A scaling approach to Caffarelli-Kohn-Nirenberg inequality

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Abstract

We consider the general Caffarelli-Kohn-Nirenberg inequality in the Euclidean and Riemannian setting. From a new parameter introduced, the proof of the former case, follows by simple interpolation arguments and Hölder’s inequality. Moreover, the ranges of this convenient parameter completely characterize the inequality. Secondly, the same techniques are used to study the Caffarelli-Kohn-Nirenberg inequality in the Riemannian case.

1 Introduction

In this paper, we first consider the general form of Caffarelli-Kohn-Nirenberg inequality in the Euclidean setting, that is to say, we study the inequality

\[
\left( \int_{\mathbb{R}^n} \|x\|^{\gamma_r} |u|^r \, dx \right)^{1/r} \leq C \left( \int_{\mathbb{R}^n} \|x\|^{\alpha_p} \|\nabla u\|^p \, dx \right)^{a/p} \left( \int_{\mathbb{R}^n} \|x\|^{\beta_q} |u|^q \, dx \right)^{(1-a)/q},
\]

where the real parameters \( p, q, r, \alpha, \beta, \gamma \), satisfy

\[
p, q \geq 1, \quad r > 0 \quad \text{and} \quad \gamma r, \alpha p, \beta q > -n. \quad (1.2)
\]

Moreover, a parameter \( \sigma \) is introduced by the following convex combination

\[
\forall a \in [0,1], \quad \gamma = a \sigma + (1-a) \beta. \quad (1.3)
\]

From a dimensional balance of (1.1), we obtain

\[
\frac{1}{r} + \frac{\gamma}{n} = a \left( \frac{1}{p} + \frac{\alpha - 1}{n} \right) + (1-a) \left( \frac{1}{q} + \frac{\beta}{n} \right). \quad (1.4)
\]

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Also, if $a > 0$, then $\sigma \leq \alpha$ and, if $a > 0$ and

$$\frac{1}{p} + \frac{\alpha - 1}{n} = \frac{1}{r} + \frac{\gamma}{n},$$

then $\sigma \geq \alpha - 1$. These are necessary and sufficient conditions for (1.1), as it was proved in [8]. Here, we be focused on the sufficiency. Furthermore, for any compact set in the parameter space, such that, (1.2), (1.4) and $(\alpha - 1) \leq \sigma \leq \alpha$, the positive constant $C$ in (1.1) is bounded.

In this paper the study of Caffarelli-Kohn-Nirenberg inequality relies in a suitable introduced parameter $s$ as defined in (2.10) below, where $0 < s \leq p^*$, and $p^*$ is the Sobolev conjugate of $p$, see (1.7). In fact, the proof of (1.4) follows by simple interpolation arguments and Hölder’s inequality, once the new parameter $s$ is considered. Moreover, we completely characterize the inequality from the ranges of $s$. In particular, for $s \in [p, p^*]$ the Caffarelli-Kohn-Nirenberg inequality is proved to be the interpolation between Hardy’s inequality (1.9) and the weighted Sobolev inequality (1.8). In this case, the constant $C > 0$, which appears in (1.1) is finite, and it is an exponential convex combination of the Hardy and the Sobolev constant inequalities. On the other hand, for $0 < s < p$ the inequality (1.1) is no more an interpolation between Hardy and (weighted) Sobolev, further the constant $C$ is not necessarily bounded. The characterization of (1.1) from the ranges of $s$ is new, and we hope clarify the understanding of it.

Before we continue to discuss the inequality (1.1) in the Euclidean setting, which contains most of the well known inequalities, we present the Caffarelli-Kohn-Nirenberg inequality in the Riemannian case, which is also one of the issues of this paper and a new result, in this generality, from authors’ knowledge. Then, we study in Section 3 the following inequality

$$(\int_U \|h\|^\gamma_r |u|^r dV)^{1/r} \leq C \left(\int_U \|h\|^\alpha_p \|\nabla u\|^p dV\right)^{a/p} \left(\int_U \|h\|^\beta_q |u|^q dV\right)^{(1-a)/q},$$

(1.5)

where $U \subset M$ is any open precompact region, $M$ is a Riemannian $n$-manifold, with $n \geq 3$, and $h$ is a special vector field which allows us, to apply the same technics used before for the Euclidean setting. The conditions on $M$, which is to say, a complete and non-compact Riemannian manifold, also with maximal volume growth and non-negative Ricci curvature, will be explained with details in Section 3. Moreover, the special vector field $h$. Those conditions are most related to weight’s homogeneity, and extra terms on the right hand side of Hardy-Sobolev type inequalities on manifolds.

The Caffarelli-Kohn-Nirenberg inequality appeared for the first time in [9], in that case $p = q = 2$ and $a = 1$. The paper [9] introduces the convenient definition of a suitable weak solution for the incompressible 3D Navier-Stokes equations with unit viscosity, and the Caffarelli-Kohn-Nirenberg inequality was used to improve the result established before by Scheffer concerning the dimension of the subset of singularities. Albeit (1.1) appears earlier in the study
of incompressible Navier-Stokes equations, it was soon understood that, this inequality is important in the theory of elliptic equations, for instance of the following type

$$- \text{div} \left( A(x) \nabla u \right) = f(x, u),$$

where $A$ is a nonnegative function that may be unbounded and $f$ is a given function.

In different works, the existence and multiplicity of positive or nodal solutions for (1.6) was established, provided the differential operator $\text{div} \left( A(x) \nabla \cdot \right)$ is uniformly elliptic (for more details, see [3] and [22]). Although, interesting and important situations are obtained in the degenerated and singular cases, respectively $\inf A(x) = 0$, $\sup A(x) = \infty$. For instance, it was studied in [17] the existence (of at least two solutions) for the following problem

$$- \text{div} \left( |x|^{-2s} \nabla u \right) = K(x) |x|^{-\sigma p} |u|^{p-2} u + \lambda g(x) \quad \text{in} \ \mathbb{R}^n \setminus \{0\},$$

where $K \in L^\infty(\mathbb{R}^n)$ (in fact, $K$ has more conditions), $\lambda$ is a parameter, and $g$ is a continuous function. The inequality (1.1) was used to show that the functional

$$J_\lambda(u) = \frac{1}{2} \int_{\mathbb{R}^n} |x|^{-2s} \|
abla u\|^2 \, dx - \frac{1}{p} \int_{\mathbb{R}^n} K(x) |x|^{-\sigma p} |u|^p \, dx - \lambda \int_{\mathbb{R}^n} g(x) u \, dx$$

is well defined among other properties, that is to say, the existence of (at least) two critical points for $J_\lambda$. Similarly, it was studied in [5] the existence of non-trivial solutions for the following problem

$$- \text{div} \left( |x|^{-2s} \nabla u \right) = \mu |u|^{-2(s+1)} u + K(x) |x|^{2^* \sigma} |u|^{2^* - 2} u + \lambda g(x) \quad \text{in} \ \mathbb{R}^n \setminus \{0\},$$

where $\mu$ is also a parameter. Again the Caffarelli-Kohn-Nirenberg inequality (1.1) was used to show that the functional

$$I_{\lambda, \mu}(u) = \frac{1}{2} \|u\|_\mu^2 - \frac{1}{2^*} \int_{\mathbb{R}^n} K(x) |x|^{-2^* \sigma} |u|^{2^*} - \lambda \int_{\mathbb{R}^n} g(x) u$$

is well defined and the existence of critical points, where

$$\|u\|_\mu^2 = \frac{1}{2} \int_{\mathbb{R}^n} \left( |x|^{-2s} \|
abla u\|^2 - \frac{\mu}{s} |u|^{-2s} \right) \, dx.$$

Therefore, the importance of Caffarelli-Kohn-Nirenberg inequality (1.1) is also shown in the two elliptic problems mentioned before. More information related to applications of this inequality in elliptic problems can be found in [10], [14] and [21]. Finally, we highlight that these singular and degenerate elliptic equations are given models (at the equilibrium) for anisotropic media, that are possibly somewhere between perfect insulators or perfect conductors, see [13].

Now, let us consider some particular values of the parameters:

1. **(Sobolev inequality)** When $a = 1$, we have by (1.3) $\gamma = \sigma$. Taking $\gamma = \alpha$, it follows by (1.3) that, $r = p^*$, and $p < n$, where

$$p^* := \frac{np}{n-p}.$$
Then, we obtain from (1.1) the weighted version of the Sobolev inequality
\[
(\int |x|^{|\alpha|p}|u(x)|^p dx)^{1/p} \leq C_S (\int |x|^{|\alpha|p} \|\nabla u(x)\|^p dx)^{1/p}.
\]
(1.8)

In particular, for \(\alpha = 0\) the weights disappear, and we get the usual Sobolev’s inequality
\[
(\int |u(x)|^p dx)^{1/p} \leq C_S (\int \|\nabla u(x)\|^p dx)^{1/p}.
\]

Since it was found by Sobolev, many studies were made to better understand this inequality in different directions (sharp version, remainder terms, bounded domains, Riemannian manifolds, etc). More information about this inequality can be found for example in [23].

2. (Hardy inequality) Again, we take \(a = 1\) (thus \(\gamma = \sigma\)) and consider \(\gamma = \alpha - 1\), hence \(r = p\). Therefore, we obtain from (1.1) the following version of Hardy’s inequality
\[
(\int |x|^{(\alpha-1)p}|u(x)|^p dx)^{1/p} \leq C_H (\int |x|^{|\alpha|p} \|\nabla u(x)\|^p dx)^{1/p}.
\]
(1.9)

In particular, for \(\alpha = 0\) we have
\[
\int \frac{|u(x)|^p}{|x|^p} dx \leq (C_H)^p \int \|\nabla u(x)\|^p dx.
\]

Information about the history of Hardy inequality can be found in [18]. This inequality has also been studied in many different directions (remainder terms, bounded and unbounded domains, singularity on the boundary, etc). One interesting application can be found in [11], where this inequality is used to show the existence of solutions for the Dirichlet problem for the p-Laplacian operator in bounded domains.

3. (Gagliardo-Nirenberg inequality) When \(\alpha = \beta = \sigma = 0\) we get \(\gamma = 0\), and it is possible to recover from (1.1) the Gagliardo-Nirenberg inequality
\[
(\int |x|^p dx)^{\frac{1}{p}} \leq C (\int \|\nabla u(x)\|^p dx)^{\frac{1}{2p}} (\int |u(x)|^q dx)^{\frac{1}{2q}}.
\]

In particular, if we consider \(a = 2/(2 + \frac{4}{n})\), \(p = 2\) and \(q = 1\), it follows the important Nash inequality
\[
(\int |u(x)|^2 dx)^{\frac{1}{2}} \leq C (\int \|\nabla u(x)\|^2 dx)^{\frac{1}{2}} (\int |u(x)|^{\frac{4}{n+4}} dx)^{\frac{4}{n+4}}.
\]

The outline of the paper is the following. First we define a new parameter that permits us to uncouple the parameter \(\gamma\) and to write the interpolation inequality in an appropriate way. Then, we use this new parameter to obtain a more simple inequality that is equivalent to (1.1) in an appropriate sense. Next, we give the proof of this inequality in the Euclidean and the Riemannian case. Finally, we place in the Appendix some information about the weighted versions of Hardy and Sobolev inequalities.
2 A simplified inequality

In this section, we define the new parameter that, as announced before at the introduction, allow us to write the original inequality in a more convenient way. From now on, all the integrals are on $\mathbb{R}^n$, and the functions that appear in the inequalities are test functions, i.e., functions in the space $C^\infty_c(\mathbb{R}^n)$.

First, we rewrite the condition (1.4) in the following way
\[
\frac{1}{r} = a \left[ \frac{1}{p} \left( \frac{\sigma - (\alpha - 1)}{n} \right) \right] + \frac{1-a}{q},
\]
and define the key parameter $s$ as
\[
s := \frac{np}{n - p(\sigma - (\alpha - 1))}.
\]

(2.10)

One observes that $s$ has some useful properties:

a. The parameter $s$ is always positive. Indeed, from (1.2) we know that the parameter $r$ is positive, hence if $s$ is supposed negative, from (2) we would have some values of $a$ for which $r$ would be negative (more precisely, it happens for values of $a$ nearby to 1). Moreover, if $s$ is zero, then for $a = 1$, we would have that $r$ would be infinite, but $r$ is a real number. In both cases, we have a contradiction with the condition that $r$ is a positive real number.

b. As a consequence of item a., we have the following inequality from (2.10)
\[
n - p(\sigma - (\alpha - 1)) > 0
\]
or
\[
n > p(\sigma - (\alpha - 1)).
\]

(2.11)

Therefore, when $\sigma = \alpha$, we have $n > p$. So, it means that this relation between $n$ and $p$ is implicit in the inequality when $\sigma = \alpha$, and it is important, because (weighted) Sobolev inequality is used exactly in the case $\sigma = \alpha$. Observe that, for $\sigma \in [\alpha - 1, \alpha)$, it does not follow necessarily $p < n$. For instance, with $\sigma = \alpha - (1/2)$ we have
\[
n > p/2
\]
and, if $p = n$, this inequality is still true. More precisely, if we write $\sigma = \alpha - \delta$ for any $\delta \in [0,1)$, we must have $p < n/(1-\delta)$. On the other hand, for $\sigma \leq (\alpha - 1)$ we do not have any relation between $n$ and $p$ that comes from (2.11). However, there exist three relations between $n$ and $p$ (also $\alpha$) that comes from (1.2). The first is
\[
\alpha p > -n.
\]
The other two relations come from the condition $\gamma r > -n$ in equation (1.2), and equations (1.3), (1.4). They are
\[
(\alpha - 1)p > -n \quad \text{and} \quad \alpha p^* > -n.
\]
These conditions, which are valid for all values of $\sigma$, allow us to use weighted versions of Hardy and Sobolev inequalities.

c. The parameter $s$ is an increasing function of $\sigma$. Let $\sigma_1, \sigma_2 \in (-\infty, \alpha]$ be such that, $\sigma_1 \leq \sigma_2$. Then, we have $\sigma_1 - \alpha + 1 \leq \sigma_2 - \alpha + 1$, and thus

$$\frac{np}{n - p(\sigma_1 - \alpha + 1)} \leq \frac{np}{n - p(\sigma_2 - \alpha + 1)},$$

that is, $s(\sigma_1) \leq s(\sigma_2)$. In particular, if $\sigma \leq \alpha$, then $s \leq p^*$, and since $s > 0$ we obtain

$$0 < s \leq p^*.$$

To follow, we use the parameter $s$ to define a new inequality. The relation between (1.1) and this new inequality is shown in the following

**Lemma 2.1.** Assume conditions (1.2) and (1.3). If there exist $C > 0$ such that

$$\left( \int \|x\|^{\sigma q} |u(x)|^s \, dx \right)^{1/s} \leq C \left( \int \|x\|^{\alpha p} \|\nabla u(x)\|^p \, dx \right)^{1/p}, \quad (2.12)$$

then the inequality (1.1) holds.

**Proof.** First, we rewrite the dimensional balance condition (1.4) in the following convenient way

$$r = (1 - b)q + bs, \quad (2.13)$$

where $b \in [0, 1]$ is defined as

$$b := \frac{aq}{aq + (1 - a)s}. \quad (2.14)$$

Therefore, for any fixed $q$ the parameter $r$ depends on the values of $s$ and $a$. Now, using (1.4) it is possible to get a convenient inequality, see (2.16), where the importance of the parameter $s$ becomes more clear. Indeed, we observe first that for $a = 0$ the inequality (1.1) turns a equality with $C = 1$. Then, we hereupon assume $a > 0$ and consequently $b > 0$. Hence from the definition of $\gamma$ and (1.4), we obtain

$$\gamma r = (1 - b)(1 - a)\beta q + (1 - b)a\sigma q + (1 - a)bs\beta + b\sigma s,$$

but, we have from (2.13), (2.14) the following relation between $b, a, q$ and $s$

$$a(1 - b)q = b(1 - a)s. \quad (2.15)$$

Consequently, replacing (2.15) in the former equality, it follows that

$$\gamma r = (1 - b)\beta q - (1 - b)a\beta q + b(1 - a)s + (1 - b)a\beta q + ab\sigma s$$

$$= (1 - b)\beta q - a(1 - b)\beta q + bs\sigma s - ab\sigma s + a(1 - b)\beta q + ab\sigma s$$

$$= (1 - b)\beta q + bs\sigma s.$$
Then, we write
\[
\left( \int \|x\|^{\gamma r} |u|^r dx \right)^{1/r} = \left( \int \|x\|^{(1-b)q+bs} |u(x)|^{(1-b)q+bs} dx \right)^{1/r} \\
\leq \left( \int \|x\|^{\beta q} |u(x)|^q dx \right)^{(1-b)/r} \left( \int \|x\|^s |u(x)|^s dx \right)^{b/r},
\]
where we have applied Hölder’s inequality for \(1/(1-b)\) and \(1/b\). Further, using \(2.13\) and the definition of \(b\), we have
\[
\left( \int \|x\|^{\gamma r} |u(x)|^r dx \right)^{1/r} \leq \left( \int \|x\|^{\beta q} |u(x)|^q dx \right)^{(1-a)/q} \times \left( \int \|x\|^s |u(x)|^s dx \right)^{a/s}. \tag{2.16}
\]
One observes that, the last inequality holds for all admissible values of the parameters. Moreover, in order to prove \((1.1)\) it is enough to show the simpler inequality \(2.12\).

**Remark 2.2.** From equations \((2.12)\) and \((2.16)\) the role of the new parameter \(s\) is clear: We pass from the analysis of the parameter \(\gamma\), that is, in a bidimensional parameter space, to the analysis of the parameter \(s\), that is, in a one dimensional parameter space.

At the end of this section, we state the principal theorem of this paper. First, we establish a useful relation between \(s, \sigma\) and \(n\). Let us recall the condition between \(\gamma, r\) and \(n\), that is
\[
\gamma r > -n. \tag{2.17}
\]
As defined in \((1.3)\), we have that
\[
\gamma = a\sigma + (1-a)\beta.
\]
On the other hand, using the relation \((1.4)\) we obtain
\[
r = sq \quad \frac{a\sigma + (1-a)\beta}{aq + (1-a)s}.
\]
Using these equalities in \((2.17)\), we have for all \(a \in [0,1]\),
\[
a[sq\sigma + nq] + (1-a)[ns + \beta sq] > 0.
\]
In particular, for \(a = 1\), we have
\[
s\sigma > -n. \tag{2.18}
\]

**Theorem 2.3.** Let \(p \geq 1, \alpha, \) and \(\sigma\) be such that \(\alpha p > -n, \sigma \leq \alpha\). Consider \(s\) as defined in \((2.10)\) satisfying \((2.18)\). Then, there exists \(C > 0\), such that \((2.12)\) holds, that is
\[
\left( \int \|x\|^s |u(x)|^s dx \right)^{1/s} \leq C \left( \int \|x\|^\alpha \|\nabla u(x)\|^p dx \right)^{1/p}.
\]
Moreover, when \(s \in [p, p^*]\), the constant \(C\) is bounded.
2.1 Proof of Theorem 2.3: The case \( s \in [p, p^*] \)

The strategy to show the inequality (2.12), it will be to interpolate the end-point values of \( s \). As observed from the definition of \( s \), this parameter only depends of the value of \( \sigma \), that is, for each value of \( \sigma \) we obtain different values of \( s \). In the following table, we summarize the values of \( \sigma \) and the corresponding values of \( s \), that we consider here.

\[
\begin{array}{|c|c|c|}
\hline
\sigma & \alpha - 1 & \alpha \\
\hline
s & p & p^* \\
\hline
\end{array}
\]

The proof of (2.12) when \( s = p \) and \( s = p^* \) can be found in the Appendix. Then, for \( s \in (p, p^*) \), first we write \( \sigma \) conveniently as

\[ \sigma = (1 - \theta)(\alpha - 1) + \theta \alpha, \]

where \( \theta \in [0, 1] \) and \( s = (1 - c)p + cp^* \), where \( c \in [0, 1] \) is given by

\[ c = \frac{\theta(n - p)}{n - \theta p}. \]

It follows that

\[ \sigma s = (1 - c)(\alpha - 1)p + \alpha cp^*, \quad (2.19) \]

since we have the following relation

\[ \theta(1 - c)p = (1 - \theta)cp^*. \]

Now using (2.19), we obtain

\[
\int \|x\|^{\sigma s} |u(x)|^s \, dx = \int \|x\|^{(1-c)(\alpha-1)p + \alpha cp^*} |u(x)|^{(1-c)p + cp^*} \, dx
\]

\[
= \int \left( \|x\|^{(\alpha-1)p} |u(x)|^p \right)^{(1-c)} \left( \|x\|^{\alpha p^*} |u(x)|^{p^*} \right)^c \, dx
\]

\[
\leq \left( \int \|x\|^{(\alpha-1)p} |u(x)|^p \, dx \right)^{1-c} \left( \int \|x\|^{\alpha p^*} |u(x)|^{p^*} \, dx \right)^c,
\]

where we have applied Hölder’s inequality. Therefore, for each \( s \in [p, p^*] \) fixed

\[
\left( \int \|x\|^{\sigma s} |u(x)|^s \, dx \right)^{1/s} \leq C \left( \int \|x\|^{\alpha p} \|\nabla u(x)\|^{p^*} \, dx \right)^{1/p},
\]

where the constant \( C \) is given by

\[ C = C_H^{(1-c)p/s} C_S^{cp^*/s}. \]
2.2 Proof of Theorem 2.3. The case \( s \in (0, p) \)

Let \( 0 < s < p \) be fixed and define

\[
\kappa := \frac{1}{s} - \frac{1}{p},
\]

which is a positive number. Then from the definition of \( s \), we could write

\[
\sigma = (\alpha - 1) - \kappa n,
\]

and we have

\[
\int \|x\|^{\sigma s} |u(x)|^s \, dx = \int \|x\|^{(\alpha-1)s} |u(x)|^s \|x\|^{-\kappa ns} \, dx
\]

\[
\leq \left( \int \|x\|^{(\alpha-1)p} |u(x)|^p \, dx \right)^{s/p} \left( \int \|x\|^{-n} \, dx \right)^{(p-s)/p},
\]

where we have used the Hölder inequality with \( 1/\tilde{p} + 1/\tilde{q} = 1 \), for \( \tilde{p} = p/s > 1 \).

Denoting \( U = \text{spt}(u) \), \( R = \sup_{x \in U} \|x\| \), \( r = \inf_{x \in U} \|x\| \), we get the following

\[
\left( \int \|x\|^{\sigma s} |u(x)|^s \, dx \right)^{1/s} \leq \left( \int \|x\|^{(\alpha-1)p} |u(x)|^p \, dx \right)^{1/p} \left( \ln \frac{R}{r} \right)^{(p-s)/sp}
\]

\[
\leq C \left( \int \|x\|^{\alpha p} \|\nabla u(x)\|^{p} \, dx \right)^{1/p},
\]

where \( C = C_H \left( \ln \frac{R}{r} \right)^{\kappa} \).

3 The Riemannian case

In this section, we study the Caffarelli-Kohn-Nirenberg inequality \((1.5)\), that is, the general inequality in the Riemannian setting. The proof may follow the same ideas as before, hence we just state Lemma 3.3 and Theorem 3.4, which are adapted versions from the Euclidean case. In fact, we focus here to describe in details, the main differences which occurs due the inequality \((1.5)\) be posed on complete and non-compact Riemannian manifolds.

First of all, the inequality \((1.1)\) in \( \mathbb{R} \) was defined using weight functions of \( \|x\|^r \) type, for some \( r \in \mathbb{R} \). The homogeneity of this type of weight functions was one of the main ingredients in the original proof of the inequality \((1.1)\). Since then, many modifications on the weights, in particular cases of the \((1.1)\), have been considered. For instance, it was considered in \( [6] \) a cylindrical weight, i.e., a weight function of the form

\[
w(y) = \|y\|,
\]
where \( y \) is the projection of a point \( x \) in \( \mathbb{R}^n \) onto \( \mathbb{R}^{n-k} \) (that is \( x = (x_0, y) \), with \( x_0 \in \mathbb{R}^k \) and \( y \in \mathbb{R}^{n-k} \)). Also, it was considered in [16] the following type of weight

\[
    w(x) = \begin{cases} 
    \log \left( \frac{1}{|x|} \right) & \text{if } n = 1, \\
    \log \left( \frac{R}{||x||} \right) & \text{if } n \geq 2,
\end{cases}
\]

where \( R > 1 \). Moreover, it can be found in [24] an approach of the inequality (1.1) with general weights and also a remainder term. In all these cases, the relationship between the parameters do not follow the conditions of the original theorem, because the weights are not necessary homogeneous. Similar condition happens for weighted inequalities on manifolds, from obvious reason. In fact, it does not have a standard way to consider weighted inequalities on manifolds.

Concerning the Sobolev inequality in its weighted version, the distance function is a commonly used weight function, see [19, 25], but it is not a consensus. In a different and interesting direction, it was used in [7] the existence of a conformal Killing vector field \( h \) (see the definition below) on a complete \( n \)-manifold \( M \), \( n \geq 3 \), to prove the following inequality

\[
    \int_M \| h \|^{-p} |u|^p \, dV \leq \left( \frac{|n-p|}{p} \right)^{-p} \int_M ||\nabla u||^p \, dV,
\]

where \( p > 1 \), \( u \in W^{1,p}(M) \) and \( M \) admits a \( C^1 \) conformal Killing vector field, such that \( \text{div}_g h = n \). In that paper, more general inequalities were proved, but the case of the weighted Sobolev inequality was left open. On the other hand, it was considered in [19] both inequalities: a weighted Sobolev and a weighted Hardy, with the distance function being the weight function. It is given in that paper the proof of both inequalities under the volume growth assumption, which is not maximal, but the volume satisfies a doubling condition.

Another important difference in the Riemannian setting (not necessarily with weights) from the Euclidean case, is concerned an extra term which appears on the right hand side, more precisely, let us consider the Sobolev inequality. It can be found in [23], Theorem 3.3.10, the following version of the Sobolev inequality: if \( M \) is a complete \( n \)-manifold, \( U \subset M \) is any open precompact region, \( u \in C_0^\infty(U) \), and \( p \in [1, n) \), then there exists a constant \( C(U, p) \) such that

\[
    \left( \int_U |u(x)|^{p^*} \, dV \right)^{1/p^*} \leq C(U, p) \left[ \left( \int_U \|\nabla u(x)\|^p \, dV \right)^{1/p} \right. \\
    \left. + \left( \int_U |u(x)|^p \, dV \right)^{1/p} \right].
\]

Therefore, we have an extra term which does not appear in the Euclidean setting. Although, as mentioned in [24], under conditions about the volume growth and the Ricci curvature (\( \text{Ric} \geq 0 \)), see Section 3.3.5, applying a pseudo-Poincaré
type inequality, we obtain Theorem 3.3.11, where there does not exist the second integral on the right hand side of the above inequality.

Finally, it is very important to observe that, as it was showed in [15], it could happen a surprising phenomena in the Riemannian setting: For any integer \( n \geq 2 \), there exist a smooth, complete \( M \) Riemannian manifold, such that, for each \( p \in [1, n) \), \( W^{1,p}(M) \) does not embed in \( L^p(M) \), see [15], Chapter 3, Proposition 3.3. Therefore, \( M \) be complete, is not a sufficient condition in order to avoid this surprising phenomena.

In this section, we consider the inequality (1.5) for functions in \( W^{1,p}(M) \), where \( M \) is a complete non-compact Riemannian \( n \)-manifold \( (n \geq 3) \) with maximal volume growth, \( Ric \geq 0 \), and the weight function is a conformal Killing vector field. In fact, we assume that the functions verifying (1.5) are in \( C^\infty_c(U) \), whereas before \( U \subset M \) is any open precompact region. The general case, that is \( W^{1,p}(U) \) can be obtained by a standard density argument. One remarks that, the existence of conformal Killing vector fields, in the case of a closed manifold (that is, a compact manifold without boundary) implies that, the Ricci curvature is non-negative, see [26].

Now, we recall the definition of a conformal Killing vector field on a Riemannian manifold.

**Definition 3.1.** Let \((M, g)\) be a complete \( n \)-dimensional Riemannian manifold \((n \geq 3)\), with \( g \) the Riemannian metric. Using local coordinates \((x^i)_{i=1}^n\), we have that \( g \) = \((g^{ij})\). A nontrivial conformal Killing vector field \( h = h^i \frac{\partial}{\partial x^i} \), is a vector field on \( M \), such that

\[
\nabla^i h^j + \nabla^j h^i = \frac{2}{n} (\text{div}_g h) g^{ij} =: \mu g^{ij}.
\]

We observe that, \( \nabla^i (\cdot) \) is the covariant derivative corresponding to the Levi-Civita connection, which is uniquely determined by the metric \( g \), \( (g^{ij}) \) is the inverse matrix of \( (g_{ij})\), and \( \text{div}_g h \) is the covariant divergence operator.

Following [23], we define the maximal volume growth condition of geodesic balls on manifolds. For this, let \( V(x, t) \) be the volume of a geodesic ball of radius \( t > 0 \) around a point \( x \) on a manifold \( M \). Then, we have the following

**Definition 3.2.** Let \((M, g)\) be a complete Riemannian manifold. We say that \( M \) has a **maximal volume growth**, if there exist a \( c > 0 \) such that

\[
\forall r > 0, \quad V(x, r) \geq cr^n.
\]

It is important to remark that, the above definition in [23] (see page 82), appears associated to the condition that \( Ric \geq 0 \), i.e., the Ricci curvature is non-negative, such that \( M \) satisfies the pseudo-Riemannian inequality.

Then, we are in condition to state the principal results of this section. First, as in the Euclidean case, using the same parameter \( s \) as defined in (2.10), we have the following
Lemma 3.3. Let $(M, g)$ be a complete non-compact Riemannian $n$-manifold, $n \geq 3$, with maximal volume growth, $\text{Ric} \geq 0$, and let $U$ be any open precompact region in $M$. Assume conditions (1.2), (1.3), and $M$ admits a conformal Killing vector field $h$, with $\text{div}_g h = n$. If there exist $C > 0$ such that

$$\left( \int_U ||h||^{\sigma s} |u(x)|^s \, dV \right)^{1/s} \leq C \left( \int_U ||h||^{\alpha p} ||\nabla u(x)||^p \, dV \right)^{1/p}, \tag{3.20}$$

then the inequality holds.

And hence we pass on the main

Theorem 3.4. Under conditions of Lemma 3.3, let $p \geq 1$, $\alpha$, and $\sigma$ be such that $\alpha p > -n$, $\sigma \leq \alpha$. Consider $s$ as defined in (2.10) satisfying (2.18). Then, there exists $C > 0$, such that (3.20) holds. Moreover, when $s \in [p, p^*]$, the constant $C$ is bounded.

4 Appendix

In this last section, we first state the Hardy type inequality. The proof follows easily combining the ideas in [7] and [23]. It is important to note that, the inequality in the Euclidean case can be recovered with $h(x) = x$.

Theorem 4.1. Let $(M, g)$ be a complete non-compact Riemannian $n$-manifold with $n \geq 3$, and $U \subset M$ any open precompact region. If $M$ admits a conformal Killing vector field, there exists a positive constant $C$, such that the following inequality holds for all $u \in C^\infty_c(U)$

$$\int_U ||h||^{p(\alpha-1)} |u(x)|^p \, dV \leq C \int_U ||h||^{\alpha p} ||\nabla u(x)||^p \, dV. \tag{4.21}$$

It was stated and proved in [7], the following result about conformal Killing vector fields: Let $\epsilon$ be an arbitrary positive real number and $h$ a conformal Killing vector field, then

$$\text{div}_g \left( \frac{h}{\epsilon + ||h||^k} \right) = \frac{1}{2} \frac{\mu}{(\epsilon + ||h||^k)^2} (n\epsilon + (n-k)||h||^k),$$

where $k \in \mathbb{R}$. This result is used in that paper to prove a Caffarelli-Kohn-Nirenberg inequality (particular case) in the Riemannian setting. In our case, we take $k = (1 - \alpha)p$ in the above identity, and the proof of (4.21) is done using the same technique that appeared in [7].

Now, we state the weighted Sobolev inequality in the Riemannian setting, and give an original proof of it. One observes that, the maximal growth condition is necessary here, since along the proof we use standard Sobolev inequality (if this condition is not assumed, then the inequality can be false, see [13]).
Theorem 4.2. Let $M$ be a complete non-compact Riemannian $n$-manifold with maximal volume growth, $\text{Ric} \geq 0$, $n \geq 3$ and let $U$ be any open precompact region in $M$. If $M$ admits a conformal Killing vector field, then there exists a positive constant $C$, such that the following inequality holds for all $u \in C_c^\infty(U)$

$$
\left( \int_U \|h\|^{\alpha p^*} |u(x)|^{p^*} dV \right)^{1/p^*} \leq C \left( \int_U \|h\|^{\alpha p} \|\nabla u(x)\|^{p} dV \right)^{1/p}.
$$

Proof. 1. First, we consider the following result:

Claim: If $h$ is a vector field on $M$, then for almost all $x \in M$

$$
h(x) \cdot \nabla(\|h(x)\|) = \|h(x)\|,
$$

where the inner product is taking with respect to the Riemannian metric $(g_{ij})$.

Proof of Claim: The proof of (4.22) follows the ideas in Cordero-Erausquin, Nazaret, Villani [12]. Given $\epsilon > 0$, we define a function $\lambda$, such that $\lambda : (1-\epsilon, 1+\epsilon) \rightarrow X(M)$, where $X(M)$ is the space of vector fields on $M$. Observe that, $\lambda$ is differentiable in $(1-\epsilon, 1+\epsilon)$, and $\lambda'(t) = h$. To follow, we define the function $\eta$ as $\eta(t) = \|\lambda(t)\|$. Then, we have for almost all $t$,

$$
\eta'(t) = \nabla \|h\| \cdot h
$$

On the other hand, using the definition of $\eta$, we obtain $\eta'(t) = \|h\|$. Thus, we obtain

$$
h \cdot \nabla \|h\| = \|h\|.
$$

2. For any $g \in C_c^\infty(U)$, we have for each open precompact region $U \subset M$

$$
\left( \int_U |g|^{p^*} dV \right)^{1/p^*} \leq C \left( \int_U \|\nabla g\|_*^{p} dV \right)^{1/p},
$$

where $C$ is a positive constant and $\|\cdot\|_*$ denotes the dual norm of a vector field. Also, we recall the following simple inequality of real numbers: If $a, b \in \mathbb{R}$ and $k \geq 1$, then

$$
(a + b)^k \leq 2^{k-1}(a^k + b^k).
$$

Now, defining $f(x) := \|h\|^{\alpha} u(x)$, we have

$$
\int_U |f(x)|^{p^*} dV = \int_U \|h\|^{\alpha p^*} |u(x)|^{p^*} dV.
$$

On the other hand

$$
\nabla f(x) = \alpha \|h\|^{\alpha-1} u(x) \nabla(\|h\|) + \|h\|^{\alpha} \nabla u(x),
$$

(4.26)
and making the inner product between \( h \) and \( \nabla f(x) \), we have
\[
 h \cdot \nabla f(x) = \alpha \|h\|^{\alpha-1} u(x) h \cdot \nabla \|h\| + \|h\|^\alpha h \cdot \nabla u(x).
\]
From the above equality and (4.22), it follows that (\( h \neq 0 \), \( h = 0 \) is trivial)
\[
 \tilde{h} \cdot \nabla f(x) \leq \alpha \|h\|^{\alpha-1} |u(x)| + \|h\|^\alpha \|\nabla u(x)\|,
\]
where \( \tilde{h} = h/\|h\| \). Replacing \( h \) by \( -h \), we do not change the defining of \( f \).
Furthermore, it is not difficult to see that, we also have the estimate
\[
 -\tilde{h} \cdot \nabla f(x) \leq \alpha \|h\|^{\alpha-1} |u(x)| + \|h\|^\alpha \|\nabla u(x)\|.
\]
Moreover, from (4.26) we observe that, there exists \( K_0 > 0 \), such that
\[
 |\cos(\tilde{h}, \nabla f)| \leq K_0 \|\nabla f(x)\|.
\]
Consequently, we obtain from (4.27), (4.28) and the definition of the dual norm
\[
 K_0 \|\nabla f(x)\|_* \leq \alpha \|h\|^{\alpha-1} |u(x)| + \|h\|^\alpha \|\nabla u(x)\|.
\]
Hence applying (4.24), it follows that
\[
 \|\nabla f(x)\|_p \leq K_1 \|h\|^{(\alpha-1)p} |u(x)|^p + K_2 \|h\|^{\alpha p} |\nabla u(x)|^p,
\]
where \( K_1 \) and \( K_2 \) are positive constants.

3. Finally, we integrate (4.29) on \( U \) to obtain
\[
 \left( \int_U \|\nabla f\|_p^p \, dV \right)^{p'/p} \leq K_3 \left( \int_U \|h\|^{(\alpha-1)p} |u(x)|^p \right)^{p'/p} + K_4 \left( \int_U \|h\|^{\alpha p} |\nabla u(x)|^p \right)^{p'/p}.
\]
In the first integral of the right hand side of the above inequality, we apply the weighted Hardy inequality (4.21), then
\[
 \left( \int_U \|\nabla f\|_p^p \, dV \right)^{p'/p} \leq K_5 \left( \int_U \|h\|^{\alpha p} |\nabla u(x)|^p \right)^{p'/p}.
\]
From (4.23), (4.25), and (4.30), we show the thesis of the theorem, that is
\[
 \left( \int_U \|h\|^{\alpha p^*} |u(x)|^{p^*} \, dV \right)^{1/p^*} \leq C \left( \int_U \|h\|^{\alpha p} \|\nabla u(x)\|_p^p \, dV \right)^{1/p}.
\]
\[\square\]

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