THE GENERALIZED SAINT VENANT OPERATOR AND INTEGRAL MOMENT TRANSFORMS

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Abstract. In this article, we work with a generalized Saint Venant operator introduced by Vladimir Sharafutdinov [8] to describe the kernel of the integral moment transforms over symmetric \( m \)-tensor fields in \( n \)-dimensional Euclidean space. We also provide an equivalence between the injectivity question for the integral moment transforms and generalized Saint Venant operator over symmetric tensor fields of Schwartz class.

Keywords: Saint Venant operator, integral moment ray transforms, integral geometry, tensor tomography

1. Introduction

The space of covariant symmetric \( m \)-tensor fields on \( \mathbb{R}^n \) with components in the Schwartz space will be denoted by \( \mathcal{S}(S^m) \). In Cartesian coordinates, an element \( f \in \mathcal{S}(S^m) \) can be written as

\[
f(x) = f_{i_1 \ldots i_m}(x) \, dx^{i_1} \cdots dx^{i_m}
\]

where \( f_{i_1 \ldots i_m} \in \mathcal{S}(\mathbb{R}^n) \) are symmetric in all indices. For repeated indices, Einstein summation convention will be assumed throughout this article. Moreover, we will not distinguish between covariant tensors and contravariant tensors since we work with the Euclidean metric.

The Saint Venant operator \( W : C^\infty(S^m) \to C^\infty(S^m \otimes S^m) \) is defined by

\[
(Wf)_{i_1 \ldots i_m j_1 \ldots j_m} = \sigma(i_1 \ldots i_m) \, \sigma(j_1 \ldots j_m) \sum_{\ell=0}^m (-1)^\ell \binom{m}{\ell} \frac{\partial^m f_{i_1 \ldots i_m-j_1 \ldots \ell}}{\partial x^{i_{\ell+1}} \cdots \partial x^{i_m} \partial x^{j_{m-\ell+1}} \cdots \partial x^{j_m}},
\]

where \( \sigma \) is the symmetrization operator defined below (please see equation (2.1)). This operator \( W \) was named after the French mathematician Barré de Saint Venant. In one dimension, the Saint Venant operator describes the unsteady water flow and simplifies the shallow water equations. This operator appears in various fields such as deformation theory, elasticity, and many more (see [2] and the references therein).

For a vector field \( f \) (\( m = 1 \)) the equation \( Wf = 0 \) gives the well known integrability condition \( \frac{\partial f_i}{\partial x^j} - \frac{\partial f_j}{\partial x^i} = 0 \) for Pfaff form. For a symmetric 2-tensor field \( f \) the condition \( Wf = 0 \) reduces to

\[
\frac{\partial^2 f_{ij}}{\partial x^i \partial x^j} + \frac{\partial f_{ik}}{\partial x^j} - \frac{\partial f_{jk}}{\partial x^i} = 0
\]

which was derived by Saint Venant and usually called as the deformations compatibility condition. This paper aims to describe the kernel of the integral moment transforms using a generalized version of the Saint Venant operator on the Schwartz class of tensor fields.

For a non-negative integer \( q \geq 0 \), the \( q \)-th integral moment transform of a symmetric \( m \)-tensor field is the function \( I^q : \mathcal{S}(S^m) \to \mathcal{S}(TS^{n-1}) \) given by \([7, 8] :\)

\[
(I^q f)(x, \xi) = \int_{-\infty}^{\infty} t^q \langle f(x + t \xi), \xi^m \rangle dt = \int_{-\infty}^{\infty} t^q f_{i_1 \ldots i_m}(x + t \xi) \xi^{i_1} \cdots \xi^{i_m} dt,
\]

where \( TS^{n-1} = \{ (x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n \mid |\xi| = 1, \langle x, \xi \rangle = 0 \} \) denotes the space of oriented lines in \( \mathbb{R}^n \). These transforms were introduced by Sharafutdinov and have been investigated by many authors (see for instance [1, 4, 5, 6, 3] and the references therein).
Observe that the right hand side (R.H.S.) of \((1.2)\) is valid even for \((x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n \setminus \{0\}\). Therefore, we also define the extended integral moment transforms \(I^q : \mathcal{S}(S^m) \to C^\infty (\mathbb{R}^n \times \mathbb{R}^n \setminus \{0\})\) by
\[
(I^q f)(x, \xi) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} t^q f(x + t\xi, \xi^{im}) dt.
\]
For any fixed integer \(k \geq 0\), the data \((I^0 f, I^1 f, \ldots, I^k f)\) and \((J^0 f, J^1 f, \ldots, J^k f)\) are equivalent, in fact, there is an explicit relation between these operators (see [4])
\[
(J^q f)(x, \xi) = |\xi|^{m-2q-1} \sum_{\ell=0}^{q} (-1)^{q-\ell} \left(\begin{array}{c} q \\ \ell \end{array}\right) |\xi|^\ell |\langle \xi, x \rangle|^{q-\ell} \left(I^{I^q f}\right) \left(x - \frac{\langle x, \xi \rangle}{|\xi|^2 \xi, |\xi|}\right).
\]
The operators \(I^q f(x, \xi)\) obey nice decay property in the first variable. On the other hand the operators \(J^q f(x, \xi)\) are smooth with respect to both variable and the partial derivatives \(\partial_x, \partial_\xi\) are well defined on \(J^q f\).

We denote the collection of first \((k + 1)\) integral moment transforms of \(f \in \mathcal{S}(S^m)\) by \(I^k f\). More specifically, the operator \(I^k : \mathcal{S}(S^m) \to \mathcal{S}(TS^{n-1})^{k+1}\) is defined by
\[
(I^k f)(x, \xi) = \left(I^0 f(x, \xi), I^1 f(x, \xi), \ldots, I^k f(x, \xi)\right), \quad \text{for } (x, \xi) \in TS^{n-1}.
\]
The case \(k = 0, I^0 = I^1 = I^0\), corresponds to the classical ray transform of symmetric \(m\)-tensor fields in \(\mathbb{R}^n\) and it is well known that \(I^0\) has a non-trivial kernel consisting of all potential tensor fields. An equivalent way to describe the kernel of \(I^0\) for compactly supported symmetric \(m\) \((m > 0)\) tensor fields was expressed in terms of Saint Venant operator by Sharafutdinov [8, Theorem 2.2.1]. Additionally, the kernel of the operator \(I^k\) was also discussed for compactly supported tensor fields [8, Theorem 2.1.7.2] in terms of generalized Saint Venant operator \(W^k\) (defined in the next section). In this article, we aim to study the operator \(W^k\) in detail to give an alternate kernel description (similar to [8, Theorem 2.1.7.2]) for the operator \(I^k\) on Schwartz class of symmetric \(m\)-tensor fields (see Theorem 3.2 for more details). The proofs are completely new and based on the ideas developed by authors in their recent article [6].

The rest of the article is organized as follows. In Section 2, we define certain differential operators (including \(W^k\)) that we use throughout the article. Then we state some known results for the integral moment transforms. Section 3 contains the main results of this article and their proofs.

2. Preliminaries

In this section, we recall some known facts (including definitions, notations, and lemmas) about the integral moment transforms and Saint Venant operator, which we will be using throughout this article. A detailed discussion for these facts can be found in [4] and also in the book [8, Chapter 2].

2.1. Some differential operators. Let \(T^m = T^m(\mathbb{R}^n)\) denotes the space of \(m\)-tensors on \(\mathbb{R}^n\). There is a natural projection of \(T^m\) onto the space of symmetric tensors \(S^m\), \(\sigma : T^m \to S^m\) given by
\[
(\sigma v)_{i_1 \ldots i_m} = \sigma(i_1 \ldots i_m)v = \frac{1}{m!} \sum_{\pi \in \Pi_m} v_{\pi(i_1) \ldots \pi(i_m)}, \quad \text{for } v \in T^m
\]
where \(\Pi_m\) is the set of permutation of order \(m\).

Using this symmetrization operator \(\sigma\), we define the operator of inner differentiation or symmetrized derivative \(d : C^\infty(S^m) \to C^\infty(S^{m+1})\) by
\[
(dv)_{i_1 \ldots i_m i_{m+1}} = \sigma(i_1, \ldots, i_m) \left(\frac{\partial u_{i_1 \ldots i_m}}{\partial x_{i_{m+1}}}\right), \quad \text{where } \sigma \text{ is defined in } (2.1).
\]
Given a symmetric $m$-tensor field, we define a symmetric $(m - \ell)$-tensor field $f^{i_1 \cdots i_\ell}$ obtained from $f$ by fixing the first $\ell$ indices $i_1, \ldots, i_\ell$. This can be done by fixing any $\ell$ indices. Due to symmetry it is enough to fix the first $\ell$ indices, that is,

\[
  f^{i_1 \cdots i_{m-\ell}} = f_{i_1 \cdots i_{m-\ell}}, \quad \text{where } i_1, \ldots, i_\ell \text{ are fixed}.
\]

Next, we introduce the generalized Saint Venant operator, the primary object of study in this article.

**Definition 2.1** (Generalized Saint Venant operator, [8]). For $m \geq 0$ and $0 \leq k \leq m$, the generalized Saint Venant operator (of order $k$) $W^k : C^\infty(S^m) \to C^\infty(S^{m-k} \otimes S^m)$ is defined by the equality

\[
  (W^k f)_{p_1 \cdots p_{m-k} q_1 \cdots q_{m-k} i_1 \cdots i_k} = \sigma(p_1 \cdots p_{m-k}) \sigma(q_1 \cdots q_{m-k} i_1 \cdots i_k) \sum_{\ell=0}^{m-k} (-1)^\ell \binom{m-k}{\ell} \frac{\partial^{m-k} f^{i_1 \cdots i_{m-k} i_{m-k+1} \cdots i_k}_{p_1 \cdots p_{m-k} q_{m-k+1} \cdots q_k}}{\partial x_{p_1} \cdots \partial x_{p_{m-k}} \partial x_{q_{m-k+1}} \cdots \partial x_{q_k}}.
\]

Note that, $W^k$ is a differential operator of order $(m - k)$. For $k = 0$, this is well known Saint-Venant operator $W$ defined above (see equation (1.1)). There is an equivalent way to define the Saint Venant operator $W$ which we discuss next. This equivalent formulation will be used to simplify several calculations.

**Definition 2.2.** [8, Chapter 2] We define the operator $R : S(S^m) \to S(T^{2m})$ as follows

\[
  (Rf)_{i_1 j_1 \cdots i_m j_m} = \alpha(i_1 j_1) \alpha(i_2 j_2) \cdots \alpha(i_m j_m) \frac{\partial^m f_{i_1 \cdots i_m}}{\partial x_{j_1} \cdots \partial x_{j_m}}
\]

where $\alpha(i_1 j_1)$ gives alternation with respect to two indices, that is,

\[
  \alpha(i_1 j_2) = \frac{1}{2} (g_{i_1 j_2} - g_{i_2 j_1}), \quad \text{for } g \in T^m(\mathbb{R}^n).
\]

The operators $R$ and $W$ are equivalent in the sense that they satisfy the following two relations [8, Equations 2.4.6 and 2.4.7]:

\[
  (W f)_{i_1 \cdots i_m j_1 \cdots j_m} = \sigma(i_1 \cdots i_m) \sigma(j_1 \cdots j_m) (Rf)_{i_1 j_1 \cdots i_m j_m},
\]

\[
  (R f)_{i_1 j_1 \cdots i_m j_m} = (m+1) \alpha(i_1 j_1) \alpha(i_2 j_2) \cdots \alpha(i_m j_m) (W f)_{i_1 \cdots i_m j_1 \cdots j_m}.
\]

### 2.2. Some known results for integral moment transforms.

The extended $q$-th integral moment ray transform of the tensor field $f^{i_1 \cdots i_\ell}$ for any fixed choice of $i_1, \ldots, i_\ell$ will be denoted by $J^q f^{i_1 \cdots i_\ell}(x, \xi)$, for any integer $q \geq 0$. The following result [6] provides a way to compute the ray transform of $f^{i_1 \cdots i_\ell}$ from the knowledge of $I^k f$ for $0 \leq k \leq m$.

**Lemma 2.3.** [6, Lemma 7] The following identity holds for any $f \in S(S^m)$:

\[
  J^0 f^{i_1 \cdots i_r} = \frac{(m-r)!}{m!} \sigma(i_1 \cdots i_r) \sum_{p=0}^{r} (-1)^p \binom{r}{p} \frac{\partial^p J^p f}{\partial x^{i_{p+1}} \cdots \partial x^{i_r} \partial x^{\xi_{p+1}} \cdots \partial x^{\xi_r}}, \quad \text{for } 1 \leq i_1, \ldots, i_r \leq n.
\]

**Lemma 2.4.** [5, Lemma 2.6] Let a function $\psi \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n \setminus \{0\})$ be positively homogeneous of degree $\lambda$ in the second argument

\[
  \psi(x, t\xi) = t^\lambda \psi(x, \xi) \quad (t > 0).
\]

Assume the restriction $\psi|_{TS^{n-1}} \in S(TS^{n-1})$. Further assume the restriction of $\langle \xi, \partial_\xi \rangle \psi$ and all its derivatives to $TS^{n-1}$ belong to $S(TS^{n-1})$, that is,

\[
  \frac{\partial^{k+i_\ell} (\xi, \partial_\xi) \psi}{\partial x^{i_1} \cdots \partial x^{i_k} \partial \xi^{j_1} \cdots \partial \xi^{j_\ell} } \bigg|_{TS^{n-1}} \in S(TS^{n-1}) \quad \text{for all } 1 \leq i_1, \ldots, i_k, j_1, \ldots, j_\ell \leq n.
\]

Then the restriction to $TS^{n-1}$ of every derivative of $\psi$ also belongs to $S(TS^{n-1})$, i.e.,

\[
  \frac{\partial^{k+i_\ell} \psi}{\partial x^{i_1} \cdots \partial x^{i_k} \partial \xi^{j_1} \cdots \partial \xi^{j_\ell} } \bigg|_{TS^{n-1}} \in S(TS^{n-1}) \quad \text{for all } 1 \leq i_1, \ldots, i_k, j_1, \ldots, j_\ell \leq n.
3. Main results and their proofs

The main result of the article provides a kernel description of the operator $I^k$ in terms of the generalized Saint Venant operator $W^k$. An equivalent kernel description is also presented in terms of potential tensor fields, but this description is a bit restrictive as discussed in the second theorem below.

**Theorem 3.1.** Let $f \in \mathcal{S}(S^m)$ in $\mathbb{R}^n$ ($n \geq 2$) and $0 \leq k \leq m$. Then $I^k f = 0$ if and only if $W^k f = 0$. That is, the operators $I^k$ and $W^k$ have the same kernel.

The proof of this theorem is completely new even for $k = 0$. Note that we do not have any restriction on the dimension which arises naturally in [6, Theorem 6]. The following result uses the dimension restriction coming from [6] to relate the above result with [6, Theorem 6]. This theorem is known for compactly supported symmetric tensor fields in the case $k = 0$ [8, Theorem 2.2.1].

**Theorem 3.2.** Let $f \in \mathcal{S}(S^m)$ and $k$ be an integer such that $1 \leq k \leq \min\{m, n - 1\}$. Then the following conditions are equivalent:

1. $I^k f = 0$.
2. $f = d_{k+1} v$, for some $(m - k - 1)$-tensor field $v$ satisfying $d^\ell v \to 0$ as $|x| \to \infty$ for $0 \leq \ell \leq k$.
3. $W^k f = 0$.

If we assume that Theorem 3.1 holds, then this theorem’s proof follows from the known chain of equivalence relations given below.

**Proof.** (1) $\iff$ (2) follows from [6, Theorem 6], (1) $\iff$ (3) follows from Theorem 3.1 and (2) $\iff$ (3) holds trivially.

The remainder of the article focuses on the proof of Theorem 3.1. The proof of this theorem is divided into several lemmas.

**Lemma 3.3 ([6]).** Let $f \in \mathcal{S}(S^m)$ and $0 \leq k \leq m$. The generalized Saint Venant operator $W^k f$ can be recovered explicitly from the knowledge of $I^k f$.

The authors proved this lemma in [6]. However, we prefer to sketch the proof here because some intermediate steps are essential for upcoming lemmas. To prove this lemma, we need to recall an important second order differential operator known as the John operator from [8, Theorem 2.10,1]. The John operator is denoted by $J_{pq} : C^\infty(\mathbb{R}^n \times \mathbb{R}^n \setminus \{0\}) \to C^\infty(\mathbb{R}^n \times \mathbb{R}^n \setminus \{0\})$, $(1 \leq p, q \leq n)$ and given by

$$(3.1) \quad J_{pq} = \frac{\partial^2}{\partial x^p \partial \xi^q} - \frac{\partial^2}{\partial x^q \partial \xi^p}.$$

Please note $J$ denotes the John operator while $I$ is used for extended integral moment transform.

**Proof.** It is sufficient to prove that $W^k f$ can be determined from $J^0 f, \ldots, J^k f$ because knowing $I^k f = (I^0 f, \ldots, I^k f)$ is equivalent to knowing $J^0 f, \ldots, J^k f$. Now, for fixed $1 \leq i_1, \ldots, i_k \leq n$, the following is known from Lemma 2.3

$$(3.2) \quad J^0 f^{i_1\cdots i_k} = \frac{(m - k)!}{m!} \sigma(i_1 \ldots i_k) \sum_{p=0}^{k} (-1)^p \binom{k}{p} \frac{\partial^k J^0 f}{\partial x^{i_1} \cdots \partial x^p \partial \xi^{p+1} \cdots \partial \xi^{i_k}}.$$

Applying $J$ to $J^0 f^{i_1\cdots i_k}$, we obtain

$$\left( J(J^0 f^{i_1\cdots i_k}) \right)_{p_1 q_1} = 2 (m - k) \alpha(p_1 q_1) \int_{\mathbb{R}} \xi^{j_1} \cdots \xi^{j_{m-k-1}} \frac{\partial f^{i_1\cdots i_k}}{\partial x^{j_1}} (x + t\xi) dt.$$

Applying the John operator $(m - k - 1)$ more times to the above equation and repeating the same arguments, we obtain

$$(3.3) \quad \left( J^{m-k} (J^0 f^{i_1\cdots i_k}) \right)_{p_1 q_1 \cdots p_{m-k} q_{m-k}} = 2^{m-k} (m - k)! \int_{\mathbb{R}} (R f^{i_1\cdots i_k})_{p_1 q_1 \cdots p_{m-k} q_{m-k}} (x + t\xi) dt.$$
The right hand side is the ray transform of scalar function \((Rf^{i_1\ldots i_k})_{p_1q_1\ldots p_mq_m}\) for all possible choices of indices \(1 \leq p_1, q_1, \ldots, p_m, q_m \leq n\). Thus \(Rf^{i_1\ldots i_k}\) can be determined explicitly by inverting X-ray transform of scalar functions [8, Theorem 2.12.2 for \(m = 0\)]. Knowing \(Rf^{i_1\ldots i_k}\) is same as knowing \(Wf^{i_1\ldots i_k}\) from the first relation of (2.5). Finally, to complete the proof of this lemma we need to connect \(Wf^{i_1\ldots i_k}\) and \(W^k f\). To this end, let us write \(Wf^{i_1\ldots i_k}\) explicitly
\[
(Wf^{i_1\ldots i_k})_{p_1\ldots p_mq_1\ldots q_m} = \sigma(p_1\ldots p_m) \sigma(q_1\ldots q_m) \sum_{\ell=0}^{m-k} (-1)^\ell \binom{m-k}{\ell} \frac{\partial^{m-k} f^{i_1\ldots i_k}}{\partial x_{p_{m-k-\ell+1}} \partial x_{p_{m-k-\ell+1}} \ldots \partial x_{q_m}}.
\]
Here, we make the following observation
\[
(W^k f)_{p_1\ldots p_mq_1\ldots q_m} = \sigma(q_1,\ldots,q_m, i_1,\ldots,i_k)(Wf^{i_1\ldots i_k})_{p_1\ldots p_mq_1\ldots q_m}.
\]
Now, right-hand side of (3.4) is completely known to us in terms of \(J^0 f, \ldots, J^k f\) as we discussed above. Therefore we know \(W^k f\), which completes the proof.

Lemma 3.4 (Main Lemma). Let \(0 \leq k < m\) and \(W^k f = 0\) for some \(f \in \mathcal{S}(S^m)\), then we have
\[
\sigma(q_1,\ldots,q_m-k, i_1,\ldots,i_k)(Wf^{i_1\ldots i_k}) = 0 \quad \text{for fixed} \quad 1 \leq i_1 \cdot i_k \leq n.
\]

Proof. Assume that \(W^k f = 0\) then from (3.4) we have
\[
\sigma(q_1,\ldots,q_m-k, i_1,\ldots,i_k)(Wf^{i_1\ldots i_k}) = 0 \quad \text{for fixed} \quad 1 \leq i_1 \cdot i_k \leq n.
\]
This together with (3.3) and the second equation of (2.5) entails
\[
\sigma(q_1,\ldots,q_m-k, i_1,\ldots,i_k) \sigma(p_1,\ldots,p_{m-k}) \left(\mathcal{J}^{m-k}(J^0 f^{i_1\ldots i_k})\right)_{p_1q_1\ldots p_mq_m} = 0.
\]
Since the symmetrization operators \(\sigma(q_1,\ldots,q_m-k, i_1,\ldots,i_k)\) and \(\sigma(p_1,\ldots,p_{m-k})\) commute with each other, this together with (3.6) implies
\[
\sigma(p_1,\ldots,p_{m-k}) \sigma(q_1,\ldots,q_m-k, i_1,\ldots,i_k) \left(\mathcal{J}^{m-k}(J^0 f^{i_1\ldots i_k})\right)_{p_1q_1\ldots p_mq_m} = 0.
\]
Now multiplying above by a symmetric \(m-k\) tensor \(\xi^{p_m-k}\) we obtain
\[
\xi^{p_m-k} \sigma(p_1,\ldots,p_{m-k}) \sigma(q_1,\ldots,q_m-k, i_1,\ldots,i_k) \left(\mathcal{J}^{m-k}(J^0 f^{i_1\ldots i_k})\right)_{p_1q_1\ldots p_mq_m} = 0
\]
\[
\sigma(p_1,\ldots,p_{m-k}) \left(\xi^{p_m-k} \sigma(q_1,\ldots,q_m-k, i_1,\ldots,i_k) \left(\mathcal{J}^{m-k}(J^0 f^{i_1\ldots i_k})\right)_{p_1q_1\ldots p_mq_m}\right) = 0.
\]
Taking summation over \(p_1,\ldots,p_{m-k}\) we get
\[
\sigma(q_1,\ldots,q_m-k, i_1,\ldots,i_k) \xi^{p_m-k} \left(\mathcal{J}^{m-k}(J^0 f^{i_1\ldots i_k})\right)_{p_1q_1\ldots p_mq_m} = 0.
\]
Since \((J^0 f^{i_1\ldots i_k})\) is the ray transform of a symmetric \(m-k\) tensor field, from the definition we have
\[
(J^0 f^{i_1\ldots i_k})(x + t\xi, \xi) = (J^0 f^{i_1\ldots i_k})(x, \xi), \quad (J^0 f^{i_1\ldots i_k})(x, \lambda\xi) = \lambda^{m-k-1}(J^0 f^{i_1\ldots i_k})(x, \xi)
\]
for \(\lambda > 0\) and \(t \in \mathbb{R}\). This immediately gives
\[
\langle \xi, \partial_x \rangle (J^0 f^{i_1\ldots i_k}) = 0,
\]
\[
\langle \xi, \partial_\xi \rangle J^{\ell} ((J^0 f^{i_1\ldots i_k})) = (m-k-1-\ell) J^{\ell} ((J^0 f^{i_1\ldots i_k})) \quad \text{for} \quad 0 \leq \ell \leq (m-k-1).
\]
Now we compute L.H.S of (3.7) without symmetrization.

\[
\xi^{p_1} \ldots \xi^{p_{m-k}} \left( J^{m-k} (J^0 f_{i_1^{m-k}}) \right)_{p_1 q_1 \ldots p_{m-k} q_{m-k}}
\]

\[
= \xi^{p_1} \ldots \xi^{p_{m-k-1}} \left[ \partial_\xi q_{m-k} \langle \xi, \partial_\xi \rangle \left( J^{m-k-1} (J^0 f_{i_1^{m-k}}) \right)_{p_1 q_1 \ldots p_{m-k-1} q_{m-k-1}} \right.

- \partial_\xi q_{m-k} \left( J^{m-k-1} (J^0 f_{i_1^{m-k}}) \right)_{p_1 q_1 \ldots p_{m-k-1} q_{m-k-1}}

\]

(3.9)

Using (3.8) and the fact that John operator commutes with \( \langle \xi, \partial_\xi \rangle \), we get

\[
\langle \xi, \partial_\xi \rangle \left( J^{m-k-1} (J^0 f_{i_1^{m-k}}) \right) = 0 \quad \text{and} \quad \langle \xi, \partial_\xi \rangle \left( J^{m-k-1} (J^0 f_{i_1^{m-k}}) \right) = 0.
\]

This together with (3.9) gives

\[
\xi^{p_1} \ldots \xi^{p_{m-k}} \left( J^{m-k} (J^0 f_{i_1^{m-k}}) \right)_{p_1 q_1 \ldots p_{m-k} q_{m-k}}
\]

\[
= \langle -1 \rangle \partial_\xi q_{m-k} \left[ \xi^{p_1} \ldots \xi^{p_{m-k-1}} \left( J^{m-k-1} (J^0 f_{i_1^{m-k}}) \right)_{p_1 q_1 \ldots p_{m-k-1} q_{m-k-1}} \right].
\]

(10)

Repeating the same analysis as in (3.9) we get

\[
\xi^{p_1} \ldots \xi^{p_{m-k-1}} \left( J^{m-k-1} (J^0 f_{i_1^{m-k}}) \right)_{p_1 q_1 \ldots p_{m-k-1} q_{m-k-1}}
\]

\[
= \xi^{p_1} \ldots \xi^{p_{m-k-2}} \left[ \partial_\xi q_{m-k-1} \langle \xi, \partial_\xi \rangle \left( J^{m-k-2} (J^0 f_{i_1^{m-k}}) \right)_{p_1 q_1 \ldots p_{m-k-2} q_{m-k-2}} \right.

- \partial_\xi q_{m-k-1} \left( J^{m-k-2} (J^0 f_{i_1^{m-k}}) \right)_{p_1 q_1 \ldots p_{m-k-2} q_{m-k-2}}

\]

(10)

This, (10) and together with (3.8) gives

\[
\xi^{p_1} \ldots \xi^{p_{m-k}} \left( J^{m-k} (J^0 f_{i_1^{m-k}}) \right)_{p_1 q_1 \ldots p_{m-k} q_{m-k}}
\]

\[
= \langle -1 \rangle \langle -2 \rangle \partial^2_{x_{q_{m-k}} x_{q_{m-k}} q_{m-k}} \left[ \xi^{p_1} \ldots \xi^{p_{m-k-2}} \left( J^{m-k-2} (J^0 f_{i_1^{m-k}}) \right)_{p_1 q_1 \ldots p_{m-k-2} q_{m-k-2}} \right].
\]

Iterating this \((m - k - 2)\) times more, we obtain

\[
\xi^{p_1} \ldots \xi^{p_{m-k}} \left( J^{m-k} (J^0 f_{i_1^{m-k}}) \right)_{p_1 q_1 \ldots p_{m-k} q_{m-k}} = (-1)^{m-k} (m-k)! \frac{\partial^{m-k}}{\partial x_{q_{m-k}} \ldots \partial x^n} (J^0 f_{i_1^{m-k}}).
\]

(11)

Combining (3.7) with (11)

\[
\sigma(q_1, \ldots, q_{m-k}, i_1, \ldots, i_k) \left[ \frac{\partial^{m-k}}{\partial x_{q_{m-k}} \ldots \partial x^n} (J^0 f_{i_1^{m-k}}) \right] = 0.
\]

This finishes the proof.

\[ \square \]

**Lemma 3.5.** Suppose the relation (3.5) holds, then we have

\[
\sigma(q_1, \ldots, q_{m-r}, i_1, \ldots, i_r) \left[ \frac{\partial^{m-r}}{\partial x_{q_{m-r}} \ldots \partial x^n} (J^0 f_{i_1^{m-r}}) \right] = 0 \quad \text{for} \quad 0 \leq r \leq k.
\]

(12)
Proof. For \( f \in S(\mathbb{R}^n) \), one can obtain the following relation for \( 0 \leq r \leq k \) by a direct computation:

\[
    \mathcal{J}^0 f^{i_1 \ldots i_r}(x, \xi) = \int_{-\infty}^{\infty} (f^{i_1 \ldots i_r})_j \ldots \xi^{j_{m-1}}(x + t \xi) \xi^{j_r} \ldots \xi^{j_{m-r}} f^{i_1 \ldots i_r}(x, \xi)\, dt = \xi^{i_1} \ldots \xi^{i_{m-r}} \mathcal{J}^0 f^{i_1 \ldots i_r}(x, \xi).
\]

(3.13)

By [8, Lemma 2.4.1], for any \( m \)-tensor \( f \), which has symmetry in the first \((m-k)\) indices and last \( k \) indices, the following symmetrization relation holds:

\[
    \sigma(q_1, \ldots, q_{m-k}, i_1, \ldots, i_k) f_{q_1 \ldots q_{m-k}i_1 \ldots i_k} = \frac{1}{m} \sigma(q_1, \ldots, q_{m-k}, i_1, \ldots, i_{k-1}) (k f_{q_1 \ldots q_{m-k}i_1 \ldots i_k} + (m - k) f_{i_k q_1 \ldots q_{m-k-1}q_{m-k}i_1 \ldots i_{k-1}}).
\]

(3.14)

Multiplying (3.5) by \( \xi^{i_1} \) and then summing over \( i_1 \) we get

\[
    \xi^{i_1} \sigma(q_1, \ldots, q_{m-k}, i_1, \ldots, i_k) \left[ \frac{\partial^{m-k}}{\partial x^{q_1} \ldots \partial x^{q_{m-k}}} (\mathcal{J}^0 f^{i_1 \ldots i_k}) \right] = 0.
\]

Using (3.14) from above we obtain

\[
    \xi^{i_1} \sigma(q_1, \ldots, q_{m-k}, i_2, \ldots, i_k) \left[ k \frac{\partial^{m-k}}{\partial x^{q_1} \ldots \partial x^{q_{m-k}}} (\mathcal{J}^0 f^{i_1 \ldots i_k}) + (m - k) \frac{\partial^{m-k}}{\partial x^{i_1} x^{q_1} \ldots \partial x^{q_{m-k-1}}} (\mathcal{J}^0 f^{q_{m-k}i_2 \ldots i_k}) \right] = 0.
\]

Since the symmetrization operator \( \sigma(q_1, \ldots, q_{m-k}, i_2, \ldots, i_k) \) is independent of \( i_1 \), this gives

\[
    \sigma(q_1, \ldots, q_{m-k}, i_2, \ldots, i_k) \left[ k \frac{\partial^{m-k}}{\partial x^{q_1} \ldots \partial x^{q_{m-k}}} (\mathcal{J}^0 f^{i_1 \ldots i_k}) \xi^{i_1} + (m - k) \langle \xi, \partial x \rangle \frac{\partial^{m-k-1}}{\partial x^{q_1} \ldots \partial x^{q_{m-k-1}}} (\mathcal{J}^0 f^{q_{m-k}i_2 \ldots i_k}) \right] = 0.
\]

Using the first relation in (3.8) and the fact that \( \langle \xi, \partial x \rangle \) commutes with constant coefficient differential operator, we obtain

\[
    \sigma(q_1, \ldots, q_{m-k}, i_2, \ldots, i_k) \left[ \frac{\partial^{m-k}}{\partial x^{q_1} \ldots \partial x^{q_{m-k}}} (\mathcal{J}^0 f^{i_1 \ldots i_k}) \xi^{i_1} \right] = 0.
\]

(3.15)

Multiplying (3.15) by \( \xi^{i_2} \ldots \xi^{i_{k-r}} \) and summing over the indices \( i_2, \ldots, i_{k-r} \) and repeating similar analysis as above we get

\[
    \sigma(q_1, \ldots, q_{m-k}, i_r + 1, \ldots, i_k) \left[ \frac{\partial^{m-k}}{\partial x^{q_1} \ldots \partial x^{q_{m-k}}} (\mathcal{J}^0 f^{i_1 \ldots i_k}) \xi^{i_1} \ldots \xi^{i_{k-r}} \right] = 0.
\]

(3.16)

After a re-indexing, combining (3.16) with (3.13) we get

\[
    \sigma(q_1, \ldots, q_{m-r}, i_r, \ldots, i_r) \left[ \frac{\partial^{m-r}}{\partial x^{q_1} \ldots \partial x^{q_{m-r}}} (\mathcal{J}^0 f^{i_1 \ldots i_r}) \right] = 0.
\]

Finally differentiating this equation with respect to \( x^{q_{m-r+1}} \ldots x^{q_{m-r}} \) and then taking \( \sigma(q_1, \ldots, q_{m-r}, i_1, \ldots, i_r) \), we get

\[
    \sigma(q_1, \ldots, q_{m-r}, i_1, \ldots, i_r) \left[ \frac{\partial^{m-r}}{\partial x^{q_1} \ldots \partial x^{q_{m-r}}} (\mathcal{J}^0 f^{i_1 \ldots i_r}) \right] = 0 \quad \text{for} \quad 0 \leq r \leq k.
\]

\[ \square \]

Proof of Theorem 3.1. The aim is to prove \( W^k f = 0 \) if and only if \( \mathcal{I}^k f = 0 \). We only need to prove the “only if” part of the statement since the other direction

\[ \mathcal{I}^k f = 0 \implies W^k f = 0, \]

follows from Lemma 3.3.
In order to prove the “only if” part, we assume $W^k f = 0$. The idea here is to use Lemma 3.5 repeatedly. For $f \in S(S^m)$, a direct application of integration by parts implies
\begin{equation}
\langle \xi, \partial_x \rangle J^k f = -k J^{k-1} f.
\end{equation}

As a first step, we put $r = 0$ in (3.12) to get
\[
\sigma(q_1, \ldots, q_m) \left[ \frac{\partial^{m-1}}{\partial x^{q_{m-1}} \cdots \partial x^{q_1}} (J^0 f) \right] = \frac{\partial^m}{\partial x^{q_{m-1}} \cdots \partial x^{q_1}} (J^0 f) = 0.
\]

We know that $J^0 f |_{TS^n-1} \in S(TS^{n-1})$ and $\langle \xi, \partial_x \rangle J^0 f = 0$ follows from substituting $k = 0$ in (3.17). Thus $J^0 f$ satisfies all the hypotheses of Lemma 2.4. This implies
\[
\frac{\partial^{m-1}}{\partial x^{q_{m-1}} \cdots \partial x^{q_1}} (J^0 f) |_{TS^n-1} \in S(TS^{n-1}).
\]

This together with $\partial_x^m \left( \frac{\partial^{m-1}}{\partial x^{q_{m-1}} \cdots \partial x^{q_1}} (J^0 f) \right) = 0$, entails $\frac{\partial^{m-1}}{\partial x^{q_{m-1}} \cdots \partial x^{q_1}} (J^0 f) = 0$. This can be proved directly (see also [5, Statement 2.12]). Proceeding in this way after finitely many steps we conclude
\begin{equation}
J^0 f (x, \xi) = 0.
\end{equation}
Next, consider the Lemma 3.5 with $r = 1$ to get
\[
\sigma(q_1, \ldots, q_{m-1}, i_1) \left[ \frac{\partial^{m-1}}{\partial x^{q_{m-1}} \cdots \partial x^{q_1}} (J^0 f^{i_1}) \right] = 0.
\]

Using $J^0 f^{i_1} = \frac{\partial^0 f}{\partial x^{i_1}} - \frac{\partial^1 f}{\partial x^{i_1}}$ (see Lemma 2.3) together with the fact $J^0 f = 0$ in the above equation which gives
\[
\sigma(q_1, \ldots, q_{m-1}, i_1) \left[ \frac{\partial^{m-1}}{\partial x^{q_{m-1}} \cdots \partial x^{q_1}} (J^0 f^{i_1}) \right] = \frac{\partial^m}{\partial x^{q_{m-1}} \cdots \partial x^{q_1}} (J^1 f) = 0.
\]

We now apply the similar argument on $J^1 f$ as it satisfies following:
\[
J^1 f |_{TS^n-1} \in S(TS^{n-1}) \quad \text{and} \quad \langle \xi, \partial_x \rangle J^1 f = -J^0 f (= 0), \quad \text{by (3.17) and (3.18)}.
\]

Repeating similar argument used above we can conclude $J^1 f = 0$. Following the same idea, assume $J^p f = 0$ for $p = 0, 1, \ldots, r - 1$

and apply Lemma 2.3 again to get
\[
J^0 f^{i_1 \cdots i_r} = \frac{(-1)^r (m - r)!}{m!} \frac{\partial^r J^r f}{\partial x^{i_1} \cdots \partial x^{i_r}}.
\]

This together with (3.12) gives
\[
\sigma(q_1, \ldots, q_{m-r}, i_1, \ldots, i_r) \left[ \frac{\partial^{m-r}}{\partial x^{q_{m-r}} \cdots \partial x^{q_1}} (J^0 f^{i_1 \cdots i_r}) \right] = \frac{\partial^m}{\partial x^{q_{m-r}} \cdots \partial x^{q_1}} (J^r f) = 0, \quad \text{for } 0 \leq r \leq k.
\]

Repeating similar analysis this implies $J^r f = 0$. Therefore $J^r f = 0$ for $0 \leq r \leq k$ or equivalently, $J^k f = 0$. Thus we have proved that
\[
W^k f = 0 \implies J^k f = 0.
\]
This completes the proof of our main theorem. 

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