Article

Conformal $\eta$-Ricci Solitons on Riemannian Submersions under Canonical Variation

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Abstract: This research article endeavors to discuss the attributes of Riemannian submersions under the canonical variation in terms of the conformal $\eta$-Ricci soliton and gradient conformal $\eta$-Ricci soliton with a potential vector field $\zeta$. Additionally, we estimate the various conditions for which the target manifold of Riemannian submersion under the canonical variation is a conformal $\eta$-Ricci soliton with a Killing vector field and a $\varphi(\text{Ric})$-vector field. Moreover, we deduce the generalized Liouville equation for Riemannian submersion under the canonical variation satisfying by a last multiplier $\Psi$ of the vertical potential vector field $\zeta$ and show that the base manifold of Riemannian submersion under canonical variation is an $\eta$-Einstein for gradient conformal $\eta$-Ricci soliton with a scalar concircular field $\gamma$ on base manifold. Finally, we illustrate an example of Riemannian submersions between Riemannian manifolds, which verify our results.

Keywords: conformal $\eta$-Ricci soliton; Riemannian submersion; Riemannian manifold; $\eta$-Einstein manifold

1. Introduction

Geometric flows analysis has become one of the most important geometrical techniques for explaining geometric structures in Riemannian geometry during the last two decades. In the study of singularities of flows as they occur as potential singularity models, a section of solutions in which the metric changes through dilations and diffeomorphisms plays an essential role. Solitons are a term used to describe these types of solutions. In 1988, Hamilton [1] proposed the notion of Ricci flow for the first time. The Ricci soliton appears as in the solution limit of the Ricci flow. Furthermore, in recent days, much emphasis has been paid to the classification of solutions that are self-similar to geometric flows. Fischer presented a novel geometric flow called conformal Ricci flow in [2], which is a modification of the standard Ricci flow equation that substitutes a scalar curvature constraint for the unit volume constraint. The conformal Ricci flow equations are extremely similar to the Navier–Stokes equations of fluid mechanics, and as a result of this analogy, the time dependent scalar field $p$ is referred to as a conformal pressure. The conformal pressure,
like the real physical pressure in fluid mechanics, supports as a Lagrange multiplier to conformally deform the metric flow in order to maintain the scalar curvature constraint. The conformal Ricci flow equations’ equilibrium points are metrics of the Einstein-type with the Einstein constant \(-\frac{1}{n}\). As a result, the conformal pressure \(p\) is zero at equilibrium and positive elsewhere.

Basu and Bhattacharyya [3] established the concept of the conformal Ricci soliton in 2015, using the equation as follows:

\[
S + \frac{1}{2} \mathcal{L}_\zeta d + \left[ \Lambda - \left( p + \frac{2}{n} \right) \right] d = 0. \tag{2}
\]

If the data \((d, \zeta, \Lambda - (p + \frac{2}{n}))\) satisfies Equation (14), then it is termed as conformal Ricci soliton [4] on \(M\). Here, \(\Lambda\) is a real constant and \(\mathcal{L}_\zeta\) is the Lie derivative operator along the vector field \(\zeta\). A conformal Ricci soliton (CRS) will be, respectively, shrinking, steady or expanding if

1. \(\Lambda < 0\),
2. \(\Lambda = 0\) and
3. \(\Lambda > 0\).

In 2018, Siddiqi [5] established a more general notion named conformal \(\eta\)-Ricci soliton (conformal \(\eta\)-RS), which is a generalization of Ricci soliton, conformal Ricci soliton, and \(\eta\)-Ricci soliton. The definition of conformal \(\eta\)-RS is given by

\[
S + \frac{1}{2} \mathcal{L}_\zeta g + \left[ \Lambda - \left( p + \frac{2}{n} \right) \right] g + \mu \eta \otimes \eta = 0, \tag{3}
\]

where \(\mathcal{L}_\zeta\) is indicates the Lie derivative with the direction of soliton vector field \(\zeta\), \(S\) is the Ricci tensor, \(n\) is the dimension of the manifold, \(p\) is the conformal pressure and \(\Lambda, \mu\) are real constant. Particularly, if \(\mu = 0\), then conformal \(\eta\)-Ricci soliton (conformal \(\eta\)-RS) reduces to the conformal Ricci soliton (CRS) [6].

Recently, A.N. Siddiqui and M.D. Siddiqi [7] presented the study based on the geometrical bearing of relativistic perfect fluid spacetime and GRW-spacetime in terms of almost Ricci–Bourguignon solitons with torse-forming vector fields.

On the other hand, from the inception of Riemannian geometry, the concept of Riemannian immersion has been thoroughly investigated. Indeed, the Riemannian manifolds that were first examined were surfaces embedded in \(\mathbb{R}^3\). Nash [8] demonstrated in 1956 that every Riemannian manifold may be isometrically immersed in any small surface of Euclidean space, which was a revolution for Riemannian manifolds. As a result, Riemannian immersions’ differential geometry is well understood.

Since the key research work of O’Neill 1966 and Gray 1967, where their fundamental equations are created in an attempt to dualize the theory of Riemannian immersions, Riemannian submersions have been a continual focus of study in differential geometry. The Hopf fibration is the most simple example.

Riemannian submersions have been intensively investigated not only in mathematics, but also in theoretical physics due to its usefulness in Kaluza Klein theory, super gravity, Yang–Mills theory, relativity, and super-string theories (see [9–13]). Singularity theory and submanifold theory are also crucially related to this subject and will be helpful for future research (for more details see [14–17]). The majority of Riemannian submersion investigations may be found in books [18,19]. In 2019, Meriç and Kılıç [20], initiated the study of Ricci solitons along Riemannian submersions. Moreover, other authors are also discussed submersion with various solitons for more details (see [21–26]). Therefore, in the present note we will determine the characteristics of conformal \(\eta\)-Ricci soliton along Riemannian submersions under canonical variation.
2. Riemannian Submersions

In this segment, the required foundation for Riemannian submersions is furnished by us. Let \((M_T, d)\) and \((N_B, d)\) be two Riemannian manifolds, and if \(\text{dim}(M_T, d) > \text{dim}(N_B, d)\). Then, a surjective map \(\psi : (M_T, d) \rightarrow (N_B, d)\) is said to be a Riemannian submersion [27] if it fulfill the following two axioms:

(A1): \(\text{Rank}(\psi) = \text{dim}(N_B)\). In this scenario, \(\psi^{-1}(s) = \psi^{-1}\) is a \(l\)-dimensional submanifold of \(M_T\) and is referred to as an \(\text{fiber}\) for each \(s \in N_B\), where \(k = \text{dim}(M) - \text{dim}(N)\).

If a vector field on \(M_T\) is always tangent (resp. orthogonal) to fibers, it is called \(\text{vertical}\)/(resp. \(\text{orthogonal}\)). If \(X^h\) is horizontal and \(\psi\)-related to a vector field \(X^h\) on \(N_B\), i.e., \(\psi_*(X^h_p) = X^h(\psi(p))\) for all \(p \in M_T\), \(\psi_*(X^h_p) = X^h(\psi(p))\) for all \(p \in M_T\), \(\psi_*(X^h_p) = X^h(\psi(p))\).

The projections on the vertical distribution \(\ker\psi_\ast\) and the horizontal distribution \(\ker\psi_\ast^\perp\) will be denoted by \(\mathcal{V}\) and \(\mathcal{H}\), respectively. The manifold \((M_T, d)\) is referred to as the \(\text{total manifold}\), whereas the manifold \((N_B, d)\) is referred to as the \(\text{base manifold}\) of the submersion \(\psi : (M_T, d) \rightarrow (N_B, d)\).

(A2): The lengths of the horizontal vectors are preserved by \(\psi_*\).

These criteria are analogous to saying that the \(\psi_*\) derivative map of \(\psi\), confined to \(\ker\psi_\ast^\perp\), is a linear isometry.

We have the following facts if \(X^h\) and \(Y^h\) are the basic vector fields, \(\psi\)-related to \(\hat{X}^h, \hat{Y}^h\):

1. \(d(X^h, Y^h) = \hat{d}(\hat{X}^h, \hat{Y}^h) \circ \psi\),
2. \(h[X^h, Y^h]\) is the basic vector field \(\psi\), which is connected to \([\hat{X}^h, \hat{Y}^h]\),
3. \(h(\nabla_{\hat{X}^h} Y^h)\) is the basic vector field \(\psi\)-connected to \(\nabla_{X^h} \hat{Y}^h\),

for any vertical vector field \([X^h, Y^h]\) is the vertical.

O’Neill’s [27] tensors \(T\) and \(A\), which are defined as follows, characterize the geometry of Riemannian submersions:

\[
T_{E^h} F^o = \mathcal{V} \nabla_{E^h} \mathcal{H} F^o + \mathcal{H} \nabla_{E^h} \mathcal{V} F^o, \quad (4)
\]

\[
A_{E^h} F^o = \mathcal{V} \nabla_{H^h} \mathcal{H} F^o + \mathcal{H} \nabla_{E^h} \mathcal{V} F^o, \quad (5)
\]

where \(\nabla\) is the Levi-Civita connection of \(d\) for any vector fields \(E^o\) and \(F^o\) on \(M\). The skew-symmetric operators \(TE^o\) and \(AE^o\) on the tangent bundle of \(M_T\) inverting the vertical and horizontal distributions are obvious. The characteristics of the tensor fields \(T\) and \(A\) are outlined. On \(M_T\), if \(E^o, F^o\) are vertical vector fields and \(X^h, Y^h\) are horizontal vector fields, we possess

\[
T_{E^h} F^o = T_{E^h} V^o, \quad (6)
\]

\[
A_{X^h} Y^h - A_{Y^h} X^h = \frac{1}{2} \mathcal{V}[X^h, Y^h]. \quad (7)
\]

On the other way, we turn up the following equations in view of Equations (5) and (4).

\[
\nabla_{E^h} F^o = T_{E^h} F^o + \hat{V}_{E^h} F^o, \quad (8)
\]

\[
\nabla_{E^h} X^h = T_{E^h} X^h + \mathcal{H} \nabla_{E^h} X^h, \quad (9)
\]

\[
\nabla_{X^h} E^o = A_{X^h} E^o + \mathcal{V} \nabla_{X^h} E^o, \quad (10)
\]

\[
\nabla_{X^h} Y^h = \mathcal{H} \nabla_{X^h} Y^h + A_{X^h} Y^h, \quad (11)
\]

where

\[
\hat{\nabla}_{E^h} F^o = \hat{V} \nabla_{E^h} F^o.
\]
Additionally, if $X^h$ is basic vector, we obtain
\[ \mathcal{H}\nabla_{E^h}X^h = A_{X^h}E^h. \]

Moreover, we have a useful lemma:

**Lemma 1.** [19] For $E, F, G \in (M)$ we have
\[ d_M(T_E F, G) + d_M(T_E G, F) = 0, \quad d_M(A_E F, G) + d_M(A_E G, F) = 0 \] (12)

It is easy to see that $T$ works on the fibers as the second fundamental form, but $A$ operates on the horizontal distribution and estimates the obstacle to its integrability. We refer to O’Neill’s work [27] and the books [18,19] for further information on Riemannian submersions.

### 3. Curvatures Axioms

This section deals with some useful curvature properties along Riemannian submersion:

**Proposition 1.** If $\psi : (M_T, d) \to (N_B, d_{N_B})$ be a Riemannian submersion admits the Riemannian curvature tensors of total manifold $(M_T, g)$, base manifold $(N_B, \tilde{g})$ and any fiber of $\psi$ denoting by $R_m, \tilde{R}_m$ and $\check{R}_m$, respectively. Then, we have
\[ R_m(E^o, F^o, G^o, H^o) = \tilde{R}_m(E^o, F^o, \tilde{G}^o, H^o) - d(T_{E^o} H^o, T_{F^o} G^o) + d(T_{F^o} H^o, T_{E^o} G^o), \] (13)
\[ R_m(X^h, Y^h, Z^h, W^h) = \check{R}_m(X^h, Y^h, Z^h, W^h) + 2d(A_{X^h}Y^h, A_{Z^h}W^h) - d(A_{Y^h}Z^h, A_{X^h}W^h) + d(A_{X^h}Z^h, A_{Y^h}W^h), \] (14)
for any $X^h, Y^h, Z^h, W^h \in \Gamma(H(M_T))$ and $E^o, F^o, G^o, H^o \in \Gamma(V(M_T))$.

On the other side, for any fiber of Riemannian submersion $\psi$, the mean curvature of horizontal vector field $H$ is provided by $rH = N$, such that
\[ N = \sum_{j=1}^{r} T_{E^j}E^j. \] (15)

Additionally, the dimension of every $\psi$ fiber is indicated by $r$, and the orthonormal basis on vertical distribution is $\{ E^1, E^2, \ldots, E^n \}$. The horizontal vector field $N$ eliminates if and only if any Riemannian submersion fiber $\psi$ is minimum, as shown.

Now, from Equation (15), we find
\[ d(\nabla_U N, X^h) = \sum_{j=1}^{r} d(\nabla_U T(E^j), X^h) \] (16)
for any $U \in \Gamma(TM_T)$ and $X^h \in \Gamma(H(M_T))$ and $Div(X^h)$ is the horizontal divergence of any vector field $X^h$ on $\Gamma(H(M_T))$, denoted by $Div(X^h)$ and given by
\[ Div(X^h) = \sum_{j=1}^{n} d(\nabla_{X^j} X^h, X^i), \] (17)
where $\{ X^1, X^2, \ldots, X^n \}$ is an orthonormal frame of horizontal space $\Gamma(H(M_T))$. Thus, considering Equation (17), we have
\[ Div(N) = \sum_{i=1}^{n} \sum_{j=1}^{r} d(\nabla_{X^j} T(E^i), E^j, X^h). \] (18)
4. Riemannian Submersion under a Canonical Variation

This section begins with the following specifications. If \( \psi : (M_T, d) \rightarrow (N_B, \tilde{d}) \) is a Riemannian submersion with totally geodesic fibers. Then, we have

**Definition 2.** ([28] pp. 191) For each positive number \( t \), let \( d_t \) be the unique Riemannian metric on \( M \) such that

(i) \( d_t(X^h, Y^h) = d(X^h, Y^h) \) for \( X^h, Y^h \in \Gamma H_p(M), p \in M_T \),
(ii) the subspaces \( H_p \) and \( V_p \) are orthogonal to each other with respect to \( g_t \) at each point \( p \) in \( M \), and
(iii) \( d_t(X^h, Y^h) = t^2 d(X^h, Y^h) \) for \( X^h, Y^h \in V_p, p \in M_T \).

Then, \( \psi : (M_T, d) \rightarrow (N_B, \tilde{d}) \) be a Riemannian submersion with totally geodesic fibers, which is called the canonical variation. For each \( t > 0 \), \( \{v_1, \cdots, t^{-1}v_{n+1}, \cdots, t^{-1}v_m\} \) is an orthonormal local frame field on \( (M_T, d) \) with \( v_i \) the horizontal lift of \( v_i \) with respect to \( d_t \) for \( 1 \leq i \leq n \), and with \( t^{-1}v_i \) vertical for \( n + 1 \leq i \leq m \). Then, the vertical (resp. horizontal) Jacobi operator \( T^v_i \) (resp. \( T^h_i \)) of the canonical variation \( \psi : (M_T, d) \rightarrow (N_B, \tilde{d}) \) satisfies [28]

\[
\hat{T}^v_i = t^{-2} T^v_i, \quad \text{and} \quad \hat{T}^h_i = T^h_i.
\] (19)

Any metric under the canonical variation makes \( \psi \) a Riemannian submersion with same horizontal distribution \( H \). The invariants of \( \psi \) with respect to \( g_t \) are denoted by \( A^i, T^i \), as well as \( \nabla^v \) stands for the Levi-Civita connection of \( (M, g) \). Therefore, after a simple computation, one obtains

\[
\nabla^v(\nabla^v E^o) = \nabla^v(\nabla^v F^o), \quad \nabla^v(\nabla^h E^o) = t \nabla^v(\nabla^h F^o),
\]

\[
\nabla^v X^h = \nabla^v X^h, \quad \nabla^h Y^h = \nabla^h Y^h,
\] (20)

\( E^o, F^o \in \nabla(M_T) \) and \( X^h, Y^h \in H(M_T) \). Thus, combining Equations (6) and (7), one has

\[
\nabla^v E^o = t \nabla^v F^o, \quad \nabla^h X^h = \nabla^h Y^h,
\]

\[
A^v X^h F^o = t A^h X^h F^o, \quad A^h X^h E^o = A^v X^h F^o.
\] (21)

Now, let the local \( d_t \)-orthonormal vertical frame \( \{t^{-\frac{1}{2}}E^o_i\}_{1 \leq i \leq r}, \{E^o_i\}_{1 \leq i \leq r} \) as a \( d \)-orthonormal one, the first equation in Equation (21) implies

\[
N = \sum_t \nabla^v E^o_i = N^t.
\]

As a result, any fiber’s mean curvature vector field is independent of \( t \), which refers to a process lemma.

**Lemma 2.** [28] The Riemannian submersion \( \psi : (M_T, d) \rightarrow (N_B, \tilde{d}) \) has minimal fibers if and only if \( \psi \) has minimal fibers for \( t \). Moreover, the fibers of \( \psi : (M_T, d) \rightarrow (N_B, \tilde{d}) \) are totally geodesic if and only if for any \( t \), the fibers of \( \psi : (M_T, d) \rightarrow (N_B, \tilde{d}) \) are totally geodesic.

Now, in light of Equations (13), (14) and (21), Lemma 2, we have

**Theorem 3.** Let \( \psi : (M_T, d) \rightarrow (N_B, \tilde{d}) \) be a Riemannian submersion with totally geodesic fibers. For any \( X^h, Y^h \in \chi^h(M) \), \( \psi \)-related to \( X^h, Y^h \) and \( E^o, F^o \in \chi^v(M) \), the Ricci tensor \( S_t \) of the metric \( d_t \) under the canonical variation of \( d \) fulfills

\[
S_t(E^o, F^o) = \hat{S}(E^o, F^o) + t^2 \sum_{i=1}^{n} d(A^v X^h_i E^o, A^h X^h_i F^o),
\] (22)
(iii) The fundamental tensor field $\mathbb{T}$
(ii) The vertical distribution $\mathcal{V}$
(i) The horizontal distribution $\mathcal{H}$

(iii) Theorem 5. Let $\mathbb{T}, \mathcal{V}, \mathcal{H}$ be a Riemannian submersion under the canonical variation. Then, the following are equivalent to each other:

(i) The horizontal distribution $\mathcal{H}$ is parallel;

(ii) The vertical distribution $\mathcal{V}$ is parallel;

(iii) The fundamental tensor field $\mathbb{T}$ and $\mathcal{A}$ vanish identically for any $X^h, Y^h \in \Gamma H(M_T)$ and $E^v, F^v \in \Gamma V(M_T)$.

Proof. Lemma 1, Equations (8) and (11) imply (i). Next, the following formulas are proved in [19]

\[
\begin{align*}
(\nabla_{X^h} \mathbb{T})_{Y^h} &= -\mathcal{A}_{X^h} Y^h, \\
(\nabla_{E^v} \mathcal{A})_{F^v} &= -\mathcal{A}_{E^v} F^v, \\
(\nabla_{X^h} \mathbb{T})_{E^v} &= -\mathcal{A}_{X^h} E^v.
\end{align*}
\]

(25) (26)

Indeed, for any $X^h, Y^h \in \Gamma H(M_T)$ and $E^v, F^v \in \Gamma V(M_T)$. Now, in light of Equations (6), (7), (25), (26), and Lemma 1, we turn up

\[
d_M(\nabla_{X^h} \mathcal{A})_{E^v} X^h = d_M(\mathbb{A}_{X^h}(\mathcal{A}_{X^h} E^v), E^v) = -d_M(\mathcal{A}_{X^h} E^v, \mathbb{A}_{X^h} E^v).
\]

Hence, if $\mathcal{A}$ is parallel, $\mathcal{A}$ vanishes on the vertical distribution and Lemma 1 also implies $\mathbb{A}_{X^h} = 0$. Then, $\mathcal{A}$ vanishes, since it is a horizontal tensor field. There is similar proof for $\mathbb{T}$, so we omit it.

5. Conformal $\eta$-RS along Riemannian Submersions

This section will focus to the investigation of conformal $\eta$-RS along Riemannian submersion $\psi : (M_T, \tilde{d}) \longrightarrow (N_B, \tilde{d})$ from Riemannian manifolds and discussed the nature of fiber of such submersion with target manifold $(N_B, \tilde{d})$.

As a consequences of Equations (8), (11), (20) and (21) in Riemannian submersion under the canonical variation, we obtain the following characteristic of $\mathcal{A}^1$ and $\mathcal{T}^1$.

Theorem 4. Let $\psi : (M_T, \tilde{d}) \longrightarrow (N_B, \tilde{d})$ be a Riemannian submersion under the canonical variation. Then, the following are equivalent to each other:

(i) The horizontal distribution $\mathcal{H}$ is parallel;

(ii) The vertical distribution $\mathcal{V}$ is parallel;

(iii) The fundamental tensor field $\mathbb{T}$ and $\mathcal{A}$ vanish identically for any $X^h, Y^h \in \Gamma H(M_T)$ and $E^v, F^v \in \Gamma V(M_T)$.

Proof. Lemma 1, Equations (8) and (11) imply (i). Next, the following formulas are proved in [19]

\[
\begin{align*}
S_l(X^h, Y^h) &= S_i(X^h, Y^h) - 2 \sum_{i=1}^n d(A_{X^h} E^v, A_{X^h} F^v), \\
&S_i(E^v, X^h) = t \sum_{i=1}^n d((\nabla_{X^h} A)(X^h, E^v), E^v) = tS(E^v, X^h),
\end{align*}
\]

(23) (24)

where Ricci curvature tensors of total manifold $(M_T, g)$, base manifold $(N_B, \bar{g})$ and any fiber of $\psi$ denoting by $S, \tilde{S}$ and $\bar{S}$, respectively.

Proof. Let $\mathbb{T}, \mathcal{V}, \mathcal{H}$ be a Riemannian submersion under the canonical variation. Then, the following are equivalent to each other:

(i) The horizontal distribution $\mathcal{H}$ is parallel;

(ii) The vertical distribution $\mathcal{V}$ is parallel;

(iii) The fundamental tensor field $\mathbb{T}$ and $\mathcal{A}$ vanish identically for any $X^h, Y^h \in \Gamma H(M_T)$ and $E^v, F^v \in \Gamma V(M_T)$.

Proof. Let $\mathbb{T}, \mathcal{V}, \mathcal{H}$ be a Riemannian submersion under the canonical variation. Then, the following are equivalent to each other:

(i) The horizontal distribution $\mathcal{H}$ is parallel;

(ii) The vertical distribution $\mathcal{V}$ is parallel;

(iii) The fundamental tensor field $\mathbb{T}$ and $\mathcal{A}$ vanish identically for any $X^h, Y^h \in \Gamma H(M_T)$ and $E^v, F^v \in \Gamma V(M_T)$.

Proof. Let $\mathbb{T}, \mathcal{V}, \mathcal{H}$ be a Riemannian submersion under the canonical variation. Then, the following are equivalent to each other:

(i) The horizontal distribution $\mathcal{H}$ is parallel;

(ii) The vertical distribution $\mathcal{V}$ is parallel;

(iii) The fundamental tensor field $\mathbb{T}$ and $\mathcal{A}$ vanish identically for any $X^h, Y^h \in \Gamma H(M_T)$ and $E^v, F^v \in \Gamma V(M_T)$.
+ \left[ \Lambda - \left( \frac{1}{p} + \frac{1}{r} \right) \right] d(E^p, F^p) + 2\mu \eta(E^p)\eta(F^p) = 0,

wherein \{X^h_i\} indicates the orthonormal basis horizontal distribution \(\mathcal{H}\) and \(\nabla^i\) is the Levi-Civita connection on \(M\). The following equation is found employing Theorem 4, and Equations (5), (8), and (28):

\[
\frac{1}{2} \left[ d(\nabla^i_{E^p} \zeta, F^p) + d(\nabla^i_{E^p} \zeta, F^p) \right] + \mathcal{S}(E^p, F^p) + \left[ \Lambda - \left( \frac{1}{p} + \frac{1}{r} \right) \right] d(E^p, F^p) + \mu \eta(E^p)\eta(F^p) = 0,
\]

for any \(E^p, F^p \in \Gamma V(M)\), which means such a fiber of \(\psi\) is a conformal \(\eta\)-RS. \(\square\)

**Theorem 6.** Let \((M_T, d, \Lambda, \mu)\) be a conformal \(\eta\)-RS with a vertical potential field \(\zeta\) and \(\psi : (M_T, d) \rightarrow (N_B, \bar{d})\) be a Riemannian submersion under the canonical variation from Riemannian manifolds with totally geodesic fibers. If the horizontal distribution \(\mathcal{H}\) is integrable, then any fiber of Riemannian submersion \(\psi\) is a conformal \(\eta\)-RS.

**Proof.** Proof is similar as in Theorem 5 with the fact that Equations (6) and (7), and Lemma 1 entail that \(\mathcal{A}\) measures the integrability of horizontal distribution. Indeed, Equations (6) and (7), Lemma 1 and condition \(\mathcal{A}_E = 0\) for \(E^v \in \Gamma H(M_T)\) imply \(\mathcal{A} = 0\) if and only if \(\mathcal{H}\) is integrable. \(\square\)

Then, we turn up the following result:

**Theorem 7.** Let \((M_T, d, \Lambda, \mu)\) be a conformal \(\eta\)-RS with a potential field \(U \in \Gamma(TM_T)\) and \(\psi\) be a Riemannian submersion from Riemannian manifolds under the canonical variation. Then, the following conditions are fulfilled if the horizontal distribution \(\mathcal{H}\) is parallel:

1. If \(\zeta\) is a vertical vector field, then \((N_B, \bar{d})\) is an \(\eta\)-Einstein manifold.
2. If \(\zeta\) is a horizontal vector field, then \((N_B, \bar{d})\) is a conformal \(\eta\)-RS with potential vector field \(\zeta_N\), such that \(\psi_\ast \zeta = \zeta_N\).

**Proof.** Since the total space \((M_T, d)\) of Riemannian submersion \(\psi\) under the canonical variation admits an almost conformal \(\eta\)-RS with potential field \(U \in \Gamma(TM_T)\), then adopting Equations (3) and (23), we turn up

\[
\frac{1}{2} \left[ d(\nabla^i_{Y^h} X^i) + d(\nabla^i_{Y^h} X^i) \right] + \mathcal{S}(X^h, Y^h) - 2 \sum_{i=1}^n d(A_{X^h_i} E^p, A_{X^h_i} F^p) \tag{29}
\]

+ \left[ \Lambda - \left( \frac{1}{p} + \frac{1}{r} \right) \right] d(X^h, Y^h) + \mu \eta(X^h)\eta(Y^h) = 0,

where \(\tilde{X}^h\) and \(\tilde{Y}^h\) are related through \(\psi\) with \(X^h\) and \(Y^h\), respectively, for any \(X^h, Y^h \in \Gamma H(M_T)\).

Applying Theorem 4 in Equation (29), we acquire

\[
\frac{1}{2} \left[ d(\nabla^i_{X^h} U, Y^h) + d(\nabla^i_{Y^h} U, X^h) \right] + \mathcal{S}(X^h, Y^h) + \left[ \Lambda - \left( \frac{1}{p} + \frac{1}{r} \right) \right] d(X^h, Y^h) + \mu \eta(X^h)\eta(Y^h) = 0. \tag{30}
\]

1. If \(\zeta\) is a vertical vector field, then Equation (10) refers that,

\[
\frac{1}{2} \left[ d(A_{X^h} \zeta, Y^h) + d(A_{Y^h} \zeta, X^h) \right] + \mathcal{S}(X^h, Y^h) + \left[ \Lambda - \left( \frac{1}{p} + \frac{1}{r} \right) \right] d(X^h, Y^h) + \mu \eta(X^h)\eta(Y^h) = 0. \tag{31}
\]

Since \(\mathcal{H}\) is parallel, we turn up

\[
\mathcal{S}(X^h, Y^h) = \alpha d(X^h, Y^h) + \beta \eta(X^h)\eta(Y^h). \tag{32}
\]
which entails that \((N_B, \bar{d})\) is an \(\eta\)-Einstein, where \(a = -\left[\Lambda - \left(\frac{1}{p} + \frac{1}{r}\right)\right]\) and \(\beta = -\mu\).

2. Let \(\zeta\) be a horizontal vector field, then Equation (30) becomes
\[
\frac{1}{2} \langle \xi, d \rangle (X^h, Y^h) + \mathcal{S}(\bar{X}^h, \bar{Y}^h) + \left[\Lambda - \left(\frac{1}{p} + \frac{1}{r}\right)\right] \bar{d}(\bar{X}^h, \bar{Y}^h) + \mu \eta(\bar{X}^h) \eta(\bar{Y}^h) = 0.
\] (33)
which shows that the base manifold \((N_B, \bar{d})\) is an conformal \(\eta\)-RS with a horizontal potential field \(\bar{X}^h\).

Now, from Equation (33) and using the fact that \(\zeta\) is a horizontal vector field, then we turn up the following:

**Lemma 3.** If \((M_T, d, \zeta, \Lambda, \mu)\) is a conformal \(\eta\)-RS on Riemannian submersion \(\psi\) under the canonical variation from Riemannian manifolds with horizontal potential field \(\zeta\), such that \(\mathcal{H}\) is parallel. Then, the vector field \(\zeta\) on the horizontal distribution \(\mathcal{H}\) is Killing.

Since \((M_T, d, \zeta, \Lambda, \mu)\) is a conformal \(\eta\)-RS and again adopting Equation (23) in (3), we find that
\[
\frac{1}{2} \langle \xi, d \rangle (X^h, Y^h) + \mathcal{S}(\bar{X}^h, \bar{Y}^h) + \left[\Lambda - \left(\frac{1}{p} + \frac{1}{r}\right)\right] \bar{d}(\bar{X}^h, \bar{Y}^h) + \mu \eta(\bar{X}^h) \eta(\bar{Y}^h) = 0,
\] (34)
where \(\left\{X^h_i\right\}\) represents an orthonormal basis of \(\mathcal{H}\), for any \(X, Y \in \Gamma\mathcal{H}(M_T)\). In view of Theorem 4, Equation (34) becomes as
\[
\frac{1}{2} \langle \xi, d \rangle (X^h, Y^h) + \mathcal{S}(\bar{X}^h, \bar{Y}^h) + \left[\Lambda - \left(\frac{1}{p} + \frac{1}{r}\right)\right] \bar{d}(\bar{X}^h, \bar{Y}^h) + \mu \eta(\bar{X}^h) \eta(\bar{Y}^h) = 0.
\] (35)
since the base manifold \((N_B, \bar{d})\) is an \(\eta\)-Einstein, we can find the \(\zeta\) is Killing. Thus, we can articulate the following:

**Theorem 8.** If \((d, \zeta, \Lambda, \mu)\) is a conformal \(\eta\)-RS on Riemannian submersion \(\psi\) under the canonical variation from Riemannian manifold to an \(\eta\)-Einstein manifold with horizontal potential field \(\zeta\), such that horizontal distribution \(\mathcal{H}\) is parallel, then the vector field \(\zeta\) on horizontal distribution \(\mathcal{H}\) is Killing.

6. **Conformal \(\eta\)-RS on Riemannian Submersion under the Canonical Variation with \(\varphi(\text{Ric})\)-Vector Field**

In this segment, we determine conformal \(\eta\)-RS on Riemannian submersion under the canonical variation with \(\varphi(\text{Ric})\)-vector field. Thus, we entail the following definition.

**Definition 9.** A vector field \(\varphi\) on a Riemannian manifold \(M\) is said to be a \(\varphi(\text{Ric})\)-vector field if it satisfies \([29]\)
\[
\nabla_{E^i} \varphi = \omega \text{Ric} E^i
\] (36)
where \(\nabla\) is the Levi-Civita connection, \(\omega\) is a constant and \((\text{Ric})\) is the Ricci operator defined by \(\mathcal{S}(E^r, F^s) = d(\text{Ric} E^r, F^s)\). If \(\omega \neq 0\) and \(\omega = 0\) in Equation (36), then the vector field \(\varphi\) is said to be a proper \(\varphi(\text{Ric})\)-vector field and covariantly constant, respectively.

As a result, the definition of Lie derivative and Equation (36) leads to the following:
\[
(\mathcal{L}_\varphi d)(E^r, F^s) = 2\omega \mathcal{S}(E^r, F^s)
\] (37)
If the vertical potential field is a \( \varphi(\text{Ric}) \)-vector field, then in light of Equations (29) and (37), we turn up

\[
\mathcal{S}(E^p, F^p) = -\frac{1}{(\omega + 1)} \left[ \Lambda - \left( \frac{p}{2} + \frac{1}{r} \right) \right] d(E^p, F^p) - \frac{\mu}{(\omega + 1)} \eta(E^p) \eta(F^p) = 0 \tag{38}
\]

for any \( E^p, F^p \in \Gamma(V_M) \). Thus, we articulate the following results.

**Theorem 10.** Let \( \varphi : (\mathcal{M}, \varphi) \rightarrow (\mathcal{N}, \varphi) \) be a Riemannian submersion under the canonical variation, whose total manifold admitting a conformal \( \eta \)-RS \( (d, \zeta, \Lambda, \mu) \) such that vertical potential field is a proper \( \varphi(\text{Ric}) \)-vector field, provided \( \omega \neq -1 \) and the vertical distribution \( \mathcal{V} \) are parallel, then any fiber of Riemannian submersion \( \varphi \) is an \( \eta \)-Einstein.

Now, in view of Theorems 7 and 10, we easily turn up the next theorem.

**Theorem 11.** Let \( \varphi : (\mathcal{M}, \varphi) \rightarrow (\mathcal{N}, \varphi) \) be a Riemannian submersion under the canonical variation, whose total manifold admitting a conformal \( \eta \)-RS \( (d, \zeta, \Lambda, \mu) \) if a vertical potential field \( \zeta \) is a proper \( \varphi(\text{Ric}) \)-vector field, provided \( \omega \neq -1 \) and the vertical distribution \( \mathcal{V} \) are parallel, then \( (\mathcal{N}, d) \) is an \( \eta \)-Einstein.

**Corollary 12.** Let a Riemannian submersion \( \varphi : (\mathcal{M}, \varphi) \rightarrow (\mathcal{N}, \varphi) \) under the canonical variation, whose total manifold admitting a conformal \( \eta \)-RS \( (d, \zeta, \Lambda, \mu) \) if a vertical potential field \( \zeta \) is a proper \( \varphi(\text{Ric}) \)-vector field, provided \( \omega \neq -1 \) and the vertical distribution \( \mathcal{V} \) are parallel, then \( (\mathcal{N}, d) \) is an \( \eta \)-Einstein.

**Corollary 13.** Let a Riemannian submersion \( \varphi : (\mathcal{M}, \varphi) \rightarrow (\mathcal{N}, \varphi) \) under the canonical variation, whose total manifold admitting a conformal \( \eta \)-RS \( (d, \zeta, \Lambda, \mu) \) if a vertical potential field \( \zeta \) is a proper \( \varphi(\text{Ric}) \)-vector field, provided \( \omega \neq -1 \) and the vertical distribution \( \mathcal{V} \) are parallel, then \( (\mathcal{N}, d) \) is an \( \eta \)-Einstein.

**7. Gradient Conformal \( \eta \)-RS on Riemannian Submersions**

In this part, we look at Riemannian submersions under canonical variation, which admits gradient conformal \( \eta \)-RS on the base manifolds \( (\mathcal{N}, \varphi) \). As a result, we needed the requested facts. If a vector field \( \zeta \) is of gradient type, i.e., \( \zeta = \nabla \gamma \), where \( \gamma \) is a smooth function, then \( (\mathcal{N}, d, \gamma, \Lambda, \mu) \) is called a gradient conformal \( \eta \)-RS [1], and in this case the Equation (3) becomes

\[
\text{Hess}(\gamma) + \mathcal{S} + \left[ \Lambda - \left( \frac{p}{2} + \frac{1}{r} \right) \right] d + \mu \eta \otimes \eta = 0, \tag{39}
\]

wherein the Hessian operator with regard to \( \varphi \) is denoted by \( \text{Hess} \). Due to the fact that \( \gamma \) is a smooth function on base manifold \( (\mathcal{N}, \varphi) \), the Hessian tensor follows. \( h_\gamma : \Gamma(T\mathcal{N}) \rightarrow \Gamma(T\mathcal{N}) \) of \( \gamma \) is defined by [18]

\[
h_\gamma(\tilde{X}^h) = \nabla_{\gamma} \tilde{X}^h, \tag{40}
\]

for \( \tilde{X}^h \in \Gamma(T\mathcal{N}) \). The Hessian form of \( \gamma \), denoted by

\[
\text{Hess}(\gamma)(\tilde{X}^h, \tilde{Y}^h) = d(h_\gamma(\tilde{X}^h), \tilde{Y}^h) \tag{41}
\]

for all \( \tilde{X}^h, \tilde{Y}^h \in \Gamma(T\mathcal{N}) \).

Now, Theorem 7 (2) entails that the base manifolds \( (\mathcal{N}, d) \) of Riemannian submersion of the canonical variation is conformal \( \eta \)-RS with horizontal potential vector field \( \zeta_{NB} \), such that \( \psi \circ \zeta = \tilde{\zeta} \). Thus, we have

\[
\frac{1}{2} (\zeta_{\tilde{\zeta}}^d)(\tilde{X}^h, \tilde{Y}^h) + \mathcal{S}(\tilde{X}^h, \tilde{Y}^h) + \left[ \Lambda - \left( \frac{1}{p} + \frac{1}{r} \right) \right] d(\tilde{X}^h, \tilde{Y}^h) + \mu \eta(\tilde{X}^h) \eta(\tilde{Y}^h) = 0, \tag{42}
\]
for all $\mathring{X}^h, \mathring{Y}^h \in \Gamma(TNB)$. Putting $\xi = \nabla^t \gamma$ in Equation (42) we turn up

$$\frac{1}{2}[d(\nabla_{\mathring{X}^h}^t \nabla^t \gamma, \mathring{Y}^h) + d(\nabla_{\mathring{Y}^h}^t \nabla^t \gamma, \mathring{X}^h)] + \mathcal{S}(\mathring{X}^h, \mathring{Y}^h) + \left[\Lambda - \left(\frac{1}{p} + \frac{1}{r}\right)\right]d(\mathring{X}^h, \mathring{Y}^h) + \mu \eta(\mathring{X}^h)\eta(\mathring{Y}^h) = 0,$$

which entails

$$\frac{1}{2}[d(\nabla_{\mathring{X}^h}^t \nabla^t \gamma, \mathring{Y}^h) + d(\nabla_{\mathring{Y}^h}^t \nabla^t \gamma, \mathring{X}^h)] + \mathcal{S}(\mathring{X}^h, \mathring{Y}^h) + \left[\Lambda - \left(\frac{1}{p} + \frac{1}{r}\right)\right]d(\mathring{X}^h, \mathring{Y}^h) + \mu \eta(\mathring{X}^h)\eta(\mathring{Y}^h) = 0.$$  (44)

In light of Equations (40) and (41), we turn up

$$\mathcal{H}ess(\gamma)(\mathring{X}^h, \mathring{Y}^h) + \mathcal{S}(\mathring{X}^h, \mathring{Y}^h) + \left[\Lambda - \left(\frac{1}{p} + \frac{1}{r}\right)\right]d(\mathring{X}^h, \mathring{Y}^h) + \mu \eta(\mathring{X}^h)\eta(\mathring{Y}^h) = 0,$$

which infers that base manifolds $(NB, \mathring{d})$ of Riemannian submersion under the canonical variation is gradient conformal $\eta$-RS with horizontal potential vector field $\xi_{NB}$. Now, one can articulate the following result.

**Theorem 14.** Let $(NB, \mathring{d}, \xi, \Lambda, \mu)$ be a conformal $\eta$-RS with horizontal potential vector field $\xi_{NB}$ and $\psi : (MT, d) \rightarrow (NB, \mathring{d})$ is a Riemannian submersion under the canonical variation with horizontal potential vector field $\xi = \text{grad}(\gamma)$ type, then the base manifolds $(NB, \mathring{d})$ of Riemannian submersion under the canonical variation admits a gradient conformal $\eta$-RS.

**Corollary 15.** Let $(\mathring{d}, \xi, \Lambda, \mu)$ be a conformal $\eta$-RS with a vertical potential vector field $\xi$ and $\psi : (MT, d) \rightarrow (NB, \mathring{d})$ is a Riemannian submersion under the canonical variation with vertical potential vector field $\xi = \text{grad}(\gamma)$ type, where $\gamma$ is a smooth function on total manifold $MT$, then any fiber of Riemannian submersion $\psi$ under the canonical variation admits a gradient conformal $\eta$-RS.

8. Some Applications

**Definition 16.** For a function $\Psi = \Psi(t, x) \in C^\infty(M)$ (depending also on time $t$) and the vector field $\rho$ corresponding to the given ODE. Consequently, a straight forward calculation gives (for more details see [30,31])

$$\text{div}(\Psi \rho) = \rho \frac{d\Psi}{dt} + \Psi \text{Div}\rho.$$  (46)

The last multiplier of vector field $\rho = \rho(x)$ is a smooth function $\Psi \in C^\infty(M)$ with respect a metric $d$ holds $\text{div}(\Psi \rho) = 0$. The corresponding equatin [32]

$$\rho \frac{d}{dt} \ln \rho = -\text{Div}(\rho)$$  (47)

is known as **generalized Liouville equation** of the vector field $\xi$ with respect to the metric $d$ [30].

Now, consider the equation of $r$-dimensional fiber of Riemannian submersion $\psi : (MT, d) \rightarrow (NB, \mathring{d})$ admitting the conformal $\eta$-RS

$$\frac{1}{2}[d(\nabla_{E^p}^t \xi, F^p) + d(\nabla_{F^p}^t \xi, E^p)] + \mathcal{S}(E^p, F^p) + \left[\Lambda - \left(\frac{1}{p} + \frac{1}{r}\right)\right]d(E^p, F^p) + \mu \eta(E^p)\eta(F^p) = 0$$  (48)

for any $E^p, F^p \in \Gamma(V(M_T))$ and with the $g$-dual of 1-form $\eta$ of the potential vector field.

In light of Equation (17) contacting (48), we turn up

$$\text{Div}(\xi) = -\mathcal{R} - r \left[\Lambda - \left(\frac{1}{p} + \frac{1}{r}\right)\right] - \mu.$$  (49)
Adopting Equations (47) and (49), we articulate the following results.

**Theorem 17.** Let $\psi : (M_T, d) \rightarrow (NB, \tilde{d})$ be a Riemannian submersion from Riemannian manifolds under the canonical variation, admitting a conformal $\eta$-RS $(M_T, d, \zeta, \Lambda, \mu)$ with a vertical potential field $\zeta$ and a smooth function $\Psi$ is the last multiplier of $\zeta$. If the vertical distribution $\mathcal{V}$ is parallel and $\eta$ be the $d$-dual $1$-form of the vertical potential field $\zeta$, then the generalized Liouville equation of Riemannian submersion under the canonical variation satisfying by $\Psi$ and $\zeta$ is

$$
\zeta \frac{d}{dt} \ln \Psi = R + r \left[ \Lambda - \left( \frac{1}{p} + \frac{1}{r} \right) \right] + \mu. 
$$

**Corollary 18.** If $\psi : (M_T, d) \rightarrow (NB, \tilde{d})$ be a Riemannian submersion from Riemannian manifold under the canonical variation, admitting a conformal $\eta$-RS $(d, \zeta, \Lambda, \mu)$ with a vertical potential field $\zeta$, and $\eta$ is the $d$-dual $1$-form of the vertical potential field $\zeta$. If the vertical distribution $\mathcal{V}$ is parallel and the vertical potential field $\zeta$ is conformal Killing, then the conformal $\eta$-RS is expanding, steady and shrinking according as

(i) $\left( \frac{1}{p} + \frac{1}{r} \right) > \frac{1}{r} (R + \mu)$,

(ii) $\left( \frac{1}{p} + \frac{1}{r} \right) = \frac{1}{r} (R + \mu)$, and

(iii) $\left( \frac{1}{p} + \frac{1}{r} \right) < \frac{1}{r} (R + \mu)$, respectively.

**Remark 1.** There are the following utilization of Liouville equations:

1. Liouville equations are equivalent to the Gauss–Codazzi equation for minimal immersion into $3$-space.
2. The Liouville equation is valid for both equilibrium and non-equilibrium systems. It is a fundamental equation of non-equilibrium statistical mechanics. It is also the key component to describe viscosity, thermal conductivity, and electrical conductivity.
3. The analog of the Liouville equation in quantum mechanics describes the time evolution of a mixed state.
4. We can also formulate the Liouville equation in terms of symplectic geometry.

**Definition 19.** [33] A scalar field $\sigma \in C^\infty(M)$ is said to be a scalar concircular field if it satisfies the equation

$$
\text{Hess}(\sigma) = \pi g 
$$

where $\pi$ is a scalar field and $g$ is the Riemannian metric. Moreover, along any geodesic with arc-length $u$, the Equation (51) becomes the ordinary differential equation

$$
\frac{d^2 \sigma}{du^2} = \pi. 
$$

Now, Equations (45) and (51) entail that

$$
\mathcal{S}(\dot{X}^h, \dot{Y}^h) = a \tilde{d}(\dot{X}^h, \dot{Y}^h) + \mu \eta(\dot{X}^h)\eta(\dot{Y}^h), 
$$

where $a = - \left[ \Lambda - \left( \frac{1}{p} + \frac{1}{r} + \pi \right) \right]$ and $b = - \mu$. Consequently, we find the following results.

**Theorem 20.** Let $(NB, \tilde{d}, \nabla^T, \Lambda, \mu)$ be a gradient conformal $\eta$-RS with horizontal potential scalar concircular field $\gamma_{NB}$ on a a Riemannian submersion $\psi : (M_T, d) \rightarrow (NB, \tilde{d})$ under the canonical variation, then the base manifolds $(NB, \tilde{d})$ of Riemannian submersion under the canonical variation is an $\eta$-Einstein manifold.
9. Some Non-Trivial Examples

Example 1. Let \( M^6 = \{(s_1, s_2, s_3, s_4, s_5, s_6)|s_6 \neq 0\} \) be a 6-dimensional differentiable manifold where \((s_i), \text{where } i = 1, 2, 3, 4, 5, 6 \) indicates the standard coordinates of a point in \( \mathbb{R}^6 \).

Now, consider

\[
\begin{align*}
    b_1 &= \partial s_1, & b_2 &= \partial s_2, & b_3 &= \partial s_3, & b_4 &= \partial s_4, & b_5 &= \partial s_5, & b_6 &= \partial s_6.
\end{align*}
\]

is a collection of linearly independent vector fields at each point of the manifold \( M^6 \), serving as the foundation for the tangent space \( T(M^6) \). We define a positive definite metric \( d \) on \( M^6 \) as \( d = \sum_{i,j=1}^{6} dx_i \otimes dx_j \). Let the 1-form \( \eta \) be defined by \( \eta(X) = d(X, P) \) where \( P = b_6 \).

Then, it is obvious that \((M^6, d)\) is a Riemannian manifold of dimension 6. Moreover, Let \( \nabla \) represent the Levi-Civita connection in terms of metric \( d \). Thus, we turn up \([b_1, b_2] = 0\). Similarly \([b_1, b_6] = b_1, [b_2, b_6] = b_2, [b_3, b_6] = b_3, [b_4, b_6] = b_4, [b_5, b_6] = b_6 [b_i, b_j] = 0, 1 \leq i \neq j \leq 5\).

The Riemannian connection \( \nabla \) of the metric \( d \) is given by

\[
2\dd\left(\nabla_E F, G\right) = E\dd(F, G) + F\dd(G, E) - G\dd(E, F) - \dd(F, [E, G]) + \dd(G, [E, F]),
\]

where \( \nabla \) denotes the Levi-Civita connection corresponding to the metric \( d \).

By using Koszul's formula and Equation (8) together, we obtain the following equations:

\[
\begin{align*}
    \nabla_{b_i} b_1 &= b_5, & \nabla_{b_i} b_2 &= b_6, & \nabla_{b_i} b_3 &= b_6, & \nabla_{b_i} b_4 &= b_6, & \nabla_{b_i} b_5 &= b_6 & \nabla_{b_i} b_6 &= 0, & \nabla_{b_i} b_i &= 0, & 1 \leq i \leq 5
\end{align*}
\]

and \( \nabla_{b_i} b_i = 0 \) for all \( 1 \leq i, j \leq 5 \). We can now determine the non-vanishing components of the Riemannian curvature tensor \( \mathring{R}_m \), Ricci curvature tensor \( \mathring{S} \) of the fiber using Equations (13) and (54).

\[
\begin{align*}
    \mathring{R}_m(b_1, b_2)b_1 &= b_2, & \mathring{R}_m(b_1, b_2)b_2 &= -b_1, & \mathring{R}_m(b_1, b_3)b_1 &= -b_3, & \mathring{R}_m(b_1, b_3)b_3 &= b_1, & \mathring{R}_m(b_1, b_4)b_1 &= -b_4, & \mathring{R}_m(b_1, b_4)b_4 &= b_1, & \mathring{R}_m(b_1, b_5)b_1 &= -b_5, & \mathring{R}_m(b_1, b_5)b_5 &= b_1, & \mathring{R}_m(b_1, b_6)b_1 &= -b_6, & \mathring{R}_m(b_1, b_6)b_6 &= -b_1, & \mathring{R}_m(b_2, b_3)b_2 &= -b_3, & \mathring{R}_m(b_2, b_3)b_3 &= b_2, & \mathring{R}_m(b_2, b_4)b_2 &= b_4, & \mathring{R}_m(b_2, b_4)b_4 &= -b_2, & \mathring{R}_m(b_2, b_5)b_2 &= b_5, & \mathring{R}_m(b_2, b_5)b_5 &= -b_2, & \mathring{R}_m(b_2, b_6)b_2 &= b_6, & \mathring{R}_m(b_2, b_6)b_6 &= -b_2, & \mathring{R}_m(b_3, b_4)b_3 &= b_4, & \mathring{R}_m(b_3, b_4)b_4 &= b_3, & \mathring{R}_m(b_3, b_5)b_3 &= b_5, & \mathring{R}_m(b_3, b_5)b_5 &= -b_3, & \mathring{R}_m(b_3, b_6)b_3 &= b_6, & \mathring{R}_m(b_3, b_6)b_6 &= -b_3, & \mathring{R}_m(b_4, b_5)b_4 &= b_5, & \mathring{R}_m(b_4, b_5)b_5 &= -b_4, & \mathring{R}_m(b_4, b_6)b_4 &= b_6, & \mathring{R}_m(b_4, b_6)b_6 &= -b_4, & \mathring{R}_m(b_5, b_6)b_5 &= b_6, & \mathring{R}_m(b_5, b_6)b_6 &= -b_5.
\end{align*}
\]
\[
\hat{S}(E_i, E_j) = \begin{bmatrix}
-3 & 0 & 0 & 0 & 0 & 0 \\
0 & -3 & 0 & 0 & 0 & 0 \\
0 & 0 & -3 & 0 & 0 & 0 \\
0 & 0 & 0 & -3 & 0 & 0 \\
0 & 0 & 0 & 0 & -3 & 0 \\
0 & 0 & 0 & 0 & 0 & -5
\end{bmatrix}.
\]

From Equation (3), we have
\[
[d(\hat{\nabla}_b, v_i) + d(\hat{\nabla}_b, b_i)] + 2\hat{S}(b_i, b_i) + \left[2\Lambda - \left(p + \frac{2}{n}\right)\right]d(b_i, b_i) + 2\mu\delta^i_j = 0
\]
for all \(i \in \{1, 2, 3, 4, 5, 6\}\). Therefore, we obtain \(\Lambda = 3 + (p/2 + 1/n)\) and \(\mu = 3\) for the data \((d, v_6, \Lambda, \mu)\) is a conformal \(\eta\)-RS, verified by Equation (3). Thus, the data \((M, \hat{g}, \Lambda, \mu)\) is admitting the expanding conformal \(\eta\)-RS.

**Example 2.** Let \(\psi: \mathbb{R}^6 \to \mathbb{R}^3\) be a submersion characterized by
\[
\psi(s_1, s_2, \ldots, s_6) = (y_1, y_2, y_3),
\]
where
\[
y_1 = \frac{s_1 + s_2}{\sqrt{2}}, \quad y_2 = \frac{s_3 + s_4}{\sqrt{2}} \quad \text{and} \quad y_3 = \frac{s_5 + s_6}{\sqrt{2}}.
\]
Then, the Jacobian matrix for \(\psi\) is rank 3. That means \(\psi\) is a submersion. A straight computations yields
\[
ker\psi_* = span\{V_1 = \frac{1}{\sqrt{2}}(-\partial s_1 + \partial s_2), V_2 = \frac{1}{\sqrt{2}}(-\partial s_3 + \partial s_4), V_3 = \frac{1}{\sqrt{2}}(-\partial s_5 + \partial s_6)\},
\]
and
\[
(ker\psi_*)^\perp = span\{H_1 = \frac{1}{\sqrt{2}}(\partial s_1 + \partial s_2), H_2 = \frac{1}{\sqrt{2}}(\partial s_3 + \partial s_4), H_3 = \frac{1}{\sqrt{2}}(\partial s_5 + \partial s_6)\},
\]
Additionally, by direct computations, it yields
\[
\psi_*(H_1) = \partial y_1, \psi_*(H_2) = \partial y_2 \quad \text{and} \quad \psi_*(H_3) = \partial y_3
\]
Hence, it is easy to see that
\[
d_{\mathbb{R}^3}(H_i, H_i) = d_{\mathbb{R}^3}(\psi_*(H_i), \psi_*(H_i)), \quad i = 1, 2, 3
\]
Hence, \(\psi\) is a Riemannian submersion.

Now, we can compute the components of Riemannian curvature tensor \(\hat{R}_m\), and Ricci curvature tensor \(\hat{S}\) for \(ker\psi_*\) (vertical space) and \(ker\psi_*^\perp\) (horizontal space), respectively. For the vertical space, we have
\[
\hat{R}_m(V_1^\parallel, V_2^\parallel)V_1^\parallel = -2V_2^\parallel, \quad \hat{R}_m(V_1^\parallel, V_2^\parallel)V_2^\parallel = 2V_1^\parallel, \quad \hat{R}_m(V_1^\parallel, V_3^\parallel)V_1^\parallel = -2V_3^\parallel
\]
\[
\hat{R}_m(V_1^\parallel, V_2^\parallel)V_3^\parallel = V_1^\parallel, \quad \hat{R}_m(V_2^\parallel, V_3^\parallel)V_2^\parallel = V_3^\parallel, \quad \hat{R}_m(V_2^\parallel, V_3^\parallel)V_3^\parallel = V_2^\parallel
\]

\[ \mathcal{S}(V^i_j, V^i_j) = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \]

Using Equation (3), we obtain \( \Lambda = (p/2 + 1/n) - 2 \) and \( \mu = (p/2 + 1/n) - 1 \). Therefore, \((\ker\psi, d)\) is admitting the expanding, shrinking and steady conformal \( \eta \)-RS according \( (\frac{p}{2} + \frac{1}{n}) > 2 \), \( (\frac{p}{2} + \frac{1}{n}) < 2 \) or \( (\frac{p}{2} + \frac{1}{n}) = 2 \), respectively.

Next, for the horizontal space, we have

\[
\begin{align*}
\check{R}_m(\psi_+(H_1), \psi_+(H_2)) \psi_+(H_1) &= \frac{1}{2} (\partial z_3 + \partial z_4), \\
\check{R}_m(\psi_+(H_1), \psi_+(H_3)) \psi_+(H_3) &= \frac{1}{\sqrt{2}} (\partial z_6 - \partial z_5), \\
\check{R}_m(\psi_+(H_2), \psi_+(H_3)) \psi_+(H_3) &= \frac{1}{\sqrt{2}} (\partial z_6, \\
\check{R}_m(\psi_+(H_2), \psi_+(H_3)) \psi_+(H_2) &= \frac{1}{\sqrt{2}} (\partial z_1 + \partial z_2).
\end{align*}
\]

and

\[ \mathcal{S}(\psi_i H_1, \psi_i H_2) = \begin{bmatrix} -\frac{3\sqrt{2}}{2} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{\sqrt{2}} \end{bmatrix}. \]

Again using Equation (3), we turn up \( \Lambda = (\frac{p}{2} + \frac{1}{n}) + \frac{3}{2\sqrt{2}} \) and \( \mu = \frac{1}{4\sqrt{2}} \). Therefore \((\ker\psi^+_+), \check{d})\) is admitting the expanding conformal \( \eta \)-RS.

10. Conclusions and Remark

The interest in the study of the problems of Ricci flows, which are evolution equations for Riemannian metrics, has grown in recent years. Actually, it was first applied in settling the century-old Poincare conjecture and after that it became an important tool for various applications in sciences (for example, in physics, in biology, chemistry) and economics. In fact, the study of Ricci flows and Ricci solitons has shown its presence in medical imaging for brain surfaces. Due to this reason, the geometry of different types of solitons on manifolds has been the focus of attention of many mathematicians during the last two decades (for example [34]).

Differential geometry is a traditional yet currently very active branch of pure mathematics with applications notably in a number of areas of physics. Until recently, applications in the theory of statistics were fairly limited, but within the last few years there has been intensive interest in the subject. For this reason, the geometric study of statistical submersions is new and has many research problems. Therefore, we believe that the present article will help in achieving new and interesting results in the geometry of statistical solitons [35] on statistical submersions [36]. In fact, some singularity theories on submanifolds can be studied [14–17].

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References

1. Hamilton, R.S. The Ricci flow on surfaces, Mathematics and General Relativity (University of California: Santa Cruz, CA, USA, 1986). *Contemp. Math.* 1988, 71, 237–262.

2. Fischer, A.E. An introduction to conformal Ricci flow. *Class. Quantum Grav.* 2004, 21, S171–S218. [CrossRef]

3. Basu, N.; Bhattacharyya, A. Conformal Ricci soliton in Kenmotsu manifold. *Glob. J. Adv. Res. Class. Mod. Geom.* 2015, 4, 159–621.

4. Aubin, T. Métriques Riemanniennes et courbure. *J. Differ. Geom.* 1970, 4, 383–424. [CrossRef]

5. Siddiqi, M.D. Conformal $η$-Ricci solitons in $δ$-Lorentzian Trans Sasakian manifolds. *Int. J. Maps Math.* 2018, 1, 15–34.

6. Siddiqi, M.D.; Siddiqi, S.A. Conformal Ricci soliton and geometrical structure in a perfect fluid spacetime. *Int. J. Geom. Methods Mod. Phys.* 2020, 17, 2050083. [CrossRef]

7. Siddiqi, A.N.; Siddiqi, M.D. Almost Ricci-Bourguignon solitons and geometrical structure in a relativistic perfect fluid spacetime. *Balkan J. Geom. Appl.* 2021, 26, 126–138.

8. Nash, J.N. The imbedding problem for Riemannian manifolds. *Annals Math.* 1956, 63, 20–63. [CrossRef]

9. Ianus, S.; Visinescu, M. Kaluza-Klein theory with scalar fields and generalized Hopf manifolds. *Class. Quantum Gravity* 1987, 4, 1317–1325. [CrossRef]

10. Ianus, S.; Visinescu, M. Space-time compaction and Riemannian submersions. In *The Mathematical Heritage of C. F. Gauss*; Rassias, G., Ed.; World Scientific: River Edge, NJ, USA, 1991; pp. 358–371.

11. Watson, B. G,-G' Riemannian submersions and nonlinear gauge field equations of general relativity. In *The Mathematical Heritage of C. F. Gauss*; Rassias, G., Ed.; World Scientific: River Edge, NJ, USA, 1991; pp. 358–371.

12. Bourguignon, J.P.; Lawson, H.B. Stability and isolation phenomena for Yang-mills fields. *Commun. Math. Phys.* 1981, 79, 189–230. [CrossRef]

13. Bourguignon, J.P.; Lawson, H.B. A mathematician’s visit to Kaluza-Klein theory. *Rend. Sem. Mat. Univ. Politec. Torino* 1989, 1, 143–163.

14. Li, Y.; Mofarreh, F.; Abdel-Baky, R.A. Timelike Circular Surfaces and Singularities in Minkowski 3-Space. *Symmetry* 2022, 14, 2014. [CrossRef]

15. Li, Y.; Alluhaibi, N.; Abdel-Baky, R.A. One-Parameter Lorentzian Dual Spherical Movements and Invariants of the Axodes. *Symmetry* 2022, 14, 2013. [CrossRef]

16. Li, Y.; Eren, K.; Ayvacı, K.H.; Ersoy, S. Simultaneous characterizations of partner ruled surfaces using Flc frame. *AIMS Math.* 2022, 7, 20213–20229. [CrossRef]

17. Li, Y.; Nazra, S.H.; Abdel-Baky, R.A. Singularity Properties of Time-like Sweeping Surface in Minkowski 3-Space. *Symmetry* 2022, 14, 1996. [CrossRef]

18. Şahin, B. *Riemannian Submersions, Riemannian Maps in Hermitian Geometry and Their Applications*; Elsevier: Amsterdam, The Netherlands ; Academic Press: Cambridge, MA, USA, 2017.

19. Falcitelli, M.; Ianus, S.; Pastore, A.M. *Riemannian Submersions, Riemannian Maps in Hermitian Geometry and Their Applications*; World Scientific: River Edge, NJ, USA, 2004.

20. Merić, Ş.E.; Kılıç, E. Riemannian submersions whose total manifolds admit a Ricci soliton. *Int. J. Geom. Methods Mod. Phys.* 2019, 16, 1950196. [CrossRef]

21. Gray, A. Pseudo-Riemannian almost product manifolds and submersion. *J. Math. Mech.* 1967, 16, 715–737.

22. Bejan, C.L.; Merić, Ş.E.; Kılıç, E. Contact-Complex Riemannian submersions. *Mathematics* 2021, 9, 2996. [CrossRef]

23. Gündüzalp, Y. Almost Hermitian submersions whose total manifolds admit a Ricci soliton. *Honam Math. J.* 2020, 42, 733–745.

24. Merić, Ş.E. Some remarks on Riemannian submersions admitting an almost Yamabe soliton. *Adiyaman Univ. J. Sci.* 2020, 10, 295–306.

25. Siddiqi, M.D.; Akyol, M.A. $η$-Ricci-Yamabe Soliton on Riemannian Submersions from Riemannian manifolds. *arXiv preprint* 2020, arXiv:2004.14124v1.

26. Chaubey, S.K.; Siddiqi, M.D.; Yadav, S. Almost $η$- Ricci-Bourguignon solitons on submersions from Riemannian submersions. *Balk. J. Geom. Its Appl.* 2022, 27, 24–38.

27. O’Neill, B. The fundamental equations of a submersion. *Mich. Math. J.* 1966, 13, 458–469. [CrossRef]

28. Bergery, L.B.; Bourguignon, J.P. Laplacian and Riemannian submersions with totally geodesic fibers. *Illinois J. Math.* 1982, 26, 181–200. [CrossRef]

29. Hinterleither, I.; Kiosak, V.A. $φ(Ric)$-vector field in Riemannian spaces. *Arch. Math.* 2008, 94, 385–390.

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30. Gregory S.E. On the statistical mechanics of non-Hamiltonian systems: the generalized Liouville equation, entropy, and time-dependent metrics. *J. Math. Chem.* 2004, 35, 29–53.

31. Popov, A.G. Exact formula for constructing solutions of the Liouville equation $\Delta u = e^u$ from solutions of the Laplace equation $\Delta v = 0$. *Dokl. Akad. Nauk.* 1993, 333, 440–441. (In Russian)

32. Liouville, J. Sur la Théorie de la Variation des constantes arbitraires. *J. Math. Pures Appl.* 1838, 3, 342–349.

33. Fialkow, A. Conformal geodesic. *Trans. Amer. Math. Soc.* 1989, 45, 443–473. [CrossRef]

34. Blaga, A.M. $\eta$-Ricci solitons on para-Kenmotsu manifolds. *Balk. J. Geom. Its Appl.* 2015, 20, 1–13.

35. Siddiqui, A.N.; Chen, B.Y.; Bahadir, O. Statistical solitons and inequalities for statistical warped product submanifolds. *Mathematics* 2019, 7, 797. [CrossRef]

36. Siddiqi, M.D.; Siddiqui, A.N.; Mofarre F.; Aytimur, H. A Study of Kenmotsu-like Statistical Submersions. *Symmetry* 2022, 14, 1681. [CrossRef]