Quantum Gravity and
Turning Points in the Semiclassical Approximation*

Esko Keski-Vakkuri

California Institute of Technology
Pasadena CA 91125, USA.
e-mail: esko@theory.caltech.edu

and

Samir D. Mathur

Center for Theoretical Physics
Massachusetts Institute of Technology
Cambridge MA 02139, USA.
e-mail: me@ctpdown.mit.edu

(To appear in Phys. Rev. D)

Abstract:

The wavefunctional in quantum gravity gives an amplitude for 3-geometries and matter fields. The four-space is usually recovered in a semiclassical approximation where the gravity variables are taken to oscillate rapidly compared to matter variables; this recovers the Schrödinger evolution for the matter. We examine turning points in the gravity variables where this approximation appears to be troublesome. We investigate the effect of such a turning point on the matter wavefunction, in simple quantum mechanical models and in a closed minisuperspace cosmology. We find that after evolving sufficiently far from the turning point the matter wavefunction recovers to a form close to that predicted by the semiclassical approximation, and we compute the leading correction (from ‘backreaction’) in a simple model. We also show how turning points can appear in the gravitational sector in dilaton gravity. We give some remarks on the behavior of the wavefunctional in the vicinity of turning points in the context of dilaton gravity black holes.

* This work was supported in part by funds provided by the U.S. Department of Energy under cooperative agreement DE-FC02-94ER40818 and by a grant DE-FG03-92-ER40701.
1 Introduction

An interesting issue in formulations of quantum gravity is to study how the long-distance physics is recovered at the semiclassical limit. If one is interested in asking questions about unitarity, one could take the point of view that it is natural to consider wavefunctions describing states; thus one might like to consider the wavefunctional giving amplitudes for 3-geometries and matter configurations on these 3-geometries. The ‘evolution’ of this wavefunctional is given by the Wheeler–DeWitt (WDW) equation. In this framework, it has been shown [1] how the usual notion of space-time is recovered in a semiclassical approximation of the above wavefunction. One takes the gravity variables to be ‘rapidly oscillating’ while the matter variables are ‘slowly varying’. A WKB approximation to the gravity wavefunction recovers the Schrödinger equation for the matter wavefunction propagating in a WKB ‘time’, which we identify as the fourth coordinate needed to describe spacetime.

This scheme would appear to be problematic when the gravity variables encounter a turning point, where they cannot be rapidly oscillating any more. What will be the effect on the matter wavefunction if the gravity variables pass through such a turning point? This is the question that we investigate in this paper.

We begin this investigation first with the simple quantum mechanical analogue of a light particle (‘matter’) coupled to a heavy particle (‘gravity’). We find two basic cases:

a) Let $\Delta t$ be the ‘time interval’ around the turning point for which the heavy mode is oscillating slower than the light mode. Suppose the light particle wavefunction does not evolve significantly over the interval $\Delta t$. Then, as we would naturally expect, there is no significant effect on the light particle wavefunction of the fact that the heavy mode passed through the turning point.

b) In the second case, the light particle wavefunction evolves significantly in the period $\Delta t$. Now the light particle wavefunction departs from the wavefunction that is predicted semiclassically, in the interval $\Delta t$. It turns out however that after the heavy particle recedes from the turning point and its wavefunction becomes rapidly oscillating again, the light particle wavefunction again approaches the form that it would have had if the turning point had been ignored. We compute the leading order correction arising from the turning point.

Turning points in gravity-matter systems are not unusual[1]. As a simple example, we discuss a minisuperspace toy model of matter in a closed cosmology. In this case there is a turning point in the gravitational sector where the expansion of the universe stops and the recollapse begins. The semiclassical approximation breaks down locally around this turning point, and we apply our analysis to find how the approximation is recovered after passing through the turning point.

$^1$For example, the De Sitter minisuperspace model [2] has a turning point in the gravitational sector, with a finite tunneling direction. This fact has been used in the suggestion of tunneling boundary condition as an initial boundary condition for the Wheeler–DeWitt wavefunction [3]. The behavior and uniqueness of the wavefunction with such an initial boundary condition was studied in [4].
Next we show how turning points arise in the description of the geometry of black holes. One puzzle which has arisen in this context is how different choices of foliations of the black hole spacetime seem to affect the evolution of quantum matter. On one hand, it has been pointed out by several authors that it is possible to foliate the black hole spacetime with hypersurfaces on which the evolution is regular all through (the so-called ‘nice slices’) leading to the loss of unitarity \[5\]. On the other hand, examples of irregular behavior and an apparent breakdown of semiclassical approximation (even outside the horizon and far away from the singularity) with a different choice of foliation have been found \[6, 7\]. This might be closely related to the appearance of large commutators in the variables near the horizon in the treatment of \[10\]. However, it has also been argued that inclusion of the backreaction of the Hawking radiation restores the validity of the semiclassical approximation \[11\].

We study different foliations of the black hole spacetime, and find that the set of spacelike hypersurfaces that occur in this spacetime fall into two categories, which we call I and II. The Wheeler–DeWitt wavefunctional cannot be evolved from category I to category II hypersurfaces without passing through a turning point. The evolution studied in \[3, 4\] takes the matter wavefunction close to such a turning point. This might then explain the apparent breakdown of the semiclassical approximation. We anticipate (although we do not prove this here) that this turning point is similar to case b) above. By contrast the evolution of a particle in Minkowski space is presumably of kind a) above. But by the above discussion of case b), we do not expect any large effect of the turning point on the matter wavefunction in the subsequent evolution after the turning point, even in the black hole geometry.

In the above language, a foliation with ‘nice slices’ corresponds to all hypersurfaces lying in category II. Then no turning point is encountered, and the above issue with the semiclassical approximation does not arise.

The plan of this paper is as follows. In section 2, we start with a simple quantum mechanical analogue of the Wheeler–DeWitt equation. We introduce a turning point into the model, and discuss the semiclassical approximation in the presence of a turning point. We identify the reasons for the breakdown of the approximation in the vicinity of the turning point, and make order of magnitude estimates for the size of this region. Then, we study if and how the semiclassical approximation is recovered after passing through a turning point, by a comparison with an exact solution of the Schrödinger equation. We calculate the leading correction to the semiclassical approximation, due to the effect of the turning point. In section 3, we apply this analysis to a simple minisuperspace model of matter in a closed cosmology, and show that there are no large violations of the semiclassical approximation (after the turning point), if the quantum matter is lighter than Planck mass. In section 4, we discuss foliations of black hole spacetime and show how turning points can arise (in dilaton gravity). We examine the behavior of the wavefunction near the turning point and show that the semiclassical approximation breaks down locally at the turning point. Finally, we make some remarks on the potential tunneling

\[2\]For related issues, see \[8\]. Other potential issues of quantum gravity in black holes were discussed in \[3\].
issues at the turning point.

The semiclassical approximation in dilaton gravity and the problem of time has been also been investigated in [12], and more recently in [13].

2 Simple Quantum Mechanical Examples

2.1 A Heavy and a Light Particle

The simplest model for a discussion of the semiclassical approximation is given by the quantum mechanics of a heavy particle coupled to a light particle. Let a heavy particle of mass \( m_1 \) move in a potential \( U(x_1) \). Let a light particle of mass \( m_2 \) be coupled to the heavy particle through a potential \( u(x_1, x_2) \). The total Hamiltonian of the system is thus

\[
H \equiv H_1 + H_{12} = \frac{p_1^2}{2m_1} + U(x_1) + \frac{p_2^2}{2m_2} + u(x_1, x_2) .
\]

We consider the quantum Hamiltonian \( \hat{H} \) and study its zero energy eigenvalue

\[
\hat{H} \Psi(x_1, x_2) = 0.
\]

This equation can be regarded as a simple analogue to the Wheeler–DeWitt equation of a gravity-matter system, if the potential \( U(x_1) \) is assumed to be of the form \( U(x_1) = m_1 V(x_1) \). Then the heavy particle is analogous to the gravitational sector and the light particle is analogous to the matter sector in quantum gravity. We will now review quickly how the semiclassical approximation is applied to find an approximate solution of (2). The standard discussion [14] goes as follows. To find an approximate solution, we start with an ansatz

\[
\Psi(x_1, x_2) = e^{\pm \hbar [m_1 S_0 + S_1 + m_1^{-1} S_2 + \ldots]}
\]

and solve the equation order by order in \( m_1 \). This procedure yields at order \( \mathcal{O}(m_1^0) \) an approximate solution \( \Psi_{s.c.}(x_1, x_2) \) to (2) with the following properties. The wavefunction \( \Psi_{s.c.} \) can be factorized as

\[
\Psi(x_1, x_2) = \psi_{\text{WKB}}(x_1) \chi(x_1, x_2) ,
\]

where \( \psi_{\text{WKB}} \) is a WKB wavefunction for the heavy particle,

\[
\psi_{\text{WKB}}(x_1) = \frac{1}{(-2V(x_1))^{1/4}} e^{\pm \frac{\hbar}{\pi} m_1 S_0(x_1)} ,
\]

with the exponent \( S_0 \) satisfying the Hamilton-Jacobi equation for the zero energy classical motion of the heavy particle

\[
\frac{1}{2} \left( \frac{dS_0}{dx_1} \right)^2 + V(x_1) = 0 .
\]
Thus,

\[ S_0(x_1) = \int_{x_1} dx'_1 \sqrt{-2V(x'_1)} . \]  

(6)

Since \( dS_0/dx_1 \) is the velocity of the heavy particle, we can find the classical zero energy trajectory \( x_1(t) \) through

\[ t(x_1) = \int_{x_1} \frac{dx'_1}{\sqrt{-2V(x'_1)}} \]  

(7)

with the pertinent initial conditions and thus obtain \( x_1(t) \). We can think that the classical motion of the heavy particle defines a 'clock' measuring time \( t \), often called a 'WKB time'. (We now see the benefit of taking the heavy particle potential of the form \( U = m_1V \): the time \( t \) is independent of the mass of the heavy particle.) The clock is useful in the following way. The function \( \chi(x_1,x_2) \) satisfies the equation

\[ i\hbar \frac{dS_0}{dx_1} \frac{\partial}{\partial x_1} \chi(x_1,x_2) = \hat{H}_{12} \chi(x_1,x_2) \]  

(8)

which can be written in a more suggestive form using the fact that

\[ \frac{dS_0}{dx_1} \frac{\partial}{\partial x_1} = \frac{\partial}{\partial t} . \]

The equation (8) is now recognized as a time dependent Schrödinger equation for the light particle,

\[ i\hbar \frac{\partial}{\partial t} \chi = \hat{H}_{12}(x_1(t),x_2) \chi . \]  

(9)

[Note that the role that \( x_1(t) \) plays in this equation resembles the situation in adiabatic approximation in quantum mechanics, except that the motion of the heavy particle does not have to be 'slow'.] Now let us summarize what was done. We started by looking at a zero energy time independent Schrödinger equation (2). We found an approximate solution, which tells us that the heavy particle motion is essentially classical whereas the light particle behaves in a quantum mechanical way. Although the starting equation does not give any time evolution for the total state, the heavy and the light particle are mutually correlated in such a way that the light particle can be thought to time evolve according to a Schrödinger equation with the time measured by the position of the heavy particle. We point out that since the exponent of the WKB part of the total wavefunction is proportional to \( m_1 \), the heavy particle part of the wavefunction is much more rapidly oscillating compared to \( \chi \) which describes the light particle.

Finally, we would like to repeat three characteristic properties of the semiclassical solution:

1. The wavefunction \( \chi \) was not assumed to be an energy eigenstate of \( \hat{H}_{12} \).

2. The heavy particle is described by a WKB wavefunction.

\[ ^3 \text{It should be noted that these properties differ from those of the Born-Oppenheimer approximation} \]
3. The light particle satisfies a time dependent Schrödinger equation with respect to a time defined by the classical motion of the heavy particle.

To be more precise, we call this the leading order semiclassical approximation to make a distinction with a treatment where higher order \((O(m^{-1}))\) corrections would be included.

We will now turn our attention to the question what happens to the semiclassical approximation if the heavy particle potential \(U\) has classical turning points. Discussions in the literature about this issue seem to be mostly concerned about tunneling. However, we will not discuss tunneling effects but rather study the effects of the existence of one turning point on the validity of the (leading order) semiclassical approximation.

2.2 Local Breakdown of the Semiclassical Approximation in the Vicinity of a Turning Point

Since the WKB approximation for the heavy particle is a part of the semiclassical approximation, the above approximation scheme must break down near a turning point of the heavy particle motion. The WKB approximation is known to be applicable as long as the fractional change in the de Broglie wavelength \(\lambda_{dB}\) of the particle is small over distances which are of the order of the wavelength itself. Since the de Broglie wave length is inversely proportional to the momentum of the particle, and by definition the momentum approaches zero at a classical turning point, in the vicinity of the turning point \(\lambda_{dB}\) grows very quickly and the WKB approximation is no longer applicable.

However, the breakdown of the semiclassical approximation can be due to other reasons than the breakdown of WKB. Let us now incorporate a turning point into the heavy particle sector and then make simple order of magnitude estimates about the size of the region where the semiclassical approximation is problematic.

In order to avoid tunneling effects, we assume that the heavy particle has only one classical turning point, at \(x_1 = a\). It is important to keep in mind that the position of a turning point depends on the available energy of the heavy particle. Let us expand the light particle wavefunction \(\chi\) in energy eigenstates \(\chi_n\)

\[
\chi = \sum_{n=0}^{m} c_n \chi_n(x_1, x_2)
\]

with eigenenergies \(E_n(x_1)\). We will assume\(^4\) that the typical separation of the eigenenergies is \(\Delta E\) and that the difference of the minimum and maximum energy levels is of the same order as \(\Delta E\). We can include the energy \(E_0\) of the reference level \(\chi_0\) into the normalization of the total energy of the system (in the choice of the zero of the total Hamiltonian \(\hat{H}\)), therefore we can

\(^4\)We will assume that \(x_1\) is sufficiently close to the turning point so that we can ignore level crossings. Thus the light particle Hamiltonian is assumed to not change significantly in the region where the WKB approximation for the heavy particle breaks down.
set \( E_0 = 0 \). This is the normalization for the zero energy classical motion of the heavy particle with a turning point at \( x_1 = a \).

Let us fix the additive constant permitted in \( S_0 \) and rewrite (6) as

\[
S_0(x_1) = \int_{x_1}^{a} dx'_1 \sqrt{-2V(x'_1)}
\]  

(10)

so the heavy particle phase factor vanishes at the turning point \( a \). Now we investigate how rapidly the heavy particle and light particle phase factors oscillate near the turning point. Recall that the time dependence in (9) arises through the classical motion \( x_1 = x_1(t) \) of the heavy particle. Near the turning point, the motion of the heavy particle will slow down. Thus the parameter \( x_1(t) \) is slowly evolving. We assume that the energy levels are sufficiently separated, and approximate the time evolution of the light particle wavefunction by an adiabatic approximation

\[
\chi(x_1(t), x_2) = \sum_{n=0}^{m} c_n \exp\{-i \frac{\bar{h}}{1} \int_{0}^{t} E_n(x_1(t')) dt'\} \chi_n(a, x_2),
\]  

(11)

where the time is taken to be zero at the turning point: \( x_1(0) = a \). Now, we can change the integration variable from \( t \) to \( x_1 \). Using \( \dot{x}_1 = dS_0/dx_1 \) with (3) we get

\[
\chi \approx \sum_{n} \exp\{-i \varphi_{l,n}(t)\} \chi_n
\]

where the phase for level \( n \) of the light particle is

\[
\varphi_{l,n} = \frac{1}{\hbar} \int_{x_1}^{a} dx'_1 \frac{E_n(x'_1)}{\sqrt{-2V(x'_1)}}.
\]  

(12)

Let us compare this with the heavy particle phase factor

\[
\varphi_h = \frac{1}{\hbar} m_1 \int_{x_1}^{a} dx'_1 \sqrt{-2V(x'_1)}.
\]  

(13)

An essential feature of the semiclassical approximation for obtaining the wavefunction \( \Psi_{sc} \) is that the heavy particle sector should oscillate more rapidly than the light particle sector, or \( \partial \varphi_h / \partial x_1 \gg \partial \varphi_l / \partial x_1 \). This is equivalent to

\[
\Delta E(x_1) \ll -2m_1V(x_1) = m_1 \left( \frac{dx_1}{dt} \right)^2,
\]  

(14)

where \( \Delta E \) is the range of energies contained in the light particle wavefunction. In other words, the semiclassical approximation is applicable in the region where the kinetic energy of the heavy particle is much larger than the range of the light particle energies. Thus, two features of the semiclassical approximation can be violated in the vicinity of a turning point:

1. The heavy particle cannot be described by the WKB approximation: \( d\lambda_{dB}/dx_1 \sim 1 \).
2. The oscillations in the light particle sector become comparable to those of the heavy particle sector.

We make now order of magnitude estimates for the sizes of the regions where the two conditions are violated. Let us assume that the heavy particle potential is smoothly varying so that we can use a linear approximation near the turning point:

\[ U(x_1) \approx m_1 V'(a) (x_1 - a), \]

and assume that \( \Delta E(x_1) \approx \Delta E(a) \). Then, the WKB approximation breaks down in a region of size

\[ \Delta x_1^{(\text{WKB})} \sim \left( \frac{\hbar^2}{m_1^2 V'(a)} \right)^{\frac{1}{3}}. \]

On the other hand, (14) is violated in a region of size

\[ \Delta x_1^{(\text{s.c.})} \sim \frac{\Delta E}{m_1 V'(a)}. \]

The larger of the two regions \( \Delta x_1^{(\text{WKB})} \), \( \Delta x_1^{(\text{s.c.})} \) is the size of the region where the semiclassical approximation cannot be trusted. The latter region becomes larger than the region of the breakdown of WKB when

\[ \Delta E \gtrsim \left( \frac{\hbar V'(a)}{m_1^2} \right)^2. \quad (15) \]

An alternative way to think is that the region \( \Delta x_1^{(\text{s.c.})} \) (where the light particle wavefunction oscillates faster than the heavy particle wavefunction) is large enough to matter only if the semiclassical evolution in this region would have caused the light particle levels to evolve through a significant phase, i.e.

\[ \varphi_l(a + \Delta x_1^{(\text{s.c.})}) - \varphi_l(a) \gtrsim 2\pi. \quad (16) \]

It turns out that the condition (16) is equivalent to (15). So, to summarize the results of this section:

1. If the range of the light particle energies is extremely small, the semiclassical approximation breaks down near a turning point due to the breakdown of the WKB approximation in the heavy particle sector. The size of the breakdown region is \( \Delta x_1^{(\text{WKB})} \).

2. However, the breakdown of the semiclassical approximation can happen in an even larger region when either the light particle energies have a wide range \( \Delta E \) or when the slope \( U'' \) of the heavy particle potential is very gentle near a turning point. In this case the breakdown region has size \( \Delta x_1^{(\text{s.c.})} \).

The above discussion locates the region where the light particle wavefunction is not well described by the semiclassical approximation. We now ask the question: After the heavy particle moves away from the turning point and the semiclassical approximation becomes good again, what is the net effect that is obtained in its wavefunction due to the passage through the turning point?
2.3 Semiclassical Approximation vs. Exact Results

We consider a simple example where everything can be calculated exactly. Thus it is possible to compare the exact wavefunction with the approximate semiclassical wavefunction and see how good an approximation the latter is. For this purpose, let us choose specific potentials $U(x_1), u(x_1, x_2)$ and consider the model

$$H = \frac{p_1^2}{2m_1} + m_1 rx_1 + \frac{p_2^2}{2m_2} + \frac{\kappa}{2}(x_1 - x_2)^2.$$  

(17)

This system could be thought to describe a heavy particle moving on an incline (with $r$ denoting the slope of the incline) and a light particle moving on a line, connected to the heavy particle by a spring with a spring constant $\kappa$. Let us now find an exact solution of the time independent Schrödinger equation

$$\hat{H} \Psi(x_1, x_2) = E_{\text{total}} \Psi(x_1, x_2),$$

(18)

where an arbitrary energy $E_{\text{total}}$ reflects the freedom in the choice of a zero energy (i.e. freedom in adding an overall constant to the total Hamiltonian). We separate variables by changing the coordinates $x_1, x_2$ to the center of mass coordinate $X = (m_1 x_1 + m_2 x_2)/(m_1 + m_2)$ and the relative separation $y = x_2 - x_1$. Now we write the eigenfunction as $\Psi(x_1, x_2) = F(X)G(y)$. This gives the two equations

$$\begin{align*}
\left\{ -\frac{\hbar^2}{2\mu} \frac{\partial^2}{\partial y^2} + \frac{\kappa}{2} \left(y - \frac{r\mu}{\kappa}\right)^2 \right\} G(y) &= E \; G(y) \quad \text{(19)} \\
\left\{ -\frac{\hbar^2}{2M} \frac{\partial^2}{\partial X^2} + m_1 r X \right\} F(X) &= (E_{\text{total}} - E) \; F(X) \; . 
\end{align*}$$

(20)

Here $M = (m_1 + m_2)$ is the total mass and $\mu = (m_1 m_2)/M$ is the reduced mass. Since $m_1 \gg m_2$, we can assume that $M \approx m_1$ and $\mu \approx m_2$. We will then speak somewhat loosely and call $X \approx x_1$ as the position of the heavy particle and $y$ as the position of the light particle. The first equation is the simple harmonic oscillator Schrödinger equation with the familiar discrete eigenenergies $E = E_n = n\hbar\sqrt{\kappa/\mu}$ and eigenfunctions $G(y) = G_n(y)$. Consider now the second equation. We introduce the wavenumber

$$k(X, E_n) = \frac{1}{\hbar} \sqrt{2rM m_1 \left( \frac{E_{\text{total}} - E_n}{m_1 r} - X \right)}$$

and write the second equation as

$$F''_n(X) + k^2(X, E_n) \; F_n(X) = 0 \; .$$

The heavy particle has a classical turning point at

$$X = \frac{E_{\text{total}} - E_n}{m_1 r} \equiv a(E_n).$$

(21)

---

5Following the convention in Section 2.1 (see the discussion before the Eqn. (10)), the zero point energy is included in $E_{\text{total}}$. 

---
We would like to emphasize that the location of the turning point depends on the energy of the light particle. This is different from what was seen in the semiclassical approximation, there the location of the turning point did not depend on what the light particle was doing. We will show that this fact may give rise to additional effects which can be missed in the leading order semiclassical approximation. The exact solution of the heavy particle equation (see e.g. [15]) is

\[ F_n(X) = \frac{1}{\sqrt{k(X, E_n)}} v_\lambda(S(X, E_n)) , \]

where

\[ S(X, E_n) \equiv \int_X^a dX' k(X', E_n) = \frac{2}{3} \sqrt{\frac{2rm_1M}{\hbar^2}} e^{i\pi} (a(E_n) - X)^{\frac{3}{2}} \]  \hspace{1cm} (22)

and \( v_\lambda \) is a function given in a complex integral form in [15] pp. 134-137. Here the only thing we need to know about \( v_\lambda \) is that there are two independent solutions, corresponding to \( \lambda = 1/6, \ 5/6 \) and that in the region far away from the turning point \( v_\lambda \) reduces to a superposition of two WKB solutions:

\[ v_\lambda \sim A \ e^{i \int_X^a dX' k(X')} + B \ e^{-i \int_X^a dX' k(X')} . \]

However, near the turning point it behaves like

\[ v_\lambda \sim k^{3\lambda} . \]

Now we take into account the joining condition at the turning point. Since the heavy particle is deep in the incline, the wavefunction \( F \) can only be exponentially decaying in the region \( X > a \). Then we know that sufficiently far in the classically allowed region \( X < a \) the function \( F \) takes the form

\[ F_n(X) = \frac{1}{\sqrt{k(X, E_n)}} \cos[-S(X, E_n) - \phi(a(E_n))] \]

where \( \phi(a(E_n)) \) is a constant phase shift due to the turning point. Its general form is

\[ \phi(a(E_n)) = \mu_j \xi(a(E_n)) \] \hspace{1cm} (23)

where \( \mu_j \) is the Maslov index of the corresponding classical trajectory of the heavy particle (crude speaking, the number of times the particle travels through the turning point - in this case \( \mu_j = 1 \)) and \( \xi(a(E_n)) \) is a number depending on the shape of the potential and the location \( a(E_n) \) of the turning point. In the case of a linear potential (and more generally, of a smooth potential which can be approximated by a linear potential at every point), \( \xi(a(E_n)) = \frac{\pi}{4} \). Therefore, for a smooth potential the constant phase shift is independent of the location of the turning point and thus also independent of the energy of the light particle.

\[ ^6 \text{We use the notation } S(X, E_n) \text{ in the context of exact solutions, and the notation } S_0(X) \text{ in the context of semiclassical solutions. The former is defined to contain the mass and } \hbar, \text{ while the latter is defined not to contain } M \text{ or } \hbar. \]
Thus, in our example the total exact wavefunction reduces to

$$\Psi = \sum_n c_n \frac{1}{\sqrt{k(X, E_n)}} \cos[-S(X, E_n) - \frac{\pi}{4}] G_n(y)$$  \hspace{1cm} (24)$$

when \(x_1\) is far from the turning point. In Appendix A we reanalyze the problem using solely semiclassical approximation. This yields an approximate solution

$$\Psi_{\text{s.c.}} = \sum_n c_n \frac{1}{\sqrt{k_0(X)}} \cos[\frac{1}{\hbar} MS_0(X) + \frac{1}{\hbar} E_n t_{\text{WKB}}(X)] G_n(y)$$  \hspace{1cm} (25)$$

where

$$\begin{align*}
S_0(X) &= \frac{2}{3} \sqrt{2r} [a(0) - X]^\frac{3}{2} \\
a(0) &\equiv \frac{E_{\text{total}}}{(Mr)}
\end{align*}$$

We can see that the exact result reproduces the phase behavior of the semiclassical solution in the WKB region. Far away from the turning point \(\sqrt{a(0) - X} \approx \sqrt{a(E_n) - X},\) so \(k_0(X) \approx k(X, E_n)\) (recall that \(M \approx m_1\)). In order to compare the oscillation rates, we expand

$$S(X, E_n) \approx S(X, 0) + E_n \frac{\partial S}{\partial E_n}(X, 0) + \frac{1}{2} E_n^2 \frac{\partial^2 S}{\partial E_n^2}(X, 0) + \cdots$$  \hspace{1cm} (26)$$

The first term \(S(X, 0) \approx -\frac{1}{\hbar} MS_0(X)\). How about the higher order terms? For a generic potential \(U(X) = m_1 V(X)\), the first derivative of \(S\) is

$$\frac{\partial S}{\partial E_n}(X, E_n) = \frac{-M}{\hbar^2} \int_X^{a(E_n)} \frac{dX'}{k(X', a(E_n))}.$$\

If we compare this with the WKB time

$$t_{\text{WKB}} = \frac{M}{\hbar} \int_X^{a(0)} \frac{dX'}{k_0(X')}.$$  \hspace{1cm} (27)$$

we see that

$$\frac{\partial S}{\partial E_n}(X, 0) = -\frac{1}{\hbar} t_{\text{WKB}}.$$  \hspace{1cm} (28)$$

This illustrates how the (WKB) time evolution of light particle states arises from the heavy particle sector. In the present example of the linear potential, all higher derivatives \(\frac{\partial^m S}{\partial E_n^m}\) decay to zero. Thus, far away from the turning point there are no lasting effects from the reflection from the turning point (other than the constant phase shift by \(\pi/2\)) and the semiclassical approximation is again sufficient.

This is not true in general. For instance, the second derivative \(\frac{\partial^2 S}{\partial E_n^2}\) does not always decay to zero but can instead approach a nonzero value. This means that there is a leftover phase factor which is of second order in the light particle energy.
We now give an example of a case where the higher order corrections do not die out. Consider the potential

$$V(X) = \begin{cases} -bX^2 & , X < 0 \\ 0 & , X \geq 0 \end{cases}.$$  

The light particle energy dependent turning point is at $X = a(E_n) = -\sqrt{(E_n - E_{\text{total}})/(bM)}$ (now we must take $E_{\text{total}} < 0$) and the wavenumber is

$$k(X, a(E_n)) = \frac{M}{\hbar} \sqrt{2b \ (X^2 - a^2(E_n))}.$$  

In the leading order semiclassical approximation, the wavenumber is

$$k_0(X) = \frac{M}{\hbar} \sqrt{2b \ (X^2 - a^2(0))}$$

with $a(0) = -\sqrt{(-E_{\text{total}})/(bM)}$. Now we get

$$\frac{\partial S}{\partial E_n}(X, E_n) = \frac{1}{\hbar \sqrt{2b}} \cosh^{-1}\left(\frac{a(0)}{a(E_n)} \cosh\left(-\sqrt{2b \ t_{\text{WKB}}}\right)\right),$$  \(29\)

with $t_{\text{WKB}}$ defined in (27). Thus, $\frac{\partial S}{\partial E_n}(X, 0)$ satisfies (28). However, for the second derivative we get

$$\frac{\partial^2 S}{\partial E_n^2}(X, E_n) = \frac{1}{2\hbar \sqrt{2b(E_n - E_{\text{total}})}} \frac{X}{\sqrt{X^2 - \frac{E_n - E_{\text{total}}}{bM}}}$$

so far away from the turning point it tends to a finite value

$$\frac{\partial^2 S}{\partial E_n^2}(-\infty, 0) = \frac{1}{2\hbar \sqrt{2b(-E_{\text{total}})}}.$$  

If the curvature of the potential well is very small ($b$ small) or if the turning point approaches $X = 0$, this residual energy dependence in the phase factor can be quite significant, and leads to additional interference effects between different energy eigenstates. This is the ‘imprint’ in the wavefunction from the turning point which we mentioned in the introduction.

To extract a result in more general terms notice that the last formula can be rewritten as

$$\frac{\partial^2 S}{\partial E_n^2} = \frac{1}{\hbar M} \sqrt{\frac{-V''(a(0))}{[V'(a(0))]^2}}.$$  \(30\)

Then we can see that for a general potential, if the turning point is near a local maximum of the potential, then for the lowest order corrections to the wavefunction arising from the turning point...
point we must use the expression (30). After the evolution has moved far from the turning point, the light particle wavefunctions far before and after the turning point are related by

\[ \chi_{\text{before}} \equiv \sum_n c_n^{\text{before}} \chi_n \]  

(31)

\[ \chi_{\text{after}} \equiv \sum_n c_n^{\text{after}} \chi_n \]  

(32)

\[ c_n^{\text{after}} = c_n^{\text{before}} e^{-\frac{i}{\hbar} E_n \Delta t_{\text{WKB}}} - i E_n^2 \frac{\partial^2 S}{\partial E_n^2} \]  

(33)

where \( \Delta t_{\text{WKB}} \) depends on the times being compared and the constant \( \frac{\partial^2 S}{\partial E_n^2} \) is determined from (30).

In the above we have seen the distortion of the wavefunction (as compared to the lowest order semiclassical approximation) created by the fact that the slope of the heavy particle potential at the turning point is different for different values of the light particle energy. Thus happens whenever the heavy particle potential is not exactly linear. The location of the turning point varies nonlinearly with the energy, and the consequence of this is that the different energy components of the light particle wavefunction pick up a phase that is not linear in their energy, and such a phase cannot be accounted for by normal evolution using a linear Schrödinger equation for the light particle.

There is another source of the distortion of the wavefunction, which can happen if the phaseshift \( \phi \) is itself different for different positions of the heavy particle turning point. A simple example of this is given by \( V \) which has the form of a finite potential step: now \( \phi(E_n) \) is not a simple constant phase shift like \( \pi/4 \), as we show in Appendix B. This means that different light particle energy levels in the full wavefunction have different phase shifts \( \phi \). Since this is a genuine quantum mechanical phase shift, it may be hard to reproduce by considering higher orders in the semiclassical approximation scheme.

In conclusion, in the simple examples that we have considered, we found that there may be small corrections which can modify the wavefunction slightly away from its leading order semiclassical form. It would be interesting to extend this analysis to systems where the heavy and light particle sectors are coupled in a more non-trivial way, or to systems with more degrees of freedom.

### 3 Example in Minisuperspace Quantum Cosmology

Now that we have examined how turning points modify the the simple phase correlations predicted by the leading order semiclassical approximation, we would like to investigate an example in quantum gravity. We discuss a simplified quantum cosmological toy model of quantum matter evolving in a closed universe and check whether the turning point effects found in the previous section would spoil the semiclassical approximation in this case. It would be quite surprising if this should happen in the case of a macroscopic universe. At least in our
simple model there will turn out to be no large violations of the semiclassical approximation if the quantum matter is lighter than Planck mass.

We start with

\[ S = S_{\text{grav,cl.matter}} = \int d^4x \sqrt{-g} \left( \frac{R}{16\pi G} - \rho_m \right), \]

where we will consider \( \rho_m \) as a function related to the energy density of the pertinent classical matter and we will specify it later. The first term in the action is the Einstein-Hilbert action of classical gravity and \( G \) is the Newton’s constant.

Using a spherically symmetric ansatz for the metric,

\[ ds^2 = \sigma^2 \left[ N^2(t) dt^2 - a^2(t) d\Omega_3^2 \right] \]

where \( N, a \) have dimensions of length, \( t \) is dimensionless and \( \sigma^2 = 2G/3\pi \), the action reduces to

\[ S = \int dtL, \]

\[ L = \frac{N}{2\sigma^2} \left\{ a(1 - \frac{\dot{a}^2}{N^2}) - 4\pi^2 \rho_m a^3 \right\}. \]

If \( \rho_m \) would be a (cosmological) constant, this would be the De Sitter minisuperspace model \[2\]. Instead, we will choose \( \rho_m \) to satisfy

\[ 4\pi^2 \rho_m a^4 \equiv C^2 = \text{constant} \]

which means that we interpret \( \rho_m \) as the energy density of a classical radiation fluid filling the universe. The constant \( C \) has the dimension of length. Now the Lagrangian becomes

\[ L = \frac{N}{2\sigma^2} \left\{ a(1 - \frac{\dot{a}^2}{N^2}) - \frac{C^2}{a} \right\} \]

and, varying with respect to \( N \), we get a classical equation of motion

\[ 1 + \frac{\dot{a}^2}{N^2} - \frac{C^2}{a^2} = 0. \]

The model is time reparametrization invariant, which means that we need to make a gauge choice and choose a particular \( N \). We choose

\[ N = \sigma \equiv \frac{1}{M}, \]

the latter notation is useful since \( M \) is essentially the Planck mass, \( M \sim m_{\text{Planck}} \). Likewise, \( \sigma \) is the Planck length \( \sigma \sim l_{\text{Planck}} \). The classical equation of motion corresponds to the dynamics of a closed radiation filled Robertson-Walker cosmology, with the solution

\[ a = \sqrt{C^2 - \left( \frac{t}{M} \right)^2}. \]
We chose the origin of time so that the expansion of the universe starts at $t = -MC$, it reaches the maximum size $a_{\text{max}} = C$ at $t = 0$, after which it recollapses to zero size at $t = +MC$. If $MC \approx a_{\text{max}}/t_{\text{Planck}} \gg 1$, the universe will expand to a macroscopic size and has a macroscopic lifetime.

When the model is quantized, we obtain the Wheeler–DeWitt equation

$$\frac{1}{2}\left\{ \frac{1}{aM^2} \left( \frac{\partial^2}{\partial a^2} + \frac{\gamma}{a} \frac{\partial}{\partial a} \right) + M^2 a \left( \frac{C^2}{a^2} - 1 \right) \right\} \Psi(a) = 0 \, ,$$

where $0 \leq \gamma \leq 1$ is a free parameter reflecting an operator ordering ambiguity in the kinetic term. The $\gamma$-dependent term is known to be irrelevant in the WKB regime. (We provide an argument for this in Appendix C). Thus, we expect that the value of $\gamma$ will not affect our discussion and will assume $\gamma = 0$ henceforth. We rewrite the WDW equation then as

$$\hat{H}_{\text{gr+c.m.}} \Psi(a) = \frac{1}{2}\left\{ -\frac{1}{aM^2} \hat{p}^2 + M^2 a \left( C^2 - a^2 \right) \right\} \Psi(a) = 0 \, ,$$

where $\hat{p} = -i\partial/\partial a$.

Let us now include additional matter in the model. We assume that the energy of the additional matter is much lower than that of the classical radiation fluid which was already included in $\hat{H}_{\text{gr+c.m.}}$. We will then treat the additional matter quantum mechanically and consider it to propagate in the background of the expanding/recollapsing universe. The WDW equation is now

$$\{ \hat{H}_{\text{gr+c.m.}} + \hat{H}_{\text{qu.matter}} \} \Psi = 0 \, .$$

We will not be more specific about what $\hat{H}_{\text{qu.matter}}$ is. We will examine only a simple case where the quantum matter Hamiltonian has eigenfunctions

$$\hat{H}_{\text{qu.matter}} \Psi_E = EM^2 \Psi_E \, ,$$

with the eigenvalue independent of $a$. (We expect that this will illustrate sufficiently well the scales involved in the problem.) We have written the eigenenergy in a way that will be convenient later. Here $E$ has the same dimension as length. We assume the energy of the quantum matter to be much less than Planck mass, $EM^2 \ll M$. Thus, the different scales are related as follows:

$$EM \ll 1 \ll CM \, .$$

As in the quantum mechanics examples, we are interested in the phase behavior of $\Psi$. The potential energy part of $\hat{H}_{\text{gr+c.m.}}$ has a turning point at the maximum size of the universe. After adding quantum matter, the location of the turning point (i.e., the maximum size of the universe) depends on $E$. The new classical turning point is at

$$a_{\text{max}}(E) = \sqrt{C^2 + E^2 + E} \, .$$
and the classically allowed region is \( a \leq a_{\text{max}}(E) \). We know that in the tunneling region the solution can only be exponentially decaying since the tunneling region is infinite. Thus, from the WKB joining conditions we know that far away in the classically allowed region the wavefunction will reduce to a superposition

\[
\Psi \sim e^{iS + i\pi/4} + e^{-iS - i\pi/4}
\]

where

\[
S = \int_{a}^{a_{\text{max}}(E)} da' p(a')
\]

with the momentum

\[
p(a) = M^2 \sqrt{C^2 + 2aE - a^2}.
\]

Notice that the gravitational ‘potential’ is smooth, and thus the constant phase shift is just \( \pi/4 \), independently of the energy of the quantum matter. Thus this phase shift does not spoil the semiclassical approximation. The result of the integration is

\[
S = \frac{(CM)^2}{2} (1 + \varepsilon^2) \left\{ \frac{\pi}{2} - \sin^{-1}\left( \frac{\bar{a} - \varepsilon}{\sqrt{1 + \varepsilon^2}} \right) - \left( \frac{\bar{a} - \varepsilon}{\sqrt{1 + \varepsilon^2}} \right) \sqrt{1 - \left( \frac{\bar{a} - \varepsilon}{\sqrt{1 + \varepsilon^2}} \right)^2} \right\},
\]

where we introduced dimensionless variables \( \bar{a} = a/C, \varepsilon = E/C \). Notice that \( S \) is proportional to the square of the maximum size \( a_{\text{max}}(0) = CM \). Following Section 2.3, we expand \( S \) in the rescaled quantum matter energy \( \varepsilon \):

\[
S(\bar{a}, \varepsilon) = S(\bar{a}, 0) + \varepsilon \frac{\partial S}{\partial \varepsilon}(\bar{a}, 0) + \frac{1}{2} \varepsilon^2 \frac{\partial^2 S}{\partial \varepsilon^2}(\bar{a}, 0) + \cdots .
\]

The first term is the solution of the classical Hamilton-Jacobi equation in the semiclassical approximation. The coefficient of the second term is

\[
\frac{\partial S}{\partial \varepsilon}(\bar{a}, 0) = (CM)^2 \sqrt{1 - \bar{a}^2}.
\]

The scale factor \( \bar{a} \) is related to the WKB time by \( \bar{a}^2 = 1 - (t/CM)^2 \), so (39) becomes

\[
\frac{\partial S}{\partial \varepsilon} = CMt ,
\]

as expected. The coefficient of the third term turns out to be

\[
\frac{\partial^2 S}{\partial \varepsilon^2} = (MC)^2 \left\{ \frac{\pi}{2} - \sin^{-1}\left[ \sqrt{1 - \left( \frac{t}{CM} \right)^2} \right] + \sqrt{\left( \frac{CM}{t} \right)^2 - 1} \right\}.
\]

As Figure 1 shows, away from the vicinity of the turning point (31) quickly decays to order \((CM)^2\).

\[\text{From (31) we see that } \varepsilon \ll 1/CM \ll 1.\]
Figure 1: Time dependence of the second derivative of $S$ as a function of $t/CM$.

Thus we see that (40) gives a phase to eigenstates proportional to their energy, so it represents quantum mechanical evolution expected from the semiclassical approximation. The correction (41) on the other hand gives a term proportional to the square of the energy, $\sim (EM)^2$, and so cannot be absorbed into a quantum mechanical evolution with the matter Hamiltonian $\hat{H}_{\text{qu.matter}}$.

The key fact is that the nonzero end value for the third term of (38) is of the order $(EM)^2$. The correction is thus negligibly small as long as the energy of the quantum matter is much less than than Planck energy (37). If the energy of the quantum matter is larger than the Planck energy, then we can still get the correction terms to be small if we consider evolution over times small compared to the age $CM$ of the Universe; the required restrictions can be found from (41).

4 Turning Points and Black Hole Evolution in Dilaton Gravity

Models of 1+1 dimensional gravity have been around for quite a while [16]. More recently, models with black hole like solutions have become popular. We are especially interested in the black hole spacetimes which are vacuum solutions in the CGHS model [17]. This model has recently been studied in the framework of Dirac quantization and solutions to the Wheeler–DeWitt equation have been found [18, 19]. [The equivalence of the solutions of [18] and [19]
is shown in [20]. Consider all hypersurfaces in a black hole spacetime. The Wheeler–DeWitt equation controls evolution from one hypersurface to the next one. In such an evolution, turning points may appear as we shall show in this section. We will examine the behavior of the (gravitational) wavefunction in the vicinity of a turning point. We need to identify the classically allowed region where the wavefunction is oscillatory. We will also investigate the possibility of having exponentially decaying (or growing) behavior in classically disallowed regions. In other words, we study if tunneling issues can arise.

4.1 Hamiltonian Formulation of Dilaton Gravities

We start with a brief review of the results presented in [18, 21, 22]. [For additional discussion, see [8].] The action of the CGHS model of dilaton gravity is most commonly presented in the form

\[
S = \int d^2x \sqrt{-g} \ e^{-2\phi} \left[ R + 4(\nabla \phi)^2 + 4\lambda^2 \right].
\]

(42)

It is possible to rescale the metric, \( (g_{\alpha\beta}) = \frac{1}{8} e^{2\phi} (\bar{g}_{\alpha\beta}) \), so that the kinetic term disappears from the action. Then the action becomes

\[
S = \int d^2x \sqrt{-\bar{g}} \left[ \bar{\phi} \bar{R} - \bar{V}(\bar{\phi}) \right],
\]

(43)

where

\[
\bar{\phi} = e^{-2\phi}.
\]

For the CGHS model, \( \bar{V}(\bar{\phi}) = -\frac{\lambda^2}{2} \). Other models of dilaton gravity give arise to different forms of the potential \( \bar{V} \). (For some more details, see Appendix D.) The action (43) is the starting point in the Hamiltonian formulation. To proceed, the metric is parametrized by introducing the lapse and shift functions \( N, N_\perp \)

\[
ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta = \frac{1}{8} e^{2\phi} \bar{g}_{\alpha\beta} dx^\alpha dx^\beta = \frac{1}{8} e^{2(\phi+\bar{\rho})} \left[ -N^2 (dx^0)^2 + (dx^1 + N_\perp dx^0)^2 \right].
\]

(44)

Using the parametrization (44) in (43) one can rewrite the action in a form from which it is easy to read out the canonical momenta

\[
\Pi_{\phi} = \frac{2}{N} (N_\perp \dot{\bar{\rho}}' + N'_\perp - \dot{\bar{\rho}})
\]

\[
\Pi_{\bar{\rho}} = \frac{2}{N} (-\dot{\bar{\phi}} + N_\perp \bar{\phi}')
\]

\[
\Pi_N = \Pi_{N_\perp} = 0
\]

where \( ' = \partial/\partial x^1 \) and \( \cdot = \partial/\partial x^0 \). The Hamiltonian turns out to be a sum of the time reparametrization and spatial diffeomorphism constraints

\[
H = \int dx^1 \left[ NH_N + N_\perp H_\perp \right]
\]

17
where

$$H = 2\ddot{\phi}'' - 2\dot{\phi}'\dot{\rho}' - \frac{1}{2}\Pi_\rho\Pi_{\bar{\phi}} + e^{2\bar{\rho}}\bar{V}(\bar{\phi})$$

$$H_\perp = \bar{\rho}'\Pi_{\bar{\phi}} + \bar{\phi}\Pi_{\bar{\rho}} - \Pi'_\rho.$$  

The time reparametrization constraint $H = 0$ can be written as a Hamilton-Jacobi (H-J) equation

$$\frac{1}{2}\left[ \frac{\partial S}{\partial \bar{\phi}} \frac{\partial S}{\partial \bar{\rho}} - g[\bar{\phi}, \bar{\rho}] \right] = 0$$

where

$$g[\bar{\phi}, \bar{\rho}] = 4\bar{\phi}'' - 4\bar{\phi}'\bar{\rho}' + 2e^{2\bar{\rho}}\bar{V}(\bar{\phi}).$$

It was found [18] that a solution to the H-J equation is given by a functional

$$S[\bar{\phi}, \bar{\rho}, \mathcal{C}] = \int dx^1 \{ Q_{\mathcal{C}} + \bar{\rho}'\ln\left[\frac{2\bar{\phi}' - Q_{\mathcal{C}}}{2\bar{\phi}' + Q_{\mathcal{C}}}\right]\}$$  

which depends on an integration constant $\mathcal{C}$ through the functional

$$Q_{\mathcal{C}} = 2\sqrt{\bar{\phi}'^2 + (\mathcal{C} + j(\bar{\phi}))e^{2\bar{\rho}}}$$  

where $j(\bar{\phi})$ is given by $dj(\bar{\phi})/d\bar{\phi} = \bar{V}(\bar{\phi})$. For the CGHS model, $j(\bar{\phi}) = -\frac{\lambda}{2}\bar{\phi}$. The constant $\mathcal{C}$ can alternatively be expressed as a functional of the canonical variables $\bar{\rho}, \bar{\phi}, \Pi_{\bar{\rho}}, \Pi_{\bar{\phi}}$, using the relations

$$\Pi_{\bar{\phi}} = \frac{\delta S}{\delta \bar{\phi}} = \frac{g}{Q_{\mathcal{C}}}$$

$$\Pi_{\bar{\rho}} = \frac{\delta S}{\delta \bar{\rho}} = Q_{\mathcal{C}}. $$

The result is

$$\mathcal{C} = e^{-2\bar{\rho}}\left(\frac{1}{4}\Pi_{\bar{\rho}}^2 - (\bar{\phi}')^2\right) - j(\bar{\phi}).$$

So far we have not identified the coordinates $x^0, x^1$. This can be done after gauge fixing. Different choices of $N, N_\perp$ correspond to different gauge choices. A common choice is the conformal gauge $N = 1, N_\perp = 0$, so (44) becomes

$$ds^2 = e^{2\phi} [(dx^0)^2 - (dx^1)^2] = \frac{1}{8} e^{2(\phi + \bar{\rho})} [(dx^0)^2 - (dx^1)^2].$$

The CGHS model has a global symmetry which allows us to additionally fix the relation of $\rho$ and $\phi$ to be $\rho = \phi$. This is called the Kruskal gauge. Now the metric (50) can be written as

$$ds^2 = -e^{2\phi} dx^+ dx^-$$
where $x^{\pm} = x^0 \pm x^1$ are the Kruskal coordinates. The gauge choices simplify the expressions of the momenta. First, in the conformal gauge

$$\Pi_{\phi} = -2\dot{\rho}$$

$$\Pi_{\rho} = -2\dot{\phi}.$$ 

Then, in Kruskal gauge $\bar{\rho} = \sqrt{8}$ (see (50), so one of the momenta vanishes identically: $\Pi_{\bar{\phi}} = 0$.

The field equations of the CGHS model have solutions which have the same properties as static black holes. In the Kruskal gauge, these are given by

$$e^{-2\rho} = e^{-2\phi} = \frac{M}{\lambda} - \lambda x^+ \lambda x^-$$

where $M$ is the mass of the black hole. We can use this to identify the integration constant $C$. First we express $Q_C$ and $\Pi_{\bar{\rho}}$ in the CGHS fields:

$$Q_C = 4\sqrt{e^{-4\phi}(\phi')^2 + (2C - \lambda^2 e^{-2\phi})}$$

and

$$\Pi_{\bar{\rho}} = 4e^{-2\phi} \dot{\phi}.$$ 

Then, from (51) we find $\phi', \dot{\phi}$ and substitute these to $Q_C, \Pi_{\bar{\rho}}$. Using (18) we can finally identify

$$C = \frac{M\lambda}{2}.$$ 

(From now on, we will set $\lambda = 1$.)

To conclude this subsection, we briefly mention some results from efforts to quantize the theory. Following the Dirac quantization approach in the Schrödinger representation, the Hamiltonian constraint is replaced by the Wheeler–DeWitt equation. In [18] a specific operator ordering was used and the WDW equation was written as

$$\hat{H} \Psi(\bar{\phi}, \bar{\rho}) = \frac{1}{2}(g - Q_C \hat{\Pi}_{\phi} Q_C^{-1} \hat{\Pi}_{\bar{\rho}}) \Psi(\bar{\phi}, \bar{\rho}) = 0.$$ 

This equation is solved by a wavefunctional $\Psi = \exp(\pm iS)$ where $S$ is the solution (15) of the H-J equation. It is curious that the exact solution should take the form of a WKB solution. This seems to be tied with the specific choice of operator ordering. As we shall discuss, $Q_C$ tends to zero at a turning point, so (54) becomes singular. This suggests that the correct quantization would have significantly different choice of operator ordering near the turning point, and such a WKB solution would not be exact. We will thus use the WKB solution in the same spirit as in the previous section to study the potential troubles with the semiclassical approximation near the turning point.

9This formula depends on the choice of the integration constant mentioned in a footnote in Appendix D. However, the important point is that after $M$ has been substituted to the expression for $Q$, the resulting expression contains no ambiguous integration constants.

10We should emphasize that these statements apply to the formulation of dilaton gravity in the variables
4.2 Turning Points in Black Hole Evolution

Let us investigate the nature of spacelike hypersurfaces that we must consider. By the diffeomorphism constraint we know that the amplitude for any hypersurface (in the quantum gravity wavefunctional) must depend only on the intrinsic geometry of the hypersurface, and not on the coordinates used to parameterize it. To describe this intrinsic geometry we consider the scalar function \( \phi \) on the hypersurface. Let \( s \) be the invariant length along the hypersurface from some fixed point \( \phi = \phi_0 \), measured in the direction of increasing \( \phi \), say. The function \( \phi(s) \) gives an intrinsic characterization of the hypersurface.

As an example, consider (in the above black hole geometry) a class of hypersurfaces \( \Sigma_\gamma \) which are straight lines in Kruskal coordinates

\[
\Sigma_\gamma : \quad x^- = -\alpha^2 x^+ + \gamma .
\]  

(55)

For a fixed slope \( \alpha^2 \), there is one free parameter \( \gamma \) which can be positive or negative. The special case \( \gamma = 0 \) corresponds to constant Schwarzschild time slices \( t = -\ln \alpha \). For \( \Sigma_\gamma \) we find

\[
|\frac{d\phi}{ds}(\phi)| = e^{\phi/2} \sqrt{\left(\frac{\gamma}{\alpha}\right)^2 + 4(e^{-2\phi} - M)} .
\]  

(56)

The equation could be integrated to give \( \phi(s) \). Note that either sign of \( \gamma \) gives the same function \( \phi(s) \). Thus each one-geometry has two embeddings in the spacetime, except for \( \gamma = 0 \), where the one-geometry has only one kind of an embedding. Note that for a given value of \( \phi \) the slope \( |\frac{d\phi}{ds}(\phi)| \) is smallest when the direction of the hypersurface through that point corresponds to \( \gamma = 0 \).

The above situation is generic. In any spacetime, pick a point \( P \) (with value of scalar \( \phi = \phi_0 \), and consider all possible infinitesimal spacelike line segments through this point. The quantity \( |\frac{d\phi}{ds}(\phi_0)| \) reaches a minimum on a certain unique direction of the line segment; call this segment \( \sigma_0 \), and let the value of \( |\frac{d\phi}{ds}(\phi_0)| \) on \( \sigma_0 \) be called \( |\frac{d\phi}{ds}(\phi_0)|_{\text{min}} \). This line segment partitions the set of all spacelike segments at our point into two connected classes. For any given value of \( |\frac{d\phi}{ds}(\phi_0)| \) at our point, with

\[
|\frac{d\phi}{ds}(\phi_0)|_{\text{min}} < |\frac{d\phi}{ds}(\phi_0)| < \infty ,
\]  

(57)

there is exactly one line segment in each of the two classes, call these segments as \( \sigma_\pm \), matching with that value of \( |\frac{d\phi}{ds}(\phi_0)| \). We label these two classes as category I and category II as shown in Figure 2.

and foliations considered in this paper. Although in the present setting we find turning points and expect the breakdown of WKB approximation in their vicinity, it does not exclude the possibility of finding a canonical transformation into some other coordinates in which the turning point issues do not arise. For instance, in a “laboratory frame” the motion of a simple harmonic oscillator has two turning points, and the WKB approximation would break down in their vicinity. However, after a canonical transformation to action-angle variables, the turning points disappear and the WKB approximation is exact. It may be interesting to note there are alternative variables for dilaton gravity \([23]\) in which the WKB approximation is again thought to be exact \([24]\). It would be interesting to investigate what happens to turning points in the context of \([24]\).
Since the ‘special direction’ given by the segment $\sigma_+$ at each spacetime point is continuous as we move the point $P$ (as is the direction in the other category given by $\sigma_-$), we can use the above split of the tangent space at each point to define two categories of hypersurfaces. A category I hypersurface has its tangent direction at each point in category I of tangent directions, and similarly for category II. (We ignore for the discussion below the case where the hypersurface alternates between categories I and II.)

Thus in the above example of the black hole geometry the surfaces $\Sigma_\gamma$ with $\gamma < 0$ belong to category I and the with $\gamma > 0$ belong to category II. The constant Schwarzschild time hypersurfaces lie at the intersection of the two categories.

Now we note that the hypersurfaces at the boundary of categories I and II are turning points in the evolution given by the Wheeler–DeWitt equation. In the simple 1-dimensional problems considered in section 2 we see that the classical trajectory passes twice through each point $x_1$, once with positive momentum (going towards the turning point) and once with negative momentum (receding from the turning point). At the turning point, the momentum is zero. The analogue of the coordinate $x_1$ is the ‘gravitational variable’, the 1-geometry, and as we saw above, at least locally the same intrinsic 1-geometry embeds in two different ways in the classical spacetime, once in category I and once in category II. We will show now that the momentum of the gravitational field tends to zero at the boundary of the two categories. Further, the absolute value of the momentum is the same when we consider the same intrinsic 1-geometry in the two categories I and II. This is analogous to the particle quantum mechanics case where the absolute value of the momentum is the same at point $x_1$ both on the trajectory leading towards the turning point and on the trajectory returning from the turning point.

For simplicity let us consider again the surfaces $\Sigma_\gamma$ of (55) (in the case of an eternal black hole). Let us rewrite the defining equation as

$$x^0 = \frac{1 - \alpha^2}{1 + \alpha^2} x^1 + \frac{\gamma}{1 + \alpha^2}.$$

\textbf{Figure 2:} Local definition of the two classes of segments.
If $\alpha^2 = 1$, these are just constant Kruskal time hypersurfaces $x^0 = \gamma/2$. In the case $\alpha^2 \neq 1$ it is convenient to use a boosted Kruskal frame $\tilde{x}^\pm$ with

$$
\begin{align*}
\tilde{x}^0 &= x^0 \cosh \Theta - x^1 \sinh \Theta \\
\tilde{x}^1 &= -x^0 \sinh \Theta + x^1 \cosh \Theta \, .
\end{align*}
$$

(59)

with a boost angle $\Theta$ given by

$$
\tanh \Theta \equiv \frac{1 - \alpha^2}{1 + \alpha^2}.
$$

In the new frame, (58) is seen to correspond to constant time slices $\tilde{x}^0 = \tilde{\gamma}/2$ with $\tilde{\gamma} = \gamma/\alpha$. Now

$$
e^{-2\phi} = M - \frac{\tilde{\gamma}^2}{4} + (\tilde{x}^1)^2
$$

so

$$
\phi' \equiv \frac{d\phi}{d\tilde{x}^1}(\phi) = -e^{2\phi} \tilde{x}^1
$$

(60)

for a $\Sigma_\gamma$ hypersurface. We already found what $Q_C$ looks like in the CGHS notation, and similarly

$$
g = 8e^{-2\phi} [2(\phi')^2 - \phi'' - e^{2\rho}]
$$

Substituting (60) to $Q_C$ and $g$ yields

$$
\begin{align*}
\Pi_\rho &= Q_C = 2|\tilde{\gamma}| \\
\Pi_\phi &= \frac{g}{Q_C} \equiv 0 .
\end{align*}
$$

(61)

The latter equation is just what we expect from our previous observation that $\Pi_\phi \equiv 0$ in Kruskal gauge. More important is that the first equation shows that as $\gamma$ changes from negative to positive (crossing the boundary of categories), the momentum $\Pi_\rho$ decreases to zero at the boundary and then increases again; the absolute value is the same for corresponding points in the two categories; i.e., surfaces described by $\pm \gamma$. This is consistent with the boundary of the two categories being a turning point in dilaton gravity. (As noted above, the turning point hypersurfaces are constant Schwarzschild time slices.)

### 4.3 Absence of Tunneling

Recall the situation in our earlier 1-dimensional quantum mechanical examples. The classical trajectory reflects off the turning point. The quantum wavefunction decays exponentially inside the barrier, and we had assumed that this barrier grows without bound as we go towards large $x$, so that there is no tunneling. This was also the case in the minisuperspace closed cosmology. Let us verify that an analogous situation holds in the dilaton gravity case too.

11Boosts do not affect the Kruskal gauge $\rho = \phi$. 

22
We illustrate in Figure 3 below the situation for a particle moving in a linear potential \( U(x) = x \). The momentum \( p \) becomes imaginary in the region \( x > E \). In the dilaton gravity case, we consider a 1-parameter family of 1-geometries foliating the spacetime; this parameter will be analogous to the coordinate \( x \). The energy \( E \) of the quantum mechanical example will turn out to be analogous to the mass \( M \) in the black hole spacetime. We will see that the wavefunctional gets increasingly suppressed as we move deeper into the classically disallowed region along the parameter labeling the 1-geometries, so there is no hint of a finite tunneling direction in this simple model.

Let us examine how the gravitational wavefunction behaves near the turning point. We rewrite \( S \) in a geometric form

\[
S = \int ds \, e^{-2\phi} \left\{ 4q - 2 \frac{d\phi}{ds} \ln \left[ \frac{d\phi}{ds} + q \right] \right\}
\]

where \( s \) is the invariant length along a one-geometry and

\[
q \equiv q[\phi(s)] = \sqrt{\left( \frac{d\phi}{ds} \right)^2 + (Me^{2\phi} - 1)}.
\]

With this notation, the wavefunctional \( \Psi = \exp iS \) explicitly depends on a one-geometry \( \phi(s) \). At a turning point, the exponent \( S \) is zero.

A possibility to consider is that at the turning point the one-geometries might tunnel into the direction where they can only fit into spacetimes of more massive black holes. For example, consider the direction of one-geometries with

\[
\frac{d\phi}{ds}(\phi) = \sqrt{1 - (\bar{M} - \frac{\gamma^2}{4})e^{2\phi}}
\]

where \( \bar{M} > M \). These can fit in the spacetime of a higher mass \( \bar{M} \) black hole as slices \( x^0 = \gamma^2/4 \). They give arise to a momentum

\[
\Pi_\beta = Q_M = 4\sqrt{M - (\bar{M} - \frac{\gamma^2}{4})}
\]

which becomes imaginary if \( \bar{M} - \frac{\gamma^2}{4} > M \). Let us check how the wavefunctional behaves in this case. Rewriting \( Q_M \) as \( Q_M = i2\delta \), the exponent of the wavefunctional is

\[
S = \int_{-\infty}^{\infty} dx \, \{ Q_M + \bar{\phi}' \ln[\frac{2\bar{\phi}' - Q_M}{2\bar{\phi}' + Q_M}] \}
\]

\[
= i \int_{-\infty}^{\infty} du \, \{ \delta - \tan^{-1}(\frac{u}{\delta}) \} = \frac{i\pi \delta^2}{2}.
\]

After resubstituting \( \delta \) we find that the wavefunctional behaves as

\[
\Psi = \exp\{-2\pi[\bar{M} - \frac{\gamma^2}{4} - M]\}.
\]

(63)
We interpret the situation as follows. Let us compare (62) with a momentum of a particle in a
one-dimensional potential, \( p = \sqrt{2m(E - U(x))} \). We can identify \( M \) in (62) as the total gravita-
tional (ADM) energy, which is the analogue of the total energy \( E \) in the particle mechanics
case. Similarly we can identify \( \bar{M} - \gamma^2/4 \) as a coordinate, analogous to \( x \). Thus the situation is
analogous to a particle moving in a linear potential \( U(x) = x \), where the momentum \( p \) becomes
imaginary in the region \( x > E \). We illustrate this in Figure 3.

![Figure 3: The classically allowed region and the tunneling region.](image)

The tunneling direction is infinite, so the one-geometries cannot escape from the classically
allowed region. Further, the gravitational ‘potential’ appears to be smooth. Therefore in this
case we cannot see any hint of a behavior which would be required for the deviations from
semiclassical approximation which were discussed in Section 2.

4.4 Summary

We have seen in this section that the gravity part of the wavefunctional has turning points in
the evolution given by the Wheeler–DeWitt equation, and that there are no finite tunneling
directions from the turning point. A consequence of these turning points is that the semiclassical
evolution [12, 13] of matter breaks down in the vicinity of the turning point. For example, in
[13] it was shown how the semiclassical expansion yields a Tomonaga–Schwinger equation

\[
\frac{-i}{2} \left( \frac{\delta S}{\delta \phi} \frac{\delta}{\delta \phi} + \frac{\delta S}{\delta \rho} \frac{\delta}{\delta \rho} \right) \chi = \mathcal{H}_m \chi
\]

(64)

for the evolution of the matter wavefunctional \( \chi \). \( S \) is the solution of the H-J equation, and \( \mathcal{H}_m \)
is the Hamiltonian density of the matter field \( f \) in the Schrödinger picture.\(^{12}\)

If there are no turning points, this equation can be formally integrated to yield the functional
Schrödinger equation for the free scalar field on the classical background spacetime. However,

\(^{12}\)Note that our conventions are opposite of those of [13]: we denote the dilaton and the conformal
factor which appear in the action (43) as \( \phi, \rho \) whereas in [13] a notation \( \phi, \rho \) is used.
there is an obstacle if there are turning points. This equation cannot evolve the matter wavefunctional from slices of category I to slices of category II. This is because if we write the wavefunctional for matter as a function of variables that give the intrinsic 1-geometry of a slice, then both before and after the turning point we encounter the same 1-geometry. We would like the matter to evolve to different values on these two different slices with the same intrinsic geometry, but a first order evolution equation does not permit that. In handling the full second order Wheeler–DeWitt equation we have to understand the physics of the turning point. But from the discussion above (in particular the absence of tunneling directions) we expect that the quantum mechanical examples of the earlier sections should be a reliable guide. Thus we would expect that while the matter wavefunction gets distorted away from its semiclassical form near the turning point, it regains its semiclassical form away from the turning point. Thus we can patch together the solution of the Wheeler–DeWitt equation in the two categories, and have a complete solution on all hypersurfaces.

5 Discussion

In this paper we have examined the effect on the evolution of matter modes when the gravity wavefunctional passes through a turning point. At such a point the usual decomposition of the complete wavefunctional, into a ‘fast’ gravity part giving a classical spacetime and a ‘slow’ part giving matter evolution on spacetime, breaks down. One then asks what effect this apparent breakdown of the semiclassical approximation creates on the matter wavefunction after we recede from the turning point. As we saw, there are two cases to analyze. In one case the ‘time’ for which the gravity mode ceases to be ‘fast’ is so small that the matter wavefunction does not evolve significantly in that time; here it is easy to conclude that there is no lasting effect of the turning point. In the other case the matter wavefunction does evolve significantly in this period, and a more detailed analysis shows that there is again no lasting effect after we recede from the turning point, provided that there are no tunneling directions available at the turning point. (If there are tunneling directions then we cannot approximate the wavefunction as being composed only of the incident and reflected parts at the turning point.)

The above issue will arise in any approach to quantum gravity that uses a wavefunctional formalism. How do we fit together parts of the solution to the Wheeler–DeWitt equation that are found in the different domains where semiclassical evolution is well defined? At the

\footnote{It would be interesting to estimate the “size” of the breakdown region, in the manner of Section 2. For example, for black holes the size of the region could depend strongly on the time of the turning point constant Schwarzschild time slice. This is suggested by the large distortion of a matter wavefunction from its semiclassical form on the S-slices of.}

\footnote{If we make a canonical transformation on the position and momentum of the heavy particle, then the turning point in the new coordinate occurs at a different point in the heavy particle trajectory. In cases where we can make such a canonical transformation, we can understand why there should be no large effects on the light particle wavefunction due to the occurrence of a turning point in the heavy particle motion.}
boundaries of such regions in superspace we must carefully check the ‘joining conditions’ to continue the wavefunctional, in particular noting any allowed tunneling directions.

In the black hole context this issue is relevant because the Schwarzschild coordinate system determines a slicing that approaches a turning point of the second kind described above. This leads to an apparent violation of the semiclassical wavefunctional near the horizon, and gives apparent large effects from gravity fluctuations in calculations in any expansion scheme that uses the Schwarzschild coordinate system. Such effects must be carefully removed before looking for possible violations of standard physics. Note that our considerations involved only matter and gravity having simple local actions; we have thus not examined the consequences of the existence of extended strings or D-branes.

Acknowledgements

We would like to thank S. Coleman, A. Guth, and J. Preskill for discussions at various stages of this investigation.

Appendix A

Consider the Schrödinger equation (18) with the Hamiltonian (17):

\[ \hat{H} = -\hbar^2 \frac{\partial^2}{2M \partial X^2} + m_1 r X - \frac{\hbar^2}{2\mu} \frac{\partial^2}{\partial y^2} + \frac{\kappa}{2} (y - \frac{r\mu}{\kappa})^2. \]  

(65)

We now use the semiclassical approximation to find an approximate solution \(\Psi_{sc}\) of (18), in a manner in which it is most convenient to compare with the exact solution. Thus, we will keep using the coordinates \(X, y\), and use the total mass \(M\) as the expansion parameter in the semiclassical approximation. So we will find the approximate wavefunction \(\Psi_{sc} = \exp\{i/\hbar[S_0(X) + MS_1(X, y)]\}\). We want the classical motion of \(X\) to be independent of the mass \(M\), so that it defines a good clock as we discussed in Section 2.1. So we rewrite the Schrödinger equation (18) in the form

\[ \{\hat{H}_X + \hat{H}_{Xy}\} \Psi = \left\{ \left[ -\frac{\hbar^2}{2M} \frac{\partial^2}{\partial X^2} + m_1 r X - E_{\text{total}} \right] + \left[ -\frac{\hbar^2}{2\mu} \frac{\partial^2}{\partial y^2} + \frac{\kappa}{2} (y - \frac{r\mu}{\kappa})^2 - m_2 r X \right] \right\} \Psi = 0. \]  

(66)

The leading order approximate solution is then

\[ \Psi_{sc}(X, y) \approx \frac{1}{(-2V(X))^{1/4}} e^{\pm \frac{i}{\hbar} MS_0(X)} \chi(X, y), \]  

(67)

where

\[ \begin{align*}
S_0(X) &= \frac{2}{3} \sqrt{2r}[a(0) - X]^{3/2} \\
a(0) &= E_{\text{total}}/(Mr) \\
V(X) &= r(X - a(0)).
\end{align*} \]  

26
The function $\chi$ satisfies
\[ i\hbar \frac{\partial S_0}{\partial X} \frac{\partial}{\partial X} \chi = \hat{H}_{Xy} \chi. \] (68)

The WKB time is defined using the classical trajectory of the center of mass:
\[ t_{WKB}(X) = \int_X^a \frac{dX'}{\partial S_0/\partial X} = \pm \sqrt{\frac{2}{r}} [a(0) - X]^\frac{1}{2}, \] (69)

note that in the leading order semiclassical approximation the location of the turning point $a$ is independent of the energy of the light particle. The overall sign choice will be chosen to give a negative (positive) $t_{WKB}$ for a motion towards (away from) the turning point.

Next, we expand $\chi$ in the eigenenergy $E_n$ eigenfunctions $G_n(y)$ of the simple harmonic oscillator; these satisfy
\[ \hat{H}_{Xy} G_n(y) = (E_n - m_2 r X) G_n(y). \]

According to (68), (69) $\chi$ then time evolves as
\[ \chi = \sum_n c_n(t_{WKB}) \chi_n = \sum_n c_n(t_{WKB,0})e^{-\frac{i}{\hbar}E_n(t_{WKB,0}-t_{WKB}) + \frac{i m_2 r}{\hbar} \int_{t_{WKB,0}}^{t_{WKB}} X(t')dt'} G_n(y). \]

The second term in the exponent is independent of the energy levels and can be dropped. However, in general one needs to be more careful. A time evolution where $X$ propagates to the turning point and back to the same point again is periodic. This may give arise to an additional Berry’s phase in general (multidimensional) models.

### Appendix B

Consider the same model as in Section 2, but change the potential $V(X)$ to be

\[ V(X) = \begin{cases} 0, & X \leq 0 \\ V_0 > 0, & 0 \leq X \end{cases} \]

Now the equation (20) becomes
\[ [-\frac{\hbar^2}{2M}\frac{\partial^2}{\partial X^2} + m_1 V(X)] F_n(X) = (E_{\text{total}} - E_n) F_n(X). \]

In the region $X \leq 0$, the exact wavefunction of the system is then
\[ \Psi = \sum_n c_n \cos[k_n X + \phi(E_n)] G_n(y) \]

where
\[ k_n = \frac{1}{\hbar} \sqrt{\frac{2M}{(E_{\text{total}} - E_n)}}. \]

We will assume that the energy of the heavy particle $E_{\text{total}} - E_n$ is less than $V_0$; thus classically the heavy particle reflects off the potential barrier at the same position $X = 0$ for all energies
$E_n$ of the light particle. Thus there is no effect of the form (30) which comes from a nonlinear change in the reflection position with changing $E_n$. But a simple calculation shows that

$$\phi(E_n) = \tan^{-1}\left(\sqrt{\frac{V_0 - (E_{\text{total}} - E_n)}{E_{\text{total}} - E_n}}\right).$$

Thus, $\phi(E_n)$ ranges from $\pi/2$ to 0 as $E_{\text{total}} - E_n$ ranges from 0 to $V_0$; the phase shift is zero when the turning point is just below the edge of the potential step. This illustrates that the phase shift (23) can indeed depend on the location of the turning point. We now get for the light particle wavefunction the relations (31), (32) with

$$c_n^{\text{after}} = c_n^{\text{before}} e^{-\frac{i}{\hbar} E_n \Delta t_{\text{WKB}}} e^{2i\phi(E_n)}$$

(70)

**Appendix C**

In this appendix, we give an argument why the $\gamma$-dependent term in the minisuperspace model (36) of Section 3 can be dropped in the WKB regime. First, let us rewrite the Wheeler–DeWitt equation (36) using dimensionless variables and parameters. We rescale the scale factor $a$ and the maximum size of the universe $C$ as

$$a \mapsto \tilde{a} \equiv Ma, \quad C \mapsto \tilde{C} \equiv MC.$$ 

In other words, $\tilde{a}$ and $\tilde{C}$ are given in Planck units. Dropping the tildes, the rescaled WDW equation reads

$$\frac{1}{2} \left\{ \frac{\partial^2}{\partial a^2} + \frac{\gamma}{a} \frac{\partial}{\partial a} + (C^2 - a^2) \right\} \Psi(a) = 0.$$

The WKB form for the wavefunctional $\Psi$ is

$$\Psi \approx \exp\{i \int_a^C da' p(a')\}$$

where $p(a)$ is the momentum

$$p(a) = \sqrt{C^2 - a^2}.$$

Substituting this ansatz to the WDW equation, we find that the kinetic terms $\partial^2/\partial a^2 + (\gamma/a)\partial/\partial a$ give a contribution

$$-ip' - p^2 - i\frac{\gamma}{a} p$$

(71)

where $p' = dp/da = -a/p$. We cannot use the WDW approximation if the universe is of the order of Planck scale or smaller. So we must consider the region $a \gg 1$. Further, the validity of the WDW approximation requires that

$$p^2 \gg |p'|.$$

In this case, we can drop the $p'$ term in (71). From the above two requirements we find that

$$p^3 \gg a \gg 1.$$
which also implies that
\[ p^2 \gg p \gg \frac{\gamma}{a} p . \]
This means that we can also drop the $\gamma$-dependent term from the kinetic term of the WDW equation.

**Appendix D**

Consider a class of two dimensional dilaton gravity models of the form
\[
S = \int d^2 x \sqrt{-g} \left[ \frac{1}{2} g^{\alpha\beta} \partial_\alpha \chi \partial_\beta \chi - V(\chi) + D(\chi) R \right] . \tag{72}
\]
where $\chi$ is the dilaton scalar field and $(g_{\alpha\beta})$ is a two dimensional metric. To rewrite the action in a form where the kinetic term disappears, one first needs to rescale the metric
\[ (g_{\alpha\beta}) = \Omega^{-2} (\bar{g}_{\alpha\beta}) \]
with a rescaling factor $\Omega$ satisfying the equation
\[ -1 + 4 \frac{dD}{d\chi} \frac{d \ln \Omega}{d\chi} = 0 , \]
then perform a field redefinition
\[ \chi \rightarrow \bar{\phi} = D(\chi) \]
and introduce a new potential
\[ \bar{V}(\bar{\phi}) = \frac{V(\chi(\bar{\phi}))}{\Omega^2(\chi(\bar{\phi}))} . \]
As a result, the action (72) becomes
\[
S = \int d^2 x \sqrt{-\bar{g}} \left[ \bar{\phi} R - \bar{V}(\bar{\phi}) \right] . \tag{73}
\]
In the case of the CGHS model, the action is most commonly presented in the form
\[
S = \int d^2 x \sqrt{-g} \ e^{-2\phi} \left[ R + 4(\nabla \phi)^2 + 4\lambda^2 \right] . \tag{74}
\]
This form of the action can be related to (72) and (73) with the identifications\footnote{Note that $\Omega$ and $\bar{V}$ are defined up to an arbitrary integration constant, we have set it equal to 1.}
\[ \begin{align*}
\bar{\phi} &= D(\chi) = \frac{1}{8} \chi^2 = e^{-2\phi} \\
\Omega^2(\chi) &= \chi^2 \\
\bar{V}(\bar{\phi}) &= -\frac{\lambda^2}{2} .
\end{align*} \]
References

[1] V. G. Lapchinsky and V. A. Rubakov, Acta Phys. Pol. B10 (1979) 1041; T. Banks, Nucl. Phys. B249 (1985) 332.

[2] J. B. Hartle and S. W. Hawking, Phys. Rev. D28 (1983) 207.

[3] For a review, see e.g. A. Vilenkin, Phys. Rev. D50 (1994) 2587.

[4] T. Vachaspati and A. Vilenkin, Phys. Rev. D37 (1988) 898.

[5] R. M. Wald, Space, Time and Gravity: the Theory of the Big Bang and Black Holes, Chicago U., EFI, 1992; D. A. Lowe, J. Polchinski, L. Susskind, L. Thorlacius and J. Uglum, Black Hole Complementarity Versus Locality, preprint NSF-ITP-95-47 [hep-th/9506138]; J. Polchinski, String Theory and Black Hole Complementarity, preprint NSF-ITP-95-63 [hep-th/9507094].

[6] S. D. Mathur, Black Hole Entropy and the Semiclassical Approximation, Talk given at 2nd TIFR International Colloquium on Modern Quantum Field Theory, Bombay, India, 5-11 Jan 1994 MIT Report No. MIT-CTP-2304 [hep-th/9404135].

[7] E. Keski-Vakkuri, G. Lifschytz, S. D. Mathur and M. E. Ortiz, Phys. Rev. D 51 (1995) 1764.

[8] G. Lifschytz, S. D. Mathur, and M. E. Ortiz, Phys. Rev. D53 (1996) 766.

[9] G. Lifschytz and M. Ortiz, Nucl. Phys. B456 (1995) 377; Black Hole Thermodynamics from Quantum Gravity, Brandeis U. Report No. BRX-TH-381 [hep-th/9510113].

[10] E. Verlinde and H. Verlinde, A Unitary S-matrix for 2D Black Hole Formation and Evaporation, Princeton U. Report No. PUPT-1380, IASSNS-HEP-93/8 [hep-th/9302022]; K. Schoutens, E. Verlinde and H. Verlinde, Phys. Rev. D48 (1993) 2670; Y. Kiem, E. Verlinde and H. Verlinde, Phys. Rev. D52 (1995) 7053.

[11] S. Bose, L. Parker, and Y. Peleg, Phys. Rev. D53 (1996) 7089.

[12] S. P. De Alwis, Phys. Lett. B317 (1993) 217; S. P. De Alwis and D. A. MacIntire, Phys. Rev. D50 (1994) 5164; Phys. Lett. B344 (1995) 110.

[13] J.-G. Demers and C. Kiefer, Decoherence of Black Holes by Hawking Radiation, preprint McGill 95-96 – Freiburg THEP-95/22, hep-th/951147.

[14] For recent reviews on the topic, see for example C. Kiefer, The Semiclassical Approximation to Quantum Gravity, Freiburg University Report No. THEP-93/27, to appear in Canonical Gravity - from Classical to Quantum, edited by J. Ehlers and H. Friedrich (Springer, Berlin 1994) [gr-qc/9312013]; C. Isham, Canonical Quantum Gravity and the Problem of Time, Lectures presented at the NATO Advanced Summer Institute “Recent Problems in Mathematical Physics”, Salamanca, June 15-27, 1992; K. Kuchař,
Time and Interpretations of Quantum Gravity, in Proceedings of the 4th Canadian Conference on General Relativity and Relativistic Astrophysics, eds. G. Kunstatter, D. Vincent and J. Williams (World Scientific, Singapore, 1992).

[15] E. Merzbacher, Quantum Mechanics, Second edition, John Wiley & Sons (1970), Chapter 7.

[16] C. Teitelboim, Phys. Lett. B126 (1983) 41 and “The Hamiltonian Structure of Two-Dimensional Space-Time and its Relation with the Conformal Anomaly”, in Quantum Theory of Gravity, S. Christensen, ed. (Adam Hilger, Bristol, 1984); R. Jackiw, “Liouville Field Theory: A Two-Dimensional Model for Gravity?”, in Quantum Theory of Gravity, S. Christensen, ed. (Adam Hilger, Bristol, 1984); R. Jackiw, Nucl. Phys. B252 (1985) 343.

[17] C. Callan, S. B. Giddings, J. A. Harvey, and A. Strominger, Phys. Rev. D45 (1992) R1005.

[18] D. Louis-Martinez, J. Gegenberg and G. Kunstatter, Phys. Lett. B321 (1993) 193.

[19] D. Cangemi and R. Jackiw, Phys. Rev. D50 (1994) 3913; Phys. Lett. B337 (1994) 271.

[20] E. Benedict, Phys. Lett. B340 (1994) 43.

[21] J. Gegenberg and G. Kunstatter, Phys. Rev. D47 (1993) R41292.

[22] J. Gegenberg, G. Kunstatter and D. Louis-Martinez, Phys. Rev. D51 (1995) 1781.

[23] D. Cangemi, R. Jackiw and B. Zwiebach, Ann. Phys. 245 (1996) 408.

[24] A. Barvinsky and G. Kunstatter, Exact Physical Black Hole States in Generic 2-D Dilaton Gravity, hep-th/9606134; D. Louis-Martinez, UBC preprint (April, 1996).