Thermodynamic Study for Conformal Phase in Large $N_f$ Gauge Theory

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We investigate the chiral phase transition at finite temperature ($T$) in colour SU($N_c = 3$) Quantum Chromodynamics (QCD) with six species of fermions ($N_f = 6$) in the fundamental representation $[1]$. The simulations have been performed by using lattice QCD with improved staggered fermions. The critical couplings $\beta^c_L$ for the chiral phase transition are observed for several temporal extensions $N_t$, and the two-loop asymptotic scaling of the dimensionless ratio $T_c/\Lambda_L$ ($\Lambda_L =$ Lattice Lambda-parameter) is found to be achieved for $N_f \geq 6$. Further, we collect $\beta^c_L$ at $N_f = 0$ (quenched), and $N_f = 4$ at a fixed $N_t = 6$ as well as $N_f = 8$ at $N_t = 6, 12$, the latter relying on our earlier study. The results are consistent with enhanced fermionic screening at larger $N_f$. The ratio $T_c/\Lambda_L$ depends very mildly on $N_f$ in the $N_f = 0 - 4$ region, begins increasing at $N_f = 6$, and significantly grows up at $N_f = 8$, as $N_f$ reaches to the edge of the conformal window. We discuss the interrelation of the results with preconformal dynamics in the light of a functional renormalization group analysis.

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1. Introduction

Emergence of a conformal symmetry and a preconformal (walking) behavior in strongly flavored non-Abelian gauge theories has received much attention. Walking dynamics near the infrared fixed point has been advocated as a basis for strongly interacting mechanisms of electroweak symmetry breaking. Lattice Monte-Carlo simulations are expected to provide a solid theoretical base to understand the (pre-)conformal nature in the gauge theory.

A second zero of the two-loop beta-function of massless QCD with $N_f$ flavours implies, at least perturbatively, the appearance of an infrared fixed point (IRFP) at $N_f \gtrsim 8.05$ [2] with the restoration of conformal symmetry before the loss of asymptotic freedom (LAF) at $N_{LAF}^f = 16.5$. Conformality should emerge when the renormalized coupling at the would be IRFP is not strong enough to break chiral symmetry. This condition provides the lower bound $N_c^f$ of a so called conformal window in the flavor space, and we find elaborated analytic predictions [3, 4]: for instance, the functional renormalization group method [5] suggests $N_c^f \sim 12$. Before the emergence of conformal symmetry, a qualitative change of dynamics is claimed at $N_f = 6$ based on instanton study [6].

Recent lattice studies [7] focused on the computation of the edge of the conformal window $N_{fc}^f$ and the analysis of the conformal window itself, either with fundamental fermions [8 – 15], or other representations [16]. Among the many interesting results with fundamental fermions, we single out the observation that QCD with three colours and eight flavours is still in the hadronic phase [9, 10], while $N_f = 12$ seems to be close to $N_{fc}^f$, with some groups favouring conformality [8, 8, 11, 12], and others chiral symmetry breaking [14]. The onset of new strong dynamics at $N_f = 6$ has been implied via an enhancement of the ratio of chiral condensate to cubed pseudoscalar decay constant [17].

Using the thermal transition as a tool for investigating preconformal dynamics has been largely inspired by a renormalization group analysis [5]. The critical temperature for the chiral phase transition has been obtained as a function of $N_f$, Then the onset of the conformal window has been estimated by locating the vanishing critical temperature. The phase transition line is almost linear with $N_f$ for small $N_f$, and clearly elucidates the universal critical behaviour at zero and non-zero temperature in the vicinity of $N_c^f$. Thus, it would be a promising direction to extend the knowledge of finite $T$ lattice QCD to the larger $N_f$ region, by using the FRG results as analytic guidance.

In this proceedings, we investigate the thermal chiral phase transition for $N_f = 6$ colour SU($N_c = 3$) QCD by using lattice QCD Monte Carlo simulations with improved staggered fermions based on our recent study [1]. $N_f = 6$ is expected to be in the important regime as suggested by the results in Refs. [3, 8]. We also compute the critical couplings for $N_f = 0$ (quenched) and $N_f = 4$ at $N_t = 6$, and use the results from Ref. [10] for $N_f = 8$. Then we investigate $N_f$ dependences of the chiral phase transition.

2. Simulation setups

Simulations have been performed in the same as in the study used for $N_f = 8$ in Ref. [11]: We have utilized the publicly available MILC code [18] with the use of an improved version of the staggered action, the Asqtad action, with a one-loop Symanzik [19, 20] and tadpole [22] improved
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gauge action. The tadpole factor $u_0$ is determined by performing zero temperature simulations on
the $12^4$ lattice, and used as an input for finite temperature simulations.

To generate configurations with mass degenerate dynamical flavours, we have used the rational
hybrid Monte Carlo algorithm (RHMC) [21]. Simulations for $N_f = 6$ have been performed by using
two pseudo-fermions, and subsets of trajectories for the chiral condensates and Polyakov loop have
been compared with those obtained by using three pseudo-fermions with the same Monte Carlo
time step $d\tau$ and total time length $\tau$ of a single trajectory. We have observed very good agreement
between the two cases for both evolution and thermalization. We have monitored the Metropolis
acceptance and reject ratio, and adjusted $\tau = 0.2 - 0.24$ and $d\tau = 0.008 - 0.016$ to realize the best
performance.

Measured observables are the expectation values of the chiral condensate and Polyakov loop,
$$a^3 \langle \bar{\psi}\psi \rangle = \frac{N_f}{4N_f^2} \langle \text{Tr}[M^{-1}] \rangle, \quad L = \frac{1}{N_c N_f^3} \sum_x \text{Re} \left\langle \text{tr}_c \prod_{i=1}^{N_t} U_{4,ix} \right\rangle,$$
where $N_t$ ($N_f$) represents the number of lattice sites in the spatial (temporal) direction, $U_{4,ix}$ is the
temporal link variable, and $\text{tr}_c$ denotes the trace in colour space. The output of this measurement is
the critical coupling $\beta_L^{\text{c}}$ for the chiral phase transition.

3. Results

All results have been obtained for a fermion bare lattice mass $am = 0.02$. In the left panel of
Figs. [1], the expectation values of the chiral condensate $a^3 \langle \bar{\psi}\psi \rangle$ are displayed as a function of $\beta_L$
for several $N_t$. It is found that different $N_t$ give a different behaviour of $a^3 \langle \bar{\psi}\psi \rangle$. The asymptotic
scaling analysis below will confirm that it corresponds to a thermal chiral phase transition (or
crossover) in the continuum limit.

All values of the critical lattice coupling $\beta_L^{\text{c}}$ are summarized in Table [1]. For larger $N_f$, the
signal for the chiral phase transition becomes less clear, hence we investigate the histogram of the
chiral condensate: The histogram for $N_f = 8$ exhibits the double-peak structure at $\beta_L^{\text{c}} = 5.2$, i.e., the
competition between chirally symmetric and broken vacua. The critical coupling can be estimated
as $\beta_L^{\text{c}} = 5.225(25)$ for $N_f = 8$. For $N_f = 12$, we also observe the double-peak structure in the
histogram of the chiral condensate around $\beta_L^{\text{c}} = 5.45$.

These results can be analyzed and interpreted in terms of the two-loop asymptotic scaling. Let
us consider the two-loop lattice beta function,
$$\beta(g) = -(b_0 g^3 + b_1 g^2), \quad (b_0, b_1) = \left(\frac{11 - 2N_f/3}{4\pi}, \frac{102 - 38N_f/3}{4\pi} \right),$$
for fundamental fermions in colour SU(3). From Eq. (3.1), we obtain the well known two-loop asymptotic scaling,
$$\Lambda_L a(\beta_L) = \left(2N_f b_0 / \beta_L^{\text{c}} \right)^{-b_1/(2b_0^2)} \exp \left[-\beta_L/(4N_f b_0) \right].$$
Here, $\Lambda_L$ is the so-called lattice Lambda-parameter, and $\beta_L = 2N_c / g^2$, with $g = \sqrt{2N_c/10 \cdot g_L}$. This
definition effectively takes account of the improvement of the staggered lattice action when comparing
to the asymptotic scaling law, see Ref. [10]. We insert $\Lambda_L$ to the definition of temperature.
\[ T \equiv [a(\beta_L)N_t]^{-1}, \]

\[ N_t^{-1} = (T_c/\Lambda_L) \times \left( \Lambda_L a(\beta_L^c) \right), \quad (3.4) \]

and extract the physical quantity \( T_c/\Lambda_L \) by substituting the simulation outputs \( \beta_L^c \) for Eq. (3.4). This ratio must be unique as long as the asymptotic scaling Eq. (3.3) is verified for a given \( \beta_L^c \).

In the right panel of Fig. 1, the slope of the line connecting the origin and the data points corresponds to \( T_c/\Lambda_L \). The \( N_t = 6, 8, \) and 12 points have a common slope to a very good approximation, while the \( N_t = 4 \) result falls on a smaller slope. The latter is interpreted as a scaling violation effect due to the use of a too small \( N_t \). The existence of a common \( T_c/\Lambda_L \) for \( N_t \geq 6 \) indicates that the data are consistent with the two-loop asymptotic scaling Eq. (3.3), confirms the thermal nature of the transition and that \( N_f = 6 \) is outside the conformal window, as expected from a previous study [10]. A linear fit provides \( T_c/\Lambda_L = 1.02(12) \times 10^3 \), which can be interpreted as the value in the continuum limit for \( N_f = 6 \) QCD.

In order to have a more complete overview, we have performed simulations for the theory with \( N_f = 0 \) (quenched) and \( N_f = 4 \), only at \( N_t = 6 \). These theories are of course very well investigated, however we have not found in the literature results for the same action as ours. We note that in a previous lattice study with improved staggered fermions [23], asymptotic scaling was observed for \( N_t \geq 6 \) for \( 0 \leq N_f \leq 4 \). Table 1 shows a summary of our results for the critical coupling \( \beta_L^c \) of the chiral phase transition at finite temperature for \( N_f = 0, 4, 6, \) and 8 - the latter from Ref. [10].

![Figure 1](https://example.com/figure1.png)

**Figure 1:** Left: The chiral condensate \( a^3 \langle \bar{\psi} \psi \rangle \) for \( N_f = 6 \) and \( am = 0.02 \) in lattice units, as a function of \( \beta_L \), for \( N_t = 4, 6, 8, \) and 12. Error-bars are smaller than symbols. Right: The thermal scaling behaviour of the critical lattice coupling \( \beta_L^c \). Data points for \( \Lambda_L a(\beta_L^c) \) at a given \( 1/N_t \) are obtained by using \( \beta_L^c \) from Table 1 as input for extracting \( \Lambda_L a(\beta_L^c) \) in the two-loop expression Eq. (3.3). The dashed line is a linear fit with zero intercept to the data with \( N_t > 4 \).

In the left panel of Fig. 2, we display the critical values of the lattice coupling \( g_c = \sqrt{2N_f/\beta_L^c} \) from Table 1 in the Miransky-Yamawaki phase diagram. Consider the \( N_t = 6 \) results: it is expected that an increasing number of flavours favors chiral symmetry restoration. Indeed, we find that, on a fixed lattice, the critical coupling increases with \( N_f \) in agreement with early studies and naive reasoning. The precise dependence of the critical coupling on \( N_f \) at fixed \( N_t \) is not known. It is, however, amusing to note that the results seem to be smoothly connected by an almost straight line: the brown line in the plot is a linear fit to the data. Comparing the trend for \( N_f = 6 \) to the one for...
$N_f = 8$ for varying $N_f$, one can infer a decreasing in magnitude (and small) step scaling function, hence a walking behaviour. Further study is needed at larger $N_f$, and by using the same action used for $N_f = 0 - 8$, to confirm or disprove it.

**Table 1:** Summary of the critical lattice couplings $\beta^c_L$ for the theories with $N_f = 0, 4, 6, 8$, $am = 0.02$ and varying $N_f = 4, 6, 8, 12$. All results are obtained using the same lattice action.

| $N_f \setminus N_f$ | 4   | 6       | 8       | 12      |
|---------------------|-----|---------|---------|---------|
| 0                   | -   | 7.88 ± 0.05 | -       | -       |
| 4                   | -   | 5.89 ± 0.03 | -       | -       |
| 6                   | 4.675 ± 0.025 | 5.025 ± 0.025 | 5.225 ± 0.025 | 5.45 ± 0.05 |
| 8                   | -   | 4.1125 ± 0.0125 | -       | 4.34 ± 0.04 |

**Figure 2:** Left: Critical values of the lattice coupling $g_c = \sqrt{2N_c/\beta^c_L}$ for theories with $N_f = 0, 4, 6, 8$ and for several values of $N_f$ in the Miransky-Yamawaki phase diagram. The dashed (brown) line is a linear fit to the $N_f = 6$ results. Right: The $N_f$ dependence of $R(N_f)/R(0)$ for several finite fixed $\beta^c_L$. Here, $R(N_f) \equiv (T_c/\Lambda_{ref})(N_f)$.

Next, we study the $N_f$ dependence of the ratio $T_c/\Lambda_L$ and related quantities. In addition to the scale $\Lambda_L$, we introduce more UV reference energy scale $\Lambda_{ref}$, which is associated with a reference coupling $\beta^c_L$. Then Eq. (3.3) is generalized as

$$
\Lambda_{ref}(\beta^c_L) a(\beta_L) = \left( \frac{b_1}{b_0} \frac{\beta_L + 2N_c b_1/b_0}{\beta^c_L + 2N_c b_1/b_0} \right)^{b_1/(2b_0)} \exp \left[ -\frac{\beta_L - \beta^c_L}{4N_c b_0} \right]. \tag{3.5}
$$

At leading order of perturbation theory $b_1 \to 0$, we find $\Lambda_{ref}/\Lambda_L = \exp[\beta^c_L/(4N_c b_0)]$. This equation would be analogous of the ratio $\Lambda_L/\Lambda_{MS}$ derived in [24] for Wilson fermions up to a further linear dependence on $N_f$ in the numerator of the exponent. In a nutshell, the difference originates from the fact that we are fixing a bare reference coupling $\beta^c_L$, which will be specified later. Notice that by construction $\Lambda_{ref}$ reproduces the lattice Lambda-parameter $\Lambda_L$ in the limit $\Lambda_{ref}(\beta^c_L \to 0) = \Lambda_L \left( 1 + \mathcal{O}(1/\beta^c_L) \right)$. 

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Let us consider first $R(N_f)\beta_{L,\text{ref}=0.0} = T_c/\Lambda_L$. The values of $T_c/\Lambda_L$ are found to be $600 \pm 34$, $620 \pm 28$, $1023 \pm 117$, and $2098 \pm 191$ for $N_f = 0$, $4$, $6$, and $8$, respectively, and represented as circles in the right panel of Fig. 2 (the vertical axis is normalized by $R(0) = (T_c/\Lambda_L)/(N_f = 0)$ for each $\beta_{L,\text{ref}}$). The ratio does not show a significant $N_f$ dependence in the region $0 \leq N_f \leq 4$, it starts increasing at $N_f = 6$, and undergoes a rapid rise around $N_f = 8$. The nearly constant nature of $T_c/\Lambda_L$ in the region $N_f \leq 4$ indicates that the role of such energy scale is not significantly changed by the variation of $N_f$ (see [25] for a detailed discussion of this point.) In turn, the increase of $T_c/\Lambda_L$ in the region $N_f \geq 6$ might well imply that the chiral dynamics becomes different from the one for $N_f \leq 4$. Indeed, a recent lattice study [17] indicates that $\Lambda_{\text{ref}}/\Lambda_L$ dependence in the region $0 < N_f < 4$ shows an increasing at $N_f = 6$, and undergoes a rapid rise around $N_f = 8$. The nearly constant nature of $T_c/\Lambda_L$ in the region $N_f \leq 4$ indicates that the role of such energy scale is not significantly changed by the variation of $N_f$ (see [25] for a detailed discussion of this point.) In turn, the increase of $T_c/\Lambda_L$ in the region $N_f \geq 6$ might well imply that the chiral dynamics becomes different from the one for $N_f \leq 4$. Indeed, a recent lattice study [17] indicates that $N_f = 6$ is close to the threshold for preconformal dynamics.

We now consider $T_c/\Lambda_{\text{ref}}$ with finite $\beta_{L,\text{ref}}$. The $N_f$ dependence of the ratio $R(N_f) \equiv (T_c/\Lambda_{\text{ref}})/(N_f = 0)$ for each $\beta_{L,\text{ref}}$. $T_c/\Lambda_{\text{ref}}$ is now a decreasing function of $N_f$ for a larger $\beta_{L,\text{ref}}$. The $\Lambda_{\text{ref}}$ associated with a $\beta_{L,\text{ref}} \gg 2N_c/g_{\text{UV}}^2$ would be less sensitive to the IR or chiral dynamics. Assuming $N_f \simeq 12$, the two-loop beta-function leads to $\beta_4 = -2N_c b_1/b_0 \simeq 0.63$. The decreasing nature of $(T_c/\Lambda_{\text{ref}})/(N_f)$ is found to start around $\beta_{L,\text{ref}} = 1.0 \simeq \beta_4$. Thus, the use of a UV reference scale leads to the decreasing $(T_c/\Lambda_{\text{ref}})/(N_f)$. This trend is consistent with the FRG study [3], where the decreasing $T_c(N_f)$ has been obtained by using the $\tau$-lepton mass $m_\tau$ as a common UV reference scale with a common coupling $\alpha_s(m_\tau)$. We note that we have constrained our analyses $\beta_{L,\text{ref}} < \beta_{\text{UV}} = \beta_{L,c}(N_f) \leq 4.1125 \pm 0.0125$.

With the use of a UV reference scale, we should observe the predicted critical behaviour [3].

$$T_c(N_f) = K |N_f - N_f^c|^{-1/\theta}.$$  \hspace{1cm} (3.6)

By choosing the critical exponent $\theta$ in the range predicted by FRG: $1.1 < 1/|\theta| < 2.5$, our data are consistent with the values $N_f^c = 9(1)$ for $\beta_{L,\text{ref}} = 4.0$ and $N_f^c = 11(2)$ for $\beta_{L,\text{ref}} = 2.0$. We plan to extend and refine this analysis in the future, and here we only notice a reasonable qualitative behaviour.

4. Summary

We have studied the chiral phase transition at finite $T$ for colour $SU(3)$ QCD with $N_f = 6$ by using lattice QCD Monte-Carlo simulations with improved staggered fermions [1]. We have determined the critical lattice coupling $\beta_{L,c}$ for several lattice temporal extensions $N_t$, and extracted the dimensionless ratio $T_c/\Lambda_L$ ($\Lambda_L =$Lattice Lambda-parameter) by using two-loop asymptotic scaling. The analogous result for $N_f = 8$ has been extracted from Ref. [10]. $T_c/\Lambda_L$ for $N_f = 0$ and $N_f = 4$ has been measured at fixed $N_t = 6$, barring asymptotic scaling violations. Then we have discussed the $N_f$ dependence of the ratios $T_c/\Lambda_L$ and $T_c/\Lambda_{\text{ref}}$, where $\Lambda_{\text{ref}}$ is a UV reference energy scale, related to $\Lambda_L$ via $\Lambda_{\text{ref}}/\Lambda_L \simeq \exp[\beta_{L,\text{ref}}/(4N_c b_0)]$. We have observed that $T_c/\Lambda_L$ shows an increase in the region $N_f = 6 - 8$, while it is approximately constant in the region $N_f \leq 4$. We have discussed this qualitative change for $N_f \geq 6$ and a possible relation with a preconformal phase. The ratio $T_c/\Lambda_{\text{ref}}$ is a decreasing function of $N_f$. This behaviour is consistent with the result obtained in the functional renormalization group analysis [3]. Next steps of the current project involve a scale setting at zero temperature by measuring a common UV observable.

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