NARUTAKA OZAWA

A SUBSTITUTE FOR KAZHDAN'S PROPERTY (T)
FOR UNIVERSAL NONLATTICES
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The well-known theorem of Shalom–Vaserstein and Ershov–Jaikin-Zapirain states that the group $\text{EL}_n(\mathcal{R})$, generated by elementary matrices over a finitely generated commutative ring $\mathcal{R}$, has Kazhdan’s property (T) as soon as $n \geq 3$. This is no longer true if the ring $\mathcal{R}$ is replaced by a commutative rng (a ring but without the identity) due to nilpotent quotients $\text{EL}_n(\mathcal{R}/\mathcal{R}^k)$. We prove that even in such a case the group $\text{EL}_n(\mathcal{R})$ satisfies a certain property that can substitute property (T), provided that $n$ is large enough.

1. Introduction

We continue and extend the scope of the study of [Kaluba et al. 2019; 2021; Netzer and Thom 2015; Nitsche 2020; Ozawa 2016], which develops the way of proving Kazhdan’s property (T) via sum of squares methods. See [Bekka et al. 2008] for a comprehensive treatment of property (T). Let $\Gamma = \langle S \rangle$ be a group together with a finite symmetric generating subset $S$. We denote by $\mathbb{R}[\Gamma]$ the real group algebra with the involution $\ast$ that extends the inverse $\ast : x \mapsto x^{-1}$ on $\Gamma$. The positive elements in $\mathbb{R}[\Gamma]$ are sums of (hermitian) squares,

$$\Sigma^2 \mathbb{R}[\Gamma] := \left\{ \sum_i \xi_i^* \xi_i : \xi_i \in \mathbb{R}[\Gamma] \right\}$$

and the combinatorial Laplacian is

$$\Delta := \frac{1}{2} \sum_{s \in S} (1-s)^*(1-s) = |S| - \sum_{s \in S} s \in \Sigma^2 \mathbb{R}[\Gamma].$$

It is proved in [Ozawa 2016] that the group $\Gamma$ has property (T) if and only if there is $\varepsilon > 0$ that satisfies

$$\Delta^2 - \varepsilon \Delta \in \Sigma^2 \mathbb{R}[\Gamma].$$

Property (T) for the so-called universal lattice $\text{EL}_n(\mathbb{Z}\langle t_1, \ldots , t_d \rangle)$, $n \geq 3$, is proved in [Shalom 2006; Vaserstein 2006; Ershov and Jaikin-Zapirain 2010]. See also [Mimura 2015] for a simpler proof and [Kassabov and Nikolov 2006; Kaluba et al. 2019] for partial results. All the proofs (save for [Kaluba et al. 2019]) rely on relative property (T) of certain semidirect products. Our interest in this paper is in the infinite index subgroup $\text{EL}_n(\mathbb{Z}\langle t_1, \ldots , t_d \rangle)$ of $\text{EL}_n(\mathbb{Z}\langle t_1, \ldots , t_d \rangle)$. Here $\mathcal{R} := \mathbb{Z}\langle t_1, \ldots , t_d \rangle$ is the commutative rng (i.e., a ring, but without assuming the existence of the identity; $\mathcal{R}$ is an ideal in the unitization $\mathcal{R}^1$) of polynomials in $t_1, \ldots , t_d$ with zero constant terms and $\text{EL}_n(\mathcal{R}) \subset \text{SL}_n(\mathcal{R}^1)$ denotes...
the group generated by the elementary matrices over the rng \( \mathcal{R} \). The elementary matrices are those \( e_{i,j}(r) \in \text{SL}_n(\mathbb{R}^1) \) with 1’s on the diagonal, \( r \in \mathcal{R} \) in the \((i, j)\)-th entry, and zeros everywhere else. The group \( \text{EL}_n(\mathcal{R}) \) does not have property (T), because it has infinite nilpotent quotients \( \text{EL}_n(\mathcal{R}/\mathbb{R}^k) \). The group does not seem to admit a good analogue of relative property (T) phenomenon, either. Still, we prove via sum of squares methods that \( \text{EL}_n(\mathcal{R}) \) satisfies a property that can substitute property (T).

**Main Theorem.** Let \( d \in \mathbb{N} \) and consider the commutative rng \( \mathcal{R} := \mathbb{Z}\langle t_1, \ldots, t_d \rangle \). Then there are \( n_0 \in \mathbb{N} \) and \( \varepsilon > 0 \) such that, for every \( n \geq n_0 \), the combinatorial Laplacians

\[
\Delta := \sum_{i \neq j} \sum_{r=1}^{d} (1 - e_{i,j}(t_r))^* (1 - e_{i,j}(t_r))
\]

for \( \text{EL}_n(\mathcal{R}) \) and

\[
\Delta^{(2)} := \sum_{i \neq j} \sum_{r,s=1}^{d} (1 - e_{i,j}(t_r t_s))^* (1 - e_{i,j}(t_r t_s))
\]

for \( \text{EL}_n(\mathcal{R}^2) \) satisfy

\[
\Delta^2 - n\varepsilon \Delta^{(2)} \in \Sigma^2 R[\text{EL}_n(\mathcal{R})].
\]

Here \( \Sigma^2 R[\Gamma] \) denotes the archimedean closure of \( \Sigma^2 R[\Gamma] \) (see Section 2). An upper bound for \( n_0 \) in the Main Theorem is in principle explicitly calculable, but we do not attempt to do that (nor attempt to optimize the proof for a better estimate). We conjecture\(^1\) that the Main Theorem holds true with \( n_0 = 3 \) (in particular \( n_0 \) should not depend on \( d \)). Our proof is inspired by the work of Kaluba, Kielak and Nowak [Kaluba et al. 2021] that proves property (T) for \( \text{Aut}(F_d) \) for \( d \geq 5 \) via computer calculations and an ingenious idea on stability. Our proof does not rely on computers, but instead on analysis by Boca and Zaharescu [2005] on the almost Mathieu operators in the rotation \( C^*\)-algebras. In fact, there is no known method of rigorously proving a result like the Main Theorem by computers. This is because the conclusion is analytic in nature—the archimedean closure is indispensable. See discussions in Section 6.

The above theorem has a couple of corollaries. The first one is reminiscent of one of the standard definitions of property (T) (see Definition 1.1.3 in [Bekka et al. 2008]).

**Corollary A.** For every \( d \), if \( n \) is large enough, then for every \( \kappa > 0 \) there is \( \delta > 0 \) satisfying the following property. For every orthogonal representation \( \pi \) of \( \text{EL}_n(\mathbb{Z}\langle t_1, \ldots, t_d \rangle) \) on a Hilbert space \( \mathcal{H} \) and every unit vector \( v \in \mathcal{H} \) with \( \max_{i,j,r} \|v - \pi(e_{i,j}(t_r))v\| \leq \delta \), there is a vector \( w \in \mathcal{H} \) such that \( \|v - w\| \leq \kappa \) and

\[
\lim_{l \to \infty} \max_{i,j,r} \|w - \pi(e_{i,j}(t_r^l))w\| = 0.
\]

We remark that a certain strengthening of the above corollary does not hold. Namely, there is an orthogonal representation \( \pi \) of \( \text{EL}_n(\mathbb{Z}\langle t_1, \ldots, t_d \rangle) \) that simultaneously admits asymptotically invariant vectors \( v_k \) and a sequence \( x_l \in \text{EL}_n(\mathbb{Z}[t_r^1, \ldots, t_r^l]) \) with \( \pi(x_l) \to 0 \) in the weak operator topology.

**Corollary B.** For every \( d \), if \( n \) is large enough, then the group \( \text{EL}_n(\mathbb{Z}\langle t_1, \ldots, t_d \rangle) \) has property \((\tau)\) with respect to the finite quotients of the form \( \text{EL}_n(S) \), where \( S \) is a finite unital quotients of \( \mathbb{Z}\langle t_1, \ldots, t_d \rangle \).

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\(^1\)NB: As the author is lame at the computer, no computer experiments have been carried out.
Property (τ) is a generalization of property (T) for finite quotients. See Section 7 for the definition and the proofs of the above corollaries. Corollary B says \( \text{EL}_n(S):S \) forms an expander family with respect to elementary generating subsets of fixed size. The novel point compared to the previously known case of the universal lattice [Kassabov and Nikolov 2006] is that the generating subsets of the finite commutative rings \( S \) need not contain the unit although the \( S \) are assumed unital. For example, for \( n \) large enough, the Cayley graphs of \( \text{SL}_n(\mathbb{Z}/q\mathbb{Z}) \) with respect to the generating subsets \( \{ e_{i,j}(p) : i \neq j \} \) form an expander family as relatively prime pairs \( (p, q) \) vary. The study of the expander property for \( \text{SL}_n(\mathbb{Z}/q\mathbb{Z}) \) and alike is a very active area. See [Breuillard and Lubotzky 2022; Helfgott 2019; Kowalski 2019] for recent surveys on this.

2. Preliminaries

Let \( \Gamma = \langle S \rangle \) be a group together with a finite symmetric generating subset \( S \). We denote by \( \mathbb{R}[\Gamma] \) the real group algebra with the involution \( * \) which is the linear extension of \( x^*: = x^{-1} \) on \( \Gamma \). The identity element of \( \Gamma \) as well as \( \mathbb{R}[\Gamma] \) is simply denoted by 1. Recall the positive cone of \textit{sums of (hermitian) squares} is given by

\[
\Sigma^2 \mathbb{R}[\Gamma] := \left\{ \sum_i \xi_i^* \xi_i : \xi_i \in \mathbb{R}[\Gamma] \right\} \subset \mathbb{R}[\Gamma]^{\text{her}} := \{ \xi \in \mathbb{R}[\Gamma] : \xi = \xi^* \}.
\]

The elements in \( \Sigma^2 \mathbb{R}[\Gamma] \) are considered positive. For \( \xi, \eta \in \mathbb{R}[\Gamma]^{\text{her}} \), we write \( \xi \leq \eta \) if \( \eta - \xi \in \Sigma^2 \mathbb{R}[\Gamma] \). It is obvious that \( \xi \geq 0 \) implies \( \xi \geq 0 \) in the full group \( C^* \)-algebra \( C^*[\Gamma] \), that is to say, \( \pi(\xi) \) is positive selfadjoint for every orthogonal (or unitary) representation \( \pi \) of \( \Gamma \) on a real (or complex) Hilbert space \( \mathcal{H} \).

The converse is true up to the \textit{archimedean closure}:

\[
\Sigma^2 \mathbb{R}[\Gamma] := \{ \xi \in \mathbb{R}[\Gamma] : \text{for all } \epsilon > 0 \xi + \epsilon \cdot 1 \geq 0 \} = \{ \xi \in \mathbb{R}[\Gamma] : \xi \geq 0 \text{ in } C^*[\Gamma] \}.
\]

See, e.g., [Cimprič 2009; Ozawa 2013; Schmudgen 2009] for this. On this occasion, we recall the basic fact that \( 0 \leq \xi \leq \eta \) (or \( 0 \leq \xi \leq \eta \)) need not imply \( 0 \leq \xi^2 \leq \eta^2 \). Note that since any orthogonal representation of \( \Gamma \) dilates to an orthogonal representation of any supergroup \( \Gamma_1 \geq \Gamma \) by induction (i.e., \( C^*[\Gamma] \subset C^*[\Gamma_1] \) in short), whether \( \xi \geq 0 \) or not does not depend on the ambient group. The same holds true for \( \xi \geq 0 \), by the coset decomposition. The \textit{combinatorial Laplacian}, with respect to the (symmetric) generating subset \( S \),

\[
\Delta := \frac{1}{2} \sum_{s \in S} (1 - s)^* (1 - s) = |S| - \sum_{s \in S} s
\]

satisfies, for every orthogonal representation \( (\pi, \mathcal{H}) \) and a vector \( v \in \mathcal{H} \),

\[
\langle \pi(\Delta) v, v \rangle = \frac{1}{2} \sum_{s \in S} \| v - \pi(s) v \|^2.
\]

3. Proof of the Main Theorem, prelude

For any rng \( \mathcal{R} \), we denote by \( \text{EL}_n(\mathcal{R}) \subset \text{SL}_n(\mathcal{R}^1) \) the group generated by the elementary matrices over the rng \( \mathcal{R} \). The elementary matrices are those \( e_{i,j}(r) \in \text{SL}_n(\mathcal{R}^1) \) with 1’s on the diagonal, \( r \in \mathcal{R} \) in the
We follow the idea of [Kaluba et al. 2021] about the stability with respect to
We note that every rng homomorphism $R \to S$ induces by entrywise operation a group homomorphism $EL_n(R) \to EL_n(S)$ and that $EL_n(R/R^k)$ is nilpotent for every $k$, where $R^k := \text{span}\{r_1 \cdots r_k : r_i \in R\}$. To ease notation, we will write
\[ E_{i,j}(r) := (1 - e_{i,j}(r))\mathsf{p}(1 - e_{i,j}(r)) = 2 - e_{i,j}(r) - e_{i,j}(r)^\ast \in R[EL_n(R)]. \]
We now consider the case $R = \mathbb{Z}(t_1, \ldots, t_d)$ and start proving the Main Theorem. Recall that the combinatorial Laplacians with respect to the generating subset $\{e_{i,j}(\pm t_r)\}$ are given by
\[ \Delta_n := \sum_{i \neq j} \sum_{r=1}^d E_{i,j}(t_r) \quad \text{and} \quad \Delta_n^{(2)} := \sum_{i \neq j} \sum_{r,s=1}^d E_{i,j}(t_r t_s). \]
We follow the idea of [Kaluba et al. 2021] about the stability with respect to $n$ of the relation like $\Delta_n^{(2)} \ll \Delta_n^2$. Here $\xi \ll \eta$ means that $\xi \leq R\eta$ for some $R > 0$ in the full group $C^\ast$-algebra. For each $n$, put $E_n := \{i, j : 1 \leq i, j \leq n, \ i \neq j\}$ and, for $e, f \in E_n$, write $e \sim f$ if $\{e \cap f\} = 1$ and $e \perp f$ if $e \cap f = \varnothing$. One has
\[ \Delta_n = \sum_{e \in E_n} \Delta_e, \]
where $\Delta_{i,j} := \sum_{r=1}^d E_{i,j}(t_r) + E_{j,i}(t_r)$. Thus
\[ \Delta_n^2 = \sum_{e} \Delta_e^2 + \sum_{e \sim f} \Delta_e \Delta_f + \sum_{e \perp f} \Delta_e \Delta_f =: \mathsf{Sq}_n + \mathsf{Adj}_n + \mathsf{Op}_n. \]
The elements $\mathsf{Sq}_n$ and $\mathsf{Op}_n$ are positive, while $\mathsf{Adj}_n$ is not and this causes trouble.
For $m < n$, we view $EL_m(R)$ as a subgroup of $EL_n(R)$ sitting at the left upper corner. The symmetric group $\text{Sym}(n)$ acts on $EL_n(R)$ by permutation of the indices. We note that
\[ |E_m| = \frac{1}{2} m(m - 1), \]
\[ |\{(e, f) \in E_m^2 : e \sim f\}| = m(m - 1)(m - 2), \]
\[ |\{(e, f) \in E_m^2 : e \perp f\}| = \frac{1}{4} m(m - 1)(m - 2)(m - 3). \]
Hence, as it is proved in [Kaluba et al. 2021], one has
\[ \sum_{\sigma \in \text{Sym}(n)} \sigma(\Delta_m^{(2)}) = m(m - 1) \cdot (n - 2)! \cdot \Delta_n^{(2)}, \]
\[ \sum_{\sigma \in \text{Sym}(n)} \sigma(\mathsf{Adj}_m) = m(m - 1)(m - 2) \cdot (n - 3)! \cdot \mathsf{Adj}_n, \]
\[ \sum_{\sigma \in \text{Sym}(n)} \sigma(\mathsf{Op}_m) = m(m - 1)(m - 2)(m - 3) \cdot (n - 4)! \cdot \mathsf{Op}_n. \]
Thus if we know there are $m \in \mathbb{N}$, $R > 0$, and $\varepsilon > 0$ such that
\[ \text{Adj}_m + R \text{Op}_m \geq \varepsilon \Delta_m^{(2)} \] holds true in $C^*[\text{EL}_m(\mathcal{R})]$, then it follows
\[ \frac{n - 2}{m - 2} \varepsilon \Delta_n^{(2)} \leq \text{Adj}_n + \frac{m - 3}{n - 3} R \text{Op}_n \leq \Delta_n^2 \]
for all $n$ such that $R(m - 3)/(n - 3) \leq 1$ and the Main Theorem is proved. This is Proposition 4.1 in [Kaluba et al. 2021]. To apply this machinery, we further expand $\text{Adj}_m$:
\[ \text{Adj}_m = \sum_{r,s} \sum_{i,j,k \text{ distinct}} (E_{i,r}(t_r) + E_{j,i}(t_r))(E_{j,k}(t_s) + E_{k,j}(t_s)) \]
\[ = \sum_{r,s} \sum_{i,j,k \text{ distinct}} (E_{i,j}(t_r)E_{j,k}(t_s) + E_{j,k}(t_s)E_{i,j}(t_r)) + E_{i,r}(t_r)E_{i,k}(t_s) + E_{j,k}(t_s)E_{i,r}(t_r) + R \text{Op}_m \geq \varepsilon E_{i,k}(t_r t_s) \]
holds true, then we obtain ($\heartsuit$) (for different $R > 0$ and $\varepsilon > 0$) by summing up this over the Sym$(m)$-orbit and over $r, s$. This is what we will prove in the next section.

### 4. The Heisenberg group and the rotation $C^*$-algebras

In this section, we will work entirely in the $C^*$-algebra setting. Let’s consider the integral Heisenberg group
\[ H := \left\{ \begin{bmatrix} 1 & a & c \\ 1 & b & 1 \end{bmatrix} : a, b, c \in \mathbb{Z} \right\} \cong \langle x, y : z := [x, y] \text{ is central} \rangle, \]
where
\[ x = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad y = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad z = \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \]
Note that every irreducible unitary representation of $H$ sends the central element $z$ to a scalar (multiplication operator) of modulus 1. For $\theta \in [0, 1)$, we consider the irreducible unitary representation $\pi_\theta$ of $H$ on $\ell_2(\mathbb{Z})$ or $\ell_2(\mathbb{Z}/q\mathbb{Z})$, depending on whether $\theta$ irrational or $\theta = p/q$ is rational with gcd$(p, q) = 1$, given by
\[ \pi_\theta(x) \delta_j = \exp(2j \pi i \theta) \delta_j, \quad \pi_\theta(y) \delta_j = \delta_{j+1}, \quad \pi_\theta(z) = \exp(2\pi i \theta). \]
By convention, if $\theta = p/q$ is rational, then gcd$(p, q) = 1$ is assumed, and if $\theta$ is irrational, we consider $q = \infty$, and $\mathbb{Z}/q\mathbb{Z}$ means $\mathbb{Z}$. Thus in either case $\pi_\theta$ is a representation on $\ell_2(\mathbb{Z}/q\mathbb{Z})$. The $C^*$-algebra $A_\theta := \pi_\theta(C^*[H])$ is called the rotation $C^*$-algebra.

We fix the notation used throughout this section. We define
\[ X := (1 - x)^* (1 - x) = 2 - x - x^* \in C^*[H]_+, \quad X_\theta := \pi_\theta(X) \in A_\theta, \]
We also write $s$. More precisely, because the statements are formulated in a different way in [Boca and Zaharescu 2005], we state associated with the left regular representation of $\tau$ that

$$H_{\theta, \lambda} := \pi_\theta \left( \frac{\lambda}{2} (x + x^*) + y + y^* \right) = (\lambda + 2) - \left( \frac{\lambda}{2} X_\theta + Y_\theta \right).$$

We also write $s = \sin \pi \theta$, $s_m = \sin 2m\pi \theta$, and $c_m = \cos 2m\pi \theta$. In particular,

$$Z_\theta = 2(1 - \cos 2\pi \theta) = 4s^2.$$

See [Boca 2001] for more information about the almost Mathieu operators and [Nitsche 2020] for some discussion in connection with the semidefinite programming.

Eventually, we will prove a certain inequality (Theorem 9) about $X$, $Y$, and $Z$ (in the full group C*-algebra of a higher-dimensional Heisenberg group) that leads to $(\Diamond)$ in the previous section. To prove inequalities about $X$, $Y$, and $Z$, it suffices to work with $X_\theta$, $Y_\theta$, and $Z_\theta$ for each $\theta \in [0, \frac{1}{2}]$ separately, thanks to the following well-known fact (Lemma 1). The critical estimate is the one for small $\theta > 0$ (Corollary 4 and Lemma 6). The rest will work out anyway.

**Lemma 1.** For any dense subset $I \subset [0, 1)$, the representation $\bigoplus_{\theta \in I} \pi_\theta$ is faithful on the full group C*-algebra $C^*\{H\}$.

**Proof.** For the readers’ convenience, we sketch the proof. Let $\tau_\theta$ denote the tracial state on $C^*\{H\}$ associated with $\pi_\theta$. That is to say, if $\theta$ is irrational, then $\tau_\theta$ arises from the canonical tracial state on the irrational rotation C*-algebra $\mathcal{A}_\theta$ and it is given by $\tau_\theta(x^i y^j) = 0$ for all $(i, j) \neq (0, 0)$. If $\theta = p/q$ is rational, then $\tau_\theta$ is given by $tr_q \circ \pi_\theta$, where $tr_q$ is the tracial state on $\mathcal{M}_q(C)$, and it satisfies $\tau_\theta(x^i y^j) = 0$ for all $(i, j) \neq (0, 0)$ in $(\mathbb{Z}/q\mathbb{Z})^2$. It follows that $\theta \mapsto \tau_\theta$ is continuous at irrational points and the assumption of the lemma implies that $\tau := \int_0^1 \tau_\theta d\theta$ is a continuous state on $\bigoplus_{\theta \in I} \pi_\theta$. It is not hard to see that $\tau$ coincides with the tracial state associated with the left regular representation of $H$, that is to say, $\tau(x^i y^j z^k) = 0$ for all $(i, j, k) \neq (0, 0, 0)$. Since $H$ is amenable, the tracial state $\tau$ is faithful on the full group C*-algebra $C^*\{H\}$. □

**Theorem 2** [Boca and Zaharescu 2005]. Let $\theta \in [0, \frac{1}{2})$. One has

$$||H_{\theta, \lambda}|| \leq \lambda + 2 - \frac{2\lambda}{\lambda + 2} \sin \pi \theta.$$

More precisely, for any real unit vector $\xi$ in $\ell_2(\mathbb{Z}/q\mathbb{Z})$,

$$||H_{\lambda, \theta} \xi||^2 = \lambda^2 + 4 + 2(1 - \tan \pi \theta) \left( \frac{\lambda}{2} \pi_\theta (x + x^*) \xi, \pi_\theta (y + y^*) \xi \right) - \sum_m |\xi_{m-1} - \xi_{m+1} - \lambda s_m \xi_m|^2.$$

**Proof.** Because the statements are formulated in a different way in [Boca and Zaharescu 2005], we replicate here the proof from that work:

$$||H_{\lambda, \theta} \xi||^2 = \sum_m |\lambda c_m \xi_m + \xi_{m-1} + \xi_{m+1}|^2$$

$$= \lambda^2 + 4 + \sum_m \left( -\lambda^2 s_m \xi_m^2 - |\xi_{m-1} - \xi_{m+1}|^2 + 2\lambda c_m \xi_m (\xi_{m-1} + \xi_{m+1}) \right)$$

$$= \lambda^2 + 4 - \sum_m |\xi_{m-1} - \xi_{m+1} - \lambda s_m \xi_m|^2 - 2\lambda \sum_m s_m (\xi_{m-1} - \xi_{m+1}) \xi_m + 2\lambda \sum_m c_m \xi_m (\xi_{m-1} + \xi_{m+1}).$$
We continue with the computation,
\[ \sum_m c_m \xi_m (\xi_{m-1} + \xi_{m+1}) = \sum_m (c_{m-1} + c_m) \xi_{m-1} \xi_m = 2 \cos \pi \theta \sum_m \xi_{m-1} \xi_m \cos (2m-1) \pi \theta \]
and similarly
\[ - \sum_m s_m (\xi_{m-1} - \xi_{m+1}) \xi_m = \sum_m (s_{m-1} - s_m) \xi_{m-1} \xi_m \]
\[ = -2 \sin \pi \theta \sum_m \xi_{m-1} \xi_m \cos (2m-1) \pi \theta \]
\[ = - \tan \theta \sum_m c_m \xi_m (\xi_{m-1} + \xi_{m+1}). \]
Thus one obtains the purported formula for \( \|H_{\lambda, \theta, \xi}\|^2 \). We also observe that
\[ \|H_{\lambda, \theta, \xi}\|^2 \leq \lambda^2 + 4 + 4\lambda (\cos \pi \theta - \sin \pi \theta) \sum_m \xi_{m-1} \xi_m \cos (2m-1) \pi \theta \]
\[ \leq \lambda^2 + 4 + 4\lambda (1 - \sin \pi \theta). \]
This yields the purported estimate for \( \|H_{\theta, \lambda}\| \). \( \square \)

**Corollary 3.** In the full group C*-algebra \( C^* [H] \), one has
\[ X + Y \geq \frac{1}{2} \sqrt{Z}. \]

**Proof.** By Lemma 1, it suffices to show the assertion in \( \mathcal{A}_\theta \) for each \( \theta \in [0, \frac{1}{2}] \). It follows from Theorem 2 with \( \lambda = 2 \) that \( X_\theta + Y_\theta = 4 - H_{\theta, 2} \geq \frac{1}{2} \sqrt{Z_\theta} \). \( \square \)

Since \( Z \) is central, \( X + Y \geq \frac{1}{2} \sqrt{Z} \) is equivalent to \( 4(X + Y)^2 \geq Z \) in \( C^* [H] \). However, there is no \( R > 0 \) such that \( R(X + Y)^2 \geq Z \) in \( \mathbb{R} [H] \). We will elaborate this in Section 6.

**Corollary 4.** Let \( R \geq 1, 0 < \kappa < 1 \), and
\[ \theta_0 := \min \left\{ \frac{1}{4 \pi}, \frac{1}{\pi} \arcsin \left( \sqrt{\frac{1}{\kappa}} \right) \right\}. \]
Then, for any \( \theta \in [0, \theta_0] \), one has
\[ RX_\theta + Y_\theta \geq \frac{\sqrt{(1 - \kappa)R}}{2} \sqrt{Z_\theta}. \]

**Proof.** We write
\[ s_0 := \sin \pi \theta_0, \quad c := \text{diag}_m c_m = \pi_\theta \left( \frac{x + x^*}{2} \right) = 1 - \frac{1}{2} X_\theta, \quad C = \sqrt{(1 - \kappa)R}. \]
Let \( \theta \in [0, \theta_0] \) and a real unit vector \( \xi \in \ell_2 (\mathbb{Z} / q \mathbb{Z}) \) be given. We need to prove \( \langle (RX_\theta + Y_\theta) \xi, \xi \rangle \geq Cs \).
For this, we may assume that \( \langle \pi_\theta (x + x^*) \xi, \pi_\theta (y + y^*) \xi \rangle > 0 \) because otherwise
\[ \langle (X_\theta + Y_\theta) \xi, \xi \rangle \geq 4 - \| \pi_\theta (x + x^* + y + y^*) \xi \| \geq 4 - 2 \sqrt{2}. \]
Put $\varepsilon := 1 - \|c\xi\|$. If $\varepsilon \geq Cs/(2R)$, then $\langle RX_{\theta}, \xi \rangle \geq 2R\varepsilon \geq Cs$ and we are done. From now on, we assume that $\varepsilon < Cs/(2R)$. By Theorem 2 for $\lambda := 2R/C$, one has

$$
\|H_{\lambda, \theta} \xi\| \leq \lambda^2 + 4\lambda(1-s)(c\xi, (H_{\theta, \lambda} - \lambda c)\xi)
\leq \lambda^2 + 4\lambda(1-s)(1-\varepsilon)\|H_{\theta, \lambda} \xi\| - 2\lambda^2(1-s)(1-\varepsilon)^2
$$

and hence

$$
(\|H_{\lambda, \theta} \xi\| - \lambda(1-s)(1-\varepsilon))^2 \leq 4 + \lambda^2(1 - 2(1-s)(1-\varepsilon)^2)\|H_{\theta, \lambda} \xi\| - 2\lambda^2(1-s)(1-\varepsilon)^2
\leq 4 + \lambda^2(s^2 + 2\varepsilon).
$$

Thus

$$
\|H_{\lambda, \theta} \xi\| \leq 2 + \lambda s\left(\frac{1}{4}\lambda s_0 + \frac{1}{2}\frac{\lambda \varepsilon}{s}\right) + \lambda(1-s).
$$

By our choices,

$$
\lambda s_0 = \frac{2R}{C} \cdot \frac{\kappa \sqrt{(1-\kappa)} \sqrt{R}}{1-\kappa} = 2\kappa
$$

and $s \varepsilon/s \leq 1$. Therefore,

$$
\|H_{\lambda, \theta} \xi\| \leq \lambda + 2 - \left(1 - \frac{1}{4} \cdot 2\kappa - \frac{1}{2}\right) \cdot 2\sqrt{\frac{R}{1-\kappa}} \leq \lambda + 2 - Cs.
$$

Since $\lambda + 2 - H_{\lambda, \theta} = (\lambda/2)X_{\theta} + Y_{\theta} \leq RX_{\theta} + Y_{\theta}$, we are done. $\square$

**Proposition 5.** In the full group $C^*$-algebra $C^*[H]$, one has

$$
(X + Y)\sqrt{Z} + \frac{1}{2}(XY + YX) \geq 0.
$$

*Proof.* By Lemma 1, it suffices to show the same for the $X_{\theta}$. We write $b_m := 1 - c_m = 1 - \cos 2m\pi \theta = 2\sin^2 m\pi \theta$. We observe that

$$
X_{\theta} = \begin{bmatrix}
\cdots & 2b_{m-1} & \cdots \\
2b_m & \cdots & 2\
\cdots & \cdots & 2
\end{bmatrix}, \quad Y_{\theta} = \begin{bmatrix}
\cdots & 2 & -1 & \cdots \\
-1 & 2 & \cdots & \cdots
\end{bmatrix},
$$

$$
\frac{1}{2}(X_{\theta}Y_{\theta} + Y_{\theta}X_{\theta}) = \begin{bmatrix}
\cdots & 4b_{m-1} & -(b_{m-1} + b_m) & \cdots \\
-(b_{m-1} + b_m) & 4b_m & \cdots & \cdots
\end{bmatrix}.
$$

These are the sums of the following 2-by-2 matrices sitting at the $(m-1)$-to-$m$-th corners:

$$
X_{\theta,m} = \begin{bmatrix}
b_{m-1} & b_m
\end{bmatrix}, \quad Y_{\theta,m} = \begin{bmatrix}
1 & -1 \\
-1 & 1
\end{bmatrix},
$$

$$
\frac{1}{2}(XY + YX)_{\theta,m} = \begin{bmatrix}
2b_{m-1} & -(b_{m-1} + b_m) \\
-(b_{m-1} + b_m) & 2b_m
\end{bmatrix}.
Thus, it suffices to show
\[ T_{\theta,m} := 2s(X_{\theta,m} + Y_{\theta,m}) + \frac{1}{2}(XY + YX)_{\theta,m} \]
\[ = \begin{bmatrix} 2(s+1)b_{m-1} + 2s & -(2s + b_{m-1} + b_m) \\ -(2s + b_{m-1} + b_m) & 2(s+1)b_m + 2s \end{bmatrix} \]
is positive in \( M_2(\mathbb{C}) \) for every \( m \). We only need to calculate the determinant:
\[ \det(T_{\theta,m}) \geq 4b_{m-1}b_m + 4s(s+1)(b_{m-1} + b_m) + 4s^2 - (2s + b_{m-1} + b_m)^2 \]
\[ = 4s^2(b_{m-1} + b_m) - (b_{m-1} - b_m)^2 \]
\[ = 8s^2(\sin^2(m-1)\pi\theta + \sin^2 m\pi\theta) - 4s^2 \sin^2(2m-1)\pi\theta \]
\[ \geq 0. \]
Here, we have used the formulas
\[ b_m = 2\sin^2 m\pi\theta, \]
\[ b_{m-1} - b_m = -2s\theta \sin(2m-1)\pi\theta, \]
\[ |\sin(2m-1)\pi\theta| \leq |\sin(m-1)\pi\theta| + |\sin m\pi\theta|. \]
\( \square \)

A similar calculation shows \( Z + \frac{1}{2}(XY + YX) \geq 0 \) in \( C^*\{H\} \). In fact, it is a sum of squares:
\[ Z + \frac{1}{2}(XY + YX) = \frac{1}{4}(X + Y)Z + \frac{1}{8} \sum (1-b)^\delta (1-a)^\varepsilon (1-a)^\ell (1-b)^\ell, \]
where \( \sum \) is over the eight terms \((a, b) \in \{(x, y), (y, x)\} \) and \((\varepsilon, \delta), (\ell, \ell) \in \{(*, *), (\cdot, *)\}\).

Now, we consider the \( C^*\)-algebra \( A_\theta \otimes A_\theta \) on \( \ell_2(\mathbb{Z}/q\mathbb{Z}) \otimes \ell_2(\mathbb{Z}/q\mathbb{Z}) \). We continue to view \( Z_\theta \) as a scalar in \( A_\theta \otimes A_\theta \). We want to find an inequality that leads to \((\otimes)\). The following does the job for small \( \theta > 0 \). We note that it fails at \( \theta_0 = \frac{1}{2}. \)

**Lemma 6.** There are \( \theta_0 > 0, R > 1, \) and \( \varepsilon > 0 \) such that, for every \( \theta \in [0, \theta_0] \), one has
\[ R(X_\theta \otimes Y_\theta + Y_\theta \otimes X_\theta) + X_\theta \otimes X_\theta + Y_\theta \otimes Y_\theta + (X_\theta Y_\theta + Y_\theta X_\theta) \otimes 1 \geq \varepsilon Z_\theta. \]

**Proof.** By Corollary 4, there are \( \theta_0 > 0 \) and \( R > 1 \) such that \( 1 \otimes (RX_\theta + Y_\theta) \geq 8s \) for every \( \theta \in [0, \theta_0] \).

By Proposition 5 and Corollary 3, it follows that the left-hand side dominates
\[ (X_\theta + Y_\theta) \cdot 8s + X_\theta Y_\theta + Y_\theta X_\theta \geq (X_\theta + Y_\theta) \cdot 4s \geq Z_\theta, \]
where we omitted writing \( \otimes 1. \)
\( \square \)

To deal with the case \( \theta \geq \theta_0 \), we need a few more auxiliary lemmas on \( A_\theta \).

**Lemma 7.** For every \( \theta \in [0, \frac{1}{2}] \), one has
\[ \| \pi_{\theta}((1-x)(1-y)) \| \leq 4 \cos(\pi \theta/2). \]
We remark that this estimate can be improved to $E$ The set $\{x(y^* + y) + x^*(z^*y^* + y)\}$ has 16 terms (counting multiplicity) and among them are $-(1+z)x$, $-(1+z)x^*$ and $x(y^* + y) + x^*(z^*y^* + y)$. One has $|1+z| = 2 \cos \pi \theta$ and

$$\|x(y^* + y) + x^*(z^*y^* + y)\| \leq \|[x y^* x^*y^*]\| \left\|\begin{bmatrix} z + y^2 \\ z^* + y^2 \end{bmatrix} \right\| \leq \sqrt{2}(z + y^2)^* (z + y^2) + (z^* + y^2)^* (z^* + y^2)\|^{1/2} = 4 \cos \pi \theta.$$ 

Hence $\|\pi_\theta((1-y)^*(1-x)^*(1-x)(1-y))\| \leq 8 + 8 \cos \pi \theta = 16 \cos^2(\pi \theta/2)$.

For a positive operator $A$, we denote by $\mathbb{P}_{A \leq \delta}$ (resp. $\mathbb{P}_{A > \delta} = 1 - \mathbb{P}_{A \leq \delta}$) the spectral projection of $A$ corresponding to the spectrum $[0, \delta]$ (resp. $(\delta, \infty)$). We also write $\mathbb{P}_{A \leq \delta} \cap \mathbb{P}_{B \leq \delta}$ etc. for the orthogonal projection onto $\text{ran} \mathbb{P}_{A \leq \delta} \cap \text{ran} \mathbb{P}_{B \leq \delta}$ etc. Note that if $A$ and $B$ commute, then so do their spectral projections and $\mathbb{P}_{A \leq \delta} \cap \mathbb{P}_{B \leq \delta} = \mathbb{P}_{A \leq \delta} \mathbb{P}_{B \leq \delta}$.

**Lemma 8.** For every $\theta \in (0, \frac{1}{2}]$ and $0 < \delta < 2(1 - \cos \pi \theta)$, one has

$$\mathbb{P}_{X_\theta \leq \delta} \mathbb{P}_{X_\theta \leq \delta} = 2 \mathbb{P}_{X_\theta \leq \delta},$$

the same with $X_\theta$ and $Y_\theta$ interchanged, and

$$\|\mathbb{P}_{Y_\theta \leq \delta} \mathbb{P}_{X_\theta \leq \delta}\| \leq \sqrt{\frac{2}{4 - \delta}}.$$ 

In particular, $\ell_2(\mathbb{Z}/q\mathbb{Z})$ is decomposed into a direct sum

$$\ell_2(\mathbb{Z}/q\mathbb{Z}) = \text{ran} \mathbb{P}_{X_\theta \leq \delta} + \text{ran} \mathbb{P}_{Y_\theta \leq \delta} + \text{ran} \mathbb{P}_{X_\theta > \delta} \cap \mathbb{P}_{Y_\theta > \delta}$$

and the corresponding (not necessarily orthogonal) projections have norm at most $\sqrt{(4 - \delta)/(2 - \delta)}$.

**Proof.** We observe that $\mathbb{P}_{X_\theta \leq \delta}$ is the projection onto $\ell_2(E)$ with

$$E := \{m : 2(1 - \cos 2m \pi \theta) \leq \delta\} \subset \{m : m \theta \in (-\theta/2, \theta/2) + \mathbb{Z}\}.$$ 

The set $E$ does not contain consecutive numbers and the first assertion follows. The second follows from the unitary equivalence of the pairs $(X_\theta, Y_\theta)$ and $(Y_\theta, X_\theta)$. Since $Y_\theta \leq \delta \mathbb{P}_{Y_\theta \leq \delta} + 4(1 - \mathbb{P}_{Y_\theta \leq \delta}) = 4 - (4 - \delta) \mathbb{P}_{Y_\theta \leq \delta}$, one has

$$2 \mathbb{P}_{X_\theta \leq \delta} \leq 4 \mathbb{P}_{X_\theta \leq \delta} - (4 - \delta) \mathbb{P}_{X_\theta \leq \delta} \mathbb{P}_{Y_\theta \leq \delta} \mathbb{P}_{X_\theta \leq \delta}$$

and $\|\mathbb{P}_{Y_\theta \leq \delta} \mathbb{P}_{X_\theta \leq \delta}\|^2 = \|\mathbb{P}_{X_\theta \leq \delta} \mathbb{P}_{Y_\theta \leq \delta} \mathbb{P}_{X_\theta \leq \delta}\| \leq 2/(4 - \delta)$. This gives the desired estimate for $\|\mathbb{P}_{Y_\theta \leq \delta} \mathbb{P}_{X_\theta \leq \delta}\|$.

We remark that this estimate can be improved to $1/\sqrt{3}$ if $\theta$ is away from $\frac{1}{2}$ and $\delta > 0$ is small enough. Indeed, the gaps of $E$ will have length at least 2 and hence any unit vectors $\xi \in \text{ran} \mathbb{P}_{X_\theta \leq \delta}$ and $\eta \in \mathbb{P}_{Y_\theta \leq \delta}$ satisfy

$$|\langle \xi, \eta \rangle| \approx \|\langle \xi, \pi_\theta(1 + y + y^*)\eta \rangle\| = \|\pi_\theta(1 + y + y^*)\xi, \eta\| \leq \frac{1}{\sqrt{3}}.$$ 

The projection onto the third subspace is orthogonal. On the other hand, any $\xi + \eta \in \text{ran} \mathbb{P}_{X_\theta \leq \delta} + \text{ran} \mathbb{P}_{Y_\theta \leq \delta}$ satisfies

$$\|\xi + \eta\|^2 \geq \|\xi\|^2 + \|\eta\|^2 - 2 \mathbb{P}_{Y_\theta \leq \delta} \mathbb{P}_{X_\theta \leq \delta} \|\xi\| \|\eta\| \geq (1 - \|\mathbb{P}_{Y_\theta \leq \delta} \mathbb{P}_{X_\theta \leq \delta}\|^2) \|\xi\|^2.$$ 

This gives the desired norm estimate.
Now, we consider this time the cubic tensor product $A_\theta \otimes A_\theta \otimes A_\theta$. This arises as an irreducible representation of the higher dimensional Heisenberg group.

$$H_3 := \begin{cases} 
1 & * & * & * \\
0 & 0 & * & \\
1 & 0 & * & \\
* & & & 1
\end{cases} \subset \text{SL}(5, \mathbb{Z}).$$

We put $x_i := e_{1,i+1}(1)$, $y_i := e_{i+1,5}(1)$, and $z := e_{1,5}(1)$ in $H_3$, where we recall that $e_{i,j}(1)$ is the elementary matrix defined in the beginning of the previous section. Note that $[x_i, y_i] = z$ and $[x_i, y_j] = 1$ for $i \neq j$.

Hence $H_3$ is isomorphic to the quotient of $H \times H \times H$ modulo $z$ are identified. As before, we write $X_i := (1 - x_i)^*(1 - x_i)$, etc. This should not be confused with $X_\theta$ in $A_\theta$.

**Theorem 9.** There are $R > 0$ and $\varepsilon > 0$ such that

$$R(X_1 Y_2 + Y_1 X_2 + X_1 Y_3 + Y_1 X_3) + X_1 X_2 + Y_1 Y_2 + X_1 Y_1 + Y_1 X_1 \geq \varepsilon Z$$

holds in $C^*[H_3]$.

**Proof.** By Lemma 1 (adapted to this case), it suffices to prove the assertion in $A_\theta \otimes A_\theta \otimes A_\theta$ for each $\theta \in [0, \frac{1}{2}]$. We write $X_{i,\theta}$ for $X_\theta$ in the $i$-th tensor component. For a unit vector

$$\zeta \in \ell_2(\mathbb{Z}/q\mathbb{Z}) \otimes \ell_2(\mathbb{Z}/q\mathbb{Z}) \otimes \ell_2(\mathbb{Z}/q\mathbb{Z}),$$

we need to prove

$$\langle (R(X_{1,\theta} Y_{2,\theta} + Y_{1,\theta} X_{2,\theta} + X_{1,\theta} Y_{3,\theta} + Y_{1,\theta} X_{3,\theta}) + X_{1,\theta} X_{2,\theta} + Y_{1,\theta} Y_{2,\theta} + X_{1,\theta} Y_{1,\theta} + Y_{1,\theta} X_{1,\theta}) \zeta, \zeta \rangle \geq \varepsilon Z_\theta.$$

By Lemma 6, we are already done for $\theta \in [0, \theta_0]$. To apply Lemma 8, fix $0 < \delta < 2(1 - \cos \pi \theta_0)$ small enough and consider $\theta \in [\theta_0, \frac{1}{2}]$. Since we may choose $R > 1$ arbitrarily large with respect to the fixed $\delta$, we may assume

$$\max\{\|P_{X_{1,\theta} Y_{2,\theta} > \delta} \zeta\|, \|P_{Y_{1,\theta} X_{2,\theta} > \delta} \zeta\|, \|P_{X_{1,\theta} X_{3,\theta} > \delta} \zeta\|, \|P_{Y_{1,\theta} Y_{2,\theta} > \delta} \zeta\|\} < \delta.$$

As described in Lemma 8, we consider the decomposition

$$\zeta = \xi + \eta + \gamma \in \text{ran} P_{X_{1,\theta} \leq \delta} + \text{ran} P_{Y_{1,\theta} \leq \delta} + \text{ran} P_{X_{1,\theta} > \delta \land Y_{1,\theta} > \delta}.$$

Note that $\max\{\|\xi\|, \|\eta\|, \|\gamma\|\} \leq 2$. By writing $\approx_{\delta}$, we will mean that the difference is at most $\delta$. Since $\xi \approx_{\delta} P_{X_{1,\theta} Y_{2,\theta} \leq \delta} \zeta$ and $P_{Y_{2,\theta} > \delta \land X_{1,\theta} Y_{2,\theta} \leq \delta} \leq P_{X_{1,\theta} \leq \delta} \land Y_{2,\theta} > \delta}$, one has

$$P_{Y_{2,\theta} > \delta} \zeta \approx_{\delta} P_{X_{1,\theta} \leq \delta} \land Y_{2,\theta} > \delta} \zeta.$$

It follows that

$$P_{Y_{2,\theta} > \delta} \eta + P_{Y_{2,\theta} > \delta} \gamma \approx_{\delta} P_{X_{1,\theta} \leq \delta} \land Y_{2,\theta} > \delta} (\xi + \eta + \gamma) - P_{Y_{2,\theta} > \delta} \xi = P_{X_{1,\theta} \leq \delta} \land Y_{2,\theta} > \delta} \eta.$$

Since $P_{Y_{2,\theta} > \delta}$ leaves $\text{ran} P_{X_{1,\theta} \leq \delta}$ and $\text{ran} P_{Y_{1,\theta} \leq \delta}$ invariant, this implies

$$P_{Y_{2,\theta} > \delta} \eta \approx_{\delta} P_{X_{1,\theta} \leq \delta} \land Y_{2,\theta} > \delta} \eta \quad \text{and} \quad P_{Y_{2,\theta} > \delta} \gamma \approx_{\delta} 0.$$

Hence, in combination with Lemma 8 that $P_{Y_1,\theta \leq \delta} P_{X_{1,\theta} > \delta} P_{Y_1,\theta \leq \delta} \geq \frac{1}{4} P_{Y_1,\theta \leq \delta}$, one obtains
\[
\delta^2 \geq \| P_{X_{1,\theta} > \delta} P_{Y_{2,\theta} > \delta} \eta \|^2 \geq \frac{1}{4} \| P_{Y_{2,\theta} > \delta} \eta \|^2,
\]
that is,
\[
\eta \approx_{28} P_{Y_{2,\theta} \leq \delta} \eta.
\]
The same consideration on $Y_{1,\theta} X_{2,\theta}$ yields
\[
P_{X_{2,\theta} > \delta} \gamma \approx_{28} 0 \quad \text{and} \quad \xi \approx_{28} P_{X_{2,\theta} \leq \delta} \xi.
\]
Thus $P_{Y_{2,\theta} > \delta} P_{X_{2,\theta} \leq \delta} \gamma \approx_{28} 0 \quad \text{and, by Lemma 8 again,}$
\[
\| \gamma \|^2 \approx_{28} \| P_{X_{2,\theta} \leq \delta} \gamma \|^2 \leq 4 \| P_{Y_{2,\theta} > \delta} P_{X_{2,\theta} \leq \delta} \gamma \|^2 \leq 16 \delta^2.
\]
Further, the same for $X_{1,\theta} Y_{3,\theta}$ and $Y_{1,\theta} X_{3,\theta}$ yields
\[
\xi \approx_{28} P_{X_{3,\theta} \leq \delta} \xi \quad \text{and} \quad \eta \approx_{28} P_{Y_{3,\theta} \leq \delta} \eta.
\]
Now a routine but tedious calculation with Lemma 8 yields
\[
\langle X_{1,\theta} X_{2,\theta} \xi, \zeta \rangle \approx_{C_\delta} C \langle X_{1,\theta} Y_{2,\theta} \xi, \zeta \rangle \approx_{C_\delta} 4 \| \xi \|^2.
\]
for some absolute constant $C$ (e.g., $C = 1000$ should be enough), and likewise
\[
\langle Y_{1,\theta} Y_{2,\theta} \xi, \zeta \rangle \approx_{C_\delta} 4 \| \xi \|^2.
\]
On the other hand, by Lemmas 7 and 8,
\[
| \langle (X_{1,\theta} Y_{1,\theta} + Y_{1,\theta} Y_{1,\theta}) \xi, \zeta \rangle | \approx_{C_\delta} 2 | \langle X_{1,\theta} Y_{1,\theta} \| X_{1,\theta} \leq \delta \wedge X_{2,\theta} \leq \delta \wedge X_{3,\theta} \leq \delta \| X_{1,\theta} \leq \delta \wedge X_{2,\theta} \leq \delta \wedge Y_{3,\theta} \leq \delta \| \eta \rangle | 
\]
\[
\leq 2 \| P_{X_{1,\theta} \leq \delta} X_{1,\theta} \| \| P_{X_{2,\theta} \leq \delta} X_{2,\theta} \| \| P_{X_{3,\theta} \leq \delta} Y_{3,\theta} \| \| \eta \|
\]
\[
\leq 16 \left( \cos \frac{\pi \theta}{2} \right) \cdot \frac{2}{4 - \delta} \| \xi \| \| \eta \|.
\]
If we have chosen $\delta > 0$ small enough, then
\[
\epsilon := 8 - 16 \left( \cos \frac{\pi \theta_0}{2} \right) \cdot \frac{2}{4 - \delta} > 4C \delta.
\]
Observe that $\delta > 0$ and $\epsilon > 0$ depends on the absolute constants $\theta_0 > 0$ and $C > 0$, but not on $\theta \in \left[ \theta_0, \frac{1}{2} \right]$. In the end,
\[
| \langle (X_{1,\theta} Y_{1,\theta} + X_{1,\theta} Y_{1,\theta}) \xi, \zeta \rangle | \leq (8 - \epsilon) \| \xi \| \| \eta \| + C \delta
\]
\[
\leq 4(1 - \epsilon/2)(\| \xi \|^2 + \| \eta \|^2) + C \delta
\]
\[
\leq (\| X_{1,\theta} X_{2,\theta} + Y_{1,\theta} Y_{2,\theta} \| \xi, \zeta \| - \epsilon + 3C \delta.
\]
This completes the proof. We remark that the above proof for $\theta \in \left[ \theta_0, \frac{1}{2} \right]$ is not as tight as it appears (and $\epsilon > 0$ can be “visible”), because if $\theta$ is around $\frac{1}{2}$, then $\cos \frac{\pi \theta}{2} \approx \frac{1}{\sqrt{2}}$, and if $\theta$ is away from $\frac{1}{2}$, then $\| P_{X_{\theta} \leq \delta} P_{Y_{\theta} \leq \delta} \|$ is bounded by $\approx \frac{1}{\sqrt{3}}$. □
We also record an easy consequence of the Hahn–Banach separation theorem (a.k.a. the Eidelheit–Kakutani separation theorem in this context). For \( x_1 = e_{1,2}(t_r), \ x_2 = e_{1,3}(t_s), \ x_3 = e_{1,4}(t_r), \ y_1 = e_{2,5}(t_s), \ y_2 = e_{3,5}(t_r), \ y_3 = e_{4,5}(t_s), \) and \( z = e_{1,5}(t_r, t_s) \) in \( \text{EL}_5(\mathcal{R}) \). This yields \( \diamondsuit \) in Section 3 and the proof of the Main Theorem is complete. \( \square \)

The terms \( X_1 Y_2 = E_{1,2}(t_r) E_{3,5}(t_r) \) and \( Y_1 X_2 = E_{2,5}(t_s) E_{1,3}(t_s) \) are diagonal with respect to \( \{t_r, t_s\} \). This causes an annoying dependence of \( R \) on \( d \) in the formula \( \bigcirc \), which results in dependence of \( n_0 \) on \( d \) in the Main Theorem.

### 6. Real group algebras and property \( H_T \)

In this section, we continue the study of [Netzer and Thom 2013; 2015; Nitsche 2020; Ozawa 2013; 2016] about positivity in real group algebras. In addition to the notation from Section 2, we denote by

\[
I[\Gamma] := \text{span}\{1 - x : x \in \Gamma\} \subset \mathbb{R}[\Gamma]
\]

the augmentation ideal. We observe that \( \Sigma^2 I[\Gamma] = I[\Gamma] \cap \Sigma^2 \mathbb{R}[\Gamma] \) and hence there is no ambiguity about the order \( \preceq \) on \( I[\Gamma] \). In [Ozawa 2016], it was observed that the combinatorial Laplacian \( \Delta \in \Sigma^2 I[\Gamma] \) is an order unit for \( I[\Gamma] \) (more precisely for \( I[\Gamma]^{\text{her}} \), but this abuse of terminology should not cause any problem). That is to say, for every \( \xi \in I[\Gamma]^{\text{her}} \), there is \( R > 0 \) such that \( \xi \preceq R\Delta \). We will indicate this by \( \xi \ll \Delta \).

We review the relation between positive linear functionals on \( I[\Gamma] \) and \( 1 \)-cocycles (with unitary coefficients). A linear functional \( \varphi \) on \( I[\Gamma] \) is said to be positive if it is selfadjoint and \( \varphi(\Sigma^2 I[\Gamma]) \subset \mathbb{R}_{\geq 0} \). One has \( \varphi(\Delta) = 0 \) if and only if \( \varphi = 0 \). Every positive linear functional \( \varphi \) gives rise to a semi-inner product \( \langle \xi, \eta \rangle := \varphi(\xi^* \eta) \) and the corresponding seminorm \( \|\xi\| := \varphi(\xi^* \xi)^{1/2} \) on \( I[\Gamma] \), with respect to which the left multiplication by an element of \( \Gamma \) is orthogonal. This is the Gelfand–Naimark construction.

The map \( b : \Gamma \to I[\Gamma], \ t \mapsto 1 - t, \) is a \( 1 \)-cocycle, i.e., it satisfies \( b(st) = b(s) + s b(t) \) for every \( s, t \in \Gamma \). We note that \( \varphi(1 - t) = \frac{1}{2} \varphi((1 - t)^*(1 - t)) = \frac{1}{2} \|b(t)\|^2 \) and \( \varphi(\Delta) = \frac{1}{2} \sum_{s \in S} \|b(s)\|^2 \). In fact, every \( 1 \)-cocycle arises in this way. See, e.g., Appendix C in [Bekka et al. 2008] and Appendix D in [Brown and Ozawa 2008] for a comprehensive treatment.

It is proved in [Ozawa 2016] that \( \Sigma^2 I[\Gamma] = I[\Gamma]^{\text{her}} \cap \Sigma^2 \mathbb{R}[\Gamma] \). That is to say,

\[
\Sigma^2 I[\Gamma] := \{ \xi \in I[\Gamma]^{\text{her}} : \text{for all } \varepsilon > 0, \ \xi + \varepsilon \Delta \geq 0 \}
= \{ \xi \in I[\Gamma]^{\text{her}} : \varphi(\xi) \geq 0 \text{ for every positive linear functional } \varphi \text{ on } I[\Gamma] \}
= \{ \xi \in I[\Gamma]^{\text{her}} : \xi \geq 0 \text{ in } C^*[\Gamma] \}.
\]

We also record an easy consequence of the Hahn–Banach separation theorem (a.k.a. the Eidelheit–Kakutani separation theorem in this context). For \( \xi, \eta \in I[\Gamma]^{\text{her}} \) (or in any real ordered vector space with an order unit \( \Delta \)), the following are equivalent:

1. \( \varphi(\xi) = 0 \) implies \( \varphi(\eta) \leq 0 \) for every positive linear functional \( \varphi \) on \( I[\Gamma] \).
2. \( -\eta \in \Sigma^2 I[\Gamma] - \mathbb{R}_\xi \).
3. For all \( \varepsilon > 0 \), there exists \( R \in \mathbb{R} \) such that \( R \xi - \eta + \varepsilon \Delta \geq 0 \).
We observe that since
\[ \varphi(\Delta^2) = \langle \Delta, \Delta \rangle = \left\| \sum_{s \in S} b(s) \right\|^2, \]
one has \( \varphi(\Delta^2) = 0 \) if and only if the corresponding 1-cocycle \( b \) is harmonic in the sense \( \sum_{s \in S} b(s) = 0 \). This observation recovers Shalom’s theorem [2000] that every finitely generated group without property (T) has a nonzero harmonic 1-cocycle. An essentially same proof was given in [Nitsche 2020].

We record the following well-known fact:

- If a 1-cocycle \( b \) vanishes on a normal subgroup \( N \triangleleft \Gamma \), then \( N \) acts trivially on \( \text{span} b(\Gamma) \) and hence \( b \) factors through the quotient \( \Gamma/N \).
- If \( b \) is a harmonic 1-cocycle on \( \Gamma \), then the center \( \mathcal{Z}(\Gamma) \) acts trivially on \( \text{span} b(\Gamma) \) and \( \Gamma \) acts trivially on \( \text{span} b(\mathcal{Z}(\Gamma)) \).
- Every harmonic 1-cocycle on an abelian group is an additive homomorphism.

The first assertion is not difficult to show. The second follows from the identity \( (1 - x)b(z) = (1 - z)b(x) \) for \( x \in \Gamma \) and \( z \in \mathcal{Z}(\Gamma) \). If \( b \) is harmonic, then \( (|S| - \sum_{s \in S} s)b(z) = 0 \) and, by strict convexity of a Hilbert space, \( b(z) = sb(z) \) for \( s \in S \) and hence for all \( s \in \Gamma \).

An additive character \( \chi : \Gamma \to \mathbb{R} \) can be viewed as a harmonic 1-cocycle. The corresponding positive linear functional \( \varphi_\chi : I[\Gamma] \to \mathbb{R} \) is given by \( \varphi_\chi(1 - t) = \frac{1}{2} \chi(t)^2 \). This should not be confused with the linear extension \( \chi : I[\Gamma] \to \mathbb{R} \) which is not even selfadjoint. The positive linear functional \( \varphi_\chi \) factors through the abelianization \( I[\Gamma^{ab}] \).

We denote the augmentation power by
\[ I^k[\Gamma] := \text{span}(I[\Gamma]^k) \subset \mathbb{R}[\Gamma]. \]
It is well-known and easy to see from the formula
\[ 1 - x y = (1 - x) + (1 - y) - (1 - x)(1 - y) \in (1 - x) + (1 - y) + I^2[\Gamma] \]
that \( I[\Gamma] \) is generated as a rng by \( \{1 - s : s \in S\} \) and that \( \Gamma \ni x \mapsto 1 - x \in I[\Gamma]/I^2[\Gamma] \) is an additive homomorphism. On the other hand, every additive homomorphism \( \chi \) vanishes on \( I^2[\Gamma] \), because \( \chi((1 - x)(1 - y)) = \chi(1 - x - y + xy) = 0 \). Hence \( I^2[\Gamma] = \bigcap_{\chi} \ker \chi \), where the intersection is taken over the additive characters \( \chi \) on \( \Gamma \). We will see that \( \Delta^2 \in \Sigma^2 I^2[\Gamma] \) need not be an order unit for \( I^4[\Gamma] \), but the element
\[ \square := \frac{1}{4} \sum_{s,t \in S} (1 - s)^* (1 - t)^* (1 - t)(1 - s) \in \Sigma^2 I^2[\Gamma] \]
is. Since \( \square = \Delta^2 \) in \( I[\Gamma^{ab}] \), one has \( \varphi_\chi(\square) = \varphi_\chi(\Delta^2) = 0 \) for every additive character \( \chi \). We will prove later that the converse is also true.

**Theorem 10.** The element \( \square \) is an order unit for \( I^4[\Gamma] \). Namely
\[ I^4[\Gamma]^{\text{her}} = \{ \xi \in \mathbb{R}[\Gamma]^{\text{her}} : \pm \xi \ll \square \} = \text{span} \Sigma^2 I^2[\Gamma] \]
and moreover \( I^4[\Gamma] \cap \Sigma^2 \mathbb{R}[\Gamma] = \Sigma^2 I^2[\Gamma] \).
Theorem 10 implies that 
\[ \eta \in \operatorname{span} \Sigma^2 I^2[\Gamma]. \]
However, we will observe here that

\[ \sum_i \eta_i^* \eta_i \ll \square \quad \text{for all } \eta_i \in I^2[\Gamma]. \]

We conjecture that the group \( \operatorname{EL} \) coincides with Kazhdan’s property (T) for unipotent \( \mathbb{G} \), which does not seem useful though.

Corollary 11. A positive linear functional \( \phi \) on \( I[\Gamma] \) satisfies \( \phi(\square) = 0 \) if and only if the associated 1-cocycle is an additive homomorphism.

Proof. We have already noted that \( \phi(\square) = 0 \) for all additive character \( \chi \). Conversely, suppose \( \phi(\square) = 0 \). Since this implies \( \phi(\Delta^2) = 0 \), the 1-cocycle \( b \) associated with \( \phi \) is harmonic. Moreover, since

\[ 1 - [x, y] = (xy - yx)x^{-1}y^{-1} = ((1 - x)(1 - y) - (1 - y)(1 - x))x^{-1}y^{-1} \in I^2[\Gamma]. \]

Theorem 10 implies that \( b = 0 \) on the commutator subgroup \([\Gamma, \Gamma]\). Thus \( b \) factors through \( \Gamma^{ab} \) and is an additive homomorphism.

We recall that a finitely generated group \( \Gamma \) is said to have Shalom’s property \( H_T \) if every harmonic 1-cocycle on \( \Gamma \) is an additive homomorphism. Property \( H_T \) coincides with Kazhdan’s property (T) for groups with finite abelianization. It is observed in [Shalom 2004] that finitely generated nilpotent groups have property \( H_T \). We conjecture that the group \( \operatorname{EL}_n(\mathbb{Z}[t_1, \ldots, t_d]) \) has property \( H_T \). By the Hahn–Banach separation theorem, one obtains the following characterization of property \( H_T \), which does not seem useful though.

Corollary 12. The finitely generated group \( \Gamma \) has finite abelianization if and only if \( \Delta \ll \square \). The finitely generated group \( \Gamma \) has property \( H_T \) if and only if for every \( \varepsilon > 0 \) there is \( R > 0 \) such that \( \square \leq R \Delta^2 + \varepsilon \Delta \).

Property \( H_T \) for nilpotent groups also follows from Corollary 3 if a commutator \( z = [x, y] \) is central, then \( (1 - z)^*(1 - z) \ll \Delta^2 \) in \( C^*[\Gamma] \). It is tempting to conjecture that every finitely generated nilpotent group \( \Gamma \) satisfies \( \square \ll \Delta^2 \). Had it been true that \( \square \ll \Delta^2 \) for a given group \( \Gamma \), it would have been able to rigorously prove this by computer calculations because \( \square \) is an order unit for \( I^4[\Gamma] \) (modulo a quantitative estimate, see [Netzer and Thom 2015]). However, we will observe here that \( \square \not\ll \Delta^2 \).
in \( \mathbb{R}[H] \). Hence, unlike property (T), property \( H_T \) is probably not characterized by a “simple”\(^2\) inequality in the real group algebra. This spoils the current methods of proving something like the Main Theorem by computer calculations. (Note that \( \text{EL}_n(\mathbb{Z}(t)) \) has the Heisenberg group \( H_{n-2} \) as a quotient and the analogous statement to the following proposition holds true for this group.)

**Proposition 13.** Let \( H \) be the integral Heisenberg group and \( z := [x, y] \) be as described in the beginning of Section 4. Then \((1 - z)^{s}(1 - z) \not\in \Delta^2\) in \( \mathbb{R}[H] \). Moreover,

\[
\Sigma^2 I^2[H] \neq I^4[H]_{\text{her}} \cap \Sigma^2 \mathbb{R}[H].
\]

The proof of \( \Sigma^2 I[\Gamma] = I[\Gamma]_{\text{her}} \cap \Sigma^2 \mathbb{R}[\Gamma] \) given in [Ozawa 2016] is based on Schoenberg’s theorem that any positive linear functional on \( I[\Gamma] \) is approximable by those that extend on \( \mathbb{R}[\Gamma] \). The above proposition says there is no good enough analogue of Schoenberg’s theorem for augmentation powers. For the proof of the proposition, we need a description of the graded vector space \( \mathbb{R}[H]/I^n[H] \).

To ease notation, we write \( \bar{x} := 1 - x \) etc. and observe that \( \bar{z} \in \mathbb{Z}(\mathbb{R}[H]) \cap I^2[H] \) and

\[
\bar{y}\bar{x} = \bar{x}\bar{y} + \bar{z} - \bar{z}\bar{x} - \bar{z}\bar{y} + \bar{z}\bar{y}\bar{x} \in \bar{x}\bar{y} + \bar{z} + I^3[H].
\]

**Lemma 14.** For every \( n \in \mathbb{N} \), the set \( \{ \bar{x}^i \bar{y}^j \bar{z}^k + I^n[H] : i, j, k \geq 0, \ i + j + 2k < n \} \) forms a basis for \( \mathbb{R}[H]/I^n[H] \). In particular,

\[
\dim I^n[H]/I^{n+1}[H] = ([n/2] + 1)(n - [n/2] + 1).
\]

**Proof.** We first observe that the asserted set spans \( \mathbb{R}[H]/I^n[H] \). Indeed, this follows from the above equation for \( \bar{y}\bar{x} \) and the general facts that

\[
1 - uv = (1 - u) + (1 - v) - (1 - u)(1 - v),
\]

\[
1 - u^{-1} = -(1 - u) + (1 - u^{-1})(1 - u)
\]

for every \( u, v \in H \). It is left to show that the asserted set is also linearly independent. Suppose that

\[
\xi := \sum_{i+j+2k<n} \alpha_{i,j,k} \bar{x}^i \bar{y}^j \bar{z}^k \in I^n[H].
\]

By considering the abelianization \( \pi^{ab} : C^*[H] \to C^*[\mathbb{Z}^2] \), one sees \( \alpha_{i,j,k} = 0 \) whenever \( k = 0 \). It follows that \( \xi \in I^n[H] \cap \bar{z}\mathbb{R}[H] \). We claim that

\[
I^n[H] \cap \bar{z}\mathbb{R}[H] = \bar{z}I^{n-2}[H] \quad \text{for } n \geq 2.
\]

Since \( \bar{z} \) is not a zero divisor in \( \mathbb{R}[H] \) (e.g., because \( \pi_\theta(\bar{z}) \) are invertible for \( \theta \in (0, 1) \)), the lemma would follow from this claim by induction.

The homomorphisms \( \mathbb{R}[\langle x \rangle] \hookrightarrow \mathbb{R}[H] \) and \( \mathbb{R}[\langle y \rangle] \hookrightarrow \mathbb{R}[H] \) extend to a linear injection

\[
\sigma : \mathbb{R}[\langle x \rangle] \otimes \mathbb{R}[\langle y \rangle] \hookrightarrow \mathbb{R}[H], \quad \xi \otimes \eta \mapsto \xi \eta,
\]

with the left inverse

\[
\pi^{ab} : \mathbb{R}[H] \twoheadrightarrow \mathbb{R}[\mathbb{Z}^2] \cong \mathbb{R}[\langle x \rangle] \otimes \mathbb{R}[\langle y \rangle].
\]

\(^2\)The quantifier elimination techniques, which the author is not familiar with, may be relevant.
We define a linear functional \( \varphi \). We say a finitely generated group \( \pi(0) \) is large enough. Corollaries A and B say this happens uniformly for any unitary representation \( \pi \) of \( \Gamma \) that factors through some \( \Gamma \to \Gamma_i \). This proves the claim. \( \square \)

**Proof of Proposition 13.** We observe that in \( I^4[H]/I^3[H] \)

\[
(\bar{x}\bar{y}\bar{z})^* = \bar{y}\bar{y}\bar{x} = \bar{y}\bar{y}\bar{x} + \bar{y}\bar{x}\bar{z} = \bar{x}\bar{y}\bar{y} + 2\bar{x}\bar{y}z + 2\bar{z}z.
\]

We define a linear functional \( \varphi \) on \( I^4[H]/I^3[H] \) by

\[
\varphi(\bar{x}^4) = \varphi(\bar{y}^4) = 1, \quad \varphi(\bar{z}^2) = -2, \quad \varphi(\bar{x}\bar{y}\bar{z}) = 1,
\]

and zero on all the other basis elements. Then, the linear functional \( \varphi \) is selfadjoint. Moreover, with respect to the basis \( \{\bar{x}, \bar{y}, \bar{x}, \bar{y}\} \) for \( I^2[H]/I^3[H] \), the bilinear form \((\xi, \eta) \leftrightarrow \varphi(\bar{x}^2\bar{y}^2)\) is represented by the matrix

\[
\begin{bmatrix}
1 & 0 & 0 & -1 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
-1 & 0 & 0 & 1
\end{bmatrix}.
\]

Since this matrix is positive semidefinite, the linear functional is positive on \( I^4[H] \), by Theorem 10. One sees that \( \varphi(\bar{x}^2\bar{y}) = -\varphi(\bar{x}\bar{y}) = 2 > 0, \varphi(\square) = 4 \), and

\[
\varphi(\Delta^2) = \varphi((\bar{x}\bar{x} + \bar{y}\bar{y})(\bar{x}\bar{x} + \bar{y}\bar{y})) = 0.
\]

Therefore there cannot be \( R > 0 \) such that \( \bar{z}^*\bar{z} \leq R\Delta^2 + \frac{1}{4} \square \). It follows that \( 4\Delta^2 - \bar{z}^*\bar{z} \notin \Sigma^2I^2[H] \), while \( 4\Delta^2 - \bar{z}^*\bar{z} \in I^4[H]^{\text{her}} \cap \Sigma^2\Sigma^2[H] \) by Corollary 3. \( \square \)

**7. Property (\( \tau \))**

We say a finitely generated group \( \Gamma = \langle S \rangle \) has property (\( \tau \)) with respect to a family \( \{\Gamma_i\} \) of finite quotients \( \Gamma \to \Gamma_i \) if there is \( \delta > 0 \) such that any unitary representation \( \pi \) of \( \Gamma \) that factors through some \( \Gamma \to \Gamma_i \) either admits a nonzero \( \pi(\Gamma)-\)invariant vector or admits no unit vector \( v \) such that max \( s \in S \|v - \pi(s)v\| \leq \delta \). This is equivalent to that the Cayley graphs of \( \{\Gamma_i\} \) with respect to the generating subset \( S \) form an expander family. In case the family \( \{\Gamma_i\} \) is the set of all finite quotients of \( \Gamma \), it is simply said \( \Gamma \) has property (\( \tau \)). See [Kowalski 2019] for a comprehensive treatment of expander graphs. By the Main Theorem, EL_{n}(S) has property (T) if \( S \) is a finitely generated irng (i.e., a rng which is idempotent, \( S = S^2 \), see [Monod et al. 2012]) and \( n \) is large enough. Corollaries A and B say this happens uniformly for finite commutative irngs with a fixed number of generators.

**Proof of Corollary A.** Let \( n_0 \) be as in the Main Theorem for \( \mathbb{Z}\langle T_1, \ldots, T_d, S_1, \ldots, S_d \rangle \) and \( n \geq n_0 \). By the Main Theorem applied to \( T_r \leftrightarrow t_r^k \) and \( S_r \leftrightarrow t_r^{k+1} \), there is \( \varepsilon > 0 \) such that

\[
\Delta_k := \sum_{i \neq j} \sum_{r=1}^d (1 - e_{i,j}(t_r^k))^*(1 - e_{i,j}(t_r^k)) \in \mathbb{R}[\text{EL}_{n_0}(\mathbb{Z}\langle t_1, \ldots, t_d \rangle)]
\]
(so $\Delta_1 = \Delta$) satisfy
\[
(\Delta_k + \Delta_{k+1})^2 \geq \varepsilon(\Delta_{2k} + \Delta_{2k+1} + \Delta_{2k+2})
\]
for all $k$. We may also assume that $\varepsilon > 0$ satisfies $\Delta_1^2 \geq \varepsilon \Delta_2$.

Let $\pi$, $\mathcal{H}$ and $v$ be given for $\text{EL}_n(\mathbb{Z} \langle t_1, \ldots, t_d \rangle)$ (but we will omit writing $\pi$ to ease notation) and put
\[
\delta := \left( \sum_{i,j,r} \|v - e_{i,j}(t_r) v\|^2 \right)^{1/2} = \langle \Delta v, v \rangle^{1/2}.
\]
We assume $\delta < \left( \frac{1}{2} \right)^{10}$ and put $\rho := \delta^{1/10}$. Recall that $\mathbb{P}_{\Delta \leq (\delta/\rho)^2}$ stands for the spectral projection of $\Delta$ for the interval $[0, (\delta/\rho)^2]$. For $v_0 := \mathbb{P}_{\Delta \leq (\delta/\rho)^2} v$, one has $\|v - v_0\| \leq \rho$ and
\[
\langle (\Delta_1 + \Delta_2) v_0, v_0 \rangle \leq \delta^2 + \varepsilon^{-1}(\delta/\rho)^4 =: \delta_0^2.
\]
Now, $v_1 := \mathbb{P}_{\Delta \leq (\delta_0/\rho)^2} v_0$ satisfies $\|v_0 - v_1\| \leq \rho^2$ and
\[
\langle (\Delta_2 + \Delta_3) v_1, v_1 \rangle \leq \varepsilon^{-1}(\delta_0/\rho^2)^4 =: \delta_1^2.
\]
We continue this and obtain $v_2 := \mathbb{P}_{\Delta \leq (\delta_1/\rho)^2} v_1$, . . . such that $\|v_k - v_{k+1}\| \leq \rho^{k+2}$ and
\[
\langle (\Delta_{2k} + \Delta_{2k+1}) v_k, v_k \rangle \leq \varepsilon^{-1}(\delta_{k-1}/\rho^{k+1})^4 =: \delta_k^2.
\]
Then the vector $w := \lim_k v_k$ satisfies $\|v_k - w\| \leq \rho^{k+1}$ (as $\rho < \frac{1}{2}$). Moreover,
\[
2^{-k} |\log \delta_k| = 2^{-(k-1)}|\log \delta_{k-1}| - 2^{-(k-1)}(k + 1)|\log \rho| + 2^{-(k+1)} \log \varepsilon
= |\log \delta_0| - \left( \sum_{m=1}^{k} 2^{-(m-1)}(m + 1) \right)|\log \rho| + \frac{1}{2}(1 - 2^{-k}) \log \varepsilon
> \frac{1}{10} |\log \delta|
\]
if $\delta > 0$ is small enough compared to $\varepsilon > 0$. Hence $\delta_k \to 0$ at a double exponential rate.

We need to show $\lim_{k} \max_{i,j,r} \|w - e_{i,j}(t_r^k) w\| = 0$. We first observe that
\[
\|w - e_{i,j}(t_r^2) w\| \leq 2 \|v_k - w\| + \delta_k \leq \rho^k + \delta_k.
\]
Let $l$ be given. Take $k = k(l)$ such that $l \in [2^k, 2^{k+1})$ and write $l = 2^k + \sum_{m=0}^{k-1} a(m) 2^m$ with $a(m) \in \{0, 1\}$. Then for $b := \sum_{m=0}^{[k/2] - 1} a(m) 2^m$, one has
\[
\|e_{i,j}(t_r^l) w - e_{i,j}(t_r^{2^k + b}) w\| \leq \sum_{m=0}^{[k/2] - 1} a(m) (\rho^m + \delta_m),
\]
which tends to 0 as $l \to \infty$. Observe that the recurrence relation
\[
p_0 := 2^{k-[k/2]}, \quad p_{m+1} := 2 p_m + a([k/2] - 1 - m)
\]
gives $p_{[k/2]} = 2^k + b$. Now by arguing as in the previous paragraph, but starting at $v_{k-[k/2]}$ and using $(\Delta_{pm} + \Delta_{pm+1})^2 \geq \varepsilon(\Delta_{pm+1} + \Delta_{pm+1+1})$, one obtains
\[
\|v_{k-[k/2]} - e_{i,j}(t_r^{2^k + b}) v_{k-[k/2]}\| \leq \rho^{k-[k/2]} + \delta_k \to 0.
\]
Since $\|v_{k-[k/2]} - w\| \to 0$ as $l \to \infty$, this completes the proof. \qed
We give a proof of the remark that was made after Corollary A. Let $\mathcal{R} := \mathbb{Z}\langle t_1, \ldots, t_d \rangle$. Since $\text{EL}_n(\mathcal{R}/\mathcal{R}^l)$ is nilpotent, there is a proper 1-cocycle $b_l$ (see Section 2.7 in [Bekka et al. 2008] or Section 12 in [Brown and Ozawa 2008]). We view $b_l$ as 1-cocycles on $\text{EL}_n(\mathcal{R})$ and consider $b := \sum b_l$, which we may assume convergent pointwise on $\text{EL}_n(\mathcal{R})$. We denote by $\pi_k$ the Gelfand–Naimark representation associated with the positive definite function $\varphi_k(x) := \exp(-\frac{1}{k} \|b(x)\|^2)$. Then, the representation $\pi := \bigoplus \pi_k$ simultaneously admits asymptotically invariant vectors and a weak operator topology null sequence $x_l \in \text{EL}_n(\mathcal{R}^l)$.

**Proof of Corollary B.** Let $\mathcal{R}^1 := [t_1, \ldots, t_d]$ denote the unitization of $\mathcal{R} := \mathbb{Z}\langle t_1, \ldots, t_d \rangle$. Any quotient map $\mathcal{R} \to S$ with $S$ unital gives rise to a group homomorphism $\text{EL}_n(\mathcal{R}^1) \to \text{EL}_n(S)$ that extends $\text{EL}_n(\mathcal{R}) \to \text{EL}_n(S)$. We need to show that an orthogonal representation of $\text{EL}_n(\mathcal{R}^1)$ which factors through $\text{EL}_n(S)$ has a nonzero invariant vector, provided that it has almost $\text{EL}_n(\mathcal{R})$ invariant vector. Since we know $\text{EL}_n(\mathcal{R}^1)$ has property (T), it suffices to show that every almost $\text{EL}_n(\mathcal{R})$ invariant vector is also almost $\text{EL}_n(\mathbb{Z}1)$ invariant. The latter is true when $S$ is finite. Indeed, the vector $w$ in Corollary A is invariant under those $e_{i,j}(t^l_{i,j})$ such that $t^l_{i,j}$ is an idempotent in the quotient $S$. Since a finite commutative ring is a direct sum of local rings (see, e.g., [Kassabov and Nikolov 2006]), the rng generated by such idempotents contains the identity of $S$ and hence $w$ is invariant under $\text{EL}_n(\mathbb{Z}1)$. □

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NARUTAKA OZAWA: narutaka@kurims.kyoto-u.ac.jp

Research Institute for Mathematical Sciences, Kyoto University, Kyoto, Japan
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YANG LI

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