Stability of SU(2) Quantum Skyrmion and Static Properties of Nucleons

A. Acus and E. Norvaišas

Institute of Theoretical Physics and Astronomy

Gostauto 12, Vilnius 2600, Lithuania

Abstract

The Skyrme model is considered quantum mechanically ab initio in various irreducible representations of the SU(2) group. The canonical quantization procedure yields negative mass correction ensuring existence of stable soliton solution even in chiral limit. The evaluated static properties of nucleons (masses, magnetic moments, radii etc.) are in a good agreement with experimental data.
Quantum chromodynamics is the fundamental theory of strong interactions. However it does not provide a simple description of low-energy hadron physics. One of the simplest models which describes the main features of low-energy physics is the SU(2) Skyrme model \[1\]. The topological solutions of the model are quantized in collective coordinates approach \[2\] and interpreted as baryons. The semiclassical quantization procedure leads to the energy expression which minimizing does not provide any stable quantum skyrmion \[3,4\]. In this letter we make use of canonical quantization procedure, developed in \[5,6\] and show for the first time existence of stable quantum skyrmion even in the chiral limit. The numerical solution of equation and predicted static nucleon observables are in a good agreement with empirical values. The pion mass is also evaluated using nucleon characteristics only.

Usually the Skyrme model is defined in the fundamental representation of the SU(2) group. The present analysis is based on the chirally symmetric Lagrangian density

\[
\mathcal{L}[U(x, t)] = -\frac{f^2}{4} \text{Tr}\{R_\mu R^\mu\} + \frac{1}{32e^2} \text{Tr}\{[R_\mu, R_\nu]^2\},
\]

where the unitary field \(U(x, t) = D^j(\alpha(x, t))\) belongs to an irreducible representation of general dimension of the SU(2) group. Here \(f_\pi\) (the pion decay constant) and \(e\) are parameters and \(R_\mu = (\partial_\mu U)U^\dagger\) is ‘right’ current. The elements of \((2j + 1)\) dimension matrices \(D^j\) are the Wigner D-functions expressed in terms of three unconstrained Euler angles \(\alpha = (\alpha^1, \alpha^2, \alpha^3)\) \[3\]. We quantize the skyrmion in collective coordinate approach

\[
U(x, q(t)) = A(q(t))U_0(x)A^\dagger(q(t))
\]

\textit{ab initio} \[3,4\]. The \(U_0\) matrix is the classical hedgehog ansatz in \(j\) representation. The collective coordinates \(q(t)\) (the Euler angles in matrix \(A\)) are dynamical variables such that \([q^a, q^b] \neq 0\). The energy of quantum skyrmion corresponding to Lagrangian density \(\mathcal{L}\) in representation \(j\) with spin-isospin \(\ell\) and chiral angle \(F(r)\), have a form

\[
E(j, \ell, F) = M(F) + \Delta M_j(F) + \frac{\ell(\ell + 1)}{2a(F)},
\]

where
\[ M(F) = \frac{f_\pi}{e} \tilde{M}(F) = 2\pi \frac{f_\pi}{e} \int d\tilde{r} \tilde{r}^2 \left[ F'^2 + \frac{\sin^2 F}{\tilde{r}^2} (2 + 2F'^2 + \frac{\sin^2 F}{\tilde{r}^2}) \right] \]

(4)

is the classical skyrmion mass,

\[
a(F) = \frac{1}{e^3 f_\pi} \tilde{a}(F) = \frac{1}{e^3 f_\pi} \frac{8\pi}{3} \int d\tilde{r} \tilde{r}^2 \sin^2 F \left[ 1 + F'^2 + \frac{\sin^2 F}{\tilde{r}^2} \right]
\]

(5)

is the classical inertial momenta and

\[
\Delta M_j(F) = e^3 f_\pi \cdot \Delta \tilde{M}_j(F) = e^3 f_\pi \frac{-2\pi}{5\tilde{a}^2(F)} \int d\tilde{r} \tilde{r}^2 \sin^2 F \left[ 5 + 2(2j - 1)(2j + 3) \sin^2 F 
+ [2j(j + 1) + 1] \frac{\sin^2 F}{\tilde{r}^2} + [8j(j + 1) - 1]F'^2 - 2(2j - 1)(2j + 3)F'^2 \sin^2 F \right]
\]

(6)

is a negative quantum mass correction, \( \tilde{r} = e f_\pi r \) being a dimensionless variable. The usual symmetric Weyl ordering for the operators are employed throughout (i.e. \( \partial_0 G(q) = 1/2 \{ q^\alpha, \frac{\partial}{\partial q^\alpha} G(q) \} \)). The operator ordering is thus fixed by the form of the Lagrangian (1). No further ambiguity associated with the ordering appears at the level of the Hamiltonian [8].

Minimizing Eq (4) for \( M(F) \), gives the usual classical solution \( F(r) \) which behaves as \( 1/\tilde{r}^2 \) at large distances. In semiclassical case [2], the quantum mass correction \( \Delta M_j(F) \) is absent, and variation of Eq (3) does not yield any stable solution [3,4]. Such a semiclassical skyrmion [2] was considered as 'rotating' rigid-body skyrmion with fixed \( F(r) \). The canonical quantization procedure in collective coordinates approach leads to the energy (3), variation of which produces a new integro-differential equation

\[
F'' \left[-2\tilde{r}^2 - 4\sin^2 F + \frac{e^4\tilde{r}^2}{\tilde{a}(F)} \sin^2 F \left\{ 2R(F) - K(F) \right\} \right]
+ F'^2 \left[-2 \sin 2F + \frac{e^4\tilde{r}^2}{\tilde{a}(F)} \sin 2F \left\{ R(F) - K(F) \right\} \right]
+ F' \left[-4\tilde{r} + \frac{e^4\tilde{r}^2}{\tilde{a}(F)} \sin^2 F \left\{ 4R(F) - 2K(F) \right\} \right]
+ \sin 2F \left[ 2 + 2\sin^2 \frac{\tilde{r}}{\tilde{r}} - \frac{e^4\tilde{r}^2}{\tilde{a}(F)} \left\{ R(F) - \frac{8j(j + 1) - 6}{5\tilde{a}(F)} + 2K(F) \right\} \right]
- \frac{e^4}{\tilde{a}(F)} \sin^2 F \left\{ 2R(F) - \frac{12j(j + 1) - 4}{5\tilde{a}(F)} \right\} = 0
\]

(7)
where

\[ R(F) = \frac{8}{3} \Delta \bar{M}_j(F) + \frac{2\ell(\ell + 1)}{3\bar{a}(F)} + \frac{8j(j + 1) - 1}{5\bar{a}(F)} \]  

(8)

and

\[ K(F) = \frac{4(2j - 1)(2j + 3)}{5\bar{a}(F)} \sin^2 F \]  

(9)

with the boundary condition \( F(0) = \pi \) and \( F(\infty) = 0 \). Contrary to the semiclassical case, the asymptotic equation for \( F(r) \) becomes at large \( r \)

\[ \tilde{r}^2 F'' + 2\tilde{r}F' - (2 + m^2\pi\tilde{r}^2)F = 0, \]  

(10)

with

\[ m^2\pi = -\frac{e^4}{3\bar{a}(F)} \left\{ 8\Delta \bar{M}_j(F) + \frac{2\ell(\ell + 1) + 3}{\bar{a}(F)} \right\}. \]  

(11)

The corresponding solution has the form

\[ F(\tilde{r}) = k \left( \frac{m_\pi}{\tilde{r}} + \frac{1}{\tilde{r}^2} \right) \exp(-m_\pi\tilde{r}). \]  

(12)

The integrals (4), (5), (6) converge ensuring stability of a quantum skyrmion only for \( m^2\pi > 0 \). The positive quantity \( m_\pi \) can be interpreted as the mass of pion. The negative quantum correction \( \Delta M_j(F) \) changes the asymptotical behaviour of \( F(r) \) radically. The quantum chiral angle \( F(r) \) falls off exponentially even in the chiral limit. We have calculated numerically the solution of equation (7) which depends on the representation \( j \), spin-isospin \( \ell \) and the parameter \( e \) of the model.

To perform numerical calculation of observables, we have made use of explicit expressions [6] for nucleon mass \( m_N \), isoscalar radius \( r_0 \) of nucleon, the axial coupling constant \( g_A \) and the magnetic moments of proton and neutron \( \mu_p \) and \( \mu_n \), respectively. The parameters of the Skyrme model \( f_\pi \) and \( e \) can be evaluated using two arbitrary empirical values. In the table (I) we adjust \( f_\pi \) and \( e \) to fit the isoscalar radius and axial coupling constant. The mass of nucleon and the magnetic moments decrease slowly with increasing dimension of
the representation. On the contrary, the mass of pion increases rapidly. The empirical values of \( m_N \) and \( r_0 \) are fixed in calculations of the table (II). Here the magnetic moments are decreasing with the dimension of representation, but the quantities \( g_A \) and \( m_\pi \) are increasing. The representation \( j \) plays a role of a new discrete 'phenomenological' parameter. The best agreement with experimental data is achieved for \( j = 1 \) in both tables. The agreement is much better then that obtained in the previous semiclassical calculations.

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TABLES

TABLE I. The predicted static nucleon observables for different representations with fixed empirical values of isoscalar radius and axial coupling constant

|        | j=1/2 | j = 1 | j = 3/2 | Exp.    |
|--------|-------|-------|---------|---------|
| $m_N$  | 986   | 948   | 882     | 939 MeV |
| $f_\pi$| 61.4  | 58.9  | 55.7    | 93 MeV  |
| $e$    | 4.37  | 4.13  | 3.92    |         |
| $r_0$  | input | input | input   | 0.72fm  |
| $\mu_p$| 2.70  | 2.53  | 2.42    | 2.79    |
| $\mu_n$| −2.14 | −1.95 | −1.84   | −1.91   |
| $g_A$  | input | input | input   | 1.26    |
| $m_\pi$| 75.8  | 179   | 259     | 138 MeV |

TABLE II. The predicted static nucleon observables for different representations with fixed empirical values of nucleon mass and isoscalar radius.

|        | j=1/2 | j = 1 | j = 3/2 | Exp.    |
|--------|-------|-------|---------|---------|
| $m_N$  | input | input | input   | 939 MeV |
| $f_\pi$| 59.8  | 58.5  | 57.7    | 93 MeV  |
| $e$    | 4.46  | 4.15  | 3.86    |         |
| $r_0$  | input | input | input   | 0.72 fm |
| $\mu_p$| 2.60  | 2.52  | 2.51    | 2.79    |
| $\mu_n$| −2.01 | −1.93 | −1.97   | −1.91   |
| $g_A$  | 1.20  | 1.25  | 1.33    | 1.26    |
| $m_\pi$| 79.5  | 180   | 248     | 138 MeV |