Young tableaux and homotopy commutative algebras

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Abstract A homotopy commutative algebra, or $C_\infty$-algebra, is defined via the Tornike Kadeishvili homotopy transfer theorem on the vector space generated by the set of Young tableaux with self-conjugated Young diagrams $\{\lambda : \lambda = \lambda'\}$. We prove that this $C_\infty$-algebra is generated in degree 1 by the binary and the ternary operations.

1 Introduction

We consider the 2-nilpotent graded Lie algebra $\mathfrak{g}$, with degree one generators in the finite dimensional vector space $V$ over a field $K$ of characteristic 0,

$$\mathfrak{g} = V \oplus [V, V].$$

The Universal Enveloping Algebra (UEA) $U\mathfrak{g}$ arises naturally in physics as the subalgebra closed by the creation operators of the parastatistics algebra. The algebra of creation and annihilation parastatistics operators was introduced by H.S. Green [6], its defining relations generalize the canonical (anti)commutation relations.

As an UEA of a finite dimensional positively graded Lie algebra, $U\mathfrak{g}$ belongs to the class of Artin-Schelter regular algebras (see e.g. [5]). As every finitely generated graded connected algebra, $U\mathfrak{g}$ has a free minimal resolution which is canonically built from the data of its Yoneda algebra $\mathcal{E} := \text{Ext}_{U\mathfrak{g}}(K, K)$. By construction the
Yoneda algebra $E$ is isomorphic (as algebra) to the cohomology of the Lie algebra (with coefficients in the trivial representation provided by the ground field $K$)

$$E = \text{Ext}^\bullet_{Ug}(K, K) \cong H^\bullet(g, K)$$ (1)

the product on $E$ being the super-commutative wedge product between cohomological classes in $H^\bullet(g, K)$.

An important result due to Józefiak and Weyman [7] implies that a basis of the cohomology $E = H^\bullet(g, K)$ is indexed by Young tableaux with self-conjugated Young diagrams (i.e., symmetric with respect to the diagonal). On the other hand according to the homotopy transfer theorem due to Tornike Kadeishvili [8] the Yoneda algebra $E$ is a $C_\infty$-algebra.

The aim of this note is to endow the cohomology $H^\bullet(g, K)$ (i.e., the vector space generated by the set of Young tableaux with self-conjugated Young diagrams $\{\lambda : \lambda = \lambda'\}$) with a $C_\infty$-structure, induced by the isomorphism (1) through the homotopy transfer.

Here we deal only with the parafermionic case corresponding to an (even) vector space $V$. To include the parabosonic degrees of freedom one have to consider $V$ in the category of vector superspaces. The supercase will be consider elsewhere.

## 2 Artin-Schelter regularity

Let $g$ be the 2-nilpotent graded Lie algebra $g = V \oplus \bigwedge^2 V$ generated by the finite dimensional vector space $V$ having Lie bracket

$$[x, y] := \begin{cases} x \wedge y & x, y \in V \\ 0 & \text{otherwise} \end{cases}$$ (2)

We denote the Universal Enveloping Algebra $Ug$ by $PS$ and will refer to it as parastatistics algebra (by some abuse). The parastatistics algebra $PS(V)$ generated in $V$ is graded

$$PS(V) := Ug = U(V \oplus \bigwedge^2 V) = T(V)/([[[V, V], V]) .$$

We shall write simply $PS$ when the space of generators $V$ is clear from the context.

Artin and Schelter [1] introduced a class of regular algebras sharing some “good” homological properties with the polynomial algebra $K[V]$. These algebras were dubbed Artin-Schelter regular algebras (AS-regular algebra for short).

**Definition 1.** (AS-regular algebras) A connected graded algebra $A = K \oplus A_1 \oplus A_2 \oplus \ldots$ is called Artin-Schelter regular of dimension $d$ if

1. $A$ has finite global dimension $d$,
2. $A$ has finite Gelfand-Kirillov dimension,
3. $A$ is Gorenstein, i.e., $\text{Ext}^d_{A}(K, A) = \delta^d K$.

1 Strictly speaking $PS(V)$ is the creation parastatistics algebra, closed by creation operators alone.
A general theorem claims that the UEA of a finite dimensional positively graded Lie algebra is an AS-regular algebra of global dimension equal to the dimension of the Lie algebra. Hence the parastatistics algebra $PS$ is AS-regular of global dimension $d = \frac{\dim V (\dim V + 1)}{2}$. In particular the finite global dimension of $PS$ implies that the ground field $K$ has a minimal resolution $P_n$ by projective left $PS$-modules

$$P_n: 0 \rightarrow P_d \rightarrow \cdots \rightarrow P_n \rightarrow \cdots \rightarrow P_2 \rightarrow P_1 \rightarrow P_0 \rightarrow \epsilon \rightarrow K \rightarrow 0.$$  

Here $K$ is a trivial left $PS$-module, the action being defined by the projection $\epsilon$ onto $PS_0 = K$. Since $PS$ is graded and in the category of graded modules projective module is the same as free module, we have $P_n \cong PS \otimes E_n$ where $E_n$ are finite dimensional vector spaces.

The minimal projective resolution is unique (up to a isomorphism). Minimality implies that the complex $K \otimes PS P_\bullet$ has “zero differentials” hence

$$H_\bullet(K \otimes PS P_\bullet) = K \otimes PS P_\bullet = E_n.$$  

One can calculate the derived functor $\text{Tor}^PS_n(K, K)$ using the resolution $P_\bullet$, it yields

$$\text{Tor}^PS_n(K, K) = E_n.$$  

The data of a minimal resolution of $K$ by free $PS$-modules provides an easy way to find $\text{Tor}^PS_n(K, K)$. Conversely if the spaces $\text{Tor}^PS_n(K, K)$ are known then one can construct a minimal free resolution of $K$.

The Gorenstein property guarantees that when applying the functor $\text{Hom}_{PS}(-, PS)$ to the minimal free resolution $P_\bullet$ we get another minimal free resolution $P^\bullet := \text{Hom}_{PS}(P_\bullet, PS)$ of $K$ by right $PS$-modules

$$P^\bullet: 0 \leftarrow K \leftarrow P'_d \leftarrow \cdots \leftarrow P'_n \leftarrow \cdots \leftarrow P'_2 \leftarrow P'_1 \leftarrow P'_0 \leftarrow 0$$  

with $P'_n \cong E^*_n \otimes PS$. Note that by construction $E^*_n = \text{Ext}^d_{PS}(K, K)$ thus one has vector space isomorphisms

$$E_n \cong E^*_n \cong \text{Tor}^PS_n(K, K) \cong \text{Ext}^d_{PS}(K, K).$$  

The Gorenstein property is the analog of the Poincaré duality since it implies

$$E^*_{d-n} \cong E_n.$$  

The finite global dimension $d$ of $PS$ and the Gorenstein condition imply that its Yoneda algebra

$$\mathcal{O}^\bullet := \text{Ext}_{PS}^d(K, K) \cong \bigoplus_{n=0}^d E^*_n$$

is Frobenius.
3 Homology and cohomology of $g$

Let us first recall that the standard Chevalley-Eilenberg chain complex $C_\bullet(g) = (Ug \otimes \mathbb{K} \wedge^p g, d_p)$ where the differential reads

$$d_p(u \otimes x_1 \wedge \ldots \wedge x_p) = \sum_i (-1)^{i+1} u x_i \otimes x_1 \wedge \ldots \wedge \hat{x}_i \wedge \ldots \wedge x_p$$

$$+ \sum_{i < j} (-1)^{j+i} u \otimes [x_i, x_j] \wedge x_1 \wedge \ldots \wedge \hat{x}_i \wedge \ldots \wedge \hat{x}_j \wedge \ldots \wedge x_p$$

provides a non-minimal projective (in fact free) resolution of $\mathbb{K}$, $C(g) \xrightarrow{\varepsilon} \mathbb{K}$ [14]. With the latter resolution $C_\bullet(g)$ one calculates homologies of the derived complex $\mathbb{K} \otimes_{PS} C_\bullet(g)$

$$E_\infty = \text{Tot}^n_{PS}(\mathbb{K}, \mathbb{K}) \cong H_n(\mathbb{K} \otimes_{PS} C_\bullet(g)) = H_n(g, \mathbb{K}) ,$$

coinciding with the homologies $H_n(\mathbb{K}, \mathbb{K})$ of the Lie algebra $g$ with trivial coefficients. The derived complex $\mathbb{K} \otimes_{PS} C_\bullet(g)$ is the chain complex with degrees $\wedge^p g = \mathbb{K} \otimes_{PS} \wedge^p g$ and differentials $\partial_p := id \otimes_{PS} d_p : \wedge^p g \to \wedge^{p-1} g$.

The differential $\partial$ is induced by the Lie bracket $[\cdot, \cdot] : \wedge^2 g \to g$ of the graded Lie algebra $g = g_1 \oplus g_2$. It identifies a pair of degree 1 generators $e_i, e_j \in g_1$ with one degree 2 generator $e_{ij} := [e_i, e_j] \in g_2$. The differential $\partial_p$ is a continuation as coderivation (see e.g. [11]) of the mapping $\partial_2 := -[\cdot, \cdot]$ on the exterior powers $\wedge^p g$. In greater details the chain degrees read

$$\wedge^p g = \bigoplus_{s+r=p} \wedge^s (\wedge^2 V) \otimes \wedge^r V \quad (9)$$

and differentials $\partial_{p=r+1} : \wedge^r (\wedge^2 V) \otimes \wedge^r (V) \to \wedge^{r+1} (\wedge^2 V) \otimes \wedge^{r-2} (V)$ are given by

$$\partial_p : e_{i_1 j_1} \wedge \ldots \wedge e_{i_r j_r} \otimes e_1 \wedge \ldots \wedge e_r \mapsto \sum_{i < j} (-1)^{i+j} e_{ij} \wedge e_{i_1 j_1} \wedge \ldots \wedge e_{i_r j_r} \otimes e_1 \wedge \ldots \wedge \hat{e}_i \wedge \ldots \wedge \hat{e}_j \wedge \ldots \wedge e_r .$$

By duality, one has the cochain complex $\text{Hom}_{PS}(C(g), \mathbb{K}) = (\wedge^\bullet g^*, \delta)$ which is a (super)commutative DGA. This cochain complex calculates the cohomology

$$E^n_\infty = \text{Ext}^n_{PS}(\mathbb{K}, \mathbb{K}) \cong H^n(\text{Hom}_{PS}(C(g), \mathbb{K})) = H^n(g, \mathbb{K}) . \quad (10)$$

The coboundary map $\delta^p : \wedge^p g^* \to \wedge^{p+1} g^*$ is transposed\footnote{In the presence of metric one has $\delta := \partial^*$ (see Proposition 1 below).} to the differential $\partial_{p+1}$

$$\delta^p : e^*_{i_1 j_1} \wedge \ldots \wedge e^*_{i_r j_r} \otimes e^*_1 \wedge \ldots \wedge e^*_r \mapsto \sum_{k=1}^s \sum_{i_k < j_k} (-1)^{i_k+j_k} e^*_{i_1 j_1} \wedge \ldots \wedge e^*_{i_k j_k} \wedge \ldots \wedge e^*_{i_r j_r} \otimes e^*_{i_k} \wedge e^*_j \wedge e^*_1 \wedge \ldots \wedge e^*_r , \quad (11)$$
it is (up to a conventional sign) a continuation of the dualization of the Lie bracket \( \delta^1 := [\cdot, \cdot] : g^* \to \Lambda^2 g^* \) by the Leibniz rule.

It is important that in the complexes \((\Lambda^p g, \partial_p)\) and \((\Lambda^p g^*, \delta^p)\) two different degrees are involved; one is the homological degree \( p := r + s \) counting the number of \( g \)-generators, while the second is the tensor degree \( t := 2s + r \). The differentials \( \partial \) and \( \delta \) preserve the tensor degree \( t \) but the spaces \( H_n(g, \mathbb{K}) \) and \( H^n(g, \mathbb{K}) \) are not homogeneous in \( t \) in general.

**4 Littlewood formula and \( PS \)**

In this section we review the beautiful result of Józefiak and Weyman [7] giving a representation-theoretic interpretation of the Littlewood formula

\[
\prod_i (1 - x_i) \prod_{i<j} (1 - x_i x_j) = \sum_{\lambda, \lambda' = \lambda} (-1)^{\frac{1}{2} (|\lambda| + r(\lambda))} s_\lambda(x). \tag{12}
\]

Here the sum is over the self-dual Young diagrams \( \lambda \), \( s_\lambda(x) \) stands for the Schur function and \( r(\lambda) \) stands the rank of \( \lambda \) which is the number of diagonal boxes in \( \lambda \).

An irreducible \( GL(V) \)-module \( V_\lambda \) is called Schur module, it has a basis labelled by semistandard Young tableaux which are fillings of the Young diagram \( \lambda \) with the numbers of the set \( \{1, \ldots, \dim V\} \). The action of the linear group \( GL(V) \) on the space \( V \) of the generators of the Lie algebra \( g \) induces a \( GL(V) \)-action on the UEA \( PS = U g \cong S(V \oplus \Lambda^2 V) \) and on the space \( \Lambda^* g \cong \Lambda^* (V \oplus \Lambda^2 V) \). The algebra \( PS(V) \) has remarkable property, it is a model of the linear group \( GL(V) \), in the sense that it contains every polynomial finite-dimensional irreducible representation \( V_\lambda \) of \( GL(V) \) once and only once

\[
PS(V) \cong \bigoplus_{\lambda} V_\lambda.
\]

A nice combinatorial proof of this fact was given by Chaturvedi [3]. The \( GL(V) \)-model \( PS(V) \) enjoys the universal property that every parastatistics Fock representation specified by the parastatistics order \( p \in \mathbb{N}_0 \) is a factor of \( PS(V) \) [4], [10].

The differential \( \partial \) commutes with the \( GL(V) \) action and the homology \( H_n(g, \mathbb{K}) \) is also a \( GL(V) \)-module. The decomposition of the \( GL(V) \)-module \( H_n(g, \mathbb{K}) \) into irreducible polynomial representations \( V_\lambda \) is given by the following theorem;

**Theorem 1 (Józefiak and Weyman [7], Sigg [13]).** The homology \( H_*(g, \mathbb{K}) \) of the 2-nilpotent Lie algebra \( g = V \oplus \Lambda^2 V \) decomposes into irreducible \( GL(V) \)-modules

\[
H_n(g, \mathbb{K}) = H_n(\bigwedge^* g, \partial) \cong \text{Tor}^n_{PS}(\mathbb{K}, \mathbb{K})(V) \cong \bigoplus_{\lambda, \lambda' = \lambda} V_\lambda \tag{13}
\]

where the sum is over self-conjugate Young diagrams \( \lambda \) such that \( n = \frac{1}{2} (|\lambda| + r(\lambda)) \).

The data \( H_n(g, \mathbb{K}) = \text{Tor}^n_{PS}(\mathbb{K}, \mathbb{K}) \) encodes the minimal free resolution \( P_* \) (cf. [3]).
The acyclicity of the complex $P_\ast$ implies an identity about the $GL(V)$-characters

$$
ch PS(V) \cdot ch \left( \bigoplus_{\lambda, \lambda' = \lambda} (-1)^{\frac{1}{2}(|\lambda'| + r(\lambda))} V_{\lambda'} \right) = 1.
$$

The character of a Schur module $V_\lambda$ is the Schur function, $ch V_\lambda = s_\lambda (x)$. Due to the Poincaré-Birkhoff-Witt theorem $ch PS(V) = ch S(V \oplus \Lambda^2 V)$ thus the identity reads

$$
\prod r \frac{1}{(1 - x_i)} \prod r < j \frac{1}{(1 - x_i x_j)} \sum_{\lambda, \lambda' = \lambda} (-1)^{\frac{1}{2}(|\lambda'| + r(\lambda))} s_\lambda (x) = 1.
$$

But the latter identity is nothing but rewriting of the Littlewood identity $\text{(12)}$. The moral is that the Littlewood identity reflects a homological property of the algebra $PS$, namely the above particular structure of the minimal projective (free) resolution of $\mathbb{K}$ by $PS$-modules.

## 5 Homotopy algebras $A_\infty$ and $C_\infty$

**Definition 2.** ($A_\infty$-algebra) A homotopy associative algebra, or $A_\infty$-algebra, over $\mathbb{K}$ is a $\mathbb{Z}$-graded vector space $A = \bigoplus_{i \in \mathbb{Z}} A^i$ endowed with a family of graded mappings (operations)

$$m_n : A^\otimes n \to A, \quad \text{deg}(m_n) = 2 - n \quad n \geq 1$$

satisfying the Stasheff identities $SI(n)$ for $n \geq 1$

$$\sum_{r + s + t = n} (-1)^{r+s+t} m_{r+1+t}(Id^{\otimes r} \otimes m_s \otimes Id^{\otimes t}) = 0 \quad SI(n)$$

where the sum runs over all decompositions $n = r + s + t$.

Here we assume the Koszul sign convention $(f \otimes g)(x \otimes y) = (-1)^{|x||y|} f(x) \otimes g(y)$.

We define the shuffle product $Sh_{p,q} : A^{\otimes p} \otimes A^{\otimes q} \to A^{\otimes p+q}$ throughout the expression

$$(a_1 \otimes \ldots \otimes a_p) \shuffle (a_{p+1} \otimes \ldots \otimes a_{p+q}) = \sum_{\sigma \in Sh_{p,q}} sgn(\sigma) a_{\sigma^{-1}(1)} \otimes \ldots \otimes a_{\sigma^{-1}(p+q)}$$

where the sum runs over all $(p,q)$-shuffles $Sh_{p,q}$, i.e., over all permutations $\sigma \in S_{p+q}$ such that $\sigma(1) < \sigma(2) < \ldots < \sigma(p)$ and $\sigma(p+1) < \sigma(p+2) < \ldots < \sigma(p+q)$.

**Definition 3.** ($C_\infty$-algebra [8]) A homotopy commutative algebra, or $C_\infty$-algebra, is an $A_\infty$-algebra $\{A, m_n\}$ with the condition: each operation $m_n$ vanishes on shuffles

$$m_n((a_1 \otimes \ldots \otimes a_p) \shuffle (a_{p+1} \otimes \ldots \otimes a_n)) = 0, \quad 1 \leq p \leq n - 1.$$

In particular for $m_2$ we have $m_2(a \otimes b \pm b \otimes a) = 0$, so a $C_\infty$-algebra such that $m_n = 0$ for $n \geq 3$ is a (super)commutative DGA.
A morphism of two $A_\infty$-algebras $A$ and $B$ is a family of graded maps $f_n : A^{\otimes n} \to B$ for $n \geq 1$ with $\deg f_n = 1 - n$ such that the following conditions hold

$$
\sum_{r+i+1+\ldots+n} (-1)^{r+st} f_{r+1+i+\ldots+n}(Id^{\otimes r} \otimes m_s \otimes Id^{\otimes n}) = \sum_{1 \leq q \leq n} (-1)^{s} m_q(f_{i_1} \otimes f_{i_2} \otimes \ldots \otimes f_{i_q})
$$

where the sum on the RHS is over all decompositions $i_1 + \ldots + i_q = n$ and the sign is determined by $S = \sum_{k=1}^{n-1} (r - q)(i_q - 1)$. The morphism $f$ is a quasi-isomorphism of $A_\infty$-algebras if $f_1$ is a quasi-isomorphism. It is strict if $f_i = 0$ for all $i \neq 1$. The identity morphism of $A$ is the strict morphism $f$ such that $f_1$ is the identity of $A$.

A morphism of $C_\infty$-algebras is a morphism of $A_\infty$-algebras with components vanishing on shuffles $f_n((a_1 \otimes \ldots \otimes a_p) \sqcup (a_{p+1} \otimes \ldots \otimes a_n)) = 0$, $1 \leq p \leq n - 1$.

### 6 Homotopy Transfer Theorem

**Lemma 1 (see e.g. [11]).** Every cochain complex $(A, d)$ of vector spaces over a field $\mathbb{K}$ has its cohomology $H^\ast(A)$ as a deformation retract.

One can always choose a vector space decomposition of the cochain complex $(A, d)$ such that $A^n \cong B^n \oplus H^n \oplus B^{n+1}$ where $H^n$ is the cohomology and $B^n$ is the space of coboundaries, $B^n = dA^{n-1}$. We choose a homotopy $h : A^n \to A^{n-1}$ which identifies $B^n$ with its copy in $A^{n-1}$ and is 0 on $H^n \oplus B^{n+1}$. The projection $p$ to the cohomology and the cocycle-choosing inclusion $i$ given by $A^n \xrightarrow{i} H^n$ are chain homomorphisms (satisfying the additional conditions $hh = 0$, $hi = 0$ and $ph = 0$). With these choices done the complex $(H^\ast(A), 0)$ is a deformation retract of $(A, d)$

$$
\xymatrix{
(A, d) \ar[r]^p & (H^\ast(A), 0) \ , & pi = Id_{H^\ast(A)} , & ip - Id_A = dh + hd .}
$$

Let now $(A, d, \mu)$ be a DGA, i.e., $A$ is endowed with an associative product $\mu$ compatible with $d$. The cochain complexes $(A, d)$ and its contraction $H^\ast(A)$ are homotopy equivalent, but the associative structure is not stable under homotopy equivalence. However the associative structure on $A$ can be transferred to an $A_\infty$-structure on a homotopy equivalent complex, a particular interesting complex being the deformation retract $H^\ast(A)$. For a friendly introduction to homotopy transfer theorems in much broader context we send the reader to the textbook [11].

**Theorem 2 (Kadeishvili [8]).** Let $(A, d, \mu)$ be a (commutative) DGA over a field $\mathbb{K}$. There exists a $A_\infty$-algebra $(C_\infty$-algebra) structure on the cohomology $H^\ast(A)$ and a $A_\infty(C_\infty)$-quasi-isomorphism $f_1 : (\otimes^{s} H^\ast(A), \{m_i\}) \to (A, \{d, \mu, 0, 0, \ldots\})$ such that the inclusion $f_1 = i : H^\ast(A) \to A$ is a cocycle-choosing homomorphism of cochain complexes. The differential on $H^\ast(A)$ is zero $m_1 = 0$ and $m_2$ is strictly associative operation induced by the multiplication on $A$. The resulting structure is unique up to quasi-isomorphism.
Kontsevich and Soibelman\cite{9} gave an explicit expressions for the higher operations of the induced $A_{\infty}$-structure as sums over decorated planar binary trees with one root where all leaves are decorated by the inclusion $i$, the root by the projection $p$ the vertices by the product $\mu$ of the (commutative) DGA $(A,d,\mu)$ and the internal edges by the homotopy $h$. The $C_{\infty}$-structure implies additional symmetries on trees. We will make use of the graphic representation for the binary operation on $H^\bullet(A)$

$$m_2(x,y) := p\mu(i(x),i(y)) \text{ or } m_2 = \mu \downarrow$$

and the ternary one $m_3(x,y,z) = p\mu(i(x),h\mu(i(y),i(z))) - p\mu(h\mu(i(x),i(y)),i(z))$ being the sum of two planar binary trees with three leaves

\textbf{Proposition 1.} The cohomology $H^\bullet(g,K) \cong \text{Ext}_K^\bullet(K,K)$ of the 2-nilpotent graded Lie algebra $g = V \otimes \bigwedge^2 V$ is a homotopy commutative algebra. The $C_{\infty}$-algebra $H^\bullet(g,K)$ is generated in degree 1, i.e., in $H^1(g,K)$ by the operations $m_2$ and $m_3$.

\textbf{Sketch of the proof.} We apply the Kadeishvili homotopy transfer Theorem\cite{2} for the commutative DGA $(\bigwedge^\bullet g^*,\mu,\delta^\ast)$ and its deformation retract $H^\bullet(\bigwedge^\bullet g^*) \cong H^\bullet(g,K)$ and conclude that the cohomology $H^\bullet(g,K)$ is a $C_{\infty}$-algebra.

Further on we need the explicit mappings in the deformation retract. Let us choose a metric $g(\cdot,\cdot) = \langle \cdot,\cdot \rangle$ on the vector space $V$ and an orthonormal basis $\langle e_i,e_j \rangle = \delta_{ij}$. The choice induces a metric on $\bigwedge^\ast g^* \cong \bigwedge^\ast g^\ast$. In the presence of metric $g$ the differential $\delta$ is identified with the adjoint of $\partial$, $\delta : \cong \partial^\ast$ while $\partial$ plays the role of a homotopy. Then the deformation retract $H^\bullet(\bigwedge^\bullet g^*,\delta^\ast)$ of $\langle \bigwedge^\bullet g^*,\delta^\ast \rangle$ looks like

$$pi = Id_{H^\bullet(\bigwedge^\bullet g^*)} \ , \ \ ip - Id_{\bigwedge^\bullet g^*} = \delta^\ast \delta + \delta^\ast \delta \ , \ \ \delta^\ast \cong \partial$$
Here the projection $p$ identifies the subspace $\ker \delta \cap \ker \delta^*$ with $H^*(\wedge^* g^*)$, which is the orthogonal complement of the space of the coboundaries $\text{im} \delta$. The cocycle-choosing homomorphism $i$ is $\text{Id}$ on $H^*(\wedge^* g^*)$ and zero on coboundaries.

Due to the isomorphisms $\text{Tor}^P_R(\mathbb{K}, \mathbb{K}) \cong \text{Ext}^P_R(\mathbb{K}, \mathbb{K})$ (see eq. (6)) induced by $V \cong V^*$ the theorem implies the decomposition of $H^*(g, \mathbb{K})$ into Schur modules

$$H^n(g, \mathbb{K}) \cong H^n(\wedge^* g^*, \delta) \cong \text{Ext}^n_R(\mathbb{K}, \mathbb{K})(V^*) \cong \bigoplus_{\lambda, \lambda = \lambda'} V_{\lambda}$$

where the sum is over self-conjugate diagrams $\lambda$ such that $n = \frac{1}{2}(|\lambda| + r(\lambda))$. The operation $m_n$ is bigraded $\text{deg}_{n', r}(m_n) = (2 - n, 0)$ by homological degree $n'$ and tensor degree $r'$ (weight). The bi-grading impose the vanishing of many higher products.

The Kontsevich and Soibelman tree representations of the operations $m_n$ provide explicit expressions. Let us take $\mu$ to be the super-commutative product $\wedge$ on the DGA $(\wedge^* g^*, \delta^*)$. The projection $p$ maps onto the Schur modules $V_{\lambda}$ with $\lambda = \lambda'$.

The binary operation on the degree 1 generators $e_i \in H^1(g, \mathbb{K})$ is trivial, one gets

$$m_2(e_i, e_j) = p(e_i \wedge e_j) = 0 \quad p(V_{(1^2)}) = 0.$$ 

Hence $H^*(g, \mathbb{K})$ could not be generated in $H^1(g, \mathbb{K})$ as algebra with product $m_2$.

The ternary operation $m_3$ restricted to $H^1(g, \mathbb{K})$ is nontrivial, indeed one has

$$m_3(e_i, e_j, e_k) = p \{ e_i \wedge (e_j \wedge e_k) - \partial (e_i \wedge e_j) \wedge e_k \} = p \{ e_j \wedge e_k - e_i \wedge e_{jk} \}$$

$$= p \{ e_j \wedge e_k + e_{jk} \wedge e_i + e_{ki} \wedge e_j - e_{ki} \wedge e_j \} = e_{ik} \wedge e_j \in H^2(g, \mathbb{K})$$

The completely antisymmetric combination in the brackets (…) spans the Schur module $V_{(1^3)}$, $p(e_i \wedge e_k + e_{jk} \wedge e_i + e_{ki} \wedge e_j) = 0$ yields a Jacobi-type identity. The monomials $e_{ij} \wedge e_k$ modulo $V_{(1^3)}$ span a Schur module $V_{(2,1)} \in H^2(g, \mathbb{K})$ with basis in bijection with the semistandard Young tableaux $e_{ik} \wedge e_j \leftrightarrow \begin{array}{c} \hline \downarrow \downarrow \downarrow \hline \end{array}$ and $e_{ij} \wedge e_k \leftrightarrow \begin{array}{c} \hline \downarrow \downarrow \downarrow \hline \end{array}$.

We check the symmetry condition on ternary operation $m_3$ in $C_\infty$-algebra; indeed $m_3$ vanishes on the (signed) shuffles $Sh_{1,2}$ and $Sh_{2,1}$

$$m_3(e_i \sqcup e_j \otimes e_k) = m_3(e_i, e_j, e_k) - m_3(e_j, e_i, e_k) + m_3(e_j, e_k, e_i) = 0 = m_3(e_i \otimes e_j \sqcup e_k).$$

On the level of Schur modules the ternary operation glues three fundamental $GL(V)$-representations $V_{(2,1)}$ into a Schur module $V_{(2,1)}$. By iteration of the process of gluing boxes we generate all elementary hooks $V_k := V_{(1^k, k+1)},$

$$m_3(V_{(2,1)} V_{(2,1)}) = V_{(2,2)}; \quad m_3 \left( V_{(2,1)} V_{(2,1)} V_{(2,1)} \right) = V_{(2,3)}; \ldots, m_3(V_0, V_k, V_0) = V_{k+1}.$$
For self-dual diagrams $\lambda = \lambda'$, i.e., $a_i = b_i$ we set $V_{a_1, \ldots, a_r} := V_{(a_1, \ldots, a_r | a_1, \ldots, a_r)}$ when $a_1 > a_2 > \ldots > a_r \geq 0$ (and set the convention $V_{a_1, \ldots, a_r} := 0$ otherwise). Any two elementary hooks $V_{a_1}$ and $V_{a_2}$ can be glued together by the binary operation $m_2$, the decomposition of $m_2(V_{a_1}, V_{a_2}) \cong m_2(V_{a_2}, V_{a_1})$ is given by

$$m_2(V_{a_1}, V_{a_2}) = V_{a_1, a_2} \oplus \bigoplus_{i=1}^{a_2} V_{a_1 + i, a_2 - i} \quad a_1 \geq a_2$$

where the “leading” term $V_{a_1, a_2}$ has the diagram with minimal height. Hence any $m_2$-bracketing of the hooks $V_{a_1}, V_{a_2}, \ldots, V_{a_r}$ yields[3] a sum of $GL(V)$-modules

$$m_2(\ldots m_2(m_2(V_{a_1}, V_{a_2}), V_{a_3}), \ldots, V_{a_r}) = V_{a_1, \ldots, a_r} \oplus \ldots$$

whose module with minimal height is precisely $V_{a_1, \ldots, a_r}$. We conclude that all elements in the $C_\infty$-algebra $H^*(\mathfrak{g}, \mathbb{K})$ can be generated in $H^1(\mathfrak{g}, \mathbb{K})$ by $m_2$ and $m_3$. □

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[3] The operation $m_2$ is associative thus the result does not depend on the choice of the bracketing.
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