Strong Coordination over Noisy Channels with Strictly Causal Encoding

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Abstract—We consider a network of two nodes separated by a noisy channel, in which the input and output signals have to be coordinated with the source and its reconstruction. In the case of strictly causal encoding and non-causal decoding, we prove inner and outer bounds for the strong coordination region and show that the inner bound is achievable with polar codes.

I. INTRODUCTION

While communication networks have traditionally been designed to reliably convey information, modern decentralized networks are introducing new challenges. More than communication by itself, what is crucial for the next generation of networks is to ensure the cooperation and coordination of the constituent devices, viewed as autonomous decision makers. The devices have to adapt their behavior to the state of the environment and to the actions of other devices, which may not be known by all players, creating information asymmetries; coordination is meant in the broad sense of enforcing a joint behavior of the devices through communication to resolve such asymmetries.

More specifically, we quantify coordination in terms of how well we can approximate a target joint distribution between the actions and signals of the devices. In particular, empirical coordination requires the joint histogram of actions and signals to approach a target distribution, while strong coordination requires their joint distribution to converge in total variation to an i.i.d. target distribution [1].

In this work, we consider a two-node network with an information source and a noisy channel in which the input and output signals should be strongly coordinated with the source and the reconstruction. This scenario presents two conflicting goals: the encoder needs to convey a message to the decoder to coordinate the actions, while simultaneously coordinating the signals coding the message. The two nodes are assisted in their task by a shared source of randomness. The case in which the encoder and the decoder are both non-causal has already been considered in [2, 3] but the problem of finding the coordination region is still open. We focus here on the setting in which the encoder is strictly causal, which has the benefit of shortening the transmission delay.

In [4] the authors provide a characterization of the empirical coordination region when the encoder is strictly causal. In [5], we proposed an explicit polar coding scheme that achieves this region. In this paper, we provide an inner and an outer bound for the strong coordination region and show that the inner bound is achievable with polar codes. Although the achievability techniques are similar to the ones used in [5], the strictly causal nature of the encoder requires a more subtle random coding scheme with a block-Markov structure.

The remainder of the paper is organized as follows. Section II introduces the notation, Section III describes the model under investigation and states the main result. Section IV proves an inner bound by proposing a random binning scheme and a random coding scheme that have the same statistics and Section V proves an outer bound. The two bounds match, except for the bound on the minimal rate of common randomness, and closing the gap between the two regions remains an open problem. Finally, we provide an explicit polar code construction achieving the inner bound in the appendix.

II. PRELIMINARIES

We define the integer interval \([a, b]\) as the set of integers between a and b. Given a random vector \(X^n := (X_1, \ldots, X_n)\), we note \(X^i\) the first i components of \(X^n\), \(X_{\sim j}\) the vector \((X_j)_{j \neq i, j \in [1, n]}\), where the component \(X_j\) has been removed and \(X[A]\) the vector \((X_j)_{j \in A}\), \(A \subseteq [1, n]\).

Given two random vectors \(A \) and \(B\), \(A \perp B\) indicates that \(A\) and \(B\) are independent. We denote with \(Q_A\) the uniform distribution over \(A\). We note \(\mathcal{V}(\cdot, \cdot)\) and \(\mathbb{D}(\cdot\|\cdot)\) the variational distance and the Kullback-Leibler divergence between two distributions. The notation \(f(\varepsilon)\) denotes a function which tends to zero as \(\varepsilon\) does, and the notation \(\delta(n)\) denotes a function which tends to zero exponentially as \(n\) goes to infinity.

We now state some useful results.

**Lemma 1** ([6] Lemma 17): \(\mathcal{V}(P_A, \hat{P}_A) = \mathcal{V}(P_A P_B | A, \hat{P}_A P_B | A)\).

**Lemma 2:** \(\mathbb{D}(P_A \| \hat{P}_A) = \mathbb{D}(P_A P_B | A \| \hat{P}_A P_B | A)\).

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Lemma 3 (Lemma 2.7): Let $P$ and $P'$ two probability mass functions on $A$ such that $\mathbb{V}(P, P') = \varepsilon \leq 1/2$, then

$$|H(P) - H(P')| \leq \varepsilon \log |A|.$$

Lemma 4 (Lemma 4): If $\mathbb{V}(P_A^n, P_{B^n|A^n}, P_{B^n|A^n}) = \varepsilon$, then there exists $a \in A^n$ such that

$$\mathbb{V}(P_{B^n|A^n=a}, P_{B^n|A^n=a}) = 2\varepsilon.$$

Lemma 5 (Lemma 6): Let $P_{A^n}$ such that $\mathbb{V}(P_{A^n}, P_{A^n})$ is smaller than $\varepsilon$, then we have

$$\sum_{i=1}^{n} I(A_i; A_{-i}) \leq nf(\varepsilon).$$

III. SYSTEM MODEL AND MAIN RESULT

We consider the model depicted in Figure 1. Two agents, the encoder and the decoder, wish to coordinate their behaviors, in the sense that the stochastic actions of the agents should follow a known and fixed joint distribution. We suppose that the encoder and the decoder have access to a shared source of uniform randomness $C \in \{1, 2^{2nR_0}\}$. Let $U^n \in U^n$ be an i.i.d. source with distribution $P_U$. At time $i \in [1, n]$, the strictly causal encoder observes the sequence $U^{i-1} \in U^n$, common randomness $C$ and selects a signal $X_i = f_i(U^{i-1}, C)$, where $f_i : U^{i-1} \times \{1, 2^{2nR_0}\} \rightarrow \mathcal{X}$ is a stochastic function. The signal $X^n = (X_1, \ldots, X_n)$ is transmitted over a discrete memoryless channel $\tilde{P}_Y | X$. Upon observing $Y^n$ and the common randomness $C$, the decoder selects an action $V^n = g^n(Y^n, C)$, where $g^n : Y^n \times \{1, 2^{2nR_0}\} \rightarrow \mathcal{V}$ is a stochastic map. Let $f^n := \{f_i\}_{i=1}^{n}$ for block length $n$. The pair $(f^n, g^n)$ constitutes a code. We introduce the definitions of achievable and strong coordination in this setting.

Definition 1: A pair $(\tilde{P}_{UXYV}, R_0)$ is achievable for strong coordination if there exists a sequence $(f^n, g^n)$ of strictly causal encoders and non causal decoders with rate of common randomness $R_0$, such that for every $\varepsilon > 0$ there exists $n \in \mathbb{N}$ and a sufficiently long sub-sequence $(U^n, X^n, Y^n, V^n)$ with $n > (1-\varepsilon)n$ that satisfies

$$\lim_{n \to \infty} \mathbb{V}(P_{U^nX^nY^nV^n}, \tilde{P}_{UXYV}^{\tilde{g}, \tilde{f}}) = 0$$

where $P$ is the joint distribution induced by the code. The strong coordination region $\mathcal{R}$ is the closure of the set of achievable pairs $(\tilde{P}_{UXYV}, R_0)$.

Remark 1: The definition for strong coordination in this setting is slightly different from the definition of strong coordination with non-causal encoder and decoder in [12, 3], which for the strictly-causal encoder would be satisfied only by trivial distributions since the last block of the source will never be observed by the encoder. Here, we avoid this issue by losing coordination in a negligible fraction of time slots.

The problem of characterizing the strong coordination region is still open, but we establish the following inner and outer bounds.

Theorem 1: Let $\tilde{P}_U$ and $\tilde{P}_{Y|X}$ be the given source and channel parameters, then $\mathcal{R}_{in} \subseteq \mathcal{R} \subseteq \mathcal{R}_{out}$

$$\mathcal{R}_{in} := (\tilde{P}_{UXYV}, R_0)$$

$$\mathcal{R}_{out} := (\tilde{P}_{UXYV}, R_0)$$

Theorem 2: The region $\mathcal{R}_{in}$ defined in [3] is achievable using polar codes, provided there exists an error-free channel of negligible rate between the encoder and decoder.

Remark 2: By the chain rule, we have

- $I(XW; U) = I(W; UX) + I(X; U) = I(W; U|X)$ since $U$ and $X$ are independent;
- $I(XW; Y) = I(W; Y|X) + I(X; Y) = I(X; Y)$ because of the Markov chain $W - X - Y$.

Hence the condition $I(WX; U) \leq I(WX; Y)$ in [1] and [2] is equivalent to $I(W; U|X) \leq I(X; Y)$.

Comparison with empirical coordination: For empirical coordination, [3], Theorem 3 gives the following characterization of the region with strictly causal encoding:

$$\mathcal{R}_{emp} := (\tilde{P}_{UXYV}, R_0)$$

Observe that in $\mathcal{R}_{in}$ and $\mathcal{R}_{out}$ the decomposition of the joint distribution and the information constraints are the same as in $\mathcal{R}_{emp}$, but for strong coordination a positive rate of common randomness is also necessary. This is consistent with the conjecture, stated in [1], that with enough common randomness the strong coordination capacity region is the same as the empirical coordination capacity region for any specific network setting.
IV. Achievability proof of Theorem 1

The key idea of the achievability proof is to define a random binning for the target joint distribution, and a random coding scheme, each of which induces a joint distribution, and to prove that the two schemes have almost the same statistics. The proof uses the same techniques as in [10] inspired by [8], to deal with the strictly causal encoder, a block Markov structure is required for the random coding scheme. Before defining the coding scheme, we state some results that we use to prove the inner bound.

The following lemma is a consequence of the Slepian-Wolf Theorem.

Lemma 6 (Source coding with side information at the decoder [11, Theorem 10.1]): Given a discrete memoryless source \((A^n, B^n)\), where \(B^n\) is side information available at the decoder, let \(\varphi_n : A^n \to [1, 2^{nR}]\) be a uniform random binning of \(A^n\), and let \(C := \varphi_n(A^n)\). Then if \(R > H(A|B)\), the decoder can recover \(A^n\) from \(C\) and \(B^n\) with:

\[
\mathbb{E}_{\varphi_n}[\mathbb{P}\{\hat{A}^n \neq A^n\}] \leq \delta(n).
\]

Lemma 7 (Channel randomness extraction for discrete memoryless sources and channels): Let \(A^n\) with distribution \(P_{A^n}\) be a discrete memoryless source and \(P_{B^n|A^n}\) a discrete memoryless channel. Let \(\varphi_n : B^n \to [1, 2^{nR}]\) be a uniform random binning of \(B^n\), and let \(K := \varphi_n(B^n)\). Then if \(R \leq H(B|A)\), there exists a constant \(\alpha > 0\) such that

\[
\mathbb{E}_{\varphi_n}[\mathbb{D}(P_{A^n|K} \| P_{A^n|Q^n(K)})] \leq 2^{-\alpha n}.
\]

We omit the proof of Lemma 7 as it follows directly from the discussion in [12, Section III.A].

A. Random binning scheme

Assume that the sequences \(U^n, X^n, W^n, Y^n\) and \(V^n\) are jointly i.i.d. with distribution

\[
P_{U^n} P_{X^n} P_{W^n|U^n, X^n} P_{Y^n|X^n} P_{V^n|W^n}.
\]

First, we consider two uniform random binnings for \(X^n\):

- \(M_1 := \varphi_1(X^n)\), where \(\varphi_1 : X^n \to [1, 2^{nR_1}]\),
- \(M_2 := \varphi_2(X^n)\), where \(\varphi_2 : X^n \to [1, 2^{nR_2}]\).

The rates \(R_1\) and \(R_2\) are chosen as follows:

- \(R_1 + R_2 < H(X)\), so that by Lemma 7 there exists one binning \((\varphi_1', \varphi_2')\) of \(X\) such that \(M_1\) and \(M_2\) are almost uniform and almost independent of each other;
- \(R_1 > H(X|Y)\), so that by Lemma 6 there exists one binning \(\varphi_1'\) of \(X\) such that it is possible to reconstruct \(X\) from \(Y\) and \(M_1\) with high probability using a Slepian-Wolf decoder via

\[
P_{X^n|M_1,Y^n},
\]

where we can use the same binning \(\varphi_1'\) for both conditions, as proved in [3, Remark 7].

Then, we consider the following uniform random binnings for \(W^n\):

- \(M_3 := \varphi_3(W^n)\), where \(\varphi_3 : W^n \to [1, 2^{nR_3}]\),
- \(M_4 := \varphi_4(W^n)\), where \(\varphi_4 : W^n \to [1, 2^{nR_4}]\),
- \(F := \psi(W^n)\), where \(\psi : W^n \to [1, 2^{nR}]\).

where the rates \(R_3, R_4\) and \(R\) are chosen as follows:

- \(R_3 + R < H(W|X^n)\), so that by Lemma 7 there exists one binning \((\varphi_3', \psi)\) of \(W\) such that \(M_3\) and \(F\) are almost uniform and almost independent of \(X^n\) and \(U^n\);
- \(R_3 + R_4 + \tilde{R} > H(W|X^n)\), so that by Lemma 6 there exists one binning \((\varphi_2', \varphi_4', \psi')\) of \(W\) such that it is possible to reconstruct \(W\) from \((M_3, M_4, F)\) with high probability using a Slepian-Wolf decoder via the conditional distribution

\[
P_{SWW^n|M_3,M_4,F,X^n,Y^n},
\]

and we can use the same binning \((\varphi_3', \psi')\) for both conditions, as proved in [3, Remark 7]. This defines a joint distribution:

\[
P_{RB} := P_{U^n} P_{X^n} P_{W^n|U^n,X^n} P_{M_1|X^n} P_{M_2|X^n} P_{M_3|X^n} P_{M_4|W^n}.
\]

In particular, the conditional distributions \(P_{RB}^{M_3|M_3,X^n,U^n}\), \(P_{RB}^{W^n|M_3,M_4,F,X^n,Y^n}\), and \(P_{RB}^{X^n|M_1,M_2,F}\) are well-defined.

B. Random coding scheme

In this section we follow the approach in [8, Section IV.E] and [10]. Suppose that encoder and decoder have access to extra randomness \(F\), where \(F\) is generated uniformly at random in \([1, 2^{nR}]\) with distribution \(Q_F\) independently of the rest of the common randomness.

1) Encoder: We use a chaining construction over \(k\) blocks of length \(n\) in which the encoder observes \(U^n(i) := (U^n(1),\ldots,U^n(i))\), where \(U^n(i)\) for \(i \in [1, k]\) are \(k\) blocks of the source. The encoder has access to common randomness \((M_1(1:k), M_3(1:k), F(1:k), K(2:k))\) and the block-Markov scheme proceeds as follows.

At time \(i = 1,\ldots,k\), the encoder does the following:

- For \(i \in [1, k]\), \(M_3(i)\) and \(F(i)\) are generated independently and uniformly over \([1, 2^{nR_3}]\) and \([1, 2^{nR}]\) using common randomness with distributions \(Q_{M_3}\) and \(Q_F\) respectively;
- \(M_1(i)\) is generated independently and uniformly over \([1, 2^{nR_1}]\) using common randomness with distribution \(Q_{M_1}\);
- In the first block, \(M_2(1)\) is generated uniformly at random using some independent local randomness;
- For \(i \in [2, k]\), \(M_4(i-1)\) is generated according to the distribution defined earlier

\[
P_{RB}^{M_{4}(i-1)|M_3(X^n,U^n), \mathbf{m}_3(i-1), \mathbf{x}_3(i-1), \mathbf{u}_3(i-1)},
\]

where \((\mathbf{m}_3(i-1), \mathbf{x}_3(i-1), \mathbf{u}_3(i-1))\) are generated at time \(i - 1\);
- For \(i \in [2, k]\), \(M_2(i) = (M_2'(i), M_2''(i))\), where

\[
M_2'(i) = M_4(i-1) \oplus K_i
\]

and \(K_i\) is generated uniformly over \([1, 2^{nR_4}]\) using common randomness, while \(M_2''(i)\) is generated uniformly at random using some independent local randomness. Thanks to the Crypto Lemma [13, Lemma 3.1], the distribution on \(M_2(i)\) is uniform and we denote it with \(Q_{M_2'}\);
Remark 3: Observe that we have imposed the condition
\[
P_{\text{RC}}^{(i)} := P_{\text{RC}}^{(i)}(U^n X^n Y^n W^n M_1 M_2 M_3 M_4 F)_{(i)}
\]
\[
= P_{U^n}(u(i)) Q_{M_1} (m_{1,(i)}) Q_{M_2} (m_{2,(i)}) Q_{M_3} (m_{3,(i)}) Q_{F} (f(i)) P_{\text{RB}}^{(i)}(X^n | M_1 M_2 M_3 M_4 F (x(i)| m_{1,(i)}, m_{2,(i)}, m_{3,(i)}, f(i)) \]
\[
P_{\text{SW}}^{(i)}(W^n | M_4 F X^n) (w(i)| m_{3,(i)}, m_{4,(i)}, f(i), x(i)) P_{\text{RC}}^{(i)} (v^n | W^n Y^n (v(i)| w(i), y(i)).
\]

\[
P_{\text{RC}}^{(i)} := P_{\text{RC}}^{(i)}(U^n X^n Y^n W^n M_1 M_2 M_3 M_4 F)_{(i)}
\]
\[
= P_{U^n}(u(i)) Q_{M_1} (m_{1,(i)}) Q_{M_2} (m_{2,(i)}) Q_{M_3} (m_{3,(i)}) Q_{F} (f(i)) P_{\text{RB}}^{(i)}(X^n | M_1 M_2 M_3 M_4 F (x(i)| m_{1,(i)}, m_{2,(i)}, m_{3,(i)}, f(i)) \]
\[
P_{\text{SW}}^{(i)}(W^n | M_4 F X^n) (w(i)| m_{3,(i)}, m_{4,(i)}, f(i), x(i)) P_{\text{RC}}^{(i)} (v^n | W^n Y^n (v(i)| w(i), y(i)).
\]

The encoder generates $X_{(i)}^n$ according to the distribution defined earlier

\[
P_{\text{RB}}^{(i)} X^n | M_1 M_2 M_3 M_4 F (x(i)| m_{1,(i)}, m_{2,(i)}, m_{3,(i)}, f(i));
\]

Note that this distribution satisfies the strictly causal constraint, since $X_{(i)}^n$ is generated knowing the common randomness and $M_{2,(i)} = M_{4,(i-1)} \oplus K_i$, where $M_{4,(i-1)}$ depends on the source at time $i - 1$.

Then, the sequence $X_{(i)}^n$ is sent through the channel.

Remark 3: Observe that we have imposed the condition $|M_{4,(i-1)}| = |M_{2,(i)}|$, which holds as long as $R_4 \leq R_2$. We have

\[
R_2 < H(X) - R_1 < H(X) - H(X|Y) = I(X; Y),
\]

Then, $R_4 \leq R_2$ implies $I(W; U|X) < I(X; Y)$.

2) Decoder: Since the decoder is non-causal, it observes $Y_{(i)}^n$ and common randomness $(M_{1,(1:k)}, M_{3,(1:k)}, F(1:k), K_{(2:k)})$ and the decoding algorithm proceeds as follows:

- The decoder reconstructs $\hat{X}_{(1:k)}^n$, where, for all $i \in [1, k]$, $X_{(1:k)}^n$ is generated via the conditional distributions

\[
P_{\text{SW}}^X (X|Y^n (x(i)| m_{1,(i)}, y(i));
\]

- The decoder recovers $\hat{M}_{2,(1:k)}$, where, for all $i \in [1, k]$, $\hat{M}_{2,(i)}$ is generated via

\[
\varphi_2(\hat{x}_{(i)}) = m_{2,(i)};
\]

where $\hat{x}_{(i)}$ is the output of the Slepian-Wolf decoder;

- For all $i \in [2, k]$, with the key of the one-time pad $K_{(i)}$ and $M_{2,(i)}$, the decoder recovers

\[
\hat{M}_{4,(i-1)} = \hat{M}_{2,(i)} \oplus K_{(i)};
\]

- Observe that at time $i$, the decoder knows an estimate of $M_{4,(i)}$ because the non-causal nature of the decoder allows us to decode in reverse order and we note its distribution $P_{\text{RC}}^M (\hat{m}_{4,(i)}^i)$. Therefore, once the decoder
has $\hat{M}_{4(i)}$, it reconstructs $W^n_{(i)}$, $i \in [1, k - 1]$, via

$$P_{W^n|M_2,\hat{M}_4,F,W_i}^\text{SW}(w_{(i)}, m_{3(i)}, \hat{m}_{4(i)}, f_{(i)}, x_{(i)});$$

Finally, the decoder generates $V^n_{(i)}$, $i \in [1, k - 1]$, letter by letter according to the distribution

$$P_{V^n|W^n,Y^n}(v_{(i)}|w_{(i)}, y_{(i)}).$$

For all $i \in [1, k - 1]$, the block-Markov coding scheme defines the joint distribution $P_{R(i)}^\text{RC}$ in (8).

Remark 4: Observe that, even though the block-Markov algorithm is over $k$ blocks, the last block is only used to convey information on the source at time $k - 1$ through $M_{4(k-1)}$ which is generated at time $k$. In fact, if $k$ is large enough, Definition 1 allows us to coordinate only the first $k - 1$ blocks.

Now, observe that we impose rate conditions $R_1 > H(X|Y)$ such that $P(\hat{X}^n_{(i)} \neq X^n_{(i)}) \leq \delta(n)$ which in turn implies $P(\hat{M}_{2(i)} \neq M_{2(i)}) \leq \delta(n)$, $P(\hat{M}_{4(i)} \neq M_{4(i)}) \leq \delta(n)$. Moreover,

$$P(\hat{X}^n_{(1:k)} \neq X^n_{(1:k)}) \leq \sum_{i=1}^{k} P(\hat{X}^n_{(i)} \neq X^n_{(i)}) \leq k\delta(n),$$

$$P(\hat{M}_{2(i)} \neq M_{2(i)}) \leq \sum_{i=1}^{k} P(\hat{M}_{2(i)} \neq M_{2(i)}) \leq k\delta(n),$$

$$P(\hat{M}_{4(i)} \neq M_{4(i)}) \leq \sum_{i=1}^{k-1} P(\hat{M}_{4(i)} \neq M_{4(i)}) \leq (k - 1)\delta(n),$$

where $k\delta(n)$ and $(k - 1)\delta(n)$ vanish due to $\delta(n)$ goes to zero exponentially fast.

We recall the definition of coupling and the basic coupling inequality for two random variables [14].

Definition 2: A coupling of two probability distributions $P_A$ and $P_{A'}$ on the same measurable space $A$ is any probability distribution $P_{AA'}$ on the product measurable space $A \times A$ whose marginals are $P_A$ and $P_{A'}$. Proposition 1 ([14] I.2.6): Given two random variables $A$, $A'$ with probability distributions $P_A$, $P_{A'}$, any coupling $P_{AA'}$ of $P_A$, $P_{A'}$ satisfies

$$\mathbb{V}(P_A, P_{A'}) \leq 2\mathbb{V}_{P_{AA'}}(A \neq A').$$

Then, we apply Proposition 1 to

$$A = (U^n X^n Y^n V^n W^n M_1 M_2 M_3 M_4 F)(i),$$

$$A' = (U^n \hat{X}^n Y^n V^n W^n M_1 M_2 M_3 \hat{M}_4 F)(i),$$

$$P_A = P_{R^C}^\text{RC}, P_{A'} = P_{A'}^\text{RC},$$

$$A = U \times X \times Y \times V \times [1, 2^{nR_2}] \times [1, 2^{nR_2}] \times [1, 2^{nR_2}] \times [1, 2^{nR_2}],$$

and because of (9) the distribution $\hat{P}_{R^C}^\text{RC(i)}$ defined in (10) has almost the same statistics of $\hat{P}_{R^C}^\text{RC(i)}$.

$$\mathbb{V}(\hat{P}_{R^C}^\text{RC(i)}, \hat{P}_{R^C}^\text{RC(i)}) \leq \delta(n).$$

C. Coordination of $(U^n, X^n, W^n, Y^n, V^n)(i)$

We want to show that the distribution $\hat{P}_{R^C}^\text{RC(i)}$ is achievable for strong coordination, i.e.

$$\lim_{n \to \infty} \mathbb{V}(P_{R^C}, \hat{P}_{R^C}^\text{RC(i)}) = 0. \quad (11)$$

Observe that

- By Lemma 1 the total variational distance remains the same without $\hat{P}_{Y^n|X^n}$ and $P_{Y^n|W^n,Y^n}$ in both $P_{R^B}$ and $\hat{P}_{R^C}^\text{RC}$.
- The random binning distribution becomes

$$P_{R^B | M_1 M_2 M_3 F}^\text{RC} = P_{M_1 M_2 M_3 F}^\text{RC}$$

and $P_{R^B | M_1 M_2 M_3 F}^\text{RC}$ can be removed in both $P_{R^B}$ and $\hat{P}_{R^C}^\text{RC}$ by Lemma 1.
- Now, (11) is satisfied if

$$\mathbb{V}(P_{R^B | M_1 M_2 M_3 F}^\text{RC}, \hat{P}_{R^C}^\text{RC(i)}) = \mathbb{V}(\hat{P}_{R^B | M_1 M_2 M_3 F}^\text{RC}, \hat{P}_{R^C}^\text{RC(i)})$$

vanishes. By Lemma 7 this would be true if

$$R_1 + R_2 + R_3 + \tilde{R} < H(W | X | U) \quad (12)$$

Since we have imposed the rate condition $R_3 + \tilde{R} < H(W | X | U)$ and $R_1 + R_2 < H(X)$ and $H(W | X | U) + H(X) = H(W | X | U)$ because $X$ and $U$ are independent, (12) holds and there exists a binning of $(W, X$) such that

$$\mathbb{V}(P_{R^B | M_1 M_2 M_3 F}^\text{RC}, \hat{P}_{R^C}^\text{RC(i)}) \leq \delta(n).$$

Then we conclude that (11) holds.

D. Coordination of $(U^n, X^n, Y^n, V^n)(i)$ by removing the extra randomness $F$

Even though the extra common randomness $F$ is required to coordinate $(U^n, X^n, Y^n, V^n, W^n)$ we will show that we do not need it in order to coordinate only $(U^n, X^n, Y^n, V^n)$. As in [8], we would like to reduce the amount of common randomness by having the two nodes agree on an instance $F = f$. To do so, we apply Lemma 7 again where $B^n = W^n$, $K = F$, $\varphi$ and $A^n = U^n X^n Y^n V^n$. If $R < H(W | U X Y V)$, there exists a fixed binning such that

$$\mathbb{V}(\hat{P}_{R^B | U^n X^n Y^n V^n, F}^\text{RC}, \hat{P}_{R^C}^\text{RC(F)}) = \delta(n). \quad (13)$$

which implies

$$\mathbb{V}(\hat{P}_{R^B | U^n X^n Y^n V^n}^\text{RC(F)}, \hat{P}_{R^C}^\text{RC(F)}) = \delta(n). \quad (14)$$

By Lemma 4 there exists an instance $f \in [1, 2^{n\tilde{R}}]$ such that

$$\mathbb{V}(\hat{P}_{R^B | U^n X^n Y^n V^n}^\text{RC(F)}, \hat{P}_{R^C}^\text{RC(F)}) = \delta(n). \quad (15)$$

Then, by fixing $F = f$ and using common randomness $C = (M_{1(i)}, M_{3(i)}, K_i)$, we have coordination for $(U^n, X^n, Y^n, V^n)$. 

E. Rate of common randomness

We have used common randomness to generate $M_1, M_3$ and the key of the one time pad, which has the same size of $M_3$. Then, upon denoting by $R_0$ the total rate of common randomness, $R_0 := R_1 + R_3 + R_4$ and

$$R_0 + R > H(X|Y) + H(W|X)$$

which implies

$$R_0 > H(X|Y) + H(W|X) - H(W|UXYV).$$

Observe that

$$H(WX|Y) = H(WX) - I(WX; Y)$$

because the Markov chain $W - X - Y$ implies $I(W; Y|X) = 0$ and therefore (16) becomes

$$R_0 > H(WX|Y) - H(W|UXYV)$$

$$= H(W|Y) + H(X|WY) - H(W|UXYV)$$

$$= I(W;UXV|Y) + H(X|WY).$$

F. Coordination of all blocks

To simplify the notation, we set

$L_i := U^n_{(i)}, X^n_{(i)} Y^n_{(i)}, V^n_{(i)}$ \quad $i \in \{1, k-1\}$

$L_{a:b} := U^n_{(a:b)}, X^n_{(a:b)} Y^n_{(a:b)}, V^n_{(a:b)}$ \quad $[a, b] \subseteq \{1, k-1\}$.

First, note that two consecutive blocks $L_{i-1}$ and $L_i$ are dependent only through $M_{4,(i-1)}$, in fact, $M_{4,(i-1)}$ is created at time $i$ using $U^n_{(i-1)}$ and $X^n_{(i-1)}$ and it is used to generate $M_{2,(i)}$, which in turn is used at the encoder to generate $X^n_{(i)}$. Hence, since $Y^n_{(i)}$ is the output of the channel and $V^n_{(i)}$ is generated using $Y^n_{(i)}$ and the auxiliary random variable, generated through an estimate of $M_{4,(i)}$, uniform common randomness and $X^n_{(i)}$, we can conclude that $L_{i-1}$ and $L_i$ are dependent only through $M_{2,(i)}$ and therefore $M_{4,(i-1)}$. However, to generate $M_{2,(i)}$, the encoder applies a one-time pad on $M_{4,(i-1)}$ as shown in (7), making $M_{4,(i-1)}$ and $M_{2,(i)}$ independent of each other and ensuring the independence of two consecutive blocks.

To conclude the proof we need the following results.

**Lemma 8**: We have

$$\forall \left( P_{L_{1:k-1}} \prod_{i=1}^{k-1} P_{L_i} \right) \leq \delta(n).$$

**Lemma 9**: We have

$$\forall \left( P_{L_{1:k-1}}, \tilde{P}^{\otimes n(k-1)}_{UXYV} \right) \leq \delta(n).$$

We omit the proofs because they are very similar to the proofs of [3] Lemma 15 and [3] Lemma 16 respectively.

V. Outerbound of Theorem[1]

Consider a code $(f^n, g^n)$ that induces a distribution $P^n_{UXY^n}$ that is $\varepsilon$-close in total variational distance to the i.i.d. distribution $P^n_{UXY^n}$. Let the random variable $T$ be uniformly distributed over the set $[1, n]$ and independent of the sequence $(U^n, X^n, Y^n, V^n, C)$. The variable $T$ will serve as a random time index. The variable $U_T$ is independent of $T$ because $U^n$ is an i.i.d. source [1].

A. Bound on $R_0$

We have

$$nR_0 = H(C) \geq H(C|Y^n) \geq I(C; U^n V^n X^n | Y^n)$$

$$= \sum_{t=1}^{n} I(U_t V_t X_t; C|U^{t-1} V^{t-1} X^{t-1} Y_{t-1} Y_t)$$

$$= \sum_{t=1}^{n} I(U_t V_t X_t; C Y_{t-1} U^{t-1} V^{t-1} X^{t-1} Y_t) - \sum_{t=1}^{n} I(U_t V_t X_t; Y_{t-1} U^{t-1} V^{t-1} X^{t-1} Y_t)$$

$$\geq \sum_{t=1}^{n} I(U_t V_t X_t; C Y_{t-1} U^{t-1} V^{t-1} X^{t-1} Y_t) - n f(\varepsilon)$$

$$= n I(U_T V_T X_T; C Y_{T+1} U^{T-1} T | Y_T) - n f(\varepsilon)$$

$$= n I(U_T V_T X_T; C Y_{T+1} U^{T-1} T | Y_T)$$

$$\geq \sum_{t=1}^{n} I(U_t V_t X_t; C Y_{t+1} U^{t-1} T | Y_T) - n f(\varepsilon)$$

where (a) comes from Lemma 5 and (b) comes from [9] Lemma VI.3.

B. Information constraint

We have

$$n I(U_T; CY^n_{T+1} U^{T-1} X_T T)$$

$$= \sum_{t=1}^{n} I(U_t; CY^n_{t+1} U^{t-1} X_T)$$

$$= \sum_{t=1}^{n} I(U_t; CY^n_{t+1} U^{t-1}) + \sum_{t=1}^{n} I(U_t; X_T; CY^n_{t+1} U^{t-1})$$

$$\leq \sum_{t=1}^{n} I(U_t; CY^n_{t+1} U^{t-1}) + \sum_{t=1}^{n} I(U_t Y^n_{t+1}; X_T | U^{t-1} C U^{t-1})$$

$$= \sum_{t=1}^{n} I(U_t; CY^n_{t+1} U^{t-1}) + \sum_{t=1}^{n} I(U_t Y^n_{t+1}; X_T | U^{t-1} C U^{t-1})$$

$$\leq \sum_{t=1}^{n} I(Y_t; U^{t-1} Y^n_{t+1} C X_T) = n I(Y_T; U^{T-1} Y^n_{T+1} C X_T | T)$$
\[ \leq nI(Y_T; U^T Y^n_{T+1} C X_T T) \]

where (a) follows from the i.i.d. nature of the source, (b) from the following Markov chain

\[ X_t \sim (C, U^{t-1}) - (U_t, Y^n_{t+1}) \]

that holds because of the strictly causal nature of the encoder. Then, (c) comes from the fact that the source is generated i.i.d. and independent of \( C \) and (d) from Csiszár’s sum identity.

We identify the auxiliary random variables \( W_t \) with \( (U_t^{t-1}, Y^n_{t+1}, C) \) for each \( t \in [1, n] \) and \( W \) with \( (W, T) = (U_t^{T-1}, Y^n_{T+1}, C, T) \).

C. Identification of the auxiliary random variable

For each \( t \in [1, n] \), \( W_t \) satisfies the following conditions:

\[ U_t \perp X_t \]

\[ Y_t - X_t - (U_t, W_t) \]

\[ V_t = (Y_t, W_t) - (U_t, X_t) . \]

Then, we have

\[ U_T \perp X_T \]

\[ Y_T - X_T - (U_T, W_T) \]

\[ V_T = (Y_T, W_T) - (U_T, X_T) , \]

and, since \( W = W_t \) when \( T = t \), it implies

\[ U \perp X \]

\[ Y = X - (U, W) \]

\[ V = (Y, W) - (U, X) . \]

We do not write all details because they follow similarly the discussion in [15], Section VIII-B. The proof of the cardinality bound is omitted since it follows the ones in [3], Appendix G.

APPENDIX

EXPLICIT POLAR CODING SCHEME

In this section, we propose a polar coding scheme that achieves the region \( \mathcal{R}_{in} \). For brevity, we only focus on the set of achievable distributions in \( \mathcal{R}_{in} \) for which the auxiliary variable \( W \) is binary. The scheme can be extended to the case of a non-binary random variable \( W \) using non-binary polar codes as long as the cardinality \( \lvert W \rvert \) is a prime number [16].

A. Polar coding scheme

Assume that the sequences \( U^n, X^n, W^n, Y^n \) and \( V^n \) are jointly i.i.d. with distribution \( \mathcal{P} \). We propose an explicit coding scheme similar to the one in [5] that enables a joint distribution close to \( \mathcal{P} \) in total variational distance.

a) Polarize \( X \): Let \( S^n = X^n G_n \) be the polarization of \( X^n \), where \( G_n \) is the source polarization transform. For some \( 0 < \beta < 1/2 \), let \( \delta_n := 2^{-n^\beta} \) and define the very high and high entropy sets:

\[ \mathcal{V}_X : = \{ j \in [1, n] \mid H(S_j | S^{j-1}) > 1 - \delta_n \} , \]

\[ \mathcal{H}_X := \{ j \in [1, n] \mid H(S_j | S^{j-1}) > \delta_n \} \]  

(21)

\[ \mathcal{H}_X[Y] := \{ j \in [1, n] \mid H(S_j | S^{j-1} Y^n) > \delta_n \} . \]

Partition the set \( [1, n] \) into four disjoint sets:

\[ A_1 := \mathcal{V}_X \cap \mathcal{H}_X[Y] , \]

\[ A_2 := \mathcal{V}_X \cap \mathcal{H}_X[Y] , \]

\[ A_3 := \mathcal{V}_X^c \cap \mathcal{H}_X[Y] , \]

\[ A_4 := \mathcal{V}_X^c \cap \mathcal{H}_X[Y] . \]

Remark 5: We have:

- \( \mathcal{V}_X \subset \mathcal{H}_X \) and \( \lim_{n \to \infty} \mathcal{H}_X[Y]/n = 0 \) \([17]\).
- \( A_1 \cup A_2 = \mathcal{V}_X \) and \( \lim_{n \to \infty} \mathcal{V}_X[Y]/n = H(X) \) \([18]\).
- \( A_1 \cup A_3 = \mathcal{H}_X[Y] \) and \( \lim_{n \to \infty} \mathcal{H}_X[Y]/n = H(X|Y) \) \([17]\).

Since \( \lim_{n \to \infty} \frac{|A_2| - |A_3|}{n} = H(X) - H(X|Y) = I(X; Y) \geq 0 \) this implies directly that for \( n \) large enough \( |A_2| \geq |A_3| \).

b) Polarize \( W \): Let \( Z^n = W^n G_n \) be the polarization of \( W^n \) and define:

\[ \mathcal{V}_{W|XU} := \{ j \in [1, n] \mid H(Z_j | Z^{j-1} X^n U^n) > 1 - \delta_n \} , \]

\[ \mathcal{H}_{W|XU} := \{ j \in [1, n] \mid H(Z_j | Z^{j-1} X^n U^n) > \delta_n \} \]  

(22)

\[ \mathcal{H}_{W|X} := \{ j \in [1, n] \mid H(Z_j | Z^{j-1} X^n) > \delta_n \} . \]

Partition the set \( [1, n] \) into four disjoint sets:

\[ B_1 := \mathcal{V}_{W|XU} \cap \mathcal{H}_{W|X} = \mathcal{V}_{W|U} \]

\[ B_2 := \mathcal{V}_{W|XU} \cap \mathcal{H}_{W|X}^c = \emptyset , \]

\[ B_3 := \mathcal{V}_{W|XU}^c \cap \mathcal{H}_{W|X} = \mathcal{V}_{W|U}^c , \]

\[ B_4 := \mathcal{V}_{W|XU}^c \cap \mathcal{H}_{W|X}^c = \emptyset . \]

Remark 6: We have:

- \( \mathcal{V}_{W|XU} \subset \mathcal{H}_{W|XU} \) and \( \lim_{n \to \infty} \mathcal{H}_{W|XU}/n = 0 \) \([17]\).
- \( B_1 = \mathcal{V}_{W|XU} \) and \( \lim_{n \to \infty} \mathcal{V}_{W|XU}/n = H(W|XU) \) \([18]\).
- \( B_4 = \mathcal{H}_{W|X}^c \) and \( \lim_{n \to \infty} \mathcal{V}_{W|XU}/n = 1 - H(W|X) \) \([18]\).
- \( B_3 \cup B_4 = \mathcal{V}_{W|XU} \) and \( \lim_{n \to \infty} \mathcal{V}_{W|XU}/n = 1 - H(W|XU) \) \([18]\).

Note that

\[ H(W|X) - H(W|XU) = I(W; U|X) = I(WX; U) \geq 0 \]

and \( |B_3|/n \) tends to \( I(WX; U) \). Since \( I(WX; Y) = I(X; Y) \), the inequity \( I(WX; U) \leq I(WX; Y) \) implies directly that for \( n \) large enough \( |B_3| \leq |A_2| - |A_3| \).

c) Encoding: The encoder observes \( U^n_{(0:k)} := (U^n_{(0)}, U^n_{(1)}, \ldots, U^n_{(k)}) \), where \( U^n_{(i)} \) is a uniform random sequence and \( U^n_{(i)} \) for \( i \in [1, k] \) are \( k \) blocks of the source. It then generates for each block \( i \in [1, k] \) random variables \( S^n_{(i)} \) and \( Z^n_{(i)} \) following the procedure described in Algorithm 1. The chaining construction proceeds as follows:

- The bits in \( A_1 \subset \mathcal{V}_X \) in block \( i \in [1, k] \) are chosen with uniform probability using a uniform randomness source \( C_i \) shared with the decoder;
- In the first block the bits in \( A_2 \subset \mathcal{V}_X \) are chosen with uniform probability using a local randomness source \( M_i \);
- Let \( B^c_i := \mathcal{V}_{W|UXXYV} \) observe that \( B^c_i \) is a subset of \( B_i \) since \( \mathcal{V}_{W|UXXYV} \subset \mathcal{V}_{W|UX} \). The bits in \( B^c_i \subset \mathcal{V}_{W|UX} \)
Algorithm 1: Encoding algorithm at Node 1

**Input:** \((U^n_0, \ldots, U^n_k)\), local randomness (uniform random bits) \(M\) and common randomness \(C = \{(C_i)_{i=1,\ldots,k}, (C'_i)_{i=1,\ldots,k}, \{K_i\}_{i=1,\ldots,k}, \{K'_i\}_{i=1,\ldots,k}\}\) shared with Node 2:
- \(C_i\) of size \(|A_i|\) and \(K_i\) of size \(|A_3|\)
- \(C'_i\) of size \(|B'_i|\), \(|C'_i|\) of size \(|B_1 \setminus B'_i|\) and \(K'_i\) of size \(|B_3|\)

**Output:** \((S^n_1, \ldots, S^n_k), (Z^n_1, \ldots, Z^n_k)\)

if \(i = 1\) then

\[ S_1[A_1] \leftarrow C_1, \quad S_1[A_2] \leftarrow M \]

for \(j \in A_3 \cup A_4\) do

\[ \text{Successively draw the bits } S_j(1) \text{ according to } \]
\[ P_{S_j(1)}|S_j(0)}(1) \]
\[ Z_1[B'_1] \leftarrow C'_1, \quad Z_1[B_1 \setminus B'_1] \leftarrow C'_1 \]

for \(j \in B_3 \cup B_4\) do

\[ \text{Given } U^n_1, \text{ successively draw the bits } Z^j(1) \text{ according to } \]
\[ P_{Z_j(1)}|Z_j(0)}(1) \]

for \(i = 2, \ldots, k\) do

\[ S_i[A_1] \leftarrow C_i, \quad S_i[A_2] \leftarrow M \]
\[ S_i[B'_1] \leftarrow Z_{i-1}[B_3] \oplus K_{i-1} \]
\[ S_i[A'_3] \leftarrow S_{i-1}[A_3] \oplus K_{i-1} \]

for \(j \in A_3 \cup A_4\) do

\[ \text{Succ. draw the bits } S_{i,j}(1) \text{ according to } \]
\[ Z_{i}(B'_1) \leftarrow C'_{i} \]
\[ Z_{i}[B_1 \setminus B'_1] \leftarrow C'_{i} \]

for \(j \in B_3 \cup B_4\) do

\[ \text{Succ. draw the bits } Z_{i,j}(1) \text{ according to } \]

in block \(i \in [1, k]\) are chosen with uniform probability using a uniform randomness source \(\tilde{C}'\) shared with the decoder, and their value is reused over all blocks:

- The bits in \(B_1 \setminus B'_1 \subset W_{|X'|}\) in block \(i \in [1, k]\) are chosen with uniform probability using a uniform randomness source \(C'_i\) shared with the decoder;

- The bits in \(A_3 \cup A_4\) and \(B_3 \cup B_4\) are generated according to the previous bits using successive cancellation encoding as in \([13]\). Note that it is possible to sample efficiently from \(P_{S_j(1)}|S_j(0)}(1)\) and \(P_{Z_j(1)}|Z_j(0)}(1)\) (given \(U^n_1\) and \(X^n\)) respectively;

- From the second block, the encoder generates the bits of \(A_2\) in the following way. Let \(A'_3\) and \(B'_3\) be two disjoint subsets of \(A_2\) such that \(|A'_3| = |A_3|\) and \(|B'_3| = |B_3|\). The existence of those disjoint subsets is guaranteed by Remark 5 and Remark 6. The bits of \(A_3\) and \(B_3\) in block \(i\) are used as \(A'_3\) and \(B'_3\) in block \(i + 1\) using one-time pads with keys \(K_i\) and \(K'_i\) respectively:

\[ S_{(i+1)[A'_3]} = S_i[A'_3] \oplus K_i \quad i = 1, \ldots, k-1, \]
\[ S_{(i+1)[B'_3]} = Z_i[A'_3] \oplus K'_i \quad i = 1, \ldots, k-1. \]

Thanks to the Crypto Lemma \([13\text{ Lemma 3.1}]\), if we choose \(K_i\) of size \(|A_3|\) and \(K'_i\) of size \(|B_3|\) to be uniform random keys, the bits in \(A'_3\) and \(B'_3\) in the block \(i + 1\) are uniform. The bits in \(A_2 := A_2 \setminus (A'_3 \cup B'_3)\) are chosen with uniform probability using the local randomness source \(M\).

The encoder then computes \(X^n_i = S^n_i G_n\) for \(i = 1, \ldots, k\) and sends it over the channel. As in \([19]\), to deal with unaligned indices, chaining also requires in the last encoding block to transmit \(S(k)[A_3] \cup Z(k)[B_3]\) to the decoder. Hence the coding scheme requires an error-free channel between the encoder and decoder which has negligible rate since \(|S(k)[A_3] \cup Z(k)[B_3]| \leq |H_X|\) and

\[ \lim_{k \to \infty} \frac{|H_X|}{kn} = \lim_{k \to \infty} \frac{H(X)}{k} = 0. \]

![Figure 3. Chaining construction for block Markov encoding with polar codes](image-url)
In every block \( i \in [1, k] \), the decoder has access to \( \hat{S}_i[A_1] \subseteq S_i[H_X|Y] \) and \( \hat{Z}_i[B_1] \subseteq Z_i[H_W|X] \) because the bits in \( A_1 \) and \( B_1 \) correspond to shared randomness \( \{C_1\}_{i=1}^{n} \). \( C'_i \), \( C''_i \).

In block \( i \in [1, k-1] \), the bits in \( A_3 \) and \( B_3 \) are obtained by successfully recovering \( A_2 \) in block \( i+1 \), which is possible because the keys of the one-time pad are part of the common randomness.

From \( Y_n^i \) and \( \hat{S}_i[A_1 \cup A_3] \) the successive cancellation decoder can retrieve \( \hat{S}_i[A_2 \cup A_4] \) and \( \hat{Z}_i[B_4] \). Note that, by [17] Theorem 3, \( Z^n \) is equal to \( \hat{Z}^n \) and \( S^n \) is equal to \( S^n \) with high probability.

The decoder computes \( W'_i = \hat{Z}_i^n G_n \).

Finally, the decoder generates \( V'_i \) by symbol by symbol using
\[
P_{V_i,\hat{W}_i,\hat{Y}_i,\hat{Y}_i}(v|w, y) = \tilde{P}_{V|W, Y}(v|w, y).
\]

Algorithm 2: Decoding algorithm at Node 2

**Input:** \( (Y_1^n, \ldots, Y_k^n), S_i(k_i)|A_3 \cup B_3 \) and a common randomness shared with Node 1

**Output:** \( (\hat{S}_i^n, \ldots, \hat{S}_{i-1}^n), (\hat{Z}_i^n, \ldots, \hat{Z}_{i-1}^n) \)

for \( i = k, \ldots, 1 \) do

\[
\begin{align*}
\hat{S}_i[A_1] &\leftarrow C_i \\
\hat{Z}_i[B_1] &\leftarrow C''_i \\
\hat{Z}_i[B_1] &\leftarrow C''_i
\end{align*}
\]

if \( i \neq k \) then

\[
\begin{align*}
\hat{S}_i[A_3] &\leftarrow \hat{S}_{i+1}[A'_i] \oplus K_i \\
\hat{Z}_i[B_3] &\leftarrow \hat{S}_{i+1}[B'_i] \oplus K'_i
\end{align*}
\]

for \( j \in A_2 \cup A_4 \) do

Successively draw the bits according to
\[
\begin{align*}
\hat{S}_i[j] = \begin{cases} 
0 & \text{if } L_n(Y_n^i, \hat{S}_i^{i-1}) \geq 1 \\
1 & \text{else}
\end{cases}
\end{align*}
\]

where
\[
L_n(Y_n^i, \hat{S}_i^{i-1}) = \frac{\hat{P}_{S_j[S_j^{i-1}]}(0|\hat{S}_i^{i-1}Y_n)}{\hat{P}_{S_j[S_j^{i-1}]}(1|\hat{S}_i^{i-1}Y_n)}
\]

for \( j \in B_4 \) do

Successively draw the bits according to
\[
\begin{align*}
\hat{Z}_i[j] = \begin{cases} 
0 & \text{if } L_n(X_n^i, \hat{Z}_i^{i-1}) \geq 1 \\
1 & \text{else}
\end{cases}
\end{align*}
\]
e) Rate of common randomness: The rate of common randomness is \( I(W; UXV|Y) + H(X|WY) \) since:
\[
\begin{align*}
&\lim_{k \to \infty} \frac{k|A_1| + (k-1)|A_3| + |B_1| + |B_3| - |B'_1|}{kn} \\
&= \lim_{n \to \infty} \frac{|A_1| + |A_3| + |B_1| + |B_3| - |B'_1|}{n} \\
&= H(X|Y) + H(W|X) - H(W|UXV)
\end{align*}
\]

\( \therefore (a) \) has been proved in [17].

B. Coordination in one block

We note with \( P \) the joint distribution induced by the encoding and decoding algorithm of the previous sections. The proof requires a few steps. Similarly to [3] Lemma 13, we first prove that in each block \( i \in [1, k] \)
\[
\mathbb{D} \left( \tilde{P}_{U|XW}^{\otimes n} \mid P_{U^n|X^nW^n}(i) \right) = 2 n \delta_n.
\]

In fact, we have
\[
\begin{align*}
\mathbb{D} \left( \tilde{P}_{U|XW}^{\otimes n} \mid P_{U^n|X^nW^n}(i) \right) &= \mathbb{D} \left( \tilde{P}_{U^n|X^nW^n} \mid P_{U^n} \right) \\
&= \mathbb{D} \left( \tilde{P}_{U^n|X^nW^n} \mid P_{U^n} \right) + \mathbb{D} \left( \tilde{P}_{U^n|X^nW^n} \mid P_{U^n} \right)
\end{align*}
\]

We call \( D_1 \) and \( D_2 \) the first and the second term. Then:
\[
D_1 \overset{(a)}{=} \mathbb{D} \left( \tilde{P}_{U|XW}^{\otimes n} \mid P_{U^n}(i) \right) \overset{(b)}{=} \mathbb{D} \left( \tilde{P}_{U^n|X^nW^n}(i) \right)
\]
\[
\begin{align*}
&\overset{(c)}{=} \sum_{j=1}^{n} \mathbb{D} \left( \tilde{P}_{U^n|X^nW^n}(i) \right) \\
&\overset{(d)}{=} \sum_{j=1}^{n} \left( 1 - H(S_j[S_j^n]) \right) \leq n \delta_n
\end{align*}
\]

where \( (a) \) follows from the fact that \( X \) is independent of \( U \), \( (b) \) from the invertibility of \( G_n \), \( (c) \) from the chain rule, \( (d) \) from [23], \( (e) \) from the fact that the conditional distribution \( P_{S_n[J^n]}[J^n] \) is uniform for \( j \in A_1 \cup A_2 \) and \( (f) \) from Definition 21.

Similarly,
\[
D_2 \overset{(a)}{=} \mathbb{D} \left( \tilde{P}_{Z^n|X^nU^n} \mid P_{Z^n(X^nU^n)} \right)
\]
\[
\begin{align*}
&\overset{(b)}{=} \sum_{j=1}^{n} \mathbb{D} \left( \tilde{P}_{Z^n|X^nU^n} \mid P_{Z^n(X^nU^n)} \right) \\
&\overset{(c)}{=} \sum_{j=1}^{n} \left( 1 - H(Z_j[Z_j^n]) \right) \leq n \delta_n
\end{align*}
\]

where \( (a) \) comes from the invertibility of \( G_n \), \( (b) \) follows from the chain rule, \( (c) \) comes from [24], \( (d) \) comes from the fact that the conditional distribution \( P_{Z^n[J^n]}[J^n] \) is uniform for \( j \in B_1 \) and \( (e) \) comes from [22]. Then \( D_1 + D_2 < 2 n \delta_n \).

Therefore, applying Pinsker’s inequality to (25) we have
\[
\mathbb{V} \left( P_{U^n|X^nW^n}(i), \tilde{P}_{U|XW}^{\otimes n} \right) \leq 2 \sqrt{\log 2 \sqrt{n \delta_n}} := \delta_n^{(1)} \to 0.
\]
Note that $Y^n_i$ is generated symbol by symbol via the channel $P_{Y^n_i|X}$. By Lemma 1 for each $i \in [1,k]$
\[ \mathbb{V}(\tilde{P}_{U^n_iW^n_iX^n_iY^n_i}, \tilde{P}_{Y^n_i|U^n_iX^n_iW^n_i}) = \mathbb{V}(\tilde{P}_{U^n_iW^n_i}, \tilde{P}_{Y^n_i|W^n_i}) \leq \delta_1(n) \]
and therefore the left-hand side of (28) vanishes.

Observe that $Y^n_i$ is generated using $W^n_i$ (i.e. the estimate of $W^n_i$ at the decoder) and not $W^n_i$. By the triangle inequality for all $i \in [1,k]$
\[ \mathbb{V}(P_{X^n_iW^n_iY^n_i}) \leq \mathbb{V}(P_{X^n_iY^n_i|W^n_i}) + \mathbb{V}(P_{X^n_iW^n_i}) \]
(29)
We have proved in (28) that the second term of the right-hand side in (29) goes to zero, we show that the first term tends to zero as well. To do so, we apply [14 1.2.6] to
\[ A = U \times X \times W \times Y \times V \]
\[ P = P_{U^n_iW^n_iX^n_iY^n_i} \]
on $A = U \times X \times W \times Y \times V$. Since it has been proven in [17] that
\[ p_c := \mathbb{P}\{W^n_i \neq W^n_i\} = O(\delta_n) \]
we find that
\[ \mathbb{V}(P_{U^n_iW^n_iX^n_iY^n_i}) \leq 2p_c \]
and therefore
\[ \mathbb{V}(P_{U^n_iW^n_iX^n_iY^n_i}) \leq 2p_c + \delta_1(n) = \delta_2(n) \to 0. \]
Since $V^n_i$ is generated symbol by symbol from $\tilde{W}^n_i$ and $Y^n_i$, we apply Lemma [1] again and find
\[ \mathbb{V}(P_{U^n_iW^n_iX^n_iY^n_i\tilde{V}^n_i}) \leq \delta_2(n) \to 0. \]

C. Coordination of all blocks

First, we want to show that two consecutive blocks are almost independent. To simplify the notation, we set
\[ L := U^n X^n Y^n V^n. \]

Lemma 10: For $i \in [2,k]$, we have
\[ \mathbb{V}(P_{L_{i-1}C^i, P_{L_{i-1}C^i} P_{L_i}}) \leq \delta(n). \]

Proof: For $i \in [2,k]$, we have
\[ \mathbb{D}(P_{L_{i-1}C^i, P_{L_{i-1}C^i} P_{L_i}}) = I(L_{i-1}C^i; L_i) + I(L_i; C^i) + I(L_{i-1}; L_{i-1}C^i) \]
(a) $I(L_{i-1}; C^i) = I(L_i L_{i-1} | C), I(L_i; C^i) = I(L_i; Z_{i+1} | B_{i+1}), I(L_i; C^i) = I(L_i; B_{i+1} | B_{i+1}), I(L_i; C^i) = I(L_i; Z_{i+1} | B_{i+1}), I(L_i; C^i) = I(L_i; B_{i+1} | B_{i+1})$
(b) $|B_{i+1} - H(Z_{i+1} | B_{i+1}) + \delta_3(n) \]
(c) $\leq |B_{i+1} - H(Z_{i+1} | B_{i+1}) + \delta_3(n) \]
We omit the proofs because they are very similar to the proofs of [3 Lemma 15] and [3 Lemma 16] respectively.

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