A MODEL FOR THE CONTINUOUS q-ULTRASPHERICAL POLYNOMIALS

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Abstract

We provide an algebraic interpretation for two classes of continuous $q$-polynomials. Rogers’ continuous $q$-Hermite polynomials and continuous $q$-ultraspherical polynomials are shown to realize, respectively, bases for representation spaces of the $q$-Heisenberg algebra and a $q$-deformation of the Euclidean algebra in these dimensions. A generating function for the continuous $q$-Hermite polynomials and a $q$-analog of the Fourier-Gegenbauer expansion are naturally obtained from these models.

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1. INTRODUCTION

The algebraic theory of \( q \)-special functions\(^1\,^2 \) is currently being actively developed. Indeed, it has been realized that quantum groups and algebras offer a unifying framework for describing and studying these functions. In a series of papers on this topic,\(^3\,^4\,^5\) we, among others, have shown that many \( q \)-orthogonal polynomials and \( q \)-functions appear as matrix elements or basis vectors of quantum algebra representations. We have further used these observations to derive and obtain in a natural way, various relations and identities that \( q \)-special functions obey. In most cases, the \( q \)-polynomials that we encountered, were orthogonal with respect to a discrete measure. There are, however, numerous sets of \( q \)-polynomials orthogonal with respect to continuous measures.\(^1\,^2 \) Two classes of these continuous \( q \)-orthogonal polynomials will be considered here and given algebraic interpretations.

Our approach to basic special functions can be illustrated by considering the connection between \( q \)-Bessel functions and the two-dimensional quantum Euclidean algebra \( \mathcal{U}_q(E(2)) \).\(^4\,^5 \) This Hopf algebra is generated by the elements \( D, D^{-1}, P_+ \) and \( P_- \) satisfying the defining relations:

\[
[P_+, P_-] = 0, \quad DP_\pm = q^{\pm 1}P_\pm D, \quad DD^{-1} = D^{-1}D = 1 . \tag{1.1}
\]

The coproduct \( \Delta \), antipode \( S \) and counit \( \varepsilon \) are determined by

\[
\begin{align*}
\Delta(D) &= D \otimes D , \quad \Delta(P_\pm) = P_\pm \otimes D^{-1/2} + D^{1/2} \otimes P_\pm , \\
S(D) &= D^{-1} , \quad S(P_\pm) = -q^{\mp 1/2}P_\pm , \\
\varepsilon(D) &= \varepsilon(1) = 1 , \quad \varepsilon(P_\pm) = 0 . \tag{1.2}
\end{align*}
\]

Representations of \( \mathcal{U}_q(E(2)) \) are characterized by two complex numbers \( \omega \) and \( m_0 \), with \( \omega \neq 0 \) and \( 0 \leq \text{Re} \, m_0 < 1 \). The representation spaces have for basis, vectors \( f_m \), with the index \( m \) running over the elements of the set \( S = \{ m_0 + n : n \in \mathbb{Z} \} \). In these bases the generators act as follows:

\[
P_\pm f_m = \omega f_{m \pm 1} , \quad D^{\pm 1} f_m = q^{\pm m} f_m . \tag{1.3}
\]

Consider now the following two \( q \)-analogs of the exponential function:\(^1 \)

\[
e_q(z) = \sum_{n=0}^{\infty} \frac{1}{(q; q)_n} z^n = \frac{1}{(z; q)_{\infty}} , \\
E_q(z) = \sum_{n=0}^{\infty} q^{n(n-1)/2} \frac{1}{(q; q)_n} z^n = (-z; q)_{\infty} . \tag{1.4}
\]

The symbols \((a; q)_{\alpha}\) are referred to as \( q \)-shifted factorials, and are defined by

\[(a; q)_{\alpha} = \frac{(a; q)_{\infty}}{(aq^{\alpha}; q)_{\infty}} , \tag{1.5a}\]
with
\[(a; q)_\infty = \prod_{k=0}^{\infty} (1 - aq^k), \quad |q| < 1. \tag{1.5b}\]

The \(q\)-exponentials \(e_q(z)\) and \(E_q(z)\) are eigenfunctions of the \(q\)-derivative operators \(D^+_z\) and \(D^-_z\) respectively. These operators are defined by
\[
D^\pm_z = \frac{1}{z}(1 - T_z^{\pm 1}), \tag{1.6}\]
where \(T_z\) is the \(q\)-shift operator \(T_z f(z) = f(qz)\). It is easy to check that
\[
D^+_z e_q(\lambda z) = \lambda e_q(\lambda z), \quad D^-_z E_q(\lambda z) = -q^{-1}\lambda E_q(\lambda z). \tag{1.7}\]

Note also that \(\lim_{q \to 1^-} e_q(z(1 - q)) = \lim_{q \to 1^-} E_q(z(1 - q)) = e^z\).

In analogy with ordinary Lie theory, we introduce for instance, the following element in the completion of \(U_q(E(2))\):
\[
U(\alpha, \beta) = E_q(\alpha P_+) E_q(\beta P_-). \tag{1.8}\]

In the limit \(q \to 1^-\), \(U[(1 - q)\alpha, (1 - q)\beta]\) goes into the element \(e^{\alpha P_+ + \beta P_-}\) of the two-dimensional Euclidean Lie group. The matrix elements \(U_{kn}(\alpha, \beta)\) of \(U(\alpha, \beta)\) are defined through
\[
U(\alpha, \beta) \, f_{m_0 + n} = \sum_{k=-\infty}^{\infty} U_{kn}(\alpha, \beta) \, f_{m_0 + k}. \tag{1.9}\]

Given the action (1.3) of the generators, these matrix elements are straightforwardly computed and one finds\(^4,5\)
\[
U_{kn}(\alpha, \beta) = q^{(k-n)^2/2} \left( -\frac{\alpha}{\beta} \right)^{(k-n)/2} J_{k-n}^{(2)} \left( 2\omega \left( -\frac{\alpha\beta}{q} \right)^{1/2} ; q \right). \tag{1.10}\]

The functions \(J^{(2)}_{\nu}(z; q)\) are \(q\)-analogos of the Bessel functions \(J_{\nu}(z)\). They are defined by the series\(^1,13\)
\[
J^{(2)}_{\nu}(z; q) = \sum_{n=0}^{\infty} q^{n(n+\nu)} \frac{(-1)^n}{(q; q)_n (q;q)_{n+\nu}} \left( \frac{z}{2} \right)^{2n+\nu}, \tag{1.11}\]
and one readily checks that \(\lim_{q \to 1^-} J^{(2)}_{\nu}(z(1 - q); q) = J_{\nu}(z)\). This relation between the \(q\)-Bessel functions and \(U_q(E(2))\) proves rather fruitful. It is for instance possible to construct a two-variable realization of \(U_q(E(2))\) where the matrix elements \(U_{kn}(\alpha, \beta)\), appear as basis vectors of the associated module. Within this model, (1.9) is shown to entail a \(q\)-analog of Graf’s addition formula for Bessel functions.\(^5\) Many properties of the \(q\)-Bessel functions can be obtained and interpreted in this fashion. Let us record for future reference the three-term recurrence relation that the functions \(J^{(2)}_{\nu}(z; q)\) obey\(^1,13\)
\[
q^\nu J^{(2)}_{\nu+1}(z; q) = \frac{2}{z} (1 - q^\nu) \, J^{(2)}_{\nu}(z; q) - J^{(2)}_{\nu-1}(z; q), \tag{1.12}\]
and the asymptotic behavior of these functions as $\nu$ goes to infinity:

\[
J^{(2)}_{\nu}(z; q) \sim \frac{(z/2)^{\nu}}{(q; q)_\infty}.
\]

In summary, our algebraic interpretation of $q$-special functions proceeds as follows. Using certain $q$-exponentials of $q$-algebra generators, $q$-analogs of Lie group elements are formed and their matrix elements in representation spaces of the deformed algebra are shown to involve $q$-special functions. Various models and realizations are then constructed and called upon to exploit these results. This approach has been applied to many situations, but so far, has not been used much to discuss the interesting continuous orthogonal polynomials. Encompassing all of these, are the Askey-Wilson polynomials $p_n(x; a, b, c, d|q)$. They form a four-parameter family and are defined by

\[
p_n(x; a, b, c, d|q) = (ab, ac, ad; q)_n a^{-n} \times \phi_3\left(\frac{q^{-n}, abcd q^{-n-1}, ae^{i\theta}, ae^{-i\theta}}{ab, ac, ad} \bigg| q; q\right).
\]

We are using the notation

\[
(a_1, a_2, \ldots, a_k; q)_\alpha = (a_1; q)_\alpha (a_2; q)_\alpha \ldots (a_k; q)_\alpha,
\]

\[
r\phi_s\left(\frac{a_1, a_2, \ldots, a_r}{b_1, \ldots, b_s} \bigg| q; z\right) = \sum_{n=0}^{\infty} \frac{(a_1, \ldots, a_r; q)_n}{(q, b_1, \ldots, b_s; q)_n} \times \left[(-1)^n q^{n(n-1)/2}\right]^{1+s-r} z^n.
\]

When $a, b, c, d$ are real, the Askey-Wilson polynomials are orthogonal over the interval $0 < \theta < \pi$ with respect to the continuous measure

\[
w(\cos \theta; a, b, c, d) = \left|\frac{(e^{2i\theta}; q)_\infty}{(ae^{i\theta}, be^{i\theta}, ce^{i\theta}, de^{i\theta}; q)_\infty}\right|^2.
\]

Particular cases of interest are obtained for special choices of the parameters. The continuous $q$-ultraspherical polynomials $C_n(x; \beta|q)$ of Rogers are obtained by setting $a = \beta^{1/2}$, $b = \beta^{1/2} q^{1/2}$, $c = -\beta^{1/2}$, $d = -\beta^{1/2} q^{1/2}$ and changing also the normalization:

\[
C_n(x; \beta|q) = \frac{\beta^2 q^n}{(\beta q^{1/2}, -\beta, -\beta q^{1/2}, q; q)_n} \times p_n(x; \beta^{1/2}, \beta^{1/2} q^{1/2}, -\beta^{1/2}, -\beta^{1/2} q^{1/2}|q).
\]

By putting $a = 0$, $b = 0$, $c = 0$, $d = 0$, one arrives at the continuous $q$-Hermite polynomials $H_n(x|q)$:

\[
H_n(x|q) = p_n(x; 0, 0, 0, 0|q).
\]
In the following, our attention will be focused on these two classes of polynomials. Note that
\[
\lim_{q \to 1} C_n(x; q^\lambda; q) = C_n^\lambda(x), \quad (1.20a)
\]
\[
\lim_{q \to 1} \left( \frac{1 - q}{2} \right)^{-n/2} H_n \left( x \sqrt{\frac{1 - q}{2}} q \right) = H_n(x), \quad (1.20b)
\]
where \(C_n^\lambda(x)\) and \(H_n(x)\) respectively denote the ultraspherical and Hermite polynomials.\(^{18}\)

Let us point out that neither the \(q\)-exponential \(e_q(x)\), nor \(E_q(x)\) proved relevant in the algebraic interpretation of the polynomials \(C_n(x; \beta|q)\) and \(H_n(x|q)\). In fact, it is the \(q\)-exponential introduced recently in Ref.[14] that turns out to be the appropriate \(q\)-analog of the exponential to use here. The properties of this function will be discussed in Section 3, after the relation between the continuous \(q\)-Hermite polynomials and the \(q\)-oscillator algebra will have been established in Section 2.

A generating function for these continuous \(q\)-Hermite polynomials will be derived in Section 4. Next, in Section 5, a \(q\)-deformation of the three-dimensional Euclidean algebra will be realized on the complex 2-sphere. This construct will be seen to provide a nice framework for studying the properties of the continuous \(q\)-ultraspherical polynomials. It will allow us to present in Section 6, an algebraic derivation of a \(q\)-analog (see (6.22)) of the Fourier-Gegenbauer expansion:\(^{14}\)

\[
e^{i\vec{k} \cdot \vec{r}} = \Gamma(\nu) \left( \frac{kr}{2} \right)^{\nu} \sum_{n=0}^{\infty} i^n (\nu + n) J_{\nu+n}(kr) C_n^\nu(x) \cos \theta, \quad (1.21)
\]
where \(\vec{k} \cdot \vec{r} = kr \cos \theta\). Concluding remarks will then end the paper.

### 2. CONTINUOUS \(q\)-HERMITE POLYNOMIALS AND THE \(q\)-HEISENBERG ALGEBRA

The continuous \(q\)-Hermite polynomials \(H_n(x|q)\) given in (1.19) as special cases of the Askey-Wilson polynomials, can also be defined as follows:\(^2\)

\[
H_n(x|q) = \sum_{k=0}^{n} \frac{(q; q)_n}{(q; q)_k (q; q)_{n-k}} e^{i(n-2k)\theta} e^{i\theta}, \quad x = \cos \theta. \quad (2.1)
\]

Let
\[
z = e^{i\theta}, \quad (2.2)
\]
and introduce the divided difference operators\(^{19}\)

\[
\tau = \frac{1}{z - z^{-1}} (T_z^{1/2} - T_z^{-1/2}) \quad (2.3)
\]
\[
\tau^* = \frac{q^{-1/2}}{z - z^{-1}} 
\left( \frac{1}{z^2} T_z^{1/2} - z^2 T_z^{-1/2} \right) \quad (2.4)
\]
These operators will be taken to act on functions of \( x = (z + z^{-1})/2 \). Notice that as \( q \to 1 \):

\[
\lim_{q \to 1^-} 2 \frac{1}{(q^{1/2} - q^{-1/2})} \tau = \frac{d}{dx},
\]

\[
\lim_{q \to 1^-} \tau^*/2 = x.
\]

Set

\[ f_n(x) = H_n(x|q), \quad n = 0, 1, 2, \ldots. \] (2.7)

One can directly verify from the definition (2.1), that

\[
\tau f_n(x) = q^{n/2}(1 - q^{-n}) f_{n-1}(x),
\]

\[
\tau^* f_n(x) = -q^{-(n+1)/2} f_{n+1}(x).
\]

It follows that \( \tau \) and \( \tau^* \) satisfy

\[
\tau^* \tau - q \tau \tau^* = -(1 - q).
\] (2.10)

These operators hence provide a realization of the \( q \)-Heisenberg algebra with the continuous \( q \)-Hermite polynomials occurring as basis vectors for a representation space of this algebra. The action on that basis of the operator

\[
\mu = \frac{1}{z - z^{-1}} \left( -\frac{1}{z} T_z^{1/2} + z T_z^{-1/2} \right),
\] (2.11)

is also readily determined:

\[
\mu f_n(x) = q^{-n/2} f_n(x).
\] (2.12)

Remark that \( \mu \) reduces to the identity operator when \( q \to 1 \) and that

\[
\tau \mu - q^{-1/2} \mu \tau = 0,
\]

\[
\tau^* \mu - q^{1/2} \mu \tau^* = 0.
\] (2.14)

It is also natural to consider the operator multiplication by \( x \). This operator is not independent of \( \tau \), \( \tau^* \) and \( \mu \) since

\[
x \mu = -\tau - q^{1/2} \tau^*.
\] (2.15)

Its action on the basis \( \{f_n\} \) can be gotten from this last relation and is in fact embodied in the three-term recurrence relation of the continuous \( q \)-Hermite polynomials: \( 1, 2 \)

\[
2x f_n(x) = f_{n+1}(x) + (1 - q^n) f_{n-1}(x).
\] (2.16)

We now want to follow the approach that we outlined in the Introduction, that is, take \( q \)-exponentials of algebra elements and compute their action on representation spaces to arrive at special function identities. To this end, we need \( q \)-exponentials that behave nicely under the action of the elementary difference operator of the models under consideration.
In the present case, we would thus wish for a $q$-exponential that is an eigenfunction of the divided difference operator $\tau$. Such a function has been introduced recently and will be the object of the next section.

3. THE $q$-EXPONENTIAL $E_q$

Consider the divided difference operator $\tau$ given in (2.3). We already observed that $[2/(q^{1/2} - q^{-1/2})] \tau$ tends to the derivative operator $d/dx$ as $q \to 1$. We therefore expect the eigenfunctions of $\tau$ to provide a certain $q$-analog of the exponential. Let us now determine these eigenfunctions.

Take the functions

$$\psi_n(a, \cos \theta) = \left( a^{(1-n)/2} e^{i\theta} ; q \right)_n \left( a^{-n/2} e^{-i\theta} ; q \right)_n.$$ (3.1)

It is easy to check that they satisfy

$$\tau \psi_n(a, \cos \theta) = a^{-n/2} (1 - q^n) \psi_{n-1}(a, \cos \theta).$$ (3.2)

Following Ref.[14], define the function

$$E_q(x; a, b) = \sum_{n=0}^{\infty} \frac{q^{n^2/4}}{(q; q)_n} \psi_n(a, \cos \theta) b^n, \quad x = \cos \theta.$$ (3.3)

It immediately follows from (3.2) that $E_q(x; a, b)$ is an eigenfunction of $\tau$:

$$\tau E_q(x; a, b) = abq^{-1/4} E_q(x; a, b).$$ (3.4)

The function $E_q(x; a, b)$ is thus the $q$-analog of the exponential that we were looking for. Indeed as $q \to 1^-$, $\psi_n(a, \cos \theta) \to (1 + a^2 - 2ax)^n$ and $(1 - q^n)/(q; q)_n \to 1/n!$. Therefore,

$$\lim_{q \to 1^-} E_q(x; a, (1 - q)b) = \exp[(1 + a^2 - 2ax)b],$$ (3.5)

and in particular, for $a = -i$,

$$\lim_{q \to 1^-} E_q(x; -i, (1 - q) b/2) = e^{ibx}.$$ (3.6)

In the following, we shall need the value of $E_q(x; -i, b/2)$ at $x = 0$. In this connection, one readily finds that

$$i^n q^{n^2/4} \psi_n(-i, \cos \theta) \bigg|_{\cos \theta = 0} = \begin{cases} (q; q^2)^2_{n/2}, & n \text{ even} \\ 0, & n \text{ odd} \end{cases}.$$ (3.7)
As a result, one has that

\[ E_q(0, -i, b/2) = \sum_{n=0}^{\infty} \frac{(q; q^2)_n}{(q^2, q^2)_n} \left( -\frac{b^2}{4} \right)^n = \frac{(-qb^2/4; q^2)_\infty}{(-b^2/4; q^2)_\infty}, \]  

with the last equality obtained from the \(q\)-binomial theorem:

\[ \sum_{n=0}^{\infty} \frac{(a; q)_n}{(q; q)_n} z^n = \frac{(az; q)_\infty}{(z; q)_\infty}. \]

4. A GENERATING FUNCTION FOR THE CONTINUOUS \(q\)-HERMITE POLYNOMIALS

We shall now use the \(q\)-exponential we described in Section 3 as well as the connection between the \(q\)-Heisenberg algebra and the continuous \(q\)-Hermite polynomials to provide an algebraic derivation of an identity obtained in Refs.[14,20]. Take the \(q\)-exponential \(E_q(x; -i, b/2)\) of the operator \(x\) and act with it on the representation space with basis \(\{f_n\}\)

\[ E_q(x; -i, b/2) f_n = \sum_{k=0}^{\infty} U_{kn}(b) f_k. \]  

In the realization of Section 2, \(f_n(x) = H_n(x|q)\) and so, \(f_0(x) = 1\). Setting \(U_{k0}(b) \equiv U_k(b)\), we thus have for \(n = 0\):

\[ E_q(x; -i, b/2) = \sum_{k=0}^{\infty} U_k(b) H_k(x|q). \]  

To determine the matrix elements \(U_k(b)\), act with \(\tau\) on both sides of (4.2). Using (3.4) on the l.h.s. and (2.8) on the r.h.s., one finds

\[ -(ib/2) q^{-1/4} E_q(x; -i, b/2) = \sum_{k=0}^{\infty} q^{k/2}(1 - q^{-k}) U_k(b) H_{k-1}(x|q). \]  

Using again (4.2), the following two-term recurrence relation is obtained

\[ (ib/2) U_k(b) = q^{-(2k+1)/4}(1 - q^{k+1}) U_{k+1}(b). \]  

It has for solution

\[ U_k(b) = \frac{q^{k^2/4}}{(q; q)_k} \left( \frac{ib}{2} \right)^k U_0(b). \]
To determine $U_0(b)$, replace $U_k(b)$ in (4.3) by this last expression and set $x = 0$. With the help of (3.8), one thus obtains

$$\frac{(-qb^2/4; q^2)_\infty}{(-b^2/4; q^2)_\infty} = U_0(b) \sum_{k=0}^{\infty} \frac{q^{k^2/4}}{(q; q)_k} \left( \frac{ib}{2} \right)^k H_k(0|q) . \quad (4.6)$$

From the three-term recurrence relation (2.16) for $H_k(x|q)$ one easily gets:

$$H_{2k}(0|q) = (-1)^k (q; q^2)_k , \quad H_{2k+1}(0|q) = 0 . \quad (4.7)$$

With this information, (4.6) is rewritten as

$$\frac{(-qb^2/4; q^2)_\infty}{(-b^2/4; q^2)_\infty} = U_0(b) \sum_{k=0}^{\infty} \frac{(q^2)^{k(k-1)/2}}{(q^2; q^2)_k} \left( \frac{qb^2}{4} \right)^k . \quad (4.8)$$

Recalling (1.4), this yields

$$U_0(b) = \frac{1}{(-b^2/4; q^2)_\infty} . \quad (4.9)$$

Putting everything together, we get the following identity\textsuperscript{14,20}

$$(-b^2/4; q^2)_\infty E_q (x; -i, b/2) = \sum_{k=0}^{\infty} \frac{q^{k^2/4}}{(q; q)_k} \left( \frac{ib}{2} \right)^k H_k(x|q) , \quad (4.10)$$

which provides a generating relation for the continuous $q$-Hermite polynomials.

5. CONTINUOUS $q$-ULTRASPHERICAL POLYNOMIALS AND A $q$-DEFORMATION OF E(3)

The continuous $q$-Hermite polynomials $H_n(x|q)$ involve no parameters. We shall now work our way one step up in the $q$-Askey scheme\textsuperscript{2} and consider a class of continuous $q$-polynomials depending on one parameter. The continuous $q$-ultraspherical polynomials $C_n(x; q^m|q)$ have already been defined in (1.18) through their relation with the Askey-Wilson polynomials. They can also be expressed as follows:\textsuperscript{1,2}

$$C_n(x; q^m|q) = \sum_{k=0}^{n} \frac{(q^m; q)_k (q^m; q)_{n-k}}{(q; q)_k (q; q)_{n-k}} e^{i(n-2k)\theta} , \quad x = \cos \theta , \quad (5.1)$$

from where it is directly seen that these polynomials are of definite parity,

$$C_n(-x; q^m|q) = (-1)^n C_n(x; q^m|q) . \quad (5.2)$$
We shall now be considering operators acting on functions of two variables \(x = (z + z^{-1})/2\) and \(t\).

Let us first introduce the operators \(J_+, J_-\) and \(K:\)

\[
J_+ = \frac{q^{1/2} t T_t^{-1/2}}{1 - q} \tau ,
\]

\[
J_- = \frac{q T_t^{-1/2}}{1 - q t} \tilde{\tau} ,
\]

\[
K = q^{-1/2} T_t ,
\]

where \(\tau = (z - z^{-1})^{-1}(T_z^{1/2} - T_z^{-1/2})\) is again the divided difference operator and

\[
\tilde{\tau} = \frac{q^{-1/2}}{z - z^{-1}} \left[ \frac{(1 - z^2 T_t)(1 - q z^2 T_t)}{z^2} T_z^{1/2} - z^2 \left( 1 - T_t T_z^{1/2} \right) \left( 1 - q T_t T_z^{1/2} \right) T_z^{-1/2} \right] .
\]

If we set \(t = e^{i\phi}\) and \(K = q^{I_0}\), we note that in the limit \(q \to 1^-:\)

\[
J_+ \to \frac{1}{2} e^{i\phi} \frac{\partial}{\partial \theta} ,
\]

\[
J_- \to -2e^{-i\phi} \left( \sin \theta \frac{\partial}{\partial \theta} - 2i \cos \theta \frac{\partial}{\partial \phi} + \cos \theta \right) ,
\]

\[
J_0 \to -i \left( \frac{\partial}{\partial \phi} + \frac{1}{2} \right) .
\]

If we set \(t = e^{i\phi}\) and \(K = q^{I_0}\), we note that in the limit \(q \to 1^-:\)

\[
J_+ \to \frac{1}{2} e^{i\phi} \frac{\partial}{\partial \theta} ,
\]

\[
J_- \to -2e^{-i\phi} \left( \sin \theta \frac{\partial}{\partial \theta} - 2i \cos \theta \frac{\partial}{\partial \phi} + \cos \theta \right) ,
\]

\[
J_0 \to -i \left( \frac{\partial}{\partial \phi} + \frac{1}{2} \right) .
\]

When \(q \to 1^-\), we therefore have a realization of \(sl(2)\) on the 2-sphere. It can be checked in fact, that the generators \(J_+, J_-\) and \(K\) satisfy the relations

\[
[J_+, J_-] = \frac{K - K^{-1}}{q^{1/2} - q^{-1/2}} ,
\]

\[
K J_\pm = q^{\pm 1} J_\pm K ,
\]

and thus provide a realization of the quantum algebra \(U_q(sl(2))\). The connection with the continuous \(q\)-ultraspherical polynomials is now made by observing that in this model, the basis vectors of the associated module are realized by

\[
Q^\ell_m(x, t) = \frac{(q; q)_{\ell-m}}{(q^2; q)_{\ell-m}} q^{m(\ell-m)/2} C_{\ell-m}(x; q^m | q) \ell^m ,
\]

\[
m \leq \ell ; \quad \ell, m = 0, 1, 2 \ldots .
\]

It is found that \(J_\pm\) and \(K\) act as follows on these basis functions:

\[
J_+ Q^\ell_m = \frac{q}{1 - q} \frac{(1 - q^{m-\ell})(1 - q^{m+\ell})}{(1 - q^{2m+1})(1 - q^m)} Q^\ell_{m+1}
\]

\[
J_- Q^\ell_m = -\frac{q^{1-m}}{1 - q} \frac{(1 - q^{2m-1})(1 + q^{m-1})}{(1 - q^m)} Q^\ell_{m-1} ,
\]

\[
K Q^\ell_m = q^{m-1/2} Q^\ell_m .
\]
Actually, it turns out possible to realize an algebra larger than $U_q((2))$. Indeed, the three independent operators

$$P_0 = x, \quad P_+ = t, \quad P_- = \frac{1}{t}(1 - x^2),$$

(5.9)

are well-defined on the space spanned by the $Q^\ell_m$. In the limit $q \to 1^-$, the six generators $J_+, J_-, J_0, P_+, P_-$ and $P_0$ satisfy the commutation relations of the Euclidean algebra in three dimensions,\textsuperscript{21,22} which we shall denote by $E(3)$. It thus follows that this set of operators defines a realization on the 2-sphere of a $q$-deformation of $E(3)$. We shall now describe how $P_0$ and $P_+$ transform the basis functions $Q^\ell_m$.

The action of $P_0$ is directly obtained from the three-term recurrence relation of the continuous $q$-ultraspherical polynomials $C_n(x; q^m|q)$:\textsuperscript{1,2}

$$2x C_n(x; q^m|q) = \frac{1 - q^{n+1}}{1 - q^{m+n}} C_{n+1}(x; q^m|q) + \frac{1 - q^{2m+n-1}}{1 - q^{m+n}} C_{n-1}(x; q^m|q).$$

(5.10)

This yields

$$P_0 Q^\ell_m = \frac{q^{-m/2}}{2} \left( \frac{1 - q^{\ell+m}}{1 - q^\ell} \right) Q^{\ell+1}_m + \frac{q^{m/2}}{2} \left( \frac{1 - q^{\ell-m}}{1 - q^\ell} \right) Q^{\ell-1}_m.$$  

(5.11)

The action of $P_+ = t$ is also straightforwardly obtained. In the limit $q \to 1^-$, $P_+$ is a $sl(2)$ spin 1 operator and for generic $q$, we know the $sl(2)$ and $U_q(sl(2))$ modules to be isomorphic. Taking into account the fact that $KP_+ = q P_+ K$, we must have therefore

$$P_+ Q^\ell_m = f_{\ell,m} Q^{\ell-1}_{m+1} + h_{\ell,m} Q^{\ell+1}_{m+1},$$

(5.12)

with $f_{\ell,m}$ and $h_{\ell,m}$ coefficients to be determined. In terms of the polynomials $C_n(x; q^m|q)$, this means that there must be an identity of the form

$$C_n(x; q^m|q) = \frac{q^{-m-1}}{(1 - q^n)(1 - q^{n-1})} C_{n-2}(x; q^{m+1}|q) f_{n,m}$$

$$+ \frac{1}{(1 - q^{2m+n})(1 - q^{2m+n+1})} C_n(x; q^{m+1}|q) h_{n,m}.$$  

(5.13)

The constants $f_{n,m}$ and $h_{n,m}$ can now be obtained as follows. The values of the $h_{n,m}$ are found first by equating the coefficients of $x^n$ on both sides of (5.13) and using

$$C_n(x; q^m|q) = 2^n \frac{(q^{m}; q)_n}{(q; q)_n} x^n + \ldots.$$  

(5.14)

With the result substituted back in (5.13), the $f_{n,m}$ are determined by evaluating at $x = 0$. The values of the continuous $q$-ultraspherical polynomials at $x = 0$ can be gotten for instance from the recurrence relation (5.10) and are

$$C_{2k}(0; q^m|q) = (-1)^k \frac{(q^{2k}; q^2)_k}{(q^k; q^2)_k},$$

$$C_{2k+1}(0; q^m|q) = 0.$$  

(5.15)
If \( n = 2k \), setting \( x = 0 \) in (5.13) will then immediately yield \( f_{2k,m} \). In order to obtain \( f_{2k+1,m} \), one first applies \( \tau \) to both sides of (5.13) knowing that

\[
\tau C_n(x; q^m|q) = -q^{-n/2} (1 - q^m) C_{n-1}(x; q^{m+1}|q),
\]

before taking \( x = 0 \). One thus supplements (5.12) with

\[
f_{\ell,m} = -q^{-(\ell-m)/2} \frac{(1 - q^m)(1 - q^{\ell-m})(1 - q^{\ell-m-1})q^{2m+1}}{(1 - q^{2m})(1 - q^{2m+1})(1 - q^\ell)}, \tag{5.17a}
\]

\[
h_{\ell,m} = q^{-(\ell-m)/2} \frac{(1 - q^m)(1 - q^{\ell+m})(1 - q^{\ell+m+1})}{(1 - q^{2m})(1 - q^{2m+1})(1 - q^\ell)}. \tag{5.17b}
\]

The action of \( P_- \) on the basis functions \( Q^\ell_m \) is similarly obtained and one has

\[
P_- Q^\ell_m = r_{\ell,m} Q^{\ell-1}_{m-1} + s_{\ell,m} Q^{\ell+1}_{m-1}, \tag{6.18}
\]

with

\[
\begin{align*}
r_{\ell,m} &= q^{(\ell-m)/2} \frac{(1 - q^{2m-1})}{(1 - q^{\ell+m-1})} \left[ 1 + \frac{(1 - q^{2m-1})(1 - q^{\ell-m+1})}{4(1 - q^{m-1})(1 - q^\ell)} \right], \tag{5.19a} \\
s_{\ell,m} &= -q^{(\ell-3m+2)/2} \frac{(1 - q^{2m-2})(1 - q^{2m-1})}{4(1 - q^{m-1})(1 - q^\ell)}. \tag{5.19b}
\end{align*}
\]

### 6. A q-ANALOG OF THE FOURIER-GEGENBAUER EXPANSION

We shall now make use of the results obtained in the previous sections to derive a q-analog of the Fourier-Gegenbauer expansion (1.21). Take the \( \mathcal{E}_q \) exponential of \( P_0 = x \) and consider its matrix elements in the basis \( Q^\ell_m \):

\[
\mathcal{E}_q(x; -i, b/2) Q^\ell_m = \sum_{\ell' m'} U^{\ell',\ell}_{m',m}(b) Q^{\ell'}_{m'}. \tag{6.1}
\]

Recalling that \( K Q^\ell_m = q^{m-1/2} Q^\ell_m \), we must have \( U^{\ell',\ell}_{m',m}(b) = U^{\ell',\ell}_{m,m}(b) \delta_{mm'} \), since \( K = q^{-1/2} T_t \) and \( P_0 \) commute. If we specialize (6.1) to the case \( m = \ell \), taking into account (5.11) and setting \( U^{\ell+k,\ell}_{\ell,\ell}(b) \equiv W^k_{\ell}(b) \), we find

\[
\mathcal{E}_q(x; -i, b/2) Q^\ell = \sum_{k=0}^{\infty} W^k_{\ell}(b) Q^{\ell+k}, \tag{6.2}
\]
where from (5.7), we know that \( Q_{q}^{\ell} = t^{\ell} \). We shall see that (6.2) contains the identity we are looking for. We shall exploit the representation of the \( q \)-deformation of \( E(3) \) given in Section 5, acting in turn with \( P_{+} \) and \( J_{+} \) on both sides of (6.2), to derive the recursion relations that the matrix elements of \( E_{q}(x; -i, b/2) \) obey and in the end, to determine these \( W_{q}^{k}(b) \). Let us start with \( P_{+} = t \). Since \( t \) \( Q_{q}^{\ell} = Q_{q}^{\ell+1} \), we have on the one hand

\[
P_{+} \ E_{q}(x; -i, b/2) \ Q_{q}^{\ell} = \ E_{q}(x; -i, b/2) \ Q_{q}^{\ell+1} = \sum_{k=0}^{\infty} W_{q}^{k}(b) \ Q_{q}^{\ell+1+k} , \tag{6.3}
\]

with the last equality following from (6.2). On the other hand,

\[
P_{+} \ E_{q}(x; -i, b/2) \ Q_{q}^{\ell} = \sum_{k=0}^{\infty} W_{q}^{k}(b) \ P_{+} \ Q_{q}^{\ell+k} ; \tag{6.4}
\]

using the action of \( P_{+} \) on the basis functions \( Q^{\ell}_{m} \), given in (5.12), we can write

\[
P_{+} \ E_{q}(x; -i, b/2) \ Q_{q}^{\ell} = \sum_{k=0}^{\infty} \left[ W_{q}^{k+2}(b) f_{\ell+k+2,\ell} + W_{q}^{k}(b) h_{\ell+k,\ell} \right] Q_{q}^{\ell+k+1} , \tag{6.5}
\]

with \( f_{\ell,m} \) and \( h_{\ell,m} \) as in (5.17). Equations (6.3) and (6.5) are then seen to imply the relation:

\[
W_{q}^{k+1}(b) = W_{q}^{k+2}(b) f_{\ell+k+2,\ell} + W_{q}^{k}(b) h_{\ell+k,\ell} . \tag{6.6}
\]

Now act similarly on both sides of (6.2) with \( J_{+} \) to find:

\[
J_{+} \ E_{q}(x; -i, b/2) \ Q_{q}^{\ell} = \sum_{k=0}^{\infty} W_{q}^{k}(b) J_{+} \ Q_{q}^{\ell+k} . \tag{6.7}
\]

From the definition (5.3a) of \( J_{+} \) and the fundamental property (3.4) of \( E_{q} \), it is immediate to check that

\[
J_{+} \ E_{q}(x; -i, b/2) \ Q_{q}^{\ell} = \frac{i}{2} \frac{q^{-(\ell-1)/2}}{1-q} \ E_{q}(x; -i, b/2) \ Q_{q}^{\ell+1} = \frac{i}{2} \frac{q^{-(\ell-1)/2}}{1-q} \sum_{k=0}^{\infty} W_{q}^{k}(b) \ Q_{q}^{\ell+k+1} , \tag{6.8}
\]

with the second equality resulting from using (6.2) anew. The action of \( J_{+} \) on \( Q_{q}^{\ell} \) is given in (5.8a), and yields

\[
\sum_{k=0}^{\infty} W_{q}^{k}(b) J_{+} \ Q_{q}^{\ell+k} = \sum_{k=0}^{\infty} W_{q}^{k+1}(b) \left[ \frac{q(1-q^{-k})(1-q^{2\ell+k+1})}{(1-q)(1-q^{2\ell+1})(1+q^{\ell})} \right] Q_{q}^{\ell+k+1} . \tag{6.9}
\]
These results imply:

\[ W_{k+1}^\ell(b) = \frac{2i}{b} q^{(2\ell+3)/4} \frac{(1 - q^{-k-1})(1 - q^{2\ell+k+1})}{(1 - q^{2\ell+1})(1 + q^2)} W_{\ell+1}^k(b) . \] (6.10)

It will now prove convenient to use instead of \( W_{\ell+1}^k(b) \), the functions \( Y_{\ell+1}^k(b) \), which differs from \( W_{\ell+1}^k(b) \) by a multiplicative constant:

\[ W_{\ell+1}^k(b) = i^k q^{k(2\ell+3)/4} \frac{(q^{2\ell}; q)_k (1 - q^{k+\ell})}{(q; q)_k} Y_{\ell+1}^k(b) . \] (6.11)

Upon replacing \( W_{\ell+1}^k(b) \) in (6.6) by the r.h.s. of (6.10) and inserting the explicit expressions for \( f_{\ell+k+2,\ell} \) and \( h_{\ell+k,\ell} \), one arrives at the following relation

\[ \frac{2}{b} (1 - q^{k+\ell+1}) Y_{\ell+1}^k(b) = q^{k+\ell+1} Y_{\ell+2}^k(b) + Y_{\ell}^k(b) . \] (6.12)

It will determine \( Y_{\ell}^k(b) \) up to a factor not indexed by \( k \). Indeed, if we cast \( Y_{\ell}^k(b) \) in the form

\[ Y_{\ell}^k(b) = Y_{k+\ell}(b) V_{\ell}(b) , \] (6.13)

the term \( V_{\ell}(b) \) factors out from (6.12) which becomes

\[ \frac{2}{b} (1 - q^{s+1}) Y_{s+1}(b) = q^{s+1} Y_{s+2}(b) + Y_{s}(b) , \quad s = k + \ell . \] (6.14)

Comparing with (1.12), we recognize in (6.14), the three-term recurrence relation of the \( q \)-Bessel function \( J_{k+2}^{(2)}(b; q) \).

As for the function \( V_{\ell}(b) \), it can be obtained by substituting (6.11) in (6.10) with

\[ Y_{\ell}^k(b) = J_{k+\ell}^{(2)}(b; q) V_{\ell}(b) . \] (6.15)

This gives

\[ \frac{b}{2(1 - q^\ell)} V_{\ell+1}(b) = V_{\ell}(b) , \] (6.16)

which is readily found to imply that

\[ V_{\ell}(b) = \left( \frac{2}{b} \right)^\ell (q; q)_{\ell-1} X(b) , \] (6.17)

with \( X(b) \), a function yet to be specified. At this point, putting everything together, equation (6.2) has been shown to entail the expansion:

\[ E_q(x; i, b/2) = X(b) \sum_{k=0}^{\infty} i^k q^{k^2/4} (1 - q^{k+\ell}) \frac{(q; q)_k}{(q^2; q)_k} \left( \frac{2}{b} \right)^\ell J_{\ell+k}^{(2)}(b) C_k(x; q^\ell | q) . \] (6.18)
The function $X(b)$ can now be determined by taking the limit $q^\ell \to 0$ in this equation. Note from (1.18) and (1.19), that the continuous $q$-ultraspherical polynomials $C_n(x; q^\ell |q)$ reduce to the continuous $q$-Hermite polynomials when $q^\ell \to 0$:

$$C_n(q; 0|q) = \frac{H_n(x|q)}{(q;q)_n}.$$  \hfill (6.19)

Recalling formula (1.13) for the large order behavior of the functions $J_\nu^{(2)}(z; q)$, (6.18) is seen to yield

$$\mathcal{E}_q(x; -i, b/2) = X(b) \sum_{k=0}^\infty i^k \left( \frac{b}{2} \right)^k \frac{q^{k^2/4}}{(q; q)_k} H_k(x|q),$$  \hfill (6.20)

when $q^\ell \to 0$. Upon comparing (6.20), with the generating relation (4.10) for the continuous $q$-Hermite polynomials which was algebraically derived in Section 4, we see that

$$V(b) = \left( -b^2/4; q \right)^{-1}_{\infty}.$$  \hfill (6.21)

We hence finally arrive at the following identity:

$$\mathcal{E}_q(x; -i, b/2) = \frac{(q; q)_\infty (2/b)^\ell}{(b^2/4; q^2)_\infty (q^\ell; q)_\infty} \times \sum_{k=0}^\infty i^k q^{k^2/4} (1 - q^{k+\ell}) J_{\ell+k}^{(2)}(b; q) C_k(x; q^\ell |q).$$  \hfill (6.22)

This expansion formula which was here constructively derived using algebraic methods, was first obtained in Ref.[14].

7. CONCLUDING REMARKS

We have undertaken in this paper a study of the continuous $q$-orthogonal polynomials from the algebraic point of view. Our strategy has been to proceed in analogy with the Lie theory approach to ordinary special functions and to construct models where the continuous $q$-polynomials occur as matrix elements or basis vectors in representations of $q$-algebras. Divided difference operators have been seen to play a fundamental role in these models as should have been expected considering the difference equations that the continuous $q$-polynomials obey. It has also been observed that the $q$-exponential $\mathcal{E}_q$ is the one to use in this context to mimic the classical exponential map.

We examined first the continuous $q$-Hermite polynomials. They have no free parameters and thus lie at the bottom of the Askey tableau. They were shown to realize a representation space for the $q$-Heisenberg algebra. It was immediate to derive the generating
relation (4.10) in this framework. We then proceeded to discuss similarly the continuous \(q\)-ultraspherical polynomials which involve one free parameter. These functions were shown to arise in the basis vectors of a representation space of a \(q\)-deformation of \(E(3)\). Using these results and “bootstrapping” from the \(q\)-Hermite to the \(q\)-ultraspherical case, we derived algebraically the \(q\)-analog of the Fourier-Gegenbauer expansion.

It would be of interest to interpret other \(q\)-continuous polynomials in an analogous fashion and in particular, working our way up in the Askey classification, to determine the symmetry algebras of the higher families of continuous \(q\)-polynomials. Work along those lines is in progress.\(^{23,24}\)

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