On the Topological Centre Problem for Weighted Convolution Algebras

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Abstract

Let $G$ be a locally compact non-compact group. We show that under a very mild assumption on the weight function $w$, the weighted group algebra $L_1(G, w)$ is strongly Arens irregular in the sense of [Dal–Lam–Lau 01]. To this end, we first derive a general factorization theorem for bounded families in the $L_\infty(G, w^{-1})^*$-module $L_\infty(G, w^{-1})$.

1 Introduction

Let $G$ be a locally compact group, and let $w : G \to (0, \infty)$ be a weight function, i.e., a positive continuous function on $G$ such that $w(st) \leq w(s)w(t)$ for all $s, t \in G$; for convenience we shall assume that $w(e) = 1$, where $e$ is the neutral element of $G$. We will consider the following spaces, normed in such a way that multiplication resp. division by the weight becomes an isometry between the unweighted and the corresponding weighted space (whose norm we will denote by $\| \cdot \|_w$):

- $L_1(G, w) = \{ f \mid wf \in L_1(G) \}$
- $L_\infty(G, w^{-1}) = \{ f \mid w^{-1}f \in L_\infty(G) \}$
- $LUC(G, w^{-1}) = \{ f \mid w^{-1}f \in LUC(G) \}$
- $C_0(G, w^{-1}) = \{ f \mid w^{-1}f \in C_0(G) \}$
- $M(G, w) = \{ \mu \mid w\mu \in M(G) \}$

Then we have $L_\infty(G, w^{-1}) = L_1(G, w)^*$ and $M(G, w) = C_0(G, w^{-1})^*$. For every $y \in G$, we define $\tilde{\delta}_y := w(y)^{-1} \delta_y$, which is an element of norm one in $M(G, w)$.

Our aim is to show that for all locally compact non-compact groups, the weighted group algebra $L_1(G, w)$ is strongly Arens irregular in the sense of Dales–Lamb–Lau (see [Dal–Lam–Lau 01]), provided the weight satisfies some very mild boundedness condition. Here, strong Arens irregularity means that the topological centre of the bidual algebra $(L_1(G, w)^{**}, \circ)$, equipped with the first Arens product, precisely equals the algebra $L_1(G, w)$ itself, i.e., it is extremally small. This is a generalization of the main result, Thm. 1, of [Lau–Los 88], where the corresponding assertion is proved for the (unweighted) group algebra $L_1(G)$, to the weighted situation. Although covering a

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by far more general case, our proof is not of higher complexity, if not even simpler, than the one given in [LAU–LOS 88].

In the following, we shall always regard $L_1(\mathcal{G}, w)^{**}$ as endowed with the first Arens multiplication. Let us briefly recall the three step construction of the latter, arising from the convolution product (denoted by “$*$”) in $L_1(\mathcal{G}, w)$ via various module actions. – For $m, n \in L_1(\mathcal{G}, w)^{**}$, $h \in L_1(\mathcal{G}, w)^*$ and $f, g \in L_1(\mathcal{G}, w)$ one defines:

$$
\langle h \circ f, g \rangle := \langle h, f * g \rangle \\
\langle n \circ h, f \rangle := \langle n, h \circ f \rangle \\
\langle m \circ n, h \rangle := \langle m, n \circ h \rangle.
$$

A fairly comprehensive exposition of the basic theory of Arens products is given in [PAL 94], §1.4. As for topological centres, an excellent source is [LAU–ÜLG 96]. We shall only need the definition of the latter, which we briefly recall here:

$$Z_t(\mathcal{L}_1(\mathcal{G}, w)^{**}) := \{m \in \mathcal{L}_1(\mathcal{G}, w)^{**} \mid n \mapsto m \circ n \text{ is } w^* - w^* - \text{continuous on } \mathcal{L}_1(\mathcal{G}, w)^{**}\}.$$  

We will use the fact (cf. [GR0 90], Prop. 1.3) that, with the natural module operation stemming from the construction of the first Arens product on $\mathcal{L}_1(\mathcal{G}, w)^{**}$, the equality $L_\infty(\mathcal{G}, w^{-1}) \odot \mathcal{L}_1(\mathcal{G}, w) = \text{LUC}(\mathcal{G}, w^{-1})$ holds. Hence, a natural module operation of $\text{LUC}(\mathcal{G}, w^{-1})^*$ on $L_\infty(\mathcal{G}, w^{-1})$ is given by

$$\langle m \circ h, g \rangle = \langle m, h \circ g \rangle,$$

where $m \in \text{LUC}(\mathcal{G}, w^{-1})^*$, $h \in L_\infty(\mathcal{G}, w^{-1})$, $g \in \mathcal{L}_1(\mathcal{G}, w)$. It is readily verified that we have $m \circ h = \bar{m} \odot h$, where $\bar{m}$ is an arbitrary Hahn-Banach extension of $m$ to $L_\infty(\mathcal{G}, w^{-1})^*$.

In the sequel, we shall denote by $\mathfrak{c}(\mathcal{G})$ the compact covering number of the group $\mathcal{G}$, i.e., the least cardinality of a compact covering of $\mathcal{G}$. For the sake of brevity, we further introduce the following terminology (the first part of the definition also appears in [DAL–LAM–LAI 01]).

**Definition 1.1.** (i) A subset $S$ of $\mathcal{G}$ will be called dispersed if $S$ is not contained in any union of a family of compact subsets of $\mathcal{G}$, the family having cardinality strictly less than $\mathfrak{c}(\mathcal{G})$.

(ii) For a subset $S \subseteq \mathcal{G}$, we say that the weight $w$ is diagonally bounded on $S$ if we have:

$$\sup_{s \in S} w(s)w(s^{-1}) < \infty.$$  

Now we can formulate the main result of the present note; we remark that it has very recently also been obtained independently by Dales–Lamb–Lau in [DAL–LAM–LAI 01], though with a different proof. In particular, our factorization result, Theorem 2.2, does not appear in [DAL–LAM–LAI 01].

**Theorem 1.2.** Let $\mathcal{G}$ be a locally compact non-compact group with compact covering number $\mathfrak{c}(\mathcal{G})$. Suppose that there is a dispersed set $S \subseteq \mathcal{G}$ on which the weight function $w$ is diagonally bounded. Then $L_1(\mathcal{G}, w)$ is strongly Arens irregular.

We wish to stress the following important points regarding our approach:

• We prove a (formal) sharpening of the interesting inclusion contained in Theorem 1.2. Namely, we will show that for an element $m \in L_\infty(\mathcal{G}, w^{-1})^*$ in order to belong to $L_1(\mathcal{G}, w)$, it suffices that left multiplication by $m$ be $w^*-w^*$-continuous on the $w^*$-closure of the set of all Hahn-Banach extensions of functionals in $\overline{\delta_{\mathcal{G}}w} \subseteq \text{Ball}(\text{LUC}(\mathcal{G}, w^{-1})^*)$ to $L_\infty(\mathcal{G}, w^{-1})^*$. Instead, the definition of the topological centre demands $w^*$-continuity on all of $L_\infty(\mathcal{G}, w^{-1})^*$. 

2
• The proof is direct and follows, once the necessary prerequisites are established (section 2), a purely Banach algebraic pattern (section 3). Our global procedure is similar to the one presented in [NEU 01a] and may thus be considered as having the same merits as the latter.

2 A factorization theorem for families of functions in \( L_\infty (\mathcal{G}, w^{-1}) \)

In the proof of our main result we will use the following two results, which are of independent interest.

**Proposition 2.1.** For an arbitrary locally compact group \( \mathcal{G} \), the space \( L_1(\mathcal{G}, w) \) enjoys Mazur’s property of level \( \mathfrak{f}(\mathcal{G}) \cdot \aleph_0 \). – This means that a functional \( m \in L_1(\mathcal{G}, w)^{**} \) actually belongs to \( L_1(\mathcal{G}, w) \) if it carries bounded \( w^* \)-converging nets of cardinality at most \( \mathfrak{f}(\mathcal{G}) \cdot \aleph_0 \) into converging nets.

**Proof.** This follows from Thm. 4.4 in [NEU 01], which states the above property for the space \( L_1(\mathcal{G}) \), and the fact that the latter is stable under isomorphism (cf. [NEU 01], Remark 4.3). \( \square \)

Next we present our crucial tool, which is a general factorization theorem for bounded families in \( L_\infty (\mathcal{G}, w^{-1}) \); it provides a generalization of Thm. 2.2 in [NEU 01a] to the weighted situation.

**Theorem 2.2.** Let \( \mathcal{G} \) be a locally compact non-compact group with compact covering number \( \mathfrak{f}(\mathcal{G}) \). Suppose that there exists a family \( (\psi_\alpha)_{\alpha \in I} \), \( |I| = \mathfrak{f}(\mathcal{G}) \), of functionals in \( \delta_\mathcal{G}^{w^*} \subseteq \text{Ball} \left( \text{LUC}(\mathcal{G}, w^{-1})^* \right) \) such that whenever \( (h_\alpha)_{\alpha \in I} \subseteq L_\infty (\mathcal{G}, w^{-1}) \) is a bounded family of functions, there exists a function \( h \in L_\infty (\mathcal{G}, w^{-1}) \) such that the factorization formula

\[
h_\alpha = \psi_\alpha \circ h
\]

holds for all \( \alpha \in I \).

**Proof.** For \( y \in \mathcal{G} \), we denote by \( r_y \) the operator of right translation, i.e., \( (r_y f)(x) = f(xy) \) whenever \( f \) is a function on \( \mathcal{G} \) and \( x \in \mathcal{G} \).

There is a covering of \( \mathcal{G} \) by open sets whose closure is compact, of minimal cardinality, i.e., of cardinality \( \mathfrak{f}(\mathcal{G}) \), and closed under finite unions; we denote the corresponding family of compacta by \( (K_\alpha)_{\alpha \in I} \). Set \( \tilde{I} := I \times I \). For \( \tilde{\alpha} = (\alpha, i) \in \tilde{I} \), put \( K_{\tilde{\alpha}} = K_{(\alpha,i)} := K_\alpha \). Then \( (K_{\tilde{\alpha}})_{\tilde{\alpha} \in \tilde{I}} \) is a covering of \( \mathcal{G} \) having the same properties than the original one. Since the set \( S \) is dispersed, by the same reasoning as in Lemma 3 of [Lau–Los 88], we see that there exists a family \( (y_{\tilde{\alpha}})_{\tilde{\alpha} \in \tilde{I}} \subseteq S \) such that

\[
K_{\tilde{\alpha}}y_{\tilde{\alpha}}^{-1} \cap K_{\tilde{\beta}}y_{\tilde{\beta}}^{-1} = \emptyset \quad \forall \tilde{\alpha}, \tilde{\beta} \in \tilde{I}, \tilde{\alpha} \neq \tilde{\beta}.
\] (1)

Set \( S' := \{ y_{\tilde{\alpha}} | \tilde{\alpha} \in \tilde{I} \} \). We define, for \( (\alpha, i), (\beta, j) \in \tilde{I} \):

\[
(\alpha, i) \preceq (\beta, j) \iff K_{(\alpha, i)} \subseteq K_{(\beta, j)} \iff K_\alpha \subseteq K_\beta \iff \alpha \preceq' \beta.
\] (2)

Let \( \mathfrak{F} \) be an ultrafilter on \( I \) which dominates the order filter. Define, for \( j \in I \),

\[
\psi_j' := w^* - \lim_{\beta \to \mathfrak{F}} y_{(\beta,j)}^{-1} \in \delta_\mathcal{G}^{w^*} \subseteq \text{Ball} \left( \text{LUC}(\mathcal{G}, w^{-1})^* \right),
\]
and let $\psi_j$ be arbitrary Hahn-Banach extensions of $\psi_j'$ to $L_\infty(G, w^{-1})^*$.

Since $w$ is diagonally bounded on $S'$, we have:

$$\sup_{s \in S'} \| w \left( s^{-1} \right) \delta_s \|_w = \sup_{s \in S'} w(s)w \left( s^{-1} \right) < \infty.$$ 

Thus, the family of functions

$$H_{(\alpha,i)} := \left( w \left( y_{(\alpha,i)}^{-1} \right) \delta_{(y_{(\alpha,i)}, h_i)} \right) = w \left( y_{(\alpha,i)}^{-1} \right) r_{y_{(\alpha,i)}} \left( \chi_{K_{(\alpha,i)}} h_i \right)$$

is bounded in $L_\infty(G, w^{-1})$, whence $(w^{-1} H_{(\alpha,i)})$ is a bounded family in $L_\infty(G)$. By (1), the projections $r_{y_{(\alpha,i)}} \chi_{K_{(\alpha,i)}} = \chi_{K_{(\alpha,i)}(y_{(\alpha,i)}^{-1})}$ are pairwise orthogonal, so that

$$H := \sum_{\alpha \in I} \sum_{i \in I} w^{-1} H_{(\alpha,i)} \quad (w^* \text{ - limits})$$

defines a function in $L_\infty(G)$. Hence, we have

$$h := \sum_{\alpha \in I} \sum_{i \in I} H_{(\alpha,i)} \in L_\infty(G, w^{-1}).$$

Using (1), we obtain for all $(\alpha,i), (\beta,j), (\gamma,k) \in \overline{I}$, where $(\gamma,k) \preceq (\beta,j)$:

$$\chi_{K_{(\gamma,k)}} r_{y_{(\beta,j)}} r_{y_{(\alpha,i)}} \left( \chi_{K_{(\alpha,i)}} h_i \right) = \chi_{K_{(\gamma,k)}} \chi_{K_{(\beta,j)}} r_{y_{(\beta,j)}} r_{y_{(\alpha,i)}} \left( \chi_{K_{(\alpha,i)}} h_i \right)$$

$$= \chi_{K_{(\gamma,k)}} \left[ r_{y_{(\beta,j)}} \left( r_{y_{(\beta,j)}} \chi_{K_{(\beta,j)}} \right) r_{y_{(\alpha,i)}} \left( \chi_{K_{(\alpha,i)}} h_i \right) \right]$$

$$= \chi_{K_{(\gamma,k)}} r_{y_{(\beta,j)}} \chi_{K_{(\gamma,k)}} h_j.$$ 

Taking into account (2), we deduce that for all $j \in I$ and $(\gamma,k) \in \overline{I}$:

$$\chi_{K_{(\gamma,k)}} \left( \psi_j \circ h \right) = w^* - \lim_{\beta \to \overline{\beta}} \sum_{\alpha \in I} \sum_{i \in I} w \left( y_{(\alpha,i)}^{-1} \right) w \left( y_{(\beta,j)}^{-1} \right)^{-1} \chi_{K_{(\gamma,k)}} r_{y_{(\beta,j)}} r_{y_{(\alpha,i)}} \left( \chi_{K_{(\alpha,i)}} h_i \right)$$

$$= \chi_{K_{(\gamma,k)}} h_j,$$

whence the desired factorization formula follows.

\[ \square \]

### 3 Strong Arens irregularity of $L_1(G, w)$

We now come to the proof of Theorem 1.2. – To establish the non-trivial inclusion, let $m \in Z_l (L_1(G, w))^{**}$. The group $G$ being non-compact, we infer from Proposition 2.1 that $L_1(G, w)$ has Mazur’s property of level $\mathfrak{p}(G)$. So in order to prove that $m \in L_1(G, w)$, let $(h_{\alpha})_{\alpha \in I} \subseteq L_\infty(G, w^{-1})$ be a bounded net converging $w^*$ to 0, where $|I| = \mathfrak{p}(G)$. Thanks to Theorem 2.2, we have the factorization

$$h_{\alpha} = \psi_{\alpha} \circ h = \widetilde{\psi}_{\alpha} \circ h \quad (\alpha \in I)$$
with \( \psi_\alpha \in \delta^w (G) \subseteq \text{Ball}(\text{LUC}(G, w^{-1})) \) and \( h \in L_\infty(G, w^{-1}) \). Here, \( \widetilde{\psi}_\alpha \) denotes some arbitrarily chosen Hahn-Banach extension of \( \psi_\alpha \) to \( L_\infty(G, w^{-1})^* \). We have to show that \( a_\alpha := \langle m, h_\alpha \rangle \to 0 \).

Due to the boundedness of \( (h_\alpha)_\alpha \), it suffices to prove that every convergent subnet of \( (a_\alpha)_\alpha \) tends to 0. Let \( (\langle m, h_{\alpha_\beta} \rangle)_\beta \) be such a convergent subnet. Furthermore, let

\[
E := w^* - \lim_{\gamma} \widetilde{\psi}_{\alpha_{\beta_\gamma}} \in \text{Ball} \left( L_\infty(G, w^{-1})^* \right)
\]

be a \( w^* \)-cluster point of the net \( \left( \widetilde{\psi}_{\alpha_\beta} \right)_\beta \subseteq \text{Ball} \left( L_\infty(G, w^{-1})^* \right) \).

We first note that \( E \circ h = 0 \), since for arbitrary \( g \in L_1(G, w) \) we obtain:

\[
\langle E \circ h, g \rangle = \langle E, h \circ g \rangle = \lim_{\gamma} \langle \psi_{\alpha_{\beta_\gamma}}, h \circ g \rangle = \lim_{\gamma} \langle \psi_{\alpha_{\beta_\gamma}} \circ h, g \rangle = \lim_{\gamma} \langle h_{\alpha_{\beta_\gamma}}, g \rangle = 0.
\]

Now we conclude, using the fact that \( m \in Z_\ell (L_1(G, w)^{**}) \):

\[
\lim_{\beta} \langle m, h_{\alpha_\beta} \rangle = \lim_{\gamma} \langle m, h_{\alpha_{\beta_\gamma}} \rangle = \lim_{\gamma} \langle m, \widetilde{\psi}_{\alpha_{\beta_\gamma}} \circ h \rangle = \lim_{\gamma} \langle m \circ \widetilde{\psi}_{\alpha_{\beta_\gamma}}, h \rangle = \langle m \circ E, h \rangle = \langle m, E \circ h \rangle = 0,
\]

which yields the desired convergence.

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