Abstract

In this paper we study rank two commuting ordinary differential operators with polynomial coefficients and the orbit space of the automorphisms group of the first Weyl algebra on such operators. We prove that for arbitrary fixed spectral curve of genus one the space of orbits is infinite. Moreover, we prove in this case that for any \( n \geq 1 \) there is a pair of self-adjoint commuting ordinary differential operators of rank two

\[
L_4 = (\partial^2_x + V(x))^2 + W(x),
\]

where \( W(x), V(x) \) are polynomials of degree \( n \) and \( n + 2 \). We also prove that there are hyperelliptic spectral curves with the infinite spaces of orbits.

1 Introduction

The group of automorphisms of the first Weyl algebra \( A_1 = \{ \sum_{j=0}^{n} u_j(x) \partial^j_x, u_j \in \mathbb{C}[x] \} \) acts on the set of solutions of the equation

\[
f(X,Y) = \sum_{j,i=0}^{n} \alpha_{ij} X^i Y^j = 0, \quad X, Y \in A_1, \alpha_{ij} \in \mathbb{C},
\]

i.e. if \( X, Y \in A_1 \) satisfy \( \square \) and \( \varphi \in Aut(A_1) \), then \( \varphi(X), \varphi(Y) \) also satisfy \( \square \). The group \( Aut(A_1) \) is generated by the following automorphisms

\[
\begin{align*}
\varphi_1(x) &= \alpha x + \beta \partial_x, & \varphi_1(\partial_x) &= \gamma x + \delta \partial_x, & \alpha, \beta, \gamma, \delta \in \mathbb{C}, & \alpha \delta - \beta \gamma = 1, \\
\varphi_2(x) &= x + P_1(\partial_x), & \varphi_2(\partial_x) &= \partial_x, \\
\varphi_3(x) &= x, & \varphi_3(\partial_x) &= \partial_x + P_2(x),
\end{align*}
\]

where \( P_1, P_2 \) are arbitrary polynomials (see \( \square \)). So, \( Aut(A_1) \) consists of tame automorphisms. A natural and important problem is to describe the orbit space of the group action of \( Aut(A_1) \) in the set of solutions of \( \square \). If one describes the orbit space it gives a chance to compare \( End(A_1) \) and \( Aut(A_1) \) (\( End(A_1) \) consists of endomorphisms \( \varphi : A_1 \to A_1 \), i.e. \( [\varphi(\partial_x), \varphi(x)] = 1 \)). Let us recall the Dixmier conjecture: \( End(A_1) = Aut(A_1) \), or in other words, if differential operators \( L_n, L_m \) with polynomial coefficients satisfy the string equation

\[
[L_n, L_m] = 1,
\]

then \( L_m, L_n \) can be obtained from \( x, \partial_x \) with the help of compositions \( \varphi_j \) above (the general Dixmier conjecture for \( A_n \) is stably equivalent to the Jacobian conjecture due to \( \square \)). Berest has proposed the following interesting conjecture:

If the Riemann surface corresponding to the equation \( f = 0 \) with generic \( \alpha_{ij} \in \mathbb{C} \) has genus \( g = 1 \) then the orbit space is infinite, and if \( g > 1 \) then there are only finite number of orbits.

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One can prove that if there are finite number of orbits for some equation (1) then $\text{End}(A_1) = \text{Aut}(A_1)$.

In this paper we consider the equation

$$Y^2 = X^{2g+1} + c_{2g}X^{2g} + \cdots + c_1X + c_0, \quad X, Y \in A_1, c_j \in \mathbb{C}. \quad (2)$$

Using Schur’s arguments [3] one can prove that if $X, Y \in A_1$ satisfy (2) then $XY = YX$. Our approach to the above problem is based on the Krichever–Novikov theory of commuting higher rank ordinary differential operators. Let us recall some basic notions and facts related to commuting differential operators. If $L_n = \sum_{j=0}^{n} v_j(x) \partial_x^j, \; L_m = \sum_{k=0}^{m} u_k(x) \partial_x^k$ commute then there is a Burchnall–Chaundy’s polynomial $F(z, w)$ which vanishes the operators, $F(L_n, L_m) = 0$.

The spectral curve $\Gamma$ defined by the equation $F = 0$ is irreducible and is completed at infinity with a unique point $q$. The spectral curve parametrizes common eigenvalues of $L_n$ and $L_m$, i.e. if $L_n \psi = z \psi, \; L_m \psi = w \psi$, then $(z, w) \in \Gamma$. The dimension of the space of common eigenfunctions for generic $P = (z, w) \in \Gamma$ is called the rank. Commutative rings of ordinary differential operators were classified by Krichever [4], [5]. In the case of rank one eigenfunctions are Baker–Akhiezer functions, found by Krichever. The case of rank $l > 1$ is very complicated. In this case the eigenfunctions can not be found explicitly. Operators of rank two corresponding to elliptic spectral curves were found by Krichever and Novikov [6], operators of fourth order have the form

$$L_{KN} = (\partial_x^2 + u)^2 + 2c_x(\wp(\gamma_2) - \wp(\gamma_1))\partial_x + (c_x(\wp(\gamma_2) - \wp(\gamma_1))x - \wp(\gamma_2) - \wp(\gamma_1),$$

where $\gamma_1(x) = \gamma_0 + c(x), \gamma_2(x) = \gamma_0 - c(x), \gamma_0 \in \mathbb{C}$ is a constant. The operator $L_{KN}$ commutes with a six order differential operator $\tilde{L}_{KN}$.

Let us formulate our main results.

**Theorem 1.1.** For arbitrary integer $m > 0$ and arbitrary spectral curve $\Gamma$ given by the equation $w^2 = z^3 + c_2 z^2 + c_1 z + c_0$ there are polynomials

$$V_m = \alpha_{m+2} x^{m+2} + \cdots + \alpha_0, \quad W_m = \beta_m x^m + \cdots + \beta_0, \quad \alpha_{m+2} \neq 0, \beta_m \neq 0$$

such that the operator

$$L_{4,m} = (\partial_x^2 + V_m(x))^2 + W_m(x)$$

commutes with a six order operator $L_{6,m}$. The spectral curve of $L_{4,m}, L_{6,m}$ coincides with $\Gamma$.

At $m = 1$ we have

$$L_{4,1} = (\partial_x^2 + \alpha_3 x^3 + \alpha_2 x^2 + \alpha_1 x + \alpha_0)^2 + 2\alpha_3 x, \quad \alpha_3 \neq 0.$$ 

At $\alpha_3 = 1, \alpha_1 = \alpha_2 = 0$ the operators $L_{4,1}, L_{6,1}$ coincide with the Dixmier operators [1]. The example of Dixmier was the first example of commutative subalgebra in $A_1$. It is an interesting problem how to obtain $L_{4,m}, L_{6,m}$ from $L_{KN}, \tilde{L}_{KN}$? At $m = 1$ the answer is given in the Grinevich’s theorem [7]:
• Operator $L_{KN}$ corresponding to the curve $w^2 = 4z^3 + g_2z + g_3$ has rational coefficients if and only if

$$c(x) = \int_{q(x)}^{\infty} \frac{dt}{\sqrt{4t^3 + g_2t + g_3}},$$

where $q(x)$ is a rational function. If $\gamma_0 = 0$ and $q(x) = x$, then $L_{KN}$ coincides with $L_{4,1}$.

Theorem 1.1 allows to prove the following theorem.

**Theorem 1.2.** The set of orbits of the group $\text{Aut}(A_1)$ in the space of solutions of arbitrary equation

$$Y^2 = X^3 + c_2X^2 + c_1X + c_0, \quad X, Y \in A_1, c_j \in \mathbb{C}$$

is infinite.

Commuting operators of rank two of order 4 and $4g + 2$ corresponding to hyperelliptic spectral curves of genus $g$ were studied in [5]. With the help of methods of [5] one can construct rank 2 operators at $g > 1$. For example

$$L_4^i = (\partial_x^3 + \alpha_3 x^3 + \alpha_2 x^2 + \alpha_1 x + \alpha_0)^2 + g(g + 1)\alpha_3 x, \quad \alpha_3 \neq 0$$

commutes with an operator $L_{4g+2}^j$. Mokhov [9] proved that if one apply elements of $\text{Aut}(A_1)$ to $L_4^i, L_{4g+2}^j$ then one can obtains operators of rank $l = 2k$ and $l = 3k$, where $k$ is a positive integer. For example if we apply the automorphism $\varphi(x) = \partial_x, \varphi(\partial_x) = -x$ to $L_4^i, L_{4g+2}^j$ we obtain rank 3 operators. Herewith

$$\varphi(L_4^i) = (\partial_x^3 + \alpha_2 \partial_x^2 + \alpha_1 \partial_x + \alpha_0 + x^2)^2 + g(g + 1)\alpha_3 \partial_x.$$ 

Another important example constructed in [10] is the following. The operator

$$L_4^i = (\partial_x^2 + \alpha_1 \cosh x + \alpha_0)^2 + \alpha_1 g(g + 1) \cosh x, \quad \alpha_1 \neq 0$$

commutes with $L_{4g+2}^j$. Using $L_4^i, L_{4g+2}^j$ Mokhov constructed examples of operators of arbitrary rank $l > 1$ [11] (we discuss this construction in section 2). Let $\Gamma^z$ be a spectral curve of $L_4^i, L_{4g+2}^j$ given by the equation

$$w^2 = z^{2g+1} + c_2^5 z^{2g} + \cdots + c_1^5 z + c_0^5.$$ 

(3)

Coefficients $c_j^5$ can be found with the help of a recurrent formula (see Lemma 1 below). Probably for all $g$ the curve $\Gamma^z$ is not singular for general set of parameters $\alpha_0, \alpha_1$. For small $g$ using Lemma 1 one can check this by direct calculation.

**Theorem 1.3.** The set of orbits of the group $\text{Aut}(A_1)$ in the space of solutions of the equation

$$Y^2 = X^{2g+1} + c_2^5 X^{2g} + \cdots + c_1^5 X + c_0^5, \quad X, Y \in A_1$$

is infinite.

It would be interesting to check the Berest conjecture at $g > 1$ for generic equation [11] having a nonconstant solution in $A_1$.

**Remark 1.1.** The group $\text{Aut}(A_1)$ acts on the set of rings of commuting differential operators with affine spectral curves considered in Theorems 1.2 and 1.3. One can prove that the space of orbits is also infinite.
2 Method of deformation of Tyurin parameters

Every ring $A$ of commuting ordinary differential operators is isomorphic to a ring of meromorphic functions on spectral curve $\Gamma$ with a pole in some point $q \in \Gamma$ (we consider in this section the case when $\Gamma$ is nonsingular, i.e. $\Gamma$ is a Riemann surface). For a meromorphic function $f(P)$, $P \in \Gamma$ with pole in $q$ of order $n$ we have $L_f \psi(x, P) = f(P)\psi(x, P)$ where $L_f \in A$ is a differential operator of order $ln$, $l$ is the rank of commuting operators, $\psi = (\psi_1, \ldots, \psi_l)$ is a vector Baker–Akhiezer function. Function $\psi$ can be reconstructed from the following spectral data (see [4])

$$\{\Gamma, q, k^{-1}, \gamma_1, \ldots, \gamma_l, \alpha_1, \ldots, \alpha_l, \omega_1(x), \ldots, \omega_{l-1}(x)\}.$$ 

Here $k^{-1}$ is a local parameter near $q$, $g$ is the genus of $\Gamma$, $\gamma_j \in \Gamma$, $\alpha_j = (\alpha_{j,1}, \ldots, \alpha_{j,l-1})$ is a vector, $\omega_j(x)$ is a smooth function. The set $(\gamma, \alpha)$ is called the Tyurin parameters. This parameters define a semi-stable holomorphic rank $l$ vector bundle on $\Gamma$ of degree $lg$ with holomorphic sections $\eta_1, \ldots, \eta_l$. The points $\gamma_1, \ldots, \gamma_l$ are points of their linear dependence of the sections

$$\eta_l(\gamma_j) = \sum_{i=1}^{l-1} \alpha_{i,j} \eta_i(\gamma_j).$$

The vector-function $\psi$ is defined by the following properties.

1. In the neighbourhood of $q$ it has the form

$$\psi(x, P) = \left(\sum_{s=0}^{\infty} \xi_s(x)k^{-s}\right) \Phi(x, k),$$

where $\xi_0 = (1, 0, \ldots, 0), \xi_i(x) = (\xi^1_i(x), \ldots, \xi^l_i(x))$, the matrix $\Phi$ satisfies the equation

$$\frac{d\Phi}{dx} = A\Phi, \quad A = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 0 & 1 \\ k + \omega_1 & \omega_2 & \omega_3 & \ldots & \omega_{l-1} & 0 \end{pmatrix}.$$ 

2. The components of $\psi$ are meromorphic functions on $\Gamma \setminus \{q\}$ with the simple poles $\gamma_1, \ldots, \gamma_l$, and

$$\text{Res}_{\gamma_i} \psi_j = \alpha_{i,j} \text{Res}_{\gamma_i} \psi_1, \quad 1 \leq i \leq lg, \quad 1 \leq j \leq l - 1.$$ 

The main difficulty to construct operators of rank $l > 1$ is the fact that the Baker–Akhiezer function is not found explicitly. In the recent paper [12] were shown that the class of Baker–Akhiezer functions contains some known special functions.

Let us recall the method of deformation of Tyurin parameters [6]. The main idea of this method is to study the linear differential operator which vanishes the common eigenfunctions. The common eigenfunctions of commuting differential operators of rank $l$ satisfy the linear differential equation of order $l$

$$\psi^{(l)}(x, P) = \chi_0(x, P)\psi(x, P) + \cdots + \chi_{l-1}(x, P)\psi^{(l-1)}(x, P).$$

The coefficients $\chi_i$ are rational functions on $\Gamma$ with the simple poles $P_1(x), \ldots, P_{lg}(x) \in \Gamma$, and with the following expansions in the neighbourhood of $q$

$$\chi_0(x, P) = k + g_0(x) + O(k^{-1}), \quad \chi_j(x, P) = g_j(x) + O(k^{-1}), \quad 0 < j < l - 1,$$

$$\chi_{l-1}(x, P) = O(k^{-1}).$$
Let \( k - \gamma_i(x) \) be a local parameter near \( P_i(x) \). Then
\[
\chi_j = \frac{c_{i,j}(x)}{k - \gamma_i(x)} + d_{i,j}(x) + O(k - \gamma_i(x)).
\]

Functions \( c_{i,j}(x), d_{i,j}(x) \) satisfy the following equations (see [4]).
\[
\begin{align*}
\alpha_{i,l-1}(x) &= -\gamma_i(x), \\
\alpha_{i,0}(x) &= \alpha_{i,0}(x) + \alpha_{i,l-2}(x) + \alpha_{i,0}(x)d_{i,l-1}(x) - \alpha_{i,0}(x), \\
\alpha_{i,j}(x) &= \alpha_{i,j}(x) + \alpha_{i,j}(x)d_{i,l-1}(x) - \alpha_{i,j}(x), j \geq 1,
\end{align*}
\]

where \( \alpha_{i,j}(x) = \frac{c_{i,j}(x)}{c_{i,l-1}(x)} \), \( 0 \leq j \leq l - 1, 1 \leq i \leq l g \). To find \( \chi_i \) one should solve the equations (4)–(6). Using \( \chi_i \) one can find coefficients of the operators. At \( g = 1, l = 2 \) Krichever and Novikov [6] solved these equations and found the operators \( L_{KN} \). Operators of Krichever–Novikov and their applications were studied in [13]–[19] Operators of rank 3 corresponding to elliptic spectral curves were found by Mokhov [20]. In [21]–[24] some examples of operators of rank 2,3 corresponding to spectral curves of genus 2–4 were constructed.

In \( \mathbb{N} \) commuting operators of rank two of order 4 and 4g+2 corresponding to hyperelliptic spectral curves were studied
\[
L_4 \psi = z \psi, \quad L_{4g+2} \psi = w \psi, \quad w^2 = F_g(z) = z^{2g+1} + c_{2g} z^{2g} + \cdots + c_0.
\]

Common eigenfunctions of \( L_4 \) and \( L_{4g+2} \) satisfy the second order differential equation
\[
\psi'' - \chi_1(x, P) \psi' - \chi_0(x, P) \psi = 0, \quad P = (z, w) \in \Gamma,
\]

where \( \chi_0(x, P), \chi_1(x, P) \) are rational functions on \( \Gamma \) satisfying equations (4)–(6).

**Theorem 2** ([5]) *The operator \( L_4 \) is formally self-adjoint if and only if*
\[
\chi_1(x, P) = \chi_1(x, \sigma(P)) \equiv \chi_1(x, P),
\]

*where \( \sigma \) is the hyperelliptic involution on \( \Gamma \).*

**Theorem 3** ([5]) *If \( L_4 \) is formally self-adjoint, i.e. \( L_4 = (\partial_x^2 + V(x))^2 + W(x) \), then*
\[
\begin{align*}
\chi_0 &= -\frac{1}{2} \frac{Q_{xx}}{Q} + \frac{w}{Q} - V, \quad \chi_1 = \frac{Q_x}{Q},
\end{align*}
\]

*where \( Q = z^g + a_{g-1}(x) z^{g-1} + \cdots + a_0(x), a_0(x), \ldots, a_{g-1}(x) \) are some functions. The function \( Q \) satisfies the equation
\[
4 F_g(z) = 4(z - W) Q^2 - 4V(\chi_x)^2 + (Q_{xx})^2 - 2Q_x Q_{xxx} + 2Q(2V_x Q_x + 4V Q_{xx} + \partial_x^4 Q).
\]

From Theorem 3 it follows

**Corollary 1** *The function \( Q \) satisfies the linear equation*
\[
\partial_x^4 Q + 4V Q_{xxx} + 6V_x Q_{xx} + 2(2z - 2W + V_{xx}) Q_x - 2W Q_x = 0.
\]
Corollary 2 If $g = 1$ then
\[
V = \frac{-16F_1(\frac{1}{2}(-c_2 - W)) + W_{xxx}^2 - 2W_xW_{xxx}}{4W_x^2},
\] (9)
where $F_1$ defines the spectral curve $w^2 = F_1(x) = z^3 + c_2z^2 + c_1z + c_0$.

With the help of Theorem 3 many examples of rank 2 operators were recently constructed (see [25–27]).

Let us consider commuting operators $L_4^2, L_{4g+2}^2$ [10]. These operators do not commute with operators of odd orders [28], hence these operators are operators of true rank 2. The polynomial $Q$ for $L_4^2, L_{4g+2}^2$ has the form (see [10])
\[
Q(x, z) = A_g(z) \cosh^g x + \cdots + A_1(z) \cosh x + A_0(z),
\]
where
\[
A_s = \frac{1}{8(2s + 1)a_1(g(g + 1) - s(s + 1))} \left( 4A_{s+5} \frac{(s + 5)!}{s!} - 8A_{s+3} \frac{(s + 3)!}{s!} (2a_0 + s^2 + 4s + 5) - 8A_{s+2} \frac{(s + 2)!}{s!} (2s + 3) a_1 + 4A_{s+1} (s + 1) ((s + 1)^2 (4a_0 + (s + 1)^2 + 4z)) \right),
\]
$0 \leq s < g$, (10)
we assume that $A_s = 0$ at $s < 0$ and $s > g$, $A_g$ is a constant.

Lemma 1 ([5]) The spectral curve $\Gamma^3$ of $L_4^2, L_{4g+2}^2$ is given by the equation
\[
w^2 = F_g(z) = \frac{1}{4} \left( 4A_0^2z - 4A_0A_1a_1 - 16A_2(a_0 + 1) + 48A_4 + 4a_0A_1^2 + 4A_2^2 - 2A_1(6A_3 - A_1) \right),
\]
where $A_j(z)$ are defined in [10].

Examples:
1) $g = 1$
\[
F_1(z) = z^3 + \left( \frac{1}{2} - 2a_0 \right) z^2 + \frac{1}{16} (1 - 8a_0 + 16a_0^2 - 16a_1^2) z + \frac{a_1^2}{4}.
\]
2) $g = 2$, let for simplicity of formulas $a_0 = 0$
\[
F_2(z) = z^5 + \frac{17}{2} z^4 + \frac{1}{16} (321 - 336a_1^2) z^3 + \frac{1}{4} (34 - 531a_1^2) z^2 + (1 - 189a_1^2 + 108a_1^4) z + 24a_1^2 + 513a_1^4.
\]
The spectral curves defined by the above equations are not singular for the general parameters.

Mokhov [11] found a remarkable change of variable
\[
x = \ln(y + \sqrt{y^2 - 1})^r, \quad r = \pm 1, \pm 2, \ldots,
\]
which reduces the operators $L_4^2, L_{4g+2}^2$ to the operators with polynomial coefficients. In particular, $L_4^2$ in new variable $y$ gets the form
\[
L_4^2 = ((1 - y^2) \partial_y^2 - 3y \partial_y + aT_r(y) + b)^2 - ar^2 g(g + 1)T_r(y), \quad a \neq 0,
\]
b is arbitrary constant, $T_r(y)$ is the Chebyshev polynomial of degree $|r|$. Recall that
\[
T_0(y) = 1, \quad T_1(y) = y, \quad T_r(y) = 2yT_{r-1}(y) - T_{r-2}(y), \quad T_{-r}(y) = T_r(y).
\]
Chebyshev polynomials are commuting polynomials, i.e.

\[ T_n(T_m(y)) = T_m(T_n(y)) = T_{n+m}(y). \]

If one applies the automorphism

\[ \varphi(y) = -\partial_y, \quad \varphi(\partial_y) = y, \quad \varphi \in Aut(A_1) \]

to the operators \( L_4^5, L_{4g+2}^5 \) written in \( y \) variable, then one gets operators of orders \( 2r, (2g+1)r \) of rank \( r \)

\[ \varphi(L_4^5) = (aT_r(\partial_y) - y^2 \partial_y^2 - 3y\partial_y + y^2 + b)^2 - arg(g+1)T_r(\partial_y). \]

### 3 Proof of Theorems 1.1–1.3

#### 3.1 Proof of theorem 1.1

Let us rewrite (9) in the form

\[ 4W_x^2V = -16F_1\left(\frac{1}{2}(c_2 - W)\right) + W_x^2 - 2W_xW_{xxx}, \] (11)

Note that from (11) it follows

\[ -4F_1\left(\frac{1}{2}(c_2 - W)\right) + 2V_xW_x + 4VW_{xx} + W_{xxxx} = 0. \] (12)

Further we assume that \( V, W \) are polynomials

\[ V = \alpha_n x^n + \ldots + \alpha_0, \quad W = \beta_m x^m + \ldots + \beta_0, \quad \alpha_n \neq 0, \beta_m \neq 0. \] (13)

Equation (11) is equivalent to the system of equations: equation (12) and the equation on free terms of (11) which is

\[ \alpha_0 \beta_1^2 = -4F_1\left(\frac{1}{2}(c_2 - \beta_0)\right) + \beta_2^2 - 3\beta_1 \beta_3. \] (14)

Let us prove the following important proposition.

**Proposition 3.1.** For any \( m > 0 \) there exists a solution of the equation (11) of the form (13), where \( n = m + 2 \).

**Proof.** Equation (12) is equivalent to a system of \( 2m + 1 \) equations in \( 2m + 4 \) variables \( \alpha_i, \beta_j \). Note that all equations have degree 2 and the set of their solutions consists of points in \( \mathbb{C}^{2m+4} \) (with coordinates \( \alpha_i, \beta_j \)) which lie in the intersection of \( 2m + 1 \) quadrics defined by these equations. By [29, Ch.1, Th.7.2] the intersection \( X \) of these quadrics in \( \mathbb{P}^{2m+4} \) (with homogeneous coordinates \( \alpha_i, \beta_j, u \)) is non-empty and each its irreducible component has dimension greater or equal to 3. By the same reason the intersection of \( X \) with the hyperplane \( Z = \{u = 0\} \) at infinity is non-empty and each its irreducible component has dimension greater or equal to 2.

To prove the proposition it is sufficient to prove that for any fixed \( m > 0 \) there is a two-dimensional irreducible component of \( X \cap Z \). From this fact we can conclude that affine part of the intersection of quadrics is non-empty.

The homogeneous parts of our equations in \( \mathbb{P}^{2m+4} \) not depending on \( u \) can be easily written: these are exactly the coefficients at \( x^i \) of the sum

\[ 4VW_{xx} + 2V_xW_x - 3W^2. \] (15)
Let us introduce the following notations:

\[ V_x W_x = \sum_{i=0}^{m+n-2} b_i x^i, \quad VW_{xx} = \sum_{i=0}^{m+n-2} c_i x^i, \quad W^2 = \sum_{i=0}^{m+n-2} d_i x^i. \]

Then the intersection \( X \cap Z \) is given by the equations

\[ 4c_i + 2b_i - 3d_i = 0, \quad i = 0, \ldots, 2m. \]  \hspace{1cm} (16)

Note that the coefficients \( b_i, c_i, d_i \) can be written in the following form:

\[ d_i = \sum_{k=0}^{i} \beta_{i-k} \beta_k, \quad b_i = \sum_{k=0}^{i} B_{k,i} \alpha_{i-k} \beta_{k+1}, \quad c_i = \sum_{k=0}^{i} C_{k,i} \alpha_{i-k} \beta_{k+2}, \]  \hspace{1cm} (17)

where \( B_{k,i} = (k+1)(i-k+1), C_{k,i} = (k+1)(k+2) \) are positive integers, and we set \( \alpha_j \equiv 0 \) if \( j > n, \beta_j \equiv 0 \) if \( j > m \).

The next observation is: equations (16), (17) always have a solution of the form

\[ P = (\alpha_n \neq 0 : \beta_m \neq 0 : 0 : \ldots : 0) \]

for any \( m > 0 \). Indeed, if \( \alpha_0 = \ldots = \alpha_{n-1} = \beta_0 = \ldots = \beta_{m-1} = 0 \), then only \( 2m \)-th equation from (16) remains to be non-trivial, and this equation becomes a quadratic homogeneous equation linear in \( \alpha_n \) and quadratic in \( \beta_m \):

\[ (2B_{m-1,2m} + 4C_{m-2,2m})\alpha_n \beta_m - 3\beta_m^2 = 0. \]

Thus, we can set \( \beta_m = 1 \) where from \( \alpha_n = 3/(2B_{m-1,2m} + 4C_{m-2,2m}) \).

Let us prove that for any fixed \( m > 0 \) any irreducible component of \( X \cap Z \) containing \( P \) has dimension 2.

If \( m = 1 \) then there are only 3 equations in (16):

\[ 4C_{0,0} \alpha_0 \beta_2 + 2B_{0,0} \alpha_1 \beta_2 - 3\beta_0^2 = 0, \]

\[ 4(C_{0,1} \alpha_1 \beta_2 + C_{1,2} \alpha_0 \beta_3) + 2(B_{0,1} \alpha_2 \beta_1 + B_{1,2} \alpha_1 \beta_2) - 6\beta_0 \beta_1 = 0, \]

\[ 4(C_{0,2} \alpha_2 \beta_2 + C_{1,2} \alpha_1 \beta_3 + C_{2,2} \alpha_0 \beta_4) + 2(B_{0,2} \alpha_3 \beta_1 + B_{1,2} \alpha_2 \beta_2 + B_{2,2} \alpha_1 \beta_3) - 3(2\beta_0 \beta_2 + \beta_1^2) = 0, \]

and their Jacobi matrix at \( P \) has the following form:

\[
\begin{pmatrix}
* & 2B_{0,0} \beta_1 & 0 & 0 & * & *
\end{pmatrix}
\]

\[
\begin{pmatrix}
* & 2B_{0,1} \beta_1 & 0 & 0 & * & *
\end{pmatrix}
\]

\[
\begin{pmatrix}
* & * & * & 2B_{0,2} \beta_1 & * & * \\
\end{pmatrix}
\]

where the first columns denote derivations with respect to \( \alpha_0, \ldots, \alpha_3 \), and the last two columns denote derivations with respect to \( \beta_0, \beta_1 \). The rank of the matrix is 3, so, these equations define a smooth variety in the neighbourhood of the point \( P \) of dimension two.

For generic \( m \) the point \( P \) might not be regular. Nevertheless, any irreducible component containing \( P \) has a dense subset of smooth points. At any such point \( Q \) the Jacobi matrix \( J \) can be written in the following form. It can be divided in two blocks: one consists of \( m + 1 \) columns (derivations of equations with respect to \( \beta_0, \ldots, \beta_m \), and another one consists of \( n + 1 \) columns (derivations of equations with respect to \( \alpha_n, \ldots, \alpha_0 \)). We shall describe only essential columns for us.

The columns of the first block are (to save the space we shall write them as rows):

2-nd column: \((j_{0,0} \alpha_1, (j_{0,1} \alpha_2 - 6\beta_0), (j_{0,2} \alpha_3 - 6\beta_1), \ldots, (j_{0,n-1} \alpha_n - 6\beta_m), 0, \ldots, 0)\)
3-d column: \((j_{1,0}\alpha_0,j_{1,1}\alpha_1,(j_{1,2}\alpha_2-6\beta_0),\ldots,(j_{1,n}\alpha_n-6\beta_m),0,\ldots,0)\)

4-th column: \((0,j_{2,1}\alpha_0,j_{2,2}\alpha_1,(j_{2,3}\alpha_2-6\beta_0),\ldots,(j_{2,n+1}\alpha_n-6\beta_m),0,\ldots,0)\)

5-th column: \((0,0,j_{3,2}\alpha_0,j_{3,3}\alpha_1,(j_{3,4}\alpha_2-6\beta_0),\ldots,(j_{3,n+2}\alpha_n-6\beta_m),0,\ldots,0)\)

\(\ldots\) \(\ldots\) \(\ldots\)

\((m+1)\)-th column: \((0,\ldots,0,j_{m-1,m-2}\alpha_0,j_{m-1,m-1}\alpha_1,(j_{m-1,m}\alpha_2-6\beta_0),\ldots,(j_{m-1,2m}\alpha_n-6\beta_m))\);

the columns of the second block are:

1-st column: \((0,\ldots,0,j_{0,m+1}\beta_1,j_{1,m+2}\beta_2,\ldots,j_{m-1,2m}\beta_m)\)

2-nd column: \((0,\ldots,0,j_{0,m}\beta_1,j_{1,m+1}\beta_2,\ldots,j_{m-1,2m-1}\beta_m,0)\)

3-d column: \((0,\ldots,0,j_{0,m-1}\beta_1,j_{1,m}\beta_2,\ldots,j_{m-1,2m-2}\beta_m,0,0)\)

\(\ldots\) \(\ldots\) \(\ldots\)

n-th column: \((j_{0,0}\beta_1,j_{1,1}\beta_2,\ldots,j_{m-1,m-1}\beta_m,0,\ldots,0)\)

\((n+1)\)-th column: \((j_{1,0}\beta_2,\ldots,j_{m-1,m-2}\beta_m,0,\ldots,0)\),

where the numbers \(j_{k,l}\) are defined as

\[j_{k,l} = 2B_{k,l} + 4C_{k-1,l},\]

where we assume \(B_{k,l} = 0\) if \(k > l\) and \(C_{k,l} = 0\) if \(k < 0\).

Without loss of generality we can assume that the point \(Q\) belongs to a sufficiently small neighbourhood of the point \(P\) (in the complex topology), such that, for fixed numbers \(j_{k,l}\), the modules of all terms of the matrix \(J\), except the terms containing \(\beta_m = 1\) and \(\alpha_n\), are comparable with some \(0 < \epsilon \ll 1\) (i.e. they are \(< \epsilon\) but \(> \epsilon^2\)). We call such terms comparable with \(\epsilon\).

We have the following possibilities now. If there is a smooth point \(Q\) such that its coordinate \(\alpha_0 \neq 0\) or \(\alpha_1 \neq 0\), then the rank of the matrix \(J\) is \(2m + 1\), i.e. the dimension of the component is two. Indeed, we can first apply the Gauss elimination algorithm to kill all terms of the right part of the matrix lying over terms containing \(\beta_m\). We can choose \(\epsilon\) small enough such that the terms of the left part of \(J\) will change, but the top non-zero elements of the first \(m - 2\) rows will remain non-zero and comparable with \(\epsilon\), and all elements over them will be comparable with \(\epsilon^2\). Applying again the Gauss elimination algorithm we can kill all elements in the columns except these top non-zero elements, thus obtaining \(2m + 1\) linearly independent rows in the matrix \(J\).

Note that the case \(\alpha_0 = 0\), \(\alpha_1 \neq 0\) (i.e. \(\alpha_0 = 0\) for all smooth points) is in fact impossible: in this case the whole component belongs to the hyperplane \(\alpha_0 = 0\). But then the dimension of the component must be 1, a contradiction.

Now we claim that there exists a smooth point such that \(\alpha_0 \neq 0\) or \(\alpha_1 \neq 0\). Indeed, if there are no such smooth points, then the whole component belongs to the intersection of hyperplanes \(\alpha_0 = \alpha_1 = 0\) (cf. [29, Ch.1, ex.1.6]). Note that in this case from 0-th equation in \([16]\) it follows \(\beta_0 = 0\), and from the 1-st equation it follows \(\alpha_2\beta_1 = 0\).

Let’s show first that \(\alpha_2 = \beta_1 = 0\). If there is a smooth point in the component with \(\alpha_2 \neq 0\), then the Jacobi matrix of our system restricted to the \(2m\)-dimensional intersection of hyperplanes \(\alpha_0 = \alpha_1 = \beta_0 = 0\) reduces to the following matrix.

The columns of the first block are (to save the space we will again write them as rows):

1-st column: \((j_{0,1}\alpha_2, (j_{0,2}\alpha_3-6\beta_1),\ldots,(j_{0,n-1}\alpha_n-6\beta_m),0,\ldots,0)\)

2-nd column: \((0, j_{1,2}\alpha_2, (j_{1,3}\alpha_3-6\beta_1),\ldots,(j_{1,n}\alpha_n-6\beta_m),0,\ldots,0)\)
the columns of the second block are:

1-st column: $(0, \ldots, 0, j_{0,m+1} \alpha_1, j_{1,m+2} \alpha_2, \ldots, j_{m-1,2m} \alpha_m)$
2-nd column: $(0, \ldots, 0, j_{0,m} \alpha_1, j_{1,m+1} \alpha_2, \ldots, j_{m-2,2m} \alpha_m, 0)$
3-d column: $(0, \ldots, 0, j_{0,m-1} \alpha_1, j_{1,m} \alpha_2, \ldots, j_{m-3,2m} \alpha_m, 0, 0)$

\[ \vdots \quad \quad \vdots \quad \quad \vdots \quad \quad \vdots \quad \quad \vdots \]

(n-1)-th column: $(j_{0,1} \beta_1, \ldots, j_{m-1,m} \beta_m, 0, \ldots, 0)$,

where the \( m \) columns of the first block denote derivations with respect to \( \beta_1, \ldots, \beta_m \), and the \( n-1 \) columns of the second block denote derivations with respect to \( \alpha_n, \ldots, \alpha_2 \).

Since \( \beta_1 \) must be equal to zero, we can apply the same arguments as above and obtain that the rank of this matrix is \( 2m \). But this is impossible, because the dimension of the component is not less than two.

If the whole component belongs to the intersection \( Y = \{ \alpha_0 = \alpha_1 = \alpha_2 = \beta_0 = 0 \} \), but there are smooth points with \( \beta_1 \neq 0 \), then the 2-th equation in (16) reduces to \( 2B_{0,2} \alpha_3 \beta_1 - 3 \beta_1^2 = 0 \), where from we see that \( \alpha_3 = 3 \beta_1/(2B_{0,2}) \neq 0 \). In this case analogously to the previous case the matrix \( J \) reduces to the following matrix.

The columns of the first block are:

1-st column: \( ((j_{0,2} \alpha_3 - 6 \beta_1), \ldots, j_{0,n-1} \alpha_n - 6 \beta_m), 0, \ldots, 0) \)
2-nd column: \( (j_{1,3} \alpha_3 - 6 \beta_1), \ldots, j_{1,n} \alpha_n - 6 \beta_m), 0, \ldots, 0) \)

\[ \vdots \quad \quad \vdots \quad \quad \vdots \quad \quad \vdots \]

m-th column: \( (0, \ldots, 0, (j_{m-1,m+1} \alpha_3 - 6 \beta_1), \ldots, (j_{m-1,2m} \alpha_n - 6 \beta_m)) \);

the columns of the second block are:

1-st column: \( (0, \ldots, 0, j_{0,m+1} \beta_1, j_{1,m+2} \beta_2, \ldots, j_{m-1,2m} \beta_m) \)
2-nd column: \( (0, \ldots, 0, j_{0,m} \beta_1, j_{1,m+1} \beta_2, \ldots, j_{m-2,2m} \beta_m, 0) \)
3-d column: \( (0, \ldots, 0, j_{0,m-1} \beta_1, j_{1,m} \beta_2, \ldots, j_{m-3,2m} \beta_m, 0, 0) \)

\[ \vdots \quad \quad \vdots \quad \quad \vdots \quad \quad \vdots \]

(n-2)-th column: \( (j_{0,2} \beta_1, \ldots, j_{m-1,m+1} \beta_m, 0, \ldots, 0) \),

where the \( m \) columns of the first block denote derivations with respect to \( \beta_1, \ldots, \beta_m \), and the \( n-1 \) columns of the second block denote derivations with respect to \( \alpha_n, \ldots, \alpha_3 \).

Now the situation differs from the first main case. If we apply the Gauss elimination algorithm to kill all terms of the right part of the matrix lying over terms containing \( \beta_m \), we can destroy the top non-zero terms. So, we must control the changes of these terms modulo \( \epsilon^2 \). Fortunately, it is not difficult: the term \( j_{m-1-k,n-1-k} \alpha_3 - 6 \beta_1 \), where \( 0 \leq k \leq m-1 \), will be changed to the term

\[
(\epsilon^2) \frac{(-1 + m)^2 m (2 + 5m + 2m^2) + 2k^2 (-1 + m^3) + k (4 + m - m^3 - 4m^4)}{m^3}.
\]
As it can be easily checked, the numerator can be equal to zero only for \( k > m - 1 \). Thus, the rank of \( J \) is equal to \( 2m - 1 = \dim Y \), a contradiction.

Now we can use the induction: suppose we have proved that the whole component belongs to the intersection \( Y = \{ \alpha_0 = \ldots = \alpha_{l-1} = 0 = \beta_0 = \ldots = \beta_{l-2} \} \). Then the \( 2(l-1) - 1 \)-th equation in (16) implies \( \alpha_l \beta_{l-1} = 0 \). If there is a smooth point with \( \alpha_l \neq 0 \), then we can apply the arguments from the first main case to show that the matrix \( J \) has the maximal rank equal to the dimension of \( Y \), a contradiction. If there is a smooth point with \( \beta_{l-1} \neq 0 \), then from the \( 2(l-1) \)-th equation we get

\[
\alpha_{l+1} = \frac{3}{l-2,2(l-1)} \beta_{l-1},
\]

and, analogously to the case \( \alpha_2 = 0 \), \( \beta_1 \neq 0 \), we can control the changes of the top non-zero terms \( (j_{m-1-k,m-1-k+l} \alpha_{l+1} - 6 \beta_{l-1}) \), \( l \leq m \). They will be changed to the terms

\[
\frac{1}{(-1+l)^2 m^3} (-1 + l - m) (-4k - 2k^2 + 12kl + 4k^2 l - 12kl^2 - 2k^2 l^2 + 4kl^3 - 2m + 5km + 2k^2 m + 6lm
-10klm - 2k^2 lm - 6l^2 m + 5kl^2 m + 2l^3 m +
3m^2 - 5km^2 - 2k^2 m^2 - 6m^2 + 5klm^2 + 3l^2 m^2 + 3m^3 + 4km^3 - 3lm^3 - 2m^4).
\]

The last expression is equal to zero only for \( k = -2 + 2l + m > m \) or

\[
k = m \frac{-1 + 2l - l^2 + m - lm + 2m^2}{2(1 - 2l + l^2 - m + lm + m^2)}.
\]

But the last expression can not be integer. Indeed, the great common divisor of \( m \) and \( (1 - 2l + l^2 - m + lm + m^2) \) must divide also the numerator, i.e. the doubled fraction must be integer. On the other hand, it is clear that the fraction is positive and less than one. It also easy to check that it can not be equal to \( 1/2 \).

At the end we obtain that the whole component belongs to the intersection \( Y = \{ \alpha_0 = \ldots = \alpha_m = 0 = \beta_0 = \ldots = \beta_{m-1} \} \) with \( \dim Y = 2 \). Then from \( (2m - 1) \)-th equation we obtain \( \alpha_{m+1} = 0 \), i.e. the component lies in \( Y \cap \{ \alpha_{m+1} = 0 \} \), whose dimension is one, a contradiction. \( \square \)

Let us prove Theorem 1.1 The intersection \( X' \) (in \( \mathbb{P}^{2m+4} \)) of \( X \) from proposition \( 3.1 \) and the cubic defined by \( (14) \) is again non-empty, and each its irreducible component has dimension greater or equal to 2; the intersection \( X' \cap Z \) with \( Z \) is non-empty and each its irreducible component has dimension greater or equal to 1. The homogeneous part of \( (14) \) not depending on \( u \) is

\[
\alpha_0 \beta_1^2 + \beta_0^3/2.
\]

It also has a solution of the form \( P \) from proposition \( 3.1 \).

To prove Theorem 1.1 it is sufficient to prove that for any fixed \( m > 0 \) any irreducible component of \( X' \cap Z \) containing \( P \) has dimension 1.

Note that if \( \alpha_0 \neq 0 \), then either \( \beta_1 \) or \( \beta_0 \) is not equal to 0. Indeed, if \( \beta_0 = \beta_1 = 0 \), then from 0-th equation it follows that \( \beta_2 = 0 \), from 1-st equation it follows that \( \beta_3 = 0 \) and, by iteration, \( \beta_m = 0 \), a contradiction.

Let \( Q \) be a smooth point on some irreducible component of \( X' \cap Z \) as in the proof of proposition \( 3.1 \). Consider the new Jacobi matrix with the first row consisting of partial derivatives of the equation \( (15) \):

\[
(3\beta_0^2, 2\alpha_0 \beta_1, 0, \ldots, 0, \beta_1^2).
\]
If $\alpha_0 \neq 0$, then it’s easy to see that this row and all other rows of the old matrix $J$ are linearly independent, i.e. the dimension of the component is one.

If $\alpha_0 = 0$, we can literally repeat the arguments from the proof of proposition 3.1. Indeed, as we have already seen, in this case even an irreducible component of $X \cap Z$ would be of dimension less or equal to 1. Theorem 1.1 is proved.

### 3.2 Proof of Theorems 1.2, 1.3

According to Theorem 1.1 an arbitrary equation

$$Y^2 = X^3 + c_2X^2 + c_1X + c_0, \quad X, Y \in A_1$$

has infinitely many solutions of the form $L_{4,m} = (\partial_x^2 + V_m(x))^2 + W_m(x)$, $L_{6,m}$, and the equation

$$Y^2 = X^{2g+1} + c_2gX^{2g} + \cdots + c_1X + c_0, \quad X, Y \in A_1$$

also has infinitely many solutions of the form $\varphi(L_4^1)$, $\varphi(L_{4g+2}^1)$, where

$$L(r) = \varphi(L_4^1) = ((1 - y^2)\partial_y^2 - 3y\partial_y + aT_r(y) + b)^2 - ar^2g(g+1)T_r(y),$$

$T_r(y)$ is the Chebyshev polynomial of degree $|r|$ (see section 2). To prove Theorem 1.2 and Theorem 1.3 it is enough to prove that at $r > 10$ and $r \neq r_1$

$$\varphi(L_{4,r}) \neq L_{4,r_1}, \quad \varphi(L(r)) \neq L(r_1),$$

for arbitrary $\varphi \in \text{Aut}(A_1)$. This facts follow from the following lemma.

**Lemma 3.1.** Consider a family of operators of order four with polynomial coefficients

$$L(r) = (a(x)\partial_x^2 + b(x)\partial_x + c_r(x))^2 + d_r(x), \quad r \in \mathbb{N},$$

where $a(x), b(x)$ are polynomials of fixed degree such that

$$\deg a(x) > \deg b(x), \quad \deg c_r(x) = r, \quad r \geq \deg d_r(x).$$

If $r > \deg a(x) + 8$, then

$$\varphi(L(r)) \neq L(r_1)$$

at $r \neq r_1$ for arbitrary $\varphi \in \text{Aut}(A_1)$.

Here we assume that $\deg b(x) = -\infty$ if $b(x) = 0$.

**Proof.** Let us assume that there is $\varphi \in \text{Aut}(A_1)$ such that at $r > \deg a(x) + 8$ we have $\varphi(L(r)) = L(r_1)$ for some $r \neq r_1$. Let

$$\varphi(x) = q_n(x)\partial_x^n + \cdots + q_0(x), \quad \varphi(\partial_x) = p_m(x)\partial_x^m + \cdots + p_0(x),$$

where $q_j, p_s$ are some polynomials. First consider the case $n = 0$. If $n = 0$, then $m = 1$ otherwise the operator $\varphi(L(r))$ has order greater than four. Further,

$$\varphi(a(x)\partial_x^2 + b(x)\partial_x) = a(q_0(x))(p_1(x)\partial_x + p_0(x))^2 + b(q_0(x))(p_1(x)\partial_x + p_0(x)) =$$

$$a(q_0(x))p_1^2(x)\partial_x^2 + a(q_0(x))(p_1(x)p_1(x) + p_0(x)) + b(q_0(x))(p_1(x))\partial_x +$$

$$+ a(q_0(x))p_0(x) + b(q_0(x)) + b(q_0(x))p_0(x).$$

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From our assumption it follows that
\[ a(q_0(x))p_1^2(x) = a(x), \quad a(q_0(x))(p_1(x)p_1'(x) + p_0(x)) + b(q_0(x))p_1(x) = b(x). \]

Hence from the first identity we get that \( p_1(x) \) is a constant and \( q_0(x) \) is a linear function. From the second identity we get that \( p_0(x) \) is a constant, otherwise the degree of the left hand side is greater than the degree of the right hand side. Thus
\[ \varphi(x) = s_1x + s_2, \quad \varphi(\partial_x) = s_3\partial_x + s_4, \quad s_j \in \mathbb{C}. \]
From this we obtain \( \varphi(L(r)) \neq L(r_1) \).

Let us consider the general case \( n \neq 0 \). We have the following identities for orders of differential operators
\[ \text{ord}\varphi(a(x)\partial_x^2) = \text{ndeg}a(x) + 2m, \quad \text{ord}\varphi(b(x)\partial_x) = \text{ndeg}b(x) + m, \quad \text{ord}\varphi(c_r(x)) = rn. \]
Let us note that
\[ \text{ord}\varphi(a(x)\partial_x^2) = \text{ord}\varphi(c_r(x)), \]
for otherwise, since \( \text{ord}\varphi(a(x)\partial_x^2) > \text{ord}\varphi(b(x)\partial_x) \) we have
\[ \text{ord}\varphi(a(x)\partial_x^2 + b(x)\partial_x + c_r(x)) = \text{ord}\varphi(a(x)\partial_x^2 + c_r(x)) = \max\{rn, n\text{deg}a(x) + 2m\} \geq r, \]
and therefore \( \text{ord}\varphi(L(r)) \geq 2r > 4 \), a contradiction. Thus,
\[ \text{ndeg}a(x) + 2m = rn. \quad (19) \]
By direct calculations one can check that
\[ \text{ad}(-x)^3(L(r)) = [[[L(r), x], x], x] = 24a^2(x)\partial_x + 12a(x)b(x) + 12a(x)a'(x), \]
hence
\[ \text{ord}\varphi(\text{ad}(-x)^3(L(r))) = 2\text{ndeg}a(x) + m. \]
On the other hand,
\[ \varphi(\text{ad}(-x)^3(L(r))) = \text{ad}(-\varphi(x))^3(\varphi(L(r))). \]
We have
\[ \text{ord}[,\varphi(L(r)), \varphi(x)] \leq n + 3, \quad \text{ord}[[,\varphi(L(r)), \varphi(x)], \varphi(x)] \leq 2n + 2, \]
\[ \text{ord}[[[,\varphi(L(r)), \varphi(x)], \varphi(x)], \varphi(x)] \leq 3n + 1. \]
Thus, using (19) and our assumption \( r > \text{deg}a(x) + 8 \), we get
\[ 3n + 1 \geq \text{ord}[,\varphi(L(r))] = 2\text{ndeg}a(x) + m = n(r + 3\text{deg}a(x))/2 > \frac{n}{2}(8 + 4\text{deg}a(x)) = 4n + 2\text{ndeg}a(x). \]
We get a contradiction.

Hence Theorems 1.2 and 1.3 are proved.

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