HOLOMORPHIC DISCS WITH DENSE IMAGES

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Abstract. Let $\Delta$ be the open unit disc in $\mathbb{C}$, $X$ a connected complex manifold and $D$ the set of all holomorphic maps $f: \Delta \to X$ with $f(\Delta) = X$. We prove that $D$ is dense in $\text{Hol}(\Delta, X)$.

1. Introduction

Let $\Delta_r = \{z \in \mathbb{C} : |z| < r\}$ and $\Delta = \Delta_1$. In [7] the second author proved that for any irreducible complex space $X$ there exists a holomorphic map $\Delta \to X$ with dense image, and he raised the question whether the set of all holomorphic maps $\Delta \to X$ with dense image forms a dense subset of the set $\text{Hol}(\Delta, X)$ of all holomorphic maps $\Delta \to X$ with respect to the topology of locally uniform convergence.

In this paper we show that the answer to this question is positive if $X$ is smooth, but negative for some singular space.

Theorem 1. For any connected complex manifold $X$ the set of holomorphic maps $\Delta \to X$ with dense images forms a dense subset in $\text{Hol}(\Delta, X)$. The conclusion fails for some singular complex surface $X$.

The situation is quite different for proper discs, i.e., proper holomorphic maps $\Delta \to X$. The paper [3] contains an example of a non-pseudoconvex bounded domain $X \subset \mathbb{C}^2$ such that a certain nonempty open subset $U \subset X$ is not intersected by the image of any proper holomorphic disc $\Delta \to X$. On the other hand, proper holomorphic discs exist in great abundance in Stein manifolds [5], [1], [2].

2. Preparations

Lemma 1. Let $W_n$ be a decreasing sequence (i.e., $W_{n+1} \subset W_n$) of open sets with $\Delta \subset W_n \subset \Delta_2$ for every $n$. Let $K = \cap_n \overline{W}_n$ and assume that the interior of $K$ coincides with $\Delta$. Furthermore assume that there are biholomorphic maps $\phi_n : \Delta \to W_n$ with $\phi_n(0) = 0$ for $n = 1, 2, \ldots$. 

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Then there exists an automorphism \( \alpha \in \text{Aut}(\Delta) \) and a subsequence \((\phi_{n_k})\) of the sequence \((\phi_n)\) such that \(\phi_{n_k} \circ \alpha^{-1}\) converges locally uniformly to the identity map \(\text{id}_\Delta\) on \(\Delta\).

Proof. Montel’s theorem shows that, after passing to a suitable subsequence, we have \(\lim_{n \to \infty} \phi_n = \alpha : \Delta \to K\) and \(\lim_{n \to \infty} (\phi_n^{-1}|_\Delta) = \beta : \Delta \to \overline{\Delta}\). Since the limit maps are holomorphic and satisfy \(\alpha(0) = 0\) and \(\beta(0) = 0\), we conclude that \(\alpha(\Delta) \subset \text{Int}K = \Delta\) and \(\beta(\Delta) \subset \Delta\). Moreover \(\alpha \circ \beta = \text{id}_\Delta = \beta \circ \alpha\), and hence both \(\alpha\) and \(\beta\) are automorphisms of \(\Delta\) (indeed, rotations \(z \to ze^{it}\)). □

We also need the following special case of a result of the first author (theorem 3.2 in [4]):

Proposition 1. Let \(X\) be a complex manifold, \(0 < r < 1\), \(E\) the real line segment \([1, 2] \subset \mathbb{C}\), \(K = \overline{\Delta} \cup E\), \(U\) an open neighbourhood of \(\Delta\) in \(\mathbb{C}\), \(S\) a finite subset of \(K\) and \(f : U \cup E \to X\) a continuous map which is holomorphic on \(U\).

Then there is a sequence of pair of open neighbourhoods \(W_n \subset \mathbb{C}\) of \(K\) and holomorphic maps \(g_n : W_n \to X\) such that:

1. \(g_n|_K\) converges uniformly to \(f|_K\) as \(n \to \infty\), and
2. \(g_n(a) = f(a)\) for all \(a \in S\) and \(n \in \mathbb{N}\).

3. Towards the main result

In this section we prove the following proposition which is the main technical result in the paper. The first statement in theorem \(\Pi\) (§1) is an immediate corollary.

Proposition 2. Let \(X\) be a connected complex manifold endowed with a complete Riemannian metric and induced distance \(d\), \(S\) a countable subset of \(X\), \(f : \Delta \to X\) a holomorphic map, \(\epsilon > 0\) and \(0 < r < 1\).

Then there exists a holomorphic map \(F : \Delta \to X\) such that

1. \(S \subset F(\Delta)\), and
2. \(d(f(z), F(z)) \leq \epsilon\) for all \(z \in \Delta_r\).

Proof. Let \(s_1, s_2, s_3, \ldots\) be an enumeration of the elements of \(S\). We shall inductively construct a sequence of holomorphic maps \(f_n : \Delta \to X\), numbers \(r_n \in (0, 1)\) and points \(a_{1,n}, \ldots, a_{n,n} \in \Delta\) satisfying the following properties for \(n = 0, 1, 2, \ldots:\)

1. \(f_0 = f\) and \(r_0 = r\),
2. \((r_n + 1)/2 < r_{n+1} < 1\),
3. \(f_n(a_{j,n}) = s_j\) for \(n \geq 1\) and \(j = 1, 2, \ldots, n\),
4. \(d(f_n(z), f_{n+1}(z)) < 2^{-(n+1)}\epsilon\) for all \(z \in \Delta_{r_n}\), and
(5) \( d_\Delta(a_{j,n}, a_{j,n+1}) < 2^{-n} \) for \( j = 1, 2, \ldots, n \) where \( d_\Delta \) denotes the Poincaré distance on the unit disc.

Assume inductively that the data for level \( n \) (i.e., \( f_n, r_n, a_{j,n} \)) have been chosen. (For \( n = 0 \) we do not have any points \( a_{j,0} \).) With \( n \) fixed we choose an increasing sequence of real numbers \( \lambda_k \) with \( \lambda_k > r_n \) and \( \lim_{k \to \infty} \lambda_k = 1 \). For every \( k \in \mathbb{N} \) the map \( \tilde{g}_k(z) \) defined by \( f_n(\lambda_k z) \in X \) is defined and holomorphic on the disc \( \Delta_{1/\lambda_k} \supset \Delta \). After a slight shrinking of its domain we can extend it continuously to the segment \( E = [1, 2] \subset \mathbb{C} \) such that the right end point 2 of \( E \) is mapped to the next point \( s_{n+1} \in S \) (this is possible since \( X \) is connected).

Applying proposition \( \mathbb{I} \) to the extended map \( \tilde{g}_k \) we obtain for every \( k \in \mathbb{N} \) an open neighbourhood \( V_k \subset \mathbb{C} \) of \( K = \Delta \cup E \) and a holomorphic map \( g_k : V_k \to X \) such that

(i) \( |g_k(z) - f_n(\lambda_k z)| < 2^{-k} \) for all \( z \in \Delta \),
(ii) \( g_k(2) = s_{n+1}, \) and
(iii) \( g_k(a_{j,n}/\lambda_k) = f_n(a_{j,n}) = s_j \) for \( j = 1, \ldots, n \).

Next we choose a decreasing sequence of simply connected open sets \( W_k \subset \mathbb{C} \) (\( k \in \mathbb{N} \)) with \( K \subset W_k \subset V_k \) and \( K = \cap_k \overline{W}_k \). Notice that \( \text{Int} K = \Delta \). By lemma \( \mathbb{I} \) there is a sequence of biholomorphic maps \( \phi_k : \Delta \to W_k \) with \( \lim_{k \to \infty} \phi_k = i_d \).

Consider the holomorphic maps \( h_k = g_k \circ \phi_k : \Delta \to X \). By our construction we know that \( \lim_{k \to \infty} h_k = f_n \) locally uniformly on \( \Delta \).

To fulfill the inductive step it thus suffices to choose \( f_{n+1} = h_k \) for a sufficiently large \( k \), \( a_{j,n+1} = a_{j,n}/\lambda_k \) (\( j = 1, \ldots, n \)), \( a_{n+1,n+1} = \phi_k^{-1}(2) \).

Finally we choose a number \( r_{n+1} \) satisfying

\[ \max\{|a_{n+1,n+1}|, \frac{r_n + 1}{2}\} < r_{n+1} < 1. \]

This completes the inductive step.

By properties (2) and (4) the sequence \( f_n \) converges locally uniformly in \( \Delta \) to a holomorphic map \( F : \Delta \to X \). Aided by property (1) we also control \( d(f(z), F(z)) \) for \( z \in \Delta_n \). Since the Poincaré metric is complete, property (5) insures that for every fixed \( j \in \mathbb{N} \) the sequence \( a_{j,n} \in \Delta \) (\( n = j, j+1, \ldots \)) has an accumulation point \( b_j \) inside of \( \Delta \), and (3) implies \( F(b_j) = s_j \) for \( j = 1, 2, \ldots \). Hence \( S \subset F(\Delta) \).

4. **Singular spaces**

We use an example of Kaliman and Zaidenberg [9] to show that for a complex spaces \( X \) with singularities the set of maps \( \Delta \to X \) with dense image need not be dense in \( \text{Hol}(\Delta, X) \). We denote by \( \text{Sing}(X) \) the singular locus of \( X \).
Proposition 3. There is a singular compact complex surface $S$, a non-constant holomorphic map $f: \Delta \to S$ and an open neighbourhood $\Omega$ of $f$ in $\text{Hol}(\Delta, S)$ such that $g(\Delta) \subset \text{Sing}(S)$ for every $g \in \Omega$.

Proof. In [6] Kaliman and Zaidenberg constructed an example of a singular surface $S$ with normalization $\pi: Z \to S$ such that $S$ contains a rational curve $C \cong \mathbb{P}^1$ while $Z$ is smooth and hyperbolic. Denote by $d_Z$ the Kobayashi distance function on $Z$. We choose two distinct points $p, q \in C$ and open relatively compact neighbourhoods $V$ of $p$ and $W$ of $q$ in $S$ such that $V \cap W = \emptyset$. The preimages $\pi^{-1}(V)$ and $\pi^{-1}(W)$ in $Z$ are also compact, and since $Z$ is hyperbolic we have

$$r = \min \{ d_Z(x, y) : x \in \pi^{-1}(V), y \in \pi^{-1}(W) \} > 0.$$

Fix a point $a \in \Delta$ with $0 < d_\Delta(0, a) < r$ and let $\Omega$ consist of all holomorphic maps $g: \Delta \to S$ satisfying $g(0) \in V$ and $g(a) \in W$. Since both $p$ and $q$ are lying on the rational curve $C$, there is a holomorphic map $g: \Delta \to C$ with $g(0) = p \in V$ and $g(a) = q \in W$; hence the set $\Omega$ is not empty. Clearly $\Omega$ is open in $\text{Hol}(\Delta, S)$.

To conclude the proof it remains to show that $g(\Delta) \subset \text{Sing}(S)$ for all $g \in \Omega$. Indeed, a holomorphic map $g: \Delta \to S$ with $g(\Delta) \subset \text{Sing}(S)$ admits a holomorphic lifting $\tilde{g}: \Delta \to Z$ with $\pi \circ \tilde{g} = g$. If $g \in \Omega$ then by construction

$$d_Z(\tilde{g}(0), \tilde{g}(a)) \geq r > d_\Delta(0, a)$$

which violates the distance decreasing property for the Kobayashi pseudometric. This contradiction establishes the claim. \hfill $\square$

In particular, we see that in this example the set of all holomorphic maps $f: \Delta \to S$ with dense image does not constitute a dense subset of $\text{Hol}(\Omega, S)$.

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