Scattering and Sparse Partitions, and their Applications

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A partition \( P \) of a weighted graph \( G \) is \((\sigma, \tau, \Delta)\)-sparse if every cluster has diameter at most \( \Delta \), and every ball of radius \( \Delta/\sigma \) intersects at most \( \tau \) clusters. Similarly, \( P \) is \((\sigma, \tau, \Delta)\)-scattering if instead for balls we require that every shortest path of length at most \( \Delta/\sigma \) intersects at most \( \tau \) clusters. Given a graph \( G \) that admits a \((\sigma, \tau, \Delta)\)-sparse partition for all \( \Delta > 0 \), Jia et al. [STOC05] constructed a solution for the Universal Steiner Tree problem (and also Universal TSP) with stretch \( O(\tau^3 \log^3 n) \). Given a graph \( G \) that admits a \((\sigma, \tau, \Delta)\)-scattering partition for all \( \Delta > 0 \), we construct a solution for the Steiner Point Removal problem with stretch \( O(\tau^3 \sigma^3) \). We then construct sparse and scattering partitions for various different graph families, receiving many new results for the Universal Steiner Tree and Steiner Point Removal problems.

CCS Concepts: • Theory of computation → Graph algorithms analysis; Random projections and metric embeddings; Facility location and clustering; Routing and network design problems.

1 INTRODUCTION

Graph and metric clustering are widely used for various algorithmic applications (e.g., divide and conquer). Such partitions come in a variety of forms, satisfying different requirements. This paper is dedicated to the study of bounded diameter partitions, where small neighborhoods are guaranteed to intersect only a bounded number of clusters.

The first problem we study is the Steiner Point Removal (SPR) problem. Here we are given an undirected weighted graph \( G = (V, E, w) \) and a subset of terminals \( K \subseteq V \) of size \( k \) (the non-terminal vertices are called Steiner vertices). The goal is to construct a new weighted graph \( M = (K, E', w') \) with the terminals as its vertex set, such that: (1) \( M \) is a graph minor of \( G \), and (2) the distance between every pair of terminals \( t, t' \) in \( M \) is distorted by at most a multiplicative factor of \( \alpha \), formally

\[
\forall t, t' \in K, \ d_G(t, t') \leq d_M(t, t') \leq \alpha \cdot d_G(t, t') ,
\]

Property (1) expresses preservation of the topological structure of the original graph. For example if \( G \) was planar, so will \( M \) be. Whereas property (2) expresses preservation of the geometric structure of the original graph, that is, distances between terminals. The question is thus: given a graph family \( \mathcal{F} \), what is the minimal \( \alpha \) such that every graph in \( \mathcal{F} \) with a terminal set of size \( k \) will admit a solution to the SPR problem with distortion \( \alpha \).

Consider a weighted graph \( G = (V, E, w) \) with a shortest path metric \( d_G \). The weak diameter of a cluster \( C \subseteq V \) is the maximal distance between a pair of vertices in the cluster w.r.t. \( d_G \) (i.e., \( \max_{u, v \in C} d_G(u, v) \)). The strong diameter is the maximal distance w.r.t. the shortest path metric in the induced graph \( G[C] \) (i.e., \( \max_{u, v \in C} d_{G[C]}(u, v) \)). A partition \( P \) of \( G \) has weak (resp. strong) diameter \( \Delta \) if every cluster \( C \in P \) has weak (resp. strong) diameter at most \( \Delta \). Partition \( P \) is connected, if the graph induced by every cluster \( C \in P \) is connected. Given a shortest path \( \mathcal{I} = \{v_0, v_1, \ldots, v_{\tau}\} \), denote by \( Z_I(P) = \sum_{C \in P \cap \mathcal{I} \neq \emptyset} 1 \) the number of clusters in \( P \) intersecting \( \mathcal{I} \). If \( Z_I(P) \leq \tau \), we say that \( \mathcal{I} \) is \( \tau \)-scattered by \( P \).

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We say that a graph is (σ, τ, Δ)-scattering if the following conditions hold:

- **P** is connected and has weak diameter Δ.
- Every shortest path I of length at most Δ/σ is τ-scattered by **P**, i.e., \( Z_I(\mathcal{P}) \leq \tau \).

We say that a graph is (σ, τ)-scatterable if for every parameter Δ, \( G \) admits a (σ, τ, Δ)-scattering partition that can be computed efficiently.

The main contribution of this paper is the finding that scattering partitions imply solutions for the SPR problem. The proof appears in Section 3.1.

**Theorem 1** (Scattering Partitions imply SPR). Let \( G = (V, E, w) \) be a weighted graph such that for every subset \( A \subseteq V \), \( G[A] \) is \((1, \tau)\)-scatterable. Let \( K \subseteq V \) be some subset of terminals. Then there is a solution to the SPR problem with distortion \( O(\tau^3) \) that can be computed efficiently.

Jia, Lin, Noubir, Rajaraman, and Sundaram [55] defined the notion of sparse partitions, which is closely related to scattering partitions. Consider a partition \( \mathcal{P} \). Given a ball \( B = B_G(v, r) \) of radius \( r \leq \Delta/\sigma \) intersects at most \( \Delta \) clusters, i.e., \( Z_B(\mathcal{P}) \leq \tau \).

**Definition 2** (Strong/Weak Sparse Partition). Given a weighted graph \( G = (V, E, w) \), we say that a partition \( \mathcal{P} \) is (σ, τ, Δ)-weak (resp. strong) sparse partition if for every parameter Δ, \( G \) admits an efficiently computable (σ, τ, Δ)-weak (resp. strong) sparse partition.

Jia et al. [55] found a connection between sparse partitions to the Universal Steiner Tree Problem (UST). Consider a complete weighted graph \( G = (V, E, w) \) (or a metric space \( (X, d) \)) where there is a special server vertex \( rt \in V \), which is frequently required to multicast messages to different subsets of clients \( S \subseteq V \). The cost of a multicast is the total weight of all edges used for the communication. Given a subset \( S \), the optimal solution is to use the minimal Steiner tree spanning \( S \cup \{rt\} \). In order to implement an infrastructure for multicasting, or in order to make routing decisions much faster (and not compute it from scratch once \( S \) is given), a better solution will be to compute a Universal Steiner Tree (UST). A UST is a tree \( T \) over \( V \), such that for every subset \( S \), the message will be sent using the sub-tree \( T(S) \) spanning \( S \cup \{rt\} \). See illustration on the right, where the set \( S \) is surrounded by red circles and \( T(S) \) is purple. The stretch of \( T \) is the maximum ratio among all subsets \( S \subseteq X \) between the weight of \( T(S) \) and the weight of the minimal Steiner tree spanning \( S \cup \{rt\} \), \( \max_{S \subseteq X} \frac{w(T(S))}{\text{Opt}(S \cup \{rt\})} \).

Jia et al. [55] proved that given a sparse partition scheme, one can efficiently construct a UST with low stretch (the same statement holds w.r.t. UTPP as well).

**Theorem 2** (Sparse Partitions imply UST, [55]). Suppose that an \( n \)-vertex graph \( G \) admits a (σ, τ)-weak sparse partition scheme, then there is a polynomial time algorithm that given a root \( rt \in V \) computes a UST with stretch \( O(\tau r^2 \log n) \).

\(^1\) In Observation 2 we argue that (σ, τ, Δ)-scattering partition is also (1, τ, Δ)-scattering. We study the general case, even though Theorem 1 requires only σ = 1. This is as we find the more general case theoretically interesting, as well as potentially applicable.

\(^2\) Awerbuch and Peleg [8] were the first to study sparse covers (see Definition 3). Their notion of sparse partition is somewhat different from the one used here (introduced by [55]).

\(^3\) A closely related problem is the Universal Traveling Salesman Problem (UTSP), see Section 1.5.
Jia et al. [55] constructed \((O(\log n), O(\log n))\)-weak sparse partition scheme for general graphs, receiving a solution with stretch polylog\((n)\) for the UST problem. In some applications the communication is allowed to flow only in certain routes. It is therefore natural to consider the case where \(G = (V, E, w)\) is not a complete graph, and the UST is required to be a subgraph of \(G\), we refer to this as the Subgraph UST problem. Busch, Jaikumar, Radhakrishnan, Rajaraman, and Srivathsan [14] proved a theorem in the spirit of Theorem 2, stating that given a \((σ, τ, γ)\)-hierarchical strong sparse partition, one can efficiently construct a subgraph UST with stretch \(O(σ^2τ^2γ\log n)\). A \((σ, τ, γ)\)-hierarchical strong sparse partition is a laminar collection of partitions \(\{P_ι\}_ι\) such that \(P_ι\) is \((σ, τ, γ)\)-strong sparse partition which is a refinement of \(P_ι+1\). Busch et al. constructed a \(2^{O(\sqrt{\log n})}, 2^{O(\sqrt{\log n})}, 2^{O(\sqrt{\log n})}\) hierarchical strong sparse partition, obtaining a \(2^{O(\sqrt{\log n})}\) stretch algorithm for the subgraph UST problem. We tend to believe that poly-logarithmic stretch should be possible. It is therefore interesting to construct strong sparse partitions, as it eventually may lead to hierarchical ones.

A notion which is closely related to sparse partitions is sparse covers.

Definition 3 (Strong/Weak Sparse cover). Given a weighted graph \(G = (V, E, w)\), a \((σ, τ, Δ)\)-weak (resp. strong) sparse cover is a set of clusters \(C \subseteq 2^V\), where all the clusters have weak (resp. strong) diameter at most \(Δ\), and the following conditions hold:

- **Cover:** \(Yu \in V\), there exists \(C \in C\) such that \(B_σ(u, Δ/σ) \subseteq C\).
- **Sparsity:** every vertex \(u \in V\) belongs to at most \(|\{C \in C \mid u \in C\}| \leq τ\) clusters.

We say that a graph \(G\) admits a \((σ, τ, Δ)\)-weak (resp. strong) cover scheme if for every parameter \(Δ\), \(G\) admits a \((σ, τ, Δ)\)-weak (resp. strong) cover that can be computed efficiently.

It was (implicitly) proven in [55] that given \((σ, τ, Δ)\)-weak sparse cover \(C\), one can construct a \((σ, τ, Δ)\)-weak sparse partition. In fact, most previous constructions of weak sparse partitions were based on sparse covers.

1.1 Previous results

**SPR.** Given an \(n\)-point tree, Gupta [47] provided an upper bound of \(8\) for the SPR problem (on trees). This result was recently reproved by the author, Krauthgamer, and Trabelsi [41] using the Relaxed-Voronoi framework. Chan, Xia, Konjevod, and Richa [16] provided a lower bound of \(8\) for trees. This is the best known lower bound for the general SPR problem. Basu and Gupta [11] provided an \(O(1)\) upper bound for the family of outerplanar graphs. For general \(n\)-vertex graphs with \(k\) terminals the author [34, 36] recently proved an \(O(\log k)\) upper bound for the SPR problem using the Relaxed-Voronoi framework, improving upon previous works by Kamma, Krauthgamer, and Nguyen [57] \((O(\log^3 k))\), and Cheung [21] \((O(\log^2 k))\) (which were based on the Ball-Growing algorithm). Interestingly, there are no results on any other restricted graph family, although several attempts have been made (see [22, 32, 62]).

**UST.** Given an \(n\)-point metric space and root \(rt\), Gupta, Hajiaghayi and Räcke [48] constructed a UST with stretch \(O(\log^2 n)\), improving upon a previous \(O(\log^4 n/\log\log n)\) result by [55]. [55] is based on sparse partitions, while [48] is based on tree covers. Jia et al. [55] proved a lower bound of \(Ω(\log n)\) to the UST problem, based on a lower bound to the online Steiner tree problem by Alon and Azar [6]. Using the same argument, they [55] proved an \(Ω(\log n/\log\log n)\) lower bound for the case where the space is the \(n \times n\) grid (using [53]). Given a space with doubling dimension \(d\text{dim}\), Jia et al. [55] provided a solution with stretch \(2^{O(d\text{dim})} \cdot \log n\), using sparse partitions. Given an \(n\) vertex planar graph, Busch, LaFortune, and Tirthapura [15] proved an \(O(\log n)\) upper bound (improving over Hajiaghayi, Kleinberg, and Leighton [51]). More generally, for graphs \(G\) excluding a fixed

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minor, both Hajiaghayi et al. [51] (implicitly) and Busch et al. [15] (explicitly) provided a solution with stretch $O(\log^2 n)$. Both constructions used sparse covers. Finally, Busch et al. [14] constructed a subgraph UST with stretch polylog$(n)$ for graphs excluding a fixed minor (using hierarchical strong sparse partitions).

**Scattering Partitions.** As we are the first to define scattering partitions there is not much previous work. Nonetheless, Kamma et al. [57] implicitly proved that general $n$-vertex graphs are $(O(\log n), O(\log n))$-scatterable.\(^7\)

**Sparse Covers and Partitions.** Awerbuch and Peleg [8] introduced the notion of sparse covers and constructed $(O(\log n), O(\log n))$-strong sparse cover scheme for $n$-vertex weighted graphs.\(^9\) Jia et al. [55] induced an $(O(\log n), O(\log n))$-weak sparse partition scheme. Hajiaghayi et al. [51] constructed an $(O(1), O(\log n))$-weak sparse cover scheme for $n$-vertex planar graph, concluding an $(O(1), O(\log n))$-weak sparse partition scheme. Their construction is based on the [59] clustering algorithm. Abraham, Gavoille, Malkhi, and Wieder [5] constructed $(O(r^2), 2^{O(r^2)} \cdot r!)$-strong sparse cover scheme for $K_r$-free graphs. Busch et al. [15] constructed a $(48, 18)$-strong sparse cover scheme for planar graphs and $(8, O(\log n))$-strong sparse cover scheme for graphs excluding a fixed minor, concluding a $(48, 18)$ and $(8, O(\log n))$-weak sparse partition schemes for these families (respectively). For graphs with doubling dimension $\text{ddim}$, Jia et al. [55] constructed an $(1, 8^{\text{ddim}})$-weak sparse scheme. Abraham et al. [3] constructed a $(2, 4^{\text{ddim}})$-strong sparse cover scheme. In a companion paper, the author [35] constructed an $(O(\text{ddim}), O(\text{ddim} \cdot \log \text{ddim}))$-strong sparse cover scheme.\(^10\) Busch et al. [14] constructed $(O(\log^4 n), O(\log^3 n), O(\log^2 n))$-hierarchical strong sparse partition for graphs excluding a fixed minor.

### 1.2 Our Contribution

The main contribution of this paper is the definition of scattering partition and the finding that good scattering partitions imply low distortion solutions for the SPR problem (Theorem 1). We construct various scattering and sparse partition schemes for many different graph families, and systematically classify them according to the partition types they admit. In addition, we provide several lower bounds. The specific partitions and lower bounds are described below. Our findings are summarized in Table 1, while the resulting classification is illustrated in Figure 1.

Recall that [55] (implicitly) showed that sparse covers imply weak sparse partitions (Lemma 5). We show that the opposite direction is also true. That is, given a $(\sigma, \tau, \Delta)$-weak sparse partition, one can construct a $(\sigma+2, \tau, (1+\frac{\sigma}{\tau})\Delta)$-weak sparse cover (Lemma 6). Interestingly, in addition we show that strong sparse partitions imply strong sparse covers, while the opposite is not true. Specifically there are graph families that admit $(O(1), O(1))$-strong sparse cover schemes, while there are no constants $\sigma, \tau, \Delta$ such that they admit $(\sigma, \Delta)$-strong sparse partitions. All our findings on the connection between sparse partitions and sparse covers, and a classification of various graph families, appears in Section 4 and are summarized in Figure 3.

The scattering partitions we construct imply new solutions for the SPR problem previously unknown. Specifically, for every graph with pathwidth $\rho$ we provide a solution to the SPR problem with distortion $\text{poly}(\rho)$, independent of the number of terminals (Corollary 8). After trees [47] and outerplanar graphs [11],\(^5\) this is the first graph family to have solution for the SPR problem independent from the number of terminals. Furthermore, we obtain solutions with constant distortion for Chordal and Cactus graphs (Corollaries 11 and 12).\(^11\)

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\(^7\)This follows from Theorem 1.6 in [57] by choosing parameters $t = \beta = O(\log n)$ and using union bounds over all $n^2$ shortest paths. Note that they assume that for every pair of vertices there is a unique shortest path.

\(^9\)More generally, for $k \in \mathbb{N}$, [8] constructed a $(4k - 2, 2k \cdot n^{1/3})$-strong sparse cover scheme.

\(^10\)More generally, for a parameter $t = \Omega(1)$, [35] constructed $(O(t), O(2\text{ddim} / \Delta \cdot \log t))$-sparse cover scheme.

\(^11\)Note that the family of cactus graph is contained in the family of outerplanar graph. Basu and Gupta [11] solved the SPR problem directly on outerplanar graphs with constant distortion. However, this manuscript was never published. See also footnote 5.
| Family                   | Partition type | Padding ($\sigma$) | #intersections ($\tau$) | Ref/Notes |
|-------------------------|----------------|------------------|------------------------|-----------|
| General $n$-vertex graphs | Weak           | $O(\log n)$      | $O(\log n)$            | [55]      |
|                         | Scattering     | $O(\log n)$      | $O(\log n)$            | [57]      |
|                         | Strong         | $O(\log n)$      | $O(\log n)$            | Theorem 4 |
|                         | Weak L.B.      | $\Omega(log n/\log \log n)$ | $O(\log n)$            | Theorem 5 |
| ddim doubling dimension | Weak           | 1                | $g^{ddim}$             | [55]      |
|                         | Strong         | $O(\text{ddim})$ | $O(\text{ddim})$       | Theorem 10 |
| Euclidean space $(\mathbb{R}^d, ||\cdot||_2)$ | Scattering     | 1                | $2d$                   | Theorem 11 |
|                         | Weak L.B.      | $O(1)$           | $\Omega(d/\log d)$     | Theorem 12 |
| Trees                   | Scattering     | 2                | 3                      | Theorem 7  |
|                         | Weak           | 4                | 3                      | Theorem 8  |
|                         | Strong L.B.    | $\log n/\log \log n$ | $\log n/2\sqrt{\log n}$ | Theorem 9 |
| Pathwidth $\rho$ (SPDdepth $1^4$) | Strong         | $O(\rho)$       | $O(\rho^2)$            | Theorem 13 |
|                         | Weak           | 8                | $5\rho$                | Theorem 14 |
|                         | Corollary 10   |                  |                        |           |
| Chordal                 | Scattering     | 2                | 3                      | Theorem 15 |
|                         | Weak           | 24               | 3                      | Corollary 10 |
| K$_\rho$ free           | Weak           | $O(r^2)$         | $2^r$                  | Corollary 3 |
| Cactus                  | Scattering     | 4                | 5                      | Theorem 16 |

Table 1. Summary of the various new/old, weak/strong scattering/sparse partitions.

Table footnotes: * More generally, there is a partition $P$ s.t. every ball of radius $\Delta$ intersects at most $O(n^{1/\alpha})$ clusters, for all $\alpha > 1$ simultaneously. ■ More generally, it must hold that $\tau \geq n^{\Omega(1/\alpha)}$. ♠ More generally, there is a partition $P$ s.t. every ball of radius $\Omega(\Delta)$ intersects at most $O(2^{\alpha \text{ddim}}/\alpha)$ clusters, for all $\alpha > 1$ simultaneously. ♦ More generally, it must hold that $\tau \geq (1 + \frac{1}{\Delta})^d$.

Note that this lower bound holds for chordal/cactus/planar/K$_\rho$-free graphs. More generally, it must hold that $\tau \geq \Omega(n^{2(\alpha+1)})$.

The weak sparse partitions we construct imply improved solutions for the UST (and UTSP) problem. Specifically, we conclude that for graphs with doubling dimension ddim a UST (and UTSP) with stretch poly(ddim) · log $n$ can be efficiently computed (Corollary 7), providing an exponential improvement in the dependence on ddim compared with the previous state of the art [55] of $2^{O(ddim)}$ · log $n$. For K$_\rho$-minor free graphs we conclude that an UST (or UTSP) with stretch $2^{O(r)}$ · log $n$ can be efficiently computed (Corollary 4), providing a quadratic improvement in the dependence on $n$ compared with the previous state of the art [48] of $O(\log^2 n)$. $^{12}$Finally, for pathwidth $\rho$ graphs (more generally, graphs with SPDdepth $\rho$) we can compute a UST (or UTSP) with stretch $O(\rho \cdot \log n)$ (Corollary 9), improving over previous solutions that were exponential in $\rho$ (based on the fact that pathwidth $\rho$ graphs are K$_{\rho^2+2}$-minor free).

$^{12}$This result is a mere corollary obtained by assembling previously existing parts together. Mysteriously, although UTSP on minor free graphs was studied before [15, 51], this corollary was never drawn, see Section 4.1.

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Before we proceed to describe our partitions we make two observations.

**Observation 1.** Every \((\sigma, \tau, \Delta)\)-strong sparse partition is also a scattering partition and a weak sparse partition with the same parameters.

**Observation 2.** Every \((\sigma, \tau, \Delta)\)-scattering partition is also \((1, \sigma\tau, \Delta)\)-scattering partition.

Observation 1 follows as every path of weight \(\sigma\Delta\) is contained in a ball of radius \(\sigma\Delta\). Observation 2 follows as every shortest path of length \(\leq \Delta\) can be assembled as a concatenation of at most \(\sigma\) shortest paths of length \(\leq \frac{\Delta}{\sigma}\).

**General Graphs:** Given an \(n\)-vertex general graph and parameter \(\Delta > 0\) we construct a single partition \(P\) which is simultaneously \(8k, O(n^{1/k} \cdot \log n), \Delta\)-strong sparse partition for all parameters \(k \geq 1\) (Theorem 4). Thus we generalize the result of [55] and obtain a strong diameter guarantee. This partition implies that general graphs are \((O(\log n), O(\log n))\)-scatterable (reproving [57] via an easier proof), inducing a solution for the SPR problem with stretch \(\text{polylog}(|K|)\) (Corollary 5). While quantitatively better solutions are known, this one is arguably the simplest, and induced by a general framework. Further, we provide a lower bound, showing that if all \(n\)-vertex graphs admit \((\sigma, \tau)\)-weak sparse partition scheme, then \(\tau \geq n^{(1/2)\left(1/d}\right) \) (Theorem 5). In particular there is no sparse partition scheme with parameters smaller than \((\Omega(\log n/\log \log n), \Omega(\log n))\). This implies that both our results and [55] are tight up to second order terms. Although we do not provide any lower bound for scattering partitions, we present some evidence that general graphs are not \((O(1), O(1))\)-scattering. Specifically, we define a stronger notion of partitions called super-scattering and show that general graphs are not \((1, \Omega(\log n))\)-super scatterable (Theorem 6).

**Trees:** Trees are the most basic of the restricted graph families. Weak sparse partitions for trees follows from the existence of sparse covers. Nevertheless, in order to improve parameters and understanding we construct \((4, 3)\)-weak sparse partition scheme for trees (Theorem 8). Further, we prove that trees are \((2, 3)\)-scatterable (Theorem 7). Finally, we show that there are no good strong sparse partition for trees. Specifically, we prove that if all \(n\)-vertex trees admit \((\sigma, \tau)\)-strong sparse partition scheme, then \(\tau \geq 1/3 \cdot n^{\frac{1}{2\tau}}\) (Theorem 9). This implies that for strong sparse partitions, trees are essentially as bad as general graphs.

**Doubling Dimension:** We prove that for every graph with doubling dimension \(\text{ddim}\) and parameter \(\Delta > 0\), there is a partition \(P\) which is simultaneously \((58\alpha, 2^{\text{ddim}/\alpha} \cdot \tilde{O}(\text{ddim}), \Delta)\)-strong sparse partition for all parameters \(\alpha \geq 1\) (Theorem 10). Note that this implies an \((\tilde{O}(\text{ddim}), \tilde{O}(\text{ddim}))\)-strong sparse partition scheme.

**Euclidean Space:** We prove that the \(d\)-dimensional Euclidean space \((\mathbb{R}^d, \|\cdot\|_2)\) is \((1, 2d)\)-scatterable \(^{13}\) (Theorem 11), while for every \((\sigma, \tau)\)-weak sparse partition scheme it holds that \(\tau > (1 + \frac{1}{\sigma})^d\) \(^{14}\) (Theorem 12). In particular, if \(\sigma\) is at most a constant, then \(\tau\) must be exponential. This provides an interesting example of a family where scattering partitions have considerably better parameters than sparse partitions.

**SPDdepth:** \(^{13}\) We prove that every graph with SPDdepth \(\rho\) (in particular graph with pathwidth \(\rho\)) admit \((O(\rho), O(\rho^2))\)-strong sparse partition scheme (Theorem 13). Further, we prove that such graphs admit \((8, 5\rho)\)-weak sparse partition scheme (Theorem 14).

**Chordal Graphs:** We prove that every Chordal graph is \((2, 3)\)-scatterable (Theorem 15).

**Cactus Graphs:** We prove that every Cactus graph is \((4, 5)\)-scatterable (Theorem 16).

\(^{13}\)See Section 8 for clarifications.

\(^{14}\)Every (weighted) path graph has an SPDdepth 1. A graph \(G\) has an SPDdepth \(\rho\) if there exist a shortest path \(P\), such that every connected component in \(G \setminus P\) has an SPDdepth \(\rho - 1\). This family includes graphs with pathwidth at most \(\rho\), and more. See [2].
1.3 Follow-Up Work

In a follow up work, Hershkowitz and Li [52] proved that series-parallel graphs are \((O(1), O(1))-scatterable.\) Using our framework (Theorem 1) they concluded that every instance of the SPR problem in series parallel graphs admit a solution with \(O(1)\) distortion. Recently, Chang et al. [17, 19] constructed shortcut partitions with constant parameters for planar, and minor free graphs. Shortcut partitions are a relaxed version of our scattering partitions. Chang et al. then generalized our Theorem 1 to shortcut partitions, and concluded that planar graphs [18], as well as fixed minor free graph [19] admit a solution with constant stretch for the SPR problem. The question of whether these graph families admit scattering partitions remains open.

Recently, the author [39] construct sparse covers, and sparse partition for fixed minor free graphs with parameters exponentially improving over our Corollary 3. Another recent work by Busch et al. [13] constructed a hierarchy of strong sparse partitions for general graphs, doubling graphs, and graphs with bounded pathwidth (building on the techniques in this paper). These partitions implied a solution to the subgraph UST problem with poly-logarithmic stretch. Finally, sparse partitions were recently used to solve the facility location problem (building on the techniques in this paper). These partitions implied a solution to the subgraph UST problem with constant stretch for the SPR problem. The question of whether these graph families admit scattering partitions remains open.

1.4 Technical Ideas

**Scattering Partition Imply SPR.** Similarly to previous works on the SPR problem, we construct a minor via a terminal partition. That is, a partition of \(V\) into \(k\) connected clusters, where each cluster contains a single terminal. The minor is then induced by contracting all the internal edges. Intuitively, to obtain small distortion, one needs to ensure that every Steiner vertex is clustered into a terminal not much further than its closest terminal, and that every shortest path between a pair of terminals intersects only a small number of clusters. However, the local partitioning of each area in the graph requires a different scale, according to the distance to the closest terminal. Our approach is similar in spirit to the algorithm of Englert et al. [32], who constructed a minor with small expected distortion \(^{15}\) using stochastic decomposition for all possible distance scales. We however, work in the more restrictive regime of worst case distortion guarantee. Glossing over many details, we create different scattering partitions for different areas, where vertices at distance \(\approx \Delta\) to the terminal set are partitioned using a \((1, \tau, \Delta)\)-scattering partition. Afterwards, we assemble the different clusters from the partitions in all possible scales into a single terminal partition. We use the scattering property twice. First to argue that each vertex \(v\) is clustered to a terminal at distance at most \(O(\tau) \cdot D(\nu)\) (here \(D(\nu)\) is the distance to the closest terminal). Second, to argue that every shortest path, where all the vertices are at similar distance to the terminal set, intersects the clusters of at most \(O(\tau^2)\) terminals.

**Miller, Peng and Xu [68] clustering algorithm.** We use [68] to create partitions for general graphs, and graphs with either bounded doubling dimension or SPDdepth. In short, there is a set of centers \(N\), where each center \(t\) samples a starting time \(\delta_t\). Vertex \(v\) joins the cluster of the center \(t\) maximizing \(f_v(t) = \delta_t - d_G(t, v)\). Denote this center by \(t_v\). The diameter guarantee obtained using this algorithm is inherently that of a strong diameter. The key observation is the following: if the cluster of the center \(t\) intersects the ball \(B_G(\nu, r)\), then necessarily \(f_v(t) \geq f_v(t_v) - 2r\). Thus in order to bound the number of intersecting clusters it is enough to bound the number of centers whose \(f_v\) values falls inside the interval \([f_v(t_v) - 2r, f_v(t_v)]\). For each family we choose the starting times \(\{\delta_t\}_{t \in N}\) appropriately.

**Strong Sparse Partition for General Graphs.** For general graphs, we use the [68] clustering algorithm with the set of all vertices as centers. The starting times \(\{\delta_t\}_{t \in N}\) are chosen i.i.d. using an exponential distribution with

\[^{15}\]A distribution \(\mathcal{D}\) over solutions to the SPR problem has expected distortion \(\alpha\) if \(\forall t, t' \in K, \ E_{M\sim \mathcal{D}}[d_M(t, t')] \leq \alpha \cdot d_G(t, t')\).
we are aiming for, it is not clear how to analyze the density of the centers in w.h.p. there are no more than \( \tilde{O}(n^{\alpha}) \) centers whose \( f_v \) value falls in \([f_v(t_v) - \Omega(\frac{\Delta}{n}), f_v(t_v)]\).

**Doubling Dimension.** In a companion paper, the author [35] constructed a strong sparse cover scheme for doubling graphs. Together with Lemma 5, this implies sparse partition with weak diameter guarantee only (and thus does not imply scattering). While both [35] and Theorem 10 are [68] based, here we have additional complications. For a graph with doubling dimension \( \text{ddim} \) we use [68], with a \( \Delta \)-net serving as the set of centers. The idea is to use the same analysis as for general graphs in a localized fashion, where each vertex is “exposed” to \( 2^{\text{ddim}} \) centers, thus replacing the \( \log n \) parameter in the number of intersections with \( \text{ddim} \). The standard solution will be to use a truncated exponential distribution.\(^\text{16}\) Although this will indeed guarantee the “locality” we are aiming for, it is not clear how to analyze the density of the centers in \([f_v(t_v) - \Omega(\frac{\Delta}{n}), f_v(t_v)]\). Instead, we are using bailed exponential distribution. This is an exponential distribution with a threshold parameter \( \lambda_T \) such that all possible values above the threshold collapse to the threshold. In order to bound the density of the centers in \([f_v(t_v) - \Omega(\frac{\Delta}{n}), f_v(t_v)]\), we first treat the distribution as a standard exponential distribution. Afterwards we argue that the actual number of “bailed” centers is small. Interestingly, we bound the number of centers whose \( f_v \) value falls in \([f_v(t_v) - \Omega(\frac{\Delta}{n}), f_v(t_v)]\), where \( t_v \) is the center with the \( s \)th largest \( f_v \) value. Then, we argue that the bound on the density in the actual interval we care about, can withstand any \( s \) occurrences of “bailing”. Eventually, we use the Lovász Local Lemma to argue that we can choose the starting times such that the density of centers in all the possible intervals is small (“intervalwise”).

**SPD.** Our strong sparse partition for a graph with SPD depth \( \rho \) is also produced using the [68] clustering algorithm. Interestingly, unlike all previous executions of [68] in the literature, there is no randomness involved. The SPD produces a hierarchy of partial partitions \( \{X_1, \ldots, X_\rho\} \), where \( \mathcal{P}_t \) is the set of shortest paths deleted from each of the connected components in \( X_t \), to create \( X_{t+1} \). The set of centers consists of \( \epsilon \Delta \)-nets \( \{N_t\}_{t=1}^\rho \) taken from all the shortest paths \( \mathcal{P}_t \), where the starting time \( \delta_t \) is equal for all centers \( t \in N_t \) with the same hierarchical depth, and decreasing with the steps \( (t \in N_t, t' \in N_{t+1} \Rightarrow \delta_t > \delta_{t'}) \). The parameters are chosen in such a way that each vertex \( v \) can join the cluster of a center \( t \in N_t \), only if \( v \) and \( t \) belong to the same connected component in \( X_t \). Consider a small ball \( B \). If a vertex from \( B \) belongs to \( \mathcal{P}_t \), then no vertex of \( B \) will join a cluster of a center in \( \mathcal{P}_{t'} \) for \( t' > t \). To conclude, in each step \( t \), the vertices of \( B \) might join the cluster of centers lying on a single shortest path from \( \mathcal{P}_t \). As all these centers have equal starting time, \( B \) can intersect only a small number of them. A bound on the number of intersections follows.

**Chordal Graphs.** The partition is created inductively using the tree decomposition (where each bag is a clique). The label of a vertex \( v \), is its distance in \( G \) to the cluster center \( t \) (w.r.t. \( d_G \)). We maintain the property that all the vertices with label strictly smaller than \( \Delta/2 \), within a single bag, belong to the same cluster. Interestingly, while the partition is connected, the diameter guarantee we obtain is only weak (in contrast to all other scattering partitions in the paper).

**Lower Bounds.** For the lower bound for strong sparse partitions in trees we use a full \( d \)-ary tree, of depth \( D \). It holds that for every partition with diameter smaller than \( 2D \), there must be an internal vertex with all its children belonging to different clusters. Thus we have a ball of radius 1 intersecting \( d + 1 \) clusters. Noting that such a tree has less than \( 2 \cdot d^{D+1} \) vertices, the theorem follows.

For the lower bound on weak partitions of general graphs we use expanders. The cluster \( A \) will intersect all radius \( \frac{\Delta}{\rho} \) balls with centers in \( B_G(A, \frac{\Delta}{\rho}) \). By expansion property, \( |B_G(A, \frac{\Delta}{\rho})| \geq \Omega(1)^{\frac{\Delta}{\rho}} \cdot |A| \). We uniformly sample a center \( v \), the expected number of clusters intersecting \( B_G(v, \frac{\Delta}{\rho}) \) is

\[^{16}\text{A truncated exponential distribution is an exponential distribution conditioned on the event that the outcome lies in a certain interval. This is usually used when the maximal possible value must be bounded. See e.g. [35].}\]
For Euclidean space the lower bound follows similar ideas, where the expansion property is replaced by the Brunn-Minkowski theorem.

## 1.5 Related Work

In the functional analysis community, the notion of Nagata dimension was studied. The Nagata dimension of a metric space \((X, d)\), \(\dim_N X\), is the infimum over all integers \(n\) such that there exists a constant \(c\) s.t. \(X\) admits a \((c, n + 1)\)-weak sparse partition scheme. In contrast, in this paper our goal is to minimize this constant \(c\). See [65] and the references therein.

A closely related problem to UST is the Universal Traveling Salesman Problem (UTSP). Consider a postman providing post service for a set \(X\) of clients with \(n\) different locations (with distance measure \(d_x\)). Each morning the postman receives a subset \(S \subseteq X\) of the required deliveries for the day. In order to minimize the total tour length, one solution may be to compute each morning an (approximation of an) Optimal TSP tour for the set \(S\). An alternative solution will be to compute a Universal TSP (UTSP) tour. This is a universal tour \(R\) containing all the points \(X\). Given a subset \(S\), \(R(S)\) is the tour visiting all the points in \(S\) w.r.t. the order induced by \(R\). Given a tour \(T\) denote its length by \(|T|\). The stretch of \(R\) is the maximum ratio among all subsets \(S \subseteq X\) between the length of \(R(S)\) and the length of the optimal TSP tour on \(S\), \(\max_{S \subseteq X} \frac{|R(S)|}{|Opt(S)|}\).

All the sparse partition based upper bounds for the UST problem translated directly to the UTSP problem with the same parameters. The first to study the problem were Platzman and Bartholdi [71], who given \(n\) points in the Euclidean plane constructed a solution with stretch \(O(\log n)\), using space filling curves. Recently, Christodoulou, and Sgouritsa [23] proved a lower and upper bound of \(\Theta(\log n / \log \log n)\) for the \(n \times n\) grid, improving a previous \(\Omega(\sqrt{\log n / \log \log n})\) lower bound of Hajajghayi, Kleinberg, and Leighton [51] (and the \(O(\log n)\) upper bound of [71]). For general \(n\) vertex graphs Gupta et al. [48] proved an \(O(\log^2 n)\) upper bound, while Gorodezky, Kleinberg, Shmoys, and Spencer [45] proved an \(O(\log n)\) lower bound. From the computational point of view, Schalekamp and Shmoys [74] showed that if the input graph is a tree, an UTSP with optimal stretch can be computed efficiently.

The A Priori TSP problem is similar to the UTSP problem. In addition there is a distribution \(\mathcal{D}\) over subsets \(S \subseteq V\) and the stretch of tour a \(R\) is the expected ratio between the induced solution to optimal \(\mathbb{E}_{S \sim \mathcal{D}} \frac{|R(S)|}{|Opt(S)|}\) (instead of a worst case like in UTSP). Similarly, A Priori Steiner Tree was studied (usually omitting rt from the problem). See [45, 54, 74] for further details. Another similar problem is the Online (or dynamic) Steiner Tree problem. Here the set \(S\) of vertices that should be connected is evolving over time, see [6, 46, 53] and references therein.

Unlike the definition used in this paper (taken from [55]), sparse partitions were also defined in the literature as partitions where only a small fraction of the edges are inter-cluster (see for example [5]). A closely related notion to sparse partitions are padded and separating decompositions. A graph \(G\) is \(\beta\)-decomposable if for every \(\Delta > 0\), there is a distribution \(\mathcal{D}\) over \(\Delta\) bounded partitions such that for every \(u, v \in V\), the probability that \(u\) and \(v\) belong to different clusters is at most \(\beta \cdot \frac{\delta(u,v)}{\Delta}\). Note that by linearity of expectation, a path \(I\) of length \(\Delta / \sigma\) intersects at most \(1 + \beta / \sigma\) clusters in expectation. For comparison, in scattering partition we replace the distribution by a single partition and receive a bound on the number of intersections in the worst case. See [1, 4, 5, 9, 33, 35, 42, 49, 59] for further details.

Englert et al. [32] showed that every graph which is \(\beta\)-decomposable, admits a distribution \(\mathcal{D}\) over solution to the SPR problem with expected distortion \(O(\beta \log \beta)\). In particular this implies constant expected distortion for graphs excluding a fixed minor, or bounded doubling dimension.

For a set \(K\) of terminals of size \(k\), Krauthgamer, Nguyen and Zondiner [62] showed that if we allow the minor \(M\) to contain at most \(\binom{k}{2}\) Steiner vertices (in addition to the terminals), then distortion 1 can be achieved. They
further showed that for graphs with constant treewidth, $O(k^2)$ Steiner points will suffice for distortion 1. Cheung, Gramoz and Henzinger [22] showed that allowing $O(k^{2+\varepsilon})$ Steiner vertices, one can achieve distortion $2t - 1$. For planar graphs, Cheung et al. achieved $1 + \varepsilon$ distortion with $O((\frac{1}{\varepsilon^2})$ Steiner points.

There is a long line of work focusing on preserving the cut/flow structure among the terminals by a graph minor. See [7, 20, 24, 32, 44, 63, 64, 66, 67, 69].

There were works studying metric embeddings and metric data structures concerned with preserving distances among terminals, or from terminals to other vertices, out of the context of minors. See [10, 25, 28–31, 40, 50, 58, 61, 73].

2 PRELIMINARIES

All the logarithms in the paper are in base 2. We use $\bar{O}$ notation to suppress constants and logarithmic factors, that is $\bar{O}(f(j)) = f(j) \cdot \text{polylog}(f(j))$.

Graphs. We consider connected undirected graphs $G = (V, E)$ with edge weights $w : E \rightarrow \mathbb{R}_{\geq 0}$. Let $d_G$ denote the shortest path metric in $G$. $B_G(u, r) = \{u \in V | d_G(u, u) \leq r\}$ is the ball of radius $r$ around $u$. For a vertex $v \in V$ and a subset $A \subseteq V$, let $d_G(x, A) := \min_{a \in A} d_G(x, a)$, where $d_G(x, \emptyset) = \infty$. For a subset of vertices $A \subseteq V$, let $G[A]$ denote the induced graph on $A$, and let $G \setminus A := G[V \setminus A]$.

Doubling dimension. The doubling dimension of a metric space is a measure of its local “growth rate”. A metric space $(X, d)$ has doubling constant $\lambda$ if for every $x \in X$ and radius $r > 0$, the ball $B(x, 2r)$ can be covered by $\lambda$ balls of radius $r$. The doubling dimension is defined as $\text{ddim} = \log \lambda$. We say that a weighted graph $G = (V, E, w)$ has doubling dimension $d_{dim}$ if the corresponding shortest path metric $(V, d_G)$ has doubling dimension $d_{dim}$. A $d$-dimensional $\ell_p$ space has $d_{dim} = \Theta(d)$, every $n$ point vertex graph has $d_{dim} = \bar{O}(\log n)$, and every weighted path has $d_{dim} = 1$. The following lemma gives the standard packing property of doubling metrics (see, e.g., [49]).

Lemma 3 (Packing Property). Let $(X, d)$ be a metric space with doubling dimension $d_{dim}$. If $S \subseteq X$ is a subset of points with minimum interpoint distance $r$ that is contained in a ball of radius $R$, then $|S| \leq \left(\frac{2R}{r}\right)^{d_{dim}}$.

Nets. A set $N \subseteq V$ is called a $\Delta$-net, if for every vertex $v \in V$ there is a net point $x \in N$ at distance at most $d_G(v, x) \leq \Delta$, while every pair of net points $x, y \in N$, is farther than $d_G(x, y) > \Delta$. A $\Delta$-net can be constructed efficiently in a greedy manner. In particular, by Lemma 3, given a $\Delta$-net $N$ in a graph of doubling dimension $d_{dim}$, a ball of radius $R \geq \Delta$, will contain at most $\left(\frac{2R}{\Delta}\right)^{d_{dim}}$ net points.

Exponential Distribution. $\text{Exp}(\lambda)$ denotes the exponential distribution with mean $\lambda$ and density function $f(x) = \lambda e^{-\lambda x}$ for $x \geq 0$. A useful property of exponential distribution is memoryless: let $X \sim \text{Exp}(\lambda)$, for every $a, b \geq 0$, $\Pr[X \geq a + b \mid X \geq a] = \Pr[X \geq b]$. In other words, given that $X \geq a$, it holds that $X - a \sim \text{Exp}(\lambda)$.

Special graph families. A graph $H$ is a minor of a graph $G$ if we can obtain $H$ from $G$ by edge deletions/contractions, and vertex deletions. A graph family $\mathcal{G}$ is $H$-minor-free if no graph $G \in \mathcal{G}$ has $H$ as a minor. Some examples of minor free graph families are planar graphs ($K_4$ and $K_{3,3}$ free), outerplanar graphs ($K_4$ and $K_{3,3}$ free), series-parallel graphs ($K_4$ free), Cactus graphs (also known as tree of cycles) ($K_3$ free), and trees ($K_3$ free).

Given a graph $G = (V, E)$, a tree decomposition of $G$ is a tree $T$ with nodes $B_1, \ldots, B_s$ (called bags) where each $B_i$ is a subset of $V$ such that the following properties hold:

- For every edge $\{u, v\} \in E$, there is a bag $B_i$ containing both $u$ and $v$.
- For every vertex $v \in V$, the set of bags containing $v$ form a connected subtree of $T$.

The width of a tree decomposition is $\max_i(|B_i| - 1)$. The treewidth of $G$ is the minimal width of a tree decomposition of $G$. A path decomposition of $G$ is a special kind of tree decomposition where the underlying tree is a path. The pathwidth of $G$ is the minimal width of a path decomposition of $G$.
Chordal graphs are unweighted graphs where each cycle of length greater than 4 contains a chordal. In other words, if the induced graph on a set of vertices \( V' \) is the cycle graph, than necessarily \(|V'| \leq 3\). Chordal graphs contain interval graphs, subtree intersection graphs and other interesting sub families. A characterization of Chordal graphs is that they have a tree decomposition such that each bag is a clique. That is, there is a tree decomposition \( T \) of \( G \) where there is no upper bound on the size of a bag, but for every bag \( B \in T \) the induced graph \( G[B] \) is a clique.

A Cactus graph (a.k.a. tree of cycles) is a graph where each edge belongs to at most one simple cycle. Alternatively it can be defined as the graph family that excludes \( K_5 \) minus an edge (\( K_5^\sim \)) as a minor.

Abraham et al. [2] defined shortest path decompositions (SPDs) of "low depth". Every (weighted) path graph has an SPDdepth 1. A graph \( G \) has an SPDdepth \( k \) if there exist a shortest path \( P \), such that every connected component in \( G \setminus P \) has an SPDdepth \( k - 1 \). In other words, given a graph, in SPD we hierarchically delete shortest paths from each connected component, until no vertices remain. See Section 9 for formal definition (and also footnote 14). Every graph with pathwidth \( \rho \) has SPDdepth at most \( \rho + 1 \), treewidth \( \rho \) implies SPDdepth at most \( O(\rho \log n) \), and every graph excluding a fixed minor has SPDdepth \( O(\log n) \). See [2, 37] for further details and applications.

2.1 Clustering algorithm using shifted starting times [68]

In some of our clustering algorithms we will use a generalized version of the clustering algorithm by Miller et al. [68]. This version has also been applied by the author in a companion paper [35]. Here we describe the algorithm and some of its basic properties. One of the main advantages of this algorithm is that it inherently produces clusters with strong diameter. Additionally, the inter-relationship between the clusters is relatively easy to analyze.

Consider a weighted graph \( G = (V, E, w) \). Let \( N \subseteq V \) be some set of centers, where each \( t \in N \) admits a parameter \( \delta_t \). The choice of the centers and the parameters differs among different implementations of the algorithm. For each vertex \( v \) set a function \( f_v : N \rightarrow \mathbb{R} \) as follows: for a center \( t, f_v(t) = \delta_t - d_G(t, v) \). The vertex \( v \) will join the cluster \( C_t \) of the center \( t \in N \) maximizing \( f_v(t) \). Ties are broken in a consistent manner.\(^{18}\) Note that it is possible that a center \( t \in N \) will join the cluster of a different center \( t' \in N \). An intuitive way to think about the clustering process is as follows: each center \( t \) wakes up at time \( -\delta_t \) and begins to spread in a continuous manner. The spread of all centers is performed in the same unit tempo. A vertex \( v \) joins the cluster of the first center that reaches it.

Claim 1. Suppose that a vertex \( v \) joined the cluster of the center \( t \). Let \( I \) be a shortest path from \( v \) to \( t \), then all the vertices on \( I \) joined the cluster of \( t \).

Proof. For every vertex \( u \in I \) and center \( t' \in N \), it holds that

\[
\delta(t) - d_G(u, t) = \delta(t) - (d_G(v, t) - d_G(v, u)) \geq \delta(t') - d_G(v, t') + d_G(v, u) \geq \delta(t') - d_G(u, t') ,
\]

where the first inequality holds as \( f_v(t) \geq f_v(t') \). We conclude that \( f_u(t) \geq f_u(t') \), hence \( u \) joins the cluster of \( t \).

By Claim 1 it follows that if the distance between every vertex to the cluster center \( v \) is bounded by \( \Delta \) w.r.t. \( d_G \), then the cluster has strong diameter \( 2\Delta \).

\(^{17}\)There are two differences between the algorithm here and the original [68] clustering algorithm. First, in [68] all the vertices are potential cluster centers (while we are allowing to use only a subset of the vertices as cluster centers). Secondly, in [68] the shifts \( \delta_t \) are sampled randomly using exponential distribution. In contrast, here we are allowing to choose the shifts in an arbitrary manner. In particular, in Theorems 4 and 10 we will sample the shifts using biailed exponential distribution, while in Theorem 13 the shifts will be chosen deterministically.

\(^{18}\)That is we have some order \( x_1, x_2, \ldots \). Among the centers \( x_i \) that maximize \( f_v(t) \), \( v \) joins the cluster of the center with minimal index.
Claim 2. Consider a ball \( B = B_G(v, r) \), and let \( t \) be the center maximizing \( f_v \). Then for every center \( t' \) such that \( f_v(t) - f_v(t') > 2r \), the intersection between \( B \) and the cluster centered at \( t' \) is empty.

Proof. For every vertex \( u \in B \) it holds that,
\[
f_v(t) = \delta(t) - d_G(u, t) \geq \delta(t) - d_G(u, t) - r > \delta(t') - d_G(u, t') + r \geq \delta(t') - d_G(u, t) = f_v(t'),
\]
where the second inequality holds as \( f_v(t) > f_v(t') + 2r \). It follows that \( t' \) is not maximizing \( f_v \) for any vertex in \( B \), the claim follows. \( \square \)

3 FROM SCATTERING PARTITIONS TO SPR: PROOF OF ??

This entire section is devoted to the proof of Theorem 1. We will assume w.l.o.g. that the minimal pairwise distance in the graph is exactly 1, otherwise we can scale all the weights accordingly. The set of terminals is denoted by \( K = \{ t_1, \ldots, t_k \} \). For every vertex \( v \in V \), denote by \( D(v) = d_G(v, K) \) the distance to its closest terminal. Note that \( \min_{v \in V \setminus K} D(v) \geq 1 \).

Similarly to previous papers on the SPR problem, we will create a minor using terminal partitions. Specifically, we partition the vertices into \( k \) connected clusters, with a single terminal in each cluster. Such a partition induces a minor by contracting all the internal edges in each cluster. More formally, a partition \( \{ V_0, \ldots, V_k \} \) of \( V \) is called a terminal partition (w.r.t to \( K \)) if for every \( 1 \leq i \leq k \), \( t_i \in V_i \), and the induced graph \( G[V_i] \) is connected. For a vertex \( v \in V_i \), we say that \( v \) is assigned to \( t_i \). See Figure 2 for an illustration. The induced minor by the terminal partition \( \{ V_1, \ldots, V_k \} \), is a minor \( M \), where each set \( V_j \) is contracted into a single vertex called (abusing notation) \( t_j \). Note that there is an edge in \( M \) from \( t_i \) to \( t_j \) if and only if there are vertices \( v_i \in V_i \) and \( v_j \in V_j \) such that \( \{ v_i, v_j \} \in E \). We determine the weight of the edge \( \{ t_i, t_j \} \in E(M) \) to be \( d_G(t_i, t_j) \). Note that by the triangle inequality, for every pair of (not necessarily neighboring) terminals \( t_i, t_j \), it holds that \( d_M(t_i, t_j) \geq d_G(t_i, t_j) \). The distortion of the induced minor is \( \max_{i,j} \frac{d_M(t_i, t_j)}{d_G(t_i, t_j)} \).

3.1 Algorithm

We create the terminal partition in an iterative manner, where initially each set \( \{ t_i \} \) is a singleton, and gradually more vertices are joining. We will denote the stage of the terminal partition after \( i \) steps, using a function \( f_i : V \rightarrow K \cup \{ \perp \} \). For a yet unassigned vertex \( v \) we write \( f_i(v) = \perp \), otherwise the vertex \( v \) will be assigned to \( f_i(v) \). Initially for every terminal \( t_j \), \( f_0(t_j) = t_j \) while for every Steiner vertex \( v \in V \setminus K \), \( f_0(v) = \perp \). In iteration \( i \) we will define \( f_i \) by "extending" \( f_{i-1} \). That is, unassigned vertices may be assigned (i.e., for \( v \) such that \( f_{i-1}(v) = \perp \), it might be \( f_i(v) = t_j \)), while the function will remain the same on the set of assigned vertices \( \{ f_{i-1}(v) \neq \perp \Rightarrow f_i(v) = f_{i-1}(v) \} \). We will guarantee that all the vertices in \( R_i \) will be assigned in \( f_i \). In particular, after \( \log(\max_v D(v)) \) steps, all the vertices will be assigned.

Denote by \( V_i \) the set of vertices assigned by \( f_i \). Initially \( V_0 = K = R_0 \). By induction we will assume that \( \bigcup_{j=1}^{R_i} R_j \subseteq V_{i-1} \). Let \( G_i = G[V \setminus V_{i-1}] \) be the graph induced by the set of yet unassigned vertices. Fix \( \Delta_i = 2^{\ell-1} \). Let \( P_i \) be an \((1, \tau, \Delta_i)\)-scattering partition of \( G_i \). Let \( C \subseteq P_i \) be the set of clusters \( C \) which contain at least one vertex \( v \in \mathcal{R}_i \). All the vertices in \( \bigcup C \) (that is the union of all the clusters in \( C \)) will be assigned by \( f_i \).

We say that a cluster \( C \subseteq C \) is at level \( l \), noting \( \delta_l(C) = 1 \), if there is an edge \( \{ v, u_C \} \) in \( G \) from a vertex \( v \in C \) to a vertex \( u_C \in V_{i-1} \) of weight at most \( 2^l \). In general, \( \delta_l(C) = l \), if \( l \) is the minimal index such that there is an edge \( \{ v, u_C \} \) from a vertex \( v \in C \) to a vertex \( u_C \in C' \) of weight at most \( 2^l \), such that \( \delta_l(C) = l - 1 \). In both cases \( u_C \) is called the linking vertex of \( C \). Next, we define \( f_i \) based on \( f_{i-1} \). For every vertex \( v \in V_{i-1} \), set \( f_i(v) = f_{i-1}(v) \). For every vertex not in \( \bigcup C \) in \( V_{i-1} \) set \( f_i(v) = \perp \). For a cluster \( C \subseteq \mathcal{C}_i \) s.t. \( \delta_l(C) = 1 \), let \( u_C \in V_{i-1} \) be its linking vertex. For every \( v \in C \) set \( f_i(v) = f_i(u_C) \). Generally, for level \( l \) suppose that \( f_i \) is already defined on all the clusters of level \( l = 1 \). Let \( C \subseteq \mathcal{C}_i \) s.t. \( \delta_l(C) = l \). Let \( u_C \) be the linking vertex of \( C \). For every \( v \in C \), set \( f_i(v) = f_i(u_C) \). Note that for every cluster, all the vertices are mapped to the same terminal. This finishes the definition of \( f_i \).
Scattering and Sparse Partitions, and their Applications

Fig. 1. Classification of various graph families according to the possibility of construction different partitions. Graphs with bounded doubling dimension or SPD (pathwidth) admit strong sparse partitions with parameters depending only on the dimension/depth. Trees, Chordal and Cactus graphs admit both $O(1), O(1)$-weak sparse and scattering partitions, while similar strong partitions are impossible. $\mathbb{R}^d$ with norm 2 admit $(1, 2d)$ scattering partition while weak sparse partition with constant padding will have an exponential number of intersections. Planar graphs admit $O(1), O(1)$-weak sparse partitions, while it is an open question whether similar scattering partitions exist. Finally, while sparse partitions for general graphs are well understood, we lack a lower bound for scattering partitions.

Fig. 2. The left side of the figure contains a weighted graph $G = (V, E)$, with weights specified in red, and four terminals $\{t_1, t_2, t_3, t_4\}$. The dashed black curves represent a terminal partition of the vertex set $V$ into the subsets $V_1, V_2, V_3, V_4$. The right side of the figure represents the minor $M$ induced by the terminal partition. The distortion is realized between $t_1$ and $t_3$, and is $\frac{d_M(t_1, t_3)}{d_G(t_1, t_3)} = \frac{12}{4} = 3$. 

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The algorithm continues until there is $f_i$ where all the Steiner vertices are assigned. Set $f = f_i$. The algorithm returns the terminal-centered minor $M$ of $G$ induced by $\{f^{-1}(t_1), \ldots, f^{-1}(t_k)\}$.

### 3.2 Basic Properties

It is straightforward from the construction that $f^{-1}(t_1), \ldots, f^{-1}(t_k)$ define a terminal partition. We will prove that every vertex $v$ will be assigned during either iteration $[\log D(v)]$ or $[\log D(v)] - 1$ (Claim 5), to a terminal at distance at most $O(\tau) \cdot D(v)$ from $v$ (Corollary 1). We begin by arguing that in each iteration, the maximum possible level of a cluster is $\tau$.

#### Claim 3. For every cluster $C \in C_i$, $\delta_i(C) \leq \tau$.

**Proof.** Consider a cluster $C \in C_i$, and let $v \in C$ be a vertex s.t. $D(v) \leq 2^i$. Let $P_v = \{v = v_0, \ldots, v_q\}$ be a shortest path from $v$ to a terminal $v_q \in K$ of total weight $D(v)$. Let $s + 1$ be a minimal index of a clustered vertex. That is $f(v_{s+1}) \neq \perp$, and for every $j \leq s$, $f(v_j) = \perp$. Such an index exists as $f(v_q) = v_q$. In particular, $v_{s+1} \in V_{i-1}$. The prefix $P = \{v = v_0, \ldots, v_q\}$ is a shortest path in $G_i$. $P_v$ is a $(1, \tau, \Delta_i)$-scattering partition. Hence the vertices of $P$ are partitioned to $\tau'$ vertices of weight at most $2^\tau' < 2^i$, while the edge $\{v_q, v_{s+1}\}$ is from $C_1$ to $V_{i-1}$ of weight at most $2^\tau$. It holds that $\delta_i(C_1) = 1$, and by induction $\delta_i(C_1) \leq j$. In particular $\delta(C) \leq \tau$.

#### Claim 4. For every vertex $v$ which is assigned during the $i$’th iteration (i.e., $v \in C_i$) it holds that $d_G(v, f(v)) \leq 3\tau \cdot 2^i$.

**Proof.** The proof is by induction on $i$. For $i = 0$ the assertion holds trivially as every terminal is assigned to itself. We will assume the assertion holds for $i$ and prove it for $i + 1$. Let $C \in C_i$ be some cluster, and let $v \in C$. Suppose first that $\delta_i(C) = 1$. Let $u_C \in V_{i-1}$ be the linking vertex of $C$. By the induction hypothesis $d_G(u_C, f(u_C)) \leq 3\tau \cdot 2^{i-1}$. As the diameter of $C$ is bounded by $2^i - 1$, and the weight of the edge towards $u_C$ is at most $2^i$ we conclude $d_G(v, f(v)) \leq d_G(v, u_C) + d_G(u_C, f(u_C)) \leq (2^{i-1} + 2^i) + 3\tau \cdot 2^{i-1} = 3\tau \cdot 2^i$. Generally, for $\delta_i(C) = \tau$, we argue by induction that for every $v \in C$ it holds that $d_G(v, f(v)) \leq (\tau - 1) \cdot 3\tau \cdot 2^{i-1} + 3\tau \cdot 2^{i-1}$. Indeed, let $u_C$ be a linking vertex of $C$. By the induction hypothesis it holds that $d_G(u_C, f(u_C)) \leq (\tau - 1) \cdot 3\tau \cdot 2^{i-1}$. Using similar arguments, it holds that $d_G(v, f(v)) \leq d_G(v, u_C) + d_G(u_C, f(u_C)) \leq (\tau - 1) \cdot 3\tau \cdot 2^{i-1} + (\tau - 1) \cdot 3\tau \cdot 2^{i-1} + 3\tau \cdot 2^{i-1} = (\tau - 1) \cdot 3\tau \cdot 2^{i-1} + 3\tau \cdot 2^{i-1}$. Using Claim 3, $d_G(v, f(v)) \leq 3\tau \cdot 2^{i-1} + 3\tau \cdot 2^{i-1} = 3\tau \cdot 2^i$ as required.

#### Corollary 1. For every vertex $v$ it holds that $d_G(v, f(v)) \leq 6\tau \cdot D(v)$.

**Proof.** Let $i \geq 0$ such that $2^{i-1} < D(v) \leq 2^i$. The vertex $v$ is assigned at iteration $i$ or earlier. By Claim 4 we conclude $d_G(v, f(v)) \leq 3\tau \cdot 2^i \leq 6\tau \cdot D(v)$.

#### Claim 5. Consider a vertex $v$ such that $2^{i-1} < D(v) \leq 2^i$ Then $v$ is assigned either at iteration $i - 1$ or $i$.

**Proof.** Clearly if $v$ remains un-assigned until iteration $i$, it will be assigned during the $i$’th iteration. Suppose that $v$ was assigned during iteration $j$. Then $v$ belongs to a cluster $C \in C_j$. In particular there is a vertex $u \in C$ such that $D(u) \leq 2^j$. As $C$ has diameter at most $2^{j-1}$, it holds that

$$2^{j-1} < D(u) \leq D(u) + d_G(v, u) \leq 2^j + 2^{j-1} = 3 \cdot 2^{j-1}.$$ 

$i, j$ are integers, hence $j \geq i - 1$.

### 3.3 Distortion Analysis

In this section we analyze the distortion of the minor induced by the terminal partition created by our algorithm. We have several variables that are defined with respect to the algorithm. Note that all these definitions are for analysis purposes only, and have no impact on the execution of the algorithm.

ACM Trans. Algor.
Consider a pair of terminals $t$ and $t'$. Let $P_{t,t'} = \{ t = v_0, \ldots, v_p = t' \}$ be the shortest path from $t$ to $t'$ in $G$. We can assume that there are no terminals in $P_{t,t'}$ other than $t$, $t'$. This is because if we will prove the distortion guarantee for every pair of terminals $t$, $t'$ such that $P_{t,t'} \cap K = \{ t, t' \}$, then by the triangle inequality the distortion guarantee will hold for all terminal pairs.

**Detours:** The terminals $t$, $t'$ are fixed. During the execution of the algorithm, for every terminal $t_j$ we will maintain a detour $D_{t_j}$ (or shortly $D_j$). A detour is a consecutive subinterval $\{a_j, \ldots, b_j\}$ of $P_{t,t'}$, where $a_j \in D_j$ is the leftmost (i.e., with minimal index) vertex in the detour and $b_j$ is the rightmost. Initially $D_j = \{ t \}$ and $D_{t'} = \{ t' \}$, while for every $t_j \notin \{ t, t' \}$, $D_j = \emptyset$. Every pair of detours $D_j, D_j'$ will be disjoint throughout the execution of the algorithm.

A vertex $v \in P_{t,t'}$ is active if and only if it does not belong to any detour. It will hold that every active vertex is necessarily assigned (while there might be unassigned vertices which are inactive). Initially, $t'$ are inactive, while all the other vertices of $P_{t,t'}$ are active. Next we consider the $i$th iteration of the algorithm, and we will redefine the detours in each iteration as follows. We go over the terminals according to any arbitrary order $\{ t_1, \ldots, t_k \}$. Consider the terminal $t_i$ with detour $D_j = \{ a_j, \ldots, b_j \}$ (which might be empty). If no active vertices are assigned to $t_i$ at the $i$th iteration we do nothing. Otherwise, let $a'_j \in P_{t,t'}$ (resp. $b'_j$) be the leftmost (resp. rightmost) active vertex that was assigned to $t_j$ during the $i$th iteration. Set $a_j$ to be vertex with minimal index between the former $a_j$ and $a'_j$ (if there was no $a_j$). Similarly $b_j$ is the vertex with maximal index between the former $b_j$ and $b'_j$. $D_j$ is updated to be $\{a_j, \ldots, b_j\}$. All the vertices in $\{a_j, \ldots, b_j\} = D_j$ become inactive. Note that a vertex might become inactive while remaining yet unassigned.

Consider an additional detour $D_j$. Before the updating of $D_j$ at iteration $i$, $D_j, D_j'$ were disjoint. Then we defined $a'_j, b'_j$ (to be leftmost/rightmost active vertex assigned to $t_j$) and updated $D_j$ to be the interval from $\min\{a_j, a'_j\}$ to $\max\{b_j, b'_j\}$. Note that as $a'_j, b'_j$ were active (assuming they existed) they cannot belong to $D_j$. Thus after the update, $a_j, b_j$ did not belong to $D_j$ as well. Nevertheless, it is possible that after the update $D_j$ and $D_j'$ are no longer disjoint. However as $D_j'$ is also an interval not containing the endpoints of the new interval $D_j$, the only such possibility is if $D_j'$ is fully contained in $D_j$. If indeed $D_j' \subset D_j$, we will nullify $D_j' \leftarrow \emptyset$ and thus maintain the disjointness property (while not changing the active/inactive status of any vertex).

After we nullify all the detours that were contained in $D_j$, we will proceed to treat the next terminals in turn. Once we finish going over all the terminals, we proceed to the $i + 1$ iteration. Eventually, all the vertices are assigned, and hence are inactive. In particular every vertex belong to some detour. In other words, as the detours are disjoint intervals, all the vertices of $P_{t,t'}$ are partitioned to consecutive disjoint detours $D_{t_1}, \ldots, D_{t_k}$.

**Intervals:** For an interval $Q = \{ v_h, \ldots, v_b \} \subseteq P_{t,t'}$, the internal length is $L(Q) = d_G(v_h, v_b)$, while the external length is $L^*(Q) = d_G(v_{h-1}, v_{b+1})$. \(^{20}\) We denote by $D(Q) = D(v_h)$ the distance from the leftmost vertex $v_h \in Q$ to its closest terminal. Set $c_{\text{int}} = \frac{1}{\sqrt{n}} \cdot \text{int}^*$ for interval. We partition the vertices in $P_{t,t'}$ into consecutive intervals $Q$, such that for every $Q \subseteq Q_r$,

$$L(Q) \leq c_{\text{int}} \cdot D(Q) \leq L^*(Q) \cdot$$ \hspace{1cm} (3.1)

Such a partition could be obtained as follows: Sweep along the path $P_{t,t'}$ in a greedy manner, after partitioning the prefix $v_h, \ldots, v_{h-1}$, to construct the next interval $Q$, simply pick the minimal index $s$ such that $L^*(\{v_h, \ldots, v_{h+s}\}) \geq c_{\text{int}} \cdot D(v_h)$. By the minimality of $s$, $L(\{v_h, \ldots, v_{h+s}\}) \leq L^*(\{v_h, \ldots, v_{h+s-1}\}) \leq c_{\text{int}} \cdot D(v_h)$ (in the case $s = 0$, trivially $L(\{v_h\}) = 0 \leq c_{\text{int}} \cdot D(v_h)$). Note that such $s$ could always be found, as $L^*(\{v_h, \ldots, v_f = t'\}) = d_G(v_{h-1}, t') \geq d_G(v_h, t') \geq D(v_h) = D(Q)$.

\(^{19}\)This is because given two intervals $[x_1, y_1]$, $[x_2, y_2]$ where $x_1, y_1 \notin [x_2, y_2]$, by case analysis it must be the case that either the intervals are disjoint, or $[x_2, y_2] \subseteq [x_1, y_1]$.

\(^{20}\)For ease of notation we will denote $v_{-1} = t$ and $v_{a+1} = t'$. 

ACM Trans. Algor.
Consider some interval $Q = \{v_0, \ldots, v_k\} \in \mathcal{Q}$. For every vertex $v \in Q$, by triangle inequality it holds that $D(Q) - L(Q) \leq D(v) \leq D(Q) + L(Q)$. Therefore,

$$\left(1 - c_{\text{int}}\right)D(Q) \leq D(v) \leq \left(1 + c_{\text{int}}\right)D(Q).$$

(3.2)

Note that the set $Q$ of intervals is determined before the execution of the algorithm, and is never changed. In particular, it is independent from the set of detours (which evolves during the execution of the algorithm).

For an interval $Q$, we denote by $i_Q$ the first iteration when some vertex $v$ belonging to the interval $Q$ is assigned.

**Claim 6.** All $Q$ vertices are assigned in either iteration $i_Q$ or $i_Q + 1$.

**Proof.** Let $u \in Q$ be some vertex which is assigned during iteration $i_Q$. Then $u$ belongs to a cluster $C \in \mathcal{C}_{i_Q}$, containing a vertex $u' \in C$ such that $D(u') \leq 2^{i_Q}$. As $C$ has diameter at most $2^{i_Q-1}$, it holds that $2^{i_Q} \geq D(u') \geq D(u) - d_G(u, u') \geq D(u) - 2^{i_Q-1}$. Hence $D(u) \leq \frac{3}{2} \cdot 2^{i_Q}$. It follows that

$$D(Q) \overset{(3.2)}{\leq} \frac{1}{1 - c_{\text{int}}} \cdot D(u) \leq \frac{3}{2} \cdot \frac{1}{1 - c_{\text{int}}} \cdot 2^{i_Q}.$$  

(3.3)

For every vertex $v \in Q$ it holds that,

$$D(v) \overset{(3.1)}{\leq} D(Q) + L(Q) \overset{(3.3)}{\leq} \left(1 + c_{\text{int}}\right) \cdot D(Q) \overset{(3.3)}{\leq} \frac{3}{2} \cdot \frac{1 + c_{\text{int}}}{1 - c_{\text{int}}} \cdot 2^{i_Q} = 2^{i_Q} - 2^{i_Q+1}.$$  

Therefore, by Claim 5, in the $i_Q + 1$ iteration, all the (yet unassigned) vertices of $Q$ will necessarily be assigned. \(\square\)

**Lemma 4.** Consider an interval $Q \in \mathcal{Q}$. Then the vertices of $Q$ are partitioned into at most $O(\tau^2)$ different detours.

**Proof.** By definition, by the end of the $i_Q - 1$'th iteration all the vertices of $Q$ are unassigned. We first consider the case where by the end $i_Q - 1$'th iteration some vertex $v \in Q$ is inactive. It holds that $v$ belongs to some detour $D_j$. As all the vertices of $Q$ are unassigned, necessarily $Q \subseteq D_j$. In particular, all the vertices of $Q$ belong to a single detour. This property will not change till the end of the algorithm, thus the lemma follows.

Next, we consider the case where by the end of the $i_Q - 1$'th iteration all the vertices of $Q$ are active. The algorithm at iteration $i_Q$ creates an $(1, \tau, \Delta_{i_Q})$-scattering partition $\mathcal{P}_{i_Q}$. The length of $Q$ is bounded by

$$L(Q) \overset{(3.1)}{\leq} c_{\text{int}} \cdot D(Q) \overset{(3.3)}{\leq} c_{\text{int}} \cdot \frac{3}{2} \cdot \frac{1}{1 - c_{\text{int}}} \cdot 2^{i_Q} = \frac{1}{4} \cdot 2^{i_Q} < \Delta_{i_Q}.$$  

(3.4)

Hence $Q$ is partitioned by $\mathcal{P}_{i_Q}$ to $\tau^2$ clusters $C_1, \ldots, C_{\tau^2} \in \mathcal{P}_{i_Q}$. It follows that by the end of the $i_Q$'th iteration, the inactive vertices in $Q$ are partitioned to at most $\tau$ detours. If all the vertices in $Q$ become inactive, then we are done, as the number of detours covering $Q$ can only decrease further in the algorithm (as a result of detour nullification). Hence we will assume that some of $Q$ vertices remain active.

A *slice* is a maximal sub-interval $S \subseteq Q$ of active vertices. The active vertices in $Q$ are partitioned to at most $\tau + 1$ slices $S_1, S_2, \ldots, S_{\tau+1}$.

By the end of the $i_Q + 1$ iteration, according to Claim 6 all $Q$ vertices will be assigned, and in particular belong to some detour. The algorithm creates a $(\Delta_{i_Q+1}, \tau, 1)$-scattering partition $\mathcal{P}_{i_Q+1}$ of the unassigned vertices. By equation (3.4) the length of every slice $S$ is bounded by $L(S) \leq L(Q) \leq \frac{1}{4} \cdot 2^{i_Q} \leq \Delta_{i_Q+1}$. Therefore the vertices $S$ intersect at most $\tau$ clusters of $\mathcal{P}_{i_Q+1}$, and thus will be partitioned to at most $\tau$ detours. Some detours might get nullified, however in the worst case, by the end of the $i_Q + 1$ iteration, the vertices in $\bigcup_i S_i$ are partitioned to at most $\tau \cdot (\tau + 1)$ detours. In particular all the vertices in $Q$ are partitioned to at most $O(\tau^2)$ detours. As the number of detours covering $Q$ can only decrease further in the algorithm, the lemma follows. \(\square\)

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21Actually, as at least one $Q$ vertex remained active, at the beginning of the $i_Q + 1$ iteration the inactive vertices of $Q$ partitioned to at most $\tau - 1$ detours. Therefore the maximal number of slices is $\tau$. 

ACM Trans. Algor.
By the end of algorithm, we will charge the intervals for the detours. Consider the detour \( D_j = \{ a_j, \ldots, b_j \} \) of \( t_j \). Let \( Q_j \in Q \) be the interval containing \( a_j \). We will charge \( Q_j \) for the detour \( D_j \). Denote by \( X(Q) \) the number of detours for which the interval \( Q \) is charged for. By Lemma 4, \( X(Q) = O(\tau^2) \) for every interval \( Q \in Q \).

Recall that by the end of the algorithm, all the vertices of \( P_{t,t'} \) are partitioned to consecutive disjoint detours \( D_t, \ldots, D_t \), where \( D_t = \{ a_t, \ldots, b_t \} \) and \( a_t, b_t \) belong to the cluster of \( t_t \). In particular \( t_t = t \) and \( t_t = t' \), as each terminal belongs to the cluster of itself. Moreover, for every \( j < s \), there is an edge \( \{ b_j, a_j s \} \) in \( G \) between the cluster of \( t_j \) to that of \( t_j s \). Therefore, in the minor induced by the partition there is an edge between \( t_j \) to \( t_j s \). We conclude

\[
d_M(t, t') \leq \sum_{j=1}^{s-1} d_G(t_j, t_j s) \leq \sum_{j=1}^{s-1} \left[ d_G(t_j, a_j) + d_G(a_j, a_j s) + d_G(a_j s, t_j s) \right] \\
\leq \sum_{j=1}^{s-1} d_G(a_j, a_j s) + 2 \sum_{j=1}^{s} d_G(t_j, a_j).
\]

Note that \( \sum_{j=1}^{s-1} d_G(a_j, a_j s) \leq d_G(t, t') \) as \( P_{t,t'} \) is a shortest path. Denote by \( Q_j \) the interval containing \( a_j \). By Corollary 1,

\[
d_G(t_j, a_j) = d_G(a_j, f(a_j)) \leq O(\tau) \cdot D(a_j) \overset{(3.2)}{=} O(\tau) \cdot D(Q_j) \overset{(3.1)}{=} O(\tau) \cdot L^+(Q_j).
\]

By changing the order of summation we get

\[
\sum_{j=1}^{s} d_G(t_j, a_j) = O(\tau) \cdot \sum_{Q \in Q} X(Q) \cdot L^+(Q) = O(\tau^2) \cdot \sum_{Q \in Q} L^+(Q).
\]

Finally, note that \( \sum_{Q \in Q} L^+(Q) \leq 2 \cdot d_G(t, t') \) as every edge in \( P_{t,t'} \) is counted at most twice. We conclude \( d_M(t, t') \leq O(\tau^2) \cdot d_G(t, t') \). Theorem 1 now follows.

4 EQUIVALENCE BETWEEN SPARSE COVERS AND WEAK SPARSE PARTITIONS

Jia et al. [55] proved (implicitly) that sparse covers imply weak sparse partitions. We provide here a formal statement and a proof, for the sake of clarity and completeness.

**Lemma 5** ([55]). Suppose that a graph \( G = (V, E, w) \) admits a \((\sigma, \tau, \Delta)\)-weak sparse cover \( C \), then \( G \) admits a \((\sigma, \tau, \Delta)\)-weak sparse partition.

**Proof.** Let \( \Delta > 0 \) be some parameter. We create a partition \( \mathcal{P} \) as follows: each vertex \( v \) joins to an arbitrary cluster \( P_C \), which corresponds to a cluster \( C \in \mathcal{C} \) covering \( v \), that is \( B_G(v, \frac{\Delta}{\sigma}) \subseteq C \). Note that every cluster \( P_C \) in \( \mathcal{P} \) is contained in cluster \( C \) in \( \mathcal{C} \). It immediately follows that \( \mathcal{P} \) has weak diameter \( \Delta \).

Next consider a ball \( B = B_G(u, \frac{\Delta}{\sigma}) \). Suppose that a cluster \( P_C \in \mathcal{P} \) intersects \( B \) at a vertex \( u \in P_C \cap B \). As \( u \) joined \( P_C \), it holds that \( B_G(u, \frac{\Delta}{\sigma}) \subseteq C \), implying \( v \in C \). Thus \( B \) intersects only clusters \( P_C \) in \( \mathcal{P} \) corresponding to clusters \( C \) in \( \mathcal{C} \) containing \( v \). We conclude that \( B \) intersects at most \( \tau \) clusters. \( \square \)

We discuss the implications of Lemma 5 to minor free graphs in Section 4.1. Note that if the given partition Lemma 5 has strong diameter guarantee, the resulting sparse partition will still have weak diameter guarantee only. Furthermore there are graphs that admit strong sparse covers but do not admit strong sparse partitions with similar parameters. One example will be trees which by Theorem 9 do not admit \((\sigma, \tau, \Delta)\)-strong sparse partition scheme for any constant \( \sigma, \tau \), while having \( (O(1), O(1))\)-strong sparse cover schemes [5, 15].

Interestingly, sparse partitions also imply sparse covers. Furthermore, unlike the previous direction, if the partition had strong diameter, so would the cover.
LEMMA 6. Suppose that a graph \( G = (V, E, w) \) admits a \((\sigma, \tau, \Delta)\)-weak sparse partition \( P \), then \( G \) admits a \((\sigma + 2, \tau, (1 + \frac{2}{\sigma})\Delta)\)-weak sparse cover \( C \). Furthermore, if \( P \) has a strong diameter guarantee, so will \( C \).

PROOF. For every cluster \( P \in P \) set \( C_P = B_G(P, \frac{\Delta}{2}) \). Let \( C = \{C_P \mid P \in P\} \) to be the resulting cover. First, note that every ball \( B_G(v, \frac{\Delta}{2}) \) is contained in the cluster \( C_P \) corresponding to the cluster \( P \in P \) containing \( v \).

Second, we argue that \( C \) has diameter \((1 + \frac{2}{\sigma})\Delta\). Consider a cluster \( C_P \in C \) and let \( u, v \in C_P \). There are \( u', v' \in P \) at distance at most \( \frac{\Delta}{2} \) from \( u \) and \( v \) respectively. By triangle inequality, \( d_G(u, v) \leq d_G(u, u') + d_G(u', v') + d_G(v', v) \leq (1 + \frac{2}{\sigma})\Delta \). Note that if \( P \) has strong diameter at most \( \Delta \), then as \( C_P \) includes the shortest path from \( u \) to \( u' \) (and \( v \) to \( v' \)) all these inequalities hold w.r.t. \( d_G[C_P] \) as well. We conclude that \( C \) is a sparse cover with the padding parameter at most \((1 + \frac{2}{\sigma})\Delta/\frac{\Delta}{2} = \sigma + 2\).

Third, we argue that \( C \) is sparse. Consider a vertex \( v \in V \). For a cluster \( C_P \in C \) that contains \( v \), necessarily \( v \in B_G(P, \frac{\Delta}{2}) \). In other words, \( B_G(v, \frac{\Delta}{2}) \cap P \neq \emptyset \). Thus \( C_P \) corresponds to a cluster \( P \in P \) intersecting \( B_G(v, \frac{\Delta}{2}) \). As the number of clusters in \( P \) intersecting \( B_G(v, \frac{\Delta}{2}) \) is bounded by \( \tau \), we conclude that the number of clusters containing \( v \) in \( C \) is also bounded by \( \tau \). \( \square \)

Lemma 6 phrased in other words: suppose that a graph \( G \) admits a \((\sigma, \tau)\)-weak/strong sparse partition scheme, these \( G \) also admits \((\sigma + 2, \tau)\)-weak/strong sparse cover scheme. Applying Lemma 6 on Theorems 10, 13 and 14, we conclude:

**Corollary 2.** Suppose that a weighted graph \( G = (V, E, w) \) has:

1. **Doubling dimension** \( \text{ddim} \), then for every \( t \geq 1 \), \( G \) admits an \((O(t), 2^{\text{ddim}/t}, \tilde{O}(\text{ddim}))\)-strong sparse cover scheme.
2. **SPD of depth** \( \rho \), then \( G \) admits a \((O(\rho), O(\rho^2))\)-strong sparse cover scheme.
3. **SPD of depth** \( \rho \), then \( G \) admits a \((10, 5\rho)\)-weak sparse cover scheme.

Strong sparse covers for doubling graphs were constructed directly by the author in a companion paper [35]. See discussion in Section 1.2. See Figure 3 for illustrations of the connections between the different notions of sparse covers and partitions.

### 4.1 Implications of Lemma 5: Minor Free Graphs

The graph family which has the most interesting implications due to sparse covers are minor free graphs. As previously mentioned in the literature (see e.g. [5]), it implicitly follows from the padded decompositions of Klein et al. [59] that \( K_{r,r} \) free minor graph admit \((O(r^2), 2^r)\)-weak sparse partition scheme. Nevertheless, as this fact was never stated as a theorem, it was overlooked by previous works on UST/UTSP. Specifically, covers with worse parameters were used by [15, 51], obtaining solutions with stretch \( \Omega(\log^2 n) \) for these problems. For the sake of clarity and completeness, we provide the explicit statements and a proof sketch.

**Theorem 3 (implicit from [33, 59]).** Every weighed graph that excludes \( K_{r,r} \) as a minor admits an \((O(r^2), 2^r)\)-weak sparse cover scheme.

**Proof Sketch.** While we will use the celebrated partition algorithm of [59], the analysis itself will be similar to the proof of Lemma 3.2 in [60]. Let \( \Delta > 0 \) be some parameter, and let \( \Delta' \) be a parameter depending on \( \Delta \) to be determined later. Fix \( rt \in V \) to be an arbitrary vertex. For \( b \in \{0, \frac{1}{2}\} \) and \( j \geq 0 \) set

\[
A_j^b = \{ v \in V \mid (j - 1 + b)\Delta' \leq d_G(rt, v) < (j + b)\Delta' \} .
\]

\[22\]Busch et al. [15] argued that their UTSP construction is deterministic. However the [59] based cover (Theorem 3) is deterministic as well, and hence implies a deterministic construction of UTSP with better parameters.
Fig. 3. The Venn diagram demonstrates the containment relations between the set of graphs admitting weak/strong sparse covers/partitions. Graphs with constant doubling dimension or SPD depth (or pathwidth) admit strong sparse partitions scheme with constant parameters (Corollary 2). All graph families excluding a fixed minor admit strong sparse covers with constant parameters [5], while no such strong sparse partitions exist (Theorem 9). The family of general graphs do not admit weak sparse partitions with constant parameters (Theorem 5). We currently lack an example of a graph family that admit weak sparse covers but do not admit strong sparse covers. Finding such an example, or alternatively proving that weak sparse covers imply the existence of strong sparse covers with similar parameters remains an open question.

Note that fixing the parameter $b$, we obtain a partition $P^b = \{A^b_i\}_{i \geq 0}$. Thus we created two partitions $P^0, P^\frac{1}{2}$. Note also, that every ball of radius smaller than $\Delta^\frac{1}{4}$ is fully contained in a cluster in one of the partitions.

Consider each connected component $C$ in each cluster of a each partition $P^b$, and apply the above process again. Continuing this way recursively to a total depth of $A$, we obtain $2^A$ partitions of $V$, such that every ball of radius smaller than $\Delta^\frac{1}{4}$ is fully contained in a cluster in one of the partitions. According to Fakcharoenphol and Talwar [33], there exists a universal constant $c > 0$ such that all the created partitions have weak diameter $c \cdot r^2 \cdot \Delta'$. Fix $\Delta' = \frac{\Delta}{c^2}$. Uniting all the $2^A$ partitions we obtain an $(O(r^2), 2^A, \Delta')$-weak sparse cover as required. □

Using Lemma 5 combined with Theorem 3, and then applying Theorem 2 we conclude:

**Corollary 3.** Every graph that excludes $K_{r,r}$ as a minor admits an $(O(r^2), 2^A)$-weak sparse partition scheme.

**Corollary 4.** Let $G = (V, E, w)$ be an $n$-vertex graph excluding $K_{r,r}$ as a minor. Then there is an efficient algorithm constructing a solution for the UST problem with stretch $O(4^r \cdot r^2 \cdot \log n)$.

Note that Corollary 4 has a quadratic improvement in the dependence on $n$ compared with previously explicitly stated results.

5 GENERAL GRAPHS

Jia et al. [55] constructed weak sparse partitions using the sparse covers of Awerbuch and Peleg [8]. This approach inherently produces unconnected clusters and therefore provides only weak diameter guarantee. The first result of this section is an efficient construction of strong sparse partition for general graphs. We construct a single partition that is good w.r.t. all ball sizes simultaneously. The proof appears in Section 5.1.
Theorem 4. Consider an $n$ vertex weighted graph $G = (V, E, w)$. For every parameter $\Delta > 0$ there is a partition $\mathcal{P}$ such that for every $\alpha \geq 1$, $\mathcal{P}$ is $(8\alpha, O(n^{1/\alpha} \cdot \log n), \Delta)$-strong sparse partition. In particular, for every $\alpha \geq 1$, $G$ admits an $(8\alpha, O(n^{1/\alpha} \cdot \log n))$-strong sparse partition scheme.

Note that by picking parameter $\alpha = \log n$, we obtain an $(O(\log n), O(\log n))$-strong sparse partition scheme. Theorem 4 is also a generalization of the scattering partitions of [57], while having a considerably simpler proof. Using Theorem 1 we can induce a solution for the SPR problem with distortion $\text{polylog}(k)$. While quantitatively better solutions are known, this one has the advantage of being induced by a general framework. Furthermore, the resulting proof is shorter than all the previous ones (and arguably simpler, even though the Relaxed Voronoi algorithm from [36] is much more elegant).

Corollary 5. Given a weighted graph $G = (V, E, w)$ with a set $K$ of terminals of size $k$, there is an efficient algorithm that returns a solution to the SPR problem with distortion $\text{polylog}(k)$.

Proof. According to Krauthgamer, Nguyen, and Zondiner [62] (Theorem 2.1), $G$ contains a minor with $O(k^4)$ Steiner points, preserving exactly all distances between the terminals.\(^{23}\) I.e., there is a minor $G'$ of $G$ with $O(k^4)$ vertices containing $K$, such that for every $t, t' \in K$, $d_G(t, t') = d_{G'}(t, t')$. Thus we can assume that $|V| = O(k^4)$. By Theorem 4, Observation 1, and Observation 2, $G$ and all its induced subgraphs are $(1, O(\log^3 k))$-scatterable. The corollary follows by Theorem 1. □

Next, we provide a lower bound on weak sparse partitions for general graphs. Quantitatively, this lower bound is essentially equivalent to the lower bound for strong sparse partition on trees. However, qualitatively it is different as it requires strong diameter. This lower bound implies that both Theorem 4 and the weak sparse partitions of [55] are tight up to second order terms. The proof appears in Section 5.2.

Theorem 5. Suppose that the all $n$-vertex graphs admit a $(\sigma, \tau)$-weak sparse partition scheme. Then $\tau \geq n^{\Omega(1/\sigma)}$.

The best scattering partitions we were able to provide are due to Theorem 4. We believe that quadratically better scattering partitions exist. Specifically, that every $n$ vertex weighted graph is $(1, O(\log n))$-scatterable, and furthermore that this is tight. See Conjecture 2 in Section 12. Unfortunately, we did not provide any lower bound on the parameters of scattering partitions. Nevertheless, we provide some evidence that no better scattering partitions from those conjectured to exist by Conjecture 2 can be constructed. Given a partition $\mathcal{P}$, we say that an edge $\{u, v\}$ is separated if its endpoints belong to different clusters $(P(u) \neq P(v))$. Given a shortest path $I = \{v_0, v_1, \ldots, v_m\}$, denote by $S_I(\mathcal{P}) = | \{ i \in [m] \mid P(v_{i-1}) \neq P(v_i) \} |$ the number of separated edges along $I$. We say that partition $\mathcal{P}$ is $(\sigma, \tau, \Delta)$-super scattering if $\mathcal{P}$ is connected, weakly $\Delta$-bounded, and every shortest path $I$ of length at most $\frac{\Delta}{\sigma}$ has at most $\tau$ separated edges $S_I(\mathcal{P}) \leq \tau$. We say that a graph $G$ is $(\sigma, \tau)$-super-scatterable if for every parameter $\Delta$, $G$ admits a $(\sigma, \tau, \Delta)$-super scattering partition.

It is straightforward to see that a $(\sigma, \tau, \Delta)$-super scattering partition is also a $(\sigma, \tau + 1, \Delta)$-scattering. However, the opposite does not always hold. Interestingly, some of the scattering partitions created in this paper are actually super-scattering. Specifically our scattering partition for trees, cactus graphs, chordal graphs and Euclidean space are actually super-scattering. Our strong sparse partition for general graph, doubling graphs, and graph with bounded SPDdepth are not necessarily super-scattering. The proof of the following theorem appears in Section 5.3.

Theorem 6. Suppose that all $n$-vertex graphs are $(1, \tau)$-super scatterable. Then $\tau = \Omega(\log n)$.

\(^{23}\)This minor is obtained by first deleting all edges which do not lie on a shortest path between two terminals, and then contracting all Steiner vertices of degree 2.
5.1 Strong Sparse Partitions for General Graphs: Proof of Theorem 4

Let \( \Delta > 0 \) be some parameter. Our partition will be created using the clustering algorithm of Miller et al. [68] described in Section 2, with the set of all vertices as centers (that is \( N = V \)). For each vertex \( t \in N \), we sample a shift \( \delta_t \) according to exponential distribution with parameter \( \lambda = \frac{\Delta}{4 \ln n} \). As a result of the execution of [68], we get a clustering \( \mathcal{P} \), where each cluster is connected and associated with some center vertex.

Denote by \( \phi \) the event that for all the vertices \( t \in V \), \( \delta_t \leq \frac{\Delta}{2} \). By union bound

\[
\Pr \left[ \bigcup_t \{ \delta_t > \frac{\Delta}{2} \} \right] = n \cdot e^{-\frac{\Delta^2}{8}} = \frac{1}{n}.
\]

Suppose that \( \phi \) indeed occurs. Consider some vertex \( v \in V \), suppose that \( v \) joined the cluster of the center \( t \in N \). Thus \( \delta_t - d_G(v, t) = f_o(t) \geq f_o(v) = \delta_t - d_G(v, v) \geq 0 \), implying \( d_G(v, t) \leq \delta_t \leq \frac{\Delta}{2} \). By Claim 1, for every vertex \( v \) in the cluster \( C \) of \( t \) it holds that \( d_G(C, v) = d_G(v, t) \). It follows that (assuming \( \phi \)) \( \mathcal{P} \) has strong diameter \( \Delta \).

We drop now any conditioning on \( \phi \). Fix some \( \alpha \geq 1 \). Let \( r_\alpha = \frac{\Delta}{8\alpha} \). Consider an arbitrary vertex \( v \) and let \( B_{0,\alpha} = B_G(v, r_\alpha) \) be the ball of radius \( r_\alpha \) around \( v \). We will bound \( Z_{B_{0,\alpha}} \), the number of clusters in \( \mathcal{P} \) intersecting \( B_{0,\alpha} \). Consider the set \( \{ f_o(t) \mid t \in N \} \), and order the values according to decreasing order, that is we denote by \( t^{(i)} \) the center corresponding to the \( i \)th largest value w.r.t. \( f_o \). Specifically \( f_o(t^{(i)}) \geq f_o(t^{(i+1)}) \geq \ldots \). Note that \( t^{(i)} \) is a random variable. Set \( s_\alpha = n^{\frac{1}{2\alpha}} \cdot 3 \ln n = O(n^{\frac{1}{2\alpha}} \cdot \log n) \). Denote by \( \psi_{\alpha,\alpha} \) the event that \( f_o(t^{(i)}) - f_o(t^{(i+1)}) > 2r_\alpha \).

**Claim 7.** For every \( \alpha \geq 1 \), \( \Pr[\psi_{\alpha,\alpha}] = \Pr[f_o(t^{(i)}) - f_o(t^{(i+1)}) \leq 2r_\alpha] \leq \frac{2}{2^\alpha} \).

**Proof.** We will use the law of total probability. Fix the center \( t^{(i+1)} \) and the set \( N = \{ t \in V \mid f_o(t) \geq f_o(t^{(i+1)}) \} \), note that we did not fix the order of the centers in \( N \). For \( t \in N \), by the memoryless property of exponential distribution it holds that

\[
\Pr[ f_o(t) - f_o(t^{(i+1)}) \leq 2r_\alpha \mid f_o(t) \geq f_o(t^{(i+1)}) ] = \Pr[ \delta_t \leq 2r_\alpha + \delta(t^{(i+1)}) - d_G(v, t^{(i+1)}) + d_G(v, t) \mid \delta_t \geq \delta(t^{(i+1)}) - d_G(v, t^{(i+1)}) + d_G(v, t) ] \leq \Pr[ \delta_t \leq 2r_\alpha ] = 1 - e^{-\frac{\Delta^2}{8}} = 1 - n^{-\frac{1}{2\alpha}}.
\]

As all the centers in \( N \) are independent, we conclude that

\[
\Pr[f_o(t^{(i)}) - f_o(t^{(i+1)}) \leq 2r_\alpha] = \Pr[ \forall t \in N, f_o(t) - f_o(t^{(i+1)}) \leq 2r_\alpha ] \leq \left(1 - n^{-\frac{1}{2\alpha}}\right)^{s_\alpha} < e^{-3\ln n} = \frac{1}{n^3}.
\]

Suppose that \( \psi_{\alpha,\alpha} \) indeed occurs. By Claim 2, \( B_{0,\alpha} \) will not intersect clusters centered in vertices \( t \) for which \( f_o(t^{(i)}) - f_o(t^{(i+1)}) > 2r_\alpha \). Therefore, at most \( s_\alpha \) clusters might intersect \( B_{0,\alpha} \). We conclude that if all the events \( \psi_{\alpha,\alpha} \) occur, then \( \mathcal{P} \) is \( (\frac{\Delta}{r_\alpha}, s_\alpha, \Delta) = (8\alpha, n^{\frac{1}{2\alpha}} \cdot 3 \ln n, \Delta) \)-strong sparse partition. Set \( A = \{ 1, 1 + \frac{1}{\log n}, 1 + \frac{2}{\log n}, \ldots, \log n \} \) be the arithmetic progression from 1 to \( \log n \) with difference between every pair of consecutive terms being \( \frac{1}{\log n} \). By union bound, the probability that for every \( \alpha \in A \), \( \mathcal{P} \) is \( (\frac{\Delta}{r_\alpha}, s_\alpha, \Delta) = (8\alpha, n^{\frac{1}{2\alpha}} \cdot 3 \ln n, \Delta) \)-strong sparse partition is at least \( 1 - \Pr[\bar{\phi}] - \sum_{\alpha \in A} \sum_{\alpha' \in V} \Pr[\bar{\psi}_{\alpha,\alpha}] \geq 1 - \frac{2}{\alpha} \). Thus we sample such a partition with high probability. We argue that the theorem holds for this partition. Indeed consider some \( \alpha \geq 1 \). If \( \alpha < \log n \), set \( \alpha' \) to be a number in \( A \) such that \( \alpha' \leq \alpha \leq \alpha' + \frac{1}{\log n} \). Else, if \( \alpha > \log n \), set \( \alpha' = \log n \). Every ball of radius \( \frac{\Delta}{r_\alpha} \) is contained in a ball of radius \( \frac{\Delta}{r_{\alpha'}} \), while the number of clusters it intersects is bounded by \( s_{\alpha'} = n^{\frac{1}{2\alpha'}} \cdot 3 \ln n \leq n^{\frac{1}{2\alpha}} \cdot 6 \ln n = O(n^{\frac{1}{2\alpha}} \cdot \log n) \) as required. The theorem follows.

\[\square\]
5.2 Lower Bound on Weak Sparse Partitions: Proof of Theorem 5

Let \( d > 1, c > 0 \) be some constants such that for every \( n \) there is an unweighted \( n \)-vertex graph \( G_n \) with maximal degree \( d \) and vertex expansion at least \( c \). That is, every subset \( A \) of \( G_n \) of cardinality at most \( n/2 \) has at least \( c \cdot |A| \) neighbors. Note that as the maximal degree is bounded by \( d \), a ball of radius \( r \) contains at most \( 1 + d + d^2 + \cdots + d^r < d^{r+1} \) vertices.

Fix \( \Delta = \frac{\log n}{d^2} \). Let \( \mathcal{P} \) be a partition with weak diameter \( \Delta \). Consider a cluster \( A \in \mathcal{P} \). Set \( A_r = B_{G}(A, r) \) to be a ball of radius \( r \) around \( A \). As \( A \) is contained in a ball of radius \( \Delta, |A_\frac{\Delta}{3}| \leq d^{(\log n) \cdot \Delta + 1} = \frac{n}{2} \). Thus we can use the expansion property in \( \frac{\Delta}{3} \) consecutive steps, and conclude

\[
|A_\frac{\Delta}{3} | \geq (1 + c) \cdot |A_\frac{\Delta}{3} - 1| \geq \cdots \geq (1 + c)^{\frac{\Delta}{3}} \cdot |A_0| = (1 + c)^{\frac{\Delta}{3}} \cdot |A| .
\]

Pick uniformly at random vertex \( v \in V \). For \( A \in \mathcal{P} \), let \( X_A \) be an indicator for the event that \( B_{G}(v, \frac{\Delta}{3}) \) the ball of radius \( \frac{\Delta}{3} \) around \( v \) intersects \( A \). It holds that

\[
\mathbb{E}_v \left[ Z_{B_{G}(v, \frac{\Delta}{3})} (\mathcal{P}) \right] = \sum_{A \in \mathcal{P}} \text{Pr} [X_A] = \sum_{A \in \mathcal{P}} \text{Pr} [v \in A_\frac{\Delta}{3}] = \sum_{A \in \mathcal{P}} \frac{|A_\frac{\Delta}{3}|}{n} = (1 + c)^{\frac{\Delta}{3}} \cdot \sum_{A \in \mathcal{P}} \frac{|A|}{n} = (1 + c)^{\frac{\Delta}{3}} .
\]

By averaging arguments we conclude that \( \tau \geq (1 + c)^{\frac{\Delta}{3}} = (1 + c)^{\frac{\Delta}{3}} \cdot \frac{\log n}{d} = n^{\Omega(\frac{1}{d^3})} \).

5.3 Lower Bound for Super Scattering Partition: Proof of Theorem 6

Let \( G = (V, E) \) be the \( d \)-dimensional hypercube. That is, the vertices \( V = \{0, 1\}^d \) are corresponding to \( 0, 1 \) strings of length \( d \), and two vertices are adjacent if their corresponding strings differ in a single bit. Set \( \Delta = k = \frac{d}{10} \). Note that every cluster of diameter \( \Delta \) includes at most \( \left( \frac{d}{10} \right)^d \leq \left( \frac{dt}{10} \right)^k = (10e)^{\frac{d}{10}} < 2^d \) vertices, as it is included in a ball of radius \( \Delta \). According to the edge isoperimetric inequality for the hypercube, given such a cluster \( A \), the number of outgoing edges from this cluster is at least \( |A| (d - \log |A|) \geq \frac{d}{2} \cdot |A| \). In particular, given a partition \( \mathcal{P} \) into \( k \)-bounded clusters, the total number of inter-cluster edges is at least \( \frac{1}{2} \sum_{A \in \mathcal{P}} \frac{d}{2} \cdot |A| = 2^d \cdot \frac{d}{4} \).

Let \( x_0 \) be a uniformly chosen random vertex. Pick \( k \) bits \( b_1, b_2, \ldots, b_k \) from \( [d] \) without repetitions where the order is important (we pick uniformly among the \( \frac{d!}{(n-k)!} \) possible choices). Set \( x_t \) to be equal to \( x_0 \) where we flip the bits \( b_1, \ldots, b_t \). Let \( X_i \) be an indicator for the event that \( x_i \) and \( x_{i+1} \) belong to different clusters \( (P(x_i) \neq P(x_{i+1})) \). Set \( X = \sum_{i=1}^{k} X_i \). Note that for every \( i, \{x_{i-1}, x_{i}\} \) is a uniformly random edge. Therefore \( \text{Pr}[X_i] \geq \frac{2^d \cdot \frac{d}{2} \cdot \frac{d}{2}}{2^d} = \frac{1}{2} \) equals to the ratio between inter cluster edges to all edges. We conclude that \( \mathbb{E}[X] \geq \frac{k}{2} \). By averaging arguments, there exists a shortest path \( I \) of length \( k \) with \( S_{f} (\mathcal{P}) \geq \frac{k}{2} = \Omega(d) \) separated edges. As the number of vertices in the hypercube is \( 2^d \), the theorem follows.

6 TREES

In this section we deal with the most basic graph family of trees. We show that trees admit scattering and weak sparse partitions with constant parameters, while no strong sparse partitions with constant parameters are possible. Thus we have a sharp contrast between the different partition types.

The first result of the section is the scattering partition. The proof appears in Section 6.1. Using Theorem 1 one can induce a (previously known [41, 47]) solution for the SPR problem on trees with distortion \( O(1) \).
Theorem 7. Every tree \( T = (V, E, \omega) \) is \((2, 3)\)-scatterable.

Next we turn our attention to weak sparse partitions. The fact that trees admit \((O(1), O(1))\)-weak sparse partition scheme actually follows by Lemma 5, as are there sparse covers constant parameters for trees [5, 15]. Nevertheless, as trees are the most basic graph family, we present a direct and elegant proof, obtaining better constants and understanding. Our constructions of scattering and weak sparse partitions are similar. Figure 4 illustrates both partitions on the full binary tree. The proof appears in Section 6.2.

Theorem 8. Every tree \( T = (V, E, \omega) \) admits a \((4, 3)\)-weak sparse partition scheme.

Finally we turn to the impossibility result. This lower bound is especially interesting as many graph families contain trees, and thus the lower bound applies for them as well. Specifically it also holds for general graphs, all minor free graphs, chordal graphs, etc. Using the strong sparse partitions for general graphs (Theorem 4), we conclude that the parameters of the lower bound are tight up to second order terms. We conclude that for strong sparse partition, trees are essentially as hard as general graphs. The proof appears in Section 6.3.

Theorem 9. Suppose that all trees with at most \( n \) vertices admit a \((f, g)\)-strong sparse partition scheme. Then \( g \geq \frac{1}{3} \cdot n \pi n^{-1} \).

We illustrate the lower bound via two different parameter choices.

Corollary 6. For every large enough parameter \( n > 1 \), there are \( n \) vertex trees \( T_1, T_2 \) such that,

- \( T_1 \) do not admit \((\log n, \log \log n)\)-strong sparse partition scheme.
- \( T_2 \) do not admit \((\sqrt{n \log n}, 2 \sqrt{n \log n})\)-strong sparse partition scheme.

6.1 Scattering Partitions for Trees: Proof of Theorem 7

Let \( \Delta > 0 \) be some parameter. Let \( rt \) be an arbitrary root vertex. Partition the graph according to distances from \( rt \). Specifically for every \( i \geq 0 \) set \( \mathcal{R}_i = \{ u \mid d_T(u, rt) \in \left[ i \cdot \frac{\Delta}{2}, (i + 1) \cdot \frac{\Delta}{2} \right) \} \). The clusters will be the connected components of each induced subgraph \( G[\mathcal{R}_i] \). See Figure 4 for illustration.

Consider some cluster \( C \subseteq \mathcal{R}_i \), and let \( v_C \in C \) be the closest vertex to \( rt \) in \( C \). Note that for every vertex \( u \in C \), its shortest path towards \( rt \) goes through \( v_C \). In particular, \( d_T(v_C, u) = d_T(rt, u) - d_T(rt, v_C) < (i + 1) \cdot \frac{\Delta}{2} - i \cdot \frac{\Delta}{2} = \frac{\Delta}{2} \).

It follows that our partition has strong diameter \( \Delta \).

Consider a path \( I = \{v_0, \ldots, v_m\} \) of length at most \( \frac{\Delta}{2} \). Let \( v_j \in I \) be the closest vertex to \( rt \) among \( I \) vertices. Let \( i \geq 0 \) be the index such that \( v_j \in \mathcal{R}_i \), and denote by \( \mathcal{C}_j \) the cluster of \( v_j \). As the distance of all the vertices in \( I \)
from \( v_j \) is at most \( \frac{A}{3} \), all \( I \) vertices belong to either \( \mathcal{R}_i \) or \( \mathcal{R}_{i+1} \). Let \( a \in [1, j] \) (resp. \( b \in [j, m] \)) be the minimal (resp. maximal) index such that \( v_a \in \mathcal{R}_i \) (resp. \( v_b \in \mathcal{R}_j \)). All the vertices \( v_a, \ldots, v_b \) are belonging to \( \mathcal{R}_j \) and lie in a single connected component. From the other hand, all the vertices \( v_0, \ldots, v_{a-1}, v_{a+1}, \ldots, v_m \) belong to \( \mathcal{R}_{i+1} \) and lie in at most two connected components. \(^{24}\) We conclude that the vertices of \( I \) intersect at most 3 different clusters. 

\[ \]

6.2 Weak Sparse Partitions for Trees: Proof of Theorem 8

Let \( \Delta > 0 \) be some parameter. Let \( rt \) be an arbitrary root vertex. Partition the graph according to distances from \( rt \). Specifically for every \( i \geq 0 \), set \( \mathcal{R}_i = \{ u \mid d_T(u, rt) \in [i \cdot \frac{\Delta}{4}, (i + 1) \cdot \frac{\Delta}{4}) \} \). Fix \( \mathcal{R}_i \), we say that \( u, v \in \mathcal{R}_i \) are equivalent \( u \sim v \) if and only if \( u \) and \( v \) have a common ancestor in \( \mathcal{R}_{i-1} \cup \mathcal{R}_i \). It is straightforward to verify that \( \sim \) is an equivalence relation. Our partition \( \mathcal{P} = \bigcup_{i \geq 0} \mathcal{R}_i / \sim \) will simply be all the equivalence classes for all indices \( i \). See Figure 4 for illustration.

First we argue that our partition has weak diameter \( \Delta \). Consider a pair of vertices \( u, v \in \mathcal{R}_i \) which are clustered together. As \( u \sim v \), they have a common ancestor \( z \in \mathcal{R}_{i-1} \cup \mathcal{R}_i \). Thus

\[
d_T(u,v) \leq d_T(u,z) + d_T(z,v) = (d_T(rt,u) - d_T(rt,z)) + (d_T(rt,v) - d_T(rt,z)) < 2 \cdot \left( (i + 1) \cdot \frac{\Delta}{4} - (i - 1) \cdot \frac{\Delta}{4} \right) = \Delta.
\]

Consider a pair of vertices such that \( u, v \in \mathcal{R}_i \) but \( u \sim v \). Their least common ancestor is at distance greater than \( \frac{\Delta}{4} \) from both \( u, v \), thus \( d_T(u, v) > \frac{\Delta}{2} \). It follows that every pair of vertices in \( \mathcal{R}_i \) at distance at most \( \frac{\Delta}{2} \) are necessarily equivalent, and hence belong to the same cluster. Let \( B = B_T(v, r) \) be some ball with radius at most \( r \leq \frac{\Delta}{4} \). The maximal pairwise distance of a pair of vertices in \( B \) is at most \( \frac{\Delta}{2} \). It follows that for every index \( i \), \( B \) can intersect at most a single cluster from \( \mathcal{R}_i \). Suppose w.l.o.g. that \( v \in \mathcal{R}_i \), then the vertices of \( B \) are contained in \( \mathcal{R}_{i-1} \cup \mathcal{R}_i \cup \mathcal{R}_{i+1} \). It follows that \( B \) intersects at most 3 clusters.

\[ \]

6.3 Lower Bound on Strong Sparse Partition for Trees: Proof of Theorem 9

Assume for contradiction that all \( n \)-vertex trees admit a \((\sigma, \tau)\)-strong sparse partition scheme where \( \tau < \frac{1}{2} \cdot n^{\frac{2}{3\tau}} \). Set \( d = \tau \) and \( D = \frac{22\sigma}{2} \). Let \( T \) be a tree rooted at \( rt \), of depth \( D \), such that the root has \( d + 1 \) children, and all other vertices at distance less than \( D \) from \( rt \) have exactly \( d \) children. The number of vertices is bounded by

\[
1 + (d + 1) + (d + 1) \cdot d + \cdots + (d + 1) \cdot d^{D-1} = 1 + (d + 1) \cdot \frac{d^D - 1}{d - 1} < 3 \cdot d^D \leq n.
\]

Fix \( \Delta = 2D - 1 \), and let \( \mathcal{P} \) be a \((\sigma, \tau, \Delta) = (2D - 1, d, \Delta)\)-strong sparse partition. Consider the cluster \( C \in \mathcal{P} \) containing the root \( rt \). If at most one child of \( rt \) belongs to \( C \), then the ball \( B_T(rt, 1) \) of radius \( 1 = \frac{\Delta}{2D - 1} \) intersects \( d + 1 > \tau \) clusters (as every other child must belong to a different cluster, because \( \mathcal{P} \) is connected). Otherwise, there must be a non-leaf vertex \( v \in C \) such that none of \( v \)'s children belong to \( C \), as otherwise \( C \) will contain a path of length \( 2D > \Delta \). All the children of \( v \) belong to different clusters. Therefore the ball \( B_T(v, 1) \) intersects \( d + 1 \) different clusters, a contradiction.

\[ \]

\(^{24}\) If \( a = 0 \) then \( v_0, \ldots, v_{a-1} \) is an empty set. Similarly for \( b = m \).
7 DOUBLING GRAPHS

In this section we construct strong sparse partitions for graphs with bounded doubling dimension. Similarly to general graphs, fixing a diameter parameter $\Delta$, we construct a single partition which is simultaneously good for all ball sizes.

**Theorem 10.** Let $G = (V, E, w)$ be a graph with doubling dimension $\text{ddim}$. For every parameter $\Delta > 0$, there is an efficiently computable partition $\mathcal{P}$, such that for every integer $\alpha \geq 1$, $\mathcal{P}$ is $(58\alpha, 2^{\text{ddim}+\alpha} \cdot \tilde{O}(\text{ddim}), \Delta)$-strong sparse partition. In particular, for every $\alpha \geq 1$, $G$ admits a $(58\alpha, 2^{\text{ddim}+\alpha} \cdot \tilde{O}(\text{ddim}))$-strong sparse partition scheme.

In Theorem 5 we prove that if all $n$ vertex graphs admit $(\sigma, \tau)$-weak sparse partition scheme, then $\tau \geq n^0(\frac{\Delta}{\text{ddim}})$. As every $n$ vertex graph has doubling dimension at most $\log n$, it implies that Theorem 10 is tight up to second order terms. In particular, even if one replaced the diameter guarantee in Theorem 10 from strong to weak, it will still be tight up to second order terms.

An exponential improvement for UST and UTSP problems in the dependence on the dimension is a direct corollary from Theorem 2.

**Corollary 7.** Let $G = (V, E, w)$ be an $n$-vertex graph with doubling dimension $\text{ddim}$. Then there is an efficient algorithm solving the UST problem with stretch $O(\text{ddim}^3 \cdot \log n)$.

Unfortunately, the doubling dimension of an induced graph $G[A]$ might be considerably larger than the doubling dimension of the original graph $G$. Thus this partition as is cannot be plugged in directly into Theorem 1. We leave the construction of a solution for the SPR problem on doubling graphs with distortion poly$(\text{ddim})$ as an open problem for future work.

The rest of this section is devoted to the proof of Theorem 10.

**Proof of Theorem 10.** Let $\Delta > 0$ be some parameter. We will construct partition which is only $O(\Delta)$ bounded. As this works for every $\Delta$, eventually one can re-adjust the parameters accordingly. Let $N \subseteq X$ be a $\Delta$-net. Our partition will be created using the modified clustering algorithm of Miller et al. [68] described in Section 2.1. Set $c_\tau = 4$. For each net point $t \in N$, we sample a shift $\delta_t$ according to *btailed* exponential distribution with parameters $(\lambda = \frac{\Delta}{\text{ddim}}, \lambda_\tau = c_\tau \cdot \Delta)$.\(^{25}\) This distribution is the same as an exponential distribution with parameter $\lambda$, where the only difference is that any value above $\lambda_\tau$ collapses to $\lambda_\tau$. Formally, a variable distributed according to a btailed exponential distribution with parameters $(\lambda, \lambda_\tau)$ get value in $[0, \lambda_\tau]$, where the density function in $[0, \lambda_\tau]$ is $f_{\lambda, \lambda_\tau}(x) = \frac{1}{\lambda_\tau} e^{-\frac{x}{\lambda_\tau}}$, and the probability to get the value $\lambda_\tau$ is $e^{-\frac{\lambda_\tau}{\lambda}}$.

As a result of the execution of [68] algorithm we get a clustering $\mathcal{P}$, where each cluster is connected and associated with some net point from $N$. Consider some vertex $v \in V$. There is a net point $t_v \in N$ at distance at most $\Delta$ from $v$. Suppose that $v$ joined the cluster of the net point $t \in N$. Hence $\delta_t - d_G(v, t) = f_v(t) = f_v(t_v) = \delta_{t_v} - d_G(v, t_v) \geq -\Delta$. Therefore $d_G(v, t) \leq \lambda_\tau + \Delta = (c_\tau + 1) \cdot \Delta$. By Claim 1, for every vertex $v$ in the cluster $C$ of $t$ it holds that $d_G(C, v, t) = d_G(v, t)$. It follows that $\mathcal{P}$ has strong diameter $2 \cdot (c_\tau + 1) \cdot \Delta = 10\Delta$.

Fix some $\alpha \geq 1$, and let $r_\alpha = \frac{\ln 2}{\alpha^2} \cdot \Delta$. Consider an arbitrary vertex $v$ and let $B_{\alpha, v} = B_G(v, \frac{r_\alpha}{2})$ be the ball of radius $\frac{r_\alpha}{2}$ around $v$. Denote by $N_0 \subseteq N$ the set of net points at distance at most $(c_\tau + 2) \cdot \Delta$ from $v$. For every $t \notin N_\alpha, f_v(t_v) - f_v(t) \geq (0 - d_G(v, t_v)) - (\lambda_\tau - d_G(v, t)) > \Delta$. By Claim 2, the cluster of a net point $t \notin N_\alpha$ will not intersect $B_G(v, \frac{r_\alpha}{4})$ and in particular $B_{\alpha, v}$. Using the packing property (Lemma 3),

$$|N_0| \leq (\frac{2}{c_\tau^2 + 2})^\Delta \text{ddim} \leq (3 \cdot c_\tau)^\Delta \text{ddim}. \quad (7.1)$$

\(^{25}\) The author is not aware of previous appearance of this distribution in the literature. The name btailed is inspired by the term "beheaded", as we cut off the tail of the distribution.

ACM Trans. Algor.
We will bound \( Z_{B_{\alpha, \delta}} \), the number of clusters in \( \mathcal{P} \) intersecting \( B_{\alpha, \delta} \). For the sake of analysis, the sample of \( \delta_t \), the shift of the net point \( t \in N \), will be computed in two steps: First sample a variable \( \delta_t \) according to exponential distributed with parameter \( \lambda \), secondly set \( \delta_t = \min(\delta_t, cT \cdot \Delta) \). Similarly to the definition of \( f_\delta(t) \), set \( \tilde{f}_\delta(t) = \tilde{d}_t - d_{c2}(o, t) \).

Consider the set \( \{ \tilde{f}_\delta(t) \mid t \in N_\alpha \} \), and order the values according to decreasing order, that is we denote by \( \tilde{i} = \min \{ \tilde{i} \} \) the net point corresponding to the \( i \)th largest value w.r.t. \( \tilde{f}_\delta \). Specifically \( \tilde{f}_\delta(\tilde{i}) \geq \tilde{f}_\delta(\tilde{i} - 1) \geq \ldots \). Note that \( \tilde{i} \) is a random variable. Fix \( s = \left[ 4 \ln \left( 4e \cdot \dim (24 \cdot c_T \cdot \dim) \right) \right] = O(\dim) \). Set \( m_\alpha = 2s \cdot 2^\dim/\alpha \).

Claim 8. \( \Pr \left[ \tilde{f}_\delta(t) - \tilde{f}_\delta(\tilde{i}(m_{\alpha} + 1)) \leq r_\alpha \right] \leq e^{-\delta} \).

Proof. We will use the law of total probability. Fix the net point \( \tilde{i}(m_{\alpha} + 1) \in N_\alpha \). Let \( N = \{ t \in N_\alpha \mid \tilde{f}_\delta(t) \geq \tilde{f}_\delta(\tilde{i}(m_{\alpha} + 1)) \} \), note that \( |N| = m_\alpha \) and that we did not fixed the inner order of the points in \( N \). For \( t \in N \) denote by \( X_t \) the event that \( \tilde{f}_\delta(t) - \tilde{f}_\delta(\tilde{i}(m_{\alpha} + 1)) > r_\alpha \). By the memoryless property of exponential distribution,

\[
\Pr \left[ \tilde{f}_\delta(t) - \tilde{f}_\delta(\tilde{i}(m_{\alpha} + 1)) > r_\alpha \mid \tilde{f}_\delta(\tilde{i}) \geq \tilde{f}_\delta(\tilde{i}(m_{\alpha} + 1)) \right] = \Pr[\tilde{f}_\delta > r_\alpha] = e^{-r_\alpha/\lambda} = e^{-r_\alpha/\dim}.
\]

Set \( X = \sum_{t \in N} X_t \). Then \( \mathbb{E}[X] \geq m_\alpha \cdot 2^{-\dim/\alpha} = 2s \). As all \( \{ \tilde{f}_\delta(t) \} \) \( t \in N \) are independent, by Chernoff inequality it holds that

\[
\Pr[X \leq \frac{1}{2} \cdot \mathbb{E}[X]] \leq e^{-\mathbb{E}[X]/2} < e^{-\delta}.
\]

In case \( X > \frac{1}{2} \cdot \mathbb{E}[X] \geq s \), it will imply that for at least \( s \) net points \( t \in N_\alpha \), \( \tilde{f}_\delta(t) > \tilde{f}_\delta(\tilde{i}(m_{\alpha} + 1)) \). In particular, \( \tilde{f}_\delta(\tilde{i}) - \tilde{f}_\delta(\tilde{i}(m_{\alpha} + 1)) > r_\alpha \). The claim now follows.

Denote by \( \phi_\alpha \) the event that for at least \( s \) different net points \( t \in N_\alpha \), it holds that \( \delta_t = \lambda_T \). As all \( \delta_t \) \( t \in N_\alpha \) are independent,

\[
\Pr[\phi_\alpha] \leq \left( \frac{|N_\alpha|}{s} \right) \cdot \left( e^{-\frac{\lambda_T}{s}} \right)^s \leq \left( 3 \cdot c_T \cdot \dim \cdot e^{-c_T \cdot \dim} \right)^s \leq e^{-s \cdot \dim}.
\]

Next, we incorporate the truncations into the analysis. Denote by \( \psi_{\alpha, \lambda} \) the event that \( \tilde{f}_\delta(\tilde{i}(s)) - \tilde{f}_\delta(\tilde{i}(m_{\alpha} + 1)) \leq r_\alpha \). By Claim 8, \( \Pr[\psi_{\alpha, \lambda}] \leq e^{-\gamma/\alpha} \). Similarly, denote by \( \psi_{\alpha, \lambda} \) the event that \( \tilde{f}_\delta(\tilde{i}(1)) - \tilde{f}_\delta(\tilde{i}(m_{\alpha} + 1)) \leq r_\alpha \). Where \( \tilde{i}(1) \) is the net point having the \( i \)th largest value w.r.t. \( \tilde{f}_\delta \) in \( N_\alpha \). We argue that \( \psi_{\alpha, \lambda} \subseteq \psi_{\alpha, \lambda} \cup \phi_\alpha \). It will be enough to show that if both \( \psi_{\alpha, \lambda} \) and \( \phi_\alpha \) did not occur, neither did \( \psi_{\alpha, \lambda} \) \( \cup \phi_\alpha \). Indeed, consider the case that both \( \psi_{\alpha, \lambda} \) and \( \phi_\alpha \) did not occur. As \( \phi_\alpha \) did not occur, there is an index \( i \in [1, s] \) such that \( \delta_t = \tilde{d}_t \). For every \( j \geq m_{\alpha} + 1 \) it holds that

\[
\tilde{f}_\delta(\tilde{i}(1)) \geq \tilde{f}_\delta(\tilde{i}(s)) \geq \tilde{f}_\delta(\tilde{i}(s)) > r_\alpha + \tilde{f}_\delta(\tilde{i}(m_{\alpha} + 1)) \geq r_\alpha + \tilde{f}_\delta(\tilde{i}(m_{\alpha} + 1)) \cdot r_\alpha + \tilde{f}_\delta(\tilde{i}(j)) \geq r_\alpha + \tilde{f}_\delta(\tilde{i}(j)) \cdot r_\alpha + \tilde{f}_\delta(\tilde{i}(j)) \cdot r_\alpha + \tilde{f}_\delta(\tilde{i}(j)) \cdot r_\alpha + \tilde{f}_\delta(\tilde{i}(j)) \cdot r_\alpha + \tilde{f}_\delta(\tilde{i}(j)).
\]

Thus for all but at most \( m_{\alpha} \) net points \( t \in N_\alpha \), it holds that \( \tilde{f}_\delta(\tilde{i}(1)) - \tilde{f}_\delta(t) > r_\alpha \). Hence \( \psi_{\alpha, \lambda} \) did not occur. By Claim 2, it follows that \( B_{\alpha, \delta} \) did not intersect the clusters of such net points \( t \in N_\alpha \). Therefore the number of clusters intersecting \( B_{\alpha, \delta} \) is bounded by \( Z_{B_{\alpha, \delta}}(\mathcal{P}) \leq m_{\alpha} \). By union bound

\[
\Pr \left[ Z_{B_{\alpha, \delta}}(\mathcal{P}) > m_{\alpha} \right] \leq \Pr[\psi_{\alpha, \lambda}] \leq \mathbb{E}[\phi_\alpha \vee \psi_{\alpha, \lambda}] \leq e^{-\lambda_T\dim} + e^{-\gamma/\alpha} \leq 2 \cdot e^{-\gamma/\alpha}.
\]

Set \( A = \{ 1, 1 + \frac{1}{\dim}, 1 + \frac{2}{\dim}, \ldots, \dim \} \) to be the arithmetic progression from 1 to \( \dim \) with difference between every pair of consecutive terms being \( 1/\dim \). Set \( \psi_{\alpha} = \bigcup_{\alpha \in A} \psi_{\alpha, \lambda} \) the event that \( \psi_{\alpha, \lambda} \) holds for some \( \alpha \in A \). By union bound \( \Pr(\psi_{\alpha}) = \sum_{\alpha \in A} \Pr(\psi_{\alpha}) \leq 2 \cdot \dim^2 \cdot e^{-\gamma/\alpha} \).

ACM Trans. Algor.
Our next goal is to show that there is a positive probability for the event that simultaneously for every vertex \(v\) and all parameters \(\alpha \geq 1\), the ball \(B_G(v, \frac{\alpha}{4})\) intersects at most \(m_\alpha\) clusters. Set \(\epsilon = \frac{\Delta}{4 \dim}\). and let \(\hat{N} \subset X\) be an \(\epsilon\)-net. Set \(\mathcal{A} = \{\Psi_v\}_{v \in \hat{N}}\). For a net point \(v \in \hat{N}\) denote \(\Gamma_v = \{u \in \hat{N} \mid d_G(v, u) \leq 3 \cdot c_\Gamma \cdot \Delta\}\). For every net point \(u \in \hat{N} \setminus \Gamma_v\), \(N_v\) and \(N_u\) are disjoint, thus \(\Psi_v\) and \(\Psi_u\) are independent. Using the packing property (Lemma 3), \(\Psi_v\) is dependent with at most

\[
|\Gamma_v| \leq \left(\frac{6 \cdot c_\Gamma \cdot \Delta}{\epsilon}\right)^{\dim} = (24 \cdot c_\Gamma \cdot \dim)^{\dim}
\]

other events from \(\mathcal{A}\). We will use the constructive version of the Lovász Local Lemma by Moser and Tardos [70].

**Lemma 7 (Constructive Lovász Local Lemma).** Let \(\mathcal{P}\) be a finite set of mutually independent random variables in a probability space. Let \(\mathcal{A}\) be a set of events determined by these variables. For \(A \in \mathcal{A}\) let \(\Gamma(A)\) be a subset of \(\mathcal{A}\) satisfying that \(A\) is independent from the collection of events \(\mathcal{A} \setminus (\{A\} \cup \Gamma(A))\). If there exist an assignment of reals \(x: \mathcal{A} \to (0, 1)\) such that

\[
\forall A \in \mathcal{A} \mid \Pr[A] \leq x(A) \cdot \prod_{B \in \Gamma(A)} (1 - x(B)),
\]

then there exists an assignment to the variables \(\mathcal{P}\) not violating any of the events in \(\mathcal{A}\). Moreover, there is an algorithm that finds such an assignment in expected time \(\sum_{A \in \mathcal{A}} \frac{x(A)}{1 - x(A)} \cdot \text{poly}(|\mathcal{A}| + |\mathcal{P}|)\).

For \(v \in \hat{N}\) set \(x(\Psi_v) = \epsilon \cdot \Pr[\Psi_v]\). For every \(v \in \hat{N}\) it holds that

\[
x(\Psi_v)\Pi_{u \in \Gamma_v}(1 - x(\Psi_u)) \geq \Pr[\Psi_v] \cdot \epsilon \cdot \left(1 - 2 \epsilon \cdot \dim \cdot e^{-\epsilon}\right)^{|\Gamma_v|}
\]

\[
\geq \Pr[\Psi_v] \cdot \epsilon \cdot \left(1 - \frac{1}{2} \cdot (24 \cdot c_\Gamma \cdot \dim)^{-\dim}\right)^{(24 \cdot c_\Gamma \cdot \dim)^{\dim}} \geq \Pr[\Psi_v].
\]

By Lemma 7 we can efficiently choose \(\{\delta_v\}_{v \in \hat{N}}\) such that none of the events \(\{\Psi_v\}_{v \in \hat{N}}\) occur. Consider the partition \(\mathcal{P}\) obtained by choosing this values \(\{\delta_v\}\). Fix some parameter \(\alpha > 1\) and some vertex \(v\). There is a net point \(v \in \hat{N}\) at distance at most \(\epsilon \leq \frac{\alpha}{4}\) from \(u\). If \(\alpha \leq \dim\), pick \(\alpha' < A\) such that \(\alpha' \leq \alpha \leq \alpha' + \frac{1}{\dim}\). Else, if \(\alpha > \dim\), set \(\alpha' = \dim\). The ball of radius \(\frac{\alpha'}{4}\) around \(u\) is contained in a ball of radius \(\frac{\alpha'}{4}\) around \(v\). As \(\Psi_{\alpha'} \in \Psi_{v}\) did not occur, it holds that \(Z_{B_{\alpha'}(\mathcal{P})} \leq m_{\alpha'}\). Therefore \(Z_{B_{\alpha'}(\mathcal{P})} \leq Z_{B_{\alpha'}(\mathcal{P})} \leq m_{\alpha'} = 2s \cdot 2^{\dim/\alpha'} \leq 4s \cdot 2^{\dim/\alpha'} = 2^{\dim/\alpha'} \cdot \hat{O}(\dim)\). The padding padding parameter is \(\frac{2^{\dim/\alpha'} \cdot \hat{O}(\dim)}{\alpha'} = \frac{40}{\alpha} \cdot \alpha \leq 58\alpha\). We conclude that for every \(\alpha \geq 1\), \(\mathcal{P}\) is \(\{58\alpha, 2^{\dim/\alpha'}, \hat{O}(\dim), 10\Delta\}\)-strong sparse partition. The theorem follows. \(\square\)

### 8 Euclidean Space

Consider the \(d\) dimensional Euclidean space \((\mathbb{R}^d, \|\cdot\|_2)\). Partitions of this space are well studied. We will study the Euclidean space from the lenses of sparse and scattering partitions. Weak sparse partitions are defined in the natural way. A cluster \(C\) is connected if for every pair of vectors \(\overrightarrow{x}, \overrightarrow{y} \in C\), there is a continuous function \(f: [0, 1] \to \mathbb{R}^d\) such that \(f(0) = \overrightarrow{x}, f(1) = \overrightarrow{y}\), and the entire image of \(f\) is inside \(C\). The shortest path between two vectors \(\overrightarrow{x}, \overrightarrow{y}\) is simply the interval between them \(\{\overrightarrow{x} + t \cdot (\overrightarrow{y} - \overrightarrow{x}) \mid t \in [0, 1]\}\). A partition \(\mathcal{P}\) is \((\sigma, \tau, \Delta)\)-scattering if each cluster is connected and has weak diameter at most \(\Delta\), and for every pair of points at distance at most \(\frac{\Delta}{2}\), the interval between them intersects at most \(\tau\) clusters.

Interestingly, we show that \((\mathbb{R}^d, \|\cdot\|_2)\) has scattering partitions with significantly better parameters compared to the weak sparse partitions it admits. Specifically, for constant padding parameter \(\sigma = \Omega(1)\) there are scattering partitions where the number of intersections \(\tau = O(d)\) is linear in \(d\), while every weak sparse partition with such
We will prove that
\[ (\mathbb{R}^d, \|\cdot\|_2) \text{ is } (1, 2d)\text{-scattering}. \]

\textbf{Theorem 11.} The Euclidean space \((\mathbb{R}^d, \|\cdot\|_2)\) is \((1, 2d)\)-scattering.

\textbf{Theorem 12.} Suppose that the space \((\mathbb{R}^d, \|\cdot\|_2)\) admits a \((\sigma, \tau)\)-weak sparse partition scheme. Then \(\tau > (1 + \frac{1}{2d})^d\).

Another way to represent the parameters in Theorem 12 is \(\sigma > \frac{1}{2} \cdot \frac{1}{e^{d-1}} > \frac{1}{2} \cdot \frac{1}{e^{\frac{1}{2\ln 2} - 1}} = \frac{d}{\ln e}, \) where the second inequality holds as \(e^{\frac{1}{2\ln 2}} < 1 + \frac{1}{2\ln 2}\) (using that \(e^x < 1 + 2x\) for \(x \in (0, 1)\)). Note that in order to create a partition with at most polynomially many intersections, the padding parameter must be essentially linear in \(d\).

\subsection{8.1 Scattering Partitions for Euclidean Space: Proof of Theorem 11}

We will prove that \((\mathbb{R}^d, \|\cdot\|_2)\) admits an \((1, 2d, \sqrt{d})\)-scattering partition. By scaling, this will imply the general theorem. Define \(P\) to be the natural partition according to axis parallel hyperplanes. That is for every \(\vec{a} = (a_1, \ldots, a_d) \in \mathbb{Z}^d\), we will have the cluster \(C_\vec{a} = \{\vec{x} = (x_1, \ldots, x_d) \mid \forall i, a_i \leq x_i < a_i + 1\}\). It is straightforward that this partition has diameter \(\sqrt{d}\). Note that in this partition, an arbitrarily small ball centered in a vector with integer coordinates will intersect \(2^d\) different clusters. Nevertheless, intervals will intersect only a small number of clusters.

Consider a pair of vectors \(\vec{a}, \vec{b}\) at distance at most \(\|\vec{a} - \vec{b}\|_2 \leq \sqrt{d}\). Denote by \(\vec{c} = \vec{b} - \vec{a}\) their difference, and thus the shortest path between them is the interval \(I = \{\vec{a} + t \cdot \vec{c} \mid t \in [0, 1]\}\). Denote by \(h^n = (x_1, \ldots, x_d)\) the \(i\)-axis parallel hyperplane at height \(n\). Denote by \(\mathcal{H}_i = \{h^n \mid n \in \mathbb{Z}\}\) the set of all \(i\)-axis parallel hyperplanes at integer heights. We say that the interval \(I\) crosses \(h^n\) if there are two vectors \(\vec{x}, \vec{y} \in I\) s.t. \(\vec{x} \leq n < \vec{y}\). Set \(X_i\) to be the number of hyperplanes in \(\mathcal{H}_i\) which are crossed by \(I\). The number of clusters intersecting \(I\) is bounded by the number of axis parallel hyperplanes \(I\) crosses, plus the initial cluster containing \(\vec{a}\). That is \(Z_i(P) \leq 1 + \sum_{i=1}^{d} X_i\).

Set \(\vec{c} = (c_1, \ldots, c_d)\). As the projection of \(I\) on the \(i\)th coordinate is an interval of length \(|c_i|\), it holds that \(X_i \leq \lceil |c_i| \rceil\). The total number of intersections is bounded by

\[ Z_i(P) \leq 1 + \sum_{i=1}^{d} X_i \leq 1 + \sum_{i=1}^{d} \lceil |c_i| \rceil < d + 1 + |\vec{c}|_1 \leq d + 1 + \sqrt{d} \cdot \|\vec{b} - \vec{a}\|_2 = 2d + 1, \]

where the inequality \((*)\) holds as for every vector \(\vec{x} \in \mathbb{R}^d, \|\vec{x}\|_1 \leq \sqrt{d} \cdot \|\vec{x}\|_2\). As the number of intersection is an integer, we conclude \(Z_i(P) \leq 2d\). The theorem follows.

\textbf{Remark 1.} The analysis of the partition above is tight. Indeed, consider the interval between the vectors \(\vec{a} = (-\epsilon, -2\epsilon, -3\epsilon, \ldots, -d \cdot \epsilon)\) and \(\vec{b} = (\epsilon, 1 + \epsilon, 1 + \epsilon, \ldots, 1 + \epsilon)\) for small enough \(\epsilon\). The number of clusters the interval \([\vec{a}, \vec{b}]\) intersects is exactly \(2d\).

\subsection{8.2 Lower Bound on Euclidean Weak Sparse Partitions: Proof of Theorem 12}

The proof is in \((\mathbb{R}^d, \|\cdot\|_2)\) with Lebesgue measure, we will omit unnecessary notation. Fix an arbitrary \(\Delta > 0\) and set \(c = 4\sigma\). Instead of partitioning the entire space, we will use the assumption to produce a \((\sigma, \tau, \Delta)\)-weak sparse partition \(P\) of \(B(\vec{0}, c \cdot \Delta)\), the origin centered ball with radius \(c \cdot \Delta\). Let \(r = \frac{\Delta}{\sigma}\). Given a set \(A\), denote by \(A_r = \{x + y \mid x \in A, y \in \mathbb{R}^d, \|y\|_2 \leq r\}\) the set of all points at distance at most \(r\) from \(A\). Note that \(A_r\) is the Minkowski sum of \(A\) and \(B(\vec{0}, r)\). By the Brun-Minkowski Theorem (see e.g. [43] Corollary 5.3.), it holds that

\[ \text{Vol}^2(A_r) \geq \text{Vol}^2(A) + \text{Vol}^2(B(\vec{0}, r)). \]
Consider a cluster $C \in \mathcal{P}$. As the diameter of $C$ is bounded by $\Delta$, $C$ is contained in a ball of radius $\Delta$. Therefore $\text{Vol}(B(\bar{0}, r)) = \left(\frac{\sigma}{\Delta}\right)^d \cdot \text{Vol}(B(\bar{0}, \Delta)) \geq \left(\frac{1}{\sigma}\right)^d \cdot \text{Vol}(C)$. We conclude, that for every $C \in \mathcal{P}$ it holds that

$$\text{Vol}(C) \geq \left(\text{Vol}^2(C) + \text{Vol}^2(B(\bar{0}, r))\right)^d \geq \left(\text{Vol}^2(C) + \frac{1}{\sigma} \cdot \text{Vol}^2(C)\right)^d = \left(1 + \frac{1}{\sigma}\right)^d \cdot \text{Vol}(C).$$

The proof of the theorem will be concluded using the probabilistic method. Set $\mathcal{P}' = \{C \cap B(\bar{0}, (c - 1)\Delta) \mid C \in \mathcal{P}\}$ to be the set consisting of $\mathcal{P}$ clusters limited to $B(\bar{0}, (c - 1)\Delta)$. Sample a point $x$ from the ball $B(\bar{0}, c \cdot \Delta)$ uniformly at random. Denote by $X$ the number of clusters in $\mathcal{P}'$ the ball $B(x, r)$ intersects. In other words, $X = \lvert\{C \in \mathcal{P}' \mid x \in C\}\rvert$. For $C \in \mathcal{P}'$, denote by $X_C$ an indicator for the event $x \in C$. Then $\Pr[X_C = 1] = \frac{\text{Vol}(C)}{\text{Vol}(B(\bar{0} : c \cdot \Delta))}$. Thus

$$\mathbb{E}[X] = \sum_{C \in \mathcal{P}'} \mathbb{E}[X_C] = \sum_{C \in \mathcal{P}'} \frac{\text{Vol}(C)}{\text{Vol}(B(\bar{0}, c \cdot \Delta))} \geq \left(1 + \frac{1}{\sigma}\right)^d \cdot \sum_{C \in \mathcal{P}'} \frac{\text{Vol}(C)}{\text{Vol}(B(\bar{0}, c \cdot \Delta))} \geq \left(1 + \frac{1}{\sigma}\right)^d \cdot \left(1 - \frac{1}{c}\right)^d > \left(1 + \frac{1}{2\sigma}\right)^d.$$

By an averaging argument, there is a point $x \in B(\bar{0}, c \cdot \Delta)$ such that $B(x, r)$ intersects more than $(1 + \frac{1}{2\sigma})^d$ clusters from $\mathcal{P}'$. In particular, this ball intersects more than $(1 + \frac{1}{2\sigma})^d$ clusters from $\mathcal{P}$, as required.

\[\Box\]

9 GRAPHS WITH BOUNDED SPD

In the preliminaries (Section 2), and in Footnote 14 we gave a recursive definition for SPD. Below we provide an alternative and more applicable definition.

**Definition 4** (Shortest Path Decomposition (SPD) [2]). Given a weighted graph $G = (V, E, w)$, an SPD of depth $\rho$ is a pair $(X, Q)$, where $X$ is a collection $X_1, \ldots, X_{\rho}$ of partial partitions of $V$,\(^\text{26}\) and $Q$ is a collection of sets of paths $Q_1, \ldots, Q_{\rho}$, where $X_1 = \{V\}$, $X_{\rho} = Q_{\rho}$, and the following properties hold:

1. For every $1 \leq i \leq \rho$ and every subset $X \in X_i$, there exists a unique path $Q_X \in Q_i$ such that $Q_X$ is a shortest path in $G[X]$.

2. For every $2 \leq i \leq \rho$, $X_i$ consists of all connected components of $G[X \setminus Q_X]$ over all $X \in X_{i-1}$.

We say that all the paths in $Q_i$ are deleted at level $i$. In this section we provide two different constructions of sparse partition for graphs with SPD depth $\rho$. First we construct an $(O(\rho), O(\rho^3))$-strong sparse partition scheme. This partition is constructed using the [68] clustering algorithm. Interestingly, unlike all previous executions of [68], the shifts $\delta_{\rho}$ are chosen deterministically. The proof appears in Section 9.1.

**Theorem 13.** Let $G = (V, E, w)$ be a graph with SPD $(X, Q)$ of depth $\rho$. Then $G$ admits a $(O(\rho), O(\rho^3))$-strong sparse partition scheme.

A graph with bounded SPD depth might have an induced subgraph with much larger SPD depth, thus we cannot apply Theorem 1. Nevertheless, graphs with pathwidth $\rho$ have SPD depth $\rho + 1$, and also all their subgraphs have pathwidth $\rho$. Using Observation 1 and Theorem 1 we conclude:

\[\text{That is for every } X \in X_i, X \subseteq V, \text{ and for every different subsets } X, X' \in X_i, X \cap X' = \emptyset.\]
Corollary 8. Given a weighted graph $G = (V, E, w)$ with pathwidth $\rho$, and a set $K$ of terminals, there is an efficient algorithms that returns a solution to the SPR problem with distortion $O(\rho^3)$.

For weak diameter guarantee we were able to construct sparse partition with improved parameters. The partition algorithm is inspired by previous constructions of padded decompositions for pathwidth graphs [4, 56].

Theorem 14. Let $G = (V, E, w)$ be a graph with SPD $\{X, Q\}$ of depth $\rho$. Then $G$ admits a $(8, 5\rho)$-weak sparse partition scheme.

By Theorem 2, a solution to the UST (and UTSP) follows.

Corollary 9. Let $G$ be a graph with SPD$\{X, Q\}$ of depth $\rho$, then there is an efficient algorithm constructing a solution for the UST problem with stretch $O(\rho \cdot \log n)$.

The following lemma will be useful for both our constructions:

Lemma 8. $G = (V, E, w)$ is a weighted graph with a shortest path $P$, $N \subseteq P$ is a set of vertices such that for every $u, z \in N$, $d_G(u, z) > \varepsilon$. Then for every vertex $v \in V$ and radius $r \geq 0$, it holds that $|B_G(v, r) \cap N| \leq \frac{2r}{\varepsilon}$.

Proof. Denote $P = v_0, v_1, \ldots, v_m$. Let $v_i$ be the vertex with minimal index among $B_G(v, r) \cap N$ (if there is no such $v_i$, the lemma holds trivially). We order the vertices in the intersection $B_G(v, r) \cap N = \{v_0, v_q, v_{q+1}, \ldots, v_{q+1}\}$ w.r.t. to the order induced by $P$. By triangle inequality, $d_G(v_q, v_{q+1}) \leq d_G(v_q, v) + d_G(v, v_{q+1}) \leq 2r$. From the other hand, as $P$ is a shortest path $d_G(v_q, v_{q+1}) = \sum_{j=q}^{q+1} d_G(v_j, v_{j+1}) > s \cdot \varepsilon$. We conclude that $s = \frac{2r}{\varepsilon}$. The lemma now follows. □

9.1 Strong Diameter for SPD Graphs: Proof of Theorem 13

Let $\Delta > 0$ be some parameter. Set $\varepsilon = \frac{1}{p}$. For every $i$, set $\alpha_i = (1 + 2\varepsilon)^{p+1-i} \cdot \Delta$ and $\beta_i = \varepsilon \cdot \alpha_{i+1}$. We will create a partition with strong diameter $2\alpha_1 < 2\varepsilon^2 \cdot \Delta$. Afterwards, the parameters could be readjusted accordingly. For a path $Q \in Q_i$, let $N_Q \subseteq P$ be a $\beta_i$-net. For every vertex $v \in N_Q$ for $Q \in Q_i$, set $\delta_v = \alpha_i$. We execute the clustering algorithm of Miller et al. [68] as described in Section 2.1, with the set $\bigcup_{i \in \{1, \rho\}, Q \in Q_i} N_Q$ as centers. As a result we get a partition $\mathcal{P}$.

We first argue that our partition has strong diameter $2\alpha_1$. Indeed, consider a vertex $v$ that belongs to a path $Q \in Q_i$. There is a center $t_Q \in N_Q$ such that $d_G(v, t_Q) \leq \beta_i$ and $\delta_{t_Q} = \alpha_i$. Suppose that $v$ joined a cluster centered in $t_v$. Then $\delta_{t_v} - d_G(v, t_v) = f_v(t_v) = \delta_{t_Q} - d_G(v, t_Q)$. Thus,

$$d_G(v, t_v) \leq \delta_{t_v} - \delta_{t_Q} + d_G(v, t_Q) \leq \delta_{t_v} - \delta_{t_Q} + \beta_i \leq \alpha_1.$$

By Claim 1, for every vertex $v$ in the cluster $C$ of $t_v$, it holds that $d_G(C, v) = d_G(v, t_v)$. It follows that $\mathcal{P}$ has strong diameter $2\alpha_1 < 2\varepsilon^2 \cdot \Delta$.

We say that vertices $u, v$ were separated at level $i$ if they belong to the same cluster of the SPD at level $i$ (in $X$), and either belong to different clusters of the SPD at the $i + 1$ level of the hierarchy, or if one of them is deleted during the $i$'th level (that is belong to a path $Q \in Q_i$). Set $r = \frac{\varepsilon}{2} \cdot \Delta$. Consider some vertex $v$, and let $B = B_G(v, r)$.

Claim 9. Consider a center $t$. Suppose that at level $i$ of the SPD some vertex $u$ on the shortest path from $v$ to $t$ was deleted, while $t$ was not. Then no vertex from $B$ will join the cluster centered at $t$.

Proof. As $t$ was not deleted on or before level $i$, $\delta_t \leq \alpha_{i+1}$. The vertex $u$ is laying on a shortest path $Q \in Q_i$. There is a center $t_Q \in N_Q$ at distance at most $\beta_i$ from $u$, while $\delta_{t_Q} = \alpha_i$. Let $z \in B$, by triangle inequality $d_G(z, t_Q) \leq d_G(z, v) + d_G(v, u) + d_G(u, t_Q) \leq r + d_G(u, v) + \beta_i$. Similarly, $d_G(z, t) \geq d_G(z, v) - d_G(v, u) \geq d_G(u, v) - r$, thus $d_G(z, t) \leq d_G(z, t) + r$. See figure on the right.
for illustration.
As \( \alpha_{i+1} + \beta_i + 2r = (1 + \epsilon) \cdot \alpha_{i+1} + \epsilon \cdot \Delta < (1 + 2\epsilon) \cdot \alpha_{i+1} = \alpha_i \), and thus \( \alpha_{i+1} < \alpha_i - \beta_i - 2r \), it holds that
\[
\delta_i - d_G(z, t) \geq \alpha_i - (r + d_G(v, u) + \beta_i) \\
\geq \alpha_i - 2r - \beta_i - d_G(z, t) > \alpha_{i+1} - d_G(z, t) \geq \delta_i - d_G(z, t) = f_\epsilon(t).
\]

The claim now follows.

\( \square \)

Let \( i_B \) be the first level in which some vertex \( u \in B \) is deleted. \( u \) belongs to a path \( Q_B \in Q_{i_B} \). There is a center \( i_B \) at distance at most \( \beta_{i_B} \) from \( u \). By triangle inequality, for every vertex \( z \in B, d_G(z, t) \leq d_G(z, u) + d_G(u, t) \leq \beta_{i_B} + 2r \). Furthermore
\[
f_\epsilon(t_B) = \delta_{i_B} - d_G(z, t_B) \geq \alpha_{i_B} - \beta_{i_B} - 2r > \alpha_{i+1}. \tag{9.1}
\]
It follows that no vertex of \( B \) will join the cluster of a center \( t \) that belongs to a path deleted at levels \( i_B + 1 \) and higher. We conclude that the ball \( B \) can be covered only by clusters with centers from the exact \( i_B \) paths in the clusters containing \( B \) in each level.

**Claim 10.** Let \( Q \in Q_j \) be the path deleted from the component \( X \) containing \( B \) at level \( j \leq i_B \). Then \( B \) intersects at most \( 2\rho + 4 \) clusters with centers in \( Q \).

**Proof.** Consider some center \( t \in N_{i_B}. \) If \( d_G[C]\{v, t\} > d_G(v, t) \) then some vertex on the shortest path from \( v \) to \( t \) was deleted in an earlier level. By Claim 9, no vertex from \( B \) will join the cluster of \( t \). Denote by \( N'_{i_B} \subseteq N \) the subset of centers for which \( d_G[X]\{v, t\} = d_G(v, t) \). The centers in \( N'_{i_B} \) lie on a shortest path (in \( G[C] \)), and all are at distance greater than \( \beta_j \) apart (w.r.t. \( d_G[X]\)).

Let \( t \in N'_{i_B} \) be a center such that some vertex \( z \in B \) joined the cluster of \( t \). Then \( d_G(v, t) \leq \alpha_j \), as otherwise \( f_\epsilon(t) = \delta_t - d_G(z, t) \leq \alpha_j - d_G(u, t) + r < r \). By equation (9.1), \( z \) will not join the cluster of \( t \), a contradiction. By Lemma 8, there are at most \( \frac{2\alpha_j}{\beta_j} = \frac{2}{\epsilon} + 4 = 2\rho + 4 \) vertices in \( N'_{i_B} \). The claim follows.

\( \square \)

To wrap up, the vertices of \( B \) can join only clusters with centers lying on \( i_B \leq \rho \) paths. In each such path, there are at most 2\( \rho + 4 \) centers, to the cluster of which a vertex from \( B \) might join. As the maximal diameter is \( e^2 \cdot \Delta \), the padding parameter is \( \frac{e^2 \cdot \Delta}{\epsilon} = \frac{2e^2}{\epsilon} = 2 \cdot e^2 \cdot \rho \). We conclude that \( Q \) is a \( (2 \cdot e^2 \cdot \rho, (2\rho + 4)\rho, e^2 \cdot \Delta) \)-strong sparse partition. Thus \( G \) admits a \( O(\rho), O(\rho^2) \)-strong sparse partition scheme as required.

\( \square \)

### 9.2 Weak Diameter for SPD Graphs: Proof of Theorem 14

Let \( \Delta > 0 \) be some parameter. The clustering will be done in two phases and described in Algorithm 1.

**First phase.** This is an iterative process. The set of active vertices will be denoted by \( A \), initially \( A = V \). In level \( i \), for every cluster \( X \in X_i \) recall that \( Q_X \in Q_i \) is a shortest path w.r.t. \( G[X] \). Set \( C_X = \{ v \in A \mid d_G[X]\{v, Q_X\} \leq \frac{\Delta}{4} \} \), that is the set of active vertices in \( X \) at distance at most \( \frac{\Delta}{4} \) from \( Q_X \) w.r.t. \( d_G[X] \). All the vertices in \( C_X \) cease to be active (\( A \leftarrow A \setminus C_X \)). By the end of the algorithm all the vertices became inactive, thus we constructed a partition \{\( C_X \\}_{i \in [k], X \in X_i} \). Note that \( Q_X \) is not necessarily contained in \( C_X \), in particular \( C_X \) might not be connected. Nevertheless, the cluster \( C_X \) is contained in \( B_{G[X]}(Q_X, \frac{\Delta}{4}) \).

**Second phase.** Next, each cluster \( C_X \) is partition into balls of bounded (weak) diameter. Let \( N_X \) be a \( \frac{\Delta}{4} \) net of \( Q_X \) w.r.t. \( G[X] \). Note that \( N_X \) might not be contained in \( C_X \). Order the vertices in \( N_X = \{ t_1, t_2, \ldots \} \) arbitrarily. For \( t_j \in N_X \), set \( C_{t_j} = \{ u \in C_X \mid d_G[X]\{t_j, u\} \leq \frac{\Delta}{4} \} \setminus \cup_{i < j} C_{t_i} \). Our final partition is simply \{\( C_{t_j} \\}_{i \in [\rho], X \in X_i, t_j \in N_X} \). Note that each vertex \( u \in X \) joins some cluster. Indeed, as \( u \in C_X \), there is \( w \in Q_X \) s.t. \( d_G[X]\{u, w\} \leq \frac{\Delta}{4} \). Furthermore,
Theorem 15.\ obtain equal to the parameters of scattering partition for trees. Nevertheless, we are able to construct scattering partitions for Chordal graphs. Interestingly, the parameters we\ distortion. Therefore the scattering partition for trees do not imply scattering partition for Chordal graphs.

Every Chordal Graph admits a \( (\Delta) \)-weak sparse partition scheme. The only question left is regarding scattering partitions. One can try the same approach above of embedding Chordal graphs into trees. Unfortunately, Chordal graph embed only into non-subgraph trees. Specifically, Prisner \[72\] showed that Chordal graphs do not embed into a spanning trees with any constant distortion. Therefore the scattering partition for trees do not imply scattering partition for Chordal graphs. Nevertheless, we are able to construct scattering partitions for Chordal graphs. Interestingly, the parameters we\ equal to the parameters of scattering partition for trees.

Theorem 15. Every Chordal graph \( G = (V, E) \) is \( (2, 3) \)-scatterable.

\( 32 \) Arnold Filtser

At least 20 papers are needed to be cited.
As every induced subgraph of a Chordal graph is also Chordal, using Theorem 1 we conclude.\footnote{Note that Chordal graphs are unweighted, while the minor constituting the solution to the SPR problem will necessarily be weighed.}

**Corollary 11.** Given a Chordal graph $G = (V, E)$ with a set $K$ of terminals, there is an efficient algorithm that returns a solution to the SPR problem with distortion $O(1)$.

**Proof of Theorem 15.** Let $\Delta \in \mathbb{N}$ be some integer parameter. We can assume that $\Delta \geq 3$, as otherwise the trivial partition where each vertex is in a separate cluster fulfill the requirements. Set $r = \frac{\Delta}{2}$. We begin by describing the algorithm creating the partition $P$. Each cluster $C \in P$ will have a center vertex $\pi(C) \in C$. Each vertex $v \in V$ will admit a label $\delta_v \in [0, r]$. We denote by $\pi(v)$ the center of the cluster containing $v$.

Let $T$ be a tree decomposition for the chordal graph $G$. We can assume that $T$ is rooted in a bag $B_1$ containing a single vertex $r_1$, and that every bag $B$ contains exactly one new vertex not belonging to its parent (w.r.t. the order defined by the root bag). We will denote the first bag introducing a vertex $v$ by $B_v$. Note that this defines a bijection between the vertices to the bags. Moreover, this induces a partial order on the vertices $V$, where $v \leq u$ if $B_v$ is a decendent of $B_u$.

The partition $P$ of $G$ is defined inductively w.r.t. this partial order. Initially we create a cluster $C_{rt}$ and set $\pi(C_{rt}) = \pi(r_t) = r$, $\delta_{rt} = 0$. Consider a vertex $v \in V$, which is introduced at bag $B_v$. By induction all the other vertices in $B_v$ are already labeled and clustered. Let $u$ be the vertex with minimal label among $B \setminus \{v\}$ (breaking ties arbitrarily).

- If $\delta_u < r$, set $\pi(v) = \pi(u)$ and $\delta_v = d_G(u, \pi(v))$.
- Else ($\delta_u = 0$), create new cluster $C_v$ centered at $v$. Set $\pi(C_v) = \pi(v) = v$ and $\delta_v = 0$.

This finishes the description of the algorithm. As each vertex joins a cluster where it has a neighbor (or starts a new cluster), the connectivity follows. For every vertex $v$ which is not a cluster center, $v$ joined a cluster centered in $\pi(u)$ for some neighbor $u$ of $v$ with label $\delta_u < r$. It holds that $d_G(v, \pi(v)) \leq d_G(u, \pi(u)) + 1 \leq r$. In other words, the distance from every cluster center to all other vertices in the cluster is bounded by $r$. It follows that the created partition has weak diameter $2r \leq \Delta$.

Next, we claim by induction on the tree decomposition, that in every bag $B$, all the vertices with labels strictly smaller than $r$, \{\{v \in B \mid \delta_v < r\}\} belong to the same cluster. Indeed consider a bag $B_v$ introducing $v$, and let $u$ be a vertex minimizing $\delta_u$ among $B' = B \setminus \{v\}$. If $\delta_u \geq r$, there is nothing to prove. Otherwise, $\pi(v) = \pi(u)$ and by the induction hypothesis all the vertices with labels strictly smaller than $r$ belong to the cluster centered at $\pi(u)$.

Finally we are ready to prove that the partition is indeed scattering. Consider a path $I = v_0, v_1, \ldots, v_q$ where $q \leq r$. We will abuse notation and denote $B_i$ by $B_i$ for $i \in [0, q]$. Suppose that $v_i$ is the maximal vertex of $I$ w.r.t. the partial order induced by $T$. In particular $v_i$ is the first vertex from $I$ to be clustered in our algorithm. Let $j > i$ be the minimal index such that $\pi(v_j) \neq \pi(v_i)$ (if exist). As $I$ is a shortest path, for every $a \in [0, q-2]$, there is no edges between $v_a, v_{a+2}$. Therefore, the path in $T$ between $B_i$ to $B_j$ must go through $B_i, B_{i+1}, \ldots, B_{j-1}$ in that order. In particular, $B_j$ necessarily contains $v_{j-1}$. As $\pi(v_{j-1}) \neq \pi(v_i)$ it follows that $\delta_{v_{j-1}} = r$. It must be that $B_{j-1}$ contains a vertex $u$ such that $\pi(u) = \pi(v_{j-1}) = \pi(v_{j-1})$ and $\delta_u = r - 1$.

We argue that $\delta_{v_j} \leq 2$. If $v_j$ is a cluster center, then $\delta_{v_j} = 0$. Otherwise, there is a vertex $u \in B_j$ such that $v_j$ joins the cluster centered in $\pi(u)$. As $\pi(u) \neq \pi(u)$, and $\delta_{\pi(u)} = 0$, necessarily $\pi(u) \notin B_{j-1}$ (as $u \in B_{j-1}$ and $\delta_u < r$). In particular, the bag $B_{\pi(u)}$ introducing $\pi(u)$ lies in $T$ on the path between $B_{j-1}$ and $B_j$. As $v_j \in B_{j-1} \cap B_j$, it must hold that $v_j \in B_{\pi(u)}$. As $G$ is a Chordal graph, it follows that $\{v_j, \pi(u)\} \in E$. We conclude that $\delta_{v_j} = d_G(v_j, \pi(u)) \leq d_G(v_j, v_{j-1}) + d_G(v_{j-1}, \pi(u)) = 2$. See the figure below for illustration.
Observe, that for every pair of neighboring vertices \( \{v, u\} \) where \( v \) was introduced before \( u \) it holds that \( \delta_u \leq \delta_v + 1 \). By induction, for every \( j' > j \), \( \delta(v_{j'}) \leq \delta(v_j) + |j' - j| \leq 2 + |j' - j| \). In particular, it follows that all the vertices \( v_j, v_{j+1}, \ldots, v_{\min\{q, j-2+r\}} \) belong to the cluster centered at \( \pi(v_j) \) (as each cluster \( B_j \) contains some vertex belonging to \( C_{\pi(v_j)} \) with label strictly less than \( r \)).

Finally, using case analysis we argue that all the vertices in \( I \) belong to at most 3 clusters. If \( 1 < i < q \), then all the vertices \( v_i, \ldots, v_q \) belong to at most two clusters \( (C_{\pi(v_i)}, C_{\pi(v_j)}) \). By a symmetric argument, all the vertices \( v_1, \ldots, v_i \) belong to at most two clusters. Thus the theorem follows. For the case \( i = 1 \), we can argue that all the vertices \( v_1, \ldots, v_{q-1} \) belong to at most two clusters, and hence again the theorem follows. The case \( i = q \) is symmetric. 

\[ \square \]

11 CACTUS GRAPHS

In this section we construct scattering partitions for Cactus graphs. Note that weak sparse partitions for this family already follow by Corollary 3, while the lower bound Theorem 9 on strong sparse partitions holds for Cactus graphs, as they contain the family of trees.

**Theorem 16.** The family of Cactus graphs is \( (4, 5) \)-scatterable.

As every subgraph of a Cactus graph is also a Cactus graph, using Theorem 1 we conclude.

**Corollary 12.** Given a Cactus graph \( G = (V, E, w) \) with a set \( K \) of terminals, there is an efficient algorithm that returns a solution to the SPR problem with distortion \( O(1) \).

As a preliminary to the proof of Theorem 16, we begin with a characterization of Cactus graphs:

**Claim 11.** Each weighted Cactus graph \( G = (V, E, w) \) can be composed as sequence of Cactus graphs \( G_0, G_1, \ldots, G_3 = G \) (here \( G_i = (V_i, E_i) \)) such that:

1. \( G_0 \) is a single vertex.
2. The graph \( G_1 \) is obtained by attaching a path \( P_i = (v_0, v_1, \ldots, v_q) \) (disjoint from \( V_{i-1} \)) to a single vertex \( u_i \) of \( G_{i-1} \) to either one or both endpoints of \( P_i \). That is \( V_i = V_{i-1} \cup \{v_0, v_1, \ldots, v_q\} \) and \( E_i \) consist of the edges in \( E_{i-1} \) the edges along the path \( P_i \), and at least one of the edges \( \{u_i, v_1\}, \{u_i, v_q\} \).

Further, it holds that the shortest path metrics of \( G_i \) and \( G \) agree on \( V_i \). In other words \( \forall u, v \in V_i, d_G(u, v) = d_G(u, v) \).

**Proof.** The proof is by induction on the number of vertices \( n \). The base case is when \( G \) is either a single vertex, a path graph, or a cycle graph. All these cases are trivial. Suppose that the claim holds for every Cactus graph with strictly less than \( n \) vertices, and consider a cactus graph \( G = (V, E) \) with \( n \) vertices, which is not covered by the base of the induction. We define an auxiliary graph as follows: Let \( C \) be the collection of cycles in \( G \). Let \( E_1 \subseteq E \) be subset of \( G \) edges that do not belong to a cycle. Our auxiliary graph \( \mathcal{T} \) will have \( \mathcal{V} = C \cup E_1 \) as its vertex set. For every \( X, Y \in \mathcal{V} \) that share a vertex, we will add an edge to \( \mathcal{G} \). Clearly, as \( G \) is a Cactus graph, \( \mathcal{T} \) is a tree. Further, every \( X, Y \in \mathcal{V} \) share at most a single vertex (as every edge belongs to at most a single cycle).

Let \( X \in \mathcal{V} \) be a leaf in the auxiliary tree \( \mathcal{T} \) (note that \( \mathcal{T} \) is not a singleton, as we are not in the base case). Let \( x \in X \) be the unique vertex that belongs to an additional node in \( \mathcal{T} \) other than \( X \). Let \( G' \) be the graph obtained
by deleting all the vertices in X other than x. Note that G′ is a cactus graph with strictly less than n vertices, and hence by the induction hypothesis it has a decomposition G0, G1, . . . , Gs = G′ as in the claim. There are two cases. Suppose first that X is an edge \{x, y\}. Then we set P_{x+1} = (y) to be a singleton vertex and attach it to Gs via x. G_{s+1} = G as required. The second case is that X is a cycle C = (v0, v1, . . . , vk) where v0 = v_k = x. Here we set P_{x+1} = (v1, . . . , v_k) and attach it to Gs using the two edges \{x, v1\}, \{v_{k-1}, x\} to obtain G_{s+1}. Clearly G_{s+1} = G, and for every y, z \in Gs, d_G(y, z) = d_{G_{s+1}}(y, z). The claim now follows.

We are now ready to proceed to the proof of Theorem 16.

**Proof of Theorem 16.** Let \( \Delta > 0 \) be some parameter. Set \( r = \frac{\Delta}{4} \). Each cluster we create \( C_T \) will have a center T. The center T might be either a singleton, or a set of size 2. The clustering is defined inductively using Claim 11. First, \( G_0 = \{v\} \), create a cluster \( C_0 \) with v as a center. In each step we will either extend previously created clusters or create new clusters. Consider the \( i \)th step in the composition procedure. Suppose that all the vertices in \( G_{i-1} \) are already clustered. A new path \( P_i = \{v_0, v_1, \ldots, v_m\} \) is attached to \( G_{i-1} \) at vertex \( u \in G_{i-1} \) which belong to a cluster \( C_T \). We consider two cases:

- \( P_i \) is attached to u via a single edge \( \{v_0, u\} \). The prefix of the vertices \( v_0, \ldots, v_{j-1} \) which are at distance at most \( r \) (w.r.t. \( d_G = d_G') \) from T join the cluster \( C_T \). Let \( v_j \) be the first vertex at distance greater then \( r \) from T (if it exist). \( v_j \) is defined as a center of a new cluster \( C_D \). The prefix of the remaining vertices \( v_j, \ldots, v_{q-1} \) at distance at most \( r \) from \( v_j \) join \( C_D \). The vertex \( v_q \) (if it exist) is defined as the center of a new cluster \( C_{D_0} \). We process in this manner until all the vertices of \( P_i \) are clustered.

- \( P_i \) is attached to \( \{u\} \) via two edges \( \{v_0, u\}, \{v_m, u\} \). The clustering of \( P_i \) has three phases:
  - The prefix \( v_0, v_1, \ldots, v_{j-1} \) (resp. the suffix \( v_{j+1}, v_{j+2}, \ldots, v_m \)) of the vertices which are at distance at most \( r \) from T join the cluster \( C_T \). If not all the vertices of \( P_i \) are clustered, denote \( t_a = v_j \) and \( t_b = v_{j'} \) (it is possible that \( t_a = t_b \)) and proceed to the next phase.
  - If \( d_G(v_j, t_a) \leq 2r \) go to the next phase. Otherwise, create two new clusters \( C_{t_a}, C_{t_b} \) centered at \( t_a, t_b \) respectively. The prefix (resp. suffix) of the remaining vertices \( v_a, \ldots, v_{p-1} \) (resp. \( v_{q+1}, \ldots, v_{q'} \)) at distance at most \( r \) from \( t_a \) (resp. \( t_b \)) joins the cluster \( C_{t_a} \) (resp. \( C_{t_b} \)). If all the vertices were clustered we are done. Otherwise, denote \( t_a = v_p \) and \( t_b = v_q \). Repeat phase two.
  - Set \( \hat{T} = \{t_a, t_b\} \). Create a new cluster \( \hat{C}_T \) with \( \hat{T} \) as a center. All the yet unclustered vertices in \( P_i \) join \( C_T \).

**Claim 12.** The partition has strong diameter 4r = \( \Delta \).

**Proof.** Consider a cluster \( C_T \). It is straightforward by induction that for every vertex \( v \in C_T \), \( d_G(C_T)(v, T) = d_G(v, T) \leq r \). Thus if \( T \) is a singleton, then \( C_T \) has strong diameter 2r. Otherwise (\( |T| = 2 \)), consider a pair of vertices \( v, u \in C_T \). There are centers \( t_a, t_u \in T \) such that \( d_G(C_T)(t_a, v) \leq r, d_G(C_T)(u, t_u) \leq r \), and \( d_G(C_T)(t_a, t_u) \leq 2r \). The claim follows by triangle inequality.

For two indices \( i, i' \) denote \( P_{i'} = \cup_{q=i}^{i+1} P_q \) and \( P_{>i} = \cup_{q>i} P_q \). For a cluster \( C_T \) created during the \( i \)th phase, denote by \( \hat{C}_T = C_T \cap P_i \) the set of vertices belonging to \( C_T \) by the end of the \( i \)th phase. Using a straightforward induction, we have the following observation.

**Observation 9.** Consider a center T of the cluster \( C_T \) created during the \( i \)th phase. Then \( B_{G(C_T)}(P_{>i})(T, r) \subseteq C_T \). In words, every vertex \( v \in P_{>i} \) for which there is a path towards T of length at most r containing vertices from \( P_{>i} \) and \( \hat{C}_T \) only, will join \( C_T \).

Consider a shortest path \( Q = \{z_0, z_1, \ldots, z_{n_0}\} \) of length at most \( r = \frac{\Delta}{4} \). Suppose that in the composition procedure, \( P_i \) is the path with minimal index which intersects \( Q \). The intersection between \( P_i \) and \( P_i \) must be an interval.
Q \cap P_t = \{z_{j+1}, \ldots, z_{j-1}\}. This is, as the shortest path between vertices in G is do not contain edges from G[P_{s+t}].

As \( w(Q) \leq r \), by case analysis, \( Q \cap P_t \) can be divided to at most 3 consecutive clusters.

Consider the suffix \( \{z_j, z_{j+1}, \ldots, z_\alpha\} \). Denote by \( C_{z_j+1} \), the cluster the vertex \( z_{j-1} \) joined to. We argue that other than to \( C_{z_{j-1}} \), the suffix vertices can join to at most one additional cluster. By the composition procedure, the vertices \( \{z_j, z_{j+1}, \ldots, z_\alpha\} \) must be added to \( G \) in paths \( P_{i1}, P_{i2}, \ldots, P_{i_q} \) where \( i_1 < i_2 < \cdots < i_q \) and \((Q \cap P_{i_t}) \circ (Q \cap P_{i_{t+1}}) \circ \cdots \circ (Q \cap P_{i_q}) = \{z_j, z_{j+1}, \ldots, z_\alpha\} \). Let \( i_p \) be the minimal index such that not all the vertices of \( P_{ip} = \{x_0, x_1, \ldots, x_p\} \) join \( C_{z_{j-1}} \). If there is no such index, we are done. Note that only \( x_0 \) and \( x_p \) might have edges to formerly introduced vertices. Hence \( x_0 \) or \( x_p \) belong to \( Q \cap P_{ip} \). W.l.o.g. \( Q \cap P_{ip} = \{x_0, x_1, \ldots, x_p\} \). Let \( q \) be the minimal index such that \( x_q \) not joining the cluster of \( z_{j+1} \). Following the construction algorithm, \( x_q \) must belong to the center of a cluster \( C_T \). As \( Q \) is of length at most \( r \), all the vertices \( \{x_q, \ldots, x_r\} \subseteq Q \) join \( C_T \). By Observation 9, all the vertices in \( Q \cap P_{ip+1}, Q \cap P_{ip+2}, \ldots, Q \cap P_{i_q} \) also join \( C_T \).

By symmetric arguments, the prefix vertices \( z_1, z_2, \ldots, z_j \) join to at most a single cluster other than the cluster of \( z_{j+1} \). The theorem follows.

12 DISCUSSION AND OPEN PROBLEMS

In this paper we defined scattering partitions, and showed how to apply them in order to construct solutions to the SPR problem. We proved an equivalence between sparse partitions and sparse covers. Finally, we constructed many sparse and scattering partitions for different graph families (and lower bounds), implying new results for the SPR, UST, and UTSP problems. An additional contribution of this paper is a considerable list of (all but question (5)) new intriguing open questions and conjectures.

(1) Planar graphs: The SPR problem is most fascinating and relevant for graph families which are closed under taking a minor. Note that already for planar graphs (or even treewidth 2 graphs), the best upper bound for the SPR problem is \( O(\log k) \) (same as general graphs), while the only lower bound is 8. The most important open question coming out of this paper is the following conjecture:

Conjecture 1. Every graph family excluding a fixed minor is \( (O(1), O(1)) \)-scatterable.

Note that proving this conjecture for a family \( F \) will imply a solution to the SPR problem with constant distortion.

(2) Scattering Partitions for General Graphs: While we provide almost tight upper and lower bounds for sparse partitions, for scattering partitions, the story is different.

Conjecture 2. Consider an \( n \) vertex weighted graph \( G \) such that between every pair of vertices there is a unique shortet path. Then \( G \) is \( (1, O(\log n)) \)-scatterable. Furthermore, this is tight.

Theorem 6 provides some evidence that Conjecture 2 cannot be pushed further. However, any nontrivial lower bound will be interesting. Furthermore, every lower bound larger than 8 for the general SPR problem will be intriguing.

(3) Doubling graphs: While we constructed strong sparse partition for doubling graphs (which imply scattering), it has no implication for the SPR problem. This is due to the fact that Theorem 1 required scattering partition for every induced subgraph. As induced subgraphs of a doubling graph might have unbounded doubling dimension, the proof fails to follow through. We leave the required readjustments to future work.

(4) Sparse Covers: We classify various graph families according to the type of partitions/cover they admit, as exhibited in Figure 3. We currently lack any example of a graph family that admits weak sparse covers but does not admit strong sparse covers. It will be interesting to find such an example, or even more so to prove that every graph that admits weak sparse cover, also has strong sparse cover with (somewhat) similar parameters.
Treewidth graphs: The parameters in some of our partitions perhaps might be improved. The most promising example in this context is treewidth $d$ graphs. As such graphs exclude $K_{d+2}$ as a minor, by Theorem 3 they admit $(O(\rho^2), 2^{\rho^2})$-weak sparse partition scheme. However, they might admit sparse partitions with parameter polynomial, or even logarithmic in $\rho$. Recently Hershkowitz and Li [52] constructed an $(O(1), O(1))$-scattering partitions for series parallel graphs (alternatively treewidth 2 graphs), answering an open question posed in the conference version of this paper [38]. However, already for treewidth 3 the question is wide open.

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