Estimates of the derivative of the entropy of Gaussian thermostats

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Abstract

We consider a variation of an Anosov geodesic flow by Gaussian thermostats and we obtain estimates of the derivative of the entropy map at the geodesic flow. In particular, we prove that the entropy of the geodesic flow is a local maximum for the entropy map.

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1. Introduction

The topological entropy of a flow is a well-known quantity that measures the complexity of the flow. It is a central theme in the theory of dynamical systems and a lot of research was done to understand and calculate it. So given a flow with certain properties it is a natural question to obtain bounds for the entropy. Moreover, given a family of flows it is nice to know how the topological entropy varies in it.

In this article, we deal with such a question in a family formed by Gaussian thermostats. These flows provide interesting models in non-equilibrium statistical mechanics, see [2, 8].

Let \((M, g)\) be a closed Riemannian manifold. We denote a point in \(TM\) by \((p, v) = \theta \in TM\), where \(p \in M\) and \(v \in T_p M\). In these coordinates, we can write the geodesic flow (see [5]) as the following equation:

\[
\frac{dp}{dr} = v, \quad \frac{Dv}{dr} = 0.
\]

(1)
Let \( t \to (\gamma(t), \gamma'(t)) \) be a curve in \( TM \). This curve is an integral curve to the geodesic field \( G(\theta) = (v, 0) \) if and only if

\[
\frac{D}{dt} \gamma'(t) = 0.
\]  

(2)

This shows that the solutions to the equation in (1) are exactly the same of the geodesic field \( G \). Analogously, we can define the Gaussian thermostat vector field as

\[
\frac{dp}{dt} = v, \quad \frac{dv}{dt} = E - \frac{\langle E, v \rangle}{|v|^2} v,
\]  

(3)

where \( E \in \chi(M) \) is a vector field over \( M \).

Now, we assume that the geodesic flow \( \phi_t : SM \to SM \) is Anosov. Let \( \lambda \mapsto E(\lambda) \) be a family of vector fields on \( M \) such that \( E(0) = 0 \).

So we can consider the vector fields \( F_\lambda \) given by \( F_\lambda(\theta) = (v, E_\lambda(p) - \frac{\langle E_\lambda(p), v \rangle}{|v|^2} v) \). Let us denote by \( \phi^\lambda_t : SM \to SM \) the flow generated by \( F_\lambda \).

We recall the concept of entropy (see [9]). If \( f : (X, d) \to (X, d) \) is a homeomorphism then let us define \( d_n(x, y) = \max \{d(f^i(x), f^i(y)) ; \text{ for } i = 0, \ldots, n - 1 \} \).

A subset \( A \) of \( X \) is said to be \((n, \varepsilon)\)-separated if each pair of distinct points of \( A \) is at least \( \varepsilon \) apart in the metric \( d_n \). Denote by \( N(n, \varepsilon) \) the maximum cardinality of an \((n, \varepsilon)\)-separated set. The topological entropy of \( f \) is defined by

\[
h_{\text{top}}(f) = \lim_{\varepsilon \to 0} \lim_{n \to \infty} \frac{1}{n} \log N(n, \varepsilon).
\]

If \( X_t \) is a flow then we set \( h_{\text{top}}(X_t) = h_{\text{top}}(X_1) \). Let us define \( h(\lambda) := h_{\text{top}}(\phi^\lambda_t) \), the entropy map associated to this variation.

We remark that by structural stability the flows \( \phi^\lambda_t : SM \to SM \) are Anosov for \( \lambda \in (-\varepsilon, \varepsilon) \), where \( \varepsilon \) is small.

We set \( E_0^\lambda = \frac{dE}{d\lambda} \big|_{\lambda=0} \). Now, we define the operator \( Z : TM \to TM \) such that for every \((p, v) = \theta \in TM \) we have

\[
Z_\theta(v) = \frac{1}{|v|^2} \left( \langle v, \cdot \rangle E_0^\lambda(p) - \langle E_0^\lambda(p), \cdot \rangle v \right).
\]

Our main result is the following estimate:

**Theorem 1.** The first derivative of the entropy map is zero, i.e. \( h'(0) = 0 \). Moreover, if \( Z \neq 0 \) then

\[
h''(0) \leq -h(0) - \int_{SM} \left( \left| E_0^\lambda(p) \right|^2 - \frac{\langle E_0^\lambda(p), v \rangle^2}{|v|^2} \right) dm < 0
\]

where \( m \) is the maximal entropy measure of the geodesic flow \( \phi^0_t \).

This immediately implies the following result.

**Corollary 2.** If \( Z \neq 0 \) then \( \lambda \mapsto h(\lambda) \) has a strict local maximum at \( \lambda = 0 \).

A particular case of the previous theorem is \( E(\lambda) = \lambda E \) where \( E \) is a fixed vector field. In this case \( E_0^\lambda = E \) and \( Z = \dot{Y} \) (defined below). The motivation of these results comes from [3], [4]. The proof also relies on Pollicott’s formula [7]. However, in the case of the Gaussian thermostat flow, the computations are a little bit different. For instance, the variational field produced by the flow has much more terms, and now they need be estimated. Also, in
the magnetic case we could consider the magnetic field (the Lorentz force) as an \((1, 1)\)-tensor and we can take covariant derive to differentiate it. However, for Gaussian thermostats this do not occur. Indeed, the Gaussian thermostat field (defined below) is not a tensor in general and this also complicates the computations.

2. Preliminaries

As in the introduction, we set \((M, g)\) as a closed Riemannian manifold. Let \(H : TM \to \mathbb{R}\) be the energy functional \(H(p, v) = \frac{1}{2}g_p(v, v)\) and \(E \in \mathfrak{X}(M)\) be a vector field on \(M\).

As we said before, the Gaussian thermostat can be defined as the flow on \(TM\) generated by the following equations.

\[
\begin{align*}
\frac{dp}{dt} &= v, \\
\frac{Dv}{dt} &= E - \frac{\langle E, v \rangle}{|v|^2}v,
\end{align*}
\]  

where \(p \in M\) and \(\frac{D}{dt}\) is the covariant derivative. This is a simple way of considering the Gaussian thermostat, but for our purposes it is better to consider an equivalent definition.

Let us consider two 1-forms on \(TM\) as follows:

\[
\alpha ((v_1, v_2)) = g_p(v_1, v_2), \quad \beta ((v_1, v_2)) = g_p(E(p), v_1)
\]

for every \((v_1, v_2) \in T_\theta TM \equiv T_pM \times T_pM\). For simplicity, we denote \(g_p(\cdot, \cdot) = \langle \cdot, \cdot \rangle\), sometimes we will also omit \(p\). Let \(Y : TM \to TM\) the operator such that for every \(u \in T_pM\) we have

\[
\tilde{Y}_\theta (u) = \frac{1}{|v|^2} (\langle v, u \rangle E - \langle E, u \rangle v).
\]

We notice that for every \(\theta \in TM\), \(\tilde{Y}_\theta : T_pM \to T_pM\) is anti-symmetric.

Let \(\kappa\) the 2-form on \(TM\) such that for every \(\theta \in TM\) we have

\[
\kappa(\cdot, \cdot) = \left\{ \tilde{Y}_\theta (d_\theta \pi(\cdot)), d_\theta \pi(\cdot) \right\}.
\]

Then, we define a vector field \(F : TM \to TTM\) such that for \(\xi \in T_\theta TM\) we have

\[
\langle \langle d_\theta H, \xi \rangle \rangle = d_\theta H(\xi) = \omega_0 (F(\theta), \xi).
\]

Here \(\langle \langle \cdot, \cdot \rangle \rangle\) is the Sasaki metric and \(\omega = \omega_0 + \kappa\), where \(\omega_0\) is the canonical symplectic form of \(TM\). Usually, we denote \(dH = i_F \omega\).

We remark that since \(k\) is only a non-degenerated 2-form we will not use the notation \(X_H\) for symplectic gradients.

In general, the Gaussian thermostats are not Hamiltonian. However they preserve the energy levels. Let \(\Sigma = H^{-1}(e)\) where \(e \in \mathbb{R}\). For every \(\theta \in \Sigma\) the orbit \(\sigma_\theta(t) \equiv \varphi_t(\theta)\) satisfies

\[
\frac{d}{dt} H(\sigma_\theta(t)) = d_{\sigma_\theta(t)} H(F(\sigma_\theta(t))) = \omega_{\sigma_\theta(t)} (F(\sigma_\theta(t)), F(\sigma_\theta(t))) = 0,
\]

since \(\omega\) is a 2-form. Thus, \(H(\sigma_\theta(t)) = H(\theta) = \Sigma\) for every \(t \in \mathbb{R}\). In particular, \(\varphi_t(\Sigma) \subset \Sigma\).

It is not difficult to see that we can define the Gaussian thermostat of \(E\) as the flow \(\varphi_t\) on \(TM\) generated by the vector field \(F\).

Lemma 3. The vector field \(F\) satisfies

\[
F(\theta) = \left( v, E - \frac{\langle E, v \rangle}{|v|^2}v \right).
\]
Proof. Let \((\xi_h, \xi_v) = \xi \in T_\theta \Sigma\), we have
\[d_\theta H(\xi) = \langle v, \xi_v \rangle.\]
On the other hand,
\[\omega_0(F(\theta), \xi) = \langle F(\theta)_h, \xi_v \rangle - \langle F(\theta)_v, \xi_h \rangle,\]
and
\[\kappa(F(\theta), \xi) = \langle \tilde{Y}_\theta (d_\theta \pi(F(\theta))), d_\theta \pi(\xi) \rangle = \langle \tilde{Y}_\theta (F(\theta)_h), \xi_h \rangle.\]
So
\[\omega(F(\theta), \xi) = \left( \langle F(\theta)_h, \xi_v \rangle - \langle \xi_h, F(\theta)_v \rangle - \frac{1}{|v|^2} \langle [v, F(\theta)_h] E - \langle E, F(\theta)_h \rangle v \rangle \right).\]
By definition, we have
\[d_\theta H(\xi) = \omega(F(\theta), \xi).\]
Thus
\[\langle v, \xi_v \rangle = \left( \langle F(\theta)_h, \xi_v \rangle - \langle \xi_h, F(\theta)_v \rangle - \frac{1}{|v|^2} \langle [v, F(\theta)_h] E - \langle E, F(\theta)_h \rangle v \rangle \right).\]
Henceforth,
\[F(\theta)_h = v \quad \text{and} \quad F(\theta)_v = E - \frac{\langle E, v \rangle v}{|v|^2}.\]

As in [10], if the vector field \(E\) has a global potential \(U\) then the Gaussian Thermostat preserves the measure \(\mu = e^{-\frac{1}{2}} \frac{1}{|v|^2} \text{d}x\), where \(\text{d}x\) denotes the standard volume element.

3. Jacobi fields and Pollicott's formula

In this section we collect two useful formulas, the first one is a Jacobi-type equation for Gaussian thermostats, the second one is due to Pollicott that describes how the entropy varies in an Anosov family.

Let \(\psi_t : SM \to SM\) a Gaussian thermostat flow as before. Let \(t \mapsto (\gamma, \dot{\gamma})\) an orbit of \(\psi_t\).

In particular,
\[\frac{D}{dt} \dot{\gamma} = \tilde{Y}_\gamma(\dot{\gamma}) = E - \langle E, \gamma \rangle \frac{\dot{\gamma}}{|\gamma|^2},\]
where \(D/dt\) denotes the covariant derivative and \(\dot{\gamma} = dy/dt\).

Let \(z : (-\epsilon, \epsilon) \to TM\) such that \(z(0) = \theta\) and \(\dot{z}(0) = \xi\). We let a variation \(f(s, t) = \pi(\psi_t(z(s)))\). Let \(J_s(t) = \partial f/\partial s(0, t), \gamma_s(t) = f(s, t),\) and \(\gamma = \gamma_0\). We recall the following identity from Riemannian geometry,
\[\frac{D}{dt} \frac{D}{ds} \frac{\partial f}{\partial s} + R \left( \frac{\partial f}{\partial t}, \frac{\partial f}{\partial s} \right) \frac{\partial f}{\partial t} = D \frac{D}{dv} \frac{\partial f}{\partial s} \frac{\partial f}{\partial t}.\]

By (6), we obtain
\[\tilde{J}_s(t) + R(\gamma, J_s) \dot{\gamma} = \frac{D}{ds} (\tilde{Y}_\gamma(\dot{\gamma})) = \frac{D}{ds} \left( E(\gamma_s) - \frac{\langle E(\gamma_s), \gamma_s \rangle}{|\gamma_s|^2} \gamma_s \right).\]
In \( s = 0 \), we obtain
\[
\ddot{J}_\xi + R(\dot{\gamma}, J_\xi)\dot{\gamma} - \nabla_{J_\xi} E + \frac{1}{|\dot{\gamma}|^2} \left( \nabla_{J_\xi} E, \dot{\gamma} + \langle E, J_\xi \rangle \dot{\gamma} \right)
- 2\langle J_\xi, \dot{\gamma} \rangle \frac{\langle E, \dot{\gamma} \rangle}{|\dot{\gamma}|^3} \langle E, \dot{\gamma} \rangle \dot{J}_\xi = 0.
\]
(7)

This is the Jacobi equation for Gaussian thermostats.

Let \( \lambda \mapsto \phi_\lambda^0 \) with \( \lambda \in (-\varepsilon, \varepsilon) \) be a \( C^\infty \) family of Anosov flows on a closed manifold \( X \).

By the structural stability theorem there exists maps \( \alpha_\lambda \in C^s(X) \), \( \Theta_1^\lambda \in C^s(X, X) \) (where \( s \) depends on the unstable and stable foliations) such that

(i) \( \alpha_0 \equiv 1; \Theta_1^0 \equiv I_X \),
(ii) \( \Theta_1^\lambda \) sends orbits of \( \phi_0^0 t \) on orbits of \( \phi_\lambda^0 t \),
(iii) \( \alpha_\lambda \) is a change of speed of \( \phi_0^0 t \) that turns \( \Theta_1^\lambda \) a conjugacy. Moreover, the maps \( \lambda \mapsto \alpha_\lambda, \Theta_1^\lambda \) are \( C^\infty \).

We consider the Taylor expansion of \( \alpha_\lambda^0 \):
\[
\lambda \mapsto \alpha_\lambda^0 = 1 + \lambda \left( D_0 \alpha_\lambda^0 \right) + \left( \alpha_\lambda^0 \right)^2 \left( D^2_0 \alpha_\lambda^0 \right) + \ldots
\]

Let \( h(\lambda) \) the topological entropy of \( \phi_\lambda^0 \). We recall that the variance of \( \phi_\lambda^0 \) is defined as follows. If \( m \) denotes the maximal entropy measure of \( \phi_\lambda^0 \) and \( F : M \to \mathbb{R} \) is a Hölder continuous function then
\[
\text{Var}(F) = \int_{-\infty}^{\infty} \left( \int F \circ \phi_\lambda^0 t . F \, dm - \left( \int F \, dm \right)^2 \right) \, dt.
\]

In [7], Pollicott obtains the following results.

**Theorem 4.** The first derivative of \( h(\lambda) \) at \( \lambda = 0 \) satisfies
\[
h'(0) = h(0) \int_X (D_0 \alpha_\lambda^0) \, dm.
\]

Moreover, the second derivative of \( h(\lambda) \) at \( \lambda = 0 \) satisfies
\[
h''(0) = h(0) \left\{ \text{Var} (D_0 \alpha_\lambda^0) + \int_X (D^2_0 \alpha_\lambda^0) \, dm + 2 \left( \int_X D_0 \alpha_\lambda^0 \, dm \right)^2 - 2 \int_X (D_0 \alpha_\lambda^0)^2 \, dm \right\},
\]
where \( m \) is the maximal entropy measure of \( \phi_\lambda^0 \) and \( \text{Var} \) is the variance of \( \phi_\lambda^0 \).

**4. Some reductions**

In this section, we will apply the results in the previous sections to reduce the proof of the theorem.

Let \( \Theta_1^\lambda : SM \to SM \) the map of the previous section which sends orbits of \( \phi_t = \phi_\lambda^0 \) on orbits of \( \phi_\lambda^0 \). Let \( \theta \in SM \) such that \( \phi_0^0 t (\theta) \) is a periodic orbit of period \( T \). Let \( \gamma_0 = \pi(\phi_0^0(\theta)) \).

Hence,
\[
\gamma_\lambda(t) := \pi \left( \phi_\lambda^0(\Theta_1^\lambda(\theta)) \right)
\]
gives a \( C^\infty \) variation of the curve \( \gamma_0 \), since \( \lambda \mapsto \Theta_1^\lambda \) is \( C^\infty \).

Moreover, the curves \( \gamma_\lambda \) of the variation are closed with period \( T_\lambda \). We will see how \( T_\lambda \) varies with respect to \( \lambda \).
Given a curve \( c : [0, a] \to M \), we denote its length by \( L(c) \). Its energy is denoted by \( E(c) \) as follows:

\[
E(c) = \int_0^a \langle \dot{c}(t), \dot{c}(t) \rangle \, dt.
\]

Let \( \Psi^\lambda \) be the reparametrization \( \varphi^\lambda \) by the change of speed \( \alpha^\lambda \). Hence, \( \Theta^\lambda \) is a conjugacy between \( \Psi^\lambda \) and \( \varphi^\lambda \). Thus the orbits of \( \Psi^\lambda \) (\( \theta \)) are closed with periods \( T^\lambda \). In particular, \( T \) and \( T^\lambda \) satisfy the following relation:

\[
T^\lambda = \int_0^T \frac{1}{\alpha^\lambda (\gamma_0(t))} \, dt.
\]

Let \( \beta^\lambda(t) := \frac{1}{\alpha^\lambda(\gamma_0(t))} \). We obtain the following result.

**Lemma 5.** For every closed geodesic \( \gamma_0 \) with period \( T \) we have

\[
\int_0^T D_0 \beta (\gamma_0(t)) \, dt = 0 \quad \text{and} \quad \int_0^T D_0^2 \beta (\gamma_0(t)) \, dt = \left. \frac{1}{2} \frac{d^2 E}{d\lambda^2} \right|_{\lambda=0}.
\]

**Proof.** Since \( |\dot{\gamma}_0| = 1 \) we have

\[
L(\gamma_0) = T^\lambda = \int_0^T \beta^\lambda (\gamma_0(t)) \, dt.
\]

Now, we consider a new family of curves \( \tilde{\gamma}_\lambda : [0, T] \to M \) given by

\[
\tilde{\gamma}_\lambda(t) := \gamma_\lambda(tT^\lambda/T).
\]

The map \( \lambda \mapsto \tilde{\gamma}_\lambda \) gives rise a \( C^\infty \) variation of the closed geodesic \( \gamma_0 \) by closed curves with period \( T \) (however, not necessarily we have \( |\dot{\gamma}_\lambda| = 1 \)). Let us denote \( E_\lambda = E(\tilde{\gamma}_\lambda) \). We obtain

\[
E_\lambda = \frac{T^2}{T}.
\]

Differentiating with respect to \( \lambda \) we obtain

\[
\frac{d E_\lambda}{d\lambda} = \frac{2T_\lambda \, dT_\lambda}{T \, d\lambda}.
\]

(11)

Taking \( \lambda = 0 \) in (11), we obtain, from (10), that

\[
\int_0^T D_0 \beta (\gamma_0(t)) \, dt = \left. \frac{d T_\lambda}{d\lambda} \right|_{\lambda=0} = 1 \left. \frac{d E}{d\lambda} \right|_{\lambda=0}.
\]

(12)

On the other hand, \( \gamma_0 \) is a closed geodesic. Thus it is a critical point of the energy. So,

\[
\left. \frac{d E}{d\lambda} \right|_{\lambda=0} = 0.
\]

(13)

This, joint with equation (12), gives the proof of the first part of the lemma.

Moreover, differentiating (11) with respect to \( \lambda \), evaluating at \( \lambda = 0 \) and using (10) we obtain

\[
\int_0^T D_0^2 \beta (\gamma_0(t)) \, dt = \left. \frac{d^2 T_\lambda}{d\lambda^2} \right|_{\lambda=0} = \left. \frac{1}{2} \frac{d^2 E}{d\lambda^2} \right|_{\lambda=0} = \left. \left( \frac{d T_\lambda}{d\lambda} \right)^2 \right|_{\lambda=0}.
\]

Using the first part of the lemma, this completes the proof.

So, we conclude that

\[
\int_{SM} D_0 \beta \, dm = 0 \quad \text{and} \quad \text{Var} (D_0 \beta) = 0,
\]

where \( \text{Var} \) denotes the variance (as in [7]).

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Indeed, since \( \int_0^T D_0 \beta(\dot{\gamma}_0(t)) \, dt = 0 \), by Livsic’s theorem [6], we have that the map \( \theta \rightarrow D_0 \beta(\theta) \) is zero up to a co-boundary. Using the properties of the variance, we obtain that \( \text{Var}(D_0 \beta) \equiv 0 \).

Moreover, since \( \beta^\lambda = \frac{1}{\sigma} \), we have
\[
D_0^2 \beta = \frac{2}{(\sigma^2)} \left( D_\lambda \alpha^\lambda \right)^2 - \frac{1}{(\sigma^2)} D_0^2 \alpha^\lambda.
\]
Thus, at \( \lambda = 0 \), we have
\[
D_0^2 \beta = 2 (D_0 \alpha)^2 - D_0^2 \alpha,
\]
once that \( \alpha^0 \equiv 1 \).

Using equations (14) and (15) of theorem 4, we obtain
\[
h'(0) = 0, \quad h''(0) = -h(0) \int_{S^M} D_0^2 \beta \, dm.
\]

Hence, to conclude the proof of theorem 4 it is enough to estimate \( \int_{S^M} D_0^2 \beta \, dm \).

5. The variational field

In this section, we obtain a useful equation for the variational field.

Lemma 6. Let \( W = \frac{\dot{\gamma}_\lambda}{\lambda} \bigg|_{\lambda=0} \) be the variational field associated to the variation \( \lambda \mapsto \dot{\gamma}_\lambda \) of the closed geodesic \( \gamma_0 \). Then
\[
W(0) = W(T),
\]
\[
W(0) = \dot{W}(T),
\]
\[
\dot{W} + R(\dot{\gamma}_0(t), W(t)) \dot{\gamma}_0(t) = \frac{dE_\lambda}{d\lambda} \bigg|_{\lambda=0} - \frac{1}{|\dot{\gamma}_0|^2} \left( \frac{dE_\lambda}{d\lambda} \bigg|_{\lambda=0}, \dot{\gamma}_0 \right) \dot{\gamma}_0.
\]

Proof. By definition of \( \dot{\gamma}_\lambda \), we have,
\[
\dot{\gamma}_\lambda(t) = \gamma_\lambda(tT_\lambda / T),
\]
where \( \gamma_\lambda(t) = \pi (\varphi_\lambda^T (\theta^T (t))) \).

Hence, \( \dot{\gamma}_\lambda(0) = \dot{\gamma}_\lambda(T) \) and \( \dot{\gamma}_\lambda(0) = \dot{\gamma}_\lambda(T) \). From the definition of \( W(t) \), we obtain \( W(0) = W(T) \). The equality \( W(0) = W(T) \) follows from the symmetry of the Riemannian connection.

Let us consider the surface given by \( (t, \lambda) \rightarrow f(t, \lambda) \). We recall the following identity from Riemannian geometry:
\[
D \frac{df}{dt} \frac{df}{d\lambda} + R \left( \frac{df}{dt}, \frac{df}{d\lambda}, \frac{df}{dt}, \frac{df}{d\lambda} \right) = D \frac{df}{dt} \left( D \frac{df}{dt} \right).
\]
Setting \( f(t, \lambda) = \dot{\gamma}_\lambda \), by the identity above, at \( \lambda = 0 \) we have
\[
\dot{W} + R(\dot{\gamma}_0, W) \dot{\gamma}_0 = \left. \frac{D}{d\lambda} \left( \frac{D}{dt} \dot{\gamma}_\lambda \right) \right|_{\lambda=0}.
\]
Now, by the definition of the Gaussian thermostat, we have
\[
\frac{D}{dt} \dot{\gamma}_\lambda = E_\lambda - \frac{\langle E_\lambda, \gamma_\lambda \rangle}{|\gamma_\lambda|^2} \gamma_\lambda.
\]
Thus,
\[
\frac{d}{dt} \hat{\gamma}_\lambda(t) = \frac{d}{dt} \left[ \gamma_\lambda \left( \frac{tT_\lambda}{T} \right) \right] = \frac{T_\lambda}{T} \left( E_\lambda \left( \hat{\gamma}_\lambda(t) \right) - \frac{\langle E_\lambda \left( \hat{\gamma}_\lambda(t) \right), \hat{\gamma}_\lambda(t) \rangle}{|\hat{\gamma}_\lambda(t)|^2} \hat{\gamma}_\lambda(t) \right).
\]

Hence
\[
\frac{d}{d\lambda} \left( \frac{d}{dt} \hat{\gamma}_\lambda \right) \bigg|_{\lambda=0} = \frac{d}{d\lambda} \left[ \frac{T_\lambda}{T} \left( E_\lambda \left( \hat{\gamma}_\lambda(t) \right) - \frac{\langle E_\lambda \left( \hat{\gamma}_\lambda(t) \right), \hat{\gamma}_\lambda(t) \rangle}{|\hat{\gamma}_\lambda(t)|^2} \hat{\gamma}_\lambda(t) \right) \right] \bigg|_{\lambda=0} = \frac{d}{d\lambda} \left( \left( E_\lambda \left( \hat{\gamma}_\lambda(t) \right), \hat{\gamma}_\lambda(t) \right) \right) \bigg|_{\lambda=0}.
\]

Indeed, lemma 5, says that \( \frac{d}{d\lambda} \left( T_\lambda / T \right) \big|_{\lambda=0} = dE/dt \big|_{\lambda=0} = 0 \) and \( T_0 = T \).

Now, we compute each term in (18).

We have
\[
\frac{d}{d\lambda} E_\lambda \left( \gamma_\lambda(t) \right) \bigg|_{\lambda=0} = \frac{dE}{d\lambda} \bigg|_{\lambda=0} + \nabla W E_0
\]
\[
= \frac{dE}{d\lambda} \bigg|_{\lambda=0}.
\]

Since, by hypothesis, \( E_0 \equiv 0 \).

We also have
\[
\frac{d}{d\lambda} \left( \frac{\langle E_\lambda \left( \hat{\gamma}_\lambda(t) \right), \hat{\gamma}_\lambda(t) \rangle}{|\hat{\gamma}_\lambda(t)|^2} \right) \bigg|_{\lambda=0} = \frac{d}{d\lambda} \left( \frac{\langle E_\lambda \left( \hat{\gamma}_\lambda(t) \rangle, \hat{\gamma}_\lambda(t) \rangle}{|\hat{\gamma}_\lambda(t)|^2} \right) \bigg|_{\lambda=0} = \frac{d}{d\lambda} \left( \frac{\langle E_\lambda \left( \hat{\gamma}_\lambda(t) \rangle, \hat{\gamma}_\lambda(t) \rangle}{|\hat{\gamma}_\lambda(t)|^2} \right) \bigg|_{\lambda=0}.
\]

and
\[
\frac{d}{d\lambda} \left( \frac{\langle E_\lambda \left( \hat{\gamma}_\lambda(t) \rangle, \hat{\gamma}_\lambda(t) \rangle}{|\hat{\gamma}_\lambda(t)|^2} \right) \bigg|_{\lambda=0} = \frac{1}{|\gamma_0|^2} \left( \frac{d}{d\lambda} \left( \frac{\langle E_\lambda \left( \hat{\gamma}_\lambda(t) \rangle, \hat{\gamma}_\lambda(t) \rangle}{|\hat{\gamma}_\lambda(t)|^2} \right) \bigg|_{\lambda=0} \right) = \frac{1}{|\gamma_0|^2} \left( \frac{dE}{d\lambda} \bigg|_{\lambda=0}, \gamma_0 \right) \hat{\gamma}_0.
\]

Thus, we conclude that
\[
\frac{d}{d\lambda} \left( \frac{d}{dt} \hat{\gamma}_\lambda \right) \bigg|_{\lambda=0} = \frac{dE}{d\lambda} \bigg|_{\lambda=0} - \frac{1}{|\gamma_0|^2} \left( \frac{dE}{d\lambda} \bigg|_{\lambda=0}, \gamma_0 \right) \hat{\gamma}_0.
\]

This completes the proof. \( \square \)

6. Proof of theorem 1

In this section we prove theorem 1.

Let \( \Lambda \) the vector space of piecewise differentiable vector fields \( V : [0, T] \rightarrow \Lambda \) along the closed geodesic \( \gamma_0 : [0, T] \rightarrow M \) such that \( V(0) = V(T) \).
We also consider the index form of $\gamma_0$, $I : \Lambda \times \Lambda \rightarrow \mathbb{R}$ given by

$$I(U, V) = \int_0^T \{ \dot{U}, \dot{V} \} - \langle R(\gamma_0, U) \gamma_0, V \rangle \ dt.$$  \hfill (19)

Since the vector fields of $\Lambda$ satisfy $V(0) = V(T)$, we obtain

$$\frac{1}{2} \left. \frac{d^2 \mathcal{E}}{d \lambda} \right|_{\lambda=0} = I(W, W),$$

whereas in lemma 6, $W$ is the variational field of the variation $\lambda \mapsto \bar{\gamma}_\lambda$.

Since $\phi_\lambda^t$ is Anosov, then the metric has no conjugated points. So, by Morse’s index theorem, we obtain

$$I(V, V) \geq 0$$

for every $V \in \Lambda$. By lemma 6, we have that $W(t) \in \Lambda$.

Obviously, $E_0'(t) - \left( \frac{E_0'(t), \gamma_0(t)}{|\gamma_0(t)|^2} \right) \gamma_0(t) \in \Lambda$, where $E_0'(t) = E'(\gamma_0(t))$.

In the following, we will omit $t$. We have that $[W + x(E_0' - \left( \frac{E_0', \gamma_0}{|\gamma_0|^2} \right) \gamma_0)] \in \Lambda$ for every $x \in \mathbb{R}$. Thus, using (21) we have

$$I \left( W + x \left( E_0' - \frac{E_0', \gamma_0}{|\gamma_0|^2} \gamma_0 \right), W + x \left( E_0' - \frac{E_0', \gamma_0}{|\gamma_0|^2} \gamma_0 \right) \right) \geq 0.$$  \hfill (22)

Therefore,

$$I(W, W) \geq -2x I \left( W, E_0' - \frac{E_0', \gamma_0}{|\gamma_0|^2} \gamma_0 \right) - x^2 I \left( E_0' - \frac{E_0', \gamma_0}{|\gamma_0|^2} \gamma_0, E_0' - \frac{E_0', \gamma_0}{|\gamma_0|^2} \gamma_0 \right).$$

Now, we compute each term of (22). We have,

$$I \left( E_0' - \frac{E_0', \gamma_0}{|\gamma_0|^2} \gamma_0, E_0' - \frac{E_0', \gamma_0}{|\gamma_0|^2} \gamma_0 \right) = I(E_0', E_0')$$

$$-2I \left( E_0', \frac{E_0', \gamma_0}{|\gamma_0|^2} \gamma_0 \right) + I \left( \frac{E_0', \gamma_0}{|\gamma_0|^2} \gamma_0, \frac{E_0', \gamma_0}{|\gamma_0|^2} \gamma_0 \right).$$

By the definition of the index form of $\gamma_0$, given by (19), we have

$$I \left( E_0', E_0' \right) = \int_0^T \left\{ \nabla_{\gamma_0'} E_0', \nabla_{\gamma_0'} E_0' \right\} - \{ R(\gamma_0, E_0') \gamma_0, E_0' \} \ dt$$

$$= \int_0^T \left\| \nabla_{\gamma_0'} E_0' \right\|^2 - \{ R(\gamma_0, E_0') \gamma_0, E_0' \} \ dt.$$  \hfill (24)

We also have

$$I \left( E_0', \frac{E_0', \gamma_0}{|\gamma_0|^2} \gamma_0 \right) = \int_0^T \left\{ \nabla_{\gamma_0'} E_0', \frac{D}{dt} \left( \frac{E_0', \gamma_0}{|\gamma_0|^2} \gamma_0 \right) \right\} - \left\{ R(\gamma_0, E_0') \gamma_0, \frac{E_0', \gamma_0}{|\gamma_0|^2} \gamma_0 \right\} \ dt$$

$$= \int_0^T \left\| \nabla_{\gamma_0'} E_0' \right\|^2 \frac{D}{dt} \left( \frac{E_0', \gamma_0}{|\gamma_0|^2} \gamma_0 \right) \ dt.$$  \hfill (25)
However,

\[
\frac{D}{dt} \left( \frac{\langle E'_0, \gamma_0 \rangle}{|\gamma_0|^2} \right) = \frac{d}{dt} \left( \frac{\langle E'_0, \gamma_0 \rangle}{|\gamma_0|^2} \right) \dot{\gamma}_0 + \frac{\langle E'_0, \gamma_0 \rangle}{|\gamma_0|^2} \frac{d}{dt} \dot{\gamma}_0
\]

\[
= \frac{1}{|\gamma_0|^2} \{ \nabla_\gamma E'_0, \dot{\gamma}_0 \} \dot{\gamma}_0.
\]

Thus, using (25), we obtain

\[
I \left( E'_0, \frac{\langle E'_0, \gamma_0 \rangle}{|\gamma_0|^2} \right) = \int_0^T \frac{\{ \nabla_\gamma E'_0, \dot{\gamma}_0 \}^2}{|\gamma_0|^2} \, dt.
\]  

(26)

Moreover,

\[
I \left( \frac{\langle E'_0, \gamma_0 \rangle}{|\gamma_0|^2}, \frac{\langle E'_0, \gamma_0 \rangle}{|\gamma_0|^2} \right) = \int_0^T \left\{ \frac{D}{dr} \left( \frac{\langle E'_0, \gamma_0 \rangle}{|\gamma_0|^2} \right) \cdot \frac{D}{dr} \left( \frac{\langle E'_0, \gamma_0 \rangle}{|\gamma_0|^2} \right) - \frac{\langle R(\gamma_0, E'_0) \rangle \gamma_0, \langle E'_0, \gamma_0 \rangle \rangle}{|\gamma_0|^2} \right\} \, dt
\]

\[
- \left\{ R(\gamma_0, E'_0) \dot{\gamma}_0, E'_0 \right\} \, dt.
\]  

(27)

Replacing these terms, (24), (26) and (27), in (23), we obtain

\[
I \left( E'_0 - \frac{\langle E'_0, \gamma_0 \rangle}{|\gamma_0|^2} \dot{\gamma}_0, E'_0 - \frac{\langle E'_0, \gamma_0 \rangle}{|\gamma_0|^2} \dot{\gamma}_0 \right) = \int_0^T \left| \nabla_\gamma E'_0 \right|^2 - \left| \nabla_\gamma \frac{\langle E'_0, \gamma_0 \rangle}{|\gamma_0|^2} \right|^2
\]

\[
- \left\{ R(\gamma_0, E'_0) \dot{\gamma}_0, E'_0 \right\} \, dt.
\]  

(28)

We also have

\[
I \left( W, E'_0 - \frac{\langle E'_0, \gamma_0 \rangle}{|\gamma_0|^2} \dot{\gamma}_0 \right) = I (W, E'_0) - I \left( W, \frac{\langle E'_0, \gamma_0 \rangle}{|\gamma_0|^2} \dot{\gamma}_0 \right).
\]  

(29)

But,

\[
I (W, E'_0) = - \int_0^T \langle \dot{W}, E'_0 \rangle + \langle R(\gamma_0, W) \gamma_0, E'_0 \rangle \, dt
\]

\[
= \int_0^T \left| E'_0 \right|^2 + \left| \frac{\langle E'_0, \gamma_0 \rangle}{|\gamma_0|^2} \right|^2 \, dt.
\]

Lemma 6 says that

\[
I \left( W, \frac{\langle E'_0, \gamma_0 \rangle}{|\gamma_0|^2} \dot{\gamma}_0 \right) = - \int_0^T \left\langle \dot{W}, \frac{\langle E'_0, \gamma_0 \rangle}{|\gamma_0|^2} \dot{\gamma}_0 \right\rangle + \left\langle R(\gamma_0, W) \gamma_0, \frac{\langle E'_0, \gamma_0 \rangle}{|\gamma_0|^2} \dot{\gamma}_0 \right\rangle \, dt
\]

\[
= \int_0^T \left\langle R(\gamma_0, W) \dot{\gamma}_0 + \frac{\langle E'_0, \gamma_0 \rangle}{|\gamma_0|^2} \dot{\gamma}_0 - E'_0, E'_0 \right\rangle \, dt
\]

\[
= 0.
\]

Therefore,

\[
I \left( W, E'_0 - \frac{\langle E'_0, \gamma_0 \rangle}{|\gamma_0|^2} \dot{\gamma}_0 \right) = \int_0^T \left| E'_0 \right|^2 + \left| \frac{\langle E'_0, \gamma_0 \rangle}{|\gamma_0|^2} \right|^2 \, dt.
\]  

(30)
On the other hand, using lemma 5 and equation (20), we obtain
\[ \int_0^T D_0^2\beta (\gamma_0) \, dt = \frac{1}{2} \frac{d^2}{dt^2} I(W, W). \]

This gives us
\[ \int_0^T D_0^2\beta (\gamma_0) \, dt \geq 2 \int_0^T \left| E_0' \right|^2 - \frac{\left| E_0', \gamma_0 \right|^2}{\left| \gamma_0 \right|^2} \, dt \]
\[ -x^2 \int_0^T \left| \nabla_0 E_0' \right|^2 - \frac{\left| \nabla_0 E_0', \gamma_0 \right|^2}{\left| \gamma_0 \right|^2} - \left| R(\gamma_0, E_0')\gamma_0, E_0' \right| \, dt. \quad (31) \]

Now, since \( \psi_t = \psi_t^0 \) is a volume-preserving Anosov flow, we have that the set of probability measures supported on periodic orbits is dense in the set of invariant probability measures. Hence, by (31) we obtain
\[ \int_{SM} D_0^2\beta (\gamma_0) \, dm \geq 2 \int_{SM} \left| E_0' \right|^2 - \frac{\left| E_0', \gamma_0 \right|^2}{\left| \gamma_0 \right|^2} \, dm \]
\[ -x^2 \int_{SM} \left| \nabla_0 E_0' \right|^2 - \frac{\left| \nabla_0 E_0', \gamma_0 \right|^2}{\left| \gamma_0 \right|^2} - \left| R(\gamma_0, E_0')\gamma_0, E_0' \right| \, dm := 2x A - x^2 B, \quad (32) \]
for every \( x \in \mathbb{R} \). Since, \( B = I(V, V) \), with \( V = E_0' - \frac{\left| E_0', \gamma_0 \right|}{\left| \gamma_0 \right|} \gamma_0 \in \Lambda \), we obtain from (21) that \( B \geq 0 \).

From the above equation, if \( Z \neq 0 \) (which is equivalent to \( A > 0 \)) then \( B > 0 \).

Now, we consider, \( f : \mathbb{R} \to \mathbb{R} \) given by \( f(x) = 2x A - x^2 B \). Then,
\[ f'(x) = 2A - 2xB \quad \text{and} \quad f''(x) = -2B. \]

We have that \( 0 = f'(\frac{A}{B}) \) and \( f''(\frac{A}{B}) = -2B < 0 \). Indeed, by hypothesis \( A > 0 \), hence, \( B > 0 \). This implies that \( f \) has a maximum at \( \frac{A}{B} \), and its value is \( f(\frac{A}{B}) = \frac{A^2}{B} > 0 \).

Hence, by (17) we have
\[ h''(0) = -h(0) \int_{SM} D_0^2\beta dm \leq -h(0)(2x A - x^2 B) = -h(0) \frac{A^2}{B} < 0, \]
where in the last equality we take \( x = A/B \).

In particular,
\[ h''(0) \leq -h(0) \frac{\left[ \int_{SM} \left( \left| E_0' \right|^2 - \frac{\left| E_0', v \right|^2}{\left| v \right|^2} \right) \, dm \right]^2}{\int_{SM} \left( \left| \nabla_0 E_0' \right|^2 - \frac{\left| \nabla_0 E_0', v \right|^2}{\left| v \right|^2} - \left| R(v, E_0')v, E_0' \right| \right) \, dm} < 0. \]

And this completes the proof.

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