On quantization of nondispersive wave packets

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Abstract

Canonical commutation relations for the Bateman-Hillion type nondispersive wave packets are constructed

1 Introduction

The packet-like asymptotic solutions of the wave equation

\[ (\partial^2_{x^2} + \partial^2_{y^2} + \partial^2_{z^2} - \partial^2_{t^2})u(x, y, z, t) = 0. \]  

(1)

trace their origin from the Bateman’s works on conformal symmetry [1, 2]. Historically such solutions were first derived approximately in terms of parabolic equation of the diffraction and are related to paraxial optical beams, see [3] for a review. The interest to the localized solutions of wave equation is encouraged by the progress in generation of ultra-short laser pulses [4], in view of the fact that large variety of optical problems can be solved in terms of scalar equation. The packet-like localized solutions of equation (1) have a general form [5]:

\[ u = \frac{f(\theta)}{\sqrt{2}\xi_+ - i\varepsilon}, \quad \theta = \sqrt{2}\xi_- + \frac{x^2 + y^2}{\sqrt{2}\xi_- - i\varepsilon}, \]  

(2)

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where \( f \) is an arbitrary localized function,

\[
\xi_{\pm} = \frac{z \pm t}{\sqrt{2}},
\]

(3)

are the light-cone coordinates, describing a particle moving at a speed of light \( (c = 1) \) along \( z \) axis, \( \varepsilon > 0 \). Their particular form with \( f(\theta) = e^{\imath q \theta} \) was called 'quasiphotons' by Babich and Ulin [6], or generally 'focus wave modes' [7], for the waves which remain focused in \( \xi_- \) coordinate.

Although the plane-wave solutions \( \exp(\imath(kx - \omega t)) \) of equation (1) have laid the basis of quantum electrodynamics [8, 9], the packet-like solutions of type (2) were considered only as a special case of classical electrodynamics. Their importance has raised after the seminal paper of Brittingham [7], who has proved the (2)-type solutions – the focus wave modes – (i) satisfy the homogeneous Maxwell’s equations, (ii) are continuous and non-singular, (iii) have a three-dimensional pulse structure, (iv) be non-dispersive for all time, (v) move at light velocity in straight lines. This stimulated a series of work indented for practical application of localized electromagnetic pulses for long-range energy transfer without dispersion, electromagnetic and acoustic bullets, Gaussian wave beams in optics and geophysics, optical information processing in microcavities [10, 11, 12, 13].

The application of the Bateman-Hillion-like solutions or weakly localized wave packets to quantum field theory was restricted to the formal studies of the localized solutions of the Klein-Gordon and the Dirac equations [14, 15]. In contrast to the solitons, the solutions localized due to nonlinearity, which have long been studied in quantum field theory, see e.g.[16], neither the Bateman-Hillion solutions of wave equations nor the Moses-Prosser wave bullets have been ever considered as operator-valued functions and have been ever subjected to canonical commutation relations.

In the present paper we consider the nondispersive solutions of wave equation, as particle-like solutions of the field equations, describing a quantum particles subjected to canonical commutation relations. This quantization condition results in certain restrictions on the amplitude and the width of the pulse-wave which makes it into quantum particle. In the next sections we will consider this problem for free scalar field theory models in \( 1 + 1 \) and \( 3 + 1 \) dimensions.
2 \quad d = 2

The nondispersive localized wave solutions in three plus one dimensions are generalizations of the traveling wave solutions of the wave equation in one plus one dimension:

\[(\partial_{z^2} - \partial_{t^2})u(z,t) = 0,\]

which is equivalent to the equation \(\partial_{\xi^2}^2 u(\xi, \xi) = 0\). The solution of the equation (4) is a superposition of two independent solutions traveling right and left along the \(z\) axis:

\[u = f(\xi^-) + g(\xi^+),\]

where \(f\) and \(g\) are arbitrary functions.

In quantum field theory, the field \(\phi\), which satisfies the massless field equation (4), is considered as an operator-valued function \(\phi(z, t)\). Using the Fourier transform, the field \(\phi\) can be cast as a sum of the positive and negative frequency components:

\[\phi^\pm(x) = \int_{k_0>0} e^{\pm ikx} \delta(k^2 - m^2)\phi(\pm k)dk.\]

In case of massless field in two dimensions, \(x = (z, t)\), this gives

\[\phi(z, t) = \int \frac{dk}{2\pi 2\omega_k} \left[ e^{-\omega_k t + ikz} \hat{u}(k) + e^{\omega_k t - ikz} \hat{u}^\dagger(k) \right],\]

where the integration over \(\frac{dkd\omega_k}{(2\pi)^2}\) is made into one dimensional integration \(\frac{dk}{2\pi 2\omega_k}\) using the mass-shell delta-function \(\delta(\omega^2 - k^2)\), which results in \(\omega_k = |k|\). The operators \(\hat{u}(k)\) and \(\hat{u}^\dagger(k)\) are referred to as the annihilation and the creation operators for the quanta with momentum \(k\). They satisfy commutation relations:

\[[\hat{u}(k), \hat{u}^\dagger(k')] = 2\pi 2\omega_k \delta(k - k').\]

The equation (5) is the basis of field quantization. However, its physical interpretation leads to counterintuitive result: if a photon is described by plane wave (5), then the absorption of photon by photographic plate should expose the whole plate: for the plane wave is present everywhere. In reality the exposure is very local. This prompts us to use some localized function, travelling at a speed of light, instead of plane waves. Same as for classical
$c$-valued fields the localization is achieved by substituting plane wave by the Fourier image of some localized function. This leads to the operator-valued solution of the massless field equation

$$
\phi(z, t) = \int \frac{dk}{2\pi 2\omega_k} \left[ e^{-\omega_k t + i k z} c(k) \hat{u}(k) + e^{\omega_k t - i k z} c^*(k) \hat{u}^\dagger(k) \right],
$$

(7)

here we assume $c^*(k) = c(-k)$.

The canonical momentum, conjugated to the field density $\phi(z, t)$ is

$$
\pi(z, t) = -\frac{i}{2} \int \frac{dk}{2\pi} \left[ e^{-\omega_k t + i k z} c(k) \hat{u}(k) - e^{\omega_k t - i k z} c^*(k) \hat{u}^\dagger(k) \right].
$$

(8)

Since we consider the wave packet $\phi$ as a quantum particle we can introduce the operator of the "mean field coordinate" $\hat{Q}(t)$ and the total momentum $\hat{P}(t)$ of the wave packet

$$
\hat{Q}(t) = \frac{1}{V} \int \phi(z, t) dz, \quad \hat{P}(t) = \int \pi(z', t) dz',
$$

where $V$ is the volume occupied by the field. Using the commutation relations (6) for the Fourier modes, we get the commutator

$$
[\hat{Q}, \hat{P}] = \frac{i}{V} \int dz dz' e^{ik(z - z')} |c(k)|^2 \frac{dk}{2\pi}.
$$

Thus to ensure canonical commutation relations for the wave packet

$$
[\hat{Q}, \hat{P}] = i
$$

(9)

we need to fulfil the constraint

$$
\int \Lambda(z - z') \frac{dz dz'}{V} = 1,
$$

where

$$
\Lambda(z) = \int e^{ikz} |c(k)|^2 \frac{dk}{2\pi}.
$$

Using the symmetry $|c(k)|^2 = |c(-k)|^2$ the constraint that ensures canonical commutation relations (9) for the operator-valued wave packet (7) can be written in the form

$$
2 \int_0^\infty dz \int_{-\infty}^\infty \cos(kz) |c(k)|^2 \frac{dk}{2\pi} = 1.
$$

(10)

For the Gaussian wave packet with $c(k) = Ae^{-k^2\sigma^2/2}$ this leads to the constraint $A^2 = 1$ independent on $\sigma$. 

4
Without loss of generality we can consider the Klein-Gordon equation in $3+1$ dimensions
\[(\partial^2_{x^2} + \partial^2_{y^2} + \partial^2_{z^2} - \partial^2_{t^2} - m^2)u(x, y, z, t) = 0\] (11)
with $m$ set to zero for the case of massless field. The Fourier image of the localized solution of the Klein-Gordon equation [15] can be written as
\[g(\sqrt{2}k_z) = \frac{B}{k_+} \int \frac{k_+}{\sqrt{2}} \delta(k^2 - m^2), \quad k_+ = \frac{k_z + \omega_k}{\sqrt{2}},\] (12)
where $B$ is a constant, $\delta = 0, \frac{1}{2}$ - for massless, and massive field, respectively, see B, C.

This leads to the operator-valued solution of the field equation
\[\phi(x, t) = \int \frac{d^3k}{(2\pi)^3 2\omega_k} \left[ e^{-i\omega_k t + ikx} g(k_z + \omega_k) \hat{u}(k) + e^{-i\omega_k t - ikx} g(-k_z - \omega_k) \hat{u}^\dagger(k) \right],\] (13)
where the annihilation and the creation operators satisfy the commutation relations
\[[\hat{u}(k), \hat{u}^\dagger(k')] = (2\pi)^3 \cdot 2\omega_k \cdot \delta(k - k'), \quad \omega_k = \sqrt{k^2 + m^2}.\] (14)

The canonical momentum, conjugated to the field density $\phi(x, t)$, is $\pi(x, t) = \frac{\partial \phi}{\partial t}$.

\[\pi(x, t) = -\frac{i}{2} \int \frac{d^3k}{(2\pi)^3} \left[ e^{-i\omega_k t + ikx} g(k_z + \omega_k) \hat{u}(k) - e^{-i\omega_k t - ikx} g(-k_z - \omega_k) \hat{u}^\dagger(k) \right].\] (15)

The "mean field coordinate" $\hat{Q}(t)$ and the total momentum $\hat{P}(t)$ of the wave packet are:
\[\hat{Q}(t) = \frac{1}{V} \int \phi(x, t) d^3x, \quad \hat{P}(t) = \int \pi(x', t) d^3x'.\]

The constraint (9) results in
\[1 = \int d^3x e^{ikx} g(k_z + \omega_k) g(-k_z - \omega_k) \frac{d^3k}{(2\pi)^3} = g(k_z + \omega_k) g(-k_z - \omega_k)|_{k=0}.\] (16)
For Gaussian packet $g(k) = Ae^{-\frac{k^2\sigma^2}{2}}$ this gives the normalization constraint

$$A^2e^{-\sigma^2m^2} = 1.$$  \quad (17)

### 4 Conclusions

The idea of construction of nondispersive wave packets, which follow a trajectory of classical particle, from the coherent superposition of harmonic oscillators can be traced back to Schrödinger [17]. In classical scales such packets of different form can be created by combination of electromagnetic pulses of equispaced frequencies. In quantum physics, where the energy levels of atoms, used for laser pulse generation, are not equispaced, the known way to suppress wave packet dispersion is to use external electromagnetic fields [18, 19]. The most intensive experimental studies have been performed in creation Trojan wave packets – nondispersive wave packets of electron density on circular orbits [20]. The creation of nondispersive packet on a circular orbit is achieved by using circularly polarized laser beams and is a technically sophisticated, however it seems that a simpler problem of creating a nondispersive wave packet moving linearly, which was studied in classical electrodynamics since Brittingham [7] and is implemented experimentally, was not implemented in quantum case, regardless an evident advantage of no need of the external field for suppressing dispersion. In present paper, using a simple scalar model, we constructed the quantization condition for such packets in purely relativistic case. If such mechanism exists in quantum electrodynamics, it may be related to the coherent electromagnetic energy transfer at mesoscopic scales, along with excitons and other nonrelativistic mechanisms, see e.g. [21]. In future we hope to apply the localized wave packets, subjected to quantization constraints described in this paper, to the scattering problems of localized quantum particles described by quantum field theory methods [22]. In particular the existence of the localized focus wave modes [7] in classical electrodynamics may be of interest for the quantum theory of gauge fields [23].

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A  The Bateman-Hillion solution of wave equation

The Bateman-Hillion solution of the wave equation (1) can be obtained by the change of variable (§4.2.1 of [3]):

\[ u = \int_{-\infty}^{\infty} \hat{u} e^{i\alpha a} \, da, \]

(18)

where \( \hat{u} = \hat{u}(x, y, \alpha, \beta) \), \( \alpha = z - t \), \( \beta = z + t \).

The new function \( \hat{u} \) satisfies the parabolic equation

\[ \frac{\partial^2 \hat{u}}{\partial x^2} + \frac{\partial^2 \hat{u}}{\partial y^2} + 4ia \frac{\partial \hat{u}}{\partial \beta} = 0, \]

which has the solution

\[ \hat{u} = \beta^{-1} \exp(ia(x^2 + y^2) / \beta) \hat{f}(a), \]

(19)

where \( \hat{f}(a) \) is arbitrary function. The Bateman-Hillion solution is obtained by substitution of (19) into (18).

B  Fourier image of the massless field solution

To evaluate the Fourier image of the solution (2) of wave equation (1) we use the light-cone variables

\[ \xi_\pm = \frac{z \pm t}{\sqrt{2}}, \quad k_\pm = \frac{k_z \pm \omega}{\sqrt{2}}. \]

In these variables

\[ \tilde{\phi}(k_x, k_y, k_+, k_-) = \int dx dy d\xi_- d\xi_+ \frac{f(\theta)}{\sqrt{2}\xi_+} e^{-i(k_+ x + k_- y + k_+ \xi_- + k_- \xi_+)} \]

\[ = \frac{1}{2} \int d\theta d\xi_+ dx dy \frac{f(\theta)}{\xi_+} \exp \left[ -i(k_x x + k_y y + k_- \xi_+ + \frac{k_+}{\sqrt{2}} \left[ \theta - \frac{x^2 + y^2}{\sqrt{2}\xi_+} \right] \right]. \]
We have the product of two identical integrals in $x$ and $y$ transversal coordinates
\[
\int_{-\infty}^{\infty} e^{ikx^2 - ikx} dx \int_{-\infty}^{\infty} e^{iky^2 - iky} dy = \frac{i\pi}{k} e^{-\frac{k^2}{4k}} \tag{20}
\]
with $k = \frac{k_+}{2\xi}$. The remaining integration in $\xi_+$ is performed by the change of variable $r = \xi_+/k_+$. This gives
\[
\tilde{\phi}(k_x, k_y, k_+, k_-) = 2i\pi^2 \tilde{f} \left( \frac{k_+}{\sqrt{2}} \right) \delta \left( \frac{k_+^2}{2} + k_+ k_- \right) \tag{21}
\]

C Fourier image of the massive solution

To evaluate the Fourier image of the localized solution for the Klein-Gordon equation we integrate the massless solution in 5d over the extra variable $z'$ [15]:
\[
\tilde{\phi}_m(k_x, k_y, k_+, k_-) = \int e^{-i(k_x x + k_y y + m z' + k_+ \xi_- + k_- \xi_+)} \times
\]
\[
\times \frac{f(\theta)}{(\sqrt{2\xi_+})^{3/2}} dx dy dz' d\xi_- d\xi_+ = \int e^{-i(k_x x + k_y y + m z' + k_+ \xi_+)} \times
\]
\[
\times e^{\frac{k_+}{\sqrt{2} \xi_+} \left( \theta - \frac{z'^2 + y^2 + z'^2}{2\xi_+} \right)} \frac{f(\theta)}{(\sqrt{2\xi_+})^{3/2}} \sqrt{2} d\xi_+ dx dy dz'.
\]
The product of three identical integrals in $x, y, z'$, cf. Eq. 20, is equal to
\[
\frac{(2\pi)^{3/2}}{\sqrt{2}} \left( \frac{\xi_+}{k_+} \right)^{3/2} e^{-\frac{1}{2}(k_+^2 + m^2) \frac{\xi_+}{k_+}}.
\]
Introducing the new variable $r = \frac{\xi_+}{k_+}$ and integrating over it we get
\[
\tilde{\phi}_m(k_x, k_y, k_+, k_-) = \frac{\tilde{f} \left( \frac{k_+}{\sqrt{2}} \right)}{\sqrt{2i k_+ 2^{3/4}}} \left( 2\pi \right)^{5/2} \delta \left( \frac{k_+^2 + m^2}{2} + k_+ k_+ \right). \tag{22}
\]