THE IDEAL BOUNDARY AND THE ACCUMULATION LEMMA

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Abstract. Let $S$ be a connected surface possibly with boundary, $\mu$ a finite Borel measure which is positive on open sets and $f : S \to S$ a homeomorphism preserving $\mu$. We prove that if $K$ is an $f$-invariant compact connected subset of $S$ and $L$ is a branch of a hyperbolic periodic point of $f$ then $L \cap K \neq \emptyset$ implies $L \subset K$. This is called the accumulation lemma. For this we develop a classification of connected surfaces with boundary and a characterization of residual domains of compact subsets with finitely many connected components in a connected surface with boundary.

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1. Introduction.

A fundamental result in the modern theory of the dynamics of area preserving maps is the Accumulation Lemma which asserts that if $K$ is a compact connected invariant set and $L$ is a branch of a hyperbolic periodic point $p$ (i.e. a connected component of $W^s(p) - \{p\}$ or $W^u(p) - \{p\}$) then $L \cap K \neq \emptyset$ implies that $L \subset K$.

The accumulation lemma is used to prove that for generic area preserving maps all the branches of a hyperbolic periodic points have the same closure. This fact in turn allows to obtain certain homoclinic orbits. For example a version of the accumulation lemma in [2]

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provided homoclinic orbits which permitted to prove the existence of Birkhoff sections for all Kupka-Smale Reeb flows.

The accumulation lemma was proved by John Mather in Corollary 8.3 of [6] for connected surfaces without boundary. A proof for the 2-sphere appears in Franks, Le Calvez [4] Lemma 6.1. Here we present a proof of the accumulation lemma for connected surfaces with boundary and partially defined maps in Theorem 4.3.

We need the version for surfaces with boundary and partially defined area preserving maps in order to be able to apply the lemma to return maps of Birkhoff sections of Reeb flows and holonomy maps on broken book decompositions of contact 3-manifolds. The lemma allows in [8] to obtain homoclinics for Kupka-Smale area preserving maps of surfaces with boundary and in [3] to prove that Kupka-Smale geodesic flows of surfaces have homoclinic in every hyperbolic closed geodesic.

If $S$ is a surface with boundary and $K \subset S$ is a compact connected set, the connected components of $S - K$ are called residual domains of $K$. The proof of the accumulation lemma uses the topology of residual domains, so we need to extend the representation of connected surfaces by Kerékjártó [5] and Richards [9] to connected surfaces with boundary which we do in Proposition 3.11.

The ideal boundary points of a surface are also known as topological ends. We use the former terminology to distinguish them from the prime end compactification of a surface which is also needed in the study of dynamics of area preserving maps. The ideal boundary gives a description of the way in which compact subsets divide a complete surface into unbounded components.

The proof of the accumulation lemma for subsets of the sphere $S^2$ in [4, 6.1] is quick because in $S^2$ a residual domain of a compact connected set is simply connected. For general surfaces the proof can be localized but requires the study of the ideal boundary compactification of residual domains.

We see in Proposition 3.12 that a relatively compact ideal boundary point of a residual domain has a neighborhood in the ideal boundary compactification which is homeomorphic to a disk in $\mathbb{R}^2$. This provides a proof of Proposition 1.1 in Mather [6] where it is attributed as a consequence of the main result in Richards [9]. This is a key element to prove the accumulation lemma.

We use exhaustions from Ahlfors and Sario [1] to obtain a method for computing ideal boundary points in Proposition 3.9. We also construct a canonical exhaustion for residual domains on manifolds with boundary in subsection § 3.4 that we use to prove the accumulation lemma 4.3. This technique allows a more elementary exposition.

As a consequence of the methods developed to deal with the ideal boundary of residual domains we obtain a characterization of residual domains on surfaces. In Corollary 3.28 we prove that for an open subset $U$ of a connected surface $S$: its frontier in $S$, $fr_S U$, is a compact set with finitely many components if and only if $U$ is a residual domain of a compact set $K$ which has finitely many connected components. We also see that this result is not valid in other topological spaces $S$ which are not surfaces.
2. Notation and definitions

If $B$ is a topological space and $A \subset B$, we use the notation $\text{int}_B A$, $\text{cl}_B A$ and $\text{fr}_B A$ for the interior, closure and the frontier of $A$ in $B$, respectively. The boundary of a surface will be denoted by $\partial S$ and we will use $S^o = S - \partial S$ for its interior as a manifold. By domain we mean a connected open subset of $S$. If $A$ is a closed subset of $S$, a connected component of $S - A$ is called a residual domain of $A$.

A set $K \subset S$ is relatively compact if its closure $\text{cl}_S K$ is compact. Observe that a residual domain of a compact subset $K \subset S$ is relatively compact if and only if it is bounded for some complete metric on $S$.

By a closed (open) disk we mean a set homeomorphic to a closed (open) ball in the plane.

3. The ideal boundary.

Let $S$ be a non-compact connected surface with a complete metric. We are going to describe a compactification of $S$ by the addition of its (topological) ends or ideal boundary points. See [1], [6], [9].

An ideal boundary component of $S$ is a decreasing sequence $V_1 \supset V_2 \supset \cdots$ of non-empty subsets of $S$ such that:

1. $V_n$ is open in $S$.
2. $V_n$ is connected.
3. $\text{cl}_S V_n$ is not compact.
4. $\text{fr}_S V_n$ is compact.
5. If $K$ is a compact subset of $S$, then there is $n_0$ such that $K \cap V_n = \emptyset$ for $n \geq n_0$.

We will refer to this property briefly by saying that the sequence $(V_n)$ leaves compact subsets of $S$.

Two ideal boundary components $V_1 \supset V_2 \supset \cdots$ and $V'_1 \supset V'_2 \supset \cdots$ are equivalent if for every $n$ there is $m$ such that $V_m \subset V'_n$ and vice versa. An ideal boundary point is an equivalence class of ideal boundary components. The set of ideal boundary points $b(S)$ is called the ideal boundary of $S$ and the disjoint union $B(S) = S \sqcup b(S)$ is called the ideal completion of $S$. The ideal boundary gives a description of the way in which compact subsets of $S$ divide $S$ into unbounded (i.e non relatively compact) residual domains.

3.1. Remark. It follows from conditions (1) through (5) of the definition of an ideal boundary component $(V_n)$ that the condition

\begin{equation}
(5') \quad \cap_n \text{cl}_S V_n = \emptyset.
\end{equation}

holds true. In fact, we could have replaced condition (5) by $(5')$ and have an equivalent definition of ideal boundary components. It is not true that condition (5) is equivalent to $\cap_n V_n = \emptyset$, as we have seen stated elsewhere.

Consider polar coordinates $(\theta, r)$ on $S = \mathbb{R}^2 - \{(0,0)\}$ and define

$$V_n = \{(\theta, r) \in S : r < \frac{1}{n}\} \cup \{(\theta, r) \in S : r < 2, \ 0 < \theta < \frac{1}{n}\}.$$
Clearly that

Then \((V_n)\) is a decreasing sequence of sets that satisfy \(\cap_n V_n = \emptyset\), (1), (2), (3) and (4) but not (5), because

\[ \cap_n \text{cl}_S V_n = \{ (\theta, r) \in S : 0 < r \leq 2, \theta = 0 \} \]

and \(V_n\) does not leave circles centered at \((0,0)\) and radius \(r\) with \(0 < r < 2\).

3.2. Remark. For an ideal boundary component \((V_n)\) it is easy to see that for every \(n\) there exists \(m > n\) such that \(fr_S V_m \cap fr_S V_n = \emptyset\). From this and \(V_m \subset V_n\) it follows that \(cl_S V_m \subset V_n\). Therefore, by taking a subsequence of \((V_n)\), we have that \((V_n)\) is equivalent to an ideal boundary component \((W_n)\) for which \(cl_S W_{n+1} \subset W_n\) for every \(n\).

Let \(A\) be an open connected subset of \(S\) with \(fr_S A\) compact and let \(A'\) be the set of ideal boundary points of \(S\) whose representing ideal boundary components \((V_n)\) satisfy \(V_n \subset A\) for sufficiently large \(n\). The collection of all sets \(A'\) where \(A\) is an open connected subset of \(S\) with \(fr_S A\) compact, forms a basis for a topology on \(b(S)\). With this topology \(b(S)\) is a compact totally disconnected space (cf. [1, ch. 1, §36–37]). On \(B(S)\) we use the topology with basis the sets \(A^* = A \cup A'\), where \(A \subset S\) is open connected with \(fr_S A\) compact. By a compactification of \(S\) we mean a compact topological space \(M\) that contains \(S\) as an open and dense subset.

A characterization of \(B(S)\) as a compactification of \(S\) is given by the following result:

3.3. Proposition.

The set \(B(S)\) is a compactification of \(S\) that satisfies the following properties:

1. \(B(S)\) is a locally connected Hausdorff space.
2. \(b(S)\) is totally disconnected.
3. \(b(S)\) is non separating on \(B(S)\),
   (i.e. if \(V \subset B(S)\) is open and connected then \(V - b(S)\) is connected).

If \(M\) is another compactification of \(S\) satisfying these properties, then there exists a homeomorphism from \(M\) onto \(B(S)\) which is the identity on \(S\).

For a proof see sections 36 and 37 of chapter 1 of [1]. Later we will provide a good description of \(B(S)\). Sometimes we will just write \(b = (V_n)\) to say that \(b\) is an ideal boundary point of \(S\) represented by the ideal boundary component \((V_n)\). Let \(V^*_n = V_n \cup V_n'\). Then \((V^*_n)\) and \((V^*_n)\) are fundamental systems of neighborhoods for \(b\) in \(b(S)\) and in \(B(S)\) respectively.

Now we describe a result of KerékJártó that gives necessary and sufficient conditions for non-compact surfaces to be homeomorphic.

Let \(b \in b(S)\) be represented by an ideal boundary component \((V_n)\). We say that \(b\) is planar if \(V_n\) is homeomorphic to a subset of the plane for all sufficiently large \(n\). We say that \(b\) is orientable if \(V_n\) is orientable for all sufficiently large \(n\). Let

\[ b'(S) = \{ \text{non planar ideal boundary points of } S \}, \]
\[ b''(S) = \{ \text{non orientable ideal boundary points of } S \}. \]

Clearly \(b''(S) \subset b'(S)\).
We say that the surface $S$ has finite genus if there exists a compact bordered surface $K$ contained in $S$ such that $S - K$ is homeomorphic to a subset of the plane. In this case the genus of $S$ is defined to be the genus of $K$. Otherwise we say that $S$ has infinite genus. The genus of a connected surface with boundary can also be defined as the maximum number of disjoint simple closed curves that can be embedded in the interior of the surface without disconnecting it.

3.4. Remark.

The surface with boundary $S$ has finite genus if and only if every ideal boundary point of $S$ is planar.

Now we are going to define four types of orientability for a non compact surface.

Assume that the surface $S$ is not orientable. We say that $S$ is finitely non orientable if there exists a compact bordered surface $K$ contained in $S$ such that $S - K$ is orientable. Otherwise we say that $S$ is infinitely non orientable. Every compact non orientable surface is the connected sum of a compact orientable surface and one or two projective planes. If $S$ is finitely non orientable and $S - K$ is orientable, we say that $S$ is of odd or even non orientability type according to $K$ being the connected sum of a compact orientable surface and one or two projective planes, respectively.

3.5. KerékJártó Theorem.

Let $S_1$ and $S_2$ be two non compact connected surfaces which have the same genus and the same orientability type. Then $S_1$ and $S_2$ are homeomorphic if and only if there exists a homeomorphism of $b(S_1)$ onto $b(S_2)$, such that $b'(S_1)$ and $b''(S_1)$ are mapped onto $b'(S_2)$ and $b''(S_2)$ respectively.

This is an old result of KerékJártó and a complete proof appears in Theorem 1 of Richards [9].

If $K$ is a compact totally disconnected subset of a compact surface $M$, it is not difficult to show that $b(M - K)$ is homeomorphic to $K$ (for instance, the final part of the proof of Theorem 2 of [9] contains a detailed proof of this fact). From this, it follows that $B(M - K)$ is homeomorphic to $M$. Clearly $b'(M - K) = b''(M - K) = \emptyset$.

Any compact, separable, totally disconnected space $X$ is homeomorphic to a subset of the Cantor ternary set. Therefore the same happens with the ideal boundary of any non compact surface.

3.6. Proposition.

Let $S$ be a non compact connected surface of finite genus. Then there exists a compact surface $M$ which contains a totally disconnected compact subset $K$ such that $S$ is homeomorphic to $M - K$ and $b(S)$ is homeomorphic to $K$.

Proof: We have that $b'(S) = b''(S) = \emptyset$. Let $M$ be a compact surface of the same genus and orientability type as $S$. The Cantor ternary set can be embedded in $M$ and therefore there exists a subset $K$ of $M$ homeomorphic to $b(S)$. It follows from KerékJártó Theorem 3.5 that $S$ is homeomorphic to $M - K$. \qed
From Proposition 3.6 we immediately have:

3.7. **Proposition.**

The ideal completion \( B(S) \) of a non compact connected surface is a surface if and only if \( S \) has finite genus.

Now we would like to describe a method of computing ideal boundary points.

An *exhaustion* of \( S \) is an increasing sequence \( F_1 \subset F_2 \subset \cdots \) of compact connected bordered surfaces contained in \( S \) such that:

(a) \( F_n \subset int_S F_{n+1} \).
(b) \( S = \cup_n F_n \).
(c) If \( W \) is a connected component of \( S - F_n \), then \( cl_S W \) is a non compact bordered surface whose boundary consists of exactly one connected component of \( \partial F_n \).

We have the following existence result:

3.8. **Proposition.**

Every non compact connected surface without boundary admits an exhaustion.

For a proof see the Theorem in section 29A of chapter 1 in Ahlfors and Sario [1].

Let \( (F_n) \) be an exhaustion of \( S \). Since \( F_n \) is a compact bordered surface, the number of connected components of \( \partial F_n \) is finite. This implies that the number of connected components of \( S - F_n \) is finite.

Denote the connected components of \( S - F_n \) by \( W_n^i \), where \( i \) runs over some finite set. If \( m > n \) then \( S - F_m \subset S - F_n \) and every \( W_m^i \) is contained in some \( W_n^j \). Besides that, every \( W_n^i \) contains at least one \( W_m^j \), otherwise \( W_n^i \subset F_m \) and \( cl_S W_n^i \) would be compact, contradicting the definition of an exhaustion.

Let \( W_1^1 \supset W_2^1 \supset \cdots \supset W_n^1 \supset \cdots \) be a decreasing sequence of connected components of the sets \( S - F_n \). It is easy to check that \( (W_n^1) \) satisfies the conditions (1)–(5) for being an ideal boundary component of \( S \).

3.9. **Proposition.**

Let \( (F_n) \) be an exhaustion of a non compact connected surface \( S \). Then every ideal boundary component of \( S \) is equivalent to another of the form \( (W_n) \), where \( W_n \) is a connected component of \( S - F_n \).

**Proof:**

Let \( (V_k) \) be an ideal boundary component of \( S \). Since \( fr_S V_n \) is compact and \( F_n \) is an exhaustion of \( S \) we have that

\[
\forall k \exists m_0 > k \forall m > m_0 \quad fr_S V_k \subset int_S F_m.
\]

Let \( W_m^* \) be the connected component of \( B(S) - F_m \) which contains the ideal boundary point \( b = (V_k) \in b(S) \). Let \( W_m := W_m^* - b(S) \). We claim that \( W_m \) is connected\(^1\).

\(^1\)This follows from proposition 3.3.(3). We include a proof for completeness.
Indeed, suppose that \( W_m \) is not connected. Let \( A_1, A_2 \) be disjoint nonempty open subsets of \( S \) such that \( W_m = A_1 \cup A_2 \). The sets \( A_i^* := A_i \cup A_i' \) are nonempty, open and disjoint in \( B(S) \). Let \( c = (U_n) \in W_m^* - W_m \subset b(S) \). Since \( W_m^* \) is open in \( B(S) \) and \( U_n^* := U_n \cup \{ c \} \) is a fundamental system of neighborhoods of \( c \) in \( B(S) \), we can choose the sets \( U_n \) satisfying \( U_n \subset W_n \). Since \( U_n \) is connected, there is one \( A_i \) such that \( U_n \subset A_i \) for infinitely many \( n \). Then \( c \in A_i^* \subset A_i^* \). This shows that \( W_m^* = A_i^* \cup A_2^* \) and also that \( W_m^* \) is disconnected, which contradicts its choice.

Observe that \( W_m^* \) contains a connected component of \( S - F_m \). Since \( W_m \) is connected, we have that
\[
W_m \text{ is a connected component of } S - F_m.
\]

Since \( W_m^* \) contains the ideal boundary point \( b = (V_n) \), we have that \( W_m \cap V_k \neq \emptyset \). Observe that \( V_k \) is a connected component of \( S - fr_S V_k \). By \( (1) \), \( S - F_m \subset S - fr_S V_k \) for all \( m > m_0(k) \). Therefore
\[
\forall m > m_0(k) \quad W_m \subset V_k.
\]

Since \( fr_S W_m \subset F_m \) is compact, the set \( W_m^* \) is open in \( B(S) \). The condition \( b \in W_m^* \) means that there exists \( \ell > m \) such that \( V_\ell \subset W_m \).

For every \( k \) we have obtained \( \ell > m > k \) and \( V_\ell \subset W_m \subset V_k \), where \( W_m \) is a connected component of \( S - F_m \). Then there is a sequence
\[
V_1 \supset W_{m_1} \supset V_{k_2} \supset W_{m_2} \supset V_{k_3} \supset \cdots
\]
where \( W_{m_i} \) is a connected component of \( S - F_m \) and in particular \( fr_S W_{m_i} \) is compact. Then \( (W_{m_i}) \) is an ideal boundary component equivalent to \( b \).

\[\Box\]

3.10. Remark.

If for every \( n \) the number of connected components of \( S - F_n \) is less than or equal to \( k \), then \( S \) has at most \( k \) ideal boundary points.

Now we would like to describe a result that shows that every non compact connected surface can be represented as a sphere with the deletion of a finite or infinite closed totally disconnected subset and the addition of a finite or infinite number of handles and crosscaps. This can be seen as a generalization of the classical representation theorem for compact surfaces.

So let \( S \) be a compact connected surface.

If \( S \) has finite genus then the above description is a consequence of Proposition 3.6. In this case \( S \) is homeomorphic to the sphere with a closed totally disconnected subset removed and the addition of finitely many handles and crosscaps.

Suppose now that \( S \) has infinite genus. This is equivalent to \( b'(S) \neq \emptyset \).

Let \( S^2 = \mathbb{R}^2 \cup \{ \infty \} \) be the one point compactification of the plane. We consider the Cantor ternary set \( C \) as the subset of \( S^2 \) consisting of all points \( (x, 0) \) such that \( x \) has a ternary expansion which contains no 1’s.

Let \( X \supset Y \supset Z \) be subsets of \( C \) homeomorphic to \( b(S) \), \( b'(S) \) and \( b''(S) \), respectively. If we look at \( C \) as obtained by the process of removing middle thirds, then \( C = \cap_{n \geq 1} J_n \)
where \((J_n)\) is a nested sequence of sets consisting of the union of pairwise disjoint closed intervals \(I_{n,k}^n\) of length \(\frac{1}{3^n}\), \(1 \leq k \leq 2^n\). For each \(n\) choose a collection of \(2^n\) pairwise disjoint open balls \(B_{n,k}^k\) such that \(I_{n,k}^n \subset B_{n,k}^k\), the centers of \(B_{n,k}^k\) and \(I_{n,k}^n\) coincide, and the balls have the same radius. We also want that either \(B_{n+1}^k \cap B_n^k = \emptyset\) or \(B_{n+1}^k \subset B_n^k\). We still have that \(C = \bigcap_{n \geq 1} \bigcup_{1 \leq k \leq 2^n} B_{n,k}^k\) and for each \(x \in C\) there exists a unique sequence \(k_n\) such that \(\cap_{n \geq 1} B_{n,k_n}^k = \{x\}\).

Every ball \(B_{n,k}^k\) contains exactly two balls \(B_{n+1,k}^{k+1}\) and \(B_{n+1,k}^{k+1}\). If \(B_{n,k}^k\) contains a point of \(Z\) then we choose a closed disk \(D\) contained in \(B_{n,k}^k\) and disjoint from the closures of \(B_{n+1,k}^k\) and \(B_{n+1,k}^{k+1}\), and make the connected sum of \(S^2\) and a projective plane along the boundary of \(D\). If \(B_{n,k}^k\) contains a point of \(Y - Z\), then we choose a closed disk \(D\) contained in \(B_{n,k}^k\) and disjoint from the closures of \(B_{n+1,k}^k\) and \(B_{n+1,k}^{k+1}\), and make the connected sum of \(S^2\) and a torus along the boundary of \(D\).

Let \(M\) be the surface obtained after these connected sums and the deletion of points of \(X\). In \(S^2\) points of \(Z\) are accumulated by crosscaps and points of \(Y - Z\) are accumulated by handles.

For each \(x \in X\) there exists a unique sequence \(k_n\) such that \(\cap_{n \geq 1} B_{n,k_n}^k = \{x\}\). This gives a one to one correspondence between points \(x \in X\) and ideal boundary components \((B_{n,k_n}^k)\) of \(b(M)\). It is not difficult to show that this correspondence gives a homeomorphism from \(b(M)\) onto \(X\) that takes \(b'(M)\) to \(Y\) and \(b''(M)\) to \(Z\). (The final part of the proof of Theorem 2 of [9] contains a detailed proof of this fact).

The surfaces \(S\) and \(M\) have the same genus and orientability type. It follows from KerékJártó Theorem 3.5 that \(S\) and \(M\) are homeomorphic. To summarize, we have the following:

3.11. Proposition.

Let \(S\) be a non compact connected surface of infinite genus. Then \(S\) is homeomorphic to a surface obtained from the sphere \(S^2\) by removing a totally disconnected closed subset \(X\), taking an infinite collection of pairwise disjoint closed balls \((B_n)\) and making the connected sum of \(S^2\) and a projective plane along the boundaries of \(B_n\). Given a neighborhood \(W\) of \(X\) in \(S^2\) all but finitely many balls \(B_n\) are contained in \(W\).

See Theorem 3 of [9] and the discussion preceding it. Although \(B(S)\) is not always a surface, the last proposition gives a very good description of what it looks like. The completion \(B(S)\) is locally homeomorphic to \(\mathbb{R}^2\) at every point not in \(b'(S)\) or \(b''(S)\), whose points are accumulated by handles or crosscaps, respectively.

3.1. More general surfaces.

The concept of ideal boundary can be generalized to non connected surfaces possibly with boundary.

Let \(S\) be a surface. The definitions of ideal boundary components, ideal boundary points, ideal boundary \(b(S)\), ideal completion \(B(S)\) and their topologies are the same. If \(S_1, S_2, \ldots\) are the connected components of \(S\), then \(b(S) = \bigcup_i b(S_i)\) and \(B(S) = \bigcup_i B(S_i)\). The completion \(B(S)\) is compact if and only if \(S\) has finitely many connected components. Obviously if \(S\) is compact then \(b(S) = \emptyset\).
An important case is that of a connected surface \( S \) with \( \partial S \) compact. In this case \( \partial S \) is the union of a finite number of curves homeomorphic to circles. All results previously developed for non compact connected surfaces without boundary, with the obvious adaptations, apply to these surfaces. The ideal completion of \( S \) is a surface if and only if \( S \) has finite genus.

Let \( S_1 \) and \( S_2 \) be two non compact connected surfaces with compact boundary which have the same genus and the same orientability type. Then \( S_1 \) and \( S_2 \) are homeomorphic if and only if they have the same number of boundary components and there exists a homeomorphism of \( b(S_1) \) onto \( b(S_2) \), such that \( b'(S_1) \) and \( b''(S_2) \) are mapped onto \( b'(S_2) \) and \( b''(S_2) \), respectively.

The definition of exhaustions \((F_n)\) for non compact connected surfaces \( S \) with compact boundary could be the same. Since \((\text{int}_SF_n)\) is an open cover of \( S \), any compact subset of \( S \) would be contained in some \( F_n \). So for surfaces with compact boundary we will further require that all the sets \( F_n \) in an exhaustion of \( S \) contain \( \partial S \) in their interiors. Let \( S^* \) be the surface obtained by gluing a closed disk \( D_i \) to each boundary component \( \xi_i \) of \( S \). We are going to think of \( S \) as a subset of \( S^* \) and write

\[
S^* = S \cup D, \quad \text{where } D = D_1 \cup \cdots \cup D_k.
\]

Obviously \( b(S) = b(S^*) \). Exhaustions of \( S \) would correspond to exhaustions of \( S^* \) whose compact sets contain \( D \) in their interiors. From this, it is clear that there exist exhaustions \((F_n)\) for \( S \) and that ideal boundary points of \( S \) can be computed by using ideal boundary components made of connected components of the sets \( S - F_n \).

3.2. The impression of an ideal boundary point of a surface contained in another surface.

Now let \( S \) be a connected boundaryless surface, let \( U \subset S \) be a surface possibly with compact boundary and \( b \) and ideal boundary point of \( U \). Let \((V_n)\) be an ideal boundary component of \( U \) representing \( b \). We define the \textit{impression of } \( b \) (relative to \( S \)) as

\[
Z(b) := \bigcap_n \text{cl}_{B(S)}V_n.
\]

The impression \( Z(b) \) is just the set of limit points in \( B(S) \) of sequences in \( U \) that converge to \( b \) in \( B(U) \). The definition does not depend on the choice of \((V_n)\) and \( Z(b) \) is a nonempty, connected, compact subset of \( \text{fr}_{B(S)}U \).

We say that \( b \) is relatively compact in \( S \) if some \( V_n \) is relatively compact in \( S \), i.e. if some \( V_n \) has compact closure in \( S \). It is easy to check that \( b \) is relatively compact in \( S \) if and only if \( Z(b) \subset S \). In this case \( Z(b) = \bigcap_n \text{cl}_SV_n \). We call \( b \) a regular ideal boundary point of \( U \) if \( b \) is relatively compact in \( S \) and \( Z(b) \) contains more than one point.

Next we show that if \( b \) is relatively compact in \( S \) then it has a neighborhood in \( B(U) \) homeomorphic to \( \mathbb{R}^2 \).

3.12. Proposition.

Let \( S \) be a non compact connected surface, let \( U \subset S \) be a connected surface possibly with compact boundary. If \( b \) is an ideal boundary point of \( U \) which is relatively compact in \( S \) then \( b \) has a neighborhood in \( B(U) \) homeomorphic to \( \mathbb{R}^2 \).
Proof:

There exists an ideal boundary component \((V_n)\) of \(U\) that represents \(b\) such that \(cl_S V_n\) is compact for some value of \(n\). Consider an exhaustion \((F_n)\) of \(S\). Then \(cl_S V_n \subset int_S F_k\) for some \(k\). Since \(F_k\) is a compact surface we have that \(V_n\) is a surface of finite genus.

We claim that \((V_i)_{i\geq n}\) is an ideal boundary component of \(V_n\). It follows from the claim and Remark 3.4 that \(V_i\) is homeomorphic to an open subset of \(\mathbb{R}^2\) for every \(i\) large enough.

By Remark 3.2 we may assume that \(cl_U V_{i+1} \subset V_i\) for every \(i\), implying that \(cl_{V_n} V_i = cl_U V_i\) and \(fr_{V_n} V_i = fr_U V_i\) for \(i > n\). The proof of the claim is a simple verification that \((V_i)_{i\geq n}\) satisfies conditions (1) through (4) and \((5')\) of the definition of ideal boundary components of \(V_n\).

Then Proposition 3.7 finishes the proof.

\[\square\]

3.3. Residual domains of compact sets whose frontier contains finitely many components.

Now we are going to present some properties of residual domains of compact subsets of \(S\) that have finitely many components. Endow \(S\) with a complete riemannian metric, so that the non relatively compact ideal boundary points of a residual domain are at infinity and its non relatively compact boundary components are unbounded.

3.13. Proposition.

Let \(K\) be a compact subset of a complete connected surface \(S\). Then the union of \(K\) and its bounded residual domains is compact.

Proof:

Let \(F\) be a compact bordered surface which is a neighborhood of \(K\) in \(S\). Let \(V\) be the union of the residual domains of \(K\) contained in \(F\). Then \(K \cup V\) is closed in \(S\), and since \(K \cup V \subset F\), we have that \(K \cup V\) is compact.

Every residual domain of \(K\) not contained in \(F\) contains at least one connected component of \(\partial F\). Therefore at most finitely many residual domains of \(K\) are not contained in \(F\). Since \(S\) is complete, any bounded set in \(S\) is relatively compact. Let \(W\) be the union of the bounded residual domains of \(K\) not contained in \(F\). Then \(K \cup V \cup W\) is closed in \(S\) and since \(W\) is a finite union of relatively compact sets, we have that \(K \cup V \cup W\) is compact.

\[\square\]

3.14. Proposition.

Let \(K\) be a compact subset of a connected surface \(S\). Then \(b(S)\) is homeomorphic to a compact subset of \(b(S-K)\).

Proof:

It follows from Remark 3.2 that any ideal boundary point \(b \in b(S)\) can be represented by an ideal boundary component \((V_n)\) such that \(cl_S V_n \subset S-K\). We have that \(cl_{S-K} V_n = (S-K) \cap cl_S V_n\) and since \(S-K\) is open in \(S\) we have that \(fr_{S-K} V_n = (S-K) \cap fr_S V_n\).
From this it follows that $cl_{S-K}V_n = cl_SV_n$, $fr_{S-K}V_n = fr_SV_n$ and the sequence of sets $(V_n)$ also defines an ideal boundary point $b' \in b(S-K)$. This correspondence provides an one-to-one mapping $\phi : b(S) \to b(S-K)$, $\phi(b) = b'$.

Basic sets for the topology of $b(S-K)$ are defined by means of connected subsets $A$ of $S-K$, such that $A$ is open in $S-K$ and $fr_{S-K}A$ is compact. Since $fr_{S}A = fr_{S-K}A \cup (fr_{S}A \cap K)$, we also have that $A$ is open in $S$, connected and $fr_{S}A$ is compact. Therefore the sets $A$ define basic open sets for the topologies of $b(S)$ and $b(S-K)$, which we denote by $A'_1$ and $A'_2$, respectively. From the definition of ideal boundary component it follows that $\phi^{-1}(A'_2) = A'_1$, hence $\phi$ is continuous. Since $b(S)$ is compact, it follows that $\phi$ is a homeomorphism from $b(S)$ onto a compact subset of $b(S-K)$.

\[ \square \]

We will just think of $b(S)$ as a subset of $b(S-K)$. Later in Corollary 3.26 we will see that if $K$ has finitely many connected components then $b(S-K) - b(S)$ is a discrete topological space and that $b(U) - b(S)$ is a finite set for every residual domain $U$ of $K$.

First we consider the case when $S$ is a compact surface possibly with boundary.

3.15. Proposition.

Let $S$ be a compact connected surface possibly with boundary and $K$ a compact subset of the interior $S^0$. Assume that $K$ has $m$ connected components. If $U$ is a residual domain of $K$, then $U$ has at most $m(g + 1)$ ideal boundary points, where $g$ is the genus of $S$.

Proof:

By taking the union of $K$ with all its residual domains different from $U$, we may assume that $K \cup U = S$.

Since $K \subset S^0$, $U$ is a submanifold of $S$. Let $(F_n)$ be an exhaustion of $U$.

We first prove that if $W$ is a connected component of $S - F_n$, then $W$ contains at least one of the connected components of $K$. Since $K \subset S - F_n$, it is enough to show that $W \cap K \neq \emptyset$. Let us assume by contradiction that $W \cap K = \emptyset$. It follows that $W \subset U$ and that $W$ is contained in a connected component $V$ of $U - F_n$. But every component of $U - F_n$ is contained in a component of $S - F_n$. From this we conclude that $W = V$. We have that $\partial U = \partial S \cap U \subset F_n^0$ and then $cl_S W \cap \partial S = \emptyset$. Also $cl_S W = W \cup C_1 \cup \cdots \cup C_\ell$, where each $C_i$ is a connected component of $\partial F_n$. Therefore $cl_S W \subset U$ and $cl_U V = U \cap cl_S V = U \cap cl_S W = cl_S W$ is compact. But since $(F_n)$ is an exhaustion of $U$, we have that a connected component $V$ of $U - F_n$ is not relatively compact in $U$, a contradiction.

Let $E_1, \ldots, E_k$ be the closures in $S$ of the connected components of $S - F_n$. Since every component of $S - F_n$ contains a component of $K$, we have that $k \leq m$.

We claim that the boundaries

\[ \partial E_i \] are disjoint.

\( (4) \)
Indeed, we have that \( \partial E_j \subset \partial (S - F_n) \subset \partial S \cup \partial F_n \). A connected component of \( \partial E_i \cap \partial E_j \) would be a component \( D \) of \( \partial F_n \) which is the boundary of both \( E_i \) and \( E_j \) (whose interior is disjoint). Then \( E_i \cup E_j \) contains a tubular neighborhood of \( D \subset F_n \). This contradicts the fact that \( F_n \) is a surface.

Let \( \nu_i \) be the number of components of \( \partial E_i - \partial S \). We claim that

\[
\nu_i \leq g + 1.
\]

In fact, if we had \( \nu_i \geq g + 2 \), then by removing \( g + 1 \) components \( C_1, \ldots, C_{g+1} \) of \( \partial E_i - \partial S \) from \( S \), we would obtain a disconnected set \( S - (C_1 \cup \cdots \cup C_{g+1}) \). On the other hand, at least one component \( C \) of \( \partial E_i - \partial S \) remains in \( S - (C_1 \cup \cdots \cup C_{g+1}) \).

By (4) the connected components \( E_j, j \neq i \), attach to \( F_n \) through boundary components which are disjoint from \( C_1, \ldots, C_{\nu_i} \). Therefore the sets \( E_i^o \cup (\partial S \cap \partial E_i) \) and

\[
F_n \cup (\bigcup_{j \neq i} E_j) - (C_1 \cup \cdots \cup C_{g+1})
\]

are connected and their union is \( S - (C_1 \cup \cdots \cup C_{g+1}) \). But inside \( S - (C_1 \cup \cdots \cup C_{g+1}) \) these sets are glued through \( C \). Therefore \( S - (C_1 \cup \cdots \cup C_{g+1}) \) is connected, a contradiction.

Let \( \eta_n \) be the number of connected components of \( \partial F_n - \partial S \). Since

\[
\partial F_n - \partial S \subset \partial (S - F_n) - \partial S \subset \bigcup_{i=1}^{k}(\partial E_i - \partial S),
\]

we have that

\[
\eta_n \leq \nu_1 + \cdots + \nu_k \leq k(g + 1) \leq m(g + 1).
\]

Observe that every component of \( U - F_n \) contains a component of \( \partial F_n - \partial S \). It follows form Remark 3.10 that \( U \) has at most \( m(g + 1) \) ideal boundary points.

\[\square\]

3.16. Remark.

If \( K \cap \partial S \neq \emptyset \) we can apply Proposition 3.15 to the augmented boundaryless surface \( S^* \) from (2) and to \( K^* = K \cup (S^* - S^o) \). The interior of the connected components of \( S - K \) and \( S^* - K^* \) are the same. If \( K \) has \( m \) connected components, \( \partial S \) has \( n \) component, \( g \) is the genus of \( S \) and \( U^* \) is a connected component of \( S^* - K^* \) (or of \( S^o - K \) then

\[
\#b(U^*) \leq (m + n)(g + 1)
\]

because \( K^* \) has at most \( m + n \) connected components.

3.4. The canonical exhaustion of an unbounded residual domain.

If \( S \) is a surface with compact boundary and \( K \subset S \) is a compact subset with \( K \cap \partial S \neq \emptyset \) then a connected \( U \) component of \( S - K \) may not be a surface with compact boundary. In this case we can use the augmented boundaryless surface \( S^* \) from (2) and \( K^* = K \cup (S^* - S^o) \) as in Remark 3.16.

Let \( S \) be a non compact connected boundaryless surface. Endow \( S \) with a complete metric. Let \( K \) be a compact subset of \( S \) and let \( U \) be a residual domain of \( K \), i.e. a connected component of \( S - K \).

We are going to construct an adapted exhaustion of \( U \) in order to express its ideal boundary points in terms of ideal boundary points of \( S \) and ideal boundary points of \( U \cap N \), where \( N \) is a compact surface which is a neighborhood of \( K \) in \( S \). The ideal boundary
points of $U$ that are relatively compact in $S$ will correspond to the ideal boundary points of $U \cap N$ and the ideal boundary points of $U$ that are not relatively compact in $S$ will correspond to the ideal boundary points of $S$.

The construction will be applicable to both cases, $U$ unbounded or not.

As a consequence, we will see that if $K$ has finitely many components then $U$ has only finitely many ideal boundary points which are relatively compact in $S$ and that the others are ideal boundary points of $S$. As a corollary we have that $frS$ has finitely many connected components, a result that is not valid in all topological spaces.

Let $(F_n)_{n \geq 0}$ be an exhaustion of $S$. By Proposition 3.13, we may assume that $K$ and its bounded residual domains are contained in $F^0_0$. Therefore a residual domain of $K$ is unbounded if and only if it is not contained in $F^0_0$.

3.17. Claim.

Any unbounded component $U$ of $S - K$ contains at least one component of $\partial F^0_0$.

Proof:

We have that $U$ intersects both $F^0_0$ (since $frS U \subset K \subset F^0_0$) and $S - F^0_0$, implying that $U$ intersects $\partial F^0_0$. Therefore some component $C$ of $\partial F^0_0$ intersects $U$, and since $\partial F^0_0 \subset S - K$ we have that $C \subset U$.

From this we conclude that $K$ has finitely many unbounded residual domains. Let

$$U_+ = U - F^0_0 \quad \text{and} \quad U_- = U \cap F_0.$$

Then $U_+$ and $U_-$ are surfaces with compact boundary and their boundary as surfaces is

$$\partial U_+ = \partial U_- = U \cap \partial F_0 = U_+ \cap U_-.$$

Observe that

$$U \cap \partial F_0 = \partial U_+ \subset U_+.$$

Denote the connected components of $\partial U_+ = \partial U_- = U \cap \partial F_0$ by $\xi_1, \ldots, \xi_s$.

3.18. Claim.

The components of $U_+ = U - F^0_0$ are the components of $S - F^0_0$ contained in $U$.

Proof:

Let $C$ be a component of $U - F^0_0$. We have that $U - F^0_0 \subset S - F^0_0$ and therefore $C$ is contained in a component $D$ of $S - F^0_0$. We will show that $C = D$. We first prove that $D$ is contained in $U$. Indeed, since $S - F^0_0 \subset S - K$ we have that $D$ is contained in a component of $S - K$ such as $U$. But $C \subset D$ and $C \subset U$ implies that $D \cap U \neq \emptyset$ and therefore $D \subset U$. Now we have that $D \subset U - F^0_0$ and therefore $D$ must be contained in a component of $U - F^0_0$. Since $C \cap D \neq \emptyset$, we have that $C = D$. This proves the claim.

We have that $F_0$ is part of an exhaustion of $S$. Therefore every component of $S - F^0_0$ is a surface whose boundary consists exactly of one component of $\partial F_0$. From Claims 3.17
we conclude that $U_+$ has $s$ connected components, each one a surface with compact boundary consisting of exactly one of the curves $\xi_k$.

3.19. **Claim.** The set $U_- = U \cap F_0$ is connected.

**Proof:**

Assume by contradiction that $U_- = A_1 \cup A_2$ with $A_1 \cap A_2 = \emptyset$ and $A_i$ closed in $U_-$ and non empty. Then $A_1$ is also closed in $U$. Also $U_-$ contains all the components $\xi_1, \ldots, \xi_s$ of $\partial U_- = U \cap \partial F_0$. Every $\xi_k$, being connected, is contained in either $A_1$ or $A_2$. Let $B_i$ be the union of the components of $U_+$ whose boundary curves $\xi_k$ are contained in $A_i$. Since every component of $U_+$ contains exactly one $\xi_k$, we have that $B_1 \cap B_2 = \emptyset$. Also each $B_i$ is closed in $U$. Therefore $A_1 \cup B_i, \ i = 1, 2$ are disjoint subsets of $U$ that are closed in $U$ whose union is $U$. That would imply that $U$ is disconnected, a contradiction.

3.20. **Claim.** $U_+ \cap F_n = U \cap (F_n - F_0^0)$ is compact.

**Proof:**

From (5), $U_+ = U - F_0^0$. This implies that $U_+ \cap F_n = U \cap (F_n - F_0^0)$. Since $F_n - F_0^0$ is compact, it is enough to prove that $U \cap (F_n - F_0^0)$ is closed in $S$. Since $K \subseteq F_0^0$ and $fr_S U \subseteq K$ we have that

$$fr_S(U) \cap (F_n - F_0^0) = \emptyset.$$ 

Therefore

$$fr_S(U \cap (F_n - F_0^0)) \subseteq [fr_S U \cap (F_n - F_0^0)] \cup [U \cap fr_S(F_n - F_0^0)] \subseteq U \cap (F_n - F_0^0).$$ 

Hence $U \cap (F_n - F_0^0)$ is closed in $S$.

We construct the exhaustion of $U$ as follows. Let $(E_n)_{n \geq 1}$ be an exhaustion of $U_-$.

We may assume that $E_1$ contains all the components of $\partial U_- = U \cap \partial F_0$. Then $E_n$ is a compact surface whose boundary contains all the components of $\partial U_- = U \cap \partial F_0$.

For $n \geq 1$, let

$$G_n = E_n \cup (U_+ \cap F_n) \subseteq E_n \cup (F_n - F_0^0).$$

Using Claim 3.20 we have that $G_n$ is compact. Also $\cup_{n \geq 1} G_n = U$.

3.21. **Claim.** Every component of $U_+ \cap F_n \subseteq F_n - F_0^0$ contains a component of $\partial F_0$.

**Proof:** Let $C$ be a component of $F_n - F_0^0$ and let $x \in C$. There exists a path $\alpha$ in $F_n$ from $x$ to a point in $\partial F_0$. Let $y$ be the first point of $\alpha$ to intersect $F_0$ starting from $x$. The restriction $\beta$ of $\alpha$ from $x$ to $y$ is a path in $F_n - F_0^0$ connecting $x$ to a component $\nu$ of $\partial F_0$. Therefore $\{x\} \cup \beta \cup \nu \subseteq C$, which proves that $C$ contains a component of $\partial F_0$.

We showed in Claim 3.18 that a component of $U_+$ is a component of $S - F_0^0$. Therefore a component of $U_+ \cap F_n$ is a component of $(S - F_0^0) \cap F_n = F_n - F_0^0$. Using the previous paragraph we conclude that every component of $U_+ \cap F_n$ contains one component of $U \cap \partial F_0$.
3.22. Claim. $G_n$ is connected.

Proof: We have that $E_n$ is connected and contains all the components $\xi_1, \ldots, \xi_s$ of $U \cap \partial F_0$. The inclusion (7) implies that

$$U \cap \partial F_0 \subset U_+ \cap F_n.$$ 

Let $C_k$ be the component of $U_+ \cap F_n$ that contains $\xi_k$, then $E_n \cup C_k$ is connected. By Claim 3.21 these $C_k$'s are all the components of $U_+ \cap F_n$. This implies that

$$G_n = E_n \cup (U_+ \cap F_n) = (E_n \cup C_1) \cup \cdots \cup (E_n \cup C_s)$$

is connected. \hfill \Box

3.23. Claim. $G_n \subset \text{int}_U G_{n+1}$.

Proof: First we prove that using (5), (8),

(9) \hspace{1cm} U_- \cap G_n \subset E_n, \quad U_+ \cap G_n \subset F_n.

The first inclusion is because $U_- \cap U_+ = \emptyset$ and the second because $U_+ \cap E_n \subset U_+ \cap U_- = \emptyset$.

Let $x \in G_n$. We will prove that there exists a neighborhood $V$ of $x$ in $S$ such that $V \subset G_{n+1}$. We have three possibilities for $x$: either $x \in U_-^o$, $x \in U_+$ or $x \in \partial U_+ = \partial U_-$.\hfill \Box

(9) \hspace{1cm} U_- \cap G_n \subset E_n, \quad U_+ \cap G_n \subset F_n.$$

(a) If $x \in U_-^o \cap G_n$ then there exists a neighborhood $V$ of $x$ in $S$ such that $V \subset U_-^o$. By (9) in this case $x \in E_n \subset \text{int}_U E_{n+1}$, and therefore there exists a neighborhood $W$ of $x$ in $S$ such that $W \cap U_- \subset E_{n+1}$. It follows that $V \cap W \subset E_{n+1} \subset G_{n+1}$.

(b) If $x \in U_+ \cap G_n$, by (9) we have that $x \in F_n \subset \text{int}_S F_{n+1}$. Then there is a neighborhood $V$ of $x$ in $S$ such that $V \subset U_+ \cap F_{n+1} \subset G_{n+1}$.

(c) Suppose that $x \in G_n$ and $x \in \partial U_+ = \partial U_- = U \cap \partial F_0$. From the choice of $E_n \supset U \cap \partial F_0$ we have that $x \in E_n \subset \text{int}_U E_{n+1}$. Then there exist a neighborhood $V$ of $x$ in $S$ such that

$$V \cap U_- \subset E_{n+1}.$$ 

By (7) $\partial U_+ \subset U_+$, therefore

$$x \in \partial U_+ = U \cap \partial F_0 \subset U_+ \cap F_n \subset U_+ \cap \text{int}_S F_{n+1}.$$ 

Then there is a neighborhood $W$ of $x$ in $S$ such that $W \subset F_{n+1}$. Thus

$$W \subset U_+ \subset F_{n+1}.$$ 

Since $x \in G_n \subset U$, we can assume that $V \cup W \subset U$. Hence

$$V \cap W = ((V \cap W) \cap U_-) \cup ((V \cap W) \cap U_+) \subset E_{n+1} \cup (U_+ \cap F_{n+1}) = G_{n+1}.$$ 

\hfill \Box

3.24. Lemma.

(1) Every component of $U - G_n$ is contained in $U_+$ or in $U_-$. 
(2) The components of $U - G_n$ contained in $U_-$ are the components of $U_- - E_n$.
(3) The components of $U - G_n$ contained in $U_+$ are the components of $S - F_n$ contained in $U$.
Proof:

Recall that $G_n = E_n \cup (U_+ \cap F_n) \subset E_n \cup (F_n - F_n^0)$.

Let $V$ be a connected component of $U - G_n$. Observe that the open surfaces $U_+ - \partial U_+$ and $U_- - \partial U_-$ are disjoint with $\partial U_+ = \partial U_-$. Since $V$ is connected, if we had that $V \cap U_+ \neq \emptyset$ and $V \cap U_- \neq \emptyset$, then $V$ would intersect

$$\partial U_+ = \partial U_- = U_+ \cap U_- = U \cap \partial F_0 \subset U_+ \cap F_n \subset G_n$$

using (7),
a contradiction with $V \subset U - G_n$. This proves item (1).

Before proceeding, note that

(i) $U_- - E_n \subset U - G_n$.

In fact, let $x \in U_- - E_n$. If we had that $x \in G_n$, then $x \in U_+ \cap F_n$ and then

$$x \in U_+ \cap U_- = U \cap \partial F_0 = \partial U_- \subset E_n.$$  

A contradiction.

(ii) We also have that every component of $S - F_n$ is contained in $U$ or is disjoint from $U$.

This follows from the fact that $U$ is a component of $S - K$ which contains $S - F_n$.

Let $V$ be a component of $U - G_n$ contained in $U_-$. Since $V \cap G_n = \emptyset$ we have that $V \cap E_n = \emptyset$. Therefore $V$ is contained in a component of $U_- - E_n$. Conversely, suppose that $W$ is a component of $U_- - E_n$. By remark (i), $U_- - E_n \subset U - G_n$. This implies that $W$ is contained in a component of $U - G_n$. This proves item (2).

Observe that to prove item (3) it is enough to prove that any component of $U - G_n$ contained in $U_+$ is contained in a component of $S - F_n$ contained in $U$ and vice versa.

Now let $V$ be a component of $U - G_n$ contained in $U_+$. Then

$$(10) \quad V \cap (U_+ \cap F_n) = \emptyset.$$  

By the choice of $V$, $V \subset U_+$. Thus from (10), $V \cap F_n = \emptyset$. Therefore $V$ is contained in a component $W$ of $S - F_n$. Since $W \cap U \neq \emptyset$, by remark (ii) we have that $W \subset U$.

Conversely, let $W$ be a component of $S - F_n$ contained in $U$. Since by (5), $U_- \subset F_n$, we have that

$$(11) \quad W \cap U_- = \emptyset \quad \text{and} \quad W \subset U_+.$$  

We claim that $W \cap G_n = \emptyset$. If we had a point $x \in W \cap G_n$, then $x \in G_n$ implies that $x \in U_+ \cap F_n$ or $x \in E_n$. Since $x \in W$, we have that $x \in U_+ \cap F_n$ is not possible. If we had $x \in E_n$, then $x \in U_-$. Therefore, using (11), $x \in W \cap U_- = \emptyset$, a contradiction. This proves the claim. Therefore $W \subset U - G_n$ and then $W$ is contained in a component of $U - G_n$. And by (11), $W \subset U_+$. This proves item (3).

\[ \square \]

Lemma 3.24 has two purposes. First, it concludes the proof that $(G_n)$ is a exhaustion of $U$ by showing that the closure in $U$ of every component of $U - G_n$ is a bordered surface whose boundary consists of exactly one component of $\partial G_n$. This follows from the fact that $(F_n)$ and $(E_n)$ are exhaustions of $S$ and $U_-$, respectively.
The second is the relation between ideal boundary points of $U$ and ideal boundary points of $S$ and $U_-$, namely,

$$b(U) = b(U_+) \cup b(U_-) \quad \text{with} \quad b(U_+) \subset b(S).$$

3.25. **Proposition.**

Let $U$ be a residual domain of a compact set $K$ contained in a connected boundaryless surface $S$ and assume that $K$ has finitely many connected components.

1. Then $U$ has only finitely many ideal boundary points that are relatively compact in $S$. Each of these points is isolated in $b(U)$.
2. The ideal boundary points of $U$ that are not relatively compact in $S$ are ideal boundary points of $S$ and their impressions relative to $S$ are disjoint from $S$.
3. The frontier of $U$ in $S$ is the union of the impressions in $S$ of the ideal boundary points of $U$ that are relatively compact in $S$.

**Proof:** Consider first the case when $U$ is relatively compact. As in Lemma 3.24, there exists a compact bordered surface $F_0$ which is a neighborhood of $K$ in $S$ and all relatively compact residual domains of $K$ are contained in $F_0$. It follows from Proposition 3.15 that $U$ has finitely many ideal boundary points. Of course, each one is isolated and relatively compact in $S$. From Proposition 3.12, the ideal boundary points of $U$ are planar and the ideal completion of $U$ is a compact surface.

We use a complete metric on $S$. Now consider the case $U$ is unbounded and let $b$ be an ideal boundary point of $U$. As before, we decompose $U$ into $U_+ \cup U_-$. From Proposition 3.9 and item (1) of Lemma 3.24, $b$ can be represented by an ideal boundary component $(V_n)$, where the sets $V_n$ are components of $U - G_n$ and either all of them are contained in $U_+$ or all of them are contained in $U_-$. Let $b_+(U)$ be the set of ideal boundary points of $U$ for which $V_n \subset U_+$ for every $n$ and let $b_-(U)$ be the set of ideal boundary points of $U$ for which $V_n \subset U_-$ for every $n$.

Recall that $F_n$ is an exhaustion of $S$. If $b \in b_+(U)$, then from item (3) of Lemma 3.24 we see that $(V_n)$ is an ideal boundary component defining a point $b' \in b(S)$. Moreover $Z(b) = \cap_n cl_{b(S)} V_n = \{b'\}$. This proves item (2).

Suppose now that $b \in b_-(U)$. In this case $(V_n)$ also defines an ideal boundary point $b'$ of $U_-$, and this correspondence defines a bijection from $b_-(U)$ onto $b(U_-)$. By Claim 3.19 the set $U_-$ is a residual domain of $K$ in the compact manifold with boundary $F_0$. From Proposition 3.15 we have that $b(U_-) \approx b_-(U)$ is finite.

Clearly every $b \in b_-(U)$ is relatively compact in $S$. Since every $b \in b_+(U)$ is not relatively compact in $S$, we have that $b_- (U)$ is the set of relatively compact ideal boundary points of $U$. An ideal boundary point $b \in b_-(U)$ can not be accumulated by ideal boundary points of $b_+(U)$, and since $b_-(U)$ is finite, we have that its points are isolated in $b(U)$. This proves item (1).

We now prove item (3), let $b = (V_n)$ be a relatively compact ideal boundary point of $U$. Then $b \in b_-(U)$ and $Z(b) = \cap_n cl_{V_n}$. If $x \in Z(b)$, then $x \in cl_{V_n}$. If we had $x \in U$, then $x \in U \cap cl_{V_n} = cl_{U/V_n}$ for every $n$, contradicting that $(V_n)$ is an ideal boundary component of $U$. Therefore $Z(b) \subset fr_{S}U$. 

It remains to show that \( fr_{S}U \subset \bigcup_{b \in b_{-}(U)} Z(b) \).

Let \( b_{1}, \ldots, b_{n} \) be the relatively compact ideal boundary points of \( U \). Each \( b_{i} \) can be represented by an ideal boundary component \( (V_{n}^{i}) \) of \( U_{-} \), where \( V_{n}^{i} \) is a component of \( U_{-} - E_{n} \). Since \( b(U_{-}) \) is a finite set with \( m \) elements, there exists \( n_{0} \) such that for \( n \geq n_{0} \) the number of components of \( U_{-} - E_{n} \) is also \( m \). For each \( i = 1, \ldots, m \), the collection \( (V_{n}^{i})_{n \geq n_{0}} \) is an ideal boundary component of \( U \) that represents \( b_{i} \), where \( V_{n}^{1}, \ldots, V_{n}^{m} \) are the components of \( U_{-} - E_{n} \).

Now let \( x \in fr_{S}U \) and let \( (x_{k}) \) be a sequence in \( U \) such that \( \lim x_{k} = x \). Given \( n_{1} \geq n_{0} \) there exists \( k_{0} = k_{0}(n_{1}) \) such that \( x_{k} \notin G_{n_{1}} \) for \( k \geq k_{0} \). Since \( fr_{S}U \subset K \subset F_{0}^{s} \), there is \( k_{1} = k_{1}(n_{1}) \geq k_{0} \) such that \( x_{k} \in U \cap F_{0} \) for \( k \geq k_{1} \). Using (5) and (8) we have that \( x_{k} \in U_{-} - E_{n_{1}} \) for \( k \geq k_{1} \). This is

\[
(12) \quad \forall n_{1} \geq n_{0} \quad \exists k_{1} = k_{1}(n_{1}) \quad \forall k \geq k_{1}(n_{1}) \quad x_{k} \in U_{-} - E_{n_{1}}.
\]

Using \( n_{1} = n_{0} \) in (12) we have that there exists \( i_{0} \) and a subsequence \( (x_{k_{j}}) \) of \( (x_{k}) \) such that \( x_{k_{j}} \in V_{n}^{i_{0}} \) for every \( j \). For \( n \geq n_{0} \) we have that \( V_{n}^{i_{0}} \) is the only component of \( U_{-} - E_{n} \) contained in \( V_{n}^{i_{0}} \). Using (12) with \( n_{1} = n \geq n_{0} \) we have that there exists \( j_{n} \) such that \( x_{k_{j}} \in V_{n}^{i_{0}} \) for all \( j \geq j_{n} \). It follows that \( x \in cl_{S}V_{n}^{i_{0}} \) for every \( n \geq n_{0} \) and then

\[
x \in \bigcap_{n \geq n_{0}} cl_{S}V_{n}^{i_{0}} = Z(b_{i_{0}}).
\]

There is a natural inclusion \( b(S) \subset b(S - K) \). Namely, if \( b = (V_{n}) \in b(S) \) then

\[
\exists n_{0} \quad \forall n \geq n_{0} \quad V_{n} \cap K = \emptyset, \quad \text{and hence} \quad b' = (V_{n})_{n \geq n_{0}} \in b(S - K).
\]

The set \( S - K \) may have infinitely many components and \( b(S - K) - b(S) \) may be infinite, but

3.26. Corollary.

Let \( S \) be a connected boundaryless surface. Let \( K \) be a compact subset of \( S \) with finitely many connected components.

(1) \( b(S - K) - b(S) \) is a discrete topological space.

(2) If \( U \) is a residual domain in \( S - K \) then \( b(U) - b(S) \) is finite.

Proof:

By item (2) in Proposition 3.25, the ideal boundary points in \( b(U) - b(S) \) are relatively compact. By item 3.25.(1), \( b(U) \) has only finitely many relatively compact ideal boundary points. Therefore \( b(U) - b(S) \) is finite. This proves item (2).

Item (2) in Proposition 3.25 implies that the ideal boundary points in \( b(S - K) - b(S) \) are relatively compact. Then

\[
(13) \quad b(S - K) - b(S) = \bigsqcup_{\lambda \in \Lambda} (b(U_{\lambda}) - b(S)),
\]

where \( \Lambda \) is the collection of residual domains in \( S - K \). By item (2) each \( b(U_{\lambda}) \) is finite and discrete. And \( b(S - K) - b(S) \) has the disjoint union topology in (13). This proves item (1).
Summarizing for future reference we have

3.27. Proposition.

Let $S$ be a connected surface without boundary. Let $K$ be a compact subset of $S$ with finitely many connected components and let $U$ be a residual domain of $K$, i.e. a connected component of $S - K$. There is a decomposition

$$U = U_+ \cup U_-,$$

where $U_+$ and $U_-$ are connected surfaces with compact boundary $\partial U_+ = \partial U_- = U_+ \cap U_-$ which satisfy

$$b(U) = b(U_+) \sqcup b(U_-) \quad \text{with} \quad b(U_+) = b(S) \cap b(U).$$

More explicitly, there are exhaustions $(E_n)$, $(F_n)$, $(G_n)$ of $U_-$, $S$ and $U$ respectively such that

1. $U_- = U \cap F_0$ and $U_+ = U - F_0^\circ$.
2. $F_0^\circ$ contains $K$ and all the relatively compact residual domains in $S - K$. In particular $K \subset \text{cl}_S U_-$.
3. Every component of $U - G_n$ is contained in $U_+$ or in $U_-$. 
4. The components of $U - G_n$ contained in $U_-$ are the components of $U_- - E_n$.
5. The components of $U - G_n$ contained in $U_+$ are the components of $S - F_n$ contained in $U$.

Moreover,

(a) The interior of the surface with boundary $F_0$ contains $K$ and all the relatively compact residual domains of $K$.
(b) The relatively compact ideal boundary points of $U$ are finitely many and are isolated in $b(U)$. Moreover, they are exactly the ideal boundary points in $b(U_-)$ of the surface $U_- = U \cap F_0$.
(c) The ideal boundary points in the complement $U_+$ of the interior of the surface $F_0$, $U_+ = U - F_0^\circ$ are in $b(S)$ and hence they are not relatively compact in $S$.
(d) Using Proposition 3.12, for any relatively compact ideal boundary point of $U$, $b \in b(U_-)$ there is a neighborhood of $b$ in $B(U)$ given by $V_n^* = V_n \cup \{b\}$, where $V_n$ is a component of $U_- - E_n$, such that $V_n^*$ is homeomorphic to a disk in $\mathbb{R}^2$.
(e) The frontier of $U$ in $S$ is the union of the impressions in $S$ of the ideal boundary points of $U$ that are relatively compact in $S$.
(f) By items (d) and (e), if $x \in \text{fr}_S U$ then $x \in Z(b)$, where $b \in b(U)$ is relatively compact and there is a neighborhood of $b$ in $B(U)$ which is a disk.

As a consequence of Proposition 3.25 we have the following:

3.28. Corollary.

Let $U$ be an open connected subset of a connected surface without boundary $S$. Then $\text{fr}_S U$ is a compact set with finitely many connected components if and only if $U$ is a residual domain of a compact set $K$ that has finitely many connected components.
Proof:

If \( fr_SU \) is a compact set with finitely many connected components, then \( U \) is a residual domain of \( S - fr_SU \). Conversely, if \( U \) is a residual domain of a compact set \( K \) that has finitely many connected components, then by item (3) of Proposition 3.25, \( fr_SU = \cup_bZ(b) \), where the union is taken over the finite set of relatively compact ideal boundary points \( b \) of \( U \).

This result is false if \( S \) is not a surface. Consider the following subset of \( \mathbb{R}^2 \):

\[
X = ([0,1] \times \{0,2\}) \cup (C \times [0,2]),
\]

where \( C \) is an infinite closed subset of \([0,1]\). If \( K = \{(x,y) \in X : y \leq 1\} \), then \( K \) is compact connected, \( U = X - K \) is connected, but \( fr_XU = C \times \{1\} \).

4. The Accumulation Lemma.

Let \( S \) be a connected surface without boundary and let \( S_0 \subset S \) be an open subset with \( fr_S S_0 \) compact. Let \( \mu \) be a Borel measure in \( S \) which is finite on compact sets and positive on open non empty sets. We call \( \mu \) the area measure. Let \( f : S_0 \to S \) be an injective area preserving homeomorphism of \( S_0 \) onto an open subset \( f(S_0) \subset S \). Let \( K \subset S_0 \) be a connected compact subsets such that \( f(K) = K \).

There is a natural inclusion \( b(S) \subset b(S - K) \) obtained as follows. If \((V_n)\) is an ideal boundary component for \( q \in b(S) \), since \( K \) is compact, there is \( n_0 \) such that \( V_n \cap K = \emptyset \) for all \( n > n_0 \). Then \((V_n)_{n>n_0} \in b(S - K)\).

Let \( p \in b(S - K) \), from Proposition 3.25.2 we have that

(14) If \( p \in b(S) \) then \( Z(p) = p \).

(15) Otherwise \( p \in b(S - K) - b(S) \), \( Z(p) \subset K \) and \( p \) is relatively compact.

We claim that the map \( f \) naturally defines an injective map

\[
f_* : b(S - K) - b(S) \to b(S - K) - b(S)\]

Indeed, if \( p = (V_n) \in b(S - K) - b(S) \) then \( Z(p) \subset K \), therefore for all \( n \), \( cl_S V_n \cap K \neq \emptyset \). Since \( fr_S S_0 \) is compact and \( fr_S S_0 \cap K = \emptyset \), \( \exists n_1, \forall n > n_1, V_n \cap fr_S S_0 = \emptyset \). Since \( V_n \) is connected and \( K \subset S_0 \), we obtain that \( \forall n > n_1, V_n \subset S_0 \). Therefore \( f V_n \) is defined for \( n > n_1 \) and \( (fV_n)_{n>n_1} \) is an ideal boundary component for a point \( f_* (p) \) which is in \( b(S - K) - b(S) \), because \( Z(f_*(p)) = f(Z(p)) \subset K \). The injectivity of \( f \) implies that \( f_* \) is injective.

4.1. Proposition. For any \( p \in b(S - K) - b(S) \) there is \( n \) such that \( f_*(p) = p \).

Proof:

By Proposition 3.13, there exists a compact bordered surface \( N \) which is a neighborhood of \( K \) in \( S_0 \), such that all relatively compact residual domains in \( S_0 - K \) are contained in \( N \). The residual domains in \( S - K \) which are not contained in \( N \), contain a component of \( \partial N \). Then there are only finitely many of them. Let \( \varepsilon \) be the minimum of the area of the residual domains in \( S - K \) which are not contained in \( N \). Then \( \varepsilon > 0 \).
Since \( \varepsilon > 0 \) and the area of \( N \) is finite, there are only finitely many residual domains in \( S - K \) with area \( \geq \varepsilon \) which are contained in \( N \). The number of residual domain not contained in \( N \) is finite. Therefore the number of residual domains in \( S - K \) with area \( \geq \varepsilon \) is finite.

If \( U \) is a residual domain in \( S - K \) with area \( < \varepsilon \). Then \( U \) is contained in \( N \subset S_0 \) and hence it is relatively compact in \( S_0 \) and \( f_{\mathcal{RS}}U \subset K \). The map \( f \) is defined in \( U \) and \( f(U) \) is a connected component of \( f(S_0) - K \) which is relatively compact in \( f(S_0) \). Then the frontier \( f_{\mathcal{RS}}f(U) \subset K \). This implies that \( f(U) \) is a connected component of \( S - K \). Since \( f \) preserves area, we have that \( f(U) \) is a residual domain in \( S - K \) with area \( < \varepsilon \), and then \( f(U) \subset N \subset S_0 \). Inductively, all the iterates \( f^n \) are defined in \( U \) and \( f^n(U) \) is a residual domain in \( S - K \) with \( f^n(U) \subset N \). Since the area of \( N \) is finite, we have that there is \( n > 0 \) such that \( f^n(U) = U \). This implies that the map \( f^n \) sends \( b(U) \) to itself.

If \( U \) is a residual domain in \( S - K \) with area \( < \varepsilon \) then it is relatively compact in \( S \) and hence by Proposition 3.25(1), \( b(U) \) is finite. Since \( b(U) \) is a periodic set for \( f_* \) and \( f_* \) is injective, we obtain that all points in \( b(U) \) are periodic for \( f_* \).

There are only finitely many residual domains \( V \) in \( S - K \) of area \( \geq \varepsilon \) and by Corollary 3.26(2) for each one \( b(V) - b(S) \) is finite. Therefore the set \( Q \subset b(S - K) - b(S) \) of points which are not periodic for \( f_* \) is at most finite. The injectivity of \( f_* \) implies that \( f_*(Q) \subset Q \). Since \( f_* \) is injective, \( Q \) is finite and \( f_*(Q) \subset Q \), we have that \( f_* \) is a bijection, i.e. a permutation of \( Q \). Then all points in \( Q \) are periodic, i.e. \( Q = \emptyset \), and all points in \( b(S - K) - b(S) \) are periodic for \( f_* \).

We will need the following result. In its proof we refer to the canonical exhaustion of \( U \) from subsection § 3.4 which is also described in Proposition 3.27.

4.2. Proposition.

Let \( U \) be a residual domain of \( K \) and \( \alpha : ]0, 1] \to U \) a path such that
\[
\lim_{t \to 0} \alpha(t) = p \in K \quad \text{in } S.
\]
Then there exists a relatively compact ideal boundary point \( b \) of \( U \) such that
\[
\lim_{t \to 0} \alpha(t) = b \quad \text{in } B(U).
\]

Proof:

Observe that \( U_- \) contains a relative neighborhood of \( K \subset f_{\mathcal{RS}}U_- \) inside \( U_- \). Then there exists \( \tau > 0 \) such that \( \alpha(t) \in U_- \) for all \( t < \tau \). As in the proof of item (3) of Proposition 3.25, if \( b_1, \ldots, b_m \) are the relatively compact ideal boundary points of \( U \), then there exists \( n_0 \) such that for \( n \geq n_0 \) the number of components of \( U_- - E_n \) is \( m \), and for each \( i = 1, \ldots, m \), the collection \((V_n^{i_0})_{n \geq n_0}\) is an ideal boundary component of \( U \) that represents \( b_i \), where \( V_n^1, \ldots, V_n^m \) are the components of \( U_- - E_n \). There exist \( 0 < t_0 < \tau \) and \( i_0 \) such that the curve \( \alpha([0, t_0]) \subset V_n^{i_0} \). For every \( n > n_0 \) we have that \( V_n^{i_0} \) is the only component of \( U_- - E_n \) contained in \( V_n^{i_0} \), and therefore there exists a sequence \( t_n \downarrow 0 \) such that \( \alpha([0, t_n]) \subset V_n^{i_0} \). For \( n > n_0 \), we have that
\[
cd_{B(U)} \alpha([0, t_{n+1}]) \subset \cd_{B(U)} V_n^{i_0} \subset (V_n^{i_0})^* := V_n^{i_0} \cup \{b_{i_0}\}.
\]
implying that
\[ \cap_{n>n_0} \text{cl}_{B(U)} \alpha([0,t_{n+1}]) \subset \cap_{n>n_0} (V_n^i)^* = \{ b_{i_0} \}. \]
From this we conclude that \( \lim_{t \to 0} \alpha(t) = b_{i_0} \) in \( B(U) \).

Let \( S \) be a connected surface with compact boundary and \( S_0 \subset S \) be an open subset with \( f \text{r}_S S_0 \) compact. Let \( f : S_0 \to S \) be an area preserving homeomorphism onto its image. Let \( p \in \text{Fix}(f^n) \) be a periodic point of \( f \), possibly in the boundary \( p \in \partial S_0 \). We say that \( p \) is a saddle point if there is an open neighborhood \( V \) of \( p \) in \( S_0 \) and a continuous system of coordinates in \( V \) such that in these coordinates we have that \( f^n(x,y) = (\lambda x, \lambda^{-1} y) \) for some \( \lambda > 1 \). In this case the branches of \( p \) are the connected components of \( W^s(p,f^n) - \{ p \} \) and of \( W^u(p,f^n) - \{ p \} \), where
\[ W^s(p,f^n) = \{ y \in S : \lim_{k \to +\infty} f^{nk}(y) = p \}, \]
\[ W^u(p,f^n) = \{ y \in S : \lim_{k \to -\infty} f^{nk}(y) = p \} \]
are the stable and unstable manifolds of \( p \). The branches of \( f \) are the branches of its saddle periodic points.

In order to use the previous results for surfaces without boundary we define the double of a surface with boundary. Let \( \sim \) be the equivalence relation defined by the partition of \( S \times \{ 0, 1 \} \) into one point sets \( \{ (p,i) \} \) if \( p \notin \partial S \), \( i \in \{ 0, 1 \} \) and two point sets \( \{ (p,0), (p,1) \} \) if \( p \in \partial S \). Let \( \overline{S} = S \times \{ 0, 1 \} / \sim \), provided with the quotient topology. Then \( \overline{S} \) is a surface without boundary and \( \overline{S}_0 = S_0 \times \{ 0, 1 \} / \sim \) is an open subset of \( \overline{S} \).

Let \( \nu \) be the measure with \( \nu(\{ 0 \}) = \nu(\{ 1 \}) = 1 \) on \( \{ 0, 1 \} \). Extend the measure \( \mu \) in \( S \) to the measure \( \overline{\mu} \) given by the push forward of the measure \( \mu \times \nu \) on \( S \times \{ 0, 1 \} \) by the quotient map. Extend the map \( f \) to \( \overline{S}_0 \) by \( \overline{f}([x,i]) = [(f(x),i)] \), where \( [(x,i)] \) is the class of \( (x,i) \). Then \( \overline{f} \) is an area preserving homeomorphism of \( \overline{S}_0 \) onto an open subset of \( \overline{S} \). If \( p \) is a saddle point for \( f \) in \( S \) and \( i \in \{ 0, 1 \} \), then the point \( [(p,i)] \in \overline{S} \) is a saddle point for \( \overline{f} \).

4.3. Theorem (The accumulation lemma).

Let \( S \) be a connected surface with compact boundary provided with a Borel measure \( \mu \) such that open non-empty subsets have positive measure and compact subsets have finite measure. Let \( S_0 \subset S \) be an open subset with \( f \text{r}_S S_0 \) compact.

Let \( f, f^{-1} : S_0 \to S \) be an area preserving homeomorphism of \( S_0 \) onto open subsets \( f(S_0), f^{-1}(S_0) \) of \( S \). Let \( K \subset S_0 \) be a compact connected invariant subset of \( S_0 \).

If \( L \subset S_0 \) is a branch of \( f \) and \( L \cap K \neq \emptyset \), then \( L \subset K \).

Proof:

By applying the theorem to the double of the surface \( \overline{S} \) and \( \overline{S}_0 \), \( \overline{f} \), we can assume that \( \partial S = \emptyset \). Observe that \( \overline{S}_0 \) is an open subset of \( \overline{S} \) and \( f \text{r}_{\overline{S}} \overline{S}_0 \) is also compact. We use \( \overline{L} = \pi_0(L) \) and \( \overline{K} = \pi_0(K) \), where \( \pi_0 : S \to \overline{S} \) is \( \pi_0(x) = [(x,0)] \). We may assume that \( L \) is an invariant stable branch. Observe that it is enough to prove the theorem for an iterate \( f^n \) of \( f \).
Suppose by contradiction that $L$ contains an arc $\beta$ whose endpoints $p_0$ and $p_1$ belong to $K$ but its interior $\beta^c$ satisfies $\beta^c \cap K = \emptyset$. We are going to think of $\beta$ also as a path $\beta : [0,1] \to U$, with $\beta(0) = p_0$ and $\beta(1) = p_1$. Since $L$ is invariant and $\beta \subset L \subset S_0$, we have that $f^n\beta$ is defined and $f^n\beta \subset S_0$ for all $n$.

Let $U$ be the connected component of $S - K$ that contains $\beta^c$. By Proposition 3.25.(2) the relatively compact ideal boundary points for $U$ are in $b(S - K) - b(S)$. By Proposition 3.25.(1) they are finitely many and by Proposition 4.1 they are periodic points for $f_*$. Therefore by taking an iterate $f^n$ if necessary, we can assume that $f_*(b) = b$ for any relatively compact ideal boundary point of $U$.

By Proposition 4.2, there exist two relatively compact ideal boundary points of $U$, $b_0$ and $b_1$, such that $\lim_{t \to i} \beta(t) = b_i$ in $B(U)$, $i = 0, 1$. If $\beta_* = cl_{B(U)}\beta^c$, then $\beta_* = \beta^c \cup \{b_0, b_1\}$. By Proposition 3.25.(1) we have that $b_0$ and $b_1$ are isolated points of $b(U)$.

Referring to the canonical exhaustion $(G_n)$ of $U$, described in Proposition 3.27, (b), (4), the point $b_0 \in b(U)$ is represented by an ideal boundary component $(V_k)$ of $U$, where $V_k$ is a component of $U_- - E_k$ and $cl_{S}V_k$ is a bordered surface whose boundary contains only one component, which we denote by $\partial(cl_{S}V_k)$. By 3.27.(b) or 3.25.(1), for every $k$ sufficiently large, $b_0$ is the only ideal boundary point of $U$ contained in $cl_{B(U)}V_k$. So we assume that this happens for every $k$. Let $V'_k := V_k \cup \{b_0\}$. By 3.27.(d), the collection $(V'_k)$ is a fundamental system of neighborhoods of $b_0$ in $B(U)$, $D_k := cl_{B(U)}V'_k$ is a closed disk and $\partial D_k = \partial(cl_{U}V_k)$.

Since $fr_SJ_0$ is compact and $K \subset S_0$, we have that $U \cap fr_SJ_0$ is compact. Then

\begin{equation}
\exists k_0 \quad \forall k > k_0 \quad V_k \cap fr_SJ_0 = \emptyset.
\end{equation}

Using (3) and (15) we have that

\begin{equation}
\emptyset \neq \bigcap_k cl_{B(S)}V_k = Z(b_0) \subset K.
\end{equation}

From (17), $\forall k \quad cl_{S}V_k \cap K \neq \emptyset$. Since $V_k$ is connected and $K \subset S_0$, from (16) we get that $\forall k > k_0 \quad V_k \subset S_0$. Fix $k_1$ such that $V_{k_1} \subset S_0$ and $D_0 := cl_{B(S)}V_{k_1}$ is a closed disk and

\begin{equation}
\forall k \geq k_1 \quad V'_k \subset int_{B(S)}D_0 \subset V_{k_1} \cup \{b_0\}.
\end{equation}

Extend the measure $\mu$ to $B(U)$ by $\mu(b(U)) = 0$. Then $f_*$ is defined in $D_0$, is continuous, preserves area and $f_*(b_0) = b_0$.

By its definition in (5) we have that $U_-$ is relatively compact in $S$, and hence it has finite measure. By Proposition 3.27.(b) $V_k \subset U_-$. Since $\mu(U'_k) = \mu(U_-) < \infty$, $V_k \subset U_-$ and $\cap_k V'_k = \{b_0\}$, we have that

\begin{equation}
\lim_{k \to \infty} \mu(V'_k) = 0.
\end{equation}

Let $d$ be a metric compatible with the topology of $S$. Since $\beta^c \subset L$, if $diam(f^n\beta)$ is the diameter of $f^n\beta$, then $\lim_{n \to \infty} diam(f^n\beta) = 0$. Therefore

\begin{equation}
\forall k \quad \exists n_k > k \quad \forall n \geq n_k \quad diam(f^n\beta) < d(\partial(cl_{U}V_k), K),
\end{equation}

where $d(\partial(cl_{U}V_k), K) = \inf\{d(x,y) : x \in \partial(cl_{U}V_k), y \in K\}$.

Since $K$ is invariant, we have that $f^n\beta(0) \in K$. Therefore $f^n\beta$ can not intersect $\partial(cl_{U}V_k)$ if $n \geq n_k$. Since $\lim_{t \to 0} f^n\beta(t) = f^n_*(b_0) = b_0$ in $B(U)$, we have that for $n \geq n_k$, $f^n\beta \subset S_0$ for all $n$.

\begin{equation}
\forall k \quad \exists n_k > k \quad \forall n \geq n_k \quad diam(f^n\beta) < d(\partial(cl_{U}V_k), K).
\end{equation}
$f^n\beta(t) \in V_k$ for some $t > 0$. This implies by connectedness that $f^n\beta = f^n\beta([0,1]) \subset V_k$ and hence $b_0 = b_1$.

For $k > k_1$, $n \geq n_k$ we have that $f^n\beta_\ast$ is a simple closed curve contained in $V_k^\ast$ which separates $D_0$ and also $B(U)$ in two connected components. Let $A_n$ be the component of $B(U) - f^n\beta_\ast$ contained in $V_k^\ast$. Let $B_n := D_0 - cl_{B(U)}A_n$ be the other component of $D_0 - f^n\beta_\ast$. We have that $A_n$ is homeomorphic to an open ball.

Since $\lim_n \diam f^n(\beta) = 0$ we have that $\lim_n \diam A_n = 0$ and then
\begin{equation}
\exists k_2 > k_1 \quad \forall n \geq n_{k_2} \quad f_* : A_n \to f_*(A_n) \subset D_0 \text{ is a homeomorphism.}
\end{equation}

Since $L$ is a stable branch, the set $cl_S(\cup_{n \geq 0} f^n \beta)$ is compact, and hence it has finite $\mu$-measure. Since the points in $fr_U A_n \subset f^n \beta \subset L$ are wandering we have that
$$\mu(fr_U A_n) \leq \mu(f^n \beta) = \mu(\beta) = 0.$$ Observe that $fr_B(U) A_n = fr_U A_n \cup \{b_0\}$, and hence $\mu(fr_B(U) A_n) = 0$. Therefore
\begin{equation}
\mu(D_0 - A_n) = \mu(D_0 - cl_B(U) A_n) = \mu(B_n).
\end{equation}

Since $V_{k_1} \subset D_0$, by (19),
\begin{equation}
\exists k_3 > k_2 \quad \forall k \geq k_3 \quad \mu(V_k) < \mu(D_0 - V_k).
\end{equation}

If $k \geq k_3$ and $n \geq n_k$, then $A_n \subset V_k^\ast$. Therefore by (19) we can extract a subsequence $A_n$ such that $\mu(A_n)$ is decreasing and
\begin{equation}
\lim_n \mu(A_n) = 0.
\end{equation}

Using (23) and (22), we have that
$$\forall n \geq n_{k_3} \quad \mu(A_n) < \mu(D_0 - A_n) = \mu(B_n) \leq \mu(B_{n+1}).$$

Using (21) we have that for $n \geq n_{k_3}$,
\begin{align*}
(25) & \quad f_* (A_n) \text{ is connected.} \\
(26) & \quad f_* (A_n) \subset B(U) - f^{n+1} \beta_\ast. \\
(27) & \quad fr_B(U) f_* (A_n) \subset f_* (fr_B(U) A_n) = f^{n+1} \beta_\ast. \\
(28) & \quad \mu(f_* (A_n)) = \mu(A_n) < \mu(B_n) \leq \mu(B_{n+1}).
\end{align*}

4.4. **Claim.** If $f_* (A_n) \not\subset A_{n+1}$ then $f_* (A_n) \cap D_0$ would contain $D_0 - cl_{B(U)} A_{n+1} = B_{n+1}$.

But the thesis in Claim 4.4 contradicts (28). Therefore
\begin{equation}
(29) \quad f_* (A_n) \subset A_{n+1}.
\end{equation}

**Proof of Claim 4.4:**

Since $fr_B(U) A_{n+1} = f^{n+1} \beta_\ast$, from (25) and (27) we have that either $f_* (A_n) \subset A_{n+1}$ or $f_* (A_n) \subset B(U) - A_{n+1}$. We are assuming that $f_* (A_n) \not\subset A_{n+1}$ then $f_* (A_n) \subset B(U) - A_{n+1}$. Moreover by (26), $f_* (A_n) \subset B(U) - cl_B(U)A_{n+1}$. Since $b_0 \in D_0 \cap cl_B(U) f_* (A_n)$, there is $x_0 \in D_0 \cap f_* (A_n) \subset D_0 \cap f_* (A_n) - cl_B(U)A_{n+1}$.
Given $x_1 \in D_0 - \text{cl}_{B(U)} A_{n+1}$ there is a path $\gamma : [0, 1] \to D_0 - \text{cl}_{B(U)} A_{n+1}$ with $\gamma(i) = x_i$, $i = 0, 1$. Let $t = \sup\{s \in [0, 1] : \gamma(s) \in f_s(A_n)\}$. Since $x_0 \in f_s(A_n)$, $t > 0$. If $t < 1$ then by (27),
\[ \gamma(t) \in fr_{B(U)} f_s(A_n) \subset f^{n+1}_s \beta_s = fr_{B(U)} A_{n+1}, \]
which contradicts the choice of $\gamma$. Therefore $x_1 \in D_0 \cap f_s(A_n)$.
\[ \triangle \]

From (29) and (24) we have that
\[ \forall n > n_k_3 \quad \mu(A_{n+1}) \geq \mu(f_s(A_n)) = \mu(A_n) > 0 \quad \text{and} \quad \lim_n \mu(A_n) = 0. \]
A contradiction. This finishes the proof of Theorem 4.3.
\[ \square \]

References

[1] Lars V. Ahlfors and Leo Sario, *Riemann surfaces*, Princeton Mathematical Series, No. 26, Princeton University Press, Princeton, N.J., 1960.

[2] Gonzalo Contreras and Marco Mazzucchelli, *Existence of Birkhoff sections for Kupka-Smale Reeb flows of closed contact 3-manifolds*, Geom. Funct. Anal. 32 (2022), no. 5, 951–979.

[3] Gonzalo Contreras and Fernando Oliveira, *Existence of Birkhoff sections for Kupka-Smale Reeb flows of closed contact 3-manifolds*, Geom. Funct. Anal. 32 (2022), no. 5, 951–979.

[4] John Franks and Patrice Le Calvez, *Regions of instability for non-twist maps*, Ergodic Theory Dynam. Systems 23 (2003), no. 1, 111–141.

[5] Béla Kerékjártó, *Vorlesungen über Topologie I*, Die Grundlehren der Mathematischen Wissenschaften, vol. 8, Springer, Berlin, Heidelberg, 1923.

[6] John N. Mather, *Invariant subsets for area preserving homeomorphisms of surfaces*, Mathematical analysis and applications, Part B, Academic Press, New York-London, 1981, pp. 531–562.

[7] ________, *Topological proofs of some purely topological consequences of Carathéodory’s theory of prime ends*, Selected studies: physics-astrophysics, mathematics, history of science, North-Holland, Amsterdam-New York, 1982, pp. 225–255.

[8] Fernando Oliveira and Gonzalo Contreras, *No elliptic points from fixed prime ends*, Preprint, 2022.

[9] Ian Richards, *On the classification of noncompact surfaces*, Trans. Amer. Math. Soc. 106 (1963), 259–260.

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