Chiral Compactifications of 6D Conformal Theories

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Abstract

We construct chiral $N = 1$ gauge theories in 4D by compactifying the 6D Blum-Intriligator $(1, 0)$ theories of 5-branes at $A_k$ singularities on $T^2$ with a nontrivial bundle of the global $U(1)$ symmetry of these theories.

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## 1 Introduction

In recent years it has been realized that many 3+1D gauge theories can be obtained as special low-energy limits of compactified 5+1D superconformal theories. Some of the known 5+1D theories are the $\mathcal{N} = (2,0)$ theory \cite{1}, the $E_8 \mathcal{N} = (1,0)$ theory \cite{2} and the Blum-Intriligator (BI) \cite{3} theories of $N$ M5-branes at an $A_{k-1}$ singularity.

Indeed, part of the appeal of these theories is that by compactification on $T^2$ we can get various gauge theories in 3+1D at low-energy. Thus, $\mathcal{N} = 4$ SYM is obtained from the $(2,0)$-
theory and $\mathcal{N} = 2$ SYM with various matter content is obtained from the $E_8$ $\mathcal{N} = (1, 0)$ theory.

Starting with the 5+1D BI theories we can compactify on $S^1$ to get, at low-energies, the $\mathcal{N} = 2$ quiver gauge theories with gauge group $SU(N)^k$ and bi-fundamental matter hypermultiplets. One can also realize a mass to the hypermultiplets by using the global $U(1)$ symmetry of the BI theories. Turning on a small background Wilson line for that $U(1)$ corresponds at low energy to turning on the mass.

In this paper we will construct chiral 3+1D theories from the BI theories. As an intermediate step, we start with a 4+1D hypermultiplet. Given a hypermultiplet in 4+1D we can construct a low-energy chiral multiplet as follows. Let us take an infinite 5th direction and let us give the fermions of the hypermultiplet a mass $m(x_5)$ that varies along the 5th direction from $m = -\infty$ at $x_5 = -\infty$ to $m = \infty$ at $x_5 = \infty$ (see [8]). As we shall review below, if we also let the scalar fields have masses $\sqrt{m^2 + \frac{dm}{dx_5}}$ then in the remaining 4 dimensions $\mathcal{N} = 1$ supersymmetry is preserved and at low energies we get a chiral multiplet localized near the point where $m(x_5) = 0$. Thus, by varying the mass of the hypermultiplets in a 5D gauge theory along the 5th direction, we can obtain, at low-energies, a chiral gauge theory in 4D.

A 5D gauge theory is only defined as a low-energy effective action. However, we can realize it as a 6D theory compactified on a circle. We would like to elevate the construction of chiral gauge theories to 6D. One motivation for that is that a 6D realization often provides insight into the strong coupling behavior of the theory. The 6D theories that we will use are the BI theories and the construction of chiral gauge theories from their compactifications is the purpose of this paper.

The paper is organized as follows. In section (2) we review the example of a 4+1D hypermultiplet. In section (3) we study the compactification of a general 5+1D theory. In section (4) we discuss the BI theories and their compactification.

## 2 A free hypermultiplet

In this section we will study a free hypermultiplet in 5+1D and 4+1D. The reason for studying this simple system is that it gives us an explicit realization of the mechanism which
produces chiral matter in 3+1D. We will later apply the same type of compactification to obtain chiral matter in 3+1D starting from 5+1D theories.

We will show that a 4+1D hypermultiplet with a mass that varies along the 5th direction preserves $\mathcal{N} = 1$ SUSY in 3+1D and gives rise to chiral multiplets. The 4+1D hypermultiplet with a varying mass can be obtained from a 5+1D hypermultiplet compactified on a circle and coupled to a background field.

2.1 A 5+1D chiral hypermultiplet

A convenient way of getting the quantum numbers of a 5+1D hypermultiplet is to start from 9+1D super Yang-Mills reduced to 5+1D. This theory comprises of a single multiplet under the $\mathcal{N} = (1, 1)$ SUSY. However, under an $\mathcal{N} = (1, 0)$ subgroup of the supersymmetry algebra it decomposes into a vector-multiplet and a hypermultiplet. The statements below follow easily by thinking about the system in this way.

A hypermultiplet in 5+1D (with $\mathcal{N} = (1, 0)$ supersymmetry) contains 4 real scalars and one chiral fermion. It is convenient to decompose the components under the Lorentz group $SO(5, 1)$, the R-symmetry group $SU(2)_R$ and the global flavor symmetry $SU(2)_F$. Under $SO(5, 1) \times SU(2)_R \times SU(2)_F$ the SUSY generators $Q^i_\alpha$ transform as $(\mathbf{4}, \mathbf{2}, \mathbf{1})$. Note that both $\mathbf{4}$ and $\mathbf{2}$ are pseudo-real representations so one can add a reality condition to have 8 real SUSY generators. Here $i = 1, 2$ is an index of the $\mathbf{2}$ of $SU(2)_R$ and $\alpha = 1 \ldots 4$ is an index of the $\mathbf{4}$ of $SO(5, 1)$. We will assume that the hypermultiplet is charged under $SU(2)_F$. The fermions of the hypermultiplet transform as $(\mathbf{4}, \mathbf{1}, \mathbf{2})$ with an added reality condition. We will denote them by $\psi^a_\alpha$ with $a = 1, 2$ an index of $SU(2)_F$. The bosons transform as $(\mathbf{1}, 2, 2)$ and will be denoted by $\phi^{ia}$. Recall that the Dirac matrices, $\Gamma^\mu_{a\beta}$ ($\mu = 0 \ldots 5$), of $SO(5, 1)$ can be chosen to be anti-symmetric. In the rest of the paper they will be anti-symmetric. We will also use the anti-symmetric $\epsilon_{ij}$ of the $\mathbf{2}$ of $SU(2)_R$ to lower and raise the indices $i, j = 1, 2$.

The reality conditions are,

$$
(\phi^{ia})^\dagger = C^a_b C^i_j \phi^{jb}, \quad (\psi^a_\beta)^\dagger = C^a_b C^\alpha_\beta \psi^a_\alpha,
$$

where $C^a_b$, $C^i_j$ and $C^\alpha_\beta$ are the charge conjugation matrices of (respectively) $\mathbf{2}$ of $SU(2)_F$. 


of $SU(2)_R$ and $4$ of $SO(5, 1)$.

The action is

$$S = \int d^6x \left( -\frac{1}{4} \epsilon_{ij} \epsilon_{ab} \partial_\mu \phi^i \partial^\mu \phi^j + \frac{1}{2} \epsilon_{ab} \psi^\alpha \Gamma^{\mu \alpha \beta} \partial_\mu \psi^b \right).$$

Our sign conventions are $\epsilon_{12} = -\epsilon_{12} = 1$. The equations of motion derived from this action are

$$\square \phi^{ja} = 0, \quad \Gamma^{\mu, \alpha \beta} \partial_\mu \psi^\alpha = 0.$$

The supersymmetry transformations are:

$$\delta \phi^{ja} = 2 \eta^\alpha \psi^a, \quad \delta \psi^a = \epsilon_{ij} \epsilon_{\beta \gamma} \Gamma^{\mu \beta \gamma} \partial_\mu \phi^j.$$

### 2.2 A 4+1D massive hypermultiplet

Now we will consider a massive hypermultiplet in 4+1D. The quantum numbers, action and supersymmetry transformations of this can easily be obtained from the 5+1D case. We consider a 5+1D hypermultiplet with a specific $x^5$ dependence.

$$\phi^a(x, x^5) = \phi^b(x) (e^{imx^5 \tau^3})^a_b$$

$$\psi^a(x, x^5) = \psi^b(x) (e^{imx^5 \tau^3})^a_b$$

(2)

Here $x$ stands for $x^0, x^1, x^2, x^3, x^4$ and

$$\tau^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

is inserted to give the right sign in the exponential. $\phi^1$ and $\phi^2$ must have different signs because of the reality condition ($\blacksquare$). The quantum numbers are the same as in 5+1D. A 4+1D massive hypermultiplet contains 4 bosons $\phi^{ia}$ in the $(1, 2, 2)$ of $SO(4, 1) \times SU(2)_R \times SU(2)_F$, where $SO(4, 1)$ is the Lorentz group and $SU(2)_R$ and $SU(2)_F$ are the R-symmetry and flavor symmetry, respectively. It also has fermions $\psi^{\alpha a}$ in the $(4, 1, 2)$. Recall that the representation $4$ of $SO(4, 1)$ has an invariant anti-symmetric form $\epsilon_{\alpha \beta}$ which we will use to lower and raise indices. From the 5+1D point of view that is just $\Gamma^5$ which commutes with $SO(4, 1)$ transformations. The action in 4+1D is obtained simply by plugging the fields in (2) into the 5+1D action.

$$S = \int d^5x \left( -\frac{1}{4} \epsilon_{ij} \epsilon_{ab} \partial_\mu \phi^i \partial^\mu \phi^j + m^2 \phi^i \phi^j + \frac{1}{2} \epsilon_{ab} \psi^\alpha \Gamma^{\mu \alpha \beta} \partial_\mu \psi^b + \frac{1}{2} \epsilon_{ab} \psi^\alpha \Gamma^{\mu \alpha \beta} \partial_\mu \psi^b \right).$$
The equations of motion follow:

\[(\Box + m^2)\phi^{ia} = 0, \quad \Gamma^\mu_{\alpha\beta} \partial_\mu \psi^a_\alpha + im(\tau^3)_b^a \Gamma^5_{\alpha\beta} \psi^b_\alpha = 0.\]

The reality conditions on the fields are the same as in 5+1D, as is obvious from the way we obtained them. The SUSY transformations are obtained from the 5+1D transformations:

\[
\delta\phi^{ia} = 2\eta^{a\alpha}\psi^a_\alpha, \\
\delta\psi^a_\alpha = \epsilon_{ij}\eta^{\beta j}\Gamma^\mu_{\alpha\beta} \partial_\mu \phi^{ia} + im\epsilon_{ij}\eta^{\beta j}\Gamma^5_{\alpha\beta}(\tau^3)_b^a \phi^{jb}. 
\] (3)

### 2.3 Variable mass

We will now discuss a reduction of the 4+1D massive hypermultiplet to 3+1D in a way that preserves half the supersymmetry (i.e. $N = 1$ in 3+1D) and can produce chiral multiplets. This reduction was also discussed in [8]. We pick a spatial direction $x^4$ and let the mass vary as a function of $x^4$ only. Let this function be $m(x^4)$. In the previous subsection we wrote down the action and supersymmetry transformations for a massive hypermultiplet. The mass, $m$, was constant. The question is what action should we use when $m$ is not constant.

The only condition the new action must fulfill is that it reduces to the usual one when $m$ is constant. However that only determines the action up to terms involving derivatives of $m$. Since we are interested in preserving some supersymmetry we will impose the condition that the action should be invariant under the transformations (3) for some $\eta$. Varying the above action, now with $m(x^4)$ a function, gives:

\[
\delta(S) = \int d^5x m'(x^4) \epsilon_{ij} \epsilon_{ab} (\tau^3)_c^b \eta^{\gamma i} (i\Gamma^4\Gamma^5)_{\gamma}^\alpha \psi^a_\alpha \phi^c.
\]

Here $m'(x_4) \equiv dm/dx^4$. Let us try adding the following term to the Lagrangian:

\[
L_{\text{new}} = \frac{1}{4} m'(x^4) \epsilon_{ij} (\tau^3)_c^b \epsilon_{ij} (\tau^3)_k^j \phi^{ia} \phi^{kc}.
\]

The supersymmetry variation of this term is:

\[
\delta(L_{\text{new}}) = \frac{1}{2} m'(x^4) \epsilon_{ij} (\tau^3)_c^b \epsilon_{ij} (\tau^3)_k^j \eta^{i\alpha} \psi^a_\alpha \phi^{kc}.
\]

We see that this term cancels $\delta(S)$ if

\[
(\tau^3)_j^i \eta^{\alpha j} = \eta^{\gamma i} (i\Gamma^4\Gamma^5)^\alpha_{\gamma}
\] (4)
This equation breaks half the supersymmetry and leaves $\mathcal{N} = 1$ in 3+1D.

We thus conclude that a sensible action for a hypermultiplet with a varying mass is:

$$S = \int d^5 x \left( -\frac{1}{4} \epsilon_{ij} \epsilon_{ab} (\partial_\mu \phi^i a \partial^\mu \phi^j b + m^2 \phi^i a \phi^j b - m' (x^4) (\tau^3)_c (\tau^3)_j \phi^i a \phi^j c) 
+ \frac{1}{2} \epsilon_{ab} \psi^a \Gamma^{\alpha \beta} \partial_\mu \psi^b + \frac{1}{2} i m \epsilon_{ab} (\tau^3)_c \psi^a \Gamma^{5 \alpha \beta} \partial_5 \psi^b \right).$$

(5)

It preserves the supersymmetry transformations (4) when $\eta$ solves (4). The equations of motion are:

$$\left( \Box + m(x^4)^2 \right) \phi^{i a} - m'(x^4) (\tau^3)_i (\tau^3)_b \phi^{j b} = 0,$$

(6)

$$\Gamma^{\mu \alpha \beta} \partial_\mu \psi^a \beta + i m (x^4) (\tau^3)_b \psi^a \beta \tau^3 = 0.$$

2.4 Chiral zero modes

As usual one can reduce the fields along the $x^4$ direction and find the modes seen from a 3+1D point of view. The $x^4$ direction is noncompact here. Later we will consider the compactified case. Let us find the massless modes in 3+1D. Since $\mathcal{N} = 1$ is preserved in 3+1D we know that the fields have to come in chiral multiplets. The bosons will have a 3+1D massless mode for every solution of (setting $y \equiv x^4$):

$$\left( -\frac{d^2}{dy^2} + m(y)^2 \right) \phi^{i a} - m'(y) (\tau^3)_i (\tau^3)_b \phi^{j b} = 0.$$

The fermions will have zero modes for every solution of:

$$(i \Gamma^4 \Gamma^5)^{\alpha \beta} \frac{d}{dy} \psi^a \beta + m(\tau^3)_b \psi^a \beta = 0.$$

We see that the bosonic equation is the square of the fermionic one in a certain sense and that the term proportional to $m'(y)$ is essential for this. The solution to the fermionic equation is:

$$\psi(y) = e^{-i \Gamma^4 \Gamma^5 \tau^3 \int_0^y m(y') dy'} \psi_0.$$

Here we use matrix notation and suppress indices. Both matrices $(i \Gamma^4 \Gamma^5)$ and $\tau^3$ have eigenvalues $+1$ and $-1$. For the solution to be normalizable it is thus necessary that either:

$$\int_0^y m(y') dy' \to \infty \quad \text{for } y \to \pm \infty.$$
or
\[ \int_0^y m(y')dy' \to -\infty \quad \text{for } y \to \pm\infty. \]

In the former case the solution is normalizable if \( \psi_0 \) has the same eigenvalue as \( i\Gamma^4\Gamma^5 \) and \( \tau^3 \) and in the latter case the eigenvalues must be opposite. In both cases we end up with two chiral spinors in 3+1D which are related by the reality condition \( (\text{I}) \) leaving one independent chiral spinor.

The solution to the bosonic equation is:
\[ \phi(y) = e^{-\tau_3^3 R} F \int_0^y m(y')dy' \phi_0, \]
where again we suppress indices. The \( \tau_3 \) matrices are written with a subindex to distinguish the R-symmetry and the flavor-symmetry. There is a normalizable solution exactly in the same two cases of \( \int_0^y m(y')dy' \) as above. In both cases there are two solutions which are related by the reality condition. So there is one massless complex boson in both cases. This one pairs up with the chiral fermion to give a massless \( \mathcal{N} = 1 \) chiral multiplet as we expect. (For a similar mechanism, see \([9]\).)

The condition on \( m(y) \) stated above implies in particular that \( m(y) \) crosses zero at some point. A particular example of an \( m(y) \) that obeys the condition is a function that goes to \( -m_0 \) for \( y \to -\infty \), crosses zero and goes to \( m_0 \) for \( y \to \infty \).

### 2.5 Flavor current multiplet

In subsection \((2.2)\) above, we generated a 4+1D mass by reduction from 5+1D requiring that the fields have a specific \( x^5 \) behavior \((2)\). If one just compactifies on a circle, the 4+1D theory will have a tower of Kaluza-Klein states with the lowest one being massless. The massless mode is the constant mode on the circle. The theory has a current, \( J_\mu \), associated with the \( U(1)_F \) symmetry. We can introduce a background gauge field, \( A_\mu \), that couples to this current. Creating a Wilson line for the background gauge field, \( A_\mu \), around the circle is equivalent to changing the periodicity condition of the 5+1D hypermultiplet fields. They will be identified with themselves up to a \( U(1)_F \) rotation. This gives them exactly the \( x^5 \) behavior of \((2)\). In a circle compactification with a Wilson line for \( A_\mu \) there will still be a Kaluza-Klein tower of states in 4+1D but their masses will be shifted with an amount
proportional to the Wilson line. The $U(1)_F$ is part of an $SU(2)_F$ symmetry. The 5+1D hypermultiplet has a current $J^A_\mu$ ($A = 1, 2, 3$ is an index of the $\mathfrak{g}$ of $SU(2)$) associated with the $SU(2)_F$ flavor symmetry. This Noether-current is easily found from the action.

By applying supersymmetry transformations to the current one finds that it is part of the following supermultiplet:

$$J^A_\mu = \frac{i}{4} \epsilon_{ij} \epsilon_{ab} (\tau^A)_c b (\phi^{jc} \partial_\mu \phi^{ja} - \partial_\mu \phi^{jc} \phi^{ja}) - i \frac{1}{2} \epsilon_{ab} (\tau^A)_c b \Gamma_\mu^{\alpha \beta} \psi_\alpha^a \psi_\beta^c,$$
$$S^j_\alpha = \epsilon_{ba} (\tau^A)_c b \phi^{ja} \psi_\alpha^c,$$
$$D^{ij}_A = \frac{1}{2} \epsilon_{ba} (\tau^A)_c b \phi^{ic} \phi^{ja}.$$  \tag{7}

Note that $D^{ij}_A$ is symmetric in $i$ and $j$. The SUSY transformations of these operators are:

$$\delta J^\mu_\mu = \epsilon_{ij} \eta^\alpha (\Gamma^\mu)_\alpha^\beta \partial_\nu S^j_\beta,$$
$$\delta S^j_\beta = \eta^\gamma \Gamma^\mu_\beta \gamma J^A_\mu + \epsilon_{ki} \eta^\gamma \Gamma^\mu_\gamma \partial_\mu D^{ij}_A,$$
$$\delta D^{ij}_A = \eta^\alpha S^j_\alpha + \eta^\alpha D^{ij}_A.$$

In the transformation of $J^A_\mu$ the equation of motion for $\psi_\alpha$ was used.

Since a mass in 4+1D comes from the component $A_5$ along the circle, a mass varying in the $x^4$ direction comes from an $A_5$ which varies along $x^4$. In other words, there is a nonzero field strength $F_{45}$. The usual way of coupling $A_\mu$ to a theory is by adding

$$\int d^6x J^A_\mu A^\mu$$

to the action plus a term proportional to $A^2$ in order to preserve gauge invariance. In the action (3) the terms proportional to $m$ and $m^2$ come from this coupling. What about the extra term needed for supersymmetry? We see that it is proportional to $D^{12(A=3)}$. Since $m'(x^4)$ is $F_{45}$ we see that the extra term is just proportional to

$$\int d^6x F_{45} D^{12(A=3)}.$$

We will apply these observations to more general systems in the next section. The important point is that the deformation of the Lagrangian can be expressed in terms of the current $J_\mu$ and its superpartner $D$ without referring to the specific fields of the theory.

8
3 Construction from 6D

We wish to analyze the situation starting from a general 5+1D theory. We start with a 5+1D theory with $\mathcal{N} = (1, 0)$ supersymmetry and a global $U(1)$ symmetry and we compactify it on $T^2$. We wish to put a background gauge field $A_\mu$ that is associated to the $U(1)$ symmetry along $T^2$ such that the first Chern class will be $c_1 = n$. The question is how do we do it while preserving half the supersymmetry.

3.1 The current multiplet

The 5+1D theory has a current $J_\mu$ associated with the $U(1)$ symmetry. The current is a member of an $\mathcal{N} = (1, 0)$ multiplet which also contains a fermionic partner $S_\alpha^i$ and a bosonic “D-term” partner $D^{ij}$ as we saw in subsection (2.5) for the free hypermultiplet. Here, $i, j = 1, 2$ are $SU(2)_R$ symmetry indices and $D^{ij}$ is symmetric. They satisfy:

\[
\begin{align*}
\delta J^\mu &= \epsilon_{ij} \eta^{\alpha i} (\Gamma^{\mu \nu})_{\alpha}^{\beta} \partial_\nu S_j^i \\
\delta S_j^i &= \eta^{i j} \Gamma_{\beta\gamma}^\mu J_\mu + \epsilon_{ki} \eta^{k \gamma} \Gamma_{\beta\gamma}^\mu \partial_\mu (D^{ij}) \\
\delta D^{ij} &= \eta^{\alpha i} S_\alpha^j + \eta^{\alpha j} S_\alpha^i
\end{align*}
\]

We claim that compactifying on $T^2$ and adding:

\[
S_1 = - \int (A_4 J_4 + A_5 J_5 + i F_{45} D^{12} + \cdots)
\]

(9)

to the action gives a supersymmetric theory with $\mathcal{N} = 1$ in the uncompactified 3+1D. The $i$ in the second term is necessary to make the action real, since $D^{12}$ is imaginary. The $(\cdots)$ represent $O(A_\mu^2)$ terms that are dictated by $U(1)$ gauge invariance. For example, if under a local $U(1)$ transformation

\[
\delta J_\mu = \partial_\mu \epsilon \Theta,
\]

we have to add $\frac{1}{2} A_\mu A^\mu \Theta$ to the Lagrangian.

In order to see that $\mathcal{N} = 1$ is unbroken we calculate the supersymmetry variation of $S_1$ using (8).
which is equal to zero if
\[(\Gamma^{45})^\alpha_\beta \eta^{\alpha_1} = i\eta^{\beta_1}, \quad (\Gamma^{45})^\alpha_\beta \eta^{\alpha_2} = -i\eta^{\beta_2}.\]

These two equations are complex conjugate of each other. We see that we are left with \(N = 1\) in 3+1D.

### 3.2 Example – a free hypermultiplet

After compactification on a \(T^2\) to 3+1D we would like to know the masses of the fields. There will be a Kaluza-Klein tower of fields. In the low energy limit we are, of course, only interested in the massless fields. Let us go back to the free hypermultiplet and calculate the Kaluza-Klein masses. We need only do it for the fermions because of \(N = 1\). The Dirac equation for the fermions reads
\[\Gamma^\mu \nabla_\mu \psi = 0\]

where \(\nabla_\mu = \partial_\mu + iA_\mu\) is the covariant derivative with respect to the \(U(1)\) symmetry. In our case the only nonzero components of \(A_\mu\) are \(A_4\) and \(A_5\). In reducing to 3+1D \(\psi\) can be written as
\[\psi = \psi_L \phi_L + \psi_R \phi_R\]

where \(\psi_L, \psi_R\) are left- and right-handed spinors in 3+1D and \(\phi_L, \phi_R\) are left- and right-handed spinors on \(T^2\). Plugging into the Dirac equation we get the following formula for the mass \(m\) in 3+1D.
\[
\begin{align*}
(\nabla_4 \Gamma_4 + \nabla_5 \Gamma_5) \phi_R &= m \phi_L \\
(\nabla_4 \Gamma_4 + \nabla_5 \Gamma_5) \phi_L &= -m^* \phi_R
\end{align*}
\]

The mass \(m\) is a complex number. The physical mass is the absolute value of \(m\). The phase can be transformed away by redefining \(\phi_L\), say. The phase would then show up in the couplings. In the free theory there is no meaning to them. We will just rotate the phase away for now and let \(m\) be real. We see that for \(m \neq 0\), \(\phi_L\) and \(\phi_R\) come in pairs. This implies that in 3+1D \(\psi_L\) and \(\psi_R\) come in pairs of the same mass. This is as it should be, since a chiral spinor that is charged under a \(U(1)\) symmetry cannot be massive. Both a
lethanded and a righthanded spinor are needed for a mass term. However for \( m = 0 \) there is no relation between a lefthanded solution and a righthanded one. For each solution of

\[
(\nabla_4 \Gamma_4 + \nabla_5 \Gamma_5) \phi_R = 0
\]

there is a massless righthanded fermion in 3+1D and for each solution of

\[
(\nabla_4 \Gamma_4 + \nabla_5 \Gamma_5) \phi_L = 0
\]

there is a massless lefthanded fermion in 3+1D.

Eq. (10) implies second order differential equations for \( \phi_L \) and \( \phi_R \):

\[
\begin{align*}
(\nabla_4^2 + \nabla_5^2 - F_{45}) \phi_L &= -m^2 \phi_L \\
(\nabla_4^2 + \nabla_5^2 + F_{45}) \phi_R &= -m^2 \phi_R
\end{align*}
\]

These equations are the same as the ones determining the boson masses. It had to be so due to the supersymmetry. In these equations \( A_\mu \) is a connection in a \( U(1) \)-bundle over \( T^2 \) and \( \phi_{L,R} \) are sections of this circle-bundle. The setup here is the same as a charged particle on a torus moving in a background magnetic field (Landau levels). For a general \( A_\mu \) the eigenvalues \( m \) are not known, to our knowledge.

We can say more about the case of \( m = 0 \). Here we find the zero modes of the Dirac equation in 2 dimensions for respectively lefthanded and righthanded spinors. The number of those will depend on the gauge field \( A_\mu \) but the difference between the number of lefthanded and righthanded zero modes is known as the index of the Dirac operator. It is equal to the first chern class, \( c_1 \), of the circle-bundle.

\[
c_1 = \frac{1}{2\pi} \int_{T^2} F_{45}
\]

For a generic gauge field there will be \( |c_1| \) solutions of one kind and 0 of the other kind. But for special gauge fields it could be different. An example of a special case is the case of \( A_\mu = 0 \). Here \( c_1 = 0 \). There is one zero mode of each chirality, namely the constant function.

We thus conclude that in the theories under consideration the hypermultiplets will give rise to \( c_1 \) massless chiral multiplets. Even in the special cases mentioned above this will also be the case, since the couplings generically will lift the accidental pairs and still leave us with \( c_1 \) massless chiral multiplets.
Now we will consider the special case of constant $F_{45}$, where the problem has an explicit solution. Let the first Chern class be $c_1 = n$. We will take the fields to obey the following boundary conditions.

$$
\phi(x_4, x_5 + 2\pi R_5) = \phi(x_4, x_5)
$$

$$
\phi(x_4 + 2\pi R_4, x_5) = e^{-in\frac{x_5}{R_5}}\phi(x_4, x_5)
$$

Here $\phi$ denotes both $\phi_R$ and $\phi_L$. The gauge field can be gauge transformed to the following form:

$$
A_4(x_4, x_5) = a_4
$$

$$
A_5(x_4, x_5) = \frac{nx_4}{2\pi R_4 R_5} + a_5
$$

Here $a_4, a_5$ are constants. On the plane they could be gauged to zero, but on the torus they are there in general. The eigenvalue equations now read

$$
\left[\left(\partial_4 + ia_4\right)^2 + \left(\partial_5 + i\frac{nx_4}{2\pi R_4 R_5} + ia_5\right)^2 \pm F_{45}\right] \phi = -m^2 \phi,
$$

where the $\pm$ refers to $\phi_R$ and $\phi_L$, respectively. The periodicity conditions above imply that we can write $\phi$ as:

$$
\phi(x_4, x_5) = \sum_{k=-\infty}^{\infty} e^{ik\frac{x_5}{R_5}}\phi_k(x_4) \quad 0 \leq x_4 \leq 2\pi R_4,
$$

with the boundary condition:

$$
\phi_k(2\pi R_4) = \phi_{k+n}(0).
$$

The equation for $\phi_k$ becomes

$$
\left[\left(\partial_4 + ia_4\right)^2 + \left(i\frac{k}{R_5} + i\frac{nx_4}{2\pi R_4 R_5} + ia_5\right)^2 \pm F_{45}\right] \phi_k(x_4) = -m^2 \phi_k(x_4), \quad 0 \leq x_4 \leq 2\pi R_4
$$

Using the boundary condition we can define $n$ functions, $f_k, k = 0, 1, ..., n - 1$ on the real line:

$$
f_k(x_4) = \phi_{k+n}(x_4 - 2\pi R_4 l) \quad \text{for } 2\pi R_4 l \leq x_4 \leq 2\pi R_4 (l + 1).
$$

It follows from (14) that $f_k$ obeys

$$
\left[\left(\partial_4 + ia_4\right)^2 + \left(i\frac{k}{R_5} + i\frac{nx_4}{2\pi R_4 R_5} + ia_5\right)^2 \pm F_{45}\right] f_k(x_4) = -m^2 f_k(x_4), \quad -\infty < x_4 < \infty.
$$

(15)
Here $k = 0, 1, \ldots, n - 1$ and $\pm$ still refers to the two chiralities. We are only interested in normalizable solutions. The norm square of a field $\phi$ in (12) is equal to the sum of the norm squares of the $n$ functions on the real line, $f_k$. This means that the eigenvalues and eigenfunctions are exactly the normalizable solutions to (13).

To solve (15) we first redefine $f_k$ by a phase to set $a_4$ to zero. This can now be done since $x_4$ runs over the real line. The equation becomes the eigenvalue problem for a one dimensional harmonic oscillator. The eigenvalues are:

$$m_j^2 = (j + \frac{1}{2} + \frac{1}{2}) \frac{n}{\pi R_4 R_5}$$

for each $k = 0, 1, \ldots, n - 1$. We see that there is a $n$-fold degeneracy of all masses. There are $n$ massless modes of one chirality and zero of the other. For the massive levels there is an equal number of solutions of each chirality. These features were general as discussed above and it is nice to see how it works in the special case of constant $F_{45}$.

We thus conclude that the free hypermultiplet compactified in this way produces $n$ chiral multiplets with zero mass as well as a tower of nonchiral (double) multiplets $\Phi_{j}^{k,\pm}$ ($j = 1, \ldots$ and $k = 0, 1, \ldots, n - 1$) with masses,

$$m_j^2 = \frac{j n}{\pi R_4 R_5}.$$

The superpotential therefore contains a term,

$$\sum_{k=1}^{n} \sum_{j=1}^{\infty} \left( \frac{j n}{\pi R_4 R_5} \right)^{1/2} \Phi_{j}^{k,+} \Phi_{j}^{k,-}.$$

### 3.3 $\sigma$-models

The previous example can be generalized to $q$ hypermultiplets describing a low-energy $\sigma$-model with a hyper-Kähler target space, $\mathcal{M}$, of dimension $4q$. Let us also assume that $\mathcal{M}$ has a $U(1)$ isometry that is related to a hyper-Kähler moment map. Recall that a hyper-Kähler manifold has a $\mathbf{CP}^1$-family of complex structures and each complex structure has its own Kähler class. The collection of Kähler 2-forms can be written as:

$$\omega = \sum_{a=1}^{3} c_a \omega_a, \quad \sum c_a^2 = 1.$$
Here, the $\omega_a$’s are (real) 2-forms and the $c_a$’s are real coefficients. They satisfy,
\[ g_{IK}\omega_a^{IJ}\omega_b^{KL} + g_{IK}\omega_b^{IJ}\omega_a^{KL} = 2\delta_{ab}\delta^{IL}, \]
where $g_{IJ}$ is the metric ($I, J, K = 1 \ldots 4q$). A hyper-Kähler moment map is a $\mathbb{CP}^1$-family of functions on $\mathcal{M}$:
\[ \mu = \sum_{a=1}^{3} c_a \mu_a. \]
They satisfy,
\[ \omega_a^{JK} \partial_K \mu_b + \omega_b^{IJ} \partial_J \mu_a = 2\delta_{ab} \xi^I, \]
where $\xi^I$ is the Killing vector for the $U(1)$ isometry.

Now, let us consider a 5+1D $\sigma$-model with target space $\mathcal{M}$ (the hypermultiplet moduli space). (See [10] and [11].) The $U(1)$ current is given by:
\[ J_\mu = \xi_I \partial_\mu \phi^I. \]
The role of the triplet of operators $D^{ij}$ from (8) is played by the triplet of moment maps $\mu_a$ ($a = 1 \ldots 3$). When we compactify on $T^2$, (9) becomes:
\[ S_1 = -\int (A_4 J_4 + A_5 J_5 + i F_{45} \mu_1 + \cdots) \]
Let us discuss the low-energy description of this model. We wish to find the dimension of the moduli space of solutions to the scalar equations of motion. The kinetic part of the $\sigma$-model:
\[ \int g_{ij}(\phi, \overline{\phi}) \partial \phi^i \overline{\partial \phi^j} + \int g_{ij}(\phi, \overline{\phi}) \partial \phi^i \overline{\partial \phi^j}, \]
leads to the following equations of motion:
\[ 0 = -\partial (g_{ij} \overline{\partial \phi^i}) - \overline{\partial} (g_{ij} \partial \phi^i) + \partial_j g_{ik} \overline{\partial \phi^k} \overline{\partial \phi^j} + \partial_j g_{ik} \overline{\partial \phi^k} \partial \phi^i. \]
We use the Kähler condition:
\[ \partial_j g_{ik} = \partial_k g_{ij} = g_{ij} \Gamma^l_{jk} \]
and obtain:
\[ (D\overline{D}\phi)^i = 0, \quad \overline{D}D\overline{\phi}^i = 0. \]
Here $D$ is the covariant derivative:
\[ (D\phi)^i = \overline{\partial} \phi^i, \quad (D\overline{D}\phi)^i = \partial \overline{\partial} \phi^i + \Gamma^i_{jk} \partial \phi^j \overline{\partial} \phi^k. \]
This implies:

\[ \overline{\partial} \phi^i = 0, \quad \partial \overline{\phi}^i = 0. \]

The zero modes are thus holomorphic curves from \( T^2 \) into the target space, as is well known. To incorporate the gauge field \( A_\mu \) we replace \( \partial \) and \( \overline{\partial} \) with the \( U(1) \)-covariant derivative:

\[ (D \phi)^i = \overline{\partial} \phi^i - i A_z \xi^i. \]

Now let us fix the complex structure that corresponds to \( \omega_1 \) (out of the 3 \( \omega_a \)'s). We can then express the Killing vector, \( \xi^i \), in terms of \( \mu_1 \) as:

\[ \xi^i = g^{jk} \partial_k \mu_1. \]

The zero modes corresponding to (16) are easily seen to satisfy:

\[ 0 = \overline{\partial} \phi^i - i A_z \xi^i. \] (18)

How many zero modes do we get? Let us assume that \( \phi^i \) is a solution and study the linearized equation:

\[ 0 = \overline{\partial} \delta \phi^i - i A_z \partial_k \xi^i \delta \phi^k - i A_z \partial_k \xi^j \delta \overline{\phi}^k. \]

Using (17) we see that:

\[ \partial \overline{k} \xi^l = \partial \overline{l} \xi^k, \]

but since \( \xi \) is assumed to be a Killing vector it must satisfy:

\[ \partial \overline{k} \xi^l + \partial \overline{l} \xi^k = 2 \Gamma^\overline{i}_{\overline{k}l} \xi^\overline{i}. \]

so

\[ \partial \overline{k} \xi^l = \Gamma^\overline{i}_{\overline{k}l} \xi^\overline{i}. \]

Also,

\[ \partial \overline{k} g^{j\overline{i}} = -g^{j\overline{n}} \partial \overline{k} g_{m\overline{n}} g^{\overline{m}l} = -g^{j\overline{n}} \Gamma^\overline{i}_{\overline{k}n}. \]

Therefore,

\[ \partial \overline{k} \xi^j = 0. \]

The linearized equations of motion are therefore:

\[ 0 = \overline{\partial} \delta \phi^i - i A_z \partial_k \xi^j \delta \phi^k. \]
To solve this we write the $2q \times 2q$ matrix with elements $A_z \partial_k \xi^j$ as:

$$-iA_z \partial_k \xi^j = (\Omega^{-1})^j_k \overline{\partial}_z \Omega^j_l,$$

where $\Omega(z, \bar{z}) \in GL(2q, \mathbb{C})$. We find that:

$$\overline{\partial}_z (\Omega^j_k \delta \phi^k) = 0.$$

Thus $\Omega \delta \phi$ is a holomorphic section of a vector-bundle. Moreover, from the Killing vector equation:

$$\partial_k \xi^j + \partial_l \xi^k = 2\Gamma^j_{kl} \xi^j + 2\Gamma^j_{lj} \xi^j = 0.$$

We therefore find:

$$\partial_l \xi^i = \partial_l g^{ik} \xi^k + g^{ik} \partial_l \xi^k = -\Gamma^i_{lk} \xi^k - g^{ik} \partial_k \xi^j.$$

Using (18) we can write:

$$(\Omega^{-1})^j_k \overline{\partial}_z \Omega^j_l = -\Gamma^j_{kl} \overline{\partial}_z \phi^l + iA_z g^{jl} \partial_l \xi^k.$$

Now $\delta \phi^j$ is a section of the pullback $\phi^* T\mathcal{M}$ of the tangent-bundle $T\mathcal{M}$ of $\mathcal{M}$ under the map $\phi : \mathbb{T}^2 \mapsto \mathcal{M}$. This vector-bundle has the connection $\Gamma^j_{kl} \overline{\partial}_z \phi^l$. Thus, the vector-bundle $V$, of which $\Omega \delta \phi$ is a holomorphic section can be described as follows. Find $\tilde{\Omega} \in GL(2q, \mathbb{C})$ such that:

$$(\tilde{\Omega}^{-1})^j_k \overline{\partial}_z \tilde{\Omega}^j_l = iA_z g^{jl} \partial_l \xi^k = iA_z g^{jl} \partial_l \partial_k \mu_1.$$

Then, $\tilde{\Omega}$ is a section of a principal bundle with the same structure group as $V$. This means the following: Let $\mathbb{T}^2$ be described by $z$, as we did, with

$$z \sim z + 1, \quad z \sim z + \tau.$$

If $s$ is a section of $V$ then the boundary conditions on $s$ are that $\tilde{\Omega}(z, \bar{z})^{-1}s$ should be continuous.

The eigenvalues of the $GL(2q, \mathbb{C})$ matrix with elements $g^{jl} \partial_l \xi^k$ pulled back to $\mathbb{T}^2$ are constants, and therefore also integers. The fact that the invariant polynomial $P(\lambda) \equiv \det(g^{jl} \partial_l \xi^k - \lambda \delta^j_k)$ is constant follows from $\partial_k \xi^l = 0$. It implies that $\partial_k P(\lambda) = 0$. Thus $P(\lambda)$ is a holomorphic function. If $\mathcal{M}$ were compact this is enough. Even if it is not compact, it still follows that the pullback of $P(\lambda)$ to $\mathbb{T}^2$ is holomorphic and therefore constant. Thus, the vector-bundle $V$ splits into a product: $\bigotimes_{i=1}^{2q} \mathcal{O}(n \lambda_i)$ where $\lambda_i$ are the eigenvalues of $P(\lambda)$. They must therefore be integers.
3.4 Coupling to a vector multiplet

Now let us start with a 5+1D hypermultiplet in the representation $\mathbb{N}$ ($\bar{\mathbb{N}}$) of $SU(N)$ and couple it to a 5+1D $SU(N)$ vector-multiplet. Although this is a nonrenormalizable interaction, we can think of it as the low-energy description of a sector of one of the little-string theories of [12]. The 5+1D coupling of the vector-multiplet to the hypermultiplet preserves $SU(2)_R \times U(1)_F$. Out of the two chiral fermions $\psi^a_\alpha (a = 1, 2)$ one transforms in the $\mathbb{N}$ of $SU(N)$ and the other transforms in the $\bar{\mathbb{N}}$ of $SU(N)$.

Let us classically reduce, as before, on $T^2$ with a global $U(1)$ background field with first Chern class $c_1 = n$. The hypermultiplet gives rise to $n$ chiral multiplets $\Phi^{(k),+}_0$ ($k = 1 \ldots n$) in the $\mathbb{N}$ of $SU(N)$ as well as a tower of massive multiplets $\Phi^{(k),\pm}_j$ ($j = 1 \ldots$) where $\Phi^{(k),+}_j$ is in the $\mathbb{N}$ of $SU(N)$ and $\Phi^{(k),-}_j$ is in the $\bar{\mathbb{N}}$. Their masses are given by the superpotential,

$$
\sum_{k=1}^n \sum_{j=1}^{\infty} \left( \frac{j n}{\pi R_4 R_5} \right)^{1/2} \Phi^{(k),+}_j \Phi^{(k),-}_j.
$$

The 5+1D vector-multiplet gives rise to an $\mathcal{N} = 1$ vector-multiplet in 3+1D and a chiral multiplet $\Phi_{ad}$ in the adjoint representation of $SU(N)$. There is also a Yukawa coupling of the fields $\Phi_{ad}, \Phi^{(k),-}_{j+1}$ and $\Phi^{(k),+}_j$.

4 Compactifying the BI theory

We will now construct a specific example that produces chiral matter in 3+1D by compactifying the Blum-Intriligator (BI) theories [3].

4.1 Preliminaries

Compactifying the BI theory of $N$ M5-branes at an $A_{k-1}$ singularity on $S^1$ of radius $R$ one obtains a low-energy description given by a gauge theory with gauge group

$$
SU(N)_1 \times SU(N)_2 \times \cdots \times SU(N)_k.
$$

The sub-indices are added for purposes of identification. There are also hypermultiplets in the ($\mathbb{N}_i, N_{i+1}$) representation (with $k + 1 \equiv 1$). On top of that there are $(k - 1)$ more $U(1)$

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vector multiplets. The scalar components set the coupling constants of the $k$ SU($N$) gauge groups. These coupling constants, $g_i$, ($i = 1 \ldots k$) satisfy

$$\sum_{i=1}^{k} \frac{1}{g_i^2} = \frac{1}{R}$$

If we compactify on another $S^1$ of radius $R'$ we obtain a 3+1D gauge theory at low-energies. The $(k-1)$ U(1) vector multiplets that set the gauge couplings decouple and the gauge couplings become background parameters. The interacting gauge theory has a gauge group $SU(N)^k$ and $(N_i, N_{i+1})$ hypermultiplets. The coupling constants and $\theta$-angles are set by $(k-1)$ background parameters (originating from the original $(k-1)$ U(1) vector multiplets) and subject to the condition that

$$\sum_{i=1}^{k} \tau_i = i \frac{R'}{R}, \quad \tau_i \equiv \frac{\theta_i}{2\pi} + \frac{8\pi i}{g_i^2}$$

4.2 Adding the background U(1) field

Now we take a specific 5+1D theory – the BI theory. Also, let the complex structure $\tau$ of $T^2$ become very large. We can take $T^2 = S^1 \times S^1$ with one $S^1$ of radius $R_4$ and the other with radius $R_5 \ll R_4$. We can first reduce the theory along $R_5$. The holonomy $W(x_4) = \int_0^{2\pi R_5} A_5(x_4, x_5) dx_5$ varies from 0 to $2\pi n$ as $x_4$ varies from 0 to $2\pi R_4$.

For a fixed $x_4$, the reduction of the BI theory along $S^1$ with Wilson line $W(x_4)$ was studied in [13, 7]. For small $W(x_4)$ and at low energies $0 \leq E \ll R_5^{-1}$ the theory is described by an effective 4+1D Lagrangian which is the quiver theory of [8] of $N$ D4-branes at an $A_{k-1}$ singularity but such that the hypermultiplets have a mass $m = W(x_4) R_4^{-1}$. For generic $x_4$ the mass is of the order of $R_4^{-1}$. There are $n$ values of $x_4$ for which $W(x_4)$ is a multiple of $2\pi$ and in the vicinity of those points the mass $m$ varies from a small negative to a small positive value. According to the discussion in subsection (2.3), the 3+1D low-energy description contains a chiral multiplet for every time the mass crosses zero. Note that the term $F_{56}D^{12}$ in [8] becomes the term proportional to $dm/dx^4$ in [8]. In subsection (2.3) the $4^{th}$ direction (counting from 0...4) was infinite and there was a continuum of massive modes with arbitrarily low mass. In our case the $4^{th}$ direction is compact and therefore we expect a discrete spectrum with the first level of order $R_4^{-1}$. The chiral mode is likely to remain massless because of arguments similar to those of [13].
The low-energy description in 3+1D will therefore contain $n$ chiral multiplets for each hypermultiplet of the quiver theory. We obtain an $SU(N)^k$ vector multiplets of $\mathcal{N} = 2$ supersymmetry together with $n$ copies of chiral multiplets (of $\mathcal{N} = 1$ supersymmetry) in the $(N_i, N_{i+1})$ representations, for each $i = 1 \ldots k$. The $\mathcal{N} = 2$ vector multiplets should be decomposed into $\mathcal{N} = 1$ vector multiplets and chiral multiplets in the adjoint representation of the fields.

Let us now discuss the issue of whether the adjoint multiplets have a superpotential or not. On the face of it, the adjoint mutliplets can receive a mass term. In the limit that we have been using, $R_4 \gg R_5$, the mass term, if it exists, might be of the order of $R_5^{-1}$. However, the 6D origin of the expectation value of the chiral multiplets is the expectation values for the $k(N-1)$ tensor multiplets of the 6D theory. Specifically, let $Φ$ be the scalar of one of those tensor multiplets and let $B_{45}$ be the component of the anti-self-dual tensor field corresponding to it. We can set $φ = 4π^2(Φ + iB_{45})R_4R_5$. In the limit that $ΦR_4R_5$ is large, we can trust the 6D low-energy description of the Coulomb branch of the BI theory and dimensionally reduce the 6D low-energy effective action to 4D on $T^2$ with twists. Because of the periodicity $φ \sim φ + 2\pi i$, a superpotential for $φ$ has to have the form $\sum a_n e^{-nφ}$. We recognize this as the contribution of instantons made from strings of the 6D BI-theory wrapped on $T^2$. To determine whether such instantons contribute to the superpotential we have to count the zero modes of the fermions in the low-energy effective action that describes the world-sheet of the string. The world-sheet theory that lives on the string of the BI theory can be deduced by dimensionally reducing the theory that lives on the M2-brane and an $A_{k-1}$ singularity on a segment between two M5-branes, setting the boundary conditions appropriately. It seems that the 1+1D effective theory always has a supermultiplet of $\mathcal{N} = (2, 2)$ supersymmetry which comprises of 4 scalars (describing transverse motion of the string inside the 5+1D space) and fermions that are uncharged under $U(1)$. Because they are uncharged, and because it is only the interaction with this global $U(1)$ that breaks the supersymmetry into $\mathcal{N} = 1$ in 3+1D, the instanton will have twice as many fermionic zero modes than required for a superpotential. It will therefore not contribute to a superpotential.
5 Discussion

We argued that chiral gauge theories can be realized as a low-energy limit of certain compactifications of 6D conformal field theories. There are several issues that we have not addressed in this paper. In section (4.2) we argued that the particular compactification of the BI theory that we studied gives an $SU(N)^k$ gauge vector multiplets of $\mathcal{N} = 2$ supersymmetry together with $n$ copies of chiral multiplets (of $\mathcal{N} = 1$ supersymmetry) in the $(\mathcal{N}_i, N_{i+1})$ representations, for each $i = 1 \ldots k$. Some questions for further study would be:

- Do the adjoint chiral multiplets get a mass term?
- Can we realize the compactifications in an M-theory setting? That is, can we find a supergravity solution with M5-brane whose low-energy is described by the compactifications we considered?
- In that case, are these models dual to other chiral gauge field constructions similar to those in [13, 14, 17] or chiral F-theory compactifications [18] (and see also [19] and refs. therein)? Are they dual to the new compactifications discovered in [20]? There

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