STOCHASTIC NONLINEAR SCHRÖDINGER EQUATIONS IN THE DEFOCUSING MASS AND ENERGY CRITICAL CASES

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Abstract. We study the stochastic nonlinear Schrödinger equations with linear multiplicative noise, particularly in the defocusing mass-critical and energy-critical cases. For general initial data, we prove the global well-posedness of solutions in both mass-critical and energy-critical cases. We also prove the rescaled scattering behavior of global solutions in the spaces $L^2, H^1$ as well as the pseudo-conformal space for dimensions $d \geq 3$ in the case of global quadratic variation of noise. Furthermore, the Stroock-Varadhan type theorem is also obtained for the topological support of the probability distribution induced by global solutions in the Strichartz and local smoothing spaces. Our proof is based on the construction of a new family of rescaling transformations indexed by stopping times and on the stability analysis adapted to the multiplicative noise.

1. Introduction

This work is devoted to stochastic nonlinear Schrödinger equations (SNLS for short) with linear multiplicative noise in the defocusing mass-critical and energy-critical cases. We consider

$$idX = \Delta X dt + \lambda F(X) dt - i\mu X dt + i \sum_{k=1}^{N} X G_k d\beta_k(t),$$

$$X(0) = X_0.$$  (1.1)

Here, the nonlinearity $F(X) = |X|^\alpha - 1 X, \alpha > 1$, $\lambda = -1$ (resp. $\lambda = 1$) corresponds to the defocusing (resp. focusing) case, $\beta_k$ are standard real-valued Brownian motions on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with normal filtration $(\mathcal{F}_t)_{t \geq 0}$, $G_k(t, x) = g_k(t) \phi_k(x)$, where $\{g_k\}$ are real-valued predictable processes, $g_k \in L^2_{\text{loc}}(\mathbb{R}^+; \mathbb{R})$ $\mathbb{P}$-a.s., $\{\phi_k\}$ are deterministic spatial functions and $\phi_k \in C^\infty(\mathbb{R}^d; \mathbb{C}), d \geq 1$. For the simplicity of exposition, we mainly focus on the case where $N < \infty$. See also Remark 2.4 below for the possible extension to the infinite dimensional noise.

Furthermore, the term $\mu$ is of the form

$$\mu(t, x) = \frac{1}{2} \sum_{k=1}^{N} |G_k(t, x)|^2.$$  

In particular, in the conservative case where $\Re G_k = 0$, $1 \leq k \leq N$, $-i\mu X dt + i \sum_{k=1}^{N} X G_k d\beta_k(t)$ is indeed the Stratonovitch differential, and, via Itô’s formula, the mass is pathwisely conserved $|X(t)|^2 = |X_0|^2, t \in [0, T]$. Hence, for the normalized initial state $|X_0|^2 = 1$, the quantum system evolves on the unit mass of $L^2$ and verifies the conservation of probability. See, e.g., [1, 2] for applications in molecular aggregates with thermal fluctuations in dimension two. See also [66] for stochastic Schrödinger equations with quintic nonlinearity in dimension one.

2010 Mathematics Subject Classification. 60H15, 35Q55, 35J10.

Key words and phrases. Critical space, global well-posedness, scattering, stochastic nonlinear Schrödinger equations, support theorem.
The non-conservative case (i.e., Re $G_k \neq 0$ for some $1 \leq k \leq N$) also plays an important role in the application to open quantum systems [9]. One of the main features is that $t \mapsto |X(t)|^2$ is a continuous martingale. This fact ensures the mean square norm conservation $E|X(t)|^2$, $t \in [0, T]$, and enables one to define a new probability on $(\Omega, \mathcal{F}_T)$ (see [9, (2.29)])

\[ \hat{P}^{T}_{X_0}(d\omega) := (E_\mathbb{P}[|X_0|^2])^{-1}|X(T, \omega)|^2 \mathbb{P}(d\omega), \]

which is the physical probability law of the events occurring in $[0, T]$, while $\hat{\psi}(t, \omega) = X(t, \omega)|X(t, \omega)|^{-1}$ is the state of the quantum system conditioned by the observation of $s \mapsto \beta_k(s, \omega)$, $0 < s < t$. We refer to [9, Chapter 2] for more details. See also [70] for more physical applications, e.g. nonlinear optics, Bose-Einstein condensation and the Gross-Pitaevskii equation.

In the literature, most previous results of SNLS center around the subcritical case. The global well-posedness was first proved by de Bouard-Debussche [28, 29], by using the theory of radonifying operators. Later, the compact manifold case was studied by Brzeźniak-Millet [18], where more general stochastic Strichartz estimates were proved. See also [19, 20]. The global well-posedness of (1.1) for the full subcritical exponents was proved in [5, 6], based on the rescaling approach which can be viewed as Doss-Dussman type transformations in Hilbert spaces. We also refer to [48] for the global well-posedness in the full mass-subcritical case with quite general multiplicative noise. See also [7, 8, 26, 77].

In the critical case, the local well-posedness of SNLS can be obtained by using Strichartz estimates with noises under suitable conditions, see, e.g., [5, 6, 28, 29, 48]. However, the global behavior of solutions is much more subtle. As a matter of fact, the main difference between the subcritical and critical cases is, that the maximal existing time of solutions depends only on the $L^2$- or $H^1$-norm of initial data in the subcritical case, while on the whole profile in the critical case. Hence, although the standard energy method works well for the global existence in the subcritical case, it fails in the critical case.

In contrast to the stochastic case, the critical deterministic nonlinear Schrödinger equations (NLS) have been extensively studied. It has been one central question in the field of dispersive PDEs that, the solutions to the defocusing mass- and energy-critical NLS exist globally and even scatter at infinity. This conjecture was first proved, via the energy induction method, by Bourgain in the seminal work [15] for the energy-critical case with radial initial data in dimensions three and four. Later, for general initial data and dimensions, it was proved by the I-team [27], Ryckman-Visan [67] and Visan [75], based on the energy induction method and interaction Morawetz estimates. See also the concentration compactness method introduced by Kenig-Merle [49]. This conjecture in the mass-critical case was proved by Dodson [33, 34, 35] for general initial data, where the key ingredients are long-time Strichartz estimates. We refer to the monographs [52, 53, 71] for detailed explanations on critical dispersive equations.

Let us mention that, there are also vast results on dispersive equations with randomized initial data below the scaling critical regularity, with which the deterministic equations are usually ill-posed. It was initiated by Bourgain [13, 14] for the periodic NLS in dimensions one and two. Subsequently, Burq-Tzvetkov [23, 24] studied the cubic nonlinear wave equation (NLW) on a three-dimensional compact manifold. For the critical random dispersive equations, the almost sure global well-posedness was obtained by Bényi-Oh-Pocovnicu [10] for the NLS, Pocovnicu [65] for the NLW in dimensions $d = 4, 5$, and extended by Oh-Pocovnicu [63] to dimension $d = 3$, where the stability arguments adapted to random initial data for the globalization in the critical case were first introduced. Regarding the asymptotic behavior of solutions, the almost sure scattering
has been recently proved by Dodson-Lührmann-Mendelson [36, 37], Killip-Murphy-Visan [50] and Bringman [16, 17] for the critical NLS and NLW, by the stability arguments and double (or triple) bootstrap arguments with the energy and the Morawetz term (and the interaction flux estimate). We refer the interested reader to [11, 21, 22, 31, 32, 60, 64] and the references therein for more results in this field.

The situations for SNLS and NLS (or with random initial data) are quite different. One key fact is that the presence of noise in (1.1) destroys the symmetries of equation and thus several conservation laws are absent in the stochastic case. As a matter of fact, the Itô evolution of the Hamiltonian consists of several stochastic integrations (see (5.32) below). The Itô formula (5.4) below also implies that the mass may even not conserved if \( \text{Re} G_k \neq 0 \) for some \( 1 \leq k \leq N \). Thus, it is difficult to obtain the sharp pathwise estimates as in the deterministic or random cases. Moreover, even if a Banach space \( \mathcal{Y} \) is compactly embedded into another one \( \mathcal{V} \), one does not generally have the compact embedding of \( L^p(\Omega; \mathcal{X}) \) into \( L^p(\Omega; \mathcal{Y}) \), \( p \geq 1 \).

Hence, the global existence of SNLS in the mass- and energy-critical cases with general initial data has been an open problem. We would like to refer to the recent progresses by Fan-Xu [39, 40] for the SNLS with multiplicative noise in the conservative mass-critical case for \( d = 1 \), and by Oh-Okamoto [61] for the SNLS with additive noise in the mass-critical case for \( d \geq 1 \) and the energy-critical case for \( 3 \leq d \leq 6 \).

In this work, we prove the global well-posedness of (1.1) in the mass-critical case for dimensions \( d \geq 1 \). Moreover, in the energy-critical case, we prove the global well-posedness for dimensions \( 3 \leq d \leq 6 \), and we also obtain the conditional global well-posedness results for high dimensions \( d > 6 \) if assuming an \( a\text{-priori} \) bound of the energy.

Thus, together with the previous works [5, 6] and the \( a\text{-priori} \) bound of the energy in the energy-critical case with \( d > 6 \), the global existence and uniqueness of solutions to (1.1) are obtained for the full subcritical and critical exponents of the nonlinearity in the defocusing case. Let us also mention that, these results apply to the non-conservative case, which is important in the physical context [9].

Our second main result, motivated by the recent work [47], is concerned with the large time scattering behavior of global solutions to (1.1), namely, solutions behave asymptotically like linear solutions.

Because of the rapid fluctuations of noise at large time, it is non-trivial to derive the global-in-time Strichartz estimates for the linear stochastic Schrödinger equations with multiplicative noise. Moreover, besides the absence of the conservation law of Hamiltonian (possibly also of the mass), the Itô evolution of Morawetz type functional also contains several stochastic integrations, which make it difficult to perform similar analysis as in the aforementioned deterministic or random cases.

Recently, the rescaled scattering behavior of stochastic solutions to (1.1) was proved in [47] for the energy-subcritical exponents \( \alpha \in [\max\{2, 1 + \frac{4}{d}\}, 1 + \frac{4}{d} - 2], \) \( 3 \leq d \leq 6 \), by using the stability arguments in the case of global quadratic variation of noise. Moreover, the non-conservative noise also has the effect to improve the scattering with high probability, even in the regime where deterministic solutions fail to scatter ([47]). The delicate energy-critical case is also studied there, however, relying on the \( a\text{-priori} \) assumption on the global existence of solutions. We also refer to the recent works [41, 42], where the scattering is proved for the three dimensional mass-critical SNLS with the noise under small decay conditions.

Here for the critical SNLS (1.1), we prove the rescaled scattering behavior of global solutions in the spaces \( L^2, H^1 \) and the pseudo-conformal space, respectively, in the case of global quadratic variation of noise. These results are new in the mass-critical case.
for dimensions $d \geq 3$, and are also new in the energy-critical case for high dimensions $d > 6$. We note that, in the energy-critical case for dimensions $3 \leq d \leq 6$, the \textit{a-priori} assumption on the global existence of solutions in [47] is removed here.

The last result of present work characterizes the topological support of the probability distributions induced by the global solutions to (1.1), in both mass- and energy-critical cases. Precisely, we obtain the Stroock-Varadhan type support theorem and prove that the law of stochastic solutions is supported on the closure of all deterministic controlled trajectories in the Strichartz and local smoothing spaces. Let us mention that, the support theorem indeed provides the intuition for the global well-posedness of the critical SNLS (see also Remark 2.10 (i) below).

Our proof is based on the refined rescaling approach and the stability analysis. The latter is inspired by the works [27, 33, 34, 35, 52, 67, 73, 75] for the critical NLS. It should be mentioned that, the stability arguments also have been used in the study of the almost sure global well-posedness and scattering for the critical dispersive equations driven by noise or with random initial data, we refer to [10, 36, 37, 40, 50, 61, 62, 63, 65].

One novelty here is that, we partition the time regime into random intervals depending on the trajectories of noises, and then compare the SNLS reduced by the rescaling transformations with the NLS with lower order perturbations, which are different from [40, 61]. We note that, in order to control the lower order perturbations, the stability arguments require the Strichartz estimates for the Laplacian with lower order perturbations, of which the proof is non-trivial and relies on the asymptotical flatness conditions as in the contexts [57, 77]. Moreover, another key analytic tool is the local smoothing estimate, which in particular permits to gain locally one half derivative of solutions, and to obtain the compatibility between the lower order perturbations and the exotic Strichartz spaces in the high dimensional energy-critical case when $d > 6$. These stability estimates are collected in Theorems 4.1, 4.4 and 4.6 below, the corresponding proofs constitute the main analytic part of the present work.

Furthermore, a new family of rescaling transformations is constructed in order to implement the stability analysis. Because of the multiplicative type of noise, the rescaling transformation is different from the Da Prato-Debussche type transformation used for the dispersive equations with additive noise or with random initial data. It is also different from the previous single rescaling transformation used in [5, 6].

The new family of rescaling transformations is indexed by a sequence of stopping times, which are chosen suitably to keep track of the growth of driving noise in order to ensure the stability analysis. It is important that, the lower order perturbations of transformed equations depend locally in time on the trajectories of noise. Moreover, the probabilistic arguments are also used to piece together the resulting local estimates to derive the crucial global space-time bounds. In this stage, the Hölder continuity of Itô’s integrations and the uniform bounds of the mass and energy implied by Itô’s evolutions in the defocusing case are exploited.

It would be also interesting to note that, for the study of scattering and support theorem, different rescaling transformations have to be constructed in order to match the underlying structures, and the stability analysis fits well with the probabilistic support theorem. Let us also mention that, in the case of global quadratic variation of noise, the scattering for (1.1) can be proved by stability analysis, without using the delicate double and triple bootstrap arguments as in the works [16, 17, 36, 37]. See also Remarks 2.8 (i) and 2.10 (iii) below.
Notation. For $z \in \mathbb{C}$, we set $F(z) := |z|^\alpha z$ with $\alpha = 1 + \frac{4}{\sigma^2}$ or $\alpha = 1 + \frac{4}{\sigma^2}$ in the mass-critical or energy-critical case, respectively. Let $F_x$ and $F_{\tau}$ denote the usual complex derivatives $F_x = \frac{i}{2}\left(\frac{\partial F}{\partial y} - i\frac{\partial F}{\partial y}\right)$, $F_{\tau} = \frac{1}{2}\left(\frac{\partial F}{\partial y} + i\frac{\partial F}{\partial y}\right)$. For any $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$ and multi-index $\alpha = (\alpha_1, \ldots, \alpha_d)$, set $|\alpha| = \sum_{j=1}^d \alpha_j$, $\langle x \rangle = (1 + |x|^2)^{1/2}$, $\partial_j = \frac{\partial}{\partial x_j}$.

Let $\mathcal{F}$ denote the space of rapid decreasing functions and $\mathcal{F}'$ be the dual space of $\mathcal{F}$. For any $f \in \mathcal{F}$, $\mathcal{F}(f)(\xi) = \int e^{-ix\cdot\xi} f(x) dx$.

Given $1 \leq p \leq \infty$, $s \geq 0$, $L^p = L^p(\mathbb{R}^d)$ is the space of $p$-integrable complex functions with the norm $\| \cdot \|_{L^p}$. Moreover, for two Banach spaces $\mathcal{X}$, $\mathcal{Y}$, the norm of $\mathcal{X} \cap \mathcal{Y}$ is $\| \cdot \|_{\mathcal{X} \cap \mathcal{Y}} = \inf\{\|u\|_{\mathcal{X}}, \|u\|_{\mathcal{Y}} : u \in \mathcal{X}, u \in \mathcal{Y}\}$.

A pair $(p, q)$ is called a Strichartz pair, if $\frac{2}{q} = d\left(\frac{1}{2} - \frac{1}{p}\right)$, $(p, q) \in [2, \infty] \times [2, \infty]$ and $(d, p, q) \neq (2, 1, 2)$. For any interval $I \subseteq \mathbb{R}^+$, the Strichartz spaces $V(I) = L^{2,4}(I; \mathbb{R}^d)$, $W(I) = L^{2,4}(I; \mathbb{R}^d)$, $W(I) = L^{(d+2)/(d-2), 4}(I; \mathbb{R}^d)$ and $\mathbb{W}(I) = L^{2,4}(I; \mathbb{R}^d)$ will be frequently used in the mass- and energy-critical spaces. Let $S_0(I) := \bigcap_{I} L^p(I; L^q)$, where the intersection is over finitely many Strichartz spaces which include $C(I; L^2)$, $V(I)$, $W(I)$ and the Strichartz spaces used in Lemma 3.6 below. Its norm is defined by $\|u\|_{S_0} := \max_{I} \|u\|_{L^p(I; L^q)}$. Let $(N_0, \cdot \cdot \cdot, N_{0}(I))$ be the dual space of $(S_0(I), \cdot \cdot \cdot, S_{0}(I))$. Similarly, let $S_1(I) = \{u \in \mathcal{F}' : \|u\|_{S_0(I)} < \infty\}$, and $N_1(I) = \{u \in \mathcal{F}' : \|u\|_{N_0(I)} < \infty\}$.

We shall use the exotic Strichartz spaces $X_0(I, \mathcal{X}(I)$ and $\mathcal{Y}(I)$ with the norms

$$
\|u\|_{X_0(I)} = \|u\|_{L^{(d+2)/(d-2), 4}(I; \mathbb{R}^d)}^{d(d-2)} \|u\|_{L^{(d+2)/(d-2), 4}(I; \mathbb{R}^d)}^{d(d-2)} ,
$$

$$
\|u\|_{X_0(I)} = \|\langle \nabla \rangle^{d(d-2)} u\|_{L^{d(d+2)/(d-2), 4}(I; \mathbb{R}^d)}^{d(d-2)} \|u\|_{L^{d(d+2)/(d-2), 4}(I; \mathbb{R}^d)}^{d(d-2)} ,
$$

$$
\|u\|_{Y(I)} := \|\langle \nabla \rangle^{d(d+2)} u\|_{L^{d(d+2)/(d-2), 4}(I; \mathbb{R}^d)}^{d(d-2)} \|u\|_{L^{d(d+2)/(d-2), 4}(I; \mathbb{R}^d)}^{d(d-2)} ,
$$

which are the inhomogeneous versions of the exotic Strichartz spaces in [52]. We also use the local smoothing spaces defined by, for $\alpha, \beta \in \mathbb{R}$, $L^2(I; H^\alpha_x) = \{u \in \mathcal{F}' : \int_I \int \langle x \rangle^{2\beta} u(x)^2 dx dt < \infty\}$.

Throughout this paper, $C(\cdot \cdot \cdot)$ denotes constants that may change from line to line.

2. Formulations of main results

Let us start with the definition of solutions to (1.1).

Definition 2.1. Fix $T > 0$. An $L^2$-solution, $H^1$-solution to (1.1) is an $L^2$-(resp., $H^1$-)valued continuous $(\mathcal{F}_{t})$-adapted process $X = X(t)$, $t \in [0, T]$, such that $|X|^\alpha \in L^1([0, T], H^1)$ and it satisfies $\mathbb{P}$-a.s.,

$$
X(t) = X_0 + \int_0^t (i\Delta X(s) + \mu X(s) + \lambda i F(X(s))) ds + \sum_{k=1}^N \int_0^t X(s) G_k(s) d\beta_k(s), \ \forall t \in [0, T],
$$

as an Itô equation in $H^{-2}$ (resp. $H^{-1}$).
We assume the asymptotically flat condition as in [5, 6, 47].

(H0) For each $1 \leq k \leq N$, $G_k(t, x) = g_k(t)\phi_k(x)$, $g_k$ are real-valued predictable processes, $g_k \in L^\infty(\Omega \times [0, T])$, $0 < T < \infty$, and $\phi_k \in C^\infty(\mathbb{R}^d, \mathbb{C})$ satisfying that for any multi-index $\gamma, \gamma \neq 0$,

$$\limsup_{|x| \to \infty} |x|^2|\partial_\gamma^k \phi_k(x)| = 0.$$  

The local well-posedness in the mass- and energy-critical cases is summarized below.

**Theorem 2.2. (Local Well-posedness)**

Consider (1.1) in the mass-(resp., energy-)critical case, i.e., $\alpha = 1 + 4/d$, $d \geq 1$ (resp., $\alpha = 1 + 4/(d - 2)$, $d \geq 3$). Assume (H0). Then, for each $X_0 \in L^2$ (resp. $X_0 \in H^1$), there exists a unique $L^2$-(resp., $H^1$-)solution $X$ to (1.1) on $[0, \tau^*)$, where the maximal existing time $\tau^*$ is an $\{\mathcal{F}_t\}$-stopping time, satisfying that $\mathbb{P}$-a.s. for any $t \in (0, \tau^*)$ and any Strichartz pair $(\rho, \gamma)$,

$$X|_{[0,t]} \in C([0, t]; L^2) \cap L^\gamma(0, t; L^\rho)$$  

$$(\text{resp., } X|_{[0,t]} \in C([0, t]; H^1) \cap L^\gamma(0, t; W^{1, \rho})).$$

Moreover, $X$ exists globally $\mathbb{P}$-a.s. if for any $0 < T < \infty$,

$$\|X\|_{L^{2+\frac{1}{2}}(\gamma, \gamma^*; L^{2+\frac{1}{2}})} < \infty, \quad \mathbb{P} \text{-a.s.}$$

$$(\text{resp., } \|X\|_{L^{2+\frac{1}{2}}(\gamma, \gamma^*; L^{2+\frac{1}{2}})} < \infty, \quad \mathbb{P} \text{-a.s.}).$$

The proofs are similar to those of [5, Proposition 5.1] and [6, Theorem 2.1], and the last assertion of global existence follows from the blow-up alternative results as in [5, 6].

The first main result of this paper is formulated below, concerning the global existence and uniqueness of solutions to (1.1) in the critical cases.

**Theorem 2.3. (Global Well-posedness)**

(i) Consider (1.1) in the defocusing mass-critical case, i.e., $\lambda = -1$, $\alpha = 1 + 4/d$, $d \geq 1$. Assume (H0). Then, for each $X_0 \in L^2$ and $0 < T < \infty$, there exists a unique $L^2$-solution to (1.1) on $[0, T]$, satisfying that for any $p \geq 1$,

$$\mathbb{E}\|X\|_{C([0, T]; L^2)}^p \leq C(p, T) < \infty,$$

and for any Strichartz pair $(\gamma, \rho)$,

$$X \in L^\gamma(0, T; L^\rho) \cap L^{\frac{1}{2}}(0, T; H^\frac{1}{2}, \mathbb{P} \text{-a.s.}).$$

(ii) Consider (1.1) in the defocusing energy-critical case, i.e., $\lambda = -1$, $\alpha = 1 + 4/(d - 2)$, $d \geq 3$. Assume (H0). In the high dimensional case where $d > 6$, assume additionally that for each $0 < T < \infty$,

$$E_T := \sup_{0 \leq t < \tau^* \wedge T} |X(t)|_{H^1} \leq C(T) < \infty, \quad \mathbb{P} \text{-a.s.}$$

Then, for each $X_0 \in H^1$ and $0 < T < \infty$, there exists a unique $H^1$-solution to (1.1) on $[0, T]$, satisfying that for any $p \geq 1$,

$$\mathbb{E}\|X\|_{C([0, T]; H^1)}^p + \mathbb{E}\|X\|_{L^{\frac{2}{d+2}}(0, T; L^{\frac{2d}{d+2}})}^p \leq C(p, T) < \infty,$$

and for any Strichartz pair $(\gamma, \rho)$,

$$X \in L^\gamma(0, T; W^{1, \rho}) \cap L^{\frac{1}{2}}(0, T; H^\frac{1}{2}, \mathbb{P} \text{-a.s.}).$$
Remark 2.4. (i) One has also the stability results in both mass- and energy-critical cases, see Theorems 4.1, 4.4 and 4.6 below. A key role there is played by the local smoothing spaces, which turn out to be quite effective to control the lower order perturbations related to the noise.

(ii) One may obtain (2.9) by using the Itô formula of the Hamiltonian (5.32) below, which can be derived directly by formal computations. However, the rigorous proof of (5.32) is technically unclear in the high dimensional case where \( d > 6 \).

(iii) We would like to mention that, the global well-posedness of SNLS has been recently proved in [39, 40] for the mass-critical case for dimension \( d = 1 \) in the conservative case, under a different spatial decay assumption on the noise. The results in [39, 40] also hold in dimensions \( d = 2, 3 \) and in the case where one has a uniform pathwise control of mass (see [40, Remark 1.7]). The global well-posedness of SNLS with additive noise is proved in [61] with the \( L^2 \) and \( H^1 \) noise, respectively, in the mass-critical case with \( d \geq 1 \) and the energy-critical case with \( 3 \leq d \leq 6 \).

(iv) The stronger smoothness condition is assumed here merely for the convenience to perform pseudo-differential calculus. It suffices to assume that \( \phi_k \in C^n \) for \( n \) large enough. It is possible to weaken the smoothness to \( C_d^k \) regularity by using the delicate local smoothing spaces constructed by Marzuola-Metcalfe-Tataru [57], see also the remark in [5] for the \( C_d^2 \) regularity of noise in the \( L^2 \) case. For the infinite dimensional noise, it is possible to prove the global well-posedness in Theorem 2.3 and also the scattering in Theorem 2.7 below under additional suitable summability conditions as in [76, Remark 2.3.13]. We also refer to [3, 4] for similar summability conditions on the noise related to Hilbert-Schmidt operators.

We also enhance the estimates (2.8) and (2.11) to the whole time regime, provided that \( g_k \in L^2(\mathbb{R}^+) \), \( 1 \leq k \leq N \), a.s. Namely, we have

\[
\text{Theorem 2.5. Consider the situations in Theorem 2.3 (i) (resp. (ii)). Assume additionally that } g_k \in L^2(\mathbb{R}^+) \text{, } 1 \leq k \leq N \text{, a.s.. In the mass-critical case assume also } d \geq 3. \text{ Then, for each } X_0 \in L^2 \text{ (resp. } X_0 \in H^1) \text{, the solution } X \text{ to } (1.1) \text{ satisfies that for any Strichartz pair } (\rho, \gamma),
\]

\[
X \in L^{\gamma}(\mathbb{R}^+; L^\rho) \cap L^2(\mathbb{R}^+; H_{-1}^{\frac{1}{2}}), \quad \mathbb{P} - \text{a.s.}
\]

\[
(\text{resp. } X \in L^{\gamma}(\mathbb{R}^+; W^{1,\rho}) \cap L^2(\mathbb{R}^+; H_{-1}^{\frac{3}{2}}), \quad \mathbb{P} - \text{a.s..}).
\]

The next result is concerned with the scattering behaviour of global solutions to (1.1).

Besides in \( L^2 \) and \( H^1 \), let us also introduce the pseudo-conformal space, i.e., \( \Sigma := \{f \in H^1 : |f(\cdot)| \in L^2\} \), in which we assume that, as in [47], the time functions \( g_k \) in (H0) have appropriate integrability and decay speed at infinity.

(H1) For each \( 1 \leq k \leq N \),

\[
\limsup_{|x| \to \infty} |x|^3 \partial_x^\gamma \phi_k(x) = 0, \quad 1 \leq |\gamma| \leq 3,
\]

\[
\text{esssup}_x \int_0^\infty (1 + t^4) g_k^2(t) dt < \infty, \quad 1 \leq k \leq N, \text{ and for } \mathbb{P}-\text{a.e. } \omega \in \Omega,
\]

\[
\lim_{t \to 1} (1 - t)^{-3} \left( \int_t^\infty g_k^2(\omega, s) ds \ln \int_t^\infty g_k^2(\omega, s) ds \right)^{\frac{1}{2}} = 0.
\]

Remark 2.6. As mentioned in [47, Remark 1.4], the \( L^\infty(\Omega) \) condition on \( \int_0^\infty (1 + t^4) g_k^2(t) dt \) can be weakened by some suitable exponential integrability. The condition (2.15) is closely related to the law of the iterated logarithm of martingales, we refer to [47, Section 5] for more details.
In order to formulate the scattering results, we introduce the rescaling function

$$\varphi_*(t) = -\sum_{k=1}^{N} \int_{t}^{\infty} G_k(s)d\beta_k(s) + \frac{1}{2} \sum_{k=1}^{N} \int_{t}^{\infty} (|G_k(s)|^2 + G_k^2(s)) ds,$$

Note that, $\varphi_* \in C(\mathbb{R}; W^1,\infty)$ if $g_k \in L^2(\mathbb{R})$, $1 \leq k \leq N$, a.s.. Then, letting

$$z_*(t) := e^{-\varphi_*(t)} X(t),$$

we have

$$i\partial_t z_* = e^{-\varphi_*(t)} \Delta(e^{\varphi_*(t)} z_*) - e^{(\alpha-1)\text{Re} \varphi_*(t)} F(z_*),$$

with $z_*(0) = X_0$. Here,

$$e^{-\varphi_*(t)} \Delta(e^{\varphi_*(t)} z_*) = (\Delta + b_*(t) \cdot \nabla + c_*(t)) z_*$$

with the coefficients of lower order perturbations

$$b_*(t) = -2 \sum_{k=1}^{N} \int_{t}^{\infty} \nabla G_k(s)d\beta_k(s) + 2 \int_{t}^{\infty} \nabla \hat{\mu}(s) ds,$$

$$c_*(t) = \sum_{j=1}^{N} \sum_{k=1}^{N} \int_{t}^{\infty} \partial_j G_k(s)d\beta_k(s) - \int_{t}^{\infty} \partial_j \hat{\mu}(s) ds^2$$

$$- \sum_{k=1}^{N} \int_{t}^{\infty} \Delta G_k(s)d\beta_k(s) + \int_{t}^{\infty} \Delta \hat{\mu}(s) ds,$$

and

$$\hat{\mu}(s, x) = \frac{1}{2} \sum_{k=1}^{N} (|G_k(s, x)|^2 + G_k(s, x)^2) = \sum_{k=1}^{N} (\text{Re} G_k) G_k(s, x).$$

It will be also convenient to introduce the evolution operators $U_*(t, s)$, $s, t \geq 0$, corresponding to equation (2.18) with $\sigma = 0$ in the homogeneous case $F \equiv 0$. Precisely, for any $v_* \in L^2(\text{or } H^1)$, $v(t) := U_*(t, s)v_*$ denotes the corresponding solution with the initial condition $v(s) = v_*$. The existence and uniqueness of solutions follow from the well posedness result in [38, Theorem 1.1]. In particular, one has $U_*(t, s)U_*(s, r) = U_*(t, r)$, $t, s, r \in \mathbb{R}^+$. We shall also use the evolution operators $\{U(t, s)\}$ related to equation (2.29) with $F \equiv 0$ and $\sigma = 0$.

We are now ready to state the scattering result.

**Theorem 2.7. (Scattering)**

(i) Consider the defocusing mass-critical case, i.e., $\lambda = -1$, $\alpha = 1 + 4/d$, $d \geq 3$. Assume (H0) and that $g_k \in L^2(\mathbb{R})$, $1 \leq k \leq N$, a.s.. Then, for each $X_0 \in L^2$, the global $L^2$-solution $X$ to (1.1) scatters at infinity, i.e., $\mathbb{P}$-a.s. there exist $v_+, u_+ \in L^2$ such that

$$e^{it\Delta}e^{-\varphi_*(t)} X(t) \to v_+, \text{ in } L^2, \text{ as } t \to \infty,$$

and

$$U_*(0, t)e^{-\varphi_*(t)} X(t) \to u_+, \text{ in } L^2, \text{ as } t \to \infty.$$

(ii) Consider the defocusing energy-critical case, i.e., $\lambda = -1$, $\alpha = 1 + 4/(d-2)$, $d \geq 3$. Assume (H0) and that $g_k \in L^2(\mathbb{R})$, $1 \leq k \leq N$, a.s.. In the high dimensional case where $d > 6$, assume additionally that

$$E_\infty := \sup_{0 \leq t < \infty} |X(t)|_{H^1} \leq C < \infty, \text{ a.s..}$$
Then, for each $X_0 \in H^1$, the global $H^1$-solution satisfies the asymptotics (2.23) and (2.24) with $H^1$ replacing $L^2$.

(iii) Consider the situations as in the defocusing energy-critical case in (ii), $d \geq 3$. Then, for each $X_0 \in \Sigma$, the asymptotic (2.23) holds with $\Sigma$ replacing $L^2$.

Remark 2.8. (i). The scattering behavior of stochastic solutions to (1.1) is related to the rescaling function $e^{-\varphi_*}$ which, actually, encodes the information of noise in (1.1).

We also note that, the rescaling transformation (2.17) for the scattering is different from (2.27) below for the global well-posedness. This is due to the fact that the reduced equation (2.29) (as in the previous works [5, 6]) is not a small perturbation around the NLS in the large time regime. In contrast, thanks to the global integrability $g_k \in L^2(\mathbb{R}^+)$, the coefficients $b_\ast, c_\ast$ of the lower order perturbations in (2.19) are of small amplitudes for large time. Thus, equation (2.19) can be regarded as a perturbation around the NLS and, intuitively, the corresponding solutions are expected to behave close to those of NLS. This fact was also used in the recent work [47].

(ii). As in the $H^1$ case in Theorem 2.3 (ii), it is possible to obtain the global pathwise bound (2.25) from the Hamiltonian (5.32) below, by using similar arguments as in the proof of [47, (1.7)]. However, the rigorous proof of (5.32) is technically unclear.

The last main result characterizes the topological support of the probability distributions induced by the global solutions to (1.1), in both mass- and energy-critical cases.

The support theorem for diffusions was initiated in the seminal papers [68, 69] and has been extensively studied in literature. See, e.g., [44] and [46] for SNLS with additive noise and with fractional noise, respectively, and [30] for the application of support theorem to the noise effect on blow-up. See also [58, 59] and references therein.

To formulate the results precisely, we let $\mathcal{H}$ denote the Cameron-Martin space associated with the Brownian motions $\beta = (\beta_1, \cdots, \beta_N)$, i.e., $\mathcal{H} = \{ h \in H^1(0, T; \mathbb{R}^N) : h(0) = 0 \}$. For any $h = (h_1, \cdots, h_N) \in \mathcal{H}$, let $X(\beta + h)$ be the solution to (1.1) with the driven process $\beta + h$ replacing the Brownian motion $\beta$. Moreover, let $S(h)$ denote the solution to the controlled equation below

$$
(2.26) \quad idS(h) = \Delta S(h)dt + \lambda F(S(h))dt - i\bar{\mu}S(h)dt + i \sum_{k=1}^{N} S(h)G_k\dot{h}_k dt,
$$

$$
S(h)(0) = X_0,
$$

where $\bar{\mu}$ is given by (2.22), and $\dot{h}_k$ is the derivative of $h_k$. We also use the notation supp$(\mathbb{P} \circ X^{-1})$ for the topological support of law of solutions to (1.1).

Theorem 2.9. (Support Theorem)

(i) Consider the defocusing mass-critical case, i.e., $\lambda = -1, \alpha = 1 + 4/d$, $d \geq 1$. Assume (H0) and that $g_k$ are deterministic and continuous, $1 \leq k \leq N$. Let $X$ be the global $L^2$-solution to (1.1) corresponding to $X(0) = X_0 \in L^2$.

Then, the support supp$(\mathbb{P} \circ X^{-1})$ in the spaces $S^0(0, T)$ and $L^2(0, T; H^2_{\Sigma})$ is the closure of the set $\{ S(h), h \in \mathcal{H} \}$.

(ii) Consider the defocusing energy-critical case, i.e., $\lambda = -1, \alpha = 1 + 4/(d - 2)$, $3 \leq d \leq 6$. Assume (H0) and that $g_k$ are deterministic and continuous, $1 \leq k \leq N$. Let $X$ be the global $H^1$-solution to (1.1) with $X(0) = X_0 \in H^1$.

Then, the support supp$(\mathbb{P} \circ X^{-1})$ in the spaces $S^1(0, T)$ and $L^2(0, T; H^2_{\Sigma})$ is the closure of the set $\{ S(h), h \in \mathcal{H} \}$.
Remark 2.10. (i) Theorem 2.9 provides the intuition for the global well-posedness of (1.1) in the critical cases. Actually, equation (2.26) can be viewed as a subcritical perturbation of the NLS (2.33) below, and thus its global well-posedness can be obtained by using the stability results in [51, 52, 72, 73]. Hence, if assuming a-priori the support theorem to hold, then, heuristically, the corresponding stochastic solution \(X\) is also expected to exist globally.

(ii). Theorem 2.9 applies in particular to SNLS in [5, 6], in which the case \(g_k \equiv 1\) was considered, \(1 \leq k \leq N\).

(iii). We note that, unlike in the case of global well-posedness and scattering, we use the different rescaling transformations (7.14) and (7.26) in order to prove the validity of asymptotics (7.12) and (7.13). The reduced equation (7.15) (resp. (7.27)) is a small perturbation of equation (7.16) (resp. (7.28)), the stability results then can be implemented to derive the desirable asymptotics.

(iv). For the infinite dimensional noise, we expect the support theorem in Theorem 2.9 to hold under additional suitable summability conditions on \(\{G_k\}\), by using the theory of Q-Wiener process and Hilbert-Schmidt operators as in [44, 46] and the similar formulation of controlled equation (or skeleton) as in [45].

In order to prove the global existence of solutions to (1.1), in view of Theorem 2.2, we only need to obtain the global bounds of the \(L^{2+\frac{1}{d}}(0, \tau^* \land T; L^{2+\frac{1}{d}})\)- and \(L^{2(d+2)}(0, \tau^* \land T; L^{2(d+2)})\)-norms of solutions in the critical cases for any \(0 < T < \infty\). Such estimates were obtained in the deterministic case by using the energy induction method or the concentration-compactness method, combined with the conservation laws (e.g., of the mass and Hamiltonian) and interaction Morawetz estimates. However, the presence of noises in (1.1) destroys the conservation laws, the corresponding Itô evolutions indeed consist of several stochastic integrals (see (5.4), (5.32) below), which make it quite hard to obtain the estimates as in the deterministic case.

Proceeding differently, we introduce a family of rescaling transformations on a random partition (depending on the growth of noise) of any bounded time interval. On each small random time piece, we apply the stability arguments to compare the resulting random equation with the standard NLS with the same initial data. The probabilistic arguments are then used to derive the global space-time bounds of solutions. More precisely, by virtue of the Hölder continuity of noise and the global bounds of mass and energy in the defocusing case, we show that the entire number of random time pieces is finite almost surely. This fact enables us to piece together the local bounds derived on the small random intervals in the previous step and thus to obtain the key global space-time estimates.

For the reader’s convenience, let us explain more precisely the procedure above on a random time interval \([\sigma, \sigma + \tau]\), where \(\sigma\) and \(\sigma + \tau\) are \((\mathcal{F}_t)\)-stopping times.

Let \(\hat{\mu}\) be as in (2.22) and introduce the rescaling transformation
\[
\varphi_\sigma(t, x) := \int_{\sigma}^{\sigma + t} G_k(s, x) d\beta_k(s) - \int_{\sigma}^{\sigma + t} \hat{\mu}(s, x) ds.
\]

The rescaling transformation can be regarded as a Doss-Sussman type transformation in Hilbert space. See, e.g., [4] for the applications of rescaling approach to general stochastic partial differential equations with coercive structure. See also [8, 78] for the
application to optimal bilinear control problems, and \[7, 77\] for other quite general stochastic dispersive equations.

The nice feature of rescaling transformation is, that it reveals the structure of the stochastic equation (1.1) by reducing it to the random equation with lower order perturbations below

\[
i \partial_t v_\sigma = e^{-\varphi_\sigma} \Delta (e^{\varphi_\sigma} v_\sigma) - e^{(\alpha-1) \Re \varphi_\sigma} F(v_\sigma),
\]

\[v_\sigma(0) = X(\sigma),\]

where

\[
e^{-\varphi_\sigma} \Delta (e^{\varphi_\sigma} v_\sigma) = \left( \Delta + b_\sigma(t) \cdot \nabla + c_\sigma(t) \right) v_\sigma,
\]

and the coefficients

\[
b_\sigma(t) = 2 \nabla \varphi_\sigma(t) = 2 \sum_{k=1}^{N} \int_{\sigma}^{\sigma+t} \nabla G_k(s) d\beta_k(s) - 2 \int_{\sigma}^{\sigma+t} \nabla \tilde{\mu}(s) ds,
\]

\[
c_\sigma(t) = \Delta \varphi_\sigma + \sum_{j=1}^{N} (\partial_j \varphi_\sigma)^2
\]

\[= \sum_{j=1}^{N} \sum_{k=1}^{N} \int_{\sigma}^{\sigma+t} \partial_j G_k(s) d\beta_k(s) - \int_{\sigma}^{\sigma+t} \partial_j \tilde{\mu}(s) ds^2
\]

\[+ \sum_{k=1}^{N} \int_{\sigma}^{\sigma+t} \Delta G_k(s) d\beta_k(s) - \int_{\sigma}^{\sigma+t} \Delta \tilde{\mu}(s) ds.
\]

The relationship between equations (1.1) and (2.29) is stated in Theorem 2.11 below, which extends the previous case where \(\sigma \equiv 0\) in [5]-[8]. The proof is contained in the Appendix.

**Theorem 2.11.** Consider the situations in Theorem 2.2. Let \(X\) be the \(L^2\)-(resp. \(H^1\)-)solution to (1.1) on \([0, \tau^*]\) with \(X(0) = X_0 \in L^2\), where \(\tau^*\) is the maximal existing time. Let \(v_\sigma\) be as in (2.27), where \(\sigma\) is any \((\mathcal{F}_t)\)-stopping time satisfying \(0 \leq \sigma < \tau^*\). Then, \(v_\sigma\) satisfies (2.29) on \([0, \tau^* - \sigma]\) in the space \(H^{-2}\) (resp, \(H^{-1}\)) almost surely.

The important fact is, that the transformed equation (2.29) is indexed by the stopping time \(\sigma\), and the amplitude of lower order perturbations (caused by the multiplicative noise) depends on the local trajectories of noise on the random interval \([\sigma, \sigma+\tau]\). These stopping times will be chosen suitably to keep track of the growth of noise such that, on the small random interval \([\sigma, \sigma+\tau]\), the resulting equation (2.29) can be regarded as a small perturbation of the NLS

\[
i \partial_t \tilde{u} = \Delta \tilde{u} - F(\tilde{u}),
\]

\[\tilde{u}(0) = v_\sigma(0) = X(\sigma).
\]

Thus, a stability-type result will fulfill the comparison procedure above. It should be mentioned that, because of the lower order perturbations, one needs to prove stability results for the equation of similar structure as in (2.29), i.e., with lower order perturbations. For this reason, equation (2.33) is reformulated as follows

\[
i \partial_t \tilde{u} = e^{-\varphi_\sigma} \Delta (e^{\varphi_\sigma} \tilde{u}) - e^{(\alpha-1) \Re \varphi_\sigma} F(\tilde{u}) + e
\]

with the error term

\[
e = -(b_\sigma(t) \cdot \nabla + c_\sigma(t)) \tilde{u} - (1 - e^{(\alpha-1) \Re \varphi_\sigma}) F(\tilde{u}),
\]
where the coefficients $b_\sigma$, $c_\sigma$ are as in (2.31) and (2.32), respectively.

The proof of the stability results in Theorems 4.1, 4.4 and 4.6 below is mainly inspired by the works [52, 72, 73]. As mentioned above, two key analytic tools are the Strichartz estimates for the Laplacian with lower order perturbations and the local smoothing estimates to control the lower order perturbations, for which the pseudo-differential calculus is performed.

The strategy above applies also to the proof of scattering in Theorem 2.7 and the Stroock-Varadhan type support theorem in Theorem 2.9. However, we note that, different kind of appropriate rescaling functions has to be constructed in order to match the underlying structures of corresponding problems. See the rescaling transformations (2.17) in the scattering case, and (7.14), (7.26) in the proof of the Stroock-Varadhan type support theorem.

The remainder of this paper is structured as follows. In Section 3, we present the preliminaries used in this paper, including the pseudo-differential operators, the Strichartz and local smoothing estimates and the exotic Strichartz spaces. Then, in Section 4 we prove the stability results in both mass- and energy-critical cases. Sections 5, 6 and 7 are mainly devoted to the proof of Theorems 2.3, 2.7 and 2.9, respectively. Finally, some technical proofs are postponed to the Appendix, i.e. Section 8.

3. Preliminaries

This section collects some preliminaries used in this paper.

3.1. Pseudo-differential operators. We recall some basic facts of pseudo-differential operators. For more details see [54, 74, 77] and references therein.

We say that $a \in C^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ is a symbol of class $S^m$, if for any multi-indices $\alpha, \beta \in \mathbb{N}^d$, $|\partial_\xi^\alpha \partial_\eta^\beta a(x, \xi)| \leq C_{\alpha, \beta} \langle \xi \rangle^{m-|\alpha|}$. The semi-norms $|a|_{S^m}$ are defined by

$$|a|_{S^m} = \max_{|\alpha + \beta| \leq l} \sup_{\mathbb{R}^d} \{|\partial_\xi^\alpha \partial_\eta^\beta a(x, \xi)| \langle \xi \rangle^{-(m-|\alpha|)}\}, \ l \in \mathbb{N}.$$ 

Let $\Psi_a$ denote the pseudo-differential operator related to the symbol $a(x, \xi)$, i.e.,

$$\Psi_a v(x) = (2\pi)^{-d} \int e^{ix \cdot \xi} a(x, \xi) \mathcal{F}(v)(\xi) d\xi, \ v \in \mathcal{S}.$$ 

In this case, we write $\Psi_a \in S^m$ when no confusion arises.

Lemma 3.1. Let $a_i \in S^m$, $i = 1, 2$. Then, $\Psi_{a_1} \circ \Psi_{a_2} = \Psi_a \in S^{m_1+m_2}$ with

$$a(x, \xi) = (2\pi)^{-d} \int e^{-iy \cdot \eta} a_1(x, \xi + \eta) a_2(x + y, \xi) dy d\eta.$$ 

Note that, the commutator $i[\Psi_a, \Psi_b] := i(\Psi_a \Psi_b - \Psi_b \Psi_a)$ is an operator with symbol in $S^{m_1+m_2-1}$, and the principal symbol is the Poisson bracket

$$H_{a,b} := \{a, b\} = \sum_{j=1}^d \partial_\xi_j a \partial_\eta_j b - \partial_\xi_j b \partial_\eta_j a.$$ 

One can also expand the composition of two pseudo-differential operators into any finite order and estimate the remainder. See Lemmas 3.1 and 3.2 in [77].

Lemma 3.2. Let $a \in S^0$, $p \in (1, \infty)$. Then, for some $C > 0$ and $l \in \mathbb{N}$,

$$\|\Psi_a\|_{L^p} \leq C |a|^{(l)}_{S^0}.$$
3.2. **Strichartz and local smoothing estimates.** We first present the Strichartz and local smoothing estimates below.

**Theorem 3.3.** Let $I = [t_0, T] \subseteq \mathbb{R}^+$. Consider the equation

\begin{equation}
 i\partial_t u = e^{-\Phi} \Delta (e^\Phi u) + f.
\end{equation}

Here, the function $\Phi = \Phi(t, x)$ is continuous in $t$ for each $x \in \mathbb{R}^d$, $d \geq 1$, and satisfies that for each multi-index $\gamma$,

\begin{equation}
 |\partial_2^\gamma \Phi(t, x)| \leq C(\gamma)g(t) \langle x \rangle^{-2}
\end{equation}

for some positive and continuous function $g$. Then, for any $u(t_0) \in L^2$ and $f \in N^0(I) + L^2(I; H^{-2}_x)$, the solution $u$ to (3.2) satisfies

\begin{equation}
 \|u\|_{S^0(I) \cap L^2(I; H^2_x)} \leq C_T(\|u(t_0)\|_2 + \|f\|_{N^0(I) + L^2(I; H^{-2}_x)}).
\end{equation}

Moreover, if in addition $u(t_0) \in H^1$, $d \geq 3$, $f \in N^1(I) + L^2(I; H^{1}_x)$, then

\begin{equation}
 \|u\|_{S^1(I) \cap L^2(I; H^2_x)} \leq C_T(\|u(t_0)\|_{H^1} + \|f\|_{N^1(I) + L^2(I; H^{1}_x)}).
\end{equation}

Below, we use the notation $C_T$ for the constant in Strichartz estimates above throughout the paper. We may assume $C_T \geq 1$ without loss of generality.

**Remark 3.4.** Estimates (3.4) and (3.5) are the so called local-in-time estimates, in that the constant $C_T$ depends on time. Quantitative estimates and $L^p(\Omega)$-integrability of $C_T$ have been obtained in [77] for quite general stochastic dispersive equations, including stochastic Schrödinger equations with variable coefficients as well as the stochastic Airy equation. See also [57] for more general situations where the Hamiltonian flows associated to Schrödinger operators are trapped.

**Proof.** Estimate (3.4) can be proved similarly as in [77, Theorem 2.11]. See also Remark 2.14 in [77]. Actually, the asymptotically flat condition (3.3) guarantees that the lower order perturbations arising in the operator $e^{-\Phi} \Delta (e^\Phi \cdot)$ can be controlled, via the Gårding inequality, by the Poisson bracket $i[\Psi_h, \Delta]$ for some appropriate symbol $h \in S^0$ (see the proof of [77, Theorem 4.1]). We refer to [77] for more details. See also [5, Lemma 4.1] and [6, Lemma 2.7] for the special case where $\Phi$ is as in (2.28) with $\sigma \equiv 0$.

Regarding (3.5), Applying the operator $\langle \nabla \rangle$ to both sides of (3.2) we get

\begin{equation}
 i\partial_t \langle \nabla \rangle u = e^{-\Phi} \Delta (e^\Phi \langle \nabla \rangle u) + \langle \nabla \rangle b \cdot \nabla + c |u| + \langle \nabla \rangle f,
\end{equation}

with the coefficients $b = 2\nabla \Phi$, $c = \Delta \Phi + \sum_{j=1}^d (\partial_j \Phi)^2$. We regard (3.6) as the equation for the unknown $\langle \nabla \rangle u$. Then, (3.4) yields

\begin{equation}
 \|u\|_{S^1(I) \cap L^2(I; H^2_x)} \leq C_T(\|u(t_0)\|_{H^1} + \|\langle \nabla \rangle b \cdot \nabla + c |u|\|_{L^2(I; H^{-1}_x)} + \|f\|_{N^1(I) + L^2(I; H^{1}_x)}).
\end{equation}

Note that, for the commutator $\langle \nabla \rangle b \cdot \nabla + c$,

\begin{equation}
 \langle x \rangle \langle \nabla \rangle^{-\frac{1}{2}} \langle \nabla \rangle b \cdot \nabla + c = \Psi_p \langle x \rangle^{-\frac{1}{2}} \langle \nabla \rangle^2,
\end{equation}

where $\Psi_p := \langle x \rangle \langle \nabla \rangle^{-\frac{1}{2}} \langle \nabla \rangle b \cdot \nabla + c \langle \nabla \rangle^{-\frac{1}{2}} \langle x \rangle$ is a pseudo-differential operator of order 0 with semi-norms depending on $\sup_{t \in I} g(t)$. By Lemma 3.2 and (3.4),

\begin{equation}
 \|\langle \nabla \rangle b \cdot \nabla + c |u|\|_{L^2(I; H^{-1}_x)} \leq C \sup_{t \in I} g(t) \|u\|_{L^2(I; H^{1}_x)}
\end{equation}

\begin{equation}
 \leq C C_T(\|u_0\|_2 + \sup_{t \in I} g(t) \|f\|_{N^0(I) + L^2(I; H^{-1}_x)}).
\end{equation}
Since \( \langle x \rangle^2 \) is a weight of Muckenhoupt class \( A_2 \) if \( d \geq 3 \) (see, e.g., [43, Lemma 2.3 (iv)]), by virtue of the boundedness of multiplier \( m(\xi) = \langle \xi \rangle^{-1} \) in the weighted space \( L^2(\langle x \rangle^2 dx) \) (see, e.g., [55, 56]), we have the embedding \( H^s_1 \hookrightarrow H^s_{1-\frac{1}{2}} \) and so \( L^2(I; H^s_1) \hookrightarrow L^2(I; H^s_{1-\frac{1}{2}}) \).

Therefore, taking into account \( N^1(I) \hookrightarrow N^0(I) \) and plugging (3.8) into (3.7) we obtain (3.5). The proof is complete.

It is known that global-in-time Strichartz and local smoothing estimates (i.e., the constant \( C_T \) is independent of \( T \)) hold for the free Schrödinger group \( \{e^{-it\Delta}\} \). See, e.g., [25, 52, 57] and references therein. This is also true for the operator \(-ie^{-\Phi} \Delta(e^{\Phi} \cdot)\) when \( g \) satisfies some smallness condition where \( d \geq 3 \), which is important in the study of scattering in Section 6 below. Precisely, we have

**Theorem 3.5.** Consider the situations as in Theorem 3.3 in the case where \( d \geq 3 \). Assume (H0). Assume additionally that for some \( T_* > 0 \), \( \sup_{t \geq T_*} g(t) \leq \varepsilon \) with \( \varepsilon \) sufficiently small. Then, the estimates (3.4) and (3.5) also hold with some constant \( C \) independent of \( t_0 \) and \( T \), and \( u(t_0) \) can be replaced by the final datum \( u(T) \).

Theorem 3.5 follows from [47, Corollary 5.3], where similar estimates were proved with \( L^2(I; H^s_{1-\frac{1}{2}}) \) and \( L^2(I; H^s_{1-\frac{1}{2}}) \) replaced by the local smoothing spaces \( LS(I) \) and \( LS'(I) \), respectively. Actually, by using the Hardy type inequality in [57, (3.5)], one has the embedding \( LS(I) \hookrightarrow L^2(I; H^s_{1-\frac{1}{2}}) \). See also [57] for more general situations.

Below we collect some estimates in the Strichartz space \( V(I) \), \( W(I) \) and \( W(I) \), which will be used frequently throughout this paper. We have

\[
(3.9) \quad \|u\|_{V(I)}^{\frac{1}{2}} \leq \|u\|_{V(I)}^{\frac{1}{2}},
\]

\[
(3.10) \quad \|u\|_{W(I)}^{\frac{1}{2}} \leq \|u\|_{W(I)}^{\frac{1}{2}},
\]

and if \( 3 \leq d \leq 6 \),

\[
(3.11) \quad \|u\|_{W(I)}^{\frac{6-d}{2}} \leq \|u\|_{W(I)}^{\frac{6-d}{2}}.
\]

Estimates (3.9)-(3.11) follow from Hölder’s inequality and the Sobolev embedding

\[
(3.12) \quad \|u\|_{W(I)}^{\frac{2d+2}{2d}} \leq \|u\|_{W(I)}^{\frac{2d+2}{2d}}.
\]

### 3.3. Exotic Strichartz estimates.

The exotic Strichartz spaces are introduced primarily to treat the non-Lipschitzness of the derivatives of nonlinearity for dimensions larger than six. Actually, for \( F(u) = |u|^\frac{4}{d-2} u, u \in \mathbb{C} \), we have (see [73, (1.3), (1.4)])

\[
(3.13) \quad |F_x(u)| + |F_{xx}(u)| \leq C|u|^\frac{4}{d-2},
\]

\[
(3.14) \quad |F_x(u) - F_x(v)| + |F_{xx}(u) - F_{xx}(v)| \leq \begin{cases} C|u - v|^\frac{4}{d-2}, & \text{if } d > 6; \\ C|u - v|(|u|^{\frac{6-d}{2}} + |v|^{\frac{6-d}{2}}), & \text{if } 3 \leq d \leq 6. \end{cases}
\]

The space \( X(0, \tau) \) allows to take \( \frac{4}{d-2} \)-derivatives of the nonlinearity, instead of taking the full derivative. Below we recall some important estimates in the exotic Strichartz spaces when \( d \geq 3 \), which are mainly proved in [52] in the homogeneous case. The arguments there apply also the inhomogeneous case considered here.
Lemma 3.6. For any compact time interval $I \subseteq \mathbb{R}^+$,

\begin{equation}
\|u\|_{X_0(I)} \leq C\|u\|_{X(I)} \leq C\|u\|_{L^{\frac{d(d+2)}{2d-2}}(I;W^{1,p_1})},
\end{equation}

\begin{equation}
\|u\|_{X(I)} \leq C\|u\|_{L^{\frac{d+2}{d}}(I;\mathbb{R}^d)} \leq C\|u\|_{L^{\frac{2d}{d-2}(d+1)(d+2)}(I;W^{1,p_2})},
\end{equation}

\begin{equation}
\leq C\|u\|_{L^{\frac{d+2}{d-2}}(I;\mathbb{R}^d)} \leq C\|u\|_{L^{\frac{2d}{d-2}(d+1)(d+2)}(I;W^{1,p_2})},
\end{equation}

where $p_1, p_2$ are such that $(p_1, \frac{d(d+2)}{2(d-2)})$ and $(p_2, \frac{2d}{d-2} \times \frac{2d+1}{3d+8})$ are Strichartz pairs.

Moreover, for some $0 < c \leq 1$,

\begin{equation}
\|u\|_{L^{\frac{2(d+2)}{d}}(I;\mathbb{R}^d)} \leq \|u\|_{X(I)} \leq C\|u\|_{L^{\frac{2d}{d-2}(d+1)(d+2)}(I;W^{1,p})},
\end{equation}

where the Strichartz pair $(p, q) = (2, \infty)$ if $d = 3$ and $(p, q) = (\frac{2d^2}{d^2 - 2d + 4}, \frac{2d}{d-2})$ if $d \geq 4$.

The proof is similar to that of [52, Lemma 3.11].

Lemma 3.7. Let $I = [t_0, T]$ be any compact interval in $\mathbb{R}^+$. We have

\begin{equation}
\|e^{-i(-t_0)\Delta}u_0\|_{X(I)} \leq C\|u_0\|_{H^1},
\end{equation}

\begin{equation}
\left\| \int_{t_0}^{\cdot} e^{-i(-t)\Delta}f(s)ds \right\|_{X(I)} \leq C\|f\|_{\dot{Y}(I)} + L^2(I;H^{\frac{1}{2}}_N + N^1(I)) .
\end{equation}

Proof. Estimate (3.18) follows from (3.15) and the homogenous Strichartz estimates. For (3.19), similar arguments as in the proof of [52, Lemma 3.10] yield

\begin{equation}
\left\| \int_{t_0}^{\cdot} e^{-i(-t)\Delta}f(s)ds \right\|_{X(I)} \leq C\|f\|_{\dot{Y}(I)} .
\end{equation}

Moreover, using (3.15) and Strichartz estimates we have

\begin{equation}
\left\| \int_{t_0}^{\cdot} e^{-i(-t)\Delta}f(s)ds \right\|_{X(I)} \leq C\left\| \int_{t_0}^{\cdot} e^{-i(-t)\Delta}f(s)ds \right\|_{S^1(I)} \leq C\|f\|_{L^2(I;H^{\frac{1}{2}}_N + N^1(I))} .
\end{equation}

Combining the estimates above together we prove (3.19). □

Lemma 3.8 below can be proved similarly as in [52, Lemma 3.12].

Lemma 3.8. For any compact time interval $I \subseteq \mathbb{R}^+$, we have

\begin{equation}
\|F(u)\|_{\dot{Y}(I)} \leq C\|u\|_{L^{\frac{d+2}{d-2}}(I;X(I))} .
\end{equation}

Moreover,

\begin{equation}
\|F_z(u + v)w\|_{\dot{Y}(I)} \leq C\|u\|_{L^{\frac{d}{d+2}}(I;X(I))} \|u\|_{L^{\frac{d}{d+2}}(I;X(I))} \|v\|_{L^\infty(I;W^{1,p})} + \|v\|_{L^{\frac{d}{d+2}}(I;X(I))} \|v\|_{L^\infty(I;W^{1,p})} \|w\|_{X(I)} ,
\end{equation}

where $(p, q)$ is as in (3.15), and similar estimate also holds for $\|F_z(u + v)w\|_{\dot{Y}(I)}$.

4. Stability

This section is devoted to the stability results in the mass- and energy-critical cases, which are crucial in the proof of global well-posedness in the next section.

To begin with, let us start with the easier mass-critical case.
4.1. Mass-critical case. The main result of this subsection is formulated below.

**Theorem 4.1. (Mass-Critical Stability Result)** Fix $I = [t_0, T] \subseteq \mathbb{R}^+$. Let $v$ be the solution to

$$i\partial_t v = e^{-\Phi} \Delta (e^\Phi v) - e^{\frac{2}{d} \text{Re} \Phi} F(v),$$

where $\Phi$ satisfies (3.3), $d \geq 1$, and $\tilde{v}$ solve the perturbed equation

$$i\partial_t \tilde{v} = e^{-\Phi} \Delta (e^\Phi \tilde{v}) - e^{\frac{2}{d} \text{Re} \Phi} F(\tilde{v}) + \epsilon$$

for some function $\epsilon$. Assume that

$$\left\| \tilde{v} \right\|_{C(I; L^2)} \leq M, \quad \left\| v(t_0) - \tilde{v}(t_0) \right\|_2 \leq M', \quad \left\| \tilde{v} \right\|_{V(I)} \leq L$$

for some positive constants $M, M'$ and $L$. Assume also the smallness conditions

$$\left\| U(\cdot, t_0)(v(t_0) - \tilde{v}(t_0)) \right\|_{V(I)} \leq \varepsilon, \quad \left\| \epsilon \right\|_{L^2(I; H^{-\frac{1}{2}}_{-1} + N^0(I))} \leq \varepsilon$$

for some $0 < \varepsilon \leq \varepsilon_*$, where $\varepsilon_* = \varepsilon_*(C_T, D_T, M, M', L) > 0$ is a small constant, $C_T$ is the Strichartz constant in Theorem 3.3, $D_T = \| e^{\frac{2}{d} \text{Re} \Phi} \|_{C(I; L^\infty)}$. Then,

$$\left\| v - \tilde{v} \right\|_{V(I)} \leq C(C_T, D_T, M, M', L)\varepsilon,$$

$$\left\| v - \tilde{v} \right\|_{S^0(I) \cap L^2(I; H^{-\frac{1}{2}}_{-1})} \leq C(C_T, D_T, M, M', L)M',$$

$$\left\| v \right\|_{S^0(I) \cap L^2(I; H^{-\frac{1}{2}}_{-1})} \leq C(C_T, D_T, M, M', L).$$

We can take $\varepsilon_*(C_T, D_T, M, M', L)$ (resp. $C(C_T, D_T, M, M', L)$) to be decreasing (resp. nondecreasing) with respect to each argument.

Theorem 4.1 means that, the two solutions will stay close to each other, if the difference between two initial data and the error term are small enough in appropriate spaces.

**Remark 4.2.** In view of Theorem 3.5, Theorem 4.1 also holds if $\Phi$ is replaced by $\varphi_*$ given by (2.16). In this case, since the Strichartz constants are independent of time and $\varphi_* \in L^\infty(\mathbb{R}^+; L^\infty)$, the constants in (4.5)-(4.7) are independent of time, i.e., depend only $M', M$ and $L$. This fact will be used in the proof of scattering in Section 6 below.

In order to prove Theorem 4.1, we first prove the short-time perturbation result below.

**Proposition 4.3. (Mass-Critical Short-time Perturbation).** Let $I = [t_0, T] \subseteq \mathbb{R}^+$ and $v, \tilde{v}$ be the solutions to equations (4.1) and (4.2), respectively. Assume that

$$\left\| \tilde{v} \right\|_{C(I; L^2)} \leq M, \quad \left\| v(t_0) - \tilde{v}(t_0) \right\|_2 \leq M'$$

for some positive constants $M, M'$. Assume also the smallness conditions

$$\left\| \tilde{v} \right\|_{V(I)} \leq \delta, \quad \left\| U(\cdot, t_0)(v(t_0) - \tilde{v}(t_0)) \right\|_{V(I)} \leq \varepsilon, \quad \left\| \epsilon \right\|_{L^2(I; H^{-\frac{1}{2}}_{-1} + N^0(I))} \leq \varepsilon$$

for some $0 < \varepsilon \leq \delta$ where $\delta = \delta(C_T, D_T, M, M') > 0$ is a small constant, and $C_T, D_T$ are as in Theorem 4.1. Then, we have

$$\left\| v - \tilde{v} \right\|_{V(I)} \leq C(C_T, D_T)\varepsilon,$$

$$\left\| v - \tilde{v} \right\|_{S^0(I) \cap L^2(I; H^{-\frac{1}{2}}_{-1})} \leq C(C_T, D_T)M',$$

$$\left\| v \right\|_{S^0(I) \cap L^2(I; H^{-\frac{1}{2}}_{-1})} \leq C(C_T, D_T)(M + M'),$$

$$\left\| e^{\frac{2}{d} \text{Re} \Phi} (F(v) - F(\tilde{v})) \right\|_{N^0(I)} \leq C(C_T, D_T)\varepsilon.$$
Thus, in view of Lemma 4.3. Divide \([t_0,T]\) into finitely many small pieces \(I_j = [t_j, t_{j+1}], 0 \leq j \leq l, \) such that \(t_{l+1} = T, \|\tilde{v}\|_{V(t_j, t_{j+1})} = \delta, 0 \leq j \leq l - 1, \) and \(\|\tilde{v}\|_{V(t_l, t_{l+1})} \leq \delta. \) Then, \(l \leq (L/\delta)^{2+\frac{4}{d}} < \infty. \)
Let $C(0) = C(C_T, D_T)$, $C(j + 1) = \max\{C(0)C_T^2(\sum_{k=0}^{j} C(j) + 2), (0)(1 + 2C_T)\}$, $0 \leq j \leq l - 1$, where $C(C_T, D_T)$ is the constant in (4.10)-(4.13). Choose $\varepsilon_\ast = \varepsilon_\ast(C_T, D_T, M, M', L)$ sufficiently small such that

\[
(4.22) \quad \left(\sum_{k=0}^{l} C(k) + 1\right)\varepsilon_\ast \leq M', \quad C_T^2\left(\sum_{k=0}^{l} C(k) + 2\right)\varepsilon_\ast \leq \delta.
\]

Below we use inductive arguments to prove that for any $0 \leq j \leq l$,

\[
(4.23) \quad \|v - \bar{\nu}\|_{V(t_j)} \leq C(j)\varepsilon,
\]

\[
(4.24) \quad \|v - \bar{\nu}\|_{S^0(t_j) \cap L^2(t_j; H^j_x)} \leq C(j)M',
\]

\[
(4.25) \quad \|v\|_{S^0(t_j) \cap L^2(t_j; H^j_x)} \leq C(j)(M + M'),
\]

\[
(4.26) \quad \|e^{\frac{t}{2}\Re \Phi} (F(v) - F(\bar{\nu}))\|_{N^0(t_j)} \leq C(j)\varepsilon.
\]

Proposition 4.3 yields that the estimates above hold for $j = 0$. Suppose that (4.23)-(4.26) are also valid for each $0 \leq k \leq j < l$. We shall apply Proposition 4.3 to show that they also hold for the case where $j + 1$ replaces $j$.

For this purpose, by Theorem 3.3, (4.3) and the inductive assumptions

\[
|v(t_{j+1}) - \bar{\nu}(t_{j+1})|_2 \leq C_T(\|v(t_0) - \bar{\nu}(t_0)\|_2 + S(t_0, t_{j+1}) + \|v\|_{L^2(t_0, t_{j+1}; H^{0}_{x})})
\]

\[
\leq C_T(M' + \sum_{k=0}^{j} C(k)\varepsilon + \varepsilon) \leq 2C_TM',
\]

where $S(t_0, t_{j+1})$ is as in the proof of Proposition 4.3, the last step is due to (4.22).

Moreover, by equation (4.15),

\[
(v - \bar{\nu})(t_{j+1}) = U(t_{j+1}, t_0)(v - \bar{\nu})(t_0) + \int_{t_0}^{t_{j+1}} U(t_{j+1}, s)(ie^{\frac{s}{2}\Re \Phi} (F(v) - F(\bar{\nu})) + ie)ds.
\]

This yields that

\[
U(t, t_{j+1})(v - \bar{\nu})(t_{j+1}) = U(t, t_0)(v - \bar{\nu})(t_0)
\]

\[
+ U(t, t_{j+1}) \int_{t_0}^{t_{j+1}} U(t_{j+1}, s)(ie^{\frac{s}{2}\Re \Phi} (F(v) - F(\bar{\nu})) + ie)ds.
\]

Then, applying Theorem 3.3 to (4.14) again we have

\[
\|U(\cdot, t_{j+1})(v(t_{j+1}) - \bar{\nu}(t_{j+1}))\|_{V(t_{j+1})}
\]

\[
\leq \|U(\cdot, t_0)(v - \bar{\nu})(t_0)\|_{V(t)}
\]

\[
+ C_T\left(\int_{t_0}^{t_{j+1}} U(\cdot, s)(ie^{\frac{s}{2}\Re \Phi} (F(v) - F(\bar{\nu})) + ie)ds\right)\bigg|_{C([t_0, t_{j+1}]; L^2)}
\]

\[
\leq \varepsilon + C_T^{2}\left(\|e^{\frac{t}{2}\Re \Phi} (F(v) - F(\bar{\nu}))\|_{N^0(t_0, t_{j+1})} + \|e\|_{N^0(t_0, t_{j+1}) + L^2(t_0, t_{j+1}; H^{0}_{x})}\right)
\]

\[
\leq \varepsilon + C_T^{2}\left(\sum_{k=0}^{j} C(k)\varepsilon + \varepsilon\right) \leq \delta,
\]

where the last step is again due to (4.22).

Thus, the conditions (4.8) and (4.9) of Proposition 4.3 are satisfied with $2C_TM'$ and $C_T^2\left(\sum_{k=0}^{j} C(k) + 2\varepsilon\right)$ replacing $M'$ and $\varepsilon$, respectively. Then, Proposition 4.3 yields that estimates (4.23)-(4.26) are valid with $j + 1$ replacing $j$. 

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Therefore, inductive arguments yield that (4.23)-(4.26) hold for all \(0 \leq j \leq l\), thereby proving Theorem 4.1. The proof is complete. \qed

4.2. Energy-critical case. The main results of this subsection are Theorems 4.4 and 4.6 below. The delicate problem here is that the derivatives of the nonlinearity in (1.1) are Lipschitz when \(3 \leq d \leq 6\), however, they are merely Hölder continuous in high dimensions when \(d > 6\). In the latter case, more delicate arguments involving the exotic Strichartz spaces as well as local smoothing spaces will be used.

To begin with, we start with the easier case when \(3 \leq d \leq 6\).

4.2.1. The case when \(3 \leq d \leq 6\). In this case, the stability result is quite similar to the previous mass-critical case.

**Theorem 4.4.** (Energy-critical Stability Result when \(3 \leq d \leq 6\)) Consider any closed interval \(I = [t_0, T] \subseteq \mathbb{R}^+\). Let \(w\) be the solution to

\[
i\partial_t w - e^{-\Phi} \Delta (e^\Phi w) - e^{-\frac{1}{2}} Re F(w)
\]

with \(\Phi\) satisfying (3.3), \(3 \leq d \leq 6\), and \(\tilde{w}\) solve the perturbed equation

\[
i\partial_t \tilde{w} - e^{-\Phi} \Delta (e^\Phi \tilde{w}) - e^{-\frac{1}{2}} Re F(\tilde{w}) + e
\]

for some function \(e\). Assume that

\[
\|\tilde{w}\|_{C(I; H^1)} \leq E, \quad \|w(t_0) - \tilde{w}(t_0)\|_{H^1} \leq E', \quad \|\tilde{w}\|_{W(I)} \leq L
\]

for some positive constants \(E, E'\) and \(L\). Assume also the smallness conditions

\[
\|U(\cdot, t_0)(w(t_0) - \tilde{w}(t_0))\|_{W(I)} \leq \varepsilon, \quad \|e\|_{N^1(I) + L^2(I; H^\frac{1}{2})} \leq \varepsilon
\]

for some \(0 < \varepsilon \leq \varepsilon_*\), where \(\varepsilon_* = \varepsilon_*(C_T, D'_T, E, E', L) > 0\) is a small constant, \(C_T\) is the Strichartz constant in Theorem 3.3 and \(D'_T = \|e^{-\frac{1}{2}} Re \Phi\|_{C(I; W^{1, \infty})}\). Then,

\[
\|w - \tilde{w}\|_{W(I)} \leq C(C_T, D'_T, E, E', L)\varepsilon,
\]

\[
\|w - \tilde{w}\|_{S^1(I) \cap L^2(I; H^\frac{3}{2})} \leq C(C_T, D'_T, E, E', L)E',
\]

\[
\|w\|_{S^1(I) \cap L^2(I; H^\frac{3}{2})} \leq C(C_T, D'_T, E, E', L).
\]

The constants \(\varepsilon_*(C_T, D'_T, E, E', L)\) and \(C(C_T, D'_T, E, E', L)\) can be taken to be decreasing and nondecreasing with respect to each argument, respectively.

As in the mass-critical case, Theorem 4.4 follows from the short-time perturbation result below.

**Proposition 4.5.** (Energy-Critical Short-time Perturbation when \(3 \leq d \leq 6\)) Let \(I = [t_0, T]\), \(w\) and \(\tilde{w}\) be as in Theorem 4.4, \(3 \leq d \leq 6\). Assume that

\[
\|\tilde{w}\|_{C(I; H^1)} \leq E, \quad \|w(t_0) - \tilde{w}(t_0)\|_{H^1} \leq E'
\]

for some positive constants \(E, E'\). Assume also the smallness conditions

\[
\|\tilde{w}\|_{W(I)} \leq \delta, \quad \|U(\cdot, t_0)(w(t_0) - \tilde{w}(t_0))\|_{W(I)} \leq \varepsilon, \quad \|e\|_{N^1(I) + L^2(I; H^\frac{1}{2})} \leq \varepsilon
\]
for some $0 < \varepsilon \leq \delta$, where $\delta = \delta(C_T, D'_T, E, E') > 0$ is a small constant, and $C_T, D'_T$ are as in Theorem 4.4. Then, we have

\begin{align}
(4.36) &\quad \|w - \tilde{w}\|_{W(I)} \leq C(C_T, D'_T)\varepsilon, \\
(4.37) &\quad \|w - \tilde{w}\|_{S^1(I) \cap L^2(I; H^\frac{3}{2}_x)} \leq C(C_T, D'_T)E', \\
(4.38) &\quad \|w\|_{S^1(I) \cap L^2(I; H^\frac{3}{2}_x)} \leq C(C_T, D'_T)(E + E'), \\
(4.39) &\quad \|e^{\frac{4}{\sigma-2}\Re \Phi}(F(w) - F(\tilde{w}))\|_{N^1(I)} \leq C(C_T, D'_T)\varepsilon.
\end{align}

**Proof.** Set $z := w - \tilde{w}$ and $S(I) := \|e^{\frac{4}{\sigma-2}\Re \Phi}(F(\tilde{w} + z) - F(\tilde{w}))\|_{N^1(I)}$. Then,

\[ S(I) \leq D'_T\|F(z + \tilde{w}) - F(\tilde{w})\|_{L^2(I; L^\frac{4+2}{2\sigma-2})} + \|\nabla(F(z + \tilde{w}) - F(\tilde{w}))\|_{L^2(I; L^\frac{4+2}{\sigma-2})}. \]

By (3.10), (3.13) and (4.35),

\[ \|F(z + \tilde{w}) - F(\tilde{w})\|_{L^2(I; L^\frac{4+2}{2\sigma-2})} \leq C\|\frac{z}{w(I)}\|_{L^\frac{4}{2\sigma-2}} + \|\tilde{w}\|_{L^\frac{4}{2\sigma-2}}\|z\|_{W(I)} \leq C\delta\frac{4+2}{\sigma-2}\|z\|_{W(I)} + C\|z\|_{L^\frac{4+2}{2\sigma-2}(I)}. \]

Moreover, since by (3.14) we have (see, e.g., [47, (3.20)])

\[ \|\nabla(F(z + \tilde{w}) - F(\tilde{w}))\| \leq C(\|\nabla \tilde{w}\|_{L^\frac{4}{2\sigma-2}} + \|\tilde{w}\|_{L^\frac{4}{2\sigma-2}}\|\nabla z\| + \|\nabla \tilde{w}\||\tilde{w}|_{L^\frac{4+2}{\sigma-2}}). \]

Taking into account (3.10), (3.11) and (4.35) we get

\[ \|\nabla(F(z + \tilde{w}) - F(\tilde{w}))\|_{L^2(I; L^\frac{4+2}{2\sigma-2})} \leq C(\|\tilde{w}\|_{W(I)}\|\frac{z}{w(I)}\|_{L^\frac{4}{2\sigma-2}} + \|\tilde{w}\|_{L^\frac{4}{2\sigma-2}}\|z\|_{W(I)} + \|z\|_{L^\frac{4+2}{2\sigma-2}(I)}). \]

Thus, combining the estimates above together we obtain

\[ S(I) \leq CD'_T\left(\delta\frac{4+2}{\sigma-2}\|z\|_{W(I)} + \delta\frac{4}{\sigma-2}\|z\|_{\frac{4+2}{\sigma-2}}(I)\right). \]

Since $1 \leq \frac{4}{\sigma-2} \leq \frac{4+2}{\sigma-2}$ when $3 \leq d \leq 6, \|z\|_{L^\frac{4}{2\sigma-2}}(I) \leq \|z\|_{W(I)} + \|z\|_{L^\frac{4+2}{\sigma-2}(I)}$, we come to

\[ (4.40) \quad S(I) \leq 2CD'_T(\delta\|z\|_{W(I)} + \|z\|_{L^\frac{4+2}{2\sigma-2}(I)}). \]

Moreover, similarly to (4.15), we have

\[ (4.41) \quad z(t) = U(t, 0)z(t_0) + \int_{t_0}^t U(t, s)(ie^{\frac{4}{\sigma-2}\Re \Phi}(F(z + \tilde{w}) - F(\tilde{w})) + ie)ds. \]

Applying Theorem 3.3 and using (4.35) we have

\[ \|z\|_{W(I)} \leq C_T(\|U(\cdot, t_0)z(t_0)\|_{W(I)} + S(I) + \|e^{\frac{4}{\sigma-2}\Re \Phi}(F(z + \tilde{w}) - F(\tilde{w})) + ie\|_{L^2(I; L^\frac{1}{\sigma-1})} + N^1(I)) \leq C_T(2\varepsilon + S(I)). \]

Thus, plugging (4.40) into the estimate above we obtain

\[ \|z\|_{W(I)} \leq 2C_T(\varepsilon + CD'_T\delta\|z\|_{W(I)} + CD'_T\|z\|_{L^\frac{4+2}{2\sigma-2}(I)}). \]

Taking $\delta = \delta(C_T, D'_T)$ small enough such that $2CD'_T\delta \leq \frac{1}{2}$ we come to

\[ \|z\|_{W(I)} \leq 4C_T\varepsilon + 4C_TD'_T\|z\|_{L^\frac{4+2}{2\sigma-2}(I)}. \]

Then, by virtue of [8, Lemma A.1], taking $\delta = \delta(C_T, D'_T)$ smaller such that $4C_T\delta < (1 - \frac{1}{\alpha})(4\alpha C_T D'_T)^{-\frac{1}{\alpha-1}}$ with $\alpha = 1 + \frac{4}{\sigma-2}$ we obtain

\[ (4.42) \quad \|z\|_{W(I)} \leq \frac{4\alpha}{\alpha - 1} C_T\varepsilon, \]
which together with (4.40) implies (4.36) and (4.39).

For (4.37), applying Theorem 3.3 to (4.41) and using (4.35), (4.39) we have

$$
\|z\|_{S^1(I) \cap L^2(\mathbb{T}^2)} \leq C_T(\|z(t_0)\|_{H^1} + S(I) + \|e\|_{N^1(I) + L^2(\mathbb{T}^2)} ) \\
\leq C_T(E' + C(C_T, D_T') \varepsilon + \varepsilon),
$$

(4.43)

which implies (4.37), provided $$C(C_T, D_T') \delta + \delta \leq E'.$$

Similarly, by (4.28),

$$
\|\tilde{w}\|_{S^1(I) \cap L^2(\mathbb{T}^2)} \leq C_T(\|\tilde{w}(t_0)\|_{H^1} + \|e^{\frac{4}{d+2}} \text{Re} \Phi(\tilde{w})\|_{N^1(I)} + \|e\|_{N^1(I) + L^2(\mathbb{T}^2)} ) \\
\leq C_T(\|\tilde{w}(t_0)\|_{H^1} + D_{T'} \|\tilde{w}\|_{W(\mathbb{T}^d)}^{\frac{d+2}{d}} + \|e\|_{N^1(I) + L^2(\mathbb{T}^2)} ) \\
\leq C_T(E + D_{T'} \delta^{\frac{d+2}{d}} + \varepsilon) \leq 2C_T E,
$$

(4.44)

if we take $$\delta$$ even smaller such that $$D_{T'} \delta^{\frac{d+2}{d}} + \delta \leq E.$$

Therefore, we obtain (4.38) from (4.37) and (4.44) and so finish the proof. □

Once Proposition 4.5 obtained, we can use the partition arguments as in the proof of Theorem 4.1 to prove Theorem 4.4. The details are omitted for simplicity.

4.2.2. The case when $$d > 6.$$

**Theorem 4.6.** (Energy-Critical Stability Result when $$d > 6$$) Consider any closed interval $$I = [t_0, T] \subseteq \mathbb{R}^+.$$ Let $$w, \tilde{w}$$ solve the equations (4.27) and (4.28), respectively, and $$\Phi$$ satisfy (3.3), $$d > 6.$$ Assume that,

$$
\|\tilde{w}\|_{C(I; H^1)} \leq E, \quad \|\tilde{w}\|_{W(I) \cap L^2(\mathbb{T}^2)} \leq L
$$

(4.45)

for some positive constants $$E$$ and $$L.$$ Assume also the smallness conditions

$$
\|g\|_{C(I; \mathbb{R}^+)} \leq \varepsilon, \quad |w(t_0) - \tilde{w}(t_0)|_{H^1} \leq \varepsilon, \quad \|e\|_{N^1(I) + L^2(\mathbb{T}^2)} \leq \varepsilon
$$

(4.46)

for some $$0 < \varepsilon \leq \varepsilon_*,$$ where $$g$$ is the time function as in (3.3), $$\varepsilon_* = \varepsilon_*(C_T, D_T', E, L) > 0$$ is a small constant, $$C_T, D_T'$$ are as in Theorem 4.4. Then, for some $$\varepsilon = \varepsilon(C_T, D_T', E, L) > 0,$$

$$
\|w - \tilde{w}\|_{L^2(\mathbb{T}^d)}^{\frac{2(d+2)}{d}} (I \times \mathbb{R}^d) \leq C(C_T, D_T', E, L) \varepsilon^c,
$$

(4.47)

$$
\|w - \tilde{w}\|_{S^1(I) \cap L^2(\mathbb{T}^2)} \leq C(C_T, D_T', E, L) \varepsilon^c,
$$

(4.48)

$$
\|w\|_{S^1(I) \cap L^2(\mathbb{T}^2)} \leq C(C_T, D_T', E, L).
$$

(4.49)

We can take the constants $$\varepsilon_*(C_T, D_T', E, L)$$ and $$C(C_T, D_T', E, L)$$ to be decreasing and nondecreasing with respect to each argument, respectively.

**Remark 4.7.** Unlike in the case $$3 \leq d \leq 6,$$ the smallness condition (4.46) is mainly to control the lower order perturbations (see (4.70) below).

We first prove the short-time perturbation result below.

**Proposition 4.8.** (Energy-Critical Short-time Perturbations when $$d > 6$$) Let $$I = [t_0, T], w, \tilde{w}$$ and $$g$$ be as in Theorem 4.6, $$d > 6.$$ Assume that

$$
\|\tilde{w}\|_{C(I; H^1)} \leq E
$$

(4.50)
for some positive constant $E$. Assume also the smallness conditions
\[(4.51) \quad \| \tilde{w} \|_{W(I) \cap L^2(I; H^{\frac{3}{2}}_1)} \leq \delta, \]
\[(4.52) \quad \| g \|_{C([t_0, T]; \mathbb{R}^+)} \leq \varepsilon, \quad \| w(t_0) - \tilde{w}(t_0) \|_{H^1} \leq \varepsilon, \quad \| e \|_{L^2(I; H^{1}_{L_2}) + N^1(I)} \leq \varepsilon \]
for some $0 < \varepsilon \leq \delta$, where $\delta = \delta(C_T, D'_T, E) > 0$ is a small constant and $C_T, D'_T$ are as in Theorem 4.6. Then, we have
\[(4.53) \quad \| w - \tilde{w} \|_{X(I)} \leq C(C_T, D'_T, E)\varepsilon, \]
\[(4.54) \quad \| w - \tilde{w} \|_{S^1(I) \cap L^2(I; H^{\frac{3}{2}}_1)} \leq C(C_T, D'_T, E)\varepsilon^\frac{4}{d-2}, \]
\[(4.55) \quad \| w \|_{S^1(I) \cap L^2(I; H^{\frac{3}{2}}_1)} \leq C(C_T, D'_T, E), \]
\[(4.56) \quad \| F(w) - F(\tilde{w}) \|_{Y(I)} \leq C(C_T, D'_T, E)\varepsilon, \]
\[(4.57) \quad \| F(w) - F(\tilde{w}) \|_{N^1(I)} \leq C(C_T, D'_T, E)\varepsilon^\frac{1}{d-2}. \]
where $X(I), Y(I)$ are exotic Strichartz spaces defined in Section 1.

**Remark 4.9.** We mention that, the exotic Strichartz space $X(I)$ and $Y(I)$ are used to deal with the non-Lipschitzness of the derivatives of nonlinearity when $d > 6$. Moreover, the local smoothing spaces are applied primarily to treat the lower order perturbations of the Laplacian arising in the operator $e^{-\Phi} \Delta (e^{\Phi})$.

In order to prove Proposition 4.8, we first prove Lemma 4.10 below.

**Lemma 4.10.** Consider the situations in Proposition 4.8. Then, for $\delta = \delta(C_T, D'_T, E)$ small enough,
\[(4.58) \quad \| \tilde{w} \|_{S^1(I) \cap L^2(I; H^{\frac{3}{2}}_1)} \leq C(C_T, D'_T, E), \]
\[(4.59) \quad \| w \|_{W(I) \cap L^2(I; H^{\frac{3}{2}}_1)} \leq C(C_T, D'_T, E)\delta, \]
\[(4.60) \quad \| w \|_{X(I)} \leq C(C_T, D'_T, E)\delta^\frac{1}{d-2}. \]

**Remark 4.11.** Unlike in [52, 72], because of the lower order perturbations in (4.27), it is more delicate here to derive the smallness bound (4.60) of $w$ in the exotic Strichartz space $X(I)$. Below we first prove (4.59) in the local smoothing space $L^2(I; H^{\frac{3}{2}}_1)$, with which we are able to control the lower order perturbations and then obtain the estimate (4.60).

**Proof.** We first prove (4.58). Applying Theorem 3.3 to (4.28) and using (3.10), (4.50) and (4.51) we have
\[(4.61) \quad \| \tilde{w} \|_{S^1(I)} \leq C_T(\| \tilde{w}(t_0) \|_{H^1} + \| e^{\frac{1}{d+2}} Re F(\tilde{w}) \|_{L^2(I; W^{\frac{d+2}{d-2}}_1)} + \| e \|_{N^1(I) + L^2(I; H^{\frac{1}{2}}_1)}) \leq C_T(E + CD'_T \| \tilde{w} \|_{W(I)}^\frac{d+2}{d-2} + \varepsilon) \leq C_T(E + CD'_T \delta^\frac{d+2}{d-2} + \varepsilon), \]
which yields (4.58) if $\delta = \delta(D'_T, E)$ is small enough such that $CD'_T \delta^\frac{d+2}{d-2} + \delta \leq E$.

In order to prove (4.59), again applying Theorem 3.3 to (4.28) and using the Hölder inequality (3.10) and (4.51) we have
\[\| U(\cdot, t_0) \tilde{w}(t_0) \|_{W(I) \cap L^2(I; H^{\frac{3}{2}}_1)} \leq \| \tilde{w} \|_{W(I) \cap L^2(I; H^{\frac{3}{2}}_1)} + CC_T D'_T \| \tilde{w} \|_{W(I)}^\frac{d+2}{d-2} + CT \| e \|_{N^1(I) + L^2(I; H^{\frac{3}{2}}_1)} \]
\[\leq \delta + CC_T D'_T \delta^\frac{d+2}{d-2} + CT \varepsilon. \]
Moreover, by the homogeneous Strichartz estimates and (4.52),
\[ \|U(\cdot, t_0)(w(t_0) - \tilde{w}(t_0))\|_{\mathcal{W}(I) \cap L^2(I; H^{\frac{3}{2}})} \leq C|w(t_0) - \tilde{w}(t_0)|_{H^1} \leq C\epsilon. \]

Thus, we obtain
\[ (4.62) \quad \|U(\cdot, t_0)w(t_0)\|_{\mathcal{W}(I) \cap L^2(I; H^{\frac{3}{2}})} \leq C_1(C_T, D_T')\delta. \]

Arguing as above and using (4.62) we deduce from equation (4.27) that
\[ \|w\|_{\mathcal{W}(I) \cap L^2(I; H^{\frac{3}{2}})} \leq \|U(\cdot, t_0)w(t_0)\|_{\mathcal{W}(I) \cap L^2(I; H^{\frac{3}{2}})} + C C_T D'_T \|w\|_{\mathcal{W}(I)}^{\frac{\frac{3}{2}}{2}} + \|C C_T D'_T \|w\|_{\mathcal{W}(I)}^{\frac{\frac{3}{2}}{2}}. \]

Then, in view of [8, Lemma A.1], taking \( \delta = \delta(C_T, D_T') \) smaller such that \( C_1(C_T, D_T') \delta < (1 - \frac{1}{\alpha})(\alpha C C_T D'_T)^{-\frac{\alpha}{\alpha - 1}} \), we obtain (4.59).

It remains to prove (4.60). For this purpose, we see from (4.28) that
\[ i\partial_t \tilde{w} = \Delta \tilde{w} + (b \cdot \nabla + c)\tilde{w} - e^{\frac{4}{\alpha - 2}} \text{Re} \Phi F(\tilde{w}) + e, \]
where \( b = 2\nabla \Phi \) and \( c = \Delta \Phi + \sum_{j=1}^{d} (\partial_j \Phi)^2 \). This yields that
\[ \|e^{-i(-t_0)\Delta} \tilde{w}(t_0)\|_{\mathcal{X}(I)} \leq \|\tilde{w}\|_{\mathcal{X}(I)} + \| \int_{0}^{t} e^{-i(-\cdot)\Delta} (b \cdot \nabla + c)\tilde{w}ds\|_{\mathcal{X}(I)} \]
\[ + \| \int_{0}^{t} e^{-i(-\cdot)\Delta} e^{\frac{4}{\alpha - 2}} \text{Re} \Phi F(w)ds\|_{\mathcal{X}(I)} + \| \int_{0}^{t} e^{-i(-\cdot)\Delta} e(s)ds\|_{\mathcal{X}(I)} =: K_0 + K_1 + K_2 + K_3. \]

Note that, by (3.16),
\[ (4.64) \quad K_0 \leq C\|\tilde{w}\|_{\mathcal{W}(I)}^{\frac{1}{2}}\|\tilde{w}\|_{S^1(I)}^{\frac{\frac{3}{2}}{2}}. \]

Moreover, (3.19) yields that
\[ K_1 \leq C\|(b \cdot \nabla + c)\tilde{w}\|_{L^2(I; H^{\frac{1}{2}})}. \]

We see that, \( \|(b \cdot \nabla + c)\tilde{w}\|_{H^{1\frac{3}{2}}} = |\Psi_q \langle x \rangle^{-\frac{3}{2}} (\nabla)\tilde{w}|_2 \), where \( \Psi_q := \langle x \rangle \langle \nabla \rangle \tilde{w} + c \langle \nabla \rangle + c \langle x \rangle \in S^0 \). Then, using Lemma 3.2 and (4.52) we have for some \( t \geq 1 \),
\[ \|(b \cdot \nabla + c)\tilde{w}\|_{H^{1\frac{3}{2}}} \leq C\sup_{t \in \mathcal{I}} \|(ib(t) \cdot \xi + c(t))\|_{S^1(I)} \langle x \rangle^{-\frac{3}{2}} \langle \nabla \rangle \tilde{w} \leq C\delta\|\tilde{w}\|_{H^{\frac{3}{2}}} \]

This yields that
\[ (4.65) \quad K_1 \leq C\|(b \cdot \nabla + c)\tilde{w}\|_{L^2(I; H^{\frac{1}{2}})} \leq C\delta\|\tilde{w}\|_{L^2(I; H^{\frac{3}{2}})}. \]

We also deduce from (3.19) that
\[ K_2 \leq C\|e^{\frac{4}{\alpha - 2}} \text{Re} \Phi F(\tilde{w})\|_{\mathcal{Y}(I)}. \]

Using the interpolation arguments (see, e.g., [12]) we have
\[ \|e^{\frac{4}{\alpha - 2}} \text{Re} \Phi F(\tilde{w})\|_{\mathcal{Y}(I)} \leq C\|e^{\frac{4}{\alpha - 2}} \text{Re} \Phi \|_{C(I; W^{1, \infty})}\|F(\tilde{w})\|_{\mathcal{Y}(I)}, \]
which along with (3.16), (3.20) implies that
\[ (4.66) \quad K_2 \leq C D'_T\|\tilde{w}\|_{\mathcal{W}(I)}^{\frac{\frac{3}{2}}{2}} \|\tilde{w}\|_{S^1(I)}^{\frac{\frac{3}{2}}{2}}. \]
Regarding $K_3$, by (3.19),

$$K_3 \leq C\|e\|_{N^1(I) + L^2(I; H^{\frac{1}{2}})}.$$  

Thus, plugging (4.64)-(4.67) into (4.63) and using (4.51), (4.52) and (4.58) yield

$$\|e^{-i(-t_0)\Delta}w(t_0)\|_{\mathcal{X}(I)} \leq C\|\bar{w}\|_{L^2(I; W^{\frac{1}{2}})} + C\delta\|\bar{w}\|_{L^2(I; H^{\frac{1}{2}})} + CD'_T\|\bar{w}\|_{W^{1,1}(I; W^{\frac{1}{2}})} + C\|e\|_{N^1(I) + L^2(I; H^{\frac{1}{2}})}$$

(4.68)

Moreover, by (3.18) and (4.52),

$$\|e^{-i(-t_0)\Delta}(w(t_0) - \bar{w}(t_0))\|_{\mathcal{X}(I)} \leq C\|w(t_0) - \bar{w}(t_0)\|_{H^1} \leq C\varepsilon.$$  

Thus, we obtain

$$\|e^{-i(-t_0)\Delta}w(t_0)\|_{\mathcal{X}(I)} \leq C_3(C_T, D'_T, E)\delta^{\frac{1}{2+\alpha}}.$$  

Now, similarly as above, we deduce from equation (4.27) that

$$\|w\|_{\mathcal{X}(I)} \leq \|e^{-i(-t_0)\Delta}w(t_0)\|_{\mathcal{X}(I)} + \| \int_0^t e^{-i(s-s')\Delta}(b \cdot \nabla + c)w(s)ds \|_{\mathcal{X}(I)}$$

$$+ \| \int_0^t e^{-i(s-s')\Delta}e^{\frac{4}{\alpha}}\text{Re} \Phi(F(w(s)))ds \|_{\mathcal{X}(I)}$$

$$\leq C_3(C_T, D'_T, E)\delta^{\frac{1}{2+\alpha}} + C\delta\|w\|_{L^2(I; H^{\frac{1}{2}})} + CD'_T\|w\|_{L^2(I; H^{\frac{1}{2}})}$$

$$\leq C_4(C_T, D'_T, E)\delta^{\frac{1}{2+\alpha}} + CD'_T\|w\|_{\mathcal{X}(I)},$$

where the last step is due to (4.59).

Therefore, taking $\delta = \delta(C_T, D'_T, E)$ even smaller such that $C_4(C_T, D'_T, E)\delta^{\frac{1}{2+\alpha}} < (1 - \frac{1}{\alpha}) (\alpha CD'_T)^{-\frac{1}{2+\alpha}}$ and using [8, Lemma A.1] we obtain (4.60). \hfill \Box

**Proof of Proposition 4.8.** We first estimate $\|w - \bar{w}\|_{\mathcal{X}(I)}$.

For this purpose, we note that $z := w - \bar{w}$ satisfies the equation

$$i\partial_t z = e^{-\Phi}\Delta(e^{\Phi} z) - e^{\frac{4}{\alpha}}\text{Re} \Phi(F(z + \bar{w}) - F(\bar{w})) - e$$

$$= \Delta z + (b \cdot \nabla + c)z - e^{\frac{4}{\alpha}}\text{Re} \Phi(F(z + \bar{w}) - F(\bar{w})) - e.$$  

(4.69)

This yields that

$$\|z\|_{\mathcal{X}(I)} \leq \|e^{-i(-t_0)\Delta}z(t_0)\|_{\mathcal{X}(I)} + \| \int_{t_0}^t e^{-i(s-s')\Delta}(b \cdot \nabla + c)zds \|_{\mathcal{X}(I)}$$

$$+ \| \int_{t_0}^t e^{-i(s-s')\Delta}e^{\frac{4}{\alpha}}\text{Re} \Phi(F(z + \bar{w}) - F(\bar{w}))ds \|_{\mathcal{X}(I)} + \| \int_{t_0}^t e^{-i(s-s')\Delta}e(s)ds \|_{\mathcal{X}(I)}$$

$$=: J_0 + J_1 + J_2 + J_3.$$  

First, Theorem 3.5, (3.15) and (4.52) yield that

$$J_0 \leq C\|e^{-i(-t_0)\Delta}z(t_0)\|_{S^1(I)} \leq C\|z(t_0)\|_{H^1} \leq C\varepsilon.$$  

Moreover, similarly to (4.65), using (4.51), (4.52) and (4.59) we have

$$J_1 \leq C\|(b \cdot \nabla + c)z\|_{L^2(I; H^{\frac{1}{2}})} \leq C\varepsilon\|z\|_{L^2(I; H^{\frac{3}{2}})} \leq C_1(C_T, D'_T, E)\varepsilon.$$  

(4.70)
We also use (3.19) and (3.21) to get that
\[
J_2 \leq C \| e^{\frac{\alpha}{2} t} \Re \Phi (F(z + \tilde{w}) - F(\tilde{w})) \|_{\mathcal{Y}(I)} \\
\leq C D_T' \| (F(z + \tilde{w}) - F(\tilde{w})) \|_{\mathcal{Y}(I)} \\
\leq C D_T' \| \tilde{w} \|_{\mathcal{F}(I)}^2 \| \tilde{w} \|_{\mathcal{S}(I)}^{\frac{8}{d+4}} \| z \|_{\mathcal{F}(I)}^{\frac{8}{d+4}} \| z \|_{\mathcal{S}(I)}^{\frac{8}{d+4}} \| z \|_{\mathcal{X}(I)}.
\]
(4.71)

Note that, by (3.16) and (4.51),
\[
\| \tilde{w} \|_{\mathcal{X}(I)} \leq C \| \tilde{w} \|_{\mathcal{F}(I)}^{\frac{1}{2}} \| \tilde{w} \|_{\mathcal{S}(I)}^{\frac{1}{2}} \leq C \delta^{\frac{1}{d+2}} \| \tilde{w} \|_{\mathcal{S}(I)}.
\]
(4.72)

Plugging (4.72) into (4.71) and using (4.58) we obtain
\[
J_2 \leq C_2 (C_T, D_T', E) \delta^{\frac{8}{d+2(d+2)^2}} \| z \|_{\mathcal{X}(I)} + C D_T' \| z \|_{\mathcal{S}(I)}^{\frac{8}{d+4}} \| z \|_{\mathcal{X}(I)}^{\frac{8}{d+4}}.
\]

Regarding $J_3$, similarly to (4.67), by (3.19),
\[
J_3 \leq C \| e \|_{N^{1}(I)+L^{2}(I;H^\frac{1}{2})} \leq C \varepsilon.
\]

Thus, combining the estimates of $J_i$ above, $0 \leq i \leq 3$, we obtain that for $\delta = \delta(C_T, D_T, E)$ small enough such that $C_2(C_T, D_T', E) \delta^{\frac{8}{d+2(d+2)^2}} \leq \frac{1}{2}$,
\[
\| z \|_{\mathcal{X}(I)} \leq 2 (C_1(C_T, D_T', E) + 2C) \varepsilon + 2C D_T' \| z \|_{\mathcal{S}(I)}^{\frac{8}{d+4}} \| z \|_{\mathcal{X}(I)}^{\frac{8}{d+4}}.
\]
(4.73)

Below we estimate $\| z \|_{\mathcal{S}(I)}$ and $\| z \|_{L^{2}(I;H^{\frac{3}{2}+})}$. Arguing as in the proof of (4.61), applying Theorem 3.3 to (4.69) and using (3.10) and (4.52) we have
\[
\| z \|_{\mathcal{S}(I)} \leq C \| z(t_0) \|_{H^{1}} + C_T \| e^{\frac{\alpha}{2} t} \Re \Phi (F(z + \tilde{w}) - F(\tilde{w})) \|_{N^{1}(I)} + C_T \| e \|_{N^{1}(I)+L^{2}(I;H^{\frac{1}{2}})}
\]
(4.74)
\[
\leq 2C_T \varepsilon + C_T D_T' \| (F(z + \tilde{w}) - F(\tilde{w})) \|_{N^{1}(I)}.
\]

Note that, by Hölder's inequality (3.10), (4.51) and (4.59),
\[
\| F(z + \tilde{w}) - F(\tilde{w}) \|_{N^{0}(I)} \leq C (\| \tilde{w} \|_{\mathcal{F}(I)}^{\frac{8}{d+4}} + \| z \|_{\mathcal{F}(I)}^{\frac{8}{d+4}}) \| z \|_{\mathcal{S}(I)} \leq C_3(C_T, D_T', E) \delta^{\frac{8}{d+2}} \| z \|_{\mathcal{S}(I)}.
\]
(4.75)

Moreover, arguing as in the proof of [52, (3.67)] and using (4.51), (4.60) we have
\[
\| \nabla (F(z + \tilde{w}) - F(\tilde{w})) \|_{N^{0}(I)} \leq C (\| \nabla \tilde{w} \|_{\mathcal{F}(I)} + \| z + \tilde{w} \|_{\mathcal{F}(I)}) \| \nabla z \|_{\mathcal{S}(I)}
\]
(4.76)
\[
\| z \|_{\mathcal{S}(I)} \leq C_4(C_T, D_T', E)(\| \nabla \tilde{w} \|_{\mathcal{F}(I)}^{\frac{8}{d+4}} + \delta^{\frac{8}{d+4}}) \| z \|_{\mathcal{S}(I)}.
\]

Thus, plugging (4.75), (4.76) into (4.74) we get
\[
\| z \|_{\mathcal{S}(I)} \leq 2C_T \varepsilon + C_5(C_T, D_T', E)(\| \nabla \tilde{w} \|_{\mathcal{F}(I)}^{\frac{8}{d+4}} + (\delta^{\frac{8}{d+4}} + \delta^{\frac{8}{d+4}})) \| z \|_{\mathcal{S}(I)}.
\]
(4.77)

Taking $\delta = \delta(C_T, D_T', E)$ small such that $C_5(C_T, D_T', E)(\delta^{\frac{8}{d+4}} + \delta^{\frac{8}{d+4}}) \leq \frac{1}{2}$ yields
\[
\| z \|_{\mathcal{S}(I)} \leq 4C_T \varepsilon + 2C_5(C_T, D_T', E) \| z \|_{\mathcal{X}(I)}^{\frac{8}{d+4}}.
\]
(4.77)

Now, plugging (4.77) into (4.73) we get that if $c_1 := \frac{8}{d+4}$, $c_2 := \frac{2d-16}{(d+2)^2} > 0$,
\[
\| z \|_{\mathcal{X}(I)} \leq C_6(C_T, D_T', E) \varepsilon + C_6(C_T, D_T', E) \varepsilon^{\frac{8}{d+4}} (\| z \|_{\mathcal{X}(I)}^{\frac{1+c_1}{d+4}} + \| z \|_{\mathcal{X}(I)}^{\frac{1+c_2}{d+4}}).
\]
(4.78)
Since \(0 < c_1 < c_2, \|z\|^{1+c_1}_H \leq \|z\|_{\mathcal{X}(I)} + \|z\|^{1+c_2}_H\), we have
\[
\|z\|_{\mathcal{X}(I)} \leq C_6(C_T, D'_T, E)(\varepsilon + \varepsilon^{\frac{4d}{3d-4}}\|z\|_{\mathcal{X}(I)} + 2\|z\|_{\mathcal{X}(I)}^{1+c_2}).
\]
Then, taking \(\delta\) very small such that \(C_6(C_T, D'_T, E)\delta^{\frac{4d}{3d-4}} \leq \frac{1}{2}\), we come to
\[
\|z\|_{\mathcal{X}(I)} \leq 2C_6(C_T, D'_T, E)\varepsilon + 4C_6(C_T, D'_T, E)\|z\|_{\mathcal{X}(I)}^{1+c_2}.
\]
Thus, taking \(\delta\) even smaller such that \(2C_6\delta < (1 - \frac{1}{\alpha})(4\alpha C_6)\delta^{\frac{1}{1+c_2}},\) we apply [8, Lemma A.1] to obtain (4.53), which along with (4.77) implies (4.54).

Finally, (4.55) follows from (4.54) and (4.58), (4.56) can be proved by (4.53) and similar estimates as in (4.71), and (4.57) follows from (4.53), (4.54) and (4.76). Therefore, the proof is complete. \(\square\)

**Proof of Theorem 4.6.** Let \(\delta = \delta(C_T, D'_T, E)\) be as in Proposition 4.8. As in the proof of Theorem 4.1, since \(\|\tilde{w}\|_{W(I)} \leq L < \infty\), we can divide \(I\) into subintervals \(I_j = [t_j, t_{j+1}],\) such that \(0 \leq j \leq l' \leq (\frac{4L^2}{\alpha})^{\frac{1}{2(d+2)}} \leq \infty,\) and \(\|\tilde{w}\|_{W(I_j')} \leq \delta\) for each \(0 \leq j \leq l'.\) Similarly, since \(\|\tilde{w}\|_{L^2(I_j', H^{\frac{3}{2}})} \leq L < \infty,\) we have another finite partition \(I_j'' = [t_j', t_{j+1}'],\) so that \(0 \leq j \leq l'' \leq (\frac{4L^2}{\delta})^2\) and on each \(I_j'',\) \(\|\tilde{w}\|_{L^2(I_j'', H^{\frac{3}{2}})} \leq \delta.\)

Let \(C(0) := C(C_T, D'_T, E), C(j+1) = C(0)(2C_T + C_TD'_T \sum_{k=0}^{j} C(k)), 0 \leq j \leq l,\) with \(C(C_T, D'_T, E)\) as in Proposition 4.8. Choose \(\varepsilon(\ast)(C_T, D'_T, E, L)\) very small such that
\[
(4.79) \quad (2C_T + C_TD'_T \sum_{k=0}^{l} C(k))\varepsilon(\ast)^{(k+1)} \leq \delta.
\]
We claim that on each \(I_j, 0 \leq j \leq l,\) estimates (4.53)-(4.57) hold with \(I, C(C_T, D'_T, E), \varepsilon\) replaced by \(I_j, C(j)\) and \(\varepsilon(\ast)^{(k+1)},\) respectively.

Actually, Proposition 4.8 implies that the claim is true for \(j = 0.\) Suppose that it is valid for each \(0 \leq k \leq j < l.\) Then, on the next interval \(I_{j+1},\) applying Theorem 3.3 to (4.69) and using the inductive assumptions and (4.79) we have
\[
|w(t_{j+1}) - \tilde{w}(t_{j+1})|_{H^1} \leq C_T|w(t_0) - \tilde{w}(t_0)|_{H^1} + C_T\varepsilon^{\frac{1}{4d-2}}|Re \Phi(F(w) - F(\tilde{w}))|_{N^1(t_0, t_{j+1})} + C_T\|\varepsilon\|_{L^2(t_0, T; H_{\frac{3}{2}})}^{\frac{1}{4d-2}}|N^1(t_0, T)|
\]
\[
\leq 2C_T\varepsilon + C_TD'_T \sum_{k=0}^{j} C(k)\varepsilon^{\frac{1}{4d-2}}^{(k+1)} \leq \delta.
\]
Thus, Proposition 4.8 yields that the claim holds on \(I_{j+1}.\)

Therefore, using the inductive arguments we prove the claim on any \(I_j, 0 \leq j \leq l.\) This yields (4.48), (4.49) and that for some \(c = c(C_T, D'_T, E, L) > 0,\)
\[
\|w - \tilde{w}\|_{\mathcal{X}(I)} \leq C'(C_T, D'_T, E)\varepsilon^c.
\]
Finally, taking into account (3.17) and (4.48), we obtain for some \(0 < c' \leq 1,\)
\[
\|w - \tilde{w}\|_{L^2(I \times [0,t])} \leq \|w - \tilde{w}\|_{\mathcal{X}(I)} \|w - \tilde{w}\|^{1-c'}_{\mathcal{X}(I)} \leq C''(C_T, D'_T, E)\varepsilon^c,
\]
thereby yielding (4.47). The proof is complete. \(\square\)
5. Global well-posedness

This section is mainly devoted to the global well-posedness of (1.1) in the mass- and energy-critical cases.

5.1. Mass-critical case. We first recall the global well-posedness and scattering results in the deterministic case, based on the works by Dodson [33, 34, 35].

**Theorem 5.1.** ([33, 34, 35]) For any \( u_0 \in L^2 \), there exists a unique global \( L^2 \)-solution \( u \) to the equation

\[
(5.1) \quad i \partial_t u = \Delta u - |u|^{\frac{4}{3}} u, \quad u(0) = u_0
\]

with \( d \geq 1 \). Moreover,

\[
(5.2) \quad \|u\|_{L^2(0,T;H^\frac{1}{2})} \leq B_0(T, |u_0|_2) < \infty,
\]

where \( B_0(T, |u_0|_2) \) is increasing in \( T \) and continuous in \( u_0 \), and \( u \) scatters at infinity, i.e., there exist \( u_\pm \in L^2 \) such that

\[
(5.3) \quad |e^{it\Delta} u(t) - u_\pm|_2 \to 0, \text{ as } t \to \pm\infty.
\]

**Remark 5.2.** The bound of \( \|u\|_{L^2(0,T;H^\frac{1}{2})} \) in (5.2) follows from that of \( \|u\|_{V(R)} \) and Strichartz estimates in Theorem 3.3, and the continuity of \( B_0(T, |u_0|_2) \) in \( u_0 \) is a consequence of the mass-critical stability result in Lemma 3.6 of [73]. In the case \( d \geq 3 \), we may apply Theorem 3.5 to obtain the global-in-time version of (5.2) with \( \infty \) replacing \( T \).

We also need the following boundedness of \( X \) in the space \( L^2 \).

**Lemma 5.3.** Assume the conditions of Theorem 2.3 (i) to hold. Then, for each \( X_0 \in L^2 \),

\[
(5.4) \quad |X(t)|_2^2 = |X_0|_2^2 + 2 \sum_{k=1}^N \int_0^t \int \text{Re} G_k(s)|X(s)|^2 dx d\beta_k(s), \quad 0 \leq t < \tau^*, \quad \mathbb{P} - a.s.,
\]

with \( \tau^* \) the maximal existing time as in Theorem 2.2. Moreover, for \( 0 < T < \infty \), \( p \geq 1 \),

\[
(5.5) \quad \mathbb{E}\|X\|_{L^p([0,\tau^* \wedge T];L^2)}^p \leq C(p,T) < \infty.
\]

In particular,

\[
(5.6) \quad M_T := \sup_{0 \leq t < \tau^* \wedge T} |X(t)|_2 \leq C(T) < \infty, \quad a.s.\]

**Proof.** The Itô formula (5.4) was obtained in [47, (6.1)]. The proof of (5.5) is similar to that of [6, Lemma 3.6], based on the Burkholder-Davis-Gundy inequality and the Gronwall inequality. We omit the details here for simplicity. \( \square \)

**Proof of Theorem 2.3 (i).** (Mass-Critical Case). Let \( X \) be the unique \( L^2 \)-solution to (1.1) on the maximal time interval \([0, \tau^*]\). In order to prove the global existence of \( X \), it suffices to prove the global bound (2.5) for any \( 0 < T < \infty \).

The proof proceeds as follows: we first treat a small random time interval \( I_1 \), determined by the smallness condition (4.4) of Theorem 4.1, and apply the rescaling transformation and the stability result to obtain the bound of \( V(I_1) \)-norm of the resulting random solution and thus of the stochastic solution \( X \). Then, using stopping times depending on the growth of noise, we construct consecutively small random subintervals \( I_j \), \( 2 \leq j \leq l \), on

\[\text{[33, 34, 35]}\] study the equation \( i \partial_t u = -\Delta u + |u|^{\frac{4}{3}} u \), which can be easily transformed into (5.1) by reversing the time. Hence, the results in [33, 34, 35] also hold for (5.1).
Hence, plugging (5.10) and (5.11) into (5.9) and using (5.2) we obtain, if
where

\[
\varepsilon_1(t) = \frac{\varepsilon_s(C_t, D_t, |X_0|_2)}{D_0(t, |X_0|_2)},
\]
where \( \varepsilon_s(C_t, D_t, |X_0|_2) := \varepsilon_s(C_t, D_t, |X_0|_2, 0, B_0(t, |X_0|_2) \) is as in Theorem 4.1, and \( D_0(t, |X_0|_2) = C_0(B_0(t, |X_0|_2) + (B_0(t, |X_0|_2))^{1+\frac{\alpha}{2}} \) with \( C_0 \) specified in (5.12) below.

Let \( \varphi \) be as in (2.28) with \( \sigma \equiv 0 \). By Theorem 2.11, \( v_1 := e^{-\varphi}X \) satisfies the random equation (2.29) with \( \alpha = 1 + \frac{\alpha}{2} \). In order to obtain the bound of \( \| v_1 \|_{V(0,\tau_1)} \), we compare \( v_1 \) with the solution \( \tilde{v}_1 \) to (2.33) (or, equivalently, (2.34)) with \( \alpha = 1 + \frac{\alpha}{2} \) and with the same initial datum, i.e., \( \tilde{v}_1(0) = v_1(0) = X_0 \).

Then, by Theorem 5.1, \( \tilde{v}_1 \) exists globally and satisfies that

\[
\| \tilde{v}_1 \|_{V(0,\tau_1)\cap L^2(0,\tau_1;H^\frac{1}{2})} \leq B_0(\tau_1, |\tilde{v}_1(0)|_2) = B_0(\tau_1, |X_0|_2) < \infty.
\]

Moreover, in order to estimate the error term (2.35), i.e.,

\[
e_1 := -(b \cdot \nabla + c)\tilde{v}_1 - (1 - e^{\frac{4}{3}Re\varphi})F(\tilde{v}),
\]
where \( b, c \) are given by (2.31) and (2.32) with \( \sigma \equiv 0 \), respectively, we note that

\[
\| e_1 \|_{V(0,\tau_1) \cap L^2(0,\tau_1;H^\frac{1}{2})} \leq \| (b \cdot \nabla + c)\tilde{v}_1 \|_{L^2(0,\tau_1;H^\frac{1}{2})} + \| (1 - e^{\frac{4}{3}Re\varphi})F(\tilde{v}) \|_{L^2(0,\tau_1 \times \mathbb{R}^d)}.
\]

In order to estimate the first term on the right-hand side above, we see that, by (2.2),

\[
\langle x \rangle \langle \nabla \rangle^{-\frac{\alpha}{2}} (b \cdot \nabla + c)\tilde{v}_1 = \Psi_p(x)^{-1} \langle \nabla \rangle^{-\frac{\alpha}{2}} \tilde{v}_1,
\]
where \( \Psi_p := \langle x \rangle \langle \nabla \rangle^{-\frac{\alpha}{2}} \langle b \cdot \nabla + c \rangle \langle \nabla \rangle^{-\frac{\alpha}{2}} \langle x \rangle \) is a pseudo-differential operator of zero order. Then, using Lemma 3.2 we get for some \( m \geq 1 \),

\[
\| (b \cdot \nabla + c)\tilde{v}_1 \|_{L^2(0,\tau_1;H^\frac{1}{2})} \leq C \sup_{0 \leq t \leq \tau_1} \| p(t) \|_{L^m(\tau_1)}^{(m)} \| \tilde{v}_1 \|_{L^2(0,\tau_1;H^\frac{1}{2})} \leq C' \sup_{0 \leq t \leq \tau_1} g(t) \| \tilde{v}_1 \|_{L^2(0,\tau_1;H^\frac{1}{2})}.
\]

Moreover, using (3.9) and the inequality \( |1 - e^x| \leq e|x| \) for \( |x| \leq 1 \), we have

\[
\| (1 - e^{\frac{4}{3}Re\varphi})F(\tilde{v}) \|_{L^2(0,\tau_1 \times \mathbb{R}^d)} \leq C'' \sup_{0 \leq t \leq \tau_1} g(t) \| \tilde{v}_1 \|_{V(0,\tau_1)}^{1+\frac{\alpha}{2}}.
\]

Hence, plugging (5.10) and (5.11) into (5.9) and using (5.2) we obtain, if \( C_0 := \max\{C', C''\} \),

\[
\| e_1 \|_{V(0,\tau_1) \cap L^2(0,\tau_1;H^\frac{1}{2}) + N^0(0,\tau_1)} \leq C_0 \sup_{0 \leq t \leq \tau_1} g(t) (\| \tilde{v}_1 \|_{L^2(0,\tau_1;H^\frac{1}{2})} + \| \tilde{v}_1 \|_{V(0,\tau_1)}^{1+\frac{\alpha}{2}}) \leq C_0 (B_0(\tau_1, |X_0|_2) + (B_0(\tau_1, |X_0|_2))^{1+\frac{\alpha}{2}}) \varepsilon_1(\tau_1) \leq \varepsilon_s(C_1, D_1, |X_0|_2).
\]

Thus, in view of Theorem 4.1, we obtain

\[
\| v_1 \|_{V(0,\tau_1)} \leq C(C_1, D_1, |X_0|_2, 0, B_0(\tau_1, |X_0|_2)) =: C(C_1, D_1, |X_0|_2),
\]

which we obtain the bound of \( \| X \|_{V(t_1)} \) by using again Theorem 2.11 and Theorem 4.1. At last, by virtue of the global \( L^2 \) bound in Lemma 5.3 and the Hölder continuity of Itô’s integrals, we show that the total number of random subintervals is finite almost surely, which consequently yields the desirable global bound (2.5).

Let us start with the first step.

**Step 1.** Set \( g(t) := \sum_{k=1}^N |f_0^{1} g_k(s) d\beta_k(s)| + \int_0^t g_k^2(s) ds, 0 \leq t \leq L \). Let \( \tau_1 := \inf\{0 < t < T \wedge \tau^* : g(t) \geq \varepsilon_1(t)\} \wedge (T \wedge \tau^*) \) with

\[
\varepsilon_1(t) = \frac{\varepsilon_s(C_t, D_t, |X_0|_2)}{D_0(t, |X_0|_2)}.
\]
which implies that
\begin{equation}
\|X\|_{V(0, \tau_1)} \leq \|e^\sigma\|_{C([0, \tau_1]; L^\infty)} C(C_{\tau_1}, D_{\tau_1}, |X_0|_2).
\end{equation}

Thus, (5.13) implies (2.5) if \( \tau_1 = T \land \tau^* \). Otherwise, we come to the next step.

**Step 2.** Set \( \sigma_0 := 0, \sigma_1 := \tau_1 \). For \( j \geq 1 \), we define random times inductively:

\[
\tau_{j+1} := \inf \{ t \in (0, (T \land \tau^*) - \sigma_j) : g_{\sigma_j}(t) \geq \varepsilon_{j+1}(t) \} \land (T \land \tau^* - \sigma_j),
\]
\[
\sigma_j := \sum_{k=1}^j \tau_k (\leq T \land \tau^*), \quad l := \inf \{ j \geq 1 : \sigma_j = T \land \tau^* \}.
\]

Here, let \( g_{\sigma_j}(t) := \sum_{k=1}^N \left| \int_{\sigma_j}^{\sigma_{j+1}} g_k(s)d\beta_k(s) \right| + \int_{\sigma_j}^{\sigma_{j+1}} g_k^2(s)ds \) and
\begin{equation}
\varepsilon_{j+1}(t) := \frac{\varepsilon_* (C_{\sigma_{j+1}, t}, D_{\sigma_{j+1}, t}, |X(\sigma_j)|_2)}{D_0(\sigma_j + t, |X(\sigma_j)|_2)}
\end{equation}
with \( \varepsilon_*(C_{\sigma_{j+1}, t}, D_{\sigma_{j+1}, t}, |X(\sigma_j)|_2) := \varepsilon_*(C_{\sigma_{j+1}, t}, D_{\sigma_{j+1}, t}, |X(\sigma_j)|_2, 0, B_0(\sigma_j + t, |X(\sigma_j)|_2)) \) as in Theorem 4.1, and \( D_0(\sigma_j + t, |X(\sigma_j)|_2) \) is defined similarly to \( D_0(t, |X_0|_2) \). We see that \( \tau_{j+1} \) (resp. \( \sigma_j \)) are given by (2.28) with \( \sigma_j + t \) replacing \( \sigma_j \) and \( \tau_{j+1} \) (resp. \( \tau(t) \)) stopping times, \( 0 \leq t \leq T \).

We use the inductive arguments to obtain the bound of \( \|X\|_{V(0, \sigma_j)} \) for any \( 1 \leq j \leq l \).

Suppose that for each \( 1 \leq k \leq j < l \),
\begin{equation}
\|X\|_{V(0, \sigma_k)} \leq \sum_{i=0}^{k-1} \|e^{\varphi_{\sigma_i}}\|_{C([0, \tau_{i+1}]; L^\infty)} C(C_{\tau_{i+1}, \sigma_{i+1}, t}, D_{\sigma_{i+1}, t}, |X(\sigma_i)|_2),
\end{equation}
where \( \varphi_{\sigma_i} \) is given by (2.28) with \( \sigma_i \) replacing \( \sigma \), and \( C(C_{\sigma_{i+1}, t}, D_{\sigma_{i+1}, t}, |X(\sigma_i)|_2) := C(C_{\sigma_{i+1}, D_{\sigma_{i+1}, t}, |X(\sigma_i)|_2, 0, D_0(\sigma_{i+1}, |X(\sigma_i)|_2)) \) as in Theorem 4.1. Below we show that (5.14) also holds when \( k \) is replaced by \( j + 1 \).

For this purpose, we apply Theorem 2.11 to obtain that
\begin{equation}
v_{j+1}(t) := e^{-\varphi_{\sigma_j}(t)}X(\sigma_j + t), \quad 0 \leq t < (T \land \tau^*) - \sigma_j,
\end{equation}
satisfies the equation
\begin{equation}
i\partial_t v_{j+1} = e^{-\varphi_{\sigma_j}} \Delta(e^{\varphi_{\sigma_j}}v_{j+1}) - e^{\frac{4}{2}\text{Re}\varphi_{\sigma_j}} F(v_{j+1}),
\end{equation}
\[v_{j+1}(0) = X(\sigma_j).\]

Similarly to Step 1, we compare (5.18) with the equation
\begin{equation}
i\partial_t \tilde{v}_{j+1} = \Delta \tilde{v}_{j+1} - F(\tilde{v}_{j+1}),
\end{equation}
or equivalently,
\begin{equation}
i\partial_t \tilde{v}_{j+1} = e^{-\varphi_{\sigma_j}} \Delta(e^{\varphi_{\sigma_j}} \tilde{v}_{j+1}) - e^{\frac{4}{2}\text{Re}\varphi_{\sigma_j}} F(\tilde{v}_{j+1}) + e_{j+1},
\end{equation}
with \( \tilde{v}_{j+1}(0) = X(\sigma_j) \) and
\begin{equation}
e_{j+1} := -(b_{\sigma_j}(t) \cdot \nabla + c_{\sigma_j}(t)) \tilde{v}_{j+1} - (1 - e^{\frac{4}{2}\text{Re}\varphi_{\sigma_j}(t)}) F(\tilde{v}_{j+1}).
\end{equation}

where \( b_{\sigma_j} \) and \( c_{\sigma_j} \) are given by (2.31) and (2.32) with \( \sigma_j \) replacing \( \sigma \), respectively.

Again, Theorem 5.1 yields that \( \tilde{v}_{j+1} \) exists globally and
\begin{equation}
\|\tilde{v}_{j+1}\|_{V(0, \tau_{j+1})} \leq B_0(\sigma_{j+1}, |\tilde{v}_{j+1}(0)|_2) = B_0(\sigma_{j+1}, |X(\sigma_j)|_2).
\end{equation}
This implies that, similarly to (5.12),
\[
\|e_{j+1}\|_{N^0(0,\tau_{j+1})+L^2(0,\tau_{j+1};H^2)} \leq C_0 \sup_{0 \leq t \leq \tau_{j+1}} g_{\sigma_j}(t)(\|\bar{v}_{j+1}\|_{L^2(0,\tau_{j+1};H^2)} + \|\tilde{v}_{j+1}\|_{V(0,\tau_{j+1})})^{1+\frac{d}{2}} \\
\leq C_0(B_0(\sigma_{j+1}, |X_{\sigma_j}|^2) + (B_0(\sigma_{j+1}, |X_{\sigma_j}|^2))^{1+\frac{d}{2}} e_{j+1}(\tau_{j+1}) \\
\leq \varepsilon_*(C_{\sigma_{j+1}}, D_{\sigma_{j+1}}, |X_{\sigma_j}|^2).
\]
(5.23)

Thus, by virtue of Theorem 4.1, we obtain
\[
\|v_{j+1}\|_{V(0,\tau_{j+1})} \leq C(C_{\sigma_{j+1}}, D_{\sigma_{j+1}}, |X(\sigma_j)|^2),
\]
(5.24)
and so
\[
\|X\|_{V(\sigma_j, \tau_{j+1})} \leq \|e^{\varepsilon_{\sigma_j}}\|_{C([0,\tau_{j+1}];L^\infty)} C(C_{\sigma_{j+1}}, D_{\sigma_{j+1}}, |X(\sigma_j)|^2).
\]
(5.25)
This along with the inductive assumptions yields (5.16) with $j + 1$ replacing $k$.

Thus, using the inductive arguments we conclude that (5.16) holds for all $1 \leq j \leq l$. This yields that
\[
\|X\|_{V(0,\sigma_l)} \leq \sum_{k=0}^{l-1} \|e^{\varepsilon_{\sigma_k}}\|_{C([0,T];L^\infty)} C(C_T, D_T, M_T),
\]
(5.26)
where $C(C_T, D_T, M_T) \triangleq C(C_T, D_T, M_T, 0, \sup_{0 \leq x \leq M_T} B_0(T, x))$ and $M_T$ are as in Theorem 4.1 and Lemma 5.3, respectively.

**Step 3.** We claim that
\[
P(l < \infty) = 1.
\]
(5.27)
To this end, we use the contradiction argument. Suppose that (5.27) is not true. We consider $\omega \in \{l = \infty\}$. For simplicity, we omit the argument $\omega$ below.

On one hand, by the definition of $\tau_{j+1}$,
\[
g_{\sigma_j}(\tau_{j+1}) = \varepsilon_*(C_{\sigma_{j+1}}, D_{\sigma_{j+1}}, |X(\sigma_j)|^2) \leq \frac{\varepsilon_*(C_T, D_T, M_T)}{D_0(T, M_T)} > 0,
\]
where $\varepsilon_*(C_T, D_T, M_T) \triangleq \varepsilon_*(C_T, D_T, M_T, 0, \sup_{0 \leq x \leq M_T} B_0(T, x))$, and $D_0(T, M_T) \triangleq C_0 \sup_{0 \leq x \leq M_T} ((B_0(T, x)) + (B_0(T, x))^{1+\frac{d}{2}}$.

On the other hand, since the processes $t \mapsto \int_0^t g_k d\beta_k(s)$ and $t \mapsto \int_0^t g_k^2 ds$ are $(\frac{1}{2} - \kappa)$-Hölder continuous for any $\kappa < \frac{1}{2}$ and $1 \leq k \leq N$, we have for some positive $C(T)$ (depending on $\omega$)
\[
g_{\sigma_j}(\tau_{j+1}) \leq C(T)(\tau_{j+1})^{2-\kappa}, \quad \forall j \geq 1.
\]
Hence, we conclude that
\[
\tau_{j+1} \geq \left( \frac{\varepsilon_*(C_T, D_T, M_T)}{C(T)D_0(T, M_T)} \right)^{\frac{2}{2-\kappa}} > 0, \quad \forall j \geq 1.
\]
Thus, for $\omega \in \{l = \infty\}$,
\[
\sigma_l(\omega) = \sum_{j=1}^{\infty} \tau_j(\omega) = \infty,
\]
which contracts the fact that $\sigma_l(\omega) \leq (T \wedge \tau^*)(\omega) \leq T < \infty$, thereby yielding (5.27), as claimed.

Now, since $\{l < \infty\} \subseteq \{\sigma_l = T \wedge \tau^*\}$, combining (5.26) and (5.27) together we obtain
\[
\|X\|_{V(0, \tau^* \wedge T)} \leq \sum_{k=0}^{l-1} \|e^{\varepsilon_{\sigma_k}}\|_{C([0,T];L^\infty)} C(C_T, D_T, M_T) < \infty, \quad a.s.
\]
(5.28)
Moreover, in the defocusing case where
\[ (5.29) \]
with \( d \geq 3 \). Moreover,
\[ (5.30) \]
where
\[ (5.31) \]
Theorem 5.4. (\cite{27, 67, 75}) For every \( u_0 \in H^1 \), there exists a unique global \( H^1 \)-solution \( u \) to the equation
\[ (5.32) \]
with \( d \geq 3 \). Moreover,
\[ (5.33) \]
where \( B_1(T, |u_0|_{H^1}) \) is increasing in \( T \) and continuous in \( u_0 \), and \( u \) scatters at infinity, i.e., there exist \( u_{\pm} \in H^1 \) such that
\[ (5.34) \]
Remark 5.5. As is the mass-critical case, the bound of \( \|u\|_{L^2(0,T; H^\frac{3}{2} - 1)} \) follows standardly from Strichartz estimates in Theorem 3.3 and the bound of \( \|u\|_{S^1(\mathbb{R})} \) and the continuous dependence in \( u_0 \) follows from the energy-critical stability result Lemma 8.3 of \cite{73}. By Theorem 3.5, one also has the global-in-time version of (5.30) with \( T \) replaced by \( \infty \).

We also need the global energy estimates below.

Lemma 5.6. Assume the conditions of Theorem 2.3 (ii) to hold, \( 3 \leq d \leq 6 \). Define the Hamiltonian of \( X \) by
\[ (5.35) \]
Then, for each \( X_0 \in H^1 \), we have \( \mathbb{P} \)-a.s., for any \( t \in (0, \tau^*) \),
\[ (5.36) \]
Moreover, in the defocusing case where \( \lambda = -1 \), for any \( 0 < T < \infty \), \( p \geq 1 \),
\[ (5.37) \]
where \( \tau^* \) is the maximal existing time as in Theorem 2.2. In particular,
\[ (5.38) \]
The proof is postponed to the Appendix.

As in the mass-critical case, although the equation studied in \cite{27, 67, 75} is \( i\partial_t u = \Delta u + |u|^\frac{4}{d} u \), the results there are also valid for (5.29) by reversing the time.
Remark 5.7. Similar formula was proved in [6] in the energy-subcritical case (i.e., \( \alpha \in (1, 1 + \frac{4}{d-2}) \), \( d \geq 3 \)), where an approximating procedure was used to derive the Itô formula of the potential energy \( |X|^{\alpha+1} \).

We are now ready to prove Theorem 2.3 (ii).

Proof of Theorem 2.3 (ii). (Energy-Critical Case). The arguments below are similar to those in the mass-critical case, however, based on the more delicate stability results Theorems 4.4 and 4.6. Below we mainly treat the case \( d > 6 \), the case \( 3 \leq d \leq 6 \) is easier and can be proved similarly by using Theorem 4.4.

Let \( X \) be the unique \( H^1 \)-solution to (1.1) with \( \alpha = 1 + \frac{4}{d-2} \) on the maximal time interval \([0, \tau^*)\). In view of Theorem 2.2, it suffices to prove (2.6) for any \( 0 < T < \infty \).

For this purpose, we define \( g \) as in Step 1 in the proof of Theorem 2.3 (i) and let \( \tau_1 := \inf\{t \in (0, T \wedge \tau^*) : g(t) \geq \varepsilon_1(t)\} \wedge (T \wedge \tau^*) \) with

\[
\varepsilon_1(t) := \frac{\varepsilon_*(C_t, D_t, |X_0|_{H^1})}{D_t(t, |X_0|_{H^1})},
\]

where \( \varepsilon_*(C_t, D_t', |X_0|_{H^1}) := \varepsilon_*(C_t, D_t', \sqrt{2H(X_0)}, B_1(t, |X_0|_{H^1})) \) is as in Theorem 4.6, and \( D_t(t, |X_0|_{H^1}) = 1 + C_1(B_1(t, |X_0|_{H^1}) + (B_1(t, |X_0|_{H^1}))^{1 + \frac{4}{d-2}} \) with \( C_1 \) given by (5.38) below.

We can take \( g \) as the time function in Theorem 4.6. Hence, \( \sup_{0 \leq t \leq \tau_1} g(t) \leq \varepsilon_*(C_{\tau_1}, D_{\tau_1}', |X_{\tau_1}|_{H^1}) \), and so the smallness condition on \( g \) in (4.46) is satisfied on \([0, \tau_1)\).

Let \( \varphi \) be as in (2.28) with \( \sigma \equiv 0 \). By Theorem 2.11, \( w_1 := e^{-\varphi}X \) satisfies (2.29) with \( \sigma \equiv 0 \) and \( \alpha = 1 + \frac{4}{d-2} \).

Moreover, let \( \tilde{w}_1 \) be the solution to (2.33) with \( \alpha = 1 + \frac{4}{d-2} \) and \( \tilde{w}(0) = w_1(0) = X_0 \).

Then, Theorem 5.1 implies that

\[
\|\tilde{w}_1\|_{\mathcal{W}(0, \tau_1) \cap L^2(0, \tau_1; H^{\frac{3}{2}})} \leq B_1(\tau_1, |\tilde{w}_1(0)|_{H^1}) = B_1(\tau_1, |X_0|_{H^1}) < \infty,
\]

and by the conservation law of Hamiltonian (i.e., \( H(\tilde{w}(t)) = H(\tilde{w}(0)) \)),

\[
\|\tilde{w}\|_{C([0, t]; H^1)} \leq \sqrt{2 \sup_{0 \leq s \leq t} H(\tilde{w}(s))} = \sqrt{2H(X_0), \ t \in [0, \tau_1].}
\]

For the error term

\[
e_1 := -(b \cdot \nabla + c) \tilde{w}_1 - (1 - e^{\frac{d}{d-2}Re\varphi})F(\tilde{w}_1),
\]

where \( b, c \) are given by (2.31) and (2.32) with \( \sigma \equiv 0 \), respectively, using (3.10) and similar arguments as in the proof of (5.10) we have

\[
\|e_1\|_{N^1(0, \tau_1) + L^2(0, \tau_1; H^{\frac{3}{2}})} \leq \|b \cdot \nabla + c\|_{L^2(0, \tau_1; H^{\frac{3}{2}})} + \|(1 - e^{\frac{d}{d-2}Re\varphi})F(\tilde{w}_1)\|_{L^2(0, \tau_1; W^{1, \frac{d+2}{d-2}})}
\]

\[
\leq C \sup_{0 \leq t \leq \tau_1} g(t) \|\tilde{w}_1\|_{L^2(0, \tau_1; H^{\frac{3}{2}})} + \|\tilde{w}_1\|_{\mathcal{W}(0, \tau_1)}^{\frac{d+2}{d-2}}
\]

\[
\leq C_1(B_1(\tau_1, |X_0|_{H^1}) + (B_1(\tau_1, |X_0|_{H^1}))^{1 + \frac{4}{d-2}}) \varepsilon_1(\tau_1)
\]

\[
(5.38)
\]

Then, applying Theorem 4.6 we obtain

\[
\|w_1\|_{\mathcal{W}(0, \tau_1)} \leq C(C_1, D_{\tau_1}', \sqrt{2H(X_0)}, B_1(\tau_1, |X_0|_{H^1})) =: C(C_1, D_{\tau_1}', |X_0|_{H^1}),
\]

which implies that

\[
\|X\|_{\mathcal{W}(0, \tau_1)} \leq \|e^{\varphi}\|_{C([0, \tau_1]; W^{1, \infty})} C(C_1, D_{\tau_1}', |X_0|_{H^1}).
\]

Thus, (2.6) follows from (5.40) if \( \tau_1 \geq \tau^* \). Otherwise, we turn to the next step.
Let \( \tau_j, \sigma_j, g_{\sigma_j} \) and \( l \) be as in Step 2 in the proof of Theorem 2.3 (i), but with

\[
\varepsilon_{j+1}(t) = \frac{\varepsilon_*(C_{\sigma_j+1}, D'_{\sigma_j+1}, |X(\sigma_j)|_{H^1})}{D_1(\sigma_j + t, |X(\sigma_j)|_{H^1})},
\]

where \( D_1(\sigma_j + t, |X(\sigma_j)|_{H^1}) \) is defined similarly to \( D_1(t, |X_0|_{H^1}) \), \( \varepsilon_*(C_{\sigma_j+1}, D'_{\sigma_j+1}, |X(\sigma_j)|_{H^1}) \)

\[
:= \varepsilon_*(C_{\sigma_j+1}, D'_{\sigma_j+1}, \sqrt{2}|X(\sigma_j)|_{H^1}, B_1(\sigma_j + t, |X(\sigma_j)|_{H^1})) \text{ is as in Theorem 4.6.}
\]

We use the inductive arguments to prove that for any \( 1 \leq j \leq l \),

\[
\tag{5.41}
\|X\|_{W(0, \sigma_j)} \leq \sum_{k=0}^{j-1} \|e^{\varphi_{\sigma_k}} C(0, \tau_{k+1}; W^{1, \infty}) C(\sigma_{k+1}, D'_{\sigma_{k+1}}, |X(\sigma_k)|_{H^1}) < \infty,
\]

where \( \varphi_{\sigma_k} \) are as in Theorem 2.11 with \( \sigma_k \) replacing \( \sigma \), and \( C(\sigma_{k+1}, D'_{\sigma_{k+1}}, |X(\sigma_k)|_{H^1}) := C(C_{\sigma_{k+1}}, D'_{\sigma_{k+1}}, \sqrt{2}|X(\sigma_k)|_{H^1}, B_1(\sigma_{k+1}, |X(\sigma_k)|_{H^1}) \) are as in Theorem 4.6.

We see from (5.40) that (5.41) holds for \( j = 1 \). Suppose that (5.41) holds for each \( 1 \leq k \leq j < l \).

In order to prove (5.41) with \( j + 1 \) replacing \( j \), we consider the rescaling transformation

\[
\tag{5.42}
\tilde{w}_{j+1}(0) := e^{-\varphi_{\sigma_j}}(0) X(\sigma_j + t), \quad 0 \leq t < (T \wedge \tau^*) - \sigma_j, \text{ and apply Theorem 2.11 to obtain}
\]

\[
i \partial_t \tilde{w}_{j+1} = e^{-\varphi_{\sigma_j}} \Delta (e^{\varphi_{\sigma_j}} w_{j+1}) - e^{\varphi_{\sigma_j}} \nabla \Re \varphi_{\sigma_j} F(w_{j+1}),
\]

\[
\tilde{w}_{j+1}(0) = X(\sigma_j).
\]

Then, we compare (5.42) with the equation

\[
\tag{5.43}
i \partial_t \tilde{w}_{j+1} = \Delta \tilde{w}_{j+1} - F(\tilde{w}_{j+1}),
\]

or equivalently,

\[
\tag{5.44}
i \partial_t \tilde{w}_{j+1} = e^{-\varphi_{\sigma_j}} \Delta (e^{\varphi_{\sigma_j}} \tilde{w}_{j+1}) - e^{\varphi_{\sigma_j}} \nabla \Re \varphi_{\sigma_j} F(\tilde{w}_{j+1}) + e_{j+1},
\]

with \( \tilde{w}_{j+1}(0) = w_{j+1}(0) = X(\sigma_j) \) and the error term

\[
e_{j+1} = -(b_{\sigma_j} \cdot \nabla + c_{\sigma_j}) \tilde{w}_{j+1} - (1 - e^{\varphi_{\sigma_j}} \nabla \Re \varphi_{\sigma_j}) F(\tilde{w}_{j+1}).
\]

Theorem 5.4 yields that \( \tilde{w}_{j+1} \) exists globally and satisfies

\[
\tag{5.45}
\|\tilde{w}_{j+1}\|_{W(0, \tau_{j+1}) \cap L^2(0, \tau_{j+1} + H^{\frac{3}{2}})} \leq B_1(\sigma_{j+1}, |\tilde{w}_{j+1}(0)|_{H^1}) = B_1(\sigma_{j+1}, |X(\sigma_j)|_{H^1}),
\]

and similarly to (5.37),

\[
\tag{5.46}
\|\tilde{w}_{j+1}\|_{C(0, \tau_{j+1}; H^1)} \leq \sqrt{2} \sup_{0 \leq s \leq \tau_{j+1}} H(\tilde{w}_{j+1}(s)) = \sqrt{2} H(X(\sigma_j)).
\]

Note that, we can take \( g_{\sigma_j} \), the time function in Theorem 4.6. Then, by the definition of \( \tau_{j+1} \), \( \sup_{0 \leq t \leq \tau_{j+1}} g_{\sigma_j}(t) \leq \varepsilon_*(C_{\sigma_{j+1}}, D'_{\sigma_{j+1}}, |X(\sigma_{j+1})|_{H^1}) \), and so the smallness condition

on \( g_{\sigma_j} \) in Theorem 4.6 is satisfied on \( [0, \tau_{j+1}] \).

Moreover, similarly to (5.38),

\[
\tag{5.48}
\|w_{j+1}\|_{W(0, \tau_{j+1})} \leq C(C_{\sigma_{j+1}}, D'_{\sigma_{j+1}}, |X(\sigma_j)|_{H^1}),
\]

Thus, by virtue of Theorem 4.6, we obtain

\[
\tag{5.49}
\|w_{j+1}\|_{W(0, \tau_{j+1})} \leq C(C_{\sigma_{j+1}}, D'_{\sigma_{j+1}}, |X(\sigma_j)|_{H^1}),
\]

and so the smallness condition
and so
\[
\|X\|_{W(\sigma_j, \sigma_{j+1})} \leq \|e^{\varphi_{\sigma_j}}\|_{C([0, \tau_{j+1}]; W^{1, \infty})} C(C_{\sigma_{j+1}}, D'_{\sigma_{j+1}}, |X(\sigma_j)|_{H^1}) < \infty, 
\]
thereby yielding (5.41) with \(j + 1\) replacing \(j\).

Therefore, the inductive arguments yield (5.41) for all \(1 \leq j \leq l\) and so
\[
\|X\|_{W(0, \sigma_j)} \leq \sum_{k=0}^{l-1} \|e^{\varphi_{\sigma_k}}\|_{C([0, \tau_{k+1}]; W^{1, \infty})} C(C_{\sigma_{k+1}}, D'_{\sigma_{k+1}}, |X(\sigma_j)|_{H^1}) \leq C \left( \sum_{k=0}^{l-1} \|e^{\varphi_{\sigma_k}}\|_{C([0, \tau_{k+1}]; W^{1, \infty})} C(C_{\sigma_{k+1}}, D'_{\sigma_{k+1}}, |X(\sigma_j)|_{H^1}) \right),
\]
(5.49)
where \(E_T\) is as in (2.9), and we also used the inequality \(\sup_{0 \leq t < \tau^*} \sqrt{2H(X(t))} \leq C' E_T\) in the last step, implied by the Sobolev embedding.

Since by (2.9), \(E_T < \infty\), a.s., and for any \(0 \leq j \leq l - 1\),
\[
g_{\sigma_j}(\tau_{j+1}) = \frac{\varepsilon_*(C_{\sigma_{j+1}}, D'_{\sigma_{j+1}}, |X(\sigma_j)|_{H^1})}{D_1(\sigma_{j+1}, |X(\sigma_j)|_{H^1})} \geq \varepsilon_*(C_{T}, D'_T, C'E_T, \sup_{0 \leq x \leq E_T} B_1(T, x)) \geq \sup_{0 \leq x \leq E_T} \frac{D_1(T, x)}{X(\tau_{j+1})},
\]
we can use similar arguments as in Step 3 in the proof of Theorem 2.3 (i) to deduce that \(l < \infty\), a.s., which together with (5.49) yields the global bound (2.6), thereby implying the global existence of \(X\) to (1.1).

Finally, the estimate (2.10) follows from Lemma 5.6, and (2.11) follows from (5.49) and Strichartz estimates. The proof of Theorem 2.3 is complete.

6. SCATTERING

This section is concerned with the scattering behavior of global solutions to (1.1). The idea here is based on the recent work [47], combined with the stability results obtained above. Precisely, we use a new rescaling transformation (2.17), i.e., \(z_* = e^{-\varphi_*} X\) with \(\varphi_*\) as in (2.16), and compare the resulting random equation (2.18) with (2.33) but after some large time \(T\), i.e.,
\[
(6.1) \quad i\partial_t u = \Delta u - |u|^{\alpha-1} u, \\
\quad u(T) = z_*(T).
\]
Let us start with the mass-critical case.

6.1. Mass-critical case. First we enhance the bounds (5.5) and (5.6) to the whole time regime, under the condition that \(g_k \in L^2(\mathbb{R}^+),\) a.s..

**Lemma 6.1.** Consider the situations in Theorem 2.7 (i). Then, for any \(p \geq 1\),
\[
(6.2) \quad \mathbb{E} \sup_{0 \leq t < \infty} |X(t)|_p^p \leq C(p) < \infty.
\]
In particular, we have the global pathwise bound
\[
(6.3) \quad M_\infty := \sup_{0 \leq t < \infty} |X(t)|_2 \leq C < \infty, \text{ a.s.}
\]

**Proof.** Estimate (6.2) can be proved by using the Itô formula (5.4) and similar arguments as in the proof of [47, (1.7)]. We omit the details here for simplicity.

Below we also have the important uniform boundedness (independent of \(T\)).
Lemma 6.2. For \( z_\ast(T) \in L^2 \), there exists a unique global \( L^2 \)-solution \( u \) (depending on \( T \)) to (6.1) with \( \alpha = 1 + \frac{4}{d}, \ d \geq 3 \), which scatters at infinity and satisfies
\[
\|u\|_{S^0(T,\infty) \cap L^2(T,\infty;H^{1\over 2})} \leq C < \infty, \ \text{a.s.,}
\]
where \( C \) is independent of \( T \).

Proof. For each \( z_\ast(T) \in L^2 \) fixed, the global well-posedness and scattering follow from Theorem 5.1. Regarding (6.4), applying the global-in-time Strichartz estimates in Theorem 3.5 to (2.34) and using the Hölder inequality (3.9) we have that for any \( t > T \),
\[
\|u\|_{L^2(T,t;H^{1\over 2})} + \|u\|_{S^0(T,t)} \leq C|u(T)|_2 + C\|u\|_{V(T,t)}^{1+\frac{4}{d}},
\]
with \( C \) independent of \( T \) and \( t \). Using Remark 5.2 and that \( u(T) = z_\ast(T) \), we get
\[
\|u\|_{L^2(T,t;H^{1\over 2})} + \|u\|_{S^0(T,t)} \leq C|z_\ast(T)|_2 + C(B_0(|z_\ast(T)|_2))^{1+\frac{4}{d}}.
\]
Since \( g_k \in L^2(\mathbb{R}^+), \) a.s., we have \( \varphi_\ast \in C(\mathbb{R}^+;L^\infty) \), and so \( |z_\ast(T)|_2 \leq C|X(T)|_2 \) with \( C \) independent of \( T \). In view of the global bound (6.3), we obtain
\[
\|u\|_{L^2(T,t;H^{1\over 2})} + \|u\|_{S^0(T,t)} \leq CM_\infty + C\sup_{0 \leq z \leq M_\infty}(B_0(x))^{1+\frac{4}{d}} < \infty
\]
with \( C, M_\infty \) independent of \( T \) and \( t \). Thus, letting \( t \to \infty \), we obtain (6.4). \( \square \)

The following result is crucial for the scattering in the mass-critical case.

Lemma 6.3. Consider the situations in Theorem 2.7 (i). Let \( u \) be the solution to (6.1) with \( u(T) = z_\ast(T) \). Then, \( \mathbb{P} \)-a.s. as \( T \to \infty \),
\[
\|z_\ast - u\|_{S^0(T,\infty) \cap L^2(T,\infty;H^{1\over 2})} \to 0.
\]

Proof. We use the idea of comparison as in the proof of Theorem 2.3 (i). Precisely, we compare the solution \( z_\ast \) to (2.18) with the solution \( u \) to (6.1).

For this purpose, we rewrite (6.1) with \( \alpha = 1 + \frac{4}{d} \) as follows
\[
i\partial_x u = -\varphi_\ast \Delta (e^{\varphi_\ast}u) - e^{4\frac{Re\varphi_\ast}{d}}F(u) + e,
\]
with the error term
\[
e = -(b_\ast \cdot \nabla + c_\ast)u - (1 - e^{4\frac{Re\varphi_\ast}{d}})F(u).
\]

Note that for each \( 1 \leq k \leq N \) and for any multi-index \( \gamma \),
\[
\sup_{t \geq T} |\partial^2 x_\gamma \varphi_\ast(t, x)| \leq CR_\gamma(x) \sum_{k=1}^N \left( \int_t^\infty g_k d\beta_k \right) + \int_t^\infty g_k^2 ds,
\]
where \( R_\gamma \) is a continuous function satisfying that \( \lim_{|x| \to \infty} |x|^2 R_\gamma(x) = 0 \). Since \( g_k \in L^2(\mathbb{R}^+) \), we infer that the right-hand side of (6.6) converges to zero almost surely, as \( T \to \infty \). Actually, since \( g_k \in L^2(\mathbb{R}^+) \), \( s \mapsto \int_t^s g_k d\beta_k, \ s \in [T, \infty], \) is a continuous martingale. Then, using the maximal inequality and that \( \int_t^\infty g_k d\beta_k = (\int_T^\infty - \int_t^T)g_k d\beta_k \), we get that for any \( s \geq T \),
\[
\mathbb{E} \sup_{T \leq t \leq s} \left\| \int_t^\infty g_k d\beta_k \right\|^2 \leq 2C\mathbb{E} \int_T^\infty g_k^2 ds \to 0, \text{ as } T \to \infty,
\]
which along with Fatou’s lemma yields that
\[
\mathbb{E} \sup_{t \geq T} \left\| \int_t^\infty g_k d\beta_k \right\|^2 \to 0, \text{ as } T \to \infty.
\]
In particular, for some sequence \( \{ T_n \} \), sup\(_{t \geq T_n} \| f \|_{L^2([-T/2, T/2])} \to 0 \) as \( n \to \infty \), a.s.. Since \( T \mapsto \sup_{t \geq T} \| f \|_{L^2([-T/2, T/2])} \) is decreasing, we have sup\(_{t \geq T} \| f \|_{L^2([-T/2, T/2])} \to 0 \) as \( T \to \infty \), a.s..

Hence, for \( T \) large enough, Theorem 3.5 yields that global-in-time Strichartz and local smoothing estimates hold for the operator \( e^{-i \varphi \Delta (e^x \cdot \cdot \cdot)} \).

Note that, for any \( t \geq T \),

\[
\| e \|_{N^0(T,t) + L^2(T,t; H^{1/2}_x)} \leq \|(b_* \cdot \nabla + c_*) u \|_{L^2(T,t; H^{1/2}_x)} + \| (e^{\Re e^x} - 1) F(u) \|_{L^{2+4/(4+2)}((T,t) \times \mathbb{R}^d)}.
\]

(6.7)

Since \( g_k \in L^2(\mathbb{R}^+) \), a.s., using similar arguments as above, we have

\[
\varepsilon_1(T) := \sup_{t \geq T} \sum_{k \in I} (| \int_t^\infty g_k d\beta_k | + \int_t^\infty g_k^2 ds) \to 0, \quad \text{as } T \to \infty, \quad \text{a.s.}
\]

Then, estimating as in (5.10) we get

\[
\|(b_* \cdot \nabla + c_*) u \|_{L^2(T,t; H^{1/2}_x)} \leq C \varepsilon_1(T) \| u \|_{L^2(T,t; H^{1/2}_x)},
\]

(6.8)

where \( C \) is independent of \( T \) and \( t \).

Moreover, using again \( g_k \in L^2(\mathbb{R}^+) \), a.s., we deduce that

\[
\varepsilon_2(T) := \sup_{t \geq T} \| \varphi_* (t) \|_{W^{1,\infty}} \to 0, \quad \text{as } T \to \infty, \quad \text{a.s.}
\]

Then, using the inequality \(| e^x - 1 | \leq e | x | \) for \(| x | \leq 1 \) and (3.9), we get

\[
\|(e^{\Re e^x} - 1) F(u) \|_{N^0(T,t)} \leq C \varepsilon_2(T) \| u \|_{V^1((T,t))}^{1+\frac{4}{2}}.
\]

(6.11)

Plugging (6.9) and (6.11) into (6.7) and using (6.4) we obtain that,

\[
\| e \|_{N^0(T,t) + L^2(T,t; H^{1/2}_x)} \leq C \varepsilon(T) \| u \|_{L^2(T,t; H^{1/2}_x)} + \| u \|_{V^1((T,t))}^{1+\frac{4}{2}} \leq C \varepsilon(T),
\]

(6.12)

where \( \varepsilon(T) := \max \{ \varepsilon_1(T), \varepsilon_2(T) \} \), and \( C \) is independent of \( T, t \), due to (6.4).

Thus, in view of Remark 4.2 and (6.4), we obtain that for \( T \) large enough,

\[
\| z_* - u \|_{N^0(T,t) + L^2(T,t; H^{1/2}_x)} \leq C \varepsilon(T),
\]

(6.13)

where \( C \) is independent of \( T \) and \( t \). (Note that, since \(| z_* (T) - u(T) | \|_2 = 0 \), we can take \( M' = \varepsilon(T) \) when applying the stability result.)

Therefore, letting \( T \to \infty \) in (6.13) and using (6.8) and (6.10) we obtain (6.5).

**Proof of Theorem 2.7** (i) (Mass-Critical Case). Let \( u \) be as in Lemma 6.2. We have \( \mathbb{P} \)-a.s. for any \( t_1, t_2 \geq T \),

\[
| e^{i t_1 \Delta} z_* (t_1) - e^{i t_2 \Delta} z_* (t_2) |_2 \leq | e^{i t_1 \Delta} (z_* - u) (t_1) - e^{i t_2 \Delta} (z_* - u) (t_2) |_2 + | e^{i t_1 \Delta} u (t_1) - e^{i t_2 \Delta} u (t_2) |_2.
\]

By lemma 6.2, the scattering of \( u \) yields

\[
| e^{i t_1 \Delta} u (t_1) - e^{i t_2 \Delta} u (t_2) |_2 \to 0, \quad \text{as } t_1, t_2 \to \infty.
\]

Hence, taking into account (6.5) we obtain

\[
\limsup_{t_1, t_2 \to \infty} | e^{i t_1 \Delta} z_* (t_1) - e^{i t_2 \Delta} z_* (t_2) |_2 \leq \limsup_{t_1, t_2 \to \infty} | e^{i t_1 \Delta} (z_* - u) (t_1) - e^{i t_2 \Delta} (z_* - u) (t_2) |_2 \leq 2 \| z_* - u \|_{C([T, \infty); L^2)} \to 0, \quad \text{as } T \to \infty, \quad \text{a.s.}
\]

This implies that \( \{ e^{i t \Delta} z_* (t) \} \) is a Cauchy sequence in \( L^2 \), thereby yielding (2.23).
Next we prove (2.24). Recall that $U_s(t, s), s, t \geq 0$, are the evolution operators related to the operators $e^{-\phi \Delta} e^{\phi \cdot t}$, $t \geq 0$. Then, by equation (2.18),
\[ z_s(t) = U_s(t, 0) X_0 + i \int_0^t U_s(t, s) e^{\frac{4}{3} \text{Re} \phi} F(z_s) ds, \quad t \geq 0. \]
Since $U_s(0, t) U_s(t, s) = U_s(0, s)$ for $s \geq 0$, applying $U_s(0, t)$ to both sides we get
\[ U_s(0, t) z_s(t) = X_0 + i \int_0^t U_s(0, s) e^{\frac{4}{3} \text{Re} \phi} F(z_s) ds, \]
which implies that for any $0 < t_1 < t_2 < \infty$,
\[ U_s(0, t_2) z_s(t_2) - U_s(0, t_1) z_s(t_1) = i \int_{t_1}^{t_2} U_s(0, s) e^{\frac{4}{3} \text{Re} \phi} F(z_s) ds \]
(6.14)
\[ = U_s(0, t_2) (i \int_{t_1}^{t_2} U_s(t_2, s) e^{\frac{4}{3} \text{Re} \phi} F(z_s) ds) =: U_s(0, t_2) w(t_2). \]
Thus, applying the homogeneous Strichartz estimates in Theorem 3.5 leads to
\[ |U_s(0, t_2) z_s(t_2) - U_s(0, t_1) z_s(t_1)| \leq \|U_s(\cdot, t_2) w(t_2)\|_{C([0, t_2]; L^2)} \leq C |w(t_2)|, \]
where $C$ is independent of $t_1, t_2$. Moreover, since $w(\cdot)$ satisfies (2.29) with the initial datum $w(t_1) = 0$, applying Theorem 3.5 again and using (3.9) we obtain
\[ |w(t_2)| \leq \|w\|_{C([t_1, t_2]; L^2)} \leq C \|e^{\frac{4}{3} \text{Re} \phi} F(z_0)\|_{L^{\frac{2d+4}{2d-4}}([t_1, t_2] \times \mathbb{R}^d)} \leq C \|z_0\|_{V(t_1, t_2)}^{1+\frac{4}{d}}, \]
where $C$ is independent of $t_1, t_2$, due to the global-in-time Strichartz estimates and that $e^{\frac{4}{3} \text{Re} \phi} \in C(\mathbb{R}^d; L^\infty)$, a.s.. Moreover, taking into account (6.4) and (6.5) we have
\[ \|z_s\|_{V(T, \infty)} \leq \|u\|_{V(T, \infty)} + \|z_s - u\|_{V(T, \infty)} < \infty, \quad a.s. \]
Thus, plugging (6.16) into (6.15) and using the global bound (6.17) we obtain
\[ |U_s(0, t_2) z_s(t_2) - U_s(0, t_1) z_s(t_1)| \leq C \|z_s\|_{V(t_1, t_2)}^{1+\frac{4}{d}} \to 0, \quad \text{as} \quad t_1, t_2 \to \infty, \quad a.s. \]
This implies that $\{U_s(0, t) z_s(t)\}$ is a Cauchy sequence in $L^2$, and so (2.24) follows. The proof of Theorem 2.7 (i) is complete.

6.2. Energy-critical and pseudo-conformal cases. As in the mass-critical case, we have the global bound of solutions $X$ and $u$ below.

Lemma 6.4. ([47]) Consider the situations in Theorem 2.7 (ii) with $3 \leq d \leq 6$. Then,
\[ \mathbb{E} \sup_{0 \leq t < \infty} |X(t)|_{H^1}^p + |X(t)|_{L^2}^{\frac{4d}{d-2}} \leq C(p) < \infty \]
for any $p \geq 1$. In particular,
\[ E_\infty := \sup_{0 \leq t < \infty} |X(t)|_{H^1} \leq C < \infty, \quad a.s.. \]

Lemma 6.5. For $z_s(T) \in H^1$, there exists a unique global $H^1$-solution $u$ (depending on $T$) to (6.1) with $\alpha = 1 + \frac{4}{d-2}$, $d \geq 3$, which scatters at infinity and satisfies
\[ \|u\|_{S^1(T, \infty) \cap L^2(T, \infty; H^\frac{3}{2})} \leq C < \infty, \quad a.s., \]
where $C$ is independent of $T$. 37
Theorem 3.5 to (6.1) and using (3.10) and Remark 5.5 we obtain that for any \( t > T \),
\[
\|u\|_{L^2(T,t;H^\frac{3}{2})}^2 + \|u\|_{\dot{H}^1(T,t)}^2 \leq C\|u(T)\|_{H^1} + C\|u\|_{\dot{W}^{\frac{4+2}{2},2}(T,t)}^\frac{4+2}{2},
\]
where \( C \) is independent of \( T \) and \( t \). Since \( \|z^*_t(T)\|_{H^1} \lesssim \|e^{\phi_t^*}u\|_{C([-1,1]^N;L^2(\mathbb{R}^d;X(T)))}, \) using (6.19) if \( 3 \leq d \leq 6 \) and (2.25) if \( d > 6 \) and then letting \( t \to \infty \) we prove (6.20).

We have the crucial asymptotics of difference between the solutions \( z^*_t \) and \( u \).

**Lemma 6.6.** Consider the situations in Theorem 2.7 (ii). Let \( u \) be the solution to (6.1) with \( \alpha = 1 + \frac{4}{d-2} \), \( u(T) = z^*_t(T), \ d \geq 3 \). Then,
\[
(6.21) \quad \|z^*_t - u\|_{S^1(T,\infty) \cap L^2(\mathbb{R}^d;H^\frac{3}{2})} \to 0, \text{ as } T \to \infty, \ a.s.,
\]

**Proof.** The case \( 3 \leq d \leq 6 \) was proved in the recent work [47] under weaker condition on \( \phi_k \). Below we consider the high dimensional case \( d > 6 \).

We shall apply Theorem 4.6 to compare the solutions \( z^*_t \) and \( u \). For this purpose, we reformulate equation (6.1) as follows
\[
i \partial_t u = e^{-\phi^*_t} \Delta (e^{\phi^*_t}u) - e^{\frac{4}{d-2}Re \phi^*_t} F(u) + e,
\]
with the error term
\[
e = -(b^*_t \cdot \nabla + c^*_t)u - (1 - e^{\frac{4}{d-2}Re \phi^*_t})F(u).
\]

We see that, (6.6) implies that for \( T \) large enough,
\[
\sup_{t \geq T} g(t) := \sup_{t \geq T} \sum_{k=1}^N (\int_t^\infty g_k d|\beta_k| + \int_t^\infty g_k^2 ds) \leq \varepsilon,
\]
so the smallness condition on \( g \) in Theorem 4.6 is satisfied on \([T,t]\) for any \( t > T \).

Regarding the error term, similarly to (6.12),
\[
\|e\|_{N^1(T,t) \cap L^2(\mathbb{R}^d;H^\frac{3}{2})} \leq C \varepsilon(T) \|u\|_{L^2(T,t;H^\frac{3}{2})} + \|u\|_{\dot{W}^{\frac{4+2}{2},2}(T,t)} \leq C \varepsilon(T) \to 0, \text{ as } T \to \infty.
\]

where \( C \) is independent of \( T \) and \( t \) due to (6.20), and \( \varepsilon(T) \) is as in (6.12).

Thus, Theorem 4.6 yields that there exist \( c, C > 0 \), independent of \( T \) and \( t \), such that
\[
\|z^*_t - u\|_{S^1(T,t) \cap L^2(\mathbb{R}^d;H^\frac{3}{2})} \leq C(\varepsilon(T))^c.
\]
Therefore, letting \( t \to \infty \) and then taking \( T \to \infty \) we obtain (6.21). \( \square \)

Now, we are ready to prove Theorem 2.7 (ii).

**Proof of Theorem 2.7 (ii).** In the case where \( 3 \leq d \leq 6 \), because of the global well-posedness of (1.1) and the estimate (2.11) in Theorem 2.3 (ii), Assumption \((H0')\) in [47] in the energy-critical case is satisfied. Thus, the asymptotics (2.23) and (2.24) with \( H^1 \) replacing \( L^2 \) follow from Theorem 1.4 in the recent work [47].

Below we consider the case where \( d > 6 \). The proof is similar to that of mass-critical case, thanks to Lemmas 6.4, 6.5 and 6.6.

Actually, let \( u \) be as in Lemma 6.6. We have that for any \( t_1, t_2 \geq T \),
\[
|e^{it_1} \Delta z^*_t(t_1) - e^{it_2} \Delta z^*_t(t_2)|_{H^1} \leq |e^{it_1} \Delta u(t_1) - e^{it_2} \Delta u(t_2)||_{H^1} + 2\|z^*_t - u\|_{C((T,\infty);H^1)}.
\]
Then, we first use the scattering of $u$ in Lemma 6.5 to pass to the limits $t_1, t_2 \to \infty$, and then we use Lemma 6.6 to take the limit $T \to \infty$. It follows that

$$\limsup_{t_1, t_2 \to \infty} \|e^{it_1 \Delta} z_* (t_1) - e^{it_2 \Delta} z_* (t_2)\|_{H^1} \leq 2\|z_* - u\|_{C([T, \infty); H^1)} \to 0, \text{ as } T \to \infty, \text{ a.s.}$$

Hence, $\{e^{it \Delta} z_* (t)\}$ is a Cauchy sequence in $H^1$, which yields (2.23) with $H^1$ replacing $L^2$.

Moreover, applying Theorem 3.5 to (6.14) with $\frac{4}{3 - 2}$ replacing $\frac{4}{7}$ we get

$$|U_s (0, t_2) z_* (t_2) - U_s (0, t_1) z_* (t_1)|_{H^1} \leq C \|e^{\frac{4}{7} \varepsilon \Re \varphi_*} F(z_*)\|_{N^1(t_1, t_2)} \leq C \|z_*\|_{H^1}^{\frac{4 + 2}{3 - 2}}$$

where $C$ is independent of $t_1$ and $t_2$.

Then, taking into account the global pathwise bound of $z_*$ implied by (6.20) and (6.21), we pass to the limits $t_1, t_2 \to \infty$ to obtain that right-hand side above tends to 0 almost surely. This yields that $\{U_s (0, t) z_* (t)\}$ is a Cauchy sequence in $H^1$, thereby implying (2.24) with $H^1$ replacing $L^2$.

Therefore, the proof of Theorem 2.7 (ii) is complete. \(\square\)

**Proof of Theorem 2.7 (iii).** In view of the global well-posedness of (1.1), we see that Assumption (H0') in [47] in the energy-critical case is satisfied. Thus, the asymptotic (2.23) with $\Sigma$ replacing $L^2$ follows from Theorem 1.3 in the recent work [47]. \(\square\)

We close this section with the proof of Theorem 2.5.

**Proof of Theorem 2.5.** In the mass-critical case, the global bound (2.12) follows from (2.8) and Lemma 6.2. In the energy-critical case, the global bound (2.13) is a consequence of (2.11) and Lemma 6.5. \(\square\)

### 7. Support theorem

In this section we prove Theorem 2.9 concerning the Stroock-Varadhan type support theorem. We shall combine the idea of [58] with the stability results in Section 4.

Recall that, for any $h = (h_1, \cdots, h_N) \in \mathcal{H}$ (i.e., the Cameron-Martin space), $X(\beta + h)$ denotes the solution to (1.1) with the driving processes $\beta_k + h_k$ replacing the Brownian motions $\beta_k$, $1 \leq k \leq N$, $S(h)$ denotes the controlled solution to (2.26). The global existence and uniqueness of $X(\beta + h)$ and $S(h)$ can be proved similarly as in Section 5.

In view of Proposition 2.2 in [58], it suffices to prove that, for any $\varepsilon > 0$,

\begin{align}
(7.1) \quad & \lim_{n \to \infty} \mathbb{P}(\|S(\beta^n) - X(\beta)\|_{\mathcal{X}(0, T)} \geq \varepsilon) = 0, \\
(7.2) \quad & \lim_{n \to \infty} \mathbb{P}(\|X(\beta^n - \beta + h) - S(h)\|_{\mathcal{X}(0, T)} \geq \varepsilon) = 0,
\end{align}

where $\mathcal{X}(0, T) = S^0(0, T) \cap L^2(0, T; H^\frac{3}{2})$ or $\mathcal{X}(0, T) = S^1(0, T) \cap L^2(0, T; H^\frac{3}{2})$ in the mass-critical or energy-critical case, respectively, and $\beta^n$ are the adapted linear interpolation of Brownian motions in [58], defined by

$$\beta^n(t) = \beta(\tilde{t}_n) + 2^n(t - \tilde{t}_n)(\beta_{\tilde{t}_n} - \beta_{\tilde{t}_n}),$$

$$\tilde{t}_n = \frac{k}{2^n} \text{ and } \tilde{t}_n = \frac{k - 1}{2^n} \lor 0 \text{ if } \frac{k}{2^n} \leq t < \frac{k + 1}{2^n}.$$  

For this purpose, we first prove the asymptotic result below.

**Lemma 7.1.** Assume $g_k$ are deterministic and continuous, $1 \leq k \leq N$. Then,

\begin{equation}
(7.3) \quad \mathbb{E} \left( \sup_{t \in [0, T]} \left( \int_0^t g_k(s) \beta^n_k(s) ds - \int_0^t g_k(s) d\beta_k(s) \right)^2 \right) \to 0, \text{ as } n \to \infty.
\end{equation}
Proof. Note that, for each $1 \leq k \leq N$ fixed,

$$
| \int_0^t g_k(s) \beta_n^k(s) ds - \int_0^t g_k(s) d\beta_k(s) | \leq | \int_0^t g_k(s) d\beta_k(s) | + | \int_0^t g_k(s) \beta_n^k(s) ds |
$$

$$
+ | \int_0^{[t^n/2^n]} g_k(s) \beta_n^k(s) ds - \int_0^{[t^n/2^n]} g_k(s) d\beta_k(s) |
$$

$$
=: J_{n,1}'(t) + J_{n,2}'(t) + J_{n,3}'(t).
$$

(7.4)

Below we estimate $J_{n,1}'$, $J_{n,2}'$, $J_{n,3}'$ respectively.

First we prove that

$$
\mathbb{E} \sup_{0 \leq t \leq T} (J_{n,1}'(t))^2 \to 0, \ as \ n \to \infty.
$$

(7.5)

To this end, we set $M_k(t) := \int_0^t g_k(s) d\beta_k(s)$. Since $g_k \in C(0, T)$, using the Burkholder-Davis-Gundy inequality we have that for any $p \geq 1$,

$$
\mathbb{E} |M_k(t) - M_k(s)|^{2p} \leq C(p) |t - s|^p.
$$

Then, in view of Kolmogorov’s continuity criterion (see, e.g., [58, Proposition 2.1]), we get that for any $\lambda > 0$, $\gamma < \frac{b-1}{2p}$,

$$
\mathbb{P}(\sup_{t \neq s} |M_k(t) - M_k(s)| > \lambda |t - s|^{\gamma}) \leq C \lambda^{-2p}.
$$

In particular, taking $p = 3$ and $\gamma = \frac{1}{4}$, we arrive at

$$
\mathbb{P}(\sup_{0 \leq t \leq T} |M_k(t) - M_k(\frac{[2^n t]}{2^n})| > \lambda) \leq C \lambda^{-6} 2^{-\frac{3}{2}n}.
$$

This yields that

$$
\mathbb{E} \sup_{0 \leq t \leq T} (J_{n,1}'(t))^2 = 2 \int_0^\infty \lambda \mathbb{P}(\sup_{0 \leq t \leq T} |M_k(t) - M_k(\frac{[2^n t]}{2^n})| > \lambda) d\lambda
$$

$$
\leq \frac{2}{n} + 2 \int_0^\infty \lambda^{-5} 2^{-\frac{3}{2}n} d\lambda = \frac{2}{n} + \frac{1}{2} n^4 2^{-\frac{3}{2}n} \to 0, \ as \ n \to \infty,
$$

which implies (7.5), as claimed.

Similarly, since $g_k \in C(0, T)$, $0 \leq t - \frac{[2^n t]}{2^n} \leq \frac{1}{2^n}$,

$$
J_{n,2}'(t) = 2^n \int_0^{[t^n/2^n]} g_k(s) ds (\beta_k(\frac{[2^n t]}{2^n}) - \beta_k(\frac{[2^n t] - 1}{2^n})) \leq C |\beta_k(\frac{[2^n t]}{2^n}) - \beta_k(\frac{[2^n t] - 1}{2^n})|.
$$

Arguing as above we have

$$
\mathbb{E} \sup_{0 \leq t \leq T} (J_{n,2}'(t))^2 \to 0, \ as \ n \to \infty.
$$

(7.6)

It remains to prove that

$$
\mathbb{E} \sup_{0 \leq t \leq T} (J_{n,3}'(t))^2 \to 0, \ as \ n \to \infty.
$$

(7.7)

For this purpose, since

$$
\int_0^{\frac{2^n}{2^n-1}} g_k(s) \beta_n^k(s) ds = \int_0^{\frac{1}{2^n-1}} (\int_0^{2^n} g_k(r) r^n dr) d\beta_k(s),
$$
Then, using the maximal inequality and the Burkholder-Davis-Gundy inequality we get

\[ J_{n,3}(t) = \left| \sum_{j=1}^{[2^n t]} \int_{-\frac{1}{2^n}}^{\frac{1}{2^n}} g_k(r)2^n dr \right| \]

\[ = \left| \sum_{j=1}^{[2^n t]-1} \int_{-\frac{1}{2^n}}^{\frac{1}{2^n}} (\int_{-\frac{1}{2^n}}^{\frac{1}{2^n}} g_k(r)2^n dr - g_k(s))d\beta_k(s) - \int_{-\frac{1}{2^n}}^{\frac{1}{2^n}} g_k(s)d\beta_k(s) \right|. \]

Let \( M_n(t) := \sum_{j=1}^{[2^n t]-1} \int_{-\frac{1}{2^n}}^{\frac{1}{2^n}} (\int_{-\frac{1}{2^n}}^{\frac{1}{2^n}} g_k(r)2^n dr - g_k(s))d\beta_k(s) \). We get

\[ J_{n,3}(t) \leq |M_n(t)| + \int_{-\frac{1}{2^n}}^{\frac{1}{2^n}} g_k(s)d\beta_k(s). \]

Estimating as in the proof of (7.5) we see that

\[ \mathbb{E} \sup_{0 \leq t \leq T} \int_{-\frac{1}{2^n}}^{\frac{1}{2^n}} g_k(s)d\beta_j(s)^2 \to 0, \text{ as } n \to \infty. \]

Moreover, since \( g_k(s) \) is deterministic, using the independence of increments of Brownian motions we have that for each \( 1 \leq n \leq N, t \mapsto M_n(t) \) is a right-continuous martingale. Then, using the maximal inequality and the Burkholder-Davis-Gundy inequality we get

\[ \mathbb{E} \sup_{0 \leq t \leq T} |M_n(t)|^2 \leq 4 \mathbb{E} |M_n(T)|^2 \leq C \sum_{j=1}^{[2^n T]-1} \int_{-\frac{1}{2^n}}^{\frac{1}{2^n}} (\int_{-\frac{1}{2^n}}^{\frac{1}{2^n}} g_k(r)2^n dr - g_k(s))^2 ds. \]

For any \( \varepsilon > 0 \), by the uniform continuity of \( g_k \) on \([0, T]\), we have that for \( n \) large enough, \( |g_k(r_1) - g_k(r_2)| \leq \varepsilon \) for any \( |r_1 - r_2| \leq 2^{1-n} \). Then, by the mean-value theorem for integrals, we get that for any \( 1 \leq j \leq [2^n T] - 1 \),

\[ |\int_{-\frac{1}{2^n}}^{\frac{1}{2^n}} g_k(r)2^n dr - g_k(s)| \leq |g_k(s_{n,j}) - g_k(s)| \leq \varepsilon, \]

where \( s_{n,j} \in \left( \frac{j-1}{2^n}, \frac{j}{2^n} \right) \). Thus the right-hand-side of (7.10) is bounded by

\[ C \sum_{j=1}^{[2^n T]-1} \int_{-\frac{1}{2^n}}^{\frac{1}{2^n}} \varepsilon^2 ds \leq CT \varepsilon^2. \]

This implies that

\[ \mathbb{E} \sup_{0 \leq t \leq T} |M_n(t)|^2 \to 0, \text{ as } n \to \infty. \]

Thus, we obtain (7.7) from (7.9) and (7.11).

Therefore, collecting (7.4), (7.5), (7.6) and (7.7) together we prove (7.3). \( \square \)

**Proof of Theorem 2.7.** We mainly prove Theorem 2.9 in the energy-critical case when \( 3 \leq d \leq 6 \). The mass-critical case can be proved similarly, based on the stability result Theorem 4.1.

It order to obtain (7.1) and (7.2), it is equivalent to prove that for any subsequence \( \{n_j\} \), there exists some subsequence \( \{n_{j_k}\} \) of \( \{n_j\} \) such that as \( k \to \infty \),

\[ \lim_{k \to \infty} \|S(\beta^n_{n_k}) - X(\beta)\|_{S^1(0,T) \cap L^2(0,T;H_{-1}^\frac{d}{2})} = 0, \text{ a.s.,} \]

\[ \lim_{k \to \infty} \|X(\beta^{n_{k_j}} + h) - S(h)\|_{S^1(0,T) \cap L^2(0,T;H_{-1}^\frac{d}{2})} = 0, \text{ a.s..} \]
In particular, for some positive constant $C$, this along with Assumption (7.19) of Theorem 4.4.9 yields that for any multi-index $\gamma$,

$$
\sup_{n \geq 1} \sup_{0 \leq t \leq T} |\partial_x^n \psi(\beta^n)| \leq C \langle x \rangle^{-2} \sup_{n \geq 1} \sup_{0 \leq t \leq T} g^n(t) \leq C \langle x \rangle^{-2}.
$$
Then, Theorem 3.3 yields that the Strichartz and local smoothing estimates hold for the operator $e^{-\psi(\beta^n)}\Delta(e^{\psi(\beta^n)} \cdot)$, and the corresponding Strichartz constants $C_T$ are uniformly bounded for all $n$.

Estimating as in (5.38) and using the global bounds of the $L^2(0, T; H^\frac{3}{2})$- and $S^1(0, T)$-norms of $\tilde{z}$, we obtain

$$\|e_n\|_{N^1(0, T) + L^2(0, T; H^\frac{1}{2})} \leq C(T) \sup_{0 \leq t \leq T} \varepsilon_n(t)(\|\tilde{z}\|_{L^2(0, T; H^\frac{3}{2})} + \|\tilde{z}\|_W^{1 + \frac{1}{2}}) \leq C'(T)\varepsilon_n(t).$$

We note that $C'(T)$ is independent of $n$, due to the uniform bound (7.19). This along with (7.18) yields that

$$\|\varepsilon_n\|_{N^1(0, T) + L^2(0, T; H^\frac{1}{2})} \to 0, \quad \text{as } n \to \infty, \; a.s.$$  

Then, by virtue of Theorem 4.4, we obtain that $\mathbb{P}$-a.s. as $n \to \infty$,

$$\|z_n - \tilde{z}\|_{S^1(0, T) \cap L^2(0, T; H^\frac{3}{2})} \leq C(T) \sup_{0 \leq t \leq T} \varepsilon_n(t) \to 0.$$  

In particular, this yields the uniform bound

$$\sup_{n \geq 1} \|z_n\|_{S^1(0, T) \cap L^2(0, T; H^\frac{3}{2})} \leq C(T) < \infty, \quad a.s..$$

We claim that (7.22) implies (7.12). Actually, we have

$$\|S(\beta^n) - X(\beta)\|_{S^1(0, T) \cap L^2(0, T; H^\frac{3}{2})} \leq \|e^{\psi(\beta^n)}(z_n - \tilde{z})\|_{S^1(0, T) \cap L^2(0, T; H^\frac{3}{2})} + \|(e^{\psi(\beta^n)} - e^{\psi(\beta)})\tilde{z}\|_{S^1(0, T) \cap L^2(0, T; H^\frac{3}{2})}.$$  

In order to pass to the limit $n \to \infty$, using (7.19) and the inequality $|e^x - e^y| \leq C|e^y||x - y|$ for $|x|, |y| \leq \frac{1}{2}$, we have, as $n \to \infty$,

$$\|e^{\psi(\beta^n)}(z_n - \tilde{z})\|_{S^1(0, T)} + \|(e^{\psi(\beta^n)} - e^{\psi(\beta)})\tilde{z}\|_{S^1(0, T)} \leq C(T)\|z_n - \tilde{z}\|_{S^1(0, T)} + \|(e^{\psi(\beta^n)} - \psi(\beta))\|_{C(0, T; W^{1, \infty})}$$

$$\leq C(T)\|z_n - \tilde{z}\|_{S^1(0, T)} + \sup_{0 \leq t \leq T} \varepsilon_n(t) \to 0,$$  

where in the last step we also used (7.18) and (7.22).

Regarding the $L^2(0, T; H^\frac{3}{2})$-norm in (7.23), we deduce from (7.20) that $e^{\psi(\beta^n)} \in S^0$, and so $\Psi_p := \langle x \rangle^{-1} (\nabla)^\frac{3}{2} e^{\psi(\beta^n)} \langle x \rangle^{-\frac{3}{2}} \langle x \rangle$ is a pseudo-differential operator of order zero. This along with Lemma 3.2 yields

$$\|e^{\psi(\beta^n)}(z_n - \tilde{z})\|_{L^2(0, T; H^\frac{3}{2})} = \|\Psi_p \langle x \rangle^{-1} (\nabla)^\frac{3}{2} (z_n - \tilde{z})\|_{L^2(0, T; L^2)} \leq C \|(z_n - \tilde{z})\|_{L^2(0, T; H^\frac{3}{2})}.$$  

Moreover, using Assumption (H0) and the inequality $|e^x - e^y| \leq C|e^y||x - y|$ for $|x|, |y| \leq \frac{1}{2}$ we have that for any multi-index $\gamma$,

$$\partial^\gamma_t (e^{\psi(\beta^n)} - e^{\psi(\beta)})(t, x) \leq C(T) \langle x \rangle^{-2} \varepsilon_n(t),$$

where $C(T)$ is independent of $n$. Then, estimating as above we get

$$\|(e^{\psi(\beta^n)} - e^{\psi(\beta)})\tilde{z}\|_{L^2(0, T; H^\frac{3}{2})} \leq C(T) \sup_{0 \leq t \leq T} \varepsilon_n(t) \|(z_n - \tilde{z})\|_{L^2(0, T; H^\frac{3}{2})} \leq C(T) \sup_{0 \leq t \leq T} \varepsilon_n(t).$$
Combing the estimates above together we conclude that, \( \mathbb{P}\text{-a.s. as } n \to \infty, \)
\[
\| e^{i(x \cdot h)} (x_n - z) \|_{L^2(0,T;H^3_{-1})} + \| (e^{i(x \cdot h)} - e^{i(x \cdot h)^{\prime}}) z \|_{L^2(0,T;H^{3}_{-1})} 
\leq C(T)(\|(x_n - z)\|_{L^2(0,T;H^3_{-1})} + \sup_{0 \leq t \leq T} \varepsilon_n(t)) \to 0.
\]
(7.25)

Therefore, plugging (7.24) and (7.25) into (7.23) we prove (7.12), as claimed.

**Proof of** (7.13). The proof is similar as above. Now we use a new transformation
\[
y_n = e^{-i(x \cdot h)} X(x_n - h)
\]
to obtain
\[
i \partial_t y_n = e^{-i(x \cdot h)} \Delta (e^{-i(x \cdot h)} y_n) - e^{\frac{t}{4}} \text{Re}(\psi(x \cdot h)) F(y_n)
\]
with \( y_n(0) = X_0 \). Arguing as above, we have the Strichartz and local smoothing estimates for the operator \( e^{-i(x \cdot h)} \Delta (e^{-i(x \cdot h)}), \) the related Strichartz constants \( C_T \) are uniformly bounded of \( n \).

Moreover, letting \( \bar{y} := e^{-i(x \cdot h)} S(h) \) we have
\[
i \partial_t \bar{y} = e^{-i(x \cdot h)} \Delta (e^{-i(x \cdot h)} \bar{y}) - e^{\frac{t}{4}} \text{Re}(\psi(x \cdot h)) F(\bar{y})
\]
(7.28)

with \( \bar{y}(0) = X_0 \) and the error term
\[
eq \left( b(\psi(h)) - b(\psi(x \cdot h)) \right) \cdot \nabla \bar{y} + (c(\psi(h)) - c(\psi(x \cdot h))) \cdot \bar{y}
\]
(7.29)

where \( b(\psi(h)) = 2 \nabla \psi(h) \), \( c(\psi(h)) = \Delta \psi(h) + \sum_{j=1}^{d} (\partial_j \psi(h))^2 \), and \( b(\psi(x \cdot h)) \), \( c(\psi(x \cdot h)) \) are defined similarly.

Similarly to (7.21), for some subsequence of \( \{n\} \) (still denoted by \( \{n\} \)), for any multi-index \( \gamma \),
\[
\sup_{0 \leq t \leq T} \| \partial_\gamma (\psi(x \cdot h) - h)(t) - \psi(h)(t) \| \leq C_\gamma \| x \|^{-1} \sup_{0 \leq t \leq T} \varepsilon_n(t),
\]
and
\[
\sup_{0 \leq t \leq T} \| \psi(x \cdot h) - h)(t) - \psi(h)(t) \|_{W^{1,\infty}} \leq C \| x \|^{-1} \sup_{0 \leq t \leq T} \varepsilon_n(t).
\]
Then, similarly to (7.21), we have that
\[
\| \varepsilon_n \|_{N^1(0,T); L^2(0,T;H^{1}_{-1})} \to 0, \quad \text{as } n \to \infty, \text{ a.s.},
\]
which along with Theorem 4.4 implies that \( \mathbb{P}\text{-a.s. as } n \to \infty, \)
\[
\| y_n - \bar{y} \|_{S^1(0,T) \cap L^2(0,T;H^{2}_{-1})} \to 0.
\]
(7.30)

In particular,
\[
\sup_{n \geq 1} \| y_n \|_{S^1(0,T) \cap L^2(0,T;H^{2}_{-1})} \leq C(T) < \infty, \quad \text{a.s.}
\]

Thus, estimating as those below (7.22) and using (7.30) we obtain (7.13). Therefore, the proof of Theorem 2.9 is complete. \( \square \)
8. Appendix

Proof of Theorem 2.11. The case where \( \sigma \equiv 0 \) can be proved similarly as in [5, Lemma 6.1] and [6, Lemma 2.4] in the \( L^2 \) and \( H^1 \) space, respectively. For the general case, we prove the \( L^2 \) case below, the \( H^1 \) case can be proved similarly.

Set \( \varphi(t) := \varphi_0(t) = \int_0^t G_k \beta_k(s) - \int_0^t \mu ds \) and \( v(t) := v_0(t) = e^{-\varphi(t)}X(t), t \in [0, \tau^*). \) For any \( 0 \leq t < \tau^* - \sigma \), we have

\[
X(t) = e^{\varphi(t)}v(t), \quad X(\sigma + t) = e^{\varphi_\sigma(t)}v_\sigma(t),
\]

and

\[
\varphi(\sigma + t) - \varphi(\sigma) = \varphi_\sigma(t).
\]

It follows that

\[
v_\sigma(t) = e^{-\varphi_\sigma(t)}X(\sigma + t) = e^{-(\varphi(\sigma + t) - \varphi(\sigma))}X(\sigma + t) = e^{\varphi(\sigma)}v(\sigma + t).
\]

Then, similar arguments as in the proof of [5, Lemma 6.1] and [6, Lemma 2.4] in the space \( H^{-2} \), with 0 replacing \( \sigma \). Hence, \( P \)-a.s. for any \( t \in [0, \tau^* - \sigma] \),

\[
iv(\sigma + t) = iv(\sigma) + \int_\sigma^{\sigma+t} e^{-\varphi(s)} \Delta(e^{\varphi(s)}v(s)) ds - \int_\sigma^{\sigma+t} e^{(\alpha - 1)\Re \varphi(s)} F(v(s)) ds,
\]

where the equation is taken in \( H^{-2} \). Plugging this into (8.3) yields that

\[
iv_\sigma(t) = i e^{\varphi(\sigma)}v(\sigma) + \int_\sigma^{\sigma+t} e^{\varphi(\sigma) - \varphi(s)} \Delta(e^{\varphi(s)}v(s)) ds
\]

\[- \int_\sigma^{\sigma+t} e^{\varphi(\sigma)} e^{(\alpha - 1)\Re \varphi(s)} F(v(s)) ds
\]

\[
= iX(\sigma) + \int_0^t e^{\varphi(\sigma) - \varphi(s)} \Delta(e^{\varphi(s)}v(\sigma + s)) ds
\]

\[- \int_0^t e^{\varphi(\sigma)} e^{(\alpha - 1)\Re \varphi(s)} F(v(\sigma + s)) ds.
\]

Note that, by (8.1) and (8.2),

\[
e^{\varphi(\sigma) - \varphi(s)} \Delta(e^{\varphi(s)}v(\sigma + s)) = e^{-\varphi_\sigma(s)} \Delta(X(\sigma + s)) = e^{-\varphi_\sigma(s)} \Delta(e^{\varphi_\sigma(s)}v_\sigma(s)).
\]

Moreover,

\[
e^{\varphi(\sigma)} e^{(\alpha - 1)\Re \varphi(s)} F(v(\sigma + s)) = e^{\varphi(\sigma) - \varphi(s)} F(X(\sigma + s)) = e^{(\alpha - 1)\Re \varphi_\sigma(s)} F(v_\sigma(s)).
\]

Thus, plugging the identities above into (8.4), we obtain \( P \)-a.s. for any \( 0 \leq t < \tau^* - \sigma \),

\[
iv_\sigma(t) = iX(\sigma) + \int_0^t e^{-\varphi_\sigma(s)} \Delta(e^{\varphi_\sigma(s)}v_\sigma(s)) ds - \int_0^t e^{(\alpha - 1)\Re \varphi_\sigma(s)} F(v_\sigma(s)) ds
\]

as an equation in \( H^{-2} \), which implies (2.29), thereby finishing the proof.

Proof of Lemma 5.6. Below, we mainly prove the Itô formula (5.32). The estimate (5.33) can be obtained from (5.32) by using similar arguments as in the proof of [8, (2.4)],[8], involving the Burkholder-Davis-Gundy inequality and the Gronwall inequality.

In order to prove (5.32), we apply the stability result to pass to the limit in the approximating procedure as in the proof of (2.4) in [6].

More precisely, we consider the solution \( X_m \) to (1.1), with the nonlinearity \( \Theta_m(|X|^{4/(d-2)}X) \) replacing \( |X|^{4/(d-2)}X \), where \( \Theta_m f := \mathcal{F}^{-1}(\theta(|\cdot|/m)) f, \theta \in C_c^\infty \) is real-valued, nonnegative, and \( \theta(x) = 1 \) for \( |x| \leq 1, \theta(x) = 0 \) for \( |x| \geq 2 \).
Moreover, similar arguments as in the proof of [6, (3.9)] yield that
the common time regime \([0, \tau^*)\), and
\[
\sup_{m \geq 1} \|X_m\|_{S^1(0,t) \cap L^2(0,t; H^{\frac{3}{2}})} \leq C(t) < \infty, \quad t \in (0, \tau^*), \text{ a.s.}
\]
Moreover, similar arguments as in the proof of [6, (3.9)] yield that
\[
H(X_m(t)) = H(X_0) - \int_0^t \text{Re} \int \nabla X_m \nabla (\mu X_m) dxds + \frac{1}{2} \sum_{k=1}^N \int_0^t |\nabla (G_k X_m)|^2 dxds
- \frac{\lambda(\alpha - 1)}{2} \sum_{k=1}^N \int_0^t \int (\text{Re} G_k)^2 |X_m|^{|\alpha|+1} dxds
- \lambda \int_0^t \text{Re} \int i\nabla((\Theta_m - 1)F(X_m))\nabla X_m dxds
+ \sum_{k=1}^N \int_0^t \text{Re} \int \nabla X_m \nabla (G_k X_m) dx d\beta_k(s) - \lambda \sum_{k=1}^N \int_0^t \int \text{Re} G_k |X_m|^{|\alpha|+1} dx d\beta_k(s).
\]
Then, in order to pass to the limit \(m \to \infty\), we only need to show that, for \(w_m := e^{-\varphi} X_m\) and \(w := e^{-\varphi} X\) with \(\varphi\) as in (2.28) with \(\sigma \equiv 0\),
\[
w_m \to w, \quad \text{in} \ S^1(0,t), \quad \text{as} \ m \to \infty, \quad t \in (0, \tau^*), \text{ a.s.}
\]
(8.5)
For this purpose, we apply the stability result in Section 4 to replace the subcritical arguments in [6]. Note that, \(w_m\) satisfies
\[
i\partial_t w_m = e^{-\varphi} \Delta (e^{\varphi} w_m) - e^{-\varphi} \Theta_m (F(e^{\varphi} z_m))
\]
with \(w_m(0) = X_0\). Moreover, \(w\) satisfies (4.27) with \(\varphi\) replacing \(\psi\), i.e.,
\[
i\partial_t w = e^{-\varphi} \Delta (e^{\varphi} w) - e^{-\varphi} \Theta_m (F(e^{\varphi} z)) + e_m
\]
with the error
\[
e_m = e^{-\varphi} (\Theta_m (F(e^{\varphi} w)) - F(e^{\varphi} w)).
\]
Since for \(p \in (1, \infty), \Theta_m f \to f \) in \(L^p\) (see [6, (3.3)]), we have for \(t \in (0, \tau^*),\)
\[
\|e_m\|_{L^\infty(0,t; W^{1, \frac{2d}{p}})} \leq C(t) \|\Theta_m (F(e^{\varphi} w)) - F(e^{\varphi} w)\|_{L^\infty(0,t; W^{1, \frac{2d}{p})}} \to 0, \quad m \to \infty,
\]
where \(C(t)\) is independent of \(m\).
Therefore, we deduce that the asymptotic (8.5) holds by using the stability result similar to Theorem 4.4 with the nonlinearity \(e^{-\varphi} \Theta_m (F(e^{\varphi} w_m))\) replacing \(e^\frac{1}{p-2} \text{Re} \Phi (F(w)),\) which can be proved similarly as in the proof of Theorem 4.4. Then, we use (8.5) to pass to the limit in the Itô formula of \(H(X_m)\) to obtain (5.32). The proof is complete. 

**Acknowledgements.** The author would like to thank Michael Röckner for warm hospitality during the visit to the University of Bielefeld in August 2018, when part of this work is done. The author is also grateful to Daniel Tataru for valuable discussions on Strichartz and local smoothing estimates during the visit to University of California, Berkeley, in October 2019. Many thanks also to Viorel Barbu, Jun Cao and Jiqiang Zheng for helpful discussions. This work is supported by NSFC (No. 11871337) and Shanghai Rising-Star Program 21QA1404500. The author also thanks for the financial support by the Deutsche Forschungsgemeinschaft (DFG, German Science Foundation) through SFB 1283/2 2021-317210226 at Bielefeld University.
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