On the Existence of General Factors in Regular Graphs

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Abstract

Let $G$ be a graph, and $H : V(G) \to 2^\mathbb{N}$ a set function associated with $G$. A spanning subgraph $F$ of $G$ is called an $H$-factor if the degree of any vertex $v$ in $F$ belongs to the set $H(v)$. This paper contains two results on the existence of $H$-factors in regular graphs. First, we construct an $r$-regular graph without some given $H^*$-factor. In particular, this gives a negative answer to a problem recently posed by Akbari and Kano. Second, by using Lovász’s characterization theorem on the existence of $(g, f)$-factors, we find a sharp condition for the existence of general $H$-factors in $\{r, r+1\}$-graphs, in terms of the maximum and minimum of $H$. The result reduces to Thomassen’s theorem for the case that $H(v)$ consists of the same two consecutive integers for all vertices $v$, and to Tutte’s theorem if the graph is regular in addition.

Keywords: $H$-factor; $\{k, r-k\}$-factor; regular graph

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1 Introduction

Let $G = (V(G), E(G))$ be a simple graph, where $V(G)$ and $E(G)$ denote the set of vertices and edges of $G$ respectively. For any vertex $v$, denote the degree of $v$ by $d_G(v)$. Let $2^\mathbb{N}$ denote the collection of sets of nonnegative integers. We call

$$H : V(G) \to 2^\mathbb{N}$$

a set function associated with $G$ if $H(v) \subseteq \{0, 1, \ldots, d_G(v)\}$. A spanning subgraph $F$ of $G$ is called an $H$-factor if $d_F(v) \in H(v)$ for all $v$. It is often that $H(v)$ coincides
with some set $H'$ for all $v$. In this case, we call $H'$ a set associated with $G$, and call $F$ an $H'$-factor without confusion. Let

$$g, f : V(G) \to \mathbb{Z}$$

be two functions such that $g(v) \leq f(v)$ for all $v$. An $H$-factor is called a $(g, f)$-factor if $H(v)$ is the interval $[g(v), f(v)]$ for all $v$. A $(g, f)$-factor is called an $[a, b]$-factor if $g(v) = a$ and $f(v) = b$ for all $v$. An $[a, b]$-factor $F$ is called an $(a, b)$-parity-factor if

$$d_F(v) \equiv a \equiv b \pmod{2}$$

for every vertex $v$.

In particular, $F$ is called a $k$-factor if $a = b = k$.

A graph is said to be $r$-regular if every vertex has degree $r$. This paper is concerned with the existence of $H$-factors in regular graphs. The study on the existence of factors in regular graphs was started, to the best of our knowledge, from Petersen [9].

**Theorem 1.1** (Petersen). Let $r$ and $k$ be even integers such that $1 \leq k \leq r$. Then any $r$-regular graph has a $k$-factor.

In contrast with even-factors in Theorem 1.1, Gallai [6] obtained the next result for odd-factors. For any graph $G$, we call the number $|V(G)|$ of vertices the order of $G$, denoted alternatively by $|G|$ as usual.

**Theorem 1.2** (Gallai). Let $r, k$ and $m$ be integers such that $r$ is even, $k$ is odd and

$$\frac{r}{m} \leq k \leq r \left(1 - \frac{1}{m}\right).$$

Then any $m$-edge-connected $r$-regular graph of even order has a $k$-factor.

It is clear that having an odd-factor implies that the order of the graph must be even. So the “even order” condition in Theorem 1.2 is not a real restriction. Removing the parity conditions for both $r$ and $k$, Tutte [12] gave the following theorem.

**Theorem 1.3** (Tutte). Let $1 \leq k \leq r - 1$. Then any $r$-regular graph has a $\{k, k + 1\}$-factor.

A graph $G$ is said to be an $\{r, r + 1\}$-graph if every vertex of $G$ has degree $r$ or $r + 1$. Thomassen [11] generalized Theorem 1.3 by considering $\{r, r + 1\}$-graphs.

**Theorem 1.4** (Thomassen). Let $1 \leq k \leq r - 1$. Then any $\{r, r + 1\}$-graph has a $\{k, k + 1\}$-factor.

For more results along this line, the reader is referred to Akiyama and Kano’s book [3]. Recently, Akbari and Kano [2] considered the existence of $\{k, r - k\}$-factors in $r$-regular graphs.
Theorem 1.5 (Akbari-Kano). Let \( r \) and \( k \) be integers such that \( r \) is odd, \( k \) is even and \( 1 \leq k \leq r \). Then any \( r \)-regular graph has a \( \{k, r-k\} \)-factor.

By Theorems 1.1, 1.3 and 1.5 any \( r \)-regular graph has a \( \{k, r-k\} \)-factor as if \( k \) is even. For odd \( k \), Akbari and Kano [2] posed the next problem for the case \( r \) is even, and a conjecture for the case that \( r \) is odd.

Problem 1.6 (Akbari-Kano). Let \( r \) and \( k \) be integers such that \( r \) is even, \( k \) is odd and \( 1 \leq k \leq \frac{r}{2} - 1 \). Is it true that every connected \( r \)-regular simple graph of even order has a \( \{k, r-k\} \)-factor?

Again, the “even order” condition is not a real restriction. On the other hand, any \( r \)-regular graph of even order has an \( \frac{r}{2} \)-factor. This can be seen immediately from Theorem 1.2 if one notices that any even-regular graph is 2-edge connected. Therefore, the condition \( 1 \leq k \leq \frac{r}{2} - 1 \) is not a real restriction either.

The first aim of this paper is to give a negative answer to Problem 1.6. In Section 2, we construct an \( r \)-regular graph \( G^* \) without \( \{k, r-k\} \)-factors for all \( 1 \leq k \leq \frac{r}{2} - 2 \), and deal with the case \( k = \frac{r}{2} - 1 \) by using the following Lovász’s characterization [8] (see also [3, Theorem 6.1]) on parity-factors. For any two subsets \( S \) and \( T \) of \( V(G) \), denote by \( E_G(S, T) \) the set of edges with one end in \( S \) and the other end in \( T \). Denote \( e_G(S, T) = |E_G(S, T)| \).

Theorem 1.7 (Lovász). Let \( G \) be a graph, and \( g, f: V(G) \to \mathbb{Z} \) be functions such that \( g(v) \leq f(v) \) and \( g(v) \equiv f(v) \pmod{2} \) for all vertices \( v \). Then \( G \) has a \( (g, f) \)-parity-factor if and only if

\[
\eta(S, T) = \sum_{s \in S} f(s) + \sum_{t \in T} (d_G(t) - g(t)) - e_G(S, T) - q(S, T) \geq 0 \quad (1.1)
\]

for all disjoint subsets \( S \) and \( T \) of \( V(G) \), where \( q(S, T) \) denotes the number of components \( C \) of the graph \( G - S - T \) such that

\[
\sum_{c \in V(C)} f(c) + e_G(V(C), T) \equiv 1 \pmod{2}. \quad (1.2)
\]

In fact, Lovász [8] presented a structural description for the degree constrained subgraph problem for the case that no two consecutive integers are missed in \( H(v) \) for every \( v \). He also showed that the problem without this restriction is NP-complete. In particular, the next theorem, which is due to Lovász [7] (see also [3, Theorem 4.1]), will be used in our deduction.

Theorem 1.8 (Lovász). Let \( G \) be a graph, and \( g, f: V(G) \to \mathbb{Z} \) be functions such that \( g(v) \leq f(v) \) for all vertices \( v \). Then \( G \) has a \( (g, f) \)-factor if and only if

\[
\gamma(S, T) = \sum_{s \in S} f(s) + \sum_{t \in T} (d_G(t) - g(t)) - e_G(S, T) - q^*(S, T) \geq 0
\]
for all disjoint subsets $S$ and $T$ of $V(G)$, where $q^*(S, T)$ denotes the number of components $C$ of the graph $G - S - T$ satisfying (1.2), and $g(v) = f(v)$ for all $v \in V(C)$.

By using Alon’s combinatorial nullstellensatz [4], Shirazi and Verstraëte [10] established the following brief result for general $H$-factors, which was originally posed by Addario-Berry et al. [1] as a conjecture.

**Theorem 1.9 (Shirazi-Verstraëte).** Let $G$ be a graph with an associated set function $H$. If

$$|H(v)| > \left\lceil \frac{d_G(v)}{2} \right\rceil$$

for all $v \in V(G)$, \hspace{1cm} (1.3)

then $G$ has an $H$-factor.

Frank et al. [5] found an elementary proof for Theorem 1.9 by using the next result on directed graphs. For any directed graph $G$, denote by $d^-_G(v)$ the in-degree of $v$.

**Theorem 1.10 (Frank et al.).** Let $G$ be a graph with an associated set function $H$. If $G$ has an orientation for which

$$d^-_G(v) \geq |\{0, 1, \ldots, d_G(v)\}\setminus H(v)|$$

for all $v \in V(G)$, \hspace{1cm} (1.4)

then $G$ has an $H$-factor.

It seems that the existence of $H$-factors in regular graphs has not been extensively investigated yet. Let $G$ be a graph, and $H$ a set function associated with $G$. Denote

$$mH = \min_{v \in G} \min H(v),$$

$$MH = \max_{v \in G} \max H(v).$$

Here is the second result of this paper.

**Theorem 1.11.** Let $G$ be an \{$r, r + 1\}$-graph with an associated set function $H$. If $mH \geq 1$, $MH \leq r$ and

$$|H(v)| \geq \frac{MH - mH + 3}{2}$$

for all $v \in V(G)$, \hspace{1cm} (1.5)

then $G$ has an $H$-factor.

The proof of Theorem 1.11 will be given in Section 3. As will be seen, the condition (1.5) is sharp. For the case

$$H(v) = \{k, k + 1\}$$

for all $v \in V(G)$,

where $1 \leq k \leq r - 1$, Theorem 1.11 reduces to Theorem 1.4. Moreover, as a result restricting to \{$r, r + 1\}$-graphs, Theorem 1.11 is stronger than Theorem 1.9 because the condition (1.3) implies (1.5) for \{$r, r + 1\}$-graphs.
2 Answer to Akbari-Kano’s problem

This section is concerned with Problem 1.6. Note that $1 \leq k \leq r/2 - 1$. The following theorem deal with the case $k \leq r/2 - 2$. For any integer $n$, denote by $[n]_{\text{odd}}$ the set of positive odd integers less than or equal to $n$. For any vertex $v$ in any graph $G$, denote by $N_G(v)$ the neighborhood of $v$ in $G$.

**Theorem 2.1.** For any even integer $r$, there exists an $r$-regular graph $G^*$ of even order such that $G^*$ has no $H^*$-factors where

$$H^* = [r]_{\text{odd}} \setminus \left\{ \frac{r}{2} - 1, \frac{r}{2}, \frac{r}{2} + 1 \right\}.$$

In particular, $G^*$ has no $\{k, r - k\}$-factors for any odd integer $k$ such that $1 \leq k \leq r/2 - 2$.

**Proof.** Let $J$ be the graph obtained by removing an edge from the complete graph $K_{r+1}$. Let $J_1, J_2, \ldots, J_r$ be pairwise disjoint copies of $J$. In each copy $J_i$, let $a_i$ and $b_i$ be the ends of the edge that removed from $K_{r+1}$. Let $G^*$ be the graph consisting of the copies $J_1, J_2, \ldots, J_r$, together with two new vertices $u$ and $v$, such that

$$N_{G^*}(u) = \{ a_1, b_1, a_2, b_2, \ldots, a_{\frac{r}{2}-1}, b_{\frac{r}{2}-1}, a_r \},$$

$$N_{G^*}(v) = \{ a_{\frac{r}{2}}, b_{\frac{r}{2}}, a_{\frac{r}{2}+1}, b_{\frac{r}{2}+1}, \ldots, a_{r-2}, b_{r-2}, b_{r-1}, b_r \}.$$  \hspace{1cm} (2.1)

Then $G^*$ is an $r$-regular graph of the even order $r(r + 1) + 2$.

Now we shall show that $G^*$ has no $H^*$-factors. Suppose to the contrary that $F$ is an $H^*$-factor of $G^*$. Let $1 \leq i \leq r$. Since $d_F(w)$ is odd for all $w \in J_i$, and the order $|J_i|$ is odd, we find

$$\sum_{w \in J_i} d_F(w) \equiv 1 \pmod{2}. \hspace{1cm} (2.2)$$

Let $F_i$ be the subgraph of $F$ induced by the vertices in $J_i$. By the Handshaking theorem, we have

$$\sum_{w \in J_i} d_{F_i}(w) \equiv 0 \pmod{2}. \hspace{1cm} (2.3)$$

Taking the difference between (2.2) and (2.3), we obtain

$$e_F(J_i, \{u, v\}) = \sum_{w \in J_i} (d_F(w) - d_{F_i}(w)) \equiv 1 \pmod{2}.$$ 

Since $e_{G^*}(J_i, u) = 2$ and $e_{G^*}(J_i, v) = 0$ for $1 \leq i \leq r/2 - 1$, we derive

$$e_F(J_i, u) = 1 \quad \text{for } 1 \leq i \leq \frac{r}{2} - 1.$$ 

By the definition (2.1) of $N_{G^*}(u)$, we get

$$d_F(u) \in \left\{ \frac{r}{2} - 1, \frac{r}{2}, \frac{r}{2} + 1 \right\},$$

contradicting the definition of $H^*$. This completes the proof.
The graph $G^*$ constructed above will be used to explain the sharpness of the condition (1.5) in the next section. Now we cope with the case $k = r/2 - 1$.

**Theorem 2.2.** Let $r$ be an even integer such that $r/2$ is even. Then any connected $r$-regular graph of even order has an $\{r/2 - 1, r/2 + 1\}$-factor.

**Proof.** We shall apply Theorem 1.7 by setting $g(v) = r/2 - 1$ and $f(v) = r/2 + 1$ for all vertices $v$. Let $G$ be a connected $r$-regular graph of even order. Let $S$ and $T$ be disjoint subsets of $V(G)$. First, we claim that

\[ e_G(S \cup T, V(G) \setminus S) \geq 2q(S, T). \] \hspace{1cm} (2.4)

In fact, if $S \cup T \in \{\emptyset, G\}$, then $q(S, T) = 0$, and (2.4) follows immediately. Otherwise, let $C$ be a component of $G - S - T$. Then both $S \cup T$ and $C$ are nonempty. Note that any even-regular graph is 2-edge-connected. So $G$ is 2-edge-connected. In particular, we have

\[ e_G(S \cup T, C) \geq 2. \]

Summing the above inequality over all components $C$, we get the desired inequality (2.4). Hence,

\[ \eta(S, T) = \left( \frac{r}{2} + 1 \right)(|S| + |T|) - e_G(S, T) - q(S, T) \]
\[ \geq \frac{r}{2}(|S| + |T|) - e_G(S, T) - \frac{1}{2}e_G(S \cup T, V(G) \setminus S) \]
\[ = e_G(S, S) + e_G(T, T) \geq 0. \]

By Theorem 1.7, $G$ has an $\{r/2 - 1, r/2 + 1\}$-factor. \hfill \qed

Combining Theorems 2.1 and 2.2, we obtain a negative answer to Problem 1.6.

3 The existence of $H$-factors in regular graphs

This section is devoted to establish Theorem 1.11. A subset $U$ of $V(G)$ is called independent if any two vertices in $U$ are not adjacent in $G$. We need the following lemma to prove Theorem 1.11.

**Lemma 3.1.** Let $r$ and $k$ be positive integers such that $1 \leq k \leq r - 1$. Let $G$ be an $\{r, r + 1\}$-graph and

\[ U = \{ v \in V(G) \mid d_G(v) = r + 1 \}. \]

If $U$ is independent, then $G$ has a $\{k, k + 1\}$-factor $F$ such that

\[ d_F(u) = k + 1 \quad \text{as if} \quad u \in U. \]
Proof. Let \( f(v) = k + 1 \) for all vertices \( v \), and

\[
g(v) = \begin{cases} 
  k + 1, & \text{if } v \in U, \\
  k, & \text{otherwise.}
\end{cases}
\]

It suffices to show that \( G \) has a \((g, f)\)-factor. Suppose to the contrary that \( G \) has no \((g, f)\)-factors. By Theorem 1.8 we have

\[
\gamma(S, T) < 0 \quad \text{for some } S, T \subseteq V(G).
\]

Let \( S \) and \( T \) be disjoint subsets of \( V(G) \) such that \( \gamma(S, T) < 0 \) and the set \( S \cup T \) is maximal. We claim that \( q^*(S, T) = 0 \).

Suppose to the contrary that \( q^*(S, T) \geq 1 \). Let \( C \) be a component of \( G - S - T \) counted by \( q^*(S, T) \). It follows that

\[
e_G(C, G - S - T) = 0. \tag{3.1}
\]

By the definition of \( q^*(S, T) \), we have

\[
g(v) = f(v) = k + 1 \quad \text{for all } v \in V(C). \tag{3.2}
\]

So \( V(C) \subseteq U \). But \( U \) is independent, we deduce that \( C \) is a single vertex, say, \( V(C) = \{a\} \). Let \( S' = S \cup \{a\} \) and \( T' = T \cup \{a\} \). Then (3.1) implies

\[
q^*(S', T) = q^*(S, T) - 1, \tag{3.3}
\]

\[
q^*(S, T') = q^*(S, T) - 1. \tag{3.4}
\]

Note that the condition (1.2) implies \( e_G(a, T) \neq k + 1 \). If \( e_G(a, T) \leq k \), then (3.1) and (3.2) yield

\[
d_G(a) - e_G(a, S) = e_G(a, T) \leq g(a) - 1.
\]

Together with (3.3), we have

\[
\gamma(S, T') - \gamma(S, T) = d_G(a) - g(a) - e_G(S, a) - q^*(S, T') + q^*(S, T) \leq 0.
\]

So \( \gamma(S, T') < 0 \), contradicting the maximality of \( S \cup T \). Otherwise \( e_G(a, T) \geq k + 2 \).

By (3.3), we deduce

\[
\gamma(S', T) - \gamma(S, T) = f(a) - e_G(a, T) - q^*(S', T) + q^*(S, T) \leq 0.
\]

So \( \gamma(S', T) < 0 \), contradicting, again, the maximality of \( S \cup T \). Thus the claim is true.
Now we can deduce
\[
\gamma(S, T) = \sum_{s \in S} d_G(s) \frac{f(s)}{d_G(s)} + \sum_{t \in T} d_G(t) \left(1 - \frac{g(t)}{d_G(t)}\right) - e_G(S, T)
\]
\[
\geq \sum_{s \in S, t \in T \atop st \in E(G)} \left(\frac{f(s)}{d_G(s)} + \left(1 - \frac{g(t)}{d_G(t)}\right)\right) - e_G(S, T)
\]
\[
= \sum_{s \in S, t \in T \atop st \in E(G)} \left(\frac{k + 1}{d_G(s)} - \frac{g(t)}{d_G(t)}\right)
\]
\[
\geq \sum_{x \in S, y \in T \atop xy \in E(G)} \left(\frac{k + 1}{r + 1} - \max \left(\frac{k}{r}, \frac{k + 1}{r + 1}\right)\right) = 0,
\]
contradicting the hypothesis \(\gamma(S, T) < 0\). This completes the proof. 

We remark that Lemma 3.1 is a generalization of Theorem 1.3. Now we are in a position to prove Theorem 1.11.

Proof. Write \(m = mH\) and \(M = MH\) for short. By Theorem 1.4, we can suppose that \(F\) is an \(\{M, M + 1\}\)-factor of \(G\) with the minimum number of edges. It follows that any two vertices of degree \(M + 1\) in \(F\), if they exist, are not adjacent. By Lemma 3.1, \(F\) has an \(\{m - 1, m\}\)-factor, say, \(F'\), such that
\[
d_{F'}(v) = m \quad \text{as if} \quad d_F(v) = M + 1. \tag{3.5}
\]
Let \(F''\) be the complemented graph of \(F'\) in \(F\). In view of (3.5), we have
\[
d_{F''}(v) \in \{M - m, M - m + 1\} \quad \text{for all} \ v. \tag{3.6}
\]
We observe that \(F''\) has an orientation such that
\[
d_{F''}(v) \geq \left\lfloor \frac{d_{F''}(v)}{2} \right\rfloor \quad \text{for all} \ v. \tag{3.7}
\]
This can be seen by orienting an eulerian tour of the graph that obtained from \(F''\) by adding a new vertex and joining it to all vertices of odd degree in \(F''\). Let
\[
H'(v) = \{h - d_{F'}(v) \mid h \in H(v)\} \quad \text{for all} \ v.
\]
Then the condition (1.5) reads
\[
|H'(v)| = |H(v)| \geq \frac{M - m + 3}{2}. \tag{3.8}
\]
By (3.6), (3.7) and (3.8), it is easy to verify that
\[
\left|\{0, 1, \ldots, d_{F''}(v)\} \setminus H'(v)\right| \leq d_{F''}(v) \quad \text{for all} \ v.
\]
By Theorem 1.10 the graph \(F''\) has an \(H\)-factor, say, \(G'\). Hence, the graph induced by the edge set \(E(F') \cup E(G')\) is an \(H\)-factor of \(G\). This completes the proof. 

In fact, the condition (1.5) is sharp. For instance, when \( r \) is even, let \( G^* \) be the graph constructed in the proof of Theorem 2.1. Consider a set \( H \) of the form
\[
H = \{m, m + 2, m + 4, \ldots, M\},
\]
where both \( m \) and \( M \) are odd, and \( M \leq r/2 - 2 \). On one hand, \( G^* \) has no \( H \)-factors by Theorem 2.1. On the other hand, it is straightforward to compute
\[
|H| = \frac{M - m + 2}{2}.
\]

Comparing it with the condition (1.5), we deduce the latter one is sharp. For other possibilities of the associated set \( H \), for example, \( mH + MH \) is odd, we mention that it is also not hard to find \( r \)-regular graphs without \( H \)-factors such that
\[
|H(v)| = \left\lfloor \frac{MH - mH + 2}{2} \right\rfloor \quad \text{for all } v \in V(G).
\]

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