EXPONENTIAL STABILITY OF SYSTEMS OF VECTOR DELAY DIFFERENTIAL EQUATIONS WITH APPLICATIONS TO SECOND ORDER EQUATIONS

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Abstract. Various results and techniques, such as Bohl-Perron theorem, a priori solution estimates, M-matrices and the matrix measure, are applied to obtain new explicit exponential stability conditions for the system of vector functional differential equations

\[
\dot{x}_i(t) = A_i(t)x_i(h_i(t)) + \sum_{j=1}^{n} \sum_{k=1}^{m_{ij}} B_{kj}(t)x_j(h_{kj}(t)) + \sum_{j=1}^{n} \int_{g_{ij}(t)}^{t} K_{ij}(t,s)x_j(s)ds, \quad i = 1, \ldots, n.
\]

Here \(x_i\) are unknown vector functions, \(A_i, B_{kj}, K_{ij}\) are matrix functions, \(h_i, h_{kj}, g_{ij}\) are delayed arguments. Using these results, we deduce explicit exponential stability tests for second order vector delay differential equations.

Keywords: Exponential stability; differential systems with matrix coefficients and a distributed delay; second order vector delay differential equations; Bohl-Perron theorem; matrix measure; M-matrices.

AMS(MOS) subject classification: 34K20, 34K06, 34K25.

1. Introduction

Exponential stability of vector, scalar and higher order delay differential equations has attracted a lot of attention, see [2, 3, 4, 8, 11, 12, 13, 14, 15, 16] and the bibliography therein. Considered models included neutral equations and systems with both concentrated and distributed delays. However, for two classes of linear functional differential equations there are only few stability results. These types are systems of several linear delay differential equations of the first order with matrix coefficients and systems, where at least one equation is of the second or higher order. Scientific interest to these models is not purely theoretical. There are many interesting real-world applications of such systems. For example, the ordinary differential vector equation of the second order

\[
\ddot{x}(t) + A(t)\dot{x}(t) + B(t)x(t) = 0
\]

arises in control theory [9, 17]. Explicit asymptotic stability conditions for vector ordinary differential equations of the second order were obtained in [9, 10, 17, 19, 22]. Delay differential vector equations of the second order are natural generalizations of (1.1) that also can be applied in control theory. However, presently this application is not practical due to lack of qualitative results for this class of equations. We can mention only the paper [20], where for the nonlinear vector equation

\[
\ddot{x}(t) + F(x(t), \dot{x}(t))\dot{x}(t) + H(x(t - \tau)) = L(t)
\]

\[
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\]
asymptotic stability of the zero solution and the boundedness of all solutions of equation (1.2) were investigated. One of the goals of this paper is to fill this gap, by studying exponential stability of linear and nonlinear vector delay differential equations of the second order.

The paper is organized as follows. Section 2 contains relevant definitions, notations and auxiliary statements. In Section 3, we obtain the main result of the paper on exponential stability for linear systems of functional differential equations with matrix coefficients. We also give several corollaries, one of them will further be applied to second order vector equations. Section 4 deals with exponential stability for a linear delay differential equation of the second order with matrix coefficients. A stability test obtained here is one of the first stability results for this class of equations. As a corollary, we deduce new stability conditions for non-delay vector equation (1.1).

In Section 5, we compare stability results of the present paper with known ones. Section 5 also includes some open problems and topics for future research.

2. Preliminaries

The main object of the present paper is (2.1)

\[ \dot{x}_i(t) = A_i(t)x_i(h_i(t)) + \sum_{j=1}^{n} \sum_{k=1}^{m_{ij}} B_{ij}^k(t)x_j(h_{ij}^k(t)) + \sum_{j=g_{ij}}^{n} \int_{t_{g_{ij}}}^{t} K_{ij}(t, s)x_j(s) ds, \quad i = 1, \ldots, n, \quad t \geq t_0 \geq 0, \]

where \( x : \mathbb{R} \rightarrow \mathbb{R}^d, \quad i = 1, \ldots, n \) are unknown vector-functions, \( A_i, B_{ij}^k, K_{ij} \) are given \( d \times d \) matrices with Lebesgue measurable essentially bounded on the semi-axis \([0, \infty)\) entries, and the delays \( h_i, h_{ij}^k, g_{ij} \) are Lebesgue measurable functions.

Let us denote by \( \| \cdot \| \) an arbitrary vector norm, the same notation will be used for the induced matrix norm, \( I \) is the identity matrix. The absolute value of the matrix is understood componentwise: if \( B = (b_{ij})_{i,j=1}^{d,d} \) then \( |B| := (|b_{ij}|)_{i,j=1}^{d,d} \), we use the same notation for vectors. Matrix or vector inequalities \( A \leq B \) or \( A < B \) are also understood componentwise: each entry satisfies the inequality. All the relations (equations and inequalities) with Lebesgue measurable functions are understood almost everywhere.

To get main results, we will utilize the matrix measure (the logarithmic norm) defined in [18, 21] and [14, Table 3.1, p. 286] as

\[ \mu(B) := \lim_{\vartheta \to 0^+} \frac{\|I + \vartheta B\| - 1}{\vartheta}. \]

In particular, for the maximum norm \( \|x\|_{\infty} := \max_{i=1,\ldots,d} |x_i| \) we have for \( B = (b_{ij})_{i,j=1}^{d,d} \)

\[ \|B\|_{\infty} := \max_{i=1,\ldots,d} \sum_{j=1}^{d} |b_{ij}|, \quad \mu_{\infty}(B) = \max_{i=1,\ldots,d} \left\{ b_{ii} + \sum_{j=1, j \neq i}^{d} |a_{ij}| \right\}. \]

The following classical definition for an \( M \)-matrix will be used.

**Definition 2.1.** [6] A matrix \( B = (b_{ij})_{i,j=1}^{n,n} \) is called a (non-singular) \( M \)-matrix if \( b_{ij} \leq 0, \quad i \neq j \), and one of the following equivalent conditions holds:

1. \( B \) is invertible, and \( B^{-1} \geq 0 \);
2. the leading principal minors of the matrix \( B \) are positive.

**Remark 2.2.** Note that for any three column vectors \( X \leq Y \) and \( Z \geq 0 \) we have \( Z^T(Y - X) \geq 0 \) as a sum of non-negative numbers. Thus, for any matrix \( A \geq 0 \) and \( X \leq Y \) with entries of arbitrary signs, we have \( AX \leq AY \).
For a fixed bounded interval \( \Omega = [t_0, t_1] \), let \( \|y\|_\Omega = \text{ess sup}_{t \in \Omega} \|y(t)\| \), also for a half-line \( \|f\|_{t_0, \infty} = \text{ess sup}_{t \geq t_0} \|f(t)\| \). Denote by \( L^d_{\infty}[t_0, t_1] \) the space of all essentially bounded on \( \Omega \) vector functions \( y \) with the norm \( \|y\|_\Omega \), the same for \( L^d_{\infty}[t_0, \infty) \) and for the spaces \( L^d_{\infty}[t_0, t_1] \), \( L^d_{\infty}[t_0, \infty) \) of matrix functions.

Assume that for a fixed \( t_0 \geq 0 \) vector-functions \( \varphi_i : (-\infty, t_0) \to \mathbb{R}^d, i = 1, \ldots, n \) are Borel measurable and bounded. Below for every \( t_0 \geq 0 \), system (2.1) will be considered with the initial condition

\[
(2.2) \quad x_i(t) = \varphi_i(t), \quad t \leq t_0, \ i = 1, \ldots, n
\]

under the hypotheses:

(i) \( A_i, B_{ij}^k, K_{ij} : [0, \infty) \to \mathbb{R}^{d \times d}, k = 1, \ldots, m_{ij}, i, j = 1, \ldots, n \) belong to \( L^d_{\infty}[0, \infty) \).

(ii) \( h_i, h_{ij}^k, g_{ij} : [0, \infty) \to \mathbb{R} \), are Lebesgue measurable functions, and there are numbers \( \tau_{ij}^k > 0, \sigma_{ij} > 0, \tau_i \geq 0 \) such that almost everywhere

\[
(2.3) \quad 0 \leq t - h_{ij}^k(t) \leq \tau_{ij}^k, \quad 0 \leq t - g_{ij}(t) \leq \sigma_{ij}, \quad 0 \leq t - h_i(t) \leq \tau_i, \quad k = 1, \ldots, m_{ij}, i, j = 1, \ldots, n, \ t \geq 0.
\]

A solution of problem (2.1), (2.2) is understood in the sense of the following definition.

**Definition 2.3.** A set of vector-functions \( x_i : \mathbb{R} \to \mathbb{R}^d, i = 1, \ldots, n \) is called a solution of problem (2.1), (2.2) if \( x_i \) satisfy (2.1) for almost all \( t \in [t_0, \infty) \) and (2.2) for all \( t \in [t_0, t_0 - \tau_i] \), where \( \tau_i = \max \left\{ \max_{j,k} \tau_{ij}^k, \max_{i,j} \tau_{ij}^k, \tau_i \right\} \), \( i = 1, \ldots, n \).

Let \( x_i, i = 1, \ldots, n \) be a solution of problem (2.1), (2.2). Introduce the matrix function \( \Phi(t) = \{\varphi_1(t), \ldots, \varphi_n(t)\} \) with \( \varphi_i \) as columns.

**Definition 2.4.** System (2.1) is called uniformly exponentially stable, if there exist positive constants \( H \) and \( \nu \), such that any solution \( X \) of (2.1), (2.2) satisfies

\[
\|x_i(t)\| \leq H e^{-\nu(t-t_0)} \sup_{t \in [t_0, t_0 - \max \tau_i]} \|\Phi(t)\|, \quad t \geq t_0 \geq 0, \ i = 1, \ldots, n,
\]

where \( H \) and \( \nu \) do not depend on \( t_0 \) and \( \Phi \).

Consider a non-homogeneous version of (2.1)

\[
\begin{align*}
\dot{x}_i(t) &= A_i(t)x_i(h_i(t)) + \sum_{j=1}^{n} \sum_{k=1}^{m_{ij}} B_{ij}^k(t)x_j(h_{ij}^k(t)) \\
&\quad + \sum_{j=1}^{n} \int_{g_{ij}(t)}^{t} K_{ij}(t,s)x_j(s)ds + f_i(t), \ i = 1, \ldots, n, \ t \geq t_0 \geq 0
\end{align*}
\]

(2.4)

and assume that \( f_i \in L^d_{\infty}[t_0, \infty) \). The definition of a solution for (2.4), (2.2) is similar to that of (2.1), (2.2).

A modification of [4, Lemma 2] will further be used.

**Lemma 2.5.** Let a function \( a : [t_0, \infty) \to [0, \infty) \) be Lebesgue measurable and \( \omega \in L^1_{\infty}[t_0, \infty) \). Then the inequality

\[
\int_{t_0}^{t} e^{-\int_{t_0}^{s} \omega(r)dr} a(s)\omega(s)ds \leq \text{ess sup}_{t \in [t_0, t_1]} \|\omega(t)\|, \quad t \in [t_0, t_1],
\]

holds for any \( t_1 > t_0 \).

Let \( x_i \) be a solution of non-homogeneous system (2.4), where \( f_i \in L^d_{\infty}[t_0, \infty) \) satisfy

\[
(2.5) \quad x_i(t) = 0, \quad t \leq t_0, \ i = 1, \ldots, n.
\]

The following Bohl-Perron type result is cited from [2, 11].
Theorem 3.1. Let \( t_0 \geq 0 \) and any \( f_i \in L^d_{\infty}[t_0, \infty), i = 1, \ldots, n \) all \( x_i \) in the solution of problem (2.4), (2.5) belong to \( L^d_{\infty}[t_0, \infty), i = 1, \ldots, n \) then system (2.1) is uniformly exponentially stable.

Below, we use the Coppel inequality [7, 21] for a system of ordinary differential equations

\[
\dot{x}(t) = D(t)x(t), \quad t \geq 0,
\]

where the columns of \( D \) are in \( L^d_{\infty}[0, \infty) \). Let \( \Phi(t, s) \) be the fundamental matrix of (2.6), i.e. a solution of the problem

\[
\dot{x}(t) = D(t)x(t), \quad t > s \geq 0, \quad x(s) = I.
\]

Lemma 2.7 (Coppel inequality). The fundamental matrix \( \Phi(t, s) \) of (2.6) satisfies

\[
\| \Phi(t, s) \| \leq \exp \left( \int_s^t \mu(D(\xi))d\xi \right), \quad t > s \geq 0.
\]

Remark 2.8. By Lemma 2.7, the condition \( \mu(D(t)) \leq d_0 < 0, t \in [t_0, \infty), \forall t_0 \geq 0 \) for the matrix measure implies uniform exponential stability of system (2.6).

The final auxiliary result gives an priori estimate for the derivative of a solution. Let \( \Omega := [t_0, t_1] \subset [t_0, \infty), t_1 > t_0 \) and

\[
C_{ij}(t) := \int_{g_{ij}(t)}^t |K_{ij}(t, s)|ds.
\]

Lemma 2.9. Let \( x_i = x_i(t), i = 1, \ldots, n \) be a solution of problem (2.4), (2.5), \( \Omega = [t_0, t_1] \). Then the derivatives \( \dot{x}_i \) satisfy

\[
\| \dot{x}_i \|_\Omega \leq \| A_i \|_{[t_0, \infty)} \| x_i \|_\Omega + \sum_{j=1}^n \left( \sum_{k=1}^{m_{ij}} \| B_{ij}^k \|_{[t_0, \infty)} + \| C_{ij} \|_{[t_0, \infty)} \right) \| x_j \|_\Omega + \| f_i \|_{[t_0, \infty)}.
\]

Proof. Let \( t \in \Omega \) and \( x = x(t) \) a solution of (2.4), (2.5). By (2.5), \( x(t) = 0 \) if \( t \leq t_0 \). Estimating the norm in the left-hand side of (2.4), we derive

\[
\| \dot{x}_i(t) \| \leq \| A_i \|_{[t_0, \infty)} \| x_i \|_\Omega + \sum_{j=1}^n \left( \sum_{k=1}^{m_{ij}} \| B_{ij}^k \|_{[t_0, \infty)} + \| C_{ij} \|_{[t_0, \infty)} \right) \| x_j \|_\Omega + \| f_i \|_\Omega.
\]

Since the right-hand side of (2.9) does not depend on \( t \in \Omega \), inequality (2.9) implies (2.8). \( \square \)

3. Main Results

3.1. The main theorem.

Theorem 3.1. Let there exist \( \alpha_i \) such that \( \mu(A_i(t)) \leq \alpha_i < 0, i = 1, \ldots, n \) and \( L = I - D \) be an \( M \)-matrix, where the entries of \( D = (d_{ij})_{i,j=1}^n \) be defined as follows:

\[
d_{ii} = \tau_i \left( \frac{A_i}{\mu(A_i)} \right)_{[t_0, \infty)} \left( \| A_i \|_{[t_0, \infty)} + \sum_{k=1}^{m_{ij}} \| B_{ik} \|_{[t_0, \infty)} + \| C_{ii} \|_{[t_0, \infty)} \right) + \sum_{k=1}^{m_{ij}} \left( \frac{B_{ik}}{\mu(A_i)} \right)_{[t_0, \infty)} + \left( \frac{C_{ii}}{\mu(A_i)} \right)_{[t_0, \infty)}, \]

\[
d_{ij} = \tau_i \left( \frac{A_i}{\mu(A_i)} \right)_{[t_0, \infty)} \left( \sum_{k=1}^{m_{ij}} \| B_{kj} \|_{[t_0, \infty)} + \| C_{ij} \|_{[t_0, \infty)} \right) + \sum_{k=1}^{m_{ij}} \left( \frac{B_{kj}}{\mu(A_i)} \right)_{[t_0, \infty)} + \left( \frac{C_{ij}}{\mu(A_i)} \right)_{[t_0, \infty)}, \quad j \neq i.
\]

Then system (2.1) is uniformly exponentially stable.
Proof. To apply Lemma 2.6, we explore boundedness of solutions to (2.4), (2.5). First, transform equation (2.4)

\[ x(t) = A(t)x(t) - \int_{t_0}^{t} x(s)ds + \sum_{j=1}^{n} \sum_{k=1}^{m_{ij}} B_{ij}^{k}(t)x_{j}(h_{ij}^{k}(t)) + \sum_{j=1}^{n} \int_{g_{ij}(t)}^{t} K_{ij}(t,s)x_{j}(s)ds + f_{i}(t). \]

Denote by \( \Phi_{i}(t,s) \) the fundamental matrix of (3.1)

\[ \dot{x}(t) = A_{i}(t)x(t). \]

Since \( \mu(A_{i}(t)) \leq \alpha_{i} < 0 \), (3.1) is uniformly exponentially stable. Continue transformations of equation (2.4)

\[ x_{i}(t) = \int_{t_{0}}^{t} \Phi_{i}(t,s) \left[ -A_{i}(s) \int_{h_{i}(s)}^{s} \dot{x}_{i}(\xi)d\xi + \sum_{j=1}^{m_{ij}} B_{ij}^{k}(s)x_{j}(h_{ij}^{k}(s)) \right] ds + \tilde{f}_{i}(t), \]

where \( \tilde{f}_{i}(t) = \int_{t_{0}}^{t} \Phi_{i}(t,s)f_{i}(s)ds \in L_{\infty}[t_{0}, \infty) \). Hence by Lemma 2.7,

\[ \|x_{i}(t)\| \leq \tau_{i} \left( \|A_{i}\|_{[t_{0}, \infty)} \left( \|\dot{x}_{i}\|_{\Omega} + \|\tilde{f}_{i}\|_{[t_{0}, \infty)} + \sum_{j=1}^{m_{ij}} \left( \|B_{ij}^{k}\|_{[t_{0}, \infty)} + \|C_{ij}\|_{[t_{0}, \infty)} \right) \|x_{j}\|_{\Omega} \right) + \|f_{i}\|_{\Omega} \right). \]

Therefore by Lemma 2.9,

\[ \|x_{i}\|_{\Omega} \leq \tau_{i} \left( \|A_{i}\|_{[t_{0}, \infty)} \left( \|x_{i}\|_{[t_{0}, \infty)} + \sum_{j=1}^{m_{ij}} \left( \|B_{ij}^{k}\|_{[t_{0}, \infty)} \|x_{j}\|_{\Omega} \right) + \|C_{ij}\|_{[t_{0}, \infty)} \right) \|x_{j}\|_{\Omega} \right) \]

\[ + \|f_{i}\|_{\Omega} \right) + \sum_{j \neq i} d_{ij}\|x_{j}\|_{\Omega} + M_{i}, \]

where \( M_{i} > 0 \) are constants, not dependent on \( t_{1} > t_{0} \).

Introduce the column vectors \( X_{\Omega} = (\|x_{1}\|_{\Omega}, \dots, \|x_{n}\|_{\Omega})^{T}, M = (\|M_{1}\|, \dots, \|M_{n}\|)^{T}. \) Then \((I-D)X_{\Omega} \leq M. \) Since \( I - D \) is an \( M \)-matrix, there exists \((I - D)^{-1} \geq 0. \) Therefore (see Remark 2.2) \( X_{\Omega} \leq (I - D)^{-1}M, \) where the right-hand side of the last inequality does not depend on the interval \( \Omega. \) Hence the solution of problem (2.4), (2.5) is bounded on \([t_{0}, \infty)\). By Lemma 2.6 system (2.1) is uniformly exponentially stable. \( \square \)
3.2. **Corollaries of the main theorem.** Consider several particular cases of (2.1).

We start with the system where the first term in each vector equation is non-delayed

\[ \dot{x}_i(t) = A_i(t)x_i(t) + \sum_{j=1}^{n} \sum_{k=1}^{m_{ij}} B_{ij}^{k}(t)x_j(h_{ij}^{k}(t)) + \sum_{j=1}^{n} \int_{g_{ij}(t)}^{t} K_{ij}(t,s)x_j(s)ds, \quad i = 1, \ldots, n. \]  

**Corollary 3.2.** Assume that \( \mu(A_i(t)) \leq \alpha_i < 0, \ i = 1, \ldots, n \) and \( L = I - D \) is an \( M \)-matrix, where \( D = (d_{ij})_{i,j=1}^{n} \) has the entries

\[ d_{ij} = \sum_{k=1}^{m_{ij}} \left\| \frac{B_{ij}^{k}}{\mu(A_i)} \right\|_{[t_0, \infty)} + \left\| \frac{C_{ij}}{\mu(A_i)} \right\|_{[t_0, \infty)}, \quad i, j = 1, \ldots, n. \]

Then system (3.2) is uniformly exponentially stable.

Next, consider a system without integral terms and with single delays in the diagonal terms

\[ \dot{x}_i(t) = A_i(t)x_i(h_i(t)) + \sum_{j=1}^{m_{ij}} B_{ij}^{k}(t)x_j(h_{ij}^{k}(t)), \quad i = 1, \ldots, n. \]

**Corollary 3.3.** Assume that \( \mu(A_i(t)) \leq \alpha_i < 0, \ i = 1, \ldots, n \), and the matrix \( I - D \) is an \( M \)-matrix, where \( D = (d_{ij})_{i,j=1}^{n} \) are defined as

\[ d_{ii} = \tau_i \left\| \frac{A_i}{\mu(A_i)} \right\|_{[t_0, \infty)} \left\| A_i \right\|_{[t_0, \infty)}, \quad d_{ij} = \tau_i \left\| \frac{A_i}{\mu(A_i)} \right\|_{[t_0, \infty)} \sum_{k=1}^{m_{ij}} \left\| \frac{B_{ij}^{k}}{\mu(A_i)} \right\|_{[t_0, \infty]} + \sum_{k=1}^{m_{ij}} \left\| \frac{B_{ij}^{k}}{\mu(A_i)} \right\|_{[t_0, \infty)}, \quad j \neq i. \]

Then system (3.3) is uniformly exponentially stable.

We proceed to the case of two vector equations:

\[ \begin{align*} 
\dot{x}_1(t) &= A_1(t)x_1(h_1(t)) + \sum_{j=1}^{m_{11}} B_{1j}^{k}(t)x_j(h_{1j}^{k}(t)) + \sum_{j=1}^{m_{12}} B_{2j}^{k}(t)x_j(h_{2j}^{k}(t)) + \sum_{j=1}^{m_{12}} \int_{g_{1j}(t)}^{t} K_{1j}(t,s)x_j(s)ds, \\
\dot{x}_2(t) &= A_2(t)x_2(h_2(t)) + \sum_{j=1}^{m_{21}} B_{2j}^{k}(t)x_j(h_{2j}^{k}(t)) + \sum_{j=1}^{m_{22}} \int_{g_{2j}(t)}^{t} K_{2j}(t,s)x_j(s)ds. 
\end{align*} \]

**Corollary 3.4.** Let \( \mu(A_i(t)) \leq \alpha_i < 0, i = 1, 2 \) and

\[ d_{11} < 1, \ d_{22} < 1, \ d_{12}d_{21} < (1 - d_{11})(1 - d_{22}), \]

where

\[
\begin{align*}
    d_{11} &= \tau_1 \left\| \frac{A_1}{\mu(A_1)} \right\|_{[t_0, \infty)} \left( \left\| A_1 \right\|_{[t_0, \infty]} + \sum_{k=1}^{m_{11}} \left\| B_{11}^{k} \right\|_{[t_0, \infty]} + \left\| C_{11} \right\|_{[t_0, \infty]} \right) + \sum_{k=1}^{m_{11}} \left\| \frac{B_{11}^{k}}{\mu(A_1)} \right\|_{[t_0, \infty]} + \left\| \frac{C_{11}}{\mu(A_1)} \right\|_{[t_0, \infty]} , \\
    d_{22} &= \tau_2 \left\| \frac{A_2}{\mu(A_2)} \right\|_{[t_0, \infty)} \left( \left\| A_2 \right\|_{[t_0, \infty]} + \sum_{k=1}^{m_{22}} \left\| B_{22}^{k} \right\|_{[t_0, \infty]} + \left\| C_{22} \right\|_{[t_0, \infty]} \right) + \sum_{k=1}^{m_{22}} \left\| \frac{B_{22}^{k}}{\mu(A_2)} \right\|_{[t_0, \infty]} + \left\| \frac{C_{22}}{\mu(A_2)} \right\|_{[t_0, \infty]} , \\
    d_{12} &= \tau_1 \left\| \frac{A_1}{\mu(A_1)} \right\|_{[t_0, \infty)} \sum_{k=1}^{m_{12}} \left\| B_{12}^{k} \right\|_{[t_0, \infty]} + \left\| C_{12} \right\|_{[t_0, \infty]} \right) + \sum_{k=1}^{m_{12}} \left\| \frac{B_{12}^{k}}{\mu(A_1)} \right\|_{[t_0, \infty]} + \left\| \frac{C_{12}}{\mu(A_1)} \right\|_{[t_0, \infty]} , \\
    d_{21} &= \tau_2 \left\| \frac{A_2}{\mu(A_2)} \right\|_{[t_0, \infty)} \sum_{k=1}^{m_{21}} \left\| B_{21}^{k} \right\|_{[t_0, \infty]} + \left\| C_{21} \right\|_{[t_0, \infty]} \right) + \sum_{k=1}^{m_{21}} \left\| \frac{B_{21}^{k}}{\mu(A_2)} \right\|_{[t_0, \infty]} + \left\| \frac{C_{21}}{\mu(A_2)} \right\|_{[t_0, \infty]} .
\end{align*}
\]

Then system (3.3) is uniformly exponentially stable.
For only two delays in each vector equation in (3.4), we get
\begin{align}
\dot{x}_1(t) &= A_1(t)x_1(h_1(t)) + B_{12}(t)x_2(h_{12}(t)), \\
\dot{x}_2(t) &= A_2(t)x_2(h_2(t)) + B_{21}(t)x_1(h_{21}(t)).
\end{align}

**Corollary 3.5.** Let \( \mu(A_i(t)) \leq \alpha_i < 0, i = 1, 2 \) and there exist constant matrices \( \tilde{A}_1, \tilde{A}_2, \tilde{B}_{12}, \tilde{B}_{21} \), such that \( |A_i(t)| \leq A_i, i = 1, 2, |B_{12}(t)| \leq B_{12}, |B_{21}(t)| \leq B_{21} \). If
\begin{align}
\tau_i||\tilde{A}_i||^2 &< |\alpha_i|, i = 1, 2, \\
\|\tilde{B}_{12}||\tilde{B}_{21}||(1 + \tau_1||\tilde{A}_1||)(1 + \tau_2||\tilde{A}_2||) &< (|\alpha_1| - \tau_1||\tilde{A}_1||^2)(|\alpha_2| - \tau_2||\tilde{A}_2||^2)
\end{align}
then system (3.6) is uniformly exponentially stable.

**Proof.** Let us check that conditions of Corollary 3.4 hold, where
\begin{align}
(3.6)
12
\dot{x}_i(t) &= A_i(t)x_i(h_i(t)) + B_{ij}(t)x_j(h_j(t)), \\
&= A_i(t)x_i(h_i(t)) + B_{ij}(t)x_j(h_j(t)),
\end{align}
then system (3.6) is uniformly exponentially stable.

\[ \text{Corollary 3.5.} \]

For brevity, we omit the interval index \([t_0, \infty)\) for the norms of matrix functions. We have
\begin{align}
d_{ii} &= \tau_i \left\| \frac{A_i}{\mu(A_i)} \right\| ||A_i||, i = 1, 2, \\
d_{12} &= \tau_1 \left\| \frac{A_1}{\mu(A_1)} \right\| ||B_{12}|| + \left\| \frac{B_{12}}{\mu(A_1)} \right\|, d_{21} = \tau_2 \left\| \frac{A_2}{\mu(A_2)} \right\| ||B_{21}|| + \left\| \frac{B_{21}}{\mu(A_2)} \right\|.
\end{align}
The inequality \( |A| \leq B \) implies \( ||A|| \leq ||B|| \) for any matrix norm. Hence
\begin{align}
d_{ii} &\leq \tau_i \left\| \frac{\tilde{A}_i}{|\alpha_i|} \right\|, i = 1, 2, \\
d_{12} &\leq \left\| \frac{\tilde{B}_{12}}{|\alpha_1|} \right\| \left(1 + \frac{\tau_1||\tilde{A}_1||}{|\alpha_1|}\right), d_{21} \leq \left\| \frac{\tilde{B}_{21}}{|\alpha_2|} \right\| \left(1 + \frac{\tau_2||\tilde{A}_2||}{|\alpha_2|}\right).
\end{align}
Inequality (3.8) and the first inequality in (3.7) imply \( d_{ii} < 1, i = 1, 2 \). By (3.9), inequality \( d_{12}d_{21} < (1 - d_{11})(1 - d_{22}) \) holds if
\begin{align}
\left\| \frac{\tilde{B}_{12}}{|\alpha_1|} \right\| \left(1 + \frac{\tau_1||\tilde{A}_1||}{|\alpha_1|}\right) \left(1 + \frac{\tau_2||\tilde{A}_2||}{|\alpha_2|}\right) < \left(1 - \frac{\tau_1||\tilde{A}_1||^2}{|\alpha_1|}\right) \left(1 - \frac{\tau_2||\tilde{A}_2||^2}{|\alpha_2|}\right),
\end{align}
which is equivalent to the second inequality in (3.7). Since all the conditions in (3.5) hold, by Corollary 3.4 system (3.6) is uniformly exponentially stable. \( \square \)

**Remark 3.6.** Theorem 3.1 and its corollaries also hold for vector differential equations if the terms \( t \int K_{ij}(t, s)x_j(s)ds \) in equation (2.1) are replaced with \( C_{ij}(t) \int g_{ij}(t) R_{ij}(t, s)x_j(s) \), where \( g_{ij}(t) \) distributed delays are of a more general type, and \( t \int g_{ij}(t) R_{ij}(t, s) = 1. \)

4. Applications to Second Order Vector Delay Differential Equations

4.1. Linear equations. In this section we consider a second order vector equation
\begin{align}
\ddot{x}(t) + A(t)x(t) + B(t)x(h(t)) = 0, \ t \geq t_0 \geq 0,
\end{align}
where for the matrices \( A, B \) and the delay \( h \) the same conditions hold as (i), (ii) for system (2.1). Definitions of solutions of an initial value problem for vector equation (4.1) and of uniform exponential stability are also similar to the definitions for (2.1), and we omit them.
Theorem 4.1. Assume that for some $t \geq t_0 \geq 0$, there exist a constant matrix $\tilde{A}$ and numbers $\alpha < 0$, $\tau \geq 0$ such that $t - h(t) \leq \tau$, $\mu(-\tilde{A}) < 0$, $\mu\left(\tilde{A} - 2A(t)\right) \leq \alpha < 0$ and

\[
(4.2) \quad \frac{1}{|\mu(-\tilde{A})|} \left( \left\| \frac{2\tilde{A}A - \tilde{A}^2 - 4B}{\mu(A - 2A)} \right\|_{[t_0, \infty]} + 2\tau \left\| \frac{\tilde{A}B}{\mu(A - 2A)} \right\|_{[t_0, \infty]} \right) < 1 - 2\tau \left\| \frac{B}{\mu(A - 2A)} \right\|_{[t_0, \infty]}.
\]

Then equation (4.1) is uniformly exponentially stable.

Proof. Equation (4.1) can be transformed to the form

\[
(4.3) \quad \ddot{x}(t) + A(t)\dot{x}(t) + B(t)x(t) - B(t) \int_{h(t)}^{t} \dot{x}(s) ds = 0.
\]

After the substitution

\[\dot{x}(t) = -\frac{\tilde{A}}{2}x(t) + y(t), \quad \ddot{x}(t) = \frac{\tilde{A}^2}{4}x(t) - \frac{\tilde{A}}{2}y(t) + \dot{y}(t),\]

equation (4.3) becomes

\[
\frac{\tilde{A}^2}{4}x(t) - \frac{\tilde{A}}{2}y(t) + \dot{y}(t) + A(t) \left[ -\frac{\tilde{A}}{2}x(t) + y(t) \right] + B(t)x(t) - B(t) \int_{h(t)}^{t} \left( -\frac{\tilde{A}}{2}x(s) + y(s) \right) ds = 0.
\]

Hence equation (4.1) is equivalent to the system

\[
\dot{x}(t) = -\frac{\tilde{A}}{2}x(t) + y(t)
\]

\[
(4.4) \quad \dot{y}(t) = \left( -\frac{\tilde{A}}{2} - A(t) \right) y(t) + \left( \frac{\tilde{A}}{2}A(t) - \frac{\tilde{A}^2}{4} - B(t) \right) x(t) - \frac{\tilde{A}}{2}B(t) \int_{h(t)}^{t} x(s) ds + B(t) \int_{h(t)}^{t} y(s) ds = 0.
\]

For system (4.4), let us apply Corollary 3.4. System (4.4) has a form of (3.4), where

\[
\begin{align*}
x_1 &= x, \quad x_2 = y, \quad A_1 = -\frac{\tilde{A}}{2}, \quad h_1(t) = t, \quad \tau_1 = 0, \quad m_{11} = 0, \quad m_{12} = 1, \\
B_{11} &= 0, \quad B_{12} = 1, \quad h_{12}(t) = t, \quad C_{11} = C_{12} = 0, \\
A_2 &= -\frac{\tilde{A}}{2} - A(t), \quad h_2(t) = t, \quad \tau_2 = 0, \quad m_{21} = 1, \quad m_{22} = 0, \quad B_{21} = \frac{\tilde{A}}{2}A(t) - \frac{\tilde{A}^2}{4} - B(t), \\
C_{21} &= \frac{\tilde{A}}{2} |B(t)(t - h(t))| \leq \frac{\tilde{A}}{2} |B(t)|, \quad C_{22} = |B(t)(t - h(t))| \leq |B(t)|\tau.
\end{align*}
\]

We have

\[
\mu(A_1) < 0, \quad \mu(A_2) < 0, \quad d_{11} = 0, \quad d_{22} \leq \tau \left\| \frac{B}{\mu(\frac{\tilde{A}}{2} - A)} \right\|_{[t_0, \infty]}, \quad d_{12} = \frac{2}{|\mu(-A)|},
\]

\[
d_{21} \leq \left\| \frac{\tilde{A}A - \tilde{A}^2 - B}{\mu(\frac{\tilde{A}}{2} - A)} \right\|_{[t_0, \infty]} + \tau \left\| \frac{\tilde{A}B}{\mu(\frac{\tilde{A}}{2} - A)} \right\|_{[t_0, \infty]}.
\]
Then \(d_{12}d_{21} \leq (1 - d_{11})(1 - d_{22})\) takes the form

\[
\frac{2}{\|\mu(-A)\|} \left( \left\| \frac{4}{2} A - \frac{3}{1} B \right\|_{\|t_0,\infty\|} + \tau \left\| \frac{4}{2} B \right\|_{\|t_0,\infty\|} \right) < 1 - \tau \left\| \frac{B}{\mu(\frac{4}{2} - A)} \right\|_{\|t_0,\infty\|},
\]

which is equivalent to (4.2). Hence all the conditions of Corollary 3.4 hold for (4.4). Thus, system (4.4) and therefore equation (4.3) are uniformly exponentially stable.

**Remark 4.2.** If \(A_1 \leq A(t) \leq A_2\) then anyone of \(A_1, A_2\) can be taken as \(\tilde{A}\), see examples below.

**Corollary 4.3.** Let there exist a constant matrix \(\tilde{A}, \alpha < 0\) and \(\tau \geq 0\) such that for \(t \geq t_0 \geq 0\), \(t - h(t) \leq \tau, \mu(-\tilde{A}) < 0, \mu(\tilde{A} - 2A(t)) \leq \alpha < 0\) and

\[
2\tilde{A}A - \tilde{A}^2 - 4B\|_{\|t_0,\infty\|} + 2\tau\tilde{A}B\|_{\|t_0,\infty\|} < \|\mu(-\tilde{A})\| (|\alpha| - 2\tau\|B\|_{\|t_0,\infty\|}).
\]

Then equation (4.1) is uniformly exponentially stable.

Consider now a second order system with constant coefficients

\[
\dot{x}(t) + Ax(t) + Bx(h(t)) = 0.
\]

Choosing \(\tilde{A} = A, \alpha = \mu(-A)\) in Corollary 4.3 we get a result for (4.6).

**Corollary 4.4.** Let \(t - h(t) \leq \tau < \infty\) for \(t \geq t_0 \geq 0, \mu(-A) < 0\) and

\[
\|\dot{A}^2 - 4B\| + 2\tau\|AB\| < \|\mu(-A)\| (|\alpha| - 2\tau\|B\|).
\]

Then equation (4.6) is uniformly exponentially stable.

Next, taking \(h(t) = t\) in Theorem 4.1 we get a result for a vector ordinary differential equation of the second order.

**Corollary 4.5.** Let there exist a constant matrix \(\tilde{A}, \alpha < 0\) and \(\tau \geq 0\) such that for \(t \geq t_0 \geq 0, \mu(-\tilde{A}) < 0, \mu(\tilde{A} - 2A(t)) \leq \alpha < 0\) and

\[
2\tilde{A}A - \tilde{A}^2 - 4B\|_{\|t_0,\infty\|} < |\alpha|\|\mu(-\tilde{A})\|.
\]

Then equation (4.1) is uniformly exponentially stable.

**Corollary 4.6.** Assume that \(\mu(-A) < 0\) and \(\|A^2 - 4B\| < (\mu(-A))^2\). Then equation (4.1) with constant matrices \(A\) and \(B\) is uniformly exponentially stable.

**Example 4.7.** Consider equation (4.1), where \(t - h(t) \leq \tau, A(t) = \begin{pmatrix} 4 & \sin^2 t \\ 2\cos^2 t & 6 \end{pmatrix}, B(t) = \begin{pmatrix} 4 \\ 1 \end{pmatrix}, \mu(-A) = -3 < 0\).

\[
\tilde{A} - 2A(t) = \begin{pmatrix} 4 & \sin^2 t \\ \cos^2 t & 6 \end{pmatrix}, \mu(\tilde{A} - 2A(t)) \leq \alpha = -3 < 0.
\]

Next,

\[
\tilde{A} = \begin{pmatrix} 17 & 10 \\ 10 & 37 \end{pmatrix}, 4B(t) = \begin{pmatrix} 16 & 8\sin^2 t \\ 8\cos^2 t & 32 \end{pmatrix},
\]

\[
\|2\tilde{A}A - \tilde{A}^2 - 4B\|_{\|t_0,\infty\|} = 7, \|\tilde{A}B\|_{\|t_0,\infty\|} = 66, \|B\|_{\|t_0,\infty\|} = 10.
\]

For \(\alpha = -3\), inequality (4.3) is

\[
7 + 132\tau < 3(3 - 20\tau), \text{ which holds if } \tau < \frac{1}{96}.
\]

Thus for \(\tau < \frac{1}{96}\) vector equation (4.1) is uniformly exponentially stable.
Next, consider equation (4.1) with a non-zero right-hand side

(4.9) \[ \ddot{x}(t) + A(t)\dot{x}(t) + B(t)x(h(t)) = L(t), \quad t \geq t_0 \geq 0. \]

The following statement can be justified for a variety of uniformly exponentially stable differential equations with bounded delays, we prove it here for second order vector equations.

**Lemma 4.8.** Let equation (4.1) be uniformly exponentially stable and \( L(t) \) have one of the following properties:

a) \( L(t) \) is bounded, \( t \in [t_0, \infty) \);

b) \( \int_{t_0}^{t_1} \|L(s)\|ds \) is bounded, \( t \in [t_0, \infty) \);

c) \( \lim_{t \to \infty} L(t) = 0 \);

d) \( \lim_{t \to \infty} \int_{t_0}^{t_1} \|L(s)\|ds \to 0. \)

Then any solution of equation (4.9) possesses the same property.

**Proof.** According to [5, page 3], (11), there are matrix functions \( X_1(t), X_2(t), X(t, s) \) such that a solution \( x \) of equation (4.9) with the initial condition \( x(t_0) = \varphi(t), t \leq t_0 \) satisfies

\[ x(t) = X_1(t)x(t_0) + X_2(t)\dot{x}(t_0) - \int_{t_0}^{t_{0+\tau}} X(t, s)B(s)\varphi(h(s))ds + \int_{t_0}^{t} X(t, s)L(s)ds, \]

where \( \varphi(h(s)) = 0, h(s) \geq t_0 \). Exponential stability of (4.1) implies for some \( M_i > 0, \lambda_i > 0 \), \( i = 1, 2, M > 0, \lambda > 0, \)

\[ \|X_1(t)\| \leq M_i e^{-\lambda_i t}, \quad i = 1, 2, \quad \|X(t, s)\| \leq M e^{-\lambda(t-s)}. \]

Assume that condition b) holds: \( \int_{t_0}^{t_1} \|L(s)\|ds \leq K \) for \( t \geq t_0 \). We have

\[ \|x(t)\| \leq M_1 e^{-\lambda_1 t}\|x(t_0)\| + M_2 e^{-\lambda_2 t}\|\dot{x}(t_0)\| + M \|B\|_{[t_0, \infty)} \|\varphi\|_{[t_0-\tau, t_0]} \int_{t_0}^{t_{0+\tau}} e^{-\lambda(t-s)}ds \]

\[ + M \int_{t_0}^{t} e^{-\lambda(t-s)}\|L(s)\|ds \]

\[ \leq M_1 \|x(t_0)\| + M_2 \|\dot{x}(t_0)\| + \frac{M}{\lambda} \|B\|_{[t_0, \infty)} \|\varphi\|_{[t_0-\tau, t_0]} e^{\lambda(t_0+\tau)} \]

\[ + M \int_{t_0}^{t} e^{-\lambda(t-s)}\|L(s)\|ds. \]

Let \( t \in [t_0 + m, t_0 + m + 1] \) for some integer \( m \geq 0 \). Then

\[ \int_{t_0}^{t} e^{-\lambda(t-s)}\|L(s)\|ds \leq \sum_{k=0}^{m} \int_{t_0+k}^{t_0+k+1} e^{-\lambda(t-s)}\|L(s)\|ds \leq \sum_{k=0}^{m} e^{-\lambda(m-k)} \int_{t_0+k}^{t_0+k+1} \|L(s)\|ds \]

\[ \leq K \sum_{k=0}^{m} e^{-\lambda(m-k)} < \frac{K}{1 - e^{-\lambda}}, \]

where the last expression does not depend on \( m \). Hence the solution \( x \) of equation (4.9) is bounded, and \( \int_{t_0}^{t_1} \|x(s)\|ds \) is bounded. Therefore a) and b) yield that solutions of equation (4.9) are bounded. Similarly, c)-d) imply convergence of all solutions to zero. \( \Box \)

**Theorem 4.9.** Let all the conditions of Theorem 4.7 hold and \( L(t) \) have one of the properties a)-d) of Lemma 4.8. Then any solution of equation (4.9) possesses the same property.
4.2. Nonlinear equations. In this section consider a nonlinear equation

\[ \ddot{x}(t) + A(t, x(t), \dot{x}(t)) \dot{x}(t) + B(t, x(t), \dot{x}(t))x(h(t)) = L(t), \ t \geq t_0 \geq 0, \]

where the matrices \( A(\cdot, u, v) \) and \( B(\cdot, u, v) \) belong to \( L^d_{\infty}([0, \infty)) \) for any pair \( (u, v) \) of locally integrable on \([0, \infty)\) vector functions, and \( L \in L^d_{\infty}([0, \infty)) \). A solution of an initial value problem for (4.11) is defined similarly to that of (4.1). We assume that a solution exists and is unique.

Consider first equation (4.10) with \( L(t) \equiv 0 \).

**Theorem 4.10.** Let for any \( t \geq t_0 \geq 0, L(t) \equiv 0, \) there exist a constant matrix \( \tilde{A} \) and \( \alpha < 0, \tau \geq 0 \) such that \( t - h(t) \leq \tau, \mu(-\tilde{A}) < 0, \) and for any pair \( (u, v) \) of constant vectors, \( \mu(\tilde{A} - 2A(t, u, v)) \leq \alpha < 0 \) and

\[
\frac{1}{|\mu(-\tilde{A})|} \left( \left\| 2\tilde{A}(\cdot, u, v) - \tilde{A}^2 - 4B(\cdot, u, v) \right\|_{[t_0, \infty)} \right) + 2\tau \left\| \tilde{A}B(\cdot, u, v) \right\|_{[t_0, \infty)} < 1 - 2\tau \left\| \frac{B(\cdot, u, v)}{\mu(A-2A(\cdot, u, v))} \right\|_{[t_0, \infty)}.
\]

Then any solution \( x \) of (4.10) tends to zero as \( t \to \infty \), together with its derivative.

**Proof.** Let \( x \) be a solution of (4.10) with \( L \equiv 0 \). Consider the linear equation with variable coefficients for this fixed \( x \):

\[ \ddot{y}(t) + A(t, x(t), \dot{x}(t)) \dot{y}(t) + B(t, x(t), \dot{x}(t))y(h(t)) = 0, \ t \geq t_0 \geq 0. \]

All conditions of Theorem 4.11 hold for (4.12). Hence this equation is uniformly exponentially stable. Then any solution \( y \) of this equation tends to zero, together with its derivative. But the vector-function \( x \) is one of these solutions. Therefore any solution \( x \) of (4.10) tends to zero as \( t \to \infty \), together with its derivative. \( \square \)

Consider now equation (4.10) with a non-zero right-hand side \( L(t) \).

**Corollary 4.11.** Let all the conditions of Theorem 4.10 hold and \( L(t) \) possess one of Properties a)-d) of Lemma 4.8. Then any solution of equation (4.10) has the same property.

**Proof.** For any solution \( x \) of (4.10), consider the linear equation

\[ \ddot{y}(t) + A(t, x(t), \dot{x}(t)) \dot{y}(t) + B(t, x(t), \dot{x}(t))y(h(t)) = L(t), \ t \geq t_0 \geq 0. \]

Assume that \( L(t) \) satisfies one of a)-d). Since equation (4.13) with \( L \equiv 0 \) is exponentially stable, by Lemma 4.8 all solutions of (4.13), as well as the solution \( x \) of (4.10), possess this property. \( \square \)

**Corollary 4.12.** Let there exist a constant matrix \( \tilde{A}, \alpha < 0 \) and \( \tau \geq 0 \) such that \( \mu(-\tilde{A}) < 0, \ t - h(t) \leq \tau \) for \( t \geq t_0 \geq 0, \) and for any pair \( (u, v) \) of constant vectors, \( \mu(\tilde{A} - 2A(t, u, v)) \leq \alpha < 0 \) and

\[
\|2\tilde{A}(\cdot, u, v) - \tilde{A}^2 - 4B(\cdot, u, v)\|_{[t_0, \infty)} + 2\tau \|\tilde{A}B(\cdot, u, v)\|_{[t_0, \infty)} < |\mu(-\tilde{A})| |(\alpha - 2\tau)\|B(\cdot, u, v)\|_{[t_0, \infty)}).
\]

Then any solution \( x \) of (4.10) with \( L \equiv 0 \) tend to zero as \( t \to \infty \), together with its derivative.

**Example 4.13.** Consider equation (4.10), where \( L(t) \equiv 0, \ t - h(t) \leq \tau, \ x = (x_1, x_2), \)

\[
A(t) = \left( \begin{array}{cc} 4 & \sin^2(tx_1(t)) \\ \cos^2(tx_2(t)) & 6 \end{array} \right), \quad B(t) = \left( \begin{array}{cc} 4 & 2\sin^2(tx_1(t)) \\ 2\cos^2(tx_2(t)) & 8 \end{array} \right).
\]

Exactly the same calculations as in Example 4.7 and condition (4.14) imply that for \( \tau < \frac{1}{90} \), any solution of (4.10) tends to zero as \( t \to \infty \), together with its derivative.
5. Discussion and Topics for Further Research

We investigated uniform exponential stability for systems of linear vector functional differential equations, as well as asymptotic stability of linear and certain nonlinear vector equations of the second order. System (2.1) has many applications to real-world models but, to the best of our knowledge, very little is known about its exponential stability. One of most common applications is to second order vector delay differential equations, which is considered in the paper for equations with one delay. However, since the general theory is developed for a more general model, the results are easily extended to equations with several bounded delays.

\begin{equation}
    \ddot{x}(t) + A(t)\dot{x}(t) + \sum_{k=1}^{m} B_k(t)x(h_k(t)) = 0, \quad t \geq t_0 \geq 0,
\end{equation}

and with a bounded distributed delay

\begin{equation}
    \ddot{x}(t) + A(t)\dot{x}(t) + \int_{h(t)}^{t} B(t,s)x(s)ds = 0, \quad t \geq t_0 \geq 0.
\end{equation}

Corollary 4.3 gives a new simple and general stability test for linear ordinary differential vector equations of the second order, compared to the papers [9, 10, 17, 19, 22] where only matrices of a specific form were considered. For example, in the recent paper [10] the following stability test was obtained. Denote \( A_R = \frac{A + A^*}{2} \), \( A_I = \frac{A - A^*}{2i} \).

**Proposition 5.1.** Let there exist \( m_A > 0 \) such that

\[ A_R(t) \geq 2m_A I, \quad T_R(t) \geq m_A I, \quad \|T_R\|_{[0,\infty)} + \|T_I\|_{[0,\infty)} \leq 2m^2_A, \]

where \( T(t) = m_A A(t) - B(t) \) and inequality \( A \geq B \) means that \( A - B \) is a positive definite matrix (all eigenvalues are non-negative). Then equation (1.1) is exponentially stable.

Actually it is not easy to check that all eigenvalues of time-variable matrix \( T(t) \) are non-negative. For a nonlinear delay vector equation of the second order

\begin{equation}
    \ddot{X}(t) + F(X(t), \dot{X}(t))\dot{X}(t) + H(X(t - \tau)) = 0,
\end{equation}

where the matrix-function \( F(u, v) \) and vector-function \( H(u) \) are continuous, an interesting explicit stability test was obtained in [20].

**Proposition 5.2.** [20] Let there exist positive numbers \( a_0, a_1, a_2 \) such that

1) the matrix \( F(u, v) \) is symmetric, and the eigenvalues of this matrix \( \lambda_i(F(u, v)) \geq a_1 \) for all pairs \((u, v) \in \mathbb{R}^d \times \mathbb{R}^d\).

2) \( H(0) = 0, \quad H(X) \neq 0, \quad X \neq 0 \), the Jacobian \( J_H(X) \) is symmetric and \( a_2 \leq \lambda_i(J_H(X)) \leq a_0 \).

If \( \tau < \frac{a_1}{a_0 \sqrt{d}} \) then the trivial solution of (5.3) is asymptotically stable.

Compare now Proposition 5.2 of [20] and the test in Theorem 4.10.

The stability result of Proposition 5.2 applies only to autonomous equations, with continuous functions \( G, H \) and constant delay \( \tau \). Our result works for non-autonomous equations, with measurable coefficients and a variable delay. We also apply different stability methods: in [20] the method of Krasovskii-Lyapunov functionals was applied, in this paper the stability results are based on Bohl-Perron theorem.

The linear equation

\begin{equation}
    \ddot{X}(t) + A\dot{X}(t) + Bx(t - \tau) = 0,
\end{equation}

where \( A, B \) are constant matrices, is a partial case of equation (5.3). Let us compare for (5.4) the results of Corollary 4.3 and Proposition 5.2. For the scalar case \((d = 1)\) Proposition 5.2 leads to \( B > 0, \quad A > 0, \quad B\tau < A \), while Corollary 4.3 reduces to \( B > 0, \quad A > 0, \quad |A^2 - 4B| < A^2 - 4AB\tau \).
In the scalar case (and presumably also for small $d$), Proposition 5.2 is better than Corollary 4.4. But for large $d$, Corollary 4.4 could be better than Proposition 5.2. Assume, for example, that in (5.3) $A = aI, B = bI, a > 0, b > 0$. Then the condition of Proposition 5.2 becomes
$$\tau < \frac{a}{\sqrt{d}},$$
where $\tau \to 0$ for fixed $a, b$ and $d \to \infty$. Proposition 5.2 gives the same condition as for the scalar case: $|a^2 - 4b| < a^2 - 4ab\tau$ and does not depend on $d$. Consider for comparison $A = aI, B = \frac{1}{4}a^2I$, where $a > 0$. Then Proposition 5.2 implies asymptotic stability for $\tau < \frac{1}{a}$ in Corollary 4.4. Thus Corollary 4.4 gives a sharper result for $d > 16$.

Together with stability results obtained here, the aim of the paper was also to attract attention to the new class of systems of functional differential equations.

It is not yet clear how to obtain exponential stability tests for equations with unbounded delays, for example, pantograph-type with $h(t) = \lambda t$, and for equations with delays in the derivative terms such as
$$\ddot{x}(t) + A(t)\dot{x}(g(t)) + B(t)x(h(t)) = 0.$$  
(5.5)

Further, let us discuss other possible problems for future research.

An interesting extension of (2.1) is a nonlinear system
$$\dot{x}_i(t) = A_i(t)x_i(h_i(t)) + \sum_{j=1}^{n} \sum_{k=1}^{m_{ij}} H_{ij}(t, x_j(h_{ij}(t))) + \sum_{j=1}^{n} F_{ij} \left( t, \int_{g_{ij}(t)}^{t} K_{ij}(t, s)x_j(s)ds \right), \quad i = 1, \ldots, n.$$  

Results for this system can be applied to a nonlinear vector delay equation of the second order
$$\ddot{x}(t) + F(t, \dot{x}(t)) + H(t, x(h(t))) = 0.$$  

Also, there are no results on asymptotic behavior of solutions to neutral systems of vector differential equations
$$\ddot{x}_i(t) - Q_i(t)\dot{x}_i(g_i(t)) = A_i(t)x_i(h_i(t)) + \sum_{j=1}^{n} \sum_{k=1}^{m_{ij}} B_{ij}^k(t)x_j(h_{ij}^k(t)) + \sum_{j=1}^{n} \int_{g_{ij}(t)}^{t} K_{ij}(t, s)x_j(s)ds,$$  
i = 1, \ldots, n

neutral vector equations of the second order
$$\ddot{x}(t) + A\dot{x}(g(t)) + Bx(h(t)) = C\ddot{x}(p(t)).$$  

It would be interesting to study stability for systems of second order equations even in the scalar case, such as
$$\ddot{x}(t) + A_1\dot{x}(g_1(t)) + B_1x(h_1(t)) = 0,$$
$$\ddot{y}(t) + A_2\dot{y}(g_2(t)) + B_2x(h_2(t)) = 0$$
and
$$\ddot{x}(t) + A_1\dot{x}(g_1(t)) + B_1x(h_1(t)) = C_1y(p_1(t)),$$
$$\ddot{y}(t) + A_2\dot{y}(g_2(t)) + B_2y(h_2(t)) = C_2x(p_2(t)),$$

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