TWISTORIAL HARMONIC MORPHISMS WITH ONE-DIMENSIONAL FIBRES ON SELF-DUAL FOUR-MANIFOLDS

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Abstract

We introduce a general notion of twistorial map and classify twistorial harmonic morphisms with one-dimensional fibres from self-dual four-manifolds. Such maps can be characterised as those which pull back Abelian monopoles to self-dual connections. In fact, the constructions involve solving a generalised monopole equation, and also the Beltrami fields equation of hydrodynamics, and lead to constructions of self-dual metrics.

Introduction

Harmonic morphisms between Riemannian manifolds are smooth maps which preserve Laplace’s equation. By the basic characterisation theorem (12, 19), they are harmonic maps which are horizontally weakly conformal (see below).

Classification results for harmonic morphisms with one-dimensional fibres can be found in [6, 8, 30, 31] and [32]. In [32] it is proved that, from an Einstein manifold of dimension at least five, there are just two types (2, 8) of harmonic morphism with one-dimensional fibres. In dimension four, the situation is different: from an Einstein 4-manifold there are precisely three types of harmonic morphism with one-dimensional fibres (30, 31) (see also 33), where the first two types are as before. It is significant that all these three types of harmonic morphism are twistorial maps in the sense of Definition 3.3, below. Moreover, by a result of 37, submersive harmonic morphisms from Einstein 4-manifolds to Riemann surfaces are twistorial maps.

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We shall see that a submersion with (nowhere degenerate) one-dimensional fibres from a four-dimensional (complex-)Riemannian manifold is twistorial if and only if it is self-dual in the sense of [9].

In this paper we classify twistorial harmonic morphisms with one-dimensional fibres from real-analytic Riemannian four-manifolds, finding precisely one more type which is related to a metric construction of [9].

In Section 1 we review some basic facts on harmonic morphisms with one-dimensional fibres. In Section 2, we introduce a notion of (almost) twistorial structure. Then, we recall ([34], [1], [18], [25]) some basic examples from twistor theory and show how they fit into our framework. In Section 3, we introduce the notion of twistorial map and then, also in Sections 4 and 5, we show that many examples and facts from twistor theory appear as natural properties of such twistorial maps. To make our definition of twistorial, we must complexify our manifolds, so that we shall often work with complex-analytic maps between complex-Riemannian manifolds (see [25]).

The main result of Section 5 is a reformulation (Theorem 5.3) of a result of [9] which gives a nice characterisation for twistorial maps with nowhere degenerate fibres from a four-dimensional oriented conformal manifold. We prove that the induced Weyl connection on the codomain coincides with the one hinted at in [9]. Such twistorial maps pull back (Abelian) monopoles to self-dual connections (cf. [10], [9]); by using Theorem 5.3, we show that this property is equivalent to the property of being twistorial. Then we show that twistorial harmonic morphisms \( \varphi : (M^4, g) \to (N^3, h, D) \) are characterised by the property that there exist nontrivial monopoles on \((N^3, h, D)\) which are pulled back to flat connections.

In Section 6 we prove (Theorem 6.3) that, from a real-analytic four-dimensional Riemannian manifold, there are, up to conformal changes with basic factor, just four types of twistorial harmonic morphism with one-dimensional fibres, where the first three types are as above with a slight extension of type 3, and type 4 is new. The proof involves solving the monopole equation (5.15) (cf. [33]). Recall [30], [31], [33] that harmonic morphisms of type 3 are determined by the Beltrami fields equation (see [21]). Here, this equation appears once more: for a harmonic morphism \( \varphi : (M^4, g) \to (N^3, h) \) of type 4, the Lee form \( \alpha \) with respect to \( h \) of the Weyl connection on \( N^3 \), with respect to which \( \varphi \) is twistorial, satisfies the Beltrami fields equation \( d\alpha = \pm \ast \alpha \) outside the zero set of a function, up to a
conformal change of \( h \) (Proposition 6.2).

Theorem 6.3 together with a result of \([9]\) (see Theorem 5.3, below), gives the classification of twistorial harmonic morphisms with one-dimensional fibres from a self-dual four-dimensional manifold with real-analytic metric (Corollary 6.4).

In \([33]\), we gave a new construction of Ricci-flat self-dual metrics based on harmonic morphisms of type 3. In Section 7, we show that harmonic morphisms are related to constructions of Einstein and self-dual metrics in \([20]\) and \([9]\).

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1. Some facts on harmonic morphisms with one-dimensional fibres

In this section we present some basic facts on harmonic morphisms with one-dimensional fibres. See \([6]\) and \([29]\) for general accounts and \([16]\) for a list of papers on harmonic morphisms.

**Definition 1.1.** A harmonic morphism is a smooth map \( \varphi : (M^m, g) \to (N^n, h) \) between Riemannian manifolds which pulls back (locally defined) harmonic functions to harmonic functions, i.e., if \( f : V \to \mathbb{R} \) is a harmonic function on an open subset of \( N \) with \( \varphi^{-1}(V) \) nonempty, then \( f \circ \varphi \) is a harmonic function on \( \varphi^{-1}(V) \).

**Definition 1.2.** A smooth map \( \varphi : (M^m, g) \to (N^n, h) \) between Riemannian manifolds is horizontally (weakly) conformal if, at each point \( x \in M \), either \( d\varphi_x = 0 \), in which case \( x \) is called a critical point of \( \varphi \), or \( d\varphi_x : T_xM \to T_{\varphi(x)}N \) is surjective and its restriction to the horizontal space \( \mathcal{H}_x = (\ker d\varphi_x)^\perp \) is a conformal (linear) map \( (\mathcal{H}_x, g_x|_{\mathcal{H}_x}) \to (T_{\varphi(x)}N, h_{\varphi(x)}) \), in which case \( x \) is called a regular point of \( \varphi \). Denote the conformality factor by \( \lambda(x) \). The resulting function \( \lambda \) is called the dilation of \( \varphi \). The dilation is smooth outside the set of critical points and can be extended to a continuous function on \( M^m \), with \( \lambda^2 \) smooth, by setting it equal to zero on the set of critical points.

A smooth map is called horizontally homothetic if it is horizontally conformal with dilation constant along horizontal curves.

A homothetic foliation is a foliation which is locally defined by horizontally homothetic submersions \([29], [30]\).

**Remark 1.3.** A map \( \varphi : (M, g) \to (N, h) \) is horizontally weakly conformal if and only if, for each \( x \in M \), the adjoint \( d\varphi_x^* : (T_{\varphi(x)}N, h_{\varphi(x)}) \to (T_xM, g_x) \) is a
weakly conformal linear map (with image \( \mathcal{H}_x \)). This formulation shows that the condition of horizontal weak conformality is dual to that of weak conformality (see also [6, §2.4]).

The basic characterisation result for harmonic morphisms is the following.

**Theorem 1.4** ([12], [19]). A smooth map between Riemannian manifolds is a harmonic morphism if and only if it is a harmonic map which is horizontally weakly conformal.

It follows that the set of regular points of a nonconstant harmonic morphism is an open dense subset of the domain [12].

The following two propositions give two of the four types of harmonic morphism with one-dimensional fibres that we shall meet (see Section 3 below).

**Proposition 1.5** ([2]). Let \( \varphi : (M^{n+1}, g) \to (N^n, h) \) be a nonconstant horizontally weakly conformal map between Riemannian manifolds of dimensions \( n + 1 \) and \( n \), respectively \( (n \geq 1) \). If \( n = 2 \), then \( \varphi \) is a harmonic morphism if and only if its fibres are geodesic at regular points. If \( n \neq 2 \), then any two of the following assertions imply the third:

(i) \( \varphi \) is a harmonic morphism;

(ii) the fibres of \( \varphi \) are geodesic at regular points;

(iii) \( \varphi \) is horizontally homothetic.

**Proof.** Let \( \varphi : (M^m, g) \to (N^n, h) \) be a horizontally weakly conformal submersion. Then, at regular points we have the following fundamental equation (see, for example, [6, §4.5]) for the tension field \( \tau(\varphi) \in \Gamma(\varphi^*(TN)) \) of \( \varphi 

\begin{equation}
\tau(\varphi) + d\varphi(\text{trace}(B^\varphi)) = -(n-2)d\varphi(\text{grad}(\log \lambda))
\end{equation}

where \( \text{trace}(B^\varphi) \) is the trace of the second fundamental form of \( V = \text{ker} \, d\varphi \) and \( \lambda \) is the dilation of \( \varphi \).

The proposition is an immediate consequence of (1.1).

**Remark 1.6.** Proposition 1.5 is true for maps with higher-dimensional fibres after replacing ‘geodesic’ with ‘minimal’.

Recall that a foliation is called *Riemannian* if it is locally defined by the fibres of Riemannian submersions; then we have the following result (see also [6, §12.3]).
**Proposition 1.7** ([8]). Let \( \varphi : (M^{n+1}, g) \to N^n \) be a submersion with connected one-dimensional fibres from a Riemannian manifold of dimension \( n + 1 \) to a smooth manifold of dimension \( n (n \geq 3) \). Suppose that the fibres of \( \varphi \) form an orientable Riemannian foliation. Then the following assertions are equivalent:

(i) There exists a Riemannian metric \( h \) on \( N^n \) with respect to which \( \varphi : (M^{n+1}, g) \to (N^n, h) \) is a harmonic morphism;

(ii) there exists a nowhere zero Killing vector field on \( (M^{n+1}, g) \) tangent to the fibres of \( \varphi \).

In general, harmonic morphisms with one-dimensional fibres have a local normal form as follows.

**Theorem 1.8** ([8]). Let \( (M^{n+1}, N^n, S^1) \) be a principal bundle with projection \( \varphi : M^{n+1} \to N^n \), endowed with a principal connection \( \mathcal{H} \subseteq TM \). Let \( h \) be a Riemannian metric on \( N^n \) and \( \lambda \) a smooth positive function on \( M^{n+1} \).

Define a Riemannian metric on \( M^{n+1} \) by

\[
(1.2) \quad g = \lambda^{-2} \varphi^* (h) + \lambda^{2n-4} \theta^2
\]

where \( \theta \) is the connection form of \( \mathcal{H} \). Then \( \varphi : (M^{n+1}, g) \to (N^n, h) \) is a harmonic morphism.

Conversely, any submersive harmonic morphism with one-dimensional fibres is locally of this form, up to isometries.

See [6, §12.2], [29], [30] (and [5] for the case \( n = 2 \)) for a proof of Theorem 1.8 and a more explicit version of the converse.

The vector field \( V \) on \( M^{n+1} \) which is the vertical dual of \( \theta \) (i.e., \( d\varphi (V) = 0 \) and \( \theta (V) = 1 \)) is the infinitesimal generator of the local \( S^1 \)-action; it is called the fundamental (vertical) vector field; up to sign, it is characterised by the property that it is vertical and \( g(V, V) = \lambda^{2n-4} \).

**Remark 1.9.** 1) By Proposition 1.5, any horizontally homothetic submersion with geodesic fibres is a harmonic morphism. In the context of Theorem 1.8 this corresponds to the case when \( \lambda \) is constant along horizontal curves; then if \( d\lambda \) is nowhere zero, the horizontal distribution is integrable and totally umbilical.

2) The Killing vector field of Proposition 1.7 is equal to (a multiple of) the
fundamental vector field. In fact, from Proposition 1.7 and Theorem 1.8 (see [8], [6, §12.3], [29]) we deduce the following: Let \( V \) be a Killing vector field on \((M^{n+1}, g)\) \((n \neq 2)\) whose integral curves are the fibres of the submersion \( \varphi : (M^{n+1}, g) \to N^n \); let \( \tilde{g} \) denote the unique metric on \( N^n \) with respect to which \( \varphi : (M^{n+1}, g) \to (N^n, \tilde{g}) \) is a Riemannian submersion and \( \tilde{\lambda} \) the positive smooth function on \( N^n \) such that \( \varphi^* (\tilde{\lambda}^{2n-4}) = g(V, V) \); then \( \varphi : (M^{n+1}, g) \to (N^n, \tilde{\lambda}^{-2} \tilde{g}) \) is a harmonic morphism. In the context of Theorem 1.8 this corresponds to the case when \( \lambda \) is constant along the fibres of \( \varphi \).

We end this section by recalling the following.

**Definition 1.10** ([30], cf. [36]). Let \( \mathcal{V} \) be (the tangent bundle of) a foliation on the Riemannian manifold \((M^m, g)\).

We say that \( \mathcal{V} \) produces harmonic morphisms if it can be locally defined by submersive harmonic morphisms.

2. Twistorial structures

In this section we shall work in the complex-analytic category. Thus, all manifolds will be complex and all maps will be complex-analytic. Examples of such manifolds and maps can be obtained by complexifying real-analytic maps between real-analytic manifolds. In particular, by a Riemannian manifold we shall mean a complex-Riemannian manifold in the sense of [25].

We define horizontally (weakly) conformal maps similarly to Definition 1.2, but \( \lambda \) may then be a complex number defined only up to sign with the square dilation \( \lambda^2 \) globally defined.

In what follows, it is convenient to work with the following definition.

**Definition 2.1.** Let \( M \) be a (complex) manifold. By an almost twistorial structure (on the manifold \( M \)) we shall mean a quadruple \((P, M, \pi, \mathcal{F})\) where \( \pi : P \to M \) is a proper surjective (complex-analytic) submersion and \( \mathcal{F} \subseteq TP \) is a distribution on \( P \) such that \((\ker d\pi) \cap \mathcal{F} = \{0\} \); we call \( \mathcal{F} \) the twistor distribution of \((P, M, \pi, \mathcal{F})\).

We shall call the almost twistorial structure \((P, M, \pi, \mathcal{F})\) integrable if its twistor distribution \( \mathcal{F} \) is integrable. An integrable almost twistorial structure will be called a twistorial structure. If \((P, M, \pi, \mathcal{F})\) is a twistorial structure then the leaf space of \( \mathcal{F} \) is called the twistor space of \((P, M, \pi, \mathcal{F})\).
Remark 2.2. 1) Let \((P,M,\pi,\mathcal{F})\) be a twistorial structure. Assume that the foliation \(\mathcal{F}\) is simple (that is, its leaf space \(Z\) is a manifold and the projection \(\pi_Z : P \to Z\) is a submersion) and that any of its leaves intersect each fibre of \(\pi\) at most once. Then \(\{\pi_Z(\pi^{-1}(x))\}_{x \in M}\) is an analytic family \([23]\) of compact submanifolds of \(Z\), which we shall call twistor submanifolds, or twistor lines when they are projective lines.

2) For all of the almost twistorial structures \((P,M,\pi,\mathcal{F})\) (\(\dim \mathcal{F} = k\)) which will appear in this paper, the map \(P \to G_k(TM), \quad p \mapsto d\pi(\mathcal{F}_p)\), is an embedding of \(P\) into the Grassmann bundle of \(k\)-dimensional planes on \(TM\). Such almost twistorial structures can be obtained as follows.

By a linear partial connection on a vector bundle \(E \to M\), over a distribution \(\mathcal{H}\) on \(M\) \([7]\), we mean a morphism \(\nabla\) from the sheaf of sections of \(E\) to the sheaf of sections of \(E \otimes \mathcal{H}^*\) such that, if \(U\) is an open set of \(M\), then \(\nabla : \Gamma(U,E) \to \Gamma(U,E \otimes \mathcal{H}^*)\) is a \(C\)-linear map which satisfies

\[
\nabla(sf) = (\nabla s)f + s \otimes df|_\mathcal{H}
\]

for any function \(f : U \to C\) and section \(s \in \Gamma(U,E)\); in a similar way to the case of usual connections, any linear partial connection over \(\mathcal{H}\) corresponds to a principal partial connection over \(\mathcal{H}\) on the frame bundle of \(E\) where by a principal partial connection, over \(\mathcal{H}\), on the principal bundle \((P,M,G)\), with projection \(\pi : P \to M\), we mean a \(G\)-invariant distribution \(\mathcal{H}\) on \(P\) such that \(\mathcal{H} \cap \ker d\pi = \{0\}\) and \(d\pi(\mathcal{H}) = \mathcal{H}\).

Now let \(\mathcal{H} \subseteq TM\) be an \(n\)-dimensional distribution endowed with a linear partial connection \(D\), over itself. Suppose that \(D\) corresponds to a principal partial connection, over \(\mathcal{H}\), on a principal subbundle \((Q,M,G)\), \((G \subseteq GL(n,C))\), of the bundle of (complex) frames of \(\mathcal{H}\). Let \(F \subseteq G_k(C^n)\) be a submanifold which is invariant under the action of \(G\) and let \(P = Q \times_G F\) be the associated bundle. Clearly, \(P \subseteq G_k(\mathcal{H})\). Also, \(D\) induces a connection \(\mathcal{D} \subseteq TP\) on \(\pi : P \to M\). Then, for each \(p \in P\) we define \(\mathcal{F}_p \subseteq T_pP\) to be the horizontal lift, with respect to \(\mathcal{D}\), at \(p \in P\) of \(p \subseteq \mathcal{H}_{\pi(p)}\).

Usually, \(G \subseteq CO(n,\mathbb{C})\) where \(CO(n,\mathbb{C})\) is the complex-conformal group in dimension \(n\), so that \(\mathcal{H}\) is endowed with a conformal structure. Then, if \(D\) corresponds to a principal partial connection on the corresponding frame bundle \((Q,M,CO(n,\mathbb{C}))\), it is called a conformal partial connection.
3) Let \((P, M, \pi, \mathcal{F})\), \((\dim M = m, \dim \mathcal{F} = k)\) be a twistorial structure where, as above, \(P\) is a subbundle of the Grassmann bundle \(G_k(TM)\) and \(\mathcal{F}\) is induced by some connection \(D\) on \(M\) which preserves \(P\). Suppose that \(D\) is torsion-free and, for any \(p \in P\), there exists a totally geodesic submanifold of \(M\), of dimension \((m - k)\), which passes through \(\pi(p)\) and which is transversal to \(p\). Then each point of \(M\) has an open neighbourhood \(U\) such that \(\mathcal{F}|_{\pi^{-1}(U)}\) is simple.

Next, we give the basic examples of almost twistorial structures with which we shall work. Recall that we are working in the complex-analytic category. First, we consider structures over 2- and 3-dimensional manifolds.

**Example 2.3** (C.R. LeBrun [25]). Let \(M = M^2\) be a two-dimensional Riemannian manifold and let \(\pi : P \to M\) be the bundle of null directions on \(M^2\). Clearly, \(P = \det(O(M))\) and hence \(\pi : P \to M\) is a \(\mathbb{Z}_2\)-covering. Furthermore, there exists a canonical one-dimensional foliation \(\mathcal{F}\) on \(P\) such that \(\pi\) locally maps leaves of \(\mathcal{F}\) to (local) null geodesics on \(M^2\). Hence any two-dimensional Riemannian (or conformal, if one prefers) manifold \(M^2\) is canonically endowed with the twistorial structure \((P, M, \pi, \mathcal{F})\).

Conversely, any (almost) twistorial structure \((P, M, \pi, \mathcal{F})\) with \(\dim M = 2\), \(\dim \mathcal{F} = 1\) and \(\pi : P \to M\) a \(\mathbb{Z}_2\)-covering such that the map \(P \to P(TM), p \mapsto \text{d}\pi(\mathcal{F}_p)\), is an embedding induced by a conformal structure as above.

If the Riemannian manifold \(M^2\) is orientable (equivalently, if the \(\mathbb{Z}_2\)-covering \(\pi : P \to M\) is trivial) then \(P = M_+ \sqcup M_-\) where \(M_+\) and \(M_-\) are copies of \(M\). Hence, there exist two foliations by null geodesics \(\mathcal{F}_+\) and \(\mathcal{F}_-\), on \(M\), which are the projections of \(\mathcal{F}\) restricted to \(M_+\) and \(M_-\), respectively. Therefore, the twistor space \(Z = Z(M)\) of the canonical twistorial structure of an oriented two-dimensional Riemannian manifold \(M^2\) (more generally, of a two-dimensional manifold endowed with an oriented conformal structure) is the *space of null geodesics* \([25]\) of \(M^2\). Then, locally, \(Z\) is the disjoint union of two curves \(Z = C_+ \sqcup C_-\) and \(M = C_+ \times C_-\). Also, note that the (complex-analytic) almost Hermitian structures \(J_\pm\) on \(M\) defined by \(J_\pm(X) = \pm iX\) for \(X \in \mathcal{F}_\pm\) are integrable.

Let \((M, g)\) be a Riemannian manifold, \(\dim M = m\). A *degenerate hyperplane* \(H \subseteq TM\) is a subspace of codimension one such that \(g|_H\) is degenerate (equivalently, \(H\) is the orthogonal complement of a null vector which is thus contained in \(H\)) \([cf. 25]\); if \(\dim M = 3\) we shall say *degenerate plane*. For fixed \(x \in M\), the
space of degenerate hyperplanes in $T_xM$ can be identified with the nonsingular quadric $Q_{m-2}$ in $P(T_xM)$.

**Example 2.4** (N.J. Hitchin [18]). Let $M^3$ be a three-dimensional Riemannian manifold. Let $D$ be a Weyl connection (that is, a torsion-free conformal connection) on $M$. Let $\pi : P \to M$ be the bundle of degenerate planes on $M^3$. Then $P = CO(M) \times_\rho \mathbb{C}P^1$ where $CO(M)$ is bundle of conformal frames of $M^3$ and $\rho$ is the action of $CO(3, \mathbb{C})$ on $\mathbb{C}P^1 = \{ p \mid p \subseteq \mathbb{C}^3 \text{degenerate plane} \}$ induced by the canonical action of $CO(3, \mathbb{C})$ on $\mathbb{C}^3$. Thus $D$ induces a connection on $\pi : P \to M$ and, for each $p \in P$, we define $F_p \subseteq T_pP$ to be the horizontal lift at $p \in P$ of $p \subseteq T_{\pi(p)}M$. Obviously, the almost twistorial structure $(P, \pi, F)$ depends only on $D$ and on the conformal class $c$ of the metric of $M^3$.

**Theorem 2.5** ([18]). The twistor distribution $F$ is integrable if and only if $D$ is Einstein–Weyl.

Furthermore, if $D$ is Einstein–Weyl then (locally) the leaf space $Z$ of $F$ contains a locally complete analytic family of projective lines each of which has normal bundle $O(2)$. Conversely, any surface $Z$ containing a projective line with normal bundle $O(2)$ is (locally) the twistor space of a 3-dimensional Riemannian manifold $M^3$ endowed with an Einstein–Weyl connection $D$. The conformal structure of $M^3$ and the Einstein–Weyl connection $D$ are uniquely determined.

Let $(P, M, \pi, F)$ be the twistorial structure corresponding, as above, to $(M^3, D)$ where $D$ is an Einstein–Weyl connection on $M^3$. Then, as just explained, $F$ is integrable and the locally complete analytic family [23] of projective lines on $Z$ of Theorem 2.5 appears as in Remark 2.2(1) (see, also, Remark 2.2(3)): each projective line represents all the (local) degenerate surfaces through a given point which are totally-geodesic with respect to $D$. The fact that the normal bundle of any twistor line in $Z$ is $O(2)$ can be proved as follows. Let $t = \mathbb{C}P^1$ be a fibre of $\pi$. Then the normal bundle of $\pi_Z(t) = \mathbb{C}P^1$ in $Z$ is (isomorphic to) $(t \times \mathbb{C}^3)/(F|_t)$. Now, $F|_t$ is the restriction to $t = Q_1 \subseteq \mathbb{C}P^2 = G_2(\mathbb{C}^3)$ of the tautological plane bundle $E$ over $G_2(\mathbb{C}^3)$ where $Q_1$ is the one-dimensional nonsingular quadric and $G_2(\mathbb{C}^3)$ is the Grassmann manifold of planes in $\mathbb{C}^3$. As $(G_2(\mathbb{C}^3) \times \mathbb{C}^3)/E = H$ where $H \to \mathbb{C}P^2$ is the hyperplane bundle and the embedding $t = \mathbb{C}P^1 = Q_1 \to \mathbb{C}P^2$ has degree two we obtain $(t \times \mathbb{C}^3)/(F|_t) = O(2)$.

Note that, locally, any leaf of $F$ is mapped by $\pi$ to a degenerate surface in
$M^3$ which is totally geodesic with respect to $D$, and so $Z$ is, locally, the space of degenerate surfaces on $M^3$ which are totally geodesic with respect to $D$.

Finally, we discuss an important example of almost twistorial structures over a four-dimensional manifold.

Example 2.6 (R. Penrose [34]; M.F. Atiyah, N.J. Hitchin, I.M. Singer [1]).
(i) Let $M^4$ be a four-dimensional Riemannian manifold. A plane $H \subseteq TM$ on $(M^4, g)$ is called null if $g|_H = 0$. Let $\pi : P \to M$ be the bundle of null planes on $M^4$. Then the Levi-Civita connection of $M^4$ induces a connection on $\pi : P \to M$ and, for each $p \in P$, we define $\mathcal{F}_p \subseteq T_p P$ to be the horizontal lift at $p \in P$ of $p \subseteq T_{\pi(p)}M$. As $\mathcal{F}$ is conformally invariant, the almost twistorial structure $(P, M, \pi, \mathcal{F})$ canonically associated to $M^4$ is conformally invariant. It is well known that $\mathcal{F}$ is integrable if and only if $M^4$ is conformally flat (a consequence of Theorem 2.7, below).

(ii) We have that $(M^4, g)$ is orientable if and only if $P = P_+ \sqcup P_-$ is the disjoint union of two $\mathbb{C}P^1$-bundles over $M^4$. In this case, with respect to a choice of orientation on $M^4$, $P_+ \to M$ is the bundle of self-dual planes on $M^4$ and $P_- \to M$ is the bundle of anti-self-dual planes on $M^4$. (Self-dual and anti-self-dual planes are sometimes called $\alpha$-planes and $\beta$-planes, see [35], [28].) Then, with $\pi_{\pm} = \pi|_{P_{\pm}}$ and $\mathcal{F}_{\pm} = \mathcal{F}|_{P_{\pm}}$, $(P_{\pm}, M, \pi_{\pm}, \mathcal{F}_{\pm})$ are almost twistorial structures on $M^4$. These almost twistorial structures are conformally invariant and so are well defined on any four-dimensional manifold endowed with an oriented conformal structure.

Theorem 2.7 ([34], [1]). The twistor distribution $\mathcal{F}_-$ is integrable if and only if $M^4$ is self-dual. (Similarly, $\mathcal{F}_+$ is integrable if and only if $M^4$ is anti-self-dual.)

Furthermore, if $M^4$ is self-dual then (locally) the leaf space $Z$ of $\mathcal{F}_-$ contains a locally complete analytic family of projective lines each of which has normal bundle $O(1) \oplus O(1)$. Conversely, any three-dimensional manifold $Z$ containing a projective line with normal bundle $O(1) \oplus O(1)$ is (locally) the twistor space of a self-dual Riemannian 4-manifold which is uniquely determined, up to a conformal change of the metric.

As the oriented $\mathbb{Z}_2$-covering of a nonorientable Riemannian manifold is canonically endowed with an orientation-reversing isometry, the oriented $\mathbb{Z}_2$-covering of a nonorientable Riemannian four-manifold $M^4$ is (anti-)self-dual if and only if
$M^4$ is conformally flat.

Let $(P_-, M, \pi_-, \mathcal{F}_-)$ be the twistorial structure corresponding, as above, to the four-dimensional self-dual Riemannian manifold $M^4$. Then, by Theorem \ref{thm:2.7} $\mathcal{F}_-$ is integrable and the locally complete analytic family of projective lines on $Z$ appears again as in Remark \ref{rem:2.2}(1) (see, also, Remark \ref{rem:2.2}(3)): each projective line represents all the (local) anti-self-dual surfaces through a given point. The fact that the normal bundle of any twistor line in $Z$ is $\mathcal{O}(1) \oplus \mathcal{O}(1)$ can be proved as follows. Let $t = \mathbb{CP}^1$ be a fibre of $\pi_-$. Then the normal bundle of $\pi_Z(t) = \mathbb{CP}^1$ in $Z$ is (isomorphic to) $(t \times \mathbb{C}^4)/(\mathcal{F}_-|_t)$. Now, $\mathcal{F}_-|_t$ is the restriction to $t = \mathbb{CP}^1 \subseteq G_2(\mathbb{C}^4)$ of the tautological plane bundle $E$ over $G_2(\mathbb{C}^4)$. As any anti-self-dual $p$ plane on $\mathbb{C}^4 = \mathbb{C}^2 \otimes \mathbb{C}^2$ is given by $p = \{ v \otimes v' \mid v \in \mathbb{C}^2 \}$ for some fixed $v' \in \mathbb{C}^2$ we have that $E|_t = \mathcal{O}(-1) \oplus \mathcal{O}(-1)$ and hence the normal bundle of $\pi_Z(t)$ in $Z$ is $\mathcal{O}(1) \oplus \mathcal{O}(1)$.

Note that, locally, any leaf of $\mathcal{F}_-$ is mapped by $\pi_-$ to an anti-self-dual surface in $M^4$ so $Z$ is, locally, the space of anti-self-dual surfaces in $M^4$.

**Remark 2.8.** The twistorial structures of Example \ref{ex:2.4} are *reductions* of the twistorial structures of Example \ref{ex:2.6} \cite{18} (see \cite{9}).

### 3. Twistorial maps

In this section, unless otherwise stated, we again work in the complex-analytic category, thus all the manifolds are complex and all the maps are complex-analytic.

In what follows, it is convenient to work with the following definitions. We start with an important special case.

**Definition 3.1.** Let $\sigma = (P, M, \pi_P, \mathcal{F})$ and $\tau = (Q, N, \pi_Q, \mathcal{G})$ be almost twistorial structures over $M$ and $N$, respectively. Let $\varphi : M \rightarrow N$ be a map. Suppose that there exists a map $\Phi : P \rightarrow Q$ which covers $\varphi$ (that is, $\pi_Q \circ \Phi = \varphi \circ \pi_P$).

Then $\varphi : (M, \sigma) \rightarrow (N, \tau)$ will be called a twistorial map (with respect to $\Phi$) if $d\Phi(\mathcal{F}_p) \subseteq \mathcal{G}_{\Phi(p)}$ for all $p \in P$.

The following lemma allows us to construct ‘substructures’ of an almost twistorial structure; its proof is omitted.
Lemma 3.2. Let $\sigma = (P, M, \pi, F)$ be an almost twistorial structure and let $P' \subseteq P$ be a closed submanifold such that $\pi(P') = M$. Suppose that $F_p \subseteq T_p P'$ for all $p \in P'$ and $\dim(T_p P' \cap \ker d\pi_p)$ does not depend on $p \in P'$.

Then $\sigma' = (P', M, \pi|_{P'}, F|_{P'})$ is an almost twistorial structure.

Next, we generalize the definition of twistorial map by allowing $\Phi$ to be a map between such substructures.

Definition 3.3. Let $\sigma = (P, M, \pi_P, F)$ and $\tau = (Q, N, \pi_Q, G)$ be almost twistorial structures over $M$ and $N$, respectively, and let $\varphi : M \to N$ be a map. Let $P' \subseteq P$ be as in Lemma 3.2 and similarly for $Q' \subseteq Q$ so that $\sigma' = (P', M, \pi_{P'}|_{P'}, T_{P'} \cap F)$ and $\tau' = (Q', N, \pi_{Q'}|_{Q'}, T_{Q'} \cap G)$ are almost twistorial structures. Suppose that $\varphi$ is covered by a map $\Phi : P' \to Q'$. Then the map $\varphi : (M, \sigma) \to (N, \tau)$ will be called a twistorial map (with respect to $\Phi$) if $d\Phi(F_p) \subseteq G_{\Phi(p)}$ for all $p \in P'$.

Often, our choice of $P'$ and $Q'$ will depend on the map $\varphi$.

If the distributions $F|_{P'}$ and $G|_{Q'}$ are integrable with leaf spaces $Z(M, \varphi)$ and $Z(N, \varphi)$, respectively, then we have an induced local map $Z(\varphi) : Z(M, \varphi) \to Z(N, \varphi)$ which we shall call the twistorial representation of $\varphi$.

Remark 3.4. 1) Any real-analytic (Riemannian) manifold has a germ-unique complexification to a complex(-Riemannian) manifold [25]. We can complexify any real-analytic map between real-analytic manifolds, i.e., extend it to (the germ of) a complex-analytic map between complexifications of those manifolds. A map between real-analytic manifolds will be called twistorial if its complexification is twistorial.

2) Note that Definition 3.1 is the case when $P' = P$ and $Q' = Q$.

3) In all our examples, $P$ is embedded as a subbundle of some Grassmann bundle $G_k(TM)$, as in Remark 2.2(2). Also, $\Phi$ will be naturally induced by $d\varphi$ so that we shall write $\Phi = d\varphi|_{P'}$.

4) In Definition 3.3 if the distributions $F$ and $G$ are integrable and $Z(M)$, $Z(M, \varphi)$, $Z(N)$, $Z(N, \varphi)$ are the leaf spaces of $F$, $F|_{P'}$, $G$, $G|_{Q'}$, respectively, then we have induced local maps $Z(M, \varphi) \to Z(M)$, $Z(N, \varphi) \to Z(N)$.

5) Most of the twistorial maps which will appear in this paper are submersive maps. We could also consider twistorial foliations, that is, foliations which are locally defined by submersive twistorial maps; most of the facts that follow can
be easily reformulated in terms of such foliations.

6) **(Compositions)** If $M_j$ ($j = 1, 2, 3$) are endowed with almost twistorial structures $\tau_j = (\mathcal{P}_j, M_j, \pi_j, \mathcal{F}_j)$ ($j = 1, 2, 3$) and $\varphi_1 : (M_1, \tau_1) \to (M_2, \tau_2)$, $\varphi_2 : (M_2, \tau_2) \to (M_3, \tau_3)$ are twistorial maps with respect to $\Phi_1 : P'_1 \to Q'_1$ and $\Phi_2 : P'_2 \to Q'_2$, such that $\Phi_1(P'_1) \subseteq P'_2$, then $\varphi_2 \circ \varphi_1 : (M, \tau_1) \to (M, \tau_3)$ is a twistorial map with respect to $\Phi_2 \circ \Phi_1$.

The following simple proposition will be useful later on.

**Proposition 3.5.** Let $M$ be a manifold endowed with a linear connection $D$ and let $H \subseteq T(G_k(TM))$ be the connection induced by $D$ on the Grassmann bundle $\pi : G_k(TM) \to M$ of $k$-dimensional planes on $M$ ($k \leq \dim M$). Let $\mathcal{F}$ be the $k$-dimensional distribution on $G_k(TM)$ defined by setting $\mathcal{F}_p$ equal to the horizontal lift at $p \in G_k(TM)$ of $p \subseteq T_{\pi(p)} M$.

Then, for a distribution $s : M \to G_k(TM)$ on $M$ the following assertions are equivalent:

(i) $\mathcal{F}_{s(x)} \subseteq T_{s(x)}(s(M))$ for all $x \in M$;

(ii) for any curve $c$ tangent to the distribution $s$ on $M$ we have that $s \circ c$ is a parallel section of $G_k(TM)$;

(iii) for any vector fields $X, Y \in \Gamma(s)$ we have $D_X Y \in \Gamma(s)$.

Furthermore, if $D$ is torsion free then, the following assertion can be added to this list:

(iv) the distribution $s$ is integrable and its leaves are totally geodesic with respect to $D$.

**Proof.** Assertion (i) is equivalent to the fact that for any $X \in s(x)$ we have $ds(X) \in \mathcal{F}_{s(x)}$; this is clearly equivalent to assertion (ii).

Assertion (ii) is equivalent to the fact that the distribution $s$ is invariant under parallel transport along curves tangent to $s$. Then, the equivalence of (ii) and (iii) follows from the fact that, if $U \subseteq \mathbb{C}$ is a domain in $\mathbb{C}$, $H$ is a Lie subgroup of the Lie group $G$ and $(a_z)_{z \in U}$ is a curve in $G$ such that $a_{z_0} \in H$ for some $z_0 \in U$ and $\theta(da_z/ dz) \in L(H)$ for all $z \in U$, where $\theta \in \Gamma(L(G) \otimes T^*G)$ is the canonical form of $G$, then $a_z \in H$ for all $z \in U$.

If $D$ is torsion free, the equivalence (iii) $\iff$ (iv) is trivial. $\square$

**Example 3.6.** We interpret the twistor lift of J. Eells and S. Salamon [11] in our framework. Let $M^2$ and $N^4$ be manifolds of dimension two and four, respectively,
endowed with oriented conformal structures and let $\varphi : M^2 \to N^4$ be an injective conformal immersion.

Endow $M^2$ and $N^4$ with the almost twistorial structures $(P, M, \pi, \mathcal{F})$ and $(P_-, N, \pi_-, \mathcal{F}_-)$ of Examples 2.3 and 2.6, respectively.

As $\varphi$ maps null directions on $M^2$ to null directions on $N^4$, we can define $\Phi_- : P \to P_-$ by $\Phi_-(p) = d\varphi(p)$, for each null direction $p$ on $M^2$. Then, $\varphi$ is twistorial with respect to $\Phi_-$ if and only if, for any null geodesic $\gamma$ on $M^2$, $\Phi_-(\gamma)$ is parallel along $\varphi(\gamma)$. If $N^4$ is self-dual then $\varphi$ is twistorial with respect to $\Phi_-$ if and only if, for any null geodesic $\gamma$ on $M^2$ there exists a (necessarily unique) anti-self-dual surface $S_\gamma \subseteq N^4$ such that $\varphi(\gamma) \subseteq S_\gamma$. Then, the map $\gamma \mapsto S_\gamma$ is the twistorial representation of $\varphi$ whose image is a pair of (local) curves in $Z(N)$.

A nowhere degenerate surface in $N^4$ will be called a $(-)$twistorial surface if the corresponding inclusion map is twistorial, in the above sense.

Similarly, we define $(+)$twistorial surfaces by using instead the almost twistorial structure $(P_+, N, \pi_+, \mathcal{F}_+)$ of Example 2.6(ii). A nowhere degenerate surface on an oriented four-dimensional conformal manifold is totally umbilical if and only if it is both $(+)$twistorial and $(-)$twistorial [11].

4. Twistorial maps to surfaces

We shall now describe the twistorial maps which we shall need, namely those with nowhere degenerate fibres.

**Example 4.1** (cf. [3], [4], [10]). Let $\varphi : M^3 \to N^2$ be a surjective submersion whose fibres are nowhere null from a Riemannian manifold onto an oriented Riemannian manifold. Let $D$ be a Weyl connection on $M^3$. We consider $M^3$ to be endowed with the almost twistorial structure $(P, M, \pi, \mathcal{F})$ of Example 2.3, and $N^2$ with the twistorial structure of Example 2.3(i) with $Q \to N$ the bundle of null directions on $N^2$.

At each $x \in M$ there are precisely two horizontal null directions $h_+(x)$ and $h_-(x)$. Thus to $\varphi$ correspond the two disjoint embeddings $M_\pm = h_\pm(M) \hookrightarrow P$ given by $x \mapsto h_\pm(x) = \text{Span}(h_\pm(x), \ker d\varphi_x)$.

As $\dim P = 4$, $\dim \mathcal{F} = 2$ and $\dim M_\pm = 3$, at each degenerate plane $p \in M_\pm$, $d_\pm(p) = \dim(\mathcal{F}_p \cap T_p M_\pm) \in \{1, 2\}$. 

By Proposition 3.5, \( d_\pm = 2 \) on \( M_\pm \) if and only if \( \varphi : M^3 \to N^2 \) is a horizontally conformal submersion with fibres which are geodesic with respect to \( D \). Therefore, \( \varphi \) is a twistorial map with \( P' = M_+ \sqcup M_- \) (and \( \Phi = d\varphi|_{P'} : P' \to Q \)) if and only if it is a horizontally conformal submersion with geodesic fibres (with respect to \( D \)). If \( D \) is Einstein–Weyl, then \( Z(M, \varphi) = C_+ \sqcup C_- \) is the leaf space of the foliation induced by \( \mathcal{F} \) on \( M_+ \sqcup M_- \) and the map \( Z(M, \varphi) \to Z(M) \) is simply the inclusion \( C_+ \sqcup C_- \hookrightarrow Z(M) \). Obviously \( C_+ \) and \( C_- \) are transversal to the twistor lines in \( Z(M) \). Also, \( Z(N, \varphi) = Z(N) \) is the space of null geodesics of \( N^2 \), which, as a consequence of the horizontal conformality of \( \varphi \) can be canonically identified with \( C_+ \sqcup C_- \), then the twistorial representation of \( \varphi \) is simply the identification map \( C_+ \sqcup C_- \to Z(N) \). To retrieve \( \varphi \) from its twistorial representation, let \( x \in M^3 \); this corresponds to the twistor line \( t_x \subseteq Z(M) \). Then, locally, \( t_x \) meets \( C_+ \) and \( C_- \) in two points which, under the identification \( C_+ \sqcup C_- = Z(N) \), correspond to two null geodesics on \( N^2 \) whose intersection is precisely \( \varphi(x) \).

If \( d_\pm = 1 \) on \( M_\pm \) then \( \mathcal{F} \) induces a one-dimensional foliation on \( M_+ \sqcup M_- \). For example, if \( \varphi : M^3 \to N^2 \) is a horizontally conformal submersion (with nowhere null fibres) whose fibres are nowhere geodesics, then \( d_\pm = 1 \) on \( M_\pm \) and the leaves of the induced foliation on \( M_+ \sqcup M_- \) project onto the horizontal null geodesics. Therefore, a horizontally conformal submersion \( \varphi : M^3 \to N^2 \) is a twistorial map if and only if its fibres are geodesics (with respect to \( D \)). For such a horizontally conformal submersion, if \( D \) is Einstein–Weyl, then, at least locally, any totally geodesic (with respect to \( D \)) degenerate surface contains precisely one horizontal null geodesic and so we have two local sections \( Z_\pm \hookrightarrow \mathcal{N}(M) \) of the canonical projection \( \mathcal{N}(M) \to Z(M) \) where \( \mathcal{N}(M) \) is the space of null geodesics of \( M^3 \) so that \( Z_+ \sqcup Z_- \) is the space of null geodesics of \( M^3 \) which are horizontal with respect to \( \varphi \). If \( \varphi \) is twistorial (with respect to \( D \)), then the fibres of \( \mathcal{N}(M) \to Z(M) \) induce a one-dimensional foliation on \( Z_+ \sqcup Z_- \) whose leaf space is \( C_+ \sqcup C_- \).

The following result (cf. Theorem 4.5 and Theorem 5.11 below), which is a consequence of Proposition 1.5 and Example 4.1, is a rephrasing of the starting point of the classification results of [3], [4] (see also [6, Chapters 1 and 6]), there given for the smooth category.

**Proposition 4.2.** Let \( \varphi : M^3 \to N^2 \) be the complexification of a real-analytic submersive map between Riemannian manifolds. Equip \( M^3 \) with the Levi-Civita
connection and endow it with the almost twistorial structure of Example 2.4; equip \( N^2 \) with the twistorial structure of Example 2.3.

Then, \( \varphi \) is a twistorial map, in the sense of Example 4.1, if and only if it is the complexification of a harmonic morphism. □

**Definition 4.3.** Let \( M^4 \) be an oriented four-dimensional Riemannian manifold considered with the almost twistorial structure \( \tau_- = (P_-, M, \pi_-, \mathcal{F}_-) \) of Example 2.6(ii), with \( \pi_- : P_- \to M \) the bundle of anti-self-dual planes. Let \( \varphi : M^4 \to N^n \) be a surjective submersive map with \( n (= \dim N) = 2 \) or \( 3 \). Let \( \tau = (Q, N, \pi, \mathcal{G}) \) be the almost twistorial structure of Example 2.3 (if \( n = 2 \)), or of Example 2.4 for some Weyl connection on \( N \) (if \( n = 3 \)).

We shall say that \( \varphi \) is \((-)\)twistorial (with respect to \( \Phi \)) if \( \varphi : (M, \tau_-) \to (N, \tau) \) is a twistorial map with respect to \( \Phi = d\varphi|_{P_-} : P' \to Q \) for some \( P' \subseteq P_- \) as in Lemma 3.2.

Similarly, we define what is meant by \( \varphi \) is \((+)\)twistorial by using instead the almost twistorial structure \( (P_+, N, \pi_+, \mathcal{F}_+) \) of Example 2.6(ii). We shall write \((\pm)\)twistorial to mean \((+)\)twistorial or \((-)\)twistorial.

**Example 4.4 (cf. [37]).** Let \( \varphi : M^4 \to N^2 \) be a surjective submersion between oriented Riemannian manifolds, whose fibres are nowhere degenerate. We consider \( M^4 \) endowed with the almost twistorial structure \( (P_-, M, \pi_-, \mathcal{F}_-) \) of Examples 2.6(ii) with \( P = P_- \) the bundle of anti-self-dual planes on \( M^4 \), and \( N^2 \) endowed with the twistorial structure \( (Q, N, \pi, \mathcal{G}) \) of Examples 2.3, with \( Q \to N \) the bundle of null directions on \( N^2 \).

At each point \( x \in M^4 \) there are precisely two horizontal null directions \( h_+(x) \), \( h_-(x) \) and two vertical null directions \( v_+(x) \), \( v_-(x) \). Let \( p_+(x) \) (respectively, \( p_-(x) \)) be the (null) plane spanned by \( h_+(x) \) and \( v_+(x) \) (respectively, \( h_-(x) \) and \( v_-(x) \)). Then, from the fact that through each null direction there passes a unique (anti-)self-dual plane, it easily follows that either both \( p_+(x) \) and \( p_-(x) \) are self-dual or both are anti-self-dual. We assume that we have chosen the orientations such that both \( p_+(x) \) and \( p_-(x) \) are anti-self-dual planes. These give two disjoint embeddings \( M_\pm = p_\pm(M) \hookrightarrow P_- \).

As \( \dim P_- = 5 \), \( \dim \mathcal{F} = 2 \) and \( \dim M_\pm = 4 \), at each null plane \( p \in M_\pm \) we have \( d_\pm(p) = \dim(\mathcal{F}_p \cap T_p M_\pm) \in \{1, 2\} \).

By Proposition 3.3, \( d_\pm = 2 \) on \( M_\pm \) if and only if the two distributions \( p_+ \)
and $p_-$ are integrable and totally geodesic. This is equivalent to the fact that the almost Hermitian structures $J_\pm$ defined by $J_\pm(X) = \pm iX$ for $X \in p_\pm$ are integrable. Then, $\varphi$ maps any leaf of $p_+$ and $p_-$ to a null geodesic on $N^2$; in particular, $\varphi$ is horizontally conformal. Thus, a necessary condition for $\varphi$ to be $(-)$twistorial is that $\varphi$ be horizontally conformal. Then, on setting $P' = M_+ \sqcup M_-$ and $\Phi = d\varphi|_{P'} : P' \to Q$, $\varphi$ is a $(-)$twistorial map with respect to $\Phi$ if and only if $J_-(= -J_+)$ is integrable.

If $\varphi$ is $(-)$twistorial, then the anti-self-dual surfaces on $M^4$ which are leaves of the distributions $p_+$ or $p_-$ are foliated by horizontal null geodesics (this follows from the fact that $\varphi$ is horizontally conformal and, by definition, at each point $x \in M^4$, the space $p_\pm(x)$ intersects the horizontal space of $\varphi$ at $x$ along a null direction). If, further, $M^4$ is self-dual, then $Z(M, \varphi) = S_+ \sqcup S_-$ is the leaf space of the foliation induced by $\mathcal{F}$ on $M_+ \sqcup M_-$, and the map $Z(M, \varphi) \to Z(M)$ is simply the inclusion $S_+ \sqcup S_- \hookrightarrow Z(M)$. Obviously $S_+$ and $S_-$ are transversal to the twistor lines. Furthermore, as the preimage of a null geodesic on $N^2$ through $\varphi$ is a hypersurface on $M^4$ foliated by anti-self-dual surfaces, the two ‘surfaces’ $S_+$ and $S_-$ in $Z(M)$ are foliated by curves with leaf spaces $C_+$ and $C_-$. Also, $Z(N, \varphi) = Z(N)$ is the space of null geodesics of $N^2$ which, as a consequence of the horizontal conformality of $\varphi$, can be canonically identified with $C_+ \sqcup C_-$. Then, the twistorial representation of $\varphi$ is simply the projection $S_+ \sqcup S_- \to Z(N)$. To retrieve $\varphi$ from its twistorial representation, let $x \in M^4$; this corresponds to the twistor line $t_x \subseteq Z(M)$. Then, locally, $t_x$ meets $S_+$ and $S_-$ in two points which are projected by $S_+ \sqcup S_- \to Z(N)$ to two points: one in $C_+$ and the other in $C_-$. These correspond to two null geodesics on $N^2$ whose intersection is precisely $\varphi(x)$.

In a similar way to Example 4.1 if $\varphi : M^4 \to N^2$ is a horizontally conformal submersion for which $J_-(= -J_+)$ is nowhere integrable then $d_\pm = 1$ on $M_\pm$ and, if $M^4$ is self-dual then, locally, each anti-self-dual surface contains precisely one horizontal null geodesic and so we have two local sections $Z_\pm \hookrightarrow \mathcal{N}(M)$ of the canonical projection (see [18]) $\mathcal{N}(M) \to Z(M)$, where $\mathcal{N}(M)$ is the space of null geodesics of $M^4$ [25]. Note that if $\varphi$ is $(-)$twistorial then the fibres of $\mathcal{N}(M) \to Z(M)$ induce on $Z_+ \sqcup Z_-$ a one-dimensional foliation whose leaf space is $S_+ \sqcup S_-$.

A horizontally conformal submersion $\varphi : M^4 \to N^2$ with nowhere degenerate
fibres is \((-\)twistorial if and only if its fibres are \((+)\)twistorial in the sense of Example 3.6. In particular, \(a \text{ horizontally conformal submersion } \varphi : M^4 \rightarrow N^2 \text{ with nowhere degenerate fibres is both } (+)\text{twistorial and } (-)\text{twistorial if and only if it has totally umbilical fibres} \) (cf. Example 3.6).

The following theorem is a reformulation of a result of [37] (cf. Proposition 4.2 and Theorem 5.11 below).

**Theorem 4.5.** Let \(M^4\) be an orientable four-dimensional Einstein manifold and let \(\varphi : M^4 \rightarrow N^2\) be the complexification of a submersive harmonic morphism.

Then, with a suitable choice of orientations, \(\varphi\) is \((-)\text{twistorial.}\)

**Remark 4.6.**

1) Let \(M\) be a manifold endowed with an almost twistorial structure \(\tau = (P, M, \pi, \mathcal{F}), \text{ (dim } \mathcal{F} = k)\), such that the map \(P \rightarrow G_k(TM)\) given by \(p \mapsto d\pi(F_p)\) is an embedding of \(P\) in the Grassmann bundle of \(k\)-dimensional planes on \(M\).

Then \(\tau\) is integrable if, locally, there are sufficiently many twistorial maps from \((M, \tau)\); for example, if for each \(p \in P\) there exists a (local) twistorial map \(\varphi\) from \((M, \tau)\) such that \(\ker d\varphi_{\pi(p)} = p\) then \(\tau\) is integrable. In the above examples this condition is also necessary; moreover other similar necessary and sufficient conditions can easily be formulated. In Theorem 5.3 below, we shall see that the existence of a single suitable twistorial map may be sufficient for a twistorial structure to be integrable.

2) Twistorial maps as in Example 4.1, Example 4.4 and as in the next section appear, in a more or less explicit way, in [26].

5. **Twistorial Maps with One-Dimensional Fibres from Four-Dimensional Riemannian Manifolds**

In this section we continue to work in the complex-analytic category. The almost twistorial structures which may appear in this section will be those of Examples 2.4 and 2.6. As usual, the results can be applied to the real-analytic category by complexification (Remark 3.4).

Let \(\varphi : M^4 \rightarrow N^3\) be a surjective submersive map with nowhere degenerate fibres from an oriented 4-dimensional Riemannian manifold to a 3-dimensional Riemannian manifold. Suppose, for the moment, that \(M^4\) is self-dual and is
TWISTORIAL MAPS, HARMONIC MORPHISMS, SELF-DUAL MANIFOLDS

endowed with the twistorial structure of Example 2.6(ii), with $P = P_-$ the bundle of anti-self-dual planes of $M^4$, and that $N^3$ is endowed with the twistorial structure of Example 2.4, which corresponds to an Einstein–Weyl connection $D$ on $N^3$, with $Q$ the bundle of degenerate planes on $N^3$. Then, it is clear that $\varphi$ is $(-)$twistorial for a suitable choice of $P' \subseteq P$ if and only if it maps some, if not all, of the anti-self-dual surfaces on $M^4$ to totally geodesic degenerate surfaces on $(N^3, D)$ (cf. [18], [20]). But, unless we introduce some extra structure, there is no reason to ignore any of the anti-self-dual surfaces on $M^4$. Moreover, if $\varphi$ maps anti-self-dual surfaces on $M^4$ to totally geodesic degenerate surfaces on $(N^3, D)$, then, in particular, $d\varphi$ maps anti-self-dual planes on $M^4$ to degenerate planes on $N^3$ which is equivalent to the condition that $\varphi$ be horizontally conformal (indeed, the differential $d\varphi_x$, $(x \in M)$, maps the orthogonal complement, in $\mathcal{H}_x$, of a null direction $l \subseteq \mathcal{H}_x$ onto the orthogonal complement of a null direction $k \subseteq T_{\varphi(x)}N$ if and only if the adjoint of $d\varphi_x$ maps $k$ onto $l$; thus, as the horizontal projection of any anti-self-dual plane is a degenerate horizontal plane and any degenerate horizontal plane is obtained in this way, $d\varphi$ maps anti-self-dual planes on $M^4$ to degenerate planes on $N^3$ if and only if the adjoint of $d\varphi$ is conformal at each point).

Therefore, given an oriented Riemannian manifold $M^4$ and a horizontally conformal submersion $\varphi : M^4 \rightarrow N^3$ with nowhere degenerate fibres onto a Riemannian manifold, we shall look for necessary and sufficient conditions under which there exists a Weyl connection $D$ on $N^3$ with respect to which $\varphi : M^4 \rightarrow (N^3, D)$ is a $(-)$twistorial map with $P' = P_-$ the bundle of anti-self-dual planes on $M^4$.

To do this, we first place the discussion in a slightly more general context. Let $M^4$ be an oriented Riemannian manifold and let $\mathcal{H}$ be a nowhere degenerate three-dimensional distribution. Denote, as usual, $\mathcal{V} = \mathcal{H}^\perp$ and assume that $(\mathcal{V}, g|_{\mathcal{V}})$ and $(\mathcal{H}, g|_{\mathcal{H}})$ are oriented so that the isomorphism $(TM, g) = (\mathcal{V}, g|_{\mathcal{V}}) \oplus (\mathcal{H}, g|_{\mathcal{H}})$ is orientation preserving. Let $I^\mathcal{H}$ be the integrability 2-form of $\mathcal{H}$ defined by $I^\mathcal{H}(X, Y) = -g(U, [X, Y])$ for all local sections $X, Y$ of $\mathcal{H}$ where $U$ is the positive unit section of $\mathcal{V}$. Also, let $B^\mathcal{H}$ be the second fundamental form of $\mathcal{H}$ defined by $B^\mathcal{H}(X, Y) = \frac{1}{2} \mathcal{V}(\nabla_X Y + \nabla_Y X)$ for any local horizontal vector fields $X$ and $Y$, where $\nabla$ is the Levi-Civita connection of $(M^4, g)$. Then we define a section $B^\mathcal{H}$ of $\mathcal{H}^* \otimes \mathcal{H}^*$ by $B^\mathcal{H}(X, Y) = g(U, B^\mathcal{H}(X, Y))$ for any
horizontal $X$ and $Y$; we shall denote by $B_0^\mathcal{H}$ the trace-free part of $B^\mathcal{H}$. Let $^*\mathcal{H},g$ be the Hodge star-operator on $(\mathcal{H}, g|_\mathcal{H})$.

Recall (see [13]) that if $D$ is a conformal connection on a conformal manifold $(M, c)$ and $g$ is a local representative of $c$ on some open set $U$, then the Lee form of $D$ with respect to $g$ is the one-form $\alpha \in \Gamma(T^*U)$ characterised by $Dg = -2\alpha \otimes g$.

The Lee form of a conformal partial connection can be defined in a similar way.

Also, if $D$ is a partial connection on $\mathcal{H}$, over $\mathcal{H}$, then its torsion $T$, with respect to $V$, is defined by $T(X, Y) = DXY - DYX - \mathcal{H}[X, Y]$ for any local sections $X$ and $Y$ of $\mathcal{H}$. We now introduce a conformal partial connection which encodes the condition of twistoriality.

**Definition 5.1** (cf. [9]). The conformal partial connections $D_\pm$ induced by $g$ on $\mathcal{H}$ are the unique conformal partial connections on $(\mathcal{H}, g|_\mathcal{H})$, over $\mathcal{H}$, whose Lee forms $\alpha_\pm$ and torsion tensors $T_\pm$, with respect to $\mathcal{V}$, are given by

$$\begin{align*}
\alpha_\pm &= \text{trace}(B^{\mathcal{H}})^\pm \pm ^*\mathcal{H},g I^{\mathcal{H}}, \\
^*\mathcal{H},g T_\pm &= \mp 2B_0^{\mathcal{H}}
\end{align*}$$

(5.1)

where we have identified $\mathcal{H} \otimes \mathcal{H}^*$ and $\mathcal{H}^* \otimes \mathcal{H}^*$ by using $g|_\mathcal{H}$.

If $\mathcal{H}$ is totally umbilical then $D_-$ is the horizontal part of a Weyl connection of [9, §4].

Let $\pi^{\mathcal{H}} : P^{\mathcal{H}} \to M$ be the bundle of degenerate planes of $\mathcal{H}$. As in Example [2.4], $D_-$ induces a twistor distribution $\mathcal{T}_-^{\mathcal{H}}$ defined by setting $\mathcal{T}_-^{\mathcal{H}}(p) \subseteq T_p P^{\mathcal{H}}$ equal to the horizontal lift at $p \in P^{\mathcal{H}}$, with respect to $D_-$, of $p \subseteq \mathcal{H}_{\pi^{\mathcal{H}}}(p)$. Assume, for simplicity, that $\mathcal{V} = \ker d\varphi$ where $\varphi : M^4 \to N^3$ is a surjective submersion.

**Proposition 5.2.** (a) Let $\gamma$ be a null geodesic on $(M^4, g)$.

1. If $\gamma$ is horizontal then it is a geodesic of $D_-$. 
2. Suppose that $\gamma$ is not horizontal at $x_0 = \gamma(0)$; let $p_{0-} \subseteq T_{x_0}M$ be the anti-self-dual plane containing $(d\gamma/dz)(0)$ and let $p_0 = \mathcal{H}(p_{0-})$. Denote by $c$ the (local) horizontal curve such that $c(0) = x_0$ and $\varphi \circ c = \varphi \circ \gamma$.

If $p$ is the field of horizontal degenerate planes along $c$ such that $p(0) = p_0$ then $(dp/dz)(0) \in \mathcal{T}_-^{\mathcal{H}}$.

(b) Moreover, if $D$ is a conformal partial connection on $(\mathcal{H}, g|_\mathcal{H})$ which has
this property for all nonhorizontal null geodesics on \((M^4, g)\), and whose torsion \(T^D\), with respect to \(\mathcal{Y}\), is such that \(*T^D\) is self-adjoint and trace-free, then \(D = D_-\). In particular, \(D_-\) is conformally invariant.

Proof. (1) Suppose that \(\gamma\) is horizontal and let \(Y\) be its velocity vector field. Then \(B_0^\mathcal{Y}(Y, Y) = 0\) and hence also \((*\mathcal{Y}, g)T_-(Y) = 0\); equivalently, \(T_-\) restricted to \(Y^\perp\) is zero, and thus \(g(Y, T_-(X, Y)) = 0\) for any \(X \in Y^\perp\).

Because \(\gamma\) is a null curve and \(D_-\) is conformal, we have that \(D_- Y \in Y^\perp\) at each point of \(\gamma\). Thus, to prove that \(\gamma\) is a geodesic of \(D_-\), it is sufficient to prove that \(g(D_- Y, X) = 0\) for any \(X \in Y^\perp\). The proof follows from the fact that \(D_- Y = \nabla Y + 2\alpha_-(Y)Y\) where \(\nabla\) is the Levi-Civita connection of \(g\).

(2) If \(\gamma\) is not horizontal then we can write \(d\gamma/dz = X + iU\) where \(X\) is horizontal, \(U\) is vertical and \(g(X, X) = g(U, U) = a^2 \neq 0\). Extend \(X\) and \(U\) to local sections of \(\mathcal{H}\) and \(\mathcal{Y}\), respectively. We can assume that \(X\) is a basic vector field and \(a^{-1}U\) is positive on \((\mathcal{Y}, g|_{\mathcal{Y}})\). Then, along \(\gamma\), we have

\[
\nabla_{X+iU}(X + iU) = \nabla X - \nabla_U U + i(\nabla X + \nabla_U X) = 0
\]

where \(\nabla\) is the Levi-Civita connection of \((M^4, g)\).

We see from \((5.2)\) that, for any horizontal vector field \(Y\), we have

\[
g(\nabla X, Y) - g(\nabla Y, U) + i g(\nabla X, Y) = 0
\]

along \(\gamma\).

Now, assume that \(Y\) is null (and horizontal) and \(g(X, Y) = 0\). Then, by using the fact that \(X\) is basic, the last relation implies that

\[
g(\nabla X, Y) = a^2 \{g(\text{trace}(B^\mathcal{Y}), Y) - i I^\mathcal{H}(a^{-1}X, Y) + 2i B^\mathcal{H}(a^{-1}X, Y)\}
\]

along \(\gamma\) where we have assumed that \(\{a^{-1}U\}\) is a positive local frame of \((\mathcal{Y}, g|_{\mathcal{Y}})\).

On the other hand, let \(D\) be a conformal partial connection on \((\mathcal{H}, g|_{\mathcal{H}})\), let \(T\) be its torsion with respect to \(\mathcal{Y}\), and let \(\alpha\) be its Lee form with respect to \(g|_{\mathcal{H}}\). Then we have

\[
g(\nabla X, Y) = g(D_X X, Y) + \alpha(Y) g(X, X) + g(X, T(X, Y))
\]

Let \(\{X_0, X_1, X_2, X_3\}\) be a positive local orthonormal frame on \((M^4, g)\) such that \(X_0 = a^{-1}U\) and \(X_1 = a^{-1}X\). Then \(\{X_1, X_2, X_3\}\) is a positive orthonormal local frame of \((\mathcal{H}, g|_{\mathcal{H}})\). Take \(Y = X_2 + iX_3\) and note that \(p_{0-}\) is spanned by \(\{X_{x_0} + iU_{x_0}, Y_{x_0}\}\) and \(p_0\) by \(\{X_{x_0}, Y_{x_0}\}\). Also, \(c\) is the integral curve of \(X\) through...
so that \( p_z \) is spanned by \( \{X_{c(z)}, Y_{c(z)}\} \) at each \( z \).

Then \( (dp/dz)(0) \) is horizontal with respect to the connection induced by \( D \) on \( \pi^\X \) if and only if \( (D_X X)_x \in p_0 \); equivalently \( g((D_X X)_x, Y_x) = 0 \).

By (5.3) and (5.4), the last condition is equivalent to

\[
\alpha(Y) - g(\text{trace}(B^\X), Y) + i I^\X(X_1, Y) = -g(X_1, T(X_1, Y)) + 2i B^\X(X_1, Y)
\]
at \( x_0 \). Condition (5.5) is equivalent to

\[
(\alpha - \text{trace}(B^\X)) + *_{\H} \text{I}^\X(Y) = -g(X_1, T(X_1, Y)) + 2i B^\X(X_1, Y)
\]
at \( x_0 \). It is easy to see that, if in (5.6) we replace \( \alpha \) and \( T \) by \( \alpha_- \) and \( T_- \), respectively, then both sides are zero. Hence, \( (dp/dz)(0) \in \mathcal{F}^\X \).

Suppose now that \( D \) satisfies condition (2) of the proposition. Then (5.6) must hold. Moreover, the relation obtained from (5.6) by replacing \( Y \) with \( \tilde{Y} = X_2 - iX_3 \) and \( X_1 \) by \(-X_1\) (so that \( \{X_0, -X_1, X_2, -X_3\} \) is positive) must also hold, viz.

\[
(\alpha - \text{trace}(B^\X)) + *_{\H} \text{I}^\X(\tilde{Y}) = -g(X_1, T(X_1, \tilde{Y})) - 2i B^\X(X_1, \tilde{Y})
\]
at \( x_0 \). By taking the sum of (5.6) and (5.7) we obtain

\[
(\alpha - \text{trace}(B^\X)) + *_{\H} \text{I}^\X(X_2) = -g(X_1, (T)(X_3)) - 2B^\X(X_1, X_3)
\]
at \( x_0 \).

From the fact that the right-hand-side of (5.8) does not depend on \( X_2 \) it follows that \( *T + 2 B^\X \) is proportional to \( g|_\H \). But \( *T \) is self-adjoint and trace-free, hence \( *T + 2 B^\X_0 = 0 \). Then, by (5.8), we also have \( \alpha = \text{trace}(B^\X) - *_{\H} \text{I}^\X \) and hence \( D = D_- \). \( \square \)

Since, up to Riemannian coverings, any horizontally conformal submersion admits global dilations, from now on, for simplicity, we shall consider only horizontally conformal submersions which admit global dilations.

The following theorem is a reformulation of results of [9] (see Definitions 4.3 and 5.1 for ‘\((-\)twistorial map’ and \( D_- \), respectively).

**Theorem 5.3.** Let \((M^4, g)\) and \((N^3, h)\) be orientable Riemannian manifolds and let \( \varphi : (M^4, g) \to (N^3, h) \) be a surjective horizontally conformal submersion with connected nowhere degenerate fibres.

Suppose that orientations on \((M^4, g)\), \((\mathcal{V}, g|_\mathcal{V})\) and \((\H, g|_\H)\) are chosen such
that the isomorphism \((TM, g) = (\mathcal{Y}, g|_{\mathcal{Y}}) \oplus (\mathcal{H}, g|_{\mathcal{H}})\) is orientation preserving.

Then, the following assertions are equivalent:

(i) there exists a Weyl connection \(D\) on \(N^3\) with respect to which \(\varphi\) is \((-)\)twistorial (with \(P' = P_+\) the bundle of anti-self-dual planes on \((M^4, g)\));

(ii) the one-form

\[
\text{trace}(B^\gamma)^b - \lambda^{-1} d^\mathcal{H} \lambda - *_{\mathcal{H}, g} I^\mathcal{H}
\]

is basic where \(\lambda\) is the dilation of \(\varphi\), \(d^\mathcal{H} = \mathcal{H} \circ d\) and \(*_{\mathcal{H}, g}\) is the Hodge star-operator on \((\mathcal{H}, g|_{\mathcal{H}})\);

(iii) the two-form \(d(\text{trace}(B^\gamma)^b + \frac{1}{3} \text{trace}(B^\mathcal{H})^\gamma)\) is self-dual.

Moreover, if any of the assertions (i), (ii) or (iii) holds then:

(a) \(\varphi^*(D)^\mathcal{H} = D_-\); in particular, \(D\) is the unique Weyl connection that satisfies (i);

(b) \(M^4\) is self-dual if and only if \(D\) is Einstein–Weyl.

If \(M^4\) is self-dual then, the following assertion can be added to the assertions (i) to (iii):

(iv) There exists a (unique) Einstein–Weyl connection \(D\) on \(N^3\) such that, for any local map \(\psi : N^3 \to P^2\) which is twistorial in the sense of Example 4.4, the map \(\psi \circ \varphi : M^4 \to P^2\) is \((-)\)twistorial (in the sense of Example 4.4).

Proof. Let \(\pi_+ : P_+ \to M\) be the bundle of anti-self-dual planes on \(M^4\) and let \(\Phi = d\varphi|_{P_+}\) be the map induced by \(\varphi\) from \(P_+\) to the bundle of degenerate planes on \(N^3\). Assertion (i) is equivalent to the existence of a Weyl connection \(D\) on \(N^3\) such that for any \(x \in M^4\) and any null geodesic \(\gamma\) on \(M^4\) with \(\gamma(0) = x\), if we denote by \(p\) the field of anti-self-dual planes along \(\gamma\) such that \((d\gamma/dz)(z) \in p(z)\) for all \(z\), then \(d\Phi(d\rho/dz)(0)) \in \mathcal{F}\), where \(\mathcal{F}\) is the twistor distribution induced by \(D\) on the bundle of degenerate planes on \(N^3\) (Example 2.4). Thus, by Proposition 5.2, assertion (i) is equivalent to the existence of a Weyl connection \(D\) on \(N^3\) such that \(\varphi^*(D)^\mathcal{H} = D_-\). Now, applying the conformal change \(\varphi^*(h)|_{\mathcal{H}} = \lambda^2 g|_{\mathcal{H}}\) to (5.1) introduces the term \(-\lambda^{-1} d^\mathcal{H} \lambda\) so that the Lee form of \(D_-\) with respect to \(\varphi^*(h)|_{\mathcal{H}}\) is equal to the one-form (5.9). Hence we have the equivalence of (i) and (ii). Furthermore, if (i) holds then assertion (a) also holds.

The equivalence (ii) \(\iff\) (iii) can be found in the proof of [9, Proposition 4.4].

The fact that, if \(M^4\) is self-dual, then (i) \(\iff\) (iv) is obvious. Also, if \(M^4\) is self-dual and \(\varphi\) is \((-)\)twistorial, then from Theorem 2.5 it follows that \(D\) is
Einstein–Weyl. Conversely, if $\varphi$ is $(-)$twistorial and $D$ is Einstein–Weyl, then by composing $\varphi$ with local twistorial maps $(N^3, D) \to P^2$ we obtain sufficiently many local twistorial maps $M^4 \to P^2$ to show that any anti-self-dual plane on $M^4$ is tangent to some anti-self-dual surface in $M^4$. Thus, by Theorem \ref{thm:2.7}, $M^4$ is self-dual. \hfill $\square$

**Remark 5.4.** 1) D.M.J. Calderbank \cite{9} calls a horizontally conformal submersion, between real-analytic Riemannian manifolds, satisfying condition (iii) of Theorem \ref{thm:5.3} **self-dual**. We have thus interpreted self-dual submersions as those which are $(-)$twistorial.

2) By using the null-tetrad formalism (see \cite{28}), the equivalence $(i) \iff (ii)$ of Theorem \ref{thm:5.3} can also be obtained after a straightforward but tedious computation.

In the following, let $\varphi : (M^4, c_M) \to (N^3, c_N)$ be a surjective horizontally conformal submersion with nowhere degenerate fibres from a four-dimensional oriented conformal manifold to a three-dimensional conformal manifold. Let $L$ be the line bundle over $N^3$ associated with the bundle of conformal frames on $N^3$ through the morphism of Lie groups $\rho : CO(3, \mathbb{K}) \to \mathbb{K} \setminus \{0\}$, $(\mathbb{K} = \mathbb{R} \text{ or } \mathbb{C})$, characterised by $a \in CO(3, \mathbb{K})$ if and only if, $a^T a = \rho(a)^2 I$ (in the smooth category, $L$ can be defined as the line bundle associated to the frame bundle of $N^3$ through the morphism of Lie groups $GL(3, \mathbb{R}) \to (0, \infty)$, $a \mapsto |\det(a)|^{1/3}$, see \cite{19}). Note that $L$ can be defined on any odd-dimensional conformal manifold (in the smooth category, on any (conformal) manifold $(N, c_N)$ and that its local sections correspond to oriented local representatives of $c_N$. Let $E$ be some line bundle over $N$ and let $\tilde{E} = \varphi^*(E)$.

To any pair $(s, \nabla)$ where $s$ is a section of $L^*$ over some open subset $U$ of $N$ and $\nabla$ is a connection on $E|_U$ there can be associated a connection $\tilde{\nabla}$ of $\tilde{E}|_{\varphi^{-1}(U)}$ as follows: assume, initially, that $s$ is nowhere zero, let $h$ be the oriented local representative of $c_N$ over $U$ corresponding to it and let $g$ be the oriented local representative of $c_M$ over $\varphi^{-1}(U)$ such that $\varphi : (\varphi^{-1}(U), g) \to (U, h)$ is a Riemannian submersion. Denote, as usual, $\mathcal{V} = \ker d\varphi$, $\mathcal{H} = \mathcal{V}^\perp$, and let $\omega \in \Gamma(\varphi^{-1}(U), \mathcal{V}^*)$ be the induced orientation on $\mathcal{V}|_{\varphi^{-1}(U)}$ such that the isomorphism $(T(\varphi^{-1}(U)), g) = (\mathcal{V}|_{\varphi^{-1}(U)}, g|_\mathcal{V}) \oplus (\mathcal{H}|_{\varphi^{-1}(U)}, \varphi^*(h))$ is orientation preserving. If $A$ is the local connection form of $\nabla$ with respect to some nowhere zero local
section \( \sigma \in \Gamma(U, E) \), then we set the local connection form of \( \tilde{\nabla} \) corresponding to \( \varphi^*(\sigma) \) equal to
\[
\tilde{A} = -\omega + \varphi^*(A) .
\]
We shall call \( \tilde{\nabla} \) the pull-back, by \( \varphi \), of the pair \((s, \nabla)\). Note that, if \( u \) is a nowhere zero function locally defined on \( N \), the pull-back by \( \varphi \) of \((us, \nabla)\) is given by \(-u \omega + \varphi^*(A)\). Hence, we can extend the definition of the pull-back by \( \varphi \) of any pair \((s, \nabla)\) where the local section \( s \) of \( L^* \) may have zeros.

The monopole equation of [20] can be written as follows:

\[
(5.10) \quad (d - \alpha)u = \ast F
\]

where \( u \) is a function on a three-dimensional oriented Riemannian manifold \((N^3, h)\), \( \alpha \) is a one-form on \( N^3 \), \( F \) is a two-form on \( N^3 \) and \( \ast \) is the Hodge star-operator of \((N^3, h)\). By interpreting \( \alpha \) as the Lee form with respect to \( h \) of a Weyl connection on \((N^3, [h])\), and \( F \) as the curvature form of some connection on some line bundle over \((N^3, h)\), the equation (5.10) can be written as follows:

**Definition 5.5** ([20] [10]). Let \( D \) be a Weyl connection on \((N^3, c_N)\). A monopole on \((N^3, c, D)\) is a pair \((s, \nabla)\) where \( s \) is a section of \( L^* \) and \( \nabla \) is a connection on a line bundle \( E \) over \( N^3 \) such that

\[
(5.11) \quad \ast_N Ds = F
\]

where \( F \) is the curvature form of \( \nabla \). A monopole is called nontrivial if \( s \neq 0 \).

**Remark 5.6.** Let \((N^3, c)\) be a three-dimensional conformal manifold endowed with a Weyl connection \( D \).

1) It is well known (see [10]) that if \((N^3, c, D)\) is Einstein–Weyl then, at least locally, there exist nontrivial monopoles on \((N^3, c, D)\).

 Conversely, let \((N^3, c)\) be a three-dimensional conformal manifold and let \( E \) be a line bundle over \( N \) endowed with a connection \( \nabla \). Also, let \( s \) be a nowhere zero (local) section of \( L^* \) and let \( h \) be the oriented (local) representative of \( c \) corresponding to \( s \). Define a one-form \( \alpha \) such that \(- \ast_h \alpha = \varphi^*(F)\) where \( F \) is the curvature form of \( \nabla \). Then \((s, \nabla)\) is a monopole on \((N^3, c, D)\) where \( D \) is the Weyl connection on \((N^3, c)\) whose Lee form with respect to \( h \) is \( \alpha \).

2) Let \( h \) be a local representative of \( c \) over some open subset \( U \) of \( N \) and let \( \alpha \) be the Lee form of \( D \) with respect to \( h \). Then, it is well known (and easy to prove)
that, on small enough open subsets \( U \), monopoles on \((N^3, c, D)\) correspond to solutions of the equation \( \Delta u = d^*(u\alpha) \) where \( \Delta \) is the Laplacian and \( d^* \) is the codifferential on \((U, h)\). Hence, if \((N^3, c, D)\) is (the complexification) of a real-analytic conformal manifold endowed with a real-analytic Weyl connection then, at least locally, there exist nontrivial monopoles on \((N^3, c, D)\).

Assertion (i) of the following proposition is essentially in [10, 9].

**Corollary 5.7.** Let \( \varphi : (M^4, c_M) \rightarrow (N^3, c_N) \) be a surjective horizontally conformal submersion with nowhere degenerate fibres and let \( D \) be a Weyl connection on \((N^3, c_N)\).

(i) If \( \varphi : (M^4, c_M) \rightarrow (N^3, c_N, D) \) is a \((-)\)twistorial map, then it pulls back monopoles on \((N^3, c_N, D)\) to self-dual connections on \((M^4, c_M)\).

(ii) Conversely, suppose that there exists a nontrivial monopole on \((N^3, c_N, D)\) which is pulled back by \( \varphi \) to a self-dual connection on \((M^4, c_M)\). Then, \( \varphi : (M^4, c_M) \rightarrow (N^3, c_N, D) \) is \((-)\)twistorial.

**Proof.** Let \((s, \nabla)\) be a nontrivial monopole. We may assume that \( s \) is nowhere zero so that it corresponds to an oriented local representative \( h \) of \( c_N \). Let \( \alpha \) be the Lee form of \( D \) with respect to \( h \). Then (5.11) is equivalent to

\[
- \ast_h \alpha = F.
\]

If \( A \) is a local connection of \( \nabla \), then the corresponding local connection form of \( \tilde{\nabla} \) is \( \tilde{A} = -\omega + \varphi^*(A) \) where \( \omega \in \Gamma(\mathcal{V}^\ast) \) is as above. Now, by using the fact that

\[
d\omega = -\text{trace}(B^\mathcal{V})\wedge\omega + I^\mathcal{H}
\]

it is easy to prove that \( d\tilde{A} \) is self-dual if and only if

\[
- \ast_h \alpha_\ast = \varphi^*(dA)
\]

where \( \alpha_\ast = \text{trace}(B^\mathcal{V}^\mathcal{H}) - \ast_{\mathcal{H}, g} I^\mathcal{H} \) is the Lee form of \( D_\ast \) with respect to \( h \).

From (5.12) and (5.14) it follows that \( \alpha_\ast = \alpha \) is basic. The result follows from Theorem 5.3. \( \square \)

Let \( M^m \) be a manifold of dimension \( m \geq 3 \) endowed with a three-dimensional distribution \( \mathcal{H} \). Suppose that \( \mathcal{H} \) is endowed with a conformal structure \( c \) and a conformal partial connection \( D \). In a similar fashion to the above, we consider
a line bundle $L$ over $M$ whose nowhere zero local sections correspond to oriented local representatives of $c$.

Suppose that there exists a foliation $\mathcal{V}$ which is complementary to $\mathcal{H}$ and assume, for simplicity, that there exists a submersion $\varphi : M^m \to N^3$ whose fibres are leaves of $\mathcal{V}$.

Then Definition 5.5 can be easily generalised by defining a monopole on $(\mathcal{H}, c, D)$ (with respect to $\varphi$) to be a pair $(s, \nabla)$ where $s$ is a section of $L^*$ and $\nabla$ a connection on a line bundle $E$ over $N^3$ such that $\ast_{\mathcal{H}} Ds = \varphi^* (F)$ where $F$ is the curvature form of $\nabla$. (One could also consider line bundles on $M$ endowed with partial connections over $\mathcal{H}$ but then the resulting equation would depend on a local section of $E$.)

Furthermore, if $(M^4, c)$ is a four-dimensional oriented conformal manifold then, a construction similar to above associates to any pair $(s, \nabla)$ a connection $\tilde{\nabla}$ on $\varphi^*(E)$. We then have the following simple fact (which may be known) whose proof is similar to the proof of Corollary 5.7.

**Proposition 5.8.** Let $\varphi : (M^4, c) \to N^3$ be a surjective submersion with nowhere degenerate fibres from an oriented four-dimensional conformal manifold. Suppose that $D$ is a conformal partial connection on $(\mathcal{H}, c|_{\mathcal{H}})$ and let $E$ be a line bundle over $N^3$. Let $s$ be a section of $L^*$ and $\nabla$ a connection on $E$.

Then any two of the following assertions imply the third:

(i) $(s, \nabla)$ is a nontrivial monopole on $(\mathcal{H}, c|_{\mathcal{H}}, D)$;

(ii) $\tilde{\nabla}$ is self-dual;

(iii) $D = D_\perp$.

Next, we show how harmonic morphisms fit into the above discussion.

**Proposition 5.9.** Let $\mathcal{V}$ be (the complexification) of a one-dimensional (nowhere degenerate) conformal foliation on a four-dimensional oriented conformal manifold $(M^4, c)$; let $\mathcal{H} = \mathcal{V}^\perp$.

Then, the following assertions are equivalent:

(i) there exist local representatives $g$ of $c$ with respect to which $\mathcal{V}$ is locally defined by harmonic morphisms;

(ii) there exist nontrivial monopoles $(s, \nabla)$ on $(\mathcal{H}, c|_{\mathcal{H}}, D_\perp)$ for which $\tilde{\nabla}$ is flat.
Proof. By Theorem 1.8, assertion (i) is equivalent to the following:

(i′) locally, there exist representatives \( g \) of \( c \) with respect to which \( \mathcal{V} \) is geodesic.

Also, from (5.13), we easily obtain that a pair \((s, \nabla)\) is as in assertion (ii) if and only if the following hold:

(a) \( \mathcal{V} \) has geodesic fibres with respect to the local representative \( g \) of \( c \) whose horizontal component corresponds to \( s \);

(b) if \( \theta \) is the local volume form induced by \( g \) on \( \mathcal{V} \) then \( d\theta \) is equal to the pull-back by \( \varphi \) of the curvature form of \( \nabla \).

We have thus shown that (ii)\( \Rightarrow \) (i).

Conversely, suppose that (i) holds and let \( s \) be a nowhere zero local section of \( L^* \) such that, with respect to the corresponding oriented local representative \( g \) of \( c \), the foliation \( \mathcal{V} \) is geodesic. Then \( I^\mathcal{H} = d\theta \) where \( \theta \) is the local volume form induced by \( g \) on \( \mathcal{V} \). Moreover, because \( \mathcal{V} \) is geodesic with respect to \( g \), we have, locally, \( d\theta = \varphi^*(d\alpha) \) for some one-form \( \alpha \) on \( N^3 \). Furthermore, by Definition 5.1, we have \( -**\mathcal{H},g \alpha_- = I^\mathcal{H} \). Hence \( -**\mathcal{H},g \alpha_- = \varphi^*(d\alpha) \); equivalently, the pair formed by \( s \) and the connection determined by \( \alpha \), on the trivial line bundle over \( N^3 \), is a monopole on \((\mathcal{H},c|_\mathcal{H},D_-)\). The proposition is proved. \( \square \)

The following result gives another characterisation for twistorial harmonic morphisms between orientable Riemannian manifolds of dimension four and three, respectively.

**Corollary 5.10.** Let \((M^4, g)\) and \((N^3, h)\) be orientable Riemannian manifolds. Let \( \varphi : (M^4, g) \to (N^3, h) \) be a surjective submersive harmonic morphism with connected nowhere degenerate fibres. Suppose that orientations on \((M^4, g), (N^3, h), (\mathcal{V}, g|_\mathcal{V})\) and \((\mathcal{H}, g|_\mathcal{H})\) are chosen such that the isomorphisms \((TM, g) = (\mathcal{V}, g|_\mathcal{V}) \oplus (\mathcal{H}|_\mathcal{H})\) and \((\mathcal{H}, g|_\mathcal{H}) = (\varphi^*(TN), \varphi^*(h))\) are orientation preserving.

Then, the following assertions are equivalent:

(i) There exists a Weyl connection \( D \) on \( N^3 \) with respect to which \( \varphi \) is \((-)\)twistorial.

(ii) There exists a basic one-form \( \alpha \) on \( M^4 \) such that

\[
(\alpha_\mathcal{H} - \alpha)(\lambda^{-2}) = *_\mathcal{H}\Omega
\]

where \( \lambda \) is a dilation of \( \varphi \), \(*_\mathcal{H} \) is the Hodge star operator on \((\mathcal{H}, \varphi^*(h))\) and \( \Omega \) is the curvature form of the horizontal distribution (i.e., in the notation of Theorem 1.8, \( \Omega = d\theta \)).
Moreover, if (i) or (ii) holds then the Weyl connection $D$ of assertion (i) is unique and the one-form $\alpha$ of assertion (ii) is the pull-back of the Lee form of $D$ with respect to $h$.

**Proof.** Comparing the fundamental equation (1.1) with (5.9) and by using the fact that $I^\mathcal{H} = \lambda \Omega$, it follows that a one-form $\alpha$ on $M^4$ satisfies (5.15) if and only if it is the Lee form of $D_-$ with respect to $\varphi^*(h)$. The proof follows from Theorem 5.3. □

From [31, Corollary 1.9] we obtain the following result on maps from Einstein manifolds (cf. Proposition 4.2 and Theorem 4.5).

**Theorem 5.11.** Let $M^4$ be an orientable four-dimensional Einstein manifold and let $\varphi : M^4 \to N^3$ be the complexification of a submersive harmonic morphism.

Then, there exists a Weyl connection on $N^3$ with respect to which, with a suitable choice of orientations, $\varphi$ is $(\pm)$twistorial. □

Another consequence of Theorem 5.3 is that a one-dimensional conformal foliation with nowhere degenerate leaves on an oriented four-dimensional Riemannian manifold is both $(\pm)$twistorial and $(\pm)$twistorial (Remark 3.4(5)) if and only if it is locally generated by conformal vector fields [9] (the ‘if’ part follows also from [20]). In particular, (the complexification of) any one-dimensional homothetic foliation locally defined by harmonic morphisms on an oriented four-dimensional Riemannian manifold is both $(\pm)$twistorial and $(\pm)$twistorial.

With the same notations as above and in Section 11, we have the following consequences of Theorem 5.3 (in which we do not claim that (i) $\iff$ (iii) is new).

**Corollary 5.12.** Let $\mathcal{V}$ be a one-dimensional conformal foliation with integrable orthogonal complement on an oriented four-dimensional conformal manifold $(M^4, c)$.

Then the following assertions are equivalent:

(i) $\mathcal{V}$ is $(\pm)$twistorial;

(ii) $\mathcal{V}$ produces harmonic morphisms with respect to any local representative of $c$ for which $\mathcal{V}$ has geodesic leaves;

(iii) locally, there exist representatives of $c$ with respect to which $\mathcal{V}$ is defined by totally geodesic Riemannian submersions. □
We end this section with a result on \((\pm)\)twistorial maps from conformally flat four-manifolds. Recall (see [35]) that, any conformally flat four-manifold can be locally embedded in the four-dimensional nonsingular quadric \(Q_4\).

**Proposition 5.13.** Let \(M^4\) be an oriented conformally flat four-manifold and let \(\varphi : M^4 \to N^3\) be a horizontally conformal submersion with nowhere degenerate fibres. Then, the following assertions are equivalent:

(i) \(\varphi\) is \((-)\)twistorial;

(ii) \(\varphi\) is \((+)\)twistorial.

Furthermore, if \(\varphi\) is \((\pm)\)twistorial and \(M^4\) is locally embedded in \(Q_4\), then the twistor space \(Z(N)\) of the induced Einstein–Weyl connection on \(N^3\) is, locally, a surface in \(Q_4\) such that the space of horizontal null geodesics of \(\varphi\) is equal to the space of null geodesics on \(Q_4\) which pass through \(Z(N) \subseteq Q_4\).

6. **The classification of twistorial harmonic morphisms with one-dimensional fibres from self-dual four-manifolds**

In the last two sections we shall work in the real-analytic category.

Firstly, we list the four types of \((-)\)twistorial harmonic morphisms with one-dimensional fibres which can be defined on a four-dimensional Riemannian manifold. Later on we shall see how they come from equation (5.15).

**Type 1** (‘Killing type’ [8]). Harmonic morphisms \(\varphi : M^4 \to N^3\) whose fibres are locally generated by nowhere zero Killing vector fields (see Section 1 above).

**Type 2** (‘warped-product type’ [2]). Horizontally homothetic submersions with geodesic fibres orthogonal to an umbilical foliation by hypersurfaces (see Remark 1.9(1)).

**Type 3** (cf. [30, 31, 33]). \(\varphi : (M^4, g) \to (N^3, h)\) is a real-analytic map which is, locally, the canonical projection \(\mathbb{R} \times N^3 \to N^3\), \((\rho, x) \mapsto x\), and

\[(6.1)\]

\[g = \rho h + \rho^{-1}(d\rho + A)^2\]

where \(A\) is a one-form on \(N^3\) which satisfies the Beltrami fields equation \(dA = -\ast A\).
Type 4 (cf. [9]). \((N^3, h)\) is endowed with a Weyl connection \(D, \varphi : (M^4, g) \to (N^3, h)\) is a real-analytic map which is, locally, the canonical projection \(\mathbb{R} \times N^3 \to N^3, (\rho, x) \mapsto x\), and

\[
g = (e^\rho + c) h + \frac{1}{e^\rho + c} (d\rho - \alpha)^2
\]

where the Lee form \(\alpha\) of \(D\) with respect to \(h\) satisfies the equation

\[
d\alpha - c \ast \alpha + \ast dc = 0
\]
on \(N^3\) with \(c : N^3 \to \mathbb{R}\) a function (if \(e^\rho + c < 0\), then we replace \(g\) with \(-g\)).

Remark 6.1. 1) Note that maps of types 3 and 4 are always harmonic morphisms by Theorem 1.8.

2) Let \(\varphi : (M^{n+1}, g) \to (N^n, h) (n \geq 1)\) be a submersive harmonic morphism with one-dimensional fibres. Then, the components of the fibres of \(\varphi\) form a homothetic foliation if and only if either \(\varphi\) is of type 1 or, up to a conformal change of \((M^{n+1}, g)\) with factor constant along the fibres, \(\varphi\) is of type 2.

3) If \(\varphi : (M^4, g) \to (N^3, h)\) is a harmonic morphism of type 3 then \(\rho\) and \(\varphi^*(A)\) are globally defined on \(M^4\). Indeed, if we denote by \(\lambda\) the dilation of \(\varphi\) then \(\rho = \lambda^{-2}\) and \(\varphi^*(A) = -d^\varphi (\lambda^{-2})\).

4) If \(\varphi\) is a harmonic morphism of type 4 with \(d\alpha = 0\) then the horizontal distribution is integrable and, after a conformal change with basic factor, \(\varphi\) is of type 2. If \(\alpha \neq 0\), then \(\rho\) and \(c \circ \varphi\) are globally defined on \(M^4\) (this follows from the fact that, up to signs, we have \(\theta = d\rho - \alpha\) and \(\lambda^{-2} = e^\rho + c \circ \varphi\) where \(\lambda\) is the dilation of \(\varphi\); in particular, \(e^\rho + c \circ \varphi\) has constant sign on \(M^4\).

In the complex-analytic category the construction of type 4 is globally defined if \(e^\rho + c \circ \varphi\) is nowhere zero. Then, if \(\alpha \neq 0\), \(e^\rho\) and \(c \circ \varphi\) are well-defined functions on \(M^4\).

Proposition 6.2. Let \(\varphi : (M^4, g) \to (N^3, h, D)\) be a harmonic morphism of type 4 with \(c \neq 0\). Then, outside the zero set of \(c\), after a conformal change with factor constant along the fibres and a suitable change of coordinates we may assume that \(c = \pm 1\) — that is, \(\varphi\) is, locally, the canonical projection \(\mathbb{R} \times N^3 \to N^3, (\rho, x) \mapsto x\), and

\[
g = (e^\rho \pm 1) h + \frac{1}{e^\rho \pm 1} (d\rho - \alpha)^2
\]
where the Lee form $\alpha$ of $D$ with respect to $h$ satisfies the Beltrami fields equation

\[(6.4) \quad d\alpha = \pm * \alpha\]
on $N^3$.

**Proof.** We shall show that $\tilde{g} = |c| g$ is as in (6.3), (6.4) after the change of coordinates $\tilde{\rho} = \rho - \log |c|$.

Assume for simplicity that $c > 0$ and let $\tilde{h} = c^2 h$. Then, the dilation $\tilde{\lambda}$, fundamental vector field $\tilde{V}$ and connection form $\tilde{\theta}$ of the harmonic morphism $\varphi: (M^4, \tilde{g}) \to (N^3, \tilde{h})$ are as follows: $\tilde{\lambda}^2 = c\lambda^2$, $\tilde{V} = V$ and $\tilde{\theta} = \theta$. Also the Lee form of $D$ with respect to $\tilde{h}$ is $\tilde{\alpha} = \alpha - d \log c$.

It is easy to see that

\[d\alpha - c * \alpha + * dc = 0\]

written with respect to $\tilde{h}$ becomes $* d\tilde{\alpha} = c \tilde{\alpha}$. But, if we denote by $\tilde{*}$ the Hodge-star operator on $(N^3, \tilde{h})$, then

\[\tilde{*}|_{\Lambda^2} = c^{-1} *|_{\Lambda^2}\]

and hence

\[(6.5) \quad \tilde{*} d\tilde{\alpha} = \tilde{\alpha}.\]

Now it is easy to see that, with respect to $\tilde{\rho}$, $\tilde{h}$ and $\tilde{\alpha}$, the metric $\tilde{g}$ has the required form. \[\square\]

Now we are able to state the main result of this section.

**Theorem 6.3.** Let $M^4$ and $N^3$ be real-analytic Riemannian manifolds of dimensions four and three, respectively. Suppose that $N^3$ is endowed with a Weyl connection $D$.

Then $\varphi: M^4 \to (N^3, D)$ is a $(-)$twistorial harmonic morphism if and only if, up to conformal changes of the metric on the domain and codomain, $\varphi$ is of type 1, 2, 3 or 4; further, the conformal factor on $M^4$ can be taken to be constant along the fibres of $\varphi$. If $\varphi$ is of type 2, then $D$ is the Levi-Civita connection of some metric in the conformal class of $N^3$, whilst if $\varphi$ is of type 3, then $D$ is the Levi-Civita connection of $N^3$.

**Proof.** By Corollary 5.10 equation (5.15) is satisfied. If $V(\lambda^{-2}) = 0$ then $\varphi$ is of type 1. From [32, Corollary 1.5] it follows that $V(\lambda^{-2})$ is real-analytic, and from now on we shall assume that $V(\lambda^{-2})$ is nowhere zero on some open set. Moreover, by replacing, if necessary, $V$ with $-V$, we can assume that $V(\lambda^{-2})$ is positive at
each point.

Let $X$ be a basic vector field. Note that the left-hand side of (5.15) must be basic and hence, using $[V, X] = 0$ we obtain $X(V(\lambda^{-2})) - V(\lambda^{-2})\alpha(X) = 0$. Thus

\begin{equation}
\alpha = d\mathcal{K}(\log V(\lambda^{-2})).
\end{equation}

This implies that $X(\log V(\lambda^{-2}))$ is basic. Hence

\begin{equation}
X(V(\log V(\lambda^{-2}))) = V(X(\log V(\lambda^{-2}))) = 0.
\end{equation}

It follows that, if $V(\log V(\lambda^{-2}))$ is nonconstant then $\mathcal{K}$ is integrable, equivalently $\Omega = 0$ on an open set, and hence, by real-analyticity, on $M^4$. This, together with (5.15), gives $\alpha = -2 d\mathcal{K}(\log \lambda)$; hence $d\mathcal{K}(\log \lambda)$ is basic, that is, $V(\mathcal{K}(\log \lambda)) = 0$ for any basic vector field $X$. Because $\mathcal{K}$ is integrable this implies that $\mathcal{K}$ is homothetic, and hence after a conformal change with basic factor we get that $\varphi$ is of type 2 (see Remark 6.1(2)). Moreover, as $\alpha$ is exact, $D$ is the Levi-Civita connection of some Riemannian metric in the conformal class of $N^3$.

From now on we shall suppose that $V(\log V(\lambda^{-2})) = a$ for some constant $a \in \mathbb{R}$. If $a = 0$ this implies that $V(\lambda^{-2})$ is basic. We can write $V = \partial/\partial \rho$ for some function $\rho$ which is zero on a chosen section of $\varphi$. Hence $\lambda^{-2} = b \rho + c$ for some basic functions $b$ and $c$. After suitable conformal changes on $N^3$ and $M^4$ we have $\lambda^{-2} = \rho + c/b$, i.e., $V(\lambda^{-2}) = 1$. By (6.6), this implies that $\alpha = 0$ and hence $D$ is the Levi-Civita connection of $N^3$. Moreover, we can locally write $\theta = d(\lambda^{-2}) + \varphi^*(A)$ for some one-form $A$ on $N^3$. Hence $d\mathcal{K}(\lambda^{-2}) = -\varphi^*(A)$ and $\Omega = d\theta = \varphi^*(dA)$ which together with (5.15) gives $-A = \ast dA$. Thus we have proved that, if $a = 0$ then, up to a conformal change with basic factor, $\varphi$ is of type 3.

It remains to consider the case when $V(\log V(\lambda^{-2})) = a$ for some nonzero constant $a$. Then, if we set $\rho = \log V(\lambda^{-2})$, we can assume that $V = a \partial/\partial \rho$. As $V(\lambda^{-2}) = e^\rho$ we have that $a\lambda^{-2} = e^\rho + c$ for some basic function $c$. Hence

\begin{equation}
\lambda^{-2} = \frac{1}{a}(e^\rho + c).
\end{equation}

From $d\mathcal{K} \rho = a \theta$ and (6.6) we get $d\rho = a \theta + \alpha$; equivalently

\begin{equation}
\theta = \frac{1}{a}(d\rho - \alpha).
\end{equation}
Now, we can write

\[ a(d^\mathcal{H} - \alpha)(\lambda^{-2}) = a(d^\mathcal{H} - d^\mathcal{H}_\rho)(\frac{1}{a}(e^\rho + c)) \]

(6.9)

\[ = e^\rho d^\mathcal{H}_\rho + d^\mathcal{H}_c - e^\rho d^\mathcal{H}_\rho - c d^\mathcal{H}_\rho \]

\[ = d^\mathcal{H}_c - c d^\mathcal{H}_\rho = dc - c d^\mathcal{H}_\rho = dc - c\alpha. \]

On the other hand, from (6.8) it follows that \( a \Omega = a d \theta = -da \) which, together with (5.15) and (6.9), gives \( dc - c\alpha = -\ast d\alpha \). Thus we have proved that, if \( \phi \) is not of type 1, 2 or 3 then

(6.10)

\[ da - c \ast \alpha + \ast dc = 0 \]

and, locally, the metric of \( M^4 \) can be written in the form

(6.11)

\[ g = \frac{1}{a} \{ (e^\rho + c) h + \frac{1}{e^\rho + c} (d\rho - \alpha)^2 \} \]

where, obviously, we can assume that \( a = \pm 1 \) and that \( c \) is nonzero; since, if \( c = 0 \), then after a suitable conformal change of basic factor, \( \phi \) is of type 2.

The following result is an immediate consequence of Theorem 5.3 and Theorem 6.3.

**Corollary 6.4.** Let \( M^4 \) be a self-dual manifold with a real-analytic metric and let \( \phi: M^4 \to (N^3, D) \) be a \((-)\)twistorial harmonic morphism.

Then \( D \) is Einstein–Weyl and, up to conformal changes of the metrics, \( \phi \) is of type 1, 2, 3 or 4; further, the conformal factor on \( M^4 \) can be taken to be constant along the fibres of \( \phi \). If \( \phi \) is of type 2, then both \( M^4 \) and \( N^3 \) are conformally flat, whilst if \( \phi \) is of type 3, then \( N^3 \) has constant curvature.

7. Constructions of self-dual metrics

In this section we show that harmonic morphisms are related to known constructions of self-dual metrics.

Firstly, we recall the following construction of P.E. Jones and K.P. Tod [20] (see [26], cf. [15, 17]).
Theorem 7.1. Let \((N^3, [h], D)\) be an Einstein–Weyl 3-manifold and let \((M^4, N^3, S^1)\) be a (local) principal bundle endowed with a (local) principal connection \(\mathcal{H} \subseteq TM\). Define a Riemannian metric \(g\) on \(M^4\) by
\[
g = v \varphi^* (h) + v^{-1} \theta^2
\]
where \(\varphi : M^4 \to N^3\) is the projection of the principal bundle \((M^4, N^3, S^1)\), \(v = u \circ \varphi\) for some positive smooth function \(u\) on \(N^3\) and \(\theta\) is the connection form of \(\mathcal{H}\).

Then \((M^4, g)\) is self-dual (respectively, anti-self-dual) if the following \(S^1\)-monopole equation holds on \(N^3\):
\[
(\text{d} - \alpha) u = \ast F \quad \text{(respectively, } (\text{d} - \alpha) u = - \ast F )
\]
where \(\alpha\) is the Lee form of \(D\) with respect to \(h\) and \(F \in \Gamma(\Lambda^2(T^* N))\) is the curvature form of \(\mathcal{H}\).

Remark 7.2. The construction of Theorem 7.1 clearly gives harmonic morphisms of type 1 and, if \(\mathcal{H}\) is flat, of type 2, up to a conformal change with basic factor.

Theorem 7.3 (Type 3, cf. [9], [33]). Let \((N^3, h)\) be a constant curvature 3-manifold and let \(A\) be a one-form on \(N^3\). Define a Riemannian metric on \((0, \infty) \times N^3\) by
\[
g = \rho h + \rho^{-1} (\text{d} \rho + A)^2 \quad (\rho \in (0, \infty))
\]
Then \(g\) is self-dual (respectively, anti-self-dual) if the following Beltrami fields equation holds on \(N^3\):
\[
\text{d} A = - \ast A \quad \text{(respectively, } \text{d} A = \ast A )
\]

Theorem 7.4 (Type 4, [9]). Let \((N^3, [h], D)\) be an Einstein–Weyl 3-manifold. Define a Riemannian metric on \((0, \infty) \times N^3\) by
\[
g = (e^\rho \pm 1) h + \frac{1}{e^\rho \pm 1} (\text{d} \rho - \alpha)^2 \quad (\rho \in (0, \infty))
\]
where \(\alpha\) is the Lee form of \(D\) with respect to \(h\).

Then \(g\) is self-dual (respectively, anti-self-dual) if the following Beltrami fields equation holds on \(N^3\)
\[
\text{d} \alpha = \pm \ast \alpha \quad \text{(respectively, } \text{d} \alpha = \mp \ast \alpha )
\]
Remark 7.5. 1) The construction of Theorem 7.1 gives Einstein metrics if \( \alpha = 0 \) and \( h \) is flat, in which case \( g \) is Ricci-flat self-dual \([15, 17]\) (see \([33]\)).

2) The construction of Theorem 7.3 gives Einstein metrics if and only if \( (N^3, h) \) has constant sectional curvature equal to 1/4, in which case \( g \) is Ricci-flat self-dual. The construction of Theorem 7.4 gives Einstein metrics if and only if \( \alpha = 0 \) and so \( h \) has constant curvature, in which case \( g \) also has constant curvature. These facts follow from \([33\), Theorem 1.5\] and are also true if we make a conformal change of \( g \) with factor constant along the fibres of \( \varphi \).

3) Let \( A \) be a solution of the Beltrami fields equation (7.4) on a three-dimensional Riemannian manifold \( (N^3, h) \) with constant curvature. Then the components of \( A \) with respect to an orthonormal basis of left invariant one-forms are eigenfunctions of the Laplace-Beltrami operator of \( (N^3, h) \) (cf. \([33]\); note that the corresponding eigenvalues are imaginary if \( (N^3, h) \) has negative sectional curvature).

4) Let \( (N^3, [h], D) \) be an Einstein–Weyl space for which there exists a surjective submersion \( \pi : Z(N) \to \mathbb{C}P^1 \) whose fibres are transversal to the twistor lines. Then, it is well known that, for any \( x_1, x_2 \in \mathbb{C}P^1 = S^2 \), the angle formed by any leaf of the foliation corresponding to \( \pi^{-1}(x_1) \) and any leaf of the foliation corresponding to \( \pi^{-1}(x_2) \) is equal to \( \text{dist}_{S^2}(x_1, x_2) \) (see \([14, 10]\)).

For example, if \( D \) is the Levi-Civita connection of a Riemannian manifold \( (N^3, h) \) with constant nonnegative sectional curvature then \( (N^3, [h], D) \) has this property (see \([6]\)). It follows that, if the codomain of a harmonic morphism of type 3 has nonnegative constant sectional curvature then its domain is hyper-Hermitian.

5) D.M.J. Calderbank \([9]\) gives the type 4 construction with the extra condition on \( (N^3, [h], D) \) that it is an Einstein–Weyl space for which there exists a surjective submersion \( \pi : Z(N) \to \mathbb{C}P^1 \) whose fibres are transversal to the twistor lines. Then the construction gives hyper-Hermitian metrics which, after a suitable conformal change, are Einstein with nonzero scalar curvature.

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