A dual-stress Bayesian Weibull accelerated life testing model

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Abstract

In this paper, a Bayesian accelerated life testing model is presented. The Weibull distribution is used as the life distribution and the generalised Eyring model as the time transformation function. This is a model that allows for the use of more than one stressor, whereas other commonly used acceleration models, such as the Arrhenius and power law models, allow for only one stressor. The generalised Eyring-Weibull model is used in an application, where MCMC methods are utilised to generate samples for posterior inference.

Keywords: Accelerated life testing; Bayes; Generalised Eyring model; Markov chain Monte Carlo; Weibull distribution.

1 Introduction

Reliability and life testing is vitally important in the manufacturing of consumer and capital goods, particularly in the engineering fields. It is crucial to determine whether a product will perform its prescribed function, without failure, for a desired period of time. Reliability and life testing provides a theoretical as well as practical framework by which the life characteristics of a product, component or system can be quantified. An extensive discussion on the objectives of, and need for, reliability and life testing can be found in Kececioglu (2002).

In this day and age, manufacturers are pressured to provide durable products, timely and at competitive prices. Performing life tests on high-reliability materials, components and systems can be a tedious and unproductive task. Many modern products are made to endure for years or even decades. Obtaining sufficient failure time data for these products at normal use conditions can prove to be very costly and time consuming. This longevity obstacle has lead to a rising interest in reliability engineering and the development of accelerated life tests (ALTs). In ALTs, products are tested in a more severe than normal use environment, by applying or fluctuating one or more stressors (stress variables) at accelerated levels, in order to induce early failures (Nelson, 1990). This failure data is then extrapolated to predict the reliability characteristics of the products at normal use conditions (Pan, 2009).

Stressors can include temperature, pressure, voltage, wattage, humidity, loads, vibration amplitude, use rate, etc. (Kececioglu, 2002; Escobar & Meeker, 2006). Appertaining to the understanding of the physics of failure, a functional relationship, known as a time transformation function (TTF), is assumed between the stressors and the parameters of the life distribution (see, for example, Singpurwalla,
1971a,b, 1973; Singpurwalla et al. , 1975). The most commonly used models for acceleration include the Arrhenius, inverse power law, Eyring, and generalised Eyring models. Various other acceleration models are discussed in Escobar & Meeker (2006), Kececioglu (2002), and Thiraviam (2010).

In this paper, a Bayesian approach to the generalised Eyring model with the Weibull distribution as life distribution will be explored. Section 2 consists of a short review of some prominent Bayesian models and methods in accelerated life testing. The generalised Eyring-Weibull (GEW) model and the general likelihood formulation for this model is given in Section 3. In Section 4, variations of the GEW model is presented by means of different choices for the prior distributions. The posterior and full conditional posterior distributions for each variation, up to at least proportionality, is also provided. Section 5 will provide an extensive application and comparison of the GEW models to a data set. Due to the mathematically intractable posteriors, the log-concavity of the models are assessed in the appendix to determine which Markov Chain Monte Carlo (MCMC) method can be utilised for posterior sampling. The paper will conclude in Section 6 with some final remarks on the GEW model.

2 Bayesian Accelerated Life Testing

The development of Bayesian methods to draw inferences from ALTs has experienced a surge in recent years. Ahmad (1990) provides a general overview of early Bayesian methods in ALTs. These methods include reducing high dimensional integration problems via semi-sufficient statistics, the use of semi-parametric inference, a Kalman filter approach, and failure rate strategies. Achcar & Louzada-Neto (1991) considers an exponential distribution with the Eyring model. Type-II censored data is used and Jeffreys’ priors are employed for the model parameters, where Laplace approximations are utilised for integrals that are difficult to solve analytically. Chaloner & Larntz (1992) examines experimental Bayesian design for ALTs, where lifetimes follow either a Weibull or a log-normal distribution. A dynamic general linear model setup for ALTs, that uses linear Bayesian methods for inference, is presented in Mazzuchi & Soyer (1992). Lifetimes are assumed to be exponential with the power law as the TTF. Achcar (1993) explores Laplace approximation methods for deriving posterior densities for various commonly used life distributions and TTFs.

Van Dorp et al. (1996) develops a Bayesian model for step-stress ALTs where the lifetimes are exponentially distributed, and provides Bayesian point estimates and credibility intervals for parameters at normal use conditions. Dietrich & Mazzuchi (1996) discusses the design of experiments in ALTs, pointing out some problems that emerge from typical regression and ANOVA techniques, and proposes an alternative analysis procedure based on Bayesian considerations. Mazzuchi et al. (1997) presents a Bayesian approach, based on general linear models, for inference from ALTs. The lifetimes are assumed to be Weibull distributed and the power law is used as TTF on the scale parameter. Erkanli & Soyer (2000) provides simulation-based designs for ALTs, using a Bayesian decision theory approach, where lifetimes follow an exponential distribution with the power law as TTF. Perdona & Louzada-Neto (2005) proposes an ALT model where lifetimes are exponential and a general log-non-linear TTF is used. A uniform prior is used and Laplace approximations are utilised to find marginal posterior distributions.

Van Dorp & Mazzuchi (2004) develops a general Bayesian inference model for ALTs, which is compatible with various types of stress loading, where the lifetimes are exponentially distributed and strict conformity to a specific TTF is not compulsory. MCMC methods are used for inference and to derive posterior quantities. This model is also used in Van Dorp et al. (2006) to compare constant stress, step-stress and profile stress ALTs within a single Bayesian inferential framework, and is extended to the Weibull distribution in Van Dorp & Mazzuchi (2005). Leijonen et al. (2007) presents Bayesian inferences from ALTs, making use of MCMC methods, where items originate from different groups with random effects.
Although the procedure is applied to an elementary example, it can also be used with multiple random effects and acceleration factors. Barriga et al. (2008) considers Bayesian methods for ALTs where the exponentiated Weibull is used as the life distribution and the Arrhenius model is the TTF. MCMC methods are used to sample from the posterior for inferences, where a mixture of normal and gamma priors are imposed on the parameters.

Soyer (2008) reviews Bayesian designs for ALTs by comparing Bayesian decision theory, linear Bayesian, and Bayesian simulation-based designs. An exponential life distribution with the power law as TTF is used throughout the review. A basic parametric Bayesian ALT model, where lifetimes follow a Weibull distribution with the power law as TTF, is discussed in Soyer et al. (2008), and then extended to allow for a more dynamic TTF. The two extensions are a hierarchical Bayesian model and a Markov model where inferences are made via MCMC methods. Pan (2009) provides a Bayesian approach, and introduces a calibration factor, which allows for a combination of field data and ALT data to be used for reliability prediction. A log-linear TTF is used with the exponential and Weibull life distributions, and MCMC methods are employed for intricate posterior distributions. Upadhyay & Mukherjee (2010) presents a Bayesian comparison between accelerated Weibull and Birnbaum-Saunders models, where the TTF is the inverse power law. A combination of independent vague priors for the model parameters are chosen and MCMC techniques are used to generate posterior samples.

Yuan et al. (2014) proposes a semi-parametric Bayesian approach for ALTs where the TTF is log-linear and no assumption is made on the form of the life distribution. The authors employ a Dirichlet process mixture model, using a Weibull kernel, to model failure times. Model fitting is performed via a simulation-based algorithm that incorporates Gibbs sampling. Mukhopadhyay & Roy (2016) considers Bayesian analysis for ALTs where log-lifetimes follow a distribution from the log-concave location-scale family and a linear TTF is used. After a general discussion, the article focuses on using the log-normal life distribution with the Arrhenius model as TTF. MCMC techniques are used to obtain posterior samples for inference. Classical and Bayesian inference on a progressive stress ALT with a generalised Birnbaum-Saunders life distribution is presented in Sha (2018). MCMC methods are used for Bayesian analysis and the author states that the proposed Bayesian method is not only efficient and accurate, but outperforms the classical likelihood-based approach to ALTs.

To our knowledge, the generalised Eyring model has primarily been implemented only in frequentistic ALT setups, and is still less commonly used than other TTFs. The generalised Eyring model allows for an ALT model with more than one stressor, which makes it a more applicable model to use in industry. It is however an intricate model due to the number of unknown parameters, which complicates both classical and Bayesian inferences. The aim of this article is to present a Bayesian ALT model that incorporates the generalised Eyring relationship, where lifetimes follow a Weibull distribution. The Weibull distribution is a more flexible and appropriate life distribution to use in practice, compared to, for example, the exponential distribution. In the event of mathematically intractable posterior distributions, MCMC methods are used to draw posterior samples for inference.

# The Generalised Eyring-Weibull Model Specification

Let $X$ be a continuous random variable that follows a Weibull distribution with scale parameter $\alpha$ and shape parameter $\beta$ ($\alpha > 0$, $\beta > 0$). The probability density function (PDF) is given by

$$f(x | \alpha, \beta) = \alpha \beta x^{\beta - 1} \exp(-\alpha x^\beta), \quad x \geq 0,$$

and the reliability function by

$$R(x) = 1 - F(x) = \exp(-\alpha x^\beta).$$
Consider two stressors, one thermal and one non-thermal. Indicate the $k$ distinct accelerated levels of the stressors by $\{T_i, S_i\}, i = 1, \ldots, k$, where $T_i, i = 1, \ldots, k$, is the accelerated levels of the thermal stressor and $S_i, i = 1, \ldots, k$ is the accelerated levels of the non-thermal stressor. An item is exposed to the constant application of a specific stress level combination $\{T_i, S_i\}$. A common assumption in the literature is that the Weibull scale parameter $\alpha$ is then dependent on the stress levels, whereas the shape parameter $\beta$ is not (see, for example, Mazzuchi et al., 1997; Soyer et al., 2008; Upadhyay & Mukherjee, 2010). The reparameterisation of $\alpha$ given by the generalised Eyring model is

$$\alpha_i = T_i \exp \left( -\theta_1 - \frac{\theta_2}{T_i} - \theta_3 V_i - \frac{\theta_4 V_i}{T_i} \right),$$

(3.3)

where $\theta_1$, $\theta_2$, $\theta_3$, and $\theta_4$ are unknown model parameters (ReliaSoft Corporation, 2015), and $V_i$ is a function of the non-thermal stressor $S_i$ (Escobar & Meeker, 2006). For a lifetime subjected to the $i^{th}$ level of the stressors, it follows from (3.1) and (3.3) that the Weibull PDF can be written as

$$f(x_i | \theta_1, \theta_2, \theta_3, \theta_4, \beta) = T_i \exp \left( -\theta_1 - \frac{\theta_2}{T_i} - \theta_3 V_i - \frac{\theta_4 V_i}{T_i} \right) \beta x_i^{\beta-1} \times \exp \left[ -T_i \exp \left( -\theta_1 - \frac{\theta_2}{T_i} - \theta_3 V_i - \frac{\theta_4 V_i}{T_i} \right) x_i^\beta \right].$$

(3.4)

From (3.2) and (3.3), the Weibull reliability function at some time $\tau$ can be written as

$$R(\tau) = \exp \left[ -T_i \exp \left( -\theta_1 - \frac{\theta_2}{T_i} - \theta_3 V_i - \frac{\theta_4 V_i}{T_i} \right) \tau^\beta \right].$$

(3.5)

Suppose that $n_i$ items are tested at each of the $k$ different stress levels and denote the failure times by $x_{ij}, j = 1, \ldots, n_i, i = 1, \ldots, k$. The likelihood function, in general, is then given by

$$L(x | \theta_1, \theta_2, \theta_3, \theta_4, \beta) = \prod_{i=1}^{k} \prod_{j=1}^{r_i} f(x_{ij} | \theta_1, \theta_2, \theta_3, \theta_4, \beta) \left[ R(\tau_i) \right]^{n_i-r_i}.$$

Note that for complete samples $r_i = n_i$. For type-I censoring $r_i$ is the number of failures that occur before censoring time $\tau_i$, where $\tau_i < \infty, i = 1, \ldots, k$ is predetermined censoring times for the $k$ different stress levels. For type-II censoring $\tau_i = x_{i(r_i)}$, where $x_{i(r_i)}$ is the failure time of the $r_i^{th}$ failure, where $r_i, i = 1, \ldots, k$ is the pre-chosen number of failures after which censoring occurs for the $k$ different stress levels. From (3.4) and (3.5) it follows that
The joint posterior distribution is then given by

\[
L(\mathbf{x} | \theta_1, \theta_2, \theta_3, \theta_4, \beta) = \prod_{i=1}^{k} \left( \prod_{l=1}^{r_i} f(x_{ij} | \theta_1, \theta_2, \theta_3, \theta_4, \beta) \right) [R(\tau_i)]^{r_i - r_i} \pi(\theta_1, \theta_2, \theta_3, \theta_4, \beta) \]

\[
= \prod_{i=1}^{k} \exp \left[ - (n_i - r_i) T_i \exp \left( -\theta_1 - \frac{\theta_2}{T_i} - \frac{\theta_3 V_i - \theta_4 V_i}{T_i} \right) \tau_i^\beta \right] \times \beta^{r_i \tau_i} T_i \exp \left( -\theta_1 r_i - \frac{\theta_2 r_i}{T_i} - \frac{\theta_3 r_i V_i - \theta_4 r_i V_i}{T_i} \right) \times \prod_{j=1}^{r_i} x_{ij}^{\beta - 1} \exp \left[ - T_i \exp \left( -\theta_1 - \frac{\theta_2}{T_i} - \frac{\theta_3 V_i - \theta_4 V_i}{T_i} \right) x_{ij}^\beta \right] \]

\[
= \beta^{\sum_{i=1}^{k} r_i} \exp \left( -\theta_1 \sum_{i=1}^{k} r_i - \theta_2 \sum_{i=1}^{k} \frac{r_i}{T_i} - \theta_3 \sum_{i=1}^{k} r_i V_i - \theta_4 \sum_{i=1}^{k} \frac{r_i V_i}{T_i} \right) \times \exp \left[ - \sum_{i=1}^{k} (n_i - r_i) T_i \exp \left( -\theta_1 - \frac{\theta_2}{T_i} - \frac{\theta_3 V_i - \theta_4 V_i}{T_i} \right) \tau_i^\beta \right] \times \exp \left[ - \sum_{i=1}^{k} \sum_{j=1}^{r_i} T_i \exp \left( -\theta_1 - \frac{\theta_2}{T_i} - \frac{\theta_3 V_i - \theta_4 V_i}{T_i} \right) x_{ij}^\beta \right] \left[ \prod_{i=1}^{k} \prod_{j=1}^{r_i} T_i x_{ij}^{\beta - 1} \right].
\] 

4 Priors and Posterior

In this section, a number of GEW models are defined, resulting from the selection of different prior distributions. Soland (1969) affirms that there is no conjugate family of continuous joint prior distributions for the two-parameter Weibull distribution, and proposes a gamma prior for the scale parameter and a discrete prior for the shape parameter. Tsokos (1972) suggests using an inverse gamma prior for the scale parameter and a uniform prior for the shape parameter. A compound inverse gamma prior on the scale parameter, with either a discrete, inverse gamma, or uniform prior on the shape parameter is proposed in Papadopoulos & Tsokos (1976). Kundu (2008) assumes a gamma prior for the scale parameter and a non-specific log-concave prior density with support \((0, \infty)\) for the shape parameter. Banerjee & Kundu (2008) considers gamma priors on both the scale and shape parameters.

Taking the above into consideration, we formulate four GEW models. Assume that the priors on the unknown parameters \(\theta_1, \theta_2, \theta_3, \theta_4\) and \(\beta\) are independent, and given by

\[
\pi(\theta_1, \theta_2, \theta_3, \theta_4, \beta) = \pi(\theta_1) \pi(\theta_2) \pi(\theta_3) \pi(\theta_4) \pi(\beta).
\]

The joint posterior distribution is then given by

\[
\pi(\theta_1, \theta_2, \theta_3, \theta_4, \beta | \mathbf{x}) \propto L(\mathbf{x} | \theta_1, \theta_2, \theta_3, \theta_4, \beta) \pi(\theta_1, \theta_2, \theta_3, \theta_4, \beta).
\]
4.1 \textit{GEW}$_1$ Model

The \textit{GEW} model where uniform priors are imposed on all the parameters, thus

\[
\begin{align*}
\theta_1 &\sim U(c_0, c_1), \quad c_1 > c_0, \quad \pi_1(\theta_1) \propto \text{constant} \\
\theta_2 &\sim U(c_2, c_3), \quad c_3 > c_2, \quad \pi_1(\theta_2) \propto \text{constant} \\
\theta_3 &\sim U(c_4, c_5), \quad c_5 > c_4, \quad \pi_1(\theta_3) \propto \text{constant} \\
\theta_4 &\sim U(c_6, c_7), \quad c_7 > c_6, \quad \pi_1(\theta_4) \propto \text{constant} \\
\beta &\sim U(c_8, c_9), \quad c_9 > c_8, \quad \pi_1(\beta) \propto \text{constant},
\end{align*}
\]

is denoted by \textit{GEW}$_1$. The joint prior for \textit{GEW}$_1$ is then given by

\[
\pi_1(\theta_1, \theta_2, \theta_3, \theta_4, \beta) \propto \text{constant}, \quad (4.1)
\]

and using (3.6) and (4.1) the joint posterior is given by

\[
\begin{align*}
\pi_1(\theta_1, \theta_2, \theta_3, \theta_4, \beta \mid x) &\propto \beta^{\sum_{i=1}^{k} r_i} \exp \left( -\theta_1 \sum_{i=1}^{k} r_i - \theta_2 \sum_{i=1}^{k} \frac{r_i}{T_i} - \theta_3 \sum_{i=1}^{k} r_i V_i - \theta_4 \sum_{i=1}^{k} \frac{r_i V_i}{T_i} \right) \\
&\quad \times \exp \left[ - \sum_{i=1}^{k} (n_i - r_i) T_i \exp \left( -\theta_1 - \frac{\theta_2}{T_i} - \theta_3 V_i - \frac{\theta_4 V_i}{T_i} \right) \tau_i^\beta \right] \\
&\quad \times \exp \left[ - \sum_{i=1}^{k} \sum_{j=1}^{r_i} T_i \exp \left( -\theta_1 - \frac{\theta_2}{T_i} - \theta_3 V_i - \frac{\theta_4 V_i}{T_i} \right) x_{ij}^\beta \right] \prod_{i=1}^{k} \prod_{j=1}^{r_i} T_i x_{ij}^\beta^{-1}.
\end{align*}
\]

The posterior is of an intractable form, therefore MCMC methods are employed in order to draw posterior samples for inferences. The full conditional posteriors for the \textit{GEW}$_1$ model are given by

\[
\begin{align*}
\pi_1(\theta_1 \mid x, \theta_2, \theta_3, \theta_4, \beta) &\propto \exp \left( -\theta_1 \sum_{i=1}^{k} r_i \right) \exp \left[ - \sum_{i=1}^{k} (n_i - r_i) T_i \exp \left( -\theta_1 - \frac{\theta_2}{T_i} - \theta_3 V_i - \frac{\theta_4 V_i}{T_i} \right) \tau_i^\beta \right] \\
&\quad \times \exp \left[ - \sum_{i=1}^{k} \sum_{j=1}^{r_i} T_i \exp \left( -\theta_1 - \frac{\theta_2}{T_i} - \theta_3 V_i - \frac{\theta_4 V_i}{T_i} \right) x_{ij}^\beta \right]
\end{align*}
\]

\[
\begin{align*}
\pi_1(\theta_2 \mid x, \theta_1, \theta_3, \theta_4, \beta) &\propto \exp \left( -\theta_2 \sum_{i=1}^{k} \frac{r_i}{T_i} \right) \exp \left[ - \sum_{i=1}^{k} (n_i - r_i) T_i \exp \left( -\theta_1 - \frac{\theta_2}{T_i} - \theta_3 V_i - \frac{\theta_4 V_i}{T_i} \right) \tau_i^\beta \right] \\
&\quad \times \exp \left[ - \sum_{i=1}^{k} \sum_{j=1}^{r_i} T_i \exp \left( -\theta_1 - \frac{\theta_2}{T_i} - \theta_3 V_i - \frac{\theta_4 V_i}{T_i} \right) x_{ij}^\beta \right]
\end{align*}
\]

\[
\begin{align*}
\pi_1(\theta_3 \mid x, \theta_1, \theta_2, \theta_4, \beta) &\propto \exp \left( -\theta_3 \sum_{i=1}^{k} r_i V_i \right) \exp \left[ - \sum_{i=1}^{k} (n_i - r_i) T_i \exp \left( -\theta_1 - \frac{\theta_2}{T_i} - \theta_3 V_i - \frac{\theta_4 V_i}{T_i} \right) \tau_i^\beta \right] \\
&\quad \times \exp \left[ - \sum_{i=1}^{k} \sum_{j=1}^{r_i} T_i \exp \left( -\theta_1 - \frac{\theta_2}{T_i} - \theta_3 V_i - \frac{\theta_4 V_i}{T_i} \right) x_{ij}^\beta \right]
\end{align*}
\]

\[
\begin{align*}
\pi_1(\theta_4 \mid x, \theta_1, \theta_2, \theta_3, \beta) &\propto \exp \left( -\theta_4 \sum_{i=1}^{k} \frac{r_i V_i}{T_i} \right) \exp \left[ - \sum_{i=1}^{k} (n_i - r_i) T_i \exp \left( -\theta_1 - \frac{\theta_2}{T_i} - \theta_3 V_i - \frac{\theta_4 V_i}{T_i} \right) \tau_i^\beta \right] \\
&\quad \times \exp \left[ - \sum_{i=1}^{k} \sum_{j=1}^{r_i} T_i \exp \left( -\theta_1 - \frac{\theta_2}{T_i} - \theta_3 V_i - \frac{\theta_4 V_i}{T_i} \right) x_{ij}^\beta \right]
\end{align*}
\]
The joint prior for

\[ \pi_1(\theta_4 | x, \theta_1, \theta_2, \theta_3, \beta) \propto \exp \left( -\theta_4 \sum_{i=1}^{k} \frac{r_i V_i}{T_i} \right) \exp \left[ -\sum_{i=1}^{k} (n_i - r_i) T_i \exp \left( -\theta_2 - \frac{\theta_2}{T_i} - \theta_3 V_i - \frac{\theta_3 V_i}{T_i} \right) \tau_i^\beta \right] \]

\[ \times \exp \left[ -\sum_{i=1}^{k} r_i \sum_{j=1}^{k} T_i \exp \left( -\theta_1 - \frac{\theta_2}{T_i} - \theta_3 V_i - \frac{\theta_4 V_i}{T_i} \right) x_{ij}^\beta \right] \]

\[ \pi_1(\beta | x, \theta_1, \theta_2, \theta_3, \theta_4) \propto \beta^{\sum_{i=1}^{k} r_i} \exp \left[ -\sum_{i=1}^{k} (n_i - r_i) T_i \exp \left( -\theta_2 - \frac{\theta_2}{T_i} - \theta_3 V_i - \frac{\theta_3 V_i}{T_i} \right) \tau_i^\beta \right] \]

\[ \times \exp \left[ -\sum_{i=1}^{k} r_i \sum_{j=1}^{k} T_i \exp \left( -\theta_1 - \frac{\theta_2}{T_i} - \theta_3 V_i - \frac{\theta_4 V_i}{T_i} \right) x_{ij}^\beta \right] \left[ \prod_{i=1}^{k} \prod_{j=1}^{r_i} x_{ij}^{\beta - 1} \right]. \]

**4.2 GEW₂ Model**

Let GEW₂ denote the GEW model where gamma priors on all the parameters are assumed, with

\[ \theta_1 \sim \Gamma(c_{10}, c_{11}) , c_{10}, c_{11} > 0 \] , \[ \pi_2(\theta_1) \propto \theta_1^{c_{10} - 1} \exp \left( -c_{11} \theta_1 \right) \]

\[ \theta_2 \sim \Gamma(c_{12}, c_{13}) , c_{12}, c_{13} > 0 \] , \[ \pi_2(\theta_2) \propto \theta_2^{c_{12} - 1} \exp \left( -c_{13} \theta_2 \right) \]

\[ \theta_3 \sim \Gamma(c_{14}, c_{15}) , c_{14}, c_{15} > 0 \] , \[ \pi_2(\theta_3) \propto \theta_3^{c_{14} - 1} \exp \left( -c_{15} \theta_3 \right) \]

\[ \theta_4 \sim \Gamma(c_{16}, c_{17}) , c_{16}, c_{17} > 0 \] , \[ \pi_2(\theta_4) \propto \theta_4^{c_{16} - 1} \exp \left( -c_{17} \theta_4 \right) \]

\[ \beta \sim \Gamma(c_{18}, c_{19}) , c_{18}, c_{19} > 0 \] , \[ \pi_2(\beta) \propto \beta^{c_{18} - 1} \exp \left( -c_{19} \beta \right). \]

The joint prior for GEW₂ is then given by

\[ \pi_2(\theta_1, \theta_2, \theta_3, \theta_4, \beta) \propto \theta_1^{c_{10} - 1} \theta_2^{c_{12} - 1} \theta_3^{c_{14} - 1} \theta_4^{c_{16} - 1} \beta^{c_{18} - 1} \exp \left( -c_{11} \theta_1 - c_{13} \theta_2 - c_{15} \theta_3 - c_{17} \theta_4 - c_{19} \beta \right), \] (4.2)

and using (3.6) and (4.2) the joint posterior is given by

\[ \pi_2(\theta_1, \theta_2, \theta_3, \theta_4, \beta | x) \propto \theta_1^{c_{10} - 1} \theta_2^{c_{12} - 1} \theta_3^{c_{14} - 1} \theta_4^{c_{16} - 1} \beta^{c_{18} - 1} \exp \left( -c_{11} \theta_1 - c_{13} \theta_2 - c_{15} \theta_3 - c_{17} \theta_4 - c_{19} \beta \right) \]

\[ \times \beta^{\sum_{i=1}^{k} r_i} \exp \left( \theta_1 \sum_{i=1}^{k} r_i - \theta_2 \sum_{i=1}^{k} \frac{r_i V_i}{T_i} - \theta_3 \sum_{i=1}^{k} r_i V_i - \theta_4 \sum_{i=1}^{k} \frac{r_i V_i}{T_i} \right) \]

\[ \times \exp \left[ -\sum_{i=1}^{k} (n_i - r_i) T_i \exp \left( -\theta_2 - \frac{\theta_2}{T_i} - \theta_3 V_i - \frac{\theta_3 V_i}{T_i} \right) \tau_i^\beta \right] \]

\[ \times \exp \left[ -\sum_{i=1}^{k} r_i \sum_{j=1}^{k} T_i \exp \left( -\theta_1 - \frac{\theta_2}{T_i} - \theta_3 V_i - \frac{\theta_4 V_i}{T_i} \right) x_{ij}^\beta \right] \left[ \prod_{i=1}^{k} \prod_{j=1}^{r_i} T_i x_{ij}^{\beta - 1} \right]. \]

Due to the complexity of the posterior, MCMC methods are used to draw posterior samples to base inferences on. For GEW₂, the full conditional posteriors are given by

\[ \pi_2(\theta_1 | x, \theta_2, \theta_3, \theta_4, \beta) \propto \theta_1^{c_{10} - 1} \exp \left( -c_{11} \theta_1 \right) \exp \left( -\theta_1 \sum_{i=1}^{k} r_i \right) \]

\[ \times \exp \left[ -\sum_{i=1}^{k} (n_i - r_i) T_i \exp \left( -\theta_1 - \frac{\theta_2}{T_i} - \theta_3 V_i - \frac{\theta_3 V_i}{T_i} \right) \tau_i^\beta \right] \]

\[ \times \exp \left[ -\sum_{i=1}^{k} r_i \sum_{j=1}^{k} T_i \exp \left( -\theta_1 - \frac{\theta_2}{T_i} - \theta_3 V_i - \frac{\theta_4 V_i}{T_i} \right) x_{ij}^\beta \right] \]
\[ \pi_2 (\theta_2 | x, \theta_1, \theta_3, \theta_4, \beta) \propto \theta_2^{c_{12}-1} \exp (-c_{13} \theta_2) \exp \left( -\theta_2 \sum_{i=1}^{k} \frac{r_i}{T_i} \right) \]
\[ \times \exp \left[ -k \sum_{i=1}^{k} (n_i - r_i) T_i \exp \left( -\theta_1 - \frac{\theta_2}{T_i} - \theta_3 V_i - \frac{\theta_4 V_i}{T_i} \right) \tau_i^{\beta} \right] \]
\[ \times \exp \left[ -k \sum_{i=1}^{k} \sum_{j=1}^{r_i} T_i \exp \left( -\theta_1 - \frac{\theta_2}{T_i} - \theta_3 V_i - \frac{\theta_4 V_i}{T_i} \right) x_{ij}^{\beta} \right] \]

\[ \pi_2 (\theta_3 | x, \theta_1, \theta_2, \theta_4, \beta) \propto \theta_3^{c_{14}-1} \exp (-c_{15} \theta_3) \exp \left( -\theta_3 \sum_{i=1}^{k} r_i V_i \right) \]
\[ \times \exp \left[ -k \sum_{i=1}^{k} (n_i - r_i) T_i \exp \left( -\theta_1 - \frac{\theta_2}{T_i} - \theta_3 V_i - \frac{\theta_4 V_i}{T_i} \right) \tau_i^{\beta} \right] \]
\[ \times \exp \left[ -k \sum_{i=1}^{k} \sum_{j=1}^{r_i} T_i \exp \left( -\theta_1 - \frac{\theta_2}{T_i} - \theta_3 V_i - \frac{\theta_4 V_i}{T_i} \right) x_{ij}^{\beta} \right] \]

\[ \pi_2 (\theta_4 | x, \theta_1, \theta_2, \theta_3, \beta) \propto \theta_4^{c_{16}-1} \exp (-c_{17} \theta_4) \exp \left( -\theta_4 \sum_{i=1}^{k} \frac{r_i V_i}{T_i} \right) \]
\[ \times \exp \left[ -k \sum_{i=1}^{k} (n_i - r_i) T_i \exp \left( -\theta_1 - \frac{\theta_2}{T_i} - \theta_3 V_i - \frac{\theta_4 V_i}{T_i} \right) \tau_i^{\beta} \right] \]
\[ \times \exp \left[ -k \sum_{i=1}^{k} \sum_{j=1}^{r_i} T_i \exp \left( -\theta_1 - \frac{\theta_2}{T_i} - \theta_3 V_i - \frac{\theta_4 V_i}{T_i} \right) x_{ij}^{\beta} \right] \]

\[ \pi_2 (\beta | x, \theta_1, \theta_2, \theta_3, \theta_4) \propto \beta^{c_{18}-1} \exp (-c_{19} \beta) \beta^{\sum_{i=1}^{k} r_i} \]
\[ \times \exp \left[ -k \sum_{i=1}^{k} (n_i - r_i) T_i \exp \left( -\theta_1 - \frac{\theta_2}{T_i} - \theta_3 V_i - \frac{\theta_4 V_i}{T_i} \right) \tau_i^{\beta} \right] \]
\[ \times \exp \left[ -k \sum_{i=1}^{k} \sum_{j=1}^{r_i} T_i \exp \left( -\theta_1 - \frac{\theta_2}{T_i} - \theta_3 V_i - \frac{\theta_4 V_i}{T_i} \right) x_{ij}^{\beta} \right] \prod_{i=1}^{k} \prod_{j=1}^{r_i} x_{ij}^{\beta-1} \]

### 4.3 GEW₃ Model

Consider a mixture of uniform and gamma priors for the model parameters, and denote this model by GEW₃. Let \( \beta \) have a gamma prior, and the other parameters all have uniform priors,

\[ \theta_1 \sim U(c_{20}, c_{21}) \quad c_{21} > c_{20} \quad , \pi_3 (\theta_1) \propto \text{constant} \]
\[ \theta_2 \sim U(c_{22}, c_{23}) \quad c_{23} > c_{22} \quad , \pi_3 (\theta_2) \propto \text{constant} \]
\[ \theta_3 \sim U(c_{24}, c_{25}) \quad c_{25} > c_{24} \quad , \pi_3 (\theta_3) \propto \text{constant} \]
\[ \theta_4 \sim U(c_{26}, c_{27}) \quad c_{27} > c_{26} \quad , \pi_3 (\theta_4) \propto \text{constant} \]
\[ \beta \sim \Gamma(c_{28}, c_{29}) \quad c_{29}, c_{29} > 0 \quad , \pi_3 (\beta) \propto \beta^{c_{28}-1} \exp(-c_{29} \beta) . \]
The joint prior for $GEW_3$ is then given by

$$
\pi_3(\theta_1, \theta_2, \theta_3, \theta_4, \beta) \propto \beta^{c_{28}-1} \exp \left( -c_{29} \beta \right),
$$

and using (3.6) and (4.3) the joint posterior is given by

$$
\pi_3(\theta_1, \theta_2, \theta_3, \theta_4, \beta | \mathbf{x}) \propto \beta^{c_{28}-1} \exp \left( -c_{29} \beta \right) \sum_{i=1}^{k} r_i 
\times \exp \left( -\theta_1 \sum_{i=1}^{k} r_i - \theta_2 \sum_{i=1}^{k} \frac{r_i}{T_i} - \theta_3 \sum_{i=1}^{k} r_i V_i - \theta_4 \sum_{i=1}^{k} \frac{r_i V_i}{T_i} \right) 
\times \exp \left[ - \sum_{i=1}^{k} (n_i - r_i) T_i \exp \left( -\theta_1 - \frac{\theta_2}{T_i} - \frac{\theta_3 V_i}{T_i} \right) \tau_i^\beta \right] 
\times \exp \left[ - \sum_{i=1}^{k} \sum_{j=1}^{r_i} T_i \exp \left( -\theta_1 - \frac{\theta_2}{T_i} - \frac{\theta_3 V_i}{T_i} \right) x_{ij}^{\beta} \right] \left[ \prod_{i=1}^{k} \prod_{j=1}^{r_i} T_i x_{ij}^{\beta-1} \right].
$$

MCMC methods is used for posterior inference since the posterior is of an unmanageable form. The full conditional posteriors for this model can be written as

$$
\pi_3(\theta_1 | \mathbf{x}, \theta_2, \theta_3, \theta_4, \beta) \propto \exp \left( -\theta_1 \sum_{i=1}^{k} r_i \right) \exp \left[ - \sum_{i=1}^{k} (n_i - r_i) T_i \exp \left( -\theta_1 - \frac{\theta_2}{T_i} - \frac{\theta_3 V_i}{T_i} \right) \tau_i^\beta \right] 
\times \exp \left[ - \sum_{i=1}^{k} \sum_{j=1}^{r_i} T_i \exp \left( -\theta_1 - \frac{\theta_2}{T_i} - \frac{\theta_3 V_i}{T_i} \right) x_{ij}^{\beta} \right] 
$$

$$
\pi_3(\theta_2 | \mathbf{x}, \theta_1, \theta_3, \theta_4, \beta) \propto \exp \left( -\theta_2 \sum_{i=1}^{k} \frac{r_i}{T_i} \right) \exp \left[ - \sum_{i=1}^{k} (n_i - r_i) T_i \exp \left( -\theta_1 - \frac{\theta_2}{T_i} - \frac{\theta_3 V_i}{T_i} \right) \tau_i^\beta \right] 
\times \exp \left[ - \sum_{i=1}^{k} \sum_{j=1}^{r_i} T_i \exp \left( -\theta_1 - \frac{\theta_2}{T_i} - \frac{\theta_3 V_i}{T_i} \right) x_{ij}^{\beta} \right] 
$$

$$
\pi_3(\theta_3 | \mathbf{x}, \theta_1, \theta_2, \theta_4, \beta) \propto \exp \left( -\theta_3 \sum_{i=1}^{k} r_i V_i \right) \exp \left[ - \sum_{i=1}^{k} (n_i - r_i) T_i \exp \left( -\theta_1 - \frac{\theta_2}{T_i} - \frac{\theta_3 V_i}{T_i} \right) \tau_i^\beta \right] 
\times \exp \left[ - \sum_{i=1}^{k} \sum_{j=1}^{r_i} T_i \exp \left( -\theta_1 - \frac{\theta_2}{T_i} - \frac{\theta_3 V_i}{T_i} \right) x_{ij}^{\beta} \right] 
$$

$$
\pi_3(\theta_4 | \mathbf{x}, \theta_1, \theta_2, \theta_3, \beta) \propto \exp \left( -\theta_4 \sum_{i=1}^{k} \frac{r_i V_i}{T_i} \right) \exp \left[ - \sum_{i=1}^{k} (n_i - r_i) T_i \exp \left( -\theta_1 - \frac{\theta_2}{T_i} - \frac{\theta_3 V_i}{T_i} \right) \tau_i^\beta \right] 
\times \exp \left[ - \sum_{i=1}^{k} \sum_{j=1}^{r_i} T_i \exp \left( -\theta_1 - \frac{\theta_2}{T_i} - \frac{\theta_3 V_i}{T_i} \right) x_{ij}^{\beta} \right].
$$
\[ \pi_3 (\beta | \theta_1, \theta_2, \theta_3, \theta_4) \propto \beta^{c_{28} - 1} \exp \left( -c_{29} \beta \right) \beta^{\sum_{i=1}^{k} r_i} \times \exp \left[ -\sum_{i=1}^{k} (n_i - r_i) T_i \exp \left( -\theta_1 - \frac{\theta_2}{T_i} - \theta_3 V_i - \frac{\theta_4 V_i}{T_i} \right) \tau_i^\beta \right] \times \exp \left[ -\sum_{i=1}^{k} \sum_{j=1}^{r_i} T_i \exp \left( -\theta_1 - \frac{\theta_2}{T_i} - \theta_3 V_i - \frac{\theta_4 V_i}{T_i} \right) x_{ij}^\beta \right] \right] . \]

4.4 \textit{GEW}_4 \text{ Model}

Denote by \textit{GEW}_4 another model where a different mixture of uniform and gamma priors are used. Impose gamma priors on the parameters \( \theta_1, \theta_2, \theta_3 \) and \( \theta_4 \), and let \( \beta \) have a uniform prior,

\[
\begin{align*}
\theta_1 &\sim \Gamma(c_{30}, c_{31}) \quad , c_{30}, c_{31} > 0 \quad , \pi_4 (\theta_1) \propto \theta_1^{c_{30} - 1} \exp(-c_{31} \theta_1) \\
\theta_2 &\sim \Gamma(c_{32}, c_{33}) \quad , c_{32}, c_{33} > 0 \quad , \pi_4 (\theta_2) \propto \theta_2^{c_{32} - 1} \exp(-c_{33} \theta_2) \\
\theta_3 &\sim \Gamma(c_{34}, c_{35}) \quad , c_{34}, c_{35} > 0 \quad , \pi_4 (\theta_3) \propto \theta_3^{c_{34} - 1} \exp(-c_{35} \theta_3) \\
\theta_4 &\sim \Gamma(c_{36}, c_{37}) \quad , c_{36}, c_{37} > 0 \quad , \pi_4 (\theta_4) \propto \theta_4^{c_{36} - 1} \exp(-c_{37} \theta_4) \\
\beta &\sim U(c_{38}, c_{39}) \quad , c_{39} > c_{38} \quad , \pi_4 (\beta) \propto \text{constant}.
\end{align*}
\]

The joint prior for \textit{GEW}_4 is then given by

\[
\pi_4 (\theta_1, \theta_2, \theta_3, \theta_4, \beta) \propto \theta_1^{c_{30} - 1} \theta_2^{c_{32} - 1} \theta_3^{c_{34} - 1} \theta_4^{c_{36} - 1} \exp(-c_{31} \theta_1 - c_{33} \theta_2 - c_{35} \theta_3 - c_{37} \theta_4), \quad (4.4)
\]

and using (3.6) and (4.3) the joint posterior is given by

\[
\begin{align*}
\pi_4 (\theta_1, \theta_2, \theta_3, \theta_4, \beta | \underline{x}) &\propto \theta_1^{c_{30} - 1} \theta_2^{c_{32} - 1} \theta_3^{c_{34} - 1} \theta_4^{c_{36} - 1} \exp(-c_{31} \theta_1 - c_{33} \theta_2 - c_{35} \theta_3 - c_{37} \theta_4) \beta^{\sum_{i=1}^{k} r_i} \\
&\times \exp \left( -\theta_1 \sum_{i=1}^{k} r_i - \theta_2 \sum_{i=1}^{k} r_i T_i - \theta_3 \sum_{i=1}^{k} r_i V_i - \theta_4 \sum_{i=1}^{k} r_i V_i \right) \tau_i^\beta \\
&\times \exp \left[ -\sum_{i=1}^{k} (n_i - r_i) T_i \exp \left( -\theta_1 - \frac{\theta_2}{T_i} - \theta_3 V_i - \frac{\theta_4 V_i}{T_i} \right) \tau_i^\beta \right] \\
&\times \exp \left[ -\sum_{i=1}^{k} \sum_{j=1}^{r_i} T_i \exp \left( -\theta_1 - \frac{\theta_2}{T_i} - \theta_3 V_i - \frac{\theta_4 V_i}{T_i} \right) x_{ij}^\beta \right] \right] .
\end{align*}
\]

The posterior is difficult to work with, consequently MCMC methods are employed to draw posterior samples for inferences. The full conditional posteriors for the \textit{GEW}_4 model is given by

\[
\begin{align*}
\pi_4 (\theta_1 | \underline{x}, \theta_2, \theta_3, \theta_4, \beta) &\propto \theta_1^{c_{30} - 1} \exp(-c_{31} \theta_1) \exp(-\theta_1 \sum_{i=1}^{k} r_i) \\
&\times \exp \left[ -\sum_{i=1}^{k} (n_i - r_i) T_i \exp \left( -\theta_1 - \frac{\theta_2}{T_i} - \theta_3 V_i - \frac{\theta_4 V_i}{T_i} \right) \tau_i^\beta \right] \\
&\times \exp \left[ -\sum_{i=1}^{k} \sum_{j=1}^{r_i} T_i \exp \left( -\theta_1 - \frac{\theta_2}{T_i} - \theta_3 V_i - \frac{\theta_4 V_i}{T_i} \right) x_{ij}^\beta \right] .
\end{align*}
\]
\[
\pi_4 (\theta_2 | x, \theta_1, \theta_3, \theta_4, \beta) \propto \theta_2^{c_{32} - 1} \exp(-c_{33} \theta_2) \exp \left(-\theta_2 \sum_{i=1}^{k} \frac{r_i}{T_i} \right)
\]
\[
	imes \exp \left[-\sum_{i=1}^{k} (n_i - r_i) T_i \exp \left(-\theta_1 - \frac{\theta_2}{T_i} - \theta_3 V_i - \frac{\theta_4 V_i}{T_i} \right) \tau_i^\beta \right]
\]
\[
	imes \exp \left[-\sum_{i=1}^{k} \sum_{j=1}^{r_i} T_i \exp \left(-\theta_1 - \frac{\theta_2}{T_i} - \theta_3 V_i - \frac{\theta_4 V_i}{T_i} \right) x_{ij}^\beta \right]
\]
\[
\pi_4 (\theta_3 | x, \theta_1, \theta_2, \theta_4, \beta) \propto \theta_3^{c_{34} - 1} \exp(-c_{35} \theta_3) \exp \left(-\theta_3 \sum_{i=1}^{k} r_i V_i \right)
\]
\[
	imes \exp \left[-\sum_{i=1}^{k} (n_i - r_i) T_i \exp \left(-\theta_1 - \frac{\theta_2}{T_i} - \theta_3 V_i - \frac{\theta_4 V_i}{T_i} \right) \tau_i^\beta \right]
\]
\[
	imes \exp \left[-\sum_{i=1}^{k} \sum_{j=1}^{r_i} T_i \exp \left(-\theta_1 - \frac{\theta_2}{T_i} - \theta_3 V_i - \frac{\theta_4 V_i}{T_i} \right) x_{ij}^\beta \right]
\]
\[
\pi_4 (\theta_4 | x, \theta_1, \theta_2, \theta_3, \beta) \propto \theta_4^{c_{36} - 1} \exp(-c_{37} \theta_4) \exp \left(-\theta_4 \sum_{i=1}^{k} r_i V_i \right)
\]
\[
	imes \exp \left[-\sum_{i=1}^{k} (n_i - r_i) T_i \exp \left(-\theta_1 - \frac{\theta_2}{T_i} - \theta_3 V_i - \frac{\theta_4 V_i}{T_i} \right) \tau_i^\beta \right]
\]
\[
	imes \exp \left[-\sum_{i=1}^{k} \sum_{j=1}^{r_i} T_i \exp \left(-\theta_1 - \frac{\theta_2}{T_i} - \theta_3 V_i - \frac{\theta_4 V_i}{T_i} \right) x_{ij}^\beta \right]
\]
\[
\pi_4 (\beta | x, \theta_1, \theta_2, \theta_3, \theta_4) \propto \beta^{\sum_{i=1}^{k} r_i} \exp \left[-\sum_{i=1}^{k} (n_i - r_i) T_i \exp \left(-\theta_1 - \frac{\theta_2}{T_i} - \theta_3 V_i - \frac{\theta_4 V_i}{T_i} \right) \tau_i^\beta \right]
\]
\[
	imes \exp \left[-\sum_{i=1}^{k} \sum_{j=1}^{r_i} T_i \exp \left(-\theta_1 - \frac{\theta_2}{T_i} - \theta_3 V_i - \frac{\theta_4 V_i}{T_i} \right) x_{ij}^\beta \right] \prod_{i=1}^{k} \prod_{j=1}^{r_i} x_{ij}^{\beta - 1}
\]

5 Application

An ALT data set in ReliaSoft Corporation (2015) is used for the application of the GEW model. The data relates to failure times (in hours) obtained from an electronics epoxy packaging ALT, where temperature and relative humidity are used as the accelerated stressors. The normal use conditions are \( T_u = 350K \) and \( S_u = 0.3 \). Table I contains the specifications for the priors used in the application. Flat uniform and gamma priors are imposed on all the parameters for the \( GEW_1 \), \( GEW_{2,1} \), \( GEW_{2,2} \), \( GEW_3 \) and \( GEW_4 \) models. Subjective gamma priors, all with mean 5 but different variances, are chosen for the parameters of the \( GEW_{2,3} \), \( GEW_{2,4} \) and \( GEW_{2,5} \) models.
Table 1: Prior specifications.

| Model     | $\theta_1$         | $\theta_2$         | $\theta_3$         | $\theta_4$         | $\beta$         |
|-----------|---------------------|---------------------|---------------------|---------------------|-----------------|
| GEW$_1$   | $U(0,100)$          | $U(0,100)$          | $U(0,100)$          | $U(0,100)$          | $U(0,100)$      |
| GEW$_{2,1}$ | $\Gamma(1,0.00001)$ | $\Gamma(1,0.00001)$ | $\Gamma(1,0.00001)$ | $\Gamma(1,0.00001)$ | $\Gamma(1,0.00001)$ |
| GEW$_{2,2}$ | $\Gamma(1,0.001)$  | $\Gamma(1,0.001)$  | $\Gamma(1,0.001)$  | $\Gamma(1,0.001)$  | $\Gamma(1,0.001)$ |
| GEW$_{2,3}$ | $\Gamma(2.5,0.5)$  | $\Gamma(2.5,0.5)$  | $\Gamma(2.5,0.5)$  | $\Gamma(2.5,0.5)$  | $\Gamma(2.5,0.5)$ |
| GEW$_{2,4}$ | $\Gamma(5,1)$       | $\Gamma(5,1)$       | $\Gamma(5.1)$      | $\Gamma(5,1)$       | $\Gamma(5,1)$      |
| GEW$_{2,5}$ | $\Gamma(25,5)$      | $\Gamma(25,5)$      | $\Gamma(25,5)$      | $\Gamma(25,5)$      | $\Gamma(25,5)$      |
| GEW$_3$    | $U(0,100)$          | $U(0,100)$          | $U(0,100)$          | $U(0,100)$          | $U(0,100)$      |
| GEW$_4$    | $\Gamma(1,0.001)$  | $\Gamma(1,0.001)$  | $\Gamma(1,0.001)$  | $\Gamma(1,0.001)$  | $U(0,100)$      |

The log-concavity of the full conditional posteriors of the GEW models is evaluated and given in the appendix. It is shown that $\pi_z(\theta_1 | \underline{x}, \theta_2, \theta_3, \theta_4, \beta)$, $\pi_z(\theta_2 | \underline{x}, \theta_1, \theta_3, \theta_4, \beta)$, $\pi_z(\theta_3 | \underline{x}, \theta_1, \theta_2, \theta_4, \beta)$, $\pi_z(\theta_4 | \underline{x}, \theta_1, \theta_2, \theta_3, \beta)$, and $\pi_z(\beta | \underline{x}, \theta_1, \theta_2, \theta_3, \theta_4)$ for $z = 1, 2, 3, 4$ are all log-concave, subject to the conditions $c_{10}, c_{12}, c_{14}, c_{16}, c_{30}, c_{32}, c_{34}, c_{36} \geq 1$, and at least one failure occurring. Under these conditions, it is possible to use the adaptive rejection sampling (ARS) method of Gilks & Wild (1992) to sample from the full conditional posteriors at each iteration of the Gibbs sampler. Alternative conditions for log-concavity can also be formulated, but these conditions are difficult to implement. Slice sampling, introduced by Neal (2003), can also be utilised.

Posterior samples are generated for the models using the Bayesian data analysis software OpenBUGS. A single Markov chain is initiated for each model with a burn-in of 50000 iterations. The modified Gelman-Rubin statistic, proposed by Brooks & Gelman (1998), and trace plots are used to confirm that the Markov chains for the above models all converge well in advance of 50000 iterations. Each chain is then run for another 200000 iterations to obtain posterior samples to base inference on.

The deviance information criterion (DIC), proposed by Spiegelhalter et al. (2002), is the most popular measure that is used to compare various models in Bayesian ALTs. The DIC not only takes into consideration the goodness-of-fit for the model, but also penalizes the model for complexity in terms of overparameterisation. The models with the smaller DIC values will typically be favoured above models with larger values for the DIC, but there are other considerations that also need to be taken into account. For a parameter vector $\theta$, with likelihood function $L(x | \theta)$, the deviance can be defined as $D(\theta) = -2 \ln[L(\underline{x} | \theta)]$. Let $\overline{D}$ be the posterior mean of the deviance, and $\hat{D}(\overline{\theta}) = -2 \ln[L(\underline{x} | \overline{\theta})]$, with $\overline{\theta}$ being the posterior mean of $\theta$, be a point estimate for the deviance. The DIC can then be calculated as DIC = $\overline{D} + p_D$, where $p_D$ is the effective number of parameters given by $p_D = \overline{D} - \hat{D}(\overline{\theta})$.

The DIC values and the effective number of parameters for the various GEW models are given in Table 2. The GEW$_1$, GEW$_{2,1}$, GEW$_{2,2}$, GEW$_3$ and GEW$_4$ models have very similar DIC values, with the GEW$_{2,1}$ and GEW$_4$ models exhibiting the lowest DIC. The GEW$_{2,3}$, GEW$_{2,4}$ and GEW$_{2,5}$ models show an increasingly worse fit to the data as subjective priors with smaller variances are implemented. This may be due to the posterior being dominated by the prior, when the prior variance is very small.
Table 2: Deviance information criterion.

| Model   | DIC  | $p_D$ |
|---------|------|-------|
| $GEW_1$ | 228.4| 2.038 |
| $GEW_{2,1}$ | 228.2| 2.033 |
| $GEW_{2,2}$ | 228.4| 2.061 |
| $GEW_{2,3}$ | 231.8| 1.890 |
| $GEW_{2,4}$ | 236.0| 1.705 |
| $GEW_{2,5}$ | 251.8| 1.237 |
| $GEW_3$   | 228.3| 2.048 |
| $GEW_4$   | 228.2| 2.029 |

The summary statistics for the marginal posterior distributions of the GEW models are provided in Table 3. For the models with flat priors, that is $GEW_1$, $GEW_{2,1}$, $GEW_{2,2}$, $GEW_3$ and $GEW_4$, very similar summary statistics are produced. When subjective priors are employed, one can note that the central location measures of the marginal posteriors are progressively, as more certainty is given by the prior, pulled towards the central location of the prior. As the variances are reduced between the priors imposed on the $GEW_{2,3}$, $GEW_{2,4}$ and $GEW_{2,5}$ models, the posterior is dominated to a greater extent by the prior.
The marginal posterior distributions for the GEW models are shown in Figure 1.

Again, it can be
observed that the \( GEW_1, GEW_{2.1}, GEW_{2.2}, GEW_3 \) and \( GEW_4 \) models produce very similar marginal posteriors. The marginal posteriors of the \( GEW_{2.3}, GEW_{2.4} \) and \( GEW_{2.5} \) models show how the density is increasingly concentrated towards the central location of the priors as more prior certainty is conveyed by means of smaller prior variances.
Figure 1: Marginal posterior distributions for the GEW models.
Finally, the aim of these Bayesian ALT models is to obtain the predictive reliability for the item being tested. The predictive reliability at normal use stress levels is given by

\[ R(x_u | x) = \int \int \int \int R(x_u | \theta_1, \theta_2, \theta_3, \theta_4, \beta) \pi(\theta_1, \theta_2, \theta_3, \theta_4, \beta | x) d\theta_1 d\theta_2 d\theta_3 d\theta_4 d\beta, \]

(5.1)

where \( R(x_u | \theta_1, \theta_2, \theta_3, \theta_4, \beta) \) is the Weibull reliability function at use stress levels \( T_u \) and \( S_u \).

To evaluate \( R(x_u | x) \), the following is done:

1. Sample \( \theta_1, \theta_2, \theta_3, \theta_4 \) and \( \beta \) from the posterior \( M \) times, where \( M \) is a large number.
2. Calculate the integral in (5.1) by the Monte Carlo average

\[ R(x_u | x) \approx \frac{1}{M} \sum_{m=1}^{M} R(x_u | \theta_1^{(m)}, \theta_2^{(m)}, \theta_3^{(m)}, \theta_4^{(m)}, \beta^{(m)}) , \]

which is the expected reliability at time \( x_u \), using the posterior sample \( \{ \theta_1^{(m)}, \theta_2^{(m)}, \theta_3^{(m)}, \theta_4^{(m)}, \beta^{(m)} \} \), \( m = 1, \ldots, M \).

Table 4 and Figure 2 show the predictive reliability of the models. Consistent with the previous findings, the predictive reliability results for the \( GEW_1, GEW_{2,1}, GEW_{2,2}, GEW_3 \) and \( GEW_4 \) models are comparable. The \( GEW \) model is not sensitive to different choices of flat priors. The models that make use of subjective priors produce significantly higher predictive reliability results in this case, compared to the models where flat priors are used. The use of subjective priors can result in either an underestimation or overestimation of the predictive reliability, compared to data driven results obtained from employing flat priors, depending on the choice of hyperparameters for the gamma priors in the \( GEW_2 \) model.

| Time | \( GEW_1 \) | \( GEW_{2,1} \) | \( GEW_{2,2} \) | \( GEW_{2,3} \) | \( GEW_{2,4} \) | \( GEW_{2,5} \) | \( GEW_3 \) | \( GEW_4 \) |
|------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|
| 1    | 0.999987    | 0.999991    | 0.999990    | 0.999990    | 0.999998    | 1.000000    | 0.999990    | 0.999998    |
| 2    | 0.999968    | 0.999976    | 0.999974    | 0.999974    | 0.999994    | 1.000000    | 0.999974    | 0.9999973   |
| 3    | 0.999943    | 0.999957    | 0.999954    | 0.999956    | 0.999989    | 1.000000    | 0.999954    | 0.9999952   |
| 500  | 0.826214    | 0.845024    | 0.836377    | 0.894309    | 0.952155    | 0.997263    | 0.840580    | 0.837681    |
| 501  | 0.825665    | 0.845191    | 0.835848    | 0.893973    | 0.951989    | 0.997251    | 0.840064    | 0.837159    |
| 502  | 0.825115    | 0.844013    | 0.835319    | 0.893638    | 0.951823    | 0.997239    | 0.839547    | 0.836637    |
| 2000 | 0.179376    | 0.193196    | 0.183626    | 0.369133    | 0.586761    | 0.946555    | 0.186418    | 0.187901    |
| 2001 | 0.179176    | 0.192981    | 0.183418    | 0.368884    | 0.586517    | 0.946499    | 0.186206    | 0.187692    |
| 2002 | 0.178977    | 0.192766    | 0.183210    | 0.368636    | 0.586273    | 0.946442    | 0.185995    | 0.187482    |

Table 4: Predictive reliability at use stress \( T_u = 350, S_u = 0.3 \).
Predictive reliability for the GEW models

![Graph showing predictive reliability for different GEW models.](image)

Figure 2: Predictive reliability at use stress $T_u = 350, S_u = 0.3$.

### 6 Conclusions

In this paper, a Bayesian ALT model is presented, where the generalised Eyring model is used as the TTF and lifetimes follow a Weibull distribution. This is a versatile ALT model which allows for more than one accelerated stressor, whereas many other TTFs only permit the use of a single accelerated stressor. A general likelihood formulation is given, which allows for complete samples, type-I censoring and type-II censoring. Different GEW models are defined by means of the choice of prior distributions for the model parameters. The full conditional posteriors for each model are given and the log-concavity for each is also assessed. Under no, to very lenient, conditions for the various models, the ARS method within the Gibbs sampler can be used to obtain posterior samples from the full conditional distributions. Alternatively, since these are complex models, slice sampling can also be used to obtain posterior samples. The models are applied to a data set concerning an electronics epoxy packaging ALT, where temperature and relative humidity are used as the accelerated stressors, and the results are compared.

Various choices for the hyper-parameters of the $GEW_2$ model are considered. OpenBUGS is used to generate posterior samples for the models, and the modified Gelman-Rubin statistic is calculated to assess the convergence of the Markov chains. The fit of the models are compared via the DIC. The $GEW_1, GEW_{2,1}, GEW_{2,2}, GEW_3$ and $GEW_4$ models show very similar DIC. The five models where flat priors are imposed on the parameters produce alike summary statistics, marginal posteriors and
predictive reliability results. The use of subjective priors in the $GEW_{2,3}$, $GEW_{2,4}$ and $GEW_{2,5}$ models lead to much different results, as noted in the summary statistics, marginal posteriors and significantly higher predictive reliability. Subjective priors can thus be utilised to adjust reliability estimates if the researcher is of the opinion that the use of flat priors either overestimates or underestimates the predictive reliability. This can be achieved by means of the choice of hyper-parameters for the gamma priors in the $GEW_2$ model. It may also be possible to adjust a single parameter that is related to a specific stress, by means of a subjective prior, if the researcher is of the opinion that the effect of the accelerated stress is not correctly expressed by a model where only flat priors are used. The use of priors with very small variance in the $GEW_2$ model is not recommended and great caution should be exercised by the reliability engineer if doing so.

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Appendix

In this appendix, the log-concavity of the full conditional posteriors for the GEW models in Section 4 will be discussed. The log-concavity of the $GEW_1$ and $GEW_2$ models are evaluated. The results can then easily be extended to the $GEW_3$ and $GEW_4$ models.

A twice-differentiable function $f(x)$ is said to be log-concave if the second derivative of its natural log is non-positive on its domain (see, for example, Bagnoli & Bergstrom, 2005), thus if

$$\frac{\partial^2 \ln[f(x)]}{\partial x^2} \leq 0 \quad \forall x.$$ 

**Theorem 1.** The full conditional posterior distributions of the $GEW_1$ model are all log-concave on their domains.
Proof. For the GEW\(_1\) model, the second derivatives of the natural logs of the full conditional posteriors are determined as follows

\[
\ell_{1, \theta_1} = \ln \left[ \pi_1 (\theta_1 | x, \theta_2, \theta_3, \theta_4, \beta) \right]
\]
\[
= -\theta_1 \sum_{i=1}^k r_i - \sum_{i=1}^k (n_i - r_i) T_i \exp \left( -\theta_1 - \frac{\theta_2}{T_i} - \theta_3 V_i - \frac{\theta_4 V_i}{T_i} \right) \tau_i^3
\]
\[
- \sum_{i=1}^k \sum_{j=1}^k T_i \exp \left( -\theta_1 - \frac{\theta_2}{T_i} - \theta_3 V_i - \frac{\theta_4 V_i}{T_i} \right) x_{ij}^\beta
\]
\[
\frac{\partial \ell_{1, \theta_1}}{\partial \theta_1} = -\sum_{i=1}^k r_i + \sum_{i=1}^k (n_i - r_i) T_i \exp \left( -\theta_1 - \frac{\theta_2}{T_i} - \theta_3 V_i - \frac{\theta_4 V_i}{T_i} \right) \tau_i^3
\]
\[
+ \sum_{i=1}^k \sum_{j=1}^k T_i \exp \left( -\theta_1 - \frac{\theta_2}{T_i} - \theta_3 V_i - \frac{\theta_4 V_i}{T_i} \right) x_{ij}^\beta
\]
\[
\frac{\partial^2 \ell_{1, \theta_1}}{\partial \theta_1^2} = -\sum_{i=1}^k (n_i - r_i) T_i \exp \left( -\theta_1 - \frac{\theta_2}{T_i} - \theta_3 V_i - \frac{\theta_4 V_i}{T_i} \right) \tau_i^3
\]
\[
- \sum_{i=1}^k \sum_{j=1}^k T_i \exp \left( -\theta_1 - \frac{\theta_2}{T_i} - \theta_3 V_i - \frac{\theta_4 V_i}{T_i} \right) x_{ij}^\beta
\]

\[
\ell_{1, \theta_2} = \ln \left[ \pi_1 (\theta_2 | x, \theta_1, \theta_3, \theta_4, \beta) \right]
\]
\[
= -\theta_2 \sum_{i=1}^k \frac{r_i}{T_i} - \sum_{i=1}^k (n_i - r_i) T_i \exp \left( -\theta_1 - \frac{\theta_2}{T_i} - \theta_3 V_i - \frac{\theta_4 V_i}{T_i} \right) \tau_i^3
\]
\[
- \sum_{i=1}^k \sum_{j=1}^k T_i \exp \left( -\theta_1 - \frac{\theta_2}{T_i} - \theta_3 V_i - \frac{\theta_4 V_i}{T_i} \right) x_{ij}^\beta
\]
\[
\frac{\partial \ell_{1, \theta_2}}{\partial \theta_2} = -\sum_{i=1}^k \frac{r_i}{T_i} + \sum_{i=1}^k (n_i - r_i) \exp \left( -\theta_1 - \frac{\theta_2}{T_i} - \theta_3 V_i - \frac{\theta_4 V_i}{T_i} \right) \tau_i^3
\]
\[
+ \sum_{i=1}^k \sum_{j=1}^k \exp \left( -\theta_1 - \frac{\theta_2}{T_i} - \theta_3 V_i - \frac{\theta_4 V_i}{T_i} \right) x_{ij}^\beta
\]
\[
\frac{\partial^2 \ell_{1, \theta_2}}{\partial \theta_2^2} = -\sum_{i=1}^k (n_i - r_i) \frac{1}{T_i} \exp \left( -\theta_1 - \frac{\theta_2}{T_i} - \theta_3 V_i - \frac{\theta_4 V_i}{T_i} \right) \tau_i^3
\]
\[
- \sum_{i=1}^k \sum_{j=1}^k \frac{1}{T_i} \exp \left( -\theta_1 - \frac{\theta_2}{T_i} - \theta_3 V_i - \frac{\theta_4 V_i}{T_i} \right) x_{ij}^\beta
\]
\[ \ell_{1,\theta_3} = \ln \left[ \pi_1 \left( \theta_3 \mid \mathbf{x}, \theta_1, \theta_2, \theta_4, \beta \right) \right] \\
= -\theta_3 \sum_{i=1}^{k} r_i V_i - \sum_{i=1}^{k} (n_i - r_i) T_i \exp \left( -\theta_1 \frac{T_i}{T_i} - \theta_3 V_i - \frac{\theta_4 V_i}{T_i} \right) \tau_i^\beta \\
- \sum_{i=1}^{k} \sum_{j=1}^{k} T_i \exp \left( -\theta_1 \frac{T_i}{T_i} - \theta_3 V_i - \frac{\theta_4 V_i}{T_i} \right) x_{ij}^\beta \\
\frac{\partial \ell_{1,\theta_3}}{\partial \theta_3} = -\sum_{i=1}^{k} r_i V_i + \sum_{i=1}^{k} (n_i - r_i) T_i V_i \exp \left( -\theta_1 \frac{T_i}{T_i} - \theta_3 V_i - \frac{\theta_4 V_i}{T_i} \right) \tau_i^\beta \\
+ \sum_{i=1}^{k} \sum_{j=1}^{k} T_i V_i \exp \left( -\theta_1 \frac{T_i}{T_i} - \theta_3 V_i - \frac{\theta_4 V_i}{T_i} \right) x_{ij}^\beta \\
\frac{\partial^2 \ell_{1,\theta_3}}{\partial \theta_3^2} = -\sum_{i=1}^{k} (n_i - r_i) T_i V_i^2 \exp \left( -\theta_1 \frac{T_i}{T_i} - \theta_3 V_i - \frac{\theta_4 V_i}{T_i} \right) \tau_i^\beta \\
- \sum_{i=1}^{k} \sum_{j=1}^{k} T_i V_i^2 \exp \left( -\theta_1 \frac{T_i}{T_i} - \theta_3 V_i - \frac{\theta_4 V_i}{T_i} \right) x_{ij}^\beta \\
\ell_{1,\theta_4} = \ln \left[ \pi_1 \left( \theta_4 \mid \mathbf{x}, \theta_1, \theta_2, \theta_3, \beta \right) \right] \\
= -\theta_4 \sum_{i=1}^{k} \frac{r_i V_i}{T_i} - \sum_{i=1}^{k} (n_i - r_i) T_i \exp \left( -\theta_1 \frac{T_i}{T_i} - \theta_3 V_i - \frac{\theta_4 V_i}{T_i} \right) \tau_i^\beta \\
- \sum_{i=1}^{k} \sum_{j=1}^{k} T_i \exp \left( -\theta_1 \frac{T_i}{T_i} - \theta_3 V_i - \frac{\theta_4 V_i}{T_i} \right) x_{ij}^\beta \\
\frac{\partial \ell_{1,\theta_4}}{\partial \theta_4} = -\sum_{i=1}^{k} \frac{r_i V_i}{T_i} + \sum_{i=1}^{k} (n_i - r_i) V_i \exp \left( -\theta_1 \frac{T_i}{T_i} - \theta_3 V_i - \frac{\theta_4 V_i}{T_i} \right) \tau_i^\beta \\
+ \sum_{i=1}^{k} \sum_{j=1}^{k} V_i \exp \left( -\theta_1 \frac{T_i}{T_i} - \theta_3 V_i - \frac{\theta_4 V_i}{T_i} \right) x_{ij}^\beta \\
\frac{\partial^2 \ell_{1,\theta_4}}{\partial \theta_4^2} = -\sum_{i=1}^{k} \frac{(n_i - r_i) V_i^2}{T_i} \exp \left( -\theta_1 \frac{T_i}{T_i} - \theta_3 V_i - \frac{\theta_4 V_i}{T_i} \right) \tau_i^\beta \\
- \sum_{i=1}^{k} \sum_{j=1}^{k} \frac{V_i^2}{T_i} \exp \left( -\theta_1 \frac{T_i}{T_i} - \theta_3 V_i - \frac{\theta_4 V_i}{T_i} \right) x_{ij}^\beta \]
\[ \ell_{1,\beta} = \ln \left[ \pi_1(\beta | x, \theta_1, \theta_2, \theta_3, \theta_4) \right] \]
\[ = \ln (\beta) \sum_{i=1}^{k} r_i - \sum_{i=1}^{k} (n_i - r_i) T_i \exp \left( -\theta_1 - \frac{\theta_2}{T_i} - \theta_3 V_i - \frac{\theta_4 V_i}{T_i} \right) \tau_i^\beta \]
\[ - \sum_{i=1}^{k} \sum_{j=1}^{r_i} T_i \exp \left( -\theta_1 - \frac{\theta_2}{T_i} - \theta_3 V_i - \frac{\theta_4 V_i}{T_i} \right) \tau_i^\beta \sum_{j=1}^{r_i} \ln (x_{ij}) \]
\[ \frac{\partial \ell_{1,\beta}}{\partial \beta} = \frac{1}{\beta} \sum_{i=1}^{k} r_i - \sum_{i=1}^{k} (n_i - r_i) T_i \exp \left( -\theta_1 - \frac{\theta_2}{T_i} - \theta_3 V_i - \frac{\theta_4 V_i}{T_i} \right) \tau_i^\beta \ln (\tau_i) \]
\[ - \sum_{i=1}^{k} \sum_{j=1}^{r_i} T_i \exp \left( -\theta_1 - \frac{\theta_2}{T_i} - \theta_3 V_i - \frac{\theta_4 V_i}{T_i} \right) \tau_i^\beta \ln (x_{ij}) + \sum_{i=1}^{k} \sum_{j=1}^{r_i} \ln (x_{ij}) \]
\[ \frac{\partial^2 \ell_{1,\beta}}{\partial \beta^2} = -\frac{1}{\beta^2} \sum_{i=1}^{k} r_i - \sum_{i=1}^{k} (n_i - r_i) T_i \exp \left( -\theta_1 - \frac{\theta_2}{T_i} - \theta_3 V_i - \frac{\theta_4 V_i}{T_i} \right) \tau_i^\beta \ln^2 (\tau_i) \]
\[ - \sum_{i=1}^{k} \sum_{j=1}^{r_i} T_i \exp \left( -\theta_1 - \frac{\theta_2}{T_i} - \theta_3 V_i - \frac{\theta_4 V_i}{T_i} \right) \tau_i^\beta \ln^2 (x_{ij}) \]

Since \( T_i \geq 0 \) (temperature measured in kelvin), \( r_i \leq n_i \) (can not be more failures than items tested), \( \tau_i \geq 0 \) (censoring time), \( x_{ij} \geq 0 \) (failure time), \( \beta > 0 \) (shape parameter of the Weibull distribution), \( \exp(\cdot) \geq 0, V_i^2 \geq 0 \) and \( \ln^2(\cdot) \geq 0 \), the full conditional posteriors for the GEW\(_1\) model are confirmed to be log-concave on their domains. \( \Box \)

**Theorem 2.** The full conditional posterior distributions of the GEW\(_2\) model are all log-concave on their domains, subject to \( c_{10}, c_{12}, c_{14}, c_{16}, \sum_{i=1}^{k} r_i \geq 1 \).

**Proof.** The second derivatives of the natural logs of the full conditional posteriors for the GEW\(_2\) model are given by

\[ \ell_{2,\theta_1} = \ln \left[ \pi_2(\theta_1 | x, \theta_2, \theta_3, \theta_4, \beta) \right] \]
\[ = (c_{10} - 1) \ln (\theta_1) - c_{11} \theta_1 - \sum_{i=1}^{k} r_i - \sum_{i=1}^{k} (n_i - r_i) T_i \exp \left( -\theta_1 - \frac{\theta_2}{T_i} - \theta_3 V_i - \frac{\theta_4 V_i}{T_i} \right) \tau_i^\beta \]
\[ - \sum_{i=1}^{k} \sum_{j=1}^{r_i} T_i \exp \left( -\theta_1 - \frac{\theta_2}{T_i} - \theta_3 V_i - \frac{\theta_4 V_i}{T_i} \right) \tau_i^\beta \]
\[ \frac{\partial \ell_{2,\theta_1}}{\partial \theta_1} = \frac{c_{10} - 1}{\theta_1} - c_{11} - \sum_{i=1}^{k} r_i + \sum_{i=1}^{k} (n_i - r_i) T_i \exp \left( -\theta_1 - \frac{\theta_2}{T_i} - \theta_3 V_i - \frac{\theta_4 V_i}{T_i} \right) \tau_i^\beta \]
\[ + \sum_{i=1}^{k} \sum_{j=1}^{r_i} T_i \exp \left( -\theta_1 - \frac{\theta_2}{T_i} - \theta_3 V_i - \frac{\theta_4 V_i}{T_i} \right) \tau_i^\beta \]
\[ \frac{\partial^2 \ell_{2,\theta_1}}{\partial \theta_1^2} = \frac{1 - c_{10}}{\theta_1^2} - \sum_{i=1}^{k} (n_i - r_i) T_i \exp \left( -\theta_1 - \frac{\theta_2}{T_i} - \theta_3 V_i - \frac{\theta_4 V_i}{T_i} \right) \tau_i^\beta \]
\[ - \sum_{i=1}^{k} \sum_{j=1}^{r_i} T_i \exp \left( -\theta_1 - \frac{\theta_2}{T_i} - \theta_3 V_i - \frac{\theta_4 V_i}{T_i} \right) \tau_i^\beta \]

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\[ \ell_{2, \theta_2} = \ln [\pi_2 (\theta_2 | x, \theta_1, \theta_3, \theta_4, \beta)] \]
\[ = (c_{12} - 1) \ln (\theta_2) - c_{13} \theta_2 - \theta_2 \sum_{i=1}^{k} \frac{r_i}{T_i} - \sum_{i=1}^{k} (n_i - r_i) T_i \exp \left( -\theta_1 - \frac{\theta_2}{T_i} - \theta_3 V_i - \frac{\theta_4 V_i}{T_i} \right) \tau_i^\beta \]
\[ - \sum_{i=1}^{k} \sum_{j=1}^{r_i} T_i \exp \left( -\theta_1 - \frac{\theta_2}{T_i} - \theta_3 V_i - \frac{\theta_4 V_i}{T_i} \right) x_{ij}^\beta \]
\[ \frac{\partial \ell_{2, \theta_2}}{\partial \theta_2} = \frac{c_{12} - 1}{\theta_2} - c_{13} - \sum_{i=1}^{k} \frac{r_i}{T_i} + \sum_{i=1}^{k} (n_i - r_i) \exp \left( -\theta_1 - \frac{\theta_2}{T_i} - \theta_3 V_i - \frac{\theta_4 V_i}{T_i} \right) \tau_i^\beta \]
\[ + \sum_{i=1}^{k} \sum_{j=1}^{r_i} \exp \left( -\theta_1 - \frac{\theta_2}{T_i} - \theta_3 V_i - \frac{\theta_4 V_i}{T_i} \right) x_{ij}^\beta \]
\[ \frac{\partial^2 \ell_{2, \theta_2}}{\partial \theta_2^2} = \frac{1 - c_{12}}{\theta_2^2} - \sum_{i=1}^{k} (n_i - r_i) \frac{1}{T_i} \exp \left( -\theta_1 - \frac{\theta_2}{T_i} - \theta_3 V_i - \frac{\theta_4 V_i}{T_i} \right) \tau_i^\beta \]
\[ - \sum_{i=1}^{k} \sum_{j=1}^{r_i} \frac{1}{T_i} \exp \left( -\theta_1 - \frac{\theta_2}{T_i} - \theta_3 V_i - \frac{\theta_4 V_i}{T_i} \right) x_{ij}^\beta \]
\[ \ell_{2, \theta_3} = \ln [\pi_2 (\theta_3 | x, \theta_1, \theta_2, \theta_4, \beta)] \]
\[ = (c_{14} - 1) \ln (\theta_3) - c_{15} \theta_3 - \theta_3 \sum_{i=1}^{k} r_i V_i - \sum_{i=1}^{k} (n_i - r_i) T_i \exp \left( -\theta_1 - \frac{\theta_2}{T_i} - \theta_3 V_i - \frac{\theta_4 V_i}{T_i} \right) \tau_i^\beta \]
\[ - \sum_{i=1}^{k} \sum_{j=1}^{r_i} T_i \exp \left( -\theta_1 - \frac{\theta_2}{T_i} - \theta_3 V_i - \frac{\theta_4 V_i}{T_i} \right) x_{ij}^\beta \]
\[ \frac{\partial \ell_{2, \theta_3}}{\partial \theta_3} = \frac{c_{14} - 1}{\theta_3} - c_{15} - \sum_{i=1}^{k} r_i V_i + \sum_{i=1}^{k} (n_i - r_i) T_i V_i \exp \left( -\theta_1 - \frac{\theta_2}{T_i} - \theta_3 V_i - \frac{\theta_4 V_i}{T_i} \right) \tau_i^\beta \]
\[ + \sum_{i=1}^{k} \sum_{j=1}^{r_i} T_i V_i \exp \left( -\theta_1 - \frac{\theta_2}{T_i} - \theta_3 V_i - \frac{\theta_4 V_i}{T_i} \right) x_{ij}^\beta \]
\[ \frac{\partial^2 \ell_{2, \theta_3}}{\partial \theta_3^2} = \frac{1 - c_{14}}{\theta_3^2} - \sum_{i=1}^{k} (n_i - r_i) T_i V_i^2 \exp \left( -\theta_1 - \frac{\theta_2}{T_i} - \theta_3 V_i - \frac{\theta_4 V_i}{T_i} \right) \tau_i^\beta \]
\[ - \sum_{i=1}^{k} \sum_{j=1}^{r_i} T_i V_i^2 \exp \left( -\theta_1 - \frac{\theta_2}{T_i} - \theta_3 V_i - \frac{\theta_4 V_i}{T_i} \right) x_{ij}^\beta \]
\[ \ell_{2, \theta_4} = \ln [\pi_2 (\theta_4 \mid x, \theta_1, \theta_2, \theta_3, \beta)] \]
\[ = (c_{16} - 1) \ln (\theta_4) - c_{17} \theta_4 - \theta_4 \sum_{i=1}^{k} \frac{r_i V_i}{T_i} - \sum_{i=1}^{k} (n_i - r_i) T_i \exp \left( -\theta_1 - \frac{\theta_2}{T_i} - \theta_3 V_i - \frac{\theta_4 V_i}{T_i} \right) \tau_i^\beta \]
\[ - \sum_{i=1}^{k} \sum_{j=1}^{k} T_i \exp \left( -\theta_1 - \frac{\theta_2}{T_i} - \theta_3 V_i - \frac{\theta_4 V_i}{T_i} \right) x_{ij}^\beta \]
\[ \frac{\partial \ell_{2, \theta_4}}{\partial \theta_4} = \frac{c_{16} - 1}{\theta_4} - c_{17} - \sum_{i=1}^{k} \frac{r_i V_i}{T_i} + \sum_{i=1}^{k} (n_i - r_i) V_i \exp \left( -\theta_1 - \frac{\theta_2}{T_i} - \theta_3 V_i - \frac{\theta_4 V_i}{T_i} \right) \tau_i^\beta \]
\[ + \sum_{i=1}^{k} \sum_{j=1}^{k} V_i \exp \left( -\theta_1 - \frac{\theta_2}{T_i} - \theta_3 V_i - \frac{\theta_4 V_i}{T_i} \right) x_{ij}^\beta \]
\[ \frac{\partial^2 \ell_{2, \theta_4}}{\partial \theta_4^2} = \frac{1 - c_{16}}{\theta_4^2} - \sum_{i=1}^{k} (n_i - r_i) \frac{V_i^2}{T_i} \exp \left( -\theta_1 - \frac{\theta_2}{T_i} - \theta_3 V_i - \frac{\theta_4 V_i}{T_i} \right) \tau_i^\beta \]
\[ - \sum_{i=1}^{k} \sum_{j=1}^{k} \frac{V_i^2}{T_i} \exp \left( -\theta_1 - \frac{\theta_2}{T_i} - \theta_3 V_i - \frac{\theta_4 V_i}{T_i} \right) x_{ij}^\beta \]
\[ \ell_{2, \beta} = \ln [\pi_2 (\beta \mid x, \theta_1, \theta_2, \theta_3, \theta_4)] \]
\[ = (c_{18} - 1) \ln (\beta) - c_{19} \beta + \ln (\beta) \sum_{i=1}^{k} r_i - \sum_{i=1}^{k} (n_i - r_i) T_i \exp \left( -\theta_1 - \frac{\theta_2}{T_i} - \theta_3 V_i - \frac{\theta_4 V_i}{T_i} \right) \tau_i^\beta \]
\[ - \sum_{i=1}^{k} \sum_{j=1}^{k} T_i \exp \left( -\theta_1 - \frac{\theta_2}{T_i} - \theta_3 V_i - \frac{\theta_4 V_i}{T_i} \right) x_{ij}^\beta + (\beta - 1) \sum_{i=1}^{k} \sum_{j=1}^{k} \ln (x_{ij}) \]
\[ \frac{\partial \ell_{2, \beta}}{\partial \beta} = \frac{c_{18} - 1}{\beta} - c_{19} + \frac{1}{\beta} \sum_{i=1}^{k} r_i - \frac{1}{\beta} \sum_{i=1}^{k} (n_i - r_i) T_i \exp \left( -\theta_1 - \frac{\theta_2}{T_i} - \theta_3 V_i - \frac{\theta_4 V_i}{T_i} \right) \tau_i^\beta \ln (\tau_i) \]
\[ - \sum_{i=1}^{k} \sum_{j=1}^{k} T_i \exp \left( -\theta_1 - \frac{\theta_2}{T_i} - \theta_3 V_i - \frac{\theta_4 V_i}{T_i} \right) x_{ij}^\beta \ln (x_{ij}) + \sum_{i=1}^{k} \sum_{j=1}^{k} \ln (x_{ij}) \]
\[ \frac{\partial^2 \ell_{2, \beta}}{\partial \beta^2} = \frac{1 - c_{18}}{\beta^2} - \frac{1}{\beta^2} \sum_{i=1}^{k} r_i - \frac{1}{\beta^2} \sum_{i=1}^{k} (n_i - r_i) T_i \exp \left( -\theta_1 - \frac{\theta_2}{T_i} - \theta_3 V_i - \frac{\theta_4 V_i}{T_i} \right) \tau_i^\beta \ln^2 (\tau_i) \]
\[ - \sum_{i=1}^{k} \sum_{j=1}^{k} T_i \exp \left( -\theta_1 - \frac{\theta_2}{T_i} - \theta_3 V_i - \frac{\theta_4 V_i}{T_i} \right) x_{ij}^\beta \ln^2 (x_{ij}) \].

Since \( T_i \geq 0 \) (temperature measured in kelvin), \( r_i \leq n_i \) (can not be more failures than items tested), \( \tau_i \geq 0 \) (censoring time), \( x_{ij} \geq 0 \) (failure time), \( \beta > 0 \) (shape parameter of the Weibull distribution), \( \exp (\cdot) \geq 0 \), \( V_i^2 \geq 0 \) and \( \ln^2 (\cdot) \geq 0 \), the full conditional posteriors for the GEW_2 model are log-concave on their domains, subject to \( c_{10}, c_{12}, c_{14}, c_{16}, \sum_{i=1}^{k} r_i \geq 1 \).

**Theorem 3.** The full conditional posterior distributions of the GEW_3 model are all log-concave on their domains, when \( \sum_{i=1}^{k} r_i \geq 1 \). The same holds for the full conditional posterior distributions of the GEW_4 model, when \( c_{30}, c_{32}, c_{34}, c_{36} \geq 1 \).
Proof. Following the same reasoning as in Theorem 1 and Theorem 2, it is easy to show that the full conditional posteriors for the GEW\(_3\) and GEW\(_4\) models are log-concave on their domains subject to the conditions \(\sum_{i=1}^{k} r_i \geq 1\) and \(c_{30}, c_{32}, c_{34}, c_{36} \geq 1\), respectively. \(\square\)