ALGEBRAIC SURFACES AND SEIBERG-WITTEN INVARIANTS

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1. Introduction.

Donaldson theory has shown that there is a deep connection between the 4-manifold topology of a complex surface and its holomorphic geometry [5], [6], [8], [9], [14], [15]. Recently Seiberg and Witten have introduced a new set of 4-manifold invariants [33]. These invariants have greatly clarified the structure of 4-manifolds and have made it possible to prove various conjectures suggested by Donaldson theory. The invariants take on an especially simple form for Kähler surfaces, as realized by Witten [34] and Kronheimer-Mrowka. For example, their arguments show that, if $X$ is a minimal algebraic surface of general type, then every orientation-preserving self-diffeomorphism $f : X \rightarrow X$ satisfies $f^*c_1(K_X) = \pm c_1(K_X)$, where $K_X$, the canonical line bundle of $X$, is the line bundle whose local holomorphic sections are holomorphic 2-forms. More generally, in case $X$ is a Kähler surface with $b_2^+ \geq 3$, the form of the invariants easily establishes some conjectures of [14], that the only smoothly embedded 2-spheres of self-intersection $-1$ in $X$ represent the classes of exceptional curves, and that the pullback of the canonical class of the minimal model of $X$ is invariant up to sign under every orientation-preserving diffeomorphism of $X$ (see Section 3 below). In this paper we shall extend these results to cover the case of $b_2^+ = 1$. Here, for a Kähler surface $X$, $b_2^+(X) = 2p_g(X) + 1$, where $p_g(X)$ is the number of linearly independent holomorphic 2-forms on $X$, i.e. $\dim H^0(X; K_X)$. More specifically, we shall prove:

Theorem 1.1. Let $X$ be a minimal surface of general type with $p_g(X) = 0$. Let $\tilde{X}$ be a blowup of $X$ at $\ell$ distinct points, and let $E_1, \ldots, E_\ell$ be the exceptional curves on $\tilde{X}$. Finally let $K_0 \in H^2(\tilde{X}; \mathbb{Z})$ be the pullback to $\tilde{X}$ of the canonical class of the minimal model $X$ of $\tilde{X}$. If $f$ is an orientation-preserving self-diffeomorphism $f : \tilde{X} \rightarrow X$, then for all $i$ there is a $j$ such that $f^*[E_i] = \pm [E_j]$ and $f^*K_0 = \pm K_0$. More generally, let $X$ and $X'$ be two minimal surfaces of general type satisfying the above hypotheses. Suppose that $\tilde{X}$ and $\tilde{X}'$ are blowups of $X$ and $X'$ respectively at distinct points, that $E_1, \ldots, E_\ell$ and $E'_1, \ldots, E'_{\ell'}$ are the exceptional curves on $\tilde{X}$ and $\tilde{X}'$ respectively and that $K_0$ and $K'_0$ are the pullbacks to $\tilde{X}$ and $\tilde{X}'$ of the canonical classes of $X$ and $X'$. If $f : \tilde{X} \rightarrow \tilde{X}'$ is an orientation-preserving diffeomorphism, then $\ell = m$, for every $i$ there exists a $j$ such that $f^*[E'_i] = \pm [E_j]$ and $f^*K'_0 = \pm K_0$.

More generally, we can replace embedded 2-spheres of square $-1$ in the above theorem by more general negative definite 4-manifolds. Here, if $N$ is a negative
definite 4-manifold and \(X\) is a Kähler surface which is orientation-preserving diffeomorphic to \(M \# N\) for some 4-manifold \(M\), then it is essentially a remark of Kotschick (see [17], [20]) that \(N\) has no nontrivial finite covering spaces and in particular \(H_1(N; \mathbb{Z}) = 0\). Thus, if \(N\) is a negative definite summand of a Kähler manifold, \(H_1(N; \mathbb{Z}) = 0\) and \(H^2(N; \mathbb{Z})\) is torsion free. Again by a theorem of Donaldson [7], \(H^2(N; \mathbb{Z})\) has a basis \(\{n_1, \ldots, n_\ell\}\) such that \(n_i^2 = -1\) and \(n_i \cdot n_j = 0\) if \(i \neq j\). Such a basis is unique up to sign changes and permutation, and we will refer to the \(n_i\) as the exceptional classes of \(N\).

**Theorem 1.2.** Let \(X\) be a minimal surface of general type with \(p_g(X) = 0\). Let \(\tilde{X}\) be a blowup of \(X\) at \(\ell\) distinct points, and let \(E_1, \ldots, E_\ell\) be the exceptional curves on \(\tilde{X}\). Let \(N\) be a closed oriented negative definite 4-manifold, and suppose that \(\{n_1, \ldots, n_\ell\}\) is a basis for \(H^2(N; \mathbb{Z})\) such that \(n_i^2 = -1\) for all \(i\) and \(n_i \cdot n_j = 0\) if \(i \neq j\). If there is an orientation-preserving diffeomorphism \(\tilde{X} \to M \# N\), then, for every \(i\), \(n_i = \pm[E_j]\) for some \(j\).

Using the above theorems, the arguments of Witten, Kronheimer, and Mrowka in case \(p_g > 0\), and standard material on algebraic surfaces (see e.g. [15]) we can deduce the following corollary:

**Corollary 1.3.** Let \(X\) be a minimal Kähler surface which is not rational or ruled, or equivalently such that the Kodaira dimension of \(X\) is at least zero. Then the conclusions of (1.1) and (1.2) hold for \(X\).

By the classification of surfaces, a Kähler surface \(X\) which is not rational or ruled either satisfies \(p_g(X) > 0\), \(p_g(X) = 0\) and \(X\) is of general type, or \(p_g(X) = 0\) and \(X\) is elliptic. To establish the corollary, the case \(p_g > 0\) is covered by the arguments of Witten and Kronheimer-Mrowka. The case \(p_g = 0\) and \(X\) of general type is covered by the above theorems. There remains the case that \(p_g = 0\) and \(X\) is elliptic. The case where \(p_g(X) = 0, X\) is elliptic, and \(b_1(X) = 0\) (essentially the case of Dolgachev surfaces) can be handled by arguments similar to the proof of the above theorems. In the remaining case \(p_g(X) = 0\) and \(b_1(X) = 2\). In this paper, we shall just show by elementary methods that, in the notation of (1.1), every orientation-preserving self-diffeomorphism \(f\) preserves \(\pm K_0\) up to torsion, and similarly for the case of two different surfaces \(X\) and \(X'\). In fact, one can also show that \(\pm K_0\) itself is preserved.

Using the above results and the general theory of complex surfaces (not necessarily Kähler), we can easily deduce that the plurigenera \(P_n(X)\) of a complex surface are smooth invariants:

**Corollary 1.4.** If \(X\) and \(X'\) are two diffeomorphic complex surfaces, then \(P_n(X) = P_n(X')\) for all \(n \geq 1\).

The method of proof of Theorem 1.1 also yields:

**Corollary 1.5.** Let \(X\) be a Kähler surface, not necessarily minimal. If there exists a Riemannian metric of positive scalar curvature on \(X\), then \(X\) is rational or ruled.

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2. Seiberg-Witten invariants for Kähler metrics.

Here we review the general theory of Seiberg-Witten invariants for Kähler metrics. None of the results in this section are original, and most should appear in [10], but we shall sketch some of the proofs for the sake of clarity. Other expositions of this material and its consequences for Kähler surfaces have appeared in [4], [24]. Let \( M \) be a a general closed oriented Riemannian 4-manifold with Riemannian metric \( g \). First we recall some of the properties of Spin\( ^c \) structures on \( M \). There is an exact sequence
\[
\{1\} \to U(1) \to \text{Spin}^c(4) \to SO(4) \to \{1\},
\]
which realizes Spin\( ^c(4) \) as a central extension of \( SO(4) \). In particular, by considering the exact sequence of cohomology sets associated to this central extension, the set of Spin\( ^c \) structures on \( M \) lifting the frame bundle (which is nonempty if and only if \( w_2(M) \) is the mod 2 reduction of an integral class) is a principal homogeneous space over \( H^1(M;U(1)) \), where \( U(1) = C^\infty(M)/\mathbb{Z} \) is the sheaf of \( C^\infty \) functions from \( M \) to \( U(1) \). Since \( H^1(M;C^\infty(M)/\mathbb{Z}) \cong H^2(M;\mathbb{Z}) \), given two Spin\( ^c \) structures \( \xi_1, \xi_2 \) on \( M \) lifting the frame bundle, their difference \( \delta(\xi_1, \xi_2) \) is a well-defined element of \( H^2(M;\mathbb{Z}) \). In dimension four, Spin\( ^c(4) \) is the subgroup of \( U(2) \times U(2) \) consisting of pairs \((T_1, T_2)\) with \( \det T_1 = \det T_2 \). Thus there are two natural homomorphisms Spin\( ^c(4) \) \to \( U(2) \), and the corresponding homomorphisms Spin\( ^c(4) \) \to \( U(1) \) given by taking the determinant agree. If \( \xi \) is a Spin\( ^c \) structure on \( M \), there are two associated \( U(2) \) bundles \( S^\pm = S^\pm(\xi) \), and \( L = c_1(S^\pm) \) is a complex line bundle which satisfies \( c_1(L) \equiv w_2(M) \mod 2 \). Thus \( L \) is characteristic, i.e. \( c_1(L) \) has mod two reduction equal to \( w_2(M) \). We shall call \( L \) the complex line bundle associated to \( \xi \). Let \((L, \xi)\) be a pair consisting of a characteristic complex line bundle \( L \) on \( M \), and a Spin\( ^c \) structure \( \xi \) whose associated line bundle is \( L \). Of course, the Spin\( ^c \) structure determines the line bundle \( L \), but we shall record both since we shall primarily be interested in \( L \) (and we shall sometimes omit the \( \xi \) from the notation). Conversely, \( L \) determines \( \xi \) up to 2-torsion in \( H^2(M;\mathbb{Z}) \), and thus \( L \) determines \( \xi \) uniquely if there is no 2-torsion in \( H^2(M;\mathbb{Z}) \). In terms of the pairs \((L_1, \xi_1)\) and \((L_2, \xi_2)\), we have \( 2\delta(\xi_1, \xi_2) = c_1(L_1) - c_1(L_2) \). Thus we may define the difference \( \frac{c_1(L_1) - c_1(L_2)}{2} = \delta(\xi_1, \xi_2) \), and this difference is well-defined in integral cohomology (not just modulo 2-torsion).

In case \( X \) is a Kähler surface, or more generally a 4-manifold with an almost complex structure, then there is a natural lifting of the reduction of the structure group of the tangent bundle of \( X \) to Spin\( ^c(4) \). Namely, we take the map \( U(2) \to U(2) \times U(2) \) defined by \((\rho_1, \rho_2)\), where \( \rho_2 = \text{Id} \) and
\[
\rho_1(T) = \begin{pmatrix} 1 & 0 \\ 0 & \det T \end{pmatrix}.
\]
Thus \( S^+ = C \oplus K_X^{-1} \) and \( S^- = T_C \), where \( K_X^{-1} \) is the inverse of the canonical line bundle of the almost complex structure and \( T_C \) is the tangent bundle, viewed as a complex 2-plane bundle. In terms of the bundles of \((p, q)\)-forms defined by the almost complex structure, we may also write this as \( S^+ = \Omega^0_X \oplus \Omega^{0,2}_X \) and \( S^- = \Omega^{0,1}_X \). For this lift \( L = K_X^{-1} \). If we replace \( \xi \) by \( \xi \otimes \Xi \), where \( \Xi \) is a \( C^\infty \) line bundle on \( X \), then we replace \( K_X^{-1} \) by \( L = \Xi \otimes K_X^{-1} \). Thus, for a characteristic complex (but not necessarily holomorphic) line bundle \( L \), a Spin\( ^c \) structure for \( X \) is the same as
the choice of a complex line bundle \( \Xi \) with \( \Xi \otimes \Xi = L \otimes K_X \). In this case, given two such Spin\(^c\) structures, the difference \( \frac{c_1(L_1) - c_1(L_2)}{2} \) is equal to \( c_1(\Xi_1) - c_1(\Xi_2) \).

Of course, the obvious choices are \( L = K_X^{-1} \) and \( L = K_X \), with \( \Xi \) the trivial bundle in the first case and \( \Xi = K_X \) in the second. We shall refer to these two choices as the natural Spin\(^c\) structures.

Recall that the Seiberg-Witten equations associated to a Spin\(^c\) structure \( \xi \) on a Riemannian manifold \( M \) are equations for a pair \((A, \psi)\) where \( A \) is a unitary connection of the determinant line bundle \( L \) of solutions to the Seiberg-Witten equations by the group of based changes of gauge, \( S \) and taking the first Chern class of the two-dimensional cohomology class \( \mu \) of solutions forms a compact smooth manifold \( F \). In the case of a \( K \)ähler metric, \( c_1(L, \xi) = 0 \) implies that \( A \) is a \((1, 1)\)-connection (and thus \( A \) defines a holomorphic structure on \( L \)) and \( c_1(L) \cdot \omega = 0 \), where \( \omega \) is the Kähler form on \( M \).

Once we have the Seiberg-Witten moduli space, we can define the Seiberg-Witten invariant for \( M \), which is a function \( SW_{M,g} \) which assigns an integer to each pair \((L, \xi)\) as above. It suffices to evaluate \( \mu^{d/2} \) over the fundamental class of \( \mathcal{M}(L, \xi) \), where \( d = \text{dim} \mathcal{M}(L, \xi) \). By definition this integer is zero if \( d \) is odd. Strictly speaking, we also need to choose an orientation for \( H^1(X; \mathbb{R}) \otimes H^2(M; \mathbb{R}) \) to determine the sign of \( SW_{M,g} \), but we shall be a little careless on this point. If \( b_2^+(M) \geq 3 \), then \( SW_{M,g} \) is independent of the choice of \( g \), whereas if \( b_2^+(M) = 1 \), then \( SW_{M,g} \) is only defined for generic \( g \) and we will describe the dependence of \( SW_{M,g} \) on \( g \) more precisely later. If \( SW_{M,g}(L, \xi) \neq 0 \) we shall say that the pair \((L, \xi)\) (or \( L \)) is a basic class (for \( g \)). This can only happen in case the index of the corresponding linearized equations is nonnegative. This index is

\[
\frac{1}{4}(L^2 - (2\chi(M) + 3\sigma(M)) ,
\]

where \( \chi(M) \) is the Euler characteristic and \( \sigma(M) \) is the signature, and we shall also refer to this index as the index of the basic class \( L \). If \( M = X \) is a complex surface, then it follows from the Hirzebruch signature formula that \( 2\chi(X) + 3\sigma(X) = K_X^2 \).

Thus if \( L \) is a basic class, then \( L^2 \geq K_X^2 \). If all of the basic classes have index zero, then \( M \) is called of simple type (for \( g \)). Finally we note that, even though the function \( SW \) is defined by perturbing the Seiberg-Witten equations, if the unperturbed Seiberg-Witten equations have no solution for \((L, \xi)\), then \( SW_{M,g}(L, \xi) = 0 \).

If on the other hand the solutions to the unperturbed Seiberg-Witten equations are transverse in an appropriate sense, then we can use the moduli space for the unperturbed equations to calculate \( SW_{M,g}(L, \xi) \).

Recall that a solution \((A, \psi)\) to the SW equations is reducible if and only if \( \psi = 0 \) and hence \( F_A^+ = 0 \), where \( F_A^+ \) is the self-dual part of the curvature of a connection on \( L \). In the case of a Kähler metric, \( F_A^+ = 0 \) implies that \( A \) is a \((1, 1)\)-connection (and thus \( A \) defines a holomorphic structure on \( L \)) and \( c_1(L) \cdot \omega = 0 \), where \( \omega \) is the Kähler form on \( M \).
is the Kähler form. By the Hodge index theorem, $L^2 \leq 0$. Conversely, if $L$ is a holomorphic line bundle and $c_1(L) \cdot \omega = 0$, then there exists a $(1,1)$-connection $A$ on $L$ with $F_A^\perp = 0$, giving a reducible solution to the SW equations, and indeed in this case all solutions will be reducible. We call a basic class $L$ reducible if all solutions to the corresponding SW equations are reducible, and irreducible otherwise. Of course, reducible solutions to the SW equations for $(L, \xi)$ do not necessarily imply that $(L, \xi)$ is basic. For a generic Kähler metric, we can assume that the Kähler form $\omega$ is not orthogonal to any nontorsion class $L \in H^2(X; \mathbb{Z})$ with $L^2 \geq K_X^2$. We shall call a Kähler metric whose associated Kähler form $\omega$ satisfies this condition generic. Thus for a generic Kähler metric, there exist reducible basic classes $L$ only if $c_1(L)$ is zero as an element of $H^2(X; \mathbb{R})$. Of course, these will give solutions of nonnegative index only if $0 = L^2 \geq K_X^2$.

In the Kähler case, we have the following criterion for the SW moduli space to be nonempty and of nonnegative formal dimension [34], [10]:

**Proposition 2.1.** Suppose that $X$ is a Kähler surface with Kähler form $\omega$. Then the pairs $(L, \xi)$ of nonnegative index admitting irreducible solutions to the SW equations are in one-to-one correspondence with holomorphic characteristic line bundles $L$, together with a choice of a holomorphic square root $(K_X \otimes L)^{1/2}$ for the line bundle $K_X \otimes L$, satisfying:

(i) $L^2 \geq K_X^2$;
(ii) Either $H^0(X; (K_X \otimes L)^{1/2}) \neq 0$ and $\omega \cdot L < 0$ or $H^0(X; (K_X \otimes L^{-1})^{1/2}) \neq 0$ and $\omega \cdot L > 0$. □

Here the choice of a square root $(K_X \otimes L)^{1/2}$ for $K_X \otimes L$ naturally gives the square root $(K_X \otimes L^{-1})^{1/2} = (K_X \otimes L)^{1/2} \otimes L^{-1}$ for $K_X \otimes L^{-1}$.

The idea behind the proof of (2.1) is that since $X$ has a complex structure, $S^+(\xi)$ splits as a sum of line bundles $(K_X \otimes L)^{1/2}$ and $\Omega_X^{0,2}((K_X \otimes L)^{1/2})$. Thus, the spinor field $\psi$ decomposes into components $(\alpha, \beta)$. The curvature part of the Seiberg-Witten equations says that

$$F^{0,2}_\omega = \bar{\alpha} \beta$$

$$F^\perp_A^{1,1} = \frac{i}{2}(|\alpha|^2 - |\beta|^2)\omega,$$

where $\omega$ is the Kähler form. The Dirac equation for a Kähler surface becomes

$$\bar{\partial}_A \alpha + \bar{\partial}_A^* \beta = 0.$$

Applying $\partial_A$ to this equation we get

$$\partial_A \bar{\partial}_A \alpha + \partial_A \bar{\partial}_A^* \beta = 0.$$

Equivalently,

$$F^{0,2}_A \cdot \alpha + \bar{\partial}_A \bar{\partial}_A^* \beta = 0.$$

Since $F^{0,2}_A = \bar{\alpha} \beta$, this equation becomes

$$|\alpha|^2 \beta + \bar{\partial}_A \bar{\partial}_A^* \beta = 0.$$
Taking the $L^2$-inner product with $\beta$ yields
\[
\int_X |\alpha|^2|\beta|^2 + \|\bar{\partial}_A \beta\|^2_{L^2} = 0.
\]

It follows that $|\alpha|^2|\beta|^2$ and $\bar{\partial}_A \beta$ are zero. Thus $F_{A}^{0,2} = \bar{\alpha}\beta = 0$. This means that $A$ is a holomorphic connection and so defines a holomorphic structure on $L$. Moreover $\bar{\partial}_A \beta = 0$ and so $\bar{\partial}_A \alpha = 0$. Hence $\alpha$ is a holomorphic section of $(K_X \otimes L)^{1/2}$ and $\bar{\beta}$ is a holomorphic section of $(K_X \otimes L^{-1})^{1/2}$. Since $\alpha$ and $\bar{\beta}$ are holomorphic, they do not vanish on any open subset unless they vanish identically. So either $\alpha = 0$ or $\beta = 0$. Furthermore,
\[
\omega \cdot L = \int_X \omega \wedge \frac{i}{2\pi} F_A^+ = \frac{1}{4\pi} \int_X (|\alpha|^2 - |\beta|^2) \omega \wedge \omega,
\]
and so $\alpha$ is not zero if and only if $\omega \cdot L < 0$ and $\beta$ is not zero if and only if $\omega \cdot L > 0$. If $\alpha \neq 0$ then $\alpha$ is a nonzero holomorphic section of $(K_X \otimes L)^{1/2}$. If $\beta \neq 0$, then $\bar{\beta}$ is a nonzero holomorphic section of $(K_X \otimes L^{-1})^{1/2}$.

We have seen that the conditions listed in Proposition 2.1 are necessary for a solution. Let us show that they are sufficient as well. We shall just consider the case where $\omega \cdot L$ is negative. The holomorphic structure on $L$ uniquely determines a connection $A$, once we have chosen a hermitian metric on $L$, by choosing the unique connection compatible with the holomorphic structure and the metric. Fix an arbitrary hermitian metric on $L$. Given $L$ and a nontrivial holomorphic section $\alpha$ of $(K_X \otimes L)^{1/2}$, we wish to change the metric on $L$ until the curvature part of the Seiberg-Witten equation is satisfied. If we think of varying the hermitian metric by $\exp \lambda$ for some real valued function $\lambda$, then the equation we need to solve for $\lambda$ is
\[
F_A^+ + (\bar{\partial}\partial \lambda)^+ = \frac{i}{2} e^\lambda |\alpha|^2 \omega.
\]
For a $(1,1)$-form $\eta$, $\eta^+ = \Lambda \eta \cdot \omega$, where $\Lambda$ is contraction with $\omega$. We take the pointwise contraction with the Kähler form $\omega$ and obtain
\[
\Delta \lambda - \frac{|\alpha|^2}{4} e^\lambda - \frac{1}{2} * (\Lambda F_A^+ \wedge \omega) = 0.
\]
Here $\Delta$ is the negative definite Laplacian on functions (in Euclidean space it would be $\sum_i \partial^2/\partial x_i^2$). According to results of Kazdan-Warner [18] (first applied in gauge theory to the vortex equation by Bradlow [3]), there is a unique solution $\lambda$ to this equation provided that $\int_X i F_A^+ \wedge \omega < 0$, which is just the condition that $\omega \cdot L$ is negative. Thus, we have seen that for each non-trivial holomorphic section of $(K_X \otimes L)^{1/2}$ we can obtain a solution to the Seiberg-Witten equations with the holomorphic section as the spinor field. This completes the sketch of the proof of the proposition.

A gauge equivalence between two solutions $(A, \psi)$ and $(A', \psi')$ will be a holomorphic isomorphism between the holomorphic structures $L$ and $L'$ determined by the two connections. It will also carry the section $\psi$ to $\psi'$. Since the only holomorphic automorphisms of a holomorphic bundle are multiplication by nonzero constant functions, this implies that under the holomorphic identification $\psi$ and $\psi'$ define
the same point of $\mathbb{P}H^0((K_X \otimes L)^{1/2})$. Conversely, it is easy to check that two sections which agree modulo $\mathbb{C}^*$ define gauge equivalent solutions. Thus we may identify the moduli space to the unperturbed equations with

$$\bigcup_L \mathbb{P}H^0((K_X \otimes L)^{1/2}),$$

where we think of $L$ as ranging over all holomorphic structures on a fixed $C^\infty$ line bundle (the set of all such structures is isomorphic to the complex torus $\text{Pic}^0 X$). The moduli space may thus be identified with an appropriate component of the Hilbert scheme of curves on $X$. Of course, there will be another moduli space corresponding to $L^{-1}$ as well.

We now divide the study of the basic classes into two cases: the case where $X$ is minimal and the case where $X$ is not minimal.

The case of a minimal surface. For a Kähler surface $X$ which is not rational or ruled, $X$ is minimal if and only if $K_X$ is nef: in other words, for every holomorphic curve $C$ on $X$, $K_X \cdot C \geq 0$. If $K_X$ is nef, then $K_X^2 \geq 0$. The case where $K_X$ is nef and $K_X^2 > 0$ is the case where $X$ is of general type. In this case, since $K_X^2 > 0$, there are no reducible basic classes for any Kähler metric.

**Proposition 2.2.** With notation as above, suppose that $X$ is a minimal surface of general type, i.e. suppose that $K_X$ is nef and that $K_X^2 > 0$, and that $\omega$ is a generic Kähler metric. Then the only pairs $(L, \xi)$ satisfying the conditions (i) and (ii) are $L = K_X^{\pm 1}$, with the natural Spin$^c$ structures, i.e. the ones corresponding to the square root $K_X$ of $K_X \otimes K_X$ and the square root 0 of $K_X \otimes K_X^{-1}$.

**Proof.** For the proof we use additive notation for holomorphic line bundles (which we could identify with divisor classes on $X$). After replacing $L$ by $-L$, we may assume that $\omega \cdot L < 0$ and that $K_X + L$ is effective. We have $(K_X + L) \cdot \omega \geq 0$, $L \cdot \omega < 0$, so there is an $a \geq 1$ such that $(K_X + aL) \cdot \omega = 0$. By the Hodge index theorem $(K_X + aL)^2 \leq 0$, with equality only if $K_X + aL$ is numerically trivial. Thus

$$K_X^2 + 2a(K_X \cdot L) + a^2 L^2 \leq 0.$$

On the other hand $K_X$ is nef, so that $(K_X + L) \cdot K_X \geq 0$. Putting together

$$2aK_X^2 + 2aK_X \cdot L \geq 0;$$

$$K_X^2 + 2a(K_X \cdot L) + a^2 L^2 \leq 0,$$

we obtain

$$(1 - 2a)K_X^2 + a^2 L^2 \leq 0,$$

or in other words

$$L^2 \leq \frac{2a - 1}{a^2} K_X^2.$$

But

$$\frac{2a - 1}{a^2} = 1 - \left(\frac{a - 1}{a}\right)^2 = 1 - \left(1 - \frac{1}{a}\right)^2,$$

which is decreasing for $a \geq 1$. Thus

$$L^2 \leq \frac{2a - 1}{a^2} K_X^2 \leq K_X^2.$$
Since $L^2 \geq K_X^2$, we have $L^2 = K_X^2$. For equality to hold we must have $a = 1$ and $K_X + L$ must be numerically trivial. In this case, $K_X + L$ has a section since $\frac{K_X + L}{2}$ has a section. Thus $K_X + L$ is the trivial divisor, so $L = -K_X$. Moreover $\frac{K_X + L}{2}$ is numerically trivial and has a section as well, so that it is trivial. Thus the Spin$^c$ structure corresponds to taking the trivial square root of $K_X + L = 0$. □

Essentially the same argument shows:

**Proposition 2.3.** With notation as above, suppose that $K_X$ is nef and that $K_X^2 = 0$. If $L$ is a line bundle satisfying the conditions (i) and (ii), then there exists a rational number $r \leq 1$ such that $L$ is numerically equivalent to $\pm rK_X$. Moreover, in case $r = \pm 1$, then in fact $L = \pm K_X$, and the Spin$^c$ structures are again the natural ones. □

Note that in all cases we have $L^2 = K_X^2$. In other words $X$ is of simple type for $g$ if $X$ is minimal and $K_X$ is nef. Of course, so far we have not actually shown that there are any basic classes. But in case $L = \pm K_X$, the value of SW is ±1. To determine the exact sign, we need to make a choice of orientation for the moduli space. The orientation convention we shall follow is this: To orient the relevant determinant line bundle for a general 4-manifold $M$, we must choose an orientation for the vector space $H^1(M; \mathbb{R}) \oplus H^2(M; \mathbb{R}) \oplus H^0(M; \mathbb{R})$, by choosing orientations on $H^1(M; \mathbb{R})$ and $H^2(M; \mathbb{R})$ and using the standard orientation on $H^0(M; \mathbb{R})$. For a Kähler surface $X$,

$$H^2(X; \mathbb{R}) \cong \mathbb{R} \cdot \omega \oplus (H^{2,0}(X) \oplus H^{0,2}(X))_{\mathbb{R}}$$

and $H^1(X; \mathbb{R}) \cong (H^{1,0}(X) \oplus H^{0,1}(X))_{\mathbb{R}}$. We choose the orientation given by taking the standard orientation on $\mathbb{R} \cdot \omega$ and using the isomorphism $(H^{i,0}(X) \oplus H^{0,i}(X))_{\mathbb{R}} \to H^{0,i}(X)$ to transfer the usual complex orientation on $H^{0,i}(X)$ to $(H^{i,0}(X) \oplus H^{0,i}(X))_{\mathbb{R}}$. We then have the following result, which follows easily by considering the linearization of the SW equations [10]:

**Proposition 2.4.** For an arbitrary complex surface $X$, if $g$ is a Kähler metric on $X$ with Kähler form $\omega$ and $\omega \cdot K_X > 0$, then the value of $SW_{X,g}$ on $-K_X$ for the natural Spin$^c$ structure is 1 and $SW_{X,g}(K_X) = (-1)^{q+ps}$ for the natural Spin$^c$ structure.

**Sketch of proof.** Let us consider the elliptic complex associated to the unique solution for $L = -K_X$. The kernel of the Dirac operator is isomorphic to $H^0(X; \mathbb{C}) \oplus H^{0,2}(X)$. The cokernel of the Dirac operator is $H^{0,1}(X)$. Of course, $H^2_+(X; i\mathbb{R})$ is, as an oriented vector space, isomorphic to $(i\mathbb{R}) \cdot \omega \oplus H^{0,2}(X)$. Clifford multiplication by the solution $(\alpha, 0) \in H^0 \oplus H^{0,2}$ induces an orientation-preserving isomorphism $H^1(X; i\mathbb{R}) \to H^{0,1}(X)$. If $Dq$ is the differential of the quadratic mapping $q$, then $-Dq$ induces an orientation-preserving mapping

$$H^{0,2}(X) \to H^{0,2}(X),$$

namely multiplication by $-\bar{\alpha}$. The map $Dq$ also induces a map

$$H^0(X; \mathbb{C}) \to (i\mathbb{R}) \cdot \omega$$
which sends $\eta$ to $-i \text{Re}(\alpha, \eta)$. Of course, the action of the stabilizer $S^1$ on the spin fields is by the opposite of the complex orientation. Hence, the solution to the equations modulo the action of the group of changes of gauge is a single point. The equations are transverse at this point and the orientation is plus one.

A similar computation shows that

$$SW_{X,\theta}(K_X) = (-1)^{\theta(X) + p_g(X)}.$$  

In fact, for any smooth 4-manifold $M$ and any Spin$^c$ structure $\xi$ there is a naturally defined opposite Spin$^c$ structure $-\xi$ and we have

$$SW_{M,\theta}(\xi) = (-1)^{(1-b_1(M) + b_2^s(M))/2}SW_{M,\theta}(-\xi).$$

To end our discussion of minimal Kähler surfaces, we consider the example of elliptic surfaces. For simplicity, and because it is the most interesting case, we shall just consider the case of simply connected elliptic surfaces, so that linear, numerical, and homological equivalence are the same. In this case the basic classes have a certain multiplicity which need not be one, but which we shall not compute here. Let $X$ be a simply connected elliptic surface, with $p_g(X) = p_g$. Then $X$ has at most two multiple fibers $F_1$ and $F_2$, of multiplicities $m_1$ and $m_2$, say. From the canonical bundle formula

$$K_X = (p_g - 1)f + (m_1 - 1)F_1 + (m_2 - 1)F_2,$$

where $f$ is the class of a general fiber: $f = m_1F_1 = m_2F_2$. Let $D$ be a divisor which is a rational multiple of $K_X$ and thus of the fiber $f$, say $D = rf$, and define $\deg D = r$. For an arbitrary Kähler metric $\omega$, normalized so that $\omega \cdot f = 1$, we have $\deg D = \omega \cdot D$. The basic classes correspond to line bundles $L$ such that either $K_X + L = 2D$, where $D$ is effective and $\omega \cdot L \leq 0$, i.e. $0 \leq \deg D \leq \frac{\deg K_X}{2}$, or $K_X - L = 2D$, where $D$ is effective and $\omega \cdot L \geq 0$, i.e. again we have $0 \leq \deg D \leq \frac{\deg K_X}{2}$. The effective divisor $D$ on $X$ can be written as $af + bF_1 + cF_2$, where $a \geq 0$, $0 \leq b \leq m_1 - 1$, and $0 \leq c \leq m_2 - 1$. If $p_g > 0$, $\deg D \leq \deg K_X$ if and only if $a \leq p_g - 1$. In this case it is clear that $D$ is effective if and only if $K_X - D$ is effective. Thus in case $K_X + L = 2D$, where $D$ is effective and $0 \leq \deg D \leq \frac{\deg K_X}{2}$, we have

$$L = 2D - K_X = K_X - 2(K_X - D) = K_X - 2D',$$

where $D'$ is effective and $\frac{\deg K_X}{2} \leq \deg D' \leq \deg K_X$. Likewise if $K_X - L = 2D$, where $D$ is effective and $\omega \cdot L \geq 0$, then $L = K_X - 2D$, where $D$ is effective and $0 \leq \deg D \leq \frac{\deg K_X}{2}$. Thus we see that in all cases the basic classes are exactly the classes of the form $K_X - 2D$, where $D$ is effective and $0 \leq \deg D \leq \deg K_X$. In other words, the basic classes are the classes $(p_g - 1 - 2a)f + (m_1 - 2b - 1)F_1 + (m_2 - 2c - 1)F_2$ for $0 \leq a \leq p_g - 1$, $0 \leq b \leq m_1 - 1$, $0 \leq c \leq m_2 - 1$. These are exactly the Kronheimer-Mrowka basic classes [21], [11], and one can show that the appropriate multiplicity, up to sign, to attach to the class $K_X - 2D$ with $D = af + bF_1 + cF_2$,
with $a \geq 0$, $0 \leq b \leq m_1 - 1$, and $0 \leq c \leq m_2 - 1$, is $\left( \begin{array}{c} p_g - 1 \\ a \end{array} \right) = \left( \begin{array}{c} h_+ + h_- \\ h_+ \end{array} \right)$, where $h_+ = h^0(D) - 1$ and $h_- = h^0(K_X - D) - 1$.

A similar analysis holds for the case of an elliptic surface with $p_g = 0$, where we must simply analyze the conditions on $L$ directly. For example, we obtain the SW classes $L$ with nonnegative fiber degree (i.e. the $L$ such that $L = rK_X$ with $r \geq 0$ in rational cohomology) by considering the effective divisors $D$ such that $L = K_X - 2D$ has nonnegative fiber degree. In this case $\frac{K_X - L}{2} = D$ and the correct multiplicity, up to sign, is one.

The case of a nonminimal surface. It is enough to consider the case where $\tilde{X} \to X$ is a single blowup.

**Proposition 2.5.** Let $X$ be a Kähler surface which is the blowup of a surface for which the canonical bundle is nef and let $g$ be a Kähler metric on $X$ with Kähler form $\omega$ such that $\omega$ is not orthogonal to any nontorsion class $L \in H^2(X; \mathbb{Z})$ with $L^2 = K_X^2$. Let $\tilde{X}$ be the blowup of $X$ at a point, with $E$ the exceptional divisor, and consider a Kähler metric $\tilde{g}$ on $\tilde{X}$ corresponding to the Kähler form $\tilde{\omega} = N\omega - E$, $N \gg 0$. Then:

(i) Every basic class on $X$ for $\tilde{g}$ is irreducible.

(ii) If $(L, \tilde{\xi})$ is a basic class on $X$ for $\tilde{g}$, then either $\tilde{L} = L \pm E$, where $L$ is an irreducible basic class on $X$ for $g$, or $\tilde{L} = L \pm E$, where $L$ is a reducible basic class on $X$ and so the image of $L$ is zero in $H^2(X; \mathbb{R})$. If $\tilde{L} = L + E$ is a basic class for $\tilde{X}$, then the corresponding Spin$^c$ structure $\tilde{\xi}$, or in other words the square root $\tilde{\Xi}$ of $\tilde{L} + K_{\tilde{X}} = L + K_X + 2E$, is $\Xi + E$, where $\Xi$ is the square root of $L + K_X$ corresponding to the Spin$^c$ structure $\xi$ on $X$, and likewise if $\tilde{L} = L - E$, then $\tilde{\Xi} = \Xi$. Moreover, the classes $L \pm E$ where $L$ is an irreducible basic class on $X$ for $g$ all give basic classes on $X$ for $\tilde{g}$, and in this case

$$SW_{X,\tilde{g}}(L \pm E, \tilde{\xi}) = \pm SW_{X,g}(L, \xi).$$

(iii) If there is a basic class on $\tilde{X}$ of the form $L \pm E$, where $L$ is a reducible basic class on $X$, then $X$ is minimal. Moreover, if $K_X$ is torsion, then $L = \pm K_X$. In this case $\pm K_X \pm E$ are basic classes on $\tilde{X}$ and $SW_{X,\tilde{g}}(\pm K_X \pm E) = \pm 1$ for the natural Spin$^c$ structures.

(iv) Let $\tilde{X}$ be a Kähler surface which is not rational or ruled. Let $\tilde{g}$ be a Kähler metric on $\tilde{X}$ whose Kähler form $\tilde{\omega} = N\omega - \sum E_i$, where the $E_i$ are the exceptional curves on $\tilde{X}$, $\omega$ is the Kähler form of a generic Kähler metric on the minimal model of $\tilde{X}$, and $N \gg 0$. For every basic class $L$ on $X$ for $\tilde{g}$, we have $L^2 = K_{\tilde{X}}^2$, so that $\tilde{X}$ is of simple type for $\tilde{g}$.

**Proof.** First we prove (i). Clearly, for all $N \gg 0$, a Kähler metric with Kähler form $N\omega - E$ is generic. Thus the only possible reducible basic classes are torsion. But a basic class is characteristic. If there were a torsion characteristic element in $H^2(\tilde{X}; \mathbb{Z})$, then every element of $H^2(\tilde{X}; \mathbb{Z})$ would have even square. This contradicts the fact that $E^2 = -1$.

Next we consider (ii) and (iii). Suppose that $\tilde{L}$ is a basic class on $\tilde{X}$. Possibly after replacing $\tilde{L}$ by $-\tilde{L}$, we can assume that $\tilde{L}^2 \geq K_{\tilde{X}}^2 = K_X^2 - 1$, $\frac{L + K_X}{2}$ is
effective, and \( \tilde{\omega} \cdot \tilde{L} < 0 \). We have \( K_{\tilde{X}} = K_X + E \), and \( \tilde{L} = L + aE \) for some characteristic \( L \in H^2(X; \mathbb{Z}) \) and odd integer \( a \). As \( \frac{\tilde{L} + K_{\tilde{X}}}{2} \) is effective, \( \frac{L + K_X}{2} \) is effective as well. Since \( \tilde{L}^2 = L^2 - a^2 \geq K_{\tilde{X}}^2 - 1 \), and \( a \) is odd, we have \( L^2 \geq K_X^2 \).

As \( \tilde{\omega} \cdot \tilde{L} < 0 \), we have \( N(\omega \cdot L) + a < 0 \) for all \( N \gg 0 \). Thus \( \omega \cdot L \leq 0 \). First consider the case where \( \omega \cdot L < 0 \). Then the line bundle \( L \) on \( X \) satisfies (i) and (ii) of (2.1). Conversely, starting with a \( L \) on \( X \) satisfying (i) and (ii) of (2.1), the class \( \tilde{L} = L \pm E \) will satisfy (i) of (2.1). If say \( L \cdot \omega < 0 \), then \( \tilde{L} \cdot \tilde{\omega} < 0 \) provided that \( N \gg 0 \). Finally \( \frac{K_{\tilde{X}} + \tilde{L}}{2} = \frac{K_X + L}{2} \) or \( \frac{K_X + L}{2} + E \), and we have natural isomorphisms

\[
H^0(\tilde{X}; \frac{K_{\tilde{X}} + \tilde{L}}{2} + E) \cong H^0(\tilde{X}; \frac{K_X + L}{2}) \cong H^0(X; \frac{K_X + L}{2}),
\]

where we have used the notation \( H^0(X; D) \) to denote the group of sections of \( \mathcal{O}_X(D) \). Thus (ii) of (2.1) is satisfied as well, and we see that the \( \text{Spin}^c \) structures are as claimed. We omit the argument that \( \text{SW}_{\tilde{X}, \tilde{g}}(L \pm E) = \pm \text{SW}_{X, g}(L) \), where \( \tilde{g} \) and \( g \) are appropriate generic Kähler metrics on \( \tilde{X} \) and \( X \) respectively. In case \( L \) is irreducible, this result is established in the general case via a general blowup formula in [10]. Note that, in case \( L = K_X \), the main case of interest, \( L + E = K_X + E = K_{\tilde{X}} \), and this case is covered by (2.4) with \( X \) replaced by \( \tilde{X} \). The case \( K_X - E \) then follows by the naturality of the function \( \text{SW} \), since \( K_X - E = R^*(K_X + E) \), where \( R: \tilde{X} \to \tilde{X} \) is the diffeomorphism corresponding to reflection in the class \( E \in H^2(\tilde{X}; \mathbb{Z}) \).

Now consider the case where \( \omega \cdot L = 0 \). By the hypothesis that the metric is generic, \( L \) is zero in rational cohomology. From \( \tilde{L}^2 = -a^2 \geq K_{\tilde{X}}^2 = K_X^2 - 1 \), we see that \( K_{\tilde{X}}^2 \leq 1 - a^2 \leq 0 \). Since \( L \) is characteristic, \( X \) must in fact be minimal. Hence \( K_{\tilde{X}}^2 \geq 0 \) and so \( K_{\tilde{X}}^2 = 0 \) and \( a = \pm 1 \). In this case \( \tilde{L}^2 = K_{\tilde{X}}^2 \), so that all of \( \tilde{L} \) constructed in this way are of index zero. We note that (ii) of (2.1) on \( \tilde{X} \) is satisfied if and only if \( H^0(X; \frac{K_X + L}{2}) \neq 0 \). Thus \( L + K_X \) is effective. If \( K_X \) is torsion, then \( L + K_X \) is also zero in rational cohomology. Thus it is the trivial divisor, and so \( L = -K_X \), and the \( \text{Spin}^c \) structure is the trivial square root of \( K_X + L \). It follows from (2.4) that \( \text{SW}_{\tilde{X}, \tilde{g}}(K_X + E) = \pm 1 \), and arguments as in the irreducible case handle show that \( \text{SW}_{\tilde{X}, \tilde{g}}(\pm K_X \pm E) = \pm 1 \) also.

Lastly we prove (iv). If \( X \) is minimal and \( L \) is irreducible, then (2.2) and (2.3) imply that \( L^2 = K_X^2 \). If \( L \) is reducible, then \( X \) is minimal and \( K_X^2 = 0 \). Since \( L \) is torsion \( L^2 = K_X^2 = 0 \) in this case as well. By induction on the number of blowups, and using the above discussion for reducible \( L \), we may assume that \( \tilde{X} \) is the blowup of a surface \( X \) at one point, where \( L^2 = K_X^2 \) for all basic classes \( L \) on \( X \). We will show that the same is true for \( \tilde{X} \). It suffices to show that basic classes of the form \( L + aE \), where \( L \) is an irreducible class on \( X \) and \( a \) is an odd integer, have square equal to \( K_X^2 - 1 \). Since \( L^2 = K_X^2 \) by induction on the number of blowups, \( a^2 \leq 1 \). Thus \( a = \pm 1 \) and \( \tilde{L}^2 = K_{\tilde{X}}^2 \).

We note that using blowups, we can calculate the invariant in case the moduli space is singular because the solutions are all reducible. For example, for a Kähler surface \( X \) with a generic metric \( g \), suppose \( X \) is a minimal surface and \( K_X \) is
torision. In this case the basic classes are exactly $\pm K_X$ with the natural Spin$^c$ structure, and the value of $SW_{g,X}$ on these classes is $\pm 1$. This is analogous to the use of blowups in Donaldson theory to define “unstable” invariants, as in Chapter III, Section 8 of [15].

3. The case where $p_g$ is nonzero.

In this section we will describe the proofs of the results corresponding to Theorems 1.1 and 1.2 in case $p_g(X) \neq 0$. In this case $SW_{g,X}$ does not depend on the choice of the metric, and the set of basic classes for $g$, which is independent of the choice of $g$, is a diffeomorphism invariant of $X$. Let $X$ be a minimal surface of general type, and let $\tilde{X}$ be a blowup of $X$. We may assume that $\tilde{X}$ is a blowup of $X$ at distinct points. If $E_1, \ldots, E_\ell$ are the exceptional classes of $\tilde{X}$, and $K_0$ is the pullback to $\tilde{X}$ of $K_X$, then the basic classes are $\pm K_0 + \sum_{i=1}^\ell \pm E_i$, with the natural Spin$^c$ structures. Consider the subset of all expressions of the form $\frac{\tilde{L}_1 - \tilde{L}_2}{2}$, where $\tilde{L}_1$ and $\tilde{L}_2$ are distinct basic classes, where we have used the Spin$^c$ structures to define the square roots as integral cohomology classes. Here, if $\tilde{L}_1 = \pm K_0 + \sum_{a \in A} E_a + \sum_{a \not\in A} (-E_a)$, then the square root $\tilde{\Xi}_1$ of $\tilde{L}_1 \otimes K_X$ corresponding to the choice of Spin$^c$ structure is

$$\tilde{\Xi}_1 = \begin{cases} K_0 + \sum_{a \in A} E_a, & \text{if } \tilde{L}_1 = K_0 + \sum_{a \in A} E_a + \sum_{a \not\in A} (-E_a), \\ \sum_{a \in A} E_a, & \text{if } \tilde{L}_1 = -K_0 + \sum_{a \in A} E_a + \sum_{a \not\in A} (-E_a). \end{cases}$$

Thus the set of difference classes consists exactly of the elements $\pm K_0, \pm K_0 + \sum_{i \in A} \pm E_i$, where $A$ is a proper subset of $\{1, \ldots, \ell\}$, or $\sum_{i \in A} \pm E_i$, where $A$ is here an arbitrary subset of $\{1, \ldots, \ell\}$. First we recover the classes $\pm K_0$ as the two elements of maximal square in this collection. The $E_i$ are then the elements of the collection orthogonal to $K_0$ of square $-1$. Thus $\pm K_0$ is preserved by every orientation-preserving self-diffeomorphism of $\tilde{X}$, and every such diffeomorphism induces a permutation of the set $\{\pm E_1, \ldots, \pm E_\ell\}$. Similar results hold for orientation-preserving diffeomorphisms between two surfaces. This establishes Theorem 1.1 in this case.

Note that, if we had only kept track of the line bundles $L_i$ in the basic classes and not the Spin$^c$ structures, we would only be able to prove Theorem 1.1 modulo 2-torsion in case $\tilde{X}$ is not minimal. This is because, if $2T = 0$ in $H^2(\tilde{X}; \mathbb{Z})$, then the sets $\pm K_0 \pm E$ and $\pm (K_0 + T) \pm (E + T)$ are equal.

Now suppose that $N$ is a closed negative definite 4-manifold such that $\tilde{X}$ is diffeomorphic to $M \# N$ for some $M$. As we have seen, $H_1(N; \mathbb{Z}) = 0$. Let $\{n_1, \ldots, n_k\}$ be a basis for $H^2(N; \mathbb{Z})$ such that $n_i^2 = -1$ for all $i$ and $n_i \cdot n_j = 0$ if $i \neq j$. The blowup formula for basic classes [10] implies that $M$ is of simple type and that the basic classes for $X$ are exactly of the form $K + \sum_{i=1}^k \pm n_i$, where $K$ is a basic class for $M$. There is also a formula for the corresponding Spin$^c$ structures which is analogous to (2.5)(ii). Thus $n_i$ is of the form $\frac{L_1 - L_2}{2}$ for two distinct basic classes $L_1$ and $L_2$, and $n_i \in \{\pm K_0, \pm K_0 + \sum_{i \in A'} \pm E_i, \sum_{i \in A} \pm E_i\}$, where $A'$ denotes a proper subset of $\{1, \ldots, \ell\}$, and $A$ denotes an arbitrary subset of $\{1, \ldots, \ell\}$. Since $n_i^2 = -1$, the only possibility is $n_i = \pm E_j$ or $n_i = \pm K_0 + \sum_{i \in A} \pm E_i$, where $\#(A) = K_0^2 + 1$. On the other hand, the isometry given by reflection in $n_i$ fixes
the set of basic classes $(L, \xi)$ and thus the set of differences of basic classes. Thus the reflection must send $K_0$ to $\pm K_0$. We may assume that $n_i = K_0 + \sum_{i \in A} \pm E_i$. Then reflection in $n_i$ applied to $K_0$ gives

$$K_0 + 2(K_0)^2(K_0 + \sum_{i \in A} \pm E_i).$$

Since $K_0^2 > 0$, $K_0 + 2(K_0)^2(K_0 + \sum_{i \in A} \pm E_i) = (1 + 2(K_0)^2)K_0 + 2(K_0)^2(\sum_{i \in A} \pm E_i)$ is never a difference of basic class. This is a contradiction. Thus we must have $n_i = \pm E_j$ for some $j$, proving Theorem 1.2 in this case.

Now let us extend the argument to handle the case $X$ is minimal and $p_g(X) \neq 0$, but where $K_X^2 = 0$. We again let $\tilde{X}$ be a blowup of $X$ with exceptional classes $E_i$ and denote by $K_0$ the image of $K_X$ in $H^2(\tilde{X}; \mathbb{Z})$. The above argument tells us how to recover the basic classes for $X$ from the basic classes for $\tilde{X}$; they appear as the differences $\frac{E_1 - E_2}{2}$ of square zero, where the $E_i$ range over the basic classes of $\tilde{X}$. Moreover the classes $\pm K_0$ are characterized among such classes by noting that, in rational cohomology, all other classes $K$ can be written as $rK_0$ with $|r| < 1$, at least if $K_0$ is not torsion, whereas if $K_0$ is torsion then the only basic classes are $\pm K_0$. Thus we see that $\pm K_0$ is preserved by every orientation preserving self-diffeomorphism of $\tilde{X}$, and similarly for diffeomorphisms between two surfaces.

We must now recover the classes of the exceptional curves, which again appear as differences of basic classes, and are of square $-1$ orthogonal to $K_0$. However in this case we have additional difference classes $\frac{K_1 - K_2}{2} \pm E_i$, where $K_i \in H^2(X; \mathbb{Z})$. Suppose that some cohomology class $\alpha$ is represented by an embedded 2-sphere, or more generally lies in $H^2(N; \mathbb{Z})$, where $N$ is a negative definite 4-manifold such that $\tilde{X}$ is diffeomorphic to $M \# N$ for some $M$. The blowup formula says that the basic classes for $\tilde{X}$ are exactly of the form $\pm \alpha \pm L$ for certain classes $L$ orthogonal to $\alpha$. In particular $\alpha$ is again an difference class, and it has square $-1$. Thus $\alpha = \frac{K_1 - K_2}{2} \pm E_i$ for some $K_1, K_2 \in H^2(X; \mathbb{Z})$. Set $T = \frac{K_1 - K_2}{2}$. After renumbering and sign change we may assume that $\alpha = T + E_1$.

Let $K$ be an arbitrary basic class for $X$. Since $K + E_1 + \sum_{i > 1} E_i$ is a basic class for $\tilde{X}$, either $K + E_1 + \sum_{i > 1} E_i = T + E_1 + L$ or $K + E_1 + \sum_{i > 1} E_i = -(T + E_1) + L$ for some class $L$. We claim that we cannot have $K + E_1 + \sum_{i > 1} E_i = -(T + E_1) + L$, for otherwise we would have

$$L = K + T + 2E_1 + \sum_{i > 1} E_i$$

and in this case

$$T + E_1 + L = K + 2T + 3E_1 + \sum_{i > 1} E_i$$

would be a basic class for $\tilde{X}$, which is clearly impossible. Thus $L = K - T + \sum_{i > 1} E_i$.

Since $-(T + E_1) + L$ is also a basic class, we see that

$$K - 2T - E_1 + \sum_{i > 1} E_i$$

is a basic class for $\tilde{X}$ whenever $K$ is a basic class for $X$. Now the basic classes for $\tilde{X}$ are of the form $K' + \sum_i \pm E_i$, for $K'$ a basic class on $X$. Since $T \in H^2(X; \mathbb{Z})$, the only such class that can equal $K - 2T - E_1 + \sum_{i > 1} E_i$ is $K' - E_1 + \sum_{i > 1} E_i$ for some basic class $K'$ on $X$. It follows that, if $K$ is a basic class for $X$, then $K - 2T$ is also a basic class for $X$. Since there are only finitely many basic classes, $T$ is torsion. Now applying the above to $K = K_0$, we see that $K_0 - 2T$ is a basic class which equals $K_0$ in rational homology. By the last sentence in (2.3), it follows that $2T = 0$. Finally, keeping track of the Spin$^c$ structures in the blowup formula for taking connected sum with a negative definite 4-manifold, one checks that the difference class

$$\frac{(L + \alpha) - (-L + \alpha)}{2} = L = K_0 - T + \sum_{i > 1} E_i.$$ 

Since $T$ is torsion, it follows from (2.3) that the only way that such a class can be of the form

$$\frac{K_1 - K_2}{2} + \sum_{i \in A} \pm E_i,$$

where $K_1$ and $K_2$ are basic classes on $X$, is if $K_1 = K_0$ and $K_2 = -K_0$. In this case the difference $\frac{K_0 - (-K_0)}{2}$ is equal to $K_0$, and so $T = 0$.

In general, using somewhat different methods, one can show the following (cf. [15], Chapter VI, Theorem 5.3 for the corresponding result in Donaldson theory):

**Proposition 3.1.** Let $M$ be a closed oriented 4-manifold with $b_2^+ (M) \geq 3$ and such that the set of basic classes for $M$ is nonempty. Suppose that $M$ is orientation-preserving diffeomorphic to $M_1 \# N_1$ and also to $M_2 \# N_2$, where the $N_i$ are negative definite 4-manifolds with $H_1(N_1) = H_1(N_2) = 0$. Let $n_1, \ldots, n_r$ be the exceptional classes for $N_1$ and $n'_1, \ldots, n'_s$ the exceptional classes for $N_2$. Then, for every $i$, $1 \leq i \leq s$, either there exists a $j$, $1 \leq j \leq r$, such that $n'_i = \pm n_j$ mod torsion, or $n'_i$ is orthogonal to the span of the $n_j$.

However, without more knowledge about the nature of the basic classes as in the case of a Kähler surface, the equality mod torsion in (3.1) seems to be the optimal statement.

Finally let us show that the basic classes determine the plurigenera of $\tilde{X}$. In all cases the basic classes determine $K_0^2$. If $K_0^2 > 0$, then $\tilde{X}$ is of general type. It is well-known that, for $n \geq 2$,

$$P_n(\tilde{X}) = \frac{n(n-1)}{2} K_0^2 + \chi(O_{\tilde{X}}).$$

Thus $P_n(\tilde{X})$ is determined from the knowledge of $K_0^2$ for all $n \geq 2$, and $P_1(\tilde{X}) = p_g(\tilde{X})$ is an oriented homotopy invariant since $b_2^+ (\tilde{X}) = 2p_g(\tilde{X}) + 1$. Hence the plurigenera are determined by $K_0^2$ as long as $K_0^2 > 0$.

For $K_0^2 = 0$, $X$ is deformation equivalent to an elliptic surface and the plurigenera are essentially determined from the knowledge of the multiple fibers (see [15], Chapter I Proposition 3.22 for a more complete discussion). We will deal here with the simply connected case, the case of at most two multiple fibers of relatively prime multiplicity. Here the smooth classification of elliptic surfaces with $p_g \geq 1$
has been worked out in Donaldson theory; see [2], [23], [22], [12], as well as [21] and [11]. The general case may be reduced to the simply connected case (in the case of finite cyclic fundamental group) or dealt with either by elementary arguments involving the fundamental group (in case the fundamental group is not finite cyclic), see [15] for this reduction. One could also use the basic classes to determine the multiplicities in the general case along the lines of what we do here for the simply connected case.

Suppose that there are two multiple fibers \( F_1 \) and \( F_2 \), with relatively prime multiplicities \( m_1 \leq m_2 \). Here to handle all cases at once we will also allow \( m_1 \) or both \( m_1 \) and \( m_2 \) to be 1. All the basic classes are of the form \( rk \), where \( k \) is a primitive integral class and \( r \in \mathbb{Q} \). The largest value of \(|r|\) is attained for \( \pm K_X \), and it is \((p_g+1)m_1m_2 - m_1 - m_2\). The next largest value is attained for \( L = \pm (K_X - 2F_2) \), and it is

\[
(p_g + 1)m_1m_2 - m_1 - m_2 - 2m_1,
\]

provided that this number is not negative. Note that since \( p_g \geq 1 \), if \( m_1 \geq 2 \), in other words if there are two multiple fibers, then as \( m_1 < m_2 \),

\[
(p_g + 1)m_1m_2 - m_1 - m_2 - 2m_1 > 4m_2 - 4m_2 = 0.
\]

Thus we determine \((p_g + 1)m_1m_2 - m_1 - m_2 \) and \( m_1 \), and so

\[
((p_g + 1)m_1 - 1)(m_2 - 1) = (p_g + 1)m_1m_2 - m_1 - m_2 - p_g m_1 + 1.
\]

From the knowledge of \( m_1 \) and \((p_g + 1)m_1 - 1)(m_2 - 1)\) we may then determine \( m_2 \).

If \( m_1 = 1 \) then \((p_g + 1)m_1m_2 - m_1 - m_2 - 2m_1 = p_g m_2 - 3 \geq 0 \) provided that \( p_g m_2 \geq 3 \). Except in these cases, we then find \( m_1 = 1 \) and can solve for \( m_2 \) as before. The remaining cases are \( p_g = 1 \), \( m_2 = 1 \) or 2 or \( p_g = 2 \) and \( m_2 = 1 \). For example if \( p_g = 1 \), the basic classes are \( \pm K_0 \) and \( K_0 \) is trivial if \( m_2 = 1 \) and nontrivial otherwise. Thus we can distinguish these cases also. So we have determined the multiplicities of the multiple fibers and thus the plurigenera.

4. The case where \( p_g \) is zero and \( X \) is of general type.

In this section we consider the case where \( p_g(X) = 0 \), i.e. \( b_2^+(X) = 1 \), and \( X \) is of general type. If \( X \) is of general type, then automatically \( b_1(X) = 0 \) since \( \chi(\mathcal{O}_X) = 1 - q(X) + p_g(X) > 0 \). For a 4-manifold \( M \) with \( b_2^+(M) = 1 \), the function \( SW_{M,g} \) is no longer independent of the metric \( g \), at least for \( b_2(M) \) sufficiently large. For the purposes of this paper, we shall only consider basic classes of index zero. Equivalently we shall restrict the function \( SW_{M,g} \) to a function defined on characteristic line bundles \( L \) with \( L^2 = 2\chi(M) + 3\sigma(M) \). In this case, if \( b_2(M) = b \geq 10 \), the orthogonal hyperplanes to characteristic cohomology classes of square \( 10 - b \) divide

\[
\mathbb{H}(M) = \{ \alpha \in H^2(M;\mathbb{R}) : \alpha^2 = 1 \}
\]

into a set of chambers \( \mathcal{C} \), and we can define \( SW_{M,C} \), for each chamber \( \mathcal{C} \), as a function on pairs of characteristic line bundles and \( \text{Spin}^c \) structures \((L,\xi)\) with \( L^2 = 2\chi(M) + 3\sigma(M) \). Here given a metric \( g \), there is a unique associated self-dual
harmonic 2-form $\eta$ for $g$, mod nonzero scalars. For a generic metric $g$, $t\eta$ will lie in the interior of a chamber $C$ for an appropriate positive real number $t$, and we let $SW_{M,C} = SW_{M,g}$ for every metric $g$ whose self-dual 2-form lies in $\mathbb{R}^+ \cdot C$. Changing $\eta$ to $-\eta$ corresponds to changing the orientation of the SW moduli space. Thus

$$SW_{M,-C} = -SW_{M,C}.$$

This procedure defines $SW_{M,C}$ for every $C$ which contains the self-dual harmonic 2-form of a Riemannian metric. However, we can define $SW_{M,C}$ in general formally once we have a wall crossing formula [10]:

**Proposition 4.1.** Suppose $b_1(M) = 0$ and that $C_0$ and $C_1$ are two chambers whose boundaries intersect in an open subset of a wall $L \perp$. Suppose further that there is a path of metrics $\{ g_t : t \in [0,1] \}$ such that the self-dual harmonic 2-form associated to $g_0$ lies in $C_0$ and that the self-dual harmonic 2-form associated to $g_1$ lies in $C_1$. Then

$$SW_{M,C_0}(L', \xi) = \begin{cases} 
SW_{M,C_0}(L', \xi), & \text{if } c_1(L') \neq \pm c_1(L) \text{ in } H^2(M; \mathbb{R}); \\
SW_{M,C_0}(L', \xi) \pm 1, & \text{if } c_1(L') = \pm c_1(L) \text{ in } H^2(M; \mathbb{R}).
\end{cases}$$

Here the sign in the wall crossing formula depends on whether $L' \cdot C_0$ is positive or negative, as well as on the general conventions we have used to orient the moduli space. Although we shall not need to know the sign precisely, let us give the correct choice of sign in the case of interest to us:

**Claim 4.2.** Suppose in the above situation that $X$ is a Kähler surface and that our orientations are chosen so that the SW moduli space has its natural complex orientation. Suppose further that $L$ is a wall of $C_+$ and $C_-$ and that $L \cdot C_+ > 0 > L \cdot C_-$. Then

$$SW_{X,C_+}(L, \xi) = SW_{X,C_-}(L, \xi) - 1.$$

Let us show that the claim holds in a special situation. Suppose that $X$ is a rational surface which is the blowup of $\mathbb{P}^2$ at $d^2$ points which lie on a smooth curve $C$ of degree $d \geq 4$. Thus $g(C) \geq 3$. We continue to denote by $C$ the proper transform of $C$ on $X$. On $X$, $C^2 = 0$ and so $K_X \cdot C + C^2 = K_X \cdot C = 2g(C) - 2 > 0$. Let $H$ denote the pullback of the positive generator of $H^2(\mathbb{P}^2; \mathbb{Z})$ to $X$ and let the classes of the exceptional curves be denoted by $E_1, \ldots, E_d$. Let $g$ be a Kähler metric on $X$ with Kähler form $\omega_0$ equal to $NH - \sum_i E_i$, for $N \gg 0$. Thus as $K_X = -3H + \sum_i E_i$, if $N \gg 0$ $\omega_0 \cdot K_X < 0$. It follows that there are no holomorphic line bundles $L$ on $X$ with $L \cdot \omega_0 \leq 0$ and positive $(K_X + L)$ effective (we would simultaneously have $(K_X + L) \cdot \omega_0 < 0$ and $(K_X + L) \cdot \omega_0 \geq 0$). Thus there are no basic classes for the chamber $C_+$ containing $\omega_0$. Since $C^2 = 0$, the curve $C$ has nonnegative intersection with every irreducible curve, so by the Nakai-Moishezon criterion $\omega = \omega_0 + tC$ is ample for every $t$. Using the fact that $K_X \cdot C > 0$, we can choose $t$ so large that $\omega \cdot K_X > 0$. It follows that $\pm K_X$ are basic classes for the chamber $C_-$ containing $\omega$, and moreover, by (2.4), $SW_{X,C_-}(-K_X) = 1$ for the natural Spin$^c$ structure and the natural choice of complex orientation. Now $C_-$ and $C_+$ are separated by the wall $(-K_X) \perp$, and $-K_X \cdot C_+ > 0 > -K_X \cdot C_-$. Thus $SW_{X,C_+}(-K_X) = SW_{X,C_-}(-K_X) - 1$. 
Corollary 4.4. The chamber basic classes are as claimed. □

Proposition 4.5. $\mathbb{C}K$ which is impossible. Thus above.

Sign associated to the wall-crossing formula and working it out in a special case as above.

Using the wall crossing formula, we can define $SW_{M,C}$ for all chambers $C$, and this definition agrees with $SW_{M,g}$ in case $g$ is a generic Riemannian metric whose associated harmonic self-dual 2-form lies in $C$. A characteristic vector $L$ of the appropriate square for which $SW_{M,C}(L) \neq 0$ will be called a basic class for the chamber $C$ (recall however that we are only considering basic classes of index zero).

We consider now the following situation: $X$ is a minimal surface of general type with $p_g(X) = 0$ and $b_1(X) = 0$. Let $n = K_X^2$, so that $1 \leq n \leq 9$. Let $\tilde{X}$ be a blowup of $X$ at $\ell$ distinct points. Let $K_{0}$ be the pullback of $K_X$ in $\tilde{X}$ and let $E_1, \ldots, E_\ell$ be the exceptional curves.

Proposition 4.3. There is a unique chamber $C_0$ containing $tK_{0}$ in its interior for some positive real number $t$. The chamber $C_0$ is invariant under reflection by the classes $E_i$. The basic classes for $C_0$ are then $\pm K_0 + \sum_{i=1}^{\ell} \pm E_i$, with the natural Spin$^c$ structures.

Proof. If $K_0$ lies on a wall, then $K_0$ is orthogonal to some characteristic cohomology
element $L = A + \sum_{i=1}^{\ell} m_iE_i$. Here $A \in H^2(X;\mathbb{Z}) \subseteq H^2(\tilde{X};\mathbb{Z})$ is characteristic, hence nonzero, and the $m_i$ are odd, hence nonzero. Since $K_0$ is perpendicular to $L$, we have $A^2 < 0$. Thus $L^2 = -\sum m_i^2 < -\ell$. But on the other hand $L^2 = n - \ell$, which is impossible. Thus $K_0$ lies on no wall, hence lies in the interior of a chamber $C_0$. In this case, it follows from (2.2) and from the blowup formula (2.5) that the basic classes are as claimed. □

Corollary 4.4. The chamber $C_0$ has the following properties:

(i) For all characteristic $L$ with $L^2 = K_0^2 = n - \ell$, $SW_{\tilde{X},C_0}(L, \xi)$ is zero or $\pm 1$. Moreover, if $(L, \xi)$ and $(L, \xi')$ are two basic classes for $C_0$, then $\xi = \xi'$.

(ii) There are $2\ell + 1$ $(L, \xi)$ as in (i) such that $SW_{\tilde{X},C_0}(L, \xi) = \pm 1$.

(iii) If $L_1 \neq L_2$ are two classes for which $SW_{\tilde{X},C_0}(L_i, \xi_i) = \pm 1$, $i = 1, 2$, then

\[
\left( \frac{L_1 + L_2}{2} \right)^2 \leq n,
\]

with equality holding for at least one such pair of $L_1$ and $L_2$.

(iv) For every $L_1$ and $L_2$ such that equality holds above, the line through $L_1 + L_2$

meets $C_0$. □

We now claim:

Proposition 4.5.

(i) Suppose that $H^2(X;\mathbb{Z})$ has no 2-torsion, or more generally that the group of 2-torsion elements of $H^2(X;\mathbb{Z})$ is not isomorphic to $\mathbb{Z}/2\mathbb{Z}$. If $C$ is any chamber satisfying (i)—(iii) of (4.4), then $C = \pm C_0$.

(ii) If the group of 2-torsion elements of $H^2(X;\mathbb{Z})$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$, and $C$ is any chamber satisfying (i)—(iii) of (4.4), then $(L, \xi)$ is a basic class for $C$ if and only if there exists a $\xi'$ such that $(L, \xi')$ is a basic class for $C_0$.

Proof. Throughout this proof we shall identify classes in $H^2(\tilde{X};\mathbb{Z})$ with their images in $H^2(\tilde{X};\mathbb{R})$, i.e. with their images mod torsion, since we are only concerned with
If we calculate the square, however, we find:

\[ \text{the chamber structure which lives inside } H^2(X; \mathbb{R}) \text{. Let } \mathcal{C} \text{ be any chamber satisfying (i)--(iii) of (4.4). Since } -\mathcal{C} \text{ also satisfies these conditions, we can assume that } \mathcal{C} \text{ and } \mathcal{C}_0 \text{ lie in the same component of } \mathbb{H}(X). \text{ Now let us show that we may assume that } K_0 + \sum E_i \text{ is positive on both } \mathcal{C}_0 \text{ and } \mathcal{C}. \]

**Lemma 4.6.** Suppose that \( \mathcal{C} \) is a chamber lying on the same component of \( \mathbb{H}(X) \) as \( \mathcal{C}_0 \). Then, possibly after replacing \( \mathcal{C} \) by \( f^*\mathcal{C} \), where \( f \) is an orientation-preserving self-diffeomorphism of \( X \) corresponding to reflection about one of the \( E_i \)'s, we may assume that \( K_0 + \sum E_i \text{ is positive on both } \mathcal{C}_0 \text{ and } \mathcal{C}. \) Thus \( SW_{X,\mathcal{C}}(K_0 + \sum E_i) \neq 0 \). Hence \( SW_{X,\mathcal{C}} \) is not identically zero for any chamber \( \mathcal{C} \). In particular \( X \) is not diffeomorphic to a rational surface and does not have a Riemannian metric of positive scalar curvature.

**Proof.** Suppose that \( x \in \mathcal{C} \). Write \( x = \sum t_i E_i \), where \( B \in H^2(X; \mathbb{R}) \). After composing by a sequence of reflections in the \( E_i \) (which leave the chamber \( \mathcal{C}_0 \) invariant) we may assume that \( t_i \leq 0 \) for all \( i \). Moreover \( x \) and \( K_0 \) lie in the same component of the positive cone of \( H^2(X; \mathbb{R}) \). Thus \( K_0 \cdot x > 0 \). It follows that \( (K_0 + \sum E_i) \cdot x = K_0 \cdot x + \sum t_i > 0 \). Thus \( K_0 + \sum E_i \) is positive on \( \mathcal{C} \). By the wall-crossing formula, since \( K_0 + \sum E_i \) does not separate \( \mathcal{C}_0 \) from \( \mathcal{C}, \) \( SW_{X,\mathcal{C}}(K_0 + \sum E_i) \neq 0 \). Hence \( SW_{X,\mathcal{C}} \) is not zero on any chamber. The final statements are then clear. \( \square \)

We note that the fact that no general type surface \( \tilde{X} \) can be diffeomorphic to a rational surface was proved via Donaldson theory in [16] (see also [19], [27], [28], [31], [32], [30], [29]). Okonek and Telemann [25] have independently observed that one can use the method of (4.6) to show that no surface of general type is diffeomorphic to a rational surface.

Returning to the situation where \( \mathcal{C} \) satisfies (i)--(iii) of (4.4), we see that we can assume that \( SW_{X,\mathcal{C}}(K_0 + \sum E_i) \neq 0 \), and that there exists an \( x \in \mathcal{C} \) of the form \( B - \sum r_i E_i \), where \( B \in H^2(X; \mathbb{R}) \) and \( r_i \geq 0 \). Now consider the walls that separate \( \mathcal{C} \) from \( \mathcal{C}_0 \). Suppose that \( L = C + \sum i (2s_i + 1)E_i \) is such a wall, where \( C \in H^2(X; \mathbb{Z}) \), and suppose that \( SW_{X,\mathcal{C}_0}(L, \xi) = 0 \) for all choices of \( \xi \). Thus \( L \) is a basic class for \( \mathcal{C} \) but not for \( \mathcal{C}_0 \). In this case we shall show that, possibly after modifying \( L \), the condition (iii) of (4.4) does not hold, in other words there is an average of classes for \( \mathcal{C}_0 \) with square greater than \( n \).

Since \( L \) is just defined up to sign, we may assume that \( L \cdot K_0 > 0 \), and so \( L \cdot \mathcal{C}_0 > 0 \). Thus \( L \cdot x < 0 \) for all \( x \in \mathcal{C} \). We claim that we can replace \( L = C + \sum i (2s_i + 1)E_i \) by the class \( L' = C - \sum i (2s_i + 1)E_i \). Indeed,

\[ x \cdot L' = (B \cdot C) - \sum i r_i (2s_i + 1) \leq (B \cdot C) - \sum i r_i (2s_i + 1) = x \cdot L < 0 < K_0 \cdot L = K_0 \cdot L'. \]

Hence the class \( L' \) also defines a wall which separates \( \mathcal{C}_0 \) and \( \mathcal{C} \). Replacing \( L \) by \( L' \), we can assume that \( L = C - \sum i (2s_i + 1)E_i \) where \( 2s_i + 1 \geq 0 \) and so \( s_i \geq 0 \). Moreover \( L \) is a basic class for \( \mathcal{C} \). It follows that \( L^2 = n - \ell \) and so \( C^2 = n - \ell + \sum i (2s_i + 1)^2 \).

Since \( 2s_i + 1 \geq 1 \), with equality only if \( s_i = 0 \), we see that \( C^2 \geq n, \) with equality only if \( s_i = 0 \) for all \( i \).

By assumption on the chamber \( \mathcal{C}, \) \( ((K_0 + \sum E_i) + L)/2 \) has square at most \( n \).

If we calculate the square, however, we find:

\[ \left( \frac{(K_0 + \sum E_i) + L}{2} \right)^2 = \left( \frac{K_0 + C}{2} - \sum s_i E_i \right)^2 = \frac{K_0^2 + C^2 + 2(K_0 \cdot C)}{4} - \sum s_i^2. \]
Now \( K_0^2 = n \) and \( C^2 = L^2 + \sum_i (2s_i + 1)^2 = n - \ell + \sum_i (2s_i + 1)^2 \geq n \). By the Hodge index theorem,
\[
| (K_0 \cdot C) | = (K_0 \cdot C) \geq \sqrt{K_0^2 \cdot C^2} \geq \sqrt{n \cdot n},
\]
with equality holding only if \( K_0 = C \). Hence
\[
\frac{K_0^2 + C^2 + 2(K_0 \cdot C)}{4} - \sum_i s_i^2 \geq \frac{n + n - \ell + \sum_i (2s_i + 1)^2 + 2n}{4} - \sum_i s_i^2,
\]
\[
= \frac{1}{4} (4n + 4 \sum_i s_i^2 + 4 \sum_i s_i) - \sum_i s_i^2 \geq n + \sum_i s_i.
\]
Thus the square of \( (K_0 + \sum_i E_i + L)/2 \) is greater than \( n \) unless \( s_i = 0 \) for all \( i \) and \( K_0 = C \). In this case \( L = K_0 - \sum_i E_i \), which is already a basic class for \( C_0 \). This contradicts the choice of \( L \).

It follows that the only walls which can separate \( C_0 \) from \( C \) are of the form \( L \perp \), where \((L, \xi)\) is a basic class for \( C_0 \). First suppose that \( H^2(X; \mathbb{Z}) \) has no 2-torsion. In this case we shall show that (ii) of (4.4) does not hold, in other words that \( \ell \) decreases, if the set of such walls is nonempty. In any case the basic classes for \( C \) must be a subset of the basic classes for \( C_0 \). The wall crossing formula then implies that, if there is such a wall, then there are fewer classes for \( C \) than for \( C_0 \). This contradicts our assumption on \( C \). (Note that, if we did not know the sign in the wall crossing formula, we would still be able to conclude at this point that either \( C \) had fewer classes than \( C_0 \) or that there existed a basic class \( L \) for \( C \) for which \( SW_{X,C}(L) = \pm 2 \). This would again contradict the choice of \( C \).)

If there is 2-torsion in \( H^2(X; \mathbb{Z}) \), then the wall crossing formula implies that we lose the basic classes \( \pm(L, \xi) \) as we cross the wall \( L \perp \) but we gain new classes of the form \( \pm(L, \xi') \) for \( \xi' \neq \xi \) (recall that for \( C_0 \) there is a unique \( \xi \) such that \((L, \xi)\) is a basic class). If the group of 2-torsion elements of \( H^2(X; \mathbb{Z}) \) is larger than \( \mathbb{Z}/2\mathbb{Z} \), then there would be two distinct \( \text{Spin}^c \) structures \( \xi_1 \neq \xi_2 \) such that \((L, \xi_1)\) and \((L, \xi_2)\) are basic classes for \( C \). Thus \( C \) would violate (i) of (4.4). In the remaining case it is clear that the basic classes for \( C \) are exactly of the form \((L, \xi')\), where \((L, \xi)\) is a basic class for \( C_0 \). \( \square \)

Note that we do not as yet claim that the integer \( n \) or equivalently \( \ell \) is specified by the 4-manifold \( X \), in the sense that there might \textit{a priori} exist other positive integers \( n' \leq 9 \) and \( \ell' \) with \( n' - \ell' = n - \ell \), and a chamber \( C_0' \) satisfying (i)—(iii) of (4.4) for \( n' \) and \( \ell' \). In fact, although we shall not really need this, our arguments show that \( \ell \) is the maximum over all possible \( \ell' \) such that there exists a chamber \( C_0' \) satisfying (i)—(iii) of (4.4) for \( \ell' \) and \( n' = K_X^2 + \ell' \).

Let us now show how to deduce Theorems 1.1 and 1.2 from Proposition 4.5, at least in the case where \( p_g(X) = 0 \) and \( X \) is of general type. First suppose that there is no 2-torsion in \( H^2(X; \mathbb{Z}) \) and that \( f: \tilde{X} \to X \) is an orientation-preserving self-diffeomorphism. Then \( f^*C_0 \) has the same properties (i)—(iii) as \( C_0 \), and so by (4.5) \( f^*C_0 = \pm C_0 \). In particular \( f^* \) leaves invariant the basic classes for \( C_0 \). As in the case \( p_g > 0 \) and \( K_X^2 > 0 \) we see that \( f^* \) preserves \( \pm K_0 \) and the span of the \( E_i \). A similar argument works in case the 2-torsion subgroup of \( H^2(X; \mathbb{Z}) \) is not isomorphic to \( \mathbb{Z}/2\mathbb{Z} \). Finally, if the 2-torsion subgroup of \( H^2(X; \mathbb{Z}) \) is isomorphic
to $\mathbb{Z}/2\mathbb{Z}$ and $f : \tilde{X} \to \tilde{X}$ is an orientation-preserving self-diffeomorphism, then $f^*C_0$ has the same properties (i)—(iii) as $C_0$, and so $f^*$ leaves invariant the set of $L$ such that there exists a $\xi$ with $(L, \xi)$ a basic class for $C_0$. Thus $f^*$ preserves $\pm 2K_0$ and so $f^*C_0 = \pm C_0$. Again we may complete the argument as in the case $p_g > 0$.

Next, let $N$ be a negative definite 4-manifold such that $\tilde{X}$ is diffeomorphic to $M \# N$ for some 4-manifold $M$. Let $x \in H^2(M; \mathbb{R}) \subseteq H^2(\tilde{X}; \mathbb{R})$ satisfy $x^2 = 1$, and assume that $x$ lies on no wall in $\mathbb{H}(M)$ or $\mathbb{H}(\tilde{X})$. Let $\mathcal{C}$ be the chamber of $\mathbb{H}(\tilde{X})$ containing $x$. Let $n_i$ be an exceptional class of $N$ and let $R_i : H^2(\tilde{X}; \mathbb{R}) \to H^2(\tilde{X}; \mathbb{R})$ be the reflection about $n_i$. Note that, as $H^2(N; \mathbb{Z})$ is torsion free, $R_i$ acts on the set of Spin$^c$ structures as well. Clearly $R_i$ fixes $\mathcal{C}$. The blowup formula says that $R_i$ also fixes the set of basic classes for $\mathcal{C}$.

We claim that, in the above situation, the basic classes for $R_i(C_0)$ are exactly those of the form $R_i(L)$, where $L$ is a basic class for $C_0$, and more precisely that $SW_{X, R_i(C_0)}(R_i(L), R_i(\xi)) = SW_{X, C_0}(L, \xi)$. (This would of course be clear if $n_i$ was represented by a smoothly embedded 2-sphere.) Thus, arguing as above, $R_i(C_0)$ satisfies (i)—(iii) of (4.4) and moreover $R_i(C_0) = \pm C_0$, indeed $R_i(C_0) = C_0$ since $R_i$ fixes the components of $\mathbb{H}(\tilde{X})$. It follows that the wall through $n_i$ passes through $C_0$ for every $i$, and that (by induction on the number of exceptional classes) we can choose an $x \in \mathbb{H}(M)$ whose image in $\mathbb{H}(\tilde{X})$ lies in $C_0$. The proof that every exceptional class $n_i$ must be equal to $\pm E_j$ for some $j$ then runs as in the case $p_g(X) > 0$ and $X$ of general type (i.e. $K^2_X > 0$). Thus it suffices to show (we omit the Spin$^c$ structures for notational simplicity):

**Proposition 4.7.** Let $Y = M \# N$ be an oriented smooth 4-manifold such that $b_2^+(M) = 1$ and $N$ is negative definite. For an exceptional class $n_i \in H^2(N; \mathbb{Z})$, let $R_i$ be the reflection about $n_i$, viewed as an automorphism of $H^2(Y; \mathbb{Z})$. Then for every chamber $C_0$ of $\mathbb{H}(Y)$, we have:

$$SW_{Y, R_i(C_0)}(R_i(L)) = SW_{Y, C_0}(L).$$

**Proof.** As above, let $x \in H^2(M; \mathbb{R}) \subseteq H^2(Y; \mathbb{R})$ satisfy $x^2 = 1$, and assume that $x$ lies on no wall in $\mathbb{H}(M)$ or $\mathbb{H}(Y)$. Let $\mathcal{C}$ be the chamber of $\mathbb{H}(Y)$ containing $x$, so that the blowup formula holds for the basic classes of $\mathcal{C}$. To see that $SW_{Y, R_i(C_0)}(R_i(L)) = SW_{Y, C_0}(L)$, we may assume that $\mathcal{C}$ and $C_0$ lie on the same component of $\mathbb{H}(Y)$. Choose an $x_0 \in C_0$ and consider a generic path $\gamma$ from $x$ to $x_0$. Let $L_1, \ldots, L_k$ be the walls crossing $\gamma$. Then if $L \neq L_i$ for any $i$, the wall-crossing formula says that

$$SW_{Y, C_0}(L) = SW_{Y, C}(L).$$

If $L = L_\alpha$ for some $\alpha$, then

$$SW_{Y, C}(L_\alpha) = SW_{Y, C}(L_\alpha) \pm 1.$$

Now consider the function $SW_{Y, R_i(C_0)}$. The point $R_i(x_0)$ lies in $R_i(C_0)$, the path $R_i(\gamma)$ joins $R_i(x)$ to $R_i(x_0)$, and the walls crossed by $R_i(\gamma)$ are the walls $R_i(L_\alpha)$. Note that the blowup formula implies that $L$ is a basic class for $\mathcal{C}$ if and only if $R_i(L)$ is a class for $\mathcal{C}$, and indeed $SW_{Y, C}(L) = SW_{Y, C}(R_i(L))$. Now to calculate $SW_{Y, R_i(C_0)}(R_i(L))$, first assume that $L \neq L_\alpha$ for any $\alpha$. Then

$$SW_{Y, R_i(C_0)}(R_i(L)) = SW_{Y, C}(R_i(L)) = SW_{Y, C}(L) = SW_{Y, C_0}(L).$$
If $L = L_\alpha$, then
\[ \text{SW}_{Y,R_i(C_0)}(R_i(L_\alpha)) = \text{SW}_{Y,C}(R_i(L_\alpha)) \pm 1 = \text{SW}_{Y,C}(L_\alpha) \pm 1. \]

Now examination of the wall crossing formula says that
\[ \text{SW}_{Y,C,0}(L_\alpha) = \text{SW}_{Y,C}(L_\alpha) \pm 1, \]
and the sign must be the same as in the above formula. Putting this together we see that $\text{SW}_{Y,R_i(C_0)}(R_i(L_\alpha)) = \text{SW}_{Y,C,0}(L_\alpha)$, so the formula holds in all cases. □

Thus we have established Theorem 1.2. In particular, the cohomology classes of embedded 2-spheres of self-intersection $-1$ span a sublattice of $H^2(X; \mathbb{Z})$ of rank exactly $\ell$. It follows that if $X'$ is another surface of general type and $f: X' \to X$ is an orientation-preserving diffeomorphism, then $X'$ can be blown up at most $\ell$ times from its minimal model. By symmetry $X$ and $X'$ are blown up the same number of times, namely $\ell$, from their minimal models. If $C_0'$ is the chamber on $X'$ corresponding to $C_0$, it then follows that $f^*C_0' = \pm C_0$. Theorem 1.1 is an immediate consequence. □

Finally let us deduce that if $Y$ is a complex surface diffeomorphic to $\hat{X}$, then $P_n(Y) = P_n(\hat{X})$ for all $n$. Note that $Y$ is Kähler since $b_1(Y) = 0$. Moreover $Y$ cannot be a rational surface by Lemma 4.6. We will rule out the case where $Y$ is elliptic in the next section. Thus $Y$ is again a surface of general type, and by Theorem 1.1 we may determine $K^2_Y$ for $Y$. It follows as in the case where $p_g > 0$ that $P_n(Y) = P_n(\hat{X})$ for all $n$.

We note that Theorem 1.1 has the following corollary:

**Corollary 4.8.** Let $\hat{X}$ be a surface of general type with $p_g(\hat{X}) = 0$, and let $D(\hat{X})$ be the image of the group of orientation-preserving diffeomorphisms of $\hat{X}$ in the automorphism group of $H^2(\hat{X}; \mathbb{Z})$. Then $D(\hat{X})$ is finite.

**Proof.** Let $\phi \in D(\hat{X})$. Up to finite index we may assume that $\phi(E_i) = E_i$ for all $i$ and that $\phi(K_0) = K_0$. Thus $\phi$ is determined by its action on $K^2_0 \cap H^2(\hat{X}; \mathbb{Z})$. Since $K^2_0 \cap H^2(\hat{X}; \mathbb{Z})$ is negative definite, there are only finitely many automorphisms of $K^2_0 \cap H^2(\hat{X}; \mathbb{Z})$. Hence there are only finitely many possibilities for $\phi$. □

5. The case where $p_g$ is zero and $X$ is not of general type.

Suppose that $X$ is a minimal Kähler surface, not of general type, rational, or ruled, such that $p_g(X) = 0$. In this case $X$ is elliptic (possibly an Enriques or hyperelliptic surface), $K_X^2 = 0$, and, since $\chi(O_X) \geq 0$, either $b_1(X) = 0$ and $X$ is elliptic or $b_1(X) = 2$. We shall first consider the parts of the analysis of the topology of a blowup of $X$ which can be handled either by elementary methods or by reduction to the case $p_g > 0$.

If $b_1(X) = 0$, then $X$ is an elliptic surface, obtained from a rational elliptic surface by a number of logarithmic transforms. Here $X$ is rational if there is just one logarithmic transform and it is simply connected (a Dolgachev surface) if there are two logarithmic transforms of relatively prime multiplicities. In all other cases $X$ has a finite covering space $Y$ which is an elliptic surface with $p_g \geq 1$. Let $\hat{X}$ be a blowup of $X$, and let $\hat{Y}$ be the induced cover, which is a blowup of $Y$. In this case,
since Theorems 1.1 and 1.2 hold for $\bar{Y}$, they also hold for $\bar{X}$ mod torsion. We can then determine the plurigenera of $X$, either via elementary arguments involving the fundamental group as in [15] if $\pi_1(X)$ is not finite cyclic or by reducing to the simply connected case with $p_g \geq 1$ in case $\pi_1(X)$ is finite cyclic (but nontrivial). Finally, since $\bar{Y}$ has no Riemannian metric with positive scalar curvature, the same is true for $\bar{X}$.

Likewise, if $X$ is a minimal Kähler surface with $p_g(X) = 0$ and $b_1(X) = 2$, or in other words $q(X) = 1$, then $X$ is an elliptic surface with Euler number zero. In this case either $X$ is ruled over an elliptic base or it is nonruled. If $X$ is not ruled it is a logarithmic transform either of an elliptic surface without singular or multiple fibers over a base curve of genus one, with nontrivial holonomy (in other words, a hyperelliptic surface), or it is a logarithmic transform of $E \times \mathbb{P}^1$ so that the corresponding base orbifold is not spherical (see [15] for more description). In both of these cases, provided that $X$ is not ruled, it again has a finite covering space $Y$ which is an elliptic surface with $p_g \geq 1$. Again, we see that Theorems 1.1 and 1.2 hold for blowups of $X$, mod torsion. The determination of the plurigenera then follows from the fundamental group as in [15], and the fact that $\bar{X}$ has no metric of positive scalar curvature, for every blowup $\bar{X}$ of $X$, follows as in the case where $b_1(X) = 0$ and $\pi_1(X)$ is not trivial.

The remaining case is the case of Dolgachev surfaces. This case was handled via Donaldson theory in [13], [1], [12] (see also [26]). It can also be handled by the methods of this paper, as we now outline. Let $\bar{X}$ be the blowup of a minimal Dolgachev surface $X$, and let $K_0$ be the image of the canonical class of $X$ in $H^2(\bar{X}; \mathbb{Z})$. Suppose that $\bar{X}$ is the blowup of the minimal surface $X$ at $\ell$ distinct points. All of the basic classes for $X$ are rational multiples $rK_0$ of $K_0$ with $|r| \leq 1$. We suppose that there are $d$ basic classes for $X$. It is easy to check that on each basic class of $X$ the value of $SW_X$, which does not depend on a choice of chamber, is $\pm 1$.

Now let $C_0$ be a chamber in $\mathbb{H}(\bar{X})$ which contains classes of the form $\omega$, where $\omega$ is the Kähler form of a generic Kähler metric on $X$. It follows that there are $2d\ell$ basic classes for $C_0$. They are exactly the classes $L + \sum_i \pm E_i$, where $L$ is a basic class for $X$ and the $E_i$ are the exceptional classes on $\bar{X}$, and the value of $SW_{\bar{X}, C_0}$ on each such class is $\pm 1$. Moreover, for every average $\frac{L_1 + L_2}{2}$ of basic classes for $C_0$, we have $\left( \frac{L_1 + L_2}{2} \right)^2 \leq 0$, with equality holding for some pair of basic classes $L_1 \neq \pm L_2$. Arguments very similar to those given in the proof of Proposition 4.5 show that $\pm C_0$ are the unique chambers with these properties. Thus every orientation-preserving self-diffeomorphism $f : \bar{X} \to \bar{X}$ satisfies $f^*C_0 = \pm C_0$, and if $n_i$ is an exceptional class for a negative definite summand of $\bar{X}$, then the reflection $R_i$ in $n_i$ preserves $C_0$. Moreover, in the above two cases, we see that both $f^*$ and $R_i$ permute the set of basic classes. In particular the wall through $n_i$ passes through $C_0$ and $n_i$ is a difference class for $C_0$. It follows then from the arguments in Section 3 for the case $K_0^2 = 0$ that $f^*$ preserves $\pm K_0$ and that $n_j = \pm E_j$ for some $j$. In particular there can be at most $\ell$ disjoint smoothly embedded 2-spheres of self-intersection $-1$ on $\bar{X}$. Thus $\bar{X}$ cannot be diffeomorphic to a blown up surface of general type. The basic classes for $C_0$ determine the the basic classes for $X$, by arguments similar to those in Section 3 for the case where $K_X^2 = 0$. Finally, the
basic classes for $X$ determine the multiplicities of the multiple fibers, by arguments along the lines of those given in Section 3 for simply connected elliptic surfaces with $p_g \geq 0$. Here, in case $p_g = 0$, there are a few extra cases to consider. Lastly, the arguments used to prove Lemma 4.6 also show that there is no chamber $C$ where the function $SW_{\hat{X},C}$ is identically zero, and thus $\hat{X}$ has no metric of positive scalar curvature. Thus we have completed the proofs of Corollary 1.4 and Corollary 1.5.

Finally we note that one can modify the proofs of the results in Section 4 to handle the non-simply connectedelliptic surfaces with $p_g = 0$, and replace equality mod torsion with equality in the non simply connected case. The main point is to handle wall crossings in case $b_1(X) = 2$. However, we shall not give these arguments here.

6. Some open questions.

The non-Kähler case. Can one generalize the above results to the non-Kähler case? The structure of non-Kähler complex surfaces of nonnegative Kodaira dimension is well-understood. In particular they are all elliptic surfaces with odd first Betti number and Euler number zero. Elementary methods [15] show that all such surfaces are $K(\pi,1)$’s and hence that the classes of exceptional curves are preserved under diffeomorphisms. It is also straightforward to show that the class of a general fiber is preserved, and so $K_X$ mod torsion, and that the plurigenera are diffeomorphism invariants. However, it does not seem possible to extend these methods to handle negative definite connected summands. On the other hand, the analysis of the SW equations for Kähler or symplectic manifolds may admit a straightforward generalization to this case (possibly under some assumptions on the metric).

Ruled surfaces. First we note that the Seiberg-Witten theory accounts for all the self-diffeomorphisms of a rational surface $X$. Let $X$ be the blowup of $\mathbb{P}^2$ at $\ell$ distinct points. Inside $\mathbb{H}(X)$ there is a distinguished convex set, the “super $P$-cell” $S_0$ defined in [13]. Its walls are certain characteristic elements of $H^2(X;\mathbb{Z})$ of square $K_X^2$. In fact one can show that $S_0$ is a chamber for the set of walls defined by primitive characteristic elements of square $K_X^2$. It is shown in [13] that an automorphism $\varphi$ of the lattice $H^2(X;\mathbb{Z})$ is the image of an orientation-preserving self-diffeomorphism of $X$ if and only if $\varphi(S_0) = \pm S_0$. This result can be established by Seiberg-Witten theory as well. It is elementary to show that every $\varphi$ such that $\varphi(S_0) = \pm S_0$ is the image of a diffeomorphism, and the difficult part of the argument is to show that, for every orientation-preserving self-diffeomorphism $\psi$ of $X$, $\psi^*(S_0) = \pm S_0$. Let $C_0$ be the chamber of $\mathbb{H}(X)$, for the set of all walls defined by characteristic elements of $H^2(X;\mathbb{Z})$ of square $K_X^2$, which contains $\omega_0$, where $\omega_0$ is a Kähler metric on $\mathbb{P}^2$, or equivalently contains Kähler metrics on $X$ with Kähler form a positive multiple of $N\omega_0 - \sum_i E_i$. Thus $C_0$ contains the period points of metrics with positive scalar curvature, and so $SW_{X,C_0}$ is identically zero (this also follows from the blowup formula). Moreover $\pm C_0$ are the unique chambers $C$ such that $SW_{X,C}$ is identically zero. Hence $\psi^*C_0 = \pm C_0$. For simplicity assume that $\psi^*C_0 = C_0$. Now the interior of $C_0$ is nonempty and is contained in the interior of $S_0$, since the walls defining $S_0$ are a subset of the walls defining $C_0$ and $\omega_0 \in C_0 \cap S_0$. Thus $\psi^*(S_0)$ is a super $P$-cell such that $\psi^*(S_0) \cap S_0 \neq \emptyset$. It follows by Lemma 5.3(e) on p. 339 of [13] that $\psi^*(S_0) = S_0$.

Similar arguments imply that, if $N$ is a negative definite summand of $X$ and $n_i$ is an exceptional class for $N$, then the reflection in $n_i$ preserves $S_0$. Using
this fact and the method of proof of [16], Theorem 1.7, one can show that every
negative definite summand \( N \) of \( X \) can be accounted for in the following sense: if
\( X \) is orientation-preserving diffeomorphic to \( M \# N \), where \( N \) is negative definite,
then there is an orientation-preserving diffeomorphism \( f: X \to \tilde{Y} \), where \( \tilde{Y} \) is the
blowup of a rational surface \( Y \), such that \( H^2(N; \mathbb{Z}) \) corresponds to the span of the
exceptional classes of the blowup \( \tilde{Y} \to Y \).

If one tries to extend the above results to surfaces \( X \) which are (not neces-
sarily minimal) ruled surfaces over a nonrational base curve, most of the results
extend with elementary proofs. For example, let \( f \) be the class of a general fiber
of \( X \) and \( E_1, \ldots, E_t \) be the classes of the exceptional curves. Then the cohomol-
ogy classes of smoothly embedded 2-spheres of self-intersection zero are exactly the
classes \( nf, n \in \mathbb{Z} \), and the cohomology classes of smoothly embedded 2-spheres of
self-intersection \( -1 \) are the classes \( nf + E_i, n \in \mathbb{Z} \). Moreover every orientation-
preserving self-diffeomorphism \( \psi \) of \( X \) satisfies \( \psi^*f = \pm f \), and indeed an automor-
phism \( \varphi \) of \( H^2(X; \mathbb{Z}) \) is the image of an orientation-preserving self-diffeomorphism
if and only if \( \psi^*f = \pm f \). However, it is not clear how to generalize these results
to arbitrary negative definite summands. Such generalizations would presumably
follow from Seiberg-Witten theory provided that we have a better understanding
of the transition formula in case \( b_1(X) \neq 0 \). Working out such transition formulas
is of course an interesting problem in its own right.

**Questions seemingly inaccessible to Seiberg-Witten theory.** The overall
moral of the above is that the sum total of the basic classes gives us information
about the obvious invariants of a complex surface, the exceptional curves and the
pullback of the canonical class of the minimal model, and no more. Presumably
the same is true of Donaldson theory. One can ask if there is more to the smooth
topology of a complex surface than this. For example, there exist minimal surfaces
of general type, say \( X_1 \) and \( X_2 \), and an isometry \( \varphi: H^2(X_1; \mathbb{Z}) \to H^2(X_2; \mathbb{Z}) \) such
that \( \varphi(K_{X_1}) = K_{X_2} \), and such that \( X_1 \) and \( X_2 \) are not deformation equivalent. As
has been pointed out by Fintushel and Stern, certain pairs of Horikawa surfaces
are among the simplest examples of surfaces with this property. In particular we
cannot distinguish such pairs \( X_1 \) and \( X_2 \) via the basic classes, and, at least in the
case of Horikawa surfaces, they have the same Donaldson polynomials as well, as
has been shown by the second author and Z. Szabó. Can one find new smooth
invariants which will show that \( X_1 \) and \( X_2 \) are not diffeomorphic?

In a similar vein, are there further restrictions on self-diffeomorphisms \( f \) of a
Kähler surface \( X \) of nonnegative Kodaira dimension beyond the conditions \( f^*K_0 =
\pm K_0 \) and \( f^*E_i = \pm E_i \)? For example, if the algebraic geometry of \( X \) imposes a
fundamental asymmetry on \( X \), is this seen by the smooth topology? One example
of this might be a surface which is the double cover of \( \mathbb{P}^1 \times \mathbb{P}^1 \) along an asymmetric
branch divisor in the linear system \( |2af_1 + 2bf_2| \), where the \( f_i \) are the fibers of the
two different projections and \( a, b \) are positive integers with \( a \neq b \). Is it true that
for general \( a \) and \( b \) every orientation-preserving self-diffeomorphism of \( X \) preserves
the pullbacks to \( X \) of \( f_1 \) and \( f_2 \) up to sign (and not just \( K_X \) which is a positive
combination of the pullbacks)? Even for simply connected elliptic surfaces, there
is a gap of finite index between those isometries of \( H^2(X; \mathbb{Z}) \) known to arise from
self-diffeomorphisms and the restrictions placed on such isometries by Donaldson
theory or Seiberg-Witten theory (see for example [15], Chapter II, Theorem 6.5 and
[13]). Can one close this gap by constructing more diffeomorphisms, or can one find
new invariants which rule out the existence of such diffeomorphisms?

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