Linear and superlinear spread for continuous-time frog model

Viktor Bezborodov $^*$ and Tyll Krueger $^†$

$^1$Wrocław University of Science and Technology

September 23, 2020

Abstract

Consider a stochastic growth model on $\mathbb{Z}^d$. Start with some active particle at the origin and sleeping particles elsewhere. The initial number of particles at $x \in \mathbb{Z}^d$ is $\eta(x)$, where $\eta(x)$ are independent random variables distributed according to $\mu$. Active particles perform a simple continuous-time random walk while sleeping particles stay put until the first arrival of an active particle to their location. Upon the arrival all sleeping particles at the site activate at once and start moving according to their own simple random walks. The aim of this paper is to give conditions on $\mu$ under which the spread of the process is linear or faster than linear. The main technique is comparison with other stochastic growth models.

Mathematics subject classification: 60K35, 82C22

1 Introduction

At time $t = 0$ there are $\eta(x)$ particles at $x \in \mathbb{Z}^d$, where the random variables $\{\eta(x)\}_{x \in \mathbb{Z}^d}$ are independent and identically distributed according to a distribution $\mu$ on $\mathbb{Z}_+ := \mathbb{N} \cup \{0\}$, and $d \in \mathbb{N}$ is the dimension. The particles at the origin are active while all other particles are dormant (sleeping). Active particles perform a simple continuous-time random walk independently of all other particles. Sleeping particles stay still until the first arrival of an active particle to their location; upon arrival they become active and start their own simple random walks. The model was originally defined in discrete time $t = 0, 1, 2, \ldots$ with particles performing a simple discrete-time random walk. In this paper we consider the continuous-time version. We exclude a trivial case and assume throughout that $\mu(0) < 1$. Denote by $A_t$ the set of sites visited by an active particle by the time $t$. In this paper we investigate the various conditions on $\mu$ ensuring that the system spreads linearly with time, or that the system spreads faster than linearly with time.

$^*$Email: viktor.bezborodov@pwr.edu.pl
$^†$Email: tyll.krueger@pwr.wroc.pl
**Definition 1.1.** We say that the spread is linear, or the spread rate is linear, or the system spreads linearly, if there exists a constant $C > 0$ such that a.s.

$$\mathcal{A}_t \subset C[-t, t]^d \quad \text{for large } t > 0. \quad (1)$$

If a.s. (1) does not hold for any $C > 0$, the spread (rate) is said to be superlinear, or faster than linear; the system spreads faster than linearly with time. In other words, the spread is superlinear if for every $C > 0$ the set $\{t \geq 0 : \mathcal{A}_t \not\subset C[-t, t]^d\}$ is unbounded a.s.

In the one-dimensional case the spread is superlinear if \(\limsup_{t \to \infty} \frac{\sup \mathcal{A}_t}{t} = \infty\). Sometimes instead of the linear spread the phrase ‘the linear growth’ (or ‘the linear growth rate’) is used to describe (1). Let $0$ be the origin of $\mathbb{Z}^d$. In the case $\eta(0) = 0$ a single active particle is added to the origin to prevent a possible absence of active particles. We collect our principal results in the next theorem.

**Theorem 1.2.** Consider the continuous-time frog model.

(i) Assume that for some $B > 1$,

$$\sum_{m \in \mathbb{N}} [\mu([B^m, \infty))]^{\frac{1}{d}} < \infty. \quad (2)$$

Then the spread is linear.

(ii) Assume that for every $B > 1$,

$$\sum_{m=1}^{\infty} \prod_{n=1}^{m} \mu([0, B^n]) < \infty. \quad (3)$$

Then the spread is superlinear.

(iii) Assume that for some $B > 1$,

$$\sum_{n \in \mathbb{N}} \mu([B^{n \ln^2 n}, \infty)) = \infty. \quad (4)$$

Then the spread is superlinear.

We postpone a discussion on the cases not covered by Theorem 1.2 to Remark 6.1. The continuous-time frog model belongs to the class of processes known as the stochastic combustion growth process [RS04], or a model of $X + Y \to 2X$ [CQR09, BR10]. The process in [RS04] is exactly the frog model with $\mu$ being the delta measure at $1$; that is, at the beginning there is exactly one particle per site. In [RS04] a shape theorem for the process is proven for the stochastic combustion growth process, and it is shown that the distribution of the number of particles in visited sites converge to the product Poisson measure with parameter 1 (see [RS04] for the precise formulation). A central limit theorem for the one-dimensional process with a
fixed number of sleeping particles per site is given in [CQR09]. For a slightly modified one-dimensional model in [CQR07] a shape theorem and a central limit theorem for the front are established. Extensions with sleeping particles replaced by moving particles of different type are treated in [KS05, KS08, BR16]. In particular, in [KS05] the linear growth is established, while in [KS08] a shape theorem is proven. Further discussion of this and related models takes place in [KRS12]. In [BR16] a central limit theorem is obtained for the front in the one-dimensional models with mobile particles of two types.

Perhaps the first particle growth model where the spread rate can be linear or superlinear depending on parameters was the branching random walk. The spread rate is linear for the branching random walk with exponential tails [Dur79] and exponential the branching random walk with polynomial tails [Dur83]. An intermediate case can result in a polynomial spread rate [Gan00]. The exact expression of the speed of a discrete-space branching random walk with an exponential moment condition can be found in [Big95]. The model in [BPKT20] is the branching random walk with the additional restriction that the birth rate at any spatial location cannot exceed one. It is shown that this model spreads linearly provided the tails of the dispersion kernel are lighter than $\frac{1}{|x|^{4+\varepsilon}}$ for some $\varepsilon > 0$.

In this paper we establish conditions for the linear or superlinear spread using comparison to certain percolation and percolation-like models. Comparing a growth process with a percolation model is not uncommon. Famously, the renormalization procedure in the proof of the shape theorem for the contact process is a stepping stone toward comparison to oriented percolation [DG82, Dur91]. The comparison of the contact process to a simpler growth model via the renormalization proves useful in various contexts [Lig99, Chapter 2]. Renormalization procedure and comparison to another process is used in percolation theory itself [ADH17].

Domination by other models is a common technique when proving a linear growth of a certain stochastic process. In [SS19] a certain aggregation process is shown to grow linearly with time via comparison with a two type first passage percolation process. In [GM08] and in this paper the growth process is compared to a greedy paths model [CGGK93] (more precisely, in [GM08] a continuous-space equivalent is used). The linear speed of the continuous-space growth models can often be deduced from the linear speed of similar discrete-space models [Dei03], [BDPK+17]. In [BDD+18] the Brownian frog model is dominated by a certain specially designed branching process. In [BPKT20] the system viewed from its tip is dominated by another more amenable to analysis process.

The (discrete-time) frog model has been an active research subject in recent years. In discrete time the model cannot grow faster than linearly as the set of visited sites is always contained in $tD$, where $D = \{(x_1, \ldots, x_d) : |x_1| + \cdots + |x_d| \leq 1\}$ and $t = 0, 1, \ldots$. The shape theorem was proven in [AMP02] and [AMPR01]. The question whether $D$ can be a limiting shape for distributions $\mu$ with sufficiently heavy tails was answered positively in [AMPR01]. Recent papers [DHL19] and [BFHM20] provide an overview of other research on this model.
The transitivity and recurrence properties of the frog model attract considerable attention [DGH+18, HJJ17, HJJ16, KZ17, GNR17]. In [Zer18] a recurrence criterion is obtained for an asymmetric frog model with particles removed after a geometric number of steps. For various finite graphs the asymptotics of the first moment when all sites are visited is established in [BFHM20]. Continuity of the asymptotic shape of the discrete-time frog model with respect to the measure $\mu$ is established in [Kub20]. The variance of the passage times is sublinear [vHCN19], which implies that a central limit theorem does not hold. In [DHL19] a possibility of co-existence in a two-type frog model was demonstrated.

In the Brownian frog model the particles perform a Brownian motion instead of a simple random walk. Naturally, the process evolves in continuous time. A shape theorem and an asymptotic density results were obtained in [BDD+18], while conditions for transience of the one-dimensional version are established in [Ros17] (active particles in [Ros17] have a leftward drift, thus it is possible that all of them escape to $-\infty$ and the origin is not being visited starting from some positive time onwards).

The paper is organized as follows. Further definitions, notation, comments, the structural description of the paper, and the proof ideas are collected in Section 2. In Section 3 the properties of an auxiliary percolation model are given. The proof of Theorem 1.2 is spread across Sections 5, 6, and 7. In Section 4 some auxiliary results used in the proofs are collected.

2 Further results, discussion, and some ideas of the proof

Theorem 1.2 is a composite of Theorems 2.1, 2.2, 2.3, and 2.7, and Remark 2.13. The four theorems are proven in the subsequent sections. The first theorem of this section gives sufficient conditions for the linear spread of the one-dimensional system.

Theorem 2.1. Let $d = 1$ and assume
\[ \sum_{k=1}^{\infty} \log k \mu(k) < \infty. \] (5)

Then the spread is linear in time.

We note that for $d = 1$, (5) is equivalent to (2). The proof of Theorem 2.1 can be found in Section 5. The next two theorems give conditions for the superlinear spread.

Theorem 2.2. Assume that (3) holds for every $B > 1$. Then the system spreads faster than linearly with time.

Theorem 2.3. Assume that (4) holds for every $B > 1$. Then the system spreads faster than linearly with time.

Remark 2.4. Series (3) and (4) have independent convergent properties (that is, the convergence or divergence of either one of them does not imply anything about the other). See Proposition 4.5 for more details and examples.
The proofs of Theorems 2.2 and 2.3 are located in Sections 5 and 6, respectively. A brief discussion on the cases not covered by Theorem 1.2 can be found at the end of Section 6 in Remark 6.1. The following proposition is useful because it shows that it is enough to prove that the spread is superlinear for $d = 1$ only. In particular, in the proofs of Theorems 2.2 and 2.3 it is sufficient to consider the dimension $d = 1$ only. A monotonicity in dimension of this kind appears in [RS04].

**Proposition 2.5.** Assume $\mu$ is such that the spread of the corresponding one-dimensional model is superlinear a.s. Then a.s. the spread is superlinear for $d \geq 2$ as well.

**Remark 2.6.** Proposition 2.5 can be extended to dimensions $d_1$ and $d_2$ with $1 \leq d_1 < d_2$.

The next theorem gives sufficient conditions for the linear spread in dimensions $d \geq 2$. As expected, the assumptions are stronger than in the case $d = 1$.

**Theorem 2.7.** Let $d \geq 2$ and assume that for every $B > 1$,

$$\sum_{m \in \mathbb{N}} \mu([B^m, \infty))^{\frac{1}{d}} < \infty. \quad (6)$$

Then the spread is linear.

The proof of Theorem 2.7 is contained in Section 7. Now we formulate a shape theorem, which in this case is a consequence to linear growth. For two sets $A, B$, let their sum be defined in the usual way $A + B = \{a + b : a \in A, b \in B\}$.

**Corollary 2.8 (Shape Theorem).** Assume $d$ and $\mu$ satisfy conditions of Theorem 2.1 or Theorem 2.7. There exists a bounded non-empty convex set $A$ such that for any $\varepsilon \in (0, 1)$,

$$(1 - \varepsilon)A \subset \frac{A + [-\frac{1}{2}, \frac{1}{2}]^d}{t} \subset (1 + \varepsilon)A \quad (7)$$

for all sufficiently large $t$.

We do not prove the shape theorem in this paper and refer instead to Section 3 of [AMPR01]. The authors of that paper point out that the shape theorem for the discrete-time frog model proven in that paper holds for the continuous-time version too, provided that the faster than linear spread is ruled out – and this is exactly the conclusions of Theorem 2.1 and Theorem 2.7.

As mentioned in the introduction, we make use of auxiliary models. In the proofs of Theorems 2.1, 2.2, and 2.3, the auxiliary process is a percolation model similar to the Poisson blob model. We call this auxiliary model totally asymmetric discrete Boolean percolation. The comparison is not carried out via renormalization, but rather the connected components in the auxiliary percolation process represent regions of space traversed quickly, while the vacant regions are traversed slowly. The totally asymmetric discrete Boolean percolation model and its properties are described in Section 3. In the proof of Theorem 2.7 the auxiliary process is
the greedy lattice animals model [Mar02, GK94, CGGK93]. Here too low values (in particular, zero) in the greedy lattice animal model represent sites that do not have any quick outgoing particles.

The idea to get some information about the spread of the process by treating certain regions of space as fast appears in [GM08], where it is applied to a continuous-time continuous-space model of growing sets introduced by Deijfen [Dei03]. To deal with the continuous-space nature of Deijfen’s model, the authors in [GM08] introduce continuous greedy paths model which is a continuous-space equivalent of the greedy lattice animals. In the present paper we treat a lattice model (the frog model), hence we work directly with the greedy lattice animals. Further discussion of the ideas of the proof can be found in Section 2.2.

Remark 2.9. We see in Theorem 2.1 and Theorem 2.7 that it is possible for $\mu$ to have infinite expectation while the speed is finite. This may be considered counter-intuitive. One heuristic explanation for this may be that the probabilities of a simple random walk traveling at high speed decline exponentially with the distance (see Lemma 5.3), and hence the conditions for the linear spread are given in terms of roughly speaking the logarithmic moments.

Remark 2.10. In this work we always start with active particles located exclusively at the origin. Starting from a finite collection of sites with active particles does not affect the asymptotic spread rate and our results still apply. This is a consequence of the following observation. Let $\zeta$ be a finite non-empty subset of $\mathbb{Z}^d$ and denote by $A_t^\zeta$ the set of sites visited by the time $t$ if at time 0 the locations of active particles are exactly $\zeta$. In the case $\eta(x) = 0$ for some $x \in \zeta$, an active particle is added to $x$ at time 0 (the addition of new particles is not necessary if active particles exist elsewhere. It is done for convenience because with the addition we get equality in (8); otherwise we would have to work with inclusions). Then for any finite $\zeta \subset \mathbb{Z}^d$ a.s.

$$A_t^\zeta = \bigcup_{x \in \zeta} A_t^{\{x\}}, \quad (8)$$

and all the conclusions about the linear or superlinear spread rate follow.

Remark 2.11. It was shown in [BPK20] that the set $A_t$ can become infinite in a finite time if the tails of $\mu$ are heavy enough. In the present paper we address the conditions for the linear and superlinear spread rates. The observation in [BPK20] raises the questions about the conditions separating the case of the superlinear spread rate such that at every moment of time $t > 0$ only finitely many sites have been visited by active particles, and the case of an explosion. By the explosion here we mean that by a certain finite time, infinitely many sites have been visited by active particles.

Remark 2.12. As mentioned in Remark 2.11, the set $A_t$ can become infinite in a finite time when the tails of $\mu$ are heavy enough. It therefore behooves us to say a few words about the
construction of the process. Define the explosion time

$$\tau_e = \sup \{ t : A_t \text{ is finite} \}. \quad (9)$$

Prior to \( \tau_e \) only finitely many events occur, hence the construction on \([0, \tau_e)\) presents no challenges (indeed, on \([0, \tau_e)\) the collection of active particles can be seen as a pure jump type Markov process, see e.g. [Kal02, Section 12]). On the event \( \{ \tau_e < \infty \} \) the construction of the process on \([\tau_e, \infty)\) may present additional challenges because the process might become a system of infinitely many interacting particles. Since we are only interested in the spread rate, there is no need to consider the process on \([\tau_e, \infty)\); since \( A_t \) is non-decreasing in \( t \) and \( \bigcup_{t < \tau_e} A_t \) is infinite, we define the spread to be superlinear on the event \( \{ \tau_e < \infty \} \).

**Remark 2.13.** Items (i) and (iii) of Theorem 1.2 are statements of the form ‘if for some \( B > 1 \), the series \( \ldots \text{converges/diverges, then} \ldots \)’. For series in (2) and (4), the convergence for every \( B > 1 \) is equivalent to the convergence for some \( B > 1 \). For instance for if \( 1 < A < B \),

$$\sum_{m \in \mathbb{N}} \frac{1}{\mu\left([B^m, \infty]\right)} \leq \sum_{m \in \mathbb{N}} \frac{1}{\mu\left([A^m, \infty]\right)}$$

and

$$\sum_{m \in \mathbb{N}} \frac{1}{\mu\left([A^m, \infty]\right)} \leq \log_A B + \log_A B \sum_{m \in \mathbb{N}} \frac{1}{\mu\left([B^m, \infty]\right)}$$

Similarly for (4)

$$\sum_{n \in \mathbb{N}} \mu\left([A^{n \ln^2 n}, \infty]\right) \simeq \sum_{n \in \mathbb{N}} \mu\left([B^{n \ln^2 n}, \infty]\right)$$

(\( \simeq \) means ‘have the same convergence properties’ and is introduced in Section 2.3). This is however not the case with the series in (3) which may have different convergent properties for different \( B > 1 \).

### 2.1 Totally asymmetric discrete Boolean percolation (TADBPP)

We now describe the auxiliary percolation model used in the proofs of Theorems 2.1, 2.2, and 2.3. It belongs to the class of discrete Boolean percolation. An overview of the earlier works related to this model can be found in [BMS05], while some connectivity properties are established in [CMG20]. Here we are interested in the case when the random connected neighborhoods, or grains in the terminology of [BMS05], are totally asymmetric in the sense that instead of the random balls \([x-r, x+r]\) with random radii \( r \), the intervals \([x, x+r]\) comprise connected components.

Let \( \{ \psi_z \}_{z \in \mathbb{Z}} \) be a collection of independent identically distributed \( \mathbb{Z}_+ \)-valued random variables with distribution \( p_k = \mathbb{P}\{ \psi_0 = k \} \). We say that \( x, y \in \mathbb{Z}, x \leq y \), are directly connected (denoted by \( x \xrightarrow{Z} y \)) if there exists \( z \leq x, z \in \mathbb{Z} \), such that \( z + \psi_z \geq y \). We say that \( x \) and \( y \) are connected (denoted by \( x \xrightarrow{Z} y \)) if they are directly connected, or if there exists \( z_1 \leq \ldots \leq z_n \in \mathbb{Z}, \)
The random variable $\ell_f$ further discussed in Section 2.2 is a continuous-time random walk on $\mathbb{Z}^d$ and $\tau_1, \tau_2, \ldots$ be its jump times, $\tau_0 = 0$. Let also $\{(S_t^{(x,j)}), t \geq 0, x \in \mathbb{Z}^d, j \in \mathbb{N}\}$ be independent copies of $\{S_t, t \geq 0\}$ assigned to individual particles. For fixed $t, x$, and $j$, $x + S_t^{(x,j)}$ represents the position of the $j$-th particle started at location $x$, $t$ units of time after the particle was activated. For each realization of $\eta$, only the walks $(S_t^{(x,j)}, t \geq 0)$ with the indices satisfying $j \leq \eta(x)$ are used. A particle is identified with its index $(x, j)$.

The ideas of the proofs of Theorems 2.1 and 2.2. The ideas discussed here are inspired by those articulated in [GM08] and applied there to Deijfen’s model. Let $d = 1$. For $x \in \mathbb{Z}$ and $A > 0$ define

$$\ell_x^{(A)} = \max\{k \in \mathbb{Z}_+: \exists t > 0, j \in \mathbb{N}, \eta(x) \text{ such that } \frac{S_t^{(x,j)}}{t} \geq A \text{ and } S_t^{(x,j)} \geq k\}.$$  

(10)

The random variable $\ell_x^{(A)}$ can be thought of as the length of the longest interval traveled toward $+\infty$ at speed at least $A$ by a particle started from $x$. 

2.2 Very brief outlines of the proofs

Let $\{S_t, t \geq 0\}$ be a simple continuous-time random walk on $\mathbb{Z}^d$ and $\tau_1, \tau_2, \ldots$ be its jump times, $\tau_0 = 0$. For each realization of $\eta$, only the walks $(S_t^{(x,j)}, t \geq 0)$ with the indices satisfying $j \leq \eta(x)$ are used. A particle is identified with its index $(x, j)$.
Consider now TADBP with $\psi_x = \ell_x^{(A)}$. Imagine that $x \xrightarrow{Z_+} \infty$. It means that for every $y \in (x, \infty)$ there exists $z < y$ such that there is a particle starting from $z$ and traveling to $y$ or farther at speed at least $A$. Thus, intuitively, the speed of the system should be at least $A$. If this is true for any $A > 1$, then the spread must be superlinear. Conversely, imagine many sites of $Z_+$ are dry. Then each of those sites is traveled at speed not greater than $A$. If such sites constitute a positive proportion of all sites in a certain sense, then we get a bound on the speed, and thus the spread is linear.

The ideas of the proof of Theorem 2.3. If (3) converges, then we are under the assumptions of Theorem 2.2. If it diverges, then the TADBP random variables $\ell_x^{(A)}$ may be in the setting of Lemma 3.4. That is, the average size of a connected component is infinity, but no site is connected to $+\infty$. Thus, almost all sites are wet, but the set of dry sites is unbounded. To deal with the dry sites, we use a bound on the differences between activation time in some ways similar to Lemma 5.2 in [RS04]. Let $\sigma_x = \min\{t \geq 0 : x \in A_t\}$ be the moment when $x$ is visited by an active particle for the first time. We show that for sufficiently large $q$

$$P\{\sigma_x - \sigma_{x-1} \geq q\} \leq c_q^{1/2}. \quad (11)$$

where $c_q < 1$, and then proceed to obtain

$$P\left\{ \max_{1 \leq y \leq x} (\sigma_y - \sigma_{y-1}) \geq C^2 \ln^2 x \text{ infinitely often} \right\} = 0 \quad (12)$$

for some constant $C > 0$. Combination of (4) and (12) is then shown to imply the superlinear spread.

The ideas of the proof of Theorem 2.7. Here the greedy lattice animals [Mar02, CGGK93] play the role of the auxiliary model instead of TADBP. Recall that $\ell_x^{(A)}$ were defined in (10), where the interpretation of $\ell_x^{(A)}$ is also briefly discussed. Now, imagine that for any infinite sequence $x_0 = 0, x_1, x_2, \ldots, x_n, \ldots, x_i \in \mathbb{Z}^d, x_i \neq x_j, i \neq j, \min_{j=1, \ldots, i-1} |x_i - x_j| = 1, i = 1, 2, \ldots, n$, with distinct points the inequality

$$\frac{1}{n} \sum_{i=0}^{n} \ell_x^{(A)} \leq \frac{1}{2}$$

holds. That means along the path $x_0 = 0, x_1, x_2, \ldots$ not more than half the distance is traveled at speed greater than $A$. The remaining one half is then traveled at speed at most $A$. If this is true uniformly across all paths, the linear spread should follow.

2.3 Notation and conventions

For two series $\sum_n a_n$ and $\sum_n b_n$ with non-negative elements we write $\sum_n a_n \simeq \sum_n b_n$ if they have the same convergence properties, that is, they either both converge or both diverge. Respectively, we write $\sum_n a_n \preceq \sum_n b_n$ if $\sum_n b_n$ diverges, or if both $\sum_n a_n$ and $\sum_n b_n$ converge. This is true for example if $a_n \leq b_n$ for large $n \in \mathbb{N}$ (but not necessarily for all $n \in \mathbb{N}$).
The minimum and maximum operators $\land$ and $\lor$ precede addition and subtraction but follow after multiplication and division; in other words, $a + b \lor cd = a + (b \lor (cd))$. For an interval $I$, $|I|$ is its length. We adopt the following convention regarding the operations over the empty set: $\sum_{q \in \emptyset} q = 0$, $\prod_{q \in \emptyset} q = 1$, $\max \emptyset = -\infty$, $\min \emptyset = +\infty$. The symbol $\mathbf{1}$ denotes an indicator.

**Remark 2.14.** The following inequalities are used extensively in the paper. For $a \in (0, 1)$, $b \geq 1$,

\[
(1 - a)^b \geq 1 - 1 \land ab, \quad (1 - a)^b \leq e^{-ab}
\]

They are consequences and extensions of Bernoulli’s inequality.

3 Totally asymmetric discrete Boolean percolation: properties

In this section we determine the fraction of wet sites and establish necessary and sufficient conditions for a node being connected to $+\infty$ with positive probability. Most of the results in this section are not new. In particular, the transience criterion in Proposition (3.6) appears in Kesten’s Appendix to [Lam70] and also later in [KW06], and the positive recurrence criterion is formulated in [KW06, Page 283]; it is also the content of [Zer18, Proposition 1.1]. The proof of our Lemma 3.4 is basically the same as the proof of Proposition 1.1 in [Zer18].

First we establish under what conditions $\mathbb{P}\{x \xrightarrow{Z} \infty\} = 1$. For $n \in \mathbb{N}$, let $r_n = \sum_{i=n}^{\infty} p_i = \mathbb{P}\{\psi \geq n\}$ be the tail of the distribution $\{p_n\}_{n \in \mathbb{Z}^+}$. Let the exclamation mark in front of the connectivity relations denote the negation, for example the event $\{-1 !\xrightarrow{Z} 0\}$ is the complement of $\{-1 \xrightarrow{Z} 0\}$. We exclude trivial cases and assume $p_0 \in (0, 1)$.

**Lemma 3.1.** Consider TADBP on $\mathbb{Z}$. We have $\mathbb{P}\{x \xrightarrow{Z} \infty\} \in \{0, 1\}$, and $\mathbb{P}\{x \xrightarrow{Z} \infty\} = 1$ if and only if $\prod_{k=1}^{\infty} (1 - r_k) = 0$. Respectively, $\mathbb{P}\{x \xrightarrow{Z} \infty\} = 0$ if and only if $\prod_{k=1}^{\infty} (1 - r_k) > 0$, and in this case a.s.

\[
\frac{\#\{k \in \{1, \ldots, n\} : k \text{ is dry}\}}{n} \xrightarrow{\mathbb{P}} \prod_{k=1}^{\infty} (1 - r_k), \quad n \to \infty. \tag{13}
\]

**Proof.** We have

\[
\mathbb{P}\{-1 !\xrightarrow{Z} 0\} = \mathbb{P}\{\psi_{-m} < m \text{ for all } m \in \mathbb{N}\} = \prod_{m=1}^{\infty} \mathbb{P}\{\psi_0 < m\} = \prod_{m=1}^{\infty} (1 - r_m). \tag{14}
\]

Hence $\mathbb{P}\{-1 \xrightarrow{Z} 0\} = 1$ if $\prod_{k=1}^{\infty} (1 - r_k) = 0$, and by translation invariance $\mathbb{P}\{x \xrightarrow{Z} x + 1\} = 1$ for every $x \in \mathbb{Z}$. Thus, a.s. every node is connected to infinity provided $\prod_{k=1}^{\infty} (1 - r_k) = 0$. 

10
Let now $\prod_{k=1}^{\infty} (1 - r_k) > 0$. Define the random variables $Z_n = \mathbb{1}\{n - 1! \xrightarrow{\mathbb{Z}} n\}$. Since $\{\psi_n\}_{n \in \mathbb{Z}}$ is a sequence of i.i.d. random variables and thus ergodic, so is $\{Z_n\}_{n \in \mathbb{Z}}$, because

$$Z_n = \prod_{k=1}^{n} \mathbb{1}\{\psi_{n-k} < k\}$$

is a functional transformation of $\{\psi_n\}$ (see e.g. [Dur10, Theorem 7.1.3]). By the ergodic theorem and (14) a.s.

$$\sum_{k=1}^{n} Z_k \rightarrow \mathbb{E}Z_1 = \prod_{k=1}^{\infty} (1 - r_k) > 0.$$  \hfill (15)

In the remaining part of the section we focus on TADBP on $\mathbb{Z}_+$.

**Lemma 3.2.** Consider TADBP on $\mathbb{Z}_+$ and let $\prod_{k=1}^{\infty} (1 - r_k) > 0$. The fraction of sites that are dry is $\prod_{k=1}^{\infty} (1 - r_k)$ in the sense that a.s.

$$\frac{\#\{k \in \{1, \ldots, n\} : k \text{ is dry}\}}{n} \rightarrow \prod_{k=1}^{\infty} (1 - r_k), \quad n \rightarrow \infty.$$  \hfill (16)

The fraction of isolated sites is $p_0 \prod_{k=1}^{\infty} (1 - r_k)$. A.s. no site is connected to $+\infty$.

**Proof.** The convergence (16) follows from (13) since a.s. $\sup_{m \in \mathbb{N}} (\psi_m - m) < \infty$, and for sites $x > \sup_{m \in \mathbb{N}} (\psi_m - m)$ the values $\psi_m$, $m \in \mathbb{N}$, do not have any effect on whether $x$ is wet or dry.

A site $x$ is isolated if and only if $x$ is dry and $\psi_x = 0$. Since $\psi_x$ is independent of $\{\psi_y\}_{y < x}$, the second statement of the lemma follows. \hfill □

For $m \in \mathbb{Z}_+$, denote by $Y_m$ the difference between the rightmost site directly connected to $m$ and $m$, that is,

$$Y_m = \max\{l : m \xrightarrow{\mathbb{Z}_+} l\} - m.$$  \hfill (17)

Recall that by definition $m \xrightarrow{\mathbb{Z}_+} m$ holds true for all $m \in \mathbb{Z}_+$, hence $Y_m \geq 0$. By construction for $m \in \mathbb{N}$

$$Y_m = \psi_m \vee (Y_{m-1} - 1) = \psi_m \vee (\psi_{m-1} - 1) \vee \ldots \vee (\psi_1 - m + 1) \vee (\psi_0 - m).$$  \hfill (18)

Note that since we have assumed $p_0 \in (0,1)$, $(Y_t, t \in \mathbb{Z}_+)$ constitutes an irreducible Markov chain on $\mathbb{Z}_+$. In essence this Markov chain appears in [Lam70, (24)] and [KW06, Page 268].

**Lemma 3.3.** Assume that

$$\sum_{n=1}^{\infty} \prod_{k=1}^{n} (1 - r_k) < \infty.$$  \hfill (19)

Then a.s. there exists $x \in \mathbb{Z}_+$ connected to $\infty$:

$$\mathbb{P}\left\{x \xrightarrow{\mathbb{Z}_+} \infty \text{ for some } x \in \mathbb{N}\right\} = 1.$$  \hfill (20)
Proof. We have
\[ P\{Y_m = 0 \} = \prod_{i=0}^{m} P\{\psi \leq m - i\} = \prod_{i=0}^{m} P\{\psi \leq i\} = \prod_{i=0}^{m} (1 - r_{i+1}). \quad (21) \]

By (19) and (21)
\[ P\{Y_m = 0 \text{ for infinitely many } m \in \mathbb{N}\} = 0. \quad (22) \]

It remains to note that if some \( m \in \mathbb{N}, Y_i \geq 1 \) for \( i \geq m \), then \( m \xrightarrow{Z_+} \infty \). \]

**Lemma 3.4.** Assume that
\[ \sum_{n=1}^{\infty} \prod_{k=1}^{n} (1 - r_k) = \infty \quad (23) \]
and
\[ \prod_{k=1}^{\infty} (1 - r_k) = 0. \quad (24) \]

Then
\[ P\bigg\{ x \xrightarrow{Z_+} \infty \text{ for some } x \in \mathbb{N}\bigg\} = 0. \quad (25) \]

**Proof.** Recall that the Markov \((Y_t, t \in \mathbb{Z}_+)\) is defined in (17). Since
\[ \{x \xrightarrow{Z_+} \infty\} = \{Y_k > 0, k = x, x+1, x+2, \ldots\}, \quad (26) \]
(25) is equivalent to \((Y_t, t \in \mathbb{Z}_+)\) being recurrent. By (23)
\[ \sum_{n=1}^{\infty} P\{Y_n = 0\} = \sum_{n=1}^{\infty} \prod_{k=1}^{n} (1 - r_k) = \infty. \quad (27) \]

Therefore \((Y_t, t \in \mathbb{Z}_+)\) is recurrent by the well known properties of Markov chains with a countable state space, see e.g. [Dur10, Theorem 6.4.2] or [Shi19, Theorem 1, Section 5, Chapter 8].

**Remark 3.5.** *Lemma 3.4 is akin to the dichotomy occurring under certain conditions in Boolean percolation when each occupied component is a.s. finite but the expected size of an occupied component is infinite, see [MR96, Corollary 3.2].*

We note that the individual assumptions of Lemmas 3.2, 3.3 and 3.4 regarding \(\{p_i\}_{i \in \mathbb{Z}_+}\),
\[ \prod_{k=1}^{\infty} (1 - r_k) > 0, \quad (28) \]
\[ \sum_{n=1}^{\infty} \prod_{k=1}^{n} (1 - r_k) < \infty, \quad (29) \]
and
\[ \sum_{n=1}^{\infty} \prod_{k=1}^{n} (1 - r_k) = \infty, \quad \prod_{k=1}^{\infty} (1 - r_k) = 0. \quad (30) \]
exhaust all options; that is, one (and only one of course) of the conditions (28), (29), and (30) always holds. Therefore, those lemmas lead to characterization of the recurrence properties of the Markov chain \((Y_t, t \in \mathbb{Z}_+)\). These properties are collected in the next proposition. It is formulated in the self-sufficient way, so that all necessary notation used in this section is reintroduced. As indicated at the beginning of the section, this proposition does not contain new results.

**Proposition 3.6.** Let \(\{\psi_k\}_{k \in \mathbb{Z}_+}\) be a sequence of \(\mathbb{Z}_+\)-valued random variables with distribution \(\{p_i\}_{i \in \mathbb{Z}_+}\). Set \(r_k = \sum_{i=k}^{\infty} p_i\) and \(Y_m = \psi_m \lor (Y_{m-1} - 1), m \in \mathbb{N}, Y_0 = \psi_0\). Then

(i) The Markov chain \((Y_t, t \in \mathbb{Z}_+)\) is positive recurrent if and only if \(\mathbb{E}\psi_1 = \sum_{k=1}^{\infty} r_k < \infty\),

(ii) \((Y_t, t \in \mathbb{Z}_+)\) is transient if and only if \(\sum_{n=1}^{\infty} \prod_{k=1}^{n} (1 - r_k) < \infty\),

(iii) The chain \((Y_t, t \in \mathbb{Z}_+)\) is null recurrent if and only if both \(\sum_{k=1}^{\infty} r_k = \infty\) and \(\sum_{n=1}^{\infty} \prod_{k=1}^{n} (1 - r_k) = \infty\).

**Proof.** As was noted in the proof of Lemma 3.4, (25) is equivalent to \((Y_t, t \in \mathbb{Z}_+)\) being recurrent. Thus Lemmas 3.2, 3.3, and 3.4 combined yield (ii).

Assume now \(\sum_{n=1}^{\infty} \prod_{k=1}^{n} (1 - r_k) = \infty\), that is, that the chain is recurrent. If \(\sum_{k=1}^{\infty} r_k = \infty\), then the chain is not positive recurrent because the expected recurrence time to 0 is greater than \(\sum_{k=1}^{\infty} r_k\), so it is infinite. On the other hand, if \(\sum_{k=1}^{\infty} r_k < \infty\), then for every \(m \in \mathbb{N}\)

\[
P\{Y_m = 0\} = P\{\psi_m \leq 0\}P\{\psi_{m-1} \leq 1\} \cdots P\{\psi_0 \leq m\} = \prod_{k=1}^{m+1} (1 - r_k) \geq \prod_{k=1}^{\infty} (1 - r_k) > 0.
\]

and hence \((Y_t, t \in \mathbb{Z}_+)\) cannot be null recurrent. \(\square\)

**Corollary 3.7.** Assume that (23) and (24) hold. Then the fraction of wet sites is 1.

The next two lemmas will be helpful in showing that the spread is superlinear. They provide a way to translate the properties of the associated TADBP to the properties of the frog model.

**Lemma 3.8.** Let \(x \in \mathbb{N}\). A.s. on \(\{x \xrightarrow{Z_{+}} \infty\}\), every site \(y > x\) is wet, and there exists a (random) sequence \(x = x_0 < x_1 < x_2 < \ldots, x_i \in \mathbb{N}\), such that for every \(i \in \mathbb{Z}_+\)

\[
x_{i+1} \leq x_i + \psi_{x_i} < x_{i+2}.
\]

In particular, every \(z \geq x\) belongs to no more than two intervals of the type \([x_i, x_i + \psi_{x_i}], i \in \mathbb{Z}_+\).

**Proof.** By definition of \(\xrightarrow{Z_{+}}\), every site \(y > x\) is wet a.s. on \(\{x \xrightarrow{Z_{+}} \infty\}\). Define the elements of the sequence \(\{x_i\}_{i \in \mathbb{Z}_+}\) consecutively setting \(x_0 = x\) and letting for \(i \in \mathbb{Z}_+\)

\[
x_{i+1} = \max\{y \in [x_i + 1, x_i + \psi_{x_i}] \cap \mathbb{N} : y + \psi_y = \max\{z + \psi_z : z = x_i + 1, \ldots, x_i + \psi_{x_i}\}\}.
\]

In other words, \(x_{i+1} \in [x_i + 1, x_i + \psi_{x_i}]\) is characterized by two properties:
(i) for every $z \in [x_i + 1, x_i + \psi_{x_i}] \cap \mathbb{N}$,
\[ x_{i+1} + \psi_{x_{i+1}} \geq z + \psi_z, \]
(ii) and for every $z' \in [x_{i+1} + 1, x_i + \psi_{x_i}] \cap \mathbb{N}$,
\[ x_{i+1} + \psi_{x_{i+1}} > z' + \psi_{z'}. \]
(here $[a, b] = \emptyset$ if $a > b$). By construction, $x_{i+1} \leq x_i + \psi_{x_i}$, so the left inequality in (31) holds.
A.s. on $\{x \overset{Z_i}{\to} \infty\}$, $x_{i+1} + \psi_{x_{i+1}} > x_i + \psi_{x_i}$, because otherwise $x_i + \psi_{x_i} + 1$ would not be wet. Hence a.s. on $\{x \overset{Z_i}{\to} \infty\}$ also $x_{i+2} + \psi_{x_{i+2}} > x_{i+1} + \psi_{x_{i+1}}$. Therefore the inequality $x_{i+2} \leq x_i + \psi_{x_i}$ is impossible a.s. on $\{x \overset{Z_i}{\to} \infty\}$ because it would contradict (i).

The next lemma replicates Lemma 3.8 for the case of a finite component. The proof is practically identical and is therefore omitted.

**Lemma 3.9.** Let $x \in \mathbb{N}$. A.s. on $\{x \overset{Z_i}{\to} y\}$, every site $z \in [x + 1, y]$ is wet, and there exists a (random) sequence $x = x_0 < x_1 < \cdots < x_m = y$, $x_i \in \mathbb{N}$, such that for every $i \in \{0, \ldots, m-2\}$
\[ x_{i+1} \leq x_i + \psi_{x_i} < x_{i+2}, \tag{33} \]
and $x_{m-1} + \psi_{x_{m-1}} = x_m$. In particular, every $z \in [x, y]$ belongs to no more than two intervals of the type $[x_i, x_i + \psi_{x_i}]$, $i \in \{0, \ldots, m-1\}$.

## 4 Convergence properties of related series

In this section some auxiliary results used in the proofs are collected. The focus is mostly on the convergence properties of related series.

**Lemma 4.1.** For $A > 1$,
\[ \sum_{k=1}^{\infty} \mu(k) \sum_{n=1}^{\infty} 1 \wedge kA^{-n} < \infty \tag{34} \]
if and only if
\[ \sum_{n=1}^{\infty} \log n \mu(n) < \infty. \tag{35} \]

**Proof.** We have
\[
\sum_{k=1}^{\infty} \mu(k) \sum_{n=1}^{\infty} 1 \wedge kA^{-n} = \sum_{k=1}^{\infty} \mu(k) \sum_{1 \leq n \leq \log_A k} 1 + \sum_{k=1}^{\infty} \mu(k) \sum_{n > \log_A k} kA^{-n} \\
\simeq \sum_{k=1}^{\infty} \mu(k) \log k + \sum_{k=1}^{\infty} \mu(k) \frac{1}{1 - A^{-1}} = (\log A)^{-1} \sum_{k=1}^{\infty} \mu(k) \log k + \frac{1}{1 - A^{-1}}.
\]
\[ \square \]
Lemma 4.2. Suppose that for some $A, B > 1$

\[ 1 - \sum_{k=0}^{\infty} \mu(k) [1 - A^{-n}]^k \leq r_n \leq 1 - \sum_{k=0}^{\infty} \mu(k) [1 - B^{-n}]^k, \quad n \in \mathbb{N}. \quad (36) \]

Then

\[
\prod_{n=1}^{\infty} (1 - r_n) = 0
\]

if and only if

\[
\sum_{n=1}^{\infty} \log n \mu(n) = \infty.
\]

Proof. Note that by (36), $r_n < 1$, $n \in \mathbb{N}$. Since $(1 + a)^b \leq e^{ab}$ for $a \geq -1, b \geq 0$, we have

\[
1 - \sum_{k=0}^{\infty} \mu(k) [1 - A^{-n}]^k \geq 1 - \sum_{k=0}^{\infty} \mu(k) e^{-kA^{-n}} = \sum_{k=0}^{\infty} \mu(k) \left[ 1 - e^{-kA^{-n}} \right]. \quad (39)
\]

By (36) and (39) and since $\inf_{a>0, b>0} \frac{1-e^{-a}}{1-b} > 0$ and $\sup_{a>0, b>0} \frac{1-e^{-a}}{1-b} = 1$ we get

\[
\sum_{n=1}^{\infty} r_n \geq \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} \mu(k) \left[ 1 - e^{-kA^{-n}} \right] = \sum_{k=0}^{\infty} \mu(k) \sum_{n=1}^{\infty} \left[ 1 - e^{-kA^{-n}} \right] \simeq \sum_{k=0}^{\infty} \mu(k) \sum_{n=1}^{\infty} 1 \& kA^{-n}. \quad (40)
\]

On the other hand, since by Bernoulli’s inequality $(1 - a)^b \geq 1 - 1 \& ab$, $a \in (0, 1)$, $b \geq 1$, we have

\[
1 - \sum_{k=0}^{\infty} \mu(k) [1 - B^{-n}]^k \leq 1 - \sum_{k=0}^{\infty} \mu(k) [1 - 1 \& kB^{-n}] = \sum_{k=0}^{\infty} \mu(k) [1 \& kB^{-n}]. \quad (41)
\]

Hence

\[
\sum_{n=1}^{\infty} r_n \leq \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \mu(k) [1 \& kB^{-n}] = \sum_{k=1}^{\infty} \mu(k) \sum_{n=1}^{\infty} 1 \& kB^{-n}. \quad (42)
\]

From (40), (42), and Lemma 4.1 it follows that

\[
\sum_{n=1}^{\infty} r_n \simeq \sum_{k=1}^{\infty} \mu(k) \sum_{n=1}^{\infty} 1 \& kA^{-n} \simeq \sum_{n=1}^{\infty} \log n \mu(n). \quad (43)
\]

The equivalence of (37) and (38) follows since (37) is equivalent to $\sum_{n=1}^{\infty} r_n = \infty$. \qed

Note that

\[
\sum_{m=1}^{\infty} \prod_{n=1}^{m} \sum_{k=0}^{\infty} \mu(k) [1 - B^{-n}]^k = \sum_{m=1}^{\infty} \prod_{n=1}^{m} M_{\mu}(1 - B^{-n}), \quad (44)
\]

where $M_{\mu}$ is the moment generating function of the distribution $\mu$. This observation is not used anywhere in the paper.

Lemma 4.3. The series in (3) converges for every $B > 1$ if and only if for every $A > 1$

\[
\sum_{m=1}^{\infty} \prod_{n=1}^{m} \sum_{k=0}^{\infty} \mu(k) [1 - A^{-n}]^k < \infty. \quad (45)
\]

15
Proof. Suppose (3) converges for every $B > 0$. It is sufficient to show that (45) holds for large $A$. Thus we can assume that $\mu([0,A]) \geq \frac{1}{2}$ and $A \geq 10$. We have

$$\sum_{k=0}^{\infty} \mu(k) \left[ 1 - A^{-n} \right]^k \leq \sum_{k=0}^{\infty} \mu(k) e^{-kA^{-n}} = \sum_{k=0}^{\infty} \mu(k) e^{-kA^{-n}} + \sum_{k=[A^{2n}]+1}^{\infty} \mu(k) e^{-kA^{-n}} \leq \mu([0,A^{2n}]) + e^{-A^n}. \quad (46)$$

For $m \in \mathbb{N}$,

$$\frac{\prod_{n=1}^{m} (\mu([0,A^{2n}]) + e^{-A^n})}{\prod_{n=1}^{m} \mu([0,A^{2n}])} \leq \prod_{n=1}^{m} \left( 1 + 2e^{-A^n} \right) < \prod_{n=1}^{\infty} (1 + 2e^{-A^n}) < \infty. \quad (47)$$

Hence

$$\sum_{m=1}^{\infty} \prod_{n=1}^{m} (\mu([0,A^{2n}]) + e^{-A^n}) \simeq \sum_{m=1}^{\infty} \prod_{n=1}^{m} \mu([0,A^{2n}]) < \infty, \quad (48)$$

and

$$\sum_{m=1}^{\infty} \prod_{n=1}^{m} \sum_{k=0}^{\infty} \mu(k) \left[ 1 - A^{-n} \right]^k \leq \sum_{m=1}^{\infty} \prod_{n=1}^{m} \left( \mu([0,A^{2n}]) + e^{-A^n} \right) < \infty. \quad (49)$$

Conversely, suppose (45) holds for every $A > 1$. Then for $B > 1$

$$\infty > \sum_{m=1}^{\infty} \prod_{n=1}^{m} \sum_{k=0}^{\infty} \mu(k) \left[ 1 - 2^{-n}B^{-n} \right]^k \geq \sum_{m=1}^{\infty} \prod_{n=1}^{m} \sum_{k=0}^{\infty} \mu(k) \left[ 1 - 2^{-n}B^{-n} \right]^k \geq \sum_{m=1}^{\infty} \prod_{n=1}^{m} \mu([0,B^{2n}]) \left[ 1 - 2^{-n} \right]^k \geq \left( \prod_{n=1}^{\infty} (1 - 2^{-n}) \right)^{-1} \sum_{m=1}^{\infty} \prod_{n=1}^{m} \mu([0,B^{2n}]) \quad (50)$$

In the rest of this section we show that the series in (ii) and (iii) of Theorem 1.2 have independent convergence properties. Thus, the content of the remaining part of the section is not used in the proof of Theorem 1.2 per se but rather addresses the logical independence of its parts. The construction in the proof of the following lemma is courtesy of Christian Remling.

Lemma 4.4. Let $\{u_m\}_{m\in\mathbb{N}}$ be an increasing sequence of non-negative numbers, $u_m \to \infty$, $m \to \infty$. Let $\{v_m\}_{m\in\mathbb{N}}$ be another sequence of positive numbers. There exists a sequence $\{g_n\}_{n\in\mathbb{N}}$, $g_n \to 0$, $g_n > 0$ such that

$$\sum_{n=1}^{\infty} g_n = \infty \quad (51)$$

and

$$\sum_{m=1}^{\infty} u_m \exp \left\{ - \sum_{i=1}^{m} g_i v_i \right\} = \infty. \quad (52)$$
Proof. Define the elements of the sequences \( \{g_n\}_{n \in \mathbb{N}}, \{M_n\}_{n \in \mathbb{N}}, \text{and} \{K_n\}_{n \in \mathbb{N}} \) consecutively as follows. Set \( K_1 = 1 \) and \( g_1 = 1 \). For \( n \in \mathbb{N} \) once \( K_n \) and \( g_i, i = 1, \ldots, K_n, \) are defined, set

\[
M_n = \min \left\{ m \geq K_n + 1 : u_m \geq e^{\exp \left( \sum_{i=1}^{K_n} v_i g_i \right)} \right\} \tag{53}
\]

Then for \( i = K_n + 1, \ldots, M_n, \) set \( g_i = h_n := \frac{1}{(M_n - K_n) \max_{K_n + 1 \leq j \leq M_n} v_j} \wedge \frac{1}{n} \). A single step is completed by setting

\[
K_{n+1} = \min \{ m \geq M_n + 1 : h_n (K_{n+1} - K_n) \geq 1 \},
\]

and \( g_i = h_n \) for \( i = M_n + 1, \ldots, K_{n+1} \). Next we define \( M_{n+1} \) as in (53) (of course with \( n \) replaced by \( n + 1 \) everywhere in (53)), and so forth.

With this construction we have

\[
\sum_{i=K_n+1}^{K_{n+1}} g_i \geq 1 \tag{54}
\]

and

\[
u_{M_n} \exp \left\{ - \sum_{i=1}^{M_n} g_i v_i \right\} = u_{M_n} \exp \left\{ - \sum_{i=1}^{K_n} g_i v_i \right\} \times \exp \left\{ - \sum_{i=K_n+1}^{M_n} g_i v_i \right\} \\
\geq e \times \exp \left\{ -h_n (M_n - K_n) \max_{K_n + 1 \leq j \leq M_n} v_j \right\} \geq ee^{-1} = 1. \tag{55}
\]

Thus both (51) and (52) hold. \( \square \)

**Proposition 4.5.** The convergence properties of the series in (3) and (4) are independent. That is, all four combinations of both series converging, either one of the two converging, and both diverging are possible.

Proof. Fix \( B > 1 \) and let \( b_n = \mu([0, B^n]) \) and \( c_n = \mu([0, B^n \ln^2 n]) \). We only consider distributions \( \mu \) with unbounded support. We have \( b_n \not\to 1 \) and

\[
b_{[n \ln^2 n]} \leq c_n \leq b_{[n \ln^2 n]}, \quad n \geq 2. \tag{56}
\]

The series in (3) and (4) can be written as \( \sum_{m=1}^{\infty} \prod_{n=1}^{m} b_n \) and \( \sum_{n=1}^{\infty} (1 - c_n) \) respectively. Note that

\[
\ln^2 k \leq (k + 1) \ln^2 (k + 1) - k \ln^2 k \leq \ln^2 k + 2 \ln k + 2, \quad k \in \mathbb{N}. \tag{57}
\]
Since the sequence \( \{b_n\} \) is monotone

\[
\sum_{m=2}^{\infty} \prod_{n=1}^{m} b_n \geq \sum_{m=2}^{\infty} \prod_{k=1}^{\infty} \prod_{i \in \mathbb{N}: k \ln^2 k \leq i < (k+1) \ln^2 (k+1)} b_i
\]

\[
\geq \sum_{m=2}^{\infty} \max\{l \in \mathbb{N}: l \ln^2 l < m\} \prod_{k=1}^{\infty} \prod_{i \in \mathbb{N}: k \ln^2 k \leq i < (k+1) \ln^2 (k+1)} c_k
\]

\[
\geq \sum_{m=2}^{\infty} \max\{l \in \mathbb{N}: l \ln^2 l < m\} \prod_{k=1}^{\infty} \prod_{i \in \mathbb{N}: k \ln^2 k \leq i < (k+1) \ln^2 (k+1)} c_k^{\ln^2 k + 2 \ln k + 2}
\]

\[
\geq \sum_{l=2}^{\infty} \ln^2 l \prod_{k=1}^{m} c_k^{\ln^2 k + 2 \ln k + 2}
\]

\[
= \sum_{l=2}^{\infty} \ln^2 l \exp \left\{ -\sum_{i=1}^{l} \gamma_i (\ln^2 i + 2 \ln i + 2) \right\},
\]

where \( \gamma_i := -\ln c_i > 0 \). Note that \( \sum_{n=1}^{\infty} (1 - c_n) \simeq \sum_{n=1}^{\infty} \gamma_n \) since \( \lim_{n \to \infty} \frac{\ln \gamma_n}{\ln n} = 1 \). By Lemma 4.4, \( \{\gamma_i\}_{i \in \mathbb{N}} \) can be chosen in such a way that simultaneously

\[
\sum_{m=1}^{\infty} \prod_{n=1}^{m} b_n = \infty \quad \text{and} \quad \sum_{n=1}^{\infty} (1 - c_n) = \infty.
\]

The other three cases are more straightforward. Before proceeding to them, note that similarly to (58)

\[
\sum_{m=2}^{\infty} \prod_{n=1}^{m} b_n \leq \sum_{m=2}^{\infty} \max\{l \in \mathbb{N}: l \ln^2 l < m\} - 1 \prod_{k=1}^{\infty} \prod_{i \in \mathbb{N}: k \ln^2 k \leq i < (k+1) \ln^2 (k+1)} b_i
\]

\[
\leq \sum_{m=2}^{\infty} \max\{l \in \mathbb{N}: l \ln^2 l < m\} - 1 \prod_{k=1}^{\infty} \prod_{i \in \mathbb{N}: k \ln^2 k \leq i < (k+1) \ln^2 (k+1)} c_k^{\ln^2 k + 2 \ln k + 2}
\]

\[
\leq \sum_{m=2}^{\infty} (\ln^2 l + 2 \ln l + 2) \prod_{k=1}^{\infty} c_k^{\ln^2 k + 2 \ln k + 2}
\]

\[
= \sum_{l=2}^{\infty} (\ln^2 l + 2 \ln l + 2) \exp \left\{ -\sum_{i=1}^{l} \gamma_i (\ln^2 i + 2 \ln i + 2) \right\}
\]

Taking \( \{\gamma_i\}_{i \in \mathbb{N}} \) very small (for example \( \gamma_i = e^{-i^2} \)) we can easily achieve

\[
\sum_{m=1}^{\infty} \prod_{n=1}^{m} b_n = \infty \quad \text{and} \quad \sum_{n=1}^{\infty} (1 - c_n) < \infty.
\]
Letting \( \{ \gamma_i \}_{i \in \mathbb{N}} \) converge to 0 very slowly, for example \( \gamma_i = \frac{1}{\ln \ln \ln i} \) for large \( i \), we get
\[
\sum_{m=1}^{\infty} \prod_{n=1}^{m} b_n < \infty \quad \text{and} \quad \sum_{n=1}^{\infty} (1 - c_n) = \infty.
\]

Letting \( \gamma_{n+1} = \frac{1}{n(\ln n)^{3/2}} \), \( n \geq 2 \), we get
\[
\sum_{n=1}^{\infty} \gamma_n < \infty \quad \text{and} \quad \sum_{n=1}^{\infty} (1 - c_n) < \infty.
\]

5 Proofs of Theorem 2.1 and Theorem 2.2

Let \( \text{tip}(t) \) be the position of the rightmost active particle at time \( t \geq 0 \). Note that it is not necessarily true that \( \text{tip}(t) = \sup A_t \) for \( t \geq 0 \), because active particles can move back toward \(-\infty\). Let us introduce a total ordering \( \prec \) on the set of indices with \( (x,i) \prec (y,j) \) if \( x < y \), or if \( x = y \) and \( i < j \).

Recall that \( \sigma_x = \min\{ t \geq 0 : x \in A_t \} \) is the activation time for particles at \( x \). At time \( t \geq 0 \), let \( X_0 \) be the particle with the smallest index with respect to \( \prec \) located at \( \text{tip}(t) \). We note here that no particle located at \( \text{tip}(t) \) at time \( t \) was activated before \( X_0 = (x_0,i_0) \) was. Let \( X_1 = (x_1,i_1) \) be the particle that activated \( X_0 \), and further define recursively \( X_{k+1} \) as the particle that activated \( X_k \), until \( X_m = (0,i_m) \) for some \( m \in \mathbb{N} \). Set \( w_i = \sigma_{X_k} \), \( k = 1, \ldots, m \).

Let us note right here that the sequence \( \{ X_i, i = 1, \ldots, m \} \) depends on \( t \). Denote by \( W_k \), \( k = 1, \ldots, m \), the interval \([w_{m-k+1}, w_{m-k}]\). If \( x_{m-k+1} < x_{m-k} \), the expression
\[
\frac{(x_{m-k} - x_{m-k+1}) \vee 0}{w_{m-k} - w_{m-k+1}} = \frac{(x_{m-k} - x_{m-k+1}) \vee 0}{|W_k|}
\]
can be seen as the speed at which the interval \([x_{m-k+1}, x_{m-k}]\) is traversed. We take non-negative part in the numerator because the sequence \( \{ x_i, i = 1, \ldots, m \} \) does not have to be non-increasing; indeed, it is possible that active particles from the origin travel leftward, activate a particle at \(-k\), \( k \in \mathbb{N} \), and that particle then moves toward \(+\infty\) very quickly overtaking every other active particle, and becomes a leading particle for some time.
Recall that for \( x \in \mathbb{Z} \) and \( A > 0 \), we have defined
\[
\ell_x^{(A)} = \max\{ k \in \mathbb{Z}_+ : \exists t > 0, j \in [1, \eta(x)] \text{ such that } \frac{S_t^{(x,j)}}{t} \geq A \text{ and } S_t^{(x,j)} \geq k \}. \tag{61}
\]

Let \( \{p_k^{(A)}\}_{k \in \mathbb{Z}_+} \) be the distribution of \( \ell_x^{(A)} \), \( p_k^{(A)} = \mathbb{P}\left\{ \ell_x^{(A)} = k \right\} \). Let \( r_k^{(A)} \) be the corresponding tail, \( r_k^{(A)} = \sum_{i=k}^{\infty} p_i^{(A)} = \mathbb{P}\{ \ell_x^{(A)} \geq k \} \). Note that \( p_0^{(A)} \in (0, 1) \).

**Proposition 5.1.** Suppose that for some \( A > 0 \)
\[
\prod_{k=1}^{\infty} (1 - r_k^{(A)}) > 0. \tag{62}
\]
Then \( \sup \mathcal{A}_t \) grows linearly in time.

**Proof.** Consider TADBP on \( \mathbb{Z} \) with interval distribution \( \{p_k\}_{k \in \mathbb{Z}_+} = \{p_k^{(A)}\}_{k \in \mathbb{Z}_+} \). By Lemma 3.1, the fraction of sites that are dry is \( u = \prod_{k=1}^{\infty} (1 - r_k) > 0 \). Therefore with high probability at least \( (1 - \varepsilon)u \cdot \text{tip}(t) \) sites among \( 1, 2, \ldots, \text{tip}(t) \) are traveled at a speed at most \( A \), where \( \varepsilon \in (0, 10^{-1}) \) is a small number. Hence a.s. for large \( t \), \( \sum_{k=1}^{m} |W_k| \geq \frac{1 - \varepsilon)}{\varepsilon} \cdot \text{tip}(t) \) and
\[
\limsup_{t \to \infty} \frac{\text{tip}(t)}{t} = \limsup_{t \to \infty} \frac{\text{tip}(t)}{\sum_{k=1}^{m} |W_k|} \leq \frac{A}{(1 - \varepsilon)u}.
\]
Since \( \sup \mathcal{A}_t \leq \sup_{s \leq t} \text{tip}(s) \), the statement of the proposition follows. \( \square \)

**Proposition 5.2.** Suppose that for all \( A > 0 \)
\[
\sum_{m=1}^{\infty} \prod_{n=1}^{m} (1 - r_n^{(A)}) < \infty. \tag{63}
\]
Then
\[
\liminf_{t \to \infty} \frac{\text{tip}(t)}{t} = \infty,
\]
and \( \sup \mathcal{A}_t \) grows faster than linearly in time.

**Proof.** Here for \( A > 0 \) we consider TADBP on \( \mathbb{Z}_+ \) with interval distribution \( \{p_k\}_{k \in \mathbb{Z}_+} = \{p_k^{(A)}\}_{k \in \mathbb{Z}_+} \). By (63) and Lemma 3.3 a.s. there is a large (random) number \( x_0 \in \mathbb{N} \) such that \( x_0 \xrightarrow{\mathbb{Z}_+} \infty \). In particular, every site \( x \in (x_0, \infty) \cap \mathbb{N} \) is wet. By Lemma 3.8, there exists a sequence \( x_0, x_1, x_2, \ldots \) such that \( x_i < x_{i+1} \leq x_i + \ell_{x_i}^{(A)} \), \( i \in \mathbb{Z}_+ \), and every site \( y \in (x_0, \infty) \cap \mathbb{N} \) is belongs to at most two intervals of the type \( [x_i, x_i + \ell_{x_i}^{(A)}], i \in \mathbb{Z}_+ \). Consequently for every \( n \in \mathbb{N} \)
\[
\sum_{i=0}^{n} \ell_{x_i}^{(A)} = \sum_{i=0}^{n} \left| [x_i, x_i + \ell_{x_i}^{(A)}] \right| \leq 2(x_n + \ell_{x_n}^{(A)} - x_0). \tag{64}
\]
Once \( x_0 \) is reached, let us consider only particles that start at \( x_i \) and travel to \( x_i + \ell(A) \) at speed at least \( A \). For \( n \in \mathbb{N} \) and \( y_n := x_n + \ell(A) \) by (64)

\[
\sigma_{y_n} - \sigma_{x_0} \leq \frac{1}{A} \sum_{i=0}^{n-1} |x_i, x_i + \ell(A)| \leq \frac{2(y_n - x_0)}{A}.
\]

Hence

\[
\limsup_{t \to \infty} \frac{\text{tip}(t)}{t} = \limsup_{n \to \infty} \frac{y_n}{\sigma_{y_n}} = \limsup_{n \to \infty} \frac{x_0 + (y_n - x_0)}{\sigma_{x_0} + (\sigma_{y_n} - \sigma_{x_0})} = \limsup_{n \to \infty} \frac{y_n - x_0}{\sigma_{y_n} - \sigma_{x_0}} \geq \frac{A}{2}. \tag{65}
\]

Since \( A > 0 \) can be arbitrary large, the statement of the proposition follows.

The next lemma helps in translating conditions (62) and (63) of Propositions 5.1 and 5.2, respectively, into conditions on \( \mu \). It is relevant for inequalities (66) and (67) that for every \( \varepsilon \in (0,1) \), \( \varepsilon^{1-\varepsilon} < 1 \).

**Lemma 5.3.** For \( n \in \mathbb{N} \) and \( \varepsilon \in (0,1) \)

\[
\frac{e^{n\varepsilon(1-\varepsilon)n}}{e^{2n\sqrt{n}}} \leq P \left\{ \exists t \geq 0 : \frac{S_t}{t} \geq \varepsilon^{-1}, S_t \geq n \right\} \leq \frac{1}{2(1-\varepsilon^2)(1-\varepsilon)} \frac{e^{n\varepsilon(1-\varepsilon)n}}{\sqrt{2\pi n}}. \tag{66}
\]

Additionally, for \( \varepsilon < e^{-9/64} \),

\[
P \left\{ \exists t \geq 0 : \frac{S_t}{t} \geq \varepsilon^{-1}, S_t \geq n \right\} \leq \frac{1}{(1-\varepsilon^{2}e^{9/32})(1-\varepsilon)} \exp \left\{ \frac{-3n}{16} \right\} \frac{e^{n\varepsilon(1-\varepsilon)n}}{\sqrt{2\pi n}}. \tag{67}
\]

**Proof.** On the one hand,

\[
P \left\{ \exists t \geq 0 : \frac{S_t}{t} \geq \varepsilon^{-1}, S_t \geq n \right\} \geq P \left\{ \frac{T_n}{n} \leq \varepsilon \right\} \frac{1}{P \left\{ S_{r_j} - S_{r_{j-1}} = 1, j = 1, \ldots, n \right\}} \geq e^{-n\varepsilon n^2} n^{-2n} \geq e^{-n\varepsilon n^2} n^{-2n} \geq \frac{e^{n\varepsilon(1-\varepsilon)n}}{e^{2n\sqrt{n}}} \tag{68}
\]

On the other hand,

\[
P \left\{ \frac{T_n+2k}{n} \leq \varepsilon \right\} = e^{-n\varepsilon} \sum_{i=0}^{n} \frac{(\varepsilon n)^i}{i!} \leq e^{-n\varepsilon e^{n+2k}} \frac{n^{n+2k}}{(n+2k)!} \sum_{i=0}^{n} \frac{(\varepsilon n)^i}{(n+2k)^n} \leq e^{-n\varepsilon e^{n+2k}} \frac{n^{n+2k}}{(n+2k)^{n+2k}} \frac{1}{\sqrt{2\pi(n+2k)}} = e^{-n\varepsilon e^{n+2k}} \frac{e^{2k}}{\sqrt{2\pi(n+2k)}} \leq e^{-n\varepsilon e^{n+2k}} \frac{e^{2k}}{\sqrt{2\pi(n+2k)}} e^{-2k} \leq \frac{1}{1-\varepsilon} e^{-n\varepsilon e^{n+2k}} \frac{e^{2k}}{\sqrt{2\pi(n+2k)}} e^{n}
\]

and hence

\[
P \left\{ \exists t \geq 0 : \frac{S_t}{t} \geq \varepsilon^{-1}, S_t \geq n \right\} \leq \sum_{k=0}^{\infty} P \left\{ \frac{T_n+2k}{n} \leq \varepsilon \right\} P \left\{ S_{r_{n+2k}} \geq n \right\} \leq \sum_{k=0}^{\infty} \frac{1}{1-\varepsilon} e^{-n\varepsilon e^{n+2k}} \frac{e^{2k}}{\sqrt{2\pi(n+2k)}} e^{n} \tag{69}
\]
Note that $S_{r_{n+2k}} \geq n$ if and only if $S_{r_j} - S_{r_{j-1}} = 1$ holds for at least $n + k$ distinct $j \in \{1, \ldots, n + 2k\}$. By a concentration inequality for a sum of independent Bernoulli random variables [CL06, Theorem 2.4]

\[
\mathbb{P}\{S_{r_{n+2k}} \geq n\} = \mathbb{P}\left\{\frac{1}{2}S_{r_{n+2k}} + \frac{1}{2}(n + 2k) \geq \mathbb{E}\left\{\frac{1}{2}S_{r_{n+2k}} + \frac{1}{2}(n + 2k)\right\} + \frac{1}{2}n\right\} 
\leq \exp\left\{-\frac{1}{4}n^2\right\} = \exp\left\{-\frac{-3n}{2(4n + k + \frac{9}{n})}\right\} = \exp\left\{\frac{-3n}{16 + 24k/n}\right\}. \tag{70}
\]

Since

\[
\exp\left\{-\frac{3n}{16 + 24k/n}\right\} \leq \exp\left\{\frac{-3n}{16 + 24k/n}\right\} \leq \exp\left\{\frac{72}{16^2}\right\} = \exp\left\{\frac{9}{32}\right\}, \tag{71}
\]

by (70)

\[
\mathbb{P}\{S_{r_{n+2k}} \geq n\} \leq \frac{1}{2} \exp\left\{-\frac{3n}{16 + \frac{9k}{32}}\right\}. \tag{72}
\]

Hence by (69) for $\varepsilon < e^{-9/64}$

\[
\mathbb{P}\left\{\exists t \geq 0: \frac{S_t}{t} \geq \varepsilon^{-1}, S_t \geq n\right\} \leq \frac{\varepsilon^n e^{(1-\varepsilon)n}}{\sqrt{2\pi n}} \sum_{k=0}^{\infty} \frac{1}{1 - \varepsilon} \frac{2k}{e^{2k}} \exp\left\{\frac{-3n}{16} + \frac{9k}{32}\right\} = \frac{\varepsilon^n e^{(1-\varepsilon)n}}{\sqrt{2\pi n}} \exp\left\{-\frac{3n}{16}\right\} \frac{1}{(1 - \varepsilon^2 e^{9/32}) (1 - \varepsilon)}, \tag{73}
\]

and (67) is proven.

The second inequality in (66) also follows from (69),

\[
\mathbb{P}\left\{\exists t \geq 0: \frac{S_t}{t} \geq \varepsilon^{-1}, S_t \geq n\right\} \leq \frac{\varepsilon^n e^{(1-\varepsilon)n}}{\sqrt{2\pi n}} \sum_{k=0}^{\infty} \frac{1}{1 - \varepsilon/2} \frac{1}{e^{2k/2}} = \frac{\varepsilon^n e^{(1-\varepsilon)n}}{2\sqrt{2\pi n}} \frac{1}{(1 - \varepsilon^2)(1 - \varepsilon)}. \tag{74}
\]

\[\square\]

**Lemma 5.4.** For $A > 1$ there exist $D_1 = D_1(A)$, $D_2 = D_2(A)$ such that

\[
1 - \sum_{k=0}^{\infty} \mu(k) \left[1 - D_1^{-n}\right]^k \leq r_n^{(A)} \leq 1 - \sum_{k=0}^{\infty} \mu(k) \left[1 - D_2^{-n}\right]^k, \quad n \in \mathbb{N}, \tag{75}
\]

and $D_1(A), D_2(A) \to \infty$, $A \to \infty$.

**Proof.** By the definition of $\ell_{x}^{(A)}$

\[
\ell_{x}^{(A)} = \mathbb{P}\left\{\ell_{x}^{(A)} \geq n\right\} = 1 - \mathbb{P}\left\{\ell_{x}^{(A)} < n\right\} = 1 - \sum_{k=0}^{\infty} \mu(k) \left[\mathbb{P}\left\{\forall t \geq \tau_n, \frac{S_t}{t} < A, \right\}\right]^k. \tag{76}
\]

Note that $\left\{\forall t \geq \tau_n, \frac{S_t}{t} < A, \right\}$ is the complement of the event on the left hand side of (67) with $A = \varepsilon^{-1}$. Thus (75) follows from Lemma 5.3. \[\square\]
Lemma 5.5. Let $A > 1$. The convergence of the infinite product
\[
\prod_{k=1}^{\infty} (1 - r_k^{(A)}) > 0
\] (77)
is equivalent to $\sum_{k=1}^{\infty} \log k \mu(k) < \infty$. Likewise,
\[
\prod_{k=1}^{\infty} (1 - r_k^{(A)}) = 0
\] (78)
if and only if $\sum_{k=1}^{\infty} \log k \mu(k) = \infty$.

Proof. It follows from Lemma 5.4 that (36) holds with $r_n = r_n^{(A)}$, and hence the statement follows from Lemma 4.2. \qed

Lemma 5.6. The convergence (63) takes place for all $A > 0$ if and only if (3) holds for all $B > 1$.

Proof. Suppose (63) takes place for all $A > 0$. By Lemma 5.4 for $B > 1$ there exists $A > 1$ such that
\[
1 - r_n^{(A)} \geq \sum_{k=0}^{\infty} \mu(k) \left[1 - B^{-n}\right].
\] (79)
Hence
\[
\infty > \sum_{m=1}^{\infty} \prod_{n=1}^{m} (1 - r_n^{(A)}) \geq \sum_{m=1}^{\infty} \prod_{n=1}^{m} \sum_{k=0}^{\infty} \mu(k) \left[1 - B^{-n}\right].
\] (80)
Since $B > 1$ is arbitrary, the convergence (3) for all $B > 1$ follows from Lemma 4.3.

Conversely, suppose (3) holds for all $B > 1$. By Lemma 5.4 for some $A_1 > 1$,
\[
\sum_{m=1}^{\infty} \prod_{n=1}^{m} (1 - r_n^{(A)}) \leq \sum_{m=1}^{\infty} \prod_{n=1}^{m} \sum_{k=0}^{\infty} \mu(k) \left[1 - A_1^{-n}\right].
\] (81)
The latter series converges by Lemma 4.3, hence (63) holds for $A > 1$. \qed

Proof of Proposition 2.5. The projections of active particles in a d-dimensional continuous-time frog model on the first coordinate axis perform a slowed down simple continuous-time random walk. The proposition is a consequence of this observation. \qed

Proof of Theorem 2.1. The statement of the theorem follows from Proposition 5.1 and and Lemma 5.5. \qed

Remark 5.7. We see from the proof of Proposition 5.1 that in the settings of Theorem 2.1 the growth toward $+\infty$ would still remain linear even if all particles left of the origin were activated at time $t = 0$. Of course, to consider such an initial configuration one would need to construct rigorously the process started with infinitely many active particles. If a.s. for all $t \geq 0$,
\[
\sup\{x + S_s^{(x,j)} : x \leq 0, j = 1, \ldots, \eta(x), 0 \leq s \leq t\} < \infty,
\]
the construction may follow the standard arguments as a.s. only finitely many new sites will be activated within finite time intervals.
Proof of Theorem 2.2. Due to symmetry it suffices to show the superlinear spread in one direction only. The theorem is then a consequence of Propositions 2.5 and 5.2 and Lemma 5.6.

6 Proof of Theorem 2.3

By Proposition 2.5 it is enough to consider the case $d = 1$ only. In this section we give the proof of Theorem 2.3 for the one-dimensional system.

Recall that $\sigma_x = \min\{t \geq 0 : x \in A_t\}$ is the moment when $x$ is visited by an active particle for the first time. For $a > 0$ denote by $\chi_a$ the first time when a simple continuous-time random walk started at 0 hits $[a, \infty)$. Set $u = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} e^{-t^2/2} dt$ and let $\varepsilon \in (0, 1)$ be a small constant. By the central limit theorem and reflection principle, for large $q$

$$u(1 - \varepsilon) \leq \mathbb{P}\left\{\chi_{q^{1/2}} \geq q\right\} \leq u(1 + \varepsilon). \tag{82}$$

Note that for $q > 1$, $x \geq q^{1/2}$, $y \in [x - q^{1/2}, x - 1]$

$$\{\sigma_x - \sigma_{x-1} \geq q\} \subset \{S_{y,j}^{q,y} \leq x - y \text{ for all } j = 1, \ldots \eta(y)\}. \tag{83}$$

Consequently for sufficiently large $q$ for $x \geq q^{1/2}$

$$\mathbb{P}\{\sigma_x - \sigma_{x-1} \geq q\} \leq \mathbb{P}\left\{S_{y,j}^{q,y} \leq x - y \text{ for all } y = x - 1, \ldots, x - \lfloor q^{1/2} \rfloor, j = 1, \ldots \eta(y)\right\}$$

$$= \prod_{y=x-\lfloor q^{1/2} \rfloor}^{x} \sum_{k=0}^{\infty} \mu(k) \left[\mathbb{P}\{\chi_{x-y} \geq q\}\right]^k = \prod_{z=1}^{\lfloor q^{1/2} \rfloor} \sum_{k=0}^{\infty} \mu(k) \left[\mathbb{P}\{\chi_{z} \geq q\}\right]^k \tag{84}$$

$$\leq \prod_{z=1}^{\lfloor q^{1/2} \rfloor} \sum_{k=0}^{\infty} \mu(k) \left[\mathbb{P}\{\chi_{q^{1/2}} \geq q\}\right]^k \leq \left(\sum_{k=0}^{\infty} \mu(k)(1 + \varepsilon)^k u^k\right)^{q^{1/2}}$$

By (84) for sufficiently large $q$

$$\mathbb{P}\{\sigma_x - \sigma_{x-1} \geq q\} \leq c_q^{1/2} = \exp\left\{-|\ln c_{\varepsilon}|q^{1/2}\right\}. \tag{85}$$

where $c_{\varepsilon} = \sum_{k=0}^{\infty} \mu(k)(1 + \varepsilon)^k u^k < 1$ for sufficiently small $\varepsilon$.

By (85) for $c > 2$

$$\sum_{x \in \mathbb{N}} \mathbb{P}\left\{\max_{1 \leq y \leq x} (\sigma_y - \sigma_{y-1}) \geq c^2 \left(\frac{\ln x}{\ln c_{\varepsilon}}\right)^2\right\} \leq \sum_{x \in \mathbb{N}} \sum_{y=1}^{x} \mathbb{P}\left\{\sigma_y - \sigma_{y-1} \geq c^2 \left(\frac{\ln x}{\ln c_{\varepsilon}}\right)^2\right\}$$

$$\lesssim \sum_{x \in \mathbb{N}} x \exp\left\{-|\ln c_{\varepsilon}|c\left|\frac{\ln x}{\ln c_{\varepsilon}}\right|\right\} = \sum_{x \in \mathbb{N}} x^{-(c-1)} < \infty. \tag{86}$$
Hence for \( c > 2 \)

\[
\mathbb{P}\left\{ \max_{1 \leq y \leq x} (\sigma_y - \sigma_{y-1}) \geq c^2 \left( \frac{\ln x}{\ln c} \right)^2 \text{ infinitely often} \right\} = 0. \tag{87}
\]

Let \( A > 1 \). Consider TADBP on \( \mathbb{Z}_+ \) with \( \psi_{x} = \ell_{x}^{(A)} \). It is possible that there exists an unbounded component, that is, there is \( y \in \mathbb{N} \) such that \( y \xrightarrow{\mathbb{Z}} \infty \). In this case we may proceed as in the proof of Proposition 5.2. In the rest of the present proof we exclude this case and assume that a.s. no site is connected to infinity. Note that under assumptions of Theorem 2.3, the existence of \( y \in \mathbb{Z}_+ \) satisfying \( y \xrightarrow{\mathbb{Z}} +\infty \) is not guaranteed; see Proposition 4.5 and characterization of connected components in TADBP in Section 3.

Denote by \( R_n \) the rightmost site of the \( n \)-th connected component (counting from the origin to the right) in the realization of TADBP on \( \mathbb{Z}_+ \) with \( \psi_{x} = \ell_{x}^{(A)} \). Recall that \( \ell_{x}^{(A)} \) is defined in (10). Note that \( R_n \) depends on \( A \). The number of site in the interval \([0, R_n]\) which are dry is \( n - 1 \) (specifically, the dry sites are the leftmost sites of every component starting from the second; recall that we consider the origin to be wet). Denote by \( l_k \) the length of \( k \)-th connected component, \( l_k = R_k - R_{k-1} - 1 \). The random variables \( \{l_k\}_{k \in \mathbb{N}} \) are the excursions from 0 of the Markov chain \( \{Y_m\}_{m \in \mathbb{Z}_+} \) defined in (17). In particular, \( l_1 \) stochastically dominates \( \ell_1 = \ell_{1}^{(A)} \).

By Lemma 5.4 for some \( D = D(A) \)

\[
\sum_{n=1}^{\infty} \mathbb{P}\{l_n > n \ln^2 n\} \geq \sum_{n=1}^{\infty} \mathbb{P}\{\ell_n > n \ln^2 n\} = \sum_{n=1}^{\infty} r_{[n, \ln^2 n]} \\
\geq \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} \mu(k) \left( 1 - \left( 1 - D^{-[n, \ln^2 n]} \right)^k \right) \\
\geq \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} \mu(k) \left( 1 - e^{-kD^{-[n, \ln^2 n]}} \right) \\
\geq \sum_{n=1}^{\infty} (1 - e^{-1}) \mu([D^{-[n, \ln^2 n]}, \infty)).
\]

Hence by (4)

\[
\sum_{n=1}^{\infty} \mathbb{P}\{l_n > n \ln^2 n\} = \infty, \tag{88}
\]

and consequently ([CR61, Lemma 1])

\[
\mathbb{P}\left\{ \limsup_{n \to \infty} \frac{l_n}{n \ln^2 n} = \infty \right\} = 1. \tag{89}
\]

In particular, a.s. \( \limsup_{n \to \infty} \frac{R_n}{n \ln^2 n} = \infty \). Since the map

\[
(10, \infty) \ni x \mapsto \frac{x}{\ln^2 x} \in \mathbb{R}_+
\]

is an increasing function, for any \( C > 1 \)

\[
\limsup_{n \to \infty} \frac{R_n}{n \ln^2 R_n^2} \geq \liminf_{n \to \infty} \frac{Cn \ln^2 n}{n \ln^2 (C^2n^2 \ln^4 n)} = \liminf_{n \to \infty} \frac{C \ln^2 n}{(2 \ln C + 2 \ln n + 4 \ln \ln n)^2} = C. \tag{90}
\]
That is,

\[ \limsup_{n \to \infty} \frac{R_n}{n \ln^2 R_n^2} = \infty. \]  

(91)

By (87) for large \( n \) for all \( y \in [0, R_n] \)

\[ \sigma_y - \sigma_{y-1} \leq c_2^2 \left( \frac{\ln R_n}{\ln c_e} \right)^2, \]  

(92)

where \( c_2 > 2 \). Using the same arguments as when Lemma 3.8 was applied in the proof of Proposition 5.2, by Lemma 3.9 we see that the \( n \)-th connected component is traversed within at most \( \frac{2\ln R}{A} \) units of time. Hence the time needed to reach \( R_n \)

\[ \sigma_{R_n} \leq \frac{2}{A} \sum_{k=1}^{n} l_k + (n - 1) c_2^2 \left( \frac{\ln R_n}{\ln c_e} \right)^2 < \frac{2R_n}{A} + nc_2^2 \left( \frac{\ln R_n}{\ln c_e} \right)^2, \]  

(93)

and by (91)

\[ \limsup_{n \to \infty} \frac{R_n}{\sigma_{R_n}} \geq \limsup_{n \to \infty} \frac{R_n}{2A R_n + nc_2^2 \left( \frac{\ln R_n}{\ln c_e} \right)^2} = \limsup_{n \to \infty} \frac{1}{\frac{2A}{n} + \frac{n \ln^2 R_n}{R_n} \left( \frac{c_2}{\ln c_e} \right)^2} = \frac{A}{\frac{2A}{n} + \left( \frac{c_2}{\ln c_e} \right)^2 \liminf_{n \to \infty} n \ln^2 R_n \frac{R_n}{R_n}} = \frac{A}{2}. \]  

(94)

Since \( A > 1 \) is arbitrary, it follows that

\[ \limsup_{m \to \infty} \frac{m}{\sigma_m} = \infty. \]  

(95)

\[ \square \]

**Remark 6.1.** Let \( d = 1 \). The case not covered by Theorem 1.2 is when the series

\[ \sum_{k=1}^{\infty} \log k \mu(k). \]  

(96)

diverges but slowly, so that

\[ \sum_{n \in \mathbb{N}} \mu \left( \left[ e^{n \ln^2 n}, \infty \right) \right) < \infty \]

and for some \( B > 1 \)

\[ \sum_{m=1}^{\infty} \prod_{n=1}^{m} \mu \left( [0, B^n] \right) = \infty. \]

One might be tempted to conjecture that the spread is superlinear once (96) diverges. Indeed, as we saw earlier in the proofs (specifically, Lemma 5.5 and Section 3), the following holds true: for every \( A > 1 \) the fraction of sites that are traversed by a particle moving toward \(+\infty\) at speed at least \( A \) (as defined in (10)) is one. It would therefore suffice to show that the time needed to traverse the rest of the sites (the ‘slow’ sites) is not too large. This is done in the proof of Theorem 2.3 under an additional assumption (4). A better bound on \[ \max_{1 \leq y \leq x} (\sigma_y - \sigma_{y-1}) \] could
allow to weaken (4), but it is unclear if (4) can be dispensed with altogether. It may be that if
(96) diverges very very slowly, the slow sites have enough of an effect to bog down the growth
and the spread is linear; or it may be that the spread is superlinear no matter how slowly (96)
diverges. Both possibilities seem plausible to the authors of this paper; if a guess (or conjecture)
had to be made, the latter would be chosen. We would also like to note that the slow sites should
still be slow if the initial configuration of particles is capped so that there are \( \eta(x) \land M \) at \( x, M \in \mathbb{N} \). Thus, understanding the slow sites for the frog model with bounded initial configuration
may prove helpful in shedding light on the cases not covered by Theorem 1.2.

The gap between the conditions for the linear spread and superlinear spread in Theorem
1.2 widens as the dimension \( d \) increases. Indeed, our proofs of the superlinear spread rely on
Proposition 2.5 and thus are essentially one-dimensional. One might therefore hope to weaken
the conditions implying the superlinear spread for \( d \geq 2 \) by using techniques that would take
into account the spatial structure of \( \mathbb{Z}^d \).

7 Proof of Theorem 2.7

Recall that \( d \geq 2 \) in the settings of Theorem 2.7. Let \( \theta_1^{(x,j)} < \theta_2^{(x,j)} < \ldots \) be the jump times of
the random walk \( \{S_t^{(x,j)}, t \geq 0\} \). Let \( A > 1 \). We will see later in the proof that we need \( A \) to
be large enough to satisfy (105) below. Define

\[
W^{(x,j)}_A = \max \left \{ n \in \mathbb{N} : \frac{\theta_n^{(x,j)}}{n} \leq \frac{1}{A} \right \} . \tag{97}
\]

Since \( A > 1 \), \( W^{(x,j)}_A \) is a.s. finite. Note how \( W^{(x,j)}_A \) defined here differs from \( \ell_x^{(A)} \) in (10): the
former is defined solely in terms of the jump moments, the latter is not. Define also

\[
W^{(x)}_A = \max \left \{ n \in \mathbb{N} : \frac{\theta_n^{(x,j)}}{n} \leq \frac{1}{A}, j \in \{1, \ldots, \eta(x)\} \right \}, \tag{98}
\]

and

\[
\rho_A = \max \left \{ n \in \mathbb{N} : \frac{\tau_n}{n} \leq \frac{1}{A} \right \}. \tag{99}
\]

Note that \( \rho_A \) is equal in distribution to \( W^{(x,j)}_A \).

Lemma 7.1. Let \( \varepsilon = \frac{1}{A} \in (0,1) \). For large \( n \)

\[
\mathbb{P} \{ \rho_A \geq n \} \leq (\varepsilon e^{1-\varepsilon} n). \tag{100}
\]
Proof. For $n$ satisfying $(1 - \varepsilon)(1 - \varepsilon e^{1-\varepsilon})\sqrt{2\pi n} > 1$ we have

$$\mathbb{P}\{\rho_A \geq n\} = \mathbb{P}\left\{ \exists m \geq n : \frac{m}{\tau_m} \geq A \right\} = \mathbb{P}\left\{ \exists m \geq n : \tau_m \leq m\varepsilon \right\} \leq \sum_{m=n}^{\infty} \mathbb{P}\{\tau_m \leq m\varepsilon\}$$

$$= \sum_{m=n}^{\infty} e^{-m\varepsilon} \sum_{j=0}^{\infty} \frac{(m\varepsilon)^j}{j!} \leq \sum_{m=n}^{\infty} e^{-m\varepsilon} \frac{(m\varepsilon)^m}{m^m} \sum_{j=0}^{\infty} \frac{(m\varepsilon)^j}{m^j} \leq \sum_{m=n}^{\infty} e^{-m\varepsilon} \frac{(m\varepsilon)^m}{m^m \sqrt{2\pi m}} \times \frac{1}{1 - \varepsilon}$$

$$= \frac{1}{1 - \varepsilon} \sum_{m=n}^{\infty} e^{-m\varepsilon} \frac{e^{m\varepsilon}}{\sqrt{2\pi m}} \leq \frac{1}{\sqrt{2\pi n}} \times \frac{e^{n e^n(1-\varepsilon)}}{1 - \varepsilon e^{1-\varepsilon}} < (\varepsilon e^{1-\varepsilon})^n.$$

The random variables $\{W_A^{(x)}\}_{x \in \mathbb{Z}^d}$ are independent and identically distributed. Let $W_A$ be a copy of $W_A^{(x)}$, $x \in \mathbb{Z}^d$, independent of the sequence $\{W_A^{(x)}\}_{x \in \mathbb{Z}^d}$. By Lemma 7.1

$$\mathbb{P}\{W_A \geq n\} = 1 - \mathbb{P}\{W_A < n\} = 1 - \sum_{k=0}^{\infty} \mu(k) \left( \mathbb{P}\{\rho_A < n\} \right)^k = 1 - \sum_{k=0}^{\infty} \mu(k) \left( 1 - \mathbb{P}\{\rho_A \geq n\} \right)^k$$

$$\leq 1 - \sum_{k=0}^{\infty} \mu(k) \left( 1 - (\varepsilon e^{1-\varepsilon})^n \right)^k \leq 1 - \sum_{k=0}^{\infty} \mu(k) \left( 1 - 1 \wedge k(\varepsilon e^{1-\varepsilon})^n \right) = \sum_{k=0}^{\infty} \mu(k) \left( 1 \wedge k(\varepsilon e^{1-\varepsilon})^n \right)$$

Letting $B = (\varepsilon e^{1-\varepsilon})^{-1} > 1$ we get

$$\sum_{n \in \mathbb{N}} \left[ \mathbb{P}\{W_A \geq n\} \right]^\frac{1}{n} \leq \sum_{n \in \mathbb{N}} \left[ \sum_{k=1}^{\infty} \mu(k) \left( 1 \wedge kB^{-n} \right) \right]^\frac{1}{n} \leq \sum_{n \in \mathbb{N}} \left[ \sum_{k=1}^{B^n} \mu(k) \left( 1 \wedge kB^{-n} \right) + \sum_{k=B^n}^{\infty} \mu(k) \left( 1 \wedge kB^{-n} \right) \right]^\frac{1}{n}$$

$$= \sum_{n \in \mathbb{N}} \left[ B^{-n} \sum_{k=1}^{B^n} k\mu(k) + \mu([B^n, \infty)) \right]^\frac{1}{n} \leq 2\frac{1}{n} \sum_{n \in \mathbb{N}} \left[ B^{-n} \sum_{k=1}^{B^n} k\mu(k) \right]^\frac{1}{n} + 2\frac{1}{n} \sum_{n \in \mathbb{N}} \left[ \mu([B^n, \infty)) \right]^\frac{1}{n}$$

Denote $b_k = \mu((B^{k-1}, B^k])$. We have

$$\sum_{n \in \mathbb{N}} \left[ B^{-n} \sum_{k=1}^{B^n} k\mu(k) \right]^\frac{1}{n} \leq \sum_{n \in \mathbb{N}} \left[ B^{-n} \sum_{m=1}^{n} b_m B^m \right]^\frac{1}{n} \leq \sum_{n \in \mathbb{N}} \left[ \sum_{m=1}^{n} b_m^\frac{1}{n} B^{-\frac{m}{n}} B^m \right]^\frac{1}{n} = \sum_{m=1}^{\infty} b_m^\frac{1}{n} B^{-\frac{m}{n}} B^m \sum_{n=m}^{\infty} B^{-\frac{n}{n}}$$

$$= \sum_{m=1}^{\infty} b_m^\frac{1}{n} \frac{B^{-\frac{m}{n}}}{1 - B^{-\frac{1}{n}}} = \frac{1}{1 - B^{-\frac{1}{n}}} \sum_{m=1}^{\infty} b_m^\frac{1}{n} \sum_{n=m}^{\infty} \left| \mu([B^m, \infty)) \right|^\frac{1}{n}.$$
Therefore by (101)

\[
\sum_{n \in \mathbb{N}} [\mathbb{P} \{W_A \geq n\}]^{\frac{1}{d}} \leq \sum_{m \in \mathbb{N}} [\mu([B^m, \infty))]^{\frac{1}{d}},
\]

(102)

and hence by (6)

\[
\sum_{n \in \mathbb{N}} [\mathbb{P} \{W_A \geq n\}]^{\frac{1}{d}} < \infty.
\]

(103)

Since for each \( n \in \mathbb{N} \), \( \mathbb{P} \{W_A \geq n\} \xrightarrow{A \to \infty} 0 \), by the monotone convergence theorem,

\[
\lim_{A \to \infty} \sum_{n \in \mathbb{N}} [\mathbb{P} \{W_A \geq n\}]^{\frac{1}{d}} = 0.
\]

(104)

Combining this with Theorem 1.1 in [Mar02] yields the existence of \( A > 1 \) such that

\[
\limsup_{n \to \infty} \sup_{x_0, \ldots, x_n} \frac{1}{n+1} \sum_{i=0}^{n} W_A(x_i) \leq \frac{1}{3}.
\]

(105)

where the supremum is taken over all connected sets of \( n+1 \) elements of \( \mathbb{Z}^d \) containing the origin, that is, \( x_0 = 0, x_1, x_2, \ldots, x_n \in \mathbb{Z}^d, x_i \neq x_j, i \neq j \), and \( \min_{j=1, \ldots, i-1} |x_i - x_j| = 1, i = 1, 2, \ldots, n \).

**Proof of Theorem 2.7.** Take an infinite sequence

\[
\{(Z_n, t_n, i_n)\}_{n \in \mathbb{Z}^d}, \quad (Z_n, t_n, i_n) \in \mathbb{Z}^d \times \mathbb{R}_+ \times \mathbb{N}, \quad n \in \mathbb{N},
\]

\( Z_0 = 0, t_0 = 0 \), such that the particles at site \( Z_{n+1} \) are activated by the particle \((i_n, Z_n)\) that started at \( Z_n \), and \( t_n \) is the activation time for \( Z_n \). Let \( z_0, z_1, z_2, \ldots, \) be the successive sites visited by the particles \((i_n, Z_n), n \in \mathbb{Z}^d \), during the time interval \([t_n, t_{n+1}]\), so that

\[
\bigcup_{j=0}^{\infty} \{z_j\} = \bigcup_{n=0}^{\infty} \{S^{(i_n, Z_n)}_t + Z_n, 0 \leq t \leq t_{n+1} - t_n\}
\]

and the sequence \( \{z_j\}_{j \in \mathbb{Z}^d} \) does not contain repeating elements (that is, if a site is already in the sequence \( \{z_j\} \), it is not appended even when visited by a particle \((i_m, Z_m)\) during \([t_m, t_{m+1}]\))

Note that \( \{Z_j\}_{j \in \mathbb{Z}^d} \subset \{z_j\}_{j \in \mathbb{Z}^d} \). By a.s. (105) for large \( m \in \mathbb{N} \),

\[
\frac{1}{m} \sum_{j=0}^{m} W_A(z_j) \leq \frac{1}{2}.
\]

(106)

Hence a.s. for large \( n \in \mathbb{N} \)

\[
\frac{1}{n} \sum_{j=0}^{n-1} W_A(z_j) \leq \frac{1}{2}.
\]

(107)

where \( \text{trav}(Z_n, Z_{n+1}) \) is the number of sites excluding \( Z_n \) visited by the particle \((i_n, Z_n)\) by the moment \( t_{n+1} \) when it reaches \( Z_{n+1} \) (for instance, if \( Z_{n+1} \) and \( Z_n \) are neighbors and \((i_n, Z_n)\)
goes directly from $Z_n$ to $Z_{n+1}$, then $\text{trav}(Z_n, Z_{n+1}) = 1$). Thus

$$\frac{t_n}{|Z_n|_1} \geq \frac{\sum_{j=0}^{n-1} (t_{j+1} - t_j)}{\sum_{j=0}^{n-1} \text{trav}(Z_j, Z_{j+1})} \geq \frac{1}{2} \frac{\sum_{j=0}^{n-1} \text{trav}(Z_j, Z_{j+1})}{\sum_{j=0}^{n-1} \text{trav}(Z_j, Z_{j+1})} = \frac{1}{2A}. \quad (108)$$

and since $|Z_n|_1 \leq \sum_{j=0}^{n-1} \text{trav}(Z_j, Z_{j+1})$, a.s. for large $n$

$$\frac{|Z_n|_1}{|t_n|} \leq 2A. \quad (109)$$

### Acknowledgements

We would like to thank Christian Remling for the construction used in the proof of Lemma 4.4 and Martin Zerner for bringing to our attention the works [Lam70] and [KW06] and thus helping to place Section 3 in the context of existing research.

### References

[ADH17] A. Auffinger, M. Damron, and J. Hanson. *50 Years of First-Passage Percolation*, volume 68. American Mathematical Soc., 2017.

[AMP02] O. S. M. Alves, F. P. Machado, and S. Y. Popov. The shape theorem for the frog model. *Ann. Appl. Probab.*, 12(2):533–546, 2002.

[AMPR01] O. S. M. Alves, F. P. Machado, S. Y. Popov, and K. Ravishankar. The shape theorem for the frog model with random initial configuration. *Markov Process. Relat. Fields*, 7(4):525–539, 2001.

[BDD+18] E. Beckman, E. Dinan, R. Durrett, R. Huo, and M. Junge. Asymptotic behavior of the Brownian frog model. *Electron. J. Probab.*, 23:19, 2018. Id/No 104.

[BDPK+17] V. Bezborodov, L. Di Persio, T. Krueger, M. Lebid, and T. Ożański. Asymptotic shape and the speed of propagation of continuous-time continuous-space birth processes. *Advances in Applied Probability*, 50(1):74–101, 2017.

[Bez20] V. Bezborodov. Non-triviality in a totally asymmetric one-dimensional boolean percolation model on a half-line. arXiv:2009.00742, 2020.

[BFHM20] I. Benjamini, L. R. Fontes, J. Hermon, and F. P. Machado. On an epidemic model on finite graphs. *Ann. Appl. Probab.*, 30(1):208–258, 02 2020.
[Big95] J. D. Biggins. The growth and spread of the general branching random walk. *Ann. Appl. Probab.*, 5(4):1008–1024, 1995.

[BMS05] M. Bilodeau, F. Meyer, and M. Schmitt, editors. *Space, structure, and randomness. Contributions in honor of Georges Matheron in the fields of geostatistics, random sets, and mathematical morphology.*, volume 183. New York, NY: Springer, 2005.

[BPK20] V. Bezborodov, L. D. Persio, and T. Krueger. Continuous-time frog model can spread arbitrary fast. *arXiv:2005.12970*, 2020.

[BPKT20] V. Bezborodov, L. D. Persio, T. Krueger, and P. Tkachov. Spatial growth processes with long range dispersion: Microscopics, mesoscopics and discrepancy in spread rate. *Ann. Appl. Probab.*, 30(3):1091–1129, 2020.

[BR10] J. Béram and A. F. Ramírez. Large deviations of the front in a one-dimensional model of $X + Y \to 2X$. *Ann. Probab.*, 38(3):955–1018, 2010.

[BR16] J. Béram and A. Ramírez. Fluctuations of the front in a one-dimensional model for the spread of an infection. *Ann. Probab.*, 44(4):2770–2816, 2016.

[CGGK93] J. T. Cox, A. Gandolfi, P. S. Griffin, and H. Kesten. Greedy lattice animals. I: Upper bounds. *Ann. Appl. Probab.*, 3(4):1151–1169, 1993.

[CL06] F. Chung and L. Lu. *Complex graphs and networks.*, volume 107. Providence, RI: American Mathematical Society (AMS), 2006.

[CMG20] C. F. Coletti, D. Miranda, and S. P. Grynberg. Boolean percolation on doubling graphs. *J. Stat. Phys.*, 178(3):814–831, 2020.

[CQR07] F. Comets, J. Quastel, and A. F. Ramírez. Fluctuations of the front in a stochastic combustion model. *Ann. Inst. Henri Poincaré, Probab. Stat.*, 43(2):147–162, 2007.

[CQR09] F. Comets, J. Quastel, and A. F. Ramírez. Fluctuations of the front in a one dimensional model of $X + Y \to 2X$. *Trans. Am. Math. Soc.*, 361(11):6165–6189, 2009.

[CR61] Y. S. Chow and H. Robbins. On sums of independent random variables with infinite moments and “fair” games. *Proc. Natl. Acad. Sci. USA*, 47:330–335, 1961.

[Dei03] M. Deijfen. Asymptotic shape in a continuum growth model. *Adv. in Appl. Probab.*, 35(2):303–318, 2003.

[DG82] R. Durrett and D. Griffeth. Contact processes in several dimensions. *Z. Wahrscheinlichkeitstheor. Verw. Geb.*, 59:535–552, 1982.
Recurrence and transience of frogs with drift on $\mathbb{Z}^d$. *Electron. J. Probab.*, 23:23, 2018. Id/No 88.

Competing frogs on $\mathbb{Z}^d$. *Electron. J. Probab.*, 24:17, 2019. Id/No 146.

Maxima of branching random walks vs. independent random walks. *Stochastic Processes and their Applications*, 9(2):117–135, 1979.

Maxima of branching random walks. *Z. Wahrsch. Verw. Gebiete*, 62(2):165–170, 1983.

The contact process, 1974-1989. Mathematics of random media, Proc. AMS-SIAM Summer Semin., Conf., Blacksburg/VA (USA) 1989, Lect. Appl. Math. 27, 1-18 (1991), 1991.

Probability. *Theory and examples. 4th ed.*, volume 31. Cambridge: Cambridge University Press, 4th ed. edition, 2010.

Random networks for communication. *From statistical physics to information systems.*, volume 24. Cambridge: Cambridge University Press, 2007.

The maximum of a branching random walk with semiexponential increments. *Ann. Probab.*, 28(3):1219–1229, 2000.

Greedy lattice animals. II: Linear growth. *Ann. Appl. Probab.*, 4(1):76–107, 1994.

Continuous first-passage percolation and continuous greedy paths model: linear growth. *Ann. Appl. Probab.*, 18(6):2300–2319, 2008.

On the range of the transient frog model on $\mathbb{Z}$. *Adv. Appl. Probab.*, 49(2):327–343, 2017.

From transience to recurrence with Poisson tree frogs. *Ann. Appl. Probab.*, 26(3):1620–1635, 2016.

Recurrence and transience for the frog model on trees. *Ann. Probab.*, 45(5):2826–2854, 2017.

*Foundations of modern probability*. Probability and its Applications. Springer-Verlag, second edition, 2002.
H. Kesten, A. F. Ramírez, and V. Sidoravicius. Asymptotic shape and propagation of fronts for growth models in dynamic random environment. In *Probability in complex physical systems. In honour of Erwin Bolthausen and Jürgen Gärtner. Selected papers based on the presentations at the two 2010 workshops*, pages 195–223. Berlin: Springer, 2012.

H. Kesten and V. Sidoravicius. The spread of a rumor or infection in a moving population. *Ann. Probab.*, 33(6):2402–2462, 2005.

H. Kesten and V. Sidoravicius. A shape theorem for the spread of an infection. *Ann. of Math. (2)*, 167(3):701–766, 2008.

N. Kubota. Continuity for the asymptotic shape in the frog model with random initial configurations. *Stochastic Processes and their Applications*, 130(9):5709–5734, 2020.

H. G. Kellerer and G. Winkler. Random dynamical systems on ordered topological spaces. *Stoch. Dyn.*, 6(3):255–300, 2006.

E. Kosygina and M. P. W. Zerner. A zero-one law for recurrence and transience of frog processes. *Probab. Theory Relat. Fields*, 168(1-2):317–346, 2017.

J. Lamperti. Maximal branching processes and ‘long-range percolation’. *J. Appl. Probab.*, 7:89–98, 1970.

T. M. Liggett. *Stochastic interacting systems: contact, voter and exclusion processes*, volume 324. Springer-Verlag, Berlin, 1999.

J. B. Martin. Linear growth for greedy lattice animals. *Stochastic Processes Appl.*, 98(1):43–66, 2002.

R. Meester and R. Roy. *Continuum percolation.*, volume 119. Cambridge: Cambridge Univ. Press, 1996.

J. Rosenberg. The frog model with drift on $\mathbb{R}$. *Electron. Commun. Probab.*, 22:14, 2017. Id/No 30.

A. F. Ramírez and V. Sidoravicius. Asymptotic behavior of a stochastic combustion growth process. *J. Eur. Math. Soc. (JEMS)*, 6(3):293–334, 2004.

A. N. Shiryaev. *Probability-2. Translated from the fourth Russian edition by R. P. Boas and D. M. Chibisov. 3rd edition of the book previously published as a single-volume edition.*, volume 95. New York, NY: Springer, 3rd edition of the book previously published as a single-volume edition, 2019.
[SS19] V. Sidoravicius and A. Stauffer. Multi-particle diffusion limited aggregation. *Invent. Math.*, 218(2):491–571, 2019.

[vHCN19] van Hao Can and S. Nakajima. First passage time of the frog model has a sublinear variance. *Electron. J. Probab.*, 24:27, 2019. Id/No 76.

[Zer18] M. P. W. Zerner. Recurrence and transience of contractive autoregressive processes and related Markov chains. *Electron. J. Probab.*, 23:24, 2018. Id/No 27.