THE TWISTOR SPINORS OF GENERIC 2- AND 3-DISTRIBUTIONS

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Abstract. Generic distributions on 5- and 6-manifolds give rise to conformal structures that were discovered by P. Nurowski resp. R. Bryant. We describe both as Fefferman-type constructions and show that for orientable distributions one obtains conformal spin structures. The resulting conformal spin geometries are then characterized by their conformal holonomy and equivalently by the existence of a twistor spinor which satisfies a genericity condition. Moreover, we show that given such a twistor spinor we can decompose a conformal Killing field of the structure. We obtain explicit formulas relating conformal Killing fields, almost Einstein structures and twistor spinors.

1. Introduction

It was found by P. Nurowski [Nur05] that one can naturally associate to every generic 2-distribution on a 5-manifold a conformal structure of signature $(2, 3)$, and a similar observation has been made by R. Bryant [Bry06], who showed that there is a natural conformal $(3, 3)$-structure associated to every generic 3-distribution in dimension 6. These treatments employ Cartan’s method of equivalence and explicitly prolong the equations defining the distributions to a Cartan connection form, which is then seen to induce a conformal class of metrics on the underlying manifold. [Nur05] and [Bry06] also show that the induced conformal structures have special holonomies $G_2 \subset \text{SO}(3, 4)$ and $\text{SO}(3, 4) \subset \text{SO}(4, 4)$. The necessary computations are not easily accessible and quite complicated when done in full detail. The approach taken here is to deal with these two constructions by descriptions as Fefferman-type constructions [Cap06]. This viewpoint is useful for focusing on the essential algebraic relations between the structure groups of the geometries in question. The treatment via parabolic geometry conveniently shows that conformal structures associated to these distributions admit non-trivial solutions to certain overdetermined systems of PDEs and it uncovers relations between solution spaces - this is similar to the classical Fefferman spaces [CG06, CG08].

Date: April 29, 2011.

2000 Mathematics Subject Classification. 34A26, 35N10, 53A30, 53B15, 53B30.

Key words and phrases. generic distributions, conformal geometry, spin geometry, twistor spinors, Fefferman-type constructions, conformal Killing fields, almost Einstein scales.
The purpose of this text is twofold. First, it provides a complete discussion of the Fefferman-type construction for a generic rank 3-distribution in dimension 6. Second, it details and extends a relation to spin geometry that was found in [Ham09] for generic rank 2-distributions: We show that the twistor spinor attached to a generic rank 2 or 3 distribution is a very convenient encoding of the distribution and makes the special properties of the induced conformal structure easily visible. The Fefferman-type construction for generic rank 2 distributions has been treated by the authors in [HS09]. The new treatment here via the relation to spin geometry yields considerable simplifications.

The structure of this paper is as follows: In section 2 we discuss conformal spin structures of signature \((2, 3)\) and \((3, 3)\) and how twistor spinors satisfying a genericity condition on such structures give rise to generic distributions and conformal holonomy reductions.

In section 3 we show that all conformal spin structures of signature \((2, 3)\) and \((3, 3)\) admitting such a generic twistor spinor are induced by a Fefferman-type construction from a generic rank 2- resp. 3- distribution.

In section 4 we show that this twistor spinor can be used to decompose the conformal Killing fields of the induced conformal spin structure into a symmetry of the distribution and, in one case an almost Einstein scale, and in the other case a twistor spinor which is orthogonal in a suitable sense.

Acknowledgements. As always, the authors have benefited from many discussions with Andreas Čap. Both authors gladly acknowledge support from project P 19500-N13 of the "Fonds zur Förderung der wissenschaftlichen Forschung" (FWF). In the addition, the first author was supported by the IK I008-N funded by the University of Vienna, and the second author was supported by a L’Oréal Fellowship "For Women in Science".

2. Generic twistor spinors on conformal spin structures of signature \((2, 3)\) and \((3, 3)\)

2.1. Conformal spin structures. A conformal structure of signature \((p, q)\) on an \(n = p + q\)-dimensional manifold \(M\) is an equivalence class \(\mathcal{C}\) of pseudo-Riemannian metrics with two metrics \(g\) and \(\hat{g}\) being equivalent if \(\hat{g} = e^{2f}g\) for a function \(f \in C^\infty(M)\). Suppose we have a manifold with a conformal structure of signature \((p, q)\). Let \(\mathcal{G}_0\) be the associated conformal frame bundle with structure group the conformal group \(\text{CO}_o(p, q) = \mathbb{R}_+ \times \text{SO}_o(p, q)\) preserving both orientations. Then a conformal spin structure on \(M\) is a reduction of structure group of \(\mathcal{G}_0\) to \(\text{CSpin}(p, q) = \mathbb{R}_+ \times \text{Spin}(p, q)\).

It will be useful to employ abstract index notation, [PR87]: we write \(\mathcal{E}_a = \Omega^1(M), \mathcal{E}^a = \mathcal{X}(M)\) and multiple indices as in \(\mathcal{E}_{ab} = T^*M \otimes T^*M\) denote tensor products. To write conformally invariant objects we will also need the conformal density bundles \(\mathcal{E}[w]\), which are the line bundles associated to the 1-dimensional representations \((c, C) \mapsto c^w \in \mathbb{R}_+\) of \(\text{CSpin}(p, q) = \mathbb{R}_+ \times \text{Spin}(p, q)\). The tensor product of a bundle \(\mathcal{V}\) with \(\mathcal{E}[w]\) will be denoted
\( \mathcal{W} [w] \). We note that the conformal class of metrics \( \mathcal{C} \) gives rise to a canonical conformal metric \( \mathfrak{g} \in \mathcal{E}_{(ab)}[2] \) which is used to identify \( \mathcal{E}^a \) with \( \mathcal{E}_a[2] \).

Let us briefly introduce the main curvature quantities of the conformal structure \( \mathcal{C} \) (cf. [Eas96].) For \( \mathfrak{g} \in \mathcal{C} \), let
\[
P_{\mathfrak{g}} := \frac{1}{n-2} \left( \text{Ric}_{\mathfrak{g}} - \frac{\text{Sc}_{\mathfrak{g}}}{2(n-1)} \mathfrak{g} \right)
\]
be the Schouten tensor; this is a trace modification of the Ricci curvature \( \text{Ric}_{\mathfrak{g}} \) by a multiple of the scalar curvature \( \text{Sc}_{\mathfrak{g}} \). The trace of the Schouten tensor is denoted \( J_{\mathfrak{g}} := \mathfrak{g}^{pq} P_{pq} \). We will omit the subscripts \( \mathfrak{g} \) hereafter when giving a formula with respect to some \( \mathfrak{g} \in \mathcal{C} \).

It is well known that the complete obstruction against conformal flatness of \((M, \mathcal{C})\) with, \( \mathcal{C} \) having signature \( p + q \geq 3 \), is the Weyl curvature
\[
C_{\mathfrak{g} ab d} := R_{\mathfrak{g} ab d} - 2 \delta_{\mathfrak{g}}^a \mathcal{P}_{b d} + 2 \mathfrak{g}_{d(a} \mathcal{P}_{b)\mathfrak{g}},
\]
where indices between square brackets are skewed over.

We now introduce the basic ingredients of tractor calculus for conformal structures ([BEG94]). The tractor bundles are to be introduced as equivalence classes now, and alternatively as associated bundles to the Cartan structure bundle of a conformal spin structure in the next section 3.

2.1.1. The standard tractor bundle. The standard tractor bundle \( \mathcal{T} \) of a conformal structure \((M, \mathcal{C})\) is defined as an equivalence class of bundles \([\mathcal{T}]_{\mathfrak{g}}, \mathfrak{g} \in \mathcal{C} \); For a given metric \( \mathfrak{g} \) in the conformal class, we define the direct sum bundle \([\mathcal{T}]_{\mathfrak{g}} := \left( \mathcal{E}[-1] \oplus \mathcal{E}[1] \right) \), and a section \([s]_{\mathfrak{g}} = \left( \begin{array}{c} \hat{\rho} \\ \hat{\varphi}_a \\ \sigma \end{array} \right) \in \Gamma([\mathcal{T}]_{\mathfrak{g}}) \)
corresponds to the section \([s]_{\hat{\mathfrak{g}}} = \left( \begin{array}{c} \hat{\rho} \\ \hat{\varphi}_a \\ \sigma \end{array} \right) = \left( \begin{array}{c} \rho - \Upsilon_a \varphi^a - \frac{1}{2} \sigma \Upsilon^b \Upsilon_b \\ \varphi_a + \sigma \Upsilon_a \\ \sigma \end{array} \right) \) for \( \hat{\mathfrak{g}} = e^{2f} \mathfrak{g} \), \( \Upsilon = df \). The standard tractor bundle carries the invariant tractor metric \([h]_{\mathfrak{g}} = \left( \begin{array}{ccc} 0 & 0 & 1 \\ 0 & \mathfrak{g} & 0 \\ 1 & 0 & 0 \end{array} \right) \), which is compatible with the standard tractor connection
\[
\left[ \nabla_c^S \begin{array}{c} \rho \\ \varphi_a \\ \sigma \end{array} \right]_{\mathfrak{g}} = \left( \begin{array}{c} D_c \rho - \mathcal{P}^b \varphi_{b} \\ D_c \varphi_a + \sigma \mathcal{P}_{ca} + \rho \mathfrak{g}_{ca} \\ D_c \sigma - \varphi_c \end{array} \right).
\]

2.1.2. The spin tractor bundle. In the case where \((M, \mathcal{C})\) is a conformal spin structure, we define the weighted conformal spin bundle of \((M, \mathcal{C})\) as
\[
S_{\mathfrak{g}}[\frac{1}{2}] := G_0 \times_{\mathcal{C} \text{Spin}(p,q)} \Delta_{\mathfrak{g}}[\frac{1}{2}].
\]
Then we have the conformal Clifford symbol \( \gamma \in \Gamma(\text{End}(TM) \otimes S[1]) \). For \( \xi \in \mathfrak{X}(M) \) and \( \chi \in \Gamma(S[\frac{1}{2}]) \) we will write Clifford multiplication also as \( \xi \cdot \chi = \gamma(\xi) \chi \).
We define the spin tractor bundle of $\mathcal{C}$ again as an equivalence class over $\mathcal{C}$: with respect to $g$ it is the direct sum of weighted spin bundles $[S]_g := \left( \frac{S[-\frac{1}{2}]}{S[\frac{1}{2}]} \right)$, and a section $[X]_g \in [S]_g$ corresponds to $[X]_g = \left( \tau + \frac{1}{2} P \gamma^p \gamma^p \chi \right) \in [S]_g$; for $\hat{g} = e^{2f} g$, $\Upsilon = df$.

Indeed, $\mathcal{S}$ is the Clifford representation of $\mathcal{T}$: This is seen by introducing the Clifford action
\[
\begin{pmatrix}
\rho \\
\varphi_a \\
\sigma
\end{pmatrix} \cdot \begin{pmatrix}
\tau \\
\chi
\end{pmatrix} = \begin{pmatrix}
-\varphi_a \cdot \tau - \sqrt{2} \rho \chi \\
\varphi_a \cdot \chi + \sqrt{2} \sigma \tau
\end{pmatrix}.
\]

(2)

It is easy to compute directly (see e.g. [Ham09]) that indeed $s \cdot s \cdot X = -h(s,s)X$ for all $s \in \Gamma(\mathcal{T})$, $X \in \Gamma(\mathcal{S})$, and that this action is well defined.

$\mathcal{S}$ carries the spin tractor connection that is induced from the standard tractor connection on $\mathcal{T}$:
\[
[\nabla^\mathcal{S} \left( \begin{array}{c}
\tau \\
\chi
\end{array} \right) ]_g = \left( \begin{array}{c}
D_c \tau + \frac{1}{\sqrt{2}} P \gamma^p \gamma^p \chi \\
D_c \chi + \frac{1}{\sqrt{2}} \gamma^c \gamma^c \chi
\end{array} \right).
\]

Definition 2.1. The conformal holonomy of a conformal spin structure $\mathcal{C}$ is defined as
\[
\text{Hol}(\mathcal{C}) := \text{Hol}(\nabla^\mathcal{T}) = \text{Hol}(\nabla^\mathcal{S}) \subset \text{Spin}(p + 1, q + 1).
\]

(3)

2.2. The twistor spinor equation and its prolonged form. For a given metric $g \in \mathcal{C}$ the Dirac operator is defined as
\[
\mathcal{D} : \Gamma(\mathcal{S}[\frac{1}{2}]) \to \Gamma(\mathcal{S}[\frac{1}{2}]),
\]
and is used to define the twistor operator (cf. e.g. [BFGK90])
\[
\Theta : \Gamma(\mathcal{S}[\frac{1}{2}]) \to \Gamma(T^* M \otimes \mathcal{S}[\frac{1}{2}]),
\]
\[
\Theta(\chi) := D\chi + \frac{1}{n} \gamma \mathcal{D} \chi.
\]

Alternatively, $\Theta$ is described as the projection of the Levi-Civita derivative of a spinor to the kernel of Clifford multiplication. The definition of $\Theta$ with respect to the appropriately weighted spinor spaces used here is independent of the choice of $g \in \mathcal{C}$, i.e., it is a conformally invariant linear differential operator. An element in the kernel of $\Theta$ is called a twistor spinor, and we denote the space of twistor spinors by $\text{Tw}_\mathcal{C} := \ker \Theta$. By the following result, the space of twistor spinors is always finite dimensional:

Proposition 2.2 ([Fri90], [BFGK90], [Bra05], [Lei08], [Ham09]). Let $\Pi_0 : \mathcal{S} \to \Delta[\frac{1}{2}]$ be the (well-defined) projection from the spin tractor bundle to the lowest slot.

$\Pi_0$ induces an isomorphism of the space of $\nabla^\mathcal{S}$-parallel sections of $\mathcal{S}$ with the space of twistor spinors $\text{Tw}_\mathcal{C}$ in $\Gamma(\mathcal{S}[\frac{1}{2}])$. Its inverse is the conformally
invariant differential splitting operator \( L_0^S : \Gamma(S[1/2]) \to \Gamma(S) \) that is defined with respect to a \( g \in C \) by

\[
\chi \mapsto \left( \frac{\sqrt{2}}{n} \mathcal{D} \chi \right).
\]

(4)

In particular, one sees that the existence of a twistor spinor reduces the holonomy of the conformal spin structure.

2.3. Conformal spin structures of signature \((2, 3)\). We take some \( g_{2,3} \) be some signature \((2, 3)\)-bilinear form on \( \mathbb{R}^5 \) and write \( \mathbb{R}^{2,3} \) for \( \mathbb{R}^5 \) endowed with this form. It is well known (due to [Car67], also cf. [Lou01], [KS09]) that the real, 4-dimensional spin representation \( \Delta_{2,3} \) of \( \text{Spin}(2,3) = \text{Spin}(g_{2,3}) \)
carries a unique skew-form \( b_{2,3} \in \Lambda^2(\Delta_{2,3})^* \) that satisfies

\[
b_{2,3}(\xi \cdot \chi, \tau) - b_{2,3}(\chi, \xi \cdot \tau) = 0
\]

for all \( \xi \in \mathbb{R}^{2,3}, \chi, \tau \in \Delta_{2,3} \). It follows in particular that \( b_{2,3} \) is invariant under \( \text{Spin}(2,3) \), and this realizes the isomorphism \( \text{Spin}(2,3) \cong \text{Sp}(4,\mathbb{R}) \). The corresponding skew-symmetric pairing to \( b_{2,3} \) on the \( \text{CSpin}(2,3) \)-associated spin bundles is denoted \( b_{2,3} \in \Gamma(\Lambda^2 S^*[1]) \).

Definition 2.3. A twistor spinor \( \chi \in \Gamma(S[1/2]) \) is called generic if it satisfies

\( b_{2,3}(\chi, \mathcal{D} \chi) \neq 0 \).

Remark 2.1. (1) It easily follows from the twistor spinor equation, that \( b_{2,3}(\chi, \mathcal{D} \chi) \in C^\infty(M) \) is constant, given that \( M \) is connected, which we will assume. It is also easy to see from skew-symmetry of \( b_{2,3} \) and the transformation behavior of spin tractors that this number doesn’t depend on the choice of \( g \in C \) used for its computation.

(2) Recall that a spinor is called pure if its kernel under Clifford multiplication is a maximally isotropic subspace - in our case this means that it has dimension 2. Since \( \text{Spin}(2,3) \) acts transitively on \( \Delta_{2,3} \setminus \{0\} \), all non-zero spinors are pure. It is a well known fact due to E. Cartan [Car67] that the canonical pairing between two pure spinors is non-trivial if and only if their kernels under Clifford multiplication are transversal. This is easily seen directly in this case, which is done in the proof below.

Proposition 2.4. Let \( \chi \in \Gamma(S[1/2]) \) be a generic twistor spinor on a conformal spin structure \((M, C)\) of signature \((2, 3)\). Denote \( \tau = \frac{\sqrt{2}}{n} \mathcal{D} \chi \) for some \( g \in C \), and assume that \( b_{2,3}(\chi, \tau) = 1 \).

(1) For every \( x \in M \) there is a local frame \( e_1, e_2, r, f_1, f_2 \in \mathcal{X}(U) \), \( U \) a neighborhood of \( x \), such that on \( U \),

\[
\ker \gamma \chi = \text{span}(e_1, e_2), \text{ker} \gamma \tau = \text{span}(f_1, f_2),
\]

(6)

\[
(\ker \gamma \chi)^\perp \cap (\ker \gamma \tau)^\perp = \mathbb{R} r,
\]

\[
g(r, r) = -1, g(e_i, r) = 0, g(f_i, r) = 0, g(e_i, f_j) = \delta_{ij},
\]

(7)
and
\[ \frac{1}{2}e_1 \cdot e_2 \cdot \tau = \chi, \quad \frac{1}{2}f_1 \cdot f_2 \cdot \chi = \tau. \] (8)

This implies that
\[ r \cdot \chi = \chi, \quad r \cdot \tau = \tau, \quad f_1 \cdot \chi = -e_2 \cdot \tau, \quad f_2 \cdot \chi = e_1 \cdot \tau. \] (9)

(2) For \( \xi \in \mathfrak{X}(M) \) arbitrary and \( \eta \in \ker \gamma \chi \subset \mathfrak{X}(M) \), one has
\[ b_{2,3}(\xi \cdot \chi, \tau) = -g(\xi, \tau), \] (10)
\[ b_{2,3}(\xi \cdot \chi, \eta \cdot \tau) = -2g(\xi, \eta). \] (11)

Proof. (1) Let \( y \in M \) be arbitrary. We claim that \( \ker \gamma \chi(y) \) and \( \ker \gamma \tau(y) \)
have transversal (2-dimensional) kernels under Clifford multiplication: assume that there exists some \( 0 \neq \xi \in \ker \gamma \chi(y) \cap \ker \gamma \tau(y) \).
Then we can take some isotropic \( \eta \) with \( g(\xi, \eta) = 1 \), and then
\[ 0 = b_{2,3}(\xi \cdot \chi, \eta \cdot \tau) = b_{2,3}(\eta \cdot \xi \cdot \chi, \tau) = -2b_{2,3}(\chi, \tau), \]
which contradicts the assumption. Therefore we can take local sections \( e_1, e_2 \) and \( f_1, f_2 \)
on some neighborhood \( U \) of \( x \in M \) which are linearly independent at all points in \( U \) and span the kernels of \( \chi \) resp. \( \tau \) under Clifford multiplication. It is clear how to choose \( r \in \mathfrak{X}(U) \) then and that we can achieve (6), (7). The additional freedom \( \alpha e_1, \frac{1}{\alpha} f_1, \alpha \in \mathbb{R}_+ \) allows us to obtain (8), and (6)-(8) automatically imply (9).

(2) To see (10), note that
\[ b_{2,3}(\xi \cdot \chi, \tau) = b_{2,3}(\xi \cdot \chi, r \cdot \tau) =
= b_{2,3}(r \cdot \xi \cdot \chi, \tau) = -2 - b_{2,3}(\xi \cdot \chi, \tau). \]

Now to (11): Let \( \eta \in \ker \gamma \chi \). If \( \xi = e_i, i = 1, 2 \) the equation holds, since \( \xi \cdot \chi = 0 \) and \( g(\xi, \eta) = 0 \). If \( \xi = \tau \), then \( \xi \cdot \chi = \chi \), and by (5) and (7) the equation holds. If \( \xi = f_i, i = 1, 2 \), then
\[ b_{2,3}(\xi \cdot \chi, \eta \cdot \tau) = b_{2,3}((\xi + \eta) \cdot \chi, (\xi + \eta) \cdot \tau) =
= b_{2,3}((\xi + \eta) \cdot \chi, (\xi + \eta) \cdot \chi, \tau) = -2g(\xi, \eta). \]

The generlicity of a twistor spinor \( \chi \in \Gamma(S^{(1,0)}_1) \) carries over to the induced distribution \( \mathcal{D}_\chi := \ker \gamma \chi \). To make this precise, we first define, for two subbundles \( \mathcal{D}_1 \subset TM \) and \( \mathcal{D}_2 \subset TM \),
\[ [\mathcal{D}_1, \mathcal{D}_2]_x := \text{span}(\{[\xi, \eta] : \xi \in \Gamma(\mathcal{D}_1), \eta \in \Gamma(\mathcal{D}_2)\}). \] (12)

Definition 2.5. A smooth rank 2 subbundle \( \mathcal{D} \) of the tangent bundle \( TM \) of a 5-manifold \( M \) is called a generic 2 distribution if \( \mathcal{D}^1 := [\mathcal{D}, \mathcal{D}] \subset TM \)
is of constant rank 3 and \([\mathcal{D}, \mathcal{D}^1] \) is already \( TM \).

Employing Proposition 2.4 we can show

Proposition 2.6. Let \((M, C, \chi)\) be a conformal spin structure of signature (2,3) with a generic twistor spinor \( \chi \). Then \( \mathcal{D}_\chi = \ker \gamma \chi \) is a generic rank 2 distribution on \( M \).
Proof. Abbreviate \( \tau = \frac{\sqrt{2}}{3} \mathcal{D} \chi \). Then \( \mathcal{D} \chi = -\frac{1}{\sqrt{2}} \tau \). Choose a local frame \( e_1, e_2, r, f_1, f_2 \) in some neighborhood \( U \subset M \) with the properties \( (0)-(3) \). Then \( e_2 \cdot \chi = 0 \), and therefore also \( b_{2,3}(e_2 \cdot \chi, \tau) = 0 \). A differentiation gives \[
abla(D_{e_1} e_2) \cdot \chi, \tau) - \frac{1}{\sqrt{2}} b_{2,3}(e_2 \cdot e_1 \cdot \tau, \tau) = 0 \]
and via \( (8) \) we see \[
abla(D_{e_1} e_2 \cdot \chi, \tau) = -\sqrt{2}, \quad b_{2,3}(D_{e_2} e_1 \cdot \chi, \tau) = \sqrt{2}. \quad (13) \]

It is also easily seen that \[
b_{2,3}(D_{e_1} e_1 \cdot \chi, \tau) = b_{2,3}(D_{e_2} e_2 \cdot \chi, \tau) = 0, \quad (14) \]
which will be needed below. Alternating \( e_1, e_2 \) in \((13)\) shows \( b_{2,3}([e_1, e_2] \cdot \chi, \tau) = -2\sqrt{2} \), but since \( b_{2,3}([e_1, e_2] \cdot \chi, \tau) = -g([e_1, e_2], r) \), we have \( g([e_1, e_2], r) = 2\sqrt{2} \). A similar calculation gives that for all \( \eta \in \Gamma(\ker \gamma \chi) \) one has \( g([e_1, e_2], \eta) = 0 \), and thus \( [e_1, e_2] \in \Gamma(\ker \gamma \chi) \), and then \( [e_1, e_2]/\mathcal{D} = -\sqrt{2}r/\mathcal{D} \).

Starting with the equation \( b_{2,3}(r \cdot \chi, \eta \cdot \tau) = 0 \) for \( \eta \in \Gamma(\ker \gamma \chi) \) and differentiating in direction \( \xi \in \Gamma(\ker \gamma \chi) \) one obtains \[
b_{2,3}(D_\xi r \cdot \chi, \eta \cdot \tau) - \frac{1}{\sqrt{2}} b_{2,3}(r \cdot \xi \cdot \tau, \eta \cdot \tau) + b_{2,3}(\chi, D_\xi \eta \cdot \chi) = 0. \]

Reversing the roles of \( r \) and \( \xi \) gives \[
b_{2,3}(D_\xi r \cdot \chi, \eta \cdot \tau) - \frac{1}{\sqrt{2}} b_{2,3}(\xi \cdot r \cdot \tau, \eta \cdot \tau) = 0. \]

One has, since \( b_{2,3} \) is skew and satisfies \((13)\), \( b_{2,3}(r \cdot \xi \cdot \tau, \eta \cdot \tau) = b_{2,3}(\xi \cdot r \cdot \tau, \eta \cdot \tau) = 0 \). Therefore a subtraction yields \[
b_{2,3}(\xi \cdot r \cdot \chi, \eta \cdot \tau) = -b_{2,3}(D_\xi \eta \cdot \chi). \]

Employing \((11)\), \((13)\) and \((14)\) then gives \[
g([e_1, r], e_1) = 0, \quad g([e_1, r], e_2) = \frac{1}{\sqrt{2}} \]
and thus \([e_1, r]/\mathcal{D}^1 = \frac{1}{\sqrt{2}} f_2/\mathcal{D}^1 \), and similarly \([e_2, r]/\mathcal{D}^1 = -\frac{1}{\sqrt{2}} f_1/\mathcal{D}^1 \). \( \square \)

We now calculate the holonomy reduction implied by the existence of a generic twistor spinor on a conformal spin structure of signature \((2, 3)\).

**Theorem 2.7.** Let \((M, C)\) be a conformal spin structure of signature \((2, 3)\) with its real 4 dimensional, conformally weighted spin bundle \( S[1/2] \), which is endowed with its skew-form \( b_{2,3} \).

Then \( \text{Hol}(C) \subset G_2 \subset \text{Spin}(3, 4) \) if and only if there exists a twistor spinor \( \chi \in \Gamma(S[1/2]) \) with \( b_{2,3}(\chi, \mathcal{D} \chi) \neq 0 \).

Here, and throughout this text, \( G_2 \) will always refer to the connected Lie group with fundamental group \( \mathbb{Z}_2 \) that has Lie algebra \( \mathfrak{g}_2 \), the split real form of the exceptional complex Lie group \( \mathfrak{g}_2^C \).
By Proposition 2.2 and the definition of conformal holonomy, we only need that the stabilizer in Spin(3,4) of an element \( X = \begin{pmatrix} \tau \\ \chi \end{pmatrix} \in \Delta^{3,4} \) with the property that \( b_{2,3}(\chi, \tau) \neq 0 \) is indeed \( G_2 \). This is the content of the next subsection.

2.3.1. Algebraic background on \( G_2 \leftrightarrow Spin(3,4) \). Recall that we have fixed some signature (2,3)-bilinear form \( g_{2,3} \) on \( \mathbb{R}^5 \), and we write \( \mathbb{R}^{2,3} = (\mathbb{R}^5, g_{2,3}) \).

Let us extend this form orthogonally to a form \( h_{3,4} \) on \( \mathbb{R}^7 \) by introducing two new directions \( e_+ \) and \( e_- \) and defining \( h_{3,4} = 2(de_+)(de_-) + g_{2,3} \). The vectors \( e_+, e_- \in \mathbb{R}^{3,4} = (\mathbb{R}^7, h_{3,4}) \) are isotropic, and \( \mathbb{R}^{2,3} \hookrightarrow \mathbb{R}^{3,4} \) is the orthogonal complement to the subspace of \( \mathbb{R}^{3,4} \) spanned by \( e_+ \) and \( e_- \).

We have \( \Delta^{3,4} = \left( \frac{\Delta^{2,3}}{\Delta^{2,2}} \right) \), where an element \( v = se_- \otimes \xi \otimes pe_+ \in \mathbb{R}e_- \oplus \mathbb{R}e_+ \oplus \mathbb{R}^{2,3} \) acts on \( X = \begin{pmatrix} \tau \\ \chi \end{pmatrix} \) by \( v \cdot X = \begin{pmatrix} -\xi \cdot \tau - \sqrt{2}r \rho \chi \\ \xi \cdot \chi + \sqrt{2}\sigma \tau \end{pmatrix} \). \( \Delta^{3,4} \) is endowed with the canonical signature (4,4) symmetric bilinear form \( B_{3,4} \), defined by

\[
B_{3,4} \left( \begin{pmatrix} \tau \\ \chi \end{pmatrix}, \begin{pmatrix} \tau' \\ \chi' \end{pmatrix} \right) = b_{2,3}(\chi, \tau') + b_{2,3}(\chi', \tau),
\]

and with respect to \( B_{3,4} \) an element \( X = \begin{pmatrix} \tau \\ \chi \end{pmatrix} \) is evidently non-null if and only if \( b_{2,3}(\chi, \tau) \neq 0 \).

Proposition 2.8. Let \( X = \begin{pmatrix} \tau \\ \chi \end{pmatrix} \in \Delta^{3,4} \) be such that \( b_{2,3}(\chi, \tau) = 1 \).

1. There exists a basis \( e_1, e_2, f_1, f_2 \) of \( \mathbb{R}^{2,3} \subset \mathbb{R}^{3,4} \) with the properties (3,4).

2. The isotropy algebra \( \mathfrak{g}_2 := \mathfrak{so}(3,4)_X \) is the split real form of \( \mathfrak{g}^C_2 \). It carries a grading, more precisely, it is the direct sum \( \mathfrak{g}_{-3} \oplus \mathfrak{g}_{-2} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \mathfrak{g}_3 \), where the individual components are spanned by the following elements: (i = 1, 2)

\[
\begin{align*}
\mathfrak{g}_{-3} &= \text{span}(e_- \wedge f_1), \\
\mathfrak{g}_{-2} &= \text{span}(e_- \wedge r - \frac{1}{\sqrt{2}} f_1 \wedge f_2), \\
\mathfrak{g}_{-1} &= \text{span}(e_- \wedge e_i + \sqrt{2} r \wedge i e_i f_1 \wedge f_2), \\
\mathfrak{g}_0 &= \text{span}(e_1 \wedge f_2, e_2 \wedge f_1, e_i \wedge f_1 + e_+ \wedge e_-), \\
\mathfrak{g}_1 &= \text{span}(e_+ \wedge f_i - \sqrt{2} r i f_1 e_1 \wedge e_2), \\
\mathfrak{g}_2 &= \text{span}(e_+ \wedge r + \frac{1}{\sqrt{2}} e_1 \wedge e_2), \\
\mathfrak{g}_3 &= \text{span}(e_+ \wedge e_i).
\end{align*}
\]

The sum \( \mathfrak{p} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \mathfrak{g}_3 \) forms a parabolic subalgebra.
(3) $\text{Spin}(3,4)_X = G_2$.

(4) The restriction of the standard representation of $\text{Spin}(3,4)$ on $\mathbb{R}^{3,4}$ to $G_2$ is the irreducible standard representation of $G_2$. The restriction of the spin representation of $\text{Spin}(3,4)$ on $\Delta^{3,4}$ to $G_2$ splits into one copy of the standard representation and $\mathbb{R}$.

**Proof.** (1) is done as in Proposition 2.4, and (2) is checked directly. To see (3): It follows from the Proposition that the isotropy subgroup of $X$ in $\text{Spin}(3,4)$ is a closed subgroup with Lie algebra $\mathfrak{g}_2$. One can verify that it is the connected subgroup with fundamental group $\mathbb{Z}_2$. The action of $\text{Spin}(3,4)$ on the space of all spinors of a fixed norm $B_{3,4}(X, X)$ is transitive, since the orbit on any non-null spinor must be open by Proposition 2.8 and dimension count. To see that $G_2 := \text{Spin}(3,4)_X$ is connected and has fundamental group $\mathbb{Z}_2$ is then a standard argument using the exact homotopy sequence for the action.

(4) is seen easily on the Lie algebra level, and carries over to the connected groups. □

This is an alternative and more detailed description of $G_2 \subset \text{Spin}(3,4)$ of [Kat99].

2.4. Conformal spin structures of signature $(3,3)$. This section runs closely parallel to the case of signature $(2,3)$ in section 2.3, and is therefore done succinctly.

Let $g_{3,3}$ be some signature $(3,3)$-form on $\mathbb{R}^6$, which is then written $\mathbb{R}^{3,3}$. Let $\Delta^{3,3}$ be the real, 8-dimensional spin representation of $\text{Spin}(3,3) = \text{Spin}(g_{3,3})$. Then $\Delta^{3,3} = \Delta^{3,3}_+ \oplus \Delta^{3,3}_-$ and the Clifford multiplication is written $\mathbb{R}^{3,3} \otimes \Delta^{3,3} \to \Delta^{3,3}_\mp$, $\xi \otimes \chi \mapsto \xi \cdot \chi$.

The canonical pairing $b_{3,3}$ (cf. e.g. [KS09]) of $\Delta^{3,3}_+ \oplus \Delta^{3,3}_-$ satisfies

$$b_{3,3}(\chi, \xi \cdot \eta) + b_{3,3}(\eta, \xi \cdot \chi) = 0$$

for all $\xi \in \mathbb{R}^{3,3}, \chi, \eta \in \Delta^{3,3}_\pm$. It follows in particular that $b_{3,3}$ is invariant under $\text{Spin}(3,3)$, and this realizes the isomorphism $\text{Spin}(3,3) \cong \text{SL}(4, \mathbb{R})$.

The corresponding skew-symmetric pairing to $b_{3,3}$ on the $\text{CSpin}(3,3)$-associated spin bundles is denoted $b_{3,3} \in \Gamma(S_+^* \otimes S_-^*[1])$.

**Definition 2.9.** A positive twistor spinor $\chi \in \Gamma(S^+[\frac{1}{2}])$ is called generic if it satisfies $b_{3,3}(\chi, D/\chi) \neq 0$.

Similarly to Proposition 2.4 one shows

**Proposition 2.10.** Let $\chi \in \Gamma(S^+[\frac{1}{2}])$ be a positive generic twistor spinor on a conformal spin structure $(M, \mathcal{C})$ of signature $(3,3)$. Denote $\tau = \frac{\sqrt{2}}{6}D/\chi$ for some $g \in \mathcal{C}$, and assume that $b_{3,3}(\chi, \tau) = 1$. 

For every \( x \in M \) there is a local frame \( e_1, e_2, e_3, f_1, f_2, f_3 \in \mathcal{X}(U) \), \( U \) a neighborhood of \( x \), such that on \( U \),
\[
\ker \gamma \chi = (e_1, e_2, e_3), \quad \ker \gamma \tau = (f_1, f_2, f_3),
\]
(\(\text{16}\))
\[
g(e_i, f_j) = \delta_{ij},
\]
(\(\text{17}\))
\[
e_1 \cdot e_2 \cdot e_3 \cdot \tau = \chi, \quad f_1 \cdot f_2 \cdot f_3 \cdot \chi = \tau.
\]
(\(\text{18}\))

\(\text{(2)}\) For \( \xi \in \mathcal{X}(M) \) arbitrary and \( \eta \in \ker \gamma \chi \subset \mathcal{X}(M) \), one has
\[
b_{3,3}(\xi \cdot \chi, \eta \cdot \tau) = -2g(\xi, \eta).
\]

**Definition 2.11.** A generic rank 3 distribution on a 6-manifold \( M \) is a smooth rank 3 subbundle \( \mathcal{D} \) of \( TM \) with \([\mathcal{D}, \mathcal{D}] = TM\).

An analogous proof to Proposition 2.6 gives

**Proposition 2.12.** Let \((M, C, \chi)\) be a conformal spin structure of signature \((3,3)\) with a positive generic twistor spinor \( \chi \). Then \( \mathcal{D}_\chi = \ker \gamma \chi \) is a generic rank 3 distribution on \( M \).

We now discuss the conformal holonomy discussion induced by a generic twistor spinor in signature \((3,3)\):

**Theorem 2.13.** Let \((M, [g])\) be a conformal spin structure of signature \((3,3)\) with its real 8 dimensional conformally weighted spin bundle \(S^{\frac{1}{2}} = S_+^{\frac{1}{2}} \oplus S_-^{\frac{1}{2}}\); there is a canonical non-degenerate pairing \(b_{3,3}: S_+^{\frac{1}{2}} \otimes S_-^{\frac{1}{2}} \to \mathbb{R}\).

Then \(\text{Hol}([g]) \subset \text{Spin}(3,4)\) if and only if there exists a generic twistor spinor \( \chi \in \Gamma(S_+^{\frac{1}{2}}) \), i.e., \(b_{3,3}(\chi, \mathcal{D}\chi) \neq 0\).

**Proof.** For this, we compute, analogously to the case of signature \((2,3)\) above, the stabilizer in \(\text{Spin}(4,4)\) of an element \( X = \begin{pmatrix} \tau \\ \chi \end{pmatrix} \in \Delta^{4,4} \) with the property that \(b_{3,3}(\chi, \tau) \neq 0\). It is easy to see (cf. formula (19) below) that the invariant \(\text{Spin}(3,3)\)-invariant form \(b_{3,3}\) can be used to define the canonical \(\text{Spin}(4,4)\)-invariant form \(b_{4,4}\) on \(\Delta^{4,4}\). Via triality, this is equivalent to the standard signature \((4,4)\)-inner product \(h_{4,4}\) on \(\mathbb{R}^8\), which is the standard representation of \(\text{Spin}(4,4)\). Since the stabilizer of a non-null element in the standard representation of \(\text{Spin}(4,4)\) is a standard embedding of \(\text{Spin}(3,4)\) into \(\text{Spin}(4,4)\), the stabilizer of a non-null \( X \in \Delta^{4,4} \) as above is just such a standard embedding composed with a triality automorphism of \(\text{Spin}(4,4)\).

Although not strictly necessary for the computation of the conformal holonomy in this signature, it will be useful to discuss the embedding \(\text{Spin}(3,4) \hookrightarrow \text{Spin}(4,4)\) as the stabilizer of a non-null \( X \in \Delta^{4,4} \) also directly again; this will provide us with an explicit form of the stabilizing Lie algebra in a canonical basis that will be useful in the next section.
2.4.1. Spin(3, 4) ⊆ Spin(4, 4). We have the 8-dimensional Clifford representation \( \Delta^{3,3} \) of \( \mathbb{R}^{3,3} = (\mathbb{R}^6, g_{3,3}) \) that splits into \( \Delta^{3,3}_+ \oplus \Delta^{3,3}_- \) under Spin(3, 3) \( \cong \text{SL}(4) \).

Let \( h_{4,4} \) be the signature \((4, 4)\) symmetric bilinear form \( h_{4,4} = 2(de_+)(de_-) + g_{3,3} \) on \( \mathbb{R}^8 = \mathbb{R}e_+ \oplus \mathbb{R}^{3,3} \oplus \mathbb{R}e_- \). The Clifford representation of \( \mathbb{R}^{4,4} = (\mathbb{R}^8, h_{4,4}) \) is defined on \( \Delta^{4,4}_+ := \Delta^{3,3}_+ \oplus \Delta^{3,3}_- \) via

\[
\mathbb{R}^{4,4} \otimes \Delta^{4,4}_+ \rightarrow \Delta^{4,4}_+, \quad \begin{pmatrix} \rho \\ \xi \\ \sigma \end{pmatrix} \cdot \begin{pmatrix} \tau \\ \chi \end{pmatrix} = \begin{pmatrix} -\xi \cdot \tau - \sqrt{2} \rho \chi \\ \xi \cdot \chi + \sqrt{2} \sigma \tau \end{pmatrix}.
\]

The Clifford-invariant symmetric split signature \((4, 4)\)-form \( B_{4,4} \), is defined on \( \Delta^{4,4}_+ \) and \( \Delta^{4,4}_- \) by

\[
B_{4,4}(\begin{pmatrix} \tau \\ \chi \end{pmatrix}, \begin{pmatrix} \tau' \\ \chi' \end{pmatrix}) = b_{3,3}(\chi, \tau') + b_{3,3}(\chi', \tau).
\]

Using this one shows

**Proposition 2.14.** Let \( X = \begin{pmatrix} \tau \\ \chi \end{pmatrix} \in \Delta^{4,4}_+ \) be such that \( b_{3,3}(\chi, \tau) = 1 \).

1. There exists a basis \( e_1, e_2, e_3, f_1, f_2, f_3 \) of \( \mathbb{R}^{3,3} \) such that (16)-(18) hold.
2. The isotropy algebra \( g := \mathfrak{so}(4, 4)_X \) is a realization of \( \mathfrak{so}(3, 4) \). It is graded as \( g_{-2} \oplus g_{-1} \oplus g_0 \oplus g_1 \oplus g_2 \), where, with \( i = 1, 2, 3 \),

\[
\begin{align*}
g_{-2} &= \text{span}(e_- \wedge f_i) \\
g_{-1} &= \text{span}(e_- \wedge e_i - \sqrt{2}i e_i \wedge f_1 \wedge f_2 \wedge f_3) \\
g_0 &= \text{span}(e_i \wedge f_j, \text{for } i \neq j \text{ and } e_i \wedge f_i + e_+ \wedge e_-) \\
g_1 &= \text{span}(e_+ \wedge f_i - \sqrt{2}i f_i e_1 \wedge e_2 \wedge e_3) \\
g_2 &= \text{span}(e_+ \wedge e_i).
\end{align*}
\]

The subspace \( p = g_0 \oplus g_1 \oplus g_2 \) forms a parabolic subalgebra.

3. \( \text{Spin}(4, 4)_X = \text{Spin}(3, 4) \).

4. The restriction of the standard representation of \( \text{Spin}(4, 4) \) to \( \text{Spin}(3, 4) \) is the spin representation \( \Delta^{3,4} \), as is the restriction of the negative spin representation of \( \text{Spin}(4, 4) \). The restriction of the positive spin representation of \( \text{Spin}(4, 4) \) to \( \text{Spin}(3, 4) \) decomposes into a copy of \( \Delta^{3,4} \) and \( \mathbb{R} \).

3. **Conformal spin structures associated to generic 2 and 3 distributions.**

3.1. **Generic distributions and conformal spin structures as parabolic geometries.** Let \( G \) be a Lie group and \( P \subset G \) a closed subgroup, denote by \( g \) and \( p \) the respective Lie algebras. A Cartan geometry (see e.g. [Sha97]) of type \((G, P)\) is given by a principal bundle \( G \rightarrow M \) with structure group \( P \) and a Cartan connection \( \omega \in \Omega^1(G, g) \), this is a \( P \)-equivariant
1-form that reproduces generators of fundamental vector fields and defines isomorphisms $\omega_u : T_u G \rightarrow \mathfrak{g}$ for each $u \in G$. The basic example of a Cartan geometry of type $(G, P)$ is the homogeneous model, i.e., the bundle $p : G \rightarrow G/P$ equipped with the Maurer Cartan form. The curvature $\kappa \in \Omega^2(G, \mathfrak{g})$ of a Cartan geometry, defined as

$$\kappa(\xi, \eta) = d\omega(\xi, \eta) + [\omega(\xi), \omega(\eta)]$$

for $\xi, \eta \in \mathfrak{X}(G)$ is a complete obstruction to local equivalence with the homogeneous model. The curvature can equivalently be described as a $P$-equivariant function $G \rightarrow \Lambda^2(\mathfrak{g}/p)^* \otimes \mathfrak{g}$ and we will often take this point of view.

Parabolic geometries are Cartan geometries of type $(G, P)$ for a semisimple Lie group $G$ and a parabolic subgroup $P \subset G$. There is by now an big amount of general theory available for geometries of this type, cf. [ˇCS09].

One of the main reasons for their special importance is that they allow uniform Lie algebraic regularity and normality conditions on the Cartan curvature. Assuming regularity a parabolic geometry determines a certain underlying structure, called a regular infinitesimal flag structure. If the parabolic geometry is also normal it is uniquely determined by its underlying structure, and one obtains an equivalence of categories in this case (cf. [ˇCS09] for the general statement and earlier versions for particular geometries.).

To describe the underlying structures note that every parabolic subalgebra $p$ of a semisimple Lie algebra $\mathfrak{g}$ determines a grading of the Lie algebra

$$\mathfrak{g} = \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k$$

such that $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$, the negative part $\mathfrak{g}_- = \mathfrak{g}_{-1} \oplus \cdots \oplus \mathfrak{g}_k$ is generated by $\mathfrak{g}_{-1}$ and $p = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k$. The positive part $\mathfrak{p}_+ = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k$ is then a nilpotent ideal in $\mathfrak{p}$, and $\mathfrak{g}_0$ a reductive subalgebra. The natural action of the corresponding subgroup $G_0$ preserves the grading on $\mathfrak{g}$, while the parabolic $P$ only preserves filtration induced by the grading.

For a parabolic geometry of type $(G, P)$ this Lie algebra filtration induces a filtration of the tangent bundle and, assuming regularity, also some additional structure: Let $T^{-1}M \subset \cdots \subset T^{-k}M = TM$ be a filtration of the tangent bundle by subbundles that is compatible with forming Lie brackets. Then the Lie bracket induces a tensorial bracket called Levi bracket $\mathcal{L} : \text{gr}(TM) \times \text{gr}(TM) \rightarrow \text{gr}(TM)$ on the associated graded $\text{gr}(TM) = \bigoplus_i T^i M/T^{i+1}M$. Suppose $(\text{gr}(TM), \mathcal{L})$ is a bundle of Lie algebras modelled on the graded Lie algebra $\mathfrak{g}_-$ and consider the natural frame bundle $\mathcal{P}$ for $\text{gr}(TM)$ with structure group the automorphisms $\text{Aut}_{\text{gr}}(\mathfrak{g}_-)$ of $\mathfrak{g}_-$ that preserve the grading. A regular infinitesimal flag structure of type $(G, P)$ consists of such a filtration and a reduction of structure group of the bundle $\mathcal{P}$ with respect to $\text{Ad} : G_0 \rightarrow \text{Aut}_{\text{gr}}(\mathfrak{g}_-)$.

3.1.1. Generic rank two distributions in dimension five. Suppose $\mathcal{D}$ is a generic rank 2-distribution on a 5-manifold, as defined via (12) and Definition 2.5. Defining $T^{-1}M = \mathcal{D}$, $T^{-2}M = [\mathcal{D}, \mathcal{D}]$ and $T^{-3}M = TM$
yields a filtered manifold, such that the Levi bracket defines isomorphisms
\[ \Lambda^2 T^{-1}M \to T^{-2}M/T^{-1}M \text{ and } T^{-1}M \otimes T^{-2}M/T^{-1}M \to T^{-3}M/T^{-2}M. \]
In particular, \( \text{Aut}_\text{gr}(\mathfrak{g}_-) \cong \text{GL}(2, \mathbb{R}) \) and the frame bundle for \( \text{gr}(TM) \) can be identified with the frame bundle for the distribution. A reduction of this frame bundle to \( \text{GL}_+(2, \mathbb{R}) \) is the same as an orientation of the distribution.

Now let \( G_2 \) be the connected Lie group with Lie algebra the split real form of the simple complex exceptional Lie algebra \( \mathfrak{g}_2^C \) and with fundamental group \( \mathbb{Z}_2 \). A grading of \( \mathfrak{g}_2 \) corresponding to a maximal parabolic subalgebra \( \mathfrak{p} \) was introduced in Proposition 2.8. It is easy to see that \( (\text{gr}(TM), \mathcal{L}) \) from above is modelled on the negative part \( \mathfrak{g}_- \) of that grading. Thus, for a parabolic subgroup \( P \subset G_2 \) with Lie algebra \( \mathfrak{p} \), a regular infinitesimal flag structure of type \( (G_2, P) \) is the same as a reduction of the frame bundle of a generic 2-distribution \( D \) to the structure group \( G_0 \). The usual choice for the parabolic is to define it as the stabilizer \( P' \) of the line through the highest weight vector \( v \) in the 7-dimensional fundamental representation of \( G_2 \). In that case the reductive subgroup \( G_0 \cong \text{GL}(2, \mathbb{R}) \), and the regular infinitesimal flag structure is just a generic rank 2 distribution. However, in the context of this paper we will use the connected parabolic subgroup \( P \), i.e., the stabilizer of the ray \( \mathbb{R}_+v \) through the highest weight vector. Then \( G_0 \) is isomorphic to the group \( \text{GL}_+(2, \mathbb{R}) \), and a regular infinitesimal flag structure encodes an oriented distribution. Thus, the equivalence result for parabolic geometries implies:

**Proposition 3.1.** With the above choice of Lie groups, there is an equivalence of categories between regular, normal parabolic geometries of type \( (G_2, P) \) and oriented generic rank 2 distributions on 5-manifolds.

### 3.1.2. Generic rank three distributions in dimension six

Now suppose \( D \) is a generic rank 3-distributions on a 6-manifold, i.e., values of sections \( \xi, \eta \in \Gamma(D) \) and their Lie brackets \( [\xi, \eta] \) span the tangent bundle \( TM \), recall (12) and Definition 2.11. Then the distribution gives rise to the filtration
\[ T^{-1}M = D \subset T^{-2}M = TM \text{ such that the Levi bracket } \mathcal{L} : \Lambda^2 T^{-1}M \to TM/T^{-1}M \text{ is an isomorphism.} \]

In this case there is a grading of \( \mathfrak{so}(3, 4) \) corresponding to a parabolic \( \mathfrak{p} \), described explicitly in Proposition 2.13 such that in every point the graded Lie algebra \( (\text{gr}(T_xM), \mathcal{L}_x) \) is isomorphic to \( \mathfrak{g}_- \). Now consider as a group with Lie algebra \( \mathfrak{p} \) the parabolic subgroup \( P \subset \text{Spin}(3, 4) \) defined as the stabilizer of a ray through a highest weight vector in the real spinor representation \( \Delta^{3,4} \). Then \( P \) does not contain the element \(-1\) acting as minus the identity on the spin representation, and one easily verifies that \( P \) is connected. It follows that the 2-fold covering \( \text{Spin}(3, 4) \to \text{SO}_0(3, 4) \) restricts to a diffeomorphism from \( P \) onto the connected component of the parabolic \( \mathcal{P}' \subset \text{SO}_0(3, 4) \) defined as the stabilizer of an isotropic 3-dimensional subspace of \( \mathbb{R}^{3,4} \). In particular, the Levi subgroup \( G_0 \subset P \subset \text{Spin}(3, 4) \) is seen to be \( \text{GL}_+(3, \mathbb{R}) \). Thus, invoking the general theory yields:
Proposition 3.2. With the above choice of Lie groups, there is an equivalence of categories between regular, normal parabolic geometries of type \((SO(3,4), P)\) and oriented generic rank 3 distributions on 6-manifolds.

3.1.3. Conformal spin structures. A conformal spin structure (see section 2.1) can be equivalently described as a normal parabolic geometry of type \((\text{Spin}(p+1, q+1), \tilde{P})\), where \(\tilde{P}\) is the stabilizer of a positive ray through a null-vector in \(\mathbb{R}^{p+1,q+1}\). For this choice of groups the parabolic subgroup \(\tilde{P}\) is connected and the reductive subgroup \(\tilde{G}_0 \subset \tilde{P}\) is precisely \(\text{CSpin}(p,q)\).

We remark that in this case the Cartan structure bundle \(\tilde{G} \rightarrow M\) can be realized as the adapted frame bundle of the standard tractor bundle \(T\) introduced in section 2.1.1, i.e., it is the frame bundle of \((T, h)\) that additionally satisfies the canonical filtration of \(T\), cf. \(\text{CG08}\). Conversely, the standard tractor bundle is the associated bundle \(T = \tilde{G} \times_{\tilde{P}} \mathbb{R}^{p+1,q+1}\).

3.1.4. Tractor bundles. More generally, there are associated vector bundles carrying canonical linear connections for all types of parabolic geometries:

Suppose \((G, \omega)\) is a regular, normal parabolic geometry of type \((G, P)\). Given a \(G\)-representation \(V\), one can form the associated bundle \(V = G \times_P V\). Such a vector bundle is called a tractor bundle. Let \(G' := G \times_P G\) be the extended Cartan bundle, which is now a \(G\)-principal bundle over \(M\). Then we can extend \(\omega\) canonically in an equivariant way to the extended \(G\)-principal bundle connection form \(\omega' \in \Omega^1(G', g)\). Since \(V = G \times_P V\) can also be written as \(G' \times_{G'} V\), we see that \(\omega'\) induces a linear connection on \(V\), which is the normal tractor connection \(\nabla^V\), cf. eg. \(\text{CG02}\).

3.2. The Fefferman-type constructions \(D \rightsquigarrow C_D\). We prove that to any oriented generic rank 2 distribution on a 5 manifold, and to any oriented rank 3 distribution on a 6 manifold there is an associated conformal spin structure. This is done via a Fefferman-type construction in the sense on A. Čap.

3.2.1. Fefferman-type constructions over the same manifold. Consider an inclusion of simple Lie groups \(G \hookrightarrow \tilde{G}\), and parabolic subgroups \(\tilde{P} \subset \tilde{G}\), \(P = \tilde{P} \cap G\), and suppose the inclusion induces a diffeomorphism of the corresponding homogeneous spaces

\[G/P \cong \tilde{G}/\tilde{P}.
\]

Then there is a is a functorial construction, see \(\text{Cap06}\), associating to a parabolic geometry \((G, \omega)\) of type \((G, P)\) a parabolic geometry of type \((\tilde{G}, \tilde{P})\): First one extends the Cartan bundle to a \(\tilde{P}\)-principal bundle

\[\tilde{G} = G \times_P \tilde{P},\]

and then one shows that there is a unique extension of \(\omega\) to a Cartan connection \(\tilde{\omega} \in \Omega^1(\tilde{G}, \tilde{g})\) on \(\tilde{G}\).

It is shown in \(\text{DS08}\) that there is a very limited number of Lie group data that give rise to such a Fefferman-type construction over the same base.
manifold; indeed, there are only three families of such constructions. We discuss the two constructions that give rise to conformal structures. In fact, assuming orientability of the distributions, we will see that we get induced conformal spin structures.

3.2.2. The Fefferman-type construction $G_2 \hookrightarrow \text{Spin}(3,4)$. Let $P \subset G_2$ and $\tilde{P} \subset \text{Spin}(3,4)$ be the parabolic subgroups from 3.1.1 and 3.1.3, i.e., the subgroups defined as the stabilizer of a positive ray through a null-vector in $\mathbb{R}^{3,4}$. Then $P = \tilde{P} \cap G_2$. Since $G_2/P$ and $\text{Spin}(3,4)/\tilde{P}$ are compact, connected and have the same dimension, the homogeneous spaces are indeed diffeomorphic: we have

$$G_2/P \cong \text{Spin}(3,4)/\tilde{P} \cong S^2 \times S^3.$$ 

The functorial construction discussed above thus assigns to a parabolic geometry $(G, \omega)$ of type $(G_2, P)$ a parabolic geometry $(\tilde{G}, \tilde{\omega})$ of type $(\text{Spin}(3,4), P)$.

**Proposition 3.3.** An oriented generic rank 2-distribution on a 5-manifold $M$ naturally induces a conformal spin structure of signature $(2,3)$ on $M$.

3.2.3. The Fefferman-type construction $\text{Spin}(3,4) \hookrightarrow \text{Spin}(4,4)$. Let $\tilde{P} \subset \text{Spin}(4,4)$ be the parabolic subgroup defined as the stabilizer of the positive ray through a null-vector in $\mathbb{R}^{4,4}$. Then, since as a $\text{Spin}(3,4)$ representation $\mathbb{R}^{4,4} = \Delta^{3,4}$, the intersection $P = \tilde{P} \cap \text{Spin}(3,4)$ is precisely the parabolic introduced in 3.2.2. Using again that generalized flag manifolds are compact and counting dimensions one obtains

$$\text{Spin}(3,4)/P \cong \text{Spin}(4,4)/\tilde{P} \cong S^3 \times S^3.$$ 

Thus, the Fefferman-type construction associates to a parabolic geometry of type $(\text{Spin}(3,4), P)$ a parabolic geometry of type $(\text{Spin}(4,4), \tilde{P})$.

**Proposition 3.4.** An oriented generic rank 3-distribution on a 6-manifold naturally induces a conformal spin structure of signature $(3,3)$ on the manifold.

3.2.4. Normality of the induced parabolic geometry. For parabolic geometries there is a uniform algebraic normalization condition, defined in terms of the $P$-equivariant Kostant codifferential

$$\partial^* : \Lambda^2((\mathfrak{g}/\mathfrak{p})^* \otimes \mathfrak{g}) \rightarrow ((\mathfrak{g}/\mathfrak{p})^* \otimes \mathfrak{g})$$

given on decomposable elements as

$$\partial^*(X \wedge Y \otimes Z) = X \otimes [Y, Z] - Y \otimes [X, Z] - [X, Y] \otimes Z.$$ 

A parabolic geometry $(\mathcal{G} \rightarrow M, \omega)$ of type $(G, P)$ is called normal if its curvature function $\kappa : \mathcal{G} \rightarrow \Lambda^2((\mathfrak{g}/\mathfrak{p})^* \otimes \mathfrak{g}$ satisfies $\partial^* \circ \kappa = 0$. The curvature of a normal parabolic geometry projects to a simpler curvature quantity, the harmonic curvature $\kappa_H$. The harmonic curvature takes values in a $G_0$-submodule that is explicitly computable via Kostant’s version of the Bott-Borel-Weil theorem [Kos61].
For applications it will be essential that the conformal parabolic geometries attached to generic distributions are normal. To verify compatibility of a Fefferman-type construction with normality is in general a non-trivial problem. Suppose $\tilde{\omega}$ is the extension to $\tilde{\mathcal{G}} = \mathcal{G} \times_P \tilde{P}$ of a regular, normal Cartan connection form $\omega$. Let $I : \Lambda^2(\mathfrak{g}/\mathfrak{p})^* \otimes \mathfrak{g} \to \Lambda^2(\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}})^* \otimes \tilde{\mathfrak{g}}$ be the induced map from the inclusion $\mathfrak{g} \hookrightarrow \tilde{\mathfrak{g}}$. Then the curvature functions $\kappa : \mathcal{G} \to \Lambda^2(\mathfrak{g}/\mathfrak{p})^* \otimes \mathfrak{g}$ of $\omega$ and $\tilde{\kappa} : \tilde{\mathcal{G}} \to \Lambda^2(\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}})^* \otimes \tilde{\mathfrak{g}}$ of $\tilde{\omega}$ are related by

$$\tilde{\kappa}(u) = I \circ \kappa(u)$$

for all $u \in \mathcal{G}$, and this determines $\tilde{\kappa}$ by equivariance, see [CZ09]. We now ask is whether the parabolic geometry $(\tilde{\mathcal{G}}, \tilde{\omega})$ is normal, i.e., whether $\tilde{\partial}^* \circ \tilde{\kappa} = 0$. By (21) we can rephrase this as to whether $\kappa$ takes values in the $(G \cap \tilde{P})$-submodule $I^{-1}(\ker(\tilde{\partial})) \subset \Lambda^2(\mathfrak{g}/\mathfrak{p})^* \otimes \mathfrak{g}$.

For Fefferman-type constructions over the same manifold, i.e., in those cases where $P = (G \cap \tilde{P})$, this problem can be considerably simplified if one uses the following strong result:

**Proposition 3.5** ([Cap05]). Suppose $E \subset \ker(\partial^*) \subset \Lambda^2(\mathfrak{g}/\mathfrak{p})^* \otimes \mathfrak{g}$ is a $P$-submodule and consider the $G_0$-module $E_0 := E \cap \ker(\Box)$. Let $(\tilde{\omega} : G \to M, \omega)$ be a regular, normal parabolic geometry, which is furthermore torsion-free. Then, if the harmonic curvature $\kappa_H$ takes values in $E_0$ the curvature function $\kappa$ takes values in $E$.

3.2.5. **Normality for Spin(3, 4) → Spin(4, 4).** The harmonic curvature $\kappa_H$ of a regular, normal parabolic geometry of type $(\text{Spin}(3, 4), P)$ takes values in an irreducible 27-dimensional $G_0 = \text{GL}_+(3, \mathbb{R})$-subrepresentation of $\Lambda^2(\mathfrak{g}/\mathfrak{p})^* \otimes \mathfrak{g}_0$. This implies that the geometry is torsion-free, and we may apply the above result to prove:

**Proposition 3.6.** The Fefferman-type construction associates to a regular, normal parabolic geometry of type $(\text{Spin}(3, 4), P)$ a normal parabolic geometry of type $(\text{Spin}(4, 4), \tilde{P})$.

**Proof.** Let $\tilde{\partial}^* : \Lambda^2(\tilde{\mathfrak{p}}_+ \otimes \tilde{\mathfrak{g}}) \to \tilde{\mathfrak{p}}_+ \otimes \tilde{\mathfrak{g}}$ be the Kostant codifferential describing the conformal normalization condition. Since a regular, normal parabolic geometry of type $(\text{Spin}(3, 4), P)$ is torsion-free, we can apply Proposition 3.5 which shows that to prove normality of $\omega$ it suffices to prove that $\kappa_H$ takes values in the $P$-module $\ker(\tilde{\partial}^* \circ I)$.

Now $\tilde{\partial}^* \circ I$ is equivariant and thus it either vanishes on $G_0$-irreducible components, or it is an isomorphism. In particular, it must contain the 27-dimensional irreducible representation where $\kappa_H$ takes its values either in its image or in its kernel. The formula for the Kostant codifferential $\tilde{\partial}^*$ shows that if we restrict $\tilde{\partial}^* \circ I$ to $\Lambda^2(\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}})^* \otimes \tilde{\mathfrak{g}}_0$ and identify $(\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}})^* \cong \tilde{\mathfrak{p}}_+$, its image is contained in $\tilde{\mathfrak{p}}_+ \otimes \tilde{\mathfrak{p}}_+$. But $\tilde{\mathfrak{p}}_+$ decomposes as a $G_0$-representation as $(\mathbb{R}^3)^* \oplus \Lambda^2(\mathbb{R}^3)^*$, and therefore $\tilde{\mathfrak{p}}_+ \otimes \tilde{\mathfrak{p}}_+$ cannot contain a 27-dimensional irreducible summand.
3.2.6. Normality for $G_2 \hookrightarrow \text{Spin}(3, 4)$. We used similar arguments in [HS09] to prove that the extension of a regular, normal Cartan connection $\omega$ of type $(G_2, P)$ to a Cartan connection $\tilde{\omega}$ of type $(\text{SO}(3, 4), \tilde{P})$ is again normal. Note that the arguments in the proof do not depend on the choice of groups $(\text{Spin}(3, 4), \tilde{P})$ or $(\text{SO}(3, 4), \tilde{P})$, respectively.

**Proposition 3.7.** The Fefferman-type construction associates to a regular, normal parabolic geometry of type $(G_2, P)$ a normal parabolic geometry of type $(\text{Spin}(3, 4), \tilde{P})$.

3.2.7. The twistor spinors of generic 2 and 3 distributions.

**Theorem 3.8.** The Fefferman-type constructions 3.2.2 and 3.2.3 for generic 2 and 3 distributions determine generic twistor spinors. The kernels of these twistor spinors recover the 2 or 3 distribution.

**Proof.** Let $D$ be a generic rank 2 or 3 distribution, $(\mathcal{G}, \omega)$ the associated parabolic geometry of type $(G, P)$ and $(\tilde{\mathcal{G}}, \tilde{\omega})$ the conformal spin geometry obtained via the Fefferman-type construction. Then the spin tractor bundle of $\mathcal{C}$ is, for $(p, q) = (2, 3)$ resp. $(3, 4)$,

$$S = \tilde{\mathcal{G}} \times_p \Delta^{p+1,q+1} = \mathcal{G} \times_P \Delta^{p+1,q+1}. \tag{22}$$

Since $G \subseteq \text{Spin}(p + 1, q + 1)$ is the isotropy subgroup of a non-null element, say $X \in \Delta^{p+1,q+1}$, the constant function $G \to \Delta^{p+1,q+1}$ onto this element defines a spin-tractor $X \in \Gamma(S)$. Propositions 3.7 and 3.6 imply that the spin tractor connection $\nabla^S$ is induced from the canonical Cartan connection $\omega \in \Omega^1(\mathcal{G}, g)$ for the distribution $D$. Thus, $\nabla^S X = 0$. By Proposition 2.2, $X$ corresponds via the map (4) to a twistor spinor $\chi$. Since $X$ is non-null for the bilinear form on $\Delta^{p+1,q+1}$, the corresponding spinor tractor is non-null for the induced form on the tractor bundle. By (15) resp. (19) this is equivalent to $b_{3,4}(\chi, \mathcal{D} \chi) \neq 0$ resp. $b_{4,4}(\chi, \mathcal{D} \chi) \neq 0$ for the underlying twistor spinor, and thus $\chi$ is generic.

Moreover $\chi$ has kernel $D$. This follows from the fact that via the identification $TM \cong \mathcal{G} \times_P g/p$ determined by the Cartan connection $\omega \in \Omega^1(\mathcal{G}, g)$, the distribution corresponds to the subbundle $\mathcal{G} \times_P g^{-1}/p$, and from the descriptions of the gradings from Propositions 2.8 and 2.14. □

Note that it is not obvious that the map that assigns to a generic twistor spinor its kernel and the map 3.2.7 from distributions to twistor spinors coming from Fefferman-type constructions discussed above are inverse bijections: a priori we don’t know that all generic twistor spinors in the right signatures are induced from generic distributions. In order to obtain a characterization of the conformal spin structures associated to generic rank 2 and 3 distributions by generic twistor spinors we will invoke the holonomy characterizations to be discussed below.

3.3. Conformal holonomy characterization of the Fefferman-type spaces.
3.3.1. Holonomy of the conformal structures associated to generic 2 and 3 distributions. We have seen in Proposition 3.5 that the conformal spin structures induced by generic distributions carry parallel spin-tractors. Thus, they have reduced conformal holonomy: Conformal spin structures associated to oriented generic rank 2-distributions in dimension 5 have conformal holonomy contained in $G_2 \subset \text{Spin}(3, 4)$, and conformal spin structures associated to oriented generic rank 3-distributions in dimension 6 have conformal holonomy contained in $\text{Spin}(3, 4) \subset \text{Spin}(4, 4)$.

Let $(\tilde{G}, \tilde{\omega})$ be a regular, normal parabolic geometry of type $(\text{Spin}(p, q), P)$ encoding a conformal structure $\mathcal{C}$ on $M$. Then $\text{Hol}(\mathcal{C})$ was defined in Definition 2.1 as the holonomy of the spin tractor connection $\nabla^S$. Since $\nabla^S$ is the induced holonomy from the extended normal Cartan connection $\tilde{\omega}' \in \Omega^1(\tilde{G}, \mathfrak{so}(p + 1, q + 1))$ (cf. subsection 3.1.4), we have that $\text{Hol}(\mathcal{C}) = \text{Hol}(\nabla^S) = \text{Hol}(\tilde{\omega}') \subset \text{Spin}(p + 1, q + 1)$. It will be useful for our purposes of reversing the Fefferman-type constructions from above to see the conformal holonomy reductions from this viewpoint:

Suppose $(\mathcal{G}, \omega)$ is a regular normal parabolic geometry of type $(G, P)$, and suppose there is a Fefferman-type construction that gives rise to a normal parabolic geometry $(\tilde{G}, \tilde{\omega})$ of type $(\text{Spin}(p + 1, q + 1), \tilde{P})$ over the same manifold. Let $\mathcal{G}' = \mathcal{G} \times_P G$ be the extended $G$-principal bundle of $\mathcal{G}$ and $\omega'$ the principal connection obtained by extension. Then we have the commuting diagram of inclusions

\[
\begin{array}{ccc}
(G', \omega') & \longrightarrow & (\tilde{G}', \tilde{\omega}') \\
\uparrow & & \uparrow \\
(G, \omega) & \longrightarrow & (\tilde{G}, \tilde{\omega})
\end{array}
\]

which shows that $\tilde{\omega}' \in \Omega^1(\tilde{G}', \mathfrak{so}(p + 1, q + 1))$ reduces to the $G$-principal bundle connection $\omega' \in \Omega^1(G', \mathfrak{g})$ and thus $\text{Hol}(\mathcal{C}) = \text{Hol}(\tilde{\omega}') = \text{Hol}(\omega') \subset G$.

3.3.2. Holonomy characterizations. Conversely, let $(\tilde{G}, \tilde{\omega})$ be a parabolic geometry of type $(\text{Spin}(p + 1, q + 1), \tilde{P})$ such that the conformal holonomy group $\text{Hol}(\tilde{\omega}')$ is contained in $G$ and suppose we have a parabolic subgroup $P \subset G$ with $G/P \cong \text{Spin}(p + 1, q + 1)/\tilde{P}$. Then, since we have a holonomy reduction, $\tilde{G}'$ reduces to a $G$-principal bundle $\mathcal{G}'$ and $\tilde{\omega}'$ reduces to $\omega' \in \Omega^1(\mathcal{G}', \mathfrak{g})$. Using that $G/P \cong \text{Spin}(p + 1, q + 1)/\tilde{P}$, one can show that $\mathcal{G}'$ intersects with $\tilde{G}$ in a $P$-principal bundle $\mathcal{G}$ and $\omega'$ restricts to a Cartan connection $\omega \in \Omega^1(G, \mathfrak{g})$. See [Ham09] and [HS09] for details.

A normal conformal Cartan connection $\tilde{\omega}$ is torsion-free, i.e. the curvature function $\tilde{\kappa} : \tilde{G} \to \Lambda^2(\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}})^* \otimes \tilde{\mathfrak{g}}$ takes values in the $P$-submodule $\Lambda^2(\mathfrak{p}/\tilde{\mathfrak{p}})^* \otimes \tilde{\mathfrak{p}}$. Since $\mathfrak{p} = \tilde{\mathfrak{p}} \cap \mathfrak{g}$ and because of the relation of the curvatures of $\omega$ and $\tilde{\omega}$, as in [21], this implies that $\omega$ is torsion-free and thus regular. This means that the geometry $(\mathcal{G}, \omega)$ obtained by reduction as explained in the previous paragraph induces an underlying generic distribution $\mathcal{D}$. 

It remains to prove that this distribution $D$ induces via the Fefferman-type construction the conformal structure encoded in $(\tilde{G}, \tilde{\omega})$. For this we need to see that $\omega$ and the normal Cartan connection $\omega_N$ for $D$, which is a priori different, induce the same conformal structure. Now, the conformal structure induced by a geometry $(G, \omega)$ does not depend on the entire Cartan connection, but only on the isomorphism $TM \cong G \times_p g/p$ defined by the Cartan connection. If the difference $\omega - \omega_N$, seen as a function $G \to \Lambda^2(g/p)^* \otimes g$ takes values in $\Lambda^2(g/p)^* \otimes p$, then the two induce the same conformal structure. It is verified in Proposition 4.1 of [Arm07] that this is the case for generic rank 3-distributions in dimension 6: Since $\omega$ is torsion-free, $\kappa$ takes values in maps $\Lambda^2(g/p)^* \otimes p$ of homogeneity $\geq 2$, where the homogeneity is defined with respect to the grading of $g$. Since $\partial^* \kappa$ preserves homogeneities, the same is true for $\partial^* \kappa$, and then also the difference $\omega - \omega_N$ maps into homogeneity $\geq 2$, see Proposition 3.1.13 in [CS09]. If $g$ is 2-graded, as it is the case for generic rank 3 distributions, this already implies that $\omega - \omega_N$ takes values in $\Lambda^2(g/p)^* \otimes p$. For generic rank 2 distributions a similar argument applies if one first directly verifies that a certain curvature component of $\kappa$ vanishes, see [Sag08] and [HS09].

Summarizing we obtain:

**Theorem 3.9.** The Fefferman-type construction for oriented generic rank 2-distributions produces exactly those conformal spin structures of signature $(2, 3)$ whose conformal holonomy is contained in $G_2$.

**Theorem 3.10.** The Fefferman-type construction for oriented generic rank 3-distributions produces exactly those conformal spin structures of signature $(3, 3)$ whose conformal holonomy is contained in Spin$(3, 4)$.

We remark that if $\text{Hol}(\mathcal{C}) \subset G$ is a proper subgroup, there may be several holonomy reductions yielding different generic distributions.

**Corollary 3.11.** A conformal spin structure $(M, \mathcal{C})$ of signature $(2, 3)$ or $(3, 3)$ is induced by a generic 2- resp 3-distribution $D \subset TM$ via a Fefferman-type construction if and only if $(M, \mathcal{C})$ carries a generic twistor spinor $\chi$.

**Proof.** The result is an immediate corollary of Theorems 3.9 and 3.10. The general holonomy correspondence between holonomy invariant elements and parallel sections implies that holonomy reductions to $G_2$ respectively Spin$(3, 4)$ correspond to the existence of parallel non-null spin tractors. Parallel spintractors are equivalent to twistor spinors via (4), and non-isotropy of the spin tractor translates into the genericity conditions expressed via the canonical forms $b_{2,3}$ resp. $b_{3,3}$.

**Remark 3.1.** (1) In particular, we have shown that the conformal structure $\mathcal{C}$ and its generic twistor spinor $\chi$ are completely determined by the generic distribution $D_\chi = \ker \gamma \chi$. 

(2) The existence of this twistor spinor should have interesting consequences for the ambient metric construction of the associated conformal structures. Indeed, for certain examples of generic 2-distributions Leistner and Nurowski [LN09] have found a corresponding parallel spinor on the ambient metric of the $(2, 3)$-conformal structure.

4. Decompositions of conformal Killing fields

The main result of this section is an explicit decomposition of infinitesimal conformal automorphisms via a transversal twistor spinor. Furthermore, we provide formulae relating almost Einstein structures with a subset of twistor spinors.

4.1. Infinitesimal automorphisms of conformal spin structures associated to 2 and 3 distributions. Infinitesimal automorphisms of a conformal structure $\mathcal{C}$ are conformal Killing fields, i.e., vector fields $\xi \in \mathfrak{X}(M)$ such that $\mathcal{L}_\xi g = fg$ for some $g \in \mathcal{C}$ and some $f \in C^\infty(M)$. Infinitesimal automorphisms of a distribution $\mathcal{D}$ are vector fields whose Lie derivatives preserve the distribution, i.e., $\xi \in \mathfrak{X}(M)$ such that $\mathcal{L}_\xi \eta = [\xi, \eta] \in \Gamma(\mathcal{D})$ for all $\eta \in \Gamma(\mathcal{D})$.

For the structures we are interested in infinitesimal automorphisms correspond to infinitesimal automorphisms of the associated Cartan geometries. Our decomposition result is based on a description of infinitesimal automorphisms of a Cartan geometry $(\mathcal{G}, \omega)$ as sections of the adjoint tractor bundle $\tilde{\mathcal{A}}M = \mathcal{G} \times \mathcal{P} \tilde{\mathfrak{g}}$ that are parallel with respect to the prolongation connection

$$\hat{\nabla}_\xi s = \nabla_\xi s - \kappa(\xi, \Pi(s)),$$  \hspace{1cm} (23)

see [Cap08]. Here $\Pi : \mathcal{A}M = \mathcal{G} \times \mathcal{P} \mathfrak{g} \to \mathcal{G} \times \mathcal{P} \mathfrak{g}/\mathfrak{p} = TM$ is the natural projection and $\kappa$ is the curvature of $\omega$ viewed as an element of $\Omega^1(M, \mathcal{A}M)$.

Let $\mathfrak{g} \subset \tilde{\mathfrak{g}}$ be either the inclusion $\mathfrak{g}_2 \subset \mathfrak{so}(3, 4)$, or $\mathfrak{so}(3, 4) \subset \mathfrak{so}(4, 4)$. Then, as a representation of $G$, i.e. $G_2$ or $\text{SO}(3, 4)$, we have a decomposition

$$\tilde{\mathfrak{g}} = \mathfrak{g} \oplus \mathbb{R}^3.$$

It follows that the conformal adjoint tractor bundle

$$\tilde{\mathcal{A}}M = \tilde{\mathcal{G}} \times \tilde{\mathcal{P}} \tilde{\mathfrak{g}} = \tilde{\mathcal{G}} \times \tilde{\mathcal{P}} \tilde{\mathfrak{g}}$$

of a conformal structure associated to a generic distribution decomposes into $\mathcal{A}M$ and a 7-dimensional complementary bundle $\mathcal{V} = \mathcal{G} \times \mathcal{P} \mathbb{R}^3$. The tractor connection is natural and decomposes accordingly.

**Proposition 4.1.** Suppose $s = s_1 + s_2 \in \Gamma(\tilde{\mathcal{A}}M) = \Gamma(\mathcal{A}M) \oplus \Gamma(\mathcal{V})$ is the decomposition of a conformal adjoint tractor according to the decomposition of $\tilde{\mathcal{A}}M$. Then $s$ is parallel with respect to the infinitesimal automorphism connection on $\tilde{\mathcal{A}}M$ if and only if $s_1$ is parallel with respect to the infinitesimal automorphism connection on $\mathcal{A}M$ and $s_2$ is parallel with respect to the tractor connection on $\mathcal{V}$. 
The proof is discussed in detail the $G_2$-case in [HS09], the Spin$(3,4)$-case is completely analogous. Briefly, one uses that the curvature of the conformal structures associated to such distributions takes indeed values in $\mathcal{AM}$, see [HS09], and that adjoint tractors in $\tilde{\mathcal{A}}M$ are parallel for the tractor connection insert trivially into the curvature, see [Gov06],[HS09].

4.2. Explicit splitting formulas. For the explicit decompositions in sections 4.3 and 4.4 it will be necessary to collect some explicit differential splitting formulas, which are particular instances of $BGG$-splitting operators [ˇCSS01, CD01].

4.2.1. Splittings of conformal Killing fields. We have discussed in section 4.1 that an infinitesimal symmetry of a conformal structure $(\mathcal{M},\mathcal{C})$ is equivalent to an adjoint tractor $s \in \Gamma(\tilde{\mathcal{A}}M)$ that is parallel with respect to (23). To make the relation between the conformal Killing fields $\xi \in \mathfrak{X}(\mathcal{M})$ and the adjoint tractor $s \in \Gamma(\tilde{\mathcal{A}}M)$ explicit, one employs the canonical differential splitting operator $L_0^\tilde{A}$, that is given by [Ham08, Ham09]

$$L_0^\tilde{A} : \mathfrak{X}(\mathcal{M}) \to \Gamma(\Lambda^2 \mathcal{T}),$$

$$\xi^a \mapsto \left(\begin{array}{c}
-\frac{1}{2n}D^pD_p\xi^a + \frac{1}{2m}D^pD_a\xi^p + \frac{1}{n^2}D_aD^p\xi^p \\
+ \frac{1}{n^2}D^a\xi_p - \frac{1}{n}J\xi_a \\
D_{[a}\xi_{|a_1]} - \frac{1}{n}g^{pq}D_p\xi_q \\
\xi_a
\end{array}\right).$$

It follows from ([ˇCap08]) that

**Lemma 4.2.** $\xi \in \mathfrak{X}(\mathcal{M})$ is a conformal Killing field if and only if $L_0^\tilde{A}(\xi)$ is parallel with respect to the connection (23).

We already have interpretations of parallel sections of the spin tractor bundle as twistor spinors and of parallel sections (with respect to a modified connection) of the adjoint tractor bundle as conformal Killing fields. We now recall the interpretation of the parallel sections of the conformal standard tractor bundle:

4.2.2. The splittings of almost Einstein scales. Let $s \in \Gamma(\mathcal{T})$, then with respect to a $g \in \mathcal{C}$ one has (recall section 2.1.1) $[s]_g = \left(\begin{array}{c}
\rho \\
\varphi_a \\
\sigma
\end{array}\right) \in \Gamma([\mathcal{T}]_g)$, and the explicit transformation behaviour of $[s]_g$ under a change of metric shows that one has a canonical projection $\Pi^\mathcal{T}_0 : \mathcal{T} \to \mathcal{E}[1]$. Moreover, it can be seen from the definition of $\nabla^\mathcal{T}$ and some simple differential consequences, [BEG94], that the section $s \in \mathcal{T}$ is $\nabla^\mathcal{T}$-parallel if and only if $\sigma = \Pi^\mathcal{T}_0$ satisfies the equation

$$(D_aD_b\sigma + P_{ab}\sigma)_0 = 0.$$
Given a solution $\sigma$ of (26), the corresponding $\nabla^T$-parallel tractor is obtained by the splitting operator $[\text{BEG94}]
abla_T^0 : \mathcal{E}[1] \to \Gamma(\mathcal{T})$, 
\[
\sigma \mapsto \left( -\frac{1}{n}g^{pq}(Dpq\sigma + P_{pq}\sigma) \quad \frac{D\sigma}{\sigma} \right).
\] 

A solution $\sigma \in \Gamma(\mathcal{E}[1])$ of (26) has been termed an almost Einstein scale by R. Gover, [Gov10], since (26) turns out to be equivalent to $\sigma^{-2}g$ being Einstein wherever $\sigma$ is non-zero.

4.3. Explicit decomposition in signature $(2,3)$. We start with conformal spin structures of signature $(2,3)$. Given a generic twistor spinor $\chi$ and its tractor spinor $X = L^3_0(\chi)$, we consider the orthogonal complement $X^\perp$ of $\mathbb{R}X$ in $\Gamma(S)$ with respect to $B_{3,4}$. It is easy to see that a twistor spinor $\eta \in \Gamma(S[\frac{1}{2}])$ splits into this complementary space if and only if $b_{2,3}(\eta, \overline{D}\chi) + b_{2,3}(\chi, \overline{D}\eta) = 0$. The space of twistor spinors satisfying this equation shall be denoted by $\text{Tw}_C^\perp(\chi) \subset \text{Tw}_C \subset \Gamma(S[\frac{1}{2}])$.

Lemma 4.3. For a fixed generic twistor spinor $\chi$, we have a bijective correspondence between almost Einstein scales and twistor spinors $\eta$ such that $b_{2,3}(\eta, \overline{D}\chi) + b_{2,3}(\chi, \overline{D}\eta) = 0$. An almost Einstein scale $\sigma \in \mathcal{E}[1]$ is mapped to
\[
\frac{2}{5}\sigma\overline{D}\chi + (D\sigma) \cdot \chi \in \text{Tw}_C^\perp(\chi).
\] and a twistor spinor $\eta \in \text{Tw}_C^\perp(\chi)$ is mapped to the almost Einstein scale $\sigma = b_{2,3}(\chi, \eta)$.

Proof. Suppose $X \in \Delta^{3,4}$ with $B_{3,4}(X, X) \neq 0$ is stabilized by $G_2$. Consider the map $\mathbb{R}^{3,4} \to \Delta^{3,4}$ given by Clifford multiplication on $X$. Then this is a non-zero, $G_2$-equivariant map. Moreover, $\mathbb{R}^{3,4}$ is an irreducible $G_2$-representation, and thus the map must be an isomorphism onto its image, which is the orthogonal complement $X^\perp$ to $\mathbb{R}X$ in $\Delta^{3,4}$ with respect to the invariant bilinear form $B_{3,4}$. The inverse maps a spinor $Y$ to the unique element $v \in \mathbb{R}^{3,4}$ such that $h_{3,4}(v, w)B_{3,4}(X, X) = B_{3,4}(w \cdot X, Y)$ for all $w \in \mathbb{R}^{3,4}$.

Passing to associated bundles yields an identification of the standard tractor bundle $\mathcal{T}$ with $X^\perp$. In order to obtain the explicit formulas relating almost Einstein scales with the subset $\text{Tw}_C^\perp(\chi)$ of the space of twistor spinors, we use the formulas (27), (4) and (2) for the splitting operators $L^T_0 : \mathcal{E}[1] \to \Gamma(\mathcal{T})$ and $L^S_0 : \Gamma(S[\frac{1}{2}]) \to \Gamma(S)$ and for the Clifford multiplication $\gamma : \mathcal{T} \otimes S \to S$.

We proceed to the decomposition of infinitesimal automorphisms in terms of $\chi$. 
Proposition 4.4. Given a conformal spin structure of signature $(2,3)$ and a generic twistor spinor $\chi \in \Gamma(S^{\downarrow}_{3}(\mathbb{R}^4))$, the space of conformal Killing fields decomposes into the space of almost Einstein scales and the space of infinitesimal automorphisms of the corresponding rank 2-distribution. Explicitly, for some $g \in \mathcal{C}$, the almost Einstein scale part of a conformal Killing field $\xi \in \mathcal{X}(M)$ is given by

$$\sigma = b_{2,3}(\chi, -\frac{4}{5} \xi \cdot \nabla \chi + (D[a\xi_b]) \cdot \chi) \in \mathcal{E}[1].$$

Conversely, an almost Einstein scale $\sigma \in \mathcal{E}[1]$ is mapped to a conformal Killing field

$$\xi_a = b_{2,3}(\gamma_a \chi, \frac{2}{5} \sigma \nabla \chi + (D\sigma) \cdot \chi) \in \mathcal{E}_a[2] = \mathcal{X}(M).$$

Proof. Proposition 4.1 implies that conformal Killing fields decompose into infinitesimal automorphisms of the distributions and almost Einstein scales. Algebraically, the projection $so(3,4) = \Lambda^2\mathbb{R}^{3,4} \rightarrow \mathbb{R}^{3,4}$ is given by action of $\Lambda^2\mathbb{R}^{3,4}$ on the non-null spinor $X$ composed with the isomorphism $X^\perp \cong \mathbb{R}^{3,4}$ from Lemma 4.3. The inverse of the map $\Lambda^2\mathbb{R}^{3,4} \rightarrow \mathbb{R}^{3,4}$ assigns to $w \in \mathbb{R}^{3,4}$ the unique element $\phi \in \Lambda^2\mathbb{R}^{3,4} = \Lambda^2(\mathbb{R}^{3,4})^*$ such that $\phi(u,v)B_{3,4}(X,X) = B_{3,4}(u \cdot v \cdot X, w \cdot X)$ for all $u, v \in \mathbb{R}^{3,4}$.

The explicit decomposition in terms of $\chi$ is then obtained using the algebraic maps and the differential BGG-splitting operators.

4.4. Explicit decomposition in signature $(3,3)$. Suppose we have a conformal spin structure of signature $(3,3)$, a generic twistor spinor $\chi \in \Gamma(S^{\downarrow}_{3}(\mathbb{R}^4))$ and its tractor spinor $X = L^S_0(\chi)$. Again, we use the notation $\text{T}\text{w}_{\mathcal{C}}^\perp(\chi) \subset \text{T}\text{w}_{\mathcal{C}} \subset \Gamma(S^{\downarrow}_{3}(\mathbb{R}^4))$ for the space of twistor spinors satisfying $b_{3,3}(\eta, \nabla \chi) + b_{3,3}(\chi, \nabla \eta) = 0$; these are mapped via the splitting operator [4] into the orthogonal complement $X^\perp$.

Proposition 4.5. Given a conformal spin structure of signature $(3,3)$ and a generic twistor spinor $\chi \in \Gamma(S^{\downarrow}_{3}(\mathbb{R}^4))$, the space of conformal Killing fields decomposes into the space of $\text{T}\text{w}_{\mathcal{C}}^\perp(\chi)$ and the space of infinitesimal automorphisms of the corresponding rank 3-distribution. Explicitly, for a $g \in \mathcal{C}$, a conformal Killing field $\xi \in \mathcal{X}(M)$ is mapped to the twistor spinor

$$\eta = -\frac{2}{3} \xi \cdot \nabla \chi + (D[a\xi_b]) \cdot \chi + \frac{1}{6}(\delta \xi)\chi \in \text{T}\text{w}_{\mathcal{C}}^\perp(\chi)$$

and a twistor spinor $\eta \in \text{T}\text{w}_{\mathcal{C}}^\perp(\chi)$ is mapped to the conformal Killing field

$$\xi_a = b_{3,3}(\chi, \gamma_a \eta) \in \mathcal{E}_a[2] = \mathcal{X}(M).$$

Proof. The map $so(4,4) = \Lambda^2\mathbb{R}^{4,4} \rightarrow \Delta^4_{4+} \mathbb{R}^{4,4}$ given by action on the Spin$(3,4)$-invariant element $X \in \Delta^4_{4+}$ provides an identification of the subrepresentation $\mathbb{R}^{3,4} \subset so(4,4)$ with $X^\perp \subset \Delta^4_{4+}$. Employing Proposition 4.1 we thus have a decomposition of conformal Killing fields into infinitesimal automorphisms of the distribution and twistor spinors $\text{T}\text{w}_{\mathcal{C}}^\perp(\chi)$. Again, the explicit
maps are obtained via simple computations using differential splitting formulas \((24)\) and \((24)\).

In this case almost Einstein scales don’t correspond to infinitesimal automorphisms, but again, they can be identified with a subset of twistor spinors. For this, note that Clifford multiplication on the non-null spinor \(X \in \Delta_4^4\) defines an isomorphism \(\mathbb{R}^{4,4} \cong \Delta_4^4\) and therefore have:

**Proposition 4.6.** There is a bijective correspondence between almost Einstein scales and negative twistor spinors. An almost Einstein scale \(\sigma \in \mathcal{E}[1]\) is mapped to the negative twistor spinor

\[
\eta = \frac{1}{3} \sigma (\partial \chi + (D\sigma) \cdot \chi) \in \Gamma(S_-(\frac{1}{2})).
\]

Conversely, a negative twistor spinor \(\eta \in \Gamma(S_-(\frac{1}{2}))\) is mapped to the almost Einstein scale \(\sigma = b_{3,3}(\chi, \eta) \in \mathcal{E}[1]\).

4.5. **Remark on the normal conformal Killing forms induced by the generic twistor spinors.** Fixing the a generic twistor spinor \(\chi\) gives rise to more maps from twistor spinors to normal conformal Killing \(k\)-forms: In signature \((2,3)\) we have an inclusion from twistor spinors into normal conformal Killing 2-forms. In particular, a generic twistor spinor \(\chi\) gives rise to \(\phi_2 := b_{2,3}(\chi, \gamma[a\gamma b]\chi) \in \mathcal{E}_{[a][b]}[3]\). In [HS09] we use the existence of such a conformal Killing 2-form to characterize the conformal structures associated to 2-distributions.

In signature \((3,3)\) we have an inclusion from negative twistor spinors into normal conformal Killing 2-forms and an inclusion of positive twistor spinors into normal conformal Killing 3-forms. In particular, in this case we always have a non-trivial normal conformal Killing 3-form, which is given by \(\phi_3 := b_{3,3}(\chi, \gamma[a\gamma b\gamma c]\chi) \in \mathcal{E}_{[a]bc}[4]\).

Since \(\phi_2\) and \(\phi_3\) correspond to to pure spinors they are decomposable and therefore insertion into the form has a 3-dimensional kernel in both cases. In signature \((3,3)\) this is already the canonical 3-distribution associated to \(\phi_3\), and in signature \((2,3)\) the canonical 2-distribution is formed by the intersection of the kernel of \(g\) with the kernel of \(\phi_2\).

**References**

[Arm07]  Stuart Armstrong. Free 3-distributions: holonomy, Fefferman constructions and dual distributions. 2007. [arXiv:0708.3027]

[BEG94]  T. N. Bailey, M. G. Eastwood, and A. Rod Gover. Thomas’s structure bundle for conformal, projective and related structures. *Rocky Mountain J. Math.*, 24(4):1191–1217, 1994.

[BFGK90]  Helga Baum, Thomas Friedrich, Ralf Grunewald, and Ines Kath. *Twistor and Killing spinors on Riemannian manifolds*, volume 108 of *Seminarcherichte [Seminar Reports]*. Humboldt Universit¨ at Sektion Mathematik, Berlin, 1990.

[Bra05]  Thomas Branson. Conformal structure and spin geometry. In *Dirac operators: yesterday and today*, pages 163–191. Int. Press, Somerville, MA, 2005.
[Bry87] Robert L. Bryant. Metrics with exceptional holonomy. *Ann. of Math. (2)*, 126(3):525–576, 1987.

[Bry06] Robert L. Bryant. Conformal geometry and 3-plane fields on 6-manifolds. *Developments of Cartan Geometry and Related Mathematical Problems, RIMS Symposium Proceedings (Kyoto University)*, 1502:1–15, 2006.

[Cap05] Andreas Čap. Correspondence spaces and twistor spaces for parabolic geometries. *J. Reine Angew. Math.*, 582:143–172, 2005.

[Cap06] Andreas Čap. Two constructions with parabolic geometries. *Rend. Circ. Mat. Palermo (2) Suppl.*, (79):11–37, 2006.

[ˇCap05] Andreas ˇCap. Correspondence spaces and twistor spaces for parabolic geometries. *J. Reine Angew. Math.*, 582:143–172, 2005.

[ˇCap06] Andreas ˇCap. Two constructions with parabolic geometries. *Rend. Circ. Mat. Palermo (2) Suppl.*, (79):11–37, 2006.

[ˇCap08] Andreas ˇCap. Infinitesimal automorphisms and deformations of parabolic geometries. *J. Eur. Math. Soc. (JEMS)*, 10(2):415–437, 2008.

[Car67] Élie Cartan. *The theory of spinors*. The M.I.T. Press, Cambridge, Mass., 1967.

[CD01] David M.J. Calderbank and Tammo Diemer. Differential invariants and curved Bernstein-Gelfand-Gelfand sequences. *J. Reine Angew. Math.*, 537:67–103, 2001.

[ˇCG02] Andreas ˇCap and A. Rod Gover. Tractor calculi for parabolic geometries. *Trans. Amer. Math. Soc.*, 354(4):1511–1548 (electronic), 2002.

[ˇCG06] Andreas ˇCap and A. Rod Gover. A holonomy characterisation of Fefferman spaces. 2006. ESI Preprint 1875.

[ˇCG08] Andreas ˇCap and A. Rod Gover. CR-tractors and the Fefferman space. *Indiana Univ. Math. J.*, 57(5):2519–2570, 2008.

[ˇČS09] Andreas ˇCap and Jan Slovák. *Parabolic Geometries I: Background and General Theory v.1*. Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI, 2009.

[ˇCSS01] Andreas ˇCap, Jan Slovák, and Vladimír Souček. Bernstein-Gelfand-Gelfand sequences. *Ann. of Math.*, 154(1):97–113, 2001.

[ˇČŽ09] Andreas ˇCap and Vojtěch Žádník. On the geometry of chains. *J. Differential Geom.*, 82(1):1–33, 2009.

[DS08] Boris Doubrov and Jan Slovák. Inclusions between parabolic geometries. 2008. [arXiv:0807.3360](http://arxiv.org/abs/0807.3360).

[Eas96] Michael Eastwood. Notes on conformal differential geometry. In *The Proceedings of the 15th Winter School “Geometry and Physics” (Srní, 1995)*, number 43, pages 57–76, 1996.

[Fri90] Thomas Friedrich. On the conformal relation between twistors and Killing spinors. In *Proceedings of the Winter School on Geometry and Physics (Srní, 1989)*, number 22, pages 59–75, 1990.

[Gov06] A. Rod Gover. Laplacian operators and Q-curvature on conformally Einstein manifolds. *Math. Ann.*, 336(2):311–334, 2006.

[Gov10] A. Rod Gover. Almost Einstein and Poincaré-Einstein manifolds in Riemannian signature. *Journal of Geometry and Physics*, 60(2):182 – 204, 2010.

[Ham08] Matthias Hammerl. Invariant prolongation of BGG-operators in conformal geometry. *Arch. Math. (Brno)*, 44(5):367–384, 2008.

[Ham09] Matthias Hammerl. Natural Prolongations of BGG-operators. Thesis, University of Vienna, 2009.

[HS09] Matthias Hammerl and Katja Sagerschnig. Conformal structures associated to generic rank 2 distributions on 5-manifolds—characterization and Killing-field decomposition. *SIGMA Symmetry Integrability Geom. Methods Appl.*, 5:Paper 081, 29, 2009. Available at [http://www.emis.de/journals/SIGMA/Cartan.html](http://www.emis.de/journals/SIGMA/Cartan.html).

[Kat99] Ines Kath. Killing spinors on pseudo-Riemannian manifolds. Habilitation thesis, Humboldt-Universität zu Berlin, 1999. [http://www-irm.mathematik.hu-berlin.de/~kath/tex/Habil.ps](http://www-irm.mathematik.hu-berlin.de/~kath/tex/Habil.ps).
[Kos61] Bertram Kostant. Lie algebra cohomology and the generalized Borel-Weil theorem. *Ann. of Math. (2)*, 74:329–387, 1961.

[KS09] Eric Korman and George Sparling. Bilinear Forms and Fierz Identities for Real Spin Representations. 2009. [arXiv:0901.0580](http://arxiv.org/abs/0901.0580).

[Lei08] Felipe Leitner. Applications of Cartan and tractor calculus to conformal and CR-geometry. Habilitation, Universität Stuttgart, 2008. [http://elib.uni-stuttgart.de/opus/volltexte/2009/3922/](http://elib.uni-stuttgart.de/opus/volltexte/2009/3922/).

[LN09] Thomas Leistner and Paweł Nurowski. Conformal structures with G2(2)-ambient metrics. 2009. [arXiv:0904.0186](http://arxiv.org/abs/0904.0186).

[Lou01] Pertti Lounesto. *Clifford algebras and spinors*, volume 286 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, second edition, 2001.

[Nur05] Paweł Nurowski. Differential equations and conformal structures. *J. Geom. Phys.*, 55(1):19–49, 2005.

[PR87] Roger Penrose and Wolfgang Rindler. *Spinors and space-time. Vol. 1*. Cambridge Monographs on Mathematical Physics. Cambridge University Press, Cambridge, 1987. Two-spinor calculus and relativistic fields.

[Sag08] Katja Sagerschnig. Weyl structures for generic rank two distributions in dimension five. Thesis, University of Vienna, 2008.

[Sha97] R.W. Sharpe. *Differential Geometry - Cartan’s Generalisation of Klein’s Erlangen Program*. Springer-Verlag, 1997.

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