Quantum Numbers of Textured Hall Effect Quasiparticles

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ABSTRACT

We propose a class of variational wave functions with slow variation in spin and charge density and simple vortex structure at infinity, which properly generalize both the Laughlin quasiparticles and baby Skyrmions. We argue that the spin of the corresponding quasiparticle has a fractional part related in a universal fashion to the properties of the bulk state, and propose a direct experimental test of this claim. We show that certain spin-singlet quantum Hall states can be understood as arising from primary polarized states by Skyrmion condensation.
1. Introduction

Almost 40 years ago Skyrme [1] introduced a model of nucleons as distributions of pion fields which has inspired much work, both in its original context and more generally in the quantum theory of solitons. In particular, a realization in quantum ferromagnets was contemplated early on [2]. More than 10 years ago Wilczek and Zee [3] discussed the novel fractional spin and quantum statistics and that can arise for what they called “baby Skyrmions” in 2+1 dimensions. These objects (which we shall here call simply skyrmions) arise in an SO(3) nonlinear $\sigma$-model, where they are described by field distributions of the type

$$\vec{n}(r, \phi) = (\sin \theta(r) \cos \phi, \sin \theta(r) \sin \phi, \cos \theta(r)),$$  \hspace{0.5cm} (1.1)

where $\vec{n}$ is a unit vector field, and $\theta(r)$ runs from $-\pi$ at $r = 0$ to 0 at $r \to \infty$.

Recently there has been a revival of interest in objects of this kind, inspired by the important realization that for some quantum Hall states – including the classic $\nu = 1$ and $\nu = 1/3$ cases – the lowest energy charged quasiparticles may be skyrmions [4,5,6,8]. There is significant numerical and experimental support for this circle of ideas.

The recent literature on skyrmions in the quantum Hall complex takes as its starting point an effective theory of the state in question which was initially deduced from a Landau-Ginzburg theory of the quantum Hall effect [5] and has since received some microscopic justification [6]. Here, by addressing the determination of quantum numbers in a more direct fashion, we refine and partially justify the effective theory. We find that the traditional Laughlin quasihole finds a natural place as a zero size texture, and that the skyrmion can be interpreted as a rotationally symmetric texture modified by a flux insertion that acts on up spins only. Most important, we find from our microscopic considerations that a parameter in the effective quantum theory, the coefficient of the Hopf term, is quantized, with a value displaced from integer by a universal constant depending on the bulk state.
This fact is reflected in quantization in the properties of the skyrmion as material parameters are varied, and specifically to a non-integral part of its spin which is in principle observable experimentally. (Note that spin in the direction of the magnetic field, here taken as the $z$ direction, is a good quantum number. In what follows, when we refer to skyrmion spin we mean this component.) We also suggest that the anomalous quantum properties of the skyrmions – their fractional charge, statistics, and spin – all come together in a hierarchical construction, by way of skyrmion condensation, of quantum Hall states involving spin degrees of freedom.

2. Wave Function for Spin Textures

We will be considering the possibility that the energy splitting between up and down spins, though non-zero due to the background magnetic field, is not so large as to preclude a dynamical role for both. In fact, as was pointed out by Halperin [7], this is the case in GaAs, where the $g$-factor is $\sim \frac{1}{60}$. We suppose that in the bulk state the spins are all aligned pointing up at infinity, with a density corresponding to an incompressible state.

To understand how it can be that the presumed smallness of the Zeeman energy for flipping an individual spin does not lead to a vast proliferation of low-energy excitations, we must recognize the possible significance of exchange energy, which makes it costly to have rapid changes in the direction of magnetization. Thus one can anticipate that locally there is a well-defined magnetization direction, and that an appropriate class of wave functions to describe the low-energy excitations should in some sense reflect that locally the physics resembles that of an incompressible single-spin fluid, but that the direction of the magnetization may slowly vary. This possibility of such a procedure is implicitly assumed in the use of a non-linear $\sigma$ model for the low-energy dynamics. However, since the Hall states are both highly correlated and rigid – described by holomorphic functions – it is not entirely obvious how, or perhaps even if, such states can be pieced together.
Our first goal is therefore to show explicitly, in enough detail to support our later considerations on quasiparticles, how to proceed in the special case of a centered geometry. We will generalize a procedure used in the spin-polarized case. Taking the center at the origin, it is appropriate to work in symmetric gauge. Then a convenient basis of single-particle wave functions in the lowest Landau level (to which, for simplicity, we restrict ourselves) takes the form

\[ f_l(z) = z^l e^{-|z|^2/4l_0^2}, \]

where \( l_0 = (eB)^{-1/2} \) is the magnetic length. This function represents a ring of charge of thickness \( l_0\sqrt{2}\pi \) at distance \( l_0\sqrt{2(l+1)} \) from the origin. Note that successive rings overlap significantly. Now to represent the possibility of a non-trivial dependence of magnetization on distance, we may consider multiplying this spatial wave function by the spinor

\[ s_l = \cos(\theta(l)) u + e^{i\phi(l)} \sin(\theta(l)) v, \]

where \( u = (1,0)^T, v = (0,1)^T \). That spinor corresponds to the magnetization vector \( (\sin \theta(l) \cos \phi(l), \sin \theta(l) \sin \phi(l), \cos \theta(l)) \). So far \( \theta(l) \) and \( \phi(l) \) are simply prescribed functions of \( l \), with no explicit space dependence. Now define the matrix of spinors

\[ M_{kl} = f_l(z_k) s_l(u_k, v_k) \quad (2.1) \]

and the spinor wave function

\[ \Psi_1(z_k) = \det M. \quad (2.2) \]

In these expressions, it is to be understood that the indices \( k \) and \( l \) run over \( 0, \ldots, N-1 \), and that the product of spinor factors is to be understood as a tensor product.

We claim that \( \Psi_1 \), as constructed in (2.2), is suitable to implement the physical requirements mentioned above. That is, it keeps the charge density uniform (at the density appropriate to a single filled Landau level) while allowing the direction of magnetization to vary in space, through the dependence of \( \theta \) and \( \phi \) on \( l \). Furthermore if these functions depend slowly on \( l \), then the exchange energy will not be unfavorable, since nearby spins will be aligned. As long as \( \theta \) approaches zero
for large $l$, it will match on to the bulk state at infinity; for $\theta$ identically zero $\Psi$ reduces, of course, to the standard polarized spin droplet. It represents, for general $\theta$ and $\phi$, a class of low-energy localized spin texture excitations.

Wave functions, $\Psi_m$, for spin texture excitations at any primary Laughlin fraction $\nu = 1/m$ may be constructed in the following way. We define

$$\Psi_m = (\det \tilde{M})^m e^{-\sum_j |z_j|^2/4l_0^2}$$

where $\tilde{M}_{kl} = z_k^l r_l(u_k, v_k)$. $r_l$ is defined only by $r_l = r_{l_1} r_{l_2} \ldots r_{l_m} = s_{l_1 + l_2 + \ldots + l_m}$ and $s_l$ is the spinor defined above. When the determinant is expanded and taken to the $m$th power, each term will have $m$ $r_n(u_k, v_k)$’s and can, hence, be rewritten in terms of the appropriate $s_{l}(u_k, v_k)$, so the wavefunction (2.3) is well defined.

As in the case of $\nu = 1$, the amplitude for two electrons to approach each other vanishes in the limit that $\theta$ and $\phi$ are very slowly varying. It is noteworthy that the wave function here does not arise by straightforward flux attachment from the $\nu = 1$ wave function, but requires taking a peculiar “$m$th root” of the spinor. Straightforward flux attachment, i.e. putting the entire spinor dependence in one factor of (2.3), would not associate a given spin direction with a definite spatial position.

Thus far, we have only constructed textures in which the spin direction is a function only of the radial variable – that is, $l$. Dependence on the azimuthal angle, consistent with the constraint of slow variation in the direction but not the magnitude of the magnetization, and with appropriate correlations and holomorphy, can be incorporated as follows. In the $l$th partial wave, instead of a constant spinor $s_l$ multiplying $f_l$, we must allow for dependence $s_l(\tilde{\phi}) = \alpha_l(\tilde{\phi})u + \beta_l(\tilde{\phi})v$ on the spatial angle $\tilde{\phi}$. This can be achieved as follows. One has approximately

$$\cos \tilde{\phi} \sim \frac{1}{2} \left( \frac{z}{R} + \frac{R}{z} \right)$$

$$\sin \tilde{\phi} \sim \frac{1}{2i} \left( \frac{z}{R} - \frac{R}{z} \right)$$

on the ring where $f_l$ is supported, where $R = l_0 \sqrt{2l + 1}$. If $\alpha_l, \beta_l$ are slowly
varying on the scale of a magnetic length, then their Fourier expansion with respect to $\tilde{\phi}$ will essentially terminate after $\sim \sqrt{l}$ terms. Thus, expressing $\alpha_l, \beta_l$ in terms of $z$ using (2.4), we do not meet overly large powers of $1/z$ – the $z^l$ factor in $f_l$ prevents any singularity from occurring in the product. Using this procedure within the partial waves, and the determinantal construction to piece the partial waves together, we can indeed accommodate the general slowly varying texture with appropriate correlations and holomorphy properties.

3. Vortices and Skyrmions in Context

The configurations we have described so far have essentially uniform charge density. In constructing localized charged excitations, we want to be sure not to change the structure of the state at long distances. Experience with the spin polarized state leads us to suspect that this must be done by adiabatically inserting a unit of flux. In making the generalization to states with non-trivial spin structure, however, we are faced with a choice: do both spin up and spin down see the flux, or only spin up? In the former case we will simply carve a hole in the charge density, just as in the spin-polarized analogue. The latter case is much more interesting. Our $l$-dependent spinor becomes

$$s_l = z \cos\left(\frac{\theta(l)}{2}\right)u + e^{i\phi(l)} \sin\left(\frac{\theta(l)}{2}\right)v.$$ (3.1)

This represents a very special case of the angle-dependent construction discussed above, and our previous discussion of how one passes from the one-particle spinor function to the correlated many-body wave function applies mutis mutandis. The resulting state is characterized by a special symmetry, in that simultaneous real space and spinor space rotation

$$z \rightarrow e^{i\gamma} z$$
$$u \rightarrow e^{-i\gamma/2} u$$
$$v \rightarrow e^{i\gamma/2} v$$

simply multiplies it by a phase. If $\theta(l)$ runs smoothly from $-\pi$ at $l = 0$ to $0$ at
$l \rightarrow \infty$, and we set $\phi(l) \equiv 0$, the spin texture associated with (3.1) is nothing but the classic skyrmion spin texture, as one sees on comparing the magnetization direction associated to (3.1) to (1.1) and recalling that $l \sim r^2$. Of course if $\theta = 0$ identically we have the classic Laughlin quasi-hole. Since the exchange energy is typically very large, we require the spin to be slowly-varying with position. In this case, an excitation of the form (3.1) must have $\theta(0) = 0$ or $\theta(0) = \pm \pi$.

By comparing the radius of the droplet for $N$ particles with a centered flux tube, one readily concludes that this configuration contains a net density deficit corresponding to $1/m$ electron, whether the flux tube affects both spins or only spin up, and whatever the detailed form of $\theta(l)$, so long as it approaches zero as $l \rightarrow \infty$.

Now we are in a position to appreciate, following [5], the possible energetic advantage of the skyrmion – or more general – textures for accommodating charge inhomogeneities. For whereas the classic Laughlin quasi-hole involves a density inhomogeneity on the scale of the magnetic length, and thus a heavy price in repulsive energy, the skyrmion texture allows the inhomogeneity to be spread over its physical radius, i.e. the size of the region over which $\theta$ differs significantly from zero. Under appropriate conditions, this advantage can be worth the price in unfavorable Zeeman and exchange energy. Note that for the quasiparticle (as opposed to the quasi-hole) it is natural to make the choice $s_l = z^{-1} \cos\left(\frac{\theta(l)}{2}\right)u + e^{i\phi(l)} \sin\left(\frac{\theta(l)}{2}\right)v$; as long as $\cos\frac{\theta(0)}{2}$ vanishes this introduces no singularity (as in the case of the skyrmion, this avoids rapid variations in the direction of the spin), and associates an antiskyrmion spin texture with the quasiparticle. In the spin-polarized case the quasiparticles cannot be implemented in quite such a simple way. Of course, the energetically favored forms for $\theta(l)$ have every reason to differ microscopically between quasiholes and quasiparticles. The antiskyrmion is also symmetric under combined real space and spinor space rotations, but the spinor space rotation is the opposite to that of the skyrmion. This symmetry has important consequences, as we shall discuss momentarily.
Let us briefly indicate how these considerations on the microscopic theory might be incorporated in an effective theory. We seek to describe the low energy excitations of the incompressible drop as spin textures, and in particular to consider how coupling of real electromagnetic and ‘fictitious’ statistical gauge fields governs the charge and statistics of the elementary excitations. Given the spinor field \( \psi_i(z) \equiv (u(z), v(z))^T \), there are two candidate conserved currents to which a \( U(1) \) gauge field could couple, \( \text{viz.} \)

\[
\begin{align*}
J^\mu_{\text{skyrm}} &= \varepsilon^{\mu\nu\rho} \partial_\nu \bar{\psi}^i \partial_\rho \psi_i, \\
\tilde{J}^\mu &= \varepsilon^{\mu\nu\rho} \varepsilon_{ij} \partial_\nu \psi_i \partial_\rho \psi_j.
\end{align*}
\]

The latter symmetry is not a suitable charge current because it is odd under the spinor space rotation \( u \to -u \). Hence, the current in an effective field theory can only be given by \( J^\mu_{\text{skyrm}} \). In particular, this theory should have a term

\[
\mathcal{L}_{\text{Hopf}} = -\frac{4\pi}{m} \left( J^\alpha_{\text{skyrm}} a_\alpha - \frac{1}{2} \varepsilon^{\alpha\beta\gamma} a_\alpha \partial_\beta a_\gamma \right)
\]

If the effective Lagrangian is written in terms of the local spin field, this term is just the Hopf invariant of the corresponding map \( S^3 \to S^2 \). Such a Lagrangian was discussed in [5] with gradient energy, Zeeman energy, and Coulomb repulsion terms. We can redefine \( \tilde{a} = \frac{4\pi}{m} a \), so that the Chern-Simons term is conventionally normalized and the quantized parameter appears explicitly as a coefficient in the Lagrangian. Its quantization is connected with the invariance of the action under large gauge transformations [10]. Given the mathematical result (3.4), the ribbon argument of [11,3] applies, and the anyon character of the skyrmion follows.
4. Quantization of the Spin

The discussion so far is incomplete in one important respect: since a generic particular texture configuration $\Psi(z_k; \alpha_k, \beta_k)$ has no definite value of the spin in the down direction, the corresponding state is always embedded in a highly degenerate continuum. One constructs states of definite spin by forming appropriate superpositions of these degenerate states, in the manner:

$$\Psi_s(z_k; \alpha_k, \beta_k) = \int_0^{2\pi} \frac{d\lambda}{2\pi} e^{-is\lambda} \Psi(z_k; \alpha_k, e^{i\lambda} \beta_k).$$  \hspace{1cm} (4.1)

The resulting state has exactly $s$ reversed spins. Now as we have seen the fundamental charged particles generically feature a localized fractional charge, as is familiar in the fully polarized case (and for the same reason). Since the total particle density has a fractional piece, and the number of reversed spins is integral, clearly the net spin relative to the ground state is fractional.

The fractional part of the spin is intimately related to the anyonic statistics of the skyrmion as a result of its symmetry (3.2) under simultaneous real space and spin space rotations. By our previous arguments, skyrmions and anti-skyrmions have statistics $-\frac{1}{m}$; as a result of the spin-statistics connection [3], they have intrinsic angular momentum $-\frac{1}{2m}$. On the other hand, the special symmetry (3.2) of the skyrmion implies that it is an eigenstate of $L - S$ with integer eigenvalue, where $L$ is the intrinsic angular momentum and $S$ is the spin – which is just an internal quantum number in this context. Hence, the fractional part of its spin is equal to the fractional part of its intrinsic angular momentum. The anti-skyrmion is also symmetric under a combined real space and spin space rotation, but one in which the spin rotation is opposite to the real space rotation, so its spin is equal and opposite to its angular momentum.

The question which $s$ is favored for low-lying charged quasiparticles in a given material is a non-universal question, whose answer depends on the detailed form
of the Hamiltonian – that is, it involves energetics, not merely topology. Indeed strictly speaking one should consider the possibility that the optimal starting wave function depends on $s$, similarly to how rotation of a molecule can affect its shape – ro-vibrational coupling – although we expect such effects to be small. In any case, one expects to find that the $s$ which minimizes the energy for a quasihole exhibits jumps as one changes the in-plane $\mathbf{B}$ field or material parameters such as density, impurity concentration, temperature, or well size in the third direction. This effect suggests a method of checking the fractional quantization of the spin. Indeed, using nuclear magnetic resonance one can measure the Knight shift induced by a skyrmion, which is proportional to its spin [8]. If the favored value of the spin jumps by an integer in response to a small change in the control parameters, then by taking the ratio of Knight shifts before and after the change one could infer the ratio, which is of course sensitive to the fractional displacement. In a material that is not perfectly homogeneous, one might find stable skyrmions with different values of $s$ at different positions; and at finite temperature one expects to find each $s$ value represented with appropriate statistical weight.

5. Skyrmion Condensation and the Hierarchy Construction

The exotic spin of the skyrmions allows us to understand spin-singlet states and, more generally, non-polarized states as hierarchical states resulting from the condensation of skyrmionic quasiparticles on a polarized parent state. To see why this is non-trivial, recall that, in the hierarchy construction, the state at $\nu = 2/5$ forms when charge $-e/3$ and statistics $-\pi/3$ quasiparticles of the polarized $\nu = 1/3$ state condense in a Laughlin state. If these quasiparticles are, in fact, skyrmionic, then the additional $\nu = \frac{2}{5} - \frac{1}{3} = \frac{1}{15}$ can cancel the spin of the $\nu = 1/3$ parent. More generally, a daughter state in which skyrmions of a Laughlin state condense will have charge and spin filling fraction:

$$\nu = \frac{1}{m} + \frac{\alpha}{m} \frac{1/m}{2p - \alpha/m} \quad (5.1)$$
\[ S_z = \frac{1}{2} \times \frac{1}{m} - \left( J - \frac{\alpha}{2m} \right) \times \frac{1/m}{2p - \alpha/m} \] \hspace{1cm} (5.2)

where \( \alpha = \pm 1 \) according to whether skyrmions or anti-skyrmions condense. \( J - \frac{\alpha}{2m} \) is the spin of the skyrmion or antiskyrmion. Observe that the fractional part of the spin is either aligned or anti-aligned with the parent, depending on whether it is particle- or hole-like \( (\alpha = \pm 1) \), but the integer part is always anti-aligned because it involve flipping spins of the parent condensate. This state will have \( S_z = 0 \) if \( J = p \). It is natural that the most favorable skyrmion size, \( J \), be determined by the skyrmion inverse density, \( p \), in the low Zeeman energy, high-density limit, where inter-skyrmion interactions are the limiting factor. This picture for the \( S_z = 0 \) states at \( \nu = \frac{2p}{2pm \pm 1} \) motivates trial wavefunctions for these states which are completely analogous to those of the polarized hierarchy but with skyrmions substituted for the Laughlin quasiparticles.

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\* Lee and Kane [4] also suggested that spin-singlet states could arise from skyrmion condensation, but the states that they construct are at denominators such as 1/2 and 1/4 whereas our states are at the same fractions as those of the usual hierarchy, such as 2/3, 2/5, etc.
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