Abstract. Recently, various extensions and variants of Bessel functions of several kinds have been presented. Among them, the \((p, q)\)-confluent hypergeometric function \(\Phi_{p,q}\) has been introduced and investigated. Here, we aim to introduce an extended \((p, q)\)-Whittaker function by using the function \(\Phi_{p,q}\) and establish its various properties and associated formulas such as integral representations, some transformation formulas and differential formulas. Relevant connections of the results presented here with those involving relatively simple Whittaker functions are also pointed out.

1. Introduction and preliminaries

We begin by recalling the classical beta function \(B(\alpha, \beta)\)

\[
B(\alpha, \beta) = \begin{cases} \int_0^1 t^{\alpha-1}(1-t)^{\beta-1} \, dt & (\min\{\Re(\alpha), \Re(\beta)\} > 0) \\ \frac{\Gamma(\alpha) \Gamma(\beta)}{\Gamma(\alpha+\beta)} & (\alpha, \beta \in \mathbb{C} \setminus \mathbb{Z}_0^-) \end{cases}
\]

(1.1)

where \(\Gamma\) denotes the familiar gamma function (see, e.g., [6, Section 1.1]). Here and in the following, let \(\mathbb{C}, \mathbb{R}^+, \mathbb{N}, \) and \(\mathbb{Z}_0^-\) be the sets of complex numbers, positive real numbers, positive integers, and non-positive integers, respectively, and let \(\mathbb{N}_0 := \mathbb{N} \cup \{0\}\) and \(\mathbb{R}_0^+ := \mathbb{R}^+ \cup \{0\}\).

The Gauss hypergeometric function \(_2F_1\) and the confluent hypergeometric function \(_1F_1\) are defined by (see, e.g., [3])

\[
_2F_1(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!} \quad (|z| < 1; \ c \in \mathbb{C} \setminus \mathbb{Z}_0^-)
\]

(1.2)

and

\[
_1F_1(a; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n} \frac{z^n}{n!} \quad (|z| < \infty; \ c \in \mathbb{C} \setminus \mathbb{Z}_0^-)
\]

(1.3)

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where \((\lambda)_n\) is the Pochhammer symbol defined (for \(\lambda \in \mathbb{C}\)) by (see [6, p. 2 and pp. 4-6])

\[
(\lambda)_n := \begin{cases} 
1 & (n = 0) \\
\lambda(\lambda + 1) \cdots (\lambda + n - 1) & (n \in \mathbb{N})
\end{cases}
\] (1.4)

Integral representations for the Gauss hypergeometric function \(\binom{2}{1}\) and confluent hypergeometric function \(\binom{1}{1}\) are recalled (see, e.g., [6, p. 65 and p. 70])

\[
\binom{2}{1} F_1 (a, b; c; z) = \frac{\Gamma(c)}{\Gamma(a) \Gamma(c - a)} \int_0^1 t^{a-1} (1 - t)^{c-a-1} (1 - zt)^{-b} dt 
\] (1.5)

\((\Re(c) > \Re(a) > 0; \ |\arg(1 - z)| < \pi)\)

and

\[
\binom{1}{1} F_1 (a; c; z) = \frac{\Gamma(c)}{\Gamma(a) \Gamma(c - a)} \int_0^1 t^{a-1} (1 - t)^{c-a-1} e^{zt} dt 
\] (1.6)

Chaudhry et al. introduced the following extended beta function (see [1, Eq. (1.7)])

\[
B(\alpha, \beta; p) = B_p(\alpha, \beta) = \int_0^1 t^{\alpha-1} (1 - t)^{\beta-1} e^{-\frac{p}{t(1-t)}} dt \quad (\Re(p) > 0). \] (1.7)

Obviously, \(B(\alpha, \beta; 0) = B(\alpha, \beta)\). Chaudhry et al. [2] introduced and investigated the following extended hypergeometric function \(F_p\) and confluent hypergeometric function \(\Phi_p\)

\[
F_p(a, b; c; z) = \sum_{n=0}^{\infty} \frac{B_p(b + n, c - b)}{B(b, c - b)} (a)_n \frac{z^n}{n!} \] (1.8)

\((p \in \mathbb{R}_0^+; \ |z| < 1; \ \Re(c) > \Re(b) > 0)\)

and

\[
\Phi_p(b; c; z) = \sum_{n=0}^{\infty} \frac{B_p(b + n, c - b)}{B(b, c - b)} \frac{z^n}{n!} \] (1.9)

including their integral representations

\[
F_p(a, b; c; z) = \frac{1}{B(b, c - b)} \int_0^1 t^{b-1} (1 - t)^{c-b-1} (1 - zt)^{-a} \exp \left[ - \frac{p}{t(1-t)} \right] dt, \] (1.10)

\((p \in \mathbb{R}_0^+; p = 0 \ \text{and} \ |\arg(1 - z)| < \pi; \ \Re(c) > \Re(b) > 0)\)

and

\[
\Phi_p(b; c; z) = \frac{1}{B(b, c - b)} \int_0^1 t^{b-1} (1 - t)^{c-b-1} \exp \left( zt - \frac{p}{t(1-t)} \right) dt
\] (1.11)

\((p \in \mathbb{R}_0^+; p = 0 \ \text{and} \ \Re(c) > \Re(b) > 0)\).

Clearly, the particular cases of (1.8), (1.9), (1.10), and (1.11) when \(p = 0\) reduce, respectively, to (1.2), (1.3), (1.5), and (1.6).
Choi et al. [3] defined the following extension of the beta function

\[ B(x, y; p, q) = B_{p,q}(x, y) = \int_0^1 t^{x-1} (1-t)^{y-1} \exp \left[ -\frac{p}{t} - \frac{q}{1-t} \right] dt \]  

(1.12)

\((\min\{\Re(p), \Re(q)\} > 0)\).

Using (1.12), they [3] defined the following \((p, q)\)-hypergeometric function and \((p, q)\)-confluent hypergeometric function, respectively, by

\[ F_{p,q}(a, b; c; z) = \sum_{n=0}^{\infty} \frac{B_{p,q}(b+n, c-b)}{B(b, c-b)} (a)_n \frac{z^n}{n!} \]  

(1.13)

\((p, q \in \mathbb{R}_0^+; \Re(c) > \Re(b) > 0)\)

and

\[ \Phi_{p,q}(b; c; z) = \sum_{n=0}^{\infty} \frac{B_{p,q}(b+n, c-b)}{B(b, c-b)} \frac{z^n}{n!} \]  

(1.14)

\((p, q \in \mathbb{R}_0^+; \Re(c) > \Re(b) > 0)\)

and presented their integral representations

\[ F_{p,q}(a, b; c; z) = \frac{1}{B(b, c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a} \]  

\[ \times \exp \left( -\frac{p}{t} - \frac{q}{1-t} \right) dt \]  

(1.15)

\((p, q \in \mathbb{R}_0^+; \Re(c) > \Re(b) > 0; |\arg(1-z)| < \pi)\)

and

\[ \Phi_{p,q}(b; c; z) = \frac{1}{B(b, c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1} \]  

\[ \times \exp \left( zt - \frac{p}{t} - \frac{q}{1-t} \right) dt \]  

(1.16)

\((p, q \in \mathbb{R}_0^+; \Re(c) > \Re(b) > 0)\).

Obviously, when \(p = q = 0\), (1.13)-(1.16) reduce, respectively, to (1.8)-(1.11). They [3, Eq. (11.4)] also obtained the following transformation formula

\[ \Phi_{p,q}(b; c; z) = e^z \Phi_{q,p}(c-b; c; -z). \]  

(1.17)

Whittaker and Watson [8, Chapter XVI] used the confluent hypergeometric function \(_1F_1\) to define the Whittaker function

\[ M_{\lambda,\rho}(z) = z^{\rho+\frac{1}{2}} \exp \left( -\frac{z}{2} \right) _1F_1 \left( \rho - \lambda + \frac{1}{2}; 2\rho + 1; z \right) \]  

(1.18)

\(\left( \Re(\rho) > -\frac{1}{2}; \Re(\rho \pm \lambda) > -\frac{1}{2}; z \in \mathbb{C} \setminus (-\infty, 0]\right)\).

This Whittaker function (1.18) is a special solution of the Whittaker’s differential equation and a modified form of the confluent hypergeometric equation to make formulas involving the solutions more symmetric (see [7]).
Nagar et al. [4] used the extended confluent hypergeometric series $\Phi_p$ in (1.9) to extend the Whittaker function $M_{\lambda,\rho}(z)$ as follows:

$$M_{p,\lambda,\rho}(z) = z^{\rho+\frac{1}{2}} \exp \left( -\frac{z}{2} \right) \Phi_p \left( \rho - \lambda + \frac{1}{2}; 2\rho + 1; z \right)$$

(1.19)

$$\left( \rho \in \mathbb{R}_0^+; \Re(\rho) > -\frac{1}{2}; \Re(\rho \pm \lambda > -\frac{1}{2}; z \in \mathbb{C} \setminus (-\infty, 0]) .$$

Here, by using the $(p, q)$-confluent hypergeometric function $\Phi_{p,q}$ in (1.14), we define the following $(p, q)$-Whittaker function

$$M_{p,q,\lambda,\rho}(z) = z^{\rho+\frac{1}{2}} \exp \left( -\frac{z}{2} \right) \Phi_{p,q} \left( \rho - \lambda + \frac{1}{2}; 2\rho + 1; z \right)$$

(1.20)

$$\left( p, q \in \mathbb{R}_0^+; \Re(\rho) > -\frac{1}{2}; \Re(\rho \pm \lambda > -\frac{1}{2}; z \in \mathbb{C} \setminus (-\infty, 0]) .$$

Clearly, $M_{p,q,\lambda,\rho}(z) = M_{p,\lambda,\rho}(z)$ in (1.19). Then we investigate a number of formulas involving the $(p, q)$-Whittaker function in (1.20), systematically, such as various integral representations, a transformation formula, a Mellin transform, and a derivative formula.

### 2. Formulas involving the $(p, q)$-Whittaker function

Here, we present a number of formulas involving the $(p, q)$-Whittaker function in (1.20), in a rather systematic way.

**Theorem 2.1.** Let $\Re(\rho) > \Re(\rho \pm \lambda > -\frac{1}{2}$ and $z \in \mathbb{C} \setminus (-\infty, 0]$. Also, let $p, q \in \mathbb{R}_0^+$. Then the following integral representations hold true:

$$M_{p,q,\lambda,\rho}(z) = \frac{z^{\rho+\frac{1}{2}} \exp \left( -\frac{z}{2} \right)}{B(\rho - \lambda + \frac{1}{2}, \rho + \lambda + \frac{1}{2})}$$

$$\times \int_0^1 t^{\rho - \lambda - \frac{1}{2}} (1 - t)^{\rho + \lambda - \frac{1}{2}} \exp \left( zt - \frac{p}{t} - \frac{q}{1 - t} \right) dt;$$

(2.1)

$$M_{p,q,\lambda,\rho}(z) = \frac{z^{\rho+\frac{1}{2}} \exp \left( -\frac{z}{2} \right)}{B(\rho - \lambda + \frac{1}{2}, \rho + \lambda + \frac{1}{2})}$$

$$\times \int_0^1 u^{\rho + \lambda - \frac{1}{2}} (1 - u)^{\rho - \lambda - \frac{1}{2}} \exp \left( -zu - \frac{p}{1 - u} - \frac{q}{u} \right) du;$$

(2.2)

$$M_{p,q,\lambda,\rho}(z) = \frac{(b-a)^{-2p} z^{\rho+\frac{1}{2}} \exp \left( -\frac{z}{2} \right)}{B(\rho - \lambda + \frac{1}{2}, \rho + \lambda + \frac{1}{2})}$$

$$\times \int_a^b (u-a)^{\rho - \lambda - \frac{1}{2}} (b-u)^{\rho + \lambda - \frac{1}{2}}$$

$$\times \exp \left[ \frac{[z(u-a) - p(b-a)]}{b-a} - \frac{q(b-a)}{b-u} \right] du$$

\( (a, b \in \mathbb{R} \text{ with } b > a);$$

(2.3)

$$M_{p,q,\lambda,\rho}(z) = \frac{z^{\rho+\frac{1}{2}} \exp \left( -\frac{z}{2} \right)}{B(\rho - \lambda + \frac{1}{2}, \rho + \lambda + \frac{1}{2})}$$

$$\times \int_0^\infty u^{\rho - \lambda - \frac{1}{2}} (1 + u)^{-(2\rho + 1)}$$

$$\times \exp \left[ \frac{zu}{1 + u} - \frac{p(1 + u)}{u} - q(1 + u) \right] du;$$

(2.4)
\[ M_{p,q;\lambda,\rho}(z) = \frac{2^{-2p} z^{p+\frac{1}{2}}}{B(\rho - \lambda + \frac{1}{2}, \rho + \lambda + \frac{1}{2})} \int_{-1}^{1} (1 + u)^{\rho - \lambda + \frac{1}{2}} (1 - u)^{\rho + \lambda - \frac{1}{2}} \]
\[ \times \exp \left[ \frac{z(u + 1)}{2} - \frac{2p}{1 + u} - \frac{2q}{1 - u} \right] \, du. \quad (2.5) \]

**Proof.** Applying the integral representation of the \((p, q)\)-confluent hypergeometric function in (1.16) to the \((p, q)\)-Whittaker function in (1.20), we obtain the integral representation (2.1). Then, by setting \(t = 1 - u, t = \frac{u - a - b}{1 + a}, \) and \(t = \frac{u + 1}{2} \) in (2.1), we get (2.2), (2.3), and (2.4), respectively. Finally, setting \(a = -1 \) and \(b = 1 \) in (2.3) yields (2.5). \[ \square \]

**Remark 2.1.** In view of (1.16), we find from (2.2) that the \((p, q)\)-Whittaker function can also be expressed in the following form
\[ M_{p,q;\lambda,\rho}(z) = z^{\rho + \frac{1}{2}} \exp \left( \frac{z}{2} \right) \Phi_{q,p} \left( \rho + \lambda + \frac{1}{2}; \rho + \lambda + 1; -z \right) \quad (2.6) \]

**Theorem 2.2.** The following transformation formula for the \((p, q)\)-Whittaker function holds true:
\[ M_{p,q;\lambda,\rho}(z) = (-1)^{p+\frac{1}{2}} M_{q,p;-\lambda,\rho}(-z) \quad (2.7) \]

**Proof.** Applying (1.17) to (1.20), we get
\[ M_{p,q;\lambda,\rho}(z) = z^{p+\frac{1}{2}} \exp \left( \frac{z}{2} \right) \Phi_{q,p} \left( \rho + \lambda + \frac{1}{2}; 2\rho + 1; -z \right). \quad (2.8) \]
Using (1.20) in (2.8), we obtain the desired result. \[ \square \]

**Theorem 2.3.** The following Mellin transformation for the \((p, q)\)-Whittaker function holds true:
\[ \mathfrak{M}\{ M_{p,q;\lambda,\rho}(z); p \to r, q \to s \} \]
\[ = z^{p+\frac{1}{2}} \exp\left(-\frac{z}{2}\right) \Gamma(r) \Gamma(s) B(\rho + r - \lambda + \frac{1}{2}, \rho + s + \lambda + \frac{1}{2}) \]
\[ \times \frac{\Gamma(r)}{B(\rho - \lambda + \frac{1}{2}, \rho + \lambda + \frac{1}{2})} \]
\[ \times \frac{1\ F_1 \left( \rho + r - \lambda + \frac{1}{2}; 2\rho + r + s + 1; z \right)}{\min\{\Re(s), \Re(r)\} > 0; \Re(\rho \pm \lambda + r) > -\frac{1}{2}} \quad (2.9) \]

**Proof.** Let \( \mathcal{L}_1 \) be the left side of (2.9). By definition of Mellin transformation, we have
\[ \mathcal{L}_1 = \int_{0}^{\infty} \int_{0}^{\infty} p^{r-1} q^{s-1} M_{p,q;\lambda,\rho}(z) \, dp \, dq. \quad (2.10) \]
Replacing $M_{p,q;\lambda,\rho}(z)$ in \[(2.10)\] by its integral representation \[(2.1)\] and interchanging the order of the integrals, which is verified under the given conditions here, we obtain
\[
\mathcal{L}_1 = \int_0^\infty \int_0^\infty p^{r-1} q^{s-1} M_{p,q;\lambda,\rho}(z) \, dp \, dq
\]
\[
= \frac{z^{\rho+\frac{1}{2}} \exp(-\frac{z}{2})}{B(\rho - \lambda + \frac{1}{2}, \rho + \lambda + \frac{1}{2})} \int_0^1 t^{\rho-\lambda-\frac{1}{2}} (1 - t)^{\rho+\lambda-\frac{1}{2}} e^{zt} \, dt
\]
\[
\times \int_0^\infty p^{r-1} \exp \left( -\frac{p}{t} \right) dp \, \int_0^\infty q^{s-1} \exp \left( -\frac{q}{1-t} \right) dq \, dt.
\] \[(2.11)\]

Using the following easily derivable formula
\[
\int_0^\infty u^{x-1} \exp(-\alpha u) \, du = \frac{\Gamma(x)}{\alpha^x} \quad (\Re(x) > 0; \alpha \in \mathbb{R}^+),
\] \[(2.12)\]
we get
\[
\int_0^\infty p^{r-1} \exp \left( -\frac{p}{t} \right) dp \, \int_0^\infty q^{s-1} \exp \left( -\frac{q}{1-t} \right) dq = t^r (1-t)^s \Gamma(r) \Gamma(s).
\] \[(2.13)\]

Setting \[(2.13)\] in \[(2.11)\], with the aid of \[(1.6)\], we obtain the desired result.

\[\square\]

**Theorem 2.4.** The following integral formula involving the $(p,q)$-Whittaker function holds true:
\[
\int_0^\infty z^{\delta-1} e^{-\alpha z} M_{p,q;\lambda,\rho}(\mu z) \, dz = \frac{\mu^{\rho+\frac{1}{2}} \Gamma(\delta + \rho + \frac{1}{2})}{(\alpha + \frac{\mu}{2})^{\delta+\rho+\frac{1}{2}}} \times F_{p,q} \left( \delta + \rho + \frac{1}{2}, \rho - \lambda + \frac{1}{2}; 2\rho + 1; 2\mu + \frac{2\mu}{2\alpha + \mu} \right)
\]
\[
(p, q \in \mathbb{R}^+_0; \alpha + \frac{\mu}{2} > 0, \mu < 0, 2\alpha + \mu > 2|\mu|; \Re(\delta + \rho) > -\frac{1}{2} \Re(\rho \pm \lambda) > -\frac{1}{2}). \tag{2.14}
\]

**Proof.** Let $\mathcal{L}_2$ be the left side of \[(2.14)\]. By using the integral representation of $M_{p,q;\lambda,\rho}(z)$ in \[(2.1)\] and interchanging the order of integrals, which is verified under the given assumptions here, we have
\[
\mathcal{L}_2 = \frac{\mu^{\rho+\frac{1}{2}}}{B(\rho - \lambda + \frac{1}{2}, \rho + \lambda + \frac{1}{2})} \int_0^1 t^{\rho-\lambda-\frac{1}{2}} (1 - t)^{\rho+\lambda-\frac{1}{2}} \exp \left( -\frac{p}{t} - \frac{q}{1-t} \right) \, dt
\]
\[
\times \left\{ \int_0^\infty z^{\delta+\rho-\frac{1}{2}} e^{-\left(\alpha + \frac{\mu}{2} - \mu\right)z} \, dz \right\} \tag{2.15}
\]

As in \[(2.12)\], we find
\[
\int_0^\infty z^{\delta+\rho-\frac{1}{2}} e^{-\left(\alpha + \frac{\mu}{2} - \mu\right)z} \, dz = \frac{\Gamma(\delta + \rho + \frac{1}{2})}{(\alpha + \frac{\mu}{2} - \mu)^{\delta+\rho+\frac{1}{2}}}
\]
\[
\left( \Re(\delta + \rho) > -\frac{1}{2}; \alpha + \frac{\mu}{2} - \mu \in \mathbb{R}^+ \right).
\] \[(2.16)\]
Setting (2.16) in (2.15), we obtain
\[ L_2 = \frac{\Gamma\left(\delta + \rho + \frac{1}{2}\right) \mu^{\rho + \frac{1}{2}}}{\left(\alpha + \frac{1}{2}\right)^{\delta + \rho + \frac{1}{2}}} B(\rho - \lambda + \frac{1}{2}, \rho + \lambda + \frac{1}{2}) \times \int_0^1 t^{\rho - \lambda - \frac{1}{2}}(1 - t)^{\rho + \lambda - \frac{1}{2}} \exp\left(-\frac{p}{1 - t} - \frac{q}{1 - t}\right) \left(1 - \frac{2\mu}{2\alpha + \mu}\right)^{-\left(\delta + \rho + \frac{1}{2}\right)} \, dt. \]  

By the generalized binomial theorem, we get
\[ \left(1 - \frac{2\mu}{2\alpha + \mu}\right)^{-\left(\delta + \rho + \frac{1}{2}\right)} = \sum_{n=0}^{\infty} \frac{\left(\delta + \rho + 1/2\right)_n}{n!} \left(\frac{2\mu}{2\alpha + \mu}\right)^n t^n \]  
which, in view of (1.13), is equal to the right side of (2.14).

\[ \text{Theorem 2.5. The following derivative formula holds true:} \]
\[ \frac{d^n}{dz^n}\left\{e^{\frac{z}{2}}z^{-\rho - \frac{1}{2}}M_{p,q;\lambda,\rho}(z)\right\} = \left(\frac{\rho - \lambda + 1/2}{2\rho + 1}\right)_n e^{\frac{z}{2}}z^{-\rho - \frac{1}{2}}M_{p,q;\lambda - \frac{1}{2},\rho + \frac{1}{2}}(z) \quad (n \in \mathbb{N}_0). \]

\[ \text{Proof. From (1.20), we have} \]
\[ e^{\frac{z}{2}}z^{-\rho - \frac{1}{2}}M_{p,q;\lambda,\rho}(z) = \Phi_{p,q}\left(\rho - \lambda + \frac{1}{2}; 2\rho + 1; z\right). \]

Recall the following formula (see [3, Eq. (9.3)]
\[ \frac{d^n}{dz^n}\left\{\Phi_{p,q}(b; c; z)\right\} = \frac{(b)_n}{(c)_n} \Phi_{p,q}(b + n; c + n; z) \quad (n \in \mathbb{N}_0). \]

Applying (2.22) to (2.21), we get the desired result.

3. Remarks and special cases

In this paper, we made a main use of the results in [3] to give certain formulas involving the \((p, q)\)-Whittaker function \(M_{p,q;\lambda,\rho}(z)\) in (1.20), whose essential factor is the function \(\Phi_{p,q}(b; c; z)\). Some important properties for the \(\Phi_{p,q}(b; c; z)\) and various formulas involving it have already been established in [3]. Yet, for a more convenient and faster use of certain properties for the \((p, q)\)-Whittaker function \(M_{p,q;\lambda,\rho}(z)\) and diverse formulas involving it, we intend to write this paper.
The results presented here, being very general, can be reduced to yield those involving relatively simple Whittaker functions. The results given here when $p = q$ reduce to the known results associated with the extended Whittaker function $M_{p, \lambda, \rho}(z)$ in (1.19) (see [4]). The special cases $p = q = 0$ of the results given here will yield the corresponding ones involving the classical Whittaker function $M_{\lambda, \rho}(z)$ in (1.18).

Among numerous special cases of the results presented here, we choose to demonstrate only one formula. Setting $\delta = 1$ and $\mu = -1$ in (2.14), we obtain the Laplace transformation of the $(p, q)$-Whittaker function $M_{p,q;\lambda,\rho}(-t)$ in (1.20)

$$
\int_0^\infty e^{-\alpha t}M_{p,q;\lambda,\rho}(-t) \, dt = \frac{(-1)^{\rho+\frac{1}{2}}\Gamma(\rho + \frac{3}{2})}{(\alpha - \frac{1}{2})^{\rho+\frac{1}{2}}} \times F_{p,q}\left(\rho + \frac{3}{2}, \rho - \lambda + \frac{1}{2}; 2\rho + 1; \frac{2}{1-2\alpha}\right)
$$

(3.1)

$p, q \in \mathbb{R}_+^+; \alpha > \frac{3}{2}; \Re(\rho) > -\frac{3}{2}; \Re(\rho \pm \lambda) > -\frac{1}{2}; \arg(-1) = \pi$.

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