Dynamical systems and $\sigma$-symmetries

G Cicogna, G Gaeta and S Walcher

1 Dipartimento di Fisica, Università di Pisa, and INFN sezione di Pisa, Largo B Pontecorvo 3, I-56127 Pisa, Italy
2 Dipartimento di Matematica, Università degli Studi di Milano, via C. Saldini 50, I-20133 Milano, Italy
3 Lehrstuhl A für Mathematik, RWTH Aachen, D-52056 Aachen, Germany

E-mail: cicogna@df.unipi.it, giuseppe.gaeta@unimi.it and walcher@mathA.rtwh-aachen.de

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Abstract

A deformation of the standard prolongation operation, defined on sets of vector fields in involution rather than on single ones, was recently introduced and christened ‘$\sigma$-prolongation’; correspondingly, one has ‘$\sigma$-symmetries’ of differential equations. These can be used to reduce the equations under study, but the general reduction procedure under $\sigma$-symmetries fails for equations of order 1. In this paper, we discuss how $\sigma$-symmetries can be used to reduce dynamical systems, i.e. sets of first-order ODEs in the form $\dot{x}^a = f^a(x)$.

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Introduction

In recent papers [6, 7], we have introduced a generalized prolongation operation, defined not on single vector fields but on sets of vector fields (in involution), and depending on a smooth matrix function $\sigma: \mathfrak{j}^1 M \to \text{Mat}(n, \mathbb{R})$. This was called $\sigma$-prolongation or joint $\lambda$-prolongation to emphasize that it is an extension of the $\lambda$-prolongation introduced a decade ago by Muriel and Romero [20] (see also [21–28] and [9, 32]). Correspondingly, we introduced the notion of $\sigma$-symmetry, or joint $\lambda$-symmetry, for systems of ODEs: Lie-point vector fields which—after being $\sigma$-prolonged up to suitable order leave a given set of equations $\mathcal{E}$ invariant—are said to be $\sigma$-symmetries for $\mathcal{E}$.

It was also shown [7] that $\sigma$-prolongations enjoy the ‘invariants by differentiation property’, and hence they may be used, essentially in the same way as standard (and $\lambda$ in the scalar case [22]) prolongations, to reduce systems of ODEs of order $q \geq 2$; see [7] for details.

These results left out the special—but relevant—case of dynamical systems (DS in the following), i.e. of systems of first-order ODEs of the form

$$\frac{dx^a}{dt} = F^a(x, t).$$

(1)
The purpose of this paper is to fill this gap, i.e. to discuss how $\sigma$-symmetries can be used in the study of DS and thus complete the discussion of [7].

Actually, by a standard procedure (based on adding a variable; see e.g. [6] for relations between the symmetries and orbital symmetries of the original and the modified system), one can always reduce to consider autonomous DS. We will therefore deal with equations of the form

$$\frac{dx^a}{dt} = f^a(x) \quad (a = 1, \ldots, n).$$

Similarly, we will later on restrict to the consideration of Lie-point time-independent vector fields, $X_i = \phi_i^a(x)(\partial/\partial x^a)$ (the Einstein summation convention over repeated indices is used throughout this paper); we prefer however to give in the following section a discussion for DS in their general, possibly a non-autonomous form (1), and for general vector fields as well, for the convenience of the reader.

It should be mentioned that we already investigated applications of these new types of symmetries to DS in a previous paper [6]; in that paper, however, we made no explicit use of prolongation for first-order systems, and rather emphasized the general framework and structural properties (and correspondingly a more abstract mathematical approach). Moreover, in [6] we were also interested in a strategy aiming at writing at least some of the (first-order) equations in higher order; this would remove the degeneracies related to having a first-order system [15, 29, 30, 33] and hence will help in the search for symmetries. Here, we will not follow this line, but, following an approach similar to the one taken in [7] (which as said above was not able to deal with DS), we just consider equations in their natural first-order form; in other words, we aim at a ‘direct’ extension of the approach and results in [7] in order to complete our study, with a view at concrete reduction results procedures.

Also, in [7] we considered the geometrical aspects of our approach; this will not be discussed here in order to avoid the duplication of material: the interested reader is referred to that paper (and similarly to [6] for algebraic aspects as well). Again in [7] (see section 6 and appendix B there), we discussed some limitations of our approach from a practical point of view, due to the difficulty of actually determining the $\sigma$-symmetries of a given system. In view of the relevance of this point for applications, we report here some points of our discussion, but we also mention how we could give constructive results in some cases.

Focusing on DS one can view our approach as just a different way of looking at something which is already well known, which is in essence the Frobenius reduction theory. It should be mentioned, however, that the same can be said for virtually any reduction procedure, and that setting this in the language of Lie symmetries leads to a wider range of concrete applications.

Finally, we would also like to mention that, as already stressed in [7] (see again there for further detail), our approach should be seen as an extension and generalization of the approach by Pucci and Saccomandi [32] to the Muriel and Romero fruitful idea of $\lambda$-symmetries [20–23].

The plan of the paper is as follows. We will first recall the definition and relevant properties of $\sigma$-prolongations, then prove a theorem showing how they can be used to reduce $\sigma$-symmetric DS and present some remarks about this reduction procedure. We will also show that this is related to the existence of $\sigma$-invariant constants of motion of the DS. We then discuss the problem of determining $\sigma$-symmetries, which is in general, as said before, beyond reach, apart from some ‘favorable’ cases. We also give a number of explicit examples in low dimensions, which illustrate the various situations, and finally we draw some conclusions.

All the objects (manifolds, functions, vector fields) considered in this paper will be assumed to be smooth.
1. Definition and properties of $\sigma$-prolongations and $\sigma$-symmetries

We denote as usual by $M = \mathbb{R} \times U$ the extended phase manifold for the DS (2), where $t \in \mathbb{R}$ and $x \in U$; here, $U$ is a smooth manifold of dimension $n$. As our considerations will be local, we can assume $U \simeq \mathbb{R}^n$ with no loss of generality; on the other hand, in this section we will consider general vector fields on $M$.

Consider a set $\mathcal{X} = \{X_1, \ldots, X_s\}$ of smooth vector fields on $M$; assume that they are in involution and their rank (assumed to be constant for the sake of simplicity) $r \leq s$ satisfies $r < n$. The involution assumption means that there are smooth functions $\mu_{ij} : M \to \mathbb{R}$ such that

$$[X_i, X_j] = \mu_{ij} X_k. \tag{3}$$

1.1. Standard prolongations and symmetries

As well known \[1, 4, 15, 29, 30, 33\], one associates with vector field, $\sigma$-prolongation of $X_i$ then the (first prolongations) are written in the natural local coordinates $(t; x; \dot{x})$ on $J^1M$ as

$$Y_i = \xi_i(x, t) \frac{\partial}{\partial t} + \psi_i^a(x, t) \frac{\partial}{\partial x^a} + \psi_i^\alpha \frac{\partial}{\partial \dot{x}^\alpha}.$$

When $Y_i$ is the standard prolongation of $X_i$, the coefficients $\psi_i^a$ are given by the (standard) prolongation formula

$$\psi_i^a = D_t \phi_i^a - \dot{x}^\alpha D_x \xi_i.$$

The vector field $X$ is said to be a symmetry for the equation $\mathcal{E}$ defined on $J^1M$ if $[Y(\mathcal{E})]_{|\mathcal{E}} = 0$, where $S(\mathcal{E})$ is the solution manifold for $\mathcal{E}$, i.e. the set of points of $J^1M$ on which $\mathcal{E}$ is satisfied. If the relation is satisfied without restricting to $S(\mathcal{E})$, i.e. if $Y(\mathcal{E}) = 0$ on all of $J^1M$, then $X$ is said to be a strong symmetry. It is well known that if $\mathcal{E}$ admits $X$ as a symmetry, then there is an equivalent equation $\tilde{\mathcal{E}}$ (i.e. an equation $\tilde{\mathcal{E}}$ with the same solutions as $\mathcal{E}$) which admits $X$ as a strong symmetry (see e.g. \[2\]); we stress that this result is obtained by working purely in $J^1M$; hence, it can be rephrased by saying that if $\mathcal{E}$ is invariant under $Y$, then there is $\tilde{\mathcal{E}}$ which is strongly invariant under $Y$ (independently of the fact that $Y$ is the prolongation of $X$).

1.2. $\sigma$-prolongations and $\sigma$-symmetries

The $\sigma$-prolongation represents a modification of the standard prolongation operation. The relation between the $\psi$ and the $\phi$ coefficients is in this case given by

$$\psi_i^a = (D_t \phi_i^a - \dot{x}^\alpha D_x \xi_i) + \sigma_{ij}(\psi_j^a - \dot{x}^\alpha \xi_j), \tag{4}$$

with $\sigma_{ij} : J^1M \to \mathbb{R}$ being smooth functions; this modified prolongation operation is thus characterized by the smooth $s \times s$ matrix function $\sigma$.

Note that $\sigma$-prolongation is defined on sets of vector fields, not on individual ones, as $\psi_i^a$ depends on the coefficients $\phi_j^a$ for the other vector fields. When the set consists of a single vector field, $\sigma$-prolongation reduces to the $\lambda$-prolongations of Muriel and Romero \[20–23\] (see also \[9\]).
It should also be noted that for a generic choice of $\sigma$, the $Y_i$ will not be in involution even if the $X_i$ are. Sufficient conditions for the involution properties to be preserved under $\sigma$-prolongation (in the form of an equation to be satisfied by $\sigma$) are discussed in [7]. We will here assume that the $Y_i$ are in involution.

Note that it may happen that the $Y_i$ are not in involution, but can be completed to an involution system by adding a finite number of vector fields; if the rank $\rho > r$ of the completed system satisfies $\rho < n$, then the reduction to be discussed below is still possible with suitable modifications (see section 3).

If the $\sigma$-prolongations $Y_i$ of the $X_i$ leave the system of equations $E$ invariant, i.e. if $[Y_i(E)]_{t \in \mathcal{E}} = 0$, then we say that the involution system $\mathcal{X} = \{X_i\}$ is a $\sigma$-symmetry for $E$. If $Y_i(E) = 0$, i.e. if the invariance condition holds without restriction to $S(E)$, then we say that $\mathcal{X}$ is a strong $\sigma$-symmetry for $E$. The same result mentioned above holding for standard and strong symmetries is immediately seen to hold for $\sigma$-symmetries and strong $\sigma$-symmetries: if $E$ admits $\mathcal{X}$ as $\sigma$-symmetry, then there is an equivalent system $\tilde{E}$ admitting $\mathcal{X}$ as a strong $\sigma$-symmetry. In the following, we will thus deal with strong symmetries.

1.3. The ‘invariants by differentiation’ property

It is well known that for standard prolongations one can generate differential invariants of all orders starting from those of lower order. This is based on the so-called invariants by differentiation property (IBDP).

The same property extends to $\lambda$-prolongations [9, 20–23] (and can be used to characterize them [32]), and also to $\sigma$-prolongations. For these we have that [7], with $D_t$ the total derivative with respect to time $t$, if the $Y$ are $\sigma$-prolonged, they satisfy

$$[Y_i, D_t] = \sigma_{ij} Y_j - (D_t \xi_i + \sigma_{ij} \xi_j) D_t .$$

(5)

Proposition 1. Let $\mathcal{Y}$ be a set of $\sigma$-prolonged vector fields, and let $\zeta_1$ and $\zeta_2$ be common independent differential invariants of order $k$ for all of them [7]. Then,

$$\Theta : = (D_t \zeta_1)/(D_t \zeta_2)$$

(6)

is a common differential invariant of order $k + 1$ for all of them.

Note that in particular, if the $X_i$ are ‘vertical’ vector fields, i.e. if $\xi_i = 0$ (see the following section and in particular remark 1 for a discussion on this point), then necessarily $t$ is an invariant for the $X_i$ (and hence also for the $Y_i$), and proposition 1 makes that given an invariant $\zeta = \zeta(\xi)$ of order zero, we obtain immediately (just choose $\zeta_2 = t$ in (6)) a first-order differential invariant in the form

$$\zeta^{(1)} : = D_t \zeta .$$

(7)

2. Reduction of DS and $\sigma$-symmetries

The reduction procedure discussed in [7] for systems of differential equations of order $q > 1$ does not apply to the special case of DS. The obstacle lies in the fact that the most general differential invariants of order $q$ can be obtained, for $q > 1$, from those of lower order (thanks to the IBDP) and in particular from those of orders 0 and 1 (the need to include differential invariants of order 1 is intuitively clear, as at order 0 there is no trace of the $\sigma$ matrix in the vector fields). But for $q = 1$ this approach fails.
We should therefore consider the special case $q = 1$, and in particular that of the DS (thus, equations solved with respect to first derivatives of dependent variables), with a different approach.

As anticipated in the introduction, we will consider autonomous DS (2) and correspondingly restrict to the consideration of Lie-point time-independent vector fields, $X_i = \varphi_i^a(x)\partial/\partial x^a$.

**Remark 1.** Note in this respect that, as well known, when dealing with a general vector field $X = \xi(\partial/\partial t) + \varphi^a(\partial/\partial x^a)$, one can equivalently pass to consider its *evolutionary representative* $X_{ev} = (\varphi^a - \xi\dot{x}^a)(\partial/\partial x^a)$ [29, 30, 33]. Upon restriction to solutions to (2), this reads $X_{ev} = (\varphi^a - \xi f^a)(\partial/\partial x^a)$; the latter is a well-defined vector field in $M$, and of the form we consider. We can also proceed in a slightly different way: on the set $S = S(\mathcal{E})$ of solutions to (2), one has $[\partial/\partial t]_S = -f^a(\partial/\partial x^a)$, and hence $X$ is again rewritten as $[X]_S = (\varphi^a - \xi f^a)\partial/\partial x^a$.

In the assumptions $\xi_1 = 0$ and $\varphi_i^a$ independent of time, the $\sigma$-determining equations, i.e. the condition for the DS (2) to be invariant under the $\sigma$-prolongations $Y_i$, where

$$Y_i = \varphi_i^a \frac{\partial}{\partial x^a} + (\sigma a^i + \sigma_i) \frac{\partial}{\partial x^a}$$

takes the form of the Lie bracket condition

$$[X_i, X_0] = \sigma_{ij} X_j \quad (i, j = 1, \ldots, s),$$

where $X_0 = f^a(x)\partial/\partial x^a$ is the vector field describing the dynamics (see also subsection 5.1 for some detail).

**Remark 2.** One can look at this construction in a different way, which is established in the literature and goes as follows. Since the vector fields $\{X_1, \ldots, X_s\}$ are in involution (regular due to our hypotheses), Condition (8) implies that the vector field $X_0$ is projectable with respect to this foliation (see e.g. [19], pp 29 ff.). Conversely, given a foliation and a projectable vector field $X_0$, consider a set of vector fields $X_i$ whose common invariants define—by means of their level sets—the leaves of the foliation; then, the $\{X_0, X_1, \ldots, X_s\}$ will satisfy a set of relations as in (8). It may be noted that such constructions are well known in symmetry reduction, see e.g. [14]. For the special case of Hamiltonian systems, the leaves are the level sets of the momentum map, see [31]. In this sense, the results contained in this paper, albeit obtained in a different way, can be seen as a recasting of this classical approach (basically going back to Frobenius) in the language of Lie symmetries.

**Remark 3.** We note that our general concept of reducibility covers the slightly broader setting of *orbital* reduction: in this case in identity (8), the summation on the rhs ranges from 0 to $s$, or, in explicit form,

$$[X_i, X_0] = \sigma_{0i} X_0 + \sigma_{ij} X_j \quad (i, j = 1, \ldots, s)$$

with some $\sigma_{0i} \neq 0$. In this case, the zero-order common invariants $\zeta_i (i = 1, \ldots, n - r)$ satisfy an *orbitally reducible* system, i.e. a system ‘reducible up to a common scalar factor’:

$$\zeta_i = \rho(x) f_i(\zeta).$$

See [6] for a proof and more detail.

**Theorem 1.** Let $\mathcal{X}$ be a set of vector fields on the $n$-dimensional manifold $M$; assume that $\mathcal{X}$ is in involution and of rank $r < n$. Let $\mathcal{Y}$ be the $\sigma$-prolongation of $\mathcal{X}$, and let $\mathcal{Y}$ be in involution and also of rank $r$. If the $n$-dimensional DS (2) is invariant under the set $\mathcal{Y}$, then the system can be locally reduced, passing to suitable symmetry-adapted coordinates, to a DS of dimension $n - r$ and a system of $r$ ‘reconstruction equations’ depending on the solution to the reduced system.
Proof. Let us first consider the situation at order zero, i.e. for the action of the vector fields $X_i$ in $\mathcal{M}$; the latter is of dimension $n+1$ and hence we will have, apart from the trivial invariant $\tau = t$, other $n-r$ independent common invariants $\zeta_i(x)$ as guaranteed by the Frobenius theorem. We can consider a change of coordinates in $\mathcal{M}$ and the set of coordinates $z_i = \zeta_i(x)$ ($i = 1, \ldots, n-r$) will be complemented by some $r$ coordinates $y_j = \eta_j(x)$ ($j = 1, \ldots, r$). The vector fields $X_i$ will be written, in the new coordinates, as

$$X_i = \Phi_i^j \left( \partial / \partial y^j \right).$$

Let us now consider the situation in $J^1 \mathcal{M}$ (of dimension $2n+1$) with the action of the first $\sigma$-prolongations $Y_i$. Now we have $(2n+1-r)$ invariants, with $n$ of them being genuinely of first order. By proposition 1, we know that $\zeta_i^{(1)} := D_t \zeta_i$ is a first-order invariant for the $Y_i$, and this allows us to identify $(n-r)$ such invariants. There will be other $r$ first-order invariants, $\beta_j$, which are not obtained in this way; these will involve the $D_t \eta$ and, in general, can also depend on the $\zeta, D_t \zeta$ (note however that any dependence on the $\dot{z}$ can be eliminated e.g. by the Gram–Schmidt-type procedure). The $Y_i$ will be written, in the new coordinates, as

$$Y_i = \Phi_i^j \left( \partial / \partial y^j \right) + \Psi_i^a \left( \partial / \partial \dot{y}^a \right).$$

We can then pass to consider, in $J^1 \mathcal{M}$, the system of coordinates $(z, y; \dot{z}, \dot{y})$. The equations of motion for the $z$ and $y$ will be obtained by rewriting (2) in the new coordinates. In full generality, these would read

$$\dot{z} = f_i(z, w); \quad \dot{y} = g_j(z, y).$$

However, the $Y$ are symmetries of the equations, and the $\dot{z}_i$ are invariant under the $Y$; it follows that the $f_i$ are also invariant, i.e. we should have $f_i = f_i(z)$. This shows that we obtain a reduced DS, of dimension $n-r$, for the invariant variables

$$dz_i / dt = f_i(z) \quad (i = 1, \ldots, n-r). \quad (10)$$

As for the other $r$ equations, $\dot{y} = g_j(z, y)$, they must also be invariant under $Y$ (at least when restricting to the solution of the equations (10)), and hence can be written in terms of the first-order invariants; in full generality, they can be given the form $\dot{\beta}_j = b_j$, where $b_j$ depends only on the $z$ and $\dot{z}$ invariants; restricting to the solutions of (10) means that we can write $f_i(z)$ for $\dot{z}_i$, and hence these equations are written in the form

$$\dot{\beta}_j = B_j(z); \quad (11)$$

the exact form of the functions $B_j$ depends of course on the initial equations (2) and on the arbitrary choices made in the definition of $\dot{\beta}_j$ (they involve coefficients being function of $z$ as well).

The set (10) represents the reduced system, and (11) are the reconstruction equations; they can be thought as a set of (generally coupled) non-autonomous equations for the $y(t)$, depending on the solution $z_i(t)$ of the reduced equations. \hfill \Box

Remark 4. The above theorem is stated and proved in a quite different setting and perspective in [6], where the reduction procedures of DS, involving also orbital symmetries, with extensions to ODEs of higher order, are discussed focusing on more algebraic aspects, generalizing in several ways the approach and the results of [13, 34].

Remark 5. Here, the ‘reconstruction equations’ will in general be (a system of coupled) differential equations in the original variables, as they involve differential invariants. Thus, at difference with the reconstruction procedure met when using standard symmetries (which requires only quadratures), solving the reconstruction equations is in general a nontrivial
task, and can turn out to be impossible. We are not aware of any reasonably general and/or
natural condition ensuring that the reconstruction equations can be solved. In other words, the
reduction based on $\sigma$-symmetries is a possible strategy to attempt in dealing with nonlinear
systems, but with no guarantee of success even when one is able to determine (some of) the
system’s $\sigma$-symmetries. On the other hand, there are cases (as will be shown in the examples
below) where one is able to solve the reconstruction equations.

Remark 6. We stress that the situation described above for $\sigma$-symmetries was already present
when dealing with the so-called $\Lambda$-symmetries [3] (also called $\rho$-symmetries [3, 9]); these are
$\mu$-symmetries [5] for the specific setting of DSs [9] and turn out to be a direct generalization
of $\lambda$-symmetries [20–23].

Remark 7. Our discussion shows that, as it often happens with symmetry considerations,
$\sigma$-symmetry properties are specially transparent when adopting symmetry-adapted
coordinates. Some of these coordinates should correspond with invariants, while our discussion
left the choice of other coordinates free. A specially convenient choice would be that of
coordinates which rectify $r$ vector fields $\{X_1, \ldots, X_r\} (r \leq s)$ in the set $\mathcal{X}$; then, the $Y_i$
take the particularly simple and significant form $Y_i = (\partial/\partial y^i) + \sigma^{ij} (\partial/\partial \dot{y}^j) (i, j = 1, \ldots, r)$, which
extends a similar result for $\lambda$-symmetries [23]. One may again see this in the light of the
Frobenius theorem and related reduction approaches, see remark 2.

Remark 8. As mentioned above, the reduction obtained through the procedure we are
considering is the restriction of the initial system to the space of invariants (under the
$\sigma$-symmetries). In this sense, the present approach can be seen as the extension to the fully
general DS of an approach (based on Michel theory [16–18]) developed some time ago for
systems in normal form [8, 10, 11] (see also [12]).

3. Completion of an involution system

The formulation of theorem 1 requiring that $\mathcal{X}$ and $\mathcal{Y}$ have the same rank does not consider
a case which can occur in applications: there is in fact the possibility that the involution
properties of the vector fields $X_i$ are different from those of the $Y_i$.

This may seem a contradictory statement: in fact, it is clear that

$$[Y_i, Y_j] = \mu_{ij}^k Y_k \quad (12)$$

(note here the $\mu_{ij}^k$ are in general functions on $J^1M$ with values in $\mathbb{R}$) requires also

$$[X_i, X_j] = \mu_{ij}^k X_k. \quad (13)$$

On the other hand, a little thinking (or some explicit examples, see below) shows that it
is well possible that starting from an involution system (13) and applying the $\sigma$-prolongation
procedure one obtains a set of vector fields $Y_i$ which do not satisfy (12); actually, this will be
the generic case for a randomly chosen $\sigma$. When (12) is not satisfied, the $\{Y_i\}$ could either not
close to a nontrivial involution system (that is, more precisely, only close once vector fields
along all directions in $J^1M$ are added to the system), or close to an involution system $\mathcal{Y}$ after
adding a certain number of auxiliary vector field, still however providing a system of rank
$r < n$. In this case, theorem 1 (which only makes reference to the prolonged vector fields, i.e.
to the vector fields in $J^1M$) still applies.

Some further considerations are in order regarding this case. As remarked above, the
involution properties (12) among the $Y_i$ imply that the same involution properties (13) are
satisfied by the $X_i$. If we start from the latter, then it is clear that the completion procedure
(i.e. adding to the set any vector field appearing in the commutators \([Y_i, Y_j]\) and so on) will produce only vector fields which have zero projection in \(M\) (i.e. they are vertical for the fibration \(J^1 M \rightarrow M\)). In other words, the new vector fields will be written in coordinates as

\[ Y_m = \psi_m^i \left( \partial / \partial x^i \right). \]

Vector fields of this form can be seen as \(\hat{\sigma}\)-prolongations (here \(\hat{\sigma}\) is an extension of the original matrix \(\sigma\), see below) of the trivial vector fields \(X_m = 0\) in \(M\). Needless to say, such vector fields commute with all the \(X_i\), and hence the auxiliary vector fields \(Y_m\) will also commute with all the \(Y_i\), and will be an Abelian subalgebra in the center of the Lie algebra \(\mathcal{Y}\).

As for the matrix \(\hat{\sigma}\), this will be written in block form as

\[ \hat{\sigma} = \begin{pmatrix} \sigma & A \\ \rho & B \end{pmatrix}, \]

where \(\rho\) embodies the relation between the \(\psi_m\) and the \(\psi_i\), while \(A\) and \(B\) are arbitrary (they act on the null \(\psi_m\) vectors) and can be set to zero; if we require to invert \(\hat{\sigma}\) (as happens when discussing the gauge meaning of \(\sigma\)-prolongations, see [7]), then we can equally well set \(A = 0\), \(B = I\).

Finally, note that the presence in \(\mathcal{Y}\) of vector fields of the form (14) can forbid the presence of some \(x_i\) in the differential invariants (this is specially clear if \(\psi_m^i\) are constant); correspondingly, in this case some of the variables in \(M_0\) do actually play the role of parameters. In other words, the invariant DS cannot have full dimension in \(M_0\), and we will be dealing with the DS of dimension \(n_0 < n\). The number of parameters \(\delta = n - n_0\) corresponds, generically, to the number of auxiliary vector fields (14) that must be introduced to complete the involutory system, and more precisely to \(\text{rank}(\mathcal{Y}) - \text{rank}(\mathcal{Y}^i)\).

In this case, it will be more convenient to distinguish between real dynamical variables and parameters; we will thus consider \(M = M_0 \times P\), where \(M_0\) represents the phase manifold and \(P\) the parameter space. The vector fields \(X_i\) (and hence \(Y_i\)) will be allowed to depend on parameters and act on them.

It turns out that also in the case one needs to complete \(\sigma\)-prolongations \(\mathcal{Y}\) in order to have an involution system \(\mathcal{Y}\); if the rank of \(\mathcal{Y}\) is sufficiently small, then we can still perform the reduction. We will now give a more precise description of the reduction procedure in this framework.

**Theorem 2.** Consider an \(n_0\)-dimensional autonomous DS (2) in \(M_0 = \mathbb{R}^n\), with \(n_0 \leq n\). Assume that it admits the set \(X = [X_1, \ldots, X_m]\) of vector fields in \(M_0\), of constant rank \(r_0 < n\), as \(\sigma\)-symmetries. Assume moreover that the completion \(\mathcal{Y}\) of the set \(\mathcal{X}\) corresponding to the \(\sigma\)-prolongation of \(\mathcal{X}\) has rank \(r = r_0 + \delta\) (with \(0 \leq \delta < 2n\)) in \(J^1 M\). Then, (2) can be reduced, passing to suitable symmetry-adapted coordinates, to an autonomous DS of dimension \(\kappa_0 = n - r_0\) and to a set of \(\theta = r_0 - \delta\) ‘reconstruction equations’ depending on the solutions to the reduced system.

**Proof.** As mentioned in the statement, we will consider only autonomous DS and autonomous vector fields; as mentioned above one can always reduce to this setting by the standard procedure of autonomization—i.e. adding a new variable \(x_0\) corresponding to \(t\) and satisfying \(\dot{x}_0 = 1\).

Let the commutation properties between the \(X_i\) be described by (13), and let \(\text{rank}(\mathcal{X}) = r_0 < n\). Thus, there are \(\kappa_0 = n - r_0\) nontrivial invariants \(\zeta_i\) of order zero in the \((n + 1)\)-dimensional phase manifold \(M\) (plus one trivial invariant, corresponding to \(t\)). By the IBDP, these generate \(\kappa_0\) differential invariants \(\zeta_i^{(1)}\) of order 1, which will be referred to as derived invariants.
On the other hand, the vector fields $Y_i$ act in $J^1 M$, which has dimension $(2n + 1)$. By assumption, the set $\hat{Y}$ has rank $r \geq r_0$; we set $r = r_0 + \delta$. Hence, $\hat{Y}$ admits $\kappa_1 = 2n + 1 - r$ invariants in $J^1 M$. Of these, $\kappa_0 + 1$ are invariants of order zero, and $\kappa_0$ are derived invariants. Thus, there are $\theta$ additional invariants $\beta_i$, with

$$\theta = (2n + 1 - r) - (2\kappa_0 + 1) = r_0 - \delta.$$ 

As the $Y$-symmetric system can be written in terms of invariants, we can write it as a DS of dimension $\kappa_1$; there are $\kappa_0 = n - r_0$ equations corresponding to derived invariants, and hence of the form

$$d\zeta_i/dt = \Phi_i(\zeta_1, \ldots, \zeta_\kappa),$$

and $\theta = r_0 - \delta$ reconstruction equations, corresponding to

$$\beta_i = h_i(\zeta_1, \ldots, \zeta_\kappa),$$

as stated. This completes the proof. \(\square\)

4. Constants of motion and $\sigma$-symmetries

In the study of the DS, special attention is given to the search for constants of motion, i.e. of functions $I : M_0 \to \mathbb{R}$ which are constants under the flow of the DS; if the latter is written in the form (2), one is looking for functions such that

$$D_t I = \left( \frac{\partial I}{\partial x^a} \right) f^a(x) = 0.$$

It is well known that strict relations exist between the standard symmetries of a DS and its constants of motion; a relation also exists between the $\sigma$-symmetries of a DS and its constants of motion.

**Theorem 3.** Let the DS (2) admit a set $\mathcal{X}$ of $\sigma$-symmetries of rank $r$; then, it admits $n - r - 1$ independent constants of motion which are simultaneously invariant under the $\sigma$-symmetry vector fields $X_i$.

**Proof.** The $\sigma$-symmetry condition in the form (8) for $\mathcal{X} = \{X_1, \ldots, X_s\}$, $X_i = \psi^a_i(\partial/\partial x^a)$, gives that the enlarged set $\hat{\mathcal{X}} = X_0 \cup \mathcal{X} = \{X_0; X_1, \ldots, X_s\}$ is a set of $\hat{s} = s + 1$ vector fields in involution in the manifold $M_0$ of dimension $n$, and its rank $\hat{r}$ satisfies $r \leq \hat{r} \leq r + 1$ (generically, $\hat{r} = r + 1$). This set of vector fields will span an $\hat{r}$-dimensional distribution and hence will have at least $n - \hat{r}$ independent common invariants. But the invariance under $X_0$ expresses just the property of being a constant of motion of the corresponding DS. \(\square\)

**Remark 9.** Let us consider separately, for completeness, the case in which the DS (2) is actually the autonomized form of a non-autonomous initial DS (1). This means that (2) contains the additional equation $\dot{x}_0 = 1$ for the new variable $x_0$, whereas the original DS (1) involves $n - 1 = \bar{n}$ dependent variables $x_j(t)$ (and $n - 1 = \bar{n}$ equations of course). Then, theorem 1 ensures that there are $\bar{n} + 1 - r$ reduced equations, or just $\bar{n} - r$ reduced equations for the $\bar{n}$ variables $x_j(t)$. Similarly, according to theorem 3, there are $\bar{n} - r$ independent symmetry-invariant constants of motion, or (excluding the trivial one $x_0 - t$) just $\bar{n} - r - 1$ constants of motion, which may depend on time. If instead the initial DS (1) is autonomous and the coefficient functions $\phi_i$ of the vector fields $X_i$ are independent of time, then also the constants of motion provided by the above theorem turn out to be independent of time.
Remark 10. To illustrate an aspect of theorem 3, consider for comparison the special case of a Hamiltonian DS, i.e. a DS of the form $i = J\nabla H$, where $J$ is the standard symplectic matrix and $H$ a given Hamiltonian (independent of time, for simplicity). Assume that a vector field $X$ admits a generating function $G$, i.e. $X = J\nabla G$, and $X$ is a $\lambda$-symmetry for the DS. Then, $G$ is in general not a constant of motion for the DS [3] (although it is trivially invariant under $X$, and—as well known—a constant of motion if $\lambda = 0$). Theorem 3 ensures that there are other constants of motion which are invariant under $X$: in the presence of $m$ degrees of freedom (i.e. $n = 2m$) and of just one vector field $X$, the expected number of these constants of motion is then $2(m - 1)$.

5. Determination of $\sigma$-symmetries

In this section, we would like to briefly discuss some points related to the determination of $\sigma$-symmetries for a given DS (see also [6, 7] for a discussion on the general case). We also give constructive results about $\sigma$-symmetries of a class of DS, see the following subsection.

5.1. General considerations

The determination of all the $\sigma$-symmetries for a given DS is in general beyond reach; this should not be surprising as also the determination of standard symmetries may be far from easy. In the same way as for standard symmetries, however, even a partial knowledge of the symmetry structure can be useful in that it allows for a reduction of the system under study; thus, one can look for special $\sigma$-symmetries, e.g. by an educated guess in view of the features of the system under study.

In general, a given set of vector fields $X_i = \xi_i(x,t)(\partial/\partial t) + \psi_i^a(x,t)(\partial/\partial x^a)$, with $\sigma$-prolongation $Y_i = X_i + \psi_i^a(\partial/\partial \dot{x}^a)$, is a $\sigma$-symmetry of the DS (2) if the $\sigma$-determining equations

$$\psi_i^a - \psi_i^b (\partial f^a/\partial \dot{x}^b) = 0$$

are satisfied on the solutions to (2) itself. These are a system of $n \cdot s$ equations for the unknown vector fields $X_i$ (i.e. for the coefficient functions $\xi_i(t, x)$ and $\psi_i(t, x)$ appearing in them) and for the $s^2$ components of the matrix $\sigma = \sigma(t, x, \dot{x})$.

If we restrict to the vector fields of the form $X_i = \varphi_i^a(x)(\partial/\partial x^a)$, then the $\sigma$-determining equations take the form (8), which can be now more conveniently rewritten in component form

$$\sigma_{ij} \psi_j^a = \psi_i^b (\partial f^a/\partial \dot{x}^b) - f_i^b (\partial \varphi_i^a/\partial \dot{x}^b),$$

where we have written $\sigma$ to emphasize that $\sigma$ is the restriction of $\sigma(t, x, \dot{x})$ to the solution manifold, i.e. $\sigma(t, x) = \sigma(t, x, f(x))$. These are again a system of equations for the unknown vector fields $X_i$, i.e. for the $\varphi_i^a(x)$, and for the $s^2$ components of the $s \times s$ matrix function $\sigma$. Thus, we have strongly under-determined systems, and one could consider further ansätze in searching for solutions. (Note that even for $s = 1$ (so for $\lambda$-symmetries) we have a system of $n$ equations for $n + 1$ unknown functions, i.e. the $\varphi^a$ and the (now, scalar) $\sigma$.)

There seems to be no algorithmic way to solve the system (15) or (16); in practice, one can look for solutions to it only by making simplifying assumptions on the functional form of the unknown functions $\xi_i$, $\psi_i^a$ and $\sigma_{ij}$, or when the DS under study has a specially favorable structure, see below.

Remark 11. In this sense, the new tool in the symmetry analysis of DS we propose here is quite different from standard Lie-point symmetry analysis in that it is not algorithmic; it should be stressed that this is not a degeneration of the DS case, and the same holds for higher
order equations [7] (so from this point of view nothing is gained by transforming the DS into a higher order form, if this is possible). Once again this feature is shared with other (related) extensions of Lie-point symmetries, such as \( \lambda \) and \( \mu \)-symmetries.

**Remark 12.** On the other hand, if we fix \( X_i \) and \( \sigma_{ij} \) (assuming the \( \sigma \)-prolongation of the set \( X \) gives a set \( Y \) in involution; see the discussion in previous sections), determining the most general DS admitting the \( X \) as a \( \sigma \)-symmetry is in principle feasible, albeit sometimes not easy in practice; for its solution one should take into account the commutation properties of the system \( X \). The possibility of determining the most general DS admitting a given involution set \( X \) as \( \sigma \)-symmetry (for a given \( \sigma \)) is shown in some of the examples below.

**Remark 13.** With reference to remark 2 above, note that the discussion of this section does also provide constructive tools to determine foliations with respect to which a given vector field is projectable.

### 5.2. Special structure of the dynamical system

We now want to discuss how a favorable structure of the DS under study can lead to a tractable problem for the determination of \( \sigma \)-symmetries of a given system.

We will consider DS in \( \mathbb{R}^n \) of the form (cf [6])

\[
\dot{x}^a = F^a(x) = f^a(x) + \sum_{k=1}^{s} \alpha_k(x) \varphi^a_k(x),
\]

where \( \alpha_k(x) \) are arbitrary functions, with the property that the ‘simplified’ system

\[
\dot{x}^a = f^a(x)
\]

admits the system \( X = \{ X_i, \ldots, X_s \} \) of vector fields

\[
X_i = \varphi^a_i \frac{\partial}{\partial x^a}
\]

as standard symmetries, and that the \( \{ X_1, \ldots, X_s \} \) thus defined are in involution,

\[
[X_i, X_j] = \beta^{i,j}_{kl} X_k
\]

Then: (i) the \( Y_i \) are in involution and actually satisfy the same involution properties as the \( X_i \); (ii) any DS of the form (17) admits \( X \) as a \( \sigma \)-symmetry and therefore can be reduced according to theorem 1.

**Proof.** We have shown in [7] (see corollary 2 there) that a sufficient (but by no means necessary) condition for the set of \( \sigma \)-prolonged vector fields \( \mathcal{Y} \) to be in involution, and actually satisfy the same involution relations as the original set \( X \), is that \( \sigma \) satisfies a certain equation. In the present case, i.e. if \( \sigma \) is of the form (21) and the \( \beta^{i,j}_{kl} \) are constant, this equation reduces—in the present notation—to

\[
X_i(\sigma_{jk}) - X_j(\sigma_{ik}) + (\sigma_{jm} \beta^{k}_{mj} - \sigma_{jm} \beta^{k}_{mi} - \beta^{m}_{ij} \sigma_{mk}) = 0.
\]

This is always satisfied for \( \sigma \) as in (21). In fact, now \( X_m(\beta^{i}_{m,j}) = 0 \) for all choices of \( i, j, k, m \), so that using (21) and recalling (20), the above equation reads

\[
(\beta^{k}_{jm} X_i(\alpha_m) - \beta^{k}_{im} X_j(\alpha_m) + \beta^{m}_{ij} X_m(\alpha_k)) + \alpha_i \left[ \beta^{m}_{ij} \beta^{k}_{mj} - \beta^{m}_{ij} \beta^{k}_{mi} - \beta^{m}_{ij} \beta^{k}_{mk} \right] + \left( X_i(\alpha_m) \beta^{m}_{mj} - X_j(\alpha_m) \beta^{m}_{mi} - X_m(\alpha_k) \beta^{m}_{ij} \right) = 0.
\]
The first and last terms cancel each other due to $\beta_{ji}^k = -\beta_{ij}^k$, and the term in square bracket vanishes due to the Jacobi identity. Using now (8) or (16) to impose the invariance of the DS (17) under the first $\sigma$-prolongation of the $X_i$ and using the fact that these are standard symmetries of (18), we obtain just the expression (21) for the $\sigma$, and then all hypotheses of theorem 1 are satisfied.

We stress that for this class of DS we have shown not only that the determining equations for $\sigma$-symmetries can be solved, but have actually provided a general class of solutions.

As a special case of (17) consider $\dot{x} = Ax$ for some matrix $A$; then, obviously $\varphi_i = B_i x$, with $B_i$ matrices such that $[A, B_i] = 0$, provide standard symmetry vector fields $X_i$ for (18). These matrices will satisfy $[B_i, B_j] = c_{ij}^k B_k$ and hence $[X_i, X_j] = -c_{ij}^k X_k$, i.e. $\beta_{ij}^k = -c_{ij}^k$, and the vector fields $X_i$ provide a $\sigma$-symmetry for the full DS

$$\dot{x}^a = (Ax)^a + \sum_{k=0}^{a} \alpha_k(x)(B_k x)^a$$

with $B_0 = A$, and with

$$\sigma_{ij} = \alpha_k(x)c_{ij}^k + (B_i x)^a(\partial\alpha_j(x)/\partial x^a).$$

6. Examples

Our examples will be of small dimension ($n \leq 4$); thus, we will slightly change our notation about dependent coordinates, avoiding indices. The original variables will be denoted by Latin letters, in particular, as $(x, y, z, w)$, and the symmetry-adapted ones by Greek letters, in particular, as $(\xi, \eta, \chi)$ for variables entering in the reduced system, and as $(\mu, \nu, \rho)$ for variables determined by the reconstruction equations. To simplify notations we will often write $\partial_i$ instead of $\partial/\partial x_i$, etc.

It can be noted that a comparison with what could be achieved in the examples below by the use of standard symmetries is not easy nor immediate, because one is not able in general to determine the full set of standard symmetries of first-order systems.

6.1. Example 1

We start by considering a nearly trivial case, i.e. with only one $X$; in other words, this is a $\lambda$-symmetry (which is a special case of $\sigma$-symmetry). We consider it in order to have a specially simple case at hand to look at through our present approach. We consider a system which is already in the symmetry-adapted form (and coordinates), so that what matters here is just the interpretation.

Let us consider the DS

$$\dot{x} = f(x, y)$$
$$\dot{y} = g(x, y)$$
$$\dot{z} = h(x, y) + f(x, y)z$$

with $f$, $g$ and $h$ being arbitrary smooth functions.

This admits as $\sigma$-symmetry the vector field $X = \partial_\tau$, with $\sigma = \dot{\tau}$; this implies that

$$Y = \partial/\partial z + \dot{x}\partial/\partial \dot{z}.$$ 

At order zero we have two invariants (apart from the trivial one, $\tau = t$) for $X$ and hence for $Y$, i.e.

$$\zeta_1 = x; \; \zeta_2 = y.$$
The total derivatives of these provide differential invariants of order 1 for \( Y \), i.e.
\[
\zeta^{(1)}_1 = D_t \zeta_1 = \dot{x}, \quad \zeta^{(1)}_2 = D_t \zeta_2 = \dot{y}.
\]

There is a third differential invariant of order 1,
\[
\beta = \dot{z} - \dot{x} z.
\]

Thus, we have a set of (invariant) reduced equations, which are just the first two of (22), and a reconstruction equation which is just the third one of these. Note that this should be thought as defined on the solution to the first set; in this sense, the reconstruction equation can be rewritten as
\[
\beta = k,
\]
where \( k \) should be seen as a constant on the solutions to the reduced set (hence as an arbitrary function of the invariants \( \zeta_i, \zeta_i^{(1)} \)). Indeed, if we write this equation in explicit form we have
\[
\dot{z} - \dot{x} z = k(x, y, \dot{x}, \dot{y});
\]
on the solutions to the reduced set this reads precisely (recall the arbitrariness of \( k \))
\[
\dot{z} = f(x, y)z + k[x, y, f(x, y), g(x, y)] = h(x, y) + f(x, y) z.
\]
Note once (and if) that a solution \( x(t), y(t) \) to the reduced equation is given; this reconstruction equation is a (non-autonomous) linear equation for \( z(t) \),
\[
\dot{z} = H(t) + F(t) z,
\]
and hence can be solved.

According to theorem 3, there is 1 = \( n - r - 1 \) constant of motion for the DS (22) which is also invariant under \( X = \partial / \partial z \). In this case, the result is trivial: it is enough to consider a constant of motion which depends only on \( x \) and \( y \).

6.2. Example 2

We now consider a situation rather similar to the previous one, but with a system which is not already in symmetry-adapted form.

Consider the three-dimensional DS
\[
\begin{align*}
\dot{x} &= - 2 z - x y^2 (x^2 + z^2) + x y z \log(y^2), \\
\dot{y} &= y^2 (y (x^2 + z^2) - z \log(y^2)), \\
\dot{z} &= 2 x - y^2 z (x^2 + z^2) + y z^2 \log(y^2).
\end{align*}
\]
(23)

If the system had a constant \emph{in lieu} of the \( \log(y^2) \) terms (or if the argument was instead e.g. \( xy \)), then it would admit as standard symmetry the vector field
\[
X = x \partial_x - y \partial_y + z \partial_z;
\]
one can look for a \( \sigma \) such that it is however a \( \sigma \)-symmetry. This is the case, i.e. \( X \) is a \( \sigma \)-symmetry (actually, having only one vector field, this is a \( \lambda \)-symmetry), e.g. with
\[
\sigma = \dot{x} y + x \dot{y} = D_t(xy).
\]
In fact, the \( \sigma \)-prolonged vector field is in this case
\[
Y = X + (\dot{x} + x \dot{y} + x^2 \dot{y}) \frac{\partial}{\partial x} - (\dot{y} + x \dot{y}^2 + xy \dot{y}) \frac{\partial}{\partial y} + (\dot{z} + x \dot{y} + xy \dot{z}) \frac{\partial}{\partial z},
\]
and one can easily check that the system (23) does indeed admit \( X \) as \( \sigma \)-symmetry.
Nontrivial invariants \( \zeta_1 \) and \( \zeta_2 \) of order zero (for \( X \) and hence for \( Y \) as well) are
\[
\zeta_1 = xy, \quad \zeta_2 = yz
\]
which will be chosen as new (symmetry-adapted) variables \( \xi = \zeta_1, \eta = \zeta_2 \); we can choose as additional variable \( \rho = 1 + y^2 \) in order to have symmetry-adapted coordinates. In the new coordinates,
\[
Y = 2(1 - \rho) \frac{\partial}{\partial \rho} - 2(\rho - (1 - \rho)\dot{\xi}) \frac{\partial}{\partial \rho}.
\]
It is immediate to check that \( \dot{\xi} = D_t \xi \) and \( \dot{\eta} = D_t \eta \) are differential invariants of order 1. Moreover, we have an additional differential invariant of order 1; for this we can pick
\[
\mu = 2(\dot{y}/y) - (\dot{y} + xy) \log(y^2).
\]
Passing to the symmetry-adapted variables, the system (23) is rewritten as
\[
\dot{\xi} = -2\eta, \quad \dot{\eta} = 2\xi; \quad \dot{\rho} = 2(\xi^2 + \eta^2).
\]
The first two represent the \( \sigma \)-symmetry reduced system, whose solution is of course
\[
\xi(t) = \alpha \cos(2t + \beta), \quad \eta(t) = \alpha \sin(2t + \beta);
\]
the third one is the reconstruction equation, which on solutions to the reduced system reads
\[
\dot{\rho} = 2\alpha^2; \quad \rho(t) = \rho_0 + 2\alpha^2 t.
\]
A related example is obtained considering the three-dimensional DS
\[
\begin{align*}
\dot{x} &= -x (1 + y(x - z) - xy^2 z \log(xz)), \\
\dot{y} &= y (1 + y(x + z) - xy^2 z \log(xz)), \\
\dot{z} &= z (1 - y(x + z) + xy^2 z \log(xz)).
\end{align*}
\]
This system admits the same \( \sigma \)-symmetry (same \( X \) and same \( \sigma \)) as above, and then the same invariants and adapted coordinates; in these, it is rewritten as
\[
\dot{\xi} = 2\xi \eta; \quad \dot{\eta} = 2\xi; \quad \dot{\rho} = -2\xi.
\]
The first two represent the \( \sigma \)-symmetry reduced system (which in this case happens to be further reducible), whose solution is
\[
\xi(t) = \exp[(e^2t - 1)\eta_0] \xi_0, \quad \eta(t) = e^{2t}\eta_0;
\]
the third one is the reconstruction equation, which on solutions to the reduced system gives
\[
\rho(t) = \rho_0 - 2\xi_0 \int_0^t \exp[(e^{2r} - 1)\eta_0] \, dr.
\]
According to theorem 3, there is here just one constant of motion which is \( X \)-invariant: it is given by \( I = x/z \).

6.3. Example 3

This is an example of orbital reduction. The DS
\[
\begin{align*}
\dot{x} &= yz, \quad \dot{y} = z + xy, \quad \dot{z} = w + xz, \quad \dot{w} = y + xw
\end{align*}
\]
admits the two vector fields
\[
X_1 = \frac{\partial}{\partial x}, \quad X_2 = y \frac{\partial}{\partial y} + z \frac{\partial}{\partial z} + w \frac{\partial}{\partial w}
\]
as a $\sigma$-orbital symmetry. Indeed, this DS satisfies the condition (9) in the form

$$[X_1, X_0] = z X_2, \quad [X_2, X_0] = X_0 + yz X_1.$$  

The two common invariants under $X_1$ and $X_2$

$$\xi_1 = y/z, \quad \xi_2 = w/z$$

satisfy the orbitally reducible system

$$\dot{\xi}_1 = y \left( 1 - \frac{\xi_2}{\xi_1} \right), \quad \dot{\xi}_2 = y \left( 1 - \frac{\xi_2}{\xi_1} \right),$$

in agreement with remark 3. The reduced equation is

$$\frac{d\xi_1}{d\xi_2} = \frac{1 - \xi_1 \xi_2}{\xi_1 - \xi_2^2}. $$

### 6.4. Example 4

As stated in section 5.1, finding the most general DS which admits a given $\sigma$-symmetry is often feasible. We consider the two (commuting) vector fields,

$$X_1 = -2z \frac{\partial}{\partial x} + \frac{\partial}{\partial z}, \quad X_2 = 8yz \frac{\partial}{\partial x} + 2w \frac{\partial}{\partial y} - 4yw \frac{\partial}{\partial w} + \frac{\partial}{\partial w};$$

and choose

$$\sigma = \begin{pmatrix} 0 & \dot{x} + 2zz^2 \\ \dot{y} - 2w \dot{w} & 0 \end{pmatrix}.$$  

The $\sigma$-prolonged vector fields will be written as $(k = 1, 2)$

$$Y_k = \psi_k^i \left( \frac{\partial}{\partial x_i} \right) + \psi_k^i \left( \frac{\partial}{\partial \dot{x}_i} \right),$$

where the prolongation coefficient vectors are

$$\psi_1 = (-2z + 8Ayzw, 2Aw, -4Ayw, A), \quad \psi_2 = (8yz \dot{z} + 2B \dot{z}, 2\dot{w}, -B, 0);$$

here, $A = 2z \dot{z} + \dot{x}, \quad B = (4w - 1) \dot{y} + 2(2y + w) \dot{w}$. Note that $[X_1, X_2] = 0$ and $[Y_1, Y_2] = 0$.

There are two nontrivial common invariants of order zero, i.e.

$$\xi_1 = x + z^2, \quad \xi_2 = y - w^2;$$

and correspondingly we obtain first-order differential invariants

$$\xi_1^{(1)} = \dot{x} + 2zz \dot{z}, \quad \xi_2^{(1)} = \dot{y} - 2w \dot{w}.$$  

The two additional first-order invariants are

$$\beta_1 = \dot{w} - (z + y^2)(\dot{x} + 2z \dot{z}), \quad \beta_2 = \dot{z} + 2y \dot{y} - w (\dot{y} - 2w \dot{w}).$$

We take as symmetry-adapted coordinates

$$\xi = \xi_1 = x + z^2, \quad \eta = \xi_2 = y - w^2, \quad \mu = z + y^2, \quad \nu = w;$$

note the first two are invariants, while the other two rectify the $X_i$. With these coordinates, we have of course $\xi_1^{(1)} = \dot{\xi}, \xi_2^{(1)} = \dot{\eta}$; moreover,

$$\beta_1 = \dot{\mu} - \xi \dot{\mu}, \quad \beta_2 = \dot{\eta} - \eta \dot{\nu}.$$
The vector fields read now \( X_1 = \partial_\mu, X_2 = \partial_\nu \); as in these coordinates
\[
\sigma = \begin{pmatrix} 0 & \dot{\xi} \\ \dot{\eta} & 0 \end{pmatrix}.
\]
the \( \sigma \)-prolonged ones are just
\[
Y_1 = (\partial/\partial \mu) + \dot{\xi} (\partial/\partial \dot{\nu}), \quad Y_2 = (\partial/\partial \nu) + \dot{\eta} (\partial/\partial \dot{\mu}).
\]
Thus, any system of the form
\[
\begin{align*}
\dot{\sigma} &= [1 + 8yzw(y^2 + z)]F + 2z[(2y - w)G - H + 4ywK] \\
\dot{\eta} &= 2(y^2 + z)wF + G + 2uK \\
\dot{z} &= -4yw(y^2 + z)F + (w - 2y)G + H - 4ywK \\
\dot{w} &= (y^2 + z)F + K
\end{align*}
\]
where we have noted \( F = f(\zeta_1, \zeta_2), G = g(\zeta_1, \zeta_2), H = h(\zeta_1, \zeta_2) \) and \( K = k(\zeta_1, \zeta_2) \), admits \( [X_1, X_2] \) as \( \sigma \)-symmetry with the \( \sigma \) considered above (this is not the most general case, but is the most general—up to multiplication by nowhere vanishing functions)—one if we require to have polynomial functions; in this case, \( f, g, h \) should of course be polynomial.

As stated by our theorems, any system in the class (24) can be reduced via \( \sigma \)-symmetry reduction. In fact, passing to symmetry-adapted coordinates as above yields the system (24) in the form
\[
\begin{align*}
\dot{\xi} &= f(\xi, \eta), \\
\dot{\eta} &= g(\xi, \eta); \\
\dot{\mu} &= h(\xi, \eta) + g(\xi, \eta) \nu, \\
\dot{\nu} &= k(\xi, \eta) + f(\xi, \eta) \mu.
\end{align*}
\]
The first two equations represent the reduced system, while the last two are the reconstruction equations.

The latter can be written, on solutions to the reduced system, in the form \( \beta_i = \kappa_i(\zeta_1, \zeta_2, \zeta_1^{(i)}, \zeta_2^{(i)}) \) (\( i = 1, 2 \)) for a suitable choice of the functions \( \kappa_i \). In fact, using the explicit form of \( \beta_i \) we would have
\[
\dot{\nu} - \dot{\xi} \mu = \kappa_1(\xi, \eta, \dot{\xi}, \dot{\eta}), \quad \dot{\mu} - \dot{\eta} \nu = \kappa_2(\xi, \eta, \dot{\xi}, \dot{\eta});
\]
on the solutions to the reduced equations these read
\[
\begin{align*}
\dot{\nu} &= \kappa_1(\xi, \eta, f(\xi, \eta), g(\xi, \eta) \mu) = k(\xi, \eta) + f(\xi, \eta) \mu, \\
\dot{\mu} &= \kappa_2(\xi, \eta, f(\xi, \eta), g(\xi, \eta) \nu) = h(\xi, \eta) + g(\xi, \eta) \nu.
\end{align*}
\]
On a given solution \( (\xi(t), \eta(t)) \) to the reduced equations, these are a system of non-autonomous linear equations
\[
\begin{align*}
\dot{\mu} &= H(t) + G(t) \nu, \\
\dot{\nu} &= K(t) + F(t) \mu,
\end{align*}
\]
in agreement to our general discussion.

As a concrete example, one can consider
\[
\begin{align*}
\dot{x} &= 1 + 4yz - w(z + y^2) \\
\dot{y} &= 1 + 2w(z + y^2) \\
\dot{z} &= w - 2y - 4yw(z + y^2) \\
\dot{w} &= z + y^2.
\end{align*}
\]
In the adapted coordinates, this reads
\[
\dot{\xi} = 1, \quad \dot{\eta} = 1; \quad \dot{\mu} = \nu, \quad \dot{\nu} = \mu.
\]
In agreement with theorem 3, there is one \( X \)-invariant constant of motion \( (n = 4, r = 2) \), which is given by \( I = x - y + z^2 + w^2 \).
6.5. Example 5

Let us now consider vector fields
\[ \mathbf{X}_1 = \frac{1}{1+2x} \left( \partial_y - \partial_x \right), \quad \mathbf{X}_2 = \partial_z; \]
and the matrix
\[ \sigma = \begin{pmatrix} 0 & \dot{x} + \dot{y} \\ \dot{x} + \dot{y} & 0 \end{pmatrix}. \]

The \( \sigma \)-prolonged vector fields are
\[ \mathbf{Y}_1 = \mathbf{X}_1 + \frac{1}{(1+2x)^2} \left[ 2x \left( \frac{\partial}{\partial x} \dot{x} - \frac{\partial}{\partial y} \dot{y} \right) + (1+2x)^2 (\dot{x} + \dot{y}) \frac{\partial}{\partial z} \right], \]
\[ \mathbf{Y}_2 = \mathbf{X}_2 + (\dot{x} - \dot{y}) \frac{\partial}{\partial \nu} + \frac{x}{1+2x} \frac{\partial}{\partial \mu} \].

It is immediate to check that \([\mathbf{X}_1, \mathbf{X}_2] = 0\) and \([\mathbf{Y}_1, \mathbf{Y}_2] = 0\).

The most general system admitting \( \mathcal{X} = \{ \mathbf{X}_1, \mathbf{X}_2 \} \) as \( \sigma \)-symmetry is easily determined, and is given by
\[ \dot{x} = -(1+2x)^{-1} \left[ (1-x-y-z) F + G \right], \]
\[ \dot{y} = (1+2x)^{-1} \left[ (3x+y+z) F + G \right], \]
\[ \dot{z} = -(1-y+x^2) F + H, \]
(25)
where \( F = f(x+y) \), \( G = g(x+y) \), \( H = h(x+y) \).

The nontrivial invariant of order zero for \( \mathbf{X}_1, \mathbf{X}_2 \), and hence also for \( \mathbf{Y}_1, \mathbf{Y}_2 \), is
\[ \zeta = x + y; \]
it follows from our general discussion that the total derivative of this, i.e. \( \zeta^{(1)} = \dot{x} + \dot{y} \), is a first-order differential invariant, and indeed one checks easily this is the case. There are two additional first-order invariants, which are
\[ \beta_1 = \dot{y} - z \dot{x}, \quad \beta_2 = \dot{z} - y \dot{x}. \]

Passing to symmetry-adapted coordinates
\[ \xi = \zeta = x + y, \quad \mu = y - x^2, \quad \nu = x + y + z, \]
the basic and \( \sigma \)-prolonged vector fields read
\[ \mathbf{X}_1 = \partial_\mu, \quad \mathbf{X}_2 = \partial_\nu; \]
\[ \mathbf{Y}_1 = \left( \partial/\partial \mu \right) + \xi \left( \partial/\partial \nu \right), \quad \mathbf{Y}_2 = \left( \partial/\partial \nu \right) + \dot{\xi} \left( \partial/\partial \mu \right). \]
The general system (25) reads, in these coordinates,
\[ \dot{\xi} = f(\xi), \]
\[ \dot{\mu} = f(\xi) \nu + g(\xi), \]
\[ \dot{\nu} = f(\xi) \mu + h(\xi). \]
The reduced system is just the first equation, the other two representing the reduction equations; these are indeed of the form \( \beta_1 = g(\zeta) \) and \( \beta_2 = h(\zeta) \).

Note that in this (and the following) example, theorem 3 does not admit the presence of \( \sigma \)-invariant constants of motion: we have indeed \( n - r - 1 = 0 \).

A more concrete case is obtained e.g. by choosing
\[ f(x) = x - x^3, \quad g(x) = x, \quad h(x) = x^2. \]
With these choices, the original system has the quite involved expression
\[\dot{x} = -(1 + 2x)^{-1}(x + y)^2(1 - x - y - z) + (x + y)(x + y + z),\]
\[\dot{y} = (1 + 2x)^{-1}[(x + y)(1 - (3x + y)(x + y)^2 + 3x + y + z(1 - x - y)(1 + x + y))],\]
\[\dot{z} = (x + y)(y - 1 - x^2) + (x + y)^2 + (1 + x)^3(1 - y + x^2).\]
Passing to symmetry-adapted coordinates this system reads
\[\dot{\xi} = \xi - \xi^3,\]
\[\dot{\mu} = (\xi - \xi^3)\nu + \xi,\]
\[\dot{\nu} = (\xi - \xi^3)\mu + \xi^2.\]

6.6. Example 6
Let us now consider a situation with a reduced system of dimension 1. We consider the four-dimensional DS
\[\dot{x} = -(x + z^2) + e^{-(y+z)} + 2z(e^{-y+w^2} + 2e^{-w}w - 2e^{-(y+z)})(x + z^2),\]
\[\dot{y} = (e^{-y+w^2} + 2e^{-w}w)(x + z^2),\]
\[\dot{z} = -(e^{-y+w^2} + 2e^{-w}w - 2e^{-(y+z)})(x + z^2) - e^{-(y+z)} w,\]
\[\dot{w} = e^{-w}(x + z^2).\] (26)

We are apparently clueless in front of such an involved DS, and this is maybe the appropriate place to show how one can proceed in trying to determine \(\sigma\)-symmetries for such a system. The recurring term \((x + z^2)\) suggests to look for vector fields which admit this term as an invariant. This requirement just selects vector fields
\[\sigma^1 = -2z\psi^3;\] we obviously have three functionally independent such fields, and we can e.g. choose the vector fields
\[X_1 = 2z\partial_x + \partial_y - \partial_z, \quad X_2 = -2z\partial_x + \partial_z, \quad X_3 = 4zw\partial_x + 2w\partial_y - 2w\partial_z + \partial_w;\]
note that these \(X_i\) have been chosen in order to be autonomous, simple, and to commute with each other; moreover, the rectifying change of coordinates will have a Jacobian with a unit determinant.

We can then look for a \(\sigma\) matrix satisfying the \(\sigma\)-determining equations (16); we thus have a system of twelve equations for the nine functions \(\sigma_{ij}\). In this case, one obtains by simple linear algebra that \(\sigma\) must be diagonal, and more precisely
\[\sigma = \text{diag}[-e^{-y+w^2}(x + z^2), \quad e^{-(y+z)}(1 - 2(x + z^2)), \quad -e^{-w}(x + z^2)].\]
This expression is not very nice, but using the equations of motion (26) we can rewrite
\[e^{-y+w^2} = \left(\frac{\dot{y} - 2w\dot{w}}{x + z^2}\right)_S, \quad e^{-(y+z)} = \left(-\frac{\dot{y} + \dot{z}}{1 - 2(x + z^2)}\right)_S, \quad e^{-w} = \left(\frac{\dot{w}}{x + z^2}\right)_S\]
(where as usual \(S = S(\mathcal{E})\) denotes the restriction to the solutions of the system); with these, we can write
\[\sigma = -\text{diag}[-2w\dot{w}, \dot{y} + \dot{z}, \dot{w}] = D_x \left[-\text{diag}[y - w^2, y + z, w]\right].\]

One can quite easily evaluate the \(\sigma\)-prolonged vector fields \(Y_i\) and verify that also commute with each other. The only common invariant of order zero for the \(X_i\) (and hence also for the \(Y_i\)) is
\[\zeta = \xi = x + z^2.\]
We can complete the system of coordinates by defining
\[ \mu = y - w^2, \quad \nu = y + z, \quad \rho = w; \]
this yields a Jacobian with unit determinant.

The total derivative of \( \zeta \) provides a differential invariant
\[ \zeta^{(1)} = D_t \zeta = \dot{x} + 2z \dot{z}. \]
There are three additional first-order differential invariants, which can be chosen as
\begin{align*}
\beta_1 &= (\dot{y} - 2w \dot{w}) \exp(y - w^2), \\
\beta_2 &= (\dot{y} + \dot{z}) \exp(y + z), \\
\beta_3 &= \dot{w} \exp(w).
\end{align*}

In the symmetry-adapted coordinates, the basic vector fields read
\[ X_1 = \partial_\mu, \quad X_2 = \partial_\nu, \quad X_3 = \partial_\rho, \]
so that these are indeed rectifying coordinates; in these same coordinates, the \( \sigma \)-prolonged vector fields read
\begin{align*}
Y_1 &= \partial_\mu - \dot{\mu} \partial_{\dot{\mu}}, \\
Y_2 &= \partial_\nu - \dot{\nu} \partial_{\dot{\nu}}, \\
Y_3 &= \partial_\rho - \dot{\rho} \partial_{\dot{\rho}}.
\end{align*}

Note that the \( \beta_i \) do now read simply
\[ \beta_1 = \dot{\mu} e^\mu, \quad \beta_2 = \dot{\nu} e^\nu, \quad \beta_3 = \dot{\rho} e^\rho. \]

Finally, the DS under study reads in these variables
\begin{align*}
\dot{\xi} &= -\xi, \\
\dot{\mu} &= e^{-\mu} \xi, \\
\dot{\nu} &= e^{-\nu} (2\xi - 1), \\
\dot{\rho} &= e^{-\rho} \xi.
\end{align*}
We thus have a one-dimensional reduced system (the equation for \( \dot{\xi} \)) and three reconstruction equations; these can be written as \( \beta_i = h_i(\xi) \), as our discussion shows to be always the case; in this case, we have \( h_1(\xi) = h_3(\xi) = \xi, h_2(\xi) = 2\xi - 1. \)

6.7. Example 7

In the examples discussed so far, we were in the situation considered by theorem 1, i.e. \( \mathcal{X} \) and \( \mathcal{Y} \) had the same involution relations, and hence in particular the same rank. We will now consider a case within the framework of theorem 2.

Let us consider the (commuting) vector fields
\[ X_1 = \partial_y, \quad X_2 = 2z \partial_t - (x + 2z^2) \partial_y + \partial_t, \]
and the matrix
\[ \sigma = \begin{pmatrix} \dot{x} - 2z \dot{z} & 0 \\ 0 & \dot{y} + x \dot{z} + z \dot{x} \end{pmatrix}. \]
The \( \sigma \)-prolonged vector fields are
\begin{align*}
Y_1 &= X_1 + (\dot{x} - 2z \dot{z})(\partial/\partial y); \\
Y_2 &= X_2 + 2(\dot{z} + z(\dot{y} + x \dot{z} + z \dot{x}))(\partial/\partial x) \\
&= (\dot{x} + 4z \dot{z} + (x + 2z^2)(\dot{y} + x \dot{z} + z \dot{x}))((\partial/\partial x)(\partial/\partial \dot{x}) + (\dot{y} + x \dot{z} + z \dot{x}))(\partial/\partial \dot{z}).
\end{align*}
These do not commute; we have instead
\[ [Y_1, Y_2] = Y_1 - 2z(x - 2z\dot{z})(\partial/\partial \dot{x}) + (x + 2\dot{z}^2)(2zz\dot{z} - \dot{x})(\partial/\partial \dot{y}) + (\dot{x} - 2xz^2)(\partial/\partial \dot{z}). \]

It is easy to check that (as stated in our general discussion, see section 3) the auxiliary vector field \( Y_3 \) commutes with the other ones: \([Y_1, Y_3] = 0 = [Y_2, Y_3].\)

With the notation of section 3, we have \(n = 3, n_0 = 2, r_0 = 2\) and \(r = 3\); it follows that the reduced DS has dimension \(\kappa_0 = n - r_0 = 1\), and the number of reconstruction equations is \(\theta = r_0 - \delta = 1\).

The nontrivial invariant of order zero and the corresponding first-order differential invariant are
\[ \zeta = \xi = x - z^2; \quad \zeta^{(1)} = D_\xi \xi = \dot{x} - 2z \dot{z}; \]
there is an additional first-order invariant, which turns out to be
\[ \beta = (\dot{y} + x \dot{z} + z \dot{x}) - (y + xz)(\dot{x} - 2z \dot{z}). \]

Any system of the form
\[ \begin{align*}
\dot{x} &= f(x - z^2) + 2z \dot{z}, \\
\dot{y} &= -zf(x - z^2) + (y + xz)f(x - z^2) + g(x - z^2) - x \dot{z}
\end{align*} \]  
(27)

is (by construction) invariant under the system \( \mathcal{Y} = \{Y_1, Y_2, Y_3\} \) of vector fields; this is the completion of \( \mathcal{Y} \), the \( \sigma \)-prolongation of \( \mathcal{X} = \{X_1, X_2\} \). Note that in (27) the variable \( z \) should be considered as a parameter; it can change in time, with speed \( \dot{z} \), or we can set this to zero, obtaining a slightly less general set of systems
\[ \begin{align*}
\dot{x} &= f(x - z^2), \\
\dot{y} &= -zf(x - z^2) + (y + xz)f(x - z^2) + g(x - z^2).
\end{align*} \]

The adapted coordinates for the vector fields we are considering are
\[ \xi = x - z^2, \quad \eta = y + xz; \quad \mu = z. \]
(Note \( \beta = \dot{\eta} - \eta \dot{\xi} \).) In terms of these, we have \( X_1 = \partial_\eta, X_2 = \partial_\mu, \) and
\[ \begin{align*}
Y_1 &= (\partial/\partial \eta) + \xi (\partial/\partial \dot{\eta}), \\
Y_2 &= (\partial/\partial \mu) + \eta (\partial/\partial \dot{\mu}), \\
Y_3 &= \dot{\xi} (\partial/\partial \dot{\mu}).
\end{align*} \]

In these variables, the system (27) reads
\[ \begin{align*}
\dot{\xi} &= f(\xi), \\
\dot{\eta} &= g(\xi) + \eta f(\xi).
\end{align*} \]

Needless to say, the first equation represents the reduced system, while the second is the reconstruction equation; on solutions to the former one, the latter can also be written as \( \beta = g(\xi) \).

6.8. Example 8

Finally, let us consider an example of the situation dealt with in section 5.2 (see in particular theorem 4), i.e. the family of DS
\[ \begin{align*}
\dot{x} &= x + ay + \alpha_1(x, y, z)x + \alpha_2(x, y, z)y \\
\dot{y} &= ax + y + \alpha_2(x, y, z)x + \alpha_1(x, y, z)y \\
\dot{z} &= bz + \alpha_1(x, y, z)z
\end{align*} \]  
(28)
with $a$, $b$ constants and $\alpha_1$, $\alpha_2$ smooth functions. The (commuting) vector fields

$$X_1 = x \partial_x + y \partial_y + z \partial_z, \quad X_2 = y \partial_x + x \partial_y$$

are standard symmetries for the linear part of the above DS; note that there would also be another standard symmetry for this linear DS, as $z \partial_z$ and $x \partial_x + y \partial_y$ are separately symmetries for it, but only $X_1$ and $X_2$ enter in the nonlinear part of the DS in the way discussed in section 5.1, with $\alpha_i$ associated with $X_i$.

The vector fields $X_1$ and $X_2$ are standard symmetries of the full DS (28) only if the $\alpha_i(x, y, z)$ depend uniquely on their common invariant $\zeta = (x^2 - y^2)/z^2$; we consider the case where they are instead arbitrary smooth functions.

According to theorem 4, they are a $\sigma$-symmetry for the full system with $\sigma$ given by equation (21); as $[X_1, X_2] = 0$, this equation yields simply $\sigma_{ij} = X_i(\alpha_j)$ or

$$\sigma = \begin{pmatrix}
(\alpha_1)_x x + (\alpha_1)_y y + (\alpha_1)_z z \\
(\alpha_2)_x y + (\alpha_2)_y y + (\alpha_2)_z z \\
\end{pmatrix}.$$ 

With this choice of $\sigma$, some standard algebra provides the $\sigma$-prolonged vector fields $Y_1$, $Y_2$ and shows that these are indeed $\sigma$-symmetries for the system (28), for any choice of the smooth functions $\alpha_1$ and $\alpha_2$.

According to theorem 1, one can introduce the common invariant

$$\zeta = \xi = (x^2 - y^2)/z^2$$

and then obtain the reduced equation, which should be in the form $\dot{\xi} = f(\xi)$; indeed, using (28) we obtain easily that

$$\dot{\xi} = 2(1 - b) \xi.$$ 

The additional first-order invariants and the reconstruction equations would of course depend on the functions $\alpha_i$.

We specialize this example to a concrete DS considering

$$\dot{x} = x - y - x^2 y - yz^2$$
$$\dot{y} = y - x - x(y^2 + z^2)$$
$$\dot{z} = 2z - xyz$$

(29)

In this case, we obtain

$$\sigma = \begin{pmatrix}
-2xy & -2z^2 \\
-(x^2 + y^2) & 0 \\
\end{pmatrix}.$$ 

The $\sigma$-prolonged vector fields are easily obtained and again one can check that these are indeed $\sigma$-symmetries for the system (29). Beside the invariants $\xi = (x^2 - y^2)/z^2$ and $\zeta^{(1)} = D_t \xi$, one has the additional first-order differential invariants

$$\mu = \frac{(x^2 - y^2)z^2 + (x \dot{y} - y \dot{x})}{x^2 - y^2}, \quad \nu = \frac{xy(x^2 - y^2) + (x \dot{x} - y \dot{y})}{x^2 - y^2}.$$ 

The system (29) is then rewritten as

$$\dot{\xi} = -2 \xi; \quad \mu = -1, \quad \nu = 1.$$ 

The first of this represents the reduced DS, and the latter two the reconstruction equations.
7. Conclusions

In a recent work [7], we have introduced a modification of the prolongation operation on sets of vector fields, called $\sigma$-prolongation; and correspondingly the concept of $\sigma$-symmetries for systems of ODEs. The formulation of that paper was not able to deal with the special but relevant case of dynamical systems (systems of first order ODEs). The aim, and the main result, of this paper is to extend our approach to encompass also the case of dynamical systems.

We were able to deal both with the case where the original and the $\sigma$-prolonged vector fields generate a foliation of the same rank (theorem 1) and the case where the ranks differ (theorem 2); in both cases, one is led to study a reduced system together with a set of reconstruction equations. The latter, at difference with the case met in the reduction based on standard symmetries, do not amount to quadrature but are a set of first-order ODEs, which might well be hard to solve in practice.

Our approach focuses on foliations generated by symmetry (in this case, $\sigma$-symmetry) vector fields; it is thus entirely natural that it falls within the general Frobenius theory. In this sense, a $\sigma$-symmetry for a dynamical vector field $X_0$ with (standard) prolongation $Y_0$ is a set of vector fields $X_i$ whose $\sigma$-prolongations $Y_i$ span a foliation $\mathcal{F}$ such that $Y_0$ commutes with vector fields in $\mathcal{F}$ modulo $\mathcal{F}$ itself. This situation can also be described by saying that $Y_0$ is projectable with respect to the foliation $\mathcal{F}$ [19].

It is thus natural that a direct application of the classical Frobenius theorem provides the relation between $\sigma$-symmetries and constants of motion (theorem 3).

The actual determination of $\sigma$-symmetries can be a highly nontrivial task, as the determining equations for these do not share the nice properties of those for standard symmetries. For dynamical systems with a special structure, one can try to use this to relate $\sigma$-symmetries to standard symmetries of a related (maybe simpler) system. We provided some constructive result in this direction (theorem 4).

It should be mentioned that our theory is able to deal not only with reduction, but also with orbital reduction [13, 34]; again this is quite natural once one adopts the Frobenius point of view, as foliations and their integral manifolds (or curves) do not depend on the parametrization of vector fields and curves.

In conclusion, dealing with dynamical systems allowed to have a clearer geometrical picture with respect to the general case (considered in previous work [7]); this suggests that one should focus on foliations generated by symmetry—or orbital symmetry—vector fields rather than on the individual symmetry vector fields; at the algebraic level, this suggests to focus on the Lie module structure rather than just on the Lie algebra one (see also [6]).

As already mentioned in the introduction (and in [7]), our approach should be seen as a development of the approach by Pucci and Saccomandi [32] to the Muriel and Romero beautiful and fruitful idea of $\lambda$-symmetries [20].

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References

[1] Alekseevsky D V, Vinogradov A M and Lychagin V V 1991 Basic Ideas and Concepts of Differential Geometry (Berlin: Springer)
[2] Carinena J F, Del Olmo M and Winternitz P 1993 On the relation between weak and strong invariance of differential equations Lett. Math. Phys. 29 151–63
[3] Cicogna G 2008 Reduction of systems of first-order differential equations via $\Lambda$-symmetries Phys. Lett. A 372 3672–7

Cicogna G 2009 Symmetries of Hamiltonian equations and \( \Lambda \)-constants of motion. *J. Nonlinear Math. Phys.* **16** 43–60

[4] Cicogna G and Gaeta G 1999 *Symmetry and Perturbation Theory in Nonlinear Dynamics* (Berlin: Springer)

[5] Cicogna G, Gaeta G and Morando P 2004 On the relation between standard and \( \mu \)-symmetries for PDEs. *J. Phys. A: Math. Gen.* **37** 9467–86

Gaeta G and Morando P 2004 On the geometry of lambda-symmetries and PDE reduction *J. Phys. A: Math. Gen.* **37** 6955–75

[6] Cicogna G, Gaeta G and Walcher S 2013 Orbital reducibility and a generalization of lambda symmetries *J. Lie Theory* **23** 357–81

[7] Cicogna G, Gaeta G and Walcher S 2013 A generalization of \( \lambda \)-symmetry reduction for systems of ODEs: \( \sigma \)-symmetries. *J. Phys. A: Math. Theor.* **45** 355205

[8] Gaeta G 1995 A splitting lemma for equivariant dynamics. *Lett. Math. Phys.* **33** 313–20

Gaeta G 1995 Splitting equivariant dynamics *Nuovo Cimento B* **110** 1213–26

[9] Gaeta G 1995 Twisted symmetries of differential equations. *J. Nonlinear Math. Phys.* **16-S** 107–36

[10] Gaeta G and Walcher S 2005 Dimension increase and splitting for Poincaré-Dulac normal forms. *J. Nonlinear Math. Phys.* **12** (Suppl. 1) 327–42

[11] Gaeta G and Walcher S 2006 Embedding and splitting ordinary differential equations in normal form. *J. Differ. Eqns* **224** 98–119

[12] Gaeta G, Grosshans F D, Scheurle J and Walcher S 2008 Reduction and reconstruction for symmetric ordinary differential equations. *J. Differ. Eqns* **244** 1810–39

[13] Hadeler K P and Walcher S 2006 Reducible ordinary differential equations. *J. Nonlinear Sci.* **16** 583–613

[14] Hermann R 1968 *Differential Geometry and the Calculus of Variations* (New York: Academic)

[15] Krasil’shchik I S and Vinogradov A M 1999 *Symmetries and Conservation Laws for Differential Equations Of Mathematical Physics* (Providence, RI: American Mathematical Society)

[16] Michel L 1971 Points critiques de fonctions invariantes sur une \( G \)-variété. *C. R. Acad. Sci. Paris* **A 272** 433–6

Michel L and Radicati L 1971 Properties of the breaking of hadronic internal symmetry. *Ann. Phys., NY* **66** 758–83

[17] Michel L 1980 Symmetry defects and broken symmetry. *Rev. Mod. Phys.* **52** 617–51

[18] Michel L 2001 Symmetry, invariants, topology. *Phys. Rep.* **341** 1–396

[19] Moerdijk I and Mrčun J 2003 *Introduction to Foliations and Lie Groupoids* (Cambridge: Cambridge University Press)

[20] Muriel C and Romero J L 2001 New methods of reduction for ordinary differential equations. *IMA J. Appl. Math.* **66** 111–25

[21] Muriel C and Romero J L 2001 \( C^\infty \) symmetries and nonsolvable symmetry algebras. *IMA J. Appl. Math.* **66** 477–98

[22] Muriel C and Romero J L 2002 Prolongations of vector fields and the invariants-by-derivation property. *Theor. Math. Phys.* **113** 1565–75

[23] Muriel C and Romero J L 2002 \( C^\infty \)-symmetries and integrability of ordinary differential equations. *Proc. I Colloquium on Lie Theory and Applications (Vigo)* pp 143–50

[24] Muriel C and Romero J L 2007 \( C^\infty \)-symmetries and nonlocal symmetries of exponential type. *IMA J. Appl. Math.* **72** 191–205

[25] Muriel C and Romero J L 2008 Integrating factors and lambda-symmetries. *J. Nonlin. Math. Phys.* **15** (Suppl. 3) 300–9

[26] Muriel C and Romero J L 2009 First integrals, integrating factors and \( \lambda \)-symmetries of second-order differential equations. *J. Phys. A: Math. Theor.* **42** 365207

[27] Muriel C, Romero J L and Olver P J 2006 Variational \( C^\infty \) symmetries and Euler–Lagrange equations. *J. Differ. Eqns* **222** 164–84

[28] Muriel C and Romero J L 2012 Nonlocal symmetries, telescopic vector fields and \( \lambda \)-symmetries of ordinary differential equations. *SIGMA* **8** 106

[29] Olver P J 1986 *Application of Lie Groups to Differential Equations* (Berlin: Springer)

[30] Olver P J 1995 *Equivalence, Invariants and Symmetry* (Cambridge: Cambridge University Press)

[31] Ortega J P and Ratiu T S 2004 *Momentum Maps and Hamiltonian Reduction* (Basel: Birkhäuser)

[32] Pucci E and Saccomandi G 2002 On the reduction methods for ordinary differential equations. *J. Phys. A: Math. Gen.* **35** 6145–55

[33] Stephani H 1989 *Differential Equations. Their Solution Using Symmetries* (Cambridge: Cambridge University Press)

[34] Walcher S 1999 Multi-parameter symmetries of first order ordinary differential equations. *J. Lie Theory* **9** 249–69