On the influence of colour magnetic currents on the confining properties of $SU(3)$ lattice gauge theory

Peter Skala, Manfried Faber and Martin Zach
Institut für Kernphysik, Technische Universität Wien
A–1040 Vienna, Austria

Abstract

We modify the standard Wilson action of $SU(3)$ lattice gauge theory by adding an extra term which suppresses colour magnetic currents. We present numerical results of simulations at zero and finite temperature and show that colour magnetic currents strongly influence the confining properties of $SU(3)$ lattice gauge theory.

1 Introduction

The mechanism which leads to permanent confinement of quarks at low temperatures is an interesting and still open question in gauge theories. A promising conjecture which could explain the existence of a non-zero string tension is the hypothesis that the QCD vacuum behaves dually to a superconductor [1]. In this model colour magnetically charged particles condense in the ground state of QCD and squeeze the colour electric field between a pair of static quarks into a narrow flux tube leading to a confining potential. A possible realization of the dual superconductor picture in non-Abelian $SU(N)$ gauge theories was proposed by 't Hooft in the framework of Abelian projection [2]. In this approach, the physically relevant degrees of freedom are identified by fixing the gauge up to the largest Abelian subgroup $U(1)^{N-1}$ of the $SU(N)$ gauge group. After gauge fixing one is left with an effective Abelian theory which contains in addition to photons magnetic monopoles. The monopoles arise as singularities in the gauge fixing condition and their condensation is expected to be crucial for the existence of a non-zero string tension. There is a lot of support for the Abelian projection approach from numerical results of lattice simulations [3]. For instance, the Abelian gauge fields obtained after gauge fixing carry almost the whole asymptotic string tension [4]. This observation is referred to as Abelian dominance. However, most of the numerical results are dependent on the choice of gauge which fixes the non-Abelian degrees of freedom [5], and a dual superconductor is convincingly observed only in one particular gauge, the so-called maximal Abelian gauge [6]. The question arises whether this apparent ambiguity in the gauge fixing procedure can be avoided by investigating the dual superconductor picture in terms of the original non-Abelian degrees of freedom.

A first attempt in this direction was made in ref. [6] where dual superconductivity in $SU(3)$ lattice gauge theory was studied numerically in a gauge invariant formulation. An operator for the magnetic current was introduced allowing the determination of its curl in the presence of a...
static quark antiquark pair without need of fixing the gauge. At finite temperature the result of a numerical simulation is the following \[6\]: In the confined phase the curl of the magnetic current is proportional to the electric field indicating the validity of a dual London relation. This seems to be in agreement with the dual superconductor picture of confinement. In the deconfined phase the curl of the magnetic current vanishes and the electric field shows Coulomb-like behaviour. At least qualitatively the results presented in \[6\] agree with the results reported in ref. \[7\] which were obtained in the maximal Abelian gauge.

In this letter, we continue the gauge invariant investigation of the dual superconductor picture of confinement and study the role of the magnetic current operator introduced in ref. \[6\] in the pure gluonic vacuum. We introduce a chemical potential \(\lambda\) and add an extra term to the standard gauge field action suppressing large magnetic currents. By calculating the string tension and the critical coupling for the finite temperature phase transition we will show that a suppression of magnetic currents strongly influences the confining properties of the considered \(SU(3)\) theory.

## 2 The Model

We consider pure \(SU(3)\) lattice gauge theory on a four-dimensional Euclidean lattice of spacing \(a\) with periodic boundary conditions in space and time direction. An action convenient for many purposes is the standard Wilson action \[8\] which for the \(SU(3)\) gauge group reads

\[
S_W = \beta \sum_{x,\mu<\nu} \left( 1 - \frac{1}{3} \text{Re} \text{Tr} U_{\mu\nu}(x) \right),
\]

where \(\beta = 6/g^2\) is the inverse coupling and \(U_{\mu\nu}(x)\) is the product of link variables \(U_\mu \in SU(3)\) around an elementary plaquette in \(\mu\nu\)-direction at lattice site \(x\):

\[
U_{\mu\nu}(x) = U_\mu(x)U_\nu(x + \hat{\mu})U_\mu^\dagger(x + \hat{\nu})U_\nu^\dagger(x).
\]

According to a derivation of Gauss’ law on the lattice \[6\] the field strength \(F_{x\mu
u}(x)\) corresponding to the action \[1\] is given by the expression

\[
ga^2 F_{\mu\nu}^x(x) = F_{\mu\nu}^x(x) = \frac{1}{2i} \left( U_{\mu\nu}(x) - U_{\mu\nu}^\dagger(x) \right)_{(tl)}, \quad (tl) \equiv \text{traceless}.
\]

The upper index \(x\) in the notation of \(F_{\mu\nu}^x(x)\) states the local colour coordinate system in which the field strength is measured. It is changed to a system defined at an arbitrary lattice site \(x'\) by performing a parallel transport of the field strength along a Schwinger line connecting the site \(x'\) with the original site \(x\).

Let us now introduce the quantity \(J_{m,\mu}^x\) given by the expression

\[
J_{m,\mu}^x(x) \equiv -\frac{1}{2} \epsilon_{\mu\nu\rho\sigma} D_\nu F_{\rho\sigma}^x(x),
\]

where we use the discretized covariant derivative of the field strength

\[
D_\nu F_{\rho\sigma}^x(x) \equiv F_{\rho\sigma}^x(x + \hat{\nu}) - F_{\rho\sigma}^x(x) = U_\nu(x)F_{\rho\sigma}^{x+\hat{\nu}}(x + \hat{\nu})U_\nu^\dagger(x) - F_{\rho\sigma}^x(x).
\]

On the lattice definition \[4\] has a simple geometrical interpretation. Each component of \(J_{m,\mu}\) corresponds to a three-dimensional cube built of six plaquettes measuring the flux out of the
cube. We therefore call \( J_{m,\mu} \) colour magnetic current. The covariant derivative in (4) guarantees all contributions to the total flux out of the cube to be taken in the colour coordinate system at \( x \). As can be easily shown [6] the current (4) fulfills a conservation law in terms of covariant derivatives.

In ref. [6] we already discussed the definition of the quantity \( J_{m,\mu} \) concerning the validity of the Bianchi identity. We think it is necessary to extend this discussion and add at this point some further comments concerning especially the general form of the Bianchi identity in a lattice formulation of SU(3) gauge theory. In a continuum formulation the right hand side of (4) corresponds to the Bianchi identity and the current \( J_{m,\mu} \) vanishes. On a lattice with finite lattice spacing \( a \), however, definition (4) may not be identified with a lattice version of the Bianchi identity. This can be understood as follows: As was shown in ref. [9] the partition function \( Z \) of SU(3) lattice gauge theory expressed in terms of link variables \( U_{\mu}(x) \) in SU(3)

\[
Z = \int D[U_{\mu}(x)] \exp(-S_W[U_{\mu}])
\]  

(6)
can be rewritten as an integral over the plaquette variables \( U_{\mu \nu}(x) \) in SU(3). The lattice Bianchi identity then automatically arises as the argument of a \( \delta \)-function in the integrand of the partition function. It essentially constrains the product of plaquettes \( U_{\mu \nu} \) covering a three-dimensional cube to the unit matrix in order to reduce the degrees of freedom (6 plaquettes per lattice site) to the original number (4 links per lattice site). On a four-dimensional lattice there are four types of three-dimensional cubes and thus four Bianchi identities. We want to emphasize that if a theory is expressed in terms of link variables the lattice Bianchi identities for the plaquette variables \( U_{\mu \nu} \) are automatically fulfilled. In the case of an Abelian theory like \( U(1) \) gauge theory the lattice Bianchi identity can be formulated also as a constraint for the plaquette phases. The sum of the phases \( \theta_{\mu \nu} \in (0, \pi] \) of the six plaquette variables \( U_{\mu \nu} = e^{i\theta_{\mu \nu}} \in U(1) \) measuring the flux out of a three-dimensional cube must be an integer multiple of \( 2\pi \). If the integer is non-zero, one speaks of a topological excitation called magnetic monopole [10]. However, if one identifies the flux through a plaquette according to (4) with \( \sin \theta_{\mu \nu} [11] \), the total flux out of a three-dimensional cube is in general non-zero. The last statement is also true in the case of a non-Abelian theory without being in contradiction to the lattice Bianchi identity. We conclude that the sum of algebra elements \( F_{\mu \nu} \) on the right hand side of (4) is in general non-zero although the product of the corresponding group elements \( U_{\mu \nu} \) is - according to the Bianchi identity - equal to the unit matrix.

It is the purpose of this paper to study the influence of the magnetic current \( J_{m,\mu} \) on the confining properties of the considered SU(3) lattice gauge theory. To be more precise, we define the following gauge invariant quantity

\[
|J_{m,\mu}(x)| = \sqrt{2 \text{ Tr} \left( J_{m,\mu}^2(x) \right)}
\]  

(7)
and call it the length in colour space of the magnetic current defined in (4). Large values of \( |J_{m,\mu}(x)| \) correspond to large fluctuations of the plaquettes covering the surface of a three-dimensional cube. We introduce a parameter \( \lambda \) and modify the Wilson action (1) in the following way

\[
S = S_W + \lambda \sum_{x,\mu} |J_{m,\mu}(x)|.
\]  

(8)
The sum extends over all three-dimensional cubes on the lattice. The parameter \( \lambda \) plays the role of a chemical potential for the magnetic current \( J_{m,\mu} \). If \( \lambda \) is chosen to be larger than zero, the
magnetic current $J_{m,\mu}$ will be suppressed. In ref. [12] the action (8) with $\lambda = \infty$ and with $J_{m,\mu}$ being the $U(1)$ monopole currents [10] was used to study four-dimensional $U(1)$ lattice gauge theory with monopoles removed.

### 3 Results

In order to study the influence of magnetic currents $J_{m,\mu}$ on physical quantities we performed simulations using the action (8) with zero and non-zero chemical potential $\lambda$. Calculations were done on symmetric lattices with size $8^4$ and $12^4$ and on asymmetric lattices with temporal extent $N_T = 4$ and spatial extent of $N_S = 8$ and 12. The case $\lambda = 0$, i.e. the standard Wilson action, was simulated by using an overrelaxed pseudo heatbath algorithm. To generate gauge field configurations at non-zero chemical potential $\lambda$ we implemented a Metropolis algorithm. Because of the additional term in the action twelve three-dimensional cubes corresponding to colour magnetic currents $J_{m,\mu}$ have to be taken into account besides the six plaquettes in the update of a single link. This procedure turned out to be very time consuming. We therefore decided to measure the physical quantities after every Metropolis sweep; this leads to large autocorrelation times especially in the critical regions of the finite temperature phase transition. As a consequence a large number of measurements up to $10^5$ had to be taken into account in order to get a reliable estimate of the considered physical quantities. To estimate the errorbars of primary quantities such as Polyakov and Wilson loops the standard procedure of calculating the integrated autocorrelation time was used. For the error analysis of secondary quantities such as Creutz ratios a simple extension of the fundamental jackknife formula [13] was applied.

We start with the presentation of results obtained on symmetric lattices of size $8^4$ and $12^4$. Fig. 1 shows the expectation value of the magnetic current $|J_m|$ (9) averaged over the space-time

![Figure 1: Expectation value of the magnetic current $|J_m|$ (9) as a function of the inverse coupling $\beta$. The measurements were taken on a $12^4$-lattice. Errorbars are omitted since they are smaller than the symbols.](image)
components and the four-dimensional lattice

$$|J_m| = \frac{1}{4N^4} \sum_{x,\mu} |J_{m,\mu}(x)|$$

(9)

for a large range of $\beta$-values and zero chemical potential $\lambda$. One can see a typical crossover of $\langle |J_m| \rangle$ between the strong and weak coupling region. For $\beta$ going to infinity the magnetic current tends to zero displaying decreasing fluctuations of the six plaquettes covering a three-dimensional cube. In the strong coupling region the current (9) surprisingly reaches its maximum at a finite value of $\beta$ and slightly decreases for $\beta$ going to zero. It is an interesting question whether the current density $\langle |J_m| \rangle$ shows scaling behaviour which means $\langle |J_m| \rangle \propto a^3(\beta)$, where $a(\beta)$ is the lattice spacing given as a function of the bare coupling constant $\beta$ which can be determined by the renormalization group equation. From the numerical data presented in fig. 1 it is clearly seen that there is no scaling behaviour of $\langle |J_m| \rangle$. Quantum fluctuations which are not of topological origin do not cancel. It should be emphasized that for the correlation function measuring the curl of the magnetic current in the vicinity of a static charge pair (see ref. [6]) this problem does not seem to occur. In this case, the magnetic current is projected to the Polyakov line which fixes a direction in colour space. Fluctuations corresponding to lattice artifacts contribute equally with positive and negative sign and thus cancel, whereas contributions of topological origin survive the averaging process. Unfortunately, the situation is different in the case of the operator $|J_m|$ which measures the magnetic current density in the gluonic vacuum. We will come back to the question of scaling of this operator and its consequences to the continuum limit in the conclusions of this letter.

To investigate the influence of the current $J_{m,\mu}$ on physical quantities we calculated Wilson loops and Creutz ratios on a $8^4$-lattice at $\beta = 6.0$ and compared the results for zero and finite chemical potential $\lambda$. In fig. 2 the ratio of Wilson loops obtained with $\lambda = 0.1$ and $\lambda = 0.0$ (standard Wilson action) is shown. It is clearly seen that a suppression of magnetic currents manifests itself in a less disordered lattice: For all sizes of Wilson loops the considered ratio is larger than one and increases with increasing size of the loops. This means that the suppression of magnetic currents corresponding to geometrical objects of the size of a unit cube especially influences the behaviour of objects being large in terms of the lattice spacing, i.e. large Wilson loops which determine the non-perturbative properties of the theory. To get a more quantitative insight we analyzed the behaviour of Creutz ratios $\chi$

$$\chi(I, \lambda) = -\ln \frac{W(I, I, \lambda)W(I - 1, I - 1, \lambda)}{W(I - 1, I, \lambda)W(I, I - 1, \lambda)}$$

(10)

for the above discussed Wilson loops estimating the string tension. Since the ratio of Wilson loops shown in fig. 2 increases stronger than linearly, the ratio $\chi(\lambda = 0.1)/\chi(\lambda = 0.0)$ decreases with the size of the loops. This behaviour is shown in fig. 3. The larger the size of Wilson loops the stronger is the influence of the suppression of magnetic currents on Creutz ratios. Taking the numerical results of $\chi(4, \lambda)$ as an asymptotic value for the string tension $\hat{\sigma}$ in lattice units we are able to estimate the decreasing of the lattice spacing $a$

$$a(\lambda = 0.1) = \sqrt{\frac{\hat{\sigma}(\lambda = 0.1)}{\hat{\sigma}(\lambda = 0.0)}} a(\lambda = 0.0) \approx 0.79 a(\lambda = 0.0).$$

(11)
Figure 2: Ratio of Wilson loops for chemical potentials $\lambda = 0.1$ and $\lambda = 0.0$. The ratio increases stronger than linearly with the size of the Wilson loops.

Figure 3: Ratio of Creutz ratios for chemical potentials $\lambda = 0.1$ and $\lambda = 0.0$ as a function of the linear extension of the considered Wilson loops.
We want to point out that for the chosen parameters $\beta = 6.0$ and $\lambda = 0.1$ the average contribution of the extra term $\lambda \sum_{x,\mu} |J_{m,\mu}|$ to the total action (8) is approximately 7%.

In addition to zero temperature investigations we also performed numerical simulations at finite temperature to study the influence of the magnetic current $J_{m,\mu}$ on the deconfinement phase transition. The order parameter of the finite temperature phase transition is the expectation value $\langle L \rangle$ of the Polyakov loop

$$L(\vec{x}) = \text{Tr} \prod_{t=1}^{N_t} U_4(\vec{x}, t).$$

(12)

In the thermodynamic limit $\langle L \rangle$ is expected to be zero in the confined and non-zero in the deconfined phase. On a finite lattice, however, $\langle L \rangle$ equals zero for all inverse couplings $\beta$ because of the non-zero probability of tunneling between different states related by the $Z(3)$-symmetry. We thus measured the finite lattice “order parameter” $\langle |L| \rangle$ given by

$$\langle |L| \rangle = \langle \left| \frac{1}{N^3} \sum_{\vec{x}} L(\vec{x}) \right| \rangle,$$

(13)

where the sum extends over the whole spatial lattice. We performed runs on lattices with size $8^3 \times 4$ and $12^3 \times 4$ for chemical potentials $\lambda = 0.00$ (standard Wilson action) and $\lambda = 0.05$. The results for $\langle |L| \rangle$ as a function of the inverse coupling constant $\beta$ are shown in fig. 4 and 5. It is clearly seen that a non-zero chemical potential $\lambda$ shifts the phase transition towards a smaller critical value of $\beta$. Hence, the addition of an extra term to the Wilson action in (8) suppressing magnetic currents for $\lambda > 0$ corresponds to a system with a larger effective inverse coupling constant. In other words, if configurations with large magnetic currents $J_{m,\mu}$ are removed from the path integral, the phase transition already occurs at smaller values of $\beta$, and it is more favourable for the system to be in the $Z(3)$ broken than in the $Z(3)$ symmetric phase. From fig. 4 and 5 one can estimate the shift of the phase transition. For $\lambda = 0.05$ we find that $\Delta \beta$ is approximately $-0.2$, whereas the current contribution $\lambda \sum_{x,\mu} |J_{m,\mu}|$ to the total action (8) is about 4% for $\beta \in [5.3, 5.7]$.

Besides the Polyakov loop we also measured the distribution of the real part of the plaquette trace. In fig. 6 we show a comparison of plaquette distributions on the $12^3 \times 4$-lattice at $\beta = 5.6$ for $\lambda = 0.00$ and $\lambda = 0.05$. The corresponding numerical values of $\langle |L| \rangle$ are displayed in fig. 5. As expected the suppression of magnetic currents causes the plaquettes to be closer to the unit matrix. Plaquette distributions corresponding to the same value of $\langle |L| \rangle$ but different values of $\beta$ for $\lambda = 0.00$ and $\lambda = 0.05$ cannot be distinguished.

By choosing the chemical potential $\lambda$ large enough the confinement phase can be completely removed for positive values of $\beta$. Starting from an ordered lattice and choosing $\lambda = 10.0$ the system is not able to evolve from the deconfined into the disordered confined phase, also not for $\beta = 0.0$, see fig. 7. This shows that for the considered lattice sizes $8^3 \times 4$ and $12^3 \times 4$ there is no confined phase without presence of magnetic currents $J_{m,\mu}$ for positive values of $\beta$.

4 Conclusion and discussion

In this letter we studied the influence of colour magnetic currents (4) originally defined in ref. 6 on the confining properties of $SU(3)$ lattice gauge theory. We introduced a chemical potential $\lambda$ and added an extra term to the standard Wilson action suppressing large magnetic currents.
Figure 4: The expectation value of the Polyakov loop as a function of the inverse coupling $\beta$ for chemical potentials $\lambda = 0.00$ and $\lambda = 0.05$ for a $8^3 \times 4$-lattice. In the case of $\lambda \neq 0$ the critical $\beta$ is shifted towards a smaller value. The symbols (diamonds for $\lambda = 0.00$ and triangles for $\lambda = 0.05$) denote the numerical results of Monte Carlo runs at different values of $\beta$ whereas the continuous curves were computed with a multi-histogram analysis [14, 15].

Figure 5: The same as in fig. 4 for a $12^3 \times 4$-lattice except that the curves are linear interpolations between the results of Monte Carlo runs drawn to guide the eye.
Figure 6: Distributions of the real part of the plaquette trace for $\beta = 5.6$ and the $\lambda$-values 0.00 (dashed line) and 0.05 (solid line). According to fig. 5 the lattice is in the confined phase for $\lambda = 0.00$ whereas it is in the deconfined phase for $\lambda = 0.05$.

Figure 7: The expectation value of the Polyakov loop as a function of the inverse coupling $\beta$ for chemical potential $\lambda = 10.0$ for a $8^3 \times 4$-lattice (left plot) and a $12^3 \times 4$-lattice (right plot). At this value of $\lambda$ there is no confined phase left for positive values of the inverse coupling $\beta$. 
With this modified action we performed numerical simulations at zero and finite temperature lattices. We found that a partial absence of magnetic currents leads to a drastic decrease in the confining string tension. At finite temperature the suppression of magnetic currents shifts the phase transition towards smaller values of the inverse coupling constant $\beta$. Additionally, for the considered lattice sizes it was shown that there exists a finite chemical potential $\lambda$ for which the confined phase is completely removed for positive values of $\beta$. We conclude that the magnetic currents strongly influence the confining properties of our considered $SU(3)$ lattice gauge theory. However, an interpretation of these currents in terms of physical objects is not directly possible since we do not observe scaling behaviour of the current density (9). As already mentioned above, the reason for this might be that there are only positive contributions to the expectation value $\langle |J_m| \rangle$ and that fluctuations which are not of topological origin do not cancel. According to our opinion, on a lattice with finite spacing there are in general two different types of contributions to the magnetic current $J_{m,\mu}$. One type of contributions is due to the finite lattice spacing. They can be regarded as lattice artifacts and vanish in the continuum. Hence, they will not contribute to the continuum limit of the expectation value $\langle |J_m| \rangle$. The other type of contributions is of topological origin. They belong to gauge field configurations which become singular in particular gauges in the continuum and thus could guarantee that the operator $J_{m,\mu}$ does not become trivial in the continuum limit. One candidate for such configurations is for instance the Euclidean pure gauge theory analogue of the ‘t Hooft-Polyakov-like monopole in the BPS limit. That this monopole configuration could play an important role in the dual superconductor picture of confinement was discussed in ref. [16]. Moreover, in ref. [17] it was argued that in an appropriate gauge such a monopole configuration becomes singular. Analogous arguments should be valid for another type of gauge field configurations, magnetic vortices. These were suggested to play a crucial role in the confinement mechanism [18], an idea which gained support from recent lattice calculations [19].

To summarize, we believe that the topological contributions to the magnetic currents strongly influence the confining properties of QCD. Unfortunately, the magnetic current operator is not able to identify its topological sources which determine the behaviour of its expectation value in the continuum. Thus, we can only speculate about this question. But for cubes of finite size, we know that the topological objects lead to magnetic currents behaving as predicted by the dual superconductor picture.

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References

[1] G. 't Hooft, in High Energy Physics, EPS International Conference, Palermo 1975, ed. A. Zichichi, S. Mandelstam, Phys. Rep. 23C (1976) 245.

[2] G. 't Hooft, Nucl. Phys. B190 (1981) 455.

[3] For a recent review of the Abelian projection see: M. Polikarpov, Nucl. Phys. B (Proc. Suppl.) 53 (1997) 134.

[4] T. Suzuki and I. Yotsuyanagi, Phys. Rev. D42 (1990) 4257.

[5] A.S. Kronfeld, M.L. Laursen, G. Schierholz and U.-J. Wiese, Phys. Lett. B198 (1987) 516.

[6] P. Skala, M. Faber, M. Zach, Nucl. Phys. B494 (1997) 293.

[7] Y. Peng, R. W. Haymaker, Phys. Rev. D52 (1995) 3030.

[8] K. Wilson, Phys. Rev. D10 (1974) 2445.

[9] G. Batrouni, Nucl. Phys. B208 (1982) 467.

[10] T.A. DeGrand, D. Toussaint, Phys. Rev. D22 (1980) 2478.

[11] M. Zach, M. Faber, W. Kainz, P. Skala, Phys. Lett. B358 (1995) 325.

[12] J.S. Barber, R.E. Schrock, R. Schrader, Phys. Lett. B152 (1985) 221.

[13] P. Gottlieb, P.B. Mackenzie, H.B. Thacker, D. Weingarten, Nucl. Phys. B263 (1986) 704.

[14] A.M. Ferrenberg and R.H. Swendsen, Phys. Rev. Lett. 61 (1988) 2635.

[15] S. Huang, K.J.M. Moriarty, E. Myers, J. Potvin, Z. Phys. C50 (1991) 221.

[16] J. Smit and A.J. van der Sijs, Nucl. Phys. B355 (1991) 603.

[17] A.J. van der Sijs, hep-th/9505019, RCNP Confinement 1995 29-36.

[18] G. 't Hooft, Nucl. Phys. B138 (1978) 1,
G. Mack, in Recent Developments in Gauge Theories, edited by G. 't Hooft et al. (Plenum, New York, 1980),
J. Ambjorn and P.Olesen, Nucl. Phys. B170(1980) 265,
J.M. Cornwall, Nucl. Phys. B157 (1979) 392.

[19] L. Del Debbio, M. Faber, J. Greensite and Š. Olejnık, Phys. Rev. D55 (1997) 2298.