Picker-Chooser fixed graph games

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Abstract

Given a fixed graph $H$ and a positive integer $n$, a Picker-Chooser $H$-game is a biased

game played on the edge set of $K_n$ in which Picker is trying to force many copies of $H$ and

Chooser is trying to prevent him from doing so. In this paper we conjecture that the value

of the game is roughly the same as the expected number of copies of $H$ in the random

graph $G(n, p)$ and prove our conjecture for special cases of $H$ such as complete graphs and
trees.

1 Introduction

A Waiter-Client game is a positional game which was first defined and studied by Beck under

the name of Picker-Chooser (see, e.g. [1]). Let $a$ and $b$ be positive integers, let $X$ be a finite

set and let $F$ be a family of subsets of $X$. A biased $(b : a)$ Waiter-Client game $(X, F)$ is

defined as follows. The game proceeds in rounds. In each round, Waiter selects exactly $a+b$

free elements of $X$ (that is, elements he has not previously selected) and offers them to Client.

Client then selects exactly $a$ of these elements which he keeps and the remaining $b$ elements

are claimed by Waiter. If at some point during the game only $r < a + b$ free board elements

remain, then Client selects $\min\{a, r\}$ of them and the remaining $\max\{0, r - a\}$ elements go
to Waiter. Waiter’s goal is to maximize the number of sets $A \in F$ whose elements were all

claimed by Client by the end of the game, whereas Client aims to minimize it. The set $X$
is referred to as the board of the game and the elements of $F$ are referred to as the winning sets.
The value of the game is the number of sets $A \in F$ whose elements were all claimed by Client
by the end of the game, assuming perfect play by both players.

The interest in such games is three-fold. Firstly, they are interesting in their own right. For
example, the case where Waiter plays randomly is the well-known Achlioptas process (without
replacement). Many randomly played Waiter-Client games were considered in the literature,
often under different names (see, e.g. [12, 14, 13]). Secondly, they exhibit a strong probabilistic

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intuition (see, e.g. [1, 2]). That is, the outcome of many natural positional games of this type is often roughly the same as it would be had both players played randomly (although, typically, a random strategy for any single player is very far from optimal). The main results of this paper form a natural new example of this intriguing phenomenon. Lastly, it is believed that these games may be useful in the analysis of the so-called Maker-Breaker games.

A biased \((a : b)\) Maker-Breaker game \((X, \mathcal{F})\) is defined as follows. Two players, called Maker and Breaker, take turns in claiming previously unclaimed elements of \(X\); usually Maker is the first player. Maker claims exactly \(a\) board elements per turn and Breaker claims exactly \(b\). Here too the value of the game is the number of sets \(A \in \mathcal{F}\) whose elements were all claimed by Maker by the end of the game, assuming perfect play by both players. Maker’s goal is to maximize the value of the game, whereas Breaker aims to minimize it.

It was suggested by Beck [1] and subsequently formally conjectured by Csernenszky, Mándity and Pluhár in [7] that “being Waiter is not harder than being Maker”. That is, whenever Maker (as the second player) has a winning strategy for the \((1 : 1)\) Maker-Breaker game \((X, \mathcal{F})\), Waiter has a winning strategy for the \((1 : 1)\) Waiter-Client game \((X, \mathcal{F})\). Though, in its full generality, this conjecture was recently refuted by Knox [11], it is still plausible that understanding Waiter-Client games is helpful in the study of Maker-Breaker games. In particular, it was proved in [3] that a version of Beck’s conjecture which applies to biased games as well, holds in certain special cases.

We remark that Waiter-Client games are also related to a well-known misère version of Maker-Breaker games, the so-called Avoider-Enforcer games, in which Enforcer aims to force Avoider to claim as many sets \(A \in \mathcal{F}\) as possible (for more information on these games see, for instance, [9, 8]).

From here on we restrict our attention to fixed graph games. Let \(H\) be a graph and let \(n\) be a positive integer. The board of the \(H\)-game is the edge set \([n]^2\) of the complete graph on \(n\) vertices and the family of winning sets \(\mathcal{F}_H\) consists of the edge sets of all copies of \(H\) in \(K_n\). Furthermore, our bias will always be of type \((b : 1)\), i.e. we shall assume that Client (or Maker) selects just one edge per round. Let us denote the value of such a Waiter-Client game by \(S(H, n, b)\) and the value of the analogous Maker-Breaker game by \(S_{MB}(H, n, b)\).

Let us first report the known results regarding \(S_{MB}(H, n, b)\). Before doing so, let us recall that, as we have already mentioned, often the outcome of a positional game is roughly the same as it would be had both players played randomly. Since the densities of the graphs built by Client and by Maker by the end of the game are the same and are equal to \(1/(1 + b)\), it would be useful to determine the number of copies of \(H\) in the random graph \(G(n, 1/(1 + b))\), where \(G(n, p)\) denotes the random graph in which each pair from \([n]^2\) is present independently with probability \(p\). It turns out that this number depends mainly on the density of \(H\). Hence, let us introduce two measures of density of a graph \(H\), both of which are crucial for Waiter-Client \(H\)-games as well. The maximum density \(m(H)\) is defined to be
\[
m(H) = \max \left\{ \frac{e(H')}{v(H')} : H' \subseteq H, v(H') \geq 1 \right\},
\]
where here and throughout the paper \(v(G)\) and \(e(G)\) denote the number of vertices and edges of \(G\) respectively. We shall also use the maximum 2-density \(m_2(H)\) of \(H\), where
\[
m_2(H) = \max \left\{ \frac{e(H') - 1}{v(H') - 2} : H' \subseteq H, v(H') \geq 3 \right\}.
\]
The following result is known (see [5, 15]).

**Theorem 1.1 ([5, 15])** For every graph \( H \) with at least one edge the following holds. If \( np^{m(H)} \to 0 \), then a.a.s. \( G(n, p) \) contains no copies of \( H \). On the other hand, if \( np^{m(H)} \to \infty \), then a.a.s. \( G(n, p) \) contains \((c_H + o(1))n^{v(H)} p^{e(H)}\) copies of \( H \), for some constant \( c_H > 0 \).

In [4] the authors studied the threshold value of \( b \) for which \( S_{MB}(H, n, b) > 0 \). As a direct consequence of the probabilistic argument presented there and of Theorem 1.1, it follows that the value of the game rapidly grows when \( b = \Theta(n^{1/m_2(H)}) \). Formally, the following result, which is implicit in [4], can be derived.

**Theorem 1.2 ([4])** For every graph \( H \) with at least three non-isolated vertices there are positive constants \( c', c'', \beta^- \), and \( \beta^+ \), such that the following holds. If \( b \geq c'n^{1/m_2(H)} \) then

\[
S_{MB}(H, n, b) = 0.
\]

On the other hand if \( b \leq c''n^{1/m_2(H)} \), then

\[
\beta^- n^{v(H)} b^{-e(H)} \leq S_{MB}(H, n, b) \leq \beta^+ n^{v(H)} b^{-e(H)}.
\]

Thus, somewhat unexpectedly, Maker cannot create even a single copy of \( H \) until the density of his graph grows to the value which would guarantee that the number of copies of \( H \) in the random graph is as large as the number of its edges. However, soon afterwards, Maker can build roughly the same number of copies of \( H \) as the expected number of such copies in \( G(n, p) \) with the same density as his graph (i.e. \( p = 1/(1 + b) \)).

The main purpose of this paper is to show that the behaviour of Waiter-Client \( H \)-games is quite different and that the value of the game grows almost exactly as suggested by the random graph heuristic. Let us start by stating the following simple corollary of Beck’s potential method which we will prove in Section 2.

**Theorem 1.3** For every graph \( H \) with at least one edge there are positive constants \( c_H \) and \( c'_H \) such that the following holds.

(i) \( S(H, n, b) \leq c_H \cdot n^{v(H)}(b + 1)^{-e(H)} \).

(ii) If \( b \geq c'_H \cdot n^{1/m(H)} \) then \( S(H, n, b) = 0 \).

We conjecture that the upper bound on \( S(H, n, b) \), given in Theorem 1.3, is tight up to a constant factor, i.e. that the following general conjecture holds for every graph \( H \).

**Conjecture 1.4** For every graph \( H \) with at least one edge, there are positive constants \( c_1, c_2, \alpha^- \), and \( \alpha^+ \) such that the following holds.

(i) If \( b \geq c_1 \cdot n^{1/m(H)} \) then \( S(H, n, b) = 0 \).
(ii) If \( b \leq c_2 \cdot n^{1/m(H)} \) then
\[
\alpha^- n^{v(H)} b^{-\varepsilon(H)} \leq S(H,n,b) \leq \alpha^+ n^{v(H)} b^{-\varepsilon(H)}.
\]

This is a rather striking conjecture since, if true, it is probably the first example where the value of the game follows so precisely the predictions given by a random heuristic. Nonetheless, we strongly believe it to be true. In fact, we do not know of any graph \( H \) and any function \( b = b(n) \) for which we cannot prove Conjecture \( \text{[1.4]} \) by some ad hoc combinations of techniques (or its more sophisticated variants) described in this paper. Unfortunately, at this moment, we cannot present a unified approach which will work for every \( H \).

In light of Theorem \( \text{[1.3]} \) in order to verify Conjecture \( \text{[1.4]} \) we need only to prove the lower bound on \( S(H,n,b) \) stated in Part (ii), that is, we need to provide a strategy for Waiter which, for appropriate values of \( b \), forces Client’s graph to contain many copies of \( H \). Probably the most natural strategy of this kind is one which constructs \( H \) for appropriate values of \( S \) bound on \( S \).

For more complicated graphs \( H \), we develop another method, which is based on counting prohibited structures. Its heart is a rather simple but very useful observation (Theorem \( \text{[3.1]} \)) asserting that Waiter can prevent Client from claiming certain structures if they are very rare. As an immediate consequence of this fact we infer that Conjecture \( \text{[1.4]} \) holds for every graph \( H \) provided that \( b \) is not too large, i.e. is bounded from above by the same function as in Theorem \( \text{[1.2]} \).

**Theorem 1.5** Let \( k \geq 2 \) and \( n \) be positive integers and let \( T \) be a tree on \( k \) vertices. Then there exists a positive constant \( \epsilon \) such that if \( 1 \leq b \leq cn^{k/(k-1)} \), then \( S(T,n,b) = \Theta (n^k \cdot (b+1)^{1-k}) \) (for \( k = 1 \) we can replace \( n^{k/(k-1)} \) with \( \infty \)).

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**Theorem 1.6** For every graph \( H \) with at least three non-isolated vertices there exist positive constants \( c, \alpha^- \), and \( \alpha^+ \) such that
\[
\alpha^- n^{v(H)} b^{-\varepsilon(H)} \leq S(H,n,b) \leq \alpha^+ n^{v(H)} b^{-\varepsilon(H)},
\]
provided that \( b \leq cn^{1/m_2(H)} \).

In fact, for graphs \( H \) that satisfy certain technical conditions which, roughly speaking, assert that \( H \) is ‘well-balanced’, we can prove much more. Namely, we can show that not only can Waiter force Client to build many copies of \( H \), but he can do it early in the game. Moreover, Waiter can offer pairs such that the following holds: for every pair of vertices \( \{v,w\} \), if \( v \) and \( w \) belong to a copy of \( H \) in Client’s graph but \( \{v,w\} \) is not an edge of \( H \), then the pair \( \{v,w\} \) belongs to exactly one copy of \( H \) and this pair has not yet been offered by Waiter. For such a graph \( H \), our strategy for verifying Conjecture \( \text{[1.4]} \) is as follows. For a given \( b = b(n) \), we delete some edges of \( H \), thus obtaining a balanced spanning subgraph \( H' \) which is sparse enough to ensure \( b \leq cn^{1/m_2(H')} \). Applying Theorem \( \text{[1.6]} \) then forces at least \( c'' n^{v(H)} b^{-\varepsilon(H')} \) copies of \( H' \).
in Client’s graph. Now Waiter can offer Client free edges which partly extend his copies of $H'$ to copies of $H$. Clearly, for every extended copy of $H'$, the number of copies of $H'$ in Client’s graph which could potentially be extended is decreased by $\Theta(b)$. Since in order to complete a copy of $H$ we need to add to $H'$ exactly $e(H) - e(H')$ edges, at the very end we are left with

$$\Theta\left(n^{v(H')}b^{-e(H')}\right) \Theta\left(b^{-(e(H)-e(H'))}\right) = \Theta\left(n^{v(H)}b^{-e(H)}\right)$$

copies of $H$ in Client’s graph, as required. This method seems to be quite effective but has one serious drawback – it only works if we can find a spanning subgraph $H'$ of $H$ which is at the same time sparse and well-balanced. Unfortunately, this is not always possible, and even when it is, the analysis of the structure of $H'$ is often technical and long. Though our method works for a large family of graphs, for the sake of clarity and simplicity, in this paper we consider only complete graphs which suffice to illustrate the technical problems one should overcome in order to apply the method. Note that the maximum density of the complete graph on $k$ vertices is equal to $m(K_k) = (k - 1)/2$.

**Theorem 1.7** Let $k \geq 3$ be an integer. Then there exists a positive constant $c_k$ such that

(i) If $k \notin \{4, 5, 6\}$ and $1 \leq b \leq c_k \cdot n^{2/(k-1)}$, then

$$S(K_k, n, b) = \Theta\left(n^k \cdot (b + 1)^{-\left(\frac{k}{2}\right)}\right).$$

(ii) If $k \in \{4, 5, 6\}$ and $b \leq c_k \cdot n^{2/(k-1)}$, then $S(K_k, n, b) > 0$.

Finally let us reiterate that typically, for any given graph $H$ and any function $b = b(n)$, one can combine the techniques described above, in order to verify Conjecture [14] for these $H$ and $b$. For instance, consider the graph $F_9$ on nine vertices which consists of two vertex disjoint copies of $K_4$ joined by a path of length two. One can check that the approach based on finding many suitable ‘well-balanced’ subgraphs $F'$ of $F_9$ which then are extended to copies of $F_9$ fails here, because no subgraph $F'$ of $F_9$ fulfills all the technical requirements which are needed to employ this method. Nevertheless, Conjecture [14] can still be verified for $F_9$ by forcing many ‘uniformly spread out’ copies of $K_4$ on, say, half the vertices and subsequently using the remaining free pairs to join them by 2-paths. Still, it is fairly hard to find and describe a general approach which applies to any graph $H$ and even for relatively small graphs $H$, proving that Conjecture [14] holds for $H$ can be long and technical.

### 1.1 Notation and terminology

Our graph-theoretic notation is standard and follows that of [17]. In particular, we use the following.

For a graph $G$, let $V(G)$ and $E(G)$ denote its sets of vertices and edges respectively, and let $v(G) = |V(G)|$ and $e(G) = |E(G)|$. For disjoint sets $A, B \subseteq V(G)$, let $E_G(A, B)$ denote the set of edges of $G$ with one endpoint in $A$ and one endpoint in $B$ and let $e_G(A, B) = |E_G(A, B)|$. For a vertex $u \in V(G)$ and a set $B \subseteq V(G)$ we abbreviate $E_G(\{u\}, B)$ under $E_G(u, B)$. For a set $S \subseteq V(G)$, let $G[S]$ denote the subgraph of $G$ which is induced on the set $S$. For a
vertex \( u \in V(G) \) and a set \( B \subseteq V(G) \), let \( N_G(u, B) = \{ v \in B : uv \in E(G) \} \) denote the set of neighbors of \( u \) in \( B \) and let \( d_G(u, B) = |N_G(u, B)| \) denote its degree in \( B \). We abbreviate \( N_G(u, V(G)) \) and \( d_G(u) \) under \( N_G(u) \) and \( d_G(u) \), respectively. Often, when there is no risk of confusion, we omit the subscript \( G \) from the notation above. Given two graphs \( G \) and \( H \) on the same set of vertices \( V \), let \( G \setminus H \) denote the graph with vertex set \( V \) and edge set \( E(G) \setminus E(H) \). If \( H \) has \( n \) vertices, the graph \( K_n \setminus H \) is the complement of \( H \), denoted by \( H^c \).

Assume that some Waiter-Client game, played on the edge set of \( K_n \), is in progress. At any given moment during this game, let \( G_C \) denote the graph spanned by Client’s edges, let \( G_W \) denote the graph spanned by Waiter’s edges, and let \( G_F \) denote the graph spanned by those edges of \( K_n \) which are neither in \( G_C \) nor in \( G_W \). The edges of \( G_F \) are called free.

The rest of this paper is organized as follows: in Section 2 we prove Theorem 1.3 and Theorem 1.5. In Section 3 we describe a general efficient strategy for Waiter to avoid rare structures. In Section 4 we prove several properties of a certain model of random graphs. Results obtained in Sections 3 and 4 are then used in Section 5 to estimate the value of Waiter-Client \( H \)-games; in particular, we prove Theorem 1.6. We consider the special case in which \( H \) is a clique in Section 6.

### 2 Tree games

Our first aim in this section is to prove Theorem 1.3. We will deduce it from the following sufficient condition for Client’s win in biased Waiter-Client games.

**Proposition 2.1** Let \( X \) be a finite set, let \( F \) be a family of subsets of \( X \) and let \( b \) be a positive integer. Playing the \((b : 1)\) Waiter-Client game \((X, F)\), Client has a strategy to ensure that, at the end of the game, the board elements he claimed will span at most \( \sum_{A \in F} (b+1)^{|A|} \) winning sets.

The proof of this proposition is a straightforward application of the potential method, whose details can be found in [2], and is therefore omitted.

**Proof of Theorem 1.3** Part (i) is an immediate corollary of Proposition 2.1 with \( X = E(K_n) \) and \( F = F_H \).

For Part (ii), let \( H' \) be a subgraph of \( H \) such that \( m(H) = e(H')/v(H') \). Then there exists a positive constant \( c \) (depending on \( H \)) such that if \( b > cn^{1/m(H)} \), then

\[
\sum_{A \in F_{H'}} (b+1)^{-|A|} \leq n^{v(H')(b+1)-e(H')} < 1.
\]

Applying Proposition 2.1 with \( X = E(K_n) \) and \( F = F_{H'} \), we conclude that, playing a \((b : 1)\) Waiter-Client game on \( E(K_n) \), Client has a strategy to avoid claiming a copy of \( H' \), thereby avoiding claiming a copy of \( H \) as well.

Our next aim is to prove Theorem 1.5; it will readily follow from the following two lemmata.
Lemma 2.2 Let \( T \) be a tree on \( k \geq 1 \) vertices. If \( n \) is sufficiently large and \( b \leq n/2^{k+6} \), then \( S(T, n, b) \geq 4^{-(k+1)/2}n^k(b+1)^{1-k} \).

Proof For every \( k \geq 1 \), let \( t_k(n, b) = 4^{-(k+1)/2}n^k(b+1)^{1-k} \). We will prove the lemma by induction on \( k \). For \( k = 1 \) the assertion of the lemma is trivially true. Fix some \( k \geq 1 \) and assume that the assertion of the lemma holds for every tree on \( k \) vertices. Let \( T \) be an arbitrary tree on \( k+1 \) vertices; we will prove that if \( b \leq n/2^{k+7} \), then Waiter can force Client to build at least \( t_{k+1}(n, b) \) copies of \( T \). Let \( v_{k+1} \) be a leaf of \( T \) and let \( v_k \) be its unique neighbor. Let \( T_k = T \setminus \{v_{k+1}\} \). We partition the vertex set \( V(K_n) \) into two subsets \( V_1 \) and \( V_2 \) such that \( |V_1| = \lfloor n/2 \rfloor \) and \( |V_2| = \lceil n/2 \rceil \). Waiter’s strategy is divided into two stages.

In the first stage, offering only edges of \( K_n[V_1] \), Waiter forces Client to build a family \( \mathcal{T} \) consisting of at least \( t_k(|V_1|, b) \) copies of \( T_k \). This is clearly possible by the induction hypothesis since \( b \leq n/2^{k+7} \leq |V_1|/2^{k+6} \).

In the second stage Waiter forces Client to extend every copy \( T^i \) of \( T_k \) he has built during the first stage, into many copies of \( T \). For every such \( T^i \), let \( u^i \) denote the vertex which corresponds to \( v_k \). Waiter offers \( b+1 \) edges of \( E(u^i, V_2) \) for \( \lceil |V_2|/(b+1) \rceil \) consecutive rounds. Client is thus forced to build at least \( t_k(|V_1|, b)|V_2|/(b+1) \) copies of \( T \). Since

\[
\frac{|V_2|}{b+1} \geq \frac{n/2-1}{b+1} > \frac{n}{4(b+1)}
\]

holds for \( b < n/4 - 2 \), the number of copies of \( T \) in Client’s graph at the end of the game is at least

\[
t_k(|V_1|, b) \cdot \frac{n}{4(b+1)} = \frac{n^k}{2^k \cdot 4^{(k+1)/2}(b+1)^{k-1}} \cdot \frac{n}{4(b+1)} \geq \frac{n^{k+1}}{4^{(k+2)/2}(b+1)^k} = t_{k+1}(n, b),
\]

as claimed. \( \square \)

Lemma 2.3 Let \( T \) be a tree on \( k \geq 1 \) vertices. If \( n \) is sufficiently large and \( n \leq b \leq n^{(k+1)/2^{k+6}} \), then playing a \( (b:1) \) Waiter-Client game on \( E(K_n) \), Waiter has a strategy to force Client to build at least \( 4^{-(k+1)/2}n^k(b+1)^{1-k} \) vertex disjoint copies of \( T \). (For \( k = 1 \) we can replace \( n^{(k+1)/2} \) with \( \infty \).)

Proof As in the previous proof we proceed by induction on \( k \) and define \( t_k(n, b), T, v_{k+1}, v_k, T_k, V_1 \) and \( V_2 \) accordingly. Assuming that \( n \leq b \leq n^{(k+1)/2^{k+7}} \), we repeat the inductive argument of that proof, with the following two differences.

The first difference is that the family \( \mathcal{T} \) of copies of \( T_k \) in Client’s graph consists of pairwise vertex disjoint graphs.

The second difference is that in the second stage Waiter forces Client to extend some of the copies of \( T_k \) in \( \mathcal{T} \) into pairwise vertex disjoint copies of \( T \) (in particular, a copy of \( T_k \) can be extended into at most one copy of \( T \)). His strategy for doing this is slightly different. Immediately before each round of the second stage, Waiter defines \( A \) to be the set of all vertices \( u \) which correspond to \( v_k \) in a copy of \( T_k \) in \( \mathcal{T} \), such that \( d_{G_C}(u, V_2) = 0 \). Moreover, he defines \( B \) to be the set of all vertices \( v \in V_2 \) such that \( d_{G_C}(v) = 0 \). If \( e_{G_F}(A, B) \geq b+1 \),
then Waiter offers Client $b+1$ arbitrary free edges of $E(A,B)$. The second stage is over as soon as $e_{G_{k}}(A,B) \geq b+1$ or $|V_2 \setminus B| \geq t_{k+1}(n,b)$ first holds.

Note that, at the end of the second stage, every edge $uv \in E(G_C)$ such that $u$ corresponds to $v_k$ in some copy $T'$ of $T_k$ and $v \in V_2 \setminus B$, extends $T'$ into a copy of $T$ in $G_C$. Moreover, the resulting copies of $T$ are pairwise vertex disjoint. Therefore, in order to complete the proof, it suffices to verify that $|V_2 \setminus B| \geq t_{k+1}(n,b)$ holds at the end of the second stage.

Suppose for a contradiction that $|V_2 \setminus B| < t_{k+1}(n,b)$ holds at the end of the second stage. It is not hard to see that if a vertex $u$ corresponds to $v_k$ in some copy of $T_k$ in $T$, then either $u \in A$ or $d_{G_C}(u,V_2) = 1$. Similarly, for every $v \in V_2$, either $v \in B$ or $d_{G_C}(v) = 1$. Therefore, the total number of rounds played in the second stage is $|V_2 \setminus B| = |T| - |A|$; denote this number by $r$. Since $|T| \geq t_k(|V_1|,b)$ holds by the induction hypothesis, we obtain

$$|A| = |T| - r > |T| - t_{k+1}(n,b) \geq t_k(n/2,b) - t_{k+1}(n,b) = \frac{n^k}{4^{(k+1)/2}(b+1)^{k-1}} \left( \frac{1}{2^k} - \frac{n}{4^{k+1}(b+1)} \right) \geq \frac{n^k}{4^{(k+1)/2}(b+1)^{k-1}} \cdot \frac{1}{2^{k+1}},$$

where the last inequality holds since $b \geq n$.

Similarly

$$|B| = |V_2| - |V_2 \setminus B| > |V_2| - t_{k+1}(n,b) > n/2 - 1 - t_{k+1}(n,b) = n/2 - 1 - \frac{n^{k+1}}{4^{(k+1)/2}(b+1)^k} > 3n/8,$$

where the last inequality holds since $b \geq n$.

Since $|V_2 \setminus B| < t_{k+1}(n,b)$ by assumption, it follows that $|A||B| - r(b+1) < b+1$ holds at the end of the second stage. Hence, using (1) and (2) we obtain

$$r + 1 > \frac{|A||B|}{b+1} = \frac{n^k}{4^{(k+1)/2}(b+1)^{k-1}} \cdot \frac{1}{2^{k+1}} \cdot \frac{3n}{8} \cdot \frac{1}{b+1} \geq \frac{1.5n^{k+1}}{4^{(k+1)/2}(b+1)^k} = 1.5t_{k+1}(n,b)$$

Since $t_{k+1}(n,b) > 2$ for every $b \leq n^{(k+1)/2k+7}$, it follows by (3) that $r > t_{k+1}(n,b)$. Since $r = |V_2 \setminus B|$, this contradicts our assumption that $|V_2 \setminus B| < t_{k+1}(n,b)$. \qed

**Proof of Theorem 1.3** The required upper bound on $S(T,n,b)$ follows immediately from Theorem 1.3(i); it thus remains to prove the lower bound.

For $b \leq n/2^{k+6}$ the desired lower bound follows from Lemma 2.2 and for $n \leq b \leq n^{k/(k-1)}/2^{k+6}$ it follows from Lemma 2.3. Assume then that $n/2^{k+6} < b < n$ and observe that Waiter-Client games are bias-monotone, that is, when proving that Waiter wins some $(b : 1)$ Waiter-Client game, we can allow Waiter to offer more than $b+1$ edges per round. We conclude that $S(T,n,b) \geq S(T,n,n) = \Omega \left( n^k \cdot (n+1)^{1-k} \right) = \Omega \left( n^k \cdot (b+1)^{1-k} \right)$. \qed

### 3 The Big Family Theorem

In this section we state and prove the main game theoretic tool of this paper, which is also of independent interest. Roughly speaking, it asserts that if almost every $M$-subset of the board
is “good”, then Waiter can force Client to claim such a set in $M$ rounds.

**Theorem 3.1** Let $X$ be a set of size $N$, let $\mathcal{H}$ be an $M$-uniform family of subsets of $X$ and let $b = b(N)$ be a positive integer. Let $0 < \alpha < 1$ be a real number such that $|\mathcal{H}| \geq (1 - \alpha^M)(N M)$ and $b + 1 \leq (1 - \alpha)N/M$. Then, playing the $(b : 1)$ Waiter-Client game $(X, \mathcal{H})$, Waiter has a strategy to force Client to fully claim some $A \in \mathcal{H}$ during the first $M$ rounds of the game.

**Proof** Let $\mathcal{H}$ be the family of sets described in the theorem and let $\mathcal{H}^c$ be its $M$-uniform complement, that is, $\mathcal{H}^c = \{A \in \binom{X}{M} : A \notin \mathcal{H}\}$. It follows by the assumption of the theorem that $|\mathcal{H}^c| \leq \alpha^M(N M)$.

In order to prove the theorem it suffices to prove that Waiter can prevent Client from claiming some $A \in \mathcal{H}^c$ during the first $M$ rounds of the game.

Let $C_0 = P_0 = \emptyset$ and, for every positive integer $i$, let $C_i$ and $P_i$ denote the set of elements of Client and of Waiter respectively, immediately after the $i$th round. Moreover, for every non-negative integer $i$ let

$$E_i = \{A \setminus C_i : A \in \mathcal{H}^c, A \cap P_i = \emptyset \text{ and } C_i \subseteq A\}.$$ 

For every $x \in X$ and non-negative integer $i$ let

$$\deg_i(x) = |\{S \in E_i : x \in S\}|$$

and let $N_i = N - i(b + 1)$. Note that $N_i$ is the number of free board elements immediately after round $i$.

The definition of $E_i$ implies that

$$|S| = M - i \text{ holds for every } S \in E_i.$$  \hspace{1cm} (4)

Moreover, if in the $i$th round Client claimed $y$, then $S \cup \{y\} \in E_{i-1}$ whenever $S \in E_i$ and thus

$$|E_i| \leq \deg_{i-1}(y).$$  \hspace{1cm} (5)

We are now ready to describe Waiter’s strategy. For every positive integer $i$, in the $i$th round, Waiter offers Client exactly $b + 1$ free board elements $x$ whose value of $\deg_{i-1}(x)$ is minimal (breaking ties arbitrarily). It remains to prove that this is a winning strategy.

Let $x$ be an element Waiter has offered Client in the $i$th round. It follows by Waiter’s strategy that

$$\deg_{i-1}(x) \leq \frac{1}{N_{i-1} - b} \sum_{u \in X \setminus (C_{i-1} \cup P_{i-1})} \deg_{i-1}(u) = \frac{|E_{i-1}|(M - (i - 1))}{N_{i-1} - b},$$  \hspace{1cm} (6)

where the last equality follows since for every $S \in E_{i-1}$, $|S| = M - (i - 1)$ holds by (4) and $S \subseteq X \setminus (C_{i-1} \cup P_{i-1})$ holds by the definition of $E_{i-1}$.

Hence, regardless of which element $y$ Client claims in round $i$, it follows by (5) and (6) that

$$|E_i| \leq \deg_{i-1}(y) \leq \frac{|E_{i-1}|(M - (i - 1))}{N_{i-1} - b}.$$  \hspace{1cm} (7)
Consequently, since $|E_0| = |\mathcal{H}^c| \leq \alpha^M \binom{N}{M}$, it follows by an iterated application of (7) that

\[
|E_M| \leq \prod_{i=0}^{M-1} \frac{M-i}{N_i-b} |E_0| \\
\leq \frac{M!}{(N_{M-1} - b)^M} |E_0| \\
\leq \frac{M!}{(N - M(b + 1))^M} \alpha^M \binom{N}{M} \\
< \left( \frac{\alpha N}{N - M(b + 1)} \right)^M \\
\leq 1,
\]

where the last inequality follows since $b + 1 \leq (1 - \alpha)N/M$ holds by assumption.

Hence, $|E_M| < 1$, which means that Client did not claim all elements of any $A \in \mathcal{H}^c$ in the first $M$ rounds and so Waiter has achieved his goal. \qed

## 4 Probabilistic tools

This section contains several results of different levels of difficulty. It is thus divided into two subsections: the first containing some useful terminology and simple facts about certain models of random graphs, and the second containing more advanced results, regarding the number of copies of a fixed graph in those random graphs.

### 4.1 Preliminaries

We begin this section with some more notation and terminology. Given a graph $H$ with $v(H) \geq 1$, its maximum density is

\[
m(H) = \max \left\{ \frac{e(H')}{v(H')} : H' \subseteq H, v(H') \geq 1 \right\}.
\]

Similarly, given a graph $H$ with $v(H) \geq 3$, its maximum 2-density is

\[
m_2(H) = \max \left\{ \frac{e(H') - 1}{v(H') - 2} : H' \subseteq H, v(H') \geq 3 \right\}.
\]

A graph $H$ on at least 3 vertices is called $m_2$-balanced if $m_2(H) = (e(H) - 1)/(v(H) - 2)$. The following simple lemma provides an alternative characterization of $m_2$-balanced graphs.

**Lemma 4.1** A graph $H$ with at least two edges and no isolated vertices is $m_2$-balanced if and only if

\[
\frac{v(H) - v(H')}{e(H) - e(H')} \leq \frac{v(H) - 2}{e(H) - 1}
\]

for every $H' \subseteq H$ with $e(H') \geq 2$.  


\textbf{Proof} Let $H' \subseteq H$ be an arbitrary subgraph with $e(H') \geq 2$. A straightforward calculation shows that

$$\frac{v(H) - v(H')}{e(H) - e(H')} \leq \frac{v(H) - 2}{e(H) - 1} \iff \frac{e(H') - 1}{v(H') - 2} \leq \frac{e(H) - 1}{v(H) - 2}.$$  

(8)

Since the right hand side of (8) clearly holds for any $H' \subseteq H$ with $e(H') \leq 1$, the assertion of the lemma follows by the definition of the maximum 2-density. \hfill \Box

For a graph $H$ on the vertex set $\{v_1, \ldots, v_t\}$ we define a graph $\mathbb{B}_H(V_1, \ldots, V_t; n)$, called the \textit{nth-blow-up} of $H$, as follows. We replace every vertex $v_i$ of $H$ with a set $V_i$ of $n$ isolated vertices and every edge $v_iv_j$ of $H$ with the corresponding complete bipartite graph, that is, with the set of edges $\{xy : x \in V_i, y \in V_j\}$. Let $H'$ be a subgraph of $H$ on the vertex set $\{v_{i_1}, \ldots, v_{i_r}\}$. A copy of $H'$ in $\mathbb{B}_H(V_1, \ldots, V_t; n)$ is said to be \textit{canonical} if $v_{i_j} \in V_{i_j}$ for every $1 \leq j \leq r$.

A family $\mathcal{F}$ of subgraphs of $\mathbb{B}_H(V_1, \ldots, V_t; n)$ is said to be a \textit{sparse $H$-family} if the following two conditions hold

(i) Every $G \in \mathcal{F}$ is a canonical copy of $H$.

(ii) $G_1^c \cap G_2^c = \emptyset$ for any two distinct graphs $G_1, G_2 \in \mathcal{F}$ (equivalently, $G_1 \cap G_2$ is either empty or a complete graph).

Let $\mathbb{G}(H, n, M)$ denote a graph obtained by selecting uniformly at random precisely $M$ edges of the $nth$-blow-up of $H$. For a graph $H$ and its subgraph $H'$ let $X_{H'}$ be the random variable counting the number of canonical copies of $H'$ in $\mathbb{G}(H, n, M)$. For every graph $H$ with at least two edges let

$$f_H(n, M) = \min \{\mathbb{E}(X_{H'}) : H' \subseteq H, e(H') \geq 2\}$$

and for every graph $H$ with at least one edge let

$$\hat{f}_H(n, M) = \min \{\mathbb{E}(X_{H'}) : H' \subseteq H, e(H') \geq 1\}.$$ 

Our next goal is to prove several properties of $f_H(n, M)$ and of $\hat{f}_H(n, M)$ that will be used in subsequent sections. Due to the well-known asymptotic equivalence of certain random graph models, it suffices to state and prove these results for the corresponding binomial model of the random graph which is easier to work with.

Let $\mathbb{G}(H, n, p)$ denote a graph obtained by randomly selecting every edge of the $nth$-blow-up of $H$ with probability $p$, independently of all other edge selections.

For a graph $H$ and its subgraph $H'$ let $Y_{H'}$ be the random variable counting the number of canonical copies of $H'$ in $\mathbb{G}(H, n, p)$; note that $\mathbb{E}(Y_{H'}) = n^{v(H')}p^{e(H')}$. Let

$$f_H(n, p) = \min \{\mathbb{E}(Y_{H'}) : H' \subseteq H, e(H') \geq 2\}$$

and let

$$\hat{f}_H(n, p) = \min \{\mathbb{E}(Y_{H'}) : H' \subseteq H, e(H') \geq 1\}.$$ 

\textbf{Lemma 4.2} Let $H$ be a graph with at least two edges and no isolated vertices.

(i) If $p \geq s^{-1/m_2(H)}$, then $\hat{f}_H(s, p) = s^2p$. 

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(ii) If $H$ is not a matching and $H_0$ is a subgraph of $H$ such that $p \geq cs^{-(v(H_0)-2)/(e(H_0)-1)}$ for some positive constant $c$ and $f_H(s,p) = s^{v(H_0)p^{e(H_0)}}$ holds for all sufficiently large $s$, then $H_0$ is connected.

(iii) Let $c_0 \leq 1$ be a positive constant. If

$$p \leq c_0^{-1} s^{v(H)-e(H')/(e(H')-1)}$$

for every $H' \subset H$ with $e(H') \geq 2$, then $f_H(s,p) \geq c_0 s^{v(H)p^{e(H)}}$.

**Proof** Starting with (i), if $H'$ is a subgraph of $H$ consisting of a single edge, then $E(Y_{H'}) = s^2 p$. Thus $\hat{f}_H(s,p) \leq s^2 p$. Conversely, let $H'$ be a subgraph of $H$ such that $f_H(s,p) = s^{v(H')}p^{e(H')}$ and suppose for a contradiction that $s^{v(H')}p^{e(H')} < s^2 p$. It follows that

$$p < s^{-v(H')/e(H')-1} \leq s^{-1/m_2(H)} ,$$

contrary to our assumption.

Next, we prove (ii). Suppose for a contradiction that $H_0$ is a disjoint union of two graphs, $H_1$ and $H_2$. For $i \in \{1,2\}$ let $v_i = v(H_i)$ and $e_i = e(H_i)$; note that $v(H_0) = v_1 + v_2$ and $e(H_0) = e_1 + e_2$. Assume without loss of generality that $v_1/e_1 \leq v_2/e_2$. It follows by the definitions of $H_0$ and $f_H(s,p)$ that $s^{v(H_0)}p^{e(H_0)} \leq s^{v_1}p^{e_1}$ for sufficiently large $s$. Combined with the assumed lower bound on $p$ this entails that

$$c s s^{v_1+v_2-2/e_1+e_2-1} \leq p \leq s^{-v_2/e_2} .$$

(9)

Since (9) holds for sufficiently large $s$, we deduce that $(v_1 - 2)/(e_1 - 1) \geq v_2/e_2$. Since $v_2/e_2 \geq v_1/e_1$ holds by assumption, it follows that $v_1/e_1 \geq 2$. We conclude that $v_2/e_2 \geq 2$ and $p \leq s^{-2}$. Now, since $H$ contains two adjacent edges, it follows that $s^{v(H_0)}p^{e(H_0)} \leq s^3 p^2$ and thus

$$c s s^{v(H_0)-2/e(H_0)-1} \leq p \leq s^{v(H_0)-3/e(H_0)-2} .$$

(10)

Since (10) holds for sufficiently large $s$, we deduce that $(v(H_0) - 2)/(e(H_0) - 1) \geq (v(H_0) - 3)/(e(H_0) - 2)$. Hence $(v(H_0) - 2)/(e(H_0) - 1) \leq 1$ and $p \geq cs^{-1}$. Therefore $c s^{-1} \leq p \leq s^{-2}$ which is a contradiction for sufficiently large $s$.

Finally, we prove (iii). Let $H_0$ be a subgraph of $H$ such that $e(H_0) \geq 2$ and $f_H(s,p) = s^{v(H_0)}p^{e(H_0)}$. If $H_0 = H$, then our assertion clearly holds; thus assume that $H_0 \neq H$. It then follows by the assumed upper bound on $p$ that $p \leq c_0^{-1} s^{v(H_0) - v(H)}/(e(H) - e(H_0))$ entailing $s^{v(H)}p^{e(H)} \geq c_0 e(H) e(H_0)/s^{v(H_0)}p^{e(H_0)}$. Since $c_0 \leq 1$ by assumption, the assertion follows. \qed

4.2 Counting copies of $H$ in $\mathbb{G}(H,n,p)$

Our first result in this section asserts that the probability that $\mathbb{G}(H,n,p)$ contains too few canonical copies of $H$ is exponentially small.
Lemma 4.3 Let $H$ be a graph, let $n$ be a positive integer and let $\omega(1) = M \leq e(H)n^2/2$. Then there exists a constant $c > 0$ such that the probability that there are at least $\mathbb{E}(X_H)/2$ canonical copies of $H$ in $\mathbb{G}(H,n,M)$ is at least $1 - \exp(-cf_H(n,M))$.

Lemma 4.3 is a direct corollary of its $\mathbb{G}(H,n,p)$ analogue which can be stated as follows.

Lemma 4.4 Let $H$ be a graph, let $n$ be a positive integer and let $\omega(n^{-2}) = p = p(n) \leq 1/2$. Then there exists a constant $c > 0$ such that the probability that there are at least $\mathbb{E}(Y_H)/2$ canonical copies of $H$ in $\mathbb{G}(H,n,p)$ is at least $1 - \exp(-cf_H(n,p))$.

In the proof of Lemma 4.4 we will make use of the following concentration inequality.

Theorem 4.5 (Theorem 2.14 in [10]) Let $\Gamma$ be a finite set, let $S$ be a family of subsets of $\Gamma$ and let $\Gamma_p$ be a random set obtained from $\Gamma$ by selecting every element of $\Gamma$ independently, with probability $p$. For every $A \in S$, let $I_A$ denote the indicator random variable for the event $A \subseteq \Gamma_p$. Let $X = \sum_{A \in S} I_A$ and let $\Delta = \sum_{A \cap B \neq \emptyset} \mathbb{E}(I_AI_B)$. Then for $0 \leq t \leq \mathbb{E}(X)$ we have

$$\Pr(X \leq \mathbb{E}(X) - t) \leq \exp\left(-\frac{t^2}{2\Delta}\right).$$

Proof of Lemma 4.4 Let $S = \{H_1, \ldots, H_m\}$ be the family of all canonical copies of $H$ in the $n$th-blow-up of $H$. For every $1 \leq i \leq m$, let $I_i$ be the indicator random variable for the event $H_i \subseteq \mathbb{G}(H,n,p)$; then $Y_H = \sum_{i=1}^m I_i$. A straightforward calculation shows that $\Delta := \sum_{H_i \cap H_j \neq \emptyset} \mathbb{E}(I_iI_j) \leq \mathbb{E}(Y_H)^2/\hat{f}_H(n,p)$. Hence, applying Theorem 4.5 with $t = \mathbb{E}(Y_H)/2$ we obtain

$$\Pr(Y_H \leq \mathbb{E}(Y_H)/2) \leq \exp\left(-\frac{\hat{f}_H(n,p)}{8}\right).$$

We are now ready to state and prove the main result of this section. It asserts that, for the right values of $M$, with very high probability $\mathbb{G}(H,n,M)$ contains a large sparse $H$-family.

Lemma 4.6 For every $\varepsilon > 0$ and every graph $H$ with at least two edges, there exist positive constants $\alpha < 1$, $\beta$ and $\delta$ such that the following holds. For every $n$ and $M \geq n/\delta$ for which $\varepsilon M \leq f_H(n,M) \leq \delta n^2$, the probability that $\mathbb{G}(H,n,M)$ contains a sparse $H$-family with at least $\beta f_H(n,M)$ copies of $H$ is greater than $1 - \alpha^M$.

Since containing a sparse $H$-family whose size is at least $\beta f_H(n,M)$ is a monotone increasing graph property and since, by Chernoff’s bound, the number of edges in the binomial random graph $\mathbb{G}(H,n,p)$ is sharply concentrated around its expectation, Lemma 4.6 is a direct corollary of its binomial analogue which can be stated as follows.

Lemma 4.7 For every $\varepsilon > 0$ and every graph $H$ with at least two edges, there exist positive constants $\alpha < 1$, $\beta$ and $\delta$ such that the following holds. For every $n$ and $p \geq (\delta n)^{-1}$ for which $\varepsilon n^2 p \leq f_H(n,p) \leq \delta n^2$, the probability that $\mathbb{G}(H,n,p)$ contains a sparse $H$-family with at least $\beta f_H(n,p)$ copies of $H$ is greater than $1 - \alpha^{n^2 p}$.  

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In the proof of Lemma 4.7 we will make use of the following well-known concentration inequality due to Talagrand [10].

**Theorem 4.8 (Theorem 2.29 in [10])** Suppose that $Z_1, \ldots, Z_N$ are independent random variables taking their values in the set $\{0,1\}$. Suppose further that $X = f(Z_1, \ldots, Z_N)$, where $f : \{0,1\}^N \to \mathbb{R}$ is a function such that there exist constants $c_1, \ldots, c_N$ and a function $\psi$ for which the following two conditions hold:

(a) If $z, z' \in \{0,1\}^N$ differ only in the $k$th coordinate, then $|f(z') - f(z)| \leq c_k$.

(b) If $z \in \{0,1\}^N$, $r \in \mathbb{R}$ and $f(z) \geq r$, then there exists a set $J \subseteq \{1, \ldots, N\}$ with $\sum_{i \in J} e_i \leq \psi(r)$, such that for all $y \in \{0,1\}^N$ with $y_i = z_i$ for every $i \in J$, we have $f(y) \geq r$.

Then for every $r \in \mathbb{R}$ and $t \geq 0$ we have

$$\Pr(X \leq r - t)\Pr(X \geq r) \leq \exp(-t^2/(4\psi(r))).$$

**Proof of Lemma 4.7** For a constant $C$ let $S_C$ denote the size of a largest family $\mathcal{F}_C$ of canonical copies of $H$ in $G(H,n,p)$ which satisfies all of the following properties:

(i) The number of edges in the union of all graphs in the family $\mathcal{F}_C$ is at most $Cn^2p$.

(ii) Every edge of $\mathcal{G}(H,n,p)$ belongs to at most $Cf_H(n,p)/(n^2p)$ graphs of $\mathcal{F}_C$.

(iii) For all graphs $H_1, H_2 \in \mathcal{F}_C$, if $V(H_1 \cap H_2) = \{x, y\}$ for some $x, y \in V(\mathcal{G}(H,n,p))$, then $xy \in E(H_1 \cap H_2)$.

(iv) If $H_1 \in \mathcal{F}_C$ and $F$ is an induced subgraph of $H_1$ such that $v(F) \geq 3$ and $e(F) \leq 1$, then $H_1 \cap G \neq F$ for every $G \in \mathcal{F}_C \setminus \{H_1\}$.

(v) If $H_1 \in \mathcal{F}_C$ and $F$ is an induced subgraph of $H_1$ such that $e(F) \geq 2$, then there are at most $C$ graphs $G \in \mathcal{F}_C \setminus \{H_1\}$ for which $H_1 \cap G = F$.

We first prove that, if $\mathbb{E}(S_C) \geq f_H(n,p)/2$ holds for sufficiently large $C$, then the assertion of the lemma holds. Let $m$ be a median of $S_C$. Since $\delta > 0$ can be chosen to be arbitrarily small in Lemma 4.7, a standard application of Theorem 4.8 shows that $m = (1 + o(1))\mathbb{E}(S_C)$ (see [10] for details). Let $e_1, \ldots, e_N$ denote the edges of the $n$th-blow-up of $H$. For every $1 \leq i \leq N$, let $Z_i$ be the indicator random variable for the event $e_i \in E(\mathcal{G}(H,n,p))$ and let $c_i = Cf_H(n,p)/(n^2p)$. Clearly $S_C$ is a function of $Z_1, \ldots, Z_N$ and, by Condition (ii) above, Part (a) of Theorem 4.8 is satisfied. Moreover, it follows by Condition (i) above that we can ‘certify’ the existence of a family $\mathcal{F}_C$ as above by revealing at most $Cn^2p$ edges. Combined with Condition (ii) above and the choice of $c_i$’s, setting $\psi \equiv Cn^2p(Cf_H(n,p)/(n^2p))^2$ we deduce that Part (b) of Theorem 4.8 is satisfied as well. It thus follows by Theorem 4.8 (with $r = m$
and \( t = \mathbb{E}(S_C)/2 \) that
\[
Pr(S_C \leq f_H(n,p)/5) \leq Pr(S_C \leq (1 + o(1))\mathbb{E}(S_C)/2) \\
\leq 2 \exp \left( \frac{(1 + o(1))(\mathbb{E}(S_C)/2)^2}{4(Cn^2p)(Cf_H(n,p)/(n^2p))^2} \right) \\
\leq 2 \exp \left( -\frac{(1 + o(1))(\mathbb{E}(S_C)/2)^2}{4(Cn^2p)(Cf_H(n,p)/(n^2p))^2} \right) \\
< \exp \left( -\frac{n^2p}{65C^3} \right).
\]

Given a family \( \mathcal{F}_C \) as above, one can construct a subfamily \( \mathcal{F}' \) such that \( H_1 \cap H_2 \) is either empty or a clique of order at most 2 for every \( H_1, H_2 \in \mathcal{F}' \). Take an arbitrary \( H_1, H_2 \in \mathcal{F}_C \), put it in \( \mathcal{F}' \), delete from \( \mathcal{F}_C \) all graphs \( G \) for which \( H_1 \) is not empty or a clique of order at most 2 and repeat this process until \( \mathcal{F}_C = \emptyset \). It follows by Conditions (iii) – (v) above that \( |\mathcal{F}'| \geq \frac{1}{2^{n/|\mathcal{F}_C|}}|\mathcal{F}_C| \). Since, in particular, \( \mathcal{F}' \) is a sparse \( H \)-family, the assertion of the lemma holds for \( \alpha = e^{-1/(65C^3)} \) and \( \beta = \frac{1}{5^{(n/|\mathcal{F}_C|)C+1}} \).

It thus remains to prove that indeed \( \mathbb{E}(S_C) \geq f_H(n,p)/2 \) for sufficiently large \( C \). Let \( \rho = \rho(n,p) = f_H(n,p)/\mathbb{E}(Y_H) \). We ‘accept’ every canonical copy of \( H \) in \( \mathbb{G}(H,n,p) \) independently with probability \( \rho \). Let \( Z_H \) denote the random variable counting the number of accepted copies of \( H \); clearly \( \mathbb{E}(Z_H) = f_H(n,p) \). Our first aim is to prove that the expected number of accepted copies of \( H \) which satisfy Conditions (ii) – (v) is at least \( 2f_H(n,p)/3 \), provided that \( C \) is sufficiently large.

Starting with (v), let \( H_1 \) be an arbitrary canonical copy of \( H \) in \( \mathbb{G}(H,n,p) \) and let \( F \) be an induced subgraph of \( H_1 \), where \( e(F) \geq 2 \). The expected number of accepted copies of \( H \) whose intersection with \( H_1 \) is precisely \( F \) is
\[
\rho n^{v(H) - e(F)} p^{e(H) - e(F)} = \frac{\rho \mathbb{E}(Y_H)}{\mathbb{E}(Y_F)} = \frac{f_H(n,p)}{\mathbb{E}(Y_F)} \leq 1, \tag{11}
\]
where the inequality follows by the definition of \( f_H(n,p) \) since \( e(F) \geq 2 \).

Let \( A_F \) denote the event that there are more than \( C \) accepted copies of \( H \) whose intersection with \( H_1 \) is precisely \( F \). It follows by (11) and by Markov’s inequality that \( Pr(A_F) < 1/C \). Summing over all choices of \( H_1 \) and \( F \) as above we conclude that the expected number of accepted copies of \( H \) which intersect more than \( C \) other accepted copies of \( H \) on a given induced subgraph \( F \) with at least two edges is at most
\[
\mathbb{E}(Z_H)2^{v(H)} / C = f_H(n,p)2^{v(H)} / C \leq f_H(n,p)/12,
\]
where the inequality holds for sufficiently large \( C \).

Next, we consider (iv). Given a canonical copy \( H_1 \) of \( H \) in \( \mathbb{G}(H,n,p) \) and an induced subgraph \( F \) of \( H_1 \) such that \( v(F) \geq 3 \) and \( e(F) \leq 1 \), the expected number of accepted copies of \( H \) whose intersection with \( H_1 \) is precisely \( F \) is at most
\[
\rho n^{v(H)-3} p^{e(H)-1} = \mathbb{E}(Z_H)/(n^3p) = f_H(n,p)/(n^3p) \leq \delta^2, \tag{12}
\]
where the inequality follows since \( f_H(n,p) \leq \delta n^2 \) and \( p \geq (\delta n)^{-1} \).
Summing over all choices of $H_1$ and $F$ as above we conclude that the expected number of accepted copies of $H$ which intersect another copy of $H$ on a given induced subgraph $F$ with at least three vertices and at most one edge is at most
\[
\mathbb{E}(Z_H)2^{v(H)}\delta^2 \leq f_H(n,p)2^{v(H)}\delta^2 \leq f_H(n,p)/12,
\]
where the last inequality holds for sufficiently small $\delta$.

The argument for (iii) is similar. Given a canonical copy $H_1$ of $H$ in $G(H,n,p)$ and two non-adjacent vertices $x,y \in V(H_1)$, the expected number of accepted copies of $H$ whose intersection with $H_1$ is precisely $\{x,y\}$ is
\[
\rho v(H) - 2p v(H) = \mathbb{E}(Z_H)/n^2 = f_H(n,p)/n^2 \leq \delta.
\]

Summing over all choices of $H_1$ and $x,y \in V(H_1)$ we conclude that the expected number of accepted copies $H_1$ of $H$ for which there is an accepted copy $H_2$ of $H$ which intersects $H_1$ on two non-adjacent vertices is at most
\[
\mathbb{E}(Z_H)\left(\frac{v(H)}{2}\right)\delta \leq f_H(n,p)v(H)\delta^2 \leq f_H(n,p)/12,
\]
where the last inequality holds for sufficiently small $\delta$.

Finally, we consider (ii). Given a canonical copy $H_1$ of $H$ in $G(H,n,p)$ and an edge $e \in E(H_1)$, the expected number of accepted copies of $H$ whose intersection with $H_1$ is precisely $e$ is
\[
\rho v(H) - 2p v(H) = \mathbb{E}(Z_H)/(n^2p) = f_H(n,p)/(n^2p).
\]

On the other hand, for sufficiently small $\delta$ it follows by [11] and [12] that the expected number of accepted copies of $H$ whose intersection with $H_1$ is a proper supergraph of $e$ is at most $2^{v(H)}$. Since $f_H(n,p)/(n^2p) \geq \epsilon$, it follows that $f_H(n,p)/(n^2p) + 2^{v(H)} \leq Kf_H(n,p)/(n^2p)$ holds for some constant $K = K(\epsilon, \delta, H)$. Therefore, by Markov’s inequality, the probability that $e$ belongs to more than $Cf_H(n,p)/(n^2p)$ accepted copies of $H$ is at most
\[
\frac{Kf_H(n,p)/(n^2p)}{Cf_H(n,p)/(n^2p)} \leq K/C.
\]

Summing over all choices of $H_1$ and $e \in E(H_1)$, we conclude that the expected number of accepted copies of $H$ which contain an edge that belongs to at least $Cf_H(n,p)/(n^2p)$ other accepted copies of $H$ is at most
\[
f_H(n,p) \cdot e(H) \cdot K/C \leq f_H(n,p)/12,
\]
where the inequality holds for sufficiently large $C$.

We conclude that the expected number of accepted copies of $H$ which satisfy Conditions (ii) – (v) is at least
\[
f_H(n,p) - 4 \cdot f_H(n,p)/12 = 2f_H(n,p)/3.
\]

In order to prove that $\mathbb{E}(S_C) \geq f_H(n,p)/2$ (thus completing the proof of the lemma), it remains to address Condition (i). Clearly $e(G(H,n,p)) \sim \text{Bin}(e(H)n^2, p)$ and thus, for sufficiently large
C, it follows by Chernoff’s bound that \( \Pr(e(G(H, n, p)) > Cn^2p) \leq e^{-n^2p} \). Therefore, conditioning on the event “\( e(G(H, n, p)) \leq Cn^2p \)” does not change much the expected number of accepted copies of \( H \) that satisfy Conditions (ii) – (v) which, in this conditional space, is simply \( \mathbb{E}(S_C) \). It follows by the above that \( \mathbb{E}(S_C \mid e(G(H, n, p)) \leq Cn^2p) \geq 4f_H(n, p)/7 \).

Since \( \Pr(e(G(H, n, p)) > Cn^2p) \) is exponentially small we conclude that \( \mathbb{E}(S_C) \geq f_H(n, p)/2 \) as claimed. \( \square \)

5 Winning criteria for the \( H \)-game

In this section we state and prove three useful corollaries of the results proven in previous sections. Each of these corollaries provides a different sufficient condition for Waiter to force Client to build many copies of \( H \) in a \((b : 1) \) Waiter-Client game on the edge set of \( \mathbb{B}_H(V_1, \ldots, V_{v(H)}; s) \). In particular, Theorem \( \ref{thm:1.6} \) will readily follow from our first result of this section and Theorem \( \ref{thm:1.3} \) (i).

Corollary 5.1 Let \( s \) be a sufficiently large integer and let \( H \) be a graph with at least two edges. Then there exist positive constants \( c \) and \( \beta \) such that if \( 2 \leq b + 1 \leq cs^{1/m_2(H)} \),

then, playing a \((b : 1) \) Waiter-Client game on the edge set of \( \mathbb{B}_H(V_1, \ldots, V_{v(H)}; s) \), Waiter has a strategy to force Client to build at least \( \beta s^{v(H)}(b + 1)^{-e(H)} \) copies of \( H \).

Proof Without loss of generality we can clearly assume that \( H \) has no isolated vertices. If \( H \) is a matching of size \( k \), then \( b = O(s) \) by assumption. Let \( X_1, Y_1, \ldots, X_k, Y_k \) be pairwise disjoint subsets of \( V_1 \cup \ldots \cup V_{v(H)} \), each of size \( \Theta(s) \). By offering the edges of \( \bigcup_{i=1}^k E(X_i, Y_i) \), it is easy to see that Waiter can force \( \Theta(n^{2k/b^k}) \) copies of \( H \) in Client’s graph. In the remainder of this proof we therefore assume that \( H \) contains two adjacent edges.

For an integer \( 1 \leq b' \leq s^{1/m_2(H)} \), let \( M = \lfloor e(H)s^2/(b' + 1) \rfloor \). Since \( H \) is not a matching, \( m_2(H) \geq 1 \) and thus \( b' \leq s \). In particular

\[ \omega(1) = M \leq e(H)s^2/2. \tag{13} \]

Moreover, it follows by Lemma \( \ref{lem:1.2} \) (i) that

\[ \hat{f}_H(s, 1/(b' + 1)) = \frac{s^2}{b' + 1} \geq \frac{M}{e(H)}. \tag{14} \]

Observe that the expected number of copies of \( H \) in \( G(H, s, M) \) is at least \( 2\hat{f}_H(s^{v(H)}(b' + 1)^{-e(H)} \) for some constant (depending on \( H \)) \( \delta > 0 \). Let \( G \) denote the family of all subgraphs of \( \mathbb{B}_H(V_1, \ldots, V_{v(H)}; s) \) with precisely \( M \) edges and let \( F \) denote the family of all graphs in \( G \) each containing at least \( \delta s^{v(H)}(b' + 1)^{-e(H)} \) copies of \( H \). Since, by \( \ref{lem:1.3} \), the conditions of Lemma \( \ref{lem:1.3} \)
are satisfied, it follows by the fact that \( \hat{f}_H(s, M) = \Theta(\hat{f}_H(s, 1/(b' + 1))) \) and by (14) that

\[
|\mathcal{F}| \geq \left(1 - \exp(-c' \hat{f}_H(s, 1/(b' + 1)))\right) |\mathcal{G}|
\]
\[
\geq (1 - \exp(-c'M/e(H))) |\mathcal{G}|
\]
\[
> (1 - \alpha^M) |\mathcal{G}|
\]  

(15)

for some positive constants \( c' \) and \( \alpha < 1 \).

Let \( b = \lfloor (1 - \alpha)(b' + 1) \rfloor - 1 \) and let \( X = E(\mathbb{B}_H(V_1, \ldots, V_{v(H)}; s)) \). Then \( N := |X| = e(H)s^2 \), \( \mathcal{F} \) is \( M \)-uniform, \( |\mathcal{F}| > (1 - \alpha^M)|\mathcal{G}| \) and \( b + 1 \leq (1 - \alpha)N/M \). It thus follows by Theorem 3.1 that, playing the \( (b : 1) \) Waiter-Client game \((X, \mathcal{F})\), Waiter has a strategy to force Client to fully claim some \( A \in \mathcal{F} \) during the first \( M \) rounds of the game. Hence, after \( M \) rounds, the subgraph of \( \mathbb{B}_H(V_1, \ldots, V_{v(H)}; s) \) built by Client, contains at least \( \delta s^{v(H)}(b' + 1)^{-e(H)} \geq \beta s^{v(H)}(b + 1)^{e(H)} \) copies of \( H \), for some constant \( \beta > 0 \).

In order to complete the proof of the corollary, it remains to observe that, since \( b + 1 \leq cs^{1/m_2(H)} \), setting \( c = (1 - \alpha)/2 \) entails \( b' \leq s^{1/m_2(H)} \).

\( \square \)

**Corollary 5.2** Let \( s \) be a sufficiently large integer and let \( H \) be a graph with two adjacent edges and no isolated vertices. Let \( H_0 \subseteq H \) be a graph with at least two edges such that \( f_H(s, 1/(b + 1)) = s^{v(H_0)}(b + 1)^{-e(H_0)} \). Then there exist positive constants (depending on \( H \)) \( c_1, c_2 \) and \( \beta \), such that for every positive integer \( b \) satisfying \( c_1 s^{v(H_0) - 2/e(H_0)} \leq b + 1 \leq c_2 s^{v(H_0) - 2/(e(H_0) - 1)} \), playing a \( (b : 1) \) Waiter-Client game on the edge set of \( \mathbb{B}_H(V_1, \ldots, V_{v(H)}; s) \), Waiter can force Client to build a sparse \( H \)-family consisting of at least \( \beta f_H(s, 1/(b + 1)) \) copies of \( H \).

**Proof** Let \( b' = b'(s) \) be a positive integer and let \( M = [e(H)s^2/(b' + 1)] \). Let \( \mathcal{G} \) denote the family of all subgraphs of \( \mathbb{B}_H(V_1, \ldots, V_{v(H)}; s) \) with exactly \( M \) edges. Since \( H \) is a fixed graph and \( s \) is sufficiently large, it follows that \( M = \Theta(s^2/(b' + 1)) \). Moreover, a straightforward calculation shows that

\[ f_H(s, M) = \Theta(f_H(s, 1/(b' + 1))) \]

Hence, if \( f_H(s, 1/(b' + 1)) \geq \varepsilon M \) for some \( \varepsilon > 0 \). Furthermore, for every \( \delta > 0 \), there exists a \( \delta' > 0 \) such that, if \( f_H(s, 1/(b' + 1)) \leq \delta' s^2 \) and \( b' < \delta' s \), then \( f_H(s, M) \leq \delta s^2 \) and \( M > s/\delta \). Hence, by Lemma 11.6 there exist positive constants \( \alpha < 1, \, \beta' \) and \( \delta' \) such that if

\[ b' < \delta' s \]

and

\[ s^2(b' + 1)^{-1} \leq f_H(s, 1/(b' + 1)) \leq \delta' s^2, \]

(16)

(17)

then there are more than \((1 - \alpha^M)|\mathcal{G}|\) graphs in \( \mathcal{G} \), each containing a sparse \( H \)-family consisting of at least \( \beta' f_H(s, 1/(b' + 1)) \) copies of \( H \).

Now, let \( b = \lfloor (1 - \alpha)(b' + 1) \rfloor - 1 \). Our next goal is to show that if \( f_H(s, 1/(b + 1)) = s^{v(H_0)}(b + 1)^{-e(H_0)} \) holds for some \( H_0 \subseteq H \) with at least two edges and if there are positive constants \( c_1 \) and \( c_2 \) such that \( c_1 s^{v(H_0) - 2/e(H_0)} \leq b + 1 \leq c_2 s^{v(H_0) - 2/(e(H_0) - 1)} \), then the inequalities (16) and (17) hold. On the one hand we have

\[ f_H(s, 1/(b + 1)) = s^{v(H_0)}(b + 1)^{-e(H_0)} \leq s^{v(H_0)} \left[ c_1 s^{v(H_0) - 2/e(H_0)} \right]^{-e(H_0)} = c_1^{-e(H_0)} s^2. \]

(18)
On the other hand we have

\[ f_H(s, 1/(b + 1)) = s^2(b + 1)^{-1} \cdot s^{e(H_0) - 2} (b + 1)^{(1 - e(H_0))} \geq c_2^{1 - e(H_0)} s^2(b + 1)^{-1} . \]  

(19)

By the definition of \( H_0 \), the assumption that \( b + 1 \leq c_2 s^{(e(H_0) - 2)/(e(H_0) - 1)} \), and Lemma 4.2(ii), we know that \( H_0 \) is connected. It follows that \( e(H_0) \geq v(H_0) - 1 \) and thus

\[ b + 1 \leq c_2 s^{(v(H_0) - 2)/(e(H_0) - 1)} \leq c_2 s . \]  

(20)

Combining 13, 14, 20 and the fact that \( f_H(s, 1/(b + 1)) = \Theta(f_H(s, 1/(b + 1))) \), we conclude that \( 16 \) and \( 17 \) hold for some suitably chosen positive constants \( c_1 \) and \( c_2 \). Hence, there are more than \( (1 - \alpha M)|\mathcal{G}| \) graphs in \( \mathcal{G} \), each containing a sparse \( H \)-family consisting of at least \( \beta' f_H(s, 1/(b' + 1)) \geq \beta f_H(s, 1/(b + 1)) \) copies of \( H \), for some positive constant \( \beta \).

In order to complete the proof, we apply Theorem 3.1 Let \( X = E(\mathbb{B}_H(V_1, \ldots, V_{v(H)}; s)) \) and let \( \mathcal{H} \) be the family of edge sets of all graphs \( G \in \mathcal{G} \) which contain a sparse \( H \)-family consisting of at least \( \beta f_H(s, 1/(b + 1)) \) copies of \( H \). Then \( N := |X| = e(H)s^2 \), \( \mathcal{H} \) is \( M \)-uniform, \( |\mathcal{H}| \geq (1 - \alpha M)|\mathcal{G}| \), and \( b + 1 \leq (1 - \alpha)(b' + 1) \leq (1 - \alpha)N/M \). It thus follows by Theorem 3.1 that, playing a \((b : 1)\) Waiter-Client game on the edge set of \( \mathbb{B}_H(V_1, \ldots, V_{v(H)}; s) \), Waiter can force Client to build a sparse \( H \)-family consisting of at least \( \beta f_H(s, 1/(b + 1)) \) copies of \( H \). \( \square \)

Before stating our last result for this section, we introduce additional notation which will be used several times in the remainder of the paper. Given a graph \( H \) with at least one edge let

\[ g(H) = \max \left\{ \frac{v(H) - v(H')}{e(H) - e(H')} : H' \subsetneq H, 2 \leq e(H') < e(H) \text{ or } (e(H') = 0 \text{ and } v(H') = 2) \right\} . \]

Corollary 5.3 Let \( s \) and \( b = b(s) \) be positive integers, where \( s \) is sufficiently large, and let \( H \) be a graph which is not a matching. Then there exist positive constants (depending on \( H \)) \( c_1 \), \( c_2 \) and \( \beta' \) such that if

(i) \( c_1 s^{g(H)} \leq b + 1 \leq c_2 s^{(v(H) - 2)/(e(H) - 1)} \) or

(ii) \( H \) is \( m_2 \)-balanced and \( c_3 s^{1/m_2(H)} \leq b + 1 \leq c_2 s^{1/m_2(H)} \) for some \( c_3 < c_2 \),

then, playing a \((b : 1)\) Waiter-Client game on the edge set of \( \mathbb{B}_H(V_1, \ldots, V_{v(H)}; s) \), Waiter has a strategy to force Client to build a sparse \( H \)-family consisting of at least \( \beta' s^{v(H)}(b + 1)^{-e(H)} \) copies of \( H \).

Proof Let \( c_1 \), \( c_2 \) and \( \beta \) be the constants whose existence follows from Corollary 5.2 Assume first that Condition (i) is satisfied. Let \( H_0 \subsetneq H \) be a graph with at least two edges such that \( f_H(s, 1/(b + 1)) = s^{v(H_0)(b + 1)^{-e(H_0)}} \). We can assume that \( H_0 \subsetneq H \) as otherwise our assertion would follow immediately from Corollary 5.2 It follows by the definitions of \( f_H(s, 1/(b + 1)) \) and \( H_0 \) that \( s^{v(H_0)(b + 1)^{-e(H_0)}} \leq s^{v(H)(b + 1)^{-e(H)}} \) and thus

\[ c_1 s^{(v(H) - 2)/(e(H) - 1)} \leq c_1 s^{g(H)} \leq b + 1 \leq s^{(v(H) - v(H_0))/e(H) - e(H_0))} . \]  

(21)

Since \( s \) is sufficiently large by assumption, it follows from (21) that

\[ \frac{v(H) - 2}{e(H)} \leq \frac{v(H) - v(H_0)}{e(H) - e(H_0)} \]  

(22)
and a straightforward calculation then yields
\[
\frac{v(H) - 2}{e(H)} \geq \frac{v(H_0) - 2}{e(H_0)}.
\] (23)

Moreover, since \( s \) is sufficiently large, it follows by Condition (i) and by the definition of \( g(H) \) that
\[
\frac{v(H) - v(H')}{e(H) - e(H')} \leq \frac{v(H) - 2}{e(H) - 1}
\]
for every \( H' \subset H \) with \( e(H') \geq 2 \). A straightforward calculation shows that
\[
\frac{v(H) - 2}{e(H) - 1} \leq \frac{v(H_0) - 2}{e(H_0) - 1}.
\] (24)

Using (i), (23), (24) and the definition of \( g(H) \) we conclude that
\[
c_1s^{(v(H_0)-2)/e(H_0)} \leq b + 1 \leq c_2s^{(v(H_0)-2)/(e(H_0)-1)}. \tag{25}
\]

It thus follows by Corollary 5.2 that, playing a \((b : 1)\) Waiter-Client game on the edge set of \( \mathbb{E}_H(V_1, \ldots, V_{v(H)}; s) \), Waiter can force Client to build a sparse \( H \)-family consisting of at least \( \beta f_H(s, 1/(b + 1)) \) copies of \( H \). Since \( b + 1 \geq c_1s^{g(H)} \), we can apply Lemma 4.2(iii) to conclude that \( \beta f_H(s, 1/(b + 1)) \geq \beta's^{v(H)(b + 1) - e(H)} \) for some \( \beta' > 0 \).

Next, assume that Condition (ii) is satisfied. By the argument used above for the case Condition (i) is satisfied, it suffices to prove that (25) holds in this case as well. Starting with the upper bound, it follows by (ii) that
\[
b + 1 \leq c_2s^{1/m_2(H)} \leq c_2s^{(v(H_0)-2)/(e(H_0)-1)}.\]

For the lower bound, as in the previous case, it follows by the definitions of \( f_H(s, 1/(b + 1)) \) and \( H_0 \) that \( b + 1 \leq s^{(v(H)-v(H_0))/(e(H)-e(H_0))} \). Moreover, \( b + 1 \geq c_3s^{1/m_2(H)} > c_1s^{(v(H)-2)/e(H)} \) holds for sufficiently large \( s \). Therefore, (22) holds again and thus we deduce (27), thereby obtaining the lower bound \( b + 1 \geq c_1s^{(v(H_0)-2)/e(H_0)} \). Hence, we infer that (25) holds as well and, as in the proof of (i), by Corollary 5.2 we obtain a sparse \( H \)-family consisting of at least \( \beta f_H(s, 1/(b + 1)) \) copies of \( H \).

Since \( H \) is \( m_2 \)-balanced, it follows by Lemma 4.1 that
\[
\frac{v(H) - v(H')}{e(H) - e(H')} \leq \frac{v(H) - 2}{e(H) - 1}
\]
holds for every \( H' \subset H \) with \( e(H') \geq 2 \). Therefore, for every such \( H' \) we have
\[
b + 1 \geq c_3s^{1/m_2(H)} = c_3s^{(v(H)-2)/(e(H)-1)} \geq c_3s^{(v(H)-v(H'))/(e(H)-e(H'))}.
\]
Applying Lemma 4.2(iii) completes the proof of (ii). \( \square \)
6 Clique games

The following lemma will play an important role in the proof of Theorem 1.7.

**Lemma 6.1** Let $k \geq 3$ be an integer and let $M_1 = \{e_1, \ldots, e_{k/2}\}$ and $M_2 = \{e_{k/2+1}, \ldots, e_{k-2}\}$ be two edge disjoint matchings of $K_k$. For every $1 \leq i \leq k - 2$, let $H_i = K_k \setminus \{e_1, \ldots, e_i\}$. Then the following two properties hold:

(i) If $k = 3$ or $k \geq 7$, then $g(H_i) = (v(H_i) - 2)/e(H_i)$ for every $1 \leq i \leq k - 2$.

(ii) For every $k \geq 3$ the graph $H_{k-2}$ is $m_2$-balanced.

**Proof** Fix an integer $k \geq 3$. It is an immediate consequence of the definition of $g(H)$ that $g(H_i) \geq (v(H_i) - 2)/e(H_i)$ holds for every $1 \leq i \leq k - 2$. Suppose that $g(H_i) > (v(H_i) - 2)/e(H_i)$ holds for some $1 \leq i \leq k - 2$. Then there exists some $H'_i \subseteq H_i$ with at least two edges such that

$$
\frac{v(H_i) - 2}{e(H_i)} < \frac{v(H_i) - v(H')}{e(H_i) - e(H')}. \quad (26)
$$

A straightforward calculation then shows that

$$
\frac{e(H')}{v(H') - 2} > \frac{e(H_i)}{v(H_i) - 2}. \quad (27)
$$

Let $t = v(H')$. By the definitions of $H_i$ and $H'$ and by (27) we have $v(H_i) = k$, $e(H_i) = \binom{k}{2} - i$ and $k > t \geq 3$.

Assume first that $H'$ is not a clique. Then (27) implies that

$$
\frac{\binom{k}{2} - 1}{t - 2} > \frac{\binom{k}{2} - i}{k - 2},
$$

and thus

$$
\frac{t + 1}{2} > \frac{k + 1}{2} - \frac{i - 1}{k - 2}. \quad (28)
$$

Since $i \leq k - 2$ and $t \leq k - 1$, it follows that

$$
k - 1 \geq t > k - \frac{2(i - 1)}{k - 2} > k - 2,
$$

where the second inequality holds by (28). This can only hold if

$$
t = k - 1 \text{ and } i > k/2. \quad (29)
$$

Consequently, $v(H_i) - v(H') = 1$ and, since $\delta(H_i) \geq k - 3$, it follows that $e(H_i) - e(H') \geq k - 3$. Thus, by (29) we have $i > k/2$ and by (26) we have

$$
k - 3 \leq e(H_i) - e(H') = \frac{e(H_i) - e(H')}{v(H_i) - v(H')} < \frac{e(H_i)}{v(H_i) - 2} = \frac{k + 1}{2} - \frac{i - 1}{k - 2}.
$$
It is easy to verify that this set of inequalities is satisfied if and only if \( k = 5 \), \( i = 3 \), and \( H' = K_4 \setminus \{e\} \) for some edge \( e \).

Assume then that \( H' \cong K_t \). Similar calculations to the ones conducted in the previous case lead to

\[
\frac{t + 1}{2} + \frac{1}{t - 2} > \frac{k + 1}{2} - \frac{i - 1}{k - 2}.
\]

If \( t = 3 \), then, since \( 1 \leq i \leq k - 2 \), it is not hard to verify that the only solutions of (30), under the additional assumption that \( K_3 \subseteq H_i \), are:

1. \( k = 6 \) and \( i = 4 \).
2. \( k = 5 \) and \( 2 \leq i \leq 3 \).
3. \( k = 4 \) and \( i = 1 \).

Therefore, from now on we assume that \( t \geq 4 \) and thus \( k \geq 5 \). Observe that, for \( t \geq 4 \), the left hand side of (30) is a nondecreasing real function. We distinguish between the following two cases:

**Case 1:** \( 1 \leq i \leq k/2 \). In this case the largest clique in \( H_i \) is of order \( k - i \). Therefore, by (30) and the aforementioned monotonicity we have

\[
\frac{k - i + 1}{2} + \frac{1}{k - i} > \frac{k + 1}{2} - \frac{i - 1}{k - 2}.
\]  

and thus

\[
k - \frac{2}{k - 2 - i} < 4.
\]

Since \( i \leq k/2 \) and \( k \geq 5 \) by assumption, it follows that \( k = 5 \) and \( i = 2 \) is the only solution to (31). However, for \( k = 5 \), the graph \( H_2 \) does not contain \( K_4 \) as a subgraph and thus there are no solutions in Case 1.

**Case 2:** \( k/2 < i \leq k - 2 \). In this case the largest clique in \( H_i \) is of order at most \((k + 1)/2\). Therefore \( 4 \leq t \leq (k + 1)/2 \) and thus \( k \geq 7 \). Applying (30) and the aforementioned monotonicity we obtain

\[
\frac{(k + 1)/2 + 1}{2} + \frac{1}{(k + 1)/2 - 2} > \frac{k + 1}{2} - \frac{i - 1}{k - 2},
\]  

which for \( i \leq k - 2 \) implies that \( k < 5 + 8/(k - 3) \). Since \( k \geq 7 \), there are no solutions in Case 2 either.

Summarizing, we have shown that:

1. If \( k \notin \{4, 5, 6\} \), then \( g(H_i) = (v(H_i) - 2)/e(H_i) \) holds for every \( 1 \leq i \leq k - 2 \); this proves (i). Moreover, it implies that for \( k \notin \{4, 5, 6\} \)

\[
\frac{v(H_i) - v(H')}}{e(H_i) - e(H')} \leq \frac{v(H_i) - 2}{e(H_i)} < \frac{v(H_i) - 2}{e(H_i) - 1}
\]

(32)

holds for every \( H' \subseteq H_i \) with at least two edges.
2. If \( k = 6 \) and \( i = 4 \), then, according to our previous calculations, for every \( H' \subsetneq H_4 \) with at least two edges, other than \( K_3 \) we have
\[
\frac{v(H_4) - v(H')}{e(H_4) - e(H')} \leq \frac{v(H_4) - 2}{e(H_4)} < \frac{v(H_4) - 2}{e(H_4) - 1}.
\]
Furthermore,
\[
\frac{v(H_4) - v(K_3)}{e(H_4) - e(K_3)} = \frac{3}{8} < \frac{2}{5} = \frac{v(H_4) - 2}{e(H_4) - 1}
\]
and thus (32) holds in this case as well.

3. If \( k = 5 \) and \( i = 3 \), then, according to our previous calculations, for every \( H' \subsetneq H_3 \) with at least two edges, other than \( K_3 \) and \( K_4 \setminus \{e\} \) we have
\[
\frac{v(H_3) - v(H')}{e(H_3) - e(H')} \leq \frac{v(H_3) - 2}{e(H_3)} < \frac{v(H_3) - 2}{e(H_3) - 1}.
\]
It is easy to verify that if \( H' \in \{K_3, K_4 \setminus \{e\}\} \), then
\[
\frac{v(H_3) - v(H')}{e(H_3) - e(H')} = \frac{1}{2} = \frac{v(H_3) - 2}{e(H_3) - 1},
\]
and thus (32) holds again.

Hence (32) holds for every \( k \geq 3 \) and for \( i = k - 2 \), which by Lemma 4.4 proves Part (ii) of our assertion.

We can now prove the main result of this section.

**Proof of Theorem 1.7.** For every positive integer \( b \) and every \( k \geq 3 \), it follows by Theorem 1.3 (i) that \( S(K_k, n, b) = O \left(n^k \cdot (b + 1)^{-\binom{k}{2}}\right)\). Note that \( n^k \cdot (b + 1)^{-\binom{k}{2}} = O(1) \) for \( b = \Theta \left(n^{k/2}\right)\).

This proves the upper bound in both (i) and (ii).

The remainder of this proof is dedicated to the lower bounds. Starting with (i), assume that \( k = 3 \) or \( k \geq 7 \). Let \( s = \lfloor n/k \rfloor \) and let \( G = (V, E) \) be a copy of \( \mathbb{B}_{K_k}(V_1, \ldots, V_k; s) \) in \( K_n \). It clearly suffices to prove that Waiter can force Client to build the required number of copies of \( K_k \) when playing on \( E \).

Since \( m_2(K_k) = \binom{k}{2} / (k - 2) \), it follows by Corollary 5.1 that there are positive constants \( c_0 \) and \( \beta \) such that if
\[
2 \leq b + 1 \leq c_0 s^{(k-2)/(\binom{k}{2})-1},
\]
then \( S(K_k, n, b) \geq \beta s^k (b + 1)^{-\binom{k}{2}} = \Theta \left(n^k \cdot (b + 1)^{-\binom{k}{2}}\right)\).

Hence, from now on we assume that \( b = \Omega \left(s^{(k-2)/(\binom{k}{2})-1}\right)\). As in Lemma 6.1 for every \( 1 \leq i \leq k - 2 \), let \( H_i = K_k \setminus \{e_1, \ldots, e_i\} \).

It follows by Corollary 5.3 (i) that for every \( 1 \leq i \leq k - 2 \) there are positive constants \( c_1^i, c_2^i \) and \( \beta_i \) such that if
\[
c_1^i s^{g(H_i)} \leq b + 1 \leq c_2^i s^{(v(H_i)-2)/(e(H_i)-1)},
\]

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then, playing on $E$, Waiter can force Client to build a sparse $H_j$-family $A_i$ consisting of at least $\beta_i s^{v(H_i)}(b+1)^{-e(H_i)}$ copies of $H_i$.

For every $1 \leq i \leq k-2$ let $e_i = u_iu'_i$ and let $U_i$ and $U'_i$ denote the corresponding pairs of vertex sets in $\mathbb{B}_{K_k}(V_1, \ldots, V_k; s)$. Fix some $1 \leq i \leq k-2$ for which (33) is satisfied (assuming such an $i$ exists). We are now ready to describe Waiter’s strategy, it is divided into the following two stages.

**Stage I:** Waiter forces Client to build a sparse $H_i$-family $A_i$ consisting of at least $\beta_i s^{v(H_i)}(b+1)^{-e(H_i)}$ copies of $H_i$.

**Stage II:** For every $1 \leq j \leq i$ and every $\ell$ which satisfies $(j-1)[|A_i|/(b+1)] + 1 \leq \ell \leq j[|A_i|/(b+1)]$, in the $\ell$th round of this stage Waiter offers Client $b+1$ free edges $x_1x'_1, \ldots, x_{b+1}x'_{b+1} \in E_G(U_j, U'_j)$, such that for every $1 \leq t \leq b+1$, $x_t$ and $x'_t$ are vertices of a copy of $H_i$ in $A_i$, corresponding to the vertices $u_t$ and $u'_t$.

As noted above, it follows by Corollary 5.3(i) that Waiter can play according to Stage I of the proposed strategy. Moreover, since no two graphs in $A_i$ share a pair of non-adjacent vertices, Waiter can play according to Stage II of the proposed strategy as well. It remains to show that, by doing so, he forces Client to build $\Omega \left( n^k \cdot (b+1)^{-\binom{k}{2}} \right)$ copies of $K_k$. At the end of Stage I, Client’s graph contains $|A_i|$ copies of $H_i$. Since no two graphs in $A_i$ share a pair of non-adjacent vertices, immediately after the $|A_i|/(b+1)$th round of Stage II, Client’s graph contains at least $|A_i|/(b+1) - 1$ copies of $H_i \cup \{e_1\}$. Repeating this argument $i$ times, we conclude that, at the end of Stage II, Client’s graph contains at least

$$\frac{|A_i|}{(b+1)^i} - i \geq \frac{\beta_i s^{v(H_i)}}{(b+1)^{e(H_i)+i}} - i = \frac{\beta_i s^k}{(b+1)^{\binom{k}{2}}} - i = \Theta \left( n^k (b+1)^{-\binom{k}{2}} \right)$$

copies of $H_i \cup \{e_1, \ldots, e_i\} \cong K_k$.

Note that by Lemma 6.1(i) the inequalities (33) are equivalent to

$$c_1 s^{\nu(H_i)/e(H_i)} \leq b+1 \leq c_2 s^{\nu(H_i)/e(H_i) - 1}. \quad (35)$$

We have thus proved that Waiter can force Client to build the required number of copies of $K_k$ provided that $b$ satisfies (33) or (35) for some $1 \leq i \leq k-2$, that is, provided that

$$b+1 \in \left[ 2, \tilde{c}_0 n^{(k-2)/(\binom{k}{2})-1} \right] \cup \bigcup_{i=1}^{k-2} \left[ 2, \tilde{c}_1 n^{(k-2)/(\binom{k}{2}-1)} \right] \cup \bigcup_{i=1}^{k-2} \left[ 2, \tilde{c}_2 n^{(k-2)/(\binom{k}{2}-i-1)} \right].$$

Since for $i = k-2$ we have $n^{(k-2)/(\binom{k}{2})-1} = n^{2/(k-1)}$, we conclude that there exists a positive constant $c$ for which the above range of $b$ covers the interval $[1, cn^{2/(k-1)}]$ with the possible exception of $b+1 \in \left( c_1 n^{(k-2)/(\binom{k}{2}-1)}, c_2 n^{(k-2)/(\binom{k}{2}-i)} \right)$ for some values of $1 \leq i \leq k-2$ and
some positive constants $c_i < C_i$. However, by the bias-monotonicity of Waiter-Client games, we can assume that Waiter is allowed to offer strictly more than $b + 1$ edges per turn, so if $b$ is of order $n^{(k-2)/(k-1)}$ for some $1 \leq i \leq k-2$ and $b' > b$ is such that $b' + 1 = \lceil C_i n^{(k-2)/(k-1)} \rceil$, then

$$S(K_k, n, b) \geq S(K_k, n, b') \geq \Theta \left( n^k \cdot (b' + 1)^{-\binom{k}{2}} \right) = \Theta \left( n^k \cdot (b + 1)^{-\binom{k}{2}} \right).$$

This concludes the proof of Part (i).

The proof of the lower bound in Part (ii) of the theorem is very similar and its details are therefore omitted. The two main differences are that we only need to consider a narrower range of $b$, namely $c_1 n^2/(k-1) \leq b \leq c_2 n^2/(k-1)$, and that we apply Part (ii) of Corollary 5.3 and of Lemma 6.1 rather than Part (i).

\[\square\]

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