Global solutions for $H^s$-critical nonlinear biharmonic Schrödinger equation

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Abstract. We consider the nonlinear biharmonic Schrödinger equation

$$i\partial_t u + (\Delta^2 + \mu \Delta) u + f(u) = 0,$$

in the critical Sobolev space $H^s(\mathbb{R}^N)$, where $N \geq 1$, $\mu = 0$ or $-1$, $0 < s < \min\{ \frac{N}{2}, 8 \}$ and $f(u)$ is a nonlinear function that behaves like $\lambda |u|^\alpha u$ with $\lambda \in \mathbb{C}, \alpha > 0$. We prove the existence and uniqueness of global solutions to (BNLS) for small initial data.

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1. Introduction

In this paper, we consider the following nonlinear biharmonic Schrödinger equation:

$$\begin{cases} i\partial_t u + (\Delta^2 + \mu \Delta) u + f(u) = 0, \\
            u(0, x) = \phi(x), \end{cases}$$

where $t \in \mathbb{R}$, $x \in \mathbb{R}^N$, $N \geq 1$, $\phi \in H^s(\mathbb{R}^N)$, $0 < s < \min\{ \frac{N}{2}, 8 \}$, $\mu = -1$ or $\mu = 0$, $u : \mathbb{R} \times \mathbb{R}^N \to \mathbb{C}$ is a complex-valued function and $f(u)$ is a nonlinear function that behaves like $\lambda |u|^\alpha u$ with $\lambda \in \mathbb{C}, \alpha > 0$. Note that if $\mu = 0$ and $f(u) = \lambda |u|^\alpha u$ with $\lambda \in \mathbb{C}, \alpha > 0$, the equation (1.1) is invariant under the scaling, $u_k(t, x) = k^{\frac{4}{\alpha}} u(k^4 t, kx), k > 0$. This means if $u$ is a solution of (1.1) with the initial datum $\phi$, so is $u_k$ with the initial datum $\phi_k = k^{\frac{2}{\alpha}} \phi(kx)$. Computing the homogeneous Sobolev norm, we get

$$\| \phi_k \|_{\dot{H}^s(\mathbb{R}^N)} = k^{s - \frac{N}{2} + \frac{2}{\alpha}} \| \phi \|_{\dot{H}^s(\mathbb{R}^N)}.$$

Hence, the scale-invariant Sobolev space is $\dot{H}^{s_c}(\mathbb{R}^N)$, with the critical index $s_c = \frac{N}{2} - \frac{4}{N}$. If $s_c = s$ (equivalently $\alpha = \frac{8}{N - 2s}$), the Cauchy problem (1.1) is known as $H^s(\mathbb{R}^N)$-critical; if in particular $s_c = 2$ (equivalently $\alpha = \frac{8}{N - 2}$), it is called energy-critical or $H^2(\mathbb{R}^N)$-critical.

The nonlinear biharmonic Schrödinger equation (1.1), also called the fourth-order Schrödinger equation, was introduced by Karpman [16], and Karpman–Shagalov [17] to take into account the role of small fourth-order dispersion terms in the propagation of intense laser beams in a bulk medium with Kerr nonlinearity. The biharmonic Schrödinger equation has attracted a lot of interest in the past decade. The sharp dispersive estimates for the fourth-order Schrödinger operator in (1.1), namely for the linear group associated with $i\partial_t + \Delta^2 + \mu \Delta$, were obtained in Ben-Artzi, Koch, and Saut [1]. In [24], Pausader established the corresponding Strichartz estimate for the biharmonic Schrödinger equation (1.1). Since then, the local and global well-posedness for (1.1) has been widely studied in recent years, see [5–9, 11–15, 18, 20, 24, 26, 28] and references therein.
We are interested in global solutions to (1.1) in the critical Sobolev space $H^s(\mathbb{R}^N)$. For $s = 2$, Pausader [24] established the global well-posedness for the defocusing energy-critical equation (1.1) (i.e., $\mu = 0$ or $-1$, $f(u) = \lambda |u|^\alpha u$ with $\lambda > 0$, $\alpha = \frac{8}{N-4}$) in a radially symmetric setting. The global well-posedness for the defocusing energy-critical problem without the radial condition and the focusing energy-critical equation (1.1) (i.e., $\mu = 0$ or $-1$, $f(u) = \lambda |u|^\alpha u$ with $\lambda < 0$, $\alpha = \frac{8}{N-4}$) were discussed in [19,20,25,26]. For general $s$, using the improved Strichartz estimate for spatially symmetric function, Wang [28] established global solutions to the biharmonic Schrödinger equation with small radial initial data. He proved the global existence of the solution for the Cauchy problem (1.1) when $N \geq 2$, $\frac{3N-2}{2N+4} < s < \frac{N}{2}$, $\alpha = \frac{8}{N-2s}$, $\mu = 0$, $f(u) = \lambda |u|^\alpha u$, $\lambda = \pm 1$, and $\phi \in H^s(\mathbb{R}^N)$ is a small radial function. Moreover, he also established the local well-posedness of (1.1) for large initial data for some negative regularity.

The goal of this paper is to establish the time global solution for (1.1) with small initial data in the critical Sobolev space $H^s(\mathbb{R}^N)$, where $N \geq 1$, $0 < s < \min \{ \frac{N}{2}, 8 \}$. Before stating our results, we define the class $\mathcal{C}(\alpha)$.

**Definition 1.1.** Let $\alpha > 0$, $f \in C^{[\alpha]+1}(\mathbb{C}, \mathbb{C})$ in the real sense, where $[\alpha]$ denotes the largest integer less than or equal to $\alpha$, and $f^{(j)}(0) = 0$ for all $j$ with $0 \leq j \leq [\alpha]$. We say that $f$ belongs to the class $\mathcal{C}(\alpha)$, if it satisfies one of the following two conditions:

(i) $\alpha \notin \mathbb{Z}$, $f^{([\alpha]+1)}(0) = 0$, and there exists $C > 0$ such that for any $z_1, z_2 \in \mathbb{C}$

$$
|f^{([\alpha]+1)}(z_1) - f^{([\alpha]+1)}(z_2)| \leq C \left( |z_1|^{\alpha-[\alpha]} + |z_2|^{\alpha-[\alpha]} \right) |z_1 - z_2|,
$$

(ii) $\alpha \in \mathbb{Z}$, and there exists $C > 0$ such that for any $z \in \mathbb{C}$

$$
|f^{([\alpha]+1)}(z)| \leq C.
$$

**Remark 1.2.** We note that the power type nonlinearities $f(u) = \lambda |u|^\alpha u$ or $f(u) = \lambda |u|^{\alpha+1}$ with $\lambda \in \mathbb{C}$ and $\alpha > 0$ are in the class $\mathcal{C}(\alpha)$, which have been widely studied in classical and biharmonic nonlinear Schrödinger equations. See [3–6,13,14,18–20,24–26] for instance.

**Remark 1.3.** For any $\alpha > 0$ and $f \in \mathcal{C}(\alpha)$, it is easy to check that there exists $C > 0$ such that for any $u, v \in \mathbb{C}$, we have

$$
|f(u) - f(v)| \leq C (|u|^{\alpha} + |v|^{\alpha}) |u - v|,
$$

|\partial_t f(u)| \leq C |u|^{\alpha} |\partial_t u|.

(1.2)

Our main result is the following. For definitions of vector-valued Besov spaces $B^0_{q,2}(\mathbb{R}, L^r(\mathbb{R}^N))$ and $B^0_{q,2}(-\sigma/4, r, 2)(\mathbb{R}, B^\sigma_{q,2}(\mathbb{R}^N))$, we refer to Sect. 2.

**Theorem 1.4.** Assume $0 < s < \min\{8, \frac{N}{2}\}$, $N \geq 1$, $\mu = 0$ or $-1$, $f \in \mathcal{C}(\alpha)$ and $\alpha = \frac{8}{N-2s} > \alpha(s)$, where

$$
\alpha(s) = \begin{cases} 0, & \text{if } 0 < s < 4, \\ \max \left\{ \frac{s}{4} - 1, s - 5 \right\}, & \text{if } 4 < s < 8. 
\end{cases}
$$

Given any $\phi \in H^s(\mathbb{R}^N)$ with $\|\phi\|_{H^s(\mathbb{R}^N)}$ sufficiently small, there exists a unique global solution $u \in C(\mathbb{R}, H^s(\mathbb{R}^N)) \cap \mathcal{X}$ to the Cauchy problem (1.1), where

$$
\mathcal{X} = \begin{cases} L^0(\mathbb{R}, B^s_{2,2}(\mathbb{R}^N)) \cap B^{s/4}_{q_1,2}(\mathbb{R}, L^{r_1}(\mathbb{R}^N)), & \text{if } 0 < s \leq 4, \\ L^0(\mathbb{R}, H^{4,\infty}(\mathbb{R}^N)), & \text{if } s = 4, \\ L^0(\mathbb{R}, B^s_{2,2}(\mathbb{R}^N)) \cap B^{s/4}_{q_3,2}(\mathbb{R}, L^{r_3}(\mathbb{R}^N)) \cap H^{1,q_3}(\mathbb{R}, B^{s-4}_{r_3,2}(\mathbb{R}^N)), & \text{if } 4 < s < 6, \\ B^{s/4}_{2,2}(\mathbb{R}, L^s(\mathbb{R}^N)) \cap B^{(s-2)/4}_{2,2}(\mathbb{R}, L^{s/2}(\mathbb{R}^N)), & \text{if } 6 \leq s < 8,
\end{cases}
$$
Remark 1.5. Note that the lower bound $\alpha(s)$ is a continuous function of $s$. Moreover, the condition $\alpha > \max \left\{ \frac{s}{4} - 1, s - 5 \right\}$ is natural for $s > 4$, since one time derivative corresponds to four spatial derivatives and the $s$-derivative of $u$ in the spatial derivative requires the $(s - 4)$-derivatives of $f(u)$ by (1.1).

Remark 1.6. Theorem 1.4 improves the result in Wang [28] in the case $0 < s < 4$, where he made an additional radial assumption on the initial datum.

Theorem 1.4 may be considered as a generalization of the corresponding results for the classical $H^s(\mathbb{R}^N)$-critical Cauchy problem for small initial data,

\[
\begin{cases}
  \ i\partial_t u + \Delta u + |u|^\alpha u = 0, \\
  u(0, x) = \phi(x) \in H^s(\mathbb{R}^N),
\end{cases}
\]  

(1.3)

for $0 \leq s < \frac{N}{2}$ and $[s] < \alpha = \frac{4}{N-2\sigma}$. The condition $[s] < \alpha$ is the required regularity for $f(u)$, which can be improved to $s - 1 < \alpha$ by applying nonlinear estimates obtained in Ginibre–Ozawa–Velo [10], and Nakamura–Ozawa [21]. Recently, Nakamura–Wada [22,23] constructed some modified Strichartz estimate and the Strichartz-type estimates in mixed Besov spaces to obtain small global solutions with less regularity assumption for the nonlinear term. More precisely, they showed that if $1 < s < 4$, $s \neq 2$, $\alpha_0(s) < \alpha = \frac{4}{N-2\sigma}$, with

\[
\alpha_0(s) := \begin{cases} 
  0, & \text{for } 0 < s < 2, \\
  \frac{s}{2} - 1, & \text{for } 2 < s < 4,
\end{cases}
\]

the Cauchy problem (1.3) admits a unique time global solution for small initial data. Theorem 1.4 extends results in [22,23] to the biharmonic Schrödinger case.

The main tool used to prove Theorem 1.4 is the following modified Strichartz estimate for fourth-order Schrödinger equation, by which we can replace the spatial derivative of order $4\theta$ with the time derivative of order $\theta$ in terms of Besov spaces. For definitions of the biharmonic admissible pairs set $\Lambda_0$, and the Chemin–Lerner-type space $l^2L^\theta L^\tau$, we refer to Sect. 2.

Proposition 1.7. Assume $0 < \theta < 1$, $0 \leq \sigma < 4 \theta$, $(q,r), (\gamma, \rho) \in \Lambda_0$ are two biharmonic admissible pairs, and $\mu = 0$ or $-1$. Assume also that $1 \leq q < q$, $1 \leq \gamma \leq \infty$ satisfy $\frac{4}{3} - N \left( \frac{\theta}{2} - \frac{\sigma}{4} \right) = 4(1 - \theta)$. Then, for any $u_0 \in H^{4\theta}$ and $f \in B^\theta_{\gamma,2}(\mathbb{R}, L^\rho) \cap l^2L^{\gamma}(\mathbb{R}, L^\tau)$, we have $e^{it(\Delta^2+\mu)\Delta}u_0, Gf \in C(\mathbb{R}, H^{4\theta})$ where

\[
(Gf)(t) = \int_0^t e^{i(t-s)(\Delta^2+\mu)\Delta} f(s)ds.
\]

Moreover, the following inequalities hold:

\[
\|e^{it(\Delta^2+\mu)\Delta}u_0\|_{L^\theta B^\theta_{\gamma,2}} \lesssim \|u_0\|_{H^{4\theta}},
\]

(1.4)

\[
\|Gf\|_{L^\theta B^\theta_{\gamma,2}} \lesssim \|f\|_{B^\theta_{\gamma,2}L^\rho} + \|f\|_{l^2L^\gamma L^\tau},
\]

(1.5)

\[
\|Gf\|_{B^\theta_{\gamma,2}L^\rho} \lesssim \|f\|_{B^\theta_{\gamma,2}L^\rho} + \|f\|_{l^2L^\gamma L^\tau}.
\]

(1.6)
Remark 1.8. The proof of Proposition 1.7 is based on the global Strichartz estimates (2.1)–(2.3). Note that we do not consider the case \( \mu = +1 \) here, since we do not have (2.1)–(2.3) when \( \mu = +1 \).

In this paper, we first establish the modified Strichartz estimates (1.4)–(1.6) for the biharmonic Schrödinger equation in the spirit of [22,23]. Then, we establish various nonlinear estimates and use the contraction mapping principle based on the modified Strichartz estimate to complete the proof of Theorem 1.4.

The rest of the paper is organized as follows. In Sect. 2, we introduce some notations and give a review of the biharmonic Strichartz estimates. In Sect. 3, we establish the modified Strichartz estimate. In Sect. 4, we give the proof of Theorem 1.4.

2. Preliminary

If \( X, Y \) are nonnegative quantities, we sometimes use \( X \lesssim Y \) to denote the estimate \( X \leq CY \) for some positive constant \( C \). Pairs of conjugate indices are written as \( p \) and \( p' \), where \( 1 \leq p \leq \infty \) and \( \frac{1}{p} + \frac{1}{p'} = 1 \). We use \( L^p(\mathbb{R}^N) \) to denote the usual Lebesgue space and \( L^\gamma(I, L^p(\mathbb{R}^N)) \) to denote the space-time Lebesgue space with the norm

\[
\|f\|_{L^\gamma(I, L^p(\mathbb{R}^N))} := \left( \int_I \|f(t)\|_{L^p(\mathbb{R}^N)}^{\gamma} dt \right)^{1/\gamma}
\]

for any time interval \( I \subset \mathbb{R} \), with the usual modification when \( \gamma \) or \( p \) is infinity. We define the Fourier transform on \( \mathbb{R}, \mathbb{R}^N \) and \( \mathbb{R}^{1+N} \) by

\[
\hat{f}(\tau) = \int_{\mathbb{R}} f(t) e^{-i \tau t} dt, \quad \tau \in \mathbb{R},
\]

\[
\hat{f}(\xi) = \int_{\mathbb{R}^N} f(x) e^{-i \xi \cdot x} dx, \quad \xi \in \mathbb{R}^N,
\]

\[
\hat{f}(\tau, \xi) = \int_{\mathbb{R}^{1+N}} f(t,x) e^{-i \tau t - i \xi \cdot x} dt dx, \quad (\tau, \xi) \in \mathbb{R} \times \mathbb{R}^N,
\]

respectively.

Next, we review the definition of Besov spaces. Let \( \phi \) be a smooth function whose Fourier transform \( \hat{\phi} \) is a non-negative even function which satisfies \( \text{supp} \hat{\phi} \subset \{ \tau \in \mathbb{R}, 1/2 \leq |\tau| \leq 2 \} \) and \( \sum_{k=-\infty}^{\infty} \hat{\phi}(\tau/2^k) = 1 \) for any \( \tau \neq 0 \). For \( k \in \mathbb{Z} \), we put \( \hat{\phi}_k(\cdot) = \hat{\phi}(\cdot/2^k) \) and \( \psi = \sum_{j=0}^{\infty} \phi_j \). Moreover, we define \( \chi_k = \sum_{j=0}^{k} \phi_j \) for \( k \geq 1 \) and \( \chi_0 = \psi + \phi_1 + \phi_2 \). For \( s \in \mathbb{R} \) and \( 1 \leq p, q \leq \infty \), we define the Besov space

\[
B^s_{p,q}(\mathbb{R}^N) = \left\{ u \in S'((\mathbb{R}^N), \|u\|_{B^s_{p,q}(\mathbb{R}^N)} < \infty) \right\},
\]

where \( S'((\mathbb{R}^N)) \) is the space of tempered distributions on \( \mathbb{R}^N \), and

\[
\|u\|_{B^s_{p,q}(\mathbb{R}^N)} = \|\psi \ast u\|_{L^p(\mathbb{R}^N)} + \left\{ \sum_{k \geq 1} \left( 2^{sk} \|\phi_k \ast u\|_{L^p(\mathbb{R}^N)} \right)^q \right\}^{1/q}, \quad q < \infty,
\]

\[
q = \infty,
\]

where \( \ast \) denotes the convolution with respect to the variables in \( \mathbb{R}^N \). Here, we use \( \phi_k \ast u \) to denote \( \phi_k(\cdot) \ast u \). We also define \( \chi_k \ast u, \psi \ast u, \chi_0 \ast u \) similarly. This is an abuse of symbol, but no confusion is likely to arise.

For \( 1 \leq q, \alpha \leq \infty \) and a Banach space \( V \), we denote the Lebesgue space for functions on \( \mathbb{R} \) to \( V \) by \( L^q(\mathbb{R}, V) \) and the Lorentz space by \( L^{q,\alpha}(\mathbb{R}, V) \). We define the Sobolev space \( H^{1,\alpha}(\mathbb{R}, V) = \{ u : u \in L^q(\mathbb{R}, V), \partial_t u \in L^q(\mathbb{R}, V) \} \). For \( 1 \leq \alpha, r, q \leq \infty \), we denote the Chemin-Lerner type space

\[
l^{\alpha} L^{q}(\mathbb{R}, L^{r}(\mathbb{R}^{N})) = \left\{ u \in L^{\alpha}_{\text{loc}}(\mathbb{R}, L^{r}(\mathbb{R}^{N})) \right\}, \|u\|_{l^{\alpha} L^{q}(\mathbb{R}, L^{r}(\mathbb{R}^{N}))} < \infty \}
\]
with the norm defined by
\[ \|u\|_{L^q(S(R, L^r(R^N)))} = \|\psi * u\|_{L^q(S(R, L^r(R^N)))} + \left( \sum_{k \geq 1} \|\phi_k * u\|_{L^q(S(R, L^r(R^N)))}^{\alpha} \right)^{1/\alpha} \]
with trivial modification if $\alpha = \infty$. We also define $l^\alpha L^{q,\infty}(R, L^r(R^N))$ similarly. Finally, we define the Besov space of vector-valued functions. Let $\theta \in R, 1 \leq q, \alpha \leq \infty$ and $V$ be a Banach space. We put
\[ B^\theta_{q, \alpha}(R, V) = \left\{ u \in S'(R, V); \|u\|_{B^\theta_{q, \alpha}(R, V)} < \infty \right\} \]
where
\[ \|u\|_{B^\theta_{q, \alpha}(R, V)} = \|\psi * u\|_{L^q(R, V)} + \left( \sum_{k \geq 1} \left( 2^{\alpha k} \|\phi_k * u\|_{L^q(R, V)} \right) \right)^{1/\alpha} \]
with trivial modification if $\alpha = \infty$. Here $*_{t}$ denotes the convolution in $R$.

In this paper, we omit the integral domain for simplicity unless noted otherwise. For example, we write $l^\alpha L^q L^r = l^\alpha L^q (R, L^r(R^N)), L^q B_{r,2}^\sigma = L^q (R, B_{r,2}^\sigma(R^N))$ and $B_{q,2}^{\beta-\sigma/4} B_{r,2}^{\sigma/4} = B_{q,2}^{\beta-\sigma/4} (R, B_{r,2}^\sigma(R^N))$ etc.

Following standard notations, we introduce the notion of Schrödinger admissible pair as well as the corresponding Strichartz estimate for the biharmonic Schrödinger equation.

**Definition 2.1.** A pair of Lebesgue space exponents $(\gamma, \rho)$ is called biharmonic Schrödinger admissible for the equation (1.1) if $(\gamma, \rho) \in \Lambda_{b}$ where
\[ \Lambda_{b} = \left\{ (\gamma, \rho) : 2 \leq \gamma, \rho \leq \infty, \frac{4}{\gamma} + \frac{N}{\rho} = \frac{N}{2}, (\gamma, \rho, N) \neq (2, \infty, 4) \right\}, \]

**Lemma 2.2.** (Strichartz estimate for BNLS, [24]) Suppose that $(\gamma, \rho), (a, b) \in \Lambda_{b}$ are two biharmonic admissible pairs, and $\mu = 0$ or $-1$. Then, for any $u \in L^2(R^N)$ and $h \in L^a(R, L^b(R^N))$, we have
\[ \|e^{it(\Delta + \mu \Delta)} u\|_{L^\gamma L^\rho} \leq C \|u\|_{L^2}, \quad (2.1) \]
\[ \| \int_{R} e^{-is(\Delta + \mu \Delta)} h(s) ds \|_{L^2} \leq C \|h\|_{L^a L^b'}, \quad (2.2) \]
\[ \| \int_{0}^{t} e^{i(t-s)(\Delta + \mu \Delta)} h(s) ds \|_{L^\gamma L^\rho} \leq C \|h\|_{L^a L^b'}, \quad (2.3) \]

**Remark 2.3.** In the case $\mu = +1$, the Strichartz estimates (2.1)–(2.3) are only valid on the integral domain $I \times R^N$ with the time interval $|I| \leq 1$. See Proposition 3.1 of [24] for the details.

### 3. Modified Strichartz estimate

In this section, we prove Proposition 1.7. First, we prepare several lemmas. We assume that functions $\phi, \chi_{0}, \psi, \phi_{j}, \chi_{j}$ are defined in Sect. 2.

**Lemma 3.1.** Assume $N \geq 1, \mu = 0$ or $-1$, and $K(t, x), K_{j}(t, x)$ for $j \geq 1 : R \times R^N \rightarrow C$ are defined by
\[ K(t, x) = \frac{1}{(2\pi)^{1+N}} \int e^{it\tau + ix \xi} \hat{\psi}(\xi) \frac{\xi^{4} - \mu |\xi|^{2}(1 - \hat{\chi}_{0}(\tau))}{i(\tau - |\xi|^{4} + \mu |\xi|^{2})} d\tau d\xi, \]
\[ K_{j}(t, x) = \frac{1}{(2\pi)^{1+N}} \int e^{it\tau + ix \xi} \hat{\phi}_{j}(\xi) \frac{\xi^{4} - \mu |\xi|^{2}(1 - \hat{\chi}_{j}(\tau))}{i(\tau - |\xi|^{4} + \mu |\xi|^{2})} d\tau d\xi. \]
Then, for any $0 < \theta < 1$, $1 \leq q \leq \infty$, $1 \leq r \leq \infty$ with $\frac{4}{q} - N(1 - \frac{1}{r}) = 4\theta$, we have
\[ \|K\|_{L^{q,r},L^{r}} \leq C, \quad \text{and} \quad \|K_{j}\|_{L^{q,r},L^{r}} \leq C2^{-j\theta}, \]
where the constant $C$ is independent of $j \geq 1$.

Proof. The method used here is inspired by the last part of Lemma 2.4 in [27]. We shall prove the estimate for $K_{j}(t,x)$, and the estimate for $K(t,x)$ can be treated similarly.

Define $\chi = \sum_{j=2}^{\infty} \phi_{j}$. To avoid the possible confusion, we remark here that $\chi$ is different with the function $\chi_{0}$ defined in the Preliminary set. Set
\[ \tilde{L}_{j}(\tau,\xi) = \frac{\mathcal{F}((\xi^{4} - \mu 2^{-j/2}|\xi|^{2})(1 - \hat{\chi}(\tau)))}{i(\tau - |\xi|^{4} + \mu 2^{-j/2}|\xi|^{2})}, \quad j \geq 1. \]

Then, by Fourier transform
\[ \tilde{K}_{j}(\tau,\xi) = \frac{\mathcal{F}((\xi^{4} - \mu 2^{-j/2}|\xi|^{2})(1 - \hat{\chi}(\tau)))}{i(\tau - |\xi|^{4} + \mu 2^{-j/2}|\xi|^{2})} \]
so that $K_{j}(t,x) = 2^{Nj/4}L_{j}(2^{j/2}t,2^{j/2}x)$. Moreover, by the change of variables, we have
\[ \|K_{j}\|_{L^{q,r},L^{r}} = 2^{j(\frac{4}{q} - 2 - \frac{1}{r})}\|L_{j}\|_{L^{q,r},L^{r}} = 2^{-j\theta}\|L_{j}\|_{L^{q,r},L^{r}}. \]
Therefore, it suffices to show that for any $m \geq 1$, there exists $C > 0$ independent of $j \geq 1$ such that
\[ |L_{j}(t,x)| \leq C(1 + |t| + |x|)^{-m}, \quad \forall (t,x) \in \mathbb{R} \times \mathbb{R}^{N}. \] (3.1)

We now prove (3.1). By Fourier inversion formula
\[ L_{j}(t,x) = \frac{1}{(2\pi)^{1+N}} \int_{\mathbb{R}^{1+N}} e^{it\tau + ix\xi} \frac{\mathcal{F}((\xi^{4} - \mu 2^{-j/2}|\xi|^{2})(1 - \hat{\chi}(\tau)))}{i(\tau - |\xi|^{4} + \mu 2^{-j/2}|\xi|^{2})} d\tau d\xi. \]
Note that on the support of the integrand of $L_{j}$, we must have $|\tau| \notin [1/4,4]$ and $|\xi|^{4} - \mu 2^{-j/2}|\xi|^{2} \in [1/2,2]$, so that $|\tau - |\xi|^{4} + \mu 2^{-j/2}|\xi|^{2}| \geq 1/4$. Therefore, we deduce that the following integral
\[ \int_{|\tau - |\xi|^{4} + \mu 2^{-j/2}|\xi|^{2}| \leq 10} e^{it\tau + ix\xi} \frac{\mathcal{F}((\xi^{4} - \mu 2^{-j/2}|\xi|^{2})(1 - \hat{\chi}(\tau)))}{i(\tau - |\xi|^{4} + \mu 2^{-j/2}|\xi|^{2})} d\tau d\xi \] (3.2)
is bounded. On the other hand, note that $|\tau - |\xi|^{4} + \mu 2^{-j/2}|\xi|^{2}| \geq 10$ and $1/2 \leq |\xi|^{4} - \mu 2^{-j/2}|\xi|^{2}| \leq 2$ imply that $|\tau| \geq 8$; so that $\hat{\chi}(\tau) = 0$. Thus, we have by the change of variables
\[ \int_{|\tau - |\xi|^{4} + \mu 2^{-j/2}|\xi|^{2}| \geq 10} e^{it\tau + ix\xi} \frac{\mathcal{F}((\xi^{4} - \mu 2^{-j/2}|\xi|^{2})(1 - \hat{\chi}(\tau)))}{i(\tau - |\xi|^{4} + \mu 2^{-j/2}|\xi|^{2})} d\tau d\xi \]
\[ = \int_{|\tau - |\xi|^{4} + \mu 2^{-j/2}|\xi|^{2}| \geq 10} e^{it\tau} \frac{\mathcal{F}((\xi^{4} - \mu 2^{-j/2}|\xi|^{2}))}{i(\tau - |\xi|^{4} + \mu 2^{-j/2}|\xi|^{2})} \int_{10/|t|}^{\infty} \frac{\sin \tau}{\tau} d\tau d\xi. \]
Since $h(x) = \int_{x}^{\infty} \frac{\sin \tau}{\tau} d\tau$ is a bounded function on $\mathbb{R}$ and $|\xi| \leq 2^{1/4}$ on the support of $\mathcal{F}((\xi^{4} - \mu 2^{-j/2}|\xi|^{2}))$, we deduce that the integral
\[ \int_{|\tau - |\xi|^{4} + \mu 2^{-j/2}|\xi|^{2}| \geq 10} e^{it\tau + ix\xi} \frac{\mathcal{F}((\xi^{4} - \mu 2^{-j/2}|\xi|^{2})(1 - \hat{\chi}(\tau)))}{i(\tau - |\xi|^{4} + \mu 2^{-j/2}|\xi|^{2})} d\tau d\xi \] (3.3)
is also bounded, which together with (3.2) proved the boundedness of $L_{j}(t,x)$.

Moreover, for $1 \leq l \leq N$, the integration by parts shows that
\[ x_{l}L_{j}(t,x) \]
\[
\begin{align*}
= \frac{1}{(2\pi)^{1+N}} \int_{\mathbb{R}^{1+N}} e^{it\tau + ix \cdot \xi} \partial_{\xi} \dot{\phi}((|\xi|^4 - \mu 2^{-j/2}|\xi|^2)(1 - \tilde{\chi}(\tau)) \, d\tau d\xi \\
= \frac{1}{(2\pi)^{1+N}} \int_{\mathbb{R}^{1+N}} e^{it\tau + ix \cdot \xi} (4|\xi|^2 \xi_i - 2\mu 2^{-j/2} \xi_i) \left( \frac{\dot{\phi}'((|\xi|^4 - \mu 2^{-j/2}|\xi|^2)(1 - \tilde{\chi}(\tau))}{\tau - |\xi|^4 + \mu 2^{-j/2}|\xi|^2} \right) d\tau d\xi + \frac{\dot{\phi}((|\xi|^4 - \mu 2^{-j/2}|\xi|^2)(1 - \tilde{\chi}(\tau))}{|\tau - |\xi|^4 + \mu 2^{-j/2}|\xi|^2} \right) d\tau d\xi. \tag{3.4}
\end{align*}
\]

The right-hand side of (3.4) is bounded as before. Similarly, \(tL_j(t, x)\) is also bounded. By repeating the latter argument, we can obtain the desired estimate (3.1). \(\square\)

**Lemma 3.2.** Let \(N \geq 1, \ 0 < \theta < 1, \ 1 \leq r_0, \tau, \varphi, \gamma \leq \infty, \ 1 < q_0, \rho < \infty.\) Assume that \(2 \leq \tau \leq \infty, \ \frac{4}{\varphi} - N\left(\frac{1}{2} - \frac{1}{\varphi}\right) = \frac{4}{q_0} - N\left(\frac{1}{2} - \frac{1}{q_0}\right) = 4(1 - \theta), (\gamma, \rho) \in \Lambda_b\) and \(r_0\) satisfies \(r^* \leq r_0 < \tau\) or \(\tau < r_0 \leq r^*\) for any function \(f \in L^2\mathcal{L}_p \cap B^{n/2}_{\gamma, 2, \rho'}\), we have
\[
\|f\|_{L^2 L_{r_0} \rightarrow L_{r_0}^\infty} \lesssim \|f\|_{L^2 \mathcal{L}_p^*} + \|f\|_{B^{n/2}_{\gamma, 2, \rho'}}.
\]

**Proof.** The proof is an obvious adaptation of Lemma 2.5 in [22]. \(\square\)

**Lemma 3.3.** Let \(s \in \mathbb{R}, \ 1 \leq p, q \leq \infty, \ \mu = 0\) or \(-1\), then the norm defined by
\[
\|u\|_{B^s_{p, q} (\mathbb{R}^n)} := \left\| \left( F_{\xi}^{-1} \left( \hat{\psi} \left( \left( |\xi|^4 - \mu |\xi|^2 \right) \right) \right) \right) \ast_x u \right\|_{L^p(\mathbb{R}^n)}
\]

and
\[
\sup_{j \geq 1} \left\| \left( F_{\xi}^{-1} \left( \hat{\psi} \left( \left( |\xi|^4 - \mu |\xi|^2 \right) \right) \right) \right) \ast_x u \right\|_{L^p(\mathbb{R}^n)}, \quad \text{if} \ q < \infty,
\]

is equivalent to the norm \(\|u\|_{B^s_{p, q} (\mathbb{R}^n)}\) for any function \(u\).

**Proof.** For the sake of convenience and completeness, we briefly sketch the proof. Indeed, readers seeking a comprehensive treatment of certain details may consult Lemma 2.3 of [22].

Firstly, we show that \(\|u\|_{B^s_{p, q} (\mathbb{R}^n)} \lesssim \|u\|_{B^s_{p, q} (\mathbb{R}^n)}\). Since
\[
\sum_{k=-6}^{7} \hat{\phi} \left( \left( |\xi|^4 - \mu |\xi|^2 \right) 2^{4j+k} \right) = 1
\]
on the support of \(\hat{\phi} \left( \left( |\xi|^4 - \mu |\xi|^2 \right) 2^{4j+k} \right)\), it follows from Young’s inequality that
\[
2^{js} \|\hat{\phi} \ast_x u\|_{L^p(\mathbb{R}^n)} = 2^{js} \left\| \left( F_{\xi}^{-1} \hat{\phi} \left( \left( |\xi|^4 - \mu |\xi|^2 \right) 2^{4j+k} \right) \right) \ast_x u \right\|_{L^p(\mathbb{R}^n)}
\]

\[
\lesssim 2^{js} \sum_{k=-4}^{5} \left\| \left( F_{\xi}^{-1} \hat{\phi} \left( \left( |\xi|^4 - \mu |\xi|^2 \right) 2^{4j+k} \right) \right) \ast_x u \right\|_{L^p(\mathbb{R}^n)}
\]

\[
\lesssim \sum_{k=-4}^{5} 2^{js} \left\| \left( F_{\xi}^{-1} \hat{\phi} \left( \left( |\xi|^4 - \mu |\xi|^2 \right) 2^{4j+k} \right) \right) \ast_x u \right\|_{L^p(\mathbb{R}^n)},
\]

where \(l = 4j + k\). Similar inequality for the low frequency part also holds. Taking the \(l^q(\mathbb{Z})\)-norm, we obtain the desired inequality \(\|u\|_{B^s_{p, q} (\mathbb{R}^n)} \lesssim \|u\|_{B^s_{p, q} (\mathbb{R}^n)}\).

Next, we show the opposite inequality. Note that \(\sum_{k=-4}^{5} \hat{\phi} \left( \left( |\xi|^4 - \mu |\xi|^2 \right) 2^{4j+k} \right) = 1\) on the support of \(\hat{\phi} \left( \left( |\xi|^4 - \mu |\xi|^2 \right) 2^{4j+k} \right)\), so that
\[
2^{js/4} \left\| F_{\xi}^{-1} \hat{\phi} \left( \left( |\xi|^4 - \mu |\xi|^2 \right) 2^{4j} \right) \ast_x u \right\|_{L^p(\mathbb{R}^n)}
\]
Taking the inverse Fourier transform, we obtain (3.6).

\[ \|u\|_{B^s_{p,q}} \approx \left\| \mathcal{F}^{-1}_\xi \left( \hat{\phi} \left( \|\xi\|^2 - \mu \right) \right) \right\|_{L^p} + \left\{ \sum_{j=1}^{\infty} \left( 2^{sj/4} \| \phi_{j/4} * u \|_{L^p} \right)^q \right\}^{1/q} \]  

(3.5)

with trivial modification if \( q = \infty \). Then, we claim that for any \( f : \mathbb{R}^N \to \mathbb{C} \), we have

\[ \phi_j * t e^{it(\Delta^2 + \mu \Delta)} f = e^{it(\Delta^2 + \mu \Delta)} \phi_{j/4} * f. \]  

(3.6)

In fact, by Fourier transform

\[ \left( \hat{\phi} * t e^{it(\Delta^2 + \mu \Delta)} f \right)(t, \xi) = \int_{-\infty}^{\infty} \phi_j(\tau) e^{i(t-\tau)(\|\xi\|^2 - \mu \|\xi\|^2)} \hat{f}(\xi) d\tau \]

\[ = e^{it(\|\xi\|^2 - \mu \|\xi\|^2)} \hat{f}(\xi) \hat{\phi}_j \left( \|\xi\|^2 - \mu \|\xi\|^2 \right). \]

Taking the inverse Fourier transform, we obtain (3.6).

We now resume the proof of Proposition 1.7. We separate the proof into three parts.

The proof of the inequality (1.4). Using the same method as that used to prove Corollary 2.3.9 in \cite{3}, we deduce that \( e^{it(\Delta^2 + \mu \Delta)} u_0 \in C \left( \mathbb{R}, H^{4\theta} \right) \) and

\[ \left\| e^{it(\Delta^2 + \mu \Delta)} u_0 \right\|_{L^q B^{\frac{\theta}{2} - \sigma/4}_{q,2}} \lesssim \| u_0 \|_{H^{4\theta}}. \]  

(3.7)

It remains to estimate \( \left\| e^{it(\Delta^2 + \mu \Delta)} u_0 \right\|_{B^{\frac{\theta}{2} - \sigma/4}_{q,2}} \). Applying (3.6) and the Strichartz estimate (2.1), we conclude that

\[ \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} 2^{(2\theta - \sigma/2)j + \sigma k/2} \left\| \phi_j * t \phi_k \ast x e^{it(\Delta^2 + \mu \Delta)} u_0 \right\|_{L^q L^r}^2 \]

\[ \lesssim \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} 2^{(2\theta - \sigma/2)j + \sigma k/2} \left\| \phi_{j/4} * x \phi_{k/4} \ast u_0 \right\|_{L^2}^2 \]

\[ \lesssim \sum_{k=1}^{\infty} 2^{2\theta k} \left\| \phi_{k/4} \ast u_0 \right\|_{L^2}^2 \lesssim \left\| u_0 \right\|_{H^{4\theta}}^2, \]

where we used (3.5) and the fact \( \hat{\phi}_j \left( \|\xi\|^2 - \mu \|\xi\|^2 \right) \hat{\phi}_k \left( \|\xi\|^2 - \mu \|\xi\|^2 \right) = 0 \) whenever \( |j - k| \geq 2 \). Since the low-frequency parts are easier to treat, we obtain

\[ \left\| e^{it(\Delta^2 + \mu \Delta)} u_0 \right\|_{B^{\frac{\theta}{2} - \sigma/4}_{q,2}} \lesssim \left\| u_0 \right\|_{H^{4\theta}}. \]

This inequality together with (3.7) yields (1.4).
The proof of the inequality (1.5). Taking Fourier transform, we get
\[(\hat{G}f)(t, \xi) = \int_0^t e^{i(t-s)(|\xi|^4 - \mu|\xi|^2)} f(s, \xi) ds\]
\[= \frac{1}{2\pi} \int_0^t e^{i(t-s)(|\xi|^4 - \mu|\xi|^2)} ds \int_{-\infty}^\infty \hat{f}(\tau, \xi) e^{i\tau s} d\tau\]
\[= \int_{-\infty}^\infty \frac{e^{it\tau} - e^{it(|\xi|^4 - \mu|\xi|^2)}}{2\pi i (\tau - |\xi|^4 + \mu|\xi|^2)} \hat{f}(\tau, \xi) d\tau. \tag{3.8}\]

We then multiply (3.8) with \(\hat{\phi}_k(|\xi|^4 - \mu|\xi|^2)\) to obtain
\[
\hat{\phi}_k(|\xi|^4 - \mu|\xi|^2)(\hat{G}f)(t, \xi) = \int_{-\infty}^\infty \frac{e^{it\tau} \hat{\phi}_k(|\xi|^4 - \mu|\xi|^2) \hat{\chi}_k(\tau)}{2\pi i (\tau - |\xi|^4 + \mu|\xi|^2)} \hat{f}(\tau, \xi) d\tau
\[+ \int_{-\infty}^\infty \frac{e^{it\tau} \hat{\phi}_k(|\xi|^4 - \mu|\xi|^2)(1 - \hat{\chi}_k(\tau))}{2\pi i (\tau - |\xi|^4 + \mu|\xi|^2)} \cdot \hat{\chi}_k(|\xi|^4 - \mu|\xi|^2) \hat{f}(\tau, \xi) d\tau
\[= \int_{-\infty}^\infty \frac{e^{it\tau} \hat{\phi}_k(|\xi|^4 - \mu|\xi|^2)(1 - \hat{\chi}_k(\tau))}{2\pi i (\tau - |\xi|^4 + \mu|\xi|^2)} \cdot \hat{\chi}_k(|\xi|^4 - \mu|\xi|^2) \hat{f}(\tau, \xi) d\tau \tag{3.9}\]

where we used the face that \(\hat{\chi}_k = 1\) on the support of \(\hat{\phi}_k\). Since
\[
\mathcal{F}_\tau^{-1}\left\{ \frac{1}{i(\tau - |\xi|^4 + \mu|\xi|^2)} \right\}(t) = \frac{1}{2\pi} P.V. \int_{-\infty}^\infty \frac{e^{it\tau}}{i(\tau - |\xi|^4 + \mu|\xi|^2)} d\tau
\[= \frac{1}{2\pi} e^{it(|\xi|^4 - \mu|\xi|^2)} P.V. \int_{-\infty}^\infty \frac{e^{it\tau}}{i\tau} d\tau
\[= \frac{1}{2} \text{sign}(t) e^{it(|\xi|^4 - \mu|\xi|^2)},
\]

it follows from (3.9) that
\[
\phi_{k/4 \ast x} (Gf) = \frac{1}{2} \int_{-\infty}^\infty \text{sign}(t-s) e^{i(t-s)(\Delta^2 + \mu\Delta)} (\phi_{k/4 \ast x} \chi_k \ast t f)(s) ds
\[+ K_{k \ast t, x} \chi_{k/4} \ast x f
\[- \frac{1}{2} e^{it(\Delta^2 + \mu\Delta)} \int_{-\infty}^\infty \text{sign}(-s) e^{is(\Delta^2 + \mu\Delta)} (\phi_{k/4 \ast x} \chi_k \ast t f)(s) ds
\[= -e^{it(\Delta^2 + \mu\Delta)} \{ K_{k \ast t, x} \chi_{k/4} \ast x f \}|_{t=0}, \tag{3.10}\]

where \(K_j\) is the function defined in Lemma 3.1. Applying (3.10), the Strichartz estimates (2.1)–(2.3), we conclude that
\[
\|\phi_{k/4 \ast x} (Gf)\|_{L^tL^r} \lesssim \|\chi_k \ast t f\|_{L^tL^r} + \|K_{k \ast t, x} \chi_{k/4} \ast x f\|_{L^tL^r \cap L^\infty L^2}. \tag{3.11}\]

Next, we estimate \(\|K_{k \ast t, x} \chi_{k/4} \ast x f\|_{L^tL^r \cap L^\infty L^2}.\) Let
\[
r_0 = \begin{cases} r, & \text{if } 1 \leq r \leq 2, \\ 2, & \text{if } r \geq 2, \end{cases}
\]\[
r_1 = \begin{cases} r, & \text{if } 1 \leq r \leq 2, \\ 2, & \text{if } r \geq 2, \end{cases}
\]

We then define \(q_0, \tilde{q}_0, \tilde{q}_0, q_1, \tilde{q}_1, q_1\) such that \(\frac{4}{q_0} - N(\frac{1}{2} - \frac{1}{r_0}) = 4(1 - \theta), 1 + \frac{1}{r} = \frac{1}{r_0} + \frac{1}{r_0}, 1 = \frac{1}{q_0} + \frac{1}{q_0}\) and \(\frac{4}{q_1} - N(\frac{1}{2} - \frac{1}{r_1}) = 4(1 - \theta), 1 + \frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_1}, 1 + \frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_1}.\) Then, it is easy to check that \(1 \leq r_0, q_0, \tilde{r}_0, \tilde{q}_0,\)
The proof of the inequality where we used Lemma 3.2 when 

Combining the above two inequalities, we obtain

Estimates (3.11), (3.12) and (3.13) imply

Similarly,

It now follows from (3.14), (3.15) and (3.5) that

where we used Lemma 3.2 when \( r \geq 2 \). This proves (1.5).

The proof of the inequality (1.6). The definition of Besov norm and (3.5), we have

where \( J = \left\{ \sum_{j=1}^{\infty} \sum_{k=1}^{2^{(2\theta-\sigma/2)j}+\sigma k/2} \| \varphi_j \ast t \varphi_{j+k} \ast x (Gf) \|_{L^2 \mathbb{R}^d} \right\}^{1/2} \). Since \( \| Gf \|_{L^2 \mathbb{R}^d} \) can be controlled by (1.5), it suffices to estimate the last two terms in (3.16).

We first estimate \( \| Gf \|_{B^{\sigma,\ell}_{q,r} \mathbb{R}^d} \). Since \( \varphi_j \ast t e^{ita} = e^{ita} \hat{\varphi}_j (a) \), for any \( a \in \mathbb{R} \), it follows from (3.8) that

so that

This together with the Strichartz estimates (2.1)–(2.3) and (3.12) implies

Similarly,

Combining the above two inequalities, we obtain

\[ \| Gf \|_{L^{p,q} \mathbb{R}^d} \lesssim \| f \|_{L^{p,q} \mathbb{R}^d} + \| f \|_{L^{p,q} \mathbb{R}^d} + \| f \|_{L^{p,q} \mathbb{R}^d}, \]

\[ \| Gf \|_{L^{p,q} \mathbb{R}^d} \lesssim \| f \|_{L^{p,q} \mathbb{R}^d} + \| f \|_{L^{p,q} \mathbb{R}^d} + \| f \|_{L^{p,q} \mathbb{R}^d}. \]
where we used Lemma 3.2 when \( r \geq 2 \).

Next, we estimate \( J \). Note that by (3.10) and (3.6)

\[
\phi_j * t \phi_{k/4} * x (Gf) \\
= \frac{1}{2} \int_{-\infty}^{\infty} \text{sign}(t - s) e^{i(t - s)(\Delta^2 + \mu \Delta)} (\phi_j * t \phi_{k/4} * x \chi_k * t f)(s) ds \\
+ K_k * t, x \phi_j * t \chi_{k/4} * x f \\
- \frac{1}{2} e^{it(\Delta^2 + \mu \Delta)} \int_{-\infty}^{\infty} \text{sign}(-s) e^{is(\Delta^2 + \mu \Delta)} (\phi_{j/4} * x \phi_{k/4} * x \chi_k * t f)(s) ds \\
- e^{it(\Delta^2 + \mu \Delta)} \{ K_k * t, x \phi_j * x \chi_{k/4} * x f \} |_{t = 0},
\]

so that

\[
J \lesssim \left( \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} 2^{(2j - \sigma/2)j + \sigma k/2} \| I_{j,k} \|_{L^q L^r}^2 \right)^{1/4} + \left( \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} 2^{(2j - \sigma/2)j + \sigma k/2} \| II_{j,k} \|_{L^q L^r}^2 \right)^{1/4} \\
+ \left( \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} 2^{(2j - \sigma/2)j + \sigma k/2} \| III_{j,k} \|_{L^q L^r}^2 \right)^{1/4} + \left( \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} 2^{(2j - \sigma/2)j + \sigma k/2} \| IV_{j,k} \|_{L^q L^r}^2 \right)^{1/4} \\
=: J_1 + J_2 + J_3 + J_4.
\]

We first consider \( J_1, J_3 \). By the Strichartz estimates (2.1)-(2.3), we get

\[
\| I_{j,k} \|_{L^q L^r} \lesssim \| \phi_j * t \chi_k * t f \|_{L^{q,f} L^{r,f}}, \quad \| III_{j,k} \|_{L^q L^r} \lesssim \| \phi_{j/4} * x \phi_{k/4} * x \chi_k * t f \|_{L^{q,f} L^{r,f}}.
\]

Since \( \phi_j * t \chi_k = 0 \) whenever \( |j - k| \geq 4 \), we have

\[
J_1^2 \lesssim \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} 2^{(2j - \sigma/2)j + \sigma k/2} \| \phi_j * t \chi_k * t f \|_{L^{q,f} L^{r,f}}^2 \\
\lesssim \sum_{j=1}^{\infty} 2^{2j} \| \phi_j * t f \|_{L^{q,f} L^{r,f}}^2 \lesssim \| f \|_{B^{\sigma/2}_{q,f} L^{r,f}}^2.
\]

Similarly, we have \( J_3 \lesssim \| f \|_{B^{\sigma/2}_{q,f} L^{r,f}}. \)

For \( J_4 \), we deduce from (2.3) and (3.12) that

\[
\| IV_{j,k} \|_{L^q L^r} \lesssim 2^{-k\theta} \| \phi_{j/4} * x \chi_{k/4} * x f \|_{L^{q,0} \rightarrow L^{r,0}}.
\]

Since \( \phi_{j/4} * x \chi_{k/4} = 0 \) whenever \( |j - k| \geq 4 \), we conclude that

\[
J_4 \lesssim \| f \|_{L^{q,0} \rightarrow L^{r,0}} \lesssim \| f \|_{B^{\sigma/2}_{q,f} L^{r,f}} + \| f \|_{L^{q,1} L^{r,1}},
\]

where we used Lemma 3.2 when \( r \geq 2 \).

Our final step is to estimate \( J_2 \). Similar to (3.13), we have

\[
\| II_{j,k} \|_{L^q L^r} \lesssim 2^{-k\theta} \| \phi_j * t \chi_{k/4} * x f \|_{L^{q,1} \rightarrow L^{r,1}}.
\]

On the other hand, by Young’s inequality

\[
\| II_{j,k} \|_{L^q L^r} \lesssim \| \phi_j * t f \|_{L^{q,f} L^{r,f}}.
\]
It follows from (3.17) and (3.18) that
\[
J_2 \lesssim \left( \sum_{j=1}^{\infty} \sum_{k=1}^{j} 2^{(2\theta-\sigma/2)j+\sigma k/2} \left\| \phi_j \ast_2 f \right\|_{L^{r_1} L^{r_1}}^2 \right)^{1/2} \\
+ \left( \sum_{j=1}^{\infty} \sum_{k=j+1}^{\infty} 2^{(2\theta-\sigma/2)(j-k)} \left\| \phi_j \ast_4 \chi_{k/4} \ast \phi_s \right\|_{L^{r_1} L^{r_1}}^2 \right)^{1/2}
\]
=: J_{2,1} + J_{2,2}.

Since \( \sum_{k=1}^{j} 2^{\sigma k/2} \lesssim 2^{\sigma j/2} \), we have \( J_{2,1} \lesssim \|f\|_{B^\theta_{r_1,2} L^{r_1}} \). To estimate \( J_{2,2} \), we interchange the order of the summation to obtain
\[
J_{2,2}^2 = \sum_{k=1}^{\infty} \sum_{j=1}^{k-1} 2^{(2\theta-\sigma/2)(j-k)} \left\| \phi_j \ast_2 \chi_{k/4} \ast \phi_s \right\|_{L^{r_1} L^{r_1}}^2
\leq \sum_{k=1}^{\infty} \left\| \chi_{k/4} \ast \phi_s \right\|_{L^{r_1} L^{r_1}}^2 \lesssim \|f\|_{B^2_{r_1,1} L^{r_1}}^2.
\]
Collecting these estimates, we obtain
\[
J \lesssim \|f\|_{B^\theta_{r_1,2} L^{r_1}} + \|f\|_{B^2_{r_1,1} L^{r_1}} + \|f\|_{B^2_{r_1,1} L^{r_1}} \lesssim \|f\|_{B^\theta_{r_1,2} L^{r_1}} + \|f\|_{B^2_{r_1,1} L^{r_1}},
\]
where we used Lemma 3.2 when \( \bar{r} \geq r \). This finishes the proof of (1.6).

Finally, the continuity of \( Gf \) in time follows from a density argument. This completes the proof of Proposition 1.7. \( \square \)

4. Proof of Theorem 1.4

In this section, we prove Theorem 1.4. Firstly, we recall two lemmas that we will need to employ a contraction argument.

**Lemma 4.1.** ([10], Lemma 3.4) Assume \( \alpha > 0, 0 \leq s < \alpha + 1, f \in C(\alpha) \). and \( 1 < p, r, \rho < \infty \) satisfy \( \frac{1}{p} = \frac{s}{\alpha} + \frac{1}{r} \). Then, for any \( u \in L^p \cap B^\rho_{r,2} \), we have
\[
(i) \text{ if } s \in \mathbb{Z}, \quad \left\| f(u) \right\|_{H^s L^r} \lesssim \left\| u \right\|_{L^\rho} \left\| u \right\|_{H^s L^r},
\]
\[
(ii) \text{ if } s \notin \mathbb{Z}, \quad \left\| f(u) \right\|_{B^\rho_{r,2}} \lesssim \left\| u \right\|_{L^\rho} \left\| u \right\|_{B^\rho_{r,2}}.
\]

**Lemma 4.2.** ([23], Lemma 2.3) Assume \( 0 < s < 8, s \neq 4, \max \left\{ 0, \frac{s}{4} - 1 \right\} < \alpha \) and \( 1 \leq \rho, r_0, r, q_0, q, r \leq \infty \). Assume also that \( 1 < \gamma, q < \infty \) and \( \gamma, \rho, q_0, r_0, q, r \) satisfy \( \frac{1}{\gamma} = \frac{\alpha}{q_0} + \frac{1}{q} + \frac{1}{r} = \frac{\alpha}{r_0} + \frac{1}{r} \). Then for any \( f \in C(\alpha) \) and \( u \in L^{q_0} L^{r_0} \cap B^\rho_{r_0,2} L^r \), we have
\[
\left\| f(u) \right\|_{B^\rho_{r_0,2} L^{r_0}} \lesssim \left\| u \right\|_{L^{q_0} L^{r_0}} \left\| u \right\|_{B^\rho_{r_0,2} L^r}.
\]

We regard the solution of the Cauchy problem (1.1) as the fixed point of the integral equation given by
\[
u(t) = (Su)(t) = e^{it(\Delta^2 + \mu \Delta)} \phi + i \int_0^t e^{i(t-s)(\Delta^2 + \mu \Delta)} f(u)(s) \, ds,
\]
for $t \in \mathbb{R}$, where $u(t) := u(t, \cdot)$. Note that $Su$ satisfies
\[
\begin{cases}
  i \partial_t (Su) + \Delta^2 (Su) + \mu \Delta (Su) + f(u) = 0, \\
  (Su)(0) = \phi,
\end{cases}
\tag{4.2}
\] and that
\[
\partial_t (Su) = ie^{it(\Delta^2 + \mu \Delta)} [((\Delta^2 + \mu \Delta) \phi + f(\phi)] + i \int_0^t e^{i(t-s)(\Delta^2 + \mu \Delta)} \partial_s f(u)(s) \, ds. \tag{4.3}
\]

We can now give the proof of Theorem 1.4. We consider four cases: $0 < s < 4$, $s = 4$, $4 < s < 6$ and $6 \leq s < 8$.

### 4.1. The case $0 < s < 4$

Throughout this subsection, we fix
\[
\gamma = \frac{2N + 8}{N}, \quad \rho = \frac{2N + 8}{N}. \tag{4.4}
\]
We then define $q, r, \beta, \tau$ such that
\[
\beta = \gamma', \quad 4 - \frac{N}{q} - N \left(\frac{1}{2} - \frac{1}{\tau}\right) = 4 - s
\tag{4.5}
\]
and
\[
\frac{1}{\beta} = \frac{1 + \alpha}{q}, \quad \frac{1}{\tau} = \alpha \left(\frac{1}{r} - \frac{s}{N}\right) + \frac{1}{r}.
\]
Since $\alpha = \frac{8}{N-2s}$, $0 < s < \min\{\frac{N}{2}, 4\}$, it is straightforward to verify that $(\gamma, \rho), (q, r) \in \Lambda_b$ are two biharmonic admissible pairs, $1 < q < 2 < \frac{N}{2}$, and $r < \frac{N}{s}$.

Assume $\|\phi\|_{H^s}$ sufficiently small such that
\[
(2C_1)^{\alpha+1} \|\phi\|_{H^s}^\alpha \leq 1, \quad C_2 (2C_1 \|\phi\|_{H^s})^\alpha \leq \frac{1}{2}, \tag{4.6}
\]
where $C_1, C_2$ are constants in (4.14) and (4.15), respectively. Set $M = 2C_1 \|\phi\|_{H^s}$ and consider the metric space (recall that $\| \cdot \|_{Y \cap Y'} = \| \cdot \|_Y + \| \cdot \|_{Y'}$ with the two norms $Y, Y'$)
\[
X_M = \left\{ u \in L^\infty H^s \cap L^q B_{r,2}^s \cap B_{q,2}^{s/4} L^r : \|u\|_{L^\infty H^s \cap L^q B_{r,2}^s \cap B_{q,2}^{s/4} L^r} \leq M \right\}.
\]
It follows that $X_M$ is a complete metric space when equipped with the distance
\[
d(u, v) = \|u - v\|_{L^\infty H^s \cap L^q B_{r,2}^s \cap B_{q,2}^{s/4} L^r}. \tag{4.7}
\]
Next, we show that the map $S$, defined in (4.1), is a contraction on the space $X_M$.

We first show that $S$ maps $X_M$ into itself. Using Proposition 1.7, we get
\[
\|Su\|_{L^\infty H^s \cap L^q B_{r,2}^s \cap B_{q,2}^{s/4} L^r} \lesssim \|\phi\|_{H^s} + \|f(u)\|_{B^{s/4}_{q',2} L^{r'}} + \|f(u)\|_{L^2 L^{r'}}.
\tag{4.8}
\]
Since $\frac{1}{\beta} = \frac{\alpha+1}{q}$ and $\frac{1}{\rho} = \alpha \frac{N+\epsilon}{N r} + \frac{1}{r}$, we deduce from Lemma 4.2 and Sobolev embedding $B_{r,2}^s(\mathbb{R}^N) \hookrightarrow L^{N \frac{N-\epsilon}{N r}}(\mathbb{R}^N)$ that
\[
\|f(u)\|_{B^{s/4}_{q',2} L^{r'}} \lesssim \|u\|_{L^{\infty} B_{q,2}^{s/4} L^r}^{\alpha} \|u\|_{L^{N \frac{N-\epsilon}{N r}} B_{q,2}^{s/4} L^r} \lesssim \|u\|_{L^\infty B_{q,2}^{s/4} L^r}^{\alpha} \|u\|_{L^N B_{q,2}^{s/4} L^r}.
\tag{4.9}
\]
Next, we estimate $\|f(u)\|_{L^2 L^{r'}}$. Since $r, \tau > 2$, we can choose $\varepsilon > 0$ sufficiently small and $\rho_\varepsilon, q_\varepsilon > 2$ such that
\[
\frac{1}{\tau} = \frac{1}{\rho_\varepsilon} - \frac{\varepsilon}{N}, \quad \frac{1}{q_\varepsilon} = \frac{1}{r} - \frac{s - \varepsilon}{N}. \tag{4.10}
\]
Then, we deduce from Minkowski inequality ($\bar{\eta} = 2\gamma \leq 2$) and Sobolev embedding $B^\infty_{p,2}(\mathbb{R}^N) \hookrightarrow B^0_{r,2}(\mathbb{R}^N)$ that
\[
\|f(u)\|_{L^2 \cap L^\infty} \lesssim \|f(u)\|_{L^{s'} \cap B^0_{r,2}} \lesssim \|f(u)\|_{L^{s'} \cap B^0_{r,2}}.
\] (4.11)
Moreover, since $\frac{1}{p'} = \alpha \left(\frac{1}{r} - \frac{N}{N\alpha} \right) + \frac{1}{q} \alpha$ by (4.4), (4.5) and (4.10), it follows from Lemma 4.1 and Sobolev embedding $B^s_{r,2}(\mathbb{R}^N) \hookrightarrow L^{\frac{N}{N-r}}(\mathbb{R}^N)$ that
\[
\|f(u)\|_{B^s_{r,2}} \lesssim \|u\|_{L^\alpha \cap B^s_{r,2}} \lesssim \|u\|_{B^s_{r,2}} \lesssim \|u\|_{B^s_{r,2}}.
\] (4.12)
Estimates (4.11), (4.12) and Hölder’s inequality imply
\[
\|f(u)\|_{L^2 \cap L^\infty} \lesssim \|u\|_{L^\alpha \cap L^2 \cap B^s_{r,2}} \lesssim \|u\|_{L^\alpha \cap L^2 \cap B^s_{r,2}},
\] (4.13)
where we used the embedding $B^s_{r,2}(\mathbb{R}^N) \hookrightarrow B^s_{q,2}(\mathbb{R}^N)$ (see (4.10)) in the second inequality. It now follows from (4.8), (4.9) and (4.13) that, for any $u \in X_M$,
\[
\|Su\|_{L^\infty \cap L^2 \cap B^s_{r,2} \cap B^{s/4}_{q,2}} \leq C_1 \|\phi\|_{H^s} + C_4 \|u\|_{L^\alpha \cap L^2 \cap B^s_{r,2} \cap B^{s/4}_{q,2}} \leq M, \tag{4.14}
\]
where we used (4.6) in the last inequality.

Our next aim is the desired Lipschitz property of $S$ with respect to the metric $d$ defined in (4.7). For any $u, v \in X_M$, we deduce from the Strichartz estimate (2.3), (1.2), (4.6), Hölder’s inequality and Sobolev embedding that
\[
d(Su, Sv) \lesssim \|\phi + |v|^\alpha - |u|^\alpha\|_{L^{s'} \cap L^r} \lesssim \left(\|u\|_{L^\alpha \cap L^2 \cap B^s_{r,2}} + \|v\|_{L^\alpha \cap L^2 \cap B^s_{r,2}}\right) \|u - v\|_{L^2 \cap L^r} \leq C_2 M^\alpha d(u, v) \leq \frac{1}{2} d(u, v). \tag{4.15}
\]

Therefore, by the Banach fixed point theorem, we conclude that the Cauchy problem (1.1) admits a unique global solution $u \in CH^s \cap L^2B^s_{r,2} \cap B^{s/4}_{q,2} \cap L^r$, where the continuity of $u$ in time follows from Proposition 1.7.

### 4.2. The case $s = 4$

Throughout this subsection, we fix
\[
\gamma = \frac{8(\alpha + 2)}{(N-8)\alpha}, \quad \rho = \frac{N(\alpha + 2)}{N + 4\alpha}.
\]
For $\phi \in H^4$ and $T > 0$, we define
\[
F(\phi, T) = \|e^{it(D^2+\mu \Delta)}(\Delta^2 + \mu \Delta)\phi\|_{L^\infty([0,T],L^\rho)} + \|e^{it(D^2+\mu \Delta)}f(\phi)\|_{L^\infty([0,T],L^\rho)} + \|e^{it(D^2+\mu \Delta)}\phi\|_{L^\infty([0,T],H^\rho,\rho)}.
\]
By the Strichartz estimate (2.1), (1.2) and Sobolev embedding $H^4(\mathbb{R}^N) \hookrightarrow L^{2\alpha+2}(\mathbb{R}^N)$, we have
\[
F(\phi, T) \lesssim \|\phi\|_{H^4} + \|f(\phi)\|_{L^2} \leq C_3 \left(\|\phi\|_{H^4} + \|\phi\|_{H^4}^{\alpha+1}\right).
\] (4.16)
We then recall the following result from [18].

**Proposition 4.3.** (Proposition 5.1 in [18]) Let $N \geq 9$, $\alpha = \frac{8}{N-8}$, $\mu = 0$ or $-1$ and $f \in C(\alpha)$. There exists $M > 0, C_4 > 0$ such that for any $T > 0$ with
\[
C_4 (1 + \|\phi\|_{H^4}^\alpha) F(\phi, T) \leq \frac{M}{2}, \tag{4.17}
\]
the Cauchy problem (1.1) admits a unique local solution \( u \in C([0, T], H^4) \cap L^\gamma([0, T], H^{4, \rho}) \) satisfying \( \|u\|_{H^{1, \gamma}([0, T], L^\rho)} \cap L^\gamma([0, T], H^{4, \rho}) \leq M \).

Let \( \|\phi\|_{H^4} \) sufficiently small such that
\[
C_3 C_4 (1 + \|\phi\|_{H^4}^\alpha) \big( \|\phi\|_{H^4}^\alpha + \|\phi\|_{H^4}^{\alpha + 1} \big) \leq \frac{M}{2},
\]
where \( C_3, C_4 \) are constants in (4.16) and (4.17), respectively. It now follows from Proposition 4.3, (4.16) and (4.17) that for any \( T > 0 \), the Cauchy problem (1.1) admits a unique local solution \( u \in C([0, T], H^4) \cap L^\gamma([0, T], H^{4, \rho}) \) with \( \|u\|_{H^{1, \gamma}([0, T], L^\rho)} \cap L^\gamma([0, T], H^{4, \rho}) \leq M \). Since \( T > 0 \) is arbitrary and \( M > 0 \) is fixed, we deduce that (1.1) admits a unique solution \( u \in C([0, \infty), H^4) \cap L^\gamma([0, \infty), H^{4, \rho}) \). By symmetry, a similar conclusion is reached in the negative time direction. Therefore, we obtain a unique solution \( u \in CH^4 \cap L^\gamma H^{4, \rho} \) to (1.1).

### 4.3. The case \( 4 < s < 6 \)

Throughout this subsection, we fix
\[
\gamma = 2, \quad \rho = \frac{2N}{N - 4}.
\]
We then define \( q, r, \overline{q}, \overline{r} \) such that
\[
\overline{q} = 2, \quad \frac{4}{\overline{q}} - N \left( \frac{1}{2} - \frac{1}{\overline{r}} \right) = 8 - s,
\]
and
\[
\frac{1}{\gamma} = \frac{\alpha + 1}{q}, \quad \frac{1}{\rho} = \frac{\alpha}{\gamma} + \frac{1}{\overline{r}}.
\]

Since \( 4 < s < 6, N > 2s \), it is straightforward to verify that \( (\gamma, \rho), (q, r) \in \Lambda_b \) are two biharmonic admissible pairs, \( 1 < \overline{r} < 2, r < \frac{N}{2} \) and \( \frac{1}{\gamma} = \alpha \left( \frac{1}{r} - \frac{s}{N} \right) + \frac{1}{\overline{r}} \).

Assume \( \|\phi\|_{H^s} \) sufficiently small such that
\[
(2C_5)^{\alpha + 1} \left( \|\phi\|_{H^s} + \|\phi\|_{H^s}^{\alpha + 1} \right)^\alpha \leq 1, \quad (C_6 + C_7) \left( 2C_5 \left( \|\phi\|_{H^s} + \|\phi\|_{H^s}^{\alpha + 1} \right) \right)^\alpha \leq \frac{1}{2},
\]
where \( C_5, C_6, C_7 \) are constants in (4.37), (4.38) and (4.42), respectively. Set \( M = 2C_5 \left( \|\phi\|_{H^s} + \|\phi\|_{H^s}^{\alpha + 1} \right) \) and consider the metric space
\[
Y_M = \left\{ u \in L^\infty H^s \cap L^q B^s_{q,2} \cap B^{s/4}_{q/2}L^r \cap H^{1, q} B_{r,2}^{s-4} : \right. \\
\left. \|u\|_{L^\infty H^s \cap L^q B^s_{q,2} \cap B^{s/4}_{q/2}L^r \cap H^{1, q} B_{r,2}^{s-4}} \leq M \right\}.
\]

It follows that \( Y_M \) is a complete metric space when equipped with the distance
\[
d(u, v) = \|u - v\|_{L^\infty L^q L^r}.
\]

Next, we show that the map \( S \), defined in (4.1), is a contraction on the space \( Y_M \).

We first show that \( S \) maps \( Y_M \) into itself. From the equation (4.2), we have
\[
\|Su\|_{L^\infty H^s} \leq \|Su\|_{L^\infty L^q} + \|Su\|_{L^\infty H^{s-2}} + \|\partial_t (Su)\|_{L^\infty H^{s-4}} + \|f(u)\|_{L^\infty H^{s-4}}
\]
and
\[
\|Su\|_{L^q B^s_{q,2}} \leq \|Su\|_{L^q L^q} + \|Su\|_{L^q B^s_{q,2}} + \|\partial_t (Su)\|_{L^q B^{s-4}_{2,2}} + \|f(u)\|_{L^q B^{s-4}_{2,2}}.
\]

By recalling that the space \( H^{s-2} \) can be viewed as the interpolation space between \( H^s \) and \( L^2 \), \( B^{s-2}_{r,2} \) as the interpolation space between \( B^s_{r,2} \) and \( B^0_{r,\infty} \), we have, see Theorem 6.4.5 of [2],
\[
(H^s, L^2)_{2/s,2} = H^{s-2}, \quad (B^s_{r,2}, B^0_{r,\infty})_{2/s,2} = B^{s-2}_{r,2}.
\]
where \((X,Y)_{q,r}\) is the notation for the real interpolation between the spaces \(X\) and \(Y\). Thus, it follows from Hölder’s inequality and Young’s inequality that

\[
\|Su\|_{L^\infty H^{s-2}} \lesssim \|Su\|_{L^\infty H^s}^{1-2/s} \|Su\|_{L^\infty L^2}^{2/s} \leq \frac{1}{2} \|Su\|_{L^\infty H^s} + C \|Su\|_{L^\infty L^2},
\]

and

\[
\|Su\|_{L^4 B^{\frac{1}{2}}_{r,2}} \lesssim \|Su\|_{L^\infty B^{\frac{1}{2}}_{r,2}}^{1-2/s} \|Su\|_{L^\infty B^{\frac{1}{2}}_{r,2}}^{2/s} \leq \frac{1}{2} \|Su\|_{L^\infty B^{\frac{1}{2}}_{r,2}} + C \|Su\|_{L^\infty L^r},
\]

where we used the embedding \(L^r(\mathbb{R}^N) \hookrightarrow B^0_{r,\infty}(\mathbb{R}^N)\) (see Theorem 6.4.4 in [2]) in (4.24). Estimates (4.21)–(4.24) imply

\[
\|Su\|_{L^\infty H^s} \leq \|Su\|_{L^\infty L^2} + \|\partial_t (Su)\|_{L^\infty H^{s-4}} + \|f(u)\|_{L^\infty H^{s-4}}
\]

and

\[
\|Su\|_{L^4 B^{\frac{1}{2}}_{r,2}} \leq \|Su\|_{L^4 L^r} + \|\partial_t (Su)\|_{L^4 B^{\frac{1}{2}}_{r,2}} + \|f(u)\|_{L^4 B^{\frac{1}{2}}_{r,2}}.
\]

From (4.25) and (4.26), we have

\[
\|Su\|_{L^\infty H^{s-4} \cap L^4 L^r} \lesssim \|\phi\|_{L^2} + \|u\|_{L^\infty B^{\frac{1}{2}}_{s/2}} \lesssim \|\phi\|_{H^s} + \|u\|_{L^\infty B^{\frac{1}{2}}_{r,2}} \|u\|_{L^4 L^r}.
\]

Next, we estimate \(|f(u)|_{L^\infty H^{s-4} \cap L^4 L^r}|. Let \(p_1 = \frac{2N}{N-8}\). Since \(\frac{1}{\gamma} = \frac{\alpha+1}{q}, \frac{1}{\rho} = \frac{1}{r} - \frac{4}{N}\) and \(\alpha + 1 > s - 4\), we deduce from Lemma 4.1 and Sobolev embedding \(H^s(\mathbb{R}^N) \hookrightarrow B^{s-4}_{p_1,2}(\mathbb{R}^N) \cap L^{\frac{N}{2N-8}}(\mathbb{R}^N)\) that

\[
|f(u)|_{L^\infty H^{s-4}} \lesssim \|u\|_{L^\infty B^{\frac{11}{12}}_{s/2}} \|u\|_{L^\infty B^{\frac{1}{2}}_{r,2}} \lesssim \|u\|_{L^\infty H^s} \|u\|_{L^\infty B^{\frac{1}{2}}_{r,2}}.
\]

Let \(p_2\) be given by \(\frac{1}{\gamma} = \frac{\alpha(N-2s)}{2N} + \frac{1}{p_2}\). Similar to (4.29), we have

\[
|f(u)|_{L^\infty B^{\frac{1}{2}}_{r,2}} \lesssim \|u\|_{L^\infty B^{\frac{11}{12}}_{s/2}} \|u\|_{L^\infty B^{\frac{1}{2}}_{r,2}} \lesssim \|u\|_{L^\infty H^s} \|u\|_{L^\infty B^{\frac{1}{2}}_{r,2}}.
\]

Finally, we claim that

\[
\|\partial_t (Su)\|_{L^\infty H^{s-4} \cap L^4 B^{\frac{1}{2}}_{r,2} \cap B^{(s-4)/4}_{r,2}} \lesssim \|\phi\|_{H^s} + \|f(u)\|_{H^{s-4}} + \|\partial_t f(u)\|_{L^\infty B^{\frac{1}{2}}_{s/2}} \|u\|_{L^\infty B^{\frac{1}{2}}_{r,2}}.
\]

In fact, form the equation (4.3) and the inequality (1.5), we have

\[
\|\partial_t (Su)\|_{L^\infty H^{s-4} \cap L^4 B^{\frac{1}{2}}_{r,2} \cap B^{(s-4)/4}_{r,2}} \lesssim \|\phi\|_{H^s} + \|f(u)\|_{H^{s-4}} + \|\partial_t f(u)\|_{L^\infty B^{\frac{1}{2}}_{s/2}} \|u\|_{L^\infty B^{\frac{1}{2}}_{r,2}}.
\]

Similar to (4.29), we have

\[
|f(u)|_{H^{s-4}} \lesssim \|\phi\|_{H^{s-1}}^{\frac{1}{s}}.
\]

Moreover, we deduce from Lemma 4.2 and Sobolev embedding \(B^{\frac{1}{s-2}}_{r,2}(\mathbb{R}^N) \hookrightarrow L^{\frac{N}{2N-8}}(\mathbb{R}^N)\) that

\[
|\partial_t f(u)|_{B^{\frac{1}{s-2}+1}_{s/2}} \lesssim |f(u)|_{B^{\frac{1}{s-2}}_{s/2}} \lesssim \|u\|_{L^\infty B^{\frac{1}{2}}_{r,2}} \|u\|_{B^{\frac{1}{s-2}}_{s/2}}.
\]
On the other hand, since $\bar{q} = \gamma' = 2$ and $1 < \bar{r} \leq 2$, it follows from Minkowski inequality and the embedding $L^q(\mathbb{R}^N) \hookrightarrow B^{\frac{N}{2}}_{q,2}(\mathbb{R}^N)$ (see Theorem 6.4.4 in [2]) that

$$
\| \partial_t f(u) \|_{L^q L^q} \lesssim \| \partial_t f(u) \|_{L^q B^{\frac{N}{2}}_{q,2}} \lesssim \| \partial_t f(u) \|_{L^q L^q}.
$$

(4.35)

Since $\frac{1}{\bar{r}} = \alpha (\frac{1}{r} - \frac{s}{N}) + \frac{1}{r} - \frac{s-4}{N}$, it follows from (1.2), Hölder’s inequality and Sobolev embedding $B^s_{r,2}(\mathbb{R}^N) \hookrightarrow L^{\frac{2N}{N-4s}}(\mathbb{R}^N), B^{s-4}_{r,2}(\mathbb{R}^N) \hookrightarrow L^{\frac{2N}{N-(s-4)r}}(\mathbb{R}^N)$ that

$$
\| \partial_t f(u) \|_{L^q} \lesssim \| u \|_{L^{\frac{2N}{N-4s}}(\mathbb{R}^N)}^\alpha \| \partial_t u \|_{L^{\frac{2N}{N-(s-4)r}}(\mathbb{R}^N)} \lesssim \| u \|_{B^s_{r,2}} \| \partial_t u \|_{B^{s-4}_{r,2}}.
$$

This inequality together with (4.35), Hölder’s inequality implies

$$
\| \partial_t f(u) \|_{L^q L^q} \lesssim \| u \|_{L^q B^s_{r,2}} \| \partial_t u \|_{L^q B^{s-4}_{r,2}}.
$$

(4.36)

The inequality (4.31) is now an immediate consequence of (4.32), (4.33), (4.34) and (4.36).

Estimates (4.27)–(4.31) imply that for any $u \in Y_M$,

$$
\| Su \|_{L^\infty H^{s-4}} \lesssim C_5 \left( \| \phi \|_{H^s} + \| \phi \|_{H^{s+1}}^2 \right) + C_6 M^{\alpha+1} \leq M,
$$

(4.37)

where we used (4.19) in the second inequality.

Our next aim is the desired Lipschitz property of $S$ with respect to the metric $d$ defined in (4.20). Similar to (4.15), we have for any $u, v \in Y_M$,

$$
d(Su, Sv) \lesssim \left( \| u \|_{L^q B^s_{r,2}} + \| v \|_{L^q B^s_{r,2}} \right) \| u - v \|_{L^q L^q}
$$

$$
\leq C_6 M^{\alpha} d(u, v) \leq \frac{1}{2} d(u, v).
$$

(4.38)

Therefore, we deduce from the Banach fixed point argument that the Cauchy problem (1.1) admits a unique global solution $u \in L^\infty H^s \cap L^q B^s_{r,2} \cap L^{q/4} L^q \cap H^{1,q} B^{s-4}_{r,2}$.

It remains to prove that $u \in C(\mathbb{R}, H^s)$. Similar to (4.25), we have

$$
\| u(t_1) - u(t_2) \|_{H^s} \lesssim \| \partial_t u(t_1) - \partial_t u(t_2) \|_{H^{s-4}} + \| u(t_1) - u(t_2) \|_{L^2}
$$

$$
+ \| f(u(t_1)) - f(u(t_2)) \|_{H^{s-4}}.
$$

(4.39)

Since $\partial_t f \in B^{s/4-1}_{r,2}(\mathbb{R}^N) \cap L^q L^q L^q$ by (4.34) and (4.35), we deduce from (4.3) and Proposition 1.7 that $u \in C^1(\mathbb{R}, H^{s-4})$, so that by (4.39) it suffices to prove $f(u) \in C(\mathbb{R}, H^{s-4})$.

To this end, we first show that $f(u) \in C(\mathbb{R}, B^0_{\rho_0,\infty})$, where $\rho_0$ is given by $\frac{1}{\rho_0} = \frac{1}{2} - \frac{s}{N}$. Indeed, using the same method as that used to derive (4.26), we obtain

$$
\| u(t_1) - u(t_2) \|_{H^{s,\rho_0}} \lesssim \| \partial_t u(t_1) - \partial_t u(t_2) \|_{L^{\rho_0}} + \| u(t_1) - u(t_2) \|_{L^{\rho_0}}
$$

$$
+ \| f(u(t_1)) - f(u(t_2)) \|_{L^{\rho_0}}.
$$

(4.40)

Moreover, it follows from (1.2), Hölder’s inequality, Sobolev embedding $H^s(\mathbb{R}^N) \hookrightarrow L^{\frac{2N}{N-s}}(\mathbb{R}^N)$ and $H^{s,\rho_0}(\mathbb{R}^N) \hookrightarrow L^{\frac{N\rho_0}{N-\rho_0}}(\mathbb{R}^N)$ that

$$
\| f(u(t_1)) - f(u(t_2)) \|_{L^{\rho_0}} \lesssim \left( \| u(t_1) \|_{L^{\frac{2N}{N-2}}(\mathbb{R}^N)}^\alpha + \| u(t_2) \|_{L^{\frac{2N}{N-2}}(\mathbb{R}^N)}^\alpha \right) \| u(t_1) - u(t_2) \|_{L^{\rho_0}}
$$

$$
\lesssim \| u \|_{L^\infty H^s} \| u(t_1) - u(t_2) \|_{H^{s,\rho_0}}.
$$

(4.41)

Combining (4.40) and (4.41), we obtain

$$
\| u(t_1) - u(t_2) \|_{H^{s,\rho_0}} \leq C_7 \| \partial_t u(t_1) - \partial_t u(t_2) \|_{L^{\rho_0}} + C_7 \| u(t_1) - u(t_2) \|_{L^{\rho_0}}
$$

$$
+ C_7 \| u \|_{L^\infty H^s} \| u(t_1) - u(t_2) \|_{H^{s,\rho_0}}.
$$

(4.42)
Since $C_7 \| u \|_{L^\infty H^s} \leq C_7 M^\alpha \leq \frac{1}{2}$ in (4.19), we have
\[
\| u(t_1) - u(t_2) \|_{H^{s-\rho_0}} \lesssim \| \partial_t u(t_1) - \partial_t u(t_2) \|_{L^{\rho_0}} + \| u(t_1) - u(t_2) \|_{L^{\rho_0}}.
\] (4.43)
On the other hand, since $u \in C^1 (\mathbb{R}, H^{s-4})$ and $H^{s-4}(\mathbb{R}^N) \hookrightarrow L^{\rho_0}(\mathbb{R}^N)$, we have $u \in C^1 (\mathbb{R}, L^{\rho_0})$. This together with (4.43) implies $u \in C (\mathbb{R}, H^{4, \rho_0})$. So by (4.41) and Sobolev embedding $L^{\rho_0}(\mathbb{R}^N) \hookrightarrow B^0_{\rho_0, \infty}(\mathbb{R}^N)$, we have $f(u) \in C(\mathbb{R}, B^0_{\rho_0, \infty})$.

We proceed to show $f(u) \in C(\mathbb{R}, H^{s-4})$. Let $\frac{1}{p_\epsilon} = \frac{1}{2} + \frac{\epsilon}{N}$ and $p_\epsilon = \frac{2N}{N - s + 2\epsilon}$, where $\epsilon > 0$ sufficiently small such that $\alpha > s - 5 + \epsilon$. We then claim that $f(u)$ is bounded in $B^{s-4+\epsilon}_{p_\epsilon, 2}$. In fact this follows from Lemma 4.2 \((\frac{1}{p_\epsilon} = \frac{\alpha N - 2s}{2N} + \frac{1}{p_\epsilon})\) and Sobolev embedding $H^s(\mathbb{R}^N) \hookrightarrow L^{\frac{N}{N-s+2\epsilon}}(\mathbb{R}^N) \cap B^{s-4+\epsilon}_{p_\epsilon, 2}(\mathbb{R}^N),$
\[
\| f(u) \|_{B^{s-4+\epsilon}_{p_\epsilon, 2}} \lesssim \| u \|^\alpha_{L^\frac{2N}{N-s+2\epsilon}} \| u \|_{B^{s-4+\epsilon}_{p_\epsilon, 2}} \lesssim \| u \|^\alpha_{H^s}.
\] (4.44)
Then, by the interpolation theorem (see Theorem 6.4.5 in [2]), we have
\[
(B^0_{\rho_0, \infty}, B^{s-4+\epsilon}_{p_\epsilon, 2})_{\theta, 2} = B^{s-4}_{2, 2} = H^{s-4}, \quad \theta = \frac{s - 4}{s - 4 + \epsilon}.
\] This together with (4.44) and the fact $f(u) \in C(\mathbb{R}, B^0_{\rho_0, \infty})$ implies $f(u) \in C(\mathbb{R}, H^{s-4})$. Combining (4.39) and $u \in C^1(\mathbb{R}, H^{s-4})$, we can immediately get that $u \in C(\mathbb{R}, H^s)$.

4.4. The case $6 \leq s < 8$

Throughout this subsection, we fix $r = \frac{2N}{N-4}$. Assume $\| \phi \|_{H^s}$ sufficiently small such that
\[
(2C_8)^{\alpha+1} \left( \| \phi \|_{H^s} + \| \phi \|_{H^s}^{\alpha+1} \right)^\alpha \leq 1, \quad C_8 \left( 2C_8 \left( \| \phi \|_{H^s} + \| \phi \|_{H^s}^{\alpha+1} \right) \right)^\alpha \leq \frac{1}{2},
\] (4.45)
where $C_8, C_9$ are constants in (6.60) and (6.61), respectively. Set $M = 2C_8 (\| \phi \|_{H^s} + \| \phi \|_{H^s}^{\alpha+1})$ and consider the metric space
\[
Z_M = \left\{ u \in L^\infty H^s \cap B^{s/2}_{2, 2} L^r \cap B^{(s-2)/4}_{2, 2} B^2_{r, 2} : \| u \|_{L^\infty H^s \cap B^{s/2}_{2, 2} L^r \cap B^{(s-2)/4}_{2, 2} B^2_{r, 2}} \leq M \right\}.
\]
It follows that $Z_M$ is a complete metric space when equipped with the distance
\[
d(u, v) = \| u - v \|_{L^\infty H^s \cap B^{s/2}_{2, 2} L^r}.
\] (4.46)
Next, we show that the map $S$, defined in (4.1), is a contraction on the space $Z_M$.

We first estimate $\| Su \|_{L^\infty H^s \cap B^{s/2}_{2, 2} L^r}$. Note that (4.25), (4.29) and (4.33) still hold in the case $6 \leq s < 8$, we have
\[
\| Su \|_{L^\infty H^s} \lesssim \| u \|^{\alpha + 1}_{L^\infty H^s} + \| Su \|_{L^\infty L^2} + \| \partial_t (Su) \|_{L^\infty H^{s-4}}.
\] (4.47)
so that
\[
\| Su \|_{L^\infty H^s \cap B^{s/2}_{2, 2} L^r} \lesssim \| u \|^{\alpha + 1}_{L^\infty H^s} + \| Su \|_{L^\infty L^2 \cap L^2 L^r} + \| \partial_t (Su) \|_{L^\infty H^{s-4} \cap B^{(s-4)/4}_{2, 2} L^r}.
\] (4.48)
From (4.2) and the Strichartz estimate (2.3), we have
\[
\| Su \|_{L^\infty L^2 \cap L^2 L^r} \lesssim \| \phi \|_{L^2} + \| f(u) \|_{L^2 L^r} \lesssim \| \phi \|_{H^s} + \| u \|_{L^\infty H^s} \| u \|_{L^2 L^r},
\] (4.49)
where we used (1.2), Hölder’s inequality and Sobolev embedding $H^s(\mathbb{R}^N) \hookrightarrow L^{\frac{2N}{N-s+2\epsilon}}(\mathbb{R}^N)$ in the second inequality. Next, we claim that
\[
\| \partial_t (Su) \|_{L^\infty H^{s-4} \cap B^{(s-4)/4}_{2, 2} L^r} \lesssim \| \phi \|_{H^s} + \| \phi \|_{H^s}^{\alpha+1} + \| u \|_{L^\infty H^s} \left( \| u \|_{B^{s/2}_{2, 2} L^r} + \| u \|_{B^{(s-2)/4}_{2, 2} B^2_{r, 2}} \right).
\] (4.50)
In fact, from the equation (4.3), inequalities (1.5) and (4.33), we have
\[
\|\partial_t (Su)\|_{L^{\infty} H^{s-4} \cap B_{2,2}^{(s-4)/4}_L} \\
\lesssim \|\phi\|_{H^s} + \|f(\phi)\|_{H^{s-4}} + \|\partial_t f(u)\|_{B_{2,2}^{(s-4)/4}_L} + \|\partial_t f(u)\|_{L^4/(8-s)}L^2 \\
\lesssim \|\phi\|_{H^s} + \|\phi\|_{H^s}^{\alpha+1} + \|f(u)\|_{B_{2,2}^{\alpha/4}_L} + \|\partial_t f(u)\|_{L^4/(8-s)}L^2 .
\]
(4.51)

From Lemma 4.2 and Sobolev embedding $H^s (\mathbb{R}^N) \hookrightarrow L^{2N/2s} (\mathbb{R}^N)$, we have
\[
\|f(u)\|_{B_{2,2}^{\alpha/4}_L} \lesssim \|u\|^\alpha_{L^{\infty} \cap L^{2N/2s}} \|u\|_{B_{2,2}^{\alpha/4}_L} \lesssim \|u\|^\alpha_{L^{\infty} H^s} \|u\|_{B_{2,2}^{\alpha/4}_L} .
\]
(4.52)

It remains to estimate $\|\partial_t f(u)\|_{L^4/(8-s)L^2}$. Note that $\frac{4}{8-s} \geq 2$, so that $B_{2,2}^{(s-6)/4}_L \hookrightarrow B_{2,4/(s-6)}^{(s-6)/4} L^2 \hookrightarrow L^{4/(8-s)}L^2$, which implies
\[
\|\partial_t f(u)\|_{L^4/(8-s)L^2} \lesssim \|\partial_t f(u)\|_{L^{2,4/(s-6)}L^2} \lesssim \|f(u)\|_{B_{2,2}^{(s-6)/4}_L} .
\]
(4.53)

Moreover, it follows from Lemma 4.2 and Sobolev embedding $H^s (\mathbb{R}^N) \hookrightarrow L^{2N/2s} (\mathbb{R}^N)$, $B_{2,2}^2 (\mathbb{R}^N) \hookrightarrow L^{2N/2s}$ that
\[
\|f(u)\|_{B_{2,2}^{(s-2)/4}_L} \lesssim \|u\|_{L^{\infty} \cap L^{2N/2s}} \|u\|_{B_{2,2}^{(s-2)/4}_L} \lesssim \|u\|^\alpha_{L^{\infty} H^s} \|u\|_{B_{2,2}^{(s-2)/4}_L} .
\]
(4.54)

The inequality (4.50) is now an immediate consequence of (4.51)–(4.54).

Estimates (4.48), (4.49) and (4.50) imply that for any $u \in Z_M$,
\[
\|Su\|_{L^{\infty} H^s \cap B_{2,2}^{\alpha/4}_L} \lesssim \|\phi\|_{H^s} + \|\phi\|_{H^s}^{\alpha+1} + M^{\alpha+1} .
\]
(4.55)

We now estimate $\|Su\|_{B_{2,2}^{(s-2)/4}_L}$. When $s = 6$, we deduce from the inequality (1.5) and the equation (4.3) that
\[
\|\partial_t (Su)\|_{L^2 B_{2,2}^1} \\
\lesssim \|\phi\|_{H^6} + \|f(\phi)\|_{H^2} + \|\partial_t f(u)\|_{B_{2,2}^{1/2}_L} + \|\partial_t f(u)\|_{L^2 L^2} \\
\lesssim \|\phi\|_{H^6} + \|\phi\|_{H^6}^{\alpha+1} + \|f(u)\|_{B_{2,2}^{\alpha/2}_L} + \|\partial_t f(u)\|_{L^2 L^2} ,
\]
(4.56)

where we used (4.33) and the embedding $L^2 L^2 \hookrightarrow l^2 L^2 L^2$ in the last inequality. When $6 < s < 8$, we deduce from the equation (4.3), the inequality (1.6) ($\sigma = 2, \theta = (s - 4)/4$) and (4.33) that
\[
\|\partial_t (Su)\|_{B_{2,2}^{(s-4)/2-4/4}_L} \\
\lesssim \|\phi\|_{H^s} + \|f(\phi)\|_{H^{s-4}} + \|\partial_t f(u)\|_{B_{2,2}^{(s-4)/4}_L} + \|\partial_t f(u)\|_{L^4/(8-s)}L^2 \\
\lesssim \|\phi\|_{H^s} + \|\phi\|_{H^s}^{\alpha+1} + \|f(u)\|_{B_{2,2}^{\alpha/4}_L} + \|\partial_t f(u)\|_{L^4/(8-s)L^2} .
\]
(4.57)

Estimates (4.56), (4.57), (4.52), (4.53) and (4.54) imply that for any $u \in Z_M$,
\[
\|\partial_t (Su)\|_{B_{2,2}^{(s-4)/4-2/4}_L} \lesssim \|\phi\|_{H^s} + \|\phi\|_{H^s}^{\alpha+1} + M^{\alpha+1} , \quad 6 \leq s < 8 .
\]
(4.58)

On the other hand, it follows from the inequality (1.5) and the embedding $L^2 L^2 \hookrightarrow l^2 L^2 L^2$ that
\[
\|Su\|_{L^2 B_{2,2}^1} \lesssim \|\phi\|_{H^2} + \|f(u)\|_{B_{2,2}^{1/2}_L} + \|f(u)\|_{L^2 L^2} \\
\lesssim \|\phi\|_{H^2} + \|u\|_{L^{\infty} H^s} \left( \|u\|_{B_{2,2}^{1/2}_L} + \|u\|_{B_{2,2}^{(s-2)/4}_L} \right) ,
\]
(4.59)

where we used (4.52) and (4.54) in the second inequality. It now follows from (4.55), (4.58), (4.59) and (4.45) that, for any $u \in Z_M$,
\[
\|Su\|_{L^{\infty} H^s \cap B_{2,2}^{\alpha/4}_L \cap B_{2,2}^{(s-2)/4}_L} \leq C_8 \left( \|\phi\|_{H^s} + \|\phi\|_{H^s}^{\alpha+1} \right) + C_8 M^{\alpha+1} \leq M .
\]
(4.60)
Our next aim is the desired Lipschitz property of $S$ with respect to the metric $d$ defined in (4.20). For any $u, v \in Z_M$, we deduce from the Strichartz estimate (2.3), the inequality (1.2), (4.45), Hölder’s inequality and Sobolev embedding $H^s(\mathbb{R}^N) \hookrightarrow L^{2\frac{2s}{s-2}}(\mathbb{R}^N)$ that

$$d(Su, Sv) \lesssim \|(|u|^\alpha + |v|^\alpha)(u - v)\|_{L^2 L'}$$

$$\lesssim (\|u\|_{L^\infty H^s} + \|v\|_{L^\infty H^s}) \|u - v\|_{L^2 L'}$$

$$\leq C_9 M^\alpha d(u, v)$$

(4.61)

Therefore, we deduce from the Banach fixed point argument that the Cauchy problem (1.1) admits a unique global solution $u \in C(\mathbb{R}, H^s) \cap B^{s/4}_{2,2} L^r \cap B^{(s-2)/4}_{2,2} B^2_{r,2}$, where the continuity of $u$ in time follows from the same argument used in the case $4 < s < 6$.

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Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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