How definitive is the standard interpretation of Gödel's Incompleteness Theorem?

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Standard interpretations of Gödel's "undecidable" proposition, \((\forall x)R(x)\), argue that, although \(\neg(\forall x)R(x)\) is PA-provable if \((\forall x)R(x)\) is PA-provable, we may not conclude from this that \(\neg(\forall x)R(x)\) is PA-provable. We show that such interpretations are inconsistent with a standard Deduction Theorem of first order theories.

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1. Introduction

In his seminal 1931 paper [Go31a], Gödel meta-mathematically argues that his "undecidable" proposition, \((\forall x)R(x)\)\(^1\), is such that (cf. [An02b], §1.6(iv)):

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\(^1\) We use square brackets to differentiate between a formal expression \([F]\) and its interpretation "F", where we follow Mendelson's definition of an interpretation \(M\) of a formal theory \(K\), and of the interpretation of a formula of \(K\) under \(M\) ([Me64], p49, §2).
If \([(A \forall x)R(x)]\) is PA-provable, then \[\neg(A \forall x)R(x)]\) is PA-provable.

Now, a standard Deduction Theorem of an arbitrary first order theory states that ([Me64], p61, Corollary 2.6):

If \(T\) is a set of well-formed formulas of an arbitrary first order theory \(K\), and if \(A\) is a closed well-formed formula of \(K\), and if \((T, [A])\vDash_K [B]\), then \(T \vdash_K ([A \Rightarrow B])\).

In an earlier paper ([An02b], Appendix 1), we implicitly assumed, without proof, that:

\[(T, [A])\vDash_K [B]\] holds if, and only if, \(T \vdash_K [B]\) holds when we assume \(T \vdash_K [A]\). (*)

In other words, we assumed that \([B]\) is a deduction from \((T, [A])\) in \(K\) if, and only if, whenever \([A]\)^2 is a hypothetical deduction from \(T\) in \(K\), \([B]\) is a deduction from \(T\) in \(K\).

We then argued, that it should follow (essentially by the reasoning in §2.2 below), that:

\[[(A \forall x)R(x) \Rightarrow \neg(A \forall x)R(x)]\] is PA-provable,

and, therefore, that:

\[\neg(A \forall x)R(x)]\) is PA-provable.

We then concluded that PA is omega-inconsistent. However, these conclusions are inconsistent with standard interpretations of Gödel’s reasoning, which, firstly, assert both \([(A \forall x)R(x)]\) and \[\neg(A \forall x)R(x)]\) as PA-unprovable, and, secondly, assume that PA can be omega-consistent. Such interpretations, therefore, implicitly deny that the PA-provability of \[\neg(A \forall x)R(x)]\ can be inferred from the above meta-argument; ipso facto, they imply that (*) is false.

\(^2\) For the purposes of this paper, we assume everywhere that \([A]\) is a closed well-formed formula of \(K\).
In the following sections, we review the Deduction Theorems used in the earlier argument, and give a meta-mathematical proof of (*). It follows that the standard interpretations of Gödel’s reasoning are inconsistent with a standard Deduction Theorem of an arbitrary first order theory ([Me64], p61, Corollary 2.6). We conclude that such interpretations cannot be accepted as definitive.

1.1 An overview

We first review, in Theorem 1, the proof of a standard Deduction Theorem, if $(T, [A]) \vdash_K [B]$, then $T \vdash_K [A \Rightarrow B]$, where an explicit deduction of $[B]$ from $(T, [A])$ is known.

We then show, in Corollary 1.2, that Theorem 1 can be constructively extended to cases where $(T, [A]) \vdash_K [B]$ is established meta-mathematically, and where an explicit deduction of $[B]$ from $(T, [A])$ is not known.

We finally prove (*) in Theorem 2.

2. A standard Deduction Theorem

The following is, essentially, Mendelson’s proof of a standard Deduction Theorem ([Me64], p61, Proposition 2.4) of an arbitrary first order theory $K$:

**Theorem 1**: If $T$ is a set of well-formed formulas of an arbitrary first order theory $K$, and if $[A]$ is a closed well-formed formula of $K$, and if $(T, [A]) \vdash_K [B]$, then $T \vdash_K [A \Rightarrow B]$.

**Proof**: Let $<[B_1], [B_2], \ldots, [B_n]>$ be a deduction of $[B]$ from $(T, [A])$ in $K$.

Then, by definition, $[B_n]$ is $[B]$ and, for each $i$, either $[B_i]$ is an axiom of $K$, or $[B_i]$ is in $T$, or $[B_i]$ is $[A]$, or $[B_i]$ is a direct consequence by some rules of inference of $K$ of some of the preceding well-formed formulas in the sequence.
We now show, by induction, that $T \vdash_K [A \Rightarrow B_i]$ for each $i < n$. As inductive hypothesis, we assume that the proposition is true for all deductions of length less than $n$.

(i) If $[B_i]$ is an axiom, or belongs to $T$, then $T \vdash_K [A \Rightarrow B_i]$, since $[B_i \Rightarrow (A \Rightarrow B_i)]$ is an axiom of K.

(ii) If $[B_i]$ is $[A]$, then $T \vdash_K [A \Rightarrow B_i]$, since $T \vdash_K [A \Rightarrow A]$.

(iii) If there exist $j$, $k$ less than $i$ such that $[B_k]$ is $[B_j \Rightarrow B_i]$, then, by the inductive hypothesis, $T \vdash_K [A \Rightarrow B_j]$, and $T \vdash_K [A \Rightarrow (B_j \Rightarrow B_i)]$. Hence, $T \vdash_K [A \Rightarrow B_i]$.

(iv) Finally, suppose there is some $j < i$ such that $[B_i]$ is $[(A \chi)B_j]$, where $\chi$ is a variable in K. By hypothesis, $T \vdash_K [A \Rightarrow B_j]$. Since $\chi$ is not a free variable of $[A]$, we have that $[(A \chi)(A \Rightarrow B_j) \Rightarrow (A \Rightarrow (A \chi)B_j)]$ is PA-provable. Since $T \vdash_K [A \Rightarrow B_j]$, it follows by Generalisation that $T \vdash_K [(A \chi)(A \Rightarrow B_j)]$, and so $T \vdash_K [A \Rightarrow (A \chi)B_j]$, i.e. $T \vdash_K [A \Rightarrow B_i]$.

This completes the induction, and Theorem 1 follows as the special case where $i = n$. ¶

2.1 A number-theoretic corollary

Now, Gödel has defined ([Go31a], p22, Definition 45(6)) a primitive recursive number-theoretic relation $x B_{(K,T)} y$ that holds if, and only if, $x$ is the Gödel-number of a deduction from $T$ of the K-formula whose Gödel-number is $y$.

We thus have:

**Corollary 1.1**: If the Gödel-number of the well-formed K-formula $[B]$ is $b$, and that of the well-formed K-formula $[A \Rightarrow B]$ is $c$, then Theorem 1 holds if, and only if:

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3 We use the symbol “¶” as an end-of-proof marker.
\[
(\exists x)xB_{K,T,[A]}b \implies (\exists z)B_{K,T}c
\]

### 2.2 An extended Deduction Theorem

We next consider the proposition:

**Corollary 1.2:** If we assume Church’s Thesis\(^6\), then Theorem 1 holds even if the premise \((T, [A])|\sim_K [B]\) is established meta-mathematically, and a deduction \(<[B_1], [B_2], \ldots, [B_n]>\) of \([B]\) from \((T, [A])\) in \(K\) is not known explicitly.

**Proof:** Since Gödel’s number-theoretic relation \(xB_{K,T}y\) is primitive recursive, it follows that, if we assume Church’s Thesis - which implies that a number-theoretic relation is decidable if, and only if, it is recursive - we can effectively determine some finite natural number \(n\) for which the assertion \(nB_{K,T,[A]}b\) holds, where the Gödel-number of the well-formed \(K\)-formula \([B]\) is \(b\).

Since \(n\) would then, by definition, be the Gödel-number of a deduction \(<[B_1], [B_2], \ldots, [B_n]>>\) of \([B]\) from \((T, [A])\) in \(K\), we may thus constructively conclude, from the meta-mathematically determined assertion \((T, [A])|\sim_K [B]\), that some deduction \(<[B_1], [B_2], \ldots, [B_n]>>\) of \([B]\) from \((T, [A])\) in \(K\) can, indeed, be effectively determined.

Theorem 1 follows. \(\square\)

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\(^4\) We note that Corollary 1.1 and Corollary 2.2 may be essentially different number-theoretic assertions, which may not be obviously equivalent; the “obvious” assumption (*), thus, may need a proof.

\(^5\) We note that this is a semantic meta-equivalence, based on the definition of the primitive recursive relation \(xB_{K,T}y\).

\(^6\) Church’s Thesis: A number-theoretic function is effectively computable if, and only if, it is recursive ([Me64], p147, footnote). We appeal explicitly to Church’s Thesis here to avoid implicitly assuming that every recursive relation is algorithmically decidable (cf. [An02c], §II(7) Corollary 14.3). In Anand ([An02g], §2.5(xii)) we show that, under a constructive interpretation of classical foundational concepts, Church’s Thesis is a Theorem; such a premise would not, then, be needed.
3. An additional deduction theorem

We finally prove (*) as an additional deduction theorem, in an arbitrary first order theory K:

**Theorem 2:** If K is an arbitrary first order theory, and if \([A]\) is a closed well-formed formula of K, then \((T, [A]) \vdash_K [B]\) if, and only if, \(T \vdash_K [B]\) holds when we assume \(T \vdash_K [A]\).

**Proof:** Firstly, if there is a deduction \(<[B_1], [B_2], ..., [B_n]>\) of \([B]\) from \((T, [A])\) in K, and there is a deduction \(<[A_1], [A_2], ..., [A_m]>\) of \([A]\) from \(T\) in K, then \(<[A_1], [A_2], ..., [A_m], [B_1], [B_2], ..., [B_n]>\) is a deduction of \([B]\) from \(T\) in K. Hence we have: if \((T, [A]) \vdash_K [B]\), then \(T \vdash_K [B]\) holds when we assume \(T \vdash_K [A]\).

Secondly, if there is a deduction \(<[B_1], [B_2], ..., [B_n]>\) of \([B]\) from \(T\) in K, then we have, trivially, that: if \(T \vdash_K [B]\) holds when we assume \(T \vdash_K [A]\), then \((T, [A]) \vdash_K [B]\).

Lastly, we assume that there is no deduction \(<[B_1], [B_2], ..., [B_n]>\) of \([B]\) from \(T\) in K. If, now, \(T \vdash_K [B]\) holds when we assume \(T \vdash_K [A]\) in any consistent extension \(K'\) of K, then, if we assume that there is a sequence \(<[A_1], [A_2], ..., [A_m]>\) of well-formed \(K'\)-formulas such that \([A_m] = [A]\) and, for each \(m \geq i \geq 1\), either \([A_i]\) is an axiom of \(K'\), or \([A_i]\) is in \(T\), or \([A_i]\) is a direct consequence by some rules of inference of \(K'\) of some of the preceding well-formed formulas in the sequence, then we can show, by induction on the deduction length \(n\), that there is a sequence \(<[B_1], [B_2], ..., [B_n]>\) of well-formed \(K\)-formulas such that \([B_1] = [A]\), \([B_n] = [B]\) and, for each \(i > 1\), either \([B_i]\) is an axiom of \(K\), or \([B_i]\) is in \(T\), or \([B_i]\) is a direct consequence by some rules of inference of \(K\) of some of the preceding well-formed formulas in the sequence.

\(^7\) \([A]\) is thus the hypothesis in the sequence; it is the only well-formed \(K\)-formula in the sequence that is not an axiom of \(K\), not in \(T\), and not a direct consequence of the axioms of \(K\) by any rules of inference of \(K\).
Hence, if there is a deduction \(<[A_1], [A_2], ..., [A_m]>\) of \([A]\) from \(T\) in \(K'\), then \(<[A_1], [A_2], ..., [A_m], [B_2], ..., [B_n]>\) is a deduction of \([B]\) from \(T\) in \(K'\). By definition, it follows that \(<[B_2], ..., [B_n]>\) is a deduction of \([B]\) from \((T, [A])\) in \(K\). We thus have: if \(T|\_K [B]\) holds when we assume \(T|\_K [A]\), then \((T, [A])|\_K [B]\). This completes the proof. ¶

In view of Corollary 1.2, we thus have:

**Corollary 2.1:** If we assume Church’s Thesis, and if \([A]\) is a closed well-formed formula of \(K\), then we may conclude \(T|\_K ([A] \Rightarrow [B])\) if \(T|\_K [B]\) holds when we assume \(T|\_K [A]\).

We note that, in the notation of Corollary 1.1, if the Gödel-number of the well-formed \(K\)-formula \([A]\) is \(a\), then Corollary 2.1 holds if, and only if:

**Corollary 2.2:** \(((\exists x)xB_{[K, \gamma x]} \Rightarrow (Eu)uB_{[K, \gamma u]} \Rightarrow (Ez)zB_{[K, \gamma z]}\).

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8 We note that there is a model-theoretic proof of Corollary 2.1. The case \(T|\_K [B]\) is straightforward.

If \(\neg T|\_K [B]\), then, as noted in Theorem 2, if \(T|\_K [B]\) holds when we assume \(T|\_K [A]\), then there is a sequence \(<[B_1], [B_2], ..., [B_n]>\) of well-formed \(K\)-formulas such that \([B_1]\) is \([A]\), \([B_n]\) is \([B]\) and, for each \(i > 1\), either \([B_i]\) is an axiom of \(K\), or \([B_i]\) is in \(T\), or \([B_i]\) is a direct consequence by some rules of inference of \(K\) of some of the preceding well-formed formulas in the sequence.

(Note: In the following, if \(T\) is the set of well-formed \(K\)-formulas \(\{[T_1], [T_2], ..., [T_i]\}\) then \((T \& [A])\)

\(\text{denotes the well-formed } K\text{-formula } [T_1 \& T_2 \& ... \& T_i \& A], \text{ and, } (T \& A)\) denotes its interpretation in \(M: T_1 \& T_2 \& ... \& T_i \& A.\)

If, now, any well-formed formula in \((T, [A])\) is false under an interpretation \(M\) of \(K\), then \((T \& A) \Rightarrow B\) is vacuously true in \(M\).

If, however, all the well-formed formulas in \((T, [A])\) are true under interpretation in \(M\), then the sequence \(<[B_1], [B_2], ..., [B_n]>\) interprets as a deduction in \(M\), since the interpretation preserves the axioms and rules of inference of \(K\) (cf. [Me64], p57). Thus \([B]\) is true in \(M\), and so is \((T \& A) \Rightarrow B\).

In other words, we cannot have \((T, [A])\) true and \([B]\) false in \(M\) as this would imply that there is some consistent extension \(K'\) of \(K\) in which \(T|\_K [A]\), but not \(T|\_K [B]\), which is contrary to the hypothesis that, in any consistent \(K\) in which we assume \(T|\_K [A]\), we also have \(T|\_K [B]\).

Hence, \((T \& A) \Rightarrow B\) is true in all models of \(K\). By a consequence of Gödel’s Completeness Theorem for an arbitrary first order theory ([Me64], p68, Corollary 2.15(a)), it follows that \(|\_K (T \& [A]) \Rightarrow [B]\), and, ipso facto, that \(T|\_K ([A] \Rightarrow [B])\).

9 We note that this, too, is a semantic meta-equivalence, based on the definition of the primitive recursive relation \(xB_{[K, \gamma x]}\).
4. Conclusion

Since standard interpretations of Gödel’s reasoning and conclusions do not admit Theorem 2 as a valid inference, such interpretations are inconsistent with the standard Deduction Theorem for an arbitrary first order theory [Me64, p61, Proposition 2.4); they cannot, therefore, be considered definitive.

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