OPTIMAL LOWER BOUND FOR THE BLOW-UP RATE OF THE MAGNETIC ZAKHAROV SYSTEM WITHOUT THE SKIN EFFECT

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Abstract. We focus on the Cauchy problem of the magnetic Zakharov system in two-dimensional space:

\[
\begin{cases}
  iE_{1t} + \Delta E_1 - nE_1 + \eta E_2 (E_1 \overline{E_2} - \overline{E_1 E_2}) = 0, \\
  iE_{2t} + \Delta E_2 - nE_2 + \eta E_1 (E_1 \overline{E_2} - \overline{E_1 E_2}) = 0, \\
  n_t + \nabla \cdot \mathbf{v} = 0, \\
  \nu_t + \nabla n + \nabla (|E_1|^2 + |E_2|^2) = 0, \\
  (E_1, E_2, n, \mathbf{v})(0, x) = (E_{10}, E_{20}, n_0, \mathbf{v}_0)(x),
\end{cases}
\]

which describes the spontaneous generation of a magnetic field without the skin effect in a cold plasma, where \(\eta > 0\) is a physical constant coefficient. The two nonlinear terms \(E_2 (E_1 \overline{E_2} - \overline{E_1 E_2})\) and \(E_1 (E_1 \overline{E_2} - \overline{E_1 E_2})\) generated by the cold magnetic field bring in a different difficulty from that for the classical Zakharov system. Assuming the initial mass satisfies the following estimates:

\[
\frac{||Q||^2_{L^2(\mathbb{R}^2)}}{1 + \eta} < \frac{||E_{10}||^2_{L^2(\mathbb{R}^2)} + ||E_{20}||^2_{L^2(\mathbb{R}^2)}}{\eta},
\]

where \(Q\) is the unique radially positive solution of the equation \(-\Delta V + V = V^3\), we prove that there is a constant \(c > 0\) depending only on the initial data such that for \(t\) near \(T\) (the blow-up time),

\[
||E_1, E_2, n, \mathbf{v}||_{H^1(\mathbb{R}^2) \times H^1(\mathbb{R}^2) \times L^2(\mathbb{R}^2) \times L^2(\mathbb{R}^2)} \geq \frac{c}{T - t}.
\]

As the magnetic coefficient \(\eta\) tends to 0, the blow-up rate recovers the result for the classical 2-D Zakharov system due to Merle.\textsuperscript{16} For any size positive \(\eta\), under the current assumption on the initial mass, we give a mathematically rigorous justification for the fact that the presence of magnetic effects without the skin effect in the cold plasma does not change the optimal lower bound for the blow-up rates.

Keywords: Blow-up rate; Magnetic Zakharov system; Skin effect; Optimal lower bound

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1. Introduction and main results

The magnetic Zakharov system

\[
\begin{align*}
\dot{E}_t + \nabla \nabla \cdot E - nE - \alpha \nabla \times (\nabla \times E) + i(E \wedge B) &= 0, \\
\partial_t n &= -\nabla \cdot v, \\
\partial_t v &= -\nabla n - \nabla |E|^2, \\
\Delta B - i\eta \nabla \times \nabla \times (E \wedge E) + A &= 0,
\end{align*}
\]

is proposed to describe the spontaneous generation of a magnetic field in a cold plasma [12]. Here, \( E = (E_1, E_2, E_3) \in \mathbb{C}^3 \) denotes the slowly varying complex amplitudes of the high-frequency electric field, \( n \) the fluctuation of the electron density from its equilibrium, \( B \) the self-generated magnetic field in a cold plasma, \( A = \delta B, \delta \leq 0, \eta > 0 \) and \( \alpha \geq 1 \) are physical constants.

It is worth noting that the Zakharov system (I) is a Schrödinger-wave coupled system with different scalings, and it keeps two conservation laws including the total mass

\[
\|E\|^2_{L^2(\mathbb{R}^2)} = \|E_1\|^2_{L^2(\mathbb{R}^2)} + \|E_2\|^2_{L^2(\mathbb{R}^2)} + \|E_3\|^2_{L^2(\mathbb{R}^2)},
\]

as well as the total energy

\[
\mathcal{H} := \|\nabla \times E\|^2_{L^2(\mathbb{R}^2)} + \|\nabla \cdot E\|^2_{L^2(\mathbb{R}^2)} + \frac{1}{2} \|n\|^2_{L^2(\mathbb{R}^2)} + \frac{1}{2} \|\nabla n\|^2_{L^2(\mathbb{R}^2)} + \int_{\mathbb{R}^2} n |E|^2 \, dx + \frac{\eta}{2} \int_{\mathbb{R}^2} \frac{1}{|\xi|^2 - \delta} \left( |\xi \cdot \mathcal{F}(E \wedge \overline{E})|^2 - |\xi|^2 |\mathcal{F}(E \wedge \overline{E})|^2 \right) \, d\xi. \tag{III}
\]

In the cold plasma, the term \( \delta B \) corresponds to the classical (collisionless) skin effect [12]. From a physical viewpoint, \( \alpha \) is relative to the velocity of electrons and the plasma frequency. In the present paper, we focus on the case for \( \delta = 0 \) and \( \alpha = 1 \), that is, the skin effect would not be involved and the velocity of electrons increases synchronously with the frequency of plasma.

From a physical point of view, the two-dimensional case for \( E \) is essential and of great importance. Let

\[
E(t, x) = (E_1(t, x), E_2(t, x), 0), \quad x \in \mathbb{R}^2.
\]

Through standard computations, one obtains

\[
B(t, x) = (0, 0, B_3(t, x)) = (0, 0, -i\eta (E_1 E_2 - E_1 \overline{E}_2))
\]

by the fact that \( \nabla \cdot (E \wedge \overline{E}) = 0 \) and the vectorial identity \( \Delta E = \nabla (\nabla \cdot E) - \nabla \times \nabla \times E \). Hence, the interaction term involving the electronic and magnetic fields is given by

\[
iE \wedge B = \eta E \wedge E \wedge \overline{E} = \eta \left( E_2 (E_1 \overline{E}_2 - E_1 \overline{E}_2), E_1 (E_1 E_2 - E_1 \overline{E}_2), 0 \right),
\]
and (I) can be rewritten by the following two-dimensional magnetic Zakharov system:

\[
\begin{align*}
    i\partial_t E_1 + \Delta E_1 - nE_1 + \eta E_2 (E_1 \overline{E}_2 - \overline{E}_1 E_2) &= 0, \\
    i\partial_t E_2 + \Delta E_2 - nE_2 + \eta E_1 (\overline{E}_1 E_2 - E_1 \overline{E}_2) &= 0, \\
    \partial_t n + \nabla \cdot \mathbf{v} &= 0, \\
    \partial_t \mathbf{v} + \nabla n + \nabla \left( |E_1|^2 + |E_2|^2 \right) &= 0,
\end{align*}
\]

which describes the spontaneous generation of a magnetic field without the skin effect in a cold plasma \cite{12}. Here, \( \eta > 0 \) is a physical constant coefficient, \((E_1, E_2, n) : \mathbb{R}^+ \times \mathbb{R}^2 \to \mathbb{C}, \ n(t, x) : \mathbb{R}^+ \times \mathbb{R}^2 \to \mathbb{R}, \ \mathbf{v}(t, x) : \mathbb{R}^+ \times \mathbb{R}^2 \to \mathbb{R}^2 \). The initial data for (1.1) is given by

\[
\begin{align*}
    E_1(0, x) &= E_{10}(x), \quad E_2(0, x) = E_{20}(x), \\
    n(0, x) &= n_0(x), \quad \mathbf{v}(0, x) = \mathbf{v}_0(x).
\end{align*}
\]

Due to the identity (III), the Hamiltonian for (1.1) can be written as

\[
\mathcal{H}(E_1, E_2, n, \mathbf{v}) = \|\nabla E_1\|_{L^2}^2 + \|\nabla E_2\|_{L^2}^2 + \frac{1}{2} \|n\|_{L^2}^2 + \frac{1}{2} \|\mathbf{v}\|_{L^2}^2
\]

\[
+ \int_{\mathbb{R}^2} n \left( |E_1|^2 + |E_2|^2 \right) dx - \frac{\eta}{2} \int_{\mathbb{R}^2} |E_1 \overline{E}_2 - E_2 \overline{E}_1|^2 dx.
\]

Clearly it is a well-defined functional on the energy space \( \mathbb{H}_1 := H^1(\mathbb{R}^2) \times H^1(\mathbb{R}^2) \times L^2(\mathbb{R}^2) \times L^2(\mathbb{R}^2) \).

The blow-up dynamics of the two-dimensional classical Zakharov system have been studied in detail by several authors, in particular, Glangetas and Merle \cite{10, 11} and Merle \cite{13, 14}. For the Zakharov systems with magnetic field effect (I), Laurey in \cite{13} proved the global existence of weak solutions for small initial data as well as local existence and uniqueness of smooth solutions in both two-dimensional and three-dimensional spaces. Based on Laurey’s work \cite{13}, over the last decade, finite time blow-up dynamics for (I) were considered. Gan, Guo and Huang in \cite{7} constructed a family of blow-up solutions in two-dimensional space and proved the existence of self-similar blow-up solutions. The instability and the concentration property of a class of periodic solution were also obtained. In \cite{6}, the authors studied the Virial type blow-up solutions of the Cauchy problem for (I). Later, the authors in \cite{8} established the space-time integral estimate of the blow-up rate for the finite time blow-up solutions to (I) in the three dimensional space. Note that (I) without the classical (collisionless) skin effect (\( \delta = 0 \)) and \( \alpha = 1 \) in two dimensional space reduces to system (1.1), these results for the system (I) are naturally true for system (1.1).

Although for the Zakharov systems have an additional magnetic field, there have been some results on the well-posedness as well as some progress on the blowup dynamics. To our best knowledge, there is no estimate on the lower-bound of the finite time blow-up rate. The aim of this paper is to establish the (essentially optimal) lower bound of the blowup rate for the finite time blowup solutions to the system (1.1).

Let us state a few preliminary results. Firstly, using the methods used in
Proposition 1.1. The two dimensional magnetic Zakharov system \( (1.1) \) is locally well-posed in the energy space \( H^1(\mathbb{R}^2) \times H^1(\mathbb{R}^2) \times L^2(\mathbb{R}^2) \times L^2(\mathbb{R}^2) \). That is, there exists a unique solution \((E_1, E_2, n, v)\) satisfying
\[
\|(E_1, E_2, n, v)(t)\|_{H^1(\mathbb{R}^2) \times H^1(\mathbb{R}^2) \times L^2(\mathbb{R}^2) \times L^2(\mathbb{R}^2)} \leq C, \quad \forall t \in [0, T),
\]
where \( T \) is the maximal existence time of the solution and constant \( C \) depends only on the initial data \((E_{10}, E_{20}, n_0, v_0)\).

Next, following Gan, Guo, Han and Zhang [8], we can establish easily the virial-type blow-up solution for the Cauchy problem \( (1.1)-(1.2) \).

Proposition 1.2. Let \( \eta > 0 \). Suppose that for all time, the solutions \((E_1, E_2, n, v)(t)\) to the Cauchy problem \( (1.1)-(1.2) \) in \( \mathbb{R}^2 \) are radially symmetric functions and \( \mathcal{H}(E_{10}, E_{20}, n_0, v_0) < 0 \). The following alternative holds:

(i) \((E_1, E_2, n, v)(t)\) blows up in finite time,

(ii) \((E_1, E_2, n, v)(t)\) blows up in the energy space \( H^1(\mathbb{R}^2) \times H^1(\mathbb{R}^2) \times L^2(\mathbb{R}^2) \) at infinity. That is, \((E_1, E_2, n, v)(t)\) is defined for all \( t \) and
\[
\lim_{t \to +\infty} \|(E_1, E_2, n, v)(t)\|_{H^1(\mathbb{R}^2) \times H^1(\mathbb{R}^2) \times L^2(\mathbb{R}^2) \times L^2(\mathbb{R}^2)} = +\infty.
\]

With these results, it is natural to consider more quantitative descriptions on the behaviour of \( \|(E_1, E_2, n, v)(t)\|_{H^1(\mathbb{R}^2) \times H^1(\mathbb{R}^2) \times L^2(\mathbb{R}^2) \times L^2(\mathbb{R}^2)} \) as \( t \) near \( T \), where \( T < \infty \) is the blow-up time. Compared with the classical Zakharov system, \( (1.1) \) contains two extra cubic coupling terms. The two extra terms \( E_2(E_1 E_2 - E_1 E_2) \) and \( E_1(E_1 E_2 - E_1 E_2) \) which are physically generated by the cold magnetic field (without the skin effect), do bring an a different difficulty from that for the classical Zakharov system. Assuming the initial mass satisfies the following estimates:
\[
\frac{\|Q\|_{L^2(\mathbb{R}^2)}^2}{1 + \eta} < \|E_{10}\|_{L^2(\mathbb{R}^2)}^2 + \|E_{20}\|_{L^2(\mathbb{R}^2)}^2 < \frac{\|Q\|_{L^2(\mathbb{R}^2)}^2}{\eta},
\]
where \( Q \) is the unique radial positive solution of the equation \(-\Delta V + V = V^3\), we prove in the present paper that there is a constant \( c > 0 \) depending only on the initial data such that for \( t \) near \( T \) (the blow-up time),
\[
\|(E_1, E_2, n, v)(t)\|_{H^1(\mathbb{R}^2) \times H^1(\mathbb{R}^2) \times L^2(\mathbb{R}^2) \times L^2(\mathbb{R}^2)} \geq \frac{c}{T - t}.
\]

The main result of the paper can be described by the following:

Theorem 1.3. Let \((E_1, E_2, n, v)(t)\) be the finite time blow-up solution of the Cauchy problem \( (1.1)-(1.2) \), and \( T < \infty \) be the blow-up time. Suppose that the initial data \((E_{10}, E_{20})\) satisfies
\[
\frac{\|Q\|_{L^2(\mathbb{R}^2)}^2}{1 + \eta} < \|E_{10}\|_{L^2(\mathbb{R}^2)}^2 + \|E_{20}\|_{L^2(\mathbb{R}^2)}^2 < \frac{\|Q\|_{L^2(\mathbb{R}^2)}^2}{\eta}, \quad (1.4)
\]
where \( Q \) is the unique radial positive solution of
\[
-\Delta V + V = V^3,
\]
then there exist constants \( c > 0, \tilde{c} > 0 \) depending only on initial data, such that as \( t \) near \( T \),
\[
\| (E_1, E_2, n, \psi) (t) \|_{H^1(\mathbb{R}^2) \times H^1(\mathbb{R}^2) \times L^2(\mathbb{R}^2) \times L^2(\mathbb{R}^2)} \geq \frac{c}{T - t},
\]
\[
\left( \| \nabla E_1(t) \|_{L^2(\mathbb{R}^2)}^2 + \| \nabla E_2(t) \|_{L^2(\mathbb{R}^2)}^2 \right)^{\frac{1}{2}} \geq \frac{\tilde{c}}{T - t},
\]
\[
\| n(t) \|_{L^2(\mathbb{R}^2)} \geq \frac{\tilde{c}}{T - t}.
\]

More precisely,
\[
\left( \| \nabla E_1(t) \|_{L^2(\mathbb{R}^2)}^2 + \| \nabla E_2(t) \|_{L^2(\mathbb{R}^2)}^2 \right)^{\frac{1}{2}} \geq \frac{\tilde{c}}{\left( \| E_{10} \|_{L^2(\mathbb{R}^2)}^2 + \| E_{20} \|_{L^2(\mathbb{R}^2)}^2 - \frac{\| Q \|_{L^2(\mathbb{R}^2)}^2}{1 + \eta} \right)^{\frac{1}{2}} (T - t)^{-\frac{1}{2}}}
\]
\[
\| n(t) \|_{L^2(\mathbb{R}^2)} \geq \frac{\tilde{c}}{\left( \| E_{10} \|_{L^2(\mathbb{R}^2)}^2 + \| E_{20} \|_{L^2(\mathbb{R}^2)}^2 - \frac{\| Q \|_{L^2(\mathbb{R}^2)}^2}{1 + \eta} \right)^{\frac{1}{2}} (T - t)^{-\frac{1}{2}}}.
\]

The assumption on the lower bound for the initial mass is natural particularly when \( \eta \) is small as it is basically the minimal mass needed for blow-up to occur. Indeed, as the magnetic coefficient \( \eta \) tends to 0, the blow-up rate recovers the result for the classical 2-D Zakharov system due to Merle [16]. On the other hand, for any size positive \( \eta \), the assumption on the upper bound is not known to be optimal nor to be necessary, but when \( \eta \) is sufficiently small this assumption is automatic for a giving finite mass initial data. For large \( \eta \) our result is in some sense more intriguing. It provides a mass band in which one can get more precise information for blow-ups. Whether or not such a statement can be generalized to more general (likely multiple bubble blow-ups) mass levels remains to be a fascinating open problem.

Remark 1.4. In [7], Gan, Guo and Huang constructed a family of blow-up solutions of the Cauchy problem (1.1)-(1.2):
\[
\begin{align*}
E_1(t, x) &= \frac{\omega}{T - t} e^{i \left( \theta + \frac{|x|^2}{4(T - t)} - \frac{2}{T - t} \right) \rho \left( \frac{T - t}{T - t} \right) \sqrt{2}}, \\
E_2(t, x) &= -i E_1(t, x), \\
n(t, x) &= \frac{\omega^2}{(T - t)^2} \tilde{N} \left( \frac{x | \omega |}{T - t} \right),
\end{align*}
\]
where \( \hat{P}(x) = \hat{P}(|x|), \hat{N}(x) = \hat{N}(|x|) \) are radial functions on \( \mathbb{R}^2 \), \( \theta \in \mathbb{R} \) and \( \omega > 0 \). Let \( \tilde{P} = \frac{P}{(1+\eta)^2}, \tilde{N} = \frac{N}{1+\eta} \), then \( (P, N) \) satisfies
\[
\begin{cases}
\Delta P - P + \frac{\eta}{\eta+1} P^3 = \frac{1}{\eta+1} NP, \\
\frac{1}{\omega^2}(r^2 N_{rr} + 6r N_r + 6N) - \Delta N = \Delta |P|^2.
\end{cases}
\]
(1.12)

Direct calculation yields
\[
\begin{align*}
\| \nabla E_1(t) \|_{L^2(\mathbb{R}^2)} &= \frac{\omega t}{T-t} \| \nabla \tilde{P} \|_{L^2(\mathbb{R}^2)}, \\
\| \nabla E_2(t) \|_{L^2(\mathbb{R}^2)} &= \frac{\omega t}{T-t} \| \nabla \tilde{P} \|_{L^2(\mathbb{R}^2)}, \\
\| n(t) \|_{L^2(\mathbb{R}^2)} &= \frac{\omega t}{T-t} \| N \|_{L^2(\mathbb{R}^2)}, \\
\| v(t) \|_{L^2(\mathbb{R}^2)} &= \frac{\omega t}{T-t}.
\end{align*}
\]
(1.13)

These estimates imply that the lower bound estimates of blow-up rate (1.6), (1.7) and (1.8) in Theorem 1.3 are optimal. On the other hand, letting \( \omega \to +\infty \) and \( (P, N) \to (Q, -Q^2) \), it yields that (1.9) and (1.10) in Theorem 1.3 are also optimal.

We note that the Zakharov system (1.1) is a Hamiltonian system which leads to conservations of the total mass and total energy:
\[
\| E_1 \|_{L^2(\mathbb{R}^2)}^2 + \| E_2 \|_{L^2(\mathbb{R}^2)}^2 = \| E_{10} \|_{L^2(\mathbb{R}^2)}^2 + \| E_{20} \|_{L^2(\mathbb{R}^2)}^2,
\]
(1.14)

\[
\mathcal{H}(E_1, E_2, n, v) = \mathcal{H}(E_{10}, E_{20}, n_0, v_0) = \mathcal{H}_0,
\]
(1.15)

where \( \mathcal{H}(E_1, E_2, n, v) \) is defined in (1.3).

In contrast to the Zakharov system without the magnetic field effect, the presence of extra nonlinear terms \( E_2 (E_1 E_2 - \overline{E_1 E_2}) \) and \( E_1 (E_1 E_2 - \overline{E_1 E_2}) \) (generated by the magnetic field) in (1.1), make the study of the lower-bound estimate for blow-up rates of finite time blow-up solution (to the Cauchy problem (1.1)-(1.2)) more difficult and complicated. Motivated by Merle’s beautiful arguments given in \([10]\), in particular, its geometrical estimate, the non-vanishing estimate and the compactness argument, we need to establish additional a priori estimates for the extra nonlinear terms. For this we also need some different techniques from those adopted in \([10]\). In particular, to obtain the optimal lower bound of blow-up rate, the initial mass is required to satisfy (1.4) so that we can establish corresponding needed a priori estimates involving the higher order nonlinear terms. As we have mentioned earlier, the condition (1.4) is natural from both the physical and mathematical point of views. Indeed, by Lemma 2.2 in Section 2, it is standard to conclude that the mild (no blow up) solution of (1.11) is globally well-defined if the initial data \( (E_{10}, E_{20}) \) satisfies
\[
\| E_{10} \|_{L^2(\mathbb{R}^2)}^2 + \| E_{20} \|_{L^2(\mathbb{R}^2)}^2 < \frac{\| Q \|_{L^2(\mathbb{R}^2)}^2}{1+\eta}.
\]
In [7] the authors also pointed out that there is no mass-concentration at a finite time provided
\[ ||E_{10}||_{L^2(\mathbb{R}^2)}^2 + ||E_{20}||_{L^2(\mathbb{R}^2)}^2 = \frac{||Q||_{L^2(\mathbb{R}^2)}^2}{1 + \eta}. \]

We note that condition (1.4) is also consistent with the blow-up dynamics of the classical Zakharov system [11] when \( \eta \to 0 \). On the other hand, it is a fascinating issue to investigate some quantitative properties of the blow-up dynamics of (1.1)-(1.2) when \( ||E_{10}||_{L^2(\mathbb{R}^2)}^2 + ||E_{20}||_{L^2(\mathbb{R}^2)}^2 = \frac{||Q||_{L^2(\mathbb{R}^2)}^2}{1 + \eta} \). For example, it may involve multiple bubbles blow-ups etc, but we have to leave it for a future study.

The rest of paper is organized as follows. In Section 2, we collect preparatory materials, mainly on some lemmas and propositions which are crucial to the proof of Theorem 1.3. Section 3 is devoted to establishing some properties of the rescaled Zakharov system. In section 4 we prove the optimal lower-bound of finite time blow-up rate (Theorem 1.3).

2. Preliminaries

In this section, we give some notations and basic estimates. First we recall a lemma in [4, 5].

\textbf{Lemma 2.1.} Let \( \Omega \) be a smooth bounded domain in \( \mathbb{R}^n \) with \( n \geq 2 \) and \( \frac{1}{p} + \frac{1}{q} = 1 \). If \( f_k \rightharpoonup f \) in \( L^p(\Omega) \), and \( g_k \to g \) in \( L^q(\Omega) \) as \( k \to +\infty \), then
\[ \int_{\Omega} f_k g_k dx \to \int_{\Omega} f g dx \text{ as } k \to +\infty. \]

\[ \square \]

\textbf{Lemma 2.2.} (Weinstein [17]) For all \( u \in H^1(\mathbb{R}^2) \),
\[ \frac{1}{2} ||u||_{L^4(\mathbb{R}^2)}^4 \leq \left( \frac{||u||_{L^2(\mathbb{R}^2)}^2}{||Q||_{L^2(\mathbb{R}^2)}^2} \right) \|\nabla u\|_{L^2(\mathbb{R}^2)}^2, \]
where \( Q \) is the unique positive solution of the following equation
\[ -\Delta V + V = V^3. \]

\[ \square \]

Since (1.1)-(1.2) and (1.1)-(1.4) have the same scaling on spatial structure but they are of different space-time structures, it will conserve (1.1)-(1.2) and (1.1)-(1.4) respectively. Taking a suitable space-time scaling to the Zakharov system (1.1) can yield a re-scaled system.
Proposition 2.3. Let \((E_1, E_2, n, v)\) be the finite-time blow-up solutions to the Zakharov system (1.1) and \(T\) be its blow-up time. For any \(t \in [0, T)\), let

\[
\begin{align*}
\tilde{E}_1(t,s,x) &= \frac{1}{\lambda(t) E_1} \left( t + \frac{s}{\lambda(t)}, \frac{x}{\lambda(t)} \right), \\
\tilde{E}_2(t,s,x) &= \frac{1}{\lambda(t) E_2} \left( t + \frac{s}{\lambda(t)}, \frac{x}{\lambda(t)} \right), \\
\tilde{n}(t,s,x) &= \frac{1}{\lambda(t)} n \left( t + \frac{s}{\lambda(t)}, \frac{x}{\lambda(t)} \right), \\
\tilde{v}(t,s,x) &= \frac{1}{\lambda(t)} v \left( t + \frac{s}{\lambda(t)}, \frac{x}{\lambda(t)} \right),
\end{align*}
\]

where \(s \in [0, \lambda(t)(T-t)]\),

\[
\begin{align*}
\lambda^2(t) &= \|(E_1,E_2,n,v)\|_{H^1(\mathbb{R}^2) \times H^1(\mathbb{R}^2) \times L^2(\mathbb{R}^2) \times L^2(\mathbb{R}^2)}^2 \\
&= \int_{\mathbb{R}^2} |\nabla E_1(t,x)|^2 dx + \int_{\mathbb{R}^2} |\nabla E_2(t,x)|^2 dx \\
&+ \frac{1}{2} \int_{\mathbb{R}^2} |n(t,x)|^2 dx + \frac{1}{2} \int_{\mathbb{R}^2} |v(t,x)|^2 dx.
\end{align*}
\]

Then \((\tilde{E}_1, \tilde{E}_2, \tilde{n}, \tilde{v})(s)\) satisfies the following re-scaled Zakharov system:

\[
\begin{align*}
\frac{1}{\lambda(t)} \tilde{E}_1 + \Delta \tilde{E}_1 - \tilde{n} \tilde{E}_1 + \eta \tilde{E}_2 \left( \tilde{E}_1 \tilde{E}_2 - \overline{\tilde{E}_1 \tilde{E}_2} \right) &= 0, \\
\frac{1}{\lambda(t)} \tilde{E}_2 + \Delta \tilde{E}_2 - \tilde{n} \tilde{E}_2 + \eta \tilde{E}_1 \left( \tilde{E}_1 \tilde{E}_2 - \overline{\tilde{E}_1 \tilde{E}_2} \right) &= 0, \\
\tilde{n}_s + \nabla \cdot \tilde{v} &= 0, \\
\tilde{v}_s + \nabla \left( \tilde{n} + |\tilde{E}_1|^2 + |\tilde{E}_2|^2 \right) &= 0.
\end{align*}
\]

In addition, there hold

\[
\begin{align*}
\lim_{s \to \lambda(t)(T-t)} \|(\tilde{E}_1, \tilde{E}_2, \tilde{n}, \tilde{v})(s)\|_{H^1(\mathbb{R}^2) \times H^1(\mathbb{R}^2) \times L^2(\mathbb{R}^2) \times L^2(\mathbb{R}^2)}^2 \\
= \lim_{s \to \lambda(t)(T-t)} \left( \|\nabla \tilde{E}_1(s)\|_{L^2(\mathbb{R}^2)}^2 + \|\nabla \tilde{E}_2(s)\|_{L^2(\mathbb{R}^2)}^2 \\
+ \frac{1}{2} \|\tilde{n}(s)\|_{L^2(\mathbb{R}^2)}^2 + \frac{1}{2} \|\tilde{v}(s)\|_{L^2(\mathbb{R}^2)}^2 \right) \\
= +\infty,
\end{align*}
\]

**(1)**

\[
\int_{\mathbb{R}^2} \left( |\nabla \tilde{E}_1(t,0,x)|^2 + |\nabla \tilde{E}_2(t,0,x)|^2 + \frac{1}{2} |\tilde{n}(t,0,x)|^2 + \frac{1}{2} |\tilde{v}(t,0,x)|^2 \right) dx = 1,
\]

**(2)**
\[ (3) \]
\[ \| \tilde{E}_1(t, s, x) \|_{L^2(\mathbb{R}^2)}^2 + \| \tilde{E}_2(t, s, x) \|_{L^2(\mathbb{R}^2)}^2 \]
\[ = \| \tilde{E}_1(t, 0, x) \|_{L^2(\mathbb{R}^2)}^2 + \| \tilde{E}_2(t, 0, x) \|_{L^2(\mathbb{R}^2)}^2 \]
\[ = \| E_{10} \|_{L^2(\mathbb{R}^2)}^2 + \| E_{20} \|_{L^2(\mathbb{R}^2)}^2, \]

and

\[ (4) \]
\[ \mathcal{H}(\tilde{E}_1, \tilde{E}_2, \tilde{n}, \tilde{v}) \]
\[ = \| \nabla \tilde{E}_1 \|_{L^2(\mathbb{R}^2)}^2 + \| \nabla \tilde{E}_2 \|_{L^2(\mathbb{R}^2)}^2 + \frac{1}{4} \| \tilde{n} \|_{L^2(\mathbb{R}^2)}^2 + \frac{1}{4} \| \tilde{v} \|_{L^2(\mathbb{R}^2)}^2 \]
\[ + \int_{\mathbb{R}^2} \tilde{n} \left( |\tilde{E}_1|^2 + |\tilde{E}_2|^2 \right) dx - \eta \int_{\mathbb{R}^2} |\tilde{E}_1|^2 |\tilde{E}_2|^2 dx \]
\[ + \frac{\eta}{2} \int_{\mathbb{R}^2} \left( (\tilde{E}_1)^2 (\tilde{E}_2)^2 + (\bar{\tilde{E}}_1)^2 (\bar{\tilde{E}}_2)^2 \right) dx \]
\[ = \frac{1}{\lambda^2(t)} \mathcal{H}(E_1, E_2, n, v) \]
\[ = \frac{1}{\lambda^2(t)} \mathcal{H}(E_{10}, E_{20}, n_0, v_0). \]

**Proof.** According to (2.1), direct calculation yields

\[ \begin{align*}
\hat{E}_{1s} &= \frac{1}{\lambda(t)} E_{1t}, \quad \Delta \hat{E}_1 = \frac{1}{\lambda(t)} \Delta E_1, \quad \nabla |\hat{E}_1|^2 = \frac{1}{\lambda(t)} \nabla |E_1|^2, \\
\hat{E}_{2s} &= \frac{1}{\lambda(t)} E_{2t}, \quad \Delta \hat{E}_2 = \frac{1}{\lambda(t)} \Delta E_2, \quad \nabla |\hat{E}_2|^2 = \frac{1}{\lambda(t)} \nabla |E_2|^2, \\
\hat{n}_s &= \frac{1}{\lambda(t)} n_t, \quad \nabla \hat{n} = \frac{1}{\lambda(t)} \nabla n, \quad \nabla \cdot \hat{v} = \frac{1}{\lambda(t)} \nabla \cdot v, \quad \hat{v}_s = \frac{1}{\lambda(t)} v_t.
\end{align*} \]
Taking (2.8) into the Zakharov system (1.1) yields the re-scaled Zakharov system (2.3). Similarly, using (2.1) one obtains

$$\left\| \left( \hat{E}_1, \tilde{E}_2, \hat{n}, \tilde{v} \right) \right\|_{H^1(\mathbb{R}^2) \times H^1(\mathbb{R}^2) \times L^2(\mathbb{R}^2) \times L^2(\mathbb{R}^2)}^2$$

$$= \left\| \nabla \hat{E}_1(t, s, x) \right\|_{L^2(\mathbb{R}^2)}^2 + \left\| \nabla \tilde{E}_2(t, s, x) \right\|_{L^2(\mathbb{R}^2)}^2$$

$$+ \frac{1}{2} \left\| \tilde{n}(t, s, x) \right\|_{L^2(\mathbb{R}^2)}^2 + \frac{1}{2} \left\| \tilde{v}(t, s, x) \right\|_{L^2(\mathbb{R}^2)}^2$$

$$= \left\| \frac{1}{\lambda^2(t)} \nabla E_1 \right\|_{L^2(\mathbb{R}^2)}^2 + \left\| \frac{1}{\lambda^2(t)} \nabla E_2 \right\|_{L^2(\mathbb{R}^2)}^2$$

$$+ \frac{1}{2} \left\| \frac{1}{\lambda^2(t)} \eta \right\|_{L^2(\mathbb{R}^2)}^2 + \frac{1}{2} \left\| \frac{1}{\lambda^2(t)} \tilde{v} \right\|_{L^2(\mathbb{R}^2)}^2$$

$$= \frac{1}{\lambda^2(t)} \int_{\mathbb{R}^2} \left( \left| \nabla E_1 \right|^2 + \left| \nabla E_2 \right|^2 + \frac{1}{2} \left| \eta \right|^2 + \frac{1}{2} \left| \tilde{v} \right|^2 \right)$$

$$\left( t + \frac{s}{\lambda(t)}, \frac{x}{\lambda(t)} \right) d \left( \frac{x}{\lambda(t)} \right).$$

Noting that $t + \frac{s}{\lambda(t)} \to T$ as $s \to \lambda(t)(T - t)$, one has

$$\lim_{s \to \lambda(t)(T - t)} \int_{\mathbb{R}^2} \left( \left| \nabla E_1 \right|^2 + \left| \nabla E_2 \right|^2 + \frac{1}{2} \left| \eta \right|^2 + \frac{1}{2} \left| \tilde{v} \right|^2 \right)$$

$$\left( t + \frac{s}{\lambda(t)}, \frac{x}{\lambda(t)} \right) d \left( \frac{x}{\lambda(t)} \right) = +\infty.$$ 

That is,

$$\lim_{s \to \lambda(t)(T - t)} \left\| \left( \hat{E}_1, \tilde{E}_2, \hat{n}, \tilde{v} \right) \right\|_{H^1(\mathbb{R}^2) \times H^1(\mathbb{R}^2) \times L^2(\mathbb{R}^2) \times L^2(\mathbb{R}^2)}^2 = +\infty,$$

this is the estimate (2.4).

Taking the inner product of (2.3a) with $\overline{E_1}$ and of (2.3b) with $\overline{E_2}$, integrating with respect to spatial variable $x$, then taking the imaginary part of the resulting equations yield

$$Im \int_{\mathbb{R}^2} \left[ \frac{i}{\lambda} \tilde{E}_{1s} \cdot \overline{E_1} + \Delta \overline{E_1} \cdot \overline{E_1} - \hat{n} \overline{E_1} \cdot \overline{E_1} + \eta \overline{E_1} \overline{E_2} \left( \overline{E_1} \overline{E_2} - \overline{E_1} \overline{E_2} \right) \right] dx$$

$$= Re \int_{\mathbb{R}^2} \frac{1}{\lambda} \tilde{E}_{1s} \cdot \overline{E_1} dx - Im \int_{\mathbb{R}^2} \eta \left( \overline{E_1} \right)^2 \left( \tilde{E}_2 \right)^2 dx$$

$$= \frac{1}{2\lambda} \frac{d}{ds} \int_{\mathbb{R}^2} \left| \tilde{E}_1 \right|^2 dx - Im \int_{\mathbb{R}^2} \eta \left( \overline{E_1} \right)^2 \left( \tilde{E}_2 \right)^2 dx$$

$$= 0.$$

That is,

$$\frac{1}{2\lambda} \frac{d}{ds} \int_{\mathbb{R}^2} \left| \tilde{E}_1 \right|^2 dx - Im \int_{\mathbb{R}^2} \eta \left( \overline{E_1} \right)^2 \left( \tilde{E}_2 \right)^2 dx = 0.$$
Similar discussion gives
\[ \frac{1}{2\lambda} \frac{d}{ds} \int_{\mathbb{R}^2} |\tilde{E}_2|^2 \, dx - Im \int_{\mathbb{R}^2} \eta \left( \tilde{E}_1 \right)^2 \left( \overline{\tilde{E}_2} \right)^2 \, dx = 0. \]  
(2.14)

(2.13) and (2.14) yield
\[ \frac{d}{ds} \int_{\mathbb{R}^2} \left( |\tilde{E}_1|^2 + |\tilde{E}_2|^2 \right) \, dx = 0. \]  
(2.15)

Here we use the conclusion: \( \forall (f, g) \in \mathbb{C}^2, \; Im \left( f^2 \overline{g}^2 + \overline{f} \overline{g} \right) = 0. \) Hence one gets
\[ \left\| \tilde{E}_1(t, s, x) \right\|^2_{L^2(\mathbb{R}^2)} + \left\| \tilde{E}_2(t, s, x) \right\|^2_{L^2(\mathbb{R}^2)} = \left\| \tilde{E}_1(t, 0, x) \right\|^2_{L^2(\mathbb{R}^2)} + \left\| \tilde{E}_2(t, 0, x) \right\|^2_{L^2(\mathbb{R}^2)}. \]  
(2.16)

In addition, from the mass conservation (1.14), it follows that
\[ \left\| \tilde{E}_1(t, 0, x) \right\|^2_{L^2(\mathbb{R}^2)} + \left\| \tilde{E}_2(t, 0, x) \right\|^2_{L^2(\mathbb{R}^2)} = \int_{\mathbb{R}^2} \left( |\tilde{E}_1(t, 0, x)|^2 + |\tilde{E}_2(t, 0, x)|^2 \right) \, dx \]  
(2.17)

Noting that
\[ \lambda^2(t) = \int_{\mathbb{R}^2} |\nabla E_1|^2 \, dx + \int_{\mathbb{R}^2} |\nabla E_2|^2 \, dx + \frac{1}{2} \int_{\mathbb{R}^2} |n|^2 \, dx + \frac{1}{2} \int_{\mathbb{R}^2} |\nabla n|^2 \, dx, \]  
(2.18)

one gets
\[ \int_{\mathbb{R}^2} \left( |\nabla \tilde{E}_1(t, 0, x)|^2 + |\nabla \tilde{E}_2(t, 0, x)|^2 + \frac{1}{2} |\tilde{n}(t, 0, x)|^2 + \frac{1}{2} |\nabla \tilde{n}(t, 0, x)|^2 \right) \, dx \]  
\[ = \frac{1}{\lambda^2(t)} \int_{\mathbb{R}^2} \left( |\nabla E_1(t, x)|^2 + |\nabla E_2(t, x)|^2 + \frac{1}{2} |n(t, x)|^2 + \frac{1}{2} |\nabla n(t, x)|^2 \right) \, dx \]  
\[ = 1. \]  
(2.19)

The above arguments imply (2.5) and (2.6).

Finally, taking the inner product of (2.3a) with \( \overline{\tilde{E}_1} \) and of (2.3b) with \( \overline{\tilde{E}_2} \), integrating with respect to spatial variable \( x \), then taking the real part of the result equations, we have
\[ Re \int_{\mathbb{R}^2} \left[ \frac{i}{\lambda} \tilde{E}_1 \cdot \overline{\tilde{E}_1} + \Delta \tilde{E}_1 \cdot \overline{\tilde{E}_1} - \tilde{n} \tilde{E}_1 \cdot \overline{\tilde{E}_1} + \eta \tilde{E}_1 \overline{\tilde{E}_2} \left( \tilde{E}_1 \overline{\tilde{E}_2} - \bar{\tilde{E}_1} \bar{\tilde{E}_2} \right) \right] \, dx = 0, \]
that is,
\[
\text{Re} \int_{\mathbb{R}^2} \left[ i \tilde{E}_{1s} \cdot \tilde{E}_{2s} + \Delta \tilde{E}_2 \cdot \tilde{E}_{2s} - \tilde{n} \tilde{E}_2 \cdot \tilde{E}_{2s} + \eta \tilde{E}_{2s} \tilde{E}_1 \left( \tilde{E}_1 \tilde{E}_2 - \tilde{E}_1 \tilde{E}_2 \right) \right] dx = 0,
\]

namely,
\[
-\frac{1}{2} \frac{d}{ds} \int_{\mathbb{R}^2} |\nabla \tilde{E}_2|^2 dx - \frac{1}{2} \int_{\mathbb{R}^2} \tilde{n} \cdot \frac{d}{ds} |\tilde{E}_2|^2 dx
+ \frac{\eta}{2} \int_{\mathbb{R}^2} |\tilde{E}_1|^2 \frac{d}{ds} |\tilde{E}_2|^2 dx - \frac{\eta}{2} \text{Re} \int_{\mathbb{R}^2} (\tilde{E}_1)^2 \frac{d}{ds} (\tilde{E}_2)^2 dx = 0. \tag{2.21}
\]

Next, taking the inner product of (2.3c) with \( \tilde{n} \) yields
\[
\int_{\mathbb{R}^2} (\tilde{n} \cdot \tilde{v} + (\nabla \cdot \tilde{v}) \tilde{n}) dx = \frac{1}{2} \frac{d}{ds} \int_{\mathbb{R}^2} |\tilde{n}|^2 dx - \int_{\mathbb{R}^2} \tilde{v} \cdot \nabla \tilde{n} dx = 0. \tag{2.22}
\]

On the other hand, taking the inner product of (2.3d) with \( \tilde{v} \) implies
\[
\int_{\mathbb{R}^2} \left[ \tilde{v} \cdot \nabla \left( \tilde{n} + |\tilde{E}_1|^2 + |\tilde{E}_2|^2 \right) \right] dx = \frac{1}{2} \frac{d}{ds} \int_{\mathbb{R}^2} |\tilde{v}|^2 dx - \int_{\mathbb{R}^2} \tilde{n} \nabla \cdot \tilde{v} dx
+ \int_{\mathbb{R}^2} \tilde{n} \left( |\tilde{E}_1|^2 + |\tilde{E}_2|^2 \right) dx = 0. \tag{2.23}
\]

Combining (2.21) with (2.22) and (2.23) gives
\[
\frac{d}{ds} \int_{\mathbb{R}^2} \left( |\nabla \tilde{E}_1|^2 + |\nabla \tilde{E}_2|^2 + \frac{1}{2} |\tilde{n}|^2 + \frac{1}{2} |\tilde{v}|^2 + \tilde{n} \left( |\tilde{E}_1|^2 + |\tilde{E}_2|^2 \right) \right) dx
- \frac{d}{ds} \int_{\mathbb{R}^2} \left[ \eta \left| \tilde{E}_1 \right|^2 \left| \tilde{E}_2 \right|^2 + \frac{\eta}{2} \left( |\tilde{E}_1|^2 \left| \tilde{E}_2 \right|^2 + |\tilde{E}_1|^2 \left| \tilde{E}_2 \right|^2 \right) \right] dx = 0,
\]

which implies
\[
\mathcal{H} \left( \tilde{E}_1, \tilde{E}_2, \tilde{n}, \tilde{v} \right) (t, s, x) = \mathcal{H} \left( \tilde{E}_1, \tilde{E}_2, \tilde{n}, \tilde{v} \right) (t, 0, x). \tag{2.24}
\]
According to conservation of Hamiltonian (1.15), we have
\[
\mathcal{H} \left( \tilde{E}_1, \tilde{E}_2, \tilde{n}, \tilde{\nu} \right) (t, 0, x)
\]
\[
= \left\| \nabla \tilde{E}_1 (t, 0, x) \right\|^2_{L^2(\mathbb{R}^2)} + \left\| \nabla \tilde{E}_2 (t, 0, x) \right\|^2_{L^2(\mathbb{R}^2)}
+ \frac{1}{2} \left| \tilde{n}(t, 0, x) \right|^2_{L^2(\mathbb{R}^2)} + \frac{1}{2} \left| \tilde{\nu}(t, 0, x) \right|^2_{L^2(\mathbb{R}^2)}
+ \int_{\mathbb{R}^2} \tilde{n}(t, 0, x) \left( \left| \tilde{E}_1 (t, 0, x) \right|^2 + \left| \tilde{E}_2 (t, 0, x) \right|^2 \right) \, dx
- \eta \int_{\mathbb{R}^2} \left| \tilde{E}_1 (t, 0, x) \right|^2 \left| \tilde{E}_2 (t, 0, x) \right|^2 \, dx
+ \frac{\eta}{2} \int_{\mathbb{R}^2} \left( \left( \tilde{E}_1 (t, 0, x) \right)^2 \left( \tilde{E}_2 (t, 0, x) \right)^2 \right) \, dx
\]
\[
= \left\| \frac{1}{\lambda^2(t)} \nabla E_1 (t) \right\|^2_{L^2(\mathbb{R}^2)} + \left\| \frac{1}{\lambda^2(t)} \nabla E_2 (t) \right\|^2_{L^2(\mathbb{R}^2)}
+ \frac{1}{2} \left\| \frac{1}{\lambda^2(t)} n(t) \right\|^2_{L^2(\mathbb{R}^2)} + \frac{1}{2} \left\| \frac{1}{\lambda^2(t)} \nu(t) \right\|^2_{L^2(\mathbb{R}^2)}
+ \int_{\mathbb{R}^2} \frac{1}{\lambda^4(t)} |n(t)| \left( |E_1 (t)|^2 + |E_2 (t)|^2 \right) \, dx
- \eta \int_{\mathbb{R}^2} \frac{1}{\lambda^4(t)} |E_1 (t)|^2 |E_2 (t)|^2 \, dx
+ \frac{\eta}{2} \int_{\mathbb{R}^2} \frac{1}{\lambda^4(t)} \left( |E_1 (t)|^2 \left| E_2 (t) \right|^2 + \left| E_1 (t) \right|^2 |E_2 (t)|^2 \right) \, dx
\]
\[
= \frac{1}{\lambda^2(t)} \mathcal{H} (E_1, E_2, n, \nu) (t)
= \frac{1}{\lambda^2(t)} \mathcal{H} (E_{10}, E_{20}, n_0, \nu_0),
\]
which is just the estimate (2.7). This finishes the proof of Proposition 2.3.

\[
\square
\]

3. Estimates for the rescaled Zakharov system (2.3)

In this section, we first establish some a priori estimates for the rescaled Zakharov system (2.3) in order to gain the optimal lower bound for the blow-up rate of the finite time blow-up solution to the Zakharov system (1.1). For simplicity, we denote \( \left( \tilde{E}_1, \tilde{E}_2, \tilde{n}, \tilde{\nu} \right) (t, s, x) \) by \( \left( \tilde{E}_1, \tilde{E}_2, \tilde{n}, \tilde{\nu} \right) (s) \).

We now claim the following conclusion concerning some a priori estimates for the solution \( \left( \tilde{E}_1, \tilde{E}_2, \tilde{n}, \tilde{\nu} \right) (s) \) to the re-scaled Zakharov system (2.3).
Theorem 3.1. (A priori estimates on \( \tilde{E}_1, \tilde{E}_2, \tilde{n}, \tilde{v} \) \( ) \)

Let \( (E_1, E_2, n, v)(t) \) be a solution of the Zakharov system (1.1), \( (\tilde{E}_1, \tilde{E}_2, \tilde{n}, \tilde{v})(s) \) be a solution of the rescaled Zakharov system (2.3) and \( T \) be the blow-up time. Suppose that the initial mass \( (E_{10}, E_{20}) \) satisfies

\[
\frac{\|Q\|^2_{L^2(\mathbb{R}^2)}}{1 + \eta} < \|E_{10}\|^2_{L^2(\mathbb{R}^2)} + \|E_{20}\|^2_{L^2(\mathbb{R}^2)} < \frac{\|Q\|^2_{L^2(\mathbb{R}^2)}}{\eta},
\]

where \( Q \) is the unique radial positive solution to the equation

\[-\Delta V + V = V^3,\]

then there are constants \( \theta_0 > 0 \), and \( A > 0 \) depending only on the initial data such that

\[\forall s \in [0, \theta_0), \left\| \left( \tilde{E}_1, \tilde{E}_2, \tilde{n}, \tilde{v} \right)(s) \right\|_{H^1} \leq A \text{ for } t \text{ near } T.\]

Furthermore, we can choose \( \theta_0 = \tilde{c} \left( \|E_{10}\|^2_{L^2(\mathbb{R}^2)} + \|E_{20}\|^2_{L^2(\mathbb{R}^2)} - \frac{\|Q\|^2_{L^2(\mathbb{R}^2)}}{1 + \eta} \right)^{-\frac{1}{2}}.\]

Remark 3.2. Theorem 3.1 is a crucial ingredient to show the main result (Theorem 1.3), whereas condition (3.1) being a key point for gaining the optimal lower bound on the blow-up rate.

 Ahead of proving Theorem 3.1, we first establish the geometrical estimates on the solution \( \left( \tilde{E}_1, \tilde{E}_2, \tilde{n}, \tilde{v} \right) \) to the re-scaled Zakharov system (2.3). These estimates concern Sobolev type estimates for \( \left( \tilde{E}_1, \tilde{E}_2, \tilde{n}, \tilde{v} \right)(0) \), nonvanishing properties of \( \left( \tilde{E}_1, \tilde{E}_2, \tilde{n} \right)(0) \) and compactness properties of \( \left( \tilde{E}_1, \tilde{E}_2, \tilde{n} \right)(0) \). We shall consider them in four portions:

\( \diamond \) 3.1 Sobolev Estimates on \( \left( \tilde{E}_1(0), \tilde{E}_2(0), \tilde{n}(0), \tilde{v}(0) \right) \) for \( t \) near \( T \);

\( \diamond \) 3.2 Non-vanishing properties of the solutions to the re-scaled Zakharov system as \( t \) near \( T \);

\( \diamond \) 3.3 Compactness of the solution to the re-scaled Zakharov system (2.3);

\( \diamond \) 3.4 Proof of Theorem 3.1.

3.1. Sobolev Estimates on \( \left( \tilde{E}_1(0), \tilde{E}_2(0), \tilde{n}(0), \tilde{v}(0) \right) \) for \( t \) near \( T \).

The Sobolev estimates on \( \left( \tilde{E}_1(0), \tilde{E}_2(0), \tilde{n}(0), \tilde{v}(0) \right) \) is given as follow.

Proposition 3.3. Let \( (E_1(t), E_2(t), n(t), v(t)) \) be the finite time blow-up solution to the Cauchy problem (1.1)-(1.2) on \( t \in [0, T] \), and \( T \) be the blow-up time. Suppose that the initial mass satisfies

\[
\frac{1}{1 + \eta} \|Q\|^2_{L^2(\mathbb{R}^2)} < \|E_{10}\|^2_{L^2(\mathbb{R}^2)} + \|E_{20}\|^2_{L^2(\mathbb{R}^2)} < \frac{1}{\eta} \|Q\|^2_{L^2(\mathbb{R}^2)},
\]

then there are constants \( \delta_1 > 0 \), \( c_1 > 0 \), \( c_2 > 0 \) and \( c_3 > 0 \) depending only on \( (E_{10}, E_{20}, n_0, v_0) \), such that for \( t \in [T - \delta_1, T] \), the solution \( \left( \tilde{E}_1, \tilde{E}_2, \tilde{n}, \tilde{v} \right)(s) \) to the re-scaled Zakharov system (2.3) admits
\[
0 < c_1 \leq \left( \left\| \nabla \tilde{E}_1(0) \right\|^2_{L^2(\mathbb{R}^2)} + \left\| \nabla \tilde{E}_2(0) \right\|^2_{L^2(\mathbb{R}^2)} \right)^{\frac{1}{2}} \leq c_2,
\]
(3.5)
\[
0 < c_1 \leq \| \tilde{n}(0) \|_{L^2(\mathbb{R}^2)} \leq c_3,
\]
(3.6)
\[
0 \leq \| \tilde{v}(0) \|_{L^2(\mathbb{R}^2)} \leq c_3.
\]
(3.7)

**Proof.** From (2.5) it follows that
\[
\left\{ \begin{aligned}
&\left( \left\| \nabla \tilde{E}_1(0) \right\|^2_{L^2(\mathbb{R}^2)} + \left\| \nabla \tilde{E}_2(0) \right\|^2_{L^2(\mathbb{R}^2)} \right)^{\frac{1}{2}} \leq 1, \\
&\| \tilde{n}(0) \|_{L^2(\mathbb{R}^2)} \leq \sqrt{2}, \quad \| \tilde{v}(0) \|_{L^2(\mathbb{R}^2)} \leq \sqrt{2}.
\end{aligned} \right.
\]
(3.8)
Note that \( \lambda(t) \rightarrow +\infty \) as \( t \rightarrow T \), by (2.7), there exists \( \delta_1 > 0 \) such that for any \( t \in [T - \delta_1, T) \),
\[
\left\| \mathcal{H} \left( \tilde{E}_1(0), \tilde{E}_2(0), \tilde{n}(0), \tilde{v}(0) \right) \right\| = \left\| \frac{H_0}{\lambda^2(t)} \right\| \leq \frac{1}{64}.
\]
(3.9)
Hence (2.5) and (2.7) yield
\[
1 = \int_{\mathbb{R}^2} |\nabla \tilde{E}_1(0)|^2 dx + \int_{\mathbb{R}^2} |\nabla \tilde{E}_2(0)|^2 dx \\
+ \frac{1}{2} \int_{\mathbb{R}^2} |\tilde{n}(0)|^2 dx + \frac{1}{2} \int_{\mathbb{R}^2} |\tilde{v}(0)|^2 dx \\
\leq \frac{1}{64} - \int_{\mathbb{R}^2} \tilde{n}(0) \left( |\tilde{E}_1(0)|^2 + |\tilde{E}_2(0)|^2 \right) dx \\
+ \frac{\eta}{2} \int_{\mathbb{R}^2} |\tilde{E}_1(0)\tilde{E}_2(0) - \tilde{E}_1(0)\bar{\tilde{E}}_2(0)|^2 dx.
\]
(3.10)
Since \( \tilde{E}_1 \tilde{E}_2 \) and \( \tilde{E}_1 \bar{\tilde{E}}_2 \) are conjugate complex-valued functions, we claim the following estimate for the quadric term \( \int_{\mathbb{R}^2} |\tilde{E}_1(0)\tilde{E}_2(0) - \tilde{E}_1(0)\bar{\tilde{E}}_2(0)|^2 dx \):
\[
\frac{1}{2} \int_{\mathbb{R}^2} |\tilde{E}_1(0)\tilde{E}_2(0) - \tilde{E}_1(0)\bar{\tilde{E}}_2(0)|^2 dx \\
\leq 2 \int_{\mathbb{R}^2} |\tilde{E}_1(0)|^2 |\tilde{E}_2(0)|^2 dx \\
\leq 2 \int_{\mathbb{R}^2} \left( \frac{|\tilde{E}_1(0)|^2 + |\tilde{E}_2(0)|^2}{2} \right)^2 dx \\
= \frac{1}{2} \int_{\mathbb{R}^2} (|\tilde{E}_1(0)|^2 + |\tilde{E}_2(0)|^2)^2 dx.
\]
(3.11)
On the other hand, it follows from the Hölder inequality that
\[
\int_{\mathbb{R}^2} -\tilde{n}(0) \left( |\tilde{E}_1(0)|^2 + |\tilde{E}_2(0)|^2 \right) dx \\
\leq \left( b^2 \int_{\mathbb{R}^2} |\tilde{n}(0)|^2 dx \right)^{\frac{1}{2}} \left( \frac{1}{b^2} \int_{\mathbb{R}^2} \left( |\tilde{E}_1(0)|^2 + |\tilde{E}_2(0)|^2 \right)^2 dx \right)^{\frac{1}{2}} \\
\leq \frac{b^2}{2} \int_{\mathbb{R}^2} |\tilde{n}(0)|^2 dx + \frac{1}{2b^2} \int_{\mathbb{R}^2} \left( |\tilde{E}_1(0)|^2 + |\tilde{E}_2(0)|^2 \right)^2 dx.
\] (3.12)

Let \( b^2 = \frac{1}{2} \). Combining (3.10) with (3.11) and (3.12) yields
\[
\frac{3}{4} \leq \frac{1}{4} \int_{\mathbb{R}^2} |\tilde{n}(0)|^2 dx + \left( \frac{\eta}{2} + 1 \right) \int_{\mathbb{R}^2} \left( |\tilde{E}_1(0)|^2 + |\tilde{E}_2(0)|^2 \right)^2 dx.
\] (3.13)

Due to \( \|\tilde{n}(0)\|_{L^2(\mathbb{R}^2)}^2 \leq 2 \), (3.13) implies
\[
\int_{\mathbb{R}^2} \left( |\tilde{E}_1(0)|^2 + |\tilde{E}_2(0)|^2 \right)^2 dx > \frac{1}{4 + 2\eta}.
\] (3.14)

Using the Gagliardo-Nirenberg inequality (Lemma 2.2), one gets the estimate for
\[
\int_{\mathbb{R}^2} \left( |\tilde{E}_1(0)|^2 + |\tilde{E}_2(0)|^2 \right)^2 dx
\]
\[
= \int_{\mathbb{R}^2} \left( |\tilde{E}_1(0)|^4 + |\tilde{E}_2(0)|^4 \right) dx + 2 \int_{\mathbb{R}^2} |\tilde{E}_1(0)|^2 |\tilde{E}_2(0)|^2 dx
\]
\[
\leq \frac{2\|\tilde{E}_1(0)\|_{L^2(\mathbb{R}^2)}^2}{\|Q\|_{L^2(\mathbb{R}^2)}^2} \int_{\mathbb{R}^2} |\nabla \tilde{E}_1(0)|^2 dx + \frac{2\|\tilde{E}_2(0)\|_{L^2(\mathbb{R}^2)}^2}{\|Q\|_{L^2(\mathbb{R}^2)}^2} \int_{\mathbb{R}^2} |\nabla \tilde{E}_2(0)|^2 dx
\]
\[
+ 2 \left( \int_{\mathbb{R}^2} |\tilde{E}_1(0)|^4 dx \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^2} |\tilde{E}_2(0)|^4 dx \right)^{\frac{1}{2}}
\]
\[
\leq \frac{2\|\tilde{E}_1(0)\|_{L^2(\mathbb{R}^2)}^2}{\|Q\|_{L^2(\mathbb{R}^2)}^2} \int_{\mathbb{R}^2} |\nabla \tilde{E}_1(0)|^2 dx + \frac{2\|\tilde{E}_2(0)\|_{L^2(\mathbb{R}^2)}^2}{\|Q\|_{L^2(\mathbb{R}^2)}^2} \int_{\mathbb{R}^2} |\nabla \tilde{E}_2(0)|^2 dx
\]
\[
+ 4 \frac{\|\tilde{E}_1(0)\|_{L^2(\mathbb{R}^2)} \|\tilde{E}_2(0)\|_{L^2(\mathbb{R}^2)}}{\|Q\|_{L^2(\mathbb{R}^2)}^2} \|\nabla \tilde{E}_1(0)\|_{L^2(\mathbb{R}^2)} \|\nabla \tilde{E}_2(0)\|_{L^2(\mathbb{R}^2)}
\]
\[
\leq \frac{2}{\|Q\|_{L^2(\mathbb{R}^2)}^2}
\]
\[
\cdot \left( \|\tilde{E}_1(0)\|_{L^2(\mathbb{R}^2)} \|\nabla \tilde{E}_1(0)\|_{L^2(\mathbb{R}^2)}^2 + \|\tilde{E}_2(0)\|_{L^2(\mathbb{R}^2)} \|\nabla \tilde{E}_1(0)\|_{L^2(\mathbb{R}^2)}^2 \\
+ \|\tilde{E}_1(0)\|_{L^2(\mathbb{R}^2)} \|\nabla \tilde{E}_2(0)\|_{L^2(\mathbb{R}^2)}^2 + \|\tilde{E}_2(0)\|_{L^2(\mathbb{R}^2)} \|\nabla \tilde{E}_2(0)\|_{L^2(\mathbb{R}^2)}^2 \right)
\]
\[
= \frac{2}{\|Q\|_{L^2(\mathbb{R}^2)}^2} \left( \|\tilde{E}_1(0)\|_{L^2(\mathbb{R}^2)}^2 + \|\tilde{E}_2(0)\|_{L^2(\mathbb{R}^2)}^2 \right)
\]
\[
\cdot \left( \|\nabla \tilde{E}_1(0)\|_{L^2(\mathbb{R}^2)}^2 + \|\nabla \tilde{E}_2(0)\|_{L^2(\mathbb{R}^2)}^2 \right).
This together with (3.14) yields
\[
\frac{1}{8 + 4\eta} \leq \frac{\|\tilde{E}_1(0)\|^2_{L^2(\mathbb{R}^2)} + \|\tilde{E}_2(0)\|^2_{L^2(\mathbb{R}^2)}}{\|Q\|^2_{L^2(\mathbb{R}^2)}} \cdot \left( \|\nabla \tilde{E}_1(0)\|^2_{L^2(\mathbb{R}^2)} + \|\nabla \tilde{E}_2(0)\|^2_{L^2(\mathbb{R}^2)} \right),
\]
and the conclusion (3.5) follows from (2.5) and the mass identity (2.6).

In the following we prove the conclusions (3.6) and (3.7). In view of condition (3.4), we can assume that there exists a sufficiently small \( \delta_0 \) with \( 0 < \delta_0 < \frac{1}{1+\eta} \) such that
\[
\|E_1(0)\|^2_{L^2(\mathbb{R}^2)} + \|E_2(0)\|^2_{L^2(\mathbb{R}^2)} < \frac{1 - \delta_0}{\eta} \|Q\|^2_{L^2(\mathbb{R}^2)}.
\]
Then from (3.8) it follows that
\[
\frac{\eta}{2} \int_{\mathbb{R}^2} \left| \tilde{E}_1(0) \tilde{E}_2(0) - \tilde{E}_1(0) \tilde{E}_2(0) \right|^2 dx \\
\leq 2\eta \int_{\mathbb{R}^2} |\tilde{E}_1(0)|^2 |\tilde{E}_2(0)|^2 dx \\
\leq 2\eta \left( \int_{\mathbb{R}^2} |\tilde{E}_1(0)|^4 dx \right)^{1/2} \left( \int_{\mathbb{R}^2} |\tilde{E}_2(0)|^4 dx \right)^{1/2} \\
\leq 4\eta \|\tilde{E}_1(0)\|_{L^2(\mathbb{R}^2)} \|\tilde{E}_2(0)\|_{L^2(\mathbb{R}^2)} \|\nabla \tilde{E}_1(0)\|_{L^2(\mathbb{R}^2)} \|\nabla \tilde{E}_2(0)\|_{L^2(\mathbb{R}^2)} \\
\leq \eta \left( \|\nabla \tilde{E}_1(0)\|^2_{L^2(\mathbb{R}^2)} + \|\nabla \tilde{E}_2(0)\|^2_{L^2(\mathbb{R}^2)} \right) \\
\cdot \left( \|\tilde{E}_1(0)\|^2_{L^2(\mathbb{R}^2)} + \|\tilde{E}_2(0)\|^2_{L^2(\mathbb{R}^2)} \right) \\
\leq 1 - \delta_0.
\]
Combining (3.18) with (2.2), (2.5) and (2.7) yields
\[
\delta_0 \leq - \int_{\mathbb{R}^2} \tilde{n}(0) \left( |\tilde{E}_1(0)|^2 + |\tilde{E}_2(0)|^2 \right) dx \\
\leq \left( \int_{\mathbb{R}^2} \tilde{n}^2(0) dx \right)^{1/2} \left( \int_{\mathbb{R}^2} \left( |\tilde{E}_1(0)|^2 + |\tilde{E}_2(0)|^2 \right)^2 dx \right)^{1/2}.\]
Recalling (3.15), (3.17) yields
\[ \int_{\mathbb{R}^2} \left( |\tilde{E}_1(0)|^2 + |\tilde{E}_2(0)|^2 \right)^2 \, dx \leq 2 \left( \|\nabla \tilde{E}_1(0)\|_{L^2(\mathbb{R}^2)}^2 + \|\nabla \tilde{E}_2(0)\|_{L^2(\mathbb{R}^2)}^2 \right) \times \left( \|\tilde{E}_1(0)\|_{L^2(\mathbb{R}^2)}^2 + \|\tilde{E}_2(0)\|_{L^2(\mathbb{R}^2)}^2 \right) \] \quad \text{(3.20)}

Note that (3.19), for any fixed small \( \delta_0 > 0 \), there exists a constant \( c_1 > 0 \) such that
\[ c_1 \leq \frac{\eta}{2} \frac{\delta_0^2}{1 - \delta_0} \leq \int_{\mathbb{R}^2} |\tilde{n}(0)|^2 \, dx. \quad \text{(3.21)} \]

Note that for \( 0 < \delta < \frac{2}{1 + \frac{1}{1 - \delta_0}} \) and \( \eta > 0 \), \( \frac{\eta^2}{2 + \frac{2}{1 - \delta}} < 2 \), the upper bounds for \( \|\nabla \tilde{E}_1(0)\|_{L^2(\mathbb{R}^2)} + \|\nabla \tilde{E}_2(0)\|_{L^2(\mathbb{R}^2)} \), \( \|\tilde{n}(0)\|_{L^2(\mathbb{R}^2)} \) and \( \|\tilde{v}(0)\|_{L^2(\mathbb{R}^2)} \) follow from (2.5). The proof of Proposition 3.3 is completed.

Due to the condition (2.5) for the re-scaled Zakharov system (2.3), using Proposition 3.3 and the following scaling properties:
\[ \|\nabla \tilde{E}_j(0)\|_{L^2(\mathbb{R}^2)} = \frac{1}{\lambda(t)} \|\nabla E_j(t)\|_{L^2(\mathbb{R}^2)}, \quad j = 1, 2, \]
\[ \|\tilde{n}(0)\|_{L^2(\mathbb{R}^2)} = \frac{1}{\lambda(t)} \|n(t)\|_{L^2(\mathbb{R}^2)}, \quad \|\tilde{v}(0)\|_{L^2(\mathbb{R}^2)} = \frac{1}{\lambda(t)} \|v(t)\|_{L^2(\mathbb{R}^2)}, \quad \text{(3.22)} \]
we claim the following Sobolev-type estimates for the solution \((E_1(t), E_2(t), n(t), v(t))\) to the Zakharov system (1.1).

**Corollary 3.4.** Under the assumptions in Proposition 3.3, there exists constants \( \delta_1 > 0 \), \( c_1^1 \), \( c_1^2 \) depending only on initial data (1.2) such that for \( t \in [T - \delta_1, T) \), there hold:
\[ c_1^1 \|n(t)\|_{L^2(\mathbb{R}^2)} \leq \left( \|\nabla E_1(t)\|_{L^2(\mathbb{R}^2)}^2 + \|\nabla E_2(t)\|_{L^2(\mathbb{R}^2)}^2 \right)^{\frac{1}{2}} \leq \frac{1}{c_1^1} \|n(t)\|_{L^2(\mathbb{R}^2)}, \quad \text{(3.23)} \]
\[ \|v(t)\|_{L^2(\mathbb{R}^2)} \leq \frac{1}{c_1^2} \|n(t)\|_{L^2(\mathbb{R}^2)}, \quad \text{(3.24)} \]
\[ c_2^2 \|n(t)\|_{L^2(\mathbb{R}^2)} \leq \|(E_1, E_2, n, v)(t)\|_{H^1(\mathbb{R}^2) \times L^2(\mathbb{R}^2) \times L^2(\mathbb{R}^2)} \leq \frac{1}{c_2^2} \|n(t)\|_{L^2(\mathbb{R}^2)}. \quad \text{(3.25)} \]

**Proof.** From Proposition 3.3 it follows that \( c_1 < c_2 \). Taking \( c_1^1 = \frac{c_1}{c_1^2} \) and \( c_1^2 = \sqrt{\frac{2\delta_1}{4\gamma^2 + c_1^1}} \) yields the conclusion of Corollary 3.4. \( \square \)
3.2. Non-vanishing Properties of the Solutions to the Re-scaled Zakharov System as $t$ near $T$.

We now consider the non-vanishing properties of the solution $\left(\tilde{E}_1(s), \tilde{E}_2(s), \tilde{n}(s)\right)$ of the rescaled magnetic Zakharov system (2.3) for $s = 0$, i.e., the non-vanishing properties of $\left(\tilde{E}_1(0), \tilde{E}_2(0), \tilde{n}(0)\right)$.

**Proposition 3.5.** For any $t \in [0, T)$, suppose that $(E_1(t), E_2(t), n(t), v(t))$ and $\left(\tilde{E}_1(0), \tilde{E}_2(0), \tilde{n}(0), \tilde{v}(0)\right)$ is the finite time blow-up solution to the Cauchy problem (1.1)-(1.2), the initial data $(E_{10}(x), E_{20}(x))$ satisfy condition (3.4) and $T$ be the blow-up time. Then we claim:

1. There exist constants $R_1 > 0$ and $\beta_1 > 0$ depending only on $\|E_{10}\|_{L^2(\mathbb{R}^2)}$ and $\|E_{20}\|_{L^2(\mathbb{R}^2)}$ such that for a sequence $x(t) \in \mathbb{R}^2$ one has

\[
\liminf_{t \to T} \left(\|\tilde{E}_1(0, x)\|^2_{L^2(|x-x(t)| \leq R_1)} + \|\tilde{E}_2(0, x)\|^2_{L^2(|x-x(t)| \leq R_1)}\right)^{\frac{1}{2}} \geq \beta_1, \tag{3.26}
\]

\[
\liminf_{t \to T} \|\tilde{n}(0, x)\|_{L^2(|x-x(t)| \leq R_1)} \geq \beta_1. \tag{3.27}
\]

2. Let $\left(\tilde{E}_{1n}, \tilde{E}_{2n}, \tilde{n}_n\right)(s)$ be a sequence satisfying the estimates as follows:

\[
\left\|\tilde{E}_{1n}(0)\right\|_{L^2(\mathbb{R}^2)}^2 + \left\|\tilde{E}_{2n}(0)\right\|_{L^2(\mathbb{R}^2)}^2 \leq \|E_{10}\|_{L^2(\mathbb{R}^2)}^2 + \|E_{20}\|_{L^2(\mathbb{R}^2)}^2, \tag{3.28}
\]

\[
c_1 \leq \int_{\mathbb{R}^2} \left|\nabla \tilde{E}_{1n}(0)\right|^2 dx + \int_{\mathbb{R}^2} \left|\nabla \tilde{E}_{2n}(0)\right|^2 dx \leq c_2, \tag{3.29}
\]

\[
c_1 \leq \int_{\mathbb{R}^2} |\tilde{n}_n(0)|^2 dx \leq c_2, \tag{3.30}
\]

\[
\limsup_{t \to T} \mathcal{H}\left(\tilde{E}_{1n}(0), \tilde{E}_{2n}(0), \tilde{n}_n(0), 0\right) \leq 0. \tag{3.31}
\]

Then there exist $\beta_1 > 0$ and $R_1 > 0$ depending only on $\|E_{10}\|_{L^2(\mathbb{R}^2)}$, $\|E_{20}\|_{L^2(\mathbb{R}^2)}$, $c_1 > 0$ and $c_2 > 0$ such that for a sequence $x_n \in \mathbb{R}^2$,

\[
\lim_{n \to +\infty} \left(\left\|\tilde{E}_{1n}\right\|_{L^2(|x-x_n| \leq R_1)}^2 + \left\|\tilde{E}_{2n}\right\|_{L^2(|x-x_n| \leq R_1)}^2\right)^{\frac{1}{2}} \geq \beta_1 > 0, \tag{3.32}
\]

\[
\lim_{n \to +\infty} \|\tilde{n}_n\|_{L^2(|x-x_n| \leq R_1)} \geq \beta_1 > 0. \tag{3.33}
\]

The proof of this proposition is similar to Proposition 3.6 in [16]. However, it is much more complicated here since the higher-order magnetic field terms are essentially involved. Proposition 3.5 will be proven step by step later.

We first claim:
Proposition 3.6. Assume there is \( m_k = m_k \left( \| E_{1k} \|_{L^2(\mathbb{R}^2)}, \| E_{2k} \|_{L^2(\mathbb{R}^2)} \right) > 0 \) such that the sequences \( \left( E_{1k}, E_{2k}, n_k, \nu_k \right) \in H^1(\mathbb{R}^2) \times H^1(\mathbb{R}^2) \times L^2(\mathbb{R}^2) \times L^2(\mathbb{R}^2) \) satisfy
\[
\| E_{1k} \|^2_{L^2(\mathbb{R}^2)} + \| E_{2k} \|^2_{L^2(\mathbb{R}^2)} = \| E_{10} \|^2_{L^2(\mathbb{R}^2)} + \| E_{20} \|^2_{L^2(\mathbb{R}^2)} > 0,
\]
and there exist constants \( R_0 > 0 \) and \( \delta'_0 > 0 \) such that
\[
\sup_{y \in \mathbb{R}^2} \int_{|x-y| < R_0} \left( |E_{1k}(x)|^2 + |E_{2k}(x)|^2 \right) dx \leq \frac{\|Q\|^2_{L^2(\mathbb{R}^2)}}{1 + \eta} - \delta'_0,
\]
or
\[
\sup_{y \in \mathbb{R}^2} \int_{|x-y| < R_0} |n_k(x)| dx \leq m_k - \delta'_0.
\]
Then there exist constants \( C_1 > 0, C_2 > 0 \) such that
\[
-C_1 + C_2 \int_{\mathbb{R}^2} \left( \| \nabla E_{1k} \|^2 + \| \nabla E_{2k} \|^2 + \frac{1}{2} n_k^2 + \frac{1}{2} \nu_k^2 \right) dx \\
\leq \mathcal{H}(E_{1k}, E_{2k}, n_k, \nu_k),
\]
where \( \mathcal{H} \) is defined by (1.3).

Proof of Proposition 3.6.

We first define two functionals as follows:
\[
\mathcal{E}(E_1, E_2) \triangleq \| \nabla E_1 \|_{L^2(\mathbb{R}^2)}^2 + \| \nabla E_2 \|_{L^2(\mathbb{R}^2)}^2 - \frac{1}{2} \int_{\mathbb{R}^2} \left( |E_1|^2 + |E_2|^2 \right)^2 dx \\
- \frac{\eta}{2} \int_{\mathbb{R}^2} |\nabla E_1 - \nabla E_2|^2 dx,
\]
\[
\mathcal{H}_1(E_1, E_2, n) \triangleq \mathcal{E}(E_1, E_2) + \frac{1}{2} \int_{\mathbb{R}^2} \left( n + |E_1|^2 + |E_2|^2 \right)^2 dx.
\]
Let
\[
\tilde{E}_{1k}(x) = \frac{1}{\lambda_k} E_{1k} \left( \frac{x}{\lambda_k} \right), \\
\tilde{E}_{2k}(x) = \frac{1}{\lambda_k} E_{2k} \left( \frac{x}{\lambda_k} \right), \\
\tilde{n}_k(x) = \frac{1}{\lambda_k} n_k \left( \frac{x}{\lambda_k} \right),
\]
where
\[
\lambda_k^2 = \| \nabla E_{1k} \|^2_{L^2(\mathbb{R}^2)} + \| \nabla E_{2k} \|^2_{L^2(\mathbb{R}^2)} + \int_{\mathbb{R}^2} \left( |E_{1k}|^2 + |E_{2k}|^2 \right) dx \quad \text{and} \quad \frac{1}{2} \int_{\mathbb{R}^2} |n_k|^2 dx.
\]
We continue the proof of Proposition 3.6 through four steps.

Step 1. A non-vanishing property of \( \left( \tilde{E}_{1k}, \tilde{E}_{2k}, \tilde{n}_k \right) \)
Lemma 3.7. For the sequences \((E_{1k}, E_{2k}, n_k)\) introduced in Proposition 3.6, assume there is a sequence \((\tilde{E}_{1k}, \tilde{E}_{2k}, \tilde{n}_k)\) such that as \(k \to +\infty\), the following estimates hold:

\[
\mathcal{H}(E_{1k}, E_{2k}, \tilde{n}_k, 0) \leq 0,
\]

\[
\int_{\mathbb{R}^2} \left( |\tilde{E}_{1k}|^2 + |\tilde{E}_{2k}|^2 \right) dx \to c_1 > 0,
\]

\[
\int_{\mathbb{R}^2} \left( |\nabla \tilde{E}_{1k}|^2 + |\nabla \tilde{E}_{2k}|^2 \right) dx + \frac{1}{2} \int_{\mathbb{R}^2} |\tilde{n}_k|^2 dx \to c_2 > 0,
\]

\[
\int_{\mathbb{R}^2} \tilde{n}_k \left( |\tilde{E}_{1k}|^2 + |\tilde{E}_{2k}|^2 \right) dx - \eta \int_{\mathbb{R}^2} |\tilde{E}_{1k}|^2 |\tilde{E}_{2k}|^2 dx
\]

\[
\quad + \frac{\eta}{2} \int_{\mathbb{R}^2} \left( |\tilde{E}_{1k}|^2 \left( \frac{\tilde{E}_{2k}}{c_0} \right)^2 + |\tilde{E}_{1k}|^2 \left( \frac{\tilde{E}_{2k}}{c_0} \right)^2 \right) dx \to -c_3 < 0.
\]

Then there exist a constant \(c_4 = c_4(c_1, c_2, c_3) > 0\) and a sequence \(x_k \in \mathbb{R}^2\) such that

\[
\int_{|x - x_k| < c_1} |\tilde{n}_k| dx > c_4.
\]

Proof. By (3.40), we claim that there exists a sequence \(x_k \in \mathbb{R}^2\) such that

\[
\int_{C_k} \tilde{n}_k \left( |\tilde{E}_{1k}|^2 + |\tilde{E}_{2k}|^2 \right) dx + \eta \int_{C_k} |\tilde{E}_{1k}|^2 |\tilde{E}_{2k}|^2 dx
\]

\[
\quad - \frac{\eta}{2} \int_{C_k} \left( |\tilde{E}_{1k}|^2 \left( \frac{\tilde{E}_{2k}}{c_0} \right)^2 + |\tilde{E}_{1k}|^2 \left( \frac{\tilde{E}_{2k}}{c_0} \right)^2 \right) dx \geq q \cdot \int_{C_k} \left[ \left( |\nabla \tilde{E}_{1k}|^2 + |\nabla \tilde{E}_{12k}|^2 \right) + \left( |\tilde{E}_{1k}|^2 + |\tilde{E}_{2k}|^2 \right) + \frac{1}{2} |\tilde{n}_k|^2 \right] dx,
\]

for \(k\) large enough, where \(C_k\) is the square of center \(x_k\) and \(q = \frac{c_3}{c_0(c_1 + c_2)}\) with \(c_0 > 1\) is a fixed constant. Otherwise, one would obtain

\[
\int_{C_k} \tilde{n}_k \left( |\tilde{E}_{1k}|^2 + |\tilde{E}_{2k}|^2 \right) dx + \eta \int_{C_k} |\tilde{E}_{1k}|^2 |\tilde{E}_{2k}|^2 dx
\]

\[
\quad - \frac{\eta}{2} \int_{C_k} \left( |\tilde{E}_{1k}|^2 \left( \frac{\tilde{E}_{2k}}{c_0} \right)^2 + |\tilde{E}_{1k}|^2 \left( \frac{\tilde{E}_{2k}}{c_0} \right)^2 \right) dx < q \cdot \int_{C_k} \left[ \left( |\nabla \tilde{E}_{1k}|^2 + |\nabla \tilde{E}_{12k}|^2 \right) + \left( |\tilde{E}_{1k}|^2 + |\tilde{E}_{2k}|^2 \right) + \frac{1}{2} |\tilde{n}_k|^2 \right] dx.
\]

Let \(k \to +\infty\), (3.48) yields \(c_3 < q(c_1 + c_2) = \frac{c_3}{c_0}(c_0 > 1)\), which is a contradiction.

We now claim the following conclusion.

Conclusion I: There exist constants

\[
c_1^* = \frac{2\sqrt{2q}^2}{1 + q\eta} ||Q||_{L^2(\mathbb{R}^2)} > 0, \quad c_2^* = \frac{4q^2}{1 + q\eta} ||Q||_{L^2(\mathbb{R}^2)} > 0,
\]

\[
c_3^* = \varepsilon c_1^* > 0 \text{ with } \varepsilon = \frac{\sqrt{2q} ||Q||_{L^2(C_k)}}{q\eta + 1},
\]
such that
\[
\left[\int_{C_k} \left( |\tilde{E}_{1k}|^2 + |\tilde{E}_{2k}|^2 \right)^2 \, dx \right]^{\frac{1}{2}} \geq c_1^* > 0, \quad (3.49)
\]
\[
\int_{C_k} -\tilde{n}_k \left( |\tilde{E}_{1k}|^2 + |\tilde{E}_{2k}|^2 \right) \, dx + \eta \int_{C_k} |\tilde{E}_{1k}|^2 \, dx \geq c_2 > 0, \quad (3.50)
\]
\[
\int_{C_k} -\tilde{n}_k \left( |\tilde{E}_{1k}|^2 + |\tilde{E}_{2k}|^2 \right) \, dx \geq c_3 > 0. \quad (3.51)
\]

**Proof of Conclusion I.**

Lemma 2.2, Cauchy-Schwartz inequality: \(2ab \leq \frac{(a+b)^2}{2} \) \((a, b > 0)\) and Young’s inequality give

\[
\int_{C_k} \left( |\nabla \tilde{E}_{1k}|^2 + |\nabla \tilde{E}_{2k}|^2 \right) \, dx + \int_{C_k} \left( |\tilde{E}_{1k}|^2 + |\tilde{E}_{2k}|^2 \right) \, dx \geq \sqrt{2} \|Q\|_{L^2(\mathbb{R}^2)} \left[ \int_{C_k} \left( |\tilde{E}_{1k}|^2 + |\tilde{E}_{2k}|^2 \right) \, dx \right]^{\frac{1}{2}}. \quad (3.52)
\]

\[
q \cdot \sqrt{2} \|Q\|_{L^2(C_k)} \left[ \int_{C_k} \left( |\tilde{E}_{1k}|^2 + |\tilde{E}_{2k}|^2 \right)^2 \, dx \right]^{\frac{1}{2}} + \frac{q}{2} \|\tilde{n}_k\|_{L^2(C_k)}^2 \leq \int_{C_k} -\tilde{n}_k \left( |\tilde{E}_{1k}|^2 + |\tilde{E}_{2k}|^2 \right) \, dx + \eta \int_{C_k} |\tilde{E}_{1k}|^2 \, dx \geq \frac{q}{2} \int_{C_k} |\tilde{n}_k|^2 \, dx + \frac{1}{2q} \int_{C_k} \left( |\tilde{E}_{1k}|^2 + |\tilde{E}_{2k}|^2 \right)^2 \, dx \quad (3.53)
\]

\[
\leq \frac{q}{2} \int_{C_k} |\tilde{n}_k|^2 \, dx + \frac{1}{2q} \int_{C_k} \left( |\tilde{E}_{1k}|^2 + |\tilde{E}_{2k}|^2 \right)^2 \, dx + \frac{\eta}{2} \int_{C_k} \left( |\tilde{E}_{1k}|^2 + |\tilde{E}_{2k}|^2 \right)^2 \, dx.
\]

That is,

\[
\left( \frac{1 + \eta q}{2q} \right) \left[ \int_{C_k} \left( |\tilde{E}_{1k}|^2 + |\tilde{E}_{2k}|^2 \right)^2 \, dx \right]^{\frac{1}{2}} \geq \sqrt{2} q \|Q\|_{L^2(C_k)}. \quad (3.54)
\]
This yields (3.49).

Similarly, (3.53) and (3.54) imply

$$\int_{C_k} -\bar{n}_k \left( |\tilde{E}_{1k}|^2 + |\tilde{E}_{2k}|^2 \right) dx + \eta \int_{C_k} |\tilde{E}_{1k}|^2 |\tilde{E}_{2k}|^2 dx$$

$$- \eta \int_{C_k} \left( (\tilde{E}_{1k})^2 (\tilde{E}_{2k})^2 + (\tilde{E}_{1k})^2 (\tilde{E}_{2k})^2 \right) dx$$

$$\geq q \sqrt{2} \|Q\|_{L^2(C_k)} \left[ \int_{C_k} \left( |\tilde{E}_{1k}|^2 + |\tilde{E}_{2k}|^2 \right)^2 dx \right]^\frac{1}{2}$$

$$\geq \frac{4q^3 \|Q\|_{L^2(C_k)}^2}{1 + q\eta} = e_2^* > 0.$$  \hspace{0.5em} (3.55)

So (3.50) is true.

In the following we prove (3.51). By (3.47), one obtains

$$q \int_{C_k} \left[ \left( |\nabla \tilde{E}_{1k}|^2 + |\nabla \tilde{E}_{2k}|^2 \right) + \left( |\tilde{E}_{1k}|^2 + |\tilde{E}_{2k}|^2 \right) \right] dx$$

$$+ \frac{q}{2} \int_{C_k} |\tilde{n}_k|^2 dx$$

$$\leq \frac{\eta}{2} \int_{C_k} \left( |\tilde{E}_{1k}|^2 + |\tilde{E}_{2k}|^2 \right) dx + \int_{C_k} -\tilde{n}_k \left( |\tilde{E}_{1k}|^2 + |\tilde{E}_{2k}|^2 \right) dx.$$  \hspace{0.5em} (3.56)

From Lemma 2.2, it follows

$$\int_{C_k} \left( |\nabla \tilde{E}_{1k}|^2 + |\nabla \tilde{E}_{2k}|^2 \right) dx \geq \frac{\|Q\|_{L^2(C_k)}^2}{2} \int_{C_k} \left( |\tilde{E}_{1k}|^2 + |\tilde{E}_{2k}|^2 \right) dx.$$

Note that

$$\int_{C_k} \left[ \left( |\nabla \tilde{E}_{1k}|^2 + |\nabla \tilde{E}_{2k}|^2 \right) + \left( |\tilde{E}_{1k}|^2 + |\tilde{E}_{2k}|^2 \right) \right] dx$$

$$\geq 2 \left( \int_{C_k} \left( |\nabla \tilde{E}_{1k}|^2 + |\nabla \tilde{E}_{2k}|^2 \right) dx \right)^\frac{1}{2} \left( \int_{C_k} \left( |\tilde{E}_{1k}|^2 + |\tilde{E}_{2k}|^2 \right) dx \right)^\frac{1}{2},$$

(3.53),(3.54),(3.55) and (3.56) give

$$\frac{\sqrt{2q} \|Q\|_{L^2(C_k)} \sqrt{2q} \|Q\|_{L^2(C_k)} + \frac{q}{2} \int_{C_k} |\tilde{n}_k|^2 dx}{\frac{\eta}{2} + \frac{q}{2}}$$

$$- \frac{\eta}{2} \int_{C_k} \left( |\tilde{E}_{1k}|^2 + |\tilde{E}_{2k}|^2 \right) dx$$

$$\leq \int_{C_k} -\tilde{n}_k \left( |\tilde{E}_{1k}|^2 + |\tilde{E}_{2k}|^2 \right) dx.$$ \hspace{0.5em} (3.57)
Assume that there exists an \( \varepsilon > 0 \) such that

\[
\frac{\sqrt{2}q}{2} \|Q\|_{L^2(C_k)} \leq \left( \int_{C_k} \left( |\tilde{E}_{1k}|^2 + |\tilde{E}_{2k}|^2 \right)^2 \, dx \right)^{\frac{1}{2}} \leq \frac{\sqrt{2}q\|Q\|_{L^2(C_k)} - \varepsilon}{\frac{2}{q}}.
\] (3.58)

This yields

\[
\eta \leq \frac{2\sqrt{2}q\|Q\|_{L^2(C_k)} - 2\varepsilon}{\left( \int_{C_k} \left( |\tilde{E}_{1k}|^2 + |\tilde{E}_{2k}|^2 \right)^2 \, dx \right)^{\frac{1}{2}}} \leq \frac{2\sqrt{2}q\|Q\|_{L^2(C_k)} - 2\varepsilon}{\sqrt{2}q\|Q\|_{L^2(C_k)}} \left( \frac{\eta}{2} + \frac{1}{2q} \right). \] (3.59)

that is,

\[
\varepsilon \leq \sqrt{2}q\|Q\|_{L^2(C_k)} \left( 1 - \frac{\eta}{\eta + \frac{1}{q}} \right) = \sqrt{2}q\|Q\|_{L^2(C_k)} \frac{1}{q\eta + 1} = \frac{\sqrt{2}q\|Q\|_{L^2(C_k)}}{q\eta + 1}. \] (3.60)

Taking \( \varepsilon = \frac{\sqrt{2}q\|Q\|_{L^2(C_k)}}{q\eta + 1} \) in (3.58), one obtains

\[
\sqrt{2}q\|Q\|_{L^2(C_k)} - \frac{\eta}{2} \left( \int_{C_k} \left( |\tilde{E}_{1k}|^2 + |\tilde{E}_{2k}|^2 \right)^2 \, dx \right)^{\frac{1}{2}} \geq \frac{\sqrt{2}q\|Q\|_{L^2(C_k)} - \sqrt{2}q\|Q\|_{L^2(C_k)}}{q\eta + 1} \cdot \frac{\eta}{2} \] (3.61)

\[
= \frac{\sqrt{2}q\|Q\|_{L^2(C_k)}}{q\eta + 1}.
\]

Note that \( \eta > 0 \), then there exists \( \varepsilon > 0 \) small enough such that the following estimate holds:

\[
\sqrt{2}q\|Q\|_{L^2(C_k)} - \frac{\eta}{2} \left( \int_{C_k} \left( |\tilde{E}_{1k}|^2 + |\tilde{E}_{2k}|^2 \right)^2 \, dx \right)^{\frac{1}{2}} \geq \varepsilon > 0. \] (3.62)

Combining (3.47), (3.52), (3.53) and (3.62) together yields...
\[
\int_{C_k} -\tilde{n}_k \left( |\tilde{E}_{1k}|^2 + |\tilde{E}_{2k}|^2 \right) dx \\
\geq \sqrt{2q} \|Q\|_{L^2(C_k)} \left( \int_{C_k} \left( |\tilde{E}_{1k}|^2 + |\tilde{E}_{2k}|^2 \right)^2 dx \right)^{\frac{1}{2}} \\
+ \frac{q}{2} \|\tilde{n}_k\|^2_{L^2(C_k)} - \frac{\eta}{2} \left( \int_{C_k} \left( |\tilde{E}_{1k}|^2 + |\tilde{E}_{2k}|^2 \right)^2 dx \right)^{\frac{1}{2}} \\
= \left[ \sqrt{2q} \|Q\|_{L^2(C_k)} - \frac{\eta}{2} \left( \int_{C_k} \left( |\tilde{E}_{1k}|^2 + |\tilde{E}_{2k}|^2 \right)^2 dx \right)^{\frac{1}{2}} \right] \\
\times \left( \int_{C_k} \left( |\tilde{E}_{1k}|^2 + |\tilde{E}_{2k}|^2 \right)^2 dx \right)^{\frac{1}{2}} + \frac{q}{2} \|\tilde{n}_k\|^2_{L^2(C_k)} \\
\geq \varepsilon \left( \int_{C_k} \left( |\tilde{E}_{1k}|^2 + |\tilde{E}_{2k}|^2 \right)^2 dx \right)^{\frac{1}{2}} + \frac{q}{2} \|\tilde{n}_k\|^2_{L^2(C_k)} \\
\geq \frac{2\sqrt{2q} \|Q\|_{L^2(C_k)}}{q\eta + 1} = \varepsilon c_3^* = c_3^* > 0.
\]

This is just the estimate (3.51). Hence Conclusion I follows from (3.54), (3.55) and (3.63).

We next finish the proof of Lemma 3.7 according to Conclusion I by contradiction.

Assume by contradiction that there exists a subsequence (still denoted by \(\tilde{n}_k\)) such that as \(k \to +\infty\),
\[
\int_{C_k} |\tilde{n}_k| dx \to 0. \tag{3.64}
\]

Let
\[
\tilde{n}_k(x_k + \cdot) \to N' \text{ in } L^2(\mathbb{R}^2), \tag{3.65}
\]
and
\[
\left( \tilde{E}_{1k}(x_k + \cdot), \tilde{E}_{2k}(x_k + \cdot) \right) \to (E'_1, E'_2) \text{ in } H^1(\mathbb{R}^2) \times H^1(\mathbb{R}^2). \tag{3.66}
\]

By Sobolev-type estimates, we have
\[
\left( \tilde{E}_{1k}(x_k + \cdot), \tilde{E}_{2k}(x_k + \cdot) \right) \to (E'_1, E'_2) \text{ in } L^4_{\text{loc}}(\mathbb{R}^2) \times L^4_{\text{loc}}(\mathbb{R}^2), \tag{3.67}
\]
and
\[
\left( |\tilde{E}_{1k}(x_k + \cdot)|^2, |\tilde{E}_{2k}(x_k + \cdot)|^2 \right) \to (|E'_1|^2, |E'_2|^2) \text{ in } L^2_{\text{loc}}(\mathbb{R}^2) \times L^2_{\text{loc}}(\mathbb{R}^2). \tag{3.68}
\]
Indeed, for a bounded open domain $C_k$ in $\mathbb{R}^2$, $H^1(C_k) \subset L^p(C_k)$, $p \in [2, +\infty)$, there holds:

$$\left\| \tilde{E}_{1k}^2 - |E'_1|^2 \right\|_{L^2(C_k)} \leq c \left( \int_{\mathbb{R}^2} |\tilde{E}_{1k} - E'_1|^4 \, dx \right)^{\frac{1}{4}} \cdot \left( \int_{\mathbb{R}^2} \left( |\tilde{E}_{1k}|^2 + |E'_1|^2 \right)^2 \, dx \right)^{\frac{1}{4}} \tag{3.69}$$

From (3.64) it follows that as $k \to \infty$,

$$\tilde{n}_k(x_k + \cdot) \to 0 \text{ in } L^2(C_k). \tag{3.70}$$

On the other hand, by Lemma 2.1 there holds as $k \to +\infty$,

$$\int_{C_k} \tilde{n}_k \left( |\tilde{E}_{1k}|^2 + |\tilde{E}_{2k}|^2 \right) \, dx \to 0,$$

which is contradictory to (3.63). Hence the Lemma 3.7.

\[\square\]

**Step 2: An alternative form for (3.37)**

Implementing similar arguments to those in the previous section, we follow from (3.63) that there exists a constant $c_1^* > 0$ such that

$$\int_{C_0} \left( |\tilde{E}_{1k}(x_k + x)|^2 + |\tilde{E}_{2k}(x_k + x)|^2 \right) \, dx \geq c_1^* > 0. \tag{3.72}$$

According to the definitions of $\mathcal{H}_1(E_1, E_2, n)$ (see (3.39)) and $\mathcal{H}(E_1, E_2, n, \nu)$ (see (1.3)), for the sake of proving (3.37), it is sufficient to show that there exist constants $C_1 > 0$ and $C_2 > 0$ such that

$$-C_1 + C_2 \int_{\mathbb{R}^2} \left( |\nabla E_{1k}|^2 + |\nabla E_{2k}|^2 + \frac{1}{2} |n_k|^2 \right) \, dx \leq \mathcal{H}_1(E_{1k}, E_{2k}, n_k). \tag{3.73}$$

We will verify (3.73) by contradiction.

Assume that (3.73) would not hold for a subsequence $(E_{1k}, E_{2k}, n_k)$. That is, for all constants $C_1$, $C_2$, there holds

$$-C_1 + C_2 \int_{\mathbb{R}^2} \left( |\nabla E_{1k}|^2 + |\nabla E_{2k}|^2 + \frac{1}{2} |n_k|^2 \right) \, dx \geq \mathcal{H}_1(E_{1k}, E_{2k}, n_k). \tag{3.74}$$

Then the following conclusions would be true provided $k \to +\infty$:

$$\lambda_k^2 := \int_{\mathbb{R}^2} \left( |\nabla E_{1k}|^2 + |\nabla E_{2k}|^2 \right) \, dx + \frac{1}{2} \int_{\mathbb{R}^2} |n_k|^2 \, dx \to +\infty, \tag{3.75}$$
\[
\limsup_{k \to +\infty} \frac{\mathcal{H}_1(E_{1k}, E_{2k}, n_k)}{\lambda_k^2} \leq 0. \tag{3.76}
\]

Otherwise,

1. If \( \lambda_k \leq C \), then

\[
|\mathcal{H}_1(E_{1k}, E_{2k}, n_k)| = \|\nabla E_{1k}\|_{L^2(\mathbb{R}^2)}^2 + \|\nabla E_{2k}\|_{L^2(\mathbb{R}^2)}^2 + \int_{\mathbb{R}^2} |n_k| \left( |E_{1k}|^2 + |E_{2k}|^2 \right) \, dx
\]

\[
+ \frac{\eta}{2} \int_{\mathbb{R}^2} |E_{1k} E_{2k} - E_{1k} \nabla E_{2k}|^2 \, dx + \frac{1}{2} \|n_k\|_{L^2(\mathbb{R}^2)}^2
\]

\[
\leq \lambda_k^2 + \frac{1}{2} \|n_k\|_{L^2(\mathbb{R}^2)}^2 + \frac{1}{2} \int_{\mathbb{R}^2} \left( |E_{1k}|^2 + |E_{2k}|^2 \right)^2 \, dx
\]

\[
+ \eta \int_{\mathbb{R}^2} \left( |E_{1k}|^2 + |E_{2k}|^2 \right)^2 \, dx
\]

\[
\leq \lambda_k^2 + \frac{1}{2} \|n_k\|_{L^2(\mathbb{R}^2)}^2 + (1 + 2\eta) \left( \frac{\|E_{1k}\|_{L^2(\mathbb{R}^2)}^2 + \|E_{2k}\|_{L^2(\mathbb{R}^2)}^2}{\|Q\|_{L^2(\mathbb{R}^2)}^2} \right)
\]

\[
\cdot \left( \|\nabla E_{1k}\|_{L^2(\mathbb{R}^2)}^2 + \|\nabla E_{2k}\|_{L^2(\mathbb{R}^2)}^2 \right)
\]

\[
\leq \lambda_k^2 + \frac{1}{2} \|n_k\|_{L^2(\mathbb{R}^2)}^2 + (1 + 2\eta) \frac{1}{\eta} \left( \|\nabla E_{1k}\|_{L^2(\mathbb{R}^2)}^2 + \|\nabla E_{2k}\|_{L^2(\mathbb{R}^2)}^2 \right)
\]

\[
\leq \left( 3 + \frac{1}{\eta} \right) \lambda_k^2 \leq C. \tag{3.76*}
\]

This implies (3.73), which is contradictory to the assumption (3.74). So (3.75) holds true.

2. If \( \lim_{k \to +\infty} \frac{\mathcal{H}_1(E_{1k}, E_{2k}, n_k)}{\lambda_k^2} = C > 0 \), then for \( k_0 > 0 \) large enough, there holds

\[
\mathcal{H}_1(E_{1k}, E_{2k}, n_k) \geq \frac{C}{2} \lambda_k^2
\]

\[
= \frac{C}{2} \left( \|\nabla E_{1k}\|_{L^2(\mathbb{R}^2)}^2 + \|\nabla E_{2k}\|_{L^2(\mathbb{R}^2)}^2 + \frac{1}{2} \|n_k\|_{L^2(\mathbb{R}^2)}^2 \right), \tag{3.77}
\]

which is a contradiction since (3.73) will be satisfied with \( C_1 = 0 \) and \( C_2 = \frac{C}{2} \).

**Step 3: Scaling discussion**
The proof continues as follow. Let
\[
\begin{align*}
\tilde{E}_{1k}(x) &= \frac{1}{\lambda_k} E_{1k} \left( \frac{x}{\lambda_k} \right), \\
\tilde{E}_{2k}(x) &= \frac{1}{\lambda_k} E_{2k} \left( \frac{x}{\lambda_k} \right), \\
\tilde{n}_k(x) &= \frac{1}{\lambda_k} n_k \left( \frac{x}{\lambda_k} \right).
\end{align*}
\]
(3.78)

Straightforward calculation gives
\[
\int_{\mathbb{R}^2} \left( |\tilde{E}_{1k}(x)|^2 + |\tilde{E}_{2k}(x)|^2 \right) dx = \int_{\mathbb{R}^2} (|E_{10}(x)|^2 + |E_{20}(x)|^2) dx,
\]
(3.79)
\[
\int_{\mathbb{R}^2} \left( |\nabla \tilde{E}_{1k}(x)|^2 + |\nabla \tilde{E}_{2k}(x)|^2 \right) dx + \frac{1}{2} \int_{\mathbb{R}^2} |\tilde{n}_k(x)|^2 dx = 1.
\]
(3.80)

In view of
\[
\limsup_{k \to \infty} \left( 1 + \int_{\mathbb{R}^2} \tilde{n}_k \left( |\tilde{E}_{1k}|^2 + |\tilde{E}_{2k}|^2 \right) dx \right.
\]
\[
- \frac{\eta}{2} \int_{\mathbb{R}^2} \left( |\tilde{E}_{1k} \tilde{E}_{2k} - \tilde{E}_{1k} \tilde{E}_{2k}|^2 \right) dx \bigg) = \limsup_{k \to +\infty} \mathcal{H}_1 \left( \tilde{E}_{1k}, \tilde{E}_{2k}, \tilde{n}_k \right) \leq 0,
\]
(3.81)

by H"{o}lder inequality one has
\[
\left| \int_{\mathbb{R}^2} \tilde{n}_k \left( |\tilde{E}_{1k}|^2 + |\tilde{E}_{2k}|^2 \right) dx - \frac{\eta}{2} \int_{\mathbb{R}^2} |\tilde{E}_{1k} \tilde{E}_{2k} - \tilde{E}_{1k} \tilde{E}_{2k}|^2 dx \right|
\leq \frac{1}{2} \left\| \tilde{n}_k \right\|_{L^2(\mathbb{R}^2)}^2 + \frac{1}{2} \int_{\mathbb{R}^2} \left( |\tilde{E}_{1k}|^2 + |\tilde{E}_{2k}|^2 \right) dx
\]
\[
+ \frac{\eta}{2} \int_{\mathbb{R}^2} \left( |\tilde{E}_{1k}|^2 + |\tilde{E}_{2k}|^2 \right) dx
\leq \frac{1}{2} \left\| \tilde{n}_k \right\|_{L^2(\mathbb{R}^2)}^2 + (1 + \eta) \frac{\left\| \tilde{E}_{1k} \right\|_{L^2(\mathbb{R}^2)}^4 + \left\| \tilde{E}_{2k} \right\|_{L^2(\mathbb{R}^2)}^4}{\left\| Q \right\|_{L^2(\mathbb{R}^2)}^4}
\cdot \left( \left\| \nabla \tilde{E}_{1k} \right\|_{L^2(\mathbb{R}^2)}^2 + \left\| \nabla \tilde{E}_{2k} \right\|_{L^2(\mathbb{R}^2)}^2 \right)
\leq C.
\]
(3.82)

Hence we can assume by (3.81) and (3.82) that as $k \to +\infty$,
\[
\int_{\mathbb{R}^2} \tilde{n}_k \left( |\tilde{E}_{1k}|^2 + |\tilde{E}_{2k}|^2 \right) dx - \frac{\eta}{2} \int_{\mathbb{R}^2} |\tilde{E}_{1k} \tilde{E}_{2k} - \tilde{E}_{1k} \tilde{E}_{2k}|^2 dx \rightarrow c \leq -1.
\]
(3.83)

On the other hand, recalling (3.35) and (3.75), we have $\forall R > 0$,
\[
\liminf_{k \to +\infty} \left( \sup_y \int_{|x-y|<R} \left( |\tilde{E}_{1k}(x)|^2 + |\tilde{E}_{2k}(x)|^2 \right) dx \right) \leq \frac{\left\| Q \right\|_{L^2(\mathbb{R}^2)}^2}{1 + \eta} - \delta_0,
\]
(3.84)
or as $R \to 0$,
\[
\liminf_{k \to +\infty} \left( \sup_y \int_{|x-y|<R} |\tilde{n}_k| dx \right) \to 0. \tag{3.85}
\]
Note that Lemma 3.7, (3.85) does not hold. Therefore we need to concern the case (3.84) only.

**Step 4: Proof of Proposition 3.6**

Recalling the definitions of $\mathcal{E}$ (see 3.38) and $\mathcal{H}_1$ (see 3.39), there hold
\[
\mathcal{H}_1 \left( \tilde{E}_{1k}, \tilde{E}_{2k}, \tilde{n}_k \right) = \mathcal{E} \left( \tilde{E}_{1k}, \tilde{E}_{2k} \right) + \frac{1}{2} \int_{\mathbb{R}^2} \left( \tilde{n}_k + |\tilde{E}_{1k}|^2 + |\tilde{E}_{2k}|^2 \right) dx, \tag{3.86}
\]
and
\[
\limsup_{k \to +\infty} \mathcal{E} \left( \tilde{E}_{1k}, \tilde{E}_{2k} \right) \leq \limsup_{k \to +\infty} \mathcal{H}_1 \left( \tilde{E}_{1k}, \tilde{E}_{2k}, \tilde{n}_k \right) \leq 0. \tag{3.87}
\]
Then by (3.79), (3.80) and Sobolev type estimates, we conclude that there exist $c_1 > 0$ and $c_2 > 0$ such that
\[
c_1 \leq \int_{\mathbb{R}^2} \left( |\tilde{E}_{1k}(x)|^2 + |\tilde{E}_{2k}(x)|^2 \right) dx \leq c_2, \tag{3.88}
\]
\[
c_1 \leq \int_{\mathbb{R}^2} \left( |\nabla \tilde{E}_{1k}(x)|^2 + |\nabla \tilde{E}_{2k}(x)|^2 + |\tilde{\psi}_{1k}(x)|^2 + |\tilde{\psi}_{2k}(x)|^2 \right) dx \leq c_2. \tag{3.89}
\]
Hence there exist a constant $\delta_1 > 0$ and a sequence $x_k^1 \in \mathbb{R}^2$ such that
\[
\int_{|x-x_k^1|<1} \left( |\tilde{E}_{1k}(x)|^2 + |\tilde{E}_{2k}(x)|^2 \right) dx \geq \delta_1. \tag{3.90}
\]
In view of Lemma 3.7 and its proof, we introduce the following dichotomy
\[
\begin{cases}
\tilde{E}_{1k}(x) = \tilde{E}_{1k}^1(x) + \tilde{E}_{1k}^{1,R}(x), \\
\tilde{E}_{2k}(x) = \tilde{E}_{2k}^1(x) + \tilde{E}_{2k}^{1,R}(x).
\end{cases} \tag{3.91}
\]
Hence, for a sequence $x_k^1$,
\[
\left( \tilde{E}_{1k}^1(x+x_k^1), \tilde{E}_{2k}^1(x+x_k^1) \right) \rightharpoonup (\psi_1, \psi_2) \text{ in } H^1(\mathbb{R}^2) \times H^1(\mathbb{R}^2), \tag{3.92}
\]
and
\[
\left( \int_{|x-x_k^1|<1} \left( |\tilde{E}_{1k}^1(x+x_k^1)|^2 + |\tilde{E}_{2k}^1(x+x_k^1)|^2 \right)^2 dx \right)^{\frac{1}{4}} \geq c > 0. \tag{3.93}
\]
By Sobolev estimates, there exists a $\delta_1 > 0$ depending only on $\|E_{10}\|_{L^2(\mathbb{R}^2)}$ and $\|E_{20}\|_{L^2(\mathbb{R}^2)}$ such that
\[
\left\| \tilde{E}_{1k}^1(x_k^1+\cdot) \right\|_{L^2(|x-x_k^1|<1)} + \left\| \tilde{E}_{2k}^1(x_k^1+\cdot) \right\|_{L^2(|x-x_k^1|<1)} \geq \delta_1 > 0. \tag{3.94}
\]
Recalling (3.84), we also obtain for $\forall R > 0$,
\[
\liminf_{k \to +\infty} \left( \|\tilde{E}_{1k}^1(x_k^1+\cdot)\|_{L^2(B_R)}^2 + \|\tilde{E}_{2k}^1(x_k^1+\cdot)\|_{L^2(B_R)}^2 \right) \leq \frac{\|Q\|_{L^2(\mathbb{R}^2)}^2}{1+\eta} - \delta_0. \tag{3.95}
\]
Furthermore, using concentration compactness method (Lions [14]), one gets for a suitable choice for \( (\tilde{E}_{1k}, \tilde{E}_{2k}) \),

\[
\left\| \tilde{E}_{1k} \right\|_{L^2(\mathbb{R}^2)}^2 + \left\| \tilde{E}_{1k}^{1, R} \right\|_{L^2(\mathbb{R}^2)}^2 + \left\| \tilde{E}_{2k} \right\|_{L^2(\mathbb{R}^2)}^2 + \left\| \tilde{E}_{2k}^{1, R} \right\|_{L^2(\mathbb{R}^2)}^2 \rightarrow \left\| E_{10} \right\|_{L^2(\mathbb{R}^2)}^2 + \left\| E_{20} \right\|_{L^2(\mathbb{R}^2)}^2,
\]

(3.96)

\[
\delta_1 \leq \lim_{k \to +\infty} \left( \left\| \tilde{E}_{1k}^1(x) \right\|_{L^2(\mathbb{R}^2)}^2 + \left\| \tilde{E}_{2k}^1(x) \right\|_{L^2(\mathbb{R}^2)}^2 \right) \leq \frac{\|Q\|_{L^2(\mathbb{R}^2)}^2}{1 + \eta} - \delta'_0,
\]

(3.97)

and

\[
\mathcal{E}(\psi_1, \psi_2) \leq \limsup_{k \to +\infty} \mathcal{E}(\tilde{E}_{1k}, \tilde{E}_{2k}) + \limsup_{k \to +\infty} \mathcal{E}(\tilde{E}_{1k}^{1, R}, \tilde{E}_{2k}^{1, R}) \\
\leq \limsup_{k \to +\infty} \mathcal{E}(\tilde{E}_{1k}, \tilde{E}_{2k}) \leq 0.
\]

(3.98)

Since

\[
\delta_1 \leq \|\psi_1\|_{L^2(\mathbb{R}^2)}^2 + \|\psi_2\|_{L^2(\mathbb{R}^2)}^2 \leq \frac{\|Q\|_{L^2(\mathbb{R}^2)}^2}{1 + \eta} - \delta'_0,
\]

(3.99)

we have

\[
\limsup_{k \to +\infty} \mathcal{E}(\tilde{E}_{1k}^{1, R}, \tilde{E}_{2k}^{1, R}) \leq -\mathcal{E}(\psi_1, \psi_2) < 0.
\]

(3.100)

We now can extract a subsequence still denoted by \( (\tilde{E}_{1k}^{1, R}, \tilde{E}_{2k}^{1, R}) \) such that

\[
\left\| \tilde{E}_{1k}^{1, R}(x) \right\|_{L^2(\mathbb{R}^2)}^2 + \left\| \tilde{E}_{2k}^{1, R}(x) \right\|_{L^2(\mathbb{R}^2)}^2 \rightarrow c_1 < \frac{\|Q\|_{L^2(\mathbb{R}^2)}^2}{1 + \eta} - \delta'_1,
\]

(3.101)

\[
\limsup_{k \to +\infty} \mathcal{E}(\tilde{E}_{1k}^{1, R}, \tilde{E}_{2k}^{1, R}) \leq -\mathcal{E}(\psi_1, \psi_2) < 0.
\]

(3.102)

Then there exists a constant \( k_0 > 0 \) such that \( \forall k \geq k_0, \)

\[
\mathcal{E}(\tilde{E}_{1k}^{1, R}, \tilde{E}_{2k}^{1, R}) \leq \frac{-\mathcal{E}(\psi_1, \psi_2)}{2} < 0.
\]

(3.103)

Note that

\[
\left\| \tilde{E}_{1k}^{1, R}(x) \right\|_{L^2(\mathbb{R}^2)}^2 + \left\| \tilde{E}_{2k}^{1, R}(x) \right\|_{L^2(\mathbb{R}^2)}^2 \leq \frac{\|Q\|_{L^2(\mathbb{R}^2)}^2}{1 + \eta},
\]

(3.104)
then

\[
E(\tilde{E}_{1k}^1, \tilde{E}_{2k}^1) = \int_{\mathbb{R}^2} \left( |\nabla \tilde{E}_{1k}^1|^2 + |\nabla \tilde{E}_{2k}^1|^2 \right) dx
\]

\[
- \frac{1}{2} \int_{\mathbb{R}^2} \left( |\tilde{E}_{1k}^1|^2 + |\tilde{E}_{2k}^1|^2 \right) dx
\]

\[
- \frac{\eta}{2} \int_{\mathbb{R}^2} \left( \tilde{E}_{1k}^1 \tilde{E}_{2k}^1 - \tilde{E}_{1k}^1 \tilde{E}_{2k}^1 \right)^2 dx
\]

\[
\geq \int_{\mathbb{R}^2} \left( |\nabla \tilde{E}_{1k}^1|^2 + |\nabla \tilde{E}_{2k}^1|^2 \right) dx
\]

\[
- \frac{(1 + \eta)}{\|Q\|_{L^2(\mathbb{R}^2)}} \left( \|\tilde{E}_{1k}^1\|^2_{L^2(\mathbb{R}^2)} + \|\tilde{E}_{2k}^1\|^2_{L^2(\mathbb{R}^2)} \right)
\]

\[
\cdot \int_{\mathbb{R}^2} \left( |\nabla \tilde{E}_{1k}^1|^2 + |\nabla \tilde{E}_{2k}^1|^2 \right) dx
\]

\[
\geq 0.
\]

This is contradictory to (3.103). Hence (3.104) is not true. Therefore, we claim

(2) If

\[
\left\| \tilde{E}_{1k}^1(x) \right\|^2_{L^2(\mathbb{R}^2)} + \left\| \tilde{E}_{2k}^1(x) \right\|^2_{L^2(\mathbb{R}^2)} > \frac{\|Q\|^2_{L^2(\mathbb{R}^2)}}{1 + \eta}.
\]

In view of (3.103), there exists a constant \( c > 0 \) such that

\[
\int_{\mathbb{R}^2} \left( |\tilde{E}_{1k}^1|^2 + |\tilde{E}_{2k}^1|^2 \right) dx > c.
\]

Indeed, by \( E(\tilde{E}_{1k}^1, \tilde{E}_{2k}^1) < 0 \), we have

\[
\int_{\mathbb{R}^2} \left( |\tilde{E}_{1k}^1|^2 + |\tilde{E}_{2k}^1|^2 \right) dx
\]

\[
> 2 \int_{\mathbb{R}^2} \left( |\nabla \tilde{E}_{1k}^1|^2 + |\nabla \tilde{E}_{2k}^1|^2 \right) dx - \eta \int_{\mathbb{R}^2} \left( \tilde{E}_{1k}^1 \tilde{E}_{2k}^1 - \tilde{E}_{1k}^1 \tilde{E}_{2k}^1 \right)^2 dx
\]

\[
> 2 \int_{\mathbb{R}^2} \left( |\nabla \tilde{E}_{1k}^1|^2 + |\nabla \tilde{E}_{2k}^1|^2 \right) dx - \eta \int_{\mathbb{R}^2} \left( |\tilde{E}_{1k}^1|^2 + |\tilde{E}_{2k}^1|^2 \right) dx.
\]

This yields the estimate (3.107). We then iterate the same procedure as above and define

\[
\begin{aligned}
\tilde{E}_{1k}^1 &= E_{1k}^2 + E_{1k}^2, \\
\tilde{E}_{2k}^1 &= E_{2k}^2 + E_{2k}^2,
\end{aligned}
\]

(3.108)

where \( E_{1k}^2 \) and \( E_{2k}^2 \) satisfy for a sequence \( x_k^2 \),

\[
\left\| E_{1k}^2(x_k^2 + \cdot) \right\|^2_{L^2(|x - x_k^2| < 1)} + \left\| E_{2k}^2(x_k^2 + \cdot) \right\|^2_{L^2(|x - x_k^2| < 1)} \geq \delta_1.
\]

Defining \( p \) such that

\[
-p\delta_1 + \|E_{10}\|^2_{L^2(\mathbb{R}^2)} + \|E_{20}\|^2_{L^2(\mathbb{R}^2)} < \frac{\|Q\|^2_{L^2(\mathbb{R}^2)}}{1 + \eta},
\]

(3.109)
applying the same procedure at most $p$ times, we find for an $i \leq p$ and $k$ large, there exists a function $(\tilde{E}_{1k}, \tilde{E}_{2k})$ such that

$$
\left\| \tilde{E}_{1k} \right\|_{L^2(\mathbb{R}^2)}^2 + \left\| \tilde{E}_{2k} \right\|_{L^2(\mathbb{R}^2)}^2 \leq \frac{\| Q \|_{L^2(\mathbb{R}^2)}^2}{1 + \eta},
$$

and

$$
\mathcal{E} \left( \tilde{E}_{1k}, \tilde{E}_{2k} \right) \leq -\mathcal{E} (\psi_1, \psi_2) < 0.
$$

Then by Lemma 2.2, (3.111) and (3.112) are contradictory.

In addition, from (3.4) one gets

$$
\left\| \nabla \tilde{E}_{1k} \right\|_{L^2(\mathbb{R}^2)}^2 + \left\| \nabla \tilde{E}_{2k} \right\|_{L^2(\mathbb{R}^2)}^2 \\
\geq \frac{\int_{\mathbb{R}^2} (|\tilde{E}_{1k}|^2 + |\tilde{E}_{2k}|^2)^2 dx \cdot \| Q \|_{L^2(\mathbb{R}^2)}^2}{2 \left( \left\| \tilde{E}_{1k} \right\|_{L^2(\mathbb{R}^2)} + \left\| \tilde{E}_{2k} \right\|_{L^2(\mathbb{R}^2)} \right)} \\
> \frac{\eta}{2} \int_{\mathbb{R}^2} (|\tilde{E}_{1k}|^2 + |\tilde{E}_{2k}|^2)^2 dx \\
\geq \frac{\eta}{2} \int_{\mathbb{R}^2} \left| \tilde{E}_{1k} \tilde{E}_{2k} - \tilde{E}_{1k} \tilde{E}_{2k} \right|^2 dx.
$$

Since

$$
\limsup_{k \to +\infty} \mathcal{H}_3 \left( \tilde{E}_{1k}, \tilde{E}_{1k}, \tilde{E}_{2k}, \tilde{n}_k \right) = \limsup_{k \to +\infty} \frac{\mathcal{H}_3 \left( \tilde{E}_{1k}, \tilde{E}_{1k}, \tilde{E}_{2k}, \tilde{n}_k \right)}{\lambda_k^2} \leq 0, \quad (3.114)
$$

then

$$
\int_{\mathbb{R}^2} \tilde{n}_k \left( |\tilde{E}_{1k}|^2 + |\tilde{E}_{2k}|^2 \right) dx \to -C < 0. \quad (3.115)
$$

It follows from Lemma 3.7 that there exist a constant $C' > 0$ and a sequence $x_k$ such that

$$
\int_{|x-x_k| < 1} |\tilde{n}_k| dx > C' > 0. \quad (3.116)
$$

Therefore, recalling (3.36) and the definition of $\tilde{n}_k$, we have as $R \to 0$,

$$
\liminf_{k \to +\infty} \left( \sup_y \int_{|x-y| < R} |\tilde{n}_k| dx \right) \to 0. \quad (3.117)
$$

(3.116) and (3.117) are contradictory.

This finishes the proof of Proposition 3.6.

We now claim the following conclusions to prove the non-vanishing properties of $(\tilde{E}_1(0), \tilde{E}_2(0), \tilde{n}(0))$.

**Proposition 3.8.** Let $(E_1, E_2, n, v)$ be the finite time blow-up solution of the Zakharov system (1.1) and $T$ be its blowup time. That is, as $t \to T$,

$$
\| E_1 \|_{H^1(\mathbb{R}^2)} + \| E_2 \|_{H^1(\mathbb{R}^2)} + \| n \|_{L^2(\mathbb{R}^2)} + \| v \|_{L^2(\mathbb{R}^2)} \to +\infty. \quad (3.118)
$$
Assume that the initial data satisfy (3.4), then

1. If $E_1, E_2, n$ are radially symmetric functions, then there exists a constant $m > 0$ such that for any $R > 0$,
\[
\liminf_{t \to T} \left( \|E_1(t,x)\|_{L^2(B(0,R))}^2 + \|E_2(t,x)\|_{L^2(B(0,R))}^2 \right) > \frac{1}{1 + \eta} \|Q\|_{L^2(\mathbb{R}^2)}^2, \quad (3.119)
\]
\[
\liminf_{t \to T} \|n(t,x)\|_{L^1(B(0,R))} \geq m. \quad (3.120)
\]

2. If $E_1, E_2, n$ are non-radially symmetric functions, then there exists a sequence $x(t) \in \mathbb{R}^2$ and a constant $m > 0$ depending only on initial data such that for any $R > 0$,
\[
\liminf_{t \to T} \left( \|E_1(t,x)\|_{L^2(B(x(t),R))}^2 + \|E_2(t,x)\|_{L^2(B(x(t),R))}^2 \right) > \frac{1}{1 + \eta} \|Q\|_{L^2(\mathbb{R}^2)}^2, \quad (3.121)
\]
\[
\liminf_{t \to T} \|n(t,x)\|_{L^1(B(x(t),R))} \geq m. \quad (3.122)
\]

**Proof.** We first show the case (1): $(E_1, E_2, n) \in H^1_1(\mathbb{R}^2) \times H^1_1(\mathbb{R}^2) \times L^2_1(\mathbb{R}^2)$.

Define two spaces:
\[
H^1_1(\mathbb{R}^2) = \{ f \in H^1(\mathbb{R}^2), f(x) = f(|x|) \},
\]
\[
L^2_1(\mathbb{R}^2) = \{ f \in L^2(\mathbb{R}^2), f(x) = f(|x|) \}.
\]

It follows from (1.3), (3.38) and (3.39) that
\[
\mathcal{E}(E_1, E_2) \leq \mathcal{H}(E_1, E_2, n, v) = \mathcal{H}(E_1, E_2, n, v) - \frac{1}{2} \|v\|^2_{L^2(\mathbb{R}^2)}, \quad (3.123)
\]
We proceed our proof by contradiction.

Assume that there exist constants $\delta_0 > 0$, $R_0 > 0$ and a sequence $t_k \to T$ ($k \to +\infty$) such that
\[
\int_{|x|<R_0} (|E_1(t_k,x)|^2 + |E_2(t_k,x)|^2) \, dx \leq \frac{1}{1 + \eta} \|Q\|_{L^2(\mathbb{R}^2)}^2 - \delta_0, \quad (3.124)
\]
or
\[
\liminf_{k \to +\infty} \int_{|x|<R_0} |n(t_k,x)| \, dx = 0. \quad (3.125)
\]

We then complete the proof of case (1) by scaling and compactness method.

Let
\[
\begin{cases}
E_{1k}(x) = \frac{1}{\lambda_k} E_1 \left( t_k, \frac{x}{\lambda_k} \right), & E_{2k}(x) = \frac{1}{\lambda_k} E_2 \left( t_k, \frac{x}{\lambda_k} \right), \\
n_k(x) = \frac{1}{\lambda_k} n \left( t_k, \frac{x}{\lambda_k} \right), & v_k(x) = \frac{1}{\lambda_k} v \left( t_k, \frac{x}{\lambda_k} \right),
\end{cases} \quad (3.126)
\]
where
\[
\lambda_k^2 = \|\nabla E_1(t_k,x)\|^2_{L^2(\mathbb{R}^2)} + \|\nabla E_2(t_k,x)\|^2_{L^2(\mathbb{R}^2)}.
\]
Direct calculation gives

\[
\begin{cases}
\int_{\mathbb{R}^2} |\nabla E_{1k}|^2 dx + \int_{\mathbb{R}^2} |\nabla E_{2k}|^2 dx = 1, \\
\int_{\mathbb{R}^2} |E_{1k}|^2 dx + \int_{\mathbb{R}^2} |E_{2k}|^2 dx = \int_{\mathbb{R}^2} |E_{01}|^2 dx + \int_{\mathbb{R}^2} |E_{20}|^2 dx, \\
\mathcal{E}(E_{1k}, E_{2k}) = \frac{1}{\lambda_k^2} \mathcal{E}(E_1(t_k, x), E_2(t_k, x)), \\
\mathcal{H}_1(E_{1k}, E_{2k}, n_k) = \frac{1}{\lambda_k^2} \mathcal{H}_1(E_1(t_k, x), E_2(t_k, x), n(t_k, x)), \\
\mathcal{H}(t_k) = \mathcal{H}(0).
\end{cases}
\]

Note that

\[
\mathcal{H}(t_k) = \mathcal{E}(E_1(t_k, x), E_2(t_k, x)) + \frac{1}{2} \int_{\mathbb{R}^2} |\mathbf{v}(t_k)|^2 dx + \frac{1}{2} \int_{\mathbb{R}^2} [n(t_k) + (|E_1(t_k)|^2 + |E_2(t_k)|^2)]^2 dx.
\]

We then conclude

\[
\mathcal{E}(E_1(t_k), E_2(t_k)) \leq \mathcal{H}_1(E_1(t_k), E_2(t_k), n(t_k)) \leq \mathcal{H}(t_k) = \mathcal{H}(0),
\]

\[
\mathcal{E}(E_{1k}, E_{2k}) \leq \mathcal{H}_1(E_{1k}, E_{2k}, n_k, \mathbf{v}_k) \leq \frac{1}{\lambda_k^2} \mathcal{H}(0)^{k+\varepsilon} 0.
\]

Especially,

\[
\limsup_{k \to +\infty} \mathcal{E}(E_{1k}, E_{2k}) \leq 0, \quad \limsup_{k \to +\infty} \mathcal{H}_1(E_{1k}, E_{2k}, n_k) \leq 0.
\]

Using Cauchy-Schwartz inequality: \(ab \leq \frac{(a+b)^2}{4}\) \((a > 0, b > 0)\), we obtain

\[
\mathcal{E}(E_1, E_2)
\geq \|\nabla E_1\|_{L^2(\mathbb{R}^2)}^2 + \|\nabla E_2\|_{L^2(\mathbb{R}^2)}^2
\geq \int_{\mathbb{R}^2} (|E_1|^2 + |E_2|^2)^2 dx - 2\eta \int_{\mathbb{R}^2} |E_1|^2 |E_2|^2 dx
\geq \int_{\mathbb{R}^2} (|E_1|^2 + |E_2|^2)^2 dx - \frac{1 + \eta}{2} \int_{\mathbb{R}^2} (|E_1|^2 + |E_2|^2)^2 dx.
\]

This together with (3.127) and (3.131) yields

\[
\liminf_{k \to +\infty} \int_{\mathbb{R}^2} (|E_{1k}|^2 + |E_{2k}|^2)^2 dx
\geq \frac{2}{1 + \eta} \liminf_{k \to +\infty} \left( \int_{\mathbb{R}^2} (|\nabla E_{1k}|^2 + |\nabla E_{2k}|^2) dx - \mathcal{E}(E_{1k}, E_{2k}) \right)
\geq \frac{2}{1 + \eta}.
\]
On the other hand, since
\[
\int_{\mathbb{R}^2} \left( n_k + (|E_{1k}|^2 + |E_{2k}|^2) \right)^2 \, dx - (1 + \eta) \int_{\mathbb{R}^2} (|E_{1k}|^2 + |E_{2k}|^2)^2 \, dx \\
\leq \int_{\mathbb{R}^2} \left( n_k + (|E_{1k}|^2 + |E_{2k}|^2) \right)^2 \, dx - \int_{\mathbb{R}^2} \left( |E_{1k}|^2 + |E_{2k}|^2 \right)^2 \, dx \\
- \eta \int_{\mathbb{R}^2} |E_{1k}E_{2k} - E_{1k}E_{2k}|^2 \, dx \\
= 2 \left( \mathcal{H}_1 (E_{1k}, E_{2k}, n_k) - \|\nabla E_{1k}\|_{L^2(\mathbb{R}^2)}^2 - \|\nabla E_{2k}\|_{L^2(\mathbb{R}^2)}^2 \right),
\]
one has
\[
\limsup_{k \to +\infty} \int_{\mathbb{R}^2} \left( n_k + (|E_{1k}|^2 + |E_{2k}|^2) \right)^2 \, dx \\
- (1 + \eta) \int_{\mathbb{R}^2} (|E_{1k}|^2 + |E_{2k}|^2)^2 \, dx \leq -2. \tag{3.134}
\]
Noting Lemma 2.2, (3.4), (3.127) and
\[
\eta \int_{\mathbb{R}^2} \left( |E_{1k}|^2 + |E_{2k}|^2 \right)^2 \, dx \leq \frac{2\eta}{\|Q\|_{L^2(\mathbb{R}^2)}^2} \left( \|\nabla E_{1k}\|_{L^2(\mathbb{R}^2)}^2 + \|\nabla E_{2k}\|_{L^2(\mathbb{R}^2)}^2 \right) \\
\leq 2,
\]
we get
\[
\limsup_{k \to +\infty} \int_{\mathbb{R}^2} \left( n_k + (|E_{1k}|^2 + |E_{2k}|^2) \right)^2 \, dx - \int_{\mathbb{R}^2} \left( |E_{1k}|^2 + |E_{2k}|^2 \right)^2 \, dx \leq 0.
\]
In view of (3.38), one obtains
\[
\frac{1}{2} \int_{\mathbb{R}^2} \left[ \left( |E_1|^2 + |E_2|^2 \right)^2 + \frac{\eta}{2} \left| E_1E_2 - E_1E_2 \right|^2 \right] \, dx \\
= \|\nabla E_1\|_{L^2(\mathbb{R}^2)}^2 + \|\nabla E_2\|_{L^2(\mathbb{R}^2)}^2 - \mathcal{E}(E_1, E_2).
\]
Hence there holds
\[
\liminf_{k \to +\infty} \left( \int_{\mathbb{R}^2} (|E_{1k}|^2 + |E_{2k}|^2)^2 \, dx + \eta \int_{\mathbb{R}^2} |E_{1k}E_{2k} - E_{1k}E_{2k}|^2 \, dx \right) \\
= 2 \liminf_{k \to +\infty} \left( \|\nabla E_{1k}\|_{L^2(\mathbb{R}^2)}^2 + \|\nabla E_{2k}\|_{L^2(\mathbb{R}^2)}^2 - \mathcal{E}(E_{1k}, E_{2k}) \right) \\
\geq 2.
\]
In addition, (3.39) yields
\[
\frac{1}{2} \int_{\mathbb{R}^2} n_k^2 \, dx \\
= \mathcal{H}_1 (E_{1k}, E_{2k}, n_k) - \|\nabla E_{1k}\|_{L^2(\mathbb{R}^2)}^2 - \|\nabla E_{2k}\|_{L^2(\mathbb{R}^2)}^2 \\
- \frac{1}{2} \int_{\mathbb{R}^2} n_k (|E_{1k}|^2 + |E_{2k}|^2) \, dx + \frac{\eta}{2} \int_{\mathbb{R}^2} |E_{1k}E_{2k} - E_{1k}E_{2k}|^2 \, dx.
\]
Now for any $\epsilon \in (0, 1)$, by Young’s inequality, Hölder inequality and Cauchy-Schwartz inequality: $ab \leq \frac{(a+b)^2}{4}$, \forall a, b > 0, we have
\[
\int_{\mathbb{R}^2} n_k^2 \, dx \leq 2 \int_{\mathbb{R}^2} -n_k (|E_{1k}|^2 + |E_{2k}|^2) \, dx \\
+ \eta \int_{\mathbb{R}^2} |E_{1k}^2 E_{2k} - E_{1k}^2 E_{2k}|^2 \, dx \\
\leq \epsilon \|n_k\|_{L^2(\mathbb{R}^2)}^2 + \frac{1}{\epsilon} \int_{\mathbb{R}^2} (|E_{1k}|^2 + |E_{2k}|^2)^2 \, dx \\
+ \eta \int_{\mathbb{R}^2} (|E_{1k}|^2 + |E_{2k}|^2)^2 \, dx.
\]
This yields that
\[
\limsup_{k \to +\infty} \int_{\mathbb{R}^2} n_k^2 \, dx \leq \frac{1 + \eta \epsilon}{(1 - \epsilon)\epsilon} \int_{\mathbb{R}^2} (|E_{1k}|^2 + |E_{2k}|^2)^2 \, dx \\
\leq \frac{2 + 2\eta \epsilon}{(1 - \epsilon)\epsilon} \|E_{1k}\|_{L^2(\mathbb{R}^2)}^2 + \|E_{2k}\|_{L^2(\mathbb{R}^2)}^2.
\]  
(3.135)

Due to (3.124), (3.125) and $\lambda_k \xrightarrow{k \to +\infty} +\infty$, one obtains \forall $R > 0$,
\[
\limsup_{k \to +\infty} \int_{|x|<R} (|E_{1k}|^2 + |E_{2k}|^2) \, dx \leq \frac{1}{1 + \eta} \|Q\|_{L^2(\mathbb{R}^2)}^2 - \delta_0,
\]  
(3.136)
or
\[
\forall R > 0, \limsup_{k \to +\infty} \int_{|x|<R} |n_k| \, dx = 0. 
\]  
(3.137)

We continue to our discussion by a compactness argument.

By (3.127) and (3.135), there exists $(E'_1, E'_2, N') \in H^1(\mathbb{R}^2) \times H^1(\mathbb{R}^2) \times L^2(\mathbb{R}^2)$ such that
\[
(E_{1k}, E_{2k}) \rightharpoonup (E'_1, E'_2) \text{ in } H^1(\mathbb{R}^2) \times H^1(\mathbb{R}^2),
\]  
(3.138)
\[
\text{and } n_k \rightharpoonup N' \text{ in } L^2(\mathbb{R}^2). 
\]  
(3.139)

Since $H^1_p(\mathbb{R}^2) \hookrightarrow L^p(\mathbb{R}^2)$ $(2 < p < +\infty)$ is compact, we obtain
\[
(E_{1k}, E_{2k}) \to (E'_1, E'_2) \text{ in } L^4(\mathbb{R}^2) \times L^4(\mathbb{R}^2). 
\]  
(3.140)

On one hand,
\[
(E_{1k}^2, E_{2k}^2) \to (E'_1^2, E'_2^2) \text{ in } L^2(\mathbb{R}^2) \times L^2(\mathbb{R}^2). 
\]  
(3.141)

On the other hand,
\[
\int_{\mathbb{R}^2} (|E'_1|^2 + |E'_2|^2)^2 \, dx \geq \frac{1}{1 + \eta}, \quad (E'_1, E'_2) \neq (0, 0).
\]  
(3.142)

Let $R \to +\infty$, it follows from (3.136) that
\[
\int_{\mathbb{R}^2} (|E'_1|^2 + |E'_2|^2)^2 \, dx < \frac{1}{1 + \eta} \|Q\|_{L^2(\mathbb{R}^2)}^2,
\]  
(3.143)
or
\[
N' = 0. 
\]  
(3.144)
Then from the boundedness of weakly convergent sequence, we have

\[
\int_{|x|<R} (|E_1'|^2 + |E_2'|^2) \, dx \leq \liminf_{k \to +\infty} \int_{|x|<R} (|E_{1k}|^2 + |E_{2k}|^2) \, dx \\
\leq \frac{1}{1 + \eta} \|Q\|_{L^2(\mathbb{R}^2)}^2 - \delta_0,
\]

or

\[
\int_{|x|<R} |N'| \, dx \leq \liminf_{k \to +\infty} \int_{|x|<R} |n_k| \, dx = 0.
\]

In addition, there hold:

\[
\lim_{k \to +\infty} \int_{\mathbb{R}^2} n_k (|E_1|^2 + |E_{2k}|^2) \, dx = \lim_{k \to +\infty} \int_{\mathbb{R}^2} N' (|E_1|^2 + |E_{2}'|^2) \, dx,
\]

\[
\lim_{k \to +\infty} \int_{\mathbb{R}^2} |E_{1k}|^2 |E_{2k}|^2 \, dx = \lim_{k \to +\infty} \int_{\mathbb{R}^2} |E_1'|^2 |E_2'|^2 \, dx,
\]

\[
\lim_{k \to +\infty} \text{Re} \int_{\mathbb{R}^2} (E_{1k})^2 (E_{2k})^2 \, dx = \lim_{k \to +\infty} \text{Re} \int_{\mathbb{R}^2} (E_{1}')^2 (E_{2}')^2 \, dx,
\]

\[
\lim_{k \to +\infty} \int_{\mathbb{R}^2} |E_{1k} E_{2k} - E_{1k} \overline{E_{2k}}|^2 \, dx = \lim_{k \to +\infty} \int_{\mathbb{R}^2} |E_1' E_2' - E_1 \overline{E_2}|^2 \, dx.
\]

Indeed, from Lemma 2.1 and (3.141) it follows that (3.147) holds.

Next, direct calculation yields

\[
\left| \int_{\mathbb{R}^2} |E_{1k}|^2 |E_{2k}|^2 \, dx - \int_{\mathbb{R}^2} |E_1'|^2 |E_2'|^2 \, dx \right| \\
= \int_{\mathbb{R}^2} (|E_{1k}|^2 - |E_1'|^2) |E_{2k}|^2 \, dx + \int_{\mathbb{R}^2} |E_1'|^2 (|E_{2k}|^2 - |E_2'|^2) \, dx \\
\leq \left\| (|E_{1k}|^2 - |E_1'|^2) \right\|_{L^2(\mathbb{R}^2)} \|E_{2k}\|_{L^2(\mathbb{R}^2)} \\
+ \|E_1'\|_{L^2(\mathbb{R}^2)} \left\| |E_{2k}|^2 - |E_2'|^2 \right\|_{L^2(\mathbb{R}^2)}.
\]

Let \(k \to +\infty\), one gets (3.148).
We next note that

\[
\left| \operatorname{Re} \int_{\mathbb{R}^2} (E_{1k})^2 (E_{2k})^2 \, dx - \operatorname{Re} \int_{\mathbb{R}^2} (E'_1)^2 (E'_2)^2 \, dx \right|
\leq \int_{\mathbb{R}^2} \left| (E_{1k})^2 (E_{2k})^2 - (E'_1)^2 (E'_2)^2 \right| \, dx
\leq \int_{\mathbb{R}^2} \left[ (E_{1k})^2 - (E'_1)^2 \right] (E_{2k})^2 + (E'_1)^2 \left[ (E_{2k})^2 - (E'_2)^2 \right] \, dx
\leq \left\| (E_{1k})^2 - (E'_1)^2 \right\|_{L^2(\mathbb{R}^2)} \left\| (E_{2k})^2 \right\|_{L^2(\mathbb{R}^2)}
+ \left\| (E'_1)^2 \right\|_{L^2(\mathbb{R}^2)} \left\| (E_{2k})^2 - (E'_2)^2 \right\|_{L^2(\mathbb{R}^2)}.
\]

As \( k \to +\infty \), we get (3.149). Moreover, according to (3.148) and (3.149), (3.150) holds. We now use estimates (3.130), (3.147) and (3.150) to get

\[
\mathcal{H}_1 (E'_1, E'_2, N') \leq \liminf_{k \to +\infty} \mathcal{H}_1 (E_{1k}, E_{2k}, n_k) \leq 0,
\]

that is,

\[
\mathcal{E} (E'_1, E'_2) + \frac{1}{2} \int_{\mathbb{R}^2} \left[ N' + \left( |E'_1|^2 + |E'_2|^2 \right) \right] ^2 \, dx \leq 0.
\]

According to \( \int_{\mathbb{R}^2} \left( |E'_1|^2 + |E'_2|^2 \right) \, dx < \frac{\|Q\|_{L^2(\mathbb{R}^2)}^2}{1 + \eta} \), (3.152) then yields

\[
\int_{\mathbb{R}^2} \left( |\nabla E'_1|^2 + |\nabla E'_2|^2 \right) \, dx
\leq \frac{1}{2} \int_{\mathbb{R}^2} \left( |E'_1|^2 + |E'_2|^2 \right) ^2 \, dx + \frac{\eta}{2} \int_{\mathbb{R}^2} |E'_1 E'_2 - E'_1 E'_1|^2 \, dx
\leq \frac{\eta + 1}{2} \int_{\mathbb{R}^2} \left( |E'_1|^2 + |E'_2|^2 \right) ^2 \, dx
\leq (1 + \eta) \frac{\|E'_1\|_{L^2(\mathbb{R}^2)}^2 + \|E'_2\|_{L^2(\mathbb{R}^2)}^2}{\|Q\|_{L^2(\mathbb{R}^2)}^2} \left( \|\nabla E'_1\|_{L^2(\mathbb{R}^2)}^2 + \|\nabla E'_2\|_{L^2(\mathbb{R}^2)}^2 \right)
\leq \|\nabla E'_1\|_{L^2(\mathbb{R}^2)}^2 + \|\nabla E'_2\|_{L^2(\mathbb{R}^2)}^2.
\]

This is a contradiction.
On the other hand, if \( N' = 0 \), (3.4) yields

\[
\mathcal{H}_1(E_1', E_2', 0) \\
\geq \int_{\mathbb{R}^2} \left( |\nabla E_1'|^2 + |\nabla E_2'|^2 \right) \, dx - 2\eta \int_{\mathbb{R}^2} |E_1|^2 |E_2'|^2 \, dx \\
\geq \int_{\mathbb{R}^2} \left( |\nabla E_1'|^2 + |\nabla E_2'|^2 \right) \, dx - \frac{\eta}{2} \int_{\mathbb{R}^2} \left( |E_1'|^2 + |E_2'|^2 \right)^2 \, dx \\
\geq \int_{\mathbb{R}^2} \left( |\nabla E_1'|^2 + |\nabla E_2'|^2 \right) \, dx \\
- \eta \frac{\left( \|E_1'|^2_{L^2(\mathbb{R}^2)} + \|E_2'|^2_{L^2(\mathbb{R}^2)} \right)}{\|Q\|^2_{L^2(\mathbb{R}^2)}} \cdot \left( \|\nabla E_1'|^2_{L^2(\mathbb{R}^2)} + \|\nabla E_2'|^2_{L^2(\mathbb{R}^2)} \right) \\
\geq \int_{\mathbb{R}^2} \left( |\nabla E_1'|^2 + |\nabla E_2'|^2 \right) \, dx \left( 1 - \frac{\eta}{\|Q\|^2_{L^2(\mathbb{R}^2)}} \left( \|E_1'|^2_{L^2(\mathbb{R}^2)} + \|E_2'|^2_{L^2(\mathbb{R}^2)} \right) \right) \\
> 0,
\]

which is contradictory to (3.151). Hence there exists a constant \( m > 0 \) depending on the initial data such that for any \( R > 0 \), (3.119) and (3.120) hold.

Now we turn to consider the non-radial case (2). Assume that there exist constants \( R_0 > 0 \), \( \delta_0 > 0 \) and a sequence \( t_k \) such that as \( t_k \to T \) \((k \to +\infty)\),

\[
\liminf_{k \to +\infty} \left( \sup_y \int_{|x-y| < R_0} \left( |E_1(t_k, x)|^2 + |E_2(t_k, x)|^2 \right) \, dx \right) \leq \frac{\|Q\|^2_{L^2(\mathbb{R}^2)}}{1 + \eta} - \delta_0,
\]

or

\[
\liminf_{k \to +\infty} \left( \sup_y \int_{|x-y| < R_0} |n(t_k, x)| \, dx \right) \leq m_n - \delta_0.
\]

Then it follows from Lemma 3.7 that as \( t_k \to T \),

\[
\int_{\mathbb{R}^2} \left( |\nabla E_1(t_k)|^2 + |\nabla E_2(t_k)|^2 + |n(t_k)|^2 + |v(t_k)|^2 \right) \, dx \leq C.
\]

This is contradictory to the assumption that \((E_1, E_2, n, v)\) blows up at a finite time \( T \). So (3.121) and (3.122) hold.

This finishes the proof of Proposition 3.8.

We are now in the position to prove Proposition 3.5 by utilizing Proposition 3.6, Lemma 3.7 and Proposition 3.8.

**Proof of Proposition 3.5.**

Due to (2.1), Proposition 3.8 implies the conclusion (1) in Proposition 3.5. In fact, let \( R_1 > 0 \) be a fixed constant, then

\[
\left\| \tilde{E}_1(0, x) \right\|_{L^2(|x-x(t)| \leq R_1)}^2 + \left\| \tilde{E}_2(0, x) \right\|_{L^2(|x-x(t)| \leq R_1)}^2 \\
= \left\| E_1(t, x) \right\|_{L^2(|x-x(t)| \leq \frac{R_1}{1+t})}^2 + \left\| E_2(t, x) \right\|_{L^2(|x-x(t)| \leq \frac{R_1}{1+t})}^2.
\]
Noting that $\lambda(t) \to +\infty$ as $t \to T$ and (3.119), we have

$$\liminf_{t \to T} \left( \|\hat{E}_1(0, x)\|_{L^2(|x-x(t)| \leq R_1)}^2 + \|\hat{E}_2(0, x)\|_{L^2(|x-x(t)| \leq R_1)}^2 \right)$$

$$\geq \liminf_{t \to T} \left( \|E_1(t, x)\|_{L^2(|x-x(t)| \leq \frac{\rho}{\lambda(t)}}^2 + \|E_2(t, x)\|_{L^2(|x-x(t)| \leq \frac{\rho}{\lambda(t)}}^2 \right)$$

$$\geq \frac{Q}{1 + \eta}.$$

On the other hand, in view of Proposition 3.8, for any fixed $R_1 > 0$, Hölder’s inequality yields

$$\liminf_{t \to T} \|\hat{n}(0, x)\|_{L^2(|x-x(t)| \leq R_1)}$$

$$\geq \liminf_{t \to T} \|\hat{n}(0, x)\|_{L^1(|x-x(t)| \leq R_1)}$$

$$\geq \liminf_{t \to T} \|n(t, x)\|_{L^1(|x-x(t)| \leq \frac{\rho}{\lambda(t)}}$$

$$\geq m_n.$$

Hence (3.26) and (3.27) hold.

We now show conclusion (2) in Proposition 3.5 by using the same scaling argument as that adopted in Proposition 3.8.

For a sequence $t_k \to T$ ($k \to +\infty$), let

$$\begin{cases}
\hat{E}_1^n = \frac{1}{\lambda_n} \hat{E}_1 \left( \frac{x}{\lambda_n} \right), & \hat{E}_2^n = \frac{1}{\lambda_n} \hat{E}_2 \left( \frac{x}{\lambda_n} \right), \\
\hat{n}_n = \frac{1}{\lambda_n^2} \hat{n} \left( t_n, \frac{x}{\lambda_n} \right), & \hat{v}_n = \frac{1}{\lambda_n^2} \hat{v} \left( t_n, \frac{x}{\lambda_n} \right)
\end{cases}$$

satisfy

$$\int_{\mathbb{R}^2} \left( \|\nabla \hat{E}_1^n\|^2 + \|\nabla \hat{E}_2^n\|^2 + \frac{1}{2} |\hat{n}_n|^2 + \frac{1}{2} |\hat{v}_n|^2 \right) dx = 1.$$
Letting $n \to +\infty$ yields (3.32) and (3.33) due to (3.121) and (3.122), which ends the proof of Proposition 3.5. \hfill \square

3.3. Compactness of the Solution to the Rescaled Zakharov System (2.3).

Here, we discuss the compactness of $\left( \tilde{E}_1(0, x), \tilde{E}_2(0, x), \tilde{n}(0, x) \right)$.

Remark 3.9. From Proposition 3.5, it follows that $\left( \tilde{E}_1(0, x), \tilde{E}_2(0, x) \right)$ is bounded and weakly compact in $H^1(\mathbb{R}^2) \times H^1(\mathbb{R}^2)$. Then we can choose a sequence $t_n \to T$, and extract a subsequence (still denoted by $t_n$). Let

$$\left( \tilde{E}_1(0, x + x(t_n)), \tilde{E}_2(0, x + x(t_n)) \right) \rightharpoonup (E'_1, E'_2) \text{ in } H^1(\mathbb{R}^2) \times H^1(\mathbb{R}^2),$$

and

$$\tilde{n}(0, x + x(t_n)) \rightharpoonup N' \text{ in } L^2(\mathbb{R}^2).$$

Note that the embedding $H^1(\mathbb{R}^2) \hookrightarrow L^2_{loc}(\mathbb{R}^2)$ is compact, from Proposition 3.5, for a bounded domain $\Omega \subset \mathbb{R}^2$, there holds

$$\|E'_1\|_{L^2(\Omega)}^2 + \|E'_2\|_{L^2(\Omega)}^2 \geq c_1 > 0.$$  

Under this case, one can not exclude the case of $N' = 0$, that is, Proposition 3.5 can not guarantee the non-vanishing property of $N'$.

To overcome the difficulty mentioned in Remark 3.9, we will investigate the relation of $(E'_1, E'_2)$ and $N'$ to obtain the compactness property of $N'$ following the related information for $(E'_1, E'_2)$.

We now claim:

**Proposition 3.10.** Let $t_n \to T$. Then there exists a subsequence (still denote $t_n$) such that for a sequence $x_n := x(t_n) \in \mathbb{R}^2$ and $(E'_1, E'_2, N') \in H^1(\mathbb{R}^2) \times H^1(\mathbb{R}^2) \times L^2(\mathbb{R}^2)$, as $n \to +\infty$, the conclusions hold as below:

$$\left( \tilde{E}_1(0, x + x_n), \tilde{E}_2(0, x + x_n) \right) \rightharpoonup (E'_1, E'_2) \text{ in } H^1(\mathbb{R}^2) \times H^1(\mathbb{R}^2), \quad (3.153)$$

and

$$\tilde{n}(0, x + x_n) \rightharpoonup N' \text{ in } L^2(\mathbb{R}^2). \quad (3.154)$$

Furthermore, there exist constants $\beta_1 > 0$ and $R_1 > 0$ depending only on $\|E_{10}\|_{L^2(\mathbb{R}^2)}$, $\|E_{20}\|_{L^2(\mathbb{R}^2)}$ such that

$$\left( \|E'_1\|_{L^2(|x| \leq R_1)}^2 + \|E'_2\|_{L^2(|x| \leq R_1)}^2 \right)^{\frac{1}{2}} \geq \beta_1, \quad (3.155)$$

and

$$\mathcal{H}(E'_1, E'_2, N', 0) \leq 0. \quad (3.156)$$

**Proof.** In view of Proposition 3.5, we will show this proposition by implementing the classical iteration technique, Concentration-compactness principle and mathematical induction.
Let $\beta_1$ be defined as in (ii) of Proposition 3.5. Assume that $\left(\hat{E}_{1n}, \hat{E}_{2n}, \tilde{n}_n, \tilde{\upsilon}_n\right) \in H^1(\mathbb{R}^2) \times H^1(\mathbb{R}^2) \times L^2(\mathbb{R}^2) \times L^2(\mathbb{R}^2)$ satisfy

$$
\begin{aligned}
\hat{E}_{1n} &= \frac{1}{\lambda_n(t_n)} \hat{E}_1 \left( t_n + \frac{s}{\lambda_n(t_n)}, \frac{x}{\lambda_n(t_n)} \right), \\
\hat{E}_{2n} &= \frac{1}{\lambda_n(t_n)} \hat{E}_2 \left( t_n + \frac{s}{\lambda_n(t_n)}, \frac{x}{\lambda_n(t_n)} \right), \\
\tilde{n}_n &= \frac{1}{\lambda_n^2(t_n)} \tilde{n} \left( t_n + \frac{s}{\lambda_n(t_n)}, \frac{x}{\lambda_n(t_n)} \right), \\
\tilde{\upsilon}_n &= \frac{1}{\lambda_n^2(t_n)} \tilde{\upsilon} \left( t_n + \frac{s}{\lambda_n(t_n)}, \frac{x}{\lambda_n(t_n)} \right).
\end{aligned}
$$

(3.157)

Direct calculation yields

$$
\lambda_n^2(t_n) = \int_{\mathbb{R}^2} |\nabla \hat{E}_1|^2 \, dx + \int_{\mathbb{R}^2} |\nabla \hat{E}_2|^2 \, dx + \frac{1}{2} \int_{\mathbb{R}^2} |\tilde{n}|^2 \, dx + \frac{1}{2} \int_{\mathbb{R}^2} |\tilde{\upsilon}|^2 \, dx.
$$

(3.158)

We will show that there exist $(E'_1, E'_2, N') \in H^1(\mathbb{R}^2) \times H^1(\mathbb{R}^2) \times L^2(\mathbb{R}^2)$ and a sequence $x_n \in \mathbb{R}^2$ such that as $n \to +\infty$,

$$
\left( \hat{E}_{1n}(x_n + x), \hat{E}_{2n}(x_n + x) \right) \to (E'_1(x), E'_2(x)) \text{ in } H^1(\mathbb{R}^2) \times H^1(\mathbb{R}^2),
$$

(3.162)

and

$$
\tilde{n}_n(x_n + x) \to N'(x) \text{ in } L^2(\mathbb{R}^2),
$$

(3.163)

and

$$
\left( \int_{|x| \leq R_1} \left( |E'_1|^2 + |E'_2|^2 \right) \, dx \right)^{\frac{1}{2}} \geq \beta_1, \quad \mathcal{H}(E'_1, E'_2, N') \leq 0.
$$

(3.164)

By Proposition 3.3 and Corollary 3.4, there exist constants $c_1 > 0$, $c_2 > 0$ such that

$$
c_1 \leq \int_{\mathbb{R}^2} \left( |\nabla \hat{E}_{1n}|^2 + |\nabla \hat{E}_{2n}|^2 \right) \, dx \leq c_2, \quad c_1 \leq \int_{\mathbb{R}^2} |\tilde{n}_n|^2 \, dx \leq c_2.
$$

(3.165)

Let the integer $k_0$ be defined by

$$
\left( \int_{\mathbb{R}^2} \left( |\hat{E}_1|^2 + |\hat{E}_{2n}|^2 \right) \, dx \right)^{\frac{1}{2}} < (k_0 + 1)\beta_1,
$$

(3.166)

where

$$
k_0 = 1, 2, ..., E \left[ \frac{\|E_{10}\|_{L^2(\mathbb{R}^2)}^2 + \|E_{20}\|_{L^2(\mathbb{R}^2)}^2}{\beta_1} - 1 \right].
$$

(3.167)
We then show Proposition 3.10 by induction on the integer $k_0$.

(1) For $k_0 = 1$, there holds
\[
\frac{\|E_{10}\|_{L^2(\mathbb{R}^2)}^2 + \|E_{20}\|_{L^2(\mathbb{R}^2)}^2}{\beta_1} = 2, \quad \text{that is},
\|E_{10}\|_{L^2(\mathbb{R}^2)}^2 + \|E_{20}\|_{L^2(\mathbb{R}^2)}^2 = 2\beta_1.
\] (3.168)

From (3.159) it follows that
\[
\left\|E_{1n}\right\|^2_{L^2(\mathbb{R}^2)} + \left\|E_{2n}\right\|^2_{L^2(\mathbb{R}^2)} \leq \|E_{10}\|_{L^2(\mathbb{R}^2)}^2 + \|E_{20}\|_{L^2(\mathbb{R}^2)}^2 = 2\beta_1.
\] (3.169)

Hence (3.166) holds for $k_0 = 1$. From compactness and the boundedness of weakly convergent sequence, similar argument to the proof of Lemma 3.7 yields
\[
\mathcal{H}(E_1', E_2', N') \leq \lim_{n \to +\infty} \mathcal{H}(\hat{E}_{1n}, \hat{E}_{2n}, \hat{n}_n) \leq 0.
\]
So (3.162)-(3.165) are true. Note that Proposition 3.5, Proposition 3.10 holds for $k_0 = 1$.

(2) Assume that (3.162)-(3.165) are true for $k_0 > 1$, we then show that they also hold for $k_0 + 1$.

Let \( (\hat{E}_{1n}, \hat{E}_{2n}, \hat{n}_n) \) be the sequence satisfying (3.164), (3.165), (3.166) and
\[
\lim_{n \to +\infty} \mathcal{H}(\hat{E}_{1n}, \hat{E}_{2n}, \hat{n}_n, 0) \leq 0.
\]
In view of Proposition 3.5, we may assume that there exist a sequence $x_n \in \mathbb{R}^2$ and a constant $R = R(c_1, c_2)$ satisfying
\[
\left(\int_{|x-x_n|<R} \left(\hat{E}_{1n}^2 + \hat{E}_{2n}^2\right) dx\right)^{\frac{1}{2}} \geq \beta_1.
\] (3.170)

and \( (\hat{E}_1, \hat{E}_2, \hat{N}) \in H^1(\mathbb{R}^2) \times H^1(\mathbb{R}^2) \times L^2(\mathbb{R}^2) \) such that
\[
(\hat{E}_{1n}(x + x_n), \hat{E}_{2n}(x + x_n)) \rightharpoonup (\hat{E}_1, \hat{E}_2) \text{ in } H^1(\mathbb{R}^2) \times H^1(\mathbb{R}^2),
\]
\[
\hat{n}_n(x + x_n) \rightharpoonup \hat{N} \text{ in } L^2(\mathbb{R}^2).
\]
We extract a subsequence still denoted by \( (\hat{E}_{1n}, \hat{E}_{2n}, \hat{n}_n) \) for simplicity. We make the following decomposition for the extracted subsequence:
\[
\begin{cases}
\hat{E}_{1n}(x + x_n) = \hat{E}_{1n,1}(x + x_n) + \hat{E}_{1n,2}(x + x_n), \\
\hat{E}_{2n}(x + x_n) = \hat{E}_{2n,1}(x + x_n) + \hat{E}_{2n,2}(x + x_n), \\
\hat{n}_n(x + x_n) = \hat{n}_{n,1}(x + x_n) + \hat{n}_{n,2}(x + x_n).
\end{cases}
\]

Let $R_n \to +\infty$ as $n \to +\infty$. This decomposition admits the following properties:

(I)
\[
\begin{align*}
\hat{E}_{1n,1}(x) &= \hat{E}_{2n,1}(x) = \hat{n}_{n,1}(x) = 0, & |x| \leq \frac{R_n}{2}, \\
\hat{E}_{1n,2}(x) &= \hat{E}_{2n,2}(x) = \hat{n}_{n,2}(x) = 0, & |x| \geq R_n.
\end{align*}
\]
(II) As \( n \to +\infty \),
\[
\int_{\mathbb{R}^2} \left( \left| \hat{E}_{1n,1} \right|^2 + \left| \hat{E}_{1n,2} \right|^2 + \left| \hat{E}_{2n,1} \right|^2 + \left| \hat{E}_{2n,2} \right|^2 \right) \, dx
- \int_{\mathbb{R}^2} \left( \left| \hat{E}_{1n} \right|^2 + \left| \hat{E}_{2n} \right|^2 \right) \, dx \to 0,
\]
\[
\int_{\mathbb{R}^2} \left( \left| \nabla \hat{E}_{1n,1} \right|^2 + \left| \nabla \hat{E}_{1n,2} \right|^2 + \left| \nabla \hat{E}_{2n,1} \right|^2 + \left| \nabla \hat{E}_{2n,2} \right|^2 \right) \, dx
- \int_{\mathbb{R}^2} \left( \left| \nabla \hat{E}_{1n} \right|^2 + \left| \nabla \hat{E}_{2n} \right|^2 \right) \, dx \to 0,
\]
\[
\int_{\mathbb{R}^2} \left( \left| \hat{n}_{n,1} \right|^2 + \left| \hat{n}_{n,2} \right|^2 \right) \, dx - \int_{\mathbb{R}^2} \left| \hat{n}_n \right|^2 \, dx \to 0,
\]

(III)
\[
\lim_{n \to +\infty} \mathcal{H} \left( \hat{E}_{1n,1}, \hat{E}_{2n,1}, \hat{n}_{n,1}, 0 \right) + \lim_{n \to +\infty} \mathcal{H} \left( \hat{E}_{1n,2}, \hat{E}_{2n,2}, \hat{n}_{n,2}, 0 \right) \leq 0.
\]

Note that as \( n \to +\infty \),
\[
\int_{\mathbb{R}^2} \left( \left| \hat{E}_{1n,1} \right|^2 + \left| \hat{E}_{2n,1} \right|^2 \right) \, dx \to \int_{\mathbb{R}^2} \left( \left| \hat{E}_1 \right|^2 + \left| \hat{E}_2 \right|^2 \right) \, dx.
\]

Especially, \( \int_{\mathbb{R}^2} \left( \left| \hat{E}_1 \right|^2 + \left| \hat{E}_2 \right|^2 \right) \, dx \geq \beta_1 \). So when \( n \) is large enough, there holds
\[
\int_{\mathbb{R}^2} \left( \left| \hat{E}_{1n,2} \right|^2 + \left| \hat{E}_{2n,2} \right|^2 \right) \, dx < k_0 \beta_1.
\]

We proceed our argument by dividing two cases:

**Case 1:**
\[
\mathcal{H} \left( \hat{E}_1, \hat{E}_2, \hat{N}, 0 \right) \leq \lim_{n \to +\infty} \mathcal{H} \left( \hat{E}_{1n,1}, \hat{E}_{2n,1}, \hat{n}_{n,1}, 0 \right) \leq 0.
\]

In this case, letting \( E'_1 = \hat{E}_1, \ E'_2 = \hat{E}_2, \ N' = \hat{N} \), one can get the conclusion of Proposition 3.10.

**Case 2:**
\[
\mathcal{H} \left( \hat{E}_1, \hat{E}_2, \hat{N}, 0 \right) > 0.
\]

In this case, let \( P_1 = \lim_{n \to +\infty} \mathcal{H} \left( \hat{E}_{1n,1}, \hat{E}_{2n,1}, \hat{n}_{n,1}, 0 \right) > 0 \), and for \( n \) large enough,
\[
\mathcal{H} \left( \hat{E}_{1n,2}, \hat{E}_{2n,2}, \hat{n}_{n,2}, 0 \right) \leq -\frac{P_1}{2} < 0.
\]

By induction assumption, there exists a sequence \( y_n \in \mathbb{R}^2 \) such that
\[
\left( \hat{E}_{1n,2}(+y_n), \hat{E}_{2n,2}(+y_n) \right) \to (E'_1, E'_2) \text{ in } H^1(\mathbb{R}^2) \times H^1(\mathbb{R}^2),
\]
\[
\hat{n}_{n,2}(+y_n) \to N' \text{ in } L^2(\mathbb{R}^2),
\]
there exists a constant $c_\ast > 0$ such that
\begin{equation}
\frac{c_\ast}{\|N'\|_{L^2(\mathbb{R}^2)}} \leq 1.
\end{equation}

**Proof.** From Proposition 3.10 it follows that
\begin{equation}
\mathcal{H}(E_1', E_2', N') = \int_{\mathbb{R}^2} \left( |\nabla E_1'|^2 + |\nabla E_2'|^2 + N' \left( |E_1'|^2 + |E_2'|^2 \right) + \frac{1}{2} N'^2 \right) dx
\end{equation}

\begin{equation}
+ \frac{\eta}{2} \int_{\mathbb{R}^2} \left[ \left( E_1' \right)^2 \left( E_2' \right)^2 + \left( E_2' \right)^2 \left( E_1' \right)^2 \right] - 2 |E_1'|^2 |E_2'|^2 \right) dx
\end{equation}

\begin{equation}
\leq 0,
\end{equation}

and
\begin{equation}
\|E_1'\|_{L^2(\mathbb{R}^2)}^2 + \|E_2'\|_{L^2(\mathbb{R}^2)}^2 \geq \beta_1.
\end{equation}

Sobolev embedding theorem ($\forall q \in [n, +\infty), W^{1,n}(\mathbb{R}^n) \subset L^q(\mathbb{R}^n)$) yields $H^1(\mathbb{R}^2) \subset L^2(\mathbb{R}^2)$. In view of (3.161), there exists $c_1 > 0$ such that
\begin{equation}
0 < (c_1)^2 \leq \int_{\mathbb{R}^2} \left( |\nabla E_1'|^2 + |\nabla E_2'|^2 \right) dx \leq 1.
\end{equation}

On one hand, by (3.172) one has
\begin{equation}
- \int_{\mathbb{R}^2} N' \left( |E_1'|^2 + |E_2'|^2 \right) dx
\end{equation}

\begin{equation}
\geq \int_{\mathbb{R}^2} \left( |\nabla E_1'|^2 + |\nabla E_2'|^2 \right) dx - \eta \int_{\mathbb{R}^2} |E_1'|^2 |E_2'|^2 dx
\end{equation}

\begin{equation}
+ \frac{\eta}{2} \int_{\mathbb{R}^2} \left( E_1'^2 \left( E_2' \right)^2 + \left( E_2' \right)^2 \left( E_1' \right)^2 \right) dx
\end{equation}

\begin{equation}
\geq \int_{\mathbb{R}^2} \left( |\nabla E_1'|^2 + |\nabla E_2'|^2 \right) dx - \eta \int_{\mathbb{R}^2} \left( |E_1'|^2 + |E_2'|^2 \right)^2 dx
\end{equation}

\begin{equation}
\geq \int_{\mathbb{R}^2} \left( |\nabla E_1'|^2 + |\nabla E_2'|^2 \right) dx
\end{equation}

\begin{equation}
- \frac{\eta}{\|Q\|_{L^2(\mathbb{R}^2)}} \int_{\mathbb{R}^2} \left( |E_1'|^2 + |E_2'|^2 \right) dx \int_{\mathbb{R}^2} \left( |\nabla E_1'|^2 + |\nabla E_2'|^2 \right) dx
\end{equation}

\begin{equation}
= \left[ 1 - \frac{\eta}{\|Q\|_{L^2(\mathbb{R}^2)}} \int_{\mathbb{R}^2} \left( |E_{10}|^2 + |E_{20}|^2 \right) dx \right]
\end{equation}

\begin{equation}
\cdot \int_{\mathbb{R}^2} \left( |\nabla E_1'|^2 + |\nabla E_2'|^2 \right) dx.
\end{equation}
Let 
\[
c_0 = 1 - \frac{\eta}{\|Q\|_{L^2(\mathbb{R}^2)}} \int_{\mathbb{R}^2} (|E_{10}|^2 + |E_{20}|^2) \, dx > 0.
\] (3.176)

According to (3.4), one has 
\[0 < c_0 < \frac{1}{1 + \eta}\]. On the other hand,

\[
- \int_{\mathbb{R}^2} N' \left( |E'_1|^2 + |E'_2|^2 \right) \, dx \\
\leq \left( \int_{\mathbb{R}^2} |N'|^2 \, dx \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^2} \left( |E'_1|^2 + |E'_2|^2 \right)^2 \, dx \right)^{\frac{1}{2}} \\
\leq \sqrt{2} \||N'||_{L^2(\mathbb{R}^2)} \frac{1}{\|Q\|_{L^2(\mathbb{R}^2)}} \left[ \int_{\mathbb{R}^2} \left( |E'_1|^2 + |E'_2|^2 \right) \, dx \right]^{\frac{1}{2}} \\
\cdot \left( \int_{\mathbb{R}^2} \left( |\nabla E'_1|^2 + |\nabla E'_2|^2 \right) \, dx \right)^{\frac{1}{2}}.
\]

Combining (3.174) with (3.175) and (3.176) yields 
\[
\|N'||_{L^2(\mathbb{R}^2)} \geq \sqrt{\frac{\eta}{2}} c_0 c_1 = c^* > 0.
\]

From (3.174) and (3.176) it follows that 
\[0 < c_0 < \frac{1}{1 + \eta}, \quad 0 < c_1 \leq 1\]. We then conclude 
\[
c^* = \sqrt{\frac{\eta}{2}} c_0 c' = \sqrt{\frac{\eta}{2}} c_0 c_1 < \sqrt{\frac{\eta}{2}} \frac{1}{1 + \eta} c_1 < 1.
\]

So far, the proof of Corollary 3.11 is completed. \(\square\)

3.4. Proof of Theorem 3.1.

In this subsection, we shall establish some estimates for \(\left( \tilde{E}_1, \tilde{E}_2, \tilde{n}, \tilde{v} \right) (0)\) based on these estimates obtained in subsection 3.1, subsection 3.2 and subsection 3.3. Next, by considering the rescaled Zakharov system (2.3c)-(2.3d):

\[
\tilde{n}_s = -\nabla \cdot \tilde{v},
\] (3.177)

\[
\tilde{v}_s = -\nabla \left( \tilde{n} + |\tilde{E}_1|^2 + |\tilde{E}_2|^2 \right),
\] (3.178)

We then finish the proof of Theorem 3.1.

Proof of Theorem 3.1.
∀ t > 0, we consider \( \left( \tilde{E}_1(s), \tilde{E}_2(s), \tilde{n}(s), \tilde{v}(s) \right) \) for \([0, \lambda(t)(T - t))\). From (2.5) it follows that

\[
\left\| (\tilde{E}_1(0), \tilde{E}_2(0), \tilde{n}(0), \tilde{v}(0)) \right\|_{H^1(\mathbb{R}^2) \times H^1(\mathbb{R}^2) \times L^2(\mathbb{R}^2) \times L^2(\mathbb{R}^2)}^2 = 1,
\]

\[
\lim_{s \to \lambda(t)(T-t)} \left\| (\tilde{E}_1(s), \tilde{E}_2(s), \tilde{n}(s), \tilde{v}(s)) \right\|_{H^1(\mathbb{R}^2) \times H^1(\mathbb{R}^2) \times L^2(\mathbb{R}^2) \times L^2(\mathbb{R}^2)}^2 = +\infty.
\]

Let \( A > 1 \) be a fixed constant. By the continuity of \( s \), there exists a \( \theta(t) > 0 \) such that \( \forall s \in [0, \theta(t)] \),

\[
\left\| (\tilde{E}_1(s), \tilde{E}_2(s), \tilde{n}(s), \tilde{v}(s)) \right\|_{H^1(\mathbb{R}^2) \times H^1(\mathbb{R}^2) \times L^2(\mathbb{R}^2) \times L^2(\mathbb{R}^2)}^2 \leq A, \quad (3.179)
\]

\[
\left\| (\tilde{E}_1(\theta(t)), \tilde{E}_2(\theta(t)), \tilde{n}(\theta(t)), \tilde{v}(\theta(t))) \right\|_{H^1(\mathbb{R}^2) \times H^1(\mathbb{R}^2) \times L^2(\mathbb{R}^2) \times L^2(\mathbb{R}^2)}^2 = A. \quad (3.180)
\]

We now claim that as \( t \to T \), there exists a uniform lower bound \( \theta_0 > 0 \) for \( \theta(t) \), that is,

\[
\theta(t) \geq \theta_0. \quad (3.181)
\]

We proceed our discussion through two sides. On one hand, some properties for \( \left( \tilde{E}_1(\theta), \tilde{E}_2(\theta), \tilde{n}(\theta), \tilde{v}(\theta) \right) \) will be established. On the other hand, in view of (3.177) and the compactness for \( \tilde{n} \), we complete the proof of (3.181) by contradiction.

Firstly we claim that:

**Proposition 3.12.** There exist constants \( c_1 > 0 \) and \( c_2 > 0 \) independent of \( t \) and \( A > 1 \) such that

(1) \( \forall s \in [0, \theta(t)] \),

\[
\left( \left\| \nabla \tilde{E}_1(s) \right\|_{L^2(\mathbb{R}^2)}^2 + \left\| \nabla \tilde{E}_2(s) \right\|_{L^2(\mathbb{R}^2)}^2 \right)^\frac{1}{2} \leq Ac_2, \quad (3.182)
\]

\[
\left\| \tilde{n}(s) \right\|_{L^2(\mathbb{R}^2)} \leq Ac_2, \quad \left\| \tilde{v}(s) \right\|_{L^2(\mathbb{R}^2)} \leq Ac_2. \quad (3.183)
\]

(2) Let \( t_n \to T \). Extracting a subsequence, still denoted by \( t_n \), such that for sequences \( x_n := x(t_n) \in \mathbb{R}^2 \), \( (E_1', E_2', N') \in H^1(\mathbb{R}^2) \times H^1(\mathbb{R}^2) \times L^2(\mathbb{R}^2) \), there hold

\[
\left( \tilde{E}_1(t_n, \theta(t_n), x - x_n), \tilde{E}_2(t_n, \theta(t_n), x - x_n) \right)
\]

\[
= \left( \tilde{E}_1(\theta(t_n), x - x_n), \tilde{E}_2(\theta(t_n), x - x_n) \right)
\]

\[
\to (E_1', E_2') \text{ in } H^1(\mathbb{R}^2) \times H^1(\mathbb{R}^2),
\]

\[
\tilde{n}(t_n, \theta(t_n), x - x_n) = \tilde{n}(\theta(t_n), x - x_n) \to N' \text{ in } L^2(\mathbb{R}^2),
\]

\[
\left( \left\| \nabla E_1' \right\|_{L^2(\mathbb{R}^2)}^2 + \left\| \nabla E_2' \right\|_{L^2(\mathbb{R}^2)}^2 \right)^\frac{1}{2} \geq Ac_1, \quad \left\| N' \right\|_{L^2(\mathbb{R}^2)} \geq Ac_1. \quad (3.186)
\]
Proof. It follows from (3.179) and (3.180) that
\[
\int_{\mathbb{R}^2} \left( |\nabla \tilde{E}_1(s)|^2 + |\nabla \tilde{E}_2(s)|^2 + \frac{1}{2} |\tilde{n}(s)|^2 + \frac{1}{2} |\tilde{v}(s)|^2 \right) dx \leq A^2. \tag{3.187}
\]

Note that (2.1), there holds
\[
\left\| (\tilde{E}_1, \tilde{E}_2, \tilde{n}, \tilde{v}) (s) \right\|^2_{H^1(\mathbb{R}^2) \times H^1(\mathbb{R}^2) \times L^2(\mathbb{R}^2) \times L^2(\mathbb{R}^2)}
\]
\[
= \int_{\mathbb{R}^2} \left( |\nabla \tilde{E}_1(t, s)|^2 + |\nabla \tilde{E}_2(t, s)|^2 + \frac{1}{2} |\tilde{n}(t, s)|^2 + \frac{1}{2} |\tilde{v}(t, s)|^2 \right) dx
\]
\[
= \frac{1}{\lambda^2(t)} \left[ \int_{\mathbb{R}^2} \left( |\nabla E_1 \left( t + \frac{s}{\lambda(t)} \right) |^2 + |\nabla E_2 \left( t + \frac{s}{\lambda(t)} \right) |^2 \right.ight.
\]
\[
+ \left. \frac{1}{2} \right] |n \left( t + \frac{s}{\lambda(t)} \right)|^2 + \left. \frac{1}{2} |v \left( t + \frac{s}{\lambda(t)} \right)|^2 \right) dx \right]
\]
\[
= \left( \frac{\lambda \left( t + \frac{s}{\lambda(t)} \right)}{\lambda(t)} \right)^2.
\]

So \( \forall s \in [0, \theta(t)] \),
\[
\frac{\lambda \left( t + \frac{s}{\lambda(t)} \right)}{\lambda(t)} \leq A, \quad \frac{\lambda \left( t + \frac{\theta(t)}{\lambda(t)} \right)}{\lambda(t)} = A. \tag{3.189}
\]

This yields that
\[
\tilde{E}_1(t, \theta(t), x) = \frac{1}{\lambda(t)} E_1 \left( t + \frac{\theta(t)}{\lambda(t)}, \frac{x}{\lambda(t)} \right)
\]
\[
= \frac{A}{A \lambda(t)} E_1 \left( t + \frac{\theta(t)}{\lambda(t)}, \frac{Ax}{A \lambda(t)} \right)
\]
\[
= \frac{A}{\lambda \left( t + \frac{\theta(t)}{\lambda(t)} \right)} E_1 \left( t + \frac{\theta(t)}{\lambda(t)}, \frac{Ax}{t + \frac{\theta(t)}{\lambda(t)}} \right)
\]
\[
= A \tilde{E}_1 \left( t + \frac{\theta(t)}{\lambda(t)}, 0, Ax \right). \tag{3.190}
\]

Similar argument to (3.190) yields
\[
\tilde{E}_2(t, \theta(t), x) = A \tilde{E}_2 \left( t + \frac{\theta(t)}{\lambda(t)}, 0, Ax \right), \tag{3.191}
\]
\[
\tilde{n}(t, \theta(t), x) = A^2 \tilde{n} \left( t + \frac{\theta(t)}{\lambda(t)}, 0, Ax \right). \tag{3.192}
\]

As \( t_n \to T \), there holds \( t_n + \frac{\theta(t_n)}{\lambda(t_n)} \to T \). Hence from Proposition 3.10 and Corollary 3.11, it follows that there exists a subsequence (still denoted by \( t_n \)) such that for
following conclusions hold:

\[
\left( \tilde{E}_1 \left( t_n + \frac{\theta(t_n)}{\lambda(t_n)}, 0, x + x_n \right), \tilde{E}_2 \left( t_n + \frac{\theta(t_n)}{\lambda(t_n)}, 0, x + x_n \right) \right) \rightarrow (E'_1, E'_2) \text{ in } H^1(\mathbb{R}^2) \times H^1(\mathbb{R}^2),
\]

(3.193)

In addition, in view of Proposition 3.10 and Corollary 3.11, there exists a constant \( c_1 > 0 \) such that

\[
\left( \|E'_1\|_{L^2(\mathbb{R}^2)}^2 + \|E'_2\|_{L^2(\mathbb{R}^2)}^2 \right)^{\frac{1}{2}} \geq c_1, \quad \|N'\|_{L^2(\mathbb{R}^2)} \geq c_1.
\]

(3.195)

Hence, combining (3.190)-(3.195) yields

\[
\left( \tilde{E}_1 \left( t_n, \theta(t_n), x + \frac{x_n}{A} \right), \tilde{E}_2 \left( t_n, \theta(t_n), x + \frac{x_n}{A} \right) \right) = \left( A\tilde{E}_1 \left( t_n + \frac{\theta(t_n)}{\lambda(t_n)}, 0, Ax + x_n \right), A\tilde{E}_2 \left( t_n + \frac{\theta(t_n)}{\lambda(t_n)}, 0, Ax + x_n \right) \right)
\]

\[
\rightarrow (AE'_1(Ax), AE'_2(Ax)) \text{ in } H^1(\mathbb{R}^2) \times H^1(\mathbb{R}^2),
\]

(3.196)

\[
\tilde{n} \left( t_n, \theta(t_n), x + \frac{x_n}{A} \right) = A^2 \tilde{n} \left( t_n + \frac{\theta(t_n)}{\lambda(t_n)}, 0, Ax + x_n \right)
\]

\[
\rightarrow A^2 N'(Ax) \text{ in } L^2(\mathbb{R}^2),
\]

(3.197)

and

\[
\left( \|\nabla [AE'_1(Ax)]\|_{L^2(\mathbb{R}^2)}^2 + \|\nabla [AE'_2(Ax)]\|_{L^2(\mathbb{R}^2)}^2 \right)^{\frac{1}{2}}
\]

\[
= A \left( \|\nabla E'_1\|_{L^2(\mathbb{R}^2)}^2 + \|\nabla E'_2\|_{L^2(\mathbb{R}^2)}^2 \right)^{\frac{1}{2}}
\]

\[
\geq A c_1,
\]

(3.198)

\[
\|A^2 N'(Ax)\|_{L^2(\mathbb{R}^2)} = A \|N'\|_{L^2(\mathbb{R}^2)} \geq A c_1.
\]

(3.199)

This finishes the proof of Proposition 3.12. \( \square \)

**Remark 3.13.** By the property of weak convergence, from (3.196)-(3.199) it follows that

\[
c_1 \leq \left( \|\nabla E'_1\|_{L^2(\mathbb{R}^2)}^2 + \|\nabla E'_2\|_{L^2(\mathbb{R}^2)}^2 \right)^{\frac{1}{2}}
\]

\[
\leq \liminf_{n \to +\infty} \left( \|\nabla E'_1\|_{L^2(\mathbb{R}^2)}^2 + \|\nabla E'_2\|_{L^2(\mathbb{R}^2)}^2 \right)^{\frac{1}{2}}
\]

\[
c_1 \leq \|N'\|_{L^2(\mathbb{R}^2)} \leq \liminf_{n \to +\infty} \|\tilde{n}\|_{L^2(\mathbb{R}^2)}.
\]

\( \square \)
Next, we fixed $A$ such that $A\Omega \geq 4$.

**Remark 3.14.** Note that $\|\tilde{v}(t, 0)\|_{L^2(\mathbb{R}^2)} \leq \sqrt{2}$, we confine the value of $A$ to distinguish $\|\tilde{v}(t, \theta(t))\|_{L^2(\mathbb{R}^2)}$ and $\|\tilde{v}(t, 0)\|_{L^2(\mathbb{R}^2)}$. 

**Remark 3.15.** By Proposition 3.12, one can obtain more delicate estimates on $\tilde{v}(s)$. Compared to the classical case, these estimates obtained are uniform. 

**Corollary 3.16.** For any $s \in [0, \theta(t)]$, there holds 

$$\|\tilde{v}(s)\|_{L^2(\mathbb{R}^2)} \leq A\sqrt{\frac{2}{\eta}}.$$

**Proof.** From (2.7) it follows that 

$$\int_{\mathbb{R}^2} \left( |\nabla \tilde{E}^1|^2 + |\nabla \tilde{E}^2|^2 \right) dx - \frac{1}{2} \int_{\mathbb{R}^2} \left( |\tilde{E}^1|^2 + |\tilde{E}^2|^2 \right)^2 dx$$

$$= \int_{\mathbb{R}^2} \left( |\nabla \tilde{E}^1|^2 + |\nabla \tilde{E}^2|^2 \right) dx - \frac{1}{2} \int_{\mathbb{R}^2} \left( |\tilde{E}^1|^2 + |\tilde{E}^2|^2 \right)^2 dx$$

$$\geq \int_{\mathbb{R}^2} \left( |\nabla \tilde{E}^1|^2 + |\nabla \tilde{E}^2|^2 \right) dx - \frac{1 + \eta}{2} \int_{\mathbb{R}^2} \left( |\tilde{E}^1|^2 + |\tilde{E}^2|^2 \right)^2 dx$$

$$= \int_{\mathbb{R}^2} \left( |\nabla \tilde{E}^1|^2 + |\nabla \tilde{E}^2|^2 \right) dx \left( 1 - \frac{1 + \eta}{\|Q\|_{L^2(\mathbb{R}^2)}} \int_{\mathbb{R}^2} (|E_{10}|^2 + |E_{20}|^2) dx \right).$$

By (3.4) and (3.179), there holds 

$$\int_{\mathbb{R}^2} \left[ \tilde{v} + \left( |\tilde{E}^1|^2 + |\tilde{E}^2|^2 \right) \right] dx + \|\tilde{v}\|_{L^2(\mathbb{R}^2)}^2$$

$$\leq \frac{2H_0}{\lambda^2} - 2 \int_{\mathbb{R}^2} \left( |\nabla \tilde{E}^1|^2 + |\nabla \tilde{E}^2|^2 \right) dx + \int_{\mathbb{R}^2} \left( |\tilde{E}^1|^2 + |\tilde{E}^2|^2 \right)^2 dx$$

$$\leq \frac{2H_0}{\lambda^2} - \int_{\mathbb{R}^2} \left( |\nabla \tilde{E}^1|^2 + |\nabla \tilde{E}^2|^2 \right) dx$$

$$\leq \frac{2H_0}{\lambda^2} + \left( 2 - \frac{2(1 + \eta)}{\|Q\|_{L^2(\mathbb{R}^2)}} \int_{\mathbb{R}^2} (|E_{10}|^2 + |E_{20}|^2) dx \right) A^2$$

$$\leq \frac{2H_0}{\lambda^2} + \frac{2(1 + \eta)}{\|Q\|_{L^2(\mathbb{R}^2)}} \int_{\mathbb{R}^2} (|E_{10}|^2 + |E_{20}|^2) dx - 2 A^2.$$
Note that $\lambda \to +\infty$ as $t \to T$, one then obtains
\[
\|\tilde{v}(s)\|_{L^2(\mathbb{R}^2)} \leq \sqrt{\frac{2}{\eta}} A.
\]
This finishes the proof of Corollary 3.16.

**Proposition 3.17.** There exists a constant $c > 0$ such that
\[
\liminf_{t \to T} \int_0^{\theta(t)} \|\tilde{v}(s)\|_{L^2(\mathbb{R}^2)} ds \geq c. \tag{3.200}
\]

**Proof.** We argue it by contradiction. Assume that as $n \to +\infty$, there exists a sequence $t_n \to T$ such that
\[
\int_0^{\theta(t_n)} \|\tilde{v}(s)\|_{L^2(\mathbb{R}^2)} ds \to 0. \tag{3.201}
\]
From (3.177) it follows that $\forall \psi(x) \in C_0^\infty(\mathbb{R}^2)$,
\[
\int_{\mathbb{R}^2} \tilde{n}(t_n, \theta(t_n))\psi dx - \int_{\mathbb{R}^2} \tilde{n}(t_n, 0)\psi dx = \int_0^{\theta(t_n)} \int_{\mathbb{R}^2} (-\nabla \cdot \tilde{v}(s)\psi) dx ds \tag{3.202}
\]
which yields
\[
\left| \int_{\mathbb{R}^2} \tilde{n}(t_n, \theta(t_n))\psi dx - \int_{\mathbb{R}^2} \tilde{n}(t_n, 0)\psi dx \right| \leq \left( \int_0^{\theta(t_n)} \|\tilde{v}(s)\|_{L^2(\mathbb{R}^2)} ds \right) \|\nabla \psi\|_{L^2(\mathbb{R}^2)}. \tag{3.203}
\]
By (3.185) and (3.186), taking a subsequence still denoted by $t_n$ yields that there exist a sequence $x_n \in \mathbb{R}^2$, and $N' \in L^2(\mathbb{R}^2)$ such that
\[
\tilde{n}(t_n, \theta(t_n), x - x_n) \rightharpoonup N' \text{ in } L^2(\mathbb{R}^2),
\]
and
\[
\|N'\|_{L^2(\mathbb{R}^2)} \geq Ac_1.
\]
Let $\psi_0(x) \in C_0^\infty(\mathbb{R}^2)$ and satisfy $\left( \int_{\mathbb{R}^2} \psi_0^2 dx \right)^{\frac{1}{2}} = 1$ and $\int_{\mathbb{R}^2} N'\psi_0 dx \geq \frac{1}{2} \left( \int_{\mathbb{R}^2} N'^2 dx \right)^{\frac{1}{2}}$. In view of the assumptions (3.201) and (3.202), we have as $n \to +\infty$,
\[
\left| \int_{\mathbb{R}^2} \tilde{n}(t_n, \theta(t_n), x)\psi_0(x + x_n) dx - \int_{\mathbb{R}^2} \tilde{n}(t_n, 0, x)\psi_0(x + x_n) dx \right| \leq \left( \int_0^{\theta(t_n)} \|\tilde{v}(s)\|_{L^2(\mathbb{R}^2)} ds \right) \|\nabla \psi_0(x + x_n)\|_{L^2(\mathbb{R}^2)} \to 0. \tag{3.204}
\]
On the other hand, one has
\[
\int_{\mathbb{R}^2} \tilde{n}(t_n, \theta(t_n), x) \psi_0(x + x_n) \, dx = \int_{\mathbb{R}^2} \tilde{n}(t_n, \theta(t_n), x - x_n) \psi_0(x) \, dx
\]
\[
\rightarrow \int_{\mathbb{R}^2} N' \psi_0 \, dx \quad (n \rightarrow +\infty)
\]
\[
\geq \frac{1}{2} \left( \int_{\mathbb{R}^2} N'^2 \, dx \right)^{\frac{1}{2}} \geq \frac{A_c}{2} \geq 2.
\]
(3.205)

However, by (2.5) one has
\[
\left| \int_{\mathbb{R}^2} \tilde{n}(t_n, 0) \psi_0 \, dx \right| \leq \left( \int_{\mathbb{R}^2} |\tilde{n}(t_n, 0)|^2 \, dx \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^2} \psi_0^2 \, dx \right)^{\frac{1}{2}} \leq \sqrt{2}.
\]
(3.206)

It is obviously contradictory to (3.205).

This finishes the proof of Proposition 3.17. \hfill \Box

Remark 3.18. (3.200) gives the estimate for \( \theta(t) \).

In fact, in view of (3.183), if
\[
\int_0^{\theta(t)} \| \tilde{v}(s) \|_{L^2(\mathbb{R}^2)} \, ds \leq \int_0^{\theta(t)} A_2 \, ds = A c_2 \theta(t),
\]
then (3.200) implies that there exists a constant \( c > 0 \) such that
\[
\liminf_{t \to T} A c_2 \theta(t) \geq c,
\]
that is,
\[
\liminf_{t \to T} \theta(t) \geq c.
\]
(3.208)

On the other hand, by Corollary 3.16 one has
\[
c \leq \int_0^{\theta(t)} \| \tilde{v} \|_{L^2(\mathbb{R}^2)} \, ds \leq \theta(t) \sqrt{\frac{2}{\eta}} A,
\]
(3.209)

namely, there exists \( \tilde{c} = \frac{c}{\sqrt{2A}} \) such that
\[
\theta(t) \geq \tilde{c} \sqrt{\eta}.
\]
(3.210)

Therefore (3.208) and (3.210) imply that there exists a constant \( \theta_0 > 0 \) such that as \( t \to T \), \( \theta(t) \geq \theta_0 \), and
\[
\forall s \in [0, \theta), \quad \left\| \left( \hat{E}_1, \hat{E}_2, \tilde{n}, \tilde{v} \right)(s) \right\|_{H^1(\mathbb{R}^2) \times H^1(\mathbb{R}^2) \times L^2(\mathbb{R}^2) \times L^2(\mathbb{R}^2)} \leq A.
\]
(3.211)

This finishes the proof of Theorem 3.1. \hfill \Box

4. Proof of the main results (Theorem 1.3)

In this section, based on these estimates obtained in Section 2 and Section 3, we prove the main result (Theorem 1.3) of the present paper.

We first show Conclusion (1) of Theorem 1.3.
By Theorem 3.1, as $t \to T$, there exist $\theta_0 = \theta_0 \left( \| E_{10} \|_{L^2(\mathbb{R}^2)}, \| E_{20} \|_{L^2(\mathbb{R}^2)} \right)$ and $A > 0$ such that

$$\forall s \in [0, \theta_0), \quad \left\| \left( \vec{E}_1, \vec{E}_2, \tilde{n}, \vec{\nu} \right)(s) \right\|_{H^1(\mathbb{R}^2) \times H^1(\mathbb{R}^2) \times L^2(\mathbb{R}^2) \times L^2(\mathbb{R}^2)} \leq A. \quad (4.1)$$

In view of (2.2) and (2.4), one gets

$$\lambda(t)(T - t) \geq \theta_0. \quad (4.2)$$

This yields the estimate (1.6).

In addition, it follows from (2.1) that

$$\left( \left\| \nabla \vec{E}_1(0) \right\|_{L^2(\mathbb{R}^2)}^2 + \left\| \nabla \vec{E}_2(0) \right\|_{L^2(\mathbb{R}^2)}^2 \right)^{\frac{1}{2}} = \frac{1}{\lambda(t)} \left( \left\| \nabla E_1(t) \right\|_{L^2(\mathbb{R}^2)}^2 + \left\| \nabla E_2(t) \right\|_{L^2(\mathbb{R}^2)}^2 \right)^{\frac{1}{2}}, \quad (4.3)$$

$$\| \tilde{n}(0) \|_{L^2(\mathbb{R}^2)} = \frac{1}{\lambda(t)} \| n(t) \|_{L^2(\mathbb{R}^2)}. \quad (4.4)$$

Going back to Proposition 3.3, (4.3) and (4.4) yields

$$\left( \left\| \nabla E_1(t) \right\|_{L^2(\mathbb{R}^2)}^2 + \left\| \nabla E_2(t) \right\|_{L^2(\mathbb{R}^2)}^2 \right)^{\frac{1}{2}} \geq c_1 \lambda(t) \geq \frac{c_1 \theta}{T - t} = \frac{\tilde{c}}{T - t}, \quad (4.5)$$

$$\| n(t) \|_{L^2(\mathbb{R}^2)} \geq c_1 \lambda(t) \geq \frac{c_1 \theta}{T - t} = \frac{\tilde{c}}{T - t}. \quad (4.6)$$

This completes the proof of (1) in Theorem 1.3.

Next we are going to prove conclusion (2). Firstly we claim the following:

**Proposition 4.1.** For $\theta_0$ in Theorem 3.1, there exists a constant $c > 0$ such that

$$\theta_0 \geq \frac{c}{\left( \| E_{10} \|_{L^2(\mathbb{R}^2)}^2 + \| E_{20} \|_{L^2(\mathbb{R}^2)}^2 - \frac{1}{\eta + 1} \| Q \|_{L^2(\mathbb{R}^2)}^2 \right)^{\frac{1}{2}}}. \quad (4.7)$$

**Proof.** Due to the Hamiltonian given by (2.7), one gains

$$
\int_{\mathbb{R}^2} \left( \nabla \vec{E}_1 \right)^2 dx + \int_{\mathbb{R}^2} \left( \nabla \vec{E}_2 \right)^2 \left[ \frac{1}{2} \int_{\mathbb{R}^2} \left( \vec{E}_1 + \vec{E}_2 \right)^2 \right] dx
+ \frac{1}{2} \int_{\mathbb{R}^2} \left( n + \vec{E}_1 \right)^2 + \vec{E}_2 \right)^2 \right] dx
\leq \frac{H_0}{\lambda^2(t)} + 2 \eta \int_{\mathbb{R}^2} \left| \vec{E}_1 \right|^2 \left| \vec{E}_2 \right|^2 dx,
\quad (4.8)
$$

which yields
\[ \int_{\mathbb{R}^2} \left( |\nabla \tilde{E}_1|^2 + |\nabla \tilde{E}_2|^2 \right) \, dx - \frac{1 + \eta}{2} \int_{\mathbb{R}^2} \left( |\tilde{E}_1|^2 + |\tilde{E}_2|^2 \right)^2 \, dx \\
+ \frac{1}{2} \int_{\mathbb{R}^2} \left( n + |\tilde{E}_1|^2 + |\tilde{E}_2|^2 \right)^2 \, dx + \frac{1}{2} \int_{\mathbb{R}^2} |\tilde{\psi}|^2 \, dx \leq \frac{H_0}{\lambda^2(t)}. \tag{4.9} \]

Direct calculation then gives

\[ \int_{\mathbb{R}^2} \left( |\nabla \tilde{E}_1|^2 + |\nabla \tilde{E}_2|^2 \right) \, dx - \frac{1 + \eta}{2} \int_{\mathbb{R}^2} \left( |\tilde{E}_1|^2 + |\tilde{E}_2|^2 \right)^2 \, dx \leq \frac{H_0}{\lambda^2(t)}. \tag{4.10} \]

\[ \frac{1}{2} \int_{\mathbb{R}^2} \left( n + |\tilde{E}_1|^2 + |\tilde{E}_2|^2 \right)^2 \, dx + \frac{1}{2} \int_{\mathbb{R}^2} |\tilde{\psi}|^2 \, dx \leq \frac{H_0}{\lambda^2(t)} - \int_{\mathbb{R}^2} \left( |\nabla \tilde{E}_1|^2 + |\nabla \tilde{E}_2|^2 \right) \, dx \tag{4.11} \]

Note that

\[ \left( 1 - \frac{(\eta + 1) \left( \|E_{10}\|_{L^2(\mathbb{R}^2)}^2 + \|E_{20}\|_{L^2(\mathbb{R}^2)}^2 \right)}{\|Q\|_{L^2(\mathbb{R}^2)}^2} \right) \cdot \int_{\mathbb{R}^2} \left( |\nabla \tilde{E}_1|^2 + |\nabla \tilde{E}_2|^2 \right) \, dx \tag{4.12} \]

\[ \leq \int_{\mathbb{R}^2} \left( |\nabla \tilde{E}_1|^2 + |\nabla \tilde{E}_2|^2 \right) \, dx - \frac{1 + \eta}{2} \int_{\mathbb{R}^2} \left( |\tilde{E}_1|^2 + |\tilde{E}_2|^2 \right)^2 \, dx, \]

then one obtains

\[ \frac{1}{2} \int_{\mathbb{R}^2} \left( n + |\tilde{E}_1|^2 + |\tilde{E}_2|^2 \right)^2 \, dx + \frac{1}{2} \int_{\mathbb{R}^2} |\tilde{\psi}|^2 \, dx \leq \frac{H_0}{\lambda^2(t)} + \left( \frac{(\eta + 1) \left( \|E_{10}\|_{L^2(\mathbb{R}^2)}^2 + \|E_{20}\|_{L^2(\mathbb{R}^2)}^2 \right)}{\|Q\|_{L^2(\mathbb{R}^2)}^2} - 1 \right) \cdot \int_{\mathbb{R}^2} \left( |\nabla \tilde{E}_1|^2 + |\nabla \tilde{E}_2|^2 \right) \, dx. \tag{4.13} \]
Since $\lambda(t) \to \infty$ as $t \to T$, taking $t \to T$ one obtains

$$
\frac{1}{2} \int_{\mathbb{R}^2} |\tilde{v}|^2 dx
\leq \frac{\eta + 1}{\|Q\|_{L^2(\mathbb{R}^2)}^2} \left( \|E_{10}\|_{L^2(\mathbb{R}^2)}^2 + \|E_{20}\|_{L^2(\mathbb{R}^2)}^2 - \frac{1}{1 + \eta} \|Q\|_{L^2(\mathbb{R}^2)}^2 \right)
\cdot \int_{\mathbb{R}^2} (|\nabla E_1|^2 + |\nabla E_2|^2) dx
\leq A^2(\eta + 1) \|Q\|_{L^2(\mathbb{R}^2)} \left( \|E_{10}\|_{L^2(\mathbb{R}^2)}^2 + \|E_{20}\|_{L^2(\mathbb{R}^2)}^2 - \frac{1}{1 + \eta} \|Q\|_{L^2(\mathbb{R}^2)}^2 \right).$$

Therefore, by Proposition 3.17, we obtain

$$
c \leq \liminf_{t \to T} \int_{0}^{t} \|\tilde{v}(s)\|_{L^2(\mathbb{R}^2)} ds
\leq A \sqrt{2(\eta + 1)} \left( \|E_{10}\|_{L^2(\mathbb{R}^2)}^2 + \|E_{20}\|_{L^2(\mathbb{R}^2)}^2 - \frac{1}{1 + \eta} \|Q\|_{L^2(\mathbb{R}^2)}^2 \right)^{\frac{1}{2}} \theta_0,
$$

and

$$
\theta_0 \geq c' \left( \|E_{10}\|_{L^2(\mathbb{R}^2)}^2 + \|E_{20}\|_{L^2(\mathbb{R}^2)}^2 - \frac{1}{1 + \eta} \|Q\|_{L^2(\mathbb{R}^2)}^2 \right)^{-\frac{1}{2}}.
$$

This finishes the proof of Proposition 4.1.

Using Proposition 4.1 and taking $\theta_0 = c' \left( \|E_{10}\|_{L^2(\mathbb{R}^2)}^2 + \|E_{20}\|_{L^2(\mathbb{R}^2)}^2 - \frac{1}{1 + \eta} \|Q\|_{L^2(\mathbb{R}^2)}^2 \right)^{-\frac{1}{2}}$ in the proof of (1.7) and (1.8), we achieve (1.9) and (1.10). This finishes the proof of Theorem 1.3.

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