On the probability of satisfying a word in nilpotent groups of class 2

Matthew Levy
Imperial College London

Abstract

Let $G$ be a finite group of nilpotency class 2 and $w$ a group word. In this short paper we show that the probability that a random $n$-tuple of elements from $G$ satisfies $w$ is at least one over the order of $G$. This answers a special case of a conjecture of Alon Amit.

1 Introduction

Let $G$ be a finite group, $w(x_1, ..., x_n)$ a group word and denote by $N(G, w = c)$ the number of $n$-tuples $g = (g_1, ..., g_n) \in G^{(n)}$ satisfying $w(g) = c$, that is

\[ N(G, w = c) = |\{ g \in G^{(n)} : w(g) = c \}|. \]

Also, denote by $P(G, w = c)$ the probability that a random $n$-tuple $g = (g_1, ..., g_n) \in G^{(n)}$ satisfies $w(g) = c$, that is

\[ P(G, w = c) = \frac{N(G, w = c)}{|G|^n}. \]

When $c = 1$ we will just write $N(G, w)$ and $P(G, w)$ and we will say that $g$ satisfies $w$ if $w(g) = 1$. If $G$ were abelian, then the word map

\[ w : G^{(n)} \to G, \]

defined by a word $w$ is a homomorphism and it is clear that

\[ N(G, w) = |\text{Ker } w| = \frac{|G|^n}{|\text{Im } w|} \geq |G|^{n-1} \]

and so $P(G, w) \geq \frac{1}{|G|}$.

It is a conjecture of Alon Amit (see [1] or [2]) that if $G$ is a nilpotent group then $P(G, w) \geq \frac{1}{|G|}$. Here we establish the result for nilpotency class 2 groups in the following theorem.

Theorem 1.1. Let $G$ be a finite group of nilpotency class 2. Then for any group word $w$, $P(G, w) \geq \frac{1}{|G|}$. 

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This result improves the bound in the nilpotency class 2 case established in a paper by Nikolov and Segal (see [3]).

Remark 1.2. If the theorem holds true for two groups $G_1$ and $G_2$ then it holds true for their direct product $G = G_1 \times G_2$ since the word values can be solved componentwise. Let $g = (g_1,...,g_n)$ be an $n$-tuple in $G$ then $g = a\cdot h$ where $a = (a_1,...,a_n)$ and $h = (h_1,...,h_n)$ are $n$-tuples in $G_1$ and $G_2$ respectively. Then it is clear that $w(g) = w(a)\cdot w(h)$. Hence $P(G, w) \geq P(G_1, w)\cdot P(G_2, w)$ and the result follows since $|G| = |G_1|\cdot |G_2|$. In fact, if $A$ is abelian and the theorem holds for a group $H$ which acts on $A$ by automorphisms then the theorem holds for $G = A \times H$.

**Proposition 1.3.** Let $G$ be a finite group such that $G = A \times H$ where $A$ and $H$ are subgroups of $G$ and $A$ is abelian. Suppose that $P(H, w) \geq \frac{1}{|H|^2}$ where $w$ is a group word then $P(G, w) \geq \frac{1}{|H|^2}$.

**Proof.** For any $g \in G$ we may write $g = ah$ for unique $a \in A$ and $h \in H$, so if $w$ is a word in $n$ variables we have, for some $a_i \in A$ and $h_i \in H$,

$$w(g_1,...g_n) = w(a_1h_1,...,a_nh_n) = \prod_{i=1}^{n} a_i^{\phi_i(h_1,...,h_n)}w(h_1,...,h_n),$$

where the $\phi_i(h_1,...,h_n)$ are automorphisms of $A$ depending on the $h_i$. Note that there are at least $|H|^{n-1}$ $n$-tuples, $h = (h_1,...,h_n) \in H^{(n)}$, satisfying $w$. Having fixed such an $h \in H^{(n)}$ consider the induced map on $A^{(n)}$,

$$T_h : A^{(n)} \longrightarrow A \quad (a_1,...,a_n) \mapsto \prod_{i=1}^{n} a_i^{\phi_i(h_i)}.$$ 

Since $A$ is an abelian group, $T_h$ is a linear map and the number of $n$-tuples $a \in A^{(n)}$ such that $T_h(a) = 1$ is at least $|A|^{n-1}$. The result follows. \qed

Since any nilpotent group is a direct product of its Sylow subgroups, by the above remark it will be enough to prove the following theorem:

**Theorem 1.4.** Let $G$ be a finite $p$-group of nilpotency class 2 where $p$ is a prime. Then for any group word $w$, $P(G, w) \geq \frac{1}{|G|^2}$.

2 Nilpotent class 2 groups

We begin by making the following definition:

**Definition 2.1.** We will say that two group words $w_1$ and $w_2$ on $n$ variables are $G$-equivalent if $N(G, w_1 = c) = N(G, w_2 = c)$ for every $c \in G$.

**Remark 2.2.** It is clear that relabelling the variables of $w$ gives $G$-equivalent words for any group $G$ since the word maps are unchanged. Suppose that $w$ and $w'$ are two group words in $n$ variables such that $w \equiv w' \mod R$ where $R$ is a normal subgroup of the free group of rank $n$, $F_n$. Then it is easy to see that $w$
and \( w' \) are \( G\text{-equivalent} \) for any group \( G \) that is a homomorphic image of \( F_n/R \) since \( v(G) = 1 \) for any \( v \in R \). Also, suppose \( w(x_1, ..., x_n) \) is a group word and that \( w(x_1, ..., x_n) = v(y_1, ..., y_n) \) where \( v \) is a group word and the \( y_i \) are words in the \( x_i \)'s. Let \( \mathfrak{N}_{2,m} \) denote the class of all finite \( p \)-groups of nilpotency class at most 2 and of exponent at most \( p^m \). If the set \( \{y_1, ..., y_n\} \) maps onto a basis for the vector space \( L_n/\Phi(L_n) \) where \( L_n \) is the free \( \mathfrak{N}_{2,m} \)-group on \( n \) generators then \( w \) is \( G\text{-equivalent} \) to the word \( v(y_1, ..., y_n) \) for any \( G \in \mathfrak{N}_{2,m} \), since for each \( i \) the image of \( x_i \) in \( L_n \) can be expressed as a word in the \( y_j \)'s. This follows from the Burnside Basis Theorem (see [4]).

The following commutator identities will be used throughout and are easy to prove:

(1) \([x,y] = x^{-1}x^y;\)

(2) \([xy,z] = [x,z][y,z] \mod \gamma_3(F_n);\)

(3) \([xz, y] = [x, y][z, y] \mod \gamma_3(F_n);\)

(4) \([x, y]^{-1} = [y, x] \equiv [x^{-1}, y] \equiv [x, y^{-1}] \mod \gamma_3(F_n);\)

(5) \((xy)^n \equiv x^ny^n[y, x]^{[n(n-1)]_2} \mod \gamma_3(F_n).\)

Let \( w(x_1, ..., x_n) \) be any group word and fix a group \( G \in \mathfrak{N}_{2,m} \). By Hall’s Collecting Process (see [4]), we can write \( w \) in the form

\[
w(x_1, ..., x_n) = x_1^{\alpha_1} ... x_n^{\alpha_n} \prod_{i=1}^{n} \prod_{1 \leq i < j} (x_i, x_j)^{\beta_{ij}} c,
\]

(1)

where \( c \in \gamma_3(F_n) \), \( F_n \) being the free group of rank \( n \) and \( \alpha_i, \beta_{ij} \in \mathbb{Z} \). We aim to show that the word map given by \( w \) is ‘equivalent over \( G \)’, in the sense of definition [2.1] to the word map given by a particular word \( w' \), where it will be easy to see that \( P(G, w') \geq \frac{1}{|G|}. \) To do this we will first prove the following lemmas.

**Lemma 2.3.** Let \( w(x_1, ..., x_n) = x_1^{\alpha_1} ... x_n^{\alpha_n} \) be an element of the free group of rank \( n \) and \( m \in \mathbb{N} \). Then, for any \( G \in \mathfrak{N}_{2,m} \), \( w \) is \( G\text{-equivalent} \) to the word

\[
v(y_1, x_2, ..., x_n) = y_1^{p_1} \prod_{1 \leq i < j} (x_i, x_j)^{\beta_{ij}} \prod_{i=2}^{n} [y_1, x_i]^{\gamma_i},
\]

for some \( l, \beta_{ij}, \gamma_i \in \mathbb{Z} \).

**Proof.** Let \( R = \gamma_3(F_n) F_n^m \) and write \( w(x_1, ..., x_n) = x_1^{p_1^{l_1}} ... x_k^{p_k^{l_k}} m_k \), where \( i_1 < i_2 < ... < i_k \), \( l_j, m_j \in \mathbb{Z}, l_j \geq 0 \) and the \( m_j \) are non-zero and coprime to \( p \) for all \( j \). Choose \( l_t \) minimal among the \( l_j \) and let

\[y_{i_t} = x_1^{p_1^{l_1}} ... x_k^{p_k^{l_k}} m_k.
\]

Note that in the above expression for \( y_{i_t} \) the exponent of \( x_{i_t} \) is \( m_t \). Then

\[y_{i_t}^{p_{i_t}} \equiv x_1^{p_1^{l_1}} ... x_k^{p_k^{l_k}} \prod_{i_t \leq i_p < i_t \leq i_k} [x_{i_p}, x_{i_k}]^{-p_{i_t}^{l_{i_t}} p_{i_k}^{l_{i_k}} m_t \cdot p_{i_t}^{l_{i_t}} - 1} \mod \gamma_3(F_n),\]
so that

\[
w(x_1, \ldots, x_n) \equiv y_{t_i}^{\beta_{ij}} \prod_{i < j} [x_i, x_j]^{\beta_{ij}} \mod \gamma_3(F_n), \tag{2}
\]

for some \( \beta_{ij} \in \mathbb{Z} \). Note that \( x_{t_i}^{m_i} = (\prod_{c, t_i} x_i^{r_{i1} - 1} m_i) - 1 \) and that since \( p \) does not divide \( m_i \) there exists a positive integer \( r_i \) such that \( x_{t_i}^{m_i r_i} \equiv x_{t_i} \mod F_n^{p^{m_i}} \). Substituting the resulting expression for \( x_{t_i} \mod F_n^{p^{m_i}} \) into (2) we have

\[
w(x_1, \ldots, x_n) \equiv y_{t_i}^{\beta_{ij}} \prod_{i < j, i \neq j} [x_i, x_j]^{\beta_{ij}} \prod_{i \geq 1} [y_i, x_i]^{\gamma_i} \mod R,
\]

for some \( \beta_{ij}, \gamma_i \in \mathbb{Z} \). The result follows in view of remark 2.2. \( \square \)

**Lemma 2.4.** Let \( w(x_1, \ldots, x_n) = \prod_{i=1}^{n} \prod_{i < j} [x_i, x_j]^{\alpha_{ij}} \) be an element of the free group of rank \( n \) and \( m \in \mathbb{N} \). Then, for any \( G \in \mathfrak{R}_{2, m} \), \( w \) is \( G \)-equivalent to the word

\[
v(y_1, \ldots, y_{2k+1}, y_{2k+2}, \ldots, x_n) = \prod_{i=1}^{k} [y_{2i-1}, y_{2i}]^{\gamma_{2i-1} \gamma_{2i}}
\]

for some \( \gamma_{ij} \in \mathbb{Z} \), where \( 2k \leq n \).

**Proof.** Let \( R = \gamma_3(F_n) F_n^{p^m} \). We are going to describe an algorithm which shows us that we may write \( w \) in the form

\[
w(x_1, \ldots, x_n) = \prod_{i=1}^{k} [y_{2i-1}, y_{2i}]^{\gamma_{2i-1} \gamma_{2i}} c
\]

for some \( \gamma_{ij} \in \mathbb{Z} \), where \( 2k \leq n \), the \( y_i \) are words in the \( x_i \)'s and \( c \in \gamma_3(F_n) F_n^{p^m} \).

In particular, each \( y_i \) is of the form \( x_{i_1}^{m_1} \cdots x_{i_d}^{m_d} \) the \( m_j \) being non-zero and coprime to \( p \) with \( l_{i_j} \geq 0 \) and \( l_{i_j} = 0 \) for some \( u_i \). Moreover, for all \( y_i, y_j \) with \( i \neq j \) we have \( i_{u_i} \neq j_{u_j} \), i.e. \( x_{i_{u_i}} \neq x_{j_{u_j}} \). The result then follows in view of remark 2.2. We proceed as follows:

Choose the first non zero \( \alpha_{ij} \), with respect to the ordering in the product, say \( \alpha_{s_1 s_1} \). Then

\[
w(x_1, \ldots, x_n) = \prod_{i < j} [x_i, x_j]^{\alpha_{ij}} = [x_{s_1}, x_{s_1}]^{\alpha_{s_1 s_1}} \prod_{s_1 < i < j} [x_i, x_j]^{\alpha_{ij}},
\]

with \( s_1 < s_11 < s_{12} < \ldots < s_{1q} \) and \( \alpha_{s_1 s_1} \) non-zero for all \( j \). Now \( \alpha_{s_1 s_j} = p^{l_{s_11}+l_{s_11}} m_{s_1 s_j} \) where \( l_{s_1 s_j} \geq 0 \) is an integer and \( m_{s_1 s_j} \) is coprime to \( p \). So choose \( l_{s_1 s_j} \) minimal among the \( l_{s_1 s_j} \), say \( l_{s_1 s_{1u_1}} \), and let

\[
y_{s_1 u_1} = x_{s_11}^{p^{l_{s_11}+l_{s_11}} m_{s_1 s_{11}}}, \ldots, x_{s_1 q}^{p^{l_{s_11}+l_{s_11}} m_{s_1 s_{1q}}},
\]

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Similarly to the previous lemma,
\[ [x_{s_1}, x_{s_11}] \ldots [x_{s_{1n}}] \equiv [x_{s_1}, y_{s_{1u_1}}, y_{s_{1u_2}}]^{l_{s_1u_1} - l_{s_1u_2}} \mod \gamma_3(F_n). \]

So
\[ w(x_1, \ldots, x_n) \equiv [x_{s_1}, y_{s_{1u_1}}]^{p_{l_{s_1u_1}}} \prod_{s_1 < i < j} [x_i, x_j]^{\alpha_{ij}} \mod \gamma_3(F_n). \quad (3) \]

If the remaining \( \alpha_{ij} \) are all zero we can stop here noting that in the expression for \( y_{s_{1u_1}} \) the exponent of \( x_{s_{1u_1}} \) is non-zero and coprime to \( p \) and \( x_{s_{1u_1}} \neq x_{s_1} \). Otherwise, as in the previous lemma, we may substitute \( x_{s_{1u_1}} \) for \( y_{s_{1u_1}} \) and that \( \alpha \) is non-zero and coprime to \( y \). We can set \( y_1 := y_{s_{1u_1}} \) and \( y_2 := y_{s_{1u_1}} \) and repeat the algorithm for the rest of the commutators.

Note that the exponent of \( x \) is non-zero and coprime to \( y \) and that \( \alpha \) is non-zero and coprime to \( y \). We can set \( y_1 := y_{s_{1u_1}} \) and \( y_2 := y_{s_{1u_1}} \). Then, similarly to before,
\[ [y_{s_{1u_1}}, x_{s_{21}} \ldots x_{s_{2r}}] \equiv [y_{s_{1u_1}}, y_{s_{2u_2}}]^{\alpha_{s_1u_1} - \alpha_{s_1u_2}} \mod \gamma_3(F_n). \]

Thus
\[ w = [x_{s_1}, y_{s_{1u_1}}]^{p_{l_{s_1u_1}}} [y_{s_{1u_1}}, y_{s_{2u_2}}]^{p_{l_{s_1u_1} - l_{s_1u_2}}} \prod_{s_1 < i < j} [x_i, x_j]^{\alpha_{ij}} \mod R. \quad (4) \]

There are two cases to consider.

Case (1a): The first case is when \( l_{s_1s_{1u_1}} \leq l_{s_{1u_1}s_{2u_2}} \). We have
\[ [x_{s_1}, y_{s_{1u_1}}]^{p_{l_{s_1u_1}}} [y_{s_{1u_1}}, y_{s_{2u_2}}]^{p_{l_{s_1u_1} - l_{s_1u_2}}} \equiv [y_{s_{1u_1}}, y_{s_{1u_1}}]^{p_{l_{s_1u_1} - l_{s_1u_2}}} \mod \gamma_3(F_n), \]
where \( y_1 = x_{s_1}^{-1} y_{s_{2u_2}}^{p_{l_{s_1u_1} - l_{s_1u_2}}} \). This gives
\[ w(x_1, \ldots, x_n) \equiv [y_{s_{1u_1}}, y_{s_{1u_1}}]^{p_{l_{s_1u_1} - l_{s_1u_2}}} \prod_{s_1 < i < j} [x_i, x_j]^{\alpha_{ij}} \mod R. \quad (5) \]

Note that in the expression for \( y_{s_{1u_1}} \) the exponent of \( x_{s_1} \) is non-zero and coprime to \( p \) as is the exponent of \( x_{s_{1u_1}} \) in \( y_{s_{1u_1}} \). In particular note that neither \( x_{s_1} \) nor \( x_{s_{1u_1}} \) appear in the rest of the expression for \( w \) mod \( R \) and that \( x_{s_{1u_1}} \neq x_{s_1} \). We can set \( y_1 := y_{s_{1u_1}} \) and \( y_2 := y_{s_{1u_1}} \) and repeat the algorithm for the rest of the commutators.
Case (2a): The second case however is when \( l_{s_1, s_{1u_1}} > l_{s_1, s_{2u_2}} \). We have

\[
[x_{s_1}, y_{s_{1u_1}}]^{l_{s_1, s_{1u_1}}'} [y_{s_{1u_1}}, y_{s_{2u_2}}]^{l_{s_1, s_{2u_2}}'} \equiv [y_{s_{1u_1}}, y_{t_{2u_2}}]^{l_{s_1, s_{2u_2}}'} \mod \gamma_3(F_n)
\]

where \( y_{t_{2u_2}} = x_{s_1}^{l_{s_1, s_{1u_1}}'} - l_{s_1, s_{2u_2}}' \). This gives

\[
w(x_1, ..., x_n) \equiv [y_{s_{1u_1}}, y_{t_{2u_2}}]^{l_{s_1, s_{2u_2}}'} \prod_{s_1 < i < j} [x_i, x_j]^{\alpha_{ij}} \mod R.
\]

Here the exponent of \( x_{s_1, s_{1u_1}} \) in \( y_{s_{1u_1}} \) is non-zero and coprime to \( p \) and the same is true for the exponent of \( x_{s_{2u_2}} \) in \( y_{t_{2u_2}} \). However, in contrast to Case (1a), whilst \( x_{s_1, s_{1u_1}} \) does not appear in the rest of the expression for \( w \) mod \( R \) it is possible that \( x_{s_{2u_2}} \) may. If it doesn’t then set \( y_1 := y_{s_{1u_1}} \) and \( y_2 := y_{t_{2u_2}} \) and repeat the algorithm for the rest of the commutators, if there are any. If it does substitute it out of the expression in (6) using

\[
y_{t_{2u_2}} = x_{s_1}^{l_{s_1, s_{1u_1}}'} - l_{s_1, s_{2u_2}}' \quad y_{s_{2u_2}} = x_{s_1}^{l_{s_1, s_{1u_1}}'} - l_{s_1, s_{2u_2}}'
\]

This gives

\[
w \equiv [y_{s_{1u_1}}, y_{t_{2u_2}}]^{l_{s_1, s_{2u_2}}'} [y_{t_{2u_2}}, x_{s_{3u_3}}^{\alpha''_{s_{2u_2}, s_{3u_3}}} \ldots x_{s_{3u_3}}^{\alpha''_{s_{2u_2}, s_{3u_3}}} \ldots x_{s_{3u_3}}^{\alpha''_{s_{2u_2}, s_{3u_3}}}] \prod_{i < j} [x_i, x_j]^{\alpha_{ij}^\prime} \mod R,
\]

where, as usual, \( \alpha''_{s_{2u_2}, s_{3u_3}} = p_{s_{2u_2}, s_{3u_3}}^{l_{s_{2u_2}, s_{3u_3}}'} m_{s_{2u_2}, s_{3u_3}}^{l_{s_{2u_2}, s_{3u_3}}'} \) for all \( j \). Now choose \( l_{s_{2u_2}, s_{3u_3}}'' \) minimal among the \( l_{s_{2u_2}, s_{3u_3}}'' \)'s and let

\[
y_{s_{3u_3}} = x_{s_{3u_3}}^{l_{s_{2u_2}, s_{3u_3}}''} - l_{s_{2u_2}, s_{3u_3}}'' \quad m_{s_{2u_2}, s_{3u_3}}^{l_{s_{2u_2}, s_{3u_3}}''} \ldots x_{s_{3u_3}}^{l_{s_{2u_2}, s_{3u_3}}''} \ldots x_{s_{3u_3}}^{l_{s_{2u_2}, s_{3u_3}}''}.
\]

Then, as before, the expression for \( w \) becomes

\[
w \equiv [y_{s_{1u_1}}, y_{t_{2u_2}}]^{l_{s_1, s_{2u_2}}'} [y_{t_{2u_2}}, y_{s_{3u_3}}]^{l_{s_{2u_2}, s_{3u_3}}'} \prod_{i < j} [x_i, x_j]^{\alpha_{ij}''} \mod R.
\]

Again there are two cases.

Case (1b): If \( l_{s_1, s_{2u_2}}' \leq l_{s_{2u_2}, s_{3u_3}}'' \) then \( w \) becomes

\[
w(x_1, ..., x_n) \equiv [y_{t_{2u_2}}, y_{s_{1u_1}}]^{l_{s_1, s_{2u_2}}'} \prod_{i < j} [x_i, x_j]^{\alpha_{ij}''} \mod R.
\]

where \( y_{s_{1u_1}} = y_{s_{1u_1}}^{-1} y_{s_{3u_3}}^{l_{s_{2u_2}, s_{3u_3}}'} - l_{s_1, s_{2u_2}}' \) and we are done as in Case (1a).
Case (2b): If however $l'_{2u_2} > l''_{2u_2}$ we can repeat the algorithm described in Case (2a) above and we will end up with an expression of the form

$$w \equiv [y_{2u_2}, y_{2u_2}] [y_{2u_3}, y_{2u_4}] \prod_{i<j} [x_i, x_j]^{a_{ij}} \mod R, \quad (9)$$

where $y_{2u_3}$ and $y_{2u_4}$ are words in the $x_i$s with the exponents of $x_{2u_3}$ and $x_{2u_4}$ non-zero and coprime to $p$ respectively. Again there are two cases. If in (9) we have $l''_{2u_2} \leq l''_{3u_2}$ then, as in Cases (1a) and (1b), we are done. If not we have $l_{2u_1, u_2} > l'_{2u_2} > l''_{2u_2} > l''_{3u_2} \geq \cdots \geq 0$ and we keep going until the algorithm stops, i.e. we are in the first case.

We will eventually end up with an expression like (5) or (8). If the remaining $a'_{ij}$, in (8) say, are all zero we can stop here. Otherwise, set $y_1 := y_{2u_2}$ and $y_2 := y_{2u_1}$ and repeat the algorithm on the rest of the commutators. Eventually, after relabelling, we have an expression for $w$ of the desired form.

We are now ready to prove the theorem. Let $w(x_1, ..., x_n)$ be any group word and fix a group $G \in \mathfrak{G}_{2-m}$. From (1) we have

$$w(x_1, ..., x_n) \equiv x_1^{a_1} ... x_n^{a_n} (\prod_{i<j} [x_i, x_j]^{\beta_{ij}}) \mod \gamma_3(F_n).$$

By Lemma 2.3 $w$ is $G$-equivalent to the word

$$w'(y_1, x_2, ..., x_n) \equiv y_1^{p'} \prod_{i<j} [x_i, x_j]^{\beta_{ij}} [y_1, h]$$

for some $l, \beta_{ij} \in \mathbb{Z}$, where $h$ is some word in the $x_i$s with $i \neq 1$. As in the proof of Lemma 2.3 we can substitute $x_1$ out of the expression above giving us

$$w'(y_1, x_2, ..., x_n) \equiv y_1^{p'} \prod_{1<i<j} [x_i, x_j]^{\beta_{ij}} [y_1, h'] \mod R$$

for some $\beta_{ij} \in \mathbb{Z}$, where $R$ denotes $\gamma_3(F_n) R_{p-m}^n$ and $h'$ is some word in the $x_i$s with $i \neq 1$. Then by Lemma 2.4 $w'$ is $G$-equivalent to the word

$$v(y_1, y_2, ..., y_{2k+1}, x_{2k+2}, ..., x_n) = y_1^{p'} \prod_{i=1}^k [y_{2i}, y_{2i+1}]^{\gamma_{2i+1}} [y_1, h'']$$

for some $\gamma_{ij} \in \mathbb{Z}$, where $2k + 1 \leq n$ and $h''$ is some word in the $y_i$s and $x_j$s for $i = 2, ..., 2k + 1$ and $j = 2k + 2, ..., n$. We write this as

$$v(y_1, y_2, z_2, ..., z_{k+1}, x_{2k+2}, ..., x_n) = y_1^{p'} \prod_{i=2}^{k+1} [y_i, z_i]^{\nu_i} [y_1, h''],$$

where $\nu_i+1 = \gamma_{2i+1}$. Consider the word map given by $v$. The commutator map from $G \times G$ to $G$ sending a pair $(x, y)$ to its commutator $[x, y]$ is a bilinear map. Fixing the $z_i$ for all $i$ and restricting $y_1$ to the derived group of $G$ we obtain a linear map

$$v' : G' \times G^{(n-1-k)} \to G'.$$
defined by \( v'(y_1, \ldots, y_{k+1}, x_{2k+2}, \ldots, x_n) = v(y_1, y_2, \ldots, y_{k+1}, z_{k+1}, x_{2k+2}, \ldots, x_n) \). Now \(|\{g \in G' \times G^{(n-1-k)} : v'(g) = 1\}| \geq |G|^{n-1-k}\) and so \(N(G, v) \geq |G|^{n-1}\) since we had \(|G|\) choices for each of the \(z_i\). Thus \( P(G, v) \geq \frac{1}{|G|} \) and the result follows since \( P(G, w) = P(G, v) \).

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