SCREW FUNCTIONS OF DIRICHLET SERIES
IN THE EXTENDED SELBERG CLASS

MASATOSHI SUZUKI

Abstract. We introduce screw functions for Dirichlet series in the extended Selberg class. Then we prove that the Grand Riemann Hypothesis for a member of the extended Selberg class is equivalent to the nonpositivity of the corresponding screw function.

1. Introduction

In [8], we introduced a screw function corresponding to the Riemann zeta-function whose nonpositivity of values is equivalent to the Riemann hypothesis for the Riemann zeta-function. As a large class of functions similar to the Riemann zeta-function, the Selberg class $S$ and the extended Selberg class $S^\sharp$ are well-known and important in number theory. The Riemann zeta-function, Dirichlet $L$-functions, and $L$-functions of holomorphic cusp forms are typical members of $S$ (cf. [6]). Therefore, it is naturally expected that a member $F$ of the extended Selberg class also corresponds to a screw function similarly to the Riemann zeta-function. This paper implements that expectation.

Let $F$ be a member of the extended Selberg class $S^\sharp$. The nontrivial zeros of $F(s)$ mean the zeros of the entire function $\xi_F(s)$, which is the Selberg class analogue of the Riemann xi-function defined in (2.3) below. We call $\gamma \in \mathbb{C}$ a nontrivial zero of $F(s)$ if $1/2 - i\gamma$ is a nontrivial zero of $F(s)$. The Grand Riemann Hypothesis (GRH, for short) for $F$ is the assertion that all nontrivial zeros of $F(s)$ lie on the line $\Re(s) = 1/2$, which is equivalent that all nontrivial zeros of $F(1/2 - iz)$ are real. We define the function $g_F(t)$ on the real line by

$$g_F(t) := -iB_F t - \frac{m_0}{2} t^2 + \sum_{\gamma \in \Gamma^\sharp_F \backslash \{0\}} m_\gamma \frac{e^{-i\gamma t} - 1}{\gamma^2}, \quad iB_F = \frac{\xi'_F}{\xi_F} \left( \frac{1}{2} \right)$$

(1.1)

for nonnegative $t$ and set $g_F(t) = g_F(-t)$ for negative $t$, where $\Gamma^\sharp_F$ is the set of all nontrivial zeros of $F(1/2 - iz)$, $m_\gamma$ is the multiplicity of $\gamma \in \Gamma^\sharp_F$, and $m_0 = 0$ if $0 \not\in \Gamma^\sharp_F$. For ease of introduction, definition (1.1) refers to [8, Proposition 2.3] rather than definition [8, (1.1)] of $\Phi(t) = -g(t)$, because we need the semi-extended Selberg class $S^{\sharp\flat}$ introduced by Smajlović [7] (and reviewed in Section 2) to define $g_F(t)$ without using zeros (cf. Theorem 4.1 below). The difference from the case of the Riemann zeta-function is the existence of the term $-m_0 t^2/2$ corresponding to $\gamma = 0$ and the lack of symmetry $\gamma \mapsto -\gamma$ in the sum over zeros. We call $g_F(t)$ the screw function of $F$ because it is a screw function in the sense of [4] assuming the GRH for $F$ is true (see Section 5 below). The main result of the present paper is the following generalization of [8, Theorem 1.2].

Theorem 1.1. Let $F$ be a member of $S^\sharp$ and let $g_F(t)$ be the screw function defined by (1.1). We assume that $F(s)$ has no real zeros except for the possible zero at $s = 1/2$. Then, the GRH for $F$ is true if and only if $\Re(-g_F(t)) \geq 0$ for all $t \geq t_0$ for some $t_0 \geq 0$. 

Date: Version of September 27, 2022.
2020 Mathematics Subject Classification. 11M26 42A82.
Key words and phrases. extended Selberg class, screw functions.
According to the proof of Theorem 1.1 in Section 3 below, if \( F \in S \), \( \Re(-g_F(t)) \geq 0 \) for all \( t \geq t_0 \), and \( s = 1 \) is neither a zero nor a pole of \( F(s) \), then there exists \( \delta_F > 0 \) such that \( F(s) \) has no zeros in the right-half plane \( \Re(s) > 1 - \delta_F \), since \( F \in S \) has no zeros in \( \Re(s) > 1 \) by the Euler product. On the other hand, \( F \in S^2 \setminus S \) may have a real zero greater than one.

In the following, we review the Selberg class in Section 2 and prove Theorem 1.1 in Section 3. Then we show that the screw function \( g_F \) allows a representation without zeros if we restrict \( F \) to the semi-extended Selberg class \( S^{2^p} \) in Section 4. Finally, we comment mainly on generalizations of other results in [S] in Section 5.

2. The Selberg class and explicit formulas

The Selberg class \( S \) introduced by A. Selberg in 1992 consists of the Dirichlet series

\[
F(s) = \sum_{n=1}^{\infty} \frac{a_F(n)}{n^s}
\]

(satisfying the following five axioms:

(S1) The Dirichlet series (2.1) converges absolutely if \( \Re(s) > 1 \).

(S2) There exists an integer \( m \geq 0 \) such that \( (s-1)^m F(s) \) extends to an entire function of finite order. The smallest \( m \) is denoted by \( m_F \).

(S3) \( F \) satisfies the functional equation

\[
\xi_F(s) = \omega \xi_F(1-s),
\]

where

\[
\xi_F(s) = s^{m_F} (s-1)^{m_F} Q^s \prod_{j=1}^{r} \Gamma(\lambda_j s + \mu_j) F(s)
\]

\[
= s^{m_F} (s-1)^{m_F} \gamma(s) F(s),
\]

\( \Gamma(s) \) is the gamma function, and \( r \geq 0, Q > 0, \lambda_j > 0, \mu_j \in \mathbb{C} \) with \( \Re(\mu_j) \geq 0 \), \( \omega \in \mathbb{C} \) with \( |\omega| = 1 \) are parameters depending on \( F \).

(S4) For every \( \varepsilon > 0 \), \( a_F(n) \ll_{\varepsilon} n^{\varepsilon} \).

(S5) \( \log F(s) = \sum_{n=1}^{\infty} b_F(n) n^{-s} \), where \( b_F(n) = 0 \) unless \( n = p^m \) with \( m \geq 1 \), and \( b_F(n) \ll n^\theta \) for some \( \theta < 1/2 \).

From (S3) and (S5), \( F \in S \) has no zeros outside the critical strip \( 0 \leq \Re(s) \leq 1 \) except for zeros in the left half-plane \( \Re(s) \leq 0 \) located at poles of the involved gamma factors. The zeros lie in the critical strip are called the nontrivial zeros. The nontrivial zeros are infinitely many unless \( F \equiv 1 \) and coincide with the zeros of the entire function \( \xi_F(s) \) of (2.2). The extended Selberg class \( S^2 \) is the class of functions satisfying (S1)–(S3) above. Note that the data (\( \omega, Q, r, \lambda_j, \mu_j \)) in (S3) are not uniquely defined by \( F \in S^2 \).

For \( F \in S^2 \), we define \( F^* (s) := \overline{F(\overline{s})} \). Then, \( F^* \in S^2 \) with \( a_{F^*}(n) = \overline{a_F(n)} \), \( m_{F^*} = m_F \), \( \omega_{F^*}(n) = \overline{\omega_F(n)} \), \( r_{F^*} = r_F \), \( Q_{F^*} = Q_F \), \( \lambda_{F^*,j} = \overline{\lambda_F,j} \), and \( \mu_{F^*,j} = \overline{\mu_F,j} \) if we write chosen \( \omega, Q, r, \lambda_j, \) and \( \mu_j \) in (S3) as \( \omega_F, Q_F, r_F, \lambda_F,j, \) and \( \mu_F,j \), respectively. In particular,

\[
\xi_{F^*}(s) = \overline{\xi_F(s)}.
\]

The survey [6] and its sequel are good introductions to the theory of the Selberg class and the extended Selberg class.

To represent the screw function \( g_F(t) \) without using zeros, we recall the subclass \( S^{2^p} \) introduced in [4, Section 2] that consists of \( F \in S^2 \) satisfying the Euler sum condition:

(S5') The logarithmic derivative of \( F \) possesses a Dirichlet series representation

\[
-\frac{F'(s)}{F(s)} = \sum_{n=2}^{\infty} \frac{c_F(n)}{n^s}
\]
converging absolutely for \( \Re(s) > 1 \).

The subclass \( S^p \), which we call the semi-extended Selberg class, contains the Selberg class \( S \) ([7, Theorem 2.1]). The Euler sum condition \((S5')\) is satisfied if \( \log F(s) \) has a Dirichlet series with suitable convergence. For a condition of Dirichlet series representation in \( \log F(s) \), see [11, Theorem 11.14] for example, but see also [16, Satz 12] for the convergence of the Dirichlet series.

3. Proof of Theorem 1.1

The central value \( i B_F = (\xi_F'/\xi_F)(1/2) \) is pure imaginary by the functional equation \( \xi_F(s) = \omega_F \overline{\xi_F(1 - s)} \). Therefore, assuming the GRH for \( F \) is true,

\[
\Re(-g_F(t)) = \frac{m_0}{2} t^2 + \sum_{\gamma \in \Gamma_F \setminus \{0\}} m_\gamma \frac{1 - \cos(\gamma t)}{\gamma^2}.
\]

The values of \(-g_F(t)\) are clearly nonnegative from the right-hand side.

To prove the converse claim, we first show that

\[
\int_0^\infty g_F(t)e^{izt} dt = \frac{1}{z^2} \xi_F \left( \frac{1}{2} - iz \right)
\]

(3.1)

holds when \( \Im(z) > 1/2 \). For \( F \in S^4 \), the function \( \xi_F(s) \) is an entire function of order one ([7, Lemma 3.3 and a note below the proof]). Therefore, the Hadamard factorization theorem implies that \( \xi_F(1/2 - iz) \) has the product formula

\[
\xi_F \left( \frac{1}{2} - iz \right) = e^{A_F + B_F z} z^{m_0} \prod_{\gamma \in \Gamma_F \setminus \{0\}} \left( 1 - \frac{z}{\gamma} \right) \exp \left( \frac{z}{\gamma} \right),
\]

where \( i B_F = (\xi_F'/\xi_F)(1/2) \) and \( \Gamma_F \) is the multiset of all zeros of \( \xi_F(1/2 - iz) \). Taking the logarithmic derivative,

\[
\frac{\xi_F'}{\xi_F} \left( \frac{1}{2} - iz \right) = i B_F + \frac{im_0}{z} + i \sum_{\gamma \in \Gamma_F \setminus \{0\}} \left( \frac{1}{z - \gamma} + \frac{1}{\gamma} \right),
\]

(3.2)

where the sum on the right-hand side converges absolutely and uniformly on every compact subset of \( \mathbb{C} \setminus \Gamma_F \). For each term on the right-hand side, we have

\[
\int_0^\infty \frac{e^{-\gamma t} - 1}{\gamma^2} e^{izt} dt = \frac{i}{z^2} \left( \frac{1}{z - \gamma} + \frac{1}{\gamma} \right) \quad \text{if} \quad \Im(z) > \Im(\gamma),
\]

and

\[
\int_0^\infty (t^2 - \frac{1}{2}) e^{izt} dt = \frac{i}{z^2} \int_0^\infty \frac{-t^2}{2} e^{izt} dt = \frac{i}{z^3} \quad \text{if} \quad \Im(z) > 0.
\]

(3.3)

Hence we obtain (3.1).

By (2.2) and (2.3), the mapping \( \gamma \mapsto -\gamma \) defines a bijection from \( \Gamma_F \) to \( \Gamma_F^* \) with multiplicity, and \( B_F = -B_{F^*} \, (\subset \mathbb{R}) \). Applying these facts to (1.1),

\[
g_F(t) = i B_F t - \frac{m_0}{2} t^2 + \sum_{0 \neq \gamma \in \Gamma_F} \frac{e^{\gamma t} - 1}{\gamma^2}
\]

(3.4)

\[
= -i B_{F^*} t - \frac{m_0}{2} t^2 + \sum_{0 \neq \gamma \in \Gamma_{F^*}} \frac{e^{-\gamma t} - 1}{\gamma^2} = g_{F^*}(t).
\]

Therefore, we obtain

\[
2 \int_0^\infty \Re(g_F(t))e^{izt} dt = \int_0^\infty (g_F(t) + g_{F^*}(t))e^{izt} dt
\]

\[
= \frac{1}{z^2} \left[ \frac{\xi_F'}{\xi_F} \left( \frac{1}{2} - iz \right) + \frac{\xi_{F^*}'}{\xi_{F^*}} \left( \frac{1}{2} - iz \right) \right] = \frac{1}{z^2} \xi_{FF^*} \left( \frac{1}{2} - iz \right)
\]

(3.5)
when $\Im(z) > 1/2$.

Let $\sigma_0 := \max\{\sigma \in \mathbb{R} \mid \xi_F(\sigma) = 0\}$. This number is the same for $F^*$ by the functional equation \(2.2\). If $\Re(-g_F(t))$ is nonnegative for $t \geq t_0$, then the integral on the left-hand side converges when $\Im(z) = \max(\sigma_0 - 1/2, 0)$ by \[\textbf{3}\] Theorem 5b of Chapter II. Hence $\xi_F(s)$ and $\xi_{F^*}(s)$ have no zeros in the right-half plane $\Re(s) > \max(\sigma_0 - 1/2, 0)$. In particular, the GRH for both $F$ and $F^*$ are true if $\xi_F(s)$ or $\xi_{F^*}(s)$ has no real zeros except for the possible zero at $s = 1/2$. \[\square\]

4. ZERO-FREE FORMULAS OF SCREW FUNCTIONS

For functions $F$ in the semi-extended Selberg class $S^\varnothing$, we obtain the following zero-
free formulas of corresponding screw functions.

**Theorem 4.1.** Let $F \in S^\varnothing$ and let $g_F(t)$ be the screw function of \[\textbf{1}\]. We chose the data $\omega$, $Q$, $r$, $\lambda_j$, and $\mu_j$ satisfying $(S3)$ and define

$$
\Phi_F(t) := 4m_F \left( e^{t/2} + e^{-t/2} - 2 \right) - \sum_{n \leq e^t} c_F(n) \frac{\sqrt{n}}{n} (t - \log n)
+ \left[ \log Q + \sum_{j=1}^r \lambda_j \frac{\Gamma'}{\Gamma} \left( \frac{\lambda_j}{2} + \mu_j \right) \right] t
+ \sum_{j=1}^r \lambda_j^2 \left[ \Phi \left( 1, 2, \frac{\lambda_j}{2} + \mu_j \right) - e^{-t(\frac{1}{2} + \mu_j)} \Phi \left( e^{-\frac{t}{2}}, 2, \frac{\lambda_j}{2} + \mu_j \right) \right]
$$

(4.1)

for nonnegative $t$ and set $\Phi_F(t) := \Phi_F(-t)$ for negative $t$, where $c_F(n)$ are numbers in $(S5')$. Then $\Phi_F(t)$ is uniquely determined from $F$ and $\Phi_F(t) = -g_F(t)$ holds for all real numbers $t$.

**Proof.** For the equality $\Phi_F(t) = -g_F(t)$, it suffices to prove that

$$
- \int_0^\infty \Phi_F(t) e^{itz} dt = \frac{1}{2\pi} \xi'_F(s), \quad s = \frac{1}{2} - iz
$$

(4.2)

holds when $\Im(z) > 1/2$ by \[\textbf{3}\] and the uniqueness of the inverse Fourier transform.

Taking the logarithmic derivative of $\Phi_F(s)$,

$$
\frac{\xi'_F}{\xi_F}(s) = \frac{m_F}{s - 1} + \frac{m_F}{s} + \frac{F'}{F}(s) + \log Q + \sum_{j=1}^r \lambda_j \frac{\Gamma'}{\Gamma} (\lambda_j s + \mu_j).
$$

On the other hand, we have

$$
\int_0^\infty 4(e^{t/2} + e^{-t/2} - 2) e^{itz} dt = \frac{1}{2\pi} \left( \frac{1}{s - 1} + \frac{1}{s} \right) \quad \text{if } \Im(z) > 1/2,
$$

(4.3)

and $\Phi_F(s)$ by direct and simple calculation. The second equality leads to

$$
\int_0^\infty \frac{\sum_{n \leq e^t} c_F(n)}{\sqrt{n}} (t - \log n) e^{itz} dt = \frac{1}{2\pi} \left( - \sum_{n=2}^\infty \frac{c_F(n)}{n^s} \right) = \frac{1}{2\pi} \frac{F'}{F}(s)
$$

(4.4)

if $\Im(z) > 1/2$ by $(S5')$. Hence, (4.2) is proved if it is shown that

$$
- \int_0^\infty \lambda \left[ \Phi \left( 1, 2, \frac{\lambda}{2} + \mu \right) - e^{-t(\frac{1}{2} + \mu)} \Phi \left( e^{-\frac{t}{2}}, 2, \frac{\lambda}{2} + \mu \right) \right] e^{itz} dt
= \frac{1}{2\pi} \Gamma' \left( \lambda s + \mu \right) - \Gamma' \left( \frac{\lambda}{2} + \mu \right)
$$

(4.5)
holds for \( \lambda > 0, \mu \in \mathbb{C} \) with \( \Re(\mu) \geq 0 \), and \( \Im(\lambda) > 0 \). We have
\[
\int_0^\infty \frac{1 - e^{-\beta t}}{\beta^2} e^{izt} dt = \frac{i}{z^2} \left( \frac{1}{z + i\beta} - \frac{1}{i\beta} \right), \quad \Im(z) > \max(0, -\Re(\beta))
\]
for a complex number \( \beta \) by direct and simple calculation. Therefore,
\[
\int_0^\infty \frac{1 - e^{-\frac{1}{2}(\xi + \mu + n)t}}{\lambda (\frac{1}{2} + \mu + n)^2} e^{izt} dt = \frac{1}{z^2} \left( -\frac{\lambda}{n} - \frac{1}{n + 1} \right)
\]
Using this and the series expansion \( \Phi(z, s, a) = \sum_{n=0}^\infty z^n (n + a)^{-s} \), the left-hand side of (4.5) is calculated as
\[
= -\int_0^\infty \left[ \sum_{n=0}^\infty \frac{1 - e^{-\frac{1}{2}(\xi + \mu + n)t}}{\lambda (\frac{1}{2} + \mu + n)^2} \right] e^{izt} dt
\]
\[
= \frac{1}{z^2} \sum_{n=0}^\infty \left( \frac{1}{\lambda s + \mu + n} - \frac{1}{n + 1} \right) - \frac{1}{z^2} \sum_{n=0}^\infty \left( \frac{1}{\frac{1}{2} + \mu + n} - \frac{1}{n + 1} \right).
\]
The right-hand side is equal to the right-hand side of (4.5) by the well-known series expansion
\[
\frac{\Gamma'}{\Gamma}(w) = -\gamma_0 - \sum_{n=0}^\infty \left( \frac{1}{w + n} - \frac{1}{n + 1} \right),
\]
where \( \gamma_0 \) is the Euler–Mascheroni constant. Hence, we complete the proof of (4.2).

On the other hand, it is known that the \( \gamma \)-factor \( \gamma(s) = Q^s \prod_{j=1}^r \Gamma(\lambda_j s + \mu_j) \) is uniquely determined up to a constant multiple ([6, Theorem 4.1]). Therefore, the logarithmic derivative \( (\gamma'/\gamma)(s) \) is uniquely determined from \( F \). Hence the values of \( \Phi_F(t) \) are uniquely determined from \( F \) by (4.2).

5. COMPLEMENTS

Finally, we make several comments on generalizations of other results in [8].

5.1. We have
\[
\int \frac{\xi_F'}{\xi_F} \left( \frac{1}{2} - iz \right) = - \left( B_F + \sum_{\gamma \in \Gamma_F \setminus \{0\}} \frac{1}{\gamma} \right) - \frac{\eta_0}{z} - \sum_{\gamma \in \Gamma_F \setminus \{0\}} \frac{1}{z - \gamma}.
\]
by (3.2), where the sum is taken over \( |\gamma| \leq T \) and then letting \( T \to \infty \) for the convergence. By the functional equation (2.2), \( \Gamma_F \) is closed under the complex conjugation, thus \( \sum_{\gamma} \gamma^{-1} \) is real. Therefore, assuming the GRH for \( F \), \( \Im(i(\xi_F'/\xi_F)(1/2 - iz)) > 0 \) when \( \Im(z) > 0 \), because \( \Im(-(z - \gamma)^{-1}) > 0 \) if \( \Im(z) > \Im(\gamma) \) for each term. Moreover, \( y^{-1}(\xi_F'/\xi_F)(1/2 + y) \ll y^{-1}\log y \) as \( y \to +\infty \) by (S5) and the Stirling formula of \( (\Gamma'/\Gamma)(w) \). Therefore, by the same argument as in [8] Introduction, after Corollary 1.1], \( g_F(t) \) is a screw function in the sense of [4].

We define the nonnegative measure \( \tau_F \) on \( \mathbb{R} \) by \( \tau_F = \sum_{\gamma \in \Gamma_F} m_\gamma \delta_{-\gamma} \), where \( \delta_\gamma \) is the Dirac measure at the point \( a \in \mathbb{R} \). Then \( \int_{-\infty}^{\infty} (1 + \lambda^2)^{-1} d\tau_F(\lambda) < \infty \) and \( g_F(t) \) has the representation
\[
g_F(t) = i C_F t + \int_{-\infty}^{\infty} \left( e^{i\lambda t} - 1 - \frac{i\lambda t}{1 + \lambda^2} \right) \frac{d\tau_F(\lambda)}{\lambda^2}
\]
on \( \mathbb{R} \) with \( C_F := B_F + \sum_{\gamma \in \Gamma_F} \gamma^{-1} (1 + \gamma^2)^{-1} \). This is nothing but the representation [4, Theorem 5.1] of screw functions on \( \mathbb{R} \).
5.2. The theory of screw functions is somewhat simpler for real-valued function \([\mathbb{H}]\). The screw function \(g_F\) for \(F \in \mathbb{S}^2\) is real-valued if \(F = F^*\) by \([4]\). In general \(F \in \mathbb{S}^2\) does not satisfy \(F = F^*\), but since \(\mathbb{S}^2\) is a multiplicative monoid, \(G := FF^*\) belongs to \(\mathbb{S}^2\) and satisfies \(G = G^*\). Since the GRH for \(F\) is equivalent to that for the product \(FF^*\) by \([2,2]\), it is sufficient to consider \(FF^*\) to study the GRH. In other words, we only need to study \(F \in \mathbb{S}^2\) satisfying \(F = F^*\).

5.3. The generalization of \([8, \text{Theorem 1.1}]\) to \(F \in \mathbb{S}^2\) follows immediately from the integral representation \((3.1)\), but generalizing \([8, \text{Theorem 1.3}]\) requires an additional assumption to prove the convergence of moments \(\mu_{F,n} = \int_0^\infty e^{-t/2}(-g_F(t)) t^n dt\) for all nonnegative integers \(n\). If we suppose that \(F \in \mathbb{S}\) and it has a zero-free region \(\Re(s) > 1 - C(\log(3 + |\Im(s)|))^{-1}\) for some \(C > 0\), then we obtain \((F^*/F)(\sigma + it) \ll \log(3 + |t|)\) in the zero-free region by standard arguments as in \([3, \text{Section 5.6}]\). With this estimate, the argument in \([8, \text{Section 4}]\) works and the convergence of the moments \(\mu_{F,n}\) is obtained.

5.4. In case \(F \in \mathbb{S}^\oplus\), the GRH for \(F\) is true if and only if
\[
\sum_{\gamma \in \Gamma_F} \int_{-\infty}^\infty \phi(x) e^{i\gamma x} dx \int_{-\infty}^\infty \overline{\phi(x)} e^{-i\gamma x} dx \geq 0
\]
for all smooth and compactly supported functions \(\phi\) on \(\mathbb{R}\) by Smajlović’s explicit formula \([7, \text{Theorem 3.1}]\), the generalization of Li’s criterion \([7, \text{Theorem 4.3}]\), and argument of the proof of \([2, \text{Theorem 1}]\). Therefore, the generalization of \([8, \text{Theorem 1.4}]\) for \(F \in \mathbb{S}^\oplus\) holds by replacing \(g\) with \(gf\). The result \([8, \text{Theorem 1.5}]\) is also expected to be generalized to the semi-extended Selberg class \(F \in \mathbb{S}^\oplus\), but detailed calculations for confirmation are left for future work.

Acknowledgments This work was supported by JSPS KAKENHI Grant Number JP17K05163.

References

[1] T. Apostol, Introduction to analytic number theory, Undergraduate Texts in Mathematics, Springer-Verlag, New York-Heidelberg, 1976.
[2] E. Bombieri, Remarks on Weil’s quadratic functional in the theory of prime numbers, I, \(\text{Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl. 11}\) (2000), no. 3, 183–233 (2001).
[3] H. Iwaniec, E. Kowalski, Analytic number theory, American Mathematical Society Colloquium Publications, 53, American Mathematical Society, Providence, RI, 2004.
[4] M. G. Kre˘ın, H. Langer, Continuation of hermitian positive definite functions and related questions, \(\text{Integral Equations Operator Theory 78}\) (2014), no. 1, 1–69.
[5] E. Landau, Über den Wertevorrat von \(\zeta(s)\) in der Halbebene \(\sigma > 1\), \(\text{Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-Physikalische Klasse}\) (1933), 81–91.
[6] A. Perelli, A survey of the Selberg class of \(L\)-functions. I, \(\text{Milan J. Math. 73}\) (2005), 19–52.
[7] L. Smajlović, On Li’s criterion for the Riemann hypothesis for the Selberg class, \(\text{J. Number Theory 130}\) (2010), no. 4, 828–851.
[8] M. Suzuki, Aspects of the screw function corresponding to the Riemann zeta function, \(\text{https://arxiv.org/abs/2206.03682}\).
[9] V. D. Widder, The Laplace Transform, Princeton Mathematical Series, vol. 6, \(\text{Princeton University Press, Princeton, N. J.}, \text{1941}\).