Estimates for entropy numbers of embedding operators of function spaces on sets with tree-like structure: some limiting cases

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1 Introduction

In [37] order estimates for entropy numbers of the embedding operator of a weighted Sobolev space on a John domain into a weighted Lebesgue space were obtained, as well as estimates for entropy numbers of a two-weighted summation operator on a tree. Here we consider some critical cases.

Recall the definition of entropy numbers (see, e.g., [6, 9, 32]).

Definition 1. Let $X, Y$ be normed spaces, and let $T : X \rightarrow Y$ be a linear continuous operator. Entropy numbers of $T$ are defined by

$$e_k(T) = \inf \left\{ \varepsilon > 0 : \exists y_1, \ldots, y_{2^k-1} \in Y : T(B_X) \subset \bigcup_{i=1}^{2^k-1} (y_i + \varepsilon B_Y) \right\}, \quad k \in \mathbb{N}.$$  

Kolmogorov, Tikhomirov, Birman and Solomyak [3, 16, 34] studied properties of $\varepsilon$-entropy (this magnitude is related to entropy numbers of embedding operators).

Denote by $l^m_p$ ($1 \leq p \leq \infty$) the space $\mathbb{R}^m$ with norm

$$\| (x_1, \ldots, x_m) \|_{l^m_p} = \begin{cases} 
\left( |x_1|^p + \cdots + |x_m|^p \right)^{1/p} & \text{for } p < \infty, \\
\max\{|x_1|, \ldots, |x_m|\} & \text{for } p = \infty.
\end{cases}$$

Estimates for entropy numbers of the embedding operator of $l^m_p$ into $l^m_q$ were obtained in the paper of Schütz [33] (see also [9]). Later Edmunds and Netrusov [7], [8] generalized this result for vector-valued sequence spaces (in particular, for sequence spaces with mixed norm).

Haroske, Triebel, Kühn, Leopold, Sickel and Skrzypczak [10, 15, 17] studied the problem of estimating entropy numbers of embeddings of weighted sequence spaces or weighted Besov and Triebel–Lizorkin spaces.

Triebel [35] and Mieth [31] studied the problem of estimating entropy numbers of embedding operators of weighted Sobolev spaces on a ball with weights that have singularity at the origin.

Lifshits and Linde [28] obtained estimates for entropy numbers of two-weighted Hardy-type operators on a semiaxis (under some conditions on weights). The similar
problem for one-weighted Riemann-Liouville operators was considered in the paper of Lomakina and Stepanov [29]. In addition, Lifshits and Linde [25–27] studied the problem of estimating entropy numbers of two-weighted summation operators on a tree.

This paper is organized as follows. In §2 we obtain the general result about upper estimates for entropy numbers of embedding operators of function spaces on a set with tree-like structure. The properties of such spaces are almost the same as properties of function spaces defined in [30,31] (see Assumptions 1, 2, 3), but there are some differences. In particular, here we suppose that (6) holds; this condition cannot be directly derived from known results for weighted Sobolev and Lebesgue spaces. Therefore, first we consider some particular cases of function spaces satisfying Assumptions A, B, D (see §3) and we prove that these spaces satisfy Assumptions 1–3. In §4 we obtain order estimates for entropy numbers of embedding operators of weighted Sobolev spaces; to this end, we prove that under given conditions on weights Assumptions A, B, D hold. In §5 we obtain order estimates for entropy numbers of two-weighted summation operators on a tree in critical cases.

2 Upper estimates for entropy numbers of embedding operators of function spaces on sets with tree-like structure

Let us give some notations.

Let \((\Omega, \Sigma, \mes)\) be a measure space. We say that sets \(A, B \subset \Omega\) are disjoint if \(\mes(A \cap B) = 0\). Let \(E, E_1, \ldots, E_m \subset \Omega\) be measurable sets, and let \(m \in \mathbb{N} \cup \{\infty\}\). We say that \(\{E_i\}_{i=1}^m\) is a partition of \(E\) if the sets \(E_i\) are pairwise disjoint and \(\mes((\bigcup_{i=1}^m E_i) \triangle E) = 0\).

Denote by \(\chi_E(\cdot)\) the indicator function of a set \(E\).

Let \(G\) be a graph containing at most countable number of vertices. We shall denote by \(V(G)\) the vertex set of \(G\). Two vertices are called adjacent if there is an edge between them. Let \(\xi_i \in V(G), 1 \leq i \leq n\). The sequence \((\xi_1, \ldots, \xi_n)\) is called a path if the vertices \(\xi_i\) and \(\xi_{i+1}\) are adjacent for any \(i = 1, \ldots, n - 1\). If all the vertices \(\xi_i\) are distinct, then such a path is called simple.

Let \((T, \xi_0)\) be a tree with a distinguished vertex (or a root) \(\xi_0\). We introduce a partial order on \(V(T)\) as follows: we say that \(\xi' > \xi\) if there exists a simple path \((\xi_0, \xi_1, \ldots, \xi_n, \xi')\) such that \(\xi = \xi_k\) for some \(k \in 0, n\). In this case, we set \(\rho_T(\xi, \xi') = \rho_T(\xi', \xi) = n + 1 - k\). In addition, we denote \(\rho_T(\xi, \xi) = 0\). If \(\xi' > \xi\) or \(\xi' = \xi\), then we write \(\xi' \geq \xi\). This partial order on \(T\) induces a partial order on its subtree.

Let \(G\) be a disjoint union of trees \((T_j, \xi_j), 1 \leq j \leq k\). Then the partial order on each tree \(T_j\) induces the partial order on \(G\).
Given \( j \in \mathbb{Z}_+ \), \( \xi \in \mathbf{V}(\mathcal{T}) \), we denote
\[
\mathbf{V}_j(\xi) := \mathbf{V}_j^T(\xi) := \{ \xi' \geq \xi : \rho_T(\xi, \xi') = j \}.
\]
For \( \xi \in \mathbf{V}(\mathcal{T}) \) we denote by \( \mathcal{T}_\xi = (\mathcal{T}_\xi, \xi) \) the subtree in \( \mathcal{T} \) with vertex set
\[
\{ \xi' \in \mathbf{V}(\mathcal{T}) : \xi' \geq \xi \}.
\]

Let \( \mathcal{G} \) be a subgraph in \( \mathcal{T} \). Denote by \( \mathbf{V}_{\text{max}}(\mathcal{G}) \) and \( \mathbf{V}_{\text{min}}(\mathcal{G}) \) the sets of maximal and minimal vertices in \( \mathcal{G} \), respectively.

Let \( \mathbf{W} \subset \mathbf{V}(\mathcal{T}) \). We say that \( \mathcal{G} \subset \mathcal{T} \) is a maximal subgraph on the vertex set \( \mathbf{W} \) if \( \mathbf{V}(\mathcal{G}) = \mathbf{W} \) and any two vertices \( \xi', \xi'' \in \mathbf{W} \) adjacent in \( \mathcal{T} \) are also adjacent in \( \mathcal{G} \). Given subgraphs \( \Gamma_1, \Gamma_2 \subset \mathcal{T} \), we denote by \( \Gamma_1 \cap \Gamma_2 \) the maximal subgraph in \( \mathcal{T} \) on the vertex set \( \mathbf{V}(\Gamma_1) \cap \mathbf{V}(\Gamma_2) \).

Let \( P = \{ \mathcal{T}_j \}_{j \in \mathbb{N}} \) be a family of subtrees in \( \mathcal{T} \) such that \( \mathbf{V}(\mathcal{T}_j) \cap \mathbf{V}(\mathcal{T}_{j'}) = \emptyset \) for \( j \neq j' \) and \( \bigcup_{j \in \mathbb{N}} \mathbf{V}(\mathcal{T}_j) = \mathbf{V}(\mathcal{T}) \). Then \( \{ \mathcal{T}_j \}_{j \in \mathbb{N}} \) is called a partition of the tree \( \mathcal{T} \). Let \( \xi_j \) be the minimal vertex of \( \mathcal{T}_j \). We say that the tree \( \mathcal{T}_s \) succeeds the tree \( \mathcal{T}_j \) (or \( \mathcal{T}_j \) precedes the tree \( \mathcal{T}_s \)) if \( \xi_j < \xi_s \) and
\[
\{ \xi \in \mathcal{T} : \xi_j \leq \xi < \xi_s \} \subset \mathbf{V}(\mathcal{T}_j).
\]

If \( \Gamma \subset \mathcal{T} \) is the maximal subgraph on the vertex set \( \mathbf{W} \), we set \( \mathcal{P}|_{\Gamma} = \{ \Gamma \cap \mathcal{T}_j \}_{j \in \mathbb{N}} \).

We consider the function spaces on sets with tree-like structure from [36, 37].

Let \( (\Omega, \Sigma, \text{mes}) \) be a measure space, let \( \hat{\Theta} \) be a countable partition of \( \Omega \) into measurable subsets, let \( (\mathcal{A}, \xi_0) \) be a tree such that
\[
\exists c_1 \geq 1 : \text{card } \mathbf{V}_1^A(\xi) \leq c_1, \quad \xi \in \mathbf{V}(\mathcal{A}),
\]
and let \( \hat{F} : \mathbf{V}(\mathcal{A}) \to \hat{\Theta} \) be a bijective mapping.

Throughout we consider at most countable partitions into measurable subsets.

Let \( 1 \leq p, q < \infty \) be arbitrary numbers. We suppose that, for any measurable subset \( E \subset \Omega \), the following spaces are defined:

- the space \( X_p(E) \) with seminorm \( \| \cdot \|_{X_p(E)} \),
- the Banach space \( Y_q(E) \) with norm \( \| \cdot \|_{Y_q(E)} \),

which all satisfy the following conditions:

1. \( X_p(\Omega) \subset Y_q(\Omega) \);
2. \( X_p(E) = \{ f|_E : f \in X_p(\Omega) \} \), \( Y_q(E) = \{ f|_E : f \in Y_q(\Omega) \} \);
3. if \( \text{mes } E = 0 \), then \( \dim Y_q(E) = \dim X_p(E) = 0 \).
Assumption 1. There exist a partition the parameters were not considered. Here we investigate one of those critical cases.

1. If the tree \( A \) is finite, then \( f \cdot \chi_E \in Y_q(\Omega) \).

Let \( \mathcal{P}(\Omega) \subset X_p(\Omega) \) be a subspace of finite dimension \( r_0 \) and let \( \| f \|_{X_p(\Omega)} = 0 \) for any \( f \in \mathcal{P}(\Omega) \). For each measurable subset \( E \subset \Omega \) we write \( \mathcal{P}(E) = \{ P|_E : P \in \mathcal{P}(\Omega) \} \). Let \( G \subset \Omega \) be a measurable subset and let \( T \) be a partition of \( G \). We set

\[
\mathcal{S}_T(\Omega) = \{ f : \Omega \to \mathbb{R} : f|_E \in \mathcal{P}(E), f|_{\Omega \setminus G} = 0 \}.
\]

If \( T \) is finite, then \( \mathcal{S}_T(\Omega) \subset Y_q(\Omega) \) (see property 5).

For any finite partition \( T = \{ E_j \}_{j=1}^n \) of the set \( E \) and for each function \( f \in Y_q(\Omega) \) we put

\[
\| f \|_{p,q,T} = \left( \sum_{j=1}^n \| f|_{E_j} \|_{Y_q(E_j)}^{\sigma_{p,q}} \right)^{1/\sigma_{p,q}}
\]

with \( \sigma_{p,q} = \min\{p, q\} \). Denote by \( Y_{p,q,T}(E) \) the space \( Y_q(E) \) with the norm \( \| \cdot \|_{p,q,T} \).

Notice that \( \| \cdot \|_{Y_q(E)} \leq \| \cdot \|_{p,q,T} \).

For each subtree \( A' \subset A \) we set

\[
\Omega_{A'} = \cup_{\xi \in \mathcal{V}(A')} \hat{F}(\xi).
\]

In upper estimates for entropy numbers of the embedding operator of the space \( X_p(\Omega) \) into \( Y_q(\Omega) \) were obtained under some conditions on these spaces (the space \( X_p(\Omega) \cong X_p(\Omega)/\mathcal{P}(\Omega) \) will be defined later). Some limiting relations between the parameters were not considered. Here we investigate one of those critical cases.

Throughout we assume that \( 1 < p < q < \infty \) and the following conditions hold.

**Assumption 1.** There exist a partition \( \{ A_{t,i} \}_{t \geq t_0, i \in J_t} \) of the tree \( A \) and a number \( c \geq 1 \) with the following properties:

1. If the tree \( A_{t,t'} \) follows the tree \( A_{t,j} \), then \( t' = t + 1 \).

2. For each vertex \( \xi \in \mathcal{V}(A) \) there exists a linear continuous projection \( P_{\xi} : Y_q(\Omega) \to \mathcal{P}(\Omega) \) such that for any function \( f \in X_p(\Omega) \) and for any subtree \( D \subset A \) rooted at \( \xi \)

\[
\| f - P_{\xi}f \|_{Y_q(\Omega_D)} \leq c \sum_{t=t_0}^\infty \sum_{i \in J_t,D} 2^{(1-\frac{2}{p})t} \| f \|_{X_p(\Omega_{D_{t,j}})}^q;
\]

here

\[
J_{t,D} = \{ i \in J_t : \mathcal{V}(A_{t,i}) \cap \mathcal{V}(D) \neq \emptyset \}, \quad D_{t,j} = A_{t,j} \cap D.
\]
Assumption 2. There exist numbers $\delta_*>0$ and $c_2 \geq 1$ such that for each vertex $\xi \in \mathbf{V}(A)$ and for any $n \in \mathbb{N}$, $m \in \mathbb{Z}_+$ there exists a partition $T_{m,n}(G)$ of the set $G = \hat{F}(\xi)$ with the following properties:

1. $\text{card} \ T_{m,n}(G) \leq c_2 \cdot 2^mn$.

2. For any $E \in T_{m,n}(G)$ there exists a linear continuous operator $P_E : Y_q(\Omega) \to \mathcal{P}(E)$ such that for any function $f \in X_p(\Omega)$

$$\|f - P_E f\|_{Y_q(E)} \leq (2^m n)^{-\delta_}* 2^{\left(\frac{1}{p} - \frac{1}{q}\right)t}\|f\|_{X_p(E)},$$

where $t \geq t_0$ is such that $\xi \in \bigcup_{j \in J_1} \mathbf{V}(A_{t,j})$.

3. For any $E \in T_{m,n}(G)$

$$\text{card} \ \{E' \in T_{m\pm 1,n}(G) : \text{mes}(E \cap E') > 0\} \leq c_2.$$  \hspace{1cm} (9)

Assumption 3. There exist numbers $\gamma_* > 0$, $c_3 \geq 1$ and an absolutely continuous function $\psi_* : (0, \infty) \to (0, \infty)$ such that $\lim_{y \to \infty} \frac{\psi_*(y)}{\psi_*(y)} = 0$ and for $\nu_t := \sum_{i \in J_1} \text{card} \ \mathbf{V}(A_{t,i})$ the following estimate holds:

$$\nu_t \leq c_3 \cdot 2^{\gamma_* \psi_*} \psi_*(2^\nu) =: c_3 \nu_t, \quad t \geq t_0.$$  \hspace{1cm} (10)

Assumption [1] together with the inequality $p < q$ implies that for any $t \geq t_0$ and for each vertex $\xi_* \in \mathbf{V}^A_{t-t_0}(\xi_0)$ there exists a linear continuous projection $P_{\xi_*} : Y_q(\Omega) \to \mathcal{P}(\Omega)$ such that for any function $f \in X_p(\Omega)$ and for any subtree $\mathcal{D} \subset A$ rooted at $\xi_*$

$$\|f - P_{\xi_*} f\|_{Y_q(\Omega_\mathcal{D})} \leq c \cdot 2^{\left(\frac{1}{p} - \frac{1}{q}\right)t}\|f\|_{X_p(\Omega_\mathcal{D})}.$$  \hspace{1cm} (11)

Hence, Assumptions 1–3 from [36, 37] hold with $\lambda_* = \mu_* = \frac{1}{p} - \frac{1}{q}$ and $u_* \equiv 1$.

In particular, there exist a linear continuous projection $\hat{P} : Y_q(\Omega) \to \mathcal{P}(\Omega)$ and a number $M > 0$ such that for any function $f \in X_p(\Omega)$ the following estimate holds:

$$\|f - \hat{P} f\|_{Y_q(\Omega)} \leq M\|f\|_{X_p(\Omega)}.$$ 

As $\hat{P}$ we take the operator $P_{\xi_0}$ (recall that $\xi_0$ is the root of $A$). Similarly as in [37], we set

$$\hat{X}_p(\Omega) = \{f - \hat{P} f : f \in X_p(\Omega)\}$$

and denote by $I$ the embedding operator of $\hat{X}_p(\Omega)$ into $Y_q(\Omega)$.

We set $\mathbf{3}_0 = (p, q, c_1, c_2, c_3, c, \delta_*, \gamma_*, \psi_*)$.

We use the following notations for order inequalities. Let $X$, $Y$ be sets, and let $f_1$, $f_2 : X \times Y \to \mathbb{R}_+$. We write $f_1(x, y) \lesssim y f_2(x, y)$ (or $f_2(x, y) \gtrsim y f_1(x, y)$) if for any $y \in Y$ there exists $c(y) > 0$ such that $f_1(x, y) \lesssim c(y)f_2(x, y)$ for any $x \in X$; $f_1(x, y) \asymp f_2(x, y)$ if $f_1(x, y) \lesssim f_2(x, y)$ and $f_2(x, y) \lesssim f_1(x, y)$.
Theorem 1. Suppose that Assumptions 1–3 hold. Then

\[ e_n(I : \hat{X}_p(\Omega) \to Y_q(\Omega)) \lesssim n^\frac{1}{q} \frac{1}{p}. \] (12)

Similarly as in [36], [37] we introduce some more notation.

- \( \hat{\xi}_{t,i} \) is the minimal vertex of the tree \( A_{t,i} \).
- \( G_t = \cup_{\xi \in V(\Gamma)} \hat{F}(\xi) = \cup_{i \in J_t} \Omega_{A_{t,i}} \).
- \( \hat{\Gamma}_t \) is the maximal subgraph on the vertex set \( \cup_{j \geq t} V(\Gamma_j), t \in \mathbb{N} \).
- \( \{\hat{A}_{t,i}\}_{i \in J_t} \) is the set of connected components of the graph \( \hat{\Gamma}_t \).
- \( \hat{\Gamma}_t = \cup_{i \in J_t} \hat{U}_{t,i} = \cup_{\xi \in V(\hat{\Gamma})} \hat{F}(\xi) \).

If \( t \geq t_0 \), then

\[ V_{\min}(\hat{\Gamma}_t) = V_{\min}(\Gamma_t) = \{\hat{\xi}_{t,i}\}_{i \in \hat{J}_t} \] (13)

(see [36]); hence, we may assume that

\[ J_t = \hat{J}_t, \quad t \geq t_0. \] (14)

The set \( \hat{J}_{t_0} \) is a singleton. Denote \( \{i_0\} = \hat{J}_{t_0} \).

In [37] the operators \( Q_t \) and \( P_{t,m} \) were defined as follows.

**Definition of the operator \( Q_t \).** For each \( t \geq t_0, i \in \hat{J}_t \) there exists a linear continuous operator \( \hat{P}_{t,i} : Y_q(\Omega) \to \mathcal{P}(\Omega) \) such that for any function \( f \in \hat{X}_p(\Omega) \) and for any subtree \( A' \subset A \) rooted at \( \hat{\xi}_{t,i} \)

\[ \|f - \hat{P}_{t,i}f\|_{Y_q(\Omega_{A'})} \lesssim 2^{-(\frac{1}{p} - \frac{1}{q})t}\|f\|_{X_p(\Omega_{A'})}. \] (15)

As \( \hat{P}_{t,i} \) we take \( P_{t,i} \). From the definition of the space \( \hat{X}_p(\Omega) \) and of the operator \( \hat{P} \) it follows that \( \hat{P}_{t_0,i_0}|_{\hat{X}_p(\Omega)} = 0 \) (see [37]).

We set

\[ Q_t f(x) = \hat{P}_{t,i}f(x) = P_{t,i}f(x) \text{ for } x \in U_{t,i}, \quad i \in \hat{J}_t, \]

\[ Q_t f(x) = 0 \text{ for } x \in \Omega \setminus \hat{U}_t, \] (16)

\[ T_t = \{U_{t+1,i}\}_{i \in \hat{J}_{t+1}}. \] (17)

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Since \( p < q \), we have for any \( f \in B\hat{X}_p(\Omega) \)

\[
\| f - Q_t f \|_{Y_q(\tilde{\varpi}_t)} \leq \| f - Q_t f \|_{Y_{p,n,\tau_{t-1} (\tilde{\varpi}_t)}} \lesssim 2^{-\left(\frac{1}{p} - \frac{1}{q}\right) t}. \tag{15}
\]

Notice that if \( t < t_0 \), then \( Q_t f = Q_{t+1} f = 0 \) (since \( \tilde{P}_{t,0} = 0 \) for \( t < t_0 \)).

Throughout we set \( \log x := \log_2 x \).

**Definition of the operators** \( P_{t,m} \). For \( t \geq t_0 \) we set

\[
m_t = \lceil \log \nu_t \rceil. \tag{19}
\]

In [36] for each \( m \in \mathbb{Z}_+ \) the set \( G_{m,t} \subset G_t \), the partition \( \tilde{T}_{t,m} \) of the set \( G_{m,t} \) and the linear continuous operator

\[
P_{t,m} : Y_q(\Omega) \to S_{\tilde{T}_{t,m}}(\Omega) \tag{20}
\]

we constructed. Here the following properties hold:

1. \( G_{m,t} \subset G_{m+1,t}, \ G_{m,t} = G_t \);

2. for any \( m \in \mathbb{Z}_+ \)

\[
\text{card } \tilde{T}_{t,m} \lesssim 2^m; \tag{21}
\]

3. for any function \( f \in \hat{X}_p(\Omega) \) and for any set \( E \in \tilde{T}_{t,m} \)

\[
\| f - P_{t,m} f \|_{Y_q(E)} \lesssim 2^{-\left(\frac{1}{p} - \frac{1}{q}\right) t} \| f \|_{X_p(E)}, \ m \leq m_t; \tag{22}
\]

\[
\| f - P_{t,m} f \|_{Y_q(E)} \lesssim 2^{-\left(\frac{1}{p} - \frac{1}{q}\right) t} \cdot 2^{-\delta_s (m-m_t)} \| f \|_{X_p(E)}, \ m > m_t; \tag{23}
\]

4. for any set \( E \in \tilde{T}_{t,m} \)

\[
\text{card } \{ E' \in \tilde{T}_{t,m+1} : \text{mes}(E \cap E') > 0 \} \lesssim 1. \tag{24}
\]

Moreover, we may assume that

\[
\tilde{T}_{m_t,t} = \{ \tilde{F}(\xi) \}_{\xi \in \mathbf{V}(\Gamma_t)}, \tag{25}
\]

\[
P_{t,m_t} f |_{F(\xi)} = P_{\xi} f |_{F(\xi)}, \ \xi \in \mathbf{V}(\Gamma_t) \tag{26}
\]

(it follows from the construction in [36, p. 37–40]).
Let
\[ t_{*}(n) = \min \{ t \in \mathbb{N} : \nu_t \geq n \} , \] (27)
\[ t_{**}(n) = \min \{ t \in \mathbb{N} : \nu_t \geq 2^n \} . \] (28)

Then
\[ 2^{t_{*}(n)} \lesssim \log n , \quad 2^{t_{**}(n)} \lesssim n \] (29)
(see [37, formula (49)]).

For any \( f \in \hat{X}_p(\Omega) \) the following equality holds:
\[
  f = t_{*}(n) - 1 \sum_{t=t_0}^{t_{*}(n)} (Q_{t+1}f - Q_tf) \chi_{\tilde{U}_{t+1}} + \sum_{t=t_0}^{t_{*}(n)-1} \sum_{m=0}^{\infty} (P_{t,m+1}f - P_{t,m}f) \chi_{G_{m,t}} + (f - Q_{t_{*}(n)}f) \chi_{\tilde{U}_{t_{*}(n)}}
\] (30)
(it can be proved similarly as formula (82) in [36]).

We set
\[
  \tilde{Q}f|_{G_t} = P_{t,m_t}f|_{G_t}, \quad t \geq t_0, \tag{31}
\]
\[
  \tilde{Q}_{n,m}f|_{G_t} = P_{t,m_t+m}f|_{G_t} \quad \text{for} \quad t_{*}(n) \leq t < t_{**}(n), \quad m \in \mathbb{Z}_+, \tag{32}
\]
\osex[\text{Since } \| f - P_{t,m_t}f \|_{Y_q(\hat{F}(\xi))} \lesssim 2^{-\left(\frac{1}{q} - \frac{1}{p} \right) t} \| f \|_{X_p(\hat{F}(\xi))} \text{ for any } \xi \in \mathcal{V}(\Gamma_t) \text{ and the space } Y_q(\Omega) \text{ is Banach, we get from the inequality } p < q \text{ that for any } f \in \hat{X}_p(\Omega) \text{ the inclusion } \tilde{Q}f \in Y_q(\Omega) \text{ holds and}
]
\[
  \| f - \tilde{Q}f \|_{Y_q(\hat{U}_{t_{**}(n)})} \lesssim 2^{\left(\frac{1}{q} - \frac{1}{p} \right) t_{**}(n)} \| f \|_{X_p(\Omega)} \lesssim \frac{1}{3_0} \frac{1}{p} - \frac{1}{q} \| f \|_{X_p(\Omega)}. \tag{33}
\]

Further,
\[
  (f - Q_{t_{*}(n)}f) \chi_{\tilde{U}_{t_{*}(n)}} = (\tilde{Q}f - Q_{t_{*}(n)}f) \chi_{\tilde{U}_{t_{*}(n)}} + \sum_{m=0}^{\infty} (\tilde{Q}_{n,m+1}f - \tilde{Q}_{n,m}f) + (f - \tilde{Q}f) \chi_{\tilde{U}_{t_{*}(n)}}; \tag{34}
\]
indeed, from (23), (31) and (32) it follows that
\[
  \sum_{m=0}^{\infty} (\tilde{Q}_{n,m+1}f - \tilde{Q}_{n,m}f) = (f - \tilde{Q}f) \chi_{\tilde{U}_{t_{*}(n)} \setminus \hat{U}_{t_{*}(n)}}.
\]
Lemma 1. There exists a sequence \( \{ k_t \} \) such that
\[
\sum_{t=0}^{t_*(n)-1} (k_t - 1) \lesssim n,
\]
(35)
\[
\sum_{t=0}^{t_*(n)-1} e_{k_t} (Q_{t+1} - Q_t : \tilde{X}_p(\Omega) \to Y_q(\tilde{U}_{t+1})) \lesssim n^{\frac{1}{p} - \frac{1}{q}}.
\]
(36)

Lemma 2. There exists a sequence \( \{ k_{t,m} \} \) such that
\[
\sum_{t,m} (k_{t,m} - 1) \lesssim n,
\]
\[
\sum_{t=0}^{t_*(n)-1} \sum_{m=0}^{t} e_{k_{t,m}} (P_{m+1} - P_{m} : \tilde{X}_p(\Omega) \to Y_q(G_{m,t})) \lesssim n^{\frac{1}{p} - \frac{1}{q}}.
\]

Lemmas 1 and 2 are proved similarly as Lemmas 6 and 7 in [37].

Lemma 3. We have
\[
e_n ((\tilde{Q} - Q_{t_*(n)}) \chi_{\tilde{U}_{t_*(n)}}) : \tilde{X}_p(\Omega) \to Y_q(\tilde{U}_t)) \lesssim n^{\frac{1}{p} - \frac{1}{q}}.
\]

Notice that for \( t_**(n) \leq t_0 \) this estimate follows from (33). Hence, throughout we assume that \( t_**(n) > t_0 \).

We need some auxiliary assertions.
For any \( \nu \in \mathbb{N} \) we denote by \( I_{\nu} \) the identity operator on \( \mathbb{R}^\nu \).

Theorem A. [9,33]. Let \( 1 \leq p \leq q \leq \infty \). Then
\[
e_k (I_{\nu} : l^p_{\nu} \to l^q_{\nu}) \sim \begin{cases} 
1, & 1 \leq k \leq \log \nu, \\
\left( \frac{\log(1 + \frac{1}{k})}{\log k} \right)^{\frac{1}{p} - \frac{1}{q}}, & \log \nu \leq k \leq \nu, \\
2^{-k \nu^{\frac{1}{p} - \frac{1}{q}}}, & \nu < k.
\end{cases}
\]

The following properties of entropy numbers are well-known (see, e.g., [9,32]):
1. if \( T : X \to Y, S : Y \to Z \) are linear continuous operators, then
\[
e_{k+1}(ST) \leq e_k(S) e_{i}(T);
\]
2. if \( T, S : X \to Y \) are linear continuous operators, then
\[
e_{k+1}(S + T) \leq e_k(S) + e_i(T).
\]
In particular, from the first property it follows that
\[
e_k(ST) \leq \| S \| e_k(T), \quad e_k(ST) \leq \| T \| e_k(S).
\]
Theorem B. 25. Let \( X, Y \) be normed spaces, and let \( V \in L(X, Y) \), \( \{ V_\nu \}_{\nu \in \mathcal{N}} \subset L(X, Y) \). Then for any \( n \in \mathbb{N} \)
\[
e_{n+\lfloor \log_2 \mathcal{N} \rfloor +1}(V) \leq \sup_{\nu \in \mathcal{N}} e_n(V_\nu) + \sup_{x \in B_X} \inf_{\nu \in \mathcal{N}} \| Vx - V_\nu x \|_Y.
\]

Lemma 4. 39. Let \((\mathcal{T}, \xi)\) be a tree with finite vertex set, let
\[
\text{card} \, V_1(\xi) \leq k \quad \text{for any vertex} \quad \xi \in V(\mathcal{T}),
\]
and let the mapping \( \Phi : 2^V(\mathcal{T}) \to \mathbb{R}_+ \) satisfy the following conditions:
\[
\Phi(V_1 \cup V_2) \geq \Phi(V_1) + \Phi(V_2), \quad V_1, V_2 \subset V(\mathcal{T}), \quad V_1 \cap V_2 = \emptyset,
\]
\( \Phi(V(\mathcal{T})) > 0 \). Then there is a number \( C(k) > 0 \) such that for any \( n \in \mathbb{N} \) there exists a partition \( \mathcal{S}_n \) of the tree \( \mathcal{T} \) into at most \( C(k)n \) subtrees \( \mathcal{T}_j \), which satisfies the following conditions:
1. \( \Phi(V(\mathcal{T}_j)) \leq \frac{(k+2)\Phi(V(\mathcal{T}))}{n} \) for any \( j \) such that \( \text{card} \, V(\mathcal{T}_j) \geq 2 \);
2. if \( m \leq 2n \), then each element of \( \mathcal{S}_n \) intersects with at most \( C(k) \) elements of \( \mathcal{S}_m \).

Lemma 5. 39. Let \( T \) be a finite partition of a measurable subset \( G \subset \Omega \), \( \nu = \dim_\mathcal{S}_T(\Omega) \) (see (3)). Then there exists a linear isomorphism \( A : \mathcal{S}_T(\Omega) \to \mathbb{R}^\nu \) such that \( \| A \|_{\mathcal{L}(\nu; \mathcal{S}_T(\Omega))} \lesssim 1, \| A^{-1} \|_{\mathcal{L}(\mathcal{S}_T(\Omega); \nu)} \lesssim 1 \).

Lemma 6. 41, formula (60)]. Let \( \Lambda_\ast : (0, +\infty) \to (0, +\infty) \) be an absolutely continuous function such that \( \lim_{y \to +\infty} \frac{\nu\Lambda_\ast(y)}{\Lambda_\ast(y)} = 0 \). Then for any \( \varepsilon > 0 \)
\[
t^\varepsilon \lesssim \frac{\Lambda_\ast(ty)}{\varepsilon \Lambda_\ast(y)} \lesssim t^\varepsilon, \quad 1 \leq y < \infty, \quad 1 \leq t < \infty.
\]

Given \( t_\ast(n) \leq t \leq t_\ast(n) \), \( i \in \hat{J}_t \), we denote
\[
\mathcal{J}_{t,i} = \begin{cases} A_{t,i} & \text{for} \quad t < t_\ast(n), \\ A_{t_\ast(n),i} & \text{for} \quad t = t_\ast(n), \\ \end{cases}
\]
\( \Gamma_t = \Gamma_t \) for \( t < t_\ast(n) \), \( \Gamma_{t_\ast(n)} = \Gamma_{t_\ast(n)} \). Let \( \mathcal{D} \) be a subtree in \( \hat{\mathcal{G}}_{t_\ast(n)} \). We set
\[
\hat{J}_{t,\mathcal{D}} = \{ i \in \hat{J}_t : V(\mathcal{J}_{t,i}) \cap V(\mathcal{D}) \neq \emptyset \}, \quad \overline{\mathcal{J}^{t,i}} = \mathcal{J}_{t,i} \cap \mathcal{D}.
\]
Throughout we take as \( \varepsilon = \varepsilon(3_0) > 0 \) a sufficiently small number (it will be chosen later by \( 3_0 \)).
Proof of Lemma 3. We set \( t'_*(n) = \max\{t_*(n), t_0\} \).

**Step 1.** Given \( t < t^{**}(n) \), we denote by \( \hat{A}_t \) the subtree in \( A \) with vertex set \( V(A) \setminus V(\hat{\Gamma}_{t+1}) \).

Let \( f \in B\hat{X}_p(\Omega) \). For each \( t'_*(n) \leq t < t^{**}(n) \) we define the mapping \( \Phi_{f,t} : 2^{V(\hat{A}_t)} \to \mathbb{R}_+ \) by

\[
\Phi_{f,t}(W) = \sum_{\xi \in W \cap V(\hat{\Gamma}_t)} \|f\|^p_{X_p(\hat{F}(\xi))}.
\]

Then for any disjoint sets \( W_1, W_2 \) we have

\[
\Phi_{f,t}(W_1 \sqcup W_2) = \Phi_{f,t}(W_1) + \Phi_{f,t}(W_2). \tag{43}
\]

For each \( t'_*(n) \leq t < t^{**}(n) \) we set

\[
\varepsilon_t = \sum_{\xi \in V(\hat{\Gamma}_t)} \|f\|^p_{X_p(\hat{F}(\xi))};
\]

\[
n_t = \lceil n \cdot 2^{-t} \varepsilon_t \rceil \quad \text{if} \quad \varepsilon_t > 0; \quad n_t = 1 \quad \text{if} \quad \varepsilon_t = 0. \tag{44}
\]

Then

\[
\sum_{t = t'_*(n)}^{t^{**}(n)-1} \varepsilon_t \leq 1, \tag{45}
\]

\[
\sum_{t = t'_*(n)}^{t^{**}(n)-1} 2^t n_t \leq \sum_{t = t'_*(n)}^{t^{**}(n)-1} n_t \varepsilon_t + \sum_{t = t'_*(n)}^{t^{**}(n)-1} 2^t \leq \hat{C}(\hat{Z}_0)n \tag{46}
\]

with \( \hat{C}(\hat{Z}_0) \in \mathbb{N} \). By (44),

\[
\Phi_{f,t}(V(\hat{A}_t)) = \varepsilon_t. \tag{47}
\]

Let \( f \neq 0 \). It follows from Lemma 4 and (1) that there exists a number \( C(\hat{Z}_0) \in \mathbb{N} \) and a family of partitions \( \{T_{f,t,l}\}_{0 \leq l \leq \log n_t} \) of the tree \( \hat{A}_t \), which satisfy the following conditions:

\[
\text{card } T_{f,t,l} \leq C(\hat{Z}_0)2^{-l}n_t, \tag{48}
\]

\[
\text{card } \{A'' \in T_{f,t,l+1} : V(A') \cap V(A'') \neq \emptyset\} \leq 1, \quad A' \in T_{f,t,l}, \tag{49}
\]

and for any subtree \( A' \in T_{f,t,l} \) such that \( \text{card } V(A') \geq 2 \)

\[
\Phi_{f,t}(V(A')) \leq C(\hat{Z}_0)2^l n_t^{-1} \varepsilon_t. \tag{50}
\]
Moreover, we may assume that
\[ T_{f,t,[\log n_t]} = \{ \hat{A}_t \}. \] (51)
For \( f \equiv 0 \) we set \( T_{f,t,t} = \{ \hat{A}_t \} \).
Given \( t < t_{**}(n) \), we denote
\[ W_t = \left\{ \xi \in V(\Gamma_t) : \{ \xi \} \in T_{f,t,0}, \| f \|^p_{X_p(F(\xi))} > C(3_0)n_t^{-1} \varepsilon_t \right\}, \] (52)
\[ \overline{W}_t = \{ \xi \in W_t : V_1^A(\xi) \cap \{ \xi_{t+1,j} \}_{j \in J_{t+1}} = \emptyset \}, \] (53)
\[ S_t = \{ \xi \in V(\Gamma_t) : \exists D \in T_{f,t,0} : \xi \in V_{\min}(D) \}, \] (54)
\[ \hat{S}_t = S_t \cup \left( \bigcup_{\xi \in \overline{W}_t} [V_1^A(\xi) \cap V(\Gamma_t)] \right) \text{ for } t < t_{**}(n), \] (55)
\[ \hat{S}_{t_{**}(n)} = \bigcup_{\xi \in \overline{W}_{t_{**}(n)-1}} [V_1^A(\xi) \cap V(\Gamma_{t_{**}(n)})], \]
\[ \hat{S} = \bigcup_{t = t_{**}(n)}^{t_{**}(n)-1} \hat{S}_t \cup \{ \xi_{t_{**}(n),j} \}_{j \in \hat{J}_{t_{**}(n)}}. \] (56)

Then
\[ \hat{S} \setminus \bigcup_{t = t_{**}(n)}^{t_{**}(n)-1} S_t \subset \bigcup_{t = t_{**}(n)}^{t_{**}(n)} \{ \xi_{t,j} : j \in \hat{J}_t \}, \] (57)
\[ W_t \subset S_t. \]

For each vertex \( \xi \in \hat{S} \) we denote by \( D(\xi) \) the tree with vertex set
\[ V(D(\xi)) = \{ \xi' \supset \xi : \{ \xi, \xi' \} \cap \hat{S} = \{ \xi \} \}. \] (58)

We set
\[ T_f = \{ D(\xi) : \xi \in \hat{S} \}. \] (59)

Then \( T_f \) is a partition of the graph \( \tilde{\Gamma}_{t_{**}(n)} \) into subtrees.

**Step 2.** We say that \( D \in T_f \) if \( D \in T_f \) and for any \( \xi \in W_t \) we have \( D \neq \{ \xi \} \).
Let \( (D, \xi) \in T_f \). Then \( \xi \in V(\tilde{\Gamma}_{t_{**}(n)}). \) From Assumption 1 and the inequality \( p < q \) it follows that
\[ \| f - P_{\xi_t} f \|^q_{Y_q(\Omega_D)} \leq C_0 \| f \|^p_{X_p(F(\xi_t))}, \] (51)
\[ \| f - \tilde{Q} f \|^q_{Y_q(\Omega_D)} \leq \sum_{t = t_{**}(n)}^{t_{**}(n)-1} \sum_{j \in J_{t,D}} \| f - P_{t,m_t} f \|^q_{Y_q(\Omega_{t_{*},j})} \leq C_0, \] (52)
and (57) we get \( \xi \). Hence,

\[
\sum_{t=t'_* (n)}^{t_* (n)} 2^t \frac{1}{\hat{f}} \sum_{j \in J_{t,D}} \| f \|^q_{X_p (\Omega_{\hat{T}_{t,j}})}.
\]

Hence,

\[
\sum_{(D, \xi) \in T_f} \| f - P_{\xi} f \|^q_{Y_q (\Omega_D)} \leq \sum_{t=t'_* (n)}^{t_* (n)} 2^t \frac{1}{\hat{f}} \sum_{j \in J_{t,D}} \| f \|^q_{X_p (\Omega_{\hat{T}_{t,j}})}
\]

(60)

\[
\sum_{(D, \xi) \in T_f} \| f - \hat{Q} f \|^q_{Y_q (\Omega_D)} \leq \sum_{t=t'_* (n)}^{t_* (n)} 2^t \frac{1}{\hat{f}} \sum_{j \in J_{t,D}} \| f \|^q_{X_p (\Omega_{\hat{T}_{t,j}})}
\]

(61)

Denote by \( \hat{T}_{f,t} \) the family of trees \( \hat{D} \in T_{f,t,0} \) such that

\[
\sum_{\xi \in V(\hat{D}) \cap V(T)} \| f \|^p_{X_p (\hat{F}(\hat{\xi}))} \leq C(3_0) n_t^{-1} \varepsilon_t.
\]

(62)

**Assertion 1.** Let \( D = T_{f,t} \), \( t'_* (n) \leq t < t_* (n) \), \( j \in J_{t,D} \). Then there exists a tree \( T \in T_{f,t} \) such that \( D_{t,j} = T_{t,j} \).

**Proof of Assertion 1.** Let \( D = T_{f,t} \), \( \xi_s \in \hat{S} \). Since \( j \in J_{t,D} \), the vertices \( \xi_s \) and \( \hat{\xi}_{t,j} \) are comparable. There exists a tree \( T \in T_{f,t,0} \) such that

\[
\max\{\xi_s, \hat{\xi}_{t,j}\} \in V(T).
\]

(63)

We claim that \( V(D_{t,j}) \subset V(T) \). Indeed, let \( \xi \in V(D_{t,j}) \setminus V(T) \). Then \( \xi > \max\{\hat{\xi}_{t,j}, \xi_s\} \).

We set

\[
\eta_s = \min\{\eta \in \max\{\xi_s, \hat{\xi}_{t,j}\}, \xi > \max\{\hat{\xi}_{t,j}, \xi_s\} \}.
\]

Then \( \eta_s \in V(T) \) and there exists a tree \( T' \in T_{f,t,0} \) such that \( \eta_s \) is the minimal vertex of \( T' \). Hence, \( \eta_s \in S_t \subset \hat{S} \); therefore, \( \xi \notin V(D) \), which leads to a contradiction.

Thus, \( V(T_{t,j}) \supset V(D_{t,j}) \neq \emptyset \).

Let us show that \( V(T_{t,j}) \subset V(D_{t,j}) \). Denote by \( \hat{\xi} \) the minimal vertex of the tree \( T \). Since \( V(T_{t,j}) \neq \emptyset \), the vertices \( \xi \) and \( \hat{\xi}_{t,j} \) are comparable. Let us show that \( \max\{\xi_s, \hat{\xi}_{t,j}\} = \max\{\xi_s, \hat{\xi}_{t,j}\} \).

Indeed, by (63) we have \( \max\{\xi_s, \hat{\xi}_{t,j}\} \geq \max\{\xi, \hat{\xi}_{t,j}\} \).

If the inequality is strict, then \( \xi_s > \hat{\xi}_{t,j} \). In addition, \( \xi_s \in V(A_{t,j}) \cap \hat{S} \), and by (54) and (57) we get \( \xi_s \in S_t \); i.e., \( \xi_s \) is the minimal vertex of some tree from the partition \( T_{f,t,0} \). By (63), \( \xi_s = \hat{\xi} \).

Thus,

\[
\max\{\xi, \hat{\xi}_{t,j}\} = \max\{\xi_s, \hat{\xi}_{t,j}\} \in V(D_{t,j}).
\]

(64)
Suppose that there exists a vertex $\xi \in V(T_{i,j}) \setminus V(D)$. We set

$$\eta_{**} = \min \left( \max \{\xi, \xi_t, j\}, \xi \right) \setminus V(D).$$

(65)

Then $\eta_{**} > \max \{\xi, \xi_t, j\}$ and $\eta_{**} \in V(A_{i,j})$. Hence, $\eta_{**} \notin \cup_{t'=t+1}^{t_t(n)} \{\xi_i : i \in J_{t'}\}$.

Further, $\eta_{**} \notin V(T_{i,j})$; therefore, $\eta_{**} \notin \cup_{t'=t+1}^{t_t(n)} \{\xi_i : i \in J_{t'}\}$ by (53). From (57) it follows that $\eta_{**} \notin \hat{S}$. This together with (58), (64), and (65) yields that $[\xi, \eta_{**}] \cap \hat{S} = \{\xi\}$; applying (58) once again, we get that $\eta_{**} \notin V(D)$, which leads to a contradiction.

It remains to check that $T \in T_{f,t}$. If $T \notin T_{f,t}$, then $V(T) \subseteq \{\xi\}$, $f \left\| X_p(\hat{S}) \right\| > C(30) n_t^{-1} \epsilon_t; i.e., $\xi \in \overline{W}_t$. Then either $\xi \in \overline{W}_t$ (in this case, $V_A^t(\xi) \subseteq S_t \cup \hat{S}_{t+1}$) or $V_A^t(\xi) \subseteq S_t$. Hence, $V_A^t(\xi) \subseteq \hat{S}$. This implies that $V(D) = \{\xi\}$ and $D \notin T_f$. This completes the proof of Assertion 1. \)

By (54), the first inclusion of (57), (58) and (59), for any tree $D \in T_f$ we have $D_{t_{**}(n),j} = A_{t_{**}(n),j}, j \in J_{t_{**}(n)}$. This together with Assertion 1 yields that

$$\sum_{t=t_{t}(n)}^{t_{t_{**}(n)}-1} 2^{t(1-q)} \sum_{(D, \xi) \in T_{f,j} \in J_{t_{t}, D}} \|f\|_{X_p(\Omega_{T_{f,j}})} \leq 2^{t_{t_{**}(n)}-1} \sum_{T \in T_f, j \in J_{t, T}} 2^{t_{t_{**}(n)}-1} \sum_{j \in J_{t_{**}(n)}} \|f\|_{X_p(\Omega_{A_{t_{**}(n),j}})} +$$

$$+ 2^{t_{t_{**}(n)}-1} \sum_{j \in J_{t_{**}(n)}} \|f\|_{X_p(\Omega_{A_{t_{**}(n),j}})} \leq \frac{30}{\epsilon_t^2} n_{t_{t}}^{-1} \epsilon_t +$$

$$+ \sum_{t=t_{t}(n)}^{t_{t_{**}(n)}-1} 2^{t_{t_{**}(n)}-1} \sum_{T \in T_f, j \in J_{t, T}} \left( \sum_{\xi \in V(T) \cap V(\Gamma_1)} \|f\|_{X_p(\hat{S})} \right) \leq \frac{30}{\epsilon_t^2} n_{t_{t}}^{-1} \epsilon_t +$$

$$\frac{45}{\epsilon_t^2} n_{t_t}^{-1} \epsilon_t^{-\frac{2}{p}} + \sum_{t=t_{t}(n)}^{t_{t_{**}(n)}-1} 2^{t_{t_{**}(n)}-1} \sum_{j \in J_{t_{**}(n)}} \epsilon_t^2 n_t^{-\frac{2}{p}} =: A. \quad \text{(66)}$$

i.e.,

$$\sum_{t=t_{t}(n)}^{t_{t_{**}(n)}-1} 2^{t_{t_{**}(n)}-1} \sum_{(D, \xi) \in T_{f,j} \in J_{t_{t}, D}} \|f\|_{X_p(\Omega_{T_{f,j}})} \leq \frac{30}{\epsilon_t^2} n_{t_{t}}^{-1} \epsilon_t + \sum_{t=t_{t}(n)}^{t_{t_{**}(n)}-1} 2^{t_{t_{**}(n)}-1} \epsilon_t^2 n_t^{-\frac{2}{p}} =: A. \quad \text{(66)}$$

If $n \cdot 2^{-t_1} \epsilon_t \geq 1$, then $n_t \leq \frac{1}{\epsilon_t}$; hence,

$$2^{t_{t_{**}(n)}-1} \epsilon_t^2 n_t^{-\frac{2}{p}} \leq 2^{t(1-q)} \epsilon_t^2 n_t^{-\frac{2}{p}} 2^{-t_1} \epsilon_t^{-\frac{2}{p}} \epsilon_t = n_t^{-\frac{2}{p}} \epsilon_t.$$
If \( n \cdot 2^{-\varepsilon t} < 1 \), then \( n_t = 1 \). From (43) it follows that \( \varepsilon t \lesssim \). Therefore,

\[
A \lesssim n^{1-\frac{q}{p}} + \sum_{t=t'_*(n)}^{t^{*+}(n)-1} n^{1-\frac{q}{p}} \varepsilon t + \sum_{t=t'_*(n)}^{t^{*+}(n)-1} 2^{t(1-\frac{q}{p})} \lesssim n^{1-\frac{q}{p}} + 2^{t^{*+}(n)(1-\frac{q}{p})} \lesssim n^{1-\frac{q}{p}}. \tag{67}
\]

From (60), (61), (66), (67) we get that

\[
\sum_{(D, \xi) \in \tilde{T}_f} \| f - P\xi f \|^q_{Y_q(\Omega_D)} + \| f - \tilde{Q} f \|^q_{Y_q(\Omega_D)} \lesssim n^{1-\frac{q}{p}};
\]

this yields

\[
\sum_{(D, \xi) \in \tilde{T}_f} \| \tilde{Q} f - P\xi f \|^q_{Y_q(\Omega_D)} \lesssim n^{1-\frac{q}{p}}. \tag{68}
\]

**Step 3.** Let \( T \) be a partition of \( \tilde{\Gamma}_t'(n) \) into subtrees. For each \( (D, \xi) \in T \) we set

\[
P_T f|_{\Omega_D} = P\xi f - Q\xi'(n)f. \tag{69}
\]

In addition, we put

\[
P_T f|_{\Omega \setminus \tilde{U}_t'(n)} = 0. \tag{70}
\]

Then

\[
\| \tilde{Q} f - Q\xi'(n)f - P_T f \|^q_{Y_q(\Omega_D)} = \sum_{(D, \xi) \in T} \| \tilde{Q} f - P\xi f \|^q_{Y_q(\Omega_D)} = \sum_{(D, \xi) \in T, \text{card} V(D) \geq 2} \| \tilde{Q} f - P\xi f \|^q_{Y_q(\Omega_D)}.
\]

In particular, from (68) and the definition of \( \tilde{T}_f \) (see the beginning of Step 2) it follows that

\[
\| \tilde{Q} f - Q\xi'(n)f - P_T f \|^q_{Y_q(\tilde{U}_t'(n))} \lesssim n^{1-\frac{q}{p}}. \tag{71}
\]

**Step 4.** Let us define the family \( N \) of partitions of the graph \( \tilde{\Gamma}_t'(n) \) into subtrees. Each of these partitions is constructed as follows. Let \( C(3_0), \tilde{C}(3_0) \) be as defined at Step 1.

1. We choose the sequence \( \{n_t\}_{t=t'_*(n)} \subset N \) such that

\[
\sum_{t=t'_*(n)}^{t^{*+}(n)-1} 2't n_t \leq \tilde{C}(3_0) n. \tag{72}
\]
2. For each \( t \in \{t'_s(n), \ldots, t_{**}(n) - 1\} \) we take \( C(3_0)n_t \) vertices in \( V(\Gamma_t) \) (some of them may coincide); we denote this set by \( U_t \).

3. For each \( t \in \{t'_s(n), \ldots, t_{**}(n) - 1\} \) we choose an arbitrary subset \( \hat{V}_t \) of \( \{\xi \in U_t : V_1^A(\xi) \cap V(\Gamma_{t+1}) \neq \emptyset\} \).

4. Let
\[
U = \bigcup_{i,j \in J_{t'_s(n)}} \{\hat{\xi}_{t'_s(n),i,j}\} \bigcup \left( \bigcup_{t=t'_s(n)} \{\xi \in V_1^A(\xi) \cap V(\Gamma_{t+1})\} \bigcup \bigcup_{t=t'_s(n)} \{\xi \in \hat{V}_t \cap \bigcup_{t=1}^{t_{**}(n)} \{D_1^C(\xi) \cap V(\Gamma_{t+1})\}\} \right).
\]

This vertex set generates the desired partition \( T \) of the graph \( \hat{\Gamma}_{t'_s(n)} \) into subtrees:
\[
T = \{D_{(\xi)}' : \xi \in U\}, \quad V(D_{(\xi)}') = \{\xi' : [\xi, \xi'] \cap U = \{\xi\}\}.
\] (73)

Let us estimate the value \( |N| \).

1. First we estimate the number of choices of \( \{n_t\}_{t=t'_s(n)}^{t_{**}(n)-1} \) (we denote this value by \( N_1 \)). Let \( 1 \leq l \leq \hat{C}(3_0)n \). The number of choices of \( n_t \in \mathbb{N} \) such that \( \sum_{t=t'_s(n)}^{t_{**}(n)-1} 2^tn_t = l \) can be estimated from above by the number of choices of numbers \( \tilde{n}_t \in \mathbb{N} \) such that \( \sum_{t=t'_s(n)}^{t_{**}(n)-1} \tilde{n}_t = l \). The last magnitude can be estimated from above by the number of partitions of \( \{1, \ldots, l\} \) into \( t_{**}(n) - t'_s(n) \) intervals. This value does not exceed \( (l + t_{**}(n) - t'_s(n))^{t_{**}(n)-t'_s(n)-1} \). Hence,
\[
N_1 \leq \sum_{1 \leq l \leq \hat{C}(3_0)n} (l + t_{**}(n) - t'_s(n))^{t_{**}(n)-t'_s(n)-1} \leq \hat{C}(3_0)n + t_{**}(n))^{t_{**}(n)-t'_s(n)-1} =: N'_1.
\]

2. Given the sequence \( \{n_t\}_{t=t'_s(n)}^{t_{**}(n)-1} \), we estimate the number of choices of a set \( \bigcup_{t=t'_s(n)}^{t_{**}(n)-1} U_t \) (we denote this magnitude by \( N_2 \)). We have
\[
N_2 \leq \prod_{t=t'_s(n)}^{t_{**}(n)-1} (c_3\mathcal{D}_t)^{C(3_0)n_t} =: N'_2.
\]

3. Let us estimate the number of choices of \( \hat{V}_t, t'_s(n) \leq t \leq t_{**}(n) - 1 \) (denote this value by \( N_3 \)). Since \( \text{card} U_t \leq C(3_0)n_t \), we have
\[
N_3 \leq \prod_{t=t'_s(n)}^{t_{**}(n)-1} 2^{C(3_0)n_t} =: N'_3.
\]
Thus, $|N| \leq N_1^1 N_2^2 N_3$, which yields
\[
\log |N| \leq t_{**}(n) \log (\hat{C}(3_0)n + t_{**}(n)) + \sum_{t=t'(n)}^{t_{**}(n)-1} C(3_0) n_t \log (c_3 \nu_t) + \sum_{t=t'(n)}^{t_{**}(n)-1} C(3_0) n_t + (\log n)^2 + 2 t_n \leq \frac{n}{30};
\]
i.e.,
\[
\log |N| \leq n. \quad \quad \text{(74)}
\]

**Step 5.** Denote by $N'$ the family of partitions $T_f$, $f \in B \hat{X}_p(\Omega)$ (see (59)). Then $N' \subset N$ (it follows from (52), (56), (58), (73) and from the estimate (46), (48)). By (74) and Theorem 5 there exists $t_*=l_*(3_0) \in \mathbb{N}$ such that
\[
e_{t_*,n} (\hat{Q} - Q_{t'_*(n)} : \hat{X}_p(\Omega) \to Y_q(\hat{U}_{t'_*(n)})) \leq \sup_{f \in B \hat{X}_p(\Omega)} \inf_{T \in N'} \| \hat{Q} f - Q_{t'_*(n)} f - P_T f \|_{Y_q(\hat{U}_{t'_*(n)})} \leq \sup_{T \in N'} \sup_{f \in B \hat{X}_p(\Omega)} \| \hat{Q} f - Q_{t'_*(n)} f - P_T f \|_{Y_q(\hat{U}_{t'_*(n)})} \leq n^{\frac{1}{2} - \frac{1}{p}} + \sup_{T \in N'} e_n(P_T : \hat{X}_p(\Omega) \to Y_q(\hat{U}_{t'_*(n)})).
\]

**Step 6.** It remains to prove that
\[
e_n(P_{T_f} : \hat{X}_p(\Omega) \to Y_q(\hat{U}_{t'_*(n)})) \leq n^{\frac{1}{2} - \frac{1}{p}}, \quad \text{(75)}
\]
where $f \in B \hat{X}_p(\Omega)$.

Recall that $T_f = \{D_\xi : \xi \in \hat{S}\}$, where $\hat{S}$ is defined by (56) and $D_\xi$ is defined by formula (58). Also we observe that if $\xi = \hat{\xi}_{t'_*(n),j}$ for some $j \in \hat{J}_{t'_*(n)}$, then
\[
P_{T_f}|_{\Omega_{P_\xi}} (P_{\xi_{t'_*(n),j}} - Q_{t'_*(n)})|_{\Omega_{P_\xi}} = 0. \quad \text{(76)}
\]

Let $t'_*(n) \leq t < t_{**}(n)$. Recall that the sets $S_t$ and $\mathcal{W}_t$ are defined by formulas (51) and (53), respectively. We set
\[
\mathcal{V}_{f,t} = S_t \cup \{\xi \in \mathcal{V}(\Gamma_{t+1}) \setminus S_{t+1} : \exists \eta \in \mathcal{W}_t : \xi \in \mathcal{V}_1^A(\eta)\}. \quad \text{(77)}
\]
Notice that $V_{f,t} \subset \hat{S}$ by (55), (56). Denote by $\hat{\Gamma}_{f,t}$ the maximal subgraph in $A$ on vertex set

$$V(\hat{\Gamma}_{f,t}) = \bigcup_{\xi \in V_{f,t}} V(\mathcal{D}(\xi)).$$

(78)

We set

$$\Omega(f,t) = \bigcup_{\xi \in V(\hat{\Gamma}_{f,t})} \hat{F}(\xi), \quad P_{(f,t)}h = P_{T_f}h \cdot \chi_{\Omega(f,t)}, \quad h \in Y_q(\Omega).$$

(79)

Denote

$$T'_{f,t} = \{D(\xi)\}_{\xi \in V_{f,t}}.$$  (80)

Then $T'_{f,t}$ is a partition of the graph $\hat{\Gamma}_{f,t}$.

We claim that

$$P_{T_f} = \sum_{t=t'_*(n)}^{t_*+n(n)-1} P_{(f,t)}.$$  (81)

Indeed, let $h \in Y_q(\Omega)$. If $x \in \bigcup_{t=t'_*(n)}^{t_*+n(n)-1} \bigcup_{\xi \in V_{f,t}} \Omega_{D(\xi)}$, then by (79) we get $P_{T_f}h(x) = \sum_{t=t'_*(n)}^{t_*+n(n)-1} P_{(f,t)}h(x)$. In other cases we have $\sum_{t=t'_*(n)}^{t_*+n(n)-1} P_{(f,t)}h(x) = 0$. On the other hand, if $P_{T_f}h(x) \neq 0$, then $x \in \Omega_{D(\xi)}$ for some $\xi \in \hat{S}$ by (59) and (70). From (55), (56), (76) and (77) it follows that $\xi \in V_{f,t}$ for some $t \in \{t'_*(n), \ldots, t_*+n(n)-1\}$. This completes the proof of (81).

Let us prove that there exists a sequence $\{k_t\}_{t'_*(n) \leq t < t_*+n(n)}$ such that

$$\sum_{t=t'_*(n)}^{t_*+n(n)-1} (k_t - 1) \lesssim n,$$

$$\sum_{t=t'_*(n)}^{t_*+n(n)-1} e_{k_t} \left( P_{(f,t)} : \hat{X}_{\mu}(\Omega) \to Y_q(\Omega) \right) \lesssim n^{\frac{1}{4}}.$$  (82)

Let $\xi \in S_t$. If $\xi \in S_t \setminus \overline{W}_t$, then we set

$$\tilde{D}(\xi) = D(\xi), \quad \tilde{P}_{(f,t)}\big|_{\Omega_{\mathcal{D}(\xi)}} = P_{(f,t)}\big|_{\Omega_{\mathcal{D}(\xi)}} \quad \text{by (69), (79)} \quad (P_{\xi} - Q_{t'_*(n)}) \big|_{\Omega_{\mathcal{D}(\xi)}}.$$  (83)

If $\eta \in \overline{W}_t$, then we denote by $\tilde{D}(\eta)$ the tree with vertex set

$$V(\tilde{D}(\eta)) = \{\eta\} \cup \left( \cup_{\xi \in V(\eta) \cap V(\Gamma_{t+1}), S_{t+1}} V(\mathcal{D}(\xi)) \right)$$  (84)
and put
\[ \tilde{P}_{(f,t)}|_{\Omega_{\tilde{P}(\Omega)}} = (P_n - Q_{t^*})|_{\Omega_{\tilde{P}(\Omega)}}. \]  

(85)

Observe that by \((78)\) we have
\[ T''_{f,t} := \{ \tilde{D}_{(\xi)} \}_{\xi \in S_t} \]  is the partition of \( \tilde{P}_{f,t} \).

(86)

Finally, we set
\[ \tilde{P}_{(f,t)}|_{\Omega_{\tilde{P}(f,t)}} = 0. \]  

(87)

Let
\[ T'_{f,t} = \{ \Omega_D : D \in T'_{f,t} \}, \quad T''_{f,t} = \{ \Omega_D : D \in T''_{f,t} \}. \]

By \((69), (79), (83), (85), (87)\) we have
\[ P_{(f,t)} \in S_{T'_{f,t}}(\Omega), \quad \tilde{P}_{(f,t)} \in S_{T''_{f,t}}(\Omega). \]

Moreover, the partition \( T'_{f,t} \) refines the partition \( T''_{f,t} \).

Let \( h \in BX_p(\Omega) \). Then
\[
\| P_{(f,t)}h - \tilde{P}_{(f,t)}h \|_{p,q,T'_{f,t}}^p \leq \sum_{n \in \mathbb{N}} \sum_{\xi \in \mathbb{N}} \| P_{(f,t)}h - \tilde{P}_{(f,t)}h \|_{Y_q(\Omega_{\tilde{P}(\Omega)})}^p \leq n^2 \leq n_0.
\]

(88)

\[
\leq \sum_{n \in \mathbb{N}} \sum_{\xi \in \mathbb{N}} \left( \| h - P_{(f,t)}h \|_{Y_q(\Omega_{\tilde{P}(\Omega)})}^p + \| h - \tilde{P}_{(f,t)}h \|_{Y_q(\Omega_{\tilde{P}(\Omega)})}^p \right) \leq 2^{-t(1 - \epsilon/4)};
\]
i.e.,
\[
\| P_{(f,t)}h - \tilde{P}_{(f,t)}h \|_{p,q,T'_{f,t}} \leq 2^{-t(1 - \epsilon/4)}.
\]  

(89)

We set \( k'_t = \lceil n \cdot 2^{-\epsilon(t - t_*(n))} \rceil \), where \( \epsilon > 0 \) is a sufficiently small number (it is chosen by \( \Omega_0 \)). For any \( t \leq t_*(n) \) we have
\[ \text{card } T'_{f,t} \leq \text{card } T_{f,t} \leq 2^{-t} n \]

(90)

This together with \((38), (88), (89)\), Theorem \(A\) and Lemma \(5\) implies that for some \( c_* = c_*(\Omega_0) > 0 \)
\[ \sum_{t = t_*(n)}^{t_*(n) - 1} e_{k'_t} (P_{(f,t)} - \tilde{P}_{(f,t)}) : X_p(\Omega) \rightarrow Y_q(\Omega) \leq \sum_{t = t_*(n)}^{t_*(n) - 1} 2^{-t(1 - \epsilon/4)} (2^{-t} n)^{\epsilon/4} \cdot 2 - c_* 2^{t - (t - t_*(n))} \leq n^{\epsilon/4 - 1/4}.
\]
We claim that there exists a sequence $\{k''_t\}_{t' | t < t^{**}(n)}$ such that

$$
\sum_{t=t'_*(n)}^{t^{**}(n)-1} (k''_t - 1) \leq n, \\
\sum_{t=t'_*(n)}^{t^{**}(n)-1} e^{k''_t}(\tilde{P}(f,t)) = \tilde{X}(\Omega) \rightarrow Y_q(\Omega) \leq n^{\frac{1}{2}}. 
$$

(91)

For each $t'_{**}(n) \leq t < t^{**}(n)$, $0 \leq l \leq \log n_t$ we consider the partition $T_{f,t,l}$ as defined at Step 1. Let $D \in T_{f,t,l}$. We set

$$
V_{f,t,D} = \{ \xi \in V(\Gamma) \cap V(D) : V^A_1(\xi) \cap V(\Gamma_{t+1}) \neq \emptyset \}
$$

and denote by $D'$ the subtree in $\mathcal{A}$ with vertex set

$$
V(D) = V(D) \cup \left( \bigcup_{\xi \in V_{f,t,D}} \bigcup_{\xi' \in V^A_1(\xi) \cap V(\Gamma_{t+1})} V(\mathcal{A}_{\xi'}) \right).
$$

(92)

Let

$$
\tilde{T}_{f,t,l} = \{ D' : D \in T_{f,t,l} \}.
$$

(93)

By (88),

$$
\text{card} \tilde{T}_{f,t,l} \lesssim 2^{-l} n_t.
$$

(94)

We claim that for any tree $D \in T_{f,t,l}$

$$
\text{card} \{ D' : D' \in T_{f,t,l+1} \cap V(D) \cap V(D') \neq \emptyset \} \lesssim \frac{1}{30}.
$$

(95)

Indeed, by (89), it is sufficient to check that if $V(D) \cap V(D') \neq \emptyset$, then $V(D) \cap V(D') \neq \emptyset$. Let $\xi \in V(D) \cap V(D')$. Then either $\xi \in V(\tilde{A})$ (therefore, $\xi \in V(D) \cap V(D')$) or $\xi \in V(\mathcal{A}_q)$, where $q \in V(\Gamma_t) \cap V(D) \cap V(D')$, $V^A_q(\xi) \cap V(\Gamma_{t+1}) \neq \emptyset$.

**Assertion 2.** We have $T_{f,t,0} | \tilde{T}_{f,t} = T''_{f,t}$. If $(T, \xi) \in T_{f,t,0}$, $\xi \notin V(\Gamma_t)$, then $V(T) \cap V(\tilde{\Gamma}_t) = \emptyset$.

**Proof of Assertion 2.**

1. Let $(T, \xi) \in T_{f,t,0}$. We claim that either there exists a tree $D \in T''_{f,t}$ such that $V(T) \cap V(\tilde{\Gamma}_f,t) = V(D)$ or $V(T) \cap V(\tilde{\Gamma}_f,t) = \emptyset$.

**Case $\xi \in V(\Gamma_t)$**. By (53), we have $\xi \in S_t$. Let us show that

$$
V(T) \cap V(\tilde{\Gamma}_f,t) = V(\tilde{D}(\xi))
$$

(96)

and apply (86).
• We claim that $V(T_+) \cap V(\hat{\Gamma}_{f,t}) \supset V(\hat{D}(\hat{\xi}))$. Since $T''_{f,t}$ is a partition of the graph $\hat{\Gamma}_{f,t}$, we have $V(\hat{D}(\hat{\xi})) \subset V(\hat{\Gamma}_{f,t})$. Let us prove that $V(\hat{D}(\hat{\xi})) \subset V(T_+)$. Indeed, let $\xi \in V(\hat{D}(\hat{\xi})) \setminus V(T_+)$. We set $\eta_* = \min \{ \eta \in [\xi, \xi] : \eta \notin V(T_+) \} > \xi$. Denote by $\zeta_*$ the direct predecessor of $\eta_*$. Then $\zeta_* \notin \cup_{\zeta \in V_{f,t} \cup \zeta \in V(\hat{\xi}) \cap V(\hat{\Gamma}_{t+1})} \hat{V}(A_{\hat{\eta}})$ (otherwise, $\eta \in V(T_+)$). Hence, $\zeta_* \in V(T_+)$ if $\eta_* \notin V(T_+)$, then $\eta_* \in V(\hat{\Gamma}_{t+1})$ (by Assumption II, condition 1); once again, we get $\eta_* \in V(T_+)$. Thus, $\eta_* \in V(T_+)$ and $\eta_*$ is the minimal vertex of some tree $\hat{T} \in T_{f,t,0}$. Therefore, $\eta_* \in S_t \subset S$, which implies $\eta_* \in V(\hat{D}(\hat{\xi}))$. On the other hand, $\eta_* \in [\hat{\xi}, \hat{\xi}] \subset V(\hat{D}(\hat{\xi}))$, which leads to a contradiction.

• We claim that $V(T_+) \cap V(\hat{\Gamma}_{f,t}) \subset V(\hat{D}(\hat{\xi}))$. We have $V(\hat{\Gamma}_{f,t}) = \cup_{\eta \in S_t} V(\hat{D}(\eta))$. If $\zeta \in V(\hat{D}(\eta)) \cap V(T_+)$, then the vertices $\xi$ and $\eta$ are comparable. Since $\xi \in S_t \subset \hat{S}$, the case $\eta < \xi$ is impossible by \[58\], \[83\], \[54\]. Consequently, $\eta \in [\hat{\xi}, \hat{\xi}] \subset V(T_+)$; in addition, $\eta \in V(T_+)$ if $\zeta \notin V(\hat{\Gamma}_{f,t})$ is the minimal vertex of some tree in $T_{f,t,0}$. Therefore, $\eta = \hat{\xi}$; i.e., $\zeta \in V(\hat{D}(\hat{\xi}))$.

This completes the proof of (96).

Case $\hat{\xi} \notin V(\hat{\Gamma}_{t})$. Then $\hat{\xi} \notin V(\hat{\Gamma}_{t})$, $t' < t$. Let us check that $V(T_+) \cap V(\hat{\Gamma}_{f,t}) = \emptyset$. Indeed, let $\xi \in V(T_+) \cap V(\hat{\Gamma}_{f,t})$. Then $\xi \in V(\hat{D}(\eta))$ for some $\eta \in S_t$; i.e., $\eta \in V(\hat{\Gamma}_{t})$ is the minimal vertex of some tree from $T_{f,t,0}$ (this tree does not coincide with $T$ since $\hat{\xi} \notin V(\hat{\Gamma}_{t})$ is the minimal vertex of $T$). In addition, the vertices $\xi$ and $\eta$ are comparable. The case $\eta < \hat{\xi}$ is impossible by Assumption II (see condition 1). Hence, $\eta \in [\hat{\xi}, \hat{\xi}] \subset V(T_+)$; i.e., $\eta \in V(T)$, which leads to a contradiction.

2. Let $\hat{D}(\hat{\xi}) \in T''_{f,t}$. We claim that there exists a tree $T \in T_{f,t,0}$ such that $V(\hat{D}(\hat{\xi})) = V(T_+) \cap V(\hat{\Gamma}_{f,t})$. Indeed, since $\hat{\xi} \notin S_t$ by \[80\], we have $\hat{\xi} \in V(\hat{\Gamma}_{t})$ and $\hat{\xi}$ is the minimal vertex of some tree $T \in T_{f,t,0}$ (see \[54\]). By \[96\], $V(T_+) \cap V(\hat{\Gamma}_{f,t}) = V(\hat{D}(\eta))$ for some $\eta \in S_t$. In addition, $\xi \in V(T_+) \cap V(\hat{\Gamma}_{f,t})$; i.e., $\xi \in V(\hat{D}(\eta))$ and $\eta = \hat{\xi}$.

This completes the proof of Assertion 2.

Let $D \in T_{f,t,0}$, $t'_*(n) \leq t < t sanitation(n)$. If the minimal vertex of $D$ does not belong to $V(\hat{\Gamma}_{t'_*(n)})$, by Assertion 2 we have $V(D) \cap V(\hat{\Gamma}_{f,t}) = \emptyset$. Hence, if $V(D) \cap V(\hat{\Gamma}_{f,t}) \neq \emptyset$, then $\hat{\Gamma}_{t'_*(n)} \cap D$ is a tree.

Denote by $T^n_{f,t,0}$ the partition of $\hat{\Gamma}_{t'_*(n)}$ formed by connected components of graphs $\hat{\Gamma}_{t'_*(n)} \cap D$, $D \in T_{f,t,0}$. We have

$$\hat{P}_{f,t}h = P_{T^n_{f,t,0}}(h \cdot \chi_{\Omega(f,t)}), \quad h \in Y_q(\Omega). \quad (97)$$
It follows from (69), (79), (83), (85), (87), (88) and Assertion 2.

Given \(0 \leq l \leq \log n\), we set
\[
\begin{align*}
k_{l,l} &= \left\lfloor n \cdot 2^{-\varepsilon(t-t_\ast(n) + l)} \right\rfloor, \\
k'_l &= 1 + \sum_{0 \leq t \leq \log n} (k_{t,l} - 1), \quad t_\ast(n) \leq t < t_\ast(n).
\end{align*}
\]

Then \(k'_l - 1 \leq n \cdot 2^{-\varepsilon(t-t_\ast(n))}, \quad \sum_{t=0}^{t_\ast(n)-1} (k'_l - 1) \leq n.

From (91), (92) and (93) it follows that \(\tilde{T}_{f,t,[\log n]} = \{A\}, \tilde{T}_{f,t,[\log n]} = \{A_{t,n}'\}_{j \in j_\ast(n)}\).

From (16) and (69) we get (101).

\[
\begin{align*}
&\text{Let us check the first inequality (the second one is proved similarly). Let } E = \Omega_{D'}, \\
&D' \in \tilde{T}_{f,t,l}, E \in \tilde{T}_{f,t,l+1} \text{, then } (P_{T_{f,t,l}} h|_{\Omega_{D'}}) \in \mathcal{P}(\Omega_{D'}), \\
&\text{and } (P_{T_{f,t,l+1}} h|_{\Omega_{D'}}) \in \mathcal{P}(\Omega_{D'}); \text{ therefore,}
\end{align*}
\]

\[
(P_{T_{f,t,l}} h - P_{T_{f,t,l+1}} h)|_{\Omega_{D',D}} \in \mathcal{P}(\Omega_{D' \cap \Omega_{D}}).
\]

(102)

We show that if \(D' \notin \tilde{T}_{f,t,l} \text{ and } D'' \notin \tilde{T}_{f,t,l+1} \text{, then } (P_{T_{f,t,l}} h - P_{T_{f,t,l+1}} h)|_{\Omega_{D',D}} = 0.

Indeed, in this case minimal vertices of the trees \(D' \text{ and } D'' \) are equal and coincide with \(\xi_{t_\ast(n),j} \) for some \(j \in j_\ast(n)\). From (91) it follows that \( (P_{T_{f,t,l}} h - P_{T_{f,t,l+1}} h)|_{\Omega_{D}} = 0.

From (11) and (102) we get (103).

We claim that for any \(E' \in \tilde{T}_{f,t,l}, E'' \in \tilde{T}_{f,t,l+1}

\[
\text{card } \{E \in \tilde{T}_{f,t,l} : E \subset E'\} \leq 1, \quad \text{card } \{E \in \tilde{T}_{f,t,l} : E \subset E''\} \leq 1.
\]

(103)

Let us check the first inequality (the second one is proved similarly). Let \(E = \Omega_{D'}, D' \in \tilde{T}_{f,t,l}, E \in \tilde{T}_{f,t,l}, E \subset E'. \text{ Since } \tilde{T}_{f,t,l} \text{ is a partition, we have } E = \Omega_{D' \cap D' \cap \Omega_{D}}, \\
D'' \in \tilde{T}_{f,t,l+1}, V(D') \cap V(D'') \neq \emptyset. \text{ There exist trees } T' \in \tilde{T}_{f,t,l} \text{ and } T'' \in \tilde{T}_{f,t,l+1}
such that $\mathcal{D}'$ and $\mathcal{D}''$ are connected components of the graphs $\mathcal{T}' \cap \tilde{\Gamma}_{t,(n)}$ and $\mathcal{T}'' \cap \tilde{\Gamma}_{t,(n)}$, respectively. Observe that the connected component of the graph $\mathcal{T}'' \cap \tilde{\Gamma}_{t,(n)}$ whose vertex set intersects with $\mathbf{V}(\mathcal{D}')$ is unique. This together with (95) implies (103).

From (94), (103) and the definition of $\hat{T}_{f,t,l}^n$ we get that

$$\text{card } \hat{T}_{f,t,l}^n \lesssim 2^{-t} n_t.$$  \hfill (104)

For any $h \in B\hat{X}_p(\Omega)$ we have

$$\|(P_{T_{f,t,l}} h - P_{T_{f,t,l+1}} h) \chi_{\Omega(f,t)}\|_{p,q,T_{f,t,l}} \lesssim 3_0,$$

$$\lesssim \|(h - Q_{t,(n)} h - P_{T_{f,t,l}} h) \chi_{\tilde{\Gamma}_{t,(n)}}\|_{p,q,T_{f,t,l}} + \|(h - Q_{t,(n)} h - P_{T_{f,t,l+1}} h) \chi_{\tilde{\Gamma}_{t,(n)}}\|_{p,q,T_{f,t,l+1}} \lesssim 5, 103 \hspace{1cm} (5), 69, 100 \hspace{1cm} (11)$$

$$\lesssim \left( \sum_{(D, \xi) \in T_{f,t,l}} \|h - P_{\xi h}\|_{Y_q(\Omega_p)}^p \right)^{1/p} + \left( \sum_{(D, \xi) \in T_{f,t,l+1}} \|h - P_{\xi h}\|_{Y_q(\Omega_p)}^p \right)^{1/p} \lesssim 3_0,$$

$$\lesssim 2 \left( \frac{1}{4} - \frac{1}{p} \right) t_{*}(n) \lesssim 2 \left( \frac{1}{4} - \frac{1}{p} \right) t_{*}(n) \lesssim 3_0 \left( \log n \right)^{\frac{1}{4} - \frac{1}{p}};$$

i.e.,

$$\|(P_{T_{f,t,l}} h - P_{T_{f,t,l+1}} h) \chi_{\Omega(f,t)}\|_{p,q,T_{f,t,l}} \lesssim 3_0 \left( \log n \right)^{\frac{1}{4} - \frac{1}{p}}. \hspace{1cm} (105)$$

From (38), (101), (104), (105) and Lemma 5 we get that

$$e_{lt,l} (P_{T_{f,t,l}} - P_{T_{f,t,l+1}} : \hat{X}_p(\Omega) \rightarrow Y_q(\Omega(f,t))) \lesssim \lesssim 3_0 \left( \log n \right)^{\frac{1}{4} - \frac{1}{p}} e_{lt,l} (I_{st,l} : l_{st,l} \rightarrow l_{st,l}^{s_{t,l}}) =: : A_{st,l},$$

where $s_{t,l} \in \mathbb{N},$

$$s_{t,l} \leq C_* \cdot 2^{-l} n_t \leq C_* \cdot 2^{-l} \left[ n \cdot 2^{-l} \right], \quad C_* = C_*(3_0) \geq 1.$$  \hfill (106)

For each $t \leq t_{*}(n)$ we have

$$\frac{s_{t,l}}{k_{t,l}} \leq \frac{C_* \cdot 2^{-l} \left[ n \cdot 2^{-l} \right]}{n \cdot 2^{-l \cdot (l + t_{*}(n))}} \leq \sigma_{t,l}'.$$
The sequence \( \{\sigma'_{t,l}\}_{t \in \mathbb{Z}_+} \) decreases not slower than some geometric progression. This together with Theorem A implies that there exists \( \gamma_0 = \gamma_0(3_0) > 0 \) such that

\[
\sum_{t=t_s(n)}^{t_{s+1}(n)-1} \sum_{0 \leq l < \log n_t} A_{t,l} \lesssim_{3_0} (\log n) \frac{1}{p} \sum_{t=t_s(n)}^{t_{s+1}(n)-1} k_{t,l}^{-\frac{1}{p}} \cdot (\sigma'_{t,l})^{-\frac{1}{p}} \cdot 2^{\frac{k_{t,l}}{n_t} - \frac{k_{t,l}}{n_t}} \lesssim_{3_0} \sum_{t=t_s(n)}^{t_{s+1}(n)-1} A_{t,l} \lesssim_{3_0} n^{\frac{1}{p} - \frac{1}{f}}.
\]

i.e.,

\[
\sum_{t=t_s(n)}^{t_{s+1}(n)-1} \sum_{0 \leq l < \log n_t} A_{t,l} \lesssim_{3_0} n^{\frac{1}{p} - \frac{1}{f}}.
\]

From (99), (106) and (108) we get (91). This together with (90) yields (82). Applying (81), we have (75). Taking into account the estimate obtained at Step 5, we complete the proof of Lemma 3. \( \square \)

It remains to prove that

\[
e_n \left( \sum_{m=0}^{\infty} (\tilde{Q}_{n,m+1} - \tilde{Q}_{n,m}) : \hat{X}_p(\Omega) \to Y_q(\Omega) \right) \lesssim_{3_0} n^{\frac{1}{p} - \frac{1}{f}}. \tag{109}
\]

In [23] there were obtained order estimates for entropy numbers of diagonal operators with weights of logarithmic type. First we give some notations.

Denote by \( \Phi_0 \) the class of non-decreasing functions \( \varphi : [1, \infty) \to (0, \infty) \) that satisfy the following condition: there exist \( c > 0 \) and \( \alpha > 0 \) such that for any \( 1 \leq s \leq t < \infty \)

\[
\frac{\varphi(t)}{\varphi(s)} \leq c \left( \frac{1 + \log t}{1 + \log s} \right)^\alpha. \tag{110}
\]

We set \( w_\varphi(s) = 1 \) for \( 0 \leq s \leq 1 \) and \( w_\varphi(s) = \varphi(s) \) for \( s > 1 \).
Let $\delta > 0$, $w : \mathbb{R}_{+} \to \mathbb{R}_{+}$ be a continuous function, and let $0 < r, p \leq \infty$. For $x = (x_{m,k})_{m,k \in \mathbb{Z}_{+}}$ we set

$$
\|x\|_{L_r(2^{\delta m}l_p(w))} := \left( \sum_{m=0}^{\infty} 2^{m\delta r} \left( \sum_{k \in \mathbb{Z}_{+}} |x_{m,k}w(2^{-m}k)|^p \right)^{\frac{1}{p}} \right)^{\frac{1}{r}}
$$

(appropriately modified if $p = \infty$ or $r = \infty$). By $l_r(2^{\delta m}l_p(w))$ we denote the space of sequences $x$ such that $\|x\|_{L_r(2^{\delta m}l_p(w))} < \infty$.

**Theorem C.** [23, p. 11]. Let $0 < p < q \leq \infty$, $0 < r, s \leq \infty$, $\delta > 0$, $\varphi \in \Phi_0$, and let (110) hold with $\alpha = \frac{1}{p} - \frac{1}{q}$. Then

$$
e_n(\text{id} : l_r(2^{\delta m}l_p(w\varphi)) \to l_s(l_q)) \succ \frac{1}{p-q-s-\varphi(2^n)}.
$$

Let us prove (110).

Given $m \geq m_t$, we denote by $\hat{T}_{t,m}$ the partition of the set $G_t$ formed by $E' \cap E''$, $E' \in \hat{T}_{t,m}$, $E'' \in \hat{T}_{t,m+1}$ (the partitions $\hat{T}_{t,m}$ are defined at page 7). Let $s''_{t,m} = \dim \mathcal{S}_{\hat{T}_{t,m}}(\Omega)$. Then there exists $M = M(3_0) \geq 1$ such that

$$s''_{t,m} \leq M \cdot 2^{m-m_t} \cdot 2^{2\gamma t^t} \psi_*(2^{2t}). \quad (111)$$

Given $t > 0$, $m' > 0$, we set

$$s_{t,m'} = \lceil M \cdot 2^{m'} \cdot 2^{2\gamma^t} \psi_*(2^{2t}) \rceil, \quad s^*_{t,m'} = \sum_{l=0}^{t-1} \hat{s}_{l,m'}.
$$

Denote $\varphi(x) = (\log(2 + x))^{\frac{1}{p} - \frac{1}{q}}$, $x \geq 0$. Let $k = s^*_{t,m'} + j$, $1 \leq j \leq \hat{s}_{t,m'}$. Then

$$2^{\gamma t^t} \psi_*(2^{2t-1}) \leq 2^{-m'k} \leq 2^{\gamma t^t} \psi_*(2^{2t}), \quad t \geq 1,
$$

$$2^{-m'k} \leq 1 \text{ if } t = 0. \text{ Hence,}
$$

$$w_{\varphi}(2^{-m'k}) \leq 2^{(\frac{1}{p} - \frac{1}{q})t}, \quad k = s^*_{t,m'} + j, \quad 1 \leq j \leq \hat{s}_{t,m'}.
$$

From Lemma [5] it follows that there is an isomorphism $\overline{A}_{t,m} : \mathcal{S}_{\hat{T}_{t,m}}(\Omega) \to \mathbb{R}^{s''_{t,m}}$ such that

$$\|\overline{A}_{t,m}\|_{L_{p,q,\hat{T}_{t,m}}(G_t) \to s''_{t,m}} \leq \frac{1}{3_0}, \quad \|\overline{A}^{-1}_{t,m}\|_{L_{q,p}^{s''_{t,m}} \to Y_q(G_t)} \leq \frac{1}{3_0}. \quad (113)$$
Let us define the operator $\overline{A} : \hat{X}_p(\Omega) \to l_\infty(2^{\delta \ast} \ell_p(\varphi))$ as follows. Consider a function $f \in \hat{X}_p(\Omega)$. Then $(\hat{Q}_{n,m'}f - \hat{Q}_{n,m}f)|_{G_t} \in \mathcal{S}^{t,m+m'}_{t,m+m'}(\Omega)$. Let $\mathcal{A}_{t,m+m'}((\hat{Q}_{n,m'}f - \hat{Q}_{n,m}f)|_{G_t}) = (c_{m',t,j})_{j=1}^{s''_{t,m+m'}}$.

We set

$$(Af)_{m',s'_{t,m}+j} = \begin{cases} c_{m',t,j} & \text{for } 1 \leq j \leq s''_{t,m+m'}, \ t \geq t_0, \\ 0 & \text{for } s''_{t,m} + 1 \leq j \leq s_{t,m'} \text{ or } t < t_0. \end{cases}$$

Then

$$\|A\|_{\hat{X}_p(\Omega) \to l_\infty(2^{\delta \ast} \ell_p(\varphi))} = \sup_{m' \geq 0} 2^{\delta \ast} \left\{ \sum_{k \in \mathbb{Z}_+} \left| \varphi(2^{-m'} k)(Af)_{m',k} \right|^p \right\}^{1/p} \lesssim \frac{Z_0}{30},$$

$$\lesssim \sup_{m' \geq 0} 2^{\delta \ast} \left( \sum_{t \geq t_0} \frac{2^{(\frac{3}{2} - 1) t}}{\mathcal{S}^{t,m+m'}_{t,m} j=1} \left| c_{m',t,j} \right|^p \right)^{1/p} =: N.$$

From (22), (23) and (24) it follows that for $m \geq m_t$

$$\|P_{t,m} - P_{t,m+1}\|_{\hat{X}_p(G_t) \to Y_{q,\tilde{t},m}^{t,m}(G_t)} \lesssim \frac{Z_0}{30} 2^{-\delta \ast (m-m_t)} \cdot 2^{\left(\frac{3}{2} - 1\right) t} \|f\|_{X_p(G_t)}.$$  

Hence, by (32), (113) we get

$$\sum_{j=1}^{s''_{t,m+m'}} \left| c_{m',t,j} \right|^p \lesssim \frac{2^{-\delta \ast p m'} 2^{(\frac{3}{2} - 1) t}}{\mathcal{S}^{t,m+m'}_{t,m} j=1} \|f\|^p_{X_p(G_t)};$$

therefore,

$$N \lesssim \frac{Z_0}{30} \left( \sum_{t \geq t_0} \|f\|^p_{X_p(G_t)} \right)^{1/p} = \|f\|_{X_p(\Omega)}.$$

Thus,

$$\|\overline{A}\|_{\hat{X}_p(\Omega) \to l_\infty(2^{\delta \ast} \ell_p(\varphi))} \lesssim 1.$$  

Let us define the operator $\overline{K} : l_1(l_q) \to Y_q(\Omega)$ by formula

$$\overline{K}((c_{m',s'_{t,m}+j})_{m' \in \mathbb{Z}_+, t \in \mathbb{Z}_+, 1 \leq j \leq s_{t,m'}}) = \sum_{m' \in \mathbb{Z}_+} \sum_{t \geq t_0} \mathcal{A}^{-1}_{t,m+m'}((c_{m',s'_{t,m}+j})_{j=1}^{s''_{t,m+m'}}).$$
Since the sets $G_t$ do not overlap pairwise, we have
\[
\|K((c_{m',s_{m',j}+j})_{m'\in\mathbb{Z}_+,t\in\mathbb{Z}_+,1\leq j \leq \delta_{t,m'}})\|_{Y_q(\Omega)} \leq \sum_{m'=0}^{\infty} \left( \sum_{t=0}^{\infty} \|A_{t,m_t+m'}((c_{m',s_{m',j}+j})_{j=1}^{\delta_{m'+m',t}})\|^{q}_{Y_q(G_t)} \right)^{1/q} 
\]
\[
\leq \sum_{m'=0}^{\infty} \left( \sum_{t\in\mathbb{Z}_+} \sum_{j=1}^{\delta_{t,m}} |c_{m',s_{m',j}+j}|^{q} \right)^{1/q} = \|(c_{m',s_{m',j}+j})_{m'\in\mathbb{Z}_+,t\in\mathbb{Z}_+,1\leq j \leq \delta_{t,m'}}\|_{l_1(l_q)}.
\]
Hence,
\[
\|K\|_{l_1(l_q)\rightarrow Y_q(\Omega)} \leq 1.
\] (115)

Let $\text{id} : l_\infty(2^{\delta_{m'}}l_p(w_\varphi)) \rightarrow l_1(l_q)$ be the identity operator. Then
\[
e_n \left( \sum_{m=0}^{\infty} (\tilde{Q}_{n,m+1} - \tilde{Q}_{n,m}) : \hat{X}_p(\Omega) \rightarrow Y_q(\Omega) \right) = e_n(K \circ \text{id} \circ \Delta : \hat{X}_p(\Omega) \rightarrow Y_q(\Omega)) \lesssim \frac{38, 114, 115}{30}
\]
\[
\lesssim e_n(\text{id} : l_\infty(2^{\delta_{m'}}l_p(w_\varphi)) \rightarrow l_1(l_q)) =: E_n.
\]
From Theorem 1 we get $E_n \lesssim n^{-\frac{1}{q} - \frac{1}{p}}$.

Applying Lemmas 1, 2, 3 together with (30), (33), (34), (109), we complete the proof of Theorem 1.

Remark 1. Suppose that instead of Assumption 2 the following condition holds: for any $\xi \in V(\mathcal{F})$ the set $\hat{F}(\xi)$ is the atom of $\mu$. Then the assertion of Theorem 7 holds as well.

3 Some particular cases

Let $(T, \xi_0)$ be a tree, and let $u, w : V(T) \rightarrow \mathbb{R}_+$. We define the summation operator $S_{u,w,T}$ by
\[
S_{u,w,T}f(\xi) = w(\xi) \sum_{\xi' \leq \xi} u(\xi') f(\xi'), \quad \xi \in V(T), \quad f : V(T) \rightarrow \mathbb{R}.
\]
Let $1 \leq p, q \leq \infty$. By $G^{p,q}_{T,u,w}$, we denote the minimal constant $C$ in the inequality
\[
\left( \sum_{\xi \in V(T)} w^q(\xi) \left| \sum_{\xi' \leq \xi} u(\xi') f(\xi') \right|^q \right)^{1/q} \leq C \left( \sum_{\xi \in V(T)} |f(\xi)|^p \right)^{1/p}, \quad f : V(T) \rightarrow \mathbb{R}
\]
There exist functions defined at the page 3. Assumptions 1–3 will be replaced by the following conditions.

\[ t \cup \Theta \text{ partition } f \text{ any function } \]

There exist numbers \( p \) and for any \( n \in \mathbb{Z}_+ \) there exists a number \( \kappa > 3 \). For any \( \tau \) and an absolutely continuous function \( \xi \) such that for any vertex \( \xi \in \mathcal{V}(A) \) and for any function \( f \in X_p(\Omega) \)

\[
\|f - P_\xi.f\|_{Y_q(\xi)} \leq c_2 w(\xi) \sum_{\xi' \leq \xi} u(\xi') \|f\|_{X_p(\hat{\xi}'(\xi'))}.
\]

**Assumption A.** There exist functions \( u, w : \mathcal{V}(A) \to (0, \infty) \) and a constant \( c_2 \geq 1 \) with the following property: for any vertex \( \xi_* \in \mathcal{V}(A) \) there exists a linear continuous projection \( P_{\xi_*} : Y_q(\Omega) \to \mathcal{P}(\Omega) \) such that for any vertex \( \xi \geq \xi_* \) and for any function \( f \in X_p(\Omega) \)

\[
\|f - P_{\xi_*}f\|_{Y_q(\hat{\xi}(\xi))} \leq c_2 w(\xi) \sum_{\xi \leq \xi' \leq \xi} u(\xi') \|f\|_{X_p(\hat{\xi}'(\xi'))}.
\]

**Assumption B.** There exists a number \( \delta_* > 0 \) such that for any vertex \( \xi \in \mathcal{V}(A) \) and for any \( n \in \mathbb{N}, m \in \mathbb{Z}_+ \) there exists a partition \( T_{m,n}(G) \) of the set \( G = \hat{\xi}(\xi) \) with the following properties:

1. \( \text{card} \ T_{m,n}(G) \leq c_2 \cdot 2^m n. \)
2. For any \( E \in T_{m,n}(G) \) there exists a linear continuous operator \( P_E : Y_q(\Omega) \to \mathcal{P}(E) \) such that for any function \( f \in X_p(\Omega) \)

\[
\|f - P_E f\|_{Y_q(E)} \leq c_2(2^m n)^{-\delta_*} u(\xi) w(\xi) \|f\|_{X_p(E)}.
\]

3. For any \( E \in T_{m,n}(G) \)

\[
\text{card} \{ E' \in T_{m\pm n}(G) : \text{mes}(E \cap E') > 0 \} \leq c_2.
\]

Let \( \mathcal{V}(A) = \{ \eta_{j,i} \}_{j,j_{\text{min}},i \in \tilde{I}_j} \), where \( j_{\text{min}} \geq 0, \tilde{I}_{j_{\text{min}}} = \{ 1 \} \). Suppose that \( \eta_{j_{\text{min}},1} \) is the minimal vertex of \( A \) and \( \mathcal{V}_{j-j_{\text{min}}}^A(\eta_{j_{\text{min}},1}) = \{ \eta_{j,i} \}_{i \in \tilde{I}_j} \) for any \( j \geq j_{\text{min}} \).

**Assumption C.** There exist numbers \( \theta > 0, \gamma \in \mathbb{R}, \kappa \geq \frac{\theta}{q}, \alpha_u, \alpha_w \in \mathbb{R}, c_3 \geq 1, m_* \in \mathbb{N} \) and an absolutely continuous function \( \tau : (0, \infty) \to (0, \infty) \) such that \( \lim_{t \to +\infty} \frac{\tau^\gamma(t)}{\tau(t)} = 0 \) and the following conditions hold.

1. If \( \kappa = \frac{\theta}{q} \), then \( \alpha_w > \frac{1-\gamma}{q} \).
2. If \( \kappa > \frac{\theta}{q} \), then \( \alpha_u + \alpha_w = \frac{1}{p} - \frac{1}{q} \); if \( \kappa = \frac{\theta}{q} \), then \( \alpha_u + \alpha_w = \frac{1}{p} \).
3. For any \( j' \geq j \geq j_{\text{min}} \) and for any vertex \( \xi \in \mathcal{V}_{j-j_{\text{min}}}^A(\eta_{j_{\text{min}},1}) \)

\[
\text{card} \ \mathcal{V}_{j'-j}(\xi) \leq c_3 \cdot 2^{\theta m_*(j'-j)} \frac{j^\gamma \tau(m_* j)}{j^\gamma \tau(m_* j')}.
\]
4. For any \( j \geq j_{\min} \), \( i \in \hat{I}_j \)
\[
u(\eta_{j,i}) = u_j = 2^{\theta m^*_s(j)} u^a, \quad w(\eta_{j,i}) = w_j = 2^{-\kappa m^*_s(j)} w^a. \tag{120}
\]

5. Let \( \kappa = \frac{\theta}{q} \). Then there exists a tree \( \hat{A} \) with the minimal vertex \( \hat{\xi}_0 \), which satisfies the following conditions.
(a) For any \( j' \geq j \geq j_{\min} \) and for any vertex \( \xi \in V^A_{j-j_{\min}}(\hat{\xi}_0) \)
\[
c_3^{-1} \cdot 2^{\theta m_s(j' - j) \frac{j}{j^*} m^*_s(j') \frac{j^*}{j^* \tau(m^*_s(j'))}} \leq \text{card } V^A_{j-j_{\min}}(\xi) \leq c_3 \cdot 2^{\theta m_s(j' - j) \frac{j}{j^*} m^*_s(j') \frac{j^*}{j^* \tau(m^*_s(j'))}}. \tag{121}
\]
(b) Let \( \{\overline{\pi}_j\}_{j \geq j_{\min}} \subset (0, \infty) \), \( \{\overline{\pi}_j\}_{j \geq j_{\min}} \subset (0, \infty) \) be arbitrary sequences. Define the functions \( \hat{\upsilon}, \hat{w} : V(A) \to (0, \infty) \), \( \hat{\upsilon}, \hat{w} : V(A) \to (0, \infty) \) by
\[
\hat{\upsilon}|_{V^A_{j-j_{\min}}(\eta_{j_{\min},1})} \equiv \overline{\pi}_j, \quad \hat{\upsilon}|_{V^A_{j-j_{\min}}(\hat{\xi}_0)} \equiv \overline{\pi}_j, \\
\hat{w}|_{V^A_{j-j_{\min}}(\eta_{j_{\min},1})} \equiv \overline{\pi}_j, \quad \hat{w}|_{V^A_{j-j_{\min}}(\hat{\xi}_0)} \equiv \overline{\pi}_j, \quad j \geq j_{\min}.
\]
Then for each \( N \geq j \geq j_{\min} \), \( i \in \hat{I}_j \) there exists a vertex \( \hat{\xi} \in V^A_{j-j_{\min}}(\hat{\xi}_0) \) such that
\[
\mathcal{G}_{[A_{0,j}]}^{p,q} \leq c_3 \mathcal{G}_{[A_{\xi}]}^{p,q} \leq c_3 \mathcal{G}_{[A_{\xi}]}^{p,q}.
\]
Denote \( Z_0 = (p, q, c_1, c_2, c_3, \theta, \gamma, \kappa, \alpha_u, \alpha_w, m_s, \delta_s, \tau) \).
Let \( \xi \in V(A) \), and let \( D \subset A \) be a subtree with the minimal vertex \( \xi \). Then from Assumption [A] it follows that
\[
\|f - P_\xi f\|_{V_p(\omega)} \leq c_2 \mathcal{G}_{D,u,w}^{p,q} \|f\|_{X_p(\omega)}.
\tag{122}
\]
From Theorem F in [36], Theorem 3.6 in [38] and Assumption [C] it follows that if \( \xi \in V^A_{j-j_{\min}}(\hat{\xi}_0) \), then in the case \( \kappa > \frac{\theta}{q} \)
\[
\mathcal{G}_{D,u,w}^{p,q} \leq \sup_{s \geq j} u_s w_s \leq (m^*_s j)^{-\alpha_u - \alpha_w} = (m^*_s j)^{\frac{1}{q} - \frac{1}{p}}, \tag{123}
\]
and in the case \( \kappa = \frac{\theta}{q} \)
\[
\mathcal{G}_{D,u,w}^{p,q} \leq \sup_{s \geq j} \left( \sum_{i = j}^s (m^*_s j)^{-\alpha_u} \cdot 2^{\theta' m^*_s j} \right)^{\frac{1}{p}} \left( \sum_{i < s} (m^*_s j)^{-\alpha_u} \cdot 2^{\theta m^*_s j} \right)^{\frac{1}{q}} \right)^{\frac{1}{q}} \leq (m^*_s j)^{-\alpha_u - \alpha_w + \frac{1}{q}} = (m^*_s j)^{\frac{1}{q} - \frac{1}{p}}. \tag{124}
\]
Notice that the proof of (124) depends on condition 5 of Assumption [C].
From Assumption [3] and conditions 2, 4 of Assumption [4] it follows that if \( \xi \in V_j^{A}(\zeta_0) \), \( G = F(\xi) \), \( E \in T_{m,n}(G) \), then

\[
\|f - P_{E}f\|_{Y_q(E)} \leq c_2 \cdot (2^{m_0}n)^{-\delta} (m_0) - \alpha_u - \alpha_w \|f\|_{X_p(E)} \leq c_2 \cdot (2^{m_0}n)^{-\delta} (m_0) \frac{1}{2 - \epsilon} \|f\|_{X_p(E)}. \tag{125}
\]

\( \text{We construct the partition} \{A_{t,i}\}_{t \geq t_0, i \in \hat{I}_t} \text{as follows.} \)

\[
t_0 = \min \{t \in \mathbb{Z}_+ : 2^t > m_* j_{\min} \}.
\]

Given \( t \geq t_0 \), we denote by \( \Gamma_t \) the maximal subgraph in \( A \) on the vertex set

\[
V(\Gamma_t) = \{\eta_{j,s} : 2^{t-1} \leq m_* j < 2^t, s \in \hat{I}_j\}, \tag{126}
\]

and by \( A_{t,i}, i \in \hat{I}_t \), the connected components of the graph \( \Gamma_t \). By \( \hat{\xi}_{t,i} \) we denote the minimal vertex of the tree \( A_{t,i} \). Then

\[
\text{card } V(\Gamma_t) \lesssim 2^{2^t} 2^{-\gamma t} (2^t)^{-1} (2^t). \tag{127}
\]

Thus, Assumption [3] holds with \( \nabla_t = 2^{2^t} 2^{-\gamma t} (2^t) \). From (125) we obtain that Assumption [2] holds.

Recall the notations \( J_{t,D} = \{i \in \hat{I}_t : V(A_{t,i}) \cap V(D) \neq \emptyset\} \), \( D_{t,i} = D \cap A_{t,i} \), where \( D \subset A \) is a subtree.

In order to obtain Assumption [1], it is sufficient to prove the following assertion.

**Lemma 7.** Let \( D \) be a subtree in \( A \) rooted at \( \xi_* \). Then

\[
\|f - P_{\xi_*}f\|_{Y_q(\Omega_D)} \leq \sum_{t = t_0}^{\infty} \sum_{i \in J_{t,D}} 2^{(1 - \frac{\kappa}{q})t} \|f\|_{X_p(\Omega_{D_{t,i}})}. \tag{128}
\]

**Proof.** By (126),

\[
\|f - P_{\xi_*}f\|_{Y_q(\Omega_D)}^q \leq \sum_{\xi \in \mathcal{V}(D)} w^q(\xi) \left( \sum_{\xi_* \leq \xi} u(\xi) \|f\|_{X_p(\hat{F}(\xi))}^q \right). \]

Let \( \kappa > \frac{q}{2} \). Then \( \alpha_u + \alpha_w = \frac{1}{p} - \frac{1}{q} \) (see condition 2 of Assumption [4]). Repeating the proof of Lemma 5.1 in [11], we get that

\[
\|f - P_{\xi_*}f\|_{Y_q(\Omega_D)}^q \leq \sum_{\xi \in \mathcal{V}(D)} w^q(\xi) u^q(\xi) \|f\|_{X_p(\hat{F}(\xi))}^q. \]

If \( \xi = \eta_{j,i}, 2^{t-1} \leq m_* j < 2^t \), then \( u^q(\xi) w^q(\xi) \leq (m_* j)^{-\frac{\kappa}{p} + 1} \lesssim 2^{(1 - \frac{\kappa}{p})t} \). This together with the condition \( p < q \) implies (128).
Now we consider the case $\kappa = \frac{p}{q}$.

Let $\xi_s \in V(A_{t,j0})$, and let $\xi \in V(D_{t+l,j1})$. Then there exists a sequence $\{\xi_{t+l+1,j1}\}_{s=1}^{l}$ such that $\xi_s < \xi_{t+l+1,j1} < \xi_{t+l+2,j2} < \cdots < \xi_{t+l+j_l}$. Denote

$$\xi_{t,j0} = \xi_s, \quad \xi_{t+s,j_s} = \xi_{t+s,j_s}, 1 \leq s \leq l. \quad (129)$$

We have

$$\sum_{\xi_s \leq \xi' \leq \xi} u(\xi') f(\xi') = \sum_{s=0}^{l-1} \sum_{\xi_{t+s,j_s} \leq \xi' < \xi_{t+s+1,j_s+1}} u(\xi') \| f \|_{X_p(\hat{F}(\xi'))} + \sum_{\xi_{t+l,j_l} \leq \xi' \leq \xi} u(\xi') \| f \|_{X_p(\hat{F}(\xi'))}. \quad (130)$$

Let $\bar{a}_w = \alpha_w - \frac{1}{q}$. By condition 2 of Assumption $\mathbb{C}$

$$\alpha_u + \bar{a}_w = \frac{1}{p} - \frac{1}{q}, \quad (131)$$

We have

$$\sum_{\xi \in V(D_{t+l,j1})} w^q(\xi) \lesssim \frac{119}{30}, \frac{120}{30}, \frac{126}{30}$$

$$\lesssim \sum_{2^{l+1} \leq m_s, j \leq 2^{l+1}} 2^{-\theta m_s, j} (m_s, j)^{-\alpha_w q} \cdot 2^{\theta (m_s, j - 2^{l+1})} \left( \frac{2^{l+1}}{m_s, j} \right)^{\gamma} \left( \frac{\tau(2^{l+1})}{\tau(m_s, j)} \right) \lesssim \frac{11}{30}$$

$$\lesssim 2^{-\kappa q 2^{l+1}} \cdot 2^{l+1} \cdot \alpha_w q \cdot 2^{-\kappa q 2^{l+1}} \cdot 2^{l+1} \cdot \bar{a}_w q,$$

which implies

$$\| f - P_\xi f \|_{Y_q(P_{D_{t+l,j1}})}^q \lesssim \frac{113}{30} \sum_{\xi \in V(D_{t+l,j1})} w^q(\xi) \left( \sum_{\xi_s \leq \xi' \leq \xi} u(\xi') f(\xi') \right)^q \lesssim \frac{119}{30} \sum_{\xi \in V(D_{t+l,j1})} w^q(\xi) \left( \sum_{s=0}^{l-1} \sum_{\xi_{t+s,j_s} \leq \xi' < \xi_{t+s+1,j_s+1}} u(\xi') \| f \|_{X_p(\hat{F}(\xi'))} \right) +$$

$$\sum_{\xi \in V(D_{t+l,j1})} w^q(\xi) \left( \sum_{\xi_{t+l,j_l} \leq \xi' \leq \xi} u(\xi') \| f \|_{X_p(\hat{F}(\xi'))} \right) \lesssim \frac{113}{30} \sum_{\xi \in V(D_{t+l,j1})} w^q(\xi) \left( \sum_{s=0}^{l-1} \sum_{\xi_{t+s,j_s} \leq \xi' < \xi_{t+s+1,j_s+1}} u(\xi') \| f \|_{X_p(\hat{F}(\xi'))} \right)$$

$$\lesssim 2^{-q \kappa 2^{l+1}} \cdot 2^{-q \bar{a}_w (l+1)} \left( \sum_{s=0}^{l-1} \sum_{\xi_{t+s,j_s} \leq \xi' < \xi_{t+s+1,j_s+1}} u(\xi') \| f \|_{X_p(\hat{F}(\xi'))} \right) +$$

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i.e.,

\[
\|f - P_\xi f\|_{Y_q(\Omega_{\tilde{D}_{t+1,i}})} \lesssim 2^{(1-\frac{q}{2})(i+l)}\|f\|_{X^q_{p}(\Omega_{\tilde{D}_{t+1,i}})} + 2^{-q_\omega(i+l)} \sum_{s=0}^{l-1} \sum_{\tilde{x}_{t+1,i,s} \leq \tilde{x}' < \tilde{x}_{t+1,i,s+1}} u(\xi') \|f\|_{X^q_{p}(\hat{F}(\xi'))}^q.
\]  

(132)

Denote

\[
\{\zeta_{t,i} : i \in I_t'\} = \{\zeta \in V_{\max}(\mathcal{D} \cap \Gamma_t) : V_1^\mathcal{D}(\zeta) \neq \emptyset\}.
\]  

(133)

By (126),

\[
\forall i \in I_t' \exists s \in \tilde{I}_{[2^l/m_\ast]-1} : \zeta_{t,i} = \eta[2^l/m_\ast]_{-1,s}.
\]  

(134)

Let us define the tree $\tilde{D}$ with vertex set $\{\xi_s\} \cup (\bigcup_{t \geq i} \{\zeta_{t,i} : i \in I_t'\})$. The partial order on $V(\tilde{D})$ is defined as follows. We set

\[
V_1^\mathcal{D}(\zeta_{t,i}) = \{\zeta_{t+1,i} : \zeta_{t+1,i} \in \mathcal{D}_{t,i}\}.
\]

If $\xi_s \notin \{\zeta_{t,i} : i \in I_t'\}$, then we set $V_1^\mathcal{D}(\xi_s) = \{\zeta_{t,i} : i \in I_t\}$.

Further, we define the functions $\varphi, \hat{u}, \hat{w} : V(\tilde{D}) \rightarrow \mathbb{R}_+$. We set

\[
\varphi(\zeta_{t,i}) = \sum_{\xi' \in V(\Gamma_t) \cap V(\tilde{D}), \xi' \leq \zeta_{t,i}} \frac{u(\xi')}{u(\zeta_{t,i})} \|f\|_{X^q_{p}(\hat{F}(\xi'))}^q;
\]  

(135)

\[
\hat{w}(\zeta_{t,i}) = 2^{-q_\omega t} \cdot 2^{-\hat{a}_w t}, \quad \hat{u}(\zeta_{t,i}) = 2^{q_\omega t} \cdot 2^{-\alpha_w t} u(\zeta_{t,i}).
\]  

(136)

If $\xi_s \notin \{\zeta_{t,i} : i \in I_t'\}$, then we set $\varphi(\xi_s) = \hat{w}(\xi_s) = \hat{u}(\xi_s) = 0$.

Let $t \geq \hat{t}, j \in J_{t,D}$. If $t > \hat{t}$, then $\hat{\zeta}_{t,j} \in V(\tilde{D})$. We take $i \in I_{t-1}'$ such that $\hat{\zeta}_{t,j} \in V_1^\mathcal{D}(\zeta_{t-1,i})$. Then

\[
\|f - P_\xi f\|_{Y_q(\Omega_{D_{t,i}})} \lesssim 2^{(1-\frac{q}{2})t} \|f\|_{X^q_{p}(\Omega_{D_{t,i}})} + \hat{w}^q(\zeta_{t-1,i}) \left( \sum_{\xi' \in V(\tilde{D}), \xi' \leq \zeta_{t,i}} \hat{u}(\xi') \varphi(\xi') \right)^q.
\]  

(137)

Notice that card $V_1^\mathcal{D}(\zeta_{t,i}) \lesssim 1$. Summing (137) over all $t \geq \hat{t}, j \in J_{t,D}$, we obtain

\[
\|f - P_\xi f\|_{Y_q(\Omega_{\tilde{D}_{t,i}})} \lesssim \sum_{t=\hat{t}}^\infty 2^{(1-\frac{q}{2})t} \sum_{j \in J_{t,D}} \|f\|_{X^q_{p}(\Omega_{D_{t,i}})} + \sum_{t=\hat{t}}^\infty \sum_{i \in I_t'} \hat{w}^q(\zeta_{t,i}) \left( \sum_{\xi' \in V(\tilde{D}), \xi' \leq \zeta_{t,i}} \hat{u}(\xi') \varphi(\xi') \right)^q.
\]  

(138)
Let us estimate the second summand (we denote it by $L$).

Let $\hat{u}(\zeta) = \hat{u}_1(\zeta)\hat{u}_2(\zeta)$, $\hat{u}_1(\zeta_{t,i}) = 2^t$, where $\varepsilon = \varepsilon(30) > 0$ is sufficiently small (it will be chosen later); if $\xi_* \notin \{\zeta_{t,i} : i \in I_t\}$, then we set $\hat{u}_1(\xi_*) = \hat{u}_2(\xi_*) = 0$. By Hölder’s inequality,

$$L = \sum_{\zeta \in \mathbf{V}(\partial)} \hat{w}^q(\zeta) \left( \sum_{\zeta' \leq \zeta} \hat{u}_1(\zeta')\hat{u}_2(\zeta')\varphi(\zeta') \right)^q \leq$$

$$\leq \sum_{\zeta \in \mathbf{V}(\partial)} \hat{w}^q(\zeta) \left( \sum_{\zeta' \leq \zeta} \hat{u}_1^q(\zeta') \right)^{\frac{q}{2}} \sum_{\zeta' \leq \zeta} \hat{u}_2^q(\zeta')\varphi^q(\zeta') \lesssim \frac{1}{30}$$

$$\lesssim \sum_{\zeta \in \mathbf{V}(\partial)} \hat{w}^q(\zeta)\hat{u}_1^q(\zeta) \sum_{\zeta' \leq \zeta} \hat{u}_2^q(\zeta')\varphi(\zeta') = \sum_{\zeta' \in \mathbf{V}(\partial)} \hat{u}_2^q(\zeta')\varphi(\zeta') \sum_{\zeta \geq \zeta'} \hat{w}^q(\zeta)\hat{u}_1^q(\zeta) =: M.$$ 

By (134), there exists $s \in \tilde{I}_{[2^t/m_*] - 1}$ such that

$$\text{card } \mathbf{V}_t(\zeta_{t,i}) = \text{card } \{i' \in I_{t+1} : \zeta_{t+1,i'} \geq \zeta_{t,i}\} \leq$$

$$\leq \text{card } \mathbf{V}^A_{[2^t/m_*]-[2^t/m_*]}(\eta[2^t/m_*]-1,s) \lesssim \frac{2 \theta^{2^t+1}2^{-\gamma(t+1)} \tau^{-1}(2^t)}{2 \theta^{2^t}2^{-\gamma(t+1)}(2^t)}.$$ 

This together with relations $\kappa = \frac{\theta}{q}$, $\alpha_w > \frac{1}{2} - \frac{t}{q}$ (see condition 1 of Assumption C) yields that for sufficiently small $\varepsilon > 0$

$$\sum_{\zeta \geq \zeta_{t,i}} \hat{w}^q(\zeta)\hat{u}_1^q(\zeta) \lesssim \frac{1}{30} \sum_{l \geq 0} 2^{-q\kappa \cdot 2^{l-t}} \cdot 2^{-q\alpha_w(t+l)} \cdot 2^{q(t+l)} \frac{2 \theta^{2^t+1}2^{-\gamma(t+1)} \tau^{-1}(2^t)}{2 \theta^{2^t}2^{-\gamma(t+1)}(2^t)} =$$

$$= 2^{-\theta \cdot 2^t} \cdot 2^{\gamma t} \sum_{l \geq 0} 2^{-q\alpha_w(t+l)} \frac{\tau(2^t)}{\tau(2^{l-t})} \lesssim \frac{1}{30} 2^{-q\alpha_w \cdot 2^t} \cdot 2^{-q(\alpha_w - \varepsilon)t} \equiv \hat{w}^q(\zeta_{t,i})\hat{u}_1^q(\zeta_{t,i}).$$ 

Thus,

$$M \lesssim \sum_{\zeta' \in \mathbf{V}(\partial)} \hat{u}_2^q(\zeta')\varphi(\zeta') = \sum_{\zeta' \in \mathbf{V}(\partial)} \hat{u}_2^q(\zeta')\varphi(\zeta') \equiv 136 \equiv 131 \equiv 136$$

$$= \sum_{t=i}^{\infty} \sum_{j \in J_{t,i}} 2^{(1 - \frac{t}{q})j} \sum_{i \in I_t : \zeta_{t,i} \in \mathbf{V}(\partial_{t,i})} \varphi(\zeta_{t,i});$$

i.e.,

$$\sum_{t=i}^{\infty} \sum_{i \in I_t'} \hat{w}^q(\zeta_{t,i}) \left( \sum_{\zeta' \in \mathbf{V}(\partial), \zeta' \leq \zeta_{t,i}} \hat{u}(\zeta')\varphi(\zeta') \right)^q \lesssim \sum_{t=i}^{\infty} \sum_{j \in J_{t,i}} 2^{(1 - \frac{t}{q})j} \sum_{i \in I_t : \zeta_{t,i} \in \mathbf{V}(\partial_{t,i})} \varphi(\zeta_{t,i}).$$
It remains to prove that for any $t \geq \hat{t}$, $j \in J_{t,D}$

$$
\sum_{i \in I_{t}': \zeta_{i,t} \in \mathcal{V}(D_{t,j})} \varphi^{q}(\zeta_{t,i}) \lesssim \|f\|^{q}_{X_{p}(\Omega_{D_{t,j}})}.
$$

(140)

Then (138), (139), (140) imply (128).

By (129) and (135),

$$
\sum_{i \in I_{t}': \zeta_{i,t} \in \mathcal{V}(D_{t,j})} \varphi^{q}(\zeta_{t,i}) = \sum_{i \in I_{t}': \zeta_{i,t} \in \mathcal{V}(D_{t,j})} \left( \sum_{\hat{\zeta}_{t,i} \leq \zeta} \frac{u(\zeta')}{u(\hat{\zeta}_{t,i})} \|f\|_{X_{p}(\hat{f}(\zeta'))} \right)^{q}
$$

$$
= \sum_{i \in I_{t}': \zeta_{i,t} \in \mathcal{V}(D_{t,j})} u^{-q}(\zeta_{t,i}) \left( \sum_{\hat{\zeta}_{t,i} \leq \zeta} \frac{u(\zeta')}{u(\hat{\zeta}_{t,i})} \|f\|_{X_{p}(\hat{f}(\zeta'))} \right)^{q} =: S.
$$

From (133) it follows that $\zeta_{t,i} \in \mathcal{V}_{\max}(A_{t,j})$.

We denote $A'_{t,j} = (A_{t,j})_{\hat{t}}$. Then $\mathcal{V}(D_{t,j}) \subset \mathcal{V}(A'_{t,j})$. Given $\sigma > 0$, we set

$$
w(\sigma)(\xi) = \begin{cases} 
  u^{-1}(\xi) & \text{for } \xi \in \mathcal{V}_{\max}(A'_{t,j}), \\
  \sigma & \text{for } \xi \in \mathcal{V}(A'_{t,j}) \setminus \mathcal{V}_{\max}(A'_{t,j}).
\end{cases}
$$

Then

$$S \lesssim \left( \mathcal{G}_{A'_{t,j},u,w(\sigma)}^{p,q} \right)^{q} \|f\|_{X_{p}(\Omega_{D_{t,j}})}.
$$

We claim that for sufficiently small $\sigma > 0$ the estimate $\mathcal{G}_{A'_{t,j},u,w(\sigma)}^{p,q} \lesssim 1$ holds. This implies (140).

Let $\zeta_{t,j} \in \mathcal{V}_{s_{t}j_{\min}}(\eta_{j_{\min},1})$. From (126) and (129) it follows that

$$2^{t-1} \leq m_{s} < 2^{t}.
$$

(141)

Moreover, $A'_{t,j} = [A_{t,j}]_{\Xi}^{c} \in [2^{t}/m_{s}] - 1 - s_{t}$ (since $\hat{\xi}_{t+1,i} \in \mathcal{V}_{[2^{t}/m_{s}]_{j_{\min}}}(\eta_{j_{\min},1})$, $i \in \hat{J}_{t}$). By condition 5b of Assumption C there exists a vertex $\zeta \in \mathcal{V}_{s_{t}j_{\min}}(\hat{\zeta}_{0})$ such that for the tree $\hat{A}_{t,j} := [\hat{A}_{t,j}]_{\Xi}^{c} \in [2^{t}/m_{s}] - 1 - s_{t}$ and for the functions $\hat{u}, \hat{w}(\sigma) : \mathcal{V}(\hat{A}_{t,j}) \rightarrow (0, \infty)$ defined by

$$
\hat{u}(\xi) = u_{s}, \quad \xi \in \mathcal{V}_{s_{t}j_{s}}(\hat{\zeta}_{0}),
$$

$$
\hat{w}(\xi) = \begin{cases} 
  u_{s}^{-1}[2^{t}/m_{s}]-1, & \xi \in \mathcal{V}_{[2^{t}/m_{s}]-1-s_{t}}(\hat{\zeta}_{0}), \\
  \sigma, & \xi \in \mathcal{V}_{s_{t}j_{s}}(\hat{\zeta}_{0}), \quad s < [2^{t}/m_{s}] - 1,
\end{cases}
$$

(142)

the inequality $\mathcal{G}_{A'_{t,j},u,w(\sigma)}^{p,q} \lesssim \mathcal{G}_{A'_{t,j},\hat{u},\hat{w}(\sigma)}^{p,q}$ holds.

From (121) it follows that we can apply Theorem 3.6 in (38) and estimate $\mathcal{G}_{A'_{t,j},\hat{u},\hat{w}(\sigma)}^{p,q}$ from above. For sufficiently small $\sigma = \sigma(3_{0}, t) > 0$ we get

$$\mathcal{G}_{A'_{t,j},\hat{u},\hat{w}(\sigma)}^{p,q} \lesssim \sup_{s_{t} \leq s < [2^{t}/m_{s}]} \left( \sum_{i=s_{t}}^{s} u_{i}^{p'} \right)^{1/p'} \times
$$

$$
\mathcal{G}_{A'_{t,j},\hat{u},\hat{w}(\sigma)}^{p,q} \lesssim \mathcal{G}_{A'_{t,j},\hat{u},\hat{w}(\sigma)}^{p,q} \lesssim \mathcal{G}_{A'_{t,j},\hat{u},\hat{w}(\sigma)}^{p,q} \lesssim 1.
$$

(34)
Since $\theta = qk$, we get the following assertion.

This completes the proof.

Thus, Assumptions \[1\] \[2\] \[3\] hold, and by Theorem \[1\] we get \[12\].

Now we suppose that instead of Assumption \[C\] the following condition holds.

Assumption D. There exist numbers $m_\ast \in \mathbb{N}$, $\kappa > 0$, $\gamma \leq 0$, $\nu \in \mathbb{R}$, $\alpha_\ast \in \mathbb{R}$, $\lambda_\ast \in \mathbb{R}$, $\lambda_\omega \in \mathbb{R}$ such that $\lambda_\ast + \lambda_\omega = \frac{1}{p} - \frac{1}{q}$ and the following assertions hold.

1. For any $j' \geq j \geq j_{\min}$ and for any vertex $\xi \in V_{j-j_{\min}}^A(\xi_0)$

   $$\text{card } V_{j-j}^A(\xi) \leq \alpha_3 \frac{(m_s j)^\gamma |\log(m_s j)|^\nu}{(m_s j')^\gamma |\log(m_s j')|^\nu}.$$  

2. For any $j \geq j_{\min}$, $i \in \tilde {I}_j$

   $$u(\eta_{j,i}) = u_j = 2^{-m_s j} (m_s j)^{\alpha_\ast} |\log(m_s j)|^{-\lambda_\ast},$$
   $$w(\eta_{j,i}) = w_j = 2^{-m_s j} (m_s j)^{\alpha_\ast} |\log(m_s j)|^{-\lambda_\omega}.$$  

The partition \{\(A_{t,i}\)\}_{t \geq t_0, i \in I_t} is defined as follows. Let

$$t_0 = \min \{ t \in \mathbb{Z}_+ : 2^t > m_s j_{\min} \}.$$  

Given $t \geq t_0$, we denote by $\Gamma_t$ the maximal subgraph in $A$ on the vertex set \{\(\eta_{j,s} : 2^{t-1} \leq m_s j < 2^t, s \in \tilde {I}_j\)\}, and by $A_{t,i}$, $i \in \tilde {I}_t$, the connected components of $\Gamma_t$.

Then

$$\text{card } V(\Gamma_t) \leq 2^{(1-\gamma)2^t-\nu}.$$  

Repeating the proof of Lemma 5.1 in \[41\] and taking into account that $p < q$, we get the following assertion.

Lemma 8. Let $D$ be a subtree in $A$, and let $\xi_\ast$ be its minimal vertex. Then

$$\|f-P_\xi f\|_{Y_p(D)} \leq \sum_{t=t_0}^{\infty} 2^{(1-\frac{q}{p})t} \sum_{i \in I_{t,D}} \|f\|_{X_p(\Omega_{\xi,i})}^q.$$  

From Assumptions B (145) and (146) we obtain Assumptions 1, 2, 3.

Remark 2. Suppose that Assumption B is replaced by the following condition: for any \( \xi \in \mathcal{V}(A) \) the set \( \hat{F}(\xi) \) is the atom of \( \text{mes} \). Then the assertion of Theorem 1 holds as well (see Remark 1).

4 Estimates for entropy numbers of embeddings of weighted Sobolev spaces

Let us define the weighted Sobolev class \( W_{p,g}^{r}(\Omega) \) and the weighted Lebesgue space \( L_{q,v}(\Omega) \).

Let \( \Omega \subset \mathbb{R}^{d} \) be a bounded domain, and let \( g, v : \Omega \to (0, \infty) \) be measurable functions. For each measurable vector-valued function \( \psi : \Omega \to \mathbb{R}^{l} \), \( \psi = (\psi_k)_{1 \leq k \leq l} \), and for each \( p \in [1, \infty) \), we put

\[
\| \psi \|_{L_p(\Omega)} = \left( \int_{\Omega} \max_{1 \leq k \leq l} |\psi_k(x)|^p \, dx \right)^{1/p}.
\]

Let \( \beta = (\beta_1, \ldots, \beta_d) \in \mathbb{Z}_+^d := (\mathbb{N} \cup \{0\})^d \), \( |\beta| = \beta_1 + \ldots + \beta_d \). For any distribution \( f \) defined on \( \Omega \) we write \( \nabla^r f = \left( \partial^r f / \partial x^\beta \right)_{|\beta|=r} \) (here partial derivatives are taken in the sense of distributions), and denote by \( l_{r,d} \) the number of components of the vector-valued distribution \( \nabla^r f \). We set

\[
W_{p,g}^{r}(\Omega) = \left\{ f : \Omega \to \mathbb{R} \mid \exists \psi : \Omega \to \mathbb{R}^{l_{r,d}} : \| \psi \|_{L_p(\Omega)} \leq 1, \nabla^r f = g \cdot \psi \right\}
\]

(we denote the corresponding function \( \psi \) by \( \nabla^r f / g \)),

\[
\| f \|_{L_{q,v}(\Omega)} = \| f \|_{q,v} = \| f \|_{L_q(\Omega)}, \quad L_{q,v}(\Omega) = \left\{ f : \Omega \to \mathbb{R} \mid \| f \|_{q,v} < \infty \right\}.
\]

We call the set \( W_{p,g}^{r}(\Omega) \) a weighted Sobolev class. Observe that if \( g \in L_{g'}^{1}(\Omega) \), then \( \nabla^r f \in L_{1}^{g'}(\Omega) \).

For \( x \in \mathbb{R}^{d} \) and \( a > 0 \) we shall denote by \( B_a(x) \) the closed Euclidean ball of radius \( a \) in \( \mathbb{R}^{d} \) centered at the point \( x \).

Definition 2. Let \( \Omega \subset \mathbb{R}^{d} \) be a bounded domain, and let \( a > 0 \). We say that \( \Omega \in FC(a) \) if there exists a point \( x_\ast \in \Omega \) such that, for any \( x \in \Omega \), there exist a number \( T(x) > 0 \) and a curve \( \gamma_x : [0, T(x)] \to \Omega \) with the following properties:

1. \( \gamma_x \in AC[0, T(x)], \left| \frac{d\gamma_x(t)}{dt} \right| = 1 \ a.e., \)
2. \( \gamma_x(0) = x, \gamma_x(T(x)) = x_\ast \),

36
3. $B_{at}(\gamma_x(t)) \subset \Omega$ for any $t \in [0, T(x)]$.

**Definition 3.** We say that $\Omega$ satisfies the John condition (and call $\Omega$ a John domain) if $\Omega \in \mathcal{FC}(a)$ for some $a > 0$.

For a bounded domain the John condition is equivalent to the flexible cone condition (see the definition in [2]). As examples of such domains we can take

1. domains with Lipschitz boundary;
2. the Koch’s snowflake;
3. domains $\Omega = \bigcup_{0 < t \leq T} \text{int} B_{ct}(\gamma(t))$, where $\gamma : [0, T] \to \mathbb{R}^d$ is a curve with natural parametrization and $c > 0$.

Domains with zero inner angles do not satisfy the John condition.

We denote by $\mathbb{H}$ the set of all nondecreasing positive functions defined on $(0, 1]$.

**Definition 4.** (see [3]). Let $\Gamma \subset \mathbb{R}^d$ be a nonempty compact set and $h \in \mathbb{H}$. We say that $\Gamma$ is an $h$-set if there are a constant $c_\ast \geq 1$ and a finite countably additive measure $\mu$ on $\mathbb{R}^d$ such that

$$c_\ast^{-1} h(t) \leq \mu(B_t(x)) \leq c_\ast h(t)$$

for any $x \in \Gamma$ and $t \in (0, 1]$.

**Example 1.** Let $\Gamma \subset \mathbb{R}^d$ be a Lipschitz manifold of dimension $k$, $0 \leq k < d$. Then $\Gamma$ is an $h$-set with $h(t) = t^k$.

**Example 2.** Let $\Gamma \subset \mathbb{R}^2$ be the Koch curve. Then $\Gamma$ is an $h$-set with $h(t) = t^{\log 4/\log 3}$ (see [30, p. 66–68]).

Let $\Omega \in \mathcal{FC}(a)$ be a bounded domain, and let $\Gamma \subset \partial \Omega$ be an $h$-set. Below we consider a function $h \in \mathbb{H}$ which has the following form near zero:

$$h(t) = t^{\theta} |\log t|^{\gamma} \tau(|\log t|), \quad 0 \leq \theta < d,$$

where $\tau : (0, +\infty) \to (0, +\infty)$ is an absolutely continuous function such that

$$\lim_{t \to +\infty} \frac{t\tau'(t)}{\tau(t)} = 0.$$

Let $1 < p \leq \infty$, $1 \leq q < \infty$, $r \in \mathbb{N}$, $\delta := r + \frac{d}{q} - \frac{d}{p} > 0$, $\beta_g, \beta_v \in \mathbb{R}$, $g(x) = \varphi_g(\text{dist}_{\|\cdot\|}(x, \Gamma))$, $v(x) = \varphi_v(\text{dist}_{\|\cdot\|}(x, \Gamma))$,

$$\varphi_g(t) = t^{-\beta_g} |\log t|^{-\alpha_g} \rho_g(|\log t|), \quad \varphi_v(t) = t^{-\beta_v} |\log t|^{-\alpha_v} \rho_v(|\log t|),$$

(150)
where \( \rho_g \) and \( \rho_v \) are absolutely continuous functions such that
\[
\frac{t \rho'_g(t)}{\rho_g(t)} \xrightarrow{t \to +\infty} 0, \quad \frac{t \rho'_v(t)}{\rho_v(t)} \xrightarrow{t \to +\infty} 0.
\]

Moreover, assume that
\[
\beta_v < \frac{d - \theta}{q} \quad \text{or} \quad \beta_v = \frac{d - \theta}{q}, \quad \alpha_v > \frac{1 - \gamma}{q}.
\]

Without loss of generality we may consider \( \Omega \subset \left(-\frac{1}{2}, \frac{1}{2}\right)^d \).

Denote \( \beta = \beta_g + \beta_v, \quad \alpha = \alpha_g + \alpha_v, \quad \rho(y) = \rho_g(y)\rho_v(y), \)
\( Z = (r, d, p, q, g, v, h, a, c), \quad Z^* = (Z, R) \), where \( c^* \) is the constant from Definition
\( 4 \) and \( R = \text{diam} \Omega. \)

Denote by \( P_{r-1}(\mathbb{R}^d) \) the space of polynomials on \( \mathbb{R}^d \) of degree not exceeding \( r - 1 \).

For a measurable set \( E \subset \mathbb{R}^d \) we put
\[
P_{r-1}(E) = \{ f \mid f \in P_{r-1}(\mathbb{R}^d) \}.
\]

Observe that \( W_{r,p,g}(\Omega) \supset P_{r-1}(\Omega) \).

In Theorems 2, 3 the conditions on weights are such that \( W_{r,p,g}(\Omega) \subset L_{q,v}(\Omega) \) and there exist \( M > 0 \) and a linear continuous operator \( P : L_{q,v}(\Omega) \to P_{r-1}(\Omega) \) such that for any function \( f \in W_{r,p,g}(\Omega) \)
\[
\| f - Pf \|_{L_{q,v}(\Omega)} \leq M \left\| \frac{\nabla^r f}{g} \right\|_{L_p(\Omega)}
\]
(see [40–43]).

Denote \( W_{p,g}(\Omega) = \text{span} W_{p,g}(\Omega), \quad W_{p,g}(\Omega) = \{ f - Pf : f \in W_{p,g}(\Omega) \} \).

Let \( W_{p,g}(\Omega) \) be equipped with norm \( \| f \|_{W_{p,g}(\Omega)} := \left\| \frac{\nabla^r f}{g} \right\|_{L_p(\Omega)} \).

Denote by \( I : W_{p,g}(\Omega) \to L_{q,v}(\Omega) \) the embedding operator. From (153) it follows that it is continuous.

First we consider the case \( 0 < \theta < d \). We set
\[
\alpha_0 := \begin{cases} \alpha & \text{for} \quad \beta_v < \frac{d - \theta}{q}, \\ \alpha - \frac{1}{q} & \text{for} \quad \beta_v = \frac{d - \theta}{q}. \end{cases}
\]

In [37] the estimates for entropy numbers \( e_n(I : \hat{W}_{p,g}(\Omega) \to L_{q,v}(\Omega)) \) were obtained under the following conditions. In the case \( \delta - \beta > \theta \left( \frac{1}{q} - \frac{1}{p} \right)_+ \) we assumed that \( \frac{\delta}{\delta - \beta} \neq \frac{\delta - \beta}{\theta} \).

In the case \( \delta - \beta = \theta \left( \frac{1}{q} - \frac{1}{p} \right)_+ \) we assumed that \( \alpha_0 \neq \frac{1}{p} - \frac{1}{q} \) for \( p < q \).

Now we obtain estimates for \( p < q, \quad \beta - \delta = 0, \quad \alpha_0 = \frac{1}{p} - \frac{1}{q}. \)
Theorem 2. Suppose that the conditions (148)–(152) hold, \( \rho_g \equiv 1, \rho_v \equiv 1 \). Let \( 0 < \theta < d, p < q, \beta = \delta \), and let \( \alpha_0 = \frac{1}{p} - \frac{1}{q} \). Then

\[
\varepsilon_n(I : \hat{W}^{pr}_{p,g}(\Omega) \to L_{q,v}(\Omega)) \asymp n^{\frac{1}{q} - \frac{1}{p}}.
\]

Now we consider the case \( \theta = 0, \beta_v < \frac{d}{q} \). We assume that

\[
\rho_g(t) = |\log t|^{-\lambda_g}, \quad \rho_v = |\log t|^{-\lambda_v}, \quad \tau(t) = |\log t|^{\nu}.
\]

(154)

Denote \( \lambda = \lambda_g + \lambda_v \).

In [37] estimates of entropy numbers were obtained in the following cases:

1. \( \delta - \beta > \theta \left( \frac{1}{q} - \frac{1}{p} \right)_+ \),
2. \( \delta - \beta = \theta \left( \frac{1}{q} - \frac{1}{p} \right)_+ \), \( \alpha > (1 - \gamma) \left( \frac{1}{q} - \frac{1}{p} \right)_+ \), \( \frac{\delta}{\gamma} \neq \frac{\alpha}{1 - \gamma} \),
3. \( \delta - \beta = \theta \left( \frac{1}{q} - \frac{1}{p} \right)_+ \), \( \alpha = (1 - \gamma) \left( \frac{1}{q} - \frac{1}{p} \right)_+ \), and if \( p < q \), then \( \lambda \neq \frac{1}{p} - \frac{1}{q} \).

Here we obtain the estimates for \( p < q, \delta - \beta = 0, \alpha = 0, \lambda = \frac{1}{p} - \frac{1}{q} \).

Theorem 3. Let the conditions (148), (150), (154) hold, and let \( \theta = 0, \beta - \delta = 0, \beta_v < \frac{d}{q}, \alpha = 0, p < q, \lambda = \frac{1}{p} - \frac{1}{q} \). Then

\[
\varepsilon_n(I : \hat{W}^{pr}_{p,g}(\Omega) \to L_{q,v}(\Omega)) \asymp n^{\frac{1}{q} - \frac{1}{p}}.
\]

Proof of Theorems 2 and 3. The lower estimates are proved similarly as in [37].

Let us obtain upper estimates. To this end, we apply Theorem 1. Consider the tree \( \mathcal{A} \) with vertex set \( \{\eta_{j,i}\}_{j \geq j_{\min}, i \in I_j} \) and the partition of the domain \( \Omega \) into subdomains \( \Omega[\eta_{j,i}] \), as defined in [40], [41]. We set \( \hat{F}(\eta_{j,i}) = \Omega[\eta_{j,i}] \). Let the number \( \bar{s} = \bar{s}(a, d) \geq 4 \) be as defined in [40]. Repeating arguments from the proof of Theorem 2 in [41] (without summation over vertices \( \xi \in V(\mathcal{A}_{\xi}) \)), we obtain that Assumption \( \Delta \) holds with \( u(\eta_{j,i}) = \varphi_g(2^{-\bar{s}}) \cdot 2^{-(r - \frac{\delta}{\gamma})\bar{s}_j}, w(\eta_{j,i}) = \varphi_v(2^{-\bar{s}}) \cdot 2^{-\frac{\epsilon m}{\gamma}} \). Assumption \( \Xi \) holds with \( \delta_* = \frac{\delta}{\gamma} \) (see [39], [36] Lemma 8, [4]). If \( \theta > 0 \), then Assumption \( \Xi \) holds; condition 5 of this assumption follows from Lemma 2 in [12]. If \( \theta = 0 \), then Assumption \( \Xi \) holds. In both cases we have \( \kappa = \frac{d}{q} - \beta_v, \alpha_u = \alpha_g, \alpha_w = \alpha_v \). In the case \( \theta = 0 \) we have \( \lambda_u = \lambda_g, \lambda_w = \lambda_v \). By Theorem 1, we obtain the desired upper estimates of entropy numbers. 

\[\square\]
5 Estimates for entropy numbers of two-weighted summation operators on a tree

Applying Remark 2, we obtain estimates for entropy numbers of weighted summation operators on trees.

Let $\mathcal{G}$ be a graph. Given a function $f : \mathcal{V}(\mathcal{G}) \to \mathbb{R}$, we set

$$
\|f\|_{l_p(\mathcal{G})} = \left( \sum_{\xi \in \mathcal{V}(\mathcal{G})} |f(\xi)|^p \right)^{1/p}
$$

for $1 \leq p < \infty$, $\|f\|_{l_\infty(\mathcal{G})} = \sup_{\xi \in \mathcal{V}(\mathcal{G})} |f(\xi)|$. (155)

Denote by $l_p(\mathcal{G})$ the space of functions $f : \mathcal{V}(\mathcal{G}) \to \mathbb{R}$ with finite norm $\|f\|_{l_p(\mathcal{G})}$.

Let $\mathcal{A}$ be a tree, $\mathcal{V}(\mathcal{A}) = \{\eta_{j,i} : j \in \mathbb{Z}_+, i \in I_j\}$, let $\eta_{0,1}$ be the minimal vertex of $\mathcal{A}$, and let $\mathcal{V}_j^\mathcal{A}(\eta_{0,1}) = \{\eta_{j,i}\}_{i \in I_j}$ for any $j \in \mathbb{Z}_+$. Suppose that for some $c_*, m_* \in \mathbb{N}$

$$
c_*^{-1} \frac{h(2^{-m_*})}{h(2^{-m_*)(j+1)})} \leq \mathcal{V}_j^\mathcal{A}(\eta_{j,i}) \leq c_* \frac{h(2^{-m_*})}{h(2^{-m_*)(j+1)})}; \quad j, l \in \mathbb{Z}_+, 
$$

where the function $h \in \mathbb{H}$ is defined by (148) near zero. Let $u, w : \mathcal{V}(\mathcal{A}) \to (0, \infty)$, $u(\eta_{j,i}) = u_j$, $w(\eta_{j,i}) = w_j$, $j \in \mathbb{Z}_+, i \in I_j$,

$$
u_j = 2^{m_*j}(m_*j + 1)^{-\alpha_u}, \quad w_j = 2^{-\kappa m_*j}(m_*j + 1)^{-\alpha_w}.
$$

In addition, we suppose that

$$
\kappa > \frac{\theta}{q} \quad \text{or} \quad \kappa = \frac{\theta}{q}, \quad \alpha_u > \frac{1 - \gamma}{q}.
$$

We set $\mathfrak{z} = (p, q, u, w, h, m_*, c_*)$.

Denote $\tilde{\alpha} = \alpha_u + \alpha_w$ for $\kappa > \frac{\theta}{q}$, $\tilde{\alpha} = \alpha_u + \alpha_w - \frac{1}{q}$ for $\kappa = \frac{\theta}{q}$.

In [37] estimates for $e_n(S_{u,w,A} : l_p(\mathcal{A}) \to l_q(\mathcal{A}))$ were obtained in the case $\tilde{\alpha} \neq \frac{1}{p} - \frac{1}{q}$. The case $\tilde{\alpha} = \frac{1}{p} - \frac{1}{q}$, $q = \infty$ was considered by Lifshits and Linde [25, 27]. To this end estimates for entropy numbers of the dual operator $S_{u,w,A}^* : l_1(\mathcal{A}) \to l'_q(\mathcal{A})$ were obtained and the result of the paper [4] was applied.

**Theorem 4.** Let $\theta > 0$, $1 < p < q < \infty$, $\tilde{\alpha} = \frac{1}{p} - \frac{1}{q}$. Then

$$
e_n(S_{u,w,A} : l_p(\mathcal{A}) \to l_q(\mathcal{A})) \lesssim \frac{n^{\frac{1}{q} - \frac{1}{p}}}{3}.
$$

Now we suppose that $\theta = 0$, $\kappa > 0$, and instead of (156) the following condition holds:

$$
u_j = 2^{m_*j}(m_*j + 1)^{\alpha \log(m_*j + 1)} - \lambda_u, \quad w_j = 2^{-\kappa m_*j}(m_*j + 1)^{-\alpha \log(m_*j + 1)} - \lambda_w
$$

with $\alpha \in \mathbb{R}$.

**Theorem 5.** Let $\theta = 0$, $\gamma \leq 0$, $1 < p < q < \infty$, $\kappa > 0$, $\lambda_u + \lambda_w = \frac{1}{p} - \frac{1}{q}$. Then

$$
e_n(S_{u,w,A} : l_p(\mathcal{A}) \to l_q(\mathcal{A})) \lesssim \frac{n^{\frac{1}{q} - \frac{1}{p}}}{3}.
$$

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