

(i) Let $A, B \in \mathcal{F}$ and $B \subset A \subset \{\xi \geq \alpha\}$. Then

$$\nu(A) - \nu(B) = (c) \int_A \xi \, d\mu - (c) \int_B \xi \, d\mu$$

$$= \int_0^\infty \mu(A \cap \{\xi \geq r\}) \, dr - \int_0^\infty \mu(B \cap \{\xi \geq r\}) \, dr$$

$$= \int_0^\alpha \mu(A \cap \{\xi \geq r\}) \, dr + \int_\alpha^\infty \mu(A \cap \{\xi \geq r\}) \, dr$$

$$- \int_0^\alpha \mu(B \cap \{\xi \geq r\}) \, dr - \int_\alpha^\infty \mu(B \cap \{\xi \geq r\}) \, dr$$

$$\geq \int_0^\alpha \mu(A \cap \{\xi \geq r\}) \, dr + \int_\alpha^\infty \mu(A \cap \{\xi \geq r\}) \, dr$$

$$- \int_0^\alpha \mu(B \cap \{\xi \geq r\}) \, dr - \int_\alpha^\infty \mu(B \cap \{\xi \geq r\}) \, dr$$

$$= \int_0^\alpha \mu(A \cap \{\xi \geq r\}) \, dr - \int_0^\alpha \mu(B \cap \{\xi \geq r\}) \, dr$$

$$= \int_0^\alpha \mu(A \cap \{\xi \geq r\}) \, dr - \alpha \mu(B).$$

Since $\{\xi \geq r\} \supseteq \{\xi \geq \alpha\}$ when $r \in [0, \alpha]$, we have

$$\mu(\{\xi \geq r\} \cap A) \geq \mu(\{\xi \geq \alpha\} \cap A).$$
Therefore,
\[
\nu(A) - \nu(B) \geq \int_0^\alpha \mu(A \cap \{\xi \geq \alpha\}) \, dr - \alpha \mu(B) \\
= \int_0^\alpha \mu(A) \, dr - \alpha \mu(B) \\
= \alpha [\mu(A) - \alpha \mu(B)].
\]

(ii) Let \( A \in \mathcal{F} \). Then

\[
\nu(A) - \nu(A \cap A_\alpha) = (c)\int_A \xi \, d\mu - (c)\int_{A \cap A_\alpha} \xi \, d\mu \\
= \int_0^\alpha \mu(A \cap \{\xi \geq r\}) \, dr - \int_0^\alpha \mu(A \cap A_\alpha \cap \{\xi \geq r\}) \, dr \\
= \int_0^\alpha \mu(A \cap \{\xi \geq r\}) \, dr + \int_0^\alpha \mu(A \cap \{\xi \geq \alpha\} \cap \{\xi \geq r\}) \, dr \\
- \int_0^\alpha \mu(A \cap \{\xi \geq \alpha\} \cap \{\xi \geq r\}) \, dr - \int_\alpha^\infty \mu(A \cap \{\xi \geq \alpha\} \cap \{\xi \geq r\}) \, dr \\
= \int_0^\alpha \mu(A \cap \{\xi \geq r\}) \, dr + \int_\alpha^\infty \mu(A \cap \{\xi \geq \alpha\} \cap \{\xi \geq r\}) \, dr \\
- \int_0^\alpha \mu(A \cap \{\xi \geq \alpha\}) \, dr - \int_\alpha^\infty \mu(A \cap \{\xi \geq \alpha\}) \, dr \\
\leq \int_0^\alpha \mu(A) \, dr - \int_0^\alpha \mu(A \cap A_\alpha) \, dr \\
= \alpha [\mu(A) - \mu(A \cap A_\alpha)].
\]

Consequently, (ii) holds and the result follows.

Now, let \( \mathcal{G} \) be a sub-\( \sigma \)-algebra of \( \mathcal{F} \), that is, \( \mathcal{G} \subset \mathcal{F} \) and \( \mathcal{G} \) is itself a \( \sigma \)-algebra. Given a nonnegative uncertain variable \( \xi \), define the capacity \( \nu \) on \( \mathcal{G} \) by \( \nu(A) = (c)\int_A \xi \, d\mu \) for all \( A \in \mathcal{G} \).

By the Radon-Nikodym Theorem for Uncertain Measures (see Theorem 3.3 above), there then exists a \( \mathcal{G} \)-measurable map \( \zeta : \Gamma \to \mathbb{R}_+ \) such that \( \nu(A) = (c)\int_A \zeta \, d\mu \) for all \( A \in \mathcal{G} \). This means that

\[
(c)\int_A \xi \, d\mu = (c)\int_A \zeta \, d\mu \quad \text{for all } A \in \mathcal{G}
\]

or equivalently, \( E[\xi 1_A] = E[\zeta 1_A] \) for all \( A \in \mathcal{G} \). We then use this nonnegative uncertain variable \( \zeta \) to define the conditional expectation of \( \xi \) with respect to \( \mathcal{G} \).

**Definition 4.2.** Let \( \mathcal{G} \) be a sub-\( \sigma \)-algebra of \( \mathcal{F} \), and let \( \xi \geq 0 \) be an uncertain variable. We say that the nonnegative uncertain variable \( \zeta \) is a conditional expectation of \( \xi \) with respect to \( \mathcal{G} \), and denote it by \( E[\xi | \mathcal{G}] \), if

(i) \( \zeta \) is \( \mathcal{G} \)-measurable; 
(ii) \( E[\xi 1_A] = E[\zeta 1_A] \) for all \( A \in \mathcal{G} \).
The uncertain variable $\zeta$ satisfying the two conditions in the definition may not be unique. Strictly speaking, there is no unique conditional expectation. However, as the next theorem shows, any two such “versions” are equal $\mu$-a.s.. Thus, as in classical probability, when we refer to “the” conditional expectation of an uncertain variable, we are referring to the equivalence class of all uncertain variables satisfying the definition. If $\zeta$ is a version of the conditional expectation, we may write $E[\xi|\mathcal{G}] = \zeta$ $\mu$-a.s., or accept the standard notation $E[\xi|\mathcal{G}] = \zeta$ used in classical probability.

**Theorem 4.3.** Let $\mathcal{G}$ be a sub-$\sigma$-algebra of $\mathcal{F}$. Then any two versions of the conditional expectation of the uncertain variable $\xi \geq 0$ are equal $\mu$-a.s.

**Proof:**
Suppose $\zeta_1$ and $\zeta_2$ both satisfy all conditions of Definition 4.2. Then

$$E[\zeta_11_A] = E[\zeta_21_A], \text{ for all } A \in \mathcal{G}.$$  

For $A_n = \{\zeta_2 - \zeta_1 \geq \frac{1}{n}\}$ with $n \in \mathbb{N}$, we have $A_n \in \mathcal{G}$. We then have

$$E[\zeta_11_{A_n}] = E[\zeta_21_{A_n}] \geq E\left[\left(\frac{\zeta_1 + \frac{1}{n}}{E}\right)1_{A_n}\right] = E[\zeta_11_{A_n}] + \frac{1}{n} \mu(A_n)$$

where in the last line, we used Proposition 2.10.(5). Therefore, $\mu(A_n) = 0$ for all $n \in \mathbb{N}$. Since $\mu$ is continuous, it follows that $\mu(\zeta_2 > \zeta_1) = \lim_{n \to \infty} \mu(A_n) = 0$. By a similar argument, we also have $\mu(\zeta_2 < \zeta_1) = 0$. Consequently, $\mu(\zeta_1 \neq \zeta_2) \leq \mu(\zeta_2 > \zeta_1) + \mu(\zeta_2 < \zeta_1) = 0$. Therefore, $\zeta_1 = \zeta_2$ $\mu$-a.s.

### 4.2. Properties of Conditional Expectations for Uncertain Measures

**Theorem 4.4.** Let $\xi, \eta \geq 0$ be uncertain variables and $\mathcal{G}$ a sub-$\sigma$-algebra of $\mathcal{F}$.

1. If $\xi$ is integrable, then $E[\xi|\mathcal{G}]$ is also integrable;
2. (Stability) If $\xi$ is $\mathcal{G}$-measurable, then $E[\xi|\mathcal{G}] = \xi$;
3. (Constant preserving) $E[a|\mathcal{G}] = a$ for all $a \geq 0$;
4. (Translation homogeneity) $E[b\xi|\mathcal{G}] = bE[\xi|\mathcal{G}]$ for all $b \geq 0$;
5. (Translation invariance) $E[\xi + c|\mathcal{G}] = E[\xi|\mathcal{G}] + c$ for all $c \geq 0$;
6. (Monotonicity) If $\xi \leq \eta$, then $E[\xi|\mathcal{G}] \leq E[\eta|\mathcal{G}]$.

**Proof:**

1. This is true since we can take $A = \Gamma$ in Definition (4.2) and so $0 \leq (C)\int E[\xi|\mathcal{G}] \, d\mu = (C)\int \xi \, d\mu < \infty$.
2. The first condition in the definition is assumed and the second holds trivially.
3. This follows from the previous item since any constant map is $\mathcal{G}$-measurable.
4. Fix $A \in \mathcal{G}$. Since $bE[\xi|\mathcal{G}]$ is $\mathcal{G}$-measurable and

$$E[E[b\xi|\mathcal{G}]1_A] = E[b\xi 1_A] = bE[\xi 1_A] = bE[E[\xi|\mathcal{G}]1_A] = E[bE[\xi|\mathcal{G}]1_A],$$

then $E[b\xi|\mathcal{G}] = bE[\xi|\mathcal{G}]$. 

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(5) Again, fix $A \in \mathcal{G}$. Observe that

$$E [E[\xi + c|\mathcal{G}]1_A] = E[(\xi + c)1_A] = E[\xi 1_A] + c\mu(A);$$

we used Proposition 2.10.(6) in the last equation. Using the same property again,

$$E((E[\xi|\mathcal{G}] + c)1_A) = E[E[\xi|\mathcal{G}]1_A + c1_A] = E[E[\xi|\mathcal{G}]1_A] + c\mu(A) = E[\xi 1_A] + c\mu(A).$$

Since $E [E[\xi + c|\mathcal{G}]1_A] = E[(E[\xi|\mathcal{G}] + c)1_A]$ and $E[\xi|\mathcal{G}] + c$ is $\mathcal{G}$-measurable, then $E[\xi + c|\mathcal{G}] = E[\xi|\mathcal{G}] + c$.

(6) We want to show that $\mu(A) = 0$ where $A := \{ \gamma \in \Gamma : E[\xi|\mathcal{G}](\gamma) > E[\eta|\mathcal{G}](\gamma) \}$. For $n \in \mathbb{N}$, define $A_n = \{ \gamma \in \Gamma : E[\xi|\mathcal{G}](\gamma) - E[\eta|\mathcal{G}](\gamma) \geq 1/n \}$. Observe that $\{A_n\}$ is an increasing sequence of events in $\mathcal{G}$, since the conditional expectations are $\mathcal{G}$-measurable, and that their limit is $A$. For a fixed $n \in \mathbb{N}$,

$$E[\xi 1_{A_n}] = E[E[\xi|\mathcal{G}]1_{A_n}] \geq E \left[ E[\eta|\mathcal{G}] + \frac{1}{n} \right] 1_{A_n} = E(E[\eta|\mathcal{G}]1_{A_n}) + \frac{1}{n}\mu(A_n)$$

$$= E[\eta 1_{A_n}] + \frac{1}{n}\mu(A_n) \geq E[\xi 1_{A_n}] + \frac{1}{n}\mu(A_n).$$

Therefore, $\mu(A_n) = 0$ for each $n \in \mathbb{N}$, and so

$$\mu(A) = \lim_{n \to \infty} \mu(A_n) = 0$$

since $\mu$ is continuous.

We end this section with definitions leading to the notion of martingales.

**Definition 4.5.** An increasing collection $\{\mathcal{F}_t\}_{t \geq 0}$ of $\sigma$-algebras in an uncertainty space $(\Gamma, \mathcal{F}, \mu)$ is called a *filtration*. The tuple $(\Gamma, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mu)$ is called a *filtered uncertainty space*.

**Definition 4.6.** Given an uncertainty space $(\Gamma, \mathcal{F}, \mu)$, a collection $\{\xi_t\}_{t \geq 0}$ of uncertain variables is said to be adapted to the filtration $\{\mathcal{F}_t\}_{t \geq 0}$ if $\xi_t$ is $\mathcal{F}_t$-measurable for every $t \geq 0$.

**Definition 4.7.** Given a filtration $\{\mathcal{F}_t\}_{t \geq 0}$, a collection $\{\xi_t\}_{t \geq 0}$ of nonnegative uncertain variables is a *martingale* if

1. $\xi_t$ is $\mathcal{F}_t$-measurable for all $t \geq 0$;
2. $E(\xi_t) < \infty$ for all $t \geq 0$; and
3. $E[\xi_t|\mathcal{F}_s] = \xi_s$ a.s. if $0 \leq s \leq t$.

5. Conclusion and Recommendations

In this paper, we have introduced the conditional expectation in an uncertainty space, specifically for nonnegative uncertain variables. We have also established some of its basic properties. These results are essential in developing the notion of martingales which was introduced in the previous section.

A more general definition of conditional expectations, one that is not just restricted to nonnegative uncertain variables, as well as the properties of martingales can also be explored.
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