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Differential equations associated with generalized Bell polynomials and their zeros

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Abstract: In this paper, we study differential equations arising from the generating functions of the generalized Bell polynomials. We give explicit identities for the generalized Bell polynomials. Finally, we investigate the zeros of the generalized Bell polynomials by using numerical simulations.

Keywords: Differential equations, Bell polynomials, Generalized Bell polynomials, Zeros

MSC: 05A19, 11B83, 34A30, 65L99

1 Introduction

Recently, many mathematicians have worked in the area of the Bernoulli numbers, Euler numbers, Genocchi numbers, and tangent numbers (see [1–9]). The moments of the Poisson distribution are well-known to be connected to the combinatorics of the Bell and Stirling numbers. As is well known, the Bell numbers \( B_n \) are given by the generating function

\[
e^{(e^t-1)} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}.
\]

The Bell polynomials \( B_n(\lambda) \) are given by the generating function

\[
e^{\lambda(e^t-1)} = \sum_{n=0}^{\infty} B_n(\lambda) \frac{t^n}{n!}.
\]

The generalized Bell polynomials \( B_n(x, \lambda) \) are defined by the generating function

\[
F = F(t, x, \lambda) = \sum_{n=0}^{\infty} B_n(x, \lambda) \frac{t^n}{n!} = e^{xt-\lambda(e^t-1)} \quad \text{(see [10])}.
\]

In particular the generalized Bell polynomials \( B_n(x, -\lambda) = E_\lambda[(Z + x - \lambda)^n] \), \( x, \lambda \in \mathbb{R}, n \in \mathbb{N} \), where \( Z \) is a Poission random variable with parameter \( \lambda > 0 \) (see [10]). The first few examples of generalized Bell polynomials are

\[
\begin{align*}
B_0(x, \lambda) &= 1, & B_1(x, \lambda) &= x, & B_2(x, \lambda) &= x^2 - \lambda, \\
B_3(x, \lambda) &= x^3 - 3x\lambda, & B_4(x, \lambda) &= x^4 - 4x\lambda - 6x^2\lambda + 3\lambda^2, \\
B_5(x, \lambda) &= x^5 - 5x\lambda - 10x^2\lambda - 10x^3\lambda + 10\lambda^2 + 15x\lambda^2, \\
B_6(x, \lambda) &= x^6 - 6x\lambda - 15x^2\lambda - 20x^3\lambda - 15x^4\lambda + 25\lambda^2 + 60x\lambda^2 + 45x^2\lambda^2 - 15\lambda^3.
\end{align*}
\]

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From (2) and (3), we see that

$$B_7(x, \lambda) = x^7 - 7x - 21x^2\lambda - 35x^3\lambda - 35x^4\lambda - 21x^5\lambda + 56\lambda^2 + 175x\lambda^2$$

$$+ 210x^2\lambda^2 + 105x^3\lambda^2 - 105\lambda^3 - 105x\lambda^3.$$

Comparing the coefficients on both sides of (4), we obtain

$$B_n(x, \lambda) = \sum_{k=0}^{n} \binom{n}{k} B_k(-\lambda)(x + \lambda)^{n-k} \quad (n \geq 0).$$

Recently, many mathematicians have studied the differential equations arising from the generating functions of special polynomials (see [11–13]). In this paper, we study differential equations arising from the generating functions of generalized Bell polynomials. We give explicit identities for the generalized Bell polynomials. In addition, we investigate the zeros of the generalized Bell polynomials with numerical methods. Finally, we observe an interesting phenomenon of ‘scattering’ of the zeros of generalized Bell polynomials.

2 Differential equations associated with generalized Bell polynomials

Differential equations arising from the generating functions of special polynomials are studied by many authors in order to give explicit identities for special polynomials (see [11–13]). In this section, we study differential equations arising from the generating functions of generalized Bell polynomials.

Let

$$F = F(t, x, \lambda) = e^{xt - \lambda(e^t - 1)} = \sum_{n=0}^{\infty} B_n(x, \lambda) \frac{t^n}{n!}, \quad \lambda, x, t \in \mathbb{C}. \quad (6)$$

Then, by (6), we have

$$F^{(1)} = \frac{d}{dt} F(t, x, \lambda) = \frac{d}{dt} \left( e^{xt - \lambda(e^t - 1)} \right) = e^{xt - \lambda(e^t - 1)}(x - \lambda(e^t - 1))$$

$$= (x + \lambda)F(t, x, \lambda) - \lambda F(t, x + 1, \lambda),$$

$$F^{(2)} = \frac{d}{dt} F^{(1)} = (x + \lambda)F^{(1)}(t, x, \lambda) - \lambda F^{(1)}(t, x + 1, \lambda)$$

$$= (x + \lambda)^2 F(t, x, \lambda) - \lambda(2x + 2\lambda + 1)F(t, x + 1, \lambda) + \lambda^2 F(t, x + 2, \lambda),$$

and

$$F^{(3)} = \frac{d}{dt} F^{(2)} = (x + \lambda)^2 F(t, x, \lambda)$$

$$+ (-1)\lambda \left( (x + \lambda)^2 + (2x + 2\lambda + 1)(x + 1 + \lambda) \right) F(t, x + 1, \lambda)$$

$$+ (-1)^2\lambda^2 (3x + 3\lambda + 3) F(t, x + 2, \lambda)$$

$$+ (-1)^3\lambda^3 F(t, x + 3, \lambda).$$

Continuing this process, we can guess that

$$F^{(N)} = \left( \frac{d}{dt} \right)^N F(t, x, \lambda) = \sum_{i=0}^{N} (-1)^i a_i(N, x, \lambda) F(t, x + i, \lambda), \quad (N = 0, 1, 2, \ldots). \quad (9)$$
Taking the derivative with respect to $t$ in (9), we have

$$F^{(N+1)} = \frac{dF^{(N)}}{dt} = \sum_{i=0}^{N} (-1)^i a_i(N, x, \lambda) F^{(1)}(t, x + i, \lambda)$$

$$= \sum_{i=0}^{N} (-1)^i a_i(N, x, \lambda) \{(x + i + \lambda) F(t, x + i, \lambda) - \lambda F(t, x + i + 1, \lambda)\}$$

$$= \sum_{i=0}^{N} (-1)^i a_i(N, x, \lambda) F(t, x + i, \lambda) + \sum_{i=0}^{N} \sum_{j=1}^{N+1} (-1)^i \lambda a_i(N, x, \lambda) F(t, x + i, \lambda)$$

$$= \sum_{i=0}^{N} (-1)^i a_i(N, x, \lambda) F(t, x + i, \lambda) + \sum_{i=1}^{N+1} (-1)^i \lambda a_{i-1}(N, x, \lambda) F(t, x + i, \lambda).$$

(10)

On the other hand, by replacing $N$ by $N + 1$ in (9), we get

$$F^{(N+1)} = \sum_{i=0}^{N+1} (-1)^i a_i(N + 1, x) F(t, x + i, \lambda).$$

(11)

Comparing the coefficients on both sides of (10) and (11), we obtain

$$a_0(N + 1, x, \lambda) = (x + \lambda)a_0(N, x, \lambda), \quad a_{N+1}(N + 1, x, \lambda) = \lambda a_N(N, x, \lambda),$$

(12)

and

$$a_i(N + 1, x, \lambda) = \lambda a_{i-1}(N, x, \lambda) + (x + i + \lambda) a_i(N, x, \lambda), \quad (1 \leq i \leq N).$$

(13)

In addition, by (9), we get

$$F(t, x, \lambda) = F^{(0)}(t, x, \lambda) = a_0(0, x, \lambda) F(t, x, \lambda).$$

(14)

By (14), we get

$$a_0(0, x, \lambda) = 1.$$  

(15)

It is not difficult to show that

$$(x + \lambda) F(t, x, \lambda) - \lambda F(t, x + 1, \lambda) = F^{(1)}(t, x, \lambda) = \sum_{i=0}^{1} (-1)^i a_i(1, x, \lambda) F(t, x + i, \lambda)$$

$$= a_0(1, x, \lambda) F(t, x, \lambda) - a_1(1, x, \lambda) F(t, x + 1, \lambda).$$

(16)

Thus, by (16), we also get

$$a_0(1, x, \lambda) = x + \lambda, \quad a_1(1, x, \lambda) = \lambda.$$ 

(17)

From (12), we note that

$$a_0(N + 1, x, \lambda) = (x + \lambda)a_0(N, x, \lambda) = \cdots = (x + \lambda)^N a_0(1, x, \lambda) = (x + \lambda)^{N+1},$$

(18)

and

$$a_{N+1}(N + 1, x, \lambda) = \lambda a_N(N, x, \lambda) = \cdots = \lambda^N a_1(1, x, \lambda) = \lambda^{N+1}.$$ 

(19)

For $i = 1, 2, 3$ in (13), we have

$$a_1(N + 1, x, \lambda) = \lambda \sum_{k=0}^{N} (x + 1 + \lambda)^k a_0(N - k, x, \lambda),$$

(20)

$$a_2(N + 1, x, \lambda) = \lambda \sum_{k=0}^{N-1} (x + 2 + \lambda)^k a_1(N - k, x, \lambda).$$

(21)
and
\[ a_3(N + 1, x, \lambda) = \lambda \sum_{k=0}^{N-2} (x + 3 + \lambda)^k a_2(N - k, x, \lambda). \] (22)

Continuing this process, we can deduce that, for \(1 \leq i \leq N\),
\[ a_i(N + 1, x, \lambda) = \sum_{k=0}^{N-i+1} (x + i + \lambda)^k a_{i-1}(N - k, x, \lambda). \] (23)

Here, we note that the matrix \( a_i(j, x, \lambda)_{0 \leq i, j \leq N + 1} \) is given by
\[
\begin{pmatrix}
1 & x + \lambda & (x + \lambda)^2 & (x + \lambda)^3 & \cdots & (x + \lambda)^{N+1} \\
0 & \lambda & \cdot & \cdot & \cdots & \cdot \\
0 & 0 & \lambda^2 & \cdot & \cdots & \cdot \\
0 & 0 & 0 & \lambda^3 & \cdots & \cdot \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & 0 & \cdots & \lambda^{N+1}
\end{pmatrix}
\]

Now, we give explicit expressions for \( a_i(N + 1, x, \lambda) \). By (20), (21) and (22), we get
\[ a_1(N + 1, x, \lambda) = \lambda \sum_{k=0}^{N} (x + 1 + \lambda)^k a_0(N - k_1, x, \lambda) = \lambda \sum_{k_1=0}^{N} (x + 1 + \lambda)^{k_1} (x + \lambda)^{N-k_1}, \]
\[ a_2(N + 1, x, \lambda) = \lambda \sum_{k_2=0}^{N-1} (x + 2 + \lambda)^{k_2} a_1(N - k_2, x, \lambda) = \lambda^2 \sum_{k_2=0}^{N-1} (x + 2 + \lambda)^{k_2}(x + 1 + \lambda)^{k_1} (x + \lambda)^{N-k_2-k_1-1}, \]
and
\[ a_3(N + 1, x, \lambda) = \lambda \sum_{k_3=0}^{N-2} (x + 3 + \lambda)^{k_3} a_2(N - k_3, x, \lambda) = \lambda^3 \sum_{k_3=0}^{N-2} (x + 3 + \lambda)^{k_3}(x + 2 + \lambda)^{k_2}(x + 1 + \lambda)^{k_1} (x + \lambda)^{N-k_3-k_2-k_1-2}. \]

Continuing this process, we have
\[ a_i(N + 1, x, \lambda) = \lambda^{i} \prod_{l=1}^{i} (x + l + \lambda)^{k_i} (x + \lambda)^{N-i+1-\sum_{l=1}^{i} k_i}. \] (24)

Therefore, by (24), we obtain the following theorem.

**Theorem 2.1.** For \( N = 0, 1, 2, \ldots \), the differential equations
\[ F^{(N)} = \sum_{i=0}^{N} (-1)^i a_i(N, x, \lambda) F(t, x + i, \lambda) = \left( \sum_{i=0}^{N} (-1)^i a_i(N, x, \lambda) e^{it} \right) F(t, x, \lambda) \]
have a solution
\[ F = F(t, x, \lambda) = e^{xt - \lambda(e^t - t - 1)}. \]
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where

\[ a_0(N, x, \lambda) = (x + \lambda)^N, \]
\[ a_N(N, x, \lambda) = \lambda^N \]
\[ a_i(N, x, \lambda) = \lambda^N \sum_{k_i=0}^{N-i} \sum_{k_{i-1}=0}^{N-i-k_i} \cdots \sum_{k_1=0}^{N-i-k_1-\cdots-k_2} \left( \prod_{l=1}^{i} (x + l + \lambda)^{k_l} \right) (x + \lambda)^{N-i-\sum_{j=1}^{i} k_j}, \quad (1 \leq i \leq N). \]

From (6), we note that

\[ F(N) = \left( \frac{d}{dt} \right)^N F(t, x, \lambda) = \sum_{k=0}^{\infty} B_{k+N}(x, \lambda) \frac{t^k}{k!}. \tag{25} \]

From Theorem 2.1 and (25), we can derive the following equation:

\[
\sum_{k=0}^{\infty} B_{k+N}(x, \lambda) \frac{t^k}{k!} = F(N) = \left( \sum_{i=0}^{N} (-1)^i a_i(N, x, \lambda) e^{it} \right) F
\]

\[
= \sum_{i=0}^{N} (-1)^i a_i(N, x, \lambda) \left( \sum_{l=0}^{\infty} \frac{t^l}{l!} \right) \left( \sum_{m=0}^{\infty} \frac{B_m(x, \lambda) t^m}{m!} \right)
\]

\[
= \sum_{i=0}^{N} (-1)^i a_i(N, x, \lambda) \left( \sum_{k=0}^{\infty} \sum_{m=0}^{k} \left( \begin{array}{c} k \\ m \end{array} \right) i^{k-m} B_m(x, \lambda) \frac{t^k}{k!} \right)
\]

\[
= \sum_{k=0}^{\infty} \sum_{i=0}^{N} \sum_{m=0}^{k} \left( \begin{array}{c} k \\ m \end{array} \right) i^{k-m} (-1)^i a_i(N, x, \lambda) B_m(x, \lambda) \frac{t^k}{k!}.
\]

By comparing the coefficients on both sides of (26), we obtain the following theorem.

**Theorem 2.2.** For \( k, N = 0, 1, 2, \ldots \), we have

\[
B_{k+N}(x, \lambda) = \sum_{i=0}^{N} \sum_{m=0}^{k} \left( \begin{array}{c} k \\ m \end{array} \right) i^{k-m} (-1)^i a_i(N, x, \lambda) B_m(x, \lambda).
\]

where

\[ a_0(N, x, \lambda) = (x + \lambda)^N, \]
\[ a_N(N, x, \lambda) = \lambda^N \]
\[ a_i(N, x, \lambda) = \lambda^N \sum_{k_i=0}^{N-i} \sum_{k_{i-1}=0}^{N-i-k_i} \cdots \sum_{k_1=0}^{N-i-k_1-\cdots-k_2} \left( \prod_{l=1}^{i} (x + l + \lambda)^{k_l} \right) (x + \lambda)^{N-i-\sum_{j=1}^{i} k_j}, \quad (1 \leq i \leq N). \]

Let us take \( k = 0 \) in (27). Then, we have the following corollary.

**Corollary 2.3.** For \( N = 0, 1, 2, \ldots \), we have

\[
B_N(x, \lambda) = \sum_{i=0}^{N} (-1)^i a_i(N, x, \lambda).
\]

For \( N = 0, 1, 2, \ldots \), the functional equations

\[
F^{(N)}(x, \lambda) = \sum_{i=0}^{N} (-1)^i a_i(N, x, \lambda) F(t, x + i, \lambda) = \left( \sum_{i=0}^{N} (-1)^i a_i(N, x, \lambda) e^{it} \right) F(t, x, \lambda)
\]

have a solution

\[
F = F(t, x, \lambda) = e^{\lambda t} - \lambda (e^{t} - 1).
\]
Here is a plot of the surface for this solution.

In Figure 1 (left), we choose \(-3 \leq x \leq 3, -1 \leq t \leq 1, \) and \(\lambda = -4.\) In Figure 1 (right), we choose \(-3 \leq x \leq 3, -1 \leq t \leq 1, \) and \(\lambda = 4.\)

**Fig. 1.** The surface for the solution \(F(t,x,\lambda)\)

3 Zeros of the generalized Bell polynomials

This section aims to demonstrate the benefit of using numerical investigation to support theoretical prediction and to discover new interesting pattern of the zeros of the generalized Bell polynomials \(B_n(x,\lambda).\) By using computer, the generalized Bell polynomials \(B_n(x,\lambda)\) can be determined explicitly. We display the shapes of the generalized Bell polynomials \(B_n(x,\lambda)\) and investigate the zeros of the generalized Bell polynomials \(B_n(x,\lambda).\) For \(n = 1, \ldots, 10,\) we can draw a plot of the generalized Bell polynomials \(B_n(x,\lambda),\) respectively. This shows the ten plots combined into one. We display the shape of \(B_n(x,\lambda), -10 \leq x \leq 10, \lambda = 4\) (Figure 2).

**Fig. 2. Zeros of \(B_n(x,\lambda)\)**

We investigate the beautiful zeros of the generalized Bell polynomials \(B_n(x,\lambda)\) by using a computer. We plot the zeros of the \(B_n(x,\lambda)\) for \(n = 5, 10, 15, 20, \lambda = 4,\) and \(x \in \mathbb{C}\) (Figure 3).

In Figure 3 (top-left), we choose \(n = 5\) and \(\lambda = 4.\) In Figure 3 (top-right), we choose \(n = 10\) and \(\lambda = 4.\) In Figure 3 (bottom-left), we choose \(n = 15\) and \(\lambda = 4.\) In Figure 3 (bottom-right), we choose \(n = 20\) and \(\lambda = 4.\) Prove that \(B_n(x,\lambda), x \in \mathbb{C},\) has \(Im(x) = 0\) reflection symmetry analytic complex functions (see Figure 3). Stacks of zeros of the generalized Bell polynomials \(B_n(x,\lambda)\) for \(1 \leq n \leq 20, \lambda = 4\) from a 3-D structure are presented (Figure 4).
Our numerical results for approximate solutions of real zeros of the generalized Bell polynomials $B_n(x, \lambda)$ are displayed (Tables 1, 2).

Plot of real zeros of $B_n(x, \lambda)$ for $1 \leq n \leq 20$ structure are presented (Figure 5).
Table 1. Numbers of real and complex zeros of $B_n(x, 4)$

| degree $n$ | real zeros | complex zeros |
|------------|------------|---------------|
| 1          | 1          | 0             |
| 2          | 2          | 0             |
| 3          | 3          | 0             |
| 4          | 4          | 0             |
| 5          | 5          | 0             |
| 6          | 6          | 0             |
| 7          | 7          | 0             |
| 8          | 6          | 2             |
| 9          | 7          | 2             |
| 10         | 8          | 2             |
| 11         | 9          | 2             |
| 12         | 10         | 2             |
| 13         | 9          | 4             |
| 14         | 10         | 4             |

Fig. 5. Real zeros of $B_n(x, \lambda)$ for $1 \leq n \leq 20$

We observe a remarkably regular structure of the complex roots of the generalized Bell polynomials $B_n(x, \lambda)$. We hope to verify a remarkably regular structure of the complex roots of the generalized Bell polynomials $B_n(x, \lambda)$ (Table 1). Next, we calculated an approximate solution satisfying $B_n(x, \lambda) = 0, x \in \mathbb{C}$. The results are given in Table 2.

Table 2. Approximate solutions of $B_n(x, 4) = 0, x \in \mathbb{R}$

| degree $n$ | $x$         |
|------------|-------------|
| 1          | 0           |
| 2          | -2.0000, 2.0000 |
| 3          | 3.62008, -3.28357, -0.336509 |
| 4          | 5.04407, -4.20888, -1.91657, 1.08138 |
| 5          | 6.34241, -4.89805, -3.12253, 2.3597, -0.681527 |
| 6          | 7.55109, -5.3997, -4.10205, 3.54357, -2.0558, 0.462889 |
| 7          | 8.69145, -5.70673, -4.95736, 4.65759, -3.19025, 1.54067, -1.03537 |
| 8          | 5.71699, -4.16654, 2.56659, -2.28486, -0.0564486 |

Finally, we shall consider the more general problems. How many zeros does $B_n(x, \lambda)$ have? Prove or disprove: $B_n(x, \lambda) = 0$ has $n$ distinct solutions (see Table 2). Find the numbers of complex zeros $C_{B_n(x, \lambda)}$ of
$B_n(x, \lambda), \ Im(x) \neq 0$. Since $n$ is the degree of the polynomial $B_n(x, \lambda)$, the number of real zeros $R_{B_n(x, \lambda)}$ lying on the real line $Im(x) = 0$ is then $R_{B_n(x, \lambda)} = n - C_{B_n(x, \lambda)}$, where $C_{B_n(x, \lambda)}$ denotes complex zeros. See Table 1 for tabulated values of $R_{B_n(x, \lambda)}$ and $C_{B_n(x, \lambda)}$. The author has no doubt that investigations along this line will lead to a new approach employing numerical method in the research field of the generalized Bell polynomials $B_n(x, \lambda)$ to appear in mathematics and physics. The reader may refer to [14, 15] for the details.

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