ARTIN’S BRAIDS, BRAIDS FOR THREE SPACE, AND GROUPS

$\Gamma_n^4$ AND $G_n^k$

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Abstract. We construct a group $\Gamma_n^4$ corresponding to the motion of points in $\mathbb{R}^3$ from the point of view of Delaunay triangulations. We study homomorphisms from pure braids on $n$ strands to the product of copies of $\Gamma_n^4$. We will also study the group of pure braids in $\mathbb{R}^3$, which is described by a fundamental group of the restricted configuration space of $\mathbb{R}^3$, and define the group homomorphism from the group of pure braids in $\mathbb{R}^3$ to $\Gamma_n^4$. In the end of this paper we give some comments about relations between the restricted configuration space of $\mathbb{R}^3$ and triangulations of the 3-dimensional ball and Pachner moves.

1. Introduction

In [6] the second named author defined a family of groups $G_n^k$ for two positive integers $n > k$, and formulated the following principle:

If dynamical systems describing a motion of $n$ particles, admit some good codimension one property governed by exactly $k$ particles, then these dynamical system have a topological invariant valued in $G_n^k$.

The main examples coming from the theory are homeomorphisms from the $n$-strand pure braid group to the groups $G_n^3$ and $G_n^4$ [5]. If we consider a motion of $n$ pairwise distinct points on the plane and choose the property “some three points are collinear”, then we get a homomorphism from the pure $n$-strand braid group $PB_n$ to the group $G_n^3$. If we choose the property “some four points belong to the same circle or line”, we shall get a group homomorphism from $PB_n$ to $G_n^4$.

In other words, in our examples we look for “walls” in the configuration space $C_n(\mathbb{R}^2)$ where some three points are collinear (or some four points are on the same circle or line). This condition can be well defined for $C_n(\mathbb{R}P^2)$.

But what is the “good” codimension one property if we try to study similar configuration spaces or braids for some other topological spaces? First, our conditions will heavily depend on the metrics: the property “three points are collinear” is metrical. On the other hand, even having some good metrics chosen, we meet other obstacles because we need to know what is a “line”. For example, there is no unique geodesics passing through two points in the general case. And when finding all possible geodesics and trying to write a word corresponding to it, we shall see that “a word will contain infinitely letters” in the case of irrational cable.

The detour for this problem will be as follows: we shall consider the condition “locally” and instead of “global configurations of spaces”. We shall consider only Voronoi tiling or Delaunay triangulations. From this point of view, we deal with the

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space of triangulations with a fixed number of triangles, where any two adjacent
triangulations are related by a Pachner move [3], which is closely related to the
group $\Gamma^4_n$ (see ahead).

The triangulation of spaces and Pachner moves are also related to Yang-Baxter
maps (see [2]). Moreover, in [13] a boundary-parabolic $PSL(2, \mathbb{C})$-representation
of $\pi_1(S^3 \setminus K)$ for a hyperbolic knot $K$ is studied by using Cluster algebras and flips –
Pachner move for 2-dimensional triangulations. In other words, it can be expected to
study not only braids, but also knots, which are obtained by closing braids.

Now we consider restricted spaces $C_n^ρ(\mathbb{R}^{k-1})$ defined as follows: a point in
$C_n^ρ(\mathbb{R}^{k-1})$ is a set of $n$ distinct points in $\mathbb{R}^{k-1}$, where every $(k-1)$ points are
in general position. Especially, for $k=3$, the only condition is that no two points
among the given $n$ points coincide and the fundamental group $\pi_1(C_n^ρ(\mathbb{R}^{3-1}))$ is pre-
cisely the Artin pure braid group. For $k=4$, for points $x_1, \ldots, x_n$ in three-space
we require that no three points are collinear (though some four points can belong
to the same plane). We call elements in $\pi_1(C_n^ρ(\mathbb{R}^{3-1}))$ braids on $n$ strand for $\mathbb{R}^3$.
We call elements in $\pi_1(C_n^ρ(\mathbb{R}^{P^{4-1}}))$ braids on $n$ strand for $\mathbb{R}^P$. In [7] the following
statement is proved:

**Proposition 1.1.** There exists the group homomorphism $f_n^k$ from $\pi_1(C_n^ρ(\mathbb{R}^{P^{k-1}}))$
to $G_n^k$.

Roughly speaking, for a path $\gamma \in \pi_1(C_n^ρ(\mathbb{R}^{P^{k-1}}))$ the mapping $f_n^k(\gamma)$ is defined
by writing $a_m$ when exactly one $k$-tuple of points belongs to a $(k-2)$-plane, where
$m$ is the set of indices for $k$ points on the $(k-2)$-plane.

The paper is organized as follows. In Section 2, we introduce basic definitions
and draw pictures describing the motion of points in $\mathbb{R}^2$ or in $\mathbb{R}^3$. In Section 3, we
define the homomorphism from $PB_n$ to $\Gamma^4_n$. In Section 4 we shall define a
homomorphism $\psi_n$ from $PB_n$ to $\Gamma^4_n \times \Gamma^4_n$, which is defined as the homomorphism
from $PB_n$ to $G^4_n$ in [3], but separating four points on the circle with respect to
the number of points inside of the circle modulo 2. In Section 5 we will define a
homomorphism $\pi_n$ from $G^4_n$ to $\Gamma^4_n \times \Gamma^4_n$ and show that the homomorphism $\psi_n$ can be
presented by the composition of the homomorphism from $PB_n$ to $G^4_n$ in [3] and
the homomorphism $\pi_n$.

2. Pictures and basic definitions

Following [3], we choose the presentation for the pure Artin braid group.

**Definition 2.1.** The pure braid group $PB_n$ of $n$ strands is the group given by
the presentation generated by $\{b_{ij}| i,j \in \{1,\cdots,n\}, i < j \}$ subject to the following
relations:

1. $b_{ij}b_{kl} = b_{kl}b_{ij}$ for $i,j,k,l \in \{1,\cdots,n\}$ such that $i < j < k < l$ or $i < k < l < j$;
2. $b_{ij}b_{ik}b_{ij} = b_{ik}b_{ij}b_{ij} = b_{ik}b_{ij}b_{ik}$ for $i,j,k \in \{1,\cdots,n\}$ such that $i < j < k$;
3. $b_{ik}b_{jk}b_{jl}b_{ik} = b_{jk}b_{ij}b_{ik}b_{jk}$ for $i,j,k,l \in \{1,\cdots,n\}$ such that $i < j < k < l$.

**Definition 2.2.** The group $G^4_n$ is the group given by group presentation generated
by $\{a_{ijkl}| \{(i,j,k,l) \subset \bar{n}, |\{(i,j,k,l)\}| = 4\}$ subject to the following relations:

1. $a_{ijkl}^2 = 1$ for $\{(i,j,k,l) \subset \bar{n}\}, |\{(i,j,k,l)\}| = 4$;
2. $a_{ijkl}a_{stuv} = a_{stuv}a_{ijkl}$ for $\{(i,j,k,l) \cap \{s,t,u,v\} \} < 3$;
3. $(a_{ijkl}a_{ijkl}a_{ijkl}a_{ijkl})^2 = 1$ for distinct $i,j,k,l$. 


We denote $a_{ijkl} := a_{\{ijkl\}}$.

Now we will recall the group homomorphism from $PB_n$ to $G^4_n$, which is defined in [8]. Pure braids can be considered as dynamical systems whose initial and final states coincide.

![Dynamical system corresponding to $b_{ij}$](image)

**Figure 1.** Dynamical system corresponding to $b_{ij}$

Let $\Gamma = \{(t, t^2) | t \in \mathbb{R}\} \subset \mathbb{R}^2$ be the graph of the function $y = x^2$. Consider a rapidly increasing sequence of positive numbers $t_1, t_2, \cdots, t_n$ and denote the point $(t_i, t_i^2) \in \Gamma$ by $P_i$.

We assume that the initial state is the configuration $\mathcal{P} = \{P_1, \cdots, P_n\}$ on the plane as described in Fig. 2. Notice that no four points of $\{P_1, \cdots, P_n\}$ are placed on the same circle, see [8] for details.

For any $i < j$ the pure braid $b_{ij}$ can be presented as the following dynamical system: the point $i$ moves along the graph $\Gamma$

1. the point $P_i(t)$ moves along the graphics $\Gamma$ and passes points $P_{i+1}(t), \cdots, P_j(t)$ from above, see the upper left of Fig. 3.
2. the point $P_j(t)$ moves from above the point $P_i(t)$, see the upper right of Fig. 3.
3. the point $P_i(t)$ moves to its initial position from above the points $P_{j-1}(t), \cdots, P_{i+1}(t)$, see the under left of Fig. 3.
Figure 2. Initial state \( \{P_1, P_2, \ldots, P_n\} \) such that \( P_i = (t_i, t_i^2) \), where \( \{t_i\}_{i=1}^n \) is a strictly increasing sequence.

(4) the points \( P_j(t) \) returns to the initial position, see the under right of Fig. 3.

Figure 3. Model of moving points, corresponding to \( b_{ij} \).
Let \( b_{ij} \in PB_n, 1 \leq i < j \leq n \) be a generator. Consider the elements

\[
\begin{align*}
\epsilon_{ij}^I &= \prod_{p=2}^{j-1} \prod_{q=1}^{p-1} a_{ijpq}, \\
\epsilon_{ij}^{II} &= \prod_{p=1}^{j-1} \prod_{q=1}^{n-j} a_{ij(p)j(p)+p}, \\
\epsilon_{ij}^{III} &= \prod_{p=1}^{n-j+1} \prod_{q=0}^{n-p+1} a_{ij(n-p)(n-q)}, \\
\epsilon_{ij} &= \epsilon_{ij}^{II} \epsilon_{ij}^{I} \epsilon_{ij}^{III}.
\end{align*}
\]

Notice that the formula (2.4) is obtained by writing every moment, when four points are on the same circle, during the point \( P_i \) turns around \( P_j \) as described in Fig. 4.

Now we define \( \phi_n : PB_n \rightarrow G^4_n \) by

\[
\phi_n(b_{ij}) = c_{i(i+1)} \cdots c_{i(j-1)} c_{ij} c_{ij}^{-1} c_{i(i-1)} \cdots c_{i(i+1)},
\]

for \( 1 \leq i < j \leq n \).

**Proposition 2.3 (8).** The mapping \( \phi_n \) is well-defined.

The above proposition can be proved by the basic principle, which is introduced in [6].

**Definition 2.4.** The group \( \Gamma^4_n \) is the group given by group presentation generated by \( \{ d_{ijkl} \mid \{ i, j, k, l \} \subset \bar{n}, |\{ i, j, k \}| = 4 \} \) subject to the following relations:

\[
\begin{align*}
\text{(1) } d_{ijkl}^2 &= 1 \text{ for } (i, j, k, l) \subset \bar{n}, \\
\text{(2) } d_{ijkl}d_{stuv} &= d_{stuv}d_{ijkl}, \text{ for } |\{ i, j, k, l \} \cap \{ s, t, u, v \}| < 3, \\
\text{(3) } d_{ijkl}d_{ijkl}d_{ijkl}d_{ijkl} &= 1 \text{ for distinct } i, j, k, l, m.
\end{align*}
\]
\[ d_{ijkl} = d_{kjl} = d_{ilkj} = d_{ijkl} = d_{ijkl} = d_{lji} = d_{ijkl}, \text{ for distinct } i, j, k, l, m. \]

The group \( \Gamma^4_n \) is naturally related to a triangulations of 2-surfaces and the Pachner moves for the two dimensional case, called “flip”. More precisely, a generator \( d_{ijkl} \) of \( \Gamma^4_n \) corresponds to the sequence of flips constituting the Pentagon relation, the most important relation for the group \( \Gamma^4_n \), see Fig. 5.

![Figure 5. Flip on a rectangle \( \square ijkl \)](image)

Especially, the relation \( d_{ijkl} d_{ijkm} d_{ijlm} d_{iklm} d_{ijkl} = 1 \) corresponds to the flips, applied on a pentagon as described in Fig. 6.

![Figure 6. Flips on a hexagon \( ijklm \)](image)

3. A GROUP HOMOMORPHISM FROM PB\(_n\) TO \( \Gamma^4_n \)

In this section we construct the group homomorphism \( \psi_n \) from \( PB_n \) to \( \Gamma^4_n \). The topological background for that is very easy: we consider codimension 1 “walls” which correspond to generators (flips) and codimension 2 relations (of the group)

\footnote{The method presented here works for arbitrary 2-surfaces as well but we restrict ourselves to the case of the plane}
Having this, we construct a map on the level of generators and prove its correctness algebraically.

**Geometric description of the mapping from PBₙ to Γₙ⁴.**

Let us consider $b_{ij}$ as the dynamical system, described in Section 2. The homomorphism $f_n$ from PBₙ to Γₙ⁴ can be defined as follows: for the above dynamical system for each generator $b_{ij}$, let us enumerate $0 < t_1 < t_2 < \cdots < t_l < 1$ such that at the moment $t_k$ four points belong to the one circle. At the moment $t_k$, if $P_s, P_t, P_u, P_v$ are positioned on the one circle as the indicated order, then $d_k = d_{(stuv)}$. With the pure braid $b_{ij}$ we associate the product $f_n(b_{ij}) = d_1d_2\cdots d_l$.

**Algebraic description of the mapping from PBₙ to Γₙ⁴.**

On the other hand, the mapping $\phi_n : PBₙ \rightarrow \Gammaₙ⁴$ can be formulated as follows:

Let us denote

$$d_{\{p,q,(r,s)\}_{j}} = \begin{cases} 
  d_{(pqrs)} & \text{if } p < q < s, \\
  d_{(prsq)} & \text{if } p < s < q, \\
  d_{(rsqp)} & \text{if } s < p < q, \\
  d_{(qprs)} & \text{if } q < p < s, \\
  d_{(qrsp)} & \text{if } q < s < p, \\
  d_{(rsqp)} & \text{if } s < q < p.
\end{cases}$$

**Remark 3.1.** Notice that the generator $d_{\{p,q,(r,s)\}_{j}}$ corresponds to four points $P_p, P_q, P_r, P_s$ such that they are placed on a circle according to the order of $p, q, s$ and the point $P_r$ is placed close to $P_s$ for the orientation $P_r$ to $P_s$ to be the counterclockwise orientation, see Fig. 7. We would like to highlight that $d_{\{p,q,(r,s)\}_{j}} \neq d_{\{p,q,(s,r)\}_{j}}$.

**Figure 7.** For $p < s < q$, $d_{\{p,q,(r,s)\}_{j}} = d_{(prsq)}$, but $d_{\{p,q,(s,r)\}_{j}} = d_{(psrq)}$.

Let $b_{ij} \in PBₙ$, $1 \leq i < j \leq n$, be a generator. Consider the elements

$$\gamma^I_{i,(i,j)} = \prod_{p=2}^{j-1} \prod_{q=1}^{p-1} d_{\{p,q,(i,j)\}},$$

$$\gamma^{II}_{i,(i,j)} = \prod_{p=1}^{j-1} \prod_{q=1}^{n-j} d_{\{(j-p),(j+p),(i,j)\}},$$

$$\gamma^{III}_{i,(i,j)} = \prod_{p=1}^{n-j} \prod_{q=0}^{n-p+1} d_{\{(n-p),(n-q),(i,j)\}},$$

$$\gamma_{i,(i,j)} = \gamma^{II}_{i,(i,j)} \gamma^I_{i,(i,j)} \gamma^{III}_{i,(i,j)}.$$
Now we define \( f_n : \text{PB}_n \to \Gamma_4^n \) by
\[
f_n(b_{ij}) = \gamma_{i,(i,(i+1))}\gamma_{i,(i,(j-1))}\gamma_{i,(i,(j))}\gamma_{i,((j-1),i)}^{-1}\gamma_{i,((i+1),i)}^{-1},
\]
for \( 1 \leq i < j \leq n \).

**Theorem 3.2.** The map \( \psi_n : \text{PB}_n \to \Gamma_4^n \), which is defined as above, is a well defined homomorphism.

**Proof.** When we consider isotopy between two pure braids, it suffices to take into account only singularities of codimension at most two. Singularities of codimension one give rise to generators, and relations come from singularities of codimension two. Now we list cases of singularities of codimension two explicitly.

1. One point moving on the plane is tangent to the circle, which passes through three points, see Fig. 8. This corresponds to the relation \( d_{ijkl}^2 = 1 \).

2. There are two sets \( A \) and \( B \) of four points, which are on the same circles such that \( |A \cap B| \leq 2 \), see Fig. 9. This corresponds to the relation \( d_{ijkl}d_{stuv} = d_{stuv}d_{ijkl} \).

3. There are five points \( \{P_i, P_j, P_k, P_l, P_m\} \) on the same circle. We obtain the sequence of five subsets of \( \{P_i, P_j, P_k, P_l, P_m\} \) with four points on the same circle, which corresponds to the flips on the pentagon, see Fig. 10. This corresponds to the relation \( d_{ijkl}d_{ijkl}d_{ijkl}d_{ijkl} = 1 \).

□

4. A GROUP HOMOMORPHISM FROM \( \text{PB}_n \) TO \( \Gamma_4^n \times \Gamma_4^n \)

In this section we construct the group homomorphism \( f_n^2 \) from \( \text{PB}_n \) to \( \Gamma_4^n \times \Gamma_4^n \).

Roughly speaking, \( f_n^2 \) will be defined by reading generators of \( \Gamma_4^n \times \Gamma_4^n \), which correspond to four points on the same circle, but we distinguish them with respect to the number of points inside of the circle.

**Geometric description of the mapping from \( \text{PB}_n \) to \( \Gamma_4^n \times \Gamma_4^n \).**

Let us consider the dynamical system for a generator \( b_{ij} \), which is described in Section 2. Assume that an orientation on the plane is given. Let us enumerate
Figure 9. Two sets $A$ and $B$ of four points on the circles such that $|A \cap B| = 2$

Figure 10. A sequence of five subsets of $\{P_1, P_j, P_k, P_l, P_m\}$, corresponding to flips on the pentagon

0 < $t_1 < t_2 < \cdots < t_l < 1$ such that at the moment $t_k$ four points are positioned on the one circle (or on the line). Notice that by the assumption of the dynamical system for a generator $b_{ij}$, there are no four points on the circle at the initial. Let us assume that at the moment $t_k$, if $P_s, P_t, P_u, P_v$ are positioned on the one circle as the indicated order. If there are even number of points inside of the circle, on which four points $P_s, P_t, P_u, P_v$ are placed, then $t_k$ corresponds to $\delta_k = (d_{stuv}, 1) \in \Gamma_n^4 \times \Gamma_n^4$. Otherwise, $t_k$ corresponds to $\delta_k = (1, d_{stuv}) \in \Gamma_n^4 \times \Gamma_n^4$. With the pure braid $b_{ij}$ we associate the product $f_n^2(b_{ij}) = \delta_1 \delta_2 \cdots \delta_l$.

Remark 4.1. From the construction of base points $\{P_1, \cdots, P_n\}$ it follows that the circle passing $P_j, P_p, P_q$ for $j < p < q$ contains points $\{P_1, \cdots, P_{j-1}\} \cup \{P_{p+1}, \cdots, P_{q-1}\}$, see Fig. 12. That is, inside of the circle passing $P_j, P_p, P_q$ for $j < p < q$ there are $j - 1 + q - p - 1$ points from $\{P_1, \cdots, P_n\}$.

Algebraic description of the mapping from $PB_n$ to $\Gamma_n^4 \times \Gamma_n^4$.

On the other hands, the mapping $f_n^2 : PB_n \rightarrow \Gamma_n^4 \times \Gamma_n^4$ can be formulated as follow: Let us denote $\text{mid}\{p, q, r\}$ if $\text{mid}\{p, q, r\} \in \{p, q, r\}$ and $\text{min}\{p, q, r\} \leq \text{mid}\{p, q, r\} < \text{max}\{p, q, r\}$. Let us define $\delta 2_{\{p, q, (i, j), l\}}$ as follow:
If $\min p, q, j < i < \text{mid} \{p, q, r\}$ and $i > \max \{p, q, j\}$, then

$$
\delta_{\{p, q, (i, j)\}} = \begin{cases} 
(d_{\{p, q, (i, j)\}}, 1), & \text{if } j + p + q \equiv 0 \mod 2, \\
(1, d_{\{p, q, (i, j)\}}), & \text{if } j + p + q \equiv 1 \mod 2.
\end{cases}
$$

If $i < \min \{p, q, j\}$ or $\text{mid} \{p, q, j\} < i < \max \{p, q, j\}$, then

$$
\delta_{\{p, q, (i, j)\}} = \begin{cases} 
(d_{\{p, q, (i, j)\}}, 1), & \text{if } j + p + q \equiv 1 \mod 2, \\
(1, d_{\{p, q, (i, j)\}}), & \text{if } j + p + q \equiv 0 \mod 2.
\end{cases}
$$

Let $b_{ij} \in PB_n$, $1 \leq i < j \leq n$ be a generator. Consider the elements
such that at the moment $t_k$ there are four points inside
the circle. If the number of points inside of the circle is $\alpha$, then just count this number of points. For the above dynamical system let us enumerate $0 < t_1 < \cdots < t_l < 1$ such that at the moment $t_k$ four points $P_s, P_l, P_u, P_v$ are positioned on the one circle. If the number of points inside of the circle is $\alpha \mod r$, on which four points $P_s, P_l, P_u, P_v$ are placed, then $t_k$ corresponds to $\delta_k = (1, \cdots , 1, d_{staw}, 1, \cdots , 1) \in \{1\} \times \cdots \{1\} \times \Gamma_{n,\alpha} \times \{1\} \times \cdots \{1\} \subset \Gamma_{n,1} \times \Gamma_{n,2} \times \cdots \times \Gamma_{n,r}$. With the pure braid $b_{ij}$ we associate the product $f_n^2(b_{ij}) = \delta_1 \delta_2 \cdots \delta_l$

Proof. This statement follows from Theorem 5.1.

5. A GROUP HOMOMORPHISM FROM $PB_n$ TO $\Gamma_n^4 \times \cdots \times \Gamma_n^4$

The homomorphism $f_n^2 : PB_n \to \Gamma_n^4 \times \cdots \times \Gamma_n^4$ can be extended to a mapping from $PB_n$ to the product of $\Gamma_n^4$ of $r$-copies for $r > 2$ as follow. We denote the product of $\Gamma_n^4$ of $r$-copies for $r > 2$ by

$$\Gamma_{n,1} \times \Gamma_{n,2} \times \cdots \times \Gamma_{n,r}.$$ 

The idea is that we can not only distinguish between “evenly many points inside the circle” or “oddly many points inside the circle”, but also just count this number of points. For the above dynamical system let us enumerate $0 < t_1 < \cdots < t_l < 1$ such that at the moment $t_k$ four points $P_s, P_l, P_u, P_v$ are positioned on the one circle. If the number of points inside of the circle is $\alpha \mod r$, on which four points $P_s, P_l, P_u, P_v$ are placed, then $t_k$ corresponds to $\delta_k = (1, \cdots , 1, d_{staw}, 1, \cdots , 1) \in \{1\} \times \cdots \{1\} \times \Gamma_{n,\alpha} \times \{1\} \times \cdots \{1\} \subset \Gamma_{n,1} \times \Gamma_{n,2} \times \cdots \times \Gamma_{n,r}$. With the pure braid $b_{ij}$ we associate the product $f_n^2(b_{ij}) = \delta_1 \delta_2 \cdots \delta_l$.

Algebraically this construction can be presented as follows:

$$D_{ij}^I = \prod_{p=2}^{j-1} \prod_{q=1}^{p-1} \delta_{\{p,q,(i,j)\}},$$

$$D_{ij}^{II} = \prod_{p=1}^{j-1} \prod_{q=1}^{n-j} \delta_{\{j-p,(j+p),(i,j)\}},$$

$$D_{ij}^{III} = \prod_{p=1}^{n-j+1} \prod_{q=0}^{n-p+1} \delta_{\{(n-p),(n-q),(i,j)\}}.$$
where
\[
\delta^r_{\{p,q,(i,j)\}} = \{1, \ldots, d_{\{p,q,(i,j)\}}, 1, \ldots, 1\} \in \{1\} \times \cdots \{1\} \times \Gamma^4_{n,\alpha} \times \{1\} \times \cdots \{1\} \\
\subset \Gamma^4_{n,1} \times \Gamma^4_{n,2} \times \cdots \times \Gamma^4_{n,r},
\]
if
\[
\min\{p, q, r\} - \mid\min\{p, q, r\}\mid + \max\{p, q, r\} - 2 =
\begin{cases}
\alpha \mod r, & \text{if } \min\{p, q, r\} < \min\{p, q, r\} \text{ or } \max\{p, q, r\}, \\
\alpha - 1 \mod r, & \text{if } \min\{p, q, r\} < \max\{p, q, r\}.
\end{cases}
\]

Proof. This statement can be proved similarly to the proof of Theorem 3.2. Let us list cases of singularities of codimension two explicitly. Notice that the image of \(f\) from four points on the circle depends on the number of points inside of the circle.

1. One point moving on the plane is tangent to the circle, which passes through three points, see the center in Fig. 8. Notice that the number of points inside of the circle does not change when the point \(P_i\) passes through the circle. It is easy to see that the image, when one point passes through the circle twice (upper left in Fig. 8), is
\[
(1, \ldots, 1, d_{(ijkl)}d_{(ijkl)}, 1, \ldots, 1) \in \Gamma^4_{n,1} \times \cdots \times \Gamma^4_{n,\alpha} \times \cdots \times \Gamma^4_{n,r},
\]
where the \(\alpha\) is the number of points inside of the circle, passing through \(P_i, P_k, P_l\). If the point does not pass through the circle (upper right in Fig. 8), then the image is
\[
(1, \ldots, 1, 1, \ldots, 1) \in \Gamma^4_{n,1} \times \cdots \times \Gamma^4_{n,\alpha} \times \cdots \times \Gamma^4_{n,r}.
\]
The equality of those two images is obtained by the relation \(d^2_{(ijkl)} = 1\).

2. There are two sets \(A = \{P_i, P_j, P_k, P_l\}\) and \(B = \{P_s, P_t, P_u, P_v\}\) of four points, which are on the same circles such that \(|A \cap B| \leq 2\), see Fig. 9. If the number of points inside circles, which passes through points \(A = \{P_i, P_j, P_k, P_l\}\) and \(B = \{P_s, P_t, P_u, P_v\}\) respectively, are same mod \(r\), then the image of \(d_{(ijkl)}\) and \(d_{(stuv)}\) from them is
\[
(1, \ldots, d_{(ijkl)}d_{(stuv)}, 1, \ldots, 1) \in \Gamma^4_{n,1} \times \cdots \times \Gamma^4_{n,\alpha} \times \cdots \times \Gamma^4_{n,r},
\]
or
\[
(1, \ldots, d_{(stuv)}d_{(ijkl)}, 1, \ldots, 1) \in \Gamma^4_{n,1} \times \cdots \times \Gamma^4_{n,\alpha} \times \cdots \times \Gamma^4_{n,r},
\]
The equality of them follows from the relation \(d_{(ijkl)}d_{(stuv)} = d_{(stuv)}d_{(ijkl)}\).

If the number of points inside circles, which passes through points \(A = \{P_i, P_j, P_k, P_l\}\) and \(B = \{P_s, P_t, P_u, P_v\}\) respectively, are different mod \(r\), then the image of \(d_{(ijkl)}\) and \(d_{(stuv)}\) from them is
\[
(1, \ldots, d_{(ijkl)}, \ldots, 1, 1, \ldots, 1, d_{(stuv)}, \ldots, 1) \in \Gamma^4_{n,1} \times \cdots \times \Gamma^4_{n,\alpha} \times \cdots \times \Gamma^4_{n,\beta} \times \cdots \times \Gamma^4_{n,r},
\]
or

\[(1, \cdots, 1, \cdots, d_{(stuv)}, \cdots, 1)(1, \cdots, d_{(ijkl)}, \cdots, 1, \cdots)\]
\[\in \Gamma^4_{n,1} \times \cdots \times \Gamma^4_{n,\alpha} \times \cdots \times \Gamma^4_{n,\beta} \times \cdots \times \Gamma^4_{n,r},\]

where \(\alpha\) and \(\beta\) depend on the numbers of points inside of the circles, which pass through \(\{P_1, P_2, P_k, P_t\}\) and \(\{P_s, P_t, P_u, P_v\}\), respectively. It is easy to obtain the equality of them.

(3) There are five points \(\{P_1, P_2, P_k, P_t, P_m\}\) on the same circle. We obtain the sequence of five subsets of \(\{P_1, P_2, P_k, P_t, P_m\}\) with four points on the same circle, which corresponds to the flips on the pentagon, see Fig. 10. Notice that the number of points inside of circles does not change. In other words, the image of the sequence has the form of

\[(1, \cdots, d_{ijkl}d_{ijkl}d_{ijkl}d_{ijkl}, 1, \cdots, 1) \in \Gamma^4_{n,1} \times \cdots \times \Gamma^4_{n,\alpha} \times \cdots \times \Gamma^4_{n,r},\]

or

\[(1, \cdots, d_{ijkl}d_{ijkl}d_{ijkl}d_{ijkl}d_{ijkl}, 1, \cdots, 1) \in \Gamma^4_{n,1} \times \cdots \times \Gamma^4_{n,\alpha} \times \cdots \times \Gamma^4_{n,r}.\]

The equality of them is obtained from the relation

\[d_{ijkl}d_{ijkl}d_{ijkl}d_{ijkl}d_{ijkl} = 1 = d_{ijkl}d_{ijkl}d_{ijkl}d_{ijkl}d_{ijkl}\]
of \(\Gamma^4_{n}\).

\[\square\]

6. BRAIDS IN \(\mathbb{R}^3\) AND GROUPS \(\Gamma^4_{n}\)

In [7], the second named author introduced the notion of braids for \(\mathbb{R}^3\) and \(\mathbb{R}P^3\). Roughly speaking, a braid for \(\mathbb{R}^3\) (or \(\mathbb{R}P^3\)) is a path in a configuration space \(C^r_{n}(\mathbb{R}^3)\) (or \(C^r_{n}(\mathbb{R}P^3)\)) with some restrictions. If the initial and end points of the path in \(C^r_{n}(\mathbb{R}^3)\) coincide, then the path is called a pure braid for \(\mathbb{R}^3(\mathbb{R}P^3)\). In the present section we will construct a group homomorphism from pure braids on \(n\) strands in \(\mathbb{R}^3\) to \(\gamma^4_{n}\).

Let us recall about the pure braids for \(\mathbb{R}^{k-1}\) for \(k > 3\). Let \(C^r_{n}(\mathbb{R}^{k-1})\) be the subset of \(C^r_{n}(\mathbb{R}^{k-1})\) of all points \(x = (x_1, \cdots, x_n)\) such that no \(k - 1\) points among \(\{x_1, \cdots, x_n\}\) are on the same \(k - 3\)–plane. We say that a point \(x \in C^r_{n}(\mathbb{R}^{k-1})\) is singular, if the set of points \(x = (x_1, \cdots, x_n)\) representing it contains some \(k\) points which are not belong to the same \((k - 2)\)–plane.

We call elements in \(\pi_n(C^r_{n}(\mathbb{R}^{k-1}))\) pure braids on \(n\) strands in \(\mathbb{R}^{k-1}\).

The group homomorphism from \(\pi_n(C^r_{n}(\mathbb{R}^{k-1}))\) to \(\Gamma^4_{n}\) is constructed as follows: Fix two non-singular points \(x, x' \in C^r_{n}(\mathbb{R}^{k-1})\). Let us consider the set of smooth path \(\gamma_{x,x'} : [0, 1] \rightarrow C^r_{n}(\mathbb{R}^{k-1})\). We call \(t \in [0, 1]\) a singular moment of \(\gamma_{x,x'}\), if \(\gamma_{x,x'}(t)\) is a singular point in \(C^r_{n}(\mathbb{R}^{k-1})\). We call a smooth path is stable and good if the following conditions hold:

1. The set of singular moments \(t\) is finite;
2. For each singular moment \(t = t_i\), there is only one subset of \(k\) points belonging to a \((k - 2)\)–plane among \(n\) points \(x_1, \cdots, x_n \in \mathbb{R}^{k-1}\) such that \(\gamma_{x,x'}(t_i) = (x_1, \cdots, x_n)\).
3. A smooth path is stable, if the number of singular moments does not change under a small perturbation.
exists a continuous family \( \{ \gamma_{x,x'}^{s} \} \) of smooth paths with endpoints fixed, such that \( \gamma_{x,x'}^{0} = \gamma \) and \( \gamma_{x,x'}^{1} = \gamma' \). A smooth path with end points \( x = (x_{1}, \cdots, x_{n}) \) and \( x' = (x'_{1}, \cdots, x'_{n}) \) is called a braid (or a pure braid) on \( n \) strands in \( \mathbb{R}^{k-1} \), if \( \{ x_{1}, \cdots, x_{n} \} = \{ x'_{1}, \cdots, x'_{n} \} \) (or \( (x_{1}, \cdots, x_{n}) = (x'_{1}, \cdots, x'_{n}) \)).

For paths \( \gamma_{x,x'} \) and \( \gamma_{x',x''} \) the concatenation operation is well-defined, that is, a smooth path \( \gamma_{x,x''} \) such that \( \gamma_{x,x'}^{s}(t) = \gamma_{x,x'}(2t) \) for \( t \in [0,\frac{1}{2}] \) and \( \gamma_{x',x''}^{s}(t) = \gamma_{x',x''}(2t-1) \) for \( t \in [\frac{1}{2},1] \) and smooth it in the neighborhood of \( t = \frac{1}{2} \) is uniquely obtained from \( \gamma_{x,x'} \) and \( \gamma_{x',x''} \) up to isotopy. It is easy to see that the set of equivalence class of pure braids on \( n \) strands in \( \mathbb{R}^{k-1} \) up to isotopy admits a group structure, moreover it is isomorphic to \( \pi_{1}(C_{n}(\mathbb{R}^{k-1})) \).

**Remark 6.1.** For a given pure braid \( \gamma \) on \( n \) strands in \( \mathbb{R}^{k-1} \) by a small perturbation we can obtain a good and stable pure braid \( \gamma' \) on \( n \) strands such that \( \gamma \) and \( \gamma' \) are isotopic. From now on we just consider good and stable pure braids.

Let \( \gamma \) be a good and stable pure braid on \( n \) strands. Let us enumerate all singular moments \( 0 < t_{1} < \cdots < t_{l} < 1 \) of \( \gamma \). For each \( t_{s} \) by definition of good pure braids on \( n \) strands there are exactly \( k \) points on \( (k-2)-plane \). Let \( m_{s} \) be the set of \( k \) indices for points \( (k-2)-plane \) at the moment \( t_{s} \). We associate \( a_{m_{s}} \) for the moment \( t_{s} \). We define a map \( f : \pi_{1}(C_{n}(\mathbb{R}^{k-1})) \rightarrow G_{n}^{s} \) by \( f(\gamma) = a_{m_{1}} \cdots a_{m_{l}} \).

**Proposition 6.2.** [7] The map \( f : \pi_{1}(C_{n}(\mathbb{R}^{k-1})) \rightarrow G_{n}^{s} \) described above is a group homomorphism.

We shall consider (good and stable) pure braids on \( n \) strands in \( \mathbb{R}^{3} \) and construct group homomorphism from pure braids on \( n \) strands in \( \mathbb{R}^{3} \) to the group \( \Gamma_{4}^{s} \). Each element of \( \Gamma_{4}^{s} \) corresponds to the moment when four points on \( (4-2)-dimensional \) plane in \( \mathbb{R}^{4-1} \), but in the case of \( \Gamma_{4}^{s} \) “the order” of four points on \( (4-2)-dimensional \) plane is very important. This order was ignored when the group homomorphism from pure braids in \( \mathbb{R}^{3} \) to \( G_{n}^{s} \) is constructed. Now we formulate more precisely how the group homomorphism from pure braids on \( n \) strands in \( \mathbb{R}^{3} \) to \( G_{n}^{s} \) is constructed.

Let \( \gamma \) be a good and stable pure braid on \( n \) strands with base point \( x = (x_{1}, \cdots, x_{n}) \). We call \( t \in [0,1] \) a special singular moment of \( \gamma \) if the followings hold:

1. At the moment \( t \) four points \( x_{p}, x_{q}, x_{r}, x_{s} \) are on the same plane \( \Pi_{t} \).
2. All of \( \{ x_{1}, \cdots, x_{n} \} \{ x_{p}, x_{q}, x_{r}, x_{s} \} \) placed in the only one of connected components of \( \mathbb{R}^{3} \setminus \Pi_{t} \).

A normal vector \( v_{t} \) of \( \Pi_{t} \) pointing to the connected component, in which the points \( \{ x_{1}, \cdots, x_{n} \} \{ x_{p}, x_{q}, x_{r}, x_{s} \} \) are placed, is called the pointing vector at the special singular moment \( t \).

**Remark 6.3.**

1. The plane \( \Pi_{t} \) admits a unique orientation with respect to \( v_{t} \).
2. On the plane \( \Pi_{t} \) there uniquely exists the quadrilateral with four vertices \( \{ x_{p}, x_{q}, x_{r}, x_{s} \} \). Naturally, the quadrilateral admits the orientation with respect to \( v_{t} \), see Fig. [7].

Let us enumerate all biased singular moments \( 0 < t_{1} < \cdots < t_{l} < 1 \) of \( \gamma \). For each \( t_{s} \) by definition of good pure braids on \( n \) strands there are exactly four points \( \{ x_{p}, x_{q}, x_{r}, x_{s} \} \) on plane \( \Pi_{t_{s}} \). As indicated in the previous remark the quadrilateral on plane \( \Pi_{t_{s}} \) with four vertices \( \{ x_{p}, x_{q}, x_{r}, x_{s} \} \) admits the orientation with respect
to \( v_t \). If four points \( x_p, x_q, x_r, x_s \) are positioned as indicated order in accordance the orientation with respect to \( v_t \), then we associate the moment \( t_s \) to \( d_{t_s} = d_{(pqrs)} \).

Let us define a map \( g : \pi_1(C'_n(\mathbb{R}^3)) \to \Gamma^4_n \) by \( g(\gamma) = d_{t_1} \cdots d_{t_l} \).

**Theorem 6.4.** The map \( g : \pi_1(C'_n(\mathbb{R}^3)) \to \Gamma^4_n \) is well-defined.

**Proof.** We consider moments of isotopy between two paths, when the path at some moment in the isotopy between two paths is not good or not stable. Let us list such cases explicitly.

1. There are four points on a , which disappears after a small perturbation, see Fig. 14. This corresponds to the relation \( d^2_{(pqrs)} = 1 \).

2. At a moment there are two sets of four points \( m \) and \( m' \) with \( |m \cap m'| < 3 \), which are placed on planes at the same moment, see Fig. 15. This corresponds to the relation \( d_{(ijkl)}d_{(stuv)} = d_{(stuv)}d_{(ijkl)} \).
Case 2: Two sets of four points \( \{x_i, x_j, x_k, x_l\} \) and \( \{x_l, x_k, x_u, x_v\} \) on planes at the same moment.

(3) At a moment five points on a plane. This is similar to the case of “five points on the circle” in the proof of Theorem 3.2. This corresponds to the relation
\[
d(ijkl) d(ijkm) d(ijlm) d(iklm) d(jklm) = 1
\]
of \( \Gamma_4^n \).

Let \( \{P_1(t), \cdots, P_n(t)\}_{t \in [0,1]} \) be \( n \) moving points in \( \mathbb{R}^3 \), corresponding to the path in \( \pi_1(C_n^3(\mathbb{R}^3)) \). We may assume that the points \( \{P_1(t), \cdots, P_n(t)\}_{t \in [0,1]} \) move inside of a sphere with sufficiently large diameter. Let us fix four points \( \{A, B, C, D\} \) on the sphere. A triangulation of 3-ball with vertices \( \{P_1(t), \cdots, P_n(t)\} \cup \{A, B, C, D\} \) can be obtained for each \( t \in [0,1] \), see Fig. 16.

On the other hand, as described in Fig. 17, the moving of a vertex of the triangulation can be described by applying the Pachner moves to the triangulation of a
Figure 17. Pachner move and moving of a vertex of the triangulation of 3-dimensional space

3 dimensional space. In other words, a path \( \{ P_0(t), \cdots, P_n(t) \} \) in \( \pi_1(C'_n(\mathbb{R}^3)) \) can be described by a finite sequence of “Pachner moves” applied to the triangulations of the sphere, see Fig. 18.

Figure 18. Applying a Pachner move to the triangulation of a 3 dimensional space and the moving of a vertex

It can be expected that by using the triangulation of a sphere with \( n + 4 \) points and the sequence of the Pachner moves, we obtain an invariant for (pure) braids.

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