The Einstein-Hilbert-Yang-Mills-Higgs Action

and

the Dirac-Yukawa Operator

by

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Abstract

In this article we show in some detail how the full action functional of the standard model of elementary particle physics can be described within the geometrical setting of generalized Dirac operators. We thereby introduce a new model building kit for (a certain class of) gauge invariant theories which provides a unified geometrical description of Einstein’s theory of gravity and Yang-Mills gauge theories on the ”classical” level. Moreover, when the gauge symmetry is spontaneously broken, the Higgs sector as well has a natural geometrical interpretation. It turns out that the Higgs field is related to the gravitational potential.

Since the full action functional of the standard model is derived in one stroke, the appropriate parameters of the model have to satisfy certain relations similar to those in the Connes-Lott approach. Likewise, this may yield some phenomenological consequences, which is illustrated by using the gauge group of the standard model in the case of \( N \)–generations of leptons and quarks.

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1 Introduction

In this article we propose a certain model building kit which permits derivation of the action functional of the standard model of elementary particle physics with gravity including in terms of generalized Dirac operators. For this we introduce the following geometrical data:

\[(G, \rho, D),\]

where \(G\) denotes a real compact semi-simple Lie group, \(\rho\) its unitary representation on a \(N\)-dimensional hermitian vector space \(V\) and \(D\) denotes a Dirac operator acting on sections into a Clifford module bundle \(\mathcal{E} := S \otimes E\), such that \(E := P \times_{\rho} V\). Here, \(S\) denotes the spinor bundle over a closed compact orientable Riemannian spin manifold \((\mathcal{M}, g)\) without boundary and of even dimension \((2n > 2); P\) is a \(G\)-principal bundle over \(\mathcal{M}\).

Having given the geometrical setting we propose the following functional

\[I_{D} := (\psi, iD\psi)_{\Gamma(\mathcal{E})} + \text{res} \zeta(D^{-2n+2}),\]

Here, \(\text{res}\) denotes the Wodzicki residue; \(\zeta\) is an element of the commutant defined by \((G, \rho)\), satisfying \([D, \zeta] = [\chi, \zeta] = 0, \zeta > 0\), with \(\chi\) the involution operator on \(\mathcal{E}\) and \((, )_{\Gamma(\mathcal{E})}\) denotes the induced hermitian product on the \(C^\infty(\mathcal{M})\)-module \(\Gamma(\mathcal{E})\) of sections \(\psi\) into \(\mathcal{E}\). In our frame this functional serves as a "general action" functional.

Using this frame the main result of this paper may be summarized in the

**Theorem:**

There exists a natural generalization (1) of the Dirac-Yukawa operator of the standard model such that the functional (2) is proportional to the full action of the standard model with gravity including.

As a consequence we obtain certain relations between the parameters involved. Especially, the mass of the Higgs field \(m_h\) is a function of the fermion masses. In the most general case the range for the electroweak angle \(\theta_w\) reads:

\[0.25 \leq \sin^2 \theta_w \leq 0.45.\]

In the case where all irreducible subspaces of the fermion representation are equally weighted we obtain the GUT preferred relations

\[\sin^2 \theta_w = \frac{3}{8},\]

\[g(3) = g(2),\]

where, respectively, \(g(3)\) and \(g(2)\) are the strong and weak coupling constants. Of course, all derived relations are expected to be scale dependent and thus the corresponding renormalization
flux must be carefully taken into account. We stress that the corresponding hypercharges of the particles involved in the model are fixed when the electrical charges are assumed to be known.

The model building kit as defined by the generalized Dirac operator (1) and the universal action (2) is motivated by the assumption that the basic objects in nature are fermions and that their dynamics is described by Dirac’s equation. Then, the appropriate dynamics of the various fields involved in the definition of the fermionic interactions should be a consequence of the latter and not independent thereof. Mathematically, this may be rephrased as follows: when fermions are geometrically described by sections into a twisted spinor bundle, the most natural operator acting on those objects is a Dirac operator. A Dirac operator, however, is but a Clifford superconnection (i.e. a certain generalization of a connection) defined by (homomorphism-valued) differential forms of various degrees. Then, fixing the admissible fermionic interactions geometrically means to fix the admissible differential forms defining a certain superconnection and thus a certain Dirac operator. The idea of the kit, proposed here, is that the functionals leading to the field equations of the differential forms defining the Dirac operator are not arbitrary but determined by this operator. As it turns out, this idea not only permits a geometrical understanding of the Higgs action but also a new geometrical interpretation of the Einstein-Hilbert and Yang-Mills functional ($I_{EH}$, $I_{YM}$). Indeed, from this point of view the former occurs - in a sense - as a natural ”companion” of the latter and both are ”consequences” of the fermionic interaction. In this (classical) description of particle physics the a priori assumption of a flat spacetime seems artificial. To make this more clear, let us assume we consider a ”free” fermion. From a geometrical viewpoint, such an object may be considered as a section into a twisted spinor bundle, where now the twisting part is assumed to possess a trivial connection. Nevertheless, the Clifford module bundle may be non-flat. Indeed, the fermion carries energy and thereby produces a non-trivial gravitational field. Hence, the corresponding spin connection is non-trivial in general. Of course, under ”normal” conditions the energy of an elementary particle is so weak that the appropriate gravitational field cannot be measured. From our point of view this simply means that in the particular case at hand the energy-momentum tensor - defined by the Dirac action - is ignorable with respect to any inertial frame and, therefore, spacetime becomes approximately flat. Though this is usually accepted by physicists the point here is that the functional $I_D$ fixes the energy-momentum tensor. Consequently, in our approach a vanishing Einstein tensor a priori makes no sense. This can strictly hold iff there are no fermions in the world (what ever such a world looks like!). Next, let us consider the case when the fermion becomes massive. Then, the interaction with the Higgs field must be taken into account. As it turns out, geometrically, the Higgs field defines a certain connection on the Clifford module bundle where the fermions live in. The corresponding curvature is non-zero, even in the case when the Higgs field represents a (classical) non-trivial vacuum. Of course, since the energy-momentum of the Higgs field is non-zero, spacetime must be curved as well. Note that this holds true even in the case when the world ”sits in the (classical) vacuum”. We stress that in our scheme the Higgs field seems intimately related to
gravity. Of course, we are only considering classical field theory and one may object that on
the level of elementary particles quantum theory has to be taken into account and then gravity
may look completely different than described by Einstein’s equation. However, as a ”first step”
towards a real understanding of the interplay between gravity and particle physics it might be
useful to have a unified geometrical description of all interactions on the level of the classical
field equations known so far.

Mathematically, it is evident that the action (2) is gauge invariant. Hence, our kit provides
a general scheme for building (a certain class of) gauge invariant theories. It therefore might
be worth remembering the input of a general Yang-Mills model and to compare both building
kits, correspondingly,. Here, we adopt the notation as given in [1].

The input of a general Yang-Mills model consists of the following data:

1. a finite dimensional real, compact Lie group G,
2. a unitary representation \( \rho_f = \rho_L \oplus \rho_R \) on the \( \mathbb{Z}_2 \)-graded hermitian vector space
   \( V_f = V_L \oplus V_R \) of the left and right handed fermions,
3. a finite set of positive constants \( \{g(k)\} \) (the gauge coupling constants), parametrizing
   the general Killing form of the Lie algebra \( \mathcal{G} \) of \( G \); the number of these constants is
   defined as the number of simple components of \( \mathcal{G} \), including \( u(1) \) factors.
4. a unitary representation \( \rho_h \) on a hermitian vector space \( V_h \) of the Higgs field,
5. a \( G \)-invariant polynomial (Higgs potential) \( V_h \xrightarrow{V} \mathbb{R} \) of order four, which is bounded
   from below,
6. one complex constant (the Yukawa coupling constant) \( g_y \) for every one dimensional
   invariant subspace in the decomposition of the representation
   \[ (V^*_L \otimes V_R \otimes V_h) \oplus (V^*_L \otimes V_R \otimes V^*_h), \]
7. an action functional
   \[ \mathcal{I} := \mathcal{I}_{EH} + \mathcal{I}_{Dirac} + \mathcal{I}_{Yukawa} + \mathcal{I}_{YM} + \mathcal{I}_{Higgs}. \]

Usually, it is assumed that spacetime is flat, so that \( \mathcal{I}_{EH} \) is ignored. In particular, the standard
model is defined by

\[ G := SU(3) \times SU(2) \times U(1) \]
with three coupling constants \((g(3), g(2), g(1))\);

\[
V_L := \bigoplus_1^3 [(1, 2, -1/2) \oplus (3, 2, 1/6)], \\
V_R := \bigoplus_1^3 [(1, 1, -1) \oplus (3, 1, -1/3) \oplus (3, 1, 2/3)],
\]

where \((n_3, n_2, n_1)\) denote the tensor product, respectively, of an \(n_3\) dimensional representation of SU(3), an \(n_2\) dimensional representation of SU(2) and a one dimensional representation of U(1) with "hypercharge" \(y\): \(\rho(e^{i\theta}) := e^{iy\theta}, \ y \in \mathbb{Q}, \ \theta \in [0, 2\pi]\);

\[
V := \lambda(\varphi\varphi^*)^2 - \mu^2\varphi\varphi^*,
\]

with \(\lambda, \mu > 0\). There are 27 Yukawa coupling constants which, however, are not all independent. In fact, the standard model can be parametrized by 18 constants, c.f. [2].

The full action functional (i.e. with gravity including) in the usual description of the standard model (N=3) reads

\[
\mathcal{I} := \frac{1}{16\pi G} \int_M \ast r_M \\
+ \int_M \ast (\bar{\psi}^{(l)} iD(\psi)^{(l)} + \bar{\psi}^{(q)} iD(\psi)^{(q)}) \\
- \int_M \ast (\sum_{i,j=1}^N (g_y)^{(q)ij} \bar{\psi}^{L_i}_L (\gamma_5 \varphi) \psi^{R_j}_R + (g_y)^{(q)} ij \bar{\psi}^{L_i}_L (\gamma_5 \varphi) \psi^{R_j}_R + \\
+ (g_y)^{(q)ij} \bar{\psi}^{d_f}_L (\gamma_5 \epsilon \varphi) \psi^{d_f}_R ) + \text{comp. conj.}
\]

\[
+ \frac{1}{g(3)} \int_M \text{tr}(C \wedge \ast C) + \frac{1}{g(2)} \int_M \text{tr}(W \wedge \ast W) + \frac{1}{2g(1)} \int_M B \wedge \ast B \\
+ \int_M \text{tr}(\nabla \varphi \wedge \ast \nabla \varphi) + \int_M \ast V
\]

\[
\equiv \mathcal{I}_{EH} + \mathcal{I}_{Dirac} + \mathcal{I}_{Yukawa} + \mathcal{I}_{YM} + \mathcal{I}_{Higgs}.
\]

The traces in the definition of the Yang-Mills action (10) are taken with respect to the corresponding fundamental representations of SU(3) and SU(2). Note that \(M\) denotes a Riemannian manifold, which explains the occurrence of the apparently wrong relative sign in front of the Higgs potential and the occurrence of \(\gamma_5\) in the Yukawa coupling term (9).

In contrast to a general Yang-Mills model our proposed kit (1) - (2) has the following input:

1. a finite dimensional real, compact Lie group \(G\),

2. a unitary representation \(\rho = \rho_L \oplus \rho_R\) on the \(\mathbb{Z}_2\)-graded hermitian vector space \(V = V_L \oplus V_R\) of the left and right handed fermions,
3. a Dirac operator $D$,

4. the general functional: $\mathcal{I}_D$.

Like in the "non-commutative approach" as introduced by Connes and Lott (c.f. [3], [4]), the Higgs representation is not arbitrary but has to lie within the fermionic representation, symbolically: $\rho_h \subset \rho_f$. In fact, this has significant consequences with respect to the relations between the various parameters involved in the model. Hence it might not be come as a surprise that there are certain similarities between the Connes-Lott model and the model introduced here in this respect (see below). Of course, the mathematical background is quite different. We mention that with respect to our physical interpretation it is quite natural that all fields involved in the model carry the same representation.

Concerning the standard model, the Dirac operator $D \equiv \tilde{D}_\phi$ is defined by a generalization of the Dirac-Yukawa operator $D_\phi$. It is well-known that the Yukawa coupling (9) together with the (standard) Dirac operator $D$ defining (8) can be considered as a new Dirac operator $D_\phi$ - the Dirac-Yukawa operator. The main feature, then, is that this Dirac operator is a "non-standard" Dirac operator (i.e. not associated with a Clifford connection, see below). In fact, such Dirac operators will play a key role in our geometrical description of the standard model. Correspondingly, we shall discuss those operators in some detail in the first part of our paper, which is totally concerned with the mathematical frame of our model. Though the larger portion of part 1 is actually not new and may be found in much more detail, e.g., in [11] we nevertheless summarize the basic mathematical notions in order for our paper to be self-contained and to permit full understanding of the issue also for those who are not familiar with the notion of non-standard Dirac operators. Also, we have emphasised the relations between non-standard Dirac operators and connections on a (general) Clifford module bundle. This is of technical significance and, moreover, explains how the Higgs field yields a certain connection on a Clifford module bundle. In part two we introduce a certain generalization of the Dirac-Yukawa operator and prove our main theorem. Moreover, we also investigate the "phenomenological" consequences of our scheme with respect to the standard model. Finally, we mention some similarities to the Connes-Lott model (c.f. [3], [4] and, concerning the new approach, [9], [13]). We conclude this paper with an outlook.

Before we start to describe the mathematical frame of our model building kit, however, some remarks concerning its similarity to the Connes-Lott approach to the standard model seem appropriate. Obviously, our notion of a generalized Dirac operator, as defined by (1), is similar to Connes' notion of a "spectral triple". Needless to say that the latter notion is more profound, mathematically, since it offers the possibility of "new mathematics", like Connes' non-commutative geometry, c.f. [4]. Also, the idea to derive specific functionals, such as in the case of the standard model of particle physics, from a "universal functional" must go back to Connes. For example, in the Connes-Lott approach to the standard model the Dixmier trace serves as the general action functional, c.f. [3], [4]. In a sense this trace can be considered as a
special case of the more general Wodzicki residue. Unfortunately, using the Dixmier trace as the universal action functional it seems hard to derive both the Einstein-Hilbert and the Yang-Mills (-Higgs) action in one stroke (for the pure EH-functional see [6]). In fact, in the Connes-Lott description of the standard model the geometric information contained in the Dirac operator is lost. But as Connes has remarked the Wodzicki residue of $D^{-2n+2}$ with $D$ a Standard Dirac operator (see below) becomes proportional to the Einstein-Hilbert action of gravity, c.f. [4]. However, this time the geometrical information contained in the Yang-Mills potential is lost. This follows immediately from $D$ being a Standard Dirac operator (see below). As a natural question one may ask whether it is possible to derive the full action functional (EHYMH-action) by considering "non-standard" Dirac-operators. In [10] this has been investigated and affirmatively answered in the case of the Einstein-Hilbert-Yang-Mills functional. In the case of the full action of the standard model and with gravity included the above question was investigated in [21]. However, in this work there is a mistake. In fact, the definition of the Dirac-Yukawa operator is wrong and as a consequence the derived functional does not coincide with the action functional of the standard model. As we shall show in the paper at hand, however, by using the right definition of the Dirac-Yukawa operator the properly corrected generalized Dirac-Yukawa operator proposed in [21] is but a special case of the generalized Dirac-Yukawa operator introduced in part two of our paper and which in fact gives rise to the full action of the standard model. Though the basic idea of our kit is already introduced in [21], and which indeed affirmatively answers the above mentioned question, the scheme in [21], however, is still not general enough to discuss physical implications of the proposed model building kit. This is because in [21] neither the considered Dirac operator - even if properly corrected - nor the proposed universal action functional is general enough. Both has been remedied in this article.

2 Part 1: The mathematical frame

The geometrical setting which we propose in order to describe gauge theories is that of a Clifford module bundle $(\mathcal{E}, c)$ over a Riemannian manifold $(\mathcal{M}, g)$ of even dimension. Within this setting there exists a distinguished class of operators called generalized Dirac operators\footnote{In the following, the term "generalized" simply means "non-standard", though by a generalized Dirac operator we actually mean the triple $(G, \rho, D)$, with $D$ a (non-standard) Dirac operator.}. We therefore start with a brief review on the notion of Clifford modules and generalized Dirac operators. More details of this issue can be found, e.g., in [11]. Afterwards we shall discuss in some length how a given generalized Dirac operator (1) determines a particular functional $I_D(\nabla^\mathcal{E}, \psi)$, using (2).

To get started, let us denote by $(\mathcal{M}, g)$ a smooth, closed compact Riemannian (spin) manifold without boundary and of even dimension: $\dim(\mathcal{M}) \equiv m := 2n (> 2)$. Moreover, let
\( \mathcal{E} := \mathcal{E}^+ \oplus \mathcal{E}^- \) be (the total space of) a \( \mathbb{Z}_2 \)-graded hermitian vector bundle \( \mathcal{E} \to M \) over \( M \). The corresponding hermitian product on \( \mathcal{E} \) is indicated by \( (,)_\mathcal{E} \). If \( \Gamma(\mathcal{E}) \) denotes the \( C^\infty(M) \)-module of smooth sections into \( \mathcal{E} \), then the induced hermitian product \( (,)_\Gamma(\mathcal{E}) \) on \( \Gamma(\mathcal{E}) \) is given by: \( (,)_\Gamma(\mathcal{E}) := f_M \star (,)_\mathcal{E} \). Here, "\( \star \)" means the Hodge map regarding the Riemannian metric \( g \) on \( M \).

**Definition 1:** A **generalized Dirac operator** \( D \) is any odd first order differential operator acting on sections \( \psi \in \Gamma(\mathcal{E}) \):

\[
D : \Gamma(\mathcal{E}^\pm) \to \Gamma(\mathcal{E}^{\mp}),
\]

so that \( D^2 \) is a **generalized Laplacian**. I.e., there exists a connection: \( \Gamma(\mathcal{E}) \xrightarrow{\nabla^\mathcal{E}} \Gamma(T^*M \otimes \mathcal{E}) \) on the vector bundle \( \mathcal{E} \) and an endomorphism \( F \in \Gamma(\text{End}(\mathcal{E})) \), both uniquely defined by \( D \), such that

\[
D^2 = \Delta^\mathcal{E} + F.
\]

Here, \( \Delta^\mathcal{E} := -ev_g(\nabla^T \otimes^\mathcal{E} \nabla^\mathcal{E}) \) denotes the horizontal (Bochner) Laplacian associated with the connection \( \nabla^\mathcal{E} \), and "\( ev_g \)" means the evaluation map regarding the metric \( g \).

**Remark 1:** \( D \) exists on \( \mathcal{E} \) iff \( \mathcal{E} \) denotes a **Clifford module** bundle over \( (M,g) \). I.e. there exists a graded (left) action on \( \mathcal{E} \)

\[
c : C(M) \times \mathcal{E} \to \mathcal{E}
\]

of the Clifford bundle \( C(M) \to M \) associated with the metric \( g \). This holds because in case that \( D \) denotes a generalized Dirac operator, this operator induces via

\[
C(M) \times \mathcal{E} \to \mathcal{E}
\]

\[
(df, \psi) \mapsto c(df)\psi := [D,f]\psi, \quad f \in C^\infty(M)
\]

a graded left action of the Clifford bundle on \( \mathcal{E} \), c.f. \[1\]. Conversely, if \( (\mathcal{E},c) \) denotes a Clifford module bundle over \( (M,g) \), then the well-known construction

\[
\Gamma(\mathcal{E}) \xrightarrow{\nabla^\mathcal{E}} \Gamma(T^*M \otimes \mathcal{E}) \hookrightarrow \Gamma(C(M) \otimes \mathcal{E}) \xrightarrow{c} \Gamma(\mathcal{E})
\]

defines an operator \( D_{\nabla^\mathcal{E}} := c(\nabla^\mathcal{E}) \) satisfying (13) and (14) for any connection \( \nabla^\mathcal{E} \) on \( \mathcal{E} \). Moreover, it is easily checked that such a defined operator also fulfils (16).

Hence, from now on \( \mathcal{E} \equiv (\mathcal{E},c) \) will always denote a Clifford module bundle over \( M \equiv (M,g) \). As a consequence, the endomorphism bundle \( \text{End}(\mathcal{E}) \) is also a Clifford module and it follows that (c.f. \[1\])

\[
\text{End}(\mathcal{E}) \simeq C(M) \otimes \text{End}_{Cl}(\mathcal{E}),
\]

\(^3\)Here, the fiber \( \tau^{-1}(x) \) is isomorphic to the Clifford algebra generated by the elements \( u,v \in T^*_xM \), using the relation: \( uv + vu := -2g_x(u,v), \forall x \in M \).
where \( \text{End}_{\text{Cl}}(\mathcal{E}) \) denotes the algebra bundle of bundle endomorphisms of \( \mathcal{E} \) supercommuting with the action of \( \mathcal{C}(\mathcal{M}) \), i.e.

\[
\text{End}_{\text{Cl}}(\mathcal{E}) := \{ \sigma \in \text{End}(\mathcal{E}) \mid [c(a), \sigma] = 0, \ \forall a \in \mathcal{C}(\mathcal{M}) \}.
\] (19)

If in addition \( \mathcal{M} \) is assumed to be a spin manifold we have: \( \mathcal{C}(\mathcal{M}) \approx \text{End}(S) \) and correspondingly \( \text{End}_{\text{Cl}}(\mathcal{E}) \approx \text{End}(E) \), where \( S \) denotes the spinor bundle and \( E \) a vector bundle over \( \mathcal{M} \). In this case \( \mathcal{E} \) is called a **twisted spinor bundle** over \( \mathcal{M} \) and we have (c.f. [11])

\[
\mathcal{E} \approx S \otimes E.
\] (20)

We say that the Clifford module bundle \((\mathcal{E}, c)\) has a **twisting graduation** if \( \text{End}_{\text{Cl}}(\mathcal{E}) \) possesses a (non-trivial) \( \mathbb{Z}_2 \)—gradation, cf. [11]. Clearly, in the case of (20) this is equivalent to saying that the vector bundle \( E = E_L \oplus E_R \) is \( \mathbb{Z}_2 \)—graded, as well.

**Definition 2:** Let \( \tilde{D} \) be a generalized Dirac operator on \( \mathcal{E} \). It is said to be compatible with the Clifford action \( c \) if it satisfies the relation (16). Then, let

\[
\mathcal{D}(\mathcal{E}) := \{ \tilde{D} \mid [\tilde{D}, f] = c(df), \ \forall f \in C^\infty(\mathcal{M}) \}
\] (21)

be the set of all generalized Dirac operators on \( \mathcal{E} \) which are compatible with the Clifford action \( c \). We have

\[
\mathcal{D}(\mathcal{E}) \approx \Omega^p(\mathcal{M}, \text{End}^-(\mathcal{E})).
\] (22)

Also, let us denote by

\[
\mathcal{A}(\mathcal{E}) := \{ \nabla^\mathcal{E} \mid \Gamma(\mathcal{E}) \xrightarrow{\nabla^\mathcal{E}} \Gamma(T^*\mathcal{M} \otimes \mathcal{E}) \}
\] (23)

the set of all (even) connections on \( \mathcal{E} \).

As a consequence of (18), there exists a natural class of connections - called **Clifford connections** - on any Clifford module bundle \( \mathcal{E} \):

\[
\mathcal{A}_{\text{Cl}}(\mathcal{E}) := \{ \nabla^\mathcal{E} \in \mathcal{A}(\mathcal{E}) \mid [\nabla^\mathcal{E}, c(a)] = c(\nabla^{\text{Cl}} a), \ \forall a \in \Gamma(\mathcal{C}(\mathcal{M})) \} \subset \mathcal{A}(\mathcal{E}),
\] (24)

where \( \nabla^{\text{Cl}} \) is the induced Levi-Civita connection on \( \mathcal{C}(\mathcal{M}) \). Note, in the case of a twisted spinor bundle (20) any Clifford connection \( \nabla^\mathcal{E} \in \mathcal{A}_{\text{Cl}}(\mathcal{E}) \) takes the form of a tensor product connection (c.f. [11])

\[
\nabla^\mathcal{E} \equiv \nabla^{S\otimes E} := \nabla^S \otimes 1_E + 1_S \otimes \nabla^E,
\] (25)

\text{Since the bundle } \mathcal{E} = \mathcal{E}^+ \oplus \mathcal{E}^- \text{ is } \mathbb{Z}_2 \text{—graded, so is the associated endomorphism bundle } \text{End}(\mathcal{E}).
\text{I.e. } \text{End}(\mathcal{E}) = \text{End}^+(\mathcal{E}) \oplus \text{End}^-(\mathcal{E}), \text{ with } \text{End}^+(\mathcal{E}) := \text{End}(\mathcal{E}^+) \oplus \text{End}(\mathcal{E}^-) \text{ and } \text{End}^-(\mathcal{E}) := \text{Hom}(\mathcal{E}^+, \mathcal{E}^-) \oplus \text{Hom}(\mathcal{E}^-, \mathcal{E}^+).
where, respectively, $\nabla^S$ denotes the spin connection on $S$ and $\nabla^E$ any connection on the vector bundle $E$. In general we have

$$\mathcal{A}(\mathcal{E}) \simeq \Omega^1(\mathcal{M}, \text{End}^+(\mathcal{E})), $$
$$\mathcal{A}_{\text{Cl}}(\mathcal{E}) \simeq \Omega^1(\mathcal{M}, \text{End}^\perp_{\text{Cl}}(\mathcal{E})), $$

(26)

where the latter isomorphism follows from (18).

**Remark 2:** Using the linear isomorphism:

$$\Lambda T^*\mathcal{M} \xrightarrow{c} \mathcal{C}(\mathcal{M})$$
$$e^{i_1} \wedge e^{i_2} \cdots \wedge e^{i_k} \mapsto e^{i_1} e^{i_2} \cdots e^{i_k}, \quad \forall \ 0 \leq k \leq m,$$

(27)

between the Grassmann- and the Clifford bundle, where $\{e^i\}_{1 \leq i \leq m}$ denotes a local orthonormal basis in $T^*\mathcal{M}$ and Clifford multiplication is indicated by juxtaposition, the Clifford action $c$ induces a linear mapping, also denoted by $c$,

$$\Omega^p(\mathcal{M}, \text{End}^\perp(\mathcal{E})) \xrightarrow{c} \Omega^p(\mathcal{M}, \text{End}^\perp(\mathcal{E}))$$
$$\alpha \mapsto c(\alpha).$$

(28)

**Lemma 1:** In the case of $p = 1$ the linear mapping (28) has a canonical right inverse defined by

$$\delta_\xi : \Omega^p(\mathcal{M}, \text{End}^\perp(\mathcal{E})) \rightarrow \Omega^{p+1}(\mathcal{M}, \text{End}^\perp(\mathcal{E}))$$
$$\alpha \mapsto \xi \wedge \alpha,$$

(29)

where $\xi \in \Omega^1(\mathcal{M}, \text{End}^-(\mathcal{E}))$ is locally given by

$$\xi := -\frac{1}{m} g(e_a, e_b) e^a \otimes c(e^b) \otimes 1_{\text{End}_{\text{Cl}}(\mathcal{E})}.$$  

(30)

The product in (29) simply means: $(\sigma \otimes \alpha) \wedge (\sigma' \otimes \alpha') := \sigma \wedge \sigma' \otimes \alpha \alpha'$ for all homogeneous elements $\sigma \otimes \alpha, \sigma' \otimes \alpha' \in \Omega^*(\mathcal{M}, \text{End}(\mathcal{E})) \simeq \Gamma(T^*\mathcal{M} \otimes \text{End}(\mathcal{E}))$; Again, $\{e^a\}_{1 \leq a \leq m}$ is a basis in $T^*\mathcal{M}$ and $\{e_a\}_{1 \leq a \leq m}$ its dual. Note that this ”wedge product” is not graded commutative and that $\Omega^*(\mathcal{M}, \text{End}(\mathcal{E}))$ is a bi-graded algebra. The map (29) is even with respect to the total grading.

**Proof:** Obviously, this follows by construction.

**Remark 3:** The form $\xi$, locally defined by (30), can be characterized via

$${\nabla}^{T^*\mathcal{M} \otimes \text{End}(\mathcal{E})} \xi \equiv 0, \quad \forall \ \nabla^\mathcal{E} \in \mathcal{A}_{\text{Cl}}(\mathcal{E}),$$

(9)

\footnote{Throughout this paper we adopt Einstein’s convention for summation.}
Lemma 2: Let $c$ be the linear mapping (28) restricted to $\Omega^1(\mathcal{M}, \text{End}^+(\mathcal{E}))$. Then we have

$$D(\mathcal{E}) \simeq A(\mathcal{E})/\ker(c).$$

(32)

Proof: For $p := \delta_\xi c : \Omega^1(\mathcal{M}, \text{End}^+(\mathcal{E})) \to \Omega^1(\mathcal{M}, \text{End}^+(\mathcal{E}))$ we get $p^2 = p$ and thus

$$\Omega^1(\mathcal{M}, \text{End}^+(\mathcal{E})) = \text{im}(p) \oplus \text{im}(q),$$

(33)

with $q := 1 - p$. Restricting the linear mapping (28) to $\Omega^1(\mathcal{M}, \text{End}^+(\mathcal{E}))$ yields the identities

$$pc \equiv p,$$

$$cp \equiv c.$$  

(34)

Since $\delta_\xi$ is a right inverse of $c$ we have: $\ker(c) \subset \text{im}(q)$. Moreover, for all $\alpha \in \text{im}(p)$ with $c(\alpha) = 0$ (34) implies $\alpha \equiv 0$. Hence

$$\ker(c) = \text{im}(q).$$

(35)

The statement follows from

$$A(\mathcal{E}) \simeq \Omega^1(\mathcal{M}, \text{End}^+(\mathcal{E})) \xrightarrow{c} \Omega^0(\mathcal{M}, \text{End}^-(\mathcal{E})) \simeq D(\mathcal{E})$$

(36)

and we are done.

Note that actually we have shown that the sequence:

$$0 \to \ker(c) \to \Omega^1(\mathcal{M}, \text{End}^+(\mathcal{E})) \xrightarrow{c} \Omega^0(\mathcal{M}, \text{End}^-(\mathcal{E})) \to 0$$

is exact and splits. Also, in this case the kernel of the mapping (28) becomes explicit.

Definition 3: Two connections $\nabla^\mathcal{E}$, $\tilde{\nabla}^\mathcal{E} \in A(\mathcal{E})$ are defined to be equivalent iff

$$\nabla^\mathcal{E} - \tilde{\nabla}^\mathcal{E} \in \ker(c).$$

(37)

By the preceding Lemma this is equivalent to

$$\nabla^\mathcal{E} \sim \tilde{\nabla}^\mathcal{E} \iff \tilde{\nabla}^\mathcal{E} = \nabla^\mathcal{E} + \omega,$$

$$\omega \in \text{im}(q) \subset \Omega^1(\mathcal{M}, \text{End}^+(\mathcal{E})).$$

(38)

Therefore, any Dirac operator $D \in D(\mathcal{E})$ is uniquely associated with an equivalence class of connections $[\nabla^\mathcal{E}]$ on $\mathcal{E}$ so that $D_{\nabla^\mathcal{E}} = D$. 

10
Remark 4: Let $\tilde{D} \in \mathcal{D}(\mathcal{E})$ be a given Dirac operator on $\mathcal{E}$. Then,

$$\tilde{\nabla}^\mathcal{E} := \nabla^\mathcal{E} + \delta_\xi (\tilde{D} - D \nabla^\mathcal{E})$$

(39)

defines a connection on $\mathcal{E}$, so that

$$D \tilde{\nabla}^\mathcal{E} = \tilde{D},$$

(40)

where $\nabla^\mathcal{E} \in \mathcal{A}(\mathcal{E})$ denotes any connection on $\mathcal{E}$. Clearly, this ambiguity simply reflects that $\mathcal{D}(\mathcal{E})$ is an affine space and thus a given Dirac operator $\tilde{D} \in \mathcal{D}(\mathcal{E})$ may be decomposed in infinitely many ways like

$$\tilde{D} = D + \Phi_D$$

(41)

with $\Phi_D := \tilde{D} - D \in \Omega^0(\mathcal{M}, \text{End}_-^\mathcal{E})$.

Definition 4: We call a Dirac operator $D \in \mathcal{D}(\mathcal{E})$ a Standard Dirac operator (SDO) if there is a Clifford connection $\nabla^\mathcal{E} \in \mathcal{A}_{Cl}(\mathcal{E})$, so that

$$D = D \nabla^\mathcal{E}.$$  

(42)

Lemma 3: Let $\nabla^\mathcal{E}$, $\tilde{\nabla}^\mathcal{E} \in \mathcal{A}_{Cl}(\mathcal{E})$ be Clifford connections on the Clifford module bundle $\mathcal{E}$ with $\nabla^\mathcal{E} \sim \tilde{\nabla}^\mathcal{E}$. Then we have: $\nabla^\mathcal{E} \equiv \tilde{\nabla}^\mathcal{E}$.

Proof: Using (38), $\nabla^\mathcal{E}$, $\tilde{\nabla}^\mathcal{E} \in \mathcal{A}_{Cl}(\mathcal{E})$ implies that there exists a $\omega := \nabla^\mathcal{E} - \tilde{\nabla}^\mathcal{E} = 1_{Cl} \otimes A$ with $A \in \Omega^1(\mathcal{M}, \text{End}^\mathcal{E}_{Cl})$. By assumption, we have

$$c(\omega) = c(e^\mu) \otimes A_\mu = 0,$$

(43)

where $\{e^\mu\}_{1 \leq \mu \leq m}$ is a local orthonormal frame. Hence, $A_\mu = 0, \forall \mu = 1, \ldots, m$ which proves the lemma.

We therefore have shown that the class of connections defining a SDO on $\mathcal{E}$ admits a canonical representative. In what follows we shall always denote by $D \in \mathcal{D}(\mathcal{E})$ a SDO and by $\nabla^\mathcal{E} \in \mathcal{A}_{Cl}(\mathcal{E})$ the appropriate Clifford connection, so that $D = D \nabla^\mathcal{E}$. In contrast, by $\tilde{D} \in \mathcal{D}(\mathcal{E})$ and $\tilde{\nabla}^\mathcal{E} \in \mathcal{A}(\mathcal{E})$, respectively, we denote an arbitrary Dirac operator and connection on $\mathcal{E}$.

Let $\tilde{D} \in \mathcal{D}(\mathcal{E})$ be an arbitrary Dirac operator on the Clifford module bundle $\mathcal{E}$ and let

$$\tilde{\nabla}^\mathcal{E} := \nabla^\mathcal{E} + \delta_\xi (\tilde{D} - D)$$

(44)
Then, in [10] it is shown that (see also [14])
\[ \tilde{D}^2 = \Delta \tilde{\nabla}^\varepsilon + \tilde{\mathcal{F}} \tilde{\nabla}^\varepsilon. \] (45)

Here, respectively, the connection \( \tilde{\nabla} \in \mathcal{A}(\varepsilon) \) and the endomorphism \( \tilde{\mathcal{F}} \tilde{\nabla}^\varepsilon \in \Gamma(\text{End}(\varepsilon)) \) are defined by

\[ \tilde{\nabla}^\varepsilon := \tilde{\nabla}^\varepsilon + \omega \tilde{\nabla}^\varepsilon \quad \text{and} \]
\[ \tilde{\mathcal{F}} \tilde{\nabla}^\varepsilon := c \left( \tilde{\nabla}^\varepsilon^2 \right) + \text{ev}_g \left( \tilde{\nabla}^T \text{End}(\varepsilon) \omega \tilde{\nabla}^\varepsilon + \omega \tilde{\nabla}^\varepsilon^2 \right), \] (47)

where the one form \( \omega \tilde{\nabla}^\varepsilon \in \Omega^1(\mathcal{M}, \text{End}^+(\varepsilon)) \) is locally given by

\[ \omega \tilde{\nabla}^\varepsilon := -\frac{1}{2} g(e_\mu, e_\nu) e^\mu \otimes c(e^\lambda) \left( [\tilde{\nabla}^\varepsilon^\lambda, c(e^\nu)] + \Gamma^\nu_{\sigma \lambda} c(e^\sigma) \right). \] (48)

The \( \Gamma \)'s denote the Christoffel symbols defined by the metric \( g \).

**Lemma 4:** The endomorphism \( \tilde{\mathcal{F}} \tilde{\nabla}^\varepsilon \) defined in (47) is independent of the representative \( \tilde{\nabla}^\varepsilon \in \mathcal{A}(\varepsilon) \) of the class of connections defining the Dirac operator \( \tilde{D} \).

**Proof:** To prove this lemma we introduce the affine mapping

\[ \varpi : \mathcal{A}(\varepsilon) \rightarrow \mathcal{A}(\varepsilon) \]
\[ \tilde{\nabla}^\varepsilon \mapsto \tilde{\nabla}^\varepsilon + \omega \tilde{\nabla}^\varepsilon \] (49)

on \( \mathcal{A}(\varepsilon) \) and show that this mapping is well-defined on \( \mathcal{A}(\varepsilon)/\ker(c) \). Let \([ \tilde{\nabla}^\varepsilon ]\) be the equivalence class of connections defining the given Dirac operator \( \tilde{D} \in \mathcal{D}(\varepsilon) \) and denote by \( \tilde{\nabla}^\varepsilon, \tilde{\nabla}^\varepsilon \in [ \tilde{\nabla}^\varepsilon ] \) two representatives of this class. Hence, \( \alpha := \tilde{\nabla}^\varepsilon - \tilde{\nabla}^\varepsilon \in \ker(c) \) and with respect to a local orthonormal frame we obtain

\[ \tilde{\nabla}^\varepsilon - \tilde{\nabla}^\varepsilon = \alpha - \frac{1}{2} \delta_{\mu \nu} e^\mu \otimes c(e^\lambda) [i_\lambda \alpha, c(e^\nu)] \] (50)
\[ = \frac{1}{2} \delta_{\mu \nu} e^\mu \otimes [c(\alpha), c(e^\nu)]_+ \] (51)
\[ = 0, \] (52)

where \( i_\mu \) is the inner derivative with respect to the local vector field \( e_\mu \) and \([ , ]_+\) means the anti-commutator. Consequently, the map: \( \tilde{D} \mapsto \Delta \tilde{\nabla}^\varepsilon \), with \( \tilde{\nabla}^\varepsilon \) given by (46), is well-defined. Hence the endomorphism \( \tilde{\mathcal{F}} \tilde{\nabla}^\varepsilon = \tilde{D}^2 - \Delta \tilde{\nabla}^\varepsilon \) only depends on the Dirac operator \( \tilde{D} \in \mathcal{D}(\varepsilon) \) which proves the lemma.

**Corollary 1:** The endomorphism \( \tilde{\mathcal{F}} \tilde{\nabla}^\varepsilon \in \Gamma(\text{End}(\varepsilon)) \) does not depend on the decomposition (44). In particular, it does not depend on the chosen Clifford connection \( \nabla^\varepsilon \).
Proof: By the preceding lemma 4 the proof is obvious.

Remark 5: The corresponding linear part of the affine map (49) has a non-trivial kernel; especially we get

\[ \varpi |_{\mathcal{A}_{\mathcal{C}I}(\mathcal{E})} = \mathbf{1}_{\mathcal{A}_{\mathcal{C}I}(\mathcal{E})}. \]  

(53)

In this case the decomposition of the square of the appropriate SDO is but the usual Lichnerowicz formula and the endomorphism \( \mathcal{F} \) takes its well-known form

\[
\mathcal{F}^{\nabla \mathcal{E}} = c[(\nabla^{\mathcal{E}})^2] \\
= \frac{1}{4} r_{\mathcal{M}} \mathbf{1}_{\mathcal{E}} + c(F^{\mathcal{E}/S}),
\]

(54)

(55)

where \( r_{\mathcal{M}} \) is the Ricci scalar curvature on the base manifold \( \mathcal{M} \) and \( F^{\mathcal{E}/S} := (\nabla^{\mathcal{E}})^2 - (\nabla^{\mathcal{C}I})^2 \otimes \mathbf{1}_{\text{End}_{\mathcal{C}I}(\mathcal{E})} \) denotes the relative curvature on the Clifford module bundle \( \mathcal{E} \). Since in this particular case the relative curvature only depends on the connection on the twisting part of the Clifford module bundle \( \mathcal{E} \), \( F^{\mathcal{E}/S} \) is also called the twisting curvature.

We now turn to the notion of superconnections which can be considered as a generalization of connections on a \( \mathbb{Z}_2 \)-graded vector bundle. As it is well-known superconnections permit to generalize the one two one correspondence between SDO and Clifford connections to arbitrary Dirac operators and Clifford superconnections on a Clifford module bundle. Hence, there is no essential difference between talking about Dirac operators and Clifford superconnections. We therefore call into mind the following

Definition 5: A superconnection on a \( \mathbb{Z}_2 \)-graded vector bundle \( \mathcal{E} \) is any odd first order differential operator

\[ \nabla^{\mathcal{E}} : [\Omega^*(\mathcal{M}, \mathcal{E})]^\pm \to [\Omega^*(\mathcal{M}, \mathcal{E})]^\mp, \]

(56)

satisfying the generalized Leibniz rule

\[ \nabla^{\mathcal{E}}(\lambda \wedge \alpha) = d\lambda \wedge \alpha + (-1)^{[\alpha]} \alpha \wedge \nabla^{\mathcal{E}} \alpha, \]

(57)

for all \( \lambda \in \Omega^*(\mathcal{M}) \) and \( \alpha \in \Omega^*(\mathcal{M}, \mathcal{E}) \). If in addition \( \mathcal{E} \) denotes a Clifford module bundle and the superconnection fulfils

\[ [\nabla^{\mathcal{E}}, c(a)] = c(\nabla^{\mathcal{C}I} a), \quad \forall a \in \Gamma(\mathcal{C}(\mathcal{M})), \]

(58)

it is called a Clifford superconnection (CSC), c.f. \[\square\]. In this case we have (c.f. loc. cit.)

\[ \nabla^{\mathcal{E}} \mapsto D^{\mathcal{E}} := c(\nabla^{\mathcal{E}}) \]

(59)

Here, \( \pm \) is understood with respect to the total grading.
is one to one. Note that because of the generalized Leibniz rule any superconnection locally
takes the form
\[ \nabla^E = d + \sum_{k=0}^{m} A[k], \]
\[ A[k] \in \left[ \Omega^k(\mathcal{M}, \text{End}(\mathcal{E})) \right]^-. \] (60)

In particular, in the case of a CSC
\[ A[1] = \omega_{\text{Cl}} \otimes 1_{\text{End}_{\text{Cl}}(\mathcal{E})} + 1_{\text{Cl}} \otimes A \]
\[ A[k] = \sum_{1 \leq i_1 < \cdots < i_k \leq m} e^{i_1} \wedge \cdots \wedge e^{i_k} \otimes 1_{\text{Cl}} \otimes B_{i_1 \cdots i_k}, \quad \forall k = 0, 2, \cdots, m \] (61)
where \( \omega_{\text{Cl}} \) denotes the induced Levi-Civita form on the Clifford bundle \( \mathcal{C}(\mathcal{M}) \) and \( A, B \in [\Omega^*(\mathcal{M}, \text{End}_{\text{Cl}}(\mathcal{E}))]^-. \) Again, \( \{e^i\}_{1 \leq i \leq m} \) is a local orthonormal frame in \( T^*\mathcal{M} \). Moreover, if \( \mathcal{M} \) denotes a spin-manifold, then any CSC is of the form (c.f. [11])
\[ \nabla^E = \nabla^S \otimes 1_E + 1_S \otimes \nabla^E. \] (62)

Clearly, the notion of a CSC completely parallels that of a Clifford connection and coincides
with the latter iff \( B \equiv 0 \). However, in general a CSC is not just defined by an element of
\( \Omega^1(\mathcal{M}, \text{End}_{\text{Cl}}^+(\mathcal{E})) \). This will be of crucial importance in what follows. Indeed, in [10] it
was shown how the combined Einstein-Hilbert-Yang-Mills (EHYM-) action functional can be
derived using non-SDO’s. Before we define a particular functional on \( \mathcal{D}(\mathcal{E}) \) we still give another

**Remark 6:** Let \( \tilde{D} \in \mathcal{D}(\mathcal{E}) \) be a Dirac operator on \( \mathcal{E} \) and \( \nabla^E \) the corresponding CSC. Then, we have
\[ \tilde{\nabla}^{\mathcal{E}} := \nabla^E + \omega, \quad \text{with} \]
\[ \omega := \omega_{\phi} + e^i \otimes \left( \sum_{i=1}^{m-1} \frac{1}{k!} \sum_{i_1, \ldots, i_k=1}^m c(e^{i_1}) \cdots c(e^{i_k}) \otimes B_{i_1 \cdots i_k} \right), \] (63)
where \( \omega_{\phi} := \delta_\xi \Phi, \Phi \equiv A[0] \in \Omega^0(\mathcal{M}, \text{End}^-(\mathcal{E})) \), so that
\[ \mathcal{D}_{\tilde{\nabla}^{\mathcal{E}}} = \tilde{D}. \] (64)

In particular, \( \tilde{\nabla}^{\mathcal{E}} \sim \tilde{\nabla}^{\mathcal{E}} \), where the latter is defined by (44). Note, the local decomposition of
the form \( \omega \in \Omega^1(\mathcal{M}, \text{End}^+(\mathcal{E})) \) may also contain a degree \([k] = 1 \) form.

After summarizing the notion of Clifford modules and generalized Dirac operators we have
also proved some lemmas, which permit an understanding of the relations between Dirac operators and connections on a Clifford module bundle. The reason to clarify these relations mainly

\footnote{Note, by abuse of notation we suppress the linear isomorphism (27).}
is motivated by the following

**Definition 6:** Let \((\mathcal{M}, g)\) be a closed, compact, orientable Riemannian manifold of even dimension \((m = 2n > 2)\) and without boundary. Also, let \((\mathcal{E}, c)\) be a Clifford module bundle over \(\mathcal{M}\) and let \(\mathcal{D}(\mathcal{E})\) be the affine space of all (generalized) Dirac operators compatible with the Clifford action \(c\) on \(\mathcal{E}\). Then, we introduce the functional

\[
\text{res} : \mathcal{D}(\mathcal{E}) \rightarrow \mathbb{C}
\]

\[
\tilde{D} \mapsto \text{res}(\tilde{D}^{-2n+2}).
\] (66)

Here, res means the Wodzicki residue, which in this case takes the explicit form (cf. [15], [16], [17], [18])

\[
\text{res}(\tilde{D}^{-2n+2}) = \frac{2}{\Gamma(n-1)} \int_{\mathcal{M}}^* \text{tr}_\mathcal{E} \left[ \frac{1}{6} r_{\mathcal{M}} - \mathcal{F}\tilde{\nabla}\mathcal{E} \right]
\] (67)

and where the endomorphism \(\mathcal{F}\tilde{\nabla}\mathcal{E} \in \Gamma(\text{End}(\mathcal{E}))\) is given by (47).

Since there exists a connection \(\tilde{\nabla}\mathcal{E} \in \mathcal{A}(\mathcal{E})\) for every \(\tilde{D} \in \mathcal{D}(\mathcal{E})\), so that \(\tilde{D} = \tilde{D}\tilde{\nabla}\mathcal{E}\), this functional may be interpreted, alternatively, as a functional defined on \(\mathcal{A}(\mathcal{E})\). However, we are interested less in the functional (66) itself than in the fact that for a given Dirac operator \(\tilde{D} \in \mathcal{D}(\mathcal{E})\) the Wodzicki residue of \(\tilde{D}^{-2n+2}\) can be considered as a functional

\[
\mathcal{I}_D(\tilde{\nabla}\mathcal{E}) := \text{res}(\tilde{D}^{-2n+2}),
\] (68)

of all connections \(\tilde{\nabla}\mathcal{E} \in \mathcal{A}(\mathcal{E})\) so that \(c(\tilde{\nabla}\mathcal{E}) = \tilde{D}\). In other words: with respect to a given Dirac operator (68) can be considered as a certain functional on the subspace of all (endomorphism valued) differential forms defining this Dirac operator. In the case that \(\tilde{D}\) denotes a SDO the functional (68) is proportional to the Einstein-Hilbert action. We again stress that this was recognized by Connes, c.f. [5] and was proved in [19]. From a more general point of view (see below) this was also discovered in [18], which in turn was the starting point to deal with non-SDO’s in [20] and [10].

There is still another motivation for (68); the connection of (68) to the heat trace associated with (the square of) a Dirac operator. For this let us remind that there is a natural functional on \(\mathcal{D}(\mathcal{E})\)

\[
\mathcal{D}(\mathcal{E}) \rightarrow \mathbb{C}
\]

\[
\tilde{D} \mapsto \text{Tr} e^{-\tau \tilde{D}^2}.
\] (69)

Though in general one is not able to calculate this functional, it is well-known that it has an asymptotic expansion:

\[
\text{Tr} e^{-\tau \tilde{D}^2} \sim (4\pi\tau)^{-n} \sum_{k \geq 0} \tau^{(k-2n)/2} \sigma_k(\tilde{D}^2),
\] (70)
where the coefficients (Seeley-DeWitt coefficients)
\[
\sigma_k(\tilde{D}^2) := \int_{\mathcal{M}} * \text{tr}_E \sigma_k(x; \tilde{D}^2)
\]  
(71)
are known to contain geometric information. In particular, the subleading term \(\sigma_2(\tilde{D}^2)\) is of the general form (c.f. [22]):
\[
\sigma_2(\tilde{D}^2) = \int_{\mathcal{M}} * \text{tr}_E \left[ \frac{1}{6} r_{\mathcal{M}} - \mathcal{F} \right]
\]  
(72)
and thus is proportional to (66). Of course, this is by no means accidental. In general, for all \((2n-k)/2 \notin \mathbb{Z}\) \((\mathcal{M} \text{ smooth})\) one has (c.f. [16])
\[
\sigma_k(\tilde{D}^2) = \Gamma((2n-k)/2) \text{ res}(\tilde{D}^{2n+k}).
\]  
(73)

From this point of view the statement of our main theorem (as given in the introduction) may be rephrased as follows:

*There exists a Hamiltonian (generalized Laplacian) \(\mathcal{H}\) such that the subleading term in the asymptotic expansion of the corresponding heat trace associated with this Hamiltonian is proportional to the classical bosonic action of the standard model with gravity including. Moreover, this Hamiltonian has a square root, \(\mathcal{H} = \tilde{D}^2\), which gives rise also to the fermionic action of the standard model.*

In what follows we assume that \(\mathcal{M}\) denotes a *spin* manifold\[^8^\]. Although this is not necessary, since we are only interested in local objects (densities), it simplifies notation. Consequently, the (total space of the) Clifford module bundle globally takes the form: \(\mathcal{E} = S \otimes E\). To get in touch with gauge theory we assume that \(E\) denotes an associated (hermitian) vector bundle: \(E = \mathcal{P} \times_\rho V\), where \(\mathcal{P}\) is a \(G\)-principal bundle over \(\mathcal{M}\) and \(\rho: G \to V\) is a unitary representation of the (real, compact and semi-simple) Lie-group \(G\) on a hermitian vector space \(V\). By \(\mathcal{G}\) we denote the corresponding Lie-algebra of \(G\) and by \(\rho'\) the induced representation of \(G\) on \(\text{End}(V)\). Consequently, any connection form \(A\) on \(E\) takes its values in \(\rho'(\mathcal{G}) \subset \text{End}(V)\). This offers the possibility of defining the slightly more general functional\[^9^\]:
\[
\mathcal{I}_D(\nabla E) := \text{res}_\zeta(\tilde{D}^{-2n+2})
\]  
(74)
Here, \(\zeta \in \Gamma(\text{End}(E))\) denotes an element of the commutant defined by \((G, \rho)\). More precisely. Let us recall the simple fact that any section \(s \in \Gamma(E)\) in an associated vector bundle \(E = \mathcal{P} \times_\rho V\) uniquely corresponds to an equivariant section \(\bar{s} \in \Gamma_{aq}(\mathcal{P}, V)\). Here, equivariant means: \(\bar{s}(pg) = \rho^{-1}(g)\bar{s}(p), \forall g \in G, p \in \mathcal{P}\). We then have the following

\[^8\]Indeed, it is widely believed by physicists that fermions are geometrically described by spinors.

\[^9\]That this functional actually is more adequate than (68) becomes clear when discussing the physical implications of the proposed model, c.f. part 2.
**Definition 7:** For a given generalized Dirac operator, consisting of the triple
\[(G, \rho, \tilde{D})\]
with \(\tilde{D} \in \mathcal{D}(\mathcal{E})\), let us denote by \(\bar{z} \in \Gamma_{aq}(\mathcal{P}, \text{End}(V))\) an element of the commutant
\[
\mathcal{C}_{\rho}(G) := \{ a \in \text{End}(V) \mid [\rho(g), a] = 0, \forall g \in G \}
\]
and by \(z \in \Gamma(\text{End}(E))\) its corresponding section in the endomorphism bundle associated with \(E\). Since \(\text{End}(S)\) is simple, this generalizes to \(\text{End}(\mathcal{E})\) via \(\zeta = \mathbf{1}_S \otimes z\). We impose the following three conditions on \(\zeta\): it is a positive operator \((\zeta > 0)\) and satisfies \([\tilde{D}, \zeta] = [\chi, \zeta] = 0\), where \(\chi \in \Gamma(\text{End}(\mathcal{E}))\) denotes the grading operator on \(\mathcal{E} = \mathcal{E}^+ \oplus \mathcal{E}^-\).

**Lemma 5:** As a consequence, \(\zeta^\alpha\) has a constant spectrum (i.e. it is independent of \(x \in \mathcal{M}\)) and the operator: \(\zeta^\alpha D^2\) is elliptic for any power \(\alpha \in \mathbb{R}\). Using this, we obtain
\[
\text{res}_\zeta(\tilde{D}^{-2n+2}) = \frac{2}{\Gamma(n-1)} \int_{\mathcal{M}} \ast tr_{\mathcal{E}} \left( \zeta \sigma_2(x; \tilde{d}^2) \right).
\]

**Proof:** Let us denote by \(\text{spec}_\zeta(x) = \{\lambda_1 \cdots \lambda_k\}|_x, x \in \mathcal{M}\) the spectrum of the positive operator \(\zeta \in \Gamma(\text{End}(\mathcal{E}))\). Here, \(k\) indicates the number of distinct eigenvalues of \(\zeta\). We first prove that the spectrum is independent of \(x \in \mathcal{M}\).
Since \(z \in \mathcal{C}_{\rho}(G)\) this section is gauge invariant. I.e. for all gauge transformations \(f \in \Gamma_{eq}(\text{Aut}(\mathcal{P})) \simeq \Gamma(\mathcal{P} \times_{\text{ad}} G)\) of \(\mathcal{P}\) we have: \(f^* z = z\). Hence, it is sufficient to consider a local situation. Using the fact that locally, any Dirac operator \(\tilde{D} \in \mathcal{D}(\mathcal{E})\) may be written as
\[
\tilde{D} = c(\mathrm{d}) + \sum_{j=0}^m c(A_{[j]})
\]
it follows that \([\tilde{D}, \zeta] = 0\) is equivalent to
\[
[c(A_{[j]}), \zeta] = 0, \quad \forall j = 0, 2, \cdots, m,
\]
\[
c(\mathrm{d}\zeta) = -[c(A_{[1]}), \zeta].
\]
However, the latter implies: \(\mathrm{d}z = 0\). Indeed, \(A_{[1]} = \omega^S \otimes 1_E + 1_S \otimes A, A \in \Omega^1(\mathcal{M}, \rho'(G))\) and thus \([A, z] = 0\). Consequently, \(\zeta\) must be constant, which yields the first assertion.

Since the spectrum of \(z\) is constant we get\(^{10}\)
\[
E = \bigoplus_{j=1}^k \ker(z - \lambda_j)
=: \bigoplus_{j=1}^k E_j.
\]

\(^{10}\)In the following we shall not distinguish between \(s\) and \(\bar{s}\).

\(^{11}\)I like to thank M. Lesch for explaining me this calculation.
Hence,

\[ \zeta = \sum_{j=1}^{k} \lambda_j \mathbf{1}_{\mathcal{E}_j}, \]
\[ D = \sum_{j=1}^{k} D_j, \]  

where \( \tilde{D}_j \) is the restriction of \( \tilde{D} \) to \( \mathcal{E}_j := S \otimes \mathbf{E}_j \). Consequently, we end up with

\[ \zeta \tilde{D}^2 = \sum_{j=1}^{k} \lambda_j \tilde{D}_j^2, \]  

which implies

\[ \text{Tr} e^{-\tau \zeta \tilde{D}^2} = \sum_{j=1}^{k} \text{Tr} e^{-\tau \lambda_j \tilde{D}_j^2} \]
\[ \sim \sum_{l \geq 0} \sum_{j=1}^{k} \frac{\tau^{(l-m)}}{l!} \lambda_j^{(l-m)/2} \int_{\mathcal{M}} \text{tr} \sigma_l(x, \tilde{D}_j^2), \]  

and thus proves the lemma, when \( \zeta \) is replaced by \( \zeta^{-n+1} \).

Since we now have fixed the mathematical frame we conclude this part by summarizing the proposed model building kit as follows: Let

\[ (G, \rho, \tilde{D}) \]  

be a given generalized Dirac operator defined on a Clifford module bundle \( \mathcal{E} \). Then, the general action functional on \( \mathcal{A}(\mathcal{E}) \times \mathcal{G}(\mathcal{E}) \) is defined as

\[ I_{\tilde{D}} \equiv I_{\text{fermionic}}(\tilde{\nabla}^{\mathcal{E}}, \psi) + I_{\text{bosonic}}(\tilde{\nabla}^{\mathcal{E}}) \]
\[ := (\psi, \tilde{D}\psi)_{\mathcal{G}(\mathcal{E})} + \text{res}_\zeta(\tilde{D}^{-2n+2}), \]  

where \( c(\tilde{\nabla}^{\mathcal{E}}) = \tilde{D} \).

## 3 Part 2: The Standard Model

In this part of the paper we are concerned with the application of the kit introduced in part 1 concerning the standard model of particle physics. We therefore shall introduce in the following section an appropriate generalization of the Dirac-Yukawa operator and prove our main theorem. Moreover, we shall discuss some consequences regarding the various parameters involved in the model.
3.1 The Dirac-Yukawa operator and the EHYMH-Action

To get started let us give the following

Definition 8: Let $\mathcal{E} := S \otimes E$ be a Clifford module bundle with a twisting graduation ($E = E_L \oplus E_R$) and denote by $\chi := \gamma_5 \otimes \chi^E$ the appropriate grading operator: $\chi^2 = 1_E$, $\chi^* = \chi$. A Dirac operator $D_\phi$ is called a (euclidean) Dirac-Yukawa operator if it takes the form

$$D_\phi := D + i \gamma_5 \otimes \begin{pmatrix} 0 & \tilde{\phi} \\ \phi^* & 0 \end{pmatrix},$$

$$\equiv D + \gamma_5 \otimes \phi$$

where $D$ is a SDO and $\tilde{\phi} \in \Gamma(\text{Hom}(E_R, E_L))$.

Since the Yukawa coupling (9) geometrically can be considered as defining a particular section $\phi$ (see below) one may try to naturally generalize the operator (86) in such a way that it not only defines the fermionic action (8-9) but also yields the bosoninc action (10-11). Here, ”naturally” means that the generalization of (86) is determined by those elements only which already determine the Dirac-Yukawa operator, i.e. by $(g, \phi, A)$.

Theorem 1: Let $\mathcal{E} = S \oplus E$ be a twisted Clifford module over $\mathcal{M}$, with $E := E_L \oplus E_R$. The functional (74) evaluated with respect to the Dirac operator

$$\tilde{D}_\phi := D + a_4 \Phi + \mathcal{J} \left( a_2 c(F^{E/S}) + a_3 c(\nabla^{\text{End}(E)} \Phi) + a_5 \Phi^2 \right)$$

defined on $\tilde{\mathcal{E}} := S \otimes \tilde{E}$ yields

$$\text{res}_\xi \left( \tilde{D}_\phi^{-2n+2} \right) = -\frac{\text{tr}(\tilde{E})}{31(n-1)} \int_{\mathcal{M}} \left\{ r_{\mathcal{M}} + a'_4 \text{tr}_E(z\phi^2) \right.$$  

$$- a'_2 \text{tr}_E(z F_{\mu\nu} F^{\mu\nu})$$  

$$+ a'_3 \text{tr}_E(z \nabla_{\mu} \phi \nabla_{\nu} \phi)$$  

$$+ a'_o \text{tr}_E(z \phi^4) \right\},$$

(88)

with

$$a'_o := \frac{24n(1 - \frac{1}{2n})}{\text{tr}_E z} a_o^2,$$

$$a'_2 := \frac{24(2n - 3)}{\text{tr}_E z} a_2^2,$$

$$a'_3 := \frac{24(n - 1)}{\text{tr}_E z} a_3^2,$$

$$a'_4 := \frac{12}{\text{tr}_E z} a_4^2.$$  

(89)

\footnote{Later we shall be mostly interested in the case $n = 2$.}
Here, $\Phi := \gamma_5 \otimes \phi$ and $F \in \Omega^2(\mathcal{M}, \rho'(\mathcal{G}))$ is the Yang-Mills curvature, induced by the gauge potential $A$ in the definition of the SDO $D \in D(\mathcal{E})$. The covariant derivative $\nabla \phi$ is defined with respect to the adjoint representation of $G$, where the $\phi$ sits in $\mathfrak{g}$. The $\alpha$'s denote arbitrary (complex) constants. The structure group $G$ is assumed to act on $\tilde{E} := E \oplus E$ via the representation $\tilde{\rho} := \rho \oplus \rho$, and the automorphism $\tilde{E} \to \tilde{E}$ denotes the corresponding ”complex structure” on $\tilde{E}$. Moreover, we have used the canonical identification $A + \mathcal{J}(B) = \begin{pmatrix} A & -B \\ B & A \end{pmatrix}$ (90) for all $A, B \in \text{End}(E)$. Note, in what follows we do not distinguish between the Clifford action $c$ on the Clifford module $\mathcal{E}$ and the corresponding action $\tilde{c} := c \oplus c$ on the (canonically) induced Clifford module $\tilde{\mathcal{E}}$. Likewise, we do not distinguish between the representation $\rho$ and $\tilde{\rho}$.

Consequently, if the constant $a_3$ is purely imaginary and the other constants are real, the functional $\text{(74)}$ with respect to the generalized Dirac-Yukawa operator $\text{(87)}$, becomes proportional to the Einstein-Hilbert-Yang-Mills-Higgs action (EHYMH) of the standard model. Moreover, if one considers ”diagonal sections”: $\tilde{\psi} := (\psi, \psi), \psi \in \Gamma(\mathcal{E})$ only, the fermionic functional in $\text{(85)}$ becomes proportional to the Dirac-Yukawa action. Note, in (85) there is still a length scale missing because the fields do not yet have the right dimensions. This will be discussed in the next section where we shall consider some constraints of our approach to the standard model.

Remark 7: The (euclidean) Dirac-Yukawa operator $\text{(86)}$, is uniquely defined by the Clifford superconnection

$$\nabla^E := \nabla^E + \epsilon,$$
$$\epsilon := i^n \ast \phi$$

(91) on $E$, where we have used the fact that the grading operator $\gamma_5$ of the spinor bundle $S$ is proportional to the volume form on $\mathcal{M}$. More precisely, we have $c(i^n \ast 1) = \gamma_5 \otimes 1_E \in \Gamma(\text{End}^+(\mathcal{E}))$.

Hence, the (euclidean) Dirac-Yukawa operator is determined by a 1-form (gauge potential) and the $2n$-form in (91). In four dimensions, however, the most general Dirac operator in addition depends on a zero form, a two form and a three form. By considering only those forms which already determine the Dirac-Yukawa operator (91) naturally yields the following ansatz

$$\nabla^{\tilde{E}} := \nabla^E + a_4 \epsilon + \mathcal{J} \left( a_2 F - a_3 \ast \nabla^{\text{End}(E)}(\ast \epsilon) + a_o (\ast \epsilon)^2 \right)$$

(92) on $\tilde{E}$. Note that the forms within the brackets are even with respect to the total degree and hence $\mathcal{J}(\cdot \cdot \cdot)$ becomes odd (c.f. also the final section.). It is easily checked that

$$\tilde{D}_{\nabla^{\tilde{E}}} = \tilde{D}_{\phi}$$

(93)
with $\nabla^\xi := \nabla^S \otimes 1_E + 1_S \otimes \nabla^E$.

There are two interesting choices for the constants $a_k$ so that (87) takes a particularly nice geometric form: First, the most natural choice is $a_k \equiv 1$, $\forall k = 0, 2 \cdots 4$. In this case the Dirac operator (87) reads

$$
\tilde{D}_\phi = c(\nabla^E + \Phi) + J \left( c(\mathcal{R}_E^E) \right)
=: c(\tilde{\nabla}^E) + J \left( c(\mathcal{R}_E^E) \right)
= D_\phi + J \left( c(\mathcal{R}_E^E) \right)
\equiv D_{\tilde{\nabla}^E},
$$

(94)

with the ”super relative curvature” $\mathcal{R}_E^E := (\tilde{\nabla}^E)^2 - (\nabla^S)^2 \otimes 1_E$.

However, there is still another nice choice: $a_o = a_3 := (1 - \frac{1}{2n})$, $a_2 = a_4 := 1$; in this case the generalization (87) of the Dirac-Yukawa operator (86) reads

$$
\tilde{D}_\phi = c(\nabla^E + \omega_\phi) + J \left( c(F_E^E + dF_\omega^E + \omega_\phi \wedge \omega_\phi) \right)
=: c(\tilde{\nabla}^E) + J \left( c(\mathcal{R}_E^E) \right)
= D_\phi + J \left( c(\mathcal{R}_E^E) \right)
\equiv D_{\tilde{\nabla}^E},
$$

(95)

with the ”Higgs-form” $\omega_\phi := \delta_\xi \Phi \in \Omega^1(M, \text{End}^+(E))$ and the relative curvature $\mathcal{R}_E^E := (\tilde{\nabla}^E)^2 - (\nabla^S)^2 \otimes 1_E$. Because of

$$
D_{\tilde{\nabla}^E} - D_{\tilde{\nabla}^E} = \frac{1}{2n} J \left( c(\mathcal{R}_E^E - F_E^E) \right),
$$

(96)

the Dirac operators $D_{\tilde{\nabla}^E}$, $D_{\tilde{\nabla}^E}$, however, are different. Therefore, strictly speaking, the definition (87) gives a whole class of Dirac operators, parametrized by the constants $a_o, \ldots, a_4$, and which all yielding (88).

After this remark let us turn to the proof of the theorem.

**Proof:** Clearly, to prove this theorem we just have to calculate the endomorphism (47) with respect to the Dirac operator (87). This tedious but straightforward calculation can most easily be achieved using the local form of (47):

$$
\mathcal{F}^E_{\xi} = \frac{1}{4} r_M 1_\xi + \frac{1}{2} \gamma^{\mu\nu} \otimes F_{\mu\nu} + \frac{1}{2} [\gamma^{\mu}[\omega_\mu, \gamma^\nu], \omega_\nu] + \gamma^{\mu\nu}(\nabla_\mu \omega_\nu) - \frac{1}{2} \gamma^{\mu}(\nabla_\nu \omega_\mu, \gamma^\nu)
+ \frac{1}{2} \gamma^{\mu\nu}[\omega_\mu, \omega_\nu] + \frac{1}{2} g_{\mu\nu}\gamma^\alpha[\omega_\alpha, \gamma^\mu][\omega_\beta, \gamma^\nu],
$$

(97)
where $\mathcal{C}(\mathcal{M})_C \rightarrow \text{End}(S)$ denotes the (chiral) representation of the complexified Clifford algebra and $\gamma^\mu \equiv c\gamma(e^\mu)$ with $\{e^\mu\}_{1 \leq \mu \leq 2n}$ a local frame in $T^*\mathcal{M}$. Also we use the shorthand notation $\gamma^{\mu \nu} := [\gamma^\mu, \gamma^\nu]/2$. Moreover, $F$ is the curvature on $E$, associated to the connection form $A$. The connection $\tilde{\nabla}^\xi \in \mathcal{A}(\tilde{\mathcal{E}})$ denotes any representative of the corresponding class defining the Dirac operator (87) and $\omega := \tilde{\nabla}^\xi - \tilde{\nabla}^\xi$, with $\nabla^\xi \in \mathcal{A}_{\text{Cl}}(\tilde{\mathcal{E}})$ arbitrarily chosen, and $'\nabla_\mu$ denotes the covariant derivative induced by this Clifford connection on $\text{End}(\tilde{\mathcal{E}})$.

In particular, we may choose $\tilde{\nabla}^\xi$ such that

$$
\omega := \omega^0 + \omega^2 + \omega^3 + \omega^4 \quad \text{with}
$$

$$
\omega^0 := a_o \delta_\xi \mathcal{J}(\Phi^2),
\omega^2 := a_2 e^\mu \otimes \gamma^\nu \otimes \mathcal{J}(F_{\mu \nu}),
\omega^3 := a_3 \mathcal{J}(\nabla\Phi),
\omega^4 := a_4 \delta_\xi(\Phi).
$$

(98)

Note, we have already omitted $\omega^1 := \omega^S \otimes 1_E + 1_S \otimes A$, which would change the Clifford connection $\nabla^\xi$ only and not the functional (88) by corollary 1. The main advantage of the local form (97) is that the whole calculation becomes purely algorithmic. Moreover, because of the trace $\text{tr}_\xi$ in (77) one only has to calculate the last four terms in (97). For the same reason most of the terms to be calculated will drop out. Indeed, using (98), it is easily checked that the two derivative terms in (97), actually, do not contribute to the functional (74). Hence, all it is left to be done is to calculate the last two quadratic terms in (97) up to "traceless" contributions. This calculation becomes even more simplified by the

**Remark 9:** Let $\mathcal{E}_j$, $j = 1, 2$ be two Clifford modules over $\mathcal{M}$ with the appropriate actions $c_j$. Then, $(\mathcal{E}, c)$ with $\mathcal{E} := \mathcal{E}_1 \oplus \mathcal{E}_1$ and $c := c_1 \oplus c_2$ is also a Clifford module. The most general Dirac operator on this Clifford module takes the form

$$
\tilde{D} = \begin{pmatrix}
\tilde{D}_1 & A_{12} \\
A_{21} & \tilde{D}_2
\end{pmatrix},
$$

(99)

where, respectively, $\tilde{D}_j \in \mathcal{D}(\mathcal{E}_j)$, $j = 1, 2$, $A_{12} \in \Gamma(\text{Hom}(\mathcal{E}_2, \mathcal{E}_1))$ and $A_{21} \in \Gamma(\text{Hom}(\mathcal{E}_1, \mathcal{E}_2))$. The Wodzicki residue of $\tilde{D}^{-2n+2}$ is such that there are no terms "mixing the diagonal with the off-diagonal", c.f. [10].

As a consequence, in calculating the corresponding quadratic terms of (97), products of the generic form $\omega^k \omega^k$, $k = 0, \cdots, 3$ may be omitted as well. As a result we end up with

$$
\frac{1}{2} \gamma^{\mu \nu} [\omega_\mu, \omega_\nu] = a_o^2 (1 - \frac{1}{2n}) \Phi^4
+ a_2^2 \text{ev}_g \left( F^\xi/S^2 \right)
+ a_4^2 (1 - \frac{1}{2n}) \Phi^2
$$

(100)
and
\[ \frac{1}{4} g_{\mu\nu} \gamma^\alpha [\omega_\alpha, \gamma^\mu] \gamma^\beta [\omega_\beta, \gamma^\nu] = 2n a_0^2 \left( 1 - \frac{1}{2n} \right)^2 \Phi^4 + 5n a_0^2 \operatorname{ev}_g \left( \frac{F^E}{S^2} \right) + 2(n - 1) a_0^2 \operatorname{ev}_g \left( \left( \nabla_{\text{End}(E)} \Phi \right)^2 \right) + \frac{1}{2n} a_0^2 \Phi^2. \]

(101)

Note that both equalities hold up only to traceless terms! Finally, if we put all together, the theorem is proven.

To summarize: we have introduced a certain Dirac operator (87), generalizing the Dirac-Yukawa operator (86), such that the functional (74) looks like that of the bosonic action of the standard model with gravity including. This result is independent of the specific structure group G and its appropriate (fermionic) representation. To get in touch with physics, however, we also have to specify the pair (G, \rho). In other words, we have to define the generalized Dirac operator of the standard model.

### 3.2 The generalized Dirac-Yukawa operator of the standard model

To begin with, we give the following

**Definition 9:** Let \( E \) be a twisted spinor bundle, so that \( E = E_L \oplus E_R \). We call

\[ (G, \rho, D) \]

(102)

the \textit{generalized Dirac-Yukawa operator of the standard model} provided the structure group \( G \) takes the form

\[ G := SU(3) \times SU(2) \times U(1) \]

(103)

and has the (fermionic) representation \( \rho := \rho_L \oplus \rho_R : G \to \text{Aut}(V) \),

\[ \rho_L(c, w, b) := \begin{pmatrix} c \otimes 1_N \otimes w & b_L^q \otimes 0 \\ 0 & 1_N \otimes w \otimes b_L^q \end{pmatrix}, \]

(104)

\[ \rho_R(c, w, b) := \begin{pmatrix} c \otimes 1_N \otimes B_R^q & 0 \\ 0 & B_R^t \end{pmatrix}, \]

(105)

on the typical fiber

\[ V := V_L \oplus V_R \]

(106)
\[ \simeq \left[ (C^6N \oplus C^2N) \right]_L \oplus \left[ (C^3N \oplus C^3N) \oplus C^1N \right]_R \]  
\[ \simeq C^8N_L \oplus C^7N_R. \]  

Moreover, the Dirac operator
\[ D := \tilde{D}_\phi \]  
is the generalized Dirac-Yukawa operator (87), where the homomorphism \( \tilde{\phi} \in \Gamma(\text{Hom}(E_R, E_L)) \) is defined by
\[ \tilde{\phi} := \begin{pmatrix} 1_3 \otimes (g^q_y \otimes \varphi, g^q_y \otimes \epsilon \bar{\varphi}) & 0 \\ 0 & g^l_y \otimes \varphi \end{pmatrix} \]  
\[ \equiv \begin{pmatrix} 1_3 \otimes \bar{\varphi}_q & 0 \\ 0 & \bar{\varphi}_l \end{pmatrix}. \]

Here, respectively, \( g^q_y, g^l_y \in M_N(\mathbb{C}) \) denote the matrices of the Yukawa coupling constants for quarks of electrical charge -1/3 and 2/3 (i.e., of quarks of "d"-type, and of "u"-type) and \( g^l_y \in M_N(\mathbb{C}) \) is the matrix of the Yukawa coupling constants for the leptons of charge -1 (i.e. of leptons of "electron" type). While \( g^q_y \) and \( g^l_y \) can be assumed to be diagonal and real, the matrix \( g^q_y \) is related to the Kobayashi-Maskawa matrix and therefore is neither diagonal nor real. The "weak hypercharges" for the left and right handed quarks (indicated by the subscript "q") and leptons (subscript "l") are defined by: \( \rho(b) := e^{iy \theta}, b \in U(1), y \in \mathbb{Q}, \theta \in [0, 2\pi] \) (c.f. the introduction). Then the two by two diagonal matrices \( B^q_R \) and \( B^l_R \) in the definition (105) are: \( B^q_R := \text{diag}(b^q_R, b^q_R) \) and \( B^l_R := b^l_R 1_N \). In (110), \( \varphi \) denotes a section into a rank two sub-bundle \( E_h \) of the vector bundle \( E \) and carries the defining representation \( \rho_h \) of the electroweak subgroup \( G_h := SU(2) \times U(1) \) of the structure group \( G \). I.e., when \( \varphi \) is considered as an element of \( \Gamma(\text{eq}(\mathcal{P}, V)) \) it transforms like \( \varphi(pg) = \rho_h(g)^* \varphi(p) \) with \( g := (w, b) \in G_h = SU(2) \times U(1) \) and \( \rho_h(w, b) := w e^{iyh \theta} \). Finally, \( \epsilon \) is the anti-diagonal matrix \( \epsilon := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \) and \( \bar{\varphi} \) here means the complex conjugate of \( \varphi \).

Hence, the Higgs field \( \varphi \) defines an element \( \Phi \) in \( \Omega^0(\mathcal{M}, \text{End}^{-1}(\mathcal{E})) \) and thus a Dirac-Yukawa operator (86). Of course, the particular form of \( \tilde{\phi} \) in (110) is such that \( \Phi \) gives the correct Yukawa coupling term (9) in the definition of the fermionic action of the standard model. However, since the section \( \phi \in \Gamma(\text{End}^{-1}(E)) \) transforms with respect to the \( (\rho \text{-induced}) \) adjoint representation of \( G \) it follows that the hypercharge \( y_h \) of the Higgs field and, respectively, the hypercharges \( y^q_L, y^q_R, y^u_R \) of the left and right handed leptons and the hypercharges \( y^d_L, y^d_R \) of the left and right handed quarks must satisfy the following well-known relations
\[ y_h = y^q_L - y^q_R = y^u_L - y^u_R. \]
\[
= y_R^u - y_L^q.
\]

In other words, when the Yukawa coupling is of the form as defined by (9) the corresponding hypercharges are not all independent but have to satisfy the relations (112). Moreover, since we know the electrical charge of the particles, the numerical values of the "y's" are fixed:

\[
(y_L^l, y_L^l) = (1/6, -1/2),
\]

\[
((y_R^d, y_R^u), y_R^l) = ((-1/3, 2/3), -1).
\]

This is a consequence of the generalized Gell-Mann-Nishijima relation

\[
Q = T_3 + Y,
\]

where \(\{iT_3 := \rho'((i\tau_3)/2), iY := \rho'(i)\}\) is a basis of the maximal Cartan subalgebra of \(\rho'(\text{su}(2) \oplus \text{u}(1))\), so that \(iQ\) generates the residual structure group - in more physical terms the "electromagnetic gauge group"

\[
U_{\text{elm}}(1) \subset \text{SU}(2) \times \text{U}(1)
\]

in the fermionic representation \(\rho\) after the mechanism of \textit{spontaneous symmetry breaking} is established. To make the latter more precise mathematically, let us remember how the notion of spontaneous symmetry breaking can be geometrically rephrased in terms of the reduction of a G-principal bundle, c.f. [23], [24].

Let therefore \(H \subset G\) be a Lie-subgroup of \(G\) and, respectively, \(P_G\) and \(P_H\) be the (total spaces of the) corresponding principal bundles over the same base manifold \(\mathcal{M}\). Then, \(P_H\) is called an H-reduction of \(P_G\) iff \(P_H \subset P_G\) is a submanifold, so that the injection \(P_H \hookrightarrow P_G\) is a bundle homomorphism. A necessary and sufficient condition for a G-principal bundle \(P\) to be H-reducible is that the \(P\)-associated fiber bundle \(P_G/H \to \mathcal{M}\), with typical fiber \(G/H\), admits a global section.

\textbf{Definition 10:} A solution \((g, A, \varphi, \Psi)\) of the Euler-Lagrange equations corresponding to the functional (12) is called a classical vacuum iff \(g\) is flat, \(A = \Psi = 0\) and the Higgs field \(\varphi = \varphi_o\) minimizes the Higgs potential

\[
V : \Gamma(E_h) \longrightarrow \mathbb{R},
\]

\[
V'|_{\varphi=\varphi_o} = 0. \quad (117)
\]

Here, \(E_h\) denotes the (total space of the) bundle where the Higgs field lives in (see below) and, in principle, \(V\) may be any gauge invariant polynomial of order less or equal then four of the

\textsuperscript{14}As usual we use \(\{E_k\}_{1 \leq k \leq 12} := \{(i\lambda_a)/2, (i\tau_b)/2, i\}_{1 \leq a \leq 8, 1 \leq b \leq 3}\) as a basis of \(\text{su}(3) \oplus \text{su}(2) \oplus \text{u}(1)\), where \(\lambda_a, a = 1 \ldots 8\) denote the Gell-Mann matrices and \(\tau_b, b = 1 \ldots 3\) the Pauli matrices.
Higgs field $\varphi$. Again, it is assumed that $V$ is bounded from below. Of course, concerning the standard model, $V$ has its well-known fashion (6).

Then, let us denote by $\Sigma \subset V_h$ the (disjoint union of) orbits of classical vacuums with respect to the representation $\rho_h$ that carries the Higgs field $\varphi$ of the structure group $G_h$. In the special case of the standard model this group is identified with the "electroweak gauge group" $SU(2) \times U(1) \subset G$ and, correspondingly, $E_h := \mathcal{P} \times_{\rho_h} V_h \subset E$ denotes the rank two subbundle describing, geometrically, the Higgs sector of the standard model. In the so-called "minimal version" of the standard model the typical fiber is $V_h \simeq \mathbb{C}^2$. Note that in the usual (non-geometrical) description of the standard model the fermion representation $\rho$, defined by (104-108), and the Higgs representation $\rho_h$ are completely independent. Let us denote by $I(\varphi_o) \subset \rho_h(G_h) \subset \text{End}(V_h)$ the isotropy group associated with a chosen classical vacuum $\varphi_o \in \Sigma' \subset \Sigma$, connected. Up to conjugation this isotropy group can be identified with some Lie-subgroup $H$ of $G_h$ and we have $\Sigma' \simeq (\rho_h(G_h)/\rho_h(H))$. Therefore, $\varphi_o$ considered as an element in $\Gamma_{eq}(\mathcal{P}_h, V_h)$ uniquely induces a section (also denoted by $\varphi_o$) $M^{\varphi_o} \mathcal{P}_h/H$. In other words, from a geometrical point of view a necessary condition for a (classical) vacuum to exist is that the $G_h-$bundle must be $H$-reducible, where the Lie-subgroup $H \subset G_h$ is identified with the isotropy group of some chosen Higgs field $\varphi_o$, minimizing the Higgs potential $V$.

**Definition 11:** The gauge symmetry is called spontaneously broken by the (classical) vacuum, represented by $\varphi_o$, iff $H \subset G_h$ is a proper subgroup.

Note that the $H$-reduction of $\mathcal{P}_h$ is only necessary but not sufficient for $\varphi_o$ to represent a classical vacuum. It is also required that $\mathcal{P}_h$ possesses a flat connection. In the particular case of the (minimal) standard model we obtain: $\Sigma = \{ \rho_h(\mathbf{w}, b)\varphi_o \} \coprod \{ 0 \}$, with $\varphi_o := (0, v/\sqrt{2})^T$, $v := \sqrt{\mu^2/\lambda} \in \mathbb{R}$ and $(\mathbf{w}, b) \in G_h = SU(2) \times U(1)$. Then, the isotropy group associated with the non-trivial $\varphi_o$ is generated by the anti-hermitian operator $iQ$, and the corresponding residual structure group $H$ can be identified with $U_{elm}(1)$, c.f. (115-116). Actually, the full residual structure group ("little group") of the standard model reads

$$H = SU(3) \times U_{elm}(1),$$

since the structure group $G$ is given by (103).

Consequently, assuming that $\mathcal{P} \equiv \mathcal{P}_G$ is $H$-reducible and $\varphi_o$ represents an appropriate (non-trivial) classical vacuum of the standard model the section $D \in \Omega^0(M, \text{End}^-(E))$, defined by

$$D \equiv \varphi_o$$

$$:= \phi|_{\varphi=\varphi_o}$$

$$=: i \begin{pmatrix} 0 & M \\ M^* & 0 \end{pmatrix},$$

(119)
is H-invariant and thus well-defined on the reduced bundle. Fixing the gauge so that \( \varphi_o = (0,v/\sqrt{2})^T \) ("unitary gauge") we may further write

\[
M \equiv \begin{pmatrix} 1_3 \otimes M_q & 0 \\ 0 & M_l \end{pmatrix}, \text{ with }
\]

\[
M_q = \begin{pmatrix} 0 & m^q_d \\ m^u & 0 \end{pmatrix}, \quad (120)
\]

\[
M_l = \begin{pmatrix} 0 \\ m^l \end{pmatrix}, \quad (121)
\]

where, respectively, the matrices \( m^l := \frac{v}{\sqrt{2}} g^l_y \in M_N(\mathbb{C}) \) and \( m^u := \frac{v}{\sqrt{2}} g^q_y \in M_N(\mathbb{C}) \) denote the "mass matrices" of the charged leptons (l) and quarks (q) of "u-type". They can be assumed to be diagonal and real. The corresponding \( N \times N \) matrix \( m^q_d := \frac{u}{\sqrt{2}} g^q_y \) of "d-type" quarks is neither diagonal nor real. It is related to the mass matrix of "d-type" quarks \( m^q_d = \text{diag}(m^q_{d1}, \ldots, m^q_{dN}) \), \( m^q_{dk} \in \mathbb{R}, k = 1, \ldots, N \) via the Kobayashi-Maskawa matrix \( V \in U(N) \):

\[
m^q_d = V m^d V^*.
\]

Obviously, \( M \) denotes the fermionic mass matrix and we have recovered the "internal Dirac operator" \( D \) of the Connes-Lott approach to the standard model, c.f. \[3\] and the corresponding references therein. Again, we stress that \( D \) minimizes the Higgs potential \( V \) but represents a classical vacuum only if \( P_H \) (and thus \( P \)) possesses a flat connection, c.f. \[24\].

As we have already mentioned, in the usual approach to the standard model the representation \( \rho_h \) of the Higgs field \( \varphi \) is independent of the fermionic representation \( \rho_f \) as defined by (104-108). However, to be consistent one has to impose the relations (112) to the hypercharges. That these relations are not accidentally and, actually, must not be chosen by hand follows from the Dirac-Yukawa operator of the standard model:

\[
(G, \rho, D_\phi), \quad (122)
\]

with \( (G, \rho \equiv \rho_f) \) like in (103-108) and \( D_\phi \) defined by (110). In other words: whenever one starts with (122) the representation of the Higgs field must be contained in the fermionic representation and then the relations between the hypercharge of the Higgs field and those of the corresponding fermions are fixed. If this does not hold, the Yukawa-coupling (9) would not define a generalized Dirac-operator.

We now turn to the consequences as implied by the generalization \( (G, \rho, \bar{D}_\phi) \) of (122). We therefore compare the functional (88), derived from the generalized Dirac-Yukawa operator (87), with the corresponding bosonic action of the standard model. Before we can do this, however, we still have to give the various fields involved in the model their right dimensions, i.e., we first have to introduce an arbitrary length scale "\( l \)". Also, we may choose to introduce a second endomorphism \( \phi' \) on \( E \) in order to define the "off-diagonal" of (87). This additional endomorphism is defined by (110), but with the Yukawa-coupling matrices, \( g^q_y, g^q_y, g^l_y \) replaced...
by arbitrary matrices \( A^q, A^q, A^l \) of the corresponding size. This freedom arises from the fact that
the form of the functional (88) does not change by this replacement and that the off-diagonal
of \( \bar{D}_\phi \) does not act on the fermions \( \Psi \), giving the matrices \( g_y \) their physical interpretation\(^{15}\). Note that this freedom is crucial for the definition of the mass matrix of the gauge bosons, see
below. Moreover, without loss of generality we may assume that \( \phi' \) is hermitian, so that all the
constants "\( a \)" are real, c.f. (89). With these replacements the "bosonic part" of the universal
functional (85) reads

\[
I_{\text{bosonic}} = I_{EH} + \alpha_2 \int_M \text{tr}_E(z F \wedge \ast F) \tag{123}
\]

\[
+ \alpha_3 \int_M \text{tr}_E(z (\nabla \phi')^* \wedge \ast(\nabla \phi')) \tag{124}
\]

\[
- \alpha_4 \int_M \ast \text{tr}_E(z \phi \ast \phi) \tag{125}
\]

\[
+ \alpha_o \int_M \ast \text{tr}_E(z (\phi \ast \phi')^2) \tag{126}
\]

with the constants

\[
\alpha_o := \frac{3n(1 - \frac{1}{2n})}{2\pi \text{tr}_E z} \left( \frac{l}{l_p} \right)^2 a_o^2, \tag{127}
\]

\[
\alpha_2 := \frac{3(2n - 3)}{\pi \text{tr}_E z} \left( \frac{l}{l_p} \right)^2 a_2^2, \tag{128}
\]

\[
\alpha_3 := \frac{3(n - 1)}{2\pi \text{tr}_E z} \left( \frac{l}{l_p} \right)^2 a_3^2, \tag{129}
\]

\[
\alpha_4 := \frac{3}{4\pi \text{tr}_E z} m_p^2 a_4^2. \tag{130}
\]

Here, \( z \) is considered as an element of \( \Gamma_{eq}(P, \text{End}(V)) \) that lies in the commutant (76) and
satisfies:

\[
[z, \chi^E] = [z, \phi] = [z, \phi'] = 0, \tag{131}
\]

as well as \( z > 0 \).

Since we use our units so that \( c = \hbar = 1 \) we have identified Newtons gravitational constant
\( G \) with the (square of the) "Planck length" \( l_p \equiv m_p^{-1} \). Moreover, we already have normalized
the Einstein-Hilbert functional \( I_{EH} \) so that

\[
I_{EH} = \frac{1}{16\pi l_p^2} \int_M \ast r_M. \tag{132}
\]

\(^{15}\)At least, when one restricts oneself to diagonal sections: \( \Psi = (\Psi, \Psi) \in \Gamma(\tilde{E}) \). The more general
case of non-diagonal sections will be discussed in a future paper, when also further phenomenological
consequences of our model and a physical interpretation of the doubling of the "internal freedoms":
\( E \to \tilde{E} = E \oplus E \) are considered, c.f. our remarks at the end of this paper.
The corresponding normalized fermionic action reads\[16\]

\[I_{\text{fermionic}} = \int_{\mathcal{M}} *(\Psi, iD\Psi)\epsilon\] (133)

\[a_4 \int_{\mathcal{M}} *(\Psi, i\Phi\Psi)\epsilon,\] (134)

with \(\Phi\) defined by (110) and \(D\) a SDO. Note that only the constant \(a_4\) carries a dimension.

We now can compare the derived functionals (123-126) and (133-134) with the corresponding bosonic and fermionic action functionals of the standard model

\[I_{EHYM} := \frac{1}{16\pi l^2} \int_{\mathcal{M}} *r_{\mathcal{M}}\] (135)

\[+ \int_{\mathcal{M}} *\left\{ \frac{1}{4} \text{tr}(C_{\mu\nu}C^{\mu\nu}) + \frac{1}{2} \text{tr}(W_{\mu\nu}W^{\mu\nu}) + \frac{1}{4} B_{\mu\nu}B^{\mu\nu} \right\}\] (136)

\[+ \int_{\mathcal{M}} *(\nabla_\mu \varphi)^*(\nabla^\mu \varphi)\] (137)

\[+ \int_{\mathcal{M}} \left[ \lambda (\varphi^*\varphi)^2 - \mu^2 \varphi^*\varphi \right]\] (138)

and

\[I_{DY} = \int_{\mathcal{M}} *(\Psi, i\gamma^\mu \nabla_\mu \Psi)\epsilon\] (139)

\[+ \int_{\mathcal{M}} *(\Psi, i\Phi\Psi)\epsilon.\] (140)

Here, respectively,

\[C_{\mu\nu} := \partial_\mu C_\nu - \partial_\nu C_\mu + ig(3) [C_\mu, C_\nu],\]

\[W_{\mu\nu} := \partial_\mu W_\nu - \partial_\nu W_\mu + ig(2) [W_\mu, W_\nu],\]

\[B_{\mu\nu} := \partial_\mu B_\nu - \partial_\nu B_\mu\] (141)

denote the \text{su}(3), \text{su}(2) and \text{u}(1) valued curvatures with respect to a local coordinate system and in the fundamental representation. Moreover, the covariant derivatives, acting on the Higgs field \(\varphi\) and on the fermions \(\Psi\), are locally defined as

\[\nabla_\mu \varphi := (\partial_\mu + ig(2) W^b_\mu \frac{\tau_b}{2} + ig(1)y_h B_\mu)\varphi,\] (142)

\[\nabla_\mu \Psi := (\partial_\mu + ig(3) C^a_\mu 1_S \otimes F_a + ig(2) W^b_\mu 1_S \otimes T_b + ig(1) B_\mu 1_S \otimes Y)\Psi,\] (143)

where again \((g(3), g(2), g(1))\) are the coupling constants and \(\{iF_a, iT_b, iY\}_{1\leq a \leq 8, 1\leq b \leq 3}\) denote the generators of \(\rho(\text{SU}(3) \times \text{SU}(2) \times \text{U}(1))\); \(\lambda, \mu^2 > 0\) are the positive real constants, parametrizing the classical vacuum. Note that the curvatures (141) are hermitian.

\[^{16}\text{Again, when restricted to diagonal sections.}\]
Lemma 6: The derived functionals (123-126) and (133-134) are identical with the bosonic and fermionic action functionals of the standard model (135-140), iff the following relations hold:

\[ N_2 g^2_{(1)} = \frac{1}{2(3y_q\lambda_q + y_l\lambda_l)}, \]  
\[ N_2 g^2_{(2)} = \frac{1}{(3\lambda_q + \lambda_l)}, \]  
\[ N_2 g^2_{(3)} = \frac{1}{4\lambda_q}, \]  
and

\[ 2\alpha_0 \left[ 3\lambda_q \text{tr}(\Lambda_q^* \Lambda_q)^2 + \lambda_l \text{tr}(\Lambda_l^* \Lambda_l)^2 \right] = \lambda, \]  
\[ 2\alpha_3 \left[ 3\lambda_q \text{tr}\Lambda_q^* \Lambda_q + \lambda_l \text{tr}\Lambda_l^* \Lambda_l \right] = 1, \]  
\[ 2\alpha_4 \left[ 3\lambda_q \text{tr} g_q^* g_q + \lambda_l \text{tr} g_l^* g_l \right] = \mu^2, \]  
\[ a_4 = 1 \]  
with the abbreviations

\[ y_q := 2(y_L^q)^2 + (y_R^d)^2 + (y_R^d)^2 \in \mathbb{R}_+, \]  
\[ y_l := 2(y_L^l)^2 + (y_R^l)^2 \in \mathbb{R}_+, \]  
\[ \Lambda_q := (\Lambda_q^q, \Lambda_q^q) \in M_{2N\times N}(\mathbb{C}), \]  
\[ g_q := (g_q^q, g_q^q) \in M_{2N\times N}(\mathbb{C}), \]  
\[ \Lambda_l := \Lambda_l^l \in M_N(\mathbb{C}), \]  
\[ g_l := g_l^l \in M_N(\mathbb{R}). \]  

Proof: The proof of this lemma mainly consists in determining \( z \) of the commutant, yielding

\[ z = \begin{pmatrix} z_L & 0 \\ 0 & z_R \end{pmatrix}, \]  
\[ z_L := \begin{pmatrix} \lambda_q 1_{6N} & 0 \\ 0 & \lambda_l 1_{2N} \end{pmatrix}, \]  
\[ z_R := \begin{pmatrix} \lambda_q 1_{6N} & 0 \\ 0 & \lambda_l 1_{N} \end{pmatrix}, \]  
with \( \lambda_q, \lambda_l \in \mathbb{R}_+ \). Then, rewriting the functionals (123-126) into the form (135-140) gives the desired relations. Clearly, the relation for \( a_4 \) simply follows by direct comparison of the corresponding fermionic functionals.
Remark 10: Note that we still have one more free parameter, \( \alpha \) say, to introduce in our model kit and to write instead of (85)

\[
I_D = (\psi, \tilde{D}\psi)_{\Gamma(E)} + \alpha \text{ res}_\zeta (\tilde{D}^{-2n+2}).
\]

Of course, this additional parameter indicates that in the definition of the universal action (2) the fermionic and the bosonic action functionals are considered as independently of each other. Alternatively, one may put \( \alpha \equiv 1 \) and then rescale both functionals independently (although we only have one over all constant!). This is what we did, actually, to obtain (132) and (133-134). Note that the right normalization of (124) crucially depends on the relations (112), which the hypercharges have to satisfy. Also note, the reason that each generation of quarks and each generation of leptons is equally weighted by the corresponding two constants \( \lambda_q \) and \( \lambda_l \), respectively, is a consequence of the arbitrariness of the matrices \( \Lambda \) used in the definition of the endomorphism \( \phi' \).

The derived relations (144-150) can be related to physically measureable parameters. First, we have

Lemma 7: The "electroweak angle" \( \theta_W \) has the range

\[
0.25 \leq \sin^2 \theta_W \leq 0.45. \quad (155)
\]

Proof: By definition, the electroweak angle \( \theta_W \) measures the portion of electromagnetism to weak force:

\[
\sin \theta_W := \frac{|T_3|}{|Q|}. \quad (156)
\]

The norm \( |.| \), used here is defined with respect to the "su(N) normalization": \( \kappa(E_a, E_b) := \frac{1}{2g_S^2} \delta_{ab} \), where \( \{E_a\}_{1 \leq a \leq \dim \text{su}(N)} \) denotes an appropriate basis in the fundamental representation of \( \text{su}(N), \ N \geq 2 \). To explain this more mathematically, we remind of the fact that the general Killing form \( \tilde{\kappa} \) on a simple Lie algebra \( \mathcal{G} \) may be written as \( \tilde{\kappa}(a, b) = \lambda \rho'(a), \rho'(b) \) \( \forall a, b \in \mathcal{G} \), where \( (.,.) \) denotes any ad-invariant scalar product on \( \rho'(\mathcal{G}) \subset \text{End}(V) \) with representation \( \mathcal{G} \overset{\rho'}{\rightarrow} \text{End}(V) \). Here, the constant \( \lambda \) depends on the scalar product and \( g \) is an arbitrarily positive constant, parametrizing the scalar product. This holds true also when \( \mathcal{G} \) is semi-simple. However, the constant \( g \) - the "coupling constant" in physical terms - may be chosen differently for each simple component. Without loss of generality we can assume that \( \lambda (\rho'(E_a), \rho'(E_b)) = \delta_{ab} \) to obtain the well-known formula

\[
\sin^2 \theta_W = \frac{g_{(1)}^2}{g_{(1)}^2 + g_{(2)}^2} \quad (157)
\]

for the electroweak angle. Hence,

\[
\sin^2 \theta_W = \frac{\lambda_l + 3\lambda_q}{(1 + 2\gamma_l)\lambda_l + 3(1 + 2\gamma_q)\lambda_q}. \quad (158)
\]
The range (155) then follows by the appropriate numerical values (113-114) of the hypercharges.

**Remark 11:** Since all norms proportional to each other give the same \( \sin \theta_W \) we may choose, alternatively, \( (\rho'(a), \rho'(b)) := \text{tr}(z\rho'(a)^*, \rho'(b)) \) to define the norm in (156). Hence,

\[
\sin^2 \theta_W = \frac{(T_3, T_3)}{(Q, Q)}
\]

which, again, leads to (158). In this form, however, it becomes evident how the range (155) does depend on the generalized Dirac-Yukawa Operator of the standard model, namely just by fixing the commutant.

The analog holds true for the ratio \( g^2(2)/g^2(3) \) of the weak and strong coupling constants, yielding

\[
g^2(2)/g^2(3) = \frac{4\lambda_q}{3\lambda_q + \lambda_l}.
\]

Note that when we disregarded the possibility to introduce the element \( \zeta \) in the definition of the bosonic action functional (74), the relations (158) and (160) were just a consequence of the fermionic representation \( \rho \) used in the derived Yang-Mills action. Obviously, the same holds true in the case of \( \lambda_q = \lambda_l \), giving the "GUT-preferred" numerical values:

\[
\sin^2 \theta_W = 3/8, \quad g(3) = g(2).
\]

On the today energy scale, however, \( \lambda_q \ll \lambda_l \) seems to be preferred, c.f. [2]. Hence, \( g(2) \ll g(3) \), which might be expected, intuitively.

A more model specific relation may be obtained concerning the ratio \( m_h/m_w \) of the "Higgs mass" and the "W-boson mass" of the electroweak interaction. By definition, having fixed a (classical) vacuum \( \varphi_o \in \Sigma \), the masses of the gauge bosons are given by the quadratic form

\[
\Omega^1(\mathcal{M}, \rho'_h(\mathcal{G})) \xrightarrow{\text{M}^2} C^\infty(\mathcal{M})
\]

\[
\alpha \mapsto \frac{1}{2} M_{ab}^2 * (\alpha^a \wedge * \alpha^b),
\]

with \( M_{ab}^2 := \varphi_o^* [\rho'_h(\mathcal{E}_a), \rho'_h(\mathcal{E}_b)]^+ \varphi_o \) and \([\ldots]\]^+ the anti-commutator; \( \{\mathcal{E}_a\}_{1 \leq a \leq \dim \mathcal{G}} \) is a basis in the (semi-simple) Lie algebra \( \mathcal{G} \) of the structure group \( \mathcal{G} \), so that \( \alpha = \alpha^a \otimes \rho'_h(\mathcal{E}_a) \). In the unitary gauge: \( \varphi_o = (0, v/\sqrt{2})^T \) we get \( m^2_w = g^2(2)v^2/4 \).

The mass (matrix) of the Higgs field is defined with respect to the quadratic form:

\[
\Gamma(\mathcal{E}_h) \xrightarrow{\text{M}^2} C^\infty(\mathcal{M})
\]

\[
h \mapsto \frac{1}{2} (h, V''|_{\varphi_o = h})_{\mathcal{E}_h},
\]

with \( h := \varphi - \varphi_o \) and \((\ldots)_{\mathcal{E}_h} \) the induced scalar product on the subbundle \( \mathcal{E}_h \subset \mathcal{E} \). In the unitary gauge this yields: \( m^2_h = 2\lambda v^2 \).
Consequently, using (147) and (149) we end up with

\[
\frac{m_h^2}{m_w^2} = \frac{4(2n-1)(2n-3)}{N\pi^2} \left( \frac{l}{l_p} \right)^4 \frac{3\lambda_q + \lambda_l}{4\lambda_q + \lambda_l} (3\lambda_q\Lambda_q^4 + \lambda_l\Lambda_l^4)(a_0a_2)^2,
\]

(164)

with the abbreviation: \( \Lambda^4 := \text{tr}((\Lambda^\dagger\Lambda)^2) \). Some further investigations similar to those in [25] are needed in order to find out whether there are sufficiently enough relations between the unknowns on the righthand side of (164) determining a range where the Higgs mass has to lie in.

We finish this section by considering the special case of \( \Lambda \equiv g \), yielding the following

**Lemma 8:** Let \( \phi' = -i\phi \) as defined by (110). Then the mass squared of the Higgs field and of the W-boson reads

\[
m_h^2 = \frac{2(2n-1)}{(n-1)} \left( \frac{a_0}{a_3} \right)^2 \frac{3M_q^4 + M_l^4}{3M_q^4 + M_l^2},
\]

(165)

\[
m_w^2 = \frac{(n-1)}{2(2n-3)N} \left( \frac{a_3}{a_0} \right)^2 \frac{3M_q^2 + M_l^2}{3\lambda_q + \lambda_l},
\]

(166)

with \( M_q^2 := \text{tr}(\lambda_q m_q^* m_q) \) and \( M_l^2 := \text{tr}(\lambda_l m_l^* m_l) \). Here, we used the abbreviation \( m_q^* m_q \equiv m_d^* m_d + m_u^* m_u \) and \( \lambda_q := \lambda_q 1_N, \lambda_l = \text{diag}(\lambda_{l_1}, \ldots, \lambda_{l_N}) \in M_N(\mathbb{R}) \). Correspondingly, \( \lambda_q := \text{tr}\lambda_q/N \), as before; However, \( \lambda_l := \text{tr}\lambda_l/N \). Note that by the commutant all irreducible subspaces of the fermionic representation space \( V \) are now independently weighted. Because of the Kobayashi-Maskawa matrix the quark sector is considered as irreducible.

**Proof:** Since \( m_h^2 = 2\lambda v^2 \) we have with (147)

\[
m_h^2 = 16\alpha_0 \frac{3M_q^4 + M_l^4}{v^2};
\]

(167)

The relation (148) then implies (155). With

\[
l^2 = \frac{1}{2(2n-1)} \left( \frac{1}{a_0} \right)^2 \frac{3M_q^2 + M_l^2}{3M_q^4 + M_l^4}, \quad \text{and}
\]

(168)

\[
v^2 = \frac{(n-1)}{\pi(2n-1)N} \left( \frac{a_3}{a_0} \right)^2 \frac{(3M_q^2 + M_l^2)^2}{3M_q^4 + M_l^4} \frac{m_p^2}{4\lambda_q + \lambda_l}
\]

(169)

an analogous calculation yields (166).

Note that in the two geometrically distinguished cases where either the Higgs field defines a certain supercurvature or a certain differential form, the Higgs-form \( \omega_\phi \), as discussed in (94-95), the Higgs mass is determined by the mass of the fermions, and its range depends on the
weights of all the irreducible subspaces of the fermionic representation. In this sense the parameter "Higgs-mass" in the standard model may be considered as derived from the generalized Dirac-Yukawa operator of the standard model. Clearly, the same holds for all other (possible) derivable relations between physical parameters. We stress that the commutant in the definition of the bosonic functional (85) is motivated mainly by the fact that absolute values of the physical parameters, like the masses or charges of the particles involved, are of no physical significance. In this sense the generalization (74) becomes very plausible. Again, one has to take into account that the parameters defining the commutant are scale dependent in general.

4 Outlook

To summarize: In this paper we have introduced a particular geometrical model building kit based on the notion of generalized Dirac operators, considered as a triple \((G, \rho, D)\). Correspondingly, the geometrical setting is given by a Clifford module bundle \((\mathcal{E}, c)\) over a (closed, compact) Riemannian (spin-) manifold \((\mathcal{M}, g)\) of even dimension \(2n > 2\). Within this geometrical frame we proposed the universal functional

\[
\mathcal{I}_D := (\Psi, i D \Psi)_{\Gamma(\mathcal{E})} + \text{res}_\zeta(D^{-2n+2})
\]

on \(\Gamma(\mathcal{E}) \times \mathcal{A}(\mathcal{E})\), \(D_V = D\), generalizing the classical action functional of the standard model. Indeed, we have shown how the action functional of the standard model - with the gravity action including - can be derived from a generalization of the Dirac-Yukawa operator. For this we have introduced a certain class of Dirac operators, parametrized by some constants. Two particular choices of these constants are distinguished geometrically. In particular, we have shown that the structure of the Yukawa coupling is such that it naturally defines a certain one form \(\omega_\phi := \delta_\xi \Phi\) on a twisted Clifford module bundle \(\mathcal{E}\) (the Higgs-form) and hence a particular connection thereof. Using this connection to define a certain Dirac operator and then evaluate the proposed functional (85) (with respect to diagonal sections) one obtains the full action of the standard model. Having derived the action we have shown how the free parameters of the model are linked to physical quantities like masses and charges of the various fields involved. Depending on the number of independent parameters of the model this may then lead to non-trivial relations between the physical quantities. For a special case we derived a relation between the mass of the Higgs field and the masses of the fermions.

The basic idea is that the fermionic interactions determine the dynamics of all the fields involved in the theory. In the case of the standard model the fermionic interaction is defined with respect to the Yukawa-coupling (besides the gauge covariant coupling), giving rise to a non-standard Dirac operator - the Dirac-Yukawa operator. Here, "non-standard" means that the appropriate class of connections, defining the Dirac operator in question, does not contain a Clifford connection as a representative and thus indicating that the basic geometrical setting is
that of a Clifford module bundle instead of the particular case of a twisted spinor bundle. Indeed, according to the Higgs form, the tensor product structure of the Clifford module is ignored. In contrast to mathematical applications, where mainly SDOs are of interest, in physics non-SDOs seem to play a dominant role. Let us assume that all particles are massless. In this case the fermionic interaction is described by a gauge potential. Accordingly, the corresponding Dirac operator $D$ is a SDO. However, in our scheme one has to consider not this operator but, instead, $	ilde{D} := D + \mathcal{J}(c(F))$, where $F$ is the relative curvature (in this case the twisting curvature) on the Clifford module bundle $\mathcal{E}$. Clearly, this new Dirac operator $\tilde{D}$ on $\tilde{\mathcal{E}}$ is non-standard. Evaluation of the functional (85) with respect to this operator leads to the Einstein-Hilbert-Yang-Mills action and hence the description of the full dynamics of the physical system under consideration, c.f. [10]. Therefore, although in the case where the dynamics of the fermions are described by a SDO it seems to be more natural, actually, to consider the non-SDO $\tilde{D}$, describing the full dynamics. As a consequence of this scheme, however, one necessarily has to double the "internal degrees of freedom" of the fermions: $E \to \tilde{E} := E \oplus E$. Moreover, both parts have to carry the same fermionic representation. Note that this doubling may be described, geometrically, as the pullback bundle $\Delta^*(E \times E)$ of $E \times E \to \mathcal{M} \times \mathcal{M}$ with respect to the diagonal map $\mathcal{M} \ni x \xrightarrow{\Delta} (x, x) \in \mathcal{M} \times \mathcal{M}$. The latter may be identified with $\mathcal{M} \times \{\pm 1\}$ and thus, implicitly, also a doubling of spacetime is involved in our scheme. Since this construction is fundamental we are left to interpret this doubling physically (c.f. discussion below).

With respect to the standard model, our model building kit somehow parallels the Connes-Lott approach. In fact, both approaches yield analogous results concerning the relations between the various parameters of the models. It therefore might be worth comparing both approaches, though the general mathematical frame work is quite different and there is no doubt that from a mathematical point of view, Connes’ non-commutative geometry is much more indepth, c.f. [4], [7] and [8] for a good review. Obviously, both kits have in common that the basic building block are Dirac operators. In the Connes-Lott scheme it is the internal Dirac operator $D$ and in the kit proposed here it is the Dirac-Yukawa operator $D_\phi$. As already mentioned in the introduction the basic ideas of our model and, especially, the approach of considering the Yukawa-coupling as the fundamental input to derive the bosonic action of the standard model with gravity including was already proposed in [21]. Indeed, also in the Chamseddine-Connes scheme the Dirac-Yukawa operator is considered as the main ingredient, c.f. [9], [13]. In this scheme the universal action is defined by the heat trace in contrast to the original Connes-Lott description of the standard model, where the action was defined via the Dixmier trace (c.f. the corresponding remarks in the introduction). Concerning the standard model, in the Chamseddine-Connes approach one has to consider all Seeley-deWitt coefficients at least up to order four in the asymptotic expansion of the heat trace since the heat kernel is defined with respect to the Dirac-Yukawa operator $D_\phi$. In our scheme, however, the gen-

\footnote{By use of the correct Yukawa coupling (9), the generalized Dirac-Yukawa operator in [21] is of the form (95) of the paper at hand, and thus gives rise to the EHYMH-action}
eralized Dirac-Yukawa operator $\tilde{D}_\phi$ is considered as the basic Dirac operator. Hence, the full action of the standard model is given by the subleading term of the asymptotic expansion of the corresponding heat trace. Note that the latter yields exactly the action of the standard model, which is not the case in the Chamseddine-Connes scheme. Since in this scheme the derived action obtained by the Dirac-Yukawa operator is more general than the action of the standard model (it also contains the well-known quadratic terms in the curvature of the base manifold), one obtains more constraints for the parameters involved in the model, c.f. [13]. However, the relations (112) for the weak hypercharges and the relations (158) and (160) for the electroweak angle and the coupling constants, respectively, are merely a consequence of the Yukawa coupling (9) and that all fields involved in the model carry the fermionic representation. Hence to this respect, the Chamseddine-Connes scheme and our kit yield similar results.

The power $-2n+2$ in the definition of the action $I_D$ proposed here is motivated by the fact that there exists a generalized Dirac operator (in our sense) that gives back the exact action of the standard model. In fact, $\sigma_2$ is the only coefficient in the asymptotic expansion of the heat trace, which is linear in the curvature of the base manifold. Also by formal analogy between the asymptotic expansion of the heat trace and the asymptotic expansion of the effective action in quantum field theory one may expect, intuitively that the "classical action" is covered by the subleading term, only. As already mentioned, while the bosonic functional (68) was already considered in [3] and in [19] in the case of the pure gravity action and in [18] from a somewhat more general perspective, the possibility to also derive the Yang-Mills and Higgs action from (68) was not taken into account.

If the former is considered as "facts" we now turn to some "fictions". As we have already mentioned, the price we have to pay for "adding even terms" to Dirac operators and thereby changing the emphasis from SDOs to non-SDOs is the additional structure $(\tilde{E}, J)$, which we have to introduce in our model. Of course, this needs some physical interpretation. A physically satisfying way to understand this additional structure may consist in interpreting the doubling of the internal degrees of fermionic freedoms by introducing the notion of "antiparticles" in our scheme. Hence, we have to incorporate the notion of "charge conjugation" within our model, which will be done in a forthcoming paper where we shall also investigate in more detail the relation (164).

Another point that we have in mind concerns spectral geometry. Since the whole dynamics of the fields involved in a physical theory should be determined by a single (generalized) Dirac operator and, moreover, the corresponding fermionic action is proposed - naturally enough - to take its well-known form, one may ask whether the bosonic action actually can be considered as "generalizing" the fermionic action. Indeed, it is well-known by physicists that - in a sense - the bosonic action can be recovered from the fermionic action by considering the former as a "one-loop" correction of the latter. For this, one has to introduce a zeta-regularized determinant of a certain operator. And this may be the point where spectral geometry comes in. In other
words, if the bosonic action is considered as a modification of the fermionic action, like above, we are left with the mathematical question of how the proposed bosonic action in our model can be expressed as a zeta-regularized determinant of a Dirac operator. Moreover, as we have seen, from a geometrical point of view the action functional of the standard model is but the subleading term of the asymptotic expansion of the heat trace of a certain Hamiltonian. Therefore, it might be natural to ask whether the higher terms in the expansion permit a physical interpretation as well. Again, so far this is but fiction. However, it might be worth investigating these points more carefully in a future paper.

Still quite another point, of course, is concerned with the fact that we are dealing with Riemannian manifolds instead of Lorentzian manifolds. Hence, the notion of gravity is just formal. However, our model may be flexible enough to work also in the case when Lorentzian manifolds are considered. The point here is that also the kernels of differential operators of "Huygens-type" have an asymptotic expansion like the heat trace of an elliptic operator, c.f. [27]. Of course, spectral geometry in this case is not so well established, but see, e.g., [20] and the corresponding references therein. Finally, since the basic geometrical setting of our model kit is a Clifford module bundle and by the fact that there is a one-to-one correspondence between Dirac operators and Clifford superconnections, it might be possible to incorporate the notion of "supersymmetry" in our scheme. Thus, one may put more emphasis on the "superformalism", as developed by Quillen et.al., c.f. [11] and [29], [28] in the case of physics. Concerning our model, this point of view was taken, e.g., in [30].

We finish this paper by a citation, which expresses the feeling of many physicists. We do, however, hope to have convinced the reader that in fact the contrary holds true:

"... this prescription [via the Yukawa-coupling] of the fermion masses is one of the least satisfactory aspects of the theory [standard model]. It is an entirely ad hoc procedure ..."[19]

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[19]This is taken from the instructive book *Gauge Theories in Particle Physics* by I. J. R. Aitchison and A. J. Hey, ADAM HILGER LTD, Bristol, page 260.
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