Binary linear codes with at most 4 weights

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Abstract

For the past decades, linear codes with few weights have been widely studied, since they have applications in space communications, data storage and cryptography. In this paper, a class of binary linear codes is constructed and their weight distribution is determined. Results show that they are at most 4-weight linear codes. Additionally, these codes can be used in secret sharing schemes.

Index Terms

Binary linear code, Weight distribution, Secret sharing.

I. INTRODUCTION AND MAIN RESULTS

Throughout this paper, let $q = 2^m$ for a positive integer $m$. Denote $\mathbb{F}_q = \mathbb{F}_{2^m}$ the finite field with $q$ elements and $\mathbb{F}_q^*$ the multiplicative group of $\mathbb{F}_q$.

Let $\mathbb{F}_2^n$ denote the vector space of all $n$-tuples over the binary field $\mathbb{F}_2$. A binary code $C$ of length $n$ is a subset of $\mathbb{F}_2^n$. Usually, the vectors in $C$ are called codewords of $C$. For codewords $\mathbf{x}$ and $\mathbf{y} \in C$, the distance $d(\mathbf{x}, \mathbf{y})$ is referred as the number of coordinates in which $\mathbf{x}$ and $\mathbf{y}$ differ. The (Hamming) distance of a code $C$ is the smallest distance between distinct codewords and is an important invariant. An $[n,k,d]$ binary linear code $C$ is defined as a $k$-dimensional subspace of $\mathbb{F}_2^n$ with distance $d$.

For a codeword $\mathbf{c} \in C$, the (Hamming) weight $wt(\mathbf{c})$ is the number of nonzero coordinate in $\mathbf{c}$. We use $A_i$ to denote the number of codewords of weight $i$ in $C$. Then $(1,A_1,\cdots,A_n)$ is called the weight distribution of $C$. And the weight enumerator is defined to be the polynomial $1 + A_1 x + A_2 x^2 + \cdots + A_n x^n$. If the number of nonzero $A_i$ ($1 \leq i \leq n$) equals $t$, then $C$ is called a $t$-weight code.

The weight distribution is an important research topic in coding theory, as it contains crucial information to compute the probability of error correcting and detection. A great deal of researchers are devoted to construct and determine specific linear codes [6, 15, 25, 27]. The weight distribution of Reed–Solomon codes were determined by Blake [1] and Kith [23]. A survey of the hamming weights in irreducible cyclic codes was given by Ding and Yang in [16]. The weight distributions of reducible cyclic codes could be found in [13, 18, 19, 20, 24, 29]. Recently, Ding [9, 14] proposed a generic construction of linear codes as follows.

Let $D = \{d_1,d_2,\ldots,d_n\} \subseteq \mathbb{F}_q^*$ and $Tr$ denote the trace function from $\mathbb{F}_q$ to $\mathbb{F}_2$. Linear codes $C_D$ of length $n$ can be constructed by

$$C_D = \{(Tr(xd_1),Tr(xd_2),\ldots,Tr(xd_n)) : x \in \mathbb{F}_q\}.$$

Here $D$ is called the defining set of $C_D$. The dimension of the code $C_D$ have been presented in [17] and is equal to the dimension of the $\mathbb{F}_2$-linear space of $\mathbb{F}_q$ spanned by $D$. This method has been widely used

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by some researchers to acquire linear codes with few weights \([8], [10], [11], [12], [28], [31], [32]\). In this paper, we will present a class of binary linear codes with at most four weights.

For \(a \in \mathbb{F}_2\), we set

\[
D_a = \{ x \in \mathbb{F}_q^* : \text{Tr}(x) = a \}.
\]

Motivated by the research work in [10], a class of binary linear codes \(C_{D_a}\) is defined by

\[
C_{D_a} = \left\{ \left( \text{Tr}(xd^{2h+1}) \right)_{d \in D_a} : x \in \mathbb{F}_2 \right\},
\]

where \(a \in \mathbb{F}_2\) and \(h < m\) is a positive factor of \(m\). The weight distribution of the presented linear codes is settled and the main results are listed as follows.

### Table I

| Weight \(w\)       | Multiplicity \(A\)          |
|--------------------|---------------------------|
| 0                  | 1                         |
| \(2m^2\)           | \(2^m - 1 - 2^{m-h}\)     |
| \(2m^2 - 2^m + 1\) | \(2m-h-1 + 2^{m-h}\)      |
| \(2m^2 - 2^{m-h} - 1\) | \(2m-h-1 - 2^{m-h}\)      |

**Theorem 1.** Let \(m/h\) be odd. Then the code \(C_{D_0}\) defined in (2) is a \([2^{m-1} - 1, m]\) binary linear code with weight distribution in Table I.

**Theorem 2.** Let \(m/h\) be odd. Then the code \(C_{D_1}\) defined in (2) is a \([2^{m-1}, m]\) binary linear code with weight distribution in Table II.

### Table II

| Weight \(w\)       | Multiplicity \(A\)          |
|--------------------|---------------------------|
| 0                  | 1                         |
| \(2m^2\)           | \(2^m - 1 - 2^{m-h}\)     |
| \(2m^2 - 2^m + 1\) | \(2m-h-1 + 2^{m-h}\)      |
| \(2m^2 - 2^{m-h} - 1\) | \(2m-h-1 - 2^{m-h}\)      |

**Theorem 3.** Let \(m/h\) be even and \(m/h > 2\). Then the code \(C_{D_0}\) defined in (2) is a \([2^{m-1} - 1, m]\) binary linear code with weight distribution in Table III.

### Table III

| Weight \(w\)       | Multiplicity \(A\)          |
|--------------------|---------------------------|
| 0                  | 1                         |
| \(2m^2 - 2^m + 1\) | \(2^{m-h-1} - (1)^{2^{m-h}} \) |
| \(2m^2 - 2^{m-h} - 1\) | \(2^{m-h} - (1)^{2m-h}\) |
| \(2m^2 - 2^m - 1\) | \(2^{m-h-1} - (1)^{2^{m-h}} \) |
| \(2m^2 - 2^{m-h} - 1\) | \(2^{m-h} - (1)^{2m-h}\) |

The above two theorems present the parameters of \(C_{D_a}\) \((a = 0, 1)\) of (2) for the case that \(m/h \equiv 1 \pmod{2}\). Next, we will assume \(m/h\) is even and \(m = 2e\). In this case, the parameters of \(C_{D_a}\) \((a = 0, 1)\) of (2) are given in the following two theorems.

**Theorem 3.** Let \(m/h\) be even and \(m/h > 2\). Then the code \(C_{D_0}\) defined in (2) is a \([2^{m-1} - 1, m]\) binary linear code with weight distribution in Table III.
Theorem 4. Let $m/h$ be even and $m/h > 2$. Then the code $C_D$ defined in (2) is a $[2^{m-1}, m]$ binary linear code with weight distribution in Table IV.

Let $D = \mathbb{F}_q^*$. If $m/h$ is odd, then gcd($2^h + 1, 2^m - 1$) = 1 (Lemma 2.1, [5]) and it is straightforward to verify that $C_D$ of (2) is a constant binary linear code. If $m/h$ is even and $m > 2$, the code $C_D$ of (2) is a 2-weight binary linear code, and the weight distribution of $C_D$ is given in Theorem 5.

Theorem 5. Let $m/h$ be even, $m > 2$ and $D = \mathbb{F}_q^*$. Then the code $C_D$ defined in (2) is a $[2^m - 1, m]$ binary linear code with weight distribution in Table V.

| Weight $w$ | Multiplicity $A$ |
|------------|------------------|
| 0          | 1                |
| $2^{m-1} + (-1)^{2^{c-1}}$ | $2^{(m-1)^2}$ |
| $2^{m-1} + (-1)^{2^{c+h-1}}$ | $2^{m-1}$ |

If $m/h$ is even, by Lemma 2.1 in [5], we know gcd($2^h + 1, 2^m - 1$) = $2^h + 1$, i.e., $2^h + 1 | 2^m - 1$. Hence $f(x) = x^{2^h+1}$ is a $(2^h + 1)$-to-1 function over $\mathbb{F}_q^*$ in the case that $m/h \equiv 0 \pmod{2}$. This implies that a binary code may be punctured from the code $C_D$ in Theorem 5.

Let $\overline{D} = \{x^{2^h+1} : x \in \mathbb{F}_q^*\}$ and

$$C_{\overline{D}} = \{ (\text{Tr}(xd))_{d \in \overline{D}} : x \in \mathbb{F}_q \}. \tag{3}$$

Then the parameters of $C_{\overline{D}}$ of (3) can be easily derived from the code $C_D$ in Theorem 5 and are given in the following corollary.

Corollary 6. Let $m/h$ be even and $m > 2$. Then the code $C_{\overline{D}}$ defined in (3) is a $[2^{m-1}/2^h+1, m]$ binary linear code with weight distribution in Table VI.

| Weight $w$ | Multiplicity $A$ |
|------------|------------------|
| 0          | 1                |
| $2^{m-1} + (-1)^{2^{c-1}}$ | $2^{(m-1)^2}$ |
| $2^{m-1} + (-1)^{2^{c+h-1}}$ | $2^{m-1}$ |

Example 1. Let $(m, h) = (5, 1)$. For $a = 0$, the code $C_{D_0}$ in Theorem 7 has parameters $[15, 5, 6]$ with weight distribution enumerator $1 + 10x^6 + 15x^8 + 6x^{10}$. For $a = 1$, the code $C_{D_1}$ in Theorem 2 has parameters $[16, 5, 8]$ with weight enumerator $1 + 6x^6 + 15x^8 + 10x^{10}$.
Example 2. Let \((m, h) = (8, 2)\). For \(a = 0\), the code \(C_{D_1}\) in Theorem 3 has parameters \([127, 8, 56]\) with weight distribution enumerator \(1 + 108x^{56} + 98x^{64} + 48x^{80} + x^{96}\). For \(a = 1\), the code \(C_{D_1}\) in Theorem 2 has parameters \([128, 8, 56]\) with weight enumerator \(1 + 96x^{56} + 109x^{64} + 48x^{80} + 2x^{96}\).

Example 3. Let \((m, h) = (6, 1)\). Then the code \(C_6\) in Theorem 5 has parameters \([63, 6, 24]\) with weight enumerator \(1 + 21x^{24} + 42x^{36}\). The code \(C_7\) in Corollary 6 has parameters \([21, 6, 8]\) with weight enumerators \(1 + 21x^8 + 42x^{12}\).

II. Preliminaries

In this section, we present some results on Weil sums, which will be needed in calculating the weight distribution of the codes defined in (2).

An additive character of \(\mathbb{F}_q\) is a group homomorphism \(\chi\) from \(\mathbb{F}_q\) to unit circle of the complex plane. Each additive character can be defined as a mapping

\[\chi_b(c) = (-1)^{\text{Tr}(bc)} \text{ for all } c \in \mathbb{F}_q,\]

with some \(b \in \mathbb{F}_q\). For \(b = 0\), the additive character \(\chi_0\) is called trivial and the other characters \(\chi_b\) with \(b \in \mathbb{F}_q^*\) are called nontrivial. For \(b = 1\), the character \(\chi_1\) is called the canonical additive character of \(\mathbb{F}_q\). And it is well-known that \(\chi_b(x) = \chi_1(bx)\) for all \(x \in \mathbb{F}_q\) [26].

Define the Weil sum

\[S_h(a, b) = \sum_{x \in \mathbb{F}_q} \chi_1(ax^{2h+1} + bx)\]

where \(a \in \mathbb{F}_q^*\) and \(b \in \mathbb{F}_q\). In this paper, we restrict that \(h\) is a proper positive divisor of \(m\). Generally, to evaluate an exponential sum over a finite field is a challenge task. At present, it has been determined only in certain cases [2], [3], [4], [5], [19], [21]. Among them is the following cases of \(S_h(a, b)\).

Lemma 7 ([5], Theorem 4.1). If \(m/h\) is odd, then \(\sum_{x \in \mathbb{F}_q} \chi_1(ax^{2h+1}) = 0\) for each \(a \in \mathbb{F}_q^*\).

Lemma 8 ([5], Theorem 4.2.). Let \(b \in \mathbb{F}_q^*\) and suppose \(m/h\) is odd. Then \(S_h(a, b) = S_h(1, bc^{-1})\), where \(c \in \mathbb{F}_q^*\) is the unique element satisfying \(c^{2h+1} = a\). Further we have

\[S_h(1, b) = \begin{cases} 0, & \text{if } \text{Tr}_h(b) \neq 1, \\ \pm 2^{m+1/h}, & \text{if } \text{Tr}_h(b) = 1, \end{cases}\]

where and hereafter \(\text{Tr}_h\) is the trace function from \(\mathbb{F}_q\) to \(\mathbb{F}_{2^h}\).

Lemma 9 ([5], Theorem 5.2). Let \(m/h\) be even and \(m = 2e\) for some integer \(e\). Then

\[S_h(a, 0) = \begin{cases} (-1)^e 2^e, & \text{if } a \neq g^{t(2^h+1)} \text{ for any integer } t, \\ -(-1)^e 2^e, & \text{if } a = g^{t(2^h+1)} \text{ for some integer } t, \end{cases}\]

where \(g\) is a generator of \(\mathbb{F}_q^*\).

Lemma 10 ([5], Theorem 5.3). Let \(b \in \mathbb{F}_q^*\) and suppose \(m/h\) is even so that \(m = 2e\) for some integer \(e\). Let \(f(x) = a^{2^h}x^{2^h} + ax \in \mathbb{F}_q[x]\). There are two cases.

1) If \(a \neq g^{t(2^h+1)}\) for any integer \(t\) then \(f\) is a permutation polynomial of \(\mathbb{F}_q\). Let \(x_0\) be the unique element satisfying \(f(x) = b^{2^h}\). Then

\[S_h(a, b) = (-1)^e 2^e \chi_1(ax_0^{2^h+1}).\]
2) If \( a = g^{(2^h+1)} \) then \( S_h(a, b) = 0 \) unless the equation \( f(x) = b^{2^h} \) is solvable. If the equation is solvable, with solution \( x_0 \) say, then

\[
S_h(a, b) = \begin{cases} 
  0 & \text{if } \Tr_h(a) = 0, \\
  -(-1)^{\frac{h}{2}} 2^{h} x_0 \chi_1 \left( ax_0^{2^h+1} \right), & \text{if } \Tr_h(a) \neq 0,
\end{cases}
\]

where \( \Tr_h \) is the trace function from \( \mathbb{F}_q \) to \( \mathbb{F}_{2^h} \).

**Lemma 11** (5, Theorem 3.1). Let \( g \) be a primitive element of \( \mathbb{F}_q \). For any \( a \in \mathbb{F}_q^* \) consider the equation

\[
a^{2^h} x^{2^h} + ax = 0 \quad \text{over} \quad \mathbb{F}_q.
\]

1) If \( m/h \) is odd then there are \( 2^h \) solutions to this equation for any choice of \( a \in \mathbb{F}_q^* \).

2) If \( m/h \) is even then there are two possible cases. If \( a = g^{(2^h+1)} \) for some \( t \), then there are \( 2^{2h} \) solutions to the equation. If \( a \neq g^{(2^h+1)} \) for any \( t \) then there exists one solution only, \( x = 0 \).

**III. The proofs of the main results**

We follow the notations fixed in Sect. 2. In this section, we will determine the length of the code \( C_{D_a} (a = 0, 1) \) of (2), and give a formula on the weight of a codeword \( c_b \ (b \in \mathbb{F}_q^*) \) in \( C_{D_a} (a = 0, 1) \) of (2). Then we give the proofs of Theorems 1 and 3.

By the definition of \( D_a \ (a = 0, 1) \) in (1), we know

\[
|D_a| = \begin{cases} 
  2^{m-1} - 1, & \text{if } a = 0, \\
  2^{m-1}, & \text{if } a = 1.
\end{cases}
\]

Define \( N(a, b) = \{ x \in \mathbb{F}_q : \Tr(x) = a \text{ and } \Tr(bx^{2^h+1}) = 0 \} \). We use \( \wt(c_b) \) to denote the Hamming weight of the codeword \( c_b \) with \( b \in \mathbb{F}_q^* \) of the code \( C_{D_a} (a = 0, 1) \) defined in (2). It can be easily checked that

\[
\wt(c_b) = 2^{m-1} - |N(a, b)|.
\]

In terms of exponential sums, for \( b \in \mathbb{F}_q^* \), we have

\[
|N(a, b)| = 2^{-2} \sum_{x \in \mathbb{F}_q} \left( \sum_{y \in \mathbb{F}_2^*} (-1)^{\Tr(x) - ya} \right) \left( \sum_{c \in \mathbb{F}_2} (-1)^{\Tr(bx^{2^h+1})} \right)
\]

\[
= 2^{-2} \sum_{x \in \mathbb{F}_q} \left( 1 + (-1)^{\Tr(x) - a} \right) \left( 1 + (-1)^{\Tr(bx^{2^h+1})} \right)
\]

\[
= 2^{m-2} + 2^{-2} \sum_{x \in \mathbb{F}_q} (-1)^{\Tr(bx^{2^h+1})} + 2^{-2} \sum_{x \in \mathbb{F}_q} (-1)^{\Tr(x+bx^{2^h+1})-a}
\]

\[
= 2^{m-2} + 2^{-2} (S_h(b, 0) + (-1)^a S_h(b, 1)).
\]

(5)

Based on the discussion above, the weight distribution of \( C_{D_a} \) of (2) can be determined by the value distribution of \( S_h(b, c) \) with \( b \in \mathbb{F}_q^* \) and \( c \in \mathbb{F}_2^* \). Combining (5) and the lemmas in preliminaries, we are ready to compute the weight distribution of the codes \( C_{D_a} (a = 0, 1) \) defined in (2).

**Proof of Theorem 1** By Lemma 7 we have \( S_h(b, 0) = 0 \) for \( b \in \mathbb{F}_q^* \). It follows from Lemma 8 that

\[
S_h(b, 1) = S_h(1, c^{-1}) = \begin{cases} 
  0, & \text{if } \Tr_h(c^{-1}) \neq 1, \\
  \pm 2^{\frac{m-h}{2}}, & \text{if } \Tr_h(c^{-1}) = 1,
\end{cases}
\]

(6)
where \( c^{2h+1} = b \) and \( c \in \mathbb{F}_q^* \). Together with equation (5), we get

\[
|N(0,b)| \in \left\{ 2^{m-2}, 2^{m-2} - 2^{m-h-4}/2, 2^{m-2} + 2^{m-h-4}/2 \right\}.
\]

Hence,

\[
\text{wt}(c_b) = 2^{m-1} - |N(0,b)| \in \left\{ 2^{m-2}, 2^{m-2} + 2^{m-h-4} \right\}.
\]

Suppose

\[
w_1 = 2^{m-2} - 2^{m-h-4}/2, \ w_2 = 2^{m-2}, \ w_3 = 2^{m-2} + 2^{m-h-4}/2.
\]

Note that \( \gcd(2^h+1, 2m-1) = 1 \) if \( m/h \) is odd. When \( b \) ranges over \( \mathbb{F}_q^* \), the element \( c \) \( (c^{2h+1} = b) \) takes on each element of \( \mathbb{F}_q^* \) exactly 1 time. Hence, for \( b \in \mathbb{F}_q^* \) we obtain

\[
\left\{ c \in \mathbb{F}_q : \text{Tr}(c^{-1}) \neq 1, \ c^{2h+1} = b \right\} = 2^m - 2^{m-h} - 1,
\]

i.e., \( A_w = 2^m - 2^{m-h} - 1 \). The first two Pless Power Moments ([22], P.260) yield the following two equations:

\[
\begin{cases}
A_w + A_w + A_w = 2^m - 1, \\
w_1A_w + w_2A_w + w_3A_w = n2^{m-1},
\end{cases}
\]

where \( n = 2^m - 1 \). Solving the system of equations in (7) gets Theorem 1.

The proof of Theorem 2 is similar to that of Theorem 1 and we omit the details.

In the sequel, we assume \( m/h \equiv 0 \pmod{2} \), \( m = 2e \) and \( g \) is a generator of \( \mathbb{F}_q^* \). In order to give the proof of Theorem 3, the following auxiliary lemma is needed. This lemma can be found in equation (10) in [10].

**Lemma 12.** Let \( T_0 = |\{x \in \mathbb{F}_q : \text{Tr}(x^{2h+1}) = 0\}| \) and \( T_1 = |\{x \in \mathbb{F}_q : \text{Tr}(x^{2h+1}) = 1\}| \). If \( m/h \) is even, then \( T_0 = 2^{m-1} - (1)^{\frac{h}{2}}2^{e+h-1} \) and \( T_1 = 2^{m-1} + (1)^{\frac{h}{2}}2^{e+h-1} \).

**Proof of Theorem 3**. If \( b \in \mathbb{F}_q^* \) and \( b \neq (g^{2h+1})^t \) for any integer \( t \), then by Lemma 9 we have

\[
S_h(b, 0) = (-1)^{\frac{h}{2}}2^e,
\]

and by Lemma 10 we get

\[
S_h(b, 1) = (-1)^{\frac{h}{2}}2^e \chi_1(bx_0^{2h+1}),
\]

where \( b^{2h}x_0^{2h} + bx_0 = 1 \).

If \( b = (g^{2h+1})^t \) for some integer \( t \), it follows from Lemma 9 that

\[
S_h(b, 0) = -(-1)^{\frac{h}{2}}2^{e+h}.
\]

Assume \( c = g^t \), then \( b = c^{2h+1} \) and by Lemma 8 \( S_h(b, 1) = S_h(1, c^{-1}) \). For the above \( c \in \mathbb{F}_q^* \), let

\[
f_c(x) = x^{2h} + x - (c^{-1})^{2h}.
\]

If \( f_c(x) \) has no root in \( \mathbb{F}_q^* \), by Lemma 10 we obtain

\[
S_h(b, 1) = S_h(1, c^{-1}) = 0.
\]

Note that \( \text{Tr}_h(1) = 0 \), since \( m/h \) is even. If \( f_c(x) \) has a root \( x_0 \) in \( \mathbb{F}_q^* \), by Lemma 10 we get

\[
S_h(b, 1) = S_h(1, c^{-1}) = -(-1)^{\frac{h}{2}}2^{e+h} \chi_1(x_0^{2h+1}).
\]
Together with (4) and (5), we know that for \( b \in \mathbb{F}_q^* \),
\[
\text{wt}(c_b) \in \left\{ 2^{m-2} + (-1)^{\hat{r}} 2^{e+h-1}, 2^{m-2} + (-1)^{\hat{r}} 2^{e-h-2}, 2^{m-2}, 2^{m-2} - (-1)^{\hat{r}} 2^{e-1} \right\}.
\]

Define
\[
w_1 = 2^{m-2} + (-1)^{\hat{r}} 2^{e+h-1}, \quad w_2 = 2^{m-2} + (-1)^{\hat{r}} 2^{e-h-2}, \quad w_3 = 2^{m-2}, \quad w_4 = 2^{m-2} - (-1)^{\hat{r}} 2^{e-1}.
\]

The next step is to determine the number \( A_{w_i} \) of codewords with weight \( w_i \). If \( f_c(x) = 0 \) (for some \( c \in \mathbb{F}_q \)) is solvable in \( \mathbb{F}_q \), by Lemma 11 there are \( 2^{2h} \) solutions of this equation over \( \mathbb{F}_q \). It can be easily checked that
\[
\left\{ x_0 \in \mathbb{F}_q : x_0^{2^2} + x = (c^{-1})^{2h}, c \in \mathbb{F}_q \right\} = \mathbb{F}_q.
\]

Hence we get
\[
\left| \left\{ c \in \mathbb{F}_q^* : x^{2^2} + x = (c^{-1})^{2h} \right\} \right| = 2^{m-2h} - 1,
\]
and
\[
\left| \left\{ c \in \mathbb{F}_q^* : x^{2^2} + x = (c^{-1})^{2h} \right\} \right| = 2^{m} - 2^{m-2h}.
\]

Since \( x^{2^2} + 1 \) is a \( (2^2 + 1) \)-to-1 function on \( \mathbb{F}_q \), there are \( 2^{m-2h} - 2^{2h} \)’s \( b = c^{2^2} + 1 \in \mathbb{F}_q^* \) such that \( S_h(b, 1) = 0 \), i.e., \( A_{w_2} = \frac{2^{m} - 2^{m-2h}}{2^{2h}} \). It follows from Lemmas 10 and 12 that
\[
\left| \left\{ c \in \mathbb{F}_q^* : S_h(1, c^{-1}) = (-1)^{\hat{r}} 2^{e+h} \right\} \right| = \frac{2^{m} - 1 + (-1)^{\hat{r}} 2^{e+h}}{2^{2h}}.
\]

Then we have
\[
\left| \left\{ b \in \mathbb{F}_q^* : b = c^{2^2} + 1 \text{ and } S_h(b, 1) = (-1)^{\hat{r}} 2^{e+h} \right\} \right| = \frac{2^{m} - 1 + (-1)^{\hat{r}} 2^{e+h}}{2^{2h}(2^2 + 1)}.
\]

Therefore, \( A_{w_3} = \frac{2^{m-2h} - 1 - (-1)^{\hat{r}} 2^{e-h-1}}{2^{2h} + 1} \). By the Pless Power Moments ([22], p. 260) we obtain the following two equations:
\[
\begin{cases}
A_{w_1} + A_{w_2} + A_{w_3} + A_{w_4} = 2^m - 1,
\medskip
w_1 A_{w_1} + w_2 A_{w_2} + w_3 A_{w_3} + w_4 A_{w_4} = 2^{m-1}(2^m - 1).
\end{cases}
\]

The solutions of (9) yield the weight distribution of Table III. The proof of Theorem 3 is completed.

We omit the proof of Theorems 4 and 5 since it is similar to that of Theorem 3.

IV. CONCLUDING REMARKS

In this paper, we present a class of binary linear codes with no more than four weights. A number of linear codes with at most five-weight codes were discussed in [7], [8], [10], [11], [12], [31], [32].

It should be remarked that the parameters of the binary linear codes in Theorem 1 are the same as those in Theorem 1 in [10]. It is open whether the two class of codes are equivalent. The readers are invited to attack this problem.

Denote the minimum and maximum nonzero weight of a linear code \( C \) over \( \mathbb{F}_p \) by \( w_{\min} \) and \( w_{\max} \), respectively. By the results in [30], if the code \( C \) satisfies the following inequality
\[
\frac{w_{\min}}{w_{\max}} > \frac{p-1}{p},
\]
then \( C \) can be employed to construct secret sharing schemes with interesting properties.
Let $m > h + 2$. Then for the codes in Theorems 1 and 2 we have

$$\frac{w_{\min}}{w_{\max}} = \frac{2m-2 - \frac{m+h-4}{2}}{2m-2 + \frac{m+h-4}{2}} > \frac{1}{2}.$$ 

If $(m,h) \neq (4,1)$ or $(6,1)$, then for the codes in Theorems 3 and 4 it can be easily checked that

$$\frac{w_{\min}}{w_{\max}} > \frac{1}{2}.$$ 

This conclusion is true also for Theorem 5 and Corollary 6.

Hence, the binary linear codes presented in this paper are suitable for constructing secret sharing schemes in many cases.

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