Semiclassical magnetotransport including the effects of the Berry curvature and Lorentz force

Seungchan Woo\textsuperscript{1,2}, Brett Min\textsuperscript{3,2} and Hongki Min\textsuperscript{1,‡}

\textsuperscript{1} Department of Physics and Astronomy, Seoul National University, Seoul 08826, Korea
\textsuperscript{2} Center for Theoretical Physics (CTP), Seoul National University, Seoul 08826, Korea
\textsuperscript{3} Department of Physics, McGill University, Montréal, Québec H3A 2T8, Canada

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In topological semimetals and insulators, negative longitudinal magnetoresistance and angle-dependent planar Hall effect have been reported arising from the Berry curvature. Using the Boltzmann transport theory, we present a closed-form expression for the nonequilibrium distribution function which includes both the effects of the Berry curvature and Lorentz force. Using this formulation, we obtain analytical expressions for conductivity and resistivity tensors in Weyl semimetals demonstrating a characteristic field dependence arising from the competition between the two effects.

I. INTRODUCTION

In topological materials with a nonvanishing Berry curvature, the positive magnetococonductance has been observed experimentally in the presence of parallel electric and magnetic fields \cite{1,2,3,4,5,6,7,8,9,10,11}. This is a unique feature to topological materials that does not occur in conventional magnetotransport which only takes classical Lorentz force effect into account.

The prevailing explanation for the positive longitudinal magnetoconductance is the so-called chiral anomaly. In 1983, Nielsen and Ninomiya suggested the chiral anomaly in Weyl fermions under a strong magnetic field regime where the chiral zeroth Landau level creates a one-dimensional conducting channel that pumps electrons from one Weyl node to another \cite{12}. In 2013, Son and Spivak discussed chiral anomaly in Weyl semimetals under weak external magnetic field using the semi-classical Boltzmann approach \cite{13}. They argued the positive magnetoconductivity that scales quadratically in magnetic field is due to topological charge pumping. One can expect possible detection of chiral anomaly between the valleys, given that the intervalley scattering is negligible compared to the intravalley scattering \cite{14}. However, chiral anomaly cannot be responsible for observed positive magnetoconductivity in topological insulators (TIs) where chiral charges are not well defined \cite{9,11,15}. In topological materials such as TIs \cite{16} and Weyl semimetals (WSMs) \cite{17}, it is suggested that the anomalous velocity induced by the non-trivial Berry curvature alone can generate an additional contribution to the conductivity that grows with the magnetic field.

To understand magnetotransport properties quantitatively, it is important to consider both the effects of the anomalous velocity due to the non-trivial Berry curvature and the classical Lorentz force effect. Most of the previous studies \cite{18,19,20} have focused on the Berry curvature effect, while only a few took the Lorentz force into considerations in describing magnetotransport behaviors \cite{21,22}. In this paper, we revisit semiclassical treatment of magnetotransport in topological materials to shed more light on the origin of the observed positive magnetoconductivity. We present a general semiclassical formula for conductivity which fully incorporates the Berry curvature and the Lorentz force. From the Boltzmann transport equation, we obtain a closed-form expression for the nonequilibrium distribution function by solving the corresponding self-consistent equation. We then apply our formula to WSMs and express the magnetoconductivity in terms of dimensionless parameters characterizing magnetic fields associated with the Lorentz force and the Berry curvature, respectively.

II. SEMICLASSICAL BOLTZMANN MAGNETOTRANSPORT THEORY FOR TOPOLOGICAL MATERIALS

The semiclassical Boltzmann transport equation governs the time evolution of a non-equilibrium distribution function \( f = f(r,k,t) \) at position \( r \) and momentum \( k \).

\[
\frac{df}{dt} = \left( \frac{\partial f}{\partial t} \right)_{\text{coll}}.
\]

In a homogeneous sample with a steady external perturbation, there are no position \( r \) nor time \( t \) dependence in the distribution function \( f \). Then, \( \frac{df}{dt} \) simplifies to

\[
\frac{df}{dt} = k \cdot \frac{\partial f_k}{\partial k}.
\]

where we have included a subscript \( k \) to the non-equilibrium distribution function \( f \) to indicate that it is only a function of \( k \). In a simple relaxation time approximation, the collision integral is replaced by the ratio between the deviation from the equilibrium Fermi-Dirac distribution function

\[
\frac{df_k}{dt} = \frac{f_k - f^{eq}_k}{\tau_k}.
\]

\( f^{eq}_k \) is the equilibrium distribution function, \( \tau_k \) is the relaxation time.

\[
\frac{df_k}{dt} = \frac{f^{eq}_k - f_k}{\tau_k}.
\]
distribution $\tilde{f}_{k}^{(0)}$ and the average time $\tau$ between successive collisions. Hence,

$$\tilde{k} \cdot \frac{\partial f_{k}}{\partial \tilde{k}} = -\frac{g_{k}}{\tau},$$

where $g_{k} = f_{k} - f_{k}^{(0)}$. Here we assume a constant transport relaxation time $\tau$ in momentum $\tilde{k}$ and magnetic field $B$ for a given chemical potential. Note that when we fully consider the collision integral in the system with a non-trivial Berry curvature, the transport relaxation time $\tau$ may show a field dependent anisotropy induced by the coupling between the Berry curvature and magnetic field [24]. However, in weak magnetic field regime, we can assume $\tau$ as a constant. Furthermore, when short-range scattering is dominant or charged impurities are fully screened, we can assume that $\tau$ does not depend on $\tilde{k}$ [25–27].

Here we provide a closed-form expression for magnetoconductivity in the presence of both a non-trivial Berry curvature and the Lorentz force within the semiclassical Boltzmann approach. The semiclassical equations of motion for electrons with a charge $q = -e$ in the presence of the Berry curvature are given by [28]

$$\dot{\tilde{r}} = \frac{\partial \epsilon_{m}(\tilde{k})}{\hbar \tilde{k}} - \tilde{k} \times \Omega_{k},$$

$$\hbar \dot{\tilde{k}} = q \tilde{E} + \frac{q}{c} \tilde{\chi} \times \tilde{B},$$

where $\epsilon_{m}(k) = \epsilon_{0}(k) - m_{k} \cdot B$, $\epsilon_{0}(k)$ is the unperturbed band energy, $m_{k}$ is an orbital magnetic moment that couples to the magnetic field and $\Omega_{k}$ is the Berry curvature. It has been reported that disorder affects not only the carrier distribution but also the semiclassical equations of motion, generating a correction to the velocity proportional to the disorder strength [29]. In this work, we neglect this correction assuming a weak disorder potential for simplicity. In the presence of a magnetic field, $\dot{\tilde{r}}$ and $\hbar \dot{\tilde{k}}$ in Eq. (4) are coupled through the Lorentz force. Combining these two equations of motion, we have

$$\dot{\tilde{r}} = D^{-1}_{k} \left[ \frac{q}{c} \tilde{E} \times \Omega_{k} - \frac{q}{\hbar c} (\Omega_{k} \cdot \tilde{v}_{k}) \tilde{B} \right],$$

$$\hbar \dot{\tilde{k}} = D^{-1}_{k} \left[ q \tilde{E} + \frac{q}{c} \tilde{v}_{k} \times \tilde{B} - \frac{q^{2}}{\hbar c} (\tilde{E} \cdot \tilde{B}) \Omega_{k} \right],$$

where $\epsilon_{m}(k) = \frac{1}{\hbar} \partial \epsilon_{m}(k) / \partial \kappa$ and $D_{k} = 1 - \frac{q^{2}}{\hbar^{2}} (\Omega_{k} \cdot \tilde{B})$ represents the modified density of states in the phase space due to the Berry curvature effect [23]. The first term in the square bracket in Eq. (5) corresponds to electric force due to an electric field $\tilde{E}$ and the second term represents Lorentz force due to a magnetic field $\tilde{B}$. The last term is anomalous electromagnetic force due to the Berry curvature which leads to positive magnetoconductivity in topological materials.

Replacing $\tilde{f}_{k} = f_{k}^{(0)} + g_{k}$ according to the relaxation time approximation in Eq. (3), we have

$$\hbar \dot{g}_{k} = \frac{1}{\hbar} \partial \left( f_{k}^{(0)} + g_{k} \right) / \partial \tilde{k} = -\frac{g_{k}}{\tau}.$$
The current is therefore computed as

\[ g_k = \mathbf{v}_k \cdot \left[ q \tau \left( - \frac{\partial f_0}{\partial \mathbf{v}_k} \right) \mathbf{E} - \frac{q \tau}{c} (\mathbf{B} \times \mathbf{M}^{-1} \mathbf{G}) \right]. \]  

(11)

From \( g_k = \mathbf{v}_k \cdot \mathbf{G} \) and Eq. (11), we obtain a self-consistent form of \( \mathbf{G} \) as

\[ \mathbf{G} = \mathbf{G}_0 - \frac{q \tau}{c} (\mathbf{B} \times \mathbf{M}^{-1} \mathbf{G}), \]  

(12)

where

\[ \mathbf{G}_0 = q \tau \left( - \frac{\partial f_0}{\partial \mathbf{v}_k} \right) \mathbf{E}. \]  

(13)

The solution for Eq. (12) can be obtained as

\[ \mathbf{G} = q \tau \left( - \frac{\partial f_0}{\partial \mathbf{v}_k} \right) \left( \mathbf{I} + \frac{q \tau}{c} \mathbf{F} \mathbf{M}^{-1} \right)^{-1} \mathbf{E}, \]  

(14)

where \( F_{ij} = \sum_i \epsilon_{ijk} B_k \) is the magnetic field strength tensor (see Appendix A for a detailed derivation). Thus, we obtain

\[ g_k = q \tau \left( - \frac{\partial f_0}{\partial \mathbf{v}_k} \right) \mathbf{v}_k \cdot \left[ \left( \mathbf{I} + \frac{q \tau}{c} \mathbf{F} \mathbf{M}^{-1} \right)^{-1} \mathbf{E} \right]. \]  

(15)

The current is therefore computed as

\[ J_\alpha = q \int \frac{d^d k}{(2\pi)^d} D_k f_k v_{\alpha}, \]  

(16)

where \( \alpha \) is the direction in which we measure the current. Let us denote the current associated with \( f_0^{(0)} \) as \( J_\alpha^{(0)} = q \int \frac{d^d k}{(2\pi)^d} D_k f_0^{(0)} v_{\alpha} \) and that associated with \( g_k \) as \( J_\alpha^{(ext)} = q \int \frac{d^d k}{(2\pi)^d} D_k g_k v_{\alpha} \). From now on, we will only focus on the extrinsic contribution \( J_\alpha^{(ext)} \) arising from scatterings.

Plugging Eq. (15) into \( J_\alpha^{(ext)} \) and using \( J_\alpha^{(ext)} = \sum_\beta \sigma_{\alpha\beta} E_\beta \) relation, we finally arrive at

\[ \sigma_{\alpha\beta} = q^2 \int \frac{d^d k}{(2\pi)^d} D_k \tau \left( - \frac{\partial f_0^{(0)}}{\partial \mathbf{v}_k} \right) \mathbf{v}_\alpha \times \left\{ \mathbf{v}_k \cdot \left[ \left( \mathbf{I} + \frac{q \tau}{c} \mathbf{F} \mathbf{M}^{-1} \right)^{-1} \mathbf{E} \right] \right\}, \]  

(17)

where \( \beta \) is the direction of an electric field.

Now let us assume that the mobility tensor \( c \tau \mathbf{M}^{-1} \) is set to a constant \( \mu \) for simplicity. Then \( \mathbf{G} \) in Eq. (14) becomes (see Appendix A)

\[ \mathbf{G} = q \tau \left( - \frac{\partial f_0^{(0)}}{\partial \mathbf{v}_k} \right) \mathbf{E} + \frac{\mu}{c} \mathbf{E} \times \mathbf{B} + \frac{\mu^2}{c^2} (\mathbf{E} \cdot \mathbf{B}) \mathbf{B}. \]  

(18)

Using Eq. (19), we finally obtain the following form for magnetoconductivity

\[ \sigma_{\alpha\beta} = q^2 \int \frac{d^d k}{(2\pi)^d} D_k \tau \left( - \frac{\partial f_0^{(0)}}{\partial \mathbf{v}_k} \right) \mathbf{v}_\alpha \times \left\{ \mathbf{v}_k \cdot \left[ \left( \mathbf{I} + \frac{q \tau}{c} \mathbf{F} \mathbf{M}^{-1} \right)^{-1} \mathbf{E} \right] \right\}, \]  

(19)

The above form is a general expression of magnetoconductivity for topological materials including the Berry curvature and Lorentz force within the semiclassical regime under the assumption that \( \mathbf{G} \) in \( g_k = \mathbf{v}_k \cdot \mathbf{G} \) is independent of \( \mathbf{v}_k \) and the mobility tensor is a constant.

### III. MAGNETOTRANSPORT IN WEYL SEMIMETALS

In this section, we study the magnetotransport properties of WSMs in three dimensions using a closed-form expression for magnetoconductivity Eq. (20) discussed in the previous section. For simplicity, we consider a single Weyl node described by the Hamiltonian \( H = \chi \hbar v_F \mathbf{k} \cdot \mathbf{\sigma} \) which has isotropic linear dispersion, where \( \chi = \pm 1 \) are for the different chiralities of Weyl fermions and \( \mathbf{\sigma} \) are the Pauli matrices.

#### A. Longitudinal magnetoconductivity

To investigate longitudinal magnetoconductivity \( \sigma_{xx}(\mathbf{B}) \) in WSMs, without loss of generality, we set the electric and magnetic field orientations as \( \mathbf{E} = E_x \hat{x} \) and \( \mathbf{B} = B_x \hat{x} + B_y \hat{y} \), respectively. Organizing terms in Eq. (20) in powers of \( \mu \) and using Eq. (19),

\[ \sigma_{xx} = q^2 \int \frac{d^d k}{(2\pi)^d} \frac{(\partial f_0^{(0)}) D_k^{-1}}{1 + \frac{\mu^2}{c^2} |\mathbf{B}|^2} \left[ \Sigma^{(0)} + \Sigma^{(1)} + \Sigma^{(2)} \right], \]  

(21)

where \( \Sigma^{(i)} \) is a sum of the terms that include \( i \)th order of \( \mu \). Due to the \( D_k^{-1} \) term in Eq. (21), it is difficult to obtain an analytic expression for \( \sigma_{xx} \) incorporating the full density of states correction. Therefore, to obtain a simple closed form result, we first assume \( D_k^{-1} \) as 1 in Eq. (21). We will discuss the correction beyond this.
approximation later. Here, $\Sigma^{(i)}_{k}$ are defined as

$$\Sigma^{(0)}_{k} = v_{x}^2 - 2 \frac{q}{\hbar c} v_{x} (v_{k} \cdot \Omega_{k}) |B| \cos \Gamma$$

$$\Sigma^{(1)}_{k} = \frac{\mu}{c} \left[ - v_{x} (v_{k} \times B)_{x} + \frac{q}{\hbar c} (v_{k} \times B)_{x} (v_{k} \cdot \Omega_{k}) |B| \cos \Gamma \right],$$

$$\Sigma^{(2)}_{k} = \frac{\mu^2}{c^2} \left[ (v_{k} \cdot B) v_{x} |B| \cos \Gamma - \frac{q}{\hbar c} (v_{k} \cdot \Omega_{k}) |B|^2 v_{x} \cos \Gamma - \frac{q}{\hbar c} (v_{k} \cdot B) (v_{k} \cdot \Omega_{k}) |B|^2 \cos^2 \Gamma + \frac{q^2}{(\hbar c)^2} (v_{k} \cdot \Omega_{k})^2 |B|^4 \cos^2 \Gamma \right],$$

where $\Gamma$ is the angle between $E$ and $B$. The first term $v_{x}^2$ in Eq. (22a) gives a well known longitudinal conductivity in the absence of magnetic field: $\sigma_{xx}(B = 0) = q^2 N_0 D \equiv \sigma_0$ where $N_0$ is the density of states at the Fermi energy and $D = v_{F}^2 \tau / d$ is the diffusion constant with $d = 3$.

Collecting terms that would give us non-zero contribution after momentum integral, Eq. (22) can be rewritten as

$$\sigma_{xx} = q^2 \int \frac{d^3k}{(2\pi)^3} \frac{\tau}{(1 + \frac{q^2}{\hbar c^2} |B|^2)} \left[ v_{x}^2 + \frac{q^2}{(\hbar c)^2} (v_{k} \cdot \Omega_{k})^2 |B|^2 \cos^2 \Gamma \right. + \left. \frac{\mu^2}{c^2} v_{x}^2 |B|^2 \cos^2 \Gamma + \frac{\mu^2}{c^2} \frac{q^2}{(\hbar c)^2} (v_{k} \cdot \Omega_{k})^2 |B|^4 \cos^2 \Gamma \right].$$

The Berry curvature in an isotropic WSM is $\Omega_{k} = \chi \frac{k}{\sqrt{k^2}}$. Therefore, $v_{x} (v_{k} \cdot \Omega_{k})$ becomes $\chi v_{F} / (2k_{F}^2)$ where $v_{F}$ and $k_{F}$ are the Fermi velocity and Fermi wave vector, respectively. Note that all the surviving terms are even functions of magnetic field $|B|$ and independent of $\chi$. Here we emphasize that there are two kinds of magnetic field effects: the Lorentz force and anomalous velocity effect due to the Berry curvature. The terms that are related to the Lorentz force comes with a $\frac{q}{\hbar c} |B|$ factor. On the other hand, the terms that are related to the Berry curvature comes with a $\frac{q^2}{(\hbar c)^2} |B|^2 \cos^2 \Gamma$ factor.

Since the terms coupled with the magnetic field are proportional to either $\frac{q}{\hbar c} |B|$ or $\frac{q^2}{(\hbar c)^2} |\Omega_{k}| |B|$, we introduce the following dimensionless parameters:

$$b_{\mu} = \frac{\mu}{c} |B| \equiv \frac{q}{\hbar c} \frac{\tau v_{F}}{k_{F}^2} |B|,$$

$$b_{BC} = \frac{q}{\hbar c} |\Omega_{k}| |B| = \frac{q}{\hbar c} \frac{1}{2k_{F}^2} |B|.$$
We emphasize that earlier studies of magnetotransport also took the $D_k$ correction into account but incompletely. For instance, Kim et al. \[20\] only took even terms in Taylor expanded $D_k^{-1}$ in Eq. (26) which gave rise to an additional positive correction to magnetoconductivity described in Eq. (25). However, numerical calculations of the full longitudinal magnetoconductivity [Eq. (21)] show a reduced value compared to Eq. (25). The reduced magnetoconductivity is due to terms that are odd orders in $|\mathbf{B}|$ in Eq. (21) coupling with odd orders in $|\mathbf{B}|$ terms in the Taylor expanded $D_k^{-1}$ in Eq. (26). As a result, the pairs of odd terms in $|\mathbf{B}|$ give rise to non-vanishing even terms in $|\mathbf{B}|$ which additionally give negative corrections to the magnetoconductivity.

By incorporating first three terms in Eq. (26), we obtain the longitudinal magnetoconductivity in WSMs up to quadratic order in $b_{BC}^2$ as

$$\sigma_{xx}(\mathbf{B}) \approx \frac{\sigma_0}{1 + b_{BC}^2} \times \{ 1 + \frac{1}{5} b_{BC}^2 + [\frac{7}{5} b_{BC}^2 + \frac{1}{5} b_{BC}^2 (1 + \frac{8}{5} b_{BC}^2)] \cos^2 \Gamma \}. \quad (27)$$

See Appendix B1 for detailed derivations. This result is well matched with the previous work which focused on specific angles between applied electric and magnetic fields \[22\].

### B. Planar Hall conductivity

To investigate the planar Hall conductivity in WSMs, we again set the electric and magnetic field orientations as $\mathbf{E} = E_x \hat{x}$ and $\mathbf{B} = B_x \hat{x} + B_y \hat{y}$, respectively. Neglecting terms that would give zero contribution to conductivity, Eq. (20) gives the following form of the planar Hall conductivity $\sigma_{yx}(\mathbf{B})$ assuming $D_k^{-1} = 1$ (see Appendix B2):

$$\sigma_{yx}(\mathbf{B}) = \sigma_0 \left[ \frac{b_{BC}^2}{1 + b_{BC}^2} + 3b_{BC}^2 \sin \Gamma \cos \Gamma \right]. \quad (28)$$

Note that $\sigma_{yx}(\mathbf{B})$ in Eq. (28) is independent of $\chi$. Equation (20) is the analytic form of the planar Hall conductivity in WSMs for an arbitrary angle of the applied magnetic field $|\mathbf{B}|$. The angle dependence of the planar Hall conductivity is well matched with the previous works $\sigma_{yx}(\Gamma) \sim \sin \Gamma \cos \Gamma$ \[22\] \[30,33\] but the field dependence shows an additional contribution from the Lorentz force in addition to a quadratic field dependence due to the Berry curvature. As shown in Fig 3, the planar Hall conductivity shows different $|\mathbf{B}|$ dependence at different $k_F l$ regimes. For large $k_F l$, the planar Hall conductivity roughly increases with $|\mathbf{B}|$ quadratically at low $|\mathbf{B}|$ field and saturates at high $|\mathbf{B}|$ field. For low $k_F l$ regime, the planar Hall conductivity shows $|\mathbf{B}|^2$ dependence with no sign of saturation in a broad range of magnetic field as reported in the previous studies \[21\].

We now come back to the approximation we made: $D_k^{-1} = 1$. Again, including the Taylor expanded $D_k^{-1}$ in Eq. (26), we obtain the following form of planar Hall conductivity in WSMs as

$$\sigma_{yx}(\mathbf{B}) \approx \sigma_0 \frac{b_{BC}^2}{1 + b_{BC}^2} \left[ 1 + \frac{1}{5} b_{BC}^2 \right] \sin \Gamma \cos \Gamma. \quad (29)$$

See Appendix B2 for detailed derivations.

### C. Hall conductivity

To investigate the Hall conductivity in WSMs, we set the electric and magnetic field orientations as $\mathbf{E} = E_x \hat{x}$ and $\mathbf{B} = B_y \hat{y}$, respectively. Then Eq. (20) gives the following form of Hall conductivity $\sigma_{xx}(\mathbf{B})$ assuming $D_k^{-1} = 1$ (see Appendix B3):

$$\sigma_{xx}(\mathbf{B}) = \sigma_0 \left( \frac{b_{BC}^2}{1 + b_{BC}^2} \right). \quad (30)$$

Note that this result is identical to the conventional magnetotransport result. This implies that there is no anomalous velocity effect in the Hall conductivity.

We now come back to the approximation we made: $D_k^{-1} = 1$. Again, including the Taylor expanded $D_k^{-1}$ in Eq. (26), we can obtain the Hall magnetoconductivity in WSMs as

$$\sigma_{xx}(\mathbf{B}) \approx \sigma_0 \frac{b_{BC}^2}{1 + b_{BC}^2} \left[ 1 + \frac{1}{5} b_{BC}^2 \right]. \quad (31)$$

See Appendix B3 for detailed derivations.

### D. Conductivity and resistivity tensors

Combining the previous results of magnetoconductivity in WSMs, we obtain the following conductivity tensor.
where $\Gamma$ is the angle between applied electric and magnetic fields. Since the resistivity tensor $\rho(B)$ is the inverse of the conductivity tensor $\sigma(B)$, we obtain $\rho(B)$ in the following form:

$$\rho(B) = \rho_0 \begin{bmatrix} 1+3b^2_{BC} \cos^2 \Gamma & -3b^2_{BC} \sin \Gamma \cos \Gamma & b_\mu \sin \Gamma \\ -3b^2_{BC} \sin \Gamma \cos \Gamma & 1+3b^2_{BC} \cos^2 \Gamma & -b_\mu \cos \Gamma \\ -b_\mu \sin \Gamma & -b_\mu \cos \Gamma & 1 \end{bmatrix},$$

(33)

where $\rho_0 = \rho_{xx}(B = 0) = \sigma_0^{-1}$. Note that although both the Lorentz force and the Berry curvature induced terms are present in $\sigma_{ij}(B)$, the Lorentz force induced terms do not appear in the resistivity $\rho_{ij}(B)$ where $i, j = x, y$. This result agrees with the previous one reporting that $\rho_{xx} = \rho_0 - \Delta \rho \cos^2 \Gamma$, $\rho_{xy} = -\Delta \rho \sin \Gamma \cos \Gamma$ where $\rho_0$ ($\rho_\perp$) is the resistivity in longitudinal (transverse) magnetic field and $\Delta \rho = \rho_\perp - \rho_0 = 3b^2_{BC}/(1+3b^2_{BC})$ is the resistivity anisotropy [30, 31]. For the conductivity and resistivity tensors including the Taylor expanded $D_k^{-1}$, see Appendix B.5.

IV. CONCLUSION

In this work, we presented a closed-form expression for the magnetoconductivity using the semiclassical magnetotransport theory that fully incorporates the Berry curvature and the Lorentz force effects. We then applied this formula to WSMs and obtained analytic expressions for the longitudinal, planar Hall and Hall conductivities in terms of dimensionless parameters $b_\mu$ and $b_{BC}$ which are normalized magnetic fields associated with the Lorentz force and the Berry curvature, respectively. From these results, we showed a non-monotonic field dependence in the longitudinal and planar Hall conductivities depending on $k_Fl$. Furthermore, we clearly demonstrated that although the Lorentz force effect is manifested in the planar Hall conductivity, its contribution vanishes in the planar Hall resistivity.

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Appendix A: Derivation of the nonequilibrium distribution function

Here we go through a detailed derivation of the nonequilibrium distribution function. Let us start with Eq. (12) in the main text:

\[ G = G_0 - \frac{qT}{c} (B \times M^{-1} G), \]  

(A1)
where \( M_{ij}^{-1} = \frac{1}{\hbar} \frac{\partial^2}{\partial \epsilon_i \partial \epsilon_j} \) is the inverse mass tensor and

\[
G_0 = q\tau \left( -\frac{\partial f_k^{(0)}}{\partial \epsilon_k} \right) E. \tag{A2}
\]

To solve \( G \) in Eq. (A1), note that Eq. (A1) is given by the following self-consistent form for a vector \( x \):

\[
x = a + (Mx) \times b, \tag{A3}
\]

where \( a \) and \( b \) are vectors and \( M \) is a matrix. Then Eq. (A3) can be rewritten as

\[
x_i = a_i + \sum_{jk} \epsilon_{ijk} (Mx)_j b_k = a_i + \sum_{jk} F_{ij} M_{jk} x_k, \tag{A4}
\]

where \( F_{ij} = \sum_k \epsilon_{ijk} b_k \). Thus, we have

\[
x = a + FMx = (1 - FM)^{-1} a. \tag{A5}
\]

Using the above result Eq. (A5), we obtain \( G \) as the following form:

\[
G = q\tau \left( -\frac{\partial f_k^{(0)}}{\partial \epsilon_k} \right) \left( I + \frac{qt}{c} FM^{-1} \right)^{-1} E. \tag{A6}
\]

where \( F_{ij} = \sum_k \epsilon_{ijk} B_k \) is the magnetic field strength tensor. Finally, the nonequilibrium distribution function \( g_k = \tilde{v}_k \cdot G \) is given by

\[
g_k = q\tau \left( -\frac{\partial f_k^{(0)}}{\partial \epsilon_k} \right) \tilde{v}_k \cdot \left( I + \frac{qt}{c} FM^{-1} \right)^{-1} E. \tag{A7}
\]

The mobility tensor is defined as

\[
\tilde{\mu}_{ij} = e\tau M_{ij}^{-1}, \tag{A8}
\]

which is in general a non-diagonal matrix. For a system with an isotropic energy dispersion in the absence of a magnetic field, the mobility tensor is given by a scalar multiple of an identity matrix. For simplicity, we assume that the mobility tensor is set to a constant \( \mu \). Then, we can rewrite Eq. (A1) as

\[
G = G_0 - \frac{\mu}{c} B \times G. \tag{A9}
\]

Note that Eq. (A9) is given by the following self-consistent form for a vector \( x \):

\[
x = a + x \times b, \tag{A10}
\]

where \( a \) and \( b \) are vectors. Then Eq. (A10) can be rewritten as

\[
x_i = a_i + \sum_{jk} \epsilon_{ijk} x_j b_k
\]

\[
= a_i + \sum_{jk} \epsilon_{ijk} \left( a_j + \sum_{lm} \epsilon_{jlm} x_l b_m \right) b_k
\]

\[
= a_i + \sum_{jk} \epsilon_{ijk} a_j b_k + \sum_{jklm} (\delta_{im}\delta_{kl} - \delta_{il}\delta_{km}) x_l b_m b_k
\]

\[
= a_i + (a \times b)_i + (x \cdot b) b_i - b^2 x_i
\]

\[
= \frac{1}{1 + b^2} [a_i + (a \times b)_i + (a \cdot b) b_i]. \tag{A11}
\]

Here, we used \( \sum_j \epsilon_{ijj} \epsilon_{mjl} = \delta_{im} \delta_{kl} - \delta_{il} \delta_{km} \) and \( x \cdot b = (a + x \times b) \cdot b = a \cdot b \). Thus, we have

\[
x = \frac{1}{1 + b^2} [a + (a \times b) + (a \cdot b) b]. \tag{A12}
\]
Using the result of Eq. (A12), we therefore obtain $G$ as the following form:

$$G = q\tau\left(-\frac{\partial f_k^{(0)}}{\partial \epsilon_k}\right)\frac{E + \frac{e}{c}E \times B + \frac{e^2}{c^2}(E \cdot B)B}{1 + \frac{\mu^2}{\tau c^2}|B|^2}. \tag{A13}$$

Note that Eq. (A6) is reduced to Eq. (A13) when the mobility tensor is given by $\tilde{\mu}_{ij} = \mu \delta_{ij}$.

Finally, the nonequilibrium distribution function $g_k = \tilde{v}_k \cdot G$ is given by

$$g = q\tau\left(-\frac{\partial f_k^{(0)}}{\partial \epsilon_k}\right)\tilde{v}_k \cdot \frac{E + \frac{e}{c}E \times B + \frac{e^2}{c^2}(E \cdot B)B}{1 + \frac{\mu^2}{\tau c^2}|B|^2}. \tag{A14}$$



### Appendix B: Magnetoconductivity of Weyl semimetals

In this section, we derive the magnetoconductivity of Weyl semimetals including the Taylor expanded $D_k^{-1}$. Here we set the magnetic field orientation as $B = B_x\hat{x} + B_y\hat{y}$.

#### 1. Longitudinal magnetoconductivity

Here we go through a detailed derivation of the longitudinal magnetoconductivity. Let us start with Eq. (21) in the main text:

$$\sigma_{xx} = q^2\int \frac{d^3k}{(2\pi)^3}\frac{\tau(-\frac{\partial f_k^{(0)}}{\partial \epsilon_k})D_k^{-1}}{1 + \frac{\mu^2}{\tau c^2}|B|^2}\left[\Sigma_k^{(0)} + \Sigma_k^{(1)} + \Sigma_k^{(2)}\right] = \sigma_{xx}^{(0)} + \sigma_{xx}^{(1)} + \sigma_{xx}^{(2)}, \tag{B1}$$

where $\Sigma_k^{(i)}$ is a sum of the terms that include $i$th order of $\mu$ described in Eq. (22) in the main text which corresponds to conductivity of $\sigma_{xx}^{(i)}$ respectively. Inserting the Taylor expanded density of states correction

$$D_k^{-1} = 1 + \frac{q}{\hbar c}(B \cdot \Omega_k) + \frac{(q}{\hbar c})^2(B \cdot \Omega_k)^2 + \cdots \tag{B2}$$

to Eq. (B1) and focusing on $\Sigma_k^{(0)}$, $\sigma_{xx}^{(0)}$ yields

$$\sigma_{xx}^{(0)} = q^2\int \frac{d^3k}{(2\pi)^3}\frac{\tau\delta(k - k_F) 1 + \frac{q}{\hbar c}(B \cdot \Omega_k) + \frac{(q}{\hbar c})^2(B \cdot \Omega_k)^2}{1 + \frac{\mu^2}{\tau c^2}|B|^2}\left[v_x^2 - \frac{2q}{\hbar c}v_x(v_k \cdot \Omega_k)|B|\cos \Gamma + \frac{q^2}{(\hbar c)^2}(v_k \cdot \Omega_k)^2|B|^2 \cos^2 \Gamma\right], \tag{B3}$$

where we replaced $(-\frac{\partial f_k^{(0)}}{\partial \epsilon_k}) = \frac{\delta(k - k_F)}{\hbar v_F}$ for zero temperature. Expanding and throwing away terms that will give zero contribution after the momentum integral due to odd order in $k_i$ $(i = x, y, z)$, we are left with the following angular integral after switching to spherical coordinates then integrating $k$ out:

$$\sigma_{xx}^{(0)} = q^2\int_0^{2\pi} d\phi \int_0^\pi d\tau \frac{k_F^2 \sin \theta}{(2\pi)^3 \hbar v_F (1 + b_F^2)}, \tag{B4}$$

where

$$I^{(0)}(\theta, \phi) = v_F^2 \left[\sin^2 \theta \cos^2 \phi + b_{BC}^2 \cos^2 \Gamma - 2b_{BC}^2 \cos^2 \Gamma \sin^2 \theta \cos^2 \phi + b_{BC}^2 (\cos^2 \Gamma \sin^4 \theta \cos^3 \phi + \sin^2 \Gamma \sin^4 \theta \cos^2 \phi \sin^2 \phi) + b_{BC}^4 (\cos^2 \Gamma \sin^2 \theta \cos^2 \phi + \cos^2 \Gamma \sin^2 \Gamma \sin^2 \theta \sin^2 \phi)\right]. \tag{B5}$$

Finally, carrying out the angular integral leads to

$$\sigma_{xx}^{(0)} = \frac{\sigma_0}{1 + b_F^2} \left[1 + \frac{b_{BC}^2}{5} (8 \cos^2 \Gamma + \sin^2 \Gamma) + b_{BC}^4 \cos^2 \Gamma\right], \tag{B6}$$
Finally, carrying out the angular integral leads to
\[
\sigma_{xx}^{(1)} = q^2 \int \frac{d^3k}{(2\pi)^3} \tau \delta(k - k_F) \frac{1}{\hbar v_F} \left[ -iv_x(v_k \times B)_x + \frac{q}{\hbar c} (v_k \times B)_x(v_k \cdot \Omega_k) |B| \cos \Gamma \right]. \tag{B7}
\]

The above whole expression vanishes after the momentum integral, because \((v_k \times B)_x = v_y B_z - v_z B_y = -v_y B_y\) as \(B_z = 0\). This will lead to a single order in \(v_x\) in every term in the integrand therefore result in zero after integration.

Finally, focusing on \(\sigma_{xx}^{(2)}\) yields
\[
\sigma_{xx}^{(2)} = q^2 \int \frac{d^3k}{(2\pi)^3} \frac{\tau \delta(k - k_F)}{\hbar v_F} \frac{1 + \frac{q}{\hbar c} (v_k \cdot \Omega_k) + \left( \frac{q}{\hbar c} \right)^2 (v_k \cdot \Omega_k)^2}{1 + \frac{\mu^2}{|B|^2}} \left[ (v_k \cdot B) v_x |B| \cos \Gamma - \frac{q}{\hbar c} (v_k \cdot \Omega_k) |B|^3 v_x \cos \Gamma \right. \\
\left. - \frac{q}{\hbar c} (v_k \cdot B)(v_k \cdot \Omega_k) |B|^2 \cos^2 \Gamma - \frac{q^2}{(\hbar c)^2} (v_k \cdot \Omega_k)^2 |B|^2 \cos^2 \Gamma \right]. \tag{B8}
\]

Simplifying the product of the Taylor expanded \(D_x^{-1}\) and terms in the square bracket while again, keeping only the non-zero contribution, we are left with the following angular integral after switching to spherical coordinates then integrating \(k\) out:
\[
\sigma_{xx}^{(2)} = q^2 \int_0^{2\pi} d\phi \int_0^\pi d\theta \frac{\tau k_F^2 \sin \theta}{(2\pi)^3} \frac{I^{(2)}(\theta, \phi)}{\hbar v_F(1 + b_\mu^2)}, \tag{B9}
\]

where
\[
I^{(2)}(\theta, \phi) = \nu_F^2 b_\mu^2 \left[ \sin^2 \theta \cos^2 \phi \cos^2 \Gamma + b_{BC}^2 \left( \cos^2 \Gamma - \cos^2 \Gamma \sin^2 \theta \cos^2 \phi - \sin^2 \theta \cos^2 \phi \cos^4 \Gamma \right. \\
- \sin^2 \theta \sin^2 \phi \sin^2 \Gamma \cos^2 \Gamma + \sin^4 \theta \sin^2 \phi \cos^2 \Gamma + 3 \sin^4 \theta \cos^2 \phi \sin^2 \phi \cos^2 \Gamma \sin^2 \Gamma) \\
+ b_{BC}^4 \left( \sin^2 \theta \cos^2 \phi \cos^4 \Gamma + \sin^2 \theta \sin^2 \phi \sin^2 \Gamma \cos^2 \Gamma \right) \right]. \tag{B10}
\]

Finally, carrying out the angular integral leads to
\[
\sigma_{xx}^{(2)} = \sigma_0 \frac{b_\mu^2}{1 + b_\mu^2} \left( \cos^2 \Gamma + \frac{8}{5} b_{BC}^2 \cos^2 \Gamma + b_{BC}^4 \cos^2 \Gamma \right). \tag{B11}
\]

Adding up the \(\sigma_{xx}^{(i)}\)’s, we have
\[
\sigma_{xx} = \frac{\sigma_0}{1 + b_\mu^2} \left[ 1 + b_{BC}^2 \left( 8 \cos^2 \Gamma + \sin^2 \Gamma \right) + b_{BC}^4 \cos^2 \Gamma + b_\mu^2 \left( \cos^2 \Gamma + \frac{8}{5} b_{BC}^2 \cos^2 \Gamma + b_{BC}^4 \cos^2 \Gamma \right) \right]. \tag{B12}
\]

2. Planar Hall conductivity

Here we go through a detailed derivation of the planar Hall conductivity. Starting with Eq. (20) in the main text for \(d = 3\), we again express it in powers of \(\mu\) while using the definition for \(\mathbf{v}_k\). We then obtain,
\[
\sigma_{yx} = q^2 \int \frac{d^3k}{(2\pi)^3} \frac{\tau \left( -\frac{\partial f_0}{\partial v_k} \right)}{1 + \frac{\mu^2}{|B|^2}} \left[ \Sigma_k^{(0)} + \Sigma_k^{(1)} + \Sigma_k^{(2)} \right] = \sigma_{yx}^{(0)} + \sigma_{yx}^{(1)} + \sigma_{yx}^{(2)}, \tag{B13}
\]

where
\[
\Sigma_k^{(0)} = v_y v_x - \frac{q}{\hbar c} v_y (v_k \cdot \Omega_k) |B| \cos \Gamma - \frac{q}{\hbar c} v_x (v_k \cdot \Omega_k) |B| \sin \Gamma + \left( \frac{q}{\hbar c} \right)^2 (v_k \cdot \Omega_k)^2 |B|^2 \cos \Gamma \sin \Gamma, \tag{B14a}
\]
\[
\Sigma_k^{(1)} = \frac{\mu}{c} \left[ -v_y v_k \cdot B + \frac{q}{\hbar c} (v_k \cdot \Omega_k)(v_k \times B)_y |B| \sin \Gamma \right], \tag{B14b}
\]
\[
\Sigma_k^{(2)} = \frac{\mu^2}{c^2} \left[ (v_k \cdot B) v_y |B| \cos \Gamma - \frac{q}{\hbar c} (v_k \cdot \Omega_k) |B|^3 v_y \cos \Gamma - \frac{q}{\hbar c} (v_k \cdot B)(v_k \cdot \Omega_k) |B|^2 \cos \Gamma \sin \Gamma \\
+ \left( \frac{q}{\hbar c} \right)^2 (v_k \cdot \Omega_k)^2 |B|^4 \cos \Gamma \sin \Gamma \right], \tag{B14c}
\]
Finally, carrying out the angular integral leads to
\[ D^{-1}_k = 1 + \frac{q}{\hbar c} (B \cdot \Omega_k) + \left( \frac{q}{\hbar c} \right)^2 (B \cdot \Omega_k)^2 + \cdots \] (B15)
to Eq. (B13) and focusing on \( \Sigma^{(0)}_k, \sigma^{(0)}_{yx} \) yields
\[ \sigma^{(0)}_{yx} = q^2 \int \frac{d^3k}{(2\pi)^3} \tau \delta(k - k_F) \left[ 1 + \frac{q}{\hbar c} (B \cdot \Omega_k) + \left( \frac{q}{\hbar c} \right)^2 (B \cdot \Omega_k)^2 \right] \frac{\left[ v_y v_x - \frac{q}{\hbar c} v_y (v_k \cdot \Omega_k) |B| \cos \Gamma - \frac{q}{\hbar c} v_x (v_k \cdot \Omega_k) |B| \sin \Gamma \right]}{1 + \frac{\mu^2}{c^2} |B|^2} \left( \frac{q}{\hbar c} \right)^2 (v_k \cdot \Omega_k)^2 |B|^2 \cos \Gamma \sin \Gamma, \right] \] (B16)
where we replaced \( (\frac{\partial \delta^{(0)}_k}{\partial k_F}) = \delta(k - k_F) \) for zero temperature. Expanding and throwing away terms that will give zero contribution after the momentum integral due to odd order in \( k_i (i = x, y, z) \), we are left with the following angular integral after switching to spherical coordinates then integrating \( k \) out:
\[ \sigma^{(0)}_{yx} = q^2 \int_0^{2\pi} d\phi \int_0^\pi d\theta \frac{k^2 \sin \theta}{(2\pi)^3} \frac{\tau I^{(0)}(\theta, \phi)}{\hbar v_F(1 + b_\mu^2)}, \] (B17)
where
\[ I^{(0)}(\theta, \phi) = v_F^2 \left[ b_{BC}^2 \cos \Gamma \sin \Gamma - \sin^2 \theta \cos^2 \phi \cos \Gamma \sin \Gamma \right. \]
\[ \left. - \sin^2 \theta \sin^2 \phi \cos \Gamma \sin \Gamma + 2 \sin^2 \theta \cos^2 \phi \sin^2 \phi \cos \Gamma \sin \Gamma \right) + b_{BC}^4 \sin^2 \theta \cos^2 \phi \sin^3 \Gamma \sin \Gamma + \sin^2 \theta \sin^2 \phi \cos \Gamma \sin^3 \Gamma \right]. \] (B18)
Finally, carrying out the angular integral leads to
\[ \sigma^{(0)}_{yx} = \frac{\sigma_0}{1 + b_\mu^2} \left( \frac{7 b_{BC}^2 + b_{BC}^4}{5} \right) \cos \Gamma \sin \Gamma. \] (B19)
Focusing now on \( \Sigma^{(1)}_k, \sigma^{(1)}_{yx} \) yields
\[ \sigma^{(1)}_{yx} = q^2 \int \frac{d^3k}{(2\pi)^3} \tau \delta(k - k_F) \left[ 1 + \frac{q}{\hbar c} (B \cdot \Omega_k) + \left( \frac{q}{\hbar c} \right)^2 (B \cdot \Omega_k)^2 \right] \frac{\mu}{c} \left[ -v_y (v_k \times B)_x + \frac{q}{\hbar c} (v_k \cdot \Omega_k) (v_k \times B)_x |B| \sin \Gamma \right]. \] (B20)
The above whole expression vanishes after the momentum integral, because \((v_k \times B)_x = v_y B_z - v_z B_y = -v_z B_y\) as \(B_z = 0\). This will lead to a single order in \( v_z\) in every term in the integrand therefore result in zero after integration.

Finally, focusing on \( \Sigma^{(2)}_k, \sigma^{(2)}_{yx} \) yields
\[ \sigma^{(2)}_{yx} = q^2 \int \frac{d^3k}{(2\pi)^3} \tau \delta(k - k_F) \left[ 1 + \frac{q}{\hbar c} (B \cdot \Omega_k) + \left( \frac{q}{\hbar c} \right)^2 (B \cdot \Omega_k)^2 \right] \frac{\mu^2}{c^2} \left[ (v_k \cdot B) v_y |B| \cos \Gamma - \frac{q}{\hbar c} (v_k \cdot \Omega_k) |B|^2 v_y \cos \Gamma \right. \]
\[ \left. - \frac{q}{\hbar c} (v_k \cdot \Omega_k) (v_k \cdot B |B| \cos \sin \Gamma + \left( \frac{q}{\hbar c} \right)^2 (v_k \cdot \Omega_k)^2 |B|^4 \cos \Gamma \sin \Gamma. \right] \] (B21)
Simplifying the product of the Taylor expanded \( D^{-1}_k \) and terms in the square bracket while again, keeping only the non-zero contribution, we are left with the following angular integral after switching to spherical coordinates then integrating \( k \) out:
\[ \sigma^{(2)}_{yx} = q^2 \int_0^{2\pi} d\phi \int_0^\pi d\theta \frac{k^2 \sin \theta}{(2\pi)^3} \frac{\tau I^{(2)}(\theta, \phi)}{\hbar v_F(1 + b_\mu^2)}, \] (B22)
where
\[ I^{(2)}(\theta, \phi) = v_F^2 b_{\mu}^2 \left[ \sin^3 \theta \sin^2 \phi \cos \Gamma \sin \Gamma + b_{BC}^2 \sin \theta \cos \Gamma \sin \Gamma - \sin^3 \theta \cos^2 \phi \cos^3 \Gamma \sin \Gamma - \sin^3 \theta \sin^2 \phi \sin \Gamma \cos \Gamma \right. \]
\[ \left. - \sin^3 \theta \sin^2 \phi \cos \Gamma \sin^3 \Gamma + \sin^5 \theta \cos^2 \phi \sin^2 \phi \cos^3 \Gamma \sin \Gamma + 2 \sin^5 \theta \cos^2 \phi \sin^2 \phi \cos^3 \Gamma \sin \Gamma \right. \]
\[ \left. + \sin^5 \theta \sin^4 \phi \cos \Gamma \sin^3 \Gamma + b_{BC}^4 \sin^3 \theta \cos^2 \phi \cos^3 \Gamma \sin \Gamma + \sin^3 \theta \sin^2 \phi \cos \Gamma \sin^3 \Gamma \right]. \] (B23)
Finally, carrying out the angular integral leads to
\[ \sigma_{yx}^{(2)} = \sigma_0 \frac{b_\mu^2}{1 + b_\mu^2} \left( 1 + \frac{8}{5} b_{BC}^2 + b_{BC}^4 \right) \cos \Gamma \sin \Gamma. \] (B24)

Adding up the \( \sigma_{yx}^{(i)} \)s, we have
\[ \sigma_{yx} = \frac{\sigma_0}{1 + b_\mu^2} \left[ \frac{7}{5} b_{BC}^2 + b_{BC}^4 \right] \cos \Gamma \sin \Gamma + b_\mu \left( 1 + \frac{8}{5} b_{BC}^2 + b_{BC}^4 \right) \cos \Gamma \sin \Gamma. \] (B25)

3. Conductivity \( \sigma_{zz} \)

Here we go through a detailed derivation of \( \sigma_{zz} \). Starting with Eq. (20) in the main text for \( d = 3 \), we again express it in powers of \( \mu \) while using the definition for \( \hat{v}_x \). We then obtain
\[ \sigma_{zz} = q^2 \int \, d^3k \, \frac{\tau (\frac{\partial \rho^{(0)}_{\mathbf{k}}}{\partial \mathbf{k}}) \delta_{\mathbf{k}}^{-1}}{(2\pi)^3} \left[ \Sigma_{\mathbf{k}}^{(0)} + \Sigma_{\mathbf{k}}^{(1)} + \Sigma_{\mathbf{k}}^{(2)} \right] = \sigma_{zz}^{(0)} + \sigma_{zz}^{(1)} + \sigma_{zz}^{(2)}, \] (B26)

where
\[ \Sigma_{\mathbf{k}}^{(0)} = v_z v_x - \frac{q}{\hbar c} (\mathbf{v}_k \cdot \mathbf{B}) v_z |\mathbf{B}| \cos \Gamma, \] (B27a)
\[ \Sigma_{\mathbf{k}}^{(1)} = - \frac{\mu}{c} v_z (\mathbf{v}_k \times \mathbf{B})_x, \] (B27b)
\[ \Sigma_{\mathbf{k}}^{(2)} = \frac{\mu^2}{c^2} \left[ (\mathbf{v}_k \cdot \mathbf{B}) - \frac{q}{\hbar c} (\mathbf{v}_k \cdot \mathbf{B})_x \right] v_z |\mathbf{B}| \cos \Gamma, \] (B27c)

each corresponds to Hall conductivity of \( \sigma_{zz}^{(i)} \) respectively. Inserting the Taylor expanded density of states correction to Eq. (B26) and focusing on \( \Sigma_{\mathbf{k}}^{(0)} \), \( \sigma_{zz}^{(1)} \) yields
\[ \sigma_{zz}^{(0)} = q^2 \int \, d^3k \, \frac{\tau \delta(k - k_F)}{(2\pi)^3} \frac{\frac{q}{\hbar c} (\mathbf{B} \cdot \mathbf{\Omega}_k) + \left( \frac{\mu}{c} \right)^2 (\mathbf{B} \cdot \mathbf{\Omega}_k)^2}{1 + \left( \frac{\mu}{c} \right)^2 |\mathbf{B}|^2} \left[ v_z v_x - \frac{q}{\hbar c} (\mathbf{v}_k \cdot \mathbf{B}) v_z |\mathbf{B}| \cos \Gamma \right], \] (B28)

where we replaced \( \left( - \frac{\partial \rho^{(0)}_{\mathbf{k}}}{\partial \mathbf{k}} \right) = \frac{\delta(k - k_F)}{\hbar v_F} \) for zero temperature. The above whole expression in Eq. (B28) vanishes after the momentum integral.

Focusing now on \( \Sigma_{\mathbf{k}}^{(1)} \), \( \sigma_{zz}^{(2)} \) yields
\[ \sigma_{zz}^{(1)} = q^2 \int \, d^3k \, \tau \delta(k - k_F) \, \frac{\frac{q}{\hbar c} (\mathbf{B} \cdot \mathbf{\Omega}_k) + \left( \frac{\mu}{c} \right)^2 (\mathbf{B} \cdot \mathbf{\Omega}_k)^2}{1 + \left( \frac{\mu}{c} \right)^2 |\mathbf{B}|^2} \left[ - \frac{\mu}{c} v_z (\mathbf{v}_k \times \mathbf{B})_x \right]. \] (B29)

Expanding and throwing away terms that will give zero contribution after the momentum integral due to odd order in \( k_i \ (i = x, y, z) \), we are left with the following angular integral after switching to spherical coordinates then integrating \( k \) out:
\[ \sigma_{zz}^{(2)} = q^2 \int_0^{2\pi} d\phi \int_0^\pi d\theta \frac{k_\theta^2 \sin \theta \tau \delta(k - k_F)}{(2\pi)^3} \frac{\mu \phi I^{(1)}(\theta, \phi)}{\hbar v_F (1 + b_\mu^2)}, \] (B30)

where
\[ I^{(1)}(\theta, \phi) = v_F^2 \left[ \cos^2 \theta \sin \theta \sin \Gamma + b_{BC}^2 \cos^2 \theta \sin^3 \theta \cos^2 \phi \cos^2 \Gamma \sin \Gamma + \cos^2 \theta \sin^3 \theta \sin^2 \phi \sin^3 \Gamma \right]. \] (B31)

Finally, carrying out the angular integral leads to
\[ \sigma_{zz}^{(1)} = \sigma_0 \frac{b_\mu}{1 + b_\mu^2} \left( 1 + \frac{1}{5} b_{BC}^2 \right) \sin \Gamma. \] (B32)
where $\sigma$ contribution is from $\Sigma$ as $B$. Note that the above expression vanishes after the momentum integral.

Adding up the $\sigma^{(i)}$s, we have

$$\sigma_{xx} = \sigma_0 \frac{b_\mu}{1 + b_\mu^2} \left( 1 + \frac{1}{5} b_{BC}^2 \right) \sin \Gamma.$$  \hspace{1cm} (B34)

Note that $\sigma_{xx}$ at $\Gamma = \frac{\pi}{2}$ in Eq. (B34) corresponds to the Hall conductivity.

### 4. Conductivity $\sigma_{zz}$

Here we go through a detailed derivation of the $\sigma_{zz}$. Starting with Eq. (20) in the main text for $d = 3$, we again express it in powers of $\mu$ while using the definition for $v_k$. We then obtain

$$\sigma_{zz} = q^2 \int \frac{d^3k}{(2\pi)^3} \frac{\tau(-\frac{\partial \mu_k}{\partial q_k}) D_k^{-1} \left[ \Sigma_k^{(0)} + \Sigma_k^{(1)} + \Sigma_k^{(2)} \right]}{1 + \frac{\mu^2}{c^2} |B|^2} \equiv \sigma_{zz}^{(0)} + \sigma_{zz}^{(1)} + \sigma_{zz}^{(2)},$$  \hspace{1cm} (B35)

where

$$\Sigma_k^{(0)} = v_z^2,$$  \hspace{1cm} (B36a)

$$\Sigma_k^{(1)} = -\frac{\mu}{c} v_z (v_k \times B)_z,$$  \hspace{1cm} (B36b)

$$\Sigma_k^{(2)} = 0,$$  \hspace{1cm} (B36c)

as $B_z = 0$. Note that after the momentum integral, $\Sigma_k^{(1)}$ will vanish due to odd order in $v_z$. The only remaining contribution is from $\Sigma_k^{(0)}$ which gives

$$\sigma_{zz}^{(0)} = q^2 \int \frac{d^3k}{(2\pi)^3} \frac{D_k^{-1} \tau(-\frac{\partial \mu_k}{\partial q_k})}{1 + \frac{\mu^2}{c^2} |B|^2} v_z^2,$$

$$= q^2 \int \frac{d\Omega}{\hbar v_F (2\pi)^3} \frac{\tau v_F^2 k_F^2}{1 + \frac{\mu^2}{c^2} |B|^2} \left( \cos^2 \theta + b_{BC}^2 \cos^2 \theta \sin^2 \theta \cos^2 \phi \cos^2 \Gamma + b_{BC}^2 \cos^2 \theta \sin^2 \theta \sin^2 \phi \sin^2 \Gamma \right) \sin \theta$$

$$= \frac{\sigma_0}{1 + b_\mu^2} \left( 1 + \frac{1}{5} b_{BC}^2 \right).$$

### 5. Conductivity and resistivity tensors

Finally, we have the following conductivity tensor including the Taylor expanded $D_k^{-1}$:

$$\sigma(B) = \frac{\sigma_0}{1 + b_\mu^2} \left[ 1 + \frac{1}{5} b_{BC}^2 + \frac{7}{5} b_{BC}^2 + \frac{b_\mu^2}{5} \left( 1 + \frac{1}{4} b_{BC}^2 \right) \right] \cos^2 \Gamma \left[ \frac{b_\mu^2}{5} \left( 1 + \frac{1}{4} b_{BC}^2 \right) + \frac{7}{5} \left( 1 + b_\mu^2 \right) b_{BC}^2 \right] \sin \Gamma \cos \Gamma$$

$$\times \left[ \frac{b_\mu^2}{5} \left( 1 + \frac{1}{4} b_{BC}^2 \right) + \frac{7}{5} \left( 1 + b_\mu^2 \right) b_{BC}^2 \right] \sin \Gamma \cos \Gamma$$

$$= \sigma_0 \left[ 1 + \frac{1}{5} b_{BC}^2 + \frac{7}{5} b_{BC}^2 \right] \cos^2 \Gamma \left[ \frac{b_\mu^2}{5} \left( 1 + \frac{1}{4} b_{BC}^2 \right) + \frac{7}{5} \left( 1 + b_\mu^2 \right) b_{BC}^2 \right] \sin \Gamma \cos \Gamma$$

$$= \sigma_0 \left[ 1 + \frac{1}{5} b_{BC}^2 \right].$$

(B38)
The resistivity tensor $\rho(\mathbf{B})$ is then obtained by the inverse of the conductivity tensor $\sigma(\mathbf{B})$ as

$$\rho(\mathbf{B}) = \rho_0 \begin{bmatrix} \frac{50+10b_2^{BC}+70b_2^{BC}\sin^2\Gamma}{50+90b_2^{BC}+16b_2^{BC}} & -\frac{35b_2^{BC}\cos\Gamma\sin\Gamma}{50+10b_2^{BC}+70b_2^{BC}\cos^2\Gamma} & \frac{5b_2\sin\Gamma}{5+b_2^{BC}} \\ -\frac{5b_2\sin\Gamma}{5+b_2^{BC}} & \frac{50+90b_2^{BC}+16b_2^{BC}}{25+15b_2^{BC}+8b_2^{BC}} & \frac{5b_2\cos\Gamma}{5+b_2^{BC}} \\ \frac{50+90b_2^{BC}+16b_2^{BC}}{25+15b_2^{BC}+8b_2^{BC}} & \frac{5b_2\cos\Gamma}{5+b_2^{BC}} & \frac{5b_2\sin\Gamma}{5+b_2^{BC}} \end{bmatrix}.$$ (B39)

Note that although both the Lorentz force and the Berry curvature induced terms are present in $\sigma_{ij}(\mathbf{B})$, the Lorentz force induced terms do not appear in the resistivity $\rho_{ij}(\mathbf{B})$ where $i,j = x,y$ even if we include the Taylor expanded $D_{k^{-1}}$. 