Vintage Factor Analysis with Varimax Performs Statistical Inference

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Abstract

Psychologists developed Multiple Factor Analysis to decompose multivariate data into a small number of interpretable factors without any a priori knowledge about those factors [Thurstone, 1935]. In this form of factor analysis, the Varimax “factor rotation” is a key step to make the factors interpretable [Kaiser, 1958]. Charles Spearman and many others objected to factor rotations because the factors seem to be rotationally invariant [Thurstone, 1947, Anderson and Rubin, 1956]. These objections are still reported in all contemporary multivariate statistics textbooks. This is an enigma because this vintage form of factor analysis has survived and is widely popular because, empirically, the factor rotation often makes the factors easier to interpret. We argue that the rotation makes the factors easier to interpret because, in fact, the Varimax factor rotation performs statistical inference. We show that Principal Components Analysis (PCA) with the Varimax rotation provides a unified spectral estimation strategy for a broad class of modern factor models, including the Stochastic Blockmodel and a natural variation of Latent Dirichlet Allocation (i.e., “topic modeling”). In addition, we show that Thurstone’s widely employed sparsity diagnostics implicitly assess a key “leptokurtic” condition that makes the rotation statistically identifiable in these models. Taken together, this shows that the know-how of Vintage Factor Analysis performs statistical inference, reversing nearly a century of statistical thinking on the topic. With a sparse eigensolver, PCA with Varimax is both fast and stable. Combined with Thurstone’s straightforward diagnostics, this vintage approach is suitable for a wide array of modern applications.

Keywords: Factor analysis, Independent Component Analysis, Spectral Clustering

Outside the language of mathematical statistics, Louis Leon Thurstone, Henry Kaiser, and other psychologists developed the first forms of Multiple Factor Analysis, or what is referred to herein as Vintage Factor Analysis [Thurstone, 1935, 1947, Kaiser, 1958]. There are two simultaneous aims of Vintage Factor Analysis. The first aim is to provide a low dimensional approximation of the observed data; in this sense, it is like Principal Components Analysis (PCA)\(^1\) The second aim is to ensure that each factor in the lower dimensional

\(^1\)PCA is not the preferred approach in Vintage Factor Analysis. See Remark 3.3 for a further discussion.
representation, each coordinate, represents a “scientifically meaningful category” [Thurstone, 1935]. A Varimax rotation of the principal components is a simple and popular way to find such meaningful dimensions [Kaiser, 1958; Jolliffe, 2002].

For example, suppose \( n \) students take an exam with \( d \) questions, producing a \( d \)-dimensional vector of data for each individual. Principal components analysis with \( k=2 \) dimensions will roughly approximate the students’ \( d \)-dimensional data; this is the first aim of factor analysis. In order to make those two dimensions more interpretable, these principal components are rotated with the Varimax rotation. In other words, Varimax provides a different coordinate basis for the two dimensional space. Selecting the basis does not change the quality of the lower dimensional approximation. However, after inspecting the \( k=2 \) Varimax coordinates, an analyst might find that one coordinate represents “linguistic intelligence” and the other coordinate represents “logical-mathematical intelligence.”

This form of data analysis is often called “exploratory” because the factor dimensions are computed from the data without needing any hypothesis that specifies them.

Factor analysis is an enigma. The key source of the controversy in Vintage Factor Analysis is the second aim, producing coordinates that correspond to “scientifically meaningful categories.” [Anderson and Rubin, 1956] formalized the concern by showing that under the Gaussian factor model, all rotations achieve the same fit. This result implies that under the Gaussian factor model, the individual Varimax coordinates cannot estimate anything meaningful. [Shalizi, 2009] gives the conventional interpretation of this result, “If we can rotate the factors as much as we like without consequences, how on Earth can we interpret them?” Contemporary multivariate analysis textbooks all discuss the result from [Anderson and Rubin, 1956], but then go on to report the empirical benefits of the factor rotation. [Ramsay and Silverman, 2007] says “It is well known in classical multivariate analysis that an appropriate rotation of the principal components can, on occasion, give components ... more informative than the original components themselves.” [Johnson and Wichern, 2007] says “A rotation of the factors often reveals a simple structure and aids interpretation.” [Bartholomew et al., 2011] says “Rotation assumes a very important role when we come to the interpretation of latent variables.” [Jolliffe, 2002] says “The simplification achieved by rotation can help in interpreting the factors or rotated PCs.” These empirical findings appear to be inconsistent with the results of [Anderson and Rubin, 1956] that are described in those same textbooks.

Varimax is the most popular way of computing a factor rotation [Kaiser, 1958]. It is discussed in all of the textbooks cited in the previous paragraph. [Ramsay and Silverman, 2005] describes Varimax as an “invaluable tool in multivariate analysis.” It is contained in the base \( R \) packages, akin to \texttt{kmeans}, and is so popular that it is often not properly cited. Given an \( n \times k \) matrix \( U \), with columns that form an orthonormal basis (e.g., as in PCA), Varimax finds a \( k \times k \) orthogonal matrix \( R \) to maximize the following function over the set
of \( k \times k \) orthonormal matrices

\[
v(R, U) = \sum_{\ell=1}^{k} \sum_{i=1}^{n} \left( UR_{ij} \right)^{4} - \left( \frac{1}{n} \sum_{q=1}^{n} \left[ UR_{i\ell} \right]^{2} \right)^{2}.
\]

(1)

Kaiser [1958] suggests normalizing each row of \( U \) such that each row has a sum of squares equal to one. For simplicity, we do not use this normalization herein.

Factor rotations have survived for nearly a century because a rotation often makes the factors more interpretable. Yet the classical theoretical results do not explain how or why. Maxwell’s Theorem resolves the enigma [Maxwell [1860] and III,4 in Feller [1971]]. It characterizes the multivariate Gaussian distribution as the only distribution of independent random variables that is rotationally invariant. This implies that the rotation is partially identifiable, so long as the factors are independent and come from any non-Gaussian distribution. As such, if the independent latent factors are generated from a non-Gaussian distribution, then the factor rotation has the potential to identify these factors as “scientifically meaningful categories.” See Figure 1 for an example in \( k = 2 \) dimensions.

![Figure 1](image)

Figure 1: Maxwell’s Theorem characterizes the multivariate Gaussian distribution (left panel) as the only rotationally invariant distribution of independent variables. The center panel and the right panel give the same data; the only difference is that the right panel gives the basis that is well estimated by Varimax.

Maxwell’s theorem and some of the core factor analysis methodologies have been rediscovered and further developed in the literature on Independent Components Analysis (ICA) [Hyvärinen et al., 2004]. More recently, [Anandkumar et al., 2014] showed how a

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2 In R, the function `varimax` has a default argument `normalize = TRUE`. Note that when \( U \) has orthogonal columns (as is the case for PCA) and normalization is not used, then the second term in Varimax is a constant function of the matrix \( R \). In such cases, this term can be ignored without changing the optimum.

3 A common point of confusion is to presume that the factors must be Gaussian if we are using PCA; see Section 3 and Remark 3.1 to see how PCA performs with non-Gaussian factors.
In this data example, the principal components (left) have radial streaks. Varimax aligns the streaks with the axes (right). Varimax rotated PCA is Vintage Sparse PCA, vsp.

Figure 2: In this example, the data is a $300,000 \times 102,660$ document-term matrix of 300,000 New York Times articles. Each small panel on the left is a scatter plot of two principal components. Each small panel on the right is a scatter plot of two Varimax rotated components. The numbers down the diagonal give the sample kurtosis of the corresponding component. See Section 1.1 for more details.

tensor decomposition can estimate a broad class of factor models that is closely related to the class studied below. This paper demonstrates that an old approach with historical precedence to ICA is sufficient; tensor methods are not required. This old approach comes with a suite of know-how and diagnostic practices that are described in Section 2. This old approach provides a unified spectral estimation strategy and diagnostic practices that can be applied to many different problems in multivariate statistics. It relates Projection Pursuit, Independent Components Analysis, Non-Negative Matrix Decompositions, Latent Dirichlet Allocation, and Stochastic Blockmodeling.

Figure 2 shows a motivating data example with a set of 300,000 New York Times articles [Dua and Graff, 2017]. In this example, the data matrix $A$ is a $300,000 \times 102,660$ document-word matrix, where $A_{ij} \in \{0,1\}$ indicates if document $i$ contains word $j$. Figure 2(a) plots nine of the leading principal components. Figure 2(b) plots these components after a Varimax rotation. Section 1 describes this procedure in more detail. See Section 1.1 for further details on the data analysis in Figure 2.
All the plots in Figure 2 display “radial streaks,” a phrase used in [Thurstone 1947] to diagnose factor rotations. In Figure 2(b), the Varimax rotation aligns the streaks with the coordinate axes. This is precisely the desired outcome of a factor rotation, to make the rotated components approximately sparse. For this reason, this paper refers to Varimax rotated PCA as Vintage Sparse PCA (vsp). Modern notions of Sparse PCA (e.g., [d’Aspremont et al. 2005]) presume that the principal components are themselves sparse. In the vintage notion of sparse PCA, it is presumed that there exists a set of sparse basis vectors for the principal component subspace. That is, perhaps the principal components are not sparse, but they become sparse after a rotation. These are two distinct notions of subspace sparsity. [Vu and Lei 2013] referred to the vintage notion of sparsity as column-wise sparsity.

Theorem 4.1 shows that, under certain conditions, vsp estimates the following semi-parametric factor model that generalizes the Stochastic Blockmodel and Latent Dirichlet Allocation.

**Definition 1.** Let $Z \in \mathbb{R}^{n \times k}$ and $Y \in \mathbb{R}^{d \times k}$ be latent factor matrices. Under the semi-parametric factor model, we observe $A \in \mathbb{R}^{n \times d}$ which has independent elements and has expectation

$$E(A|Z,Y) = ZBY^T,$$

where $B \in \mathbb{R}^{k \times k}$ is not necessarily diagonal. (2)

Importantly, in the semi-parametric factor model, the columns of $Z$ are not the principal components. However, if the elements of $Z$ are independently generated from a “leptokurtic” distribution, then a Varimax rotation of the principal components estimates the columns of $Z$. This leptokurtic condition this is the key identifying assumption for Varimax and vsp.

**Definition 2.** For a random variable $X \in \mathbb{R}$ with four finite moments, let $\eta = E(X)$ and define the $j$th centered moment as $\eta_j = E((X - \eta)^j)$ for $j = 2, 4$. The kurtosis of $X$ is $\kappa = \eta_4/\eta_2^2$. The random variable $X$ and its distribution are leptokurtic if $\kappa > 3$.

Kurtosis was originally named and used by Pearson around 1900 to measure whether a symmetric distribution was Gaussian [Fiori and Zenga 2009]. For any Gaussian random variable, $\kappa = 3$. As such, $\kappa \neq 3$ indicates a non-Gaussian distribution. Roughly speaking, when $\kappa > 3$, the distribution has a heavier tail than Gaussian.

Section 1 describes the vsp algorithm and some variations on the algorithm. Section 2 reinterprets the sparsity diagnostics developed in [Thurstone 1935, 1947] to show that they implicitly assess the key identifying assumption for vsp to estimate the semi-parametric factor model (i.e., whether the factors in the columns of $Z$ appear leptokurtic). In particular, Section 2.1 discusses Thurstone notion of “simple structure” (a form of sparsity), his conjecture that simple structure resolves the rotational invariance, and his sparsity diagnostics that are described in modern textbooks, built into the base R packages for factor
analysis, and used routinely in practice. Then, Theorem 2.1 shows that any random variable $X$ that satisfies $P(X = 0) > 5/6$ (i.e., it is sparse) is necessarily leptokurtic. In this way, Thurstone’s sparsity diagnostics and the know-how of Vintage Factor Analysis can be reinterpreted as assessing a key identifiability assumption for Varimax.

Sections 3 gives intuition for why $vsp$ can estimate the latent factors by giving population results. The first results show that the column space of the principal components of $\mathcal{A} = E(A|Z,Y) = ZBY^T$ equals to the column space of $Z$. Then, a Varimax rotation of the principal components specifies a new set of basis vectors for that column space. Under the identifying assumption where the elements of $Z$ are generated independently from a leptokurtic distribution and the entire distribution of $Z$ is known (i.e., infinite sample size), Theorem 3.1 shows that each of the new basis vectors is estimating an individual column of $Z$ (up to a sign change). If the elements of $Y$ satisfy the same conditions required for $Z$, then $Y$ and $B$ can also be estimated, even when $B$ is not diagonal. Section 4 gives the main theoretical result, Theorem 4.1, which shows that $vsp$ can estimate $Z$, when the matrix $A$ is high dimensional and random. This result allows for $A$ to be sparse and is enabled by recent technical developments that provide “element-wise” eigenvector bounds for random graphs [Erdős et al., 2013, Cape et al., 2019a, Mao et al., 2018]. Section 5 describes how the broad class of semi-parametric factor models includes the Stochastic Blockmodel, several of its generalizations, and a natural extension of Latent Dirichlet Allocation. Corollaries 5.1 and 5.2 extend Theorem 4.1 to these models.

**Key Notation:** Let $\mathcal{O}(k) = \{ R \in \mathbb{R}^{k \times k} : R^T R = RR^T = I_k \}$ denote the set of $k \times k$ orthonormal matrices. Let $1_a \in \mathbb{R}^a$ be a column vector of ones. Let $I_d$ denote the $d \times d$ identity matrix. For $x \in \mathbb{R}^d$, let $\text{diag}(x) \in \mathbb{R}^{d \times d}$ be a diagonal matrix with $\text{diag}(x)_{ii} = x_i$. For $M \in \mathbb{R}^{a \times b}$, define $M_i \in \mathbb{R}^b$ as the $i$th row of $M$ and $\| M \|_{p \rightarrow \infty} = \max_i \| M_i \|_p$, for $p \geq 1$ and $\ell_p$ norm for vectors $\| \cdot \|_p$. Let $\| M \|_F$ be the Frobenius norm, $\| M \|$ be the spectral norm, $\| M \|_\infty$ be the maximum absolute row sum of $M$, and $\| M \|_{\max}$ be the maximum element of $M$ in absolute value. For sequences $x_n, y_n \in \mathbb{R}$, define $x_n \asymp y_n$ to mean that $x_n \rightarrow \infty$ and $y_n \rightarrow \infty$ and there exists an $N, \epsilon, c$ all in $(0, \infty)$ such that $x_n / y_n \in (\epsilon, c)$ for all $n > N$. Define $x_n \succeq y_n$ to mean that for any $\epsilon \in (0, \infty)$, there exists an $N < \infty$ such that for all $n > N$, $x_n / y_n > \epsilon > 0$. Define $[k] = \{1, \ldots, k\}$.

1 **vsp: Vintage Sparse PCA**

This section describes the methodological details of Vintage Sparse PCA ($vsp$). First, the algorithm is stated. Then, Remarks 1.1, 1.2, and 1.3 describe ways in which $vsp$ can be modified for certain settings; Table 1 summarizes these settings. Section 1.1 illustrates the algorithm with a corpus of New York Times articles; this is the analysis that generated Figure 2 above.

Algorithm: $vsp$
- Input $A \in \mathbb{R}^{n \times d}$ and desired number of dimensions $k$.

1. **Centering** (optional). Define row, column, and grand means,

$$
\hat{\mu}_r = A 1_d / d \in \mathbb{R}^n, \quad \hat{\mu}_c = 1_n^T A / n \in \mathbb{R}^d, \quad \hat{\mu} = 1_n^T A 1_d / (nd) \in \mathbb{R}.
$$

Here $\hat{\mu}_r$ is a column vector and $\hat{\mu}_c$ is a row vector. Define

$$
\tilde{A} = A - \hat{\mu}_r 1_d^T - 1_n \hat{\mu}_c + \hat{\mu} 1_d 1_n^T \in \mathbb{R}^{n \times d}.
$$

(3)

If $A$ is large and sparse, step 1 and 2 can be accelerated. See Remark 1.2.

2. **SVD**. If centering is being used, then compute the top $k$ left and right singular vectors of $\tilde{A}$, $\hat{U} \in \mathbb{R}^{n \times k}$ and $\hat{V} \in \mathbb{R}^{d \times k}$. These are the principal components and their loadings. Let $\hat{D} \in \mathbb{R}^{k \times k}$ be a diagonal matrix containing the corresponding singular values. So, $\tilde{A} \approx \hat{U} \hat{D} \hat{V}^T$. If centering is not being used, then use the original input matrix $A$ instead of $\tilde{A}$.

3. **Varimax**. Compute the orthogonal matrices that maximize Varimax, $v(R, \hat{U})$ and $v(R, \hat{V})$. Define them as $R_{\hat{U}}, R_{\hat{V}} \in \mathbb{O}(k)$ respectively.

- Output:

$$
\tilde{Z} = \sqrt{n} \hat{U} R_{\hat{U}}, \quad \tilde{Y} = \sqrt{d} \hat{V} R_{\hat{V}}, \quad \text{and} \quad \hat{B} = R_{\hat{U}}^T \hat{D} R_{\hat{V}} / \sqrt{nd}
$$

(4)

In modern applications where the row sums (or column sums) of $A$ are highly heterogeneous, the scaling step in the next remark is often considered before a spectral decomposition. One can apply this step before `vsp` and input the scaled matrix $L$ into `vsp`. This step does not appear as part of Vintage Factor Analysis. Rather, it has emerged from recent work on spectral clustering [Chaudhuri et al., 2012; Amini et al., 2013].

**Remark 1.1.** [Optional scaling step] Define the row “degree”, the row regularization parameter, and the diagonal degree matrix as

$$
deg_r = A 1_d \in \mathbb{R}^n, \quad \tau_r = 1_n^T \deg_r / n \in \mathbb{R}, \quad D_r = \text{diag}(\deg_r + \tau_r 1_n) \in \mathbb{R}^{n \times n}.
$$

Similarly, define the column quantities $\deg_c, \tau_c, D_c$ with $\deg_c = 1_n^T A \in \mathbb{R}^d$ and $\tau_c = \deg_c 1_d / d$. Define the scaled (or normalized) adjacency matrix as $L = D_r^{-1/2} A D_c^{-1/2}$. Then, input $L$ to `vsp` (instead of $A$). When using $L$, `vsp` estimates a scaled version of $Z$ and $Y$.

To undo this, the output of `vsp` could be “rescaled” as $D_r^{1/2} \tilde{Z}$ and $D_c^{1/2} \tilde{Y}$. Even when $L$ is used, this paper never rescales the output.

Normalizing the adjacency matrix with the regularizer $\tau$ improves the statistical performance of spectral estimators derived from a sparse random matrix [Le et al., 2017]. In many empirical examples, the $\tau_r$ and $\tau_c$ prevent large outliers in the elements of the singular vectors that are created as an artifact of noise in sparse matrices [Zhang and Rohe, 2018]. In this paper, the scaling step is used for the analysis of the New York Times data, but it is not studied in the main theorem.
Remark 1.2. [Fast computation for sparse data matrices] In many contemporary applications, $A$ is sparse (i.e., most elements $A_{ij}$ are zero). In this case, the SVD step should be computed with power methods. These methods are faster and require less memory because they only require matrix-vector multiplication. Moreover, if step 1 is being used, then the centered matrix $\tilde{A}$ should not be explicitly computed. Instead, the matrix-vector multiplications can be computed as the right hand side of the following equality,

$$\tilde{A}x = Ax - \mu_r(1^T_d x) - 1_n(\mu_c x) + \mu_1 (1^T_d x), \tag{5}$$

and similarly for $yA$. When computed naively, the left hand side of Equation (5) requires $O(nd)$ operations. However, the right hand size requires $O(nnz)$ operations, where $nnz$ is the number of nonzero elements in $A$. In the New York Times example displayed in Figure 2, $nnz$ is three orders of magnitude smaller than $nd$. Using Equation (5) also dramatically reduces the amount of memory required to store the matrices. This can be used in conjunction with the scaling step in Remark 1.1. This is implemented in an R package available on GitHub [Rohe et al., 2020] using the R packages Matrix and rARPACK [Bates and Maechler, 2017, Qiu et al., 2016].

The optional centering step (step 1 of vsp) plays a surprising role. In particular, Proposition 3.1 in Section 3 shows that if $A$ is centered in step 1, then vsp estimates the centered factors in the semi-parametric factor model (i.e., $Z - E(Z)$). See Remark 3.2 for more discussion. To estimate $Z$, recenter $\hat{Z}$ as follows.

Remark 1.3. [Optional recentering step] After running vsp with the centering step, it is possible to use the quantities already computed to recenter the estimated factors $\hat{Z}$ and $\hat{Y}$ as a post-processing step. This enables vsp to estimate $Z$ instead of $Z - E(Z)$. Define

$$\hat{\mu}_Z = \sqrt{n} \hat{\mu}_c \hat{V} \hat{D}^{-1} \hat{U}, \quad \text{and} \quad \hat{\mu}_Y = \sqrt{d} \hat{\mu}_t \hat{U} \hat{D}^{-1} \hat{V} \tag{6}$$

and recenter the estimated factors as follows: $\hat{Z} + 1_n \hat{\mu}_Z$ and $\hat{Y} + 1_d \hat{\mu}_Y$. If the rescaling in Remark 1.1 is also used, then recenter before rescaling. Section 3 and Appendix B.1 justify the estimator $\hat{\mu}_Z$.

Table 1 below lists the variations of vsp that are defined above and discussed in this paper.

1.1 Data example

In Figure 2, the data matrix $A$ is a $300,000 \times 102,660$ document-term matrix from a collection of 300,000 New York Times articles. In this example, the row and column sums of $A$ are highly heterogeneous, ranging several orders of magnitude. As such, the matrix $A$
Table 1: The motivation for each of the optional steps in \texttt{vsp}.

was scaled as in Remark \ref{rem:scaling} and \texttt{vsp} was given \( L \). In \texttt{vsp}, the centering step (step 1) and the recentering step \ref{rem:recentering} were used. Given that the signs of the principal components and the factors are arbitrary, the sign of each principal component and each Varimax factor was chosen to make the third sample moment (i.e., skew) positive.

After computing the leading \( k = 50 \) principal components, twelve were removed because they localized on a relatively few number of articles (i.e., these twelve principal components were dominated by a few outliers) \cite{ZhangRohe2018}. Figure 3 shows the screeplot of the remaining 38 singular values and a gap at \( k = 8 \). The Varimax rotation for these \( k = 8 \) principal components was recomputed. These are the eight principal components and eight Varimax factors displayed in Figure 2. To prevent overplotting, the display only shows a sample of 5000 points. The inclusion probability for point \( i \) is proportional to \( \| \hat{Z}_i \|_2 \), where \( \hat{Z}_i \) is the \( i \)th row of \( \hat{Z} \).

With \( k = 50 \) dimensions \texttt{vsp} takes roughly two minutes in \texttt{R} on a 3.5 GHz 2017 MacBook Pro with the packages \texttt{Matrix} for sparse matrix calculations and \texttt{rARPACK} for sparse eigencomputations \cite{BatesMaechler2017,Qiu2016}. Recomputing the Varimax rotation for the leading \( k = 8 \) principal components takes roughly two seconds. This example is documented at \url{github.com/RoheLab/vsp-paper}. The \texttt{R} package is available at \url{github.com/RoheLab/vsp} \cite{Rohe2020}.

2 Rotational invariance, simple structure, Thurstone’s diagnostics, kurtosis, and sparsity

Any rotation of the factors fits the data equally well; this is what is meant by “rotational invariance.” Thurstone proposed using sparsity to remove this invariance. His sparsity diagnostics are still used routinely in practice. Theorem
2.1 shows that sparsity implies the key leptokurtic condition that is sufficient for Varimax to identify the rotation. In this way, Vintage Factor Analysis performs statistical inference.

Step 2 of \texttt{vap} approximates $\tilde{A}$ with the leading $k$ singular vectors, $\tilde{A} \approx \hat{U} \hat{D} \hat{V}^T$. Step 3 computes the Varimax rotations of $\hat{U}$ and $\hat{V}$. However, for any rotation matrices $R_1, R_2 \in O(k)$, rotating $\hat{U}$ and $\hat{V}$ does not change the approximation to $\tilde{A}$,
\[
\hat{U} \hat{D} \hat{V}^T = (\hat{U} R_1)(R_1^T \hat{D} R_2)(\hat{V} R_2)^T,
\]
where the rotated factor matrices $\hat{U} R_1$ and $\hat{V} R_2$ still have orthonormal columns. As such, no rotation can improve the approximation to $\tilde{A}$. Many have interpreted this to imply that we can never estimate factor rotations from data. This is the misunderstanding of rotational invariance.

In an attempt to resolve the rotational invariance, Thurstone developed a new type of data analysis to find rotations $R_\hat{U} \in O(k)$ such that $\hat{U} R_\hat{U}$ is sparse [Thurstone, 1935, 1947]. He developed a suite of tools and diagnostics to assess this sparsity and many of these remain in use today. They are described in modern textbooks, built into the base R packages for factor analysis, and used routinely in practice. Section 2.1 describes these diagnostic practices. Section 2.2 and Theorem 2.1 show how these diagnostics can be reinterpreted as assessing whether the factors come from a leptokurtic distribution which is a key condition for Varimax to be able to identify the correct factor rotation in Theorems 3.1 and 4.1.

2.1 Thurstone’s simple structure and diagnostics

Thurstone [1935] and [Thurstone, 1947] propose using sparsity to remove the rotational invariance. “In numerical terms this is a demand for the [rotation which provides] the smallest number of non-vanishing entries in each row of the ... factor matrix. It seems strange indeed, and it was entirely unexpected, that so simple and plausible an idea should meet with a storm of protest from the statisticians” [p333 Thurstone, 1947]. Thurstone refers to this sparsity in the rotated factor matrix as simple structure. Thurstone’s use of sparsity is analogous to the modern use of sparsity in high dimensional regression and underdetermined systems of linear equations. In these more modern problems, without any sparsity constraint, there is a large space of plausible solutions. However, under certain conditions, the sparse solution is unique. This intuition is analogous to Thurstone’s intuition for resolving rotational invariance.

Thurstone implemented techniques to find rotations which produce sparse solutions, but he struggled to find any assurance that the computed solution is the sparsest solution. “When [a solutions has] been found which produces a simple structure, it is of considerable scientific interest to know whether the simple structure is unique... The necessary and sufficient conditions for uniqueness of a simple structure need to be investigated. In the
absence of a complete solution to this problem, five criteria will here be listed which
probably constitute sufficient conditions for the uniqueness of a simple structure” [p334
Thurstone [1947]]. Thurstone’s five conditions motivate his “radial streaks” diagnostic,
illustrated in Figure 2. In the quote below, Thurstone’s original mathematical notation
has been replaced with the notation in this paper.

Five rules for simple factor structure; quoted from Thurstone [1947] p335
We shall describe five useful criteria by which the $k$ reference vectors [i.e., the columns
of $\hat{R}^U$] can be determined. These are as follows:

1. Each row of the .... matrix $\hat{U} R^{U}$ should have at least one zero.

2. For each column $\ell$ of the factor matrix $\hat{U} R^{U}$ there should be a distinct set of $k$
   linearly independent [rows] whose factor loadings $[\hat{U} R^{U}]_{j\ell}$ are zero. [sic]

3. For every pair of columns of $\hat{U} R^{U}$ there should be several [rows] whose entries
   $[\hat{U} R^{U}]_{jp}$ vanish in one column but not in the other.

4. For every pair of columns of $\hat{U} R^{U}$, a large proportion of the tests should have
   zero entries in both columns. This applies to factor problems with four or five
   or more common factors.

5. For every pair of columns there should preferably be only a small number of
   [rows] with non-vanishing entries in both columns.

When these [five] conditions are satisfied, the plot of each pair of columns shows (1)
a large concentration of points in two radial streaks, (2) a large number of points at
or near the origin, and (3) only a small number of points off the two radial streaks.
For a configuration of $k$ dimensions there are $\frac{1}{2}k(k - 1)$ diagrams. When all of them
satisfy the three characteristics, we say that the structure is ‘compelling,’ and we
have good assurance that the simple structure is unique. In the last analysis it is the
appearance of the diagrams that determines, more than any other criterion, which
of the hyperplanes of the simple structure are convincing and whether the whole
configuration is to be accepted as stable and ready for interpretation.*

*Ever since I found the simple-structure solution for the factor problem, I have never at-
ttempted interpretation of a factorial result without first inspecting the diagrams. [footnote
original to text]

*There cannot be $k$ linearly independent vectors in a $k - 1$ dimensional hyperplane.

An example of the diagrams (i.e., plots) that Thurstone proposes are given in Figure
2 for the New York Times data. Each of those plots displays radial streaks. After the
Varimax rotation, those radial streaks align with the coordinate axes, making the rotated factors approximately sparse.

If the diagnostic plots do not show radial streaks, Thurstone suggests that one should proceed more cautiously. A few pages after the quote above, Thurstone gives a diagram with points evenly spaced inside a circle (i.e., rotationally invariant) and explains what happens when you have loadings that appear to come from a rotationally invariant distribution. “A figure such as [this] leaves one unconvinced, no matter where the axes are drawn, unless an interpretation can be found that seems right. Random configurations like this seldom yield clear interpretations, but they are not, of course, physically impossible.”

The current paper creates a statistical theory around Thurstone’s key ideas by presuming that the factors are generated as random variables from a statistical model and using the Varimax estimator. Thurstone does not presume the latent factors are generated from a probability distribution per se, and as such, does not cite or recognize the importance of Maxwell’s Theorem. Moreover, Thurstone computed rotations by hand and human judgement. Only after Thurstone’s death in 1955 did it become popular to compute factor rotations such as Varimax on “electronic computers” with numerical optimization techniques.

2.1.1 Simple structure in contemporary multivariate statistics

Contemporary textbooks on multivariate statistics still suggest that the rotated factors or the rotated principal components should be inspected to see if they are sparse [Mardia et al., 1979; Jolliffe, 2002; Johnson and Wichern, 2007; Bartholomew et al., 2011]. These textbooks all share the empirical observation that it is often easier to interpret factors which have been rotated for sparsity. The given reason is that sparse factors are “simpler.” While this appears to use Thurstone’s word, these texts do not discuss whether or not this simple structure might resolve the problem of rotational invariance. Rather, it is an empirical observation that sparse and simple solutions are easier to interpret. For example, “The simplification achieved by rotation can help in interpreting the factors or rotated PCs” [Jolliffe, 2002]. Similarly, “A rotation of the factors often reveals a simple structure and aids interpretation” [Johnson and Wichern, 2007]. The notion that the data analyst should inspect the factors for sparsity is built into the print function for factor loadings in the base R packages; if a loading is less than the print argument cutoff then instead of printing a number, it appears as a whitespace.

This paper shows that sparsity does not merely make the factors simpler; sparsity enables statistical identification and inference. Sparsity and “radial streaking” are two distinctively non-Gaussian patterns. As such, Thurstone’s visualizations and diagnostics can be reinterpreted as assessing whether the factors are generated from a non-Gaussian distribution and thus, by Maxwell’s theorem, whether the rotation is statistically identifiable. The next section shows that if a distribution is sufficiently sparse, then it is leptokurtic.
2.2 Kurtosis and sparsity

The next theorem shows that sparsity implies leptokurtosis. In this way, Thurstone’s sparsity diagnostics can be reinterpreted as assessing an identifying assumption for Varimax. Moreover, sparsity can replace leptokurtosis in the identifying assumptions for Varimax.

**Theorem 2.1.** Any random variable $X$ that satisfies $\frac{2}{6} < P(X = 0) < 1$ and has four finite moments is leptokurtic.

This theorem does not make any parametric assumptions and the moment assumptions are only so that kurtosis is defined. See Section C.1 in the Appendix for a proof. This theorem assumes “hard sparsity” (i.e., $P(X = 0) > 0$) for technical convenience. See Appendix C.2 for a discussion about softer forms of sparsity.

3 Gaining intuition for vsp with the population results

This section studies each of the three steps in vsp by studying their population behavior. Statistical convergence around the population quantities is rigorously treated in Theorem 4.1 in Section 4.

The semi-parametric factor model is a latent variable model with two sequential layers of randomness. In the first layer of randomness, the latent variables $Z$ and $Y$ are generated. In the second layer, the observed matrix $A$ is generated, conditionally on the latent variables. To parallel these two layers, there are two types of population results given in this section.

The first two steps of vsp compute the principal components. Propositions 3.1 and 3.2 study these steps applied to the population matrix

$$\mathcal{A} = \mathbb{E}(A|Z,Y) = ZBY^T,$$

instead of $A$. These propositions imply that the population principal components can be expressed as $\tilde{Z}R$, where $\tilde{Z} \in \mathbb{R}^{n \times k}$ is $Z$ after column centering and $R \in \mathbb{R}^{k \times k}$ is defined below. If the $nk$ many random variables in $Z \in \mathbb{R}^{n \times k}$ are mutually independent, then $R$ converges to a rotation matrix. These results allows for the randomness in $Z$ and $Y$, but they remove the second layer of randomness by using $\mathcal{A}$ instead of $A$. Then, Theorem 3.1 studies the population version of the Varimax step. To do this, take the expectation of the Varimax objective function, evaluated at the population principal components (i.e., $\tilde{Z}R$), where the expectation is over the distribution of $Z$. This expectation removes the randomness in $Z$. Under the identification assumptions for Varimax defined below, Theorem 3.1 shows that the rotation that maximizes this function is $R^T \in \mathcal{O}(k)$. So, rotating the population principal components with the population Varimax rotation yields the original factors, $(\tilde{Z}R)R^T = \tilde{Z}$.

Define $\bar{Z} \in \mathbb{R}^{n \times k}$ such that $\bar{Z}_{ij}$ equals the sample mean of the $j$th column of $Z$. Similarly for $\bar{Y} \in \mathbb{R}^{d \times k}$. Define

$$\tilde{Z} = Z - \bar{Z} \quad \text{and} \quad \tilde{Y} = Y - \bar{Y}.$$
Proposition 3.1. [Step 1 of vsp] Centering $\mathcal{A}$ to get $\widetilde{\mathcal{A}}$ as in Equation (3), has the effect of centering $Z$ and $Y$.

$$\widetilde{\mathcal{A}} = \widetilde{Z}B\widetilde{Y}^T$$

This does not require any distributional assumptions on $Z$ or $Y$.

A proof is given in Appendix B. The next proposition gives the SVD of $\widetilde{\mathcal{A}} = \widetilde{Z}B\widetilde{Y}^T$.

Define

$$\hat{\Sigma}_Z = \frac{\widetilde{Z}^T\widetilde{Z}}{n}, \quad \hat{\Sigma}_Y = \frac{\widetilde{Y}^T\widetilde{Y}}{d},$$
and define $\widetilde{R}_U, \widetilde{R}_V \in \mathcal{O}(k)$, and diagonal matrix $\widetilde{D}$ to be the SVD of $\hat{\Sigma}_Z^{1/2}B\hat{\Sigma}_Y^{1/2} \in \mathbb{R}^{k \times k}$,

$$\hat{\Sigma}_Z^{1/2}B\hat{\Sigma}_Y^{1/2} = \widetilde{R}_U^T\widetilde{D}\widetilde{R}_V.$$

The next proposition shows that the rotation matrices $\widetilde{R}_U$ and $\widetilde{R}_V$ convert the factor matrices $\widetilde{Z}$ and $\widetilde{Y}$ into the principal components and loadings $U$ and $V$.

Proposition 3.2. [Step 2 of vsp] Define the following matrices,

$$U = n^{-1/2}\widetilde{Z} \hat{\Sigma}_Z^{-1/2} \widetilde{R}_U^T, \quad D = \sqrt{nd}\widetilde{D}, \quad V = d^{-1/2}\widetilde{Y} \hat{\Sigma}_Y^{-1/2} \widetilde{R}_V^T. \quad (9)$$

Then, $\widetilde{\mathcal{A}} = UDV^T$, where $U$ and $V$ contain the left and right singular vectors of $\widetilde{\mathcal{A}}$ and $D$ contains the singular values of $\widetilde{\mathcal{A}}$. This does not require any distributional assumptions on $Z$ or $Y$.

The proof requires demonstrating the equality $\widetilde{\mathcal{A}} = UDV^T$ and showing that $U$ and $V$ have orthonormal columns. Substituting in the definitions reveals this result. Taken together, Propositions 3.1 and 3.2 show that the first two steps of vsp on $\mathcal{A}$ compute $U \propto \widetilde{Z} \hat{\Sigma}_Z^{-1/2} \widetilde{R}_U^T$; these are the principal components of $\mathcal{A}$.

Remark 3.1. [Relationship between PCA and the factors] Proposition 3.2 relates PCA on the population matrix $\mathcal{A}$ to the factors $Z$. This is because the population principal components are the columns of the matrix

$$U = n^{-1/2}\widetilde{Z} \hat{\Sigma}_Z^{-1/2} \widetilde{R}_U^T. \quad (10)$$

So, the principal components are the centered latent factors $\widetilde{Z}$, “whitened” with $\hat{\Sigma}_Z^{-1/2}$, and rotated by a $k \times k$ nuisance matrix $\widetilde{R}_U^T$. Despite the fact that PCA is typically considered a second order technique, this result implies that the principal components themselves do not retain any first or second order information about the latent factors, but retain all other distributional information. With Maxwell’s Theorem, this suggests that higher order techniques such as Varimax hold the possibility of identifying the nuisance matrix. In fact, Theorem 3.1 below shows that Varimax can identify the nuisance matrix.
The Varimax problem applied to the population principal components $U$ in Equation (10) is

$$\arg \max_{R \in O(k)} v(R, \tilde{Z} \Sigma_{Z}^{-1/2} \tilde{R}_{U}^{T}).$$

(11)

Despite the fact that these are the population principal components, this is still a sample quantity because $Z$ is random. This randomness is from the first stage of randomness in the semi-parametric factor model. The next theorem gives a population result for the M-estimator in (11) by studying the expected value of $v$ over $Z$, to show that it can identify $\tilde{R}_{U}$. Assumption 1 gives the identification assumptions on the distribution of $Z$ that will be used in both the population result for Varimax (Theorem 3.1) and the main theorem (Theorem 4.1).

Assumption 1. [The identification assumptions for Varimax] The matrix $Z \in \mathbb{R}^{n \times k}$ satisfies the identification assumptions for Varimax if all of the following conditions hold on the rows $Z_{i} \in \mathbb{R}^{k}$ for $i = 1, \ldots, n$:

i) the vectors $Z_{1}, Z_{2}, \ldots, Z_{n}$ are iid,

ii) each vector $Z_{i} \in \mathbb{R}^{k}$ is composed of $k$ independent random variables (not necessarily identically distributed),

iii) $\text{Var}(Z_{ij}) = 1$ for all $j$ \(^4\) and

iv) the elements of $Z_{i}$ are leptokurtic.

Let $\tilde{Z}_{1}$ be the first row of $\tilde{Z}$. Define $Z^{o} = Z_{1} - \mathbb{E}(Z_{1}) \in \mathbb{R}^{k}$. Theorem 3.1 shows that the rotation matrix $R$ that maximizes the expected Varimax objective function, $\mathbb{E}(v(R, Z^{o} \tilde{R}_{U}^{T}))$, is $\tilde{R}_{U}$. In this formulation, several quantities from the sample maximization problem (11) have been replaced. First, the sample objective function $v$ in Equation (11) has been replaced with its expectation over the distribution of $Z$. Then, $\tilde{Z}$ has been replaced by $\mathbb{E}(Z_{1})$ and $\Sigma_{Z}^{-1/2}$ has been replaced with its limiting quantity under Assumption 1 (i.e., the identity matrix).

Because the Varimax objective function does not change if the estimated factors are reordered or if some of the estimated factors have a sign change, the maximizer of Varimax is actually an equivalence class that allows for these operations. Define the set

$$\mathcal{P}(k) = \{ P \in O(k) : P_{ij} \in \{-1, 0, 1\} \}.$$  

(12)

It is the full set of matrices that allow for column reordering and sign changes.

---

\(^4\)The third assumption in Varimax is not restrictive because the matrix $B$ can absorb a rescaling of the variables. That is, let $Z^{\text{rescaled}} \in \mathbb{R}^{n \times k}$ satisfy the first two conditions and presume that $\mathcal{A} = Z^{\text{rescaled}} B^{\text{rescaled}} Y^{T}$. Define $\Sigma_{Z} = \text{Cov}(Z_{i}^{\text{rescaled}})$, $Z = Z^{\text{rescaled}} \Sigma_{Z}^{-1/2}$, and $B = \Sigma_{Z}^{1/2} B^{\text{rescaled}}$. Because $Z^{\text{rescaled}}$ satisfies the second condition, $\Sigma_{Z}$ is diagonal. So, $Z = Z^{\text{rescaled}} \Sigma_{Z}^{-1/2}$ retains independent components and now satisfies the third condition. Moreover, $\mathcal{A} = Z B Y^{T}$.
Theorem 3.1. [step 3] Suppose that $Z \in \mathbb{R}^{n \times k}$ satisfies the identification assumptions for Varimax (Assumption 1). Let $Z_1 \in \mathbb{R}^k$ be the first row of $Z$. Define $Z^0 = Z_1 - \mathbb{E}(Z_1)$. For any nuisance rotation matrix $\tilde{R} \in O(k)$,

$$\arg \max_{R \in O(k)} \mathbb{E}(v(R, Z^0 \tilde{R}^T)) = \{ \tilde{R}P : P \in \mathcal{P}(k) \} \quad (13)$$

The output step of vsp right multiplies the principal components $\sqrt{n}U \approx \tilde{Z}\tilde{R}^T$ with a matrix which maximizes Varimax. In the population results, this matrix is $\tilde{R}U_P$. Thus, the Varimax rotation reveals the unrotated factors, $(\tilde{Z}\tilde{R}^T U P) = \tilde{Z}P$.

Remark 1.3 describes a method to recenter the factors $\tilde{Z}$ to get $Z$. Section B.1 in the appendix gives a population justification for this recentering step.

Remark 3.2. [The role of centering] A version of Proposition 3.2 still holds for the SVD of $\mathcal{A}$ (without centering) by replacing $\hat{\Sigma}_Z$ with $Z^T Z/n$ and replacing $\hat{\Sigma}_Y$ with $Y^T Y/d$ in Equation (9). Even if the elements of the matrix $Z$ are independent and have unit variance, then the columns of $Z$ will not be asymptotically orthogonal (unless $\mathbb{E}(Z) = 0$). As such, right multiplying $U = Z(Z^T Z/n)^{-1/2} \tilde{R}_U^T$ with an orthogonal rotation (i.e., the one estimated by Varimax) cannot reveal $Z$. This highlights the role of centering in vsp; centering $\mathcal{A}$ has the effect of centering the latent variables, which in turn makes the latent factors asymptotically orthogonal under the assumption of independence. This allows Varimax to unmix them with an orthogonal matrix.

Remark 3.3. PCA is not the standard approach in Vintage Factor Analysis. To see why, define $\mathcal{A} = \mathbb{E}(A|Z,Y) = ZBY^T$ and notice that the diagonal elements of $n^{-1} \mathcal{A}^T \mathcal{A}$ are less than or equal to the diagonal elements of the expected sample covariance matrix $n^{-1} \mathbb{E}(A^T A|Z,Y)$. PCA does not adjust for this excess along the diagonal of the sample covariance matrix and this makes PCA biased. However, as Theorem 4.1 shows in the next section, the estimates from PCA with a Varimax rotation converge to the desired quantities. Thus, in the asymptote studied herein, PCA with a Varimax rotation is asymptotically unbiased. It is possible that a different approach could have increased statistical efficiency, but this is not studied in this paper.

4 The main theorem

Theorem 4.1 is the main result for this paper. This theorem does not presume a parametric form for the random variables in $Z$ or $A$. Instead, it uses the identifying assumptions for Varimax (Assumption 1) and two further assumptions on the tails of these distributions.

Recall that $\tilde{\mu}_Z$ estimates the column means of $Z$ defined in Remark 1.3. Let $\tilde{Z}_i$ be the $i$th row of $\tilde{Z}$. Theorem 4.1 shows that for every $i \in 1, \ldots, n$, $\tilde{Z}_i + \tilde{\mu}_Z$ converges to $Z_i$ (after allowing for a permutation and sign flip).
**Assumption 2.** Each column of $Z$ and $Y$ is generated from a distribution that does not change asymptotically and has a moment generating function in some fixed $\epsilon > 0$ neighborhood around zero.

Let $\mathcal{A}$ be defined in Equation (7). Define the mean and maximum of $\mathcal{A}$ as

$$\rho_n = \frac{1}{nd} \sum_{i,j} \mathcal{A}_{ij} \quad \text{and} \quad \bar{\rho}_n = \max_{i,j} |\mathcal{A}_{ij}|.$$  (14)

Theorem 4.1 allows for $A$ to contain mostly zeros by assuming that as $n$ and $d$ grow, $B_n = \rho_n B$ for some fixed matrix $B \in \mathbb{R}^{k \times k}$. If $\rho_n \to 0$, then $A$ is sparse. This is analogous to the asymptotics in Bickel and Chen [2009] for the Stochastic Blockmodel.

**Assumption 3.** For any valid subscripts $i$ and $j$, eventually in $n$,

$$E[(A_{ij} - \mathcal{A}_{ij})^m] \leq \max\{(m - 1)!(\bar{\rho}_n)^{m/2}, \bar{\rho}_n\}, \quad \text{for all } m \geq 2,$$

where this expectation is conditional on $Z,Y$.

Assumption 3 controls the tail behavior of the random variables in the elements of $A$. This assumption is more inclusive than sub-Gaussian. For example, this assumption is satisfied when $A$ contains Poisson random variables, as happens in Latent Dirichlet Allocation in Section 5.4. This assumption is also satisfied if $A$ contains Bernoulli random variables, as happens in Stochastic Blockmodeling. See Sections F.1.5 and F.2.1 in the Appendix for further discussion.

The quantity

$$\Delta_n = n\rho_n$$

controls the asymptotic rate in Theorem 4.1. So, it is helpful to have some sense for it. For example, suppose that (i) $A$ contains Bernoulli elements, (ii) each row and column sum of $\mathcal{A}$ grows at a similar rate, (iii) $n \asymp d$, and (iv) $\rho_n \to 0$, then $\Delta_n$ is roughly the expected number of ones in each row and column of $A$.

**Theorem 4.1.** Suppose that $A \in \mathbb{R}^{n \times d}$ is generated from a semi-parametric factor model that satisfies Assumptions 1, 2, and 3. Presume that asymptotically, $\mathcal{A} = \rho_n ZBY^T$ for some fixed and full rank matrix $B$. In the asymptotic regime where $n \asymp d$ and $\Delta_n \geq \log^{11.1} n$,

$$||\hat{Z} + \mathbf{1}_n \hat{\mu}_Z - ZP_n||_2 \to \infty = O_P(\Delta_{n,24} \log^{2.75} n),$$  (15)

where $\hat{Z}$ is the estimate produced by vsp (with step 1) applied to $A$ and $\hat{\mu}_Z$ is the estimate defined in Equation (6).

Theorem 4.1 shows convergence in $2 \to \infty$ norm. This means that every row of $\hat{Z} + \mathbf{1}_n \hat{\mu}_Z$ converges to the corresponding row of $Z$ in $\ell_2$. The $P_n$ matrix accounts for the fact that we do not attempt to identify the order of the columns in $Z$, or their sign. If $\hat{Z}$ is used
without recentering by $\mathbf{1}_n \hat{\mu}_Z$, then a similar result holds for estimating $\tilde{Z}$. By symmetry, if $Y$ satisfies the identification assumptions for Varimax, then $\mathsf{vsp}$ can also estimate $Y$. If both $Z$ and $Y$ satisfy the identification assumptions for Varimax, then $B$ can also be recovered, even when it is not diagonal. The proof for Theorem 4.1 begins in Appendix C.3.

5 Modern factor models as semi-parametric factor models

The semi-parametric factor model is related to ICA, the Stochastic Blockmodel, and Latent Dirichlet Allocation. Corollaries 5.1 and 5.2 show that with some slight variations on the preprocessing of $A$, $\mathsf{vsp}$ can estimate the Stochastic Blockmodel and Latent Dirichlet Allocation.

5.1 Relationship to Independent Components Analysis

Independent Components Analysis (ICA) uses a type of semi-parametric factor model that is motivated by blind-source separation in signal processing. In the typical formulation of ICA, we observe a multivariate time series $\mathcal{A}_t = Z_t M \in \mathbb{R}^k$ for $t = 1, \ldots, n$, where $Z_t \in \mathbb{R}^k$ contains independent and non-Gaussian random variables. The aim is to estimate $M^{-1}$, to unmix the observed signals in $\mathcal{A}_t$, and reveal the independent components $Z_t$.

There are multiple ICA results that share some similarities to Theorems 3.1 and 4.1 (e.g. Comon [1994], Hyvärinen et al. [2004], Chen and Bickel [2005, 2006], Wei [2015], Miettinen et al. [2015], Samworth and Yuan [2012]). To see the connection to the current paper, let $M \in \mathbb{R}^{k \times d}$ be potentially rectangular and defined as $M = BY^T$. To enable the regime $d = k$, the results for ICA typically presume that $\mathcal{A} = ZM$ is observed with little or no noise. In contrast, Theorem 4.1 covers situations where (i) $d$ grows at the same rate as $n$, (ii) there is an abundance of noise in $A$, and (iii) $A$ is mostly zeros (i.e., sparse). This allows the theorem to cover the contemporary factor models in Section 5.

5.2 Tensor decompositions

Motivated in part by the issue of rotational invariance of PCA, Kruskal [1977] showed how a tensor decomposition called the CP decomposition is unique; it decomposes a tensor into a set of factors that are not rotationally invariant. In Section 4, Kruskal discusses how this three way decomposition does not suffer from the same problem of rotation that “consumes considerable attention and effort” in factor analysis. In an elegant formulation, Anandkumar et al. [2014] showed how these tensor spectral methods could be applied to estimate the latent factors in a model class similar to the semi-parametric factor model.

Where the principal components of $A$ are the eigenvectors of a matrix that contains the second order moments, $n^{-1} \mathbb{E}(\hat{A}^T A)_{uv} = \mathbb{E}(\hat{A}_{iu} \hat{A}_{iv})$, the elements of this higher order tensor contain the third order (or higher) moments; for example, $T \in \mathbb{R}^{d \times d \times d}$ with $T_{u,v,w} =$
\(E(A_{iu}A_{iw})\). Then, for various formulations of \(T\) and latent variable models, the CP tensor decomposition of \(T\) has components that are equal to the latent factors [Janzamin et al., 2019].

The issue of rotational invariance motivates for the extension from matrices to tensors. For example, in a recent book on using tensors for latent variable modeling, [Janzamin et al. 2019] writes in the abstract “PCA and other spectral techniques applied to matrices have several limitations. By limiting to only pairwise moments, they are effectively making a Gaussian approximation on the underlying data.” However, despite the fact that PCA is typically imagined as a second order technique, the principal components of \(A\) retain the higher-order distributional properties of the latent variables (see Remark 3.1). As such, we need not consider the higher order moments of the manifest variables \(A\) in the tensor \(T\). vsp uses the higher order moments of the principal components themselves, by applying Varimax directly to the principal components. Given our heuristics around rotational invariance, it is surprising that this can work.

5.3 Stochastic Blockmodels

In social network analysis, \(A \in \{0, 1\}^{n \times n}\) is the adjacency matrix of a graph on \(n\) people.

\[
A_{ij} = \begin{cases} 
1 & i \text{ friends with } j \\
0 & \text{o.w.}
\end{cases}
\]

The Stochastic Blockmodel [Holland et al., 1983] is a semi-parametric factor model for generating a random adjacency matrix. Under this model, each individual \(i\) is assigned to a single block \(z(i) \in \{1, \ldots, k\}\) and the probability that \(i\) and \(j\) are friends is

\[
\mathbb{P}(A_{ij} = 1 \mid z(i), z(j)) = B_{z(i), z(j)}, \text{ where } B \in [0, 1]^{k \times k}.
\]

Define \(\mathcal{A} = E(A \mid Z, B, Y)\). To express \(\mathcal{A}\) in the factor model as \(ZBZ^T\), define \(Z \in \{0, 1\}^{n \times k}\) such that \(Z_{ij} = 1\) when \(z(i) = j\) and \(Z_{ij} = 0\) otherwise. When friendships are symmetric, so is \(A\); in this setting \(Y = Z\) and the elements above the diagonal of \(A\) are independent. There are four popular generalizations of the Stochastic Blockmodel that have the structure \(ZBZ^T\), and are thus other types of semi-parametric factor models. The Degree-Corrected Stochastic Blockmodel includes an additional degree parameters \(\theta_{i,z(i)} > 0\) for each individual \(i\). The probability of friendship becomes

\[
\mathcal{A}_{ij} = \theta_{i,z(i)} \theta_{j,z(j)} B_{z(i), z(j)} \quad \text{[Karrer and Newman, 2011].}
\]

To express this model as \(ZBZ^T\), define \(Z_{ij} = \theta_{i,z(i)} \mathbb{I}\{z(i) = j\}\), where \(\mathbb{I} \in \{0, 1\}\) is the indicator function. In the Overlapping Stochastic Blockmodel, \(Z \in \{0, 1\}^{n \times k}\) is sparse [Latouche et al., 2011]. In the mixed-membership Stochastic Blockmodel, each row of \(Z\) is an independent sample from the Dirichlet distribution [Airoldi et al., 2008]. Later, Zhang et al. [2014] and Jin and Ke [2017] generalized these models to

\[5\text{The original paper paper on the overlapping Stochastic Blockmodel is not exactly the factor model used here because it includes a logistic link function, } \mathbb{P}(A_{ij} = 1) = \text{logit}(Z_i B Z_j^T).\]
only presume that \( Z_i \in \mathbb{R}^k \) is element-wise non-negative. Table 2 summarizes all of these models. While this discussion focuses on unipartite and undirected graphs, graphs that are “two-way,” “bipartite,” or “directed,” can also be modeled in the form \( \mathcal{A} = ZBY^T \) [Rohe et al., 2016].

| SBM                              | the vector \( Z_i \in \mathbb{R}^k \) contains                                | distribution of \( Z_i \)         |
|----------------------------------|--------------------------------------------------------------------------------|----------------------------------|
| 0) Standard SBM                  | a single one, the rest zeros                                                  | multinomial                      |
| 1) Degree-Corrected              | a single positive entry, the rest zeros                                       | not specified                    |
| 2) Overlapping                   | a mix of 1s and 0s                                                             | independent Bernoulli            |
| 3) Mixed Membership             | non-negative entries that sum to one                                           | Dirichlet                        |
| 4) Degree-Corrected, Mixed       | non-negative entries                                                           | not specified                    |
| Membership                        |                                                                                       |                                  |

Table 2: Restrictions on the factor matrix \( Z \) create variations on the Stochastic Blockmodel (SBM). There are further differences between these models that are not emphasized by this table.

**Estimating the Degree-Corrected Stochastic Blockmodel with vsp.** Under the Stochastic Blockmodel and the Degree Corrected version, each node \( i \) belongs to exactly one cluster. In such “hard clustering” models, the elements in the same row of \( Z \) cannot be independent. This implies that \( Z \) cannot satisfy Assumption 1 of Theorem 4.1. The next corollary shows that vsp without the centering step can estimate these models.

Let \( \pi \in \mathbb{R}^k \) be a probability distribution on \([k]\). Suppose that \( z(1), \ldots, z(n) \sim Multinomial(\pi) \), independently. For each block \( j \), suppose that \( \theta_{1, j}, \ldots, \theta_{n, j} \in \mathbb{R} \) are independent random variables generated from a bounded probability distribution \( f_j \). The scale of this distribution is unidentifiable; so for technical convenience, it is presumed that \( \mathbb{E}(Z_{ij}^2) = 1 \), or equivalently, that \( \mathbb{E}(\theta_{ij}^2) = 1/\pi_j \). This is akin to the third assumption in the Varimax assumption. This scaling ensures that \( \mathbb{E}(Z^T Z)/n \) (i.e. without centering) converges to the identity matrix. If each \( f_j \) is a point mass, then this model is equivalent to the SBM.

**Corollary 5.1.** Suppose that \( A_n \in \mathbb{R}^{n \times n} \) is generated from the Degree Corrected Stochastic Blockmodel with \( \mathbb{E}(A_n|Z_n) = Z_nB_nZ_n \), where \( Z_n \) is generated as described in the proceeding paragraph. Suppose that the probability distributions \( f_j \) for \( j \in [k] \) are bounded. Define \( \rho_n \) as in Equation (14) and suppose that there exists a fixed matrix \( B \in \mathbb{R}^{k \times k} \) such that \( B_n = \rho_n B \).

Define \( \hat{Z} \in \mathbb{R}^{n \times d} \) as the output of vsp without centering (i.e. skip step 1). In the asymptotic regime where \( \Delta_n \geq \log^{11.1} n \), there exists a sequence permutation and sign-flip matrices \( P_n \in \mathcal{P}(k) \) such that

\[
\|\hat{Z} - ZP_n\|_{2 \rightarrow \infty} = O_P(\log^{2.4} n) = O_P(\Delta_n^{1/2} \log^{2.75} n).
\]
A proof is contained in Appendix F.

To see why the centering step creates bias for vsp under a hard clustering model, note that vsp with the centering step (step 1) estimates \( \tilde{Z} \) (i.e., \( Z \) after centering). By construction, \( \tilde{Z} \) contains orthogonal columns. However, under the Stochastic Blockmodel, \( Z \) does not. Interestingly, \( Z \) without centering does contain orthogonal columns and vsp without centering can estimate it.

**Overlapping and Mixed Membership.** Under the Overlapping SBM,

\[ Z_{ij} \sim \text{Bernoulli}(p_j) \]

independently for all \( i \) and \( j \). This will satisfy the identification assumptions for Varimax so long as \( p_j \notin [1/2 \pm 1/\sqrt{12}] \) for \( j = 1, \ldots, k \). This rather strange condition ensures that \( Z_{ij} \) is leptokurtic and thus Varimax can identity the rotation. If Varimax were replaced with an alternative rotation from the ICA literature, then one could remove the awkward condition on the \( p_j \)'s.

Under the Mixed Membership SBM, \( Z \) is on the simplex. As such, its elements must sum to one and cannot be statistically independent. This restriction to the simplex also limits the ability of the Mixed Membership model to create a large amount of degree heterogeneity, a common property in empirical networks. As discussed in Section 5.4, this problem also arrises for Latent Dirichlet Allocation (LDA). Section 5.4 discusses a natural generalization of LDA that allows for more heterogeneous document lengths. A similar generalization could be applied to the Mixed Membership SBM. This would create a “Degree-Corrected Mixed Membership model.” Under such a model, a result analogous to Corollary 5.2 could be derived.

**Degree-Corrected Mixed Membership.** The papers which proposed the Degree-Corrected Mixed Membership model only presume that \( Z_i \) is element-wise non-negative \cite{Zhang2014, Jin2017}. As such, if the elements of \( Z_i \) are sampled in a way which satisfy the identification assumptions for Varimax, then Theorem 4.1 shows that vsp can estimate this model.

5.4 Latent Dirichlet Allocation

In the setting of text analysis and natural language processing, let \( A \in \mathbb{R}^{n \times d} \) be a document-term matrix on \( n \) documents and \( d \) unique words,

\[ A_{ij} = \text{number of times that word } j \text{ appears in document } i \]  

(17)

Latent Dirichlet Allocation (LDA) is a popular generative model for \( A \) that is used for modeling the topics of documents \cite{Blei2003}.

The LDA model has parameters \( \xi > 0, \alpha \in \mathbb{R}_+^k \), and \( \beta \in \mathbb{R}_+^{d \times k} \) with \( 1_d^T \beta = 1_k \). The rows of \( \beta \) index the unique words \( 1, \ldots, d \). Because the elements of \( \beta \) are positive and each
column sums to one, each column makes a probability distribution on the unique words. LDA generates a single document \( i = 1, \ldots, n \) with the following steps, (1) choose \( Z_i \sim \text{Dirichlet}(\alpha) \) to be the topic distribution for that document, (2) sample \( N_i \sim \text{Poisson}(\xi) \) to be the number of words in the document, (3) for each of the words in the document \( w = 1, \ldots, N_i \), choose the topic for that word \( z_w \sim \text{Multinomial}(Z_i) \in \{1, \ldots, k\} \), and then sample the word \( w \) as multinomial with probabilities specified by the \( z_w \) column of \( \beta \) (i.e., \( w \) is the \( j \)th unique word with probability \( \beta_{j,z_w} \)).

**Lemma 5.1.** Under the LDA model, conditionally on the Dirichlet variables \( Z_1, \ldots, Z_n \), the document-term matrix \( A \) has independent Poisson entries with

\[
\mathbb{E}(A|Z) = \xi Z \beta^T,
\]

where \( Z \in \mathbb{R}_{+}^{n \times k} \) has rows \( Z_1, \ldots, Z_n \).

A short proof in Section B.2 relies upon the Poisson-Multinomial relationship. While Equation (19) has the form of the semi-parametric factor model (e.g. set \( B = I \) and \( Y = \beta \)), it does not satisfy the identification assumptions for Varimax because the elements in \( Z_i \) sum to one and as such, they must be dependent. Moreover, this has the unnatural consequence of making \( \mathbb{E}(A|Z) \) have rank \( k - 1 \) or less. However, the following modification makes \( \mathbb{E}(A|Z) \) have rank \( k \) and enables the application of Theorem 4.1.

In the original formulation of LDA, the number of words in document \( i \) is \( N_i \sim \text{Poisson}(\xi) \), for \( \xi \in \mathbb{R}_{+} \). About this step, Blei et al. [2003] says, “more realistic document length distributions can be used as needed.” If document lengths are more heterogenous than what is modeled by Poisson(\( \xi \)), then a convenient way to increase the heterogeneity is to use Poisson overdispersion; first sampling \( \xi_i \), then sampling \( N_i \sim \text{Poisson}(\xi_i) \).

**Natural modification to LDA:** Sample \( N_i \), the number of words in document \( i \), as overdispersed Poisson via (1) \( \xi_i \sim \text{Gamma}(\sum \alpha_i, s) \) for some scale parameter \( s > 0 \) and (2) \( N_i \sim \text{Poisson}(\xi_i) \).

This “Gamma-Poisson mixture” is a well studied model of Poisson overdispersion; under this model, \( N_i \) has the negative binomial distribution. Define \( \Xi \in \mathbb{R}^{n \times n} \) as a diagonal matrix with \( \Xi_{ii} = \xi_i \).

**Lemma 5.2.** Under the LDA model with the natural modification to \( N_i \), conditionally on \( Z_1, \ldots, Z_n \) and \( \Xi \), the document-term matrix \( A \) has independent Poisson entries satisfying

\[
\mathbb{E}(A|\Xi, Z) = (\Xi Z) \beta^T.
\]

Moreover, each element \( (\Xi Z)_{ij} \) is independent Gamma(\( \alpha_j, s \)) and this distribution is leptokurtic. Define \( \Sigma \) as a diagonal matrix with \( \Sigma_{jj} = \alpha_j s^2 \), the variance of Gamma(\( \alpha_j, s \)). Then, the factor matrix

\[
Z_* = (\Xi Z) \Sigma^{-1/2}
\]

satisfies the identification assumptions for Varimax.
See Section B.2 for a short proof. The next result shows that \texttt{vsp} applied to the column centered version of $A$ (i.e., $\hat{A} = A - 1_n(1_n^T A/n)$) can estimate the LDA model with the natural modification. Similar to $\hat{A}$, define $\hat{A}'$ be the column centered version of $A$.

**Corollary 5.2.** Let $A$ be generated from the natural modification to LDA given above with $k$ topics and let $\mathcal{A} = \mathbb{E}(A|X, Z)$. Define $Z_*$ as in Equation (20). Let $\hat{Z}$ be the output of \texttt{vsp} using $\hat{A}$ as input (and skipping step 1). In the asymptotic regime where

$$\Delta_n \geq \log^{15.1} n, \quad \sigma_{\min}(\beta) \geq c_1,$$

for universal constant $c_1 \in (0, 1)$, almost surely there exists $P_n \in \mathcal{P}(k)$ s.t.

$$||\hat{Z} - (Z_* - \mathbb{E}(Z_*))P_n||_{2 \rightarrow \infty} = O_p(\Delta_n^{-24} \log^{2.75} n).$$

(21)

Define the matrix $\Phi = \hat{Z}^T \hat{A} \in \mathbb{R}^{k \times d}$ and estimate $\hat{\beta} = (\Lambda_b^{-1} \Phi)^T \in \mathbb{R}^{d \times k}$, where $\Lambda_b$ is a diagonal matrix with $i$th diagonal element equals to $\ell_1$-norm of $i$th row of $\Phi$. Under this construction,

$$||\hat{\beta}^T - P_n^T \beta^T||_{\infty} = O_p(\Delta_n^{-24} \log^{3.75} n).$$

(22)

The elements of $Z_*$ are independent Gamma random variables that have been rescaled by the diagonal matrix $\Sigma^{-1/2}$ to ensure that they have unit variance. Corollary 5.2 shows that \texttt{vsp} using the column-centered matrix $\hat{A}$ estimates $Z_* - \mathbb{E}(Z_*)$; similar to the previous results, this $2 \rightarrow \infty$ convergence implies that each row of $\hat{Z}$ converges to the corresponding row of $Z_* - \mathbb{E}(Z_*)$. Using $\hat{Z}$, the corollary constructs $\hat{\beta} \in \mathbb{R}^{d \times k}$, a simple estimator for the probability distribution of words within each of the $k$ topics. Each of the $k$ estimated topic distributions converges in $\ell_1$ norm just a little slower than $\Delta_n^{-1/4}$. A proof of Corollary 5.2 is given in Section F.2.

### 6 Discussion

PCA with Varimax is a vintage data analysis technique. Theorem 4.1 shows that it provides a unified spectral estimation strategy for a broad class of semi-parametric factor models. The reason for this is that (1) the principal components have the same column space as the latent factors and (2) under the identification assumptions for Varimax, Varimax specifies a basis for that column space in which each basis vector corresponds to a latent factor; this is the intuition gained in Section 3 and Theorem 4.1. Leptokurtosis is a key identifiability condition in the identification assumptions for Varimax. This condition is satisfied if the factors are sparse. Moreover, this condition can be examined in the data. In fact, Section 2 reinterprets the diagnostics practices developed in [Thurstone 1935, 1947] as examining that leptokurtic condition. Taken together, the results in this paper show that the Vintage Factor Analysis know-how developed by Thurstone and Kaiser performs statistical inference. This know-how has survived for nearly a century, despite the conventional wisdom that the factor rotation cannot perform statistical inference.
Things don’t necessarily happen for a reason; but things survive for a reason.
Nassim Nicholas Taleb.

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A Scree plot for New York Times analysis

![Scree plot](image)

There is an eigengap at the 8th singular value.

Leading singular values

Figure 3: This is a scree plot of the singular values \(1, \ldots, 50\) of the matrix \(L\) (defined in Remark 1.1) after centering (i.e., step 1 of vsp). A classical rule for selecting \(k\) is to look for a “gap” in the scree plot. Here, there is a gap after the 8th singular value that is illustrated with a vertical line. This value is used for the illustrative analysis. However, it should be noted that there is often more information contained in the principal components that correspond to singular values beyond the elbow [Wang and Rohe, 2016].

B Proofs for Proposition 3.1 and Theorem 3.1

The following is a proof of Proposition 3.1.

**Proof.** With input \(\mathcal{A}\), recall that the row, column, and grand means are

\[
\mu_r = \mathcal{A} \mathbf{1}_d / d \in \mathbb{R}^n, \quad \mu_c = \mathbf{1}_n^T \mathcal{A} / n \in \mathbb{R}^d, \quad \mu = \mathbf{1}_n^T \mathcal{A} \mathbf{1}_d / (nd) \in \mathbb{R}.
\]

The centered version of \(\mathcal{A}\) is defined to be \(\mathcal{A}_{\text{cent}} = \mathcal{A} - \mu_r \mathbf{1}_d - \mu_c \mathbf{1}_n + \mu \mathbf{1}_n \mathbf{1}_d / (nd) \in \mathbb{R}^{n \times d}\). Define the column means of \(Y\) and \(Z\) as

\[
\mu_Y = Y^T \mathbf{1}_d / d \quad \text{and} \quad \mu_Z = \mathbf{1}_n^T \mathbf{Z} / n.
\]

Note that

\[
\mu_r = ZB \mu_Y \quad \mu_c = \mu_Z B Y^T \quad \text{and} \quad \mu = \mu_Z B \mu_Y.
\]
Also note that $\tilde{Y} = 1_d \mu_1^T$ and $\tilde{Z} = 1_n \mu_2$. Putting the pieces together gives the result.

\[ \tilde{\mathcal{A}} = \mathcal{A} - \mu_r 1_d - 1_n \mu_c + \mu_1 1_d^T \]
\[ = ZBY^T - ZB \mu_Y 1_d^T - 1_n \mu_Z BY + 1_n \mu_Z B \mu_Y 1_d^T \]
\[ = ZBY^T - ZB \mu_Y^T - \tilde{Z} BY^T + \tilde{Z} \mu_Y^T \]
\[ = (Z - \tilde{Z}) B (Y - \tilde{Y})^T \]

The following is a proof of Theorem 3.1.

**Proof.** By the Assumption 1,

\[ E(Z_o) = 0, E((Z_o^2)^2) = 1, E((Z_o^4)^4) = \eta_i > 3, \forall i \in [k]. \]

Thus,

\[ E(v(R, Z_o \tilde{R}^T)) = \sum_{j=1}^k E[Z_o \tilde{R}^T R]_j^4. \]

To simplify notation, the proof reparameterizes the optimization parameter $R$ as follows. For the rotation matrix $R$, define $O = \tilde{R}^T R \in O(k)$. We want to choose $O \in O(k)$ to optimize the quantities $\sum_j E[Z_o O]_j^4 = \sum_j E(Z_o O_{.j})^4$, where $O_{.j} \in \mathbb{R}^k$ is the $j$th column of $O$. Notice elements of $Z_o$ are independent and each has zero-mean. We have

\[ \sum_j E(Z_o O_{.j})^4 = \sum_{j=1}^k \left( \sum_i E((Z_o^4) O_{ij}^4) + 3 \sum_{i \neq \ell} E((Z_o^2)(z_i^o)^2) O_{ij}^2 O_{ij}^2 \right) \]
\[ = \sum_{j=1}^k \left( \sum_i \eta_i O_{ij}^4 + 3 \sum_{i \neq \ell} O_{ij}^2 O_{ij}^2 \right). \quad (23) \]

The above equation only depends on the squared elements of $O$. Define $O^{(2)} \in \mathbb{R}^{k \times k}$ such that $O_{ij}^{(2)} = O_{ij}^2$. Because $O \in O(k)$, $O^{(2)}$ is a doubly stochastic matrix, where each element is non-negative and all row and column sums are equal to one. Define

\[ F_{\eta}(Q) = \sum_{j=1}^k \left( \sum_i \eta_i Q_{ij}^2 + 3 \sum_{i \neq \ell} Q_{ij} Q_{ij} \right) \quad (24) \]

and define $S(k)$ as the set of $k \times k$ doubly stochastic matrices. Note that

\[ \sum_j E(Z_o O_{.j})^4 = F_{\mu}(O^{(2)}) \leq \max_{Q \in S(k)} F_{\eta}(Q). \]
In this way, the Varimax problem relaxes from orthonormal matrices to doubly stochastic matrices.

The rest of the proof will show that

$$\max_{Q \in S(k)} F_\eta(Q) = \sum_{i=1}^{k} \eta_i. \quad (25)$$

Because $$\sum_j \mathbb{E}(Z^a O_{ij})^4$$ evaluated with $$O$$ as the identity matrix, is equal to $$\sum_{i=1}^{k} \eta_i$$, it follows that $$O = I$$ or $$R = \tilde{R}$$ obtains the maximum. Moreover, for $$P \in \mathcal{P}(k)$$, $$O = P$$ (i.e. $$R = \tilde{R} P$$) obtains the maximum value. It only remains to show Equation (25).

\[
F_\eta(Q) = \sum_{j=1}^{k} \left( \sum_{i=1}^{k} \eta_i Q_{ij}^2 + 3 \sum_{i \neq \ell} Q_{ij} Q_{\ell j} \right) \\
= \sum_{j=1}^{k} \left( \sum_{i=1}^{k} \eta_i Q_{ij}^2 + 3 \left( \sum_{i=1}^{k} Q_{ij} \right)^2 - 3 \sum_{i=1}^{k} Q_{ij}^2 \right) \\
= \sum_{j=1}^{k} \left( \sum_{i=1}^{k} \eta_i Q_{ij}^2 - 3 \sum_{i=1}^{k} Q_{ij}^2 \right) + 3k \\
= \sum_{i=1}^{k} (\eta_i - 3) \sum_{j=1}^{k} Q_{ij}^2 + 3k \\
\leq \sum_{i=1}^{k} (\eta_i - 3) \sum_{j=1}^{k} Q_{ij} + 3k \\
= \sum_{i=1}^{k} (\eta_i - 3) + 3k \\
= \sum_{i=1}^{k} \eta_i.
\]

The inequality is because $$Q_{ij} \in [0, 1], \forall i, j$$ (this is because $$Q \in S(k)$$).

To see that the maximum of $$\sum_j \mathbb{E}(Z^a O_{ij})^4$$ is only attained by matrices in $$\mathcal{P}(k)$$, note that for any rotation matrix $$O \notin \mathcal{P}(k)$$, then

$$\sum_j \mathbb{E}(Z^a O_{ij})^4 = F_\mu(O^{(2)}) = \sum_{i=1}^{k} (\eta_i - 3) \sum_{j=1}^{k} O_{ij}^4 + 3k < \sum_{i=1}^{k} (\eta_i - 3) \sum_{j=1}^{k} O_{ij}^2 + 3k = \sum_{i=1}^{k} \eta_i,$$

where the inequality is now strict.
B.1 A justification for the recentering step described in Remark 1.3

This section demonstrates how \( \hat{\mu}_Z = \sqrt{n} \mu_c \hat{V} \hat{D}^{-1} \hat{R}_U \) can estimate \( \mu_Z = \frac{1}{T} Z / n \) under the Varimax assumptions on \( Z \) by studying the population behavior of \( \hat{\mu}_Z \). Define the population version of the estimator as

\[
\mu_Z^* = \sqrt{n} \mu_c V D^{-1} \tilde{R}_U,
\]

where \( \mu_c = \frac{1}{T} \mathcal{A} / n = \frac{1}{T} Z BY^T / n \), \( V \) and \( D \) are defined in Proposition 3.2 with the SVD of \( \hat{\mathcal{A}} \) as

\[
D = \sqrt{nd} \tilde{D}, \quad V = d^{-1/2} \tilde{Y} \tilde{\Sigma}_Y^{-1/2} \tilde{R}_V,
\]

and \( R_U \) is the population Varimax rotation \( \tilde{R}_U \) (as justified by Theorem 3.1). In the steps below, it is presumed that \( Z \) satisfies the Varimax assumptions. It is only presumed that \( Y \) is full rank. For simplicity, the \( \approx \) correspond to approximating \( \hat{\Sigma}_Z \) as the identity matrix; under the Varimax assumptions, this is a reasonable approximation for large \( n \). Recall that

\[
B \hat{\Sigma}_Z^{1/2} \approx \tilde{R}_U \tilde{D} \tilde{R}_V.
\]

Thereby,

\[
\mu_Z^* = \sqrt{n} \mu_c V D^{-1} \tilde{R}_U
= \sqrt{n} \mu_Z (BY^T) V D^{-1} \tilde{R}_U
\approx \sqrt{n} \mu_Z \left( \tilde{R}_U^T \tilde{D} \tilde{R}_V \tilde{\Sigma}_Y^{-1/2} Y^T \right) d^{-1/2} \tilde{Y} \tilde{\Sigma}_Y^{-1/2} \tilde{R}_U \tilde{D}^{-1} \tilde{R}_U.
\]

Then, \( \tilde{\Sigma}_Y^{-1/2} Y^T \tilde{Y} \tilde{\Sigma}_Y^{-1/2} \) is \( d \) multiplied by the identity matrix. Substituting for \( D \) and canceling out several terms yields the result,

\[
\mu_Z^* \approx (nd)^{1/2} \mu_Z \tilde{R}_U^T \tilde{D} \tilde{R}_V \tilde{R}_U \tilde{R}_V \tilde{D}^{-1} \tilde{R}_U
= (nd)^{1/2} \mu_Z \tilde{R}_U^T \tilde{D} \tilde{R}_V \tilde{R}_U \tilde{R}_V \tilde{D}^{-1} \tilde{R}_U
= \mu_Z \tilde{R}_U^T \tilde{D} \tilde{R}_V \tilde{R}_U \tilde{D}^{-1} \tilde{R}_U
= \mu_Z.
\]

The rigorous proof will be shown later in Proposition C.6 and Proposition C.7.

B.2 Proofs for Lemmas 5.1 and 5.2

The following is a proof of Lemma 5.1.

Proof. For ease of notation, refer to the topic for word \( w \) as \( z \) (instead of \( w \)),

\[
\mathbb{P}(w = j|Z_i) = \sum_{z=1}^k \mathbb{P}(w = j|z, Z_i) \mathbb{P}(z|Z_i) = \sum_{z=1}^k \beta_{j,z} Z_{i,z} = (\beta_{j,.}, Z_i).
\]
So, step 3 in the LDA model is equivalent to choosing word \( w \) to be word \( j \) with probability \([\beta Z_i]_j\). So, conditional on \( N_i \) and \( Z_i \), the \( i \)th row of \( A \) is \( \text{Multinomial}(N_i, \beta Z_i) \). Then, unconditional on \( N_i \), due to the Poisson-Multinomial relationship, each element in the \( i \)th row of \( A \) is independent, with the distribution \( A_{ij} \sim \text{Poisson}(\xi[\beta Z_i]_j) \). So, \( E(A|Z) = \xi Z \beta^T \).

The following is a proof of Lemma 5.2.

**Proof.** There are three elements of Lemma 5.2. **Part 1:** conditionally on \( Z_1, \ldots, Z_n \) and \( \Xi \), we need to show that the document-term matrix \( A \) has independent Poisson entries satisfying

\[ E(A|\Xi, Z) = (\Xi Z)^T \beta. \tag{26} \]

The proof of this is equivalent to the proof of Lemma 5.1.

**Part 2:** The second part is that each element \( (\Xi Z)_{ij} \) is independent Gamma\((\alpha_j, s)\). To see this, let \( X_i \in \mathbb{R}_+^k \) have independent Gamma elements, \( X_{ij} \sim \text{Gamma}(\alpha_j, s) \). Define \( \xi'_i = \sum_j X_{ij} \) and

\[ Z'_i = \frac{X_i}{\xi'_i}. \]

It is well known that (1) \( Z'_i \sim \text{Dirichlet}(\alpha) \), (2) \( \xi'_i \sim \text{Gamma}(\sum_j \alpha_j, s) \), and (3) \( \xi'_i \) is independent of \( Z'_i \). So,

\[ (\Xi Z)_i = \xi_i Z_i \overset{d}{=} \xi'_i Z'_i = X_i. \]

**Part 3:** We need to show that \( \Xi Z \Sigma^{-1/2} \) satisfies the identification assumptions for Varimax. From part 2 above, each row contains \( k \) independent random variables and each row is iid. Then, each element of \( \Xi Z \) is leptokurtic because the Gamma distribution is always leptokurtic. Scaling by a constant \( \Sigma^{-1/2} \) does not change this. The fourth piece of the identification assumptions for Varimax is ensured by the scaling \( \Sigma^{-1/2} \).

\[ \square \]
Proof. Define the random variable $B \in \{0, 1\}$ to be equal to 1 when $X \neq 0$ and equal to 0 when $X = 0$. For some $0 < p < 1/6$, $B \sim \text{Bernoulli}(p)$. Define random variable $S$ such that when $B = 1$, $S = X$ and when $B = 0$, then $S$ is equal in distribution to $X$ on the set $X \neq 0$. So, $X = SB$.

Under the conditions of the theorem and the construction above, $S$ has some arbitrary distribution with finite 4th moment and is also independent of $B$.

Let $\mu_i = \mathbb{E}(S^i)$. Then

$$\theta := \mathbb{E}(X) = (1 - p)0 + p\mu_1 = p\mu_1,$$

$$\mathbb{E}[(X - \theta)^2] = p\mu_2 - p^2\mu_1^2,$$

$$\mathbb{E}[(X - \theta)^4] = p\mu_4 - 4p^2\mu_3\mu_1 + 6p^3\mu_2\mu_1^2 - 3p^4\mu_1^4.$$

So, in order to show that $\mathbb{E}[(X - \theta)^4] > 3\mathbb{E}[(X - \theta)^2]^2$, it is enough to show that

$$(\mu_4 - 3p\mu_2^2) + 6p^2\mu_1^2(\mu_2 - p\mu_1^2) > 4p\mu_3\mu_1 - 6p^2\mu_2\mu_1^2.$$ (27a)

Using Lemma C.1 with $g = S^2, h = 2pS, f$ being $S$'s pdf, we have

$$\mu_4 - \mu_2^2 + 4p^2\mu_1^2(\mu_2 - \mu_1^2) \geq 4p\mu_3\mu_1 - 4p\mu_1^2\mu_2.$$ (27b)

Subtract Equation (27a) from Equation (27b) we only need to show

$$(1 - 3p)\mu_2^2 + p^2\mu_1^2(2\mu_2 + 4\mu_1^2 - 6p\mu_1^2) > (4p - 6p^2)\mu_1^2\mu_2.$$ (27c)

Notice $p < 1/6$, thus $(6p - 1)(p - 1) > 0 \Rightarrow 1 - 3p > 4p - 6p^2$. Thus by Jensen’s Inequality

$$(1 - 3p)\mu_2^2 \geq (4p - 6p^2)\mu_2^2 \geq (4p - 6p^2)\mu_1^2\mu_2.$$

The first inequality is strict as long as $\mu_2 > \mu_1^2$. Also with $p < 1/6$ we have

$$p^2\mu_1^2(2\mu_2 + 4\mu_1^2 - 6p\mu_1^2) \geq p^2\mu_1^2(2\mu_2 + 3\mu_1^2) \geq 0.$$

The second inequality is strict as long as $p \geq 0, \mu_1 \neq 0$.

If $\mu_2 = \mu_1 = 0$ then $\mathbb{P}(X = 0) = 1$, contradiction. Thus $X$ is leptokurtic.

Lemma C.1. Suppose $f$ is any distribution pdf. $g, h$ is any integrable functions. Then

$$\int g^2 f dx - (\int g f dx)^2 + (\int h f dx)^2(\int h^2 f dx - (\int h f dx)^2) \geq (\int gh f dx - \int g f dx \int h f dx) \int h f dx.$$
Let $\tilde{g} = g - \int g f dx$, $\tilde{h} = h - \int h f dx$. Then
\[
\int g^2 f dx - (\int g f dx)^2 = \int \tilde{g}^2 f dx,
\]
\[
\int h^2 f dx - (\int h f dx)^2 = \int \tilde{h}^2 f dx,
\]
\[
\int gh f dx - \int g f dx \int h f dx = \int \tilde{g} \tilde{h} f dx.
\]

By Cauchy-Schwartz Inequality,
\[
\int g^2 f dx - (\int g f dx)^2 + (\int h f dx)^2 (\int h^2 f dx - (\int h f dx)^2)
\]
\[
= \int \tilde{g}^2 f dx + (\int h f dx)^2 \int \tilde{h}^2 f dx
\]
\[
\geq |\int h f dx| \sqrt{\int \tilde{g}^2 f dx \int \tilde{h}^2 f dx}
\]
\[
\geq |\int h f dx| \int |\tilde{g} \tilde{h}| f dx
\]
\[
\geq |\int h f dx| (|\int gh f dx - \int g f dx \int h f dx|)
\]
\[
\geq (\int gh f dx - \int g f dx \int h f dx) \int h f dx.
\]

C.2 Leptokurtosis with soft sparsity

The random variable $X$ in Theorem 2.1 satisfies a hard sparsity condition. Imagine $X$ as satisfying the conditions of Theorem 2.1. The next proposition studies $X + W$, where $W$ is any independent random variable with a small variance. So, if $W$ has a probability density, then $P(X + W = 0) \neq 0$, yet when $W$ has expectation zero, then $X + W$ is still close to zero with high probability. In this regime, the next proposition shows that if $X$ has a sufficiently large kurtosis, then $X + W$ is still leptokurtic, no matter the kurtosis of $W$.

**Proposition C.1.** Let $X$ and $W$ be any independent random variables with four finite moments. Let $\eta_{x,j} = \mathbb{E}(X - \mathbb{E}(X))^j$ and $\eta_{w,j} = \mathbb{E}(W - \mathbb{E}(W))^j$. Let $\eta_{x,2} = 1$. For any $\epsilon > 0$, if $\eta_{w,2} < \epsilon$, and $\eta_{x,4} \geq 3(1 + \epsilon)^2$, then $X + W$ is leptokurtic.

Note that both $X$ and $W$ can be rescaled to satisfy the assumption $\eta_{x,2} = 1$. In this way, it does not restrict the generality of the result. It only simplifies the notation.
Proof. Note that $\eta_{x,1} = \eta_{w,1} = 0$. Using that fact,

$$\mathbb{E}(X + W - \mathbb{E}(X + W))^2 = \eta_{x,2} + \eta_{w,2} < 1 + \epsilon$$

and

$$\mathbb{E}(X + W - \mathbb{E}(X + W))^4 = \eta_{x,4} + 6\eta_{x,2}\eta_{w,2} + \eta_{w,4} > 3(1 + \epsilon)^2.$$ 

The result follows from the definition of leptokurtic.

C.3 Proofs for the main results, Theorem 4.1

Proof. We need six propositions listed below to prove Theorem 4.1. Before the proof we clarify some notations. For a generic random matrix $X$, let $R_X$ be its sample Varimax rotation, i.e.

$$R_X \in \arg\max_{R \in \Theta(k)} v(R, X),$$

where $v(R, X)$ is defined in Equation (1). Then, let $R^*_X$ the population Varimax rotation, i.e.

$$R^*_X \in \arg\max_{R \in \Theta(k)} V_X(R),$$

where the expectation in $V_X(R) = \mathbb{E}(v(R, X \tilde{R}))$ is defined over the distribution of $X$ and the nuisance rotation $\tilde{R}$ can be understood from the context. Define

$$W = \arg\min_{W_0 \in \Theta(k)} \|\hat{U} - UW_0\|_{2 \to \infty}.$$ 

$\mathcal{P}(k)$ is defined in Equation (12). $P_n = P^{(1)}_n P^{(2)}_n P^{(3)}_n$ where $P^{(i)}_n \in \mathcal{P}(k)$, $i = 1, 2, 3$ are defined in Proposition C.3. C.4 C.5 respectively. Let $\mu_Z = 1_n Z/n$. $J_n$ is $n$ by $n$ matrix with every entry equal to 1. $X^\dagger$ is the pseudo-inverse of $X$. Define $\xi = 1 + \epsilon$ for some small positive $\epsilon < 0.01$ for notation consistency with Cape et al. [2019a]. Recall $\Delta_n = n\rho_n$, $\bar{\Delta}_n = n\bar{\rho}_n$.

Define

$$\gamma_{ij}^{(n)} = \sup_{s \geq 2} \left( \frac{\mathbb{E}[(A_{ij} - \omega_{ij})^s]}{s!} \right)^{1/s} \quad \text{and} \quad \gamma_i^{(n)} = \sup_{ij} \gamma_{ij}^{(n)}.$$ 

The $\gamma^{(n)}$ reveals the tail behaviors of sub-exponential random variables. It is useful in deriving matrix concentration results for sub-exponential random matrices later (Lemma C.4).

See Sections C.3.1 through C.3.6 for proofs to the following propositions C.2 through C.7. Several lemmas and technical details for these proofs are then delayed further into
Sections D and E

Proposition C.2. Let $\tilde{\Sigma}_Z = \tilde{Z}^T \tilde{Z} / n$. Under the settings of Theorem 4.1,
$$\|U \tilde{R}_U - U \tilde{R}_U \tilde{\Sigma}^{1/2}_Z\|_{2 \to \infty} = O_p \left( \frac{\log n}{n} \right). \quad (34)$$

Proposition C.3. Under the settings of Theorem 4.1, there exists $P^{(1)}_n \in \mathcal{P}(k)$ s.t.
$$\|\tilde{U} R_{\tilde{U}W}^* - U \tilde{R}_U P^{(1)}_n\|_{2 \to \infty} = O_p \left( (n \rho_n)^{-1/2} n^{-1/2} \log^{5/2} n \right). \quad (35)$$

Proposition C.4. Under the settings of Theorem 4.1, there exists $P^{(2)}_n \in \mathcal{P}(k)$ such that for any $\delta > 0$,
$$\|\tilde{U} R_{\tilde{U}W}^* - \tilde{U} R_{\tilde{U}W}^* P^{(2)}_n\|_{2 \to \infty} = O_p \left( n^{\delta/2 - 3/4} \log n \right). \quad (36)$$

Proposition C.5. Under the settings of Theorem 4.1, there exists $P^{(3)}_n \in \mathcal{P}(k)$ s.t.
$$\|\tilde{U} R_{\tilde{U}W}^* - \tilde{U} R_{\tilde{U}W} P^{(3)}_n\|_{2 \to \infty} = O_p \left( (n \rho_n)^{-1/4} n^{-1/2} \log^{11/4} n \right). \quad (37)$$

Proposition C.6. Define $P_n = P^{(1)}_n P^{(2)}_n P^{(3)}_n$ with $P^{(1)}_n, P^{(2)}_n, P^{(3)}_n$ defined in Proposition C.3, C.4, C.5 respectively. Under the settings of Theorem 4.1, for any $\delta > 0$,
$$\|J_n(A \sqrt{\tilde{U}} D^{-1} P_n - \tilde{\Psi} V D^{-1} \tilde{U} P_n)\|_{2 \to \infty} = O_p \left( n^{\delta/2 + 1/4} + (n \rho_n)^{-1/4} n^{1/2} \log^{7/2} n \right). \quad (38)$$

Proposition C.7. Under the settings of Theorem 4.1
$$\|J_n(\sqrt{n} \tilde{\Psi} V D^{-1} \tilde{U} - Z)\|_{2 \to \infty} = O_p(\sqrt{n} \log n). \quad (39)$$
We are going to show the bound for \(\|\sqrt{n}U\tilde{R}_U - \tilde{Z}P_n\|_{2\rightarrow\infty}\) by splitting it into four parts using triangle inequalities. Proposition C.2, C.3, C.4, C.5 give the bound for each split component. Similarly, we show the bound for \(\|1_n\tilde{m}_Z - 1_n\tilde{m}_Z P_n\|_{2\rightarrow\infty}\) by decomposing it into two parts and use Proposition C.6, C.7 to give bounds. The proofs of these propositions are shown after the proof of Theorem 4.1. The propositions that justify the equalities below are numbered on the left side of the equalities.

\[
\begin{align*}
(\text{Proposition C.2}) & \quad \|\sqrt{n}U\tilde{R}_U - \sqrt{n}U\tilde{R}_U \tilde{\Sigma}_Z^{1/2} P_n\|_{2\rightarrow\infty} \\
& = \|\sqrt{n}U\tilde{R}_U - \sqrt{n}UR_{UW} P_n^{(3)} + \sqrt{n}UR_{UW} P_n^{(3)} - \sqrt{n}UR_{UW} P_n^{(2)} P_n^{(3)} + \sqrt{n}UR_{UW} P_n^{(2)} P_n^{(3)} - \sqrt{n}UR_{UW} P_n^{(2)} P_n^{(3)} \|_{2\rightarrow\infty} \\
& + \|\sqrt{n}UR_{UW} P_n^{(2)} P_n^{(3)} - \sqrt{n}UR_{UW} P_n^{(2)} P_n^{(3)} \|_{2\rightarrow\infty} + \|\sqrt{n}UR_{UW} P_n^{(2)} P_n^{(3)} - \sqrt{n}UR_{UW} P_n^{(2)} P_n^{(3)} \|_{2\rightarrow\infty} \\
& \leq \|\sqrt{n}UR_{UW} P_n^{(3)} \|_{2\rightarrow\infty} + \|\sqrt{n}UR_{UW} P_n^{(3)} - \sqrt{n}UR_{UW} P_n^{(2)} P_n^{(3)} \|_{2\rightarrow\infty} \\
& + \|\sqrt{n}UR_{UW} P_n^{(2)} P_n^{(3)} - \sqrt{n}UR_{UW} P_n^{(2)} P_n^{(3)} \|_{2\rightarrow\infty} + O_p\left(\frac{\log n}{\sqrt{n}}\right).
\end{align*}
\]

(Proposition C.2) \[
\begin{align*}
(\text{Proposition C.3}) & \quad \|\sqrt{n}U\tilde{R}_U - \sqrt{n}UR_{UW} P_n^{(3)} \|_{2\rightarrow\infty} + \|\sqrt{n}UR_{UW} P_n^{(3)} - \sqrt{n}UR_{UW} P_n^{(2)} P_n^{(3)} \|_{2\rightarrow\infty} \\
& + O_p\left((n\rho_n)^{-1/2} \log^{\frac{5}{2}} n\right) + O_p\left(\frac{\log n}{\sqrt{n}}\right).
\end{align*}
\]

(Proposition C.4) \[
\begin{align*}
(\text{Proposition C.5}) & \quad O_p\left((n\rho_n)^{-1/4} \log^{\frac{11}{4}} n\right) + O_p\left((n\rho_n)^{-1/2} \log^{\frac{5}{2}} n\right) + O_p\left(\frac{\log n}{\sqrt{n}}\right) \\
& = O_p\left((n\rho_n)^{-1/4} \log^{\frac{11}{4}} n\right) + O_p\left((n\rho_n)^{-1/2} \log^{\frac{5}{2}} n\right) \\
& = O_p\left(\Delta_n^{-1/4} \log^{\frac{11}{4}} n\right).
\end{align*}
\]

For the recentering part, by Proposition C.6, C.7

\[
\begin{align*}
\|1_n\tilde{m}_Z - 1_n\tilde{m}_Z P_n\|_{2\rightarrow\infty} & = \frac{1}{n} \|J_n^T(\sqrt{n}A\tilde{D}^{-1}R_U - ZP_n)\|_{2\rightarrow\infty} \\
& \leq \frac{1}{n} \|J_n(\sqrt{n}A\tilde{D}^{-1}R_U - \sqrt{n}\mathcal{A}VD^{-1}\tilde{R}_U P_n)\|_{2\rightarrow\infty} + \frac{1}{n} \|J_n(\sqrt{n}\mathcal{A}VD^{-1}\tilde{R}_U P_n - ZP_n)\|_{2\rightarrow\infty} \\
& = O_p\left((n\rho_n)^{-1/4} + (n\rho_n)^{-1/2} \log^{\frac{5}{2}} n + \frac{\log n}{\sqrt{n}}\right) \\
& = O_p\left(\Delta_n^{-1/4} + \log^{\frac{7}{4}} n\right).
\end{align*}
\]
Take $\delta = 0.2$. Equation (40), (41) and triangle inequality accomplish the proof. □

Before the proofs for the six propositions, two useful lemmas are given. Lemma C.2 gives bound for the maximum absolute value of $Z$’s elements. Lemma C.3 borrows matrix $2 \to \infty$ norm’s property from Cape et al. [2019b].

**Lemma C.2.**

$$
\max_{i,j} |Z_{ij}| = O_p(\log n), \quad \max_{i,j} |\tilde{Z}_{ij}| = O_p(\log n).
$$

$$
\max_{i,j} |Y_{ij}| = O_p(\log d), \quad \max_{i,j} |\tilde{Y}_{ij}| = O_p(\log d).
$$

**Proof.** Assumption 2 indicates $Z$’s columns are sub-exponential variables. Thus there exists $C_0, \lambda_j > 0, j \in [k]$’s s.t.

$$
P(|Z_{ij} - {\mathbb{E}}Z_{ij}| > t) \leq C_0 \exp(-\lambda_j t) \leq C_0 \exp(-\lambda t),
$$

(42)

with $\lambda = \min_{j \in [k]} \lambda_j$. Then

$$
P(\max_{i,j} |Z_{ij} - {\mathbb{E}}Z_{ij}| > t) \leq \sum_{i,j} P(|Z_{ij} - {\mathbb{E}}Z_{ij}| > t)
$$

$$
\leq \sum_{i,j} C_0 \exp(-\lambda t)
$$

$$
\leq knC_0 \exp(-\lambda t).
$$

$$
\Rightarrow 
\max_{i,j} |Z_{ij}| = O_p(\log n), \quad \max_{i,j} |\tilde{Z}_{ij}| = O_p(\log n).
$$

Similar conclusion also applies to $Y$. □

With Lemma C.2, it could be trivially inferred that

$$
\bar{\rho}_n = O(\rho_n \log^2 n).
$$

(43)

**Lemma C.3.** Suppose $X_1 \in \mathbb{R}^{n_1 \times n_2}, X_2 \in \mathbb{R}^{n_2 \times n_3}$ are real matrices. Then

$$
\|X_1X_2\|_{2 \to \infty} \leq \|X_1\|_{2 \to \infty} \|X_2\|.
$$

(44)

This is a direct conclusion from Proposition 6.5 in Cape et al. [2019b].
C.3.1 Proof of Proposition C.2

Proof. The \((i,j)\) entry of \(\hat{\Sigma}_Z \in \mathbb{R}^{k \times k}\) is

\[
\hat{\Sigma}_Z[i,j] = \begin{cases} 
\frac{1}{n} \sum_{q=1}^{n} (Z_{qi} - \hat{\mu}_Z[i])^2 & \text{if } i = j, \\
\frac{1}{n} \sum_{q=1}^{n} (Z_{qi} - \hat{\mu}_Z[i])(Z_{qj} - \hat{\mu}_Z[j]) & \text{if } i \neq j.
\end{cases}
\]

By LLN, \(\|\hat{\Sigma}_Z - I\|_{\text{max}} = O_p(\frac{k^2}{\sqrt{n}}) = O_p(\frac{1}{\sqrt{n}})\), thus \(\|\hat{\Sigma}_Z - I\| \leq \sqrt{k^2} \|\hat{\Sigma}_Z - I\|_{\text{max}} = O_p(\frac{1}{\sqrt{n}})\).

Suppose eigendecomposition of \(\hat{\Sigma}_Z\) is \(\hat{\Sigma}_Z = \Psi \Lambda \Psi^T\). Then

\[
\|\hat{\Sigma}_Z - I\| = \|\Lambda - I\| = O_p(\frac{1}{\sqrt{n}}) \Rightarrow \|\hat{\Sigma}_Z^{-1/2} - I\| = O_p(\frac{1}{\sqrt{n}}).
\]

Also \(\|\hat{\Sigma}_Z - I\| = O_p(\frac{1}{\sqrt{n}})\) implies \(\|\hat{\Sigma}_Z^{-1/2}\| = O_p(1)\). By Proposition C.2 and Lemma C.3

\[
\|U\|_{2-\infty} = \frac{1}{\sqrt{n}} \|\tilde{Z}\hat{\Sigma}_Z^{-1/2}\|_{2-\infty} \leq \frac{1}{\sqrt{n}} \|\tilde{Z}\|_{2-\infty} \|\hat{\Sigma}_Z^{-1/2}\| = O_p(\frac{\log n}{\sqrt{n}}).
\]

Putting the above pieces together provides a bound on the quantity of interests.

\[
\|U \tilde{R}_U - U \tilde{R}_U \hat{\Sigma}_Z^{1/2}\|_{2-\infty} \leq \|U \tilde{R}_U\|_{2-\infty} \|\hat{\Sigma}_Z^{1/2}\| = \|U\|_{2-\infty} \|\hat{\Sigma}_Z^{1/2}\| = O_p(\frac{\log n}{\sqrt{n}}).
\]

\(\square\)

C.3.2 Proof of Proposition C.3

We give the statement of Lemma C.4, C.5 below and use them to prove proposition C.3. The proof of these two lemmas will be shown in Section D.

Lemma C.4. Define the symmetrized adjacent matrix as \(\tilde{A}_{sym} = \begin{pmatrix} 0 & \tilde{A} \\ \tilde{A}^T & 0 \end{pmatrix}\) and its population version as \(\tilde{A}_{sym} = \begin{pmatrix} 0 & \tilde{A} \\ \tilde{A}^T & 0 \end{pmatrix}\). Under the settings in Theorem 4.1

\[
\|A_{sym} - \tilde{A}_{sym}\| = O_p((n\rho_n \log^3 n)^{1/2}).
\]

Lemma C.5. Presume the conditions in Theorem 4.1. There exists \(W \in \mathcal{O}(k)\), such that

\[
\|\hat{U} - UW\|_{2-\infty} = O_p\left((n\rho_n)^{-1/2} n^{-1/2} \log^2 n\right).
\]
Lemma C.5 gives a row-wise bound for the eigenvectors’ fluctuations. This lemma follows from Theorem 1 in Cape et al. [2019a], which requires several conditions. Lemma C.4 is used for one of the conditions. The other conditions are either already satisfied by the assumptions of Theorem 4.1 or checked inside the proof of Lemma C.5.

Proof. Notice the fact that $2 \rightarrow \infty$ norm is invariant to rotations. From Theorem 3.1 there exist $P_n^{(1)} \in \mathcal{P}(k)$ s.t. $R_{UW}^{*} P_n^{(1)} = W^T \tilde{R}_U P_n^{(1)}$. Therefore

$$\|\tilde{U} R_{UW}^* U P_n^{(1)}\|_2 \rightarrow \infty = \|\tilde{U} W^T \tilde{R}_U P_n^{(1)} - U \tilde{R}_U P_n^{(1)}\|_2 \rightarrow \infty$$

$$= \|\tilde{U} W^T - U\|_2 \rightarrow \infty$$

$$= \|\tilde{U} - UW\|_2 \rightarrow \infty$$

(Lemma C.5) $= O_p\left((n \rho_n)^{-1/2} n^{-1/2} \log^{5/4} n\right).$

\[\square\]

C.3.3 Proof of Proposition C.4

The proof of Proposition C.4 uses the following lemma to bound the distance between sample and population Varimax solutions (modulo permutation and sign flip).

Lemma C.6. Recall that $R_{\tilde{Z}} \in \arg \max_{R_0 \in \mathcal{O}(k)} v(R_0, \tilde{Z})$. There exists $P_n^{(2)} \in \mathcal{P}(k)$ s.t. for $\forall \delta > 0$

$$\|R_{\tilde{Z}} - P_n^{(2)}\|_2 \rightarrow \infty = O_p(n^{\delta/2 - 1/4}).$$

The proof of Lemma C.6 is in Section D.

Proof. With some previous lemmas,

$$\|\tilde{U} R_{UW} - \tilde{U} R_{UW}^{*} P_n^{(2)}\|_2 \rightarrow \infty$$

$$= \|\tilde{U} (R_{UW} - R_{UW}^{*} P_n^{(2)})\|_2 \rightarrow \infty$$

(Lemma C.3) $\leq \|\tilde{U}\|_2 \rightarrow \infty \|R_{UW} - R_{UW}^{*} P_n^{(2)}\|$

(Lemma C.6) $= O_p(n^{\delta/2 - 1/4}\|\tilde{U}\|_2 \rightarrow \infty)$

$$\leq O_p(n^{\delta/2 - 1/4}\|\tilde{U} - UW\|_2 \rightarrow \infty + n^{\delta/2 - 1/4}\|UW\|_2 \rightarrow \infty)$$

(Lemma C.5) $= O_p\left((n \rho_n)^{-1/2} n^{-3/4} \log^{5/4} n\right) + O_p(n^{\delta/2 - 1/4}\|UW\|_2 \rightarrow \infty)$

$$\leq O_p\left((n \rho_n)^{-1/2} n^{-3/4} \log^{5/4} n\right) + O_p(n^{\delta/2 - 1/4}\|U\|_2 \rightarrow \infty)$$

(Equation 45) $\leq O_p\left((n \rho_n)^{-1/2} n^{-3/4} \log^{5/4} n\right) + O_p(n^{\delta/2 - 3/4} \log n)$

$(n \rho_n \geq \log^{2\delta} n)$ $= O_p(n^{\delta/2 - 3/4} \log n)$.

\[\square\]
C.3.4 Proof of Proposition C.5

This proposition shows that \( R_{\hat{U}} \) converges to \( R_{UW} \). The proof of Proposition C.5 is contained in Section D. This proof uses the fact that the Varimax objective function is smooth and each row of \( \hat{U} \) converges to the corresponding row of \( UW \) (i.e. \( \| \hat{U} - UW \|_{2\to\infty} \to 0 \)). This implies that the Varimax solution computed with \( \hat{U} \) (i.e. \( R_{\hat{U}} \)) converges to the Varimax solution computed with \( UW \) (i.e. \( R_{UW} \)).

C.3.5 Proof of Proposition C.6

Proof.

\[
\| J_n(A\hat{V}\hat{D}^{-1}R_{\hat{U}} - \mathcal{A}V\mathcal{D}^{-1}\hat{R}_U P_n) \|_{2\to\infty} \\
\leq \sqrt{n}\| A\hat{V}\hat{D}^{-1}R_{\hat{U}} - \mathcal{A}V\mathcal{D}^{-1}\hat{R}_U P_n \| \\
(WR_{UW} = \hat{R}_U P_n^{(1)}) \\
\leq \sqrt{n}\| A\hat{V}\hat{D}^{-1}R_{\hat{U}} - AVD^{-1}WR_{\hat{U}} + AVD^{-1}WR_{\hat{U}} - AVD^{-1}WR_{UW} P_n^{(3)} + AVD^{-1}WR_{UW} P_n^{(2)} P_n^{(3)} - \mathcal{A}V\mathcal{D}^{-1}WR_{UW} P_n^{(2)} P_n^{(3)} \| \\
\leq \sqrt{n}(\| A\hat{V}\hat{D}^{-1}R_{\hat{U}} - AVD^{-1}WR_{\hat{U}} \| + \| AVD^{-1}WR_{\hat{U}} - AVD^{-1}WR_{UW} P_n^{(3)} \| + \| AVD^{-1}WR_{UW} - AVD^{-1}WR_{UW} P_n^{(2)} \| + \| AVD^{-1}WR_{UW} - \mathcal{A}V\mathcal{D}^{-1}WR_{UW} \| ).
\] (48)

The fact that \( WR_{UW} = \hat{R}_U P_n^{(1)} \) is a direct result of Theorem 3.1. The remaining part of the proof wants to show the bounds for each term of RHS of Equation (48).

First term of Equation (48) is \( \| A\hat{V}\hat{D}^{-1}R_{\hat{U}} - AVD^{-1}R_{\hat{U}} \|. \) By Lemma C.5

\[
\| \hat{U} - UW \|_{2\to\infty} = O_p\left( (n\rho_n)^{-1/2}n^{-1/2}\log^{\frac{5}{2}} n \right),
\]
and by the same virtue (notice \( Y \) also satisfies Assumption 2, it could be shown by transposing the adjacency matrix) there exists \( W_2 \in \mathcal{O}(k) \) s.t.

\[
\| \hat{V} - VW_2 \|_{2\to\infty} = O_p\left( (n\rho_n)^{-1/2}n^{-1/2}\log^{\frac{5}{2}} n \right).
\]

By assumptions and Lemma C.4

\[
\| D^{-1} \| = O_p((n\rho_n)^{-1}), \| A - \mathcal{A} \| = O_p((n\rho_n\log^3 n)^{1/2}).
\]

Notice that \( \| X \| \leq \sqrt{m_1}\| X \|_{2\to\infty} \) for \( \forall X \in \mathbb{R}^{m_1 \times m_2} \) and \( \| V \| = 1. \) Therefore
\[ \| A \hat{V} \hat{D}^{-1} R_\theta - AVD^{-1} W R_\theta \| \\
= \| A \hat{V} \hat{D}^{-1} - AVD^{-1} W \| \\
\leq \| A \| \| \hat{V} \hat{D}^{-1} - VD^{-1} W \| \\
= \| A - \mathcal{A} + \mathcal{A} \| \| \hat{V} \hat{D}^{-1} - VW_2 \hat{D}^{-1} + VW_2 \hat{D}^{-1} - VD^{-1} W \| \\
\leq (\| A - \mathcal{A} \| + \| \mathcal{A} \|) (\| \hat{V} \hat{D}^{-1} - VW_2 \hat{D}^{-1} \| + \| VW_2 \hat{D}^{-1} - VD^{-1} W \|) \\
\leq (\| A - \mathcal{A} \| + \| \mathcal{A} \|) (\| \hat{V} - VW_2 \| \| \hat{D}^{-1} \| + \| VW_2 \hat{D}^{-1} - D^{-1} W \|) \\
= O_p(\| \mathcal{A} \| \times \| D^{-1} \|) \times |O_p((n\rho_n)^{-1/2} n^{-1/2} \log^{2} n) + O_p((n\rho_n)^{-1/2} \log^{2} n)| \\
= O_p \left( (n\rho_n)^{-1/2} \log^{2} n \right). \tag{49} \]

The third equation employs the bound of \( \| W_2 \hat{D}^{-1} - D^{-1} W \| \) from the following deduction:

\[
\| W_2 \hat{D}^{-1} - D^{-1} W \| \\
= \| \hat{D}^{-1} - W_2^T D^{-1} W \| \\
= \| \hat{V} \hat{D}^{-1} \hat{U}^T - \hat{V} W_2^T D^{-1} W \hat{U}^T \| \\
= \| \hat{V} \hat{D}^{-1} \hat{U}^T - VD^{-1} U^T + (V - \hat{V} W_2^T) D^{-1} W U + \hat{V} W_2^T D^{-1} (U^T - W \hat{U}^T) \| \\
\leq \| \hat{V} \hat{D}^{-1} \hat{U}^T - VD^{-1} U^T \| + \| (V - \hat{V} W_2^T) D^{-1} W U \| + \| \hat{V} W_2^T D^{-1} (U^T - W \hat{U}^T) \| \\
\leq \| A^\dagger - \mathcal{A}^\dagger \| + \| \mathcal{A} \| \| \mathcal{A} \| + \sqrt{d} \| V - \hat{V} W_2^T \| \| D^{-1} \| + \sqrt{n} \| D^{-1} \| \| U^T - W \hat{U}^T \| \| U^T - W \hat{U}^T \| \\
= O_p(\| D^{-1} \|) \times \left( O_p \left( (n\rho_n)^{-1/2} \log^{2} n \right) + O_p \left( (n\rho_n)^{-1/2} \log^{2} n \right) \right) \\
= O_p(\| D^{-1} \|) \times O_p \left( (n\rho_n)^{-1/2} \log^{2} n \right). \]

Second term of Equation (48) is \( \| AVD^{-1} R_\theta - AVD^{-1} W R_\theta W P_n(3) \| \). According to Equation (93) (in the proof of Proposition C.5), there exists a \( P_n(3) \in \mathcal{P}(k) \) s.t.

\[ \| R_\theta - R_{UW} P_n(3) \| \rightarrow \infty = O_p \left( (n\rho_n)^{-1/2} \log^{2} n \right). \]

Therefore,

\[
\| AVD^{-1} W R_\theta - AVD^{-1} W R_{UW} P_n(3) \| \\
\leq \| A \| \| V \| \| D^{-1} W \| \sqrt{k} \| R_\theta - R_{UW} P_n(3) \| \rightarrow \infty \\
= O_p(1) \times \| R_\theta - R_{UW} P_n(3) \| \rightarrow \infty \\
= O_p \left( (n\rho_n)^{-1/4} \log^{2} n \right). \tag{50} \]
Third term of Equation (48) is $\|AVD^{-1}WR_{UW} - AVD^{-1}WR_{UW}^*P_n\|$. Recall Proposition C.4, Theorem 3.1, there is $P_n^{(2)} \in \mathcal{P}(k)$, s.t. for any $\delta > 0$,

$$\|R_{UW} - R_{UW}^*P_n^{(2)}\|_{2 \to \infty} = O_p(n^{\delta/2-1/4}).$$

Therefore,

$$\begin{align*}
\|AVD^{-1}WR_{UW} - AVD^{-1}WR_{UW}^*P_n\| &\leq \|A\|\|V\|\|D^{-1}W\|\sqrt{k}\|R_{UW} - R_{UW}^*P_n\|_{2 \to \infty} \\
&= O_p(1) \times \|R_{UW} - R_{UW}^*P_n^{(2)}\|_{2 \to \infty} \\
&= O_p(n^{\delta/2-1/4}).
\end{align*}$$

Fourth term of Equation (48) is $\|AVD^{-1}WR_{UW}^* - AVD^{-1}WR_{UW}^*P_n^{(1)}\|$. Reusing Lemma C.4, we have

$$\begin{align*}
\|AVD^{-1}WR_{UW}^* - AVD^{-1}WR_{UW}^*\| &\leq \|A - AVD^{-1}\| \\
&\leq O_p((n\rho_n)^{-1/2} \log^2 n).
\end{align*}$$

Plugging (49), (50), (51), (52) into (48) arrives our combined bound:

$$\begin{align*}
\|J_n(A\hat{V}\hat{D}^{-1}R_{U} - AVD^{-1}\hat{R}_U P_n)\|_{2 \to \infty} &\leq O_p((n\rho_n)^{-1/2} \log^2 n + (n\rho_n)^{-1/4} \log^7 n + n^{\delta/2-1/4} + (n\rho_n)^{-1/2} \log^2 n) \\
&= O_p\left(n^{\delta/2+1/4} + (n\rho_n)^{-1/4} n^{1/2} \log^2 n\right).
\end{align*}$$

\(\square\)
C.3.6 Proof of Proposition C.7

Proof. With Lemma C.3 and Proposition 3.2

\[ \| J_n(\sqrt{n} \alpha V D^{-1} \tilde{R}_U - Z) \|_{2 \rightarrow \infty} \]
\[ \leq \sqrt{n} \| \sqrt{n} \alpha V D^{-1} \tilde{R}_U - Z \| \]
\[ = \sqrt{n} \| \sqrt{n} Z B Y^T V D^{-1} \tilde{R}_U - Z \| \]
\[ = \sqrt{n} \| Z B (Y^T \tilde{Y} / d) \tilde{\Sigma}_Y^{-1/2} \tilde{R}_U - Z \| \]
\[ = \sqrt{n} \| Z B (Y^T \tilde{Y} / d) \tilde{\Sigma}_Y^{-1} B^{-1} \tilde{\Sigma}_Z^{-1/2} - Z \| \]
\[ \leq \sqrt{n} \| Z \| \| (B(Y^T \tilde{Y} / d) \tilde{\Sigma}_Y^{-1} B^{-1} \tilde{\Sigma}_Z^{-1/2} - I) \| \]
\[ \leq \sqrt{n} \| Z \| \| (B(Y^T \tilde{Y} / d) \tilde{\Sigma}_Y^{-1} B^{-1} \tilde{\Sigma}_Z^{-1/2} - B(Y^T \tilde{Y} / d) \tilde{\Sigma}_Y^{-1} B^{-1}) \| + \| B(Y^T \tilde{Y} / d) \tilde{\Sigma}_Y^{-1} B^{-1} - I \| \]
\[ \leq \sqrt{n} \| Z \| \| B \| \| B^{-1} \| \| (Y^T \tilde{Y} / d) \tilde{\Sigma}_Y^{-1} B^{-1} - I \| \]

By Lemma C.2, \[ \| Z \| \leq \sqrt{n} k \max |Z_{ij}| = O_p(\sqrt{n} \log n). \] Conditions in main theorem statement imply \[ \| B \| \| B^{-1} \| = O_p(1). \] Using LLN results (similar to proofs in Proposition C.2) there are \[ \| \tilde{\Sigma}_Y^{-1} \| = O_p(1), \| \tilde{\Sigma}_Z^{-1/2} - I \| = O_p(1/\sqrt{n}). \]

Notice that the \((i, j)\) entry of \(\tilde{\Sigma}_Y^{-1}B^{-1}\tilde{\Sigma}_Z^{-1/2} - I\) is \[ \frac{1}{d} \hat{\mu}_Y(i) \sum_{q=1}^{n} (Y_{iq} - \hat{\mu}_Y[j]) = 0. \] By LLN

\[ \| Y^T \tilde{Y} / d \| \leq \| \tilde{Y}^T \tilde{Y} / d \| + \| \tilde{Y}^T \tilde{Y} / d \| = O_p(1) \]

and

\[ (Y^T \tilde{Y} / d) \tilde{\Sigma}_Y^{-1} - I = (\tilde{Y}^T \tilde{Y} / d) \tilde{\Sigma}_Y^{-1}, \]

is the zero matrix. Summarize these results and simplify the bounds give the desired conclusion,

\[ \| J_n(\sqrt{n} \alpha V D^{-1} \tilde{R}_U - Z) \|_{2 \rightarrow \infty} = O_p(\sqrt{n} \log n). \]

\[ \square \]

D Technical Proofs

Proof of Lemma C.4

This part of proof needs a matrix concentration bound for sub-exponential random variables. Here we cite an existing result shown below.

Lemma D.1 [Tropp 2012]. Let \(X_1, X_2, \ldots, X_n\) be independent random \(N \times N\) self-adjoint matrices. Assume that \(E(X_i) = 0\) for all \(i\), and \(E(X_i^p) \leq \frac{p}{2} R^{p-2} A_i^2\) for \(p \geq 2\). Compute the variance parameter

\[ \sigma^2 := \| \sum_k A_k^2 \|. \]

45
Then for any $t > 0$,

$$\mathbb{P}(\| \sum_{i=1}^{n} X_i \| \geq t) \leq n \times \exp(-\frac{t^2}{2\sigma^2 + 2Rt}). \quad (54)$$

Now we make use of Lemma D.1 to prove Lemma C.4.

**Proof.** Let $E^{i,j}$ be the $(n + d) \times (n + d)$ matrix with 1 in the $(i, j)$ and $(j, i)$ entries and 0 elsewhere. $\gamma_{ij}, \gamma$ are defined in Equation (33) (for simplicity we ignore the $(n)$-superscripts). To utilize Lemma D.1, we express \( \tilde{A}_{sym} - \Phi_{sym} \) as the sum of matrices,

$$Y_{i,n+j} = (A_{ij} - \Phi_{ij})E^{i,n+j}, i = 1, \ldots, n, j = 1, \ldots, d.$$ 

Notice that

$$\| \tilde{A}_{sym} - \Phi_{sym} \| = \| \sum_{i=1}^{n} \sum_{j=1}^{d} Y_{i,n+j} \|,$$

and $\mathbb{E}(Y_{i,n+j}) = 0$. Moreover,

$$(E^{i,n+j})^p = E^{i,i} + E^{n+j,n+j}, p = 2, 4, \ldots$$

$$(E^{i,n+j})^p = E^{i,n+j}, p = 3, 5, 7, \ldots,$$

and $\mathbb{E}[(A_{ij} - \Phi_{ij})^p] \leq \gamma_{ij}^p p! \leq \gamma^p p!$, for $\forall i, j, p \geq 2$. These relations indicate

$$\mathbb{E}(Y_{i,n+j}^p) \leq \frac{p!}{2} \gamma^p (\gamma_{ij}^2 (E^{i,i} + E^{n+j,n+j}) \leq \frac{p!}{2} \gamma^p (\gamma_{ij}^2 (E^{i,i} + E^{n+j,n+j})), \forall p \geq 2. \quad (55)$$

We can treat $A_i$'s in Lemma D.1 as $\frac{\gamma^2}{2} (E^{i,i} + E^{n+j,n+j})$ in our scenario. By Assumption 3

$$\sigma^2 \leq \frac{n + d}{4} \bar{p}_n.$$
Therefore, by Lemma D.1, the bound for \( \| \tilde{A}_{\text{sym}} - \tilde{A}_{\text{sym}} \| \) is obtained.

\[
P(\| \tilde{A}_{\text{sym}} - \tilde{A}_{\text{sym}} \| \geq t) \leq (n + d) \exp(-\frac{t^2}{(n+d)d + 2\gamma t}).
\]

With Assumption 3 and Equation (43) this also implies,

\[
\| \tilde{A}_{\text{sym}} - \tilde{A}_{\text{sym}} \| = O((n \rho n \log n)^{\frac{1}{2}}).
\]

Before we show the proof of Lemma C.5, we illustrate the following lemma that shows important property of matrix with special structure and could be utilized to convert bounds of eigenvectors’ perturbation of symmetized matrices to original adjacency matrices’.

**Lemma D.2.** Suppose \( M = \begin{pmatrix} M_1 & M_2 \\ M_2 & M_1 \end{pmatrix} \) is a blockwise symmetric matrix \((M_1, M_2 \in \mathbb{R}^{k \times k})\). Let \( M = U_M D_M V_M^T \) be \( M \)’s singular vector decomposition. Then \( N = U_M V_M^T \) has the same blockwise symmetric structure: \( N = \begin{pmatrix} N_1 & N_2 \\ N_2 & N_1 \end{pmatrix} \).

**Proof.** Let \( M_1 + M_2 = S_1 \Sigma_1 T_1^T, M_1 - M_2 = S_2 \Sigma_2 T_2^T \) be the singular decompositions of them. Then

\[
M_1 = \frac{1}{2} (S_1 \Sigma_1 T_1^T + S_2 \Sigma_2 T_2^T), M_2 = \frac{1}{2} (S_1 \Sigma_1 T_1^T - S_2 \Sigma_2 T_2^T).
\]

Plug in the equations,

\[
M = \begin{pmatrix} M_1 & M_2 \\ M_2 & M_1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} (S_1 \Sigma_1 T_1^T + S_2 \Sigma_2 T_2^T) & \frac{1}{2} (S_1 \Sigma_1 T_1^T - S_2 \Sigma_2 T_2^T) \\ \frac{1}{2} (S_1 \Sigma_1 T_1^T - S_2 \Sigma_2 T_2^T) & \frac{1}{2} (S_1 \Sigma_1 T_1^T + S_2 \Sigma_2 T_2^T) \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{1}{2}} S_1 & \frac{\sqrt{1}}{2} S_2 \\ \frac{\sqrt{1}}{2} S_1 & -\sqrt{\frac{1}{2}} S_2 \end{pmatrix} \begin{pmatrix} \Sigma_1 & \Sigma_2 \\ \Sigma_2 & -\Sigma_2 \end{pmatrix} \begin{pmatrix} \sqrt{\frac{1}{2}} T_1 & \frac{\sqrt{1}}{2} T_2 \\ \frac{\sqrt{1}}{2} T_1 & -\sqrt{\frac{1}{2}} T_2 \end{pmatrix}.
\]

Then \( U_M = \frac{\sqrt{1}}{2} \begin{pmatrix} S_1 & S_2 \\ S_1 & -S_2 \end{pmatrix}, V_M = \frac{\sqrt{1}}{2} \begin{pmatrix} T_1 & T_2 \\ T_1 & -T_2 \end{pmatrix}, D_M = \begin{pmatrix} \Sigma_1 & \Sigma_2 \\ \Sigma_2 & -\Sigma_2 \end{pmatrix}. \) We prove the result by the following observation.

\[
U_M V_M^T = \frac{1}{2} \begin{pmatrix} S_1 T_1^T + S_2 T_2^T & S_1 T_1^T - S_2 T_2^T \\ S_1 T_1^T - S_2 T_2^T & S_1 T_1^T + S_2 T_2^T \end{pmatrix}.
\]

\[
\square
\]

Lemma C.4 obtains the bound of spectral norm of \( \tilde{A}_{\text{sym}} - \tilde{A}_{\text{sym}} \). Next lemma makes use of this result to show the bound for distance between eigen-spaces of \( \tilde{A}_{\text{sym}} \) and \( \tilde{A}_{\text{sym}} \). Some theoretical results from [Cape et al. 2019a] is borrowed to show row-wise bounds.
Proof of Lemma \[ \text{C.5} \]

**Proof.** Let \( \mu_s = \mu_t 1_d^T + 1_n \mu_c^T - \mu 1_n 1_d^T, \) \( \tilde{\mu}_s = \tilde{\mu}_t 1_d^T + 1_n \tilde{\mu}_c^T - \tilde{\mu} 1_n 1_d^T, \) we symmetrize centered adjacent matrix as before: \( \tilde{A}_{sym} = \begin{pmatrix} 0 & \tilde{A} \\ \tilde{A}^T & 0 \end{pmatrix}, \) \( \tilde{\mathcal{A}}_{sym} = \begin{pmatrix} 0 & \tilde{\mathcal{A}} \\ \tilde{\mathcal{A}}^T & 0 \end{pmatrix}. \)

\[
\tilde{A}_{sym} - \tilde{\mathcal{A}}_{sym} = \begin{pmatrix} 0 & \tilde{A} \\ \tilde{A}^T & 0 \end{pmatrix} - \begin{pmatrix} 0 & \tilde{\mathcal{A}} \\ \tilde{\mathcal{A}}^T & 0 \end{pmatrix} = \begin{pmatrix} (A - \tilde{\mu}_s)^T & A - \tilde{\mu}_s \\ (A - \tilde{\mu}_s^T & 0 \end{pmatrix} - \begin{pmatrix} 0 & \tilde{A} - \mu_s \\ (\tilde{A} - \mu_s^T & 0 \end{pmatrix} + \begin{pmatrix} 0 & \tilde{\mathcal{A}} - \tilde{\mu}_s \\ (\tilde{\mathcal{A}} - \tilde{\mu}_s^T & 0 \end{pmatrix} = \begin{pmatrix} 0 & \tilde{\mathcal{A}} - \mu_s \\ (\tilde{\mathcal{A}} - \mu_s^T & 0 \end{pmatrix}
\]

\( = A_1 - A_2 + A_2 - A_3. \)

From Proposition \[ \text{3.2} \] we know \( \mathcal{A} - \mu_s \) is of rank \( k. \) Suppose the eigen-decomposition of \( A_3 \) is \( U_3 D_3 U_3^T. \) Then \( U_3 \in \mathbb{R}^{(n+d) \times 2k}. \) \( D_3 \) is diagonal matrix with \( 2k \) non-zero elements. Let \( U_i \in \mathbb{R}^{(n+d) \times 2k} \) be \( A_i \)’s eigenvectors corresponding to \( A_i \)’s \( k \) largest and \( k \) smallest eigenvalues and \( D_i \) being diagonal matrix contains these eigenvalues, \( i = 1, 2. \)

Define

\[
W_{(3\rightarrow 1)} := \arg \min_{W_0 \in \mathcal{O}(2k)} \| U_1 - U_3 W_0 \|_{2 \rightarrow \infty},
\]

\[
W_{(2\rightarrow 1)} := \arg \min_{W_0 \in \mathcal{O}(2k)} \| U_1 - U_2 W_0 \|_{2 \rightarrow \infty},
\]

\[
W_{(3\rightarrow 2)} := \arg \min_{W_0 \in \mathcal{O}(2k)} \| U_2 - U_3 W_0 \|_{2 \rightarrow \infty},
\]

then

\[
\| U_1 - U_3 W_{(3\rightarrow 1)} \|_{2 \rightarrow \infty} \leq \| U_1 - U_3 W_{(3\rightarrow 2)} W_{(2\rightarrow 1)} \|_{2 \rightarrow \infty} = \| U_1 - U_2 W_{(2\rightarrow 1)} + U_2 W_{(2\rightarrow 1)} - U_3 W_{(3\rightarrow 2)} W_{(2\rightarrow 1)} \|_{2 \rightarrow \infty} \leq \| U_1 - U_2 W_{(2\rightarrow 1)} \|_{2 \rightarrow \infty} + \| U_2 W_{(2\rightarrow 1)} - U_3 W_{(3\rightarrow 2)} W_{(2\rightarrow 1)} \|_{2 \rightarrow \infty} = \| U_1 - U_2 W_{(2\rightarrow 1)} \|_{2 \rightarrow \infty} + \| U_2 - U_3 W_{(3\rightarrow 2)} \|_{2 \rightarrow \infty}. \quad (56)
\]

In the following steps, we deal with the two terms in \[ (56) \] separately.

We bound \( \| U_1 - U_2 W_{(2\rightarrow 1)} \|_{2 \rightarrow \infty} \) by using Theorem 1 in Cape et al. \[ 2019a. \] There are four conditions of Cape’s Theorem that should be evaluated. The first two conditions are
followed trivially from the statements of Theorem 4.1. Let $\tilde{\sigma}_{\text{max}}$ and $\tilde{\sigma}_{\text{min}}$ represent the largest and $k$th largest singular values of $\tilde{A}$. Then with Equation (9), it is obvious to show

$$\tilde{\sigma}_{\text{min}} \geq C_1 \Delta_n = C_1 n \rho_n, \quad \tilde{\sigma}_{\text{max}} / \tilde{\sigma}_{\text{min}} \leq C_2,$$

for some positive constants $C_1, C_2$.

The following part checks the fourth condition stated below. After confirming the Cape’s fourth condition, the proof will address the third condition.

**Cape’s 4th Condition:** Write $E_1 = A_1 - A_2$. There exist constants $C_{E_1}, v > 0, \nu > 0, \xi > 1$, such that for all integers $1 \leq s \leq s(n) := \lceil \log n / \log(n\tilde{\rho}_n) \rceil$, for each fixed standard basis vector $e_i$ and any fixed unit vector $u$, with probability at least $1 - \exp(-\nu \log \xi n)$ (provided $n \geq n_0(C_{E_1}, \nu, \xi)$),

$$|\langle E_1^s u, e_i \rangle| \leq C_{E_1} n \frac{(n\tilde{\rho}_n)^{s/2}(\log n)^{s\xi}}{\|u\|_\infty}.$$

Using an argument in Lemma 7.10 of [Erdős et al. 2013, Mao et al. 2017] shows that the following Upper Bound Condition is sufficient for Cape’s 4th Condition. That appears as Lemma 5.5 of [Mao et al. 2017]. The key idea to show Cape’s 4th assumption is applying the inequality of Upper Bound Condition to upper bound the number of non-zero terms in the summation via a multigraph construction for paths counting. The difference is that our main theorem allows sub-exponential random variables, which is possibly unbounded. Thus, the only thing that needs to be checked is the following upper bound condition, with the order of magnitude $m$ ranges from 2 to the number of vertices in constructed multigraph.

**Upper Bound Condition** Let $H = \frac{A - \sqrt{n\tilde{\rho}_n}}{\sqrt{n\tilde{\rho}_n}}$ and $H_{ij}$ represents its element on $(i,j)$th entry. Then there exists a positive constant $C_{E_1}$, such that eventually in $n$,

$$\mathbb{E}(|H_{ij}|^m) \leq \frac{C_{E_1}}{n}, \quad \forall \ 2 \leq m \leq \log^\xi n.$$  

For any positive even number $m \leq \log^\xi n$. If $\tilde{\rho}_n \geq (m - 1)! \tilde{\rho}_n^{m/2}$, by Assumption 3,

$$\mathbb{E}(|H_{ij}|^m) = \mathbb{E}(H_{ij}^m) \leq \frac{\tilde{\rho}_n}{(n\tilde{\rho}_n)^{m/2}},$$

therefore, choosing $C_{E_1} = 1$, Equation (59) is implied by $n\tilde{\rho}_n \geq 1$, which is true under the main theorem’s assumptions.

If $\tilde{\rho}_n \leq (m - 1)!(\tilde{\rho}_n)^{m/2}$, by Assumption 3

$$\mathbb{E}(|H_{ij}|^m) = \mathbb{E}(H_{ij}^m) \leq \frac{(m - 1)!\tilde{\rho}_n^{m/2}}{(n\tilde{\rho}_n)^{m/2}} = \frac{(m - 1)!}{n^{m/2}}.$$  

49
The target boils down to \((m - 1)! \leq C E_1 n^{m/2-1}\). Notice that for all positive integers \(N\), there is \(N! \leq N^{N+1/2} \exp\{-N + 1\}\. Then,

\[
(m - 1)! \leq n^{m/2-1} \iff (m - 1)^{m-1/2} \exp\{-m + 2\} \leq n^{m/2-1} C E_1 \\
\iff (m - 1)^{m-1/2} \exp\{-m + 2\} \leq (m - 1)^{m/2-1} \log n + \log(C E_1) \\
\quad \text{(since } \log n^\xi \geq m) \iff (m - 1)^{m-1/2} \exp\{-m + 2\} \leq (m - 1)^{m/2-1} \log n + \log(C E_1). (60)
\]

Since \(1/\xi > 0\), there exists a positive integer \(M\), such that \((m - 1)^{1/\xi} \geq (m - 1)^{1/\xi}\) for all integer \(m > M\). Choose \(C E_1\) such that \(\log(C E_1) > (m - 1)^{1/\xi}\) for all integer \(2 \leq m \leq M\). Then Equation (60) is proved.

Hence the upper bound condition holds for even \(m\). For odd number \(m \geq 3\, by\ Cauchy-Schwartz\ inequality,

\[
(\mathbb{E}(\|H_{ij}\|^m))^2 \leq \mathbb{E}(\|H_{ij}\|^{m-1})\mathbb{E}(\|H_{ij}\|^{m+1}) \leq \frac{1}{n^2}.
\]

Thus the upper bound condition holds for all integer \(m \geq 2\ and\ therefore\ Cape’s\ fourth\ condition\ is\ valid.

We claim that Cape’s third condition could be relaxed from \(\|E\| = O_p((n \rho_n)^{1/2})\) to \(\|E\| = O_p((n \rho_n \log^3 n)^{1/2})\) as inferred from Lemma \(\text{C.4}\) with only slight modifications in Cape’s converge rate result. In the proof of Theorem 1 of Cape et al. \(\text{[2019a]}\), the bound of LHS of (5) comes from three quantities: \(\|E \tilde{U} \tilde{\Lambda}^{-1}\|_{2 \to \infty}, \|R^{(1)}\|_{2 \to \infty}, \|R^{(2)}_W\|_{2 \to \infty}\) (these three terms’ notations are from \(\text{Cape et al. [2019a]}\). Our relaxation of \(\|E\|\ adds\ an\ extra\ \log^2 n\ term\ to\ \|R^{(1)}\|_{2 \to \infty}\’s\ bound,\ while\ the\ third\ term\ remains\ the\ same\ because\ of\ Cape’s\ 4th\ assumption\ (we\ have\ already\ checked).\ Therefore\ by\ Theorem\ 1\ of\ \text{Cape et al. [2019a]},\ we\ arrive\ the\ following\ conclusion.

\[
\|U_1 - U_2 W_{(2 \to 1)}\|_{2 \to \infty} = \mathcal{O}_p\left(\left(\frac{n \rho_n}{n \rho_n \log^3 n}\right)^{1/2} \log^2 n \times \frac{\|E\|_{2 \to \infty}}{n} \right) \\
= \mathcal{O}_p\left(\left(\frac{n \rho_n}{n \rho_n \log^3 n}\right)^{1/2} \log^2 n \times \frac{\|E\|_{2 \to \infty}}{n} \right). (61)
\]

To bound \(\|U_2 - U_3 W_{(3 \to 2)}\|_{2 \to \infty}\, we\ employ\ Theorem\ 4.2\ in\ \text{Cape et al. [2019b]}.

**Theorem D.1.** ([Theorem 4.2 in Cape et al. \(\text{[2019b]}\)] Suppose the diagonal elements of \(D_3\) are sorted in descending order. If \(|D_3[k]| > 4\|E_2\|_{2 \infty}\), where \(E_2 = A_2 - A_3\, D_3[j]\ is\ the\ j-th\ diagonal\ element\ of\ D_3\). Then there exists \(W_3 \in \mathcal{O}(k)\) such that

\[
\|U_2 - U_3 W_{(3 \to 2)}\|_{\max} \leq 14\left(\frac{\|E_2\|_{2 \infty}}{|D_3[k]|}\right)\|U_3\|_{2 \to \infty}. (62)
\]
Before applying Theorem D.1, we should check its assumptions. Reuse the notation for the singular values of $\mathcal{A}$ as in Equation (57). Notice $\sigma_{\min} \geq c_1 n \rho_n$ and $\mu_\ast = \mu_f 1_d^T + 1_n \mu_c^T - \mu_1 n 1_d^T$. Denotes $\mu_r = \mathcal{A} 1_d / d := \tilde{T}$. Similarly define $\tilde{T} := A 1_d$. By Hoeffding’s Concentration Inequality,

$$\mathbb{P}( |\tilde{T}_i - T_i| \geq a ) \leq 2 \exp \left\{ - \frac{a^2}{2 \sum_{j=1}^d \text{var}(A_{ij})} \right\} \leq 2 \exp \left\{ - \frac{a^2}{d \rho_n} \right\}.$$ 

Therefore, $\tilde{T}_i - T_i = O_p((n \rho_n)^{\frac{1}{2}})$. The same bound applies to $\| \tilde{\mu}_c - \mu_c \|, \| \tilde{\mu} - \mu \|$. These imply that any entry of $E_2$ could be bounded by $O_p(\sqrt{n \rho_n} / n)$. Thus $\| E_2 \|_\infty = O_p((n \rho_n \log^2 n)^{\frac{1}{2}})$. Then with large enough $n$, there must be $|D_3[k]| > 4 \| E_2 \|_\infty$.

With Theorem D.1,

$$\| U_2 - U_3 W_{(3\rightarrow 2)} \|_{2 \rightarrow \infty} \leq \sqrt{k} \| U_2 - U_3 W_{(3\rightarrow 2)} \|_{\max} = O_p((n \rho_n)^{-\frac{1}{2}} \log n \| U_3 \|_{2 \rightarrow \infty}).$$

Combining (56), (61), (63) gives

$$\| U_1 - U_3 W_{(3\rightarrow 1)} \|_{2 \rightarrow \infty} \leq \| U_1 - U_2 W_{(2\rightarrow 1)} \|_{2 \rightarrow \infty} + \| U_2 - U_3 W_{(3\rightarrow 2)} \|_{2 \rightarrow \infty} = O_p \left( (n \rho_n)^{-1/2} \log^\frac{3}{2} n \times \| U_2 \|_{2 \rightarrow \infty} + (n \rho_n)^{-\frac{1}{2}} \log n \| U_3 \|_{2 \rightarrow \infty} \right).$$

$$\| U_2 - U_3 W_{(3\rightarrow 2)} \|_{2 \rightarrow \infty} \leq O_p \left( (n \rho_n)^{-1/2} \log^\frac{3}{2} n \times (\| U_2 - U_3 W_{(3\rightarrow 2)} \|_{2 \rightarrow \infty} + \| U_3 \|_{2 \rightarrow \infty}) + (n \rho_n)^{-\frac{1}{2}} \log n \| U_3 \|_{2 \rightarrow \infty} \right) = O_p \left( (n \rho_n)^{-1/2} \log^\frac{3}{2} n \times \| U_3 \|_{2 \rightarrow \infty} \right).$$

The next step is to convert the symmetrized adjacency matrices’ eigenvectors’ perturbation bound to that of original adjacency matrices. Suppose the $k$-rank singular value decompositions (only retains the top $k$ singular values and corresponding eigenvectors) of $A - \tilde{\mu}_\ast$, $\mathcal{A} - \tilde{\mu}_\ast$ and $\mathcal{A} - \mu_\ast$ are $F_1 L_1^T, F_2 L_2^T, F_3 L_3^T$ respectively. Then, for $i = 1, 2, 3$ there must be

$$U_i = \frac{1}{\sqrt{2}} \begin{pmatrix} F_i & -F_i \\ L_i & L_i \end{pmatrix}.$$

Suppose

$$W_{(3\rightarrow 1)} = \begin{pmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{pmatrix}$$

with each sub-matrix having $k \times k$ dimension.

Arguments in Cape et al. [2019a] (proof of Theorem 1) implies that if we have singular value decomposition $U_3^T U_1 = U_o D_o V_o^T$, then $W_{(3\rightarrow 1)} = U_o V_o^T$. It is trivial to see that $U_3^T U_1$ has special block-wise structure

$$U_3^T U_1 = \frac{1}{2} \begin{pmatrix} F_3^T F_1 + L_3^T L_1 & -F_3^T F_1 + L_3^T L_1 \\ -F_3^T F_1 + L_3^T L_1 & F_3^T F_1 + L_3^T L_1 \end{pmatrix}.$$
Thus Lemma D.2 indicates

\[ W^{11} = W^{22}, \quad W^{12} = W^{21}. \]  

(65)

Notice \( W_{(3 \rightarrow 1)} \) is orthogonal matrix, therefore

\[
W_{(3 \rightarrow 1)} W_{(3 \rightarrow 1)}^T = I \Rightarrow \begin{pmatrix} W^{11} & W^{12} \\ W^{21} & W^{22} \end{pmatrix} \begin{pmatrix} W^{11} & W^{12} \\ W^{21} & W^{22} \end{pmatrix}^T = I
\]

\[
W_{(3 \rightarrow 1)} W_{(3 \rightarrow 1)}^T = I \Rightarrow \begin{pmatrix} W^{11} & W^{12} \\ W^{12} & W^{11} \end{pmatrix} \begin{pmatrix} W^{11} & W^{12} \\ W^{12} & W^{11} \end{pmatrix}^T = I
\]

(Equation (65))

\[
(W^{21} - W^{22})(W^{21} - W^{22})^T = I, \quad (W^{11} - W^{12})(W^{11} - W^{12})^T = I.
\]

These indicate \( W^{21} - W^{22} \) and \( W^{11} - W^{12} \) are both orthogonal matrices.

Notice \( \| U \|_{2 \rightarrow \infty} = O_p(\log n / \sqrt{n}) \) from Equation (45), similarly there is same upper bound for \( \| V \|_{2 \rightarrow \infty} \). Therefore

\[
\inf_{W \in O(k)} \| \hat{U} - UW \|_{2 \rightarrow \infty} = \inf_{R \in O(k)} \| F_1 - F_3 R \|_{2 \rightarrow \infty}
\]

\[
\leq \max\{\| F_1 - F_3(W^{11} - W^{21}) \|_{2 \rightarrow \infty}, \| F_1 - F_3(W^{12} - W^{22}) \|_{2 \rightarrow \infty}\}
\]

\[
\leq \| (F_1, -F_1) - (F_3, -F_3) W_{(3 \rightarrow 1)} \|_{2 \rightarrow \infty}
\]

\[
\leq \| U_1 - U_3 W_{(3 \rightarrow 1)} \|_{2 \rightarrow \infty}
\]

\[
= O_p\left((np_n)^{-1/2} \log^{2} n \times \| U_3 \|_{2 \rightarrow \infty}\right)
\]

\[
= O_p\left((np_n)^{-1/2} \log^{2} n \times (\| U \|_{2 \rightarrow \infty} + \| V \|_{2 \rightarrow \infty})\right)
\]

\[
= O_p\left((np_n)^{-1/2} n^{-1/2} \log^{5} n\right).
\]

**Proof for requisite results of Lemma C.6**

The proof of Lemma C.6 is decomposed into Lemmas D.3, D.4, D.5, D.6, and D.7 All of which are stated below.

Lemmas D.3, D.4, D.5 bound the tail behavior of the sample Varimax objective function. Lemmas D.6 and D.7 bound the difference between the sample and population versions of the Varimax objective function, uniformly over the space of orthogonal matrices. Lemma C.6 puts these pieces together with the first and second order conditions for Varimax described in Section E to show that the optimum of the sample Varimax objective function must be close to the optimum of the population Varimax function (modulo permutation and sign-flip).
The next few lines show the existence of moment generating function (MGF) of linear inner-product of sub-exponential random vectors. Recall $Z_i \in \mathbb{R}^k$ contains sub-exponential random variables. Following the notations in the proof of Lemma C.2 (Equation (42)) the tail property of $Z$ could be shown as

$$P(Z_{ij} - \mathbb{E}Z_{ij} > t) \leq C_0 \exp(-\lambda t),$$

(66)

then for $\forall r \in \mathbb{R}^k$, by independency

$$\mathbb{E} \exp(t\langle \tilde{Z}_i, r \rangle) = \mathbb{E} \exp(t \sum_{j=1}^{k} \tilde{Z}_{ij} r_j) = \prod_{j=1}^{k} \mathbb{E} \exp(t \tilde{Z}_{ij} r_j),$$

$\Rightarrow \langle \tilde{Z}_i, r \rangle$ also has MGF.

**Lemma D.3.** $\tilde{Z}_i \in \mathbb{R}^k$ is $i$-th row of $\tilde{Z}$. $r \in \mathbb{R}^k$ is arbitrary. $\lambda$ is defined in Equation (66).

Let

$$J_i = \langle \tilde{Z}_i, r \rangle^4 - \mathbb{E}[(\tilde{Z}_i, r)^4],$$

then for $\forall t > 0$, there exists a positive constant $C_1$ s.t.

$$P(J_i > t) \leq C_1 \exp(-\lambda t^{\frac{1}{4}}).$$

(67)

**Proof.** Write $X_i = \langle \tilde{Z}_i, r \rangle$. By Markov Inequality,

$$P(J_i > t) = P(X_i^4 - \mathbb{E}(X_i^4) > t) = P(X_i^4 > t + \mathbb{E}(X_i^4)) \leq C_0^\prime \exp(-\lambda((t + \mathbb{E}(X_i^4))^{\frac{1}{4}} - \mathbb{E}X_i)) \leq C_0^\prime \exp(\lambda \mathbb{E}X_i) \exp(-\lambda t^{\frac{1}{4}}) = C_1 \exp(-\lambda t^{\frac{1}{4}}).$$

The following lemma makes use of Lemma D.3 and gives bound to the sum of sequence $\sum_{i=1}^{n} J_i$.

**Lemma D.4.** With previous definitions, then for any $\delta > 0$,

$$\left| \sum_{i=1}^{n} J_i \right| = O_p(n^{\frac{1}{2} + \delta}).$$
Proof. For sequence \( \alpha_n \uparrow \infty \). Define \( \bar{J}_i = J_i 1(\bar{J}_i \leq \alpha_n) \). Then \( \bar{J}_i \)'s are independent bounded random variables. Let \( \mathfrak{A} = \{ \bigcap_{i=1}^{n} \{ J_i = \bar{J}_i \} \} \) and \( \mathfrak{B} = \{ | \sum_{i=1}^{n} J_i | > t \} \). Then

\[
P(\mathfrak{B}) = P(\mathfrak{B} \cap \mathfrak{A}) + P(\mathfrak{B} \cap \mathfrak{A}^c)
\]

\[
\leq P(\{ | \sum_{i=1}^{n} \bar{J}_i | > t \}) + P(\mathfrak{A}^c). \tag{68}
\]

Notice \( \bar{J}_i \in [-c, \alpha_n] \) with \( c = -\mathbb{E}(X_1^4) \) is a bounded random variable. Therefore it is sub-gaussian with domain interval length \( \sigma \leq \alpha_n + c \leq 2\alpha_n \) for large enough \( n \). By Hoeffding Concentration Inequality,

\[
P(\{ | \sum_{i=1}^{n} \bar{J}_i | > t \}) \leq 2 \exp(-\frac{2t^2}{\sum_{j}^{\sigma^2}}) \leq 2 \exp(-\frac{t^2}{2n\alpha_n^2}). \tag{69}
\]

By Lemma D.3 there is,

\[
P(\mathfrak{A}^c) = P(\{ \bigcap_{i} \{ J_i \leq \alpha_n \} \}^c)
\]

\[
= P(\bigcup_{i} \{ J_i > \alpha_n \})
\]

\[
\leq nP(J_i > \alpha_n)
\]

\[
\leq nC_2 \exp(-\lambda \alpha_n^{\frac{1}{4}}). \tag{70}
\]

Plugging \( \tag{69} \tag{70} \) in \( \tag{68} \) and choosing \( \varepsilon > \delta, t = n^{1/2 + \varepsilon}, \alpha_n = n^{\delta} \) gives

\[
| \sum_{i=1}^{n} J_i | = O_p(n^{\frac{1}{2} + \delta}). \tag{71}
\]

Similar conclusion applies to second moment terms.

**Lemma D.5.** With same notations as Lemma D.3. Define \( Y_i = \langle \bar{Z}_i, r \rangle^2 - \mathbb{E}[\langle \bar{Z}_i, r \rangle^2] \), then

\[
| \sum_{i=1}^{n} Y_i | = O_p(n^{\frac{1}{2} + \delta}). \tag{72}
\]

**Proof.** This part employs the same strategy as the proof of Lemma D.4. The only difference is the bound of \( \tag{70} \). But the dominating bound \( \tag{69} \) is the same. Thus we obtain the similar bound for \( | \sum_{i=1}^{n} Y_i | \).

\[\square\]
Lemma D.6. Suppose \( r_1, r_2, \ldots, r_n \in \mathbb{R}^k \). Denote
\[
\mathbb{J}_\ell = |\sum_{i=1}^{n} (\bar{Z}_i, r_\ell)^4 - n\mathbb{E}[(\bar{Z}_i, r_\ell)^4]|.
\]
Assume \( n_0 = an^b \) with some positive constant \( a, b \). Then for any \( \delta > 0 \),
\[
\max_\ell \mathbb{J}_\ell = O_p(n^{\frac{1}{2}+\delta}). \tag{73}
\]
Similarly if we define
\[
\mathbb{Y}_\ell = (\bar{Z}_i, r_\ell)^2 - \mathbb{E}[(\bar{Z}_i, r_\ell)^2],
\]
we have
\[
\max_\ell \mathbb{Y}_\ell = O_p(n^{\frac{1}{2}+\delta}). \tag{74}
\]
**Proof.** Take \( \varepsilon > \delta, t = n^{1/2+\varepsilon}, \alpha_n = n^{\delta} \). Applying the same strategy in Lemma D.4 (Equation (69), (70)) and basic probability rules,
\[
\mathbb{P}(\max_\ell \mathbb{J}_\ell > t) \leq 2n_0 \exp\left(-\frac{t^2}{2n\alpha_n^2}\right) + n_0 nC_2 \exp(-\lambda\alpha_n^{\frac{1}{4}})
\]
\[
= 2n_0 \exp\left(-\frac{n^{1+2\varepsilon}}{2n^{1+2\delta}}\right) + n_0 nC_2 \exp(-\lambda n^{\delta/4})
\]
Since
\[
\log(2n_0) - \frac{1}{2}n^{2\varepsilon-2\delta} = \log 2a + b \log n - n^{2\varepsilon-2\delta} \to -\infty,
\]
\[
\log(C_2n_0n) - \lambda n^{\delta/4} = \log C_2 \log a + (b + 1) \log n - \lambda n^{\delta/4} \to -\infty,
\]
they could be reduced to \( \max_\ell \mathbb{J}_\ell = O_p(n^{\frac{1}{2}+\delta}) \). Similar approach gives \( \max_\ell \mathbb{Y}_\ell = O_p(n^{\frac{1}{2}+\delta}) \).

Lemma D.7. Recall notations in equation (1), Section 2.1 and Section 4. For readability we slightly abuse the notations and write \( \tilde{V}(R) = v(R, \bar{Z}) \). And
\[
V(R) := V_\mathcal{U}(R) = \sum_{j=1}^{k} \text{Var}((\bar{Z}_i \bar{U}_R)_{ij}).
\]
Then for \( \forall \delta > 0 \), there is a uniform bound between these two quantities,
\[
\sup_{O \in \mathcal{O}(k)} |\tilde{V}(O) - V(O)| = O_p(n^{\delta-1/2}). \tag{75}
\]
Proof. This part of the proof adapts the covering balls strategy to give this uniform bound. Let $\mathbb{R} = \{R_1, R_2, \ldots, R_N\}$ be a $\varepsilon$-cover for orthogonal matrices. This means for $\forall O \in \mathcal{O}(k)$, there exists $\ell$ s.t. $d(O, R_\ell) < \varepsilon$ where $d(X, Y)$ is the sin $\Theta$ distance. Let $D(\varepsilon, \mathcal{O}(k), d)$ be the $\varepsilon$-packing number. Using the notes in [Van de Geer 2000] and Lemma 4.1 in [Pollard 1990] we have,

$$N \leq D(\varepsilon, \mathcal{O}(k), d) \leq D(\varepsilon, \mathcal{O}(k), d_F) \leq \left(\frac{6}{\varepsilon}\right)^k := N_0. \quad (76)$$

$d_F$ is Frobenius norm distance. The second inequality is true because for any $O_1, O_2 \in \mathcal{O}(k)$,

$$\|\sin(O_1, O_2)\|_F^2 \leq \inf_{Q \in \mathcal{O}(k)} \|O_1 - O_2 Q\|_F^2 \leq \|O_1 - O_2\|_F^2.$$

Let $\varepsilon = 1/n$ then $N \leq N_0 = (6n)^k$. For $\forall O \in \mathcal{O}(k)$, choose $R_\ell \in \mathbb{R}$ such that $d(O, R_\ell) < \varepsilon$. Then by Triangle Inequality,

$$|\hat{V}(O) - V(O)| \leq |\hat{V}(O) - \hat{V}(R_\ell)| + |\hat{V}(R_\ell) - V(R_\ell)| + |V(O) - V(R_\ell)|. \quad (77)$$

Lemma [D.4, D.6] indicates

$$|\hat{V}(R_\ell) - V(R_\ell)| \leq \max_j |\hat{V}(R_j) - V(R_j)| = O_p(n^{\delta - \frac{1}{2}}). \quad (78)$$

Notice $\hat{V}(R) = \sum_{j=1}^k \left(\frac{1}{n} \sum_{i=1}^n [\tilde{Z}R]_{ij}^4 - \left(\frac{1}{n} \sum_{i=1}^n [\tilde{Z}R]_{ij}^2\right)^2\right)$ and $\max_{ij} |\tilde{Z}_{ij}| = O(\log n)$ (Lemma [C.2]). Also for $\forall O, R \in \mathcal{O}(k)$ such that $d(O, R) < \varepsilon$, there is fact that

$$d(O, R) \geq \frac{1}{\sqrt{2}} \|O - R\|_F, \forall O, R \in \mathcal{O}(k),$$

56
then
\[
| \sum_{ij} (\tilde{Z}O_{ij}^4 - \tilde{Z}R_{ij}^4) | = | \sum_{ij} [(\tilde{Z}O_{ij}^2 + \tilde{Z}R_{ij}^2)(\tilde{Z}O_{ij} + \tilde{Z}R_{ij})(\tilde{Z}O_{ij} - \tilde{Z}R_{ij})] |
\]
\[
\leq \sum_{ij} |[(\tilde{Z}O_{ij}^2 + \tilde{Z}R_{ij}^2)(\tilde{Z}O_{ij} + \tilde{Z}R_{ij})(\tilde{Z}O_{ij} - \tilde{Z}R_{ij})]|
\]
\[
\leq O_p(\log^2 n) \sum_{ij} |(\tilde{Z}O_{ij} + \tilde{Z}R_{ij})(\tilde{Z}O_{ij} - \tilde{Z}R_{ij})|
\]
\[
\leq O_p(\log^3 n) \sum_{ij} |\tilde{Z}O_{ij} - \tilde{Z}R_{ij}|
\]
\[
\leq O_p(n \log^3 n) \| \tilde{Z}(O - R) \|_{1 \to \infty}
\]
\[
\leq O_p(n \log^3 n) \| \tilde{Z} \|_{\text{max}} \left( \sum_{ij} |O_{ij} - R_{ij}| \right)
\]
\[
= O_p(n \log^4 n) \left( \sum_{ij} |O_{ij} - R_{ij}| \right)
\]
\[
\leq O_p(n \log^4 n) \times k \sqrt{\sum_{ij} |O_{ij} - R_{ij}|^2}
\]
\[
= O_p(n \log^4 n) \| O - R \|_F
\]
\[
\leq O_p(n \log^4 n) \times \sqrt{2d(O, R)}
\]
\[
= O_p(\varepsilon n \log^4 n)
\]
\[
= O_p(\log^4 n),
\]
(79)
and

\[
\sum_j \left[ (\sum_i [\tilde{Z}O]_{ij}^2)^2 - (\sum_i [\tilde{Z}R]_{ij}^2)^2 \right] \leq \sum_j \left[ (\sum_i (\tilde{Z}O)_{ij}^2 + [\tilde{Z}R]_{ij}^2) (\sum_i (\tilde{Z}O)_{ij}^2 - [\tilde{Z}R]_{ij}^2)) \right]
\]

\[
\leq \sum_j \left[ (2n\|\tilde{Z}\|_{\text{max}}^2) (\sum_i (\tilde{Z}O)_{ij}^2 - [\tilde{Z}R]_{ij}^2)) \right]
\]

\[
\leq O_p(n \log^3 n) \sum_j \sum_i (\tilde{Z}O)_{ij} - [\tilde{Z}R]_{ij}
\]

\[
\leq O_p(n \log^3 n) \sum_j \sum_i (\tilde{Z}O)_{ij} - [\tilde{Z}R]_{ij}
\]

\[
\leq O_p(n \log^4 n) \times k \|O - R\|_{1 \rightarrow \infty}
\]

\[
\leq O_p(n \log^4 n) \times \sqrt{2} \times d(O, R)
\]

\[
= O_p(n \log^4 n)
\]

With Equation (79), (80),

\[
|\hat{V}(O) - \hat{V}(R)|
\]

\[
\leq \frac{1}{n} \sum_{ij} (\tilde{Z}O)_{ij}^4 - [\tilde{Z}R_d]_{ij}^4) + \frac{1}{n^2} \sum_j \left[ (\sum_i [\tilde{Z}O]_{ij}^2)^2 - (\sum_i [\tilde{Z}R_d]_{ij}^2)^2 \right] \leq O_p\left(\frac{\log^4 n}{n}\right).
\]
Assume $\mathbb{E}(\tilde{Z}_{1\ell}) = \mu^{(i)}_j$ refers to $j$-th moment of $Z$’s $\ell$-th column, similar to the proof of Theorem 3.1 the population Varimax function could be expressed as,

$$V(Q) = \sum_j (\mathbb{E}([\tilde{Z}Q]_j^4) - \mathbb{E}([\tilde{Z}Q]_j^2)^2) = \sum_{i=1}^k \mu^{(i)}_i - 3\|Q_i\|_4^4 + 3k.$$ 

Write $\xi_i = \mu^{(i)}_i - 3\mu^{(i)}_2 = \mu^{(i)}_4 - 3$ (is positive by leptokurtic assumption) and $\xi_0 = \max_i \xi_i$ (is a finite positive constant). For $\forall O, R \in \mathcal{O}(k)$ such that $d(O, R) < \varepsilon$, there is

$$|V(O) - V(R)| = \left| \sum_{i=1}^k \xi_i(\|O_i\|_4^4 - \|R_i\|_4^4) \right| \leq \sum_{i=1}^k |\xi_i(\|O_i\|_4^4 - \|R_i\|_4^4)| \leq \xi_0 \sum_{i=1}^k \|\|O_i\|_4^4 - \|R_i\|_4^4\| \leq \xi_0 \sum_{i=1}^k \sum_{j=1}^k |O_{ij} - R_{ij}| \leq 4\xi_0 \sum_{i=1}^k \sum_{j=1}^k |O_{ij} - R_{ij}| \leq 4\xi_0 k\|O - R\|_F = O_p(\frac{1}{n}).$$

Summing up three bounds of Equation (77) obtains a uniform bound of $|\hat{V}(O) - V(O)|$: 

$$\sup_{O \in \mathcal{O}(k)} |\hat{V}(O) - V(O)| = O_p(n^{\delta - \frac{1}{2}}). \quad (81)$$
Proof of Lemma C.6

This part of the proof needs first/second order condition of population varimax function here. These two conditions are stated as Corollary D.1, D.2 below. For now the notations of sample & population varimax functions follow the proofs of Lemma D.7.

**Corollary D.1** (FOC). If identity matrix $I$ is a stationary point of $V_I(R_0)$ then $Z$ satisfies the following condition,

$$EZ_i^2 E(Z_i Z_j) - E(Z_i^3 Z_{1j}) = EZ_j^2 E(Z_1 Z_j) - E(Z_1^3 Z_{1i}), \forall i \neq j.$$  \hspace{1cm} (82)

**Corollary D.2** (SOC). Notate $O = \tilde{Z}^T \tilde{Z}$, if $I$ is a local maximum of the population Varimax $V_I(R_0)$, then the following condition is true,

$$3E[tr(diag(OK)^2)] \leq E(diagO, KK^T),$$ \hspace{1cm} (83)

for any skew-symmetric matrix $K$.

The proof of Corollary D.1 is trivial. Corollary D.2 is a direct result of Theorem E.2 in Section E. Now we could proceed to the proof of Lemma C.6.

**Proof.** By Proposition C.4, $R_{\tilde{Z}}$ is converging to elements of $P(k)$, WLOG we may assume $P_n^{(2)} = I$ and $R_{\tilde{Z}} \to I$ (i.e. let $P_n^{(2)} = \tilde{R}_n^T$), since elements of $P(k)$ are isolated to each other ($\forall P_1 \neq P_2 \in P(k), \|P_1 - P_2\| \geq 2/\sqrt{k}$). We may constrain our analysis on a fixed neighborhood of $I, B(I, \delta_c) s.t. \{I\} = B(I, \delta_c) \cap P(k)$. Now we want to show,

$$\|R_{\tilde{Z}} - I\|_{2 \to \infty} = O_p(n^{\delta/2-1/4}).$$

By Lie algebra theory there is a $k \times k$ skew-symmetric matrix $K$ s.t. $R_{\tilde{Z}} = \exp(K)$. Define

$$\gamma(t) = \exp(tK).$$

Then $\gamma(0) = I, \gamma(1) = R_{\tilde{Z}}$. We want to evaluate population varimax function’s first order and second order condition at $I$ (global optimal solution). To achieve that we should show: $R \to I \Rightarrow K \to 0$. This could be proved by using matrix logarithm algebra.

$$\|K\| = \|\log R_{\tilde{Z}} - \log I\| \leq \sum_{i=1}^{\infty} \frac{(-1)^{i+1}}{i} \|R_{\tilde{Z}} - I\|^i \to 0.$$  \hspace{1cm} (84)

Differential calculations indicates:

$$\frac{d}{dt} V(\gamma(t))|_{t=0} = \nabla V(\gamma(t))^T \frac{d\gamma}{dt}|_{t=0} = \langle \nabla V(I), K \rangle,$$  \hspace{1cm} (84)
\[ \frac{d^2}{dt^2} V(\gamma(t))|_{t=0} = \langle \nabla^2 V \left( \frac{d\gamma(t)}{dt} \right) |_{t=0}, K\gamma(t) \rangle + \langle \nabla V, K^2 \gamma(0) \rangle = \langle \nabla^2 V \cdot K, K \rangle + \langle \nabla V, K^2 \rangle. \]

Equation (2) of Chu and Trendafilov [1998] is a reformulated version of Varimax function. Taking expectation of it (Lemma E.1 allows exchanging differential and expectation), and notate \( E = \tilde{Z}^T \tilde{Z} - n\tilde{Z}_1^T \tilde{Z}_1 \). The varimax function could be rewritten as:

\[ V(Q) := \mathbb{E}[^{\text{trace}}(\text{diag}(Q^T EQ))] \Rightarrow \nabla V(Q) = 4\mathbb{E}[EQ\text{diag}(Q^T EQ)], Q \in \mathcal{G}(k). \quad (85) \]

Let \( O = \tilde{Z}_1^T \tilde{Z}_1 \), then

\[ \nabla V(I) = 4\mathbb{E}[(I - O)\text{diag}(I - O)] = 4\mathbb{E}(\text{diag}(O)) - 4I. \quad (86) \]

Thus by Corollary D.1, \( \nabla V(I) \) is symmetric \( \Rightarrow \langle \nabla V(I), K \rangle = 0 \).

By Frechet derivatives, for any \( H \in \mathbb{R}^{k \times k} \) and \( t > 0 \),

\[
\nabla V(Q + tH) - \nabla V(Q) = 4\mathbb{E}[E(Q + tH)\text{diag}((Q + tH)^T E(Q + tH)] - 4\mathbb{E}[EQ\text{diag}(Q^T EQ)]
\]

\[
= 4\mathbb{E}[EH\text{diag}(Q^T EQ) + 2EQ\text{diag}(H^T EQ)] + O(t^2),
\]

choose \( H = K, Q = I \), which means the derivative of \( \nabla V(Q) \) evaluated at \( I \) in the direction of \( K \) is

\[ \nabla^2 V(I) \cdot K = -4K + 4\mathbb{E}[OK\text{diag}(O)] + 8\mathbb{E}[\text{diag}(OK)]. \quad (87) \]

Theorem 3.1 indicates identity matrix \( I \) is one of the global maximas of population Varimax function. Applying Second Order Condition result (Corollary D.2), Lemma E.2 and reusing \( \langle \nabla V(I), K \rangle = 0 \) obtains

\[ \langle \nabla^2 V(I) \cdot K, K \rangle = -4\langle K, K \rangle + 4\mathbb{E}[OK\text{diag}(O), K] + 8\mathbb{E}[\text{diag}(OK), K]
\]

\[ = -4\|K\|^2_F + 12\mathbb{E}[\text{trace}(\text{diag}(OK)^2)]
\]

\[ \leq -4\|K\|^2_F + 4\mathbb{E}[\text{diag}(O), KK^T], \]

and

\[ \langle \nabla V(I), K^2 \rangle = -4\mathbb{E}[\text{diag}(O), KK^T] + 4\|K\|^2_F. \quad (88) \]

Let \( K^u = K/\|K\| \), then

\[ \frac{\partial^2 V}{\partial t^2} |_{t=0} = \langle \nabla^2 V(I) \cdot K, K \rangle + \langle \nabla V(I), K^2 \rangle
\]

\[ = -4\|K\|^2_F + 12\mathbb{E}[\text{trace}(\text{diag}(OK)^2)] - 4\mathbb{E}[\text{diag}(O), KK^T] + 4\|K\|^2_F
\]

\[ = 4 \times (3\mathbb{E}[\text{trace}(\text{diag}(OK)^2)] - \mathbb{E}[\text{diag}(O), KK^T])
\]

\[ = 4 \times (3\mathbb{E}[\text{trace}(\text{diag}(OK^u)^2)] - \mathbb{E}[\text{diag}(O), K^u K^u])\|K\|^2
\]

\[ \leq -C\|K\|^2. \]
Here
\[ -C_s = \max_{\|K^u\| = 1, K^u \in O(k)} 4 \times (3 \mathbb{E}[\text{trace}(\text{diag}(OK^u)^2) - \mathbb{E}[\text{Odiag}(O), K^uK^u^T])], \quad (89) \]
is a negative constant (thus $C_s$ is a positive constant) since RHS of Equation (89) is upperbounded and the set of skewed symmetric matrix with unit Frobenius norm is a bounded, closed and compact space. With derived results and Taylor expansion,
\[
V(R_\tilde{Z}) = V(I) + \langle \nabla V(I), K \rangle + \langle \nabla^2 V(I) \cdot K, K \rangle + \langle \nabla V(I), K^2 \rangle + o(\|K\|^2)
\]
\[
= V(I) + \langle \nabla^2 V(I) \cdot K, K \rangle + \langle \nabla V(I), K^2 \rangle + o(\|K\|^2)
\]
\[
\leq V(I) - C_s\|K\|^2 + o(\|K\|^2).
\]
By Lemma D.7 there exists $\varepsilon_0 = O_p(n^{\delta-1/2})$ s.t.
\[
|\hat{V}(R_\tilde{Z}) - V(R_\tilde{Z})| < \varepsilon_0, \quad |\hat{V}(I) - V(I)| < \varepsilon_0,
\]
\[
\Rightarrow \quad \hat{V}(R_\tilde{Z}) - \varepsilon_0 < V(R_\tilde{Z}) < V(I) < \hat{V}(I) + \varepsilon_0,
\]
\[
\Rightarrow \quad V(I) - V(R_\tilde{Z}) < 2\varepsilon_0.
\]
These implies
\[
\|K\|^2 < \frac{2\varepsilon_0 + o(\|K\|^2)}{C_s}. \quad (90)
\]
Therefore $\|K\| = O_p(n^{\delta/2-1/4})$. By matrix exponential algebra,
\[
\|R_\tilde{Z} - I\|_{2\to\infty} \leq \|R_\tilde{Z} - I\| = \left\| \sum_{i=1}^{\infty} \frac{K_i^i}{i!} \right\| \leq \sum_{i=1}^{\infty} \frac{\|K\|^i}{i!} = O_p(n^{\delta/2-1/4}). \quad (91)
\]

**Detailed Proof of Proposition C.5**

*Proof.* Write
\[
V_1(O) = \sum_{\ell=1}^{k} \left( \frac{1}{n} \sum_{i=1}^{n} [\sqrt{n}UO]_{i\ell}^4 - \left( \frac{1}{n} \sum_{i=1}^{n} [\sqrt{n}UO]_{i\ell}^2 \right)^2 \right),
\]
\[
V_2(O) = \sum_{\ell=1}^{k} \left( \frac{1}{n} \sum_{i=1}^{n} [\sqrt{n}UWO]_{i\ell}^4 - \left( \frac{1}{n} \sum_{i=1}^{n} [\sqrt{n}UWO]_{i\ell}^2 \right)^2 \right).
\]
Thus the columns of $\hat{V}$ are input. $V_2$ is sample version of Varimax function with true eigenvectors rotated with $W$ (specified in Lemma C.5). The proof of Proposition C.5 could be described as two parts. First part shows the uniform upper bound for difference between $V_1$, $V_2$ (Equation (92)). Similar to the proof of Lemma C.6, the second part explores the first and second order condition of Equation (94) to obtain the bound for the difference between solutions of $V_1$ and $V_2$ (modulo permutation and sign-flip).

Mathematically speaking, the first part (uniform upper bound for difference between $V_1$, $V_2$) is equivalent to

$$
\sup_{O \in O(k)} |V_1(O) - V_2(O)| \leq O_p \left( (n \rho_n)^{-1/2} \log^{7/2} n \right). \tag{92}
$$

In the proof of Proposition C.5 let $X_i$ be the $i$th row of $\sqrt{n}\hat{U}$, and $i$th row of $\sqrt{n}UW$ be $X_i + \epsilon_i$. From Lemma C.5, for any unit length vector $r \in \mathbb{R}^k$, we have

$$
|\langle (X_i, r) - \langle X_i + \epsilon_i, r \rangle \rangle| \leq \|\epsilon_i\||r|| \leq \sqrt{n}\|\hat{U} - UW\|_{2 \rightarrow \infty} = O_p((n \rho_n)^{-1/2} \log^{5/2} n).
$$

Therefore,

$$
\begin{align*}
&\sum_{i=1}^n |(X_i, r)^4 - (X_i + \epsilon_i, r)^4| \\
&\leq \sum_{i=1}^n |(X_i, r)^2 + (X_i + \epsilon_i, r)^2|\langle (X_i, r) + (X_i + \epsilon_i, r) \rangle - \langle (X_i, r) - (X_i + \epsilon_i, r) \rangle | \\
&\leq \sum_{i=1}^n (X_i, r)^2 + (X_i + \epsilon_i, r)^2)(\|X_i\|_2 + \|X_i + \epsilon_i\|_2)(\langle X_i, r \rangle - \langle X_i + \epsilon_i, r \rangle) \\
&\leq \sum_{i=1}^n (X_i, r)^2 + (X_i + \epsilon_i, r)^2)O_p(\log n)(\langle X_i, r \rangle - \langle X_i + \epsilon_i, r \rangle) \\
&\leq \sum_{i=1}^n (X_i, r)^2 + (X_i + \epsilon_i, r)^2)O_p(\log n)\sqrt{n}\|\hat{U} - UW\|_{2 \rightarrow \infty} \\
&\leq \sum_{i=1}^n (X_i, r)^2 + (X_i + \epsilon_i, r)^2)O_p \left( (n \rho_n)^{-1/2} \log^{7/2} n \right).
\end{align*}
$$

Notice that columns of $\hat{U}$ and $UW$ have unit length and $R$ is an orthogonal matrix. Thus the columns of $UR$ and $UWR$ are all of unit length. Therefore

$$
\left( \frac{1}{n} \sum_{i=1}^n [\sqrt{n}\hat{U}R]_{i\ell}^2 \right)^2 = \left( \sum_{i=1}^n [UR]_{i\ell}^2 \right)^2 = \left( \sum_{i=1}^n [UWR]_{i\ell}^2 \right)^2 = \left( \frac{1}{n} \sum_{i=1}^n [\sqrt{n}UWR]_{i\ell}^2 \right)^2.
$$

63
Let \( O_\ell \) be the \( \ell \)th column of \( O \). Then for any \( O \in \mathcal{O}(k) \),

\[
|V_1(O) - V_2(O)| \\
\leq \sum_{\ell=1}^{k} \left| \left( \frac{1}{n} \sum_{i=1}^{n} [\sqrt{n}\hat{U}O]_{i\ell}^4 - \left( \frac{1}{n} \sum_{i=1}^{n} [\sqrt{n}\hat{U}O]_{i\ell}^2 \right)^2 \right) - \left( \frac{1}{n} \sum_{i=1}^{n} [\sqrt{n}UWO]_{i\ell}^4 - \left( \frac{1}{n} \sum_{i=1}^{n} [\sqrt{n}UWO]_{i\ell}^2 \right)^2 \right) \right|
\]

\[
\leq \frac{1}{n} \sum_{\ell=1}^{k} \left| \sum_{i=1}^{n} (\sqrt{n}\hat{U}O)_{i\ell}^4 - [\sqrt{n}UWO]_{i\ell}^4 \right|
\]

\[
\leq \frac{1}{n} \sum_{\ell=1}^{k} \sum_{i=1}^{n} \left( \langle X_i, O_\ell \rangle^4 - \langle X_i + \epsilon_i, O_\ell \rangle^4 \right)
\]

\[
\leq \frac{1}{n} \sum_{\ell=1}^{k} \sum_{i=1}^{n} \left( \langle X_i, O_\ell \rangle^2 + \langle X_i + \epsilon_i, O_\ell \rangle^2 \right) O_p \left( (n\rho_n)^{-1/2} \log^2 n \right)
\]

\[
= \sum_{\ell=1}^{k} O_p \left( (n\rho_n)^{-1/2} \log^2 n \right)
\]

\[
= O_p \left( (n\rho_n)^{-1/2} \log^2 n \right).
\]

Since the orthogonal matrix \( O \) here is arbitrary, therefore the Equation \( (92) \) is proved.

For the next step, we want to show the upper bound of \( 2 \rightarrow \infty \) norm distance between \( \hat{R}_U \) and \( R_{UW}P_n^{(3)} \) (\( P_n^{(3)} \in \mathcal{P}(k) \) is defined in Proposition C.5),

\[
\|\hat{R}_U - R_{UW}P_n^{(3)}\|_{2 \rightarrow \infty} = O_p \left( (n\rho_n)^{-1/4} \log^{7/2} n \right). \quad (93)
\]
For simplicity notate $R_1 = R_{U1}$, $R_2 = R_{U2}$. There are $k \times k$ skew-symmetric matrices $K_1$, $K_2$ s.t. $R_1 = \exp(K_1)$, $R_2 = \exp(K_2)$. Define

$$\gamma_2(t) = \exp((1 - t)K_2 + tK_1),$$

(94)

then $\gamma_2(0) = R_2$, $\gamma_2(1) = R_1$. Again, as in the proof of Lemma C.6 we assume

$$I = \arg\min_{P_0 \in \mathcal{P}(k)} ||R_1 - R_2 P_0||_{2 \to \infty},$$

and we constrain our analysis on a neighborhood of $R_2$: $B(R_2, \delta_p) := \{ P \in \mathcal{G}(k) ||P - R_2|| < \delta_p \}$, such that

$$B(R_2, \delta_p) \cap \{ R_2 P_0 | P_0 \in \mathcal{P}(k) \} = \{ R_2 \}.$$ 

This indicates for any $R \in B(R_2, \delta_p)$ there is $V_2(R) \leq V_2(R_2)$.

Before Taylor expansion analysis, we should check that $||R_1 - R_2|| \xrightarrow{P} 0$ is true. After that we should show $||K_1 - K_2|| \xrightarrow{P} 0$ is also true. By definition,

$$V_1(R_1) \geq V_1(R_2) - o_p(1), |V_1(R_2) - V_2(R_2)| \xrightarrow{P} 0 \Rightarrow V_1(R_1) \geq V_2(R_2) - o_p(1).$$

(95)

Then,

$$V_2(R_2) - V_2(V_1) \leq V_1(R_1) - V_2(R_1) + o_p(1)$$

$$\leq \sup_{R \in \mathcal{O}(k)} |V_1(R) - V_2(R)| + o_p(1) \xrightarrow{P} 0.$$ 

(96)

(97)

By conditions, for any $\epsilon_0 > 0, \eta_0 > 0$ such that $V_2(R) < V_2(R_2) - \eta_0$ for every $R \in \mathcal{G}(k)$ with $||R - R_2|| \geq \epsilon_0$. Thus the event $\{ ||R_2 - R_1|| \}$ is contained in the event $\{ V_2(R_1) < V_2(R_2) - \eta_0 \}$. The probability of the latter event goes to 0. Therefore $||R_1 - R_2|| \xrightarrow{P} 0$.

By Lemma C.6 with high probability, $R_2$ and $R_1 R_2^T$ are both converging to $I$ as $n$ grows. Variant of Baker-Cambell-Hausdorff formula gives

$$||K_1 - K_2|| = ||\log R_1 R_2^T R_2 - \log R_2||$$

$$= ||\log R_1 R_2^T + \frac{1}{2} [\log R_1 R_2^T, \log R_2] + \cdots ||$$

$$\leq ||\log (I + R_1 (R_2^T - R_1^T))||_F + o_p(||\log R_2||) \xrightarrow{P} 0.$$ 

With differential calculation results in Chu and Trendafilov [1998],

$$\frac{d}{dt} V_2(\gamma_2(t)) |_{t=0} = \nabla V_2(\gamma_2(t))^T \frac{d\gamma_2}{dt} |_{t=0} = \langle \nabla V_2(R_2), \gamma_2(R_2(K_1 - K_2)) \rangle,$$

(98)
\[
\frac{d^2}{dt^2} V_2(\gamma_2(t))_{t=0} = \langle \nabla^2 V_2 \frac{d\gamma_2(t)}{dt} \rangle_{t=0, \gamma_2(0) = (K_1 - K_2)} + \langle \nabla V, \gamma_2(0)(K_1 - K_2)^2 \rangle
\]
\[
= \langle \nabla^2 V_2 \cdot R_2(K_1 - K_2), R_2(K_1 - K_2) \rangle + \langle \nabla V, R_2(K_1 - K_2)^2 \rangle.
\]

By Equation (7), (8) of Chu and Trendafilov [1998],
\[
V_2(Q) = n^{-3} \text{trace} \left[ \sum_{i=1}^{n} \text{diag}(Q^T E_i Q)^2 \right],
\]
where \( E_i = (UW)^T UW - n(X_i + \epsilon_i)(X_i + \epsilon_i)^T \). And
\[
\nabla V_2(Q) = 4n^{-3} \left[ \sum_{i=1}^{n} E_i Q \text{diag}(Q^T E_i Q) \right].
\]

Theorem 3.1 in Chu and Trendafilov [1998] implies
\[
R_2^T \nabla V_2(R_2) = \sum_{i=1}^{n} R_2^T E_i R_2 \text{diag}(R_2^T E_2 Q_2)
\]
is symmetric, thus
\[
\langle \nabla V_2(R_2), (K_1 - K_2) R_2 \rangle = \text{trace}[\nabla V_2(K_1 - K_2)^T R_2^T]
\]
\[
= \text{trace}[R_2^T \nabla V_2(K_1 - K_2)^T]
\]
\[
= \langle R_2^T \nabla V_2, K_1 - K_2 \rangle
\]
\[
= 0.
\]

The last equality is because \( K \) is skew-symmetric and \( R_2^T \nabla V_2(R_2) \) is symmetric.

By Frechet derivatives, for any \( H \in \mathbb{R}^{k \times k} \) and \( t > 0 \), we have
\[
\nabla V_2(Q + tH) - \nabla V_2(Q)
\]
\[
= 4n^{-3} \left[ \sum_{i=1}^{n} E_i(Q + tH) \text{diag}((Q + tH)^T E_i(Q + tH)) \right] - 4n^{-3} \left[ \sum_{i=1}^{n} E_i Q \text{diag}(Q^T E_i Q) \right]
\]
\[
= 4n^{-3} \left[ \sum_{i=1}^{n} (E_i H \text{diag}(Q^T E_i Q) + E_i Q (\text{diag}(H^T E_i Q) + \text{diag}(Q^T E_i H)) \right] + O(t^2).
\]
Choosing $H = R_2(K_1 - K_2), Q = R_2$ gives the derivative of $\nabla V_2(Q)$ evaluated at $R_2$ in the direction of $R_2(K_1 - K_2)$

$$\nabla^2 V_2(R_2) \cdot R_2(K_1 - K_2) = 4n^{-3} \sum_{i=1}^{n} \langle E_i R_2(K_1 - K_2) \text{diag}(R_2^T E_i R_2), R_2(K_1 - K_2) \rangle$$

$$+ \langle E_i R_2 \text{diag}((K_1 - K_2)^T R_2^T E_i R_2), R_2(K_1 - K_2) \rangle$$

$$+ \langle E_i R_2 \text{diag}(R_2^T E_i R_2, R_2(K_1 - K_2)^2) \rangle.$$  

Applying Corollary 3.3 of Chu and Trendafilov [1998],

$$\langle \nabla^2 V_2(R_2) \cdot R_2(K_1 - K_2), R_2(K_1 - K_2) \rangle + \nabla V, R_2(K_1 - K_2)^2)$$

$$= 4n^{-3} \sum_{i=1}^{n} \langle (E_i R_2(K_1 - K_2) \text{diag}(R_2^T E_i R_2), R_2(K_1 - K_2) \rangle$$

$$+ \langle E_i R_2 \text{diag}((K_1 - K_2)^T R_2^T E_i R_2), R_2(K_1 - K_2) \rangle$$

$$+ \langle E_i R_2 \text{diag}(R_2^T E_i R_2, R_2(K_1 - K_2)^2) \rangle$$

$$= 4n^{-3} \sum_{i=1}^{n} \langle (R_2^T E_i R_2(K_1 - K_2) \text{diag}(R_2^T E_i R_2), (K_1 - K_2) \rangle$$

$$+ 2\langle R_2^T E_i R_2 \text{diag}(R_2^T E_i R_2(K_1 - K_2)), K_1 - K_2 \rangle$$

$$+ \langle R_2^T E_i R_2 \text{diag}(R_2^T E_i R_2, (K_1 - K_2)^2) \rangle \leq 0. \quad (102)$$

Let $K_0^u = (K_1 - K_2)/\|K_1 - K_2\|_F$, then

$$\langle \nabla^2 V_2(R_2) \cdot R_2(K_1 - K_2), R_2(K_1 - K_2) \rangle + \nabla V, R_2(K_1 - K_2)^2)$$

$$= 4n^{-3} \|K_1 - K_2\|_F \sum_{i=1}^{n} \langle (R_2^T E_i R_2 K_0^u \text{diag}(R_2^T E_i R_2), K_0^u) \rangle$$

$$+ 2\langle R_2^T E_i R_2 \text{diag}(R_2^T E_i R_2 K_0^u), K_0^u \rangle$$

$$+ \langle R_2^T E_i R_2 \text{diag}(R_2^T E_i R_2, (K_0^u)^2) \rangle \leq -C_{ss} \|K_1 - K_2\|_F. \quad (103)$$

Here

$$-C_{ss} = \max_{\|K\|_F = 1, K \in O(k)} \sum_{i=1}^{n} \langle (R_2^T E_i R_2 K \text{diag}(R_2^T E_i R_2), K) \rangle$$

$$+ 2\langle R_2^T E_i R_2 \text{diag}(R_2^T E_i R_2), K \rangle + \langle R_2^T E_i R_2 \text{diag}(R_2^T E_i R_2), K^2) \rangle$$

is a negative constant. With Taylor expansion and Equation (101), (102), (103), there is
\begin{align*}
V_2(R_1) &= V_2(R_2) + \langle \nabla V_2(R_2), (K_1 - K_2)R_2 \rangle + \langle \nabla^2 V_2(R_2) \cdot R_2(K_1 - K_2), R_2(K_1 - K_2) \rangle + \\
&\quad (\nabla V, R_2(K_1 - K_2)^2) + o(||K_1 - K_2||^2_F) \\
&= V_2(R_2) + \langle \nabla^2 V_2(R_2) \cdot R_2(K_1 - K_2), R_2(K_1 - K_2) \rangle + \\
&\quad \nabla V, R_2(K_1 - K_2)^2) + o(||K_1 - K_2||^2_F) \\
&\leq V_2(R_2) - C_{ss}||K_1 - K_2||^2_F + o(||K_1 - K_2||^2_F). 
\end{align*}
\tag{104}

With Equation (92), there exists \( \varepsilon_1 = O_p((n\rho_n)^{-1/2}\log^2 n) \), s.t.

\[ |V_2(R_1) - V_1(R_1)| < \varepsilon_1, \quad |V_2(R_2) - V_1(R_2)| < \varepsilon_1, \]

\[ \Rightarrow \quad V_1(R_1) - \varepsilon_1 < V_2(R_1) < V_2(R_2) < V_1(R_2) + \varepsilon_1, \]

\[ \Rightarrow \quad V_2(R_2) - V_2(R_1) < 2\varepsilon_1. \]

Then from Equation (104) there is

\[ ||K_1 - K_2||^2_F < \frac{2\varepsilon_1 + o(||K_1 - K_2||^2_F)}{C_{ss}}. \tag{105} \]

Thus \( ||K_1 - K_2||_F = O_p \left( (n\rho_n)^{-1/4}\log^2 n \right) \). By Lie Product Formula, for any \( k \times k \) matrices \( S_1, S_2 \) the exponential of their sum could be expressed as

\[ \exp(S_1 + S_2) = \lim_{m \to \infty} \left( \exp \left( \frac{S_1}{m} \right) \exp \left( \frac{S_2}{m} \right) \right)^m. \]

Thus

\[ R_1 - R_2 \]

\[ = \exp(K_2 + K_1 - K_2) - \exp(K_2) \]

\[ = \lim_{m \to \infty} \left\{ \exp \left( \frac{K_2}{m} \right) \exp \left( \frac{K_1 - K_2}{m} \right) \right\}^m - \left( \exp \left( \frac{K_2}{m} \right) \right)^m \]

\[ = \lim_{m \to \infty} \left[ \exp \left( \frac{K_2}{m} \right) \exp \left( \frac{K_1 - K_2}{m} \right) - \exp \left( \frac{K_2}{m} \right) \right] \]

\[ \times \sum_{i=1}^{m} \left( \exp \left( \frac{K_2}{m} \right) \exp \left( \frac{K_1 - K_2}{m} \right) \right)^i \left( \exp \left( \frac{K_2}{m} \right) \right)^{m-i}. \]
Since $K_1, K_2$ are skew-symmetric matrices, we have $\| \exp(\frac{K_1 - K_2}{m}) \| = \| \exp(\frac{K_2}{m}) \| = 1$, and

\begin{align*}
\| R_1 - R_2 \|_2 & \rightarrow \infty \\
& \leq \| R_1 - R_2 \| \\
& = \| \exp(K_2 + K_1 - K_2) - \exp(K_2) \| \\
& = \lim_{m \rightarrow \infty} \| [\exp(\frac{K_2}{m}) \exp(\frac{K_1 - K_2}{m}) - \exp(\frac{K_2}{m})] \sum_{i=1}^{m} (\exp(\frac{K_2}{m}) \exp(\frac{K_1 - K_2}{m}))^i (\exp(\frac{K_2}{m}))^{m-i} \| \\
& \leq \lim_{m \rightarrow \infty} \| \exp(\frac{K_2}{m}) \| \cdot \| \exp(\frac{K_1 - K_2}{m}) - I \| \sum_{i=1}^{m} \| \exp(\frac{K_2}{m}) \|^{m-i} \cdot \| \exp(\frac{K_1 - K_2}{m}) \|^i \\
& = \lim_{m \rightarrow \infty} \| \exp(\frac{K_1 - K_2}{m}) - I \| (m - 1) \\
& = \lim_{m \rightarrow \infty} \| \sum_{i=1}^{\infty} (\frac{K_1 - K_2}{m})^i \| (m - 1) \\
& \leq \lim_{m \rightarrow \infty} \sum_{i=1}^{\infty} \| \frac{K_1 - K_2}{m} \|^i (m - 1) \\
& = O_p(\| K_1 - K_2 \|) \\
& \leq O_p(\| K_1 - K_2 \|_F) \\
& = O_p\left( (n\rho_n)^{-1/4} \log \frac{7}{4} n \right). \quad (106)
\end{align*}

Therefore,

\begin{align*}
\| \sqrt{n} \hat{U} R_\hat{U} - \sqrt{n} \hat{U} R_U W P_n^{(3)} \|_2 & \rightarrow \infty \\
& \leq \sqrt{n} \| \hat{U} \|_2 \rightarrow \infty \| R_\hat{U} - R_U W P_n^{(3)} \| \\
& \leq \sqrt{n} (\| \hat{U} - U W \|_2 \rightarrow \infty + \| U W \|_2 \rightarrow \infty) \| R_\hat{U} - R_U W P_n^{(3)} \| \\
& = O_p(\log n) \times \| R_\hat{U} - R_U W P_n^{(3)} \| \\
& = O_p\left( (n\rho_n)^{-1/4} \log \frac{7}{4} n \right) \quad (Equation \ 45) \\
& = O_p\left( (n\rho_n)^{-1/4} \log \frac{7}{4} n \right) \quad (Equation \ 106).
\end{align*}
E First and Second Order Condition for Population Varimax

This section exploits the first and second order condition of the population Varimax function based on the similar results of the sample Varimax function (Sherin [1966], Neudecker [1981], Chu and Trendafilov [1998]). This section is self-contained and only reuses the notations of (1) and the definition of Assumption 1. We redefine some notations here.

**Assumption 4.** $U = ZR^T_U$ with $R_U \in \mathbb{O}(k), Z \in \mathbb{R}^{n \times k}$ with $Z$ satisfying Assumption 4. Let $z_0$ represents first row of $Z$. $O = z_0z_0^T$. $z_i$ is the $i$th element of $z_0$ with $\mathbb{E}(z_i) = 0, \forall i \in [k]$. Population Varimax function is $\mathcal{V}(R) = \mathbb{E}(v(R, U))$.

Optimization conditions for population Varimax function borrows conclusions from Chu and Trendafilov [1998]. The math requires switching the order of expectation and differential operations. Lemma E.1 shows that this is valid for Varimax function. The proof of the lemmas and Theorems in current section are all contained in Section E.3.

**Lemma E.1.** Under Assumption 4, the expectation operator and differential operator of Varimax function are exchangeable,

$$\frac{\partial \mathbb{E}v(R, U)}{\partial R} = \mathbb{E} \frac{\partial v(R, U)}{\partial R}.$$
E.2 Second Order Condition (SOC)

Chu and Trendafilov [1998] shows SOC result for sample Varimax function. The current subsection is deriving counterpart results of the population Varimax function. To describe the SOC on sample data, Chu and Trendafilov [1998] reformulate the Varimax criterion and express the problem in a simultaneously diagonalizing symmetric matrices form (ten Berge [1984]). The detailed SOC statement for sample Varimax is shown below.

Write $E_i = U^T U - nu_i u_i^T$ with $u_i^T$ being $U$’s $i$-th row. The sufficient (necessary) SOC of $v(R, U)$ is:

$$
\sum_{i=1}^n (\langle U^T E_i R \text{diag}(R^T E_i R), K^2 \rangle + \langle R^T E_i R \text{diag}(R^T E_i R), K \rangle + 2 \langle R^T E_i R \text{diag}(R^T E_i R K), K \rangle) < (\leq) 0,
$$

(109)

for any non-zero skew-symmetric matrix $K$. Since Varimax condition gives us a special covariance structure of $z_0$ (e.g. $\text{Cov}(z_0) = I$), we could derive SOC for population Varimax function from (109).

Theorem E.2 (SOC). Under Assumption 4, a sufficient (necessary) condition for $R_U$ to be one of the maxima of the population Varimax is

$$
3\mathbb{E}[\text{tr}(\text{diag}(OK)^2)] < (\leq) \mathbb{E}(\text{OdiagO}, KK^T).
$$

(110)

E.3 Proofs in Section E

E.3.1 Proof of Lemma E.1

Proof. The main idea of the proof is applying Dominant Converge Theorem (DCT). For simplicity, write $\mathbb{E}[U_{ij}] = \mu^{(j)}_q$ as $q$-th moment of $U$’s $j$-th column and $G_i = U^T U - nu_i u_i^T$, $i \in [n]$, with $u_i^T$ being the $i$-th row of $U$. By (8) of Chu and Trendafilov [1998],

$$
\frac{\partial v(R, U)}{\partial R} = \frac{4}{n^3} \sum_{i=1}^n G_i R \text{diag}(R^T G_i R).
$$

(111)

The goal is to bound the spectral norm of RHS of Equation (111). Notice for $\forall i \in [n]$,

$$
\|G_i R \text{diag}(R^T G_i R)\| \leq \|G_i R\| \cdot \|\text{diag}(R^T G_i R)\|
= \|G_i\| \cdot \|\text{diag}(R^T G_i R)\|
= \|U^T U - nu_i u_i^T\| \cdot \|\text{diag}(R^T U^T U R) - \text{diag}(nR^T u_i u_i^T R)\|
\leq \left(\|U^T U\| + n\|u_i u_i^T\|\right) \times
\left(\|\text{diag}(R^T U^T U R)\| + n\|\text{diag}(R^T u_i u_i^T R)\|\right).
$$

(112)
Basic matrix algebra implies
\[ \|U^T U\| \leq \|U^T\| \cdot \|U\| = \|U\|^2 \leq \|U\|_F^2, \quad \|u_i u_i^T\| \leq \|u_i\|^2. \]

Notice that the \(i\)-th diagonal element of \(R^T U U^T R\) is
\[
\sum_{s=1}^{k} \left( \sum_{t=1}^{k} (U_{ts})^2 \right) \leq \sum_{s=1}^{k} \left( \sum_{t=1}^{k} U_{ts}^2 \right) (\sum_{s=1}^{k} R_{st}^2) = k \sum_{s=1}^{k} U_{is}^2.
\]

\[
\Rightarrow \quad \|\text{diag}(R^T U U^T R)\| \leq \sum_{i=1}^{n} k \left( \sum_{s=1}^{k} U_{is}^2 \right) = k \|U\|_F^2.
\]

Similarly,
\[
\|\text{diag}(R^T u_i u_i^T R)\| \leq k \|u_i\|^2.
\]

Plugging Equation (113), (114) into Equation (112) yields
\[
\|G_i R \text{diag}(R^T G_i R)\| \leq (\|U\|_F^2 + n\|u_i\|^2) (\|U\|_F^2 + nk\|u_i\|_F^2)
= k \|U\|_F^4 + 2kn\|u_i\|^2 \|U\|_F^4 + kn^2 \|u_i\|^4.
\]

Getting back to Equation (111), we have
\[
\left\| \frac{\partial v(R, U)}{\partial R} \right\| \leq \frac{4}{n^3} \sum_{i=1}^{n} \|G_i R \text{diag}(R^T G_i R)\|
\leq \frac{4}{n^3} \sum_{i=1}^{n} (k \|U\|_F^4 + 2kn\|u_i\|^2 \|U\|_F^2 + kn^2 \|u_i\|^4)
:= F_n.
\]

The \(F_n\) is a random variable (depends on norms of random matrix and vectors). It will be
sufficient to give constant bound to the expectation of each term of $F_n$. Notice

$$E\|U\|_F^4 = E[(\sum_{ij} U_{ij}^2)^2]$$

$$= n \sum_{j=1}^{k} \mu_4^{(j)} + \left( \frac{n}{2} \right) \sum_{j=1}^{k} \mu_2^{(j)2} + n^2 \sum_{1 \leq \ell \neq j \leq k} \mu_2^{(\ell) \mu_2^{(j)}},$$

$$E(\|u_i\|^2 \|U\|_F^2) = E[(\sum_{j=1}^{k} U_{ij}^2)(\sum_{i,j} U_{ij}^2)]$$

$$= \sum_{j=1}^{k} \mu_4^{(j)} + (n-1) \sum_{j=1}^{k} \mu_2^{(j)2} + n \sum_{1 \leq \ell \neq j \leq k} \mu_2^{(\ell) \mu_2^{(j)}}.$$ 

$$E(\|u_i\|^4) = E[(\sum_{j=1}^{k} U_{ij}^2)^2] \sum_{j=1}^{k} \mu_4^{(j)} + \sum_{1 \leq \ell \neq j \leq k} \mu_2^{(\ell) \mu_2^{(j)}}.$$

Therefore

$$E(F_n) = \frac{4k}{n^2} E\|U\|_F^4 + \frac{8k}{n^2} \sum_{i=1}^{n} E(\|u_i\|^2 \|U\|_F^2) + \frac{4k}{n} \sum_{i=1}^{n} E(\|u_i\|^4)$$

$$= (4k + \frac{12k}{n}) M_1 + \frac{17k(n-1)}{n} M_2 + 16k M_3$$

$$\leq 5k M_1 + 17k M_2 + 16k M_3$$

$$< \infty.$$ 

Here $M_1 = \sum_{j=1}^{k} \mu_4^{(j)}$, $M_2 = \sum_{j=1}^{k} \mu_2^{(j)2}$, $M_3 = \sum_{1 \leq \ell \neq j \leq k} \mu_2^{(\ell) \mu_2^{(j)}}$ are all constants in our settings. Then DCT accomplishes our proof.

$\square$

**E.3.2 Proof of Theorem E.2**

The following Lemma is useful in the proof of Theorem E.2.

**Lemma E.2.** For any symmetric matrix $S = vv^T$ where $v$ is a $k$-dimension vector. Any $k \times k$ matrix $P$. We have

$$\langle S \text{diag}(SP), P \rangle = \langle SP \text{diag}(S), P \rangle.$$  

(115)

**Proof.** Let $(S)_{i,j} = v_i v_j$, $(P)_{i,j} = P_{ij}$. We only need to prove $S \text{diag}(SP) = SP \text{diag}(S)$. For $\text{diag}(SP)$. Its j-th diagonal element is $v_j \sum_k v_k P_{kj}$. Multiplying a diagonal matrix on right side is equal to multiplying i-th diagonal element to i-th column. Thus
\[(S_{\text{diag}}(SP))_{ij} = v_i v_j^2 \sum_k v_k P_{kj}.\]

For \(SP_{\text{diag}}(S)\) we have \((SP)_{ij} = v_i \sum_k v_k P_{kj} \Rightarrow (SP_{\text{diag}}(S))_{ij} = v_i v_j^2 \sum_k v_k P_{kj}.\)

Now we return to the proof of Theorem E.2

\[\text{Proof.} \quad \text{By Lemma E.1 and Slutsky Theorem,} \]
\[
\mathbb{E}[R_U^T E_i R_U \text{diag}(R_U^T E_i R_U)] = n^2 \mathbb{E}[(I - O) \text{diag}(I - O)],
\]
\[
\mathbb{E}[R_U^T E_i R_U K_{\text{diag}}(R_U^T E_i R_U)] = n^2 \mathbb{E}[(I - O) K_{\text{diag}}(I - O)],
\]
\[
\mathbb{E}[R_U^T E_i R_U \text{diag}(R_U^T E_i R_U K)] = n^2 \mathbb{E}[(I - O) \text{diag}(K - OK)].
\]

The expectation of the first term of Equation (109) equals to
\[
n^2 \mathbb{E}\langle (I - O) \text{diag}(I - O), K^2 \rangle = n^2 \mathbb{E}\langle I - O - \text{diag}O + \text{Odiag}O, K^2 \rangle = n^2 (\mathbb{E}\langle \text{Odiag}O, K^2 \rangle - \langle I, K^2 \rangle). \quad (116)
\]

Similarly, the expectation of the second term of Equation (109) is
\[
n^2 \mathbb{E}\langle (I - O) K_{\text{diag}}(I - O), K \rangle = n^2 \mathbb{E}\langle OK_{\text{diag}}(O), K \rangle - \langle K, K \rangle, \quad (117)
\]
and the expectation of the third term of Equation (109) is
\[
n^2 \mathbb{E}\langle (I - O) \text{diag}((I - O) K), K \rangle = n^2 \mathbb{E}\langle \text{Odiag} (OK), K \rangle - \langle \text{diag}(K), K \rangle. \quad (118)
\]

Notice \(K\) is skew-symmetric, there are
\[
\langle I, K^2 \rangle = tr(K^2) = -tr(KK^T) = -\langle K, K \rangle, \langle \text{diag}(K), K \rangle = 0. \quad (119)
\]

By Lemma E.2
\[
\langle \text{Odiag} (OK), K \rangle = \langle OK_{\text{diag}}(O), K \rangle. \quad (120)
\]

Properties of trace operator indicate that \(tr(Y \text{diag}(Y)) = tr((\text{diag}(Y))^2)\) for any square matrix \(Y\). Then with Equation (116), (117), (118), (119), (120), the second order condition (109) boils down to
\[
3 \mathbb{E}[tr(\text{diag}(OK)^2)] < \langle \leq \rangle \mathbb{E}\langle \text{Odiag}O, KK^T \rangle, \quad (121)
\]

for any non-zero skewed symmetric matrix \(K\).
F Proofs of Corollaries 5.1 and 5.2

As we pointed out in Section 5.3, independent columns assumption does not hold in the degree-corrected stochastic block model (DC-SBM). We could still make use of the first and second order condition to show that vsp could estimate $Z$ correctly. Similar to Proposition 3.2 we have

$$U = \frac{1}{\sqrt{n}} Z(Z^T Z/n)^{-\frac{1}{2}} \tilde{R}_U, \quad \tilde{R}_U \in \mathcal{O}(k).$$

(122)

The proof of Corollary 5.1, 5.2 will focus on validating some key conditions and assumptions.

F.1 Proof of Corollary 5.1

Proof. To borrow the conclusion from Theorem 4.1, it will be sufficient to check some results. Notice we don’t have the centering step and symmetrized adjacency matrices in SBM, such difference only simplifies the proof without introducing extra layers of perturbation. The only things we have to check (because of the dependency of $Z$’s columns) are conclusions of Theorem 3.1, Theorem E.1, E.2, Lemma C.4, Assumption 3 and the arguments in the proof of Lemma C.6 that shows the existence of moment generating function of linear inner-product of $Z_i$ ($Z$’s $i$th row).

F.1.1 Theorem 3.1 Under DC-SBM

Recall that in current setting,

$$V_{\tilde{R}_U}(Q) = \sum_j (\mathbb{E}([Z\tilde{R}_U Q]^4_j) - \mathbb{E}([Z\tilde{R}_U Q]^2_j)^2).$$

We want to show

$$\arg \max_{Q \in \mathcal{O}(k)} V_{\tilde{U}}(Q) = \{\tilde{U}^T P | P \in \mathcal{P}(k)\}.$$

Let $X = Z_1 - \mathbb{E}(Z_1), \mathbb{E}(X_i^j) = \mu_j^{(i)}$. Since there is exactly one non-zero entry in $X$’s elements, we have

$$\sum_{j=1}^k \mathbb{E}([XQ]^2_j) = \sum_{j=1}^k \left(\sum_{i=1}^k X_i^2 Q^2_{ij}\right)^2 = \sum_{j=1}^k \sum_{i=1}^k \mu_j^{(i)^2} Q^4_{ij},$$

and

$$\sum_{j=1}^k \mathbb{E}([XQ]^4_j) = \sum_{j=1}^k \sum_{i=1}^k \mathbb{E}(X_i^4 Q^4_{ij}) = \sum_{i=1}^k \mu_j^{(i)} Q^4_{ij}.$$
Therefore
\[
\mathcal{V}_I(Q) = \sum_j (\mathbb{E}([XQ]^4_j) - \mathbb{E}([XQ]^2_j)^2) \\
= \sum_{i=1}^k (\mu_4^{(i)} - \mu_2^{(i)2})\|Q_v\|^4_4.
\]

Jensen Inequality indicates that \(\mu_4^{(i)} - \mu_2^{(i)2} > 0\) for \(\forall i \in [k]\). The remaining part follows the same approach as in the proof of Theorem 3.1.

**F.1.2 Theorem E.1 Under DC-SBM**

For \(\forall i \in [n]\) and \(j, \ell \in [k], j \neq \ell\), we have
\[
Z_{ij}Z_{i\ell} = 0, Z_{ij}^3Z_{i\ell} = 0 \Rightarrow \mathbb{E}[Z_{ij}Z_{i\ell}] = 0, \mathbb{E}[Z_{ij}^3Z_{i\ell}] = 0.
\]

**F.1.3 Theorem E.2 Under DC-SBM**

To show SOC. Let \(E_i = U^TU - nu_iu_i^T\) is the \(i\)th row of \(U\). It is sufficient to show
\[
\mathbb{E}[(\tilde{R}_UE_i\tilde{R}_U^T\text{diag}(\tilde{R}_UE_i\tilde{R}_U^T), K^2) + (\tilde{R}_UE_i\tilde{R}_U^TK\text{diag}(\tilde{R}_UE_i\tilde{R}_U^T), K) ] + 2(\tilde{R}_UE_i\tilde{R}_U^T\text{diag}(\tilde{R}_UE_i\tilde{R}_U^T)K, K) \leq 0,
\]  
(123)

Let \(E^{i,j}\) be \(k\) by \(k\) matrix with 1 in \((i, j)\) entry and 0’s elsewhere. Then
\[
\tilde{R}_UE_i\tilde{R}_U^T = \tilde{R}_U\tilde{R}_U^T - n\tilde{R}_Uu_iu_i^T\tilde{R}_U^T = I - nZ_i^TZ_iE^{z(i),z(i)}.
\]  
(124)

Let \(K = \begin{pmatrix} 0 & K_{1,2} & \ldots & K_{1,k} \\ K_{2,1} & 0 & \ldots & K_{2,k} \\ \ldots & \ldots & \ldots & \ldots \\ K_{k,1} & K_{k,2} & \ldots & 0 \end{pmatrix}\) with \(K_{i,j} = -K_{j,i}\) for \(i > j\). Then \(\text{diag}(K^2) = \text{diag}(-\sum_{i \neq 1} K_{1,i}^2, -\sum_{i \neq 2} K_{2,i}^2, \ldots, -\sum_{i \neq k} K_{k,i}^2)\). Now we examine each term of \((123)\).
\[
\langle \tilde{R}_U E_i \tilde{R}_U^T \text{diag}(\tilde{R}_U E_i \tilde{R}_U^T), K^2 \rangle \\
= \langle \text{diag}(1, 1, \ldots, (1 - n\theta_{i,z(i)})^2, \ldots, 1), K^2 \rangle \\
= \sum_{\ell,j} K^2_{\ell j} - [(n\theta^2_{i,z(i)})^2 - 2n\theta^2_{i,z(i)}] \sum_{\ell=1}^k K^2_{z(i)\ell}, \quad (125)
\]
and
\[
\langle \tilde{R}_U E_i \tilde{R}_U^T K \text{diag}(\tilde{R}_U E_i \tilde{R}_U^T), K \rangle \\
= \langle \text{diag}(1, 1, \ldots, 1 - n\theta^2_{i,z(i)}, \ldots, 1) K \text{diag}(1, 1, \ldots, 1 - n\theta^2_{i,z(i)}, \ldots, 1), K \rangle \\
= \sum_{\ell,j} K^2_{\ell j} - 2n\theta^2_{i,z(i)} \sum_{\ell=1}^k K^2_{z(i)\ell}. \quad (126)
\]

Since \( K \)'s diagonal elements are all zero's, \( \text{diag}(\tilde{R}_U E_i \tilde{R}_U^T K) \) will be zero matrix.
\[
\langle \tilde{R}_U E_i \tilde{R}_U^T \text{diag}(\tilde{R}_U E_i \tilde{R}_U^T K), K \rangle = 0 \quad (128)
\]

From Equations (125), (127), (128), to prove Equation (123) it will be suffice to show:
\[
\sum_{i} (\sum_{\ell,j} K^2_{\ell j} + [(n\theta^2_{i,z(i)})^2 - 2n\theta^2_{i,z(i)}] \sum_{\ell=1}^k K^2_{z(i)\ell}) \geq \sum_{i} (\sum_{\ell,j} K^2_{\ell j} - 2n\theta^2_{i} \sum_{j} K^2_{z(i)\ell}),
\]
\[
\Leftrightarrow \sum_{i} ([(n\theta^2_{i,z(i)})^2 - 2n\theta^2_{i,z(i)}] \sum_{\ell=1}^k K^2_{z(i)\ell} + 2n\theta^2_{i,z(i)} \sum_{j} \sum_{\ell=1}^k K^2_{z(i)\ell}) \geq 0,
\]
\[
\Leftrightarrow \sum_{i} ((n\theta^2_{i,z(i)})^2 \sum_{\ell=1}^k K^2_{z(i)\ell}) \geq 0. \quad (129)
\]

The last inequality is strict as long as \( K \) is not zero matrix and \( \theta_i \)'s are all positive. We conclude that (123) is true.

**F.1.4 Lemma C.4 Under DC-SBM**

Under DC-SBM, elements of \( A \) are sub-gaussian variables. Thus we could utilize a simpler concentration matrix inequality than Lemma D.1. We apply the following lemma to show the bound for perturbation between \( A \) and \( \mathcal{A} \).
**Lemma F.1** (Matrix Bernstein Inequality, [Tropp 2012]). Let $X_1, X_2, ..., X_m$ be independent random $N \times N$ symmetric matrices. Assume $\|X_i - E(X_i)\| \leq M, \forall i$. Write $v^2 = \|\sum_i \text{var}(X_i)\|, X = \sum_i X_i$. Then for any $a > 0$,\[ \mathbb{P}(\|X - E(X)\| \geq a) \leq 2N \exp(-\frac{a^2}{2v^2 + 2Ma/3}). \]

Let $E^{ij}$ be a $n$ by $n$ matrix with 1 in the $(i, j)$ and $(j, i)$ entries and 0 elsewhere. Write $p_{ij} = \mathcal{A}_{ij}$. Then we could express $A - \mathcal{A}$ as sum of matrices,\[ Y_{ij} = (A_{ij} - p_{ij})E^{ij}, i < j. \]

Notice that\[ \|A - \mathcal{A}\| = \| \sum_{1 \leq i < j \leq n} Y_{ij} \|, \]

and\[ \|Y_{ij}\| \leq \|E^{ij}\| = 1. \]

Moreover,\[ \mathbb{E}(Y_{ij}) = 0 \text{ and } \mathbb{E}(Y_{ij}^2) = (p_{ij} - p_{ij}^2)(E^{ii} + E^{jj}), \forall i < j. \]

Then we could get an upper bound for $v^2$,\[ v^2 = \| \sum_{1 \leq i < j \leq n} \mathbb{E}[Y_{ij}^2] \| \]
\[ = \| \sum_{1 \leq i < j \leq n} (p_{ij} - p_{ij}^2)(E^{ii} + E^{jj}) \| \]
\[ = \frac{1}{2} \| \sum_{1 \leq i < j \leq n} (p_{ij} - p_{ij}^2)(E^{ii} + E^{jj}) \| \]
\[ = \frac{1}{2} \| \sum_{i=1}^{n} \sum_{j \neq i} (p_{ij} - p_{ij}^2)E^{ii} \| \]
\[ \leq \frac{1}{2} \max_{1 \leq i \leq n} \left( \sum_{j \neq i} (p_{ij} - p_{ij}^2) \right) \]
\[ \leq \frac{1}{2} \max_{1 \leq i \leq n} \left( \sum_{j \neq i} p_{ij} \right) \]
\[ \leq \frac{n}{2}. \]

From Lemma F.1 we obtain,\[ \mathbb{P}(\|A - \mathcal{A}\| > a) \leq 2N \exp(-\frac{a^2}{n + 2a/3}). \] (130)
F.1.5 Assumption 3 with Bernoulli Random Variables

Suppose $A_{ij}$ has $m$-th central moment being $\mu_{m,ij}$ and $m$-th moment being $\mu'_{m,ij}$. Since $A_{ij} \sim \text{Bernoulli}(\phi_{ij})$, then $\bar{\rho}_n \leq 1$. For any $m$,  
\[ \mathbb{E}[(A_{ij} - \phi_{ij})^m] = \mu_{m,ij} \leq |\mu'_{m,ij}| = |\phi_{ij}| \leq \bar{\rho}_n, \forall i, j. \tag{131} \]

F.1.6 Arguments of Lemma C.6 under DC-SBM

Since each $Z_{i,i} \in \mathbb{R}^k$, $i \in [k]$, has only one non-zero entry, for $\forall r \in \mathbb{R}^k$, $i \in [k]$, we have  
\[ \mathbb{E}\exp(t(Z_i, r)) = \mathbb{E}\exp(t \sum_{j=1}^{k} Z_{ij} r_j) = \mathbb{E}\exp(tZ_{i,z(i)} r_{z(i)}) = \prod_{j=1}^{k} \mathbb{E}\exp(tZ_{ij} r_j). \]

F.2 Proofs for Corollary 5.2

Proof. In current LDA settings, we need Assumption 2 in Theorem 4.1 on $\tilde{Z}_s = Z_s - \mathbb{E}(Z_s)$, which is already implied in Corollary 5.2 setup. Recall that:
\[ \mathbb{E}(\tilde{A}|\Xi) = \bar{Z}_s(\sqrt{n} \Sigma^{1/2}) (n^{-1/2} \beta^T). \tag{132} \]

Compared with the semi-parametric factor model in Definition 1, $\sqrt{n} \Sigma^{1/2}$ plays the role of block matrix $B$ and satisfies all the conditions in Theorem 4.1. Other than that Assumption 3 needs to be checked to prove Equation (21) and the $2 \to \infty$ norm of $Y$ (in Equation 132), this is $(n \rho_n)^{-1} \Sigma^{1/2} \beta^T$ needs to be bounded by $O(\log d) \asymp O(\log n)$. After that we will show the error bound for topics estimation.

Notice that $s$ controls the scaling of the term-document matrix, the following inference reflects its relation to $\Delta_n$. Recall that  
\[ \rho_n = \frac{1}{nd} \sum_{i,j} \phi_{ij}; \quad \Delta = n \rho_n. \]

And,  
\[ 1_n^T \phi 1_d = 1_n^T \Xi Z \beta^T 1_d = 1_n^T \Xi Z 1_k = 1_n^T \Xi 1_n. \tag{133} \]

Equation (133) implies that  
\[ \Delta_n = n \rho_n = \frac{n}{nd} 1_n^T \phi 1_d = \frac{1}{d} 1_n^T \Xi 1_n \asymp s. \tag{134} \]

79
F.2.1 Assumption 3 Under Poisson Random Variables

Suppose $A_{ij}$ has $m$-th central moment being $\mu_{m,ij}$. Since $A_{ij} \sim \text{Poisson}(\varphi_{ij})$, recall the recurrence relation of poisson distribution (Riordan [1937]),

$$
\mu_{m+1,ij} = \varphi_{ij}(\frac{d\mu_{m,ij}}{d\varphi_{ij}} + m\mu_{m-1,ij}), \quad \mu_1 = 0, \mu_2 = \varphi_{ij}, \forall i,j.
$$

It could be shown by induction that

$$
\mu_{m,ij} \leq \frac{(m-1)!}{\bar{\rho}_n \rho_n} \max\{\varphi_{ij}^{[m]}, \varphi_{ij}\}.
$$

Thus,

$$
\mathbb{E}[(A_{ij} - \varphi_{ij})^m] \leq \max\{(m-1)!\bar{\rho}_n^{[m]}, \bar{\rho}_n\} \leq \max\{(m-1)!\bar{\rho}_n^{[m]}, \bar{\rho}_n\}. \quad (135)
$$

F.2.2 Upper Bound for $\|n^{-1/2}\|_{2\rightarrow\infty}$

Notice for arbitrary $j$-th row of $n^{-1/2}\beta^T$, it has $\ell_2$-norm

$$
n^{-1/2} \sqrt{\sum_{\ell=1}^{k} \beta_{\ell j}^2} \leq n^{-1/2} \sqrt{\sum_{\ell=1}^{k} \beta_{\ell j}} = n^{-1/2}.
$$

Therefore, $\|n^{-1/2}\beta^T\|_{2\rightarrow\infty} = O(n^{-\frac{1}{2}})$, which is much smaller than $O(\log n)$.

F.2.3 Topics Estimation

For technical convenience, this proof uses an equivalent construction of $\hat{\beta}$. Define $\Omega = (\bar{Z}^T\bar{Z})^{-1}\bar{Z}^T\bar{A} = \Phi/n$ and $\hat{\beta} = (\Lambda^{-1}_o\Omega)^T \in \mathbb{R}^{d \times k}$, where $\Lambda_o$ is a diagonal matrix with $i$th diagonal element equals to $\ell_1$-norm of $i$th row of $\Omega$.

For the topic estimation $\hat{\beta}$, from Equation (132) there is,

$$
\varphi^{T}\varphi = n\beta \Sigma^{1/2} (\bar{Z}^T \bar{Z} / n) \Sigma^{1/2} \beta^T.
$$

By LLN we have $(\bar{Z}^T \bar{Z} / n)[i,j] = 1 \{i = j\} + O(1/\sqrt{n})$. Notice that the $j$th diagonal element of $\Sigma_{jj} = \alpha_j s^2 \geq n\rho_n$. Also $\sigma_{\min}(\beta) > c_1 > 0$, and

$$
\sigma_{\max}(\beta) = \|\beta\| < \|\beta\|_F = (\sum_{ij} \beta_{ij}^2)^{1/2} \leq (\sum_{ij} \beta_{ij})^{1/2} = k^{1/2}
$$

80
is upper bounded. Therefore

$$\sigma_{\min}(\mathcal{A}) \approx \sigma_{\min}((n\beta\Sigma\beta^T)^{1/2}) \approx \sqrt{ns} \geq n\rho_n.$$ 

With conclusions of Proposition [C.2], Lemma [C.3], C.4, Davis-Kahan sin Θ Theorem, Equation (21) and triangle inequality, there exists $P_n \in P(k)$ (similar to the $P_n$ in Equation (21)) s.t. for any $\delta, \epsilon > 0$, 

\[
\|\Omega - P_n^T\Sigma^{1/2}\beta^T\|_{2\to\infty} \leq \frac{1}{n} \left[ \| (\hat{Z}^T (\tilde{A} - \mathcal{A})\|_{2\to\infty} + \| (\hat{Z}^T \tilde{Z}_* - P_n^T)\Sigma^{1/2}\beta^T\|_{2\to\infty} \right]
\]

\[
\leq \frac{1}{n} \left[ \| \hat{Z}^T \| \| \tilde{A} - \mathcal{A} \| + \| \hat{Z}^T \| \| \tilde{Z}_* - \tilde{Z} \| \| P_n \| \| \Sigma^{1/2}\beta^T\| \right]
\]

\[
= \frac{1}{n} \left[ \| \hat{Z}^T \| \| \tilde{A} - \mathcal{A} \| + \| \hat{Z}^T \| \| \tilde{Z} - \tilde{Z}_* P_n \| \| \Sigma^{1/2}\beta^T\| \right]
\]

\[
\leq \frac{1}{n} \left[ \| \hat{Z}^T \| \| \tilde{A} - \mathcal{A} \| + \sqrt{n} \| \hat{Z}^T \| \| \tilde{Z}_* \| \| P_n \| \| \Sigma^{1/2}\beta^T\| \right]
\]

\[
= O_p(\frac{1}{n} \Delta_3^{1/2} \log^{5/2} n) + O_p(\frac{\Delta_3^{3/4+\delta/2} \log^{15/4} n}{\sqrt{n}})
\]

\[
= O_p(\frac{\Delta_3^{3/4+\delta/2} \log^{15/4} n}{\sqrt{n}}).
\]

Let $\Omega_\ell$ be the $\ell$th row of $\Omega$, $\zeta_\ell$ be the $\ell$th row of $P_n^T\Sigma^{1/2}\beta^T$. Then for $\forall \ell \in [k]$ there exists $\varepsilon_n = O_p((\Delta_3^{3/4+\delta/2} \log^{15/4} n)/\sqrt{n})$ s.t. with high probability,

\[
\|\Omega_\ell - \zeta_\ell\| \leq \varepsilon_n \Rightarrow \|\Omega_\ell - \zeta_\ell\|_1 \leq s\sqrt{d}\varepsilon_n. \tag{136}
\]

Notice any $\ell$-th column of $\beta$ has unit norm: $\|\beta_\ell\|_1 = 1, \forall \ell \in [k]$. Denotes $\alpha_{\min} = \min_j \alpha_j$, $\alpha_{\max} = \max_j \alpha_j$, then RHS of Equation (136) reflects

\[
s\sqrt{\alpha_{\min}} - \sqrt{d}\varepsilon_n \leq \|\Omega_\ell\|_1 \leq s\sqrt{\alpha_{\max}} + \sqrt{d}\varepsilon_n. \tag{137}
\]

With Equation (137) and notice the $j$-th diagonal element of $\Sigma^{1/2}$ is $s\sqrt{\alpha_j}$, we also have

\[
\max_{j, \ell \in [k]} |\Sigma_{jj}^{1/2} - \|\Omega_\ell\|_1| \leq \sqrt{d}\varepsilon_n. \tag{138}
\]

Since LHS of (137) is greater than 0 with large $n$. Let $[X]_\ell$ represents the $\ell$-th row of
matrix $X$. Then for $\forall \ell \in [k],$

$$\| \tilde{\beta}_\ell^T - [P_n^T \beta^T]_\ell \|_1 = \| \frac{\Omega_\ell}{\| \Omega_\ell \|_1} - [P_n^T \beta^T]_\ell \|_1$$

$$\leq \frac{1}{\| \Omega_\ell \|_1} \| \Omega_\ell - \zeta_\ell \|_1 + \| [P_n^T (\Sigma^{1/2} \frac{1}{\| \Omega_\ell \|_1} - 1) \beta^T]_\ell \|_1$$

$$\leq \frac{\sqrt{d \varepsilon_n}}{s \sqrt{\alpha_{\min}} - \sqrt{d \varepsilon_n}} + \frac{1}{\| \Omega_\ell \|_1} \max_{j \in [k]} | \Sigma^{1/2} - \| \Omega_\ell \|_1 |$$

$$\leq \frac{2\sqrt{d \varepsilon_n}}{s \sqrt{\alpha_{\min}} - \sqrt{d \varepsilon_n}}$$

$$= O_p(\sqrt{d \varepsilon_n}/s)$$

$$= O_p(\Delta_n^{-1/4+\delta/2} \log^{15/4} n).$$

$\Box$