MULTIDIMENSIONAL CLASSICAL AND QUANTUM WORMHOLES IN MODELS WITH COSMOLOGICAL CONSTANT

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Abstract

A multidimensional cosmological model with space-time consisting of \( n(n \geq 2) \) Einstein spaces \( M_i \) is investigated in the presence of a cosmological constant \( \Lambda \) and a homogeneous minimally coupled scalar field \( \varphi(t) \) as a matter source. Classical and quantum wormhole solutions are obtained for \( \Lambda < 0 \) and all \( M_i \) being Ricci-flat. Classical wormhole solutions are also found for \( \Lambda < 0 \) and only one of the \( M_i \) being Ricci-flat for the case of spontaneous compactification of the internal dimensions with fine tuning of parameters.

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INTRODUCTION

In quantum cosmology instantons, solutions of the classical Einstein equations in Euclidean space, play an important role giving the main contributions to the path integral \([1]\). Among them classical wormholes are of special interest, because they are connected with processes changing the topology of the models \([2, 3]\). We remind that classical wormholes are usually Riemannian metrics consisting of two large regions joined by a narrow throat (handle). They exist for special types of matter \([2] - [4]\) and do not exist for pure gravity. In quantum cosmology it is generally assumed that on Planck scale processes with topology changes should take place. For this reason Hawking and Page \([5]\) introduced the notion of quantum wormholes as a quantum extension of the classical wormhole paradigm. They proposed to regard quantum wormholes as solutions of the Wheeler-DeWitt (WDW) equation with the following boundary conditions:

(i) the wave function is exponentially damped for large spatial geometry,

(ii) the wave function is regular when the spatial geometry degenerates.

The first condition expresses the fact that space-time should be Euclidean at spatial infinity. The second condition should reflect the fact that space-time is nonsingular when spatial geometry degenerates. For example, the wave function should not oscillate an infinite number of times.

The given approach extends the number of objects which can be treated as wormholes \([5, 6]\).

We believe that for the description of quantum gravitational processes at high energies the multidimensional approach is more adequate. Modern theories of unified physical interactions use ideas of hidden (or extra) dimensions. In order to study different phenomena at early stages of the universe one should use these theories or at any rate models keeping their main characteristics. But more reliable conclusions may be done only on the basis of exact solutions which are usually obtained in rather simple cases.

Therefore, at the beginning we consider a cosmological model with \(n\) (\(n > 1\)) Einstein spaces containing a massless minimally coupled scalar field and a cosmological constant \(\Lambda\). The gauge covariant form of the WDW equation was proposed in \([7]\). This model is integrable in the case with only one of the Einstein spaces being not Ricci-flat and vanishing cosmological constant. The general properties of this particular model were investigated in \([8]\) while classical as well as quantum wormhole solutions were found for different models.
in [9] - [12]. (In particular integrable cases with multicomponent perfect fluid were considered).

The present paper is devoted to the case of nonzero cosmological constant. For the model with one space of positive constant curvature in four dimensional space-time with cosmological constant and axionic matter (which is equivalent to a free minimally coupled scalar field, see, for example, [2, 3] and the paper by Brown et al. in [4]) classical wormhole solutions were obtained in [3, 13]. In the case of four dimensional space-time with nontrivial topology $\mathbb{R} \times S^1 \times S^2$ and non-zero cosmological constant this type of solution exists, too [15].

Here, we investigate the case with at least one of the spaces, say $M_1$ being Ricci-flat. If all the other spaces $M_i, i = 2, \ldots, n$ are also Ricci-flat this model is fully integrable in the classical [12] as well as in the quantum cases [7]. For $\Lambda < 0$ a family of quantum wormhole solutions with continuous and discrete spectra exist. Classical wormholes can be found in this case only in the presence of a scalar field. In the presence of a scalar field classical wormhole solutions exist also for another particular case with fine tuning of the parameters of the model, if $\Lambda < 0$ and all $M_i, i = 2, \ldots, n$ are not Ricci-flat and have the same sign of the curvature. In this case only $M_1$ has a dynamical behaviour and is considered as our external space. All the other internal spaces $M_i, i = 2, \ldots, n$ are freezeed with fixed scale factors $a(0)_i$ which are fine tuned to values determined by the cosmological constant. This type of solutions belongs to the class of models with spontaneous compactification. In the case of models without a cosmological constant and with only one non-Ricci-flat factor space solutions with spontaneous compactification were also found in [16].

We would like to note that solutions of the WDW equation in four dimensional models with $\Lambda \neq 0$ and with a conformal scalar field were first obtained in [17] and [18] respectively (see also [19]). They include possibly the first quantum wormhole type solutions in four dimensions as well as DeWitt’s solution for the Friedman universe with dust (1967). Vacuum quantum cosmological solutions in four dimensions may be found in [20]. The path integral approach to quantum cosmology [1] for models with cosmological constant in four and five dimensions with nontrivial topology was developed in [21].

The paper is organized as follows. In section 2 the general description of the models considered is given. In section 3 classical and quantum wormholes are obtained for all spaces being Ricci-flat. In section 4 classical wormholes are considered in the model with spontaneous compactification of extra dimensions. Conclusions and an extensive list of
2 GENERAL DESCRIPTION
OF THE MODEL

The metric of the model
\[ g = -\exp[2\gamma(\tau)]d\tau \otimes d\tau + \sum_{i=1}^{n} \exp[2\beta^i(\tau)]g^{(i)}, \quad (2.1) \]
is defined on the manifold
\[ M = \mathbb{R} \times M_1 \times \ldots \times M_n, \quad (2.2) \]
where the manifold \( M_i \) with the metric \( g^{(i)} \) is an Einstein space of dimension \( d_i \), i.e.
\[ R_{m,n}(g^{(i)}) = \lambda^i g^{(i)}, \quad (2.3) \]
i = 1, \ldots, n; \ n \geq 2. The total dimension of the space-time \( M \) is \( D = 1 + \sum_{i=1}^{n} d_i \).

Here we investigate the general model with cosmological constant \( \Lambda \) and a homogeneous minimally coupled field \( \varphi(t) \) with a potential \( U(\varphi) \).

The action of the model is adopted in the following form
\[ S = \frac{1}{2} \int d^D x \sqrt{|g|} \{ R[g] - \partial_M \varphi \partial_N \varphi g^{MN} - 2U(\varphi) - 2\Lambda \} + S_{GH}, \quad (2.4) \]
where \( R[g] \) is the scalar curvature of the metric \( g = g_{MN}dx^M \otimes dx^N \) and \( S_{GH} \) is the standard Gibbons-Hawking boundary term \([22]\). The field equations, corresponding to the action \((2.4)\), for the cosmological metric \((2.1)\) in the harmonic time gauge \( \gamma \equiv \sum_{i=1}^{n} d_i \beta^i \) are equivalent to the Lagrange equations, corresponding to the Lagrangian
\[ L = \frac{1}{2} \sum_{i,j=1}^{n} (G_{ij} \dot{\beta}^i \dot{\beta}^j + \varphi^2) - V, \quad (2.5) \]
with the energy constraint imposed
\[ E = \frac{1}{2} \sum_{i,j=1}^{n} (G_{ij} \dot{\beta}^i \dot{\beta}^j + \varphi^2) + V = 0. \quad (2.6) \]
Here, the overdot denotes differentiation with respect to the harmonic time \( \tau \). The components of the minisuperspace metric read
\[ G_{ij} = d_i \delta_{ij} - d_i d_j. \quad (2.7) \]
and the potential is given by

\[ V = V(\beta, \varphi) = \exp(2 \sum_{i=1}^{n} d_i \beta^i) \left[ \frac{1}{2} \sum_{j=1}^{n} \theta_j e^{-2\beta_j} + U(\varphi) + \Lambda \right], \tag{2.8} \]

where \( \theta_i = \lambda_i d_i \). If the \( M_i \) are spaces of constant curvature, then \( \theta_i \) may be normalized in such a way that \( \theta_i = k_i d_i (d_i - 1) \), \( k_i = \pm 1, 0 \). We may also consider the generalization of the model with the potential (2.8) modified by the substitution

\[ U(\varphi) \mapsto \tilde{U}(\varphi, \beta). \tag{2.9} \]

This gives us the possibility to investigate models with an arbitrary scalar field potential \( \tilde{U}(\varphi, \beta) \equiv U(\varphi) \) as well as (for \( \varphi = \text{const} \)) models with an arbitrary potential \( \tilde{U}(\varphi, \beta) \equiv U(\beta) \). Effective potentials of the form \( U(\beta) \) may have their origin in an ideal fluid matter source. In special cases the general form \( \tilde{U}(\varphi, \beta) \) of the potential leads us to new integrable models. An example of this kind of potential will be presented.

With the general potential \( \tilde{U}(\varphi, \beta) \) the equations of motion are

\[ -d_i \ddot{\beta}^i + d_i \sum_{k=1}^{n} d_k \ddot{\beta}^k + e^2 \sum_{k=1}^{n} d_k \dot{\beta}^k \left[ (d_i - 1) \theta_i e^{-2\beta_i} + d_i \sum_{k \neq i} \theta_k e^{-2\beta_k} \right] \]

\[ -\frac{\partial \tilde{U}}{\partial \beta^i} - 2d_i \ddot{\varphi} - 2d_i \Lambda \right] = 0, \quad i = 1, \ldots, n, \tag{2.10} \]

\[ \ddot{\varphi} + \frac{\partial \tilde{U}}{\partial \varphi} \exp \left( 2 \sum_{i=1}^{n} d_i \beta^i \right) = 0. \tag{2.11} \]

with the constraint (2.6).

At the quantum level the constraint (2.6) is modified into the WDW equation (see [7])

\[ \left[ \frac{1}{2} \left( G^{ij} \frac{\partial}{\partial \beta^i} \frac{\partial}{\partial \beta^j} + \frac{\partial^2}{\partial \varphi^2} \right) - V(\beta, \varphi) \right] \Psi(\beta, \varphi) = 0, \tag{2.12} \]

where \( \Psi = \Psi(\beta, \varphi) \) is the wave function of the universe, \( V \) is the potential (2.8) and

\[ G^{ij} = \delta^{ij} + \frac{1}{2 - D} \tag{2.13} \]

are the components of the matrix inverse to the matrix \( G_{ij} \) (2.7). The minisuperspace metric \( G = G_{ij} dx^i \otimes dx^j \) (2.7) was diagonalized in [8, 23]

\[ G = -dz^0 \otimes dz^0 + \sum_{i=1}^{n-1} dz^i \otimes dz^i, \tag{2.14} \]
where

\[ z^0 = q^{-1} \sum_{j=1}^{n} d_j \beta^j, \]

\[ z^i = \left[d_i/\Sigma_i \Sigma_{i+1}\right]^{1/2} \sum_{j=i+1}^{n} d_j (\beta^j - \beta^i), \quad (2.15) \]

\[ i = 1, \ldots, n - 1, \] where

\[ q = [((D-1)/(D-2))]^{1/2}, \quad \Sigma_i = \sum_{j=i}^{n} d_j. \quad (2.16) \]

The WDW equation (2.12) takes in variables (2.15) the following form

\[ \left[ -\frac{\partial}{\partial z^0} \frac{\partial}{\partial z^0} + \sum_{i=1}^{n-1} \frac{\partial}{\partial z^i} \frac{\partial}{\partial z^i} + \frac{\partial^2}{\partial \varphi^2} - 2V(z, \varphi) \right] \Psi = 0. \quad (2.17) \]

3 WORMHOLES FOR RICCI-FLAT SPACES

In this chapter we consider the Ricci-flat case \((\theta_i = \lambda_i d^i = 0, \ i = 1, \ldots, n)\), with \(\Lambda \neq 0\) and \(U(\varphi) = 0\). If the \(M_i\) are internal spaces they should be compact. This compactness is a necessary condition also for the Hartle-Hawking boundary condition (see below). The compactness of Ricci-flat spaces may be achieved by appropriate boundary conditions. The d-dimensional tore is the simplest example.

3.1 Classical solutions

In the considered case the Lagrangian (2.3) may be written in the following form

\[ L = \frac{1}{2} \sum_{I,J=1}^{n+1} G_{IJ} \dot{\beta}^I \dot{\beta}^J - \Lambda \exp\left(\sum_{I=1}^{n+1} u_I \beta^I\right), \quad (3.1) \]

where \(\beta^{n+1} = \varphi, \ u_i = 2d_i, \ i = 1, \ldots, n, \ u_{n+1} = 0\) and

\[ (G_{IJ}) = \begin{pmatrix} G_{ij} & 0 \\ 0 & 1 \end{pmatrix} \quad (3.2) \]
the matrix \( G_{ij} \) is defined in eq. (2.7)). We consider the coordinates \( z^A = (z^a, z^n = \beta^{n+1} = \varphi) \), where \( z^a, a = 0, \ldots, n - 1 \), are defined in (2.13). It is clear that

\[
z^A = \sum_{I=1}^{n+1} V^A_I \beta^I, \tag{3.3}
\]

\( A = 0, \ldots, n \), where

\[
(V^A_I) = \begin{pmatrix}
V^a_i & 0 \\
0 & 1
\end{pmatrix} \tag{3.4}
\]

and the matrix \( (V^a_i) \) is defined in (2.15). This introduced matrix diagonalizes the minisuperspace metric

\[
G_{IJ} = \sum_{A,B=0}^n \eta_{AB} V^A_I V^B_J, \tag{3.5}
\]

\( I, J = 1, \ldots, n + 1 \) and \( (\eta_{AB}) = (\eta^{AB}) = \text{diag}(-1, +1, \ldots, +1) \).

In the coordinates (3.3) the Lagrangian (3.1) reads (\( q \) is defined in (2.16))

\[
L = \frac{1}{2} \sum_{A,B=0}^n \eta_{AB} \dot{z}^A \dot{z}^B - \Lambda \exp(2qz^0). \tag{3.6}
\]

The Lagrange equations for the Lagrangian (3.6)

\[
-\ddot{z}^0 + 2q\Lambda \exp(2qz^0) = 0, \tag{3.7}
\]

\[
\ddot{z}^A = 0, \quad A = 1, \ldots, n, \tag{3.8}
\]

with the energy constraint

\[
E = \frac{1}{2} \sum_{A,B=0}^n \eta_{AB} \dot{z}^A \dot{z}^B + \Lambda \exp(2qz^0) = 0 \tag{3.9}
\]

can be readily solved. First integrals of (3.8) are

\[
\dot{z}^A = p^A, \quad A = 1, \ldots, n, \tag{3.10}
\]

where \( p^A \) are arbitrary constants of integration. Then the constraint (3.9) may be rewritten

\[
-\frac{1}{2} (\dot{z}^0)^2 + \mathcal{E} + \Lambda e^{2qz^0} = 0 \tag{3.11}
\]

with

\[
2\mathcal{E} = \sum_{A=1}^n (p^A)^2. \tag{3.12}
\]
We obtain the following solution

\[ z^A = p^A \tau + q^A, \quad A = 1, \ldots, n, \]  

(3.13)

where \( p^A \) and \( q^A \) are constants and

\[ 2qz^0 = \ln[\mathcal{E} / \{\Lambda \sinh^2(q\sqrt{2\mathcal{E}}(\tau - \tau_0))\}], \quad \mathcal{E} \neq 0, \quad \Lambda > 0, \]  

(3.14)

\[ = \ln[1 / \{2q^2 \Lambda(\tau - \tau_0)^2\}], \quad \mathcal{E} = 0, \quad \Lambda > 0, \]  

(3.15)

\[ = \ln[-\mathcal{E} / \{\Lambda \cosh^2(q\sqrt{2\mathcal{E}}(\tau - \tau_0))\}], \quad \mathcal{E} > 0, \quad \Lambda < 0, \]  

(3.16)

Here \( \tau_0 \) is an arbitrary constant.

### 3.2 Kasner-like parametrization

First we consider the case \( \mathcal{E} > 0 \). In this case the relations (3.14) and (3.16) may be written in the following form

\[ 2qz^0 = \ln[\mathcal{E} / \{\Lambda f_\delta(q\sqrt{2\mathcal{E}}(\tau - \tau_0))\}], \]  

(3.17)

where \( \delta \equiv \Lambda / |\Lambda| = \pm 1 \) and

\[ f_\delta(x) \equiv \frac{1}{2}(e^x - \delta e^{-x}) = \sinh x, \quad \delta = +1, \]  

\[ = \cosh x, \quad \delta = -1. \]  

(3.18)

We introduce a new time variable by the relation

\[ t = \frac{T}{\sqrt{\delta}} \ln \frac{\exp(q\sqrt{2\mathcal{E}}(\tau - \tau_0)) + \sqrt{\delta}}{\exp(q\sqrt{2\mathcal{E}}(\tau - \tau_0)) - \sqrt{\delta}}, \]  

(3.19)

where

\[ T \equiv [(D - 2) / 2|\Lambda|(D - 1)]^{1/2}. \]  

(3.20)

It is not difficult to verify that the following relations take place

\[ \sinh(t\sqrt{\delta}/T) / \sqrt{\delta} = 1 / f_\delta(q\sqrt{2\mathcal{E}}(\tau - \tau_0)), \]  

(3.21)

\[ \tanh(t\sqrt{\delta}/2T) / \sqrt{\delta} = \exp(-q\sqrt{2\mathcal{E}}(\tau - \tau_0)), \]  

(3.22)

\[ dt = -Tq\sqrt{2\mathcal{E}}d\tau / f_\delta(q\sqrt{2\mathcal{E}}(\tau - \tau_0)). \]  

(3.23)

Now, we introduce the following dimensionless parameters

\[ \bar{\alpha}^I \equiv -\sum_{A=1}^{n} V^I_A p^A / q\sqrt{2\mathcal{E}}, \]  

(3.24)
where \((\bar{V}_i^I) = (V_i^A)^{-1}\) (see (3.3)). It is clear that
\[
(\bar{V}_i^I) = \left(\begin{array}{cc}
\bar{V}_i^a & 0 \\
0 & 1
\end{array}\right)
\] (3.25)
where \((\bar{V}_i^a) = (V_i^a)^{-1}\). The relation (3.24) is equivalent to the following relations
\[
\bar{\alpha}^i = -\sum_{a=1}^{n-1} \bar{V}_a^i p^a / q \sqrt{2 \mathcal{E}}, \quad i = 1, \ldots, n,
\]
\[
\bar{\alpha}^{n+1} = -p^n / q \sqrt{2 \mathcal{E}}. \tag{3.27}
\]
It follows from eq. (3.24), that
\[
\bar{V}_i^I = \sum_{J=1}^{n+1} \sum_{B=0}^{n} G^{IJ} V_J^B \eta_{BA}
\] (3.28)
where
\[
(G^{IJ}) \equiv (G_{IJ})^{-1} = \left(\begin{array}{cc}
G^{ij} & 0 \\
0 & 1
\end{array}\right)
\] (3.29)
From (2.13) and (3.28) we have
\[
\bar{V}_0^i = -G^{ij} u_j / 2q = (q(D - 2))^{-1}; \quad \bar{V}_0^{n+1} = 0. \tag{3.30}
\]
Using the relations (3.13), (3.17), (3.21)-(3.23), (3.24) and (3.30) we get the following expressions for the solution of field equations
\[
g = -dt \otimes dt + \sum_{i=1}^{n} a_i^2(t) g_{(i)}, \tag{3.31}
\]
\[
a_i(t) = \exp(\beta^i(t)) = A_i [\sinh(rt/T)/r]^\sigma [\tanh(rt/2T)/r]^\alpha^i, \tag{3.32}
\]
\[
\exp(\varphi(t)) = \exp(\beta^{n+1}(t)) = A_{n+1} [\tanh(rt/2T)/r]^{\alpha^{n+1}}, \tag{3.33}
\]
where \(t > 0, r = \sqrt{\delta} = \sqrt{|\Lambda|/|\Lambda|} = \sqrt{\pm 1}, \sigma = (D - 1)^{-1}, A_i \neq 0\) are constants, \(i = 1, \ldots, n\), and the parameters \(\alpha^i\) satisfy the relations
\[
\frac{1}{2} \sum_{I=1}^{n+1} u_I \bar{\alpha}^I = \sum_{i=1}^{n} d_i \bar{\alpha}^i = 0, \tag{3.34}
\]
\[
\sum_{I,J=1}^{n+1} G_{IJ} \bar{\alpha}^I \bar{\alpha}^J = \sum_{i=1}^{n} d_i (\bar{\alpha}^i)^2 + (\bar{\alpha}^{n+1})^2 = (D - 2)/(D - 1). \tag{3.35}
\]
The first relation (3.34) can be easily proved, using the definition (3.24) and the following identity (we remind that due to (2.15) and (3.3) \( u_I = 2qV_I^0 \))

\[
\sum_{i=1}^{n+1} u_I V_A^I = 2q \sum_{i=1}^{n+1} V_I^0 V_A^I = 2q \delta_A^0 = 0
\]  

(3.36)

for \( A > 0 \). The relation (3.35) follows immediately from (2.7), (3.5), (3.12), (3.24) and (3.34).

Now we consider the case \( \mathcal{E} = 0 \). In this case for \( \Lambda > 0 \) there exist also an exceptional solution with the following scale factors in (3.31)

\[ a_i(t) = \tilde{A}_i \exp(±\sigma t/T), \]  

(3.37)

\( i = 1, \ldots, n \), and \( \varphi(t) = \text{const.} \). (This solution can be readily obtained using the formulas (3.13) and (3.15).)

It is interesting to note that for \( \Lambda > 0 \) the solution (3.37) with the sign "+" is an attractor for the solutions (3.32), i.e.

\[ a_i(t) \sim \tilde{A}_i \exp(\sigma t/T), \quad i = 1, \ldots, n, \]  

(3.38)

and \( \varphi(t) \sim \text{const} \) for \( t \to +\infty \). The relation (3.38) is the isotropization condition. We note that the solution (3.31)-(3.35) with \( \tilde{\alpha}^{n+1} = 0 \) was considered previously in [26]. The special case of this solution with \( n = 2 \) was considered earlier in [27].

The volume scale factor corresponding to (3.32) has the form

\[ v = \prod_{i=1}^{n} a_i^{d_i} = \left( \prod_{i=1}^{n} A_i^{d_i} \right) \sinh(rt/T)/r \]  

(3.39)

It oscillates for negative value of cosmological constant and exponentially increases as \( t \to +\infty \) for positive value. For positive \( A_i \) and \( \mathcal{E} \) the following identity takes place

\[ \prod_{i=1}^{n} A_i^{d_i} = \sqrt{\mathcal{E}/|\Lambda|}. \]  

(3.40)

For small time values we have the following asymptotical relations

\[ a_i(t) \sim c_i t^{\alpha_i}, \quad \exp \varphi(t) \sim c_{n+1} t^{\alpha_{n+1}} \]  

(3.41)

as \( t \to 0, i = 1, \ldots, n \), where

\[ \alpha_i = \tilde{\alpha}_i + \sigma, \quad \alpha_{n+1} = \tilde{\alpha}^{n+1}, \]  

(3.42)
are Kasner-like parameters, satisfying (see (3.34), (3.35)) the relations
\[ \sum_{i=1}^{n} d_{i} \alpha_{i} = \sum_{i=1}^{n} d_{i} \alpha_{i} + \alpha_{n+1}^2 = 1. \] (3.43)

The behaviour of the scale factors near the singularity coincides with that for the case \( \Lambda = 0 \) [28, 29]. For \( \alpha_{n+1} = 0 \) see also [12].

We note also that in terms of \( \alpha_{i} \)-parameters the solution (3.31) - (3.35) reads
\[ a_{i}(t) = \bar{A}_{i} \left[ \frac{\sinh(rt/2T)}{r} \right]^{\alpha_{i} \pm}, \] (3.44)
\[ \exp(\varphi(t)) = A_{n+1} \left[ \frac{\tanh(rt/2T)}{r} \right]^{\alpha_{n+1}}, \] (3.45)

Let us apply these solutions to the case of two spaces \( (n = 2) \). From (3.43) and (3.44) we find for the non-exceptional solutions
\[ a_{1}^{\pm} = A_{1} \left[ \frac{\sinh(rt/2T)}{r} \right]^{\alpha_{1} \pm} \left[ \cosh(rt/2T) \right]^{2\sigma - \alpha_{1}}, \] (3.46)
\[ a_{2}^{\pm} = A_{2} \left[ \frac{\sinh(rt/2T)}{r} \right]^{\alpha_{2} \pm} \left[ \cosh(rt/2T) \right]^{2\sigma - \alpha_{1}}, \] (3.47)

where
\[ \alpha_{1}^{\pm} = \frac{d_{1} \pm \sqrt{R}}{d_{1}(d_{1} + d_{2})}, \quad \alpha_{2}^{\pm} = \frac{d_{2} \pm \sqrt{R}}{d_{2}(d_{1} + d_{2})}, \] (3.48)

and
\[ R = d_{1}d_{2}[(d_{1} + d_{2})(1 - \alpha_{3}^{2}) - 1]. \] (3.49)

Graphically these solutions are presented in Figs. 1-2.

In the Euclidean case after the Wick rotation \((t \to -it)\) we get the following instanton solutions
\[ g = dt \otimes dt + \sum_{i=1}^{n} a_{i}^{2}(t) g_{(i)}, \] (3.50)
\[ a_{i}(t) = \bar{A}_{i} \left[ \frac{\sinh(ts/2T)}{s} \right]^{\alpha_{i} \pm} \left[ \cosh(ts/2T) \right]^{2\sigma - \alpha_{i}}, \] (3.51)
\[ \exp(\varphi(t)) = \bar{A}_{n+1} \left[ \frac{\tanh(ts/2T)}{s} \right]^{\alpha_{n+1}}. \] (3.52)

where \( T \) is defined by (3.20), \( s = \sqrt{-\lambda/|\lambda|} \) and the parameters \( \alpha_{i} \) satisfy the relations (3.43). For \( \Lambda < 0 \) we have the special solution \((E = 0)\)
\[ a_{i}(t) = \bar{A}_{i} \exp(\pm \sigma t/T). \] (3.53)

We note that for the Euclidean case the scale factors may be obtained from the corresponding Lorentzian ones by the substitution \( \Lambda \to -\Lambda \). For \( n = 3, \ d_{1} = d_{2} = d_{3} = 1, \ \Lambda > 0 \) the
special case of the solution (3.50) - (3.52) was considered in [30]. We note that for \( \Lambda < 0 \) there are wormhole-like sections of the total metric (3.50). This takes place, for example, if \( n = 2, \alpha_3^2 \leq 1 - d_2^{-1}, 1 < d_1 < d_2 \), (see Fig. 2). In this case the scalar field is real in Euclidean region.

Now, we consider the solutions of the field equations with complex scalar field and real metric. In this case \( \mathcal{E}, p^1, \ldots, p^{n-1} \) are real and hence (see (3.12)) \( p^n \) is either real or pure imaginary. The case of real \( p^n \) was considered above.

For pure imaginary \( p_n \) we have three subcases: a) \( \mathcal{E} > 0 \), b) \( \mathcal{E} = 0 \), c) \( \mathcal{E} < 0 \). In the first case a) \( \mathcal{E} > 0 \) after the reparametrization (3.19), (3.20) we get the solutions (3.32)-(3.35) with an imaginary value of \( \bar{\alpha}^{n+1} \). The cases b) and c): \( \mathcal{E} \leq 0 \) take place only for \( \Lambda > 0 \).

Let us consider the case c) \( \mathcal{E} < 0 \). Here, we have (see (3.26), (3.27)) imaginary \( \bar{\alpha}_k \):

\[
\bar{\alpha}^k = i \sigma_k, \quad k = 1, \ldots, n, \quad \bar{\alpha}^{n+1} = \sigma_{n+1}.
\]

The solution may be obtained from (3.32)-(3.35) substituting (3.54) and \( t/T \mapsto t/T + i\frac{\pi}{2} \):

\[
g = -dt \otimes dt + \sum_{i=1}^{n} a_i^2(t) g_{(i)}, \quad (3.55)
\]
\[
a_i(t) = \hat{A}_i [\cosh(t/T)]^{\sigma_i} [f(t/2T)]^{\sigma_i}, \quad (3.56)
\]
\[
\varphi(t) = c + 2i\sigma_{n+1} \arctan e^{-t/T}, \quad (3.57)
\]

where \( c, \hat{A}_i \neq 0 \) are constants, \( i = 1, \ldots, n, \sigma = (D - 1)^{-1}, T \) is defined in (3.20), \( \Lambda > 0 \) and the real parameters \( \sigma_i \) satisfy the relations

\[
\sum_{i=1}^{n} d_i \sigma_i = 0, \quad (3.58)
\]
\[
-\sum_{i=1}^{n} d_i \sigma_i^2 + \sigma_{n+1}^2 = (D - 2)/(D - 1). \quad (3.59)
\]

Here

\[
f(x) \equiv [\tanh(x + i\frac{\pi}{4})]^i = \exp(-2 \arctan e^{-2x}) \quad (3.60)
\]
is smooth monotonically increasing function bounded by its asymptotics:

\[ e^{-\pi} < f(x) < 1; \quad f(x) \to 1 \text{ as } x \to +\infty \text{ and } f(x) \to e^{-\pi} \text{ as } x \to -\infty \] (see Fig. 3).

The solution (3.55)-(3.59) may be also obtained from formulas (3.13), (3.14). The relation between the harmonic and the proper times (3.21) is modified for our case \( \mathcal{E} < 0 \)

\[
\cosh(t/T) = 1/\sin(q\sqrt{2|\mathcal{E}|}(\tau - \tau_0)). \quad (3.61)
\]
For the volume scale factor we have

\[ v = \prod_{i=1}^{n} a_i^{d_i} = (\prod_{i=1}^{n} \hat{A}_i^{d_i}) \cosh(t/T). \tag{3.62} \]

The scalar field varies \( \varphi(t) \) varies from \( c + i\pi \sigma_{n+1} \) to \( c \) as \( t \) varies from \( -\infty \) to \( +\infty \). The solution (3.55)-(3.59) is non-singular for \( t \in (-\infty, +\infty) \). Any scale factor \( a_i(t) \) has a minimum for some \( t_{0i} \) and

\[ a_i(t) \sim \hat{A}_i^{\pm} \exp(\sigma|t|/T), \tag{3.63} \]

for \( t \to \pm\infty \).

The Lorentzian solutions considered above have also Euclidean analogues for \( \Lambda < 0 \)

\[ g = dt \otimes dt + \sum_{i=1}^{n} a_i^2(t)g_{(i)}, \tag{3.64} \]

\[ a_i(t) = \hat{A}_i[\cosh(t/T)]^{\sigma_i} [f(t/2T)]^{\sigma_i}, \tag{3.65} \]

\[ \varphi(t) = c + 2i\sigma_{n+1} \arctan e^{-t/T}, \tag{3.66} \]

with the parameters \( \sigma_I \) satisfying the relations (3.58)-(3.59). This solution may be interpreted as classical Euclidean wormhole solution. An interesting special case of solution (3.64)-(3.66) occurs for \( \sigma_i = 0, i = 1, \ldots, n \), (this corresponds to \( p^i = 0 \))

\[ a_i(t) = \hat{A}_i[\cosh(t/T)]^{\sigma}, \tag{3.67} \]

\[ \varphi(t) = c \pm 2i\sigma^{-1} \arctan e^{-t/T}. \tag{3.68} \]

All scale factors (3.67) have a minimum at \( t = 0 \) and are symmetric with respect to time inversion: \( t \mapsto -t \). We want to stress here that wormhole solutions take place only in the presence of an imaginary scalar field in the Euclidean region. Analytic continuation of the solutions (3.67), (3.68) into the Lorentzian region leads to real geometry and real scalar field there.

### 3.3 Quantum wormholes

The model introduced above leads to the WDW equation (2.17)

\[ -2\hat{H}\Psi \equiv \left[ -\frac{\partial}{\partial z^0} \frac{\partial}{\partial z^0} + \sum_{i=1}^{n} \frac{\partial}{\partial z^i} \frac{\partial}{\partial z^i} - 2\Lambda \exp(2qz^0) \right] \Psi = 0. \tag{3.69} \]
We are seeking the solution of (3.69) in the form
\[ \Psi(z) = \exp(i\vec{p}\vec{z})\Phi(z^0), \] (3.70)
where \(\vec{p} = (p^1, \ldots, p^n)\) is a constant vector (generally from \(\mathbb{C}^n\)), \(\vec{z} = (z^1, \ldots, z^{n-1}, z^n = \varphi)\), \(\vec{p}\vec{z} \equiv \sum_{i=1}^{n} p_i z^i\) and \(p_i = \sum_{j=1}^{n} \eta_{ij} p^j = p^i\). The substitution of (3.70) into (3.69) gives
\[ \left[-\frac{1}{2}(\frac{\partial}{\partial z^0})^2 + V_0(z^0)\right]\Phi = E\Phi, \] (3.71)
where \(E = \frac{1}{2}\vec{p}\vec{p}\) and \(V_0(z^0) = -\Lambda e^{2qz^0}\). The potential \(V_0(z^0)\) is plotted on fig. 4 and fig. 5 for \(\Lambda > 0\) and \(\Lambda < 0\) respectively. The classically allowed (Lorentzian) and forbidden (Euclidean) regions are shown there with respect to the energy levels \(E\). Solving (3.71), we get
\[ \Phi(z^0) = B_i \sqrt{\frac{2}{E}} e^{i\sqrt{\frac{2}{E}} e^{qz^0}}, \] (3.72)
where \(i\sqrt{\frac{2E}{q}} = i|\vec{p}|/q\), and \(B = I, K\) are modified Bessel functions. We note, that
\[ v = \exp qz^0 = \prod_{i=1}^{n} a_i^{d_i} \] (3.73)
is proportional to the spatial volume of the universe.

The general solution of Eq. (3.69) has the following form
\[ \Psi(z) = \sum_{B=I,K} \int d^n\vec{p} \ C_B(\vec{p}) e^{i\vec{p}\vec{z}} B_i|\vec{p}|/q(\sqrt{-2\Lambda q^{-1}} e^{qz^0}), \] (3.74)
where functions \(C_B\) \((B = I, K)\) belong to an appropriate class. Similar solutions were found for the two-component model \((n = 2)\) and \(\Lambda > 0\) in [31].

The solutions (3.74) are the eigenstates of the quantum-mechanical operators \(\hat{H}_z = -(i/N)\partial/\partial z^i, i = 1, \ldots, n\) with the eigenvalues \((1/N)p_i\) where \(N = 1\) for the Lorentzian space-time region and \(N = i\) for the Euclidean one.

Due to the well known time problem in quantum cosmology the WDW equation is not really the Schrödinger equation. There is no generally accepted procedure to overcome this problem but for our particular model we can introduce some time coordinate into the quantum equations in analogy to [8].

We split the WDW operator \(\hat{H}\) (3.69) into two parts
\[ \hat{H} = -\hat{H}_0 + \hat{H}_1, \] (3.75)
where
\[ \hat{H}_0 = -\frac{1}{2} \frac{\partial^2}{\partial z_0^2} - \Lambda e^{2qz^0}, \] (3.76)
and
\[ \hat{H}_1 = -\frac{1}{2} \sum_{i=1}^{n} \frac{\partial^2}{\partial z_i^2}. \] (3.77)

Then the WDW equation (3.69) becomes
\[ \hat{H}_0 \Psi = \hat{H}_1 \Psi. \] (3.78)

Applying \( \hat{H}_1 \) to the wave function (3.70) one gets
\[ \hat{H}_1 \Psi = \mathcal{E} \Psi. \] (3.79)

Now, we take \( \mathcal{E} \) to be real. Then, equation (3.79) shows that \( \mathcal{E} \) can be treated as the energy of the subsystem \( \hat{H}_1 \) and equation (3.79) becomes the Schrödinger equation. From this point of view \( \Psi \) gives the stationary states of the subsystem described by the wave equation (in Lorentzian region)
\[ i \frac{\partial \tilde{\Psi}}{\partial \tau} = \hat{H}_1 \tilde{\Psi}, \] (3.80)
where
\[ \tilde{\Psi} = e^{-i\mathcal{E} \tau} \Psi. \] (3.81)

It can be easily seen that the wave equation
\[ i \frac{\partial \tilde{\Psi}}{\partial \tau} = \hat{H}_0 \tilde{\Psi} \] (3.82)
is reduced to equation (3.71).

In the semiclassical limit for the wave function (3.81) equations (3.80) and (3.82) are reduced to the classical equations (3.10), (3.11). Indeed, the wave function (3.81) can be rewritten in the form
\[ \tilde{\Psi} = e^{-i\mathcal{E} \tau} e^{iS_1} \Phi(z^0) \] (3.83)
with \( S_1 = \sum_{i=1}^{n} p_i z^i \). In the semiclassical limit the wave function \( \Phi(z^0) \) takes the form
\[ \Phi(z^0) = C(z^0) e^{iS_0} \] (3.84)
with \( C(z^0) \) being a slowly varying function and \( S_0(z^0) \) being a rapidly varying phase. Time is defined in the semiclassical limit as an affine parameter along integral curves
\[ \frac{\partial}{\partial \tau} = \sum_{i=0}^{n} \frac{\partial(S_0 + S_1)}{\partial z_i} \frac{\partial}{\partial z^i} \] (3.85)
where  \( z_0 = -z^0, \ z_i = z^i, \ i = 1, \ldots, n \). As result we find the equations  \( \dot{z}^i = p^i, \ i = 1, \ldots, n \), and these coincide with the classical equations (3.10). For this reason we used the same notation for the constants of integration in (3.10) and for the momenta in the wave function (3.70). The velocity along  \( z^0 \) is found to be  \( \dot{z}_0 = -\dot{z}^0 = \partial S_0/\partial z^0 \). Using this relation and putting the wave function (3.83), (3.84) into equation (3.82) we reproduce the classical equation (3.11).

As shown above the parameter  \( \mathcal{E} \) can be interpreted as energy. So we may treat the state  \( \mathcal{E} = 0 \) as the ground state of the system. The demand of reality of the geometry leads to real momenta  \( p^i \ (i = 1, \ldots, n - 1) \) in the Lorentzian region. The scalar field can be real or imaginary there. In the ground state we put all momenta  \( p^i \ (i = 1, \ldots, n) \) equal to zero and the ground state wave function reads

\[
\Psi_0 = B_0 \left( q^{-1} \sqrt{-2\Lambda e^{qz^0}} \right).
\]

(3.86)

It is interesting to note that  \( \Psi_0 \) is invariant with respect to the rotation group O(n) in the space of vectors  \( \vec{z} = (z^1, \ldots, z^n) \).

In eq. (3.86)  \( B_0 \) denotes the Bessel functions of order zero. A particular solution may be specified by boundary conditions. For example, quantum wormhole boundary conditions were presented in the Introduction. Among the different types of boundary conditions for wave functions describing the universe the most popular is the Hartle-Hawking (HH) boundary condition [1]. According to the HH proposal the ground-state wave function of the universe  \( \Psi_0^{(HH)} \) is given by a path integral over all compact Euclidean geometries and the regular matter fields:

\[
\Psi_0^{(HH)} = \int d[g] d[\varphi] e^{-I_E},
\]

(3.87)

where  \( I_E \) is the Euclidean action. For our model the Euclidean action in harmonic time gauge and in z-coordinates reads

\[
I_E = \frac{1}{2} \int_{\tau^*}^{\tau} \, d\tau \left[ -(\dot{z}^0)^2 + \sum_{i=1}^{n} (\dot{z}^i)^2 + 2\Lambda e^{2qz^0} \right] - \frac{1}{2 \nu} \left. \dot{v} \right|_{\tau^*}.
\]

(3.88)

where  \( \nu \) denotes the spacial volume (3.73) up to a numerical factor. The upper limit  \( \tau \) corresponds to the boundary of the D-dimensional manifold, where the  \( z^i \ (i = 0, \ldots, n) \) have values indicated by the arguments of  \( \Psi_0^{(HH)} \). The lower limit  \( \tau^* \) corresponds to the point where the D-dimensional manifold closes in a smooth way. The origin of the second term in (3.88) was explained in detail, e.g. by Louko [21]. In the semiclassical limit the
wave function is given by

$$\Psi_0^{(HH)} \approx e^{-I_{cl}^E},$$  \hspace{1cm} (3.89)

where $I_{cl}^E$ should be calculated on the classical Euclidean solutions with boundary conditions defined by the concrete scheme of geometry closing at $\tau = \tau^*$. Now, we find the relationship between the HH wave function (3.87), (3.89) and our ground-state wave functions (3.86). Let us first consider the case of a negative cosmological constant $\Lambda < 0$. Then, we have the classical Euclidean equations

$$\dot{z}^0 = \pm \sqrt{2|\Lambda|} e^{qz^0}. \hspace{1cm} (3.90)$$

The spacial volume $v$ may be presented in the form

$$v = e^{qz^0} = -\left(q\sqrt{2|\Lambda|} \tau \right)^{-1}, \hspace{0.5cm} -\infty < \tau < 0. \hspace{1cm} (3.91)$$

Formula (3.91) shows that the geometry closes at the harmonic time $\tau \to -\infty$. It is easy to see from (3.91) that the second term in (3.88) contributes nothing to $I_E$. So, on these classical solutions the Euclidean action $I_{cl}^E$ reads

$$I_{cl}^E = \frac{1}{q^2} \frac{1}{\tau} = -\frac{\sqrt{2|\Lambda|}}{q} e^{qz^0} \hspace{1cm} (3.92)$$

and we get the semiclassical HH wave function (3.89)

$$\Psi_0^{(HH)} \approx \exp \left(\frac{\sqrt{2|\Lambda|}}{q} e^{qz^0} \right) = \exp \left(\frac{\sqrt{2|\Lambda|}}{q} \prod_{i=1}^{n} a_i^{d_i} \right). \hspace{1cm} (3.93)$$

Eqn. (3.93) shows that (for the class of real Euclidean geometries)

$$\Psi_0^{(HH)} \to +\infty, \hspace{0.5cm} z^0 \to +\infty. \hspace{1cm} (3.94)$$

This condition provides us with the possibility to chose a solution of equation (3.69) corresponding to the HH ground state:

$$\Psi_0^{(HH)} = I_0 \left(\frac{\sqrt{2|\Lambda|}}{q} e^{qz^0} \right). \hspace{1cm} (3.95)$$

The vacuum solution (3.95) has the asymptotic form

$$\Psi_0^{(HH)} \to \exp \left(\frac{\sqrt{2|\Lambda|}}{q} e^{qz^0} \right), \hspace{0.5cm} z^0 \to +\infty, \hspace{1cm} (3.96)$$
which coincides with (3.93).

A similar procedure can be performed for a positive cosmological constant \( \Lambda > 0 \) (see, e.g. [8]). In this case the classical Euclidean equation is

\[
\left( \frac{\dot{v}}{v} \right)^2 + 2q^2\Lambda v^2 = 0
\]

and gives an imaginary geometry. This reflects the fact that the geometry should be purely Lorentzian in the case \( \Lambda > 0 \) for \( \mathcal{E} \geq 0 \). The action \( I_{E}^{cl} \) (3.88) is indefinite in this case. We can avoid this problem if we perform the analytic continuation \( v \rightarrow iv \). After this continuation the action (3.88) is formally reduced to the action in the case \( \Lambda < 0 \). Again, it leads to the following asymptotic for the HH wave function: \( \Psi_{0}^{(HH)} \rightarrow +\infty, \ v \rightarrow +\infty \), which shows that the wave function \( I_{0}[\sqrt{2\Lambda}/q,v] \) is a solution of eq. (3.69) corresponding to the HH ground state for the class of Euclidean solutions considered here. Thus, after analytic continuation to real values of \( v \) the vacuum state corresponding to the HH boundary condition is

\[
\Psi_{0}^{(HH)} = J_0 \left( \frac{\sqrt{2\Lambda}}{q} v \right).
\]  

(3.98)

This solution has the asymptotic form

\[
\Psi_{0}^{(HH)} \rightarrow \cos \left( \frac{\sqrt{2\Lambda}}{q} v \right) = \cos \left( \frac{\sqrt{2\Lambda}}{q} \prod_{i=1}^{n} a_i \right), \quad v \rightarrow +\infty.
\]  

(3.99)

For the Bianchi I universe \((n = 3, d_1 = d_2 = d_3 = 1)\) eq. (3.99) is reduced to

\[
\Psi_{0}^{(HH)} \sim \cos \left( 2\sqrt{\Lambda/3a_1a_2a_3} \right), \quad v \rightarrow +\infty.
\]  

(3.100)

Similar results for the Bianchi I universe were obtained earlier in papers by Laflamme and Louko [21].

Now, let us turn to quantum wormholes. We restrict our consideration to real values of \( p_i \). This corresponds to real geometries in the Lorentzian region. In this case we have \( \mathcal{E} \geq 0 \).

If \( \Lambda > 0 \) the wave function \( \Psi \) (3.70) is not exponentially damped when \( v \rightarrow \infty \), i.e. the condition (i) for quantum wormholes (see the Introduction) is not satisfied. It oscillates and may be interpreted as corresponding to the classical Lorentzian solution.

For \( \Lambda < 0 \), the wave function (3.70) is exponentially damped for large \( v \) only, when \( B = K \) in (3.72). But in this case the function \( \Phi \) oscillates an infinite number of times,
when $v \to 0$. So, the condition (ii) is not satisfied. The wave function describes the transition between Lorentzian and Euclidean regions.

The functions

$$\Psi_{\vec{p}}(z) = e^{i\vec{p} \cdot \vec{z}} K_{i|\vec{p}|/q}(\sqrt{-2\Lambda} q^{-1} e^{qz^0}),$$

(3.101)

may be used for constructing quantum wormhole solutions. Like in [9, 10, 24] we consider the superpositions of singular solutions

$$\hat{\Psi}_{\lambda, \vec{n}}(z) = \frac{1}{\pi} \int_{-\infty}^{+\infty} dk \Psi_{qk\vec{n}}(z) e^{-ik\lambda},$$

(3.102)

where $\lambda \in \mathbb{R}$, $\vec{n}$ is a unit vector ($\vec{n}^2 = 1$) and the quantum number $k$ is connected with the quantum number $\mathcal{E} = \frac{1}{2} |\vec{p}|^2$ by the formula $2\mathcal{E} = q^2 k^2$. The calculation gives

$$\hat{\Psi}_{\lambda, \vec{n}}(z) = \exp[-\frac{\sqrt{-2\Lambda}}{q} e^{qz^0} \cosh(\lambda - q\vec{z} \vec{n})].$$

(3.103)

It is not difficult to verify that the formula (3.103) leads to solutions of the WDW equation (3.69), satisfying the quantum wormholes boundary conditions.

We also note that the functions

$$\Psi_{m, \vec{n}} = H_m(x^0) H_m(x^1) \exp\left[-\frac{(x^0)^2 + (x^1)^2}{2}\right],$$

(3.104)

where

$$x^0 = (2/q)^{1/2}(-2\Lambda)^{1/4} \exp(qz^0/2) \cosh(\frac{1}{2} q\vec{z} \vec{n}),$$

(3.105)

$$x^1 = (2/q)^{1/2}(-2\Lambda)^{1/4} \exp(qz^0/2) \sinh(\frac{1}{2} q\vec{z} \vec{n}),$$

(3.106)

$m = 0, 1, \ldots$, are also solutions of the WDW equation with the quantum wormhole boundary conditions. Solutions of such type were previously considered in [5, 9, 10]. They are called discrete spectrum quantum wormholes.

It is clear from the equation (3.71) and fig. 4 that in the case $\Lambda < 0$ a Lorentzian region exists as well as an Euclidean one for $\mathcal{E} > 0$. In the case $\Lambda > 0$ only the Lorentzian region occurs for $\mathcal{E} \geq 0$ and for $\mathcal{E} < 0$ both of these regions exist (see fig. 5). The condition $\mathcal{E} < 0$ leads for pure gravity to a complex geometry in the Lorentzian region. We can avoid this problem by the help of a free scalar field, because in this case $2\mathcal{E} = \sum_{i=1}^{n-1} p_i^2 + p_n^2$ and we can achieve $\mathcal{E} < 0$ for real $p_i (i = 1, \ldots, n-1)$ and imaginary $p_n$, i.e. for an imaginary scalar field in the Lorentzian region. The wave functions (3.70), (3.72) with $\Lambda > 0$ and $\mathcal{E} < 0$ describe the transitions between Euclidean and Lorentzian regions, i.e. tunneling universes.
4 CLASSICAL WORMHOLES, FINE TUNING OF PARAMETERS

Classical wormhole solutions exist in our model also for another interesting case. This is the case with spontaneous compactification of the internal dimensions. Let the factor space $M_1$ be our dynamical external space. All the other factor spaces $M_i (i = 2, \ldots, n)$ are considered as internal and static. They should be compact and the internal dimensions have the size of order of Planck’s length $L_{PL} \sim 10^{-33}$ cm. The scale factors of the internal factor spaces should be constant: $a_i = e^{\beta_i} \equiv a_{(0)i} (i = 2, \ldots, n)$. It is not difficult to show that in the case of fine tuning of the parameters due to

$$\frac{\theta_i}{d_i a_{(0)i}^2} = \frac{2\Lambda}{D-2} \equiv C_0, \quad i = 2, \ldots, n,$$

(4.1)

all dynamical equations (2.10) are reduced to one for the scale factor $a_1 = e^{\beta_1}$ and this equation reads

$$\ddot{\beta}_1 = -e^{2\sum_{k=1}^n d_k \beta_k} \left[ \frac{\theta_1}{d_1} e^{-2\beta_1} - C_0 - \frac{1}{d_1-1} \left( \frac{1}{d_1} \frac{\partial \tilde{U}}{\partial \beta_1} + 2\tilde{U} \right) \right].$$

(4.2)

The constraint (??) has form

$$d_1 (d_1 - 1) \beta_1^2 = \dot{\varphi}^2 - \theta_1 e^{2(d_1 - 1)\beta_1} e^{2\sum_{k=1}^n d_k \beta_k} + e^{2d_1 \beta_1} e^{2\sum_{k=1}^n d_k \beta_k} \left[ 2\tilde{U} + C_0 (d_1 - 1) \right].$$

(4.3)

From the equations (4.4) it follows that all internal spaces should be non-Ricci-flat and $\text{sign} \theta_i = \text{sign} \Lambda, \quad (i = 2, \ldots, n)$. We remind here that overdot denotes differentiation with respect to the harmonic time $\tau$ [7]. The minimally coupled scalar field has the specific potential $\tilde{U}(\beta, \varphi) = U(\varphi) \exp \left[ -2 \sum_{i=1}^n d_i \beta_i \right]$. For this potential it is easy to get the first integral of equation (2.11)

$$\dot{\varphi}^2 + 2U(\varphi) = \nu^2 = \text{const}.$$  

(4.4)

This gives

$$\varphi = \nu \tau + \text{const}$$  

(4.5)

for $U(\varphi) = 0$ and

$$\varphi = \varphi_0 \cos m(\tau - \tau_0)$$  

(4.6)
for $U(\varphi) = \frac{m^2 \varphi^2}{2}$, where $\nu = m \varphi_0$ here.

Let us investigate the model where our external space $M_1$ is Ricci-flat, i.e. $\theta_1 = 0$. Then we can rewrite the equation (4.3) as follows

$$(\dot{\beta}_1)^2 = \tilde{\nu}^2 + \tilde{\Lambda} e^{2d_1 \beta^1},$$

(4.7)

where the constants are

$$\tilde{\nu}^2 = \frac{\nu^2}{d_1(d_1 - 1)},$$

(4.8)

and

$$\tilde{\Lambda} = \frac{2\Lambda}{d_1(\sum_{k=1}^{n} d_k - 1)} \prod_{k=2}^{n} a_{(0)k}^{2d_k}.$$  

(4.9)

It is clear from equation (4.7) that the dynamical behavior of the scale factor $a_1$ depends on the signs of $\nu^2$ and $\Lambda$. If $\Lambda > 0$ and $\nu^2 \geq 0$ then $a_1$ expands from zero to infinity. For $\Lambda > 0$ and $\nu^2 < 0$, $a_1$ has the turning point at some minimum and this case may be realized for an imaginary scalar field in the Lorentzian region. For a real scalar field in the Lorentzian region (i.e. $\nu^2 > 0$) and $\Lambda < 0$ the scale factor $a_1$ expands from zero to its maximum and after the turning point shrinks again to zero. For the latter case the solution has a continuation into the Euclidean region with the topology of a wormhole, that means, two asymptotic regions which are connected with each other through a throat.

Let us investigate the case with $\Lambda < 0$ in more details. As $\text{sign} \Lambda = \text{sign} \theta_i \quad (i = 2, \ldots, n)$ then for $\Lambda < 0$ the curvatures $\theta_i < 0$ also. (As a special case the internal spaces $M_i \quad (i = 2, \ldots, n)$ may be compact spaces of constant negative curvature.) The solution of equation (4.7) (the Lorentzian region) has the form

$$a_1(\tau) = \left[\frac{\tilde{\nu}^2}{|\tilde{\Lambda}|}\right]^{1/2d_1}[\cosh d_1 \tilde{\nu} \tau]^{-1/d_1}, \quad -\infty < \tau < +\infty.$$  

(4.10)

The synchronous time $t$ and the harmonic time $\tau$ are connected by the differential equation

$$e^{\gamma(\tau)}d\tau = dt,$$

(4.11)

where

$$\gamma(\tau) = \sum_{i=1}^{n} d_i \beta^i.$$  

(4.12)

It is not difficult to get the connection

$$\cosh(d_1 \tilde{\nu} \tau) = \left[\cos \left(\sqrt{C_0 d_1} t + \text{const}\right)\right]^{-1}. $$  

(4.13)
With the help of this connection we obtain the expression for the scale factor $a_1$ with respect to the synchronous time

$$a_1(t) = \left[ \bar{\nu}^2 / |\Lambda| \right]^{1/2d_1} \left[ \sin \sqrt{C_0 d_1} \, t \right]^{1/d_1}, \quad 0 \leq t \leq \frac{\pi}{\sqrt{C_0 d_1}},$$

(4.14)

where the constant in (4.13) was fixed by condition $a(t = 0) = 0$. For $t \to 0$ we have $a_1 \sim t^{1/d_1}$. Thus the external space $M_1$ has the behavior of a FRW-universe filled with radiation for $d_1 = 2$ and with ultrastiff matter for $d_1 = 3$. The Lorentzian metric (2.1) in synchronous time gauge reads as

$$g = -dt \otimes dt + a_1^2(t) g_{(1)} + \sum_{i=2}^n a_{(0)i}^2 g_{(i)},$$

(4.15)

with $a_1$ given by (4.14).

Our next step is to get the wormhole-type solution for $M_1$ in the Euclidean region. The transition into the Euclidean space is performed by the Wick rotation $t \to -it$. The exact form of the transformation from the Lorentzian time $t_L$ to the Euclidean "time" $t_E$ can be obtained demanding the existence of wormholes being symmetric with respect to the throat (see Zhuk in [4]). In what follows, for the expression (4.14) we should perform the analytic continuation $t_L = \frac{\pi}{2 \sqrt{C_0 d_1}} - it_E$. This gives us

$$a_1(t) = \left[ \bar{\nu}^2 / |\Lambda| \right]^{1/2d_1} \left[ \cosh \sqrt{C_0 d_1} t \right]^{1/d_1}, \quad -\infty < t < +\infty.$$

(4.16)

Of course, this formula can be obtained also as a solution to the Euclidean analog of the equations (4.3), (4.7) for an imaginary scalar field in the Euclidean region. The metric of the Euclidean region is given by

$$g = dt \otimes dt + a_1^2(t) g_{(1)} + \sum_{i=2}^n a_{(0)i}^2 g_{(i)},$$

(4.17)

where $a_1(t)$ is described by (4.16). Thus, in Euclidean space we have two asymptotic regions $t \to \pm \infty$ connected through a throat of the size $\left[ \bar{\nu}^2 / |\Lambda| \right]^{1/2d_1}$ and this object is a wormhole by definition.

It is clear that the Lorentzian solution (4.14) and its Euclidean analog (4.16) take place only in the presence of a real scalar field in the Lorentzian region (i.e $\nu^2 > 0$) or equivalently an imaginary scalar field in the Euclidean region.
5 CONCLUSIONS

In this paper we investigated multidimensional cosmological models with \( n(n > 1) \) Einstein spaces \( M_i \) in the presence of the cosmological constant \( \Lambda \) and a homogeneous minimally coupled scalar field \( \varphi(t) \) as a matter source. The problem was to find classical and quantum wormhole solutions. Classical wormholes are solutions of the classical Einstein equations describing Riemannian metrics with two large regions joined by a throat. Quantum wormholes are solutions of the Wheeler-DeWitt (WDW) equation with the proper boundary conditions proposed by Hawking and Page [5].

The model was investigated where one of the factor spaces, say \( M_1 \), is Ricci-flat. In the case when all other factor spaces \( M_i, i = 2, \ldots, n, \) are Ricci-flat too, the classical Einstein as well as the WDW equations are integrable. For a negative cosmological constant \( \Lambda < 0 \) quantum wormhole solutions were constructed. These solutions exist for pure gravity as well as for the model with a free minimally coupled scalar field. Classical wormhole solutions exist in the Euclidean region for \( \Lambda < 0 \) in the presence of an imaginary as well as real scalar field.

Classical wormhole solutions were also obtained in models with spontaneous compactification. In this case the Ricci-flat factor space \( M_1 \) was considered as our external dynamical space. All other factor spaces \( M_i, i = 2, \ldots, n \) are static with constant scale factors \( a_{(0)i} = \text{const} \) and all of them are fine tuned to each other and to the cosmological constant:

\[
\frac{\theta_i}{a_{(0)i}} = \frac{\theta_k}{a_{(0)k}} = \frac{2\Lambda}{D-2}, \quad i, k = 2, \ldots, n.
\]

As in the previous model, wormhole solutions exist for a negative cosmological constant \( \Lambda < 0 \). But there are important differences. Firstly, all inner spaces \( M_i, i = 2, \ldots, n, \) are non-Ricci-flat and have negative curvature. Secondly, the wormhole solution for the later case exists only in the presence of an imaginary scalar field in the Euclidean region. Thirdly, it seems hardly to be possible in the case \( \theta_2, \ldots, \theta_n \neq 0 \) to integrate the Einstein equations as well as the WDW equation without the demand of spontaneous compactification with fine tuning.

In models with one scale factor having a turning point (at the minimum) the production of the Lorentzian space-time is treated as a quantum tunneling process [32] (“birth from nothing”). The universe appears spontaneously going through the potential barrier with size equal to the size of the Lorentzian universe at the turning point. In our case of multidimensional models this kind of interpretation becomes more complicated. It follows
from (3.56) that the factor spaces $M_i$ in general reach their minimum expansion positions at different times. The "birth from nothing" for each factor space takes place at a different value of time. If the difference between these events goes to infinity the extra dimensions are in the classically forbidden region forever. This interpretation is in the spirit of the Rubakov-Shaposhnikov idea stating that extra dimensions are unobservable because they are hidden from us by a potential barrier.

ACKNOWLEDGMENT
The work was sponsored by KAI e.V. Berlin through the WIP project 016659/p and partly by the Russian Ministry of Science. A. Z. was supported by DFG grant 436 RUS 113-7-1, V.I. and V. M. by DFG grant 436 RUS 113-7-2. V. I., V. M. and A. Z. also thank the colleagues of the WIP gravitation project group at Potsdam University for their hospitality.

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Figures:

**Fig. 1** The behaviour of the scale factors $a_1^\pm(t), a_2^\pm(t)$ (see (3.46), (3.47)) for $\Lambda < 0$, $n = 2, 1 < d_1 < d_2$ and different values of the integration constant $\alpha_3$. The left figures correspond to the sign “+” in (3.46), (3.47).

Fig. 1.1 $0 \leq \alpha_3^2 < 1 - \frac{1}{d_1}$

Fig. 1.2 $\alpha_3^2 = 1 - \frac{1}{d_1}$

Fig. 1.3 $1 - \frac{1}{d_1} < \alpha_3^2 < 1 - \frac{1}{d_2}$
Fig. 1.4  \[ \alpha_3^2 = 1 - \frac{1}{d_2} \]

Fig. 1.5  \[ 1 - \frac{1}{d_2} < \alpha_3^2 < 1 - \frac{1}{d_1 + d_2} \]

Fig. 1.6  \[ \alpha_3^2 = 1 - \frac{1}{d_1 + d_2} \]
The behaviour of the scale factors $a_1^\pm(t), a_2^\pm(t)$ (see (3.46), (3.47)) for $\Lambda > 0$, $n = 2, 1 < d_1 < d_2$ and different values of the integration constant $\alpha_3$. The left figures correspond to the sign “+” in (3.46), (3.47).

Fig. 2.1  $0 \leq \alpha_3^2 < 1 - \frac{1}{d_1}$

Fig. 2.2  $\alpha_3^2 = 1 - \frac{1}{d_1}$

Fig. 2.3  $1 - \frac{1}{d_1} < \alpha_3^2 < 1 - \frac{1}{d_2}$
Fig. 2.4 \[ \alpha_3^2 = 1 - \frac{1}{d_2} \]

Fig. 2.5 \[ 1 - \frac{1}{d_2} < \alpha_3^2 < 1 - \frac{1}{d_1 + d_2} \]

Fig. 2.6 \[ \alpha_3^2 = 1 - \frac{1}{d_1 + d_2} \]
Fig. 3. Representation of the function $f(x) = \exp(-2 \arctan e^{-2x})$.

Fig. 4. The potential $V_0(z^0)$ for $\Lambda > 0$ (solid line) and the energy levels $\mathcal{E}$ (dashed lines). For $\mathcal{E} \geq 0$ the Lorentzian region exists, only. For $\mathcal{E} < 0$ both regions, the Lorentzian as well as the Euclidean one exist. In this case quantum transitions with topology changes take place (tunneling universe).
Fig. 5. The potential $V_0(z^0)$ for $\Lambda < 0$ (solid line) and the energy levels $\mathcal{E}$ (dashed lines). For $\mathcal{E} \leq 0$ the Euclidean region exists, only. For $\mathcal{E} > 0$ both regions, the Lorentzian as well as the Euclidean one exist. In this case quantum transitions with topology changes take place (quantum wormholes).
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