Strong fractional choice number of series-parallel graphs

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Abstract

The strong fractional choice number of a graph $G$ is the infimum of those real numbers $r$ such that $G$ is $(\lceil rm \rceil, m)$-choosable for every positive integer $m$. The strong fractional choice number of a family $\mathcal{G}$ of graphs is the supremum of the strong fractional choice number of graphs in $\mathcal{G}$. We denote by $Q_k$ the class of series-parallel graphs with girth at least $k$. This paper proves that for $k = 4q - 1, 4q, 4q + 1, 4q + 2$, the strong fractional number of $Q_k$ is exactly $2 + \frac{1}{q}$.

Keywords: strong fractional choice number; series-parallel graph

1 Introduction

A $b$-fold colouring of a graph $G$ is a mapping $\phi$ which assigns to each vertex $v$ of $G$ a set $\phi(v)$ of $b$ colours so that adjacent vertices receive disjoint colour sets. An $(a, b)$-colouring of $G$ is a $b$-fold colouring $\phi$ of $G$ such that $\phi(v) \subseteq \{1, 2, \cdots, a\}$ for each vertex $v$. The fractional chromatic number of $G$ is

$$\chi_f(G) = \inf\{\frac{a}{b} : G \text{ is } (a,b)\text{-colourable}\}.$$ 

An $a$-list assignment of $G$ is a mapping $L$ which assigns to each vertex $v$ a set $L(v)$ of $a$ permissible colours. A $b$-fold $L$-colouring of $G$ is a $b$-fold colouring $\phi$ of $G$ such that $\phi(v) \subseteq L(v)$ for each vertex $v$. We say $G$ is $(a,b)$-choosable if for any $a$-list assignment $L$ of $G$, there is a $b$-fold $L$-colouring of $G$. The fractional choice number of $G$ is

$$ch_f(G) = \inf\{\frac{a}{b} : G \text{ is } (a,b)\text{-choosable}\}.$$ 

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It was proved by Alon, Tuza and Voigt \cite{1} that for any finite graph $G$, $\chi_f(G) = \text{ch}_f(G)$ and moreover the infimum in the definition of $\text{ch}_f(G)$ is attained and hence can be replaced by minimum. This implies that if $G$ is $(a,b)$-colourable, then for some integer $m$, $G$ is $(am,bm)$-choosable. The integer $m$ depends on $G$ and is usually a large integer. A natural question is for which $(a,b)$, $G$ is $(am,bm)$-choosable for any positive integer $m$. This motivated the definition of strong fractional choice number of a graph \cite{5}.

**Definition 1** Assume $G$ is a graph and $r$ is a real number. We say $G$ is strongly fractional $r$-choosable if for any positive integer $m$, $G$ is $([rm],m)$-choosable. The strong fractional choice number of $G$ is

$$\text{ch}_f^*(G) = \inf\{r : G \text{ is strongly fractional } r\text{-choosable}\}.$$  

The strong fractional choice number of a class $\mathcal{G}$ of graphs is

$$\text{ch}_f^*(\mathcal{G}) = \sup\{\text{ch}_f^*(G) : G \in \mathcal{G}\}.$$  

It follows from the definition that for any graph $G$, $\text{ch}_f^*(G) \geq \text{ch}(G) - 1$. The variant $\text{ch}_f^*(G)$ is intended to be a refinement for $\text{ch}(G)$. However, currently we do not have a good upper bound for $\text{ch}_f^*(G)$ in terms of $\text{ch}(G)$. It was conjectured by Erdős, Rubin and Taylor \cite{3} that if $G$ is $k$-choosable, then $G$ is $(km,m)$-choosable for any positive integer $m$. If this conjecture were true, then we would have $\text{ch}_f^*(G) \leq \text{ch}(G)$. But this conjecture is refuted recently by Dvořák, Hu and Sereni in \cite{2}. Nevertheless, it is possible that for any $k$-choosable graph $G$, for any positive integer $m$, $G$ is $(km+1,m)$-choosable. If this is true, we also have $\text{ch}_f^*(G) \leq \text{ch}(G)$. In any case, $\text{ch}_f^*(G)$ is an interesting graph invariant and there are many challenging problems concerning this parameter. The strong fractional choice number of planar graphs were studied in \cite{5} and \cite{4}. Let $\mathcal{P}$ denote the family of planar graphs and for a positive integer $k$, let $\mathcal{P}_k$ be the family of planar graphs containing no cycles of length $k$. It was proved in \cite{5} that $5 \geq \text{ch}_f^*(\mathcal{P}) \geq 4 + 4/9$ and prove in \cite{4} that $4 \geq \text{ch}_f^*(\mathcal{P}_3) \geq 3 + 1/17$.

In this paper, we consider the strong fractional choice number of series-parallel graphs. For a positive integer $k$, let

$$\mathcal{Q}_k = \{G : G \text{ is a series-parallel graph with girth at least } k\}.$$  

This paper proves the following result.

**Theorem 1** Assume $q$ is a positive integer. For $k \in \{4q-1, 4q, 4q+1, 4q+2\}$, $\text{ch}_f^*(\mathcal{Q}_k) = 2 + \frac{1}{q}$.

## 2 The proof of Theorem \cite{1}

Series-parallel graphs is a well studied family of graphs and there are many equivalent definitions for this class of graphs. For the purpose of using induction, we adopt the definition that recursively construct series-parallel graphs from $K_2$ by series parallel constructions.
Definition 2 A two-terminal series-parallel graph \((G; x, y)\) is defined recursively as follows:

- Let \(V(K_2) = \{0, 1\}\). Then \((K_2; 0, 1)\) is a two-terminal series-parallel graph.
- (The parallel construction) Let \((G; x, y)\) and \((G'; x', y')\) be two vertex disjoint two-terminal series-parallel graphs. Define \(G''\) to be the graph obtained from the union of \(G\) and \(G'\) by identifying \(x\) and \(x'\) into a single vertex \(x''\), and identifying \(y\) and \(y'\) into a single vertex \(y''\). Then \((G''; x'', y'')\) is a two-terminal series-parallel graph.
- (The series construction) Let again \((G; x, y)\) and \((G'; x', y')\) be two vertex disjoint two-terminal series-parallel graphs. Define \(G''\) to be the graph obtained from the union of \(G\) and \(G'\) by identifying \(y\) and \(x'\) into a single vertex \(x''\). Then \((G''; x, y')\) is a two-terminal series-parallel graph.

A graph is a series-parallel graph if there exist some two vertices \(x, y\) such that \((G; x, y)\) is a two-terminal series-parallel graph.

Lemma 1 Assume \(m, l\) are positive integers, \(\epsilon > 0\) is a real number, \(P_t = (v_0, v_1, ..., v_t)\) is a path and \(L\) is a list assignment of path \(P_t\) with \(|L(v_0)| = m, |L(v_t)| = 2m + \epsilon m\) \((1 \leq i \leq l)\). For \(0 \leq j \leq l\), there is a subset \(T_j\) of \(L(v_j)\), for which the following holds:

- If \(j = 2t + 1\) is odd, then \(|T_j| = m + \epsilon m\); If \(j = 2t\) is even, then \(|T_j| = m\).
- For any \(m\)-subset \(B_j\) of \(L(v_j)\) for which \(|B_j \cap T_j| \geq (1 - \epsilon t)m\), there exists an \(m\)-fold \(L\)-colouring \(\phi\) for \(P_j = (v_0, v_1, ..., v_j)\) such that \(\phi(v_j) = B_j\).

Proof. By induction on \(j\). If \(j = 0\), then let \(T_0 = L(v_0)\); if \(j = 1\), then let \(T_1 = L(v_1) - T_0\). The conclusion is obviously true.

Assume \(j \geq 2\) and the lemma holds for \(j' < j\).

Case 1 \(j = 2t\) is even.

By induction hypothesis, there is an \((m + \epsilon m)\)-subset \(T_{2t-1}\) of \(L(v_{2t-1})\), such that for any \(m\)-subset \(B_{2t-1}\) of \(L(v_{2t-1})\) for which \(|B_{2t-1} \cap T_{2t-1}| \geq (1 - (t - 1)\epsilon)m\), there exists an \(m\)-fold \(L\)-colouring \(\phi\) for \(P_{2t-1}\) such that \(\phi(v_{2t-1}) = B_{2t-1}\).

As \(|L(v_{2t})| = (2 + \epsilon)m\), we have \(|L(v_{2t}) - T_{2t-1}| \geq m\). Let \(T_{2t}\) be any \(m\)-subset of \(L(v_{2t}) - T_{2t-1}\). Assume \(B_{2t}\) is an \(m\)-subset of \(L(v_{2t})\) with \(|B_{2t} \cap T_{2t}| \geq (1 - \epsilon t)m\). We shall show that there exists an \(m\)-fold \(L\)-colouring \(\phi\) for \(P_{2t}\) such that \(\phi(v_{2t}) = B_{2t}\).

Note that \(|B_{2t} \cap T_{2t-1}| \leq m - (1 - \epsilon t)m = \epsilon m t\). So \(|T_{2t-1} - B_{2t}| \geq (1 - (t - 1)\epsilon)m\). Let \(B_{2t-1}\) be an \(m\)-subset of \(L(v_{2t-1}) - B_{2t}\) containing at least \((1 - (t - 1)\epsilon)m\) colours from \(T_{2t-1}\). By induction hypothesis, there exists an \(m\)-fold \(L\)-colouring \(\phi\) of \(P_{2t-1}\) with \(\phi(v_{2t-1}) = B_{2t-1}\). Now \(\phi\) extends to an \(m\)-fold \(L\)-colouring \(\phi\) of \(P_{2t}\) with \(\phi(v_{2t}) = B_{2t}\).

Case 2 \(j = 2t + 1\) is odd.
By induction hypothesis, there is an \( m \)-subset \( T_{2t} \) of \( L(v_{2t}) \), such that for any \( m \)-subset \( B_{2t} \) of \( L(v_{2t}) \) for which \( |B_{2t} \cap T_{2t}| \geq (1 - t\epsilon)m \), there exists an \( m \)-fold \( L \)-colouring \( \phi \) for \( P_{2t} \) such that \( \phi(v_{2t}) = B_{2t} \).

As \( |L(v_{2t+1})| = (2 + \epsilon)m \), we have \( |L(v_{2t+1}) - T_{2t}| \geq (1 + \epsilon)m \). Let \( T_{2t+1} \) be any \((1 + \epsilon)m\)-subset of \( L(v_{2t+1}) - T_{2t} \). Assume \( B_{2t+1} \) is an \( m \)-subset of \( L(v_{2t+1}) \) with \( |B_{2t+1} \cap T_{2t+1}| \geq (1 - t\epsilon)m \). We shall show that there exists an \( m \)-fold \( L \)-colouring \( \phi \) for \( P_{2t+1} \) such that \( \phi(v_{2t+1}) = B_{2t+1} \).

Note that \( |B_{2t+1} \cap T_{2t}| \leq m - (1 - t\epsilon)m = tem \). So \( |T_{2t} - B_{2t+1}| \geq (1 - t\epsilon)m \). Let \( B_{2t} \) be an \( m \)-subset of \( L(v_{2t}) - B_{2t+1} \) containing at least \((1 - t\epsilon)m \) colours from \( T_{2t} \). By induction hypothesis, there exists an \( m \)-fold \( L \)-colouring \( \phi \) of \( P_{2t} \) with \( \phi(v_{2t}) = B_{2t} \). Now \( \phi \) extends to an \( m \)-fold \( L \)-colouring \( \phi \) of \( P_{2t+1} \) with \( \phi(v_{2t+1}) = B_{2t+1} \).

**Corollary 1** Assume \( m \), \( l \) are positive integers. Let

\[
\epsilon = \begin{cases} 
\frac{2}{l-1}, & \text{if } l \text{ is odd} \\
\frac{2}{l}, & \text{if } l \text{ is even.}
\end{cases}
\]

If \( P_l = (v_0, v_1, \ldots, v_l) \) is a path of length \( l \) and \( L \) is a list assignment of path \( P_l \) with \( |L(v_0)| = |L(v_l)| = m \), \( |L(v_i)| = 2m + em \) for \( 1 \leq i \leq l - 1 \), then there is an \( m \)-fold \( L \)-colouring of \( P_l \).

**Proof.** We divide the proof into two cases.

**Case 1** \( l = 2t \) is even.

By Lemma 1, there is an \((m + em)\)-subset \( T_{l-1} \) of \( L(v_{l-1}) \), such that for any \( m \)-subset \( B_{l-1} \) of \( L(v_{l-1}) \), for which \( |B_{l-1} \cap T_{l-1}| \geq (1 - (t - 1)\epsilon)m \), there exists an \( m \)-fold \( L \)-colouring \( \phi \) for \( P_{l-1} = (v_0, v_1, ..., v_{l-1}) \), such that \( \phi(v_{l-1}) = B_{l-1} \).

As \( tem = m \), we have \( |T_{l-1} - L(v_l)| \geq em = (1 - (t - 1)\epsilon)m \). So there is an \( m \)-subset \( B_{l-1} \) of \( L(v_{l-1}) - L(v_l) \) containing at least \((1 - (t - 1)\epsilon)m \) colours from \( T_{l-1} \). By Lemma 1 there exists an \( m \)-fold \( L \)-colouring \( \phi \) of \( P_{l-1} \) with \( \phi(v_{l-1}) = B_{l-1} \). Now \( \phi \) can be extended to an \( m \)-fold \( L \)-colouring \( \phi \) of \( P_l \) with \( \phi(v_l) = L(v_l) \).

**Case 2** \( l = 2t + 1 \) is odd.

Let \( B \) be an \( m \)-subset of \( L(v_{l-1}) - L(v_l) \). Let \( L'(v_i) = L(v_i) \) for \( 1 \leq i \leq l-2 \) and \( L'(v_{l-1}) = B \). By Case 1, \( P_{l-1} = (v_0, v_1, ..., v_{l-1}) \) has an \( m \)-fold \( L' \)-colouring \( \phi \) with \( \phi(v_{l-1}) = B \). Now \( \phi \) can be extended to an \( m \)-fold \( L \)-colouring \( \phi \) of \( P_l \) with \( \phi(v_l) = L(v_l) \).

**Lemma 2** If \((G; x, y)\) is a series-parallel graph of girth \( k \) and \( l = \lceil k/2 \rceil \), then either \( G \) itself is a path or \( G \) contains a path \( P = (v_0, v_1, \ldots, v_l) \) of length \( l \) such that all the vertices \( v_1, v_2, \ldots, v_{l-1} \) are degree 2 vertices of \( G \), and none of them is a terminal vertex \( x \) or \( y \).
Lemma 2. If \( G \) contains a cycle \( C \), then the conclusion is true as \( C \) has length at least \( k \). Otherwise, \( (G;x,y) \) is obtained from \((G_1;x_1,y_1)\) and \((G_2;x_2,y_2)\) by a series construction or a parallel construction. If one of \((G_1;x_1,y_1)\) and \((G_2;x_2,y_2)\) contains a cycle, then \( G_1 \) or \( G_2 \) contains a required path. Otherwise, since \( G \) contains a cycle, we conclude that \( (G;x,y) \) is obtained from \((G_1;x_1,y_1)\) and \((G_2;x_2,y_2)\) by a parallel construction, and for \( i = 1, 2 \), \( G_i \) is a path connecting \( x_i \) and \( y_i \). Then \( G \) is a cycle, and the conclusion holds. \( \blacksquare \)

Theorem 2. Assume \( q, m \) are positive integers, for any series-parallel graph \( G \) with girth at least \( k \), where \( k \in \{4q-1, 4q, 4q+1, 4q+2\} \), \( G \) is \((\lceil 2 + \frac{k}{q} \rceil m)\)-choosable.

Proof. Assume \( L \) is \((\lceil (2 + \frac{1}{q})m \rceil)\)-list assignment of \( G \). We need to show that \( G \) has an \( m \)-fold \( L \)-colouring. The proof is by induction on the number of vertices of \( G \). If \( G \) is a path, then \( G \) is \((2m, m)\)-choosable, and we are done. Assume \( G \) is not a path. By Lemma 2, \( G \) has a path \( P = \{v_0, v_1, \ldots, v_l\} \) of length \( l \) (where \( l = 2q \) when \( k \in \{4q-1, 4q\} \); \( l = 2q+1 \) when \( k \in \{4q+1, 4q+2\} \)), such that all the vertices \( v_1, v_2, \ldots, v_{l-1} \) are degree 2 vertices of \( G \). Let \( G' = G - \{v_1, v_2, \ldots, v_{l-1}\} \). Then \( G' \) is also a series-parallel graph of girth at least \( 4q-1 \) or \( G' \) is a path. If \( G' \) is a series-parallel graph of girth at least \( 4q-1 \), then by induction hypothesis, \( G' \) has an \( m \)-fold \( L \)-colouring \( \phi \); if \( G' \) is a path, as path is \((2m, m)\)-choosable, so \( G' \) also has an \( m \)-fold \( L \)-colouring \( \phi \). Let \( L' \) be the list assignment of the path \( P = \{v_0, v_1, \ldots, v_l\} \) with \( L'(v_0) = \phi(v_0), L'(v_l) = \phi(v_l) \) and \( L'(v_i) = L(v_i) \) for \( 1 \leq i \leq l-1 \). By Corollary \( \#1 \) \( P \) has an \( m \)-fold \( L' \)-colouring \( \psi \). Then the union of \( \phi \) and \( \psi \) is an \( m \)-fold \( L \)-colouring of \( G \). \( \blacksquare \)

By Theorem 2, for \( k \in \{4q-1, 4q, 4q+1, 4q+2\} \), the strong fractional choice number of \( Q_k \) is at most \( 2 + \frac{1}{q} \). In order to show that equality holds, we need to construct, for each positive integer \( m \), a graph belongs to \( Q_k \), which is not \((\lceil (2 + \frac{1}{q})m - 1 \rceil, m)\)-choosable.

Lemma 3. Assume \( m, l \) are positive integers and assume that \( \epsilon \) is a positive real number such that \( em \) is an integer and

\[
\epsilon < \begin{cases} \frac{2}{l-1}, & \text{if } l \text{ is odd} \\ \frac{2}{2}, & \text{if } l \text{ is even}. \end{cases}
\]

Let \( P_l = \{v_0, v_1, \ldots, v_l\} \) be a path. Let \( M_1, M_2 \) be \( m \)-sets. Then there exists a list assignment \( L \) of \( P_l \) for which the following holds:

- \( L(v_0) = M_1 \) and \( L(v_l) = M_2 \).
- \( |L(v_i)| = 2m + \epsilon m \) for \( 1 \leq i \leq l-1 \),
- there is no \( m \)-fold \( L \)-colouring of \( P_l \).
**Proof.** Let $A_r$ (for $r = 1, 3, 5, \ldots, 2q - 3$), $B_s$ (for $s = 2, 4, 6, \ldots, 2q - 2$), $Z_t$ (for $t = 1, 3, 5, \ldots, 2q - 1$) be disjoint colour sets, where $|A_r| = |B_s| = m$, $|Z_t| = \epsilon m$. Let $L$ be the list assignment of $P_t$ defined as follows:

- $L(v_0) = M_1$, $L(v_i) = M_2$, $L(v_1) = M_1 \cup A_1 \cup Z_1$.
- $|L(v_{2i+1})| = B_{2i} \cup A_{2i+1} \cup Z_{2i+1}$, for $i = 1, 2, 3, \ldots, q - 2$.
- $|L(v_{2j})| = B_{2j} \cup A_{2j-1} \cup Z_{2j-1}$, for $j = 1, 2, 3, \ldots, q - 1$.
- If $l = 2q$, then $L(v_{2q-1}) = M_2 \cup B_{2q-2} \cup Z_{2q-1}$; if $l = 2q + 1$, then $L(v_{2q}) = M_2 \cup A_{2q-1} \cup Z_{2q-1}$.

We shall show that $P_t$ is not $L$-colourable.

**Claim 1** For any $j \in \{2, 3, 4, \ldots, q\}$, if $\phi$ is an $m$-fold $L$-colouring of $P_{2j-2}$, then $|\phi(v_{2j-2}) \cap B_{2j-2}| \geq m - (j - 1)\epsilon m$.

**Proof.** We shall prove the claim by induction on the index $j$. Assume $j = 2$ and $\phi$ is an $m$-fold $L$-colouring of $P_2$. As $\phi(v_0) = M_1$, we conclude that $\phi(v_1) \subseteq A_1 \cup Z_1$. Therefore $|\phi(v_2) \cap (A_1 \cup Z_1)| \leq \epsilon m$. Hence $|\phi(v_2) \cap B_2| \geq m - \epsilon m$.

Assume $j \geq 3$ and the claim holds for $j' < j$ and $\phi$ is an $m$-fold $L$-colouring of $P_{2j-2}$. Apply induction hypothesis to the restriction of $\phi$ to $P_{2j-4}$, we conclude that

$$|\phi(v_{2j-4}) \cap B_{2j-4}| \geq m - (j - 2)\epsilon m.$$  

Hence $|\phi(v_{2j-3}) \cap B_{2j-4}| \leq (j - 2)\epsilon m$. This implies that

$$|\phi(v_{2j-3}) \cap (A_{2j-3} \cup Z_{2j-3})| \geq m - (j - 2)\epsilon m.$$  

Hence

$$|\phi(v_{2j-2}) \cap (A_{2j-3} \cup Z_{2j-3})| \leq (j - 1)\epsilon m.$$  

So $|\phi(v_{2j-2}) \cap B_{2j-2}| \geq m - (j - 1)\epsilon m$. $\blacksquare$

Assume $l = 2q$ is even and $\phi$ is an $m$-fold $L$-colouring of $P_{2q}$. Then $|\phi(v_{2q-2}) \cap B_{2q-2}| \geq m - (q - 1)\epsilon m$. As $\phi(v_{2q}) = M_2$, we conclude that $\phi(v_{2q-1}) \subseteq (B_{2q-2} - \phi(v_{2q-2})) \cup Z_{2q-1}$. But $|B_{2q-2} - \phi(v_{2q-2}) \cup Z_{2q-1}| \leq (q - 1)\epsilon m + \epsilon m = q\epsilon m < m$, a contradiction.

Assume $l = 2q + 1$ is odd and $\phi$ is an $m$-fold $L$-colouring of $P_{2q+1}$. By claim 1, $|\phi(v_{2q-2}) \cap B_{2q-2}| \geq m - (q - 1)\epsilon m$. Hence $|\phi(v_{2q-1}) \cap B_{2q-2}| \leq (q - 1)\epsilon m$. This implies that $|\phi(v_{2q-1}) \cap (A_{2q-1} \cup Z_{2q-1})| \geq m - (q - 1)\epsilon m$. As $\phi(v_{2q-1}) = M_2$, we conclude that $\phi(v_{2q}) \subseteq (A_{2q-1} \cup Z_{2q-1}) - \phi(v_{2q-1})$. But $|(A_{2q-1} \cup Z_{2q-1}) - \phi(v_{2q-1})| \leq m + \epsilon m - (m - (q - 1)\epsilon m) = q\epsilon m < m$, a contradiction. $\blacksquare$

For disjoint two-terminal series-parallel graphs $(G_1; l_1, r_1)$ and $(G_2; l_2, r_2)$, we use $G_1 \parallel G_2$ to denote the parallel composition of $G_1$ and $G_2$ and $G_1 \bullet G_2$ to denote the series composition of $G_1$ and $G_2$, respectively. For a two-terminal series-parallel graph $G$, we let $G^\times^n$ denote the series composition of $n$ copies of $G$, and let $G_{\parallel^n}$ denote the parallel composition of $n$ copies of $G$. 

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Theorem 3 Assume \( m, q \) are positive integers and assume that \( \epsilon \) is a positive real number such that \( \epsilon m \) is an integer and \( \epsilon < \frac{1}{q} \). For \( k \in \{4q - 1, 4q, 4q + 1, 4q + 2\} \), there exists a graph \( G \in Q_k \), such that \( G \) is not \(((2 + \epsilon)m, m)\)-choosable.

Proof. Let \( p = \left( \frac{(2 + \epsilon)m}{m} \right)^2 \).

Let graph \( G \) be obtained by making parallel composition of \( p \) paths with length \( \left\lfloor \frac{k}{2} \right\rfloor \). Denote the two terminals by \( x \) and \( y \). Then \( G \in Q_k \).

We shall show that \( G \) is not \(((2 + \epsilon)m, m)\)-choosable. Let \( X \) and \( Y \) be two \((2m + \epsilon m)\)-sets. Let \( L(x) = X \) and \( L(y) = Y \). There are \( p \) possible \( m \)-fold \( L \)-colourings of \( x \) and \( y \). Each such a colouring \( \phi \) corresponds to one path with length \( \left\lfloor \frac{k}{2} \right\rfloor \). In that path, define the list assignment \( L \) as in the proof of Lemma 3, by replacing \( M_1 \) with \( \phi(x) \) and \( M_2 \) with \( \phi(y) \). Then Lemma 3 implies that no \( m \)-fold \( L \)-colouring of \( x \) and \( y \) can be extended to \( G \).

By theorem 3 for \( k \in \{4q - 1, 4q, 4q + 1, 4q + 2\} \), the strong fractional choice number of \( Q_k \) is at least \( 2 + \frac{1}{q} \). Combining with theorem 2 this completes the proof of theorem 1.

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