Research Article

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\(L^p(\cdot) - L^q(\cdot)\) boundedness of some integral operators obtained by extrapolation techniques

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Abstract: Given a matrix \(A\) such that \(A^M = I\) and \(0 \leq \alpha < n\), for an exponent \(p\) satisfying \(p(Ax) = p(x)\) for a.e. \(x \in \mathbb{R}^n\), using extrapolation techniques, we obtain \(L^p(\cdot) \rightarrow L^q(\cdot)\) boundedness, \(\frac{1}{q(\cdot)} = \frac{1}{p(\cdot)} - \frac{\alpha}{n}\), and weak type estimates for integral operators of the form

\[T_Af(x) = \int \frac{f(y)}{|x - A_1y|^{\alpha_1} \cdots |x - A_my|^{\alpha_m}} dy,\]

where \(A_1, \ldots, A_m\) are different powers of \(A\) such that \(A_i - A_j\) is invertible for \(i \neq j\), \(\alpha_1 + \cdots + \alpha_m = n - \alpha\). We give some generalizations of these results.

Keywords: Variable exponents, fractional integrals

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1 Introduction

Given a measurable set \(\Omega \subset \mathbb{R}^n\) and a measurable function \(p(\cdot) : \Omega \rightarrow (1, \infty)\), let \(L^{p(\cdot)}(\Omega)\) denote the Banach space of measurable functions \(f\) on \(\Omega\) such that for some \(\lambda > 0\),

\[\int_\Omega \left(\frac{|f(x)|}{\lambda}\right)^{p(x)} dx < \infty,\]

with norm

\[\|f\|_{p(\cdot)} = \inf \left\{ \lambda > 0 : \int_\Omega \left(\frac{|f(x)|}{\lambda}\right)^{p(x)} dx \leq 1 \right\}.\]

These spaces are known as variable exponent spaces and are generalizations of the classical Lebesgue spaces \(L^p(\Omega)\). In the last years many authors have extended the machinery of classical harmonic analysis to these spaces, see [1, 2, 4]. The first step was to determine sufficient conditions on \(p(\cdot)\) for the boundedness on \(L^{p(\cdot)}(\Omega)\) of the Hardy–Littlewood maximal operator

\[M(f(x)) = \sup_B \frac{1}{|B|} \int_B |f(y)| dy,\]

where the supremum is taken over all balls \(B\) containing \(x\). Let \(p_- = \text{ess inf } p(x)\) and \(p_+ = \text{ess sup } p(x)\). In [2], Cruz-Uribe, Fiorenza and Neugebauer proved the following result.

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Theorem. Given an open set $\Omega \subset \mathbb{R}^n$, let $p(\cdot) : \Omega \to [1, \infty)$ be such that $1 < p_\ast \leq p_+ < \infty$. Suppose further that $p(\cdot)$ satisfies

$$|p(x) - p(y)| \leq \frac{c}{-\log|x - y|}, \quad x, y \in \Omega, \ |x - y| < \frac{1}{2},$$

(1)

and

$$|p(x) - p(y)| \leq \frac{c}{\log(e + |x|)}, \quad x, y \in \Omega, \ |y| \geq |x|. \quad (2)$$

Then the Hardy–Littlewood maximal operator is bounded on $L^{p(\cdot)}(\Omega)$.

We recall that a weight $\omega$ is a locally integrable and non-negative function. The Muckenhoupt class $A_p$, $1 < p < \infty$, is defined as the class of weights $\omega$ such that

$$\sup_Q \left( \frac{1}{|Q|^p} \int_Q |\omega|^p \, dx \right)^{1/p} \leq C,$$

where $Q$ is a cube in $\mathbb{R}^n$. For $p = 1$, $A_1$ is the class of weights $\omega$ having the property that there exists $c > 0$ such that

$$M_\omega(x) \leq c \omega(x) \quad \text{for a.e. } x \in \mathbb{R}^n.$$

Theorem. Let $A$ be an invertible $n \times n$ matrix such that $A^M = I$, and also suppose that $M$ is such that if $A^M = I$ for some $N \in \mathbb{N}$, then $M \leq N$. Let $m \in \mathbb{N}$, $1 < m \leq M$. Let $0 \leq \alpha < n$. Let $a_1, \ldots, a_m$ be real numbers such that

$$a_1 + \cdots + a_m = n - \alpha.$$ 

Let $T_{a}$ be the integral operator given by

$$T_{a}f(x) = \int k(x, y) f(y) \, dy,$$

(3)

with

$$k(x, y) = \frac{1}{|x - A_i y|^{a_i} \cdots |x - A_m y|^{a_m}},$$

where, for $1 \leq i \leq m$, the matrices $A_i$ are certain power of $A$, $A_i = A^{k_i}$, $k_i \in \mathbb{N}$, $1 \leq k_i \leq M$.

In [6], Riveros and Urciuolo studied integral operators with kernels given by

$$k(x, y) = \frac{1}{|x - A_1 y|^{a_1} \cdots |x - A_m y|^{a_m}},$$

(4)

where $A_1, \ldots, A_m$ are invertible matrices such that $A_i - A_j$ is invertible for $i \neq j$, $1 \leq i, j \leq m$. They obtained weighted $(p, q)$ estimates, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$, for weights $w \in A(p, q)$ such that $w(A_i x) \leq c w(x)$. We want to use extrapolation techniques to obtain $p(\cdot) - q(\cdot)$ and weak type estimates. In [7], Rocha and Urciuolo proved the following theorem that involves more general matrices $A_i$, with the additional hypothesis $p(A_i x) = p(x)$ for a.e. $x \in \mathbb{R}^n$.

**Theorem (Strong type).** Let $0 \leq \alpha < n$ and let $T_{a}$ be the integral operator with kernel given by (4), with $A_i$ orthogonal matrices such that $A_i - A_j$ is invertible for $i \neq j$, $1 \leq i, j \leq m$. Let $h : \mathbb{R} \to [1, \infty)$ be such that $1 < h_\ast \leq h_\ast < \frac{n}{\alpha}$ and satisfying (1) and (2). Let $p : \mathbb{R}^n \to [1, \infty)$ given by $p(x) = h(|x|)$. Then $T_{a}$ is bounded from $L^{p(\cdot)}(\mathbb{R}^n)$ into $L^{q(\cdot)}(\mathbb{R}^n)$ for $\frac{1}{p(x)} - \frac{1}{q(x)} = \frac{\alpha}{n}$.
In this paper we prove a similar result using extrapolation techniques that allow us to replace the log-Hölder conditions about the exponent \( p(\cdot) \) by a more general hypothesis concerning the boundedness of the maximal function \( M \). We obtain the following result.

**Theorem 1.** Let \( T_\alpha \) be the integral operator given by (3) such that \( A_1 - A_j \) is invertible for \( i \neq j, 1 \leq i, j \leq m \). Let \( p: \mathbb{R}^n \to [1, \infty) \) be such that \( 1 < p_- \leq p_+ < \frac{n}{n-2} \) and \( p(Ax) = p(x) \) for a.e. \( x \in \mathbb{R}^n \). Let \( q(\cdot) \) be defined by \( \frac{1}{p(x)} - \frac{1}{q(x)} = \frac{n}{2} \). If the maximal operator \( M \) is bounded on \( L^{(\frac{np}{2}-q(\cdot))'}(\mathbb{R}^n) \), then \( T_\alpha \) is bounded from \( L^{p(\cdot)}(\mathbb{R}^n) \) into \( L^{q(\cdot)}(\mathbb{R}^n) \).

In [7], Rocha and Urciuolo obtained weak type estimates with the additional hypothesis \( p(0) = 1 \).

**Theorem (Weak type).** Let \( 0 \leq \alpha < n \), and let \( h: \mathbb{R} \to [1, \infty) \) be a function satisfying (1) and (2), with \( h(0) = 1 \) and \( h_+ < \infty \). Let \( p: \mathbb{R}^n \to [1, \infty) \) given by \( p(x) = h(|x|) \). Let \( T_\alpha \) be the integral operator with kernel given by (4), with \( A_1 \) orthogonal matrices such that \( A_1 - A_j \) is invertible for \( i \neq j, 1 \leq i, j \leq m \). If \( \frac{1}{p(x)} - \frac{1}{q(x)} = \frac{n}{2} \), then there exists \( C > 0 \) such that

\[
\sup_{\lambda > 0} \lambda \| |x|: T_\alpha f(x) : q(\cdot) \| = C \| f \|_{p(\cdot)}.
\]

We obtain a weak type estimate for the operator given by (3), without that additional hypothesis. Our result is the following.

**Theorem 2.** Let \( T_\alpha \) be the integral operator given by (3) such that \( A_1 - A_j \) is invertible for \( i \neq j, 1 \leq i, j \leq m \). Let \( p: \mathbb{R}^n \to [1, \infty) \) be such that \( 1 \leq p_- \leq p_+ < \frac{n}{n-2} \) and \( p(Ax) = p(x) \) a.e. \( x \in \mathbb{R}^n \). Let \( q(\cdot) \) be defined by \( \frac{1}{p(x)} - \frac{1}{q(x)} = \frac{n}{2} \). If the maximal operator \( M \) is bounded on \( L^{(\frac{np}{2}-q(\cdot))'}(\mathbb{R}^n) \), then there exists \( C > 0 \) such that

\[
\| T_\alpha f(x) : q(\cdot) \| = C \| f \|_{p(\cdot)}.
\]

We will also show that this technique applies in the case when each of the matrices \( A_j \) is either a power of an orthogonal matrix \( A \) or a power of \( A^{-1} \).

## 2 Proofs of the results

**Proof of Theorem 1.** We denote \( q_0 = \frac{np}{2} - q \). In [6], Riveros and Urciuolo obtained a weighted \((p_-, q_0)\) estimate for weights \( w \in A(p, q) \) such that \( w(Ax) \leq cw(x) \). We let \( \overline{q}(x) = \frac{q(x)}{q_0} \) and define

\[
\mathcal{R}h(x) = \sum_{k=0}^{\infty} \frac{M^k h(Ax)}{2^k \| M^\frac{k}{q(\cdot)} \|_{q(\cdot)'}} + \cdots + \sum_{k=0}^{\infty} \frac{M^k h(A^M x)}{2^k \| M^\frac{k}{q(\cdot)} \|_{q(\cdot)'}}.
\]

(5)

It is easy to check that

1. for all \( x \in \mathbb{R}^n \), \( |h(x)| \leq \mathcal{R}h(x) \),
2. \( \mathcal{R} \) is bounded on \( L^{\overline{q}(\cdot)'}(\mathbb{R}^n) \) and \( \| \mathcal{R}h \|_{q(\cdot)'} \leq 2 M \| h \|_{q(\cdot)'} \),
3. \( \mathcal{R} h \in A_1 \) and \( \| \mathcal{R} h \|_{A_1} \leq 2 CM \| h \|_{\overline{q}(\cdot)'} \),
4. \( \mathcal{R}h(Ax), x \in \mathbb{R}^n \).

Indeed, (1) is evident; (2) is verified as follows. Let \( l \in \mathbb{N}, l \leq M \). Then

\[
\| M^k h(A^{l'} \cdot) \|_{q(\cdot)'} = \inf_{r > 0} \left\{ \lambda > 0 : \left( \frac{M^k h(A^{l'} x)}{\lambda} \right)^{\overline{q}(\cdot)'} dx \leq 1 \right\}.
\]

But

\[
\int_{\mathbb{R}^n} \left( \frac{M^k h(A^{l'} x)}{\lambda} \right)^{\overline{q}(\cdot)'} dx = \int_{\mathbb{R}^n} \left( \frac{M^k h(y)}{\lambda} \right)^{\overline{q}(A^{-l'} y)'} dy = \int_{\mathbb{R}^n} \left( \frac{M^k h(y)}{\lambda} \right)^{\overline{q}(y)'} dy,
\]

where the first equality follows from a change of variables, using that \( |\det A| = 1 \). The second equality holds because \( q(A^{l'} x) = q(x) \) for a.e. \( x \in \mathbb{R}^n \). Then we conclude that

\[
\| M^k h(A^{l'} \cdot) \|_{q(\cdot)'} = \| M^k h \|_{\overline{q}(\cdot)'}.
\]
Thus, we obtain (2) by subadditivity of the norm:
\[ \|M h(A(\cdot))\|_{q} \leq \sum_{k=0}^{\infty} \left( \frac{M^{k} h(A(\cdot))}{2^{k} \|M\|_{q}} + \cdots \right) \leq 2 M \|h\|_{q}. \]

Now, it is easy to check that there exists \( C > 0 \) such that for \( f \in L_{\text{loc}}^{1}(\mathbb{R}^{n}) \), \( M(f \circ A)(x) \leq CM f(Ax) \). So, (3) follows as in [3, p. 157]:
\[ \mathcal{M}(\mathbb{R}h)(x) \leq C \left( \sum_{k=0}^{\infty} \frac{M^{k+1} h(Ax)}{2^{k} \|M\|_{q}} + \cdots \right) \leq 2 C \|\mathbb{R}h\|_{q}. \]

and (4) follows by definition. So, \( \mathbb{R}h \) is a weight in \( A_{1} \) such that \( \mathbb{R}h(A_{1}x) \leq \mathbb{R}h(x), \ x \in \mathbb{R}^{n} \).

We now take a bounded \( f \) with compact support. We will check later that \( \|T_{af}\|_{q} < \infty \), so, as in [3, Theorem 5.24],
\[ \|T_{af}\|_{q}^{q} = \|(T_{af})^{q}\|_{q} = c \sup_{1h_{q+1} = 1} \int (T_{af})^{q}(x) h(x) dx \]
\[ \leq c \sup_{1h_{q} = 1} \int (T_{af})^{q}(x) \mathbb{R}h(x) dx \]
\[ \leq c \sup_{1h_{q} = 1} \int |f(x)|^{p} \mathbb{R}h(x) dx, \]

since \( \mathbb{R}h \in A(p_{+}, q_{0}) \). Hölder’s inequality gives
\[ \|(T_{af})^{q}\|_{q} \leq c \left( \|f\|_{p}^{p} \sup_{1h_{q+1} = 1} \|\mathbb{R}h\|_{q} \right) \leq c \left( \|f\|_{p}^{q} \|\mathbb{R}h\|_{q} \right) \leq 2 M c \|\mathbb{R}h\|_{q} \|h\|_{q}. \]

where the last inequality follows as in [3, p. 211].

Now we show that \( \|T_{af}\|_{q} < \infty \). By [3, Proposition 2.12, p. 19], it is enough to check that \( \int_{\mathbb{R}^{n}} T_{af} < \infty \). We have
\[ |Tf(x)|^{q} \leq |Tf(x)|^{q} \cdot 1_{x:T_{af}(x) > 1} + |Tf(x)|^{q} \cdot 1_{x:T_{af}(x) \leq 1}, \]

and now \( f \) is bounded and with compact support, so \( T_{af} \in L^{q}(\mathbb{R}^{n}) \) for \( \frac{n}{p-s} < s < \infty \), thus \( \int_{\mathbb{R}^{n}} |Tf(x)|^{q} dx < \infty \).

The theorem follows since bounded functions with compact support are dense in \( L^{p}(\mathbb{R}^{n}) \).

Proof of Theorem 2. We consider first the case \( p_{+} = 1 \). We denote \( q_{0} = \frac{n}{n-a} \) and \( q_{1} = \frac{1}{q_{0}} \). Theorem 3.2 of [6] implies that if \( \omega \in A(1, q_{0}) \) is such that \( \omega(Ax) \leq c \omega(x) \), then
\[ \sup_{\lambda} \lambda^{q_{0}} \omega^{q_{0}}(x) \chi_{|f|^{q_{0}}(x)}(\lambda|x|^{q_{0}}(x)) \leq C \int |f(x)| \omega(x) dx^{q_{0}}. \]

Now, let \( F_{A} = \lambda^{q_{0}} \chi_{|f|^{q_{0}}(x)}(\lambda|x|^{q_{0}}(x)). \) Then
\[ \|\chi_{|f|^{q_{0}}(x)}(\lambda|x|^{q_{0}}(x))\|_{q_{0}} \leq \|\chi_{|f|^{q_{0}}(x)}(\lambda|x|^{q_{0}}(x))\|_{q_{0}} = C \sup_{1h_{q} = 1} \int F_{A}(x) h(x) dx. \]

As in the previous theorem, we define \( \mathbb{R}h \) by (5). Since \( \mathbb{R}h \in A_{1} \), \( \mathbb{R}h \in A(1, q_{0}) \). So,
\[ \|\chi_{|f|^{q_{0}}(x)}(\lambda|x|^{q_{0}}(x))\|_{q_{0}} \leq C \sup_{1h_{q} = 1} \int F_{A}(x) \mathbb{R}h(x) dx \]
\[ \leq C \sup_{1h_{q} = 1} \int F_{A}(x) \mathbb{R}h(x) \mathbb{R}h(x|\mathbb{R}h|^{q_{0}}) \]
and, as in the previous theorem, we get
\[
\|\lambda_{\{x: T_\alpha f(x) > \lambda\}}\|_{L^q_p} \leq C\|f\|_{L^q_p} \sup_{|h|_{l^{q'}} = 1} \|\mathcal{R}h(\cdot)\|_{L^q_p}^{\frac{1}{q'}} \leq C\|f\|_{L^q_p} \sup_{|h|_{l^{q'}} = 1} \|\mathcal{R}h(\cdot)\|_{L^q_p}^{\frac{1}{q'}} \leq 2M\|f\|_{L^q_p} \sup_{|h|_{l^{q'}} = 1} \|h(\cdot)\|_{L^q_p} = 2M\|f\|_{L^q_p}.
\]

If \( p_- > 1 \), then we use that \( T_\alpha \) is of weak type \((p_-, q_0)\) and we proceed as before to get the statement of the theorem.

**Remark 3.** Theorems 1 and 2 still hold if \( m = 1 \) and \( \alpha > 0 \). In this case, if \( \frac{1}{p} - \frac{1}{q} = \frac{\alpha}{m} \) and \( \omega \in A(p, q) \) is such that \( \omega(Ax) \leq \omega(x) \) for a.e. \( x \in \mathbb{R}^n \), then
\[
T_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x - Ay|^n} dy = I_\alpha(f \circ A^{-1})(x),
\]
where \( I_\alpha \) is the classical fractional integral operator. Thus,
\[
\int_{\mathbb{R}^n} (T_\alpha f(x))^q \omega^q(x) dx = \int_{\mathbb{R}^n} (I_\alpha(f \circ A^{-1})(x))^q \omega^q(x) dx \\
\leq C\left( \int_{\mathbb{R}^n} (f \circ A^{-1})(x)^p \omega^p(x) dx \right)^{\frac{q}{p}} \leq C\left( \int_{\mathbb{R}^n} (f(x))^p \omega^p(A^{-1}x) dx \right)^{\frac{q}{p}} \leq C\left( \int_{\mathbb{R}^n} (f(x))^p \omega^p(x) dx \right)^{\frac{q}{p}}.
\]
So,
\[
\|T_\alpha f\|_{L^{q, \omega^q}} \leq C\|f\|_{L^{q, \omega^p}},
\]
In a similar way we obtain the corresponding weak type estimate and we proceed as in the previous theorems.

**Remark 4.** Let \( A \) be a orthogonal matrix and let \( T_\alpha \) be as in (3), where the matrix \( A_j \) is either a power of \( A \) or a power of \( A^{-1} \). If \( A_j - A_j \) is invertible and \( p(\cdot) \) is as in Theorem 2, we also obtain strong and weak type estimates. We simply define \( \mathcal{R} \) as follows:
\[
\mathcal{R}h(x) = \sum_{j=0}^{\infty} \frac{1}{2^j} \left( \sum_{k=0}^{\infty} \frac{M^k h(A^j x)}{2^k\|M\|_{l^{q'}}} \right) + \sum_{k=0}^{\infty} \frac{M^k h((A^{-1})^j x)}{2^k\|M\|_{l^{q'}}},
\]
and the proof follows as in the proofs of Theorems 1 and 2.

**Example 5.** We take \( r \) satisfying (1) and (2), with \( 1 < r_- \leq r_+ < \frac{2}{\alpha} \) and
\[
A = \begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix}.
\]
So, \( A^4 = I \) and \( A^i - A^j \) is invertible for \( 1 \leq i, j \leq 4, i \neq j \). We let \( p(x) = \frac{1}{4}(r(Ax) + r(A^2 x) + r(A^3 x) + r(A^4 x)) \).

**Example 6.** We take an even function \( p \) satisfying (1) and (2), with \( 1 < p_- \leq p_+ < \frac{2}{\alpha} \) and \( A = -I \).

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