INITIAL STABILITY ESTIMATES FOR RICCI FLOW AND THREE DIMENSIONAL RICCI-PINCHED MANIFOLDS

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Abstract. This paper investigates the question of stability for a class of Ricci flows which start at possibly non-smooth metric spaces. We show that if the initial metric space is Reifenberg and locally bi-Lipschitz to Euclidean space, then two solutions to the Ricci flow whose Ricci curvature is uniformly bounded from below and whose curvature is bounded by \( c \cdot t^{-1} \) converge to one another at an exponential rate once they have been appropriately gauged. As an application, we show that smooth three dimensional, complete, uniformly Ricci-pinched Riemannian manifolds with bounded curvature are either compact or flat, thus confirming a conjecture of Hamilton and Lott.

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1. Introduction

1.1. Overview. In this paper, we consider smooth solutions \((M^n, g(t))_{t \in (0, T)}\) to Ricci flow defined on smooth, connected manifolds satisfying for \( t \in (0, T) \),

\[
|\text{Rm}(g(t))| \leq \frac{c_0}{t} \quad \text{and} \quad \text{Ric}(g(t)) \geq -g(t),
\]

(1.1)
where \( c_0 \) is a positive time-independent constant. This setting has been shown to occur in many situations, one prominent one being that of expanding solitons with non-negative curvature operator coming out of cones with non-negative curvature operator: see for example [SS13], [Der16], [ST21], [BCRW19].

Using Lemma A.1 of Simon-Topping, we see that the above setting guarantees that the distances \( d_t := d(g(t)) \) on \( M \) converge locally in a strong sense. More explicitly, assuming (1.1), we have:

\[
e^{t-s}d_s \geq d_t \geq c_0\sqrt{t-s} \text{ for } 0 < s \leq t < S(x_0), \text{ on } U_{x_0},
\]

and hence there exists a unique limit \( d_0 := \lim_{t \to 0} d_t \) on \( U_{x_0} \), which is attained uniformly, such that \((U_{x_0}, d_0)\) is a metric space. Note that if for different points \( x_0, y_0 \in M \) we obtain \( d_0 = \lim_{t \to 0} d_t \) on \( U_{x_0} \) and \( \tilde{d}_0 = \lim_{t \to 0} d_t \) on \( U_{y_0} \), then we have \( d_0 = \tilde{d}_0 \) on \( U_{x_0} \cap U_{y_0} \), since they both are obtained as the uniform limit of \( d(g(t)) \) as \( t \to 0 \). For this reason we do not include a quantifier depending on \( x_0 \) in the definition of \( d_0 := \lim_{t \to 0} d_t \) on \( U_{x_0} \). We are interested in the following problem:

**Problem 1.1.** Let \((M_i^n, g_i(t))_{t \in (0,T)}, i = 1, 2\) be two smooth, connected (possibly incomplete) Ricci flows satisfying (1.1) and (1.2) and converging locally to the same metric space, up to an isometry as \( t \) tends to 0, that is: \( \lim_{t \to 0} d(g_1(t)) = d_{0,1} \) on \( U_{p,1} \), \( \lim_{t \to 0} d(g_2(t)) = d_{0,2} \), on \( U_{p,2} \) and \( \psi_0(U_{p,1}) = U_{p,2} \), where \( \psi_0 : U_{p,1} \to U_{p,2} = \psi_0(U_{p,1}) \) is a homeomorphism with \( \psi_0^*(d_{0,2}) = d_{0,1} \). Then we are concerned with the following problems:

What further assumptions on the regularity of \( d_0 \) ensure that

1. there is a suitable gauge in which we can compare the solutions \( g_1 \) and \( g_2 \) effectively?

2. the solutions \((g_1(t))_{t \in (0,T)}\) and \((g_2(t))_{t \in (0,T)}\) remain close to one another for \( t \) close to zero in this gauge?

Our fundamental regularity assumption on \( d_0 \) is the following Reifenberg property:

\[
\text{(1.3)} \quad \text{for all } p \in M, \text{ for all } x \in U_p, \text{ every tangent cone at } x \text{ of } (U_p, d_0) \text{ is } (\mathbb{R}^n, d(\delta)),
\]

where \( d(\delta) \) stands for Euclidean distance. In fact, if \((M, d_0)\) is the local limit of a non-collapsing sequence of complete, smooth Riemannian manifolds with bounded curvature and Ricci curvature uniformly bounded from below, this condition can be turned into a uniform Reifenberg condition (see Lemma 2.1),

\[
\text{(1.4)} \quad \text{such that } d_{GH}(B_{s^{-1}d_0}(x,1), B(0,1)) < \varepsilon, \text{ for all } s < r, \text{ and for all } x \in U_p \quad \text{such that } B_{d_0}(x,s) \subseteq U_p.
\]

Assumption (1.3) is infinitesimal and means that there are no singular points in \((U_p, d_0)\) for any \( p \in M \). However, it does not necessarily mean that the distance is induced by a Hölder continuous Riemannian metric, let alone a Lipschitz Riemannian metric: see [CN13, Theorem 1.2] for such a counterexample. The same paper does however show, under the assumption that \((U_p, d_0)\) is a complete, uniformly Reifenberg,
non-collapsed Ricci limit space, that there does exist a bi-Lipschitz embedding into $\mathbb{R}^N$ for a sufficiently large $N$ around each point $x \in U_p$.

The Reifenberg condition allows us to show that a uniform Pseudolocality estimate holds: see Lemma 2.2. In the case that the solution is complete and has bounded curvature, we use Perelman’s Pseudolocality and the uniform Reifenberg result from Lemma 2.1 to prove this. In the general case, we use Lemma A.2, which is valid in the setting we consider here.

We assume as a further regularity assumption on $d_0$ that there is a bi-Lipschitz chart around each point in $(U_p, d_0)$ given by distance coordinates, and that the Lipschitz constant is $\varepsilon_0$ close to 1:

For any $x_0 \in U_p$, there is a radius $R = R(x_0) > 0$ such that $B_{d_0}(x_0, 4R) \subseteq U_p$ and points $a_1, \ldots, a_n \in B_{d_0}(x_0, 3R)$ such that the map

$$D_0: \begin{cases} B_{d_0}(x_0, 4R) \to \mathbb{R}^n \\ x \mapsto (d_0(a_1, x) - d_0(a_1, x_0), \ldots, d_0(a_n, x) - d_0(a_n, x_0)) \end{cases}$$

is a $(1 + \varepsilon_0)$ bi-Lipschitz homeomorphism on $B_{d_0}(x_0, 2R)$.

Assumption (1.5) is local and always holds true in the case that $(U_p, d_0)$ is an Alexandrov space with curvature bounded from below satisfying (1.3): see [BB10, Theorem 10.8.4]. For example, this would be the case, if we assume (1.3) and replace the lower bound on the Ricci curvature in (1.1), by $\sec g(t) \geq -1$, $t \in (0, T)$, where $\sec$ refers to sectional curvature.

We establish the following initial stability estimate which addresses the question posed in Problem 1.1.

**Theorem 1.2.** Let $(M^i_t, g_i(t))_{t \in (0, T)}$, $i = 1, 2$, be smooth solutions to Ricci flow (not necessarily complete) both satisfying (1.1) and (1.2), such that the locally well defined metrics at time zero agree (up to an isometry), that is for arbitrary $p \in M_1$, $\lim_{t \to 0} d(g_1(t)) = 0$, on $U_{p_1}$, and $\lim_{t \to 0} d(g_2(t)) = 0$, on $U_{p_2}$ for an isometry $\psi_0: (U_{p_1}, d_{0,1}) \to (U_{p_2}, \psi_0(U_{p_1}), d_{0,2})$, $\bar{p} := \psi_0(p)$, that is $\psi_0: U_{p_1} \to U_{\bar{p}_2} = \psi_0(U_{p_1})$ is a homeomorphism with $\psi_0^*(d_{0,2}) = d_{0,1}$. We assume (1.3) and for fixed but arbitrary $p \in M_1$, and $x_0 \in U_{p_1}$ that (1.5) holds true for the locally defined metric $d_{0,1}$ on $U_{p_1}$ (and hence also $d_{0,2}$ on $U_{\bar{p}_2}$). Then there exists an $R_0 \in (0, 1)$ and $T_0 > 0$ depending on $n, \varepsilon_0$ and $x_0$ and solutions $(\tilde{g}_1(t))_{t \in (0, T_0)}$ and $(\tilde{g}_2(t))_{t \in (0, T_0)}$ to $\delta$-Ricci-DeTurck flow defined on $B(0, R_0) \times (0, T_0)$ satisfying the following:

1. The metrics $\tilde{g}_1(t)$ and $\tilde{g}_2(t)$ are $(1 \pm \varepsilon_0)$ close to the $\delta$-metric for $t \in (0, T_0)$,
2. there exist $C_0 > 0$ depending on $n, \varepsilon_0$ and $x_0$ such that if $t \in (0, T_0)$:
3. Moreover, $\tilde{g}_1(t) = (F_1(t))_t g_1(t)$ and $\tilde{g}_2(t) = (F_2(t))_t g_2(t)$, where $(F_1(t))_{t \in (0, T_0)}$ and $(F_2(t))_{t \in (0, T_0)}$ are smooth families of bi-Lipschitz maps on $B_{d_{0,1}}(x_0, \frac{3}{2} R_0)$ respectively $B_{d_{0,2}}(\psi_0(x_0), \frac{3}{2} R_0)$ that satisfy the following:
   a. The family of maps $(F_1(t))_{t \in (0, T_0)}$ (respectively $(F_2(t))_{t \in (0, T_0)}$) is a solution to the Ricci-harmonic map flow with respect to the Ricci flow solution $(g_1(t))_{t \in (0, T_0)}$ (respectively $(g_2(t))_{t \in (0, T_0)}$).
(b) The initial value for $F_1$ is $D_0$ and for $F_2$ is $D_0 \circ (\psi_0)^{-1}$ and the convergence rate at $t = 0$ is

$$|F_1(t) - D_0| \leq C_0 \sqrt{t}, \quad t \in (0, T_0), \quad \text{on } B_{d_{0,1}}(x_0, \frac{3}{2} R_0),$$

$$|F_2(t) - D_0 \circ (\psi_0)^{-1}| \leq C_0 \sqrt{t}, \quad t \in (0, T_0), \quad \text{on } B_{d_{0,2}}(\psi_0(x_0), \frac{3}{2} R_0),$$

where $D_0$ are the distance coordinates on $B_{d_{0,1}}(x_0, \frac{3}{2} R_0)$ coming from (1.5).

(c) The distances $\tilde{d}_1(t) := d(\tilde{g}_1(t))$ and $\tilde{d}_2(t) := d(\tilde{g}_2(t))$ converge, uniformly to the same distance $(D_0), d_{0,1}$ on $B(0, R_0)$ as $t$ approaches 0.

The convergence rate obtained in Theorem 1.2 is expected to be sharp when comparing with the behavior at $t = 0$ of solutions to the heat equation on Euclidean space coming out of initial data vanishing on a ball: see Proposition 6.1 in Section 6.

As an application of this result, we show that the approach of Lott [Lot19] leads to a full resolution of a conjecture posed by Hamilton [CLN06, Conjecture 3.39] and Lott [Lot19, Conjecture 1.1].

Recall that a Riemannian manifold $(M^n, g)$ is uniformly Ricci pinched if $\text{Ric}(g) \geq 0$ and there exists a constant $c > 0$ such that on $M$,

$$\text{Ric}(g) \geq c R_g g,$$

in the sense of quadratic forms where $R_g$ denotes the scalar curvature of the metric $g$. Notice that such a condition is invariant under rescalings.

**Theorem 1.3.** Let $(M^3, g)$ be a smooth complete Riemannian manifold with bounded and uniformly pinched Ricci curvature. Then $(M^3, g)$ is either smoothly isotopic to a spherical space form or flat. In particular, if $M$ is non-compact then $(M^3, g)$ is flat.

Hamilton introduced the Ricci flow in [Ham82], and in the case that $(M^3, g)$ is compact with non-negative uniformly pinched Ricci curvature and the scalar curvature is positive at least at one point, the paper shows that the volume preserving Ricci flow of $(M^3, g)$ exists for all time and converges smoothly to a spherical space form. In the case that $(M^3, g)$ is compact and has non-negative uniformly pinched Ricci curvature and the scalar curvature is zero everywhere, then $M^3$ is Ricci-flat and hence flat. That is, the results of Hamilton imply Theorem 1.3 immediately in the case that $M^3$ is compact.

In case $(M^3, g)$ is non-compact with bounded curvature, Lott has proven Theorem 1.3 under the assumption that the sectional curvature of $g$ has a lower bound decaying at least quadratically in the distance from a fixed point, improving a result of Chen-Zhu [CZ00] where it is assumed that the metric $g$ has non-negative sectional curvature. Finally, Theorem 1.3 can be interpreted as an extension of Myers’ theorem in dimension 3.

Lott’s approach to Hamilton’s conjecture has two major parts, which we briefly explain here. The proof is by contradiction and the first part deals with Ricci flow and it does not use the lower bound on the sectional curvature. Lott shows the following crucial result:

**Theorem 1.4 ([Lot19]).** Let $(M^3, g_0)$ be a smooth, complete non-compact Riemannian 3-manifold which is uniformly Ricci pinched with bounded curvature. Then there exists
a complete Ricci flow solution \((M^3, g(t))\) that exists for all \(t \geq 0\) with \(g(0) = g_0\) such that:

1. the solution is Type III, i.e. \(|\text{Rm}(g(t))| \leq \frac{C}{t}\) for all \(t > 0\) and some uniform positive constant \(C\),
2. the solution is uniformly \(c\)-pinched, i.e. there exists \(c > 0\) such that for \(t > 0\), 
   \[
   \text{Ric}(g(t)) \geq c R_{g(t)} g(t)
   \]
   on \(M\),
3. If \(R_{g_0}(x_0) > 0\) at some point, then the solution is uniformly non-collapsed for all positive time, i.e.
   \[
   \text{AVR}(g(t)) := \lim_{r \to +\infty} r^{-3} \text{Vol}_{g(t)}(B_{g(t)}(x_0, r)) = v_0 > 0,
   \]
   for some (hence any) \(x_0 \in M, t > 0\) and some time-independent positive constant \(v_0\).

See Section 8 for details.

The second part of Lott’s work does not use the Ricci flow. In the following exposition, we only consider solutions from Theorem 1.4 where (3) holds. If one further assumes that the sectional curvature of the initial space is bounded from below by \(-\frac{A}{r^2}\) with \(r\) being the distance, and one blows down the solution, one obtains a solution having the same properties, in some weak sense, coming out of any asymptotic cone of \((M^3, g_0)\). Based on results of Lebedeva-Petrunin [LP22] using methods of Alexandrov geometry, it is then a static problem to show that such asymptotic cones must be flat which leads to a contradiction.

This is where our approach differs, as we will now explain. After blowing down Lott’s Ricci flow solution, we first notice that the initial condition is a metric cone with no cone points outside the apex. This follows from a splitting theorem due to Hochard, see Proposition B.1, for Type III non-collapsed solutions to the Ricci flow with non-negative Ricci curvature coming out of a metric space splitting a line. This result holds in any dimension. In particular, assumption (1.3) is satisfied. Invoking results on RCD spaces (see Section 8 for references), the initial metric cone turns out to be an Alexandrov space with non-negative curvature. Therefore, Assumption (1.5) is satisfied as well. In particular, the link of the cone is a two dimensional Alexandrov space with curvature not less than one, and may be approximated by smooth spaces with curvature larger than one, in view of results from the theory of Alexandrov spaces: See the references in Section 8. Now, a consequence of the work of the first author [Der16] (alternatively the second two authors [SS13], after approximating the cone by a smooth space with non-negative curvature operator, as in [Sch14, Section 3.2]) ensures the existence of a self-similarly expanding solution coming out of the same initial metric cone that will serve as a comparison solution. This solution is an expanding gradient Ricci soliton with non-negative curvature operator which is not necessarily Ricci-pinched. If this would have been the case, we would be done by the work of [CZ00]. Using Theorem 1.2 to compare this solution with the original Ricci pinched solution, we see that this non-negatively curved expanding soliton solution is almost Ricci pinched. The error to be an exact Ricci-pinched expander is shown to be exponentially decaying at infinity: see Section 7 for details. This allows us to conclude that the expander is Euclidean which implies that the initial metric cone is flat, leading to a contradiction.
Notice that in Lott’s approach and that of the present paper, the main difficulty lies in the problem that one lacks regularity on any of the asymptotic cones of $(M^3, g_0)$, which is partially due to the fact that such an asymptotic cone does not a priori satisfy any kind of (elliptic) equation, and thus one struggles to make sense of the pinching on the initial metric cone. To circumvent this issue, we work directly with Lott’s Ricci flow solution coming out of the cone, and we use the parabolic nature of the Ricci flow equation to derive geometric properties for the initial condition.

Based on this result, a higher dimensional counterpart to Hamilton and Lott’s conjecture would be:

**Question 1.5.** Let $(M^n, g)$ be a connected non-compact complete Riemannian manifold with 2-non-negative curvature operator. Assume $(M^n, g)$ is 2-pinched. Is it true that $(M^n, g)$ is flat?

Recall that a Riemannian manifold $(M^n, g)$ has 2-non-negative curvature operator if the sum of the two lowest eigenvalues $\lambda_i(g)$, $i = 1, 2$, of the curvature operator is non-negative, i.e. $\lambda_1(g) + \lambda_2(g) \geq 0$ on $M$. Similarly, a Riemannian manifold $(M^n, g)$ is 2-pinched if there exists a constant $c > 0$ such that $\lambda_1(g) + \lambda_2(g) \geq c R_g$ on $M$.

An attempt to tackle Question 1.5 with the help of the Ricci flow, would most likely require one to first answer the following question:

**Question 1.6.** Let $(M^n, g(t))_{t \in (0, +\infty)}$ be a complete non-compact Type III Ricci flow with 2-non-negative curvature operator. Assume $(M^n, g(t))_{t \in (0, +\infty)}$ is uniformly 2-pinched and uniformly non-collapsed at all scales, i.e. there exists $c > 0$ such that for $t > 0$, $\lambda_1(g(t)) + \lambda_2(g(t)) \geq c R_g(t)$ on $M$ and there exists $V_0 > 0$ such that for $t > 0$, $\text{AVR}(g(t)) \geq V_0$. Is it true that $(M^n, g(t))_{t > 0}$ is isometric to Euclidean space?

1.2. **Outline of paper.** Sections 2 to 6 are devoted to the proof of the main result of this paper, Theorem 1.2. Section 2 explains how one improves the curvature decay as $t$ approaches 0 in the setting of Theorem 1.2. Section 3 is purely static, i.e. it is independent of the Ricci flow and describes how to adjust two Riemannian metrics originally close in the Gromov-Hausdorff topology with the help of a suitable diffeomorphism in order to make them coincide at the level of metrics. Based on Section 3, Section 4 is devoted to the adjustment of two solutions to $\delta$-Ricci-DeTurck flow coming out of the same metric space so that they are comparable at the level of Riemannian metrics along a sequence of times approaching 0, this is Corollary 4.3. Sections 5 and 6 are the core of this paper and they establish a polynomial (respectively exponential) decay rate for the difference of the two (adjusted) solutions to $\delta$-Ricci-DeTurck flow obtained in the previous section, thereby ending the proof of Theorem 1.2. Section 7 proves a rigidity result on expanding gradient Ricci soliton with non-negative Ricci curvature and almost Ricci pinched curvature. Section 8 gives a proof of Theorem 1.3.

1.3. **Notation.** We collect notation used throughout this paper.

(1) For a connected Riemannian manifold $(M^n, g)$, $x, y \in M$, $r \in \mathbb{R}^+$:

(1a) $(M, d(g))$ refers to the associated metric space,

$$d(g)(x, y) = \inf_{\gamma \in G_{x, y}} L_g(\gamma),$$

where $G_{x, y}$ is the set of all geodesics connecting $x$ and $y$ in $(M, g)$.
where $G_{x,y}$ refers to the set of smooth regular curves $\gamma : [0, 1] \to M$, with $\gamma(0) = x$, $\gamma(1) = y$, and $L_2(\gamma)$ is the length of $\gamma$ with respect to $g$.

(1b) $B_d(x, r) := B_d(g)(x, r) := \{y \in M \mid d(g)(y, x) < r\}$.
(1c) If $g$ is locally in $C^2$: Ric$(g)$ is the Ricci Tensor, Rm$(g)$ is the Riemann curvature tensor, and $R_g$ is the scalar curvature.

(2) For a one parameter family $(g(t))_{t \in (0, T)}$ of Riemannian metrics on a manifold $M$, the distance induced by the metric $g(t)$ is denoted either by $d(g(t))$ or $d_t$ for $t \in (0, T)$.

(3) For a metric space $(X, d)$, $A \subseteq X$, $r \in \mathbb{R}^+$, $B_d(A, r) := \{y \in X \mid d(y, A) < r\}$.

(4) $\mathbb{B}(x, r)$ refers to an Euclidean ball with radius $r > 0$ and centre $x \in \mathbb{R}^n$.

(5) For a metric space $(X, d)$, $\mathcal{H}^n_d(V)$ refers to the $n$-dimensional Hausdorff measure of a subset $V$ of $X$ with respect to $d$.

(6) For a Riemannian manifold $(M^n, g)$, $A \subseteq M$, Vol$_g(A)$ stands for the Riemannian volume of $A$ with respect to the metric $g$.

1.4. Acknowldgements. The first author is supported by grants ANR-17-CE40-0034 of the French National Research Agency ANR (Project CCEM) and ANR-AAPG2020 (Project PARAPLUI). The third author is supported by a grant in the Programm ‘SPP-2026: Geometry at Infinity’ of the German Research Council (DFG).

We thank Christian Ketterer, Alexander Lytchak and Stephan Stadler for answering questions related to and explaining to us their results on Alexandrov and RCD spaces.

2. Curvature decay

To simplify the setup we will in addition to (1.1)-(1.3) fix $x_0 \in M$ and $R > 0$, reducing $T > 0$ if necessary, such that

$$B_{d_0}(x_0, 200R + c_0T) \subseteq M.$$  

From Lemma A.1, this implies that $B_{d_t}(x_0, 200R) \subseteq M$ for all $t \in (0, T)$ and $B_{d_0}(x_0, 200R) \subseteq M$, where $d_t := d(g(t))$.

We record the following consequence of the Bishop-Gromov volume comparison, volume and convergence, and that almost volume cone implies almost metric cone structure.

**Lemma 2.1.** Let $(N^n_i, g_i, p_i)_{i \in \mathbb{N}}$, be a smooth sequence of complete manifolds, without boundary, with Ric$(g_i) \geq -g_i$ on $N_i$ and let $(X, d, x_0)$ be a non-collapsed Gromov-Hausdorff limit of $(N^n_i, g_i, p_i)_{i \in \mathbb{N}}$. Assume $K \subseteq X$ is compact and that all tangent cones of $X$ for $p \in K$ are Euclidean. Then for every $\varepsilon > 0$, there exists $r_\varepsilon > 0$ such that for any $p \in K$ and $0 < r \leq r_\varepsilon$ the ball $B_d(p, r)$ is, after scaling $r$ to one, $\varepsilon$-close in Gromov-Hausdorff distance to a Euclidean ball $\mathbb{B}(0, 1) \subseteq \mathbb{R}^n$, i.e. $(K, d)$ is uniformly Reifenberg.

**Proof.** Let $\delta_1 > 0$ be given. For all $p \in K$, there exists $0 < r_p \leq 1$ such that for all $0 < r \leq 2r_p$ we have that $B_d(p, r)$, after scaling $r$ to one, is $\delta_1/2$-close in Gromov-Hausdorff distance to $\mathbb{B}(0, 1)$, in view of the assumption that all tangent cones at points in $K$ of $X$ are Euclidean. Since $K$ is compact we can select finitely many points $p_1, \ldots, p_N$ such that $(B_d(p_i, r_{p_i}))_{i=1}^N$ cover $K$. Fix arbitrary $i \in \{1, \ldots, N\}$.
and let \( p \) be arbitrary in \( B_d(p_i, r_i) \). Then \( B_d(p, r_{p_i}) \subseteq B_d(p_i, 2r_{p_i}) \). This implies that 
\( B_d(p, r_{p_i}) \) is, after scaling \( r_{p_i} \) to one, \( \delta_1 \)-close in Gromov-Hausdorff distance to \( \mathbb{B}(0, 1) \).
Thus given \( \delta_2 > 0 \), by volume convergence under a uniform lower Ricci curvature bound on smooth spaces ([CC97, Theorem 5.9]; see also [Che01, Theorem 9.31]), and the fact that \((N^i_t, g_i, p_i)\) \( \text{GH} \)-converges to \((X, d, x_0)\) as \( i \to \infty \) we can assume that 
\( \delta_1 = \delta_1(\delta_2, n) > 0 \) is chosen sufficiently small so that 
\[
 r_{p_i}^{-n} \mathcal{H}^n_d(B_d(p, r_{p_i})) \geq (1 - \delta_2) \text{Vol}_d(\mathbb{B}(0, 1)).
\]

Now given \( \delta_3 > 0 \) we can apply Bishop-Gromov volume comparison and choose 
\( \delta_2 = \delta_2(\delta_3, n) > 0 \) sufficiently small so that 
\[
 r^{-n} \mathcal{H}^n_d(B_d(p, r)) \geq (1 - \delta_3) \text{Vol}_d(\mathbb{B}(0, 1)),
\]
for all \( 0 < r \leq r_{p_i} \), for arbitrary \( p \in B_d(p_i, r_i) \). Given \( \varepsilon > 0 \) we can thus apply [CC97, Theorem 5.9] to the smooth approximating spaces followed by [CC96, Theorem 4.85] (see also [Che01, Theorem 9.45]) to see that for \( \delta_3(\varepsilon, n) > 0 \) sufficiently small we have that for all \( 0 < r \leq r_{p_i} \) that \( B_d(p, r) \), after scaling \( r \) to one, is \( \varepsilon \)-close in Gromov-Hausdorff distance to \( \mathbb{B}(0, 1) \) for arbitrary \( p \in B_d(p_i, r_i) \).

Now we choose 
\[
r_{\varepsilon} := \min_{i \in \{1, \ldots, N\}} r_{p_i},
\]
to get the desired statement.

The following Lemma shows a uniform Pseudolocality type result, in a Reifenberg setting.

**Lemma 2.2.** Assume \((M^n, g(t))\) for \( t \in (0, T)\) satisfy (1.1), (1.2), (1.3) and (2.1) for a fixed \( x_0 \in M \). Then there exists \( \varepsilon(t) = \varepsilon_{x_0}(t) > 0 \) depending on \( x_0 \) such that \( \varepsilon(t) \to 0 \) as \( t \to 0 \) and
\[
|\text{Rm}(g(t))| \leq \varepsilon(t)^2/t \quad \text{on} \quad B_{d_0}(x_0, 150R) \quad \text{for all} \quad t \in (0, T).
\]
Furthermore,
\[
|\nabla^g(t)^k \text{Rm}(g(t))|^2 \leq C(k, n, \varepsilon(t))/t^{2+k} \quad \text{on} \quad B_{d_0}(x_0, 100R) \quad \text{for all} \quad t \in (0, T),
\]
where \( C(k, n, \varepsilon(t)) \to 0 \) as \( \varepsilon(t) \to 0 \).

**Proof.** To illustrate the main idea, we present first a proof in the case that there is a complete solution \((N, h(t))_{t \in (0, T)}\) with bounded curvature and \( \text{Ric}(h(t)) \geq -h(t) \) for all \( t \in (0, T) \) and \( M \subseteq N \) is open, with \( g(t) = h(t)|_M \). Let \( p \in M \) be fixed and \( \varepsilon > 0 \) be given. By scaling \( d_0 \) by a constant \( \lambda \) sufficiently large, we see that \((U_p, d_{\lambda} := \lambda d_0)\) has the property that 
\[
d_{\text{GH}}(B_{d_0}(x, 1), \mathbb{B}(0, 1)) \leq \varepsilon^2
\]
for all \( x \in B_{d_0}(x_0, 199R) \), in view of the uniform Reifenberg property, (1.4) which holds in view of (1.3) and Lemma 2.1. Hence, using (1.2), we see that
\[
d_{\text{GH}}(B_{d_{\lambda}(g(t))}(x, 1), \mathbb{B}(0, 1)) \leq \varepsilon,
\]
for all \( x \in B_{d_0}(x_0, 199R) \) for all \( t \leq \varepsilon^2 \), where \( g_\lambda(t) = \lambda^2 g\left(\frac{t}{\lambda}\right) \).

Furthermore, \( \text{Ric}(g_\lambda(t)) \geq -\varepsilon g_\lambda(t) \) on \( U_p \) if \( \lambda \) is large enough. Using the volume convergence theorem of Cheeger-Colding for spaces with curvature bounded below, we see that 
\[
\text{Vol}_{g_\lambda(t)}(B_{g_\lambda(t)}(x, 1)) \geq \omega_n(1 - \psi(\varepsilon)) \quad \text{for all} \quad x \in B_{d_0}(x_0, 199R) \quad \text{for all}
\]

\[
t \leq \varepsilon^2,
\]
and this completes the proof.
Lemma 3.1 (cf. [x]) for all $x \in B_{d_0}(x_0, 199R)$, for all $0 < t \leq \varepsilon^2$, where $\psi(x) \to 0$ as $\varepsilon \searrow 0$.

Thus Perelman’s Pseudolocality theorem [Per02, Theorem 10.1] implies that given $\varepsilon > 0$ we can choose $\lambda$ sufficiently large such that

$$|\text{Rm}(g_\lambda(1))(x)| \leq \varepsilon^2,$$

for all $x \in B_{d_0}(x_0, 150R)$. Scaling back implies the first statement of the lemma. The second statement follows as in [DSS19, Lemma 3.1].

In the general case, $|\text{Rm}(g(t))| \leq \frac{\varepsilon(t)}{t}$ on $B_{d_0}(x_0, 150R)$ follows from Lemma A.2, and the second statement follows as in [DSS19, Lemma 3.1].

Lemma 2.3. Assuming the setting of Lemma 2.2, one has for all $t \in [r, T)$ and $0 \leq r < s < T$,

$$e^{r-s}d_r \geq d_t \geq d_r - \varepsilon(t)\sqrt{T-r},$$

on $B_{d_0}(x_0, 100R)$, where $\varepsilon(t) \to 0$ as $t \searrow 0$.

Proof. This follows from the previous lemma together with the scaling of the distance estimates as in Lemma A.1.

3. Existence of an Adjustment Map

The following lemma shows the existence of a so called adjustment map. If two Riemannian metrics, and their derivatives on a ball in Euclidean space are uniformly bounded with respect to the standard metric, and the distances are close to one another, up to an isometry, then there is a diffeomorphism, a so called adjustment of the isometry, such that the pull back of the Riemannian metric with respect to this adjustment map is close in the $C^0$ sense to the other.

Lemma 3.1 (cf. [Gro07] and [Fuk87]). Let $c_0 \geq 1$ and an integer $n \geq 1$ be given. Then for all $\varepsilon > 0$, there exists a constant $c(\varepsilon, n, c_0) > 0$ with the property that $c(\varepsilon, n, c_0) \to 0$ as $\varepsilon \searrow 0$ such that the following holds. Let $K > 1$ and let $(M^n_i, g_i), i = 1, 2$, be two connected Riemannian manifolds (not necessarily complete), with $M_i \subseteq \mathbb{R}^n$ (where $\mathbb{R}^n$ is given the standard topology and differentiable structure) such that

$$\frac{1}{c_0} \delta \leq g_i \leq c_0 \delta, \quad i = 1, 2,$$

$$|D^k(g_1)|^2 + |D^k(g_2)|^2 \leq c_0 K^k, \quad \forall k \in \{1, 2, \ldots, 8\},$$

where here $D$ refers to Euclidean derivatives. Let $d(g_1)$ be the metric on $M_1$ induced by $g_1$ and $d(g_2)$ be the metric on $M_2$ induced by $g_2$. Assume furthermore that $\varphi : B_{d(g_1)}(0, s) \subseteq M_1 \to \varphi(B_{d(g_1)}(0, s)) \subseteq M_2$ for $s \geq 100K^{-\frac{1}{2}}$ is a homeomorphism which satisfies

$$|\varphi_*d(g_1) - d(g_2)| \leq \varepsilon K^{-\frac{1}{2}}.$$

Then there exists a smooth bi-Lipschitz diffeomorphism

$$\tilde{\varphi} : B_{d(g_1)}(0, \frac{s}{100K}) \to \tilde{\varphi}(B_{d(g_1)}(0, \frac{s}{100K})) \subseteq B(0, R),$$

$$\frac{1}{c_0} \delta \leq g_i \leq c_0 \delta, \quad i = 1, 2,$$

$$|D^k(g_1)|^2 + |D^k(g_2)|^2 \leq c_0 K^k, \quad \forall k \in \{1, 2, \ldots, 8\},$$

where here $D$ refers to Euclidean derivatives. Let $d(g_1)$ be the metric on $M_1$ induced by $g_1$ and $d(g_2)$ be the metric on $M_2$ induced by $g_2$. Assume furthermore that $\varphi : B_{d(g_1)}(0, s) \subseteq M_1 \to \varphi(B_{d(g_1)}(0, s)) \subseteq M_2$ for $s \geq 100K^{-\frac{1}{2}}$ is a homeomorphism which satisfies

$$|\varphi_*d(g_1) - d(g_2)| \leq \varepsilon K^{-\frac{1}{2}}.$$
such that
\[(1 - c(n, \varepsilon, c_0))g_2 \leq \tilde{\varphi}_s g_1 \leq (1 + c(n, \varepsilon, c_0))g_2, \quad (3.3)\]
\[|d(\tilde{\varphi}_s g_1)(x, y) - d(g_2)(x, y)| \leq c(n, \varepsilon, c_0)K^{-\frac{1}{2}}, \quad d(g_2)(\tilde{\varphi}(z), \varphi(z)) \leq c(n, \varepsilon, c_0)K^{-\frac{1}{2}},\]
for all \(x, y \in \tilde{\varphi}(B_{d(g_1)}(0, \frac{\varepsilon}{10c_0})), \) and all \(z \in B_{d(g_1)}(0, \frac{\varepsilon}{10c_0}).\)

A proof of Lemma 3.1 can be adapted either from Gromov’s book [Gro07, Section D, Chapter 8] or Fukaya [Fuk87]: there the setting is more general. We provide a somewhat alternative proof in our setting based on the notion of almost isometries.

Before we prove Lemma 3.1, we define the notion of *almost isometry* which shall be used in this paper.

**Definition 3.2.** Let \((W, d)\) be a metric space. We call \(Z : (W, d) \to \mathbb{R}^n\) an \(\varepsilon_0\) almost isometry if
\[(1 - \varepsilon_0)d(x, y) - \varepsilon_0 \leq |Z(x) - Z(y)| \leq (1 + \varepsilon_0)d(x, y) + \varepsilon_0\]
for all \(x, y \in W.\)

We state the following useful observation taken from [DSS19, Lemma 3.5]:

**Lemma 3.3.** For all \(\sigma > 0,\) there exists \(\gamma = \gamma(\sigma) \in (0, \sigma)\) small with the following property: if \(L : (\mathbb{B}(0, \gamma^{-1}), d) \to \mathbb{R}^n\) is a \(\gamma\) almost isometry fixing 0, then there exists an \(S \in O(n)\) such that \(|L - S|_{L^\infty(\mathbb{B}(0, \sigma^{-1}))} \leq \sigma.\)

**Proof of Lemma 3.1.** Let \(\alpha > 0\) be given. Without loss of generality, \(\frac{1}{\alpha} \geq c_0\) and \(\alpha \leq 10^{-10}.\) Assume \(\varepsilon \leq \alpha^2.\) Rescaling \(\hat{g}_1 = \frac{K}{\alpha}g_1\) and \(\hat{g}_2 = \frac{K}{\alpha}g_2\) and \(h = \frac{K}{\alpha} \delta\) we are now in the setting (denoting \(\hat{g}_1\) by \(g_1\) once again and \(\hat{g}_2\) by \(g_2\) once again) that
\[\frac{1}{c_0} h \leq g_i \leq c_0 h, \quad (3.4)\]
\[|D^k(g_1)|^2 + |D^k(g_2)|^2 \leq \alpha^k, \quad \forall k \in \{1, 2, \ldots, 8\},\]
\[|\varphi_s d(g_1) - d(g_2)| \leq \varepsilon \alpha^{-\frac{1}{2}} \leq \alpha,\]
on \(B_{d(g_1)}(0, \frac{\varepsilon_0}{\alpha^{-\frac{1}{2}}})\) where here now \(D\) refers to the covariant derivative with respect to \(h,\) and \(|\cdot|\) is the norm with respect to \(h.\) There is an isometry \(Z_i : (M_i, h) \to (K^{\frac{1}{\alpha}} \alpha^{-\frac{1}{2}} M_i, \delta)\) given by \(x \to \frac{K}{\alpha} x.\) Pushing everything forward by \(Z_i,\) we see that we may assume that we are in the setting above, with \(h = \delta\) and \(D\) is the usual derivative in \(\mathbb{R}^n\) and \(|\cdot|\) is the norm with respect to \(\delta.\) Taking any \(p \in B_{d(g_1)}(0, \frac{\varepsilon_0}{\alpha^{-\frac{1}{2}}})\), let \(a_{ij} = (g_1)_{ij}(p)\) be the metric at \(p.\) From the estimates (3.4) we see that
\[\sum_{k=0}^{6} |D^k((g_1)_{ij}(\cdot) - a_{ij})|_{C^0(\mathbb{B}(p, \gamma^{-1}))} \leq \gamma,\]
where \(\gamma = \gamma(\alpha, n, c_0) \to 0\) as \(\alpha \to 0.\) Constants of this type shall be denoted by \(\gamma\) and can change from line to line, but always satisfy \(\gamma(\alpha, n, c_0) \to 0\) as \(\alpha \to 0.\) By
performing a linear transformation $V_1 : \mathbb{R}^n \to \mathbb{R}^n$, and setting $T_1(q) = V_1(p - q) + p$, we see that in the coordinates $T_1$, that is $\tilde{g}_1 := (T_1)_* g_1$ we have, $(\tilde{g}_1)_{ij}(0) = \delta_{ij}$ and
\[
\sum_{k=0}^{6} |D^k((\tilde{g}_1)_{ij}(\cdot) - \delta_{ij})|_{C^0(\mathbb{B}(0,\gamma^{-1}))} \leq \gamma,
\]
for some $\gamma = \gamma(\alpha, n, c_0) \to 0$. Similarly for $g_2$ at $\varphi(p)$ we find an affine map $T_2$, such that for $\tilde{g}_2 := (T_2)_* g_1$, we have $(\tilde{g}_2)_{ij}(0) = \delta_{ij}$ and
\[
\sum_{k=0}^{6} |D^k((\tilde{g}_2)_{ij}(\cdot) - \delta_{ij})|_{C^0(\mathbb{B}(0,\gamma^{-1}))} \leq \gamma,
\]
for some $\gamma = \gamma(\alpha, n, c_0) \to 0$. Defining $L := T_2 \circ \varphi \circ (T_1)^{-1}$, we see that $L : (\mathbb{B}(0,\gamma^{-1}), d(\tilde{g}_1)) \to \mathbb{R}^n$ is a $\gamma$ almost isometry fixing 0. Therefore, Lemma 3.3 ensures there must exist an $\tilde{S} \in O(n)$ such that $|\tilde{S} - L|_{C^0(\mathbb{B}(0,\gamma^{-1}))} \leq \gamma(\alpha, n, c_0)$ where $\gamma \to 0$ as $\alpha \to 0$. In particular, setting $S_p := (T_2)^{-1} \circ \tilde{S} \circ T_1$, and using $(T_2)^{-1} \circ L \circ T_1 = \varphi$, we obtain
\[
|\varphi - S_p|_{C^0(\mathbb{B}(p,\gamma^{-1}))} \leq \gamma(\alpha, n, c_0) \to 0,
\]
as $\alpha \to 0$, and for $\tilde{\delta} := (T_2)_* \tilde{\delta}$, (note $\frac{1}{C(\alpha, n, \gamma)} \leq \tilde{\delta} \leq C(\alpha, n) \tilde{\delta}$),
\[
|(S_p)_* g_1 - g_2|_{C^0(\mathbb{B}(\varphi(p), 200), \delta))} = |\tilde{S}_* \tilde{g}_1 - \tilde{g}_2|_{C^0(\mathbb{B}(\varphi(p), 200), \delta))}
\]
\[
\leq C(\alpha, n)|\tilde{S}_* (\tilde{g}_1 - \delta) + \tilde{S}_* \delta - \delta - (\tilde{g}_2 - \delta)|_{C^0(\mathbb{B}(0,R(\alpha, n)))}
\]
\[
= C(\alpha, n)|\tilde{S}_* (\tilde{g}_1 - \delta) - (\tilde{g}_2 - \delta)|_{C^0(\mathbb{B}(0,R(\alpha, n)))}
\]
\[
\leq C(\alpha, n)|\tilde{S}_* (\tilde{g}_1 - \delta)|_{C^0(\mathbb{B}(0,R(\alpha, n)))} + |\tilde{g}_2 - \delta|_{C^0(\mathbb{B}(0,R(\alpha, n)))}
\]
\[
= C(\alpha, n)|\tilde{g}_1 - \delta|_{C^0(\mathbb{B}(0,R(\alpha, n)))} + |\tilde{g}_2 - \delta|_{C^0(\mathbb{B}(0,R(\alpha, n)))}
\]
\[
\leq \gamma(\alpha, n, c_0) \to 0 \quad \text{as} \quad \alpha \to 0,
\]
where we used that fact that $\tilde{S}_* \delta = \delta$ in the third line.

In particular,
\[
(1 - c(\alpha, n, c_0))(S_p)_* g_1 \leq g_2 \leq (1 + c(\alpha, n, c_0))(S_p)_* g_1, \quad \text{on} \quad \mathbb{B}(\varphi(p), 100),
\]
with $c(\alpha, n, c_0) \to 0$ as $\alpha \to 0$.

From the definition of $S_p$, we have $D S_p = D(T_2)^{-1} \circ D \tilde{S} \circ DT_1 = V_2^{-1} \circ \tilde{S} \circ V_1$ and
\[
|V_1|_{C^0(\mathbb{R}^n, \mathbb{R}^n)} + |(V_1)^{-1}|_{C^0(\mathbb{R}^n, \mathbb{R}^n)} + |V_2|_{C^0(\mathbb{R}^n, \mathbb{R}^n)} + |(V_2)^{-1}|_{C^0(\mathbb{R}^n, \mathbb{R}^n)} \leq C(n, c_0),
\]
so that $|D S_p|_{C^0(\mathbb{R}^n)} \leq C(n, c_0)$. We are ready to define our adjustment map.

We take a covering by balls $(\mathbb{B}(p_i, 1))_{i \in I}$ of radius one of $B_{d(g_1)}(0, s \alpha^{-\frac{1}{2}})$ for which $(\mathbb{B}(p_i, \frac{1}{3}))_{i \in I}$ are pairwise disjoint and $(\mathbb{B}(p_i, \frac{2}{3}))_{i \in I}$ still covers, and take a partition of unity $(\eta_i : M_1 \to \mathbb{R})_{i \in I}$ subordinate to this covering: $\eta_i = 1$ on the balls of radius $\mathbb{B}(p_i, \frac{1}{3})$, supp($\eta_i$) $\subset \mathbb{B}(p_i, \frac{2}{3})$. We let $S_i := S_{p_i}$ and define
\[
\tilde{\varphi} := \sum_{i \in I} \eta_i S_i.
\]
As we saw above, \(|\varphi - S_i|_{C^0(B(p_i, \gamma^{-1}))} \leq \gamma \to 0\). Furthermore,

\(|S_i - S_j|_{C^0(B(p, 100))} \leq |S_i - \varphi|_{C^0(B(p, 100))} + |S_j - \varphi|_{C^0(B(p, 100))} \leq \gamma(\alpha, c_0, n) \to 0,

if \(p_i, p_j \in B(p, 100)\) per construction. Hence

\[|DS_j - DS_i|^2_{C^0(B(p, 100))} = |S_j - S_i|^2_{C^0(B(p, 100))} \leq \gamma(\alpha, c_0, n) \to 0,\]

if \(p_i, p_j \in B(p, 100)\). Let \(x \in B(p_i, 10)\), and let \(J_i\) denote the indices \(j \in \mathbb{N}\) for which \(B(p_j, 1) \cap B(p_i, 10) \neq \emptyset\). Note that we can assume that there exists a uniform constant \(c(n)\) such that \(|J_i| \leq c(n)\). Furthermore, \(\tilde{\varphi}|_{B(p_i, 10)} = \sum_{j \in J_i} \eta_j S_j\) and hence

\[
|D\tilde{\varphi} - DS_i|_{C^0(B(p_i, 10))} = \left| \sum_{j \in J_i} D(\eta_j \cdot S_j) - DS_i \right|_{C^0(B(p_i, 10))} \\
= \left| \sum_{j \in J_i} D(\eta_j \cdot S_j) - D \left( \sum_{j \in J_i} \eta_j \right) S_i - \sum_{j \in J_i} (\eta_j \cdot DS_i) \right|_{C^0(B(p_i, 10))} \\
= \left| \sum_{j \in J_i} (D\eta_j)(S_j - S_i) - \eta_j(DF_i - DS_j) \right|_{C^0(B(p_i, 10))} \\
\leq \gamma(\alpha, n, c_0) \to 0, \quad \text{as } \alpha \to 0.
\]

That is

\[(3.7) \quad |D\tilde{\varphi} - DS_i|_{C^0(B(p_i, 10))} \leq \gamma(\alpha, n, c_0) \to 0, \quad \text{as } \alpha \to 0.\]

Using the fact that \(\tilde{\varphi}|_{B(p_i, 10)} = \sum_{j \in J_i} \eta_j S_j\), we also obtain

\[
|D^2(\tilde{\varphi} - S_i)|_{C^0(B(p_i, 10))} + |D^3(\tilde{\varphi} - S_i)|_{C^0(B(p_i, 10))} \leq C(n, c_0),
\]

and hence, using standard interpolation inequalities, see for example [SSS11, Appendix A, Lemma A.5 and A.6], we see that

\[
|D^2(\tilde{\varphi} - S_i)|_{C^0(B(p_i, 10))} \leq |D(\tilde{\varphi} - S_i)|_{C^0(B(p_i, 10))}|D^3(\tilde{\varphi} - S_i)|_{C^0(B(p_i, 10))} \leq \gamma(\alpha, n, c_0).
\]

We perform a Taylor expansion in each component and obtain for \(x, y \in B(p_i, 10)\) and \(k = 1, \ldots, n\),

\[
(\tilde{\varphi} - S_i)^k(x) = (\tilde{\varphi} - S_i)^k(y) + D\alpha(\tilde{\varphi} - S_i)^k(x - y) + C^k(x, y),
\]

where \(|C^k(x, y)| \leq \gamma(\alpha, n, c_0)|x - y|^2\). Hence

\[
|\tilde{\varphi}(x) - \tilde{\varphi}(y)| \geq |S_i(x) - S_i(y)| - \gamma(\alpha, c, c_0)|x - y| \\
= ||(T_{2,i})^{-1} \circ \tilde{S}_i \circ T_{1,i}(x) - (T_{2,i})^{-1} \circ \tilde{S}_i \circ T_{1,i}(y)| - \gamma(\alpha, c, c_0)|x - y| \\
= ||(T_{2,i})^{-1} \circ \tilde{S}_i \circ T_{1,i}(x - y)| - \gamma(\alpha, c, c_0)|x - y| \\
\geq \beta(n, c_0)|x - y|,
\]

for \(x, y \in B(p_i, 10)\), where \(\beta(n, c_0) > 0\).

In view of (3.5) and (3.7) we have

\[
(1 - c(\alpha, n, c_0))(\tilde{\varphi})_g1 \leq g_2 \leq (1 + c(\alpha, n, c_0))(\tilde{\varphi})_g1, \quad \text{on } B(p_i, 1),
\]

with \(c(\alpha, c, c_0) \to 0\) as \(\alpha \to 0\), and hence on all of \(B_{d(g_1)}(0, s \alpha^{-\frac{1}{2}})\). Combining this with the fact that \(\tilde{\varphi}\) is a diffeomorphism on Euclidean balls of radius 10, we see

\[
(1 - c(\alpha, n, c_0))d(g_1)(x, y) \leq d(g_2)(\tilde{\varphi}(x), \tilde{\varphi}(y)) \leq (1 + c((\alpha, n, c_0))d(g_1)(x, y),
\]
on any ball of radius 10. On the other hand, for \( x \in B(p_i, 1) \) and using the notation above, we know
\[
|\varphi(x) - \varphi(x)| = \left| \sum_{j \in J_i} \eta_j S_j(x) - \varphi(x) \right| = \left| \sum_{j \in J_i} \eta_j (S_j - \varphi)(x) \right| \leq c(\alpha, n, \delta),
\]
that is
\[
|\varphi(x) - \varphi(x)| \leq c(\alpha, n, \delta) \to 0, \quad \text{for } \alpha \to 0.
\]
This implies for points \( x, y \) with \( |x - y| \geq 10 \),
\[
d(g_2)(\varphi(x), \varphi(y)) \leq d(g_2)(\varphi(x), \varphi(y)) + d(g_2)(\varphi(x), \varphi(y)) + d(g_2)(\varphi(y), \varphi(y)) \]
\[
\leq (1 + c(\alpha, n, \delta))d(g_2)(\varphi(x), \varphi(y)) \]
and similarly,
\[
d(g_2)(\varphi(x), \varphi(y)) \geq (1 - c(\alpha, n, \delta))^2 d(g_1)(x, y).
\]
Hence, the map \( \tilde{\varphi} \) satisfies
\[
(1 - c(\alpha, n, \delta))d(g_1)(x, y) \leq d(g_2)(\tilde{\varphi}(x), \tilde{\varphi}(y)) \leq (1 + c(\alpha, n, \delta))d(g_1)(x, y)
\]
on \( B_{d(g_1)}(0, \frac{1}{2} \alpha^{-\frac{1}{2}}) \).

4. Rough convergence rate

We consider the following setup: \((M^n, g(t)), t \in (0, T)\) is a smooth solution to Ricci flow (not necessarily complete) such that (1.1), (1.2), and (1.3) hold, where \( d_0 = \lim_{t \to 0} d(g(t)) \) is locally, uniquely, well defined on \( U_p \) for all \( p \in M \) by Theorem A.1, as explained in the introduction. In particular: for any fixed \( x_0 \in M \), Lemmata 2.2 and 2.3 imply that there exists an \( R = R(x_0, n) > 0, T = T(x_0, n) \in (0, 1] \) such that:

(a) \( |\text{Rm}(g(t))| \leq \varepsilon(t)/t \) on \( B_{d_0}(x_0, 200R) \subseteq U_p \subseteq M \) for all \( t \in (0, T) \),

(b) \( \text{Ric}(g(t)) \geq -g(t) \) on \( B_{d_0}(x_0, 200R) \subseteq U_p \subseteq M \) for all \( t \in (0, T) \),

(c) \( |d_t - d_0| \leq \varepsilon(t)\sqrt{t} \) on \( B_{d_0}(x_0, 100R) \) for all \( t \in [0, T) \),

where \( 0 \leq \varepsilon(t) \to 0 \) as \( t \to 0 \). In the following we show that we can construct solutions to the Dirichlet problem to the Ricci-Harmonic map heat flow for a suitable possibly non-smooth class of initial data \( F_0 \), where the background Ricci flows are of the type considered above. More precisely, we consider initial data \( F_0 \) satisfying

\[
F_0 : \left\{ \begin{array}{c}
B_{d_0}(x_0, 4R) \to \mathbb{R}^n \\
x \mapsto (\text{((F_0)_1(x), \ldots, (F_0)_n(x)))}
\end{array} \right.
\]
is a \((1 + \varepsilon_0)\) bi-Lipschitz homeomorphism on \( B_{d_0}(x_0, 2R) \).

We also consider the special case that \( F_0 := D_0 \) where \( D_0 \) are \((1 + \varepsilon_0)\) bi-Lipschitz distance coordinates, i.e.

there are points \( a_1, \ldots, a_n \in B_{d_0}(x_0, 3R) \), such that the map

\[
D_0 : \left\{ \begin{array}{c}
B_{d_0}(x_0, 4R) \to \mathbb{R}^n \\
x \mapsto (d_0(a_1, x) - d_0(a_1, x_0), \ldots, d_0(a_n, x) - d_0(a_n, x_0))
\end{array} \right.
\]
is a \((1 + \varepsilon_0)\) bi-Lipschitz homeomorphism on \(B_{d_0}(x_0, 2R)\).

We recall that we say that two metrics \(g, h\) on a set \(\Omega \subset M\) are \(\varepsilon\)-close, for \(\varepsilon > 0\), provided

\[(1 + \varepsilon)^{-1} g \leq h \leq (1 + \varepsilon) g\]
on \(\Omega\).

We recall from [DSS19, Theorem 3.11] the following existence result and estimates for the \(\delta\)-Ricci-DeTurck flow in these settings

**Theorem 4.1.** For \(\alpha_0 \in (0, 1)\), and an integer \(n \geq 2\), there exists \(\varepsilon_0(n, \alpha_0) > 0\) and \(S = S(n, \alpha_0) > 0\) such that the following holds. Let \((M^n, g(t))_{t \in (0, T]}\), \(T \leq 1\), be a smooth solution to Ricci flow, \(p \in M\) and \(d_0\) be the locally well defined metric coming from Lemma A.1, with \(d_0 : U_p \times U_p \to \mathbb{R}_0^+\) satisfying (a), (b) and (c) for an \(R \geq 201\) and a fixed \(p \in M\) and fixed arbitrary \(x_0 \in U_p\). Let \(F_0 : B_{d_0}(x_0, R) \to \mathbb{R}^n\) be a \((1 + \varepsilon_0)\) bi-Lipschitz map with respect to \(d_0\) as in (d), where \(\varepsilon_0 \leq \varepsilon_0\) and assume that one of the following is satisfied:

(i) \(F_0 = \lim_{t \to \infty} F_t\) uniformly on \(B_{d_0}(x_0, R)\) where \(F_t\) is a \((1 + \varepsilon_0)\) bi-Lipschitz map with respect to \(d_t\) on \(B_{d_t}(x_0, R)\), for some sequence \(t_i \to 0\) with \(t_i \to 0\),

(ii) \((M^n, g(t))_{t \in (0, T]}\) can be smoothly extended to \((M^n, g(t))_{t \in [0, T]}\),

(iii) \(F_0 = D_0\) with \(D_0\) as in (d).

Then, for any \(m_0 \in B_{d_0}(x_0, R/2)\), there exist maps

\[F(t) : B_{d_0}(m_0, 3/2) \to E_t := F(t)(B_{d_0}(m_0, 3/2)) \subseteq \mathbb{R}^n,\]

which are solutions to the Ricci-harmonic map flow with initial condition the map \(F_0|_{B_{d_0}(m_0, 3/2)}\), with \(\mathbb{B}(\tilde{m}_0, 1) \subseteq E_t\) for \(\tilde{m}_0 = F_0(m_0)\), which are smooth \(1 + \alpha_0\) bi-Lipschitz diffeomorphisms for all \(0 < t \leq \tilde{S} := \min(S, T)\) and such that

\[
\partial_t F(x, t) = \Delta_{g(t)} F(x, t), \quad \text{for all } (x, t) \in B_{d_0}(m_0, 3/2) \times (0, \tilde{S}),
\]

\[
(1 - \alpha_0)d_t(x, y) \leq |F(x, t) - F(y, t)| \leq (1 + \alpha_0)d_t(x, y)
\]

for all \((x, t), (y, t) \in B_{d_0}(m_0, 3/2) \times [0, \tilde{S}),\)

\[
|F(x, t) - F_0(x)| \leq c(n)/\sqrt{t}, \quad \text{for all } (x, t) \in B_{d_0}(m_0, 3/2) \times [0, \tilde{S}),
\]

for some positive constant \(c(n)\). Setting \(\tilde{g}(t) = (F(t))_*(g(t))\) for \(t \in (0, T)\), on \(\mathbb{B}(\tilde{m}_0, 1)\), we further have that \((\tilde{g}(t))_{t \in (0, \tilde{S})}\) is a smooth family of metrics which solve the \(\delta\)-Ricci-DeTurck flow and are \(\alpha_0\)-close to the \(\delta\) metric:

\[
(1 + \alpha_0)^{-1} \delta \leq \tilde{g}(t) \leq (1 + \alpha_0)\delta,
\]

satisfying

\[
|D^k \tilde{g}(t)|^2 \leq \frac{c(k, n)}{t^k}, \quad k \geq 0,
\]

for all \(t \in (0, \tilde{S})\). The metric \(d_t := d(\tilde{g}(t))\) satisfies, \(d_t \to d_0 := (F_0)_*d_0\) uniformly on \(\mathbb{B}(\tilde{m}_0, 1)\) as \(t \to 0\).

**Proof.** Constants depending on \(n\) shall be denoted by \(c(n)\) and can change from line to line within the proof.
• (i) $F_i$ is $(1 + \varepsilon_0)$ bi-Lipschitz with respect to $g(t_i)$ implies that $|\nabla^{g(t_i)} F_i|_{g(t_i)} \leq c(n)$. Let $Z_i : B_{d_0}(m_0, 20) \times [t_i, T] \to \mathbb{R}^n$ be the solution to the Dirichlet problem with initial and boundary data given by $Z_i(t_i) = F_i$ and $Z_i|_{\partial B_{d_0}(m_0, 20)} = F_i|_{\partial B_{d_0}(m_0, 20)}$ coming from [DSS19, Theorem 2.1] $\partial_t Z_i(x, t) = \Delta_{g(t)} Z_i(x, t)$ for all $t \in [t_i, T]$. The solution satisfies $|\nabla^{g(t_i)} Z_i(t)| \leq c(n)$ for all $t \in [t_i, T]$ on $B_{d_0}(x_0, 20)$ as shown in [DSS19, Theorem 2.1]. Hence, all the conditions required to apply [DSS19, Theorem 3.8] are satisfied, and so the stated consequences there (except for the inequalities (4.3)) hold: for example

$$(1 - \alpha_0) d_i(x, y) \leq |Z_i(x, t) - Z_i(y, t)| \leq (1 + \alpha_0) d_i(x, y),$$

(4.4)

$$|\nabla^2 Z_i(x, t)| \leq \frac{c(n)}{\sqrt{t}},$$

$$|Z_i(x, t) - Z_i(x, t_i)| \leq c(n) \sqrt{t},$$

for all $t \in (2t_i, T/2)$. Taking a limit $i \to \infty$ we obtain a smooth limit $F(x, t) := \lim_{i \to \infty} Z_i(x, t), F : B_{d_0}(x_0, 10) \times (0, T] \to \mathbb{R}^n$ being a smooth solution to harmonic map heat flow. $\partial_t F(x, t) = \Delta_{g(t)} F(x, t)$ for $(x, t) \in B_{d_0}(x_0, 10) \times (0, T]$ satisfying all the stated inequalities except (4.3). The inequalities (4.3) follow from [Sim02, Lemma 2.2].

• (ii) Since $(g(t))_{t \in [0, T]}$ is smooth, we know $(1 - \varepsilon_0)g(s) \leq g(0) \leq (1 + \varepsilon_0)g(s)$ for all $s \in [0, \sigma]$ for $\sigma > 0$ sufficiently small. Hence, since $F_0$ is $(1 + \varepsilon_0)$ bi-Lipschitz with respect to $g(0)$, we must have $F_i := F_0$ is $(1 + 2\varepsilon_0)$ bi-Lipschitz with respect to $g(t_i)$ for a sequence $t_i \searrow 0$. Hence we may apply (i).

• (iii) The results in this setting are obtained in [DSS19, Theorem 3.11] and the proof thereof: the $\alpha_0$-closeness of $\hat{g}(t) := F(t) \cdot g(t)$ to $\delta$ follow from [Sim02, Lemma 4.2] as explained in the proof of [DSS19, Theorem 3.11]. The smoothness of the solution $F$ (for $t > 0$) to the harmonic map heat flow follows from the smoothness of $\hat{g}$ and $g$ (for $t > 0$) and the fact that $F(t) : B_{d_0}(m_0, 3/2) \to F(t)(B_{d_0}(m_0, 3/2))$ is a $C^1$ diffeomorphism: see [DSS19, Theorem 3.11] and the proof thereof.

The main result of this section is Proposition 4.2, which is used in the proof of the main theorem, Theorem 1.2.

**Proposition 4.2.** For $\alpha_0 \in (0, 1)$ and an integer $n \geq 2$, let $\varepsilon_0(n, \alpha_0) > 0$ and $\hat{S} = \hat{S}(n, \alpha_0) > 0$ be the constants from Theorem 4.1. Let $(M^n_1, g(t))_{t \in [0, T]}, (M^n_2, g(t))_{t \in [0, T]}, T \leq 1$, be two solutions to Ricci flow satisfying (a), (b) and (c) for an $R \geq 201$, and $x_0 \in U_{\hat{p}, 1}$, $\hat{x}_0 \in U_{\hat{p}, 2}$, $p \in M_1$, $\hat{p} \in M_2$, $(m_0$ from Theorem 4.1 equal to $x_0$, respectively $\hat{x}_0$) and let $d_{0, 1}, d_{0, 2}$ be the locally well defined metrics at time zero given by Lemma A.1. Assume $(\hat{d})$ holds for $d_{0, 1}$, with $\varepsilon_0 \leq \varepsilon_0$ and that there is an isometry $\psi_0 : U_{p, 1} \to U_{\hat{p}, 2} \subseteq U_{\hat{p}, 2}$, with $\psi(x_0) = \hat{x}_0$. Let $F_1 : B_{d_{0, 1}}(x_0, \frac{3}{2}) \times (0, \hat{S}) \to \mathbb{R}^n$, $F_2 : B_{d_{0, 2}}(\hat{x}_0, \frac{3}{2}) \times (0, \hat{S}) \to \mathbb{R}^n$, be the two smooth, $1 + \alpha_0$ bi-Lipschitz solutions to the Ricci-harmonic map heat flow provided by Theorem 4.1 with initial value the map $D_0$, respectively $D_0 \circ (\psi_0)^{-1}$, where $D_0$ comes from $(\hat{d})$ for $d_{0, 1}$. Denote the corresponding
solutions to the \(\delta\)-Ricci-DeTurck flow by \(\tilde{g}_i(t)\), \(i = 1, 2\). Let \((t_k)_k\) be any sequence of positive times with \(t_k \to 0\) as \(k \to \infty\). Then there exists a family of diffeomorphisms \(\varphi_k\) defined on \(B_{d(\tilde{g}_i(t_k))}(E^1_{t_k}(\sqrt{t_k})\) where \(E^1_{t_k} := F_1(t_k)(B_{d_0}(x_0, 3/2))\) such that
\[
(1 - \varepsilon(t_k))\tilde{g}_2(t_k) \leq (\varphi_k)_*(\tilde{g}_1(t_k)) \leq (1 + \varepsilon(t_k))\tilde{g}_2(t_k),
\]

\[
|\varphi_k - \text{Id}| \leq c(n)\sqrt{t_k},
\]

where \(\varepsilon(t_k) \to 0\) as \(t_k \to 0\).

Proof. Let \((t_k)_k\) be any sequence of positive times with \(t_k \to 0\) as \(k \to \infty\) that we fix once and for all. Let \(\varphi_k := F_2(t_k) \circ \psi_0 \circ (F_1(t_k))^{-1}\) defined on \(E^1_{t_k} := F_1(t_k)(B_{d_0}(x_0, \frac{2}{3}))\).

We first note that if \(q = F_1(t_k)^{-1}(p)\),
\[
d(\tilde{g}_2(t_k))(\varphi_k(p), \text{Id}(p)) = d(\tilde{g}_2(t_k))(F_2(t_k) \circ \psi_0(q), F_1(t_k)(q))
\leq d(\tilde{g}_2(t_k))(F_2(t_k) \circ \psi_0(q), D_0(q))
\leq d(\tilde{g}_2(t_k))(F_1(t_k)(q), D_0(q))
\leq 2|F_2(t_k) \circ \psi_0(q) - D_0(q)| + 2|F_1(t_k)(q) - D_0(q)|
\leq c\sqrt{t_k} + 2|F_2(0) \circ \psi_0(0) - D_0(0)| + 2|F_1(0)(0) - D_0(0)| + c\sqrt{t_k}
\]

\[
= c\sqrt{t_k} \to 0 \text{ as } t_k \to 0,
\]

which yields
\[
|\varphi_k(\cdot) - \text{Id}| \leq c\sqrt{t_k}.
\]

Define \(\tilde{g}_1^k := \tilde{g}_1(t_k)\), and \(\tilde{g}_2^k := \tilde{g}_2(t_k)\), \(k \geq 0\). Then, the sequences \((\tilde{g}_1^k)_{k \geq 0}\) and \((\tilde{g}_2^k)_{k \geq 0}\) satisfy the estimates \((4.2)\) and \((4.3)\) with \(t = t_k\). Also, the distance distortion estimates \((c)\) imply:
\[
|d(g_1^k)(x, y) - d(g_2^k)(x, y)| = |g_1^k \circ g_0^k \circ d(g_1^k)(x, y) - g_2^k(x, y)|
= |d(g_1^k)(F_2(t_k) \circ \psi_0)\circ d(g_1^k)(x, y) - d(g_2^k)(x, y)|
\leq |d_0((F_2(t_k) \circ \psi_0)^{-1}(x), (F_2(t_k) \circ \psi_0)^{-1}(y)) - d_0((F_2(t_k) \circ \psi_0)^{-1}(x), (F_2(t_k) \circ \psi_0)^{-1}(y))|
\leq \varepsilon(t_k)\sqrt{t_k}
\]

\[
= \varepsilon(t_k)\sqrt{t_k}.
\]

These facts let us apply Lemma 3.1 to the sequences of metrics \(g_1^k\), and \(g_2^k\), defined on a Euclidean ball of radius 1, with \(K := 100\sqrt{t_k}^{-1}\), \(c_0 = 100/99\), \(s = 1\) and \(\varphi = \varphi_k\).

We obtain the existence of a family of diffeomorphisms \(\varphi_k\) defined on \(\mathbb{R}(0, \frac{1}{2})\) having the required properties: the estimate \((4.6)\) follows from the third estimate of \((3.3)\) and \((4.7)\).

As a consequence of Proposition 4.2, we can measure the difference of the two corresponding solutions to \(\delta\)-Ricci-DeTurck flow:
Corollary 4.3. For $\alpha_0 \in (0,1)$, and an integer $n \geq 2$, assume the setting and notation of Proposition 4.2. In particular $\hat{g}_i(t) := (F_i(t)\big)_* g_i(t)$, $t \in (0,T)$, $i = 1, 2$ are the associated solutions to $(g_i(t))_{t \in (0,T)}$ to $\delta$-Ricci-DeTurck flow coming out of $(D_0), d_{0,1}$ in the distance sense provided by Theorem 4.1 (iii), where $D_0$ is from (4.1). Let $(t_k)_{k \in \mathbb{N}}$ be any sequence of positive times with $t_k \to 0$ as $k \to \infty$. Then there exists a $\beta_0(\alpha_0, n) > 0$ such that $\beta_0(\alpha_0, n) \to 0$ as $\alpha_0 \to 0$, and a solution $(\hat{g}_k(t) = (\hat{F}_k(t))_* g_1(t))_{t \in [t_k, S]}$ to $\delta$-Ricci-DeTurck flow associated to $(g_i(t))_{t \in [t_k, S]}$ which is $\beta_0$-close to the $\delta$ metric on $\mathbb{B}(0, R_0)$ for some $R_0 \in (0,1)$ and $\beta_0 = \beta_0(\alpha_0, n) \in (0, 1)$ and such that

\begin{equation}
\lim_{t_k \to 0^+} |\hat{g}_k^k(t_k) - \hat{g}_2^k(t_k)| = 0, \quad \text{on } \mathbb{B}(0, R_0),
\end{equation}

(4.8) $|\hat{F}_k^k(t) - D_0| \leq c(n)\sqrt{t}$, \quad on $B_{d_0}(x_0, \frac{1}{2})$, for $t \in [t_k, S]$.

**Proof.** Let $\hat{F}_1(t_k) := \hat{\varphi}_k \circ F_1(t_k)$ where the family of maps $\hat{\varphi}_k$ is obtained from Proposition 4.2. Then (4.5) implies the first statement in (4.8). Furthermore, $\hat{F}_1(t_k)$ is $(1 + \alpha_0)$ bi-Lipschitz with respect to $g_1(t_k)$ in view of (4.2) and (4.5). Thus we can apply Theorem 4.1 to $\hat{F}_1(t_k)$ and $(M^1_0, g_1(t) \big)_{t \in [t_k, T]}$ to obtain solutions $\hat{F}_k^k(t)$ for $t \in [t_k, T)$ to Ricci harmonic map heat flow which are $(1 + \beta_0)$ bi-Lipschitz and diffeomorphisms onto their image. Hence $(\hat{g}_1^k(t) := \hat{F}_1^k(t) \big)_* g_1(t) \big)_{t \in [t_k, T]}$ satisfy $(1 - \beta_0)\delta \leq \hat{g}_k^k(t) \leq (1 + \beta_0)\delta$ as required. The second estimate in (4.8) is a direct consequence of previously established estimates: if $t \in [t_k, S]$,

\begin{align*}
|\hat{F}_k^k(t) - D_0| &\leq |\hat{F}_k^k(t) - \hat{F}_k^k(t_k)| + |\hat{F}_k^k(t_k) - D_0| \\
&\leq c\sqrt{t} - t_k + |\hat{F}_k^k(t_k) - D_0| \\
&\leq c\sqrt{t} + |(\hat{\varphi}_k - \text{Id})(F_1(t_k))| + |F_1(t_k) - D_0| \\
&\leq c\sqrt{t},
\end{align*}

which is the second inequality of (4.8). Here we have used (4.1) from Theorem 4.1 in the second and the last inequality, and (4.6) in the last inequality. \hfill \Box

5. POLYNOMIAL CONVERGENCE RATE

We start by establishing a (faster than) polynomial convergence rate for the difference of two solutions to $\delta$-Ricci-DeTurck flow in the $L^2_{d\text{-loc}}$ sense.

**Lemma 5.1** (Faster-than-polynomial $L^2$ convergence rate). Let $n \geq 2$ be an integer. Then there exists an $\varepsilon(n) > 0$ such that the following is true. Let $(\hat{g}_i(t))_{t \in (0,T)}$, $i = 1, 2$, be two smooth solutions to $\delta$-Ricci-DeTurck flow on $\mathbb{B}(0, R) \times (0, T)$ which are $\varepsilon(n)$-close to the Euclidean metric,

\begin{equation}
(1 + \varepsilon(n))^{-1} \leq \hat{g}_i(t) \leq (1 + \varepsilon(n))\delta \quad \text{on } \mathbb{B}(0, R) \times (0, T),
\end{equation}

for $i = 1, 2$. Then for each $k \in \mathbb{N}$, $r > 0$, there exists $C_k(n) = C(n, k) > 0$ and $V(n, k, r)$ which is non-decreasing in $r$, such that if $|\hat{g}_2(0) - \hat{g}_1(0)|^2 \leq \tau_0$ on $\mathbb{B}(0, R),$

\begin{equation}
\int_{\mathbb{B}(0, 2^{-k}R)} |\hat{g}_1(t) - \hat{g}_2(t)|^2 \, dx \leq \tau_0 \cdot V(n, k, \frac{t}{R^2}) + \left(\frac{t}{R^2}\right)^{\frac{k}{2}} \cdot C_k(n) \quad t \in (0, T).
\end{equation}
Proof of Lemma 5.1. Following the statement and the proof of [DSS19, Lemma 6.1], consider the function

\[ v(t) := |h(t)|^2 \left( 1 + \lambda(|\tilde{g}_1(t) - \delta|^2 + |\tilde{g}_2(t) - \delta|^2) \right), \quad \lambda > 0, \]

where \( h(t) = \tilde{g}_1(t) - \tilde{g}_2(t) \). Define

\[ \tilde{h}^{ab}(t) := \frac{1}{2} \left( \tilde{g}_1^{ab}(t) + \tilde{g}_2^{ab}(t) \right) \quad \text{and} \quad \hat{h}^{ab}(t) := \frac{1}{2} \left( \tilde{g}_1^{ab}(t) - \tilde{g}_2^{ab}(t) \right). \]

We record the fact that \( v \) satisfies the following differential inequality if \( \lambda^{-1} \in \left[ c^{-1} \sqrt{\varepsilon(n)}, c \sqrt{\varepsilon(n)} \right] \) for some universal positive constant \( c \):

\[
\frac{\partial}{\partial t} v \leq \tilde{h}^{ab} \partial_a \partial_b v - \lambda |h|^2 \left( |D\tilde{g}_1|^2 + |D\tilde{g}_2|^2 \right) - |Dh|^2 \\
+ \left( 1 + \lambda (|\tilde{g}_1 - \delta|^2 + |\tilde{g}_2 - \delta|^2) \right) \left( h \ast \hat{h} \ast D^2(\tilde{g}_1 + \tilde{g}_2) \right) \\
+ \lambda |h|^2 \sum_{I=1}^2 (\hat{h} \ast (\tilde{g}_I - \delta) \ast D^2 \tilde{g}_I).
\]

(5.1)

Also, by choosing \( \varepsilon(n) > 0 \) smaller if necessary, we have \( \frac{1}{2}|h(t)|^2 \leq v(t) \leq 2|h(t)|^2 \).

Now, choose a smooth cut-off function \( \eta : \mathbb{B}(0, R) \to [0, 1] \) such that \( \eta \equiv 1 \) on \( \mathbb{B}(0, R/2) \) with support in \( \mathbb{B}(0, R) \) and such that \( |D\eta| \leq c \cdot R^{-1} \) for some positive constant \( c \). Then multiplying (5.1) by \( \eta \) and integrating over \( \mathbb{B}(0, R) \) gives:

\[
\frac{\partial}{\partial t} \int_{\mathbb{B}(0, R)} \eta v \leq \int_{\mathbb{B}(0, R)} \eta \tilde{h}^{ab} \partial_a \partial_b v - \lambda \int_{\mathbb{B}(0, R)} \eta |h|^2 \left( |D\tilde{g}_1|^2 + |D\tilde{g}_2|^2 \right) - \int_{\mathbb{B}(0, R)} \eta |Dh|^2 \\
+ \int_{\mathbb{B}(0, R)} \eta \left( 1 + \lambda (|\tilde{g}_1 - \delta|^2 + |\tilde{g}_2 - \delta|^2) \right) \left( h \ast \hat{h} \ast D^2(\tilde{g}_1 + \tilde{g}_2) \right) \\
+ \lambda \int_{\mathbb{B}(0, R)} \eta \sum_{I=1}^2 |h|^2 (\hat{h} \ast (\tilde{g}_I - \delta) \ast D^2 \tilde{g}_I).
\]

Integrating the first and last two terms on the right hand side of the above inequality by parts, we get:

\[
\frac{\partial}{\partial t} \int_{\mathbb{B}(0, R)} \eta v \leq - \int_{\mathbb{B}(0, R)} \partial_a \left( \eta \hat{h}^{ab} \right) \partial_a v - \lambda \int_{\mathbb{B}(0, R)} \eta |h|^2 \left( |D\tilde{g}_1|^2 + |D\tilde{g}_2|^2 \right) - \int_{\mathbb{B}(0, R)} \eta |Dh|^2 \\
+ \int_{\mathbb{B}(0, R)} D \left( \eta \left( 1 + \lambda (|\tilde{g}_1 - \delta|^2 + |\tilde{g}_2 - \delta|^2) \right) \ast h \ast \hat{h} \right) \ast D(\tilde{g}_1 + \tilde{g}_2) \\
+ \lambda \int_{\mathbb{B}(0, R)} \sum_{I=1}^2 D \left( \eta |h|^2 (\hat{h} \ast (\tilde{g}_I - \delta)) \right) \ast D\tilde{g}_I \\
=: A + B + C + D + E.
\]

We can now mimick the estimates of each integral quantity as in the proof of [DSS19, Lemma 6.1]: recalling that \( |D\tilde{g}_i| \leq c(n)t^{-1/2} \) for \( i = 1, 2 \) and taking into account the
additional presence of the cut-off function $\eta$, we see
\[
|A| \leq c(n) \int_{B(0,R)} \eta(|D\tilde{g}_1| + |D\tilde{g}_2|) |h| |Dh| + \lambda \varepsilon(n) |h|^2 (|D\tilde{g}_1| + |D\tilde{g}_2|)^2 + c \int_{B(0,R)} |Dv||D\eta|
\leq c(n) \int_{B(0,R)} \eta(|D\tilde{g}_1| + |D\tilde{g}_2|) |h| |Dh| + \lambda \varepsilon(n) |h|^2 (|D\tilde{g}_1| + |D\tilde{g}_2|)^2 + c \int_{B(0,R)} \left[ 2|h||Dh| + |h|^2 \lambda \varepsilon(n) (|D\tilde{g}_1| + |D\tilde{g}_2|) \right] |D\eta|
\leq c(n) \int_{B(0,R)} \eta(|D\tilde{g}_1| + |D\tilde{g}_2|) |h| |Dh| + \lambda \varepsilon(n) |h|^2 (|D\tilde{g}_1| + |D\tilde{g}_2|)^2 + c \sqrt{t} \int_{B(0,R)} |D\eta||h|^2 + c \left( \int_{B(0,R)} |D\eta||h|^2 \right)^{\frac{1}{2}} \left( \int_{B(0,R)} |D\eta||Dh|^2 \right)^{\frac{1}{2}},
\]
where constants $c$, for which the dependence is not explicitly noted, may depend on $\lambda$, which itself depends on $n$. In particular, the first integral on the right-hand side of the previous estimate can be absorbed by the integrals $B$ and $C$ with the help of Young's inequality.

A similar analysis can be performed for the integrals $D$ and $E$ so that:
\[
\frac{\partial}{\partial t} \int_{B(0,R)} \eta v + c^{-1} \int_{B(0,R)} \lambda \eta |h|^2 (|D\tilde{g}_1|^2 + |D\tilde{g}_2|^2) + \eta |Dh|^2 \leq c \sqrt{t} \int_{B(0,R)} |D\eta||h|^2 + c \left( \int_{B(0,R)} |D\eta||h|^2 \right)^{\frac{1}{2}} \left( \int_{B(0,R)} |D\eta||Dh|^2 \right)^{\frac{1}{2}},
\]
for some positive constant $c$ which is uniform in time and space.

By dividing by $\mathcal{H}_{d(\delta)}^n(B(0,R))$ and using the facts that
\[
|D\eta| \leq cR^{-1}, \quad |h|^2 \leq \varepsilon(n), \quad |Dh|^2 \leq \frac{c(n)}{t},
\]
one gets a first rough convergence rate at $t = 0$:
\[
\frac{\partial}{\partial t} \int_{B(0,R)} \eta v + c^{-1} \int_{B(0,R)} \lambda \eta |h|^2 (|D\tilde{g}_1|^2 + |D\tilde{g}_2|^2) + \eta |Dh|^2 \leq c R^{-1} \sqrt{t},
\]
for $t \in (0, T)$. By integrating in time (5.3) and by the assumption on the initial condition:
\[
\int_{\mathbb{B}(0,2^{-1}R)} |h(t)|^2 + \int_0^t \int_{\mathbb{B}(0,2^{-1}R)} \lambda |h|^2 (|D\tilde{g}_1|^2 + |D\tilde{g}_2|^2) + |Dh|^2 \leq c r_0 + c R^{-1} \sqrt{t},
\]
for $t \in (0, T)$, where we have used that $\frac{1}{2} |h(t)|^2 \leq v(t) \leq 2|h(t)|^2$, which is guaranteed for our choice of $\varepsilon(n)$, as we noted at the beginning of the proof.
Claim: For every \( k \in \mathbb{N} \), we have
\[
\int_{B(0,2^{-k}R)} |h(t)|^2 + \int_{0}^{t} \int_{B(0,2^{-k}R)} |Dh|^2 \leq V\left(n, k, \frac{t}{R^2}\right) \tau_0 + C_k R^{-k} t^{\frac{k}{2}},
\]
for \( t \in (0, T) \), where \( V\left(n, k, \frac{t}{R^2}\right) \) denotes a non-negative function which is non-decreasing in the third variable.

This follows by induction, assuming that the statement holds for \( k \), by using the induction assumption in (5.2) applied to a cut-off function \( \eta \) such that \( \eta \equiv 1 \) on \( B(0, 2^{-k+1}R) \) with support in \( B(0, 2^{-k+1}R) \), and \( |D\eta| \leq \frac{c_k}{R^2} \):
\[
\int_{B(0,2^{-k-1}R)} |h(t)|^2 + \int_{0}^{t} \int_{B(0,2^{-k-1}R)} |Dh|^2 \leq c\tau_0 + \int_{0}^{t} \frac{c_k R^{-1}}{\sqrt{s}} \int_{B(0,2^{-k}R)} |h|^2 + c_k R^{-1}\left( \int_{0}^{t} \int_{B(0,2^{-k}R)} |h|^2 \right)^{\frac{1}{2}} \left( \int_{0}^{t} \int_{B(0,2^{-k}R)} |Dh|^2 \right)^{\frac{1}{2}} ds
\]
\[\leq c\tau_0 + \int_{0}^{t} \frac{c_k R^{-1}}{\sqrt{s}} \left( c_k V(n, k, \frac{c_k}{R^2}) \tau_0 + c_k C_k \left( \frac{c_k}{R^2} \right)^{\frac{k}{2}} \right) ds
\]
\[+ c_k R^{-1}\left( \int_{0}^{t} V(n, k, \frac{c_k}{R^2}) \tau_0 + C_k \left( \frac{c_k}{R^2} \right)^{\frac{k}{2}} \right) \left( V(n, k, \frac{c_k}{R^2}) \tau_0 + C_k \left( \frac{c_k}{R^2} \right)^{\frac{k}{2}} \right)^{\frac{1}{2}}
\]
\[\leq c\tau_0 + c_k R^{-1} \sqrt{t} V(n, k, \frac{c_k}{R^2}) \tau_0 + c_k C_k R^{-(k+1)} t^{\frac{k+1}{2}} + c_k V(n, k, \frac{c_k}{R^2}) \sqrt{\tau_0} \left( \frac{c_k}{R^2} \right)^{\frac{k+1}{2}}
\]
\[\leq V\left(n, k + 1, \frac{c_k}{R^2}\right) \tau_0 + C_{k+1} R^{-(k+1)} t^{\frac{k+1}{2}}
\]
for \( t \in (0, T) \), where \( V(n, k + 1, \frac{c_k}{R^2}) \) denotes a non-negative function which is non-decreasing in the third variable and which may vary from line to line. A similar remark applies to \( C_k \).

We have used the induction assumption in the second inequality together with the elementary inequality \( \sqrt{a + b} \leq \sqrt{a} + \sqrt{b} \) for real numbers \( a, b \geq 0 \) in the third inequality. Finally, Young’s inequality is invoked in the last inequality.

We are now in a position to prove the main result of this section:

**Proposition 5.2** (Almost faster-than-polynomial \( L^\infty \) convergence rate). Under the same assumptions of Lemma 5.1 and for each \( k \in \mathbb{N} \), and \( r > 0 \), there exists \( C_k = C(n, k) > 0 \) and \( a = V = V(n, k, r) \) where \( V \) is non-decreasing in \( r \), such that if \( B(p, \sqrt{t}) \times (2^{-1} t, 2t) \subset B(0, 2^{-k}R) \times (0, T) \), then:
\[
(5.5) \quad |\tilde{g}_1(t) - \tilde{g}_2(t)|^2 \leq \tau_0 \cdot R^n t^{-\frac{n}{2}} \cdot V(n, k, \frac{t}{R^2}) + \left( \frac{t}{R^2} \right)^{\frac{k-n}{2}} \cdot C(n, k), \quad \text{on } B(p, \sqrt{t}/2).
\]

In particular, if \( k \in \mathbb{N} \), \( l \in \mathbb{N} \setminus \{0\} \), and \( r > 0 \), there exists \( C(n, k, l) > 0 \) and \( V = V(n, k, l, r) \) which is non-decreasing in \( r > 0 \), such that if \( B(p, \sqrt{l}) \times (2^{-1} t, 2t) \subset
\( \mathbb{B}(0, 2^{-k}R) \times (0, T) \), then on a smaller ball \( \mathbb{B}(p, \sqrt{t}/4) \),

\[
t^{\frac{1}{2}} |D^i (\bar{g}_1(t) - \bar{g}_2(t))|_\delta \leq \left( R^n t^{-\frac{2}{4}} V (n, k, \frac{t}{R^2}) \sqrt{r_0} + C(n, k) \left( \frac{t}{R^2} \right)^{\frac{n-2}{2}} \right)^{1-\frac{1}{m}}.
\]

**Remark 5.3.** The estimates on the covariant derivatives of the difference of two solutions to \( \delta \)-Ricci-DeTurck flow satisfying the setting of Lemma 5.1 obtained in Proposition 5.2 are not sharp but they will be sufficient for the proof of Theorem 1.2.

**Proof of Proposition 5.2.** With the same notations as those of the proof of Lemma 5.1, and according to the proof of [DSS19, Lemma 6.1], recall that the function \(|h|^2\) satisfies the following differential inequality:

\[
\frac{\partial}{\partial t} |h|^2 \leq \hat{h}^{ab} \partial_a \partial_b |h|^2 - \frac{2}{1 + \varepsilon} |Dh|^2 + h \ast \hat{h} \ast D^2 (\bar{g}_1 + \bar{g}_2)
+ h \ast \hat{h} \ast \bar{g}_1^{-1} \ast Dg_1 \ast Dg_1 + h \ast \bar{g}_2^{-1} \ast \hat{h} \ast Dg_1 \ast Dg_1
+ h \ast \bar{g}_2^{-1} \ast \bar{g}_2^{-1} \ast Dh \ast Dg_1 + h \ast \bar{g}_2^{-1} \ast \bar{g}_2^{-1} \ast Dg_2 \ast Dh.
\]

In particular, since there exists \( C > 0 \) such that \( t|D\bar{g}_i|^2 + t|D^2 \bar{g}_i| \leq C \), \( i = 1, 2 \), for \( t \in (0, T) \), one gets:

\[
\frac{\partial}{\partial t} |h|^2 \leq \hat{h}^{ab} \partial_a \partial_b |h|^2 - \frac{2}{1 + \varepsilon} |Dh|^2 + C \frac{|Dh|^2}{t} + \frac{C}{\sqrt{t}} |h||Dh|
\leq \hat{h}^{ab} \partial_a \partial_b |h|^2 - |Dh|^2 + C \frac{|Dh|^2}{t} + \frac{C}{\sqrt{t}} |h|^2,
\]

where \( C \) is a positive constant depending on \( n \) that may vary from line to line. Here we have used Young’s inequality in the second line to absorb the gradient term \(|Dh|\).

In particular, based on the definition of the coefficients \( \hat{h}^{ab} \) in terms of the two solutions, given at the beginning of the proof of Lemma 5.1,

\[
\frac{\partial}{\partial t} |h|^2 \leq \partial_a \left( \hat{h}^{ab} \partial_b |h|^2 \right) - \partial_a \hat{h}^{ab} \partial_b |h|^2 - |Dh|^2 + \frac{C}{t} |h|^2
\leq \partial_a \left( \hat{h}^{ab} \partial_b |h|^2 \right) + C |h| |Dh| (|Dg_1| + |Dg_2|) - |Dh|^2 + \frac{C}{t} |h|^2
\leq \partial_a \left( \hat{h}^{ab} \partial_b |h|^2 \right) - \frac{1}{2} |Dh|^2 + \frac{C}{t} |h|^2, \quad \text{on} \ \mathbb{B}(0, R) \times (0, T).
\]

As a first conclusion, there exists a positive constant \( C \) such that the function \( t^{-C} |h|^2 \) satisfies:

\[
\frac{\partial}{\partial t} (t^{-C} |h|^2) \leq \partial_a (\hat{h}^{ab} \partial_b (t^{-C} |h|^2)).
\]

Choose \( k > 2C + n \) and perform a local Nash-Moser iteration on each ball \( \mathbb{B}(p, \sqrt{t}) \times (t, 2t) \subset \mathbb{B}(0, 2^{-k}R) \times (0, T) \) to get for each \( \theta \in (0, 1) \),

\[
\sup_{\mathbb{B}(p, \sqrt{t}) \times (t(1+\theta), 2t)} t^{-C} |h|^2 \leq C(n, \theta) \int_t^{2t} \int_{\mathbb{B}(p, \sqrt{s})} (s^{-C} |h|^2) \ dx \, ds.
\]

See for instance [Lie96, Theorem 6.17] with ‘\( k = 0 \)’ for a proof.
Now apply Lemma 5.1 so that if \( t < 2^{-2k} R^2 \), the previous inequality (5.8) leads to the pointwise bound:

\[
\sup_{B(p, \sqrt{\theta t}) \times (t(1+\theta), 2t)} t^{-C} |h|^2 \leq C(n, k, \theta) \left( \frac{R^2}{t} \right)^{\frac{2}{n}} \int_{t}^{2t} s^{-C} \int_{B(0, 2^{-k} R)} |h(s)|^2 \, ds
\]

\[
\leq C(n, k, \theta) \left( \frac{R^2}{t} \right)^{\frac{2}{n}} \int_{t}^{2t} \left( V(n, k, \frac{R}{\sqrt{\theta t}}) \tau_0 + \left( \frac{1}{\sqrt{\theta t}} \right)^{\frac{k}{2}} \right) \cdot s^{-C} \, ds
\]

\[
\leq C(n, k, \theta) R^n t^{-(C+\frac{n}{2})} \int_{t}^{2t} \left( V(n, k, \frac{1}{\sqrt{\theta t}}) \tau_0 + \left( \frac{1}{\sqrt{\theta t}} \right)^{\frac{k}{2}} \right) \cdot ds
\]

\[
= C(n, k, \theta) R^n t^{-(C+\frac{n}{2})} V(n, k, \frac{1}{\sqrt{\theta t}}) \tau_0 + C(n, k, \theta) t^{-C} \left( \frac{1}{\sqrt{\theta t}} \right)^{\frac{k-n}{2}}
\]

for \( t \in (0, 2^{-2k} R^2) \).

Here we have used that \( V(n, k, r) \) is non-decreasing in \( r \). This estimate in turn implies the desired pointwise convergence rate, i.e.

\[
\sup_{B(p, \sqrt{\theta t}) \times (t(1+\theta), 2t)} |h|^2 \leq C(n, k, \theta) R^n t^{-(C+\frac{n}{2})} V(n, k, \frac{1}{\sqrt{\theta t}}) \tau_0 + C(n, k, \theta) \left( \frac{1}{\sqrt{\theta t}} \right)^{\frac{k-n}{2}}
\]

as long as \( B(p, \sqrt{\theta t}) \times (t, 2t) \subset B(0, 2^{-k} R) \times (0, T) \).

In order to prove the bounds on the derivatives, we recall the following standard local interpolation inequalities on Euclidean space:

\[
(5.9) \quad \sup_{\mathbb{R}^n} |D^j u| \leq C(n, j, m) \sup_{\mathbb{R}^n} |u|^{1-\frac{j}{m}} \cdot \sup_{\mathbb{R}^n} |D^m u|^{\frac{1}{m}},
\]

where \( u \) is any smooth function on \( \mathbb{R}^n \) with compact support and \( 0 \leq j \leq m \). See for example [Aub98, Theorem 3.70], or inductively apply Lemma A.5 of [SSS11] for a proof.

Now, by interior Bernstein-Shi’s estimates on solutions to \( \delta \)-Ricci DeTurck flow as stated in [SSS08, Corollary 5.4] together with (5.5), the previous interpolation inequalities applied to the coordinates of the tensor \( \eta \cdot h \) where \( \eta \) is a smooth cut-off function such that \( \eta \equiv 1 \) on \( B(p, \sqrt{\theta'} \cdot t) \) with support in \( B(p, \sqrt{\theta} \cdot t) \) for \( 0 < \theta' < \theta < 1 \) and such
that $t^{\frac{k}{2}} |D^k \eta|$ is bounded on $\mathbb{R}^n$ for all $k \geq 0$, imply for $0 \leq j \leq m$

$$
\sup_{\mathbb{B}(p, \sqrt{\eta t})} |D^j h((3 + \theta')t/2)| \leq \sup_{\mathbb{R}^n} |D^j (\eta \cdot h((3 + \theta')t/2))| \leq C(n, j, m) \sup_{\mathbb{R}^n} |\eta \cdot h((3 + \theta')t/2)|^{1 - \frac{j}{m}} \cdot \sup_{\mathbb{R}^n} |D^m (\eta \cdot h((3 + \theta')t/2))|^{\frac{j}{m}}
$$

(5.10)

$$
\leq C(n, j, m, \theta) \sup_{\mathbb{B}(p, \sqrt{\eta t})} |h((3 + \theta')t/2)|^{1 - \frac{j}{m}} \left( \frac{C(n, m)}{t^{\frac{m}{2}}} \right)^{\frac{j}{m}}
$$

$$
= C(n, j, m, \theta) \sup_{\mathbb{B}(p, \sqrt{\eta t})} |h((3 + \theta')t/2)|^{1 - \frac{j}{m}} \left( \frac{C(n, m)}{t^{\frac{m}{2}}} \right)^{\frac{j}{m}}
$$

$$
\leq C(n, j, m, \theta) \cdot t^{-\frac{j}{2}} \cdot \left( R^2 t^{-\frac{n}{2}} V(n, k, \theta, \frac{1}{R^2}) \sqrt{\tau_0} + C(n, k, \theta) \left( \frac{1}{t^{\frac{m}{2}}} \right)^{k-n} \right)^{1 - \frac{j}{m}}
$$

where $C(n, j, m, \theta)$ is a positive constant that may vary from line to line. This ends the proof of the desired estimate once one fixes $j \geq 1$ and let $m$ be $j^2$, and $\theta' = \frac{1}{2} < \theta = \frac{1}{2} < 1$. \hfill \square

As a first consequence of Proposition 5.2, we derive the following result which is an intermediate but essential step towards the proof of Theorem 1.2.

**Corollary 5.4.** There exists an $\bar{\varepsilon}_0(n) > 0$ such that the following holds. For $i = 1, 2$, let $(M^n_i, g_i(t))_{t \in (0,T)}$ be smooth solutions to Ricci flow (not necessarily complete) both satisfying (1.1) and (1.2) for $0 < \varepsilon_0 \leq \bar{\varepsilon}_0$ such that the locally well defined metrics at time zero agree (up to an isometry), that is for arbitrary $p \in M_1$, $\lim_{t \to 0} d(g_1(t)) = d_{0,1}$ on $U_{p,1}$, and $\lim_{t \to 0} d(g_2(t)) = d_{0,2}$, on $U_{p,2}$, and there exists an isometry $\psi_0 : (U_{p,1}, d_{0,1}) \to (U_{p,2} = \psi_0(U_{p,1}), d_{0,2})$, $\hat{p} := \psi_0(p)$, that is $\psi_0 : U_{p,1} \to U_{\hat{p},2} = \psi_0(U_{p,1})$ is a homeomorphism with $\psi_0'(d_{0,2}) = d_{0,1}$. We assume (1.3) and for arbitrary $p \in M_1$, and $x_0 \in U_{p,1}$ that (1.5) hold true for the locally defined metric $d_{0,1}$ on $U_{p,1}$ (and hence also $d_{0,2}$ on $U_{\hat{p},2}$). Then for each $x_0 \in U_{p,1}$ there exists an $R_0 > 0$ and $T_0 > 0$ depending on $n, \varepsilon_0$ and $x_0$ and solutions $(\hat{g}_1(t))_{t \in (0,T_0)}$ and $(\hat{g}_2(t))_{t \in (0,T_0)}$ to $\delta$-Ricci-DeTurck flow defined on $\mathbb{B}(0, R_0) \times (0, T_0)$ satisfying the following:

1. The metrics $\hat{g}_1(t)$ and $\hat{g}_2(t)$ are $\beta_0(\varepsilon_0)$-close to the $\delta$-metric for $t \in (0, T_0)$, where $\beta_0(\varepsilon_0) \to 0$ for $\varepsilon_0 \searrow 0$.

2. For each $k \in \mathbb{N}$, there exists $C_k = C(n, k, x_0) > 0$ and $T_k = T(n, k, x_0) > 0$ such that if $t \in (0, T_k)$:

$$
|\hat{g}_1(t) - \hat{g}_2(t)|_\delta \leq C_k t^k, \quad \text{on } \mathbb{B}(0, \sqrt{T_k}).
$$

3. Moreover, $\hat{g}_1(t) = (\hat{F}_1(t))_{t \in (0, T_0)} g_1(t)$ and $\hat{g}_2(t) = (F_2(t))_{t \in (0, T_0)} g_2(t)$, where $(\hat{F}_1(t))_{t \in (0, T_0)}$ and $(F_2(t))_{t \in (0, T_0)}$ are smooth families of bi-Lipschitz maps on $B_{d_{0,1}}(x_0, \frac{3}{2} R_0)$ respectively $B_{d_{0,2}}(\psi_0(x_0), \frac{3}{2} R_0)$ that satisfy the following:

   a. The family of maps $(\hat{F}_1(t))_{t \in (0, T_0)}$ (respectively $(F_2(t))_{t \in (0, T_0)}$) is a solution to the Ricci-harmonic map flow with respect to the Ricci flow solution $(g_1(t))_{t \in (0, T_0)}$ (respectively $(g_2(t))_{t \in (0, T_0)}$).
(b) \(|\tilde{F}_1(t) - D_0| \leq C_0 \sqrt{t}, \quad t \in (0, T_0)\) on \(B_{d_0,1}(x_0, \frac{3}{2} R_0)\)
and \(|F_2(t) - D_0 \circ (\psi_0)^{-1}| \leq C_0 \sqrt{t}, \quad t \in (0, T_0)\) on \(B_{d_0,2}(\psi_0(x_0), \frac{3}{2} R_0)\),
where here \(D_0\) are the distance coordinates on \(B_{d_0,1}(x_0, \frac{3}{2} R_0)\) coming from
(1.5).

(c) The distances \(\tilde{d}_1(t) := d(\tilde{g}_1(t))\) and \(\tilde{d}_2(t) := d(\tilde{g}_2(t))\) converge, uniformly
to the same distance \((D_0)_{d_0,1}\) on \(\mathbb{B}(0, R_0)\) as \(t\) approaches 0.

Remark 5.5. We have denoted the solution to the Ricci-harmonic map heat flow from
\((M_0^n, g_1(t))_{t \in (0, T_0)}\) to \(\mathbb{R}^n, \delta\) appearing in the statement of this theorem by \(\tilde{F}_1\) in order to
make it clear that is not necessarily the same solution to the one considered in
Proposition 4.2 (which comes from Theorem 4.1). The solution \(\tilde{F}_1\) appearing here is
obtained by modifying the \(F_1\) from Theorem 4.1 with the help of Corollary 4.3, as is
explained in the proof below.

Proof. Let \(p, x_0 \in M_1, \hat{p} = \psi_0(p), \hat{x}_0 = \psi_0(x_0) \in M_2\) be as in the statement of
the theorem. By scaling the solutions once, we may assume that \(R(x_0, n) = R(\hat{x}_0, n) > 400\).
We prove the estimates in this setting: scaling back to the original setting implies the
estimates for the original solution.

Let us start by proving part (3). To this end, let \((t_k)_{k \in \mathbb{N}}\) be an arbitrary sequence of
positive times with \(t_k \to \infty\) for \(k \to \infty\). According to Corollary 4.3, there exists a smooth family of \((1 + \beta_0)\)-bi-Lipschitz maps \((F^k_1(t))_{t \in [t_k, \hat{S}]}\) on \(B_{d_0,1}(x_0, \frac{3}{2} R_0)\) for some \(\beta_0 \in (0, 1)\)
solving the Ricci-harmonic map heat flow equation such that \(|\tilde{F}^k_1(t) - D_0| \leq c(n) \sqrt{t}\)
on \(B_{d_0}(x_0, \frac{3}{2} R_0)\), for \(t \in [t_k, \hat{S}]\), and, \(\lim_{t_k \to 0^+} |\tilde{g}^k(t_k) - \tilde{g}_2(t_k)| = 0\), on \(\mathbb{B}(0, R_0)\), for \(\tilde{g}^k(t_k) = (F^k_1)_{\ast}(g_1)(t_k)\), in view of (4.8).

The Arzela-Ascoli theorem guarantees the existence of a subsequence of \((\tilde{F}^k_1(t))_{t \in [t_k, \hat{S}]}\)
that converges locally uniformly on \(B_{d_0,1}(x_0, \frac{3}{2} R_0) \times [t_k, \hat{S}]\) to a continuous family of
\((1 + \beta_0)\)-bi-Lipschitz maps \((\tilde{F}_1(t))_{t \in [0, \hat{S}]}\). For the rest of the proof we fix this subsequence.
By interior parabolic regularity (see for instance (4.4) in the proof of Theorem
4.1 and the proof of [DSS19, Theorem 3.8]), the convergence takes place in the
\(C^1_{t, x}\) topology and this ensures that the family of maps \((\tilde{F}_1(t))_{t \in [0, \hat{S}]}\) is a solution to the Ricci-
harmonic map heat flow equation with respect to the Ricci flow solution \((g_1(t))_{t \in [0, \hat{S}]}\).
Moreover, estimates [(4.6), Proposition 4.2] and [(4.1), Theorem 4.1] ensure the maps
\((\tilde{g}_k \circ F_1(t_k))_{k \in \mathbb{N}}\) converge to \(D_0\) at rate \(\sqrt{t_k}\). Passing to the limit in estimate [(4.8),
Corollary 4.3] gives us the second estimate (3b) for \(\tilde{F}_1(t)\).

Now, let \((F_2(t))_{t \in [0, \hat{S}]}\) on \(B_{d_0,2}(\psi_0(x_0), \frac{3}{2} R_0)\) be the smooth family of bi-Lipschitz
maps solving the Ricci-harmonic map heat flow equation with initial value the map
\(D_0 \circ (\psi_0)^{-1}\) provided by Theorem 4.1. Estimate (3b) for \(F_2(t)\) follows from [(4.1),
Theorem 4.1].

In order to prove (3c), let us notice that both distances \(\tilde{d}_1(t) := d(\tilde{g}_1(t))\) and \(\tilde{d}_2(t) := d(\tilde{g}_2(t))\) converge to the same distance \((D_0)_{d_0,1}\) as \(t\) approaches 0 thanks to condition
(1.2) and estimates (3b).

We are now in a position to prove part (1) and (2).

According to Corollary 4.3, the family of metrics \((\tilde{g}^k(t) = (F^k_1(t))_{\ast}(g_1(t))_{t \in [t_k, \hat{S}]}\) is a solution to \(\delta\)-Ricci-DeTurck flow associated to \((g_1(t))_{t \in [t_k, \hat{S}]}\) which is \((1 \pm \beta_0)\) close to
the $\delta$ metric on $\mathbb{B}(0, R_0)$. Moreover, if $\tilde{g}_2(t) := (F_2(t))_*g_2(t)$, $t \in (0, T)$, is the associated solution to $(g_2(t))_{t \in (0, S)}$ to $\delta$-Ricci-DeTurck flow provided by Theorem 4.1,

$$
(5.12) \quad \lim_{t_k \to 0^+} |\tilde{g}_1^k(t_k) - \tilde{g}_2(t_k)|_\delta = 0.
$$

Denote an upper bound on $|\tilde{g}_1^k(t_k) - \tilde{g}_2(t_k)|_\delta$ on $\mathbb{B}(0, R_0)$ by $\varepsilon_k$. Now, let us apply Proposition 5.2 to $(\tilde{g}_1^k(\tau + t_k))_{\tau \in [0, \bar{S}/2]}$ and $(\tilde{g}_2(\tau + t_k))_{\tau \in [0, \bar{S}/2]}$ for $t_k$ sufficiently small to get for each $l \geq 0$, with $\theta = 4^{-1}$ and $R = 1$,

$$
|\tilde{g}_1^k(t) - \tilde{g}_2(t)|_\delta \leq V \left(n, l, \bar{S}\right) \varepsilon_k(t - t_k)^{-\frac{n}{2}} + C(n, l)(t - t_k)^l,
$$

on $\mathbb{B} \left(p, \sqrt{\frac{\theta - t_k}{t_k}}\right) \times (\bar{S}/2(t - t_k), 2(t - t_k))$,

in view of (4.8), as we pointed out above.

whenever $\mathbb{B}(p, \sqrt{1 - t_k}) \times (t - t_k, 2(t - t_k)) \subset \mathbb{B}_0 \times (0, \bar{S})$. Interior Bernstein-Shi’s estimates for solutions to the $\delta$-Ricci-DeTurck flow $(1 \pm \beta_0)$ close to $\delta$ metric as stated in [SSS08, Corollary 5.4] let us pass to the limit as $t_k$ tends to $0^+$, i.e. for each $l \geq 1$, there exists $\bar{T}_l > 0$ such that the solution $(\tilde{g}_1^k(t))_{t \in (t_k, \bar{T}_l)}$ converges in $C^0_{\text{loc}}(\mathbb{B}_0 \times (0, \bar{T}_l))$ to a solution $\hat{g}_1(t)$ to $\delta$-Ricci-DeTurck flow which is $(1 \pm \beta_0)$ close to the $\delta$ metric and which satisfies thanks to (5.13) the expected $C^0$-closeness to the solution $\tilde{g}_2(t)$ as stated in (5.11). Finally, by construction, $\hat{g}_1(t) = \lim_{k \to \infty} (\tilde{F}_1^k)_*g_1(t) = (\hat{F}_1)_*g_1(t) =: \hat{g}_1(t)$, which completes the proof of (2) and the theorem. \[\square\]

6. Exponential convergence rate

Before stating the main result of this section, we take the opportunity of explaining the behaviour at $t = 0$ of a solution to the heat equation on Euclidean space starting from an initial data vanishing on a ball. Although this result (and the short proof thereof) will not be used in the rest of the paper, we present it here, as it indicates what one may expect in more general cases. More precisely,

**Proposition 6.1.** Let $u_0 \in L^1(\mathbb{R}^n)$ with compact support. Assume $u_0$ vanishes on $\mathbb{B}(0, R)$. Then the **standard** solution $u$ to the heat equation with initial data $u_0$ satisfies:

$$
|u(x, t)| \leq (4\pi t)^{-\frac{n}{2}} \exp \left(-\frac{(R - |x|)^2}{4t}\right) \|u_0\|_{L^1(\mathbb{R}^n)}, \quad |x| \leq R, \quad t > 0.
$$

**Proof.** The proof is a simple consequence of the fact that $u(\cdot, t) = K_t * u_0$ on $\mathbb{R}^n$ for $t > 0$ where $K_t$ is the Euclidean heat kernel defined by $K_t(x) := (4\pi t)^{-\frac{n}{2}} \exp \left(-\frac{|x|^2}{4t}\right)$. 

\[\]
for $t > 0$ and $x \in \mathbb{R}^n$. Indeed, if $|x| \leq R$,

$$|u(x,t)| = |K_t * u_0(x)| \leq \int_{\mathbb{R}^n} K_t(x - y)|u_0(y)| dy$$

$$\leq \int_{\mathbb{R}^n \setminus B(0,R)} (4\pi t)^{-\frac{n}{2}} \exp \left(-\frac{|x - y|^2}{4t}\right) |u_0(y)| dy$$

$$\leq (4\pi t)^{-\frac{n}{2}} \int_{\mathbb{R}^n \setminus B(0,R)} \exp \left(-\frac{(R - |x|)^2}{4t}\right) |u_0(y)| dy$$

$$\leq (4\pi t)^{-\frac{n}{2}} \exp \left(-\frac{(R - |x|)^2}{4t}\right) \|u_0\|_{L^1(\mathbb{R}^n)}, \quad t > 0.$$}

Here we have used the assumption on the support of $u_0$ in the second line and the triangular inequality in the third line. \qed

The following theorem can be interpreted as a non-linear version of Proposition 6.1 and as such, it proves a sharp qualitative convergence rate.

**Theorem 6.2.** Let $n \geq 2$ be an integer. Then there exists $\varepsilon(n) > 0$ such that the following holds true. Let $R > 0, T > 0$ and $(\tilde{g}_i(t))_{t \in (0,T)}, i = 1, 2,$ be two smooth solutions to $\delta$-Ricci-DeTurck flow on $\mathbb{B}(0,R) \times (0,T)$ which are $\varepsilon(n)$-close to the Euclidean metric,

$$(1 + \varepsilon(n))^{-1} \delta \leq \tilde{g}_i(t) \leq (1 + \varepsilon(n)) \delta \quad \text{on } \mathbb{B}(0,R) \times (0,T),$$

for $i = 1, 2$. Assume furthermore that $\lim_{t \to 0^+} \sup_{\mathbb{B}(0,R)} |\tilde{g}_2(t) - \tilde{g}_1(t)|_\delta = 0$.

Then there exist positive constants $T_0, C_0$ and $R_0$ depending on $n, R, T$ such that

$$|\tilde{g}_1(t) - \tilde{g}_2(t)|_\delta \leq \exp \left(-\frac{C_0}{t}\right), \quad \text{on } \mathbb{B}(0,R_0) \times (0, \min\{T_0, T\}).$$

The proof of Theorem 6.2 relies on the following crucial result due to Aronson [Aro67]:

**Theorem 6.3 ([Aro67]).** Let $a^{ij} \in L^\infty(\mathbb{R}^n \times (0,T)), 1 \leq i, j \leq n,$ such that on $\mathbb{R}^n \times (0,T)$,

$$\lambda^{-1} |\xi|^2 \leq a^{ij}(x,t)\xi_i\xi_j \leq \lambda |\xi|^2, \quad \forall \xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n,$$

for some positive uniform $\lambda$. Then there exist positive constants $C = C(n, T, \lambda)$ and $\alpha = \alpha(n, T, \lambda)$ such that the fundamental solution of the divergence structure parabolic equation $\partial_t u - \partial_i \left(a^{ij}\partial_j u\right) = 0$ on $\mathbb{R}^n \times (0,T)$ satisfies:

$$K(x,t,y,s) \leq C(t - s)^{-\frac{n}{2}} \exp \left\{-\alpha \frac{|x - y|^2}{t - s}\right\}, \quad 0 \leq s < t, \quad x, y \in \mathbb{R}^n.$$  

**Remark 6.4.** In [Aro67], a lower bound for the fundamental solution is also provided. However, we will only use an upper bound in the sequel.

Let us now prove Theorem 6.2.
Proof of Theorem 6.2. Recall that equation (5.7) from the proof of Lemma 5.2 implies that the weighted norm of the difference of the solutions \( u(x, t) = \frac{1}{t} |h(t)|^2 \) satisfies for some large positive constant \( C \) on \( B(0, R) \times (0, T) \),

\[
\partial_t u \leq \partial_a \left( \tilde{h}^{ab} \partial_b u \right).
\]

The next step consists in localizing the previous differential inequality (6.2) in order to apply Theorem 6.3. To do so, let \( \psi : \mathbb{R}^n \to [0, 1] \) be any smooth cut-off function with support in \( B(0, R) \) such that \( \psi \equiv 1 \) on \( B(0, R/2) \) and \( |D\psi| \leq c/R \) on \( B(0, R) \). Then the function \( \psi u \) satisfies on \( \mathbb{R}^n \times (0, T) \):

\[
\partial_t (\psi u) \leq \partial_a \left( \tilde{h}^{ab} \partial_b (\psi u) \right) + C |D\psi| |Du| + C \left( \frac{|D\psi|}{\sqrt{t}} + |D^2 \psi| \right) |u|,
\]

where the coefficients \( \tilde{h}^{ab} \) are extended arbitrarily by continuity on \( \mathbb{R}^n \times (0, T) \) so that 
\[ (1 - \varepsilon(n))|\xi|^2 \leq \tilde{h}^{ab}(x, t)\xi_a\xi_b \leq (1 + \varepsilon(n))|\xi|^2 \] for all \( \xi \in \mathbb{R}^n \). Here we have used the fact that \( |Dg_0|, i = 1, 2 \) (and hence \( |Dh| \)) is uniformly bounded by \( C/\sqrt{t} \).

The parabolic maximum principle applied to \( \psi u \) shows that for \( 0 < t_0 < t \) and \( x \in \mathbb{R}^n \), \( (\psi u)(x, t) \leq \ell(x, t) \) where \( \ell \) is the standard solution to \( \partial_t \ell = \partial_a \left( \tilde{h}^{ab} \partial_b \ell \right) + S(t) \) with \( \ell(t_0) = u(t_0) \) and where the source term is defined as \( S(t) := C |D\psi| |Du| + C(t^{-\frac{1}{2}} |D\psi| + |D^2 \psi|) |u| \): The maximum principle can be applied as \( \ell(t) \geq 0 \) and \( \psi u(t) = 0 \) outside the ball \( B(0, R) \). In particular,

\[
(\psi u)(x, t) \leq \int_{\mathbb{R}^n} K(x, t, y, t_0)(\psi u)(y, t_0) \, dy + \int_{t_0}^t \int_{\mathbb{R}^n} K(x, t, y, s) S(y, s) \, dy \, ds, \quad x \in \mathbb{R}^n.
\]

Taking into account the definition of \( S \) and the support of \( \psi \) leads to:

\[
u(x, t) \leq \int_{B(0, R)} K(x, t, y, t_0) u(y, t_0) \, dy 
+ \frac{C(R)}{\sqrt{t}} \int_{t_0}^t \int_{\overline{B}(0, R) \setminus \overline{B}(0, R/2)} K(x, t, y, s) \left( |Du|(y, s) + \left( 1 + \frac{1}{\sqrt{s}} \right) |u|(y, s) \right) \, dy \, ds,
\]

if \( x \in \mathbb{B}(0, R/2) \) and \( t_0 < t \).

Now, Proposition 5.2 applied to the time interval \( [r_i, T] \) for any sequence of positive times \( r_i \) with \( r_i \to 0 \) as \( i \to \infty \), and the fact that \( \tau_i := \sup_{B(0, R)} |\tilde{g}_2(r_i) - \tilde{g}_1(r_i)| \to 0 \) tells us that \( |\tilde{g}_2(t) - \tilde{g}_1(t)| \leq C(n, k, R) t^k \) for \( t \in (0, T) \) and hence \( u(\cdot, t) \) satisfies 
\[ \lim_{t \to 0^+} \left( 1 + t^{-\frac{k}{2}} \right) u(\cdot, t) = 0 \] by replacing \( R \) with \( R/2^k =: R_0 \) with \( k \in \mathbb{N} \) large enough if necessary. Similarly, Proposition 5.2, ensures that the gradient of \( u \) is uniformly bounded on \( \mathbb{B}(0, R_0) \times (0, T) \) and approaches zero uniformly as \( t \searrow 0 \). In particular, by letting \( t_0 \) go to \( 0^+ \), Inequality (6.3) together with Theorem 6.3 applied to \( u(\cdot, t) \) imply that there exist \( R_0, C_0 \) and \( \alpha_0 \) depending on \( n, \varepsilon(n) \) and \( T \) such that if \( x \in \mathbb{B}(0, R_0/4) \)
and $t \in (0, T)$:

$$u(x, t) \leq C_0 \int_0^t \int_{B(0, R_0) \setminus B(0, R_0/2)} (t - s)^{-\frac{n}{2}} \exp \left\{ -\alpha_0 \frac{|x - y|^2}{t - s} \right\} dy ds$$

$$\leq C_0 \int_0^t \int_{B(0, R_0) \setminus B(0, R_0/2)} (t - s)^{-\frac{n}{2}} \exp \left\{ -\alpha_0 \frac{3R_0^2}{16(t - s)} \right\} dy ds$$

$$\leq C_0 R_0^n \int_0^t \hat{s}^{-\frac{n}{2}} \exp \left\{ -\alpha_0 \frac{3R_0^2}{16\hat{s}} \right\} d\hat{s} = C_0 R_0^2 \int_{R_0^2}^{+\infty} \tau^{-\frac{n}{2}} \exp \left\{ -\alpha_0 \frac{3\tau}{16} \right\} d\tau$$

$$\leq \frac{32}{3\alpha_0} C_0 R_0^2 \left( \frac{R_0^2}{t} \right)^{\frac{n}{4} - 2} \exp \left\{ -\alpha_0 \frac{3R_0^2}{16t} \right\} ,$$

if $t \leq T_0 = \min \{ T, \varepsilon_0(n) \alpha_0 R_0^2 \}$, where $\varepsilon_0(n)$ denotes a positive constant depending on $n$ only. Indeed, an integration by parts show that:

$$\int_{R_0^2}^{+\infty} \tau^{-\frac{n}{2}} \exp \left\{ -\alpha_0 \frac{3\tau}{16} \right\} d\tau = \frac{16}{3\alpha_0} \left( \frac{R_0^2}{t} \right)^{\frac{n}{4} - 2} \exp \left\{ -\alpha_0 \frac{3R_0^2}{16t} \right\}$$

$$+ \frac{8(n - 4)}{3\alpha_0} \int_{R_0^2}^{+\infty} \tau^{-\frac{n}{2} - 3} \exp \left\{ -\alpha_0 \frac{3\tau}{16} \right\} d\tau$$

$$\leq \frac{16}{3\alpha_0} \left( \frac{R_0^2}{t} \right)^{\frac{n}{4} - 2} \exp \left\{ -\alpha_0 \frac{3R_0^2}{16t} \right\}$$

$$+ \frac{8(t^n - 4)}{3\alpha_0 R_0^2} \int_{R_0^2}^{+\infty} \tau^{-\frac{n}{2} - 2} \exp \left\{ -\alpha_0 \frac{3\tau}{16} \right\} d\tau,$$

which implies the expected result by absorption if $t$ is chosen small enough compared to $n$, $\alpha_0$ and $R_0$.

By unravelling the definition of $u$ in terms of the norm of the difference of the solutions to $\delta$-Ricci DeTurck flow, this proves the expected exponential decay. \( \square \)

Combining Corollary 5.4 and Theorem 6.2 leads to the proof of Theorem 1.2:

**Proof of Theorem 1.2.** Thanks to Corollary 5.4, there exist solutions $(\tilde{g}_1(t))_{t \in (0, \tilde{t})}$ and $(\tilde{g}_2(t))_{t \in (0, \tilde{t})}$ to $\delta$-Ricci-DeTurck flow associated to $(g_i(t))_{t \in (0, \tilde{t})}$, $i = 1, 2$, which satisfy all the conditions stated in Theorem 1.2 but the exponential decay (1.6).

Now, we can apply Theorem 6.2 to the two aforementioned solutions to $\delta$-Ricci-DeTurck flow in order to guarantee an exponential convergence rate as expected. \( \square \)

7. **Almost Ricci-pinched expanding gradient Ricci solitons**

In this section, we prove that if an expanding gradient Ricci soliton has non-negative Ricci curvature and if it is almost Ricci-pinched in a suitable sense then it is Euclidean. Recall that a triple $(M^n, g, \nabla^g f)$ is an expanding gradient soliton if $(M^n, g)$ is a smooth Riemannian manifold, $f : M \to \mathbb{R}$ is smooth and $\nabla^{g.2} f = \text{Ric}(g) + \frac{4}{n} g$. We use the following definition, compare [Der17, Definition 1.1], to describe exponential closeness of an expanding gradient Ricci soliton to a smooth cone at infinity.
Definition 7.1. An expanding gradient Ricci soliton \((M^n, g, \nabla^g f)\) is asymptotic to a metric cone \((C(\Sigma), g_{C(\Sigma)} := dr^2 + r^2 g_\Sigma, r \partial_r/2)\) over a smooth Riemannian link \((\Sigma, g_\Sigma)\) at an exponential rate if there exists a compact \(K \subset M\), a positive radius \(R\) and a diffeomorphism \(\varphi : M \setminus K \to C(\Sigma) \setminus \overline{B_{d(g_{C(\Sigma)})}(0, R)}\) such that
\[
\sup_{\partial B(o, r)} |\nabla^k (\varphi^* g - g_{C(\Sigma)})|_{g_{C(\Sigma)}} = O(r^{-n+k} e^{-\frac{r^2}{4}}), \quad \forall k \in \mathbb{N},
\]
\[
f(\varphi^{-1}(r, x)) = \frac{r^2}{4}, \quad \forall (r, x) \in C(\Sigma) \setminus \overline{B_{d(g_{C(\Sigma)})}(0, R)},
\]
as \(r \to +\infty\).

We are now in a position to state the main rigidity result of this section.

Proposition 7.2. Let \((M^n, g, \nabla^g f)\) be a complete expanding gradient Ricci soliton with non-negative Ricci curvature. Assume there exist \(c > 0\) and \(\beta \in (0, 1)\) such that
\[
(7.1) \quad c R_g g \leq \text{Ric}(g) + \exp\left(-f^\beta\right) g, \quad \text{on } M.
\]
Then,

1. \((M^n, g, \nabla^g f)\) is asymptotic to a Ricci flat metric cone \((C(\Sigma), g_{C(\Sigma)}, \frac{1}{2}r \partial_r)\) at an exponential rate in the sense of Definition 7.1, where \(\Sigma\) is diffeomorphic to \(\mathbb{S}^{n-1}\).

2. If \(n \in \{3, 4\}\), \((M^n, g, \nabla^g f)\) is isometric to the Gaussian soliton \((\mathbb{R}^n, \delta, \frac{1}{2}r \partial_r)\).

3. If \(g\) has 2-non-negative curvature operator, then \((M^n, g, \nabla^g f)\) is isometric to the Gaussian soliton \((\mathbb{R}^n, \delta, \frac{1}{2}r \partial_r)\).

Remark 7.3. (1) This proposition was proved in [Der17, Proposition 3.5] under the assumption that the metric is Ricci pinched, i.e. \(c R_g g \leq \text{Ric}(g)\) on \(M\) for some positive constant \(c\). See the references therein for related results.

(2) The proof in dimension 3 can be simplified from the one given in [Der17] since in this particular dimension, the Ricci curvature controls pointwise the whole curvature tensor. Indeed, as soon as the Ricci curvature is shown to decay exponentially fast, then it follows that the norm of the full curvature tensor, as well as the norms of the higher covariant derivatives of the curvature tensor also decay exponentially fast, in view of Bernstein-Shi type estimates applied to the curvature tensor: See or example [CLN06, Chapter 6].

Remark 7.4. We also mention the paper [Cha20] that shows that any expanding gradient Ricci soliton coming out of Euclidean space is Euclidean and which relies on the positive mass theorem due to [SY17] in all dimensions.

Remark 7.5. The assertion (3) from Proposition 7.2 supports Question 1.5.

We underline the fact that the rigidity part of the proof of Proposition 7.2 relies on the rigidity of the Bishop-Gromov inequality only.

Before we start the proof of Proposition 7.2, we gather well-known soliton identities holding on a self-similar expander:
Lemma 7.6. Let \((M^n, g, \nabla^g f)\) be an expanding gradient Ricci soliton. Then the trace and first order soliton identities are:

\[
\Delta_g f = R_g + \frac{n}{2},
\]

\[
\nabla^g R_g + 2\text{Ric}(g)(\nabla^g f) = 0,
\]

\[
|\nabla f|^2_g + R_g = f + \text{cst}.
\]

See for instance [CLN06, Chapter 4, Section 1] for a proof.

Proof of Proposition 7.2. Recall that the potential function of an expanding gradient Ricci soliton with non-negative Ricci curvature satisfies:

\[
\nabla^g 2f \geq \frac{g}{2}, \quad \text{on } M.
\]

In particular, \( f \) is strictly convex and by integrating along geodesics, \( f \) is proper. More precisely,

(7.2) \[ f(x) \geq \min_M f + \frac{d_g(p, x)^2}{4}, \quad \forall x \in M, \]

where \( p \) is the unique critical point of \( f \) which necessarily corresponds to the minimum of \( f \). In particular, by the Morse lemma, this implies that the level sets \( \{ f = t \} \) of \( f \), for \( t > \min_M f \), are all diffeomorphic to \( S^{n-1} \) and that \( M \) is diffeomorphic to Euclidean space \( \mathbb{R}^n \).

By the first order identity \( |\nabla f|^2_g + R_g = f \) that holds on \( M \) after subtracting a constant from \( f \) if necessary (see Lemma 7.6) one also gets \( |\nabla f|^2_g \leq f \) on \( M \) which implies:

(7.3) \[ f(x) \leq \left( \frac{\min_M f + d_g(p, x)}{2} \right)^2, \quad \forall x \in M. \]

Let us consider the Morse flow associated to \( f \) which is well-defined outside a compact set thanks to the first order identity mentioned above and the lower bound (7.2) on \( f \), i.e.

(7.4) \[ \partial_t \varphi_t = \left( \frac{1}{|\nabla f|^2_g} \nabla^g f \right) \circ \varphi_t, \quad \varphi_{t_0} = \text{Id}_M, \quad \text{for } t \geq t_0 \geq \min_M f + 1. \]

Then, \( f(\varphi_t(x)) = f(x) + t - t_0 = t, \ t \geq t_0 \) and \( x \in f^{-1}(\{t_0\}) \). Now, thanks to the contracted Bianchi identity, \( \nabla^g R_g + 2\text{Ric}(g)(\nabla^g f) = 0 \) on \( M \) from Lemma 7.6. Contracting this identity with \( \nabla^g f \) and invoking (7.1), we see

\[
2eR_g |\nabla g f|^2_g + g(\nabla^g R_g, \nabla^g f) \leq 2e^{-t^2} |\nabla^g f|^2_g, \quad \text{on } M.
\]

Reinterpreting the previous differential inequality in terms of the Morse flow associated to \( f \), one gets:

\[
\frac{\partial}{\partial t} \left( e^{2ct} R_g(\varphi_t(x)) \right) \leq 2e^{-t^2 + 2ct}, \quad t \geq t_0.
\]
In particular, by integrating from \( t_0 \) to \( t \):
\[
e^{2ct} R_g(\varphi_t(x)) \leq e^{2ct_0} R_g(\varphi_{t_0}(x)) + 2 \int_{t_0}^{t} e^{-s^\beta + 2cs} \, ds
\]
\[
= e^{2ct_0} R_g(x) + 2 \int_{t_0}^{t} e^{-s^\beta + 2cs} \, ds
\]
\[
\leq e^{2ct_0} R_g(x) + \frac{1}{c} e^{-2ct - \nu},
\]
if \( t_0 \) is chosen sufficiently large. Here we have used \( \beta \in (0,1) \) and \( c > 0 \) in the last inequality.

As a first conclusion, one gets
\[
R_g(\varphi_t(x)) \leq e^{-2c(t-t_0)} R_g(x) + \frac{1}{c} e^{-t^\beta},
\]
for any \( x \in f^{-1}\{t_0\} \). By definition of the Morse flow, this gives
\[
R_g(y) \leq \max_{x \in f^{-1}\{t_0\}} R_g \cdot e^{2ct_0 - f(y)} + \frac{1}{c} e^{-f(y)^\beta},
\]
for any \( y \in M \) such that \( f(y) \geq t_0 \geq \min_M f + 1 \).

This shows that \( R_g \) decays as fast as \( e^{-f^\beta} \) at infinity. So does the Ricci curvature since \( g \) has non-negative Ricci curvature.

The rest of the proof of (1) and (2) is [Der17, Proposition 3.5] verbatim.

In order to prove (3), we invoke (1) to ensure that the link is diffeomorphic to \( S^{n-1} \) and is endowed with an Einstein metric such that the cone over it has 2-non-negative curvature operator. The results of [BW08] then shows the expected rigidity result: the cone is isometric to Euclidean space which forces the soliton to be the expanding Gaussian soliton by Bishop-Gromov volume comparison.

\[\square\]

8. Compactness of Ricci-pinched manifolds in dimension three

We consider \((M^3,g_0)\) a smooth, complete Riemannian manifold with uniformly pinched, non-negative, bounded Ricci curvature. Assuming by contradiction that \((M^3,g_0)\) is non-compact and non-flat from now on, the work of Lott [Lot19] ensures there exists a smooth, complete Ricci flow \((M^3,g(t))_{t \in [0,\infty)}\) with \( g(0) = g_0 \), satisfying the following properties:

(1) [Lot19, Propositions 2.1, 2.13] there exists \( C > 0 \) such that
\[
|Rm(g(t))| \leq \frac{C}{t}, \text{ for all } t > 0,
\]

(2) [Ham82, Theorem 9.4], [Lot19, Proposition 2.1] the flow has \( \text{Ric}(g(t)) > 0 \) and is uniformly Ricci pinched, i.e. there exists \( \lambda_0 > 0 \) such that
\[
\text{Ric}(g(t)) \geq \lambda_0 \text{R}(g(t) \cdot g(t)), \text{ for all } t \geq 0,
\]

(3) [Lot19, Propositions 1.5, 3.1 and 4.1] the flow is uniformly volume non-collapsed, i.e. there exists \( v_0 > 0 \) such that
\[
\text{Vol}_g(t)(B_{g(t)}(x, r)) \geq v_0 r^3,
\]
for all \( x \in M, r > 0 \) and \( t > 0 \).
By performing a blow-down we can instead assume that the flow \((M^n, g(t))\) is only smooth for \(t > 0\) and converges in the metric sense uniformly as \(t \to 0\) to a metric space \((M, d_0)\), since \(d_0 \geq d_t \geq d_0 - c_0 \sqrt{t}\) for all \(t > 0\) by Lemma A.1. In fact \((M, d_0)\) is a metric cone over a 2-dimensional metric space \((\Sigma^2, \tilde{d}_0)\) from Cheeger-Colding [Che01, Theorem 9.79], and by Lemma A.1, \((M, d_0)\) is homeomorphic to \((M, d_t)\) for all \(t > 0\). Furthermore \((M, d_0, \mathcal{H}^{n}_{d_0})\) is also an \(\text{RCD}^\ast(0,3)\) space, as it is the limit (in the Gromov-Hausdorff and distance sense) of smooth spaces with non-negative Ricci curvature and uniform Euclidean volume growth, i.e. \(\text{Vol}_{g(t)}(B_{g(t)}(p_0, r)) \geq \delta_0 r^n, r > 0\): see [GMS15, Theorem 7.2] for instance.

Finally, the result of Ketterer [Ket15] implies that \((\Sigma, \tilde{d}_0, \mathcal{H}^{n-1}_{\tilde{d}_0})\) is an \(\text{RCD}^\ast(1,2)\) space. Furthermore \((M, d_0)\) is uniformly volume non-collapsed.

**Lemma 8.1.** Let \(o\) be the tip of the cone \((M, d_0)\). Then for \(r' > r > 0\), \((A_{d_0}(o, r, r')) := B_{d_0}(o, r') \setminus B_{d_0}(o, r, d_0) \subseteq M\) is Reifenberg (and hence uniformly Reifenberg due to Lemma 2.1).

**Proof.** We first show that every tangent cone to \((M, d_0)\) at any point away from the vertex \(o\) is 3-dimensional Euclidean space.

Let \(p \in M \setminus \{o\}\) and consider a blow-up sequence of the flow \((M^n, g(t))_{t \in (0,\infty)}\) around \((p,0)\). Since \((M, d_0)\) is a cone, this converges to a flow \((M', (g'(t))_{t \in (0,\infty)})\), with the same properties \((1), (2), (3)\) that the original solution had, coming out of a cone \((C', d'_0)\) which splits a line, i.e. \((C', d'_0)\) is isometric to \((C'' \times \mathbb{R}, d''_0 \oplus ds^2)\). But then Proposition B.1 implies that \((M', (g'(t))_{t \in (0,\infty)})\) splits a Euclidean factor. Since \((M', (g'(t))_{t \in (0,\infty)})\) is still uniformly Ricci pinched and 3-dimensional, this implies that \((M', (g'(t))_{t \in (0,\infty)})\) is static flat Euclidean \((\mathbb{R}^3, \delta)\) and thus \((C', d'_0)\) is isometric to \((\mathbb{R}^3, d(\delta))\).

Considering \(A_{d_0}(o, r, r') \subseteq X\), Lemma 2.1 then shows that \((A_{d_0}(o, r, r'), d_0)\) is uniformly Reifenberg. \(\square\)

We consider the induced distance \(\hat{d}_0\) on the link \(\Sigma\).

**Lemma 8.2.** The metric space \((\Sigma, \hat{d}_0)\) is an Alexandrov space of curvature at least 1.

**Proof.** As we noted above, \((C(\Sigma), d_0, \mathcal{H}^n_{d_0}) = (M, d_0, \mathcal{H}^n_{d_0})\) is an \(\text{RCD}^\ast(0,3)\) space, and so the result of Ketterer [Ket15] implies that \((\Sigma, \hat{d}_0, \mathcal{H}^{n-1}_{\hat{d}_0})\) is an \(\text{RCD}^\ast(1,2)\) space. Now, by [CM21, Corollary 13.7], \((\Sigma, \hat{d}_0, \mathcal{H}^{n-1}_{\hat{d}_0})\) is an \(\text{RCD}(1,2)\) space. But then the result of Lytchak-Stadler [LS18] implies that \((\Sigma, \hat{d}_0)\) is a two dimensional Alexandrov space of curvature at least 1. \(\square\)

**Lemma 8.3.** There exists a smooth structure on \(\Sigma\) and a sequence of metrics \((g_i)_{i \in \mathbb{N}}\) on \(\Sigma\), which are smooth with respect to this structure, such that \((\Sigma, g_i)\) has sectional curvature greater than one, and \((\Sigma, d(g_i))\) converges to \((\Sigma, \hat{d}_0)\) as \(i \to \infty\) in the Gromov-Hausdorff sense. In particular, \(\Sigma\) is a 2-sphere.

**Proof.** According to [BBI01, Corollary 10.10.3] and the references therein and thanks to Lemma 8.2, \(\Sigma\) is a topological surface without boundary which therefore admits a unique smooth structure: compare also the proof of [Lot19, Proposition 6.4]. Now, [IRV15, Lemmata 2.3, 2.4] ensure that the set of smooth Riemannian metrics on \(\Sigma\)
with curvature bounded from below by 1 is dense in this space with respect to the Gromov-Hausdorff topology. In particular, the Gauss-Bonnet formula guarantees that $\Sigma$ is a 2-sphere and that there exists a sequence of smooth metrics $(g_i)_{i \in \mathbb{N}}$ on $\Sigma$ with $K_{g_i} \geq 1$ such that $(\Sigma, d(g_i))_{i \in \mathbb{N}}$ converges to $(\Sigma, d_0)$ in the Gromov-Hausdorff sense. Moreover, by stretching the sequence $(g_i)_{i \in \mathbb{N}}$ by a factor $(1 - \varepsilon_i)$ where $\varepsilon_i$ tends to 0, we can assume that $K_{g_i} > 1$. This ends the proof of the lemma. \hfill \Box

Remark 8.4. Some of the proofs of the results cited in the paper [IRV15, Lemmata 2.3, 2.4] may be replaced by alternative proofs from [CR21]. For example, an alternative proof to the theorem of Alexandrov and Zalgaller from [AZ67], of the fact that any 2-dimensional manifold with bounded integral curvature can be decomposed into a finite union of geodesic triangles whose interiors are disjoint, can be found in [CR21].

Lemma 8.5. There exists a smooth self-similar expanding solution $(\mathbb{R}^3, g_e, \nabla^{g_e} f_e)$ with bounded non-negative curvature operator which is uniformly volume non-collapsed, such that its tangent cone at infinity is $C = (C(\Sigma), d_0)$. That is: $(\mathbb{R}^3, d_{g_e}) \to (C(\Sigma), d_0)$ in the Gromov-Hausdorff sense as $\varepsilon \searrow 0$.

Proof. By [Der16], alternatively [SS13] after approximating the cone by a smooth space with non-negative curvature operator as in [Sch14, Proposition 3.2.6], we know that there exists smooth, positively curved expanders $(\mathbb{R}^3, g^i_{C}(t))_{t \in (0, \infty)}$ coming out of $(C(\Sigma), d^i_C, o)$, where $d^i_C$ denotes the cone metric induced by the Riemannian distance $d(g_i)$ on $\Sigma$. Here, and in the following, when we say that a solution to Ricci flow $(X, \ell(t))_{t \in (0,T)}$ is 'coming out of' a metric space $(Z, d_0)$, then this means that $(X, \ell(t)) \to (Z, d_0)$ in the Gromov-Hausdorff sense, as $t \searrow 0$. By the continuity of Hausdorff measure under Gromov-Hausdorff convergence for sequences of Alexandrov spaces with curvature uniformly bounded from below ([BBI01, Theorem 10.10.10]), the asymptotic volume ratios

$$\text{AVR}(g^i_e) := \lim_{r \to +\infty} \frac{\text{Vol}_{g^i_e}(B_{g^i_e}(p_i, r))}{r^3} = \frac{1}{2} \text{Vol}_{g_i}(\Sigma), \quad p_i \in \mathbb{R}^3,$$

are uniformly bounded from below by a positive constant $V_0$ say. Now, this implies that there is a uniform bound, $|Rm(g^i_e(t))|_{g^i_e} \leq \frac{B_0(V_0)}{t}$ on the curvature of the metrics $g^i_e(t)$, where $B_0(V_0)$ is a constant depending on $V_0$, as explained in [SS13] or in the proof of [Der17, Theorem 4.7]. This allows us to take a limit in the smooth Cheeger-Gromov topology as $i \to \infty$ to obtain an expanding gradient Ricci soliton $(\mathbb{R}^3, g_e)$. Moreover, as explained in the same aforementioned references, $(\mathbb{R}^3, g_e)$ must come out of $(C(\Sigma), d_0)$, i.e. its asymptotic cone must be $(C(\Sigma), d_0)$. \hfill \Box

We are now in a position to prove Hamilton’s conjecture in dimension 3:

Proof of Theorem 1.3. Assume $(M^3, g)$ is a complete Riemannian manifold with non-negative uniformly pinched bounded Ricci curvature. Assume furthermore that $M^3$ is non-compact and non-flat. As explained in the beginning of this section this implies that there exists an immortal solution to Ricci flow, $(M^3_t, g_1(t))_{t \in (0, \infty)}$ satisfying (1), (2) and (3), whose singular initial condition, achieved in the distance sense, is by Lemma 8.2 a metric cone $(C, d_1(0), o)$ over an Alexandrov space $(\Sigma, d_1(0))$ with curvature larger than or equal to 1. By Lemma 8.1 the cone $(C, d_1(0), o)$ satisfies (1.3) away from $o$. 

\begin{itemize}
  \item [1.3]
Moreover, Lemma 8.5 ensures the existence of a self-similar solution \((M^3, g_2(t))_{t \in (0, \infty)}\) coming out of the metric cone \((\mathcal{C}, d_2(0), \delta)\), which is isometric to \((\mathcal{C}, d_1(0), o)\) through some map \(\psi_0\), where \((\mathcal{C}, d_2(0), \delta)\) is a cone over \((\Sigma, d_2(0))\) where \((\Sigma, d_1(0))\) is isometric to \((\Sigma, d_1(0))\) and \((M^3, g_2(t))_{t \in (0, \infty)}\) is an expanding gradient Ricci soliton with non-negative curvature operator.

Let \(x_0 \in \mathcal{C}\) such that \(d_1(0)(o, x_0) \geq 2\) so that \(B_{d_1(0)}(x_0, 1) \subseteq \mathcal{C} \setminus \{o\}\). Then Theorem 1.2 guarantees that there exist \(0 < T_0 < 1\), \(0 < R_0 < 1\) and solutions \(F_1 : B_{d_1(0)}(x_0, \frac{2}{3} R_0) \times (0, T_0) \to \mathbb{R}^n\), \(F_2 : B_{d_2(0)}(\psi_0(x_0), \frac{2}{3} R_0) \times (0, T_0) \to \mathbb{R}^n\) such that

\[
\tilde{g}_1(t) = (F_1(t))_* g_1(t) \quad \text{and} \quad \tilde{g}_2(t) = (F_2(t))_* g_2(t), \quad t \in (0, T_0],
\]

are solutions to the \(\delta\)-Ricci-De Turck flow on \(B(0, R_0)\) \(i = 1, 2\), which are \(\varepsilon_0\)-close to the \(\delta\) metric and such that there exist \(C_0 > 0\) such that if \(t \in (0, T_0]\):

\[
|\tilde{g}_1(t) - \tilde{g}_2(t)|_\delta \leq \exp \left( -\frac{C_0}{t} \right), \quad \text{on } B(0, \sqrt{T_0}).
\]  

(8.1)

In the following \(C_0\) denotes a constant, which may change from line to line, but remains positive. Remembering that \((\tilde{g}_1(t))_{t \in (0, T_0)}\) is uniformly Ricci-pinched, we see that (8.1), Shi’s type estimates on covariant derivatives of \(\tilde{g}_1(t) - \tilde{g}_2(t)\) and standard local interpolation inequalities imply:

\[
\text{Ric}(\tilde{g}_2(t)) \geq \lambda_0 \text{R} \tilde{g}_2(t) \tilde{g}_2(t) - \exp \left( -\frac{C_0}{t} \right) \tilde{g}_2(t), \quad t \in (0, T_0],
\]

on \(B(0, \sqrt{T_0})\) for some uniform \(\lambda_0 > 0\).

Since \(\tilde{g}_2(t) = (F_2(t))_* g_2(t)\) with \(F_2(t)\) \((1 \pm \varepsilon_0)\) bi-Lipschitz, the same estimate holds on \(B_{d_2(0)}(\psi_0(x_0) = \bar{x}_0, \sqrt{T_0}/2)\):

\[
\text{Ric}(g_2(t)) \geq \lambda_0 \text{R} g_2(t) g_2(t) - \exp \left( -\frac{C_0}{t} \right) g_2(t), \quad t \in (0, T_0].
\]

Recall that the solution \(g_2(t) = t \Phi_t^* g_2(1)\) is an expanding gradient Ricci soliton with \(\partial_t \Phi_t = -t^{-1} \nabla g_2 f \circ \Phi_t\) and \(\Phi_t|_{t=1} = \text{Id}_M\), where \(f : M \to \mathbb{R}\) is the associated potential function, therefore the previous estimate can be further simplified to:

\[
\text{Ric}(g_2(1)) \geq \lambda_0 \text{R} g_2(1) g_2(1) - \exp \left( -\frac{C_0}{t} \right) g_2(1),
\]

on \(\Phi_t(B_{d_2(0)}(\bar{x}_0, \sqrt{T_0}/2))\), \(t \in (0, T_0]\).

Without loss of generality, we can assume \(\bar{x}_0\) to lie in the annulus \(A_{d_2(0)}(\hat{o}, 2, 4) = \{x \in \mathcal{C} \mid 2 \leq d_2(0)(\hat{o}, x) \leq 4\}\) so that the previous estimate holds on \(\Phi_t(A_{d_2(0)}(\hat{o}, 1, 5))\), \(t \in (0, T_0]\), by reducing \(T_0\) if necessary by a compactness argument. We can also assume \(\hat{o}\) to be the unique critical point of \(f\).

We claim that there exists \(c_1 > 0\) and \(c_2 > 0\) such that for \(t \in (0, T_0]\),

\[
\left[ c_1^{-1} t, c_1 t \right] \subseteq f \left( \Phi_t \left( A_{d_2(0)}(\hat{o}, 1, 5) \right) \right) \subseteq \left[ c_2^{-1} t, c_2 t \right].
\]

(8.2)

Indeed, by invoking the bounds of \(f\) in terms of \(d(g_2(1))(\hat{o}, \cdot)\) given in (7.2) and (7.3) and since \(\hat{o}\) is fixed by the diffeomorphisms \(\Phi_t\), \(t \in (0, T_0]\), one gets on the first hand,

\[
\min_M f + \frac{d(g_2(t))(\hat{o}, x)^2}{4t} \leq f(\Phi_t(x)) \leq \left( \sqrt{\min_M f + \frac{d(g_2(t))(\hat{o}, x)^2}{2t}} \right)^2, \quad t > 0.
\]
On the other hand, thanks to distance distortion estimates from (1.2),
\[
\sqrt{f(\Phi_t(x))} \leq \sqrt{\min_M f + \frac{d_2(0)(\hat{o}, x)}{2\sqrt{t}}},
\]
\[
\frac{d_2(0)(\hat{o}, x)}{2\sqrt{t}} \leq \frac{d(g_2(t))(\hat{o}, x) + c_0\sqrt{t}}{2\sqrt{t}} \leq \sqrt{f(\Phi_t(x)) - \min_M f + \frac{c_0}{2}}, \quad t > 0,
\]
where \(c_0\) is a positive constant uniform in space and time.

In particular, if \(\gamma \in f(\Phi_t(A_{d_2}(\hat{o}, 1, 5)))\) then, for all \(t > 0\),
\[
\frac{1}{2\sqrt{t}} - \frac{c_0}{2} \leq \sqrt{\gamma - \min_M f} \leq \sqrt{\gamma} \leq \sqrt{\min_M f} + \frac{5}{2\sqrt{t}}.
\]
This implies the second inclusion in (8.2) for a suitable positive \(c_2\) for all \(t \in (0, T_0]\) by reducing \(T_0\) if necessary.

Now, if \(\gamma \in [\frac{c_1}{t}, \frac{c_1}{t}]\) for some positive constant \(c_1\), since \(f\) is proper and bounded from below, there must be some point \(y \in M\) such that \(f(y) = \gamma\). Moreover, since \(\Phi_t\) is a diffeomorphism for all \(t > 0\), \(y = \Phi_t(x)\) for some unique \(x \in M\) if \(t > 0\) is fixed and it must satisfy:
\[
\sqrt{\gamma} - \sqrt{\min_M f} \leq \frac{d_2(0)(\hat{o}, x)}{2\sqrt{t}} \leq \sqrt{\gamma - \min_M f} + \frac{c_0}{2}.
\]
This implies the first expected inclusion in (8.2) for a suitable positive constant \(c_1\) for all \(t \in (0, T_0]\) by reducing \(T_0\) again if necessary. This shows that (7.1) in Proposition 7.2 holds, i.e.
\[
\text{Ric}(g_2(1)) \geq \lambda_0 R_{g_2(1)} g_2(1) - \exp(-C_0 f) g_2(1), \quad \text{on } M.
\]
As a conclusion, [2, Proposition 7.2] implies that \((M^3, g_2)\) is isometric to Euclidean 3-space which in turn implies that the cone \((C, d_2(0))\) is itself isometric to Euclidean 3-space. Therefore, the asymptotic volume ratio \(\text{AVR}(g_2) := \lim_{r \to +\infty} r^{-3} \text{Vol}_{g_2}(B_{g_2}(p, r))\) (independent of the base point \(p \in M\)) equals the volume \(\omega_3\) of the unit ball \(B(0, 1) \subseteq \mathbb{R}^3\). Now \(\text{AVR}(g_2) = \text{AVR}(g_1)\) since the two corresponding solutions share the same cone at \(t = 0\) (together with Colding’s continuity of the volume), one gets \(\text{AVR}(g_1) = \omega_3\) which implies that \((M^3, g_1)\) is isometric to Euclidean 3-space by the rigidity part of Bishop-Gromov volume comparison. This ends the proof of Theorem 1.3. \(\square\)
Appendix A. Volume and Distance Convergence, for Solutions to Ricci Flow with $\text{Ric} \geq -1$ and Curvature Bounded by $c_0/t$

We recall the following result of Simon-Topping.

**Lemma A.1** (Simon-Topping, [ST21, Lemma 3.1]). Let $(M^n, g(t))_{t \in (0,T)}$, $T \leq 1$, be a smooth Ricci flow, satisfying $\text{Ric}(g(t)) \geq -g(t)$, $|\text{Rm}(g(t))| \leq c_0/t$, where $M$ is connected but $(M^n, g(t))$ not necessarily complete. Assume furthermore that $B_{g(t)}(x_0,1)$ is compactly contained in $M$ for all $t \in (0,T)$.

Then $X := (\bigcap_{s \in (0,T)} B_{g(s)}(x_0, \frac{1}{2}))$ is non-empty and there is a well defined limiting metric $d_t \to d_0$ as $t \searrow 0$, where

$$e^t d_0 \geq d_t \geq d_0 - \gamma(n) \sqrt{c_0 t} \quad \text{for all } t \in [0,T) \text{ on } X.$$  

Furthermore, there exists $R = R(c_0,n) > 0, S = S(c_0,n) > 0$ such that $B_{d_0}(x_0,r) \subset X \subset X$ and $B_{g(t)}(x_0,r) \subset X$ for all $r \leq R(c_0,n)$ and $t \leq S$ where $X$ is the connected component of $X$ which contains $x_0$, and the topology of $B_{d_0}(x_0,r)$ induced by $d_0$ agrees with that of the set $B_{g(t)}(x_0,r) \subset M$ induced by the topology of $M$.

In the setting of solutions $(M^n, g(t))_{t \in (0,T)}$ with $\text{Ric}(g(t)) \geq -1$ and curvature bounded by $c_0/t$ which are Reifenberg, at time zero, one has a Pseudolocality type theorem, even when $(M^n, g(t))$ is not necessarily complete. This is in contrast to the general setting, where completeness is required, as can be seen by the example of Giesen-Topping [GT13].

**Lemma A.2** (Pseudolocality for Reifenberg regions). Let $(M^n, g(t))_{t \in (0,T)}$, $T \leq 1$, be a smooth Ricci flow, satisfying $\text{Ric}(g(t)) \geq -g(t)$, $|\text{Rm}(g(t))| \leq c_0/t$, where $M$ is connected but $(M^n, g(t))$ not necessarily complete. Assume furthermore that $B_{g(t)}(x_0,1)$ is compactly contained in $M$ for all $t \in (0,T)$, and let $X := (\bigcap_{s \in (0,T)} B_{g(s)}(x_0, \frac{1}{2}))$ be as in A.1 endowed with the metric $d_0$. Assume that $B_{d_0}(x_0, r)$ is as in Theorem A.1, and furthermore that all tangent cones of $B_{d_0}(x_0, r)$ are isometric to $(\mathbb{R}^n, d(\delta))$. Let $\varepsilon > 0$ be given. Then there exists a $\hat{\delta} > 0$ depending on $\varepsilon, n, x_0$ and the solution, such that if $0 < t \leq \hat{\delta}$ then

$$|\text{Rm}(g(t))| \leq \frac{\varepsilon}{t}, \quad \text{on } B_{d_0}(x_0, \frac{r}{4}).$$

**Proof.** Let $\varepsilon > 0$ be fixed. Assume we find a sequence of points $y_i \in B_{d_0}(x_0, r/4)$ and times $0 < t_i \to 0$ as $i \to \infty$ so that $t_i |\text{Rm}(g(t_i))|(y_i) \in [\varepsilon, c_0]$.

After taking a subsequence we may assume without loss of generality that $y_i \to z \in B_{d_0}(x_0, r/2)$. We consider different cases, Case 1.2 being the most difficult.

**Case 1:** $d_0^2(y_i, z) \geq A t_i$ for some $A > 0$ for all $i \in \mathbb{N}$.

We scale solutions and initial data so that $\tilde{d}_i(0) := \lambda_i d_i(0)$, for $\lambda_i := (d_0(y_i, z))^{-1} \to \infty$ so that $\tilde{d}_i(0)(y_i, z) = 1$, and $(M, \tilde{d}_i(0), z) \to (\mathbb{R}^n, d(\delta), 0)$ in the Gromov-Hausdorff sense as $i \to \infty$. Then the new times $\tilde{t}_i := t_i \lambda_i^2$ for the scaled solution $(M, \tilde{g}_i(t) = \lambda_i^2 g(\lambda_i^{-2}t))_{t \in (0, \lambda_i^2)}$ satisfy $0 < \tilde{t}_i = (\tilde{d}_i(0)(y_i, z))^{-2} \tilde{t}_i = (d_i(0)(y_i, z))^{-2} t_i \leq A^{-1} < \infty$. In particular, $(M, \tilde{d}_i(0), z)$ approaches $(\mathbb{R}^n, d(\delta), 0)$ in the Gromov-Hausdorff sense as $i \to \infty$, and the solution $(M, \tilde{g}_i(t), z)_{t \in (0, \lambda_i^2)}$ (we denote the variable $t$ by $\tilde{t}$ again, for ease of
reading) approaches a solution $(X, h(t), z)_{t \in (0, \infty)}$ with $\text{Ric}(h(t)) \geq 0$, $|\text{Rm}(h(t))| \leq c_0/t$, and

$$d_s \geq d_t \geq d_s - c\sqrt{t-s}, \quad 0 \leq s \leq t,$$

such that $(X, d(h(t)), z) \to (\mathbb{R}^n, d(\delta), 0)$ in the Gromov-Hausdorff sense as $t \searrow 0$.

Hochard’s splitting result (Proposition B.1) then tells us that $(X, h(t), z)_{t \in (0, \infty)}$ is isometric for all times to $(\mathbb{R}^n, d(\delta), 0)$ (alternatively, scaling the solution by $\eta \searrow 0$ leads to solutions also coming out of $(\mathbb{R}^n, d(\delta))$, and hence the volume convergence theorem of Cheeger-Colding, [Che01, Theorem 9.45] with the distance estimates (A.2), shows for the solution $(X, h(t))$ for any fixed $t > 0$, that $r^{-n}\text{Vol}_{h(t)}(B_{h(t)}(x, r)) \to \omega_n$ as $r \to \infty$, and so Bishop-Gromov volume comparison tells us that $r^{-n}\text{Vol}_{h(t)}(B_{h(t)}(x, r)) = \omega_n$ for all $r > 0$ which implies the solution is isometric to $(\mathbb{R}^n, d_\delta)$ (by the equality case of Bishop-Gromov volume comparison). From the construction, we know that $\tilde{t}_i \leq A^{-1}$.

**Case 1.1:** For a subsequence of $\tilde{t}_i$ we have $\tilde{t}_i \to \tau > 0$, as $i \to \infty$.

Then,

$$|\text{Rm}(\tilde{g}_i(\tilde{t}_i))(y_i) = \frac{1}{\tilde{t}_i}|\text{Rm}(\tilde{g}_i(\tilde{t}_i))(y_i)| \in \left[\varepsilon_0/\tilde{t}_i, c_0/\tilde{t}_i\right] \to [\varepsilon_0/\tau, c_0/\tau],$$

leads to a contradiction, since the limiting solution is flat.

**Case 1.2:** $\tilde{t}_i \to 0$ as $i \to \infty$.

For $\eta > 0$, we can find $N(\eta) \in \mathbb{N}$ large so that

$$\text{Vol}_{\tilde{g}_i(\eta)}(B_{\tilde{g}_i(\eta)}(y_i, 1)) \geq \omega_n(1 - \eta), \quad \text{for all } i \geq N(\eta).$$

In the following, $\gamma(\eta) > 0$ can change from line to line, but always satisfies $\gamma(\eta) \to 0$ as $\eta \to 0$. Hence, with the help of Bishop-Gromov volume comparison and the fact that $\text{Ric}(\tilde{g}_i(t)) \geq -\varepsilon(i)$ for all $t > 0$ (in particular for $t = \eta$), where $\varepsilon(i) \to 0$ as $i \to \infty$, we see $\text{Vol}_{\tilde{g}_i(\eta)}(B_{\tilde{g}_i(\eta)}(y_i, \ell)) \geq \omega_n(1 - \gamma(\eta))\ell^n$ for all $i \geq N(\eta), \ell \in (0, 1)$, after adjusting $N(\eta)$ if necessary. Using the distance estimates (A.2), we know for arbitrary $r \in (0, \eta)$ that $B_{\tilde{g}_i(r)}(y_i, 1) \supset B_{\tilde{g}_i(\eta)}(y_i, 1 - \gamma(\eta))$ and hence,

$$\text{Vol}_{\tilde{g}_i(\eta)}(B_{\tilde{g}_i(\eta)}(y_i, 1)) \geq \text{Vol}_{\tilde{g}_i(\eta)}(B_{\tilde{g}_i(\eta)}(y_i, 1 - \gamma(\eta))) \geq \omega_n(1 - \gamma(\eta))$$

for $i \geq N$. For arbitrary $r \in (0, \eta)$, let $\Omega := B_{\tilde{g}_i(r)}(y_i, 1)$ for $i \geq N(\eta)$. Using $\text{Ric}(\tilde{g}_i(t)) \geq -\varepsilon(i)$, with $\varepsilon(i) \to 0$ as $i \to \infty$ and

$$\frac{d}{dt} \int_{\Omega} dm_{\tilde{g}_i(t)} = -\int_{\Omega} \text{Ric}(\tilde{g}_i(t)) dm_{\tilde{g}_i(t)} \leq \varepsilon(i) \int_{\Omega} dm_{\tilde{g}_i(t)},$$

we see that

$$\omega_n(1 - \gamma(\eta)) \leq \int_{\Omega} dm_{\tilde{g}_i(\eta)} \leq e^{\varepsilon(i)\eta} \int_{\Omega} dm_{\tilde{g}_i(r)},$$

that is $\text{Vol}_{\tilde{g}_i(r)}(B_{\tilde{g}_i(r)}(y_i, 1)) \geq \omega_n(1 - \gamma(\eta))$ for arbitrary $r \in (0, \eta), i \geq N(\eta)$. Once again, $\text{Ric}(\tilde{g}_i(t)) \geq -\varepsilon(i)$ for all $t \geq 0$ coupled with Bishop-Gromov volume comparison tells us that $\text{Vol}_{\tilde{g}_i(r)}(B_{\tilde{g}_i(r)}(y_i, \ell)) \geq \omega_n(1 - \gamma(\eta))\ell^n$ for all $\ell \leq 1$, for all $i \geq N(\eta)$, where we assume without loss of generality, $\varepsilon(i) \leq \eta$ for all $i \geq N(\eta)$.
[\text{Rm}(\hat{g}_i(1))](y_i) \in [\varepsilon, c_0] and on the other hand, \( \text{Vol}_{\hat{g}_i(1)}(B_{\hat{g}_i(t)}(y_i, \ell)) \geq \omega_n(1 - \gamma(\eta))\ell^n \) for all \( t \in (0, \eta^{-1}) \), for all \( i \geq N(\eta) \). Taking a subsequence if necessary, we can assume that \((M, \hat{g}_i(t), y_i)_{t \in (0, T_i)} \to (Y, \hat{g}(t), p)_{t \in (0, \infty)} \) as \( i \to \infty \) where \( T_i := \eta^{-1} \). Moreover, \((Y, \hat{g}(t), p) \) satisfies \( \text{Ric}(\hat{g}(t)) \geq 0, |\text{Rm}(\hat{g}(t))| \leq c_0/t \) for all \( t > 0 \) and \( \text{Vol}_{\hat{g}(t)}(B_{\hat{g}(t)}(p, \ell)) \geq \omega_n(1 - \gamma(\eta))\ell^n \) for all \( l > 0, t > 0 \). Thus letting \( \eta \to 0 \) we have \( \text{Vol}_{\hat{g}(t)}(B_{\hat{g}(t)}(p, \ell)) \geq \omega_n\ell^n \), and hence Bishop-Gromov volume comparison implies \( \text{Vol}_{\hat{g}(t)}(B_{\hat{g}(t)}(p, \ell)) = \omega_n\ell^n \), for all \( l > 0 \) and so \((Y, \hat{g}(t), p) \) is flat and isometric to \((\mathbb{R}^n, \delta, 0) \) for each \( t \in (0, \infty) \). This is a contradiction to \( |\text{Rm}(\hat{g}_i(1))|(y_i) \in [\varepsilon, c_0] \).

**Case 2:** There exists a subsequence of \((y_i, t_i)_{i \in \mathbb{N}} \) so that \( d^g_0(y_i, z)/t_i \to 0 \) as \( i \to \infty \).

We scale solutions and initial data so that \( \tilde{d}_i(0) := \lambda_i d(0) \), for \( \lambda_i := t_i^{-1/2} \) so that \( \tilde{d}^2_0(0)(y_i, z) = d^2_0(y_i, z)/t_i \to 0 \). Since by assumption, \((M, \tilde{d}_i(0), z_i) \to (\mathbb{R}^n, d(\delta), 0) \) in the Gromov-Hausdorff sense as \( i \to \infty \), so does the sequence \((M, \tilde{d}_i(0), y_i) \). Then the new times \( \tilde{t}_i := t_i \lambda_i^2 \) for the scaled solution \((M, \tilde{g}_i(\tilde{t}) = \lambda_i^{-2}g(\lambda_i^{-2}\tilde{t})\)\) satisfies \( \tilde{t}_i = 1 \), and as before, \((M, \tilde{g}_i(\tilde{t}), y_i)_{\tilde{t} \in (0, \lambda_i^2)} \to (X, h(t), p)_{t \in (0, \infty)} \) as \( i \to \infty \), where \((X, h(t), p)_{t \in (0, \infty)} \) is a solution to Ricci flow satisfying \( \text{Ric}(h(t)) \geq 0 \) and \( |\text{Rm}(h(t))| \leq c_0/t \) for all \( t > 0 \), which approaches \((\mathbb{R}^n, d(\delta), 0) \) in the Gromov-Hausdorff sense as \( t \to \infty \), 0, in view of the distance estimates (A.2). Hochard’s splitting result (Proposition B.1), or the alternative argument presented in the proof above, show us that \((X, h(t), p) \) is isometric to \((\mathbb{R}^n, d(\delta), 0) \) for each \( t > 0 \). But \( |\text{Rm}(\tilde{g}_i(\tilde{t}_i))|(y_i) = |\text{Rm}(\tilde{g}_i(1))|(y_i) \in [\varepsilon, c_0] \) which is a contradiction for large enough \( i \). \( \square \)
Appendix B. A splitting theorem of R. Hochard for the Ricci flow in a singular setting

We recall an unpublished result due to Hochard [Hoc19].

**Proposition B.1** ([Hoc19, Lemma I.3.12]). Let \((M^n, g(t))_{t \in (0,T)}\) be a complete Ricci flow such that for \(t \in (0,T)\) and for some \(K > 0\),

\[
|\text{Rm}(g(t))| \leq \frac{K}{t}, \quad \text{inj}_x(g(t)) \geq \sqrt{\frac{t}{K}}, \quad \text{Ric}(g(t)) \geq 0.
\]

Let \((M, d_0) := \lim_{t \to 0^+} (M, d_{g(t)})\) denotes the limit metric space obtained from Lemma A.1 and assume it is non-collapsed at all scales, i.e. \(\mathcal{H}^n(B_{d_0}(x,r)) \geq v r^n\) for all \(r > 0\) and \(x \in M\) for some \(v > 0\). Assume furthermore that there exists a metric space \((X, d_X)\) and for some \(1 \leq m \leq n\) an isometry

\[
\varphi : (X \times \mathbb{R}^m, d_X \times \mathbb{R}^m) \to (M, d_0),
\]

where \(d_X \times \mathbb{R}^m\) denotes the product metric on \(X \times \mathbb{R}^m\). Then \(X\) is homeomorphic to a smooth manifold \(N\) of dimension \(n - m\) and there exists a smooth, complete Ricci flow \((N^{n-m}, h(t))_{t \in (0,T)}\), satisfying (B.1), such that

\[
\varphi^* g(t) = h(t) \oplus g^{\mathbb{R}^m}.
\]
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