Linearized modified gravity theories with a cosmological term: advance of perihelion and deflection of light

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Abstract
Two different ways of generalizing Einstein’s general theory of relativity with a cosmological constant to Brans–Dicke type scalar–tensor theories are investigated in the linearized field approximation. In the first case a cosmological constant term is coupled to a scalar field linearly whereas in the second case an arbitrary potential plays the role of a variable cosmological term. We see that the former configuration leads to a massless scalar field whereas the latter leads to a massive scalar field. General solutions of these linearized field equations for both cases are obtained corresponding to a static point mass. Geodesics of these solutions are also presented and solar system effects such as the advance of the perihelion, deflection of light rays and gravitational redshift were discussed. In general relativity a cosmological constant has no role in these phenomena. We see that for the Brans–Dicke theory, the cosmological constant also has no effect on these phenomena. This is because solar system observations require very large values of the Brans–Dicke parameter and the correction terms to these phenomena becomes identical to GR for these large values of this parameter. This result is also observed for the theory with arbitrary potential if the mass of the scalar field is very light. For a very heavy scalar field, however, there is no such limit on the value of this parameter and there are ranges of this parameter where these contributions may become relevant in these scales. Galactic and intergalactic dynamics is also discussed for these theories at the latter part of the paper with similar conclusions.

Keywords: Brans–Dicke theory, weak solutions, cosmological constant, deflection of light, advance of perihelion

(Some figures may appear in colour only in the online journal)
1. Introduction

The remarkable observation at the end of 20th century showed that we live in an accelerating universe [1–3]. According to the well tested theory of gravitation, namely Einstein’s general relativity (GR) theory, this cosmic accelerating expansion is caused by a mysterious component of the universe, called dark energy. The most clear candidate of dark energy is Einstein’s cosmological constant, since, this observed behavior of the universe is compatible with a very small positive cosmological constant, i.e. $\Lambda \sim 10^{-52}$ m$^{-2}$ [1–3]. Another candidate is quintessence in the form of a minimally coupled scalar field which varies slowly along its potential [4–6]. For a review on quintessence, see, for example [7]. Considering thus the fact that we live in an asymptotically de Sitter universe, it might be reasonable to investigate the possible effects of a positive cosmological constant into local and global behavior of the universe. Therefore, it is logical to investigate whether such a cosmological constant, despite its smallness, affects the local gravitational phenomena such as the bending of light from distant objects or the advance of perihelion of objects in bound orbits.

Alternative theories of GR has been a very popular field of research, especially over recent decades. There are several theoretical or observational motivations that exist for this active field of research. One of them is to understand the mathematical structure, physical predictions and behaviour of GR by studying its alternatives. Another one is the quantization of gravitational interaction, and the fact that it may require some modifications to GR [8, 9]. One more reason is the idea of unification of fundamental interactions, generalizing the Kaluza–Klein idea of unifying gravity and electromagnetism [10, 11] into all interactions, such as string theory [12]. Such attempts require the ideas of the existence of extra dimensions and compactification. Such a compactification of higher dimensional theories into four dimensions usually produces a scalar field called dilaton into the four dimensional effective theories [10–12]. Apart from these, modified gravity theories, such as $f(R)$ theory, are also popular to investigate the possibility that the accelerating universe may be explained by large scale modifications to GR, without needing a dark energy. We refer to the latest reviews for the further motivations and developments of these theories [13–17].

Brans–Dicke (BD) theory [18–20] is one of the most simple modifications to GR and usually considered as a suitable test bed for investigating the effects of possible modifications to GR. After its presentation more than half a decade ago, the properties and outcomes of this theory is investigated in great detail [21–23]. For example, its weak field solution for point particle is obtained and two most interesting weak field phenomena, namely the perihelion precession of Mercury and light deflection by the Sun is investigated in the original papers of this theory [20, 21]. In its original form, as we will discuss in the next section, BD theory does not involve neither a cosmological constant or a potential term. However, in the latter years those extensions were also discussed, mostly in the cosmological scheme.

In this paper, we investigate weak field solutions of theories which generalizes Einstein’s general relativity with a cosmological constant to the Brans–Dicke type scalar-tensor theory. Since, as we will discuss in the next section, this generalization can be made, at least, in two different ways, we will consider both cases, separately. There are many works considering weak field solutions, properties of these solutions and astrophysical implications for different modified gravity theories [24–36]. However, in most of the works, except for example [30–32], asymptotic flatness is assumed. In [31] a post-Newtonian extension of the BD theory with a potential was presented. Our motivation in this paper is to investigate the weak field solutions of BD theory in the presence of an asymptotically de Sitter background. This will enable us to shed light into the effects of background curvature on the dynamics of the space time in the presence of a positive cosmological constant in these theories. We transform the
linearized field equations in a known suitable gauge which makes scalar and tensor equations decouple from each other and makes it easier to obtain the solutions. We will solve these equations for a static point particle in the coordinates where this gauge is valid for both cases and transform the obtained solutions into isotropic or Schwarzschild-like coordinates where this gauge is not valid. In order to obtain physical properties of these solutions, we will discuss the geodesics of both solutions in Schwarzschild type coordinates. Advance of the perihelion of test particles around this point particle, deflection of light rays by this point particle in the presence of a curvature background and also the gravitational redshift and galaxy rotation curves and intergalactic dynamics will be discussed for both solutions. Contribution of the mass of the source, the cosmological term or the minimum of the potential to these phenomena will be derived using appropriate methods. The paper is organized as follows. In the next section, we will discuss two different ways of generalizing GR with a cosmological constant to BD theory and obtain the weak field equations in a chosen gauge for both cases. In section 3 we will present a static point particle as a source, solve the field equations for both cases in the chosen gauge and transform the solutions to the isotropic and Schwarzschild type coordinates. In section 4 we will obtain radial geodesic equations in Schwarzschild coordinates. We will investigate solar system effects such as the advance of perihelion in section 5, deflection of light rays in section 6 and gravitational redshift in section 7 for both of the theories. Galactic and intergalactic dynamics is considered in section 8. The paper ends with a brief discussion.

2. Weak field equations

According to Einstein’s general theory of relativity (GR), the gravitational phenomena can be explained by the following action

\[ S_{\text{GR}} = \int \sqrt{|g|} d^4x \left[ \frac{1}{2\kappa} (R - 2\Lambda) + L_{\text{matter}} \right]. \]

(1)

Here, Einstein’s famous modification of adding a cosmological constant term to the action is already included. We may call this theory as GR\(\Lambda\) theory. Here \(\kappa = 8\pi G/c^4\) is the gravitational coupling constant, \(T_{\mu\nu}\) is the energy–momentum tensor, \(R\) is the Ricci scalar and \(\Lambda\) is the cosmological constant term and we choose the units where \(c = G = 1\) in this paper. One of the most studied alternative of GR is the Brans–Dicke (BD) scalar–tensor theory [20], where the Newton gravitational constant \(\kappa\) is replaced by a scalar function as \(\kappa \rightarrow 8\pi\phi^{-1}\) together with addition of a kinetic term for this scalar field coupled by a dimensionless constant known as the BD parameter \(\omega\). In the original derivation of the BD theory, cosmological constant \(\Lambda\) is set to zero. However, if one wants to extend GR theory with a cosmological constant to scalar–tensor theories, the most straightforward way is to replace \(\kappa \rightarrow 8\pi\phi^{-1}\) in action (1), similar to the original BD theory. This yields the following action in Jordan frame

\[ S_{\text{BDA}} = \int \sqrt{|g|} d^4x \left\{ \frac{1}{16\pi} \left[ \phi (R - 2\Lambda) - \frac{\omega}{\phi} R_{\mu\nu} \partial_\mu \phi \partial_\nu \phi \right] + L_{\text{matter}} \right\}. \]

(2)

This action where all curvature related terms \(R\) and \(\Lambda\) is coupled with scalar field \(\phi\) in the same manner, is known as the Brans–Dicke theory with a cosmological constant [37–39] and its cosmological [40–46], cylindrical [47–49] and other [50] applications were discussed in previous works. Here the value of \(\Lambda\) in BDA theory may be different from its value in GR\(\Lambda\) theory. We call the action (2) as the BD\(\Lambda\) action. Note that as \(\phi\) becomes a constant, this theory reduces to GR\(\Lambda\) theory. However, this action is not the only action which reduces to GR\(\Lambda\)
when $\phi$ is set to a constant. One can replace $2\Lambda\phi$ term with an arbitrary potential term $V(\phi)$ to obtain the following action

$$S_{\text{BDV}} = \int \sqrt{|g|} d^4x \left\{ \frac{1}{16\pi} \left[ \frac{\dot{\phi}}{\phi} \partial_\mu \phi \partial_\nu \phi - V(\phi) \right] + L_{\text{matter}} \right\}.$$  

which can be called the BD action with a potential, i.e. BD theory. Here, $V(\phi)$ plays the role of a variable cosmological term and when $\phi$ is set to a constant, the action (3) also reduces to GR $\Lambda$ theory. There are various works considering the addition of such a potential term and its various implications to BD theory [23, 27, 35, 36, 51–55] in various contexts. In general BDA and BDV theories and also all possible BDV theories having different potentials, may have different characteristics and lead to different physics. Hence, there is an arbitrariness in the generalization of GR $\Lambda$ theory to scalar–tensor theories. One possible method of identifying differences of different gravitation theories is to obtain their weak field solutions and compare with the results of GR. Therefore, here we want to discuss the weak field solutions of the theories (2) and (3) in the presence of a constant curvature background. Since these different choices may have different characteristics, we have to consider these theories separately, though we try to use a unified treatment as much as possible in the text. For example, when we discuss the field equations and their linearization of the actions (2) and (3) below, to avoid repeated similar equations, we will present those equations for the action (3) but keep in mind that for the case (2) we have to replace $V(\phi) = 2\Lambda\phi$ in the relevant equations. We will present the result of both theories separately whenever their distinction is important.

The extended BD actions (2) and (3) we are considering can also be expressed in other frames [22, 23], such as Einstein or string frames, by considering appropriate conformal transformations. For example, the following conformal transformation,

$$\tilde{g}_{\mu\nu} = \phi g_{\mu\nu},$$  

and the redefinition of the scalar field

$$\tilde{\phi} = \sqrt{\frac{2\omega + 3}{16\pi}} \ln \phi,$$

bring the BDV action into the Einstein frame as given by

$$S_{\text{EV}} = \int \left\{ \sqrt{|\tilde{g}|} d^4x \left[ \frac{\tilde{R}}{16\pi} - \frac{1}{2} \tilde{g}^{\mu\nu} \partial_\mu \tilde{\phi} \partial_\nu \tilde{\phi} + \frac{V(\tilde{\phi})}{(\phi(\tilde{\phi}))^2} + \frac{(\phi(\tilde{\phi}))}{(\phi(\tilde{\phi}))^2} L_{\text{matter}} \right] \right\}.$$  

In this frame, the role of the scalar field is changed from being a part of gravitational interaction to a canonical scalar-field matter–energy distribution permeating all points of the spacetime. Moreover, it also couples to the matter Lagrangian nonminimally. Hence, the manifold is no longer Riemannian and the test particles do not follow geodesics. In the Jordan frame, however, the scalar field is a part of gravitational interaction. Therefore, the theory is a metric theory and test particles follow geodesics in this frame. Actually, there was a debate on which of these frames are physical or are they equivalent or not. For a review of this debate see, for example, [56]. Here we share the original idea of transformation of units [20] of Brans and Dicke which says that both frames are equivalent and give same physical results. After some works concerning this debate, there seems to be a concencus on that all these frames are mathematically and physically equivalent [57–62]. Namely, a physical quantity measured on a certain frame does not depend on the chosen frame, if the transformations between frames is properly used. Hence, the choice of frame is a matter of convenience since some calculations
can be more easily performed in a particular frame. Therefore, in this work, since we will investigate the motion of test particles in the later stages of this paper, we prefer to work in the Jordan frame. This is because this frame has a calculational advantage, since in this frame test particles follow geodesics and we do not want to deal with the fifth force arising from the modifications of geodesics equations in the Einstein frame.

The action of a modified gravity theory ($f(R)$ theory) that is very popular in recent years [13–17] is
\[ S = \frac{1}{2\kappa} \int \sqrt{|g|} d^4x \left( f(R) + L_{\text{matter}} \right). \] (7)

It is well known [13–17] that under a Legendre transformation the $f(R)$ theories become equivalent to the BDV theory (3) for specific values of $\omega$, namely $\omega = 0$ for metric and $\omega = -3/2$ for Palatini $f(R)$ theories. Thus, BDV theory with arbitrary $\omega$ leads to a more general treatment, includes both $f(R)$ theories as special cases. Therefore, we will consider the theory (3) without any restriction on $\omega$ in this paper, except for $\omega = -3/2$ case. In this case, the scalar field becomes an auxiliary field with no dynamics and therefore we will not discuss the $\omega = -3/2$ case in this paper. The results for metric $f(R)$ theory can be recovered from BDV theory by setting $\omega = 0$ in the resulting expressions.

The Jordan frame field equations of the action (3) can be written as
\[ G_{\mu\nu} = 8\pi \phi T_{\mu\nu} + \omega \phi^2 \left( \nabla_\mu \phi \nabla_\nu \phi - \frac{1}{2} g_{\mu\nu} \nabla_\alpha \phi \nabla_\alpha \phi \right) + \frac{1}{\phi} \left( \nabla_\mu \nabla_\nu \phi - g_{\mu\nu} \Box g \phi \right) - V(\phi) g_{\mu\nu}, \] (8)

\[ \Box \phi = \frac{1}{2\omega + 3} \left( 8\pi T + \phi \frac{dV(\phi)}{d\phi} - 2V(\phi) \right), \] (9)

where here $T$ is the trace of the matter energy–momentum tensor $T_{\mu\nu}$ and $\Box$ is the D’Alembertian operator with respect to the full metric. Now, let us consider a weak field expansion of the above field equations. Hence, the space time metric and the BD scalar field can be expanded as
\[ g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \quad \phi = \phi_0 + \varphi, \] (10)

where $\eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1)$ is the Minkowski metric, $h_{\mu\nu}$ is the tensor representing small deviation from flatness, $\phi_0$ is a constant value of the scalar field and $\varphi$ is a small perturbation to the scalar field i.e. $|h_{\mu\nu}| \ll 1$ and $\varphi \ll 1$. Using the above expansion, defining a new tensor [27]
\[ \theta_{\mu\nu} = h_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} h - \eta_{\mu\nu} \frac{\varphi}{\phi_0}, \] (11)

together with the gauge
\[ \theta^{\mu\nu}_\nu = 0, \] (12)

the weak field BD field equations, up to second order, become [27]
\[ \Box \theta_{\mu\nu} = -\frac{16\pi}{\phi_0} \left( T_{\mu\nu} + \tau_{\mu\nu} \right) + \frac{V_{\phi \phi}}{\phi_0^2} g_{\mu\nu}, \] (13)
\[ \Box \varphi = 16\pi S. \] (14)
Here $\tau_{\mu\nu}$ is the energy–momentum pseudo tensor involving quadratic terms and $\Box_\eta = \eta^{\mu\nu} \partial_\mu \partial_\nu$ is the D’Alembertian of the Minkowski spacetime. The term $S$ is given by

$$S = \frac{1}{4\omega + 6} \left[ T \left( 1 - \frac{\theta}{2} - \frac{\varphi}{\phi_0} \right) + \frac{1}{8\pi} \left( \phi \frac{dV}{d\phi} - 2V \right) \right]_{\text{lin}} + \frac{1}{16\pi} \left( \eta^{\mu\nu} \varphi_{,\mu\nu} + \frac{\varphi_{,\mu}\varphi_{,\nu}}{\phi_0} \right).$$

(15)

where in deriving $S$ the relation between Minkowski and curved D’Alembertian operators is used

$$\Box_\xi = \left( 1 + \frac{\theta}{2} + \frac{\varphi}{\phi_0} \right) \Box_\eta - \theta^{\mu\nu} \varphi_{,\mu\nu} - \frac{\varphi_{,\mu}\varphi_{,\nu}}{\phi_0} + O(\text{higher terms}).$$

(16)

Here subtext lin in equations (13) and (15) means that these terms must also be properly linearized. In our setting, we want to discuss the case where the space time is not asymptotically flat but asymptotically de Sitter, therefore we need an effective cosmological constant in the linearized field equations. Since this term can be obtained for either of the actions (2) and (3) differently, now we have to discuss these cases separately.

### 2.1. Linearized field equations of Brans–Dicke theory with a cosmological constant (BD\(\Lambda\) case)

For the action (2), the terms involving the potential $V$ can be expanded as

$$V(\phi) \approx 2\Lambda \phi_0, \quad \left( \phi \frac{dV}{d\phi} - 2V \right) \approx -2\Lambda \phi_0,$$

(17)

where the terms linear in $\Lambda$ or $\varphi$ are kept and the terms of the order $\Lambda \varphi$ or higher are ignored. Using these, we obtain the following linearized field equations

$$\Box_\eta \theta_{\mu\nu} = -\frac{16\pi}{\phi_0} T_{\mu\nu} + 2\Lambda \eta_{\mu\nu},$$

(18)

$$\Box_\eta \varphi = \frac{8\pi}{2\omega + 3} \frac{T}{\phi_0} - \frac{2}{2\omega + 3} \Lambda \phi_0.$$

(19)

In the chosen parametrization (11) and gauge (12), the tensor equation (18) have similar structure to weak field $\text{GR}\Lambda$ equations [63]. Thus, all the differences from $\text{GR}$, namely the effects of the $BD$ scalar field will be originated from scalar field equation (19). The first important observation is that for this case the scalar field is massless. This means that $\text{BD}\Lambda$ theory has a similar structure with $BD$ theory where the scalar field has a long range and the existence of a cosmological constant does not change this behavior. Hence, in $\text{BD}\Lambda$ theory the cosmological constant does not change the local behavior of the scalar field and acts as a background curvature similar to $\text{GR}$ and responsible for the asymptotical nonflatness. Therefore, $\text{BD}\Lambda$ theory is a natural generalization of $\text{GR}\Lambda$ theory to BD theory, merging the properties of both theories into a single unified theory.

### 2.2. Linearized field equations of Brans–Dicke theory with an arbitrary potential (BDV case)

Now we consider the action (3). We suppose that the arbitrary potential $V$ is a well behaving function of its argument and it is Taylor expandable around a constant value of the scalar field, namely $\phi = \phi_0$ as:
\[ V(\phi) = V(\phi_0) + V'(\phi_0)\varphi + \frac{1}{2}V''(\phi_0)\varphi^2 + \ldots \] (20)

Here \( ' \) means partial derivative with respect to the scalar field. In previous work considering this action [27, 35], asymptotic flatness was assumed, which requires vanishing of the first two terms in the expansion. Here we do not impose such a condition, but it might be reasonable to expect \( \phi_0 \) to be a minimum of this potential for stability, which require \( V'(\phi_0) \) to be vanishing. Then, the relevant terms in the linearized field equations can be written as

\[ V(\phi)g_{\mu\nu} \approx V(\phi_0)\eta_{\mu\nu} \left( \frac{dV}{d\varphi} - 2V \right) \approx \phi_0 V''(\phi_0)\varphi - 2V(\phi_0) \] (21)

and the field equations (13) and (14), in the first order in \( \theta, \varphi \) and \( V_0 \), become

\[ \Box_\eta g_{\mu\nu} = -\frac{16\pi}{\phi_0} T_{\mu\nu} + \frac{V_0}{\phi_0}\eta_{\mu\nu} \] (22)

\[ (\Box_\eta - m_s^2)\varphi = \frac{8\pi T}{2\omega + 3} - \frac{2V_0}{2\omega + 3} \] (23)

Here we used the abbreviations

\[ V_0 \equiv V(\phi_0), \quad m_s^2 \equiv \frac{\phi_0}{2\omega + 3}V''(\phi_0) > 0. \] (24)

In the parametrization (11) and gauge (12) considered, the tensor equation (22) still has the same structure with the weak field equations of GR\Lambda theory [63]. The effect of the scalar field and arbitrary potential is encoded into the scalar field equation (23). It is clear that the minimum of potential \( V_0 \) plays the role of a constant curvature background or cosmological constant which is responsible for the asymptotic non-flatness. However, there is a slight difference between \( \Lambda \) in BD\Lambda theory and \( V_0 \) in BDV theory. The terms in the tensor equations of both theories can be made similar by defining \( V_0 = 2\Lambda_0\phi_0 \) (we put the subscript \( \Lambda_0 \) to distinguish these theories). However, even using this redefinition, the coefficients related to cosmological constant \( \Lambda \) and \( \Lambda_0 \) of the scalar field equations (19) and (23) will have different coefficients, \( 2\Lambda\phi_0/(2\omega + 3) \) versus \( 4\Lambda_0\phi_0/(2\omega + 3) \). This is because the \( \Lambda \) term in BD\Lambda theory is linear in the scalar field whereas \( V_0 \) is zeroth order term in the expansion of \( V(\phi) \). The effects of this difference will be seen in the solutions of both theories presented in the next section and also at the physical quantities such as the advance of perihelion due to these terms.

Another important difference of these theories is that, this theory leads to a massive scalar field [27, 35] with the mass term \( m_s \) defined as (24) proportional to second derivative of the arbitrary potential \( V(\phi) \). The effect of the mass term is to make the scalar field short-ranged. As we will see in the next section, solutions representing isolated systems, such as a point particle, contain Yukawa-like terms, which gives a characteristic range \( l_c \sim 1/m_s \) where the scalar field related to this source freezes out and the physical properties due to this source becomes indistinguishable from GR outside this range. This behavior is in contrast to BD theory where the field has a long range. Hence, the introduction of an arbitrary potential changes the range of the scalar field. Since metric \( f(R) \) theory is equivalent to BDV theory for \( \omega = 0 \), this behaviour persists in this theory too. Note that, the behaviour of scalar field related to minimum of the potential is still long range and persists outside \( l_c \). Hence, asymptotically nonflat weak field solutions make it possible to open a new window to test these massive BD theories, otherwise they are indistinguishable from GR outside this range. For example, as we will see in the next section, the advance of perihelion has corrections due to minimum of potential, \( V_0 \).
This correction is negligible for very light scalar field where \( l_c \to \infty \) and solar system tests require a very large \( \omega \) for this case. For a heavy scalar case, however, since \( l_c \to 0 \), scalar field due to mass of the isolated source is already frozen and solar system tests have become insensitive to \( \omega \). Hence, since \( \omega \) can take arbitrary values, the correction terms involving \( \omega \) due to a minimum of potential can be very different from corresponding \text{GRA} solutions.

3. Solutions to linearized field equations for a point mass

Having obtained linearized metric and scalar field equations in the chosen gauge for both theories, the next step would be to obtain a physically relevant solution to these theories. Hence, in the following, we consider a static point mass solution as a source for both theories. Note that, as far as we know, weak field equations of BD\( \Lambda \) theory with nonzero \( \Lambda \) for a point particle as a source is not discussed before. For BDV theory however, due to its equivalence with \( f(R) \) theory, its weak field solutions derived for a nonvanishing \( V_0 \) \cite{30–32} using a slightly different method. The physical applications we will consider of both theories for nonzero \( \Lambda \) or \( V_0 \), namely, the advance of perihelion, deflection of light, gravitational redshift, and galactic and inter-galactic dynamics were not discussed before.

3.1. A point mass term as a source for BD\( \Lambda \) theory

We now consider a point particle located at \( \vec{r} = 0 \), where \( \vec{r}^2 = \vec{x}^2 + \vec{y}^2 + \vec{z}^2 \), described by

\[
T_{\mu\nu} = m \delta(\vec{r}) \text{diag}(1, 0, 0, 0) .
\]  

(25)

Then, the scalar field equation (19) has the solution

\[
\varphi(\vec{r}) = \frac{2m}{(2\omega + 3)} \frac{1}{\vec{r}} - \frac{\Lambda \phi_0}{3(2\omega + 3)} \vec{r}^2 .
\]  

(26)

The advantage of using the gauge (11) and (12) is that the resulting tensor equations are decoupled from the scalar field. The tensor equation (18) for a diagonal metric ansatze, together with the gauge condition (12) yield the following non vanishing components for the solution

\[
\theta_{00} = \frac{4m}{\phi_0} \frac{1}{\vec{r}} - \frac{\Lambda}{3} \vec{r}^2 , \quad \theta_{xx} = \frac{\Lambda}{2} (\vec{x}^2 + \vec{z}^2) , \\
\theta_{yy} = \frac{\Lambda}{2} (\vec{y}^2 + \vec{z}^2) , \quad \theta_{zz} = \frac{\Lambda}{2} (\vec{x}^2 + \vec{y}^2) .
\]  

(27)

Using the trace of \( \theta = \theta_{\mu\mu} \) given by

\[
\theta = -4m \frac{1}{\phi_0} \frac{1}{\vec{r}} + \frac{4\Lambda}{3} \vec{r}^2 ,
\]  

(28)

and the inverse of (11), the nonzero components of the metric perturbation term becomes

\[
h_{00} = \frac{2m}{\phi_0} + \frac{\Lambda}{3} \vec{r}^2 + \frac{\varphi}{\phi_0} , \\
h_{ij} = \frac{2m}{\phi_0} \Lambda (\vec{x}^2 + 3 \vec{x}_j^2) - \frac{\varphi}{\phi_0} \delta_{ij} \quad (i, j = 1, 2, 3) .
\]  

(29)

Clearly, the effects of the nonminimally coupled scalar field reveal themselves as the last terms in the metric perturbation tensor. The presence of this nontrivial scalar field may have
some physical consequences such as it can modify test particle trajectories compared to corresponding GR results discussed, for example, in [63].

The solution presented above in equation (29) is not in isotropic coordinates. To express this solution in isotropic coordinates we may consider the following coordinate transformations [63]

\[ \bar{x}^i = x'^i + \frac{\Lambda}{12} x^{i3}. \]  

(30)

Under these transformations (30) from barred to primed coordinates, the metric perturbation terms, up to linear order in \( M \) and \( \Lambda \) become

\[ h'_{00} = \frac{2m}{\phi_0 r^2} + \frac{\Lambda}{3} r^2 + \frac{\phi'}{\phi_0}, \quad \text{(31)} \]

\[ h'_{ij} = \left( \frac{2m}{\phi_0 r^2} - \frac{\Lambda}{6} r^2 - \frac{\phi'}{\phi_0} \right) \delta_{ij}, \quad \text{(32)} \]

\[ \phi' = \frac{2m}{(2\omega + 3) r^2} \left( 1 - \frac{\Lambda}{3(2\omega + 3)} r^2 \right). \quad \text{(33)} \]

The explicit expressions of the field variables, with the help of (33), can be expressed as

\[ g'_{00} = -1 + \frac{2m}{\phi_0 r^2} \left( 1 + \frac{1}{2\omega + 3} \right) + \frac{\Lambda r^2}{3} \left( 1 - \frac{1}{2\omega + 3} \right), \quad \text{(34)} \]

\[ g'_{ij} = \left[ 1 + \frac{2m}{\phi_0 r^2} \left( 1 - \frac{1}{2\omega + 3} \right) - \frac{\Lambda r^2}{6} \left( 1 - \frac{2}{2\omega + 3} \right) \right] \delta_{ij}, \quad \text{(35)} \]

\[ \phi' = \phi_0 \left( 1 + \frac{2m}{(2\omega + 3)\phi_0 r^2} - \frac{\Lambda r^2}{3(2\omega + 3)} \right). \quad \text{(36)} \]

As is well-known [20], the mass term in \( g_{00} \) must be related with weak field GR or Newton potential of a point mass, then \( \phi_0 \) must be equal to

\[ \phi_0 = \frac{2\omega + 4}{2\omega + 3}. \quad \text{(37)} \]

This implies that, since it is defined for an asymptotically flat space time, when \( \Lambda = 0 \), the post-Newtonian parameter \( \gamma \) is

\[ \gamma_{BD} = \frac{h_{ij}^i}{h_{00}^i} = \frac{\omega + 1}{\omega + 2}. \quad \text{(38)} \]

This result, together with the observational result of the Cassini mission [64], i.e. \( \gamma_{\text{observed}} - 1 = (2.1 \pm 2.3) \times 10^{-5} \) which sets \( \gamma \sim 1 \), implies that BD parameter must satisfy \( \omega > 40,000 \) for BD theory. For these large values of \( \omega \), the above solution given by (34) and (35) becomes indistinguishable from the corresponding GR solution [63].

Note that, in the case of vanishing \( \Lambda \), the metric (34) and (35) and scalar field (36) reduce to the linearized BD solution [20, 21]. These solutions reduce to corresponding linearized GRA solutions presented in [63] in the limit \( \omega \to \infty \), \( \phi_0 \to 1 \). Hence the above solution has correct limits. Note that the limit \( \omega \to \infty \) does not always reduce the theory to GR. For a
further discussion of the GR limit of BD theory, see, for example, [42, 65, 66–69] and references therein.

### 3.2. A point mass term in the BDV theory

For a point mass $m$ given in (25) the corresponding solution of the scalar field equation (23) reads:

$$\varphi(\bar{r}) = \frac{2m}{(2\omega + 3)\bar{r}} - \frac{V_0}{3(2\omega + 3)}\bar{r}^2. \quad (39)$$

Note that if we set

$$\frac{V_0}{2\phi_0} = \Lambda_0, \quad (40)$$

then the first order metric equation (22) becomes exactly the same as (18) when $\Lambda$ is replaced by $\Lambda_0$. Thus, we can directly use the solutions (27) and also for the metric perturbation terms $h_{\mu\nu}$ given in (29) in this case as well. Moreover, with this choice (40), we can use exactly the same transformations (30) to bring the metric solution into isotropic coordinates. With this notation, the differences between both theories are encoded in the $\varphi'$ term, which has the same form with (39) for this case with $\bar{r}$ replaced with $r'$.

The full metric and scalar field of weak field equation becomes

$$g'_{00} = -1 + \frac{2m}{\phi_0 r'} \left( 1 + \frac{e^{-m'r}}{2\omega + 3} \right) + \frac{V_0 r'^2}{6\phi_0} \left( 1 - \frac{2}{2\omega + 3} \right), \quad (41)$$

$$g'_{ij} = \delta_{ij} \left[ 1 + \frac{2m}{\phi_0 r'} \left( 1 - \frac{e^{-m'r}}{2\omega + 3} \right) - \frac{V_0 r'^2}{12\phi_0} \left( 1 - \frac{4}{2\omega + 3} \right) \right], \quad (42)$$

$$\phi' = \phi_0 \left( 1 + \frac{2m e^{-m'r}}{(2\omega + 3)\phi_0 r'} - \frac{V_0 r'^2}{3\phi_0 (2\omega + 3)} \right). \quad (43)$$

Note that this solution is discussed before by using a slightly different approach [30–32]. The vanishing $V_0$ case is known as massive BD theory and its weak field solutions were presented before [27, 28, 35]. Also for vanishing $V_0$, the post-Newtonian parameter $\gamma$ becomes position dependent [30–33]:

$$\gamma(r) = 1 - \frac{e^{-m'r}}{\omega + 3}, \quad (44)$$

The observational result of the Cassini mission, namely $\gamma_{\text{observed}} - 1 = (2.1 \pm 2.3) \times 10^{-5}$, which sets $\gamma(r) \sim 1$, can be applied to $\gamma(r)$ in several ways. The first one is to set $e^{-m'r} \to 0$, which requires $m_s \to \infty$, namely the mass of the scalar field must be very heavy. For this case $\gamma = 1$ irrespective of the value of $\omega$. If this is not the case, i.e. if the mass is not large and if $e^{-m'r}$ is $O(1)$ which requires very light scalar mass as $m_s \to 0$, then $\gamma(r)$ reduces to $\gamma_{\text{BD}}$ given in equation (38) and the limit $\omega > 40,000$ is again emerged. For intermediate values of $m_s$, however, a numerical investigation is required to find the observationally allowed regions of the parameter space $(m_s, \omega)$ and this analysis is made in the work [33].

Let us now discuss the effective gravitational constant $\phi_0$ for these theories. In the vanishing of the $V_0$ term the value of $\phi_0$ is fixed by the requirement that the theory must have a
correct Newtonian limit, which requires the investigation of the term containing the mass of the source in $g_{00}$. The exponential term spoils this expression but for very light or heavy scalar field mass cases $\phi_0$ can be fixed as discussed below \cite{27, 33, 35}:

- For a very massive potential, i.e. $m_s \gg 1$, we can ignore the terms with exponential factor and can set $\phi_0 = 1$.
- For a very light scalar mass case, i.e. $m_s \ll 1$, we can expand $e^{-m_s/r}$ term in $g_{00}$ in series, keep the first term and compare with the Newtonian potential of a point mass. This procedure yields that we must have $\phi_0 = (2\omega + 4)/(2\omega + 3)$, as in the original BD theory \cite{20}.

For the intermediate values of $m_s$, the above prescription does not work to fix $\phi_0$, but one can define an effective gravitational coupling term involving the exponential term as

$$G(r) = \left(1 + \frac{e^{-m_s r}}{2\omega + 3}\right) \frac{1}{\phi_0}. \quad (45)$$

A numerical investigation of such a case is given in \cite{33} by using the powerful PPN approach, which requires an asymptotically flat spacetime so that $V_0$ must be vanishing. In our work we will not discuss such an investigation since we want to discuss the case where the spacetime is not asymptotically flat.

As we have discussed in the previous section, another difference between the weak field solutions of BD$\Lambda$ and BDV theories is that the factors involving $\omega$ in the metric and scalar field expressions of $\Lambda$ or $V_0$ have some differences. The result reflects the fact that their couplings with the scalar field are different. Namely $V_0$ is constant whereas the term involving $\Lambda$ is linear in the scalar field.

### 3.3. Solutions in Schwarzschild Coordinates

Here we will bring both of the solutions given in sections 3.1 and 3.2 to Schwarzschild type coordinates. This can be done by the following transformations and definitions

$$r' = r \left(1 - \frac{m}{\phi_0 r} + \frac{1}{12} \Lambda r^2 + \frac{\varphi}{\phi_0} \right), \quad (46)$$

with the result

$$ds^2 = -\left(1 - \frac{2m}{\phi_0 r} - \frac{\Lambda r^2}{3} - \frac{\varphi}{\phi_0} \right) dt^2 + \left(1 + \frac{2m}{\phi_0 r} + \frac{\Lambda r^2}{3} + \frac{\alpha r}{\phi_0} \right) dr^2 + r^2 d\Omega_2^2. \quad (47)$$

Here $\varphi$ is given by

$$\varphi = \frac{2m}{(2\omega + 3) r} - \frac{\Lambda \phi_0}{3(2\omega + 3)} r^2, \quad (48)$$

for BD$\Lambda$ theory and

$$\varphi = \frac{2m e^{-m_s r}}{(2\omega + 3) r} - \frac{V_0 r^2}{3(2\omega + 3)}, \quad (49)$$

for BDV theory. The function $\alpha(r)$ in (47) is defined as

$$\alpha = \frac{d\varphi}{dr}. \quad (50)$$
This form of the metric (47) is suitable to represent both solutions with the same metric. Difference from corresponding GRA solution is encoded in $\phi_0$, $\varphi$ and $\alpha$.

4. Motion of test particles

Now we will discuss the effects of the point mass and the presence of the cosmological term on the motions of test particles and photons. In order to discuss these effects and compare with the previous results exist on the literature, we choose to work in the Schwarzschild like coordinates, which can be written as

$$ds^2 = -A(r)\, dt^2 + B(r)\, dr^2 + r^2 \left( d\theta^2 + \sin^2 \theta \, d\Phi^2 \right).$$  \hspace{1cm} (51)

We also consider equatorial motion by setting $\theta = \pi/2$, then the Lagrangian of the test particles or photons can be written as

$$2L = -A \dot{t}^2 + B \dot{r}^2 + r^2 \dot{\Phi}^2.$$  \hspace{1cm} (52)

Here overdot means derivative with respect to proper time for time-like particles and an affine parameter for photons. Symmetries of this space time results two first integrals of motion, given by

$$\dot{t} = -\frac{E}{A}, \quad \dot{\Phi} = \frac{L}{r^2}.$$  \hspace{1cm} (53)

Here $E$ and $L$ are related with specific energy and angular momentum of test particles. We can use these results into the metric itself to obtain a radial equation of motion

$$\dot{r}^2 = \frac{1}{B} \left( \varepsilon + \frac{E^2}{A} - \frac{L^2}{r^2} \right),$$  \hspace{1cm} (54)

where $\varepsilon = 0$ for photons and $\varepsilon = -1$ for timelike particles. We can obtain an equation for orbit by dividing this expression by $\dot{\Phi}^2$ with the result

$$\left( \frac{dr}{d\Phi} \right)^2 = \frac{i^2}{\dot{\Phi}^2} = r^4 \left( \frac{\varepsilon}{L^2 B} + \frac{E^2}{L^2 AB} - \frac{1}{Br^2} \right).$$  \hspace{1cm} (55)

Hereafter we may analyze the equations (54) or (55) for different types of motion for the two different solutions we have obtained. We should also read the appropriate metric functions $A$ and $B$ (51) from (47) for the solutions we are discussing and keep the terms in the linear order in mass and cosmological terms in the expressions. It is customary to use inverse radial coordinate defined by

$$u = \frac{1}{r}$$  \hspace{1cm} (56)

to obtain a modified Binet equation and solve the resulting equation. Then, using the transformation (56), the resulting equation can be written in the linearized order as

$$\left( \frac{du}{d\Phi} \right)^2 + u^2 = \left( \frac{E^2 + \varepsilon}{L^2} \right) + \frac{\Lambda}{3} - \frac{2\varepsilon m u}{\phi_0 L^2} + \frac{2m u^3}{\phi_0^3} - \frac{\varepsilon \Lambda}{3L^2 u^2} \frac{E^2}{L^2} - \frac{\varepsilon}{L^2} \frac{\phi_0^3}{u^2} \frac{\alpha(u)}{L^2 \phi_0^2} - (E^2 + \varepsilon - L^2 u^2) \frac{\alpha(u)}{L^2 \phi_0^2}.$$  \hspace{1cm} (57)

By differentiating this equation with respect to $\Phi$, one obtains a modified Binet equation as
\[
\frac{d^2 u}{d\Phi^2} + u = - \frac{\varepsilon m}{\phi_0 L^2} + \frac{3m u^2}{\phi_0} + \frac{\varepsilon \Lambda}{3L^2 u^3} + \frac{1}{2} \frac{\partial}{\partial u} \left[ \frac{E^2 \varphi(u)}{L^2} - \left( E^2 + \varepsilon - L^2 u^2 \right) \frac{\alpha(u)}{L^2 u \phi_0} \right].
\]  
(58)

Hence, we have obtained a modified Binet equation which is useful for both BDΛ and BDV theories. One can further analyze this equation by appropriately choosing the values of the functions \( \varphi, \alpha \) and constant \( \varepsilon \).

5. Precession of the perihelion of the planets

In order to derive advance of perihelion for these theories, we may try to use the usual perturbative approach to solve equation (58) for time like particles \( \varepsilon = -1 \), together with appropriate values for \( \varphi \) and \( \alpha \). For example, for BDΛ theory, we obtain the following differential equation

\[
\frac{d^2 u}{d\Phi^2} + u = \frac{2m(E^2 + \omega + 1)}{(2\omega + 3)\phi_0 L^2} + \frac{6m(\omega + 1)u^2}{(2\omega + 3)\phi_0} - \frac{(E^2 + 2\omega + 1)\Lambda}{3(2\omega + 3)L^2 u^3}.
\]  
(59)

One can consider the Newtonian elliptical solution as a zeroth order solution of the equation (59). However, this approach needs a perturbation extension of geodesic equation (59) second order in mass \( m \). But, this is beyond our linearized approximation. Hence, we cannot use the perturbation approach to derive the perihelion advance for both of the theories. But, there are alternative methods and we will use one of them to derive perihelion shift terms due to mass of the source and cosmological terms.

5.1. Review of calculation of advance of perihelion by integration of perturbation potential

The calculation is based on the principle that in the weak field of a gravitation theory, we may have a potential which has usual Newtonian term for a central mass distribution as well as other correction terms originated from the theories modifying Newtonian theory. Using different methods such as directly integrating geodesics equations [70], precession of a cousin of the Runge–Lenz vector, i.e. the Hamilton vector in the modified potential background [71], or using a modification of Landau–Lifshitz method [72] one obtains perihelion shift \( \Delta \) as a one dimensional integral of the form

\[
\Delta = \frac{-2L}{me^2} \int_{-1}^{1} \frac{z dz}{\sqrt{1 - z^2}} \frac{dV(z)}{dz},
\]  
(60)

where the coordinate transformation \( r = L/(1 + e z) \) is employed. Here \( e \) is eccentricity of elliptic motion. For details of the derivation of this integral we refer to the works [70, 73]. This perihelion shift implies that the orbit equation has the following form

\[
u = 1/r = \frac{m}{L^2} \left[ 1 + e \cos (1 - \epsilon) \Phi \right], \quad \Delta = 2\pi e.
\]  
(61)

In equation (60), \( V(z) \) contains the modification terms to the Newtonian potential for central motion which can be read from the full potential of the gravity theory considered of a point mass as given by

\[
U(r) = -\frac{m}{r} + \frac{L^2}{2r^2} + V(r).
\]  
(62)
So here we just need to find the \( \tilde{V}(r) \) term and replace into the integral (60). To do this we consider the radial geodesic equation (54), which can be written as

\[
\frac{r^2}{2} + U(r) = \tilde{E}
\]

where \( \tilde{E} = E^2/2 \), and

\[
U(r) = \frac{1}{2} \left( 1 + \frac{L^2}{r^2} \right) \left( 1 - \frac{2m}{\phi_0 r} - \frac{\Lambda r^2}{3} - \frac{\alpha r}{\phi_0} \right) + \left( \alpha r - \varphi \right) \frac{E^2}{2\phi_0}.
\]

Hence, we need to evaluate \( U(r) \) for the solutions we have found to read \( \tilde{V}(r) \). We will do this in the following subsections for both of the theories we consider.

5.2. Advance of perihelion for BD\(\Lambda \) theory

Using (47), replacing the value of \( \phi_0 \) given in (37) and the fact that for a nonrelativistic motion energy per unit mass has the value \( E = 1 \) [74], we find that

\[
\tilde{V}(r) = -\left( \frac{\omega + 1}{\omega + 2} \right) \frac{mL^2}{r^2} - \left( \frac{2\omega + 2}{2\omega + 3} \right) \frac{\Delta r^2}{6}.
\]

Here, some constant terms are discarded because they do not affect the advance of perihelion. Since the result of integral (60) is calculated for power–law potentials [70, 73], from their result, the only difference is factors involving \( \omega \) in \( \tilde{V}(r) \), so we see that the perihelion shift can be written as

\[
\Delta_{BD\Lambda} = \frac{\omega + 1}{\omega + 2} \Delta_E + \frac{2\omega + 2}{2\omega + 3} \Delta_\Lambda,
\]

where \( \Delta_E \) is the usual Einstein value of perihelion shift due to mass [75] and \( \Delta_\Lambda \) [70, 73, 76, 77] is GR\(\Lambda \) perihelion shift due to cosmological constant. Their expressions are given by

\[
\Delta_E = \frac{6\pi m}{a(1 - e^2)};
\]

\[
\Delta_\Lambda = \frac{\pi \Lambda m}{a^3} \sqrt{1 - e^2},
\]

and \( \Delta_\Lambda \) agrees with the one found in [78] only for \( e \to 0 \) as discussed in [73]. The perihelion shifts due to the mass of the source and the cosmological constant of BD\(\Lambda \) theory have similar structures with corresponding GR\(\Lambda \) theory with the same multiplicative factors involving \( \omega \).

The discussion of some observational consequences of these results will be in the section 5.4.

5.3. Advance of perihelion for BDV theory

For this case, the potential \( U \) given by equation (64) becomes

\[
U(r) = \frac{1}{2} \left( 1 + \frac{L^2}{r^2} \right) \left[ 1 - \frac{2m}{\phi_0 r} - \frac{\Lambda r^2}{3} - \frac{2m}{(2\omega + 3)\phi_0} \left( m_s + \frac{1}{r} \right) e^{-m_s r} - \frac{2V_0 r^2}{3\phi_0} \right] - \frac{E^2 m e^{-m_s r}}{(2\omega + 3)\phi_0} \left( m_s - \frac{2}{r} \right) + \frac{V_0 E^2 r^2}{6(2\omega + 3)\phi_0}.
\]
This expression is complicated and the resulting $\tilde{V}(r)$ which involve terms containing $e^{-m sr}$ factor cannot be integrated \cite{70} to obtain analytical results. However, for the following special cases it is possible to obtain analytic results for the advance of perihelion.

- For a very heavy scalar field, since as $m_s \to \infty$, $e^{-m sr} \to 0$, the perturbation potential becomes

\begin{equation}
\tilde{V}(r)_{m_s \to \infty} = -\frac{mL^2}{r^3} - \left(\frac{2\omega + 1}{2\omega + 3}\right) \frac{\Lambda_0 r^2}{6}.
\end{equation}

From this potential, since $\phi_0 = 1$ for a heavy scalar field, the advance of perihelion is calculated as

\begin{equation}
\Delta_{\text{BDV-heavy}} = \Delta_E + \left(\frac{2\omega + 1}{2\omega + 3}\right) \Delta_\Lambda_0,
\end{equation}

where $\Delta_E$ is the Einstein value \cite{67} for perihelion shift and $\Delta_\Lambda_0$ is the GR $\Lambda$ value \cite{68} with $\Lambda$, is replaced with $\Lambda_0$. Hence, for a very heavy scalar, when the minimum of the potential is zero, then the perihelion shift is indistinguishable from the GR value and independent of $\omega$. This is a well-known result that when the scalar field becomes very short range field, weak field tests yield the same results with GR. However, when the minimum of the potential $V_0$ is not zero, then the resulting perihelion shift has a term due to the minimum of the potential having a factor involving $\omega$. If the value of $V_0$ or $\Lambda_0$ would be fixed by a future observation, then one could put on bounds on $\omega$ even for very massive BD theory. For example for metric $f(R)$ case $\omega = 0$, the perihelion shift due to mass is the same with GR whereas the corresponding term due to the cosmological term is $1/3$ of its GR $\Lambda$ value with $\Lambda$ is replaced with $\Lambda_0$.

- For a very light scalar field, as $m_s \to 0$ we can expand $e^{-m sr}$ in a series and since $m_s$ and $m$ are small we can ignore the terms such as $m \times m_s$, and using the value of $\phi_0$ given in \eqref{37} for this case, then the perturbation term becomes

\begin{equation}
\tilde{V}(r)_{m_s \to 0} = -\left(\frac{\omega + 1}{\omega + 2}\right) \frac{mL^2}{r^3} - \left(\frac{2\omega + 1}{2\omega + 3}\right) \frac{\Lambda_0 r^2}{6}.
\end{equation}

This implies the perihelion shift as

\begin{equation}
\Delta_{\text{BDV-light}} = \frac{\omega + 1}{\omega + 2} \Delta_E + \left(\frac{2\omega + 1}{2\omega + 3}\right) \Delta_\Lambda_0.
\end{equation}

Hence, for the case when the mass of the scalar field is very light, the scalar field becomes a long range one, similar to original BD scalar. Thus, the advance of perihelion term of light BDV theory of a point mass becomes exactly the same as the result of BD theory given in \cite{20}. For both heavy or light BDV theories, the effect of the minimum of the potential has the same $\omega$ dependent factor, which is slightly different from the factor of the result of BD $\Lambda$ theory given in equation \eqref{66}. For $f(R)$ theory with very heavy light mass, the perihelion shift due to the mass of the source becomes one half of corresponding $GR$ value and the corresponding term due to the minimum of the potential is $1/3$ of the similar term due to cosmological constant for the GR $\Lambda$ solution. Hence a very light scalar field cannot be compatible with solar system tests but one can circumvent this result with some ideas such as chameleon mechanism \cite{16, 80, 81}. We will discuss some
other observational consequences for both light or heavy scalar field mass of the BDV theory in the following subsection.

5.4. Observability

We have obtained advance of perihelion for both $BD\Delta$ theory given in (66) and for heavy or light $BDV$ theories given in equations (71) and (73), due to mass of the source and cosmological constant or minimum of potential, respectively. These expressions have a similar structure to the corresponding GR ones given in (67) and (68). As we have discussed before, the difference is the different numerical factors involving $\omega$ multiplying these terms. The multiplicative factors due to mass are the PPN parameters $\gamma$ of these theories. For $BD\Delta$ theory and light $BDV$ theory these parameters are the same as in the original $BD$ theory as given in (38). For these cases, the results agree with corresponding GR results for large $\omega$ since for these cases we have $\omega > 40,000$. For a very massive scalar mass case, however, this term is independent of $\omega$ and equal to GR value, 1. Hence for heavy $BDV$ theory, solar system tests will be satisfied for any value of $\omega$ except $\omega = -3/2$.

When we take into account the effects due to the cosmological constant or minimum of the potential, we see that these terms have a similar structure to the corresponding $GR\Theta$ theory. The differences are the existence of multiplicative terms involving $\omega$. The behaviour of these factors can be seen from graph figure 1. As it is clear from this graph, for positive values of $\omega$, these numerical factors are in the intervals $[2/3, 1)$ for $BD\Delta$ theory and $[1/3, 1)$ for $BDV$ theories for $0 \leq \omega < \infty$. Therefore, in these intervals, the correction factors cannot make significant order of magnitude changes to these terms. For negative values of $\omega$, these factors may have significant effects, as seen from the graph. These factors even vanish at $\omega = -1$ for $BD\Delta$ and $\omega = -1/2$ for $BDV$ theories or take negative values for $-2/3 < \omega < -1$ for $BD\Delta$ and $-2/3 < \omega < -1/2$ for $BDV$ theories. These factors take positive values for both theories for $\omega < -3/2$. They blow up as seen from graph as $\omega \to -3/2$. Therefore, for small and negative values of $\omega$, the deviation from GR can be observed for these theories, in principle.

In summary, the result of the Cassini experiment sets a lower bound for BD parameter $\omega$ as $\omega > 40,000$ for BD theory and this behavior is also valid for $BD\Delta$ and light $BDV$ theories. Hence, for both $BD\Delta$ and light $BDV$ theories, the multiplying factors of $\omega$ for the terms contributing to the perihelion precession due to the mass of the source and cosmological or minimum potential terms approaches to one as $\omega$ approaches to 40,000, as can be seen from figure 1. Thus, these terms cannot have an effect on the advance of perihelion. It is also known that, at solar system scales, the effect of a cosmological constant is too small to be observable [77, 79]. Conversely, one can put lower bounds on $\Lambda$ or $\Lambda_0$ using the results of [77, 79], which is, $\Lambda \leq 10^{-41}m^{-2}$ or the same limit for $\Lambda_0$. For heavy $BDV$ theory, however, the local behaviour does not fix $\omega$ and in principle this term can take any value. Most significant effects of the multiplicative factor is at the negative values of $\omega$. Namely, for negative values of $\omega$ there are regions where the factor $(2\omega + 1)/(2\omega + 3)$ becomes zero, negative, or takes unbounded negative or positive values as $\omega \to -3/2$ from left or right. For example, as $\omega \to -3/2$ a very small minimum potential term $\Lambda_0$, smaller than current observed value of $\Lambda$, can be compatible with observations. Or conversely, if $\omega \to 1/2$, as this factor approaches to zero, a very large minimum potential compared to observed cosmological constant, may be compatible with observations on perihelion precession. As a result, there may be an observational window to test heavy $BDV$ theories with solar system tests if for example the contribution of advance of perihelion due to cosmological constant can be measured with enough sensitivity in the future observations.
6. Deflection of light rays

In this part we will discuss the deflection of light rays for both BDΛ and BDV theories using the geodesic equations derived in section 4. The effect of the cosmological constant on the deflection angle was a topic with opposing views with works confirming [82–97] or denying [98–104] this effect. Therefore, here we first give a short summary of this topic in the discussion below. Then, we will focus on the effects for the theories we are considering.

6.1. Calculation of deflection angle for GRΛ theory using the Rindler–Ishak method

Here we review deflection of light rays from a compact object in GRΛ theory in the linear approximation. This requires geodesics of photons in the corresponding space-time. We will again use Schwarzschild type coordinates (47) for this discussion as well, hence we can use the orbit equation (58) for photons $\varepsilon = 0$. Note that, whether the cosmological constant affects the light deflection angle has became a source of debate and a lot of work is devoted to clarify this issue using different techniques. The reason for this is the fact that, for GRΛ theory, the geodesic equation for photons (58) can be reduced to

$$\frac{d^2u}{d\Phi^2} + u = 3mu^2. \quad (74)$$

The fact that this equation is independent of cosmological constant term lead to the conclusion [98] that cosmological constant has no effect in the light deflection. This is because the solution of this equation, given by [78]

$$u(\Phi) \equiv u_{GR}(\Phi) = \frac{1}{r} = \frac{\sin \Phi}{R} + \frac{3m}{2R^2} \left( 1 + \frac{1}{3} \cos 2\Phi \right) \quad (75)$$

does not involve the cosmological constant explicitly, hence orbit is independent of $\Lambda$. Note that the relation between integration constant $R$ and the closest approach distance $r_0$, given by setting $\Phi = \pi/2$ in (75), is

$$\frac{1}{r_0} = \frac{1}{R} + \frac{m}{R^2}. \quad (76)$$
This means that in the orbit equation (75) we can replace $R$ with $r_0$ in the linearized approximation. Here $r_0$ is the solution of the equation $\frac{dr}{d\Phi} = 0$ and from (75) it is given by

$$\frac{1}{b^2} + \frac{\Lambda}{3} = \frac{1}{r_0^2} - \frac{2m}{r_0^3}. \quad (77)$$

This means that we can express integration constant $R$ in terms of either $r_0$ or $b$ and $\Lambda$ and the latter choice produce a $\Lambda$ dependence in the bending angle expressions. For an asymptotically flat spacetime, the solution (75) implies half bending angle for Schwarzschild spacetime, as $r \to \infty$,

$$\alpha_E = \frac{2m}{R} = \frac{2m}{b} \quad (78)$$

where the last equality is valid in the linearized order only. Note that the asymptote $r \to \infty$ is not valid for Schwarzschild–de Sitter space time because this space time is not asymptotically flat. One might attempt to obtain a $\Lambda$ dependence by using the relations (76) and (77). However, this was criticized in [90] and argued that despite this dependence, the orbit is not affected by $\Lambda$.

In a pioneering work proposed in [82], if one considers the measurement of angles which depends on both the local and global geometry of the space-time, the bending angle can be shown to be affected by the cosmological constant as well. It turns out that, this problem depends both on how to define and measure bending angle and also how to specify physical parameters such as impact parameter. However, this is not a generic conclusion and both depends exactly to the setup used to perform observation to measure, and also the initial conditions as well. Since the spacetime obtained is not asymptotically flat, the above measurements and definitions will be different from the Schwarzschild case and may lack a universal understanding. We refer the latest works for a more complete review of this topic [83, 90], and in the latter part of the paper we consider bending of light phenomena for BD $\Lambda$ and BDV theories. There are many different approaches to this problem but here we only consider Rindler–Ishak method presented in [82, 83] in this work. Now, let us review their method and results here.

Now, consider the case where both source and observer are static. The cosine of the angle between two coordinate directions $d$ and $\delta$ given in figure 2 is given by

$$\cos \psi = g_{0\delta} \delta d / \sqrt{g_{00} d^2 \sqrt{g_{\delta\delta} \delta d^2}}$$

where $g_{0\delta}$ is the two dimensional submanifold obtained by setting $t = \text{constant}, \theta = \pi/2$ from GR $\Lambda$ solution of the metric (51) given by

$$A = B^{-1} = 1 - \frac{2m}{r} - \frac{\Lambda r^2}{3}. \quad (79)$$

Also, here $d = (dr, d\Phi) = (\beta, 1)d\Phi$, with $\beta = dr/d\Phi$, is the direction of orbit of the photon whereas $\delta = (\delta r, 0)$ is the direction of coordinate line $\Phi = \text{constant}$. One can obtain $\beta$ from (75) as

$$\beta \equiv \frac{dr}{d\Phi} = \beta \sqrt{m \sin 2\Phi - \cos \Phi}. \quad (80)$$

Using this and GR expressions of metric (51), one finds $\cos \psi = |\beta| / \sqrt{\beta^2 + r^2/B}$ and from the relation $\tan \psi = \sqrt{\sec^2 \psi - 1}$ one finds

$$\tan \psi = \frac{r}{|\beta| \sqrt{B}}. \quad (81)$$
From this expression, one can find that the one-sided bending angle is $\alpha = \psi - \Phi$ and one can immediately calculate the bending angle for small $\Phi = \Phi_0 \ll 1$. For the angle $\Phi_0$ to occur, we need from (75)

$$\frac{1}{r} = \frac{\Phi_0}{R} + \frac{2m}{R^2},$$

and using this $r$ value, from (80) one finds

$$\beta = \frac{r^2}{R} \left( \frac{2m}{R} \Phi_0 - 1 \right) \approx -\frac{r^2}{R}. \quad (83)$$

Moreover, using the weak field GR value for $B$, whose exact form is given in equation (79), namely $B = 1 + \frac{2m}{r} + \frac{\Lambda r^2}{R}$, and from (81), one finds [83]

$$\alpha_{GR\Lambda} = \frac{2m}{R} - \frac{\Lambda R}{6} \approx \frac{2m}{R} - \frac{\Lambda R^3}{6(R \Phi_0 + 2m)}, \quad (84)$$

where $r$ is given by (82) and this expression reduces for $\Phi_0 = 0$ to the result given in [82] as

$$\alpha_{GR\Lambda} = \alpha_E - \alpha_\Lambda, \quad \alpha_E = \frac{2m}{R}, \quad \alpha_\Lambda = \frac{\Lambda R^3}{12m}. \quad (85)$$

Therefore, $\Lambda$ contributes to the deflection angle and this contribution has the opposite sign compared to the contribution due to mass of the source. Here, we have reviewed the deflection angle for a special case where the source and observer are static. For a more general treatment of calculation of this angle when source or observer may not be static, we refer to [90].

Having reviewed the Rindler–Ishak method for GR$\Lambda$ theory, let us now apply this method to both BD$\Lambda$ and BDV theories in the following.

### 6.1.1 Deflection of light rays in BD$\Lambda$ theory.

For BD$\Lambda$ theory, from (58), the orbit equation becomes

$$\frac{d^2 u}{d\phi^2} + u = \frac{2m}{(2\omega + 3)\phi_0 b^2} + \frac{6m(\omega + 1)u^2}{(2\omega + 3)\phi_0} - \frac{\Lambda}{3(2\omega + 3)b^2 u^4}, \quad (86)$$

where here $b = LE$ is a constant of motion. The most important observation of this equation is as follows. Unlike in the GR case [98] where the corresponding equation, i.e. equation (75), is
\[ u(\Phi) = \frac{\sin(\Phi)}{R} + \frac{2m}{R^2(2\omega + 3)} \left\{ 1 + \frac{\omega + 1}{2} (3 + \cos 2\Phi) \right\} - \frac{\Lambda R \cos^2 \Phi}{6(2\omega + 3)} \sin \Phi. \]  

In comparison with the GR solution (75), the solution (A.6) involves \( \Lambda \) explicitly. In the GR limit \( \omega \to \infty \), this \( \Lambda \) dependent term vanishes and the solution (87) reduces to (75) in GR limit. Hence, unlike GR\( \Lambda \) case, the orbit of the photons depends on \( \Lambda \) and this dependence vanishes in the GR limit. This implies that, in addition to the Einstein deflection angle multiplied by a multiplicative factor of \( [20] \), there should be an extra contribution involving \( \Lambda \). Hence, using the same method in [82] we will calculate deflection angle and for clarity we present calculation details in appendix, see equations in appendix (A.9)–(A.12). The result, at most in the linear order in \( m, \Phi_0 \) and \( \Lambda \), is

\[ \alpha_{\text{BDA}} = \frac{2m}{\phi_0 R} - \frac{\Lambda R^2}{6(2\omega + 3)\Phi_0} - \frac{\Lambda R^2}{6(2\omega + 3)} \left[ \frac{2\omega + 1}{R} \frac{r}{\Phi_0} + \frac{R}{\Phi_0^2} \right], \]  

where \( r \) is given in (A.10). The first term in (88) is due to mass, the second one is the effect of \( \Lambda \) on orbit and the last term is due to effect of metric on the measurement of angles. This result reduces to special cases such as GR\( \Lambda \) one (84) [82–84] in GR limit \( \omega \to \infty \) or BD deflection angle [20, 27] for \( \Lambda = 0 \). Also for large \( \omega \) values where the Cassini mission yields the gravitational deflection angle is indistinguishable from corresponding GR\( \Lambda \) expression.

6.1.2. Deflection of light rays in for BDV theory. For this case, from (58), corresponding differential equation becomes

\[ \frac{d^2 u}{d\Phi^2} + u = \frac{3mu^2}{\phi_0} - \frac{2\Lambda_0}{3(2\omega + 3)br^2u^{\frac{1}{3}}} + \frac{me^{-m/u}}{(2\omega + 3)L^2\phi_0} \left[ 2E^2 - 3L^2 u^2 + O(m_s, m^2_s) \right]. \]  

Due to the exponential term, this equation is complicated. However, this equation can be analyzed for very massive or light scalar cases as follows:

- For a very massive scalar, \( m_s \gg 1 \), the exponential term can be ignored and together with \( \phi_0 = 1 \) for this case, the equation (89) reduces to

\[ \frac{d^2 u}{d\Phi^2} + u = 3mu^2 - \frac{2\Lambda_0}{3(2\omega + 3)br^2u^{\frac{1}{3}}}. \]
which is exactly the same with GR case except for the last term in the equation. Hence the linearized solution would be a mixture of GR and BDΛ solutions given by

$$u(\Phi) = u_{GR}(\Phi) - \frac{\Lambda_0 R^3 \cos^2 \Phi}{3 b^3 (2\omega + 3) \sin \Phi},$$  \hspace{1cm} (91)

where $u_{GR}$ is solution of GR case given in (75). Thus, the minimum of potential, $V_0 = 2\Lambda_0$, enters in the orbit equation and will affect the light deflection. Repeating similar calculations, one finds that the bending angle in the linearized order becomes

$$\alpha_{BDmassive} = \alpha_E - \frac{\Lambda_0 R^2}{3(2\omega + 3)\Phi_0} - \frac{\Lambda_0 R^2}{3(2\omega + 3)} \left[ \frac{(2\omega - 1)r}{2R} + \frac{R}{\Phi_0^2 r} \right].$$  \hspace{1cm} (92)

Here $r$ is given by (A.10) with $\Lambda$ to be replaced by $2\Lambda_0$ together with $\phi_0 = 1$. Therefore, for a very massive scalar field, as it is well known, light deflection due to mass is exactly the same with GR, and the effect of minimum of the potential acting as a cosmological constant has a slightly different $\omega$ dependence compared with the result of BDΛ theory given in (88).

- For a very light scalar, $m_s \ll 1$, we can expand the exponential term in equation (89) and find the following equation:

$$\frac{d^2 u}{d\Phi^2} + u = \frac{2m}{(2\omega + 3)\phi_0 b^2} + \frac{6m(\omega + 1) u^2}{(2\omega + 3)\phi_0} - \frac{2\Lambda_0}{3(2\omega + 3)b^2 u^3} + O(m \times m_s).$$  \hspace{1cm} (93)

Here we see that the resulting differential equation resembles the same form with the equation (86) of the BDΛ case except $\Lambda$ is replaced by $2\Lambda_0$. Therefore its linearized solution (A.6) will have same form except $\Lambda$ to be replaced by $2\Lambda_0$. The effect of minimum of the potential acting as a cosmological constant defined by $V_0 = 2\Lambda_0\phi_0$ on the total deflection angle can be seen from the total deflection angle expression given by

$$\alpha_{BDV-lightsc.} = \frac{2m}{\phi_0 R} - \frac{\Lambda_0 R^2}{3(2\omega + 3)\Phi_0} - \frac{\Lambda_0 R^2}{3(2\omega + 3)} \left[ \frac{(2\omega - 1)r}{2R} + \frac{R}{\Phi_0^2 r} \right].$$  \hspace{1cm} (94)

Here $r$ is given by (A.10) with $\Lambda$ to be replaced by $2\Lambda_0$. In comparison with the result of BDΛ theory given in (88), there are some slight differences for corresponding results of both light scalar (92) and massive scalar (94) cases. These differences originate from small differences of the metric functions $A$ and $B$ for BDΛ and BDV theories. Due to observational results, the very light scalar mass case cannot deviate from corresponding GRΛ expression since $\omega > 40,000$ limit is also valid for this case. But for BDV theory with a very massive scalar field, there is no such limit on $\omega$ and the deflection angle due to minimum of potential can be very different from corresponding GRΛ expression. However, there is no observational data measuring the deflection angle due to the cosmological constant, yet. Hence, there is a possibility that one can limit the parameters of these theories or eliminate them if such an observation is made in the future.
7. Gravitational redshift

The spacetime described by (51) and (47), is stationary. Hence it admits a timelike Killing vector. In these coordinates, the ratio of the measured frequency $\nu$ of a light passing through different positions is given by

$$\frac{\nu_0}{\nu} = \sqrt{\frac{A(r)}{A(r_0)}}. \quad (95)$$

Reading metric function $A(r)$ from (47) and considering the fact we are working in the linear level, the equation (95) becomes

$$\frac{\nu_0}{\nu} = 1 + \frac{m}{\rho_0 r_0} - \frac{m}{\rho_0 r} - \frac{\Lambda}{6} \left( r^2 - r_0^2 \right) - \frac{\phi(r_0) - \phi(r)}{2 \phi_0}. \quad (96)$$

In the GR limit this expression reduces to the one given in [77]. The effects of the scalar field to the gravitational redshift is given by the last term. Let us evaluate these for BD$\Lambda$ and BD$V$ theories, separately.

7.1. Gravitational redshift for BD$\Lambda$ case

For this case, reading $\phi$ from (48), and $\phi_0$ from (37) we find

$$\frac{\nu_0}{\nu} = 1 + \frac{m}{\rho_0 r_0} - \frac{m}{\rho_0 r} - \frac{2 \omega + 2 \Lambda}{2 \omega + 3} \left( r^2 - r_0^2 \right), \quad (97)$$

hence the gravitational redshift due to mass is the same as in GR whereas there is a correction term involving BD parameter $\omega$ for the gravitational redshift due to a cosmological constant term. Since the result of the Cassini mission requires large $\omega$, this factor approaches to one and the expression becomes identical to GR one for BD$\Lambda$ theory.

7.2. Gravitational redshift for BD$V$ case

For this case, by considering (49) we have

$$\frac{\nu_0}{\nu} = 1 + \frac{m}{\rho_0 r_0} \left( 1 + \frac{e^{-m r_0}}{2 \omega + 3} \right) - \frac{m}{\rho_0 r} \left( 1 + \frac{e^{-m r}}{2 \omega + 3} \right) - \frac{2 \omega + 1}{2 \omega + 3} \frac{V_0}{12 \phi_0} \left( r^2 - r_0^2 \right). \quad (98)$$

Hence the gravitational redshift due to mass term is modified since each term is multiplied by a position dependent effective gravitational term. The term for the minimum of the potential has a similar contribution but the multiplicative factor involving the BD parameter is slightly different than the BD$\Lambda$ case. We can expand the terms involving mass terms for a very light or very heavy scalar field mass cases as follows.

- For a very heavy scalar, $m_s \to \infty$, since $e^{-m r} \to 0$ and $\phi_0 = 1$ we find that

$$\frac{\nu_0}{\nu} = 1 + \frac{m}{\rho_0 r_0} - \frac{m}{\rho_0 r} - \frac{2 \omega + 1}{2 \omega + 3} \frac{V_0}{12 \phi_0} \left( r^2 - r_0^2 \right). \quad (99)$$

Similar to the previous case, the gravitational redshift due to mass is the same as in GR and the term due to minimum of the potential gets a multiplicative factor, same as in the advance of perihelion for this theory. For BD$V$ theory with heavy scalar field mass, there is no lower limit for the value of $\omega$. Hence if we regard $V_0$ as a cosmo-
logical constant using the equation \( V_0 = 2\Lambda_0 \), depending on the value of \( \omega \), the redshift term due to \( V_0 \) can take any value in the interval \((-\infty, \infty)\) which can be seen from the behavior of multiplicative factor given in figure 1. Using the argument given in [77] which considers the result of the Gravity Probe-A experiment [105] one can put a bound \(|(2\omega + 1)\Lambda_0/(2\omega + 3)| \leq 10^{-28} \text{ m}^{-2} \) but this bound is much larger than the current value of the cosmological constant. If future experiments will reach enough sensitivity, then one can use this phenomena to restrict the parameter space \((\omega, \Lambda_0)\) of this theory.

- For a very light scalar, \( m_\nu \to 0 \), \( e^{-m_\nu r} \sim 1 - m_\nu r \), ignoring multiplication of \( m \) with \( m_s \) and using the value of \( \phi_0 \) given in equation (37) for this case, we find
  \[
  \frac{\nu_0}{\nu} = 1 + \frac{m}{r_0} - \frac{m}{r} - \frac{2\omega + 1}{2\omega + 3} \frac{V_0}{12\phi_0} \left( r^2 - r_0^2 \right). \tag{100}
  \]

Here again the gravitational redshift due to mass is the same with GR case [77], and the gravitational redshift due to the minimum of the potential contains a numerical factor involving \( \omega \) similar to heavy scalar field mass case. However, unlike from heavy case, for a very light scalar, this factor approaches to one since the Cassini mission requires \( \omega \geq 40.000 \) for light BDV theory. Hence, there is no deviation from GR \( \Lambda \) results for this case.

8. Galaxy dynamics

In GR it is well-known that the effects of the cosmological constant or dark energy on the solar system scales or galactic scales are too weak to be observable. However, when the scales comparable or bigger than 1 Mpc, its effects cannot be ignored anymore. Here, with the help of using the results we have obtained in the previous sections, we will discuss the effects of the cosmological constant or minimum of the potential of BD\( \Lambda \) and BDV theories on the local dynamics of the universe and whether the results agree with GR.

8.1. Galaxy rotation curves

To discuss these effects in the galactic scale, we can consider galaxy rotation curves. It was observed [106, 107] that the rotation curves of gas at the outer regions of galaxies show a nearly constant velocity up to several galactic luminous radii. To apply our results to this phenomena, now, first let us calculate the rotational velocity of stars around the center of a static, spherically symmetric galaxy. We can express radial geodesics equation on equatorial plane (54) for timelike particles as
  \[
  \dot{r}^2 + U(r) = 0, \tag{101}
  \]

where
  \[
  U(r) = \frac{1}{B} \left( 1 - \frac{E^2}{A} + \frac{L^2}{r^2} \right). \tag{102}
  \]

The conditions for the existence of stable circular orbits are:
  \[
  \dot{r} = 0 \quad (U(r) = 0), \quad U'(r) = 0, \quad U''(r) > 0. \tag{103}
  \]

Here \( ' \) denotes derivative with respect to \( r \). From the first two conditions with a little algebra one finds
\[ E^2 = \frac{2A^2}{2A - rA'}, \quad L^2 = \frac{r^2A'}{2A - rA'} \]  
(104)

Moreover, the second derivative of the potential becomes
\[ U'' = \frac{2}{rB} \left[ \frac{rA'' + A'(3 - 2rA')}{2A - rA'} \right] . \]  
(105)

The above conditions were already obtained in previous work, for example in [108]. A numerical investigation showed that the last condition in (103) is satisfied in the relevant values of \( r \). From the proper time expression
\[ d\tau^2 = -ds^2, \]  
considering the definition of four velocity
\[ U^\mu = \frac{dx^\mu}{d\tau} = (\dot{t}, \dot{r}, \dot{\theta} = 0, \dot{\phi}) \]  
we find that
\[ 1 = A(U^0)^2 \left( 1 - v^2 \right), \]  
(106)

where \( U^0 \) is the time component of the four velocity of the particle and \( v \) is the spatial velocity defined as
\[ v^2 = \frac{1}{A} \left[ B \left( \frac{dr}{dt} \right)^2 + r^2 \left( \frac{d\phi}{dt} \right)^2 \right] = (v')^2 + (v^\phi)^2, \]  
(107)

where \( v' \) and \( v^\phi \) are the components of the spatial velocity \( v \) which is observed in an orthonormal coordinate system. Its \( \phi \) component is given by
\[ \dot{v}^\phi = \frac{r}{\sqrt{A}} \Omega, \quad \Omega = \frac{d\phi}{dt}. \]  
(108)

From the first integrals of the geodesics equation and equation (104) we can calculate \( \Omega \) as
\[ \Omega = \frac{d\phi}{dt} = \frac{\dot{\phi}}{\dot{t}} = \frac{A L}{r^2 E} = \sqrt{\frac{A'}{2r}}. \]  
(109)

Using this value, we find the tangential velocity of a particle in a stable circular motion as follows
\[ (v^\phi)^2 = \frac{rA'}{2A}. \]  
(110)

In the linearized approximation, we find that
\[ (v^\phi)^2 = \frac{m}{\phi_0 r} - \frac{\Lambda r^2}{3} - \frac{r \varphi'}{2\phi_0}. \]  
(111)

Thus, the effects of the BD scalar field reveals itself in the last term as well as in the constant \( \phi_0 \) for this phenomena. Let us discuss this term for the theories we consider separately.

- For BDA theory we have
\[ (v^\phi)^2 = \frac{m}{r} - \frac{2\omega + 2\Lambda r^2}{2\omega + 3} \]  
(112)

In order that these expression can have somewhat constant behaviour, the sign of the last term after the minus sign must be negative, which is possible for the interval \(-3/2 < \omega < -1\) for positive \( \Lambda \), as can be seen in figure 1. This is however ruled by the result of the Cassini mission with the requirement that \( \omega > 40000 \). Hence, the cosmological constant term cannot explain the flat rotation curves of galaxies for BDA theory.
For BDV theory we find that

\[
(\phi^2)_{\text{BDV}}^{\Phi} = \frac{m}{\phi_0 r} \left[ 1 + \frac{e^{-m_\omega r}}{(2\omega + 3)} (m_\omega r + 1) \right] \frac{2\omega + 1}{2\omega + 3} \frac{V_0 r^2}{6\phi_0}.
\]  

(113)

We can again look for special cases for this expression. For a heavy scalar, exponential term vanishes and the first term in tangential velocity becomes similar to BDA case since \( \phi_0 = 1 \). For a very light scalar the term involving mass becomes again the same as in (112) and for the term containing \( V_0 \) we have to take \( \phi_0 \) as in (37). In all these cases we see that the numerical factor involving \( \omega \) does not change the order of magnitude of the term related to \( \Lambda \) or \( V_0 \), for positive \( \omega \). Hence, these terms cannot explain flat rotation curves of stars in a galaxy for positive \( \omega \). For negative values of \( \omega \), the factors containing \( \omega \) may be negative and there may be regions where nearly flat rotation curves possible in principle. However, similar to BDA theory, BDV theory with light scalar mass, solar system tests require large positive values of \( \omega \) and this possibility is ruled out. For a heavy scalar field mass, however, there is no restriction on \( \omega \) by solar system tests and for \(-3/2 < \omega < -1/2\), the coefficient of the last term of (112) after minus sign becomes negative for positive \( V_0 \), making this term an increasing function of \( r \). Hence, for this range, the minimum of potential can contribute to the flat rotational curves of galaxies. For the values of \( \omega \) outside this range, however, the minimum of the potential cannot contribute to flat rotation curves. In GR the flat rotation curves is explained by the existence of dark matter, usually modeled as a dust or perfect fluid surrounding the galactic core which interacts with other particles only via gravity. This behaviour can also be explained by the existence of exotic sources such as a global monopole behaving as a galactic dark matter [109, 110]. It might be interesting to consider a dust or perfect fluid source for BDA and BDV theories as a candidate of dark matter. We are currently working on this problem and we will present our results elsewhere.

For intermediate values of \( m_\omega \) where these approximations are not valid, the Yukawa type term in (113) may also explain the flat rotation curves. This is because Sanders showed in [111] that a Yukawa type phenomenical gravitational potential can explain the behavior of galaxy rotation curves. In that work the following expression for rotational velocity is obtained

\[
(\phi^2) = \frac{G_\infty m}{r} \left[ 1 + \alpha e^{-r_0 r} \left( \frac{r}{r_0} + 1 \right) \right] .
\]  

(114)

In this expression \( G_\infty \) is the gravitational constant measured at infinity, \( r_0 \) is a length scale of this potential and \( \alpha \) is a coupling constant of this Yukawa type term. Sanders showed that in the presence of a Yukawa type gravitational potential, for \(-0.95 < \alpha < -0.92\) there is a region where the general properties of extended galactic rotation curves are reproduced. Comparing our expression (113) with (114), we see that they are similar if we identify \( \alpha = (2\omega + 3)^{-1} r_0 = 1/m_\omega, G_\infty = 1/\phi_0 \). Then, the above limit on \( \alpha \) is equivalent to \(-2.04 < \omega < -2.02\). Hence, a generic BDV theory can explain the observed galaxy rotation curves without needing a dark matter if BD parameter \( \omega \) is in this interval. This result is also discussed in [112] for a generic \( f(R, \phi) \) gravity including BDV theory as a special case. The problem here is that the ranges of \( \omega \) where the observed galaxy rotation curves were reproduced are very restricted negative and unfavourable values of it. The contribution of minimum of the potential to rotational velocity is in the reducing sense since the multiplicative factor is positive for this value of \( \omega \). In summary for a very restricted and negative value of \( \omega \), rotational curves of galaxies can be explained by the mass of the scalar field of BD theory leading to a Yukawa type term.
8.2. Inter-galactic dynamics

Now let us turn our attention to inter-galactic scales. By using the radial geodesics equation \( \ddot{r} + \Gamma^{r}_{\mu
u} \dot{x}^\mu \dot{x}^\nu = 0 \), and the first integrals of motion given in (53), we find an equation describing the radial accelerations of galaxies towards each other as

\[
\frac{d^2 r}{dt^2} = -\frac{A'}{2B} = -\frac{m}{\phi_0 r^2} + \frac{\Lambda r}{3} + \frac{\varphi'}{2\phi_0},
\]

(115)

where \( r \) describes radial separation between two galaxies and \( m \) is the total mass of the galaxies. Here inner structures and relative rotations of galaxies are ignored, merely by treating them as two point particles. Now we evaluate this equation for the theories we are considering in this paper.

- For BDA theory, the acceleration equation (115) has the form

\[
\frac{d^2 r}{dt^2} = -\frac{m}{r^2} + \frac{2\omega + 2\Lambda r}{2\omega + \frac{3}{2}}
\]

(116)

and the only difference with respect to corresponding GR \( \Lambda \) expression [113] is the factor involving \( \omega \) in front of the cosmological constant. In order to better understand the effects of the mass and cosmological constant on the dynamics of the galaxies let us calculate the ratio of both terms in (118) and denote by \( q \), which is given by

\[
q_{BDA} = \frac{2\omega + 2\Lambda r}{2\omega + \frac{3}{2}} = \frac{2\omega + 2}{2\omega + \frac{3}{2}} q_{GR}\Lambda.
\]

(117)

where at the last step the corresponding GR \( \Lambda \) expression of \( q \), discussed in detail in [114] is identified. Hence, it is clear that the difference between corresponding equation of GR \( \Lambda \) theory is the factor involving \( \omega \) in front of the cosmological constant. In order to better understand the effects of the mass and cosmological constant on the dynamics of the galaxies let us calculate the ratio of both terms in (118) and denote by \( q \), which is given by

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q_{BDA} = \frac{2\omega + 2\Lambda r}{2\omega + \frac{3}{2}} = \frac{2\omega + 2}{2\omega + \frac{3}{2}} q_{GR}\Lambda.
\]

(117)

where at the last step the corresponding GR \( \Lambda \) expression of \( q \), discussed in detail in [114] is identified. Hence, it is clear that the difference between corresponding equation of GR \( \Lambda \) theory is the factor involving \( \omega \) in front of the cosmological constant. Let us now discuss the behavior of this factor. This factor has the range \((2/3, 1)\) for \( \omega > 0 \), hence it does not change the order of \( q \) in this range of \( \omega \). The behaviour of this factor is more complicated for negative values of \( \omega \), which can be seen at the graph figure 1. This factor even vanishes for \( \omega = -1 \) where the cosmological constant has no effect on galaxy dynamics. It even takes negative values in the range \(-3/2 \leq \omega \leq -1\), where in this range the effect of positive cosmological constant is attractive rather than repulsive. However, the value of \( \omega \) is fixed by solar system tests as \( \omega \approx 40,000 \) and the dramatic changes of the behaviour of \( \Lambda \) for negative values of it is ruled out. Therefore, we can have similar conclusions given in [114], namely if we take the value of \( \Lambda \) as its the recent observed value, then cosmological constant does not affect interplanetary and galactic scales and its effects becomes significant at the cluster scales for BDA theory. This is because for solar system \( q_{GR}\Lambda \sim 10^{-20} \), for galactic scale \( q_{GR}\Lambda \sim 10^{-4} \) but for cluster scale \( q_{GR}\Lambda \sim O(1) \) [114]. However, for the theories where this factor is not fixed by solar system tests, these extreme behaviors can still be possible.

- For BDV theory the acceleration equation (115) becomes

\[
\frac{d^2 r}{dt^2} = -\frac{m}{\phi_0 r^2} \left[ 1 + \frac{(1 + m_gr)e^{-m_gr}}{2\omega + 3} \right] + \frac{2\omega + 1}{2\omega + \frac{3}{2}} \frac{V_0 r}{6\phi_0},
\]

(118)

including a Yukawa like term in the expression. For the case where \( m_r \gg 1 \) we have \( e^{-m_gr} \rightarrow 0 \) and \( \phi_0 = 1 \), we have...
\[
\frac{d^2 r}{dt^2} = \frac{m}{r^2} + \frac{2\omega + 1}{2\omega + 3} \frac{V_0 r}{6}. \tag{119}
\]

Hence for this case the mass term is the same with the GR case and the term related to the minimum of the potential has the same factor involving \(\omega\) as in other cases of this theory. For a very light scalar, \(m_s \ll 1\), we can ignore terms involving \(m \times m_s\) and we find

\[
\frac{d^2 r}{dt^2} = -\frac{m}{r^2} + \frac{2\omega + 1}{2\omega + 3} \frac{V_0 r}{6}. \tag{120}
\]

Here again the term containing \(V_0\) has a numerical factor involving \(\omega\), different from both BD and massive BDV cases. For both theories the \(q\) factor becomes

\[
q_{\text{BDV}} = \frac{2\omega + 1}{2\omega + 3} \frac{V_0 r^3}{6m}. \tag{121}
\]

For a light scalar, the value of \(\omega\) is fixed by solar system tests as \(\omega > 40,000\) and the numerical factor involving \(\omega\) of (121) has no effect. Hence, similar to BDΛ theory, the effects of the cosmological constant becomes relevant at the cluster scales for BDV theory with very light scalar field mass. For very heavy scalar field mass case, however, solar system tests do not fix the value of \(\omega\) and the numerical factor may become important for small or negative values of it. This fact may have two consequences for BDV theory with heavy scalar field mass: 1) Since \(\omega\) is not fixed, significant deviations from the results of GRΛ theory can be possible to observe in principle for negative values of \(\omega\), since the behaviour of the factor for negative values of \(\omega\) may be quite large as seen in the figure 1. 2) Phenomena at ranges larger then solar system scale may help to limit BD parameter \(\omega\) for this theory compared to GRΛ theory if independent measurements determine the value of \(V_0\) and \(m\) in (121).

9. Conclusions

In this paper, we have discussed weak field equations of the Brans–Dicke scalar–tensor theory extended by the presence of either a cosmological constant term coupled linearly to the scalar field or a generic potential in the Jordan frame. The linearized field equations of both cases are obtained in the gauge choice which makes the scalar field terms decouple from the metric field equations. The most important differences of both theories is that the former leads to a massless scalar field with a source whereas the latter has a massive scalar field where mass term is proportional to second derivative of the potential in the Taylor expansion as usual. To our knowledge, the linearized expansion of the former case is not present in the literature.

In the second part of the paper, we have considered the weak field solutions for a massive point particle for both theories in the linear approximation. The solutions have been first obtained in the gauge employed and then transformed to some physically relevant coordinates such as isotropic and Schwarzschild type coordinates. As a physical application, particle motion of test particles has been investigated with the focus on the solar system effects such as the advance of perihelion, deflection of light rays and gravitational redshift. The effect of mass of the source, cosmological term or minimum of the potential on these phenomena were derived in the linear order. The effect of the mass of the scalar field is also determined for BDV theory, which contains Yukawa like terms, but analytic solutions were derived only for very light or very massive scalar field. The effects of the terms responsible for asymptotical
nonflatness, namely $\Lambda$ or $V_0$ are similar to the cosmological constant in $\Lambda$GR theory except some factors involving BD parameter $\omega$, which are different for both theories. This might imply a new observational window in the future, for example to limit $\omega$ for $\Lambda$BD or BDV theories. However, the Casini mission limits the BD parameter to $\omega > 40,000$ [64] for original BD theory, and this limit is also valid for $\Lambda$BD theory and BDV theory with very light scalar. Hence, we conclude that the effects of the cosmological constant or the minimum of the potential are indistinguishable for these theories. For BDV theory with a very heavy mass, however, since the scalar field has a very short range and freezes out outside this range, the effect of mass of the source to this phenomena becomes identical to corresponding GR one. Hence solar system test are satisfied irrespective of the value of $\omega$. Therefore, the correction to these phenomena due to the minimum of potential has a factor involving $\omega$, whose value can take much larger and smaller values then $O(1)$ as given in figure 1. Hence, for BDV theory with a very heavy mass, the effect of minimum of potential may be different from corresponding GR one even if one uses the same observed value of the cosmological constant for the minimum of the potential. This fact may even lead to put some bounds on $\omega$ for very massive $BDV$ theory if these phenomena will be measured with enough sensitivity in the future.

The latter part of this work is devoted to galactic and intergalactic dynamics of these theories. For the galaxy scale we have calculated rotational velocity of stars in a galaxy and see that the nearly flat region of the galaxy rotation curve cannot be explained by the cosmological constant of $\Lambda$BD theory as well as the minimum of potential for BDV theory with light scalar field mass. Moreover, the effects of mass and the cosmological constant or minimum of potential becomes indistinguishable from corresponding GR ones since the factors involving $\omega$ becomes equal to one for the observed limit of $\omega$. For a very heavy scalar field mass, however, since there is no limit on $\omega$ due to solar system tests, there is a range of $\omega$ where the correction factor becomes negative. Hence the contribution of the minimum of potential becomes an increasing function of $r$, which may contribute to the flat rotation curves for $-3/2 < \omega < -1/2$. Outside this range the minimum of the potential cannot contribute to such behavior for BDV theory with a very heavy scalar field mass. For generic values of the mass of the scalar field, the flat rotation curves can also be explained by the effect of the mass of the scalar field for a very limited negative range of $\omega$. This is because the mass of the scalar field introduces a Yukawa like term in rotation velocity expression and this term can produce such a behaviour if $-2.04 < \omega < -2.02$. For the intergalactic scale, we have generalized the $\Lambda$GR expression corresponding to the acceleration of two galaxies towards each other where we have treated galaxies as point particles. We have obtained a factor $q$ which can determine the scale where the contribution of the cosmological constant starts to become relevant when this factor becomes of the order of unity. Similar to other phenomena we have discussed, this factor becomes indistinguishable for $\Lambda$BD or BDV theory with a very light scalar mass from corresponding $\Lambda$GR case, due to the large value the solar system tests sets on the BD parameter $\omega$. For BDV theory with heavy scalar mass, the scale where the factor $q$ becomes at the order of unity can be very different than corresponding $\Lambda$GR theory even if we use the minimum of potential equal to the observed value of the cosmological constant. Hence, these phenomena may lead to test the BDV theory with a very heavy scalar field mass or to limit the range the parameter $\omega$ compatible with observations if in the future there will be observations with enough sensitivity to determine the other parameters of the theory.

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Appendix. Deflection of light rays in BD\(\Lambda\) theory

Here let us find a solution to the orbit equation (86) in the linearized level. In order to find the effect of the point mass and cosmological constant on a light ray coming from the very far region of spacetime, we consider a perturbative approach, and consider the following ansatz

\[ u(\Phi) = u_0(\Phi) + mu_1(\Phi) + \Lambda u_2(\Phi). \]  
\[(A.1)\]

Replacing the (A.1) into equation (86) we obtain following set of equations in the zeroth order, orders linear on \(m\) and \(\Lambda\) as

\[ u''_0 + u_0 = 0, \]  
\[(A.2)\]

\[ u''_1 + u_1 = \frac{2}{b^2(2\omega + 3)\phi_0} + \frac{6(\omega + 1)u^2_0}{(2\omega + 3)\phi_0}. \]  
\[(A.3)\]

\[ u''_2 + u_2 = -\frac{1}{3b^2(2\omega + 3)u^2_0}. \]  
\[(A.4)\]

The solution of the first equation yields a photon following a straight line with

\[ u_0(\Phi) = \frac{\sin(\Phi)}{R}. \]  
\[(A.5)\]

Replacing this into remaining equations, one obtains the solution

\[ u(\Phi) = \frac{1}{r(\Phi)} = \frac{\sin(\Phi)}{R} + m \left\{ \frac{2}{b^2(2\omega + 3)\phi_0} + \frac{(\omega + 1)[3 + \cos(2\Phi)]}{(2\omega + 3)\phi_0R^2} \right\} - \frac{\Lambda R^3 \cos^2 \Phi}{6b^2(2\omega + 3)\sin \Phi}. \]  
\[(A.6)\]

In comparison with the GR case, the solution (A.6) involves \(b\) and \(\Lambda\) and these parts vanish in the GR limit \(\omega \to \infty\). This solution implies that, in addition to the Einstein deflection angle multiplied by a multiplicative factor of \(\omega\), an extra contribution comes from the cosmological term. This extra deflection angle is due to the interaction of cosmological term \(\Lambda\) and the scalar field and vanishes for the GR limit. Hence, unlike GR case, the orbit of the photons depend on \(\Lambda\). Note that the relation between integration constant \(R\) and closest approach distance \(r_0\) can be found by setting \(\Phi = \pi/2\) in (A.6). This yields

\[ \frac{1}{r_0} = \frac{1}{R} + \frac{2m}{(2\omega + 3)\phi_0} \left[ \frac{1}{b^2} + \frac{(\omega + 1)}{R^2} \right]. \]  
\[(A.7)\]

The equation satisfied by closest approach distance \(dr/d\Phi = 0\) is given by

\[ \frac{1}{b^2} + \frac{(2\omega + 2)\Lambda}{3(2\omega + 3)} = \frac{1}{r_0^2} - \frac{2m(2\omega + 4)}{(2\omega + 3)\phi_0 r_0^3}. \]  
\[(A.8)\]

These relations show that in the linearized order in the orbit equation (A.6) we can take \(R = r_0 = b\) interchangeably. Thus the solution can be expressed in terms of only one of these constants, such as \(R\). Using this fact, the solution (A.6) simplifies to (87).

In order to calculate the deflection angle, we use the method developed in [82] by Rindler and Ishak. The deflection angle can be calculated from (81) where here

\[ \beta = \frac{dr}{d\Phi} = \frac{mr^2}{R^3} \frac{2(\omega + 1)}{(2\omega + 3)\phi_0} \sin 2\Phi - \frac{r^2}{R} \cos \Phi - \frac{\Lambda R^2}{6(2\omega + 3)} \cos \Phi \left( 1 + \frac{1}{\sin^2 \Phi} \right). \]  
\[(A.9)\]
Here, unlike [82], there is singularity in the solutions for $\Phi = 0$, so we measure the deflection angle at $\Phi = \Phi_0 \ll 1$ where deflection for mass is already achieved. We will use small angle approximations $\sin \Phi_0 \approx \Phi_0$, $\cos \Phi_0 \approx 1$. Then, from (A.6) we have

$$u = \frac{1}{r} \frac{\Phi_0}{R} + \frac{2m}{\phi_0 R^2} - \frac{\Lambda R}{6(2\omega + 3)\Phi_0}.$$  \hspace{1cm} (A.10)

The value of $\beta$ in (A.9) becomes

$$|\beta| = \frac{r^2}{B} \left[ 1 - \frac{2m(2\omega + 2)}{(2\omega + 3)\phi_0 R} \frac{\Lambda R^2}{6(2\omega + 3)} \left( 1 + \frac{1}{\Phi_0^2} \right) \right] \left[ 1 - \frac{(2\omega + 2)m}{(2\omega + 3)\phi_0 R} - \frac{(2\omega + 1)\Lambda R^2}{6(2\omega + 3)} \right].$$  \hspace{1cm} (A.11)

Note that the angle $\Phi_0$ should be at the same order of magnitude as the other parameters in (A.6), namely we can choose $\Phi_0 \approx O(m/R)$.

Using these results and the metric function $B$ evaluated at the $r$ value (A.10), and $\beta$ given in (A.11), the expression (81) yields the following result, at most the linear order of the parameters $m, \Lambda$ and $\Phi_0$, as

$$\psi = \frac{r}{|\beta|^{1/B}} \frac{R}{r} \left[ 1 - \frac{2m(2\omega + 2)}{(2\omega + 3)\phi_0 R} \frac{\Lambda R^2}{6(2\omega + 3)} \left( 1 + \frac{1}{\Phi_0^2} \right) \right]^{-1} \left[ 1 - \frac{(2\omega + 2)m}{(2\omega + 3)\phi_0 R} - \frac{(2\omega + 1)\Lambda R^2}{6(2\omega + 3)} \right].$$  \hspace{1cm} (A.12)

In deriving this we have supposed that $\Lambda$ is much smaller than the other parameters $m$ and $\Phi_0$. This expression yields our result given in equation (88) for half deflection angle for BDA theory defined as $\alpha = \psi - \Phi_0$.

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32