ON THE WIENER-PITT PHENOMENON FOR ALGEBRA OF RAJCHMAN MULTIPLIERS ON HARDY SPACE

PRZEMYSŁAW OHRYSKO, MICHAŁ WOJCIECHOWSKI, AND BARTŁOMIEJ ZAWALSKI

Abstract. We show that any positive Rajchman measure of Minkowski dimension 0 has a non-natural spectrum as an element of the multiplier algebra of $H^1_0(\mathbb{T})$. The proof is based on the estimation of the norm of the convolution operator given by a singular measure on $H^1_0(\mathbb{T})$.

1. Introduction

Let $M(\mathbb{T})$ denote the Banach algebra of all complex-valued Borel measures on the circle group equipped with the total variation norm and the convolution product. Let us recall that the Minkowski dimension of a measure $\mu \in M(\mathbb{T})$ is defined as the infimum of Minkowski dimensions of all Borel sets $A \subset \mathbb{T}$ with $\mu(A) \neq 0$. For $\beta \in [0, 1)$ we set $c_\beta(\mu) = \sup\{|\mu|(A) : \dim_M(\mu) \leq \beta\}$. We are going to discuss the properties of convolution operators $T_\mu$ associated with $\mu \in M(\mathbb{T})$ and defined on Hardy spaces via standard formula: $T_\mu(f) = \mu \ast f$. The classical (complex) Hardy space is:

$$H^1_0(\mathbb{T}) = \{f \in L^1(\mathbb{T}) : \int_{\mathbb{T}} f(t)e^{-int}dt = 0 \text{ for } n \leq 0\}.$$

Our main result is the following proposition.

Proposition 1. $\|T_\mu : H^1_0(\mathbb{T}) \to H^1_0(\mathbb{T})\| > C(\beta)c_\beta(\mu)$ for every real measure $\mu \in M(\mathbb{T})$ with Minkowski dimension $\beta \in [0, 1)$.

It is well-known that the set of all real parts of functions from $H^1_0(\mathbb{T})$ form the so-called 'real Hardy space' which will be abbreviated with $\text{Re}H^1(\mathbb{T})$. Moreover, for every $f \in \text{Re}H^1(\mathbb{T})$ there exists a unique $\tilde{f} \in \text{Re}H^1(\mathbb{T})$ such that $f + i\tilde{f} \in H^1_0(\mathbb{T})$ and $\|\tilde{f}\|_1 \leq C\|f\|_1$. Proposition 1 is an easy corollary of its 'real' counterpart.

Proposition 2. $\|T_\mu : \text{Re}H^1(\mathbb{T}) \to \text{Re}H^1(\mathbb{T})\| > C'(\beta)c_\beta(\mu)$ for every measure $\mu \in M(\mathbb{T})$ with Minkowski dimension $\beta \in [0, 1)$.

Indeed, let $f \in \text{Re}H^1(\mathbb{T})$ with $\|f\|_1 = 1$ be such that $\|T_\mu f\|_1 > C'(\beta)c_\beta(\mu)$. Then $\|f + i\tilde{f}\|_1 \leq 1 + C$ while $\|T_\mu(f + i\tilde{f})\|_1 \geq \|T_\mu(f)\|_1$ (as $iT_\mu(f)$ is purely imaginary). Therefore $\|T_\mu : H^1_0(\mathbb{T}) \to H^1_0(\mathbb{T})\| \geq C'(\beta)c_\beta(\mu) \cdot (1 + C)^{-1}$. We will mainly work with the space $\text{Re}H^1(\mathbb{T})$ as this framework permits the use of the atomic decomposition and various combinatorial arguments on contrary to the analytic

2010 Mathematics Subject Classification. Primary 43A10; Secondary 43A25.

Key words and phrases. Natural spectrum, Wiener–Pitt phenomenon, multipliers, Hardy spaces.
world of $H^1_0(\mathbb{T})$.

The main motivation for the investigation in this direction is the problem of the natural spectrum. In the most classical formulation for $M(\mathbb{T})$ the measure $\mu \in M(\mathbb{T})$ is said to have a natural spectrum, if

$$\sigma(\mu) = \overline{\hat{\mu}(\mathbb{Z})}$$

i.e. the spectrum (the set of all $\lambda \in \mathbb{C}$ such that $\mu - \lambda \delta_0$ is not invertible) is equal to the closure of the range of its Fourier-Stieltjes transform (this terminology was introduced by M. Zafran in [Zaf1]). However, it is well known that not all measures have a natural spectrum and this situation was called Wiener-Pitt phenomenon (see [WP], [Sc] and [Wi]). Moreover, there are also examples (check the Chapter on Riesz products in [GM]) of measures with a non-natural spectrum with the additional property that the Fourier-Stieltjes coefficients vanish at infinity (such measures are called Rajchman and the set of all such measures will be abbreviated $M_0(\mathbb{T})$).

The main value of Proposition 1 and Proposition 2 is establishing a measure-theoretic property which ensures that a measure $\mu \in M_0(\mathbb{T})$ does not have a natural spectrum as a multiplier on $H^1_0(\mathbb{T})$ ($\sigma_{B(H^1_0(\mathbb{T}))}(T_\mu) \neq \overline{\hat{\mu}(\mathbb{N})}$). The proof of the existence of this phenomenon was given by M. Zafran in [Zaf2] but his argument did not expose any particular measure-theoretic property responsible for it and the example was given via random construction. Noteworthy, if we drop the assumption $\mu \in M_0(\mathbb{T})$ we can give an explicit example of a measure giving a multiplier on $H^1(\mathbb{T})$ with a non-natural spectrum via classical argument of C.C.Graham from [Grah].

**Theorem 3.** Let $\mu = \prod_{k=1}^{\infty} \left(1 + \cos(3^k t)\right) \in M(\mathbb{T})$ be the standard Riesz product. Then $\sigma_{B(H^1(\mathbb{T}))}(T_\mu) \neq \overline{\hat{\mu}(\mathbb{N})}$.

**Proof.** Assume toward the contradiction that

$$\sigma_{B(H^1(\mathbb{T}))}(T_\mu) = \overline{\hat{\mu}(\mathbb{N})} = \{0\} \cup \left\{ \frac{1}{2^n} : n \in \mathbb{N} \right\} \cup \{1\}.$$  

Let $A, B \subset \mathbb{C}$ be two open disjoint subsets of the complex plane such that $\sigma_{B(H^1(\mathbb{T}))}(T_\mu) \subset A \cup B$, $\frac{1}{4} \in A$ and $\sigma_{B(H^1(\mathbb{T}))}(T_\mu) \setminus \left\{ \frac{1}{4} \right\} \subset B$. Define a function $f : A \cup B \to \mathbb{C}$ by $f|_A = 1$ and $f|_B = 0$. Then by the functional calculus the multiplier $T := f(\mu)$ is an idempotent multiplier and $\hat{T}(n) = 1$ iff $\hat{\mu}(n) = \frac{1}{4}$ (here $\hat{T}(n)$ is simply the value of the multiplier associated to $T$ at the point $n$). By the construction of Riesz product we clearly see that:

$$\left\{ n : \hat{\mu}(n) = \frac{1}{4} \right\} = \left\{ n \in \mathbb{N} : n = \pm 3^{k_1} \pm 3^{k_2}, \text{ for distinct } k_1, k_2 \in \mathbb{N} \right\}.$$  

However, idempotent multipliers on $H^1(\mathbb{T})$ were identified by Klemes in [K] as $0 - 1$ sequences $(a_n)_{n \in \mathbb{N}}$ such that the set $\{ n : a_n = 1 \}$ belongs to the coset ring generated by arithmetic progressions and lacunary sets. As our set is not of that form we get the desired contradiction. □
2. Combinatorial arguments

Let \( \mu \in M(\mathbb{T}) \) be a probability Borel measure on the circle identified with the segment \([0, 1]\). For \( n = 0, 1, 2, \ldots \), let \( \Sigma_n \) denote the algebra of sets generated by dyadic intervals of length \( 2^{-n} \) and let \( F_n \) denote the set of its atoms. Let \( \mathcal{M}(\mu) \) denote the dyadic martingale defined by \( m(\omega) = \int_\omega d\mu = \mu(\omega) \) for \( \omega \in F_n, n = 0, 1, \ldots \). We identify \( \mathcal{M}(\mu) \) with a non-negative function on the set of vertices of a binary tree. By \( \gamma(\omega) \) denote the finite path of the tree joining \([0, 1] \in F_0 \) with \( \omega \). For any fixed \( k \in \mathbb{N} \) we denote by \( \mathcal{M}(\beta, k) \) the class of positive bounded measures such that

\[
\#\{\omega \in F_k : m(\omega) \neq 0\} \leq 2^{\beta k}.
\]

Let \( 0 > \alpha > -1 \) and \( 0 < \rho < 1 \) satisfy \( \alpha \rho + \beta < 0 \). We say that a vertex \( \omega \) is turbulent provided

\[
m(\omega_i) < 2^\alpha m(\omega) \quad \text{for} \quad i = 0, 1
\]

where \( \omega_0 \) and \( \omega_1 \) denote the left and right descendants of \( \omega \). By \( \Xi_n \) we denote the set of all turbulent vertices of \( F_n \).

**Lemma 4** (Mountain river lemma). For every \( k > k_0(\beta, \alpha, \rho) \) and every measure from \( \mathcal{M}(\beta, k) \) there exists \( n > \theta k \), where \( \theta = \theta(\beta, \alpha, \rho) \), such that

\[
\sum_{\omega \in \Xi_n} m(\omega) < \rho \|\mu\|.
\]

**Proof.** Without the loss of generality we are allowed to assume \( \|\mu\| = 1 \). Let us denote by \( S_r(\gamma(\omega)) \) the number of turbulent vertices \( \omega \in \bigcup_{j=r+1}^{k-1} F_j \) on the branch \( \gamma(\omega) \) not counting vertex \( \omega \). Then we have

\[
(1) \quad \sum_{n=r+1}^{k-1} \sum_{\omega \in \Xi_n} m(\omega) = \sum_{\omega \in F_k} m(\omega) S_r(\gamma(\omega)) \quad \text{for} \quad r \in \{0, \ldots, k-2\}.
\]

Setting \( \rho' = \rho'(\alpha, \beta, \rho) < \rho \) such that still \( \alpha \rho' + \beta < 0 \) we have

\[
\sum_{\omega \in F_k} m(\omega) S_r(\gamma(\omega)) = \sum_{j=0}^{[\rho'(k-r)]} \sum_{S_r(\gamma(\omega))=j} m(\omega) + \sum_{j=0}^{[\rho'(k-r)]} \sum_{S_r(\gamma(\omega))=j} m(\omega) = I + II.
\]

By definition the sets \( A_j = \{\omega \in F_k : S_r(\omega) = j\} \) for \( j \in \{0, \ldots, [\rho'(k-r)]\} \) are pairwise disjoint which gives the bound:

\[
I \leq [\rho'(k-r)] \sum_{j=0}^{[\rho'(k-r)]} \sum_{\omega \in A_j} m(\omega) \leq [\rho'(k-r)] \sum_{\omega \in F_k} m(\omega) \leq \rho'(k-r).
\]
If \( S_r(\gamma(\omega)) = j \), then \( m(\omega) \leq 2^{\alpha j} \) which enables us to estimate \( II \):

\[
II \leq \sum_{j=|\rho'(k-r)|+1}^{k} j^{2^{\alpha j}} \# \{ \omega \in F_k : S_r(\gamma(\omega)) = j \text{ and } m(\omega) \neq 0 \} \leq k \cdot 2^{\beta k} \sum_{j=|\rho'(k-r)|+1}^{k} 2^{\alpha j} = k \cdot 2^{\beta k} \cdot \frac{2^{\alpha(|\rho'(k-r)|+1)} - 2^{\alpha(k+1)}}{1 - 2^{\alpha}} \leq \frac{k2^\alpha}{1 - 2^{\alpha}} \cdot 2^{\alpha\rho'(k-r)+\beta k}.
\]

In the estimation we have essentially used two facts: measure belongs to \( M(\beta, k) \) and \( \alpha < 0 \).
Substituting \( r = \frac{k\alpha\rho + \beta}{\alpha\rho'} \) the exponent takes value \( 2k(\alpha\rho' + \beta) < 0 \) by the choice of \( \rho' \). Thus

\[
2^{\alpha\rho'(k-r)+\beta k} = 2^{2k(\alpha\rho' + \beta)} = \left(2^{2(\alpha\rho' + \beta)}\right)^k
\]

tends to zero exponentially with \( k \to \infty \). Therefore for sufficiently big \( k \) we obtain

\[
\frac{k2^\alpha}{1 - 2^{\alpha}} \cdot 2^{\alpha\rho'(k-r)+\beta k} \leq (\rho - \rho') \cdot (k - r).
\]

Using equation (1) and the bounds for \( I \) and \( II \) we finally get:

\[
\frac{1}{k - r} \sum_{n=r+1}^{k-1} \sum_{\omega \in \Xi_n} m(\omega) < \rho' + (\rho - \rho') = \rho.
\]

Hence there exists \( n > r \) such that \( \sum_{\omega \in \Xi_n} m(\omega) < \rho \). \( \square \)

**Remark 5.** The careful inspection of the argument used in the last proof shows that \( \theta \) can be chosen arbitrarily close to \( 1 - \frac{\beta}{|\alpha\rho'|} \).

The next lemma will be used in the proof of Proposition 2 to isolate the singular part of a measure. Let \( \dim_M(\mu) < \beta < 1 \). Then there exists a set \( A \subset [0,1] \) with \( \dim_M(A) \leq \beta \) such that \( \mu(A) \geq c_\beta(\mu) > 0 \). It follows that there exists a collection of disjoint intervals \( C_\delta \) satisfying:

1. \( |\omega| = \delta \) for \( \omega \in C_\delta \),
2. \( \sum_{\omega \in C_\delta} \mu(\omega) = c_\beta(\mu) + o(1) \) as \( \delta \to 0 \).

**Lemma 6.** Let \( \nu_1, \nu_2 \in M(\mathbb{T}) \) be two mutually singular positive measures with \( \dim_M(\nu_1) < \beta \). Then for every \( \tau > 0 \) there exists \( \delta > 0 \) and a family \( D \) of dyadic intervals such that:

1. \( |\omega| < \delta \) for \( \omega \in D \),
2. \( \sum_{\omega \in D} |\omega|^\beta < 1 \),
3. \( \sum_{\omega \in D} \nu_1(\omega) > c_\beta(\mu) - \tau \),
4. \( \nu_2(\bigcup_{\omega \in D} \omega + [-\delta, \delta]) < \tau \).

**Proof.** Let \( A \) be a compact set such that \( \nu_1(A) > c_\beta - \frac{\tau}{2} \) and \( \nu_2(A) = 0 \). Let \( D_n \) be a sequence of finite families of dyadic left open and right closed intervals such that \( \sum_{\omega \in D_n} |\omega|^\beta < 1 \) for \( n \in \mathbb{N} \) and

\[
\bigcap_n \left( \operatorname{int} \bigcup_{\omega \in D_n} \omega \right) = A.
\]
Also, let \( \delta_n \to 0 \) be any sequence such that \( \delta_n > \sup\{|\omega| : \omega \in D_n \} \). Then we have

\[
\bigcap_{n \in \mathbb{N}} \left( \text{int} \bigcup_{\omega \in D_n} \omega + (-\delta_n, \delta_n) \right) = A.
\]

By the regularity of measures \( D := D_n \) satisfies the condition of the lemma for sufficiently big \( n \).

Combining now two preceding lemmas we get the following statement.

**Lemma 7.** Let \( \nu_1, \nu_2 \in M(\mathbb{T}) \) be two mutually singular positive measures and \( \dim_M(\nu_1) < \beta \). Then for every \( \tau > 0 \) there exists \( \delta > 0 \) and a family \( A \) of dyadic intervals of equal length \( d \) such that:

1. \( \#A \cdot d^3 < 1 \),
2. \( \sum_{\omega \in A} \nu_1(\omega) > \frac{1}{2} c_\beta(\nu_1) - \tau \),
3. \( \nu_2 \left( \bigcup_{\omega \in A} \omega + [-\delta, \delta] \right) < \tau \).

**Proof.** It is enough to notice that the family constructed in Lemma 6 is contained in \( \bigcup_{j=1}^k \omega_k + (-\delta_0, \delta_0) \).

3. **The proof of the main result**

Let us start the proof of Proposition 2 with a technical preparation. By the regularity of measures and the assumption, for every \( \eta > 0 \) there exists a set \( A_\eta \subset \mathbb{T} \) with the following properties:

1. \( A_\eta \) is compact,
2. \( \mu|_{A_\eta} \geq 0 \),
3. \( \mu(A_\eta) > \frac{1}{2} c_\beta(\mu) - \eta \).

Let \( \mu_p := \mu|_{A_\eta} \) and \( \mu_r = |\mu||_{\mathbb{T}\setminus A_\eta} \).

It follows from Lemma 7 applied to \( \mu_p \) and \( \mu_r \) that for any fixed \( \eta > 0 \) there exists an infinite set \( \Lambda \subset \mathbb{N} \) such that for every \( k \in \Lambda \) there is a finite collection \( D_k \) of dyadic intervals of length \( 2^{-k} \) satisfying:

1. \( \sum_{\omega \in D_k} \mu_p(\omega) = \frac{1}{2} c_\beta(\mu) - \eta \),
2. \( \#D_k < 2^{\beta k} \),
3. \( \mu_r \left( \bigcup_{\omega \in D_k} \omega + [-2^{-\theta k+1}, 2^{-\theta k+1}] \right) < \eta \), where \( \theta \) is the constant from Lemma 4.

It follows from our construction that the measure \( \mu_k = \mu_p|_{\bigcup D_k} \) belongs to the class \( M(\beta, k) \) for \( k \in \Lambda \). Applying Lemma 4 for the martingale \( M(\mu_k) \) we get that there exists \( n > \theta k \) such that \( \sum_{\omega \in \Xi_n} m(\omega) < \frac{1}{2} c_\beta(\mu) \rho \). Let \( F_n = A_n \cup B_n \cup \Xi_n \), where \( A_n \) is the set of those \( \omega \in F_n \setminus \Xi_n \) that \( m(\omega_0) > m(\omega_1) \) and similarly \( B_n \) is the set of those \( \omega \in F_n \setminus \Xi_n \) that \( m(\omega_0) < m(\omega_1) \). Using Lemma 4 we may assume without the loss of generality:

\[
\sum_{\omega \in A_n} \left( m(\omega_0) - m(\omega_1) \right) > (2^{\alpha+1} - 1) \cdot \frac{1}{2} c_\beta(\mu) \frac{1-\rho}{2}.
\]

\(^1|\mu||_{\mathbb{T}\setminus A_\eta} \) is the restriction of \( |\mu| \) to the set \( \mathbb{T}\setminus A_\eta \).
Let $\varepsilon < 2^{-n-1}$ and put $h_n(t) = h_n'(-t)$, where

$$h_n = 2^{2n}(2^{-n-1} - \varepsilon)\chi_{[-\varepsilon,2^{-n-1}]} + 2^{2n}(2^{-n-1} + \varepsilon)\chi_{[2^{-n-1},2^{-n-1} - \varepsilon]}.$$  

Clearly $\|h_n\|_1 = 2^{-n} + \varepsilon^2 2^{2n+1}$, $\int_0^1 h_n = 0$ and $\|h_n\|_{\infty} \leq 2^n$ and $h_n$ is supported on the interval of length $2^{-n}$. Hence by Ch. Fefferman atomic decomposition we get $\|h_n\|_{H^1} \leq A$ for some universal constant $A$. Let $\omega = [b_\omega, e_\omega]$ for $\omega \in \mathcal{F}_n$. Define

$$E = \bigcup_{\omega \in A_n} [b_\omega, b_\omega + \varepsilon].$$  

We have

$$\|h_n \ast \mu\|_1 = \int_0^1 \left| \int_0^1 h_n(t-x)d\mu(x) \right| dt \geq \int_E \left| \int_0^1 h_n(t-x)d\mu(x) \right| dt = \sum_{\omega \in A_n} \int_{b_\omega}^{b_\omega + \varepsilon} \left| \int_0^1 h_n(t-x)d\mu(x) \right| dt = \sum_{\omega \in A_n} \int_{b_\omega}^{b_\omega + \varepsilon} \left| 2^{2n}(2^{-n-1} - \varepsilon)\mu([t - \varepsilon, t + 2^{-n-1}]) - 2^{2n}(2^{-n-1} + \varepsilon)\mu([t + 2^{-n-1}, t + 2^{-n} - \varepsilon]) \right| dt \geq \sum_{\omega \in A_n} \int_{b_\omega}^{b_\omega + \varepsilon} \left| 2^{2n}(2^{-n-1} - \varepsilon)\mu_k([t - \varepsilon, t + 2^{-n-1}]) - 2^{2n}(2^{-n-1} + \varepsilon)\mu_k([t + 2^{-n-1}, t + 2^{-n} - \varepsilon]) \right| dt + \int_0^\varepsilon 2^n \mu_t \left( \bigcup_{\omega \in A_n} ([t + b_\omega - \varepsilon, t + b_\omega + 2^{-n} - \varepsilon]) \right) dt.$$  

The last inequality follows from the fact that $\mu \geq \mu_k$ and $\mu = \mu_k$ on the intervals $[t + 2^{-n-1}, t + 2 \cdot 2^{-n-1} - \varepsilon]$, for $t \in [b_\omega, b_\omega + \varepsilon]$, $\omega \in A_n$. Since $\omega_0 \subset [t - \varepsilon, t + 2^{-n-1}]$ and $[t + 2^{-n-1}, t + 2^{-n} - \varepsilon] \in \omega_1$ for $\omega \in A_n$ and $t \in [b_\omega, b_\omega + \varepsilon]$, we get (below we denote by $\tilde{\omega}$
any fixed element of $D_k$ contained in $\omega$:

$$
\| h_n * \mu \|_1 \geq 2^n \varepsilon \sum_{\omega \in A_n} \left| (2^{-1} - \varepsilon 2^n) m(\omega_0) - (2^{-1} + \varepsilon 2^n) m(\omega_1) \right| + 2^n \varepsilon \cdot \mu_T \left( \bigcup_{\omega \in A_n} \omega + [-\varepsilon, 0] \right) \geq \\
\geq 2^n \varepsilon \left( 2^{-1} \sum_{\omega \in A_n} m(\omega_0) - m(\omega_1) \right) - \varepsilon 2^n \sum_{\omega \in A_n} m(\omega_0) - m(\omega_1) + 2^n \varepsilon \mu_T \left( \bigcup_{\omega \in D_k} [\omega - 2^{-\theta k + 1}, 2^{-\theta k + 1}] \right) \geq \\
\geq 2^n \varepsilon \left( 2^{a+1} - 1 \right) \cdot \frac{c_\beta(\mu) \cdot (1 - \rho)}{8} - \eta - \varepsilon 2^n c_\beta(\mu) + 2^n \varepsilon \eta.
$$

Passing with $\eta$ to 0 and taking $\varepsilon = (2^{a+1} - 1) \cdot \frac{1}{4}(1 - \rho)2^{-n-3}$ we get

$$
\| h_n * \mu \|_1 \geq 2^{-6}(2^{a+1} - 1)^2 \left( \frac{1}{4}(1 - \rho) \right)^2 \frac{1}{2} c_\beta(\mu).
$$

Setting $\alpha$ arbitrary close to $-\sqrt{\beta}$ and $\rho = |\alpha|$ we get Proposition 2 with a (non-optimal constant) $C'(\beta) = A^{-1} \cdot 2^{-11}(2^{1-\sqrt{\beta}} - 1)^2(1 - \sqrt{\beta})^2$.

**Remark 8.** The above estimation is attained for infinitely many $n$'s as for every $k \in \Lambda$ we get some $n > \theta k$.

### 4. Applications

Now we present an application od Proposition 1 to the spectral theory of Banach algebra of Rajchman multipliers on $H^1_0(\mathbb{T})$. Let us denote by $M(H^1_0(\mathbb{T}))$ the algebra of Fourier multipliers on $H^1_0(\mathbb{T})$. Obviously $(\hat{\mu}(n))_{n=1}^\infty$ is such a multiplier for every $\mu \in M(\mathbb{T})$.

**Corollary 9.** For every probability Borel measure $\mu \in M_0(\mathbb{T})$ supported on the set of lower Minkowski dimension equal to zero, the multiplier $\hat{\mu} = (\hat{\mu}(n))_{n=1}^\infty \in M(H^1_0(\mathbb{T}))$ has no natural spectrum, i.e. $\sigma_{M(H^1_0(\mathbb{T}))}(T_\mu) \neq \hat{\mu}(\mathbb{N})$.

**Proof.** It is well-known that there exists $c < 1$ such that $|\hat{\mu}(n)| < c$ for $n \in \mathbb{N}_+$. Hence $\hat{\mu}(\mathbb{N}) \subset B(0, c)$. On the other hand, the lower Minkowski dimension of $\mu^n$ equals 0 for every $n \in \mathbb{N}$. Since a convolution of probability measures is a probability measure, Proposition 1 gives $\|\hat{\mu}^n\|_{M(H^1_0(\mathbb{T}))} \geq c_1 > 0$. It follows that the spectral radius of $\hat{\mu}$ equals to 1, which is a contradiction. \[\square\]
Remark 10. The above corollary is meaningful: the construction of a Rajchman measure of Minkowski dimension 0 was presented in [B]. For completeness, let us give a sketch of the main result from [B]. Let \( \mathbb{L} \) be the set of Liouville numbers:

\[
\mathbb{L} = \{ x \in \mathbb{R} \setminus \mathbb{Q} : \forall n \in \mathbb{N} \exists q \in \mathbb{N} \quad \| q x \| \leq q^{-n} \},
\]

where \( \| x \| \) is the distance from \( x \) to the nearest integer.

Let \( \mathbb{P}_M \) be the set of prime numbers between \( M \) and \( 2M \), where \( M \in \mathbb{N} \). We choose a sequence of positive integers \( (M_k)_{k \in \mathbb{N}} \) with \( M_1 < 2M_1 < M_2 < 2M_2 < M_3 < 2M_3 < \ldots \) and define the set

\[
S_\infty = \bigcap_{k=1}^{\infty} \bigcup_{p \in \mathbb{P}_M} \{ x \in [0, 1] : \| px \| \leq p^{-1-k} \}.
\]

It is shown in [B] that \( S_\infty \) is a Cantor set of Hausdorff dimension zero. Later on the author proves that for suitable choice of a sequence \( (M_k)_{k \in \mathbb{N}} \) the set \( S_\infty \) supports a positive Rajchman measure. Moreover, \( S_\infty \setminus \mathbb{Q} \subset \mathbb{L} \).

We can proceed even further. For \( \Lambda \subset \mathbb{Z} \) let \( L^1_\Lambda(\mathbb{T}) \) denote the subspace of summable functions whose Fourier coefficients vanish outside \( \Lambda \). Let \( M_0(L^1_\Lambda(\mathbb{T})) \) denote the algebra of multipliers on \( L^1_\Lambda(\mathbb{T}) \). Obviously \( (\hat{\mu}(n))_{n \in \Lambda} \) is such a multiplier for every \( \mu \in M(\mathbb{T}) \). A refinement of the previous argument allows us to construct much smaller set \( \Lambda \) for which there is a multiplier with no natural spectrum.

Proposition 11. There exists \( b > 0 \) such that for every measure \( \mu \in M(\mathbb{T}) \) with Minkowski dimension \( \beta \in (0, 1) \), there is an arbitrary big \( n \in \mathbb{Z} \) such that if \( \Lambda = [n, bn^3] \), then \( \| T_\mu : L^1_\Lambda(\mathbb{T}) \to L^1_\Lambda(\mathbb{T}) \| > C''(\beta)c_\beta(\mu) \).

Proof. Fix \( \eta > 0 \). We show first that there are \( a, b > 0 \) such that for sufficiently big \( n \),

\[
\sum_{|j| \notin [a2^{2n}, b2^{2n}]} |\hat{h}_n(j)|^2 < \eta^2,
\]

where \( h_n \) is the function from Proposition 2 and \( n \) is chosen to satisfy norm estimation for the convolution operator on \( \text{Re}H^1(\mathbb{T}) \). Indeed, the distributional derivative of \( h_n \) is a discrete measure

\[
\nu_n = 2^{2n}(2^{-n-1} - \varepsilon)\delta_{-\varepsilon} - 2^n\delta_{2^{-n-1}} + 2^{2n}(2^{-n-1} + \varepsilon)\delta_{2^{-n-1} - \varepsilon}.
\]

Clearly \( |\nu_n| < 3 \cdot 2^n \) and therefore

\[
\sum_{|j| > b2^{2n}} |\hat{h}_n(j)|^2 < 9 \cdot 2^{2n} \sum_{|j| > b2^{2n}} j^{-2} < 18b^{-1}.
\]

On the other hand we have

\[
|\hat{h}_n(j)| = \left| \int_{-\varepsilon}^{2^{-n-\varepsilon}} h_n(t)(e^{2\pi i jt} - 1)dt \right| \leq 2^n \int_{-\varepsilon}^{2^{-n-\varepsilon}} |e^{2\pi i jt} - 1|dt \leq 2^n \int_{-\varepsilon}^{2^{-n-\varepsilon}} 2\pi |jt|dt < 2\pi |j|2^{-n}.
\]
Hence
\[ \sum_{|j|<a^{2^n}} |\hat{h}_n(j)|^2 < 4\pi^2 2^{-2n} \sum_{|j|<a^{2^n}} |j|^2 < 8\pi^2 \cdot a^3. \]
Choosing now \(a\) and \(b\) such that \(18b^{-1} + 8\pi^2a^3 < \eta^2\) we get our claim. Setting \(\phi_n(t) = \sum_{|j|\leq [a^{2^n},b2^{2n}]} \hat{h}_n(j)e^{2\pi j t}\), we get \(\|\phi_n\|_2 < \eta\). Therefore
\[ \|\hat{h}_n - \hat{\phi}_n\|_2 \leq \|\hat{h}_n\|_2 + \|\hat{\phi}_n\|_2 < C_1 \|h_n\|_2 + C_2 \|\phi_n\|_2 < C_1 + C_2 \eta, \]
where \(C_1\) and \(C_2\) are the norms of trigonometric conjugation operator in \(\text{Re}H^1(\mathbb{T})\) and \(L^2(\mathbb{T})\) (respectively). Clearly \(p_n = (h_n - \phi_n) + i(\hat{h}_n - \hat{\phi}_n)\) is a trigonometric polynomial with Fourier transform supported on the interval \([a^{2^n},b2^{2n}]\) with norm \(\|p_n\|_1 \leq 1 + C_1 + (1 + C_2)\eta\). We have now
\[ \|T_\mu \ast p_n\|_1 \geq \|T_\mu \ast (h_n - \phi_n)\|_1 \geq \|T_\mu \ast h_n\|_1 - \|T_\mu \ast \phi_n\|_1 \geq C'(\beta)c_\beta(\mu) - \eta. \]
Therefore
\[ \|T_\mu : L^1_{[a2^n,b2^{2n}]}(\mathbb{T}) \hookrightarrow L^1_{[a2^n,b2^{2n}]}(\mathbb{T})\| > \frac{C'(\beta)c_\beta(\mu) - \eta}{1 + C_1 + (1 + C_2)\eta}, \]
which is equivalent to the statement of the proposition. \(\square\)

**Corollary 12.** For every probability Borel measure \(\mu \in M_0(\mathbb{T})\) of the lower Minkowski dimension equal zero, there exists an arbitrarily rapidly increasing sequence \((k_n)_{n=1}^\infty\) such that for \(\Lambda = \bigcup_{n=1}^\infty [k_n,bk_n^3]\), the multiplier \(\hat{\mu} = (\hat{\mu}(n))_{n\in\Lambda} \in M(L_\Lambda^1(\mathbb{T}))\) has no natural spectrum, i.e. \(\sigma_{M(L_\Lambda^1(\mathbb{T}))}(\hat{\mu}) \neq \hat{\mu}(\Lambda)\).

**Proof.** Let \(k_n\) be such that according to Proposition 6,
\[ \|T_\mu \ast n : L^1_{[k_n,bk_n^3]}(\mathbb{T}) \hookrightarrow L^1_{[k_n,bk_n^3]}(\mathbb{T})\| > C(0). \]
Then
\[ \|T_\mu \ast n : L_\Lambda^1(\mathbb{T}) \hookrightarrow L_\Lambda^1(\mathbb{T})\| > C(0). \]
It follows that \(\hat{\mu}(\Lambda) \subset B(0,c)\) for some \(c < 1\) and we derive the spectral radius of \(T_\mu\) equals 1. This leads to a contradiction if we choose \(\Lambda\) in such a way that \(0 \notin \Lambda\). \(\square\)

**Definition 13.** We say that set \(\Lambda \subset \mathbb{Z}\) is a set of natural spectrum provided any Rajchman multiplier from \(M_0(L_\Lambda^1(\mathbb{T}))\) has a natural spectrum.

The above proofs clearly shows that any superset of a set \(\Lambda\) from Corollary 6 is not a set of a natural spectrum. The only program may occure when 0 belongs to the superset. But then it is enough to replace \(\hat{\mu}\) by \(\hat{\mu} - \hat{\lambda}\) where \(\lambda\) is the normalized Haar measure. Then \((\mu - \lambda)^{\ast n} = \mu^{\ast n} - \lambda\) and \(\lambda \ast h_n = 0\).

We present now another interesting application of Lemma 3. Recall that for a finite subset \(A \subset \mathbb{N}\) the Walsh function is defined by the following formula: \(w_A = \prod_{i \in A} r_i\), where \(r_i\) is the \(i\)-th Rademacher function. For \(\mu \in M(\mathbb{T})\) let \(\mu \sim \sum \hat{\mu}(A)w_A\) be the expansion of \(\mu\) in
a Walsh-Stieltjes series and let $A_1, A_2, \ldots, A_{2^n}$ be the ordering of the set $\{A: \max A = n\}$ such that $|\hat{\mu}(A_1)| \geq |\hat{\mu}(A_2)| \geq \ldots \geq |\hat{\mu}(A_{2^n})|$. Then we define $G_\lambda(\mu, n) = \sum_{j=1}^{\lambda n} |\hat{\mu}(A_j)|$.

**Theorem 14.** Let $\dim_M(\mu) = \beta < 1$ and $\lambda < \min(1, \frac{\beta}{1-\beta})$. Then the sum of $2^{\lambda n}$ biggest coefficients from the group $\{ |c_j| : 2^n \leq j < 2^{n+1} \}$ is at least $C(\beta)c_\beta(\mu)$ for infinitely many $n$’s.

**Proof.** Indeed, for $a = (a_1, a_2, \ldots, a_n)$ let us denote the Lorentz norm $\|a\|_{W(k)} = \sum_{j=1}^{k} a_j^*$ where $a^* = (a_1^*, a_2^*, \ldots, a_n^*)$ is the non-increasing rearrangement of $a$. Then the assertion of Theorem 14 says that $\| (c_j)_{j=2^n+1}^{2^{n+1}} \|_{W(2^{n+1})} \geq C(\lambda)c_\beta(\mu)$ for infinitely many $n$’s. The key observation is that the sequence $(c_j)_{j=2^n+1}^{2^{n+1}}$ is an image of a linear transformation of the sequence $(\hat{\mu}(A))_{\max A=n+1}$ by the Walsh operator, which is a contraction in both $l^1$ and $l^\infty$ norms. Since $\| \cdot \|_{W(k)}$ as a Lorentz norm is an interpolation norm between those two, we immediately derive that $\| (\hat{\mu}(A))_{\max A=n+1} \|_{W(2^{n+1})} \geq \| (c_j)_{j=2^n+1}^{2^{n+1}} \|_{W(2^{n+1})}$. Now observe that $c_n = \int \tilde{h}_n d\mu = m(\omega_1) - m(\omega_2)$, where $\tilde{h}_n$ is a Haar function normalized in $L^\infty$ and $\omega$ is the support of $h_n$. By Lemma 3 (as in the proof of Proposition 2), $\sum_{\omega \in A_n} m(\omega_1) - m(\omega_2) > C(\beta)c_\beta(\mu)$. Since every $\omega \in A_n$ has to contain some $\tilde{\omega} \in D_k$ and each $\tilde{\omega} \in D_k$ is contained in no more then one $\omega \in A_n$, then $\#A_n \leq \#D_k \leq 2^{\beta k}$. This finishes the proof because $k < \left(1 - \frac{\beta}{|\alpha| \rho} \right)^{-1} n$ and $|\alpha| \cdot \rho$ may be chosen arbitrarily close to 1.

One can reformulate Theorem 11 in the following way:

**Corollary 15.** Let $\mu \sim \sum \hat{\mu}(A)w_A$ be the expansion of $\mu \in M(0,1)$ in a Walsh-Stieltjes series. If $\lim_{n \to \infty} \| (\hat{\mu}(A))_{\max A=n+1} \|_{W(2^{n+1})} = 0$ then $\dim_M(\mu) > \frac{\lambda}{\lambda + 1}$.

**Remark 16.** The careful inspection of the proof of Lemma 4 shows that the following inequality holds true:

$$\sum_{n=(1-\beta)k}^{k} \sum_{\omega \in F_n \setminus \Xi_n} |m(\omega)| > \beta'kC(\beta)c_\beta(\mu)$$

for $\beta'$ arbitrary close to $\beta$, which in turn gives

$$\sum_{n=(1-\beta)k}^{k} \sum_{\omega \in F_n, m(\omega) \neq 0} |m(\omega) - m(\omega_1)| > \beta'kC(\beta)c_\beta(\mu).$$

Exploiting this and following the lines of the argument used in the proof of Theorem 14 we obtain

$$\sum_{n=(1-\beta)k}^{k} \| (c_j)_{j=2^n-1}^{2^n+1} \|_{W(2^{n+1})} \geq \beta'kC(\beta)c_\mu(\beta)$$

and

$$\sum_{n=(1-\beta)k}^{k} \| (\hat{\mu}(A))_{\max A=n} \|_{W(2^{n+1})} \geq \beta'kC(\beta)c_\mu(\beta)$$
We can now draw our last corollary.

**Corollary 17.** Let $\mu \in M(0,1)$ and let $\mu \sim \sum \hat{\mu}(A)w_A$ be its expansion into Walsh-Stieltjes series. If

$$\liminf_{k \to \infty} \frac{1}{k} \sum_{n=(1-\beta')k}^{k} \| \hat{\mu}(A) \|_{W(2^{\beta n})} = 0,$$

then $\dim_M(\mu) > \beta$.

### 5. Open problems

1. What is the spectrum of the classical Riesz product mentioned in Theorem 3 treated as the multiplier on $H^1(\mathbb{T})$? It is well known that its spectrum in $M(\mathbb{T})$ is the whole unit disc (check chapter in [GM] on independent power measures).

2. Can we find an example of a multiplier on $H^1(\mathbb{T})$ with a non-natural spectrum within the class of Riesz products in $M_0(\mathbb{T})$?

3. Can we find a Rajchman probability measure whose spectrum as an operator on $H^1_0(\mathbb{T})$ is the unit circle or the unit disc?

4. Can one weaken the assumption of Lemma 4 by considering the Hausdorff dimension instead of the Minkowski one?

5. How far from optimal is the estimation of the norm given in the proof of Proposition 1? In particular what is its growth when $\beta$ approaches 1?

### Acknowledgements

The authors thank Fedor Nazarov for valuable discussion during the work on this paper.

### References

[B] Ch. E. Bluhm: *Liouville Numbers, Rajchman Measures, and Small Cantor Sets*, Proc. Amer. Math. Soc., Vol. 128, No. 9, pp. 2637-2640, 2000.

[Grah] C. C. Graham: *A Riesz product proof of the Wiener–Pitt theorem*, Proc. Amer. Math. Soc., vol. 44, no. 2, pp. 312–314, 1974.

[K] I. Klemes: *Idempotent multipliers of $H1(T)$*. Canad. J. Math. 39 (1987), no. 5, 1223–1234.

[GM] C. C. Graham, O. C. McGhee: *Essays in Commutative Harmonic Analysis*, Springer-Verlag, New York, 1979.

[R2] W. Rudin: *Fourier analysis on groups*, Wiley Classics Library, 1990.

[Sc] Y. A. Sreider: *The structure of maximal ideals in rings of measures with convolution*, Mat. Sbornik, vol. 27, pp. 297–318, 1950; Amer. Math. Soc. Trans., no. 81, 1953.
[WP] N. Wiener, H.R. Pitt: Absolutely convergent Fourier–Stieltjes transforms, Duke Math. J., vol. 4, no. 2, pp. 420–436, 1938.

[Wi] J.H. Williamson: A theorem on algebras of measures on topological groups, Proc. Edinburh Philos. Soc., vol. 11, pp. 195–206, 1959.

[Zaf1] M. Zafran: On the spectra of multipliers, Pacific Journal of Mathematics, vol. 47, no. 2, pp. 609–626, 1973.

[Zaf2] M. Zafran: Measures as convolution operators $H^1$ and Lip$\alpha$, J. Functional Analysis 29 (1978), no. 2, 160–183.

[Ż] W. Żelazko: Banach algebras, Elsevier Science Ltd and PWN, February 1973.