THE DUAL RADON - NIKODYM PROPERTY
FOR FINITELY GENERATED BANACH
C(K)-MODULES

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Abstract. We extend the well-known criterion of Lotz for the
dual Radon-Nikodym property (RNP) of Banach lattices to finitely
generated Banach C(K)-modules and Banach C(K)-modules of
finite multiplicity. Namely, we prove that if X is a Banach space
from one of these classes then its Banach dual X* has the RNP iff
X does not contain a closed subspace isomorphic to ℓ₁.

1. Introduction

Let us start with reminding the reader about the following equiva-
lences in the class of Banach lattices.

Theorem 1.1. (Lozanovsky [14] - Lotz [12], see also [17, Theorem
2.4.15, p.94]) Let X be a Banach lattice. Then the following conditions
are equivalent.

(1) X is reflexive.
(2) X does not contain a copy ℓ₁ of either c₀ or ℓ₁.
(3) X does not contain a copy of either c₀ or ℓ₁ as a sublattice. ²

Theorem 1.2. (Lozanovsky [15], see also [17, Theorem 2.5.6, p.104
and Theorem 2.4.12, p. 92]) Let X be a Banach lattice. Then the
following conditions are equivalent.

(1) X is weakly sequentially complete.
(2) X does not contain a copy of c₀.
(3) X does not contain a copy of c₀ as a sublattice.

²If X and Y are Banach spaces we say that X contains a copy of Y if there is a
closed subspace of X linearly isomorphic to Y.
²If X and Y are Banach lattices we say that X contains a copy of Y as a sublattice
if there is a closed sublattice of X lattice isomorphic to Y.
Before we remind the reader the next result in this direction let us recall the following definition.

**Definition 1.3.** A Banach space $X$ is said to have the Radon-Nikodym property (RNP) if for every finite measure space $(\Omega, \Sigma, \lambda)$ and for every bounded linear operator $T : L^1(\lambda) \to X$ there exists a strongly measurable $g \in L^\infty(\lambda, X)$ such that

$$Tf = \int_\Omega fg d\lambda, \quad f \in L^1(\lambda),$$

where the integral is the Bochner integral.

**Theorem 1.4.** (Lotz, see [17, Theorem 5.4.14, p.367]) Let $X$ be a Banach lattice. The following conditions are equivalent.

1. The Banach dual $X^*$ of $X$ has RNP.
2. $X$ does not contain a copy of $\ell^1$.
3. $X^*$ does not contain a copy of either $c_0$ or $L^1[0,1]$ as a sublattice.

The statements of Theorems 1.1 - 1.4 become false if instead of Banach lattices we consider arbitrary Banach spaces. For Theorems 1.1 and 1.2 a counterexample is provided by the famous James’ space [6] while in the case of Theorem 1.4 we need to use another example of James in [7] where he constructed a separable Banach space $X$ not containing a copy of $\ell^1$ and such that $X^*$ is not separable. It follows from a later result of Stegall [23] that the space $X$ from [7] does not have the dual RNP.

But, if instead of the class of all Banach spaces we consider the much smaller classes of finitely generated Banach $C(K)$-modules or Banach $C(K)$-modules of finite multiplicity, which while not contained in the class of all Banach lattices can be considered as its nearest relatives, the analogues of Theorems 1.1 - 1.4 remain true.

We refer the reader for precise definitions of the notions used in the next two statements to the next section of the current paper.

**Theorem 1.5.** ([10, Theorems 1 and 2]) Let $X$ be a finitely generated Banach $C(K)$-module or a Banach $C(K)$-module of finite multiplicity. Then the following conditions are equivalent.

1. $X$ is reflexive.
2. $X$ does not contain a copy of either $c_0$ or $\ell^1$.
3. Every cyclic subspace of $X$ does not contain a copy of either $c_0$ or $\ell^1$.

3The James’ space from [6], being quasi-reflexive, has the dual RNP.
Theorem 1.6. ([11, Theorems 3.1 and 3.8]) Let $X$ be a finitely generated Banach $C(K)$-module or a Banach $C(K)$-module of finite multiplicity. Then the following conditions are equivalent.

1. $X$ is weakly sequentially complete.
2. $X$ does not contain a copy of $c_0$.
3. Every cyclic subspace of $X$ does not contain a copy of $c_0$.

Remark 1.7. Every cyclic subspace of a Banach $C(K)$-module can be endowed in the unique way with the structure of a Banach lattice compatible with its structure of $C(K)$-submodule (see [10, 11] for more details). Therefore we can state that conditions of Theorems 1.5 and 1.6 are equivalent to any cyclic subspace not containing either $c_0$ or $ℓ^1$ as a sublattice in the case of Theorem 1.5 and not containing $c_0$ as a sublattice in the case of Theorem 1.6.

The goal of the present paper is to show that similar to Theorems 1.5 and 1.6 the result of Theorem 1.4 can be extended to the classes of finitely generated $C(K)$-modules and $C(K)$-modules of finite multiplicity.

We will need the following important characterization of the dual RNP property in Banach spaces.

Theorem 1.8. (Stegall - Uhl). Let $X$ be a Banach space. The following conditions are equivalent.

1. $X^*$ has RNP.
2. For every separable subspace $Y$ of $X$ its conjugate $Y^*$ is also separable.

Remark 1.9. The implication (1) $\Rightarrow$ (2) was proved by Stegall (see [23, Theorem 2]) and the implication (2) $\Rightarrow$ (1) by Uhl (see [24, Corollary 3]).

At the end of the introduction we want to emphasize that one of the reasons to study the dual RNP is the following important result.

Theorem 1.10. ([2, Theorem 1, p.98]) Let $(Ω, Σ, μ)$ be a finite measure space, $1 ≤ p < ∞$, and $X$ be a Banach space. Then $L^p(μ, X)^* = L^q(μ, X^*)$, where $1/p + 1/q = 1$, if and only if $X^*$ has the Radon-Nikodym property with respect to $μ$.

Remark 1.11. (1) We remind the reader that $L^p(μ, X)$, $1 ≤ p < ∞$, is the space of (classes) of all Bochner integrable functions on $Ω$ endowed with the norm $\|f\| = \left(\int_Ω \|f(ω)\|_X^p dμ\right)^{1/p}$. 
(2) If $X^*$ does not have the RNP the description of $L^p(\mu, X)^*$ becomes rather involved. For a detailed discussion of it and for the background of Theorem 1.10 we refer the reader to [2].

2. Preliminaries

All the linear spaces will be considered either over the field of real numbers $\mathbb{R}$ or the field of complex numbers $\mathbb{C}$. If $X$ is a Banach space we will denote its Banach dual by $X^*$.

Let us recall some definitions.

**Definition 2.1.** Let $K$ be a compact Hausdorff space and $X$ be a Banach space. We say that $X$ is a Banach $C(K)$-module if there is a continuous unital algebra homomorphism $m$ of $C(K)$ into the algebra $L(X)$ of all bounded linear operators on $X$.

**Remark 2.2.** Because $\ker m$ is a closed ideal in $C(K)$ by considering, if needed, $C(\tilde{K}) = C(K)/\ker m$ we can and will assume without loss of generality that $m$ is a contractive homomorphism (see [10]) and $\ker m = 0$. Then (see [5, Lemma 2 (2)]) $m$ is an isometry. Moreover, when it does not cause any ambiguity we will identify $f \in C(K)$ and $m(f) \in L(X)$.

**Definition 2.3.** Let $X$ be a Banach $C(K)$-module and $x \in X$. We introduce the cyclic subspace $X(x)$ of $X$ as $X(x) = \text{cl}\{fx : f \in C(K)\}$.

The following proposition was proved in [25] (see also [20]) in the case when the compact space $K$ is extremally disconnected and announced for an arbitrary compact Hausdorff space $K$ in [8]. It follows as a special case from [5, Lemma 2 (2)].

**Proposition 2.4.** Let $X$ be a Banach $C(K)$-module, $x \in X$, and $X(x)$ be the corresponding cyclic subspace. Then, endowed with the cone $X(x)_+ = \text{cl}\{fx : f \in C(K), f \geq 0\}$ and the norm inherited from $X$, $X(x)$ is a Banach lattice with positive quasi-interior point $x$.

Our next proposition follows from Theorem 1 (3) in [18]

**Proposition 2.5.** The center $Z(X(x))$ of the Banach lattice $X(x)$ is isometrically isomorphic to the weak operator closure of $m(C(K))$ in $L(X(x))$.

Now we can introduce one of the two main objects of interest in the current paper.
Definition 2.6. Let $X$ be a Banach $C(K)$-module. We say that $X$ is finitely generated if there are an $n \in \mathbb{N}$ and $x_1, \ldots, x_n \in X$ such that the set $\sum_{i=1}^{n} X(x_i)$ is dense in $X$.

Definition 2.7. Let $X$ be a Banach space and $\mathcal{B}$ be a Boolean algebra of projections on $X$. The algebra $\mathcal{B}$ is called Bade complete (see [3 VII.3.4, p.2197] and [22 V.3, p.315]) if
(1) $\mathcal{B}$ is a complete Boolean algebra.
(2) Let $\{\chi_\gamma\}_{\gamma \in \Gamma}$ be an increasing net in $\mathcal{B}$, $\chi$ be the supremum of this net, and $x \in X$. Then the net $\{\chi_\gamma x\}$ converges to $\chi x$ in norm in $X$.

Definition 2.8. Let $\mathcal{B}$ be a Bade complete Boolean algebra of projections on $X$. $\mathcal{B}$ is said to be of uniform multiplicity $n$, if there exist a set of nonzero pairwise disjoint idempotents $\{e_\alpha\}$ in $\mathcal{B}$ with $\text{sup } e_\alpha = 1$ such that for any $e_\alpha$ and for any $e \in \mathcal{B}$, $e \leq e_\alpha$ the $C(K)$-module $eX$ has exactly $n$ generators.

We will need the following result of Rall [21] (for a proof, see Lemma 2 in [19]).

Lemma 2.9. Let $\mathcal{B}$ be of uniform multiplicity one on $X$. Then $X$ may be represented as a Banach lattice with order continuous norm such that $\mathcal{B}$ is the Boolean algebra of band projections on $X$.

Definition 2.10. A Bade complete Boolean algebra of projections $\mathcal{B}$ on $X$ is said to be of finite multiplicity on $X$ if there exists a collection of disjoint idempotents $\{e_\alpha\}$ in $\mathcal{B}$ such that, for each $\alpha$, $e_\alpha X$ is $n_\alpha$-generated and $\text{sup } e_\alpha = 1$.

Remark 2.11. The collection $\{n_\alpha\}$ of positive integers in Definition 2.10 need not be bounded.

Theorem 2.12. (Bade [3 XVIII.3.8, p. 2267]). Let $X$ be a Banach $C(K)$-module of finite multiplicity. Then there exists a sequence of disjoint idempotents $\{e_n\}$ in $\mathcal{B}$ such that, for each $n$, $\mathcal{B}$ is of uniform multiplicity $n$ on $e_nX$ and $\text{sup } e_n = 1$. Also the norm closure of the sum of the sequence of the spaces $\{e_nX\}$ is equal to $X$.

3. The main Results

Theorem 3.1. Let $X$ be a finitely generated Banach $C(K)$-module. Then the following are equivalent:
(1) $X^*$ has the Radon-Nikodym property.
(2) $X$ does not contain any copy of $\ell^1$. 

Proof. The implication (1) ⇒ (2) follows from Theorem 1.8. It can also be proved in the following way.

Suppose (1) holds. Then it is well known that $X^*$ does not contain any copy of $L^1(0,1)$ (see [17, 5.4.4(ii), p.363 and 5.4.8, p.365]). Then by a result of Hagler [4], we have that $X$ does not contain a copy of $\ell^1$. Hence (1) implies (2) for all Banach spaces $X$.

Conversely, suppose that (2) holds. We will complete the proof by induction on the number of generators of $X$. Let $X$ have one generator. That is $X$ is cyclic. Then it follows from Proposition 2.4 and Theorem 1.4 that $X^*$ has the RNP. Now suppose that whenever $X$ has $r$ generators for some fixed $r \geq 1$ and satisfies (2), then $X^*$ has the RNP. Let $X$ have $r + 1$ generators and satisfy (2). Suppose $\{x_0, x_1, \ldots, x_r\}$ is a set of generators of $X$. Let $Y = X(x_1, x_2, \ldots, x_r)$. Then $Y$ is a Banach $C(K)$-module with $r$ generators and as a subspace of $X$, it satisfies (2). Hence, by the induction hypothesis, $Y^*$ has the RNP. Now consider the cyclic space $X/Y = X/Y([x_0])$ where $[x_0] = x_0 + Y$. We have that $(X/Y)^* = Y^0 \subset X^*$ where $Y^0$ is the polar (annihilator) of $Y$ in $X^*$. Since $X$ satisfies (2), by Hagler’s Theorem [4], $X^*$ does not contain a copy of $L^1(0,1)$ and the same is true for $(X/Y)^*$ as a subspace of $X^*$. Again by Hagler’s Theorem, this means that $X/Y$ satisfies (2). Since $X/Y$ is a cyclic Banach space, it is representable as a Banach lattice. Therefore, by Lotz’s Theorem 1.4, $Y_0 = (X/Y)^*$ has the RNP. Recall also that $X^*/Y^0 = Y^*$.

Now consider the cyclic space $X/Y = X/Y([x_0])$ where $[x_0] = x_0 + Y$. We have that $(X/Y)^* = Y^0 \subset X^*$ where $Y^0$ is the polar (annihilator) of $Y$ in $X^*$. Since $X$ satisfies (2), by Hagler’s Theorem [4], $X^*$ does not contain a copy of $L^1(0,1)$ and the same is true for $(X/Y)^*$ as a subspace of $X^*$. Again by Hagler’s Theorem, this means that $X/Y$ satisfies (2). Since $X/Y$ is a cyclic Banach space, it is representable as a Banach lattice. Therefore, by Lotz’s Theorem 1.4, $Y_0 = (X/Y)^*$ has the RNP. Recall also that $X^*/Y^0 = Y^*$. Since the RNP is a three space property [11, 6.5.b, p. 202], we have that $X^*$ has the Radon-Nikodym property.

It is possible to sharpen the result of Theorem 3.1 if we assume that the cyclic subspaces of $X$ have order continuous norm when represented as Banach lattices. Initially, we need a result which guarantees that a Banach lattice $X$ which does not contain a copy of $\ell^1$ as a sublattice does not contain a copy of $\ell^1$ as a subspace. This is not true in general. For example, $\ell^1$ is not isomorphic to any sublattice of $\ell^\infty$, however any separable Banach space (hence in particular $\ell^1$) may be embedded isometrically into $\ell^\infty$.

To proceed we need to recall the following important result by Lotz and Rosenthal.

**Theorem 3.2.** (Lotz and Rosenthal [13], see also [17, Theorem 5.2.15, p. 345])

Let $X$ be a Banach lattice. The following conditions are equivalent.

1. For each $x \in X_+$ the order interval $[0,x]$ is weakly sequentially precompact.
2. $X^*$ does not contain any copy of $L^1[0,1]$ as a sublattice.
The Lotz - Rosenthal theorem leads to the following lemma.

**Lemma 3.3.** Let $X$ be a Banach lattice with order continuous norm then $X$ contains a copy of $\ell^1$ if and only if $X$ contains a copy of $\ell^1$ as a sublattice.

**Proof.** Suppose $X$ does not contain a copy of $\ell^1$ as a sublattice. Then $X^*$ does not contain any copy of $c_0$ as a sublattice [17, Proposition 2.3.12, p. 83]. Also since $X$ has order continuous norm any order interval in $X$ is weakly compact [17, Theorem 2.4.2, p. 86]. Hence by the Eberlein - Šmulian theorem any order interval in $X$ is weakly sequentially compact. Then, by Theorem 3.2 of Lotz and Rosenthal $X^*$ does not contain any copy of $L^1(0,1)$ as a sublattice. Finally, by the equivalence (2) $\Leftrightarrow$ (3) in Theorem 1.4 by Lotz we conclude that $X$ does not contain any copy of $\ell^1$. □

**Theorem 3.4.** Let $K$ be a hyperstonian compact space and let $X$ be a finitely generated Banach $C(K)$-module such that the algebra $B$ of the idempotents in $C(K)$, is a Bade complete Boolean algebra of projections on $X$. Then the following conditions are equivalent

1. $X^*$ has the Radon-Nikodym property.
2. $X$ does not contain any copy of $\ell^1$.
3. No cyclic subspace of $X$ contains a copy of $\ell^1$.
4. No cyclic subspace of $X$, when represented as a Banach lattice, contains a copy of $\ell^1$ as a sublattice.

**Proof.** It is clear that (2) $\Rightarrow$ (3). Let us show that (3) $\Rightarrow$ (2). We will use induction on the number of generators of $X$. When $n = 1$, $X$ is cyclic, so (3) $\Rightarrow$ (2) trivially. Suppose for some $r \geq 1$, for all $X$ with $r$ generators we have (3) $\Rightarrow$ (2). Suppose $X$ has $r + 1$ generators $\{x_0, x_1, \ldots, x_r\}$ and satisfies (3). Let $Y = X(x_1, \ldots, x_r)$. Then $Y$ satisfies (2) by the induction hypothesis. On the other hand, $X/Y = X/Y([x_0])$ where $[x_0] = x_0 + Y$. Since $B$ is Bade complete on $X$, by [10, Lemma 1], $B$ is Bade complete on the cyclic space $X/Y$. By Lemma 2.9, $X/Y$ can be represented as a Banach lattice with order continuous norm and quasi-interior point $[x_0]$ such that the algebra $B$ is isometrically isomorphic to the algebra of band projections on the Banach lattice $X/Y$. Now (3) implies that $\ell^1$ is not contained in $X/Y$ as a sublattice (see the proof of Theorem 1 in [10, p. 480]). Then by Lemma 3.3 we have that $X/Y$ does not contain a copy of $\ell^1$. Since not containing $\ell^1$ is a three space property [11, Theorem 3.2.d, p.96], we have that $X$ does not contain any copy of $\ell^1$.

The implication (3) $\Rightarrow$ (4) is trivial.
Assume (4). Every cyclic subspace of $X$ when represented as a Banach lattice has order continuous norm. Thus the implication $(4) \Rightarrow (3)$ follows from Lemma 3.3. It completes the proof. □

The purpose of the next two examples is to show that condition (3) in Theorem 3.4 cannot be weakened as follows. Let $x_1, \ldots, x_n$ be some system of generators of $X$. Assume that none of the cyclic subspaces $X(x_i), i = 1, \ldots, n$ contains a copy of $\ell_1$. Then $X^*$ has RNP. (Compare Examples 1 and 2 on page 482 in [10])

**Example 3.5.** Let $X = L^1(0,1) \oplus L^2(0,1)$ considered as a $L^\infty(0,1)$ Banach module. Let $x_1 = (0,1)$ and $x_2 = (-1,1)$. Then $\{x_1, x_2\}$ is a system of generators of $X$. Clearly $X(x_1) = L^2$ and $X(x_2)$ is isomorphic to $L^2$ while $X^*$ does not have RNP.

In Example 3.5 $X$ is a non-atomic Banach lattice. It is easy to provide a similar example when $X$ is a discrete BL.

**Example 3.6.** Let $w_n$ be an increasing sequence of positive real numbers such that $\sum_{n=1}^{\infty} \frac{1}{w_n} < \infty$. We consider the Hilbert space

$$\ell^2(w_n) = \{\{a_n\} : \sum_{n=1}^{\infty} |a_n|^2 w_n^2 < \infty\},$$

with the norm $\|\{a_n\}\| = \sqrt{\sum_{n=1}^{\infty} |a_n|^2 w_n^2}$. Then $\ell^2(w_n) \subset \ell^1$. Let $X = \ell^1 \oplus \ell^2(w_n)$ considered as a $\ell^\infty$-module. Let $x_1 = (0, \{\frac{1}{nw_n}\})$ and $x_2 = (\{-\frac{1}{nw_n}\}, \{\frac{1}{nw_n}\})$. Then, as in Example 3.5 we see that the system $\{x_1, x_2\}$ generates $X$ and that the cyclic subspaces $X(x_1)$ and $X(x_2)$ are reflexive. However $X^*$ does not have RNP.

**Remark 3.7.** Already mentioned example of James in [7] shows that, in general, the condition that $X$ is a finitely generated $C(K)$-module cannot be dropped. Indeed, every Banach space can be considered as a Banach module over $C$. Still there is a large class of Banach $C(K)$-modules which are in general not finitely generated but for which the conclusion of Theorem 3.4 remains valid.

We proceed now to extend the result of Theorem 3.4 to Banach $C(K)$-modules of finite multiplicity.

**Lemma 3.8.** Let $X$ be a Banach $C(K)$-module of uniform multiplicity $n$ and let $x \in X$. Let $\{e_\alpha\}$ be the system of pairwise disjoint idempotents from Definition 2.8. Then the set $\{\alpha : e_\alpha x \neq 0\}$ is at most countable.
Proof. Consider the cyclic subspace $Y$ of $X$ generated by $x$. By Lemma 2.9 $Y$ can be represented as a Banach lattice with order continuous norm. Assume, contrary to our claim, that the set $\{ \alpha : e_\alpha x \neq 0 \}$ is uncountable. Then for some $k \in \mathbb{N}$ there are $e_1, e_2, \ldots \in \{ e_\alpha \}$ such that $\| e_n x \| \geq 1/k$, $n \in \mathbb{N}$. Notice that the elements $e_n x$ are pairwise disjoint in the Banach lattice $Y$. Then, by Proposition 0.5.5 in [26, page 36] $Y$ contains $\ell^\infty$ as a sublattice. But then by the well known result of Lozanovsky [16] (see also [9, Theorem 4, page 295]) $Y$ cannot have order continuous norm, a contradiction. □

Corollary 3.9. Let $X$ be a Banach $C(K)$-module of uniform multiplicity $n$ and let $Y$ be a separable closed subspace of $X$. Then the set $\{ \alpha : e_\alpha Y \neq 0 \}$ is at most countable.

The proof of the following lemma repeats verbatim the proof of statement (⋆) on page 485 in [10].

Lemma 3.10. Let $X$ be a Banach $C(K)$-module of finite multiplicity such that any cyclic subspace of $X$ does not contain a copy of $\ell^1$. Suppose that $\{ e_n \}$ is a system of pairwise idempotents in $C(K)$ such that $\sup_n e_n = 1$. Let $\chi_n = e_1 + \ldots + e_n$. Then for any $f \in X^*$ we have $\| f - \chi_n f \| \to 0$.

Lemma 3.11. Let $X$ be a Banach $C(K)$-module of uniform multiplicity $n$. Assume that no cyclic subspace of $X$, represented as a Banach lattice, contains $\ell^1$ as a sublattice. Then $X^*$ has RNP.

Proof. By Theorem 1.8 it is enough to prove that for any closed separable subspace $Y$ of $X$ the conjugate $Y^*$ is also separable. Thus, let $Y$ be a closed separable subspace of $X$. By Corollary 3.9 there is an at most countable subset $\{ e_i, i \in \mathbb{N} \}$ of $\{ e_\alpha \}$ such that $e_i Y \neq 0$, $i \in \mathbb{N}$ and $e_\alpha Y = 0$ for any $\alpha$ not in this subset. Then $Y \subseteq Z \subseteq X$ where $Z$ is the closure in $X$ of the direct sum $\sum e_i Y$ (we remind the reader that the idempotents $e_i$ are pairwise disjoint in $C(K)$). For every $i$, $e_i X$ is a finitely generated Banach $e_i C(K)$-module (with $n$ generators) and by Theorem 3.4 its conjugate $(e_i X)^*$ has RNP. Therefore by Theorem 1.8 the conjugate $(e_i Y)^*$ is separable.

Before we go to next step in the proof let us notice that for any $i \in \mathbb{N}$ we have $(e_i Y)^* = e_i^* Z^*$. Indeed, $(e_i Y)^* = Z^*/(e_i Y)^0$ and $e_i$ is a projection on $Z$ with $e_i Z = cl(e_i Y)$. Consequently, $e_i^*$ is a projection on $Z^*$ and $(1 - e_i^*) Z^* = (e_i Y)^0$. Therefore $(e_i Y)^* = e_i^* Z^*$. Because no cyclic subspace of $X$ contains a copy of $\ell^1$, Lemma 3.10 guarantees...
that $Z^* = cl \sum_i (e_i Y)^*$. Indeed, let $f \in Z^*$, and $m \in \mathbb{N}$. Then $\chi_m f \in \sum_{i=1}^m (e_i Y)^*$ and $\chi_m f$ converges to $f$ by norm. Thus $Z^*$ is separable. Then $Y^*$, being a factor of $Z^*$ by $Y^0$, the annihilator of $Y$ in $Z^*$, is separable as well.

\[\square\]

**Theorem 3.12.** Let $X$ be a Banach $C(K)$-module of finite multiplicity. Then the following conditions are equivalent.

1. $X^*$ has RNP.
2. $X$ does not contain a copy of $\ell^1$.
3. Any cyclic subspace of $X$ does not contain a copy of $\ell^1$.
4. Any cyclic subspace of $X$ represented as a Banach lattice does not contain $\ell^1$ as a sublattice.

**Proof.** The implication $(1) \Rightarrow (2)$ has been already established in the proof of Theorem 3.4.

The implications $(2) \Rightarrow (3) \Rightarrow (4)$ are trivial.

The implication $(4) \Rightarrow (3)$ follows from Lemma 3.3.

It remains to prove that $(3) \Rightarrow (1)$. Assume (3). Let $Y$ be a closed separable subspace of $X$. Let $e_n$ be the idempotents from Theorem 2.12 and let $Z = cl \sum\limits_n e_n Y$. Then $Z$ is a separable closed subspace of $X$ and, applying Theorem 2.12, we see that $Y \subseteq Z$. Because any cyclic subspace of $X$ does not contain a copy of $\ell^1$ by Lemma 3.10 as explained in the proof of Lemma 3.11, $Z^* = cl \sum\limits_n (e_n Y)^*$.

Next notice that by Lemma 3.11 the space $(e_n X)^*$ has RNP and by Theorem 1.8 the space $(e_n Y)^*$ is separable. Therefore $Z^*$ is separable, $Y^*$ is separable as a factor of $Z^*$, and $X^*$ has RNP by Theorem 1.8.

\[\square\]

**Remark 3.13.** A slight modification of Example 4.2(3) on page 752 in [11] (where one replaces $\ell^p$ by $c_0$) provides an example of a Banach $C(K)$-module $X$ with the following properties.

1. $X$ is of uniform multiplicity $n$, $n > 1$.
2. $X$ is not separable.
3. Every cyclic subspace of $X$ is separable and has separable dual. In particular, $X$ cannot be finitely (or even countably) generated.
4. There are cyclic subspaces of $X$ that are not weakly sequentially complete.

Thus, while Theorem 3.4 cannot be applied, by Theorem 3.12 $X$ has dual RNP.
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