Staggered $\mathcal{PT}$-symmetric ladders with cubic nonlinearity

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We introduce a ladder-shaped chain with each rung carrying a $\mathcal{PT}$-symmetric gain-loss dipole. The polarity of the dipoles is staggered along the chain, meaning that a rung bearing gain-loss is followed by one bearing loss-gain. This renders the system $\mathcal{PT}$-symmetric in both horizontal and vertical directions. The system is governed by a pair of linearly coupled discrete nonlinear Schrödinger (DNLS) equations with self-focusing or defocusing cubic onsite nonlinearity. Starting from the analytically tractable anti-continuum limit of uncoupled rungs and using the Newton’s method for identifying solutions and parametric continuation in the inter-rung coupling for following the associated branches, we construct families of $\mathcal{PT}$-symmetric discrete solitons and identify their stability regions. Waveforms stemming from a single excited rung, as well as ones from multiple rungs are identified. Dynamics of unstable solitons is presented too.

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I. INTRODUCTION

A vast research area, often called discrete nonlinear optics, is dealing, at the theoretical and experimental level alike, with evanescently coupled arrayed waveguides that feature conspicuous material nonlinearity [1]. Effectively discrete arrays of optical waveguides have drawn a great deal of interest not only because they introduce a vast phenomenology of the nonlinear light propagation, such as e.g. the prediction [2] and experimental creation [3] of discrete vortex solitons, but also due to the fact that they offer a unique platform for emulating the transmission of electric signals in solid-state devices, which is obviously interesting for both fundamental studies and applications [1-4]. Furthermore, the flexibility of techniques used for the creation of virtual (photoinduced) [5] and permanently written [6] guiding arrays enables the exploration of effects which can be difficult to directly observe in other physical settings, such as the Anderson localization [7].

Another field in which arrays of quasi-discrete waveguides find a natural application is the realization of the optical $\mathcal{PT}$ (parity-time) symmetry [8]. On the one hand, a pair of coupled nonlinear waveguides, which carry mutually balanced gain and loss, make it possible to realize $\mathcal{PT}$-symmetric spatial or temporal solitons (if the waveguides are planar ones or fibers, respectively), which admit an exact analytical solution, including their stability analysis [9]. On the other hand, a $\mathcal{PT}$-symmetric dimer, i.e., the balanced pair of gain and loss elements, can be embedded, as a defect, into a regular guiding array, with the objective to study the scattering of incident waves on the dimer [10] [11]. Discrete solitons pinned to a nonlinear $\mathcal{PT}$-symmetric defect have been reported too [12]. Generally, such systems, although governed by discrete nonlinear Schrödinger (DNLS) equations corresponding to non-Hermitian Hamiltonians, generate real eigenvalue spectra (at the linear level), provided that the gain-loss strength does not exceed a critical level, above which the $\mathcal{PT}$ symmetry is broken [13].

One-dimensional lattices built of $\mathcal{PT}$ dimers were considered in Refs. [14] and [15]. Discrete solitons, both quiescent and moving ones, were found in this system [14]. In the continuum limit, those solitons go over into those in the above-mentioned $\mathcal{PT}$-symmetric coupler [9]. Accordingly, a part of the soliton family is stable, and the other one is unstable. Pairs of dimers in the form of $\mathcal{PT}$-symmetric plaquettes were also examined (motivated by the consideration of these as building blocks for two-dimensional chains) in the work of [16].

The objective of the present work is to introduce a staggered chain of $\mathcal{PT}$-symmetric dimers, with the orientations of the dimers alternating between adjacent sites of the chain. This can also be thought of as an extension of one of the plaquettes of [16] towards a lattice. While our ladder is not a genuinely two-dimensional lattice, sometimes similar extensions of a one-dimensional, yet not fully two-dimensional chains are referred to as one-and-a-half dimensional systems [17].

As shown in Section II, where the model is introduced, the fundamental difference from the previously studied one is the fact that such a system, although being quasi-one-dimensional, actually realizes the $\mathcal{PT}$ symmetry in the two-dimensional form, with respect to both horizontal and vertical directions. Unlike its unstaggered counterpart, the present system does not admit a nontrivial continuum limit. Therefore, in Section III we start the analysis from the solvable anti-continuum limit, i.e., the limit where the rungs of the ladder are uncoupled. Using a parametric continuation from this limit makes it possible to construct families of discrete solitons in a numerical form. In a way similar to what is known e.g. for the DNLS limit [18], both solutions initiated at a single “node” (in our case, rung), as well as ones with initial support on multiple rungs are explored and are found to yield associated branches. The soliton stability
is systematically analyzed in Section III too and when the structures are identified as unstable, their dynamical evolution is examined to observe how the relevant instability is manifested. The paper is concluded by Section IV, where also some directions for future study are presented.

II. THE MODEL

We consider a ladder configuration governed by the DNLS system with intersite coupling constant $C$,

$$
i \frac{d \Psi_n}{dt} + \frac{C}{2} (\Psi_{n+1} + \Psi_{n-1} - 2\Psi_n) + \sigma |\Psi_n|^2 \Psi_n = i\gamma \Psi_n - \kappa \Phi_n, \quad (1)$$

$$i \frac{d \Phi_n}{dt} + \frac{C}{2} (\Phi_{n+1} + \Phi_{n-1} - 2\Phi_n) + \sigma |\Phi_n|^2 \Phi_n = -i\gamma \Phi_n - \kappa \Psi_n,$$

where the evolution variable $t$ represents the propagation distance, in terms of the optical realization. Coefficients $+i\gamma$ and $-i\gamma$ with $\gamma > 0$ represent $\mathcal{PT}$-symmetric gain-loss dimers with whose orientation is staggered (alternates) along the ladder, the sites carrying gain and loss being represented by amplitudes $\Psi_n(t)$ and $\Phi_n(t)$, respectively. Cubic nonlinearity with coefficient $\sigma$ is present at every site, and $\kappa > 0$ accounts for the vertical coupling along the ladder’s rungs, each of which represents a $\mathcal{PT}$-symmetric dimer. The ladder is displayed in Fig. 1. As seen from the figure, the quasi-one-dimensional ladder realizes the $\mathcal{PT}$ symmetry in the two-dimensional form, with respect to the horizontal axis, running between the top and bottom strings, and, simultaneously, with respect to any vertical axis running between adjacent rungs.

By means of obvious rescaling, we can fix $|\sigma| = 1$, hence the nonlinearity coefficient takes only two distinct values, which correspond, respectively, to the self-focusing and defocusing onsite nonlinearity, $\sigma = +1$ and $\sigma = -1$. The usual DNLS equation admits the sign reversal of $\sigma$ by means of the well-known staggering transformation [18]. However, once we fix $\gamma > 0$ (and also $\kappa > 0$) in Eq. (1), this transformation cannot be applied, as it would also invert the signs of $\gamma$ and $\kappa$.

The formal continuum limit of system (1), corresponding to $C \to \infty$ and the replacement of the finite-difference derivative by the one with respect to the respective continuous coordinate, $x \equiv n/\sqrt{C}$, is

$$
i \frac{\partial \Psi}{\partial t} + \frac{1}{2} \frac{\partial^2 \Psi}{\partial x^2} + C (\Phi - \Psi) + \sigma |\Psi|^2 \Psi = i\gamma \Psi - \kappa \Phi,$$

$$i \frac{\partial \Phi}{\partial t} + \frac{1}{2} \frac{\partial^2 \Phi}{\partial x^2} + C (\Psi - \Phi) + \sigma |\Phi|^2 \Phi = -i\gamma \Phi - \kappa \Psi. \quad (2)$$

This limit-form system is not a self-consistent one, as it implies that the difference between the two components, $\Phi - \Psi$, must be vanishingly small (in the regime of $C \to \infty$), which contradicts the competition between the gain and loss terms in the system. The inconsistency of the continuum limit is an apparent corollary of the staggered structure of the system. As shown below, a natural limit for examining the underlying discrete system is the opposite, anti-continuum one, which corresponds to $C \to 0$, i.e., the limit of uncoupled rungs (each corresponding to a $\mathcal{PT}$-symmetric dimer).

Stationary solutions to Eqs. (1) with real propagation constant $\Lambda$ are sought in the usual form, $\Psi_n = e^{i\Lambda t} u_n$ and $\Phi_n = e^{i\Lambda t} v_n$, where functions $u_n$ and $v_n$ obey the stationary equations,

$$-\Lambda u_n + \frac{C}{2} (u_{n+1} + u_{n-1} - 2u_n) + \sigma |u_n|^2 u_n = i\gamma u_n - \kappa v_n,$$

$$-\Lambda v_n + \frac{C}{2} (v_{n+1} + v_{n-1} - 2v_n) + \sigma |v_n|^2 v_n = -i\gamma v_n - \kappa u_n. \quad (3)$$

Numerical solutions of these equations for discrete solitons are produced in the next section. To analyze the stability of the solutions, we add small perturbations with (a formally small) amplitude $\varepsilon$ and frequencies $\nu$,

$$\Psi_n(t) = (u_n + \varepsilon (a_n e^{i\nu t} + b_n e^{-i\nu t})) e^{i\Lambda t},$$

$$\Phi_n(t) = (v_n + \varepsilon (c_n e^{i\nu t} + d_n e^{-i\nu t})) e^{i\Lambda t}. \quad (4)$$

The linearization of Eq. (1) leads to the eigenvalue problem,

$$M \begin{bmatrix} a_n \\ b_n \\ c_n \\ d_n \end{bmatrix} = \nu \begin{bmatrix} a_n \\ b_n \\ c_n \\ d_n \end{bmatrix}, \quad (5)$$

where $M$ is a $4N \times 4N$ matrix for the ladder of length $N$. Using standard indexing, $N \times N$ submatrices of $M$
are defined as
\[
\begin{align*}
M_{11} &= M_{33} = \text{diag}(q_n - C), \\
M_{22} &= M_{44} = \text{diag}(p_n + C), \\
M_{12} &= -M_{21} = \text{diag}(\sigma u_n^2), \\
M_{34} &= -M_{43} = \text{diag}(\sigma u_n^2), \\
M_{13} &= M_{31} = -M_{24} = -M_{42} = \frac{C}{2} G + \text{diag}(\kappa),
\end{align*}
\]  
(6)

where \( G \) is an \( N \times N \) matrix of zero elements, except for the super- and sub-diagonals that contain all ones.

For the zero solution of the stationary equation \( u_n = v_n = 0 \), the matrix \( M \) has constant coefficients, and perturbation eigenmodes can be sought for as \( a_n = A e^{i k n}, b_n = 0, c_n = B e^{i k n}, d_n = 0 \). Then Eq. (5) becomes a \( 2 \times 2 \) system, whose eigenvalues can be found explicitly:
\[
\nu = -(\Lambda + C) \pm \sqrt{(\cos k + \kappa)^2 - \gamma^2},
\]
so that \( \nu \) is real only for \( C \leq \kappa - \gamma \). In other words, the \( \mathcal{PT} \)-symmetry is broken, \( iv \) has a positive real part, and the perturbations grow exponentially for
\[
\gamma > \gamma_{cr}^{\text{(1)}}(C) \equiv \kappa - C.
\]
(9)

It is interesting to observe here how the coupling between the rungs decreases the size of the interval of exact \( \mathcal{PT} \)-symmetry, well-known in the case of the dimer \([8, 13]\). In the stability region, Eq. (6) demonstrates that real perturbation frequencies take values in the following intervals:
\[
\begin{align*}
-\left(\Lambda + C\right) - \sqrt{(\cos k + \kappa)^2 - \gamma^2} &< \nu < -\left(\Lambda + C\right) - \sqrt{(\cos k - \kappa)^2 - \gamma^2}, \\
-\left(\Lambda + C\right) + \sqrt{(\cos k - \kappa)^2 - \gamma^2} &< \nu < -\left(\Lambda + C\right) + \sqrt{(\cos k + \kappa)^2 - \gamma^2}.
\end{align*}
\]
(10)

Similarly, for the perturbations in the form of \( a_n = 0, b_n = A e^{i k n}, c_n = 0, d_n = B e^{i k n} \) the negatives of expressions (9) are also eigenvalues of the zero stationary solution, and at \( \gamma > \gamma_{cr}^{\text{(1)}}(C) \), they fall into the negatives of intervals (10).

Simultaneously, Eq. (5) and its negative counterpart give the dispersion relation for plane waves (“phonons”) in the linearized version of Eq. (1). Accordingly, intervals (10), along with their negative counterparts, represent phonon bands of the linearized system.

In Section III we produce stationary solutions in the form of discrete solitons. This computation begins by finding exact solutions in the anti-continuum limit, \( C = 0 \), of uncoupled rungs and then the solutions are continued numerically to \( C > 0 \), by means of the Newton’s method for each \( C \) and using parametric continuation (i.e., utilizing the converged iterate of the previous step in \( C \) as an initial seed for the Newton in a nearby value of the parameter). As suggested by Eq. (9), we take parameter values \( 0 < \gamma \leq \kappa \), so as at \( C = 0 \) to remain within the \( \mathcal{PT} \)-symmetric region. Subsequently, the stability interval of the different constructed solutions is explored.

**III. DISCRETE SOLITONS AND THEIR STABILITY**

**A. The anti-continuum limit, \( C = 0 \)**

To construct stationary soliton solutions of Eqs. (1) at \( C = 0 \), when individual rungs are decoupled, it is possible to substitute
\[
\begin{align*}
u_n &= e^{i k_n} u_n
\end{align*}
\]  
(11)
FIG. 3: Profiles of discrete solitons for $C = 0.4$. The configurations of the initial ($C = 0$) solution and other parameters follow the same pattern as in Figure [2].

FIG. 4: (Color online) Width diagnostic $w$, defined as per Eq. [20], corresponding to each of the plots in Fig. [2].

with real $\delta_n$ in Eq. [3], which yields relations

$$\gamma = -\kappa \sin \delta_n, \quad \sigma |v_n|^2 = -\kappa \cos \delta_n + \Lambda. \quad (12)$$

For the uncoupled ladder, one can specify e.g. either a single-rung solution, with fields at all sites set equal to zero except for $u_1, v_1$ satisfying Eqs. (11) and (12), or a double-rung solution with nonzero fields $u_1, v_1$ and $u_2, v_2$ satisfying the same equations. We focus on these two possibilities in the anticontinuum limit, although larger-size solutions are possible too. These are the canonical analogues of the single-node and two-node solutions that have been extensively studied in the Hamiltonian form of the DNLS model both in one and in higher dimensions [13].

We take parameters satisfying the constraints

$$\sigma > 0, \quad \Lambda > \kappa, \quad \gamma > 0. \quad (13)$$

to make the second equation (12) self-consistent. Then, two solution branches for $\delta_n$ are possible. The first branch satisfies $-\pi/2 \leq \delta_n = \arcsin(-\gamma/\kappa) \leq 0$ and $\cos(\delta_n) \geq 0$. Choosing a solution with $\delta_n = \delta_{in}$ at the rung carrying nonzero fields, we name it an in-phase rung, as the phase shift between the two components is smaller than $\pi/2$, namely, $|\arg(v_n u_n^*)| \in [0, \pi/2]$. The second branch satisfies $-\pi \leq \delta_{out} \equiv -\pi + |\delta_n| \leq -\pi/2$ and $\cos \delta_{out} \leq 0$. The rung carrying the solution with $\delta_n = \delta_{out}$ is called an out-of-phase one, as its phase shift between the components exceeds $\pi/2$, viz., $|\arg(v_n u_n^*)| \in [\pi/2, \pi]$. The two branches meet and disappear when $\gamma = \kappa = \gamma_{cr}^{(1)}(0)$ [see Eq. (9)] is the boundary of the $\mathcal{P}\mathcal{T}$-symmetric region for $C = 0$. This designation can
be also principally thought to stem from the Hamiltonian limit of $\gamma = 0$, where indeed $\delta_{\text{in}} = 0$ corresponds to the in-phase solution of the single Hamiltonian dimer, while $\delta_{\text{out}} = \pi$ corresponds to the out-of-phase one.

The stability eigenfrequencies for stationary solitons at $C = 0$ can be calculated analytically in this uncoupled limit. In this case, $M$ has the same eigenvalues as submatrices

$$m_0 = \begin{bmatrix} -\Lambda - i\gamma & 0 & \kappa & 0 \\ 0 & \Lambda - i\gamma & 0 & -\kappa \\ \kappa & 0 & -\Lambda + i\gamma & 0 \\ 0 & -\kappa & 0 & \Lambda + i\gamma \end{bmatrix}, \quad (14)$$

which is associated with zero-amplitude (unexcited) rungs, and

$$m_n = \begin{bmatrix} q_n^* & \sigma u_n^2 & \kappa & 0 \\ -\kappa (u_n^*)^2 & p_n^* & 0 & -\kappa \\ \kappa & 0 & q_n & 0 \\ 0 & -\kappa & -\sigma (v_n)^2 & p_n \end{bmatrix}, \quad (15)$$

associated with the excited rungs (those carrying nonzero stationary fields), with $v_n$, $u_n$ taken as per Eqs. (11) and (12). In other words, each of the four eigenvalues of $m_0$,

$$\nu = \pm \Lambda \pm \sqrt{\kappa^2 - \gamma^2}, \quad (16)$$

is an eigenvalue of $M$ with multiplicity equal to the number of zero-amplitude rungs, while each of the four eigenvalues of $m_n$,

$$\nu = \pm 0, \pm 2\Lambda \sqrt{2\alpha_{\text{in}}^2 - \alpha_{\text{out}}}, \quad (17)$$

appears as an eigenvalue of $M$ with multiplicity equal to the number of excited rungs. Here $\alpha_{\text{in}} = \alpha_{\text{out}} = \kappa \cos(\delta_{\text{in}})/\Lambda \equiv \sqrt{(\kappa^2 - \gamma^2)/\Lambda^2}$, and $\alpha_{\text{in}} = \alpha_{\text{out}} = \kappa \cos(\delta_{\text{out}})/\Lambda \equiv -\sqrt{(\kappa^2 - \gamma^2)/\Lambda^2}$ for an in- and out-of-phase rung, respectively.

Equation (17) shows that the out-of-phase excited rung is always stable, as it has $\text{Re}(\nu) = 0$. Similarly, the in-
increasing in steps of $\Delta \alpha$, the phase rung is now associated to negative $\alpha$ and so $\alpha$ is always stable, while its out-of-phase counterpart is unstable for $0 < \alpha < 1 - \alpha$. For the top right and middle right plots, the soliton’s field is nonzero at one or two in-phase rungs, while all others are out-of-phase. For the top left and middle left plots, the soliton’s field is nonzero at one or two central rungs, and all rungs remain out-of-phase with the increase of $C$. In the bottom plot, only the $n = 1$ rung remains in-phase, while all others are out-of-phase.

phase excited rung is stable for $\kappa^2 - \gamma^2 \geq \Lambda^2/4$, and unstable for $0 < \kappa^2 - \gamma^2 < \Lambda^2/4$. Thus, for solutions that contain a nonzero in-phase rung in the initial configuration at $C = 0$, there are the two critical values, $\gamma_{\text{cr}}^{(1)}(C = 0) = \kappa$ given by Eq. (9), and the additional one,

$$\gamma_{\text{cr}}^{(2)}(C = 0) = \sqrt{\kappa^2 - \Lambda^2/4}. \tag{18}$$

A choice alternative to Eq. (13) is

$$\sigma < 0, \quad \Lambda < -\kappa. \tag{19}$$

In this case, the analysis differs only in that the sign of $\alpha$, in Eq. (17) is switched. That is, the in-phase rung is now associated to negative $\alpha = \alpha_{\text{in}} = \kappa \cos(\delta_{\text{in}})/\Lambda = -\sqrt{(\kappa^2 - \gamma^2)/\Lambda^2}$, while the out-of-phase rung – to the positive one, $\alpha = \alpha_{\text{out}} = \kappa \cos(\delta_{\text{out}})/\Lambda \equiv \sqrt{(\kappa^2 - \gamma^2)/\Lambda^2}$. In the present case, the in-phase rung is always stable, while its out-of-phase counterpart is unstable at $\gamma > \gamma_{\text{cr}}^{(2)}(C = 0)$, see Eq. (18).

B. Discrete solitons at $C > 0$

To construct soliton solutions for coupling constant $C$ increasing in steps of $\Delta C$, we write Eq. (9) as a system of $4N$ equations for $4N$ real unknowns $w_n, x_n, y_n, z_n$, so that $u_n = w_n + ix_n, \quad v_n = y_n + iz_n$. Then we apply Newton’s method with the initial guess at each step taken as the soliton solution found at the previous value of $C$, as indicated above. Thus, the initial guess at $C = \Delta C$ is the analytical solution for $C = 0$ given by Eqs. (11)-(12) with parameters taken according to either Eq. (13) or Eq. (19).

Figure 2 shows $|u_n|^2$ for the solutions identified by this process on a logarithmic scale as a function of $C$ for parameters taken as per Eq. (13). The use of the logarithmic scale has been chosen for clarity, as we have found it to yield a clearer sense of the solution’s spatial width variation as the parameter $C$ is changed. The different solutions illustrated in the figure involve a single rung involving an in-phase, as well as an out-of-phase excitation within the rung (top row) and then two-rung excitations for which there are three possibilities: an in-phase in both and out-of-phase in both (second row), as well as a mixed phase involving one rung initially excited in-phase.
and one excited out-of-phase (bottom row). In Figure 3 we plot $|u_n|^2$ for fixed $C$ across the various configurations. A point that is clearly illustrated by this figure that is not evident in the logarithmic scale of Figure 2 is the asymmetric spatial distribution of the mixed phase solution of the bottom row. Equations (11)-(12) show that for $C = 0$, since the out-of-phase case corresponds to $\cos(\delta_{out}) \leq 0$, the amplitude for an out-of-phase rung $|v_n| = |u_n|$ with $\sigma = 1$ is larger in comparison to an in-phase rung where $\cos(\delta_{in}) \geq 0$. The asymmetry for the mixed phase solution persists for $C > 0$; thus Fig. 3 shows an example of this. Then, Fig. 4 shows more explicitly/quantitatively the increasing width of the soliton, using a second moment (of the density distribution) diagnostic in the form

$$w(C) \equiv \sqrt{\frac{\sum_n n^2 |u_n|^2}{\sum_n |u_n|^2}}$$

as a function of $C$ for the solutions shown in Fig. 2. It is interesting to point out that the variation of this width-measuring quantity is observed to be fairly minimal in the case of out-of-phase solutions (or mixed ones), while it is more significant in the case of in-phase excited rungs (single or multiple).

In Figure 5, the absolute value of the phase difference between fields $u_n$ and $v_n$ taken at two sides of the ladder is shown. In other words, Fig. 5 shows whether each rung of the ladder is (closer to) in-phase or out-of-phase, as a function of $C$. This figure reveals that as $C$ increases there is a progressive spatial expansion (across $n$) in the number of sites supporting a phase difference that develops against the backdrop of the principal excited sites.

Figures 6, 7, 8, 9 are similar to their counterparts 2, 3, 4, 5 respectively, but with the parameters taken as per Eq. (19) instead of Eq. (13). Comparing Figs. 5 and 6 shows that the solutions for $\sigma = 1$ favor in-phase rungs, as $C$ increases, and $\sigma = -1$ favors the solutions with out-of-phase rungs. That is to say the progressive expansion of the background rung relative phase is closer to the in-phase value for the $\sigma = 1$ case and to the out-of-phase values for $\sigma = -1$. Also by (12), since $\sigma = -1$ the asymmetry of the mixed phase solution is switched in comparison to the $\sigma = 1$ case. That is, the in-phase rung has larger magnitude than the out-of-phase rung.
deform the relevant stability region. The additional instability stemming from the in-phase rungs in Fig. 10 and the out of phase ones in Fig. 11 can be separately observed in the left panels of the former figure and the right panels of the latter. Given that this critical point stems from a single-rung analysis, we have no $C$ dependence, but clearly this delimits (some times fairly dramatically as in the middle row left panel of Fig. 10 and the middle right row of Fig. 11 i.e., for the two-site, same-phase excitations susceptible to this instability mechanism) the stability of the resulting state. Although the precise stability thresholds may be fairly complex and arise from the interplay (in the presence of nonlinearity) of localized and extended modes, a general conclusion is that the above instabilities play a critical role for the stability of the ladder states (see also the discussion below), and another is that the higher the coupling, the less robust the corresponding solutions are likely to be, some times even fairly dramatically so.

The values of $\nu \gamma$ whose maximum real part is represented in Figs. 10 and 11 are computed numerically via an appropriate numerical eigenvalue solver. At $C = 0$, the eigenvalues agree with Eqs. (16) and (17). As shown in Figs. 12 and 13 following the variation of $C$ and $\gamma$, eigenvalues (16), associated with zero-amplitude sites, vary in accordance with the prediction of Eq. (8), and eigenvalues (17), associated with excited rungs, also shift. In the case of the mixed phase solutions, there are parametric intervals (across $C > 0$ for fixed $\gamma$) for which solutions have been identified with different phase profiles than the ones initialized (in the fixed point iteration) at the anti-continuum limit of $C = 0$. It is these distinct branches of solutions that cause the apparent “streaks” in the bottom middle panel of Fig. 10. Such alternate solutions have similar profiles to those shown in Fig. 3 and the gain in width as a function of $C$ is similar to the examples we show in Figs. 2 4. Also, the mechanisms by which solutions become unstable for these alternate solutions follow the guidelines we outline below.

The most obvious type of instability is associated with initializing a solution at $C = 0$ with a single unstable rung, i.e. with $\gamma > \gamma_c^{(2)}(0)$ in (18). Eigenvalues for this type of instability are shown in the top two panels of Fig. 13. There are three other ways in which solutions become unstable as $C$ increases, each corresponding to a particular type of critical events. We demonstrate these critical events in Figs. 12 13. The first type occurs when $\nu$ associated with an excited rung collides with one of the intervals in Eq. (10). This weak instability generates an eigenfrequency quartet and is represented in Figs. 10 and 11 where the green boundary deviates (as $C$ increases from 0) from the threshold of Eq. (18). Figures 12 and 13 illustrate this first type of critical event in more detail by plotting eigenvalues directly in the complex plane.

The second type of the critical event takes place when the intervals in Eq. (10) come to overlap at $\gamma = \gamma_c^{(1)}(C)$, see Eq. (9). This is the background instability associated to zero sites, which is shown in Figs. 10 and 11 as

C. Stability of the discrete solitons

Figure 10 shows two-parameter stability diagrams for the solitons by plotting the largest instability growth rate (if any), $\max(\text{Re}(\nu\gamma))$, as a function of $C$ and $\gamma$ for parameter values taken as per Eq. (13), and Fig. 11 shows the same for taken according to Eq. (19). When the relevant quantity is non-vanishing (the relevant numerical threshold used is outlined by the green –white in the black and white version line), this signals the instability of the corresponding waveform. Some comments are relevant to add here. Recall from Eqs. (19) and (18) that some stability limitations are present, respectively, from the point of view of the uniform (vanishing) solutions in the former case from that of the single site excitations in the latter case. The former “background stability” condition indicates that the (parallel to the antidiagonal, cyan in Fig. 10) line of $\gamma = \kappa - C$ poses an upper bound on the potential stability of any excitation (since this is the condition for the stability of the background on top of which any solitary wave is constructed). It can be seen in both Figs. 10 and 11 that especially so in cases like the right panels of the former and the left panels of the latter (where the instability of (18) is less relevant), the relevant threshold can be clearly observed as outlining the stability region of the solutions. Of course additional instabilities due to the localized part of the solution are possible and can be observed in these panels to somewhat
bright spots originating from the corners of the diagrams
where \( \gamma = \kappa = \gamma^{(1)}_{\text{cr}} (C = 0) \). A more detailed plot of
these eigenvalues and the corresponding collisions in the
complex eigenvalue plane is presented in Fig. 12.

A third type of instability appears to occur for essen-
tially all values of \( C \) in the case of two in-phase rungs
for \( \sigma > 0 \) and of two out-of-phase ones for \( \sigma < 0 \). I.e.,
this can be thought of as a localized instability due to
the combined presence of two potentially unstable
elements due to the instability of Eq. (18). At \( C > 0 \), it
is seen as the bright spots in Figs. 10 and 11 originating
from \( \gamma^{(2)}_{\text{cr}} (C = 0) = \sqrt{\kappa^2 - \Lambda^2 / 4} \). The eigenvalues
emerge from corresponding zero eigenvalues at \( C = 0 \).
That is, in the middle row left plot of Fig. 10 at \( C = 0 \)
for \( \gamma < \gamma^{(2)}_{\text{cr}} (C = 0) \) there are four zero eigenvalues; as
\( C \) increases two of the four eigenvalues emerge from zero
onto the real axis in the complex plane. A similar effect
is observed for \( \gamma < \gamma^{(2)}_{\text{cr}} (C = 0) \) in the middle row right
plot of Fig. 11.

Finally, it is worth making one more observation in
connection e.g. to Fig. 13 and the associated jagged
lines in top right panel of Fig. 11. Notice that as \( C \) is
increased there is an initial stabilization of the mode un-
stable due to the criterion of Eq. (18), but then a collision
with the continuous spectrum on the imaginary axis pro-
vides anew a destabilization. It is this cascade of events
that accounts for the jaggedness of the curve in the top
right of Fig. 11 and in other similar occurrences (e.g. the
top left of Fig. 10). We add this explanation to the
elaborate set of possible instabilities discussed above to
shed a partial light to the complex form of the stability
boundaries encountered in our two-dimensional plots.

D. The evolution of discrete solitons

To verify the above stability predictions, we simulated
evolution of the stationary solutions in the framework of
Eq. (1) by means of the standard Runge-Kutta 4th order
integration scheme. In Figs. 14, 15, and 16 we display
elements of the evolution of each of the three instability
types which were identified in Section 3.3.

For the first type, when the instability arises from col-
sion of eigenvalues associated with the excited and unex-
cited rungs, the corresponding unstable eigenmode arises
in the form of a quartet of eigenfrequencies. In Fig. 14,
we demonstrate that this instability leads to the increase
of the solution amplitudes and oscillations at the cen-
tral rung. The corresponding (chiefly localized, although
with a weakly decaying tail) eigenvector is shown in the
top panels of the figure, while the bottom panels show
how the initial condition evolves in time through an os-
cillatory growth in accordance with the expectation of an
unstable, genuinely complex eigenfrequency.

For the second type, with the instability arising from
the collision of eigenvalues in intervals (10) which are all
associated with zero-amplitude rungs, the corresponding
unstable eigenmode is delocalized. It is shown in Fig. 15
that the corresponding unstable soliton does not preserve
its shape. Instead, the instability delocalizes the solution
which acquires a tail reminiscent of the spatial profile of the
corresponding unstable eigenvector.

Lastly, the third type of instability is shown in the
example in Fig. 16. This shows the case of two excited
in-phase rungs with \( \sigma = 1 \). Other examples of the same
type look similar as e.g. in the case of \( \sigma = -1 \) with
two out-of-phase excited rungs. The instability appears
to have a localized manifestation with the gain nodes
of each rung growing and the corresponding loss nodes
decaying in amplitude.

IV. CONCLUSIONS

We have introduced the lattice of the ladder type with
staggered pairs of mutually compensated gain and loss
elements at each rung, and the usual onsite cubic non-
linearity, self-focusing or defocusing. This nearly-one-
dimensional system, which, nevertheless, features the
two-dimensional \( PT \) symmetry, may be realized in non-
linear optics. We have constructed families of discrete

![FIG. 14: (Color online) Evolution of the soliton whose instability is predicted is in the top-most left panel of Fig. 10 for \( C = 1 \) and \( \gamma = 0.5 \). The complex plane of all the eigenvalues for this solution is shown in the bottom left plot of Fig. 12. The right plot shows the squared absolute values of perturbation amplitudes \( a_n, b_n \) (higher amplitudes) and \( c_n, d_n \) (lower amplitudes) defined in Eq. (11). The top left plot shows the solution at \( t = 0 \), and the bottom left plot shows the solution at \( t = 122 \) with \( |\Psi_n(122)|^2 \) in blue and \( |\Phi_n(122)|^2 \) in green. In the course of the evolution, the soliton maintains its shape, while the amplitude on the central rung \( (n = 1) \) grows with oscillations; the growth on the gain side associated to the function \( \Psi \) is ultimately dominant. The quantities \( D_1(t) \equiv |\Psi_1(t)|^2 - |\Phi_1(t)|^2 \) and \( D_2(t) \equiv |\Psi_1(t)|^2 + |\Phi_1(t)|^2 \) are shown in the bottom right plot, in order to better demonstrate the growing oscillations.](image-url)
stationary solitons stemming from a single, as well as two excited rungs starting from the anti-continuum limit uncoupled rungs. We also identified their stability via the calculation of eigenfrequencies for modes of small perturbations, across the system’s parameter space. A part of the solitary wave families is found to be dynamically stable, while unstable solitons may exhibit three principal scenarios of the instability development, which have been identified too. These stem roughly from interactions of localized with extended modes, from extended modes alone or from localized modes alone, respectively.

A natural extension of the work may be the consideration of mobility of the discrete solitons. In particular, here we have constructed static solitary waves, but understanding their dynamical properties upon imparting a momentum on these would be of interest in its own right. A challenging perspective is the development of a two-dimensional extension of the system. Effectively, this would entail adding further alternating ladder pairs along the transverse direction and examining further configurations not only (perhaps) in the form of discrete solitons but also conceivably in that of discrete vortices, analogously to what has been earlier done in the DNLS system.

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FIG. 15: (Color online) The evolution of the discrete soliton whose instability is predicted is in the top-most left panel of Fig. [10] for C = 1.5 and γ = 0.5. The complex plane of all the eigenvalues for this solution is shown in the bottom right plot of Fig. [12]. The top right plot has the same meaning as in Fig. [14] where here |a_n|^2, |b_n|^2 are mostly zero while |a_0|^2, |b_0|^2 have nonzero amplitudes. In the course of the evolution the soliton does not maintain its shape. In particular, the solution profiles at t = 22 are shown in the left panel at the bottom bearing the apparent signature of the delocalized, unstable eigenmode; the delocalization is stronger in the Ψ solution associated with gain. Similar to Fig. [14] we plot D_1(t), D_2(t) in the bottom right plot. This plot shows that the central node experiences oscillations similar to Fig. [14], but the oscillatory effect is dominated by the delocalization seen in the bottom left plot, which grows and exceeds past the shorter peaks in the center.
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