Title page

Variable-Order Fractional Dynamic Behavior of Viscoelastic Damping Material

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Abstract: Viscoelastic damping material has been widely used in engineering machinery to absorb vibration and noise. In engineering, the dynamic behavior of the viscoelastic material is mainly affected by temperature and frequency. Classical dynamic behavior equations of the viscoelastic damping material have complex structures with multiple and ambiguous parameters. So a novel variable-order fractional constitutive model (VOFC) is established based on the variable-order fractional operator. Then the viscoelastic dynamic equations are derived by the Laplace transform of the VOFC model. The DMA test by the three-point bending mode is carried out at variable temperatures and frequencies and the frequency spectrum of the dynamic behaviors, i.e., the loss modulus, the storage modulus and the loss factor are obtained. Against the test data, the VOFC model is compared with classical models such as the integer-order Maxwell model (IOM), the constant fractional-order Kelvin-Voigt model (CFK), the constant fractional-order Maxwell model (CFM) and the constant fractional-order standard linear solid model (CFS). Through the comparison, it can be found that the VOFC model can describe dynamic behaviors of the viscoelastic damping material at different temperatures and frequencies more accurately. Furthermore, the VOFC model has simpler structure and only two parameters with clearly-physical meanings.

Keywords: Viscoelasticity • Variable-order • Constitutive model • Dynamic behavior • Damping

1 Introduction

Vibration and noise not only arouse environment problems, but also shorten the service life of the equipments. Sometimes they even cause the fatigue driving, which might lead to serious outcomes [1–2]. Thus, the viscoelastic damping material that absorb vibration effectively has been widely used in the fields of construction vehicle, railway and building. The dynamic behavior of the viscoelastic damping material is affected by many factors, such as temperature, frequency and strain amplitude. In order to make full use of the viscoelastic material, it is vital to propose a viscoelastic constitutive model considering these factors with high precision, few parameters and clearly-physical meanings. The viscoelasticity of the damping material is caused by the slipping and the internal friction among the molecular chains. Under the alternating load, the strain lags behind the stress and the hysteresis is emerged. As a result, researches on the viscoelastic theory were rapidly developed in the 1940s and some researchers established various viscoelastic constitutive models to describe the mechanical behavior. The integer-order Maxwell model was composed of a dashpot unit and a spring unit in parallel [3], and the integer-order Kelvin model was consisted of those in series [4]. Based on these models, the generalized Maxwell model, the generalized Kelvin model, the Burgers model and the standard linear solid model were proposed [5]. The dynamic equations of the loss modulus, the storage modulus, and the loss factor of these models were obtained using the time-frequency transformation method [6]. The integer-order constitutive model could qualitatively describe the evolution of the
viscoelastic damping material from the elastic to the fluid. However, it was hard to quantitatively describe the material constitutive behavior in the large-strain deformation and the varying temperatures and frequencies.

As the fractional calculus developing [7], the constant fractional-order constitutive model [8], which could describe the evolution of the viscoelastic damping material quantitatively, was established to predict the constitutive behavior. Ren et al. [9] presented a nonlinear fractional-order creep model and the creep test was performed on the Chengdu clay. This model was proved to predict the whole process of the clay creep more reasonable and accurate. Li [10] compared the viscoelastic characteristics of the rubber in the time domain and the frequency domain and established the friction type belt transmission thermal coupling model of the V-belt transmission system. Based on the constant fractional-order viscoelastic constitutive model and the WLF equation, Li [11] proposed a fractional time-temperature superposition principle model with high precision and great frequency-extended capacity. Mei et al. [12] established a modified fractional-order Maxwell model to characterize the quasi-static and the dynamic behavior of the concrete. Amabili et al. [13] applied the fractional-order linear solid model to predict the viscoelastic material behavior. The silicone rubber rectangular plate was tested to verify this model. Qin et al. [14] constructed a fractional-order Maxwell model to describe the impact behavior of the viscoelastic material. A fractional-order viscoelastic model was introduced by Paola et al. [15] and demonstrated under the time-dependent loading. Koomson et al. [16] presented a method to determine the response of the viscoelastic damping material at different temperatures and strain rates. Bouras et al. [17] proposed a non-linear thermo-viscoelastic rheological model for high temperature creep in the concrete based on the fractional calculus. Cao [18] established a generalized fractional-order Maxwell model to describe the asphalt cements behaviors.

In 1990s, Lorenzo and Hartley first presented the variable-order fractional operator. After a long period of research by the mathematicians, the theories of the variable-order fractional operator have made considerable progress [19-21]. Based on the piecewise integral quadratic spline interpolation, Keshi et al. [22] developed a mathematical method to estimate the variable-order fractional functional integral equations. Ortigueira et al. [23] introduced several definitions of the variable-order fractional operators, such as Ross-Samko, Lorenzo-Hartley, Coimbra, Valério-Sá da Costa and applied these methods into the dynamic system. Sahoo et al. [24] proposed a new technique to solve the continuously variable-order mass-spring-damping system. Coimbra [25] presented a mathematical concept of the variable-order fractional operator and a numerical method to solve the variable-order differential equations. Compared to the constant fractional-order operator, the variable-order fractional operator is more suitable to explore the viscoelastic constitutive behavior. Lorenzo and Hartley [26] made a deep exploration on the concept of the variable-order fractional operator, and its properties of the time invariance, the linearity and the initialization were analyzed. Meng et al. [27-29] established a variable-order fractional constitutive model of the polymers under the uniaxial load at the constant strain rate, predicting the properties of the polymers positively. It was also demonstrated that the model could describe the time-dependent mechanical behavior of the polymers across the glass transition.

In conclusion, classical constant fractional-order constitutive models can describe the viscoelastic dynamic behavior, but they are hard to describe the viscoelastic dynamic behavior at variable temperatures and frequencies. Therefore, the variable-order fractional operator is needed and the variable-order fractional constitutive model (VOFC) is proposed. After the Laplace transform of the VOFC model, the viscoelastic dynamic equations are acquired. The DMA test is done to obtain the frequency spectrum of the dynamic behaviors, i.e., the loss modulus, the storage modulus and the loss factor. Then the comparative study is made between the VOFC model and the classical models such as the integer-order Maxwell model (IOM), the constant fractional-order Kelvin-Voigt model (CFK), the constant fractional-order Maxwell model (CFM) and the constant fractional-order standard linear solid model (CFS).

2 Variable-Order Fractional Constitutive Model

The constitutive behavior of the viscoelastic damping material is between the pure elastomer and the Newtonian fluid. The zero-order-model is for the pure elastomer and the first-order-model is for the Newtonian fluid. The dynamic behavior of the viscoelastic damping material has the temperature-frequency effect and the order should be variable. Hence, a variable-order fractional constitutive (VOFC) model is presented by replacing the spring and the dashpot units by a single variable-order dashpot (VOD)
where \( \sigma(t) \) is the stress, \( \varepsilon(t) \) is the strain, \( t \) is the time, \( \delta \) is the material factor, \( \alpha(t) \) is the viscoelastic factor and \( 0 \leq \alpha(t) \leq 1 \), \( \partial D^{\alpha(t)}(t) \) is the variable-order fractional operator and its Coimbra definition is adopted as

\[
\partial D^{\alpha(t)}(t) = \frac{1}{\Gamma(1-\alpha(t))} \int_0^t (t-x)^{-\alpha(t)} \varepsilon(x) dx 
\]

\[ (1) \]

where \( \Gamma(*) \) is the Gamma function.

The VOFC model only has two parameters that are the material factor \( \delta \) and the viscoelastic factor \( \alpha(t) \). The material factor \( \delta \) means the stiffness of the viscoelastic damping material and equals the slope rate in the small deformation of the strain-stress curve. The viscoelastic factor \( \alpha(t) \) is a variate, which means the distribution of viscosity to elasticity.

When \( \alpha(t) = 0 \), Eq.(1) degrades to

\[
\sigma(t) = \delta \varepsilon(t) 
\]

\[ (3) \]

In this case, the VOFC model is similar to Hooke’s law and shows the pure elasticity.

When \( \alpha(t) = 1 \), Eq.(1) evolves to

\[
\sigma(t) = \delta \varepsilon'(t) 
\]

\[ (4) \]

In this case, the VOFC model is similar to Newton’s law and shows the pure viscosity.

When \( 0 < \alpha(t) < 1 \), the variable-order dashpot unit is between the elasticity and the viscosity. The relationship of the time \( t \) and the variable-order \( \alpha(t) \) is depicted in Fig.2 and their functions are

\[
\alpha(t) = \begin{cases} 
0 & \text{if } 0 \leq \varepsilon < \varepsilon_1 \\
\alpha_i + b_i \varepsilon & \text{if } \varepsilon_1 \leq \varepsilon < \varepsilon_2 \\
\alpha_2 + b_2 \varepsilon & \text{if } \varepsilon_2 \leq \varepsilon < \varepsilon_{\text{max}} 
\end{cases} 
\]

\[ (5) \]

where \( \varepsilon_1 \) and \( \varepsilon_2 \) are the demarcation point, \( \varepsilon_{\text{max}} \) is the maximum strain, \( \alpha_i \) and \( b_i \) \((i=1,2)\) are the constant coefficients.

\section{3 Dynamic Behaviors}

The inferred integral equation of the VOFC model is derived as

\[
\partial D^{\alpha(t)}(t) \sigma(t) = \delta \varepsilon(t) 
\]

\[ (6) \]

where \( \partial D^{\alpha(t)}(t) \sigma(t) \) is the initialized variable-order fractional integral and written as

\[
\partial D^{\alpha(t)}(t) \sigma(t) = \partial D^{\alpha(t)}(\varepsilon_{\text{init}}) \varepsilon(t) + \psi(-\alpha, a, 0, t) 
\]

\[ (7) \]

where \( \partial D^{\alpha(t)}(t) \sigma(t) \) is the variable-order integral, \( \psi(-\alpha, a, 0, t) \) is the initialization function and has no physical meaning for the VOFC model. Meanwhile, according to the definition of the variable-order in Eq.(5), Eq.(7) is rewritten as

\[
\partial D^{\alpha(t)}(t) \sigma(t) = \partial D^{\alpha(t)}(\varepsilon_{\text{init}}) \varepsilon(t) 
\]

\[ (8) \]

Making the time varying Laplace transform on Eq.(8) by using the form of \( \alpha(t) = \alpha(\tau) \), we obtain the following

\[
L\{d^{\alpha(t)}(\varepsilon_{\text{init}}) \sigma(t)\} = \int_{0}^{\infty} \sigma(s)s^{-\alpha(t)}e^{-st} dx 
\]

\[ (9) \]

where \( \{g(t)\} \) is the Laplace transform of the variable-order fractional function \( g(t) \), \( s \) is the transformation parameter and \( s = i\omega \), \( \omega \) is the circular frequency.

Further, the Laplace transform of the VOFC is

\[
\overline{\sigma(s)}(s) = \delta \overline{\varepsilon(s)}(s) 
\]

\[ (10) \]

where \( \overline{\sigma(s)} \) is the Laplace transform of \( \sigma(t) \), \( \overline{\varepsilon(s)} \) is the Laplace transform of \( \varepsilon(t) \).

Then the complex modulus \( E^* \) of the VOFC model becomes
\begin{equation}
E^*(\omega) = \frac{\sigma(t)}{\varepsilon(t)} = \delta(i\omega) \gamma (i\omega + \ln(i\omega))
\end{equation}

(11)

where \( i^n = \cos \frac{\alpha \pi}{2} + i \sin \frac{\alpha \pi}{2} \), \( i = \sqrt{-1} \), \( \alpha \) is the constant fractional-order and \( 0 \leq \alpha \leq 1 \).

The storage modulus \( E'(\omega) \), the loss modulus \( E''(\omega) \) and the loss factor \( \lambda(\omega) \) of the VOFC model could be deduced by Eq.(11).

The storage modulus is
\begin{equation}
E'(\omega) = \delta \omega^\alpha [b_i \ln(\omega) \cos(\frac{\pi}{2} a_i) - (\omega + \frac{\pi}{2} b_i) \sin(\frac{\pi}{2} a_i)]
\end{equation}

(12)

The loss modulus is
\begin{equation}
E''(\omega) = \delta \omega^\alpha [b_i \ln(\omega) \sin(\frac{\pi}{2} a_i) + (\omega + \frac{\pi}{2} b_i) \cos(\frac{\pi}{2} a_i)]
\end{equation}

(13)

The loss factor is
\begin{equation}
\lambda(\omega) = \frac{E''(\omega)}{E'(\omega)} = \frac{b_i \ln(\omega) \sin(\frac{\pi}{2} a_i) + (\omega + \frac{\pi}{2} b_i) \cos(\frac{\pi}{2} a_i)}{b_i \ln(\omega) \cos(\frac{\pi}{2} a_i) - (\omega + \frac{\pi}{2} b_i) \sin(\frac{\pi}{2} a_i)}
\end{equation}

(14)

4 Dynamic Test

To investigate the performance of the VOFC model, the dynamic test is made on the rubber specimen (\( L \times B \times H \)= 50 mm \times 10 mm \times 5 mm) by DMA 242C (Netzsch, Germany, Fig.3) in the three-point bending mode.

![DMA 242C](image)

Fig. 3 DMA 242C

The excitation frequencies are 0.5Hz, 1Hz, 2Hz, 3.3Hz, 5Hz and 10Hz. The test temperatures are -35℃, -25℃, -5℃, 5℃, 25℃, 45℃ and 60℃, maintained by the liquid nitrogen. The dynamic load 4N on the specimen is applied to prevent the specimen from bouncing off the holder. Experiment results and predictions of the VOFC model are displayed in Fig.4.

As shown in the Fig.4, the viscoelastic dynamic behavior is temperature- and frequency-dependent. In the test range, the the loss modulus, storage modulus and the loss factor are all increased with the frequency and decreased with the temperature. At the same temperature, the viscoelastic damping material is ‘hardening’ at the high frequency and ‘softening’ at the low frequency, indicating the viscoelastic temperature-frequency effect.

The relative error is used to evaluate the predicting
accuracy and written as

\[ E_{re} = \frac{1}{n} \sum_{j=1}^{n} \left| E_{jth} - E_{jte} \right| \]  

(15)

where \( E_{re} \) is the relative error, \( n \) is the fitting number, \( E_{jth} \) and \( E_{jte} \) are the fitting result and the test data of the \( j \)th, respectively. The relative errors of the VOFC model are shown in Table 1.

| \( T(\degree C) \) | \( E'(\omega) \) | \( E''(\omega) \) | \( \lambda(\omega) \) |
|------------------|---------------|---------------|----------------|
| -35              | 0.32\%        | 0.29\%        | 0.22\%         |
| -25              | 0.31\%        | 0.83\%        | 0.76\%         |
| -5               | 0.31\%        | 0.73\%        | 1.56\%         |
| 5                | 0.14\%        | 0.61\%        | 0.66\%         |
| 25               | 0.08\%        | 0.59\%        | 0.65\%         |
| 45               | 0.04\%        | 0.97\%        | 0.95\%         |
| 60               | 0.2\%         | 0.58\%        | 0.73\%         |

It is obviously seen in Fig.4 and Table 1 that the curves of the loss modulus, the storage modulus and the loss factor of the VOFC model have good agreement with the test data at all temperatures. All of the relative errors are within 2\%, indicating that the VOFC model can describe the viscoelastic dynamic behavior at different temperatures and frequencies.

5 Comparison and Discussion

In order to verify the VOFC model, the classical constitutive models, i.e., IOM model, CFK model, CFM model and CFS model are also fitting with the test data at 45\( \degree \)C.

The functions of the storage modulus and loss modulus of the IOM model are

\[
\begin{align*}
E' &= \frac{p_1q_1\omega^2}{1 + p_1^2\omega^2} \\
E'' &= \frac{q_1\omega}{1 + p_1^2\omega^2}
\end{align*}
\]  

(16)

The CFK model is formed of a spring and a constant fractional-order dashpot in parallel. The functions are

\[
\begin{align*}
E' &= k + c\omega^\beta \cos\left(\frac{\pi}{2} \alpha\right) \\
E'' &= c\omega^\beta \sin\left(\frac{\pi}{2} \alpha\right)
\end{align*}
\]  

(17)

The CFM model is composed of a spring and a constant fractional-order dashpot in series and gives the forms

\[
\begin{align*}
E' &= \frac{c\omega^\beta \cos\left(\frac{\alpha}{2}\right) + c\left(c/k \right)\omega^\beta \cos\left(\frac{\beta\pi}{2}\right)}{1 + c^2/k^2} \\
E'' &= \frac{c\omega^\beta \sin\left(\frac{\alpha}{2}\right) + c\left(c/k \right)\omega^\beta \sin\left(\frac{\beta\pi}{2}\right)}{1 + c^2/k^2}
\end{align*}
\]  

(18)
From Fig. 5, the relative errors of the storage modulus and loss modulus of the IOM model are 21.93% and 37.66%, respectively. The IOM model is effective within 6Hz ~ 10Hz. Therefore, the IOM model will not be analyzed further. The CFK model, the CFM model, the CFS model, and the VOFC model can predict the trends of the loss modulus, the storage modulus and the loss factor in the full frequency. The relative errors are obtained in Table 2.

| Model | $E'(\omega)$ | $E''(\omega)$ | $\lambda(\omega)$ |
|-------|--------------|---------------|-------------------|
| CFK   | 0.69%        | 1.73%         | 1.55%             |
| CFM   | 0.98%        | 3.87%         | 2.90%             |
| CFS   | 0.47%        | 2.21%         | 2.03%             |
| VOFC  | 0.04%        | 0.97%         | 0.95%             |

From Table 2, the relative errors of those models are all less than 5% meeting the engineering requirement. The VOFC model displays the smallest relative errors of 0.04%, 0.97%, 0.95% for the storage modulus, loss modulus and loss factor, respectively. The CFK model, the CFM model and the CFS model have many parameters with ambiguous physical meanings and their identification is also a complex process. However, the VOFC model allows a more accurate representation of the test data with only two parameters including the material factor $\delta$ and the viscoelastic factor $\alpha(t)$. The material factor $\delta$ means the stiffness of the viscoelastic damping material and the viscoelastic factor $\alpha(t)$ means the distribution of elasticity and viscosity.

6 Conclusion

A novel VOFC model is established to predict the viscoelastic constitutive behavior. The dynamic equations are derived by making Laplace transform of the VOFC model. Compared with the IOM model, the CFK model, the CFM model, the CFS model, the results are as follows.

1) The material factor in the VOFC model is a constant that means the stiffness of the viscoelastic damping material. Its value equals the slope rate of the strain-stress curve in the small deformation. The viscoelastic factor is a variable order, presenting as the distributed functions of the strain and meaning the distribution of the elasticity and viscosity.

2) The VOFC model has the smallest relative errors among the models. The errors of the storage modulus, the loss modulus, and the loss factor are 0.04%, 0.97%, 0.95%, respectively. The VOFC model can describe the viscoelastic dynamic behavior at different temperatures and frequencies and only has two parameters with the clearly-physical meaning and the simple structure.

3) Further research will concentrate on the viscoelastic dynamic behavior under high strain impact load and the application to the vibration isolation field.

7 Declaration

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Availability of data and materials
The data supporting the conclusions is reported in this manuscript. Therefore, any additional data is not required to be attached.

Competing interests
The authors declare that they have no competing interests.

**Authors’ contributions**
ZL conceived the idea and made the experiment plan. ZD completed the whole experiment and wrote the manuscript. YW built the VOFC model and calculated it. ZZ and YQ processed the experimental data. All authors read and approved the final manuscript.

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