One-body reduced density matrix of trapped impenetrable anyons in one dimension

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Abstract. We study the one-body reduced density matrix of a system of \( N \) one-dimensional impenetrable anyons trapped by a harmonic potential. To this purpose we extend two methods developed to tackle related problems, namely the determinant approach and the replica method. While the former is the basis for exact numerical computations at finite \( N \), the latter has the advantage of providing an analytic asymptotic expansion for large \( N \). We show that the first few terms of such expansion are sufficient to reproduce the numerical results to an excellent accuracy even for relatively small \( N \), thus demonstrating the effectiveness of the replica method.

Keywords: correlation functions, cold atoms, Lieb–Liniger model
1. Introduction

Quantum statistics is among the most fundamental concepts in physics and is at the basis of the existence of formidable physical systems at all scales, from Bose–Einstein condensates to neutron stars. While in three (and higher) spatial dimensions, bosonic and fermionic statistics exhaust all the possibilities, physicists have postulated and then investigated the existence of (quasi-)particles with generalized statistics, or anyons, in lower dimensions for some decades [1]. In two-dimensions, the fractional statistics of the elementary excitation of the quantum Hall effect has certainly represented a huge physical motivation in this direction [2]. In one-dimension, numerous models of anyons have been proposed ([3–18] among others), but for quite a long time most of them have been thought to be essentially playgrounds for theoreticians. This perspective seems bound to change in view of the new experimental achievements, in particular concerning cold atoms in optical lattices. In fact, in the cold atom framework a number of proposals have been formulated that indicate the realization of one-dimensional anyons as feasible within the current experimental techniques. Among those proposals, the one elaborated by Keilmann et al [19] and subsequently refined by Greschner and Santos [20] appears to be particularly promising. Their idea starts from ordinary bosonic or fermionic atoms and employs a Raman-assisted tunneling process to induce an effective occupation-dependent hopping; the system obtained in this way can be shown to be equivalent to the anyonic Hubbard model, in which the statistical parameter and the on-site interaction can be changed continuously. More recently, another very interesting experimental scheme has been suggested by Sträter et al [21], which makes use of...
the so-called lattice shaking technique combined with a static tilt of the potential in order to realize the occupation-dependent tunneling. This scheme does not require any additional lasers other than the ones that form the lattice, nor any assumptions on the internal atomic structure.

These kinds of developments give reasonable expectations that the knowledge of generalized statistics and its implication on quantum many-body physics collected in extensive studies of one-dimensional models can be progressively tested against experiments. In fact, one-dimensionality brings about a wealth of theoretical tools that are unavailable in the study of quantum many-body physics in higher dimensions, the Bethe Ansatz being a prominent example. However, despite the fact that the Bethe Ansatz solution provides information about the spectrum, the thermodynamic properties, etc, in general it does not make the calculation of correlation functions straightforward, so that one has to ultimately rely on extensive numerical computation \[22\].

Among the most interesting quantities is the one-body reduced density matrix (or off-diagonal correlation), that is defined as

\[ \rho_N(t, t') = N \int \mathcal{D}x^{N-1} \overline{\psi_N(t, x_2, \ldots, x_N)} \psi_N(t', x_2, \ldots, x_N), \]

where \( \psi_N \) is the \( N \)-body ground-state wavefunction, and gives access to fundamental physical observables such as momentum distribution, one-particle entanglement entropy, natural orbitals and their occupation, etc. Whereas its analytic determination remains a challenging task in general (see \[22, 23\]), there is at least a class of models, namely models of impenetrable particles (or Tonks–Girardeau models) in various geometries (circle, interval with Dirichlet or Neumann boundary conditions, harmonic well), in which this problem has enjoyed a sensible progress over the last years. This model is a limiting case of the Bethe Ansatz integrable Lieb–Liniger model of bosons \[24\] (or anyons) in which the zero-range mutual interaction (\( \delta \)-interaction) is infinite (impenetrable, or hard-core, limit). The relative simplicity of the Tonks–Girardeau model lies mainly in the fact that the \( N \)-body ground-state wavefunction can be related to the one of free spinless fermions via the boson-fermion \[25\] (or anyon-fermion \[26\]) mapping. Starting from this, it has been shown that \( \rho_N(t, t') \) in the bosonic case can be exactly expressed as the determinant of a matrix whose symmetry depends on the geometry, e.g. Toeplitz type for circular geometry, Hankel type for harmonic well, etc \[27\]. The generalization of this construction to anyons on a circle has been performed in \[8, 28, 29\]. Besides, in a remarkable work \[30\], Gangardt was able to find the full large distance expansion of \( \rho_N(t, t') \) in a closed form for bosons on a circle and in a harmonic well; this expansion, which is valid for distances larger than \( r_0/N \), where \( r_0 \) is a characteristic length depending on the geometry, has the advantage of being analytic and, in practice, yielding very accurate results also for finite \( N \). Building upon this, a prescription to apply the replica method to anyons in circular geometry has been proposed and tested in \[31\].

The purpose of this paper is to extend the study of the one-body reduced density matrix to Tonks–Girardeau anyons trapped by a harmonic potential within the framework described above, possibly providing a theoretical tool for future cold atom experiments which are indeed mainly performed in the presence of the harmonic trapping potential. First we derive the exact expression of the one-body reduced density matrix

\[ \rho_N(t, t') = N \int \mathcal{D}x^{N-1} \overline{\psi_N(t, x_2, \ldots, x_N)} \psi_N(t', x_2, \ldots, x_N), \]
in terms of a Hankel determinant in section 2. Afterwards, in section 3, we determine its complete asymptotic expansion by combining the replica method of [30] and the anyonic prescription of [31]. Section 4 is devoted to the presentation of the numerical results and the comparison of the two approaches for some choices of the parameters. Concluding remarks are given in section 5.

2. The one-body reduced density matrix as a Hankel determinant

Anyonic statistics in one dimension can be defined by introducing field operators with the following commutation relations:

\[
\Psi_A^\dagger(x_1)\Psi_A^\dagger(x_2) = e^{i\kappa\pi(x_1-x_2)}\Psi_A^\dagger(x_2)\Psi_A^\dagger(x_1),
\]

\[
\Psi_A(x_1)\Psi_A^\dagger(x_2) = e^{-i\kappa\pi(x_1-x_2)}\Psi_A^\dagger(x_2)\Psi_A^\dagger(x_1) + \delta(x_1-x_2),
\]

where \(\epsilon(z) = -\epsilon(-z) = 1\) for \(z > 0\) and \(\epsilon(0) = 0\). \(\kappa\) is called the statistical parameter and equals 0 for bosons and 1 for fermions. We will be interested in anyons interacting with a repulsive \(\delta\)-interaction and subject to an external harmonic potential. A system of \(N\) such particles is described, in the first quantized language, by

\[
H = \sum_i \left( -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x_i^2} + \frac{1}{2} m\omega^2 x_i^2 \right) + 2c \sum_{1 \leq i < j \leq N} \delta(x_i - x_j).
\]

More specifically, we will deal with the Tonks–Girardeau limit \(c \to \infty\). By taking the distance and energy units as \((\hbar/m\omega)^{1/2}\) and \(\hbar\omega/2\) respectively, the single particle eigenstates of the Hamiltonian equation (2) can be written as

\[
\phi_k(x) = \frac{2^{-k}}{\sqrt{c_k}} e^{-x^2/2} H_k(x), \quad k = 0, 1, 2, \ldots,
\]

where \((c_k^H)^2 = 2^{-k} \pi^{1/2} k!\) and \(H_k(x)\) is the \(k\)th Hermite polynomial. In the case of fermions, it is well-known that the \(N\)-body ground-state wavefunction is

\[
\psi_N^F(x_1, \ldots, x_N) = \frac{1}{\sqrt{N!}} \det[\phi_{j-1}(x_k)]_{k=1,\ldots,N} = \frac{1}{\sqrt{N!}} \frac{C_N^H}{C_N^H} \prod_{i=1}^N e^{-x_i^2/2} \prod_{1 \leq i < j \leq N} (x_i - x_j), \quad C_N^H = \prod_{k=0}^{N-1} c_k^H,
\]

where \(\prod_{1 \leq i < j \leq N}(x_i - x_j) \equiv \Delta_N(x)\) is a Vandermonde determinant, which appears because \(2^{-k} H_k(s)\) is a monic polynomial and the determinant is a multilinear alternating form. The ground-state wavefunctions of \(N\) anyons with statistical parameter \(\kappa\) can be found from here by applying the anyon-fermion mapping [26]

\[
\psi_N^A(x_1, \ldots, x_N) = \left[ \prod_{1 \leq i < j \leq N} A(x_j - x_i) \right] \psi_N^F(x_1, \ldots, x_N),
\]

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with

\[ A^\kappa(x_j - x_i) = \begin{cases} e^{i\pi(1-\kappa)} & x_j < x_i, \\ 1 & x_j > x_i \end{cases} \]  

(6)

Note that \( \psi^F_N(x_1, \ldots, x_N) = \psi^J_N(x_1, \ldots, x_N) \). The relation

\[ \psi^\kappa_{N+1}(t, x_1, \ldots, x_N) = \frac{e^{-x^2/2}}{\sqrt{N+1}} \prod_{i=1}^{N} A^\kappa(x_i - t)(x_i - t) \psi^\kappa_N(x_1, \ldots, x_N), \]  

(7)

allows us to express the one-body reduced density matrix of this system as

\[
\rho^\kappa_{N+1}(t, t') = (N+1) \int d^N x \overline{\psi}^\kappa_{N+1}(t, x_1, \ldots, x_N) \psi^\kappa_{N+1}(t', x_1, \ldots, x_N) \\
= \frac{2^N}{N!} \frac{e^{-t^2/2}e^{-t'^2/2}}{\sqrt{\pi}} \int d^N x \prod_{i=1}^{N} A^\kappa(x_i - t)(x_i - t) \\
\times A^\kappa(x_i - t')(x_i - t')[\psi^F_N(x_1, \ldots, x_N)]^2
\]  

(8)

where we have used \( |\psi^\kappa_N(x_1, \ldots, x_N)|^2 = |\psi^F_N(x_1, \ldots, x_N)|^2 \). Plugging equation (4) into equation (8), we see that the latter is suitable for the application of the identity (see equation (74) in [32])

\[
\frac{1}{N!} \prod_{l=1}^{N} \int_{-\infty}^{\infty} dx_l g(x_l) \left( \det[f_{j-1}(x_l)]_{j,k=1,\ldots,N} \right)^2 = \det \left[ \int_{-\infty}^{\infty} ds g(s) f_{j-1}(s)f_{k-1}(s) \right]_{j,k=1,\ldots,N},
\]

(9)

from which

\[
\rho^\kappa_{N+1}(t, t') = \frac{2^N}{N!} \frac{e^{-t^2/2}e^{-t'^2/2}}{\sqrt{\pi}} \det \left[ \int_{-\infty}^{\infty} ds A^\kappa(s - t)(s - t) \\
\times A^\kappa(s - t')(s - t')\phi_j(s)\phi_k(s) \right]_{j,k=0,\ldots,N-1} \\
\equiv \frac{2^N}{N!} \frac{e^{-t^2/2}e^{-t'^2/2}}{\sqrt{\pi}} \det \left[ \frac{2^{i+j/2}}{2\sqrt{\pi}\Gamma(i+j)} a^\kappa_{ij}(t, t') \right]_{j,k=1,\ldots,N},
\]

(10)

where in the last line we have relabeled as \( i, j \rightarrow i+1, j+1 \), and employing the multilinearity of the determinant we can write

\[
a^\kappa_{ij}(t, t') = \int_{-\infty}^{\infty} ds A^\kappa(s - t)(s - t)A^\kappa(s - t')(s - t')s^{i+j-2}e^{-s^2},
\]

(11)

which is manifestly a Hankel matrix. Let us express the above equation in terms of (incomplete) gamma functions [32]. Using the definition equation (6)

\[
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\]

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\[
\alpha_{jk}(t, t') = \int_{-\infty}^{\infty} ds (s-t)(s-t')s^{j+k-2}e^{-s^2}
- (1 - e^{-i\pi(1-k)}\epsilon(t'-t))\epsilon(t'-t) \int_t^{t'} ds (s-t)(s-t')s^{j+k-2}e^{-s^2}
\equiv f_{j,k}(t, t') - (1 - e^{-i\pi(1-k)}\epsilon(t'-t))\epsilon(t'-t)
\times [tt'\mu_{j+k-2}(t, t') - (t+t')\mu_{j+k-1}(t, t') + \mu_{j+k}(t, t')].
\]

(12)

In the last step we have introduced the functions

\[
f_{j,k}(t, t') = \int_{-\infty}^{\infty} ds (s-t)(s-t')s^{j+k-2}e^{-s^2}
= \begin{cases} 
\Gamma\left(\frac{j+k-1}{2}\right)tt' + \Gamma\left(\frac{j+k+1}{2}\right) & j+k \text{ even} \\
-\Gamma\left(\frac{j+k}{2}\right)(t+t') & j+k \text{ odd}
\end{cases}
\]

(13)

\[
\mu_{m}(t, t') = \int_t^{t'} ds m e^{-s^2} = \frac{(\epsilon(t'))^{m+1}}{2\gamma\left(m+1, t'^2\right)} - \frac{(\epsilon(t))^{m+1}}{2\gamma\left(m+1, t^2\right)}
\]

(14)

where \(\gamma\) is the lower incomplete gamma function, \(\gamma(m, x) = \int_0^x ds s^{m-1}e^{-s}\).

2.1. Symmetries

It is important to note the symmetry properties of \(\rho^\kappa_{N+1}(t, t')\). In particular

(i) \(\rho^\kappa_{N+1}(t', t) = \overline{\rho^\kappa_{N+1}(t, t')}\) (coordinate exchange);

(ii) \(\rho^\kappa_{N+1}(-t, -t') = \overline{\rho^\kappa_{N+1}(t, t')}\) (center reflection).

Looking at equation (11) the first property is evident. As for the second one, changing the integration variable from \(s\) to \(-s\) and recalling that \(\tilde{A}^\kappa(-x)A^\kappa(-x') = A^\kappa(x)\tilde{A}^\kappa(x')\) we get \(\tilde{a}_{jk}(-t, -t') = (-1)^{j+k}\overline{\tilde{a}_{jk}(t, t')}\); the factor \((-1)^{j+k}\), however, does not change the determinant. It is also interesting to mention that it holds \(\rho^{-\kappa}_{N+1}(t', t) = \overline{\rho^\kappa_{N+1}(t, t')}\).

3. One-body reduced density matrix in the replica approach

We have showed that the one-body reduced density matrix can be expressed exactly as a determinant of a Hankel matrix whose coefficients can be written in terms of special functions. This form is particularly suitable for numerical computation and we will provide some examples in section 4. For increasing \(N\), however, the computational cost becomes considerably larger, which is one of the main motivations why it is interesting to seek an analytic expression that can provide accurate results under well-defined
conditions. It has been proved in several cases, including impenetrable bosons \cite{30} and anyons \cite{31} in a circular geometry and impenetrable bosons in a harmonic potential \cite{30}, that by using a replica trick one can find the full asymptotic expansion of $\rho_N(t, t')$ for large $N$ which is valid in the domain $|t - t'| > r_0/N$ and agrees to an excellent degree with the available numerical results; $r_0$ is the typical length scale of the problem, e.g. the length of the circle in a circular geometry or the Fermi–Thomas radius in a harmonic well. Other models have also been studied by means of the same replica approach \cite{33}. The purpose of this section is to extend the replica method to impenetrable anyons subject to harmonic trapping.

## 3.1. Review of the bosonic case

In this section we briefly review the replica-type calculation of the one-body density matrix in the bosonic case, essentially following the work of Gangardt \cite{30} (which was based on a general trick introduced by Kurchan \cite{34}). Distances are measured in units of half of the Fermi–Thomas radius, $R_{FT}/2 = \sqrt{\hbar N/2m\omega}$. The original units of equation (2) can be recovered by rescaling $x \to (R_{FT}/2)x$; the ones of equation (10) simply by $x \to \sqrt{N/2} x$. One-particle eigenstates are now given by (with a little abuse of notation)

$$\phi_m(x) = \frac{1}{b_m} H_m \left( \sqrt{N/2} x \right) e^{-\frac{N}{4} x^2}, \quad b_m^2 = \left( \frac{2\pi}{N} \right)^{\frac{3}{4}} 2^m m!,$$

and the $N$-body ground-state wavefunction reads

$$\psi_N^0(x_1, \ldots, x_N) = \frac{1}{\sqrt{N!}} \det \phi_{k-1}(x_k) \frac{1}{\sqrt{S_N(N)}} \Delta_N(x) e^{-\frac{N}{4} \sum_i x_i^2},$$

where $\Delta_N(x)$ is the usual Vandermonde determinant and the normalization constant $S_N(N)$ is expressed by the Selberg integral of Hermite type

$$S_N(\lambda) = \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} \prod_{i=1}^{N} \Delta_N(x) e^{\frac{N}{2} \sum_{i=1}^{N} x_i^2} = \lambda^{\frac{N^2}{4}} (2\pi)^{\frac{N}{2}} \prod_{j=1}^{N} \Gamma(1 + j).$$

Let us define the replicated average

$$Z_m(t_1, \ldots, t_m) = \frac{1}{S_N(N)} \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} \prod_{a=1}^{m} \prod_{i=1}^{N} (t_a - y_i).$$

Then the one-body reduced density matrix can be formally obtained by taking the limit $n \to 1/2$:

$$\rho_{N+1}^0(t, t') = (N + 1) S_N(N) e^{\frac{N}{4} (t^2 + t'^2)} \lim_{n \to \frac{1}{2}} Z_{4n}(t, \ldots, t', t', \ldots, t').$$

$$\equiv (N + 1) S_N(N) e^{\frac{N}{4} (t^2 + t'^2)} \lim_{n \to \frac{1}{2}} Z_{4n}(t, t').$$

\[\text{doi:10.1088/1742-5468/2016/07/073106}\]
$Z_{4n}(t, t')$ is more conveniently expressed after a duality transformation (see [30] and references therein)

$$
Z_{4n}(t, t') = \frac{1}{S_{2n}(N)} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} d^{2n}x d^{2n}x' \Delta_{2n}^2(x) \Delta_{2n}^2(x') \prod_{a, a'=1}^{2n}(x_a - x'_a) \prod_{a, a'=1}^{2n}(x'_a - x'a) e^{-N\sum_{a=1}^{2n} S(x_a, t)} e^{-N\sum_{a'=1}^{2n} S(x'_a, t')}, \tag{20}
$$

where we introduced the ‘action’

$$
S(x, t) = \frac{(x - it)^2}{2} - \log x + \frac{\pi i}{2}. \tag{21}
$$

The saddle points of (21) are given by

$$
x_\pm = \frac{it}{2} \pm \sqrt{1 - \frac{t^2}{4}} = \pm e^{\pm i\phi}, \quad \sin \phi = \frac{t}{2}, \tag{22}
$$

for $t \in ]-2, 2[$. At these points

$$
S(x_\pm, t) \equiv S_\pm = e^{+2i\phi} \mp i\phi \pm \frac{\pi i}{2},
$$

$$
S''(x_\pm, t) \equiv \sigma_\pm = 2e^{+2i\phi} \cos \phi. \tag{24}
$$

It is convenient to define the two functions

$$
\Theta(t) = 2\phi + \sin 2\phi + \pi = 2\pi \int_{-2}^{t} \rho(s) \, ds, \tag{25}
$$

$$
\rho(t) = \frac{1}{\pi} \cos \phi = \frac{1}{\pi} \sqrt{1 - \frac{t^2}{4}}. \tag{26}
$$

The latter is the well-known Wigner semi-circle law for the mean density of particles in the large $N$ limit. We will use primed symbols, namely $x'_\pm, \phi', \Theta', S'_\pm, \sigma'_\pm$, for functions of $t'$ (as opposed to $t$). We will be interested in the regime in which the saddle points are well separated: on the one hand $|t - t'|$ must be of order of the cloud size, namely $|t - t'| = O(1)$ (or $|t - t'| \sim R_{FT} \sim \sqrt{N}$ in the old units); on the other hand, $t, t'$ must be sufficiently far from the edges of the cloud, that is $|t|, |t'| = O(1)$ and similarly for $t'$. In order to take into account all the saddle points of equation (20) we must consider all the possible ways to distribute the variables between the neighborhoods of $x_-$ and $x_+$, namely

$$
x_a = x_- + \xi_a/\sqrt{N} \quad a = 1, \ldots, l
$$

$$
x_b = x_+ + \xi_b/\sqrt{N} \quad b = l + 1, \ldots, 2n
$$

$$
x'_a = x'_- + \xi_a'/\sqrt{N} \quad a' = 1, \ldots, l'
$$

$$
x'_b = x'_+ + \xi_b'/\sqrt{N} \quad b' = l' + 1, \ldots, 2n. \tag{27}
$$
When \( l, l' \in \{0, 2n\} \) the replica symmetry is preserved, whereas all the other cases are symmetry-breaking. The Vandermonde determinants in equation (20) vanish at the saddle points and are expanded in their vicinity as

\[
\Delta_{2n}^2(x) = \left( \frac{1}{\sqrt{N}} \right)^{l(l-1)+(2n-l)(2n-l-1)} (x_- - x_+)^{2l(2n-l)} \Delta_l^2(\xi_0) \Delta_{2n-l}^2(\xi_0),
\]

and similarly for \( \Delta_{2n}^2(x') \). The double product in equation (20) evaluated at a saddle point is

\[
\prod_{a=1}^{2n} \prod_{a'=1}^{2n} (x_a - x_{a'}) = 1^{4n^2} \prod_{l=0}^{l'=2n} |t - t'|^{2n^2} \prod \left[ \frac{\cos^2 \frac{\phi + \phi'}{2}}{\sin^2 \frac{\phi - \phi'}{2}} \right]^{(l-n)(l'-n)} e^{-2in(l-n)\phi + 2in(l'-n)\phi'}. \tag{29}
\]

We can now apply the saddle point method to equation (20) and calculate both the contribution from the stationary value of the action and the one from the fluctuations; the latter can be found by using the Selberg integral of equation (17). Combining all together we obtain

\[
Z_{4n}(t, t') = |t - t'|^{-2n^2} \sum_{l=0}^{2n} \sum_{l'=0}^{2n} (-1)^{n(l+l'-2n)} \left( N^l(l-l) + l'N(l-l) \right) \left[ \frac{\cos \frac{\phi + \phi'}{2}}{\sin \frac{\phi - \phi'}{2}} \right]^{2(l-n)(l'-n)}
\]

\[
\times F^n_{2n}(x_+ - x_-)^{2l(2n-l)} \left( \sqrt{\sigma_+} \right)^{2(l-l')^2} e^{-N S_+ - N(2n-l)S_- - 2in(l-n)\phi}
\]

\[
\times F^n_{2n}(x'_+ - x'_-)^{2l'(2n-l')} \left( \sqrt{\sigma_-} \right)^{2(l-l')^2} e^{-N S'_+ - N(2n-l')S'_- + 2in(l'-n)\phi'}, \tag{30}
\]

The \( F \)-symbols \( F^n_{2n} \) are defined as

\[
F^n_{2n} = \left( \frac{2n}{l} \right) \prod_{a=1}^{l} \Gamma(a+1) \prod_{b=1}^{2n-l} \Gamma(b+1) \prod_{c=1}^{2n} \Gamma(c+1)
\]

\[
= \prod_{a=1}^{l} \frac{\Gamma(a)}{\Gamma(2n+1-a)} = \frac{G(l+1)G(2n-l+1)}{G(2n+1)}, \tag{31}
\]

where \( G(x) \) is the Barnes \( G \)-function. For integer \( n \) we can extend the double sum in equation (30) to all integers because in this case \( F^n_{2n} \) vanishes for \( l < 0 \) and \( l > 2n \). This step might appear insignificant at this stage, but is actually important when we take the limit \( n \to 1/2 \) and break the replica symmetry, because in that case \( F^n_{l} \) is non-zero for any \( l \) and the two sums in the replicated average are genuinely infinite. At this point we change the summation variables in equation (30) to
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\[ m = l - n, \quad m' = l - n. \]  \quad (32)

This is actually a crucial step, that will allow to obtain the correct analytic continuation, or the correct replica symmetry breaking, for the bosonic statistics. Using equations (22)–(26), after quite a lot of algebra one arrives to

\[
Z_{4n}(t, t') = (2\pi N)^{2n^2} e^{-N n(2 - (t^2 + t'^2) / 2)} \frac{[\rho(t)\rho(t')]}{|t - t'|^{2n^2}} \times \sum_{m = -\infty}^{\infty} \sum_{m' = -\infty}^{\infty} \frac{\cos \frac{\phi + \phi'}{2}}{\sin \frac{\phi - \phi'}{2}}^{2nm'} \frac{(-1)^{nm} F_{2n}^{n+m}}{[8N\pi^3 \rho^3(t)]^{n^2m^2}} e^{-iNm\Theta - 4inm\phi} \times (-1)^{nm'} F_{2n}^{n+m'} \frac{[8N\pi^3 \rho^3(t')]^{n^2m'^2}}{[8N\pi^3 \rho^3(t)]^{m'^2}}. \quad (33)
\]

We are now in the position to take the limit as indicated in equation (19). Also, noting that

\[
\frac{S_N(N)}{S_{N+1}(N)} = \frac{N^{N+\frac{1}{2}}}{(2\pi)^{N+1}} \frac{N^{N+\frac{1}{2}}}{(2\pi)^{N+1}N!} = e^{N} \frac{1 + \mathcal{O}(\frac{1}{N})}{2\pi N}, \quad (34)
\]

we finally arrive to

\[
\rho_{N+1}^0(t, t') = (2\pi)^{\frac{1}{2}} \frac{[\rho(t)\rho(t')]}{|t - t'|^{1/2}} \sum_{m = -\infty}^{\infty} \sum_{m' = -\infty}^{\infty} (-1)^{(m+m')/2} \times F_{1}^{\frac{1}{2}+m} F_{1}^{\frac{1}{2}+m'} \frac{\cos \frac{\phi + \phi'}{2}}{\sin \frac{\phi - \phi'}{2}}^{2nm'} e^{-iN\Theta + 2\phi + i(N\Theta' + 2\phi')} \times \frac{[8N\pi^3 \rho^3(t)]^{n^2m^2}}{[8N\pi^3 \rho^3(t')]^{m'^2}}. \quad (35)
\]

It is important to note that the zero mode \((m, m' = 0)\) selected by the choice in equation (32), which gives the leading order in \(1/N\), coincides exactly with the asymptotic formula of [32]. This confirms that correctness of the analytic continuation performed above. The zero mode also coincides with the asymptotic prediction from the so called trap-size scaling [36].

3.2. Anyons: macroscopic limit

For a system of particles with generalized anyonic statistics in a harmonic potential we will employ the replica method of the previous section with the modified prescriptions found and applied to the case of circular geometry in [31].

First of all let us assume that the statistical parameter is a rational number, \(\kappa = q/p\). This is only part of the replica construction and does not pose any restriction to the final results, which will be valid for any real \(\kappa\) between 0 and 1. The one-particle density matrix for anyons is defined through the replicated average as

\[
\rho_{N+1}^\kappa(t, t') = (N + 1) \frac{S_N(N)}{S_{N+1}(N)} e^{-N/4(t^2 + t'^2)} \lim_{2np \to 1} Z_{4np}(t, t'), \quad (36)
\]

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where \( q \) indicates which branch should be taken in order to recover the appropriate analytic continuation as it will be clear below. In order to find the correct replicated average we do not need to repeat all the steps of section 3.1, but we can just start from equation (30) with the replacement \( n \rightarrow np \). Now the correct change of summation variables is given by

\[
m + nq = l - np, \quad m' + nq = l - np.
\]

which leads to

\[
Z^q_{4np}(t, t') = (2\pi N)^{2(np)^2} e^{-N np (2 - (t^2 + t'^2)/2)} \frac{[\rho(t)\rho(t')]^{np^2}}{|t - t'|^{2(np)^2}} \times \sum_{m=-\infty}^{\infty} \sum_{m'=-\infty}^{\infty} \left| \frac{\cos \frac{\phi + \phi'}{2} }{\sin \frac{\phi - \phi'}{2}} \right|^{2(np^2)(nq + m')}
\]

\[
\times (-1)^{np(nq + m)} \frac{F_{np + nq + m}}{2np} \frac{e^{-2N(nq + m)\Theta - 4inp(nq + m)\phi}}{[8N\pi^3 \rho^3(t)]^{nq + m^2}}
\]

\[
\times (-1)^{np(nq + m')} \frac{F_{np + nq + m'}}{2np} \frac{e^{2N(nq + m')\Theta + 4inp(nq + m')\phi'}}{[8N\pi^3 \rho^3(t')]^{nq + m'^2}}.
\]

Now we can analytically continue by taking the limit \( 2np \rightarrow 1 \)

\[
\lim_{2np \rightarrow 1} Z^q_{4np}(t, t') = (2\pi N)^{1/2} e^{-N + \frac{N}{4}(t^2 + t'^2)} \frac{[\rho(t)\rho(t')]^{1/2}}{|t - t'|^{1/2}} \times \sum_{m=-\infty}^{\infty} \sum_{m'=-\infty}^{\infty} \left| \frac{\cos \frac{\phi + \phi'}{2} }{\sin \frac{\phi - \phi'}{2}} \right|^{2(\frac{1}{2} + m)(\frac{1}{2} + m')}
\]

\[
\times (-1)^{1/2(nq + m)} \frac{F_{1/2 + \frac{1}{2} + m}}{[8N\pi^3 \rho^3(t)]^{1/2 + m^2}} e^{-2N(\frac{1}{2} + m)\Theta - 2i\frac{1}{2} + m)\phi}
\]

\[
\times (-1)^{1/2(nq + m')} \frac{F_{1/2 + \frac{1}{2} + m'}}{[8N\pi^3 \rho^3(t')]^{1/2 + m'^2}} e^{2N(\frac{1}{2} + m')\Theta + 2i\frac{1}{2} + m')\phi'}.
\]

Recalling equation (34) we end up with

\[
\rho_{N+1}^\kappa(t, t') = \left( \frac{N}{2\pi} \right)^{1/2} \frac{[\rho(t)\rho(t')]^{1/2}}{|t - t'|^{1/2}} \sum_{m=-\infty}^{\infty} \sum_{m'=-\infty}^{\infty} (-1)^{(\kappa + m + m')/2} \frac{F_{\frac{1}{2} + \frac{1}{2} + m}}{F_{\frac{1}{2}} F_{\frac{1}{2} + \frac{1}{2} + m'}} \left| \frac{\cos \frac{\phi + \phi'}{2} }{\sin \frac{\phi - \phi'}{2}} \right|^{2(\frac{1}{2} + m)(\frac{1}{2} + m')}
\]

\[
\times e^{-i(\frac{1}{2} + m)(N\Theta + 2\phi) + i(\frac{1}{2} + m')(N\Theta' + 2\phi')}/[8N\pi^3 \rho^3(t)]^{1/2 + m^2} [8N\pi^3 \rho^3(t')]^{1/2 + m'^2}.
\]

\[\text{doi:10.1088/1742-5468/2016/07/073106}\]

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3.3. Anyons: mesoscopic limit

In the so-called mesoscopic regime $t, t'$ are comparable with the mean inter-particle distance $1/N\rho(t)$ but such that $|t - t'| \gg 1/N$, because this implies that there is still a large number of particle between them and we can safely focus on the asymptotic behavior of correlations. We concentrate on the region around the center of the potential, setting $t + t' = 0$, and define the scaling variable $t - t' = x/N$. For small $t, t'$

$$\left| \frac{\cos \frac{\phi + \phi'}{2}}{\sin \frac{\phi - \phi'}{2}} \right|^{2(nq + m)(nq' + m')} \approx \left( \frac{4N}{x} \right)^{2(nq + m)(nq' + m')}.$$

Combining with the powers of $N$ in (38) we get a factor of $N^{-(m - m')^2}$, meaning that only diagonal terms $m = m'$ give the leading contribution (other terms are at least $1/N$ smaller). Also, as discussed in [30], the two limits $N \to \infty$ and $t - t' \to 0$ commute, which comes from the details of the saddle point integration; therefore, this result is insensitive to the introduction of a generalized statistical parameter (technically to the choice of branch of the replicated average) and we can safely use (38) with $\rho(t), \rho(t') \approx 1/\pi, \Theta - \Theta', 4\phi - 4\phi' \approx 2\pi/N$:

$$Z^q_{4np}(x) = \left( \frac{2N^2}{x} \right)^{2(np)^2} e^{-N\rho(p)(2 - (t'^2 + t^2)/2)} \times \sum_{m = -\infty}^{\infty} (-1)^{2np(nq + m)} \left[ F^q_{2np} \right]^2 e^{-2i(nq + m)(1 + np/N)x} \frac{e^{-2i(nq + m)(1 + np/N)x}}{(2x)^{2(nq + m)^2}},$$

$$\lim_{2np \to 1} Z^q_{4np}(x) = \left( \frac{2N^2}{x} \right)^{\frac{1}{2}} e^{-N - \frac{N}{4}(t^2 + t'^2)} \times \sum_{m = -\infty}^{\infty} (-1)^{2(nq + m)} \left[ F^q_{1} \right]^2 e^{-2i(nq + m)(1 + 1/2N)x} \frac{e^{-2i(nq + m)(1 + 1/2N)x}}{(2x)^{2(nq + m)^2}},$$

$$\rho^q_{N + 1}(x) = \frac{N}{\sqrt{2\pi} x^{1/2}} \sum_{m = -\infty}^{\infty} (-1)^{nq + m} \left[ F^q_{1} \right]^2 e^{-2i(nq + m)(1 + 1/2N)x} \frac{e^{-2i(nq + m)(1 + 1/2N)x}}{(2x)^{2(nq + m)^2}}.$$

This expansion has exactly the same form as the one obtained in [35] from the bosonization approach to a system of anyons on a circle when the Luttinger parameter equals 1 (Tonks–Girardeau limit). The different geometry is not an issue here, because in the mesoscopic regime near the center of the cloud the curvature of the trap plays a minor role. The theoretically interesting aspect concerns the exponents of the $1/N$ series, which the replica method generates correctly. Even more striking is the fact that the same method also produces all the correct coefficients in equations (40) and (43), as confirmed below in section 4.
3.4. Symmetries

Let us check that \( \rho_{N+1}^\kappa(t, t') \) in equation (40) correctly possesses the symmetry properties identified in section 2.1. The behavior under coordinate exchange, that is \( \rho_{N+1}^\kappa(t', t) = \overline{\rho_{N+1}^\kappa(t, t')} \), is evident. As for the center inversion, we need to consider that from the definitions equations (22), (25) and (26) we have \( \phi(-t) = -\phi(t) \), \( \Theta(-t) = -\Theta(t) + 2\pi \) and \( \rho(-t) = \rho(t) \); the last numerator in equation (40) is then transformed to its complex conjugate up to a factor \( e^{2N\pi i (\nu'-\nu)} = 1 \), which is enough to prove \( \rho_{N+1}^\kappa(-t, -t') = \overline{\rho_{N+1}^\kappa(t, t')} \).

4. Numerics and comparison with the replica method

In this section we present a sample calculation of \( \rho_{N+1}^\kappa(t, t') \) using the exact representation in equation (10) and we carefully compare the numerical results with the asymptotic expansion equation (40). The unit of length is set to \( R_{FT}/2 \) as in section 3. We will see that the first few terms in the expansion are sufficient to realize an excellent approximation in the regions where the saddle point treatment is justified (see discussion in section 3.1).

In figure 1 we display the full one-body reduced density matrix for \( N = 20 \) and \( \kappa = 0.1, 0.5, 0.9 \) as a density plot over the \([t, t']\)-plane. For \( \kappa \neq 0, 1 \) a non-zero imaginary part develops, which is a general feature of generalized statistics independently of the geometry; a striking consequence of this is that the momentum distribution function, defined by \( n_v^\kappa(k) = (1/2\pi) \int dt \int dt' e^{ik(t-t')} \rho_{N+1}^\kappa(t, t') \), is asymmetric for reflections about \( k = 0 \), as already known for circular geometry [28, 29]. Because of the harmonic trapping, \( \rho_{N+1}^\kappa(t, t') \) vanishes very quickly for \( |t|, |t'| > 2 \), namely outside of the Fermi–Thomas radius, even for relatively small \( N \). Also, the symmetry properties identified in section 2.1 are manifest in the actual calculation.

In order to make a comparison with the replica method, we first need to observe the structure of equation (40). Each term in the double sum has an amplitude that scales with a certain power of \( 1/N \), namely \((\kappa/2 + m)^2 + (\kappa/2 + m')^2\), and an oscillatory factor (complex exponential) with a characteristic frequency. In particular, the \((m, m') = (0, 0)\) term, or zero mode, is always the one with the largest amplitudes and the slowest oscillations, while higher terms are suppressed by a non-integer power of \( 1/N \) and oscillate faster (except in the limiting case \( \kappa = 1 \)). Since the power series depends on \( \kappa \), one must be careful in choosing an appropriate truncation of the double sum. In figure 2 we plot the powers of \( 1/N \) corresponding to the first few terms as a function of \( \kappa \); clearly the term \((m', m)\) shares the same power with the term \((m, m')\) and the two must be always considered together. It is easy to see that for any \( 0 < \kappa < 1 \) the next-to-leading term is given by \((0, -1)\) and \((-1, 0)\), which have to be taken into account when we want to refine the first approximation given by the zero mode. In contrast, the terms to be chosen at the next level depend on \( \kappa \); for low \( \kappa \), that is \( \kappa \lesssim 0.2 \), the \((0, 1)\) and \((1, 0)\) terms must be added; for high \( \kappa \), that is \( \kappa \gtrsim 0.4 \), the \((-1, -1)\) is more relevant instead; finally, for intermediate \( \kappa \) both must be included to have a consistent truncation because they
are essentially of the same order (note that the corresponding powers of $1/N$ cross at $\kappa = 1/3$). In figure 2(b) the behavior of the first few $F$-symbols is reported to make sure that the previous considerations are not affected by any singular behavior of the numerical coefficients.

We define a truncation of equation (40) by $\bar{\rho}_{N+1}^\kappa(t, t'; D)$, where $D$ is a certain subset of indexes $(m, m')$. In particular, given the above observations, the relevant subsets will be

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While it is possible to include further terms in the truncation without much effort, it is only of relative interest to do so both for theoretical and practical reasons. On the one hand, one must keep in mind, as shown in [30] for the bosonic case, that each term in equation (40) would acquire a whole series of corrections from a standard perturbation theory around the saddle points equation (22); the first perturbative corrections of the first terms (specifically (0, 0), (0, −1), (−1, 0)), although suppressed by a factor $1/N$, will set the actual limitation of the approximation, rather than the higher terms in the double sum. On the other hand, we will see that the truncations introduced in equation (44) are sufficient to achieve a precision of order $10^{-2}$ (except near the edges or the $t = t'$ line).

Let us analyze the relative difference

$$
\frac{\tilde{\rho}_N^{\kappa}(t, t'; D) - 1}{\rho_N^{\kappa}(t, t')}
$$

between the density matrix calculated numerically from equation (10) and the various truncations of the asymptotic expansion obtained with the replica method. For the sake of clearness we focus only on one direction in the two dimensional plane, namely (0, 0); furthermore, by exploiting the symmetries we can restrict ourselves to $t > 0$. Other choices do not show any qualitative difference in the analysis.

In figure 3 we consider the case of $\kappa = 0.1$ for $N = 25$. The zero mode itself already provides a very good approximation; this fact is expected for low $\kappa$ and is visible also in circular geometry [31]. Adding the next-to-leading term, namely considering $D_1$, makes evident the preliminary discussion made above. In fact, oscillation are sensibly suppressed, as it appears especially in the real part, but the average does not benefit very much; this is because an improvement in the average actually requires perturbative corrections to the zero mode. By proceeding to the more refined truncation $D_l$ (appropriate for this value of $\kappa$) we note a further improvement in the oscillations, particularly

Figure 3. Real and imaginary part of $\tilde{\rho}_{25}^{0.1}(0, t; D)/\rho_{25}^{0.1}(0, t) - 1$ for $D = D_0$ (black), $D = D_1$ (blue) and $D = D_l$ (red).
in the imaginary part, and overall a very good agreement which stays within 0.02 unless $t$ is very close to the edges.

In figure 4 we present the analysis for $\kappa = 0.3$. In this case the zero mode gives a slightly worse, but still remarkable, approximation, which is explained by the fact that its importance and that of the next-to-leading term are getting closer as $\kappa$ increases (see figure 2). The $D_1$ truncation, instead, is to some extent more efficient than in the $\kappa = 0.1$ case, which is due to the higher terms being comparatively more suppressed. It is however important to take the truncation $D_{\text{m}}$, consistent with this intermediate value of $\kappa$, to further get rid of oscillations, especially in the region of small $t$ (close to $t \sim 1/\sqrt{N}$) where the saddle point treatment is only marginally valid.

Finally in figure 5, which refers to $\kappa = 0.7$, we immediately note that both the zero-mode and the $D_1$ truncation are worse than in the previous cases by an order of magnitude. This is explained by the fact that the $(-1, -1)$ term is actually quite close to the first two modes for large $\kappa$ (see figure 2) and should be consistently included; indeed the truncation $D_h$ provides the same quality of approximation as achieved by $D_1$ or $D_{\text{m}}$ for lower statistical parameter.

5. Conclusions

In conclusion, we have presented two approaches to the calculation of the one-body reduced density matrix $\rho^2_{25}(t, t')$ of a gas of $N$ harmonically trapped anyons with repulsive
δ-interaction in the Tonks–Girardeau, or hard-core, limit. In the first one we find an exact representation as the determinant of a Hankel matrix of dimension \( N - 1 \); this generalizes the corresponding construction for bosons. In the second approach, we use the replica method with the correct anyonic prescription for the analytic continuation and find a complete asymptotic expansion in analytic form. We showed that, even for relatively small \( N \), the truncation to the first few terms (appropriately chosen according to \( \kappa \)) gives an extremely precise approximation (within a few percent except near the \( t = t' \) line and when \(|t| \) or \(|t'| \) get close to the Fermi–Thomas radius). Even though this result can be in principle improved by performing an ordinary perturbation theory on top of the saddle point integration of section 3, already in the present form it represents a very accurate prediction even for relatively small \( N \).

There are several possible generalizations of our work that are worth mentioning here. First, it could be theoretically interesting and in principle doable to extend our work to a system of hard-core anyons in a finite linear geometry with Dirichlet or Neumann boundary conditions (the first is physically related to trapping in an infinite square well). Another promising line of research concerns the study of the non-equilibrium properties of anyon gases. There are already a few manuscripts in this direction [37–39]. For example, [37] generalizes the results for the free expansion of a bosonic Tonks–Girardeau gas released from a harmonic trap [40] to the anyonic statistics. On the same lines, it should be possible to study the behavior of anyonic observables following the release from a trap to a finite circle, generalising the bosonic and fermionic results of [41]. A more ambitious problem would be to understand the behavior of the anyonic one-body reduced density matrix after an interaction quench to the impenetrable limit, as done for the bosonic counterpart in [42].

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Note added. This paper has taken a very long time to see the light of day. All the results in section 2 were already present in the master thesis of one of us (MP) which dates back to October 2008, see the link https://etd.adm.unipi.it/t/etd-10012008-115712/. When this manuscript was practically completed, the work [43] appeared, which deals with the same problem as the present work with the approach of section 2. In fact all the most relevant formulae in that section, equations (10)–(12), have a correspondence in [43]. Also figure 1 is essentially similar to figures 1 and 2 in [43], the difference being the choice of the parameters \( \kappa, N \).

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