THE DISTRIBUTION OF THE LOGARITHMIC DERIVATIVE OF THE RIEMANN ZETA-FUNCTION

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Abstract. We investigate the distribution of the logarithmic derivative of the Riemann zeta-function on the line \( \Re(s) = \sigma \), where \( \sigma \) lies in a certain range near the critical line \( \sigma = 1/2 \). For such \( \sigma \), we show that the distribution of \( \zeta'(s)/\zeta(s) \) converges to a two-dimensional Gaussian distribution in the complex plane. Upper bounds on the rate of convergence to the Gaussian distribution are also obtained.

1. Introduction

Let \( \zeta(s) \) denote the Riemann zeta-function with \( s = \sigma + it \) a complex variable. Throughout we let \( T \) denote a sufficiently large parameter.

In unpublished work, A. Selberg proved that the logarithm of the Riemann zeta-function on and near the line \( \sigma = 1/2 \) is normally distributed. For \( 0 \leq (2\sigma - 1) = o(1) \) as \( T \to \infty \), \( \Psi(\sigma) = \frac{1}{2} \sum_{p \leq T} p^{-2\sigma} \), and real numbers \( a < b \), he showed that

\[
\text{meas}\left\{ t \in [0, T] : \log|\zeta(\sigma + it)|\Psi(\sigma)^{-1/2} \in [a, b] \right\} = \frac{T}{2\pi} \int_a^b e^{-x^2/2} \, dx + O\left( T \frac{(\log \Psi(\sigma))^2}{\sqrt{\Psi(\sigma)}} \right),
\]

and

\[
\text{meas}\left\{ t \in [0, T] : \arg\zeta(\sigma + it)\Psi(\sigma)^{-1/2} \in [a, b] \right\} = \frac{T}{2\pi} \int_a^b e^{-x^2/2} \, dx + O\left( T \frac{\log \Psi(\sigma)}{\sqrt{\Psi(\sigma)}} \right),
\]

where meas denotes Lebesgue measure (see [14]). Although Selberg did not publish proofs of these results, his student K. M. Tsang gave the details of Selberg’s argument in his PhD thesis [17]. These theorems may also be proved, albeit with larger error terms, by the method of A. Ghosh in [2] and [3].

The purpose of this article is to investigate the distribution of the logarithmic derivative of the Riemann zeta-function near the critical line \( \sigma = 1/2 \).

The distribution of \( \zeta'(s)/\zeta(s) \) was also studied by C. R. Guo [5], who showed the following. Write \( Q(x, y) = \hat{Q}(x + iy) \), where \( \hat{Q} : \mathbb{C} \to \mathbb{R} \) is infinitely differentiable in \( x \) and \( y \) and has compact support. Then for any \( 0 < \epsilon < 1/6 \) and \( \sigma \in [1/2 + (\log T)^{-1/6-\epsilon}, 2] \) Guo showed that

\[
\frac{1}{T} \int_0^T \hat{Q}\left( \frac{\zeta'}{\zeta}(\sigma + it) \right) \, dt = \int_{\mathbb{R}^2} Q(x, y) f(x, y) \, dxdy + E(T, Q, \sigma, \epsilon).
\]

Here \( f(x, y) \) is the Fourier transform of

\[
\prod_p \left( \int_0^1 \exp\left( 2\pi iu \log p \sum_m \frac{\cos(2\pi mt)}{p^{m\sigma}} - 2\pi iv \log p \sum_m \frac{\sin(2\pi mt)}{p^{m\sigma}} \right) \, dt \right)
\]

and

\[
E(T, Q, \sigma, \epsilon) \ll \exp\left( -\frac{1}{4}(\log T)^{3(1/6-\epsilon)} \right) \int_A |\hat{Q}(\alpha, \beta)| \, d\alpha d\beta + \int_{\mathbb{R}^2 \setminus A} |\hat{Q}(\alpha, \beta)| \, d\alpha d\beta,
\]

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where $A$ is the square

$$A = [-(\log T)^{3\epsilon - \eta}, (\log T)^{3\epsilon - \eta}] \times [-(\log T)^{3\epsilon - \eta}, (\log T)^{3\epsilon - \eta}]$$

with $\eta > 0$ arbitrary.

In this paper we calculate the distribution function of $\zeta'/(\zeta + \epsilon)$ for $\epsilon = 1/2 + \psi(T)/\log T$ with $\psi(T)$ any positive function tending to infinity such that $\psi(T) = o(\log T)$ as $T \to \infty$. We also show that in this range the distribution of $\zeta'/\zeta(\epsilon + i\epsilon)$ converges to a two-dimensional Gaussian distribution in the complex plane. Finally, for rectangles with sides parallel to the coordinate axes and for disks centered at the origin, we give explicit upper bounds on the rate of convergence to the normal distribution. We should mention that our results are consistent with Guo’s work: one can show that the probability density function $\pi(x,y)$ in Guo’s theorem does indeed converge to a two-dimensional Gaussian when $2\epsilon - 1 = o(1)$ as $T \to \infty$.

Throughout we write $\mathbf{u} = (u,v) \in \mathbb{R}^2$ and, if $z \in \mathbb{C}$, we use the non-standard notation $\mathbf{u} \cdot z = u\Re(z) + v\Im(z)$. We define $1_A(\alpha)$ to be the indicator function of the set $A$, which is equal to one if $\alpha \in A$ and is equal to zero if $\alpha \notin A$. If $\theta = (\theta_1,...,\theta_n) \in [0,1]^n$, we write

$$\int_{\mathbb{R}^n} F(\theta) d\theta = \int_0^1 \cdots \int_0^1 F(\theta_1,...,\theta_n) d\theta_1 \cdots d\theta_n.$$

Finally, for $f \in L^1(\mathbb{R}^n)$ we define the Fourier transform of $f$ by

$$\hat{f}(x) = \int_{\mathbb{R}^n} f(\xi)e(-x \cdot \xi) d\xi,$$

where $e(x) = e^{2\pi i x}$.

2. MAIN RESULTS AND A SUMMARY OF THE METHOD

Let $\psi(T) = (2\epsilon - 1) \log T$ and for $\psi(T) \geq 1$ define

$$V = V(\epsilon) = \frac{1}{2} \sum_{n=2}^{\infty} \Lambda^2(n)/n^{2\epsilon}.$$ 

When $\psi(T)$ tends to infinity with $T$, we may think of $V$ as the variance of $\zeta'/\zeta(\epsilon + i\epsilon)$. For if $\epsilon$ is in this range and the Riemann hypothesis is true, it follows from equation (1.2) of Selberg [11] that

$$\frac{1}{T} \int_0^T \left| \frac{\zeta'}{\zeta}(\theta + i\epsilon) \right|^2 d\theta \sim \sum_{n=2}^{\infty} \frac{\Lambda^2(n)}{n^{2\epsilon}}.$$ 

With this in mind, we consider the normalized function $\zeta'/\zeta(\epsilon + i\epsilon)V^{-1/2}$. Our first main result is

**Theorem 1.** Let $\psi(T) = (2\epsilon - 1) \log T$, and

$$\Omega = e^{-10 \min(V^{3/2},(\psi(T)/\log \psi(T))^{1/2})}.$$ 

Suppose that $\psi(T) \to \infty$ with $T$, $\psi(T) = o(\log T)$, and that $R$ is a rectangle in $\mathbb{C}$ whose sides are parallel to the coordinate axes and have length greater than $\Omega^{-1}$. Then we have

$$\operatorname{meas}\left\{ t \in (0,T) : \frac{\zeta'}{\zeta}(\theta + i\epsilon)V^{-1/2} \in R \right\} = \frac{T}{2\pi} \int_R e^{-(x^2+y^2)/2} dx dy + O\left(T\left(\frac{\operatorname{meas}(R) + 1}{\Omega}\right)\right).$$
In the range
\[
\left( \frac{\log \log T}{\log T} \right)^{1/7} \ll (2\sigma - 1) = o(1),
\]
the error term is of order \( T(\text{meas}(R) + 1)V^{-3/2} \), while for \( \sigma \) closer to \( 1/2 \), it is of size \( T(\text{meas}(R) + 1)(\log \psi(T)/\psi(T))^{1/2} \). At the cost of a longer proof, the condition that the length of each side of the rectangle \( R \) should be greater than \( \Omega^{-1} \) could be removed.

It does not seem possible to prove a distribution theorem for \( \zeta'/\zeta(\sigma + it) \) when \((2\sigma - 1)\log T \ll 1\) without the assumption of some unproven hypothesis. For in this range the moments of \( \zeta'/\zeta(\sigma + it) \) depend on correlations of the zeros of the Riemann zeta-function. Goldston, Gonek, and Montgomery [4] proved, under the assumption of the Riemann hypothesis, that for \( T^{-1} \log T \leq a \ll 1 \),
\[
\int_0^T \left| \frac{\zeta'}{\zeta}(it) + \frac{a}{\log T} + it \right|^2 dt \sim \left( 1 - e^{-2a} + \int_1^\infty (F(\alpha, T) - 1)e^{-2a\alpha} d\alpha \right) T \log^2 T.
\]
Here \( F(\alpha, T) \) is defined by
\[
F(\alpha, T) = \frac{1}{2\pi \log T} \sum_{0 < \gamma, \gamma' \leq T} T^{i\alpha(\gamma - \gamma')} w(\gamma - \gamma'),
\]
where \( w(x) = 4/(4 + x^2) \) and the sum is over pairs of ordinates of zeros of the Riemann zeta-function. (For more on \( F(\alpha, T) \) see [5].) Moreover, D. W. Farmer et al. [1] have recently proved that if the Riemann hypothesis and some additional plausible hypotheses about the zeros are true, then for \((2\sigma - 1)\log T \approx 1\), the even moments of \( |\zeta'/\zeta(\sigma + it)| \) may be expressed in terms of correlations of the zeros of \( \zeta(s) \). The form of the answers suggests that \( \zeta'/\zeta(\sigma + it)V^{-1/2} \) is unlikely to be normally distributed when \((2\sigma - 1)\log T \approx 1\).

In applications it is useful to have an analogue of Theorem [1] in a disk. For this reason, we also prove

**Theorem 2.** Let \( \psi(T) = (2\sigma - 1)\log T \),
\[
\Omega = e^{-10} \min \left( V^{3/2}, (\psi(T)/\log \psi(T))^{1/2} \right).
\]
Suppose that \( \psi(T) \to \infty \) with \( T \), \( \psi(T) = o(\log T) \), and \( r \) is a real number such that \( r\Omega \geq 1 \). Then we have
\[
\text{meas} \left\{ t \in (0, T) : \left| \frac{\psi'}{\psi}(\sigma + it) \right| \leq \sqrt{r} \right\} = T(1 - e^{-r^2/2}) + O \left( T \left( \frac{r^2 + r}{\Omega} \right) \right).
\]
If, in addition, we let \( \tilde{\Omega} = \min \left( (2\sigma - 1)e^\gamma/(2\sigma - 1), e^{-10}(\psi(T)/\log \psi(T))^{1/2} \right) \), then we have for \( r\tilde{\Omega} \geq 1 \)
\[
\text{meas} \left\{ t \in [0, T] : \left| \frac{\psi'}{\psi}(\sigma + it) \right| \leq \sqrt{r} \right\} \ll Tr^2.
\]

One of the two main components of the proofs of Theorem [1] and Theorem [2] is an approximate formula for the characteristic function (c.f.) of \( \zeta'/\zeta(\sigma + it)V^{-1/2} \).

**Theorem 3.** Let \( \psi(T) = (2\sigma - 1)\log T \), and
\[
\Omega = e^{-10} \min \left( V^{1/2}, (\psi(T)/\log \psi(T))^{1/2} \right).
\]
Suppose that \( \psi(T) \to \infty \) with \( T \), \( \psi(T) = o(\log T) \), and \( |u|, |v| \leq \Omega \). Then
\[
\frac{1}{T} \int_0^T e \left( \frac{u}{\psi}(\sigma + it)V^{-1/2} \right) dt = e^{-2\pi^2(u^2 + v^2)}(1 + \mathcal{E}_A(u, v)) + \mathcal{E}_B,
\]
where
\[
\mathcal{E}_A(u, v) \ll \frac{|u| + |v|^3}{V^{3/2}} + \frac{u^2 + v^2}{\psi(T)^10} \quad \text{and} \quad \mathcal{E}_B \ll \psi(T)^{-10}.
\]
Observe that in Theorem 3 we may take $u$ or $v$ to be zero and obtain approximate formulas for the ch.f. of $\Re \zeta'/\zeta(\sigma + it)$ and $\Im \zeta'/\zeta(\sigma + it)$. Using these, we could easily prove that analogues of Theorem 1 hold for both the real and imaginary parts of $\zeta'/\zeta(s)$.

Theorem 3 implies that if $(2\sigma - 1) \log T$ tends to infinity with $T$ and is also $o(\log T)$, then the ch.f. of $\zeta'/\zeta(\sigma + it)V^{-1/2}$ converges pointwise, as $T \to \infty$, to a two-dimensional Gaussian. Hence, for $\sigma$ in this range, it immediately follows from standard probability theory that for any Borel measurable $S \subset \mathbb{C}$ with positive Jordan content,

$$
\lim_{T \to \infty} \frac{1}{T} \meas \left\{ t \in [0, T] : \frac{\zeta'}{\zeta}(\sigma + it)V^{-1/2} \in S \right\} = \frac{1}{2\pi} \iint_S e^{-(x^2 + y^2)/2} \, dx \, dy.
$$

To obtain Theorem 1 from Theorem 3 we use Beurling-Selberg functions, which are analytic approximations of the signum function. They are also integrable along the real axis and have Fourier transforms that vanish outside of an interval. In the 1930’s, Beurling discovered these functions but never published his findings. Independently, Selberg rediscovered them and used them in several contexts, one of which was the study of the distribution of $\log \zeta(s)$. For a discussion of these functions see Selberg [13].

The proof of Theorem 2 is similar. However, in this case we use Beurling-Selberg functions that approximate the indicator function of a disk. These functions were introduced by Holt and Vaaler in [6], and their existence is a special case of a general theorem on Beurling-Selberg functions for balls in Euclidean space.

To prove Theorem 3 we start with a formula for $\zeta'/\zeta(s)$ that was proved by Selberg in [12]. For $\sigma + it$ not too close to a zero of the Riemann zeta-function, this formula expresses $\zeta'/\zeta(\sigma + it)$ as, essentially, the Dirichlet polynomial $-\sum_{n \leq x} \Lambda(n)n^{-\sigma-it}$. If $\psi(T) = (2\sigma - 1) \log T$ tends to infinity with $T$, this formula holds for most $t$. This allows us to reduce our problem to calculating the ch.f. of $-\sum_{n \leq x} \Lambda(n)n^{-s}$, and we accomplish that by computing its moments.

3. The Characteristic Function of $\zeta'/\zeta(\sigma + it)$

Our initial step is to express the ch.f. of $\zeta'/\zeta(\sigma + it)V^{-1/2}$ in terms of the ch.f. of the Dirichlet polynomial $-V^{-1/2} \sum_{n \leq x} \Lambda(n)n^{-\sigma-it}$. Near a zero of the Riemann zeta-function $\zeta'/\zeta(s)$ cannot be approximated by a Dirichlet polynomial, hence we need to bound the possible contribution of $t \in (0, T)$, where $\sigma + it$ is close to a zero of $\zeta(s)$. In [12] Selberg discovered a very clever way to do this. Selberg’s approach begins with an explicit formula for $\zeta'/\zeta(s)$.

The explicit formula contains two important terms, the first of which is a sum over primes similar to the Dirichlet polynomial above, while the second is a sum over zeros of the Riemann zeta-function. Selberg showed that for certain values of $t$, the contribution of the sum over zeros can be bounded in terms of a sum over primes. Using this formula, we will prove

**Lemma 1.** Suppose that $10 \leq x \leq T^{1/18}$ and $1/2 + 4/\log x \leq \sigma \leq 2$. Then

$$
\frac{1}{T} \int_0^T e \left( -\bar{u} \cdot \frac{\zeta'}{\zeta}(\sigma + it)V^{-1/2} \right) \, dt = \frac{1}{T} \int_0^T e \left( \bar{u} \cdot \sum_{n \leq x} \Lambda(n)n^{-\sigma+it}V^{-1/2} \right) \, dt + E_1
$$

where

$$
E_1 \ll (|u| + |v|)V^{-1/2}x^{(1/2-\sigma)/2} \log T + T^{-(\sigma-1/2)/3} \log T \log x.
$$

Throughout we let $\rho = \beta + i\gamma$ denote a zero of the Riemann zeta-function. To state the explicit formula for $\zeta'/\zeta(s)$, we first define the number

$$
\sigma_{x,t} = \frac{1}{2} + 2 \max \left( \beta - \frac{1}{2}, \frac{2}{\log x} \right), \quad (3.1)
$$
where \( x \geq 2 \) and \( t > 0 \). Here the maximum is taken over zeros \( \rho \) satisfying \( |t - \gamma| \leq x^{3[\beta - 1/2]} \log x \). For \( \sigma \geq \sigma_{x,t} \) and \( 2 \leq x \leq t^2 \), A. Selberg proved (see equation 4.9 in \([12]\)) that

\[
-\frac{\zeta'}{\zeta}(\sigma + it) = \sum_{n \leq x^3} \frac{\Lambda(n)}{n^{\sigma+it}} w_x(n) + O \left( x^{1/2-\sigma/2} \left| \sum_{n \leq x^3} \frac{\Lambda(n)}{n^{\sigma_x,t+it}} w_x(n) \right| \right) + O \left( x^{1/2-\sigma/2} \log t \right),
\]

where

\[
w_x(n) = \begin{cases} 
1 & \text{if } n \leq x, \\
\log^2(x^3/n) - 2 \log^2(x^2/n) & \text{if } x \leq n \leq x^2, \\
\log^2(x^3/n) & \text{if } x^2 \leq n \leq x^3, \\
0 & \text{if } n > x^3.
\end{cases}
\]

One can easily modify Selberg’s proof to show that an analogue of (3.2) holds where \( \sigma_{x,t} \) is replaced by \( \sigma \) in the error term and we shall use this later in the proof of Lemma 1. (This fact merely simplifies the proof.)

We will now show that if \((2\sigma - 1) \log T\) tends to infinity with \( T \) and is also \( o(\log T) \), then the measure of the set of \( t \in (0, T) \) for which \( \sigma_{x,t} > \sigma \) is \( o(T) \). Hence, for \( \sigma \) in this range, (3.2) holds almost everywhere, in the sense that the proportion of \( t \in (0, T) \) for which the formula does not hold tends to zero as \( T \to \infty \). We prove this by using a zero-density estimate of Jutila \([7]\), that is, an estimate for the number of zeros of \( \zeta(s) \) with \( \beta > \sigma \) and \( 0 < \gamma < T \). Jutila’s result is that for any \( \epsilon > 0 \),

\[
N(\sigma, T) = \sum_{\beta > \sigma} \sum_{0 < \gamma < T} 1 \ll T^{1-(1-\epsilon)(\sigma-1/2)} \log T.
\]

We are now ready to prove

**Lemma 2.** Let \( 1/2 + 4/ \log x \leq \sigma \leq 2 \) and, for any fixed \( 0 < \epsilon < 1 \), let \( 10 \leq x \leq T^{\epsilon/3} \). Then

\[
\text{meas} \{ t \in [2, T] : \sigma_{x,t} > \sigma \} \ll T^{1-(1/2-\epsilon)(\sigma-1/2)} \frac{\log T}{\log x},
\]

where \( \sigma_{x,t} \) is the number defined in (3.1), and the implied constant depends only on \( \epsilon \).

**Proof.** By the definition of \( \sigma_{x,t} \), if for some \( t \geq 2 \) we have that \( \sigma_{x,t} > \sigma \), then there is a zero \( \rho_0 \) such that \( \beta_0 > (\sigma - 1/2)/2 + 1/2 \) and \( |t - \gamma_0| \leq x^{3[\beta_0-1/2]} \log x \). Furthermore, for each such \( \rho_0 \) we have

\[
\text{meas} \left\{ t \in [2, T] : |t - \gamma_0| \leq \frac{x^{3[\beta_0-1/2]}}{\log x} \right\} \leq x^{3[\beta_0-1/2]} \log x.
\]

We also observe that if \( 2 \leq t \leq T \) and \( x \leq T^{\epsilon/3} < T^{1/3} \), then \(-T^{1/2}/\log T < \gamma_0 < T + T^{1/2}/\log T \). Now let \( \sigma' = (\sigma - 1/2)/2 + 1/2 \). Combining our observations, we find that

\[
\text{meas} \{ t \in [2, T] : \sigma_{x,t} > \sigma \} \ll \sum_{\frac{x^{1/2}}{\log x} \leq \gamma \leq T + \frac{T^{1/2}}{\log x}} \frac{x^{3[\beta-1/2]}}{\log x}. \tag{3.4}
\]
Using (3.3), we see that
\[
\sum_{0 < \gamma \leq 2T \atop \beta > \sigma'} x^{3(\beta - 1/2)} = - (N(v, 2T)x^{3(v - 1/2)}) \left|_{\sigma'}^{1} \right. + 3 \log x \int_{\sigma'}^{1} x^{3(v - 1/2)} N(v, 2T) \, dv
\]
\[
\ll (T^{1-(1-\epsilon)(\sigma' - 1/2)} \log T)(T^{\epsilon(\sigma' - 1/2)}) + x^{-3/2} T^{1+(1-\epsilon)/2} \log(\log T) \int_{\sigma'}^{1} (x^{3T^{\epsilon-1}})^{v} \, dv
\]
\[
\ll T^{1-(1-\epsilon)(\sigma' - 1/2)} \log T.
\]

The zeros of \(\zeta(s)\) are symmetric about the real axis, so by our previous estimate,
\[
\sum_{-\tau^{1/2} \leq \gamma \leq T + \tau^{1/2} \atop \beta > \sigma'} x^{3(\beta - 1/2)} \log x \leq \frac{2}{\log x} \sum_{0 < \gamma \leq 2T \atop \beta > \sigma'} x^{3(\beta - 1/2)} \ll T^{1-(1/2-\epsilon)(\sigma-1/2)} \log T \log x.
\]
The result follows from this and (3.4). \(\square\)

We now are ready to prove Lemma \[\[\]

**Proof of Lemma \[\[\]** Let \(B = \{ t \in [0, T] : \sigma \geq \sigma_{x,t} \}\). By our comment after (3.2), for \(t \in B\) we may replace \(\sigma_{x,t}\) by \(\sigma\) in the first error term in (3.2). Also note that \(|e(\alpha) - e(\beta)| = 2\pi |\int_{\alpha}^{\beta} e(x) \, dx| \leq 2\pi |\beta - \alpha|\). Hence, by these observations and (3.2), we see that
\[
\int_{B} e\left(-\bar{u} \cdot \frac{\zeta'(\sigma + it)}{\zeta(\sigma + it)} V^{-1/2}\right) \, dt = \int_{B} e\left(-\bar{u} \cdot \sum_{n \leq x} \frac{\Lambda(n)}{n^{\sigma+it}} w_{x}(n) V^{-1/2}\right) \, dt + E_{2},
\]
where
\[
E_{2} \ll (|u| + |v|) V^{-1/2} x^{(1/2-\sigma)/2} \left( \int_{0}^{T} \left| \sum_{n \leq x} \frac{\Lambda(n)}{n^{\sigma+it}} w_{x}(n) \right| \, dt + T \log T \right).
\]
Recall that \(w_{x}(n) = 1\) for \(n \leq x\) and that \(0 \leq w_{x}(n) \leq 1\) always. Estimating the integral using Cauchy’s inequality and then applying Montgomery and Vaughan’s mean value theorem for Dirichlet polynomials [10], we see that
\[
E_{2} \ll (|u| + |v|) V^{-1/2} x^{(1/2-\sigma)/2} \left( T \left( \sum_{n \leq x} \frac{\Lambda^{2}(n)}{n^{2\sigma}} \right)^{1/2} + T \log T \right)
\]
\[
\ll T \left( |u| + |v| \right) V^{-1/2} x^{(1/2-\sigma)/2} \log T,
\]
where the estimate of the sum follows from a calculation using the Prime Number Theorem. Noting that \(x \leq T^{1/18}\), we have
\[
\int_{B} e\left(-\bar{u} \cdot \frac{\zeta'(\sigma + it)}{\zeta(\sigma + it)} V^{-1/2}\right) \, dt = \int_{B} e\left(-\bar{u} \cdot \sum_{n \leq x} \frac{\Lambda(n)}{n^{\sigma+it}} w_{x}(n) V^{-1/2}\right) \, dt
\]
\[
+ O(T(|u| + |v|) V^{-1/2} x^{(1/2-\sigma)/2} \log T).
\]
Next, note that \(|e(x)| = 1\) and, by Lemma \[\[\] that
\[
\text{meas}(\{0, T \setminus B\} \leq T^{1-(\sigma - 1/2)/3} \log T \log x.
\]
Thus,
\[
\int_0^T e\left(-\vec{u} \cdot \zeta'((\sigma+it)V^{-1/2}\right) dt = \int_B e\left(-\vec{u} \cdot \zeta'((\sigma+it)V^{-1/2}\right) dt + O\left(T^{1-(\sigma-1)/3} \log T \log x \right).
\]

An analogue of this formula also obviously holds for \( V^{-1/2} \sum_{n \leq x} \Lambda(n)n^{-(\sigma+it)} \). Hence, we obtain
\[
\int_0^T e\left(-\vec{u} \cdot \zeta'((\sigma+it)V^{-1/2}\right) dt = \int_0^T e\left(-\vec{u} \cdot \sum_{n \leq x} \Lambda(n)n^{-(\sigma+it)}w_x(n)V^{-1/2}\right) dt
+ O\left(T(|u| + |v|)V^{-1/2}(1/2 - \sigma) \log T\right)
+ O\left(T^{1-(\sigma-1)/3} \log T \log x \right).
\]

We would next like to replace the weight \( w_x(n) \) by 1. Write
\[
\int_0^T e\left(\vec{u} \cdot \sum_{n \leq x} \Lambda(n)n^{\sigma+it}w_x(n)V^{-1/2}\right) dt = \int_0^T e\left(\vec{u} \cdot \sum_{n \leq x} \Lambda(n)n^{\sigma+it}V^{-1/2}\right) dt + E_3.
\]

Using the estimate \(|e(\alpha) - e(\beta)| \ll |\beta - \alpha|\), we see that
\[
E_3 \ll (|u| + |v|)V^{-1/2} \int_0^T \left| \sum_{n \leq x} \Lambda(n)n^{\sigma+it}w_x(n) - \sum_{n \leq x} \Lambda(n)n^{\sigma+it}\right| dt.
\]

By Cauchy’s inequality and Montgomery and Vaughan’s mean value theorem for Dirichlet polynomials [10], we find that
\[
E_3 \ll T(|u| + |v|)V^{-1/2}\left( \sum_{x \leq n \leq x^3} \frac{\Lambda^2(n)}{n^2}\right)^{1/2}
\ll T(|u| + |v|)V^{-1/2}\frac{x^{1/2 - \sigma}}{(2\sigma - 1)^{3/2}} \log^{1/2} x \ll T(|u| + |v|)V^{-1/2}x^{1/2 - \sigma} \log T
\]
(the estimate of the sum follows from the Prime Number Theorem). The result now follows on combining this estimate and (3.5). \( \square \)

### 3.1. A Dirichlet Polynomial Calculation

Our next goal is to show that the ch.f. of \( \sum_{n \leq x} \Lambda(n)n^{-(\sigma+it)} \) is essentially a Gaussian.

**Lemma 3.** Let \( 1/2 + 4/\log x \leq \sigma \leq 2 \), and \( x \leq T^{1/(5N)} \) be sufficiently large with \( N \) an even integer. Then for \(|u|, |v| \leq V^{1/2}/100\), we have
\[
\frac{1}{T} \int_0^T e\left(\vec{u} \cdot \sum_{n \leq x} \Lambda(n)n^{\sigma+it}V^{-1/2}\right) dt = e^{-2\pi^2(u^2 + v^2)}(1 + E_4(u, v)) + E_5(u, v),
\]
where
\[
E_4(u, v) \ll \frac{(|u| + |v|)^3}{V^{3/2}} + (u^2 + v^2)(x^{1/2 - \sigma}(2\sigma - 1) \log x + (2\sigma - 1)^2)
\]
and
\[
E_5(u, v) \ll \frac{(6\sqrt{2}\pi(|u| + |v|))^N}{(N/2)!} + T^{-1/3}.
\]

Before proving Lemma 3 we require several additional lemmas.
Lemma 4. Let $2 \leq x \leq T$. Also let $k$ be a natural number such that $x^k \leq T/\log T$. Then for any complex numbers $a_p$ we have
\[
\int_0^T \left| \sum_{p \leq x} a_p e^{-it} \right|^{2k} dt \ll T k! \left( \sum_{p \leq x} |a_p|^2 \right)^k.
\]

Proof. This is due to Soundararajan in [15]. □

The next lemma is a generalization of Montgomery and Vaughan’s mean value theorem for Dirichlet polynomials [10].

Lemma 5. Let $T \in \mathbb{R}$ and $m, k \in \mathbb{N}$. For any complex numbers $a_n, b_n$
\[
\int_0^T \left( \sum_{n} a_n e^{-int} \right)^m \left( \sum_{n} b_n e^{-int} \right)^k dt = T \sum_{n} A_n B_n + O\left( \left( \sum_{n} |A_n|^2 \right)^{1/2} \left( \sum_{n} |B_n|^2 \right)^{1/2} \right),
\]
where
\[
A_n = \sum_{n_1 \cdots n_m = n} a_{n_1} \cdots a_{n_m} \quad \text{and} \quad B_n = \sum_{n_1 \cdots n_k = n} b_{n_1} \cdots b_{n_k}.
\]

Proof. This is proved in K.M. Tsang’s PhD thesis [17]. A proof may also be found in [18]. □

We now make an observation about the main term in Lemma 5. Let $g(n)$ be a multiplicative function defined by $g(1) = 1$ and $g(n) = \alpha_1 \theta_{p_1} + \cdots + \alpha_r \theta_{p_r}$, for $n = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$ and indeterminates $\theta_{p_1}, \ldots, \theta_{p_r}$. By the unique factorization of the integers, and since for $m \in \mathbb{Z}$ we have
\[
\int_0^1 e^{m \theta} d\theta = \begin{cases} 1 & \text{if } m = 0, \\ 0 & \text{if } m \neq 0, \end{cases}
\]
we see that
\[
\sum_n A_n B_n = \int_0^1 \cdots \int_0^1 \left( \sum_n a_n e(g(n)) \right) \left( \sum_n b_n e(g(n)) \right) \prod_p d\theta_p. \tag{3.6}
\]

To state our next lemma write
\[
S(\theta) = V^{-1/2} \sum_{p^n \leq x} \frac{\log p}{p^{n \sigma}} e(n \theta_p) \quad \text{and} \quad f(t) = V^{-1/2} \sum_{p^n \leq x} \frac{\log p}{p^{n(\sigma + it)}}.
\]

Lemma 6. Let $1/2 \leq \sigma \leq 1$, $m, k = 0, 1, 2, \ldots$, and $e^{20} \leq x \leq T^{1/(5(m+k))}$. Then
\[
\frac{1}{T} \int_0^T f(t)^m f(t)^k dt = \int_{\mathbb{T}^n} S(\theta)^m S(\theta)^k d\theta + O\left( V^{-(m+k)/2} T^{-1/3} (m!)^{1/2} \right), \tag{3.7}
\]

Moreover,
\[
\frac{1}{T} \int_0^T |f(t)|^{2k} dt, \quad \int_{\mathbb{T}^n} |S(\theta)|^{2k} d\theta \ll 18^k k!.
\]

If $m = k = 0$, then (3.7) holds without an error term.
Proof. The last assertion of the lemma is obvious. We may therefore assume that at least one of \( m, k \neq 0 \). Without loss of generality, we assume that \( m \neq 0 \). Applying Lemma 5 and (3.6), we find that

\[
V^{(m+k)/2} \int_0^T f(t)^m f(t)^k \, dt = V^{(m+k)/2} \int_{T^{r(x)}} S(\theta)^m S(\theta)^k \, d\theta
\]

\[+ O\left( \left( \sum_n n|A_n|^2 \right)^{1/2} \left( \sum_n n|B_n|^2 \right)^{1/2} \right), \tag{3.8}
\]

where, for \( m, k \neq 0 \),

\[A_n = \sum_{p_1^{n_1} \ldots p_m^{n_m} = n} \frac{\log p_1 \ldots \log p_m}{(p_1^{n_1} \ldots p_m^{n_m})^\sigma} \quad \text{and} \quad B_n = \sum_{p_1^{n_1} \ldots p_k^{n_k} = n} \frac{\log p_1 \ldots \log p_k}{(p_1^{n_1} \ldots p_k^{n_k})^\sigma}.
\]

If \( k = 0 \), then \( B_1 = 1 \) and \( B_n = 0 \) for \( n = 1, 2, \ldots \). By Cauchy’s inequality,

\[
\sum_{n \leq x^m} n|A_n|^2 = \sum_{n \leq x^m} n \left( \sum_{p_1^{n_1} \ldots p_m^{n_m} = n} \frac{\log p_1 \ldots \log p_m}{(p_1^{n_1} \ldots p_m^{n_m})^\sigma} \right)^2 \leq \sum_{n \leq x^m} n \left( \sum_{p_1^{n_1} \ldots p_m^{n_m} = n} \log^2 p_1 \ldots \log^2 p_m \right) \left( \sum_{p_1^{n_1} \ldots p_m^{n_m} = n} (p_1^{n_1} \ldots p_m^{n_m})^{-2\sigma} \right).
\]

Now, given \( n, n_1, \ldots, n_m \), the equation \( n = p_1^{n_1} \ldots p_m^{n_m} \) has at most \( m^m \) solutions in \( (p_1, \ldots, p_m) \). Therefore,

\[
\sum_{p_1^{n_1} \ldots p_m^{n_m} = n} (p_1^{n_1} \ldots p_m^{n_m})^{-2\sigma} \leq m^m \sum_{p_1, \ldots, p_m \leq x} \sum_{n_1, \ldots, n_m = 1}^{\infty} (p_1^{n_1} \ldots p_m^{n_m})^{-2\sigma},
\]

which by Mertens’ theorem is

\[
\ll 6^m m! \left( \sum_{p \leq x} \frac{1}{p^{2\sigma}} \right)^m \ll (7 \log log x)^m m!.
\]

Hence,

\[
\sum_{n \leq x^m} n|A_n|^2 \ll (7 \log log x)^m m! \sum_{n \leq x^m} \sum_{p_1^{n_1} \ldots p_k^{n_k} = n} p_1^{n_1} \ldots p_m^{n_m} \log^2 p_1 \ldots \log^2 p_m
\]

\[= (7 \log log x)^m m! \left( \sum_{n \leq x} n \Lambda^2(n) \right)^m.
\]

Since \( \sum_{n \leq x} n \Lambda^2(n) \leq 2x^2 \log x \) and \( 7 \log log x < \log x \) for \( x > e^{20} \), we see that

\[
\sum_{n \leq x^m} n|A_n|^2 \ll m!(2x \log x)^{2m}.
\]

Since \( e^{20} \leq x \leq T^{1/(5(m+k))} \), we easily see that \( (2x \log x)^{2m} < T^{2/3} \). Hence,

\[
\sum_{n \leq x^m} n|A_n|^2 \ll m!T^{2/3}.
\]
Now, if \( k = 0 \), (3.7) follows since \( \sum_{n \leq x} n \left| B_n \right|^2 = 1 \). If \( k \neq 0 \), we similarly have
\[
\sum_{n \leq x} n \left| B_n \right|^2 \ll k! T^{2/3}.
\]

Thus, the error term in (3.8) is
\[
\ll (m!k!)^{1/2} T^{2/3},
\]
and (3.7) follows.

To prove the second assertion of the lemma we start with the observation that
\[
\left| \sum_{p^j \leq x} \frac{\log p}{p^{(\sigma + it)}} \right|^{2k} \leq 9^k \left( \left| \sum_{p \leq x} \frac{\log p}{p^{\sigma + it}} \right|^{2k} + \left| \sum_{p^2 \leq x} \frac{\log p}{p^{2(\sigma + it)}} \right|^{2k} + \frac{1}{\zeta(3/2)^2} \right).
\]

By Lemma 4
\[
\frac{1}{T} \int_0^T \left| \sum_{p \leq x} \frac{\log p}{p^{\sigma + it}} \right|^{2k} dt \ll k!(2V)^k.
\]

Making the change of variable, \( u = 2t \), we also see that
\[
\frac{1}{T} \int_0^T \left| \sum_{p \leq x} \frac{\log p}{p^{2\sigma + 2it}} \right|^{2k} dt \ll k!.\]

It follows that
\[
\frac{1}{T} \int_0^T |f(t)|^{2k} dt \ll 18^k k!.
\]

To obtain the analogous bound for \( \int_{\mathbb{T}^n} |S(\theta)|^{2k} \, \theta d\theta \), we apply (3.7) and note that the error term is \( \ll 18^k k! \).

Applying the preceding lemma, we can now prove

**Lemma 7.** Let \( 1/2 + 4/\log x \leq \sigma \leq 1 \) and \( x \leq T^{1/(5N)} \) be sufficiently large with \( N \) an even integer. Then

\[
\frac{1}{T} \int_0^T e(\bar{u} \cdot f(t)) \, dt = \int_{\mathbb{T}^n} e(\bar{u} \cdot S(\theta)) \, d\theta + O\left( \frac{(6\sqrt{2}\pi |u| + |v|)^N}{(N/2)!} \right) + O\left( T^{-1/3} \sum_{k=0}^{N-1} (6\sqrt{2}\pi V^{-1/2}(u^2 + v^2)^{1/2})^k \right).
\]

(3.9)

**Proof.** We begin by noting that for any complex number \( z \),
\[
u \Re z + v \Im z = \frac{1}{2} (u - v_i) + \frac{1}{2} (u + v_i) \Re.
\]

Define \( C_1 = \frac{1}{2}(u - vi) \) and \( C_2 = \frac{1}{2}(u + vi) \). By expanding the exponential we see that the left-hand side of (3.9) is equal to
\[
\sum_{k=0}^{N-1} \frac{(2\pi i)^k}{k!} \sum_{j=0}^{k} \binom{k}{j} C_1^j C_2^{k-j} \frac{1}{T} \int_0^T f(t)^j \overline{f(t)^{k-j}} \, dt + O\left( \frac{(2\pi)^N}{N!} (|u| + |v|)^N \right) T \int_0^T |f(t)|^N \, dt.
\]

(3.10)
Similarly, expanding the main term on the right-hand side of the (3.9), we see that

\[
\sum_{k=0}^{N-1} \frac{(2\pi i)^k}{k!} \sum_{j=0}^{k} \binom{k}{j} C_1^j C_2^{k-j} \int_{T^N(s)} S(\theta)^j (S(\theta))^{k-j} d\theta + O\left(\frac{(2\pi)^N}{N!} (|u| + |v|)^N \int_{T^N(s)} |S(\theta)|^N d\theta \right).
\]  

(3.11)

By Lemma 6

\[
\frac{1}{T} \int_0^T |f(t)|^N dt, \quad \int_{T^N(s)} |S(\theta)|^N d\theta \ll 18^{N/2} (N/2)!
\] .

Thus, the error terms in (3.10) and (3.11) are

\[
\ll \left(6\sqrt{2\pi (|u| + |v|)}\right)^N \left(\frac{N}{N_2}\right)!
\] .

Next we difference the main terms of (3.10) and (3.11) and apply Lemma 6 to see that

\[
\sum_{k=0}^{N-1} \frac{(2\pi i)^k}{k!} \sum_{j=0}^{k} \binom{k}{j} C_1^j C_2^{k-j} \left(\frac{1}{T} \int_0^T f(t)^j (f(t))^{k-j} dt - \int_{T^N(s)} S(\theta)^j (S(\theta))^{k-j} d\theta \right)
\ll T^{-1/3} \sum_{k=0}^{N-1} \frac{6\sqrt{2\pi V^{-1/2} (u^2 + v^2)^{1/2}}}{k!} \sum_{j=0}^{k} \left(\binom{k}{j} (j! (k-j)!)^{1/2} \right)
\ll T^{-1/3} \sum_{k=0}^{N-1} \frac{6\sqrt{2\pi V^{-1/2} (u^2 + v^2)^{1/2}}}{k!}.
\]

The result now follows. \square

Finally, we prove the following lemma, which, when combined with Lemma 7, implies Lemma 8.

**Lemma 8.** Let \(10 \leq x \leq T\), \((2\sigma - 1) \log x \geq 1\) and \(|u|, |v| < V^{1/2}/100\). Then

\[
\int_{T^N(s)} e(\bar{u} \cdot S(\theta)) \, d\theta = e^{-2\pi^2 (u^2 + v^2)} \left(1 + E_6(u, v)\right),
\]  

(3.12)

where

\[
E_6 \ll \left(\frac{|u| + |v|^3}{V^{3/2}} + (u^2 + v^2)(x^{1-2\sigma}((2\sigma - 1) \log x + 1))\right).
\]

**Proof.** We first note that the \(\theta_p\) are independent variables, so

\[
\int_{T^N(s)} e(\bar{u} \cdot S(\theta)) \, d\theta = \prod_{p \leq x} \int_0^1 e\left(-\bar{u} \cdot \log p V^{1/2} \sum_{n \leq \log_p x} e(n\theta_p)p^{-n\sigma}\right) \, d\theta_p,
\]

where \(\log_p x\) denotes the logarithm of \(x\) with respect to base \(p\). Expanding the exponential in the integrand on the right-hand side of the equation above and integrating the first three
terms, we find that
\[
\int_{\mathbb{T}^m(x)} e(\tilde{u} \cdot S(\theta)) \, d\theta = \prod_{p \leq x} \left( \prod_{\ell=0}^{\infty} \left( \frac{\tilde{u} \cdot 2\pi i \log p \sum_{n \leq \log_p x} e(n\theta_p)p^{-n\sigma}}{\ell! \sqrt[\ell]{V^\ell}} \right)^\ell \right) d\theta_p
\]
\[
= \prod_{p \leq x} \left( 1 - \frac{\pi^2(u^2 + v^2)}{V} \sum_{n \leq \log_p x} \frac{\log^2 p}{p^{2n\sigma}} \right)
\]
\[
+ \int_{0}^{1} \sum_{\ell=3}^{\infty} \left( \frac{\tilde{u} \cdot 2\pi i \log p \sum_{n \leq \log_p x} e(n\theta_p)p^{-n\sigma}}{\ell! \sqrt[\ell]{V^\ell}} \right)^\ell d\theta_p.
\]

Now write the right-hand side of this equation as \( \prod_{p \leq x} (1 - M_p + R_p) \). Since we are assuming that \( |u|, |v| < V^{1/2}/100 \), we have that
\[
M_p = \frac{\pi^2(u^2 + v^2)}{V} \sum_{n < \log_p x} \frac{\log^2 p}{p^{2n\sigma}} < \frac{\pi^2(u^2 + v^2) \log^2 p}{V(p^{2\sigma} - 1)} < \frac{2\pi^2}{100^2} < 1/3. \tag{3.13}
\]

Here we have used the fact that \( \log^2 x/(x^{2\sigma} - 1) < 1 \) for \( x \geq 2. \) For \( |u|, |v| < V^{1/2}/100 \) we also have that
\[
\left| \int_{0}^{1} \sum_{\ell=3}^{\infty} \left( \frac{\tilde{u} \cdot 2\pi i \log p \sum_{n \leq \log_p x} e(n\theta_p)p^{-n\sigma}}{\ell! \sqrt[\ell]{V^\ell}} \right)^\ell \right|
\]
\[
\leq \sum_{\ell=3}^{\infty} \left( \frac{2\pi(|u| + |v|) \log p \sum_{n \leq \log_p x} p^{-n\sigma}}{\ell! \sqrt[\ell]{V^\ell}} \right)^\ell \tag{3.14}
\]
\[
< \sum_{\ell=3}^{\infty} \left( \frac{12\pi/100}{\ell!} \right)^\ell < 1/3.
\]

It follows that we may expand the logarithm of \( 1 - M_p + R_p \) in powers of \( -M_p + R_p \).

Now, from the estimates in (3.13) and (3.14), it is not difficult to see that
\[
|R_p| \ll \frac{(|u| + |v|)^3 \log^3 p}{V^{3/2} p^{3\sigma}} \quad \text{and} \quad |M_p| \ll \frac{(u^2 + v^2) \log^2 p}{V^{2\sigma}}.
\]

Hence,
\[
\prod_{p \leq x} (1 - M_p + R_p) = \prod_{p \leq x} \exp(\log(1 - M_p + R_p))
\]
\[
= \exp \left( \sum_{p \leq x} (-M_p + R_p + O((M_p + R_p)^2)) \right)
\]
\[
= \exp \left( \sum_{p \leq x} -M_p + O\left( \frac{(|u| + |v|)^3 \log^3 p}{V^{3/2} p^{3\sigma}} \right) \right).
\]

Thus,
\[
\int_{\mathbb{T}^m(x)} e(\tilde{u} \cdot S(\theta)) \, d\theta = \prod_{p \leq x} \exp \left( -\frac{\pi^2(u^2 + v^2)}{V} \sum_{n \leq \log_p x} \frac{\log^2 p}{p^{2n\sigma}} + O\left( \frac{(|u| + |v|)^3 \log^3 p}{V^{3/2} p^{3\sigma}} \right) \right)
\]
\[
= \exp \left( -\frac{\pi^2(u^2 + v^2)}{V} \sum_{p^n \leq x} \frac{\log^2 p}{p^{2n\sigma}} \left( 1 + O\left( \frac{(|u| + |v|)^3}{V^{3/2}} \right) \right) \right). \tag{3.15}
\]
Finally, a short calculation using the Prime Number Theorem reveals that
\[ V^{-1} \sum_{p^n \leq x} \frac{\log^2 p}{p^{2n\sigma}} = 2 \cdot \frac{\sum_{m=2}^{\infty} \frac{\Lambda^2(m)}{m^{2\sigma}}} {\sum_{m=2}^{\infty} \frac{\Lambda^2(m)}{m^{2\sigma}}} + O\left(\frac{\sum_{m>x} \frac{\Lambda^2(m)}{m^{2\sigma}}}{\sum_{m=2}^{\infty} \frac{\Lambda^2(m)}{m^{2\sigma}}} \right). \]

Using this estimate on the last line of \((3.15)\), we obtain \((3.12)\).

3.2. The Proof of Theorem 3. By Lemma 1 and Lemma 3 it follows that, for \(|u|, |v| < V^{1/2}/100\),
\[ \frac{1}{T} \int_0^T e\left(-\bar{u} \cdot \zeta'(\sigma + it)V^{-1/2}\right) dt = e^{-2x^2(u^2+v^2)}(1 + E_4(u, v)) + E_5(u, v) + E_1 \]
where
\[ E_1 \ll (|u| + |v|)V^{-1/2}x^{(1/2-\sigma)/2} \log T + T^{-(\sigma-1/2)/3} \log T \]
\[ E_4(u, v) \ll \frac{(|u| + |v|)^3}{V^{3/2}} + (u^2 + v^2)(x^{1-2\sigma}((2\sigma - 1) \log x + 1)), \]
\[ E_5(u, v) \ll \frac{(6\sqrt{2\pi}(|u| + |v|)^N)}{(N/2)!} + T^{-1/3}. \]

Now take \(N = 2\lfloor \psi(T)/(800 \log \psi(T)) \rfloor\), where \(\psi(T)\) is any function that tends to infinity with \(T\) and is also \(o(\log T)\). Also, let \(x = T^{1/(5N)}, \sigma = 1/2 + \psi(T)/(2 \log T)\), and \(|u|, |v| < \min(V^{1/2}, 1/5N^{1/2})/100\). We note that \(\sigma > 1/2 + 4/\log x\) and \(2V = 1/(2\sigma - 1)^2 + O(1)\).

With these choices we find that
\[ E_4(u, v) \ll \frac{(|u| + |v|)^3}{V^{3/2}} + \frac{(u^2 + v^2)}{\psi(T)^{10}}, \]
and that
\[ E_1, E_5(u, v) \ll \psi(T)^{-10}. \]

This gives Theorem 3.

3.3. An Upper Bound for the Characteristic Function. Before proving Theorem 1 we prove a lemma that will be needed in the proof of Theorem 2. This lemma enables us to get an upper bound on the c.f. of \(\zeta'/\zeta(\sigma + it)V^{-1/2}\) for \(|u|\) or \(|v|\) larger than \(\Omega = e^{-10} \min(V^{1/2}, (\psi(T)/\log \psi(T))^{1/2})\). One may compare this to Theorem 3 where we obtained an asymptotic formula for the c.f., but only for \(|u|, |v| \leq \Omega\).

Lemma 9. Let \(\psi(T) = (2\sigma - 1) \log T\) be any positive function that tends to infinity with \(T\) and is also \(o(\log T)\). There exists a positive constant \(K\) such that if \(T\) is sufficiently large,
\[ \bar{\Omega} = \min\left(e^{-10\left(\frac{\psi(T)}{\log \psi(T)}\right)^{1/2}}, Ke^{\sigma/(2\sigma - 1)}\right), \]
and \(|u|, |v| \leq \bar{\Omega}\), then we have that
\[ \frac{1}{T} \int_0^T e\left(-\bar{u} \cdot \zeta'(\sigma + it)V^{-1/2}\right) dt \ll e^{-c(u^2+v^2)} + \psi(T)^{-10}. \]

Here \(c\) is a positive absolute constant.
Proof. As in the proof of Theorem \ref{thm1} we let $N = 2 \lfloor \psi(T)/(800 \log \psi(T)) \rfloor$. Also, let $x = T^{1/(5N)}$ and $\sigma = 1/2 + \psi(T)/(2 \log T)$. It then follows that $\sigma > 1/2 + 4/\log x$. With these choices we find from Lemma \ref{lem1} that
\[
\frac{1}{T} \int_0^T e \left( -\bar{u} \cdot \zeta'(\sigma + it)V^{-1/2} \right) dt = \frac{1}{T} \int_0^T e \left( \bar{u} \cdot \sum_{n \leq x} \frac{\Lambda(n)}{n^{\sigma+it}V^{-1/2}} \right) dt + \psi(T)^{-10}.
\]
Applying Lemma \ref{lem7}, we have that for $|u|, |v| \leq \tilde{\Omega}$, the right-hand side of this equation equals
\[
\int_{T^{\pi(x)}} e(\bar{u} \cdot S(\theta)) d\theta + \psi(T)^{-10}.
\]
Hence, to prove the lemma it suffices to show that
\[
\int_{T^{\pi(x)}} e(\bar{u} \cdot S(\theta)) d\theta \ll e^{-c(u^2+v^2)},
\]
for $|u|, |v| \leq \tilde{\Omega}$.

If $|u|$ and $|v|$ are less than $V^{1/2}/100$ we are done, by Lemma \ref{lem8}. Thus, we may assume that $|u|$ or $|v|$ is greater than or equal to $V^{1/2}/100$. First observe that for any $D \geq 2$,
\[
\left| \int_{T^{\pi(x)}} e(\bar{u} \cdot S(\theta)) d\theta \right| \leq \prod_{D \leq p \leq x} \left( \int_0^1 e(\bar{u} \cdot S(\theta_p)) d\theta_p \right). \tag{3.16}
\]
Now take
\[
D = C^{1/\sigma} \left( \frac{|u| + |v|}{\sqrt{V}} \right)^{1/\sigma} \left( \log \left( C \frac{|u| + |v|}{\sqrt{V}} \right) \right)^{1/\sigma},
\]
where $C \geq 200$ is an absolute constant to be chosen later. We expand the exponential on the right-hand side of \ref{lem10} and integrate term-by-term, as in the proof of Lemma \ref{lem8}. We thus find that
\[
\int_0^1 e(\bar{u} \cdot S(\theta_p)) d\theta_p = 1 - \frac{\pi^2(u^2+v^2)}{V} \sum_{n \leq \log_p x} \frac{\log^2 p}{p^{2n\sigma}}
\]
\[
+ \int_0^1 \frac{\bar{u} \cdot 2\pi i \log p \sum_{n \leq \log_p x} e(n \theta_p) p^{-n\sigma} \ell}{\ell! V^{\ell/2}}, \tag{3.17}
\]
where $\log_p x$ means the logarithm of $x$ with respect to base $p$. Next notice that $\log x/x^\sigma$ is decreasing for $x \geq 10$. Thus, for $p \geq D$ we have that $\log p/p^\sigma \leq \log D/D^\sigma$. That is,
\[
\frac{\log p}{p^\sigma} \leq \sqrt{V} \log \left( C \frac{(|u| + |v|)}{\sqrt{V}} \log \left( C \frac{(|u| + |v|)}{\sqrt{V}} \right) \right). \tag{3.18}
\]
From this we easily see that
\[
\frac{\log p}{\sqrt{V} p^\sigma} \leq \frac{2}{C} \cdot \frac{\log \left( C \frac{(|u| + |v|)}{\sqrt{V}} \right) + \log \log \left( C \frac{(|u| + |v|)}{\sqrt{V}} \right)}{\log \left( C \frac{(|u| + |v|)}{\sqrt{V}} \right)} \leq \frac{4}{C}.
\]
Consequently,
\[
\frac{\pi^2(u^2+v^2)}{V} \sum_{n \leq \log_p x} \frac{\log^2 p}{p^{2n\sigma}} \leq \frac{\pi^2(|u| + |v|)^2 \log^2 p}{V (p^{2\sigma} - 1)} \leq \frac{32\pi^2}{C^2}.
\]
Next note that
\[
\left| \sum_{\ell=3}^{\infty} \frac{(\bar{u} \cdot 2\pi i \log p \sum_{n \leq \log x} e(n\theta_p)p^{-n\sigma})^\ell}{\ell!V^{\ell/2}} \right| \leq \sum_{\ell=3}^{\infty} \frac{(2\pi(|u| + |v|)\log p \sum_{n \leq \log x} p^{-n\sigma})^\ell}{\ell!V^{\ell/2}} \\
\leq \sum_{\ell=3}^{\infty} \frac{(8\pi(|u| + |v|)\log p)^\ell}{\ell!(\sqrt{V}p^{\sigma})^\ell}.
\]

By (3.18) this is
\[
\leq \sum_{\ell=3}^{\infty} \frac{(32\pi/C)^\ell}{\ell!} \leq \frac{e^{32\pi/C}(32\pi)^3}{C^3}.
\]

Hence, if \( p \geq D \) we may expand the logarithm of
\[
1 - \frac{\pi^2(u^2 + v^2)}{V} \sum_{n \leq \log x} \frac{\log^2 p}{p^{2n\sigma}} + \int_0^1 \sum_{\ell=3}^{\infty} \frac{(\bar{u} \cdot 2\pi i \log p \sum_{n \leq \log x} e(n\theta_p)p^{-n\sigma})^\ell}{\ell!V^{\ell/2}} d\theta
\]
whenever \( C \) is sufficiently large. Write this as \( 1 - M_p + R_p \). Thus, we obtain that
\[
\prod_{D \leq p \leq x} \int_0^1 e(\bar{u} \cdot S(\theta)) d\theta = \exp \left( \sum_{D \leq p \leq x} (-M_p + R_p + O(M_p^2 + R_p^2)) \right).
\]

Since \( M_p \ll (|u| + |v|)^2 \log^2 p/(V^{3/2}p^{3\sigma}) \) and \( R_p \ll (|u| + |v|)^3 \log^2 p/(V^{3/2}p^{3\sigma}) \), by (3.18) we have
\[
R_p + M_p^2 + R_p^2 \ll \frac{(|u| + |v|)^3 \log^3 p}{V^{3/2}p^{3\sigma}} + \frac{(|u| + |v|)^4 \log^4 p}{V^2p^{4\sigma}} + \frac{(|u| + |v|)^6 \log^6 p}{V^3p^{6\sigma}}
\]
\[
= \frac{(|u| + |v|)^2 \log^2 p}{CV^{p^{2\sigma}}} \ll \frac{(u^2 + v^2) \log^2 p}{CV^{p^{2\sigma}}}
\]

So, in particular, for \( C \) large enough it follows that in (3.19) the term \( R_p + O(M_p^2 + R_p^2) \) is \( \leq (\pi^2/2) \cdot (u^2 + v^2) \log^2 p/(V^{3/2}p^{3\sigma}) \). Thus,
\[
\prod_{D \leq p \leq x} \int_0^1 e(\bar{u} \cdot S(\theta)) d\theta \ll \exp \left( -\sum_{D \leq p \leq x} M_p + \frac{\pi^2(u^2 + v^2)}{2V} \sum_{D \leq p \leq x} \frac{\log^2 p}{p^{2\sigma}} \right).
\]

We now choose \( C \) to be large enough so that this holds, \( |R_p| < 1/3 \), and \( |M_p| < 1/3 \).

Next observe that \( M_p \geq \pi^2(u^2 + v^2) \log^2 p/(V^{3/2}p^{3\sigma}) \) so that
\[
\sum_{D \leq p \leq x} M_p \geq \frac{\pi^2(u^2 + v^2)}{V} \sum_{D \leq p \leq x} \frac{\log^2 p}{p^{2\sigma}}.
\]

Applying this in (3.20), we have
\[
\prod_{D \leq p \leq x} \int_0^1 e(\bar{u} \cdot S(\theta)) d\theta \ll \exp \left( -\frac{\pi^2}{2}(u^2 + v^2)V^{-1} \sum_{D \leq p \leq x} \frac{\log^2 p}{p^{2\sigma}} \right).
\]

By the Prime Number Theorem, there is an absolute constant \( c_1 > 0 \) such that
\[
\sum_{D \leq p \leq x} \frac{\log^2 p}{p^{2\sigma}} = \frac{1}{(2\sigma - 1)^2} \left( D_1^{(2\sigma - 1)} \log D + 1 \right) + O \left( (2\sigma - 1)e^{-c_1\sqrt{\log D}} \right)
\]
\[
- x^{(2\sigma - 1)}((2\sigma - 1) \log x + 1) + O \left( (2\sigma - 1)e^{-c_1\sqrt{\log x}} \right).
\]
This follows from the Prime Number Theorem with error term.

By our choice of \(x\) we note that \(x^{1-2\sigma}((2\sigma - 1) \log x + 1) = o(1)\). Also, both error terms are \(o(1)\). Next observe that there is a positive absolute constant \(K\) such that whenever \(|u|, |v| \leq \tilde{\Omega}\), we have

\[
D \leq \left( \frac{2KC}{V^{1/2}} \right)^{1/\sigma} e^{1/(2\sigma - 1)} \left( \log \left( \frac{2KC}{V^{1/2}} e^{\sigma/(2\sigma - 1)} \right) \right)^{1/\sigma}
\]

\[
\leq \left( \frac{2KC}{V^{1/2}} \right)^{1/\sigma} e^{1/(2\sigma - 1)} \left( \log e^{\sigma/(2\sigma - 1)} \right)^{1/\sigma}
\]

\[
\leq \left( \frac{2KC\sigma}{(2\sigma - 1)V^{1/2}} \right)^{1/\sigma} e^{1/(2\sigma - 1)} \leq e^{1/(2\sigma - 1)},
\]

where the last estimate follows from the Prime Number Theorem. Hence, \(D^{1-2\sigma} \geq e^{-1}\) and the right-hand side of (3.22) is \(\gg 1/(2\sigma - 1)^2 \gg V\). Combining this with (3.16) and (3.21) completes the proof.

\[\Box\]

### 4. The Rate of Convergence to the Normal Distribution

#### 4.1. Beurling-Selberg Functions

These functions allow us to obtain bounds on the rate of convergence of the distribution of \(\zeta'/\zeta(\sigma + it)V^{-1/2}\) to the normal distribution.

**Lemma 10.** Let \(\delta\) be a positive real number, let \(a, b \in \mathbb{R}\), and let \(x = z + iy\). There exists an entire function \(F(z)\) with the following properties:

i) \(0 \leq (F(x) - 1_{[a,b]}(x)) \ll \frac{\sin^2(\pi \delta(x-a))}{(\pi \delta(x-a))^2} + \frac{\sin^2(\pi \delta(x-b))}{(\pi \delta(x-b))^2} ;\)

ii) \(\int_{-\infty}^{\infty} (F(x) - 1_{[a,b]}(x)) \, dx \ll 1/\delta;\)

iii) \(\hat{F}(\xi) = 0, \text{ for } \xi \in \mathbb{R} \text{ with } |\xi| \geq \delta;\)

iv) \(\hat{F}(\xi) \ll |b-a| + 1/\delta, \text{ for } \xi \in \mathbb{R}.\)

**Proof.** Property i) follows from Lemma 5 of [9], and property ii) follows from property i). Property iii) follows from [9] (see the argument directly after the proof of Lemma 5). To obtain iv), note that that by i) the \(L^1\) norm of \(F(x)\) is \(\ll |b-a| + 1/\delta.\)

\[\Box\]

**Lemma 11.** Let \(r\) and \(\delta\) be positive real numbers with \(r\delta \geq 1\). Also let \(z \in \mathbb{C}^2\) and \(\vec{x} = (x_1, x_2) \in \mathbb{R}^2\). Then there exist entire functions \(F_+(\vec{z})\) and \(F_-(\vec{z})\), with the following properties:

i) \(F_-(\vec{x}) \leq 1_{[0,r]}(|\vec{x}|) \leq F_+(\vec{x});\)

ii) \(\int_{\mathbb{R}^2} (F_+(\vec{x}) - F_-(\vec{x})) \, d\vec{x} \ll r/\delta;\)

iii) \(\hat{F}_\pm(\vec{\xi}) = 0, \text{ for } \vec{\xi} \in \mathbb{R}^2 \text{ with } |\vec{\xi}| \geq \delta;\)

iv) \(\hat{F}_\pm(\vec{\xi}) \ll r^2, \text{ for } \vec{\xi} \in \mathbb{R}^2,\)

where \(|\vec{x}| = \sqrt{x_1^2 + x_2^2}\), and \(d\vec{x} = dx_1dx_2.\)

**Proof.** Properties i) and ii) follow from Theorem 3 of [9]. By the same theorem, \(F_+(z)\) and \(F_-(z)\) are of exponential type at most \(2\pi \delta\) (for the definition of exponential type see [6]). Thus iii) follows by the Paley-Wiener Theorem (see Chapter III, Theorem 4.9 of [1]). Finally, to obtain iv), note that by i) and ii) the \(L^1\) norm of \(F_\pm(\vec{x})\) is \(\ll r^2 + r/\delta \ll r^2, \text{ since } r\delta \geq 1. \)

\[\Box\]

#### 4.2. The Proof of Theorem 1

Let \(\psi(T), \Omega\) and \(R\) be as in the statement of Theorem 1. Also, let \(\tilde{u} = (u, v) \in \mathbb{R}^2, \tilde{d}u = du dv, \tilde{\Omega}\) be as in Lemma 9, and \(\Omega\) be as in Theorem 9. We take \(F\) to be the analytic function from Lemma 10 that approximates \(1_{[a,b]}(x)\) along the real
axis, and $G$ to be the one that approximates $1_{[c,d]}(x)$. We set $\delta = \tilde{\Omega}$ in both functions. Also, let

$$H(x) = \frac{\sin^2(\pi \tilde{\Omega} x)}{(\pi \tilde{\Omega} x)^2}.$$ 

By property i) of Lemma 10 we have

$$F(x)G(y) = 1_R(x,y) + O\left(H(x-a) + H(x-b) + H(y-c) + H(y-d)\right).$$

(4.1)

Now, note that

$$\frac{1}{T}\text{meas}\left\{ t \in [0,T] : \frac{\zeta'}{\zeta}(\sigma + it) V^{-1/2} \in R \right\} = \frac{1}{T} \int_0^T 1_R\left(\frac{\zeta'}{\zeta}(\sigma + it) V^{-1/2}\right) dt. \tag{4.2}$$

By (4.1) the right-hand side of (4.2) equals

$$\frac{1}{T} \int_0^T F\left(\Re\frac{\zeta'}{\zeta}(\sigma + it) V^{-1/2}\right) G\left(\Im\frac{\zeta'}{\zeta}(\sigma + it) V^{-1/2}\right) dt \tag{4.3}$$

plus an error that is

$$\ll \frac{1}{T} \int_0^T H\left(\Re\frac{\zeta'}{\zeta}(\sigma + it) V^{-1/2} - a\right) dt + \frac{1}{T} \int_0^T H\left(\Re\frac{\zeta'}{\zeta}(\sigma + it) V^{-1/2} - b\right) dt$$

$$+ \frac{1}{T} \int_0^T H\left(\Im\frac{\zeta'}{\zeta}(\sigma + it) V^{-1/2} - c\right) dt + \frac{1}{T} \int_0^T H\left(\Im\frac{\zeta'}{\zeta}(\sigma + it) V^{-1/2} - d\right) dt.$$

Since,

$$H(x) = \frac{2(1 - \cos(2\pi \tilde{\Omega} x))}{(2\pi \tilde{\Omega} x)^2} = \frac{2}{\tilde{\Omega}^2} \int_0^{\tilde{\Omega}} (\tilde{\Omega} - u) \cos(2\pi xu) du,$$

the first term in the error above is

$$\ll \frac{1}{\tilde{\Omega}^2} \int_0^{\tilde{\Omega}} (\tilde{\Omega} - u)(e^{-c(u^2 + \psi(T)^{10})} du \ll 1/\tilde{\Omega}.$$ 

In the inner integral $|u| \leq \tilde{\Omega}$, so we may apply Lemma 9 with $v = 0$ to see that this is

$$\ll \frac{1}{\tilde{\Omega}^2} \int_0^{\tilde{\Omega}} (\tilde{\Omega} - u)(e^{-c(u^2 + \psi(T)^{10})} du \ll 1/\tilde{\Omega}.$$ 

Clearly, the other error terms can be bounded similarly. Hence, upon applying Fourier inversion to (4.3) we have that the left-hand side of (4.2) equals

$$\int_{\mathbb{R}^2} \hat{F}(u)\hat{G}(v) \frac{1}{T} \int_0^T e\left(\bar{u} \cdot \frac{\zeta'}{\zeta}(\sigma + it) V^{-1/2}\right) dtdu + O(1/\tilde{\Omega}).$$

By property iii) of Lemma 10 the integral equals

$$\int_{-\tilde{\Omega}}^{\tilde{\Omega}} \int_{-\tilde{\Omega}}^{\tilde{\Omega}} \hat{F}(u)\hat{G}(v) \frac{1}{T} \int_0^T e\left(\bar{u} \cdot \frac{\zeta'}{\zeta}(\sigma + it) V^{-1/2}\right) dtdu.$$

We now apply Theorem 3 and Lemma 9 to see that this is

$$\int_{-\tilde{\Omega}}^{\tilde{\Omega}} \int_{-\tilde{\Omega}}^{\tilde{\Omega}} \hat{F}(u)\hat{G}(v) e^{-2\pi(u^2 + v^2)} (1 + \mathcal{E}_A(u,v) + \mathcal{E}_B) d\bar{u}$$

$$+ O\left(\int_{[-\tilde{\Omega},\tilde{\Omega}]^2 \setminus [-\tilde{\Omega},\tilde{\Omega}]^2} |\hat{F}(u)||\hat{G}(v)|(e^{-c(u^2 + v^2)} + \psi(T)^{10}) d\bar{u}\right), \tag{4.4}$$
where $\mathcal{E}_A(u,v)$ and $\mathcal{E}_B$ are as in Theorem $[3]$. We first estimate the integral in the $O$-term. Note that we are assuming that $(2\sigma - 1) = o(1)$. This implies that $\Omega \leq \bar{\Omega}$, so that $|b - a|, |d - c| \geq \bar{\Omega}^{-1}$. Hence, by property $iv$ of Lemma $[10]$, the integral in the $O$-term is

$$
\ll \int_{[-\bar{\Omega},\bar{\Omega}] \setminus [-\Omega,\Omega]} |b - a||d - c|(e^{-c(u^2 + v^2)} + \psi(T)^{-10})d\bar{u}
$$

$$
\ll |b - a||c - d| \left( \int_{\Omega}^{\infty} e^{-cu^2} du \right)^2 + \frac{\bar{\Omega}^2}{\psi(T)^{10}}.
$$

Since $\Omega \ll \Omega^2$ and $\bar{\Omega} \leq \psi(T)^{1/2}$, this is easily seen to be $\ll |b - a||c - d|/\Omega$. We now write the first integral in (4.4) as $I_1 + I_2 + I_3$, where $I_1$ is the integral of $\hat{F}(u)\hat{G}(v)e^{-2\pi(u^2 + v^2)}$ over $[-\Omega,\Omega]^2$, $I_2$ is the integral of $\tilde{F}(u)\tilde{G}(v)e^{-2\pi(u^2 + v^2)}\mathcal{E}_A(u,v)$ over $[-\Omega,\Omega]^2$, and $I_3$ is the rest. Then to prove Theorem $[1]$ it suffices to show that $I_1 + I_2 + I_3 = \int_a^b \int_c^d e^{-(x^2 + y^2)/2} dx dy + O((|b - a||d - c| + 1)/\Omega)$. By Theorem $[3]$, $\mathcal{E}_A(u,v) \ll (|u| + |v|)^3/V^{3/2} + (u^2 + v^2)/\psi(T)$ and $\mathcal{E}_B \ll \psi(T)^{-10}$. As in our treatment of the $O$-term in (4.4), we apply property $iv$ of Lemma $[10]$ and find that $I_2 \ll |b - a||d - c|/V \ll |b - a||d - c|/\Omega$ (4.5) and $I_3 \ll |b - a||d - c|/\psi(T)^9 \ll |b - a||d - c|/\Omega$. (4.6)

To estimate $I_1$, we extend the integral to all of $\mathbb{R}^2$ with a small error that is easily seen to be $\ll |b - a||d - c|/\Omega$. Next we apply Plancherel’s theorem to see that

$$
\int_{\mathbb{R}^2} \hat{F}(u)\hat{G}(v)e^{-2\pi(u^2 + v^2)} d\bar{u} = \left( \int_{\mathbb{R}} \hat{F}(u)e^{-2\pi u^2} du \right) \left( \int_{\mathbb{R}} \hat{G}(v)e^{-2\pi v^2} dv \right)
$$

$$
= \frac{1}{2\pi} \left( \int_{\mathbb{R}} F(x)e^{-x^2/2} dx \right) \left( \int_{\mathbb{R}} G(y)e^{-y^2/2} dy \right).
$$

By property $ii$) of Lemma $[10]$

$$
\int_{\mathbb{R}} F(x)e^{-x^2/2} dx = \int_a^b e^{-x^2/2} dx + O(1/\bar{\Omega}).
$$

An analogous result holds for $G$, so we have

$$
I_1 = \frac{1}{2\pi} \left( \int_a^b e^{-x^2/2} dx + O \left( \frac{|b - a||d - c| + 1}{\Omega} \right) \right) \left( \int_c^d e^{-y^2/2} dy + O \left( \frac{|b - a||d - c| + 1}{\Omega} \right) \right)
$$

$$
= \frac{1}{2\pi} \int_a^b \int_c^d e^{-(x^2 + y^2)/2} dx dy + O((|b - a||d - c| + 1)/\Omega).
$$

This combined with (4.5) and (4.6) yields

$$
I_1 + I_2 + I_3 = \int_a^b \int_c^d e^{-(x^2 + y^2)/2} dx dy + O((|b - a||c - d| + 1)/\Omega).
$$
4.3. The Proof of Theorem 2. Let $\psi(T)$, $\Omega$, and $r$ be as in the statement of Theorem 2. Also, let $\tilde{u} = (u, v) \in \mathbb{R}^2$, $\tilde{d}u = du dv$, $\Omega$ be as in Lemma 9 and let $\tilde{\Omega}$ be as in Theorem 3. We also let $D_{\Omega}(0)$ denote the disk of radius $r_1$ centered at the origin. We now consider $F_+(z)$ from Lemma 11 with $\delta = \tilde{\Omega}$ (note $\Omega \leq \tilde{\Omega}$ so $\tilde{\Omega} r \geq 1$).

By Fourier inversion

$$
\frac{1}{T} \int_0^T F_+ \left( \Re \frac{\zeta'}{\zeta}(\sigma + it)V^{-1/2}, \Im \frac{\zeta'}{\zeta}(\sigma + it)V^{-1/2} \right) dt
= \int F_+(\tilde{u}) \frac{1}{T} \int_0^T e \left( \tilde{u} \cdot \frac{\zeta'}{\zeta}(\sigma + it)V^{-1/2} \right) dt d\tilde{u}.
$$

By property $iii$ of Lemma 11, this is

$$
= \int_{D_{\Omega}(0)} \hat{F}_+(\tilde{u}) \frac{1}{T} \int_0^T e \left( \tilde{u} \cdot \frac{\zeta'}{\zeta}(\sigma + it)V^{-1/2} \right) dt d\tilde{u},
$$

and by Theorem 3 and Lemma 9, this equals

$$
\int_{D_{\Omega}(0)} \hat{F}_+(\tilde{u}) \left( e^{-2\pi^2(u^2+v^2)}(1 + E_A(u,v)) + E_B \right) d\tilde{u}
+ O \left( \int \hat{F}_+(\tilde{u}) \left| e^{-c(u^2+v^2) + \psi(T)^{-10}} \right| d\tilde{u} \right),
$$

where $A_{\Omega,\tilde{\Omega}}(0)$ is the annulus with radii $\Omega$ and $\tilde{\Omega}$ centred at the origin. By property $iv$ of Lemma 9, the integral over the annulus is

$$
\ll r^2 \int_{A_{\tilde{\Omega},\tilde{\Omega}}(0)} \left( e^{-c(u^2+v^2)} + \psi(T)^{-10} \right) d\tilde{u} \ll r^2/\Omega.
$$

We now write

$$
\int_{D_{\Omega}(0)} \hat{F}_+(\tilde{u}) \left( e^{-2\pi^2(u^2+v^2)}(1 + E_A(u,v)) + E_B \right) d\tilde{u} = I_1 + I_2 + I_3,
$$

where $I_1$ is the integral of $\hat{F}_+(\tilde{u})e^{-2\pi^2(u^2+v^2)}$ over $D_{\Omega}(0)$, $I_2$ is the integral of $\hat{F}_+(\tilde{u})e^{-2\pi^2(u^2+v^2)}E_A(u,v)$ over $D_{\Omega}(0)$, and $I_3$ is the rest. By Theorem 3, $E_A(u,v) \ll (|u| + |v|)^5/V^{3/2} + (u^2 + v^2)/\psi(T)^{10}$ and $E_B \ll \psi(T)^{-10}$. We first estimate $I_3$. By property $iv$ of Lemma 11

$$
I_3 \ll r^2\Omega^2/\psi(T)^{10} \ll r^2/\Omega.
$$

Similarly, we have

$$
I_2 \ll r^2/\Omega + r^2/\psi(T)^{10} \ll r^2/\Omega.
$$

To estimate $I_1$, we note that by property $iv$ of Lemma 11

$$
\int_{D_{\Omega}(0)} \hat{F}_+(\tilde{u})e^{-2\pi^2(u^2+v^2)} d\tilde{u} = \int_{\mathbb{R}^2} \hat{F}_+(\tilde{u})e^{-2\pi^2(u^2+v^2)} d\tilde{u} + O(r^2/\Omega).
$$

By Plancherel’s Theorem the integral on the right-hand side equals

$$
\frac{1}{2\pi} \int_{\mathbb{R}^2} F_+(\tilde{x})e^{-(x_1^2+x_2^2)/2} d\tilde{x}.
$$
Collecting our estimates, we have that

\[
\frac{1}{T} \int_0^T F_+ \left( \Re \frac{C'}{\zeta} (\sigma + it)V^{-1/2}, \Im \frac{C'}{\zeta} (\sigma + it)V^{-1/2} \right) dt = \frac{1}{2\pi} \int_{\mathbb{R}^2} F_+ (\vec{x}) e^{-\left( x_1^2 + x_2^2 \right)/2} d\vec{x} + O\left( r^2 / \Omega \right). \tag{4.7}
\]

Now, by property \textit{i)} of Lemma \[11\]

\[
\int_0^T F_- \left( \Re \frac{C'}{\zeta} (\sigma + it)V^{-1/2}, \Im \frac{C'}{\zeta} (\sigma + it)V^{-1/2} \right) dt \leq \int_0^T 1_{D_r(0)} \left( \Re \frac{C'}{\zeta} (\sigma + it)V^{-1/2} \right) dt \leq \int_0^T F_+ \left( \Re \frac{C'}{\zeta} (\sigma + it)V^{-1/2}, \Im \frac{C'}{\zeta} (\sigma + it)V^{-1/2} \right) dt.
\]

By this, \textit{(4.7)}, the analogue of \textit{(4.7)} for \( F_- (u) \), and property \textit{i)} of Lemma \[11\] we have that

\[
\int_0^T 1_{D_r(0)} \left( \Re \frac{C'}{\zeta} (\sigma + it) \right) dt = \frac{1}{2\pi} \int_{D_r(0)} e^{-\left( x_1^2 + x_2^2 \right)/2} d\vec{x} + O\left( r^2 / \Omega + \int_{\mathbb{R}^2} (F_+ (\vec{x}) - F_- (\vec{x})) e^{-\left( x_1^2 + x_2^2 \right)/2} d\vec{x} \right).
\]

By property \textit{ii)} of Lemma \[11\] the integral is \( \ll r / \tilde{\Omega} \ll r / \Omega \). The first assertion of the theorem now follows upon noting that

\[
\frac{1}{2\pi} \int_{D_r(0)} e^{-\left( x_1^2 + x_2^2 \right)/2} d\vec{x} = 1 - e^{-r^2/2}.
\]

As for the second assertion, by Fourier inversion

\[
\frac{1}{T} \text{ meas} \left\{ t \in (0, T) : \left| \frac{C'}{\zeta} (\sigma + it) \right| \leq \sqrt{V} r \right\} \leq \int_{\mathbb{R}^2} \hat{F}_+ (\vec{u}) \frac{1}{T} \int_0^T e \left( \vec{u} \cdot \frac{C'}{\zeta} (\sigma + it)V^{-1/2} \right) dt d\vec{u}.
\]

By property \textit{iii)} of Lemma \[11\] we may remove the portion of the integral with \( u^2 + v^2 > \tilde{\Omega}^2 \). We then apply Lemma \[9\] and property \textit{iv)} of Lemma \[11\] to see that the right-hand side of the above inequality is

\[
\ll \int_{D_r(0)} r^2 \left( e^{-c(u^2 + v^2)} + \psi(T)^{-10} \right) d\vec{u} \ll r^2.
\]

Noting that \( r \geq 1 / \tilde{\Omega} \), we obtain the result.

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REFERENCES

[1] D. W. Farmer, S. M. Gonek, Y. Lee, and S. J. Lester, ‘Mean values of $\zeta'/\zeta(s)$, correlation of zeros, and the distribution of almost primes’, Q. J. Math., to appear.

[2] A. Ghosh, ‘On Riemann’s zeta function—sign changes of $S(T)$’, Recent Progress in Analytic Number Theory, vol. 1, (Academic Press, New York, 1981) 29-46.

[3] D. A. Goldston, S. M. Gonek, and H.L. Montgomery, ‘Mean values of the logarithmic derivative of the Riemann zeta-function with applications to primes in short intervals’, J. reine. angew. Math. 537 (2001), 105-126.

[4] C. R. Guo, ‘The Distribution of the Logarithmic Derivative of the Riemann Zeta Function’, Proc. London Math. Soc. (3) 72 (1996), 1-27.

[5] J. J. Holt and J. D. Vaaler, ‘The Beurling-Selberg extremal functions for a ball in Euclidean space’, Duke Math. Journal. 83 (1996) no. 1, 203-248.

[6] M. Jutila, ‘Zeros of the zeta function near the critical line’, Studies in pure mathematics to the memory of Paul Turán, (Birkhäuser Verlag, Basel-Stuttgart, 1982), 385-394.

[7] H. L. Montgomery, ‘The pair correlation of zeros of the zeta function’, Proc. Sympos. Pure Math. 24, ( Amer. Math. Soc., Providence, R.I., 1973), 181-193.

[8] ———, ‘The analytic principle of the large sieve’, Bull. Amer. Math. Soc. 84 (1978), no. 4, 547-567.

[9] H. L. Montgomery and R.C. Vaughan, ‘Hilbert’s inequality’, J. London Math. Soc. (2), 8 (1974), 73-82.

[10] A. Selberg, ‘On the remainder in the formula for $N(T)$, the number of zeros of $\zeta(s)$ in the strip $0 < t < T$’, Avh. Norske Vid. Akad. Oslo. I 1 (1944).

[11] ———, ‘Contributions to the theory of the Riemann zeta-function’, Arch. Math. Naturvid. 48 (1946) no. 5, 89-155.

[12] ———, with a forward by K. Chandrasekharan, Collected papers, vol. II (Springer, Berlin, 1991).

[13] ———, ‘Old and new conjectures and results about a class of Dirichlet series’, Proceedings of the Amalfi Conference on Analytic Number Theory, (ed. E. Bombieri, Università di Salerno, Maiori, 1992), 367-385.

[14] K. Soundararajan, ‘Moments of the Riemann zeta-function’, Ann. of Math. (2) 170 (2009), no. 2, 981-993.

[15] E. M. Stein and G. Weiss, Fourier Analysis on Euclidean Space, (Princeton Univ. Press, Princeton, 1971).

[16] K. M. Tsang, ‘The distribution of the values of the Riemann zeta-function’, PhD Thesis, Princeton University, Princeton, 1984.

[17] ———, ‘Some $\Omega$-theorems for the Riemann zeta-function’, Acta Arith., 46 (1986), no. 4, 369-395.

[18] J.D. Vaaler, ‘Some extremal functions in Fourier analysis’, Bull. Amer. Math. Soc., 12 (1985), no. 2, 183-216.

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