ON THE DERIVED CATEGORY OF QUASI-HEREDITARY
ALGEBRAS WITH TWO SIMPLE MODULES

BY

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Abstract. We describe the derived Picard groups and two-term silting complexes
for quasi-hereditary algebras with two simple modules. We also describe by quivers with
relations all algebras derived equivalent to a quasi-hereditary algebra with two simple
modules.

1. Introduction. Derived categories play an important role in many
branches of mathematics. According to [22], if two derived categories of al-
gebras over a field are equivalent, then there is an equivalence induced by ten-
sor product with a tilting complex of bimodules. Such equivalences are closed
under composition, and thus autoequivalences induced by tilting complexes
of bimodules form a subgroup of the group of derived autoequivalences. This
group is called the derived Picard group of an algebra and was first introduced
in [24] and [28]. Later it was shown in [29] that this group is locally algebraic.
It was studied for many classes of algebras. For example, it was described for
hereditary algebras in [18], for selfinjective Nakayama algebras in [26, 27, 30]
and for preprojective algebras of Dynkin quivers in [4, 19]. The question if
this group coincides with the group of all autoequivalences is still open.

As already mentioned, equivalences of derived categories of algebras are
strongly connected with tilting complexes. A generalization of a tilting com-
plex is a silting complex, which in some sense gives an equivalence between
the derived category of the original algebra and the derived category of some
DG algebra. It was shown recently (see [3, 5, 7, 13]) that two-term silting
complexes constitute an interesting and deep object related to other im-
portant notions such as \( \tau \)-tilting modules and bricks. These structures were
studied, for example, for Nakayama algebras in [1], for the Auslander algebra
of \( k[x]/(x^n) \) in [14], for Brauer graph algebras in [2] and for preprojective
algebras associated with symmetrizable Cartan matrices in [10].

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Quasi-hereditary algebras are finite-dimensional algebras introduced in \cite{6, 25}. Examples of such algebras arise in the representation theory of groups and Lie algebras. Also, they include all finite-dimensional algebras of global dimension 2 (see \cite{8}).

This paper is devoted to quasi-hereditary algebras with two simple modules. Such algebras were classified in \cite{17}. Moreover, it was shown in \cite{9} that these algebras are exactly the finite-dimensional algebras of global dimension 1 or 2 having two simple modules.

Let us recall some results of \cite{9}. The author showed how one can obtain all tilting modules over a finite-dimensional algebra of global dimension 2 with two simple modules. It was also proved that any algebra whose derived category is generated by an exceptional pair is derived equivalent to one of the algebras considered in that paper. Moreover, all algebras derived equivalent to an algebra of global dimension 2 with two simple modules were described as endomorphism algebras of some tilting modules described there. All of these algebras are shown to be of global dimension 3. We will continue the study initiated in \cite{9}.

Using the results of \cite{9} we will describe the derived Picard groups of all quasi-hereditary algebras with two simple modules. We will also describe all two-term silting complexes over these algebras and using this we will get a description of all algebras derived equivalent to quasi-hereditary algebras with two simple modules. Our description will be more concrete than the one in \cite{9}: we will describe all algebras from our classification by quivers with relations. Note also that due to results of \cite{16} we thus obtain a description of all algebras with two simple modules that are not derived simple.

2. Silting and tilting complexes. We fix an algebraically closed field \(k\). All vector spaces and algebras under consideration are over \(k\). By a module we always mean a right module. For an algebra \(\Lambda\), we denote by \(C^b_\Lambda\), \(K^b_\Lambda\), \(K^{b,p}_\Lambda\) and \(D^b_\Lambda\) the category of bounded complexes of finitely generated \(\Lambda\)-modules, the bounded homotopy category of finitely generated \(\Lambda\)-modules, the bounded homotopy category of finitely generated projective \(\Lambda\)-modules and the bounded derived category of finitely generated \(\Lambda\)-modules respectively. We denote by \(J_\Lambda\) the Jacobson radical of \(\Lambda\). All complexes in this paper are equipped with a differential of degree 1, and for \(X \in C^b_\Lambda\) the complex \(X[r] (r \in \mathbb{Z})\) has terms \(X[r]_i = X_{i+r}\) and differential \((-1)^rd_X\), where \(d_X\) is the differential of \(X\).

Recall that \(X \in K^{b,p}_\Lambda\) is called presilting if \(\text{Hom}_{K^b_\Lambda}(X, X[i]) = 0\) for \(i > 0\). If this property is satisfied for all \(i \neq 0\), then \(X\) is called pretilting. If \(X\) is presilting [pretilting] and additionally the smallest full triangulated subcategory of \(K^{b,p}_\Lambda\) which contains \(X\) and is closed under direct summands coincides
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with $K^b_{\Lambda}$, then $X$ is called silting [tilting]. The complex $X$ is called basic if there are no complexes $Y$ and $X'$ such that $Y$ is not isomorphic to zero in $K^b_{\Lambda}$ and $X \cong Y^2 \oplus X'$ in $K^b_{\Lambda}$.

It was proved in [21] that if algebras $\Lambda$ and $\Gamma$ are derived equivalent, then there exists a tilting complex $X \in K^b_{\Lambda}$ such that $\text{End}_{K^b_{\Lambda}}(X)$ is isomorphic to $\Gamma$ as a $k$-algebra. In the same paper it is explained how to construct an equivalence from $D^b_{\Gamma}$ to $D^b_{\Lambda}$ sending $\Gamma$ to $X$ using the tilting complex $X$ and an algebra isomorphism $\Gamma \cong \text{End}_{K^b_{\Lambda}}(X)$. One can also find in [26, 27] a construction of an equivalence from $K^b_{\Gamma}$ to $K^b_{\Lambda}$ using the same data.

In the current paper we will use the fact that if $U = (U_i \to \cdots \to U_j)$ is an object of $K^b_{\Gamma}$, then the corresponding equivalence sends $U$ to a totalization of a bicomplex whose $k$th column is the images of $U_k$ under this equivalence, while the image of $U_k$ can be calculated using the fact that $U_k$ is a direct sum of direct summands of $\Gamma$.

Equivalences that can be constructed using the algorithm just mentioned are called standard. Standard equivalences from $K^b_{\Lambda}$ to itself considered modulo natural isomorphisms constitute a group under composition, which is called the derived Picard group of $\Lambda$ and is denoted by $\text{TrPic}(\Lambda)$ (see [26]). This group was first introduced in [24, 28] as a group of isomorphism classes of two-sided tilting complexes of $\Lambda$-bimodules under the operation of derived tensor product.

Let us also recall some basic facts about two-term silting complexes. A two-term complex in this paper is always concentrated in degrees $-1$ and $0$. Since we need to deal only with finite-dimensional algebras having two simple modules, we restrict our discussion to such algebras. Then any basic two-term silting complex has two indecomposable presilting direct summands.

Let $X = X_0 \oplus X_1$ be a two-term silting complex, where $X_0$ and $X_1$ are indecomposable direct summands. Then for $i = 0, 1$ one can define complexes $\mu^\pm_{X_i}(X_{1-i})$ in the following way. Suppose the classes of $f_1, \ldots, f_n \in \text{Hom}_{K^b_{\Lambda}}(X_{1-i}, X_i)$ constitute a basis of

$$\text{Hom}_{K^b_{\Lambda}}(X_{1-i}, X_i)/(J_{\text{End}_{K^b_{\Lambda}}(X_i)} \text{Hom}_{K^b_{\Lambda}}(X_{1-i}, X_i)),$$

and the classes of $g_1, \ldots, g_m \in \text{Hom}_{K^b_{\Lambda}}(X_i, X_{1-i})$ form a basis of

$$\text{Hom}_{K^b_{\Lambda}}(X_i, X_{1-i})/(\text{Hom}_{K^b_{\Lambda}}(X_i, X_{1-i})J_{\text{End}_{K^b_{\Lambda}}(X_i)}).$$

Then we define $\mu^+_{X_i}(X_{1-i})$ and $\mu^-_{X_i}(X_{1-i})$ by the triangles

$$X_{1-i} \xrightarrow{(f_1 \cdots f_n)} X_i \xrightarrow{\mu^+_{X_i}(X_{1-i})} \mu^+_{X_i}(X_{1-i}) \to X_{1-i}, \quad \mu^-_{X_i}(X_{1-i}) \to X_{1-i} \xrightarrow{(g_1 \cdots g_m)} X_{1-i}.$$
Thus, $\mu_{X_i}^\pm(X_{1-i})$ is a cone of a minimal right approximation and $\mu_{X_i}^\pm(X_{1-i})$ a cone of a minimal left approximation of $X_{1-i}$ with respect to $X_i$. Then $X_i \oplus \mu_{X_i}^+(X_{1-i})$ and $\mu_{X_1-i}^-(X_i) \oplus X_{1-i}$ are silting complexes. It follows from the results of \cite{3} that if $X \neq \Lambda, \Lambda[1]$, then for exactly one $i \in \{0, 1\}$ both the resulting complexes can be represented by a two-term complex in $\mathbb{K}^{bp}_\Lambda$. We fix that $i$ and introduce two-term silting complexes $\mu^+(X) = X_i \oplus \mu_{X_i}^+(X_{1-i})$ and $\mu^-(X) = \mu_{X_1-i}^-(X_i) \oplus X_{1-i}$. We will call $\mu^+(X)$ and $\mu^-(X)$ mutations of the complex $X$. Suppose that $\Lambda = P_1 \oplus P_2$, where $P_1$ and $P_2$ are indecomposable projective $\Lambda$-modules. The mutations of $\Lambda$ are by definition the complexes $P_1 \oplus \mu_{P_1}^+(P_2)$ and $\mu_{P_2}^+(P_1) \oplus P_2$ and we denote both by $\mu^+(\Lambda)$. Analogously, the mutations of $\Lambda[1]$ are $P_1 \oplus \mu_{P_1}^-(P_2)$ and $\mu_{P_2}^-(P_1) \oplus P_2$ and we denote both by $\mu^-(\Lambda[1])$. For any two-term silting complex $X$, one has $\mu^+ \mu^-(X) = \mu^- \mu^+(X) = X$ whenever everything is well defined after an appropriate choice of $\mu^-$ and $\mu^+$ if this is needed.

Any two-term silting complex $X$ is a sum of two indecomposable two-term presilting complexes, each of the form $X_i = P_{1_i}^{a_i} \oplus P_{2_i}^{b_i} \rightarrow P_{1_i}^{c_i} \oplus P_{2_i}^{d_i}$ ($i = 0, 1$), where either $a_0 = a_1 = 0$ or $c_0 = c_1 = 0$, and either $b_0 = b_1 = 0$ or $d_0 = d_1 = 0$. We will call the vector $(c_i - a_i, d_i - b_i)$ the $g$-vector of $X_i$ and denote it by $g_{X_i}$. Then we get the point $O_{X_i} = g_{X_i} / \lvert g_{X_i} \rvert$ on the unit circle. The points $O_{X_0}$ and $O_{X_1}$ divide the unit circle into two arcs. We denote by $L_X$ the shorter of these arcs. Due to \cite{3} there are two-term silting complexes $Y^0$ and $Y^1$ that can be obtained via the mutation process from $X$ such that $L_{Y_i} \cap L_X = O_{X_i}$ for $i = 0, 1$. Moreover, it is shown in \cite{7} that for a two-term silting complex $Y \neq X, Y^0, Y^1$ one has $L_X \cap L_Y = \emptyset$.

### 3. Quasi-hereditary algebras with two simple modules.

Let $Q^{p,q}$ ($p, q > 0$) be the quiver with two vertices 1 and 2, $p$ arrows $\alpha_1, \ldots, \alpha_p$ from vertex 1 to vertex 2 and $q$ arrows $\beta_1, \ldots, \beta_q$ from vertex 2 to vertex 1. Let $I^{p,q}$ be the ideal of $kQ^{p,q}$ generated by the paths $\beta_j \alpha_i$ ($1 \leq i \leq p, 1 \leq j \leq q$).

\begin{center}
\begin{tikzpicture}
  \node (1) at (0,0) {1};
  \node (2) at (1,0) {2};
  \draw[->] (1) -- node[above] {$\alpha_1$} (2);
  \draw[->] (1) -- node[below] {$\alpha_p$} (2);
  \draw[->] (2) -- node[above] {$\beta_1$} (1);
  \draw[->] (2) -- node[below] {$\beta_q$} (1);
\end{tikzpicture}
\end{center}

Our main objects of interest are the algebras $\Lambda^{p,q} = kQ^{p,q} / I^{p,q}$. It is proved in \cite{17} that all the algebras $\Lambda^{p,q}$ are quasi-hereditary and that any basic quasi-hereditary algebra with two simple modules over an algebraically closed field is isomorphic to $\Lambda^{p,q}$ for some integers $p$ and $q$. The algebra $\Lambda^{0,0}$ is semisimple and does not deserve our attention. The algebra $\Lambda^{p,0}$ is the $p$-Kronecker
We also classify two-term silting complexes over an algebra of length not greater than 2 based on the fact that any tilting complex can be sent to a tilting complex of length not greater than 2 via the Serre functor on the category $K^b_{\Lambda^{p,q}}$. Our description is based on the fact that any tilting complex can be sent to a tilting complex of length not greater than 2 via the Serre functor on the category $K^b_{\Lambda^{p,q}}$. We also classify two-term silting complexes over $\Lambda^{p,q}$ and describe all the algebras derived equivalent to $\Lambda^{p,q}$.

As a quasi-hereditary algebra, $\Lambda^{p,q}$ has a sequence of standard objects and a sequence of costandard objects. One can easily check that the sequence of standard objects for $\Lambda^{p,q}$ is $(S_2, P_1)$ and the sequence of costandard objects is $(I_1, S_2)$, where we denote by $S_i$, $P_i$, and $I_i$ the simple, projective and injective modules corresponding to vertex $i$ respectively. We will use the same notation for modules over the algebra $\Lambda^{p,q}$ and for modules over $\Lambda^{q,p}$. Which category is meant will each time be clear from the context.

4. Derived Picard group. In this section we describe the derived Picard group of $\Lambda^{p,q}$. Let us first recall the definition of Ringel duality, specialize it to $\Lambda^{p,q}$ and give a consequence of the results of [9] that will play a crucial role in the current paper.

For a quasi-hereditary algebra $\Lambda$, there exists a quasi-hereditary algebra $\Gamma$ with a tilting $\Gamma$-module $T$ such that $\mathop{End}_{\Gamma}(T) \cong \Lambda$ and $\mathop{Filt}(\Delta) \cap \mathop{Filt}(\nabla) = \mathop{add} T$, where $\mathop{Filt}(\Delta)$ is the subcategory of $\Gamma$-modules admitting a filtration by standard modules and $\mathop{Filt}(\nabla)$ is the subcategory of $\Gamma$-modules admitting a filtration by costandard modules (see [23]). This tilting module gives the Ringel duality functor $\omega : \mathop{D^b}_\Gamma \to \mathop{D^b}_\Gamma$ that sends standard objects of $\Lambda$ to costandard objects of $\Gamma$. The algebra $\Gamma$ is called the Ringel dual of $\Lambda$. If $\Lambda = \Lambda^{p,q}$, then a direct calculation shows that $\Lambda^{op} = \Lambda^{q,p}$ is the Ringel dual of $\Lambda$ (see [16]) and the $\Lambda^{q,p}$-module $T$ is a direct sum of the module $S_2$ and the cokernel of the map $P_1^{q-1} \to P_2^q$ that sends the $n$th standard generator of $P_1$ with $n = qi + j < q^2$ ($0 \leq i \leq q - 1$, $1 \leq j \leq q$) to $\nu_i(\alpha_j)$ if $i \neq j$ and to $\nu_i(\alpha_j) - \nu_q(\alpha_q)$ if $i = j$. Here we denote by $\nu_i : P_2 \to P_2^q$ the canonical $i$th embedding. We will write $\omega_{p,q}$ instead of simply $\omega$ to emphasize that we mean the Ringel duality functor for $\Lambda^{p,q}$. Note that $S_2$ is the cokernel of the map $P_1^q \xrightarrow{(\alpha_1, \ldots, \alpha_q)} P_2$. Thus, $\omega_{p,q}$ sends $P_1$ to $P_1^q \to P_2$ and $P_2$ to $P_2^q \to P_2^q$. It can also be shown that the inverse of the Ringel duality functor from $\Lambda^{q,p}$ to $\Lambda^{p,q}$, which we will denote by $\omega_{q,p}^{-1}$, sends $P_1$ to $P_2 \to P_1^p$ and $P_2$ to $P_2^p \to P_1^{p^2 - 1}$. In the first case the corresponding tilting complex is
concentrated in degrees 0 and −1 and in the second case it is concentrated in degrees 0 and 1, but this is not very important for us at this point. Note also that \( \omega_{q,p}^{-1}(S_2) = P_1 \).

Let us introduce the derived autoequivalence \( \nu_{p,q} \) of \( \Lambda^{p,q} \) by the equality \( \nu_{p,q} = \omega_{q,p} \omega_{p,q} \). It follows (for example) from [15] that \( \nu_{p,q} \) is a Serre functor, and the following lemma follows directly from [9, Theorem 3.3].

**Lemma 4.1.** Let \( X \) be a tilting complex. Then there is some \( m \in \mathbb{Z} \) such that \( \nu_{p,q}^m(X) \) can be represented by a two-term complex of projective modules.

**Proof.** By [9, Theorem 3.3], the complex \( \nu_{p,q}^m(X) \) is isomorphic to a \( \Lambda^{p,q} \)-module for some \( m \in \mathbb{Z} \). By the same theorem, either projective or injective dimension of the resulting module equals 1. If the projective dimension is 1, then we are done. In the other case, \( \nu_{p,q}^{-1}(X) \) can be represented by a two-term complex of projective modules because \( \nu_{p,q}^{-1} \) sends injective modules to projective ones.

Let us now classify all two-term tilting complexes giving autoequivalences of the derived category of \( \Lambda^{p,q} \). Note that such a complex has an indecomposable direct summand with endomorphism algebra \( \mathbf{k} \). Let us recall that any object of \( K_{b,p}^{b,p} \) can be represented by a unique (up to isomorphism in the category \( C_{b}^{b} \)) complex \((X, d)\) such that \( \text{Im} d \subset XJ_{\Lambda} \). Such a complex \( X \) is called a **radical complex**. We will use the general fact that if \( X \) is a radical silting complex with nonzero components in the range \([m, n]\) with \( m < n \), then \( X_m \) and \( X_n \) cannot contain a common nonzero direct summand. Indeed, if \( \iota : P \rightarrow X_n \) is a direct embedding and \( \pi : X_m \rightarrow P \) is a projection on a direct summand, then the morphism from \( X \) to \( X[n-m] \) that equals \( \iota \pi \) in degree \( n \) and equals 0 in all other degrees represents a nonzero morphism in \( K_{b}^{b} \). It follows from the fact that \( \text{Im}(\iota \pi) \not\subset X_mJ_{\Lambda} \) and any null homotopic morphism has to have image in \( XJ_{\Lambda} \).

**Lemma 4.2.** Suppose that \( X \) is a radical indecomposable pretilting complex over \( \Lambda = \Lambda^{p,q} \) concentrated in degrees 0 and −1 such that \( \text{End}_{K_{b}^{b}}(X) \cong \mathbf{k} \). Then \( X \) is one of the complexes \( P_1, P_1[1], \omega_{q,p}(P_1) \) or \( \omega_{p,q}^{-1}(P_1)[1] \).

**Proof.** Suppose first that \( X \) has the form \( P_1^a \rightarrow P_2^b \). We may assume that \( b > 0 \). Since \( \text{Hom}_{K_{b}^{b}}(X, X[1]) = \text{Hom}_{K_{b}^{b}}(X, X[-1]) = 0 \), by [12] we have

\[
1 = \dim_{\mathbf{k}} \text{End}_{K_{b}^{b}}(X) = a^2 + b^2(1 + pq) - ab(p + q).
\]

Note now that \( \dim_{\mathbf{k}} \text{Hom}_{C_{b}^{b}}(X, X[1]) = \dim_{\mathbf{k}} \text{Hom}_{\Lambda}(P_1^a, P_2^b) = abp \). On the other hand, any radical map from \( P_2^b \) to \( P_2^b \) is annihilated by the differential of \( X \), and hence the number of linearly independent null homotopic maps from \( X \) to \( X[1] \) is not greater than \( a^2 + b^2 - 1 \). We subtract 1 because there is the identity map from \( X \) to \( X \) that gives the zero null homotopic map.
Thus, we have $ab p \leq a^2 + b^2 - 1$. Subtracting (4.1) from this inequality, we get $bp \leq a$.

Note that the map from $P^a_1$ to $P^b_2$ has to be injective because in the opposite case its kernel would be a direct summand of $X$. We have $a(q + 1) = \dim_k P^a_1 \leq \dim_k P^b_2 J = bp(q + 1)$, and hence $a \leq bp$. Thus, $a = bp$ and it follows from (4.1) that $X = \omega_{q,p}(P_1)$.

If $X$ has the form $P^a_2 \rightarrow P^b_1$, then the complex $\text{Hom}_{\Lambda_{p,q}}(X[-1], \Lambda^{p,q})$ of $\Lambda^{q,p}$-modules has the form $P^a_1 \rightarrow P^b_2$, and hence it is isomorphic to $P_1[1]$ or $\omega_{p,q}(P_1)$ by the argument above. Then $X$ is isomorphic to $P_1$ or $\omega_{p,q}^{-1}(P_1)[1]$. ■

Recall that, for a finite-dimensional algebra $\Lambda$, the Picard group $\text{Pic}(\Lambda)$ is the group of autoequivalences of the category of $\Lambda$-modules modulo natural isomorphisms. If $\Lambda$ is basic, then this group is isomorphic to the group of outer automorphisms, $\text{Out}(\Lambda) = \text{Aut}(\Lambda)/\text{Inn}(\Lambda)$. Here $\text{Aut}(\Lambda)$ is the group of all automorphisms of $\Lambda$ and $\text{Inn}(\Lambda)$ is the group of inner automorphisms. This isomorphism is induced by the map $\text{Aut}(\Lambda) \rightarrow \text{Pic}(\Lambda)$ that sends an automorphism $\theta : \Lambda \rightarrow \Lambda$ to the autoequivalence $- \otimes \Lambda \theta^{-1}$. Here $\Lambda_{\theta^{-1}}$ is the bimodule coinciding with $\Lambda$ as a left module and having the right multiplication $\ast$ by elements of $\Lambda$ defined by $x \ast a = x\theta^{-1}(a)$, where the multiplication on the right side is the original multiplication of $\Lambda$. The group $\text{Pic}(\Lambda)$ is a subgroup of $\text{TrPic}(\Lambda)$ in a natural way. Moreover, an element of $\text{TrPic}(\Lambda)$ belongs to $\text{Pic}(\Lambda)$ if and only if the radical representative of the corresponding tilting complex is concentrated in degree 0. In fact, the autoequivalence $- \otimes \Lambda \Lambda_{\theta^{-1}}$ can be defined by the tilting complex $\Lambda$ and the isomorphism from $\Lambda$ to $\text{End}_\Lambda(\Lambda)$ that sends $x \in \Lambda$ to left multiplication by $\theta(x)$.

**Corollary 4.3.** Suppose that $\Lambda = \Lambda^{p,q}$ for some $p, q > 0$. Let $H \cong \text{Pic}(\Lambda) \times \mathbb{Z}$ be the subgroup of $\text{TrPic}(\Lambda)$ generated by the Picard group of $\Lambda$ and the shift and $K \cong \mathbb{Z}$ be the subgroup generated by $\omega_{p,p}$ if $p = q$ and by $\nu_{p,q}$ if $p \neq q$. Then $K \cap H = \{\text{Id}\}$ and $KH = \text{TrPic}(\Lambda)$.

**Proof.** Let us prove that $K \cap H = \{\text{Id}\}$. Since for any $F \in H$ the radical representative of $F(\Lambda)$ is concentrated in one degree, it is enough to prove for the generator $\rho$ of $K$ that $\rho^t(\Lambda)$ has length more than 1 for any $t > 0$. Note that the radical representative of $\omega_{p,q}(\Lambda)$ is concentrated in degrees $-1$ and 0 and has the form $P^a_1 \rightarrow P^b_2$. Then it is enough to prove that, for any radical complex $X$ concentrated in the interval $[-l, 0]$ such that $X_0 = P^b_2$ and $X_{-l} = P^a_1$ for some $a, b > 0$, the radical representative $Y$ of $\omega_{p,q}(X)$ is concentrated in the interval $[-l - 1, 0]$ and has $Y_0 = P^b_2$ and $Y_{-l-1} = P^d_1$ for some $c, d > 0$. Everything is clear from the definition of $\omega_{p,q}$ except that $c, d > 0$; but this follows from the fact that $\text{Hom}_{K^b_{\Lambda}}(\omega_{p,q}^{-1}(\Lambda), X)$ and $\text{Hom}_{K^b_{\Lambda}}(X, \omega_{p,q}^{-1}(\Lambda[l + 1]))$ are nonzero.
It remains to prove that $KH = \text{TrPic}(\Lambda)$. Pick some $F \in \text{TrPic}(\Lambda)$ and denote by $X$ the radical representative of $F(\Lambda)$. It follows from Lemma 4.1 that, for some $m \in \mathbb{Z}$, the radical representative of $\nu_{p,q}^m(X)$ has length not greater than 2. Thus, we may assume that $X$ has length not greater than 2. Since $X$ has the direct summand $F(P_1)$ with endomorphism algebra $k$ in the homotopy category, we can apply some shift and one of the equivalences $\omega_{p,q}$ or $\omega_{q,p}^{-1}$ to $X$ and get a tilting complex with a direct summand isomorphic to $P_1$ by Lemma 4.2. Thus, we may assume that either $F(P_1) = P_1$ or $\omega_{p,q}F(P_1) = P_1$.

Let $Y$ be the radical representative of $F(P_2)$ or $\omega_{p,q}F(P_2)$ respectively. Suppose that $Y$ is concentrated in the interval $[r, s]$ such that $Y_r, Y_s \neq 0$. If $Y_r$ is $P_1^a$ with $a > 0$, then $\text{Hom}_{K^b}(Y, P_1[-r]) \neq 0$. If $Y_r$ has the form $P_2^a$ with $a > 0$, then the map $\alpha_1 : P_1 \to P_2$ gives a nonzero element of $\text{Hom}_{K^b}(P_1[-r], Y)$ because the kernel of any radical map from $P_2$ to a projective module contains the image of $\alpha_1$. Here $\Lambda'$ denotes either $\Lambda$ or $\Lambda^{\text{op}}$ depending on what category the complex $Y$ belongs to. Thus, we have $r = 0$. Analogously, $s = 0$. Thus, $X$ or $\omega_{p,q}(X)$ is concentrated in degree 0. In the first case we have $F \in \text{Pic}(\Lambda)$. In the second case we have an equivalence $\omega_{p,q}F : K^b_{\Lambda} \to K^b_{\Lambda^{\text{op}}}$ that sends $\Lambda$ to $\Lambda^{\text{op}}$. Thus, this case is possible only when $\Lambda \cong \Lambda^{\text{op}}$, i.e. $p = q$. If $p = q$ and $\omega_{p,p}(X)$ is concentrated in degree 0, then $\omega_{p,p}F \in \text{Pic}(\Lambda)$ and the corollary is proved.

It remains to calculate $\text{Pic}(\Lambda^{p,q}) \cong \text{Out}(\Lambda^{p,q})$ and derive the commutation formulas for $\omega_{p,q}$, $\omega_{q,p}$ and elements of $\text{Pic}(\Lambda^{p,q})$ and $\text{Pic}(\Lambda^{q,p})$. The next lemma implements the first part of this plan. Let us introduce the vector spaces $A = A^p = e_2^p\Lambda^{p,q}e_1 = \bigoplus_{i=1}^p k\alpha_i$ and $B = B^q = e_1^q\Lambda^{p,q}e_2 = \bigoplus_{i=1}^q k\beta_i$. We identify $A$ and $B$ with the corresponding subspaces of $\Lambda^{p,q}$. Moreover, we identify the elements of $\text{GL}(A) \times \text{GL}(B)$ with the automorphisms of $\Lambda^{p,q}$ induced by them. We also denote by $D = D^{p,q}$ the subgroup of $\text{GL}(A) \times \text{GL}(B)$ formed by the elements $(\lambda \text{Id}_A, \lambda^{-1} \text{Id}_B)$ for $\lambda \in k^*$.

**Lemma 4.4.** $\text{Out}(\Lambda^{p,q}) \cong (\text{GL}(A) \times \text{GL}(B))/D$. This isomorphism is induced by the canonical projection $\text{Aut}(\Lambda^{p,q}) \twoheadrightarrow \text{Out}(\Lambda^{p,q})$.

**Proof.** It follows from [20] and [11] that

$$\text{Out}(\Lambda^{p,q}) \cong \text{Aut}_S(\Lambda^{p,q})/(\text{Inn}_S(\Lambda^{p,q}) \cap \text{Aut}_S(\Lambda^{p,q})), $$

where $\text{Aut}_S(\Lambda^{p,q})$ denotes the set of automorphisms of $\Lambda^{p,q}$ that stabilize the subalgebra generated by $e_1$ and $e_2$. It is clear that all such automorphisms act identically on $e_1$ and $e_2$, and hence have the form $(g, h)$ for some $g \in \text{GL}(A)$ and $h \in \text{GL}(B)$.

It remains to show that $\text{Inn}_S(\Lambda^{p,q}) \cap \text{Aut}_S(\Lambda^{p,q}) = D$. But this follows from the fact that any invertible $x$ such that $x^{-1}e_1x = e_1$ and $x^{-1}e_2x = e_2$ has the form $x = \lambda_1e_1 + \lambda_2e_2 + \sum_{1 \leq i \leq p, 1 \leq j \leq q} \kappa_{i,j}\alpha_i\beta_j$ for some $\lambda_1, \lambda_2 \in k^*$.
and $\kappa_{i,j} \in k$. Then modulo the center of $\Lambda^{p,q}$ the element $x$ has the form $x = \lambda e_1 + e_2$ for some $\lambda \in k^*$. It is easy to see that the inner automorphisms induced by such elements $x$ are exactly the automorphisms from $D$. ■

For the commutator formula, we will need the description of $\omega_{p,q}$ on morphisms. We will use for this a description of $\omega_{p,q}$ that is a little different from the one we used before. Let us introduce the $\Lambda^{p,q}$-module $\Xi = \Xi^{p,q}$. As usual, to do this we describe the spaces $\text{Hom}_{\text{GL}(A)}(\Lambda^{p,q})$ and $\text{End}_{\text{GL}(A)}(\Lambda^{p,q})$ induced by multiplication by the arrows of $Q^{p,q}$. We define $\Xi_1 = k$ and $\Xi_2 = A^* \oplus B = \text{Hom}_k(A,k) \oplus B$. We define $\phi_{1,2}(v)(1) = (0,v)$ and $\phi_{2,1}(u)(f,v) = f(u)$ for $u \in A$, $v \in B$ and $f \in A^*$. Now we have natural isomorphisms $\text{Hom}_{\text{GL}(A)}(\Xi_2,\Xi_1) \cong B$ and $\text{Hom}_{\text{GL}(A)}(\Xi_1,\Xi_2) \cong (A^*)^* \cong A$. We will identify the corresponding spaces via these isomorphisms. In particular, $\text{GL}(A)$ acts on $\text{Hom}_{\text{GL}(A)}(\Xi_2,\Xi_2)$ and $\text{GL}(B)$ acts on $\text{Hom}_{\text{GL}(A)}(\Xi_1,\Xi_2)$.

Now it is easy to see that actually $\omega_{p,q}(P_1)$ is a minimal $\Lambda^{q,p}$-projective resolution of $S_2$ and $\omega_{p,q}(P_2)$ is a minimal $\Lambda^{p,q}$-projective resolution of $\Xi^{p,q}$. Thus, if we replace $\omega_{p,q}$ by a naturally isomorphic equivalence, we may assume that $\omega_{p,q}(\Lambda^{p,q}) = S_2 \oplus \Xi^{p,q}$ and also that the isomorphism $\Lambda^{p,q} \cong \text{End}_{\text{GL}(A)}(\Xi_2,\Xi_1)$ induced by $\omega_{p,q}$ is induced by the isomorphisms $A^p \cong B^p = \text{Hom}_{\text{GL}(A)}(\Xi_2,\Xi^{q,p})$ and $B^q \cong A^q = \text{Hom}_{\text{GL}(A)}(\Xi^{q,p},S_2)$. Here the isomorphisms $A^p \cong B^q$ and $B^p \cong A^q$ are the renaming isomorphisms sending $\alpha_i$ to $\beta_i$ and $\beta_j$ to $\alpha_j$ for $1 \leq i \leq p$ and $1 \leq j \leq q$. Note that the renaming isomorphisms induce an isomorphism

$$\Phi_{p,q} : \text{GL}(A^p) \times \text{GL}(B^q) \cong \text{GL}(A^q) \times \text{GL}(B^p)$$

that induces an isomorphism from $\text{Pic}(\Lambda^{p,q})$ to $\text{Pic}(\Lambda^{q,p})$. We denote the induced isomorphism by $\Phi_{p,q}$ too. Note that the composition $\Phi_{p,q} \Phi_{p,q}$ is the identity automorphism. In the case $p = q$ we get an automorphism $\Phi_{p,q}$ of $\text{GL}(A^p)$ of order 2 interchanging two copies of $\text{GL}(A^p)$ and the corresponding semidirect product $\text{Pic}(\Lambda^{p,p}) \rtimes_{\Phi_{p,q}} \mathbb{Z}$ with multiplication defined by $(F,a) * (G,b) = (F \Phi_{p,q}(G), a + b)$ for $a,b \in \mathbb{Z}$ and $F,G \in \text{Pic}(\Lambda^{p,p})$.

**Lemma 4.5.** $\omega_{p,q} F \cong \Phi_{p,q}(F) \omega_{p,q}$ for any $F \in \text{Pic}(\Lambda^{p,q})$.

**Proof.** Let $\Lambda$ denote $\Lambda^{p,q}$ and $\Phi$ denote $\Phi_{p,q}$. We need to construct an isomorphism $\phi : \omega_{p,q} F(\Lambda) \cong \Phi(F) \omega_{p,q}(\Lambda)$ such that $\phi \circ \omega_{p,q} F(x) = \Phi(F) \omega_{p,q}(x) \circ \phi$ for any $x \in \text{End}(\Lambda) = \Lambda$. Let us introduce $\theta \in \text{Aut}(\Lambda)$ such that $F \cong -\otimes \Lambda \Lambda_{\theta^{-1}}$. By our definitions, $\omega_{p,q} F(\Lambda) = S_2 \oplus \Xi^{q,p}$ and $\omega_{p,q} F(x) = \omega_{p,q} \theta(x) = \Phi(\theta) \omega_{p,q}(x)$ for $x \in A^p \oplus B^q$. Here $\omega_{p,q}$ is the renaming isomorphism from $A^p \oplus B^q$ to $A^q \oplus B^p = \text{Hom}_{\text{GL}(A)}(\Xi^{q,p},S_2) \oplus \text{Hom}_{\text{GL}(A)}(S_2,\Xi^{q,p})$ and $\Phi$ is the conjugation by this isomorphism.
Applying again our definitions we get $\Phi(F)\omega_{p,q}(\Lambda) = (S_2 \oplus \Xi^{q,p})\Phi(\theta^{-1})$ and $\Phi(F)\omega_{p,q}(x) = \omega_{p,q}(x)$ for $x \in A^p \oplus B^q$. Here $\omega_{p,q}(x) : (S_2 \oplus \Xi^{q,p})\Phi(\theta^{-1}) \to (S_2 \oplus \Xi^{q,p})\Phi(\theta^{-1})$ is the map obtained by the identification of the linear spaces $(S_2 \oplus \Xi^{q,p})\Phi(\theta^{-1})$ and $S_2 \oplus \Xi^{q,p}$.

Let us now introduce $\phi : S_2 \oplus \Xi^{q,p} \to S_2 \oplus \Xi^{q,p}$ that is identical on $S_2$ and is defined on $\Xi^{q,p} = (A^q)^* \oplus B^p \oplus k$ by the equality $\phi(f, v, \lambda) = (f \circ \Phi(\theta), \Phi(\theta^{-1})(v), \lambda)$ for $(f, v, \lambda) \in (A^q)^* \oplus B^p \oplus k$. Now, for $x = (u, w)$ in $A^q \oplus B^p \subset \Lambda^{q,p}$, we get

$$\phi((f, v, \lambda)x) = \phi(0, \lambda w, f(u)) = (0, \lambda \Phi(\theta^{-1})(u), f(u))$$

i.e. $\phi$ induces an isomorphism $S_2 \oplus \Xi^{q,p} \cong (S_2 \oplus \Xi^{q,p})\Phi(\theta^{-1})$. It remains to check that $\phi \circ \Phi(\theta)\omega_{p,q}(x) = \omega_{p,q}(x) \circ \phi$ for $x \in A^p \cup B^q$. Let $e$ be the generator of $S_2$. For $u \in A^p$, $w \in B^q$, and $(f, v, \lambda) \in (A^q)^* \oplus B^p \oplus k$, we have

$$(\phi \circ \Phi(\theta)\omega_{p,q}(u))(e) = \phi(0, \Phi(\theta)\omega_{p,q}(u), 0) = (0, \omega_{p,q}(u), 0) = \omega_{p,q}(u)(e) = (\omega_{p,q}(u) \circ \phi)(e),$$

$$(\phi \circ \Phi(\theta)\omega_{p,q}(w))(f, v, \lambda) = (f \Phi(\theta)\omega_{p,q}(w))\phi(e) = (f \Phi(\theta)\omega_{p,q}(w))e = \omega_{p,q}(w)(f \Phi(\theta), \Phi(\theta^{-1})(v), \lambda) = (\omega_{p,q}(w) \circ \phi)(f, v, \lambda),$$

and thus the lemma is proved. ■

Now we are ready to formulate one of our main results. Let us introduce the group $G_{p,q}(k) = (\text{GL}_p(k) \times \text{GL}_q(k))/D_{p,q}(k)$, where $D_{p,q}(k)$ is the subgroup formed by the elements $(\lambda \text{Id}_k, \lambda^{-1} \text{Id}_k)$ for $\lambda \in k^*$. For $p = q$ we denote by $\Phi$ the automorphism of $G_{p,q}(k)$ induced by interchanging the two copies of $\text{GL}_p(k)$. As before, this automorphism gives an action of $Z$ on $G_{p,q}(k)$, and hence determines the semidirect product $G_{p,q}(k) \rtimes \Phi Z$.

**Theorem 4.6.** Let $p \neq q$ be positive integers. Then

$$\text{TrPic}(\Lambda^{p,q}) \cong G_{p,q}(k) \times Z \times Z \quad \text{and} \quad \text{TrPic}(\Lambda^{p,p}) \cong (G_{p,p}(k) \rtimes \Phi Z) \times Z.$$

**Proof.** This follows directly from Corollary 4.3 and Lemmas 4.4 and 4.5. ■

5. Two-term silting complexes and derived equivalences. In this section we describe all two-term silting complexes over $\Lambda^{p,q}$. By Lemma 4.1, this will allow us to describe all the algebras derived equivalent to $\Lambda^{p,q}$. Note that the last point can be achieved using [9, Theorem 4.7], but in any case to get the description in terms of quivers with relations, one has to obtain projective resolutions of the modules $\nu^m_{p,0}(\Lambda^{p,0})$, which is as difficult as a direct description of two-term silting complexes over $\Lambda^{p,q}$. 

Fix nonnegative integers \( p, q \). Assume \( p \geq 2 \). We introduce the sequence of \( k \)-linear spaces \( A_i = A_i^p \) with monomorphisms \( \kappa_k = \kappa_{i,k} : A_i \hookrightarrow A_{i+1} \) \((i \geq 0, 1 \leq k \leq p)\) by induction. We set \( A_0 = k \), \( A_1 = \bigoplus_{i=1}^p k \alpha_i \cong k^p \) and define \( \kappa_{0,k} : A_0 \to A_1 \) by \( \kappa_{0,k}(1) = \alpha_k \) for \( 1 \leq k \leq p \). Suppose that we have already defined \( A_{m-1}, A_m \) and \( \kappa_{m-1,k} : A_{m-1} \hookrightarrow A_m \) for some integer \( m \) and all \( 1 \leq k \leq p \). Let \( \iota_m \) denote the monomorphism

\[
A_{m-1} \cong A_0 \otimes A_{m-1} \xrightarrow{\sum_{k=1}^p \kappa_{0,k} \otimes \kappa_{m-1,k}} A_1 \otimes A_m.
\]

Here and below we write simply \( \otimes \) instead of \( \otimes_k \). We set \( A_{m+1} = \operatorname{Coker} \iota_m \) and denote by \( \pi_m : A_1 \otimes A_m \twoheadrightarrow A_{m+1} \) the canonical projection. Now we define \( \kappa_{m,k} \) for \( 1 \leq k \leq p \) as the composition

\[
A_m \cong A_0 \otimes A_m \xrightarrow{\kappa_{0,k} \otimes \text{Id}_A} A_1 \otimes A_m \xrightarrow{\pi_m} A_{m+1}.
\]

For convenience, we also set \( A_{-1} = 0 \). Let \( a_m = a_{p,m} := \dim_k A_m \). The numbers \( a_k \) satisfy the recursive formula \( a_{m+1} = pa_m - a_{m-1} \) and an induction argument shows that \( a_m^2 + a_{m-1}^2 - pa_{m-1}a_m = 1 \). Moreover, it is not difficult to show that \( a_m/a_{m-1} \) is a decreasing sequence with limit \( (p + \sqrt{p^2 - 4})/2 \).

Let \( C_m = C_m^{p,q} \) \((m \geq 0)\) be the two-term \( \Lambda^{p,q} \)-complex

\[
A_{m-1} \otimes P_1 \xrightarrow{\sum_{k=1}^p \kappa_k \otimes \kappa_{m-1,k}} A_m \otimes P_2
\]

concentrated in degrees \(-1\) and 0. Note that

\[
C_0 = P_2, \quad C_1 = \left( P_1 \xrightarrow{\begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_p \end{pmatrix}} P_2^p \right) = \mu_{P_2}^+(P_1),
\]

and hence \( C_0 \oplus C_1 \) is a silting \( \Lambda^{p,q} \)-complex. Moreover, it is not difficult to see that \( C_0 \oplus C_1 \) is a tilting complex. We also set \( C_{-1} := P_1 \) for convenience.

Let us now introduce the algebras \( \Lambda_m^{p,q} \) \((m \geq 0)\) in the following way. The algebra \( \Lambda_m^{p,q} \) has two primitive orthogonal idempotents \( e_1 \) and \( e_2 \) such that \( e_1 + e_2 = 1 \) and its bimodule structure over the semisimple algebra generated by \( e_1 \) and \( e_2 \) is defined by the equalities

\[
e_1 \Lambda_m^{p,q} e_1 = k e_1 \bigoplus A_{m+1} \otimes B \otimes A_m^*, \quad e_1 \Lambda_m^{p,q} e_2 = A_m \otimes B \otimes A_m^*,
\]

\[
e_2 \Lambda_m^{p,q} e_1 = A_1 \bigoplus A_{m+1} \otimes B \otimes A_{m-1}^*, \quad e_2 \Lambda_m^{p,q} e_2 = k e_2 \bigoplus A_m \otimes B \otimes A_m^*,
\]

where, as before, \( B = \bigoplus_{i=1}^q k \beta_i \). The products that do not follow from the \( k e_1 \oplus k e_2 \)-bimodule structure are all zero except the products induced by the maps

\[
A_1 \otimes (A_m \otimes B \otimes A_i^*) \xrightarrow{\pi_m \otimes \text{Id}_B \otimes \text{Id}_{A^*}} A_{m+1} \otimes B \otimes A_i^* \quad (i = m, m-1),
\]

\[
(A_i \otimes B \otimes A_m^*) \otimes A_1 \xrightarrow{\text{Id}_{A_i} \otimes B \otimes \iota_m^*} A_i \otimes B \otimes A_{m-1}^* \quad (i = m, m+1).
\]
Here we identify $A^*_m \otimes A_1$ with $(A_1 \otimes A_m)^*$ via the canonical isomorphism $A^*_m \otimes A^*_1 \cong (A_1 \otimes A_m)^*$ and the isomorphism $A_1 \cong A^*_1$ that sends $\alpha_i$ to $\alpha^*_i$, where $\alpha^*_1, \ldots, \alpha^*_p$ is the basis dual to $\alpha_1, \ldots, \alpha_p$. Note that $A^{p,q}_m$ is the path algebra of a quiver with vertices 1 and 2, $p$ arrows from 1 to 2 associated with some basis of $A_1$ and $q a^2_m$ arrows from 2 to 1 associated with some basis of $A_m \otimes B \otimes A^*_m$ modulo the ideal generated by the images of the maps

$$A_{m-1} \otimes B \otimes A^*_m \xrightarrow{\iota_m \otimes \text{Id}_B \otimes A^*_m} A_1 \otimes (A_m \otimes B \otimes A^*_m),$$

$$A_m \otimes B \otimes A^*_m+1 \xrightarrow{\text{Id}_A \otimes B \otimes \pi^*_m} (A_m \otimes B \otimes A^*_m) \otimes A_1.$$

In particular, the algebra $A^{p,q}_m$ is quadratic for any $m \geq 0$. Indeed, since $\iota^*_m$ and $\pi^*_m$ are surjective, it is clear that $J_{A^{p,q}_m}/J_{A^{p,q}_m}^2 \cong (A_m \otimes B \otimes A^*_m) \oplus A_1$ and it remains to show that $A^*_m \otimes \text{Im} \iota_m + \text{Im} \pi^*_m \otimes A_m = A^*_m \otimes A_1 \otimes A_m$. Since

$$\dim_k (A^*_m \otimes A_{m-1}) + \dim_k (A^*_m+1 \otimes A_m) = \dim_k (A^*_m \otimes A_1 \otimes A_m),$$

the required equality is equivalent to $(\iota^*_m \otimes \text{Id}_{A^*_m})(\text{Id}_{A^*_m} \otimes \iota_m)$ being an isomorphism and to $(\text{Id}_{A^*_m} \otimes \pi^*_m)(\pi^*_m \otimes \text{Id}_{A^*_m})$ being an isomorphism. Thus, we can proceed by induction on $m$ using the equality

$$(\iota^*_m \otimes \text{Id}_{A^*_m})(\text{Id}_{A^*_m} \otimes \iota_m) = (\text{Id}_{A^*_m-1} \otimes \pi^*_m-1)(\pi^*_m-1 \otimes \text{Id}_{A^*_m-1}).$$

Since $(\iota^*_1 \otimes \text{Id}_{A^*_1})(\text{Id}_{A^*_1} \otimes \iota_1)$ is an isomorphism, we are done. Note that $A^{p,q}_0 = A^{p,q} = \text{End}_{K^b_{A^{p,q}}}(C_m \oplus C_0)$.

**Lemma 5.1.** For any $m \geq 0$, the complex $C_{m-1} \oplus C_m$ is a tilting $A^{p,q}$-complex such that $\text{End}_{K^b_{A^{p,q}}}(C_{m-1} \oplus C_m) \cong A^{p,q}_m$.

**Proof.** Set $\Lambda = A^{p,q}$. We proceed by induction on $m$. Suppose that $C_{m-1} \oplus C_m$ is a tilting complex. Note that any map from $A_{m-2} \otimes P_1$ to $A_{m-1} \otimes P_1$ is of the form $f \otimes \text{Id}_{P_1}$ for some $f \in \text{Hom}_k(A_{m-2}, A_{m-1})$, while any map from $A_{m-1} \otimes P_2$ to $A_m \otimes P_2$ is a sum of a map of the form $f \otimes \text{Id}_{P_2}$ and of maps of the form $f \otimes \alpha_i \beta_j$, where $f \in \text{Hom}_k(A_{m-1}, A_m)$, $1 \leq i \leq p$ and $1 \leq j \leq q$. The maps $f \otimes \alpha_i \beta_j$ automatically give maps from $\text{Hom}_{C^b_{A}}(C_{m-1}, C_m[-1])$, while the maps of the form $(f_1 \otimes \text{Id}_{P_1}, f_2 \otimes \text{Id}_{P_2})$ have to give $p a_{m-2} a_m$ null homotopic elements of $\text{Hom}_{C^b_{A}}(C_{m-1}, C_m[1])$. Thus, there are $a_{m-2} a_{m-1} + a_{m-1} a_m - p a_{m-2} a_m = p$ linearly independent maps of the form $(f_1 \otimes \text{Id}_{P_1}, f_2 \otimes \text{Id}_{P_2})$ from $C_{m-1}$ to $C_m$ in $K^b_{A}$. Then it is easy to see that these maps are linear combinations of the maps $(\kappa_k \otimes \text{Id}_{P_1}, \kappa_k \otimes \text{Id}_{P_2})$ ($1 \leq k \leq p$). Analogously, one can show that there are no nonzero elements of the form $(f_1 \otimes \text{Id}_{P_1}, f_2 \otimes \text{Id}_{P_2})$ in $\text{Hom}_{C^b_{A}}(C_m, C_{m-1})$. 


Now the map \((0, \sum_{i=1}^{p} \sum_{j=1}^{q} f_{i,j} \otimes \alpha_i \beta_j) \in \text{Hom}_{C_{\Lambda}^{b}}(C_{m-1}, C_{m}[1])\) is nonzero in \(K_{\Lambda}^{b}\) if and only if its image does not lie in the image of
\[
\sum_{k=1}^{p} \kappa_{k} \otimes \alpha_{k} : A_{m-1} \otimes P_{1} \to A_{m} \otimes P_{2}.
\]
Since the image of \(\sum_{i=1}^{p} \sum_{j=1}^{q} f_{i,j} \otimes \alpha_i \beta_j\) is a submodule of the socle of \(A_{m} \otimes P_{2}\), nonzero elements of \(\text{Hom}_{K_{\Lambda}^{b}}(C_{m-1}, C_{m})\) that can be represented by a map of the form \((0, \sum_{i=1}^{p} \sum_{j=1}^{q} f_{i,j} \otimes \alpha_i \beta_j)\) correspond to maps from \(A_{m-1} \otimes P_{2}\) to
\[
\text{Coker} \left( A_{m-1} \otimes A_{0} \otimes B \otimes S_{2} \to \sum_{k=1}^{p} \kappa_{m-1,k} \otimes \kappa_{0,k} \otimes \text{Id}_{B} \otimes S_{2} \right) \cong A_{m+1} \otimes B \otimes S_{2},
\]
i.e. can be naturally parametrized by the elements of \(A_{m+1} \otimes B \otimes A_{m}^{*}\). Analogously, \(\text{Hom}_{K_{\Lambda}^{b}}(C_{m}, C_{m-1}) \cong A_{m} \otimes B \otimes A_{m}^{*}\).

In the same manner one can show that, for \(t = m-1, m\), the elements of \(\text{End}_{K_{\Lambda}^{b}}(C_{t})\) are linear combinations of \(\text{Id}_{C_{t}}\) and maps that can be represented as \((0, \sum_{i=1}^{p} \sum_{j=1}^{q} f_{i,j} \otimes \alpha_i \beta_j)\) with \(f_{i,j} \in \text{End}_{k}(A_{t})\) and that nonzero maps of the second type are in one-to-one correspondence with \(A_{t+1} \otimes B \otimes A_{t}^{*}\). Thus, we have isomorphisms \(\text{End}_{K_{\Lambda}^{b}}(C_{t}) \cong k \oplus (A_{t+1} \otimes B \otimes A_{t}^{*})\). Sending \((\kappa_{k} \otimes \text{Id}_{P_{1}}, \kappa_{k} \otimes \text{Id}_{P_{2}})\) to \(\alpha_{k}\), we also get the isomorphism \(\text{Hom}_{K_{\Lambda}^{b}}(C_{m-1}, C_{m}) \cong A_{1} \oplus (A_{m+1} \otimes B \otimes A_{m}^{*})\). It is clear that the products not involving \(\text{Id}_{C_{m-1}}\) and \(\text{Id}_{C_{m}}\) are all zero except the products
\[
\mu_{1} : A_{m} \otimes B \otimes A_{m}^{*} \times A_{1} \to A_{m} \otimes B \otimes A_{m}^{*},
\]
\[
\mu_{2} : A_{1} \times A_{m} \otimes B \otimes A_{m}^{*} \to A_{m+1} \otimes B \otimes A_{m}^{*}.
\]
By our definitions, \(\mu_{1}(u \otimes v \otimes f, \alpha_{k}) = u \otimes v \otimes f \kappa_{k}\) and \(\mu_{2}(u \otimes v \otimes f, \alpha_{k}) = \kappa_{k}(u) \otimes v \otimes f\) for \(u \in A_{m}, v \in B, f \in A_{m}^{*}\) and \(1 \leq k \leq p\). It remains to note that the map from \(A_{m}^{*} \otimes A_{1}\) to \(A_{m}^{*}\) that sends \(f \otimes \alpha_{k}\) to \(f \kappa_{k}\) is exactly \(\iota_{m}\), while the map from \(A_{1} \otimes A_{m}\) to \(A_{m+1}\) that sends \(\alpha_{k} \otimes u\) to \(\kappa_{k}(u)\) is exactly \(\pi_{m}\).

Thus, we have proved that \(\text{End}_{K_{\Lambda}^{b}}(C_{m-1} \oplus C_{m}) \cong A_{m}^{p+q}\), where the arrows from \(C_{m-1}\) to \(C_{m}\) correspond to the maps \((\kappa_{k} \otimes \text{Id}_{P_{1}}, \kappa_{k} \otimes \text{Id}_{P_{2}})\) \((1 \leq k \leq p)\). Then \(\mu_{C_{m}}(C_{m-1})\) is the cone of the map
\[
C_{m-1} \xrightarrow{\left( \begin{array}{cc}
(\kappa_{1} \otimes \text{Id}_{P_{1}}, \kappa_{1} \otimes \text{Id}_{P_{2}}) \\
\vdots \\
(\kappa_{p} \otimes \text{Id}_{P_{1}}, \kappa_{p} \otimes \text{Id}_{P_{2}})
\end{array} \right)} (C_{m})^{p},
\]
which in turn is isomorphic to \(C_{m+1}\). Hence, \(C_{m} \oplus C_{m+1}\) is a two-term silting complex. Since the nonzero component of its differential is injective, it is easy to see that \(C_{m} \oplus C_{m+1}\) is tilting and the induction step is finished. ■
If $q \geq 2$, then applying the functor $\Hom_{\Lambda^q,p}(-,\Lambda^q,p) : C^b_{\Lambda^q,p} \to C^b_{\Lambda^p,q}$ to the tilting complex $C^q_{m}[1] (m \geq -1)$, we obtain the tilting $\Lambda^p,q$-complex
\[(A^q_{m})^* \otimes P_2 \sum_{k=1}^q \kappa_k \otimes \beta_k \to (A^q_{m-1})^* \otimes P_1,\]
which we will denote by $\overline{C}_m = \overline{C}_{m}^{p,q}$. By [21] Proposition 9.1 the complex $\overline{C}_{m-1} \oplus \overline{C}_m$ realizes a derived equivalence between $\Lambda^p,q$ and $(\Lambda^p_{m})^{op}$.

Let $p \geq 2$ again. Suppose that we have a two-term complex
\[C = \begin{pmatrix} w_{1,1} & \cdots & w_{1,a} \\ \vdots & \ddots & \vdots \\ w_{b,1} & \cdots & w_{b,a} \end{pmatrix} \xrightarrow{P_1} P_2^b\]
with $w_{i,j} \in \Hom_{\Lambda^p,q}(P_1, P_2) = e_2 \Lambda^p,q e_1$. We define
\[C^* := \begin{pmatrix} w_{1,1} & \cdots & w_{1,1} \\ \vdots & \ddots & \vdots \\ w_{1,a} & \cdots & w_{b,a} \end{pmatrix} \xrightarrow{P_1} P_2^a\]
It is easy to see that $(C^*)^* = C$ and that $C^*$ is indecomposable, radical or presilting if and only if $C$ satisfies the same condition. This assertion is based on the fact that to verify the presilting condition for $C$ and $C^*$ one has to deal only with identity morphisms on $P_1$ and $P_2$ and morphisms from $P_1$ to $P_2$.

On the other hand, the same assertion for the pretilting property fails. Indeed, $C^* \oplus C^*_1 = P_1[1] \oplus \omega_{q,p}(P_1)$ is not a tilting complex if $q > 0$. In fact, $\End_{K^b_{\Lambda^p,q}}(C^*_0) = \End_{K^b_{\Lambda^p,q}}(C^*_1) = k$, $\Hom_{K^b_{\Lambda^p,q}}(C^*_0, C^*_1[i]) = 0$ for all $i \in \mathbb{Z}$, $\Hom_{K^b_{\Lambda^p,q}}(C^*_1, C^*_0) = k^p$ and $\Hom_{K^b_{\Lambda^p,q}}(C^*_1, C^*_0[-1]) = k^q$, and hence $C^*_0 \oplus C^*_1$ is related to a derived equivalence between the algebra $\Lambda^p,q$ and the graded $(p+q)$-Kronecker algebra $\delta_{p,q}$ with $p$ arrows of degree 0 and $q$ arrows of degree $1$ (see [16]).

Further, for $m \geq 1$, the radical representative of $\omega_{q,p}^{-1}(C^*_m)$ coincides as a graded $\Lambda^p,q$-module with $\overline{C}^q_{m-2}$. One can see this, for example, by representing $C^*_m$ as a cocone of a morphism from the complex $(P_1^p \to P_2)^{a_{m-1}}$ to $P_1^{a_{m-2}}[1]$ and noting that $\omega_{q,p}^{-1}(C^*_m)$ is a cocone of a morphism from $P_1^{a_{m-1}}$ to $P_2^{a_{m-2}} \to P_1^{b_{m-2}}$. Since an indecomposable two-term presilting complex is determined by its $g$-vector, we have $C^*_m = \omega_{q,p} \Hom_{\Lambda^p,q}(C_{m-1}, \Lambda^p,q)$ for $m \geq 1$, and in particular $(C_{m-1} \oplus C_m)^*$ realizes a derived equivalence between $\Lambda^p,q$ and $(\Lambda^p_{m-2})^{op}$ for $m \geq 2$. Note that $C^*_m$ is the complex
\[A^*_m \otimes P_1 \sum_{k=1}^p \kappa_k \otimes \alpha_k \to A^*_{m-1} \otimes P_2.\]
Finally, suppose again that $q \geq 2$. Note that the complex
\[A^{m-1} \otimes P_2 \sum_{k=1}^q \kappa_k \otimes \beta_k \to A^{m} \otimes P_1\]
equals $\text{Hom}_{\Lambda_{q,p}}((C_{m}^{q,p})^{*}, \Lambda_{q,p})[1]$. We will denote this complex by $\overline{C}_{m}^{*} = (\overline{C}_{m}^{q,p})^{*}$. By [21, Proposition 9.1] the complex $\overline{C}_{m-1}^{*} \oplus \overline{C}_{m}^{*}$ realizes a derived equivalence between $\Lambda_{p,q}^{*}$ and $\Lambda_{m-2}^{p,q}$ for $m \geq 2$ while $\overline{C}_{0}^{*} \oplus \overline{C}_{1}^{*} \cong P_{1} \oplus \omega_{q,p}^{-1}(P_{1}[1])$ is a two-term silting complex related to a derived equivalence between the algebra $\Lambda_{p,q}$ and the graded $(p + q)$-Kronecker algebra $\delta_{q,p}$ with $q$ arrows of degree 0 and $p$ arrows of degree $-1$.

Define

$$\text{silt}_{p,q} = \begin{cases} \{C_{m-1}^{p,q} \oplus C_{m}^{p,q}\} \cup \{(C_{m-1}^{p,q} \oplus C_{m}^{p,q})^{*}\} & \text{if } p \geq 2, \\
\{\Lambda, P_{2} \oplus (P_{1} \rightarrow P_{2}), (P_{1} \rightarrow P_{2}) \oplus P_{1}[1]\} & \text{if } p = 1, \\
\{\Lambda, P_{2} \oplus P_{1}[1]\} & \text{if } p = 0. \end{cases}$$

We will also denote $\{\text{Hom}_{\Lambda_{q,p}}(C, \Lambda_{q,p}) | C[1] \in \text{silt}_{q,p}\}$ by $\overline{\text{silt}}_{q,p}$. Now we can formulate and prove our second main result.

**Theorem 5.2.** Let $p > 0$ and $q$ be integers. Then the set of basic silting complexes is $\text{silt}_{p,q} \cup \overline{\text{silt}}_{q,p}$. All of these complexes are tilting except for the complexes $P_{1}[1] \oplus \omega_{q,p}(P_{1})$ and $P_{1} \oplus \omega_{p,q}^{-1}(P_{1}[1])$. In particular, the derived equivalence class of $\Lambda_{p,q}$ is the set

$$[\Lambda_{p,q}] = \begin{cases} \{\Lambda_{q,p}\} & \text{if } q = 0 \text{ or } p = q = 1, \\
\{\Lambda_{p,q}\} \cup \{\Lambda_{q,p}, (\Lambda_{q,p})^{\text{op}}\} & \text{if } q > p = 1, \\
\{\Lambda_{q,p}\} \cup \{\Lambda_{p,q}, (\Lambda_{p,q})^{\text{op}}\} & \text{if } p > q = 1, \\
\{\Lambda_{p,q}\} \cup \{\Lambda_{q,p}, (\Lambda_{q,p})^{\text{op}}\} & \text{if } p = q \geq 2, \\
\{\Lambda_{p,q}, (\Lambda_{p,q})^{\text{op}}, \Lambda_{q,p}, (\Lambda_{q,p})^{\text{op}}\} & \text{if } p, q \geq 2 \text{ and } p \neq q. \end{cases}$$

**Proof.** The second assertion follows from the first one and the discussion above. Here we consider the case $p, q \geq 2$. The other cases are easier and can be considered in the same way.

We have already proved that $\text{silt}_{p,q} \cup \overline{\text{silt}}_{q,p}$ consists of two-term silting complexes. Moreover, the pairs of $g$-vectors of complexes from $\text{silt}_{p,q}$ are

$$((1, 0), (0, 1)), \quad ((-a_{p,m-1}, a_{p,m}), (-a_{p,m}, a_{p,m+1})) \quad (m \geq 0),$$

$$((-a_{p,m}, a_{p,m-1}), (-a_{p,m+1}, a_{p,m})) \quad (m \geq 0),$$

while for the set $\overline{\text{silt}}_{q,p}$ we get the pairs

$$((-1, 0), (0, -1)), \quad ((a_{q,m-1}, -a_{q,m}), (a_{q,m}, -a_{q,m+1})) \quad (m \geq 0),$$

$$((a_{q,m}, -a_{q,m-1}), (a_{q,m+1}, -a_{q,m})) \quad (m \geq 0).$$

As we said before, the limits of $a_{p,m+1}/a_{p,m}$ and $a_{q,m+1}/a_{q,m}$ as $m \rightarrow \infty$ are $(p + \sqrt{p^{2} - 4})/2$ and $(q + \sqrt{q^{2} - 4})/2$. This means that the arcs $L_{C}$ with $C \in \text{silt}_{p,q} \cup \overline{\text{silt}}_{q,p}$ cover the whole unit circle except two arcs: the minimal
arc cut by the rays \((-1, \frac{p+\sqrt{p^2-4}}{2})\) and \((-1, \frac{p-\sqrt{p^2-4}}{2})\) and the minimal arc cut by the rays \((\frac{q+\sqrt{q^2-4}}{2}, -1)\) and \((\frac{q-\sqrt{q^2-4}}{2}, -1)\) (see Fig. 1).

Recall that any indecomposable two-term presilting complex \(X\) such that \(O_X \in \bigcup_{C \in \text{silt}_{p,q} \cup \text{silt}_{q,p}} L_C\) belongs to \(\text{silt}_{p,q} \cup \text{silt}_{q,p}\). In particular, if there exists some indecomposable two-term presilting complex \(X\) that does not belong to \(\text{silt}_{p,q} \cup \text{silt}_{q,p}\), then its \(g\)-vector belongs to one of the gray sectors in Fig. 1. This means that either \(X = (P_1^a \to P_2^b)\) with \((p-\sqrt{p^2-4})/2 \leq b/a \leq (p + \sqrt{p^2-4})/2\) or \(X = P_2^a \to P_1^b\) with \((q-\sqrt{q^2-4})/2 \leq b/a \leq (q + \sqrt{q^2-4})/2\). One has \(a^2 + b^2 - pab \leq 0\) in the first case, and \(a^2 + b^2 - qab \leq 0\) in the second. Consider the case where \(X = P_1^a \to P_2^b\) with \(pab \geq a^2 + b^2\). Then \(\dim_k \text{Hom}_{C_{\Lambda_{p,q}}}(X, X[1]) = pab\) while morphisms from \(P_1^a\) to \(P_1^a\) and
from $P_2^b$ to $P_2^b$ with components in the Jacobson radical of $\Lambda^{p,q}$ and the identity morphism on $X$ give zero morphisms from $X$ to $X[1]$. Thus, the dimension of the space of null homotopic maps from $X$ to $X[1]$ is not greater than $a^2 + b^2 - 1$. Thus, $X$ cannot be silting. The case $X = P_2^a \to P_1^b$ can be considered in the same manner.

Figure 1 illustrates the discussion in this section. The rays are denoted by indecomposable two-term presilting complexes and correspond to their $g$-vectors. Each pair of consecutive rays gives a silting complex and the corresponding derived equivalent algebra is written in the cone bounded by these rays. Cones corresponding to two-term silting complexes that are not tilting are marked by $\delta_{p,q}$ and $\delta_{q,p}$.

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