POLYNOMIAL PARTITIONING FOR SEVERAL SETS OF VARIETIES

PAVLE V. M. BLAGOJEVIĆ, ALEKSANDRA S. DIMITRIJEVIĆ BLAGOJEVIĆ, AND GÜNTER M. ZIEGLER

Abstract. We give a new, systematic proof for a recent result of Larry Guth and thus also extend the result to a setting with several families of varieties: For any integer $D \geq 1$ and any collection of sets $\Gamma_1, \ldots, \Gamma_i$ of low-degree $k$-dimensional varieties in $\mathbb{R}^n$ there exists a non-zero polynomial $p \in \mathbb{R}[X_1, \ldots, X_n]$ of degree at most $D$ such that each connected component of the complement $\mathbb{R}^n \setminus Z(p)$ intersects $O(D^{k-n}|\Gamma|)$ varieties of $\Gamma_i$, simultaneously for every $1 \leq i \leq j$. For $j = 1$ we recover the original result by Guth. Our proof, via an index calculation in equivariant cohomology, shows how the degrees of the polynomials used for partitioning are dictated by the topology, namely by the Euler class being given in terms of a top Dickson polynomial.

1. Introduction

The celebrated work [5] by Larry Guth and Nets Hawk Katz on the Erdős distinct distances problem in the plane brought to light the following beautiful partitioning result:

**Theorem 1.1** (Guth and Katz 2015 [5, Thm. 4.1]). Let $X$ be a finite set of points in $\mathbb{R}^n$, and let $D \geq 1$ be an integer. Then there exists a non-zero polynomial $p \in \mathbb{R}[X_1, \ldots, X_n]$ of degree at most $D$ such that each connected component of the complement $\mathbb{R}^n \setminus Z(p)$ contains at most $C_n D^{-n}|X|$ points of $X$, where $C_n$ is a constant that may depend on $n$.

Here $Z(p)$ denotes the set of zeroes in $\mathbb{R}^n$ of the polynomial $p$, that is

$$Z(p) = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : p(x_1, \ldots, x_n) = 0\}.$$  

In his recent paper [4], Guth used equivariant topology to prove the following extended polynomial partitioning result:

**Theorem 1.2** (Guth, 2015 [4, Thm. 0.3]). Let $\Gamma$ be a finite set of $k$-dimensional varieties in $\mathbb{R}^n$, each of them defined by at most $m$ polynomial equations of degree at most $d$. Then for any $D \geq 1$ there exists a non-zero polynomial $p \in \mathbb{R}[X_1, \ldots, X_n]$ of degree at most $D$ such that each connected component of the complement $\mathbb{R}^n \setminus Z(p)$ intersects at most $C(d, m, n) D^{k-n}|\Gamma|$ varieties in $\Gamma$, where $C(d, m, n)$ is a constant that may depend on the parameters $d$, $m$, and $n$.

In this paper, based on the set-up from the proof of the previous theorem and the use of the Fadell–Husseini index theory [3] for the proof of a necessary Borsuk–Ulam type theorem, we make the next extension step by proving the following “colored” generalization of Theorem 1.2.

**Theorem 1.3.** Let $j \geq 1$ be an integer. For $1 \leq i \leq j$, let $\Gamma_i$ be a finite set of $k_i$-dimensional varieties in $\mathbb{R}^n$, each of them defined by at most $m_i$ polynomial equations of degree at most $d_i$. Then for any $D \geq 1$ there exists a non-zero polynomial $p \in \mathbb{R}[X_1, \ldots, X_n]$ of degree at most $D$ such that each connected component of the complement $\mathbb{R}^n \setminus Z(p)$ for every $1 \leq i \leq j$ intersects at most $C(d_i, m_i, n) D^{k_i-n}|\Gamma_i|$ varieties in $\Gamma_i$, where $C(d_i, m_i, n)$ is a constant that may depend on parameters $d_i$, $m_i$, and $n$.

In a concrete example, this says the following: There are constants $C_1 = C(1, 2, 3)$ and $C_2 = C(1, 3, 3)$ such that if we have (large) collections $\Gamma_1$ of red lines and $\Gamma_2$ of blue points in $\mathbb{R}^3$, then for every $D \geq 1$ there is a nonzero polynomial $p(x, y, z) \in \mathbb{R}[x, y, z]$ of degree at most $D$ such that each connected component of $\mathbb{R}^3 \setminus Z(p)$ meets at most $2C_1 \frac{|\Gamma_1|}{D^2}$ red lines, and at most $2C_2 \frac{|\Gamma_2|}{D^2}$ blue points. (This is the special case when we have $j = 2$ families of varieties in $\mathbb{R}^3$, so $n = 3$, where the first family consists of lines, so $k_1 = 1$ and e.g. $m_1 = 2, d_1 = 1$, and the second one of points, so $k_2 = 0, m_2 = 3, d_2 = 1$.)
Note added in revision. In his recent paper [8], Ben Yang used Theorem 1.3 to obtain new upper bounds on the number of generalized joints for (possibly higher-dimensional) varieties.

2. Proof of Theorem 1.3

The proof of Theorem 1.3 will have several separate components that at the end of the proof merge into the final argument. We also rely on two particular results by József Solymosi and Terence Tao [7, Thm. A.2] and by Guth [4, Lemma 3.1].

2.1. Let $P_n^δ$ be the vector space of polynomials in $n$ variables of degree at most $δ$ with real coefficients,

$$P_n^δ = \{ p \in \mathbb{R}[X_1, \ldots , X_n] : \deg p \leq δ \}.$$

The dimension of this vector space is $\dim P_n^δ = \binom{δ+n}{n}$. For every integer $ℓ \geq 1$ choose the smallest integer $δ_ℓ$ with the property that

$$j 2^{ℓ−1} ≤ \frac{δ_ℓ}{n} < j 2^n2^{ℓ−1},$$

or equivalently

$$(n!)^{\frac{1}{2(2+\frac{1}{δ})}} ≤ δ_ℓ < 2(n!)\frac{1}{2(2+\frac{1}{δ})} j 2^n2^{ℓ−1}. \tag{1}$$

In particular, $\dim P_n^δ > j 2^{ℓ−1}$. Next, let $s$ be the smallest integer such that

$$\sum_{ℓ=1}^s δ_ℓ ≤ D < \sum_{ℓ=1}^{s+1} δ_ℓ. \tag{2}$$

The inequalities (1) and (2) imply that

$$D < \sum_{ℓ=1}^{s+1} δ_ℓ < 2(n!)\frac{1}{2(2+\frac{1}{δ})} \sum_{ℓ=1}^{s+1} 2^{ℓ−1} = 2(n!)\frac{1}{2(2+\frac{1}{δ})} \frac{2^{s+1}−1}{2−1} < \frac{2(n!)\frac{1}{2(2+\frac{1}{δ})}}{2−1} j 2^n2^{s+1} = \frac{2^{s+1}−1}{2−1} j 2^n2^{s+1}. \tag{3}$$

Consequently, the inequality (3) gives

$$D^n < \frac{2^{n+1}n!}{(2+1−1)n} j 2^n \implies \frac{1}{2^n} < \frac{2^{n+1}n!}{(2+1−1)n} \frac{D}{j} = C_n \frac{j}{D^n}, \tag{4}$$

where $C_n$ depends only on $n$.

2.2. For every $1 ≤ ℓ ≤ s$ we have that $\dim P_n^{δ_ℓ} ≥ j 2^{ℓ−1} + 1$. Let $V_ℓ$ denote an arbitrary vector subspace of $P_n^{δ_ℓ}$ of dimension $j 2^{ℓ−1} + 1$. The unit sphere $S(V_ℓ)$ in the vector space $V_ℓ$ is equipped with the free $\mathbb{Z}/2 = \langle ω_ℓ \rangle$ action given by $ω_ℓ \cdot p = −p$ for $p \in S(V_ℓ)$.

We will use the product space

$$Y := \prod_{ℓ=1}^s S(V_ℓ) ≃ \prod_{ℓ=1}^s S^j2^{ℓ−1}.$$

The elementary abelian group $(\mathbb{Z}/2)^s = \langle ω_1, \ldots , ω_s \rangle$ acts on $Y$ componentwise, that is, for $1 ≤ ℓ ≤ s$ and $(p_1, \ldots , p_t, \ldots , p_s) \in Y$ the generator $ω_ℓ$ acts as follows:

$$ω_ℓ \cdot (p_1, \ldots , p_t, \ldots , p_s) = (p_1, \ldots , −p_t, \ldots , p_s). \tag{5}$$

2.3. Consider the vector space $\mathbb{R}((\mathbb{Z}/2))^r$ and the vector subspace of codimension 1 given by

$$U_s = \left\{ (y_α)_{α \in (\mathbb{Z}/2)^r} \in \mathbb{R}((\mathbb{Z}/2))^r : \sum_{α \in (\mathbb{Z}/2)^r} y_α = 0 \right\}.$$

We introduce the following action of $(\mathbb{Z}/2)^s$ on $\mathbb{R}((\mathbb{Z}/2))^r$: The element $(β_1, \ldots , β_s) ∈ (\mathbb{Z}/2)^s$ acts on the vector $(y_α)_{α \in (\mathbb{Z}/2)^r} ∈ \mathbb{R}((\mathbb{Z}/2))^r$ by acting on its index set

$$(β_1, \ldots , β_s) \cdot (α_1, \ldots , α_s) = (β_1 + α_1, \ldots , β_s + α_s), \tag{6}$$

where the addition is assumed to be in $\mathbb{Z}/2$. With respect to the introduced action the vector subspace $U_s$ is a $(\mathbb{Z}/2)^s$-subrepresentation of $\mathbb{R}((\mathbb{Z}/2))^r$ of dimension $2^s − 1$. 
2.4. Any non-constant polynomial $p \in \mathbb{R}[X_1, \ldots, X_n]$ determines two disjoint open regions in $\mathbb{R}^n$, possibly one of them empty, which we denote by

$$D_p^0 = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : p(x_1, \ldots, x_n) > 0\} \quad \text{and} \quad D_p^1 = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : p(x_1, \ldots, x_n) < 0\}.$$ 

Thus $\mathbb{R}^n \setminus \{p\} = D_p^0 \cup D_p^1$ and $D_p^0 \cap D_p^1 = \emptyset$.

Let $(p_1, \ldots, p_s) \in Y$ be an ordered tuple of polynomials in $Y$, and let $\alpha = (\alpha_1, \ldots, \alpha_s) \in (\mathbb{Z}/2)^s = \{0, 1\}^s$. The sign pattern domain determined by the tuple $(p_1, \ldots, p_s)$ and the element $\alpha \in (\mathbb{Z}/2)^s$ is the intersection of open regions

$$\mathcal{O}_\alpha^{(p_1, \ldots, p_s)} = D_{p_1}^{\alpha_1} \cap \cdots \cap D_{p_s}^{\alpha_s}.$$ 

An sign pattern domain can be empty. Moreover

$$\mathbb{R}^n \setminus (p_1 \cdots p_s) = \bigcup_{\alpha \in (\mathbb{Z}/2)^s} \mathcal{O}_\alpha^{(p_1, \ldots, p_s)}, \quad (7)$$

where the union is disjoint union. Observe that $\deg(p_1 \cdots p_s) \leq \sum_{\ell = 0}^{s} \delta_\ell \leq D$. Furthermore, the sign pattern domains $\mathcal{O}_\alpha^{(p_1, \ldots, p_s)}$ are unions of connected components of the complement $\mathbb{R}^n \setminus (p_1 \cdots p_s)$.

2.5. For every $\alpha \in (\mathbb{Z}/2)^s$ and every variety $\gamma \subset \mathbb{R}^n$ we define the function $\phi_{\alpha, \gamma} : Y \rightarrow \mathbb{R}$ by

$$\phi_{\alpha, \gamma}(p_1, \ldots, p_s) = \begin{cases} 1, & \text{if } \mathcal{O}_{\alpha}^{(p_1, \ldots, p_s)} \cap \gamma \neq \emptyset \\ 0, & \text{if } \mathcal{O}_{\alpha}^{(p_1, \ldots, p_s)} \cap \gamma = \emptyset, \end{cases}$$

where $(p_1, \ldots, p_s) \in Y$. The functions $\phi_{\alpha, \gamma}$ are not continuous, but as Guth showed in [4, Lemma 3.1], they can be approximated by sequences of continuous functions:

**Lemma 3.1.** For $\varepsilon > 0$, $\gamma \subset \mathbb{R}^n$ and $\alpha \in (\mathbb{Z}/2)^s$, we define functions $\phi_{\alpha, \gamma}^\varepsilon : Y \rightarrow \mathbb{R}$ with the following properties.

1. The functions $\phi_{\alpha, \gamma}^\varepsilon : Y \rightarrow \mathbb{R}$ are continuous.
2. $0 \leq \phi_{\alpha, \gamma}^\varepsilon \leq 1$.
3. If $\mathcal{O}_{\alpha}^{(p_1, \ldots, p_s)} \cap \gamma = \emptyset$, then $\phi_{\alpha, \gamma}^\varepsilon = 0$.
4. If $\varepsilon_i \rightarrow 0$ and $(p_1, \ldots, p_i) \rightarrow (p_1, \ldots, p_s)$ in $Y$, and $\mathcal{O}_{\alpha}^{(p_1, \ldots, p_s)} \cap \gamma \neq \emptyset$, then

$$\lim_{i \rightarrow \infty} \phi_{\alpha, \gamma}^\varepsilon(p_1, \ldots, p_i) = 1.$$ 

In other words, $\phi_{\alpha, \gamma}(p_1, \ldots, p_s) \leq \liminf_{i \rightarrow \infty} \phi_{\alpha, \gamma}^\varepsilon(p_1, \ldots, p_i)$.

In order to simplify the presentation we postpone the typical compactness argument applied to $Y$ to the last step in the proof of 2.3 and for now continue to work with the functions $\phi_{\alpha, \gamma}$ as if they were continuous.

2.6. Let $1 \leq i \leq j$ be fixed. By assumption $\Gamma_i$ is a finite set of $k_i$-dimensional varieties in $\mathbb{R}^n$, each defined by at most $m_i$ polynomial equations of degree at least $d_i$. Consider the following map from the space $Y$ to the representation $U^{\mathbb{Z}/2}_s$ associated to the collection of finite sets of varieties $\Gamma_1, \ldots, \Gamma_j$ given in the theorem:

$$\Phi : Y \rightarrow U^{\mathbb{Z}/2}_s,$$

$$(p_1, \ldots, p_s) \mapsto \left( \left( \sum_{\gamma \in \Gamma_i} \phi_{\alpha, \gamma}(p_1, \ldots, p_s) - \frac{1}{2^s} \sum_{\beta \in (\mathbb{Z}/2)^s} \sum_{\gamma \in \Gamma_i} \phi_{\beta, \gamma}(p_1, \ldots, p_s) \right)_{\alpha \in (\mathbb{Z}/2)^s} \right)_{i \in \{1, \ldots, j\}}. \quad (8)$$

The sum $\sum_{\gamma \in \Gamma_i} \phi_{\alpha, \gamma}(p_1, \ldots, p_s)$ counts the number of varieties in $\Gamma_i$ that intersect sign pattern domain $\mathcal{O}_{\alpha}^{(p_1, \ldots, p_s)}$. The map $\Phi$ is continuous and $(\mathbb{Z}/2)^s$-equivariant with respect to the actions given by $\Phi$ and $\Phi$; assuming the diagonal action on the direct sum $U^{\mathbb{Z}/2}_s$.

Let us assume that $\Phi^{-1}(0) \neq \emptyset$, and pick $(p_1, \ldots, p_s) \in \Phi^{-1}(0)$. Then each of $2^s$ sign pattern domains $\mathcal{O}_{\alpha}^{(p_1, \ldots, p_s)}$, $\alpha \in (\mathbb{Z}/2)^s$, determined by the tuple $(p_1, \ldots, p_s)$ in $Y$ intersects the same number of varieties in the set $\Gamma_i$, for every $1 \leq i \leq j$. We use the following result of Solymosi and Tao [7, Thm. A.2], stated as in [3, Thm. 0.2]:

**Theorem A.2.** Suppose that $\gamma$ is a $k$-dimensional variety in $\mathbb{R}^n$ defined by $m$ polynomial equations each of degree at most $d$. If $p$ is a polynomial of degree at most $D$, then $\gamma$ intersects at most $C(d, m, n)D^k$ different connected components of $\mathbb{R}^n \setminus \{p\}$, where $C(d, m, n)$ is a constant that may depend on the parameters $d_i$, $m_i$, and $n$. 

**Proof.** (Sketch)

1. Let $\gamma$ be a $k$-dimensional variety in $\mathbb{R}^n$.
2. $p$ is a polynomial of degree at most $D$.
3. $\gamma$ intersects at most $C(d, m, n)D^k$ different connected components of $\mathbb{R}^n \setminus \{p\}$.
4. $C(d, m, n)$ is a constant that may depend on the parameters $d_i$, $m_i$, and $n$.

**Conclusion:** Suppose that $\gamma$ is a $k$-dimensional variety in $\mathbb{R}^n$ defined by $m$ polynomial equations each of degree at most $d$. If $p$ is a polynomial of degree at most $D$, then $\gamma$ intersects at most $C(d, m, n)D^k$ different connected components of $\mathbb{R}^n \setminus \{p\}$, where $C(d, m, n)$ is a constant that may depend on the parameters $d_i$, $m_i$, and $n$. 

**End of Proof.**
Hence, each variety $\gamma \in \Gamma_1$ intersects at most $C'(d_i, m_i, n)D^{k_i}$ connected components of the complement $\mathbb{R}^n \setminus Z(p_1 \cdots p_s)$. Since each sign pattern domain $O^{(p_1, \cdots, p_s)}_0$ is a disjoint union of connected components of $\mathbb{R}^n \setminus Z(p_1 \cdots p_s)$, each variety $\gamma \in \Gamma_1$ intersects at most $C'(d_i, m_i, n)D^{k_i}$ sign pattern domains. As we are looking at a point $(p_1, \ldots, p_s) \in \Phi^{-1}(0)$, where we get the same number of varieties $\gamma \in \Gamma_1$ intersecting each sign pattern domain $O^{(p_1, \cdots, p_s)}$; this number is at most $\frac{1}{2^n |\Gamma_1|} C'(d_i, m_i, n)D^{k_i}$. The inequality (4) implies that

$$\frac{1}{2^n |\Gamma_1|} C'(d_i, m_i, n)D^{k_i} < C_n C'(d_i, m_i, n)D^{k_i-n} |\Gamma_1|.$$ 

Each connected component of $\mathbb{R}^n \setminus Z(p_1 \cdots p_s)$ is contained in a unique sign pattern domain, and therefore the number of varieties $\gamma \in \Gamma_1$ intersecting a connected component of $\mathbb{R}^n \setminus Z(p_1 \cdots p_s)$ cannot exceed

$$C(d_i, m_i, n) \cdot jD^{k_i-n} |\Gamma_1|,$$

where $C(d_i, m_i, n)$ is a constant that may depend on the parameters $d_i$, $m_i$, and $n$. This concludes the proof of Theorem 4.3 except that it remains to be verified that the map $\Phi$ indeed has a zero.

2.7. We still need to prove that the $(\mathbb{Z}/2)^s$-equivariant map $\Phi: Y \rightarrow U_{s \mathbb{Z}}^{\mathbb{Z}}$, defined in (8), has a zero. Indeed, in the spirit of the usual resolution of “configuration space/test map schemes” for discrete geometry problems [6], we will show that there is no continuous $(\mathbb{Z}/2)^s$-equivariant map $Y \rightarrow U_{s \mathbb{Z}}^{\mathbb{Z}}$ at all that avoids zero.

Let us assume for the contrary that there is such a map, then this induces a continuous $(\mathbb{Z}/2)^s$-equivariant map $Y \rightarrow S(U_{s \mathbb{Z}}^{\mathbb{Z}})$, where $S(U_{s \mathbb{Z}}^{\mathbb{Z}})$ denotes the unit sphere in the vector spaces $U_{s \mathbb{Z}}^{\mathbb{Z}}$. Using the Fadell–Husseini ideal-valued index theory [2] for the group $(\mathbb{Z}/2)^s$ and $F_2$ coefficients, we will prove that such an equivariant map cannot exist, obtaining the required contradiction.

The cohomology of the group $(\mathbb{Z}/2)^s$ with $F_2$ coefficients is given by $H^*(((\mathbb{Z}/2)^s; F_2) \cong F_2[u_1, \ldots, u_s]$, where $\deg(u_i) = 1$ and the variable $u_i$ corresponds to the generator $\omega_i$ for $1 \leq i \leq s$. According to [2 Ex. 3.3],

$$\text{Index}_{(\mathbb{Z}/2)^s}(Y; F_2) = \langle u_1^{i_1} u_2^{i_2} \cdots u_s^{i_s} \mid 0 \leq i_1, \ldots, i_s \leq 2^{s-1} \rangle.$$ 

Furthermore, from [2 Prop. 13.1] (see also [2 Prop. 3.12 and Prop. 3.13]) we have that

$$\text{Index}_{(\mathbb{Z}/2)^s}(S(U_{s \mathbb{Z}}^{\mathbb{Z}}); F_2) = \langle \prod_{(a_1, \ldots, a_s) \in (\mathbb{Z}/2)^s \setminus \{0\}} (a_1 u_1 + \cdots + a_s u_s) \rangle.$$ 

Since a continuous $(\mathbb{Z}/2)^s$-equivariant map $Y \rightarrow S(U_{s \mathbb{Z}}^{\mathbb{Z}})$ exists, the basic property of the Fadell–Husseini index [3 Sec. 2] implies that

$$\text{Index}_{(\mathbb{Z}/2)^s}(S(U_{s \mathbb{Z}}^{\mathbb{Z}}); F_2) \subseteq \text{Index}_{(\mathbb{Z}/2)^s}(Y; F_2),$$

and consequently that

$$\prod_{(a_1, \ldots, a_s) \in (\mathbb{Z}/2)^s \setminus \{0\}} (a_1 u_1 + \cdots + a_s u_s) \in \langle u_1^{j_1} u_2^{j_2} \cdots u_s^{j_s} \rangle. \tag{9}$$

The polynomial

$$q := \prod_{(a_1, \ldots, a_s) \in (\mathbb{Z}/2)^s \setminus \{0\}} (a_1 u_1 + \cdots + a_s u_s) \in F_2[u_1, \ldots, u_s].$$

is the Dickson polynomial of maximal degree [1 Sec. III.2]. It can be presented in the form

$$q = \sum_{\pi \in \mathfrak{S}_s} u_{\pi(1)}^{2^{s-1}} u_{\pi(2)}^{2^{s-2}} \cdots u_{\pi(s)}^{2^0}.$$ 

Now the $j$-th power of the Dickson polynomial $q^j$ can be decomposed as follows

$$q^j = \left( \prod_{(a_1, \ldots, a_s) \in (\mathbb{Z}/2)^s \setminus \{0\}} (a_1 u_1 + \cdots + a_s u_s) \right)^j = \left( \sum_{\pi \in \mathfrak{S}_s} u_{\pi(1)}^{2^{s-1}} u_{\pi(2)}^{2^{s-2}} \cdots u_{\pi(s)}^{2^0} \right)^j$$

$$= \langle u_s^{2^{s-1}} u_{s-1}^{2^{s-2}} \cdots u_1^{2^0} \rangle + \text{Rest},$$

where “Rest” denotes a polynomial that does not contain the monomial $u_s^{2^{s-1}} u_{s-1}^{2^{s-2}} \cdots u_2^{2^0} u_1^0$. Hence,

$$q^j \notin \langle u_1^{j_1} u_2^{j_2} \cdots u_s^{2^{s-1}+1} \rangle, \tag{10}$$

in contradiction to relation (9).
2.8. We just proved that there cannot be any continuous \((\mathbb{Z}/2)^s\)-equivariant map \(Y \to S(U^{|J|})\). Therefore, every continuous \((\mathbb{Z}/2)^s\)-equivariant map \(Y \to U^{|J|}\) has a zero.

More is true: Since \(Y\) is compact, every \((\mathbb{Z}/2)^s\)-equivariant map \(Y \to U^{|J|}\), not necessarily continuous, which is a limit of a sequence of continuous \((\mathbb{Z}/2)^s\)-equivariant maps \(Y \to U^{|J|}\), will also have a zero. Indeed, let \(\Psi := \lim_{i \to \infty} \Psi^i\) where \(\Phi^i: Y \to U^{|J|}\) are continuous \((\mathbb{Z}/2)^s\)-equivariant maps. Since the maps \(\Psi^i\) are continuous, for every \(i\) there exists a \(y_i \in Y\) such that \(\Psi^i(y_i) = 0\). The compactness of \(Y\) yields the existence of a converging subsequence \(\lim_{i \to \infty} y_{k(i)} = y \in Y\), where \(k: \mathbb{N} \to \mathbb{N}\) is a strictly increasing function. Thus \(\Psi(y) = \lim_{i \to \infty} \Psi^k(i)(y_{k(i)}) = 0\).

We have proved in (2.5) that the maps \(\phi_{\alpha, \gamma}\), used for the definition of the map \(\Phi\), are limits of sequences of continuous maps. Thus the \((\mathbb{Z}/2)^s\)-equivariant map \(\Phi: Y \to U^{|J|}\), even not continuous by construction, still has a zero. The proof of Theorem 1.3 is now complete.

Remark. The choice of the degree bounds \(\delta_\ell\) for the polynomials \(p_\ell\) used for partitioning, and consequently of the vector spaces \(V_\ell\) etc., which in the special case \(j = 1\) already appeared in Guth’s work [4], can now be seen as very natural if one tries to show that at least one monomial in the power of the Dickson polynomial \(q\) does not belong to the index of the configuration space \(Y\) of \(s\)-tuples of polynomials \((p_1, \ldots, p_s)\), and thus to obtain a contradiction in (10).

Acknowledgements. We are grateful to Josh Zahl and to the referee of JFPTA for very valuable remarks.

References

[1] Alejandro Adem and James R. Milgram, Cohomology of Finite Groups, 2nd ed., Springer, Berlin, 2004.
[2] Pavle V. M. Blagojević and Günter M. Ziegler, The ideal-valued index for a dihedral group action, and mass partition by two hyperplanes, Topology and its Applications 158 (2011), no. 12, 1326–1351.
[3] Edward R. Fadell and Sufian Y. Hussein, An ideal-valued cohomological index theory with applications to Borsuk–Ulam and Bourgin–Yang theorems, Ergodic Theory Dynam. Systems 8* (1988), 73–85.
[4] Larry Guth, Polynomial partitioning for a set of varieties, Mathematical Proceedings of the Cambridge Philosophical Society 150 (2015), 459–469.
[5] Larry Guth and Nets Hawk Katz, On the Erdős distinct distances problem in the plane, Annals of Mathematics (2) 181 (2015), no. 1, 155–190.
[6] Jiří Matoušek, Using the Borsuk–Ulam Theorem. Lectures on Topological Methods in Combinatorics and Geometry, Universitext, Springer, Heidelberg, 2003.
[7] József Solymosi and Terence Tao, An incidence theorem in higher dimensions, Discrete and Computational Geometry 48 (2012), no. 2, 255–280.
[8] Ben Yang, Generalizations of joints problem, Preprint, 20 pages, June 2016, arXiv:1606.08529