HYPERGEOMETRIC EXPRESSIONS OF $L$-VALUES FOR A
BORWEINS THETA PRODUCT OF WEIGHT 3

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Abstract. In this paper, we consider a modular form of weight 3, which is
a product of the Borweins theta series, and express its $L$-values at $s = 1, 2$
and 3 in terms of special values of Kampé de Fériet hypergeometric functions,
which are two-variable generalization of generalized hypergeometric functions.

1. Introduction and Main Results

For a modular form $f$ of weight $k$ with $q$-expansion $f(q) = \sum_{n=0}^{\infty} a_n q^n$ ($q = e^{2\pi i \tau}$,
$\text{Im}(\tau) > 0$), its $L$-function $L(f, s)$ is defined by

$$L(f, s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}, \quad \text{Re}(s) > k + 1.$$ 

The function $L(f, s)$ has meromorphic continuation to the whole complex plane
with a possible simple pole at $s = k$ when the Fricke involution image $f^\sharp$ of $f$
is also a modular form. Furthermore, if $f^\sharp(0) = 0$, then $L(f, s)$ is entire (cf. [17]).
In this paper, we consider the case when $f(q)$ is a product of the Borweins theta
series [4, 5]

$$a(q) := \sum_{m,n \in \mathbb{Z}} q^{m^2+mn+n^2},$$
$$b(q) := \sum_{m,n \in \mathbb{Z}} \omega^{m-n} q^{m^2+mn+n^2},$$
$$c(q) := \sum_{m,n \in \mathbb{Z}} q^{(m+\frac{1}{3})^2+(m+\frac{1}{3})(n+\frac{1}{3})+(n+\frac{1}{3})^2},$$

which are modular forms of weight 1. Here $\omega$ denotes a primitive cube root of unity.
These are cubic analogues of the Jacobi theta series and satisfy the cubic identity
[4, (2.3)]

$$a^3(q) = b^3(q) + c^3(q).$$

In 2010s, it was proved that some $L$-values for certain modular forms can be
expressed in terms of special values of generalized hypergeometric functions

$$A+1 F_A \left[ \begin{array}{c} a_1, a_2, \ldots, a_{A+1} \\ a'_1, \ldots, a'_A \end{array} \middle| x \right] := \sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n \cdots (a_{A+1})_n}{(a'_1)_n \cdots (a'_A)_n} \frac{x^n}{(1)_n}.$$
and, their two-variable generalization, Kampé de Fériet hypergeometric functions [1, 19]

\[
\begin{align*}
F_{A:B:C}^{A':B':C'}(a_1, \ldots, a_A, b_1, \ldots, b_B, c_1, \ldots, c_C; a'_1, \ldots, a'_{A'}, b'_1, \ldots, b'_{B'}, c'_1, \ldots, c'_{C'}) & x, y
\end{align*}
\]

where \(a_i, a'_i, b_i, b'_i, c_i, c'_i\) are complex parameters with \(a'_i, b'_i, c'_i \not\in \mathbb{Z}_{\leq 0}\), and \((a)_n := \Gamma(a + n)/\Gamma(a)\) denotes the Pochhammer symbol. We list some known cases.

1. For some theta products \(f(q)\) of weight 2, Otsubo [12] expressed \(L(f, 2)\) in terms of \(3F_2(1)\) via regulators.
2. Rogers [13], Rogers-Zudilin [15], Zudilin [20] and the author [8] expressed \(L(f, 2)\) for some theta products \(f(q)\) of weight 2 in terms of \(3F_2(1)\) by an analytic method. Furthermore, for the Jacobi theta product which corresponds to the elliptic curve of conductor 32, Zudilin [20] expressed \(L(f, 3)\) in terms of \(4F_3(1)\).
3. For some quotients \(f(q)\) of the Dedekind eta function \(\eta(q) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n)\) of weight 3 (resp. 4, 5), Rogers-Wan-Zucker [14] expressed \(L(f, 2)\) (resp. \(L(f, 3), L(f, 4)\)) in terms of special values of the gamma function or generalized hypergeometric functions by an analytic method. The author [9] expressed \(L(f, 1)\) (hence the values \(L(f^3, 2)\) by the functional equation) for some theta products \(f(q)\) of weight 3 in terms of \(3F_2(1)\) by the Rogers-Zudilin method.
4. Samart [16] expressed \(L(f, 3)\) for some eta quotients \(f(q)\) of weight 3 in terms of \(5F_3(1)\) via Mahler measures.
5. The author [10] expressed \(L(f, 3)\) and \(L(f, 4)\) for some Jacobi theta products \(f(q)\) of weight 3 in terms of \(F_{1:1:B:1:C}^{1:1:B+1:C+1}(1, 1)\) by the Rogers-Zudilin method.
6. For certain binary theta series \(f\) of odd weight \(k \geq 3\), Osburn and Straub [11] expressed \(L(f, k - 1)\) in terms of special values of the gamma function by an analytic method.

In this paper, we consider the Borweins theta product of weight 3

\[
f(q) := \frac{1}{3} b^2(q) c(q^3),
\]

which satisfies the condition \(f^2(0) = 0\) (so \(L(f, s)\) is entire), and express its \(L\)-values \(L(f, 1), L(f, 2)\) and \(L(f, 3)\) in terms of special values of Kampé de Fériet hypergeometric functions.

The main result is the following.
Theorem 1. We have the following hypergeometric expressions:

\[
\begin{align*}
(1.1) \quad L(f, 1) &= \frac{1}{27} F_{1;1;1}^{1;2;2} \left[ \begin{array}{c} 1, \frac{4}{3}, \frac{1}{3} ; 2, 1 \end{array} \right] 1, 1 \right), \\
(1.2) \quad L(f, 2) &= \frac{4\pi}{81\sqrt{3}} \left( F_{1;1;1}^{1;2;2} \left[ \begin{array}{c} 1, \frac{4}{3}, \frac{1}{3} ; 2, 1 \end{array} \right] 1, 1 \right) - \frac{1}{4} F_{1;1;1}^{1;2;2} \left[ \begin{array}{c} 2, \frac{2}{3} ; 3, \frac{3}{3} ; 1 \end{array} \right] 1, 1 \right), \\
L(f, 3) &= \frac{2\pi^2}{27} \left( F_{1;1;1}^{1;2;2} \left[ \begin{array}{c} 1, \frac{4}{3}, \frac{1}{3} ; 2, 1 \end{array} \right] 1, 1 \right) - \frac{1}{4} F_{1;1;1}^{1;2;2} \left[ \begin{array}{c} 2, \frac{2}{3} ; 3, \frac{3}{3} ; 1 \end{array} \right] 1, 1 \right) \\
&\quad + \frac{1}{27} F_{1;1;1}^{1;3;2} \left[ \begin{array}{c} 1, \frac{4}{3}, \frac{1}{3} ; 2, 2, 1 \end{array} \right] 1, 1 \right) - \frac{2}{27} F_{1;1;1}^{1;3;2} \left[ \begin{array}{c} 1, \frac{4}{3}, \frac{1}{3} ; 2, 2, 1 \end{array} \right] 1, 1 \right). 
\end{align*}
\]

Note that the double series \( F_{A,B+1:C+1}^{A:B:C} (x, y) \) converges absolutely on \(|x| \leq 1\) and \(|y| \leq 1\) when the parameters satisfy the three conditions [7]

\[
\Re \left( \sum_{i=1}^{A} a'_i + \sum_{i=1}^{B} b'_i - \sum_{i=1}^{A} a_i - \sum_{i=1}^{B+1} b_i \right) > 0, \\
\Re \left( \sum_{i=1}^{A} a'_i + \sum_{i=1}^{C} c'_i - \sum_{i=1}^{A} a_i - \sum_{i=1}^{C+1} c_i \right) > 0, \\
\Re \left( \sum_{i=1}^{A} a'_i + \sum_{i=1}^{B} b'_i + \sum_{i=1}^{C} c'_i - \sum_{i=1}^{A} a_i - \sum_{i=1}^{B+1} b_i - \sum_{i=1}^{C+1} c_i \right) > 0. 
\]

To prove the main result, we use the Rogers-Zudilin method. Its strategy is as follows. We start with the Mellin transformation of \( f(q) \): For \( n \in \mathbb{Z}_{\geq 1} \),

\[
L(f, n) = \frac{(-1)^{n-1}}{3(n-1)!} \int_0^1 b^3(q) c(q^3) (\log q)^{n-1} \frac{dq}{q^1}. 
\]

Set \( \alpha = c^3(q)/a^3(q) \). Note that we have \( 1 - \alpha = b^3(q)/a^3(q) \) by the cubic identity. The key formulas to give a hypergeometric expression of \( L(f, n) \) are the following:

\[
a(q) = 2 F_1 \left[ \begin{array}{c} 1, \frac{2}{3} \end{array} \right] \left[ \begin{array}{c} \alpha \end{array} \right], \quad a^2(q) \frac{dq}{q} = \frac{d\alpha}{\alpha(1-\alpha)}. 
\]

The former is [3, p.97, (2.26)], and the latter follows from the former and [2, p.87, Entry 30]. By these transformation formulas and some computations, we can reduce (1.4) to an integral of the form

\[
\int_0^1 P(\alpha) A_{B+1} F_{A} \left[ \begin{array}{c} a_1, a_2, \ldots, a_{A+1} \end{array} \right] \left[ \begin{array}{c} a'_1, \ldots, a'_A \end{array} \right] 2 F_1 \left[ \begin{array}{c} \frac{1}{3}, \frac{2}{3} \end{array} \right] \left[ \begin{array}{c} \alpha \end{array} \right] \frac{d\alpha}{\alpha(1-\alpha)}. 
\]

Here \( P(\alpha) \) denotes a polynomial in \( \alpha^k (1-\alpha)^l \) for various \( k \) and \( l \). Then the formulas are obtained by the integral expression

\[
\Gamma(\alpha) \Gamma(a') - \Gamma(a') \int_0^1 b_{B+1} \left[ \begin{array}{c} b_1, \ldots, b_{B+1}, c_1, \ldots, c_{C+1} \end{array} \right] \left[ \begin{array}{c} a, \ldots, a \end{array} \right] x, y 
\]

\[
= \int_0^1 a^1 (1-t)^{a'-a} \int_{B+C} \left[ \begin{array}{c} b_1, \ldots, b_{B+1} \end{array} \right] \left[ \begin{array}{c} x, y \end{array} \right] \frac{dt}{(1-t)} 
\]

which easily follows from the series expansion of \( m+1 F_m(x) \) and termwise integration.
2. Proof

First, we show (1.1). We have [5, (2.1)]

\[ c(q^3) = \frac{a(q) - b(q)}{3}, \]

hence

\[
L(f, 1) = \frac{1}{3} \int_0^1 b^2(q)c(q^3) \frac{dq}{q} = \frac{1}{9} \int_0^1 b^2(q)(a(q) - b(q)) \frac{dq}{q}.
\]

By the transformation formulas (1.5), the integral above becomes

\[
\frac{1}{9} \int_0^1 (1 - \alpha)^{\frac{2}{3}} (1 - (1 - \alpha)^{\frac{2}{3}}) \frac{d\alpha}{\alpha} - 1 \left( \frac{1 - \alpha}{\alpha} \right) \frac{d\alpha}{\alpha}.
\]

If we use

\[
(1 - x)^{-a} - 1 = ax_2F_1 \left[ \frac{1, a + 1}{2} \right] x,
\]

which follows from (1 - x)^{-a} = 1F_0 \left[ \frac{a}{x} \right], then we obtain, by (1.6),

\[
\frac{1}{9} \int_0^1 (1 - \alpha)^{\frac{2}{3}} (1 - (1 - \alpha)^{\frac{2}{3}}) \frac{d\alpha}{\alpha} = \frac{1}{27} \int_0^1 \alpha(1 - \alpha) \frac{d\alpha}{\alpha}.
\]

Next, we show (1.2). By applying (1.4) to \( n = 2 \) and changing the variable \( q = e^{-2\pi u} \), we have

\[
L(f, 2) = \frac{4\pi^2}{3} \int_0^{\infty} b^2(e^{-2\pi u})c(e^{-6\pi u}) u du.
\]

If we use the involution formula

\[
b(e^{-2\pi u}) = \frac{1}{\sqrt{3u}} c(e^{-\frac{2\pi}{3u}}),
\]

which follows from \( b(q) = \eta^3(q)/\eta(q^3) \), \( c(q) = 3\eta^3(q^3)/\eta(q) \) and an involution formula of \( \eta(q) \), then we obtain

\[
L(f, 2) = \frac{4\pi^2}{27\sqrt{3}} \int_0^{\infty} c^2(e^{-\frac{2\pi}{3u}})b(e^{-\frac{2\pi}{9u}}) \frac{d\alpha}{u^2}.
\]

By the variable transformations \( u \rightarrow 1/u, q = e^{-2\pi u} \) and \( q \rightarrow q^3 \), the integral above becomes

\[
\frac{2\pi}{3\sqrt{3}} \int_0^1 c^2(q^3)b(q) \frac{dq}{q}.
\]
Applying (2.1) and the transformation formulas (1.5), we obtain

\[
L(f, 2) = \frac{2\pi}{27\sqrt{3}} \int_0^1 b(q)(a(q) - b(q))^2 dq \\
= \frac{2\pi}{27\sqrt{3}} \int_0^1 (1 - \alpha)^{\frac{1}{3}} (1 - (1 - \alpha)^{\frac{1}{3}})^2 \frac{1}{\alpha(1 - \alpha)} d\alpha \\
= \frac{2\pi}{27\sqrt{3}} \int_0^1 \left( (1 - \alpha)^{-\frac{2}{3}} - 2(1 - \alpha)^{-\frac{1}{3}} + 1 \right) \frac{1}{\alpha(1 - \alpha)} d\alpha.
\]

We have

\[
(1 - \alpha)^{-\frac{2}{3}} - 2(1 - \alpha)^{-\frac{1}{3}} + 1 = \frac{2}{3} \alpha \left( 2F_1 \left[ \frac{5}{3}, 1 \left| 1 \right| \alpha \right] - 2F_1 \left[ \frac{4}{3}, 1 \left| 1 \right| \alpha \right] \right),
\]

by (2.2), hence the formula follows from (1.6).

Finally, we prove (1.3). If we apply (1.4) to \( n = 3 \) and change the variable \( q = e^{-2\pi u} \), we have

\[
L(f, 3) = \frac{4\pi^3}{3} \int_0^\infty b^3(e^{-2\pi u})c(e^{-6\pi u})u^2 du \\
= \frac{4\pi^3}{3\sqrt{3}} \int_0^\infty b(e^{-2\pi u})c(e^{-6\pi u}) \cdot c(e^{-\frac{2\pi u}{3}}) u^2 du.
\]

Here we used the involution formula (2.3) for the last equality. We know the following Lambert series expansions [6, Theorem 3.19, (3.36)] and [15, (23)]:

\[
c(q) = 3 \sum_{r,s=1}^{\infty} \chi_{-3}(r) \left( q^{\frac{r}{3}} - q^{r*} \right),
\]

(2.4)

\[
b(q)c(q^3) = 3 \sum_{n,k=1}^{\infty} \chi_{-3}(nk)kq^{nk},
\]

where \( \chi_{-3} \) denotes the primitive Dirichlet character of conductor 3. By these series expansions and the variable transformation \( u \mapsto su/k \), the integral above becomes

\[
4\sqrt{3}\pi^3 \int_0^\infty \left( \sum_{n,s=1}^{\infty} \chi_{-3}(n)s^2 e^{-2\pi uns} \right) \left( \sum_{k,r=1}^{\infty} \chi_{-3}(kr) \frac{1}{k} \left( e^{-\frac{2\pi kr}{9u}} - e^{-\frac{2\pi kr}{3u}} \right) \right) u du.
\]

The first series is the Borweins theta product [6, Theorem 3.35]:

\[
c^3(q) = 27 \sum_{n,s=1}^{\infty} \chi_{-3}(n)s^2 q^{ns},
\]

which implies

\[
L(f, 3) = \frac{4\pi^3}{9\sqrt{3}} \int_0^\infty c^3(e^{-2\pi u}) \sum_{k,r=1}^{\infty} \chi_{-3}(kr) \frac{1}{k} \left( e^{-\frac{2\pi kr}{9u}} - e^{-\frac{2\pi kr}{3u}} \right) u du.
\]

Using (2.3) and changing the variables \( u \mapsto 1/u, q = e^{-2\pi u} \) and \( q \mapsto q^3 \), we obtain

\[
L(f, 3) = \frac{2\pi^2}{27} \int_0^1 b^3(q) \sum_{k,r=1}^{\infty} \chi_{-3}(kr) \frac{1}{k} \left( q^{\frac{r}{3}} - q^{kr} \right) dq.
\]
By Lemma 2 (below) and the transformation formulas (1.5), the integral above becomes

\[
\frac{2\pi^2}{27} \int_0^1 (1 - \alpha) \left( \frac{1}{3} \alpha \frac{1}{2} F_1 \left[ \frac{1}{3}, 1 \bigg| \frac{1}{3} \alpha \right] - \frac{1}{6} \alpha \frac{2}{3} F_1 \left[ \frac{2}{3}, 1 \bigg| \frac{2}{3} \alpha \right] \right. \\
+ \left. \frac{\alpha}{27} \frac{1}{2} F_2 \left[ 1, 1, \frac{4}{3} \bigg| \frac{2}{3} \alpha \right] - 2\alpha \frac{1}{27} \frac{1}{2} F_2 \left[ 1, 1, \frac{5}{3} \bigg| \frac{2}{3} \alpha \right] \right) \frac{d\alpha}{\alpha(1 - \alpha)}.
\]

Then the formula follows from (1.6). \qed

**Lemma 2.**

\[
\sum_{k,r=1}^{\infty} \chi_{-3}(kr) \left( \frac{1}{3} q^{kr} - q^{kr} \right) = \frac{1}{3} \alpha \frac{1}{2} F_1 \left[ \frac{1}{3}, 1 \bigg| \frac{1}{3} \alpha \right] - \frac{1}{6} \alpha \frac{2}{3} F_1 \left[ \frac{2}{3}, 1 \bigg| \frac{2}{3} \alpha \right] \\
+ \frac{\alpha}{27} \frac{1}{2} F_2 \left[ 1, 1, \frac{4}{3} \bigg| \frac{2}{3} \alpha \right] - 2\alpha \frac{1}{27} \frac{1}{2} F_2 \left[ 1, 1, \frac{5}{3} \bigg| \frac{2}{3} \alpha \right].
\]

**Proof.** Denote the left hand side by \( E_0(q) \). Then, by (2.4),

\[
q \frac{d}{dq} E_0(q) = \sum_{k,r=1}^{\infty} \chi_{-3}(kr) \left( \frac{1}{3} q^{kr} - q^{kr} \right) = \frac{1}{9} b(q^x) c(q) - \frac{1}{3} b(q) c(q^3) \\
= \frac{1}{9} (a(q) c(q) - c^2(q) - a(q) b(q) + b^2(q)).
\]

Here, for the last equality, we used (2.1) and

\[
b(q^x) = a(q) - c(q),
\]

which follows from [5, Lemma 2.1 (ii), (iii)]. Hence, by the transformation formulas (1.5), we have

\[
E_0(q) = \frac{1}{9} \int_0^q \left( a(q) c(q) - c^2(q) - a(q) b(q) + b^2(q) \right) \frac{dq}{q} \\
= \frac{1}{9} \int_0^\alpha \left( \alpha^x - \alpha^{3x} - (1 - \alpha)^x + (1 - \alpha)^{3x} \right) \frac{d\alpha}{\alpha(1 - \alpha)} \\
\left( = \frac{1}{9} \int_0^\alpha \left( x^x - x^{3x} - (1 - x)^x + (1 - x)^{3x} \right) \frac{dx}{x(1 - x)} \right).
\]

We divide the integral above into the three integrals

\[(2.5) \quad \int_0^\alpha \frac{dx}{x(1 - x)},\]

\[(2.6) \quad \int_0^\alpha \frac{dx}{x(1 - x)},\]

\[(2.7) \quad \int_0^\alpha \left( (1 - x)^x - (1 - x)^{3x} \right) \frac{dx}{x(1 - x)},\]
and show that each integral can be written as hypergeometric functions. First we compute (2.5). If we change the variable \( x \mapsto \alpha x \), then
\[
\int_0^\alpha \frac{x^{\frac{1}{3}}}{x(1-x)} \, dx = \alpha^{\frac{1}{3}} \int_0^{\alpha \frac{1}{3}} \frac{x^{\frac{1}{3}}}{x(1-\alpha x)} \, dx
\]
\[
= \alpha^{\frac{1}{3}} \int_0^1 x^{\frac{1}{3}}(1-x)(1-\alpha x)^{-1} \, dx \quad \frac{dx}{x(1-x)}
\]
\[
= \alpha^{\frac{1}{3}} \frac{\Gamma \left( \frac{1}{3} \right) \Gamma \left( 1 \right)}{\Gamma \left( \frac{4}{3} \right)} \, F_1 \left[ \left. \frac{2}{3},1 \right| \alpha \right] = 3 \alpha^{\frac{1}{3}} F_1 \left[ \left. \frac{2}{3},1 \right| \alpha \right].
\]
Here we used the integral expression of generalized hypergeometric functions [18, p.108, (4.1.2)]
\[
\Gamma(a_1) \Gamma(a_1' - a_1) \frac{1}{A+1} \left. A \right|_{A} \left. F_A \right|_{A} \left[ \left. a_1, a_2, \ldots, a_{A+1} \right| \begin{array}{c} a_1', \ldots, a_A' \end{array} \right] x
\]
\[
= \int_0^1 x^{a_1'}(1-x)^{a_1'-a_1} A \left. F_{A-1} \right|_{A} \left[ \left. a_2, \ldots, a_{A+1} \right| \begin{array}{c} a_1', \ldots, a_A' \end{array} \right] x \, \frac{dx}{(1-x)}. \tag{2.8}
\]
for the last equality. By similar computations, one can show that (2.6) coincides with
\[
\frac{3}{2} \alpha^{\frac{1}{3}} F_1 \left[ \left. \frac{2}{3},1 \right| \alpha \right].
\]
Finally, by (2.2), the variable transformation \( x \mapsto \alpha x \) and (2.8), the integral (2.7) becomes
\[
\int_0^\alpha \left( (1-x)^{-\frac{1}{3}} - (1-x)^{-\frac{1}{3}} \right) \frac{dx}{x}
\]
\[
= \int_0^\alpha \left( \frac{2}{3} x \, F_1 \left[ \left. 1,\frac{5}{2} \right| x \right] - \frac{1}{3} x \, F_1 \left[ \left. 1,\frac{4}{2} \right| x \right] \right) \frac{dx}{x}
\]
\[
= \frac{\alpha}{3} \int_0^1 x(1-x) \left( 2 \, F_1 \left[ \left. 1,\frac{5}{2} \right| \alpha x \right] - 2 \, F_1 \left[ \left. 1,\frac{4}{2} \right| \alpha x \right] \right) \, \frac{dx}{(1-x)}
\]
\[
= \frac{2\alpha}{3} F_2 \left[ \left. 1,1,\frac{5}{2} \right| 2,2,2 \right] \alpha - \frac{\alpha}{3} F_2 \left[ \left. 1,1,\frac{5}{4} \right| 2,2,2 \right] \alpha
\].
This proves the lemma. \( \square \)

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