Anisotropic Born-Infeld Cosmologies

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Abstract

Anisotropic cosmological spacetimes are constructed from spherically symmetric solutions to Einstein’s equations coupled to nonlinear electrodynamics and a positive cosmological constant. This is accomplished by finding solutions in which the roles of $r$ and $t$ are interchanged for all $r > 0$ (i.e. $r$ becomes timelike and $t$ becomes spacelike). Constant time hypersurfaces have topology $R \times S^2$ and in all the spacetimes considered the radius of the two sphere vanishes as $t$ goes to zero. The scale factor of the other dimension diverges as $t$ goes to zero in some solutions and vanishes (or goes to a constant) in other solutions. At late times local observers would see the universe to be homogeneous and isotropic.
Introduction

Over the last few years Born-Infeld theory \[1\] has undergone a revival due to its appearance in string theory \[2\]. In this paper some exact cosmological solutions are found to the Einstein field equations coupled to nonlinear electrodynamics and a positive cosmological constant. These solutions are constructed from spherically symmetric solutions with \( g_{tt} = 1 / g_{rr} = -(1 - 2m(r)/r) \). If \( m(r) > \frac{1}{2}r \) for \( 0 < r < \infty \) then \( r \) and \( t \) interchange roles and the solutions describe cosmological spacetimes with a singularity at \( t = 0 \) (instead of at \( r = 0 \)). Constant time hypersurfaces have topology \( R \times S^2 \) and the radius of the two sphere goes to zero as \( t \) goes to zero. The scale factor of the other dimension diverges as \( t \) goes to zero in some solutions and vanishes (or goes to a constant) in other solutions. The Schwarzschild solution with a cosmological constant leads to a cosmological solution as does Born-Infeld theory. However, Maxwell’s theory does not as it is not possible to satisfy \( m(r) > \frac{1}{2}r \) for all \( r \) on \((0, \infty)\). Some other Born-infeld cosmologies can be found in \[3, 4\].

Born-Infeld Theory

In nonlinear electrodynamics the Maxwell Lagrangian

\[
L = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} = \frac{1}{2}(E^2 - B^2)
\]

is replaced by

\[
L = L(F^2, G^2)
\]

where \( F^2 = \frac{1}{2}F^{\mu\nu}F_{\mu\nu}, \) \( G^2 = \frac{1}{2}F^{\mu\nu}F^*_{\mu\nu}, \) \( F^*_{\mu\nu} \) is the dual of \( F_{\mu\nu}, \) and \( L \) is any function that reduces to (1) in the weak field limit. Born and Infeld took \( L \) to be given by

\[
L = -\frac{1}{a^2}\left[\sqrt{1 + a^2F^2} - 1\right]
\]

For the solutions considered in this paper \( \vec{B} = 0 \) so that \( G^2 = 0 \). Thus, all \( G^2 \) dependences will be dropped.

The field equations are

\[
\nabla_{\mu}P^{\mu\nu} = 0
\]

and

\[
\nabla_{\mu}F^{*\mu\nu} = 0,
\]

where

\[
P^{\mu\nu} = \frac{\partial L}{\partial F_{\mu\nu}}
\]

The energy-momentum tensor is

\[
T^{\mu\nu} = -2P^{\mu\alpha}F^*_{\alpha\nu} + g^{\mu\nu}L
\]
and the “Hamiltonian”, which is a function of $P^{\mu\nu}$, is

$$H = P^{\mu\nu} F_{\mu\nu} - L.$$  \hfill (8)

For the Born-Infeld Lagrangian

$$T^{\mu\nu} = \left[ \frac{F^{\mu\alpha} F_{\nu}^{\alpha}}{\sqrt{1 + a^2 F^2}} - \frac{1}{a^2} g^{\mu\nu} \left( \sqrt{1 + a^2 F^2} - 1 \right) \right]$$  \hfill (9)

and

$$H(P^2) = \frac{1}{a^2} \left[ \sqrt{1 + a^2 P^2} - 1 \right]$$  \hfill (10)

where $P^2 = -2P^{\alpha\beta} P_{\alpha\beta}$.

**Cosmologies from Spherically Symmetric Solutions**

Birkhoff’s theorem holds for nonlinear electrodynamic theories and the general spherically symmetric solution is 

$$ds^2 = -\left[ 1 - \frac{2m(r)}{r} \right] dt^2 + \left[ 1 - \frac{2m(r)}{r} \right]^{-1} dr^2 + r^2 d\Omega^2$$  \hfill (11)

and

$$P = \frac{Q}{r^2} dt \wedge dr$$  \hfill (12)

and

$$\frac{dm(r)}{dr} = 4\pi r^2 H(P^2) + \frac{1}{2} r^2 \Lambda$$  \hfill (13)

where $P^2 = Q^2 / r^4$ and $\Lambda$ is the cosmological constant.

If $m(r) > \frac{1}{2} r$ for $0 < r < \infty$ then $r$ is a timelike coordinate and $t$ is a spacelike coordinate. Relabeling $r$ and $t$ and denoting the spacelike variable by $x$ gives

$$ds^2 = -\left[ \frac{2m(t)}{t} - 1 \right]^{-1} dt^2 + \left[ \frac{2m(t)}{t} - 1 \right] dx^2 + t^2 d\Omega^2$$  \hfill (14)

and

$$P = \frac{Q}{t^2} dx \wedge dt$$  \hfill (15)

and

$$\frac{dm(t)}{dt} = 4\pi t^2 H \left[ \frac{Q^2}{t^4} \right] + \frac{1}{2} t^2 \Lambda.$$  \hfill (16)

Constant timelike surfaces have topology $R \times S^2$ and the two sphere has radius $t$. The Ricci scalar is given by

$$R = -2 \left[ \frac{\dot{t} \ddot{m} + 2 \dot{m}}{t^2} \right]$$  \hfill (17)
and \( R \) generically diverges as \( t \) goes to zero.

Equation (16) can be written as

\[
\frac{dm(t)}{dt} = 4\pi t^2 H \left[ \frac{Q^2}{t^4} \right] + \frac{1}{2} t^2 \Lambda
\]

Integrating gives

\[
\frac{2m(t)}{t} - 1 = 8\pi \int t^2 H \left[ \frac{Q^2}{t^4} \right] dt + \frac{2m_0}{t} + \frac{1}{3} \Lambda t^2 - 1
\]

where \( m_0 \) is a constant. It is important to remember that the constraint

\[
\frac{2m(t)}{t} - 1 > 0
\]

must be satisfied for all \( t > 0 \).

First consider the case \( Q = 0 \) and take \( H(0) = 0 \). The constraint becomes

\[
\frac{2m_0}{t} + \frac{1}{3} \Lambda t^2 - 1 > 0.
\]

This will be satisfied if \( \Lambda > 0 \) and \( m_0 > \frac{1}{3} \Lambda^{-1/2} \). Even though \( R \) remains finite as \( t \) goes to zero the scalar \( R_{\mu\nu\alpha\beta} R^{\mu\nu\alpha\beta} \) diverges, so that \( t = 0 \) is an initial singularity. Thus, Schwarzschild with a positive cosmological constant can be converted into a cosmological solution with metric

\[
ds^2 = -\left[ \frac{2m_0}{t} + \frac{1}{3} \Lambda t^2 - 1 \right]^{-1} dt^2 + \left[ \frac{2m_0}{t} + \frac{1}{3} \Lambda t^2 - 1 \right] dx^2 + t^2 d\Omega^2.
\]

As \( t \to 0 \) the two sphere collapses but the x direction blows up. For large \( t \) the metric is

\[
ds^2 = -d\tau^2 + \exp \left[ 2 \sqrt{\frac{\Lambda}{3}} \tau \right] \left[ d\bar{x}^2 + d\Omega^2 \right],
\]

where \( \tau = \sqrt{3/\Lambda} \ln t \) and \( \bar{x} = \sqrt{\Lambda/3} x \). Thus, at late times we have inflationary behaviour and the scale factor of the two sphere is the same as the scale factor for the x direction.

Next consider Maxwell’s theory with \( H(P^2) = 1/2P^2 = Q^2/2t^4 \). The constraint is

\[
\frac{2m_0}{t} - \frac{4\pi Q^2}{t^2} + \frac{1}{3} \Lambda t^2 - 1 > 0
\]

which cannot be satisfied for all \( t > 0 \). The problem is that the \( Q^2 \) term diverges faster than the \( m_0 \) term and has the wrong sign. This can be modified in nonlinear electrodynamics by including a more divergent term with the correct sign or by eliminating the
divergence. If Maxwell’s theory is modified so that \( H(P^2) = \frac{1}{2}P^2 - \alpha^2 P^4 \), the constraint becomes
\[
\frac{2m_0}{t} - \frac{4\pi Q^2}{t^2} + \frac{8\pi \alpha^2 Q^4}{5t^6} + \frac{1}{3} \Lambda t^2 - 1 > 0.
\] (25)
This inequality is satisfied for a wide range of values of the parameters \( m_0, Q, \Lambda, \) and \( \alpha \). Here the additional term diverges more rapidly than the Maxwell term and has the correct sign. This spacetime behaves in a similar fashion to the case with \( Q = 0 \).

Finally consider the Born-Infeld Lagrangian. The constraint is
\[
\frac{2m_0}{t} + \frac{1}{3} \Lambda t^2 - 1 + \frac{8\pi}{a^2 t} \int_0^t \sqrt{a^2 Q^2 + t^4} dx > 0.
\] (26)
Since the integral is greater than zero for all \( t > 0 \) the inequality will certainly be satisfied if \( m_0 > \frac{1}{3} \Lambda^{-1/2} \). In Born-Infeld theory the electric contribution remains finite and does not present a problem as \( t \) goes to zero. For \( m_0 > 0 \) this spacetime has similar properties to the case with \( Q = 0 \). It is possible to take \( m_0 = 0 \). For small \( t \)
\[
\frac{2m(t)}{t} - 1 \simeq \frac{1}{3} \left( \Lambda - \frac{8\pi}{a^2} \right) t^2 - \frac{8\pi}{a^2} |Q|.
\] (27)
Thus, we require that \( |Q| \geq a/8\pi \). Now let \( f(t) = t(2m(t)/t - 1) \). The derivative of \( f(t) \) is given by
\[
f'(t) = \left( \Lambda - \frac{8\pi}{a^2} \right) t^2 - 1 + \frac{8\pi}{a^2} \sqrt{a^2 Q^2 + t^4}.
\] (28)
If \( \Lambda \geq 8\pi/a^2 \) and \( |Q| \geq a/8\pi \) then \( f'(t) > 0 \) for \( t > 0 \) and \( 2m(t)/t - 1 > 0 \) for \( t > 0 \).

Equation (19) determines the spacetime metric given \( H(P^2) \). The reverse process is also possible. For a metric of the form
\[
d s^2 = -\frac{dt^2}{a(t)^2} + a(t)^2 dx^2 + t^2 d\Omega^2
\] (29)
the Hamiltonian is given by
\[
H \left[ \frac{Q^2}{t^4} \right] = \frac{1}{4\pi t^2} \left[ \frac{d}{dt} (ta^2) - \Lambda t^2 + 1 \right].
\] (30)
To be physically reasonable \( H \) must reduce to the Maxwell Hamiltonian in the weak field limit.

**Conclusion**

Exact cosmological solutions to the Einstein field equations coupled to nonlinear electrodynamics, including Born-Infeld theory, were constructed. These solution were produced by considering spherically symmetric solutions in which the roles of \( r \) and \( t \) are reversed.
These spacetimes have an initial singularity and constant time hypersurfaces have topology $R \times S^2$. The radius of the two sphere is $t$ and the scale factor of the other dimension diverges in some cases as $t$ goes to zero and vanishes (or goes to a constant) in other cases. At late times local observers would see the universe to be homogeneous and isotropic. Such solutions can be constructed from the Schwarzschild solution with a positive cosmological constant and from Born-Infeld theory. Maxwell theory does not lead to a cosmological solution because the roles of $r$ and $t$ cannot be reversed for all $r > 0$.

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