Ordinary Grothendieck groups of a Frobenius $P$-category

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Abstract: In [5] we have introduced the Frobenius categories $\mathcal{F}$ over a finite $p$-group $P$, and we have associated to $\mathcal{F}$ — suitably endowed with some central $k^*$-extensions — a “Grothendieck group” as an inverse limit of Grothendieck groups of categories of modules in characteristic $p$ obtained from $\mathcal{F}$, determining its rank. Our purpose here is to introduce an analogous inverse limit of Grothendieck groups of categories of modules in characteristic zero obtained from $\mathcal{F}$, determining its rank and proving that its extension to a field is canonically isomorphic to the direct sum of the corresponding extensions of the “Grothendieck groups” above associated with the centralizers in $\mathcal{F}$ of a suitable set of representatives of the $\mathcal{F}$-classes of elements of $P$.

1 Introduction

1.1 Let $p$ be a prime number and $\mathcal{O}$ a complete discrete valuation ring with a field of quotients $K$ of characteristic zero and a residue field $k$ of characteristic $p$; we assume that $k$ is algebraically closed and that $K$ contains “enough” roots of unity for the finite family of finite groups we will consider. Let $G$ be a finite group, $b$ a block of $G$ — namely a primitive idempotent in the center $Z(\mathcal{O}G)$ of the group $\mathcal{O}$-algebra — and $(P,e)$ a maximal Brauer $(b,G)$-pair [5, 1.16]; recall that the Frobenius $P$-category $\mathcal{F}(b,G)$ associated with $b$ is the subcategory of the category of finite groups where the objects are all the subgroups of $P$ and, for any pair of subgroups $Q$ and $R$ of $P$, the morphisms $\varphi$ from $R$ to $Q$ are the group homomorphisms $\varphi: R \to Q$ induced by the conjugation of some element $x \in G$ fulfilling

$$(R, g) \subset (Q, f)^x$$

where $(Q, f)$ and $(R, g)$ are the corresponding Brauer $(b, G)$-pairs contained in $(P, e)$ [5, Ch. 3].

1.2 In [5, Ch. 14] we consider a suitable inverse limit of Grothendieck groups of categories of modules in characteristic $p$ obtained from $\mathcal{F}(b,G)$, which, according to Alperin’s Conjecture, should be isomorphic to the Grothendieck group of the category of finitely dimensional $kGb$-modules. As announced in the title, our purpose here is to introduce an analogous inverse limit of Grothendieck groups of categories of modules in characteristic zero obtained from $\mathcal{F}(b,G)$, which again, according to Alperin’s Conjecture, should be isomorphic to the Grothendieck group of the category of finitely dimensional $KGb$-modules.

1.3 More explicitly, recall that a Brauer $(b, G)$-pair $(Q, f)$ is called self-centralizing if, for any $x \in G$ such that $(P, e)^x$ contains $(Q, f)$, we have $C_{P^x}(Q) = Z(Q)$ or, equivalently, if the image $\bar{f}$ of $f$ in $Z(kC_G(Q))$, where
where automorphisms \( F \) mapping any natural isomorphisms between the corresponding functors \( \Delta \) over the same simplex \([5, A2.8]\) — we prove in \([5, \text{Ch. 11}]\) that the canonical functor \( \Delta \) \( n \) Conjecture actually states \([5, I32]\)

\[
\mathcal{F}_{(b,G)}(Q) = \mathcal{F}_{(b,G)}(Q, Q) \cong N_{(Q, f)} / C_{(Q, f)}(Q)
\]

1.3.1.

1.4 Then, considering the proper category of \( \mathcal{F}_{(b,G)}^{ce} \) -chains \( \mathcal{F}^n_{(b,G)} \) — namely, the category of functors \( q : \Delta_n \to \mathcal{F}^{ce}_{(b,G)} \) from any ordered simplex \( \Delta_n \) where the morphisms are defined by the order preserving maps between simplexes and the natural isomorphisms between the corresponding functors over the same simplex \([5, A2.8]\) — we prove in \([5, \text{Ch. 11}]\) that the canonical functor \( \text{aut}^{ce}_{\mathcal{F}_{(b,G)}} : \mathfrak{ch}^* \left( \mathcal{F}^{ce}_{(b,G)} \right) \to \mathfrak{Gr} \)

1.4.1

mapping any \( \mathcal{F}^{ce}_{(b,G)} \) -chain \( q : \Delta_n \to \mathcal{F}^{ce}_{(b,G)} \) on the group \( \mathcal{F}^{ce}_{(b,G)}(q) \) of natural automorphisms of \( q \) \([5, \text{Proposition A2.10}]\) can lifted to a functor \( \tilde{\text{aut}}^{ce}_{\mathcal{F}_{(b,G)}} : \mathfrak{ch}^* \left( \mathcal{F}^{ce}_{(b,G)} \right) \to k^* \cdot \mathfrak{Gr} \)

1.4.2

where \( \mathfrak{Gr} \) and \( k^* \cdot \mathfrak{Gr} \) respectively denote the categories of finite groups and of central \( k^* \)-extensions of finite groups, called finite \( k^* \)-groups.

1.5 At this point, denoting by \( g_k : k^* \cdot \mathfrak{Gr} \to \mathcal{O} \cdot \text{mod} \) the contravariant functor sending any \( k^* \)-group \( \hat{G} \) to the \( \mathcal{O} \)-extension of the Grothendieck group of the category of finitely dimensional \( k^* \)-modules — noted \( G_k(\hat{G}) \) — in \([5, \text{Ch. 4}]\) we introduce the inverse limit

\[
G_k(\mathcal{F}_{(b,G)}), \tilde{\text{aut}}^{ce}_{\mathcal{F}_{(b,G)}} = \lim_{\leftarrow} (g_k \circ \tilde{\text{aut}}^{ce}_{\mathcal{F}_{(b,G)}})
\]

1.5.1

— called the (modular) Grothendieck group of \( \mathcal{F}_{(b,G)} \) — which, strictly speaking, depends only not on the Frobenius \( P \)-category \( \mathcal{F}_{(b,G)} \) but on the lifting \( \tilde{\text{aut}}^{ce}_{\mathcal{F}_{(b,G)}} \). There, we also prove that

\[
\text{rank}_{\mathcal{O}} \left( G_k(\mathcal{F}_{(b,G)}), \tilde{\text{aut}}^{ce}_{\mathcal{F}_{(b,G)}} \right) = \sum_{(q, \Delta_n)} (-1)^n \text{rank}_{\mathcal{O}} \left( G_k(\mathcal{F}^{ce}_{(b,G)}(q)) \right)
\]

1.5.2

where \( (q, \Delta_n) \) runs over a set of representatives for the set of isomorphism classes of regular \( \mathfrak{ch}^* \left( \mathcal{F}^{ce}_{(b,G)} \right) \)-objects \([5, 14.31] \) which shows that Alperin’s Conjecture actually states \([5, I32] \)

\[
G_k(\mathcal{F}_{(b,G)}), \tilde{\text{aut}}^{ce}_{\mathcal{F}_{(b,G)}} \cong G_k(G, b)
\]

1.5.3.
1.6 Since we have the well-known isomorphism [1]
\[ \mathcal{K} \mathcal{G}_k(G, b) \cong \bigoplus_{(u, g) \in \mathcal{U}} \mathcal{K}_g(C_G(u), g) \]
1.6.1
where \( \mathcal{U} \) is a set of representatives for the set of \( G \)-conjugacy classes of
\text{Breuer} \((b, G)\)-elements [5, 1.10], where \( \mathcal{G}_k(G, b) \) denotes the \( \mathcal{O} \)-extension of the
Grothendieck group of the category of finitely dimensional \( KGB \)-modules and where we set
\( \mathcal{K}_g \mathcal{G}_k(G, b) = \mathcal{K} \otimes \mathcal{O} \mathcal{G}_k(G, b) \) for short, as we said in [5, I51
and I52] the direct sum
\[ \bigoplus_{(u, g) \in \mathcal{U}} \mathcal{K}_g \mathcal{G}_k \left( \mathcal{F}_{(g, C_G(u))}, \mathcal{aut}_{\mathcal{F}_{(g, C_G(u))}} \right) \]
1.6.2
already provides a reasonable definition for the ordinary Grothendieck group
of \( \mathcal{F}_{(b, G)} \), at least extended to \( \mathcal{K} \). But, as we mention there, there is a more
reasonable definition as an inverse limit, analogous to definition 1.5.1.

1.7 Firstly notice that it does not suffice to replace \( g_k \) by the \textit{contravariant}
functor \( \mathcal{G}_k : k^* \mathcal{G} \rightarrow \mathcal{O} \text{-mod} \) sending any \( k^* \)-group \( G \) to the \( \mathcal{O} \)-extension
of the Grothendieck group of the category of finitely dimensional \( K_\mathcal{g} \)-modules
where, considering the canonical group homomorphisms \( k^* \rightarrow \mathcal{O} \subset \mathcal{K}^* \)
and \( k^* \rightarrow \hat{G} \), we set
\[ \mathcal{K}_* \hat{G} = \mathcal{K} \otimes_{k^*} \mathcal{K} \hat{G} \]
1.7.1
indeed, it suffices to consider the case of the blocks of the \( p \)-solvable groups —
 discussed in [6], for instance — to understand that the groups \( \mathcal{F}_{(b, G)}^{(e)}(q) \)
above have to be replaced by the corresponding \textit{localizers} [5, 18.3] and, more
generally, that the functor \( \mathcal{aut}_{\mathcal{F}_{(b, G)}^{(e)}} \) has to be replaced by the
\( \mathcal{F}_{(b, G)}^{(e)} \)-\textit{localizing functor} introduced in [5, Ch. 18].

1.8 Precisely, recall that for any \( \mathcal{F}_{(b, G)}^{(e)} \)-object \( Q \) such that \( \mathcal{F}_P(Q) \) is a
Sylow \( p \)-subgroup of \( \mathcal{F}_{(b, G)}^{(e)}(Q) \), the group \( N_P(Q) \) — viewed as an extension
of \( \mathcal{F}_P(Q) \) by \( Z(Q) \) — determines an extension \( \mathcal{L}_{(b, G)}(Q) \) of \( \mathcal{F}_{(b, G)}^{(e)}(Q) \) by
\( Z(Q) \), containing \( N_P(Q) \) — called the \textit{localizer} of \( Q \) [5, Theorem 18.6]. Then,
considering the category \( \mathcal{L}_{\mathcal{O}c} \) where the objects are the pairs \((L, Z)\) formed by
a finite group \( L \) and a normal \( p \)-subgroup \( Z \) of \( L \), and where the morphisms
from \((L, Z)\) to \((L', Z')\) are the \( Z' \)-conjugacy classes of group monomorphisms
\( f : L \rightarrow L' \) fulfilling \( f(Z) \subset Z' \) [5, 18.12], in [5, Proposition 18.19] we prove
the existence, and the uniqueness up to isomorphisms, of a functor
\[ \mathcal{L}_{\mathcal{O}c}^{(e)}(\mathcal{F}_{(b, G)}^{(e)}) : \mathcal{G}^*(\mathcal{F}_{(b, G)}^{(e)}) \rightarrow \mathcal{L}_{\mathcal{O}c} \]
1.8.1
which lifts \( \mathcal{aut}_{\mathcal{F}_{(b, G)}^{(e)}} \) above via the functor \( \mathcal{L}_{\mathcal{O}c} \rightarrow \mathcal{G} \) sending \((L, Z)\) to \( L/Z \),
and maps any \( \mathcal{F}_{(b, G)}^{(e)} \)-chain \( q : \Delta_n \rightarrow \mathcal{F}_{(b, G)}^{(e)} \) such that \( \mathcal{F}_P(q(n)) \) is a Sylow
\( p \)-subgroup of \( \mathcal{F}_{(b, G)}^{(e)}(q(n)) \), on the pair \((\mathcal{L}_{(b, G)}(q), Z(q(n)))\) where \( \mathcal{L}_{(b, G)}(q) \)
is the converse image of \( \mathcal{F}_{(b, G)}^{(e)}(q) \) in \( \mathcal{L}_{(b, G)}(q(n)) \).
1.9 Mutatis mutandis, we can consider the category $k^*\text{-}\mathcal{Loc}$ where the objects are the pairs $(\hat{L}, Z)$ formed by a finite $k^*$-group $\hat{L}$ and a normal $p$-subgroup $Z$ of $\hat{L}$, and then the lifting $\hat{\text{aut}}_{F_{(b,G)}}$ of $\text{aut}_{F_{(b,G)}}$ above determines, via pull-backs, a functor

$$\hat{\text{loc}}_{F_{(b,G)}} : \text{ch}^*(F_{(b,G)}) \rightarrow k^*\text{-}\mathcal{Loc}$$

lifting $\hat{\text{loc}}_{F_{(b,G)}}$. At this point, still denoting by $g_K : k^*\text{-}\mathcal{Loc} \rightarrow \mathcal{O}\text{-}\text{mod}$ the functor sending $(\hat{L}, Z)$ to $G_K(\hat{L})$, we define the ordinary Grothendieck group of $F_{(b,G)}$ as the following inverse limit

$$G_K(F_{(b,G), \hat{\text{aut}}_{F_{(b,G)}}}) = \lim_{\leftarrow} (g_K \circ \hat{\text{loc}}_{F_{(b,G)}})$$

which once again depends on the lifting $\hat{\text{aut}}_{F_{(b,G)}}$. Note that, since we have $G_k(\hat{L}) \cong G_k(\hat{L}/Z)$, we also have

$$g_k \circ \hat{\text{loc}}_{F_{(b,G)}} \cong g_k \circ \hat{\text{aut}}_{F_{(b,G)}}$$

1.10 Our purpose here is to prove that an equality analogous to equality 1.5.2 holds, namely that we have

$$\text{rank}_\mathcal{O}(G_K(F_{(b,G), \hat{\text{aut}}_{F_{(b,G)}}})) = \sum_{(q, \Delta_n)} (-1)^n \text{rank}_\mathcal{O}(G_K(\hat{L}_{\hat{L}/Z}(q)))$$

where $(q, \Delta_n)$ runs over a set of representatives for the set of isomorphism classes of regular $\text{ch}^*(F_{(b,G)})$-objects (cf. 8.3 below) and we set

$$\hat{L}_{\hat{L}/Z}(q) = \hat{F}_{\hat{L}/Z}(q) \times F_{\hat{L}/Z}(q) \times L_{\hat{L}/Z}(q)$$

and that the direct sum 1.6.2 coincides with the extension to $\mathcal{K}$ of the ordinary Grothendieck group of $F_{(b,G)}$, namely that we still have

$$\mathcal{K}G_K(F_{(b,G), \hat{\text{aut}}_{F_{(b,G)}}}) \cong \bigoplus_{(u, g) \in U} \mathcal{K}G_k(F_{(g,CG)(u), \hat{\text{aut}}_{F_{(g,CG)(u)}}})$$

1.11 A remarkable fact is that neither these statements nor our arguments for proving them need to assume that the Frobenius $P$-category $F$ we are dealing with comes from a block of a finite group, but only need the choice of a lifting

$$\hat{\text{aut}}_{F_{(b,G)}} : \text{ch}^*(F_{(b,G)}) \rightarrow k^*\text{-}\mathcal{Gr}$$

of the functor $\text{aut}_{F_{(b,G)}}$ [5, Proposition A2.10]. Thus, as in [5] for the modular Grothendieck group, we will carry out our purpose over such a triple $(P, F, \hat{\text{aut}}_{F_{(b,G)}})$ that we call a folded Frobenius $P$-category. Actually, the reader may ask himself why the present material has not been included in [5].
The answer is quite simple: because when finishing [5] we had not at all it and our question in [5, I52] was not yet answered! Naturally, this means that some arguments here are definitely not contained in [5]. On the one hand, our proof of isomorphism 1.10.3 needs the rather technical Lemma 9.4 below which in some sense “explains” why the localizers of the $\mathcal{F}$-selfcentralizing subgroups of $P$ are powerful enough to compute the complete Grothendieck group. On the other hand, even our proof of equality 1.10.1 needs a more sophisticated machinery than the proof in [5] of equality 1.5.2 above.

1.12 This paper is divided on nine sections; notation, terminology and results in [5] are our main reference. In all the paper $P$ is a finite $p$-group; in section 2 we recall the main facts we need here on Frobenius $P$-categories and give a sufficient condition to have a folded Frobenius $P$-category. In section 3 we give equivalent definitions of the ordinary Grothendieck group of a folded Frobenius $P$-category. Section 4 is devoted to the functoriality of both, the ordinary and the modular Grothendieck groups of folded Frobenius $P$-categories, a subject that in [5] has only been partially discussed in the framework of blocks; on the converse, it is reasonable to hope that the analogous reduction results in [5, Ch. 15] will admit a translation to the ordinary Grothendieck group of the Frobenius $P$-categories of blocks, but this has not been done here.

1.13 In section 8 we prove equality 1.10.1, and the previous sections 5, 6 and 7 play an auxiliary role in that proof. In section 5 we develop a canonical decomposition of the ordinary Grothendieck group analogous to the decomposition of the modular Grothendieck group in [5, Ch. 14], except on the fact, pointed out in 1.7 above, that we have to replace the functor $\text{aut}_{F_{sc}}$ by the $\mathcal{F}$-localizing functor. Section 6 is devoted to some formal transformation of each term of our decomposition, which facilitates the application of a vanishing cohomological result. In section 7 we prove this vanishing cohomological result which generalizes [5, Theorem 6.26]; the existence of such a generalization is a critical point in our paper. Finally, in section 9 we prove isomorphism 1.10.3; our proof needs Lemma 9.4 as mentioned above, and a sophisticated counting argument to show the equality of dimensions.

2 The folded Frobenius $P$-categories

2.1 Let $P$ be a finite $p$-group and denote by $\text{i}\mathfrak{G}$ the category formed by the finite groups and by the injective group homomorphisms, and by $\mathcal{F}_{P}$ the subcategory of $\text{i}\mathfrak{G}$ where the objects are all the subgroups of $P$ and the morphisms are the group homomorphisms induced by conjugation by elements of $P$.

2.2 Recall that a Frobenius $P$-category $\mathcal{F}$ is a subcategory of $\text{i}\mathfrak{G}$ containing $\mathcal{F}_{P}$ where the objects are all the subgroups of $P$ and the morphisms fulfill the following three conditions [5, 2.8 and Proposition 2.11]
2.2.1 For any subgroup $Q$ of $P$ the inclusion functor $(\mathcal{F})_Q \to (\mathfrak{G})_Q$ is full.

2.2.2 $\mathcal{F}_p(P)$ is a Sylow $p$-subgroup of $\mathcal{F}(P)$.

2.2.3 For any subgroup $Q$ of $P$ such that we have $\xi(C_p(Q)) = C_p(\xi(Q))$ for any $\mathcal{F}$-morphism $\xi: Q \to C_p(Q)$, any $\mathcal{F}$-morphism $\varphi: Q \to P$ and any subgroup $R$ of $N_p(\varphi(Q))$ containing $\varphi(Q)$ such that $\mathcal{F}_P(Q)$ contains the action of $\mathcal{F}_R(\varphi(Q))$ over $\varphi$, there is an $\mathcal{F}$-morphism $\xi: R \to P$ fulfilling $\xi(\varphi(u)) = u$ for any $u \in Q$.

As in [5, 1.2], for any pair of subgroups $Q$ and $R$ of $P$, we denote by $\mathcal{F}(Q,R)$ the set of $\mathcal{F}$-morphisms from $Q$ to $R$ and set $\mathcal{F}(Q) = \mathcal{F}(Q,Q)$; moreover, recall that, for any category $\mathfrak{C}$ and any $\mathfrak{C}$-object $C$, $\mathfrak{C}_C$ (or $(\mathfrak{C})_C$ to avoid confusion) denotes the category of “$\mathfrak{C}$-morphisms to $C$” [5, 1.7].

2.3 Given a Frobenius $P$-category $\mathcal{F}$, a subgroup $Q$ of $P$ and a subgroup $K$ of the group $\text{Aut}(Q)$ of automorphisms of $Q$, we say that $Q$ is fully $K$-normalized in $\mathcal{F}$ if we have [5, 2.6]

$$\xi(N^K_p(Q)) = N^K_p(\xi(Q))$$

for any $\mathcal{F}$-morphism $\xi: Q \to N^K_p(Q)$, where $N^K_p(Q)$ is the converse image of $K$ in $N_p(Q)$ via the canonical group homomorphism $N_p(Q) \to \text{Aut}(Q)$ and $^K$ is the image of $K$ in $\text{Aut}(\xi(Q))$ via $\xi$. Recall that if $Q$ is fully $K$-normalized in $\mathcal{F}$ then we have a new Frobenius $N^K_p(Q)$-category $N^K_p(Q)$ where, for any pair of subgroups $R$ and $T$ of $N^K_p(Q)$, $(N^K_p(Q))(R,T)$ is the set of group homomorphisms from $T$ to $R$ induced by the $\mathcal{F}$-morphisms $\psi: Q \cdot T \to Q \cdot R$ which stabilizes $Q$ and induces on it an element of $K$ [5, 2.14 and Proposition 2.16].

2.4 We say that a subgroup $Q$ of $P$ is $\mathcal{F}$-selfcentralizing if we have

$$C_p(\varphi(Q)) \subset \varphi(Q)$$

for any $\varphi \in \mathcal{F}(P,Q)$, and we denote by $\mathcal{F}^{sc}$ the full subcategory of $\mathcal{F}$ over the set of $\mathcal{F}$-selfcentralizing subgroups of $P$. From the case of the Frobenius $P$-categories associated with a block of a finite group, we know that only makes sense to consider central $k^*$-extensions of $\mathcal{F}(Q)$ whenever $Q$ is $\mathcal{F}$-selfcentralizing [5, 7.4]; but, if $U$ is a subgroup of $P$ fully $K$-normalized in $\mathcal{F}$ for some subgroup $K$ of $\text{Aut}(U)$, a $N^K_p(U)$-selfcentralizing subgroup of $N_P(U)$ needs not be $\mathcal{F}$-selfcentralizing, which is a handicap when comparing choices of central $k^*$-extensions in $\mathcal{F}$ and in $N^K_p(U)$. In order to overcome this difficulty, we consider the $\mathcal{F}$-radical subgroups of $P$; we say that a subgroup $R$ of $P$ is $\mathcal{F}$-radical if it is $\mathcal{F}$-selfcentralizing and we have

$$\mathfrak{O}_P(\tilde{\mathcal{F}}(R)) = \{1\}$$

where $\tilde{\mathcal{F}}(R) = \mathcal{F}(R)/\mathcal{F}_R(R)$ [5, 1.3]; we denote by $\mathcal{F}^{rd}$ the full subcategory of $\mathcal{F}$ over the set of $\mathcal{F}$-radical subgroups of $P$. 

Lemma 2.5 Let $F$ be a Frobenius $P$-category, $U$ a subgroup of $P$ and $K$ a subgroup of $\text{Aut}(U)$ containing $\text{Int}(U)$. If $U$ is fully $K$-normalized in $F$ then any $N^K_F(U)$-radical subgroup $R$ of $N^K_F(U)$ contains $U$ and, in particular, it is $F$-selfcentralizing.

Proof: It is quite clear that the image of $N_{U,R}(R)$ in $(N^K_F(U))(R)$ is a normal $p$-subgroup and therefore it is contained in $O_p((N^K_F(U))(R))$, so that $N_{U,R}(R) = R$ which forces $U \cdot R = R$. Moreover, for any $F$-morphism $\psi : R \to P$, it is clear that $\psi(U)$ is a normal subgroup of $\psi(R) \cdot C_P(\psi(R))$ and therefore, since $U$ is also fully centralized in $F$ [5, Proposition 2.12], it follows from 2.2.3 that there is an $F$-morphism

$$\zeta : \psi(R) \cdot C_P(\psi(R)) \longrightarrow P$$

fulfilling $\zeta(\psi(u)) = u$ for any $u \in U$, so that the group homomorphism from $R$ to $N^K_F(U)$ mapping $v \in R$ on $\zeta(\psi(v))$ is a $N^K_F(U)$-morphism; in particular, $\zeta(\psi(R))$ is also $N^K_F(U)$-selfcentralizing and therefore we get

$$\zeta(C_P(\psi(R))) \subset \zeta(\psi(R))$$

which forces $C_P(\psi(R)) \subset \psi(R)$. We are done.

2.6 As a matter of fact, the $F$-radical subgroups of $P$ admit a description in terms of the dominant $F$-morphisms; let us say that an $F$-morphism $\varphi : Q \to R$ is dominant if it fulfills

$$\varphi \circ F(Q) \subset F(R) \circ \varphi$$

it is quite clear that the $F$-isomorphisms are dominant and that the composition of dominant $F$-morphisms is a dominant $F$-morphism. Setting $Q' = \varphi(Q)$ and denoting by $F(R)_{Q'}$ the stabilizer of $Q'$ in $F(R)$, this condition is equivalent to saying that $\varphi$ determines a surjective group homomorphism

$$F(R)_{Q'} \longrightarrow F(Q)$$

moreover, if $Q$ is $F$-selfcentralizing then the kernel of this homomorphism coincides with $F_{Z(Q)}(R)$ [5, Corollary 4.7]; in this case, the choice of a central $k^*$-extension of $F(R)$ clearly determines one of $F(Q)$. On the other hand, since a Sylow $p$-subgroup of $F(R)_{Q'}$ maps onto a Sylow $p$-subgroup of $F(Q)$, if $Q$ is $F$-selfcentralizing and $R$ is fully normalized in $F$ then we may assume that $F_P(R)$ contains a Sylow $p$-subgroup of $F(R)_{Q'}$; in this case $Q'$ is also fully normalized in $F$ [5, Proposition 2.12] and it is easily checked that $\varphi$ can be extended to an $F$-morphism

$$\hat{\varphi} : N_P(Q) \longrightarrow N_P(Q') = N_P(R)_{Q'} \subset N_P(R)$$
Proposition 2.7 Let $\mathcal{F}$ be a Frobenius $P$-category, $R$ a subgroup of $P$ fully centralized in $\mathcal{F}$ and $Q$ the converse image of $O_p(\mathcal{F}(R))$ in $N_P(R)$. Then, the inclusion map $i_Q^R: R \to Q$ is dominant. In particular, $R$ is $\mathcal{F}$-radical if and only if any dominant $\mathcal{F}$-morphism from $R$ is an isomorphism.

Proof: Recall that $R$ is also fully $O_p(\mathcal{F}(R))$-normalized [5, 2.10] and therefore $N_P(R)$ contains $O_p(\mathcal{F}(R))$ [5, Proposition 2.12]; in particular, for any $\tau \in \mathcal{F}(R)$, the composition $i_Q^R \circ \tau: R \to P$ can be extended to an $\mathcal{F}$-morphism $\zeta: Q \to P$ [5, statement 2.10.1]; since $\zeta(C_P(R)) = C_P(R)$, it is quite clear that $\zeta(Q) = Q$ and denoting by $\sigma: Q \cong Q$ the automorphism determined by $\zeta$, we have $i_Q^R \circ \tau = \sigma \circ i_Q^R$; and, according to condition 2.2.1, $\sigma$ belongs to $\mathcal{F}(Q)$. This proves that $i_Q^R: R \to Q$ is dominant.

In particular, if we assume that any dominant $\mathcal{F}$-morphism from $R$ is an isomorphism, then we get $R = Q$ which proves that $R$ is $\mathcal{F}$-radical.

Conversely, assume that $R$ is $\mathcal{F}$-radical and let $\varphi: R \to Q$ be a dominant $\mathcal{F}$-morphism; for any $\tau \in \mathcal{F}(R)$ there is $\sigma \in \mathcal{F}(Q)$ fulfilling $\varphi \circ \tau = \sigma \circ \varphi$ and then, setting $R' = \varphi(R)$, it is easily checked that

$$\sigma(N_Q(R')) = N_Q(R')$$

2.7.1

in particular, the image $\tau'$ of $\tau$ in $\mathcal{F}(R')$ via the $\mathcal{F}$-isomorphism $R \cong R'$ determined by $\varphi$ (cf. condition 2.2.1) normalizes the image $\mathcal{F}_Q(R')$ of $N_Q(R')$ in $\mathcal{F}(R')$; consequently, $\mathcal{F}_Q(R')$ is a normal $p$-subgroup of $\mathcal{F}(R')$ and therefore it is contained in $O_p(\mathcal{F}(R'))$; but, since $R'$ is also $\mathcal{F}$-radical, $O_p(\mathcal{F}(R'))$ is just the image of $R'$ in $\mathcal{F}(R')$ and therefore we have $N_Q(R') = R'$ which forces $R' = Q$. We are done.

2.8 But, in our general setting, we have to deal with $\mathcal{F}^e$-chains and coherent choices of central $k^*$-extensions for their $\mathcal{F}^e$-automorphisms group. Recall that we call $\mathcal{F}^e$-chain any functor $q: \Delta_n \to \mathcal{F}^e$ where the $n$-simplex $\Delta_n$ is considered as a category with the morphisms defined by the order [5, A2.2]; then, we consider the category $\mathbf{ch}^*(\mathcal{F}^e)$ where the objects are all the $\mathcal{F}^e$-chains $(q, \Delta_n)$ and the morphisms from $q: \Delta_n \to \mathcal{F}^e$ to another $\mathcal{F}^e$-chain $r: \Delta_m \to \mathcal{F}^e$ are the pairs $(\nu, \delta)$ formed by an order preserving map $\nu$, or, equivalently, a functor $\delta: \Delta_m \to \Delta_n$ and by a natural isomorphism $\nu: q \circ \delta \cong r$, the composition being defined by the composition of maps and of natural isomorphisms [5, A2.8]; the point is that we have a canonical functor

$$\text{aut}_{\mathcal{F}^e} : \mathbf{ch}^*(\mathcal{F}^e) \longrightarrow \mathfrak{G}r$$

2.8.1

mapping any $\mathcal{F}^e$-chain $q: \Delta_n \to \mathcal{F}^e$ to the group of natural automorphisms of $q$, simply noted $\mathcal{F}(q)$ [5, Proposition A2.10]. We define a folded Frobenius
The functor $\hat{\text{aut}}_{\mathcal{F}^{sc}} : \text{ch}^* (\mathcal{F}^{sc}) \to k^* - \mathfrak{Gr}$ lifting $\text{aut}_{\mathcal{F}^{sc}}$. *Mutatis mutandis*, we consider the category $\text{ch}^* (\mathcal{F}^{rd})$ and the canonical functor $\text{aut}_{\mathcal{F}^{rd}} : \text{ch}^* (\mathcal{F}^{rd}) \to \mathfrak{Gr}$. 

**Theorem 2.9** Any functor $\hat{\text{aut}}_{\mathcal{F}^{sc}}$ lifting $\text{aut}_{\mathcal{F}^{rd}}$ to the category $k^* - \mathfrak{Gr}$ can be extended to a unique functor lifting $\text{aut}_{\mathcal{F}^{sc}}$. 

**Proof:** Let $\mathfrak{X}$ be a set of $\mathcal{F}$-selfcentralizing subgroups of $P$ which contains all the $\mathcal{F}$-radical subgroups of $P$ and is stable by $\mathcal{F}$-isomorphisms; denoting by $\mathcal{F}^\times$ the full subcategory of $\mathcal{F}$ over $\mathfrak{X}$, assume that $\hat{\text{aut}}_{\mathcal{F}^{sc}}$ can be extended to a unique functor $\hat{\text{aut}}_{\mathcal{F}^{sc}} : \text{ch}^* (\mathcal{F}^\times) \to k^* - \mathfrak{Gr}$ 

assuming that $\mathfrak{X}$ does not coincide with the set of all the $\mathcal{F}$-selfcentralizing subgroups of $P$, let $V$ be a maximal $\mathcal{F}$-selfcentralizing subgroup which is not in $\mathfrak{X}$ and $\rho : V \to W$ a dominant $\mathcal{F}$-morphism such that $\rho(V)$ is a proper normal subgroup of $W$ (cf. Proposition 2.7); according to 2.6, we may assume that $W$ and $\rho(V)$ are fully normalized in $\mathcal{F}$. Then, denoting by $\mathfrak{Y}$ the union of $\mathfrak{X}$ with all the subgroups of $P$, $\mathcal{F}$-isomorphic to $V$, it is clear that it suffices to prove that $\hat{\text{aut}}_{\mathcal{F}^{sc}}$ admits a unique extension to $\text{ch}^* (\mathcal{F}^\mathfrak{Y})$. 

For any chain $q : \Delta_n \to \mathcal{F}^\mathfrak{Y}$, denote by $\hat{q} : \Delta_{n+1} \to \mathcal{F}^\mathfrak{Y}$ the chain extending $q$ such that either $q(n) \cong V$ and we set $\hat{q}(n+1) = W$ and $q(n \cdot n + 1)$ is the composition of $\rho$ with an isomorphism $q(n) \cong V$, or $q(n) \not\cong V$ and we set $\hat{q}(n+1) = q(n)$ and $q(n \cdot n + 1) = \text{id}_q(n)$; in both cases, we have an obvious $\text{ch}^* (\mathcal{F}^\mathfrak{Y})$-morphism 

$$ (\text{id}_q, \delta_n^{n+1}) : (\hat{q}, \Delta_{n+1}) \to (q, \Delta_n) $$ 

and the functor $\hat{\text{aut}}_{\mathcal{F}^\mathfrak{Y}}$ maps $(\text{id}_q, \delta_n^{n+1})$ on a group homomorphism 

$$ \mathcal{F}(\hat{q}) \to \mathcal{F}(q) $$ 

which is surjective since any $\sigma \in \mathcal{F}(q) \subset \mathcal{F}(q(n))$ can be extended to an $\mathcal{F}$-automorphism of $\hat{q}(n + 1)$ (cf. 2.6.2); moreover, since $V$ is $\mathcal{F}$-selfcentralizing, the kernel of this homomorphism is a $p$-group (cf. 2.6); then, the functor $\hat{\text{aut}}_{\mathcal{F}^{sc}}$ and the structural inclusion $\mathcal{F}^{\hat{q}} \subset \mathcal{F}(\hat{q}(n + 1))$ determine a $k^*$-group $\hat{\mathcal{F}}(\hat{q})$ and this $k^*$-group induces a unique central $k^*$-extension $\hat{\mathcal{F}}(q)$ of $\mathcal{F}(q)$ such that we have a $k^*$-group homomorphism 

$$ \hat{\mathcal{F}}(\hat{q}) \to \hat{\mathcal{F}}(q) $$ 

lifting homomorphism 2.9.4.
Now, for any $\mathfrak{ch}^*(F^\nu)$-morphism $(\nu, \delta): (\tau, \Delta_m) \to (q, \Delta_n)$, we have to exhibit a $k^*$-group homomorphism $\hat{F}(\tau) \to \hat{F}(q)$ lifting $\hat{\text{aut}}_{F^\nu}(\nu, \delta)$. Firstly, denoting by $Q$ and $R$ the respective images of $r(\delta(n))$ and $r(m)$ in $\hat{\tau}(m+1)$ by the group homomorphisms $\hat{r}(\delta(n)\cdot m + 1)$ and $\hat{r}(m\cdot m + 1)$, assume that $Q$ is normal in $R$; in this case, either $Q \cong V$, and, since $\rho(V)$ is fully normalized in $F$, according to 2.6.3 there is an $F$-morphism $\hat{\rho}: N_P(Q) \to N_P(W)$ extending the composition with $\rho$ of such an isomorphism, which allows us to set $U = \hat{\rho}(R)\cdot W$, or $Q \not\cong V$ and we set $U = R$; then, we consider the chains

$$\hat{q}^R: \Delta_{n+2} \to F^\nu$$ and

$$\hat{q}^Q: \Delta_{m+2} \to F^\nu$$ 2.9.6

respectively extending $q$ and $\hat{r}$, fulfilling

$$\hat{q}^R(n + 2) = U = \hat{r}^Q(m + 2)$$ 2.9.7

and respectively mapping $(n + 1 \cdot n + 2)$ and $(m + 1 \cdot m + 2)$ either on the inclusion map $W \to \hat{\rho}(R)\cdot W$ and, whenever $Q \not\cong R$, on the $F$-morphism $R \to \hat{\rho}(R)\cdot W$ induced by $\hat{\rho}$ if $Q \cong V$, or on the group homomorphism $q(n) \to R$ determined by $(\nu_n)^{-1}$ and on $\text{id}_R$ whenever $Q \not\cong V$.

Then, we have evident $\mathfrak{ch}^*(F^\nu)$-morphisms

$$(\hat{q}^R, \Delta_{n+2}) \to (\hat{q}, \Delta_{n+1})$$ and $$(\hat{r}^Q, \Delta_{m+2}) \to (\hat{r}, \Delta_{m+1})$$ 2.9.8

and, considering the maps

$$\Delta_{n+2} \xleftarrow{\sigma_n} \Delta_1 \xrightarrow{\sigma_m} \Delta_{m+2} \quad \text{and} \quad \Delta_{n+1} \xleftarrow{\tau_n} \Delta_0 \xrightarrow{\tau_m} \Delta_{m+1}$$ 2.9.9

induced by the sum with $n+1$ and $m+1$ respectively, the $\mathfrak{ch}^*(F^\nu)$-morphisms above determine the following $\mathfrak{ch}^*(F^\nu)$-morphisms

$$(\hat{q}^R \circ \sigma_n, \Delta_1) \to (\hat{q} \circ \tau_n, \Delta_0)$$ and $$(\hat{r}^Q \circ \sigma_m, \Delta_1) \to (\hat{r} \circ \tau_m, \Delta_0)$$ 2.9.10.

Thus, the functor $\hat{\text{aut}}_{F^x}$ maps these morphisms on $k^*$-group homomorphisms

$$\hat{F}(\hat{q}^R \circ \sigma_n) \to \hat{F}(\hat{q} \circ \tau_n)$$ and $$\hat{F}(\hat{r}^Q \circ \sigma_m) \to \hat{F}(\hat{r} \circ \tau_m)$$ 2.9.11.

But note that $F(\hat{q}^R)$, $F(\hat{q})$, $F(\hat{r}^Q)$ and $F(\hat{r})$ are respectively contained in $F(\hat{q}^R \circ \sigma_n)$, $F(\hat{q} \circ \tau_n)$, $F(\hat{r}^Q \circ \sigma_m)$ and $F(\hat{r} \circ \tau_m)$, and that, considering the corresponding $k^*$-subgroups, the homomorphisms 2.9.11 induce $k^*$-group homomorphisms

$$\hat{F}(\hat{q}^R) \to \hat{F}(\hat{q})$$ and $$\hat{F}(\hat{r}^Q) \to \hat{F}(\hat{r})$$ 2.9.12.
moreover, the right-hand one is surjective and it is quite clear that

\[ \hat{\mathcal{F}}(\hat{r}^Q) \subset \hat{\mathcal{F}}(\hat{q}^R) \subset \hat{\mathcal{F}}(U) \]  

Consequently, we get a unique \( k^* \)-group homomorphism \( \hat{\text{Aut}}_{\mathcal{F}^\oplus} (\nu, \delta) \) from \( \hat{\mathcal{F}}(\nu) \) to \( \hat{\mathcal{F}}(\delta) \) such that the following diagram is commutative

\[
\begin{array}{ccc}
\hat{\mathcal{F}}(\hat{r}) & \leftarrow & \hat{\mathcal{F}}(\hat{r}^Q) \\
\downarrow & & \downarrow \\
\hat{\mathcal{F}}(\nu) & \rightarrow & \hat{\mathcal{F}}(\delta)
\end{array}
\]

Indeed, if \( Q \not\cong V \) or \( Q = R \), these statements are clear; if \( Q \cong V \) and \( Q \neq R \) then, since \( \mathcal{F}(\hat{r}) \) stabilizes \( Q \), the action on \( Q \) of any \( \sigma \in \mathcal{F}(\hat{r}) \) can be extended to an \( \mathcal{F} \)-automorphism \( \hat{\sigma} \) of \( W \) (cf. 2.6.2); moreover, since \( W \) is normal in \( \hat{\rho}_Q(R) \cdot W \) and \( W \) is fully normalized in \( \mathcal{F} \), \( \hat{\nu}^\nu \circ \hat{\sigma} \) can still be extended to an \( \mathcal{F} \)-morphism

\[ \hat{\zeta} : \hat{\rho}_Q(R) \cdot W \rightarrow P \]  

which, \( \hat{r} \), induces an \( \mathcal{F} \)-morphism \( \psi : R \rightarrow P \); then, the restriction of this homomorphism to \( Q \) coincides with the restriction of \( \sigma \) and therefore, since \( Q \) is \( \mathcal{F} \)-selfcentralizing, it follows from Proposition 4.6 in [5] that \( \psi \) and \( \hat{\nu}^\nu \circ \hat{\sigma} \) are \( Z(Q) \)-conjugate. In particular, we get

\[ \hat{\zeta}(\hat{\rho}_Q(R) \cdot W) = \hat{\rho}_Q(R) \cdot W \]  

and thus, up to modifying our choices, we may assume that the corresponding \( \mathcal{F} \)-automorphism \( \hat{\sigma} \) of \( \hat{\rho}_Q(R) \cdot W \) extends \( \sigma \) and \( \hat{\sigma} \); then, \( \hat{\sigma} \) belongs to \( \hat{\mathcal{F}}(\hat{r}^Q) \) and to \( \hat{\mathcal{F}}(\hat{q}^R) \), which proves the surjectivity of the right-hand homomorphism in 2.9.12 and easily implies the left-hand inclusion in 2.9.13.

Consider another \( \hat{\varphi}^* (\mathcal{F}^\oplus) \)-morphism \( (\mu, \varepsilon) : (t, \Delta_\ell) \rightarrow (\nu, \Delta_m) \), and respectively denote by \( T \), \( R \) and \( Q \) the images of \( t(\ell) \), \( t(m) \) and \( t((\varepsilon \circ \delta)(n)) \) in \( \hat{t}(\ell + 1) \); assume that \( R \) and \( Q \) are both normal in \( T \); as above, we consider the chains \( \hat{q}^R \), \( \hat{q}^T \), \( \hat{q}^Q \), \( \hat{t}^T \), \( \hat{t}^Q \) and \( \hat{t}^R \), and moreover we need the chains

\[
\begin{align*}
\hat{q}^{R,T} : & \Delta_{n+3} \rightarrow \mathcal{F}^\oplus \\
\hat{t}^{T,Q} : & \Delta_{m+3} \rightarrow \mathcal{F}^\oplus \\
\hat{t}^{R,Q} : & \Delta_{\ell+3} \rightarrow \mathcal{F}^\oplus
\end{align*}
\]
respectively extending $\hat{q}^R$, $\hat{r}^T$ and $\hat{r}^R$, fulfilling
\[
\hat{q}^{R,T}(n+3) = \hat{r}^{T,Q}(m+3) = \hat{r}^{R,Q}(\ell+3) = \hat{q}^T(n+2)
\] 2.9.18,
and respectively mapping $(n+2 \cdot n+3)$, $(m+2 \cdot m+3)$ and $(\ell+2 \cdot \ell+3)$ on the inclusion $\hat{q}^R(n+2) \subset \hat{q}^T(n+2)$ in both cases, either on the homomorphism
\[
T \to \hat{\rho}_Q(T) \cdot W
\]
defined by $\hat{\rho}_Q$ if $Q \cong V$ and $Q \neq R$, or on $\id_{\hat{q}^T(n+2)}$ if $Q \ncong V$ or $Q = R$, and once again, either on the homomorphism $T \to \hat{\rho}_Q(T) \cdot W$ induced by $\hat{\rho}_Q$ if $Q \cong V$ and $Q \neq R$, or on $\id_{\hat{q}^T(n+2)}$ if $Q \ncong V$ or $Q = R$.

As above, it is easily checked that applying the functor $\hat{\aut}_{x}^{\cdot}$ to the evident $\ch^*(\mathcal{F}^x)$-morphisms, we get $k^\ast$-group homomorphisms
\[
\begin{align*}
\hat{\aut}_{x}^{\cdot} : \mathcal{F}(\hat{q}^{R,T}) & \longrightarrow \mathcal{F}(\hat{q}^R) \\
\hat{\aut}_{x}^{\cdot} : \mathcal{F}(\hat{r}^{T,Q}) & \longrightarrow \mathcal{F}(\hat{r}^Q) \\
\hat{\aut}_{x}^{\cdot} : \mathcal{F}(\hat{r}^{R,Q}) & \longrightarrow \mathcal{F}(\hat{r}^R)
\end{align*}
\]
2.9.19
and moreover it is quite clear that $\mathcal{F}(\hat{r}^{R,Q}) = \mathcal{F}(\hat{r}^Q)$. Consequently, the functoriality of $\hat{\aut}_{x}^{\cdot}$ guarantees the commutativity of the following diagram
\[
\begin{array}{cccc}
\hat{\mathcal{F}}(i) & \longrightarrow & \hat{\mathcal{F}}(\hat{r}^{R,Q}) & \longrightarrow & \hat{\mathcal{F}}(\hat{q}^T) \\
\downarrow & & \downarrow & & \downarrow \\
\hat{\mathcal{F}}(i) & \longrightarrow & \hat{\mathcal{F}}(\hat{r}^R) & \longrightarrow & \hat{\mathcal{F}}(\hat{q}^T) \\
\downarrow & & \downarrow & & \downarrow \\
\hat{\mathcal{F}}(t) & \longrightarrow & \hat{\mathcal{F}}(\hat{r}^Q) & \longrightarrow & \hat{\mathcal{F}}(\hat{q}^T)
\end{array}
\]
2.9.20;
thus, by uniqueness, in this case we obtain
\[
\hat{\aut}_{x}^{\cdot}(\nu, \delta) \circ \hat{\aut}_{x}^{\cdot}(\mu, \varepsilon) = \hat{\aut}_{x}^{\cdot}((\nu, \delta) \circ (\mu, \varepsilon))
\]
2.9.21.

Secondly, assume that the image of $t^{(\delta(n))}$ by $t^{(\delta(n) \cdot m)}$ is not normal in $t(m)$; let $m'$ be the maximal element in $\Delta_m - \Delta_{\delta(n)-1}$ such that the image of $t^{(\delta(n))}$ by $t^{(\delta(n) \cdot m')}$ is normal in $t(m')$ and denote by $R_{(\nu, \delta)}$ the normalizer of the image of $t^{(\delta(n))}$ in $t(m'+1)$, by $t_{(\nu, \delta)} : \Delta_{m+1} \rightarrow \mathcal{F}^0$ the functor fulfilling
\[
t_{(\nu, \delta)} \circ \delta_{m'+1}^X = t \quad \text{and} \quad t_{(\nu, \delta)}(m'+1) = R_{(\nu, \delta)}
\]
2.9.22
and mapping \((m' + 1 \cdot m' + 2)\) on the inclusion map \(R_{(\nu, \delta)} \to t(m' + 1)\), and by \(t'_{(\nu, \delta)}\) the restriction of \(t_{(\nu, \delta)}\) to \(\Delta_{m' + 1}\); then, it is quite clear that \(\mathcal{F}(t_{(\nu, \delta)}) = \mathcal{F}(t)\) and it is easily checked that \(\hat{\mathcal{F}}(t_{(\nu, \delta)}) = \hat{\mathcal{F}}(t)\); moreover, we have an evident \(\text{ch}^*(\mathcal{F}^\varnothing)\)-morphism

\[
(\nu', \delta') : (\nu', \delta') \mapsto (q, \Delta)
\]

such that

\[
(\nu', \delta') \circ (\text{id}_{\nu'}, \nu') = (\nu, \delta) \circ (\text{id}_{\nu'}, \nu')
\]

where \(\nu_{m' + 1} : \Delta_{m' + 1} \to \Delta_{m + 1}\) denotes the natural inclusion, and in 2.9.14 we already have defined \(\hat{\text{aut}}_{\varnothing}^r((\text{id}_{\nu'}, \nu')_{m + 1}) = \text{id}_{\varnothing}(\nu', \delta')\); on the other hand, arguing by induction on \(|t(m)/|q(n)|\), we may assume that \(\hat{\text{aut}}_{\varnothing}^r((\text{id}_{\nu'}, \nu')_{m + 1})\) is already defined and then we set

\[
\hat{\text{aut}}_{\varnothing}^r(\nu, \delta) = \hat{\text{aut}}_{\varnothing}^r(\nu', \delta') \circ \hat{\text{aut}}_{\varnothing}^r((\text{id}_{\nu'}, \nu')_{m + 1})
\]

For another \(\text{ch}^*(\mathcal{F}^\varnothing)\)-morphism \((\mu, \varepsilon) : (t, \Delta_t) \to (t, \Delta_m)\), we claim that

\[
\hat{\text{aut}}_{\varnothing}^r(\nu, \delta) \circ \hat{\text{aut}}_{\varnothing}^r(\mu, \varepsilon) = \hat{\text{aut}}_{\varnothing}^r((\nu, \delta) \circ (\mu, \varepsilon))
\]

we argue by induction firstly on \(|t(t)/|q(n)|\) and after on \(|t(t)/|t(m)|\). First of all, we assume that the image of \(t(\delta(n))\) in \(t(m)\) by \(t(\delta(n) \cdot m)\) is not normal; with the notation above, denote by \(\ell'\) the maximal element in \(\Delta_{(e \circ \delta)}(n)\) such that the image of \(t(\delta(n) \cdot m(\varepsilon)\) by \(t(\delta(n) \cdot \ell')\) is normal in \(t(\ell')\); then, it is clear that \(\varepsilon(m') \leq \ell' < \varepsilon(m)\) and easily checked that we have a \(\text{ch}^*(\mathcal{F}^\varnothing)\)-morphism

\[
(\mu_{(\nu, \delta)}, \varepsilon(\nu, \delta)) : (t(\nu, \delta) \circ (\mu, \varepsilon)), \Delta_{\ell + 1} \mapsto (t(\nu, \delta), \Delta_{m + 1})
\]

such that

\[
(\text{id}_{\nu'}, \delta_{m'} + 1) \circ (\mu_{(\nu, \delta), \varepsilon(\nu, \delta)}) = (\mu, \varepsilon) \circ (\text{id}_{\nu'}, \delta_{\ell' + 1})
\]

that \(\varepsilon(\nu, \delta)(m' + 1) = \ell' + 1\) and that \((\mu_{(\nu, \delta)})(m' + 1)\) is determined by \(m_{\ell + 1}\) and \(t(\ell' + 1 \cdot \varepsilon(m' + 1))\); moreover, we consider the corresponding restriction

\[
(\mu'_{(\nu, \delta), \varepsilon(\nu, \delta)} : (t'_{(\nu, \delta) \circ (\mu, \varepsilon)}), \Delta_{\ell' + 1} \mapsto (t'_{(\nu, \delta)}, \Delta_{m' + 1})
\]

which obviously fulfills

\[
(\text{id}_{\nu', \delta_{m'}} + 1) \circ (\mu_{(\nu, \delta), \varepsilon(\nu, \delta)}) = (\mu'_{(\nu, \delta), \varepsilon(\nu, \delta)} \circ (\text{id}_{\nu', \delta_{(\nu, \varepsilon)}}, \ell_{\ell'})
\]

\[
2.9.27
\]

\[
2.9.28
\]

\[
2.9.29
\]

\[
2.9.30
\]
Now, it is easily checked that the composition \((\nu', \delta') \circ (\mu'_{(\nu, \delta)}, \varepsilon'_{(\nu, \delta)})\) coincides with the corresponding morphism 2.9.23 for the \(\mathfrak{cb}^*(\mathcal{F})\)-morphism \((\nu, \delta) \circ (\mu, \varepsilon)\) and therefore, by the very definition 2.9.25, we have

\[
\widehat{\text{aut}}_{X^\Theta}((\mu, \varepsilon)) = \widehat{\text{aut}}_{X^\Theta}((\nu', \delta') \circ (\mu'_{(\nu, \delta)}, \varepsilon'_{(\nu, \delta)})) \circ \widehat{\text{aut}}_{X^\Theta}(\text{id}_{(\nu, \delta) = (\mu, \varepsilon)}, 1_{\ell'})
\]

2.9.31;

but, since \(|R_{(\nu, \delta)}|/|q(n)| < |t(\ell)|/|q(n)|\), it follows from the induction hypothesis that

\[
\widehat{\text{aut}}_{X^\Theta}((\nu', \delta') \circ (\mu'_{(\nu, \delta)}, \varepsilon'_{(\nu, \delta)})) = \widehat{\text{aut}}_{X^\Theta}((\nu', \delta') \circ \widehat{\text{aut}}_{X^\Theta}(\mu'_{(\nu, \delta)}, \varepsilon'_{(\nu, \delta)}))
\]

2.9.32;

similarly, since we have \(|t(\ell)|/|R_{(\nu, \delta)}| < |t(\ell)|/|q(n)|\) and

\[
\widehat{\text{aut}}_{X^\Theta}(\mu_{(\nu, \delta)}, \varepsilon_{(\nu, \delta)}) = \widehat{\text{aut}}_{X^\Theta}(\mu, \varepsilon)
\]

2.9.33,

we still get

\[
\begin{align*}
\widehat{\text{aut}}_{X^\Theta}((\mu, \varepsilon)) & = \widehat{\text{aut}}_{X^\Theta}((\nu', \delta') \circ \widehat{\text{aut}}_{X^\Theta}(\mu'_{(\nu, \delta)}, \varepsilon'_{(\nu, \delta)})) \circ \widehat{\text{aut}}_{X^\Theta}(\text{id}_{(\nu, \delta) = (\mu, \varepsilon)}, 1_{\ell'}) \\
& = \widehat{\text{aut}}_{X^\Theta}((\nu', \delta') \circ \widehat{\text{aut}}_{X^\Theta}(\text{id}_{(\nu, \delta) = (\mu, \varepsilon)}, \ell''_{\mu'} \circ (\mu_{(\nu, \delta)}, \varepsilon_{(\nu, \delta)}))) \\
& = \widehat{\text{aut}}_{X^\Theta}(\nu, \delta) \circ \widehat{\text{aut}}_{X^\Theta}(\mu, \varepsilon).
\end{align*}
\]

2.9.34.

Finally, we may assume that the image of \(r(\delta(n))\) by \(r(\delta(n) \cdot m)\) is normal in \(r(m)\), so that the image of \(t((\varepsilon \circ \delta)(n))\) by \(t((\varepsilon \circ \delta)(n) \cdot \varepsilon(m))\) is normal in \(t(\varepsilon(m))\); in particular, denoting by \(\ell'\) the maximal element in \(\Delta_t - \Delta_{(\varepsilon \circ \delta)(n)-1}\) such that the image of \(t((\varepsilon \circ \delta)(n))\) by \(t((\varepsilon \circ \delta)(n) \cdot \ell')\) is normal in \(t(\ell')\), we have \(\varepsilon(m) \leq \ell'\). If \(\ell'' = \ell\) then we may assume that the image of \(t(\varepsilon(m))\) is not normal in \(t(\ell)\) and, denoting by \(\ell'' \geq \varepsilon(m)\) the maximal element in \(\Delta_t\) such that the image of \(t(\varepsilon(m))\) by \(t(\varepsilon(m) \cdot \ell'')\) is normal in \(t(\ell'')\), by our very definition (cf. 2.9.25) we have

\[
\widehat{\text{aut}}_{X^\Theta}(\mu, \varepsilon) = \widehat{\text{aut}}_{X^\Theta}(\mu', \varepsilon') \circ \widehat{\text{aut}}_{X^\Theta}(\text{id}_{(\nu, \delta) = (\mu, \varepsilon)}, 1_{\ell''})
\]

2.9.35;

but, according to equality 2.9.21, we have

\[
\widehat{\text{aut}}_{X^\Theta}(\nu, \delta) \circ \widehat{\text{aut}}_{X^\Theta}(\mu', \varepsilon') = \widehat{\text{aut}}_{X^\Theta}((\nu, \delta) \circ (\mu', \varepsilon'))
\]

2.9.36;
hence, since in the compositions of \((\nu, \delta)\) with \((\mu, \varepsilon)\) and of \(((\nu, \delta) \circ (\mu', \varepsilon'))\) with \((\text{id}_{\nu', \delta'}, \ell'_{\nu', \delta'})\) the first induction indices coincide with each other and the second ones strictly decrease, it follows from the induction hypothesis that

\[
\hat{\text{aut}}_{\mathcal{F}^\nu}(\nu, \delta) \circ \hat{\text{aut}}_{\mathbb{F}^\nu}(\mu, \varepsilon)
= \hat{\text{aut}}_{\mathcal{F}^\nu}(\nu, \delta) \circ \hat{\text{aut}}_{\mathbb{F}^\nu}(\mu', \varepsilon') \circ \hat{\text{aut}}_{\mathbb{F}^\nu}(\text{id}_{\nu', \delta'}, \ell'_{\nu', \delta'})
= \hat{\text{aut}}_{\mathcal{F}^\nu}(\nu', \delta) \circ \hat{\text{aut}}_{\mathbb{F}^\nu}(\text{id}_{\nu', \delta'}, \ell'_{\nu', \delta'})
= \hat{\text{aut}}_{\mathcal{F}^\nu}(\nu, \delta) \circ (\mu, \varepsilon)
\]

2.9.37.

Otherwise, we have a \(\text{ch}^*\mathcal{F}^\nu\)-morphism

\[
(\mu'_{\nu, \delta}, \varepsilon'_{\nu, \delta}) : \left(\left(\text{id}_{\nu, \delta} \circ (\mu, \varepsilon), \Delta^\nu + 1\right), (\tau, \Delta^\nu)\right) \rightarrow (\mathcal{T}, \Delta^\nu)
\]

2.9.38

fulfilling

\[
(\mu'_{\nu, \delta}, \varepsilon'_{\nu, \delta}) \circ (\text{id}_{\nu, \delta} \circ (\mu, \varepsilon)) = (\mu, \varepsilon) \circ (\text{id}_{\nu, \delta} \circ (\mu, \varepsilon))
\]

2.9.39;

as above, it is easily checked that the composition \((\nu, \delta) \circ (\mu', \varepsilon')\) coincides with the corresponding morphism 2.9.23 for the \(\text{ch}^*\mathcal{F}^\nu\)-morphism \((\nu, \delta) \circ (\mu, \varepsilon)\) and therefore, by the very definition 2.9.25, we have

\[
\hat{\text{aut}}_{\mathcal{F}^\nu}(\nu, \delta) \circ (\mu, \varepsilon) = \hat{\text{aut}}_{\mathcal{F}^\nu}(\nu, \delta) \circ (\mu', \varepsilon') \circ \hat{\text{aut}}_{\mathbb{F}^\nu}(\text{id}_{\nu', \delta'}, \ell'_{\nu', \delta'})
\]

2.9.40;

since \(\ell' \neq \ell\) and \(\hat{\text{aut}}_{\mathbb{F}'^\nu}(\text{id}_{\nu, \delta}, \ell) = \text{id}_{\mathcal{F}'^\nu}\), it follows from the induction hypothesis applied to the composition of \((\nu, \delta)\) with \((\mu'_{\nu, \delta}, \varepsilon'_{\nu, \delta})\) that

\[
\hat{\text{aut}}_{\mathcal{F}^\nu}(\nu, \delta) \circ (\mu'_{\nu, \delta}, \varepsilon'_{\nu, \delta}) = \hat{\text{aut}}_{\mathcal{F}^\nu}(\nu, \delta) \circ \hat{\text{aut}}_{\mathcal{F}^\nu}(\mu'_{\nu, \delta}, \varepsilon'_{\nu, \delta})
\]

2.9.41;

moreover, if \(|q(n)| = |\tau(m)|\), we can apply the induction hypothesis to both members of equality 2.9.39 and then we get

\[
\hat{\text{aut}}_{\mathcal{F}^\nu}(\mu'_{\nu, \delta}, \varepsilon'_{\nu, \delta}) \circ \hat{\text{aut}}_{\mathbb{F}^\nu}(\text{id}_{\nu, \delta} \circ (\mu, \varepsilon), \ell'_{\nu, \delta'}) = \hat{\text{aut}}_{\mathcal{F}^\nu}(\mu, \varepsilon)
\]

2.9.42;

consequently, once again we have

\[
\hat{\text{aut}}_{\mathcal{F}'^\nu}(\nu, \delta) \circ (\mu, \varepsilon) = \hat{\text{aut}}_{\mathcal{F}'^\nu}(\nu, \delta) \circ \hat{\text{aut}}_{\mathbb{F}'^\nu}(\mu, \varepsilon)
\]

2.9.43.

If \(|q(n)| = |\tau(m)|\) then it follows from the definition of \(\hat{\text{aut}}_{\mathcal{F}^\nu}(\mu, \varepsilon)\) and \(\hat{\text{aut}}_{\mathcal{F}^\nu}(\nu, \delta) \circ (\mu, \varepsilon)\) (cf. 2.9.25) that \(\ell'\) coincides with both indices, that we get \(\ell'_{(\mu, \varepsilon)} = (\mu, \varepsilon)\) and that the homomorphism 2.9.23

\[
(\mu'_{(\mu, \varepsilon)}, \Delta_{\mu'_{\nu, \delta}} + 1) \rightarrow (\mu, \varepsilon)
\]
corresponding to the composition \((\nu, \delta) \circ (\mu, \varepsilon)\) coincides with \((\nu, \delta) \circ (\mu', \varepsilon')\): at this point, we can apply equality 2.9.21 to obtain
\[
\hat{\text{aut}}_{\mathcal{F}t}(\nu, \delta) \circ \hat{\text{aut}}_{\mathcal{F}t}(\mu', \varepsilon') = \hat{\text{aut}}_{\mathcal{F}t}\left( (\nu, \delta) \circ (\mu', \varepsilon') \right)
\]
then, composing this equality with \(\hat{\text{aut}}_{\mathcal{F}t}(\text{id}_{(\nu, \delta)}, \ell_{\mu}')\), from definition 2.9.25 we get
\[
\hat{\text{aut}}_{\mathcal{F}t}(\nu, \delta) \circ \hat{\text{aut}}_{\mathcal{F}t}(\mu, \varepsilon) = \hat{\text{aut}}_{\mathcal{F}t}\left( (\nu, \delta) \circ (\mu, \varepsilon) \right)
\]
We are done.

3 Ordinary Grothendieck group of a folded Frobenius \(P\)-category

3.1 Our setting is a finite \(p\)-group \(P\), a Frobenius \(P\)-category \(\mathcal{F}\) and a functor
\[
\hat{\text{aut}}_{\mathcal{F}t} : \text{ch}^\ast(\mathcal{F}^\ast) \to k^\ast\text{-Gr}
\]
lifting the functor \(\text{aut}_{\mathcal{F}t}^\ast\) (cf. 2.8); if \(q : \Delta_n \to \mathcal{F}^\ast\) is an \(\mathcal{F}^\ast\)-chain, we simply denote by \(\hat{\mathcal{F}}(q)\) the image of \((q, \Delta_n)\) via \(\hat{\text{aut}}_{\mathcal{F}t}^\ast\). Note that, if \(P'\) is a second finite \(p\)-group, \(\mathcal{F}'\) a Frobenius \(P'\)-category and \(\alpha : P' \to P\) an \((\mathcal{F}', \mathcal{F})\)-functorial group homomorphism [5, 12.1] mapping any \(\mathcal{F}'\)-radical subgroup of \(P'\) (cf. 2.4) on a \(\mathcal{F}\)-self-centralizing subgroup of \(P\), the Frobenius functor \(\tilde{\ell}_\alpha : \mathcal{F}' \to \mathcal{F}\) [5, 12.1] and the lifting \(\hat{\text{aut}}_{\mathcal{F}t}^\ast\) determine a functor
\[
\hat{\text{aut}}_{\mathcal{F}t, \alpha} : \text{ch}^\ast(\mathcal{F}'^\ast) \to \text{ch}^\ast(\mathcal{F}^\ast) \to k^\ast\text{-Gr}
\]
lifting \(\text{aut}_{\mathcal{F}t, \alpha}^\ast\) and then, according to Theorem 2.9, this lifting can be uniquely extended to a functor \(\hat{\text{aut}}_{\mathcal{F}t}^\ast\) lifting \(\text{aut}_{\mathcal{F}t}^\ast\); in particular, for any subgroup \(U\) of \(P\) fully \(K\)-normalized in \(\mathcal{F}\) for some subgroup \(K\) of \(\text{Aut}(U)\), it follows from Lemma 2.5 and Theorem 2.9 that our folded Frobenius \(P\)-category induces a folded Frobenius \(N_K^P(U)\)-category formed by \(N_K^P(U)\) and by \(\hat{\text{aut}}_{N_K^P(U)}\).

3.2 For any \(\mathcal{F}\)-selfcentralizing subgroup \(Q\) of \(P\) fully normalized in \(\mathcal{F}\), in [5, Theorem 18.6] we prove the existence, and the uniqueness up to isomorphisms, of a finite group \(\mathcal{L}(Q)\) — the \(\mathcal{F}\)-localizer of \(Q\) — such that \(N_P(Q)\) is a Sylow \(p\)-subgroup of \(\mathcal{L}(Q)\) and \(Z(Q)\) a normal subgroup fulfilling \(\mathcal{L}(Q)/Z(Q) \cong \mathcal{F}(Q)\). More generally, in [5, Proposition 18.19] we prove the existence of the \(\mathcal{F}\)-localizing functor
\[
\hat{\text{loc}}_{\mathcal{F}t}^\ast : \text{ch}^\ast(\mathcal{F}^\ast) \to \hat{\text{Loc}}\]
which lifts \(\text{aut}_{\mathcal{F}t}^\ast\) via the canonical functor \(\hat{\text{Loc}} \to \text{Gr}\) sending any \(\hat{\text{Loc}}\)-object \((L, Z)\) to \(L/Z\), and maps any \(\mathcal{F}\)-chain \(q : \Delta_n \to \mathcal{F}\) such that \(q(n)\) is fully normalized in \(\mathcal{F}\), on the pair \((\mathcal{L}(q), Z(q(n)))\) where \(\mathcal{L}(q)\) denotes the converse image of \(\mathcal{F}(q)\) in \(\mathcal{L}(q(n))\); actually, for any \(\mathcal{F}\)-chain \(q : \Delta_n \to \mathcal{F}\) we set \(\hat{\text{loc}}_{\mathcal{F}t}^\ast(q, \Delta_n) = (\mathcal{L}(q), Z(q(n)))\) and identify \(N_P(q)\) with the converse image of \(\mathcal{F}_P(q)\) in \(\mathcal{L}(q)\) [5, Proposition 18.16].
3.3 As in 1.9 above, we consider the category $k^{*}\tilde{\mathcal{Loc}}$ where the objects are the pairs $(\hat{L}, Z)$ formed by a finite $k^{*}$-group $\hat{L}$ and a normal $p$-subgroup $Z$ of $L$ and where, coherently, the morphisms from $(\hat{L}, Z)$ to $(\hat{L}', Z')$ are the $Z'$-conjugacy classes of $k^{*}$-group homomorphisms $\hat{f}: \hat{L} \to \hat{L}'$ fulfilling $\hat{f}(Z) \subset Z'$. Then, consider the functor
\[
\hat{\text{loc}}_{F^{sc}} : k^{*}(\mathcal{F}^{\text{sc}}) \to k^{*}\tilde{\mathcal{Loc}}
\]
mapping any $\mathcal{F}^{\text{sc}}$-chain $q: \Delta_{n} \to \mathcal{F}^{\text{sc}}$ such that $q(n)$ is fully normalized in $\mathcal{F}$ on the pair formed by the $k^{*}$-group
\[
\hat{L}(q) = L(q) \times_{\mathcal{F}(q)} \bar{F}(q)
\]
and by the $p$-subgroup lifting $Z(q(n))$, and any $\text{ch}^{*}(\mathcal{F}^{\text{sc}})$-morphism
\[
(\nu, \delta) : (\tau, \Delta_{m}) \to (q, \Delta_{n})
\]
where $q(n)$ and $\tau(m)$ are fully normalized in $\mathcal{F}$, on the $Z(q(n))$-conjugacy class of
\[
\hat{\text{loc}}_{(\nu, \delta)} = \text{loc}_{\mathcal{F}}(\nu, \delta) \times_{\text{aut}_{\mathcal{F}^{\text{sc}}}(\nu, \delta)} \hat{\text{aut}}_{\mathcal{F}^{\text{sc}}}(\nu, \delta)
\]

3.4 For any central $k^{*}$-extension $\hat{G}$ of a finite group $G$, recall that we respectively denote by $G_{K}(\hat{G})$ and $G_{k}(\hat{G})$ the scalar extensions from $\mathbb{Z}$ to $\mathcal{O}$ of the Grothendieck groups of the categories of finitely dimensional $K, \hat{G}$- and $k, \hat{G}$-modules; it is well-known that we have contravariant functors
\[
g_{K} : k^{*}\mathcal{Gr} \to \mathcal{O}\mathcal{mod} \quad \text{and} \quad g_{k} : k^{*}\mathcal{Gr} \to \mathcal{O}\mathcal{mod}
\]
respectively mapping $\hat{G}$ on $G_{K}(\hat{G})$ and $G_{k}(\hat{G})$, and any $k^{*}$-group homomorphism $\hat{\varphi}: \hat{G} \to \hat{G}'$ on the corresponding restriction maps; it is clear that these restriction maps only depend on the $\hat{G}'$-conjugacy class of $\varphi$; in particular, these functors have an obvious extension to the category $k^{*}\tilde{\mathcal{Loc}}$ respectively mapping any $k^{*}\tilde{\mathcal{Loc}}$-object $(\hat{L}, Z)$ on $G_{K}(\hat{L})$ and $G_{k}(\hat{L})$. Moreover, recall that the Brauer decomposition maps define a natural map
\[
\partial : g_{K} \to g_{k}
\]
admitting a natural section.

3.5 Now, we consider the composed functor
\[
\text{ch}^{*}(\mathcal{F}^{\text{sc}}) \xrightarrow{\hat{\text{loc}}_{\mathcal{F}^{\text{sc}}}} k^{*}\tilde{\mathcal{Loc}} \xrightarrow{g_{K}} \mathcal{O}\mathcal{mod}
\]
and we define the ordinary Grothendieck group of the folded Frobenius $P$-category $(\mathcal{F}, \hat{\text{aut}}_{\mathcal{F}^{\text{sc}}})$ as the inverse limit
\[
G_{K}(\mathcal{F}, \hat{\text{aut}}_{\mathcal{F}^{\text{sc}}}) = \lim_{\leftarrow} (g_{K} \circ \hat{\text{loc}}_{\mathcal{F}^{\text{sc}}})
\]
it is a finitely generated free \( \mathcal{O} \)-module. Since the natural map from \( \hat{\text{loc}}_{\mathcal{F}^{\text{exc}}} \) to \( \hat{\text{aut}}_{\mathcal{F}^{\text{exc}}} \) induces a natural isomorphism \( g_k \circ \hat{\text{loc}}_{\mathcal{F}^{\text{exc}}} \cong g_k \circ \hat{\text{aut}}_{\mathcal{F}^{\text{exc}}} \), the natural map 3.4.2 determines a decomposition map

\[
\partial_{(\mathcal{F}, \hat{\text{aut}}_{\mathcal{F}^{\text{exc}}})} : \mathcal{G}_K(\mathcal{F}, \hat{\text{aut}}_{\mathcal{F}^{\text{exc}}}) \to \mathcal{G}_k(\mathcal{F}, \hat{\text{aut}}_{\mathcal{F}^{\text{exc}}})
\]

admitting a section. Note that, if \( \mathcal{F} \) admits an \( \mathcal{F} \)-normal \( \mathcal{F} \)-selfcentralizing subgroup \( Q \) of \( P \) then we have evident isomorphisms [5, Proposition 19.5]

\[
\mathcal{G}_K(\mathcal{F}, \hat{\text{aut}}_{\mathcal{F}^{\text{exc}}}) \cong \mathcal{G}_K(\hat{\mathcal{L}}(Q)) \quad \text{and} \quad \mathcal{G}_k(\mathcal{F}, \hat{\text{aut}}_{\mathcal{F}^{\text{exc}}}) \cong \mathcal{G}_k(\hat{\mathcal{L}}(Q))
\]

and \( \partial_{(\mathcal{F}, \hat{\text{aut}}_{\mathcal{F}^{\text{exc}}})} \) is the corresponding Brauer decomposition map.

3.6 With the notation in 3.1 above, in order to relate the ordinary Grothendieck groups \( \mathcal{G}_K(\mathcal{F}, \hat{\text{aut}}_{\mathcal{F}^{\text{exc}}}) \) and \( \mathcal{G}_K(\mathcal{F}', \hat{\text{aut}}_{\mathcal{F}^{\text{exc}}}) \) we prove below that, denoting by \( i : \mathcal{F}^{\text{ex}} \to \mathcal{F}^{\text{exc}} \) the inclusion functor, the ordinary Grothendieck group \( \mathcal{G}_K(\mathcal{F}, \hat{\text{aut}}_{\mathcal{F}^{\text{exc}}}) \) can also be defined from the functor

\[
\hat{\text{loc}}_{\mathcal{F}^{\text{ex}}} = \hat{\text{loc}}_{\mathcal{F}^{\text{exc}}} \circ \text{ch}^+(i)
\]

as in the case of the modular Grothendieck group [5, Proposition 14.6]. Recall that, for any category \( \mathcal{C} \) and any contravariant functor \( m : \text{ch}^+(\mathcal{C}) \to \mathcal{O}\text{-mod} \), the inverse limit \( \varprojlim m \) coincides with the 0-cohomology group of the differential complex given by the differential maps [5, A3.11.2]

\[
d_n : \prod_q m(q, \Delta_n) \to \prod_r m(r, \Delta_{n+1})
\]

where \( q \) and \( r \) respectively run over the sets of functors \( \mathfrak{Xt}(\Delta_n, \mathcal{C}) \) and \( \mathfrak{Xt}(\Delta_{n+1}, \mathcal{C}) \), sending any family \( a = (a_q)_q \) to the family \( d_n(a) = (d_n(a)_r)_r \)

defined by

\[
d_n(a)_r = \sum_{i=0}^{n-1} (-1)^i m(\text{id}_{\Delta_0}, \delta_i^n)(a_{\Gamma_0\delta_i^n})
\]

Proposition 3.7 With the notation above, the functor \( \text{ch}^+(i) \) induces an \( \mathcal{O} \)-module isomorphism

\[
\mathcal{G}_K(\mathcal{F}, \hat{\text{aut}}_{\mathcal{F}^{\text{exc}}}) \cong \varprojlim (g_k \circ \hat{\text{loc}}_{\mathcal{F}^{\text{ex}}})
\]

Proof: First of all, we prove that the homomorphism from \( \varprojlim (g_k \circ \hat{\text{loc}}_{\mathcal{F}^{\text{ex}}}) \) to \( \varprojlim (g_k \circ \hat{\text{loc}}_{\mathcal{F}^{\text{exc}}}) \) induced by \( \text{ch}^+(i) \) is injective; consider a family \( X = (X_Q)_Q \), where \( Q \) runs over the set of all the \( \mathcal{F} \)-selfcentralizing subgroups of \( P \) and \( X_Q \) belongs to \( \mathcal{G}_K(\hat{\mathcal{L}}(Q)) \), and assume that \( d_0^{\text{ex}}(X) = 0 \) where \( d_0^{\text{ex}} \) denotes the corresponding differential map (cf. 3.6.3).
For such a $Q$, consider a dominant $F$-morphism $\rho: Q \to R$ such that $|R|/|Q|$ is maximal and the chain $r: \Delta_1 \to \mathcal{F}^\ell$ mapping 0 on $Q$, 1 on $R$ and 0•1 on $\rho$; then, we get (cf. 3.6.3)

$$0 = \text{res}_{\mathcal{F}_{\rho}(R)} \{ \text{id}_{\text{res}(0, 0, 0)} \} (X_R) - \text{res}_{\mathcal{F}_{\rho}(R)} \{ \text{id}_{\text{res}(0, 0, 0)} \} (X_Q)$$

3.7.2;

but, it follows from Proposition 2.7 that $R$ is $F$-radical and we know that the homomorphism $\mathcal{F}(R)_{\rho(Q)} \to \mathcal{F}(Q)$ induced by $\rho$ is surjective (cf. 2.6.2); in particular, the functor $\mathcal{F}_{\rho}$ provides a $k^*$-group isomorphism $\hat{\mathcal{L}}(r) \cong \hat{\mathcal{L}}(Q)$, so that we also get the isomorphism

$$\text{res}_{\mathcal{F}_{\rho}(Q)} \{ \text{id}_{\text{res}(0, 0, 0)} \} : \mathcal{G}_{\mathcal{K}}(\hat{\mathcal{L}}(Q)) \cong \mathcal{G}_{\mathcal{K}}(\hat{\mathcal{L}}(r))$$

3.7.3;

thus, if $X$ belongs to the kernel of the homomorphism induced by $\text{h}^+(i)$, we get $X_Q = 0$.

In order to prove the surjectivity, assume that a family $Y = (Y_R)_R$, where $R$ runs over all the $F$-radical subgroups of $P$ and $Y_R \in \mathcal{G}_{\mathcal{K}}(\hat{\mathcal{L}}(R))$, belongs to the kernel of the corresponding differential map $d_0^\ell$; we have to extend $Y$ to a family $X = (X_Q)_Q$, where $Q$ runs over the set of all the $F$-selfcentralizing subgroups of $P$ and $X_Q \in \mathcal{G}_{\mathcal{K}}(\hat{\mathcal{L}}(Q))$, fulfilling $d_0^\ell (X) = 0$. For any $F$-selfcentralizing subgroup $Q$ of $P$, with the notation above we can define

$$X_Q = (\text{res}_{\mathcal{F}_{\rho}(Q)} \{ \text{id}_{\text{res}(0, 0, 0)} \})^{-1}(\text{res}_{\mathcal{F}_{\rho}(Q)} \{ \text{id}_{\text{res}(0, 0, 0)} \} (Y_R))$$

3.7.4;

then, we claim that $d_0^\ell (X) = 0$.

Let $q: \Delta_1 \to \mathcal{F}^\ell$ be a chain, set $Q = q(0)$, $Q' = q(1)$ and $\varphi = q(0 \cdot 1)$; arguing by induction on $|P: Q|$ and then on $|Q'|/|Q|$, we will prove that we have $d_0^\ell (X)_q = 0$. If $Q = P$ then $\varphi$ is an $F$-semidirect-group, so that $q$ is an $F$-semidirect-chain, and therefore we have

$$d_0^\ell (X)_q = d_0^\ell (Y)_q = 0$$

3.7.5;

moreover, if we have $Q \neq P$ and $\varphi$ is an isomorphism, we may assume that we have chosen $\rho \circ \varphi^{-1}: Q' \to R$ as a dominant homomorphism for $Q'$ and considered the corresponding chain $r': \Delta_1 \to \mathcal{F}^\ell$, so that we have defined

$$X_Q' = (\text{res}_{\mathcal{F}_{\rho}(Q)} \{ \text{id}_{\text{res}(0, 0, 0)} \})^{-1}(\text{res}_{\mathcal{F}_{\rho}(Q)} \{ \text{id}_{\text{res}(0, 0, 0)} \} (Y_R))$$

3.7.6;

then, the functoriality of $\mathcal{F}_{\rho}$ forces $X_Q = \text{res}_{\mathcal{F}_{\rho}}(X_Q')$ and therefore we get (cf. 3.6.3)

$$d_0^\ell (X)_q = \text{res}_{\mathcal{F}_{\rho}(Q)} \{ \text{id}_{\text{res}(0, 0, 0)} \} (X_Q') - \text{res}_{\mathcal{F}_{\rho}(Q)} \{ \text{id}_{\text{res}(0, 0, 0)} \} (X_Q) = 0$$

3.7.7.
From now on, we assume that \(|P:Q| \neq 1 \neq |Q'|/|Q|\); firstly note that, for another \(\mathcal{F}\)-selfcentralizing subgroup \(Q''\) of \(P\) and an \(\mathcal{F}\)-morphism \(\varphi':Q'\to Q''\), considering the chains

\[
q': \Delta_1 \to \mathcal{F}_c^e, \quad q'' : \Delta_1 \to \mathcal{F}_c^e \quad \text{and} \quad q^{Q''} : \Delta_2 \to \mathcal{F}_c^e
\]

respectively mapping 0 on \(Q', Q\) and \(Q', Q''\) and \(Q', 0\) on \(\varphi', \varphi'\circ \varphi\) and \(\varphi\), 2 on \(Q''\), and 1•2 on \(\varphi'\), the equality \(d_1^e \circ d_0^e = 0\) implies that (cf. 3.6.3)

\[
\text{res}_{\text{loc}}^e (\rho_{Q''} \circ \delta_1, \delta_1' d_1) (d_0^e (X)_{q''}) = \text{res}_{\text{loc}}^e (\rho_{Q''} \circ \delta_2, \delta_2' d_1) (d_0^e (X)_{q}) + \text{res}_{\text{loc}}^e (\rho_{Q''} \circ \delta_1' d_1) (d_0^e (X)_{q'})
\]

but, by our induction hypothesis, we already know that \(d_0^e (X)_{q'} = 0\); hence, we get

\[
\text{res}_{\text{loc}}^e (\rho_{Q''} \circ \delta_1, \delta_1' d_1) (d_0^e (X)_{q''}) = \text{res}_{\text{loc}}^e (\rho_{Q''} \circ \delta_1' d_1) (d_0^e (X)_{q'})
\]

On the one hand, set \(N = N_R (\rho(Q))\) and consider the chains

\[
n : \Delta_1 \to \mathcal{F}_c^e, \quad i^R_N : \Delta_1 \to \mathcal{F}_c^e \quad \text{and} \quad r : \Delta_1 \to \mathcal{F}_c^e
\]

where \(r\) is defined as above, \(i^R_N\) by the inclusion \(N \subset R\) and \(n\) by the restriction of \(\rho\) from \(Q\) to \(N\); then, we have \(d_0^e (X)_n = 0\) by definition 3.7.4. If \(\rho\) is an isomorphism then \(n = r\) and therefore \(d_0^e (X)_n = 0\). Otherwise, it follows from the induction hypothesis that we still have \(d_0^e (X)_{i^R_N} = 0\); but, since the image in \(\mathcal{F}(Q)\) of any element \(\sigma\) of \(\mathcal{F}(N)_{\rho(Q)}\) can be lifted to and element \(\sigma\) of \(\mathcal{F}(R)\) which stabilizes \(\rho(Q)\) and therefore it stabilizes \(N\), the difference between \(\sigma\) and the restriction of \(\sigma\) to \(N\) belongs to \(\mathcal{F}(N)_{\rho(Q)}(N)\) [5, Corollary 4.7]; consequently, any element of \(\mathcal{F}(N)_{\rho(Q)}\) can be lifted to and element of \(\mathcal{F}(R)\).

Thus, \textit{mutatis mutandis}, considering the chain \(\mathfrak{n}^R : \Delta_2 \to \mathcal{F}_c^e\) extending \(n\) and mapping 2 on \(R\) and 1•2 on the inclusion map \(N \to R\), we have a \(k^*\)-\(\mathfrak{L}\)-iso-

\[
\text{iso}_{\mathcal{F}} (\rho_{N} \circ \delta_1, \delta_1' d_1) : \tilde{\mathcal{L}}(n^R) \cong \tilde{\mathcal{L}}(n)
\]

hence, since \(n^R \circ \delta_1' = r\), by equality 3.7.10 we also have \(d_0^e (X)_n = 0\).

On the other hand, set \(N' = N_{Q'} (\varphi(Q))\) and consider the chains

\[
n' : \Delta_1 \to \mathcal{F}_c^e, \quad i^{Q'}_N : \Delta_1 \to \mathcal{F}_c^e \quad \text{and} \quad q : \Delta_1 \to \mathcal{F}_c^e
\]
where \( n' \) is defined by the restriction of \( \varphi \) from \( Q \) to \( N' \) and \( i_Q^{N'} \) by the inclusion \( N' \subset Q' \); by our induction hypothesis we have \( d_0^n(X)_{Q'} = 0 \), and it is clear that any element of \( \mathcal{F}(Q')_{\varphi(Q)} \) stabilizes \( N' \). As above, considering the chain \( n(Q') : \Delta_2 \rightarrow \mathcal{F}^{\infty} \) extending \( n' \), and mapping 2 on \( Q' \) and \( 1 \cdot 2 \) on the inclusion map \( N' \rightarrow Q' \), we still have a \( k^* \)-group exoisomorphism

\[
\hat{\mathfrak{c}}_{\mathcal{F}^{\infty}}(\text{id}_{n(Q') \circ \delta^1}, \delta^1_1) : \hat{\mathcal{L}}(n(Q')) \cong \hat{\mathcal{L}}(q) \quad 3.7.14;
\]

hence, according to equality 3.7.10, in order to prove that \( d_0^n(X)_q = 0 \), it suffices to prove that we have \( d_0^n(X)_{q''} = 0 \).

That is to say, we may assume that \( Q' = N' \) normalizes \( \varphi(Q) \); in this case, it follows from [5, Proposition 2.7] that there is an \( \mathcal{F} \)-morphism \( \zeta' : Q' \rightarrow P \) such that \( \zeta' (\varphi(Q)) \) is fully normalized in \( \mathcal{F} \) and then, from [5, condition 2.8.2] that there are an \( \mathcal{F} \)-morphism \( \eta : N \rightarrow P \) and an element \( \sigma \in \mathcal{F}(Q) \) fulfilling \( \eta (\rho (\sigma (u))) = \zeta' (\varphi(u)) \) for any \( u \in Q \); actually, up to modifying our choice of \( \zeta' \), we may assume that \( \sigma = \text{id}_Q \). Now, \( \eta(N) \) and \( \zeta'(Q') \) normalize \( \varphi(Q) \) and we consider the group \( N'' = \langle \eta(N), \zeta'(Q') \rangle \) and the chains

\[
\epsilon : \Delta_1 \rightarrow \mathcal{F}^{\infty} \quad \text{and} \quad n'' : \Delta_1 \rightarrow \mathcal{F}^{\infty} \quad \text{and} \quad n^{N''} : \Delta_2 \rightarrow \mathcal{F}^{\infty} \quad 3.7.15
\]

where \( \epsilon \) and \( n'' \) are respectively defined by the homomorphisms \( N \rightarrow N'' \) and \( Q \rightarrow N'' \) determined by \( \eta \) and \( \zeta' \circ \varphi \), and \( n^{N''} \) extends \( n \) mapping 2 on \( N'' \) and \( 1 \cdot 2 \) on the homomorphism \( N \rightarrow N'' \) determined by \( \eta \).

We already know that \( d_0^n(X)_n = 0 \) and it follows from our induction hypothesis that \( d_0^n(X) \epsilon = 0 \); but, any element of \( \mathcal{F}(Q) \) can be lifted via \( \rho \) to an element of \( \mathcal{F}(R)_{\rho(Q)} \) (cf. 2.6.2) which stabilizes \( N \). In particular, for any element \( \sigma \) in

\[
\mathcal{F}(N''), \zeta'(\varphi(Q))) = \mathcal{F}(n'') \quad 3.7.16,
\]

we have an element \( \hat{\sigma} \) in \( \mathcal{F}(N) \) such that \( \eta \circ \hat{\sigma} \) coincides with \( \sigma \circ \eta \) over \( \rho(Q) \), and therefore it follows from [5, Proposition 4.6] that it suffices to modify our choice of \( \hat{\sigma} \) by composing it with the conjugation by a suitable element of \( Z(\rho(Q)) \subset N \) to get the equality \( \eta \circ \hat{\sigma} = \sigma \circ \eta \). In conclusion, we get the \( k^* \)-isomorphism

\[
\hat{\mathfrak{c}}_{\mathcal{F}^{\infty}}(\text{id}_{n^{N''} \circ \delta^1}, \delta^1_1) : \hat{\mathcal{L}}(n^{N''}) \cong \hat{\mathcal{L}}(n'') \quad 3.7.17;
\]

consequently, by equality 3.7.10, we still get \( d_0^n(X)_{q''} = 0 \).

Similarly, consider the chains

\[
\delta' : \Delta_1 \rightarrow \mathcal{F}^{\infty}, \quad n'' : \Delta_1 \rightarrow \mathcal{F}^{\infty} \quad \text{and} \quad q^{N''} : \Delta_2 \rightarrow \mathcal{F}^{\infty} \quad 3.7.18
\]
where \( z' \) is defined by the homomorphism \( Q' \to N'' \) determined by \( \zeta' \), and \( q^{N''} \) extends \( q = n' \) mapping 2 on \( N'' \) and \( 1 \cdot 2 \) on the homomorphism \( Q' \to N'' \) determined by \( \zeta' \); it follows from our induction hypothesis that \( d_0^{n'}(X)_{z'} = 0 \) and we already know that \( d_0^{n''}(X)_{z''} = 0 \). Since \( N'' \) contains

\[
\eta\left(Z(\rho(Q))\right) = Z\left(\zeta'(\varphi(Q))\right)
\]

it follows from [5, statement 2.10.1] that the automorphism of \( \zeta'(\varphi(Q)) \) determined by any \( \sigma' \in F(Q')_{\varphi(Q)} \) via \( \zeta' \) can be extended to an \( F'' \)-automorphism \( \sigma'' \) of \( N'' \) and then, arguing as above, it follows from [5, Proposition 4.6] that we may choose \( \sigma'' \) fulfilling

\[
\sigma''(\zeta'(u')) = \zeta'(\sigma'(u'))
\]

for any \( u' \in Q' \). In conclusion, we get the \( k^* \)-isomorphism

\[
\tilde{\text{loc}}_{F''} (\text{id}_{q^{N''} \circ \delta_2}, \delta_2^1): \tilde{\mathcal{L}}(q^{N''}) \cong \tilde{\mathcal{L}}(q)
\]

consequently, again by equality 3.7.10, we get \( d_0^{n''}(X)_{q} = 0 \). We are done.

4 Functoriality of the Grothendieck groups of folded Frobenius \( P \)-categories

4.1 With the notation in §3, let \( P' \) be a second finite \( p \)-group, \( \mathcal{F}' \) a Frobenius \( P' \)-category and \( \alpha: P' \to P \) an \( (\mathcal{F}', \mathcal{F}) \)-functorial group homomorphism [5, 12.1] mapping any \( \mathcal{F}' \)-radical subgroup of \( P' \) on a \( \mathcal{F} \)-selfcentralizing subgroup of \( P \). More generally, let \( \mathcal{X}' \) be a set of subgroups \( Q' \) of \( P' \) such that \( \alpha(Q') \) is \( \mathcal{F} \)-selfcentralizing, which contains the \( \mathcal{F}' \)-radical subgroups of \( P' \) and is stable by \( \mathcal{F}' \)-isomorphisms, and denote by \( \mathcal{F}'^{\mathcal{X}'} \) the full subcategory of \( \mathcal{F}' \) over \( \mathcal{X}' \). Then the restriction \( f_{\alpha}^{\mathcal{X}'}: \mathcal{F}'^{\mathcal{X}'} \to \mathcal{F}'^{\mathcal{X}} \) of the Frobenius functor \( f_{\alpha}: \mathcal{F}' \to \mathcal{F} \) induces a natural map

\[
\text{aut}_{f_{\alpha}^{\mathcal{X}'}}, \quad \text{aut}_{\mathcal{F}'^{\mathcal{X}'}} \to \text{aut}_{\mathcal{F}^{\mathcal{X}}} \circ \text{ch}^{*}(f_{\alpha}^{\mathcal{X}'})
\]

and therefore the pull-back

\[
\begin{array}{ccc}
\text{aut}_{f_{\alpha}^{\mathcal{X}'}} & \to & \text{ch}^{*}(f_{\alpha}^{\mathcal{X}'}) \\
\uparrow & & \uparrow \\
\text{aut}_{\mathcal{F}'^{\mathcal{X}'}}, \quad \text{aut}_{\mathcal{F}^{\mathcal{X}}} \circ \text{ch}^{*}(f_{\alpha}^{\mathcal{X}'}) & \to & \text{ch}^{*}(f_{\alpha}^{\mathcal{X}'})
\end{array}
\]
lifts the functor $\text{aut}_{\mathcal{F}_x}$ to the category $k^*-\mathcal{G}_x$ (cf. 3.1). Moreover, we already know from Theorem 2.9 that this lifting can be extended to a unique functor $\widehat{\text{aut}}_{\mathcal{F}_{x'}}$ lifting $\text{aut}_{\mathcal{F}_{x'}}$ and therefore we have an ordinary and a modular Grothendieck groups

$$G_k(\mathcal{F}^I, \widehat{\text{aut}}_{\mathcal{F}_{x'}}) = \lim_{\leftarrow} (g_k \circ \hat{\text{loc}}_{\mathcal{F}_{x'}})$$

$$G_k(\mathcal{F}^I, \text{aut}_{\mathcal{F}_{x'}}) = \lim_{\leftarrow} (g_k \circ \text{loc}_{\mathcal{F}_{x'}})$$

4.2 Now, according to Proposition 3.7, in order to define the restriction $\mathcal{O}$-module homomorphisms from the Grothendieck groups of the folded Frobenius $P$-category $(\mathcal{F}, \widehat{\text{aut}}_{\mathcal{F}_{x'}})$ to these ones, we have to exhibit a natural map

$$\text{loc}_{f_a} : \text{loc}_{\mathcal{F}_x} \longrightarrow \text{loc}_{\mathcal{F}_{x'}} \circ \text{ch}^*(\hat{f}_a)$$

lifting the natural map $\text{aut}_{f_a}$; explicitly, setting $\mathcal{L}'(q') = \text{loc}_{\mathcal{F}_x}(q', \Delta_n)$ for any $\mathcal{F}_{x'}$-chain $q' : \Delta_n \rightarrow \mathcal{F}_{x'}$, we have to exhibit a suitable group homomorphism

$$\lambda_{q'} : \mathcal{L}'(q') \longrightarrow \mathcal{L}(f_a \circ q')$$

sending $Z(q'(n))$ to a subgroup of $Z(\alpha(q'(n)))$ and lifting the group homomorphism

$$(\text{aut}_{f_a})_{(q', \Delta_n)} : \mathcal{F}'(q') \longrightarrow \mathcal{F}(f_a \circ q')$$

which maps $\sigma' \in \mathcal{F}'(q')$ on $f_a \circ \sigma'$.

**Proposition 4.3** With the notation above, for any $\mathcal{F}_{x'}$-chain $q' : \Delta_n \rightarrow \mathcal{F}_{x'}$ such that $q'(n)$ is fully normalized in $\mathcal{F}$ and that $\mathcal{F}_{p^r}(q'(n))$ contains a Sylow $p$-subgroup of $\mathcal{F}(q')$, there exists a group homomorphism

$$\lambda_{q'} : \mathcal{L}'(q') \longrightarrow \mathcal{L}(f_a \circ q')$$

lifting $(\text{aut}_{f_a})_{(q', \Delta_n)} : \mathcal{F}'(q') \rightarrow \mathcal{F}(f_a \circ q')$ and mapping $u \in N_{\mathcal{F}_{p^r}}(q')$ on $\alpha(u)$. Moreover, the group $Z(\alpha(q'(n)))^{\alpha(N_{\mathcal{F}_{p^r}}(q'))}$ acts transitively on the set of such homomorphisms.

**Proof:** In order to apply [5, Lemma 18.8], let us consider the groups $\mathcal{L}'(q')$ and $\mathcal{L}(f_a \circ q')$, and the group homomorphisms

$$(\text{aut}_{f_a})_{(q', \Delta_n)} \circ \pi_{q'} : \mathcal{L}'(q') \longrightarrow \mathcal{F}(f_a \circ q')$$

$$\alpha_{q'} : N_{\mathcal{F}_{p^r}}(q') \rightarrow N_{\mathcal{F}_{p^r}}(f_a \circ q') \subset \mathcal{L}(f_a \circ q')$$

4.3.2
where $\pi_q : \mathcal{L}'(q') \to \mathcal{F}'(q')$ denotes the structural homomorphism and $\alpha(q')$ the restriction of $\alpha$ from $N_{P'}(q')$ to $N_P(\tilde{f}_n \circ q')$; let $R'$ be a subgroup of $P'$ and, setting $Q' = q'(n)$, $x'$ an element of $\mathcal{L}'(q') \subset \mathcal{L}'(Q')$ such that $R'^{\text{loc}} \subset P'$; then, $\tilde{f}_n^{x'} \circ \pi_q(x')$ belongs to $\mathcal{F}(\tilde{f}_n^{x'} \circ q')$ which is contained in $\mathcal{F}(\alpha(Q'))$, and $x'$ actually determines an $\mathcal{F}_{\mathcal{L}'(Q'),Q'}$-morphism from $R'$ to $N_{P'}(Q')$ [5, 17.2]; in particular, according to [5, Theorem 18.6], $x'$ also determines an $N_{\mathcal{F}',Q'}(Q')$-morphism $\xi'$ from $R'$ to $N_{P'}(Q')$ [5, 17.6].

Moreover, it follows from [5, Proposition 2.7] that there is an $\mathcal{F}$-morphism

$$
\zeta : N_P(\alpha(Q')) \longrightarrow P
$$

such that $Q = \zeta(\alpha(Q'))$ is fully normalized in $\mathcal{F}$; in particular, we can consider the $\mathcal{F}$-localizer $\mathcal{L}(Q)$ of $Q$, the Frobenius $N_P(Q)$-category $N_{\mathcal{F}}(Q)$ and the $\mathcal{F}_Q(Q)$-locality $N_{\mathcal{F},Q}(Q)$ [5, 17.6]. Then, since $\alpha$ is $(\mathcal{F}', \mathcal{F})$-functorial, denoting by $\alpha_Q'$ the restriction of $\alpha$ from $N_{P'}(Q')$ to $N_P(\alpha(Q'))$, it is quite clear that the composition

$$
\zeta \circ \alpha_Q' : N_{P'}(Q') \longrightarrow N_P(Q)
$$

is an $(N_{\mathcal{F}',Q'}(Q'), N_{\mathcal{F},Q}(Q))$-functorial group homomorphism [5, 12.1] which induces a functor

$$
l_{\zeta \circ \alpha_Q'} : N_{\mathcal{F}',Q'}(Q') \longrightarrow N_{\mathcal{F},Q}(Q)
$$

Thus, $\xi'$ determines an $N_{\mathcal{F},Q}(Q)$-morphism $\xi$ from $R = \zeta(\alpha(R'))$ to $N_P(Q)$.

But, according to [5, Theorem 18.6] again, the categories $N_{\mathcal{F},Q}(Q)$ and $N_{\mathcal{F}_{\mathcal{L}(Q),Q}}$ [5, 17.2] are equivalent to each other and therefore there is $x \in \mathcal{L}(Q)$ such that $\xi(u) = u^x$ for any $u \in R$ and that we have $\pi_Q(x) = \zeta Q(\pi_Q'(x'))$ where $\pi_Q : \mathcal{L}(Q) \to \mathcal{F}(Q)$ and $\pi_Q' : \mathcal{L}'(Q') \to \mathcal{F}'(Q')$ denote the structural homomorphisms. Finally, considering the $\mathcal{F}$-chain $q : \Delta_n \to \mathcal{F}^\ast$ mapping $i \in \Delta_{n-1}$ on $\alpha(q'(i))$, $n$ on $Q$, any $\Delta_{n-1}$-morphism $j \cdot i$ on $(\tilde{f}_n^{x'} \circ q')(j \cdot i)$ and $n-1 \cdot n$ on the composition of $(\tilde{f}_n^{x'} \circ q')(n-1 \cdot n)$ with the isomorphism $\alpha(Q') \cong Q$ determined by $\zeta$, we have an obvious $\mathfrak{ch}^* (\mathcal{F}^\ast)$-isomorphism

$$
(\nu_\zeta, \text{id}_\Delta_n) : (q, \Delta_n) \longrightarrow (\tilde{f}_n^{x'} \circ q', \Delta_n)
$$

which the functor $\text{loc}_{\mathcal{F}^\ast}$ sends to a group isomorphism

$$
\lambda_\zeta : \mathcal{L}(q) \cong \mathcal{L}(\tilde{f}_n^{x'} \circ q')
$$

At this point, it is easily checked that $\alpha(u'^{x'}) = \alpha(u')^{\lambda_\zeta(x)}$ for any $u' \in R'$ and that we have

$$
\pi_{\alpha(Q')}(\lambda_\zeta(x)) = \alpha_{Q'}(\pi_Q'(x'))
$$
that is to say, in our setting of [5, Lemma 18.8], condition 18.8.1 holds and therefore there is a group homomorphism

\[\lambda_{q'} : \mathcal{L}'(q') \longrightarrow \mathcal{L}(f_a^{x'} \circ q')\]

fulfilling the announced conditions. Moreover, always by the same lemma, the Abelian group \(Z((f_a^{x'} \circ q')(n))^{\alpha(N_{P'}(q'))}\) acts transitively over the set of such group homomorphisms.

**Theorem 4.4** With the notation and the hypothesis above, there is a unique natural map

\[\text{loc}_{x'} : \text{loc}_{F^{x'}} \longrightarrow \text{local} \circ \hat{c}^*(f_a^{x'})\]

sending any \(F^{x'}\)-chain \(q' : \Delta_n \rightarrow F^{x'}\) such that \(q'(n)\) is fully normalized in \(F\) and that \(F_{P'}(q'(n))\) contains a Sylow \(p\)-subgroup of \(F'(q')\), to the \(Z(\alpha(q'(n)))\)-conjugacy class of

\[\lambda_{q'} : \mathcal{L}'(q') \longrightarrow \mathcal{L}(f_a^{x'} \circ q')\]

In particular, this natural map lifts \(\text{aut}_{x'}\) and, if \(t' : \Delta_m \rightarrow F^{x'}\) is a chain, then we have \(\lambda_{t'}(u') = \alpha(u')\) for a suitable representative \(\lambda_{t'}\) of \((\text{loc}_{i_{x'}})(t', \Delta_m)\) and any \(u' \in N_{P'}(t')\).

**Proof:** For any \(F^{x'}\)-chain \(q' : \Delta_n \rightarrow F^{x'}\), it follows from [5, Proposition 2.7] that there is an \(F\)-morphism \(\zeta' : N_{P'}(q'(n)) \rightarrow P'\) such that \(\zeta'(q'(n))\) is fully normalized in \(F\) and we may assume that \(F_{P'}(q'(n))\) contains a Sylow \(p\)-subgroup of \(F'(q')\); consider the \(F^{x'}\)-chain \(q' : \Delta_n \rightarrow F^{x'}\) which coincides with \(q'\) over \(\Delta_{n-1}\) and maps \(n\) on \(q'(n)\) and \((n-1 \bullet n)\) on the composition of \(q'(n-1 \bullet n)\) with the \(F\)-isomorphism \(\zeta' : q'(n) \cong \zeta'(q'(n))\) determined by \(\zeta'\); we have an obvious \(\hat{c}^*(F^{x'})\)-isomorphism

\[(\nu_{\zeta'} : \text{id}_{\Delta_n}) : (q', \Delta_n) \cong (q', \Delta_n)\]

which the functor \(\text{loc}_{F^{x'}}\) sends to a class of group isomorphisms

\[\lambda_{q'} : \mathcal{L}'(q') \cong \mathcal{L}'(q')\]

and we may assume that \(\lambda_{q'}(u') = \zeta'(u')\) for any \(u' \in N_{P'}(q')\).
On the other hand, we have the $\mathcal{F}$-morphism

$$f_\alpha^{x'}(\zeta') : \alpha\left(N_{P'}(q'(n))\right) \to \alpha(P') \quad 4.4.5$$

which can be restricted to the $\mathcal{F}$-isomorphism

$$f_\alpha^{x'}(\zeta') : \alpha(q'(n)) \cong \alpha\left(\zeta'(q'(n))\right) \quad 4.4.6$$

then, considering the $\mathcal{F}''$-chains $f_\alpha^{x'} \circ q'$ and $f_\alpha^{x'} \circ \hat{q}'$, this $\mathcal{F}$-isomorphism induces an obvious $\mathfrak{ch}^*(\mathcal{F}^{x'})$-isomorphism

$$(\nu_{f_\alpha^{x'}}(\zeta'), \text{id}_{\Delta_n}) : (f_\alpha^{x'} \circ q', \Delta_n) \cong (f_\alpha^{x'} \circ \hat{q}', \Delta_n) \quad 4.4.7$$

which the functor $\text{loc}_{\mathcal{F}''}$ sends to a class of group isomorphisms

$$\lambda_{f_\alpha^{x'}}(\zeta') : \mathcal{L}(f_\alpha^{x'} \circ q') \cong \mathcal{L}(f_\alpha^{x'} \circ \hat{q}') \quad 4.4.8$$

and we may assume that $\lambda_{f_\alpha^{x'}}(\zeta')(u) = (f_\alpha^{x'}(\zeta'))(u)$ for any $u \in N_{P'}(f_\alpha^{x'} \circ q')$.

Moreover, Proposition 4.3 provides a group homomorphism

$$\lambda_{q'} : \mathcal{L}'(\hat{q}') \to \mathcal{L}(f_\alpha^{x'} \circ \hat{q}') \quad 4.4.9$$

fulfilling $\lambda_{q'}(u') = \alpha(u')$ for any $u' \in N_{P'}(\hat{q}')$, and we define $\text{loc}_{f_\alpha^{x'}}$ as the map sending $(q', \Delta_n)$ to the $Z(\alpha(q'(n)))$-conjugacy class of

$$\lambda_{q'} = (\lambda_{f_\alpha^{x'}}(\zeta'))^{-1} \circ \lambda_{\hat{q}'} \circ \lambda_{\zeta'} \quad 4.4.10$$

first of all, note that for any $u' \in N_{P'}(q')$ we get

$$\lambda_{q'}(u') = ((\lambda_{f_\alpha^{x'}}(\zeta'))^{-1} \circ \lambda_{\hat{q}'})(\lambda_{\zeta'}(u')) = (\lambda_{f_\alpha^{x'}}(\zeta'))^{-1}\left((\lambda_{\hat{q}'}(\zeta')(u'))\right)$$

$$= (\lambda_{f_\alpha^{x'}}(\zeta'))^{-1}\left((\alpha \circ \zeta')(u')\right) \quad 4.4.11$$

$$= (\lambda_{f_\alpha^{x'}}(\zeta'))^{-1}\left((\alpha \circ f_\alpha(q')(\zeta') \circ \alpha)(u')\right) = \alpha(u')$$

In order to prove the naturality of this correspondence, let

$$(\mu', \delta) : (r', \Delta_m) \to (q', \Delta_n) \quad 4.4.12$$

be a $\mathfrak{ch}^*(\mathcal{F}^{x'})$-morphism and consider an $\mathcal{F}'$-morphism $\xi' : \tau'(m) \to P'$ such that $\xi'((r'(m)))$ is fully normalized in $\mathcal{F}'$ and that $\mathcal{F}_{P'}(q'(n))$ contains a Sylow
p-subgroup of $\mathcal{F}'(q')$, and the $\mathcal{F}'^\lambda$-chain $\xi' : \Delta_m \to \mathcal{F}'^\lambda$ which coincides with $\xi'$ over $\Delta_{m-1}$ and maps $m$ on $\xi'(\xi'(m))$ and $(m-1 \cdot m)$ on the composition of $\xi'(m-1 \cdot m)$ with the $\mathcal{F}'$-isomorphism $\xi'(\xi'(m))$ determined by $\xi'$. Then, the $\text{ch}^*(\mathcal{F}'^\lambda)$-isomorphisms $(\nu'_\xi, \text{id}_{\Delta_m})$ and $(\nu'_\xi, \text{id}_{\Delta_m})$ induce the commutative $\text{ch}^*(\mathcal{F}'^\lambda)$-diagram

$$
\begin{array}{ccc}
(\xi', \Delta_m) & \xrightarrow{\mu'_\xi, \delta} & (\xi', \Delta_n) \\
\| & & \| \\
(\bar{\xi}', \Delta_m) & \xrightarrow{\mu'_\xi, \delta} & (\bar{\xi}', \Delta_n)
\end{array}
$$

4.4.13.

Moreover, $\xi'_\alpha$ maps this $\text{ch}^*(\mathcal{F}'^\lambda)$-diagram on the commutative $\text{ch}^*(\mathcal{F}'^\lambda)$-diagram

$$
\begin{array}{ccc}
(\xi'_\alpha \circ \xi', \Delta_m) & \xrightarrow{\lambda'_{\xi'_\alpha} \circ \mu'_\xi} & (\xi'_\alpha \circ \xi', \Delta_m) \\
\| & & \| \\
(\xi'_\alpha \circ \bar{\xi}', \Delta_m) & \xrightarrow{\lambda'_{\xi'_\alpha} \circ \mu'_\xi} & (\xi'_\alpha \circ \bar{\xi}', \Delta_m)
\end{array}
$$

4.4.14.

Now, it is quite clear that it suffices to prove the commutativity up to $Z(\alpha(\xi'(n)))$-conjugation of the following diagram of group homomorphisms

$$
\begin{array}{ccc}
\mathcal{L}(\xi'_\alpha \circ \xi') & \xrightarrow{\lambda'_{\xi'_\alpha} \circ \mu'_\xi} & \mathcal{L}(\xi'_\alpha \circ \bar{\xi}') \\
\lambda' \circ & & \lambda' \circ \\
\mathcal{L}'(\xi') & \xrightarrow{\lambda'_{\xi'_\alpha} \circ \mu'_\xi} & \mathcal{L}'(\bar{\xi}')
\end{array}
$$

4.4.15.

where $\lambda_{\xi'_\alpha} \circ \mu'_\xi$ and $\lambda'_{\xi'_\alpha}$ are respective representatives of $\text{loc}_{\mathcal{F}'}(\xi'_\alpha \circ \mu'_\xi, \delta)$ and $\text{loc}_{\mathcal{F}'}(\mu'_\xi, \delta)$; this commutativity up to $Z(\alpha(\xi'(n)))$-conjugation is a consequence of the uniqueness part of [5, Lemma 18.8] and of [5, Remark 18.9] applied to the groups $\mathcal{L}'(\xi')$ and $\mathcal{L}(\xi'_\alpha \circ \bar{\xi}')$, and to the group homomorphisms $\lambda_{\xi'_\alpha} \circ \mu'_\xi$, $\lambda'_{\xi'_\alpha}$, and $\lambda'_{\xi'_\alpha} \circ \mu'_\xi$; indeed, denoting by

$$
\pi_{\xi'_\alpha \circ \bar{\xi}'} : \mathcal{L}(\xi'_\alpha \circ \bar{\xi}') \to \mathcal{F}(\xi'_\alpha \circ \bar{\xi}')
$$

4.4.16.

the structural homomorphisms, it follows from [5, Proposition 18.16] and
from Proposition 4.3 that we have (cf. 4.1.1)

\[ \pi_{\hat{f}_n' \circ \hat{q}'} \circ \lambda_{\hat{x}'} \circ \lambda_{\hat{\mu}'} = \text{aut}_{\hat{F}}(\hat{f}_n' \circ \hat{\mu}' \circ \delta) \circ \pi_{\hat{f}_n' \circ \hat{\nu}'} \circ \lambda_{\hat{\nu}'} \]

\[ = \text{aut}_{\hat{F}}(\hat{f}_n' \circ \hat{\mu}' \circ \delta) \circ (\text{aut}_{\hat{F}}(\hat{\mu}') \circ \pi_{\hat{\nu}'}) \]

\[ = (\text{aut}_{\hat{f}_n})(\hat{q}' \circ \text{aut}_{\hat{F}}(\hat{\mu}') \circ \pi_{\hat{\nu}'}) \]

\[ = (\text{aut}_{\hat{f}_n})(\hat{q}' \circ \lambda_{\hat{\mu}' \circ \lambda_{\hat{\nu}'}}) \]

Finally, set \( \hat{Q}' = \hat{q}'(n) \) and consider an \( F \)-morphism

\[ \zeta : N_P(\alpha(\hat{Q}')) \rightarrow P \]

such that \( \hat{Q} = \zeta(\alpha(\hat{Q}')) \) is fully normalized in \( F \) [5, Proposition 2.7] and that \( \zeta(N_P(\hat{f}_n' \circ \hat{\nu}')) \) is contained in \( N_P(\hat{Q}) \); denote by \( \zeta_\ast : \alpha(\hat{Q}') \cong \hat{Q} \) the \( F \)-isomorphism determined by \( \zeta \), and by \( \lambda_{\zeta_\ast} \) a representative of \( \text{loc}_{\hat{F}}(\nu_\ast, \iota_n) \) where

\[ (\nu_\ast, \iota_n) : (\hat{f}_n' \circ \hat{\nu}', \Delta_n) \rightarrow (\hat{Q}, \Delta_0) \]

is the \( \text{ch}(\hat{F}) \)-morphism formed by the map \( \iota_n : \Delta_0 \rightarrow \Delta_n \) sending \( 0 \) to \( n \) and by the natural map \( \nu_\ast \) determined by \( \zeta_\ast \); moreover, according to [5, Proposition 18.16], we may assume that \( \lambda_{\zeta_\ast}(u) = \zeta(u) \) for any \( u \in N_P(\hat{f}_n' \circ \hat{\nu}') \).

On the other hand, denoting by \( \hat{R}' \) the image by \( \hat{\nu}'(\delta(n) \bullet m) \) of \( \hat{\nu}'(\delta(n)) \) in \( \hat{r}'(m) \), it follows from [5, statement 2.10.1] that there is an \( F \)-morphism \( \hat{\xi}' : N_P(\hat{R}') \rightarrow N_P(\hat{Q}') \) extending the composition \( \hat{\nu}' : \hat{R} \cong \hat{Q}' \) of the inverse of the isomorphism \( \hat{\nu}(\delta(n)) \cong \hat{R}' \) induced by \( \hat{\nu}'(\delta(n) \bullet m) \) with the isomorphism \( \hat{\mu}'_\ast : \hat{\nu}'(\delta(n)) \cong \hat{Q}' \); then, always from [5, Proposition 18.16], we know that, for some element \( x' \) of \( L'(\hat{q}') \) such that \( \lambda_{\hat{\mu}'_\ast}(N_P(\hat{\nu}')) \subset N_P(\hat{q}') \) and any \( u' \in N_P(\hat{\nu}') \), we may assume that we have \( \lambda_{\hat{\mu}'_\ast}(u') = \xi'(u') \). Moreover, since we have (cf. 4.2.3)

\[ (\text{aut}_{\hat{f}_n})(\hat{q}', \Delta_n) \subset \text{aut}_{\hat{F}}(\hat{q}') \]

setting \( x = \lambda_{\zeta_\ast}(\lambda_{\hat{q}'}(x')) \), we get (cf. Proposition 4.3)

\[ (\lambda_{\zeta_\ast} \circ \lambda_{\hat{q}'} \circ \lambda_{\hat{\mu}'_\ast})(N_P(\hat{\nu}')) \subset \lambda_{\zeta_\ast}(\lambda_{\hat{\mu}'_\ast}(N_P(\hat{q}'))) \]

\[ \subset \lambda_{\zeta_\ast}(N_P(\hat{f}_n' \circ \hat{\nu}')) \subset N_P(\hat{Q}) \]
Similarly, setting $\hat{R} = \alpha(\hat{R}')$, it follows from [5, statement 2.10.1] that there is an $F$-morphism $\xi : N_P(\hat{R}) \to N_P(\hat{Q})$ extending the composition $\hat{\varphi} : \hat{R} \cong \hat{Q}$ of the inverse of the isomorphism $\alpha(\hat{v}(\delta(n))) \cong \hat{R}$ determined by $f_0(\hat{v}(\delta(n) \bullet m))$ with the isomorphism
\[
\zeta_n \circ (f_{\alpha}' \ast \mu')_n : \alpha(\hat{v}(\delta(n))) \cong \hat{Q}
\]
by [5, Proposition 18.16], since $\lambda_{\zeta_n} \circ \lambda_{f_{\alpha}' \ast \mu'}$ is a representative of the composition $\lambda_{\varphi} \circ (\zeta_n, \iota_n) \circ (f_{\alpha}' \ast \mu', \delta)$, up to modifying our choice of $\zeta$ according to [5, Remark 18.17], we may assume that
\[
(\lambda_{\zeta_n} \circ \lambda_{f_{\alpha}' \ast \mu'})(u) = \xi(u)^x
\]
for any $u \in N_P(f_{\alpha}' \circ \hat{v})$.

Moreover, since we have $\hat{\varphi} = \zeta_n \circ f_{\alpha}'(\hat{\varphi}')$, it is easily checked that, denoting by $\alpha_{\hat{Q}}$ and $\alpha_{\hat{R}}$ the respective restrictions of $\alpha$ from $N_{P'}(\hat{Q}')$ to $N_P(\hat{Q})$, and from $N_{P'}(\hat{R}')$ to $N_P(\hat{R})$, the compositions
\[
\zeta \circ \alpha_{\hat{Q}} \circ \xi' \quad \text{and} \quad \xi \circ \alpha_{\hat{R}}
\]
restricted to $\hat{R}'$ coincide with each other. But, $\zeta \circ \alpha_{\hat{Q}} \circ \xi'$ maps $N_{P'}(\hat{R}')$ on a subgroup of $N_P(\hat{Q})$; hence, since $\hat{R}$ is $F$-selfcentralizing, $\xi \circ \alpha_{\hat{R}}$ maps $N_{P'}(\hat{R}')$ on the same subgroup of $N_P(\hat{Q})$ and it follows from [5, Proposition 4.6] that a suitable modification of our choice of $\xi'$ suffices to guarantee that we get
\[
\zeta \circ \alpha_{\hat{Q}} \circ \xi' = \xi \circ \alpha_{\hat{R}}
\]
At this point, for any $u' \in N_{P'}(\hat{v}')$ we have (cf. 4.4.23 and 4.4.25)
\[
(\lambda_{\zeta_n} \circ \lambda_{\hat{Q}'} \circ \lambda_{\hat{R}'}')(u') = (\lambda_{\zeta_n} \circ \lambda_{\hat{Q}'})((\lambda_{\hat{R}'}(u'))^x)
\]
\[
= \lambda_{\zeta_n}(\alpha(\lambda_{\hat{Q}'}(u'))^{\lambda_{\hat{R}'}(x)}) = (\xi \circ \alpha_{\hat{R}}(u'))^x
\]
\[
= (\xi \circ \alpha_{\hat{R}}(u'))^x = (\lambda_{\zeta_n} \circ \lambda_{\hat{R}' \ast \mu'})(\alpha(u'))
\]
\[
= (\lambda_{\zeta_n} \circ \lambda_{\hat{R}' \ast \mu'} \circ \lambda_{\hat{R}})(u)
\]
in particular, the compositions
\[
\lambda_{\hat{Q}'} \circ \lambda_{\hat{R}'}' \quad \text{and} \quad \lambda_{\hat{R}' \ast \mu'} \circ \lambda_{\hat{R}}
\]
restricted to $N_{P'}(\hat{v}')$ coincide with each other. Now, as announced above, according to this statement and to equality 4.4.17, it suffices to apply the uniqueness part of [5, Lemma 18.8] and [5, Remark 18.9]. We are done.
4.5 Now, it follows from 3.3 that the natural map in 4.1.2

\[ \hat{\text{aut}}_{\hat{f}_{\alpha}^{'}} : \hat{\text{aut}}_{\hat{f}^{'}} \rightarrow \hat{\text{aut}}_{F^{'}} \circ \text{ch}^*(f^{x^{'}}_{\alpha}) \]  

and the natural map in theorem 4.4 above

\[ \hat{\text{loc}}_{\hat{f}_{\alpha}^{'}} : \hat{\text{loc}}_{\hat{f}^{'}} \rightarrow \hat{\text{loc}}_{F^{'}} \circ \text{ch}^*(f^{x^{'}}_{\alpha}) \]

determine a new natural map

\[ \hat{\text{loc}}_{\hat{f}_{\alpha}^{'}} : \hat{\text{loc}}_{\hat{f}^{'}} \rightarrow \hat{\text{loc}}_{F^{'}} \circ \text{ch}^*(f^{x^{'}}_{\alpha}) \]

then, composing this natural map with the contravariant functors \( g_K \) and \( g_k \) in 3.4 above, we get the \( \mathcal{O}\)-mod-valued natural maps

\[
\begin{align*}
\text{Res}_{f_{\alpha}} : G_K(F', \hat{\text{aut}}_{F'}) & \rightarrow G_K(F, \hat{\text{aut}}_{F}) \\
\text{res}_{f_{\alpha}} : G_k(F', \hat{\text{aut}}_{F'}) & \rightarrow G_k(F, \hat{\text{aut}}_{F})
\end{align*}
\]

4.6 But, from Proposition 3.7 above and from [5, Proposition 14.6] together with Proposition 2.7 above, we know that

\[
\begin{align*}
\lim_{\leftarrow}(g_K \circ \hat{\text{loc}}_{F^{'}} \circ \text{ch}^*(f^{x^{'}}_{\alpha})) & \cong G_K(F', \hat{\text{aut}}_{F'}) \\
\lim_{\leftarrow}(g_k \circ \hat{\text{loc}}_{F^{'}} \circ \text{ch}^*(f^{x^{'}}_{\alpha})) & \cong G_k(F', \hat{\text{aut}}_{F'})
\end{align*}
\]

Consequently, we get \( \mathcal{O}\)-module homomorphisms
which are clearly independent of our choice of \( \mathcal{X}' \); moreover, the natural map 
\[ \partial: \mathcal{G}_K \to \mathcal{G}_k \]
induces a commutative diagram
\[
\begin{array}{ccc}
\mathcal{G}_K(\mathcal{F}, \widetilde{\alpha}) & \to & \mathcal{G}_K(\mathcal{F}', \widetilde{\alpha}') \\
\mathcal{G}_k(\mathcal{F}, \widetilde{\alpha}) & \to & \mathcal{G}_k(\mathcal{F}', \widetilde{\alpha}')
\end{array}
\]

4.7 Let \( P'' \) be a third finite \( p \)-group, \( \mathcal{F}'' \) a Frobenius \( P'' \)-category and 
\( \alpha': P'' \to P' \) an \( (\mathcal{F}'', \mathcal{F}') \)-functorial group homomorphism, so that \( \alpha \circ \alpha' \) is 
an \( (\mathcal{F}'', \mathcal{F}) \)-functorial group homomorphism. Assume that \( f_{\alpha} \) and \( f_{\alpha' \alpha''} \) map 
any \( \mathcal{F}'' \)-radical subgroup of \( P'' \) on an \( \mathcal{F}' \)- and an \( \mathcal{F} \)-selfcentralizing subgroups 
of \( P' \) and \( P \) respectively, and let \( \mathcal{X}'' \) be a set of subgroups \( \mathcal{Q}'' \) of \( P'' \) such 
that \( \alpha'(\mathcal{Q}'') \) belongs to \( \mathcal{X}' \), which contains the \( \mathcal{F}'' \)-radical subgroups of \( P'' \) 
and is stable by \( \mathcal{F}'' \)-isomorphisms. Then, it easily follows from Theorem 4.4 
that we have natural maps 
\[
\begin{align*}
\text{loc}_{\alpha''} \circ \text{loc}_{\alpha' \alpha''} : \text{loc}_{\mathcal{F}''} \circ \text{ch}^*(f_{\alpha''}) & \to \text{loc}_{\mathcal{F}'} \circ \text{ch}^*(f_{\alpha'}) \\
\text{loc}_{\alpha''} : \text{loc}_{\mathcal{F}''} & \to \text{loc}_{\mathcal{F}'} \circ \text{ch}^*(f_{\alpha''})
\end{align*}
\]

and therefore from 4.5 we get the \( \mathcal{O} \)-module homomorphisms
\[
\begin{align*}
\text{Res} f_{\alpha''} : \mathcal{G}_K(\mathcal{F}', \widetilde{\alpha}) & \to \mathcal{G}_K(\mathcal{F}', \widetilde{\alpha}') \\
\text{res} f_{\alpha''} : \mathcal{G}_k(\mathcal{F}', \widetilde{\alpha}) & \to \mathcal{G}_k(\mathcal{F}', \widetilde{\alpha}')
\end{align*}
\]

Proposition 4.8 With the notation and the hypothesis above, we have
\[
\text{Res} f_{\alpha''} = \text{Res} f_{\alpha'} \circ \text{Res} f_{\alpha''} \\
\text{res} f_{\alpha''} = \text{res} f_{\alpha'} \circ \text{res} f_{\alpha''}
\]

Proof: It is quite clear that
\[
\begin{align*}
f_{\alpha''} & = f_{\alpha'} \circ f_{\alpha''} \\
\text{loc}_{\alpha''} & = \text{loc}_{\alpha'} \circ \text{loc}_{\alpha''}
\end{align*}
\]
so that \( \text{ch}^*(f_{\alpha''}) = \text{ch}^*(f_{\alpha'}) \circ \text{ch}^*(f_{\alpha''}) \); in particular, from 4.4.1 we get a natural map
\[
\text{loc}_{\alpha''} \circ \text{ch}^*(f_{\alpha''}) : \text{loc}_{\mathcal{F}''} \circ \text{ch}^*(f_{\alpha''}) \to \text{loc}_{\mathcal{F}'} \circ \text{ch}^*(f_{\alpha''})
\]
and we claim that
\[
\text{loc}_{\alpha''} = (\text{loc}_{\alpha'} \circ \text{ch}^*(f_{\alpha''})) \circ \text{loc}_{\alpha''}
\]
Indeed, by the uniqueness part of Theorem 4.4, it suffices to prove that, for any $F''\mapsto$-chain $q'': \Delta_n \to F''\mapsto$ such that $q''(n)$ is fully normalized in $F''$ and that $F''_{\alpha'',(q''(n))}$ contains a Sylow $p$-subgroup of $F''(q'')$, both members of equality 4.8.4 map $(q'', \Delta_n)$ on the same $Z((\alpha \circ \alpha')(q''(n)))$-conjugacy class of group homomorphisms from $L''(q'') = \text{loc}_{F'', \alpha''}(q'', \Delta_n)$ to $L(f_{\alpha'', \alpha''} \circ q'')$; that is to say, it suffices to prove that

$$
\lambda_{q''} : L''(q'') \to L(f_{\alpha'', \alpha''} \circ q'')
$$

is $Z((\alpha \circ \alpha')(q''(n)))$-conjugate to the composition of a representative $\lambda_{f_{\alpha'', \alpha''} \circ q''}$ of

$$(\text{loc}_{f_{\alpha''}, \alpha''}(q''))_{(q'', \Delta_n)} = (\text{loc}_{f_{\alpha''}, \alpha''}(q'', \Delta_n))$$

with the corresponding group homomorphism (cf. Proposition 4.3)

$$
\lambda'_{q''} : L''(q'') \to L(f_{\alpha'', \alpha''} \circ q'')
$$

But, we already know that $\lambda_{q''}$, $\lambda_{f_{\alpha'', \alpha''} \circ q''}$ and $\lambda'_{q''}$ respectively lift

$$(\text{aut}_{f_{\alpha'', \alpha''}}(q'', \Delta_n)), \ (\text{aut}_{f_{\alpha'', \alpha''}}(q'', \Delta_n)) \text{ and } (\text{aut}_{f_{\alpha'', \alpha''}}(q'', \Delta_n))$$

and therefore the composition $\lambda_{f_{\alpha'', \alpha''} \circ q''} \circ \lambda'_{q''}$ also lifts $(\text{aut}_{f_{\alpha'', \alpha''}}(q'', \Delta_n))$; moreover it follows from Theorem 4.4 that, for any $u'' \in N_{\alpha''}(q'')$, we have

$$
\lambda_{q''}(u'') = (\alpha \circ \alpha')(u'') \quad \text{and} \quad \lambda'_{q''}(u'') = \alpha'(u'')
$$

and that we may assume that $\lambda_{f_{\alpha'', \alpha''} \circ q''}(\alpha'(u'')) = \alpha'(\alpha'(u''))$. Now, our claim follows from Proposition 4.3.

From the surjectivity of the Brauer decomposition natural map $\partial$ (cf. 3.4) and from the commutativity of diagram 4.6.3, it follows that in 4.8.1 above it suffices to prove the top equality; but, from our claim and from 4.5 we get the commutative diagram of natural maps

$$
\begin{array}{ccc}
\text{g}_K \circ \text{loc}_{F'', \alpha''} \circ \text{ch}^*(f_{\alpha'', \alpha''}) & \xrightarrow{\text{g} \ast \text{loc}_{f_{\alpha'', \alpha''}}^*} & \text{g}_K \circ \text{loc}_{F'', \alpha''} \\
\text{g} \ast \text{loc}_{f_{\alpha'', \alpha''}}^* \circ \text{ch}^*(f_{\alpha''}) & \downarrow & \text{g} \ast \text{loc}_{f_{\alpha'', \alpha''}}^* \\
\text{g}_K \circ \text{loc}_{F'', \alpha''} \circ \text{ch}^*(f_{\alpha''}) & \xrightarrow{\text{g} \ast \text{loc}_{f_{\alpha'', \alpha''}}^*} & \text{g}_K \circ \text{loc}_{F'', \alpha''}
\end{array}
$$

which forces the commutative diagram of the corresponding inverse limits;
moreover, we have the obvious commutative $O$-$mod$-diagram
\[
\begin{align*}
\varprojlim (g_K \circ \text{loc}_{F^{\text{sec}}} \circ \text{ch}^* (f_\alpha')) & \longrightarrow \varprojlim (g_K \circ \text{loc}_{F^{\text{sec}}} \circ \text{ch}^* (f_{\alpha''}')) \\
\downarrow & \downarrow \\
\varprojlim (g_K \circ \text{loc}_{F,x'}^{\text{sec}}) & \longrightarrow \varprojlim (g_K \circ \text{loc}_{F,x'}^{\text{sec}} \circ \text{ch}^* (f_{\alpha''}'))
\end{align*}
\]
and the canonical $O$-module homomorphisms (cf. Proposition 3.7)
\[
\begin{align*}
G_K (F', \widehat{\text{aut}}_{F^{\text{sec}}}) & \longrightarrow \lim (g_K \circ \text{loc}_{F^{\text{sec}}} \circ \text{ch}^* (f_\alpha')) \\
G_K (F', \widehat{\text{aut}}_{F^{\text{sec}}}) & \cong \lim (g_K \circ \text{loc}_{F,x'}^{\text{sec}}) \\
G_K (F'', \widehat{\text{aut}}_{F^{\text{sec}}}) & \cong \lim (g_K \circ \text{loc}_{F,x''}^{\text{sec}})
\end{align*}
\]
Finally, by the very definition of the ordinary Grothendieck group in 4.6, we get the commutative $O$-$mod$-diagram
\[
\begin{align*}
G_K (F, \widehat{\text{aut}}_{F^{\text{sec}}}) & \xrightarrow{\text{Res}_\alpha} G_K (F', \widehat{\text{aut}}_{F^{\text{sec}}}) \\
\downarrow & \downarrow \\
\varprojlim (g_K \circ \text{loc}_{F^{\text{sec}}} \circ \text{ch}^* (f_{\alpha''}')) & \longrightarrow \varprojlim (g_K \circ \text{loc}_{F,x''}^{\text{sec}} \circ \text{ch}^* (f_{\alpha''}'))
\end{align*}
\]
We are done.

4.9 We apply these results to the normalizers and the centralizers of the subgroups of $P$. Let $Q$ be a subgroup of $P$ and $K$ a subgroup of $\text{Aut}(Q)$ containing $\text{Int}(Q)$, assume that $Q$ is fully $K$-normalized in $F$ and consider the Frobenius $N^K_F (Q)$-category $N^K_F (Q)$; it follows from Lemma 2.5 that any $N^K_F (Q)$-radical subgroup $R$ of $N^K_F (Q)$ contains $Q$; consequently, since the inclusion $i_Q^K : N^K_F (Q) \rightarrow P$ is $(N^K_F (Q), F)$-functorial, we have $O$-module homomorphisms
\[
\begin{align*}
\text{Res}_{K} : G_K (F, \widehat{\text{aut}}_{F^{\text{sec}}}) & \longrightarrow G_K (N^K_F (Q), \widehat{\text{aut}}_{N^K_F (Q)^{\text{sec}}}) \\
\text{res}_{\hat{K}} : G_K (F, \widehat{\text{aut}}_{F^{\text{sec}}}) & \longrightarrow G_K (N^K_F (Q), \widehat{\text{aut}}_{N^K_F (Q)^{\text{sec}}})
\end{align*}
\]
and a commutative diagram
\[
\begin{align*}
\varprojlim (g_K \circ \text{loc}_{F^{\text{sec}}} \circ \text{ch}^* (f_\alpha')) & \longrightarrow \varprojlim (g_K \circ \text{loc}_{F^{\text{sec}}} \circ \text{ch}^* (f_{\alpha''}')) \\
\downarrow & \downarrow \\
G_K (F, \widehat{\text{aut}}_{F^{\text{sec}}}) & \longrightarrow G_K (N^K_F (Q), \widehat{\text{aut}}_{N^K_F (Q)^{\text{sec}}})
\end{align*}
\]
4.10 Moreover, let $R$ and $J$ be respective subgroups of $N^K_P(Q)$ and $\text{Aut}(R)$, denote by $I$ the subgroup of automorphisms of $Q\cdot R$ which stabilize $Q$ and $R$, and act on them via elements of $K$ and $J$ respectively, and assume that $R$ is fully $J$-normalized in $N^K_P(Q)$; thus, $Q\cdot R$ is fully $I$-normalized in $\mathcal{F}$ [5, Lemma 2.17] and we have
\[ N^I_P(Q\cdot R) = N^J_{N^K_P(Q)}(R) \quad \text{and} \quad N^I_{\mathcal{F}}(Q\cdot R) = N^J_{N^K_{\mathcal{F}}(Q)}(R) \] for $N, J, K, \mathcal{F}$. Then, according to Proposition 4.8 above, we have the obvious commutative diagram
\[ \begin{array}{ccc}
\mathcal{G}_K(\mathcal{F}, \hat{\text{aut}}_{\pi^\infty}) & \longrightarrow & \mathcal{G}_K(N^I_{\mathcal{F}}(Q\cdot R), \hat{\text{aut}}_{N^J_{\mathcal{F}}(Q\cdot R)^{\pi^\infty}}) \\
\downarrow & & \downarrow \\
\mathcal{G}_k(\mathcal{F}, \hat{\text{aut}}_{\pi^\infty}) & \longrightarrow & \mathcal{G}_k(N^I_{\mathcal{F}}(Q\cdot R), \hat{\text{aut}}_{N^J_{\mathcal{F}}(Q\cdot R)^{\pi^\infty}})
\end{array} \]

5 Character decomposition of the functor $\mathfrak{g}_K$

5.1 In order to determine the $O$-rank of $\mathcal{G}_K(\mathcal{F}, \hat{\text{aut}}_{\pi^\infty})$, we need a suitable decomposition of the functor $\mathfrak{g}_K$ — analogous to the decomposition of $\mathfrak{g}_k$ in [5, 14.22] — that we develop below. First of all, for any $h \in \mathbb{N} - \{0\}$, fix a primitive $h$-th root of unity $\xi_h$ in $K$ and set $U_h = (\xi_h)$ and $\hat{U}_h = k^* \times U_h$; note that the kernel of the canonical group homomorphism $\text{Aut}_{k^*}(\hat{U}_h) \longrightarrow \text{Aut}(U_h)$ can be identified with $\text{Hom}(U_h, k^*)$ and that, denoting by $U'_h$ the subgroup of $p'$-elements of $U_h$, $\xi_h$ determines a group isomorphism
\[ \text{Hom}(U_h, k^*) \cong U'_h \]

5.2 Let $\hat{G}$ be a $k^*$-group with finite $k^*$-quotient $G$; for any injective $k^*$-group homomorphism $\hat{\eta} : \hat{U}_h \rightarrow \hat{G}$, the so-called ordinary characters determine an $O$-module homomorphism $\mathcal{G}_K(\hat{G}) \rightarrow O$ mapping the class in $\mathcal{G}_K(\hat{G})$ of a $\mathcal{K}_s \hat{G}$-module $M$ on the linear trace $\text{tr}_M(\hat{\eta}(1, \xi_h))$; actually, this
\( \mathcal{O} \)-module homomorphism only depends on the \( G \)-conjugacy class of \( \hat{\eta} \). Moreover, if we have \( \hat{\sigma}(1, \xi_h) = (\lambda, \xi_h) \) for some \( k^* \)-automorphism \( \hat{\sigma} \) of \( \hat{U}_h \) inducing the identity on \( \hat{U}_h \) then we still have

\[
\text{tr}_M ((\hat{\eta} \circ \hat{\sigma})(1, \xi_h)) = \hat{\lambda} \cdot \text{tr}_M (\hat{\eta}(1, \xi_h))
\]

where \( \hat{\lambda} \) denotes the corresponding lifting of \( \lambda \in k^* \) to \( \mathcal{O}^* \) (cf. 1.7). That is to say, denoting by \( \text{Mon}_k^* (\hat{U}_h, \hat{G}) \) the set of injective \( k^* \)-homomorphisms from \( \hat{U}_h \) to \( \hat{G} \), \( G \) acts by conjugation on this set and \( U_h^* \) acts on it via isomorphism 5.1.2 centralizing the action of \( G \), and on \( \mathcal{O} \) via the inclusion \( U_h^* \subset \mathcal{O}^* \); moreover, equality 5.2.1 shows that these actions are preserved by our correspondence. In conclusion, denoting by \( \mathcal{Fct}_{U_h^*} (\text{Mon}_k^* (\hat{U}_h, \hat{G}), \mathcal{O}) \) the set of \( \mathcal{O} \)-valued functions which preserve the corresponding \( U_h^* \)-actions, we have obtained an \( \mathcal{O} \)-module homomorphism

\[
\mathcal{G}_k^* (\hat{G}) \to \mathcal{Fct}_{U_h^*} (\text{Mon}_k^* (\hat{U}_h, \hat{G}), \mathcal{O})^G
\]

5.3 On the other hand, since we have \( \hat{G} = k^* \cdot G' \) for a suitable finite subgroup \( G' \) of \( \hat{G} \) and then, setting

\[
Z' = k^* \cap G' \quad \text{and} \quad e' = \frac{1}{|Z'|} \sum_{z' \in Z'} z'
\]

we have \( \mathcal{G}_k^* (\hat{G}) \cong \mathcal{G}_{k}(kG'e') \) [4, Proposition 5.15], it is not difficult to prove that homomorphisms 5.2.2 when \( h \) runs over \( \mathbb{N} - \{0\} \) determine a \( \mathcal{K} \)-module isomorphism

\[
\mathcal{K} \mathcal{G}_k^* (\hat{G}) \cong \prod_{h \in \mathbb{N} - \{0\}} \mathcal{Fct}_{U_h^*} (\text{Mon}_k^* (\hat{U}_h, \hat{G}), \mathcal{K})^G
\]

where \( \mathcal{Fct}_{U_h^*} (\text{Mon}_k^* (\hat{U}_h, \hat{G}), \mathcal{K}) \) is the set of \( \mathcal{K} \)-valued \( U_h^* \)-invariant functions. Note that any function \( \chi \in \mathcal{Fct}_{U_h} (\text{Mon}_k^* (\hat{U}_h, \hat{G}), \mathcal{K}) \) vanish over the \( k^* \)-group monomorphisms \( \hat{\eta} : \hat{U}_h \to \hat{G} \) mapping \( (1, \xi_h) \) on an element \( \hat{x} \in \hat{G} \) such that, setting \( x = k^* \cdot \hat{x} \), the image of \( C_G (\hat{x}) \) in \( G \) is a proper subgroup of \( C_G (x) \).

5.4 Pushing it further, denote by \( \text{Mon}(U_h, G) \) the set of injective group homomorphisms from \( U_h \) to \( G \) and by

\[
\varpi_{h, \hat{G}} : \text{Mon}_k^* (\hat{U}_h, \hat{G}) \to \text{Mon}(U_h, G)
\]

the canonical map, so that \( U_h^* \) acts regularly on the fibers (cf. 5.1.2); thus, we still have the obvious decomposition

\[
\mathcal{Fct}_{U_h^*} (\text{Mon}_k^* (\hat{U}_h, \hat{G}), \mathcal{O}) \cong \prod_{\eta \in \text{Mon}(U_h, G)} \mathcal{Fct}_{U_h^*} ((\varpi_{h, \hat{G}})^{-1}(\eta), \mathcal{O})
\]
and therefore we get

$$K_G(\hat{G}) \cong \prod_{h \in \mathbb{N} - \{0\}} \left( \prod_{\eta \in \text{Mon}(U_h, G)} \text{Fct}_{U'_h}(\{(\varpi_{h,G})^{-1}(\eta), K\}) \right)^G \quad \text{5.4.3.}$$

Note that, for any $\eta \in \text{Mon}(U_h, G)$, the term $\text{Fct}_{U'_h}(\{(\varpi_{h,G})^{-1}(\eta), O\})$ is a free $O$-module of rank one.

5.5 As in [5, 14.16], isomorphisms 5.3.2 and 5.4.3 are actually natural and, in order to show this naturality, we have to develop a suitable functorial framework. Moreover, in our present situation, we have to extend our construction to the respective subcategories $k^*\mathcal{ILoc}$ and $\mathcal{ILoc}$ of $k^*\mathcal{Loc}$ (cf. 3.3) and $\mathcal{Loc}$ (cf. 1.8) formed by the same objects and by the classes of injective homomorphisms. As in [5, 14.16], denote by $U'_h\mathbb{N}$ the category of finite sets endowed with a $U'_h$-action and by

$$\text{res}_1^{U'_h} : U'_h\mathbb{N} \rightarrow \mathbb{N} \quad \text{5.5.1}$$

the corresponding forgetful functor; then, we consider the evident functors

$$\hat{u}_h : k^*\mathcal{ILoc} \rightarrow U'_h\mathbb{N} \quad \text{and} \quad u_h : \mathcal{ILoc} \rightarrow \mathbb{N} \quad \text{5.5.2}$$

mapping any $k^*\mathcal{ILoc}$-object $(\bar{L}, Z)$ on the $U'_h$-set $\text{Mon}_{k^*}(\hat{U}_h, \hat{L})$ of $Z$-conjugacy classes in $\text{Mon}_{k^*}(\hat{U}_h, \hat{L})$, and any $\mathcal{ILoc}$-object $(L, Z)$ on the corresponding set $\text{Mon}(U_h, G)$ respectively; moreover, denoting by $\text{qt} : k^*\mathcal{ILoc} \rightarrow \mathcal{ILoc}$ the obvious $k^*$-quotient functor, we clearly have a natural map

$$\varpi_h : \text{res}_1^{U'_h} \circ \hat{u}_h \rightarrow u_h \circ \text{qt} \quad \text{5.5.3.}$$

sending $\hat{G}$ to $\varpi_{h,\hat{G}}$ (cf. 5.4.1).

5.6 Furthermore, identifying the category of sets $\mathbb{N}$ with the full subcategory of the category of small categories $\mathcal{CC}$ [5, A1.6] over the small categories which have no other morphisms than the corresponding identity morphisms, we can consider the functor

$$u_h \circ \text{qt} : k^*\mathcal{ILoc} \rightarrow \mathbb{N} \subset \mathcal{CC} \quad \text{5.6.1}$$

as a so-called representation of $k^*\mathcal{ILoc}$ [5, A2.2] and then we can consider the corresponding semidirect product [5, A2.7]

$$k^*(u_h \times \mathcal{ILoc}) = (u_h \circ \text{qt}) \times (k^*\mathcal{ILoc}) \quad \text{5.6.2}$$
where the objects are the pairs \((\bar{\eta}, \bar{\mathcal{L}})\) formed by a \(k^*\text{-}\imath\mathcal{O}_c\text{-object} \bar{\mathcal{L}} = (\bar{L}, Z)\) and, setting \((L, Z) = \text{qt}(\bar{\mathcal{L}})\), by the \(Z\)-conjugacy class of an injective group homomorphism \(\eta: U_h \to L\), and the morphisms from \((\bar{\eta}, \bar{\mathcal{L}})\) to a \(k^*\text{-}\imath\mathcal{O}_c\text{-object} \((\bar{\eta}', \bar{\mathcal{L}}')\) are the \(k^*\text{-}\imath\mathcal{O}_c\text{-morphisms} \bar{\varphi}: \bar{\mathcal{L}} \to \bar{\mathcal{L}}'\) fulfilling \(\bar{\eta}' = \text{qt}(\bar{\varphi}) \circ \bar{\eta}\).

5.7 Coherently, for any \(h'\) dividing \(h\), choosing the identification between \(\text{Hom}(U_{h'}, k^*)\) and \(U_{h'}\) determined by \((\xi_h)_{h'h'}\) (cf. 5.1), the inclusion \(U_{h'} \subset U_h\) induces a functor and two natural maps

\[
\text{res}^{U_{h'}}_{U_h}: U_{h'} \mathcal{N} \longrightarrow U_h \mathcal{N} \quad (5.7.1)
\]

\[
\hat{\varphi}_{h'h}: \text{res}^{U_{h'}}_{U_h} \circ \bar{u}_{h} \longrightarrow \bar{u}_{h'} \quad \text{and} \quad \rho_{h',h} : \bar{u}_{h} \longrightarrow \bar{u}_{h'}
\]

and it is easily checked that \([5, A1.5.1]\)

\[
\varpi_{h'} \circ (\hat{\varphi}_{h'h} \ast \text{res}^{U_{h'}}_{U_h}) = (\rho_{h',h} \ast \text{qt}) \circ \varpi_h \quad (5.7.2)
\]

Moreover, the natural map \(\rho_{h',h}\) above determines a functor \([5, \text{Proposition A2.17}]\)

\[
(\rho_{h',h} \ast \text{qt}) \times \mathfrak{d}_{k^*\text{-}\imath\mathcal{O}_c}: k^*-(u_h \times \imath\mathcal{O}_c) \longrightarrow k^*-(u_{h'} \times \imath\mathcal{O}_c) \quad (5.7.3)
\]

mapping \((\bar{\eta}, \bar{\mathcal{L}})\) on \((\bar{\eta}', \bar{\mathcal{L}}')\) where \(\bar{\eta}'\) denotes the \(Z\)-conjugacy class of the restriction of \(\eta \in \bar{\eta}\).

**Proposition 5.8** With the notation above, we have a functor

\[
\check{w}_h: k^*-(u_h \times \imath\mathcal{O}_c) \longrightarrow U_h \mathcal{N} \quad (5.8.1)
\]

mapping any \(k^*-(u_h \times \imath\mathcal{O}_c)\text{-object} \((\check{\eta}, \check{\mathcal{L}})\) on the regular \(U_h\text{-set} \((\varpi_{h'})^{-1}(\check{\eta})\) and mapping any \(k^*-(u_h \times \imath\mathcal{O}_c)\text{-morphism} \check{\varphi}: (\check{\eta}, \check{\mathcal{L}}) \to (\check{\eta}', \check{\mathcal{L}}')\) on the bijective map

\[
((\varpi_{h'})^{-1}(\check{\eta})) \cong (\varpi_{h'})^{-1}(\check{\eta}') \quad (5.8.2)
\]

determined by \(\hat{u}_h(\check{\varphi})\). Moreover, \(\check{u}_h\) is the direct image of \(\check{w}_h\) via the structural functor

\[
\check{p}_h: k^*-(u_h \times \imath\mathcal{O}_c) \longrightarrow k^*\text{-}\imath\mathcal{O}_c \quad (5.8.3)
\]

**Proof:** As we mention above, the \(k^*-(u_h \times \imath\mathcal{O}_c)\text{-objects} are the pairs formed by a \(k^*\text{-}\imath\mathcal{O}_c\text{-object} \check{\mathcal{L}} = (\check{L}, \check{Z})\) and, setting \((L, Z) = \text{qt}(\check{\mathcal{L}})\), by an object of the “category” \(\overline{\text{Mon}(U_h, L)}\), namely by the \(Z\)-conjugacy class \(\check{\eta}\) of an injective group homomorphism \(\eta: U_h \to L\); then, in the map (cf. 5.4.1)

\[
\varpi_{h,\mathcal{L}}: \overline{\text{Mon}}(\check{U}_h, \check{L}) \longrightarrow \overline{\text{Mon}(U_h, L)} \quad (5.8.4)
\]
$U'_h$ acts on the left end, stabilizing and acting regularly on the fibers, so that
$(\varpi_{h,\xi})^{-1}(\tilde{\eta})$ is indeed a $U'_h$-set. Analogously, the $k^*-(u_h \times i\overline{\Lloc})$-morphisms between two objects $(\tilde{\eta}, \hat{\mathcal{L}})$ and $(\tilde{\eta}', \hat{\mathcal{L}}')$ are the pairs formed by a $k^*\overline{\Lloc}$-morphism $\bar{\varphi} : \hat{\mathcal{L}} \to \hat{\mathcal{L}}'$ and by a "$\overline{\text{Mon}(U_h, \hat{\mathcal{L}}')}$"-morphism from the image of $\tilde{\eta}$ by the functor $(u_h \circ \text{qt})(\bar{\varphi})$ to $\tilde{\eta}'$, which actually forces the equality $\text{qt}(\bar{\varphi}) \circ \tilde{\eta} = \tilde{\eta}'$. Then, it is quite clear that the map

$$\hat{\eta} : \overline{\text{Mon}_{k^*}(\hat{U}_h, \hat{\mathcal{L}})} \to \overline{\text{Mon}_{k^*}(\hat{U}_h, \hat{\mathcal{L}}')}$$

sends $(\varpi_{h,\xi})^{-1}(\tilde{\eta})$ bijectively onto $(\varpi_{h,\xi})^{-1}(\tilde{\eta}')$, determining a $U'_h$-set map. The proofs of the functoriality and of the last statement are straightforward.

**Remark 5.9** Note that, for any $\xi \in U_h$, the inner $k^*$-group automorphism of $\hat{\mathcal{L}}$ determined by an element of $\hat{\mathcal{L}}$ lifting $\eta(\xi)$ acts trivially on $(\varpi_{h,\xi})^{-1}(\tilde{\eta})$.

**Proposition 5.10** With the notation above, for any $h'$ dividing $h$ we have a natural map

$$\tilde{\tau}_{h', h} : \text{Res}^{U'_h}_{U'_h} \circ \hat{\varpi}_h \to \hat{\varpi}_{h'} \circ ((\rho_{h', h} \ast \text{qt}) \times \text{id}_{k^*\overline{\Lloc}})$$

5.10.1

which sends any $k^*-(u_h \times i\overline{\Lloc})$-object $(\tilde{\eta}, \hat{\mathcal{L}})$ to the $U'_h$-morphism

$$\text{Res}^{U'_h}_{U'_h}((\varpi_{h,\xi})^{-1}(\tilde{\eta})) \to (\varpi_{h',\xi})^{-1}(\text{Res}^{U'_h}_{h'}(\tilde{\eta}))$$

5.10.2

mapping any $\tilde{\eta} \in (\varpi_{h,\xi})^{-1}(\tilde{\eta}) \subset \overline{\text{Mon}_{k^*}(\hat{U}_h, \hat{\mathcal{L}})}$ on its restriction to $U'_h$.

**Proof:** For any $k^*\overline{\Lloc}$-morphism $\bar{\varphi} : \hat{\mathcal{L}} \to \hat{\mathcal{L}}'$, setting $\bar{\varphi} = \text{qt}(\bar{\varphi})$, the functor $\text{Res}^{U'_h}_{U'_h} \circ \hat{\varpi}_h$ maps the $k^*-(u_h \times i\overline{\Lloc})$-morphism $(\tilde{\eta}, \hat{\mathcal{L}}) \to (\bar{\varphi} \circ \tilde{\eta}, \hat{\mathcal{L}}')$ on the $U'_h$-set map

$$\text{Res}^{U'_h}_{U'_h}((\varpi_{h,\xi})^{-1}(\tilde{\eta})) \to \text{Res}^{U'_h}_{U'_h}((\varpi_{h',\xi})^{-1}(\bar{\varphi} \circ \tilde{\eta}))$$

5.10.3

sending $\tilde{\eta} \in (\varpi_{h,\xi})^{-1}(\tilde{\eta})$ to $\bar{\varphi} \circ \tilde{\eta}$; whereas $\hat{\varpi}_{h'} \circ ((\rho_{h', h} \ast \text{qt}) \times \text{id}_{k^*\overline{\Lloc}})$ maps this morphism on the analogous $U'_{h'}$-set map

$$(\varpi_{h',\xi})^{-1}(\tilde{\eta}') \to (\varpi_{h',\xi})^{-1}(\bar{\varphi} \circ \tilde{\eta}')$$

5.10.4

where we are setting $\tilde{\eta}' = \text{Res}^{U'_h}_{h'}(\tilde{\eta})$; thus, the corresponding diagram is indeed commutative since the restriction to $U'_h$ is compatible with the composition with $\bar{\varphi}$ on the left. We are done.
5.11 We are ready to discuss the naturality of isomorphisms 5.3.2 and 5.4.3. As in [5, 14.21], consider the evident contravariant functor

\[ \text{Fct}_{U_h'} : U_h' \text{-} \text{mod} \to \mathcal{O} \text{-} \text{mod} \] 

mapping any finite \( U_h' \)-set \( X \) on the \( \mathcal{O} \)-module \( \text{Fct}_{U_h'}(X, \mathcal{O}) \) of the \( \mathcal{O} \)-valued functions over \( X \) preserving the \( U_h' \)-actions; note that if \( \xi \cdot x = x \) for some \( x \in X \) and some \( \xi \in U_h - \{1\} \) then we have \( f(x) = 0 \) for any \( f \in \text{Fct}_{U_h'}(X, \mathcal{O}) \).

On the other hand, note that if we have a contravariant functor

\[ m : k^* - \text{iLoc} \to \mathcal{K} - \text{mod} \]

for any \( k^* - \text{iLoc} \)-object \( \hat{\mathcal{L}} = (\hat{L}, Z) \), setting \( (L, Z) = q\hat{t}(\hat{\mathcal{L}}) \), \( m(\hat{\mathcal{L}}) \) has an obvious \( \mathcal{K}L \)-module structure, so that it makes sense to consider

\[ h^0(L, m(\hat{\mathcal{L}})) = m(\hat{\mathcal{L}})^L \]

further, if \( \hat{\varphi} : \hat{\mathcal{L}} \to \hat{\mathcal{L}}' = (\hat{L}', Z') \) is a \( k^* - \text{iLoc} \)-morphism, it is easily checked that \( m(\hat{\varphi}) \) maps \( m(\hat{\mathcal{L}})^{L'} \) on an \( \mathcal{K} \)-submodule of \( m(\hat{\mathcal{L}})^L \); that is to say, we get a new contravariant functor from \( k^* - \text{iLoc} \) to \( \mathcal{K} - \text{mod} \) — noted \( h^0(m) \) — mapping \( \hat{\mathcal{L}} \) on \( m(\hat{\mathcal{L}})^L \) and \( \hat{\varphi} \) on the map from \( m(\hat{\mathcal{L}}')^{L'} \) to \( m(\hat{\mathcal{L}})^L \) induced by \( m(\hat{\varphi}) \).

5.12 Finally, still denote by \( g_{\mathcal{K}} : k^* - \text{iLoc} \to \mathcal{O} - \text{mod} \) the obvious functor mapping any \( k^* - \text{iLoc} \)-object \( \hat{\mathcal{L}} = (\hat{L}, Z) \) on \( G_{\mathcal{K}}(\hat{L}) \). With all this notation, it is now quite clear that isomorphism 5.3.2 actually defines a natural isomorphism

\[ \mathcal{K}g_{\mathcal{K}} \cong \prod_{h \in \mathbb{N} - \{0\}} h^0(\mathcal{K}\text{Fct}_{U_h'} \circ \hat{u}_h) \]

where \( \mathcal{K}g_{\mathcal{K}} \) and \( \mathcal{K}\text{Fct}_{U_h'} \) denote the respective compositions of \( g_{\mathcal{K}} \) and \( \text{Fct}_{U_h'} \) with the scalar extension from \( \mathcal{O} \) to \( \mathcal{K} \). Consequently, considering the composition \( \mathcal{K}g_{\mathcal{K}} \circ \text{iLoc}_{x^e} \) in 3.5.1 above, from definition 3.5.2 we have

\[ \mathcal{K}g_{\mathcal{K}}(\mathcal{F}, \text{aut}_{x^e}) = \lim_{\tau} \left( \prod_{h \in \mathbb{N} - \{0\}} h^0(\mathcal{K}\text{Fct}_{U_h'} \circ \hat{u}_h) \circ \text{iLoc}_{x^e} \right) \]

that is to say, we still have

\[ \mathcal{K}g_{\mathcal{K}}(\mathcal{F}, \text{aut}_{x^e}) = \prod_{h \in \mathbb{N} - \{0\}} \mathcal{K}g_{\mathcal{K}}(\mathcal{F}, \text{aut}_{x^e})_h \]
where, for any \( h \in \mathbb{N} - \{0\} \), we set

\[
K^c F_k(F, \hat{\omega}t_{\mathcal{F}^\text{sec}})_h = \lim_{\leftarrow} (\hat{h}^0 (K\mathcal{F}t_{U_h} \circ \hat{\omega}_h) \circ \hat{\omega}_{\mathcal{F}^\text{sec}}) \\
\cong \lim_{\leftarrow} (K\mathcal{F}t_{U_h} \circ \hat{\omega}_h \circ \hat{\omega}_{\mathcal{F}^\text{sec}})
\]

5.12.4,

the last isomorphism being obvious.

5.13 But, according to Proposition 5.8 above, the functor

\[
\hat{\omega}_h : k^* - \mathcal{I} \mathcal{O} \mathcal{C} \longrightarrow U_h' \mathcal{N}
\]

5.13.1

is the **direct image** of the functor

\[
\hat{\omega}_h : k^* - (u_h \times i \mathcal{I} \mathcal{O} \mathcal{C}) \equiv \longrightarrow U_h' \mathcal{N}
\]

5.13.2

throughout the structural functor

\[
\hat{p}_h : k^* - (u_h \times i \mathcal{I} \mathcal{O} \mathcal{C}) = (u_h \circ \text{qt}) \times (k^* - i \mathcal{I} \mathcal{O} \mathcal{C}) \longrightarrow k^* - i \mathcal{I} \mathcal{O} \mathcal{C}
\]

5.13.3.

Moreover, considering the semidirect product

\[
u_h \text{ch}^*(\mathcal{F}^\text{sec}) = (u_h \circ \text{loc}_{\mathcal{F}^\text{sec}}) \times \text{ch}^*(\mathcal{F}^\text{sec})
\]

5.13.4,

we have the evident commutative diagram of functors

\[
\begin{array}{ccc}
k^* - (u_h \times i \mathcal{I} \mathcal{O} \mathcal{C}) & \xrightarrow{\hat{p}_h} & k^* - i \mathcal{I} \mathcal{O} \mathcal{C} \\
\xrightarrow{id_{u_h \times i \mathcal{I} \mathcal{O} \mathcal{C}}} & & \xrightarrow{\hat{\omega}_{\mathcal{F}^\text{sec}}} \text{loc}_{\mathcal{F}^\text{sec}} \\
\xrightarrow{u_h \text{ch}^*(\mathcal{F}^\text{sec})} & & \text{ch}^*(\mathcal{F}^\text{sec})
\end{array}
\]

5.13.5.

Then, it is not difficult to check that the composition \( \hat{\omega}_h \circ \text{loc}_{\mathcal{F}^\text{sec}} \) is also the **direct image** of the composition

\[
\hat{\omega}_h = \hat{\omega}_h \circ (id_{u_h \circ \text{loc}_{\mathcal{F}^\text{sec}}} \times \text{loc}_{\mathcal{F}^\text{sec}}) : u_h \text{ch}^*(\mathcal{F}^\text{sec}) \longrightarrow U_h' \mathcal{N}
\]

5.13.6

throughout the bottom functor in diagram 5.13.5; further, the **direct image** is clearly compatible with the functor \( \mathcal{F} t_{U_h} : U_h' \mathcal{N} \longrightarrow \mathcal{O}:{\text{mod}} \). Thus, we finally get [5, 1.6]

\[
\lim_{\leftarrow} (\mathcal{F} t_{U_h} \circ \hat{\omega}_h \circ \text{loc}_{\mathcal{F}^\text{sec}}) \cong \lim_{\leftarrow} (\mathcal{F} t_{U_h} \circ \hat{\omega}_h)
\]

5.13.7.
6 An equivalence of categories

6.1 With the notation above, the point is that the semidirect product
\[
\text{u} \ast \text{ch}^* (\mathcal{F}^e) = (\text{u} \ast \text{loc}^e \mathcal{F}^e) \rtimes \text{ch}^* (\mathcal{F}^e)
\]

admits another description in terms of the following category \(^{h^c}(\mathcal{F}^e)\). The \(^{h^c}(\mathcal{F}^e)\)-objects are the pairs \(Q^g\) formed by an \(\mathcal{F}\)-selfcentralizing subgroup \(Q\) of \(P\) and by a \(Z(Q)\)-conjugacy class of injective group homomorphisms \(\tilde{\rho} : U_h \to \mathcal{L}(Q)\) or, equivalently, an \(i\mathfrak{loc}\)-morphism
\[
\tilde{\rho} : (U_h, 1) \to (\mathcal{L}(Q), Z(Q))
\]

whereas the \(^{h^c}(\mathcal{F}^e)\)-morphisms from another \(^{h^c}(\mathcal{F}^e)\)-object \(R^\tilde{\sigma}\) to \(Q^\tilde{\rho}\) are the \(\mathcal{F}^e\)-morphisms \(\varphi : R \to Q\) such that, denoting by \(L(\varphi)\) the localizer of the \(\mathcal{F}^e\)-chain \(\Delta_1 \to \mathcal{F}^e\) determined by \(\varphi\), there is an \(i\mathfrak{loc}\)-morphism
\[
\tilde{\alpha} : (U_h, 1) \to (L(\varphi), Z(Q))
\]

such that we have the following commutative \(i\mathfrak{loc}\)-diagram
\[
\begin{array}{ccc}
(L(R), Z(R)) & \xleftarrow{\tilde{\sigma}} & (U_h, 1) \\
\downarrow_{\text{loc}^e \rightarrow (\text{id}_Q, \delta^0_0)} & \searrow_{\tilde{\rho}} & \nearrow_{\text{loc}^e \rightarrow (\text{id}_R, \delta^0_1)}
\end{array}
\]
\[
(L(\varphi), Z(Q))
\]

note that such an \(i\mathfrak{loc}\)-morphism \(\tilde{\alpha}\) is unique and that \(L(\varphi)\) determines well-defined subgroups of \(L(Q)\) and \(L(R)\).

6.2 The composition of the \(\mathcal{F}\)-morphisms induces a composition in \(^{h^c}(\mathcal{F}^e)\) since, for a third \(^{h^c}(\mathcal{F}^e)\)-object \(T^\tilde{\tau}\) and an \(^{h^c}(\mathcal{F}^e)\)-morphism \(\psi : T^\tilde{\tau} \to R^\tilde{\sigma}\), denoting by \(L(\psi, \varphi)\) the localizer of the \(\mathcal{F}^e\)-chain \(\Delta_2 \to \mathcal{F}^e\) determined by \(\varphi\) and \(\psi\), we have the \(i\mathfrak{loc}\)-pull-back
\[
\begin{array}{ccc}
(L(\psi), Z(R)) & \xleftarrow{\text{loc}^e \rightarrow (\text{id}_R, \delta^0_0)} \\
\downarrow_{\text{loc}^e \rightarrow (\text{id}_Q, \delta^0_0)} & \searrow_{\text{loc}^e \rightarrow (\text{id}_Q, \delta^0_1)} & \nearrow_{\text{loc}^e \rightarrow (\text{id}_Q, \delta^0_1)}
\end{array}
\]
\[
(L(\psi, \varphi), Z(Q))
\]

and therefore, denoting by \(\tilde{\beta} : (U_h, 1) \to (L(\psi), Z(R))\) the corresponding \(i\mathfrak{loc}\)-morphism, the equalities (cf. diagram 6.1.4)
\[
\text{loc}^e \rightarrow (\text{id}_Q, \delta^0_0) \circ \tilde{\alpha} = \tilde{\sigma} = \text{loc}^e \rightarrow (\text{id}_T, \delta^0_1) \circ \tilde{\beta}
\]
force the existence of an \( \widetilde{\iota} \)-morphism \( \varepsilon : (U_h, 1) \to (\mathcal{L}(\psi, \varphi), Z(Q)) \) fulfilling

\[
\mathcal{I} \circ (\text{id}, \delta_0^1) \circ \varepsilon = \alpha \
\text{ and } \mathcal{I} \circ (\text{id}, \delta_1^0) \circ \varepsilon = \beta
\]

so that, setting \( \gamma = \mathcal{I} \circ (\text{id}_{\psi \circ \varphi}, \delta_1^0) \), we finally get

\[
\mathcal{I} \circ (\text{id}_T, \delta_0^0) \circ (\gamma \circ \varepsilon) = \mathcal{I} \circ (\text{id}_R, \delta_1^0) \circ \mathcal{I} \circ (\text{id}_\varphi, \delta_0^1) \circ \varepsilon = \tilde{\rho}
\]

\[
\mathcal{I} \circ (\text{id}_Q, \delta_1^0) \circ (\gamma \circ \varepsilon) = \mathcal{I} \circ (\text{id}_R, \delta_2^0) \circ \mathcal{I} \circ (\text{id}_\psi, \delta_2^1) \circ \varepsilon = \tilde{\tau}
\]

6.3 Note that we have a \( h \)-faithful \( \text{forgetful} \) functor \( h^*(\mathcal{F}_c^e) \to \mathcal{F}_c^e \); for short, we denote by

\[
\nu_h = \nu_{h^*(\mathcal{F}_c^e)} : \chi^* (h^!(\mathcal{F}_c^e)) \to h^*(\mathcal{F}_c^e)
\]

the corresponding evaluation functor. Moreover, for any \( h' \) dividing \( h \), it is clear that the inclusion \( U_{h'} \subset U_h \) induces a faithful functor

\[
\tau_{h', h} : h^*(\mathcal{F}_c^e) \to h'^*(\mathcal{F}_c^e)
\]

On the other hand, since the category of chains \( \chi^*(\mathcal{F}_c^e) \) is already a \emph{semidirect product} \([5, \text{A2.8}]\), the \( \chi^*(\mathcal{F}_c^e) \)-objects can be identified with the triples \((\eta, q, \Delta_n)\) formed by an \( \mathcal{F}_c^e \)-chain \( q : \Delta_n \to \mathcal{F}_c^e \) and by an \( \widetilde{\iota} \)\(\text{morphism} \)

\[
\eta : (U_h, 1) \to (\mathcal{L}(q), \text{Ker}(\pi_q))
\]

for short, for any \( i \in \Delta_n \), denote by

\[
i_i^2 : (\mathcal{L}(q), \text{Ker}(\pi_q)) \to (\mathcal{L}(q(i)), Z(\pi_q(i)))
\]

the image by the functor \( \mathcal{I} \circ \chi^* \) of the \( \chi^*(\mathcal{F}_c^e) \)-morphisms from \((q, \Delta_n)\) to \((q(i), \Delta_n)\) determined by the identity map of \( q(i) \).

**Proposition 6.4** For any \( h \in \mathbb{N} - \{0\} \), we have an equivalence of categories

\[
h : u^* \chi^*(\mathcal{F}_c^e) \cong \chi^*(h^!(\mathcal{F}_c^e))
\]

which maps any \( u^* \chi^*(\mathcal{F}_c^e) \)-object \((\eta, q, \Delta_n)\) on the chain \( q^0 : \Delta_n \to h^!(\mathcal{F}_c^e) \)

sending any \( i \in \Delta_n \) to the \( h^!(\mathcal{F}_c^e) \)-object \( q(i)^{\tau_{\psi, \varphi}} \) and any \( \Delta_n \)-morphisms \((j \ast i)\)

to \( q(j \ast i) \). Moreover, for any \( h' \) dividing \( h \), we have the commutative diagram

\[
\begin{array}{ccc}
u_{h'} & \cong & \chi^* (h'^!(\mathcal{F}_c^e)) \\
(\rho_{h', h} \circ \mathcal{I} \circ \chi^* \circ (\text{id}, \delta_0^1)) & \uparrow & \chi^*(\tau_{h', h}) \\
u_{h'} & \cong & \chi^*(h^!(\mathcal{F}_c^e))
\end{array}
\]
Proof: Considering the \( \mathcal{F} \)-chain \( \Delta_1 \cong \{ j, i \} \to \mathcal{F}^e \) obtained from the restriction of \( \mathfrak{q} \), and the \( \mathfrak{ch}^*(\mathcal{F}^e) \)-morphism from \( (\mathfrak{q}, \Delta_n) \) to the \( \mathfrak{ch}^*(\mathcal{F}^e) \)-object determined by this \( \mathcal{F} \)-chain, it is easily checked that \( \mathfrak{q}(j \cdot i) \) is indeed an \( h(\mathcal{F}^e) \)-morphism from \( \mathfrak{q}(j) \circ \delta \circ \eta \) to \( \mathfrak{q}(i) \circ \delta \circ \eta \). Moreover, a \( \mathfrak{lch}^*(\mathcal{F}^e) \)-morphism to \( (\eta, \mathfrak{q}, \Delta_n) \) from a \( \mathfrak{ch}^*(\mathcal{F}^e) \)-object \( (\bar{\theta}, \tau, \Delta_m) \) is defined by a \( \mathfrak{ch}^*(\mathcal{F}^e) \)-morphism

\[
(\mu, \delta) : (\tau, \Delta_m) \to (\mathfrak{q}, \Delta_n)
\]

6.4.3 fulfilling \( \mathfrak{lch}_{\mathcal{F}^e}(\mu, \delta) \circ \bar{\theta} = \bar{\eta} \) [5, condition A2.6.2], and therefore \( (\mu, \delta) \) is also a \( \mathfrak{ch}^*(h(\mathcal{F}^e)) \)-morphism from \( (\bar{\theta}, \Delta_m) \) to \( (\mathfrak{q}, \Delta_n) \). Thus, we have obtained a functor

\[
j_{\mathfrak{h}} : \mathfrak{lch}^*(\mathcal{F}^e) \to \mathfrak{ch}^*(h(\mathcal{F}^e))
\]

6.4.4.

On the other hand, any \( h(\mathcal{F}^e) \)-chain \( \hat{\mathfrak{q}} : \Delta_n \to h(\mathcal{F}^e) \) clearly determines a \( \mathcal{F}^e \)-chain \( \mathfrak{q} : \Delta_n \to \mathcal{F}^e \); consequently, by the very definition of \( h(\mathcal{F}^e) \), we have \( \hat{\mathfrak{q}}(i) = \mathfrak{q}(i)^{\delta_i} \) where

\[
\hat{\mathfrak{q}} : (U_{\mathfrak{h}}, 1) \to (\mathcal{L}(\mathfrak{q}(i)), \mathcal{Z}(\mathfrak{q}(i))
\]

6.4.5 is an \( i\mathfrak{E}^e\mathfrak{c} \)-morphism for any \( i \in \Delta_n \) and, since \( \hat{\mathfrak{q}}(j \cdot i) \) is a morphism from \( \mathfrak{q}(j)^{\delta_j} \) to \( \mathfrak{q}(i)^{\delta_i} \) for any \( 0 \leq j \leq i \leq n \), arguing by induction on \( n \) it is not difficult to prove that there is a unique \( i\mathfrak{E}^e\mathfrak{c} \)-morphism

\[
\hat{\eta} : (U_{\mathfrak{h}}, 1) \to (\mathcal{L}(\mathfrak{q}), \text{Ker}(\pi_{\mathfrak{q}}))
\]

6.4.6 fulfilling \( \iota^i \circ \hat{\eta} = \bar{\eta} \) for any \( i \in \Delta_n \).

Similarly, if \( \hat{\tau} : \Delta_m \to h(\mathcal{F}^e) \) is a chain and, for any \( j \in \Delta_m \), we have \( \hat{\tau}(j) = \tau(j)^{\delta_j} \) for a suitable \( i\mathfrak{E}^e\mathfrak{c} \)-morphism

\[
\hat{\theta} : (U_{\mathfrak{h}}, 1) \to (\mathcal{L}(\tau), \text{Ker}(\pi_{\tau}))
\]

6.4.7, then any \( \mathfrak{ch}^*(h(\mathcal{F}^e)) \)-morphism \( (\mu, \delta) : (\hat{\tau}, \Delta_m) \to (\hat{\mathfrak{q}}, \Delta_n) \) induces a \( \mathfrak{ch}^*(\mathcal{F}^e) \)-morphism \( (\mu, \delta) : (\tau, \Delta_m) \to (\mathfrak{q}, \Delta_n) \) such that

\[
\mathfrak{lch}_{\mathcal{F}^e}(\mu, \id_{\Delta_m}) \circ \iota_{\delta(i)} \circ \hat{\theta} = \hat{\eta} = \iota^i \circ \hat{\eta}
\]

6.4.8 for any \( i \in \Delta_n \), and therefore, since we have

\[
\mathfrak{lch}_{\mathcal{F}^e}(\mu, \id_{\Delta_m}) \circ \iota_{\delta(i)} = \iota^i \circ \mathfrak{lch}_{\mathcal{F}^e}(\mu, \delta)
\]

6.4.9, we get \( \mathfrak{lch}_{\mathcal{F}^e}(\mu, \delta) \circ \hat{\theta} = \hat{\eta} \) by the uniqueness of \( \hat{\eta} \). Hence, the functor \( j_{\mathfrak{h}} \) is an equivalence of categories. The commutativity of diagram 6.4.2 is easily checked. We are done.
Proposition 6.5. For any $h \in \mathbb{N} - \{0\}$, we have a factorization of $\hat{\omega}_h$

\[
\begin{align*}
\hat{\omega}_h, \hat{\omega}_h'^* (\mathcal{F}_{x^e}) & \xrightarrow{\hat{\omega}_h} U'_h \mathbb{N} \\
\hat{\omega}_h'' & \xrightarrow{\hat{\omega}_h} h \hat{\omega}_h'' (\mathcal{F}_{x^e}) \xrightarrow{\hat{\omega}_h'} U'_h \mathbb{N}
\end{align*}
\]

throughout a functor $t_h : h (\mathcal{F}_{x^e}) \to U'_h \mathbb{N}$ mapping any $h (\mathcal{F}_{x^e})$-object $Q^\rho$ on the regular $U'_h$-set $(\varpi_{h, (\mathcal{L}(Q), \mathbb{Z}(Q))})^{-1}(\bar{\sigma})$ and any $h (\mathcal{F}_{x^e})$-morphism $\varphi : R^\sigma \to Q^\rho$ on a $U'_h$-set bijection

\[
(\varpi_{h, (\mathcal{L}(R), \mathbb{Z}(R))})^{-1}(\bar{\sigma}) \cong (\varpi_{h, (\mathcal{L}(Q), \mathbb{Z}(Q))})^{-1}(\bar{\sigma})
\]

In particular, we have

\[
\mathcal{K}G_h (\mathcal{F}, \hat{\omega}_{\mathcal{F}_{x^e}}) \cong \mathbb{H}^0 (h (\mathcal{F}_{x^e}), \mathcal{K}\mathcal{F}_{\mathcal{U}^*_h} \circ t_h)
\]

Moreover, for any $h'$ dividing $h$ we have a natural map

\[
\theta_{h', h} : \text{res}_{U'_{h'}}^{U'_h} \circ t_h \to t_{h'} \circ \theta_{h', h}
\]

sending $Q^\rho$ to the $U'_h$-set map

\[
\text{res}_{U'_{h'}}^{U'_h} ((\varpi_{h, (\mathcal{L}(Q), \mathbb{Z}(Q))})^{-1}(\bar{\rho})) \to (\varpi_{h', (\mathcal{L}(Q), \mathbb{Z}(Q))})^{-1}(\text{Res}_{U'_{h'}}^{U'_h}(\bar{\rho}))
\]

induced by the restriction throughout the inclusion $U_{h'} \subset U_h$.

**Proof:** Let us denote by $R$ and $Q$ the obvious $\mathcal{F}_{x^e}$-chains and by $\varphi$ the $\mathcal{F}_{x^e}$-chain mapping $0$ on $R$, $1$ on $Q$ and the $\Delta_1$-morphism $(0 \bullet 1)$ on $\varphi$; thus, we have evident $\hat{\omega}_h (\mathcal{F}_{x^e})$-morphisms

\[
\begin{align*}
(R, \Delta_0) & \leftarrow (\varphi, \Delta_1) \to (Q, \Delta_0)
\end{align*}
\]

and then the natural map $\hat{\omega}_h (\mathcal{F}_{x^e}) \to \hat{\omega}_h (\mathcal{F}_{x^e})$ (cf. 3.3) sends these $\hat{\omega}_h (\mathcal{F}_{x^e})$-morphisms to a commutative diagram

\[
\begin{array}{ccc}
(\mathcal{L}(R), \mathbb{Z}(R)) & \xrightarrow{(\varphi, \mathcal{Z}(Q))} & (\mathcal{L}(Q), \mathbb{Z}(Q)) \\
\uparrow & & \uparrow \\
(\hat{\mathcal{L}}(R), \mathbb{Z}(R)) & \xrightarrow{(\hat{\varphi}, \mathbb{Z}(Q))} & (\hat{\mathcal{L}}(Q), \mathbb{Z}(Q))
\end{array}
\]

hence, the bottom $k^* - \hat{\omega}_h (\mathcal{F}_{x^e})$-morphisms induce a $U'_h$-set bijection (cf. diagram 6.1.4)

\[
t_h (\varphi) : (\varpi_{h, (\mathcal{L}(R), \mathbb{Z}(R))})^{-1}(\bar{\sigma}) \cong (\varpi_{h, (\mathcal{L}(Q), \mathbb{Z}(Q))})^{-1}(\bar{\rho})
\]
Now, we claim that the correspondence sending the \( h(\mathcal{F}^c) \)-morphism \( \varphi \) to the \( U_1' \)-set bijection \( t_h(\varphi) \) defines a functor; indeed, for a third \( h(\mathcal{F}^c) \)-object \( T^\sigma \) and a \( h(\mathcal{F}^c) \)-morphism \( \psi: T^\sigma \to R^\sigma \), we have the following evident commutative \( \text{ch}^*(\mathcal{F}^c) \)-diagram

\[
\begin{array}{ccc}
(Q, \Delta_0) & = & (Q, \Delta_0) \\
\uparrow & & \uparrow \\
(\varphi, \Delta_1) & \to & (R, \Delta_0) \\
\uparrow & & \uparrow \\
(\epsilon, \Delta_2) & \to & (\psi, \Delta_1) \\
\downarrow & & \| \\
(\varphi \circ \psi, \Delta_1) & \to & (T, \Delta_0)
\end{array}
\]

where the \( \mathcal{F}^c \)-chain \( \epsilon: \Delta_2 \to \mathcal{F}^c \) maps 0 on \( T \), 1 on \( R \), 2 on \( Q \), \((0 \bullet 1)\) on \( \psi \) and \((1 \bullet 2)\) on \( \varphi \); once again, the functor \( \text{loc}_{\mathcal{F}^c} \) maps this \( \text{ch}^*(\mathcal{F}^c) \)-diagram on the commutative \( \text{loc} \)-diagram

\[
\begin{array}{ccc}
(\mathcal{L}(Q), Z(Q)) & \leftarrow & (\mathcal{L}(\varphi), Z(Q)) \\
\uparrow & & \uparrow \\
(\mathcal{L}(\epsilon), Z(Q)) & \to & (\mathcal{L}(\psi), Z(R)) \\
\downarrow & & \downarrow \\
(\mathcal{L}(\varphi \circ \psi), Z(Q)) & \to & (\mathcal{L}(T), Z(T))
\end{array}
\]

at this point, considering the corresponding commutative diagrams 6.5.7, it is easily checked that

\[
t_h(\varphi \circ \psi) = t_h(\varphi) \circ t_h(\psi)
\]

Moreover, it follows from Proposition 5.8 and from definition 5.13.6 above that the functor \( \hat{\eta} \) maps the \( \text{ch}^*(\mathcal{F}^c) \)-object \((\bar{\eta}, q, \Delta_n)\) on the \( U_1' \)-set \( (\varpi_{h,\mathcal{L}(q),\text{Ker}(\pi_q)})^{-1}(\bar{\eta}) \) but, the image by \( \text{loc}_{\mathcal{F}^c} \) of the \( \text{ch}^*(\mathcal{F}^c) \)-morphism \((q, \Delta_n) \to (q(i), \Delta_0)\) determines a lifting of \( \hat{i}_i^q \)

\[
\hat{i}_i^q: \hat{\mathcal{L}}(q) \to \hat{\mathcal{L}}(q(i))
\]

then, it is quite clear that

\[
\hat{i}_i^q \circ (\varpi_{h,\mathcal{L}(q),\text{Ker}(\pi_q)})^{-1}(\bar{\eta}) = (\varpi_{h,\mathcal{L}(q(i)),Z(q(i))})^{-1}(\hat{i}_i^q \circ \bar{\eta})
\]

where the left member denotes the set of compositions of \( \hat{i}_i^q \) with all the elements of the \( U_1' \)-set \( (\varpi_{h,\mathcal{L}(q),\text{Ker}(\pi_q)})^{-1}(\bar{\eta}) \). On the other hand, by the very definition of \( t_h \), we actually have

\[
(\varpi_{h,\mathcal{L}(q(0)),Z(q(0))})^{-1}(\hat{i}_0^q \circ \bar{\eta}) = (t_h \circ \varpi_h \circ \varpi_h)(\bar{\eta}, q, \Delta_n)
\]
From these equalities it is easily checked that we have a natural isomorphism

\[ \tilde{\mu}_i \cong t_i \circ v_i \circ j_i \]  

6.5.15.

Finally, the \( K \)-module isomorphism 6.5.3 follows from the \( O \)-module isomorphism 5.13.7, from Proposition 6.4 and from this natural isomorphism [5, A3.9]. The proof of the last statement is straightforward.

**Remark 6.6** By Remark 5.9, for any \( \xi \in U_h \) the functor \( t_h \) maps the \( h(F^w) \)-automorphism \( \pi_Q(\rho(\xi)) \) of \( Q^\rho \), where \( \rho \in \tilde{\rho} \), on the identity map of \( (\pi_{h,(\xi(Q),Z(Q))}^{-1}(\tilde{\rho}) \); in particular, \( t_h \) factorizes via the exterior quotient \( \hat{h}(F^w) \) of \( h(F^w) \) determined by the correspondence mapping any \( h(F^w) \)-object \( Q^\rho \) on the group of \( h(F^w) \)-automorphisms of \( Q^\rho \) induced by \( \rho(U_h) \) [5, 6.3].

6.7 Note that the category \( h(F^w) \) also admits an interior structure [5, 1.3] which maps any \( h(F^w) \)-object \( Q^\rho \) on the group \( F_{Q^\rho(U_h)}(Q) \) of \( h(F^w) \)-automorphisms of \( Q^\rho \) where we are choosing \( \rho \in \tilde{\rho} \); then, denoting by \( \hat{h}(F^w) \) the corresponding exterior quotient, it is easily checked that the functor \( t_h \) admits a factorization

\[ \tilde{t}_h : \hat{h}(F^w) \longrightarrow U_h N \]  

6.7.1

and, in particular, we still have (cf. 6.5.3)

\[ \mathcal{K}_G^h(F, \text{aut}_{F^w})_h \cong \mathbb{H}^0(\hat{h}(F^w), \mathcal{K}_G h U_h \circ \tilde{t}_h) \]  

6.7.2.

6.8 On the other hand, for any \( h' \in \mathbb{N} - p\mathbb{N} \), recall that in [5, 6.3 and 14.25] we have defined an analogous category \( h'(\tilde{F}^w) \) where the \( h'(\tilde{F}^w) \)-objects are the pairs \( Q^\rho \) formed by an \( F \)-selfcentralizing subgroup \( Q \) of \( P \) and by an injective group homomorphism \( \rho : U_{h'} \rightarrow F(Q) \) and, for a second \( h'(\tilde{F}^w) \)-object \( R^\sigma \), the \( h'(\tilde{F}^w) \)-morphisms from \( R^\sigma \) to \( Q^\rho \) are the \( \tilde{F}^w \)-morphisms \( \tilde{\varphi} : R \rightarrow Q \) fulfilling

\[ \rho(\xi) \circ \tilde{\varphi} = \tilde{\varphi} \circ \rho(\xi) \]  

6.8.1

for any \( \xi \in U_{h'} \), and that in [5, Proposition 14.28] we have obtained a factorization analogous to 6.5.1, via a functor \( s_{h'} : h'(\tilde{F}^w) \rightarrow U_{h'} N \) which maps any \( h'(\tilde{F}^w) \)-object \( Q^\rho \) on the \( U_{h'} \)-set \( (\varpi_{h', \tilde{F}(Q)})^{-1}(\rho) \) [5, 14.18.4] and any \( h'(\tilde{F}^w) \)-morphism \( R^\sigma \rightarrow Q^\rho \) on a suitable \( U_{h'} \)-set bijection

\[ (\varpi_{h', \tilde{F}(R)})^{-1}(\sigma) \cong (\varpi_{h', \tilde{F}(Q)})^{-1}(\rho) \]  

6.8.2.
6.9 Moreover, respectively denoting by $h^{p'}$ and $h^{p}$ the $p'$- and the $p$-part of any $h \in \mathbb{N} - \{0\}$, let us consider the functor

$$\tau_h : h^{p'}(\hat{F}^{se}) \longrightarrow \mathbb{N}$$

which maps any $h^{p'}(\hat{F}^{se})$-object $Q^p$ on the set $\text{Mon}(U_{h^p}, Q^p(U_{h^{p'}}))$ of $Q^p(U_{h^{p'}})$-conjugacy classes of injective group homomorphisms $U_{h^p} \rightarrow Q^p(U_{h^{p'}})$, where we choose a lifting $\rho(U_{h^{p'}})$ of $\rho(U_{h^p})$ to $F(Q)$ and denote by $Q^p(U_{h^{p'}})$ the subgroup of $\hat{U}^{p'}_h$-fixed elements of $Q$, and sends any $\cdot$-morphism from $R^\sigma$ to $Q^p$, determined by a $\hat{F}^{se}$-morphism $\hat{\varphi} : R \rightarrow Q$, to the map

$$\text{Mon}(U_{h^p}, R^{\sigma}(U_{h^{p'}})) \longrightarrow \text{Mon}(U_{h^p}, Q^p(U_{h^{p'}}))$$

determined by the group homomorphism $R^{\sigma}(U_{h^{p'}}) \rightarrow Q^p(U_{h^{p'}})$ induced by a suitable representative of $\hat{\varphi}$. Finally, the map $U_{h^p} \times \mathbb{N} \rightarrow U_{h^{p'}} \mathbb{N}$ induced by the direct product determines a new functor

$$\sigma_{h^{p'}} \times \tau_h : h^{p'}(\hat{F}^{se}) \longrightarrow U_{h^{p'}} \mathbb{N}$$

**Proposition 6.10** For any $h \in \mathbb{N} - \{0\}$ we have a functor

$$\varphi^h : h(F^{se}) \longrightarrow h^{p'}(\hat{F}^{se})$$

mapping any $h(F^{se})$-object $Q^p$ such that $Q$ is fully normalized in $F$ and that, choosing $\rho \in \tilde{\rho}$, $\rho(U_{h^p})$ is contained in $N^F(Q)$, on the $h^{p'}(\hat{F}^{se})$-object formed by $Q = Q \cdot \rho(U_{h^p})$ endowed with the $U_{h^{p'}}$-action $\hat{\rho} : U_{h^{p'}} \rightarrow \hat{F}(Q)$ induced by $\rho$, and mapping the class of any $h(F^{se})$-morphism $\varphi : R^\sigma \rightarrow Q^p$ on the unique $h^{p'}(\hat{F}^{se})$-morphism $\hat{\tilde{\rho}}^\cdot \rho : \hat{F}(Q) \rightarrow Q^p$ extending $\tilde{\rho} : R \rightarrow Q$. Moreover, we have

$$(\varphi^h)_* (\text{Fct}_{U_{h^{p'}}} \circ \hat{\tilde{\rho}}^\cdot) = \text{Fct}_{U_{h^{p'}}} \circ (\sigma_{h^{p'}} \times \tau_h)$$

and, in particular, for any $n \in \mathbb{N}$ we get a group isomorphism

$$\mathbb{H}^n(h(F^{se}), \text{Fct}_{U_{h^{p'}}} \circ \hat{\tilde{\rho}}^\cdot) \cong \mathbb{H}^n(h^{p'}(\hat{F}^{se}), \text{Fct}_{U_{h^{p'}}} \circ (\sigma_{h^{p'}} \times \tau_h))$$

**Proof:** First of all, consider the analogous functor

$$\eta_h : h^{p'}(\hat{F}^{se}) \longrightarrow \mathbb{N}$$

mapping any $h^{p'}(\hat{F}^{se})$-object $Q^p$ on the set $\text{Mon}(U_{h^p}, Q^p(U_{h^{p'}}))$ of $Q^p(U_{h^{p'}})$-conjugacy classes of injective group homomorphisms $U_{h^p} \rightarrow Q^p(U_{h^{p'}})$, where
\[ \rho' \in \bar{\rho} \] and \( Q^{\bar{\rho}(U_{h'})} \) denotes the subgroup of fixed points of \( \rho'(U_{h'}) \subset L(Q) \) in \( Q \subset L(Q) \), and mapping the class of any \( h_{\bar{\rho}}^*(F^c) \)-morphism \( \varphi : R^{\bar{\rho}} \to Q^{\bar{\rho}'} \) to the map
\[
\overline{\text{Mon}}(U_{h'}, R^{\bar{\rho}}(U_{h'})) \to \overline{\text{Mon}}(U_{h'}, Q^{\bar{\rho}(U_{h'})}) \]
determined by the group homomorphism \( R^{\bar{\rho}}(U_{h'}) \to Q^{\bar{\rho}(U_{h'})} \) induced by \( \varphi \), where \( \sigma' \in \bar{\sigma} \). As above, the functor \( \eta_h \) can be viewed as a representation of the category \( h_{\bar{\rho}}^*(F^c) \) [5, A2.2] and we consider the corresponding semidirect product \( \eta_h \rtimes h_{\bar{\rho}}^*(F^c) \) [5, A2.7]; then, we define an adjoint pair of functors
\[
f_h : \eta_h \rtimes h_{\bar{\rho}}^*(F^c) \to h_{\bar{\rho}}^*(F^c) \quad \text{and} \quad g_h : h_{\bar{\rho}}^*(F^c) \to \eta_h \rtimes h_{\bar{\rho}}^*(F^c)
\]
as follows.

An \( \eta_h \rtimes h_{\bar{\rho}}^*(F^c) \)-object is a pair formed by a \( h_{\bar{\rho}}^*(F^c) \)-object \( Q^{\bar{\rho}} \) and by a \( Q^{\bar{\rho}}(U_{h'}) \)-conjugacy class \( \rho'' \) of injective group homomorphisms
\[
\rho'' : U_{h'} \to Q^{\bar{\rho}(U_{h'})}
\]
it is quite clear that \( \rho' \) and \( \rho'' \) define an injective group homomorphism \( \rho : U_h \to L(Q) \) and therefore they determine a \( \mathcal{L}\text{-}\mathcal{O}\text{-}\text{morphism} \)
\[
\bar{\rho} : (U_h, 1) \to (L(Q), Z(Q))
\]
hence, we get a \( h_{\bar{\rho}}^*(F^c) \)-object \( Q^{\bar{\rho}} \) and then, we define \( f_h(\bar{\rho}'', Q^{\bar{\rho}}) = Q^{\bar{\rho}} \) for a choice of \( \rho'' \in \bar{\rho}'' \). Similarly, a \( \eta_h \rtimes h_{\bar{\rho}}^*(F^c) \)-morphism to \( (\bar{\rho}'', Q^{\bar{\rho}}) \) from another \( \eta_h \rtimes h_{\bar{\rho}}^*(F^c) \)-object \( (\tilde{\sigma}'', R^{\tilde{\sigma}}) \) is the class of an \( h_{\bar{\rho}}^*(F^c) \)-morphism \( \varphi : R^{\bar{\rho}'} \to Q^{\bar{\rho}} \) such that, denoting by \( \tilde{\varphi} \) the restriction of \( \varphi \) to \( R^{\sigma'(U_{h'})} \) for a choice \( \sigma' \in \bar{\sigma} \), we have \( \tilde{\varphi} \circ \tilde{\sigma}'' = \bar{\rho}'' \) and we may assume that \( \tilde{\varphi} \circ \sigma'' = \rho'' \); it is then clear that \( \varphi \) still determines an \( h_{\bar{\rho}}^*(F^c) \)-morphism from \( R^{\bar{\rho}} \) to \( Q^{\bar{\rho}} \) and, coherently, we define \( f_h \) mapping the class of \( \varphi \) in \( h_{\bar{\rho}}^*(F^c) \) on the class of \( \varphi \) in \( h_{\bar{\rho}}^*(F^c) \). The functoriality of this correspondence is clear.

Conversely, note that any \( h_{\bar{\rho}}^*(F^c) \)-object admits an \( h_{\bar{\rho}}^*(F^c) \)-isomorphic one \( Q^{\bar{\rho}} \) such that \( Q \) is fully normalized in \( F \) and, identifying \( N_{P}(Q) \) with its structural image in \( L(Q) \), \( \rho(U_{h'}) \) is contained in \( N_{P}(Q) \) for \( \rho \in \bar{\rho} \); thus, in order to define the functor \( g_h \), we may replace \( h_{\bar{\rho}}^*(F^c) \) by its full subcategory over those objects.

If \( Q^{\bar{\rho}} \) is such an \( h_{\bar{\rho}}^*(F^c) \)-object, the product \( \tilde{Q} = Q \cdot \rho(U_{h'}) \) is a subgroup of \( N_{P}(Q) \) and it is not difficult to check that the localizer \( L(i_{Q}^{\bar{\rho}}) \) of the
\( \mathcal{F} \)-chain determined by the inclusion map from \( Q \) to \( \hat{Q} \) can be identified with the normalizer \( N_{L(\hat{Q})}(\hat{Q}) \) which clearly contains \( \rho(U_{h'}) \); thus, choosing a representative \( \mu \) of the \( \tilde{\mathfrak{Loc}} \)-morphism

\[
\text{loc}_{\mathfrak{F}^c} (\text{id}_{\hat{Q}}, \delta^0_0) : (L(\hat{Q}), Z(\hat{Q})) \rightarrow (L(\hat{Q}), Z(\hat{Q})) \quad \text{6.10.9},
\]

the composition of \( \mu \) with the restriction \( \rho' \) of \( \rho \) to \( U_{h'} \) determines a \( \tilde{\mathfrak{Loc}} \)-morphism

\[
\hat{\rho} : (U_{h'} , 1) \rightarrow (L(\hat{Q}), Z(\hat{Q})) \quad \text{6.10.10}
\]

and therefore we get an \( \hat{h}'(\mathcal{F}^c) \)-object \( \hat{Q}^{\hat{\rho}} \); now, since \( \rho(U_{h'}) \) is contained in \( \hat{Q} \), the restriction \( \rho'' \) of \( \rho \) to \( U_{h'} \) determines an injective group homomorphism \( \rho'' : U_{h'} \rightarrow \hat{Q}^{\rho(U_{h'})} \); then, we define

\[
g_h(Q^{\hat{\rho}}) = (\hat{\rho}'' , \hat{Q}^{\hat{\rho}}) \quad \text{6.10.11}
\]

where \( \hat{\rho}'' \) denotes the \( \hat{Q}^{\rho(U_{h'})} \)-conjugacy class of \( \rho'' \).

If \( R^{\hat{\sigma}} \) is also such an \( \hat{h}(\mathcal{F}^c) \)-object and \( \varphi : R^{\hat{\sigma}} \rightarrow Q^{\hat{\rho}} \) is an \( \hat{h}(\mathcal{F}^c) \)-morphism, we know that there is a \( \tilde{\mathfrak{Loc}} \)-morphism (cf. 6.1)

\[
\tilde{\alpha} : (U_h , 1) \rightarrow (L(\varphi), Z(Q)) \quad \text{6.10.12}
\]

fulfilling (cf. 6.1.4)

\[
\text{loc}_{\mathfrak{F}^c} (\text{id}_R , \delta^0_0) \circ \tilde{\alpha} = \tilde{\sigma} \quad \text{and} \quad \text{loc}_{\mathfrak{F}^c} (\text{id}_Q , \delta^0_0) \circ \tilde{\alpha} = \tilde{\rho} \quad \text{6.10.13};
\]

then, choosing representatives \( \lambda_0 \) of \( \text{loc}_{\mathfrak{F}^c} (\text{id}_Q , \delta^0_0) \) and \( \lambda_1 \) of \( \text{loc}_{\mathfrak{F}^c} (\text{id}_R , \delta^0_0) \) such that, identifying \( \hat{R} \) with its structural image in \( L(\varphi) \), we have \( \lambda_0(v) = v \) and \( \lambda_1(v) = \varphi(v) \) for any \( v \in \hat{R} \), there are representatives \( \rho \in \hat{\rho} , \sigma \in \hat{\sigma} \) and \( \alpha \in \tilde{\alpha} \) fulfilling

\[
\lambda_0 \circ \alpha = \sigma \quad \text{and} \quad \lambda_1 \circ \alpha = \rho \quad \text{6.10.14}
\]

and, in particular, inducing group isomorphisms

\[
\sigma(U_{h'}) \cong \alpha(U_{h'}) \cong \rho(U_{h'}) \quad \text{6.10.15};
\]

since clearly \( \varphi^{-1}(Z(Q)) \subset Z(R) \), it is easily checked that \( \varphi \) and these isomorphisms determine an injective group homomorphism

\[
\hat{\varphi} : \hat{R} = R \sigma(U_{h'}) \rightarrow Q \rho(U_{h'}) = \hat{Q} \quad \text{6.10.16}
\]

which agree with \( \lambda_0 \) and \( \lambda_1 \); then, it follows from [5, Proposition 18.16] that \( \hat{\varphi} \) is actually an \( \mathcal{F} \)-morphism.
Now, $\mathcal{L}(\varphi)$ contains $\hat{R}$ and it is clear that
\[
\sigma(U_{h^{\varphi}}) \subset \lambda_{1}(N_{\mathcal{L}(\varphi)}(\hat{R})) \subset N_{\mathcal{L}(\hat{R})}(\hat{R}) \cong \mathcal{L}(i_{\hat{R}}^{\hat{R}}) \quad 6.10.17;
\]
moresover, choosing a representative $\nu$ of the $\mathfrak{L}\mathfrak{c}$-morphism
\[
\text{loc}_{\mathfrak{F}^{uc}}(\text{id}_{\hat{R}}, \delta_{1}^{0}) : (\mathcal{L}(i_{\hat{R}}^{\hat{R}}), Z(\hat{R})) \rightarrow (\mathcal{L}(\hat{R}), Z(\hat{R})) \quad 6.10.18,
\]
it is easily checked that $(\nu \circ \lambda_{1})(N_{\mathcal{L}(\varphi)}(\hat{R}))$ is contained in the image of $\mathcal{L}(\hat{\varphi})$ in $\mathcal{L}(\hat{R})$ via any representative $\lambda_{1}$ of the $\mathfrak{L}\mathfrak{c}$-morphism
\[
\text{loc}_{\mathfrak{F}^{uc}}(\text{id}_{\hat{R}}, \delta_{1}^{0}) : (\mathcal{L}(\hat{\varphi}), Z(\hat{\varphi})) \rightarrow (\mathcal{L}(\hat{R}), Z(\hat{R})) \quad 6.10.19
\]
and therefore the restriction of $\alpha$ to $U_{h^{\rho'}}$ and the composition $\nu \circ \lambda_{1}$ determine a $\mathfrak{L}\mathfrak{c}$-morphism
\[
\hat{\alpha}' : (U_{h^{\rho'}}, 1) \rightarrow (\mathcal{L}(\hat{\varphi}), Z(\hat{\varphi})) \quad 6.10.20
\]
fulfilling the corresponding equalities
\[
\text{loc}_{\mathfrak{F}^{uc}}(\text{id}_{\hat{R}}, \delta_{1}^{0}) \circ \hat{\alpha}' = \hat{\sigma}' \quad \text{and} \quad \text{loc}_{\mathfrak{F}^{uc}}(\text{id}_{\hat{\varphi}}, \delta_{1}^{0}) \circ \hat{\alpha}' = \hat{\rho}' \quad 6.10.21,
\]
so that $\hat{\varphi}$ is also an $h^{\varphi}_{\mathfrak{F}^{uc}}$-morphism from $\hat{R}^{\hat{\sigma}'}$ to $\hat{Q}^{\hat{\rho}'}$; it is easy to check that the $h^{\varphi}_{\mathfrak{F}^{uc}}$-morphism determined by $\hat{\varphi}$ does not depend on our choice.

Finally, respectively denoting by
\[
\rho'' : U_{h^{\rho'}} \rightarrow \hat{Q}^{\rho'(U_{h^{\rho'}})} \quad \text{and} \quad \sigma'' : U_{h^{\rho'}} \rightarrow \hat{R}^{\sigma(U_{h^{\rho'}})} \quad 6.10.22
\]
the restrictions to $U_{h^{\rho'}}$ of $\rho$ and $\sigma$, by the very definition of $\hat{\varphi}$ we obtain $\hat{\varphi}(\sigma''(\xi)) = \rho''(\xi)$ for any $\xi \in U_{h^{\rho'}}$; in conclusion, denoting by $\hat{\rho}''$ and $\hat{\sigma}''$ the respective $\hat{Q}^{\rho(U_{h^{\rho'}})}$ and $\hat{R}^{\sigma(U_{h^{\rho'}})}$-conjugacy classes of $\rho''$ and $\sigma''$, the class of $\hat{\varphi}$ still defines a $\eta_{h} \times h^{\varphi}_{\mathfrak{F}^{uc}}$-morphism from $(\hat{\rho}'', \hat{R}^{\hat{\sigma}'})$ to $(\hat{\rho}'', \hat{Q}^{\hat{\rho}'})$ and we define $\eta_{h}$ mapping the class of $\varphi$ in $h^{\varphi}_{\mathfrak{F}^{uc}}$ on the class of $\hat{\varphi}$ in $h^{\varphi}_{\mathfrak{F}^{uc}}$. Once again, the proof of the functoriality of $\eta_{h}$ is straightforward.

At this point, it is quite clear that
\[
\eta_{h} \circ \hat{f}_{h} \cong \text{id} \quad 6.10.23;
\]
moreover, it is not difficult to verify that the correspondence sending the $h^{\varphi}_{\mathfrak{F}^{uc}}$-object $Q^{\hat{\rho}}$ above to the $h^{\varphi}_{\mathfrak{F}^{uc}}$-morphism from $Q^{\hat{\rho}}$ to $\hat{Q}^{\hat{\rho}}$ induced by the inclusion $Q \subset \hat{Q}$ defines a natural map
\[
\eta : \text{id}_{h^{\varphi}_{\mathfrak{F}^{uc}}} \rightarrow \hat{f}_{h} \circ \eta_{h} \quad 6.10.24
\]
which fulfills $\eta \ast f_h = \text{id}_{f_h}$ and $g_h \ast \eta = \text{id}_{g_h}$; this proves that $f_h$ and $g_h$ form an adjoint pair and, in particular, we get [5, 1.6]

\[(g_h)_\ast (\mathcal{F}c_{U_{\beta'}}) \circ \tilde{t}_h = \mathcal{F}c_{U_{\beta'}} \circ \tilde{t}_h \circ f_h \quad 6.10.25.\]

On the other hand, for any $h' \in \mathbb{N} - p\mathbb{N}$ note that the categories $\mathcal{F}(\mathcal{F}^\beta)$ and $\mathcal{F}(\mathcal{F}^\alpha)$ are equivalent. Indeed, for any $\mathcal{F}(\mathcal{F}^\beta)$-object $Q^\beta$, it is quite clear that the $Q$-conjugacy class of the group homomorphism $\rho' : U_{h'} \rightarrow \mathcal{L}(Q)$, where we choose $\rho' \in \tilde{\rho}'$, is determined by the composition

\[\tilde{\pi}_Q \circ \rho' : U_{h'} \longrightarrow \tilde{F}(Q) \cong \mathcal{L}(Q)/Q \quad 6.10.26,\]

namely by the $\mathcal{F}(\mathcal{F}^\beta)$-object $Q^{\tilde{\pi}_Q \circ \rho'}$. Similarly, any $\mathcal{F}(\mathcal{F}^\alpha)$-morphism from an $\mathcal{F}(\mathcal{F}^\beta)$-object $R^\beta$ to $Q^\beta$ is the $Q^\beta(U_{h'})$-conjugacy class of an $\mathcal{F}$-morphism $\varphi : R \rightarrow Q$ which admits a group homomorphism $\alpha' : U_{h'} \rightarrow \mathcal{L}(\varphi)$ fulfilling

\[\lambda_0 \circ \alpha' = \sigma' \quad \text{and} \quad \lambda_1 \circ \alpha' = \rho' \quad 6.10.27\]

for suitable representatives $\lambda_0$ of $\mathbf{Lc}_{\mathcal{F}^\beta}(\text{id}_Q, \delta_1^Q)$ and $\lambda_1$ of $\mathbf{Lc}_{\mathcal{F}^\beta}(\text{id}_R, \delta_1^R)$; hence, according to [5, Proposition A2.10], for any $\xi' \in U_{h'}$ we have

\[
\pi_Q(\rho'(\xi')) \circ \varphi = (\pi_Q \circ \lambda_1)(\alpha'(\xi')) \circ \varphi = (\alpha'(\xi')) \circ \varphi = (\omega \circ \alpha')(\xi') = (\omega \circ \pi_{h'})(\alpha'(\xi')) = (\omega \circ \pi_{h'})(\alpha'(\xi')) = (\omega \circ \pi_{h'})(\alpha'(\xi')) = \varphi(\alpha'(\xi'))
\]

which proves that $\varphi$ also induces an $\mathcal{F}(\mathcal{F}^\beta)$-morphism from $R^{\tilde{\pi}_Q \circ \rho'}$ to $Q^{\tilde{\pi}_Q \circ \rho'}$.

Conversely, any representative $\psi$ of an $\mathcal{F}(\mathcal{F}^\alpha)$-morphism

\[\tilde{\psi} : R^{\tilde{\pi}_Q \circ \rho'} \longrightarrow Q^{\tilde{\pi}_Q \circ \rho'} \quad 6.10.29\]

determines a group homomorphism

\[U_{h'} \longrightarrow \tilde{F}(\psi) \cong \mathcal{L}(\psi)/R \quad 6.10.30\]

which clearly can be lifted to a group homomorphism $\beta : U_{h'} \rightarrow \mathcal{L}(\psi)$, and it is easily checked that this group homomorphism fulfills equalities 6.10.27 for a suitable choice of the representatives $\lambda_0$ and $\lambda_1$; hence, the $\mathcal{F}$-morphism $\psi : R \rightarrow Q$ also determines an $\mathcal{F}(\mathcal{F}^\beta)$-morphism from $R^\beta$ to $Q^\beta$, completing the proof of the equivalence

\[\mathcal{F}(\mathcal{F}^\alpha) \cong \mathcal{F}(\mathcal{F}^\beta) \quad 6.10.31.\]
Moreover, the functors $\eta_h$ and $\gamma_h$ agree with this equivalence; consequently, for any $h \in \mathbb{N} - \{0\}$, we also get an equivalence of categories

$$
\varepsilon_h : \eta_h \times h^{pr}(\mathcal{F}^e) \cong \gamma_h \times h^{pr}(\tilde{\mathcal{F}}^e)
$$

6.10.32.

Finally, denoting by

$$p^h_{\mathcal{F}} : \eta_h \times h^{pr}(\mathcal{F}^e) \longrightarrow h^{pr}(\tilde{\mathcal{F}}^e)$$

6.10.33

the structural functor, we set

$$d^h_{\mathcal{F}} = p^h_{\mathcal{F}} \circ \varepsilon_h \circ g_h$$

6.10.34

and we claim that the direct image of $\mathcal{F}ct_{U \gamma_h \circ \tilde{t}_h}$ throughout this composition coincides with $\mathcal{F}ct_{U \gamma_h \circ \tilde{t}_h \circ (s_{h^{pr}} \times \gamma_h)}$; indeed, according to equality 6.10.25, for any $h^{pr}(\tilde{\mathcal{F}}^e)$-object $Q^{pr}$, we have

$$\left( (p^h_{\mathcal{F}} \circ \varepsilon_h)_*(\mathcal{F}ct_{U \gamma_h \circ \tilde{t}_h \circ f_h})(Q^{pr}) \right) = \prod_{\rho^{pr} \in Mon(U \gamma_h \circ \tilde{t}_h \circ (s_{h^{pr}} \times \gamma_h))} \mathcal{F}ct_{U \gamma_h \circ \tilde{t}_h \circ (s_{h^{pr}} \times \gamma_h)}(Q^{pr})$$

$$= \mathcal{F}ct_{U \gamma_h \circ \tilde{t}_h \circ (s_{h^{pr}} \times \gamma_h)}(Q^{pr})$$

6.10.35

Then, isomorphism 6.10.3 follows from this fact and from Lemma 6.11 below.

**Lemma 6.11** Let $\mathcal{C}$ be a small category, $m : \mathcal{C} \to \mathcal{C} \mathcal{C}$ a representation such that, for any $\mathcal{C}$-object $C$, the category $mC$ only has the identity morphisms, and $a : m \times \mathcal{C} \to \mathcal{A}b$ a contravariant functor. Denote by $a^m : \mathcal{C} \to \mathcal{A}b$ the contravariant functor mapping any $\mathcal{C}$-object $C$ on $\prod X a(X, C)$, where $X$ runs over the set of $mC$-objects, and any $\mathcal{C}$-morphism $f : C \to C'$ on the group homomorphism

$$a^m(C') = \prod_{X'} a(X', C') \longrightarrow \prod_X a(X, C) = a^m(C)$$

6.11.1

sending $(a_{X'})_{X'} \in a^m(C')$ to $\left( (a(id_{f(X)}), f)(a_{f(X)}) \right)_{X} \in a^m(C)$. Then, for any $n \in \mathbb{N}$ the structural functor $m \times \mathcal{C} \to \mathcal{C}$ induces a group isomorphism

$$\mathbb{H}^n(m \times \mathcal{C}, a) \cong \mathbb{H}^n(\mathcal{C}, a^m)$$

6.11.2.
Proof: By its very definition in [5, A3.8], we know that $H_n(m \rtimes C, a)$ is the $n$-th homology group of the functor

$$(a \circ v^m_{m \times C})^\circ \circ \circ \circ : \Delta \longrightarrow Ab$$

where $\Delta$ denotes the simplicial 2-category [5, A1.7], $st : \Delta \to \mathcal{C}$ the standard representation of $\Delta$ [5, A2.2], $\mathcal{Fct}(st, m \rtimes C) \circ \circ : \Delta \longrightarrow \mathcal{C}$ the naive $m \times C$-dual representation of $st$ mapping $\Delta_n$ on $\mathcal{Fct}(\Delta_n, m \times C)$ [5, A2.5], and $v^m_{m \times C} : \mathcal{Fct}(st, m \rtimes C) \circ \circ \circ \circ \to \mathcal{C}$ the corresponding evaluation functor [5, A3.7].

But, any functor $\hat{q} : \Delta_n \to m \times C$ determines a functor $q : \Delta_n \to C$ and then $\hat{q}$ is determined by $q$ and by $\hat{q}(0) = (X_0, q(0))$ since, according to our hypothesis on $m$ and setting $X_i = m(q(0 \cdot i))(X_0)$, for any $i \in \Delta_n - \{0\}$ we have

$$\hat{q}(i) = (X_i, q(i)) \quad \text{and} \quad \hat{q}(i - 1 \cdot i) = (id_{X_i}, q(i - 1 \cdot i))$$

Consequently, it is quite clear that

$$(a \circ v^m_{m \times C})^\circ \circ \circ \circ (\Delta_n) = \prod_{\hat{q} \in \mathcal{Fct}(\Delta_n, m \times C)} a(\hat{q}(0))$$

$$= \prod_{q \in \mathcal{Fct}(\Delta_n, C)} \! a^m(q(0))$$

$$= ((a^m \circ v^m_{m \times C})^\circ \circ \circ \circ (\Delta_n))$$

and it is easily checked the coincidence of the functors $(a \circ v^m_{m \times C})^\circ \circ \circ \circ$ and $(a^m \circ v^m_{m \times C})^\circ \circ \circ \circ$. We are done.

7 A vanishing cohomological result

7.1 As in [5, Corollary 14.32] for the determination of the $O$-rank of the modular Grothendieck group $G_k(F, \hat{\text{aut}})$, the determination of the $O$-rank of the ordinary Grothendieck group $G_K(F, \hat{\text{aut}})$ ultimately depends on a vanishing cohomological result. However, the general result [5, Theorem 6.26] we employ there it is not powerful enough, as it stands, to discuss our present situation; but, as a matter of fact, essentially the same arguments prove a sufficiently general result. Nevertheless, even if our proof below mainly repeats the proof of [5, Theorem 6.26], we write it completely in order to clarify some arguments.
7.2 In this section, our setting is just a finite $p$-group $P$ and a Frobenius $P$-category $\mathcal{F}$. Let $K$ be a finite $p'$-group and, as in [5, 6.3], consider the category $K\text{ac}(\mathcal{F}^e)$ of the $K$-objects of $\text{ac}(\mathcal{F}^e)$ [5, 6.2]; recall that this category admits direct products and pull-backs [5, Propositions 6.14 and 6.21]. As in [5, 6.25], consider the object $\bigoplus_{x \in K} P$ of the category $\text{ac}(\mathcal{F}^e)$ [5, 6.2] endowed with the $K$-action $\pi$ defined by the regular action of $K$ on itself and by the identity on $P$ between the corresponding terms, so that $(\bigoplus_{x \in K} P)^\pi$ is an indecomposable $K$-object of $\text{ac}(\mathcal{F}^e)$ [5, 6.3].

7.3 If $\mathfrak{F}$ is a full subcategory of $K\text{ac}(\mathcal{F}^e)$ and $\mathfrak{m}: \mathfrak{F} \to \mathcal{O}\text{-}\mathfrak{mod}$ is a contravariant functor, we say that a functor $\mathfrak{m}^\circ: \mathfrak{F} \to \mathcal{O}\text{-}\mathfrak{mod}$ is a right-hand sectional functor of $\mathfrak{m}$ if it coincides with $\mathfrak{m}$ over indecomposable $K$-objects of $\mathcal{F}$ induced by $\mathfrak{F}$ over indecomposable $K$-objects and, for any $\mathfrak{F}$-morphism $\tilde{\varphi}: R^\pi \to Q^\rho$, we have

$$\mathfrak{m}(\tilde{\varphi}) \circ \mathfrak{m}^\circ(\tilde{\varphi}) = \text{id}_{\mathfrak{m}(R^\pi)}$$  \hspace{1cm} 7.3.1.

Note that $\text{ac}(\mathfrak{F})$ still can be identified to a subcategory of $K\text{ac}(\mathcal{F}^e)$. Moreover, recall that the center $Z(K)$ defines an exterior quotient $\mathcal{K}\text{ac}(\mathcal{F}^e)$ of $K\text{ac}(\mathcal{F}^e)$ [5, 6.3]; then, since $Z(K)$ is invertible in $\mathcal{O}$, if $\mathfrak{m}$ factorizes via the image $\hat{\mathfrak{F}}$ of $\mathfrak{F}$ in $K\text{ac}(\mathcal{F}^e)$ throughout a contravariant functor $\hat{\mathfrak{m}}: \hat{\mathfrak{F}} \to \mathcal{O}\text{-}\mathfrak{mod}$, it follows from [5, Proposition A4.13] that, for any $n \in \mathbb{N}$, we have

$$\mathbb{H}^n(\hat{\mathfrak{F}}, \hat{\mathfrak{m}}) = \mathbb{H}^n(\mathfrak{F}, \mathfrak{m})$$  \hspace{1cm} 7.3.2

since, considering the subcategory $\mathfrak{F}$ of $\hat{\mathfrak{F}}$ formed by the same objects and by the automorphisms of the $\mathfrak{F}$-objects induced by $Z(K)$, $\mathbb{H}^n(\hat{\mathfrak{F}}, \hat{\mathfrak{m}})$ clearly coincides with the $\mathfrak{F}$-stable $n$-cohomology group of $\mathfrak{F}$ over $\mathfrak{m}$ [5, A3.18].

**Theorem 7.4** With the notation above, let $\mathfrak{F}$ be a full subcategory of $K\text{ac}(\mathcal{F}^e)$ over indecomposable $K$-objects of $\text{ac}(\mathcal{F}^e)$ including $(\bigoplus_{x \in K} P)^\pi$, such that the subcategory $\text{ac}(\hat{\mathfrak{F}})$ is closed by direct products and pull-backs. For any contravariant functor $\mathfrak{m}: \hat{\mathfrak{F}} \to \mathcal{O}\text{-}\mathfrak{mod}$ admitting a right-hand sectional functor $\mathfrak{m}^\circ: \hat{\mathfrak{F}} \to \mathcal{O}\text{-}\mathfrak{mod}$, we have $\mathbb{H}^n(\hat{\mathfrak{F}}, \mathfrak{m}) = \{0\}$ for any $n \geq 1$.

**Proof:** First of all, we prove the statement assuming that $p \cdot \mathfrak{m} = 0$ or, equivalently, that $\mathfrak{m}$ is a contravariant functor from $\mathfrak{F}$ to $k\text{-}\mathfrak{mod}$; coherently, $\mathfrak{m}^\circ$ is a functor from $\mathfrak{F}$ to $k\text{-}\mathfrak{mod}$. Moreover, it follows from [5, Proposition A4.11] that for any $n \geq 1$ we have

$$\mathbb{H}^n(\mathfrak{F}, \mathfrak{m}) \cong \mathbb{H}^n(\text{ac}(\mathfrak{F}), \text{ac}(\mathfrak{m}))$$  \hspace{1cm} 7.4.1

so that it suffices to prove that $\mathbb{H}^n(\text{ac}(\mathfrak{F}), \text{ac}(\mathfrak{m})) = \{0\}$ for any $n \geq 1$.

Set $S = \bigoplus_{x \in K} P$; it follows from [5, Proposition 6.14] that the direct product by $S^\pi$, or the exterior intersection with $S^\pi$ [5, definition 6.13.3], defines a functor

$$\text{int}_{S^\pi}: \text{ac}(\mathfrak{F}) \rightarrow \text{ac}(\mathfrak{F})$$  \hspace{1cm} 7.4.2.
then, the existence of the structural \( \text{ac}(\mathfrak{g}) \)-morphism \( Q^\rho \cap S^\pi \to S^\pi \) for any \( \text{ac}(\mathfrak{g}) \)-object \( Q^\rho \) shows that \( \text{int}_{S^\pi} \) factorizes throughout the evident \textit{forgetful} functor \([5, 1.7]\)
\[
\text{fg}_{S^\pi} : \text{ac}(\mathfrak{g})_{S^\pi} \to \text{ac}(\mathfrak{g})
\]
explicitly, it suffices to consider the functor \( \text{ac}(\mathfrak{g}) \to \text{ac}(\mathfrak{g})_{S^\pi} \) mapping any \( \text{ac}(\mathfrak{g}) \)-object \( Q^\rho \) on the structural \( \text{ac}(\mathfrak{g}) \)-morphism above and any \( \text{ac}(\mathfrak{g}) \)-morphism \( \tilde{\alpha} : R^\sigma \to Q^\rho \) on \( \tilde{\alpha} \cap \tilde{id}_{S^\pi} \) \([5, \text{Proposition 6.14}]\). But, since the category \( \text{ac}(\mathfrak{g})_{S^\pi} \) has the final object \( \tilde{id}_{S^\pi} : S^\pi \to S^\pi \), it follows from \([5, \text{Corollary A4.8}]\) that for any \( n \geq 1 \) we have
\[
\mathbb{H}^n(\text{ac}(\mathfrak{g})_{S^\pi}, \text{ac}(m) \circ \text{fg}_{S^\pi}) = \{0\}
\]
and therefore, we still have \([5, \text{A3.10.4}]\)
\[
\mathbb{H}^n(\text{ac}(\mathfrak{g}), \text{ac}(m) \circ \text{int}_{S^\pi}) = \{0\}
\]
Moreover, the existence of the structural morphism \( \tilde{\omega}_{Q^\rho} : Q^\rho \cap S^\pi \to Q^\rho \) for any \( \text{ac}(\mathfrak{g}) \)-object \( Q^\rho \) shows the existence of a natural map
\[
\omega : \text{int}_{S^\pi} \to \text{id}_{\text{ac}(\mathfrak{g})}
\]
 sending \( Q^\rho \) to \( \tilde{\omega}_{Q^\rho} \); thus, in order to prove that \( \mathbb{H}^n(\text{ac}(\mathfrak{g}), \text{ac}(m)) = \{0\} \), it suffices to prove that the natural map \( \text{ac}(m) \circ \omega \) admits a \textit{natural section}
\[
\theta : \text{ac}(m) \circ \text{int}_{S^\pi} \to \text{ac}(m)
\]
so that \( \text{ac}(m) \) becomes a direct summand of \( \text{ac}(m) \circ \text{int}_{S^\pi} \).

Explicitly, for any \( \mathfrak{g} \)-object \( Q^\rho = (\bigoplus_{i \in I} Q_i)^\rho \), we have \([5, 6.13]\)
\[
Q^\rho \cap S^\pi = \left( \bigoplus_{(i,x) \in I \times K} \bigoplus_{(\tau,T,\tilde{\tau}) \in \check{T}_{Q_i,P}} T_i \right)^{\hat{\rho}}
\]
for a set of representatives \( \check{T}_{Q_i,P} \) of \( \check{T}_{Q_i,P} \) in \( \check{T}_{Q_i,P} \) \([5, 6.9]\) and a suitable action \( \hat{\rho} \) of \( K \) on the \( \text{ac}(\check{F}^\phi) \)-object \( Q \cap S^\pi \); in particular, \( K \) acts freely on the disjoint union
\[
\hat{I} = \bigcup_{(i,x) \in I \times K} \check{T}_{Q_i,P}
\]
and let us denote by \( \hat{I}/K \) the set of \( K \)-orbits on \( \hat{I} \) and, for any \( O \in \hat{I}/K \), by \( (T_o)^{\hat{\rho}_o} \) the corresponding indecomposable \textit{“direct summand”} of \( Q^\rho \cap S^\pi \)
and by \( \check{T}_o \) the composition
\[
\check{T}_o : (T_o)^{\hat{\rho}_o} \to Q^\rho \cap S^\pi \to Q^\rho
\]
of the structural ac(\tilde{\mathfrak{g}})-morphism with \tilde{\omega}_{Q^\rho}. Moreover, we denote by \tilde{I}^o/K the set of special orbits O ∈ \tilde{I}/K where the \tilde{\mathcal{F}}\sigma^-\text{-morphisms determining } \tilde{\tau}_o \text{ are isomorphisms; note that, according to } [5, \text{ Proposition 6.14}], we have a canonical bijection
\[
\tilde{I}^o/K \cong \bigsqcup_{i \in I} \tilde{\mathcal{F}}(P_i, Q_i)
\] 7.4.11.

Then, we consider the homomorphism
\[
\theta_{Q^\rho} : \ (\text{ac}(m))(Q^\rho \cap S^\pi) \to m(Q^\rho)
\]
\[
\prod_{O \in \tilde{I}/K} m((T_o)^{/o})
\]
sending an element \(m = (m_o)_{O \in \tilde{I}/K}\) of this product to
\[
\theta_{Q^\rho}(m) = |\tilde{I}/K|^{-1} \sum_{O \in \tilde{I}/K} m^\sigma(\tilde{\tau}_o)(m_o)
\] 7.4.13;

since for any special orbit \(O\) the composition \(\tilde{\tau}_o\) above is an isomorphism and since \(m(\tilde{\tau}_o) \circ m^\sigma(\tilde{\tau}_o) = \text{id}_{(T_o)^{/o}}\), we actually have \(m^\sigma(\tilde{\tau}_o) = m(\tilde{\tau}_o)^{-1}\) and therefore we clearly get
\[
\theta_{Q^\rho} \circ (\text{ac}(m) \ast \omega)_{Q^\rho} = \text{id}_{m(Q^\rho)}
\] 7.4.14.

By the distributivity of the exterior intersection [5, 6.13], we easily can extend this correspondence to all the ac(\tilde{\mathfrak{g}})-objects and then we claim that the extended correspondence is a natural map from ac(m) \ast \text{int}_{S^\pi} to ac(m); actually, it suffices to consider an \tilde{\mathfrak{g}}-morphism \(\tilde{\alpha} : R^\sigma \to Q^\rho\) and to prove the commutativity of the following diagram
\[
\begin{array}{ccc}
(\text{ac}(m))(Q^\rho \cap S^\pi) & \xrightarrow{\theta_{Q^\rho}} & m(Q^\rho) \\
(\text{ac}(m))(\tilde{\alpha} \cap \text{id}_{S^\pi}) & \downarrow & m(\tilde{\alpha})
\end{array}
\]
\[
\begin{array}{ccc}
(\text{ac}(m))(R^\sigma \cap S^\pi) & \xrightarrow{\theta_{R^\sigma}} & m(R^\sigma)
\end{array}
\]
7.4.15.

Explicitly, if \(R = \bigoplus_{j \in J} R_j\) is the structural decomposition of \(R\), then \(\tilde{\alpha}\) is given by a \(K\)-compatible map \(f : J \to I\) and by a \(K\)-compatible family of \(\tilde{\mathcal{F}}\)-morphisms \(\tilde{\alpha}_j : R_j \to Q_{f(j)}\) where \(j\) runs over \(J\), and as above we have
\[
R^\sigma \cap S^\pi = \bigoplus_{(j,x) \in J \times K} \bigoplus_{(\tilde{e},U,i,U)} \hat{\mathcal{F}}_{R_j,P}
\]
7.4.16.

for a set of representatives \(\hat{\mathcal{F}}_{R_j,P}\) of \(\hat{\mathcal{F}}_{R_j,P}\) in \(\mathcal{T}_{R_j,P}\) [5, 6.9] and a suitable action \(\hat{\sigma}\) of \(K\) on the ac(\tilde{\mathcal{F}}\sigma^-)-object \(R \cap S\); again, we set
\[
\hat{j} = \bigsqcup_{(j,x) \in J \times K} \hat{\mathcal{F}}_{R_j,P}
\] 7.4.17.
and denote by $\hat{J}/K$ the set of $K$-orbits on $\hat{J}$, by $\hat{\mathcal{F}}/K$ the set of special $K$-orbits on $\hat{J}$ and, for any $O \in \hat{J}/K$, by $(U_o)^{\mathcal{F}_O}$ the corresponding indecomposable “direct summand” of $R^\sigma \cap S^\sigma$ and by $\hat{\varphi}_O : (U_o)^{\mathcal{F}_O} \rightarrow R^\sigma$ the analogous composition 7.4.10; moreover, it is clear that the map $f$ and the family $\{\hat{\alpha}_j\}_{j \in J}$ determine a $K$-compatible map $\hat{f} : \hat{J} \rightarrow \hat{I}$ and, for any $O \in \hat{J}/K$, an $\mathcal{F}$-morphism

$$\hat{\alpha}_O : (U_o)^{\mathcal{F}_O} \longrightarrow (T_{f(I)})^\hat{\varphi}_{f(O)}$$ 7.4.18.

It is easily checked from [5, Propositions 6.14 and 6.21] that [5, 6.18.2]

$$R^\sigma \cap \mathcal{F}_{Qf} (Q^\eta \cap S^\pi) \cong R^\sigma \cap S^\pi$$ 7.4.19

and, by the distributivity property, we may assume that the exterior intersection $R^\sigma \cap S^\pi$ coincides with

$$\left( \bigoplus_{(j,x) \in J \times K} (\tau, \sigma) \in \mathcal{F}_{Q(j)} \right)$$ 7.4.20.

Then, for any $(j, x) \in J \times K$ and any $t = (\tau, T, i_T^C) \in \mathcal{F}_{Q(j)}, p$, we choose representatives $\tau \in \hat{T}$ and $\alpha_{f(j)} \in \hat{\alpha}_{f(j)}$, and a set of representatives $W_{(j, x, t)}$ in $Q_{f(j)}$ for the set of double classes $\tau(T) \backslash Q_{f(j)}/\alpha_j(R_j)$; we denote by $W_{(j, x, t)}^\cap$ the set of $w \in W_{(j, x, t)}$ such that the subgroup

$$U_w = (\kappa_{Q(j)}(w) \circ \alpha_j)^{-1}(\tau(T))$$ 7.4.21

remains $F$-selfcentralizing, and by $\hat{\beta}_{(j, x, t, w)} : U_w \rightarrow T$ the $\hat{\mathcal{F}}$-morphism determined by the compositions $\kappa_{Q(j)}(w) \circ \alpha_j$, where $\kappa_{Q(j)}(w)$ denotes the corresponding conjugation by $w$.

Now, with all this notation, it follows from [5, Proposition 6.19] that, for any $(j, x) \in J \times K$ and any triple $t = (\hat{\tau}, T, i_T^C) \in \mathcal{F}_{Q(j)}, p$, we have

$$R_j \hat{\alpha}_j \mathcal{F}_{T} = \bigoplus_{w \in W_{(j, x, t)}^\cap} U_w$$ 7.4.22

and isomorphism 7.4.19 determines a “graded” bijection between the disjoint union

$$\bigcup_{(j,x) \in J \times K} \bigcup_{t \in \mathcal{F}_{Q(j)}, p} W_{(j, x, t)}^\cap$$ 7.4.23

and $\hat{J}$; moreover, the $\kappa(\mathcal{F}^w)$-morphism

$$\hat{\alpha} \cap \text{id}_{S^\pi} : R^\sigma \cap S^\pi \longrightarrow Q^\eta \cap S^\pi$$ 7.4.24
is the “direct sum” over the set \( \bigcup_{(j,x) \in J \times K} \tilde{\mathcal{F}} Q_{f(j),P} \) of the \( \mathfrak{ac}(\tilde{\mathcal{F}}^w) \)-morphisms

\[
\tilde{\beta}_{(j,x,t)} : \bigoplus_{w \in W^{(j,x,t)}} U_w \to T_t
\]

7.4.25

defined by the \( \tilde{\mathcal{F}}^w \)-morphisms \( \tilde{\beta}_{(j,x,t,w)} : U_w \to T_t \) above.

Furthermore, \( K \) acts on all this situation, and let us denote by \( O_{(j,x,t,w)} \) the \( K \)-orbit — which is actually regular — of the element of \( \tilde{J} \) determined by \( (j,x,t,w) \), by \( O_{(f(j),x,t)} \) the image in \( \tilde{I} \) via \( \tilde{f} \) of \( O_{(j,x,t,w)} \) — which actually does not depend on \( w \) — and by

\[
\tilde{\tau}_{O_{(f(j),x,t)}} : (T_{O_{(f(j),x,t)}}) \tilde{\beta}_{O_{(f(j),x,t)}} \to Q^p
\]

7.4.26

\[
\tilde{\beta}_{O_{(j,x,t,w)}} : (U_{O_{(j,x,t,w)}}) \tilde{\beta}_{O_{(j,x,t,w)}} \to (T_{O_{(f(j),x,t)}}) \tilde{\beta}_{O_{(f(j),x,t)}}
\]

\[
\tilde{\nu}_{O_{(j,x,t,w)}} : (U_{O_{(j,x,t,w)}}) \tilde{\nu}_{O_{(j,x,t,w)}} \to R^\sigma
\]

the \( \tilde{\mathfrak{S}} \)-morphisms respectively determined by the \( K \)-orbits of \( \tilde{\tau}_1, \tilde{\beta}_{(j,x,t,w)} \) and \( i^{\mathfrak{R}^1}_w \); then, the naturality of \( \mathfrak{ac}(m) \ast \omega \) forces

\[
\mathfrak{m}(\tilde{\nu}_{O_{(j,x,t,w)}}) \circ \mathfrak{m}(\tilde{\alpha}) = \mathfrak{m}(\tilde{\beta}_{O_{(j,x,t,w)}}) \circ \mathfrak{m}(\tilde{\tau}_{O_{(f(j),x,t)}})
\]

7.4.27

so that we still have

\[
\mathfrak{m}(\tilde{\nu}_{O_{(j,x,t,w)}}) \circ \mathfrak{m}(\tilde{\alpha}) \circ \mathfrak{m}^\circ(\tilde{\tau}_{O_{(f(j),x,t)}}) = \mathfrak{m}(\tilde{\beta}_{O_{(j,x,t,w)}})
\]

7.4.28

Now, we are ready to prove the commutativity of the diagram 7.4.15; according to our definition, the composition \( \mathfrak{m}(\tilde{\alpha}) \circ \theta_{Q^p} \) sends the element \( m = (m_0)_{O \in \bar{I}/K} \) where \( m_0 \in \mathfrak{m}((T_0)\beta_0) \), to the sum

\[
|\bar{I}^p/K|^{-1} \sum_{O \in \bar{I}^p/K} \mathfrak{m}(\tilde{\alpha})(\mathfrak{m}^\circ(\tilde{\tau}_{O})(m_0))
\]

7.4.29

on the other hand, we have

\[
((\mathfrak{ac}(m))(\tilde{\alpha} \tilde{\tau} \text{id}_{S^p}))(m)
\]

7.4.30

\[
= \sum_{j \in J} \sum_{t \in \mathfrak{F}_{O_{(j,t)}}} \sum_{w \in W^{(j,t)}} \mathfrak{m}(\tilde{\beta}_{O_{(j,t,w)}})(m_{O_{(j,t,w)}})(m_{O_{(j,t,1)}})
\]
and therefore, denoting by $W_{(j,1,t)}^{R_j}$ the set of $w \in W_{(j,1,t)}^x$ such that $U_w = R_j$, so that then we have $m^\circ (\tilde{v}_{O_{(j,x,1,w)}}) = m(\tilde{v}_{O_{(j,x,1,w)}})^{-1}$, it follows from our definition of $\theta_{R^*}$ and from equality 7.4.28 that

$$
(\theta_{R^*} \circ (\mathbf{ac}(m))(\tilde{\alpha} \tilde{\cap} \text{id}_{G^*}))(m) = \sum_{j \in J} \sum_{t \in \hat{T}_{Q_{j(t)},P}} \frac{|W_{(j,1,t)}^{R_j}|}{|J^o/K|} m(\tilde{\alpha})(m^\circ(\tilde{\tau}_{O_{(j,1,t)}})(m_{O_{(j,1,t)}})) \quad 7.4.31.
$$

But, note that if $O_{(j,1,t)}$ belongs to $\hat{I}^o/K$ then we have $|W_{(j,1,t)}^{R_j}| = 1$; moreover, it follows from [5, Corollary 4.9] that $\tilde{\alpha}_j$ induces an injective map from $\tilde{\mathcal{F}}(P, Q_{j(t)})$ to $\tilde{\mathcal{F}}(P, R_j)$ and it is clear that $|f^{-1}(i)| = |J|/|I|$ for any $i \in I$; furthermore, according to [5, 6.7.2] and to bijection 7.4.11 above, we have

$$
|\hat{I}^o/K| \equiv |I||\tilde{\mathcal{F}}(P)| \quad \text{and} \quad |\hat{J}^o/K| \equiv |J||\tilde{\mathcal{F}}(P)| \quad (\text{mod } p) \quad 7.4.32;
$$

hence, the sum of all the corresponding terms in the second member of equality 7.4.31 coincides with the sum 7.4.29 above.

Consequently, in order to show the commutativity of diagram 7.4.15, it suffices to prove that, for any $j \in J$ and any $t \in \hat{T}_{Q_{j(t)},P}$ such that $\hat{\tau}_t: T_1 \to Q_{j(t)}$ is not an isomorphism, $p$ divides $|W_{(j,1,1)}^{R_j}|$; but, it is clear that $W_{(j,1,1)}^{R_j}$ is a set of representatives for the quotient set

$$
\tau_t(T_1) \setminus \hat{T}_{Q_{j(t)},P}(\tau_t(T_1), \alpha_j(R_j)) \quad 7.4.33
$$

and that the nontrivial $p$-group $N_{Q_{j(t)}}(\tau_t(T_1))$ acts freely on this set. This completes the proof of the naturality of $\theta$ and therefore the proof of the theorem in the case where $p \cdot m = 0$.

In the general case, there is a subfunctor $m^{\text{tor}}: \mathfrak{F} \to \mathcal{O}\text{-mod}$ mapping any $\mathfrak{F}$-object $Q^o$ on the torsion $\mathcal{O}$-submodule of $m(Q^o)$ and then we have the quotient functor

$$
m/m^{\text{tor}}: \mathfrak{F} \to \mathcal{O}\text{-mod} \quad 7.4.34
$$

which maps any object on a free $\mathcal{O}$-module; consequently, since we have an exact sequence [5, A3.11.4]

$$
\mathbb{H}^n(\mathfrak{F}, m^{\text{tor}}) \to \mathbb{H}^n(\mathfrak{F}, m) \to \mathbb{H}^n(\mathfrak{F}, m/m^{\text{tor}}) \quad 7.4.35,
$$

for any $n \in \mathbb{N}$, we already may assume that either $m = m^{\text{tor}}$ or $m^{\text{tor}} = 0$. 
In the first case we have \( p^\ell \cdot m = 0 \) for some \( \ell \in \mathbb{N} - \{0\} \) and, considering the exact sequence [5, A3.11.4]
\[
\mathbb{H}^n(\mathfrak{g}, p \cdot m) \rightarrow \mathbb{H}^n(\mathfrak{g}, m) \rightarrow \mathbb{H}^n(\mathfrak{g}, m/p \cdot m) = \{0\}
\]
for any \( n \in \mathbb{N} \), it suffices to argue by induction on \( \ell \). In the second case, if \( c_0 \) is an \( n \)-cochain, for \( n \geq 1 \), we already have proved that
\[
c_0 \equiv d_{n-1}(a_0) \pmod{p}
\]
for a suitable \((n-1)\)-cochain \( a_0 \), so that we have \( c_0 - d_{n-1}(a_0) = p \cdot c_1 \) for a suitable \( n \)-cochain \( c_1 \) since we are dealing with free \( \mathcal{O} \)-modules; thus, inductively, we can define \( n \)-cochains \( c_i \) and \((n-1)\)-cochains \( a_i \) fulfilling
\[
c_i \equiv d_{n-1}(a_i) \pmod{p} \quad \text{and} \quad c_i - d_{n-1}(a_i) = p \cdot c_{i+1}
\]
and then, according to the completeness of \( \mathcal{O} \), it is quite clear that
\[
c_0 = d_{n-1} \left( \sum_{i \in \mathbb{N} - \{0\}} p^i \cdot a_i \right)
\]
We are done.

8 The \( \mathcal{O} \)-rank of the Grothendieck groups of a folded Frobenius \( P \)-category

8.1 As in §3, let \( P \) be a finite \( p \)-group, \( \mathcal{F} \) a Frobenius \( P \)-category and
\[
\hat{\mathfrak{aut}}_{\mathcal{F}^\text{ex}} : \mathfrak{ch}^* (\mathcal{F}^\text{ex}) \rightarrow k^* \cdot \mathfrak{Gr}
\]
a functor lifting \( \mathfrak{aut}_{\mathcal{F}^\text{ex}} \) (cf. 2.8); we are ready to determine the \( \mathcal{O} \)-rank of \( G_\mathcal{K}(\mathcal{F}, \hat{\mathfrak{aut}}_{\mathcal{F}^\text{ex}}) \). As in [5, Corollary 14.32] for the determination of the \( \mathcal{O} \)-rank of \( G_\mathcal{K}(\mathcal{F}, \mathfrak{aut}_{\mathcal{F}^\text{ex}}) \), our argument is an easy consequence of the character decomposition of the functor \( \mathfrak{g} \mathcal{K} \) obtained in §5, and of the following vanishing cohomological result which, setting \( h' = h^p \) for any \( h \in \mathbb{N} - \{0\} \), involves the quotient category \( \hat{h}(\mathcal{F}^\text{ex}) \) and the functor
\[
\tilde{i}_h : \hat{h}(\mathcal{F}^\text{ex}) \rightarrow U_{h'}^\mathfrak{N}
\]
introduced in 6.7 from the factorization in Proposition 6.5.

**Theorem 8.2** For any \( h \in \mathbb{N} - \{0\} \) and any \( n \geq 1 \) we have
\[
\mathbb{H}^n \left( \hat{h}(\mathcal{F}^\text{ex}), \mathcal{F} \mathfrak{t} U_{h'} \circ \tilde{i}_h \right) = \{0\}
\]
Proof: According to Proposition 6.10, for any \( n \geq 1 \) it suffices to prove that we have
\[
\mathbb{H}^n(h' (\tilde{\mathcal{F}}_r)), \mathcal{Fct}_{U_{h'}} \circ (s_{h'} \times y_h) = \{0\} \quad \text{8.2.2.}
\]

In order to apply Theorem 7.4, let us consider the full subcategory \( h' (\tilde{\mathcal{F}}_r) \) of the category \( U_{h'} \)-objects of \( \mathcal{Fct}(\tilde{\mathcal{F}}_r) \) [5, 6.2] over the set of faithful indecomposable \( U_{h'} \)-objects, namely over the indecomposable \( U_{h'} \)-objects \( Q^\sigma \) of \( \mathcal{Fct}(\tilde{\mathcal{F}}_r) \) (cf. 6.24) such that the group homomorphism \( \rho : U_{h'} \rightarrow \tilde{\mathcal{F}}(Q) \) is injective. Note that the indecomposable \( U_{h'} \)-object \( (\bigoplus_{u \in U_{h'}} P)^\sigma \), defined by the regular action of \( U_{h'} \) on itself, is faithful.

Moreover, if \( Q^\rho = (\bigoplus_{i \in I} Q_i)^\rho \) and \( R^\sigma = (\bigoplus_{j \in J} R_j)^\sigma \) are faithful indecomposable \( U_{h'} \mathcal{ac}(\tilde{\mathcal{F}}_r) \)-objects then, according to 6.11 and 6.13, the exterior intersection of \( Q = \bigoplus_{i \in I} Q_i \) and \( R = \bigoplus_{j \in J} R_j \) in \( \mathcal{ac}(\tilde{\mathcal{F}}_r) \) yields
\[
Q \cap R = \bigoplus_{(i,j) \in I \times J} \bigoplus_{(\tilde{\alpha},T,\tilde{\beta}) \in \tilde{\mathcal{I}}_{Q_i,R_j}} T \quad \text{8.2.3}
\]
and, for any \( \xi \in U_{h'} \), \( \rho(\xi) \) and \( \sigma(\xi) \) induce an automorphism of this intersection. Thus, they induce a permutation of the disjoint union \( \bigcup_{(i,j) \in I \times J} \tilde{\mathcal{I}}_{Q_i,R_j} \) and if \( \rho(\xi) \) and \( \sigma(\xi) \) respectively fix \( i \) and \( j \), and \( (\tilde{\alpha},T,\tilde{\beta}) \in \tilde{\mathcal{I}}_{Q_i,R_j} \) is a fixed element, then \( \rho(\xi) \) and \( \sigma(\xi) \) induce \( \tilde{\mathcal{F}} \)-automorphisms \( \rho_1(\xi) \) of \( Q_i \), \( \sigma_1(\xi) \) of \( R_j \) and \( \tau(\xi) \) of \( T \) fulfilling
\[
\rho_1(\xi) \circ \tilde{\alpha} = \tilde{\alpha} \circ \tau(\xi) \quad \text{and} \quad \sigma_1(\xi) \circ \tilde{\beta} = \tilde{\beta} \circ \tau(\xi) \quad \text{8.2.4.}
\]

In particular, if \( \tau(\xi) \) is trivial then it follows from Corollary 4.9 that \( \rho_1(\xi) \) and \( \sigma_1(\xi) \) are trivial too and therefore we get \( \xi = 1 \); thus, the exterior intersection \( Q^\rho \cap R^\sigma \) in the category \( U_{h'} \mathcal{ac}(\tilde{\mathcal{F}}_r) \) is a direct sum of faithful indecomposable \( U_{h'} \)-objects. Consequently, since an indecomposable direct summand of a pull-back is also a direct summand of some exterior intersection [5, 6.18], the subcategory \( \mathcal{ac}(h' (\tilde{\mathcal{F}}_r)) \) of \( U_{h'} \mathcal{ac}(\tilde{\mathcal{F}}_r) \) is closed by direct products and pull-backs.

On the other hand, it is clear that \( h' (\tilde{\mathcal{F}}_r) \) is a full subcategory of \( h' (\tilde{\mathcal{F}}_r) \) and we claim that the contravariant functor (cf. 6.9)
\[
n_h : \mathcal{Fct}_{U_{h'}} \circ (s_{h'} \times y_h) : h' (\tilde{\mathcal{F}}_r) \rightarrow \mathcal{O}\text{-mod} \quad \text{8.2.5}
\]
can be extended to a contravariant functor \( m_h : h' (\tilde{\mathcal{F}}_r) \rightarrow \mathcal{O}\text{-mod} \) admitting a right-hand sectional functor \( m_h^\sigma \). Indeed, for any faithful indecomposable \( U_{h'} \mathcal{ac}(\tilde{\mathcal{F}}_r) \)-object \( Q^\rho = (\bigoplus_{i \in I} Q_i)^\rho \) choose \( i \in I \); it is clear that \( \rho \) induces a
group homomorphism $\rho_i : U_h \to \hat{F}(Q_i)$ where $U_h$ denotes the stabilizer of $i$ in $U_{h'}$, and then we define

$$m_h(Q^p) = \mathcal{F}ct(U_h) \left( (\varpi_{h_j}, \hat{\chi}_{(Q_j)})^{-1}(\rho_i) \times \widehat{\text{Mon}}(U_{h^p}, (Q_i)^{\rho_i(U_{h^p})}, \mathcal{O}) \right) \quad 8.2.6$$

for the lifting $\widehat{\rho_i(U_{h^p})} \subset \mathcal{F}(Q_i)$ of $\rho_i(U_h)$ chosen in 6.9.

For any faithful indecomposable $U_{h'}$-object $R^\sigma = (\bigoplus_{j \in J} R_j)^\sigma$ and any $h^\sigma$-morphism $\hat{\phi} : R^\sigma \to Q^p$, $\hat{\phi}$ determines a necessarily surjective $U_{h^p}$-set map $f : J \to I$ and an $\hat{F}^\sigma$-morphism $\hat{\phi}_j : R_j \to Q_j$, where $j \in J$ is the chosen element and we set $i' = f(j)$; in particular, it is clear that $U_{h_j} \subset U_{h^p}$ or, equivalently, that $h_j$ divides $h_{i'}$ and therefore, denoting by $\rho_j$ the restriction to $U_{h_j}$ of the group homomorphism

$$\rho_j : U_{h_j} \longrightarrow \hat{F}(Q_{i'}) \quad 8.2.7,$$

$\hat{\phi}_j$ becomes an $h^j$($\hat{F}^\sigma$)-morphism from $(R_j)^{\rho_j}$ to $(Q_{i'})^{\rho_j}$, so that from [5, Proposition 14.28] we get a $U_{h_j}$-set bijection

$$\hat{\sigma}_{h_j}(\hat{\phi}_j) : (\varpi_{h_j}, \hat{\chi}_{(R_j)})^{-1}(\sigma_j) \cong (\varpi_{h_j}, \hat{\chi}_{(Q_j)})^{-1}(\rho_j) \quad 8.2.8.$$

On the other hand, it is clear that the inclusion $U_{h_j} \subset U_{h^p}$ determines the commutative diagram [5, 14.16.3]

$$\begin{array}{ccc}
\text{Mon}_{k'}(U_{h_j}, \hat{F}(Q_{i'})) & \xrightarrow{\varpi_{h_j}, \hat{\chi}_{(Q_{i'})}} & \text{Mon}(U_{h_j}, \hat{F}(Q_{i'})) \\
\uparrow & & \uparrow \\
\text{Mon}_{k'}(U_{h^p}, \hat{F}(Q_{i'})) & \xrightarrow{\varpi_{h^p}, \hat{\chi}_{(Q_{i'})}} & \text{Mon}(U_{h^p}, \hat{F}(Q_{i'}))
\end{array} \quad 8.2.9.$$

Similarly, for the chosen lifting $\sigma_j(U_{h_j}) \subset \mathcal{F}(R_j)$ of $\sigma_j(U_{h_j})$, since we assume that $\hat{\phi}$ is an $h^\sigma$-morphism, a suitable representative $\varphi_j$ of $\hat{\phi}_j$ induces a group homomorphism from $(R_j)^{\sigma_j(U_{h_j})}$ to $(Q_{f(j)})^{\rho_j(U_h)}$, which defines a map

$$\text{Mon}(U_{h^p}, (R_j)^{\sigma_j(U_{h^p})}) \longrightarrow \text{Mon}(U_{h^p}, (Q_{i'})^{\rho_j(U_{h^p})}) \quad 8.2.10.$$

Then, applying the functor $\mathcal{F}ct_{U_{h_j}}$ and the inclusion $\mathcal{F}ct_{U_{h^p}} \subset \mathcal{F}ct_{U_{h_j}}$ of $\text{res}_{U_{h^p}}$, to diagram 8.2.9 and to its bottom map respectively, the bijection 8.2.8 and the map 8.2.10 determine an $\mathcal{O}$-module homomorphism

$$\mathcal{F}ct_{U_{h_j}} \left( (\varpi_{h_j}, \hat{\chi}_{(R_j)})^{-1}(\sigma_j) \times \text{Mon}(U_{h^p}, (R_j)^{\sigma_j(U_{h^p})}, \mathcal{O}) \right) \quad 8.2.11$$

which admits a section $\text{pro}_{\hat{\phi}_j}$ extending the $\mathcal{O}$-valued functions by zero.
Moreover, there is $\xi \in U_{h'}$ such that $\rho(\xi)$ maps $i' = f(j)$ on the chosen element $i \in I$ and therefore we have $h_{i'} = h_i$ and $\xi$ induces an $h_i(F^e)$-morphism
\[
\rho(\xi)_{i'} : (Q_{i'})^{\rho_{i'}} \to (Q_i)^{\rho_i}
\]
so that it follows again from [5, Proposition 14.28] that we get a $U_{h_i}$-set bijection
\[
\text{Fct}_{U_{h_i}} \left( (\varpi_{h_i}, \hat{\mathcal{F}}(Q_i))^{-1}(\rho_i) \times \text{Mon}(U_{h_{i'}}, (Q_i)^{\rho_i(U_{h_i})}), \mathcal{O} \right)
\]
\[
\text{Fct}_{U_{h_i'}} \left( (\varpi_{h_i'}, \hat{\mathcal{F}}(Q_i'))^{-1}(\rho_{i'}) \times \text{Mon}(U_{h_{i'}}, (Q_i')^{\rho_{i'}(U_{h_i'})}), \mathcal{O} \right)
\]
which clearly does not depend on the choice of $\xi$. Finally, we consider the compositions
\[
m_h(\hat{\varphi}) = \text{res}_{\varphi_j} \circ n_{h, h_{i'}}(\rho(\xi)_{i''}) : m_h(Q_i) \to m_h(R^e)
\]
\[
m_h^T(\hat{\varphi}) = n_{h, h_{i'}}(\rho(\xi)_{i''})^{-1} \circ \text{pro}_{\varphi_j} : m_h(R^e) \to m_h(Q_i)
\]
we claim that the correspondence $m_h$ is the announced contravariant functor and that $m_h^T$ is a right-hand sectional functor of $m_h$.

Indeed, it is clear that $m_h$ extends $n_h$ and that we have
\[
m_h(\hat{\varphi}) \circ m_h^T(\hat{\varphi}) = \text{id}_{m_h(R^e)}
\]
further, for any faithful indecomposable $U_{h'}$-module $\hat{\mathcal{F}}(\hat{\mathcal{F}}^e)$-object $T^e = (\bigoplus_{\ell \in L} T_\ell)^{\tau}$ and any $h_i$-morphism $\hat{\psi} : T^e \to R^e$, as above we have a surjective $U_{h_{i'}}$-set map $g : L \to J$, a chosen element $\ell \in L$ and, setting $j' = g(\ell)$ and denoting by $\sigma_{j'}$ the restriction of $\sigma_{j'}$ to $U_{h_{i'}}$, an $h_{i'}(\hat{\mathcal{F}}^e)$-morphism $\hat{\psi}_{j'} : (T_{\ell})^{\tau_{j'}} \to (R_{j'})^{\sigma_{j'}}$, together with a $U_{h_{i'}}$-set bijection and a map
\[
\hat{\delta}_{h_{i'}}(\hat{\psi}_{j'}) : (\varpi_{h_{i'}}, \hat{\mathcal{F}}(T_{\ell}))^{-1}(\tau_\ell) \cong (\varpi_{h_{i'}}, \hat{\mathcal{F}}(R_{j'}))^{-1}(\sigma_{j'})
\]
\[
\text{Mon}(U_{h_{i'}}, (T_{\ell})^{\tau_{j'}}(U_{h_{i'}})) \to \text{Mon}(U_{h_{i'}}, (R_{j'})^{\sigma_{j'}}(U_{h_{i'}}))
\]
for the chosen lifting $\tau_{j'}(U_{h_{i'}}) \subset \mathcal{F}(T_{\ell})$ of $\tau_{j'}(U_{h_{i'}})$. Analogously, we have an $\mathcal{O}$-module isomorphism
\[
\text{Fct}_{U_{h_i}} \left( (\varpi_{h_i}, \hat{\mathcal{F}}(T_{\ell}))^{-1}(\tau_\ell) \times \text{Mon}(U_{h_{i'}}, (T_{\ell})^{\tau_{j'}}(U_{h_{i'}})), \mathcal{O} \right)
\]
\[
\text{Fct}_{U_{h_{i'}}} \left( (\varpi_{h_{i'}}, \hat{\mathcal{F}}(R_{j'}))^{-1}(\sigma_{j'}) \times \text{Mon}(U_{h_{i'}}, (R_{j'})^{\sigma_{j'}}(U_{h_{i'}})), \mathcal{O} \right)
\]
Moreover, choosing $\zeta \in U_{h'}$ such that $\sigma(\zeta)$ maps $g(\ell)$ on $j$, as above we have $h_{j'} = h_j$ and $\zeta$ induces an $h_j(\hat{F}^{\sigma})$-isomorphism

$$\sigma(\zeta)_{j'}^j : (R_{j'})^{\sigma_{j'}} \cong (R_j)^{\sigma_j}$$

8.2.18,

and therefore we get a $U_{h_j}$-set bijection (cf. Proposition 14.27)

$$\mathcal{Fct}_{U_{h_j}}\left(\left(\varpi_{h_j, \hat{F}(R_{h_j})}\right)^{-1}(\sigma_j) \times \text{Mon}(U_{h^{p}}, (R_j)^{\sigma_j(U_{h_j})}, O)\right)$$

and

$$\mathcal{Fct}_{U_{h_{j'}}}\left(\left(\varpi_{h_{j'}, \hat{F}(R_{h_{j'}})}\right)^{-1}(\sigma_{j'}) \times \text{Mon}(U_{h^{p}}, (R_{j'})^{\sigma_{j'}(U_{h_{j'}})}, O)\right)$$

Finally, we also consider

$$m_h(\tilde{\psi}) = \text{res}_{\tilde{\psi}_\ell} \circ n_{h, h^p}\left(\sigma(\zeta)_{j'}^j\right) : m_h(R^\sigma) \to m_h(T^\sigma)$$

8.2.20.

and

$$m_h^o(\tilde{\psi}) = n_{h, h^p}\left(\sigma(\zeta)_{j'}^j\right)^{-1} \circ \text{pro}_{\tilde{\psi}_\ell} : m_h(T^\sigma) \to m_h(R^\sigma)$$

Consequently, we get

$$m_h(\tilde{\psi}) \circ m_h(\tilde{\phi}) = \text{res}_{\tilde{\psi}_\ell} \circ n_{h, h^p}\left(\sigma(\zeta)_{j'}^j\right) \circ \text{res}_{\tilde{\phi}_\ell} \circ n_{h, h^p}\left(\rho(\xi)^{f(j')}\right)$$

8.2.21

$$m_h^o(\tilde{\psi}) \circ m_h^o(\tilde{\phi}) = n_{h, h^p}\left(\rho(\xi)^{f(j')}\right)^{-1} \circ \text{pro}_{\tilde{\phi}_\ell} \circ n_{h, h^p}\left(\sigma(\zeta)_{j'}^j\right)^{-1} \circ \text{pro}_{\tilde{\psi}_\ell}$$

and in order to prove our claim it suffices to prove that

$$n_{h, h^p}\left(\sigma(\zeta)_{j'}^j\right) \circ \text{res}_{\tilde{\phi}_\ell} = \text{res}_{\tilde{\phi}_{o\zeta\ell}} \circ n_{h, h^p}\left(\rho(\xi)^{f(j')}\right)$$

8.2.22

$$\text{pro}_{\tilde{\phi}_\ell} \circ n_{h, h^p}\left(\rho(\xi)^{f(j')}\right)^{-1} = n_{h, h^p}\left(\rho(\xi)^{f(j')}\right)^{-1} \circ \text{pro}_{\tilde{\phi}_{j'}}$$

since it is easily checked that

$$\text{res}_{\tilde{\psi}_{o\zeta\ell}} \circ \text{res}_{\tilde{\phi}_{j'}} = \text{res}_{(\tilde{\phi}_{o\zeta\psi})_{\ell}} \quad \text{and} \quad \text{pro}_{\tilde{\phi}_{j'}} \circ \text{pro}_{\tilde{\psi}_\ell} = \text{pro}_{(\tilde{\phi}_{o\zeta\psi})_{\ell}}$$

8.2.23

In order to prove equalities 8.2.22 note that, since $f$ is a $U_{h'}$-set map, $\rho(\zeta)$ maps $i'' = f(j')$ on $f(j) = i'$ and we have $h_{i''} = h_{i'} = h_j$; thus, $\zeta$ induces an $h_j(\hat{F}^{\sigma})$-isomorphism

$$\rho(\zeta)^{i''} : (Q_{i''})^{\rho_{i''}} \cong (Q_{i'})^{\rho_{i'}}$$

8.2.24

and denoting by $\rho_{j'}$ the restriction to $U_{h_{j'}}$ of the group homomorphism $\rho_{i''}$, the $h'_F$-morphism $\tilde{\phi} : R^\sigma \to Q^\sigma$ forces the following commutative $h'_F$-$\mathfrak{F}$-diagram

$$(R_j)^{\sigma_j} \xrightarrow{(\hat{\phi})_{j'}} (Q_{i'})^{\rho_{i'}}$$

8.2.25.
hence, we get the commutative $U_{h_j}$-diagram

\[
\begin{array}{c}
\left(\varpi_{h_j,\hat{\varphi}(R_j)}\right)^{-1}(\sigma_j) \\ \downarrow
\end{array}
\begin{array}{c}
\left(\varpi_{h_j,\hat{\varphi}(Q_{j'})}\right)^{-1}(\rho_j)
\end{array}
\]

\text{8.2.26.}

On the other hand, the natural map $\theta_{h_k,h_j}$ in [5, Proposition 14.28] applied to the $h_j(\hat{E}^{\varphi})$-morphism $\tilde{\varphi}_j: (R_j)^{\sigma_j} \to (Q_{j'})^{\rho_{j'}}$ yields the following commutative $U_{h_j}$-diagram

\[
\begin{array}{c}
\left(\varpi_{h_j,\hat{\varphi}(R_j)}\right)^{-1}(\sigma_j) \\ \downarrow
\end{array}
\begin{array}{c}
\left(\varpi_{h_j,\hat{\varphi}(Q_{j'})}\right)^{-1}(\rho_{j'})
\end{array}
\]

\text{8.2.27.}

where $\rho_{j'}$ is the restriction of the group homomorphism $\rho_{i''}$ to $U_{h_{j'}}$.

Similarly, from the commutative $\tilde{\Phi}$-diagram

\[
\begin{array}{c}
(R_j)^{\sigma_j(U_{h_j})} \\ \downarrow
\end{array}
\begin{array}{c}
(Q_{j'})^{\rho_{j'}(U_{h_{j'}})}
\end{array}
\]

\text{8.2.28.}

we get the commutative diagram

\[
\begin{array}{c}
\tilde{\text{Mon}}(U_{h^p}, (R_j)^{\sigma_j(U_{h_j})}) \\ \downarrow
\end{array}
\begin{array}{c}
\tilde{\text{Mon}}(U_{h^p}, (Q_{j'})^{\rho_{j'}(U_{h_{j'}})})
\end{array}
\]

\text{8.2.29.}

At this point, considering the direct product of this diagram with the square diagram obtained by “composing” diagrams 8.2.26 and 8.2.27, and applying the functor $\mathcal{F}ct_{U_{h_j}}$, the equalities 8.2.22 follow easily, proving our claim.

Hence, it follows from Theorem 7.4 that, for any $n \geq 1$, we have

\[
\mathbb{H}^n(h', \mathfrak{F}, m_h) = \{0\}
\]

\text{8.2.30.}

If $h' = 1$ then $1^{\mathfrak{F}} = 1(\hat{E}^{\varphi}) = \hat{E}^{\varphi}$ and we are done. Otherwise, we consider the full subcategory $h' \mathcal{E}$ of $h' \mathfrak{F}$ over the faithful indecomposable $U_{h'} \alpha(\hat{E}^{\varphi})$-objects $Q^\rho = (\bigoplus_{i \in I} Q_i)^\rho$ such that $|I| > 1$, namely over all the $h' \mathfrak{F}$-objects which are not $h'(\hat{E}^{\varphi})$-objects; note that there is no $h' \mathfrak{F}$-morphism
from an \(h'(F^{sc})\)-object to an \(h'E\)-object. Denoting by \(l_{h} : h'E \to \mathcal{O}\text{-mod}\) the restriction of \(m_{h}\) to \(h'E\), it is quite clear that Theorem 7.4 applies again to \(h'E\) and \(l_{h}\), so that for any \(n \geq 1\) we get

\[
\mathbb{H}^{n}(h'E, l_{h}) = \{0\}
\] 8.2.31.

Recall that, denoting by \(\Delta\) the simplicial 2-category \([5, A1.7]\), the cohomology groups \(\mathbb{H}^{n}(h'\mathcal{F}, m_{h})\), \(\mathbb{H}^{n}(h'E, l_{h})\) and \(\mathbb{H}^{n}(h'(\tilde{F}^{sc}), n_{h})\) are nothing but the homology groups of the respective evident functors \(c_{m_{h}}\), \(c_{l_{h}}\) and \(c_{n_{h}}\) from \(\Delta\) to \(\mathcal{O}\text{-mod}\), mapping \(\Delta_{n}\) on \([5, A3.8]\)

\[
\mathbb{C}^{n}(h'\mathcal{F}, m_{h}) = \prod_{\tilde{q} \in \mathcal{Fct}(\Delta_{n}, h'\mathcal{F})} m_{h}(\tilde{q}(0))
\]

\[
\mathbb{C}^{n}(h'E, l_{h}) = \prod_{\tilde{q} \in \mathcal{Fct}(\Delta_{n}, h'E)} m_{h}(\tilde{q}(0))
\] 8.2.32;

\[
\mathbb{C}^{n}(h'(\tilde{F}^{sc}), n_{h}) = \prod_{\tilde{q} \in \mathcal{Fct}(\Delta_{n}, h'(\tilde{F}^{sc}))} m_{h}(\tilde{q}(0))
\]

then, the inclusion \(h'(\tilde{F}^{sc}) \subset h'\mathcal{F}\) clearly determines a surjective natural map

\[
\mu_{h} : c_{m_{h}} \longrightarrow c_{n_{h}}
\] 8.2.33,

so that we obtain a fourth functor

\[
\mathcal{R}\mathfrak{e}r(\mu_{h}) : \Delta \longrightarrow \mathcal{O}\text{-mod}
\] 8.2.34

Moreover, since there is no \(h'\mathcal{F}\)-morphism from an \(h'(\tilde{F}^{sc})\)-object to any \(h'E\)-object, we have an \(\mathcal{O}\text{-module isomorphism}

\[
(\mathcal{R}\mathfrak{e}r(\mu_{h}))(\Delta_{n}) \cong \prod_{\tilde{q}} m_{h}(\tilde{q}(0))
\] 8.2.35,

where \(\tilde{q}\) runs over the set \(\mathcal{E}^{h'}(\Delta_{n})\) of functors from \(\Delta_{n}\) to \(h'\mathcal{F}\) such that \(\tilde{q}(0)\) is a \(h'E\)-object and therefore the inclusion \(h'E \subset h'\mathcal{F}\) also determines a surjective natural map

\[
\lambda_{h} : \mathcal{R}\mathfrak{e}r(\mu_{h}) \longrightarrow c_{l_{h}}
\] 8.2.36.

But on the one hand, we already know that \(\mathbb{H}_{n}(c_{m_{h}}) = \{0\} = \mathbb{H}_{n}(c_{l_{h}})\) for any \(n \geq 1\) (cf. equalities 8.2.30 and 8.2.31) and on the other hand, setting

\[
\mathbb{H}_{-n}(c_{n_{h}}) = \mathbb{H}_{-n}(c_{m_{h}}) = \mathbb{H}_{-n}(\mathcal{R}\mathfrak{e}r(\mu_{h})) = \{0\}
\] 8.2.37
for any \( n > 0 \), there is a 1-graded connecting homomorphism [5, A3.3.4]

\[
\delta : \bigoplus_{n \in \mathbb{Z}} \mathbb{H}_n(\epsilon_{n_h}) \rightarrow \bigoplus_{n \in \mathbb{Z}} \mathbb{H}_n(\hat{\text{ker}}(\mu_h)) \quad 8.2.38
\]
such that we have the following exact triangle [5, A3.3.5]

\[
\bigoplus_{n \in \mathbb{Z}} \mathbb{H}_n(\epsilon_{n_h}) \xrightarrow{\delta} \bigoplus_{n \in \mathbb{Z}} \mathbb{H}_n(\hat{\text{ker}}(\mu_h)) \rightarrow \bigoplus_{n \in \mathbb{Z}} \mathbb{H}_n(\epsilon_{\lambda_h}) \quad 8.2.39.
\]

Consequently, in order to prove that \( \mathbb{H}_n(\epsilon_{n_h}) = \{0\} \) for any \( n \geq 1 \), it suffices to show that \( \mathbb{H}_n(\lambda_h) \) is injective. Actually, since the functors \( \hat{s}_h \) and \( \xi_h \) factorize throughout the exterior quotient \( \tilde{\kappa}'(\tilde{\mathcal{F}}^\mu) \) of \( \kappa'(\tilde{\mathcal{F}}^\mu) \) [5, Remark 14.29], it is easily checked that the contravariant functors \( \mathfrak{n}_h, \mathfrak{m}_h \) and \( \mathfrak{l}_h \) respectively determine contravariant functors

\[
\tilde{n}_h : \tilde{\kappa}'(\tilde{\mathcal{F}}^\mu) \rightarrow \mathcal{O}\text{-mod} \quad 8.2.40,
\]

\[
\tilde{m}_h : \tilde{\kappa}' \tilde{\mathfrak{F}} \rightarrow \mathcal{O}\text{-mod} \quad \text{and} \quad \tilde{l}_h : \tilde{\kappa}' \tilde{\mathfrak{E}} \rightarrow \mathcal{O}\text{-mod}
\]

where \( \tilde{\kappa}' \tilde{\mathfrak{F}} \) and \( \tilde{\kappa}' \tilde{\mathfrak{E}} \) denote the corresponding exterior quotients [5, 6.3]. Coherently, we get the corresponding functors \( \epsilon_{\tilde{n}_h}, \epsilon_{\tilde{m}_h} \) and \( \epsilon_{\tilde{l}_h} \) (cf. 8.2.32), and the corresponding natural maps (cf. 8.2.33 and 8.2.36)

\[
\hat{\mu}_h : \epsilon_{\tilde{m}_h} \rightarrow \epsilon_{\tilde{n}_h} \quad \text{and} \quad \hat{\lambda}_h : \hat{\text{ker}}(\hat{\mu}_h) \rightarrow \epsilon_{\tilde{l}_h} \quad 8.2.41.
\]

Then, it follows from equality 7.3.2 that, for any \( n \geq 1 \), it suffices to prove that \( \mathbb{H}_n(\lambda_{\tilde{l}_h}) \) is injective. First of all note that, up to isomorphisms, any \( \tilde{\kappa}' \tilde{\mathfrak{F}} \)-object has the canonical form \( Q^\rho = \left( \bigoplus_{\xi \in U_{h'}} U_{h'} Q_{\xi} \right)^\rho \) for a suitable subgroup \( U \) of \( U_{h'} \), and, denoting by \( \hat{\iota}_{h'} : \tilde{\kappa}' \tilde{\mathfrak{E}} \rightarrow \tilde{\kappa}' \tilde{\mathfrak{F}} \) the inclusion functor, we claim that we have a functor and a natural map

\[
\hat{\epsilon}_{h'} : \tilde{\kappa}' \tilde{\mathfrak{F}} \rightarrow \tilde{\kappa}' \tilde{\mathfrak{E}} \quad \text{and} \quad \hat{\epsilon}_{h'} : \hat{\iota}_{h'} \circ \hat{\epsilon}_{h'} \rightarrow \text{id}_{\tilde{\kappa}' \tilde{\mathfrak{F}}} \quad 8.2.42
\]

which respectively map \( Q^\rho \) on the \( U_{h'} \text{-ac}(\tilde{\mathcal{F}}^\mu) \)-object \( \tilde{Q}^\delta \) formed by

\[
\tilde{Q} = \bigoplus_{\xi \in U_{h'}} U_{h'} Q_{\xi} \quad 8.2.43
\]

and by the group homomorphism \( \hat{\rho} : U_{h'} \rightarrow \tilde{\kappa}' \tilde{\mathfrak{F}}(\tilde{Q}) \) defined by the regular action of \( U_{h'} \) on itself, together with the \( \tilde{\mathcal{F}}^\mu \)-isomorphisms from the \( \xi \)-summand to the \( \xi' \)-summand induced by \( \rho(\xi'\xi^{-1}) \), and on the \( \tilde{\kappa}' \tilde{\mathfrak{F}} \)-morphism

\[
(\hat{\epsilon}_{h'})^\rho : \tilde{Q}^\rho \rightarrow Q^\rho \quad 8.2.44
\]
defined by the canonical map \( U_{h'} \rightarrow U_{h'}/U \) and by the identity automorphism of \( Q_{\xi} \) for any \( \xi \in U_h \).
Indeed, if $R^\sigma = (\bigoplus_{\xi \in U_{\mathfrak{k}'}, V} R_{\xi})^\sigma$ is another $\mathfrak{h}'$-$\mathfrak{g}$-object, an $\mathfrak{h}'$-$\mathfrak{g}$-morphism from $R^\sigma$ to $Q^\rho$ forces the inclusion $V \subset U$ and admits a canonical representative formed by the canonical map $U_{\mathfrak{k}'}/V \to U_{\mathfrak{k'}/U}$ and by an $\mathcal{F}_{\mathfrak{c}'}$-morphism $\tilde{\varphi}_{\xi}: R_{\xi} \to Q_{\xi}$ for any $\xi \in U_{\mathfrak{k}'}/V$; then, the functor $\tilde{\epsilon}_{\mathfrak{c}'}$ above maps this $\mathfrak{h}'$-$\mathfrak{g}$-morphism on the $\mathfrak{h}'$-$\mathfrak{c}$-morphism

$$
\tilde{R}^\sigma = (\bigoplus_{\xi \in U_{\mathfrak{k}'}, V} R_{\xi})^\sigma \to \tilde{Q}^\rho = (\bigoplus_{\xi \in U_{\mathfrak{k}'}, V} Q_{\xi})^\rho
$$

8.2.45

admitting a canonical representative formed by the canonical map $U_{\mathfrak{k}'}$ and by the $\mathcal{F}_{\mathfrak{c}'}$-morphism $\tilde{\varphi}_{\xi}: R_{\xi} \to Q_{\xi}$ for any $\xi \in U_{\mathfrak{k}'}$; it is quite clear that this correspondence preserves the composition of $\mathfrak{h}'$-$\mathfrak{g}$-morphisms and is compatible with the $\mathfrak{h}'$-$\mathfrak{c}$-morphisms 8.2.44.

Secondly, we consider the contravariant functor $\tilde{m}_h \circ \tilde{\iota}_{\mathfrak{c}'} \circ \tilde{\epsilon}_{\mathfrak{c}'} = \tilde{l}_h \circ \tilde{\epsilon}_{\mathfrak{c}'}$, which, as above, determines a functor

$$
ci_{h, \circ \tilde{\epsilon}_{\mathfrak{c}'}}: \Delta \longrightarrow \mathcal{C} \text{-mod}
$$

8.2.46

mapping $\Delta_n$ on

$$
\mathcal{C}^n(\mathfrak{h}', \tilde{l}_h \circ \tilde{\epsilon}_{\mathfrak{c}'}) = \prod_{\tilde{q} \in \tilde{\mathfrak{c}}(\Delta_n, \tilde{\mathfrak{g}})} \tilde{m}_h(\tilde{\epsilon}_{\mathfrak{c}'}(\tilde{q}(0)))
$$

8.2.47,

and then we get a natural map $\kappa_{\tilde{\epsilon}_{\mathfrak{c}'}}: \tilde{c}_{\tilde{m}_h} \rightarrow ci_{h, \circ \tilde{\epsilon}_{\mathfrak{c}'}}$, sending $\Delta_n$ to the $\mathcal{C}$-module homomorphism

$$
(\kappa_{\tilde{\epsilon}_{\mathfrak{c}'}})_n: \prod_{\tilde{q} \in \tilde{\mathfrak{c}}(\Delta_n, \tilde{\mathfrak{g}})} \tilde{m}_h(\tilde{q}(0)) \longrightarrow \prod_{\tilde{q} \in \tilde{\mathfrak{c}}(\Delta_n, \tilde{\mathfrak{g}})} \tilde{m}_h(\tilde{\epsilon}_{\mathfrak{c}'}(\tilde{q}(0)))
$$

8.2.48

mapping $m = (m_{\tilde{q}})_{\tilde{q} \in \tilde{\mathfrak{c}}(\Delta_n, \tilde{\mathfrak{g}})}$ on $m_{\tilde{c}_{\tilde{m}_h} \circ \tilde{\epsilon}_{\mathfrak{c}'}}$. Note that if $m$ belongs to the kernel $\text{Ker}(\lambda_{\tilde{h}'\Delta_n})$ (cf. 8.2.41) then, for any $\tilde{q} \in \tilde{\mathfrak{c}}(\Delta_n, \tilde{\mathfrak{g}})$, we have $m_{\tilde{q}} = 0$ and therefore we get $(\kappa_{\tilde{\epsilon}_{\mathfrak{c}'}})_n(m) = 0$. Moreover, according to the very definition of $\tilde{m}_h$ (cf. 8.2.14 and 8.2.40), the natural map (cf. 8.2.42)

$$
\tilde{m}_h * \tilde{\epsilon}_{\mathfrak{c}'}: \tilde{m}_h \longrightarrow \tilde{l}_h \circ \tilde{\epsilon}_{\mathfrak{c}'}
$$

8.2.49

determines a natural isomorphism

$$
\kappa_{\tilde{\epsilon}_{\mathfrak{c}'}}: \tilde{c}_{\tilde{m}_h} \cong ci_{h, \circ \tilde{\epsilon}_{\mathfrak{c}'}}
$$

8.2.50

sending $\Delta_n$ to the $\mathcal{C}$-module isomorphism

$$
(\kappa_{\tilde{\epsilon}_{\mathfrak{c}'}})_n: \prod_{\tilde{q} \in \tilde{\mathfrak{c}}(\Delta_n, \tilde{\mathfrak{g}})} \tilde{m}_h(\tilde{q}(0)) \cong \prod_{\tilde{q} \in \tilde{\mathfrak{c}}(\Delta_n, \tilde{\mathfrak{g}})} \tilde{m}_h(\tilde{\epsilon}_{\mathfrak{c}'}(\tilde{q}(0)))
$$

8.2.51

mapping $m = (m_{\tilde{q}})_{\tilde{q} \in \tilde{\mathfrak{c}}(\Delta_n, \tilde{\mathfrak{g}})}$ on $(\tilde{m}_h((\tilde{\epsilon}_{\mathfrak{c}'})_{\tilde{q}(0)}))(m_{\tilde{q}})_{\tilde{q} \in \tilde{\mathfrak{c}}(\Delta_n, \tilde{\mathfrak{g}})}$. 

At this point, for any \( n \geq 1 \), following the notation introduced in \([5, \text{Lemma A4.2}]\), we consider the \( \mathcal{O} \)-module homomorphism

\[
h_{n-1} : \epsilon_{d_n}(\Delta_n) \rightarrow \epsilon_{d_n \circ \tilde{h}'}(\Delta_{n-1})
\]

mapping \( m = (m_{\tilde{q}})_{\tilde{q} \in \tilde{\mathfrak{g}} \mathfrak{t}(\Delta_n, \tilde{\mathfrak{g}})} \in \epsilon_{d_n}(\Delta_n) \) on

\[
h_{n-1}(m) = \left( \sum_{i=0}^{n-1} (-1)^i m_{h_i^{n-1}(\tilde{e}_{h'} \cdot \tilde{r})} \right)_{\tilde{r} \in \tilde{\mathfrak{g}} \mathfrak{t}(\Delta_n, \tilde{\mathfrak{g}})}
\]

Then, respectively denoting by \( d_n \) and \( \tilde{d}_n \) the differential maps for the functors \( \epsilon_{d_n} \) and \( \epsilon_{d_n \circ \tilde{h}'} \) \([5, \text{A3.2}]\), we claim that

\[
(\kappa_{\tilde{e}_{h'}})_n(m) - (\kappa_{\tilde{e}_{h'}})_n(m) = (\tilde{d}_{n-1} \circ h_{n-1} + h_n \circ d_n)(m)
\]

Indeed, setting \( \theta = m_{h}(\tilde{e}_{h'}(\tilde{q}(0 \cdot 1))) \), for any \( \tilde{q} \in \tilde{\mathfrak{g}} \mathfrak{t}(\Delta_n, \tilde{\mathfrak{g}}) \) we have

\[
\tilde{d}_{n-1}(h_{n-1}(m)) = \sum_{i=0}^{n-1} (-1)^i \tilde{d}_{n-1}\left( (m_{h_i^{n-1}(\tilde{e}_{h'} \cdot \tilde{r})})_{\tilde{r}} \right)
\]

\[
= \sum_{i=0}^{n-1} (-1)^i \left( \theta(m_{h_i^{n-1}(\tilde{e}_{h'} \cdot (\tilde{q} \circ \delta_{\tilde{r}}^{n-1}))) + \sum_{j=1}^{n} (-1)^j m_{h_i^{n-1}(\tilde{e}_{h'} \cdot (\tilde{q} \circ \delta_{\tilde{r}}^{j-1}))} \right)
\]

where \( \tilde{r} \) runs over \( \tilde{\mathfrak{g}} \mathfrak{t}(\Delta_{n-1}, \tilde{\mathfrak{g}}) \); analogously, we still have

\[
h_n(d_n(m)) = h_n\left( (\theta(m_{\tilde{t} \circ \delta_{\tilde{r}}})_{\tilde{r}}) \right)_{\tilde{r}} + \sum_{j=1}^{n+1} (-1)^j h_n(\left( m_{\tilde{r} \circ \delta_{\tilde{r}}^j} \right)_{\tilde{r}}
\]

\[
= \sum_{i=0}^{n} (-1)^i \left( \theta(m_{h_i}(\tilde{e}_{h'} \cdot \tilde{q} \circ \delta_{\tilde{r}}^{i})) + \sum_{j=1}^{n+1} (-1)^j m_{h_i}(\tilde{e}_{h'} \cdot (\tilde{q} \circ \delta_{\tilde{r}}^{j-1})) \right)
\]

where \( \tilde{r} \) runs over \( \tilde{\mathfrak{g}} \mathfrak{t}(\Delta_{n+1}, \tilde{\mathfrak{g}}) \). But from \([5, \text{Lemma A4.2}]\) we know that

\[
h_i^{n+1}(\tilde{e}_{h'} \cdot \tilde{q}) \circ \delta_{i+1}^n = h_i^n(\tilde{e}_{h'} \cdot \tilde{q}) \circ \delta_{i+1}^n
\]

\[
h_i^{n-1}(\tilde{e}_{h'} \cdot (\tilde{q} \circ \delta_{\tilde{r}}^{n-1})) = \begin{cases} h_i^{n+1}(\tilde{e}_{h'} \cdot \tilde{q}) \circ \delta_{i+1}^n & \text{if } j \leq i \\ h_i^n(\tilde{e}_{h'} \cdot \tilde{q}) \circ \delta_{i+1}^n & \text{if } i < j \end{cases}
\]

Consequently, in equality 8.2.56 the terms where \( j = i \) and \( j = i + 1 \) cancel with each other for any \( 1 \leq j \leq n \); moreover, the term \((i, j)\) in equality 8.2.55 cancel either with the term \((i+1, j)\) if \( 1 \leq j \leq i \leq n-1 \), or with the term \((i, j+1)\) if \( 0 \leq i < j \leq n \) in equality 8.2.56. Finally, the term \((i, 0)\) in equality 8.2.52 cancel with the term \((i+1, 0)\) in equality 8.2.56 for any \( 0 \leq i \leq n-1 \), whereas the terms \((0, 0)\) and \((n, n+1)\) respectively coincide with \((\kappa_{\tilde{e}_{h'}})_n(m)\) and \(-((\kappa_{\tilde{e}_{h'}})_n(m)\), which proves the claim.
In conclusion, if \( m \in \ker(d_n) \) lifts an element of \( \mathbb{H}_n(\hat{\lambda}_n) \) then we may assume that \( m \) belongs to \( \ker((\lambda_h)_\Delta_n) \), so that we get \((\kappa_{\hat{\varepsilon}_n})_n(m) = 0\), and therefore it follows from isomorphism 8.2.50 and from equality 8.2.54 that

\[
(\kappa_{\hat{\varepsilon}_n})_n(m) = \hat{d}_{n-1}(\hat{h}_{n-1}(m)) = \hat{d}_{n-1}
\left(\left((\kappa_{\hat{\varepsilon}_n})_{n-1} \circ ((\kappa_{\hat{\varepsilon}_n})_{n-1})^{-1} \circ h_{n-1}\right)(m)\right) = \left((\kappa_{\hat{\varepsilon}_n})_n \circ \hat{d}_{n-1}\right)
\left(\left((\kappa_{\hat{\varepsilon}_n})_{n-1} \circ h_{n-1}\right)(m)\right)
\]

which proves that \( m \) belongs to \( \text{Im}(d_{n-1}) \). We are done.

8.3 Let us say that a \( \mathfrak{ch}^*({\mathcal{F}}^\vee) \)-object \((q, \Delta_n)\) is regular if \( q(i-1, i) \) is not an isomorphism for any \( 1 \leq i \leq n \) [5, A5.2]; note that there is a canonical bijection between a set of representatives for the set of isomorphism classes of regular \( \mathfrak{ch}^*({\mathcal{F}}^\vee) \)-objects and a set of representatives for the set of \( \mathcal{F} \)-isomorphism classes of nonempty sets of \( \mathcal{F} \)-selfcentralizing subgroups of \( P \), which are totally ordered by the inclusion.

**Corollary 8.4** With the notation above, we have

\[
\text{rank}_O\left(\mathcal{G}_K(\mathcal{F}, \mathfrak{aut}_{\mathcal{F}^\vee})\right) = \sum_{(q, \Delta_n)} (-1)^n \text{rank}_O\left(\mathcal{G}_K(\hat{\lambda}(q))\right)
\]

where \((q, \Delta_n)\) runs over a set of representatives for the set of isomorphism classes of regular \( \mathfrak{ch}^*({\mathcal{F}}^\vee) \)-objects.

**Proof:** It follows from the decomposition 5.12.3 and from Proposition 6.5 that we have

\[
\mathcal{K}\mathcal{G}_K(\mathcal{F}, \mathfrak{aut}_{\mathcal{F}^\vee}) \cong \prod_{h \in \mathbb{N} - \{0\}} \mathbb{H}^0\left(\mathfrak{h}^v(\mathcal{F}^\vee), \mathcal{K}\mathcal{F}\mathfrak{d}t_{U_h} \circ t_h\right)
\]

where we set \( h' = h^\nu \). On the other hand, it is more or less well-known that the cohomology groups over functors to \( \mathcal{K}\text{-mod} \) coincide with the corresponding cohomology groups computed from the regular chains; more precisely, in our situation let us denote by \( \mathfrak{n}_h \) the extension of \( \mathfrak{n}_h = \mathcal{F}\mathfrak{d}t_{U_h} \circ t_h \) from \( O \) to \( \mathcal{K} \) for any \( h \in \mathbb{N} - \{0\} \); then, it follows from Propositions A4.13 and A5.7 in [5] that, with the notation there, for any \( n \in \mathbb{N} \) we have

\[
\mathbb{H}^n\left(\mathfrak{h}(\mathcal{F}^\vee), \mathcal{K}\mathfrak{n}_h\right) \cong \mathbb{H}^n\left(\mathfrak{h}(\mathcal{F}^\vee), \mathcal{K}\mathfrak{n}_h\right) \cong \mathbb{H}^n\left(\mathfrak{h}(\mathcal{F}^\vee), \mathcal{K}\mathfrak{n}_h\right)
\]

In particular, it follows from Theorem 8.2 that for any \( n \geq 1 \) we get

\[
\mathbb{H}^n\left(\mathfrak{h}(\mathcal{F}^\vee), \mathcal{K}\mathfrak{n}_h\right) = \{0\}
\]
which amounts to saying that we have an infinite exact sequence
\[ 0 \rightarrow \mathbb{H}^0 (\overline{h(F^\infty)}, K_n h) \rightarrow \cdots \rightarrow C^n \rightarrow C^{n+1} \rightarrow \cdots \]
where \( C^n = \mathbb{C}^n (\overline{h(F^\infty)}, K_n h) \) is the set of elements (cf. Proposition 6.4)
\[ (\chi_{\overline{q}h}^{\overline{q}})^{-1} \in \prod_{\overline{q}h \in \mathfrak{D} \Delta (\Delta_n, \overline{h(F^\infty)})} K_n h (\overline{q}h(0)) \]
such that, for any natural isomorphism \( \tilde{v} : \overline{q}h \cong \overline{r} \) between \( h(F^\infty) \)-valued \( n \)-chains \( \overline{q}h \) and \( \overline{r} \), \( K_n h (\tilde{v}) \) maps \( \chi_{\overline{q}h}^{\overline{q}} \) on \( \chi_{\overline{r}}^{\overline{r}} \). That is to say, since we have a bijection between the sets of isomorphism classes of \( h(F^\infty) \)- and \( \overline{h(F^\infty)} \)-objects, we actually have
\[ \mathbb{C}^n (\overline{h(F^\infty)}, K_n h) \cong \prod_{\overline{q}h} K \mathfrak{D} \mathfrak{U} h (t_h (\overline{q}h(0)))^{F(q)_h} \]
where \( q^h \) runs over a set of representatives for the set of isomorphism classes of \( h(F^\infty) \)-objects \([5, A5.3]\) and then \( F(q)_h \) denotes the stabilizer of \( \overline{q}h \) in \( F(q) \).

On the other hand, it is clear that for \( n \) big enough there are no \( \overline{h(F^\infty)} \)-valued \( n \)-chains and therefore, in the exact sequence above only finitely many terms are not zero; thus, we still get
\[ \dim_k \left( \mathbb{H}^0 (\overline{h(F^\infty)}, K_n h) \right) \]
\[ = \sum_{(q^h, \Delta_n)} (-1)^n \dim_k \left( K \mathfrak{D} \mathfrak{U} h (t_h (\overline{q}h(0)))^{F(q)_h} \right) \]
where \((q^h, \Delta_n)\) runs over a set of representatives for the set of isomorphism classes of \( \overline{h(F^\infty)} \)-\( \mathfrak{h} \)-objects \([5, A5.3]\). Consequently, since the functor \( \hat{w}_h \) in Proposition 5.8 maps the \( k^* \)-i\( \mathfrak{U} \)-morphism
\[ \hat{v}^q : (\hat{\mathcal{L}}(q), \text{Ker}(\pi_q)) \rightarrow (\hat{\mathcal{L}}(q(0)), Z(q(0))) \]
which lifts the corresponding i\( \mathfrak{U} \)-morphism 6.3.4, on a \( U_{\mathfrak{h}} \)-set bijection (cf. 5.8.2)
\[ (\varphi_{\mathfrak{h}, \hat{\mathcal{L}}(q(0))})^{-1} (\hat{v}^q \circ \hat{\eta}) \cong (\varphi_{\mathfrak{h}, \hat{\mathcal{L}}(\overline{q}))^{-1}(\overline{\eta}) \]
where, for short, we write \( \hat{\mathcal{L}}(q(0)) \) and \( \hat{\mathcal{L}}(\bar{q}) \) instead of \( (\hat{\mathcal{L}}(q(0)), Z(q(0))) \) and \( (\hat{\mathcal{L}}(q), \text{Ker}(\pi_q)) \), it follows from equality 8.4.2 and from Propositions 6.4 and 6.5 that we actually have
\[
\text{rank}_\mathcal{O}(\mathcal{G}_K(\mathcal{F}, \text{aut}_{\mathcal{F}^{\text{sec}}})) = \sum_{h} \sum_{(\bar{n}, q, \Delta_n)} (-1)^n \dim_K \left( \mathcal{K}_{\mathcal{F}^{\text{ct}}, U_h'} \left( \left(\mathcal{Z}_{h, \hat{\mathcal{L}}(q)}\right)^{-1}(\bar{n}) \right) \right)^{\mathcal{F}(q)_0} \tag{8.4.11}
\]
where \( h \) runs over \( \mathbb{N} - \{0\} \) and \( (\bar{n}, q, \Delta_n) \) over a set of representatives for the isomorphism classes of \( \mathfrak{h}_0 \text{ch}^*(\mathcal{F}') \)-objects such that \( (q, \Delta_n) \) is regular.

On the other hand, for any \( \text{ch}^*(\mathcal{F}') \)-object \( (q, \Delta_n) \), it follows from isomorphism 5.4.3 that
\[
\bigoplus_{h \in \mathbb{N} - \{0\}} \bigoplus_{\bar{n}} \mathcal{K}_{\mathcal{F}^{\text{ct}}, U_h'} \left( \left(\mathcal{Z}_{h, \hat{\mathcal{L}}(q)}\right)^{-1}(\bar{n}) \right) \right)^{\mathcal{F}(q)_0} \cong \bigoplus_{h \in \mathbb{N} - \{0\}} \left( \bigoplus_{\bar{n}} \mathcal{K}_{\mathcal{F}^{\text{ct}}, U_h'} \left( \left(\mathcal{Z}_{h, \hat{\mathcal{L}}(q)}\right)^{-1}(\bar{n}) \right) \right)^{\mathcal{F}(q)} \tag{8.4.12}
\]
where \( \bar{n} \) runs over a set of representatives for the orbits of \( \mathcal{L}(q) \) on the set \( \text{Mon}(U_h, \mathcal{L}(q)) \). We are done.

9 General decomposition maps in a folded Frobenius \( P \)-category

9.1 With the notation of §8, let us choose a set of representatives \( \mathcal{P} \subset P \) for the set of \( \mathcal{F} \)-isomorphism classes of the elements of \( P \) in such a way that, for any \( u \in \mathcal{P} \), the subgroup \( \langle u \rangle \) is fully centralized in \( \mathcal{F} \) [5, Proposition 2.7]. For any \( u \in \mathcal{P} \), we have the Frobenius \( C_{\mathcal{F}}(u) \)-category \( C_{\mathcal{F}}(u) \) [5, Proposition 2.16] and we know that the inclusion \( \iota_u : C_{\mathcal{F}}(u) \to P \) is \( (C_{\mathcal{F}}(u), \mathcal{F}) \)-functorial [5, 12.1]; since a \( C_{\mathcal{F}}(u) \)-selfcentralizing subgroup of \( C_{\mathcal{F}}(u) \) contains \( u \), it is also an \( \mathcal{F} \)-selfcentralizing subgroup of \( P \); coherently, we write \( C_{\mathcal{F}^{\text{sec}}}(u) \) instead of \( C_{\mathcal{F}}(u)^\circ \) and, according to 4.9 above, we have the \( \mathcal{O} \)-module homomorphism
\[
\text{Res}_{\iota_u} : \mathcal{G}_K(\mathcal{F}, \hat{\text{aut}}_{\mathcal{F}^{\text{sec}}}) \to \mathcal{G}_K(C_{\mathcal{F}}(u), \hat{\text{aut}}_{C_{\mathcal{F}^{\text{sec}}}(u)}) \tag{9.1.1}
\]

9.2 Following Broué [2, Appendice], for any finite group \( G \) and any central \( p \)-element \( z \) of \( G \), we consider the \( z \)-twist
\[
\omega^z_G : \mathcal{G}_K(G) \cong \mathcal{G}_K(G) \tag{9.2.1}
\]
determined by the \( z \)-translation map in the \( \mathcal{O} \)-valued functions \( \mathcal{F}^{\text{ct}}(G, \mathcal{O}) \) induced by the multiplication by \( z \); explicitly, if \( \chi \) is an irreducible ordinary
character of \( G \), the \( z \)-translated function maps \( x \in G \) on \( \chi(xz) \) and therefore it coincides with \( \frac{d\chi}{d\tilde{\chi}} \) \( \chi \) which still belongs to the image of \( G_\chi(G) \); actually, this definition can be easily extended to the finite \( k^* \)-groups \([4, \text{Proposition 5.15}]\). Thus, for any \( u \in P \) and any \( C_{F^{sc}}(u) \)-chain \( q: \Delta_n \to C_{F^{sc}}(u) \) we can consider the \( u \)-twist
\[
\omega_u^u: G_\chi \circ \widehat{\loc}_{C_{F^{sc}}(u)}(q) \cong G_\chi \circ \widehat{\loc}_{C_{F^{sc}}(u)}(q)
\]
then, it is clear that we get a natural automorphism
\[
\omega_u: g_\chi \circ \widehat{\loc}_{C_{F^{sc}}(u)}(u) \cong g_\chi \circ \widehat{\loc}_{C_{F^{sc}}(u)}(u)
\]
and therefore an \( O \)-module automorphism
\[
\Omega_u: G_\chi(C_F(u), \widehat{\aut}_{C_{F^{sc}}(u)}) \cong G_\chi(C_F(u), \widehat{\aut}_{C_{F^{sc}}(u)})
\]
Finally, we can define the \( u \)-general decomposition map
\[
\partial_{(F,\widehat{\aut}_{F^{sc}})}^u: G_\chi(F, \widehat{\aut}_{F^{sc}}) \to G_\chi(C_F(u), \widehat{\aut}_{C_{F^{sc}}(u)})
\]
as the composition \( \partial_{(C_F(u), \widehat{\aut}_{C_{F^{sc}}(u)})} \circ \Omega_u \circ \text{Res}_u \) (cf. 3.4.2).

**Theorem 9.3** The family of general decomposition maps \( \{\partial_{(F,\widehat{\aut}_{F^{sc}})}^u\}_{u \in P} \) determines a \( K \)-module isomorphism
\[
K G_\chi(F, \widehat{\aut}_{F^{sc}}) \cong \bigoplus_{u \in P} K G_\chi(C_F(u), \widehat{\aut}_{C_{F^{sc}}(u)})
\]
**Proof:** According to decomposition 5.12.3, to Proposition 6.5 and to isomorphism 6.10.3, setting \( h' = h^p \) for any \( h \in \mathbb{N} - \{0\} \), we have an injective \( O \)-module homomorphism
\[
G_\chi(F, \widehat{\aut}_{F^{sc}}) \to \prod_{h \in \mathbb{N} - \{0\}} \prod Q^\rho \left((\sigma_{h'}, \eta_h)(Q^\rho'), K\right)
\]
where \( Q^\rho' \) runs over a set of representatives for the set of \( h'(\widehat{\aut}_{F^{sc}}) \)-isomorphism classes of \( h'(\widehat{\aut}_{F^{sc}}) \)-objects; thus, in order to prove the injectivity of the map determined by the family of general decomposition maps, it suffices to prove that, for any \( X \in G_\chi(F, \widehat{\aut}_{F^{sc}}) \) in the kernel of this map, any \( h \in \mathbb{N} - \{0\} \) and any \( h'(\widehat{\aut}_{F^{sc}}) \)-object \( Q^\rho' \), for the chosen lifting \( \rho(U_{h'}) \) to \( F(Q) \) of \( \rho'(U_{h'}) \) (cf. 6.9) the corresponding projection
\[
\chi_{X,h,Q^\rho'}: (\varpi_{h',\widehat{\chi}(Q)})^{-1}(\rho') \times \text{Mon}(U_{h'}, Q^\rho'(U_{h'})) \to K
\]
is the zero function, namely that \( \chi_{X,h,Q^\rho'} \) vanish over \( (\varpi_{h',\widehat{\chi}(Q)})^{-1}(\rho') \times \{\hat{\rho}'\} \)
for any injective group homomorphism \( \rho'' : U_{h'} \to Q^\rho'(U_{h'}) \).
Setting \( u = \rho''(\xi_{h'}) \), we may assume that \( u \) belongs to \( \mathcal{P} \) and then we are actually assuming that \( \partial_u^{\mathcal{F}, \text{aut}_{\mathcal{F}^{\text{sc}}}}(X) = 0 \); consider the subgroup \( C_Q(u) \) which is clearly selfcentralizing in \( Q \) and, for a suitable \( n \in \mathbb{N} \) and any \( i \in \Delta_{n-1} \), set \( R_n = C_Q(u) \) and \( R_i = N_Q(R_{i+1}) \) in such a way that \( R_0 = Q \); it is clear that the lifting \( \sigma \in \tilde{\rho}'(U_{h'}) \) of \( \rho'(\xi_{h'}) \) stabilizes the family \( \{ R_i \}_{i \in \Delta_n} \) and then it follows from Lemma 9.4 below that there are a family \( \{ Q_i \}_{i \in \Delta_n} \) of \( \mathcal{F} \)-selfcentralizing subgroups of \( P \) such that \( Q_0 = Q \), a family of \( \mathcal{F} \)-morphisms \( \theta_i : Q_i \to P \), where \( i \) runs over \( \Delta_{n-1} \), such that \( \theta_i(Q_i) \) normalizes \( Q_{i+1} \), and a family of \( p' \)-elements

\[
\sigma_i \in \mathcal{F}(\theta_i(Q_i), Q_{i+1})
\]

where \( i \) runs over \( \Delta_{n-1} \), stabilizing \( \theta_i(Q_i) \) and \( Q_{i+1} \), in such a way that the order of \( \sigma_i \) over \( Q_i \) defined via \( \theta_i \) coincides with \( \sigma \) if \( i = 0 \) or with the action of \( \sigma_{i-1} \) otherwise, and that, setting \( T_{i,0} = R_i \) for any \( i \in \Delta_n \) and then \( T_{i,j+1} = \theta_j(T_{i,j}) \) for any \( j < i \), we have \( Q_i = T_{i,n} \cdot C_Q(T_{i,i}) \) for any \( i \in \Delta_n \).

In particular, for any \( i \in \Delta_{n-1} \), respectively denoting by

\[
\tilde{\mathcal{F}}(\theta_i(Q_i), Q_{i+1})_{\theta_i(Q_i)} \quad \text{and} \quad \tilde{\mathcal{F}}(\theta_i(Q_i), Q_{i+1})_{Q_{i+1}}
\]

the stabilizers of \( \theta_i(Q_i) \) and \( Q_{i+1} \) in \( \tilde{\mathcal{F}}(\theta_i(Q_i), Q_{i+1}) \), we already know that the restriction via \( \theta_i \) and the ordinary restriction respectively induce group homomorphisms

\[
\alpha_i : \tilde{\mathcal{F}}(\theta_i(Q_i), Q_{i+1})_{\theta_i(Q_i)} \to \tilde{\mathcal{F}}(Q_i)
\]

\[
\beta_i : \tilde{\mathcal{F}}(\theta_i(Q_i), Q_{i+1})_{Q_{i+1}} \to \tilde{\mathcal{F}}(Q_{i+1})
\]

such that their kernels are \( p \)-groups [5, Corollary 4.7]; hence, arguing by induction, the order of \( \sigma \) coincides with the order of \( \sigma_i \) for any \( i \in \Delta_{n-1} \), and therefore we have an injective group homomorphism

\[
\rho'_i : U_{h'} \to \tilde{\mathcal{F}}(\theta_i(Q_i), Q_{i+1})
\]

mapping \( \xi_{h'} \) on \( \tilde{\sigma}_i \), so that we get

\[
\alpha_i \circ \rho'_i = \begin{cases} 
\beta_{i-1} \circ \rho'_i & \text{if } i \neq 0 \\
\rho' & \text{if } i = 0 
\end{cases}
\]

That is to say, setting \( \eta'_0 = \rho' \) and \( \eta'_{i+1} = \beta_i \circ \rho'_i \) for any \( i \in \Delta_{n-1} \), and considering the \( h'(\tilde{\mathcal{F}}^{\mathcal{F}^{\text{sc}}}) \)-objects \( (Q_i)_{\eta'_i} \) and \( (\theta_i(Q_i), Q_{i+1})_{\eta'_i} \), the homomorphism \( \theta_i \) and the inclusion map \( Q_{i+1} \to \theta_i(Q_i), Q_{i+1} \) respectively determine \( h'(\tilde{\mathcal{F}}^{\mathcal{F}^{\text{sc}}}) \)-morphisms

\[
(Q_i)_{\eta'_i} \to (\theta_i(Q_i), Q_{i+1})_{\eta'_i} \quad \text{and} \quad (Q_{i+1})_{\eta'_{i+1}} \leftarrow (Q_{i+1})_{\eta'_{i+1}}
\]
In conclusion, \( K \) mines the \( \chi \) which send the \( K \) \( \hat{\chi} \) for the corresponding chosen liftings \( \chi \) and therefore, for any \( i \in \Delta_{n-1} \), which determines injective group homomorphisms

\[
\eta''_i : U_{h^r} \rightarrow (Q_i)^{\eta'(U_{h^r})}
\]

\[
\rho''_i : U_{h^r} \rightarrow (\theta_i(Q_i)Q_{i+1})^{\rho'(U_{h^r})}
\]

for the corresponding chosen liftings \( \eta'_i(U_{h^r}) \) and \( \rho'_i(U_{h^r}) \) (cf. 6.9).

More precisely, the functor \( s_{h^r} \times \hat{\chi} \) maps the \( h'(\hat{F}^\circ) \)-morphisms 9.3.9 above on \( U_{h^r} \)-set maps (cf. 6.8 and 6.9)

\[
(\omega_{h^r},\hat{\chi}(Q_i))^{-1}(\eta'_i) \times \text{Mon}(U_{h^r}, (Q_i)^{\eta'(U_{h^r})})
\]

\[
\downarrow
\]

\[
(\omega_{h^r},\hat{\chi}(\theta_i(Q_i)Q_{i+1}))^{-1}(\rho'_i) \times \text{Mon}(U_{h^r}, (\theta_i(Q_i)Q_{i+1})^{\rho'(U_{h^r})})
\]

\[
\uparrow
\]

\[
(\omega_{h^r},\hat{\chi}(Q_{i+1}))^{-1}(\eta''_i) \times \text{Mon}(U_{h^r}, (Q_{i+1})^{\eta''(U_{h^r})})
\]

which send the \( K \)-valued function \( \chi_{X,h,\theta_i(Q_i)Q_{i+1}}^{\chi'} \) to the \( K \)-valued functions \( \chi_{X,h,\theta_i(Q_i)Q_{i+1}}^{\chi''} \) and \( \chi_{X,h,\theta_i(Q_i)Q_{i+1}}^{\chi''_{i+1}} \), and the functor \( K/Fc_{U_{h^r}} \) actually determines the \( K \)-module isomorphisms

\[
Fc_{U_{h^r}} \left( (\omega_{h^r},\hat{\chi}(Q_i))^{-1}(\eta'_i) \times \{\eta''_i\} \right) \cong K
\]

\[
Fc_{U_{h^r}} \left( (\omega_{h^r},\hat{\chi}(\theta_i(Q_i)Q_{i+1}))^{-1}(\rho'_i) \times \{\rho''_i\} \right) \cong K
\]

\[
Fc_{U_{h^r}} \left( (\omega_{h^r},\hat{\chi}(Q_{i+1}))^{-1}(\eta''_i) \times \{\eta''_{i+1}\} \right) \cong K
\]

In conclusion, \( \chi_{X,h,Q^{\chi'}} \) vanish over \( (\omega_{h^r},\hat{\chi}(Q))^{-1}(\rho'_i) \times \{\rho''_i\} \) if and only if \( \chi_{X,h,Q^{\chi''}} \) vanish over \( (\omega_{h^r},\hat{\chi}(Q_{i+1}))^{-1}(\eta''_i) \times \{\eta''_{i+1}\} \).

Finally, since \( T_{n,0} = C_Q(u) \) and \( Q_n = T_{n,n}C_Q(T_{n,n}) \), the element \( u_n \) belongs to \( Z(Q_n) \); but, since \( \langle u \rangle \) is fully centralized in \( F \), there is an
\( F \)-morphism \( \theta_n : C_p(u_n) \to P \) fulfilling \( \theta_n(u_n) = u \) [5, Proposition 2.7]; then, it is easily checked that \( Q_{n+1} = \theta_n(Q_n) \) is a \( C_F(u) \)-selfcentralizing subgroup of \( C_p(u) \) and therefore, denoting by

\[
\eta_{n+1}' = \theta_n(\eta_n') : U_{h'} \longrightarrow C_{\hat{\mathcal{F}}}(\theta_n(Q_n))
\]

9.3.14

the corresponding action of \( U_{h'} \) on the group \( Q_{n+1} \), we have the \( C_{\hat{\mathcal{F}}}(u) \)-object \( (Q_{n+1})^{\eta_{n+1}'} \) and, denoting by \( \sigma_n^h \) and \( \tau_n^h \) the functors defined in 6.8 and 6.9 for the Frobenius \( C_p(u) \)-category \( C_F(u) \), the corresponding projection map

\[
\mathcal{G}_K(C_F(u), \tilde{\text{aut}}_{C_{\hat{\mathcal{F}}}(u)}) \longrightarrow \mathcal{K} \text{ct}_{U_{h'}}((\sigma_n^h \times \tau_n^h)((Q_{n+1})^{\eta_{n+1}'}))
\]

9.3.15

sends \( \text{Res}_{\eta}(X) \) to following the \( U_{h'} \)-set map \( \chi_h = \chi_{\text{Res}_{\eta}(X), h, (Q_{n+1})^{\eta_{n+1}'}} \)

\[
(\varpi_{h', (C_F(u))(Q_{n+1}))^{-1}(\eta_{n+1}')) \times \tilde{\text{Mon}}(U_{h'}, (Q_{n+1})^{\eta_{n+1}'(U_{h'})}) \to \chi_h \mathcal{K}
\]

9.3.16.

Moreover, the element \( u = \theta_n(u_n) \) determines an injective group homomorphism

\[
\eta_{n+1}'' : U_{h'} \longrightarrow (Q_{n+1})^{\eta_{n+1}'(U_{h'})}
\]

9.3.17

and the condition \( \partial_{(\mathcal{F}, \tilde{\text{aut}}_{C_{\hat{\mathcal{F}}}})}(X) = 0 \) clearly implies that the \( \mathcal{K} \)-valued function \( \chi_h \) vanish over the \( U_{h'} \)-set

\[
(\varpi_{h', (C_F(u))(Q_{n+1}))^{-1}(\eta_{n+1}') \times \{\eta_{n+1}'' \}
\]

9.3.18.

But, from the inclusion functor \( C_F(u) \to \mathcal{F} \) and from the isomorphism \( Q_{n+1} \cong Q_n \) determined by \( \theta_n \), we get an injective \( k^* \)-group homomorphism

\[
(C_{\hat{\mathcal{F}}}(u))(Q_{n+1}) \longrightarrow \hat{\mathcal{F}}(Q_n)
\]

9.3.19

inducing a \( U_{h'} \)-set bijection [5, Proposition 14.18]

\[
(\varpi_{h', (C_F(u))(Q_{n+1}))^{-1}(\eta_{n+1}') \cong (\varpi_{h', \hat{\mathcal{F}}(Q_n)})^{-1}(\eta_n')
\]

9.3.20

and it is clear that the above isomorphism sends \( \eta_{n+1}'' \) to \( \eta_n'' \). At last, it is easily checked that the corresponding \( U_{h'} \)-set bijection

\[
(\varpi_{h', (C_F(u))(Q_{n+1}))^{-1}(\eta_{n+1}') \times \{\eta_{n+1}'' \} \cong (\varpi_{h', \hat{\mathcal{F}}(Q_n)})^{-1}(\eta_n') \times \{\eta_n'' \}
\]

9.3.21

sends the restriction of \( \chi_{X,h,(Q_n)^{\eta_n'}} \) to the restriction of \( \chi_h \), so that \( \chi_{X,h,Q^n} \) vanish over \( (\varpi_{h', \hat{\mathcal{F}}(Q)})^{-1}(\rho') \times \{\tilde{\rho}' \} \), proving the injectivity in 9.3.1.
At this point, it suffices to prove that both members of isomorphism
9.3.1 have the same dimension, namely that the following equality holds
\[ \text{rank}_O \left( \mathcal{G}_K(\mathcal{F}, \widehat{\text{aut}}_{\mathcal{F}^{sc}}) \right) = \sum_{u \in P} \text{rank}_O \left( \mathcal{G}_k \left( \hat{C}_\mathcal{F}(u), \widehat{\text{aut}}_{\hat{C}_\mathcal{F}(u)} \right) \right) \] 9.3.22;
we already know that we have (cf. Corollary 8.4)
\[ \text{rank}_O \left( \mathcal{G}_K(\mathcal{F}, \widehat{\text{aut}}_{\mathcal{F}^{sc}}) \right) = \sum_{(q, \Delta_n)} (-1)^n \text{rank}_O \left( \mathcal{G}_K(\hat{L}(q)) \right) \] 9.3.23
where \((q, \Delta_n)\) runs over a set of representatives for the set of \(\mathcal{F}\)-isomorphism
classes of \(\text{ch}^*_F(\mathcal{F}^\sim)\)-objects [5, A5.3] which are fully normalized in \(\mathcal{F}\) [5, 2.18].
But, denoting by \(P_q\) a set of representatives for the set of \(N_{\mathcal{F}}(q)\)-isomorphism
classes of elements of \(N_{\mathcal{F}}(q)\) in such a way that, for any \(u \in P_q\), the subgroup
\(\langle u \rangle\) is fully centralized in \(N_{\mathcal{F}}(q)\), and identifying \(N_{\mathcal{F}}(q)\) with its structural
image in \(\hat{L}(q)\), it easily follows from [5, Proposition 19.5] that \(N_{\mathcal{F}}(q)\) coincides
with the Frobenius category associated with \(\hat{L}(q)\) [5, 1.8] and then, it is well-
known that we have (cf. isomorphism 1.6.1)
\[ \text{rank}_O \left( \mathcal{G}_K(\hat{L}(q)) \right) = \sum_{u \in P_q} \text{rank}_O \left( \mathcal{G}_k \left( C_{\hat{L}(q)}(u) \right) \right) \] 9.3.24.
On the other hand, for any \(u \in P\), we also have [5, Corollary 14.32]
\[ \text{rank}_O \left( \mathcal{G}_k \left( C_{\mathcal{F}}(u), \widehat{\text{aut}}_{C_{\mathcal{F}}(u)} \right) \right) = \sum_{(q_u, \Delta_n)} (-1)^n \text{rank}_O \left( \mathcal{G}_k \left( \left( C_{\mathcal{F}(u)} \right)(q_u) \right) \right) \] 9.3.25
where \((q_u, \Delta_n)\) runs over a set of representatives for the set of \(C_{\mathcal{F}}(u)\)-isomorphism
classes of \(\text{ch}^*_F(C_{\mathcal{F}}(u))\)-objects [5, A5.3] which are fully normalized in \(C_{\mathcal{F}}(u)\) [5, 2.18] and \(\left( C_{\mathcal{F}(u)} \right)(q_u)\) is the converse image of \(\left( C_{\mathcal{F}(u)} \right)(q_u)\)
in \(\mathcal{F}(q_u)\).
Consequently, the left-hand member in 9.3.22 is equal to
\[ \sum_{(q, \Delta_n)} \sum_{u \in P_q} (-1)^n \text{rank}_O \left( \mathcal{G}_k \left( C_{\hat{L}(q)}(u) \right) \right) \] 9.3.26
where \((q, \Delta_n)\) runs over a set of representatives for the set of \(\mathcal{F}\)-isomorphism
classes of \(\text{ch}^*_F(\mathcal{F}^\sim)\)-objects which are fully normalized in \(\mathcal{F}\) [5, 2.18], whereas
the right-hand member is equal to
\[ \sum_{u \in P} \sum_{(q_u, \Delta_n)} (-1)^n \text{rank}_O \left( \mathcal{G}_k \left( \left( C_{\mathcal{F}(u)} \right)(q_u) \right) \right) \] 9.3.27
where, for any \(u \in P\), \((q_u, \Delta_n)\) runs over a set of representatives for the set of
\(C_{\mathcal{F}}(u)\)-isomorphism classes of \(\text{ch}^*_F(C_{\mathcal{F}}(u))\)-objects which are fully normalized in \(C_{\mathcal{F}}(u)\) [5, 2.18]. But, it is clear that the element \(u\) belongs to \(Z(q_u(n))\).
and then it is easily checked that \((C_{\mathcal{F}}(u))(q_u)\) coincides with the stabilizer of \(u\) in \(\mathcal{F}(q_u)\), so that we have

\[
\mathcal{G}_k\left((C_{\mathcal{F}}(u))(q_u)\right) = \mathcal{G}_k\left(C_{\mathcal{F}}(q_u)\right) \quad 9.3.28.
\]

Moreover, if \((q, \Delta_n)\) is a \(\text{ch}_n^*\(\mathcal{F}^e\))-object fully normalized in \(\mathcal{F}\) such that all the group homomorphisms \(q(j \cdot i)\) are inclusion maps then, for any element \(u\) of \(Z(q(n))\) such that \(\langle u \rangle\) is fully centralized in \(\mathcal{F}\), it is quite clear that \((q, \Delta_n)\) remains a \(\text{ch}_n^*\(\mathcal{F}^e(u)\))-object which is fully normalized in \(C_{\mathcal{F}}(u)\).

Hence, the sum 9.3.27 above coincides with

\[
\sum_{(q, \Delta_n) \in \mathcal{P}_q} \sum_{u \in \mathcal{Z}_q} (-1)^n \text{rank}_\mathcal{O} \left( \mathcal{G}_k\left(C_{\mathcal{F}}(q)(u)\right) \right) \quad 9.3.29
\]

where \((q, \Delta_n)\) runs over a set of representatives for the set of \(\text{ch}_n^*\(\mathcal{F}^e\))-isomorphism classes of \(\text{ch}_n^*\(\mathcal{F}^e\))-objects which are fully normalized in \(\mathcal{F}\) and, for such a \(\text{ch}_n^*\(\mathcal{F}^e\))-object \((q, \Delta_n)\), \(\mathcal{Z}_q\) is a set of representatives for the set of orbits of \(\mathcal{F}(q)\) in \(Z(q(n))\). Finally, we may assume that \(\mathcal{P}_q\) contains \(\mathcal{Z}_q\) and then equality 9.3.22 above is equivalent to the following one

\[
0 = \sum_{(q, \Delta_n) \in \mathcal{P}_q \setminus \mathcal{Z}_q} \sum_{u \in \mathcal{P}_n \setminus \mathcal{Z}_q} (-1)^n \text{rank}_\mathcal{O} \left( \mathcal{G}_k\left(C_{\mathcal{F}}(q)(u)\right) \right) \quad 9.3.30
\]

where \((q, \Delta_n)\) runs over a set of representatives for the set of \(\text{ch}_n^*\(\mathcal{F}^e\))-isomorphism classes of \(\text{ch}_n^*\(\mathcal{F}^e\))-objects which are fully normalized in \(\mathcal{F}\).

Since any \(\text{ch}_n^*\(\mathcal{F}^e\))-object is \(\text{ch}_n^*\(\mathcal{F}^e\))-isomorphic to one which is fully normalized in \(\mathcal{F}\), actually we are considering a set of representatives \(\mathcal{C}\) for the set of isomorphism classes of pairs formed by a \(\text{ch}_n^*\(\mathcal{F}^e\))-object \((q, \Delta_n)\) such that all the group homomorphisms \(q(j \cdot i)\) are inclusions, and an element \(u \in \mathcal{P}\) which normalizes but does not centralize \(q\). Thus, it suffices to exhibit an involutive permutation \(t\) of \(\mathcal{C}\) such that, setting

\[
t((q, \Delta_n), u) = ((q', \Delta_n'), u') \quad 9.3.31
\]

and assuming that \((q, \Delta_n)\) and \((q', \Delta_n')\) are both fully normalized in \(\mathcal{F}\), we have

\[
\mathcal{G}_k\left(C_{\mathcal{F}}(q')(u')\right) \cong \mathcal{G}_k\left(C_{\mathcal{F}}(q)(u)\right) \quad n' \neq n \quad (\text{mod } 2) \quad 9.3.32.
\]

First of all, we consider the set \(\mathcal{C}'\) of pairs \(((q, \Delta_n), u) \in \mathcal{C}\) such that \(u\) does not belong to \(q(0)\); in this case, let \(i\) be the last element of \(\Delta_n\) such that \(u\) does not belong to \(q(i)\). If \(i = n\) or \(q(i)\)-\(\langle u \rangle \neq q(i + 1)\) then we set \(n' = n + 1\) and consider the functor \(q' : \Delta_{n'} \to \mathcal{F}^e\) mapping any \(0 \leq \ell \leq i\) on \(q(\ell)\), \(i + 1\) on \(q(i)\)-\(\langle u \rangle\), any \(i + 2 \leq \ell \leq n'\) on \(q(\ell - 1)\), and all the \(\Delta_{n'}\)-morphisms on the corresponding inclusions.
In this case, up to replacing the \( \mathfrak{ch}_n^\ast(\mathcal{F}^\cong) \)-object \((q', \Delta_{n'})\) by a \( \mathfrak{ch}_n^\ast(\mathcal{F}^\cong) \)-isomorphic one fully normalized in \( \mathcal{F} \), we get the \( \mathfrak{ch}_n^\ast(\mathcal{F}^\cong) \)-morphism

\[
(\nu, \delta_{i+1}^n): (q', \Delta_{n'}) \longrightarrow (q, \Delta_n)
\] 9.3.33

for a suitable natural isomorphism \( \nu: q' \circ \delta_{i+1}^n \cong q \), and thus we still get the \( k^\ast \text{-iLoc} \)-morphism

\[
\hat{\text{loc}}_{\mathcal{F}^\cong}(\nu, \delta_{i+1}^n): \hat{\mathcal{L}}(q') \longrightarrow \hat{\mathcal{L}}(q)
\] 9.3.34.

Moreover, since we have [5, 14.8]

\[
q \circ \delta_{i+1}^n \circ \delta_{0}^q \circ \delta_{0}^q \circ \delta_{0}^q = q \circ \delta_{i+1}^n \circ \delta_{0}^q \circ \delta_{0}^q \circ \delta_{0}^q = q \circ \delta_{i+1}^n \circ \delta_{0}^q \circ \delta_{0}^q \circ \delta_{0}^q
\]

which clearly contains the image of the stabilizer \( \mathcal{F}(q) \) of \( u \) in \( \mathcal{F}(q) \), a suitable representative of the exomorphism \( \hat{\text{loc}}_{\mathcal{F}^\cong}(\nu, \delta_{i+1}^n) \) induces a \( k^\ast \)-group isomorphism

\[
C_{\hat{\mathcal{L}}(q')} (u') \cong C_{\hat{\mathcal{L}}(q)} (u)
\] 9.3.37

where we are setting \( u' = (\nu_0)^{-1}(u) \).

If \( i+1 \leq n \) and \( q(i) \cdot u = q(i+1) \), we set \( n' = n - 1 \) and consider a chain \( q': \Delta_{n'} \rightarrow \mathcal{F}^\cong \) mapping any \( 0 \leq \ell \leq i \) on \( q(\ell) \), any \( i + 1 \leq \ell \leq n' \) on \( q(\ell + 1) \) and, as before, all the \( \Delta_{n'} \)-morphisms on the corresponding inclusions. In this case, up to replacing the \( \mathfrak{ch}_n^\ast(\mathcal{F}^\cong) \)-object \((q', \Delta_{n'})\) by a \( \mathfrak{ch}_n^\ast(\mathcal{F}^\cong) \)-isomorphic one fully normalized in \( \mathcal{F} \), we get the \( \mathfrak{ch}_n^\ast(\mathcal{F}^\cong) \)-morphism

\[
(\nu', \delta'_{i+1}^n): (q, \Delta_n) \longrightarrow (q', \Delta_{n'})
\] 9.3.38

for a suitable natural isomorphism \( \nu': q \circ \delta'_{i+1}^n \cong q' \), and thus we still get the \( k^\ast \text{-iLoc} \)-morphism

\[
\hat{\text{loc}}_{\mathcal{F}^\cong}(\nu', \delta'_{i+1}^n): \hat{\mathcal{L}}(q') \longrightarrow \hat{\mathcal{L}}(q')
\] 9.3.39.

Moreover, since we have [5, 14.8]

\[
(\text{aut}_{\mathcal{F}^\cong} (\nu', \delta'_{i+1}^n))(\hat{\mathcal{F}}(q')) = \hat{\mathcal{F}}(q')_{q(i+1)}
\] 9.3.40

where as above \( \hat{\mathcal{F}}(q')_{q(i+1)} \) denotes the set of \( \hat{\sigma} \in \hat{\mathcal{F}}(q') \) fulfilling

\[
\hat{\sigma}_{q(i+1)} \circ \hat{\sigma} = \hat{\mathcal{F}}(q(i + 1)) \circ \hat{\sigma}_{q(i+1)}
\] 9.3.41,
which clearly contains the image of the stabilizer $\mathcal{F}(q')_u$ of $u$ in $\mathcal{F}(q')$, a suitable representative of the $k^*-i\Sigma c$-morphism $\hat{\mathcal{L}}_{\mathcal{F}}(\nu', \delta_{i+1})$ induces a $k^*$-group isomorphism

$$C_{\hat{\mathcal{L}}(q)}(u) \cong C_{\hat{\mathcal{L}}(q')}(u')$$

where we are setting $u' = \nu'_u(u)$. Finally, since in both cases $q'(0) = q(0)$, we may assume that the pair $((q', \Delta_{n'}), u)$ still belongs to $C'$ and, defining $t((q, \Delta_n), u) = ((q', \Delta_{n'}), u)$, mutatis mutandis it is easily checked that

$$t((q', \Delta_{n'}), u) = ((q, \Delta_n), u)$$

9.3.43.

From now on, we consider the set $C''$ of pairs $((q, \Delta_n), u) \in C$ such that $u$ belongs to $q(0)$; note that the product

$$R' = C_{q(0)}(u).[q(n), C_{q(0)}(u)]$$

9.3.44.

is a normal subgroup of $q(n)$ and therefore it follows from [5, Proposition 2.7] that, up to replacing the pair $((q, \Delta_n), u)$ by its image throughout a suitable $\mathcal{F}$-morphism $\varphi : q(n) \to P$, we may assume that $R'$ is fully normalized and fully centralized in $\mathcal{F}$; then, $Q' = R'.C_P(R')$ is $\mathcal{F}$-selfcentralizing [5, 4.10].

Since we may assume that $Q' \neq \{1\}$, setting $q(-1) = \{1\}$, let $i$ be the last element in $\Delta_n \cup \{-1\}$ such that $Q' \nsupseteq q(i)$, so that we have $q(i) \neq Q'q(i)$. If $i = n$ or $Q'q(i) \neq q(i + 1)$ then we set $n' = n + 1$ and consider the $\mathcal{F}$-chain $q' : \Delta_{n'} \to \Delta_n$ mapping any $0 \leq \ell \leq i$ on $q(\ell)$, $i + 1$ on $Q'q(i)$, any $i + 2 \leq \ell \leq n'$ on $q(\ell - 1)$ and all the $\Delta_{n'}$-morphisms on the corresponding inclusions.

As above, up to replacing the $\mathfrak{ch}^*(\mathcal{F})$-object $(q', \Delta_{n'})$ by a $\mathfrak{ch}^*(\mathcal{F})$-isomorphic one fully normalized in $\mathcal{F}$, we get the $\mathfrak{ch}^*(\mathcal{F})$-morphism

$$(\nu, \delta_{i+1}) : (q', \Delta_{n'}) \to (q, \Delta_n)$$

9.3.45.

for a suitable natural isomorphism $\nu : q' \circ \delta_{i+1} \cong q'$, and thus we still get the $k^*-i\Sigma c$-morphism

$$\hat{\mathcal{L}}_{\mathcal{F}}(\nu', \delta_{i+1}) : \hat{\mathcal{L}}(q') \to \hat{\mathcal{L}}(q)$$

9.3.46.

Moreover, as above we have [5, 14.8]

$$(\text{aut}_{\mathcal{F}}(\nu, \delta_{i+1}))(\mathcal{F}(q')) = \mathcal{F}(q')q(i)$$

9.3.47.

where $\mathcal{F}(q')q(i)$ denotes the set of $\sigma \in \mathcal{F}(q)$ fulfilling

$$i_q^{Q'q(i)} \circ \sigma_0 \in \hat{\mathcal{F}}(Q'q(i)) \circ i_{q(0)}^{Q'q(i)}$$

9.3.48.

actually, we may assume that $\mathcal{F}_P(q)$ contains a Sylow $p$-subgroup of $\mathcal{F}(q)_u$ and then, denoting by $C_{\mathcal{F}(q)_u}(R')$ the kernel of the action of $\mathcal{F}(q)_u$ on $R'$, $\mathcal{F}_Q(q)$ is a Sylow $p$-subgroup of the product $\mathcal{F}_Q(q)\cdot C_{\mathcal{F}(q)_u}(R')$; thus, by the Frattini argument we get

$$\mathcal{F}(q)_u \subset C_{\mathcal{F}(q)_u}(R').\mathcal{F}(q)_Q(q)$$

9.3.49.
On the other hand, since we have $C_{q(0)}(R') \subset R'$, a $p'$-subgroup of $\mathcal{F}(q)_u$ which acts trivially on $R'$ is necessarily trivial [3, Ch. 5, Theorem 3.4], so that $C_{\mathcal{F}(q)_u}(R')$ is a $p$-group. Consequently, setting $u' = (\nu_0)^{-1}(u)$, the $k^*\mathfrak{L}c\mathfrak{c}$-morphism $\widetilde{\mathfrak{L}}c_{\mathcal{F}^e}(\nu, \delta_{n+1}^u)$ induces a group isomorphism

$$\mathcal{G}_k(C_{\mathcal{L}(q)}(u')) \cong \mathcal{G}_k(C_{\mathcal{L}(q)}(u))$$ 9.3.50.

Finally, if $i + 1 \leq n$ and $Q'q(i) = q(i + 1)$, we set $n' = n - 1$ and consider the $\mathcal{F}^e$-chain $q': \Delta_{n'} \to \mathcal{F}^e$ mapping any $0 \leq \ell \leq i$ on $q(\ell)$, any $i + 1 \leq \ell \leq n'$ on $q(\ell + 1)$ and, as before, all the $\Delta_n$-morphisms on the corresponding inclusions. Once again, up to replacing the $\text{ch}_q^*(\mathcal{F}^e)$-object $(q', \Delta_{n'})$ by a $\text{ch}_q^*(\mathcal{F}^e)$-isomorphic one fully normalized in $\mathcal{F}$, we get the $\text{ch}_q^*(\mathcal{F}^e)$-morphism

$$(\nu', \delta_{n'+1}^u) : (q, \Delta_u) \to (q', \Delta_{n'})$$ 9.3.51

for a suitable natural isomorphism $\nu': q \circ \delta_{n'+1}^u \cong q'$, and thus we still get the $k^*\mathfrak{L}c\mathfrak{c}$-morphism

$$\widetilde{\mathfrak{L}}c_{\mathcal{F}^e}(\nu', \delta_{n'+1}^u) : \hat{\mathcal{L}}(q) \to \hat{\mathcal{L}}(q')$$ 9.3.52;

as above, we have [5, 14.8]

$$(\text{aut}_{\mathcal{F}^e}(\nu', \delta_{n'+1}^u))(\mathcal{F}(q)) = \mathcal{F}(q')_{Q'q(i)}$$ 9.3.53

where $\mathcal{F}(q')_{Q'q(i)}$ denotes the set of $\sigma \in \mathcal{F}(q')$ fulfilling

$$\tilde{i}_{q(0)}^q \circ (\nu_0)^{-1} \circ \tilde{i}_{0} \in \tilde{\mathcal{F}}(Q'q(i)) \circ \tilde{i}_{q(0)}^q \circ (\nu_0)^{-1}$$ 9.3.54.

Moreover, setting $u' = \nu_0(u)$, we may assume that $\mathcal{F}_P(q')$ contains a Sylow $p$-subgroup of $\mathcal{F}(q')_{u'}$ and then, denoting by $C_{\mathcal{F}(q')_{u'}}(R')$ the kernel of the action of $\mathcal{F}(q')_{u'}$ on $R'$ via the group isomorphism $\nu_{u'} : q(n) \cong q'(n')$, $\mathcal{F}_{\nu_{u'}(q')}(q')$ is a Sylow $p$-subgroup of the product $\mathcal{F}_{\nu_{u'}(q')}(q') \cdot C_{\mathcal{F}(q')_{u'}}(R');$

once again, by the Frattini argument we get

$$\mathcal{F}(q')_{u'} \subset C_{\mathcal{F}(q')_{u'}}(R') \cdot \mathcal{F}(q')_{Q'q(i)}$$ 9.3.55.

On the other hand, since we have $C_{q(0)}(R') \subset R'$, a $p'$-subgroup of $\mathcal{F}(q')_{u'}$ which acts trivially on $R'$ is necessarily trivial [3, Ch. 5, Theorem 3.4], so that $C_{\mathcal{F}(q')_{u'}}(R')$ is a $p$-group. Consequently, the $k^*\mathfrak{L}c\mathfrak{c}$-morphism $\widetilde{\mathfrak{L}}c_{\mathcal{F}^e}(\nu, \delta_{n+1}^u)$ induces a group isomorphism

$$\mathcal{G}_k(C_{\mathcal{L}(q)}(u')) \cong \mathcal{G}_k(C_{\mathcal{L}(q)}(u))$$ 9.3.56.

Since $u$ belongs to $R'$, in both cases $u$ belongs to $q'(0)$ and we may assume that the pair $(q', \Delta_{n'})$, $u$ still belongs to $C'$; in this situation, we define $t((q, \Delta_n), u) = ((q', \Delta_{n'}), v)$ and claim that

$$t((q', \Delta_{n'}), u) = ((q, \Delta_n), u)$$ 9.3.57.
Indeed, set \( t((q', \Delta_{n'}), u) = ((q'', \Delta_{n''}), u) \); if \( q'(n') = q(n) \) then *mutatis mutandis* we consider

\[
R'' = C_{q''(0)}(u) \cdot [q'(n'), C_{q'(0)}(u)]
\]

9.3.58;
since \( C_{q(0)}(u) \subset R' \subset Q' \), we also have \( C_{q'(0)}(u) = C_{q(0)}(u) \) and therefore we get

\[
R'' = R' \quad \text{and} \quad Q'' = R'' \cdot C_P(R'') = Q'
\]

9.3.59;
moreover, once again setting \( q'(-1) = \{1\}, i \) is also the last element in \( \Delta_{n'} \cup \{-1\} \) such that \( Q' \) is not contained in \( q'(i) = q(i) \); in this situation, the product \( Q' \cdot q'(i) \) is different from \( q'(n') = q(n) \) and equality 9.3.57 is easily checked.

If \( q'(n') \neq q(n) \) then we have either \( q'(n') = Q' \cdot q(n) \) or

\[
q(n) = Q' \cdot q(n - 1) \quad \text{and} \quad q'(n') = q(n - 1)
\]

9.3.60;
note that in both cases we have \( q'(0) = q(0) \); in the first case, we have \( q'(n') = C_P(R') \cdot q(n) \) and, since \( R' \) contains \( C_{q(0)}(u) = C_{q'(0)}(u) \), we get

\[
[q'(n), C_{q'(0)}(u)] = [q(n), C_{q(0)}(u)]
\]

9.3.61;
consequently, equalities 9.3.59 still hold and therefore equality 9.3.57 is easily checked; in the second case, since \([q(n), C_{q(0)}(u)]\) is contained in the Frattini subgroup of \( q(n) \), we similarly obtain

\[
q(n) = C_P(R') \cdot C_{q(0)}(u) \cdot q(n - 1) = C_P(R') \cdot q'(n')
\]

9.3.62;
once again, we get

\[
[q(n), C_{q(0)}(u)] = [q'(n), C_{q'(0)}(u)]
\]

9.3.63,
equalities 9.3.59 still hold and equality 9.3.57 is easily checked. We are done.

**Lemma 9.4** Let \( Q \) be an \( F \)-selfcentralizing subgroup of \( P \) and \( \{R_i\}_{i \in \Delta_n} \) a family of selfcentralizing subgroups of \( Q \) such that \( R_0 = Q \) and \( R_{i+1} \triangleleft R_i \) for any \( i \in \Delta_{n-1} \). Then, for any \( p' \)-element \( \sigma \) in \( F(Q) \) stabilizing this family, there are a family \( \{Q_i\}_{i \in \Delta_n} \) of \( F \)-selfcentralizing subgroups of \( P \) such that \( Q_0 = Q \), a family of \( F \)-morphisms \( \theta_i : Q_i \rightarrow P \) where \( i \) runs over \( \Delta_{n-1} \), such that \( \theta_i(Q_i) \) normalizes \( Q_{i+1} \), and a family of \( p' \)-elements

\[
\sigma_i \in F(\theta_i(Q_i) \cdot Q_{i+1})
\]

9.4.1
where \( i \) runs over \( \Delta_{n-1} \), stabilizing \( \theta_i(Q_i) \) and \( Q_{i+1} \), in such a way that the action of \( \sigma_i \) over \( Q_i \) defined via \( \theta_i \) coincides with \( \sigma \) if \( i = 0 \) or with the action of \( \sigma_{i+1} \) otherwise, and that, setting \( T_{i,0} = R_i \) for any \( i \in \Delta_n \) and then \( T_{i,j+1} = \theta_j(T_{i,j}) \) for any \( j < i \), we have \( Q_i = T_{i,i} \cdot C_{Q_i}(T_{i,i}) \) for any \( i \in \Delta_n \).
Proof: We argue by induction on $n$ and may assume that $n \neq 0$; thus, we assume the existence of a family $\{Q_i\}_{i \in \Delta_{n-1}}$ of $\mathcal{F}$-selfcentralizing subgroups of $P$ such that $Q_0 = Q$, a family of $\mathcal{F}$-morphisms $\theta_i : Q_i \to P$, where $i$ runs over $\Delta_{n-2}$, such that $\theta_i(Q_i)$ normalizes $Q_{i+1}$, and a family of $\mathcal{F}$-elements $\sigma_i \in \mathcal{F}(\theta_i(Q_i)\cdot Q_{i+1})$ where $i$ runs over $\Delta_{n-2}$, stabilizing $\theta_i(Q_i)$ and $Q_{i+1}$, which fulfill the corresponding conditions above; thus, setting $T_{i,0} = R_i$ for any $i \in \Delta_{n-1}$ and then $T_{i,j+1} = \theta_j(T_{i,j})$ for any $j < i$, the following diagram summarizes our situation

\[
\begin{array}{ccccccccc}
\sigma & Q & \xrightarrow{\theta_0} & \theta_0 Q & Q_1 & \xrightarrow{\theta_1} & \ldots & \\
\downarrow & \downarrow & \uparrow & \downarrow & \uparrow & & & \\
\sigma & Q \cdot Q_1 & \xrightarrow{\theta_0} & Q_1 & \xrightarrow{\theta_1} & \ldots & \\
\downarrow & \downarrow & \uparrow & \downarrow & \uparrow & & & \\
R_1 & \cong T_{1,1} & \cong \ldots & Q_{n-2} & \xrightarrow{\theta_{n-2}} & \theta_{n-2} Q_{n-2} Q_{n-1} & 9.4.2 & \\
\downarrow & \downarrow & \uparrow & \downarrow & \uparrow & & & \\
R_{n-2} & \cong T_{n-2,1} & \cong \ldots & T_{n-2,n-2} & \xrightarrow{Q_{n-2}} & Q_{n-2} Q_{n-1} Q_{n-1} & \\
\downarrow & \downarrow & \uparrow & \downarrow & \uparrow & & & \\
R_{n-1} & \cong T_{n-1,1} & \cong \ldots & T_{n-1,n-2} & \xrightarrow{Q_{n-1}} & T_{n-1,n-1} & \\
\end{array}
\]

where we set $\theta_i Q_i = \theta_i(Q_i)$ and we have $Q_i = T_{i,0} C_{Q_i}(T_{i,1})$ for any $i \in \Delta_{n-1}$.

Setting $T_{n,0} = R_n$, inductively define $T_{n,j+1} = \theta_j(T_{n,j})$ for any $j < n-1$, which makes sense since inductively we get $T_{n,j} \subset T_{j,j} \subset Q_j$; moreover, since we have

$$Q_{n-1} = T_{n-1,n-1} C_{Q_{n-1}}(T_{n-1,n-1}) \quad \text{and} \quad T_{n,n-1} \trianglelefteq T_{n-1,n-1} \quad 9.4.3,$$

the group $Q_{n-1,0} = Q_{n-1}$ contains and normalizes

$$Q_{n,0} = T_{n,n-1} C_{Q_{n-1,0}}(T_{n,n-1}) \quad 9.4.4,$$

and, in particular, $Q_{n,0}$ is selfcentralizing in $Q_{n-1,0}$; then, according to [5, Corollary 2.21], there is an $\mathcal{F}$-morphism $\theta_{n-1,0} : Q_{n-1,0} \to P$ such that $\theta_{n-1,0}(Q_{n-1,0})$ and $\theta_{n-1,0}(Q_{n,0})$ are both fully centralized in $\mathcal{F}$, and we set

$$Q_{n,1} = \theta_{n-1,0}(Q_{n,0}) \cdot N_{C_P(\theta_{n-1,0}(Q_{n,0}))}(\theta_{n-1,0}(Q_{n-1,0})) \quad 9.4.5;$$

once again, $Q_{n,1}$ is selfcentralizing in $Q_{n-1,1}$. 
On the other hand, we denote by $\tau_{i,0}$ the image of $\sigma$ in $F(R)$ and, for any $i \in \Delta_{n-1}$ and any $j < i$, we inductively denote by $\tau_{i,j+1}$ the image of $\tau_{i,j}$ in $F(T_{i,j+1})$ via $\theta_j$; from our conditions, it is easily checked that the action $\bar{\sigma}_{n-2}$ of $\sigma_{n-2}$ over $Q_{n-1}$ stabilizes $T_{n-1,n-1}$ and induces $\tau_{n-1,n-1}$ over this group, so that it stabilizes $T_{n,n-1}$ and then it stabilizes $Q_{n,0}$; thus, the action of $\sigma_{n-2}$ over $\theta_{n-1,0}(Q_{n-1,0})$ defined via $\theta_{n-1,0}$ stabilizes $\theta_{n-1,0}(Q_{n,0})$ and it follows from [5, statement 2.10] that this action can be extended to an $F$-morphism $Q_{n-1,1} \rightarrow P$; but, the image of the group

$$N_{C,\theta_{n-1,0}(Q_{n,0})}(\theta_{n-1,0}(Q_{n-1,0}))$$

clearly normalizes $\theta_{n-1,0}(Q_{n-1,0})$ and centralizes $\theta_{n-1,0}(Q_{n,0})$; hence, this $F$-morphism determines an $F$-automorphism $\sigma_{n-1,0}$ of $Q_{n-1,1}$ which stabilizes $\theta_{n-1,0}(Q_{n-1,0})$ and $Q_{n,1}$, and, since $\sigma_{n-2}$ is a $p'$-element, we may assume that $\sigma_{n-1,0}$ is also a $p'$-element.

Now, arguing by induction on $j \in \mathbb{N}$, assume that we have two families \{\$Q_{n-1,j'}(j' \leq j)\} \$ of subgroups of $P$ such that $Q_{n,j}$ is a normal and a selfcentralizing subgroup of $Q_{n-1,j'}$, a family of $F$-morphisms

$$\theta_{n-1,j'} : Q_{n-1,j'} \rightarrow P$$

where $j'$ runs over $\Delta_{j-1}$, such that

$$\theta_{n-1,j'}(Q_{n-1,j'}) \leq Q_{n-1,j'+1}$$

$$Q_{n-1,j'+1} = \theta_{n-1,j'}(Q_{n-1,j'}) \ast C_{Q_{n-1,j'+1}}(\theta_{n,j'}(Q_{n,j'}))$$

and a family of $p'$-elements $\sigma_{n-1,j'} \in F(Q_{n-1,j'+1})$, where $j'$ runs over $\Delta_{j-1}$, stabilizing $\theta_{n-1,j'}(Q_{n-1,j'})$ and $Q_{n,j'}$, and, if $j' \neq 0$, inducing $\sigma_{n-1,j'-1}$ on $Q_{n-1,j'}$ via $\theta_{n-1,j'}$; once again, the following diagram summarizes our situation

$$\begin{array}{cccccc}
Q_{\bar{n}} & \theta_{n,0}Q_{n,0} \cdot N_{n,0} & \theta_{n,1}Q_{n,1} \cdot N_{n,1} & \cdots & \theta_{n,j}Q_{n,j} \cdot N_{n,j} \\
\bar{\sigma}_{n-2}^{} & \downarrow \sigma_{n,1} & \downarrow \sigma_{n,j} & \downarrow & \downarrow \\
Q_{n,0} & \theta_{n,0} & \sigma_{n,1} & \sigma_{n,j} & \cdots & \sigma_{n,j} \\
\downarrow & \downarrow & \downarrow & \cdots & \downarrow \\
Q_{n,0} & \theta_{n,0} & \sigma_{n,1} & \cdots & \sigma_{n,j} \\
\uparrow & \uparrow & \uparrow & \cdots & \uparrow \\
T_{n,\bar{n}} \cdot C_{Q_{n}}(T_{n,\bar{n}}) & \theta_{n,0}Q_{n,0} \cdot N_{n,0} & \theta_{n,1}Q_{n,1} \cdot N_{n,1} & \cdots & \theta_{n,j}Q_{n,j} \cdot N_{n,j}
\end{array}$$

where we are setting $\bar{n} = n - 1$, $j = j - 1$ and

$$N_{n,j'} = N_{C,\theta_{n-1,j'-1}(Q_{n,j'-1})}(\theta_{n-1,0}(Q_{n-1,j'-1}))$$
As above, by [5, Corollary 2.21], there is an $\mathcal{F}$-morphism

$$ \theta_{n-1,j} : Q_{n-1,j} \rightarrow P $n.11

such that $\theta_{n-1,j}(Q_{n-1,j})$ and $\theta_{n-1,j}(Q_{n,j})$ are both fully centralized in $\mathcal{F}$; then, we set

$$ Q_{n,j+1} = \theta_{n-1,j}(Q_{n,j}) \cdot N_{C_P(\theta_{n-1,j}(Q_{n,j}))}(\theta_{n-1,j}(Q_{n-1,j})) $$n.12
$$ Q_{n-1,j+1} = \theta_{n-1,j}(Q_{n-1,j}) \cdot N_{C_P(\theta_{n-1,j}(Q_{n,j}))}(\theta_{n-1,j}(Q_{n-1,j})) $$n.13

and it is clear that $Q_{n,j+1}$ is again a normal and a selfcentralizing subgroup of $Q_{n-1,j+1}$; similarly, if $j \neq 0$, the action of $\sigma_{n-1,j-1}$ over $\theta_{n-1,j}(Q_{n-1,j})$ defined via $\theta_{n-1,j}$ stabilizes $\theta_{n-1,j}(Q_{n,j})$ and, again from [5, statement 2.10.1], this action can be extended to an $\mathcal{F}$-morphism $Q_{n-1,j+1} \rightarrow P$; but, the image of the group

$$ N_{C_P(\theta_{n-1,j}(Q_{n,j}))}(\theta_{n-1,j}(Q_{n-1,j})) $$n.14

clearly normalizes $\theta_{n-1,j}(Q_{n-1,j})$ and centralizes $\theta_{n-1,j}(Q_{n,j})$; hence, this $\mathcal{F}$-morphism determines an $\mathcal{F}$-automorphism $\sigma_{n-1,j}$ of $Q_{n-1,j+1}$ which stabilizes $\theta_{n-1,j}(Q_{n-1,j})$ and $Q_{n,j+1}$, and, since $\sigma_{n-1,j-1}$ is a $p'$-element, we may assume that $\sigma_{n-1,j}$ is also a $p'$-element.

Finally, if we have $Q_{n,j+1} = \theta_{n-1,j}(Q_{n,j})$ then we still have

$$ Q_{n-1,j+1} = \theta_{n-1,j}(Q_{n-1,j}) $$n.15
$$ = N_{\theta_{n-1,j}(Q_{n-1,j}) \cdot C_P(\theta_{n-1,j}(Q_{n,j}))}(\theta_{n-1,j}(Q_{n-1,j})) $$n.16

and therefore $\theta_{n-1,j}(Q_{n-1,j})$ contains $C_P(\theta_{n-1,j}(Q_{n,j}))$; but, since $Q_{n,j}$ is selfcentralizing in $Q_{n-1,j}$, $\theta_{n-1,j}(Q_{n,j})$ is selfcentralizing in $\theta_{n-1,j}(Q_{n-1,j})$ and thus $\theta_{n-1,j}(Q_{n,j})$ contains $C_P(\theta_{n-1,j}(Q_{n,j}))$, so that $Q_{n,j+1}$ is $\mathcal{F}$-selfcentralizing; in this situation, it suffices to consider $Q_n = Q_{n,j+1}$, to define

$$ \theta_{n-1} : Q_{n-1} \rightarrow P $$n.17

mapping $u = u_{n-1,0} \in Q_{n-1}$ on $u_{n-1,j+1}$ where we inductively set

$$ u_{n-1,j'+1} = \theta_{n-1,0}(u_{n-1,j'}) $$n.18

for any $j' \in \Delta_j$, and to denote by $\sigma_n$ the restriction of $\sigma_{n-1,j+1}$ to the product $\theta_{n-1}(Q_{n-1})$. Indeed, since $\theta_{n-1}(Q_{n-1})$ is contained in $Q_{n-1,j+1}$, this group normalizes $Q_n$; moreover, arguing by induction on $j' \in \Delta_j$, it is easily checked that $\sigma_{n-1,j+1}$ stabilizes $\theta_{n-1}(Q_{n-1})$ and that the action over $Q_{n-1}$ defined
via $\theta_{n-1}$ coincides with the action of $\sigma_{n-1}$; similarly, it is clear that for any $j' \in \Delta_j$ we have

$$Q_{n,j'+1} = \theta_{n-1,j'}(Q_{n,j'}) \cdot C_{Q_{n,j'+1}}(\theta_{n-1,j'}(Q_{n,j'}))$$ 9.4.17

and by induction we get $Q_{n} = \theta_{n-1}(Q_{n,0}) \cdot C_{Q_{n}}(\theta_{n-1}(Q_{n,0}))$; thus, setting $T_{n,n} = \theta_{n-1}(T_{n,n-1})$, it follows from equality 9.4.4 that

$$Q_{n} = T_{n,n} \cdot C_{Q_{n}}(T_{n,n})$$ 9.4.18.

We are done.

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