Covering and 2-degree-packing numbers in graphs

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Abstract

In this paper, we give a relationship between the covering number of a simple graph $G$, $\beta(G)$, and a new parameter associated to $G$ which is called 2-degree-packing number of $G$, $\nu_2(G)$. We prove that

$$\left\lceil \frac{\nu_2(G)}{2} \right\rceil \leq \beta(G) \leq \nu_2(G) - 1,$$

for any connected simple graph $G$, with $|E(G)| > \nu_2(G)$, and we give a characterization of simple connected graphs which attains the inequalities.

Key words. Covering number, independence number, 2-degree-packing number.

1 Introduction

In this paper, we consider finite undirected simple graphs. For any undefined terms see [6]. Let $G$ be a graph, we call $V(G)$ the vertex set of $G$ and denote

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by $E(G)$ the edge set of $G$. For a subset $A \subseteq V(G) \cup E(G)$, $G[A]$ denotes the subgraph of $G$ which is induced by $A$. The distance between two vertices $u$ and $v$ in a graph $G$ is the number $d_G(u, v)$ of edges in any shortest $v - u$ path in $G$ that joins $u$ and $v$; if $u$ and $v$ are not joined in $G$, then $d_G(u, v) = \infty$.

The neighborhood of a vertex $u \in V(G)$, denoted by $N_G(u)$, is a subset of $V(G)$ adjacent to $u$ in $G$. The set of edges incident to $u \in V(G)$ is denoted by $L_u$. Hence, the degree of $u$, denoted by $\deg(u)$, is $\deg(x) = |L_u|$. The minimum and maximum degree of a graph $G$ is denoted by $\delta(G)$ and $\Delta(G)$, respectively. Let $H$ be a subgraph of $G$. The restricted degree of a vertex $u \in V(H)$, denoted by $\deg_H(u)$, is defined as $\deg_H(u) = |L_u \cap E(H)|$.

An independent set of a graph $G$ is a subset $I \subseteq V(G)$ such that any two vertices of $I$ are not adjacent. The independence number of $G$, denoted by $\alpha(G)$, is the maximum order of an independent set. A vertex cover of a graph $G$ is a subset $T \subseteq V(G)$ such that all edges of $G$ has at least one end in $T$. The covering number of $G$, denoted by $\beta(G)$, is the minimum order of a vertex cover of $G$. This invariant is well known and intensively studied in a more general context and with different names, see for example [2, 8, 11, 12, 13]. On the other hand, a $k$-degree-packing set of a graph $G$ ($k \leq \Delta(G)$), is a subset $R \subseteq E(G)$ such that $\Delta(G[R]) \leq k$. The $k$-degree-packing number of $G$, denoted by $\nu_k(G)$, is the maximum order of a $k$-degree-packing set. We are interested when $k = 2$, since $k = 1$ is the matching number of a graph.

The 2-degree-packing number is studied in [1, 8, 12, 13] on a more general context, but with a different name, as 2-packing number. It is important to say that the definition of 2-packing in graphs has different meaning: A set $X \subseteq V(G)$ is called a 2-packing if $d_C(u, v) > 2$ for any different vertices $u$ and $v$ of $X$, that is, the 2-packing is a subset $X \subseteq V(G)$ in which all the vertices are in distance at least 3 from each other, see for example [11]. Therefore, we call 2-degree-packing instead of 2-packing just in case of graphs.
In [5], it was proved for any simple graph $G$ it satisfies

$$\left\lceil \frac{\nu_2(G)}{2} \right\rceil \leq \beta(G). \quad (1)$$

In this paper, we prove that for any simple graph $G$, with $|E(G)| > \nu_2(G)$, it satisfies

$$\beta(G) \leq \nu_2(G) - 1. \quad (2)$$

Hence, by Equations (1) and (2), we have the following:

**Theorem 1.1.** If $G$ is a simple connected graph with $|E(G)| > \nu_2(G)$, then

$$\left\lceil \frac{\nu_2(G)}{2} \right\rceil \leq \beta(G) \leq \nu_2(G) - 1.$$  

The main result of this paper is to give a characterization of simple connected graphs that attain the upper and lower inequality of the Theorem 1.1.

## 2 Some results

In the remainder of this note, for the terminology, notation and missing basic definitions related to graphs, the reader may consult [6]. Only connected graphs with $|E(G)| > \nu_2(G)$ are considered, since $|E(G)| = \nu_2(G)$ if and only if $\Delta(G) \leq 2$. Moreover, we assume $\nu_2(G) \geq 4$, since in [5] was proved the following:

**Proposition 2.1.** [5] Let $G$ be a simple connected graph with $|E(G)| > \nu_2(G)$, then $\nu_2(G) = 2$ if and only if $\beta(G) = 1$.

**Proposition 2.2.** [5] Let $G$ be a simple connected graph with $|E(G)| > \nu_2(G)$. If $\nu_2(G) = 3$, then $\beta(G) = 2$.

If $G$ satisfies the hypothesis of Proposition 2.1, then $G$ is the complete bipartite graph $K_{1,m}$. If $G$ satisfies the hypothesis of Proposition 2.2, then $G$ is one of the graphs shown in Figure 1 (see [5]).
The following Proposition 2.3 show simple consequences of the definitions presented before, and some results are well known.

**Proposition 2.3.**

1. If \( R \) is a maximum 2-degree-packing of a graph \( G \), then the components of \( G[R] \) are either cycles or paths.

2. If \( G \) is either a cycle or a path, both of even length, and \( T \) is a minimum vertex cover of \( G \), then \( T \) is an independent set.

3. If \( G \) is cycle of length odd and \( T \) is a minimum vertex cover of \( G \), then there exists an unique \( u \in T \) such that \( T \setminus \{u\} \) is an independent set. On the other hand, if \( G \) is a path of length odd, then either there exists an unique \( u \in T \) such that \( T \setminus \{u\} \) is an independent set or \( T \) is an independent and \( \text{deg}_T(u) = 1 \).

4. If \( G \) is either a path or a cycle of length \( k \), then \( \beta(G) = \lceil \frac{k}{2} \rceil \).

5. \( \beta(K_n) = \nu_2(K_n) - 1 \).

**Remark 2.1.** Let \( R \) be a maximum 2-degree-packing of a simple connected graph \( G \). It is clear the number of components of \( G[R] \) is at most \( \nu_2(G) - 1 \). Moreover, if \( T \) is a minimum vertex cover of \( G[R] \), then \( \beta(G) \leq k + p \), where \( k \) is the number of components of \( G[R] \) of a single edge, and \( p = |\{v \in V(G[R]) : \text{deg}_R(v) = 2\}| \). Hence, \( \beta(G) \leq k + p \leq \nu_2(G) \).
Figure 2: Graphs with $\nu_2(G) = 4$ and $\beta(G) = 3$

**Proposition 2.4.** If $G$ is a simple connected graph with $|E(G)| > \nu_2(G)$, then $\beta(G) \leq \nu_2(G) - 1$.

**Proof.** Using the remark 2.1, we have that $\beta(G) \leq k + p \leq \nu_2(G)$. If $k \geq 1$, then it is not complicate to see that $\beta(G) \leq \nu_2(G) - 1$. On the other hand, if $k = 0$, then any component of $G[R]$ is a cycle, since if $G[R]$ has a path (of length at least 2) as a component, then $\beta(G) \leq \nu_2(G) - 1$. Hence $p = \nu_2(G)$.

We assume $V(G[R]) = V(G)$, otherwise if $u \in V(G) \setminus V(G[R])$ and $e_u = uv \in E(G) \setminus R$, where $v \in V(G[R])$, then the following set $(R \setminus \{e_u\}) \cup \{e_v\}$, where $e_v \in R$ is incident to $v$, is a maximum 2-degree-packing of $G$ with a path as a component, which implies that $\beta(G) \leq \nu_2(G) - 1$. Therefore $\{v \in V(G[R]) : \deg_R(v) = 2\} \setminus \{u\}$, for any $u \in V(G[R])$, is a vertex cover of $G$, implying that $\beta(G) \leq \nu_2(G) - 1$. 

Hence, we have the following:

**Theorem 2.1.** If $G$ is a simple connected graph with $|E(G)| > \nu_2(G)$, then

$$\left\lfloor \frac{\nu_2(G)}{2} \right\rfloor \leq \beta(G) \leq \nu_2(G) - 1.$$ 

**3 Graphs with $\beta = \nu_2 - 1$**

To begin with, some terminology is introduced in order to simplify the description of simple connected graphs $G$ such that $\beta(G) = \nu_2(G) - 1$.

In [5], as a particular case, was proved the following:
Figure 3: In (a) depict the Graph $T_{s,t}^r$ and in (b) depict the graph $G_{s,t}^r$.

**Proposition 3.1.** If $G$ is a simple connected graph $G$ with $\nu_2(G) = 4$ and $|E(G)| > 4$, then $\beta(G) \leq 3$.

Moreover, in these same paper [5], was given all simple connected graphs $G$ with $\nu_2(G) = 4$ and $\beta(G) = 3$, these graphs are certain subgraphs from Figure 2 (see [5]). Hence, by Proposition 3.1 we assume $\nu_2(G) \geq 5$.

In [14], was defined the graph $T_{s,t}$, with $s \geq 1$ and $t \geq 2$, as follow (see Figure 3 (a)):

$$V(T_{s,t}) = \{p_1, \ldots, p_s\} \cup \{q_1, \ldots, q_s\} \cup \{w_1, \ldots, w_t\},$$
$$E(T_{s,t}) = \{p_1q_i : i = 1, \ldots, s\} \cup \{vp_i : i = 1, \ldots, s\} \cup \{vw_i : i = 1, \ldots, t\}.$$  

And we define $G_{s,t}$, with $s \geq 1$ and $t \geq 2$, as follow (see Figure 3 (b)):

$$V(G_{s,t}) = V(T_{s,t}),$$
$$E(G_{s,t}) = E(T_{s,t}) \cup \{vq_i : i = 1, \ldots, s\}.$$  

As a consequence of Corollary 2.4 of [14]:

**Corollary 3.1.** [14] $\beta(T_{s,t}) = \nu_2(T_{s,t}) - 1 = s + 1$, for every $s \geq 1$ and $t \geq 2$.

Since the graph $T_{s,t}$ is a spanning graph of $G_{s,t}$, and any minimal vertex cover of $T_{s,t}$ is a vertex covering of $G_{s,t}$, we have the following:
**Corollary 3.2.** $\beta(G_{s,t}^r) = \nu_2(G_{s,t}^r) - 1 = s + 1$, for every $s \geq 1$ and $t \geq 2$.

**Corollary 3.3.** If $T_{s,t}$ is a spanning subgraph of a graph $G$ and $G$ is a spanning subgraph of $G_{s,t}$, then $\beta(G) = \nu_2(G) - 1 = s + 1$.

Let $R_1, \ldots, R_s, R_{s+1}, \ldots, R_k$ be the components of a simple connected graph $G$, where $|R_i| = 1$, for $i = 1, \ldots, s$ and $|R_j| > 1$, for $j = s+1, \ldots, k$. It is not difficult to see that $s \leq \nu_2(G) - 2$. If $s = \nu_2(G) - 2$, implies that $k = \nu_2(G) - 1$ and $|E(G[R_k])| = 2$. Hence, any edge from $E(G) \setminus E(G[R])$ is incident with the only one vertex $v \in V(G[R_k])$ with $\deg_R(v) = 2$. Hence, if $R_i = p_iq_i$, for $i = 1, \ldots, s$, $R_k = w_0w_1$, and $V(G) \setminus V(G[R]) = \{w_3, \ldots, w_t\}$ (an independent set), if $t \geq 3$, then $T_{s,t}$ is a spanning subgraph of a graph $G$ and $G$ is a spanning subgraph of $G_{s,t}$. Therefore, $\beta(G) = \nu_2(G) - 1 = s + 1$.

Let $R_1, \ldots, R_s, R_{s+1}, \ldots, R_k$ be the components of a simple connected graph $G$, with $k$ as small as possible, where $|R_i| = 1$, for $i = 1, \ldots, s$ and $|R_j| > 1$, for $j = s+1, \ldots, k$. Then, it is clear that $\beta(G) = s + \beta(H)$ and $\nu_2(G) = s + \nu_2(H)$, where $H$ is the graph defined as follow

$$V(H) = \bigcup_{i=1}^{k} V(G[R_i]) \cup (V(G) \setminus V(G[R])),$$

$$E(H) = E(G) \setminus \{R_1, \ldots, R_s\}.$$ 

Hence, if $\tau(G) = \nu_2(G) - 1$, then $\tau(H) = \nu_2(H) - 1$. Therefore, we assume that any simple connected graph $G$, with $|E(G)| > \nu_2(G)$, has a maximum 2-degree-packing $R$ of $G$, where each component of $G[R]$ has at least 2 edges; and as consequence, the set $T = \{u \in V(G[R]) : \deg_{G[R]}(u) = 2\}$ is a vertex cover of $G$.

Let $K_{n}^{\perp}$ be the simple connected graph defined as follow:

$$V(K_{n}^{\perp}) = \{x_1, \ldots, x_n\} \cup \{u\},$$

$$E(K_{n}^{\perp}) = \{x_ix_j : 1 \leq i < j \leq n\} \cup \{ux_1\}.$$

The graph $K_{n}^{\perp}$ is the complete graph of $n$ vertices joined with an edge. It is easy to see that $\beta(K_{n}^{\perp}) = \nu_2(K_{n}^{\perp}) - 1 = n - 1$. 

7
**Proposition 3.2.** Let $G$ be a simple connected graph with $|E(G)| > \nu_2(G)$, $\nu_2(G) \geq 5$ and $\beta(G) = \nu_2(G) - 1$. If $R$ is a maximum 2-degree-packing of $G$ with $V(G[R]) = V(G)$, then either $G$ is the complete graph $K_{\nu_2}$ or $G$ is $K^{1}_{\nu_2}$, where $\nu_2 = \nu_2(G)$.

**Proof.** Let $R$ be a maximum 2-degree-packing of $G$ with $V(G[R]) = V(G)$, and let $R_1, \ldots, R_k$ be the components of $G[R]$, with $k$ as small as possible. Then

Case (i) If $k = 1$, then $G[R]$ is either a spanning path or a spanning cycle of the graph $G$. Let suppose that $R = u_0 u_1 \cdots u_{\nu_2 - 1} u_0$ is a spanning cycle: If there are two non-adjacent vertices $u_i, u_j \in V(G[R])$, then $T = V(G[R]) \setminus \{u_i, u_j\}$ is a vertex cover of $G$ of cardinality $\nu_2(G) - 2$, which is a contradiction. Therefore, any different pair of vertices of $G$ are adjacent. Hence, the graph $G$ is the complete graph of $\nu_2(G)$ vertices.

On the other hand, if $R = u_0 u_1 \cdots u_{\nu_2}$ is a path, then $T = \{u_1, \ldots, u_{\nu_2 - 1}\}$ is a minimum vertex cover of $G$. Let assume that either $u_0 u_j \in E(G)$ or $u_{\nu_2} u_j \in E(G)$, for all $u_j \in T^* = T \setminus \{u_1, u_{\nu_2 - 1}\}$, otherwise, $T \setminus \{u_j\}$ is a vertex cover of $G$ of cardinality $\nu_2(G) - 2$, which is a contradiction. Without loss of generality, let suppose that $u_0 u_j \in E(G)$, for all $u_j \in T^* = T \setminus \{u_1, u_{\nu_2 - 1}\}$. If $u_j u_{\nu_2} \in E(G)$, for some $u_j \in T^*$, then $R^* = (R \setminus \{u_j, u_{j+1}\}) \cup \{u_j u_{\nu_2}\}$ (since $\nu_2(G) \geq 5$) is a 2-degree-packing with $G[R^*]$ as a cycle, which is a contradiction. Hence $u_j u_{\nu_2} \not\in E(G)$, for all $u_j \in T^*$, which implies that $\deg(u_{\nu_2}) = 1$. On the other hand, if there are two vertices $u_i \neq u_j \in T^*$ non-adjacent, then $(T \setminus \{u_i, u_j\}) \cup \{u_0\}$ is a vertex cover of $G$ of size $\nu_2(G) - 2$, which is a contradiction. Also, $u_1 u_j \in E(G)$ and $u_j u_{\nu_2 - 1} \in E(G)$, for all $u_j \in T^*$, otherwise there exists $u_j \in T^*$ such that either $(T \setminus \{u_1, u_j\}) \cup \{u_0\}$ or $(T \setminus \{u_j, u_{\nu_2 - 1}\}) \cup \{u_0\}$ is a vertex cover of $G$ of size $\nu_2(G) - 2$, which is a contradiction. Therefore, the graphs $G$ is the graph $K^1_{\nu_2}$.

Case (ii) Let suppose that $k \geq 2$ and $T = \{v \in V(G[R]) : \deg_R(v) = 2\}$. If there is at least one component as a paths (of length at least 2), say
If there are two vertices \( u, v \in V(G[R]) \) such that \( uv \not\in E(G) \), then \( T \{ u, v \} \) is a vertex cover of \( G \) with \( \beta(G) \leq \nu_2(G) - 2 \), which is a contradiction. Then, any two vertices \( u, v \in V(G[R]) \) are adjacent, which implies that \( k = 1 \), a contradiction. Therefore, \( G[R] \) is the complete graph of \( \nu_2(G) \) vertices.

\[ \beta(G) \leq |T| = (|E(R_1)| - 2) + \sum_{i=2}^{k} |E(R_i)| = \sum_{i=1}^{k} |E(R_i)| - 2 = \nu_2(G) - 2, \]

which is a contradiction. Hence, \( G[R_i] \) is a cycle, for all \( i = 1, \ldots, k \).

\[ \text{Theorem 3.1.} \text{ Let } G \text{ be a simple connected graph with } \nu_2(G) \geq 5 \text{ and } \beta(G) = \nu_2(G) - 1. \text{ Then either } G \text{ is the complete graph } K_{\nu_2} \text{ or } G \text{ is } K_{\nu_2}^{1}, \text{ where } \nu_2 = \nu_2(G). \]

\[ \text{Proof.} \text{ Let } R \text{ be a maximum 2-degree-packing of } G \text{ and } I = V(G) \setminus V(G[R]). \text{ Let assume that } I \neq \emptyset, \text{ otherwise, the theorem holds by Proposition 3.2. Hence, if } I \neq \emptyset, \text{ then } I \text{ is an independent set of vertices.} \]

Case (i): Let suppose that \( G[R] \) is the complete graph of \( \nu_2(G) \) vertices. We claimed that, if \( u \in I \), then \( \deg(u) = 1 \). To verify the claim, let suppose on the contrary, \( u \) is incident to at least two vertices of \( V(G[R]) \), say \( v \) and \( w \). If \( V(G[R]) = \{ u_1, \ldots, u_{\nu_2} \} \), then without loss of generality, we suppose \( u_1 = v \) and \( u_j = w \), for some \( j \in \{ 2, \ldots, \nu_2 \} \). Since \( G[R] \) is a complete graph, then

\[ (R \setminus \{ u_1u_{\nu_2}, u_{j-1}u_{j} \}) \cup \{ uu_1, uu_j, u_{j-1}u_{\nu_2} \} \]

is a 2-degree-packing of \( G \) of size \( \nu_2(G) + 1 \), which is a contradiction. Hence, if \( u \in I \), then \( \deg_G(u) = 1 \).
On the other hand, if $|I| > 1$, let $u, v \in I$. Without loss of generality, let suppose that $u$ is incident to $u_1$ and $v$ is incident to $u_j$, for some $j \in \{2, \ldots , \nu_2\}$. Since $G[R]$ is a complete graph, then

$$(R \setminus \{u_1u_{\nu_2}, u_{j-1}u_j\}) \cup \{uu_1, u_{j-1}u_{\nu_2}, vu_j\}$$

is a 2-degree-packing of size $\nu_2(G) + 1$, which is a contradiction. If $u$ and $v$ are adjacent to $u_1$, then

$$(R \setminus \{u_1u_2, u_1u_{\nu_2}\}) \cup \{uu_1, vu_1, u_2u_{\nu_2}\}$$

is a 2-degree-packing of size $\nu_2(G) + 1$, which is contradiction. Hence, $I = \{u\}$ with $\text{deg}(u) = 1$, which implies that the graph $G$ is $K^1_{\nu_2}$.

Case (ii): Let suppose that $G[R]$ is the graph $K^1_{\nu_2}$. Let $v \in V(G)$ such that the $G[R] - v$ is the complete graph of size $\nu_2(G)$. If $u \in I$ is such that $uw \in E(G)$, whit $w \in V(G[R])$, then there exists a 2-degree-packing of $G$ of size $\nu_2(G) + 1$ (see proof of Proposition 3.2), which is a contradiction. Then $uw \not\in E(G)$, for all $w \in V(G[R]) \cup \{v\}$, which implies that $G$ is a disconnected graph, unless $I = \emptyset$, and the theorem holds by Proposition 3.2.

\[
\blacksquare
\]

4 Graphs with $\beta = \lceil \nu_2/2 \rceil$

To begin with, some terminology and results are introduced in order to simplify the description of the simple connected graphs $G$ which satisfy $\beta(G) = \lceil \nu_2(G)/2 \rceil$.

**Proposition 4.1.** Let $G$ be a simple connected graph and $R$ be a maximum 2-degree-packing of $G$.

1. If $\nu_2(G)$ is an even integer and $\beta(G) = \frac{\nu_2(G)}{2}$, then the components of $R$ has even length.
2. If \( \nu_2(G) \) is an odd integer and \( \beta(G) = \frac{\nu_2(G) + 1}{2} \), then there is an unique component of \( R \) of odd length.

Proof. We will prove 1., since the proof of 2. is completely analogous to the proof of 1.: Let \( R \) be a maximum 2-degree-packing of \( G \) and let \( R_1, \ldots, R_k \) be the components of \( G[R] \). If \( T \) is a minimum vertex cover of \( G \), then

\[
\nu_2(G) = \beta(G) = |T| = \sum_{i=1}^{k} |T \cap V(R_i)| \geq \sum_{i=1}^{k} \beta(R_i) = \sum_{i=1}^{k} \left\lceil \frac{\nu_2(R_i)}{2} \right\rceil.
\]

Hence, if \( R_1 \) have a odd number of edges, then

\[
\sum_{i=1}^{k} \left\lceil \frac{\nu_2(R_i)}{2} \right\rceil = \frac{\nu_2(R_1)}{2} + \sum_{i=2}^{k} \left\lceil \frac{\nu_2(R_i)}{2} \right\rceil \geq \frac{1}{2} + \sum_{i=1}^{k} \frac{\nu_2(R_i)}{2} = \frac{1}{2} + \frac{\nu_2(G)}{2},
\]

which is a contradiction. Therefore, each component of \( G[R] \) has an even number of edges.

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Let \( A \) and \( B \) be two sets of vertices. The complete graph whose set of vertices is \( A \) is denoted by \( K_A \). The graph whose set of vertices is \( A \cup B \) and whose set of edges is \( \{ab : a \in A, b \in B\} \) is denoted by \( K_{A,B} \). On the other hand, let \( k \geq 3 \) be a positive integer. The cycle of length \( k \) and the path of length \( k \) are denoted by \( C_k \) and \( P_k \), respectively.

If \( A \) and \( B \) are two sets of vertices from \( V(C_k) \) and \( V(P_k) \) (not necessarily disjoint) and \( I \) be an independent set of vertices different from \( V(C_k) \) and \( V(P_k) \) then \( C_{A,B,I}^k = (V(C_{A,B,I}^k), E(C_{A,B,I}^k)) \) and \( P_{A,B,I}^k = (V(P_{A,B,I}^k), E(P_{A,B,I}^k)) \) are denoted to be the graphs with \( V(C_{A,B,I}^k) = V(C_k) \cup I \) and \( V(P_{A,B,I}^k) = V(P_k) \cup I \), respectively, and \( E(C_{A,B,I}^k) = E(C_k) \cup E(K_A) \cup E(K_{A,B}) \cup E(K_{A,I}) \) and \( E(P_{A,B,I}^k) = E(P_k) \cup E(K_A) \cup E(K_{A,B}) \cup E(K_{A,I}) \), respectively. In an analogous way, we denote by \( C_I^k \) to be the graph with \( V(C_I^k) = V(C_k) \cup I \) and \( E(C_I^k) = E(C_k) \) and we denote by \( P_I^k \) to be the graph with \( V(P_I^k) = V(P_k) \cup I \) and \( E(P_I^k) = E(P_k) \). We define \( C_{A,B,I}^k \) be the family of connected graphs \( G \) such that \( C_I^k \) is a subgraph of \( G \) and \( G \) is a subgraph of \( C_{A,B,I}^k \). Similarly, we define \( P_{A,B,I}^k \) be the family of
connected graphs $G$ such that $P^k_I$ is a subgraph of $G$ and $G$ is a subgraph of $P^k_{A,B,I}$. That is

$$C^k_{A,B,I} = \{ G : C^k_I \subseteq G \subseteq C^k_{A,B,I} \text{ where } G \text{ is a connected graph} \}$$

$$P^k_{A,B,I} = \{ G : P^k_I \subseteq G \subseteq P^k_{A,B,I} \text{ where } G \text{ is a connected graph} \}$$

**Proposition 4.2.** Let $k \geq 4$ be an even integer, $T$ be a minimum vertex cover of $C^k$ and $I$ be an independent set of vertices different from $V(C^k)$. If $\hat{T} = V(C^k) \setminus T$ and $G \in C^k_{T,\hat{T},I}$, then $\beta(G) = k/2$ and $\nu_2(G) = k$.

**Proof.** It is clear that, if $G \in C^k_{T,\hat{T},I}$, then $\beta(G) = k/2$. On the other hand, since $C^k$ is a 2-degree-packing of $G$, then $\nu_2(G) \geq k$. Moreover, since $[\nu_2(G)/2] \leq \beta(G) = k/2$, then $\nu_2(G) = k$. ■

**Corollary 4.1.** Let $k \geq 4$ be an even integer, $T$ be a minimum vertex cover of $P^k$ and $I$ be an independent set of vertices different from $V(P^k)$. If $\hat{T} = V(P^k) \setminus T$ and $G \in P^k_{T,\hat{T},I}$, then $\beta(G) = k/2$ and $\nu_2(G) = k$.

Now, let $\hat{C}^k_{A,B,I}$ be the family of simple connected graphs $G$ with $\nu_2(G) = k$, such that $C^k_I$ is a subgraph of $G$ and $G$ is a subgraph of $C^k_{A,B,I}$. Similarly, let $\hat{P}^k_{A,B,I}$ be the family of simple connected graphs $G$ with $\nu_2(G) = k$ such that $P^k_I$ is a subgraph of $G$ and $G$ is a subgraph of $P^k_{A,B,I}$. That is

$$\hat{C}^k_{A,B,I} = \{ G : C^k_I \subseteq G \subseteq C^k_{A,B,I} \text{ where } G \text{ is connected and } \nu_2(G) = k \},$$

$$\hat{P}^k_{A,B,I} = \{ G : P^k_I \subseteq G \subseteq P^k_{A,B,I} \text{ where } G \text{ is connected and } \nu_2(G) = k \}.$$

Hence if $k \geq 4$ is an even integer, $T$ is a minimum vertex cover of either $C^k$ or $P^k$, and $I$ is an independent set different from either $V(C^k)$ or $V(P^k)$, then by Proposition 4.2 and Corollary 4.1, we have

$$\hat{C}^k_{T,\hat{T},I} = C^k_{T,\hat{T},I} \text{ and } \hat{P}^k_{T,\hat{T},I} = P^k_{T,\hat{T},I},$$

12
However, if \( k \geq 5 \) is an odd integer, \( T \) is a minimum vertex cover of either \( C^k \) or \( P^k \) and \( I \) is an independent set different from either \( V(C^k) \) or \( V(P^k) \), then

\[
\hat{C}_{T,T,I}^k \neq C_{T,T,I}^k \quad \text{and} \quad \hat{P}_{T,T,I}^k \neq P_{T,T,I}^k.
\]

To see this, let \( R \) be the cycle of length \( k \) and \( u, v \in T \) adjacent. Hence, if \( G \) is such that \( V(G) = V(C^k) \cup \{w\} \), where \( w \in I \) and \( E(G) = E(C^k) \cup \{uw, vw\} \), then \( G \in C_{T,T,I}^k \). However, it is clear that \( \nu_2(G) = k + 1 \), which implies that \( G \notin \hat{C}_{T,T,I}^k \). A similar argument is used to prove that \( \hat{P}_{T,T,I}^k \neq P_{T,T,I}^k \).

**Proposition 4.3.** Let \( k \geq 5 \) be an odd integer, \( T \) be a minimum vertex cover of \( C^k \) and \( I \) be an independent set of vertices different from \( V(C^k) \). If \( \hat{T} = V(C^k) \setminus T \) and \( G \in \hat{C}_{T,T,I}^k \), then \( \beta(G) = \frac{k+1}{2} \).

**Proof.** It is clear that

\[
\frac{k+1}{2} = \lceil \nu_2(C^k)/2 \rceil \leq \lceil \nu_2(G)/2 \rceil \leq \beta(G) \leq |T| = \frac{k+1}{2},
\]

which implies that \( \beta(G) = \frac{k+1}{2} \).

**Corollary 4.2.** Let \( k \geq 5 \) be an odd integer, \( T \) be a minimum vertex cover of \( P^k \) and \( I \) be an independent set of vertices different from \( V(P^k) \). If \( \hat{T} = V(P^k) \setminus T \) and \( G \in \hat{P}_{T,T,I}^k \), then \( \beta(G) = \frac{k+1}{2} \).

**Proposition 4.4.** Let \( G \) be a connected graph with \( |E(G)| > \nu_2(G) \) and \( R_1, \ldots, R_k \) be the components of a maximum 2-degree-packing of \( G \). If \( \beta(G) = \lfloor \nu_2(G)/2 \rfloor \), then \( \beta(G) = \sum_{i=1}^{k} \beta(R_i) \).

**Proof.** Let \( R \) be a maximum 2-degree-packing of \( G \) and \( R_1, \ldots, R_k \) be the components of \( G[R] \). Since \( R_i \) is a cycle or a path of length \( \nu_2(R_i) \), then \( \beta(R_i) = \lfloor \nu_2(R_i)/2 \rfloor \), for \( i = 1, \ldots, k \). If \( \beta(G) = \lfloor \nu_2(G)/2 \rfloor \), then by Proposition 4.1, we have

\[
\lfloor \nu_2(G)/2 \rfloor = \beta(G) = \sum_{i=1}^{k} \beta(R_i) = \sum_{i=1}^{k} \lfloor \nu_2(R_i)/2 \rfloor = \lfloor \nu_2(G)/2 \rfloor.
\]
Therefore \( \beta(G) = \sum_{i=1}^{k} \beta(R_i) \).

By Proposition 4.1 and Proposition 4.4, we have:

**Theorem 4.1.** Let \( G \) be a connected graph with \(|E(G)| > \nu_2(G)\) and \( R_1, \ldots, R_k \) be the components of a maximum 2-degree-packing of \( G \). Then \( \beta(G) = \lceil \nu_2(G)/2 \rceil \), if and only if, \( \beta(G) = \sum_{i=1}^{k} \beta(R_i) \), being

1. \(|R_i|\) an even integer, for \( i = 1, \ldots, k \), if \( \nu_2(G) \) an even number.
2. \(|R_1|\) is an odd integer and \(|R_i|\) is an even integer, for \( i = 2, \ldots, k \), if \( \nu_2(G) \) is an odd number.

**Proposition 4.5.** Let \( G \) be a simple connected graph with \( \nu_2(G) \geq 4 \), \(|E(G)| > \nu_2(G)\) and \( R_1, \ldots, R_k \) be the components of a maximum 2-degree-packing \( R \) of \( G \), with \( k \) as small as possible. If \( \beta(G) = \lceil \nu_2(G)/2 \rceil \), then

\[ I = I_1 \cup \cdots \cup I_k = V(G) \setminus V(G[R]), \]

where either \( I_i = \emptyset \) or for every \( u \in I_i \) satisfies \( N(u) \subseteq V(R_i) \), for \( i = 1, \ldots, k \).

**Proof.** Let suppose that there exists \( u \in I_i, w_i \in V(R_i) \) and \( w_j \in V(R_j) \), for some \( i \neq j \in \{1, \ldots, k\} \), such that \( uw_i, uw_j \in E(G) \). Hence \((R \setminus \{e_{w_i}, e_{w_j}\}) \cup \{uw_i, uw_j\}\), where \( e_{w_i}, e_{w_j} \in E(R_i) \) and \( w_j \in e_{w_j} \in E(R_j) \), is a maximum 2-degree-packing with less components than \( R \), which is a contradiction. Therefore \( I = I_1 \cup \cdots \cup I_k \), where either \( I_i = \emptyset \) or for every \( u \in I_i \) satisfies \( N(u) \subseteq V(R_i) \), for \( i = 1, \ldots, k \).

**Corollary 4.3.** Let \( G \) be a simple connected graph with \( \nu_2(G) \geq 4 \), \(|E(G)| > \nu_2(G)\), \( R_1, \ldots, R_k \) be the components of a maximum 2-degree-packing \( R \) of \( G \), with \( k \) as small as possible, and \( I = I_1 \cup \cdots \cup I_k = V(G) \setminus V(G[R]) \), where either \( I_i = \emptyset \) or for every \( u \in I_i \) satisfies \( N(u) \subseteq V(R_i) \), for \( i = 1, \ldots, k \). If

\( \beta(G) = \lceil \nu_2(G)/2 \rceil \), then \( \beta(G[R_i]) = \lceil \nu_2(G[R_i])/2 \rceil \), for \( i = 1, \ldots, k \).
**Proposition 4.6.** Let $G$ be a simple connected graph with $\nu_2(G) \geq 4$, $|E(G)| > \nu_2(G)$ and $R$ be a maximum 2-degree-packing of $G$, such that $G[R]$ is a connected graph. If $\beta(G) = \lceil \nu_2(G)/2 \rceil$, then either $G \in \mathcal{C}_{T,\hat{T},I}^k$ or $G \in \hat{\mathcal{P}}_{T,\hat{T},I}^k$, where $T$ is a minimum vertex cover of either $C^k$ or $P^k$, $\hat{T} = V(G[R]) \setminus T$ and $I = V(G) \setminus V(G[R])$.

**Proof.** By Proposition 4.1 we have either $\hat{C}_T^k$ is a subgraph of $G$ or $\hat{P}_T^k$ is a subgraph of $G$. Let $T$ be a minimum vertex cover of $G$ (hence, a minimum vertex cover of $G[R]$), by Proposition 4.4. Hence, by definition, if $e \in E(G) \setminus E(G[R])$, then $e$ has an end in $T$, which implies that $G$ is a subgraph of $\hat{C}_T^k$ or $\hat{P}_T^k$. Therefore, either $G \in \mathcal{C}_{T,\hat{T},I}^k$ or $G \in \hat{\mathcal{P}}_{T,\hat{T},I}^k$. 

By Proposition 4.4 Proposition 4.6 and Corollary 4.3 we have:

**Corollary 4.4.** Let $G$ be a simple connected graph with $\nu_2(G) \geq 4$, $|E(G)| > \nu_2(G)$, $R_1, \ldots, R_k$ be the components of a maximum 2-degree-packing $R$ of $G$, with $k$ as small as possible, and $I = I_1 \cup \cdots \cup I_k = V(G) \setminus V(G[R])$, where either $I_i = \emptyset$ or for every $u \in I_i$ satisfies $N(u) \subseteq V(R_i)$, for $i = 1, \ldots, k$. If $\beta(G) = \lceil \nu_2(G)/2 \rceil$, then either $G[V_i] \in \mathcal{C}_{T_i,\hat{T}_i,\hat{I}_i}^{k_i}$ or $G[V_i] \in \hat{\mathcal{P}}_{T_i,\hat{T}_i,\hat{I}_i}^{k_i}$, where $V_i = V(G[R_i]) \cup I_i$, $k_i = \nu_2(G[R_i])$, $T_i$ is a minimum vertex cover of either $C^{k_i}$ or $P^{k_i}$ and $\hat{T}_i = V(G[R_i]) \setminus T_i$.

Hence, by Proposition 4.2 Proposition 4.6 Corollary 4.1 and Corollary 4.4 we have

**Theorem 4.2.** Let $G$ be a simple connected graph with $\nu_2(G) \geq 4$, $|E(G)| > \nu_2(G)$, $R_1, \ldots, R_k$ be the components of a maximum 2-degree-packing $R$ of $G$, with $k$ as small as possible, and $I = I_1 \cup \cdots \cup I_k = V(G) \setminus V(G[R])$, where either $I_i = \emptyset$ or for every $u \in I_i$ satisfies $N(u) \subseteq V(R_i)$, for $i = 1, \ldots, k$. Then $\beta(G) = \lceil \nu_2(G)/2 \rceil$, if and only if, either $G[V_i] \in \mathcal{C}_{T_i,\hat{T}_i,\hat{I}_i}^{k_i}$ or $G[V_i] \in \hat{\mathcal{P}}_{T_i,\hat{T}_i,\hat{I}_i}^{k_i}$, where $V_i = V(G[R_i]) \cup I_i$, $k_i = \nu_2(G[R_i])$, $T_i$ is a minimum vertex cover of either $C^{k_i}$ or $P^{k_i}$ and $\hat{T}_i = V(G[R_i]) \setminus T_i$, being

1. $|R_i|$ an even integer, for $i = 1, \ldots, k$, if $\nu_2(G)$ an even number.
2. \(|R_1|\) is an odd integer and \(|R_i|\) is an even integer, for \(i = 2, \ldots, k\), if \(v_2(G)\) is an odd number.

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