Tempered modules in exotic Deligne-Langlands correspondence

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April 27, 2010

Abstract

The main purpose of this paper is to produce a geometric realization for tempered modules of the affine Hecke algebra of type $C_n$ with arbitrary, non-root of unity, unequal parameters, using the exotic Deligne-Langlands correspondence ([Ka09]). Our classification has several applications to the structure of the tempered Hecke algebra modules. In particular, we provide a geometric and a combinatorial classification of discrete series which contain the $\text{sgn}$ representation of the Weyl group, equivalently, via the Iwahori-Matsumoto involution, of spherical cuspidal modules. This last combinatorial classification was expected from [HO97], and determines the $L^2$-solutions for the Lieb-McGuire system.

Introduction

The main purpose of this paper is to study tempered modules of the affine Hecke algebra $\mathbb{H}$ of type $C_n$ (Definition 1.1) using the framework in [Ka09]. Here $\mathbb{H}$ is an algebra over $\mathcal{A} = \mathbb{C}[[q_0^{\pm 1}, q_1^{\pm 1}, q_2^{\pm 1}]]$, where $q_0, q_1, q_2$ are three indeterminate parameters.

The main case we deal with in this paper is the affine Hecke algebra $\mathbb{H}_{n,m}$ of type $C_n$ with unequal parameters (see §1.3). The algebra $\mathbb{H}$ specializes to $\mathbb{H}_{n,m}$ by the specialization $(q_0, q_1, q_2) \mapsto (-1, q^m, q)$ with $q \in \mathbb{C}^\times$ and $m \in \mathbb{R}$. These algebras, for special values of $m$, appear as convolution algebras in the theory of $p$-adic groups. For example, if $m = 1$ or $m = 1/2$ (and an appropriate value of $q$), they correspond to Iwahori-Hecke algebras for split $p$-adic $SO(2n+1)$ or $PSp(2n)$, respectively. More generally, when $m \in \mathbb{Z}_{\geq 0} + \epsilon$, where $\epsilon \in \{0, 1/2, 1/4\}$, (graded versions of) these algebras appear from representations of $p$-adic groups with unipotent cuspidal support, in the sense of [Lu95a].

Set $G = Sp(2n, \mathbb{C})$, let $T$ denote its diagonal torus, and let $W = N_G(T)/T$. The algebra $\mathbb{H}$ has a large abelian subalgebra $\mathcal{A} \otimes R(T)$, and the tempered and discrete series $\mathbb{H}$-modules are defined by the Casselman criterion for the generalized $R(T)$-weights (Definition 1.7). By a result of Bernstein and Lusztig, the

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center of $H$ is $A \otimes R(T)^W$. Therefore, the central characters of irreducible $H$-modules are in one-to-one correspondence with $W$-conjugacy classes of semisimple elements of the form $a = (s, q_0, q_1, q_2)$, where $s \in T$. The $H$-action on a module with central character $a$ factors through a finite-dimensional algebra $H_a$. We say that the central character $a$ is positive real if $-q_0, q_1 \in \mathbb{R}_{>0}, q_2 \in \mathbb{R}_{>1}$, and $s$ is hyperbolic (see Definition 1.3). The algebra $H_a$ contains a copy of $\mathbb{C}[W]$, for positive real $a$ (see [Lu89]).

We are interested in the identification of tempered $H$-modules for non-root of unity parameters. Via Lusztig’s reduction ([Lu89]), this can essentially be reduced to the determination of the tempered modules and discrete series with positive real central character for the case of $H_{a,m}$. For the algebra $H_{n,m}$, we say that the parameter $m \in \mathbb{R}$ is generic if $m \notin \frac{1}{2} \mathbb{Z}$. By [Op04, OS08] (also [Lu02a] for $m \in \mathbb{Z} + 1/4$), we know that the central characters which allow discrete series in the positive real generic range are in one-to-one correspondence with partitions $\sigma$ of $n$ (see Definition 3.1).

Example $\Lambda$ ($n = 2$ case). There are two central characters which afford discrete series corresponding to $\sigma_1 = (2)$ and $\sigma_2 = (1^2)$. The first discrete series is always the Steinberg module (corresponding to the sgn representation of the Weyl group), but the dimension and the $W$-module structure of the second discrete series depend on $m$.

We call a $s_{\sigma} \in T$ which affords discrete series distinguished. Our main result is the description of parameters corresponding to discrete series $H_{n,m}$-modules within the framework of the exotic Deligne-Langlands correspondence (eDL for short). To be more precise, let us recall it briefly (c.f. [L3] or [Ka09]). Let $V_1 = \mathbb{C}^{2n}$ be the vector representation of $G$, and $V_2 = \wedge^2 V_1$. Set $\mathcal{V} = V_1 \otimes V_2$. This is a representation of $G = G \times (\mathbb{C}^\times)^3$, where $G$ acts diagonally on $(V_1, V_1, V_2)$ and $(c_0, c_1, c_2) \in (\mathbb{C}^\times)^3$ acts by multiplication by $(c_0^{-1}, c_1^{-1}, c_2^{-1})$. The eDL correspondence is stated as a one-to-one correspondence

$$G(a) \backslash \mathcal{V}^a \longleftrightarrow \text{irrep} H_a, \quad X \mapsto L_{(a,X)}^a, \quad (0.1)$$

where $G(a)$ is the centralizer of $s$ in $G$ and $\mathcal{V}^a$ is the set of $a$-fixed points in $\mathcal{V}$.

There is a combinatorial parameterization of the left hand side in terms of marked partitions (see [L3]). The right hand side is the set of irreducible $H$-modules with central character $a$. The irreducible module $L_{(a,X)}^a$ is obtained as a quotient of a standard geometric module $M_{(a,X)}$. We call $(a,X)$ the eDL parameter of $L_{(a,X)}^a$. There are some remarks to be made at this point:

1. There are no local systems in this picture: the isotropy group of every $X$ is connected;

2. There exists an exotic Springer correspondence ([Ka09]). Moreover, the homology groups of (classical) Springer fibers of $Sp(2n)$ and $SO(2n + 1)$ can be realized via the homology of a suitable exotic Springer fiber (c.f. Corollary 1.23);

3. For the Hecke algebras which are known to appear in the representation theory of $p$-adic groups, the Deligne-Langlands-Lusztig correspondence (DLL for short) was established in [KL87, Lu89, Lu95b, Lu02a]. The connection between the eDL and DLL correspondences is non-trivial. In particular, the “lowest $W$-types” of a fixed irreducible module differ between the eDL/DLL correspondences;
4. In the DLL correspondence, we can specify a discrete series by its “lowest $W$-type”. However, our “eDL lowest $W$-type” does not single out discrete series. Hence, we need the full eDL parameter in order to specify discrete series.

The third phenomenon gives a restriction on the $W$-character of discrete series which seems far from trivial (c.f. §4.7).

In §3.2, we give an effective, simple algorithm which produces from a distinguished central character $s_{\sigma}$, an element $X_{\text{out}(\sigma)}$ in $V^a$.

**Theorem B** ($=\text{Theorem 3.5}$). Let $\sigma$ be a partition of $n$. Let $a = (s_{\sigma}, \bar{q})$ be the positive real generic central character, where $\bar{q} = (-1, q^m, q)$, and $s_{\sigma}$ is the distinguished semisimple element corresponding to $\sigma$. Then $(a, X_{\text{out}(\sigma)})$ is the eDL parameter of a unique discrete series $H$-module with central character $a$.

We also give a geometric characterization of $\text{out}(\sigma)$ in §3.5, as the minimal orbit with respect to certain conditions.

More generally, one can describe the tempered spectrum of $H_{n,m}$ with real positive generic parameter as follows.

**Theorem C** ($=\text{Theorem 4.7 and Corollary 4.23}$). If $n_1 + n_2 = n$, we set $H_{n,m}^S := H_{n_1,m}^A \otimes H_{n_2,m} \subset H_{n,m}$, where $H_{n_1,m}^A$ is an affine Hecke algebra of $GL(n_1)$.

(i) Let $L_1^A$ and $L_2$ be a tempered $H_{n_1,m}^A$-module with real central character and a discrete series $H_{n_2,m}$-module with positive real generic parameter, respectively. Then, $L := \text{ind}_{H_{n,m}^S}^{H_{n,m}} (L_1^A \boxtimes L_2)$ is an irreducible tempered $H_{n,m}$-module.

(ii) Every irreducible tempered $H_{n,m}$-module with positive real generic central character can be realized in a unique way as in (i).

The classification of tempered modules for the Hecke algebras of type A is well-known from [Ze80] and [KL87]. Theorem C was established before, by different methods, in the work of Opdam [Op04], Delorme-Opdam [DO03], and Slooten [Sl08].

**Example D** (Table of discrete series for $n = 2$). Let $\{\epsilon_1 - \epsilon_2, \epsilon_1, \epsilon_2\}$ denote the positive roots (of $SO(5, \mathbb{C})$). Let $v_\beta$ denote a $T$-eigenvector of weight $\beta$ in $V$. Assume the notation for $W$-representations in Remark 1.25. We have:

| $m$       | $(0, 1/2)$ | $(1/2, 1)$ | $(1, \infty)$ |
|-----------|------------|------------|---------------|
| $X_{\text{out}(\sigma_1)}$ | $v_{\epsilon_1 - \epsilon_2} + v_{\epsilon_2}$ | $v_{\epsilon_1 - \epsilon_2} + v_{\epsilon_2}$ | $v_{\epsilon_1 - \epsilon_2} + v_{\epsilon_2}$ |
| $X_{\text{out}(\sigma_2)}$ | $v_{\epsilon_1 - \epsilon_2}$ | $v_{\epsilon_1 - \epsilon_2} + v_{\epsilon_2}$ | $v_{\epsilon_2}$ |
| $\text{ds(\text{out}(\sigma_2))}_{W}$ | $\{(1^2)(0)\}$ | $\{(1)(1) + \{(0)(1^2)\}\}$ | $\{(0)(2)\}$ |

To transfer our description from generic parameters to special parameters, we prove a continuity result of tempered modules, which is an algebraic analogue of a result of [OS08].
Theorem E (=Corollary 2.15). Let \( a^t = a \exp(\gamma t) \) be a one-parameter family of positive real central characters depending on \( t \in \mathbb{R} \) by

\[
\gamma \in \mathfrak{t} \oplus \{0\} \oplus \mathbb{R}_{\geq 0}^2 \subset \text{Lie}(T \times (\mathbb{C}^\times)^3).
\]

Let \( X_t \in \mathbb{V}^{a^t} \) be a family of exotic nilpotent elements corresponding to the same marked partition \( \tau \). We assume that \( a^t \) is generic for all \( t \in (-\epsilon, \epsilon) \setminus \{0\} \) for some small \( \epsilon > 0 \).

(i) The module \( L_{(a^t, X_t)} \) is an irreducible quotient of both of the two limit modules \( \lim_{t \to 0^+} L_{(a^t, X_t)} \).

(ii) The module \( L_{(a^t, X_t)} \) is tempered if \( L_{(a^t, X_t)} \) is a tempered module in at least one of the regions \( 0 < t < \epsilon \) or \( -\epsilon < t < 0 \).

(iii) The module \( L_{(a^t, X_t)} \) is a discrete series if \( L_{(a^t, X_t)} \) are discrete series for \( t \in (-\epsilon, \epsilon) \setminus \{0\} \).

Remark F. In general, the limit tempered module \( \lim_{t \to 0^+} L_{(a^t, X_t)} \) is reducible. For example, let us consider \( \mathbb{H}_{n,m} \) with \( n = 2 \), and \( 1/2 < m < 1 \). There is one tempered \( \mathbb{H}_{2,m} \)-module \( L_m \) with its \( W_2 \)-structure \( \{(0)(1^2)) + \{(0)(2)) + \{(1)(1)\} \). We have

\[
\lim_{m \to 1} L_m \cong U_1 \oplus U_2, \quad U_1|W \cong \{(0)(1^2)\} + \{(1)(1)\}, \text{ and } U_2|W \cong \{(0)(2)\},
\]

where \( U_1, U_2 \) are tempered modules of \( \mathbb{H}_{2,1} \) (the affine Hecke algebra with equal parameters of type \( B_2 \)). In the usual DLL correspondence, the tempered modules \( U_1 \) and \( U_2 \) are parameterized by the same nilpotent adjoint orbit in \( \mathfrak{so}(5) \), i.e., they are in the same L-packet.

The result of [OS08] guarantees that every tempered module arises as \( L_{(a^t, X_t)} \) via Theorem E. This completes the description of tempered modules in the DLL correspondence. It may be worth mentioning that the classification of [OS08] in the case of \( \mathbb{H}_{n,m} \) is basically the same as a conjecture of Slooten [St08] 4.2.1 (i). It also shares certain aspects with the classification of the tempered and discrete series \( \mathbb{H}_{n,m} \)-modules by [KL87, Lu89, Lu95b, Lu02a]. We provide a combinatorial identification of lowest \( W \)-types algorithms between [St06] and (the \( \text{Spin}(t) \)-case of) Lu95a in §4.4 for the sake of completeness.

In addition, we include certain applications of this “exotic” classification for the \( W \)-structure of tempered modules in §4. Among these applications, we give a combinatorial classification of discrete series containing the \( W \)-type \( \text{sgn} \) (Theorem 1.2). This description follows from Theorem E in conjunction with:

Theorem G (=Theorem 1.23). Let \( \sigma \) be a partition of \( n \). The discrete series of \( \mathbb{H} \) with central character \( a = (s_\sigma, \bar{q}) \) contains the \( \text{sgn} \) \( W \)-type if and only if \( G(s_\sigma)X_{out(\sigma)} \subset \mathbb{V}^a \) is open dense.

Remark H. Taking into account Theorem 1.3 and Heckman-Opdam (see [HO97] Theorem 1.7 and the discussion around it), this completes the classification of the \( L^2 \)-solutions of the Lieb-McGuire system with attractive coupling parameters. Theorem 1.2 itself fills out a missing piece in [HO97] (the unequal parameter case of Theorem 1.7 when the root system is of type \( B_n \)) and confirms a special part of a conjecture of Slooten [St08] 4.2.1 (iii). Recall that the affine
Hecke algebra has the Iwahori-Matsumoto involution which interchanges modules containing the sgn $W$-type and spherical modules, i.e., modules containing the triv $W$-type. The images of discrete series containing the sgn $W$-type under the Iwahori-Matsumoto involution are called spherical cuspidal in [HO97]. Therefore one may view Theorem 4.2 as a classification of spherical cuspidal modules for the affine Hecke algebra $H_{n,m}$ of type $B_n/C_n$ with arbitrary parameter $m$. When the Hecke algebra comes from a $p$-adic group, these should be examples of Arthur representations (in the sense of [Ar89]).

The organization of this paper is as follows. In §1, we fix notation and recall the basic results. Some of the material (like Corollary 1.24) is new in the sense that it was not included in [Ka09]. In §2, we present various technical lemmas needed in the sequel. In §3, we formulate and prove our main result, Theorem B. Namely, after recalling some preliminary facts in §3.1, we present our main algorithm $σ \rightarrow \text{out}(σ)$ and state Theorem B in §3.2. We analyze the weight distribution of certain special discrete series in §3.3. Then, we use the induction theorem repeatedly to prove that the module $L(σ, X_{\text{out}(σ)})$ must be a discrete series for all $σ$. We also give an alternate characterization of $\text{out}(σ)$ in §3.5. The last section §4 has various applications: we characterize those discrete series $L(σ, X_{\text{out}(σ)})$ which contain sgn, and analyze their deformations in §4.1. We prove Theorem C in §4.2. In §4.3, we prove that for generic parameter $m$, the $R(T)$-characters of irreducible $H_{n,m}$-modules are linearly independent. We explain a relation between the viewpoints of Lusztig and Slooten-Opdam-Solleveld in §4.4. We deduce the $W_n$-independence of tempered modules in §4.5, and characterize the tempered $H_{n,m}$-modules which are irreducible as $W$-modules. We finish the paper by presenting several constraints on the $W$-structure of tempered $H_{n,m}$-module coming from the comparison of the eDL correspondence with the DLL correspondence (c.f. [Lu02a]).

Acknowledgments. This work grew out of discussions with Peter Trapa during the second author’s visit at Utah (February 2008). We thank Peter Trapa for his kind invitation and support. We are indebted to Eric Opdam, who drew our attention to the paper [HO97]. We thank George Lusztig, Klaas Slooten, Maarten Solleveld, Toshiaki Shoji for their interest and comments. D.C. was supported by the NSF grant DMS-0554278. S.K was partly supported by the postdoctoral fellowship at MSRI special semester program “Combinatorial Representation Theory”, and by the Grant-in-Aid for Young Scientists (B) 20-74011.

1 Preliminaries

1.1 Affine Hecke algebra $H$

In this paper, $G$ will denote the group $Sp(2n, \mathbb{C})$. Fix a Borel subgroup $B$, and a maximal torus $T$ in $B$, and let $W = N_G(T)/T$ be the Weyl group. We denote the character lattice of $T$ by $X^*(T)$. Fix a root system $R$ of $(G, T)$ with positive roots $R^+$ given by $B$, and simple roots $\Pi$. In coordinates, the roots are

$$R^+ = \{ \epsilon_i \pm \epsilon_j \}_{i < j} \cup \{ 2\epsilon_i \} \subset \{ \pm \epsilon_i \pm \epsilon_j \} \cup \{ \pm 2\epsilon_i \} = R, \quad (1.1)$$

and $\Pi = \{ \alpha_i \}_{i=1}^n$ with $\alpha_i = \epsilon_i - \epsilon_{i+1} (i \neq n), 2\epsilon_n (i = n)$. For every $S \subset \Pi$, we let $P_S = L_S U_S$ denote the minimal parabolic subgroup containing $B$ and the
one-parameter unipotent subgroups corresponding to $(-S)$. We sometimes add a subscript $n'$ in order to indicate that the corresponding objects are obtained by replacing $n$ with $n'$ (e.g., $W_{n'}$ represents the Weyl group of type $B_{n'}$).

We set $\mathcal{A}_\mathbb{Z} := \mathbb{Z}[q_0^{\pm 1}, q_1^{\pm 1}, q_2^{\pm 1}]$ and $\mathcal{A} := \mathbb{C} \otimes_{\mathbb{Z}} \mathcal{A}_\mathbb{Z} = \mathbb{C}[q_0^{\pm 1}, q_1^{\pm 1}, q_2^{\pm 1}]$.

**Definition 1.1** (Affine Hecke algebras of type $C_n$). An affine Hecke algebra of type $C_n$ with three parameters is an associative algebra $\mathbb{H}_n = \mathbb{H}$ over $\mathcal{A}$ generated by $\{T_i\}_{i=1}^n$ and $\{e^\lambda\}_{\lambda \in \mathcal{X}^*(T)}$ subject to the following relations:

(Toric relations) For each $\lambda, \mu \in X^*(T)$, we have $e^\lambda \cdot e^\mu = e^{\lambda + \mu}$ (and $e^0 = 1$);

(The Hecke relations) We have

$$(T_i + 1)(T_i - q_2) = 0 \quad (1 \leq i < n) \quad \text{and} \quad (T_n + 1)(T_n + q_0 q_1) = 0; \quad (1.2)$$

(The braid relations) We have

$$T_i T_j = T_j T_i \quad \text{(if } |i - j| > 1), \quad (T_n T_{n-1})^2 = (T_{n-1} T_n)^2, \quad T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1} \quad \text{(if } 1 < i < n - 1); \quad (1.3)$$

(The Bernstein-Lusztig relations) For each $\lambda \in X^*(T)$, we have

$$T_i e^\lambda - e^{s_i \lambda} T_i = \begin{cases} (1 - q_2) \frac{e^{s_i \lambda} - e^{\lambda}}{(1 + q_0 q_1 - q_0 q_1 e^{s_i \lambda})} & \text{(if } i \neq n) \\ (e^\lambda - e^{s_n \lambda}) & \text{(if } i = n) \end{cases}. \quad (1.4)$$

**Definition 1.2** (Parabolic subalgebras of $\mathbb{H}$). Let $S \subset \Pi$ be a subset. We define $\mathbb{H}^S$ to be the $\mathcal{A}$-subalgebra of $\mathbb{H}$ generated by $\{T_i; \alpha_i \in S\}$ and $\{e^\lambda; \lambda \in X^*(T)\}$. If $S = \Pi - \{\alpha\}$, then we may denote $\mathbb{H}^S = \mathbb{H}_n^S$ by $\mathbb{H}_n^S = \mathbb{H}_n^A$.

**Remark 1.3.**

1) The standard choice of parameters $(t_0, t_1, t_n)$ is: $t_1^2 = q_2, t_n^2 = -q_0 q_1,$ and $t_n(t_0 - t_0^{-1}) = (q_0 + q_1)$. This yields

$$T_n e^\lambda - e^{s_n \lambda} T_n = \frac{1 - t_n^2 - t_n(t_0 - t_0^{-1})e^{s_n \lambda}}{e^{s_n \lambda} - 1} (e^\lambda - e^{s_n \lambda});$$

2) An equal parameter extended affine Hecke algebra of type $B_n$ is obtained by requiring $q_0 + q_1 = 0$ and $q_1^2 = q_2$. An equal parameter affine Hecke algebra of type $C_n$ is obtained by requiring $q_2 = -q_0 q_1$ and $(1 + q_0)(1 + q_1) = 0$;

3) The extended affine Hecke algebra with unequal parameters of type $B_n$ with the parameters

\[ \begin{array}{cccccc}
\circ & q & q & q & q & q^m \\
\end{array} \]

is obtained as

$$\mathbb{H}_{n,m} := \mathbb{H}/(q_0 + q^{m/2}, q_1 - q^{m/2}, q_2 - q).$$

The affine Hecke algebra with unequal parameters of type $C_n$ with the parameters

\[ \begin{array}{cccccc}
q^m & q & q & q & q^m \\
\end{array} \]

is obtained as

$$\mathbb{H}_{n,m} := \mathbb{H}/(q_0 + 1, q_1 - q^{m}, q_2 - q).$$
Set $\mathcal{G} = G \times (C^\times)^3$. We recall the following well-known description of the center of $\mathbb{H}$.

**Theorem 1.4** (Bernstein-Lusztig [La90]). The center of the Hecke algebra $\mathbb{H}$ is

$$Z(\mathbb{H}) \cong A[e^\lambda; \lambda \in X^*(T)]^{W} \cong A \otimes_{R(C^\times)^3} R(\mathcal{G}). \quad (1.5)$$

A semisimple element $a \in \mathcal{G}$ is a pair $a = (s, \vec{q}) = (s, q_0, q_1, q_2)$, for some $s \in G$ semisimple, and $\vec{q} \in (C^\times)^3$. In view of the previous theorem, each semisimple $a \in \mathcal{G}$ defines a character $\mathcal{C}_a$ of $Z(\mathbb{H})$ by taking traces of elements of $R(\mathcal{G})$ at $a$.

We can define therefore the specialized Hecke algebras

$$\mathbb{H}_a := \mathcal{C}_a \otimes_{Z(\mathbb{H})} \mathbb{H} \text{ and } \mathbb{H}_a^S := \mathcal{C}_a \otimes_{Z(\mathbb{H})} \mathbb{H}_a^S, \quad (1.6)$$

where $S \subset \Pi$.

A $\mathbb{H}_a$-module $M$ is said to be a $\mathbb{H}$-module with central character $a$ or $s$. By Theorem 1.4, having central character $s$ is equivalent to having central character $v \cdot s$ for each $v \in W$. (Here $v \cdot s$ is the action induced by the adjoint action of $NC(T)$ on $T$ inside $G$.)

We recall next the geometric construction of irreducible $\mathbb{H}_a$-modules from [Ko99], under the assumption in the following definition.

**Definition 1.5.** A semisimple element $a = (s, \vec{q}) = (s, q_0, q_1, q)$ in $\mathcal{G}$ is called positive real if $-q_0, q_1 \in \mathbb{R}_{>0}, q \in \mathbb{R}_{>1}$ and $s$ has only positive real eigenvalues on $V_1$. An element $\vec{q} = (q_0, q_1, q) \in (C^\times)^3$ is called generic if it satisfies:

$$q_0^2 \neq q^{m'}, \text{ for all } m' \in \mathbb{Z}. \quad (1.7)$$

A positive real element $a = (s, \vec{q})$ is called generic if $\vec{q}$ is generic. The set of semisimple positive real generic elements of $\mathcal{G}$ will be denoted by $\mathcal{G}_0$.

Whenever $\vec{q}$ is generic, we fix a maximal subset $S \subset \mathbb{R}_{>0}$ with the following properties:

- We have $q_{0}^{\pm 1} \not\in S$ and $q_1 \in S$;
- $S$ is stable by $q_{-}^2$-action;
- We have $\{\xi, \xi^{-1}\} \not\subset S$ for all $\xi \in \mathbb{R}_{>0}$.

We call $a = (s, \vec{q}) \in \mathcal{G}_0$ to be $S$-positive if we have $s \in T$ and $\langle \epsilon_i, s \rangle \in S$ for each $i = 1, \ldots, n$. The set of $S$-positive elements of $\mathcal{G}_0$ is denoted by $T_0$.

**Example 1.6.** In the case of the Hecke algebra $\mathbb{H}_{n,m}$ of type $C_n$ (as is defined in Example [1.3]), if we assume $q \in \mathbb{R}_{>1}$ and $m \in \mathbb{R}$, then the condition (1.7) turns into $m \not\in \frac{1}{2}\mathbb{Z}$.

Here we remark that the $G$-conjugates of $T_0$ exhaust $\mathcal{G}_0$.

We have $R(T) \subset H$. Thus, we can consider the set of $R(T)$-weights of $V$ for a finite dimensional $\mathbb{H}_a$-module $V$. We denote it by $\Psi(V) \subset T$. It is well known $\Psi(V) \subset W \cdot s$ whenever $a = (s, \vec{q})$.

**Definition 1.7.** A $\mathbb{H}_a$-module $V$ is called tempered, if for all $\chi \in \Psi(V)$, one has

$$\langle \varpi_j, \chi \rangle \leq 1, \text{ for all } 1 \leq j \leq n, \quad (1.8)$$

where $\varpi_j = \epsilon_1 + \cdots + \epsilon_j$. The module $V$ is called a discrete series, if the inequalities in (1.8) are strict.
1.2 Reduction to positive real character

We briefly recall Lusztig’s reduction to positive real central character. The original reference is [Lu89] (see also [BM93] §6, [Lu02a] §4, and [Lu02b] §3), and there is a complete recent exposition relative to the tempered spectrum in [OS08]. In this subsection alone, we denote the Hecke algebra defined in §1.1 by \( H_{R,X}^{\lambda_0,\lambda_1,\lambda_n} \), where, as before, \( X = X^*(T) \), \( T \) is the torus for \( Sp(2n,\mathbb{C}) \), \( R \) denotes the roots of type \( C_n \) (with coroots \( \check{R} \) of type \( R_n \)), and \( t_i = q^{\lambda_i} \) (as in Remark 1.3). \( H_{R,X}^{\lambda_0,\lambda_1,\lambda_n} \) is the set of isomorphism classes of irreducible modules on which \( q \) acts by some element in \( R_{>1} \). Let us also denote by \( \text{Irrep}_{W,s_n}^{t_i} H_{R,X}^{\lambda_0,\lambda_1,\lambda_n} \), the set of isomorphism classes of irreducible modules on which \( q \) acts by \( \nu_0 \), and the central character is \( s \in T \). (The emphasis on \( W \) in the notation will be justified by the reduction procedure.)

Every \( s \in T \) has a unique decomposition \( s = s_e \cdot s_h \), into an elliptic part, and a hyperbolic part: \( s_e \in S^1 \otimes \mathbb{Z} X_e(T) \), \( s_h \in R_{>0} \otimes \mathbb{Z} X_e(T) \). Note that \((s_h, \check{q}) \in \mathcal{G}_0 \).

Fix a central character \( s = s_e \cdot s_h \). Define

\[
R_{s_e} = \left\{ \alpha \in R : \frac{s_e(\alpha)}{s_h(\alpha)} = 1, \quad \text{if } \alpha \notin \{\pm e_n\} \right\},
\]

Then \( R_{s_e} \subset R \) is a root subsystem. Let \( W_{s_e} \subset W \) denote the subgroup generated by the reflections in the roots of \( R_{s_e} \).

**Definition 1.8.** Let \( R' \subset E' \), \( \check{R}' \subset \check{E}' \) be root (and coroot) systems in the usual sense, and denote by \( E'_C \) and \( \check{E}'_C \) the complexifications. Let \( E' \subset R' \) be a set of simple roots, and \( W' \) be the Coxeter group. Let \( \mu \) be a \( W' \)-invariant function on \( E' \). Define the graded (or degenerate) affine Hecke algebra \( \mathbb{H}_{R',E'}^{\mu} \) to be the associative \( \mathbb{C}[r] \)-algebra with unity generated by \( \{t_w : w \in W'\} \), \( \omega \in \check{E}'_C \) subject to the relations:

\[
t_{w}t_{w'} = t_{ww'}, \quad \text{for all } w, w' \in W';
\]

\[
\omega w' = w' \omega, \quad \text{for all } \omega, w' \in \check{E}'_C;
\]

\[
\omega t_{s_{\alpha}} - t_{s_{\alpha}} \omega = r_{s_{\alpha}}(\mu(\alpha)) \langle \alpha, \omega \rangle, \quad \text{for all } \alpha \in E', \omega \in \check{E}'_C.
\]

**Remark 1.9.** The center of \( \mathbb{H}_{R',E'}^{\mu} \) is \( \mathbb{C}[r] \otimes S(\check{E}'_C)^{W'} \), where \( S( ) \) denotes the symmetric algebra ([Lu89]). Therefore, the central characters of irreducible modules, on which \( r \) acts by a certain scalar, are given by \( W'-\)conjugacy classes of elements in \( \check{E}'_C \). Denote by \( \text{Irrep}_{W',\check{E}'_C}^{t_i} \mathbb{H}_{R',E'}^{\mu} \) the class of irreducible modules with central character \( s \in E'^*_C \), and on which \( r \) acts by \( r_0 \).

Fix \( r_0 \) such that \( e^{r_0} = \nu_0 \). Recall that every hyperbolic element \( s_h \in T \) has a unique logarithm \( \log s_h \in t \).

**Theorem 1.10** ([Lu89]). There are natural one-to-one correspondences:

\[
\text{Irrep}_{W,s_n}^{t_i} \mathbb{H}_{R,X}^{\lambda_0,\lambda_1,\lambda_n} \cong \text{Irrep}_{W_{s_n},\check{E}'_C}^{t_i} \mathbb{H}_{R',E'}^{\mu_n} \cong \text{Irrep}_{W_{s_n},\check{E}'_C}^{t_i} \mathbb{H}_{R',E'}^{\mu_1,\mu_n} \cong \text{Irrep}_{W_{s_n},\check{E}'_C}^{t_i} \mathbb{H}_{R',E'}^{\mu_n,\mu_1},
\]

where \( \mu_1 = \lambda_1, \mu_n = \lambda_n + \langle e_n, s_e \rangle \lambda_0 \).

These correspondences follow from an isomorphism of certain completions of these Hecke algebras. Applying Theorem 1.10 for \( \text{Irrep}_{W_{s_n},\check{E}'_C}^{t_i} \mathbb{H}_{R',E'}^{\mu_n/2,\mu_1,\mu_n/2} \), we see that:
Corollary 1.11. There is a natural one-to-one correspondence

\[ \text{irrep}_{W, \sigma, v_0} \cong \text{irrep}_{W, \sigma, v_0, v_0}^{2\mu_n/2}, \]

where \( \mu_1 = \lambda_1, \mu_n = \lambda_n \).

We should mention that, for a general affine Hecke algebra (not necessarily of type \( C_n \)), a similar correspondence as in Theorem 1.10 holds, but the affine graded algebra \( \mathbb{H} \) needs to be replaced with an extension of it by a group of diagram automorphisms.

Corollary 1.12. In the correspondence of Corollary 1.11, tempered modules and discrete series modules correspond, respectively.

This is easily seen from the isomorphism of algebra completions we alluded to above (see for example [BM93] §6, [Lu02] §3, or [OS08] §2).

Definition 1.13. We define \( \mathbb{H}_{n,m} \) to be the affine graded Hecke algebra corresponding to \( R' = C_n, \mu(\epsilon_i - \epsilon_{i+1}) = 1, \mu(2\epsilon_n) = m \) (notation as in Definition 1.8).

Finally, recall that, again with the notation as in Definition 1.8, there is an isomorphism between the affine graded Hecke algebra of type \( R' = B_n \) with parameters \( \mu(\epsilon_i - \epsilon_{i+1}) = \mu_1, \mu(\epsilon_n) = 2\mu_2 \) and the affine graded Hecke algebra of type \( R' = C_n \) with parameters \( \mu(\epsilon_i - \epsilon_{i+1}) = \mu_1, \mu(2\epsilon_n) = \mu_2 \). Therefore, when we use the affine graded Hecke algebra later in the paper, in particular in §4, we will consider (as we may) only the affine graded Hecke algebra of type \( C_n \) with unequal parameters as in Definition 1.13.

1.3 Irreducible \( \mathbb{H} \)-modules

We use the notation of 1.1. In addition, we introduce the following notation. For an algebraic group \( H \), an element \( h \in H \), and an algebraic \( H \)-variety \( X \) we denote by \( X^h \) and \( X^H \), the subvariety of \( h \)-fixed and \( H \)-fixed points in \( X \), respectively. For \( x \in X \), we define \( \text{Stab}_H x := \{ h \in H : hx = x \} \). We use German letters to denote Lie algebras (e.g. \( \mathfrak{h} = \text{Lie} H \)).

Let \( V_2 = \mathbb{C}^{2n} \) denote the vector representation of \( G \). Set \( V_2 = \wedge^2 V_1 \) and \( V = V_1 \oplus V_2 \). Then \( V \) is a representation of \( G \) as follows: \( G \) acts diagonally, and an element \( (c_0, c_1, c_2) \in (\mathbb{C}^x)^3 \) acts on \( (V_1, V_1, V_2) \) via multiplication by \( (c_0^{-1}, c_1^{-1}, c_2^{-1}) \).

For every nonzero weight \( \beta \in X^*(T) \) of the \( G \)-module \( V_1 \oplus V_2 \), we fix a non-zero \( T \)-eigenvector \( v_\beta \). In coordinates, these nonzero weights are \( \{ \pm \epsilon_i : 1 \leq i \leq n \} \cup \{ \pm (\epsilon_i \pm \epsilon_j) : 1 \leq i < j \leq n \} \). The corresponding weight spaces are one-dimensional, so \( v_\beta \) is unique up to scalar.

We denote by \( \mathbb{V}^+ \) the sum of \( B \)-positive \( T \)-weight spaces in \( V \). For each \( S \subset \Pi \), we will denote by \( \mathbb{V}_S \) the sum of \( T \)-weight spaces for the weights in the \( \mathbb{Q} \)-span of \( S \). We define the collapsing map (an analogue of the moment map)

\[ \mu : F := G \times B \mathbb{V}^+ \longrightarrow \mathbb{V}, \quad \mu(g, v^+) = g \cdot v^+, \quad g \in G, \quad v^+ \in \mathbb{V}^+, \]

and denote the image of \( \mu \) by \( \mathcal{R} \). For each positive real \( a = (s, \tilde{q}) \), we denote by \( \mu^a \) the restriction of \( \mu \) to the \( a \)-fixed points of \( F \). We denote by \( G(s) \) or \( G(a) \) the
centralizer $Z_G(s)$. This is a connected (reductive) subgroup, since $G$ is simply connected. Its action on $\mathcal{R}$ has finitely many orbits. We will describe this in more detail in §1.4.

Let $\text{pr}_B : F \to G/B$ be the projection $\text{pr}_B(g,v^+) = gB$. We define

$$E_X^a := \text{pr}_B(\mu^{-1}(X)^a) \subset G/B,$$

and call it an exotic Springer fiber.

**Definition 1.14 (Standard module).** Let $a \in G$ be a positive real element and let $X \in \mathcal{R}$. The total Borel-Moore homology space

$$M_{(a,X)} := \bigoplus_{m \geq 0} H_m(E_X^a, \mathbb{C})$$

admits a structure of finite dimensional $H_a$-module. We call this module a standard module. Fix $S \subset \Pi$. If $a \in L_S \times (\mathbb{C}^*)^3$ and $X \in V_S$, then

$$M^S_{(a,X)} := \bigoplus_{m \geq 0} H_m(E_X^a \cap P_S/B, \mathbb{C})$$

admits a $H_S$-module structure.

If $S = \Pi - \{n\}$, then we may denote $M^S_{(a,X)}$ by $M^A_{(a,X)}$.

Let $V^S$ be the unique $T$-equivariant splitting of the map $V^+ \to V^+/ (V^+ \cap V_S)$. If $X \in V_S$, then we have necessarily $u_S X \subset V^S$. The induction theorem is the following:

**Theorem 1.15 ([Ka09] Theorem 7.4).** Let $S \subset \Pi$ be given. Let $a = (s, \vec{q}) \in G$ be a positive real element and let $X \in \mathcal{R}$. Assume $s \in L_S$ and $X \in (\mathcal{R} \cap V_S)$. If we have

$$(V^S)^a \subset u_S X,$$

then we have an isomorphism

$$\text{Ind}_{H_S}^{H_a} M^S_{(a,X)} \cong M_{(a,X)}$$

as $H$-modules.

Fix the semisimple element $a_0 = (1, -1, 1, 1) \in G$. The following result is an exotic version of the well-known Springer correspondence.

**Theorem 1.16 ([Ka09] Theorem 8.3).** Let $X \in \mathcal{R}^{a_0}$ be given. Then, the space

$$L_X := H_{2d_X} (E_X^{a_0}, \mathbb{C}), \text{ where } d_X := \dim E_X^{a_0}$$

admits a structure of irreducible $W$-module.

Moreover, the map $X \mapsto L_X$ defines a one-to-one correspondence between the set of orbits $G\backslash \mathcal{R}^{a_0}$ and $\text{Irrep } W$.

In this correspondence, if $X$ is in the open dense $G(a_0)$-orbit of $\mathcal{R}^{a_0}$ then $L_X$ is the $\text{sgn } W$-representation. If $X = 0$, then $L_0$ is the $\text{triv } W$-representation.

In the following proposition, $Z = F \times_{\mathbb{R}} F$ denotes the exotic Steinberg variety, and $H_a^*(\bullet)$ is equivariant (Borel-Moore) homology with respect to the group $A$. 

10
Proposition 1.17 ([Ka09] Theorem 9.2). Let $a = (s, q_0, q_1, q) \in G$ be a positive real element. We set $\mathfrak{a} := (\log s, r_1, r) \in \mathbb{R}^2$, where $r_1 = \log q_1, r = \log q$. Let $A$ be a connected subtorus of $G$ which contains $(s, 1, q_1, q)$. If $X \in \mathbb{R}^A$, then $H^*_A(\mathbb{Z}^\mathfrak{a})$ acquires a structure of a $C[a]$-algebra which we denote by $H^*_a$. We have:

1. The quotient of $\mathbb{H}_a^+$ by the ideal generated by functions of $C[a]$ which are zero along $\mathfrak{a}$ is isomorphic to $\mathbb{H}_a$.

Moreover, we have

$$\mathbb{C}[a] \otimes H_*(E_X) \cong H_*(E_X)$$

for each $X \in \mathbb{R}^A$

as a compatible $(C[W], C[a])$-module, where $W$ acts on $C[a]$ trivially. $\square$

Corollary 1.18. Keep the setting of Proposition 1.17. We have

$$M_{a,X} \cong \bigoplus_{m \geq 0} H_m(E^a_{X}, C).$$

(1.16)

as $C[W]$-modules. $\square$

In ([Ka09], Theorem 10.2), the irreducible $W$-module $L_X$ appears exactly once in the decomposition of $M_{a,X}$. There is a unique irreducible quotient of $M_{a,X}$, denoted $L_{a,X}$, with the property that $L_{a,X}$ contains $L_X$.

Theorem 1.19 ([Ka09] Theorem 10.2). Let $a = (s, q) \in G$ be a positive real element. We have $\mathbb{R}^a \subset \mathbb{R}^a$. Then, we have a one-to-one correspondence

$$G(a) \backslash \mathbb{V}^a \leftrightarrow \text{Irr} \mathbb{H}_a, \quad X \mapsto L_{a,X}.$$  
(1.17)

The module $L_{a,X}$ is a $\mathbb{H}_a$-module quotient of $M_{a,X}$. Moreover, if $L_{a,Y}$ appears in $M_{a,X}$, then we have $X \in G(a)Y$.

We need to emphasize that, unlike the case of [KLS7], there are no nontrivial local systems appearing in the parameterization of $\text{Irr} \mathbb{H}_a$ in Theorem 1.19 for positive real $a \in G$ (see also Corollary 1.33).

Corollary 1.20. If $a \in G$ is a positive real element, then the set of $G(a)$-orbits of $\mathbb{R}^a$ is finite. In particular, there exists a unique dense open $G(a)$-orbit $O^a_0$ of $\mathbb{R}^a$.

It is useful to remark that $L_{X_0} \cong \text{sgn} (X_0 \in O^a_0)$ appears with multiplicity one in every standard module $M_{a,X}$.

From Theorem 1.19 together with the DLL correspondence of type A, we deduce:

Theorem 1.21. Keep the same setting as Theorem 1.19. Let $S \subset \Pi$ and assume $s \in L_S$. Then, we have a one-to-one correspondence

$$L_S(s) \backslash \mathbb{V}_S \leftrightarrow \text{Irr} \mathbb{H}_a^S, \quad X \mapsto L^S_{a,X}.$$  
(1.18)

The module $L^S_{a,X}$ is a $\mathbb{H}_a^S$-module quotient of $M^S_{a,X}$. Moreover, if $L^S_{a,Y}$ appears in $M^S_{a,X}$, then we have $X \in L_S(a)Y$. 

Malgebra of $H$

This forces $G$ points of $H$

By an argument of Lusztig \cite{Lu95b}, we deduce that an element $e$ has eigenvalues. The Bala-Carter theory implies that there exists a semisimple element which acquires a structure of irreducible $W$-modules (without grading). Therefore, Corollary 1.18 implies the result.

Remark 1.25. Before presenting an example of the correspondence in Corollary 1.24, let us recall the parameterization of Irrep $W_n$ in terms of bipartitions. Recall that $W = W_n \cong S_n \ltimes (\mathbb{Z}/2\mathbb{Z})^n$. Let $\xi = (\text{triv}, \ldots, \text{triv}, \text{sgn}, \ldots, \text{sgn})$ be a character of $(\mathbb{Z}/2\mathbb{Z})^n$, and let $S_\xi := S_{\xi} \times S_{n-k} = \text{Stab}_{S_n}(\xi)$. The representations of symmetric groups are parameterized by partitions. Let $\alpha$ be a partition of $k$ and $\beta$ be a partition of $n-k$, and $(\alpha), (\beta)$ the corresponding representations of...
\(\mathfrak{S}_k\) and \(\mathfrak{S}_{n-k}\), respectively. We denote by \(\{(\alpha)|\beta\}\) (and call it a bipartition) the irreducible representation of \(W_n\) obtained by induction from \((\alpha)\boxtimes(\beta)\boxtimes\xi\) on \(\mathfrak{S}_k \times \mathfrak{S}_{n-k} \times (\mathbb{Z}/2\mathbb{Z})^n\). All elements of \(\text{Irr}(W_n)\) are obtained by this procedure, hence the one-to-one correspondence with bipartitions of \(n\).

**Example 1.26.** We explain [1,24] in the case \(n = 3\). There are 7 nilpotent adjoint orbits for \(SO(7, \mathbb{C})\), and 8 for \(Sp(6, \mathbb{C})\). There are 10 exotic orbits (the same as the number of irreducible representations for \(W_3\)). Let us denote the representatives of these 10 orbits as follows:

| \(X_1\) | \(v_{e_1} + v_{e_2} \) | \{0\}(1^3)\} |
|---|---|---|
| \(X_2\) | \(v_{e_1} + v_{e_1} - e_3 \) | \{0\}(1^4) + \{0\}(1^4)\} |
| \(X_3\) | \(v_{e_1} + v_{e_2} - e_3 \) | \{(1^4)(1)\} + \{(1^4)(1)\} + \{(0)(1^3)\} |
| \(X_4\) | \(v_{e_1} + v_{e_2} \) | \(\text{Ind}_{W}^W(\{0\}(1^2))\} |
| \(X_5\) | \(v_{e_1} + v_{e_1} - e_3 \) | \(\text{Ind}_{W}^W((1)(1)) + \{(0)(1^2)\}) |
| \(X_6\) | \(v_{e_1} + v_{e_2} - e_3 \) | \(\text{Ind}_{W}^W(\{0\}(1))\} |
| \(X_7\) | \(v_{e_1} + v_{e_2} \) | \(\text{Ind}_{W}^W((0)(1))\} |
| \(X_8\) | \(v_{e_1} \) | \(\text{Ind}_{W}^W(\text{triv})\} |
| \(X_9\) | \(v_{e_1} - e_2 \) | \(\text{Ind}_{W}^W(\text{triv})\} |
| \(X_{10}\) | 0 | \(\text{Ind}_{W}^W(\text{triv})\} |

The last column gives the \(W_3\)-structure of \(H_\ast(\mathcal{E}_X)\) in every case. With this notation, the correspondences from Corollary [1,24] are as follows:

\[
SO(7) \begin{array}{cccccccc}
(7) & (51^2) & (32^1) & (31^2) & (31^4) & (21^2) & (1^3) \\
\hline
X_1 & X_2 & X_3 & X_7 & X_5 & X_9 & X_{10} \\
\end{array}
\]

\[
Sp(6) \begin{array}{cccccccc}
(6) & (42) & (41^2) & (3^2) & (2^3) & (2^3) & (21^2) & (1^3) \\
\hline
X_1 & X_2 & X_4 & X_6 & X_7 & X_5 & X_8 & X_{10} \\
\end{array}
\]

The notation for the parameterization for classical nilpotent orbits is as [Ca85].

### 1.4 A parameterization of exotic orbits \(G(a) \setminus \mathfrak{n}^a\)

We recall the combinatorial parameterization of \(G(a)\)-orbits in \(\mathfrak{n}^a\) following [Ka09]. Fix \(q > 1, m \in \mathbb{R}\) such that \(q = (-1, q_1, q)\) is generic, and \(a = (s, q) \in T_0\).

**Definition 1.27 (Marked partition).** A segment adapted to \(a\) is a subset \(I \subset [1, n]\) such that for every \(i \in I\), we have either

- there exists no \(j \in I\) such that \(\langle e_i, s \rangle > \langle e_j, s \rangle\);
- there exists a unique \(j \in I\) such that \(\langle e_i, s \rangle = q \langle e_j, s \rangle\).

A marked partition adapted to \(a\) is a pair \(\{\{I_i\}_{i=1}^k, \delta\}\) where

1. \(I_1 \sqcup I_2 \sqcup \cdots \sqcup I_k = [1, n]\) is a division of the set of integers \([1, n]\) into a union of segments (adapted to \(a\));
2. \(\delta : [1, n] \to \{0, 1\}\) such that \(\delta(i) = 0\) whenever \(\langle e_i, s \rangle \neq q_1\).

We refer \(\{I_i\}_{i=1}^k\) as the support of \(\{\{I_i\}_{i=1}^k, \delta\}\).
Let us denote by $\text{MP}(a)$ the set of marked partitions adapted to $a$.

**Proposition 1.28** ([Ka09]). The map $\Upsilon : \text{MP}(a) \to G(a) \backslash \mathcal{R}^a$, given by

$$
\left\{ \left\{ I^k_{\ell=1}, \delta \right\} \mapsto \sum_{\ell=1}^k v_{I^k} + \sum_{i=1}^n \delta(i) v_{\epsilon_i}, \text{ where } v_{I^k} = \sum_{i,j \in I^k, (\epsilon, \delta) = q(\epsilon, \delta)} v_{\epsilon_i - \epsilon_j}, \right. 
$$

(1.19)

is a surjection.

For each $\tau \in \text{MP}(a)$, we put $v_\tau := \Upsilon(\tau)$ and $O_\tau := G(a)v_\tau$.

In order to describe the fibers of the map $\Upsilon$, we define first a partial order on the set of subsets of $[1, n]$. Assume $I, I' \subset [1, n]$. Set $I = \{ (\epsilon, \delta) : i \in I \}$, and similarly for $I'$. Then we define

$$
I < I' \iff \min I \leq \min I' \leq \max I \leq \max I'.
$$

(1.20)

If $I < I'$, we say that $I$ is dominated by $I'$. We also introduce another partial ordering on $\mathbf{I}$, weaker than $<$:

$$
I < I' \iff \min I \leq \min I'.
$$

(1.21)

If $\left\{ \left\{ I^k_{\ell=1}, \delta \right\} \right\} \in \text{MP}(a)$ is given, we define $\left\{ \left\{ \tilde{I}^k_{\ell=1}, \tilde{\delta} \right\} \right\}$ by modifying $\delta$ as follows. If some $i$ such that $\langle \epsilon, \delta \rangle = q_1$ belongs to an $I^k$ which is dominated by a marked $I'_\ell$ (i.e., $\delta(I'_\ell) = \{0, 1\}$), then we set $\tilde{\delta}(i) = 1$ (i.e., we mark $I'_\ell$ as well).

A permutation $w \in \mathfrak{S}_n$ is said to be adapted to $a$ if we have $w \cdot s = s$. Let $\mathfrak{S}_n^a$ denote the subgroup of $\mathfrak{S}_n$ formed by permutations adapted to $a$.

It is straightforward that $\mathfrak{S}_n^a$ acts on $\text{MP}(a)$ by applying $w$ to $\left\{ \left\{ I^k_{\ell=1}, \delta \right\} \right\} \in \text{MP}(a)$ as $\left\{ \left\{ I^k_{\ell=1}, \delta \right\} \right\} \mapsto \left\{ \left\{ w(I^k) \right\}_{\ell=1}^k, w^* \delta \right\}$, where $(w^* \delta)(i) = \delta(w^{-1}(i))$.

**Proposition 1.29** ([Ka09]). Let $\Upsilon$ be the map defined in Proposition 1.28. Then $\Upsilon(\left\{ \left\{ I^k_{\ell=1}, \delta \right\} \right\}) = \Upsilon(\left\{ \left\{ I'_k_{\ell=1}, \delta' \right\} \right\})$ if and only if $\left\{ \left\{ I^k_{\ell=1} \right\} \right\} = \left\{ \left\{ I'_k_{\ell=1} \right\} \right\}$ and $\delta = \delta'$ up to $\mathfrak{S}_n^a$-action.

The marked partition corresponding to the open $G(a)$-orbit in $\mathcal{R}^a$ is obtained by extracting the longest possible $I_1$ subject to $a v_{I_1} = v_{I_1}$, then the longest possible $I_2$ from $[1, n] \setminus I_1$ subject to the same condition etc. Then we mark all $I_j$ such that $q_1 \in I_j$.

Let $\text{MP}_0(a)$ be the set of marked partitions with trivial markings. (i.e. $\tau = (I, \delta) \in \text{MP}(a)$ with $\delta \equiv 0$. ) The following result is re-interpretation of the closure relations of type A quiver orbits with uniform orientation.

**Theorem 1.30** ([AD50], [Ze51]). Let $\tau = (I, 0) \in \text{MP}_0(a)$ be given. Let $\tau' = (I', 0) \in \text{MP}_0(a)$ be obtained from $\tau$ by the following procedure:

(1) For some two segments $I_k, I_l \in I$ such that

$$
\min I_k < \min I_l \leq q \min I < q \max I,
$$

we define $I'$ to be the set of segments obtained from $I$ by replacing $\{ I_k, I_l \}$ with $\{ I^+, I^- \}$, where $I^+$ are segments such that $I^+ \cup I^- = I_k \cup I_l$, $I^+ = I_k \cup I_l$, and $I^- = I_k \cap I_l$. ($I^-$ may be an empty set.)
Then, we have $O_\tau \subset \overline{O_\tau}$. Moreover, every $G(a)$-orbit which is larger than $O_\tau$ and parameterized by $MP_0(a)$ is obtained by a successive application of the procedure (♠). 

**Convention 1.31.** For each $\tau \in MP(a)$, we sometimes denote $L_{(a,\nu)}$ by $L_{(a,\tau)}$ or just $L_\tau$ when the central character is clear. We may use similar notation like $M_{(a,\tau)}$ or $M_\tau$.

Each of $\tau = \{(I_l), 0\} \in MP_0(a)$ defines a representation $R_\tau$ of type $A$-quiver corresponding to the multisegment $(I_l)_l$ in the sense of Zelevinsky. (Here we identify $G(a) \subset V_{\lambda_\nu}^\vee$ with the representation space of type $A$-quiver of an appropriate dimension vector.) In particular, $R_\tau$ is a direct sum of indecomposable modules $R_{I_l}$ corresponding to a segment $I_l$ (or rather $I_l^{-1}$).

**Lemma 1.32.** Let $\tau = \{(I_l), \delta\} \in MP_0(a)$ be given. We have a non-zero map $R_{I_l} \rightarrow R_{I_{l'}}$ (as modules of type $A$-quivers) if and only if $I_l \lessdot I_{l'}$. Moreover, such a non-zero map is unique up to scalar.

**Proof.** Straight-forward.

**Theorem 1.33** (c.f. Brion [Br08 Proposition 2.29]). Let $a \in T_0$ and $\tau = (I, 0) \in MP_0(a)$ be given. The group of automorphisms of $R_\tau$ as type $A$-quiver representation is isomorphic to $Stab_{G(a)}v_\tau$.

**Corollary 1.34.** Keep the setting of Theorem 1.33. Let $r_\tau$ be the number of segments of $I$, and let $u_\tau$ be the set of distinct pairs of segments $I, I'$ in $I$ such that $I \lessdot I'$. Then, $Stab_{G(a)}v_\tau$ is a connected algebraic group of rank $r_\tau$ and dimension $(r_\tau + u_\tau)$.

**Proof.** Taking into account Theorem 1.33 and the fact that an automorphism group of a module over an algebra is connected, the assertion is a straightforward corollary of Lemma 1.32.

## 2 Some weight calculations

### 2.1 Varieties corresponding to weight spaces

In this section, we use the language of perverse sheaves (corresponding to middle perversity) on complex algebraic varieties. Some of the standard references for this theory are Beilinson-Bernstein-Deligne [BBD], Kashiwara-Schapira [KS90], Gelfand-Mannin [GM94], and Hotta-Tanisaki [HT08].

For a variety $X$, we denote by $\mathbb{C}$ the constant sheaf (shifted by $\dim X$). For a locally closed subset $O \subset X$, we have an embedding $j_O : O \rightarrow X$. We have a locally constant sheaf $(j_O)_!\mathbb{C}$ obtained by extending the constant sheaf on $O$ by zero to $X$. We have an intermediate extension object $IC(O)$ (simple object in the category of perverse sheaves) obtained by appropriately truncating $(j_O)_!\mathbb{C}$.

Fix $a \in T \times (\mathbb{C}^*)^d$. For every $w \in W$, let $\dot{w}$ denote a representative in $N_G(T)$. We put $w\lambda^+ := \dot{w}^{-1}\lambda^+$ and $w\lambda(a) := \lambda^a \cap w\lambda^+$. We denote $Ad(\dot{w}^{-1})B(s)$ by $wB(s)$. It is clear that these definitions do not depend on the choice of $\dot{w}$.

Recall the restriction of the collapsing map $\mu^a : F^a \rightarrow \mathfrak{h}^a$. Set 

$$F^a_w = G(s) \times ^wB(s) \, w\lambda(a). \quad (2.1)$$
Let $W_s$ be the reflection subgroup of $W$ corresponding to the subroot system of $R$ defined by the roots $\alpha$ such that $\alpha(s) = 1$. Following Lemma 3.6 in [Ka09], we have a decomposition

$$F^\alpha = \bigcup_{w \in W/W_s} F^\alpha_w. \quad (2.2)$$

Denote by $\mu^\alpha_w$ the restriction of $\mu^\alpha$ to a piece $F^\alpha_w$, where $w$ is a representative in $W/W_s$.

For each $u = w \cdot s^{-1} \in W \cdot s^{-1}$, let $E^X[u]$ denote the preimage of $X \in \Omega^a$ under $\mu^\alpha_w$, projected to $G/B$:

$$E^X[u] = \{ gw^{-1}B; gs = sg, X \in g^a \Omega^a \}. \quad (2.3)$$

Notice that replacing $w$ by $uw'$ ($w' \in W_s$) in this construction gives the same variety, hence $E^X[u]$ only depends on $w \in W/W_s$.

**Proposition 2.1.** Let $\tau \in \text{MP}(a)$ be given. For $w \in W/W_s$, the $u = (w \cdot s^{-1})$-weight space of the standard module $M_{(a,v)}$ is $H_*(E^X[u])$. Moreover, $u$ is a $R(T)$-weight of $L_{(a,v)}$ if and only if $(\mu^\alpha_w)_u \subset \Omega^a$ contains $\text{IC}(\mathcal{O}_\tau)$.

**Proof.** See Chriss-Ginzburg [CG97] §8.6. An important consequence is that we can characterize certain $R(T)$-weights of $L_{(a,v)}$.

**Corollary 2.2.** If $\mathcal{O}_\tau$ meets $w \Omega^a$ in a dense open subset, then $w \cdot s^{-1}$ is a $R(T)$-weight of $L_{(a,v)}$.

**Proof.** We have $(\mu^\alpha_w)^{-1}(X) \neq \emptyset$ only if $X \in \overline{\mathcal{O}_\tau}$. We have $\dim H_*(\mu^\alpha_w)^{-1}(X)) \neq 0$ when $X \in \mathcal{O}_\tau$. It follows that there exist a simple constituent of $(\mu^\alpha_w)_u \subset \Omega^a$ which has support contained in $\overline{\mathcal{O}_\tau}$. By the BBD-Gabber theorem and [Ka09] Theorem 4.10, we have $\text{IC}(\mathcal{O}_\tau)$ as a direct summand of $(\mu^\alpha_w)_u \subset \Omega^a$ (up to degree shift). Now Proposition 2.1 implies the result.

### 2.2 Conditions on weights of $\mathbb{H}_{n,m}$-modules

Later in this section, we work in the same setting as in [11].

**Proposition 2.3.** Let $\tau = (\mathbf{i}, \delta) \in \text{MP}(a)$. Let $w \in W$. If $w \Omega^a$ meets $\mathcal{O}_\tau$, then there exists $v \in \mathcal{E}^a_{\mathbf{i}}$ which satisfies:

$(\heartsuit)_w$ For every $i, j \in I_k \in \mathbf{i}$ such that $\langle \epsilon_i, s \rangle \in q^{Z^a} \langle \epsilon_j, s \rangle$, we have either

$$wv(i) < wv(j) \text{ or } wv(i) > 0 > wv(j).$$

**Proof.** The space $w \Omega^a$ is stable under the action of $wB(s)$. It follows that the space $G(s)^w \Omega^a$ is a closed subset of $\Omega^a$. In particular, a $G(s)$-orbit $\mathcal{O}_\tau \subset \Omega^a$ meets $w \Omega^a$ if and only if we have $\mathcal{O}_\tau \subset \overline{\mathcal{O}_\tau}$, where $\mathcal{O}$ is the open dense $G(s)$-orbit of $G(s)^w \Omega^a$.

Condition $(\heartsuit)_w$ is independent of the marking. Consider $\tau' := (\mathbf{i}, 0) \in \text{MP}_0(a)$. We have $\mathcal{O}_{\tau'} \subset \overline{\mathcal{O}_{\tau'}}$. Hence, it suffices to verify $(\heartsuit)_w$ in the case $\tau = \tau'$. We put $T_\tau := \text{Stab}_{T^{\tau'}}$. It is easy to verify that $T_\tau$ is a connected torus of rank $\# \mathbf{i}$ (the number of segments of the support of $\tau$). Since we have $\langle \epsilon_i, s \rangle \neq \langle \epsilon_j, s \rangle$
for each distinct $i, j \in I_k \in \mathbf{I}$, it follows that $Z_{G(s)}(T_r) = T$ and $T v_r = O_r^T$. The set of $^w T_r$-fixed points of

$$F_w^a := G(s) \times ^w B(s) \ w \mathcal{V}(a) \ \leftarrow G(s) / B$$

is concentrated on the fiber of $(G(s) / B)^T$ for every $w \in W$ and $v \in W_s$. The image of $\mu^n_w$ contains all of $O_r$ if and only if $O_r \subset \mathfrak{R}^n$ meets $w \mathcal{V}(a)$. Therefore, $w \mathcal{V}(a)$ meets $O_r$ if and only if

$$\dim (w \mathcal{V}(a)) = \dim T v_r$$

for some $v' \in W_s$.

Since $a \in T_0$, we have $\mathcal{V}_a \subset \mathfrak{g}(n) \oplus \mathbb{C}^n$. Moreover, we have $W_s \cong \mathfrak{S}_n^n$. Now \( \mathfrak{S} \) is equivalent to the fact that $v t_k \in w \mathcal{V}(a)$ for each $I_k \in \mathbf{I}$. 

**Corollary 2.4.** Keep the setting of Proposition 2.3. We have $w s^{-1} \in \Psi(M_{(a,v_s)})$ only if there exists some $v \in \mathfrak{S}_n^n$ satisfying \( \mathfrak{S} \). In particular, $w s^{-1} \in \Psi(L_{(a,v_s)})$ only if there exists some $v \in \mathfrak{S}_n^n$ satisfying \( \mathfrak{S} \).

**Proof.** By definition, we have $E_v^a[w s^{-1}] \neq \emptyset$ if and only if $G(a) \cap w \mathcal{V} \neq \emptyset$. It follows that $E_v^a[w s^{-1}] \neq \emptyset$ only if there exists some $v \in \mathfrak{S}_n^n$ satisfying \( \mathfrak{S} \). Therefore, Proposition 2.3 implies the result.

**Theorem 2.5** (see e.g. Barbasch-Moy [BM03 §6.4]). Fix $n = n_1 + n_2$ with $n_1, n_2 \in \mathbb{Z}_{\geq 0}$. Let $L_1$ be a \( \mathbb{H}_{n_1} \)-module and let $L_2$ be a \( \mathbb{H}_{n_2} \)-module. We form

$$M := \text{Ind}_{\mathbb{H}_{n_1} \otimes \mathbb{H}_{n_2}}^{\mathbb{H}_n} L_1 \boxtimes L_2.$$ 

If an irreducible \( \mathbb{H}_n \)-module $L$ is a subquotient of $M$, then we have

$$\Psi(L) \subset \Psi(M) = \bigcup_w w \cdot (\Psi(L_1^A) \times \Psi(L_2)),$$

(2.4)

where $w \in W$ runs over the maximal length representatives of $W / (\mathfrak{S}_{n_1} \times \mathfrak{S}_{n_2})$.

**Proposition 2.6.** Keep the setting of Theorem 2.5. Assume that $L_1 = L_{(a_1,v_{r_1})}$ holds for $r_1 = (1,0) \in M \mathcal{P}_0(a_1)$, where $a_1 = (s_1, \bar{q})$. Then, we have

$$\Psi(L) \subset \bigcup_w w \cdot (\Psi(M_{(a_1,v_{r_1})}) \times \Psi(L_2)),$$

(2.5)

where $w \in \mathfrak{S}_n$ runs over the minimal length representatives of $\mathfrak{S}_n / (\mathfrak{S}_{n_1} \times \mathfrak{S}_{n_2})$.

**Proof.** Thanks to Theorem 2.5, we have

$$\Psi(M_{(a_1,v_{r_1})}) = \bigcup_v v \cdot \Psi(M_{(a_1,v_{r_1})}),$$

$$\Psi(\text{Ind}_{\mathbb{H}_{n_1}}^{\mathbb{H}_n} M_{(a_1,v_{r_1})}) \supset \Psi(\text{Ind}_{\mathbb{H}_{n_1}}^{\mathbb{H}_n} L_{(a_1,v_{r_1})}),$$

where $v$ runs over the minimal cost representatives of $W_{n_1} / \mathfrak{S}_{n_1}$. A minimal length representative $w \in W / (\mathfrak{S}_{n_1} \times \mathfrak{S}_{n_2})$ decomposes uniquely into $w = w_1 w_2$, where $w_1$ is a minimal length representative of $W_{n_1} / \mathfrak{S}_{n_1}$ and $w_2$ is a minimal length representative in $\mathfrak{S}_n / (\mathfrak{S}_{n_1} \times \mathfrak{S}_{n_2})$. Therefore, the comparison between (2.5) and (2.4) implies the result.

**Remark 2.7.** Notice that in Proposition 2.6, the module $M_{(a_1,v_{r_1})}$ is a standard \( \mathbb{H}_{n_1} \)-module and not a standard \( \mathbb{H}_{n_1} \)-module. This is an important point when using Corollary 2.3.
2.3 Nested component decomposition

Let $\tau = (I, \delta), I = \{I_i\}_{i=1}^k$, be a marked partition adapted to $a = (s, \vec{q}) \in T_0$. Assume that $I$ can be split into a disjoint union of two collections of segments $I_1$ and $I_2$ with the property

\[ I \sqsupseteq I', \quad \text{for every } I \in I_1, I' \in I_2, \text{ or} \]
\[ I' \sqsubset I, \quad \text{for every } I \in I_1, I' \in I_2, \]

where $I \sqsupseteq I'$ means

\[ \min I < \min I' \text{ and } \max I < \max I'. \]

Let $n_1$ and $n_2$ be the sums of cardinalities of segments of $I_1$ and $I_2$, respectively. By applying an appropriate permutation, we assume that

- $I_1$ and $I_2$ are divisions of $[1, n_1]$ and $(n_1, n]$, respectively.

Then we can regard $\tau_1 = (I_1, \delta|_{I_1})$ and $\tau_2 = (I_2, \delta|_{I_2})$ as marked partitions for $G_1 = Sp(2n_1)$ and $G_2 = Sp(2n_2)$ respectively, where $Sp(2n_1) \times Sp(2n_2)$ is embedded diagonally in $Sp(2n)$.

The marked partition $\tau$ parameterizes a $G(s)$-orbit $O_\tau$ on $\mathfrak{N}^a$. We define semisimple elements $s_1 \in Sp(2n_1)$, $s_2 \in Sp(2n_2)$ to be the projections of $s$ onto the $Sp(2n_1), Sp(2n_2)$ factors, respectively. We set $a_1 := (s_1, \vec{q})$, and $a_2 := (s_2, \vec{q})$. The marked partitions $\tau_1$ and $\tau_2$ define orbits $O_{\tau_1}, O_{\tau_2}$ of $G_1 = G_1(s_1), G_2 = G_2(s_2)$ respectively on the corresponding exotic nilcones.

**Lemma 2.8.** Every $G(s)$-orbit $O$ of $\mathfrak{N}^a$ which contains $O_\tau$ in its closure can be written as $G(s)(O_1 \times O_2)$, where $O_1$ and $O_2$ are exotic nilpotent orbits of $G_1$ and $G_2$, respectively. In addition, the marked partitions corresponding to $O_1$ and $O_2$ are nested in the sense of $\sqsupseteq$.

**Proof.** The condition $\sqsupseteq$ is independent of markings. Thus, it suffices to prove the assertion for all orbits with no markings. In the algorithm of Theorem 1.30, it is straightforward to see that we cannot choose $I_i \in I_1$ and $I_i \in I_2$. Therefore, the above procedure preserves $\bigcup_{i \in I_1} I_i$ and $\bigcup_{i \in I_2} I_i$, respectively. Moreover, it preserves the nestedness of the modified $I_1$ and $I_2$, which implies the result.

Let $V^{(1)}$ and $V^{(2)}$ be the exotic representations of $G_1$ and $G_2$, respectively. We set

\[ O_\tau^i := \bigcup_{O \in O_\tau} O, \]

where $O$ runs over all $G(s)$-orbits of $\mathfrak{N}^a$. This is a $G(s)$-stable open subset of $\mathfrak{N}^a$. Similarly, for $i = 1, 2$, we define $O_\tau^{i,j}$ to be the union of $G_i(s_i)$-orbits of $\mathfrak{N}^a \cap V^{(i)}$ which contain $O_\tau^i$ in their closure.

**Corollary 2.9.** Keep the setting of Lemma 2.8. We have

\[ G(s)(O_\tau^1 \times O_\tau^2) = O_\tau^3. \]

**Lemma 2.10.** For each $x = x_1 \times x_2 \in O_\tau^1 \times O_\tau^2 \subset O_\tau^3$, we have

\[ \text{Stab}_{G(s)x} = \text{Stab}_{G_1(s_1)x_1} \times \text{Stab}_{G_2(s_2)x_2}. \]
Proof. Without loss of generality, we can assume that \( x = \nu \tau \) with \( \tau = (I, \delta) \). We have \( \text{Stab}(G_n^\pm x) \subset \text{Stab}(G_n^\pm x') \) if \( \delta = \nu \tau \delta' \) with \( \delta' = (I, 0) \). Therefore, it suffices to prove the result when \( \delta = 0 \). In that case, the assertion follows by Lemma 1.3.2 and Theorem 1.3.3. \( \square \)

**Corollary 2.11.** Keep the setting of Lemma 2.10. Then, we have

\[
G(s) \times (G_1(s_1) \times G_2(s_2)) \left( \Omega_i^* \times \Omega_j^* \right) \xrightarrow{a,X} \Omega_{i,j}^*.
\]

Let \( M \) and \( L \) be two \( \mathbb{H}_n \)-modules with \( L \) simple. Let \( [M : L] \) denote the multiplicity of \( L \) in \( M \) in the Grothendieck group of \( \mathbb{H} \text{-mod} \).

We have a geometric multiplicity formula of irreducible modules in a standard module (Kazhdan-Lusztig conjectures). It is obtained in [Ka09] as an application of the Ginzburg theory ([CG97]).

**Theorem 2.12** ([Ka09] Theorem 11.2). Let \( a \in G \) be a positive real element. Let \( X, X' \in \mathfrak{U}^* \) be given. We set \( O := G(a)X \) and \( O' := G(a)X' \). Then we have

\[
[M(a,X) : L(a,X')] = \dim H^*_\bullet(G(O')).
\]

In particular, if \( O \subset O' \) is smooth, then \( [M(a,X) : L(a,X')] = 1 \). \( \square \)

**Corollary 2.13.** Keep the setting of Lemma 2.10. Then, for every \( y = y_1 \times y_2 \in \Omega_i^* \times \Omega_j^* \), we have

\[
[M(a,x) : L(a,y)] = [M(a_1,x_1) : L(a_1,y_1)][M(a_2,x_2) : L(a_2,y_2)].
\]

**Proof.** We have \( [M(a,x) : L(a,y)] = \dim H^*_\bullet(G(a)x, \text{IC}(G(a)y)) \) for \( * = 0, 1, 2 \) in a uniform fashion by Theorem 2.12. Hence, we deduce

\[
\dim H^*_\bullet(G(a)x, \text{IC}(G(a)y)) = \dim H^*_\bullet(G_1(a_1) \times G_2(a_2), \text{IC}(G_1(a_1) \times G_2(a_2)y))
\]

\[
= (\dim H^*_\bullet(G_1(a_1), \text{IC}(G_1(a_1)y))) (\dim H^*_\bullet(G_2(a_2), \text{IC}(G_2(a_2)y))),
\]

which implies the assertion. \( \square \)

### 2.4 Specialization of parameters

We start with a corollary of a well-known result as is presented in Lemma 2.3.3 of [CCG97], for example.

**Proposition 2.14.** Let \( R \) be a \( \mathbb{C}[t] \)-algebra of finite rank. Let \( M \) be a \( R \)-module which is free as a \( \mathbb{C}[t] \)-module. Assume that we have an \( R \)-submodule \( N \subset M \) whose localization \( \mathbb{C}[t] \otimes_{\mathbb{C}[t]} N \) is a free \( \mathbb{C}[t] \)-module. Then, there exists an \( R \)-submodule \( N' \subset M \) such that

\[
\mathbb{C}[t] \otimes_{\mathbb{C}[t]} N = \mathbb{C}[t] \otimes_{\mathbb{C}[t]} N' \subset \mathbb{C}[t] \otimes_{\mathbb{C}[t]} M
\]

and the quotient \( M/N' \) is free over \( \mathbb{C}[t] \). \( \square \)

**Corollary 2.15.** Let \( a' = a \exp(\gamma t) \) be a one-parameter family depending on \( t \in \mathbb{R} \), with \( a' \in \mathcal{T}_0 \) for all but finitely many values of \( t \), and where

\[
\gamma \in \mathfrak{g} \oplus \{0\} \oplus \mathbb{R}^3_{\geq 0} \subset \text{Lie}(T \times (\mathbb{C}^\times)^3).
\]
Let $\tau$ be a marked partition adapted to each of $a^i$. Then, we have

$$\Psi(L_{(a^0,v_{r_{v}})}) \subset \lim_{t \to 0} \Psi(L_{(a^i,v_{r_{v}})}).$$

In particular, the module $L_{(a^0,v_{r_{v}})}$ is tempered if $L_{(a^i,v_{r_{v}})}$ defines a tempered module in (at least) one of the region

$$\epsilon > t > 0 \text{ or } -\epsilon < t < 0 \text{ for some positive number } \epsilon \ll 1.$$

Proof. Let $a^i \in t \oplus \mathbb{R}^2$ be the element defined from $a^i$ via the statement of Proposition 1.17. We have $a^i = a + t\gamma$. We choose $A$ in Proposition 1.17 so that $a^i \in A$. Let $\ell \subset a$ denote the line $\{a^j\}_{j \in \mathbb{R}}$. We have the corresponding surjection $\mathbb{C}[a] \to \mathbb{C}[t]$. Therefore, we have a family of $\mathbb{H}_t := \mathbb{C}[t] \otimes_{\mathbb{C}[a]} \mathbb{H}_a$-modules $M_t := \mathbb{C}[t] \otimes_{\mathbb{C}[a]} H^a_\tau(E_{r_{v}})$. We apply Proposition 2.14 to the family of maximal $\mathbb{H}_t$-submodules ($t \neq 0$) of $M_{(a^0,v_{r_{v}})}$ for which the corresponding quotients are $L_{(a^0,v_{r_{v}})}$. Every such $\mathbb{H}_a$-submodule extends to an $\mathbb{H}_t$-submodule $N \subset M_t$ whose quotient specializes to $L_{(a^0,v_{r_{v}})}$ unless $t = 0$. Since a finite-dimensional $W$-module is rigid under flat deformation, it follows that the $W$-module structure of $N$ must be constant along $t \in \mathbb{R}$. Therefore, $\mathbb{C} \otimes_{\mathbb{C}[a]} M_t/N$ contains a non-trivial $\mathbb{H}_a$-module which contains $L_{v_{r_{v}}}$ (as $W$-modules). This must be $L_{(a^i,v_{r_{v}})}$. Since $M_t$ is an algebraic family of $\mathbb{H}_a$-modules, we have

$$\Psi(L_{(a^0,v_{r_{v}})}) \subset \lim_{t \to 0} \Psi(C_t \otimes_{\mathbb{C}[t]} M_t/N) = \lim_{t \to 0} \Psi(L_{(a^i,v_{r_{v}})}),$$

where $C_t$ is the quotient of $\mathbb{C}[t]$ by the ideal $(t - t)$. The rest of the assertions are clear.

Corollary 2.16. With the notation from Corollary 2.14, assume that $L_{(a^i,v_{r_{v}})}$ is a discrete series for $t \in (-\epsilon, \epsilon) \setminus \{0\}$. Then $L_{(a^0,v_{r_{v}})}$ is a discrete series.

Proof. By 2.15, $\Psi(L_{(a^0,v_{r_{v}})}) \subset \lim_{t \to 0} \Psi(L_{(a^i,v_{r_{v}})}).$ Let $w \cdot a^i$ be a one-parameter family of weights, $w \in W$, such that $w \cdot a^i \in \Psi(L_{(a^i,v_{r_{v}})})$ and $w \cdot a^0 \in \Psi(L_{(a^0,v_{r_{v}})}).$ By the discrete series condition, $(w_j, (w \cdot a^i) < 1$, for all $1 \leq j \leq n$, and for all $t \in (-\epsilon, \epsilon) \setminus \{0\}$. Since $(w_j, w \cdot a^i)$ is continuous and linear in $t$, it follows that $(w_j, w \cdot a^0) < 1$ (for every $j$) as well.

3 Parameters corresponding to discrete series

Recall that for any finite dimensional $\mathbb{H}_a$-module $V$, we denote by $\Psi(V) \subset T$ the set of its $R(T)$-weights.

3.1 Distinguished marked partitions

We restrict now to the case of the specialized affine Hecke algebra of type $C_n$ with $\tilde{q} = (-1, q^m, q), q \in \mathbb{R}_{>1}, m \in \mathbb{R}$, and we assume the genericity condition, i.e., $m \notin \mathbb{Z}$.

Let $a = (s, \tilde{q}) \in T_0$ be given.

Definition 3.1. We say that $a$ (or $s$) is distinguished if the dense $G(a)$-orbit on $\mathfrak{H}_a$ is parameterized by a marked partition $(\{I_j\}_{j=1}^N, \delta)$ which satisfies:

1. $\max I_1 > \max I_2 > \cdots > \max I_N$;
2. \( \min I_1 < \min I_2 < \cdots < \min I_k; \)

3. \( \delta(I_j) = \{0, 1\}, \) for all \( j \) (which in particular means \( q_1 \in I_j \) for all \( j \)).

We call such a marked partition distinguished as well.

Notice that the distinguished marked partitions are in one to one correspondence with partitions of \( n \) by a “folding” procedure: for every \( J \in \{I_j\}_{j=1}^k \), define \( \# J \) to be the number of elements in \( J \) strictly smaller than \( q_1 \), and \( \# J' \) to be the number of elements in \( J' \) greater than or equal to \( q_1 \). If \( \text{mp}(\sigma) \) is a distinguished marked partition, then one can build a left-justified nondecreasing partition (tableau) \( \sigma \) of \( n \), as follows: put \( \# I_1 \) boxes on the first row and \( \# I_1 \) boxes on the first column below the first row (so the \( I_1 \) looks bent like a hook), then add \( \# I_2 \) boxes on the second row and \( \# I_2 \) on the second column, below the second row etc. Remark that, in the end, the diagonal of the tableau \( \sigma \) has boxes exactly corresponding to the markings of \( \text{mp}(\sigma) \) (see figure 3.1).

![Figure 1: The correspondence \( \text{mp}(\sigma) \leftrightarrow \sigma \), for \( \sigma = (4, 3, 3, 2, 1) \).](image)

**Theorem 3.2** ([Op04], Lemma 3.31). Assume \( s \in T \), and \( a = (s, \vec{q}) \) is as above. Then there exists a discrete series module with central character \( s \) if and only if \( s \) is distinguished in the sense of Definition 3.1.

In particular, a distinguished semisimple \( a \) (or \( s \)) corresponds to a partition \( \sigma \) of \( n \). We write \( a_\sigma \) and \( s_\sigma \) to emphasize this dependence. (We remark that \( a_\sigma \) and \( s_\sigma \) are well-defined up to \( S_n \)-action even if we require \( a_\sigma \in T_0 \).) Notice that, by [1.2] the marked partition \( \text{mp}(\sigma) \) above parameterizes the open \( G(a_\sigma) \)-orbit in \( \mathfrak{P}^{a_\sigma} \). The goal of this section is to identify which \( G(a_\sigma) \)-orbit in \( \mathfrak{P}^{a_\sigma} \) parameterizes the discrete series \( H_{a_\sigma} \)-module under Theorem 1.19. By Propositions 1.28 and 1.29 we need to describe a marked partition, denoted \( ds(\sigma) \) or \( ds(s_\sigma) \), which we may regard as a representative of an orbit via the map \( Y \).

### 3.2 Algorithm

We start with a distinguished marked partition \( \text{mp}(\sigma) \) corresponding to a partition \( \sigma \) of \( n \) as in [3.1] and let \( s_\sigma \) denote the corresponding semisimple element. We put integer coordinates \( (i, j) \) in the boxes of \( \sigma \) such that the boxes on the first row have coordinates: \((1, -1), (2, -1), (3, -1) \) etc., the boxes on the second row: \((1, -2), (2, -2), (3, -2) \) etc., the numbering starting from the left. Note that the boxes of the diagonal have coordinates \((i, -i)\).

We define a function on the boxes of \( \sigma \), which we call an \( e \)-function. For a box \((i, j)\), we set

\[
e(i, j) = \log_q(q_1^{q_1^i + j}) = m + i + j.
\]

Let \( av(I) \) denote the sum of all \( e \)-values of \( I \subset \sigma \).
Given $\sigma$, the following algorithm gives a marked partition $\text{out}(\sigma)$ which turns out to parameterize the discrete series with central character $s_\sigma$ (i.e., $\text{out}(\sigma) = \text{ds}(\sigma)$).

**Algorithm 3.3.**

1. Set $\ell = 0$, $\sigma(\ell) = \sigma$, $L^+ = L^- = \emptyset$. ($L^+$ and $L^-$ will be collections of subsets of $\sigma$.)

2. Find the unique $(i, j) \in \sigma(\ell)$ such that $e(i, j)$ or $-e(i, j)$ attains the maximum in the set \{\$e(i, j)\$ : $(i, j) \in \sigma(\ell)\}$.

   (a) If the maximum is at $e(i, j)$, append the set (horizontal strip) \{(i - k, j) \in \sigma(\ell) : k \geq 0\} to $L^+$.

   (b) If the maximum is at $-e(i, j)$, append the set (vertical strip) \{(i, j + k) \in \sigma(\ell) : k \geq 0\} to $L^-$.

Remove the horizontal or vertical string as above from $\sigma(\ell)$ and call the resulting partition $\sigma(\ell + 1)$. If $\sigma(\ell + 1) \neq \emptyset$, increase $\ell$ to $\ell + 1$ and go back to the beginning of step 2.

3. Set $L = \emptyset$. (This will be a collection of sets.) For every $-n \leq k \leq n$, form

\[
L^+_k = \{I \in L^+ : \text{min}_{(i, j) \in I} e(i, j) = m + k\}, \quad L^-_k = \{I \in L^- : \text{max}_{(i, j) \in I} e(i, j) = m + k - 1\}.
\]

(3.2)

For every $k$, order the elements in $L^+_k$, respectively $L^-_k$ decreasingly with respect to their cardinality: $I^+_k, 1, \ldots, I^+_k, h_1$ and $I^-_k, 1, \ldots, I^-_k, h_2$. By adding empty sets at the tail of the appropriate sequence, we may assume $h_1 = h_2$. Then for $j = 1, \ldots, h_1$, form the segment $I^+_k, j \cup I^-_k, j$, and append it to $L$.

(Notice that $I^+_k, j \cup I^-_k, j$ is a segment since we have an identification of $\text{mp}(\sigma)$ with $\sigma$.)

Then $L$ is the collection of segments in the marked partition $\text{out}(\sigma)$. We specify the marking $\delta$ next.

4. Define a temporary marking $\delta'$ first. For every $I \in L$, let $e(I)$ denote the set of $e(i, j)$ for $(i, j) \in I$. Recall that $I$ could be marked only if $m \in e(I)$, and if so, the marking could only be on the box $(i, j)$ with $e(i, j) = m$. Set

\[
\delta'(I) = \begin{cases} 1, & \text{if } m \in e(I) \text{ and } a\nu(I) > 0, \\ 0, & \text{otherwise}. \end{cases}
\]

(3.3)

We refine $\delta'$ to $\delta$ by removing the marking of any segment $I$ which is dominated by marked segment $I'$.

**Remark 3.4.**

1. The hypothesis that $a = (s, \bar{q})$ is generic is essential for the uniqueness of the box $(i, j)$ realizing the maximum in step 2 of the algorithm.

2. The first two steps of the algorithm are identical with the algorithm conjectured by Slooten [106] for a generalized Springer correspondence for the graded Hecke algebra $\mathcal{H}_{n,m}$ (Definition 1.13) with generic unequal labels. We will see that this algorithm is equivalent with the one described by Lusztig-Spaltenstein [155] for $\mathbb{H}_{n,m}$ with (representative) generic unequal labels constructed from cuspidal local systems in Spin groups. We explain this in more detail in [133].

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3. To clarify the algorithm, we offer an example. Consider $n = 14$, and the partition $\sigma = (4,3,3,2,1)$, assuming that $2 < m < \frac{5}{2}$, see figure 3 (in the figure an entry $k$ in the box means the $e$-value is $\log(q^k) = m+k$).

\[
\begin{array}{cccc}
0 & 1 & 2 & 3 \\
\times & 0 & 1 & 1 \\
3 & 2 & 1 & 0 \\
2 & & & \\
\end{array}
\]

Figure 2: Partition $\sigma = (4,3,3,2,1)$ and $2 < m < \frac{5}{2}$

If we identify $I$ with $e(I)$ for a segment $I$, then we find

$L^+ = \{[m, m + 1, m + 2, m + 3], [m - 1, m, m + 1], [m - 2, m - 1, m], [m - 2]\}$

$L^- = \{[m - 4, m - 3]\}$.

We separate the segments based on where they begin or end: $L^+_0 = \{[m, m + 1, m + 2, m + 3]\}$, $L^+_1 = \{[m - 1, m, m + 1]\}$, $L^-_2 = \{[m - 2, m - 1, m], [m - 2]\}$, and $L^-_3 = \{[m - 4, m - 3]\}$. Next we may combine the segment in $L^+_2$ with the longest segment in $L^-_2$. The resulting marked partition $\tau$ has support $I$ given by the segments $I_1, I_2, I_3, I_4$, such that $e(I_1) = [m, m + 1, m + 2, m + 3]$, $e(I_2) = [m - 1, m, m + 1]$, and $e(I_3) = [m - 4, m - 3, m - 2, m - 1, m]$, $I_4 = [m - 2]$. According to the algorithm, we temporarily mark the first three segments at $m$, but then since $I_3 \prec I_2 \prec I_1$, we remove the markings on $I_2$ and $I_3$. In conclusion, the output of the algorithm is the marked partition $\text{out}(\sigma)$ (see figure 3 with support given by $\{I_1, I_2, I_3, I_4\}$ and a single marking on $I_1$. (This marked partition is in the same orbit with the one where all three $I_1, I_2, I_3$ are marked.)

\[
\begin{array}{cccc}
0 & 1 & 2 & 3 \\
\times & 0 & 1 & 1 \\
3 & 2 & 1 & 0 \\
2 & & & \\
\end{array}
\]

or, after aligning the rows,

\[
\text{out}(\sigma) = \begin{pmatrix}
\times & 1 \\
\end{pmatrix}
\]

Figure 3: Output of Algorithm 3.3 when $\sigma = (4,3,3,2,1)$ and $2 < m < \frac{5}{2}$

The main result of this section is:

**Theorem 3.5.** Let $\sigma$ be a partition of $n$, and let $a_\sigma, s_\sigma$ be the semisimple elements constructed from $\sigma$ in §3.7. The discrete series $H_{a_\sigma}$-module (with central character $s_\sigma$) is $L(a_\sigma, \Upsilon(\text{out}(\sigma)))$, where $\text{out}(\sigma)$ is the marked partition constructed in Algorithm 3.3. In other words, $\text{out}(\sigma) = ds(\sigma)$. 

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The proof will be divided into several parts in the next sections.

Example 3.6. (One hook partitions.) Before proving Theorem 3.5 let us present a particular case of the algorithm. Assume $\sigma$ is a partition given by a single hook, i.e., $\sigma = (k,1^{n-k})$, for some $1 \leq k \leq n$. This means that the semisimple element is $s_\sigma = (q_1 q^{k-1}, q_1 q^{k-2}, \ldots, q_1, q_1 q^{k-n})$. In this case,

$$\mathfrak{H}_\sigma = \bigoplus_{i=1}^n \mathbb{C} v_{e_i - e_{i+1}} \oplus \mathbb{C} v_{e_k},$$

and $G(s_\sigma) = (\mathbb{C}^\times)^n$ (i.e., the maximal torus),

there are $2^n$ orbits in $G(s_\sigma) \setminus \mathfrak{H}_\sigma$, each orbit corresponding to a subset $S$ of $\{v_{e_i - e_{i+1}}, 1 \leq i < n, v_{e_k}\}$. To $S$, there corresponds a marked partition $\tau_S = (1, \delta)$, as follows: for every maximal string of consecutive weight vectors $\{v_{e_i - e_{i+1}}, \ldots, v_{e_{l+i-1} - e_{l+i}}\}$ in $S$, we attach a segment $I = [i, i+1, \ldots, i+l-1] \in I$, of length $t$ with $e$-values $(m+k-i, \ldots, m+k-i-t+1)$. In addition, we mark at $q_1$, and write $\delta = 1$, if $v_{e_k} \in S$, and we don’t mark, and write $\delta = 0$, otherwise.

We remark that, as a consequence of the results about weights, one sees that any $R(T)$-weight space of an irreducible $\mathbb{L}_{\alpha_\sigma}$-module with central character $s_\sigma$, i.e., parameterized by a marked partition of the form $\tau_S$, is one-dimensional.

By applying the algorithm explicitly, as a corollary of Theorem 3.5, we find that the discrete series $ds(\sigma)$ is parameterized by the marked partition $\tau = (1, \delta)$ as follows:

(a) if $q_1 q^{k-n} > 1$, then $\tau$ has $I = \{[1,2,\ldots,k],[k+1],[k+2],\ldots,[n]\}$ and $\delta = 1$;

(b) if $q_1 q^{k-n} < 1$, and

(b1) $(q_1 q^{k-n})^{-1} > q_1 q^{k-1}$, then $\tau$ has $I = \{[1,2,\ldots,n]\}$ and $\delta = 0$;
(b2) $q_1 q^{k-1} < (q_1 q^{k-n})^{-1} < q_1 q^{k-1}$, then $\tau$ has $I = \{[1,2,\ldots,n]\}$ and $\delta = 1$;
(b3) there exists $l < 0$ with the property that $q_1 q^{l-1} < (q_1 q^{k-n})^{-1} < q_1 q^l$, then $\tau$ had $I = \{[1,2,\ldots,k],[k+1],\ldots,[l-1],[l,l+1,\ldots,n]\}$ and $\delta = 1$.

Convention 3.7. A bijection $c : \sigma \to [1,n]$ is called a $c$-function for $\sigma$ if it satisfies the following condition

$$c(I) > c(J) \text{ if } c(I) < c(J) \text{ for } I \neq J \in \sigma.$$

In the following, using an appropriate $c$-function if necessary, we identify the set of boxes $\square \in \sigma$ with an interval $[1,n]$. Moreover, we define $s_\sigma$, so that $c(\square) = \log q_s(\sigma_{c(\square)}, s_\sigma)$ for each $\square \in \sigma$. In addition, we identify a segment $I$ (adapted to $s_\sigma$) as a set of boxes in $\sigma$. Notice that we have $c(I) = \log q_s(I)$ in this setting.

3.3 A particular case: ($\pm$)-ladders

Before we begin, we present a corollary of Theorem 2.12 which is used repeatedly in the proofs.
Corollary 3.8. Assume $n = n_1 + n_2$, and let $\mathbb{H}_P$ be the Hecke algebra for $GL(n_1) \times Sp(2n_2)$. Consider $\alpha = (s, \vec{q}) \in T_0$ which decomposes $s = s_1 \times s_2$ with $s_1 \in GL(n_1)$ and $s_2 \in Sp(2n_2)$. Let $\tau_1 \in MP_0(s_1, \vec{q})$, $\tau_2 \in MP(s_2, \vec{q})$, and $\tau \in MP(\alpha)$ be given. We assume:

- For every $\tilde{\tau}_2 \in MP(s_2, \vec{q})$, which gives a strictly larger orbit than $\tau_2$, we have $v_{\tau_1} \oplus v_{\tilde{\tau}_2} \not\in G\tau$;
- The induction condition (1.14) in Theorem 1.15 is satisfied.

Then, we have

$$[\text{Ind}_{\mathbb{H}_P}(M_{\tau_1}^A \boxtimes M_{\tau_2}^A) : L_{\tau}] = [\text{Ind}_{\mathbb{H}_P}(M_{\tau_1}^A \boxtimes L_{\tau_2}) : L_{\tau}],$$

(3.5)

and the same formula holds with $L_{\tau_1}^A$, $L_{\tau_1}^A$, or $M_{\tau_1}^A$ in place of $M_{\tau_1}^A$.

Proof. This is an obvious consequence of Theorem 2.12 and of the exactness of the parabolic induction functor.

In the following, whenever we need to apply Corollary 3.8, we omit the details of the verification of $v_{\tau_1} \oplus v_{\tilde{\tau}_2} \not\in G\tau$ since they are easily checked by inspection.

We begin with a particular instance of Algorithm 3.3: the cases when the algorithm produces $L^- = \emptyset$ or $L^+ = \emptyset$.

Definition 3.9 ((±)-ladder). Let $a_\sigma$ be a distinguished semisimple element as in §3.1. A positive ladder corresponding to $a_\sigma$ is a marked partition $\tau = (I, \delta)$ adapted to $a_\sigma$ which satisfies the following conditions:

1. We have $I = \{I_1, I_2, \ldots\}$ such that $e(I_i) = \{m + 1 - i, \ldots, m + \lambda_i - i\};$

2. We have $\delta(\square) = 1$ if $e(\square) = m$ and $\square \in I_1$, and $\delta(\square) = 0$ otherwise.

A negative ladder corresponding to $a_\sigma$ is a marked partition $\tau = (I, \delta)$ adapted to $a_\sigma$ which satisfies the following conditions:

1. We have $I = \{I_1, I_2, \ldots\}$ such that $e(I_i) = \{m + i - \lambda_i, \ldots, m + i - 1\};$

2. We have $\delta \equiv 0.$

For every distinguished $\sigma$ there are unique (±)-ladders: the positive ladder has the collection of segments $I$ as the rows of $\sigma$, and every $I \in I$ with $m \in e(I)$ is marked, while the negative ladder has the collection of segments $I$ as the columns of $\sigma$, and has no marking.

Recall that in general, the weights $\Psi(L_{\tau})$ are a subset of $W \cdot s_\sigma^{-1}$. If $\tau$ is a (±)-ladder, then the weights have a particular form:

Proposition 3.10. Choose a $c$-function for $\sigma$ (see Convention 3.7) in order to fix $s_\sigma \in T$. Then, we have

1. Assume that $\tau$ is the positive ladder for $\sigma$. Then, we have $\Psi(L_{\tau}) \subset S_n \cdot s_\sigma^{-1}$. 

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2. Assume that \( \tau \) is the negative ladder for \( \sigma \). Then, we have \( \Psi(L_\tau) \subset \mathfrak{S}_n \cdot \sigma \).

**Proof.** The proofs of the two assertions are completely analogous, therefore we only present the details when \( \tau \) is the positive ladder. The proof is by induction on \( k \), the number of rows of \( \sigma \), or equivalently, the number of segments \( J_j \) in the support of \( \tau \).

In the base case, \( k = 1 \), the orbit corresponding to \( \tau \) is \( G \)-regular in \( \mathfrak{R} = \mathfrak{R}_1 \), and the corresponding module is the Steinberg module. It follows that \( \Psi(L_\tau) = \{ s_\tau^{-1} \} \), which proves the assertion in this case.

Assume the result holds for all \( \sigma' \) with less than \( k \) rows, and assume \( \sigma \) has \( k \) rows. We want to show that for every weight \( w^{-1} \cdot s_\tau^{-1} \), we have \( w^{-1}j > 0 \) for \( 1 \leq i \leq n \) (which implies that \( w \in \mathfrak{S}_n \)), or, equivalently, that \( w^{-1}\epsilon(j) > 0 \) for every box \( j \) of \( \tau \).

Let \( \tau_1 \) and \( \tau_2 \) be the positive ladder partitions corresponding to the last row, respectively the first \( k-1 \) rows, of \( \sigma \), and let \( s_1, s_2 \) be the corresponding semisimple elements. We form the \( H_{\lambda_k}^A \)-module (one dimensional) \( L_{\lambda_k}^A = M_{\lambda_k}^A \) corresponding to \( (s_1, q, v_{\tau_1}) \), and let \( M_{\tau_2} \) be the standard module of \( H_{n-\lambda_k} \).

Theorem 1.15 applies, and we have

\[
M_\tau \cong \text{Ind}_{H^{\lambda_k}_{\lambda_k} \times H_{n-\lambda_k}}^{\mathfrak{S}_n \times \mathfrak{S}_{n-\lambda_k}} (L_{\tau_1}^A \boxtimes M_{\tau_2}).
\]  

(3.6)

(The notation is as in Convention 1.22.) Using Corollary 3.8, we deduce that

\[
[\text{Ind}_{H^{\lambda_k}_{\lambda_k} \times H_{n-\lambda_k}}^{\mathfrak{S}_n \times \mathfrak{S}_{n-\lambda_k}} (L_{\tau_1}^A \boxtimes M_{\tau_2}) : L_{\tau}] > 0.
\]

For every minimal length coset representative \( w \) of \( W_n / (\mathfrak{S}_n \times W_{n-\lambda_k}) \), we analyze the homology \( H_\bullet(\mathcal{E}_{w}^{\alpha, \sigma}[w \cdot s_\tau^{-1}]) \) to see if the \( w \cdot s_\tau^{-1} \)-weight space is nonempty. By the induction hypothesis and Theorem 2.5, we have \( \text{wcl}(j) > 0 \) for all \( j \in \tau_2 \). It remains to show that the same holds for all \( j \in \tau_1 \).

Notice that the minimal \( e \)-value \( e_{\text{min}} \) in \( \tau \) is attained by an element \( \text{min} \) of \( \tau_2 \). Recall that this makes \( e(\text{min}) = n \). There are at most two elements in \( \tau \) which have \( e \)-value equal to \( (e_{\text{min}} + 1) \): one in \( \tau_2 \), denoted \( \tau_1 \), and, if \( \tau_1 \) is not a singleton, one in \( \tau_1 \), denoted \( \tau_2 \).

If \( \text{wcl}(\text{min}) > 0 \), then in order to have \( v_{\tau_1} \in w \mathcal{V}(a) \), one must have \( \text{wcl}(j) > 0 \), for all \( j \in \tau_1 \) by Proposition 2.7 and Corollary 2.4.

If \( \text{wcl}(\text{min}) < 0 \), then we have \( v_{\tau_1} \in w \mathcal{V}(a) \) for \( \alpha = e_{\text{min}} - e_{n} \), where we set \( e(j) = n - j \) for \( j = 1, 2 \) (the first case follows by the induction hypothesis, and the second case is by Proposition 2.6 and Corollary 2.4, and appears if \( \tau_1 \) is not a singleton). But this implies that in order for

\[
H_\bullet(\mathcal{E}_{w}^{\alpha, \sigma}[w \cdot s_\tau^{-1}]) = H_\bullet((\mu_{w}^\alpha)^{-1}(v_{\tau}))
\]

to contribute a non-trivial weight space of \( L_\tau \), the orbit \( O_\tau \) must meet \( \text{Hom}_C(\mathbb{C}, \mathbb{C}^2) \subset \mathfrak{R}^a \) if \( \tau_1 \) is not a singleton, respectively \( \text{Hom}_C(\mathbb{C}, \mathbb{C}) \subset \mathfrak{R}^a \) if \( \tau_1 \) is a singleton, in its open dense part. But since \( \text{min} \) and \( \tau_1 \) are not in the same segment of \( \tau \), this is not the case for \( \tau \).
Corollary 3.11. 1. Assume Algorithm 3.3 produces \( L^- = \emptyset \) for \( \sigma \). Then the output of the algorithm \( \text{out}(\sigma) \), which is the positive ladder, is a discrete series. In particular, this is the case when \( q_1 > q^{n-1} > 1 \).

2. Assume Algorithm 3.3 produces \( L^+ = \emptyset \) for \( \sigma \). Then the output of the algorithm \( \text{out}(\sigma) \), which is the negative ladder, is a discrete series. In particular, this is the case when \( q_1 < q^{1-n} < 1 \).

Proof. Assume \( L^- = \emptyset \), so that \( \text{out}(\sigma) \) is the positive ladder \( \tau \). Then any weight \( w \cdot s_{\sigma}^{-1} \) of \( L_\tau \) is given by a permutation of the entries of \( s_{\sigma}^{-1} \) by Proposition 6.10. I.e., we have \( w \in \mathcal{S}_n \). It follows that
\[
\langle \epsilon_k, w \cdot s_{\sigma}^{-1} \rangle = \langle \epsilon_{w^{-1}k}, s_{\sigma}^{-1} \rangle \leq q_1^{-1}q^{n-1} < 1 \text{ for each } k = 1, \ldots, n.
\]
Therefore, we have \( \langle \pi_j, w \cdot s_{\sigma}^{-1} \rangle = \prod_{k=1}^j \langle \epsilon_k, w \cdot s_{\sigma}^{-1} \rangle < 1 \) for all \( 1 \leq j \leq n \). The case \( L^+ = \emptyset \) is analogous.

3.4 Proof of the main theorem

We continue with the proof of Theorem 3.3. Recall that \( \sigma \) is a partition of \( n \). We wish to prove that \( \text{out}(\sigma) \) is tempered, or equivalently \( \text{out}(\sigma) = ds(\sigma) \).

Assume that in the first two steps of Algorithm 3.3 the segments produced are \( L^+ \sqcup L^- = \{I_1, \ldots, I_N\} \). Section 3.3 proves the claim when either \( L^+ = \emptyset \) or \( L^- = \emptyset \). We may now assume that both \( L^+ \) and \( L^- \) are nonempty. We first prove two more structural results on weights in particular cases.

Proposition 3.12. Assume that the algorithm runs as
\[ I_1, \ldots, I_r \in L^+, I_{r+1}, \ldots, I_{r+t} \in L^-, \]
for some \( 0 \leq t \leq r \). Choose a c-function for \( \sigma \) (see Convention 3.3) in order to fix \( s_\sigma \in T \). If \( w \cdot s_\sigma^{-1} \in \Psi(L_{\text{out}(\sigma)}) \) for \( w \in W \), then \( wc(w) > 0 \) for all \( w \in \tau \) such that \( e(w) \geq m + t - r \).

Proof. The proof is by induction. For every \( 0 \leq u \leq t \), define the marked partitions \( \tau_1^{(u)} \) and \( \tau_2^{(u)} \) as follows: the support of \( \tau_1^{(u)} \) is \( \{I_{r+u+1}\} \) and it is unmarked, while the support of \( \tau_2^{(u)} \) is \( \{I_1, \ldots, I_{r-u}, I_{r-u+1} \sqcup I_{r+u+1}, \ldots, I_r \sqcup I_{r+1}\} \), and every \( e(w) \) such that \( e(w) \geq m \) is marked. Note that \( e(w) < m \) for all \( w \in \tau_1^{(u)} \). For \( u = 0 \), \( \tau_1^{(0)} \) has support \( \{I_{r+1}\} \), while \( \tau_2^{(0)} \) has support \( L^+ \), and in fact it is a positive ladder. We have \( \tau_2^{(0)} = \text{out}(\sigma') \) for some smaller partition \( \sigma' \). By Proposition 3.10 we deduce the assertion for \( t = 0 \). Notice that \( \tau_1^{(0)} = \emptyset \), and \( \tau_2^{(0)} = \text{out}(\sigma) \).

We proceed by induction on \( u \) to prove that \( \tau_2^{(u)} \), with \( u = t \), satisfies the assertion. As just mentioned, this holds for \( u = 0 \). Let \( u > 0 \) be fixed, and assume the theorem holds for all smaller \( u' \leq u \), and we will prove it for \( u + 1 \). We define \( \sigma^{(u)} \) as the partition such that \( \text{out}(\sigma^{(u)}) = \tau_2^{(u)} \).

Let \( n_1^{(u)} \) and \( n_2^{(u)} \) be the sizes of \( \tau_1^{(u)} \) and \( \tau_2^{(u)} \), respectively. Let \( \mathbb{H}_p \) be the subalgebra of \( \mathbb{H}_{n(u+1)} \) corresponding to \( GL(n_1^{(u)}) \times Sp(2n_2^{(u)}) \subset Sp(2n_2^{(u+1)}) \).

We regard \( v_{\tau_1^{(u)}} \) as a regular nilpotent Jordan normal form of \( gl_{n_1^{(u)}} \) and \( v_{\tau_2^{(u)}} \) as a normal form (see Proposition 12.28) of an exotic representation of \( Sp(2n_2^{(u)}) \).

We have \( M_{\tau_1^{(u)}} = L_{\tau_1^{(u)}} \) as (one-dimensional) modules for \( \mathbb{H}_{n_1}^A \).

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Claim A. We have $O_{\tau_1^{(u)} \times \tau_2^{(u)}} \subset O_{\tau_2^{(u+1)}}$ and
\[ 1 + \dim O_{\tau_1^{(u)} \times \tau_2^{(u)}} = \dim O_{\tau_2^{(u+1)}}, \]
where $\hat{\tau}$ denotes the marked partition obtained from $\tau$ by removing the markings.

Proof. We set $I^* := I_{r+u+1} \cup I_{r-u}$. The segment $J := I_{r+u+1}$ satisfies $J \cap I$ or $\bigcap \mathbb{Z} = \emptyset$ for each $I \in \tau_2^{(u)}$. We have $I_{r-u} \not\subset I$ or $I_{r-u} \not\supset I$ if $I \in \tau_2^{(u)}$ if and only if $I^* \not\subset I$ or $I^* \not\supset I$, respectively. It follows that $u_{\tau_2^{(u+1)}} = u_{\tau_2^{(u)}} + u_{\tau_1^{(u)}}$. (The definition of $u_\tau$ is as in Corollary 1.34.) Using Corollary 1.34 we conclude the dimension estimate. The existence of closure relation is straightforward since we have an attracting map from $v_{\tau_2^{(u+1)}}$ to $v_{\tau_2^{(u)}} + v_{\tau_1^{(u)}}$ defined as the scalar multiplication of the $T$-component of $v_{\tau_2^{(u+1)}}$ which does not appear in $v_{\tau_2^{(u)}} + v_{\tau_1^{(u)}}$.

We return to the proof of Proposition 3.12. Notice that both of $O_{\tau_1^{(u)} \times \tau_2^{(u)}}$ and $O_{\tau_2^{(u+1)}}$ are open subsets of vector bundles over their projections to $V_2^{(s,q)}$ with their fibers isomorphic to $V_1^{(s,q)}$. It follows that the regularity of the orbit closure $O_{\tau_1^{(u)} \times \tau_2^{(u)}} \subset O_{\tau_2^{(u+1)}}$ is equivalent to the regularity of the corresponding orbit closure in $V_2^{(s,q)}$. We identify $V_2^{(s,q)}$ with some type $A$-quiver representation space as in [13]. By the Abreas-Del Fra-Kraft theorem [ADK81], $O_{\tau_2^{(u+1)}}$ is normal along $O_{\tau_1^{(u)} \times \tau_2^{(u)}}$ since its projection to $V_2^{(s,q)}$ is so. Since normality implies regularity in codimension one, it follows that $\dim H^2_{O_{\tau_1^{(u)} \times \tau_2^{(u)}}(IC(O_{\tau_2^{(u+1)}})) = 1$.

Hence, Theorem 3.12 implies
\[ [M_{\tau_1^{(u)} \times \tau_2^{(u)}} : L_{\tau_2^{(u+1)}}] = [\text{Ind}_{M \tau_1^{(u)}}^{M \tau_2^{(u)}} (\text{Ind}_{\tau_1^{(u)}}^{\tau_2^{(u)}}): L_{\tau_2^{(u+1)}}] = 1 > 0. \tag{3.7} \]
Now we choose a $c$-function for $\sigma^{(u+1)}$ to fix $s_{\sigma^{(u+1)}}$. Let $w \cdot s_{\sigma^{(u+1)}} \in \Psi(L_{\tau_2^{(u+1)}})$ ($w \in W$) be given. Taking into account the fact that we have no $\tau_2^{(u)} \in \tau_1^{(u)}$ such that $c(\tau_2^{(u)}) \geq m + t - r$, we deduce
\[ wc(x) \geq 0 \text{ if } e(x) \geq m + t - r. \tag{3.8} \]
by Theorem 3.12.

Corollary 3.13. Keep the setting of Proposition 3.12. Let $\sigma'$ be a partition so that the corresponding algorithm runs as
\[ I_1, \ldots, I_r \in L^+; I_{r+1}, \ldots, I_{r+t-1} \in L^-, \quad 0 \leq t \leq r. \]
Let $\tau_1 = (\{I_{r+1}\}, 0)$ be the (unmarked) marked partition adapted to a semisimple element determined by $e(I_{r+1})$. Let $n'$ and $n''$ ($n = n'+ n''$) be the sizes of $\sigma'$ and $\tau_1$, respectively. Then, we have
\[ [\text{Ind}_{M \tau_1^{(u)} \otimes \mathbb{H}_{\sigma'}}^{M \tau_1^{(u)}} (t^{\tau_1^{(u)}} \otimes I_{\text{out}(\sigma')}) : L_{\text{out}(\sigma)}] > 0. \]

Proof. Let us assume that we have $u = t - 1$ and (3.7) as in the proof of Proposition 3.12. In the notation therein, we have $r_2^{(t-1)} = \text{out}(\sigma')$, $\tau_1^{(t-1)} = \tau_1$, and $\tau_2^{(t)} = \text{out}(\sigma)$. Then the claim follows from (3.7) by verifying the hypothesis of Corollary 3.13.

\[ \square \]
Proposition 3.12 and Corollary 3.13 have the following counterparts, with the analogous proofs.

**Proposition 3.14.** Assume that the algorithm runs as

\[ I_1, \ldots, I_r \in L^-, I_{r+1}, \ldots, I_{r+t} \in L^+, \]

for some \( 0 \leq t \leq r \). Choose a \( c \)-function for \( \sigma \) (see Convention 3.7) in order to fix \( s_\sigma \in T \). If \( w \cdot s_\sigma^{-1} \in \Psi(L_{\text{out}(\sigma)}) \) for some \( w \in W \), then \( \text{we}([x]) < 0 \) for all \( x \in \text{out}(\sigma) \) such that \( e_i(x) \leq m + r - t \).

**Proof.** The proof (of Proposition 3.12) works by changing the definition of \( \tau_2^{(u)} \), so that the support is

\[ I_1, \ldots, I_{r-u}, (I_{r-u+1} \cup I_{r+u}), \ldots, (I_r \cup I_{r+t}), \]

and set the support of \( \tau_1^{(u)} \) to be \( I_{r+u+1} \).

**Corollary 3.15.** Keep the setting of Proposition 3.14. Let \( \sigma' \) be a partition so that the corresponding algorithm runs as

\[ I_1, \ldots, I_r \in L^- \cup \sigma', I_{r+1}, \ldots, I_{r+t-1} \in L^+, \]

\( 0 \leq t \leq r \).

Let \( \tau = (\{I_{r+1}\}, 0) \) be the (unmarked) marked partition adapted to a semisimple element determined by \( c(I_{r+1}) \). Let \( n' \) and \( n'' \) (\( n = n' + n'' \)) be the sizes of \( \sigma' \) and \( \tau \), respectively. Then, we have

\[ \text{Ind}_{H^0_{\mathbb{A}^{\sigma'}}}^{H_{\mathbb{A}^{\sigma'}}}(M_1^A \boxtimes L_{\text{out}(\sigma')} : L_{\text{out}(\sigma)}) > 0. \]

**Theorem 3.16 (also Theorem 3.5).** The output \( \text{out}(\sigma) \) of Algorithm 3.3 defines a tempered module.

**Proof.** Algorithm 3.3 begins with \( I_1 \in L^+ \) or \( I_1 \in L^- \). We present the case \( I_1 \in L^+ \), the other situation being analogous. We fix some \( c \)-function of \( \text{out}(\sigma) \).

Denote \( \text{out}(\sigma) = (I, \delta) \). Assume that the algorithm runs as

\[ I_1, \ldots, I_r \in L^+, I_{r+1}, \ldots, I_{r+t} \in L^-, \ldots \]

\( (0 \leq t \leq r) \)

and if \( I_{r+t-1} \neq \emptyset \), then it belongs to \( L^+ \) if \( 0 < t < r \) and either \( L^+ \) or \( L^- \) if \( t = r \). (Note that Proposition 3.10 considers the situation \( L^- = \emptyset \), which means \( t = 0 \).) From step 3 of Algorithm 3.3, we see that the first \( t \) segments in \( I \) (with respect to \( \sigma \)) are \( I_r \sqcup I_{r+1} \sqcup I_{r-1} \sqcup I_{r-2} \sqcup \ldots \sqcup I_{r-t+1} \sqcup I_{r-t} \).

Set \( \tau_1 = (\{I_r \sqcup I_{r+1}\}, \delta_{\{I_r \sqcup I_{r+1}\}}) \), and let \( \tau_2 \) be the marked partition obtained from \( \text{out}(\sigma) \) by removing the segment \( I_r \sqcup I_{r+1} \). In terms of the partitions, \( I_r \) and \( I_{r+1} \) correspond to some row part and column part of the partition obtained by extracting \( I_1, \ldots, I_r \) from \( \sigma \). So if \( \sigma_2 \) is the partition obtained from removing these two pieces (which form a hook of \( \sigma \)), then we have \( \tau_2 = \text{out}(\sigma_2) \). By induction, we may assume that \( \tau_2 \) is tempered. Denote by \( n_1, n_2 \) the sizes of \( \tau_1 \) and \( \tau_2 \) respectively. If \( n_1 + n_2 = 1 \). Let \( H_P \) be the Hecke algebra for \( GL(n_1) \times Sp(2n_2) \). It is clear that \( I_r \sqcup I_{r+1} \) attains the minimal \( e \)-value, and also attains the maximal \( e \)-value if \( r = 1 \). Hence, Theorem 3.1 is applicable and we conclude that

\[ \text{Ind}_{H_P}^{H_{\mathbb{A}^{\sigma}}}(M_{\tau_1^A} \boxtimes M_{\tau_2}) : L_{\text{out}(\sigma)}]) = [M_{\text{out}(\sigma)} : L_{\text{out}(\sigma)}] = 1 \]

\[ \text{Ind}_{H_P}^{H_{\mathbb{A}^{\sigma}}}(M_{\tau_1^A} \boxtimes M_{\tau_2}) : L_{\text{out}(\sigma)}]) = [M_{\tau_2^A} : L_{\text{out}(\sigma)}] = 1, \text{ if } r = 1, \]

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where \( \tau' \) is a marked partition obtained from \( \text{out}(\sigma) \) by removing its marking on \( \tau_1 \) (and hence \( O_\tau \subset \text{out}(\sigma) \) is smooth). In particular, we applied Theorem 2.12 when \( r = 1 \).

By verifying in addition the hypothesis of Corollary 3.3, we find:

\[
\text{Ind}_{H_{\mu}}^H (M_\lambda^A \boxtimes L_{\tau_2}) : L_{\text{out}(\sigma)} = 1 \quad \text{if } r > 1, \hspace{1cm} (3.9)
\]

\[
\text{Ind}_{H_{\mu}}^H (\tau^* M_\lambda^A \boxtimes L_{\tau_2}) : L_{\text{out}(\sigma)} = 1 \quad \text{if } r = 1. \hspace{1cm} (3.10)
\]

Here \( \tau_1 \) is not a tempered module of \( GL(n_1) \) in general, so we need to check that \( L_{\tau_1} \)'s contribution to \( \Psi(L_{\text{out}(\sigma)}) \) satisfies the temperedness condition.

To do this, we define the subset \( I^* \subset I_r \cup I_{r+1} \) by

\[
x \in I^* \quad \text{if } e(x) > -e(y), \quad \text{for every } y \in I_{r+1}.
\]

(Since \( I_r \in L^+ \) is picked in the algorithm before \( I_{r+1} \in L^- \), we have \( I^* \neq \emptyset \).) If \( x \in I^* \), then we have

\[
e(x) > \max \{|e(y)| : y \in I_{r+k}\} \quad \text{for all } k \geq 1,
\]

(equivalently, \( e(x) < e(y) \) for every \( y \in I_{r+k} \)) since the maximal \(|e|-\text{value in }
\cup_{k \geq 1} I_{r+k} \) is in \( I_{r+1} \). Moreover, we deduce \( e(x) > m + t - r \) for every \( x \in I^* \) since otherwise we have

\[
\max \{|e(y)| : y \in I_{r+i+1}\} > \max \{|e(y)| : y \in I_{r+1}\},
\]

which contradicts Algorithm 3.3.

The reason for defining \( I^* \) is that it gives a criterion for checking that \( L_{\text{out}(\sigma)} \) is tempered (assuming by induction that \( L_{\text{out}(\tau_2)} \) is tempered).

**Claim B.** The \( H_{\mu} \)-module \( L_{\text{out}(\sigma)} \) is tempered if

\[
w(I^*) > 0, \quad \text{for every } w \cdot s_{\sigma}^{-1} \in \Psi(L_{\text{out}(\sigma)}). \hspace{1cm} (3.12)
\]

**Proof.** By Proposition 2.5, we deduce that \( w \cdot s_{\sigma}^{-1} \in \Psi(L_{\text{out}(\sigma)}) \) can be written as \( w \cdot s_{\sigma}^{-1} = v(s_1 \times s_2) \) with \( s_1 \in \Psi(M_{\tau_1}) \) and \( s_2 \in \Psi(L_{\text{out}(\sigma_2)}) \), where \( v \in \mathcal{S}_n/(\mathcal{S}_{n_1} \times \mathcal{S}_{n_2}) \) is the minimal coset representative in \( \mathcal{S}_n \). Let \( I_1 := v([1, 2, \ldots, n_1]) \) and \( I_2 := v([n_1, \ldots, n]) \). Notice that the ordering of numbers is preserved by applying \( v \). For every \( k \geq 1 \), we set

\[
\omega_k^i := \sum_{1 \leq j \leq k, j \neq i} e_j \quad \text{for } i = 1, 2.
\]

Notice that \( \omega_k^1 + \omega_k^2 = \omega_k \). We will show that \( \langle \omega_k^i, w \cdot s_{\sigma}^{-1} \rangle \leq 1, \quad i = 1, 2 \), which means in particular that \( \langle \omega_k, w \cdot s_{\sigma}^{-1} \rangle \leq 1 \). The condition for \( i = 2 \) follows immediately from the temperedness hypothesis for \( L_{\text{out}(\sigma_2)} \). It remains to prove the condition for \( i = 1 \). Set \( I := I_r \cup I_{r+1} \). We can assume that \( w(I) = I_1 \) (up to sign change) and either \( w(I) > 0 \) or there exists \( l \in I \) so that

\[
w(\{i \in I \mid i < l\}) > 0 \quad \text{and} \quad w(\{i \in I \mid i \geq l\}) < 0.
\]

Let \( e_1 < e_2 < \cdots < e_{n_1} \) be the list of \( e \)-values of \( I \). By Corollary 2.4, we deduce that there exists \( 0 \leq k_1 \leq k \) such that

\[
\log_q \langle \omega_k^1, w \cdot s_{\sigma}^{-1} \rangle = \log_q \langle \omega_k^1, v \cdot (s_1 \times s_2) \rangle = \sum_{p=1}^{k_1} e_p - \sum_{q=n_1-k_1+1}^{n_1} e_q. \hspace{1cm} (3.13)
\]
Assumption \((3.12)\) implies that every \(e_q > -e_l\) that appears in the right hand side of \((3.13)\), must appear in the form \(-e_q\). It follows that the quantity in \((3.14)\) must be nonpositive, which concludes the proof of the claim.

We return to the proof of Theorem \((5.16)\). If the algorithm stops at \(r + t\), i.e., \(I_{r+1} = \emptyset\), then Proposition \((3.12)\) implies condition \((3.12)\), and therefore the proof is complete in this case. Assume this is not the case. We define two smaller marked subpartitions \(\tau_j^+\) and \(\tau^+\) of \(\tau = \text{out}(\sigma)\) corresponding to two smaller subpartitions \(\sigma_j^+\) and \(\sigma^+\) of \(\sigma\) such that \(\tau_j^+ = \text{out}(\sigma_j^+)\) and \(\tau^+ = \text{out}(\sigma^+)\). It will be sufficient to show that

\[(C1) \text{ condition } (3.12) \text{ holds for } \tau_j^+ \text{ by Proposition } (3.12) \text{ applied to } \sigma_j^+;\]
\[(C2) \text{ if } (3.12) \text{ holds for } \tau_j^+ \text{ (and } \sigma_j^+) \text{ then it holds for } \tau^+ \text{ (and } \sigma^+) \text{ (using parabolic induction);}\]
\[(C3) \text{ if } (3.12) \text{ holds for } \tau^+ \text{ (and } \sigma^+) \text{ then it holds for } \tau \text{ (and } \sigma) \text{ (using parabolic induction).}\]

Let \(\tau_j^+\) be the output of Algorithm \((5.3)\) steps 3 and 4 applied to \(I_1, I_2, \ldots, I_{r+t}\), and let \(\tau_1^+\) be the set of segments \(\{I_k\}_{k \geq r+t}\) in Algorithm \((5.3)\) step 2 which are glued to \(I_1, \ldots, I_{r-t}\) in step 3. By examining \(e\)-values, \(\tau_1^+\) cannot be marked. Here \(\text{Claim C1}\) holds for \(\tau_j^+\) by Proposition \((3.12)\).

Define \(\tau^+\) to be the marked subpartition of \(\text{out}(\sigma)\), whose support consists of all segments of the form \(I_k\) or \(I_k \cup I_{t}\) (for some \(t\)), \(0 \leq k \leq r\) produced by step 3 in Algorithm \((5.3)\). The marking in \(\tau^+\) is set to be the one inherited from \(\tau\). Let \(\tau^- := (\Gamma^-, \delta^-)\) be the complementary marked subpartition of \(\tau^+\) in \(\text{out}(\sigma)\). Let \(n^+\) and \(n^- (n = n^+ + n^-)\) be sizes of \(\tau^+\) and \(\tau^-\), respectively. By Algorithm \((5.3)\) we deduce that \(\tau^+ = \text{out}(\sigma^+)\) and \(\tau^- = \text{out}(\sigma^-)\) for some \(\sigma^+, \sigma^-\).

To prove \(\text{Claim C2}\), by successive applications of Corollary \((3.13)\) we deduce that

\[
\text{Ind}^{}(\tau)_{\tau_1+}^{} \otimes L_{\tau^+} > 0;
\]
notice that here \(M^A_{\tau_1^+}\) and \(L_{\tau^+}\) are standard and irreducible modules of smaller Hecke algebras. We have no \(x \in \tau_j^+\) such that \(e(x) \in e(I^+)\) by \((3.11)\). Applying Proposition \((2.9)\) and Corollary \((2.7)\) we conclude that \((3.12)\) must hold for \(\tau^+\) since it holds for \(\tau_j^+\).

Finally, we verify \(\text{Claim C3}\). Let \(\tau_0 := (\Gamma^-, 0)\) be obtained from \(\tau^-\) by removing the markings, and let \(\tau_0\) be the marked partition of \(n\) obtained as the union of \(\tau_0^-\) and \(\tau^+\). We have \([M_{\tau_0} : L_{\tau^+}] = 1\) by Theorem \((2.12)\) since \(\mathcal{O}_{\tau_0} \subset \mathcal{O}_{\tau}\) is smooth. We would like to realize \(M_{\tau_0}\) as an induced module from \(M^A \otimes M_{\tau^+}\), for some standard \(H^A\)-module \(M^A\) (defined from \(\tau_0^-\)). In order for the induction theorem \((1.15)\) to be applicable, we need the following construction. We divide \(\tau_0^-\) into two (unmarked) marked partitions \(\tau_1^-\) and \(\tau_2^-\), where \(\tau_1^-\) consists of segments of \(\Gamma^+\) which are not dominated by a segment in the support of \(\tau^+\), and \(\tau_2^-\) consists of segments of \(\Gamma^-\) which are dominated by a segment in the support of \(\tau^+\). We define \(M^A\) to be the induction of \(\tau M^A \otimes M^A_{\tau^+}\) to \(H^A_{\tau^-}\).

Then, we can apply Theorem \((1.15)\) again, to deduce that

\[
M_{\tau_0} \cong \text{Ind}^{}_{\mathcal{H}_{\tau^-}}(M^A \otimes M_{\tau^+}), \text{ and so } [\text{Ind}^{}_{\mathcal{H}_{\tau^-}}(M^A \otimes M_{\tau^+}) : L_{\tau^+}] > 0,
\]

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where $\mathbb{P} := \mathbb{P}_- \times \mathbb{P}_+ \subset \mathbb{H} := \mathbb{H}_n \times \mathbb{H}_n$. By verifying in addition the hypothesis of Corollary 3.8 we find

$$\text{Ind}_{\mathbb{P}}^\mathbb{H}(M^A \otimes L_{\tau^+}) : L_{\tau^+} > 0.$$ 

Applying Theorem 2.5, we conclude that (3.12) follows from the corresponding statement for $\tau^+$ as desired.

### 3.5 A characterization of $\text{ds}(\sigma)$

We finish this section with certain combinatorial properties that the output $\text{out}(\sigma)$ must satisfy. By examining Algorithm 3.3 we deduce that $\text{out}(\sigma)$ acquires a nested component decomposition whenever the sums of $e$-values in the hooks of $\sigma$ are not uniformly greater than 0 or not uniformly less than 0. By Lemma 2.8, the same is true for every other $G(s_\sigma)$-orbit which contains $G(s_\sigma) \cdot \text{out}(\sigma)$ in its closure. Let us refer to this decomposition here as the "$\sigma$-hook nested components" decomposition. The hooks of $\sigma$ which contribute to a given $\sigma$-hook nested component have the sum of $e$-values uniformly greater than 0, in which case we call the component positive, or uniformly less than 0, in which case we call it negative. As an application of Theorem 3.5 and Algorithm 3.3 we obtain a combinatorial characterization of $\text{out}(\sigma) = \text{ds}(\sigma)$. Consider the following properties for a marked partition $\tau = (J', \delta')$:

1. (p1) for every $i \geq 0$, there are at most $i$ segments in $J'$ with all $e$-values greater than $m - i + 1$.

2. (n1) for every $i \geq 0$, there are at most $i$ segments in $J'$ with all $e$-values less than $m + i - 1$.

3. (p2) for every segment $J' \in J'$, we have $\text{av}(J') > 0$.

4. (n2) for every segment $J' \in J'$, we have $\text{av}(J') < 0$.

5. (p3) for every $J' \in J'$, if $m \in e(J')$, then $\delta(J') = 1$.

**Corollary 3.17.** The $G(s_\sigma)$-orbit $\text{out}(\sigma) = \text{ds}(\sigma)$ is minimal among all $\tau \in \text{MP}(a_\sigma)$ admitting the $\sigma$-hook nested decomposition and satisfying the properties:

1. (p1), (p2), (p3) on every positive $\sigma$-hook nested component.

2. (n1), (n2) on every negative $\sigma$-hook nested component.

**Proof.** It is sufficient to check the claim when $\sigma$ has only one hook nested component. The case when $\sigma$ consists of a single hook is easily verified directly (see Example 3.6).

Since the proofs in both cases are similar, we provide a proof only when (p1)–(p3) hold (i.e. the case $\sigma$-hook nested component is positive) for the unique $\sigma$-hook nested component.

Let us assume that the second step of the algorithm runs as $I_1, I_2, \ldots$. There are two situations with respect to $I_1$: either there exists $k \geq 2$ such that $I_k$ combines with $I_1$ in the third step of the algorithm, or if not, then $I_1$ appears in the support of $\text{out}(\sigma) = (J, \delta)$ by itself.

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• In the first case, we can assume further that the algorithm for \( \sigma \) runs as \( I_1, I_2, \ldots, I_{2M} \) for some \( M \geq 1 \), and such that \( \#\{I_1, \ldots, I_{2M}\} \cap L^+ = M = \#\{I_1, \ldots, I_{2M}\} \cap L^- \). Then \( \text{out}(\sigma) \) has exactly \( M \) segments in its support all of the form \( I_j \sqcup I_j' \) (see figure 3.5).

\[
\sigma = \begin{bmatrix}
\ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast
\end{bmatrix} \quad \mapsto \quad \text{out}(\sigma) = \begin{bmatrix}
\ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast
\end{bmatrix}
\]

Figure 4: Output of Algorithm 3.3 when \( \sigma = (4,4,4,3,3) \) and \( 1 < m < \frac{3}{2} \)

From the algorithm we see that every segment \( J \in \mathbf{J} \) in \( \text{out}(\sigma) = (\mathbf{J}, \delta) \) contains an \( e \)-value \( m \) and \( \delta(J) = 1 \) (by \( \text{av}(J) > 0 \)). We claim that there is no \( \tau \) in the closure of \( \text{out}(\sigma) \) which can satisfy the required conditions. Let \( J, J' \) be two segments in \( \mathbf{J} \), and we assume that \( J \prec J' \). Since \( e(J) \cap e(J') \neq \emptyset \) (because \( m \) is in the intersection), there are only two cases: either \( J \prec J' \), or else \( J' \sqsubseteq J \). If \( J \prec J' \), by the closure relations of section 1.4 we see that \( J, J' \) cannot combine to give a smaller orbit. Assume \( J' \sqsubseteq J \). Then from step 2 of the algorithm one sees that necessarily \( -\min e(J) > \max e(J') \). If they combine to give a smaller orbit \( \tau \), then \( \tau \) must contain \( J_1, J_1' \) such that \( e(J_1) = \{\min e(J), 1 + \min e(J), \ldots, \max e(J')\} \), and \( e(J_1') = \{\min e(J'), 1 + \min e(J'), \ldots, \max e(J)\} \).

But then condition (p2) fails for \( J_1 \). By a similar argument, one may also see that if a single segment \( J \in \mathbf{J} \) is broken into two pieces such (p1) holds, then the smaller segment with respect to \( \prec \) has to fail (p2).

• In the second case, \( I_1 \) forms a segment in \( \mathbf{J} \) by itself. If \( \tau = (J', \delta') \) is in the closure of \( \text{out}(\sigma) \) and satisfies the required assumptions, then we see that \( I_1 \in J' \).

(This is because of the conditions (p1,2), the segment \( I_1 \) cannot be broken into two pieces to yield such a \( \tau \), and it is also clear that if it is combined with some other segment, the resulting marked partition would not be in the closure of \( \text{out}(\sigma) \). So one can ignore the segment \( I_1 \) from consideration. This amounts to analyzing a smaller partition \( \sigma' \) which is obtained from \( \sigma \) by removing the first row and replacing \( m \) by \( m - 1 \). Then one proceeds by induction.

\[ \square \]

4 Applications of the classification

We present some consequences of the classification to the structure of discrete series. Recall that \( H_{n,m} \) denotes the affine Hecke algebra of type \( C_n \) with parameter \( \tilde{q} = (-1, q^m, q) \) with \( q > 1 \) and \( m \in \mathbb{R} \). The generic values of \( m \) are all positive real numbers except half integers.

4.1 Discrete series and deformations

One immediate corollary of the algorithm is the classification of discrete series which contain the \( \text{sgn} \) \( W \)-representation, including for nongeneric values \( m \). (At generic values of \( m \), the inequalities in Theorem 1.2 are all strict.)

Definition 4.1. Let \( \sigma \) be a partition of \( n \), which is identified with the Young diagram as in 3.2 or the RHS of Fig. 3.1. Let \( \{k\} \) denote the fractional part
of $k$ (which is 0 or $1/2$ if $k$ is a critical value). The extremities of $\sigma$ at $k$ is the set $E(\sigma,k) := E(\sigma,k)^+ \cup E(\sigma,k)^-$ defined by the procedure: put in $E(\sigma,k)^+$ the maximal entry in every row above or on the $\{k\}$-diagonal. Also put in $E(\sigma,k)^-$ the negative of the minimal entry in every column below or on the $-\{k\}$-diagonal. One allows repetitions in this set, if they exist.

**Theorem 4.2.** Let $\sigma$ be a partition of $n$. The discrete series $ds(\sigma)$ contains the sgn $W$-representation if and only if one of the following equivalent conditions hold:

1. $ds(\sigma)$ parameterizes the open $G(a)$-orbit in $H^a$;

2. The support $\{I_j\}_{j=1}^k$ of $\sigma$ satisfies:

\[
\max e(I_1) \geq -\min e(I_1) \geq \max e(I_2) \geq -\min e(I_2) \geq \ldots 
\]  
(4.1)

3. We have a sequence $e_1, e_2, e_3, \ldots \in E(\sigma, m)^+$ and $e_2, e_4, e_6, \ldots \in E(\sigma, m)^-$ such that

\[
e_1 \geq e_2 \geq e_3 \geq e_4 \geq \cdots \geq 0.
\]

**Proof.** By Theorem 1.23 we know that $mp(\sigma) = ds(\sigma)$ is equivalent to $\text{sgn} \subset \mathcal{L}_{\text{out}(\sigma)}$ as $W$-submodule. The condition (4.1) implies that we have $I_p \supset I_l$ for every $p > l$. Hence, we cannot apply the algorithm of Theorem 1.20. It follows that $ds(\sigma)$ defines the dense open orbit when projected to $V^{(s,q)}_2$. Moreover, we have $\text{av}(I_l) > 0$ for every $l$. This implies that the temporary marking $\delta$ in Algorithm 3.3 is uniformly marked. Therefore, we deduce that $mp(\sigma) = ds(\sigma)$ if and only if 2) holds. Finally, the condition 3) is equivalent to condition 2), since $e_1 = \max e(I_1'), e_2 = \min e(I_2), e_3 = \max e(I_3), \ldots$ in these cases by Algorithm 3.3 where $I_1', I_2', \ldots$ denote the output (from $\sigma$) in the second step of the algorithm.

Recall that the discrete series for $H_{n,m}$ are in one-to-one correspondence $\sigma \leftrightarrow ds(\sigma)$ with partitions $\sigma$ of $n$. To every $\sigma$ and $m$, one attached a semisimple element $s_{\sigma,m}$. The following corollary describes the properties of this family of $H_{n,m}$-modules, as $m$ varies in appropriate intervals.

**Theorem 4.3.** Consider $k < m < k + 1/2$ for some $k \in \mathbb{Z}$. Let $ds(\sigma)$ be the parameter of discrete series of $H_{n,m}$ attached to $\sigma$. Let $a_m$ be a family of semisimple elements attached of $\sigma$ with $\bar{q} = (-1, q^m, q)$. Assume that $L(a_m, s_{\sigma,m})$ contain $\text{sgn}$ as $\mathbb{C}[W]$-modules. Then, the modules in the family $\{L(a_m, s_{\sigma,m})\}_m$ in the region $k \leq m \leq k + 1/2$:

1. have the same dimension;

2. are all simple tempered modules;

3. are all isomorphic as $W$-representations.

**Proof.** Taking into account Corollary 2.15 and Theorem 1.23 it suffices to prove that $v_{ds(\sigma)}$ defines an open dense orbit of $\mathcal{V}^m$ when $m = k, k + 1/2$. By the description of the orbit structure of $\mathcal{V}^m$ in [Ka09], we deduce that we obtain no new orbits by the specialization process (inside $V_2$). It follows that the condition (4.1) with $>$ replaced by $\geq$ is already enough to guarantee that $C_{ds(\sigma)} \subset \mathcal{V}^m$ is dense open for every $k \leq m \leq k + 1/2$. 

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Notice that when \( k < m < k + 1/2 \) with \( k \) critical value, every \( L_{(a, m, v_{b(\sigma)})} \) is in fact a discrete series. But at the endpoints of the interval, \( m \in \{ k, k + 1/2 \} \) they could be just tempered. On the other hand, Corollary 2.10 effectively says that if \( L_{(a, m, v_{b(\sigma)})} \) is a discrete series in the interval \( k < m < k + 1/2 \), but also in the interval \( k - 1/2 < m < k \), then it is a discrete series at \( m = k \). This gives a combinatorial condition on \( \sigma \), viewed as a tableau for \( m \), as follows. The idea is due to [Sh05] §4.5.

**Corollary 4.4.** Assume \( k \) is a critical value. If the family \( \{ L_{(a, m, v_{b(\sigma)})} \} \) consists of discrete series in the interval \( k - 1/2 < m < k + 1/2 \), then the set \( E(\sigma, k) \) does not have repetitions.

**Proof.** In combinatorial terms, the condition that \( \{ L_{(a, m, v_{b(\sigma)})} \} \) consists of discrete series in the interval \( k - 1/2 < m < k + 1/2 \) means that the output of Algorithm 3.3 is the same for \( \sigma \) when \( k - 1/2 < m < k \) or \( k < m < k + 1/2 \). This is equivalent to the fact that step 2 of the algorithm is the same in these two intervals, which implies that step 2 of the algorithm is well-defined at \( m = k \) as well. From this, it is easy to see that \( E(\sigma, k) \) must not allow repetitions. \( \square \)

### 4.2 Tempered modules in generic parameters

Let us assume that \( m \) is generic. Let \( \mathbb{H}_{n,m}^{A} \) be the image of \( \mathbb{H}_{n}^{A} \) under the specialization map \( \mathbb{H}_{n} \to \mathbb{H}_{n,m} \). This is an affine Hecke algebra for \( GL(n) \).

**Theorem 4.5 ([KL87], [Ze80]).** The set of tempered modules with positive real central character of \( \mathbb{H}_{n,m}^{A} \) is in one-to-one correspondence with the set of partitions of \( n \). Fix a partition \( \sigma = (\sigma_{1}, \sigma_{2}, \ldots, \sigma_{\ell}) \) of \( n \). The corresponding tempered module \( L_{(a, \nu_{\tau})}^{A} \) is obtained by the following procedure:

1. Form a sequence \( I = \{ I_{k} \} \) of subsets of \([1, n]\) by setting \( I_{1} = [1, \sigma_{1}], \ldots, I_{k} = [\sigma_{1} + \cdots + \sigma_{k-1} + 1, \sigma_{1} + \cdots + \sigma_{k}] \).

2. Set the trivial labeling \( \delta \equiv 0 \) and form \( \tau := (I, \delta) \).

3. Form a semi-simple element \( s \in T \) such that \( \tau \) is adapted to \( a = (s, \overline{q}) \) and \( \{(\epsilon_{i}, s) : i \in I_{k}\} = \{(\epsilon_{i}, s)^{-1} : i \in I_{k}\} \) for each \( k \).

More precisely, the set of tempered modules with positive real central character for type \( A_{n-1} \) are in one to one correspondence with nilpotent adjoint orbits of type \( A_{n-1} \), and this is in turn are parameterized by partitions of \( n \). If \( \sigma \) is a partition as above, then one forms the (block upper triangular) parabolic subgroup \( P \) with Levi subgroup \( GL(\sigma_{1}) \times \cdots \times GL(\sigma_{\ell}) \). Let \( \mathbb{H}_{P} = \mathbb{H}_{m,n}^{A} \times \cdots \times \mathbb{H}_{m,n}^{A} \) be the Hecke subalgebra of \( \mathbb{H}_{n,m}^{A} \) corresponding to \( P \). Let \( St_{\sigma_{j}} \) denote the Steinberg \( \mathbb{H}_{\sigma_{j}, m}^{A} \)-module. The induced module \( Ind_{\mathbb{H}_{P}^{A}}^{\mathbb{H}_{m,n}^{A}}(St_{\sigma_{1}} \boxtimes \cdots \boxtimes St_{\sigma_{\ell}}) \) is irreducible and tempered, and this is \( L_{(a, \nu_{\tau})}^{A} \) in the notation of [115]. The element \( s \) corresponds to the middle element of the nilpotent orbit parameterized by \( \sigma \). Explicitly, we have \( s = \exp(\frac{\sigma_{1}}{2}, \ldots, -\frac{\sigma_{1}}{2}, \ldots, \frac{\sigma_{\ell}}{2}, \ldots, -\frac{\sigma_{\ell}}{2}) \).

When we have \( S_{0} = \Pi - \{ a_{n_{1}} \} \), then we have \( L_{S_{0}} \cong GL(n_{1}) \times Sp(2n_{2}) \). We have \( M \cap V_{S_{0}} \subset Gf(n_{1}) \oplus V_{(2)} \) (as \( L_{S_{0}} \)-varieties), where \( V_{(2)} \) is the exotic representation of \( Sp(2n_{2}) \).
Theorem 4.6. Set $S_0 = \Pi - \{\alpha_{n_1}\}$. Let $a = (s, 0) \in G_0$ be given. We define $s_1, s_2$ to be the projections of $s$ to $GL(n_1)$ and $Sp(2n_2)$, respectively. Fix $X \in \mathfrak{N}^a$ and a decomposition $X = X_1 \oplus X_2 \in \mathfrak{g}(n_1) \oplus \mathbb{V}(2)$. We assume

1. $L_{(s_1, q, X_1)}$ and $L_{(s_2, q, X_2)}$ define irreducible modules of $\mathbb{H}^A_{n_1, m}$ and $\mathbb{H}^A_{n_2, m}$, respectively;

2. We have $\{(\epsilon_i, s_1) ; 1 \leq i \leq n_1\} \subset q^{1/2}$ and $\{(\epsilon_i, s_2) ; n_1 < i \leq n\} \subset q_1 q^{2}$.

Then, we have an isomorphism

$$\text{Ind}_{\mathbb{H}^A_{n_1, m}}^{\mathbb{H}^A_{n_2, m}} (L_{(s_1, q, X_1)} \boxtimes L_{(s_2, q, X_2)}) \cong L_{(a, X)}$$

(4.2)
as $\mathbb{H}_{n, m}$-modules.

Proof. We have $q^{1/2} \cap q_1 q^{2} = \emptyset$. If follows that $(\mathbb{V}^{S_0})^a = \{0\}$, hence the induction theorem is applicable. Since we have $L_{S_0}(s) = G(s)$ and $\mathfrak{N}^a = \mathfrak{N} \cap (\mathfrak{g}(n_1) \oplus \mathbb{V}(2))$, it follows that the isomorphism classes of irreducible $\mathbb{H}_a$-modules and irreducible $\mathbb{H}^A_{S_0}$-modules are in one-to-one correspondence through the identification of parameters. For a point $X$ in the open dense $G(s)$-orbit of $\mathfrak{N}^a$, Theorem 2.2 implies that both $M^A_{(a, X)}$ and $M_{(a, X)}$ are irreducible modules of $\mathbb{H}^A_{S_0}$ and $\mathbb{H}_a$, respectively. Hence, the assertion holds in this case. We prove the assertion by induction on the closure relation of orbits. Let $O \subset \mathfrak{N}^a$ be a $G(s)$-orbit. Assume that (4.2) holds for all $\mathbb{H}^A_{S_0}$-module such that the orbit closure of the corresponding $G(s)$-orbit contains $O$. Fix $X \in O$. By the induction theorem, we have

$$\text{Ind}_{\mathbb{H}^A_{n_1, m}}^{\mathbb{H}^A_{n_2, m}} M_{S_0, (a, X)} \cong M_{(a, X)}.$$  

Moreover, Theorem 2.2 asserts that the multiplicities of each irreducible module inside $M^A_{(a, X)}$ and $M_{(a, X)}$ (as $\mathbb{H}_{n, m}$ and $\mathbb{H}_{n, m}$-modules, respectively) are the same, under the correspondence between the irreducibles of $\mathbb{H}^A_{n_1, m}$ and $\mathbb{H}_{n, m}$. Hence we deduce

$$\text{Ind}_{\mathbb{H}^A_{n_1, m}}^{\mathbb{H}^A_{n_2, m}} L_{S_0, (a, X)} \cong L_{(a, X)},$$

where $L_{S_0, (a, X)}$ is the unique $\mathbb{H}^A_{n_1, m}$-module corresponding to $O$. This is nothing but (4.2). Hence, the induction proceeds and we conclude the result.

Theorem 4.7. Set $S_0 = \Pi - \{\alpha_{n_1}\}$. Let $a = (s, 0) \in G_0$ be given. We define $s_1, s_2$ to be the projections of $s$ to $GL(n_1)$ and $Sp(2n_2)$, respectively. Fix $X \in \mathfrak{N}^a$ and a decomposition $X = X_1 \oplus X_2 \in \mathfrak{g}(n_1) \oplus \mathbb{V}(2)$. We assume

1. $L_{(s_1, q, X_1)}$ is a tempered module of $\mathbb{H}^A_{n_1, m}$;

2. $L_{(s_2, q, X_2)}$ is a discrete series of $\mathbb{H}_{n_2, m}$.

Then, $L_{(a, X)}$ is a tempered $\mathbb{H}_{n, m}$-module.

Proof. The proof of the induction theorem (see [Ka09], §7 or [KL87], §7) claims that we have an isomorphism

$$H_\bullet (\mu^{-1}(X)^a) \cong \bigoplus_w H_\bullet (\mu^{-1}(X)^a \cap P_{S_0, \tilde{w}^{-1} B/B}) \text{ as vector spaces},$$

(4.3)
where \( w \in W/(\mathfrak{S}_n \times W_{n_2}) \) denotes its minimal length representative in \( W \). Here we have \((P_\mathfrak{S}_n \wedge B/B)\) is in fact a direct sum decomposition as \( \mathbb{H}^S_{n,n_2} \)-modules (up to semi-simplification). It follows that the weight \( u = w \cdot s^{-1} \in \Psi(L(a,X)) \) is written as \( v \cdot ((w_1 \cdot s_1^{-1}) \times s'_2) \), where \( v \in \mathfrak{S}_n/(\mathfrak{S}_1 \times \mathfrak{S}_{n_2}) \) is a minimal length representative, \( w_1 \in W_{n_1} \), and \( s'_2 \in \Psi(L(s_2,q,\mathfrak{S}_2)) \). Let us take a subdivision \( I = \{I_j\}_j \) of \([1,n_1]\) (i.e. \( \cup_j I_j = [1,n_1] \)) so that \( I_j = \{i^0_1, \ldots, i^0_k\} \) satisfies \( i^0_1 < i^0_2 < \cdots < i^0_k \), \( \langle \epsilon_i, s \rangle = q \langle \epsilon_i, s \rangle \) and \( \langle \epsilon_i, s \rangle = \langle \epsilon_i, s \rangle^{-1} \) holds for each \( 1 \leq l \leq k \). Let \( s \), be a semi-simple element such that \( G(s) = L_{s} \). Then, we have necessarily \( s(X) = X \). By rearranging \( I \) and taking \( L_{s} \)-conjugate if necessary, we can assume \( X_1 = \sum_j v_{I_j} \). Since \( \text{Stab}_{G(L_{s})} X \) contains a torus of rank \#I, we need \( v_{I_j} \in w \Psi(a) \) (for all \( j \)) in order for \( \Psi \) to be non-empty. This implies that

\[
\langle w_p, s^{-1} \rangle \leq 1, \text{ where } w_p = \sum_{|w_I(i^0_j)| < p} \epsilon |w_I(i^0_j)| \text{ for each } p.\
\]

By the minimality of \( v \), we deduce

\[
\langle w''_p, s^{-1} \rangle = \langle w_p', s'_2 \rangle \leq 1, \text{ for each } p,
\]

where \( w''_p = w_p - \sum_j w'_p \) and \( p' = p - \sum_j \# \{l \in I_j \mid |w_I(i^0_l)| < p \} \). This implies that

\[
\langle w, u \rangle \leq 1 \text{ for every } u = w^{-1} \cdot s \in \Psi(L(a,X)) \text{ and every } 1 \leq i \leq n,
\]

which implies that \( L_{s} \) is tempered as desired. \( \square \)

### 4.3 Linear independence of \( R(T) \)-characters

**Construction 4.8.** Let \( a \in \mathcal{T}_0 \) and \( \tau = (I, \delta) \in MP(a) \) be given. We divide \( I \) into four sets \( D^1_+, D^1_-, D^2_+, D^2_- \) as follows:

- If \( \max I < q_1 \), we put \( I \in I \) into \( D^2_+ \);
- If \( \min I > q_1 \), we put \( I \in I \) into \( D^2_- \);

Notice that all segments \( I \) in \( D^2_+ \cup D^2_- \) are unmarked, since \( q_1 \not\in I \). Now we consider only segments in \( I \setminus (D^2_+ \cup D^2_-) \).

- If there exists some \( I' \) such that \( \delta(I') = \{0,1\} \) (i.e., \( I' \) is marked) and \( I \subseteq I' \), then we put \( I \) into \( D^1_+ \);
- If we have \( \delta(I) = \{0\} \) and there exists no \( I' \) such that \( \delta(I') = \{0,1\} \) and \( I \subseteq I' \), then we put \( I \) into \( D^1_- \).

We denote \( D_+ := (D^1_+ \cup D^2_+) \) and \( D_- := (D^1_- \cup D^2_-) \).

Notice that \( D_+ \cup D_- \) exhausts \( I \). One sees immediately that in this construction, \( I \in I \) satisfies \( \delta(I) = \{0,1\} \) if and only if \( I \in D_+ \) and \( q_1 \in I \). We change the marking of \( \tau \) so that every \( I \in D_+ \), with \( q_1 \in I \) is marked. By Proposition 1.22, this procedure does not change the \( G(s) \)-orbit of \( v_{\tau} \).

The following proposition is a criterion for finding some special weight of each simple \( \mathbb{H}_a \)-module. The notation \( w(j) \) refers to the usual action of \( W_n \) by permutations and sign changes on \([-n,n]\).
Proposition 4.9. Keep the setting of Construction 4.8. Assume that we have \( \langle \epsilon_i, s \rangle > \langle \epsilon_j, s \rangle \) for every \( i < j \). Assume that \( w \in W \) satisfies the following conditions:

- Assume \( I \in D_+ \). Then, we have \( w(j) > 0 \) for all \( j \in I \). Moreover, we have \( w(i) < w(j) \) for each \( i, j \in I \) such that \( \langle \epsilon_i, s \rangle > \langle \epsilon_j, s \rangle \);
- Assume \( I \in D_- \). Then, we have \( w(j) < 0 \) for all \( j \in I \). Moreover, we have \( w(i) < w(j) \) for each \( i, j \in I \) such that \( \langle \epsilon_i, s \rangle > \langle \epsilon_j, s \rangle \);
- Assume \( I, I' \in D_+ \) or \( I, I' \in D_- \). If we have \( I < I' \), then we have
  \[
  w(j) < w(j') \text{ for every } (j, j') \in I \times I'.
  \]

- If \( I, I' \in I \) and \( \min I = \min I' \), then we have either
  \[
  w(j) > w(j') \text{ for every } (j, j') \in I \times I', \quad \text{or}
  \]
  \[
  w(j) < w(j') \text{ for every } (j, j') \in I \times I'.
  \]

Then \( V_\tau \) meets \( wV(a) \) densely. In particular, we have \( w \cdot s^{-1} \in \Psi(L(a, v_\tau)) \).

Proof. The first two conditions imply \( \mathcal{W}_I \in \mathcal{V}_+ \), for all \( I \in I \), and \( \mathcal{W}_{I_1} \in \mathcal{V}_+ \) (if \( \delta(i) > 0 \)). Therefore, we deduce \( v_\tau \in w\mathcal{V}^+ \), which implies \( v_\tau \in w\mathcal{V}(a) \).

For each ordered pair \( (l, r) \in \mathbb{Z}^2 \), we define

\[
p^{l,r}_\tau := \bigoplus_{i \in I_r, j \in I_l : (\star)} (g(s) \cap g[\epsilon_i - \epsilon_j]),
\]

where \((\star)\) denotes the condition \( \epsilon_{w(i)} - \epsilon_{w(j)} \in R^+ \), and \( g[\epsilon_i - \epsilon_j] \) are the weight spaces. The condition \((\star)\) is also rephrased as:

- \((\star)_1\) If \( w(i)w(j) > 0 \), then we have \( w(i) < w(j) \);
- \((\star)_2\) If \( w(i) < 0 \), then we have \( w(j) < 0 \).

It is straight-forward to see that \( p^{l,r}_\tau \) is an abelian subalgebra of \( g \).

Since \( \{l_r\}_r \) exhaustive \([1, n] \), condition \((\star)\) implies

\[
p_\tau := t \oplus \bigoplus_{l,r} p^{l,r}_\tau = (\mathcal{W}^{-1} \cap g(s)).
\]

Hence, the Lie algebra \( p_\tau \) preserves \( w\mathcal{V}^+ \). Since \( p_\tau \subset g(s) \), it preserves \( \mathcal{V}^o \). Thus, \( p_\tau \) acts on \( w\mathcal{V}(a) \). Moreover, the connected algebraic subgroup \( P_\tau \subset G(s) \) with \( \text{Lie}P_\tau = p_\tau \) acts on \( w\mathcal{V}(a) \). We wish to prove that \( P_\tau v_\tau \) is dense in \( w\mathcal{V}(a) \).

We will be able to deduce this from the following claim, which is proved by computations.

Claim C. \( p_\tau v_\tau = w\mathcal{V}(a) \).

Proof. Since \( p^{l,r}_\tau \) is a direct sum of \( T \)-weight spaces, we deduce that \((t \oplus p^{l,r}_\tau)\) is again a Lie subalgebra of \( g(s) \). We set \( t' := \bigoplus_{i \in I_r} C\epsilon_i \), where \( \epsilon_i \in t' \) is identified with the dual basis \( \epsilon_i \in t \) by the pairing \( \langle \epsilon_i, \epsilon_j \rangle = \delta_{i,j} \). We have

\[
t' v_\tau = t'(v_\tau + \sum_{i \in I_r} \delta(i)v_{\epsilon_i}) = \bigoplus_{i, j \in I_r : \langle \epsilon_i, s \rangle = q(\epsilon_j, s)} C\epsilon_i - \epsilon_j \oplus \bigoplus_{i \in I_r : \delta(i) = 1} C\epsilon_i
\]

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by a simple calculation. (Here we used the fact the the weights appearing in
the RHS are linearly independent.)

By the first two conditions on \( w \), the signs of the entries in \( w(I_t) \) and \( w(I_m) \)
are constant on each segment. We calculate \( p_r^{r'}v_r \) in each of the four possible
cases of signs.

**Case 1** \((w(I_t), w(I_r)) > 0\) This means \( I_t, I_r \in D_+ \). We have either \( 0 < w(I_t) < w(I_t) \) or \( 0 < w(I_t) < w(I_t) \). If we have \( 0 < w(I_t) < w(I_t) \), then we have
\( \epsilon_j - \epsilon_j \not\in w^{-1}R^+ \) for every \( i \in I_t, j \in I_r \). Therefore, \( p_r^{r'} = \{0\} \) in this case.

Now we assume \( 0 < w(I_t) < w(I_t) \). We have \( \min I_t \geq \min I_t \) by assumption.
We have \( \epsilon_j - \epsilon_j \not\in \Psi(p_r^{r'}) \) if and only if \( i \in I_t, j \in I_r \) and \( \langle \epsilon_i - \epsilon_j, s \rangle = 1 \). By the
definition of segments, we deduce that

\[
p_r^{r'} = \bigoplus_{b \in I_t \cap I_r} g[\epsilon_{ib} - \epsilon_{jb}],
\]

where \( i_b \in I_t, j_b \in I_r \) satisfy \( \langle \epsilon_{ib}, s \rangle = b = \langle \epsilon_{jb}, s \rangle \). By an explicit computation,
we have

\[
g[\epsilon_{ib} - \epsilon_{jb}]v_r = C(v_{\epsilon_j - \epsilon_j} + v_{\epsilon_j - \epsilon_j}).
\]

Let \( b^- \) be the minimal element of \( I_t \cap I_r \). Then, the number \( j_{q_{2}^{-1}} \) does not exist. It follows that

\[
p_r^{r'}v_r = \sum_{b \in I_t \cap I_r} g[\epsilon_{ib} - \epsilon_{jb}]v_r = \sum_{b \in I_t \cap I_r} \forall[\epsilon_{ib} - \epsilon_{i_{q_{2}^{-1}}}],
\]

where \( v_r \) be the \( V_2 \)-part of \( v_r \).

Here \( I_t \) and \( I_r \) are marked if \( q_1 \in \overline{I_t} \) and \( q_1 \in \overline{I_r} \), respectively. Hence, we conclude that

\[
(t \oplus t' \oplus p_r^{r'})v_r = \sum_{b \in I_t \cap I_r} g[\epsilon_{ib} - \epsilon_{jb}]v_r = t'v_r + t'v_r \oplus V,
\]

where

\[
V = \begin{cases}
\sum_{b \in I_t \cap I_r} \forall[\epsilon_{ib} - \epsilon_{i_{q_{2}^{-1}}}] & (w(I_t) > w(I_t)) \\
\{0\} & (w(I_t) < w(I_t)).
\end{cases}
\]

**Case 2** \((w(I_t), w(I_t)) < 0\) This means \( I_t, I_r \in D_- \). This case is exactly the
same as **Case 1** if we uniformly take the inverse of every weight. Therefore, we conclude that

\[
p_r^{r'} = \begin{cases}
\bigoplus_{b \in I_t \cap I_r} g[\epsilon_{ib} - \epsilon_{jb}] & (w(I_t) < w(I_t)) \\
0 & (w(I_t) > w(I_t)).
\end{cases}
\]

and

\[
(t \oplus t' \oplus p_r^{r'})v_r = \sum_{b \in I_t \cap I_r} g[\epsilon_{ib} - \epsilon_{jb}]v_r = t'v_r + t'v_r \oplus V,
\]

where

\[
V = \begin{cases}
\sum_{b \in I_t \cap I_r} \forall[\epsilon_{ib} - \epsilon_{i_{q_{2}^{-1}}}] & (w(I_t) < w(I_t)) \\
\{0\} & (w(I_t) > w(I_t)).
\end{cases}
\]

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Case 3) \((w(I_l) < 0, w(I_r) > 0)\) This means \(I_l \in D_-, I_r \in D_+\). We have 
\(\epsilon_i - \epsilon_j \not\in w^{-1}R^+\) when \(i \in I_l\) and \(j \in I_r\). It follows that \(p_{\tau}^{I_r} = \{0\}\). Therefore we have 
\[(t^i \oplus t^j \oplus p_{\tau}^{I_r})v_\tau = t^i v_\tau \oplus t^j v_\tau.\]

Case 4) \((w(I_l) > 0, w(I_r) < 0)\) This means \(I_l \in D_+, I_r \in D_-\). We have 
\(\epsilon_i - \epsilon_j \not\in w^{-1}R^+\) when \(i \in I_l\) and \(j \in I_r\). By a similar argument as in Case 1, we deduce that 
\[p_{\tau}^{I_r} = \bigoplus_{b \in I_l \cap I_r} g[\epsilon_{i_b} - \epsilon_{j_b}],\]
where \(i_b \in I_l, j_b \in I_r\) satisfies \((\epsilon_{i_b}, s) = b = (\epsilon_{j_b}, s)\). By assumption, we have 
\(I_l \supset I_r\) only if \(I_l \subset I_r\). If \(\min I_l \leq \min I_r\), then we have \(j_q^{-1} = \emptyset\) for 
\(b^+ = \min(I_l \cap I_r)\). If \(\min I_l \geq \min I_r\), then we have \(i_q b^+ = \emptyset\) for \(b^+ = \max(I_l \cap I_r)\).

The segment \(I_l\) is marked if \(q \in I_l\), while \(I_r\) is never so. In particular, the vector \(\sum_{i \in I_l \cup I_r} \delta(i)v_{i}\) is annihilated by \(p_{\tau}^{I_r}\).

Therefore, by a similar argument as in Case 1, we have 
\[(t^i \oplus t^j \oplus p_{\tau}^{I_r})v_\tau = t^i v_\tau \oplus t^j v_\tau \oplus V,\]
where 
\[V = \left\{ \begin{array}{ll}
\sum_{b \in I_l \cap I_r} g[\epsilon_{i_b} - \epsilon_{j_b}] & (\min I_l \geq \min I_r) \\
\sum_{b \in I_l \cap I_r} g[\epsilon_{i_b} - \epsilon_{j_b}] & (\min I_l \leq \min I_r)
\end{array} \right. \]

We can rephrase the conclusion of the above case-by-case calculations as follows:

- \(t^i v_\tau\) is a sum of \(T\)-weight spaces of \(wV(a)\) of weight \(\epsilon_i\) or \(\epsilon_i - \epsilon_j\) such that 
  \(i, j \in I_l\);

- \(p_{\tau}^{I_r} v_\tau\) is a sum of \(T\)-weight spaces of \(wV(a)\) of weight \(\epsilon_i - \epsilon_j\) such that 
  \(i \in I_l\) and \(j \in I_r\).

From this, we deduce that 
\[p_{\tau} v_\tau = t \oplus \sum_{I_l \tau} p_{\tau}^{I_r} v_\tau = wV(a)\]

We have a natural identification \(p_{\tau} v_\tau = T_v, (P_{\tau} v_\tau)\) (the RHS must be read as the tangent space of \(P_{\tau} v_\tau\) at \(v_\tau\)). We deduce that 
\[\dim P_{\tau} v_\tau = \dim wV(a).\]

Since \(P_{\tau} v_\tau \subset wV(a)\), this forces \(P_{\tau} v_\tau = wV(a)\), which implies the result. \(\square\)

**Definition 4.10.** Let \(L\) be an irreducible \(\mathfrak{g}\)-module. We define the \(R(T)\)-character of \(L\) as a formal linear combination 
\[\text{ch} L := \sum_{s \in \Phi(L)} (\dim R(T)_s \otimes R(T) L) \langle s \rangle,\]  
(4.7) 
where \(\langle s \rangle\) is a formal symbol for each \(s \in T\) and \(R(T)_s\) is the localization of \(R(T)\) with respect to the kernel of the evaluation map at \(s\).
Corollary 4.11. Let $\mathcal{M}_m$ be the set of isomorphism classes of irreducible $\mathbb{H}_{n,m}$-modules. Assume that $m$ is generic. Then, the set $\{\text{ch}\, L | L \in \mathcal{M}_m\}$ is linearly independent.

Proof. Since $\Psi(L)$ is contained in the $W$-conjugacy class of a central character of $L$ for each $L \in \mathcal{M}_m$, it follows that we can argue by fixing one central character $a = (s, \vec{q})$. Let $L, L'$ be irreducible $\mathbb{H}_a$-modules with the corresponding $G(a)$-orbits $O, O'$, respectively (via the eDL correspondence). By Proposition 4.9, there exists $w_L \in W$ such that $w_L \cdot V_a \cap O$ is an open dense subset of $w_L \cdot V_a$. By Proposition 2.1, we deduce that $w_L \cdot s^{-1} \in \Psi(L')$ only if $O' \subset O$. We introduce a linear order $\succ$ on the set of irreducible $\mathbb{H}_a$-modules such that $L \succ L'$ if $O' \subset O$. Thanks to Corollary 2.2, we conclude that $\{w_L \cdot s^{-1} | L \in \mathcal{M}_m\}$ appears in $\{\Psi(L) | L \in \mathcal{M}_m\}$ triangularly with respect to $\succ$, which implies the result.

4.4 Cuspidal local systems in Spin groups

In this section, we explain the constructions and algorithms of Lusztig and Slooten and the relation with our setting.

Let $\mathcal{M}_n,m$ be the affine graded Hecke algebra as in Definition 1.13, with parameters normalized as:

\begin{equation}
1 1 1 \cdots 1 1 1 \quad m, \quad (4.8)
\end{equation}

where $4m \equiv 1$ or 3 mod 4.

Define $X_\ell$ to be the set of nilpotent orbits in $\text{so}(\ell)$ parameterized by partitions containing odd parts with multiplicity one, and even parts with even multiplicity. For every nilpotent orbit $O \subset \text{so}(\ell)$ given by the partition $(a_1, \ldots, a_s)$ define the defect of $O$

\begin{equation}
d(O) := \sum_{i=1}^{s} d(a_i), \quad \text{where} \quad d(a_i) = \begin{cases} 1, & \text{if } a_i \equiv 1 \mod 4 \\ 0, & \text{if } a_i \equiv 0, 2 \mod 4 \\ -1, & \text{if } a_i \equiv 3 \mod 4. \end{cases} \quad (4.9)
\end{equation}

For every $d \in \mathbb{Z}$, set $X_{\ell,d}$ to be the set of elements in $X_\ell$ of defect equal to $d$. Then one has

\begin{equation}
X_\ell = \bigcup_{d \in \mathbb{Z}, A(2d-1)} X_{\ell,d}. \quad (4.10)
\end{equation}

The generalized Springer correspondence ([Lu85]) for the cuspidal local systems in $\text{Spin}(\ell)$ which do not factor through $SO(\ell)$ takes the following combinatorial form.

Theorem 4.12 ([Lu85], [LS85]). There is a one to one correspondence

\[ X_{\ell,d} \leftrightarrow \text{Irrep} W_n, \quad \text{where} \quad n = \frac{\ell - d(2d-1)}{4}. \]

Remark 4.13. 1. It is not hard to see using a generating functions argument ([Lu85]) that the two sets in Theorem 4.12 have the same cardinality. Moreover, a slight modification of that argument shows that the number of distinguished orbits in $X_{\ell,d}$ equals $P(n)$, the number of partitions of $n$. 41
2. In the generalized Springer correspondence in this setting, there is a unique local system on each orbit in $X_{\ell,d}$ which enters, and this is why the correspondence can be regarded as one between orbits and Weyl group representations.

3. The relation between $\mathfrak{H}_{n,m}$ and $X_{\ell,d}$ is given by

$$4n + d(2d - 1) = \ell, \quad d = -d(4m)[m + 1/4]. \quad (4.11)$$

The left to right map in Theorem 4.12 is given by an explicit algorithm which we recall now. We use the notation for Irrep $W_n$ from Remark 1.25.

**Algorithm 4.14** ([LS85]). Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ be a partition of $\ell$ of defect $d$. Here, the convention is $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_k$. We will produce inductively a bipartition $\rho(\lambda)$ of $n = \ell - d(2d - 1)/4$, which parameterizes an element of $\text{Irrep}W_n$.

Define the (smaller) partition $\mu$ as follows:

(i) if $\lambda_p$ is odd, then set $\mu = (\lambda_1, \ldots, \lambda_{p-1})$;

(ii) if $\lambda_p$ is even, then set $\mu = (\lambda_1, \ldots, \lambda_{p-2})$.

By induction $\rho(\mu)$ is known, say it is of the form $\rho(\mu) = \{\gamma(\delta)\}$ for some (ordered) pair of partitions $(\gamma, \delta)$.

(a) $d(\lambda_p) = 0$ ($\lambda$ is even). Set $e = \lfloor(\lambda_p + 2)/4\rfloor - d(\mu)$ and $f = \lfloor\lambda_p/4\rfloor + d(\mu)$.

(Note that $e + f = \frac{\lambda_p}{4}$.)

(a1) If $d(\mu) > 0$, set $\rho(\lambda) = \{\gamma(e)f, \delta\}$.

(a2) If $d(\mu) \leq 0$, set $\rho(\lambda) = \{\gamma(e), \delta, f\}$.

(b) $d(\lambda_p) = 1$ ($\lambda_p \equiv 1 \mod 4$). Set $e = \frac{\lambda_p - 1}{4} = d(\mu)$.

(b1) If $d(\mu) > 0$, set $\rho(\lambda) = \{\gamma(e)f, \delta\}$.

(b2) If $d(\mu) = 0$, set $\rho(\lambda) = \{\delta, e(\gamma)e\}$.

(b3) If $d(\mu) < 0$, set $\rho(\lambda) = \{\gamma(e), \delta, f\}$.

(c) $d(\lambda_p) = -1$ ($\lambda_p \equiv 3 \mod 4$). Set $e = \frac{\lambda_p - 3}{4} + d(\mu)$.

(c1) If $d(\mu) > 1$, set $\rho(\lambda) = \{\gamma(e), \delta, f\}$.

(c2) If $d(\mu) = 1$, set $\rho(\lambda) = \{\delta, e(\gamma)e\}$.

(c3) If $d(\mu) < 1$, set $\rho(\lambda) = \{\gamma(e), \delta, f\}$.

**Theorem 4.15** ([Lu02a]). The tempered modules of $\mathfrak{H}_{n,m}$ with positive real central character are parameterized by the orbits in $X_{\ell,d}$. The discrete series of $\mathfrak{H}_{n,m}$ with positive real central character are parameterized by the distinguished orbits in $X_{\ell,d}$. In particular, there are $\#P(n)$ discrete series.

In [Sl06], a conjecture relating discrete series of $\mathfrak{H}_{n,m}$, partitions of $n$, and Weyl group representations (a Springer correspondence) was proposed. We explain this next.

**Partitions of $n$ to distinguished orbits.** Let $\sigma$ be a partition of $n$. We think of $\sigma$ as left justified Young tableau, with the length of rows decreasing,
same as in [35.2]. Fill out the boxes of \( \sigma \) starting at the left upper corner with \( m \) and increase by one to the right, and decrease by one down. In this way, all the boxes on the diagonal have the entry \( m \). Recall that \( 4m \equiv 1 \) or \( 3 \) mod 4. Let \( \bar{s}_\sigma \) denote the collection of the absolute values of the entries of \( \sigma \) (with multiplicities), ordered nonincreasingly. We think of \( \bar{s}_\sigma \) as being a central character for \( \mathbb{H}_{n,m} \). (The connection with the previous sections is that \( s_\sigma = q^{h_\sigma} \) is a distinguished semisimple element.)

To \( \bar{s}_\sigma \), we attach a distinguished nilpotent orbit \( O_\sigma \) in \( X_{\ell,d} \), \( \ell = 4n + d(2d - 1) \), as follows (we are thinking of \( O_\sigma \) as a partition of \( \ell \) with defect \( d \)). Let \( \{m\} = m - \lfloor m \rfloor \) denote the fractional part of \( m \). This is either \( 1/4 \) or \( 3/4 \). Start with the cuspidal part \( \lambda_c = \{4m - 2, 4m - 6, \ldots, 4 - 4\{m\}\} \). This is of the form \( \{3, 7, 11, \ldots\} \) or \( \{1, 5, 9, \ldots\} \), depending if \( \{m\} = 1/4 \) or \( 3/4 \), respectively. When \( m = 1/4 \), we have \( \lambda_c = \emptyset \). Note that the defect of \( \lambda_c \) is \( d \), and the sum of entries in \( \lambda_c \) is \( 2(m + 1/4)(m - 1/4) = d(2d - 1) \). Let \( \lambda = \lambda_c \). For every hook in \( \sigma \), we will modify \( \lambda \) so that the defect remains the same, and the sum of entries increases by the four times the length of the hook. Assume there are \( h \) hooks, \( h \geq 1 \). For every hook \( j \) starting from the exterior, denote by \( r_j \) the entry in the right extremity, and by \( r_j' \) the bottom extremity. Note that

\[
\ell > r_2 > \cdots > r_h \geq m \geq r_h' > r_{h-1}' > \cdots > r_1',
\]

and that the length of the hook is \( r_j - r_j' + 1 \). Starting with the most interior hook, for every hook \( j \), there are two cases:

1. if \( r_j' \leq 1/4 \), add to \( \lambda \), \( 4r_j + 2 \) and \( -4r_j' + 2 \) (they have opposite defect);
2. if \( r_j' > 1/4 \), add to \( \lambda \), \( 4r_j + 2 \), and remove \( 4r_j' - 2 \) (they have same defect).

The end partition \( \lambda \) is \( O_\sigma \). We summarize the obvious properties of this construction.

**Claim D.** The above procedure \( \sigma \mapsto O_\sigma \) is well-defined, and gives a distinguished orbit in \( X_{\ell,d} \). Moreover, two different partitions give different elements of \( X_{\ell,d} \).

**Example 4.16.** Let us consider the example \( n = 13, m = 9/4 \), and the partition \( \sigma = (4, 3, 3, 2, 1) \). Then \( d = -2 \) and \( \ell = 62 \). We view the partition as:

\[
\begin{array}{cccccccc}
3 & 3 & 3 & 3 & 1 & 1 & 1 & 1 \\
4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 \\
5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 \\
6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 \\
7 & 7 & 7 & 7 & 7 & 7 & 7 & 7 \\
8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 \\
9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 \\
10 & 10 & 10 & 10 & 10 & 10 & 10 & 10 \\
11 & 11 & 11 & 11 & 11 & 11 & 11 & 11 \\
12 & 12 & 12 & 12 & 12 & 12 & 12 & 12 \\
13 & 13 & 13 & 13 & 13 & 13 & 13 & 13 \\
\end{array}
\]

Figure 5: Partition \((4, 3, 3, 2, 1)\) for \( \mathbb{H}_{13, \frac{9}{4}} \).

By the algorithm, we start with \( \lambda_c = (3, 7) \). There are three hooks. The most interior hook has \( r_3 = r_3' = 9/4 \), so we add 11 and remove 7, and get
\( \lambda = (3,11) \). Next \( r_2 = 13/4 \), \( r'_2 = 1/4 \), so we add 15 and 1, and get \( \lambda = (1,3,11,15) \). Finally, \( r_1 = 21/4 \) and \( r'_1 = -7/4 \), so we add 23 and 9. Therefore \( \mathcal{O}_\sigma = (1,3,9,11,15,23) \), which is in \( X_{62,-2} \).

Behind the reasoning for this algorithm is the fact that the middle element of the nilpotent \( \mathcal{O}_\sigma \) is obtained from the central character \( \bar{s}_\sigma \) and the middle element for the cuspidal part.

**Partitions, distinguished orbits, and \( W \)-representations.** Let us recall the conjecture of [Sl06], and show that it is equivalent to the [LS85] algorithm presented above.

**Algorithm 4.17.** Start with \( \sigma \) a partition of \( n \) viewed as before. We form a bipartition \( S_n(\sigma) = (\{\gamma\})(\delta) \) of \( n \) as follows. Begin by setting \( \gamma = \delta = \emptyset \). Then find the largest in absolute value entry in \( \sigma \). (This is necessarily one of the extremities of the first hook.) Remove all the boxes to the left of it in the same row (including it), or all boxes above it in the same column (including it). Let \( x \) be the number of boxes removed. If they were in the same row, append \( x \) to \( \gamma \), if they were in the same column, append \( x \) to \( \delta \). Repeat the process until there are no boxes left, or until there is a single box left. In the latter case, if the entry in the single box left is positive, append 1 to \( \gamma \), if it is negative, append 1 to \( \delta \).

**Example 4.18.** In Example 4.16 first we remove the boxes to the left of 21/4, append 4 to \( \gamma \), then the boxes to the left of 13/4, append 3 to \( \gamma \), then the boxes to the left of 9/4, append 3 to \( \gamma \), then the remaining boxes above \( -7/4 \), append 2 to \( \delta \), finally, the remaining box 1/4, so append 1 to \( \gamma \). So the bipartition \( S_{24}(4,3,3,2,1) \) equals \( \{(4,3,3,1)(2)\} \).

**Claim E.** For every \( \sigma \) a partition of \( n \), and \( 4m \equiv 1,3 \mod 4 \), the \( W_n \)-representations (or rather, bipartitions of \( n \)) \( S_m(\sigma) \) and \( \rho(\mathcal{O}_\sigma) \) coincide.

In other words, the algorithm for the Springer correspondence of Lusztig-Spaltenstein coincides with the algorithm of Slooten in the case when \( 4m \equiv 1,3 \mod 4 \).

**Proof.** We prove this statement by induction on \( n \), the size of the Young tableau. The base \( n = 2 \) is straightforward. Let \( \sigma \) be a partition of \( n > 2 \) viewed as a Young tableau, and \( \mathcal{O}_\sigma = (\lambda_1 < \cdots < \lambda_k) \) the orbit constructed before. Since this is a distinguished orbit, only cases (b) and (c) of Algorithm 4.14 enter. The largest entry in absolute value \( \text{max} \) is given by one of the extremities \( r_1 \) or \( r'_1 \) (if \( r'_1 < 0 \)) of the first hook. It corresponds to \( \lambda_k : \lambda_k = 4\text{max} + 2 \). There are two cases.

a) Assume \( \text{max} = r_1 \). Then \( d(\lambda_k) = -d(4m) \). Let \( e' \) be the number of boxes on the first row in \( \sigma \), so \( e' = r_1 - m + 1 \). In 4.13 one forms \( r = [\lfloor \frac{e'}{2} \rfloor] - d(\lambda_k)(d(\mathcal{O}_\sigma) - d(\lambda_k)) = [r_1 + \frac{1}{2}] + d(4m)(-d(4m))(m + \frac{1}{2}) - 1 = [e' + m - 1 + \frac{1}{2}] - [m + \frac{1}{2}] + 1 = e' + [m + \frac{1}{2}] - [m + \frac{1}{2}] = e' \). So \( e = e' \). In 4.17 \( e' \) is placed in the left side of the bipartition. We check that in 4.13 \( r \) is also placed in the left side of the bipartition. There are two subcases: if \( d(4m) = 1 \), then \( d(\lambda_k) = -1 \), and so, in 4.13 (c), \( d(\mu) = -[m + \frac{1}{2}] + 1 \leq 1 \), so we are in the cases (c2,c3); if \( d(4m) = -1 \), so \( m \geq \frac{4}{3} \), \( d(\lambda_k) = 1 \), and so, in 4.14 (b), \( d(\mu) = [m + \frac{1}{2}] - 1 \geq 0 \), so we are in cases (b1,b2).

Now let \( \tau \) be the Young tableau obtained after removing the first row. The entry in the left upper corner is \( m - 1 \), which is positive unless \( m = 1/4,3/4 \). If
$m > 1$, we can regard $\tau$ as a partition of $n - e$ and with $m - 1$ instead of $m$. It is immediate that the corresponding $O_\tau$ is the same as $\mu$ in 4.14. By induction if $SL(\tau) = \{(\gamma)(\delta)\}$, then $\rho(\mu) = \{(\gamma)(\delta)\}$, and we are done.

So consider the cases $m = 1/4$ or $m = 3/4$. Then $m - 1 < 0$. Let $\tau$ be the Young tableau which is obtained from $\tau$ by first taking the transpose tableau and then multiplying all the entries by $(-1)$. The left upper corner of $\tau$ has entry $1 - m > 0$, so we can regard $\tau$ as a partition of $n - e$ for $1 - m$ (not $m$), and associate $O_\tau$. Note that if $SL(\tau) = \{(\delta)(\gamma)\}$, then $SL(\tilde{\tau}) = \{(\gamma)(\delta)\}$. The only observation left to make is that $O_\tau = \mu$, where $\mu = (\lambda_1, \ldots, \lambda_{k-1})$. This follows easily from the algorithm for $O_\sigma$ and $O_\tau$. b) Assume $m = -q_1' > 0$. Then $d(\lambda_k) = d(4m)$. By the same argument as in case a), one shows that $r'$, the number of boxes in the first column, equals $r$ from 4.14. In 4.14 $r'$ is placed in the right side of the bipartition. We check that in 4.14 $e$ is also placed in the right side of the bipartition. There are two subcases: if $d(4m) = -1$, so that $m \geq 4$, then $d(\lambda_k) = -1$, and so, in 4.14 (c), $d(\mu) = [m + \frac{1}{4}] + 1 > 1$, so we are in the cases (c1); if $d(4m) = 1$, $d(\lambda_k) = 1$, and so, in 4.14 (b), $d(\mu) = -[m + \frac{1}{4}] - 1 < 0$, so we are in case (b3).

If $\sigma$ is the Young tableau obtained after removing the first column, the entry in the upper left corner is $m + 1 > 0$, so we can regard $\sigma$ as a partition for $n - e$ and with $m + 1$ instead of $m$. It is immediate that the corresponding $O_\sigma$ is the same as $\mu$ in 4.14. By induction if $SL(\sigma) = \{(\gamma)(\delta)\}$, then $\rho(\mu) = \{(\gamma)(\delta)\}$, and this concludes this case.

Since we showed these algorithms yield the same $W$-representation, let us denote it by $\rho(\sigma)$. There are two particular cases worth mentioning.

Assume that $(\sigma, m)$ are such that the Springer correspondence algorithm picks only rows, and so $\rho(\sigma) = \{(a_1, \ldots, a_k)(\emptyset)\}$, or only columns, and so $\rho(\sigma) = \{\emptyset(b_1, \ldots, b_k)\}$. These are the cases we referred as positive, respectively negative, ladders, and so by the previous results, the discrete series representation with central character $s_p$ is irreducible as an $W$-module, and equals $\rho(\sigma) \otimes \text{sgn}$. (The tensoring with $\text{sgn}$ is a normalization, so that the Steinberg module is $\text{sgn}$ as a $W$-representation.)

The second particular case is that for discrete series which contain the $\text{sgn}$ $W$-representation. By the previous results, if the hook extremities of $\sigma$ are $r_1 > r_2 > \cdots > r_h = m > r'_h > r'_{h-1} > \cdots > r'_1$, then in order for $s_{s_p}$ to contain the $\text{sgn}$, they must satisfy:

$$r_1 > -r'_1 > r_2 > -r'_2 > \cdots > r_h > -r'_h.$$ 

This means that $\rho(\sigma)$ is obtained as follows: remove the first row, then the first column in the remaining tableau, then remove the first row remaining, then the first column etc.

We remark that in these two cases, the $W$-representation attached to $\sigma$ by the exotic Springer correspondence coincides with the $W$-representation attached to $\sigma$ by the algorithms of Lusztig-Spaltenstein and Slooten.

### 4.5 $W$-independence of tempered modules

Using the geometric realization and results of Lusztig for the graded Hecke algebras arising from cuspidal local systems, one was able to prove in [CIOS]...
a certain independence result for tempered modules with positive real central characters. This is a generalization of the similar result of Barbasch-Moy for Hecke algebras with equal parameters, and it is a Hecke algebra analogue of Vogan’s lowest K-types.

Retain the notation from Section 4.14. We formulate this result in the setting of the graded Hecke algebra $H_{n,m}$ from Section 4.14 with $4m \equiv 1, 3 \pmod{4}$. For every tempered module $\pi$ with positive real central character, which by Theorem 4.19 corresponds to an orbit $O_\pi \in X_{l,d}$, let $\rho(\pi)$ be the generalized Springer correspondence $W$-representation attached to $O_\pi$. The following result follows from the fact that any other $W$-type appearing in the restriction $\pi|_W$ is attached in the generalized Springer correspondence to an orbit larger than $O_\pi$ in the closure ordering.

**Proposition 4.19** (cf. [CM08]).

1. There is a bijection $\pi \mapsto \rho(\pi)$ between tempered modules $H_{n,m}$, $4m \equiv 1, 3 \pmod{4}$ with positive real central character and $\text{Irr}_{W}$.\[1.5]

2. The set of positive real tempered $H_{n}$-modules viewed in $R(W)$ is linearly independent. Moreover, in the ordering coming from the closure ordering in $X_{l,d}$, the change of basis matrix to $\text{Irr}_{W}$ is uni-triangular.

Theorem 4.19 allows us to extend this result to all generic positive real parameters $q = (-1, q^m, q)$, for the affine Hecke algebra of type $C_n$. (Here we use implicitly the correspondence between the affine Hecke algebra and the graded Hecke algebra for positive real central characters.)

**Corollary 4.20.** The set of tempered $H_{n}$-modules for generic positive real $a$ is $W$-independent in $R(W)$. Moreover, in the ordering coming from the generalized Springer correspondence, the change of basis matrix to $\text{Irr}_{W}$ is uni-triangular.

**Proof.** Let $m$ be in the open interval $(\frac{k}{2}, \frac{k+1}{2})$, for some integer $k \geq 0$ and $a$ be the corresponding generic parameter. Let $\text{MP}(m)$ be the set of (exotic) marked partitions which parameterizes the set of tempered $H_{n}$-modules. Set $m_0 = \frac{2k+1}{2}$, and fix $\tau \in \text{MP}(m_0)$. The results in this paper imply that the set $\text{MP}(m)$ is the same for all $m \in \left(\frac{k}{2}, \frac{k+1}{2}\right)$. Moreover, if we denote by $\text{temp}_{m}(\tau)$ the tempered module parameterized by $\tau$ at the parameter $m$, then

$$\text{temp}_{m}(\tau) \cong \text{temp}_{m'}(\tau)$$

as $W$-modules, (4.13) for any $m, m' \in \left(\frac{k}{2}, \frac{k+1}{2}\right)$. In particular, $\text{temp}_{m}(\tau) \equiv \text{temp}_{m_0}(\tau)$, for all $\tau \in \text{MP}(m) = \text{MP}(m_0)$. Then the claim follows from Proposition 4.19.\[1.8]

**Remark 4.21.** In the preprint [So08], the first part of Corollary 4.20 the $W$-independence (but not the uni-triangularity), is proved independently for arbitrary graded Hecke algebras by homological methods.

**Remark 4.22.** One can ask naturally if a similar uni-triangular correspondence as in Corollary 4.20 holds if one considers instead the exotic Springer correspondence (see [Ka08] for an explicit algorithm). This is not the case however: in general, the map assigning to a tempered $H_{n}$-module its exotic Springer representation is not one-to-one, as one can see in the example $n = 4$, and $0 < m < 1/2$ for the partitions of $n$, $\sigma_1 = (1, 1, 2)$ and $\sigma_2 = (1, 1, 1, 1)$. The exotic Springer map assigns the $W_3$-representation $\{(1^4)(0)\}$ to both $\text{ds}(\sigma_1)$ and $\text{ds}(\sigma_2)$, while the generalized Springer map assigns $\{(1^3)(1)\}$ to $\text{ds}(\sigma_1)$ and $\{(1^4)(0)\}$ to $\text{ds}(\sigma_2)$.
In particular, the “lowest $W$-type” correspondence of Corollary 4.20 shows that the construction of Theorem 4.7 exhausts all tempered modules in the real positive generic range.

**Corollary 4.23.** Every tempered $\mathbb{H}_a$-module for generic positive real $a$ is obtained by induction as in Theorem 4.7.

**Proof.** It is sufficient to show that Theorem 4.7 produces $\#\text{Irrep}_{W_n}$ distinct tempered modules. Let $P(k)$ denote the number of partitions of $k$, and $P_2(k) = \#\text{Irrep}_{W_k}$. For every $1 \leq n_1 \leq n$, a tempered $\mathbb{H}_a$-module is constructed from a tempered $GL(n_1)$ module and a discrete series of $Sp(2n_2)$, where $n_2 = n - n_1$. There are $P(n)$ tempered modules of $GL(n_1)$ and $P_2(n)$ discrete series of $Sp(2n_2)$. Therefore we get $\sum_{n_1=1}^{n} P(n_1)P(n - n_1) = P_2(n)$ tempered $\mathbb{H}_a$-modules. These are all distinct $\mathbb{H}_a$-modules since they are nonisomorphic as $W_n$-modules.

### 4.6 One $W$-type discrete series

We show that the only tempered $\mathbb{H}_{n,m}$-modules with real positive generic parameter which are irreducible as $W$-modules are the $(\pm)$-ladder representations (see [BM99]). Any tempered module which is not a discrete series is obtained by parabolic unitary induction from a discrete series module of a proper parabolic Hecke subalgebra. Therefore, no such module could be $W$-irreducible, so we can restrict to the case of discrete series, and we can restrict to the equivalent setting of $\mathbb{H}_{n,m}$-modules.

Let $\mathbb{H}^A_n$ be the one-parameter graded Hecke algebra for $GL(n)$, viewed as a subalgebra of $\mathbb{H}_{n,m}$. We have that $\mathbb{H}^A_n$ is generated by $\{\epsilon_1, \ldots, \epsilon_n\}$ and $\{t_{i,i+1} : 1 \leq i \leq n - 1\}$, where $t_{ij}$ denotes the generator corresponding to the reflection $s_{\epsilon_i - \epsilon_j}$. The following lemma is well-known and easy to prove by direct computation.

**Lemma 4.24.** There is a surjective algebra map $\phi : \mathbb{H}^A_{n-1} \mapsto \mathbb{C}S_n$, given on generators by

$$\phi(t_{i,i+1}) = s_{i,i+1},$$

$$\phi(\epsilon_j) = s_{j,j+1} + s_{j,j+2} + \cdots + s_{j,n}. \quad (4.14)$$

Note that $\phi$ allows us to lift any irreducible $S_n$-representation to an irreducible $\mathbb{H}^A_n$-module. For $\sigma$ a partition of $n$, let $\phi^*(\sigma)$ denote the irreducible $\mathbb{H}^A_{n-1}$-module obtained in this way from lifting $\sigma \otimes \text{sgn}$.

A simple modification of $\phi$ lifts any irreducible $S_n$-representation to an irreducible $\mathbb{H}_{n,m}$-module. The following statement can be viewed as a particular case of the construction in [BM99].

**Lemma 4.25.** Let $\eta \in \{+1, -1\}$ be given and let $\sigma$ be a fixed partition of $n$. The assignment

$$t_{i,i+1} \mapsto \phi^*(\sigma)(t_{i,i+1}), \quad 1 \leq i \leq n - 1,$$

$$\epsilon_i \mapsto \eta \text{Id} + \phi^*(\sigma)(\epsilon_i), \quad 1 \leq i \leq n - 1,$$

$$t_n \mapsto \eta \text{Id},$$

$$\epsilon_n \mapsto \eta \text{Id}, \quad (4.15)$$
gives an irreducible $W_{n,m}$-module, $\pi(\sigma, \eta)$.

Proof. By Lemma 4.24 we only need to check that the Hecke relations

\[
\begin{align*}
t_n \cdot \epsilon_n &= -\epsilon_n t_n + 2m, \\
t_{n-1} \cdot \epsilon_n &= \epsilon_{n-1} t_{n-1} - 1,
\end{align*}
\]

are satisfied for this assignment. This is straightforward.

Note that $\pi(\sigma, +)$ equals $\{(\sigma)(\emptyset)\} \otimes \text{sgn}$ as a $W_n$-representations, while $\pi(\sigma, -)$ equals $\{(\sigma)(\emptyset)\}$, in the bipartition notation of $W_n$-representations from \[.4.19\] We show that these are precisely the $(\pm)$-ladder representations from \[.4.3\]

**Theorem 4.26.** Let $s_\sigma$ be a distinguished central character for $\mathbb{H}_{n,m}$, where $m$ is generic. Then $\mathfrak{ds}(s_\sigma)$ is irreducible as a $W_n$-representation if and only if it is a $(\pm)$-ladder representation.

Proof. We prove the claims in the equivalent setting of $\mathbb{H}_{n,m}$. In one direction, let us assume that $\mathfrak{ds}(s_\sigma)$ is a positive ladder. The proof for the other case is analogous. We wish to show that $\mathfrak{ds}(s_\sigma) = \pi(\sigma, +)$. A direct proof of this fact would be to compute the central character of $\pi(\sigma, +)$ and show that it is $s_\sigma$. We give an indirect proof. In the bijection of Corollary 4.20 $\mathfrak{ds}(s_\sigma)$ (contains and) corresponds to the $W_n$-representation $\rho(\mathfrak{ds}(s_\sigma)) = \{(\sigma)(\emptyset)\} \otimes \text{sgn}$. By the Lusztig classification \[.4.3\], $\pi(\sigma, +)$ is the unique irreducible quotient of a standard module $M = M_{\sigma, \emptyset}$. There is a continuous deformation of $s_\sigma \rightarrow s_0$, and $M \rightarrow M_0$ such that $M_0$ is a tempered module at the semisimple element and $M_0|_{W_n} = M|_{W_n}$. Moreover, the tempered module $M_0$ must contain $\rho(\mathfrak{ds}(s_\sigma))$, and in Proposition 4.19 $\rho(M_0) = \rho(\mathfrak{ds}(s_\sigma))$. But this implies that $M_0 = \mathfrak{ds}(s_\sigma)$. Since this is a discrete series, we then have $\pi(\sigma, +) = M = M_0 = \mathfrak{ds}(s_\sigma)$.

To verify the converse claim, recall that in \[.4.3\], one determined which $W_n$-representations can be extended to hermitian graded Hecke algebra modules. When the Hecke algebra is $\mathbb{H}_{n,m}$, the only cases are $W$-representations of the form $\{(\gamma)(\emptyset)\}$, $\{(\emptyset)(\emptyset)\}$, or $\{(d^k)(f^l)\}$, where $k - d = l - f + m$.

Note that, using Algorithm 4.17 it is immediate that there is no discrete series $\mathfrak{ds}(s_\sigma)$ such that $\rho(\mathfrak{ds}(s_\sigma)) = \{(d^k)(f^l)\}$, for $k > 1$ and $l > 1$. We check the case $k = l = 1, d > 0, f > 0$. In order to have $\rho(\mathfrak{ds}(s_\sigma)) = \{(d)(f)\}$, by Algorithm 4.17 we must have $\sigma$ a one hook partition, with the largest two entry values at the two extremities of the hook. If the largest entry is the right extremity of the hook, then by \[.4.2\] $\mathfrak{ds}(s_\sigma)$ also contains the sgn $W_n$-representation, so it is not $W_n$-irreducible. So it remains to consider the case when the largest entry is in the bottom extremity of the hook. In that case, by Example 3.6 the exotic Springer correspondence attaches to $\mathfrak{ds}(s_\sigma)$, the $W_n$-representation given by the bipartition $\{(\emptyset)(n)\}$. So again $\mathfrak{ds}(s_\sigma)$ contains at least two irreducible $W_n$-representations.

**Corollary 4.27.** Let $s_\sigma$ be a distinguished central character for $\mathbb{H}_{n,m}$.

1. If $m > n - 1$, then we have $\mathfrak{ds}(s_\sigma)|W_n = \{(\sigma)(\emptyset)\} \otimes \text{sgn}$.

2. If $m < -n + 1$, then we have $\mathfrak{ds}(s_\sigma)|W_n = \{(\sigma)(\emptyset)\}$.

Proof. By Corollary 3.11 $\mathfrak{ds}(s_\sigma)$ is a positive ladder, if $m > n - 1$, and it is a negative ladder, if $m < -n + 1$. The proof of Theorem 4.26 implies that $\mathfrak{ds}(s_\sigma) = \pi(\sigma, +)$, when $m > n - 1$ and $\mathfrak{ds}(s_\sigma) = \pi(\sigma, -)$, when $m < -n + 1$. Then the claims follow from the remark after Lemma 4.20.
4.7 Closure relation of orbits

Fix a generic \( m \). Let \( \sigma \) be a partition of \( n \). Attached to \( \rho(\sigma) \) and \( \text{ds}(\sigma) \), we have irreducible \( W \)-modules \( L_\sigma \) and \( E_\sigma \), respectively. For an irreducible \( W \)-module \( K \), we denote by \( O(K) \) the nilpotent orbit of \( \text{Spin}(\ell) \) corresponding to \( K \) via (the inverse of) a generalized Springer correspondence (c.f. \[1.19\]). Let \( C \) be the pair \((O, \mathcal{L}) \) of \( \text{Spin}(\ell) \)-orbit of \( \mathfrak{so}(\ell) \) and the local system which contribute to the generalized Springer correspondence. Let \( \mathcal{O}(K) \subset \mathfrak{M}^{\text{ss}} \) be the \( G \)-orbit corresponding to \( K \) via Theorem \[1.16\]. Let \( \text{pr} : \mathcal{V} \to \mathcal{V}_2 \) be the \( G \)-equivariant projection map.

**Theorem 4.28.** In the above setting:

1. We have \([\text{ds}(\sigma) : L_\sigma] = 1 = [\text{ds}(\sigma) : E_\sigma]\) as \( W \)-modules;
2. For each irreducible \( W \)-submodule \( K \) of \( \text{ds}(\sigma) \), we have \( \mathcal{O}(L_\sigma) \subset \mathcal{O}(K) \) and \( \mathcal{O}(E_\sigma) \subset \mathcal{O}(K) \). In particular, we have \( \mathcal{O}(L_\sigma) \subset \mathcal{O}(E_\sigma) \) and \( \mathcal{O}(E_\sigma) \subset \mathcal{O}(L_\sigma) \);
3. Let \( E \) be an irreducible \( W \)-module such that \( \text{pr}(\mathcal{O}(E)) = \text{pr}(\mathcal{O}(E_\sigma)) \). Then, we have \([\text{ds}(\sigma) : E] \leq 1 \) as \( W \)-modules.

**Proof.** Recall that both of the constructions of Lusztig \[Lu95b\] and \[Ka08\] depend on the realization of \( \mathbb{H}_{\text{ss}} \) in terms of the self-extension algebras of certain complexes. Let \( \mathcal{IC}(\mathcal{O}, \mathcal{M}) \) be the minimal extension of a local system \( \mathcal{M} \) on \( \mathcal{O} \). In Lusztig’s case, we have (by \[Lu95b\]):

i) There exists a semi-simple element \( a_\sigma \in \text{Spin}(\ell) \times \mathbb{C}^\times \). Let us denote by \( C_\sigma \) the set of pairs \((\mathcal{O}, L|_\mathcal{O}) \) obtained from \( C \) by taking (connected component of) \( a_\sigma \)-fixed points;

ii) Define \( G^L := Z_{\text{Spin}(\ell) \times \mathbb{C}^\times}(a_\sigma) \). Then, each element of \( C_\sigma \) is a single \( G^L \)-orbit with a local system. The set \( C_\sigma \) is in bijection with \( \text{Irrep}^L \mathbb{H}_{\text{ss}} \);

iii) For each \((\mathcal{O}, L|_\mathcal{O}) \in C_\sigma \), we define \( \mathcal{O}^\sim := \text{Spin}(\ell) \mathcal{O} \subset \mathfrak{so}(\ell) \). It defines a \( W \)-representation \( \rho(\mathcal{O}) \) via a generalized Springer correspondence;

iv) The standard module \( M(\mathcal{O})(\mathcal{O}) \) contains an irreducible \( W \)-module \( K \) with multiplicity (as \( W \)-modules) equal to \( \dim H^*_\mathfrak{g}(\mathcal{IC}(\mathcal{O}(K), \mathcal{L})) \);

v) The standard module \( M(\mathcal{O}) \) has a unique simple quotient \( L(\mathcal{O}) \) and we have \([M(\mathcal{O}) : L(\mathcal{O})] = \dim H^*_\mathfrak{g}(\mathcal{IC}(\mathcal{O}^\sim, \mathcal{L}^\sim)) \).

Now the assertion \([\text{ds}(\sigma) : L_\sigma] = 1 \) follows by the combination of iv) and v). The assertion \([\text{ds}(\sigma) : E_\sigma] = 1 \) follows by the construction of \( L_{(a,X)} \) and Theorem \[1.19\]. We have \( \mathcal{O}(E_\sigma) \subset \mathcal{O}(L_\sigma) \) by the combination of iv) and v). We have \( \mathcal{O}(L_\sigma) \subset \mathcal{O}(E_\sigma) \) by Corollary \[1.18\] and Theorem \[2.12\] applied to \( a = a_0 \). This proves 1) and 2). We prove 3). Notice that \( \text{pr}(\mathcal{O}(E)) = \text{pr}(\mathcal{O}(E_\sigma)) \) implies either \( \mathcal{O}(E)^\bigcap\mathcal{O}(E_\sigma) = \emptyset \) or \( \mathcal{O}(E) \) is a (open dense subset of a) vector bundle over \( \mathcal{O}(E_\sigma) \). By 2), it suffices to consider the latter case. By Corollary \[1.19\] and Theorem \[2.12\] applied to \( a = a_0 \), we have \([M_{(a,X)} : L_Y] = 1 \) as \( W \)-modules for \( \mathcal{V} \in \mathcal{O}(E) \) as required.

**Remark 4.29.** To use Theorem \[4.28\] one needs to know the Weyl group representation attached to each orbit and the closure relations between orbits. These are contained in \[Ka08\] and Achar-Henderson \[AH08\], respectively.
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