Space-Time Quantization and Matrix Model

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ABSTRACT

In order to get the general framework describing a nonlocalizable object beyond the bilocal field theory early proposed by Markov and Yukawa, the quantization of space-time is reconsidered and further developed. Space-time quantities are there not only noncommutative with $U$-field describing the nonlocalizable object, as in the bilocal field theory, but also become noncommutative among themselves. Under the $U$-field representation, where the basis vectors of representation are chosen to be eigenvectors of operator $U$, space-time quantities get a matrix representation of infinite dimension in general. Field equation is considered, which determines the relation between space-time quantities and $U$-field. The possible inner relation between the recent topics of matrix model in superstring theory and the present approach is discussed.

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1. Introduction

Fifty years ago, Yukawa\textsuperscript{[1]} proposed the nonlocal field theory, according to the preceding idea by Markov\textsuperscript{[2]}, discarding the restriction that the field quantities are simply the functions of space-time coordinates. The field concept is there extended so that the field, let us call it $U$ in general, is no longer commutative with the space-time coordinates;

$$[A] \quad [U, x_\mu] \neq 0. \quad (1.1)$$

The motivation for this attempt was, on the one hand, to remove the long-pending problem of the so-called divergence difficulty inherent in the local field theory, and on the other hand, to give the unified description of elementary particles, in the face with the unexpected discovery of the mu-meson in the development of the meson theory, i.e., the first appearance of generation structure of elementary particles. It was expected that the condition (1.1) will enable us to provide the space-time uncertainty or the nonlocal extension to the field $U$, beyond the local field concept based on the point particle model of elementary particles.

In fact, Yukawa showed that the $U$-field obeying (1.1) becomes the so-called bilocal field described by the two-point function as

$$\langle x'_{\mu}|U|x''_{\mu}\rangle \equiv U(x'_{\mu}, x''_{\mu}) \equiv U(X_{\mu}, r_{\mu}), \quad (1.2)$$

with $X_{\mu} \equiv (x''_{\mu} + x'_{\mu})/2$ and $r_{\mu} \equiv x''_{\mu} - x'_{\mu}$, by taking the space-time coordinate representation where the basis vectors of representation consist of states with the definite values of four space-time coordinates like $|x'_{\mu}\rangle$ or $|x''_{\mu}\rangle$. He further asserted that the different modes of functional behavior with respect to the new internal degrees of freedom of the relative coordinate $r_{\mu}$ have the possibility of explaining the origin of the variety of elementary particles with different masses and (integer) spins, although he did not succeed to find the satisfactory way of introducing the interactions among bilocal fields.
In the present paper, we start with presenting an elementary, but very serious
question why the proposition [A] in (1.1) which asserts in the general form the non-
commutativity of the field $U$ with the space-time coordinates remains to provide
only the bilocal field suitable for the description of the system like the two-quark
system, but not the multilocal fields. In fact, the latter fields which correspond
to the existing multi-quark systems had to be introduced by hand merely as the
formal extension of the bilocal field. We wonder why the proposition (1.1) does
not cover the extended objects more in general such as string, membrane and so
forth.

The answer, however, seems to be rather simple. In fact, it turns out that from
the assumption (1.1) one arrives at the bilocal field immediately after one takes
the space-time coordinate representation, where the four space-time coordinates
are presumed to be commutative with each other and all described by diagonal
matrices at the same time. Therefore, in order to go over the limit of the bilocal
field under the general proposition (1.1), let us consider the possibility that space-
time coordinates are noncommutative:

$$ [B] \quad [x_\mu, x_\nu] \neq 0, \quad (1.3) $$

which clearly makes impossible to take the space-time coordinate representation
naively as was done in (1.2).

In the present paper, we wish to investigate the general framework to describe
the extended object under the propositions [A] in (1.1) and [B] in (1.3). The latter
proposition [B] is well-known in the old attempts of the space-time quantization.
Snyder$^3$ once proposed the idea of “Quantized Space-Time”, nearly at the same
time when the nonlocal field theory was proposed by Yukawa. Later on, Yang$^4$ gave
the special attention to the Snyder’s work and presented an argument to modify
it.

In the next section 2, we reconsider the Snyder’s work as one possible realization
of the proposition [B], by supplementing the argument given by Yang. In the
section 3, we study the propositions [A] and [B] together and present, instead of the space-time coordinate representation, newly $U$-field representation as a realization of algebra obeying [A] and [B]. In this representation, $U$-field becomes a diagonal matrix, while space-time quantities tend to the infinite dimensional matrices in general. We further investigate the field equation, which relates $U$-field to the quantized space-time. The final section 4 is devoted to discussions in which we consider the possible relation between the present approach and the matrix model proposed in the recent string theory.

2. Space-Time Quantization

As was mentioned in Introduction, with the aim of getting the finite and Lorentz-invariant theory, Snyder \[^3\] presumed the five-dimensional (de Sitter) space in the background of real space-time and attempted to express space-time quantities as linear differential operators on the former space, which are noncommutative with each other. Nearly two decades later, Yang \[^4\] gave special attention to the Snyder’s work in the point that space coordinates have discrete eigenvalues and time coordinate continuous, but the theory is Lorentz-invariant.

In what follows, let us reconsider the Snyder’s theory in supplementing Yang’s argument. Yang proposed to modify the background space, from the original five-dimensional (de Sitter) space to six-dimensional one ($\xi_0, \xi_1, \xi_2, \xi_3, \eta, \tau$), constrained as

$$\xi_0^2 + \xi_1^2 + \xi_2^2 + \xi_3^2 + \eta^2 + \tau^2 = \text{const.} \quad (2.1)$$

Space-time coordinates are defined:

$$X_i = i(\xi_i \frac{\partial}{\partial \eta} - \eta \frac{\partial}{\partial \xi_i}), \quad (2.2)$$

$$X_0 = i(\xi_0 \frac{\partial}{\partial \eta} + \eta \frac{\partial}{\partial \xi_0}) \quad (2.3)$$

in the same way as Snyder. The four-momentum coordinates are similarly defined,
replacing $\eta$ with $\tau$ in the above equations as

\[ P_i = -i(\xi_i \frac{\partial}{\partial \tau} - \tau \frac{\partial}{\partial \xi_i}), \quad (2.4) \]

\[ P_0 = -i(\xi_0 \frac{\partial}{\partial \tau} + \tau \frac{\partial}{\partial \xi_0}), \quad (2.5) \]
differently from Snyder. Actually, Snyder considered them, not as quantized ones, but in the following way; $P_i = \xi_i/\eta$ and $P_0 = \xi_0/\eta$. In the above expressions, one easily finds that both $X_i$ and $P_i (i = 1, 2, 3)$ have integer eigenvalues because they are angular momentum operators with respect to $\xi_i - \eta$ and $\xi_i - \tau$ planes, respectively.

At this point, one wonders how the Lorentz invariance is guaranteed in conformity with the discrete eigenvalues of each $i$-th component as well as the continuous eigenvalues of 0-th component. The clue to this question lies in the fact that they are all noncommutative quantities. This thing is suggested from the familiar quantum-mechanical angular momenta, which have discrete eigenvalues, but remain to be of rotation-invariant character.

In fact, one can confirm the above fact by calculating the following commutation relations;

\[ [X_i, X_j] = -iL_{ij}, \quad (2.6) \]

\[ [X_i, X_0] = iM_i, \quad (2.7) \]

and

\[ [P_i, P_j] = -iL_{ij}, \quad (2.8) \]

\[ [P_i, P_0] = iM_i, \quad (2.9) \]

with $L_{ij}$ and $M_i$ defined by

\[ L_{ij} = i(\xi_i \frac{\partial}{\partial \xi_j} - \xi_j \frac{\partial}{\partial \xi_i}), \quad (2.10) \]
\[ M_i = i(\xi_0 \frac{\partial}{\partial \xi_i} + \xi_i \frac{\partial}{\partial \xi_0}). \]  

Further,

\[ [X_i, P_j] = i\delta_{ij}N, \]  

\[ [X_0, P_0] = -iN, \]  

\[ [X_i, P_0] = [P_i, X_0] = 0 \]  

with \( N \) defined by

\[ N = i(\eta \frac{\partial}{\partial \tau} - \tau \frac{\partial}{\partial \eta}). \]

It is quite interesting here to note that the operator \( N \), i.e., angular momentum with respect to \( \eta - \tau \) plane, concerns the origin of Planck constant \( h \) in quantum mechanics, under the suitable choice of scale units of \( X_{i,0} \) and \( P_{i,0} \) such as Planck length, which were omitted so far. Furthermore, one notices that \( N \) plays the role of junction between \( X \) and \( P \);

\[ [X_i, N] = -iP_i, \ldots \]

With respect to \( L_{ij} \) and \( M_i \), one finds that they are nothing but the six generators of Lorentz transformation, which constitute the well-known Lorentz-Algebra with space-time quantities. It turns out that the transformation corresponds to the special transformation in the six-dimensional space with \( \eta \) and \( \tau \) fixed in (2.1) and guarantees the Lorentz invariance of the present theory.

In closing this section, one should remark that the fifteen operators

\[ \mathcal{R}_{15} \equiv (X_\mu, P_\mu; L_{ij}, M_i; N) \]

with \( \mu = (i,0) \), constitute as a whole a Lie ring, which characterizes the structure of our quantized space-time.
3. $U$-field Representation and Matrix Model of Space-Time

In the preceding section, we have studied the modified Snyder theory as one interesting model of space-time quantization according to the proposition [B] stated in Introduction. In the present section, we consider the proposition [A], i.e., the noncommutative structure of field $U$ with space-time quantities. If we assume, for the sake of explanation, the space-time structure to be expressed by $\mathcal{R}_{15}$, (1.1) becomes written more explicitly as

$$[A'] \quad [U, \mathcal{R}_{15}] \neq 0. \quad (3.1)$$

One sees, however, that the above equation by itself is insufficient to qualify the operator $U$, which is expected as a unified field describing ultimately all the elementary particles and fundamental forces, as the present string field theory aims. In the conventional field theory, the field equation plays the role of determining the relation between the field and space-time structure, although the space-time structure is already fixed as given, except for the gravitational equation. In the present case, it is desirable that the quantized space-time structure is determined simultaneously with $U$, as in general theory of relativity.

Before entering into the discussion of this problem, it is important to notice the general feature characteristic of the proposition [A']. As was remarked in Introduction, it is now impossible to take the space-time coordinate representation to express the noncommutative relations such as $\mathcal{R}_{15}$. Therefore, instead of this, we take a novel representation, let us call it $U$-field representation, in which the basis vectors of representation are chosen to be eigenstates of the operator $U$. Let us assume that eigenvalues are discrete, for the sake of simplicity, i.e.,

$$U|n\rangle = u_n|n\rangle. \quad (3.2)$$

* We do not exclude the possibility that some ones of $\mathcal{R}_{15}$, for instance, $N$, are commutative with $U$. 

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$U$ is represented now by a diagonal matrix of infinite-dimension, in general;

$$U = \text{diag}(u_1, u_2, u_3, \cdots). \quad (3.3)$$

Here the (ortho-normal) eigenvector $|n\rangle (n = 1, 2, 3, \cdots)$ with eigenvalue $u_n$ may be imagined to describe the $n$-th excitation mode of $U$, whose space-time structure becomes explicit through the matrix representation of the space-time Algebra such as $\mathcal{R}_{15}$. Each space component $X_i$, for instance, is represented by the matrix

$$X_i = \begin{bmatrix} X_{i'n''} \end{bmatrix} \quad (3.4)$$

with the matrix element

$$X_{i'n''} = \langle n'|X_i|n'' \rangle. \quad (3.5)$$

The diagonal matrix element $\langle n|X_i|n \rangle$ must denote the space coordinate of the $n$-th excitation mode of $U$ with fluctuation $\Delta X_i^{(n)}$ defined through

$$\langle \Delta X_i^{(n)} \rangle^2 \equiv \langle n|(X_i - X_i^{(mn)})^2|n \rangle = \sum_{n' \neq n} |X_i^{nn'}|^2. \quad (3.6)$$

Now we are in a position to consider the field equation of $U$, which more explicitly relates $U$ to the space-time structure, as stated above. Unfortunately, we have no reliable principle to determine the form of the equation, but the past nonlocal field theories such as the bilocal or the string field theory, though they are not necessarily based on the quantized space-time, seem strongly to suggest the following form, by neglecting the interaction terms;

$$[P_\mu, [P_\mu, U]] + \mathcal{M}^2 U = 0. \quad (3.7)$$

With respect to the mass-squared operator $\mathcal{M}^2$, string theory suggests that it
involves the angular momentum in the Lorentz-invariant form

\[ L_{ij}^2 - M_i^2 (= [X_\mu, X_\nu]^2). \]  

Furthermore, if we want to get the half-integer spin mode of \( U \)-field, it becomes necessary to introduce relativistic spin-angular momentum quantities beside \( L_{i,j}, M_i \), which are known to be described in terms two kinds of Euler’s angles (left and right) \[^5\] and serve to linearize the field equation as Dirac equation. Needless to say, the above field equation can be transformed into the matrix form, which works to constrain the matrix elements of space-time operators as well as the eigenvalues \( u_n \)'s.

4. Discussions

In this paper, we started with the propositions [A] and [B] so as to get the general framework describing the nonlocalizable object \( U \) beyond the bilocal field theory and arrived at the noncommutative matrix representation of space-time quantities under \( U \)-field representation.

At this point, it is quite interesting to notice the recent development in the string theory,\[^6\] which also aims a unified theory of everything. It is noticeable there that the apparently different string models investigated so far seem to be equivalently connected with each other in terms of duality, and further surprisingly space-coordinates appear in the form of noncommutative and infinite-dimensional matrices, likely in our present theory, although the string theory seems to start with the conventional space-time concept.

Therefore, it is quite important to clarify the origin of the matrix-representation in both approaches. In the string theory, it appears as \( N \) infinite limit of \( N \times N \) matrix, and \( N \) denotes the number of the so-called D(dirichlet)0-branes as the fundamental constituents of the string system. It strongly suggests that the \( n \)-th basis vector in our \( U \)-field representation, \( |n\rangle \) (the \( n \)-th excitation mode of
nonlocalizable object $U$ characterized by its eigenvalue $u_n$), in general, corresponds to the various composite state of the infinite number of D0-branes (or D-particles) as the fundamental constituents of the string system. It is interesting to conjecture that there exists some kind of transformation which connects both representations beyond their apparent difference.

Furthermore, in the present paper, the argument of several important problems remains to be done, such as the second quantization of $U$-field or the fundamental equation of $U$-field involving interactions.

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