AN INTRODUCTION TO NIGEL KALTON’S WORK ON DIFFERENTIALS OF COMPLEX INTERPOLATION PROCESSES FOR KÖTHE SPACES

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Our aim in this note is to offer one kind of introduction to Nigel Kalton’s remarkable paper [24], and to share a few thoughts about possible further sequels to it. We hope to at least capture something of the spirit of the paper. Inevitably we will bypass many of its subtleties. We will oversimplify and ignore details.

At first sight, the main topics of [24] and the objects which arise within them may seem quite exotic and even maybe “far fetched”. But it turns out that there are connections with and applications to quite a range of other topics in analysis. We shall be able to at least briefly hint at some of these below. Of course this is not the only instance where Nigel Kalton’s bold and deep explorations along paths far from the “beaten track” have bounced back with unexpected implications in more familiar settings.

For other kinds of introductions to the same paper, we warmly encourage the interested reader to consult the surveys [17] and [18]. In [17] one can find a very extensive discussion of a wide range of Nigel’s research. In particular, its Section 5 deals, among other things, with the material that we discuss here, as does [18]. Each of these surveys provides many very illuminating insights and offers quite different perspectives from ours, and mentions quite a number of those connections with other topics to which we just alluded.

Of course we also warmly recommend a somewhat longer and more detailed account of these matters written by Nigel himself, together with Stephen Montgomery-Smith in the later sections of their survey [25].

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1. Interpolation

1.1. Classical Interpolation. Classically, interpolation methods for Banach spaces are techniques for starting with a pair of spaces, \((A_0, A_1)\), and constructing an interpolation space \(A_s\) with the property that, if a linear operator \(T\) is bounded on \(A_i, i = 0, 1\) then \(T\) is also bounded on \(A_s\).

One classical way of doing this is Alberto Calderón’s complex method of interpolation (cf. [6]) which goes roughly as follows: With \((A_0, A_1)\) given, define

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\[ \mathcal{F} = \mathcal{F}(A_0, A_1) \] to be the space of holomorphic vector valued functions, \( F(z) \), defined on the strip \( S = \{ z \in \mathbb{C} : 0 < \text{Re} z < 1 \} \) for which
\[
\| F \|_F = \max_{u=0,1} \sup \{ \| F(u + iv) \|_{A_u} : -\infty < v < \infty \} < \infty.
\]

For \( 0 < t < 1 \) one then defines the complex interpolation space \( A_t \) and its norm by
\[
A_t = [A_0, A_1]_t = \{ F(t) : F \in \mathcal{F} \}
\]
(1.1)
\[
\| v \|_{A_t} = \inf \{ \| F \|_F : F(t) = v \},
\]
or, equivalently, \( A_t \) is identified isometrically with the quotient space
\[
A_t = \mathcal{F} / \{ F \in \mathcal{F} : F(t) = 0 \}.
\]
The basic interpolation theorem is that if \( T \) is bounded on \( A_i \), \( i = 0, 1 \) then \( T \) is bounded on \( A_t \).

1.2. Differentials and Commutators. Associated to the construction of \( A_t \) is a special "differential" map \( \Omega = \Omega(A_0, A_1, t) \) from \( A_t \) into \( A_0 + A_1 \). For each \( v \in A_t \), \( \Omega(v) \) is the derivative at \( t \) of the function in \( \mathcal{F} \) which attains the infimum in (1.1) for defining \( \| v \|_{A_t} \). That is, define \( F_{t,v} \) by
\[
F_{t,v} \in \mathcal{F}, \quad v = F_{t,v}(t), \quad \| v \|_{A_t} = \| F_{t,v} \|_F,
\]
and set
\[
\Omega v = F_{t,v}'(t).
\]
(1.2)

(A sample of the details we are ignoring here is the issue of whether the above-mentioned infimum is attained and, if so, whether \( F_{t,v} \) is unique, and what to do otherwise.)

Operators such as \( \Omega \) are the main topic of Nigel’s paper. It is easy to see that they are homogeneous, i.e., \( \Omega(\alpha f) = \alpha f \) for all \( \alpha \in \mathbb{C} \). It is not much harder to see that for some constant \( C \) and for all \( f, g \in A_t \)
\[
\| \Omega(f + g) - \Omega(f) - \Omega(g) \|_{A_t} \leq C(\| f \|_{A_t} + \| g \|_{A_t}).
\]
(1.3)

Even though these operators are generally unbounded and nonlinear, they interact well with the interpolation process. If \( T \) is a linear operator bounded on \( A_i \), \( i = 0, 1 \) then, in addition to the boundedness of \( T \) on \( A_t \), we have a commutator theorem (cf. [32]): there is a \( C \) so that for all \( v \in A_t \)
\[
\| [\Omega, T] v \| = \| \Omega(T(v)) - T(\Omega(v)) \| \leq C \| v \|.
\]
(1.4)

Here are some examples in a classical context of some other operators which have similar properties to those of \( \Omega \). Let \( L^2 = L^2(T, \mu) \) be the Lebesgue space on \( T = \{ e^{it} : 0 \leq t < 2\pi \} \) where \( \mu \) is arc length measure. The role of the operator \( T \) will be played by \( P : L^2 \to L^2 \) which is the orthogonal projection of \( L^2 \) onto the Hardy space, \( H^2 \), the subspace of \( L^2 \) consisting of functions whose Fourier coefficients with negative indices vanish; \( H^2 = \{ f \in L^2 : \langle f, e^{int} \rangle_{L^2} = 0, \ n < 0 \} \).

For each \( f \in L^2 \) and each \( \tau = e^{it} \in T \) set
\[
(\Omega_1 f)(\tau) = f(\tau) \log(1 - \tau),
\]
\[
(\Omega_2 f)(\tau) = f(\tau) \log \frac{|f(\tau)|}{\| f \|_{L^2}},
\]
\[
(\Omega_3 f)(\tau) = f(\tau) \log \mu(\{ \sigma \in T : |f(\sigma)| > |f(\tau)| \}).
\]
In fact the above formula for $\Omega_3 f$ has to be replaced by a more elaborate variant when $|f|$ assumes any constant values on sets of positive measure.) Each of these operators has a bounded commutator with $P$. That is, for each $i = 1, 2, 3$ there is a $C$ so that for all $f$ in $L^2$

$$||[\Omega_i, P]f|| = ||(\Omega_i P - P \Omega_i)f|| \leq C \|f\|.$$ 

Note that none of the $\Omega_j$ are bounded on $L^2$. Also $\Omega_2$ and $\Omega_3$ are nonlinear and yet we are claiming linear space estimates. In fact $\Omega_1$ and $\Omega_2$ are both differential maps $\Omega = \Omega(A_0, A_1, t)$ for suitable choices of $A_0$, $A_1$ and $t$.

1.3. More Generally. There are other methods for constructing interpolation spaces; some of the classical ones are discussed in, for example, [2], [4], [5], and the earlier sections of [25]. Associated with many of those methods are operators, similar to the special map $\Omega(A_0, A_1, t)$ defined above, but obtained by quite different constructions. For example the operator $\Omega_3$ mentioned above, can be obtained via an analogue of $\Omega(A_0, A_1, t)$ for one of the versions of the method of real interpolation. $\Omega_1$, $\Omega_2$, $\Omega_3$ and other such operators are discussed in [12], [21], [22], [25] (in its later sections) [27], [39] and other places.

Work to develop a more unified theory of such operators is in [21], [38], [33] and [33]. Some discussion of the applicability of these constructions is in [19] and [33].

In the paper [24] Nigel pursues a different direction; he focuses on complex interpolation and on a class of Banach spaces that are amenable to more refined analysis. He can then explore more deeply the relation between the interpolation construction and the associated differentials. In fact, in [24] he shows that for Köthe function spaces, one can develop a systematic theory for differentials $\Omega$, and that the theory has interesting applications. We will discuss that in the next section.

In recent years our understanding of interpolation theory has expanded and several very interesting new interpolation constructions have been introduced, for example in [10], [40] and in many papers by Zbigniew Slodkowski. (One might begin by looking at Slodkowski’s papers [42] and [43] and then, in each case proceeding to the two subsequent similarly titled papers which are their respective sequels.) These newer methods focus less on boundedness results for linear operators and more on understanding the role of convexity in Banach space theory; particularly the relation with maximum principles and differential inequalities. This focus on convexity and its variants (pseudoconvexity, quasiconvexity, quasi-affine functions,...) in the theory of linear spaces was a major theme in Nigel’s research programs and his ideas in [24] resonate with these newer views of interpolation. There is some brief discussion of this in some of the comments which we offer in the final section.

2. Kalton’s Paper

The work in [24] has strong connections with earlier results in [22]. Its point of departure is a “reasonable” topological space $S$ equipped with a $\sigma$-finite Borel measure $\mu$. Nigel focuses on complex interpolation of a large class of Banach spaces, Köthe function spaces, whose elements are Borel measurable functions on $S$. This class includes many Banach spaces which arise naturally in various contexts and, consequently, the results of [24] have quite a number of interesting applications. Within this class there are two basic tools which are not available for general families of Banach spaces. First, each space automatically carries with it a rich natural family of bounded maps, namely multiplication by bounded functions. Second, the
underlying assumptions in this context ensure that the dual of each space is also a
space of functions on the same underlying set and one can invoke the Lozanovsky’s
duality theorem [28] (see also [16] and [34]) and the associated factorization theory
for functions.

Remark: Given that Lozanovsky’s above-mentioned result plays such a crucial
and recurring role in what we are discussing here, we see fit to mention the books
[31, 29, 30] where the reader may discover other ideas of Lozanovsky. Some, maybe
even many of these may yet remain to be brought to fruition.

2.1. The Setup. As we shall see, the main result of [24] follows from an elaborate
study of the interplay between several kinds of mappings and functionals, in par-
icular, derivations, centralizers and indicators, which are defined on various Köthe
function spaces, or other subsets of the space $E^0$ of all measurable functions on the
underlying measure space.

To each space $A$, one can associate a new Banach space, Nigel’s derived space
$\mathcal{d}A$, of couples $(u, v) \in A \times (A_0 + A_1)$ for which the following norm is finite:

$$\| (u, v) \|_{\mathcal{d}A} = \inf \{ \| F \| : F \in \mathcal{F}; F(t) = u, F'(t) = v \}. \tag{2.1}$$

It is relatively straightforward to see that this space coincides with the twisted sum
(or twisted direct sum) $A_0 \oplus_\Omega A_1$ which is the space of pairs $(u, v)$ for which the
following functional

$$\| (u, v) \|_{A_0 \oplus_\Omega A_1} = \| u \|_{A_0} + \| v - \Omega u \|_{A_1}, \tag{2.2}$$

is finite. This functional is in fact a quasi-norm (in view of (1.3)) and it is equivalent
to the norm (2.1). (Cf. the proof of Lemma 2.9 of [39, p. 325]. Note that here $\Omega$ is the particular map $\Omega(A_0, A_1)$ defined by (1.2).) Using this fact it is not hard
to check that the commutator estimate, (1.4) for $T$, is equivalent to knowing that the
map of $(u, v)$ to $(Tu, Tv)$ is bounded on $\mathcal{d}A$ or, equivalently, on $A_0 \oplus_\Omega A_1$. In
fact Nigel also deals with these notions in a broader context, via certain mappings
$\Omega : A \to E^0$ which he calls derivations, which generalize $\Omega(A_0, A_1, t)$. (Twisted
sums arise in still more general contexts and in some of them, the quasi-norm (2.2)
may fail to be equivalent to a norm. See e.g. [15], Chapter 16 of [3] and the references therein.)

The basic question of [24] is: Does every such $\Omega$ arise in this way? That is, given a
Köthe function space $A$ and an $\Omega$ which satisfies various natural conditions, is
there a couple $(A_0, A_1)$ and value of $t$ so that $A = A_t$ and $\Omega = \Omega(A_0, A_1, t)$?

To approach that question we first identify three natural necessary conditions
which $\Omega$ must satisfy. For any function $b$ let $M_b$ be the operator of multiplication
by $b$.

If $\Omega = \Omega(A_0, A_1, t)$ for some $t \in (0, 1)$ and some Köthe function spaces $A_0, A_1$
and $A = [A_0, A_1]$, then it is not difficult to show that there must exist a positive
constant $\rho(\Omega)$ such that, for all $u, v \in A$, all $b \in L_\infty$, and all $\alpha \in \mathbb{C}$, the map $\Omega$
satisfies

$$\begin{align*}
(i) & \quad \Omega(\alpha u) = \alpha \Omega(u) \\
(ii) & \quad \Omega(B_A) \text{ is bounded in } E^0 \\
(iii) & \quad \| \Omega, M_b \|_A u \|_A \leq \rho(\Omega) \| b \|_\infty \| u \|_A
\end{align*} \tag{2.3}$$

(Here $E^0$ is equipped with its usual topology defined via convergence in measure
on sets of finite measure, and $B_A$, as usual, denotes the unit ball of $A$.)
In particular, the third requirement in (2.3) arises because the $M_b$ are automatically bounded on all Köthe function spaces, hence, for $\Omega$ to come from interpolation it must satisfy commutator estimates with $M_b$.

Nigel uses the term centralizer (or sometimes homogeneous centralizer) to describe any abstract map $\Omega : A \to L^0$ satisfying (2.3) where $A$ is some Köthe function space (and there is no mention of any $A_0$ or $A_1$). He had already considered similar maps in [22] and [23] (using the same terminology but without imposing condition (ii)) and showed that they are automatically also derivations in the more abstract sense alluded to above.

2.2. The Results. The main result of [24] (Theorem 7.6 on page 511) is, roughly, that if $\Omega$ satisfies (2.3) then it can be obtained, to within a certain natural kind of equivalence, by complex interpolation. That is, given $A$ and $\Omega$ one can select $A_0$, $A_1$ and $t$ so that $A = A_t$ and $\Omega$ is equivalent to $\Omega(A_0, A_1, t)$.

The proof involves an interesting associated construct, motivated by Gillespie’s alternative proof [16] of the Lozanovsky factorization theorem, namely the indicator of the Köthe function space $A$. This is the functional $\Phi_A$ defined initially only on those positive elements $f$ of $L^1$ for which it is finite, by

$$\Phi_A(f) = \sup_{x \in A, \|x\|_A \leq 1} \int_S f \log |x| \, d\mu.$$  

(In other papers it is sometimes called the entropy function.) Nigel extends the proof of Lemma 3 of [16] to obtain that the supremum in the above formula is attained, for any given $f$, by a positive function $x = x_f$ determined via the Lozanovsky factorization theorem. This will enable him, at a rather later stage, to extend the definition of $\Phi_A$ to a larger class of complex valued functions $f$ by setting $\Phi_A(f) = \int_S f \log |x| \, d\mu$. (This is ultimately done in Lemma 5.6 (on p. 499), but note that there are misprints in the formula for $\Phi_X(f)$ on the third line of the statement of that lemma, namely, the integral sign and "$d\mu$" have been omitted.) In parallel with his study of the indicator functional $\Phi_A$, Nigel studies another more general class of “indicator-like” functionals $\Phi$ defined on suitable subsets $I$ of non-negative functions in $L^1$. His definition of these functionals requires $\Phi(f)$ to be real when $f$ is real valued, and also to satisfy certain continuity conditions. Furthermore, using his notation

$$\Delta_f(f, g) = \Phi(f) + \Phi(g) - \Phi(f + g)$$

he requires that

(2.4) \hspace{1cm} \Phi(\alpha f) = \alpha \Phi(f) \hspace{0.5cm} \forall \alpha \geq 0

and also that, for some positive constant $\delta(\Phi)$ and all $f, g \in I$,

(2.5) \hspace{1cm} 0 \leq \Delta_f(f, g) \leq \delta(\Phi) (\|f\|_{L^1} + \|g\|_{L^1}).

For each Köthe function space $A$, the indicator $\Phi_A$ has all these properties, and, conversely, any $\Phi$ having all these properties and also satisfying $\Delta_f(f, g) \leq \Delta_{\Phi_{L^1}}(f, g)$ for all $f, g \in I$ is necessarily the indicator of some Köthe function space $A$.

We will now become even more informal. Much of the technical work in [24] is done using the indicators.
This “change of variable”, to working with the functionals $\Phi$ rather than the interpolation spaces, “linearizes” the interpolation process: The indicator of $A_t = [A_0, A_1]_t$ is given by the formula:

$$\Phi_{A_t} = (1 - t)\Phi_{A_0} + t\Phi_{A_1}$$

and the Lozanovsky factorization for each Köthe function space $A$ can be expressed by the formula

$$\Phi_{L^1} = \Phi_A + \Phi_{A^*}.$$ 

The fundamental technical conclusion of Nigel’s paper is that one can close the loop. Given a Köthe function space $A$ and a $t$, one can find $A_0$ and $A_1$ so that the indicator of $A$ satisfies $\Phi_{A_t} = (1 - t)\Phi_{A_0} + t\Phi_{A_1}$. It then follows that $A$ is the interpolation space $A_t$.

For the final steps towards his main result, Nigel has to reveal and exploit a connection between centralizers and indicators. One should keep in mind here that, while each Köthe function space $A$ has a unique indicator, there are infinitely many different centralizers $\Omega$ which can be defined on $A$. Given any one such centralizer $\Omega$, Nigel (as he already did in [22]) uses it to define an auxiliary centralizer $\Omega^{[1]}$ defined on the positive functions in $L^1$. Once more, ideas associated with the Lozanovsky factorization play a central role, i.e., for each non negative $x$ in $B_{L^1}$,

$$\Omega^{[1]}(x) := \Omega(a)a^*$$

where $x = aa^*$ with $a$ in $A$, $a^*$ in the dual space $A^*$, and with $\|a\|_A = \|a^*\|_{A^*} = 1$. Now it is possible to define a new functional $\Phi^\Omega$ on a suitable subset of functions $f \in L^1$ by setting

$$\Phi^\Omega(f) := \int_S \Omega^{[1]}(f)d\mu.$$ 

Results from [22] show that $\Phi^\Omega$ belongs to the general class of “indicator-like” functionals mentioned above. Therefore Nigel can apply his technical results about indicators: Knowing that any indicator can be obtained by complex interpolation insures that any centralizer can be obtained, to within an appropriate equivalence, by complex interpolation.

2.3. Furthermore. Once the basic ideas are in place it is relatively straightforward to obtain analogous results for rearrangement invariant spaces. That is, if $A$ is a rearrangement invariant space and if given centralizers or indicator functions interact appropriately with the linear operator induced by rearrangements, then the new spaces constructed, $A_0$ and $A_1$, can be chosen to be rearrangement invariant. In particular, by using rearrangement invariant spaces it is possible to obtain results for the Schatten ideals. Those are spaces of compact operators on Hilbert space which are normed by rearrangement invariant norms on the operator’s sequence of singular numbers. (Those are the numbers which quantify the rate of approximation of a compact operator by finite rank operators.)

Nigel also considers a converse question. If an operator is bounded on a scale of spaces $A_t$, $0 < t < 1$ then the commutator with $\Omega$, the associated derivation, is bounded on, say, $A_{1/2}$. In the other direction, what if we are given $A$, $T$, and a derivation $\Omega$, with both $T$ and $[\Omega, T]$ bounded on $A_{1/2}$; must it follow that $T$ is bounded on the scale $\{A_t\}$, or, perhaps, at least for $t$ near $1/2$? It is satisfying that the answer is shown to be yes, at least in the case (in Theorems 9.7 and 9.8 [24].
where $T$ is the (vector valued) Riesz transform. It is intriguing that this answer can be used to give an easy proof of a nontrivial result in harmonic analysis: A fundamental fact about the “good weights” (i.e. $A_p$ weights) in the theory of singular integral operators is that their logarithms are in the dual space of the Hardy space $H^1$, that is, they are in $BMO$; and, conversely, if the logarithm of a weight is in $BMO$ and has sufficiently small norm then it is a good weight. The first of these facts can be derived using the theory of commutators that we have been discussing; in fact the boundedness of $[P, \Omega_1]$ which we discussed earlier is a special case of that result. Nigel can deduce the fact that exponentials of functions in $BMO$ are good weights as a consequence (Corollary 9.9 [24, p. 525]) of the above-mentioned theorems. A classical presentation of these topics and their uses is in Chapters 4 and 5 of [44]. Other relations between the theory of commutators and classical analysis are also developed in [44]. Other results indicating the interplay between spaces which are analogues of $H^1$, and of $BMO$ with classes of weights which are analogues of $A_p$, can be found, for example, in [4].

Theorems 9.7 and 9.8 also enable Nigel to obtain new results about UMD-spaces.

### 3. Looking Forward

Some of the following observations seem to invite further research. We will usually be brief and cryptic. Of course it is quite possible that some of the questions that we ask here have been answered in some publication which has not come to our attention.

1. Some of the recent approaches to interpolation view interpolation families $\{A_t : 0 \leq t \leq 1\}$ as “geodesics” in a space or “manifold” of possible Banach structures; [10], [11], [36], [40] and [41, Ch. 11]. From that point of view, the construction of scales of complex interpolation spaces is a method for constructing “geodesics” between two given points; that is, solving a boundary value problem for geodesics. The results in [24] show how to solve an initial value problem for “geodesics”; that is, given a “point” (i.e., a Köthe function space $X$) and a “tangent vector” (i.e., a centralizer defined on $X$), find a matching “geodesic”.

2. It is a theorem in classical analysis that, with $P$ denoting the projection operator from our first example, then the bilinear map $B(f, g) = f.Pg + g.Pf$, which at first glance maps $L^2 \times L^2$ into $L^1$ actually has better properties; it maps into the real variable Hardy space $\text{Re} H^1$. We mention two further facts; some relations between them are developed in [19] and perhaps there is still more to learn. First, the properties of $B$ and related maps can be used as the basis of a theory of compensated compactness which is of great use when studying partial differential equations ([9]). Second, this property of $B$ is a result in interpolation theory. In particular it is equivalent, via a duality argument, to commutator results of the type in (1) for $i = 1$. This point of view is developed systematically by Nigel in [22] and [24] where he develops a theory of a space $H^1_{\text{sym}}$, the symmetrized Hardy space, a space which, in the classical case, is closely related to the rearrangement invariant hull of the real variable Hardy space. It is further shown, in [23], that the Schatten class associated to $H^1_{\text{sym}}$ plays a fundamental role in describing operator ideals ([14]).
Interpolation constructions are generally both nonlinear and very abstract. The passage to indicator functions replaces classical complex interpolation with an explicit linear construction. It is not known how indicators interact with other interpolation methods.

Analytic semigroups of operators can be used to generate complex interpolation families of Banach spaces ([39, Sec. 4B], [38]). The passage from a scale of spaces to its differential seems analogous to passing from a semigroup to its generator. Perhaps that analogy could be taken further. There are interesting further thoughts developing this relation in the final section of [13].

The theory of differentials and commutators associated to interpolation extends to notions of higher differentials and associated commutators ([37], [32]), but the formalism becomes relatively intricate. Perhaps it is cleaner for Köthe function spaces.

Also, some interpolation methods include an analysis of sub- and super-interpolation families ([35], [40], [10], [11]) which satisfy various maximum or minimum principles. In some cases those are related to curvature-like expressions involving the second derivatives of the norming function. Perhaps the higher differentials and commutators provide a natural language in which to present such ideas. Perhaps Köthe function spaces provide an area in which those ideas can be explored more fully.

Nigel’s papers on commutators make very heavy use of the Lozanovsky factorization. Are there implicit hints in his papers about how to use the commutators to turn the machine around and run it in the other direction? Could a rich theory of interpolation and commutators substitute for Lozanovsky’s duality and factorization theory in contexts far removed from Köthe function spaces?

In a series of papers, H. König and V. Milman have studied operators that satisfy certain functional equations. For example, they show (cf. [26, Theorem 1]) that an operator \( L : C^1(\mathbb{R}) \to C(\mathbb{R}) \) (not necessarily linear or continuous) that satisfies the classical Leibniz rule for differentiating a product must be of the form \( Lf = af' + b\Omega f \), where \( a, b \in C(\mathbb{R}) \), and \( \Omega f = f \ln |f| \). It could be of interest to study in detail the connection between the König-Milman theory and the theory of commutators and its applications. Here we make a few quick comments. König and Milman’s characterization allows them to conclude that for a Leibniz operator to act on higher order spaces, e.g. \( L : C^2(\mathbb{R}) \to C^1(\mathbb{R}) \), the “cancellation condition” \( b = 0 \) must hold. Likewise, they show that Leibniz operators on “lower order spaces”, e.g. \( L : C(\mathbb{R}) \to C(\mathbb{R}) \), must be of the form \( Lf = c\Omega f \), for some \( c \in C(\mathbb{R}) \). Moreover, the solution of functional equations of the form \( Lf = g \), has already appeared (as an auxiliary topic) in the study of higher order logarithmic Sobolev inequalities (cf. Feissner [15, p. 58]) and commutator inequalities (cf. [12]).

A key step in Nigel’s journey to his main result is Theorem 6.6 on p. 507. For a deeper understanding of this theory one might try to determine whether the limiting case of either of these theorems holds, i.e., when \( \epsilon = 0 \). Does the best value of the relevant constant \( C(\epsilon) \) in this theorem necessarily have to tend to \( \infty \) as \( \epsilon \) tends to 0? The same question could be asked
perhaps more conveniently, with regard to Theorem 1.1 on p. 481, which is a finite dimensional “model” of Theorem 6.6 which Nigel formulates in his introduction to help prepare the reader for what is to follow. In connection with this question we also remark that the constant \( \log 2 \) plays a special role in these theorems. Furthermore, whenever \( \Phi \) is the indicator of a Köthe space, the optimal (smallest) constant \( \delta(\Phi) \) for which (2.6) holds satisfies \( \delta(\Phi) \leq \log 2 \). Equality holds when the Köthe space is \( L^1 \). For what other spaces, if any, does equality hold?

(10) As Nigel points out on p. 510, the centralizer \( \Omega[1] \) obtained when \( \Omega = \Omega(A_0, A_1, t) \) is the same for all values of \( t \in (0, 1) \). Can we somehow interpret this to mean that, when \( A_0 \) and \( A_1 \) are both Köthe function spaces, the “geodesic curve” \( \{[A_0, A_1]_t : 0 \leq t \leq 1 \} \) which joins them is in some sense a “straight line” or has some kind of zero “curvature”. Within a whole “manifold” of Banach spaces (which should all be compatibly contained in some Hausdorff topological vector space) does it make sense to think of the Köthe function spaces, or Banach lattices as forming a “flat” “submanifold”. Is there some “geometric” way to characterize that “submanifold”?

(11) Suppose that \( A_1, A_2, A_3 \) and \( A_4 \) are Banach spaces which satisfy \( A_2 = [A_1, A_3]_{\rho_2} \) and \( A_3 = [A_2, A_4]_{\rho_2} \). Then Wolff’s Theorem [15] (see also [20]) ensures that \( A_2 = [A_1, A_4]_{\alpha_2} \) and \( A_3 = [A_1, A_4]_{\alpha_2} \) for suitable \( \alpha_1 \) and \( \alpha_2 \). In other words the “geodesic curves” from \( A_1 \) to \( A_3 \) and from \( A_2 \) to \( A_4 \) can be “glued together” to form a single such “curve” from \( A_1 \) to \( A_4 \). It is intriguing to note that, when all of the above four spaces are Köthe function spaces, this result emerges from a trivial calculation using two instances of the formula (2.6).

(12) If \( \Omega \) is a centralizer acting on some Köthe function space, then so is every constant multiple \( r\Omega \) of \( \Omega \), with the constant \( \rho(\Omega) \) replaced by \( r\rho(\Omega) \). On the other hand, if a centralizer is of the special kind \( \Omega = \Omega(A_0, A_1, t) \) then it is not at all clear whether \( r\Omega \) is exactly of this kind, even if one renorms either or both of the spaces \( A_0, A_1 \). Suppose that \( A_0 \) and \( A_1 \) are uniformly convex so that \( \Omega(A_0, A_1, t) \) is uniquely defined, and that \( B_0 = A_0 \) and \( B_1 = A_1 \) but with new norms \( \|x\|_{B_0} = r_0 \|x\|_{A_0} \) and \( \|x\|_{B_1} = r_1 \|x\|_{A_1} \) for each \( x \in A_0 \) or \( A_1 \) respectively. Then it follows very easily that \( [B_0, B_1]_t = [A_0, A_1]_t \) with \( \|x\|_{[B_0, B_1]_t} = r_0^{1-t} r_1^t \|x\|_{[A_0, A_1]_t} \) for each \( x \in [A_0, A_1]_t \). But a simple calculation gives an explicit formula which shows that \( \Omega(A_0, A_1, t) \) cannot be a scalar multiple of \( \Omega(B_0, B_1, t) \). However these two maps are equivalent in the sense that the inequality

\[
\|\Omega(A_0, A_1, t)x - c_1 \Omega(B_0, B_1, t)x\|_{[A_0, A_1]_t} \leq c_2 \|x\|_{[A_0, A_1]_t}
\]

holds for suitable constants \( c_1 \) and \( c_2 \) and all \( x \in [A_0, A_1]_t \). These remarks show that it is not surprising that the precise formulation of the main result Theorem 7.6 of [24] includes some requirements on the size of the constant \( \rho(\Omega) \). They also indicate that there is no obvious way of obtaining a version of the theorem where there is equality rather than merely equivalence of the associated centralizers.

(13) In most of the natural applications of the results of [24], the underlying measure space \( (S, \mu) \) has a topological structure, as is required by Nigel at the beginning of his exposition. However, we suspect that most or maybe
even all of the results of [24] can be obtained in the context of a “reasonable”
measure space without any topology.

(14) Since the paper [24] has such a wealth of ideas and powerful methods, there
are surely many more items that could be included here. Should we have
future thoughts about [24], we may perhaps share them with you, at least
informally via the arXiv.

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