Magnetic monopoles over topologically non trivial Riemann Surfaces

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Abstract An explicit canonical construction of monopole connections on non trivial $U(1)$ bundles over Riemann surfaces of any genus is given. The class of monopole solutions depend on the conformal class of the given Riemann surface and a set of integer weights. The reduction of Seiberg-Witten 4-monopole equations to Riemann surfaces is performed. It is shown then that the monopole connections constructed are solutions to these equations.

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1 Introduction

The duality symmetry, associated to a $U(1)$ gauge symmetry, is one of the symmetries that lately has risen high expectations in QFT mainly because it offers a way of analysing field theories describing strong-coupling interactions and weak-coupling interactions on an equal footing. In fact, recently Seiberg and Witten [1] obtained the full effective action for the fields at any coupling, from knowledge of its weak and strong coupling limit, considering the breaking of the $SU(2)$ gauge down to $U(1)$ in the $N = 2$ supersymmetric $SU(2)$ Yang Mills theory.

It has long been known that the monopole solutions, i.e., solutions to some vacuum equations in QFT with a "magnetic charge" different from zero always appear whenever there is a breaking of a compact gauge symmetry down to a $U(1)$ gauge [2]. These solutions resemble Dirac monopoles at infinity. The Seiberg-Witten procedure mentioned before gave rise to some new equations where monopoles play a significant role once again. These Seiberg-Witten equations have also interested the mathematics community mainly because, as a Topological QFT, they give some new topological invariants that may provide a new approach to the classification of 4-manifolds given by the Donaldson invariants. Most recently, it has been proven [3] an equivalence between Seiberg-Witten invariants of a symplectic 4-manifold and a set of Gromov invariants defined using pseudo-holomorphic submanifolds of dimension 2.

In this article, we give a canonical construction of monopole connections of a non-trivial $U(1)$ bundle over Riemann surfaces of any genus. The solutions constructed here should appear as soliton solutions to the low energy heterotic string field equations over non-trivial backgrounds [4]. They may provide some insight on the non-perturbative semiclassical structure of strings and membrane theories. Also, in this article, we project the Seiberg-Witten equations locally to the complex plane, impose conditions on the components of the fields normal to the plane and then extend them globally to build up the equations over compact Riemann surfaces. We end up showing that the canonical monopole connections obtained earlier are solutions of the projected Seiberg-Witten equations over compact Riemann surfaces. We expect these solutions, following the ideas of Taubes [3], to be 'grafted' into solutions of the Seiberg-Witten equations over a not necessarily symplectic 4-manifold. They also may help in the search of Gromov type invariants.
using pseudo-holomorphic submanifolds with degenerate symplectic forms.

In section 2, we give a construction of the known Dirac monopole connection mainly to compare it with the new solutions over Riemann surfaces of any genus given in section 3 and show that our solutions include the Dirac monopole for genus zero. In section 4, we prove that the new monopole connections are solutions to the reduced Seiberg-Witten equations over compact Riemann surfaces.

2 Dirac monopoles over $S_2$

We first describe the Hopf fiber bundle over $S_2$ and the connection 1-form and curvatures 2-form of the Dirac monopole. In the next section we generalize the solutions to any compact Riemann surface.

The 3-dimensional sphere $S_3$ may be defined by $z_0, z_1 \in \mathbb{C}$, the complex numbers, satisfying

$$z_0 \bar{z}_0 + z_1 \bar{z}_1 = 1$$

The group $U(1)$ acts on $S_3$ by

$$(z_0, z_1) \longrightarrow (z_0 u, z_1 u)$$

where $u\bar{u} = 1$, $\bar{u}$ defines the complex conjugate to $u \in \mathbb{C}$.

The projection $S_3 \longrightarrow S_2$ is defined by the composition of

$$(z_0, z_1) \longrightarrow \left\{ \begin{array}{ll}
z_1/z_0 & z_0 \neq 0 \\
z_0/z_1 & z_1 \neq 0
\end{array} \right.$$

with the stereographic projection

$$\mathbb{C} \longrightarrow S_2$$

such that for

$$z \in \mathbb{C}, \quad z = \rho e^{i\phi}$$

the stereographic projection associates $(\theta, \phi)$ over $S_2$ with

$$\rho = \frac{\sin(\theta)}{1 - \cos(\theta)}$$
Over the Hopf fiber bundle, there is a natural connection which may be obtained from the line element of $S_3$

$$ds^2 = 4(d\bar{z}_0 dz_0 + d\bar{z}_1 dz_1) \quad (2.7)$$

this can be decomposed in an unique way into the line element of $S_2$ and the tensorial square of the 1-form

$$\omega = d\chi + \cos(\theta)d\phi \quad (2.8)$$

That is

$$ds^2 = (d\theta^2 + \sin^2(\theta)d\phi^2) + \omega^2 \quad (2.9)$$

where the Euler angles have been used

$$z_0 = [\exp^{\frac{1}{2}i}(\chi + \phi)] \cos(\theta/2)$$
$$z_1 = [\exp^{\frac{1}{2}i}(\chi - \phi)] \sin(\theta/2) \quad (2.10)$$

$\frac{1}{2}\omega$ defines a connection over the fiber bundle $S_3$. To obtain the $U(1)$ connection 1-form over $S_2$, one may consider the local section

$$\hat{z}_0 = e^{i\phi} \cos(\theta/2), \quad \hat{z}_1 = \sin(\theta/2) \quad (2.11)$$

over $S_2$ with the point $\theta = 0$ removed, which we denote $U_+$. The $U(1)$ connection 1-form over $U_+$ is then

$$A_+ = \frac{1}{2}(1 + \cos(\theta))d\phi \quad (2.12)$$

which is regular on $U_+$.

If instead one considers the local section

$$\tilde{z}_0 = \cos(\theta/2), \quad \tilde{z}_1 = e^{-i\phi} \sin(\theta/2) \quad (2.13)$$

over $S_2$ with the point $\theta = \pi$ removed, denoted $U_-$, we obtain

$$A_- = \frac{1}{2}(-1 + \cos(\theta))d\phi \quad (2.14)$$

regular on $U_-$
In the overlapping region $U_+ \cap U_-$, we have

$$(\tilde{z}_0, \tilde{z}_1) = (e^{-i\phi} \hat{z}_0, e^{-i\phi} \hat{z}_1)$$

(2.15)

according to the action of $U(1)$ on the fibers over $S_2$ and

$$A_+ = A_- + d\phi$$

(2.16)

The curvature 2-form $\Omega$ arising from (12) and (14) is

$$\Omega = \frac{1}{2} \sin(\theta) d\phi \wedge d\theta$$

(2.17)

Notice that by applying an exterior derivative to (2.12) one obtains in addition to (2.17) a $\delta$-function at $\theta = \pi$ from $dd\phi$ not being defined on this point, however, it is annihilated because its coefficient becomes zero at $\theta = \pi$. The same occurs with (2.14).

This construction of the Dirac monopole with charge $g = 1/2$ may be generalized [5] to obtain other nontrivial Hopf fibring over $S_2$ and curvatures associated to monopoles with charge $g=n/2$. In [5] the $U(1)$ Hopf fibering

$$S_{2n+1} \rightarrow \mathcal{CP}_n$$

(2.18)

with base manifold $\mathcal{CP}_n$ and fiber bundle space $S_{2n+1}$ was considered.

Let us introduce

$$\xi_a = \frac{z_a}{z_0}, \quad z_0 = \rho e^{ix}, \quad a = 1 \cdots n$$

(2.20)

where $z_0 \neq 0$, we then have

$$\rho^2[1 + \xi_a \bar{\xi}_a] = 1.$$  

(2.21)

The line element of $S_{2n+1}$
may then be rewritten in an unique way as

\[ ds^2 = (\rho^2 \delta_{ab} - \rho^4 \bar{\xi}_a \xi_b) d\xi_a d\bar{\xi}_b + \omega^2 \]  

(2.23)

where

\[ \omega = d\chi + \frac{i}{2} \rho^2 (\xi_a d\bar{\xi}_a - \bar{\xi}_a d\xi_a) \]  

(2.24)

and the first term on the right hand side of (2.24) is a positive definite line element on \( \mathbb{C}P_n \), defining a metric which we denote \( h_{ab} \). The curvature 2-form over \( \mathbb{C}P_n \) is then given by

\[ \Omega = (\rho^2 \delta_{ab} - \rho^4 \bar{\xi}_a \xi_b) i d\xi_a \wedge d\bar{\xi}_b. \]  

(2.25)

It is a solution of Maxwell equations because

\[ \Omega \wedge \cdots \wedge \Omega \approx \underbrace{h_{a_1 \bar{b}_1} \cdots h_{a_n \bar{b}_n} d\xi_{a_1} \wedge d\bar{\xi}_{b_1} \wedge \cdots}_{(n-1)-\text{factors}} \]

\[ = (\text{deth})^2 d\xi_1 \wedge \cdots \wedge d\xi_n \wedge d\bar{\xi}_1 \wedge \cdots \wedge d\bar{\xi}_n \]  

(2.26)

hence

\[ ^*\Omega \approx \Omega \wedge \cdots \wedge \Omega \]  

(2.27)

and then

\[ d^*\Omega = 0 \]  

(2.28)

We may now consider as in [5] embedding \( \mathbb{C}P_1 \) into \( \mathbb{C}P_n \)
\[ \xi_1 = \left( \frac{n}{1} \right)^{\frac{1}{2}} \xi \]
\[ \xi_2 = \left( \frac{n}{2} \right)^{\frac{1}{2}} \xi^2 \]
\[ \vdots \]
\[ \xi_n = \left( \frac{n}{n} \right)^{\frac{1}{2}} \xi^n \]

(2.29)

After substitution of (2.29) into (2.25) and several calculations, we obtain

\[ \Omega_n = n(1 + \xi \bar{\xi})^{-2} i d\xi \wedge d\bar{\xi} \]  
(2.30)

For \( n = 1 \) we get

\[ \Omega_1 = \frac{1}{(1 + \xi \bar{\xi})^2} i d\xi \wedge d\bar{\xi} \]  
(2.31)

and by changing variables

\[ \xi = \rho e^{i\phi}, \quad \rho = \frac{\sin \theta}{1 - \cos \theta} \]  
(2.32)

as in (2.6),

\[ \Omega_1 = \frac{2}{(1 + \rho^2)^2} \rho d\rho \wedge d\phi \]  
(2.33)

\[ = -\frac{1}{2} \sin \theta d\theta \wedge d\phi \]

as in (2.17).

In general, we obtain

\[ \Omega_n = \frac{1}{2} n \sin \theta d\phi \wedge d\theta \]  
(2.34)

and the corresponding connection 1-forms on \( U_+ \) and \( U_- \) are

\[ A_{n+} = \frac{n}{2} (1 + \cos \theta) d\phi \]  
(2.35)

\[ A_{n-} = \frac{n}{2} (-1 + \cos \theta) d\phi \]

respectively.
3 Monopoles over Riemann surfaces of any genus

We construct in this section a canonical monopole connection on any non trivial $U(1)$ bundle over compact Riemann surfaces of genus $g$. The connection 1-form and curvature 2-form describing the monopoles are expressed in terms of the abelian differential of the third kind with real normalization and integer weights.

The abelian differential $d\tilde{\phi}_{ab}$ of the third kind is a holomorphic 1-form on the compact Riemann surface except for poles of residue $+1$ and $-1$ at points $a$ and $b$ respectively, with real normalization, that is with pure imaginary periods when one circles around the canonical cycles. From $d\tilde{\phi}_{ab}$ one can construct the abelian integral $\tilde{\phi}_{ab}$. Its real part $G(z, \bar{z}, a, b, t)$ is a harmonic univalent function over the Riemann surface with logarithmic behavior around $a$ and $b$

\[
\ln \frac{1}{|z-a|} + \text{regular terms} , \\
\ln |z-b| + \text{regular terms} .
\]  

It is a conformal invariant geometrical object and was explicitly constructed by Burnside in [6] using a Schottky uniformization of the Riemann surface. $z$ denotes local coordinates on the Riemann surface, and $t$ the set of $3g-3$ parameters describing the moduli space of Riemann surfaces.

Let $a_i, i = 1, \cdots, m$ be $m$ points over the compact Riemann surface, we associate to them integer weights $\alpha_{a_i}, i = 1, \cdots, m$, such that

\[\sum_{i=1}^{m} \alpha_{a_i} = 0 \quad (3.2)\]

We define

\[\phi = \sum_{i=1}^{m} \alpha_{a_i} G(z, \bar{z}, a_i, b, t) . \quad (3.3)\]

$\phi$ defines a Morse function over the Riemann surface, it is exactly the light cone time in the formulation of string theory.
\[ \phi \rightarrow -\infty \] at the points \( a_i \) with negative weights and

\[ \phi \rightarrow +\infty \] at the points \( a_i \) with positive weights.

\( \alpha_i \) are integers in order to have univalent transition functions over the non-trivial fiber bundle we will consider. It corresponds to the Dirac quantization condition of the magnetic charge.

Let us consider now a \( \phi = \text{cte} \) curve over the Riemann surface. It is a closed curve homologous to zero. It divides the Riemann surface into two regions \( U_+ \) and \( U_- \), where \( U_+ \) contains all the points \( a_i \) with negative weights and \( U_- \) the ones with positive weights.

We now generalize the connection 1-form (2.35). We define over \( U_+ \) and \( U_- \) the connection 1-forms

\begin{align*}
A_+ &= \frac{1}{2} (1 + \tanh \phi)(\frac{1}{2} \phi_z dz - \frac{1}{2} \bar{\phi}_z d\bar{z}) = \frac{1}{2} (1 + \tanh \phi) d(\text{Im} \bar{\phi}) \\
A_- &= \frac{1}{2} (-1 + \tanh \phi)(\frac{1}{2} \bar{\phi}_z dz - \frac{1}{2} \phi_z d\bar{z}) = \frac{1}{2} (-1 + \tanh \phi) d(\text{Im} \bar{\phi})
\end{align*}

\[ (3.4) \]

respectively.

\( \bar{\phi} \) denotes as before the abelian integral whose real part is \( \phi \), \( \bar{\phi}_z \) denotes its derivative with respect to a local coordinate \( z \) over the Riemann surface. \( A_+ \) and \( A_- \) are globally defined in \( U_+ \) and \( U_- \) respectively and regular therein since the singularity coming from the factor

\[ d(\text{Im} \bar{\phi}) \]

at \( a_i, i = 1, \cdots, m \), is cancelled by the corresponding coefficient

\[ (\pm 1 + \tanh \phi) \]

\[ (3.5) \]

in the same way as the factor

\[ (\pm 1 + \cos \theta) \]

\[ (3.6) \]

cancelled the singularities at \( \theta = 0 \) and \( \pi \) in (2.35).

In the overlapping \( U_+ \cap U_- \) we have
\[ A_+ = A_- + d(\text{Im} \tilde{\phi}) \]  

(3.8)

We now discuss the term in \( \text{Im} \tilde{\phi} \) more explicitly. There is a one to one correspondence between \((LC)\) light cone diagrams and the moduli space of punctured Riemann surfaces, and associated to each \(LC\) diagram with given weights there is a unique abelian differential \( d\tilde{\phi} \). It is then enough to discuss all possible transition functions on the fiber bundle in terms of \(LC\) diagrams.

In the \(LC\) diagrams the punctures \( a_i \) with positive weights are at the right of the diagrams, corresponding to \( \phi \to +\infty \), while the \( a_i \) with negative weights are at the left of the diagram, corresponding to \( \phi \to -\infty \). For example in the following figure we have six punctures \( a_i, i = 1, \cdots, 6 \), four of them with positive weights. We denote them \( \alpha_1, \alpha_2, \alpha_3, \alpha_4 > 0 \).

Diagram of the figure corresponds to a torus with 6 punctures, see figure 2.

are integers satisfying \( \sum_{a=1}^{6} \alpha_i = 0 \). The curves \( \phi = \text{cte.} \) correspond to vertical lines with the corresponding identification of the end points.

At \( \phi = C_1 \) the transition function is \( e^{i\nu} \) with

\[ n = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 \]  

(3.9)

and \( \nu \) is the angular variable describing the circle \( \phi = C_1 \).

If instead we consider \( \phi = C_2 \) the overlapping between \( U_+ \) and \( U_- \) occurs at three circles, one enclosing \( a_3 \) and \( a_4 \), another enclosing \( a_2 \) and the third enclosing \( a_1 \). The transitions are now \( e^{i(\alpha_3 + \alpha_4)\nu} \), \( e^{i\alpha_2\nu} \), and \( e^{i\alpha_1\nu} \) respectively, where \( \nu \) describes the angular coordinate at each circle.

We do not consider \( \phi = \text{cte} \) curves like \( \phi = A \) in figure 1, which correspond to two circles once the identifications of the end points are performed, because the transition function on each circle are not necessarily univalent.

The curvature 2-form associated to the connection 1-form (3.4) is given by

\[ F = \frac{1}{4} \tilde{\phi}_+ \tilde{\phi}_- d\bar{z} \wedge dz = \frac{1}{4} \frac{d\tilde{\phi} \wedge d\widetilde{\phi}}{\cosh^2 \phi} \]  

(3.10)

Let us now consider the particular case of the Riemann sphere \( S_2 \), with two punctures \( a \) and \( b \) with weights \( n \) and \(-n\) respectively. Without loss of generality we can take them to be at \( z = \infty \) and \( z = 0 \) of the complex plane \( \mathbb{C} \). In this case we have

9
\[ \phi = n \ln |z| \]

We rewrite

\[ z = \rho e^{i\varphi} \]

and use (2.6) to go to the coordinates on the sphere. We obtain

\[ \tanh \phi = \frac{\rho^2 - 1}{\rho^2 + 1} = \cos \theta \]

\[ \tilde{\phi} = n \ln z \]

\[ \text{Im} \tilde{\phi} = n \arg z = n\varphi, \]

(3.4) then exactly agrees with (2.35) and (3.10) with (2.34).

We have then constructed a canonical monopole connection on non-trivial $U(1)$ fiber bundles on Riemann surface of any genus. The Chern class $c_1$ of the fiber bundle is determined by the summation of positive integer weights at the punctures.

These are all the non trivial bundles that can be constructed over a Riemann surface. In fact, any complex vector bundle over an open Riemann surface is trivial, hence any complex vector bundle over $U_+$ or $U_-$ is trivial and we have constructed all the possible transitions between fibers over $U_+ \cap U_-$. 

4 Seiberg-Witten equations over Riemann surfaces

We obtain in this section the reduction of the Seiberg-Witten 4-monopole equations [7] to 2-dim Riemann surfaces. We then show that the canonical connections over compact Riemann surfaces we constructed in section 3 are solutions to these equations. It is convenient to use complex notation.
\[ x^\mu \sigma_\mu = z \sigma_z + \bar{z} \sigma_{\bar{z}} + y \sigma_y + \bar{y} \sigma_{\bar{y}} \]  

(4.1)

Where

\[ \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \sigma_{\bar{z}} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \]  

(4.2)

\[ \sigma_y = \begin{pmatrix} 0 & 0 \\ i & 0 \end{pmatrix} \quad \sigma_{\bar{y}} = \begin{pmatrix} 0 & i \\ 0 & 0 \end{pmatrix} \]

The 4-monopole equations [7] are

\[ F^{\mu\nu} \sigma_{\mu\nu\alpha\beta} = \bar{M}_{(\alpha M_{\beta})} \]  

(4.3a)

\[ \sigma^{\mu} \mathcal{D}_\mu M = 0 \]  

(4.3b)

We consider the reduction to 2-dim in an open neighborhood by taking \( \partial_y = \partial_{\bar{y}} = 0 \) and then extend the resulting equations to compact Riemann surfaces. We obtain after the reduction of (4.3a)

\[ -F^{z\bar{z}} = M_1^* M_1 - M_2^* M_2 \]  

(4.4)

\[ F^{y\bar{z}} = M_2 M_1^* \]  

(4.5)

Where \( F^{y\bar{z}} = F_{y\bar{z}} = -\partial_z A_{\bar{y}}, \) and * denotes complex conjugation, for (4.3b) we get

\[ \mathcal{D}^z M_1 + i A_y M_2 = 0^i \]  

(4.6a)

\[ \mathcal{D}^z M_2 + i A_y M_1 = 0 \]  

(4.6b)

We now look for a conformal extension over a compact Riemann surfaces. From (4.4) \( M_1 \) and \( M_2 \) must transform as one forms under a change of chart.
From (4.5) $A_y$ must consequently change also as a one form. However this change is not compatible with (4.6) unless

$$\partial_z A_y = 0 \quad (4.7)$$

Which implies

$$M_2 M_1^* = 0 \quad (4.8)$$

We end up with the following set of equations

$$- F_{zz} = M_1^* M_1 - M_2^* M_2 \quad (4.9)$$

and

$$\mathcal{D}_z M_1 = 0, M_2 = 0 \quad (4.10a)$$

or

$$\mathcal{D}_z M_2 = 0, M_1 = 0 \quad (4.10b)$$

Where $M_2 d\bar{z}$ $(M_1 dz)$ is a one form over the compact Riemann surfaces. We will call equations (4.9) and (4.10) the monopole equations over Riemann surfaces. We will compare (4.9), (4.10) with the Hitchin equations over compact Riemann surfaces, which arise from a reduction of the $SU(2)$ selfdual equations in 4 dim, in a next communication.

We now show that the canonical connections we constructed in section 3 give rise to a class of regular solutions to (4.9), (4.10). We consider the non trivial $U(1)$ bundle over a compact Riemann surfaces of genus g, with transitions as indicated in the figure 3. If we take the transitions as indicated in the diagram the transition functions are single valued over the compact Riemann surfaces as a consequence of our assumption of integer weights $\alpha_i$, $i = 1, \ldots, m$, $\sum_{i=1}^{m} \alpha_i = 0$, at the points $a_i$. The canonical connection of section 3 can now be reexpressed as:
\[ A_{+z} = \frac{1}{2}(1 + \tanh(\phi))\tilde{\phi}_z dz \quad \text{in } U_+, \]
\[ A_z = \frac{1}{2}\tanh(\phi)\tilde{\phi}_z dz \quad \text{in } U, \quad (4.11) \]
\[ A_{-z} = \frac{1}{2}(-1 + \tanh(\phi))\tilde{\phi}_z dz \quad \text{in } U_- . \]

It is easiest to solve (4.9) and (4.10) first at \( U \). We obtain

\[ M_2 = h(\phi)\partial_z \tilde{\phi}, \quad (4.12) \]

\[ \frac{1}{2}\partial_z \tilde{\phi} h' + A_z h = 0 , \]

We thus have for \( h \neq 0 , \)

\[ A_z = -\frac{1}{2}\partial_z \tilde{\phi} \frac{h'}{h} \quad (4.13) \]

\[ F_{zz} = -\frac{1}{4}\partial_z \tilde{\phi} \partial_z \tilde{\phi} \ln(h\tilde{h})'' \]

where \( \partial_z \tilde{\phi} = \tilde{\phi}_z . \)

We finally obtain

\[ h = \frac{1}{\sqrt{2} \cosh \phi} \quad \text{in } U ; \quad (4.14) \]

We now go back to (4.11) and notice that the solutions are

\[ h = \frac{\exp -\hat{\phi}}{\sqrt{2} \cosh \phi} \quad \text{in } U_+ ; \quad (4.15) \]

and
\[ h = \frac{\exp \hat{\phi}}{\sqrt{2} \cosh \phi} \quad \text{in } U_-, \quad (4.16) \]

where \( \hat{\phi} = \text{Im} \tilde{\phi} \).

We notice that \( h \) is a single valued object at \( U_+ \) and \( U_- \) in spite of the presence of \( \hat{\phi} \) which is multi-valued. This is so because of the integer weights we have considered. \( \exp \hat{\phi} \) is not single valued in \( U \), when we go around a handle. This is the reason why we have considered the transition as in figure 3.

We could construct a similar solution in terms of \( M_2 \). The only change is the reverse in the sign of \( A_z \). These are monopoles with opposite magnetic charge.

We have thus constructed a class of regular solutions to the monopole equations over Riemann surfaces of genus \( g \). The class of solutions depend on the conformal class of the given Riemann surfaces as well as in \( m \) points over the surface with integer weights \( \alpha_i, i = 1, \cdots, m, \sum_{i=1}^m \alpha_i = 0 \). The Chern class of the fiber bundle is given by \((\pm 1)\times\) the sum of the positive weights.

## 5 Conclusions

We obtained an explicit canonical construction of monopoles solutions for any non trivial bundle over any compact Riemann surface of genus \( g \). The solutions depend on the conformal class of the Riemann surface and on a set of integer weights. We gave the explicit solutions for the Weyl spinors of the Seiberg-Witten monopole equations on the non trivial bundle and reduced them onto compact Riemann surfaces of any genus \( g \). It is interesting to point out that the same reduction but for the \( SU(2) \) self-dual equations over 4 dimensions yields the Hitchin’s equations [8] over Riemann surfaces. The latter have a similar structure to the equations (4.9), (4.10) but in terms of Higgs fields which behave as 1-forms over the Riemann surface instead of the Weyl spinors. This distinction makes an important difference between the two sets of equations. In particular, the Dirac monopole is not a solution of Hitchin’s equations [9] but it is a solution of the reduced Seiberg-Witten equations.
References

[1] N.Seiberg and E.Witten, *Nucl. Phys. B426* (1994) 19; Erratum, *B430* (1994) 485; *B431* (1994) 484.

[2] P.Goddard and D.Olive *Rep. Prog. Phys. 41* (1978) 1357.

[3] C.H.Taubes, Preprint Harvard University;"The Seiberg-Witten and the Gromov invariants"(1995); *Math. Res. Lett.1* (1994) 809.

[4] I.Martin and A.Restuccia *Phys.Lett.***B271**(1991) 361; M.LLedo, I.Martin, A.Restuccia and A.Mendoza *Lett. Math. Phys.***24** (1992) 275.

[5] A.Trautman *Int. Jour. Theor.Phys.***16** (1977) 561.

[6] W.Burnside *Proc. London Math. Soc.***23** (1881) 49.

[7] E.Witten *Math.Res. Lett.***1** (1994) 769.

[8] N.Hitchin*Proc. London Math. Soc.***55** (1987) 59;Lectures al LASSF 89, Caracas, Venezuela.

[9] I.Martin, R.Martinez and A. Restuccia, preprint USB/SF/96-236.
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