Study of the AdS$_2$/CFT$_1$ correspondence with the contribution from the Weyl anomaly

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We consider the Almheiri–Polchinski model of the AdS$_2$ back-reaction coupled with a Liouville field, which is necessary for quantum consistency. In this model, the Liouville field is determined classically by a bulk conformal transformation. The boundary time is also reparametrized by this transformation. It is shown that the on-shell action on the boundary for the gravity sector is given by a bulk integral containing the Liouville field. This integral stems from Weyl anomaly and is SL(2,R) invariant. A prescription is given for computing correlation functions of the operators dual to massless scalars. The generating function of the correlation functions of these operators is given by a sum of matter action and the bulk integral containing the Liouville field. The latter integral leads to extra contributions to $n(\geq 6)$ point functions.

Subject Index

1. Introduction

The anti-de-Sitter/conformal field theory (AdS/CFT) correspondence [1–3] provides a relationship between effective gauge theories of the brane dynamics and string theory on the near-horizon AdS gravity. Two-dimensional nearly anti-de Sitter gravity (NAdS$_2$) appears in higher dimensions near extremal black hole geometry [4–11]. Gravity in AdS$_2$ space is difficult to understand, because there are no finite energy excitations above the vacuum [8]. Asymptotic symmetry of the AdS$_2$ space is the reparametrization of the boundary time $\tau$. However, it was argued that it is spontaneously broken to SL(2,R) symmetry by AdS$_2$ space itself, and explicitly because we need to deform the boundary due to the back-reaction of matter fields on the geometry. In Ref. [10], a toy model, the Almheiri–Polchinski (AP) model, based on the Jackiw–Teitelboim (JT) model [12,13], was introduced to study back-reaction to AdS$_2$. It was concluded that the time variable on the boundary becomes a dynamical variable and its dynamics is governed by a Schwarzian derivative action. In Refs. [14–17], this model was further analyzed based on the interest related to the SYK model [18–20]. In Ref. [15], a prescription was proposed to integrate correlation functions of operators on the boundary over reparametrizations of the boundary time by using a Schwarzian action as the effective action.

The AdS$_{d+1}$/CFT$_d$ correspondence is expected to provide a precise definition of quantum gravity in terms of a boundary non-gravitational field theory, and the quantum gravity must include the effect of the Weyl anomaly. Among quantum gravity theories the two-dimensional case is well studied. Although in two-dimensional gravity there are no physical degrees of freedom in the gravity sector, the conformal mode is related to the Weyl anomaly and plays an important role. This anomaly in the bulk is represented by the Polyakov action and in the conformal gauge it reduces to the local Liouville action. Although this anomaly is a quantum effect, this is already included in the definition of the boundary field theory. In the AdS$_2$/CFT$_1$ correspondence, the anomaly is represented by the action of a Liouville field.
of the measure for the path integral of the bulk gravity theory. In the study of the holographic dictionary, the Liouville field is to be treated semi-classically.

In the literature a boundary interaction and quantum effects due to the Weyl anomaly are studied in Ref. [16], where the Polyakov action is not introduced from the beginning, but added later. The purpose of this paper is to show that even if a Liouville field is present, the principle of holography still applies. In this paper we will start with a 2d model of gravity that is coupled to the Polyakov action in order to make the 2d gravity theory quantum mechanically consistent, and investigate the effect of including the Liouville action into the AP model on the dynamical reparametrization of the boundary, and on the correlation functions of operators on the boundary. Higher-dimensional quantum gravity such as 4d is also interesting, but it is difficult to deal with the modes related to the anomaly and this is beyond the scope of this paper.

For reparametrization of the boundary we will adopt a prescription that is distinct from that in Refs. [10,15,16]. Boundary conditions on matter fields at the time-like boundary will introduce to the boundary theory an explicit dependence on a boundary time. Then by back-reaction of matter the spacetime itself will be deformed and become dependent on time. This will induce a conformal transformation \( \zeta = z + i\tau \rightarrow F(\zeta) \) in the bulk, and the Liouville field is determined by this function \( F(\zeta) \). The boundary time is also reparametrized as \( \tau \rightarrow -iF(i\tau) \). Other fields such as dilaton and matter fields are also determined in terms of \( F(\zeta) \). By substituting these solutions to the equations of motion into the action, the boundary on-shell action is obtained. The Liouville action in AdS\(_2\) space, however, breaks conformal symmetry, and the dilaton field that corresponds to the reparametrized boundary time is not simply related by a conformal transformation of the original dilaton field corresponding to the undeformed boundary time.\(^1\) Nonetheless, it can be shown that the constraint equation of 2d gravity can be solved and, by substitution into the action, we will derive an on-shell action on the boundary. It is shown that the Schwarzian derivative action found in Refs. [15,16,18,19] does not appear. Instead, a new term (44) given by a 2d bulk integral appears, which comes from the Liouville self-interaction in the bulk. This term is an effect of the Weyl anomaly. The appearance of a bulk integral might be puzzling. As is well known, however, the on-shell action for an interacting scalar field in AdS space is given by a bulk integral that corresponds to Witten diagrams [2]. After the integration over the radial coordinate, the on-shell action is given by an integral over the boundary. Similarly, in the case of the Liouville–dilaton gravity, integration over the radial coordinate of the effective action yields a boundary on-shell action.

We will then give a prescription to obtain correlation functions of boundary operators that takes into account the back-reaction from the massless scalars. The calculation can be done in a similar way to the case [10] without the Liouville field. One difference is that in addition to the on-shell matter action the above-mentioned bulk integral gives rise to contributions to the correlation functions, which also break conformal symmetry. This is an effect of the Weyl anomaly. In this way, the holographic principle also applies to 2d gravity with a Liouville field.

The on-shell boundary action can be also derived for a black hole spacetime. In this case, in addition to the bulk integral of the Liouville field, an additional term related to the solution of the dilaton field appears. For a static black hole solution this term becomes a term linear in the temperature \( T \), and an entropy \( S \) that is linear in \( T \) is obtained. This agrees with the result in Ref. [15].

\(^1\) This is because the background metric is not a flat one but an AdS one.
AdS$_2$ appears in near-horizon geometries of various extremal black hole solutions as a factored geometry. Hence one could UV-complete the present model by embedding it into higher-dimensional AdS spacetimes. In this paper, however, we will not consider UV completion of AdS$_2$ by a higher-dimensional asymptotically AdS spacetime.

This paper is organized as follows. In Sect. 2 the JT model and nearly AdS$_2$/CFT$_1$ based on it are briefly reviewed. In Sect. 3, a Liouville action is coupled to the JT action and solutions to the classical equations of motion and constraints for the stress tensor are studied. In Sect. 4 an on-shell action is obtained. This on-shell action is a 2d bulk integral of the solution for the Liouville field. In Sect. 5 the boundary on-shell action for a black hole is studied. In Sect. 6 it is shown that correlation functions of operators dual to the massless scalar fields can be computed by imposing a condition (42) that determines a form of the function $F(\zeta)$. Hence, holography also applies to Liouville theory. We conclude with a summary in Sect. 7.

2. Nearly AdS$_2$/CFT

We start with the Jackiw–Teitelboim (JT) model [12,13] for two-dimensional AdS gravity:

$$S_{JT} = \frac{1}{16 \pi G} \int d^2x \sqrt{g} \Phi (R + 2) + \frac{1}{8 \pi G} \int d\tau \sqrt{\gamma} \Phi K. \quad (1)$$

Here we work in the Euclidean signature, and $g_{\mu\nu}$ is a bulk metric, $R$ the scalar curvature, $\gamma$ an induced metric on the boundary at infinity, $K$ is an extrinsic curvature of the boundary, $G$ is a two-dimensional Newton constant, and $\Phi$ is a dilaton field. We set the AdS length $\ell_{AdS} = 1$. The equation of motion for $\Phi$ is

$$R = -2, \quad (2)$$

which can be solved as $g_{\mu\nu} = \hat{g}_{\mu\nu}$, where

$$d^2s_{\text{Poincaré}} = \hat{g}_{\mu\nu} dx^\mu dx^\nu = \frac{1}{z^2} (dz^2 + d\tau^2) \quad (3)$$

is the pure AdS$_2$ metric in the Poincaré patch, $z$ is the radial coordinate ($z > 0$), and the boundary is located at $z = 0$.

The equations of motion for $g_{\mu\nu}$,

$$\mathcal{T}_{\mu\nu}^{\Phi} = -\frac{1}{8\pi G} \left( \partial_\mu \Phi \partial_\nu \Phi - \hat{g}_{\mu\nu} \left( \hat{\nabla}^2 \Phi \right)^2 + g_{\mu\nu} \Phi \right) = 0, \quad (4)$$

are constraints, and solved as

$$\Phi = \frac{1}{z} (a + b\tau + c(\tau^2 + z^2)). \quad (5)$$

Here, $\hat{\nabla}_\mu$ is a covariant derivative associated with $\hat{g}_{\mu\nu}$, and $a$, $b$, and $c$ are constants. This fixes the boundary condition for $\Phi$: $^3$

$$\Phi \to \frac{a}{z} \quad (z \to 0). \quad (6)$$

$^2$ The sign convention of $R_{\mu\nu}$ is $R_{\mu\nu} = \partial_\mu \Gamma^\lambda_{\nu\sigma} + \Gamma^\lambda_{\mu\sigma} \Gamma^\sigma_{\lambda\nu} - \cdots$.

$^3$ For simplicity we will set $b = c = 0$. 

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When a matter field $\chi$ is coupled to Eq. (1), the dilaton field receives a back-reaction from the matter,

$$T_{\mu \nu}^\Phi = - T_{\mu \nu}^\chi,$$

where $T_{\mu \nu}^\chi$ is a stress tensor for $\chi$, and the dilaton no longer satisfies the asymptotic behavior in pure AdS$_2$ (6). To reimpose the boundary condition, it is necessary to deform the boundary by a reparametrization of the time variable $\tau \rightarrow u$. For details, see Refs. [10,15,16].

In the above prescription, the Weyl anomaly is not properly taken into account. In the next section we will incorporate a Liouville field into the system.

### 3. Coupling to a Liouville field

Two-dimensional gravity has the Weyl anomaly, and to take account of this anomaly we need to introduce dynamics of the conformal factor of the metric tensor. Let us introduce a conformal factor into the metric:

$$g_{\mu \nu} = e^{2\rho} \hat{g}_{\mu \nu},$$

where $\hat{g}_{\mu \nu}$ is a reference metric (3) and $\rho$ is a Liouville field. Then the JT action (1) depends on $\rho$. When the massless matter fields are conformally coupled to gravity, the matter action does not depend on the conformal factor of the metric. However, if the matter fields are integrated out in the path integral, then the Polyakov action appears:

$$S_P = \frac{-1}{4\pi \alpha^2} \int d^2x \sqrt{\hat{g}} \frac{1}{\Box} R.$$ (9)

Here, $\alpha$ is a constant related to the central charge of matter fields, $C_M = N$ ($N$ is the number of scalar fields), as

$$\alpha^2 = \frac{3}{C_M - 24}. (10)$$

The appearance of Eq. (9) is due to the fact that the integration measure for the matter and other fields defined with respect to the metric $g_{\mu \nu}$ is not Weyl invariant [21,22]. To rewrite Eq. (9) into a local form it is necessary to split the metric tensor into a product of a reference metric and a conformal factor, as in Eq. (1). For consistency, the partition function of the gravity theory must be invariant under the following Weyl transformation, which is just a redundancy in Eq. (8):

$$\hat{g}_{\mu \nu} \rightarrow e^{\sigma} \hat{g}_{\mu \nu}, \quad \rho \rightarrow \rho - \frac{1}{2} \sigma.$$ (11)

This requires the cancelation of the total conformal anomaly for $\hat{g}_{\mu \nu}$. By substituting Eq. (8) into Eq. (9), the Polyakov action is transformed into

$$S_P (\rho, \hat{g}) = \frac{-1}{8\pi \alpha^2} \int d^2x \sqrt{\hat{g}} \left( \left(\nabla_\mu \rho\right)^2 + \hat{R} \rho + \mu e^{2\rho} \right) - \frac{1}{4\pi \alpha^2} \int d\tau \sqrt{\hat{g}} \rho \hat{K}. (12)$$

Here, a boundary term is introduced. A cosmological term is also introduced, and $\mu$ is a constant to be determined later by requiring consistency of the equations of motion. Its value can be adjusted...
by a local counter-term, because the anomaly is defined in terms of Eq. (9), which is computed as a functional determinant.

We will add this action to the JT action (1). By substituting Eq. (8) into Eq. (1) and adding a suitable boundary term to make the variational problem well defined, we obtain an action for Liouville–dilaton gravity:

\[ S_{LD} = \frac{1}{16\pi G} \int d^2x \sqrt{\hat{g}} \left( \Phi \left( \hat{R} + 2e^{2\rho} \right) + 2\nabla \Phi \cdot \nabla \Phi \right) + \frac{1}{8\pi G} \int d\tau \sqrt{\hat{g}} \Phi \hat{K} \]

\[ - \frac{1}{8\pi \alpha^2} \int d^2x \sqrt{\hat{g}} \left( \left( \nabla \rho \right)^2 + \hat{K} \rho + \mu e^{2\rho} \right) - \frac{1}{4\pi \alpha^2} \int d\tau \sqrt{\hat{g}} \rho \hat{K}. \] (13)

Furthermore, a matter action must be added. For real, free, massless scalar fields \( \chi^i \) \((i = 1, \ldots, N)\), it reads

\[ S_M = \int d^2x \sqrt{\hat{g}} \sum_{i=1}^{N} \frac{1}{2} \left( \nabla \chi^i \right)^2 \] (14)

Since the action is rewritten in terms of a reference metric \( \hat{g}_{\mu\nu} \), we may also transform the measure of the functional integral into a new one defined with respect to \( \hat{g}_{\mu\nu} \). In this transformation a Jacobian will appear. It is assumed that the effect of this Jacobian is to simply modify the parameter \( \alpha \) in Eq. (13). Actually, the modification of this value is not important here, as long as \( C_M \) is large [21].

The total action \( S = S_{LD} + S_M \) by itself is in fact not invariant under the Weyl transformation (11). However, the functional integral is invariant, provided a functional measure for fields in the path integral defined with respect to \( \hat{g} \) transforms in the following way:

\[ [\mathcal{D}\rho]_{\hat{g}} [\mathcal{D}\chi^i]_{\hat{g}} [\mathcal{D}(\text{ghost})]_{\hat{g}} = e^{S_p(\sigma/\hat{g})} [\mathcal{D}\rho]_{\hat{g}} [\mathcal{D}\chi^i]_{\hat{g}} [\mathcal{D}(\text{ghost})]_{\hat{g}}. \] (15)

This prescription worked in \( d < 1 \) gravity [21]. This is also assumed to be the case here. To recapitulate, the introduction of the Liouville field and functional integration over it is necessary for consistent treatment of the Weyl anomaly.

Let us now turn to the analysis of the equations of motion. By variation of \( \Phi \) and \( \rho \), we obtain the equations

\[ \hat{R} + 2e^{2\rho} = 0, \] (16)

\[ \frac{1}{16\pi G} \left( \hat{\nabla}^2 \Phi - 2e^{2\rho} \Phi \right) - \frac{1}{8\pi \alpha^2} \left( \hat{\nabla}^2 \rho - \frac{1}{2} \hat{R} - \mu e^{2\rho} \right) = 0. \] (17)

The scalar curvature of Eq. (3) is \( \hat{R} = -2 \). If we set \( \mu = 1 \), these equations are decomposed as

\[ \hat{\nabla}^2 \rho - e^{2\rho} + 1 = 0, \] (18)

\[ \hat{\nabla}^2 \Phi - 2e^{2\rho} \Phi = 0. \] (19)

Henceforth we will choose this value for \( \mu \). Matter equations of motion are given by

\[ \hat{\nabla}^2 \chi^i = 0. \] (20)

The constraint equations that come from \( \hat{g}_{\mu\nu} \) are

\[ \frac{1}{8\pi G} \left( \hat{\nabla}_\mu \hat{\nabla}_\nu \Phi - \hat{\nabla}_\mu \Phi \hat{\nabla}_\nu \rho - \hat{\nabla}_\nu \Phi \hat{\nabla}_\mu \rho - \text{trace} \right) = T^\rho_{\mu\nu} + T^\chi_{\mu\nu} \] (21)
in the case of the variation of the traceless part of the metric. Here, $T^\rho$ and $T^\chi$ are traceless energy–momentum tensors for $\rho$ and $\chi$:

$$T^\rho_{\mu\nu} = \frac{-1}{4\pi\alpha^2} \left( \partial_\mu \rho \partial_\nu \rho - \hat{\nabla}_\mu \hat{\nabla}_\nu \rho - \text{trace} \right), \quad (22)$$

$$T^\chi_{\mu\nu} = \sum_{i=1}^{N} \partial_\mu \chi^i \partial_\nu \chi^i - \text{trace}. \quad (23)$$

For the trace part, we have

$$\frac{1}{2} \hat{\Phi}^2 \Phi - \Phi - \hat{\Phi}^2 \rho + \frac{1}{\alpha^2} G \left[ e^{2\rho} - \hat{\Phi}^2 \rho \right] = 0. \quad (24)$$

This last equation is not actually complete, because the variation of the functional measure for $\rho$, (15), is not taken into account. From Eq. (15), we obtain an extra term $G/\alpha^2$ on the right-hand side of Eq. (24). Then, Eq. (24) modified in this way is consistent with Eqs. (18), (19), and (20). Hence, with the choice $\mu = 1$, independent equations of motion are Eqs. (18), (19), (20), and (21).

Now, we will set $\chi^i = 0$ for simplicity. If we choose a solution $\rho = 0$ for Eq. (18), then the solution to Eq. (19) is given by Eq. (5). On the other hand, a general solution to Eq. (18) that does not change the boundary at $z = 0$ is

$$\rho_F(z, \tau) \equiv \frac{1}{2} \log \frac{4z^2 F'(i\tau + z) F'(i\tau - z)}{[F(i\tau + z) - F(i\tau - z)]^2}. \quad (25)$$

Here, $F(x)$ is a real and monotonic function. A map $z + i\tau \to F(z + i\tau)$ is a conformal transformation in the bulk. This is different from the prescription of deforming the regularized boundary at $(z = \epsilon, \tau)$ to $(z = \epsilon F'(i\tau), -iF(i\tau))$, while keeping the bulk metric [10,15,16]. In this paper, the position of the boundary is fixed at $z = \epsilon$ with a small $\epsilon$.

On the other hand, the solution $\Phi$ to Eqs. (19) and (21) is not obtained by a conformal transformation $i\tau \pm z \to F(i\tau \pm z)$ of Eq. (5). Actually, if we define

$$\hat{\Phi}(z, \tau) \equiv \frac{1}{F(i\tau + z) - F(i\tau - z)} \left[ 2a - ib\{F(i\tau + z) + F(i\tau - z)\} - (c/2)(F(i\tau + z) + F(i\tau - z))^2 + (c/2)(F(i\tau + z) - F(i\tau - z))^2 \right], \quad (26)$$

it turns out that the constraint (21) is not satisfied. This is because the background metric (3) is not a flat metric. Indeed, if we introduce complex coordinates $\zeta$ and $\bar{\zeta}$ by

$$\zeta = z + i\tau, \quad \bar{\zeta} = z - i\tau, \quad (27)$$

we obtain

$$\frac{-1}{8\pi G} \left( \hat{\nabla}_{\zeta}^2 \hat{\Phi} - 2 \hat{\nabla}_{\zeta} \hat{\Phi} \hat{\nabla}_{\zeta} \rho_F \right) + T^\rho_{\zeta\zeta} |_{\rho = \rho_F} = \frac{1}{8\pi \alpha^2} \{F(\zeta), \zeta\}. \quad (28)$$

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4 In Euclidean space, $F(x)$ must also be an odd function in order for $\rho_F$ to be real. Then, out of the $\text{SL}(2,\mathbb{R})$ transformations only a dilatation satisfies this condition. In Lorentzian space with $t = -i\tau$ this condition is not necessary. Henceforth, this condition is not imposed on $F(x)$, even if $\rho$ becomes complex, because this problem disappears after Wick rotation.
Here,

\[ S(\zeta) \equiv \{ F(\zeta), \zeta \} \equiv \frac{F''''(\zeta)}{F''(\zeta)} - \frac{3}{2} \left( \frac{F''(\zeta)}{F'(\zeta)} \right)^2 \]  

(29)

is a Schwarzian derivative. Hence, only for the SL(2,R) transformation \( \zeta \to F(\zeta) = (a_1 \zeta + a_2)/(a_3 \zeta + a_4), (a_1a_4 - a_2a_3 = 1) \) do (25) and (26) satisfy the constraint (21).

Instead, given Eq. (25) we can solve Eq. (19) and the constraint (21). We will now solve the following equations:

\[ \hat{\nabla}^2_\zeta /\Phi_1 - 2 \hat{\nabla}_\zeta \rho_F \cdot \hat{\nabla}_\zeta \Phi = -\frac{2G}{\alpha^2} \left( (\partial_\zeta \rho_F)^2 - \hat{\nabla}^2_\zeta \rho_F \right), \]  

(30)

\[ \left( \hat{\nabla}^2 - 2e^{2\rho_F} \right) \Phi = 0. \]  

(31)

The right-hand side of Eq. (30) is given by \( G\alpha^{-2}S(\zeta) \). By substituting \( \rho_F \), the above equations are made more explicit:

\[ \partial^2_\zeta \Phi + \left( \frac{1}{z} - \frac{2}{3}zS(i\tau) - \frac{1}{3}z^2S'(i\tau) + O(z^3) \right) \partial_\zeta \Phi = \frac{G}{\alpha^2}S(\zeta), \]  

(32)

\[ \left( z^2(\partial^2_\zeta + \partial^2_\tau) - 2 \right) \Phi = \left( \frac{4}{3}z^2S(i\tau) + O(z^3) \right) \Phi. \]  

(33)

By defining

\[ \Phi(z, \tau) = \sum_{n=-1}^{\infty} z^n \Phi_n(\tau) \]  

(34)

and solving the above equations, we have \( \Phi_0(\tau) = 0 \) and a set of equations that determine \( \Phi_{-1}(\tau) \) and \( \Phi_1(\tau) \):

\[ \frac{d^3 \Phi_{-1}}{d\tau^3} - 2S(i\tau) \frac{d \Phi_{-1}}{d\tau} - \frac{4}{3} \frac{dS(i\tau)}{d\tau} \Phi_{-1} = 0, \]  

(35)

\[ \Phi_1 = \frac{1}{2} \frac{d^2 \Phi_{-1}}{d\tau^2} - \frac{2}{3} S(i\tau) \Phi_{-1}. \]  

(36)

Explicit solutions to these equations are not obtained yet. Three independent solutions which correspond to \( F(\zeta) = \zeta + \epsilon(\zeta) \), which is infinitesimally close to an identity conformal transformation, are the following:

\[ \Phi_{-1}^{(0)}(\tau) = 1 + \frac{4}{3} \epsilon'(i\tau) + \cdots, \]

\[ \Phi_{-1}^{(1)}(\tau) = \tau + 2i\epsilon(i\tau) + \frac{4}{3} \tau \epsilon'(i\tau) + \cdots, \]

\[ \Phi_{-1}^{(2)}(\tau) = \tau^2 + \frac{4}{3} \tau^2 \epsilon'(i\tau) + 4i\tau \epsilon(i\tau) - 4i \int^\tau \epsilon(i\tau_1)d\tau_1 + \cdots. \]  

(37)

Here, \( \epsilon'(\zeta) = (d/d\zeta)\epsilon(\zeta) \). Hence the solution \( \Phi \) breaks SL(2,R).
4. On-shell action for the gravity sector

The on-shell action is obtained by substituting solutions (25) and (26) into the action $S_{LD}$ (13). After using equations of motion (18), it is simplified as follows:

$$
S_{on-shell}^{LD} = -\frac{1}{8\pi G} \int d\tau \Phi \partial_\tau \rho_F|_{z=\varepsilon} + \frac{1}{8\pi G} \int d\tau \left( \frac{1}{\varepsilon} \Phi \hat{K}|_{z=\varepsilon} \right)
- \frac{1}{4\pi \alpha'^2} \int d\tau \frac{1}{\varepsilon} \rho_F \hat{K}|_{z=\varepsilon} + \int d\tau \frac{1}{8\pi \alpha'^2} \rho_F \partial_\tau \rho_F|_{z=\varepsilon}
- \frac{1}{8\pi \alpha'^2} \int d\tau \int_\varepsilon^\infty \frac{dz}{z^2} \left( -\rho_F + (1 - \rho_F) e^{2\rho_F} \right).
$$

Here, the location of the boundary is regularized as $z = \varepsilon$, where $\varepsilon$ is a small positive constant. Notice that the boundary is flat. The extrinsic curvature at the boundary is $\hat{K} = 1$. Since near the boundary $z \to 0$, $\rho_F$ and $\Phi$ behave as

$$
\rho_F(z, \tau) \sim \frac{1}{3} z^2 \{F(i\tau), i\tau\}, \quad \text{(39)}
$$

$$
\Phi(z, \tau) \sim \frac{1}{z} \Phi_{-1}(\tau) + z \Phi_1(\tau), \quad \text{(40)}
$$

the third and fourth terms in Eq. (38) vanish, and the first integral is finite as $\varepsilon \to 0$. The last bulk integral diverges in the UV limit $\varepsilon \to 0$. However, the divergence is proportional to $\int d\tau \int_{\varepsilon}^{\infty} dz \sqrt{\hat{g}}$ and behaves as a constant $\frac{1}{8\pi \alpha'^2} \int d\tau 1$ for a small, but finite, $\varepsilon$. On the other hand, the second boundary term is divergent as $\varepsilon \to 0$ for the solution (40):

$$
\frac{1}{8\pi G} \int d\tau \left[ \frac{\Phi_{-1}(\tau)}{\varepsilon^2} + \Phi_1(\tau) \right].
$$

The first term is large for small $\varepsilon$ and, in general, $\Phi_{-1}$ is not a constant. In a cut-off theory we cannot renormalize this divergence. We can, however, make such a term a constant one by imposing the condition

$$
\Phi_{-1}(\tau) = a,
$$

where $a$ is a constant. As will be discussed at the end of this section, this condition implies a trivial conformal transformation, $F(\zeta) = \zeta$. In Sect. 6 this condition will be imposed after turning on the source terms $j$ for the boundary operators dual to massless scalar fields. Then, $F(\zeta)$ will be a non-trivial function and depend on the source functions.

Continuing the analysis of the on-shell action, with the condition (42) we obtain the following on-shell action:

$$
S_{on-shell}^{LD} = \frac{1}{12\pi G} \int_{-\infty}^{\infty} d\tau \Phi_{-1}(\tau)\{F(i\tau), i\tau\} + \frac{1}{8\pi G} \int_{-\infty}^{\infty} d\tau \Phi_1(\tau)
- \frac{1}{8\pi \alpha'^2} \int_{-\infty}^{\infty} d\tau \int_0^\infty dz \left[ \sqrt{\hat{g}} \left( -\rho_F + (1 - \rho_F) e^{2\rho_F} - 1 \right) \right] + \text{const.}
$$

\text{footnote}{In Ref. [15], this problem is handled by introducing a boundary-value problem with condition $\Phi|_{\text{bndry}} = \frac{1}{2} \phi'(\tau)$ and making $\Phi$, independent of $F$. Then the divergence is a constant. From the equation of motion for $F$ in the Schwarzian derivative theory, the solution for $\Phi$ (5) is reproduced.}
The first term in Eq. (43) is the Schwarzian derivative action [15,16,18,19] with the extra factor $\Phi_1$. There are two other terms in Eq. (43). The bulk integral stems from the conformal anomaly. Interestingly, it turns out that the first and second terms cancel out due to the relation (36):

$$S_{\text{on-shell}}^{\text{on-shell}} = -\frac{1}{8\pi \alpha'} \int_{-\infty}^{\infty} \int_{0}^{\infty} d\tau d\rho \left[ \sqrt{\hat{g}} \left( -\rho_F + (1 - \rho_F) e^{2\rho_F} - 1 \right) \right] + \text{const.}$$

Hence, the terms that break SL(2,R) symmetry canceled out, and the action (44) is invariant.

This action appeared due to the Weyl anomaly, because there is a pre-factor $1/\alpha'^2$. The appearance of a bulk integral might be puzzling. As is well known, however, the on-shell action for an interacting scalar field in AdS space is given by a bulk integral that corresponds to Witten diagrams [2]. After the integration over the radial coordinate, the on-shell action is given by an integral over the boundary. Similarly, in the case of the Liouville–dilaton gravity, integration on the radial coordinate of the effective action yields a boundary on-shell action.

Let us discuss solutions to Eq. (42), $\Phi_1(\tau) = a$. When the matters $\chi^i = 0$, by Eq. (35) the Schwarzian derivative $S(\tau)$ must be a constant. Then, up to SL(2,R) transformations the function $F(\zeta)$ is one of the following:

$$F(\zeta) = \zeta, \tan(A\zeta), \tan(\zeta),$$

where $A$ is a real constant. For the first solution the on-shell action vanishes. The second solution corresponds to a black hole metric, and the third for a special value of $A$ to an AdS$_2$ metric in the global coordinates. In these cases, although the on-shell action does not vanish, it diverges inside the bulk. It is not known how to avoid this divergence, and at present it is not clear if these solutions are meaningful. In the next section the black hole solution will be studied by choosing the black hole metric as the reference metric $\hat{g}$. In Sect. 6 this on-shell action is used to obtain the generating function of correlation functions after turning on the matter field.

It will be interesting, if it is possible, to evaluate the $z$-integral in Eq. (44) explicitly without carrying out an expansion into a series. This is not attempted in this paper. Instead, here, this integral is studied for $F(\zeta)$ near the identity transformation. The result below will be used in Sect. 6. For the pure gravity in AdS$_2$ ($F(\zeta) = \zeta$), the Liouville field (25) is $\rho_F = 0$. In this case, the action (44) vanishes. For a perturbation $F(\zeta) = \zeta + \epsilon(\zeta)$, where $\epsilon(\zeta)$ is infinitesimal, Eq. (25) is expanded as

$$\rho_F = \frac{1}{2} [\epsilon'(\zeta) + \epsilon'(-\bar{\zeta})] - \frac{1}{2z} [\epsilon(\zeta) - \epsilon(-\bar{\zeta})] + O(\epsilon^2).$$

This vanishes as $z \to 0$. It also vanishes for $\epsilon(\zeta) = 1, \zeta, \zeta^2$. Because the integrand of Eq. (43) behaves for small $\rho_F$ as

$$-1 - \rho_F + (1 - \rho_F) e^{2\rho_F} \sim -\frac{2}{3} (\rho_F)^3,$$

the on-shell action (44) is $O(\epsilon^3)$.

5. AdS$_2$ black hole

In Sect. 4 we obtained the on-shell action for the dynamical reparametrization. In this action the Schwarzian derivative term found in Refs. [15,16] does not appear, and furthermore a bulk 2d integral

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We note that Eq. (25) is invariant under SL(2,R).
appears. If the on-shell action were also the same for the AdS black hole, then the entropy of the black hole would vanish.

In this section we will consider an on-shell action in the case of an AdS2 black hole. We will choose the reference metric as
\[ \hat{g}_{\mu\nu} = \frac{4\pi^2}{\beta^2 \sinh^2 \frac{2\pi z}{\beta}} \left[ dz^2 + d\tau^2 \right]. \] (48)

Here, \( \beta = \frac{1}{T} \) is the inverse temperature of the black hole. The solution to the Liouville equation (18) in this case is given by
\[ \rho_F(z, \tau) = \frac{1}{2} \log \frac{\beta^2 \sinh^2 \left( \frac{2\pi z}{\beta} \right) F'(\zeta) F'(-\bar{\zeta})}{\pi^2 [F(\zeta) - F(-\bar{\zeta})]^2}. \] (49)

If \( F(\zeta) \neq \tanh(\pi \zeta / \beta) \), then the spacetime is not static and fields are not, in general, periodic under \( \tau \to \tau + \beta \). In this case, Lorentzian time \( t = -i\tau \) must be used and the integration region is \( -\infty < t < \infty \). The equation of motion for \( \Phi_1 \) (19) and the constraint (21) are solved in a similar way to that in Sect. 3. It is, however, necessary to notice that the stress tensor of the Liouville field is modified to
\[ T^\rho_{\zeta\zeta} = \frac{1}{8\pi \alpha^2} \left[ (F(\zeta), \zeta) + \frac{2\pi^2}{\beta^2} \right]. \] (50)

By expanding \( \Phi \) as Eq. (34), we obtain equations (32) and (33), with the right-hand side of Eq. (33) modified due to the above change. The equations for \( \Phi_{-1}, \Phi_0, \) and \( \Phi_1 \) are the same as Eqs. (35), (36), and \( \Phi_0 = 0 \). The change in \( T^\rho_{\zeta\zeta} \) only modifies \( \Phi_2 \) and higher-coefficient functions.

The on-shell action is obtained by repeating the arguments in Sect. 4. A difference comes from the extrinsic curvature. For the metric (48), it is given by
\[ \hat{K} = \cosh \frac{2\pi z}{\beta} = 1 + \frac{2\pi^2}{\beta^2} z^2 + \cdots. \] (51)

The second term in the near-boundary expansion above modifies the calculation of the second term of Eq. (38):
\[
\frac{1}{8\pi G} \int_{-\infty}^{\infty} d\tau \left[ \frac{1}{\epsilon} \Phi \hat{K} \right]_{|z=\epsilon} = \frac{1}{8\pi G} \int_{-\infty}^{\infty} d\tau \frac{1}{\epsilon} \left( \frac{\Phi_{-1}}{\epsilon} + \Phi_1 \epsilon + \cdots \right) \left( 1 + \frac{2\pi^2}{\beta^2} \epsilon^2 + \cdots \right)
= \frac{1}{8\pi G} \int_{-\infty}^{\infty} d\tau \left( \frac{1}{\epsilon^2} \Phi_{-1} + \Phi_1 + \frac{2\pi^2}{\beta^2} \Phi_{-1} + \cdots \right),
\] (52)

and a new term remains after the cancelation of \( \Phi_0 \) and \( \Phi_1 \{F, \epsilon \tau\} \) in the first and second terms of Eq. (38). To summarize, the on-shell action for a black hole is given by
\[ S_{\text{on-shell}}^{\text{LD}} = \frac{\pi}{4G\beta^2} \int_{-\infty}^{\infty} dt \Phi_{-1} - \frac{1}{8\pi \alpha^2} \int_{-\infty}^{\infty} dt \int_{0}^{\infty} dz \left[ \sqrt{g} \left( -\rho_F + (1 - \rho_F) e^{2\rho_F} - 1 \right) \right]. \] (53)
For a black hole solution we have \( F(\zeta) = \tanh \frac{\pi \zeta}{\beta} \), and the Euclidean time can be used. The Liouville field is trivial, \( \rho_F(z, \tau) = 0 \), and the bulk integral vanishes. As for the first term, by noting that

\[
\{F(t), i\tau\} = -2 \left( \frac{\pi}{\beta} \right)^2
\]

and using Eq. (35), we have

\[
\Phi''_1 + \frac{4\pi^2 \Phi'}{\beta^2} = 0.
\]  

Then we can choose a solution \( \Phi_1 = a \) (constant) and the on-shell action is given by

\[
S_{\text{on-shell}}^{\text{LD}} = \frac{\pi a}{4G} = \frac{\pi a}{4G} T.
\]

Then an entropy \( S = \frac{4\pi}{2G} T \) is obtained. This linear-in-\( T \) behavior coincides with the result of Ref. [15].

When the spacetime is deformed away from the stationary black hole, the second term in Eq. (53) is non-vanishing. In this case, this term may be evaluated by expanding \( F(z+t) \) in a power series of \( z \). After integration over \( z \), the on-shell action can be expanded in powers of the Schwarzian derivative and its derivatives. Up to now, however, a systematic method for the expansion is not available.

6. Correlation functions of scalar operators

When matter fields are turned on, it is possible to regard a sum of Eq. (44) and the matter action as the generating functional for correlation functions dual to the matter. When massless fields \( \chi^i \) are turned on, the equation of motion for \( \Phi \) (19) and the constraint equation (21) are modified, and so are their solutions. We show that the condition (42) determines \( F \), and captures the back-reaction of the matter. This solution \( F \) depends on the boundary sources \( j \) of operators. In particular, when the boundary conditions \( j \) for matter fields vanish, the solution \( F \) coincides with the identity \( F(\zeta) = \zeta \). By substituting the solution for non-vanishing \( j \) into the on-shell action, the generating function for the correlation functions is obtained. In this way, in the presence of a Liouville field the principle of holography also applies and the partition function of 2d AdS2 gravity in the semi-classical approximation yields a generating function of conformal quantum mechanics.

In this section we will show that the constraint equation (21) can be solved even in the presence of the matter fields, and we discuss computation of the correlation function of operators dual to a massless field. We can use Eq. (42) to determine the function \( F(\zeta) \) and substitute this into the on-shell action. For simplicity we will keep only one massless field \( \chi \) and set the boundary conditions of the other massless scalars to zero.

The solution to the Klein–Gordon equation \( \hat{\nabla}^2 \chi = 0 \) with boundary condition \( \chi(z, \tau) \rightarrow j(\tau) \) as \( z \rightarrow 0 \) is given by

\[
\chi(z, \tau) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{z}{z^2 + (\tau - \tau')^2} j(\tau') d\tau'.
\]
This solution has an expansion
\[ \chi(z, \tau) = \frac{1}{2\pi} \sum_{n=0}^{\infty} z^n j_n(\tau), \] (58)

with
\[ j_0(\tau) = 2\pi j(\tau), \quad j_1(\tau) = 2 \int d\tau' \frac{P}{(\tau - \tau')^2} j(\tau'). \] (59)

Here, \( P \) represents a principal value. The on-shell value of the matter action is
\[ S_M = -\int d\tau \frac{1}{2} \sqrt{\hat{g} g} \partial_z \chi |_{z=0} \]
\[ = -\frac{1}{2\pi} \int d\tau d\tau' j(\tau) \frac{P}{(\tau - \tau')^2} j(\tau'). \] (60)

In the new coordinates \((\tilde{z}, \tilde{\tau}) = ((F(\xi) - F(-\tilde{\xi}))/2, (F(\xi) + F(-\tilde{\xi}))/2i), \) the source \( j \) transforms as a scalar field \( \tilde{j}(-iF(i\tau)) = j(\tau), \) and the on-shell action is given by
\[ \tilde{S}_M = -\frac{1}{2\pi} \int d\tau_1 d\tau_2 F''(i\tau_1) F''(i\tau_2) \frac{P}{[-iF(i\tau_1) + iF(i\tau_2)]^2} j(\tau_1) j(\tau_2). \] (61)

When the solution to the Liouville equation is choosen to be \( \rho = \rho_F \) (25) instead of \( \rho = 0, \) the massless scalar is given by
\[ \tilde{\chi}(z, \tau) = \chi(\tilde{z}, \tilde{\tau}) = \frac{1}{2\pi} \sum_{n=0}^{\infty} 2^n j_n(\tau) = \frac{1}{2\pi} \sum_{n=0}^{\infty} z^n \tilde{j}_n(\tau). \] (62)

For example,
\[ \tilde{j}_0(\tau) = j_0(-iF(i\tau)), \quad \tilde{j}_1(\tau) = F'(i\tau) j_1(-iF(i\tau)). \] (63)

The constraint equation for \( \Phi \) (21) and equation of motion for \( \Phi \) (19) are given by
\[ \hat{\nabla}^2 \Phi - 2\hat{\nabla}_\xi \Phi = 8\pi G T^\rho_{\xi\xi} |_{\rho = \rho_F} + 8\pi G (\partial_\xi \chi)^2, \] (64)
\[ \nabla^2 \Phi - 2 e^{2\rho_F} \Phi = 0. \] (65)

The solution \( \Phi \) is obtained as a series \( \Phi = \sum_{n=-1}^{\infty} z^n \Phi_n(\tau). \) The first few coefficients are determined by the following equations:
\[ \Phi_0 = 0, \] (66)
\[ \Phi_{-1}' - \frac{4}{3} S(i\tau) \Phi_{-1} = 2\Phi_1, \] (67)
\[ \Phi_1' = \frac{1}{3} S(i\tau) \Phi_{-1} + \frac{G}{\pi} j_1 \gamma_0. \] (68)

Here, a prime stands for a derivative with respect to \( \tau. \) By combining Eq. (67) and Eq. (68), \( \Phi_{-1} \) is obtained by solving
\[ (\Phi_{-1})'' - 2S(i\tau)(\Phi_{-1})' - \frac{4}{3} \frac{dS(i\tau)}{d\tau} \Phi_{-1} = \frac{2G}{\pi} j_1 \gamma_0. \] (69)

Then, Eq. (67) determines \( \Phi_1. \)
By the procedure explained at the beginning of this section a generating functional for correlation functions of operators can be obtained. Equation (42) defines a new time $\tilde{\tau}$. By Eq. (42), $\Phi_{-1} = a$ and $F$ is determined. Up to $O(j^2)$, $F$ is given by

$$F(i \tau) = i \tau + \frac{G}{2a} \int d\tau_1 \int d\tau_2 \frac{j'(\tau_1)j'(\tau_2)}{\tau_1 - \tau_2} \left[ (\tau - \tau_1)^3 \theta (\tau - \tau_1) - (\tau - \tau_2)^3 \theta (\tau - \tau_2) \right] + O(j^4).$$  

(70)

This agrees with the result of Ref. [10] for the back-reaction.

The full boundary action is given, after using Eq. (67), by

$$S_{\text{on-shell}}^{\text{LD}} = -\frac{1}{8\pi a^2} \int d\tau \int dz \sqrt{\hat{g}} \left[ -\rho_F + (1 - \rho_F) e^{2\rho_F} - 1 \right] + \tilde{S}_M.$$  

(71)

The two terms on the right-hand side are functionals of $F$. By substituting Eq. (70), the generating functions of the operator dual to $\chi$ are obtained. In addition to a generating function from $\tilde{S}_M$, which was obtained in Ref. [10], a new contribution is obtained by inserting Eq. (25) into the bulk integral. Due to Eq. (47), this is a generating function of $n(\geq 6)$ point functions. This represents a contribution of the Weyl anomaly. If $j(\tau)$ has a finite support, (71) will be finite. To carry out the radial integral it is necessary to Wick rotate to Lorentzian time and then split the integration region into many pieces. The result is messy and will not be presented here.

7. Summary

In this paper, a Liouville field is introduced into AdS$_2$ gravity to take into account the effect of the Weyl anomaly, and its effect on the nearly AdS$_2$/CFT$_1$ correspondence is analyzed. An on-shell boundary action is studied by changing not only the bulk time but also the bulk metric, and by keeping the location of the boundary at $z = \epsilon$. It is given by a bulk integral of some function of the Liouville field and is entirely due to the Weyl anomaly. With this on-shell action, it is shown that even if the Liouville field is introduced, the ordinary prescription of AdS/CFT works and correlation functions of boundary operators dual to matters can be obtained. The correlation functions receive contributions from the Weyl anomaly. It is interesting that although the Schwarzian derivative action is absent from the boundary on-shell action, correlation functions similar to those in the theory without the Liouville field are obtained. In this paper, correlation functions of only massless scalar fields are considered. Extension of the holographic dictionary to massive scalar fields is left for a future work.

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