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Various Doublings of Hopf Algebras
Algebras of Operators on Quantum Groups and Complex Cobordisms

Let us consider the pair of the dual Hopf algebras $X$, $X^*$ over the commutative associative ring $k$ with $1 \in k$ (or field) with the diagonals (co-multiplications) $\Delta$, antipods $s$ and augmentation $\epsilon$. Let the algebra $M$ be an $X$-module compatible with $\Delta$:

$$x(uv) = \sum x^1_i(u)x^{11}_i(v)$$

$$\Delta(x) = \sum x^1_i \otimes x^{11}_i$$

We call such module $M$ a Milnor module (who discovered from that the Hopf property of the Steenrod algebra about 1957 [1]). It was a starting point of the algebraic theory of the Hopf algebras. They were introduced by A. Borel in 1954 as a result of the algebraic analysis of the Hopf theorem on the cohomology rings of H-spaces, which he extended to the finite fields.

We introduce the operator algebras generated by the left multiplications $u$, right multiplications $v$ and by the action of $X$: $A = (M \otimes M^1)X$ (left modules), $B = X(M^1 \otimes M)$ (right modules):

$$m \rightarrow ux(m)v$$

with the commuting relations:

$$x(u \otimes v) = \sum (x^1_q(u) \otimes x_q^{11}(v))x_q^{11}$$

(left modules)

$$(u \otimes v)x = \sum x_q^{11}(x^1_q(u) \otimes x_q^{11}(v))$$

(right modules).

Here $\Delta^2 = \sum x_q^1 \otimes x_q^{11} \otimes x_q^{111}$, $M^1$ is the "inverted algebra" with the multiplication $u \cdot v = vu$.

These algebras are associative if $X$, $X^*$, $M$ are. We always suppose that. The subalgebras $A_L = (M \otimes 1)X$, $B_R = X(1 \otimes M)$ will be considered here. We call them the O-doubles.
Basic examples: let $M = X^*$. We use the general notations for the right and left multiplication operators $L_a(b) = R_b(a) = ab$. Let $L_x^*$, $R_x^*$ be the adjoint operators of $X$ on the space $X^*$.

Lemma 1. $M = X^*$ is the left (corr. right) Milnor module for the representations $R^*_{s^k x}$, $L^*_{s^k x}$ of the Hopf algebra $X$; it is the left (corr. right) Milnor module of the Hopf algebra $X^t$ for the representations $L^*_{s^{2k+1} x}$, $R^*_{s^{2k+1} x}$.

Here $X^t$ is the same algebra with the "inverted diagonal" $\Delta^t = \sum x_i^{11} \otimes x_i^1$ (it might be only bialgebra if $s^{-1}$ does not exist).

We have $X^{t*} = X^{*1}$. The representations $L^*$ and $R^*$ commutes with each other. Therefore the algebra $X^* = M$ is the left Milnor module over the Hopf algebra $X \otimes X^t$.

In the full left operator algebra $A_L = (X^* \otimes 1)(X \otimes X^t)$ there is an important subalgebra $C = (X \otimes 1)\Delta(X)$ with the relations

$$xu = \sum [\rho_1 x^1 \rho_2 x^{11}, u] x_i^{11}$$

Here $\rho_1$ and $\rho_2$ are the commuting representations of $X, X^t$. They determine the "ad-modules" over the algebra $X$ on the diagonal $\Delta(X) \subset X \otimes X^t$:

$$\rho_x(u) = \sum \rho_1 x^1 \rho_2 x_i^{11} (u)$$

such that

$$\rho_x(uv) = \sum [\rho_1 x^1 \rho_2 x_i^{11}, u] \rho_2 x_i^{11} (v)$$

For the cocommutative $X$ ad-modules are the Milnor modules; for $M = X^*$, $\rho_1 = R_x^*$, $\rho_2 = L_x^*$ the algebra $C$ above is exactly the Drinfel'd Quantum Double [1,2,3], which is a Hopf algebra with the comultiplication $\Delta(ux) = \Delta^t(u)\Delta(x)$.

O-Doubles appear as the Operator algebras (differential operators on the Lie groups and difference operators on the discreet groups).

An important example for the ring $k = Z$ gives an algebra $A^U = \Lambda X$, of all "cohomological" operations in the Complex Cobordism Theory. It was computed by the author in 1966 [5].

Here $X$ is the so-called "Landweber-Novikov" algebra found in [5,6], $\Lambda$ is the $U$-cobordism ring "for the point" found by Milnor and author about 1960.

The representation of the Hopf algebra $X$ on the ring $\Lambda$ was found geometrically [5]. In fact we have $\Lambda \subset X^*$, $\Lambda \otimes Q = X^* \otimes Q$. The Hopf
properties of the pair $\Lambda \subset X$ and the relation of the algebra $A^U$ to the differential operators on some infinite-dimensional Lie group were discovered in [7] by Bukhstaber and Shokurov.

**Proposition:** The ”geometric” representation of the algebra $X$ on $\Lambda$ is exactly $R_\times^*$ for the algebra $A^U$.

The present author found this recently and proved as a consequence from [7] (it was not observed by the authors of [7] but may be easily deduced from this paper).

O-doubles are not the Hopf algebras, but they have some ”almost Hopf” properties for the representations $\rho = L^*, R^*$. For $A_L = X^*X$ the adjoint space is obviously $A^* = XX^*$. The multiplication $\Psi : A_L \otimes A_L \to A_L$ generates the comultiplication $\Psi^* : A^* \to A^* \otimes A^*$.

**Lemma 2:** For the representation $\rho = R_\times^*$ the following formula is valid:

$$\Psi^*(ux) = \Delta(x)R\Delta(u)$$

Here $\Psi^*(1) = R = \sum e^i \otimes e_i$, $(e^i, e_j) = \delta^i_j$, $(u, x) = \epsilon R_\times^*(u)$- the canonical scalar product, $e_j$ is the basis of $X$.

**Lemma 3:** For the representations $\rho_1 = R_{s^{2k+1}_x}^*$ and $\rho_2 = L_{s^{2k}_u}^*$ the corresponding O-doubles $A_L = X^*X = B_L$ exactly coincide. The same for the representations $\rho_1 = L_{s^{2k+1}_x}^*$ and $\rho_2 = R_{s^{2k+1}_u}^*$: $A_L^1 = X^*X^t = B_L^1$.

**Lemma 4:** The following antihomomorphisms are well-defined

$$X^*X \to X^*X^t \to X^*X$$

where $ux \to s^{2m+1}(x)s^{2l+1}(u)$, $\rho_1 = R_{s^{2k+1}_x}^*$, $\rho_2 = L_{s^{2k+1}_x}^*$, $\rho_3 = R_{s^{2k+1}_x}^*$, $m + l + 2 = k - n$ for the first arrow and $k^1 - n = -(l + m)$ for the second arrow.

**Theorem 1:** Let the antipod $s$ is invertible.
1. The antiisomorphisms are well-defined:
   a) **Formally adjoint operators**

   $$A_L(= X^*X) \to A_L^1(= X^*X^t) \to A_L$$

   where $\rho_1 = \rho_2 = \rho_3 = R_{s^1}_x$, $ux \to s^{-1}(x)u$ for the first arrow and $s$ should be replaced by $s^{-1}$ for the second arrow.

   b) **Hermitian adjoint operators**

   $$A_L(= X^*X) \to A_L^+(= X^*X^t) \to A_L$$
where \( \rho_1 = \rho_3 = R_x^*, \rho_2 = L_{s-1}^*, ux \to s^{-1}(x)s(u) \) or \( ux \to s(x)s^{-1}(u) \) for the both arrows.

2. The algebras \( A^+_L = X^*X^t, \rho = L_{s-1}^*x \) and \( A^* = XX^*, \rho = R_u^* \) are canonically isomorphic.

The proof uses the lemmas 3 and 4.

The lemma 3 shows also that the right analog of the \( p \)-representation is the action of the algebra \( X^*X \) on \( X^* \):

\[ u \to L_u \text{(left multiplication)}, \quad x \to R_x^* \text{(r-differentiation)} \]

The right analog of the \( x \)-representation is the action of the same algebra on \( X \):

\[ x \to R_x \text{(right multiplication)}, \quad u \to L_u^* \text{(l-differentiation)} \]

The multiplication and differentiation are Fourier-dual. The element \( R \) from the lemma 2 is equal to \( \exp[p \otimes x] \) for the commutative Hopf algebra with the single primitive generator \( x \); \( p \) is the dual primitive generator, \( sx = -x, sp = -p \).

For the cocommutative Hopf algebras \( X \) the most important problems involving the "almost Hopf" properties of the \( O \)-doubles (for example, \( A^U \) in the complex cobordism theory) were connected with the "Lie-semigroup" containing the all multiplicative elements \( a(uv) = a(u)a(v) \), where \( a \) may belong to some formal completion of the \( O \)-double, defined in such a way that the action on the dual \( X^* \) is well-defined (the right \( U \)-analogs of the Adams operations, Chern character and Projectors \( a^2 = a \) -see [5,7,8]).

The analogous problems are very interesting also for the almost cocommutative or quasitriangular "Hopf-Drinfeld" algebras (quantum groups with well-defined abstract \( R \)-matrix determining the commutation rules in the algebra \( X^* \)).

**References**

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Remark: After the discussion with L. Faddeev I realized that some construction of the special O-double was found independently in the recent paper of Faddeev and Alexeev in the special important examples (of course they did not know the cobordism theory). Their paper does not use the the Hopf-algebraic terminology at all (the relations in that algebra were written by the complicated special formulas).

M. Semenov-Tyanshansky wrote now the paper more closed to our ideas but still he does not use some elementary algebraic definitions replacing them by the complicated formulas in the special examples (his paper will be published very soon in the Theor Math Phys, November 1992).

Yu. Manin wrote a paper "Note on Quantum groups and Quantum De Rham complexes", MPI fur mathematics, Bonn (1991), dedicated mainly to some analogs of differential forms. There is a discussion concerning the rings of differential operators and skew tensor products in his paper. Some of his ideas are parallel to ours, but the main direction is different.

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