Magnetization curves and ac susceptibilities in type-II superconductors: geometry-independent similarity and effect of irreversibility mechanisms

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The magnetic characteristics for superconducting samples of different geometries (plates and films of various widths) and orientation with respect to external field are compared. A similarity is established between the magnetization curves $M(H)$ and the corresponding susceptibilities. A sign reversal is predicted for the 3rd harmonic of the ac susceptibility when the irreversibility mechanism changes from bulk pinning to edge barrier. A mutual influence of the edge barrier and bulk pinning on the magnetization curves and field-dependent ac susceptibility is studied on the basis of an exactly solvable model (narrow plate in a parallel field and/or narrow film in a perpendicular magnetic field).

I. INTRODUCTION

Study of the magnetic characteristics of type-II superconductors and their behaviour in a magnetic field varying in time is of active interest from both the fundamental and applied standpoints. A large number of theoretical and experimental works have been devoted to investigation into the magnetization of samples of various geometries, hysteresis losses and ac susceptibilities $\chi_n^{ac} = \chi_n' + i\chi_n''$. Two mechanisms are known to be responsible for these characteristics, specifically, bulk pinning and edge barrier that prevents vortex entry into (exit from) a smooth-surface sample. The effects produced by these mechanisms on
the shape of the magnetization - and/or hysteresis curves have been considered by different authors \[2,4,8–11\]. However, the influence of an edge barrier on field dependences of the hysteresis losses and the behaviour of higher harmonics of susceptibility have not been studied adequately so far. In particular, as reported in our recent publication \[12\], the allowance for the edge barrier for a narrow film in a transverse magnetic field results in a sign reversal of the 3d harmonics of susceptibility $\chi_3'(H)$ and $\chi_3''(H)$. This is in stark contrast with the situation in which bulk pinning is the major mechanism of irreversibility. Therefore, it would be extremely important to analyse a joint influence of an edge barrier and bulk pinning on the irreversible characteristics of type-II superconducting samples. For bulk superconductors a similar problem was studied in \[1\] by Clem who has formulated a generalized local model of the critical state. This model provides an adequate basis for calculation of the magnetization curves and ac susceptibility for bulk superconductors \[13,14\]. Yet, in low-dimensional superconductors (single crystals with high demagnetization factor, films) the effect of edge barrier and bulk pinning on magnetic characteristics is still not fully understood. For example, in \[15\] the authors take account of the joint influence of edge barrier and bulk pinning on the shape of the magnetization curve for a wide film in a perpendicular magnetic field. Unfortunately, the dependences $M(H)$ obtained therein cannot be expressed in an analytical form, which does not allow to find the field and temperature dependences of magnetic susceptibility. It is, therefore, highly desirable to formulate an analytically solvable generalized model of the critical state for superconductors of arbitrary geometry, both longitudinal (as shown in Fig.1a) and transverse (see Fig.1b) with respect to the external magnetic field.

In this work we report a study on the behavior of magnetization and the harmonics of ac susceptibility as a function of the amplitude of an applied low-frequency magnetic field. As an example, we used an exactly solvable model of a narrow film with an edge barrier. This model accounts for arbitrary bulk pinning which is characterized by a certain dependence of the depinning current density $j_p$ on magnetic field $H$. For specific calculations, two model dependences $j_p(H)$ were used. One of them assumes that the depinning current density $j_p$
is a constant value independent of $H$ (the Bean model). In the other case the magnetic characteristics were obtained for the Kim-Anderson (KA) model dependence $j_p(H)$.

The paper is organized as follows. In Section II we formulate a generalized model of the critical state in a narrow film/plate. On the basis of this model we obtain current densities and vortex distributions in the mixed state taking into account both edge barrier and bulk pinning as major irreversibility mechanisms in type-II superconductors. Two specific depinning models were studied: the Bean model (see II.A) and the Kim-Anderson model (see II.B). The magnetization curves for a narrow film/plate are obtained and the harmonics of ac susceptibility are calculated in Sec. III. In Sec. IV a comparison is made of the magnetic characteristics for superconducting samples of different geometries (plates and films of various widths) and orientation with respect to external field. A similarity is found out between the magnetization curves $M(H)$ and the corresponding susceptibilities. A sign reversal effect is predicted for the 3d harmonic of ac susceptibility, following a change of irreversibility mechanisms (from bulk pinning to edge barrier). In Sec. V we discuss the applicability conditions of the approach used. The conditions of the barrier-to-pinning crossover in the temperature-dependent ac susceptibilities are analysed. A brief summary of our results is presented in Sec. VI.

II. GENERALIZED MODEL OF THE CRITICAL STATE

Let us consider a narrow plate (Fig.1a) or a film (Fig.1b) of width $W$ and thickness $d < \lambda$ ($W < \lambda$ for the plate, $W < \lambda^2/d$ for the film; $\lambda$ being the London penetration depth), which is infinite in the $x$-direction, placed in a magnetic field $H = (0,0,H)$. In view of the symmetry of the problem, the screening current has one component $j_x = j(y)$, which satisfies the equation (see Appendix)

$$\frac{1}{C_0} \frac{dj}{dy} = H - n(y)\phi_0. \quad (1)$$

Here $C_0 = cW/(8\pi\lambda^2)$, $n$ is the vortex density, $\phi_0$ is the magnetic flux quantum, and the
dimensionless coordinate $|y| \leq 1$ is measured in units $W/2$; the origin of the coordinates is assumed to be in the sample centre ($y = 0$). Equation (1) can be derived from the full version of the Maxwell-London equation (see e.g. ref. [3]) by neglecting the nonlocal (integral) term.

Since in the absence of transport current $n(y) = n(-y)$, and $j(y) = -j(-y)$, we may reduce our consideration to only one half of the film (for definiteness, the right-hand one, $y > 0$). It is assumed that a superconductor is placed in a magnetic field which is first quasistatically increasing to some value $H_0$ and then decreasing down to $-H_0$. As is known, given an edge barrier the vortices penetrate a sample when the near-the-edge current density reaches some critical value $j_s$ (for ideal surface $j_s$ coincides with the Ginzburg-Landau depairing current density $j_{GL}$). Therefore, up to the field $H = H_s = j_s/C_0$ the film remains in the Meissner state (see Fig. 2a):

$$j(y) = yHC_0, \quad 0 < y < 1,$$

Further increase of the magnetic field will cause penetration of vortices that will move inside a superconductor until the Lorentz force $F = j\phi_0/c$ acting on each vortex becomes equal to the pinning force $F_p = j_p\phi_0/c$.

A. Field-independent bulk pinning

First consider a model in which the depinning current density is independent of $H$ (the Bean model). The vortices that have penetrated into the superconductor at $H > H_s$ will be distributed with the density $n(y) = H/\phi_0$ in a region $a(H) < y < b(H)$, where $a(H) = H_p/H$ defines the depth of vortex penetration, $H_p = j_p/C_0$. Unlike the classical Bean model, the field of complete penetration ($a = 0$) here is $H = \infty$, which is a general property of the nonlocal model of the critical state both in bulk [16,17] and quasi-two-dimensional [18] superconductors. The parameter $b(H) = 1 - (H_s - H_p)/H$ determines the size of the vortex-free region (of width $1 - b = (H_s - H)/H$), from which the vortices are "driven" by a strong near-edge current $j > j_p$. The corresponding distribution of the current density has the form
We now start reducing the magnetic field from $H_0$ to $-H_0$. Until the current density in the vortex-occupied region $a(H_0) \leq y \leq b(H_0)$ has reached the critical value, $j = -j_p$, the distribution of vortices will be "frozen" and the distribution of current density will be described by the expression

$$j(y) = \begin{cases} 
  yHC_0 & 0 < y < a, \\
  j_p & a < y < b, \\
  HC_0(y - 1) + j_s & b < y < 1.
\end{cases} \quad (3)$$

where $a_0 = a(H_0)$, $b_0 = b(H_0)$. The corresponding distributions $j(y)$, $n(y)$ are shown in Fig. 2c.

At the flux-defreezing field $H_{df} = H_0(H_0 - H_s - H_p)/(H_0 - H_s + H_p)$, the current density $j(b_0)$ becomes equal to $-j_p$ and the vortices start moving towards the film edge. The magnetic flux related to the vortices will stay in the film until the vortex-exit field, $H_{ex} = H_0 - H_s - H_p$, is reached. So, in the field interval $H_{ex} \leq H \leq H_{df}$ the current density and vortex distribution will have the following form (see Fig. 2d)

$$j(y) = \begin{cases} 
  yHC_0 & 0 < y < a_0, \\
  y(H - H_0)C_0 + a_0H_0C_0 & a_0 < y < b_1, \\
  -j_p & b_1 < y < b_2, \\
  yHC_0 + H_0C_0(a_0 - b_0) & b_2 < y < 1.
\end{cases} \quad (5a)$$
\[ n(y) = \begin{cases} 
0 & 0 < y < a_0, \\
H_0/\Phi_0 & a_0 < y < b_1, \\
H/\Phi_0 & b_1 < y < b_2, \\
0 & b_2 < y < 1, 
\end{cases} \quad (5b) \]

where \( b_1 = 2H_p/(H_0 - H) \), \( b_2 = (H_0 - H_s - H_p)/H \). A further decrease in the magnetic field starting from \( H_{ex} \) will cause the vortices exit from the film at \( 0 \leq H \leq H_{ex} \), as well (see Fig. 2e):

\[ j(y) = \begin{cases} 
yHC_0 & 0 < y < a_0, \\
y(H - H_0)C_0 + a_0H_0C_0 & a_0 < y < b_1, \\
-J_p & b_1 < y < 1; 
\end{cases} \quad (6a) \]

\[ n(y) = \begin{cases} 
0 & 0 < y < a, \\
H_0/\Phi_0 & a < y < b_1, \\
H/\Phi_0 & b_1 < y < 1. 
\end{cases} \quad (6b) \]

Note that at \( H = 0 \) the remaining vortices occupy only the region \( a_0 \leq y \leq b_1 \) (see Fig. 2f). The edge current density in this case exceeds \(-j_s\) (which prevents entry of antivortices into the film), therefore, with a further decrease of the field the distribution

\[ j(y) = \begin{cases} 
yHC_0 & 0 < y < a_0, \\
y(H - H_0)C_0 + a_0H_0C_0 & a_0 < y < b_1, \\
yHC_0 - H_0C_0(a_0 - b_1) & b_1 < y < 1; 
\end{cases} \quad (7a) \]

\[ n(y) = \begin{cases} 
0 & 0 < y < a_0, \\
H_0/\Phi_0 & a_0 < y < b_1, \\
0 & b_1 < y < 1, 
\end{cases} \quad (7b) \]

will be valid right down to the field \( H_s^{(-)} = (H_0 - H_s - H_p)/2 - \sqrt{(H_0 - H_s - H_p)^2/4 + H_0(H_s - H_p)} \), at which vortices of the opposite sign ('antivortices') will start entering into the film. In the field range \(-H_0 \leq H \leq H_s^{(-)} \) (see Fig. 2g) one finds:
\[
\begin{align*}
  j(y) &= \begin{cases} 
    yHC_0 & 0 < y < a, \\
    y(H - H_0)C_0 + a_0H_0C_0 & a_0 < y < b_1, \\
    -j_p & b_1 < y < b_3, \\
    yHC_0 - (H + H_s)_0 & b_3 < y < 1;
  \end{cases} \\
  n(y) &= \begin{cases} 
    0 & 0 < y < a, \\
    H_0/\Phi_0 & a_0 < y < b, \\
    H/\Phi_0 & b_1 < y < b_3, \\
    0 & b_3 < y < 1.
  \end{cases}
\end{align*}
\]

(8a)

(8b)

where \( b_3 = 1 + (H_s - H_p)/H \).

B. Field-dependent pinning (Kim-Anderson model)

Let us find the distribution of vortices and a current in a film with a magnetic-field-dependent bulk density of the depinning current \( j_p(H) = C_0H^2_k/(H_k + |H|) \) (KA model), taking edge barrier into account. We shall follow the same procedure as considered above. Thus, for the magnetic field increased to the magnitude \( H_0 \), \( j(y) \) is determined by expressions (3), where \( a(H) = H^2_k/(H(H_k + |H|)) \), \( b(H) = a(H) + (H - H_s)/H \). When the field is decreasing from \( H_0 \) to \( H_{df} \), the current density and vortex distributions are similar to those in (4) with new values for \( a = a(H_0) \), \( b = b(H_0) \), and the field \( H_{df} \) being determined from the equation

\[
b_0(H - H_0) + a_0H_0 + \frac{H^2_k}{H_k + |H|} = 0
\]

Then, with the field decreasing from \( H_{df} \) to \( H_{ex} \), \( j(y) \) and \( n(y) \) are defined by the dependence (5) in which now we have

\[
b_1 = \frac{H_k}{H_0 - H} \left( \frac{1}{H_k + H_0} + \frac{1}{H_k + |H|} \right), \\
b_2 = \frac{H_0b_0 + b_1(H - H_0)}{H},
\]

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and field $H_{ex}$ is determined from the condition $b_2(H_{ex}) = 1$. If the field amplitude $H_0$ satisfies the condition $H_0 < \sqrt{2}H_{k1}$, then with a further decrease of the magnetic field down to $-H_0$ the evolution of $j(y), n(y)$ distributions is described by formulae (6-8) in which $b_3$ now is expressed as

$$b_3 = \frac{H_s + H}{H} - \frac{H^2_{k2}}{H(H_{k1} + |H|)},$$

and field $H_{s}^{(-)}$ satisfies the equation

$$H + H_0(a_0 - b_1(H)) + H_s = 0.$$

For the amplitudes that meet the condition $H_0 > \sqrt{2}H_{k1}$, as can be easily shown, the dependence $b_1(H)$ posesses a minimum at some field $H_{t}^{(+)} = (\sqrt{2} - 1)(H_0 + \sqrt{2}H_{k1}) > 0$. In this case expressions (6) will hold only at $H \geq H_{t}^{(+)}$. A detailed examination shows that with a still further decrease of the field the distribution of the current density and vortices density in a film takes the form

$$j(y) = \begin{cases} yHC_0 & 0 < y < a_0, \\ y(H - H_0)C_0 + a_0H_0C_0 & a_0 < y < b^*, \\ j_0(y) & b^* < y < b_{2a}, \\ -j_p(H) & b_{2a} < y < 1; \end{cases} \quad (9a)$$

$$n(y) = \begin{cases} 0 & 0 < y < a_0, \\ H_0/\Phi_0 & a_0 < y < b_1, \\ (H_{k1} + H_{2a})(b^* - y)/2b^*\Phi_0 + H_{2a}/\Phi_0 & b^* < y < b_{2a}, \\ H/\Phi_0 & b_{2a} < y < 1, \end{cases} \quad (9b)$$

where $b^*$ is determined from the condition $b^* = b_1(H_{t}^{(+)}), b_{2a}$ from the equality $j_0(b_{2a}) = -j_p(H)$, and $j_0(y)$ is defined as

$$j_0(y) = C_0(H - H_{t}^{(+)})y - C_0\frac{H_{k1} + H_{t}^{(+)}}{2b^*}(b^* - y)^2 + C_0b^*(H_{t}^{(+)} - H_0) + a_0H_0C_0$$

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This current distribution is valid in the field range \(0 < H < H_t^{(+)}\). If the field keeps decreasing, then, by analogy with the Bean model (provided \(H_s > H_{k2}/H_{k1}\)), antivortices at first will not be able to enter the film, and the current/vortex distribution in the film will be described by the following expressions

\[
\begin{align*}
    j(y) &= \begin{cases} 
        yHC_0 & 0 < y < a_0, \\
        y(H - H_0)C_0 + a_0H_0C_0 & a_0 < y < b^*, \\
        j_0(y) & b^* < y < b_{2a}, \\
        yHC_0 - j_p(H) - b_{2a}HC_0 & b_{2a} < y < 1;
    \end{cases} \\
    n(y) &= \begin{cases} 
        0 & 0 < y < a_0, \\
        H_0/\Phi_0 & a_0 < y < b_1, \\
        (H_{k1} + H_{2a})(b^* - y)/2b^*\Phi_0 + H_{2a}/\Phi_0 & b^* < y < b_{2a}, \\
        0 & b_{2a} < y < 1.
    \end{cases}
\end{align*}
\]

With a further decrease of the magnetic field two situations are possible, depending on the problem parameters. In one case, at first \(b_{2a}\) reduces to \(b^*\) (which occurs at \(H = H_t^{(-)}\)), following which the current density at the edge reaches the antivortices-entry threshold \(-j_s\). In the other case, the edge current density first falls down to \(-j_s\) (it takes place at some field \(H = \tilde{H}\)), and then \(b_{2a}\) decreases to \(b^*\). For definiteness, we shall consider only the first situation, \(|H_t^{(-)}| < |\tilde{H}|\). Here expressions (10) will hold right down to the field \(H = H_t^{(-)}\) which is determined by the condition \(b_{2a} = b^*\). Further, down to the field \(H = -H_0\), \(j(y)\) and \(n(y)\) are found from (7,8) with the \(b_3\) and \(H_s^{(-)}\) that have been introduced above.

Note that a similar method can be used to obtain the current/vortex density distribution for an arbitrary function \(j_p(H)\). For a monotonically decreasing function \(j_p(H)\) the resulting distributions will have a qualitative similarity to those obtained above. In the peak-effect conditions characterized by a nonmonotonic behaviour of \(j_p(H)\) the issue on the form of \(j(y)\) and \(n(y)\) distributions needs to be addressed separately.
III. MAGNETIZATION CURVES AND AC-SUSCEPTIBILITY HARMONICS

FOR NARROW FILMS

A. Magnetization curves

The above expressions for the current density allow to find the magnetization dependence for superconducting plates (films) in different ranges of field and obtain a hysteresis curve. For the Bean model given $H_0 \geq H_s + H_p$ the magnetization curve $M(H)$ is defined as follows

$$M(H) = \gamma \begin{cases} H_0(b_0^3 - a_0^3)/2 + 3H_0(a_0 - b_0)/2 + H, & H_{df} < H < H_0, \\ b_1^3(H_0 - H)/2 - H_0a_0^3/2 + Hb_3^2/2 + H + 3H_0(a_0 - b_0)/2, & H_{ex} < H < H_{df}, \\ 0 < H < H_{ex}, \\ H_0b_1^3 - a_0^3)/2 + 3H_0(a_0 - b_1)/2 + H, & H_s^{(-)} < H < 0, \\ b_1^3(H_0 - H)/2 - H_0a_0^3/2 + H - 3(H + H_s)/2 + Hb_3^2/2, & -H_0 < H < H_s^{(-)}, \end{cases}$$

where $\gamma = -C_0W/12c$.

At a relatively low field magnitude, $H_0 < H_s + H_p$, the mode described by expressions (4) will immediately change for the mode (7). Thus, the dependence $M(H)$ in the fields interval $H_s < H_0 < H_s + H_p$ will have the form

$$M(H) = \gamma \begin{cases} H_0(b_0^3 - a_0^3)/2 + 3H_0(a_0 - b_0)/2 + H, & H_{df} < H < H_0, \\ H_0b_1^3 - a_0^3)/2 + 3H_0(a_0 - b_1)/2 + H, & H_s^{(-)} < H < H_{df}, \\ b_1^3(H_0 - H)/2 - H_0a_0^3/2 + H - 3(H + H_s)/2 + Hb_3^2/2, & -H_0 < H < H_s^{(-)}, \end{cases}$$

In Fig. 3 the dependence $M(H)$ is shown for different values of fields $H_p$ and $H_s$. It is seen that the edge barrier causes peaks to arise on the magnetization curve $M(H)$ at $H \approx \pm H_s$. With an increase of the ratio $H_s/H_p$ the amplitude of the peak grows and the latter shifts towards higher fields. Such a behavior of magnetization is similar to the corresponding dependence reported for wide films in [15].
Within the Kim-Anderson model, if $H_s > H_{k2}^2/H_{k1}$, $H_0 > H_s + H_{k2}^2/H_{k1}$, $H_0 > \sqrt{2}H_{k1}$, $|H_{l}^\ell(-)| < |\tilde{H}|$, the dependence $M(H)$ is written in the form

$$M(H) = \gamma \begin{cases} 
H_0(b_0^3 - a_0^3)/2 + 3H_0(a_0 - b_0)/2 + H, & H_{df} < H < H_0, \\
b_1^3(H_0 - H)/2 - H_0a_0^3/2 + Hb_2^3/2 + H + 3H_0(a_0 - b_0)/2, & H_{ex} < H < H_{df}, \\
b_1^3(H_0 - H)/2 - H_0a_0^3/2 + 3b_1(H - H_0)/2 + 3H_0a_0/2, & H_{l}^{(+)} < H < H_{ex}, \\
H_0^3 + (H - H_0)((b^*)^3 - a_0^3) + 3H_0a_0((b^*)^2 - a_0^2)/2, & 0 < H < H_{l}^{(+)}, \\
3H_{k2}^2(1 - b_2^3)/2(H_{k1} + |H|) + m_0, & 0 < H < H_{l}^{(+)}, \\
H_0^3 + (H - H_0)((b^*)^3 - a_0^3) + 3H_0a_0((b^*)^2 - a_0^2)/2, & 0 < H < H_{l}^{(+)}, \\
3H_{k2}^2(1 - b_2^3)/2(H_{k1} + |H|) + m_0 + Hb_2^3/2 + H(1 - 3b_2a/2), & H_{l}^{(-)} < H < 0, \\
H_0(b_1^3 - a_0^3)/2 + 3H_0(a_0 - b_1)/2 + H, & H_{s}^{(-)} < H < H_{l}^{(-)}, \\
b_1^3(H_0 - H)/2 - H_0a_0^3/2 + H - 3(H + H_s)/2 + Hb_3^3/2, & -H_0 < H < H_{s}^{(-)}, \\
\end{cases}$$

where $m_0 = 6/(3W) \int b_2^2 yj_0(y) dy$.

When $H_0 < \sqrt{2}H_{k1}$, the dependence $M(H)$ will be fully identical to (11) with the appropriate values of parameters $a, b_1, b_2, b_3, H_{df}, H_{ex}, H_{s}^{(-)}$ obtained using the KA approach.

Fig.4 shows dependence (13) for different values of the parameters related to the Kim-Anderson depinning model and edge barrier. Unlike in the Bean model, the magnetization curve here has a maximum even without an edge barrier. Given an edge barrier, the $M(H)$ peak amplitude increases and the peak itself shifts towards the higher fields.

**B. ac - susceptibility**

If an external magnetic field $H$ changes harmonically in time, $H(t) = H_0 \cos(\omega t)$, magnetization $M(t)$ is also a periodical function with a period $T = 2\pi/\omega$, whose Fourier components of the magnetization $M(H)$ determine the ac-susceptibility harmonics

$$\chi_n^{ac} = -\frac{2\gamma}{\pi H_0} \int_0^{\pi} M(\Theta)e^{in\Theta} d\Theta = \chi_n^' + i\chi_n^\prime\prime. \quad (14)$$
where $\Theta = \omega t$.

It follows from the symmetry of the problem ($M(H) = -M(-H)$) that all of the even harmonics are zero. Fig. 5a-5b shows the imaginary and the real parts of the first (i.e. fundamental) and third harmonics of ac susceptibility, calculated by formulae (13) both with- and without an edge barrier (curves 1 and 2, respectively). One can see that existence of an edge barrier produces a quantitative (first harmonic) and a qualitative (change of sign in the real part of the 3rd harmonic) effect on the form of the dependence $\chi_n(H_0)$. We believe that the sign reversal effect (SRE) which was for the first time predicted by the authors in [12] is a vivid manifestation of how an edge barrier affects the $\chi_3(H_0)$ behavior. It should be mentioned that a similar behavior is typical for a specific set of higher harmonics, as well: $\chi_{4k-1}'(H_0), \chi''_{4k-1}(H_0)$ ($k = 2, 3$). Note also that the SRE is not observed for the harmonics $\chi_{4k+1}'(H_0), \chi''_{4k+1}(H_0)$ ($k = 1, 2$).

IV. GEOMETRY EFFECT ON MAGNETIC CHARACTERISTICS OF TYPE-II SUPERCONDUCTORS

As follows from our analysis, the magnetic characteristics of various-width plates ($W \leq \lambda$ and $W \gg \lambda$) in a parallel magnetic field and also of both narrow ($W \leq \lambda^2/d$) and wide ($W \gg \lambda^2/d$) films in a transverse magnetic field are quite similar. The analogous similarity properties of the magnetization curves were pointed out by [18] where curves $M(H)$ were compared for macroscopic samples of different geometries possessing no edge/surface 2barrier.

In Fig. 6a the magnetization curves are shown for samples without a bulk pinning. Curve 1 was obtained for a narrow film (formula (11) with $H_p = 0$), curve 2 corresponds to a wide plate in a parallel field (as calculated within the nonlocal model [16]), curve 3 is for a wide film in a transverse magnetic field [8]. The magnetic field is measured in units of $H_s$, essentially the first-vortex entry field which is different in either case:

1) $H_s = \Phi_0/2\pi \xi W$ [13, 20],
2) $H_s = \Phi_0/2\pi \xi \sqrt{W\lambda^2/d}$ [19, 21],
3) $H_s = \Phi_0/2\sqrt{2}\pi \lambda \xi$ [22].
The magnetization is normalized so as to ensure the condition $M(H) = H$ in the Meissner state. As is seen from Fig.6a, the dependences $M(H)$ in the appropriate units are qualitatively alike, therefore, the magnetization harmonics $\chi^{ac}_n(H_0)$ (Fig. 7a-10a) also feature a similarity. (Note that this property is exhibited by susceptibilities with $n > 3$, as well).

A second set of figures, 6b-10b, describes the situations in which an account is taken only of a bulk pinning ($j_p(H) = const$), and an edge barrier is neglected. Curves 1 correspond to narrow films (see dependence (11) at $H_s = H_p$), curves 2 are for wide films in the longitudinal geometry \[^{[16]}\], curves 3 were obtained for wide films in the transverse geometry \[^{[4]}\]. The magnetic field is measured in units of $H^*$ which for narrow films is equal to $j_p/C_0$, for curves 2 - $H^* = 2\pi j_p W/c$, for curves 3 - $H^* = 4j_p d/c$. Magnetization is measured in units of $j_p W/8c$. Note that in all cases considered the curves $M(H)$ never saturate, which is a direct consequence of the nonlocality effect. A similarity between the corresponding dependences $M(H)$ obtained in different geometries of the problem is more conspicuous here than in the case of a zero-bulk-pinning (see above). Besides, it is obvious that the third harmonic (both the imaginary and the real parts) changes sign in transition from the edge-barrier to bulk-pinning mode.

Performing simple calculations (for a solvable case of narrow films/plates) under condition $j_p \ll j_s$ one obtains the following expressions for $\chi'_3(H_0)$ and $\chi''_3(H_0)$ in the limit $H_0 \gg H_s$

$$\chi'_3(H_0 \gg H_s) = -\frac{2\gamma}{\pi H_0} \left(4H_p\sqrt{\frac{2H_p}{H_0}} - \frac{3\pi H_s^2}{4H_0}\right) , \quad (15a)$$

$$\chi''_3(H_0 \gg H_s) = -\frac{2\gamma}{\pi H_0} \left(\frac{3H_s^2}{2H_0} \ln \frac{H_0}{H_s} - H_p\right) . \quad (15b)$$

As follows from above expressions in the case of a superconductor without bulk pinning $\chi'_3(H_0 \gg H_s) > 0$ and $\chi''_3(H_0 \gg H_s) < 0$. However, presence of bulk pinning leads to a sign reversal of $\chi'_3(H_0)$ for sufficiently large amplitudes $H_0 > H_s(j_s/j_p)^3$ and of $\chi''_3(H_0)$ for amplitudes $H_0 > H_s(j_s/j_p) \ln(j_s/j_p)$. 

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We wish to point out that for large amplitudes of the magnetic field \( H_0 \), the numerical results related to the case of a narrow film/plate coincide with the analytically obtained asymptotics (15 a,b) for \( \chi'_3(H_0) \) and \( \chi''_3(H_0) \). For an additional control of the calculation accuracy we have also employed a twice smaller numerical step; the results practically have not changed (the difference did not exceed 3%). In our opinion, the above arguments fully confirm reliability of our calculations.

The third situation we have examined corresponds to the \( j_p(H) \) by the Kim-Anderson model (ignoring edge barrier). Magnetization curves and ac-susceptibilities are shown in Figs 6c-10c. Here we consider only one case for wide films, specifically, the longitudinal geometry, since there are no analytical expressions to describe the dependence \( M(H) \) for wide films in the transverse geometry. In [23] the results of numerical calculations of the \( M(H) \) dependence are reported for a wide film. A similar dependence is obtained (given the appropriate choice of parameters) also for a wide plate in a parallel field. Curves 1 in Fig. 6c-10c correspond to a narrow plate (film), curves 2 are for a wide plate. The magnetic field is measured in units of \( H^* \) which is equal to \( j_p(0)/C_0 \) for narrow films and to \( 2\pi j_p(0)W/c \) for wide films. In both the cases magnetization is measured in units of \( j_p(0)W/8c \). The dependence \( M(H) \) corresponding to curve 1 (Fig.6c) is obtained by substitution of the expression \( H_s = H_{k2}^2/(H_{k1} + |H|) \) in (13), while \( M(H) \) for curve 2 shown in the same figure is obtained using the critical state model for a wide plate, in the same way as it was done in [11] for an infinite cylinder in a parallel magnetic field. Note a practically absolute similarity of the curves at \( H_0 \gg H^* \); however, at lower \( H_0 \) the differences become noticeable. Therefore, at large amplitudes \( H_0 \) the susceptibility harmonics for these two cases almost coincide, whereas at small amplitudes they differ in quantity.

We should emphasize that the KA model also demonstrates sign reversal in the third harmonic of ac susceptibility (only in the real part, though) in transition from the edge-barrier to bulk-pinning irreversibility mode. As is shown by numerical analysis of narrow samples, in a physically interesting case \( H_0 > H_{k1} \) this transition occurs when \( H_{k2}^2/H_{k1} > 2H_s \). The latter corresponds to the condition that the depinning current density in a zero
magnetic field $j_p(0)$ be twice higher than the density of the edge-barrier suppression current. Note that such a situation is practically impossible in superconductors with an ideal barrier ($j_s = j_{GL} \approx 10^8 A/cm^2$). Therefore, the edge barrier effect has to be taken into account in study of the magnetic characteristics of type II superconductors and of higher harmonics of ac susceptibility.

V. DISCUSSION

Let us discuss the applicability conditions for a generalized model of critical state. A continuum approximation to describe vortex distribution in narrow samples is valid, if $W \gg \xi$ (here $\xi$ is the coherence length). Introduction of the function $n(y)$ implies essentially the averaging of vortex distribution on a scale exceeding the intervortex distance $\sim n^{-1/2}$. It is easily shown that this averaging procedure is correct in the field range $\Phi_0/W^2 < H < \Phi_0/\xi^2$. For example, the applicability condition of this model ($\xi \ll W < \lambda^2/d$) was met near the critical temperature ($T \to T_c$) for a Sn film in the experiment [24]. Similar conditions can be provided also in HTSC films based, for instance, on Y BaCuO with parameters $\lambda \sim 0.2 \mu m$ and $d \sim 0.1 \mu m$ at $T \approx 77 K$.

The results for magnetic susceptibilities, obtained in this work are valid on condition of quasistatical variation of an external field. The slowness of the magnetic field variation implies that there is sufficient time for a quasiequilibrium distribution of current and vortex density to set in a sample, and the viscous losses related to vortex motion can be neglected. As follows from formula (14), in this case $\chi_{ac}^n$ is frequency independent. Absence of frequency effects on magnetic susceptibilities was reported in refs [3][14] in the frequency domain $\omega/2\pi = 30 \div 1000 Hz$ and at amplitudes $H_0 = 0.1 \mu T \div 1 m T$.

Consider now the temperature dependence of $\chi_{ac}^n$. Note that the similarity described in Section 4 is mostly featured by field dependences $\chi_{ac}^n(H_0)$. The temperature dependences of $\chi_n(T)$ will differ appreciably for samples of different geometries and orientation. This is predetermined by a different character of the temperature dependence of fields, $H_s(T)$
and $H_p(T)$ in wide and narrow samples. Nevertheless, it is quite possible to observe a sign reversal on the temperature dependence of $\chi_3'(T)$ due to edge barrier effect. Indeed, let $H_p(T) \sim (1 - T/T_c)^\alpha$, and $H_s \sim (1 - T/T_c)^\beta$ and, besides, $\alpha > \beta$ (see e.g. [3,24]). Then, even if at low temperatures $H_p(T) \gg H_s(T)$ (bulk pinning dominates), at $T \sim T_c$ $H_p(T) \ll H_s(T)$, i.e., the edge barrier mechanism of irreversibility will dominate. Such an interchange of the irreversibility mechanisms results in a sign reversal of $\chi_3'$, which takes place at a certain crossover temperature $T^\ast$. Note that a similar crossover of magnetization was described in [25].

It should be noted that a change of irreversibility mechanism may be followed by a sign reversal not only in $\chi_3'(H_0)$, as is demonstrated by the KA model (see Fig. 9a,9c), but also in $\chi_3''(H_0)$ (within the Bean model; see Fig. 9a, 10a-9b, 10b). This brings up a question about how sensitive the $\chi_3(H_0)$ behavior is to a bulk-pinning mechanism. We have considered the dependence $j_p(H)$ which corresponds to a collective mechanism of vortex pinning [24] in the form

$$j_p(H) = \begin{cases} j_{p0} \sqrt{H_k/H} & |H| > H_k, \\ j_{p0} & |H| < H_k. \end{cases}$$

(16)

The behavior of $M(H)$ and $\chi_{nac}^3(H_0)$, found using Eq.(16) agrees qualitatively with the curves shown in Figs. 6c-10c (the KA model). Apparently, these data should also be valid qualitatively for an arbitrary function $j_p(H)$ monotonically decreasing with $H$. Therefore, we may conclude that the sign reversal effect in the real part of the third susceptibility harmonics, following a change of irreversibility mechanism, is a general property of the mixed state, which is only slightly affected by specific features of edge barrier.

Let us discuss now an unexpected similarity of the magnetization curves and ac susceptibilities. Indeed, in order to calculate the magnetic characteristics of a specific sample one should solve a second-order differential equation (see (A7) in the Appendix) (for a bulk superconductor case), the integral Bio-Savart equation (A5) (for a wide film in a perpendicular geometry) and/or the first-order differential equation (A6) or (A8) (for a narrow
film/plate). Surprisingly, the magnetization curves obtained on the basis of the solutions of these apparently different equations look quite similar (on condition that the same irreversibility mechanism, i.e., an edge barrier or a specific type of the fluxoid pinning by the lattice defects, is employed) [27]. Such geometry-independent similarity obviously follows from the fact that current densities and vortex distributions in a narrow superconducting film/plate (see Fig. 2) and in samples of other geometries are qualitatively similar in corresponding ranges of the magnetic field. The difference in samples geometry and size results in mere quantitative modifications of the characteristic parameters (e.g., $a, b, H_d, H_ex$ etc.; for specific examples see Refs. [15–17]) and in specific dependences $j_x(y)$, $n(y)$ corresponding to a sample of selected geometry. Thus, taking into account the above results, a narrow film/plate may be suggested as a basic model to study qualitative features of the magnetic properties of practically used type-II superconductors.

The geometry-related similarity revealed here seems to be caused by the one-parametric description of each irreversibility mechanism by means of introduction of a phenomenological current density for the barrier suppression $j_s$ or for the depinning $j_p(H)$. The microscopic basis of such universality still awaits a more detailed study.

VI. SUMMARY

In this work the magnetic characteristics of superconducting plates and films of various widths in an external quasistatistical magnetic field $H = H_0 \cos(\omega t)$ have been calculated. The study was carried out within a generalized model of the critical state, which accounts for both edge barrier and bulk pinning. To demonstrate the effect of a surface (edge) barrier, particular models were considered, describing individual influence of either irreversibility mechanism (edge barrier or bulk pinning) on the magnetization and ac-susceptibility of type II superconductors. The obtained results have led us to the following conclusions:

1. The magnetic characteristics of type II superconductors are determined by the type of irreversibility mechanism (bulk pinning or surface/edge barrier) rather than geometrical
parameters of samples (plates, films) and their orientation relative to an external magnetic field. The latter factors bring about only quantitative changes in the dependences $M(H)$, $\chi_n^{ac}(H_0)$, which thus prove to be similar (for a specified mechanism of vortex depinning). 2. A sign change effect in the real part of the third (7-th, 11-th) harmonics of the magnetic susceptibility is predicted to follow a change of irreversibility mechanism. A generalized model of the critical state for narrow films has been used to determine the conditions at which a crossover of the $\chi_3^{ac}(H_0)$ dependence takes place.

VII. ACKNOWLEDGMENTS

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VIII. APPENDIX

Let us consider thin superconducting film in a perpendicular magnetic field (see fig. 1b) in a mixed state. The relation between current density and vector potential in the London model reads

$$j = \frac{c}{4\pi \lambda^2} (A - \frac{\phi_0}{2\pi} \nabla \varphi). \quad (A1)$$

Here $\varphi$ is a phase of the order parameter satisfying the equation

$$\nabla \times \nabla \varphi(r) = 2\pi \delta(r - r'), \quad (A2)$$

where $r' = (x', y')$ is the vortex coordinates. The Ampere law (in a gauge $\nabla \cdot A = 0$) reads
\[ \Delta \mathbf{A} = -\frac{4\pi}{c} \mathbf{j}. \quad (A3) \]

By employing a Green function for the three-dimensional Laplace operator \( \Delta \) we rewrite (A3) in the integral form

\[ \mathbf{A}(\mathbf{r}) = \mathbf{A}_0(\mathbf{r}) + \frac{1}{c} \int \int \int \frac{\mathbf{j}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} dx'dy'dz', \quad (A4) \]

where \( \mathbf{A}_0(\mathbf{r}) \) is the vector potential of the applied magnetic field \( \mathbf{H} = \nabla \times \mathbf{A}_0 \); the integration in (A4) is performed over entire sample. With the help of Eqs. (A1)-(A4) we derive the Maxwell-London equation for the average current density \( j(y) = d^{-1} \int_{-d/2}^{d/2} j_x(y, z) dz \) in a thin-film limit \( d \ll \lambda \)

\[ \frac{8\pi \lambda_{eff}}{cW} \frac{dj(y)}{dy} + \frac{2}{c} \int_{-1}^{1} \frac{j(y')}{y' - y} dy' = d^{-1} [H - n(y)\phi_0]. \quad (A5) \]

Here \( n(y) \) is the vortex density and distance is scaled in units of \( W/2 \). Equation (A5) was derived for the first time in Ref. [28]. For the case of a narrow film \( \lambda_{eff}/W \gg 1 \) when the self-field of currents is neglected Eq. (A5) reduces to

\[ \frac{4\pi \lambda^2}{c} \frac{dj_x}{dy} = H - n(y)\phi_0. \quad (A6) \]

The numerical solution of the equation (A5) shows that the integral term becomes insignificant for sufficiently narrow films \( W \leq \lambda_{eff} \). In the opposite case of wide films, the differential term in (A5) can be neglected (everywhere inside the film except for areas near edges). In the latter limit Eq. (A5) reduces to the canonical version of the Bio-Savart equation that is conventionally studied in a quasi-two-dimensional situation [29,30].

Consider now the mixed state of a superconducting plate in a parallel magnetic field (see fig. 1). By taking curl from both parts of the London relation (A1) with account for the relation (A2) and Maxwell equations \( \nabla \times \mathbf{H} = 4\pi \mathbf{j}/c \) and \( \nabla \cdot \mathbf{B} = 0 \) we obtain the nonlocal equation for the distribution of a magnetic field (see [16,17])
\[ H_l - \lambda^2 \frac{d^2 H_l}{dy^2} = n(y)\phi_0, \]  

(A7)

where \( H_l \) is a \( z \)-component of a local magnetic field \( \mathbf{H}_l = (0, 0, H_l) \) (all other components vanish due to the symmetry of the problem). In the limiting case \( \lambda \to 0 \) (A7) reduces to the conventional relation \( H_l = n\phi_0 \).

In case of narrow plates \( \lambda/W \gg 1 \) the differential term in (A7) is dominant and, moreover, \( H_l \sim H \) since the self-field of currents can be neglected. Employing relation \( dH_l/dy = 4\pi j_x/c \), valid for longitudinal geometry, we obtain the equation for the current density \( j_x \)

\[ \frac{4\pi \lambda^2}{c} \frac{dj_x}{dy} = H - n(y)\phi_0. \]  

(A8)

Thus, the equations describing the current density distribution for narrow films (see (A6)) and for narrow plates (see (A8)) are equivalent. The mathematical equivalence of these equations follows from the fact that the current-induced self field may be neglected in these two extreme cases.

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**Figure captions**

**Fig.1**

Problem geometry: a) plate in a parallel magnetic field; b) film in a perpendicular magnetic field.

**Fig.2**

Current densities and vortex distributions in a narrow superconducting film/plate at different magnetic fields $H$. a) $0 < H \uparrow < H_s$, b) $H_s < H \uparrow < H_0$, c) $H_{df} < H \downarrow < H_0$, d) $H_{ex} < H \downarrow < H_{df}$, e) $0 < H \downarrow < H_{ex}$, f) $H_s^{(-)} < H \downarrow < 0$, g) $-H_0 < H \downarrow < H_s^{(-)}$

↑ indicates increasing field and ↓ indicates decreasing field.

**Fig.3**

Magnetization curves for a narrow thin film with an edge barrier and bulk pinning (Bean model): 1) $j_p = j_s$, 2) $j_p = 0.6j_s$, 3) $j_p = 0.3j_s$, 4) $j_p = 0$.

**Fig.4**

Magnetization curves for a narrow thin film with an edge barrier and bulk pinning (Kim-Anderson model): 1) $j_p(0) = j_s$, 2) $j_p(0) = 0.6j_s$, 3) $j_p(0) = 0.3j_s$.

**Fig.5**

Dependence of magnetic susceptibilities on the amplitude of a quasistatical applied magnetic field ($H = H_0 \cos(\omega t)$) of a narrow-film superconductor (Kim-Anderson model of bulk pinning: $j_p(0)/j_s = 1/5$, $H_{k1} = H_s$)).

Curve 1 - a sample with only bulk pinning.

Curve 2 - a sample with edge barrier and bulk pinning.

a) $\chi_1(H_0)$, b) $\chi_3''(H_0)$, c) $\chi_3'(H_0)$, d) $\chi_3''(H_0)$.

**Fig.6**

Magnetization curves for superconductors of various widths and geometries with single mechanism of irreversibility.

a) - superconductor with edge barrier, b) superconductor with bulk pinning (Bean model),

c) - superconductor with bulk pinning (Kim-Anderson model).

Curve 1 - narrow film (plate), curve 2 - wide plate, curve 3 - wide film.
Fig. 7
Dependences of the real part of the first harmonic of magnetic susceptibility on the amplitude of an applied magnetic field $H_0$.

a) - superconductor with edge barrier, b) superconductor with bulk pinning (Bean model),
c) - superconductor with bulk pinning (Kim-Anderson model).
Curve 1 - narrow film (plate), curve 2 - wide plate, curve 3 - wide film.

Fig. 8
Dependences of the imaginary part of the first harmonic of magnetic susceptibility on the amplitude of an applied magnetic field $H_0$.

a) - superconductor with edge barrier, b) superconductor with bulk pinning (Bean model),
c) - superconductor with bulk pinning (Kim-Anderson model).
Curve 1 - narrow film (plate), curve 2 - wide plate, curve 3 - wide film.

Fig. 9
Dependences of the real part of the third harmonic of magnetic susceptibility on the amplitude of an applied magnetic field $H_0$.

a) - superconductor with edge barrier, b) superconductor with bulk pinning (Bean model),
c) - superconductor with bulk pinning (Kim-Anderson model).
Curve 1 - narrow film (plate), curve 2 - wide plate, curve 3 - wide film.

Fig. 10
Dependences of the imaginary part of the third harmonic of magnetic susceptibility on the amplitude of an applied magnetic field $H_0$.

a) - superconductor with edge barrier, b) superconductor with bulk pinning (Bean model),
c) - superconductor with bulk pinning (Kim-Anderson model).
Curve 1 - narrow film (plate), curve 2 - wide plate, curve 3 - wide film.
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fig.5cd G.M.Maksimova et. al.
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fig. 10 G.M. Maksimova et. al.