The Zubair-Inverse Lomax Distribution with Applications

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Author’s contribution

The sole author designed, analyzed, interpreted and prepared the manuscript.

Article Information

DOI: 10.9734/AJPAS/2020/v8i330206

Editor(s):

(1) Dr. Belkacem Chaouchi, Khemis Miliana University, Algeria.

Reviewer(s):

(1) Uzma Jan, University of Kashmir, India.
(2) Tabassum Sindhu, Quaid-i-Azam University, Pakistan.

Complete Peer review History: http://www.sdiarticle4.com/review-history/59438

Received: 20 May 2020
Accepted: 27 July 2020
Published: 11 August 2020

Original Research Article

Abstract

In this article, an extension of Inverse Lomax (IL) distribution with the Zubair-G family is considered. Various statistical properties of the new model were derived, including moment generating function, Rényi entropy, and order statistics. A Monte Carlo simulation study was presented to evaluate the performance of the maximum likelihood estimators. The new model can be skew to the right, constant, and decreasing functions depending on the parameter values. We discussed the estimation of the model parameters by maximum likelihood method. The application of the new model to the data sets indicates that the new model is better than the existing competitors as it has minimum value of statistics criteria.

Keywords: Zubair-G; Inverse Lomax; Simulation; Entropy; Monte Carlo.

2010 Mathematics Subject Classification: 00-01, 99-00.

1 Introduction

The Inverse Lomax (IL) distribution is used in different fields, like stochastic modeling, Economics, actuarial sciences and lifestyle testing; it is one of the leading predictive life-time models. Inverse
Lomax distribution is part of a Beta form distribution. Other family members include Singh maddala, Pareto, log-Logistics, Dagum, generalized second-type beta distributions among others as in [1]. Since then, IL distribution has gained a lot of coverage in many fields such as Actuarial Science and Economics (see [1]), Geophysical data (see [2]), Survival analysis (see [3]), and Medical Science (see [4] and [5]). Recently, [6] considered multicomponent stress-strength reliability for the IL distribution with different shape parameters.

The cdf and pdf of IL distribution are given by

\[ G(x; \beta, \lambda) = \left( 1 + \frac{\lambda}{x} \right)^{-\beta} \quad x > 0, \beta, \lambda > 0 \]  

\[ g(x; \beta, \lambda) = \beta \lambda x^{-2} \left( 1 + \frac{\lambda}{x} \right)^{-(1+\beta)} \quad x > 0, \beta, \lambda > 0 \]

The motivations for this research are; to provide a better fits than the other models of IL distribution with the same number of parameters, and to improve the flexibility and characteristics of the IL distribution.

2 The Zubair-G Family

Recently, [7] introduced the Zubair-G family of distributions with the cumulative density function (cdf) and probability density function (pdf) given by

\[ F_{ZG}(x; \alpha, \kappa) = \exp\left\{ \exp\left( \frac{\alpha G(x; \kappa)^2}{\exp(\alpha)} - 1 \right) \right\} \quad x > 0, \alpha > 0 \]

\[ f_{ZG}(x; \alpha, \kappa) = \frac{2\alpha g(x; \kappa)G(x; \kappa)\exp(\alpha G(x; \kappa)^2)}{\exp(\alpha) - 1} \quad x > 0, \alpha > 0 \]

Where \( \kappa \) is a vector of parameter/parameters for any baseline distribution.

3 The Zubair-Inverse Lomax Distribution

Based on Eqns. (2.1) and (2.2), we can insert Eqns. (1.1) and (1.2) and come up with the Zubair-Inverse Lomax (Z-IL) distribution. Below are the cdf, pdf, survival (\( S(x) \)), hazard (\( h(x) \)), reverse hazard (\( r(x) \)), and cumulative hazard (\( H(x) \)) functions of Z-IL distribution, respectively.

\[ F_{Z-IL}(x; \alpha, \beta, \lambda) = \exp\left\{ \frac{\exp(\alpha (1 + \frac{1}{\lambda})^{-\beta}) - 1}{\exp(\alpha)} \right\} \quad x > 0, \alpha, \beta, \lambda > 0 \]

\[ f_{Z-IL}(x; \alpha, \beta, \lambda) = \frac{2\alpha \beta x^{-2}(1 + \frac{1}{\lambda})^{-(1+\beta)} \exp(\alpha (1 + \frac{1}{\lambda})^{-\beta})}{\exp(\alpha) - 1} \quad x > 0, \alpha, \beta, \lambda > 0 \]

\[ S_{Z-IL}(x; \alpha, \beta, \lambda) = \frac{\exp(\alpha) - \exp(\alpha (1 + \frac{1}{\lambda})^{-\beta})}{\exp(\alpha) - 1} \quad x > 0, \alpha, \beta, \lambda > 0 \]

\[ h_{Z-IL}(x; \alpha, \beta, \lambda) = \frac{2\alpha \beta x^{-2}(1 + \frac{1}{\lambda})^{-(1+\beta)} \exp(\alpha (1 + \frac{1}{\lambda})^{-\beta})}{\exp(\alpha) - \exp(\alpha (1 + \frac{1}{\lambda})^{-\beta})} \quad x > 0, \alpha, \beta, \lambda > 0 \]
\begin{equation}
\tau_{Z-IL}(x; \alpha, \beta, \lambda) = \frac{2\alpha\beta^2 \lambda x^{-2}(1 + \frac{x}{\lambda})^{-(1+2\beta)} \exp\{\alpha (1 + \frac{x}{\lambda})^{-2\beta}\}}{\exp\{\alpha (1 + \frac{x}{\lambda})^{-2\beta}\} - 1}
\end{equation}
x > 0, \alpha, \beta, \lambda > 0 \quad (3.5)

and

\begin{equation}
H_{Z-IL}(x; \alpha, \beta, \lambda) = -\log \left[ \frac{\exp\{\alpha\} - \exp\{\alpha (1 + \frac{x}{\lambda})^{-2\beta}\}}{\exp\{\alpha\} - 1} \right]
\end{equation}

4 Statistical Properties of Zubair-Inverse Lomax (Z-IL) Distribution

In this section, we considered some of the statistical properties of the Z-IL distribution like moments, mgf, Rényi entropy, and order statistics.
4.1 Moments

Suppose \( X \) is a random variable with density function defined in Eqn. (3.2), the \( r \)th non-central moments of \( X \) is given by

\[
E(X^r) = \int_{-\infty}^{\infty} x^r f(x) \, dx
\]

\[
= 2 \sum_{i=0}^{\infty} \frac{\alpha^{i+1} \beta \lambda^r}{(\exp(\alpha) - 1)i!} \int_0^\infty x^{r-2} \left(1 + \frac{\lambda}{x}\right)^{-[2\beta(1+i)+1]} \, dx
\]

by letting \( y = \frac{x}{\alpha} \) and some simplifications, we have

\[
E(X^r) = 2 \sum_{i=0}^{\infty} \frac{\alpha^{i+1} \beta \lambda^r}{(\exp(\alpha) - 1)i!} \int_0^\infty y^{r-2} \left(1 + \frac{\lambda}{y}\right)^{-[2\beta(1+i)+1]} \, dy
\]

\[
= 2 \sum_{i=0}^{\infty} \frac{\alpha^{i+1} \beta \lambda^r}{(\exp(\alpha) - 1)i!} B((1-r),(2\beta(1+i)+r)) \tag{4.1}
\]

where \( \int_0^\infty \frac{t^{n-r}}{(1+t)^{n+r}} \, dt = B(a,b) \) is Beta function of second kind.

4.2 Moment generating function

The moment generating function (mgf) of the Z-II distribution can be given in terms of 4.1 as

\[
M_x(x) = 2 \sum_{i,r=0}^{\infty} \frac{t^r \alpha^{i+1} \beta \lambda^r}{(\exp(\alpha) - 1)i!} B((1-r),(2\beta(1+i)+r)) \tag{4.2}
\]

4.3 Rényi entropy

If \( X \) is random variable with density function \( f(x) \) defined in Eqn. (3.2), then the Rényi entropy of \( X \) is given by

\[
R_\tau(x) = \frac{1}{1-\tau} \left[ \int_{-\infty}^{\infty} f(x)^\tau \, dx \right], \quad \tau > 0, \tau \neq 1; x \in \mathbb{R} \tag{4.3}
\]

the function \( f(x)^\tau \) in Eqn. (4.3) can be presented as

\[
f(x)^\tau = \left[ \frac{2\alpha \beta \lambda}{(\exp(\alpha) - 1)} \right]^{\tau} x^{-2\tau} \left(1 + \frac{\lambda}{x}\right)^{-\tau(1+2\beta)} \exp(\alpha \tau \left(1 + \frac{\lambda}{x}\right)^{-2\beta})
\]

By inserting Eqn. (4.4) back in Eqn. (4.3), we have

\[
\int_{-\infty}^{\infty} f(x)^\tau \, dx = \left[ \frac{2\alpha \beta \lambda}{(\exp(\alpha) - 1)} \right]^{\tau} \int_0^\infty x^{-2\tau} \left(1 + \frac{\lambda}{x}\right)^{-\tau(1+2\beta)} \exp(\alpha \tau \left(1 + \frac{\lambda}{x}\right)^{-2\beta}) \, dx
\]

\[
= 2 \sum_{i=0}^{\infty} \frac{t^\tau \alpha^{i+1} \beta \lambda^r}{(\exp(\alpha) - 1)i!} \int_0^\infty x^{-2\tau} \left(1 + \frac{\lambda}{x}\right)^{-\tau(1+2\beta)+2\beta i} \, dx \tag{4.5}
\]

Let \( w = \frac{\lambda}{x} \), after some simplifications we have

\[
\int_{-\infty}^{\infty} f(x)^\tau \, dx = 2 \sum_{i=0}^{\infty} \frac{t^\tau \alpha^{i+1} \beta \lambda^r}{(\exp(\alpha) - 1)i!} \int_0^\infty \frac{w^{2\tau-1-1}}{(1+w)^{\tau(1+2\beta)+2\beta i}} \, dw \tag{4.6}
\]

\[
= 2 \sum_{i=0}^{\infty} \frac{t^\tau \alpha^{i+1} \beta \lambda^r}{(\exp(\alpha) - 1)i!} \int_0^\infty \frac{w^{2\tau-1-1}}{(1+w)^{(\tau(1+2\beta)+2\beta i)}} \, dw \tag{4.7}
\]
By taking partial derivatives of Eqn. (3.2), the log-likelihood function of Eqn. (4.1) can be written as

\[ L(\eta) = n \log(2\alpha \beta \lambda) - 2 \sum_{i=1}^{n} \log(x_i) - (1 + 2\beta) \sum_{i=1}^{n} \log\left(1 + \frac{\lambda}{x_i}\right) \]

\[ + \alpha \sum_{i=1}^{n} \left(1 + \frac{\lambda}{x_i}\right)^{-2\beta} - n \log(\exp(\alpha) - 1) \]  

5 Estimation

In this section, the parameters of the proposed Z-IL distribution will be estimated using maximum likelihood method. Let \( x_1, x_2, x_3, \ldots, x_n \) be random samples of \( n \) observations drawn from the Z-IL distribution with parameter vector \( \eta = (\alpha, \beta, \lambda)^T \). The log-likelihood function of Eqn. (3.2) denoted by \( L(\eta) \) can be written as

\[ L(\eta) = n \log(2\alpha \beta \lambda) - 2 \sum_{i=1}^{n} \log(x_i) - (1 + 2\beta) \sum_{i=1}^{n} \log\left(1 + \frac{\lambda}{x_i}\right) \]

\[ + \alpha \sum_{i=1}^{n} \left(1 + \frac{\lambda}{x_i}\right)^{-2\beta} - n \log(\exp(\alpha) - 1) \]  

By taking partial derivatives of Eqn. (5.1) with respect to \( \alpha, \beta, \) and \( \lambda \), we derived the components of score vector \( U(\eta) \) presented as follows

\[ U_\alpha(\eta) = \frac{n}{\alpha} + \frac{n}{\beta} \sum_{i=1}^{n} \left(1 + \frac{\lambda}{x_i}\right)^{-2\beta} - \frac{n \exp(\alpha)}{(\exp(\alpha) - 1)} \]  

\[ U_\beta(\eta) = \frac{n}{\beta} - 2 \sum_{i=1}^{n} \log\left(1 + \frac{\lambda}{x_i}\right) + \alpha \sum_{i=1}^{n} \left(1 + \frac{\lambda}{x_i}\right)^{-2\beta} \log\left(\frac{\alpha n}{\sum_{i=1}^{n} \left(1 + \frac{\lambda}{x_i}\right)}\right) \]  

\[ U_\lambda(\eta) = \frac{n}{\alpha} - \alpha \sum_{i=1}^{n} \left(1 + \frac{\lambda}{x_i}\right)^{-2\beta} \]
\[ U_\lambda(\eta) = \frac{n}{N} + \sum_{i=1}^{n} \frac{(1 + 2\beta)}{(x_i + \lambda)} - 2\alpha\beta \sum_{i=1}^{n} \left(1 + \frac{\lambda}{x_i}\right)^{-2\beta-1} \] (5.4)

By setting Eqs. (5.2), (5.3), and (5.4) to zero and also solving them simultaneously yields the maximum likelihood estimators of the Z-IL distribution. However, the above equations are nonlinear they cannot be solved analytically. As such, the statistical software can be used to solve them numerically using numerical algorithm.

6 Monte Carlo Simulation

Here, a simulation study is conducted and presented to show the estimates’ performance at various parameter values. Monte Carlo method is any computational technique using pseudo-random numbers to solve mathematical problems as defined by [8]. The numerical study is as follows:

1. For known parameter values i.e \( \eta = (\alpha, \beta, \lambda)^T \), we simulated a random sample of size \( n \) from the Z-IL distribution using Eqn. (6.2).
2. We then estimate the parameters of the Z-IL distribution by using MLE.
3. Perform 10,000 replications of steps 1 through 2.
4. For each of the three (3) parameters of the Z-IL, we compute the Bias, MSE, Variance, and Estimates of the parameters from the 10,000 parameter estimates. We considered two (2) cases; case 1 \( (a = .6, \beta = .4, and \lambda = .5) \) whereas in case 2 \( (a = 1, \beta = .4, and \lambda = .5) \). The statistics are given by

\[ \hat{\eta} = \frac{1}{10,000} \sum_{i=1}^{10,000} \hat{\eta}_i, \quad Bias(\hat{\eta}) = \hat{\eta} - \eta, \]
\[ var(\hat{\eta}) = \frac{\sum_{i=1}^{10,000} (\hat{\eta}_i - \hat{\eta})^2}{10,000} \]
\[ MSE(\hat{\eta}) = var(\hat{\eta}) + (Bias(\hat{\eta}))^2 \] (6.1)

where \( \hat{\eta} = (\hat{\alpha}, \hat{\beta}, \hat{\lambda}) \) are the mle for each iteration \((n = 30, 50, 75, 125, 200, 300, 500, 600, 700)\). The quantile function for Z-IL is given by

\[ Q_{Z-IL}(u) = \left[ \log\left(U(exp(\alpha) - 1) + 1\right) - \frac{\lambda}{\alpha} \right]^{\frac{1}{-1}} - 1 \] (6.2)

The numerical results are presented in Table 1. The simulation study has shown that irrespective of the parameter values chosen, the Bias and MSE of the parameter estimates (for both cases) decreases as the sample size \( n \) increases. Thus, the larger the sample size, the more accurate are the estimates of the parameters. The estimates are good in both cases, as they approaches the true parameter values as the sample sizes increases.

7 Application

We illustrate the applicability of the Z-IL distribution to three (3) data sets. The first data set represents the sum of skin folds in 202 athletes collected at the Australian Institute of Sports as in [9], and [10]. The summary of the data set includes: Minimum = 28, Maximum = 200, Mean = 69.02, Median = 58.60, Skewness = 1.1747, and Kurtosis = 4.3651. The second data set represents the airborne communication transceiver as reported by [11]. The summary of the data set includes:
Table 1. A simulation results of Z-IL distribution for the 2 cases

| n   | Bias | MSE  | Est. | Case 1 | Bias | MSE  | Est. | Case 2 |
|-----|------|------|------|--------|------|------|------|--------|
|     |      |      |      | $\alpha=0.6$ |      |      |      | $\beta=0.4$ |      |      | $\lambda=0.5$ | $\alpha=1$ | $\beta=0.4$ | $\lambda=0.5$ |
| 30  | 0.2689 | 0.1594 | 0.8689 | 0.056 | -0.0633 | 0.3158 | 0.1286 |
| 50  | 0.3823 | 2.0148 | 0.9822 | 0.0379 | 0.0085 | 0.0778 | 0.1159 |
| 75  | 0.4536 | 2.2726 | 1.0536 | 0.0276 | 0.1555 | 0.0185 | 0.0984 |
| 125 | 0.4087 | 2.2569 | 1.0086 | 0.0123 | 0.1312 | -0.0029 | 0.0918 |
| 200 | 0.3966 | 2.0441 | 0.9966 | 0.0181 | 0.1277 | -0.0106 | 0.0835 |
| 300 | 0.3586 | 1.7518 | 0.9588 | 0.0145 | 0.1489 | -0.0149 | 0.0734 |
| 500 | 0.2349 | 1.1479 | 0.8349 | 0.0161 | 0.0544 | -0.0017 | 0.0683 |
| 600 | 0.1921 | 0.9771 | 0.7921 | -0.0136 | 0.0199 | -0.0098 | 0.0659 |
| 700 | 0.1749 | 0.8399 | 0.7749 | 0.0165 | 0.0093 | -0.0094 | 0.0632 |

Minimum = .5, Maximum = 24.5, Mean = 4.013, Median = 2.1, Skewness = 2.717, and Kurtosis = 10.543. The Third data set represents the stress-rupture life of kevlar 49/epoxy strands that are subjected to constant sustained pressure at the 90 per cent stress level until all have failed as reported by [12], [13], and [14]. The summary of the data set includes: Minimum = 1, Maximum = 789, Mean = 102.5, Median = 8, Skewness = 3, and Kurtosis = 16.71. The plots of the three data sets are in Figs. (3), (4), and (5), respectively. The data sets are presented in the Appendix B.

We used AdequacyModel package by [15] in R developed by [16]. The goodness of fit statistic used in comparing the performances includes Akaike information criteria (AIC), Bayesian information criteria (BIC), consistent akaike information criterion (CAIC), and Hannan Quinn information Criteria (HQIC). The smaller the value of the goodness of fit measures the better the fit to the data.
Fig. 3. The Density and CDF plots for the Sum of Skins Data Set

Fig. 4. The Density and CDF plots for the Airborne Data Set
Empirical density

Cumulative distribution

Fig. 5. The Density and CDF plots for the Failure Time Data Set

Table 2. Competing Models with Z-IL distribution

| Models | Reference |
|--------|-----------|
| IPL    | [17]      |
| APTIL  | [18]      |
| MOIL   | [19]      |

Table 3. The MLEs, statistics, and -log likelihood for the Z-IL distribution with its competitors

| Data Sets | Models | Estimates | Statistics | -log likelihood |
|-----------|--------|-----------|------------|-----------------|
|           |        | $\alpha$  | $\beta$  | $\gamma$  | $\delta$  | AIC  | BIC  | CAIC | HQIC | -ll  |
| Data Set1 | Z-IL   | 0.1232    | 0.1893   | 0.1893   | 4388.526 | 4398.451 | 4388.647 | 4392.541 | 2191.263 |
|           | IPL    | 0.1129    | 0.1058   | 0.1714   | 4422.094 | 4432.528 | 4422.725 | 4426.619 | 2208.302 |
|           | APTIL  | 0.1806    | 0.1988   | 0.1981   | 5048.348 | 5058.273 | 5048.47  | 5052.364 | 2521.174 |
|           | MOIL   | 0.1977    | 0.1977   | 0.1925   | 5323.575 | 5333.499 | 5323.666 | 5327.59  | 2658.787 |
|           | IL     | 0.1981    | 0.198    | 0.198    | 4654.587 | 4661.204 | 4654.647 | 4657.264 | 2325.294 |
| Data Set2 | Z-IL   | 0.1812    | 0.1992   | 0.1897   | 351.0375 | 356.1041 | 351.7041 | 352.8694 | 172.5187 |
|           | IPL    | 0.1976    | 0.1976   | 0.1643   | 376.5536 | 381.6202 | 377.2202 | 378.3855 | 185.2768 |
|           | APTIL  | 0.1769    | 0.1819   | 0.1773   | 503.1921 | 508.2587 | 503.8502 | 505.024  | 248.596  |
|           | MOIL   | 0.1695    | 0.1695   | 0.1778   | 572.3807 | 577.4473 | 573.0474 | 574.2126 | 283.1903 |
|           | IL     | 0.1987    | 0.1987   | 0.198    | 469.654  | 463.0318 | 469.9783 | 470.8753 | 202.827  |
| Data Set3 | Z-IL   | 0.1602    | 0.1828   | 0.1807   | 2140.866 | 2148.713 | 2141.15  | 2144.044 | 1067.434 |
|           | IPL    | 0.0113    | 0.1058   | 0.0171   | 2178.36  | 2186.205 | 2178.607 | 2181.536 | 1086.18  |
|           | APTIL  | 0.1933    | 0.1883   | 0.1947   | 2462.68  | 2470.525 | 2462.927 | 2465.856 | 1228.34  |
|           | MOIL   | 0.1977    | 0.1977   | 0.1925   | 2594.969 | 2602.814 | 2595.216 | 2598.145 | 1294.484 |
|           | IL     | 0.1981    | 0.198    | 0.198    | 2258.795 | 2263.025 | 2259.917 | 2261.912 | 1127.857 |
The competing models for the Z-IL model are presented in Table (2). They are the Inverse power lomax (IPL), the Aplha Power Transformed Inverse Lomax (APTIL), and the Marshall-Olkin Inverse Lomax. As shown in Table (3), the Z-IL model is the best with a minimum values of all the statistics. Its ranked number 1 outperforming the other models.

8 Conclusion

In this paper, a new model based on Zubair-G family called the Zubair-Inverse Lomax (Z-IL) distribution was proposed and studied. Some of its statistical properties include Moments, Moment generating function, entropy and order statistics was investigated. The parameters of ZIL distribution were estimated using maximum likelihood method. The pdf plots in Fig. (1) indicates that the shape can be skew to the right whereas, the hazard function plots explains the shape as constant, skew to the right, and decreasing. Moreover, the cdf converges to one as in Fig. (2). An example of real world data sets empirically shows the importance and value of the new model.

Competing Interests

Authors have declared that no competing interests exist.

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Appendices

#### Appendix A ####

#### Plots ####

PDF plot of Z-IL DISTRIBUTION

```r
library(zipfR)
rm(list=ls(all=TRUE))
x=seq(0,1.5,0.0001)
y=function(x,alpha=.7,beta=.8,lambda=.1) (2*alpha*beta*lambda*x^(-2)*(1+(lambda/x))^(1-(2*beta))*exp(alpha*(1+(lambda/x))^(-2*beta)))/(exp(alpha)-1)
plot(x,y(x),"l",col=1,lwd=2,ylab="f(x)",lty=1,ylim=c(0,3))
y=function(x,alpha=.7,beta=.8,lambda=.1) (2*alpha*beta*lambda*x^(-2)*(1+(lambda/x))^(1-(2*beta))*exp(alpha*(1+(lambda/x))^(-2*beta)))/(exp(alpha)-1)
curve(y,add=T,col=2,lwd=2,lty=2)
y=function(x,alpha=1,beta=.8,lambda=.1) (2*alpha*beta*lambda*x^(-2)*(1+(lambda/x))^(1-(2*beta))*exp(alpha*(1+(lambda/x))^(-2*beta)))/(exp(alpha)-1)
curve(y,add=T,col=3,lwd=2,lty=3)
```
\[ y(x) = \frac{(2\alpha \beta \lambda x^{-2} \times \left(1 + \frac{\lambda}{x}\right)^{-1 - 2\beta} \times e^{\alpha \left(1 + \frac{\lambda}{x}\right)^{-2\beta}})}{e^{\alpha} - 1} \]

\[ y(x) = \frac{(2\alpha \beta \lambda x^{-2} \times \left(1 + \frac{\lambda}{x}\right)^{-1 - 2\beta} \times e^{\alpha \left(1 + \frac{\lambda}{x}\right)^{-2\beta}})}{e^{\alpha} - e^{\alpha \left(1 + \frac{\lambda}{x}\right)^{-2\beta}}} \]

\[ \text{legend("topright", legend=c(} \]
\[ \text{expression(paste(alpha==.8,"",beta==.9,"",lambda==.1))}, \]
\[ \text{expression(paste(alpha==1,"",beta==.4,"",lambda==.4))}, \]
\[ \text{expression(paste(alpha==.7,"",beta==.8,"",lambda==.2))}, \]
\[ \text{expression(paste(alpha==1.7,"",beta==.5,"",lambda==.9))}, \]
\[ \text{lw=2,col=c(1,2,3,4,6),text.width=.5,cex=0.7,fill=c(1,2,3,4,6)}) \]

### Hazard Plot of Z-IL DISTRIBUTION

```r
library(zipfR)
rm(list=ls(all=TRUE))
x=seq(0,1.5,0.0001)
y=function(x,alpha=.8,beta=.9,lambda=.1) (2*alpha*beta*lambda*x^(-2)*\left(1+\frac{\lambda}{x}\right)^{-1-2\beta} \times e^{\alpha \left(1 + \frac{\lambda}{x}\right)^{-2\beta}})/(e^{\alpha} - e^{\alpha \left(1 + \frac{\lambda}{x}\right)^{-2\beta}})
pplot(x,y(x),"l",col=1,lwd=2,ylab="h(x)",lty=1,ylim=c(0,3))
```
expression(paste(alpha==1.7","","beta==.5","","lambda==.9")),
lwd=2,col=c(1,2,3,4,6), text.width=5,cex=0.7,fill=c(1,2,3,4,6))

library(zipfR)
m(1:5,0.0001)

x=seq(0,1.5,0.0001)
y=function(x,alpha=.8,beta=.9,lambda=.1) (exp(alpha *
*(1+(lambda/x)))^(-2*beta)-1)/(exp(alpha)-1)
plot(x,y(x),"l",col=1,lwd=2,ylab="c(x)",lty=1,ylim=c(0,1.5))

y=function(x,alpha=1,beta=.4,lambda=.4) (exp(alpha *
*(1+(lambda/x)))^(-2*beta)-1)/(exp(alpha)-1)
curve(y,add=T,col=3,lwd=2,lty=3)

y=function(x,alpha=.7,beta=.8,lambda=.2) (exp(alpha *
*(1+(lambda/x)))^(-2*beta)-1)/(exp(alpha)-1)
curve(y,add=T,col=4,lwd=2,lty=4)

y=function(x,alpha=1.7,beta=.5,lambda=.9) (exp(alpha *
*(1+(lambda/x)))^(-2*beta)-1)/(exp(alpha)-1)
curve(y,add=T,col=6,lwd=2,lty=5)

legend("topleft",legend=c(
expression(paste(alpha==.8","","beta==.9","","lambda==.1")),
expression(paste(alpha==.7","","beta==.5","","lambda==.6")),
expression(paste(alpha==1","","beta==.4","","lambda==.4")),
expression(paste(alpha==.7","","beta==.8","","lambda==.2")),
expression(paste(alpha=1.7","","beta==.5","","lambda==.9"))),
lwd=2,col=c(1,2,3,4,6), text.width=5,cex=0.7,fill=c(1,2,3,4,6))
#

### Appendix B ######

#### Data Sets #####

# airborne communication transceiver data.
x=c(0.50, 0.60, 0.60, 0.70, 0.70, 0.70, 0.80, 0.80, 1.00, 1.00, 1.00,
1.00, 1.10, 1.30, 1.50, 
1.50, 1.50, 2.00, 2.00, 2.20, 2.50, 2.70, 3.00, 3.00, 3.30,
4.00, 4.00, 4.50, 4.70,5.00, 5.40, 5.40, 7.00, 7.50, 8.80, 9.00,
10.20, 22.00, 24.50)

# failure times in hours stress-rupture life of kevlar 49/epoxy strands
x=c(001,001,002,002,003,003,004,005,006,007,007,008,009,009,010,010,
011,012,013, 018,019,020,023,
024,024,029,035,036,038,040,042,043,052,054,056,060,060,063,065,067,
068,072,072, 073,079,079,080,080,083,085,090,092,095,099,100,101,102, 103,105,110,111,115,118, 120,129,131,133,134,140,143,145,150,151,152, 153,154,154,155,158,160,163,164, 180,180,181,202,205,214,217,233,303,303, 334,420,469,789)

#sum of skins
x<-c(28.0, 98, 89.0, 68.9, 69.9, 109.0, 52.3, 52.8, 46.7, 82.7, 42.3, 109.1, 96.8, 98.3, 103.6, 110.2, 98.1, 57.0, 43.1, 71.1, 29.7, 96.3, 102.8, 80.3, 122.1, 71.3, 200.8, 80.6, 65.3, 78.0, 65.9, 38.9, 56.5, 104.6, 74.9, 90.4, 54.6, 131.9, 68.3, 52.0, 40.8, 34.3, 44.8, 105.7, 126.4, 83.0, 106.9, 88.2, 33.8, 47.6, 42.7, 41.5, 34.6, 30.9, 100.7, 80.3, 91.0, 156.6, 95.4, 43.5, 61.9, 35.2, 50.9, 31.8, 44.0, 56.8, 75.2, 76.2,101.1, 47.5, 46.2, 38.2, 49.2, 49.6, 34.5, 37.5, 75.9, 87.2, 52.6, 126.4, 55.6, 73.9, 43.5, 61.8, 88.9, 31.0, 37.6,52.8, 97.9, 111.1, 114.0, 62.9, 36.8, 56.8, 46.5, 48.3, 32.6, 31.7, 47.8, 75.1, 110.7, 70.0, 52.5, 67, 41.6, 34.8, 61.8, 31.5, 36.6, 76.0, 65.1, 74.7, 77.0, 62.6, 41.1, 58.9, 60.2, 43.0, 32.6, 48, 61.2, 171.1, 113.5, 148.9, 49.9,59.4, 44.5, 48.1, 61.1, 31.0, 41.9, 75.6, 76.8, 99.8, 80.1, 57.9, 48.4, 41.8, 44.5, 43.8, 33.7, 30.9, 43.3, 117.8, 80.3, 156.6, 109.6, 50.0, 33.7, 54.0, 54.2, 30.3, 52.8, 49.5, 90.2, 109.5, 115.9, 98.5, 54.6, 50.9, 44.7, 41.8, 38.0, 43.2,70.0, 97.2, 123.6, 181.7, 138.3, 42.3, 40.5, 64.9, 34.1, 55.7, 113.5, 75.7, 99.9, 91.2, 71.6, 103.6, 46.1, 51.2, 43.8, 30.5, 37.5, 96.9, 57.7, 125.9, 49.0, 143.5, 102.8, 46.3, 54.4, 58.3, 34.0, 112.5, 49.3, 67.2, 56.5, 47.6, 60.4, 34.9)

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