Integrability properties of a symmetric 4 + 4-dimensional heavenly-type equation

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Abstract
We demonstrate that the dispersionless \( \partial \)-dressing method developed before for a general heavenly equation is applicable to the 4 + 4 and \( 2N + 2N \)-dimensional symmetric heavenly-type equations. We introduce a generating relation and derive the two-form, defining the potential and equation for it. We develop the dressing scheme, calculate a class of special solutions and demonstrate that reduction from a 4 + 4-dimensional equation to a four-dimensional general heavenly equation can be effectively performed on the level of the dressing data. We consider also the extension of the proposed scheme to the \( 2N + 2N \)-dimensional case.

Keywords: dispersionless integrable equations, heavenly type equations, hyper-Kähler hierarchies

1. Introduction

A simple and symmetric integrable 4 + 4-dimensional (TED) equation

\[
\begin{align*}
&\left( \Theta_{x_1x_2} - \Theta_{x_2x_1} \right) \left( \Theta_{x_3x_4} - \Theta_{x_4x_3} \right) \\
&\quad + \left( \Theta_{x_2x_3} - \Theta_{x_3x_2} \right) \left( \Theta_{x_1x_4} - \Theta_{x_4x_1} \right) \\
&\quad + \left( \Theta_{x_3x_1} - \Theta_{x_1x_3} \right) \left( \Theta_{x_2x_4} - \Theta_{x_4x_2} \right) = 0
\end{align*}
\] (1)

introduced in [1] represents a natural generalisation of the four-dimensional general heavenly equation [2–4]. Equation (1) possesses interesting interpretations in terms of differential geometry of Kähler spaces [1]. The Lax pair for equation (1) reads

\[
\begin{align*}
&\left( \Theta_{x_1x_2} - \Theta_{x_2x_1} \right) D_1 \Psi + \left( \Theta_{x_3x_4} - \Theta_{x_4x_3} \right) D_2 \Psi + \left( \Theta_{x_2x_3} - \Theta_{x_3x_2} \right) D_1 \Psi = 0, \\
&\left( \Theta_{x_3x_1} - \Theta_{x_1x_3} \right) D_2 \Psi + \left( \Theta_{x_2x_4} - \Theta_{x_4x_2} \right) D_1 \Psi = 0
\end{align*}
\] (2)
where \( D_i := \partial_{y^i} - \lambda \partial_{x^i} \). The existence of potential \( \Theta \) providing given representation of coefficients of vector fields in the Lax pair corresponds to the vanishing divergence condition for vector fields.

It is possible to consider two-dimensional involutive distribution corresponding to the Lax pair. A symmetric set of divergence-free vector fields of the form

\[
V_{ijk} = (\Theta_{x^i y^j} - \Theta_{y^i x^j}) D_k + (\Theta_{x^j y^k} - \Theta_{y^j x^k}) D_i + (\Theta_{x^k y^i} - \Theta_{y^k x^i}) D_j,
\]

where \( i, j, k \) is an arbitrary substitution of (distinct) values 1, 2, 3, 4, belonging to this distribution.

The goal of the present paper is to apply the technique of integrable dispersionless hierarchies [7–10] to equation (1), to introduce a dressing scheme and construct a class of special solutions of equation (1) and its \( 2N + 2N \)-dimensional generalisation. The dressing scheme and generating equations developed in this paper are closely related to that of multidimensional generalisation of six-dimensional heavenly equation hierarchy with four degenerate wave functions introduced in [5], and equation (1) is obtained for a special choice of the set of times (see also [6]). The dressing scheme for equation (1) corresponds to the reduction of the dressing scheme for general eight-dimensional integrable dispersionless hierarchy [9], and the functional freedom of the dressing data consists of functions of seven variables in accordance with eight-dimensional integrability of equation (1).

The paper is organized as follows. The applicability of the technique of integrable dispersionless hierarchies to equation (1) is demonstrated in section 2. The dressing scheme is developed and exact solutions are constructed in section 3. The \( 2N + 2N \)-dimensional TED equation and its solutions are considered in section 4.

2. Integrability properties of equation (1)

2.1. Wave functions

The structure of the wave function is defined by linear problems (2). The set of wave functions contains four trivial wave functions

\[
\phi^1 = x^1 + \lambda y^1, \quad \phi^2 = x^2 + \lambda y^2, \quad \phi^3 = x^3 + \lambda y^3, \quad \phi^4 = x^4 + \lambda y^4.
\]

(4)

The presence of degenerate wave functions makes the situation similar to the six-dimensional heavenly equation hierarchy, where we have two functions exactly of type (4) [5].

To complete a basic set, which is six-dimensional for integrable distribution with the basis (2), we also introduce two generic wave functions

\[
\Psi^1 = q + \tilde{\Psi}^1, \quad \tilde{\Psi}^1 = \sum_{n=1}^{\infty} \Psi^1_n(p, q, x, y) \lambda^{-n}, \quad u := \Psi^1_1
\]

\[
\Psi^2 = p + \tilde{\Psi}^2, \quad \tilde{\Psi}^1 = \sum_{n=1}^{\infty} \Psi^2_n(p, q, x, y) \lambda^{-n}, \quad v := \Psi^2_1.
\]

(5)

Until some moment we will consider \( p, q \) just as parameters (constants) not entering the equations, and later we will use them as variables in the description of the general framework of the hierarchy.

**Remark.** Wave functions for vector fields, understood as first-order differential operators, are defined up to the addition of a constant, in our case up to a function of \( \lambda \), playing the role of parameter, so at this stage \( p \) and \( q \) seem to be not very important. Introducing them now, we have in mind the picture of the hierarchy [5, 9] and the dressing scheme (see below), where
they will play a role of extra independent variables, which until some moments are frozen. For a solution defined by the dressing scheme, the dependence on all independent variables (including \( p, q \)) is switched on. It is defined in an obviously consistent manner, and the generating relation produces compatible differential relations for an extended set of independent variables.

2.2. Generating relation

The structure of wave functions for the Lax pair (2) corresponds exactly to the multidimensional extension of the six-dimensional heavenly equation hierarchy considered in [5], so we will briefly remind of some general results. Integrable (involutive) distribution corresponding to the set of wave functions (4), (5) (which represents a reduction of the wave functions for the general hierarchy [9]) can be defined through the differential six-form

\[
\Omega = \left( (d\Psi^1 \wedge d\Psi^2) \wedge (d\phi^1 \wedge d\phi^2 \wedge d\phi^3 \wedge d\phi^4) \right),
\]

where the differentials are taken with respect to independent variables \( x, y \), imposing the generating relation

\[
\Omega = \left( (d\Psi^1 \wedge d\Psi^2) \wedge (d\phi^1 \wedge d\phi^2 \wedge d\phi^3 \wedge d\phi^4) \right) = 0,
\]

meaning that the projection of \( \Omega \) to negative powers of \( \lambda \) equals zero, thus \( \Omega \) in our case is polynomial (and may be meromorphic in a more general setting). The form (6) satisfying relation (7) defines an integrable distribution with polynomial coefficients corresponding to the volume-preserving case (the basic vector fields can be chosen divergence-free). The functions \( \phi \) and \( \Psi \) are wave functions for this distribution.

There are different ways to derive the basis of the distribution and compatibility conditions (equations of the hierarchy) using generating equation (7). First we will give a direct derivation of the basic vectors (3) in the spirit of the dressing method. Then we will modify the generating relation to provide a simple and elegant way of direct derivation of equation (1) using the language of differential forms. We will also introduce extra variables \( p, q \) and give the interpretation of equation (1) as a kind of superposition principle for a set of six-dimensional heavenly equations.

2.3. Lax pair—direct derivation

The set of wave functions defines a two-dimensional involutive distribution annulating them, vector fields in this case can be taken in the form

\[
V = \sum V_i D_i, \quad D_i := \partial_y - \lambda \partial_x,
\]

where \( V_i \) are in general polynomial in \( \lambda \). Generating relation (7) implies an important property

\[
\left( \frac{(V\Psi^1)_+}{(V\Psi^2)_+} \right) = 0 \Rightarrow V_{\Psi^1} = 0,
\]

allowing us to construct polynomial vector fields belonging to the integrable distribution explicitly, eliminating ‘singular terms’ in the result of the action of vector fields on the wave functions.
Using this property and relations (compare (5))

\[ (D_i \Psi_1^+) + (D_i \Psi_2^+) = -\partial_x v, \]

we construct vector fields belonging to the distribution,

\[ V_{ijk} = ((\partial_x u)(\partial_y v) - (\partial_x u)(\partial_y v))D_k + ((\partial_x u)(\partial_y v) - (\partial_x u)(\partial_y v))D_i + ((\partial_x u)(\partial_y v) - (\partial_x u)(\partial_y v))D_j, \]

where \( i, j, k \) is an arbitrary substitution of (distinct) values 1, 2, 3, 4. The vanishing divergence (or anti-self-adjointness) condition for vector fields (9) implied by generating relation (7),

\[ (\partial_x u)(\partial_y v) - (\partial_x u)(\partial_y v) = 0, \]

leads to the existence of potential \( \Theta \),

\[ ((\partial_x u)(\partial_y v) - (\partial_x u)(\partial_y v)) = \Theta_{x y} = \Theta_{x y}, \]

and vector fields (9) take exactly the form (3),

\[ V_{ijk} = (\Theta_{x y} - \Theta_{x y})D_k + (\Theta_{x y} - \Theta_{x y})D_i + (\Theta_{x y} - \Theta_{x y})D_j, \]

corresponding to two-dimensional integrable distribution connected with equation (1); the basis (Lax pair) is given by an arbitrary pair of distinct vector fields \( V_{ijk} \).

2.4. Second form of generating relation

Another form of generating relation more suitable for vector fields of the form \( V = \sum V_i D_i \) is

\[ (d^n \Psi^1 \wedge d^n \Psi^2) = 0, \]

where we use the notations

\[ w^j = y^j - \lambda^{-1}x^j, \quad \tilde{w}^j = y^j + \lambda^{-1}x^j = \lambda^{-1}p^j \]

\[ \partial_{w^j} = \frac{1}{2}(\partial_x - \lambda \partial_y), \quad \tilde{\partial}_{\tilde{w}^j} = \frac{1}{2}(\partial_x + \lambda \partial_y), \]

\[ d^n w + d^n \tilde{w} = d, \quad d\phi^j \wedge dw^j = 2dx^j \wedge dy^j, \]

d\( w \) and d\( \tilde{w} \) are differentials with respect to the subsets of variables \( w_i \) and \( \tilde{w}_i \), and the projection for the 2-form

\[ \omega := d^n \Psi^1 \wedge d^n \Psi^2 = \sum_n \omega^{(n)}_{ij}(x, y)\lambda^n dw^i \wedge dw^j \]

is understood as

\[ \omega_\lambda = \sum_{n>0} \omega^{(n)}_{ij}(x, y)\lambda^n dw^i \wedge dw^j, \]

where the coefficients \( \omega^{(n)}_{ij} \) do not depend on \( \lambda \).

**Proposition 1.** Generating relation (7)

\[ \{ (d^1 \Psi^1 \wedge d^2 \Psi^2) \wedge (d^1 \phi^1 \wedge d^2 \phi^2 \wedge d^3 \phi^3 \wedge d^4 \phi^4) \} = 0, \]
considered for the set of independent variables $x_i, y_i$, is equivalent to the relation (10)

$$(d^w \Psi^1 \wedge d^w \Psi^2)_- = 0,$$

where the projection in the second relation is defined by (12).

**Proof.** First, using the relations $d = d^w + d^\tilde{w}$ and $\tilde{w}^i = \lambda^{-1} \phi^i$, we get

$$d\Psi^1 \wedge d\Psi^2 \wedge (d\phi^1 \wedge d\phi^2 \wedge d\phi^3 \wedge d\phi^4) = d^w \Psi^1 \wedge d^w \Psi^2 \wedge (d\phi^1 \wedge d\phi^2 \wedge d\phi^3 \wedge d\phi^4).$$

Having in mind expression (11) and taking into account that

$$d\phi^i \wedge d\phi^i = 2 dy_i \wedge dx_i,$$

it is easy to check necessary and sufficient conditions of the proposition.

Then we have

$$\omega = d^w \Psi^1 \wedge d^w \Psi^2 = (d^w \Psi^1 \wedge d^w \Psi^2)_+ = \omega_y dw^j \wedge dw^j,$$

where $\omega_y$ is independent of $\lambda$. Evidently,

$$\omega \wedge \omega = 0, \quad d^w \omega = 0,$$

and for $\omega_y$ we get the relations obtained in [1].

Second relation in (13) implies the existence of potential (see [1] for more detail)

$$\omega_{ij} = \Theta_{x^i y^j} - \Theta_{x^j y^i},$$

and first relation (13) gives equation (1) for potential $\Theta$.

2.5. Connection with the six-dimensional heavenly equation—the hierarchy framework

Considering more general wave functions $\phi'$ containing higher times

$$\phi' = x' + \lambda y' + \sum_{n=2}^{N} \lambda^n,$$

and treating $q, p$ in wave functions (5) as independent variables, we put ourselves to the context of a multidimensional extension of the six-dimensional heavenly equation hierarchy considered in [5] (which is in turn a reduction of general dispersionless hierarchy [9]). Generating relation (7) preserves its form for functions (14) and extended set of independent variables,

$$\Omega_- = (d\Psi^1 \wedge d\Psi^2) \wedge (d\phi^1 \wedge d\phi^2 \wedge d\phi^3 \wedge d\phi^4)_- = 0.$$

First, this relation implies that $u = -\Theta_p, v = \Theta_q$, and potential $\Theta$ satisfies six-dimensional heavenly equations

$$\Theta_{x^i y^j} - \Theta_{x^j y^i} = \{\Theta_{x^i}, \Theta_{y^j}\}_{(qp)},$$

with the Lax pair

$$D_i \Psi + \{\Theta_{x^i}, \Psi\}_{(qp)} = 0,$$

$$D_j \Psi + \{\Theta_{y^j}, \Psi\}_{(qp)} = 0,$$

(16)
for any pair of distinct \( i, j \in 1, 2, 3, 4 \). Here we use the Poisson bracket
\[
\{f_1, f_2\}_\theta := \frac{\partial f_1}{\partial q} \frac{\partial f_2}{\partial p} - \frac{\partial f_1}{\partial p} \frac{\partial f_2}{\partial q}.
\]
Taking into account that \( u = -\Theta p \), \( v = \Theta q \) and using six-dimensional heavenly equations we get
\[
((\partial_x u)(\partial_x v) - (\partial_x v)(\partial_x u)) = \{\Theta_x, \Theta_{xy}\}_\theta = \Theta_{xy},
\]
thus providing another proof that vector fields (9) take the form corresponding to the Lax pair (2). Vector fields (9) can be obtained as combinations of three vector fields of the form (16) with an eliminated Hamiltonian part. And equation (1) may be considered as a kind of superposition principle (intertwining equation) for a set of consistent six-dimensional heavenly equation (15) with different \( i, j \in 1, 2, 3, 4 \).

3. Dressing scheme and special solutions

Dressing scheme to construct solutions of generating relations of the type (7) was introduced in [5] as a reduction of a dressing scheme for general multidimensional dispersionless hierarchies [9]. It is formulated in terms of the Riemann–Hilbert problem on the unit circle (or the boundary of some region \( G \))
\[
\begin{align*}
\Psi_{\text{in}}^1 &= F_1^1(\lambda, \Psi_1^1, \Psi_2^1; \phi^1, \phi^2, \phi^3, \phi^4)_{\text{out}}, \\
\Psi_{\text{in}}^2 &= F_1^2(\lambda, \Psi_1^1, \Psi_2^1; \phi^1, \phi^2, \phi^3, \phi^4)_{\text{out}},
\end{align*}
\]
where the diffeomorphism defined by \( F_1, F_2 \) for the case of Hamiltonian reduction should be area-preserving with respect to the variables \( \Psi_1^1, \Psi_2^1 \). Alternatively, it is possible to use the \( \bar{\theta} \) problem in the unit disk (or some region \( G \))
\[
\begin{align*}
\bar{\partial}\Psi_1 &= W_2(\lambda, \bar{\lambda}, \Psi_1^1, \Psi_2^1; \phi^1, \phi^2, \phi^3, \phi^4), \\
\bar{\partial}\Psi_2 &= -W_1(\lambda, \bar{\lambda}, \Psi_1^1, \Psi_2^1; \phi^1, \phi^2, \phi^3, \phi^4),
\end{align*}
\]
where the Hamiltonian reduction is taken into account explicitly. We search for the solutions of the form
\[
\begin{align*}
\Psi_1 &= q + \tilde{\Psi}_1, \\
\Psi_2 &= p + \tilde{\Psi}_2
\end{align*}
\]
where \( \tilde{\Psi}_1, \tilde{\Psi}_2 \) are analytic outside \( G \) and go to zero at infinity. The series for these functions at infinity give a solution to generating relation (7) at infinity. The form \( \Omega \) defined through these functions due to the problem (17) is analytic in the complex plane (may be meromorphic in more general setting).

The functional freedom of the dressing data consists of functions of seven variables, which corresponds to the functional freedom for the general solution of equation (1), indicating eight-dimensional integrability of equation (1). Below we will construct a class of solutions for equation (1). We will go along the lines of similar calculations for a general heavenly equation presented in [4] and for a six-dimensional heavenly equation in [5].
3.1. Special solutions

A class of solutions for equation (1) in terms of implicit functions (similar to solutions of hyper-Kähler hierarchy [11, 12]) can be constructed using the choice

\[
\frac{1}{2\pi i} W(\lambda, \bar{\lambda}, \Psi^1, \Psi^2; \phi^1, \phi^2, \phi^3, \phi^4) = \sum_{i=1}^{M} \delta(\lambda - \mu_i) F_i(\Psi^1; \phi^1, \phi^2, \phi^3, \phi^4) + \sum_{i=1}^{M} \delta(\lambda - \nu_i) G_i(\Psi^2; \phi^1, \phi^2, \phi^3, \phi^4),
\]

where \( \delta(\lambda - \mu_i), \delta(\lambda - \nu_i) \) are two-dimensional delta functions in the complex plane characterized by the relation \( \delta \lambda^{-1} = 2\pi i \delta(\lambda) \) (the constant defines normalization of two-dimensional delta function), and \( F_i, G_i \) are arbitrary (complex-analytic) functions of three variables.

The \( \bar{\partial} \) problem (17) in this case reads

\[
\bar{\partial} \Psi^1 = 2\pi i \sum_{i=1}^{M} \delta(\lambda - \nu_i) G_i'(\Psi^2; \phi^1, \phi^2, \phi^3, \phi^4), \quad G_i' = \frac{\partial G_i}{\partial \Psi^2},
\]

\[
\bar{\partial} \Psi^2 = 2\pi i \sum_{i=1}^{M} \delta(\lambda - \mu_i) F_i'(\Psi^1; \phi^1, \phi^2, \phi^3, \phi^4), \quad F_i' = \frac{\partial F_i}{\partial \Psi^1},
\]

where, due to delta functions, \( G_i', F_i' \) are taken at \( \lambda = \nu_i, \lambda = \mu_i \) and inverting operator \( \bar{\partial} \), we get

\[
\Psi^1 - q = \sum_{i=1}^{M} (\lambda - \nu_i)^{-1} G_i'(\Psi^2(\nu_i); \phi^1(\nu_i), \phi^2(\nu_i), \phi^3(\nu_i), \phi^4(\nu_i)),
\]

\[
\Psi^2 - p = -\sum_{i=1}^{M} (\lambda - \mu_i)^{-1} F_i'(\Psi^1(\mu_i); \phi^1(\mu_i), \phi^2(\mu_i), \phi^3(\mu_i), \phi^4(\mu_i)).
\]

The solutions of the \( \bar{\partial} \) problem are then of the form

\[
\Psi^1 = q + \sum_{i=1}^{M} \frac{f_i}{\lambda - \nu_i}, \quad \Psi^2 = p + \sum_{i=1}^{M} \frac{g_i}{\lambda - \mu_i},
\]

and from (19) the functions \( f_i, g_i \) are defined as implicit functions,

\[
f_i(x, y) = G_i' \left( p + \sum_{k=1}^{M} \frac{f_k(x, y)}{\nu_i - \mu_k} ; \phi^1(\nu_i), \phi^2(\nu_i), \phi^3(\nu_i), \phi^4(\nu_i) \right),
\]

\[
g_i(x, y) = -F_i' \left( q + \sum_{k=1}^{M} \frac{f_k(x, y)}{\mu_i - \nu_k} ; \phi^1(\mu_i), \phi^2(\mu_i), \phi^3(\mu_i), \phi^4(\mu_i) \right).
\]

where equation (21) represent a closed system of \( 2M \) equations for \( 2M \) functions \( f_i, g_i \), defining them as functions of \( x, y, p, q \). The potential \( \Theta \) is then given by the general formula (see [7])

\[
\Theta(x, y) = \frac{1}{2\pi i} \int_G \left( \bar{\partial} \Psi^1 - W(\lambda, \bar{\lambda}, \Psi^1, \Psi^2; \phi^1, \phi^2, \phi^3, \phi^4) \right) d\lambda \wedge d\bar{\lambda},
\]

it depends on the set of arbitrary functions of five variables \( F_{\nu}, G_{\nu} \).
\[ \Theta(x, y) = \sum_{i=1}^{M} F_i(\Psi^1; \phi^1, \phi^2, \phi^3, \phi^4) \big|_{\lambda=\mu_i} + \sum_{i=1}^{M} G_i(\Psi^2; \phi^1, \phi^2, \phi^3, \phi^4) \big|_{\lambda=\nu_i} + \sum_{i=1}^{M} \sum_{j=1}^{M} f_{ij} g_{ij} \big|_{\nu_i - \mu_j}, \tag{23} \]

where \( \Psi^1 \) and \( \Psi^2 \) are given by (20), \( \phi^1, \phi^2, \phi^3, \phi^4 \) are of the form (4) and the functions \( f_i \) and \( g_i \) are defined as implicit functions by equations (21). Formula (23) corresponds to the special solution of hyper-Kähler hierarchies presented in [12], however, it is important to note that in our case the solution depends on the set of arbitrary functions of five variables, in contrast to the set of functions of one variable in [12] and functions of three variables in [5].

**Remark.** It is possible to prove directly that ansatz (20), taking into account relations (21), gives a solution to generating equation (7).

### 3.2. From equation (1) to the general heavenly equation

The connection between equation (1) and four-dimensional general heavenly equation is described in [1] in a simple and elegant way, namely as a traveling wave reduction \( \partial_y = \lambda_i \partial_x \).

It is remarkable that this reduction can be rather easily performed in terms of the dressing data and, specifically, for the special solution of the type (23), after some minor modification. This reduction gives a general way of introducing a vertex variable (corresponding to simple pole at some point) instead of a pair of variables of \( x, y \), thus providing different types of generating equations for lower-dimensional systems.

First, we slightly modify the definition of the wave functions (5),

\[
\Psi^1 = q + 4 \sum_{i=1}^{4} a_i y_i + \tilde{\Psi}^1, \\
\Psi^2 = p + 4 \sum_{i=1}^{4} b_i y_i + \tilde{\Psi}^2 \tag{24}
\]

leaving the functions \( \phi^j \) intact. It is easy to check that this modification leads to equation (1) and six-dimensional heavenly equations for the potential \( \Theta \) containing vacuum terms,

\[
\Theta = \Theta_0 + \tilde{\Theta}, \quad \Theta_0 = \frac{1}{2} \sum (a_i b_j - a_j b_i) x^i y^j + \sum (a_i p x^i - b_i q x^i).
\]

To perform a transition from the eight-dimensional to the four-dimensional case, let us consider \( \partial \)-data of the form

\[
W(\lambda, \tilde{\lambda}, \tilde{\Psi}^1, \Psi^2; \phi^1, \phi^2, \phi^3, \phi^4) = F \left( \lambda, \tilde{\lambda}, \Psi^1 + \sum \frac{a_i \phi^j}{\lambda - \lambda_0}, \Psi^2 + \sum \frac{b_i \phi^j}{\lambda - \lambda_1} \right), \tag{25}
\]

where \( F \) vanishes in the neighborhoods of infinity and points \( \lambda = \lambda_i \). First, this data evidently corresponds to some solution of equation (1) with the vacuum background, and, taking into account that
the solution is of the form
$$
\Theta = \sum (a_ib_j - a_jb_i)x^iy^j/\lambda_i - \lambda_j + \tilde{\Theta}(x^1, x^2, x^3, x^4),
$$
(27)
where $\tilde{\Theta}$ corresponds to the traveling waves reduction. Then the four-dimensional potential (with the same $\tilde{\Theta}$)
$$
\Theta = \frac{1}{2} \sum_{i \neq j} (a_ib_j - a_jb_i)x^iy^j/\lambda_i - \lambda_j + \tilde{\Theta}(x^1, x^2, x^3, x^4),
$$
(28)
satisfies the general heavenly equation
$$
(\lambda_2 - \lambda_3)(\lambda_1 - \lambda_4)\Theta_{x^3 x^4} - \Theta_{x^1 x^2} + (\lambda_1 - \lambda_3)(\lambda_4 - \lambda_2)\Theta_{x^1 x^3} - \Theta_{x^2 x^4} + (\lambda_1 - \lambda_2)(\lambda_3 - \lambda_4)\Theta_{x^1 x^2} - \Theta_{x^3 x^4} = 0.
$$
(29)

Remark 1. Dressing data of the form (26) (in terms of wave variables) exactly correspond to the dressing scheme developed in [4] for general heavenly equation. And in the process of reduction from equation (1) to general heavenly equation (29) we obtain the same vacuum term that was obtained in [4] directly from the dressing scheme for general heavenly equation.

To get a solution of the form (23) corresponding to the traveling wave reduction, we should use modified wave functions (24) and the dressing data of the form
$$
\epsilon_{i_1 \cdots i_{2N}} \omega_{i_1 i_2} \cdots \omega_{i_{2N-1} i_{2N}} = 0
$$
(31)
or pf$(\omega) = 0$, where pf$(\omega)$ is a Pfaffian and $\omega$ is $2N \times 2N$ skew-symmetric matrix with the entries
$$
\omega_{ij} = \Theta_{x^i x^j} - \Theta_{x^j x^i}.
$$
(32)
Another elegant way to describe equation (31) is to use the differential two-form \( \omega \),
\[
\omega = \omega_{ij}(x, y) dx^i \wedge dy^j,
\]
where coefficients \( \omega_{ij} \) are independent of \( \lambda \). For 4 + 4-dimensional case equation (1) is equivalent to conditions (13) for the form \( \omega \). Equation (31) is equivalent to the conditions
\[
\wedge^N \omega = 0, \quad d^n \omega = 0,
\]
where the second condition, independent of dimensionality, implies the existence of potential \( \Theta \) (32), and the first condition gives equation (31) for the potential generalising equation (1) to multidimensions (see [1] for more detail).

The analogue of generating relation (7) for this case is
\[
\Omega_\omega = \left( (d \Psi^1 \wedge d \Psi^2 + d \Psi^3 \wedge d \Psi^4 + \cdots + d \Psi^{2N-3} \wedge d \Psi^{2N-2})
\right.
\wedge \left( (d \phi^1 \wedge d \phi^2 \wedge \cdots \wedge d \phi^{2N}) \right)_\omega = 0,
\]
where functions \( \phi^i \) are of the form (4) and the series the functions \( \Psi^i \) are of the type (5),
\[
\Psi_{2k-1} = q^k + \tilde{\Psi}_{2k-1}, \quad \tilde{\Psi}_{2k-1} = \sum_{n=1}^{\infty} \Psi_{2k-1}^{-1}(p, q, x, y) \lambda^{-n},
\]
\[
\Psi_{2k} = p^k + \tilde{\Psi}_{2k}, \quad \tilde{\Psi}_{2k} = \sum_{n=1}^{\infty} \Psi_{2k}^{-1}(p, q, x, y) \lambda^{-n}, \quad 1 \leq k \leq N - 1.
\]
The analogue of two-form \( \omega \) (11) is
\[
\omega = d^n \Psi^1 \wedge d^n \Psi^2 + d^n \Psi^3 \wedge d^n \Psi^4 + \cdots + d^n \Psi^{2N-3} \wedge d^n \Psi^{2N-2},
\]
and the second form of the generating relation reads
\[
\omega_\omega = \left( d^n \Psi^1 \wedge d^n \Psi^2 + d^n \Psi^3 \wedge d^n \Psi^4 + \cdots + d^n \Psi^{2N-3} \wedge d^n \Psi^{2N-2} \right)_\omega = 0.
\]
Generating relation (35) implies that
\[
\omega = \omega_{ij}(x, y) dx^i \wedge dy^j,
\]
where coefficients \( \omega_{ij} \) are independent of \( \lambda \), and equation (33) for this two-form.

Dressing scheme (17) requires an obvious modification
\[
\partial \Psi_{2k-1} = W_{2k}(\lambda, \tilde{\lambda}, \Psi^1, \ldots, \Psi^{2N-2}, \phi^1, \ldots, \phi^{2N}), \quad W_{2k} := \frac{\partial W}{\partial \Psi_{2k}},
\]
\[
\partial \Psi_{2k} = W_{2k-1}(\lambda, \tilde{\lambda}, \Psi^1, \ldots, \Psi^{2N-2}, \phi^1, \ldots, \phi^{2N}), \quad W_{2k-1} := \frac{\partial W}{\partial \Psi_{2k-1}},
\]
where \( 1 \leq k \leq N - 1 \). The calculation of special solutions of the type (23) is completely analogous. We start from the dressing data
\[
\frac{1}{2 \pi i} W = \sum_{i=1}^{M} \delta(\lambda - \mu_i) F_i(\Psi^1, \Psi^3, \ldots, \Psi^{2N-3}; \phi^1, \ldots, \phi^{2N})
\]
\[
+ \sum_{i=1}^{M} \delta(\lambda - \nu_i) G_i(\Psi^2, \Psi^4, \ldots, \Psi^{2N-2}; \phi^1, \ldots, \phi^{2N}),
\]
and solutions to the \( \partial \) problem (36) are of the form

\[
\Psi^{2k-1} = q^k + \sum_{i=1}^{M} a_i^k y_i + \sum_{i=1}^{M} \frac{f_{ik}}{\lambda - \nu_i}, \quad \Psi^2 = p^k + \sum_{i=1}^{M} b_i y_i + \sum_{i=1}^{M} \frac{g_{i}^k}{\lambda - \mu_i},
\]  

(37)

where we take into account vacuum terms. The functions \( f_{ik}^k, g_{i}^k \) are defined as implicit functions by the relations

\[
f_{ik}^k(x, y) = G_{i2k}(\Psi^2, \Psi^4, \ldots, \Psi^{2N-2}; \phi^1, \ldots, \phi^{2N})\big|_{\lambda = \nu_i},
\]

\[
g_{i}^k(x, y) = -F_{i2k-1}(\Psi^1, \Psi^3, \ldots, \Psi^{2N-3}; \phi^1, \ldots, \phi^{2N})\big|_{\lambda = \mu_i},
\]

(38)

where equation (38) represent a closed system of \( 2M(N-1) \) equations for \( 2M(N-1) \) functions \( f_{ik}^k, g_{i}^k \). The potential \( \Theta \) contains a sum of vacuum and regular terms,

\[
\Theta = \Theta_0 + \bar{\Theta}, \quad \Theta_0 = \frac{1}{2} \sum_{i,k} (a_i^k b_i^k - a_i^k b_i^k) x_i y_j + \sum_{i,k} (a_i^k p^k x_i - b_i^k q^k x_i).
\]

(39)

The regular term is defined by multidimensional extension of general formula (22),

\[
\Theta(x, y) = \int_\mathcal{G} \frac{d\lambda \wedge d\lambda}{2\pi i} \left( \sum_k \bar{\Theta} \partial_\lambda \Psi^{2k-1} - W(\lambda, \bar{\lambda}, \Psi^1, \ldots, \Psi^{2N-2}; \phi^1, \ldots, \phi^{2N}) \right)
\]

(40)

and extension of formula (23) reads

\[
\Theta(x, y, p, q) = \sum_{i=1}^{M} F_i(\Psi^1, \Psi^3, \ldots, \Psi^{2N-3}; \phi^1, \ldots, \phi^{2N})\big|_{\lambda = \mu_i}
\]

\[
+ \sum_{i=1}^{M} G_i(\Psi^2, \Psi^4, \ldots, \Psi^{2N-2}; \phi^1, \ldots, \phi^{2N})\big|_{\lambda = \nu_i} + \sum_{i=1}^{M} \sum_{j=1}^{M} f_{ik} g_{ij}^k
\]

(41)

the potential depends on the set of arbitrary functions of \( 3N-1 \) variables.

The traveling wave reduction \( \partial \gamma = \lambda \partial \gamma \) for some pair of variables \( x', y' \) corresponds to a special dependence of the \( \partial \) data on the function \( \phi' \), when it enters the data only in the combination with functions \( \Psi^k \), namely \( \Psi^{2k-1} = \sum_{i=1}^{M} \frac{d\phi_i'}{\lambda - \nu_i}, \Psi^{2k} = \sum_{i=1}^{M} \frac{\phi_i'}{\lambda - \mu_i} \) (compare (25), (30)). The traveling wave reduction for all pairs \( x', y' \) leads to the solution of \( 2N \)-dimensional extension of the general heavenly equation [4].

The analogues of six-dimensional heavenly equations are now \( 2 + 2N \)-dimensional,

\[
\Theta_{x'y'} = \{ \Theta_{x'}, \Theta_{x'} \}_{(q, p)}
\]

(42)

with the Lax pair

\[
D_\gamma \Psi + \{ \Theta_{x'}, \Psi \}_{(q, p)} = 0,
\]

\[
D_\Psi + \{ \Theta_{x'}, \Psi \}_{(q, p)} = 0,
\]

where the Poisson bracket is

\[
\{ f_1, f_2 \}_{(q, p)} := \sum_{k=1}^{N-1} \frac{\partial f_1}{\partial q^k} \frac{\partial f_2}{\partial p^k} - \frac{\partial f_1}{\partial p^k} \frac{\partial f_2}{\partial q^k}
\]
Potentials given by expressions (39), (41) provide special solutions to equation (42).

Generally, $2N + 2N$-dimensional generalisation of integrable structures connected with equation (1) is similar to the generalisation of the second heavenly equation hierarchy to hyper-Kähler hierarchy [12].

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