LINEARITY CONDITIONS LEADING TO COMPLETE POSITIVITY

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The reduced dynamics of an open quantum system $S$, interacting with its environment $E$, is not completely positive, in general. In this paper, we demonstrate that if the two following conditions are satisfied, simultaneously, then the reduced dynamics is completely positive: (1) the reduced dynamics of the system is linear, for arbitrary system-environment unitary evolution $U$; and (2) the reduced dynamics of the system is linear, for arbitrary initial state of the system $\rho_S$.

I. INTRODUCTION

In the axiomatic approach to quantum operations, as legitimate maps describing the (reduced) dynamics of a quantum system $S$, a quantum operation $E_S$ is defined as a linear trace-preserving completely positive map [1]. At first glance, requiring that $E_S$ is linear seems admissible, since the unitary evolution of a closed quantum system is linear, and we may expect similar property for open quantum systems too. In addition, nonlinear evolution may lead to superluminal signaling [2].

But, instead of being trace-preserving completely positive, one may expect that $E_S$ must be solely a trace-preserving positive map, since the only general requirement seems to be that $E_S$ must map density operators to density operators. It seems that there are two major reasons, for the usual use of completely positive maps, instead of the positive ones, in quantum information theory [1], and in the theory of open quantum systems [3-5]: First, there exists a simple operator sum representation, for each trace-preserving completely positive (CP) map $E_S$, as

$$E_S(\rho_S) = \sum_i E_i \rho_S E_i^\dagger, \quad \sum_i E_i^\dagger E_i = I_S,$$  \hspace{1cm} (1)

where $E_i$ are linear operators and $I_S$ is the identity operator, on the Hilbert space of the system $H_S$ [1].

Second, in the theory of open quantum systems, it is common to consider the set of initial states of the system-environment as $S = \{\rho_{SE} = \rho_S \otimes \omega_E\}$, where $\rho_S$ is an arbitrary state (density operator) on $H_S$ and $\omega_E$ is a fixed state on the Hilbert space of the environment $H_E$ [3-5]. Then, for such an initial set $S$, it is famous that the reduced dynamics of the system is CP, for arbitrary system-environment unitary evolution $U$ [1].

The main question of this paper is to investigate whether it is possible to result the CP-ness of the reduced dynamics, from its positivity, or even from the less restrictive condition of its linearity.

Unlike the reduced dynamics, for which, in general, its positivity is not equivalent to its CP-ness, there exists an important map for which it is so. This important map is the inverse of the partial trace over the environment, and is called the assignment map [6, 7]. It can be shown that if there exists a positive assignment map, then there exists a CP one too, which results in the CP-ness of the reduced dynamics [8].

As we will see, in Sec. IV, only requiring that the reduced dynamics is linear, for arbitrary unitary evolution of the system-environment $U$ and arbitrary initial state of the system $\rho_S$, results in the positivity of the assignment map, and so the CP-ness of the reduced dynamics.

The paper is organized as follows. In the next section, we review some introductory points, on the reduced dynamics of an open quantum system. The assignment map, and its role in representing the reduced dynamics as a linear map, is introduced in Sec. III. Our main results are given in Sec. IV, and the paper is ended in Sec. V, with a summary of our results.

II. REDUCED DYNAMICS OF AN OPEN SYSTEM

Let us denote the set of all linear operators on $H_S$ as $L_S$, and the set of all density operators on $H_S$ as $D_S$. Now, by a Hermitian map, we mean a linear trace-preserving map on $L_S$, which maps each Hermitian operator to a Hermitian operator. A Hermitian map is called positive, if it maps each density operator, in $D_S$, to a density operator. Both, Hermitian maps and positive ones, have operator sum representations as

$$\Phi_S(\rho_S) = \sum_i e_i \tilde{E}_i \rho_S \tilde{E}_i^\dagger, \quad \sum_i e_i \tilde{E}_i^\dagger \tilde{E}_i = I_S, \hspace{1cm} (2)$$

where $\tilde{E}_i$ are linear operators on $H_S$, and $e_i$ are real coefficients [9-11]. When all of the coefficients $e_i$ in Eq. (2) are positive, we can define $E_i = \sqrt{e_i} \tilde{E}_i$, and Eq. (2) can be rewritten as Eq. (1). Then, the map is called CP. It is also worth noting that the CP-ness of the map $E_S$, in Eq. (1), is equivalent to the positivity of the map $\text{id}_W \otimes E_S$, where the witness $W$ is an arbitrary (finite dimensional) quantum system, distinct from the system $S$ (and the environment $E$), and $\text{id}_W$ is the identity map on $L_W$ [1]. ($L_W$ is the set of all linear operators on the Hilbert space of the witness $H_W$.)

For the open quantum system $S$, interacting with its environment $E$, we can consider the whole system-environment as a closed quantum system, which evolves
been proven, only for some restricted sets
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amatics of the system can be given by a map, this map
t that environment, at time
deals with the factorized initial states of the system-
initial states of the system-environment [16–22].

given as a function of the initial state of the system
not be represented by a map [9, 12], i.e., $\rho_S$ cannot be

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operators $H$ and $S$

where $S$ is a CP map, depending on $U$ and $\sigma_\alpha$ [28]. In other words, in this case, the reduced dynamics is given by a set of CP maps $\{E_S^{(\alpha)}\}$, instead of only one CP map.

Second, consider the case that set of initial states of the system-environment is given by $S = \{\rho_{SE} = \sum_\alpha \tilde{w}_\alpha Q_\alpha \otimes \tilde{\sigma}_\alpha\}$, where the linear operators $Q_\alpha \in L_S$ vary, by changing $\rho_{SE}$, but $\tilde{\sigma}_\alpha$ are fixed density operators on $H_E$, and the (positive) weights $\tilde{w}_\alpha$ are also fixed. Then, the reduced dynamics of the system $S$, in Eq. (4), for arbitrary system-environment unitary evolution $U$, is given by

$$\rho'_S = \sum_\alpha \tilde{w}_\alpha E^{(\alpha)}_S(Q_\alpha),$$  

where, though $E_S(t_1,0)$ and $E_S(t_2,0)$ are CP, but $\Phi_S(t_2,t_1)$, i.e., the Hermitian map which maps $\sigma_S(t_1)$ to $\sigma_S(t_2)$, is non-CP, in general. So, changing the initial
time, from $t = 0$ to $t = t_1$, results that the reduced dynamics of the system is given by the non-CP map $\Phi_S(t,t_1)$, for $t > t_1$.

In addition to simplicity and experimental relevance, which were mentioned above and in the Introduction, one can give a rather general discussion, leading to the CP-ness of the reduced dynamics: always, in addition to the system under study $S$, one can consider another quantum system, the witness $W$, which does not interact with $S$, and, during the evolution of $S$, it does not evolve. Now, assuming that the evolution of the witness-system is given by a local map $id_W \otimes E_S$, results in the CP-ness of $E_S$. Note that the initial state of the witness-system $\rho_{WS}$ can be entangled. Now, the CP-ness of $E_S$, and so the positivity of the $id_W \otimes E_S$, is necessary to ensure that the final state $\rho_{WS} = id_W \otimes E_S(\rho_{WS})$ is a valid density operator [1]. However, one can find situations in which, though the dynamics of the witness-system is local (and the reduced state of the witness does not change, during the evolution), it cannot be written as $id_W \otimes E_S$ (see, e.g. [27]). So, the reduced dynamics of the system $S$ can be non-CP, in general, as we have seen for $\Phi_S(t,t_1)$, in the previous paragraph.

At the end of this section, we mention that the utilization of the completely positive maps, for describing the reduced dynamics of the system $S$, can be extended, at least, through the two following ways. First, consider the case that the set of initial states of the system-environment is given by $S = \{\rho_{SE} = \sum_\alpha \tilde{w}_\alpha Q_\alpha \otimes \tilde{\sigma}_\alpha\}$, where the linear operators $Q_\alpha \in L_S$ vary, by changing $\rho_{SE}$, but $\tilde{\sigma}_\alpha$ are fixed density operators on $H_E$, and the (positive) weights $\tilde{w}_\alpha$ are also fixed. Then, the reduced dynamics of the system $S$, in Eq. (4), for arbitrary system-environment unitary evolution $U$, is given by

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time, from $t = 0$ to $t = t_1$, results that the reduced dynamics of the system is given by the non-CP map $\Phi_S(t,t_1)$, for $t > t_1$.
III. ASSIGNMENT MAP

Consider the set \( S = \{ \rho_{SE} \} \) of initial states of the system-environment. The set \( S \) includes all initial \( \rho_{SE} \) which are prepared (chosen), through the preparation step of the experiment. Obviously, in general, \( S \) is a subset of \( D \), the set of all density operators on \( H_S \otimes H_E \).

The set of initial states of the system is given by \( S_S = \text{Tr}_E S \). Assuming that the system \( S \) is finite dimensional, of dimension \( d_S \), only a finite number \( m \) of the members of \( S_S \), where the integer \( m \) is \( 0 < m \leq (d_S)^2 \), are linearly independent. Let us denote this linearly independent set as \( S'_S = \{ \rho_{S}^{(1)} , \rho_{S}^{(2)} , \ldots , \rho_{S}^{(m)} \} \). Therefore, any \( \rho_S \in S_S \) can be expanded as

\[
\rho_S = \sum_{i=1}^{m} a_i \rho_{S}^{(i)}, \tag{9}
\]

where \( a_i \) are real coefficients. Note that \( \rho_S \) is a Hermitian operator. So, \( \sum (a_i - a_i^*) \rho_{S}^{(i)} = 0 \). Now, since all \( \rho_{S}^{(i)} \in S'_S \) are linearly independent, all \( a_i \) must be real.

In general, there may be more than one state in \( S \) such that tracing over the environment gives \( \rho_{S}^{(i)} \). However, we choose only one of them and denote it as \( \rho_{S}^{(i)} \). Linear independence of \( \rho_{S}^{(i)} \in S'_S \) results in linear independence of \( \rho_{SE}^{(i)} \). We denote this linearly independent set as \( S' = \{ \rho_{SE}^{(1)} , \rho_{SE}^{(2)} , \ldots , \rho_{SE}^{(m)} \} \) \[33\]. So, each \( \rho_{SE} \in S \), for which \( \rho_{S} = \text{Tr}_E(\rho_{SE}) \) is expanded in Eq. (9), can be written as

\[
\rho_{SE} = \sum_{i=1}^{m} a_i \rho_{SE}^{(i)} + Y(\rho_{SE}), \tag{10}
\]

where \( a_i \) are the same as those in Eq. (9), and \( Y \) is a Hermitian operator, on \( H_S \otimes H_E \), such that \( \text{Tr}_E(Y) = 0 \).

In other words, Eq. (9) results that \( \rho_{SE} \) and \( \sum a_i \rho_{SE}^{(i)} \) can differ with each other up to a Hermitian operator \( Y \), for which \( \text{Tr}_E(Y) = 0 \). In general, \( Y \) is a function of \( \rho_{SE} \). This dependence is explicitly given in Eq. (10), by writing it as \( Y(\rho_{SE}) \).

The subspaces \( \mathcal{V} \) and \( \mathcal{V}_S \) are defined as \[9\]

\[
\mathcal{V} = \text{Span}_\mathbb{C} S, \tag{11}
\]

and

\[
\mathcal{V}_S = \text{Tr}_E \mathcal{V} = \text{Span}_\mathbb{C} S_S. \tag{12}
\]

Therefore, each \( X \in \mathcal{V} \) can be written as \( X = \sum c_l \tau_{SE}^{(l)} \), where \( \tau_{SE}^{(l)} \in S \), and \( c_l \) are complex coefficients. Using Eq. (10), we can expand each \( \tau_{SE}^{(l)} \) as \( \tau_{SE}^{(l)} = \sum a_i \rho_{SE}^{(i)} + Y(\tau_{SE}^{(l)}) \) where the coefficients \( d_i \) are the same as those in Eq. (13). In Fig. 1, the sets \( S_S \) and \( D_S \), the subspace \( \mathcal{V}_S \), and the vector space \( \mathcal{L}_S \) are given, in a Venn diagram.

Now, we can define the linear trace-preserving assignment map \( \Lambda_S \), as follows: first, we define \( \Lambda_S(\rho_{S}^{(i)}) = \rho_{SE}^{(i)} \).

Then, we extend the definition of \( \Lambda_S \) to the whole \( \mathcal{V}_S \), as a linear map. So, for any \( x \in \mathcal{V}_S \), in Eq. (14), we have

\[
\Lambda_S(x) = \sum_{i=1}^{m} d_i \Lambda_S(\rho_{S}^{(i)}) = \sum_{i=1}^{m} d_i \rho_{SE}^{(i)}. \tag{15}
\]

The assignment map \( \Lambda_S \) maps \( \mathcal{V}_S \) to (a subspace of) \( \mathcal{V} \), and is Hermitian, by construction. (When \( x \) is a Hermitian operator, all \( d_i \), in Eq. (14), are real. So, \( \Lambda_S(x) \) is also a Hermitian operator.) Comparing Eqs. (13) and (15) shows that \( \Lambda_S \) does not necessarily map \( x \) to \( X \), unless \( Y(X) = 0 \). In addition, note that the assignment map \( \Lambda_S \), in Eq. (15), is defined on the subspace \( \mathcal{V}_S \). This definition can be extended, to the whole \( \mathcal{L}_S \), simply, i.e., one can find a Hermitian map \( \Lambda'_S \), on the whole \( \mathcal{L}_S \), such that, for each \( x \in \mathcal{V}_S \), it acts as \( \Lambda_S \) \[8\]. But, only for each \( x \in \mathcal{V}_S \), not necessarily for arbitrary \( f \in \mathcal{L}_S \), we
and the assignment map Λ. The map Φ_S is defined as Φ_S = Tr_E \circ \text{Ad}_U \circ \Lambda_S. According to Eq. (16), Φ_S gives \rho'_S, if the U-consistency condition Tr_E \circ \text{Ad}_U(Y) = 0 is satisfied. Then, rounding the diagram clockwise, from \rho_S to \rho'_S, is equivalent to rounding it counterclockwise, through the Hermitian map Φ_S.

have Tr_E \circ \Lambda'_S(x) = Tr_E \circ \Lambda_S(x) = x. In other words, the extension \Lambda'_S of the assignment map \Lambda_S is self-consistent only on \mathcal{V}_S, not necessarily on the whole \mathcal{L}_S.

Now, using Eqs. (4), (9), (10) and (15), the reduced dynamics of the system, for each \rho_{SE} \in \mathcal{V}, is given by

\[ \rho'_S = \text{Tr}_E \circ \text{Ad}_U(\rho_{SE}) = \sum_{i=1}^{m} a_i \text{Tr}_E \circ \text{Ad}_U(\rho_{SE}^{(i)}) + \text{Tr}_E \circ \text{Ad}_U(Y) \tag{16} \]

where Φ_S = Tr_E \circ \text{Ad}_U \circ \Lambda_S. The map Φ_S is a (linear) Hermitian map on \mathcal{V}_S, since Tr_E and \text{Ad}_U are CP [1], and the assignment map \Lambda_S is Hermitian on \mathcal{V}_S, as we have seen in Eq. (15). When Tr_E \circ \text{Ad}_U(Y) = 0, the subspace \mathcal{V} is called U-consistent [9]. The reduced dynamics of the system, for each \rho_{SE} \in \mathcal{V}, is given by the linear Hermitian trace-preserving map \Phi_S, if and only if \mathcal{V} is U-consistent [9, 34]. In Fig. 2, we represent when the Hermitian map Φ_S gives the reduced dynamics of the system, in a commutative diagram. It is also worth noting that, in the theory of open quantum systems, one usually approximates the reduced dynamics as a linear map, utilizing some simplifying assumptions (about \mathcal{V}) [3–5, 35].

CP-ness of Tr_E and \text{Ad}_U results that only the assignment map \Lambda determines whether Φ_S is CP or not. If \Lambda_S is Hermitian, then Φ_S can be either Hermitian, positive or CP. But, when the extension \Lambda'_S of the assignment map \Lambda_S is positive, then Φ_S is necessarily CP [8].

We end this section, with the following point. Assuming unitary dynamics for the whole system-environment, the (non)linearity of the reduced dynamics is only a consequence of U-(in)consistency of the subspace \mathcal{V}. In other words, it is only a consequence of how we choose (construct) the initial set \mathcal{S}, and there is no fundamental reasoning behind it [34]. In addition, as discussed in Ref. [34], non-linearity of the reduced dynamics does not lead to superluminal signaling.

### IV. MAIN RESULT

Assume that the reduced dynamics of the system, for each \rho_S \in \mathcal{S}_S is given by a dynamical map Ψ_S, i.e., the final state \rho'_S, in Eq. (4), is given by Ψ_S(\rho_S). As discussed in the Introduction, in the axiomatic approach to quantum operations, postulating that the dynamical map Ψ_S is linear seems more natural than postulating it as a CP map. In addition, it can be shown simply [34] that when the map Ψ_S is linear, on the subspace \mathcal{V}_S, then it is equal to Φ_S, in Eq. (16). Now, we ask, under what circumstances, does only requiring that Ψ_S is linear (and so is equal to Φ_S, in Eq. (16)) result that it is also CP? Such circumstances are given in the following Proposition.

**Proposition 1.** Requiring that the reduced dynamics of the system, for each \rho_S \in \mathcal{D}_S, and for arbitrary system-environment unitary evolution U, is a linear function of \rho_S, results in the CP-ness of the assignment map \Lambda_S. Thus, the reduced dynamics of the system S is CP, as Eq. (1).

**Proof.** First, we require that the reduced dynamics of the system, for arbitrary system-environment unitary evolution U, is linear. So, the reduced dynamics is given by the map Φ_S, in Eq. (16), for arbitrary U [34]. In other words, the subspace \mathcal{V}, in Eq. (11), is U-consistent, for arbitrary U. This results in the one to one correspondence between the subspaces \mathcal{V} and \mathcal{V}_S = \text{Tr}_E \mathcal{V} [9]. Hence, for each X, Z, \mathcal{V}, Tr_E(X) = Tr_E(Z) if and only if X = Z. It indicates that Y(\rho_{SE}), in Eq. (10), and so Y(X), in Eq. (13), are zero. Therefore, \Lambda_S(\rho_S) = \rho_{SE} and \Lambda_S(x) = X, where the linear assignment map \Lambda_S is defined in Eq. (15), and \rho_S, \rho_{SE}, X and x are given in Eqs. (9), (10), (13) and (14), respectively. Second, we require that the reduced dynamics of the system is linear, for arbitrary initial state of the system \rho_S \in \mathcal{D}_S. This means that we choose the set of initial states of the system-environment \mathcal{S} such that \mathcal{S}_S = \mathcal{D}_S. Therefore, since one can find (4\mathcal{S})^2 linearly independent states in \mathcal{D}_S (see, e.g., [36]), we have \mathcal{V}_S = \text{Span}_C \mathcal{D}_S = \mathcal{L}_S.

Note that we want to find the conditions which ensure the positivity of (the extension of) the assignment map \Lambda_S in Eq. (15). Requiring that, for a given U, the reduced dynamics is linear, for arbitrary initial state \rho_S \in \mathcal{D}_S, results that \mathcal{S}_S = \mathcal{D}_S (and so \Lambda'_S = \Lambda_S, since \mathcal{V}_S = \mathcal{L}_S) and Tr_E \circ \text{Ad}_U(Y) = 0, where Y is given in Eq. (10). But, it does not necessitate that Y = 0. So, the assignment map \Lambda_S, which maps \rho_S, in Eq. (9), to
\[ Z = \sum_{i=1}^{m} a_i \rho_{SE}^{(i)} \] is not necessarily positive, since \( Z \) is not necessarily a positive operator. But, if we add the first requirement too, which ensures that \( Y = 0 \), then we conclude that \( \Lambda_S = \Lambda_S' \) is positive.

On the other hand, only assuming the first requirement, though results in the positivity of \( \Lambda_S \) on \( S_S \), but it does not necessarily lead to the positivity of the extension \( \Lambda_S' \) of the assignment map \( \Lambda_S \), on the whole \( D_S \left( L_S \right) \). But, if we add the second requirement too, which states that \( S_S = D_S \), we ensure that \( \Lambda_S' = \Lambda_S \) is positive, on the whole \( D_S \left( L_S \right) \).

Consequently, assuming that both the first and the second requirements are satisfied simultaneously, results that \( \Lambda_S' = \Lambda_S \) is positive, on the whole \( D_S \). Now, it has been shown that when there is a positive extension \( \Lambda_S' \) of the assignment map \( \Lambda_S \), on the whole \( D_S \left( L_S \right) \), then there exists a CP assignment map \( \Lambda_S^{(CP)} \) too [8]. In fact, in this case, where \( S_S = D_S \) and so \( \Lambda_S \subseteq \Lambda_S \), and, in addition, there is a one to one correspondence between the subspaces \( V \) and \( Y \), there is a unique way to define (the extension of) the assignment map. So, the CP assignment map \( \Lambda_S^{(CP)} \) is the same as our positive \( \Lambda_S = \Lambda_S' \), with the explicit form

\[
\Lambda_S \left( \rho_S \right) = \Lambda_S^{(CP)} \left( \rho_S \right) = \rho_S \otimes \omega_E, \tag{17}
\]

where \( \omega_E \) is a fixed state on \( H_E \) [6, 8, 11]. This fact that \( \omega_E \) is a fixed state is a consequence of assuming that the assignment map is a self-consistent positive map, on the whole \( D_S \left( L_S \right) \) [6, 8, 11]. The assignment map \( \Lambda_S^{(CP)} \), given in Eq. (17), is, in fact, the famous Pechukas’s one, first introduced in Ref. [6]. Finally, the CP-ness of \( \Lambda_S^{(CP)} \) leads to the CP-ness of the reduced dynamics \( \Phi_S = \text{Tr}_E \circ \text{Ad}_U \circ \Lambda_S = \text{Tr}_E \circ \text{Ad}_U \circ \Lambda_S^{(CP)} \).

In the axiomatic approach to quantum operations, it is more appropriate to postulate that the dynamical map \( \Psi_S \) is convex-linear, instead of considering it linear. A convex-linear map is defined as follows.

**Definition 1.** When \( \Psi_S \) is convex-linear, on \( D_S \), then we have \( \Psi_S \left( \rho_S + (1-p) \tau_S \right) = p \Psi_S \left( \rho_S \right) + (1-p) \Psi_S \left( \tau_S \right) \), where \( \rho_S, \tau_S \in D_S \) and \( 0 \leq p \leq 1 \).

In the following Proposition, we refer to the convexity of the set \( S_S \). This property is defined as below.

**Definition 2.** When \( S_S \) is convex, if \( \rho_S, \tau_S \in S_S \), then, also, \( \omega_S = \rho_S + (1-p) \tau_S \in S_S \), where \( 0 \leq p \leq 1 \).

In Proposition 1, we have seen that requiring the reduced dynamics of the system \( S \) is linear, leads to its CP-ness. Now, we want to go further and show that requiring the reduced dynamics is convex-linear, results in the CP-ness of the reduced dynamics too.

**Proposition 1.** Requiring that the reduced dynamics of the system, for each \( \rho_S \in D_S \), and for arbitrary system-environment unitary evolution \( U \), is a convex-linear function of \( \rho_S \), results in the CP-ness of the assignment map \( \Lambda_S \), as Eq. (17). Thus, the reduced dynamics is CP, as Eq. (1), for arbitrary \( U \) and arbitrary \( \rho_S \in D_S \).

**Proof.** Since, as before, we have \( S_S = D_S \), the set \( S_S \) is convex. Thus, we can show that the convex-linearity of the reduced dynamics results in its linearity, following a similar procedure as Ref. [34].

Note that some of the real coefficients \( a_i \), in Eq. (9), are positive, and the others are negative. Let us denote the positive ones as \( a_i^{(+)} \), and the negative ones as \( a_i^{(-)} \). So, from Eq. (9), we have

\[
\rho_S + \sum_i |a_i^{(-)}| \rho_S^{(i)} = \sum_i a_i^{(+)} \rho_S^{(i)}. \tag{18}
\]

Tracing from both sides, we have \( 1 + \sum_i |a_i^{(-)}| = \sum_i a_i^{(+)} \equiv b \). Dividing both sides of Eq. (18) into \( b \) results in

\[
\frac{1}{b} \left( \rho_S + \sum_i |a_i^{(-)}| \rho_S^{(i)} \right) = \frac{1}{b} \left( \sum_i a_i^{(+)} \rho_S^{(i)} \right) = \omega_S, \tag{19}
\]

where \( \omega_S \in D_S = S_S \). Therefore, assuming that \( \Psi_S \) is convex-linear, on \( S_S \), we have

\[
\Psi_S \left( \omega_S \right) = \Psi_S \left( \frac{1}{b} \left( \rho_S + \sum_i |a_i^{(-)}| \rho_S^{(i)} \right) \right) = \Psi_S \left( \frac{1}{b} \sum_i a_i^{(+)} \rho_S^{(i)} \right) = \frac{1}{b} \left( \sum_i a_i^{(+)} \Psi_S \left( \rho_S^{(i)} \right) \right),
\]

which leads to

\[
\Psi_S \left( \rho_S \right) = \sum_{i=1}^{m} a_i \Psi_S \left( \rho_S^{(i)} \right). \tag{21}
\]

So, noting Eq. (9), we conclude that \( \Psi_S \) is linear. Hence, if \( \Psi_S \) is convex-linear, for arbitrary \( U \) and arbitrary \( \rho_S \in D_S \), then it is also linear, for arbitrary \( U \) and arbitrary \( \rho_S \in D_S \). Now, Proposition 1 shows that the assignment map \( \Lambda_S \) is CP, as Eq. (17), and so the reduced dynamics of the system \( \Psi_S = \Phi_S \) is also CP.

**V. SUMMARY**

Requiring that the reduced dynamics of the system \( S \), interacting with its environment \( E \), is (convex) linear means that (1) the reduced dynamics is (convex) linear, for arbitrary system-environment evolution \( U \), and (2) the reduced dynamics is (convex) linear, for arbitrary initial state of the system \( \rho_S \in D_S \).

In Proposition 1 (1'), it has been shown that the above requirement results in the CP-ness of the reduced dynamics. So, in the axiomatic approach to quantum operations, there is no need to consider the CP-ness as a
distinct postulate. It is only a consequence of (convex) linearity.
In addition, when the reduced dynamics is (convex) linear, for arbitrary $U$ and arbitrary $\rho_S$, then the set of initial states of the system-environment is as $\mathcal{S} = \{ \rho_S \otimes \tilde{\omega}_E \}$, where $\rho_S$ is an arbitrary state of the system, and $\tilde{\omega}_E$ is a fixed state of the environment. In other words, under such circumstances, the assignment map is as the Pechukas’s one [6], given in Eq. (17).

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