Quantum random walk in periodic potential on a line

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Abstract

We investigated the discrete-time quantum random walks on a line in periodic potential. The probability distribution with periodic potential is more complex compared to the normal quantum walks, and the standard deviation $\sigma$ has interesting behaviors for different period $q$ and parameter $\theta$. We studied the behavior of standard deviation with variation in walk steps, period, and $\theta$. The standard deviation increases approximately linearly with $\theta$ and decreases with $1/q$ for $\theta \in (0, \pi/4)$, and increases approximately linearly with $1/q$ for $\theta \in [\pi/4, \pi/2)$. When $q = 2$, the standard deviation is lazy for $\theta \in [\pi/4 + n\pi, 3\pi/4 + n\pi], n \in \mathbb{Z}$.

Keywords: quantum random walk, quantum scattering walk

1. Introduction

Quantum walks, as the quantum version of the classical random walks, were first introduced in 1993 [1]. Recently, quantum walks have attracted great attention from mathematicians, computer scientists, physicists, and engineers. (For an introduction, see Ref. [2, 3]). Some new quantum algorithms based on quantum walks have already been proposed [5, 7, 8, 9, 4, 6]. They proved that a discrete time quantum walk can be used to perform an oracle search on a database of $N$ items with $O(\sqrt{N})$ calls to the oracle [5], and also can be used for universal computation [10, 11].

Quantum walks in many different situations have been studied extensively. For example, the quantum walks in graph [12], on a line with a moving boundary [13], with multiple coins [18] or decoherent coins [19].

However, quantum walks in periodic potential has not been studied yet. This kind of quantum walks are popular in physics. For example, the motion of the atom in the double well lattice [14] and the propagation of photon in periodically varying the coupling in waveguide lattice [15, 16] with different waveguide periods or in the beam splitters array [17] with two kinds of BS at periodic vertices. In this paper, we will present the behaviors of quantum walks in periodic potential. We will discuss the probability distribution and the standard deviation for different periods, potentials and steps.
2. Normal quantum walks and quantum scattering walks

In this paper, we concern with the discrete-time quantum walks. To be consistent, we adopt analogous definitions and notations as those outlined in [20]. The total Hilbert space is given by \( \mathcal{H} = \mathcal{H}_P \otimes \mathcal{H}_C \), where \( \mathcal{H}_P \) is spanned by the orthonormal vectors \( \{ | x \rangle \} \) which representing the position of the walker and \( \mathcal{H}_C \) is the two-dimensional coin space spanned by two orthonormal vectors which are denoted as \( |\downarrow\rangle \) and \( |\uparrow\rangle \).

Each step of the quantum walk can be split into two operations: the flip of a coin and the position motion of the walker according to the coin state.

Here, for simplicity, we choose a Hadamard coin as the normal quantum walk’s coin, so the coin operator can be written as

\[
\hat{H} |\downarrow\rangle = \frac{1}{\sqrt{2}}(|\downarrow\rangle + |\uparrow\rangle), \quad \hat{H} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}. \tag{1}
\]

The position displacement operator is given by

\[
\hat{S} = e^{i\hat{p}\sigma_z} = \sum_x \hat{S}_x, \tag{2}
\]

where \( \hat{p} \) is the momentum operator, \( \sigma_z \) is the Pauli-\( z \) operator,

\[
\hat{S}_x = | x+1 \rangle \langle x | \otimes |\uparrow\rangle \langle\uparrow | + | x-1 \rangle \langle x | \otimes |\downarrow\rangle \langle\downarrow |. \tag{3}
\]

Therefore, the state of the walker after \( N \) steps is given by

\[
|\Psi_N\rangle = \left[ \hat{S} (\hat{I}_P \otimes \hat{H}_C) \right]^N |\Psi_0\rangle = \left[ \sum_x \hat{S}_x (\hat{I}_P \otimes \hat{H}_C) \right]^N |\Psi_0\rangle, \tag{4}
\]

where \( |\Psi_0\rangle \) is the initial state of the system.

The rule of quantum scattering walks we use here was described in Ref. [21, 22]. Suppose that the state is in \( | j+1, j \rangle \), which means that in the last step the walker walked from \( | j + 1 \rangle \) to \( | j \rangle \). In the next step if it is transmitted, it will be in the state \( | j, j - 1 \rangle \), and if it is reflected it will be in the state \( | j, j + 1 \rangle \). Then we have the transition rule

\[
\hat{U} | j + 1, j \rangle = t | j, j - 1 \rangle + r | j, j + 1 \rangle, \tag{5}
\]

where \( t \) and \( r \) are the transmission and reflection coefficients respectively, the unitarity implies that \( | t |^2 + | r |^2 = 1 \).

If we use \( \{ |\downarrow\rangle, |\uparrow\rangle \} \) to represent the direction that the walker just walked, Eq. (5) can be written as:

\[
\hat{U} | j, \downarrow \rangle = t | j, \downarrow \rangle + r | j, \uparrow \rangle \equiv \hat{S}_j \hat{C} | j, \downarrow \rangle. \tag{6}
\]

The unitarity of the scattering gives the transformation matrixes

\[
C_1 = \begin{pmatrix} t & r^* \\ r & -t^* \end{pmatrix} \quad \text{or} \quad C_2 = \begin{pmatrix} t & -r^* \\ r & t^* \end{pmatrix}. \tag{7}
\]

From Eq. (6), we can know that the quantum scattering walk is the same as the coined quantum walk [22]. Without loss of generality, we choose \( t = \sin \theta \) and \( r = \cos \theta \), and use the form of \( C_1 \). Then the scattering matrix can be written as

\[
\hat{C} = \begin{pmatrix} \sin \theta & \cos \theta \\ \cos \theta & -\sin \theta \end{pmatrix}. \tag{8}
\]

The case of \( \theta = \pi/4 \) corresponds to the discrete quantum walks with Eq. (4).
3. Quantum walks in periodic potential

We will consider the case that a normal quantum walker walks in periodic potential like in Fig. 1.

As a model, we consider the situation that the walker walks as scattering quantum walk at the positions with potential, and walks as normal discrete quantum walk for the rest. The potential is described by parameter $\theta$. Then, the operator can be written as

$$U = \sum_{x=nq, n \in \mathbb{Z}} \hat{S}_x \hat{C} + \sum_{x=nq, n \in \mathbb{Z}} \hat{S}_x \hat{H},$$

(9)

where $q$ is the period, $n$ is an integer, and the final state after $N$ steps is given by

$$| \Psi_N \rangle = U^N | \Psi_0 \rangle.$$

(10)

Here the initial state we use is

$$| \Psi_0 \rangle = \frac{1}{\sqrt{2}} | 0 \rangle (| \downarrow \rangle + i | \uparrow \rangle).$$

(11)

Fig. 2 shows the probability distribution after $N = 100$ steps of the quantum walk starting from $| \Psi_0 \rangle$ with and without periodic potential. In the first, we notice that the existence of the periodic potential does not change the symmetric of probability distribution. In the second, the behavior of probability distribution of quantum walks in periodic potential is more complex than normal quantum walks.

Fig. 3 and Fig. 4 show the standard deviation $\sigma = \sqrt{\langle (x - \langle x \rangle)^2 \rangle}$ for quantum walks in periodic potential with different periods when $\theta = \pi/6$ and $\theta = \pi/3$ respectively. Firstly, we can know that, regardless of the existence of
period potential with different period $q$ and $\theta$, the standard deviation still increases approximately linearly with $N$ (number of steps). Secondly, when $\theta = \pi/6$, the standard deviation increases with the period increasing (Fig. 3), but if $\theta = \pi/3$ the standard deviation decreases with the period increasing (Fig. 4).

Fig. 5 shows the standard deviation $\sigma$ for different $\theta \in (0, \pi/4)$, with period $q = 1$ (black star) and 2 (red rectangle), after 200 steps of quantum walk. Then we can know that the standard deviations are nearly the same when $\theta \in (0, \pi/4)$.

Fig. 6 and Fig. 7 show the standard deviation $\sigma$ for different periods with different $\theta \in (0, \pi/2)$, after 200 steps of quantum walk. From the figures, we can know the standard deviation decreases approximately linearly with $1/q$ when $\theta \in (0, \pi/4)$, $2 \leq q \leq 10$ (Fig. 6), and increases approximately linearly with $1/q$ when $\theta \in [\pi/4, \pi/2)$, $1 \leq q \leq 10$ (Fig. 7).

Fig. 8 shows the standard deviation for the quantum walks in periodic potential with $\theta$ from 0 to $2\pi$ and different periods. We can know that when $\theta \in [0, \pi/4]$, standard deviation increases approximately linearly with $\theta$ for different period. For all periods, the line
$\theta = \pi$ is a symmetry axis, so $\sigma(\theta) = \sigma(2\pi - \theta)$.

When the period $q > 3$, and $\theta \in [\pi/4, \pi]$, the standard deviation decreases with the increasing of $\theta$. The case of period $q = 2$, is the transition state between $q = 1$ and $q = 3$. The standard deviation is nearly the same as $\theta \in [\pi/4, 3\pi/4]$, while the transmission coefficient $t = \sin \theta$ is larger than the reflection coefficient $r = \cos \theta$.

When the period $q = 1$, i.e., there is no Hadamard walk but the scattering walk. With the increasing of $\theta$ from 0 to $\pi/2$, the standard deviation will increase approximately linearly [23], and from $\pi/2$ to $\pi$, it will decrease approximately linearly. The standard deviation function has a period of $\pi$. With the increasing of $\theta \in [0, \pi/2]$, the transmission coefficient $t = \sin \theta$ increases and reflection coefficient $r = \cos \theta$ decreases, then the diffusion velocity increases, so the standard deviation increases. Conversely, $t$ decreases and $r$ increases while $\theta$ increases from $\pi/2$ to $\pi$, then the standard deviation decreases with the increasing of $\theta \in [\pi/2, \pi]$. The case that $\theta \in [\pi, 2\pi]$ is the same
where $\theta$ equals to as the function $\sigma$ is the standard deviation and $N$ is the number of steps, and $f(\theta) = 1 - |\cos \theta|$.

as $\theta \in [0, \pi]$. From Ref. [23], we can know

$$\sigma^2 \approx (1 - \sin \theta')N^2 = (1 - \cos \theta)N^2, \quad (12)$$

where $\theta'$ has been defined in Ref. [23] that is equals to $\pi/2 - \theta$, when $\theta' \in [0, \pi/2]$.

Fig. 9 shows $\sigma^2/N^2$ as a function of $\theta$, and the function $f(\theta) = 1 - |\cos \theta|$. From the figure, we can know that

$$\sigma \approx \sqrt{1 - |\cos \theta|}N, \quad (13)$$

for any $\theta$, when $q = 1$. Eq. (13) is more precise than linear approximation $\sigma \approx \begin{cases} (2\theta/\pi)N, & \theta \in [2n\pi, (2n + 1)\pi] \\ (2 - 2\theta/\pi)N, & \theta \in [(2n - 1)\pi, 2n\pi] \end{cases}, n \in \mathbb{Z}$.

4. Conclusion

In this paper, we have discussed the one-dimensional quantum walks in the presence of a periodic potential. The behavior of probability distribution, standard deviation of quantum walks in periodic potential is different from the quantum walks without periodic potential.

When $\theta = \pi/4$, the quantum walks with periodic potential are the same as the normal quantum walks. The case that period of the potential $q = 1$ and $\theta = \pi/2$ corresponds to free particle propagation, then after $t$ steps, $|\Psi(t)⟩ = 1/\sqrt{2}(|-tL⟩ + i |tR⟩)$. If $\theta = n\pi, n \in \mathbb{Z}$, the transmission probability decreases to 0, and the reflection probability is 1, this situation implies the walker walks between two infinite high walls which are $q$ apart, and this situation is similar to quantum walks in cycle graph. Next, we can know that the standard deviation increases approximately linearly with $\theta$ and decreases with $1/q$ if $\theta \in (0, \pi/4)$, and increases approximately linearly with $1/q$ if $\theta \in [\pi/4, \pi/2)$. When $q = 2$, the quantum walk is lazy for $\theta \in [\pi/4 + n\pi, 3\pi/4 + n\pi], n \in \mathbb{Z}$, while the transmission coefficient is larger than the reflection coefficient. Then we can know some property about quantum walks in double well optical lattice [14] and periodic waveguide lattice [15] consists of different waveguides.

Acknowledgments

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