THE SCHNAKENBERG MODEL WITH PRECURSORS

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Abstract. In this paper, we mainly consider the following Schnakenberg model with a precursor \( \mu(x) \) on the interval \((-1,1)\):

\[
\begin{align*}
\frac{\partial u}{\partial t} &= D_1 u''(x) - \mu(x) u + v u^2, \\
\frac{\partial v}{\partial t} &= D_2 v''(x) + B - v u^2,
\end{align*}
\]

where \( D_1 > 0, \ D_2 > 0, \ B > 0 \).

We establish the existence and stability of \( N \)-peaked steady-states in terms of the precursor \( \mu(x) \) and the diffusion coefficients \( D_1 \) and \( D_2 \). It is shown that \( \mu(x) \) plays an essential role for both existence and stability of the above pattern. Similar result has been obtained for the Gierer-Meinhardt system by Wei and Winter [21].

1. Introduction. Since the work of Turing [11] in 1952, a lot of models have been established and studied to explore the so-called Turing diffusion-driven instability.

One of the most interesting models in biological pattern formation is the Schnakenberg model [10] on a one-dimension interval, which can be stated as follows:

\[
\begin{align*}
\frac{\partial u}{\partial t} &= D_1 u''(x) - u + v u^2, \\
\frac{\partial v}{\partial t} &= D_2 v''(x) + B - v u^2,
\end{align*}
\]

where \( D_1 > 0, \ D_2 > 0, \ B > 0 \) are positive constants. Substituting \( u = \hat{u}/(2B), \ v = 2B\hat{v} \), and dropping hats we obtain the following form of the system

\[
\begin{align*}
\frac{\partial u}{\partial t} &= D_1 u''(x) - u + v u^2, \\
\frac{\partial v}{\partial t} &= D_2 v''(x) + B - v u^2,
\end{align*}
\]

where \( b^{-1} = 4B^2 \). To find a scaling appropriate for spike solutions we assume that \( u \) diffuses more slowly than \( v \), so that

\[
D_1 = \varepsilon^2, \ D_1 = D,
\]

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where \( \varepsilon^2 << D << 1 \). We then introduce the new variables

\[
D = \frac{\tilde{D}}{\varepsilon}, \ v = \varepsilon \tilde{v}, \ u = \frac{\tilde{u}}{\varepsilon}.
\]

Substituting (1.3) into (1.1), and dropping the variables, we obtain the following singularly perturbed reaction-diffusion system of interest:

\[
\begin{align*}
\varepsilon_t + \varepsilon^2 u_{xx} - u + \varepsilon^2 u^2 v & = 0 \quad x \in (-1, 1), \ t > 0, \\
D v_{xx} + \frac{1}{2} - \frac{b}{\varepsilon} v^2 u & = 0 \quad x \in (-1, 1), \ t > 0, \\
\frac{\partial u}{\partial x}(\pm 1, t) = v_x(\pm 1, t) & = 0.
\end{align*}
\] (1.4)

The stationary solution to (1.4) satisfies

\[
\begin{align*}
\varepsilon^2 u_{xx} - u + \varepsilon^2 u^2 v & = 0 \quad x \in (-1, 1), \\
D v_{xx} + \frac{1}{2} - \frac{b}{\varepsilon} v^2 u & = 0 \quad x \in (-1, 1), \\
\frac{\partial u}{\partial x}(\pm 1) = v_x(\pm 1) & = 0.
\end{align*}
\] (1.5)

We note that the Schnakenberg model has been widely studied by analytical and numerical methods. In the one-dimension case. For problem (1.2), the existence and stability of symmetric \( N \)-peaked solution were established by Wei and Winter [7] using asymptotic analysis. They mainly consider the stability of symmetric \( N \)-peaked solutions to problem (1.2). In this case, the parameter \( D \) effect the stability. For \( D \) small, the \( N \)-spike solution is stable, while for \( D \) large, the \( N \)-spike solution is unstable for \( N \geq 2 \). For problem (1.5), using asymptotic expansions, Ward and Wei studied the existence and stability of asymmetric equilibrium spike patterns for the Schnakenberg model [13]. In this article, as \( \varepsilon \to 0 \), they constructed an asymmetric \( k \)-spike equilibrium solution to problem (1.5) in the form of a sequence of spikes of different heights. Moreover, they considered the stability of the asymmetric \( k \)-spike equilibrium solution.

In two dimension case. We refer to [2] and references therein-in which the Schnakenberg model is posed in a two-dimensional square.

In the present paper we will consider the following Schnakenberg model with a precursor \( \mu(x) \):

\[
\begin{align*}
\varepsilon^2 u'' - \mu(x) u + \varepsilon^2 u^2 v & = 0 \quad \text{in } (-1, 1), \\
D v'' + \frac{1}{2} - \frac{b}{\varepsilon} v^2 u & = 0 \quad \text{in } (-1, 1), \\
\frac{\partial u}{\partial x}(\pm 1) = v_x(\pm 1) & = 0,
\end{align*}
\] (1.6)

where \( 0 < \varepsilon << 1 \), \( D > 0 \) is a parameter, \( b > 0 \) is fixed.

Our interest is to consider the existence and stability of \( N \)-peaked solutions to the Schnakenberg model (1.6). For the existence of \( N \)-peaked solutions, we must consider the effect of the precursor \( \mu(x) \). Since the precursor \( \mu(x) \) may be not symmetric in \((-1, 1)\), we can’t consider this problem in symmetric space. That is the solution of (1.6) may be not symmetric. Hence, we need to construct solution by a new method. In this paper, we will construct \( N \)-peaked solutions by using the method of Liapunov-Schmidt reduction which has been used for the one-dimension Schrödinger equation [3][8][9] and then extended to the higher-dimensional Cahn-Hilliard equation [15][16] and semilinear elliptic equations[1][5][6]. This method has also been applied to the Schnakenberg model[7][13][22]. For the stability of \( N \)-peaked solutions, we study it by using the asymptotic analysis, which has been used to study the Gierer-Meinhardt system [17][18][19][20]. In [12], the authors have used the Lyapunov-Schmidt reduction to consider the effect of precursor for the Gierer-Meinhardt system. In this paper, we will employ the same idea to deal with the Schnakenberg model.
It turns out that unlike the homogeneous case for which the $N$—spike solution is stable for $D$ small, the precursor may effect the stability of the spike solutions.

Before we state our main results in Section 2, we introduce some notation. Throughout this paper, we always assume $\Omega = (−1, 1)$, $\Omega_ε = (−\frac{1}{2}, \frac{1}{2})$. With $L^2(\Omega)$ and $H^2(\Omega)$ denote the usual Sobolev spaces. The function $\omega$ we denote the solution of the following problem:

$$
\begin{align*}
\begin{cases}
\omega'' - \omega + \omega^2 = 0 & \text{in } R^1, \\
\omega > 0, & \omega(0) = \max_{y \in R} \omega(y), \\
\omega(y) \to 0 & \text{as } |y| \to \infty.
\end{cases}
\end{align*}
$$

(1.7)

An explicit representation is

$$
\omega(y) = \frac{3}{2} \cosh^{-2}(\frac{y}{2}).
$$

We list some properties of $\omega$:

$$
\begin{align*}
\begin{cases}
\omega \text{ is a even function on } R^1; \\
\omega'(y) < 0, & \text{if } y > 0; \\
\int_\mathcal{R} \omega^3(y)dy = 7.2, \int_\mathcal{R} (\omega')^2(y)dy = 1.2, \int_\mathcal{R} \omega^2(y)dy = 6.
\end{cases}
\end{align*}
$$

(1.8)

We assume that the precursor $\mu(x)$ satisfies

$$
\mu(x) \in C^2(\Omega), \mu(x) > 0.
$$

(1.9)

Let $G_D(x, z)$ be Green’s function given by

$$
\begin{align*}
\begin{cases}
DG''_D(x, z) + \frac{1}{2} - \delta_z = 0 & \text{in } (-1, 1), \\
\int_{-1}^{1} G_D(x, z)dx = 0, \\
G_D(-1, z) = G_D(1, z) = 0.
\end{cases}
\end{align*}
$$

(1.10)

We easy calculate

$$
G_D(x, z) = \begin{cases}
\frac{1}{2D} \left[ \frac{1}{3} - \frac{(x+1)^2}{4} - \frac{(1-z)^2}{4} \right], & -1 < x \leq z; \\
\frac{1}{2D} \left[ \frac{1}{3} - \frac{(z+1)^2}{4} - \frac{(1-x)^2}{4} \right], & z \leq x < 1.
\end{cases}
$$

(1.11)

We decompose

$$
G_D(x, z) = \frac{1}{2D} |x - z| + H_D(x, z).
$$

(1.12)

where $H_D(x, z) = \frac{1}{2D} \left[ -\frac{1}{3} - \frac{x^2}{2} - \frac{z^2}{2} \right]$ is the regular part of $G_D$, furthermore $H_D$ is $C^\infty$ of $x$ and $z$. We denote the singular part of $G_D$ by $K_D(|x - z|) = G_D - H_D$.

The paper is organized as follows. In Section 2, we will state the main existence and stability result for the Schnakenberg model. Section 3-Section 6 concerns the existence part. In Section 7-Section 8, we prove the stability result. Section 9 contains some technical computations and the analysis of the Green’s function is contained in the Appendix.

We use the notation $e.s.t.$ to denote an exponentially small term of order $O(e^{-\frac{d}{\varepsilon}})$ for some $d > 0$ in the corresponding norm. By $C$ we denote a generic constant which may change from line to line.

2. Main results: Existence and stability of N-peak solutions. In this paper, we always consider the following situations:

$$
0 < \varepsilon << 1 \text{ and } D > 0.
$$

(2.1)

Let

$$
\eta_j^0 = -1 + \frac{2j - 1}{N}, \quad j = 1, \ldots, N, N \geq 2.
$$

(2.2)

be $N$ points in $(-1, 1)$ and $\mu_j^0 = \mu(\eta_j^0), i = 1, \cdots, N$. 

We define
\[ \omega_a(y) = a \omega(a \frac{1}{2} y), \text{ for } a > 0, \] (2.3)
where \( \omega \) satisfies (1.7), is the unique solution of the following problem:
\[
\begin{cases}
\omega'' - a \omega_a + \omega_a^2 = 0 & \text{in } R^1, \\
\omega_a > 0, \omega_a(0) = \max_{y \in R} \omega_a(y), \\
\omega_a(y) \to 0 \text{ as } |y| \to \infty.
\end{cases}
\] (2.4)

By some simple calculation, we have the following relations
\[
\begin{align*}
\int_R \omega_a^2(y) dy &= a \int_R \omega^2(y) dy, \\
\int_R \omega_a^3(y) dy &= a \int_R \omega^3(y) dy, \\
\int_R (\omega^2_a)^2(y) dy &= a \int_R (\omega^2)^2(y) dy.
\end{align*}
\] (2.5)

We introduce several matrices for later use: For \( t = (t_1, \ldots, t_N) \in (-1,1)^N \), let
\[ \mathcal{G}_D(t) = (G_D(t_i,t_j)). \] (2.6)

Recall that \( G_D(t_i,t_j) = K_D(|t_i - t_j|) + H_D(t_i,t_j) \).

Let us denote \( \frac{\partial}{\partial y} \) as \( \nabla_{t_i} \). When \( i \neq j \), we can define \( \nabla_{t_i} G_D(t_i,t_j) \) in the classical way. When \( i = j \), \( K_D(t_i,t_j) = K_D(0) = 0 \). We define
\[
\nabla_{t_i} G_D(t_i,t_i) := \left. \frac{\partial}{\partial x} \right|_{x=t_i} H(x,t_i).
\]

Similarly, we define
\[
\nabla_{t_i} \nabla_{t_j} G_D(t_i,t_j) = \begin{cases} 
\left. \frac{\partial}{\partial x} \right|_{x=t_i} \left. \frac{\partial}{\partial y} \right|_{y=t_j} H_D(x,y), & \text{if } i = j, \\
\nabla_{t_i} \nabla_{t_j} G_D(t_i,t_j), & \text{if } i \neq j.
\end{cases}
\] (2.7)

Now the derivatives of \( G_D(t_i,t_j) \) are defined as follows:
\[
\nabla \mathcal{G}_D(t) = (\nabla_{t_i} G_D(t_i,t_j)), \ \nabla^2 \mathcal{G}_D(t) = (\nabla_{t_i} \nabla_{t_j} G_D(t_i,t_j)).
\] (2.8)

Next we state the first assumption:

\textbf{(H1)} There exists a solution \((\xi_1^0, \ldots, \xi_N^0)\) of the equation
\[
\begin{align*}
\xi_i - \frac{1}{N} \sum_{j=1}^N \xi_j = 6b \sum_{j=1}^N G_D(t_i^0,t_j^0) \left( \frac{\mu^0}{\xi_j^0} \right)^{\frac{3}{2}} - \frac{\lambda_i}{N}, & \text{ for } i = 1, \ldots, N, \\
6b \sum_{j=1}^N \left( \frac{\mu^0}{\xi_j^0} \right)^{\frac{3}{2}} = 1,
\end{align*}
\] (2.9)

where \( \lambda_1 = -\frac{1}{6DN} \). By Appendix (9.42), we know that \( \sum_{i=1}^N G_D(t_i^0,t_j^0) = -\frac{1}{6DN} = \lambda_1 \), for \( j = 1, \ldots, N \).

Next we introduce some matrices:
\[
\begin{align*}
\mathcal{G}_0 &= \mathcal{G}_D(t_i^0), \ \mu_i^{3/2} = ((\mu(t_i^0)^{3/2} \delta_{ij}), \ \mathcal{H}_0^{2} = ((\xi_j^0)^{-2} \delta_{ij}) \\
I &= (\delta_{ij}), \ E = ee^T, \ e = (1, \ldots, 1)^T, D = \frac{6b}{D^0} \mu_i^{3/2} \mathcal{H}_0^{2}.
\end{align*}
\] (2.10)
\( \mathcal{C} = \frac{N}{2} \begin{pmatrix} 1 & -1 & 0 & \cdots & 0 & 0 & 0 \\ -1 & 2 & \ddots & \ddots & \ddots & 0 & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & \ddots & \ddots & \ddots & 2 & -1 \\ 0 & 0 & 0 & \cdots & 0 & -1 & 1 \end{pmatrix} \). \quad (2.11)

Our second assumption is the following:

\( (H2) \) It holds that

\[
I - \frac{1}{N} E + 6b \xi_0 \mu_0 \frac{3}{2} \mathcal{H}_0^2
\]

and

\[
I - \frac{1}{N} E - 6b \xi_0 \mu_0 \frac{3}{2} \mathcal{H}_0^2
\]

is invertible.

**Remark 2.1.** In [7], Iron, Wei and Winter studied the Schnakenberg model with precursor \( \mu \equiv 1 \). In this special situation: \( \xi_0 = bN \int_R \omega^2 = 6bN \), \( i = 1, \ldots, N \), satisfies the assumption \( (H1) \). It is essential that \( I + \frac{1}{6bN^2} \mathcal{G}_D \) and \( I - \frac{1}{6bN^2} \mathcal{G}_D \) are invertible, which makes the assumption \( (H2) \) trivial.

According to \( (H1), (H2) \) and the implicit function theorem, for \( t = (t_1, \ldots, t_N) \) near \( t^0 = (t_1^0, \ldots, t_N^0) \) and \( \mu_j = \mu(t_j) \), \( j = 1, \ldots, N \), there exists a (locally) unique solution \( \xi(t) = (\xi_1(t), \ldots, \xi_N(t)) \) of the following equation

\[
\xi_i - \frac{1}{N} \sum_{j=1}^N \xi_j = 6b \sum_{j=1}^N \mathcal{G}_D(t_i, t_j) \left( \frac{\mu_j}{\xi_j} \right)^{3/2} - 6b \sum_{i=1}^N \sum_{j=1}^N \mathcal{G}_D(t_i, t_j) \left( \frac{\mu_j^0}{\xi_j^0} \right)^{3/2},
\]

\( i = 1, \ldots, N. \)

Moreover, \( \xi(t) \) is \( C^1 \) in \( t \).

Define

\[
\mathcal{H}(t) = \left( \frac{1}{\xi_i(t)} \delta_{ij} \right), \quad \mu(t) = (\mu(t_i) \delta_{ij}), \quad \mu'(t) = (\mu_j' \delta_{ij}), \quad \mu_j' = \mu_j'(t_j).
\]

We introduce the following vector field:

\[
F(t) = (F_1(t), \ldots, F_N(t)),
\]

where

\[
F_i(t) = \frac{5}{4} \xi_i \mu_i^{-1} \mu_i'(t_i) - 6b \sum_{j=1}^N \nabla_i \mathcal{G}_D(t_i, t_j) \xi_j^{-1} \mu_j^2, \quad i = 1, \ldots, N.
\]

Set

\[
\mathcal{M}(t) = \left( \frac{\partial F_i(t)}{\partial t_j} \right).
\]

Our final assumption states as follows:

\( (H3) \) Assume that at \( t^0 = (t_1^0, \ldots, t_N^0) \)

\[
F(t^0) = 0, \quad \det \mathcal{M}(t^0) \neq 0
\]

**Remark 2.2.** Since the matrix \( \mathcal{M} \) is of the form \( \mathcal{S} \mathcal{D} \), where \( \mathcal{S} \) is symmetric and \( \mathcal{D} \) is diagonal matrix, it follows that the eigenvalues of \( \mathcal{M} \) are all real.
Let us now calculate $\mathcal{M}(t^0)$: We first compute the derivatives of $\xi(t)$. According to (2.13), for $t \in B_{\varepsilon}(t^0) := \{ t \mid |t - t^0| < \varepsilon \}$, we obtain

For $i \neq j$,

$$
\nabla_t \xi_i - \frac{1}{N} \sum_{m=1}^{N} \nabla_t \xi_m = 6b \nabla_t G_D(t_i, t_j) \frac{(\mu_j)^{\frac{3}{2}}}{\xi_j} - 6b \sum_{m=1}^{N} G_D(t_i, t_m) \frac{(\mu_m)^{\frac{3}{2}}}{\xi_m^2} \nabla_t \xi_m \\
- 6b \frac{N}{N} \sum_{m=1}^{N} \nabla_t G_D(t_j, t_m) \frac{(\mu_m^0)^{\frac{3}{2}}}{\xi_m^0} + 9b G_D(t_i, t_j) \frac{(\mu_j)^{\frac{3}{2}}}{\xi_j} \mu'_j \\
= 6b \nabla_t G_D(t_i, t_j) \frac{(\mu_j)^{\frac{3}{2}}}{\xi_j} - 6b \sum_{m=1}^{N} G_D(t_i, t_m) \frac{(\mu_m)^{\frac{3}{2}}}{\xi_m} \nabla_t \xi_m \\
- 6b \sum_{m=1}^{N} \nabla_t G_D(t_j, t_m) \frac{(\mu_m)^{\frac{3}{2}}}{\xi_m} + 9b G_D(t_i, t_j) \frac{(\mu_j)^{\frac{3}{2}}}{\xi_j} \mu'_j + O(\varepsilon).
$$

For $i = j$,

$$
\nabla_t \xi_i - \frac{1}{N} \sum_{m=1}^{N} \nabla_t \xi_m = 6b \nabla_t G_D(t_i, t_i) \frac{(\mu_j)^{\frac{3}{2}}}{\xi_i} - 6b \sum_{m=1}^{N} G_D(t_i, t_m) \frac{(\mu_m)^{\frac{3}{2}}}{\xi_m} \nabla_t \xi_m \\
- 6b \frac{N}{N} \sum_{m=1}^{N} \nabla_t G_D(t_i, t_m) \frac{(\mu_m^0)^{\frac{3}{2}}}{\xi_m^0} + 9b G_D(t_i, t_i) \frac{(\mu_i)^{\frac{3}{2}}}{\xi_i} \mu'_i \\
+ 6b \frac{N}{N} \sum_{m=1}^{N} \nabla_t G_D(t_i, t_m) \frac{(\mu_m)^{\frac{3}{2}}}{\xi_m} \\
= 6b \nabla_t G_D(t_i, t_i) \frac{(\mu_i)^{\frac{3}{2}}}{\xi_i} - 6b \sum_{m=1}^{N} G_D(t_i, t_m) \frac{(\mu_m)^{\frac{3}{2}}}{\xi_m} \nabla_t \xi_m \\
- 6b \sum_{m=1}^{N} \nabla_t G_D(t_i, t_m) \frac{(\mu_m)^{\frac{3}{2}}}{\xi_m} + 9b G_D(t_i, t_i) \frac{(\mu_i)^{\frac{3}{2}}}{\xi_i} \mu'_i \\
+ 6b \sum_{m=1}^{N} \nabla_t G_D(t_i, t_m) \frac{(\mu_m)^{\frac{3}{2}}}{\xi_m} + O(\varepsilon).
$$

Hence, by the definition of (2.15),

$$
\nabla_t \xi_i - \frac{1}{N} \sum_{m=1}^{N} \nabla_t \xi_m = 6b \nabla_t G_D(t_i, t_i) \frac{(\mu_i)^{\frac{3}{2}}}{\xi_i} - 6b \sum_{m=1}^{N} G_D(t_i, t_m) \frac{(\mu_m)^{\frac{3}{2}}}{\xi_m} \nabla_t \xi_m \\
- \frac{1}{N} \frac{5}{4} \xi_i \mu^{-1}_i \mu'(t_i) + 9b G_D(t_i, t_i) \frac{(\mu_i)^{\frac{3}{2}}}{\xi_i} \mu'_i \\
+ \frac{5}{4} \delta_j \frac{N}{N} \xi_i \mu^{-1}_i \mu'(t_i) + O(\varepsilon + \sum_{j=1}^{N} |F_i(t)|).
$$

Note that

$$
(\nabla_t G_D(t_i, t_j)) = (\nabla G_D)^T.
$$
Therefore, by introducing matrix notation
\[ \nabla \xi = (\nabla_{t_i} \xi_i), \quad P = (I - \frac{1}{N} E + 6b G_D H^2 \mu^{3/2})^{-1}, \]
\[ \mathcal{H}^2(t) = \left( \frac{1}{\xi^2(t)} \delta_{ij} \right), \quad \mu^{3/2}(t) = ((\mu(t_i))^3/2 \delta_{ij}), \]
we have
\[ \nabla \xi = P \left[ 6b (\nabla G_D)^T H \mu^{3/2} + 9b G_D H H^{1/2} \mu + (I - \frac{1}{N} E) \frac{5}{4} H^{-1} \mu^{-1} \mu' \right] + \mathcal{O}(\varepsilon + \sum_{j=1}^N |F_i(t)|). \]
\[ \tag{2.18} \]
(2.19)

Note that by Appendix (9.38) and (9.39), we have
\[ \nabla_{t_i} (\nabla H(t_i, x))|_{x=t_i} = 0, \quad \nabla_{t_i} (\nabla G_D(t_i, t_j)) = 0, \quad \text{if} \ i \neq j. \]

Let
\[ Q = (\gamma \delta_{ij}), \quad \gamma = \frac{1}{2D} \sum_{j=1}^N \frac{\mu_m^{3/2}}{\xi_m}. \]
\[ \tag{2.20} \]

Using (2.19), we can compute \( \mathcal{M}(t^0) \):
For \( i \neq j \),
\[ \frac{\partial F_i(t)}{\partial t_j} = \frac{5 \mu_i'}{4 \mu_i} \nabla_{t_j} \xi_i + 6b \sum_{m=1}^N \nabla_{t_i} G_D(t_i, t_m) \mu_m^{3/2} \xi_m \nabla_{t_j} \xi_m - 9b \nabla_{t_i} G_D(t_i, t_j) \mu_j^{1/2} \xi_j. \]

For \( i = j \),
\[ \frac{\partial F_i(t)}{\partial t_i} = \frac{5 \mu_i'}{4 \mu_i} \nabla_{t_i} \xi_i + \frac{5 \mu_i''}{4 \mu_i} \xi_i - \frac{5 (\mu_i')^2}{4 \mu_i^2} \nabla_{t_i} \xi_i \\
+ 6b \sum_{m=1}^N \nabla_{t_i} G_D(t_i, t_m) \mu_m^{3/2} \xi_m - 9b \nabla_{t_i} G_D(t_i, t_i) \mu_i^{1/2} \xi_i \\
- 6b \sum_{m=1}^N \nabla_{t_i} \nabla_{t_i} (G_D(t_i, t_m)) \mu_m^{3/2} \xi_m - 6b \frac{\partial}{\partial x} (\nabla_{t_i} H(t_i, x))|_{x=t_i} \frac{\mu_i^{3/2}}{\xi_i}. \]

Therefore, we can obtain
\[ \mathcal{M}(t^0) = 6b Q + 6b \nabla G_D \nabla \xi \mu^{3/2} \mathcal{H}^2 - 9b \nabla G_D \mu^{1/2} \mathcal{H} \mu' + \frac{5}{4} \left[ \mathcal{H}^{-1} \mu^{-1} \mu'' - \mathcal{H}^{-1} \mu^{-2} (\mu')^2 + \mu^{-1} \mu' \nabla \xi \right]. \]
\[ \tag{2.21} \]

By (H1), moreover, we have
\[ \mathcal{M}(t^0) = \frac{1}{2D} I + 6b \nabla G_D \nabla \xi \mu^{3/2} \mathcal{H}^2 - 9b \nabla G_D \mu^{1/2} \mathcal{H} \mu' + \frac{5}{4} \left[ \mathcal{H}^{-1} \mu^{-1} \mu'' - \mathcal{H}^{-1} \mu^{-2} (\mu')^2 + \mu^{-1} \mu' \nabla \xi \right]. \]

To study the stability, we define
\[ B = \mathcal{D} (\mathcal{C} + \mathcal{D})^{-1}, \]
where \( \mathcal{D} \) and \( \mathcal{C} \) are given by (2.10) (2.11) respectively.
Remark 2.3. Since $D + C$ is diagonally dominant, $D + C$ is invertible. Thus, $B$ is well-defined.

Remark 2.4. By the same reasoning as for the matrix $M$, the eigenvalues of $B$ are real.

Our first result can be stated as follows:

**Theorem 2.5.** Assume that assumptions $(H1)$, $(H2)$ and $(H3)$ hold. Then for $\varepsilon \ll 1$, problem (1.6) has an $N$-peak solution centered at $t_1^{\varepsilon}, \ldots, t_N^{\varepsilon}, N \geq 2$. Moreover, it satisfies

$$u_\varepsilon(x) \sim \sum_{i=1}^{N} \xi_i^{-1} \omega_i \left( \frac{x - t_i^{\varepsilon}}{\varepsilon} \right), \quad (2.22)$$

where $\omega_i$ is given by (2.3) for $a = \mu(t_i^{\varepsilon})$,

$$v(t_i^{\varepsilon}) \sim \xi_i, \quad i = 1, \ldots, N, \quad (2.23)$$

$$t_i^{\varepsilon} \to t_i^0, \quad t_i^0 = -1 + \frac{2i - 1}{N}, \quad i = 1, \ldots, N. \quad (2.24)$$

The next theorem reduces the stability to the conditions on the matrices $M$ and $B$.

**Theorem 2.6.** Let $(u_\varepsilon, v_\varepsilon)$ be the solution constructed in Theorem 2.5. Assume that $0 < \varepsilon \ll 1$.

- *(stability)* If

$$\min_{\sigma \in \sigma(B)} \sigma > \frac{1}{2} \quad (2.25)$$

and

$$\sigma(M(t^0_i)) \subset \{ \sigma | Re(\sigma) \geq c > 0 \}, \quad (2.26)$$

then $(u_\varepsilon, v_\varepsilon)$ is linearly stable.

- *(instability)* If

$$\min_{\sigma \in \sigma(B)} \sigma < \frac{1}{2} \quad (2.27)$$

or there exists

$$\sigma \in \sigma(M(t^0_i)), Re(\sigma) < 0, \quad (2.28)$$

then $(u_\varepsilon, v_\varepsilon)$ is linearly unstable.

We end this section with a few remarks.

**Remark 2.7.** Generally speaking, if $\mu \not\equiv$ constant, $\xi_i \neq \xi_j$ for $i \neq j$. Thus the height of different peaks may be different. This is strikingly different from the solutions constructed by Iron, Wei and Winter in [7].

**Remark 2.8.** For $\mu \equiv 1$, the eigenvalues of matrices $B$ and $M$ have been computed explicitly in [7]. But for general precursor $\mu$, the computation may be more complicated.

**Remark 2.9.** Generally speaking, the choice of the precursor $\mu$ and the parameter $D$ will effect the stability of the spike solutions, since the eigenvalues of $M$ and $B$ are dependent on the precursor $\mu$ and the parameter $D$.

Let us consider the following case:

$$\mu(x) = sin[A(x + \frac{1}{2})(x - \frac{1}{2}) + \frac{\pi}{2}] + 2, \quad A > 0,$$
Then we have $\mu(t_0^1) = \mu(t_0^2) = 3$, $\mu'(t_0^1) = \mu'(t_0^2) = 0$ and $\mu''(t_0^1) = \mu''(t_0^2) = -A^2$. Let $\xi^0 = (\xi_0^1, \xi_0^2)$, $\xi_0^1 = \xi_0^2 = 36\sqrt{3}b$, then $\xi^0$ is a solution of (2.9). The matrix $M$ becomes

$$M = M_1 + M_2,$$

where

$$M_1 = \frac{1}{2D}I + 36b^2 m^3/\xi^3 \nabla G_D (I - \frac{1}{N}E + 6b m^{3/2}/\xi G_D)^{-1}(\nabla G_D)^T, (\xi = 36\sqrt{3}b, m = 3),$$

$$M_2 = \frac{5}{4}H^{-1} \mu^{-1} \mu''.$$

Note that $M$ is symmetric, $H = \xi I$ and $\mu = mI$. So the eigenvalues of $M$ are all real,

$$M_2 = -\frac{5m}{4\xi} A^2 (\delta_{ij})_{2 \times 2}.$$

The first matrix $M_1$ does not depend on $A$. Thus, if $A$ is sufficiently large, the eigenvalues of $(M)$ are all negative. Hence, by Theorem 2.6, we obtain $(u_\varepsilon, v_\varepsilon)$ is unstable. We conclude that precursors may give rise to instability. This new effect is not present in the homogeneous case.

3. Some preliminaries. In this section, we study a system of nonlocal linear operators. We first recall

**Theorem 3.1.** [14]: Consider the following nonlocal eigenvalue problem

$$\Delta \phi - \phi + 2\omega \phi - \gamma \int_R \frac{\omega \phi}{\omega^2} \omega^2 = \alpha \phi, \; \phi \in H^2(R). \quad (3.1)$$

1. If $\gamma < 1$, then there is a positive eigenvalue to (3.1).
2. If $\gamma > 1$, then for any nonzero eigenvalue $\alpha$ to (3.1), we have

$$Re(\alpha) \leq -c_0 < 0, \quad \text{for some } c_0 > 0.$$

3. If $\gamma \neq 1$ and $\alpha = 0$, then

$$\phi = c_0 \frac{\partial \omega}{\partial y}$$

for some constant $c_0$.

Next, we consider the following system of linear operators

$$L \Phi := \Delta \Phi - \Phi + 2\omega \Phi - 2 \left( \int_R \omega B \Phi \right) \left( \int_R \omega^2 \right)^{-1} \omega^2, \quad (3.2)$$

where

$$\Phi = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_N \end{pmatrix} \in (H^2(R))^N.$$

Set

$$L_0 u := \Delta u - u + 2\omega u,$$

where $u \in H^2(R)$. 
Then the conjugate operator of $L$ under the scalar product in $L^2(R)$ is
\[
L^\ast \Psi := \Delta \Psi - \Psi + 2\omega \Psi - 2(\int_R \omega^2 B^T \Psi)(\int_R \omega^2)^{-1} \omega,
\] (3.3)
where
\[
\Psi = \begin{pmatrix}
\psi_1 \\
\psi_2 \\
\vdots \\
\psi_N
\end{pmatrix} \in (H^2(R))^N.
\]

**Lemma 3.2.** (Lemma 3.2 of [21]) If $\frac{1}{2} \notin \sigma(B)$, then
\[
Ker(L) = X_0 \oplus X_0 \oplus \cdots \oplus X_0,
\] (3.4)
where
\[
X_0 = \text{span} \{ \frac{\partial \omega}{\partial y} \}
\]
and
\[
Ker(L^\ast) = X_0 \oplus X_0 \oplus \cdots \oplus X_0,
\] (3.5)

As a consequence of Lemma 3.2, we have

**Lemma 3.3.** (Lemma 3.3 of [21]) The operator
\[
L : (H^2(R))^N \to (L^2(R))^N
\]
is an invertible operator if it is restricted as follows
\[
L := (X_0 \oplus \cdots \oplus X_0)^\perp \cap (H^2(R))^N \to (X_0 \oplus \cdots \oplus X_0)^\perp \cap (L^2(R))^N.
\]

Moreover, $L^{-1}$ is bounded.

### 4. Study of approximate solutions.

Let $\xi^0 = (\xi_1^0, \ldots, \xi_N^0)$ be the locally unique solution of (2.9). Recall that $\mu_i^0 = \mu(t_i^0)$ and
\[
t_i^0 = (t_{i1}^0, \ldots, t_{iN}^0), \quad t_i^0 = -1 + \frac{2i - 1}{N}, i = 1, \ldots, N.
\]

We now construct an approximate solution of (1.6) which concentrates near those points $t_i^0$.

Let $-1 < t_1 < t_2 < \ldots < t_N < 1$ be such that $t = (t_1, \ldots, t_N) \in B_\varepsilon(t^0)$. We introduce a smooth cut-off function $\chi : R \to [0, 1]$ such that
\[
\chi(x) = 1 \text{ for } |x| < \frac{1}{4N} \text{ and } \chi(x) = 0 \text{ for } |x| > \frac{1}{2N}.
\] (4.2)

We define approximate solution
\[
\tilde{\omega}_i(x) = \omega_i(x) \chi(x - t_i).
\] (4.3)

Then $\tilde{\omega}_i(x)$ satisfies the following equation:
\[
\varepsilon^2 \Delta \tilde{\omega}_i - \mu_i \tilde{\omega}_i + \tilde{\omega}_i^2 + e.s.t. = 0.
\] (4.4)
Thus, by Appendix (9.42), we have
\[
\xi = \frac{1}{N} \sum_{j=1}^{N} \xi_j = 6b \sum_{j=1}^{N} G_D(t_i, t_j) \frac{\mu_j^2}{\xi_j} - 6b \sum_{i=1}^{N} \sum_{j=1}^{N} G_N(t_i, t_j) \frac{\mu_j^0}{\xi_j^0}. \tag{4.5}
\]

Moreover, \(\xi(t)\) is a \(C^1\) function in \(t\).

Let
\[
\omega_{\varepsilon, t}(x) = \sum_{j=1}^{N} \xi_j^{-1} \hat{\omega}_j(x). \tag{4.6}
\]

For \(u \in H^2(-1, 1)\), we define \(T[u]\) to be the solution of
\[
\begin{cases}
D \Delta T[u] + \frac{1}{\varepsilon} T[u] u^2 = 0 & \text{in } (-1, 1), \\
T[u]'(\pm 1) = 0.
\end{cases} \tag{4.7}
\]

Next, we will calculate \(T[\omega_{\varepsilon, t}](t_i)\) and \(T[\omega_{\varepsilon, t}](x) - T[\omega_{\varepsilon, t}](t_i)\), where \(t \in B_{\varepsilon}(t^0)\).

Denote by
\[
T[\omega_{\varepsilon, t}](t_i) := \tau_i.
\]

By equation (4.7) and Green’s function (1.10), we have
\[
T[\omega_{\varepsilon, t}](x) - T[\omega_{\varepsilon, t}](x) = b \int_{-1}^{1} G_D(x, z)(T[\omega_{\varepsilon, t}](z))dz, \tag{4.8}
\]

where \(T[\omega_{\varepsilon, t}](x) = \frac{1}{2} \int_{-1}^{1} T[\omega_{\varepsilon, t}](z)dz\).

Hence, we have
\[
\tau_i - T[\omega_{\varepsilon, t}] = 6b \sum_{j=1}^{N} G_D(t_i, t_j) \frac{\mu_j^2}{\xi_j^2} \tau_j + O(\varepsilon). \tag{4.9}
\]

Thus, by Appendix (9.42), \(\sum_{i=1}^{N} G_D(t_i^0, t_j^0) = \lambda_1 = -\frac{1}{\varepsilon DN}\),
\[
\sum_{i=1}^{N} \tau_i - NT[u] = 6b \sum_{i=1}^{N} \sum_{j=1}^{N} G_D(t_i^0, t_j^0) \frac{(\mu_j)^2}{\xi_j^2} \tau_j + O(\varepsilon), \tag{4.10}
\]

On the other hand, integrating (4.7), we obtain
\[
1 = 6b \sum_{j=1}^{N} (\mu_j)^2 \xi_j^2 \tau_j + O(\varepsilon).
\]

Hence, by (2.9) and (4.10), we obtain
\[
T[\omega_{\varepsilon, t}] = \frac{1}{N} \sum_{j=1}^{N} \tau_j - 6b \sum_{i=1}^{N} \sum_{j=1}^{N} G_D(t_i, t_j^0) \frac{(\mu_j)^3}{\xi_j^3} + O(\varepsilon).
\]

Substituting \(T[\omega_{\varepsilon, t}]\) into (4.9), we have
\[
\tau_i - \frac{1}{N} \sum_{j=1}^{N} \tau_j = 6b \sum_{j=1}^{N} G_D(t_i, t_j) \frac{\mu_j^2}{\xi_j^2} \tau_j - 6b \sum_{i,j=1}^{N} G_D(t_i, t_j^0) \frac{(\mu_j^0)^3}{\xi_j^3} + O(\varepsilon). \tag{4.11}
\]
By assumption (H2) and the implicit function theorem, the equation (4.11) has a unique solution

\[ T[\omega_{e,t}](t_i) = \tau_i = \xi_i + O(\varepsilon). \]

Now let \( x = t_i + \varepsilon y \), we have

\[
\begin{align*}
T[\omega_{e,t}](x) - T[\omega_{e,t}](t_i) &= \frac{b}{\varepsilon} \int_{-1}^{1} [G_D(x, z) - G_D(t_i, z)](T[\omega_{e,t}]\omega^2_{e,t})(z)dz \\
&= \xi_i^{-2} \frac{b}{\varepsilon} \int_{-1}^{1} [G_D(x, z) - G_D(t_i, z)](T[\omega_{e,t}]\omega^2_{e,t})(z)dz \\
&\quad + \sum_{j \neq i} \xi_j^{-2} \frac{b}{\varepsilon} \int_{-1}^{1} [G_D(x, z) - G_D(t_i, z)](T[\omega_{e,t}]\omega^2_{e,t})(z)dz \\
&= \xi_i^{-2} \frac{b}{\varepsilon} \int_{-1}^{1} [K_D(|\varepsilon y + t_i - z|) - K_D(|t_i - z|)](T[\omega_{e,t}]\omega^2_{e,t})(z)dz \\
&\quad + \sum_{j \neq i} \xi_j^{-2} \frac{b}{\varepsilon} \int_{-1}^{1} [K_D(|\varepsilon y + t_i - z|) - K_D(|t_i - z|)](T[\omega_{e,t}]\omega^2_{e,t})(z)dz \\
&\quad + \sum_{j \neq i} \xi_j^{-2} \frac{b}{\varepsilon} \int_{-1}^{1} [G_D(x, z) - G_D(t_i, z)](T[\omega_{e,t}]\omega^2_{e,t})(z)dz \\
&\quad \text{(writing } z = \hat{z} + t_i) \\
&= \xi_i^{-1} \frac{b}{\varepsilon} \int_{R} \frac{1}{2D} ||y - \hat{z}|| - |\hat{z}| \omega^2_{e}(t_i + \varepsilon \hat{z})d\hat{z} \\
&\quad + \xi_i^{-1} \frac{b}{\varepsilon} \int_{R} \omega^2_{e}(\hat{z})d\hat{z} \\
&\quad + \sum_{j \neq i} \xi_j^{-1} \frac{b}{\varepsilon} \int_{R} \omega^2_{e}(\hat{z})d\hat{z} + O(\varepsilon y^2) \\
&= \xi_i^{-1} \frac{b}{\varepsilon} \int_{R} \frac{1}{2D} ||y - \hat{z}|| - |\hat{z}| \omega_1(\varepsilon \hat{z} + t_i)d\hat{z} \\
&\quad \text{where } p_i(\cdot) = \frac{1}{2D} \int_{R} ||y - \hat{z}|| - |\hat{z}| \omega_1(\varepsilon \hat{z} + t_i) d\hat{z}.
\end{align*}
\]

Let us define

\[ S_\varepsilon[\omega_{e,t}] := \varepsilon^2 \Delta \omega_{e,t} - \mu(x)\omega_{e,t} + T[\omega_{e,t}]\omega^2_{e,t}, \quad (4.13) \]

where \( T[\omega_{e,t}] \) is given by (4.9).

We now compute \( S_\varepsilon[\omega_{e,t}] \) as follows:

\[
S_\varepsilon[\omega_{e,t}] = \varepsilon^2 \Delta \omega_{e,t} - \mu(x)\omega_{e,t} + T[\omega_{e,t}]\omega^2_{e,t} \quad \text{(by (4.6))}
\]

\[
= \sum_{i=1}^{N} \xi_i^{-1} (\varepsilon^2 \Delta \omega_i - \mu(x)\omega_i) + T[\omega_{e,t}]\omega^2_{e,t} \quad \text{(by (4.4))}
\]
\[ = \sum_{i=1}^{N} \xi_i^{-1}(\mu(t_i) - \mu(x))\tilde{\omega}_i + T[\omega_{\varepsilon,t}]\omega_{\varepsilon,t}^2 - \sum_{i=1}^{N} \xi_i^{-1}\tilde{\omega}_i^2 + \text{e.s.t.} \]

\[ = E_1 + E_2 + \text{e.s.t.}, \]

where

\[ E_1 = \sum_{i=1}^{N} \xi_i^{-1}(\mu(t_i) - \mu(x))\tilde{\omega}_i \quad (4.14) \]

and

\[ E_2 = T[\omega_{\varepsilon,t}]\omega_{\varepsilon,t}^2 - \sum_{i=1}^{N} \xi_i^{-1}\tilde{\omega}_i^2. \quad (4.15) \]

We first estimate \( E_1 \):

\[ E_1 = -\sum_{i=1}^{N} \left\{ \mu'(t_i)(x-t_i) + \frac{1}{2} \mu''(t_i)(x-t_i)^2 + o(|x-t_i|^2) \right\} \xi_i^{-1}\omega_i \chi(x-t_i). \quad (4.16) \]

Hence

\[ ||E_1||_{L^2(-\frac{1}{\varepsilon}, \frac{1}{\varepsilon})} = O(\varepsilon). \quad (4.17) \]

For \( E_2 \), we have

\[ E_2 = T[\omega_{\varepsilon,t}]\omega_{\varepsilon,t}^2 - \sum_{i=1}^{N} \xi_i^{-1}\tilde{\omega}_i^2 \]

\[ = \sum_{i=1}^{N} (T[\omega_{\varepsilon,t}](x) - T[\omega_{\varepsilon,t}](t_i))\xi_i^{-2}\tilde{\omega}_i^2 \]

\[ + \sum_{i=1}^{N} (T[\omega_{\varepsilon,t}](t_i)\xi_i^{-1} - 1)\xi_i^{-1}\tilde{\omega}_i^2 \]

\[ + \sum_{i=1}^{N} \xi_i^{-2}b\xi_i\tilde{\omega}_i^2 \left\{ \xi_i^{-1}p_i(y) + 6y[\xi_i^{-1}\mu_i^{3/2}\nabla_x HD(x,t_i)|_{x=x_i}, \right\} \]

\[ + \sum_{j\neq i}^{N} \xi_j^{-1}\mu_j^{3/2}\nabla_x G_D(x,t_j)|_{x=x_i} \} + O(\varepsilon^2 \sum_{j=1}^{N} \tilde{\omega}_j^2). \quad (4.18) \]

This implies that

\[ ||E_2||_{L^2(-\frac{1}{\varepsilon}, \frac{1}{\varepsilon})} = O(\varepsilon). \quad (4.19) \]

Combining (4.17) and (4.19), we have

\[ ||S_{\varepsilon}[\omega_{\varepsilon,t}]||_{L^2(-\frac{1}{\varepsilon}, \frac{1}{\varepsilon})} = O(\varepsilon). \quad (4.20) \]

The estimates derived in this section will enable us to carry out the existence proof in the next two sections.

5. The Liapunov-Schmidt reduction method. In this section, we use Liapunov-Schmidt method to solve the problem

\[ S_{\varepsilon}[\omega_{\varepsilon,t} + \phi] = \sum_{i=1}^{N} \alpha_i \frac{d\tilde{\omega}_i}{dx} \quad (5.1) \]
for real constants $\alpha_i$ and some function $\phi \in H^2(-\frac{1}{\varepsilon}, \frac{1}{\varepsilon})$ that is small in the corresponding normal, where $\omega_{\varepsilon, t}$ is given by (4.6) and $\omega$ by (4.3).

First, we need to study the following linear operator
$$\hat{L}_{\varepsilon, t} : H^2(\Omega_{\varepsilon}) \to L^2(\Omega_{\varepsilon})$$
defined by
$$\hat{L}_{\varepsilon, t}\phi := S_{\varepsilon}[\omega_{\varepsilon, t}]\phi = \varepsilon^2 \Delta \phi - \mu(\phi) + 2\omega_{\varepsilon, t}T(\omega_{\varepsilon, t})\phi + \varepsilon^2 T'([\omega_{\varepsilon, t}]\phi), \quad (5.2)$$
where $\Omega_{\varepsilon} = (-\frac{1}{\varepsilon}, \frac{1}{\varepsilon})$, $T[\omega_{\varepsilon, t}]$ is given by (4.9), and for given $\phi \in L^2(\Omega)$ the function $T'[\omega_{\varepsilon, t}]\phi$ is defined as the uniqueness solution of
$$\begin{cases}
D\Delta(T'([\omega_{\varepsilon, t}]\phi) - \frac{\partial}{\partial \varepsilon}(T'([\omega_{\varepsilon, t}]\phi))\omega_{\varepsilon, t}^2 - 2\varepsilon\omega_{\varepsilon, t}T[\omega_{\varepsilon, t}]\phi = 0 & \text{in } \Omega, \\
(T'([\omega_{\varepsilon, t}]\phi)(1) = (T'([\omega_{\varepsilon, t}]\phi)(1) = 0.)
\end{cases} \quad (5.3)$$

We define the approximate kernel and co-kernel of the operator $\hat{L}_{\varepsilon, t}$, respectively, as follows:
$$K_{\varepsilon, t} := \text{span}\{\frac{d\tilde{\omega}_i}{dx} | i = 1, \ldots, N\} \subset H^2(\Omega_{\varepsilon}),$$
and
$$C_{\varepsilon, t} := \text{span}\{\frac{d\tilde{\omega}_i}{dx} | i = 1, \ldots, N\} \subset L^2(\Omega_{\varepsilon}).$$

Recall the definition of the following system of linear operators from (3.2):
$$L\Phi := \Delta \Phi - \Phi + 2\omega\Phi - 2\left(\int_R\omega B\Phi\right)\left(\int_R\omega^2\right)^{-1}\omega^2, \quad (5.4)$$
where
$$\Phi = \begin{pmatrix}
\phi_1 \\
\phi_2 \\
\vdots \\
\phi_N
\end{pmatrix} \in (H^2(R))^N.$$
prove our main existence result, Theorem 2.5. The reduction problem.

Lemma 5.2. There exist \( \epsilon, \delta \) such that for any pair of \( \epsilon, t \) with \( 0 < \epsilon < \epsilon \) and \( t \in B_\epsilon(t^0) \), \( |1 + t_1| + |1 - t_N| + \min_{i \neq j} |t_i - t_j| > \delta \) there is a unique \( \phi_{\epsilon,t} \in K^{-1}_{\epsilon,t} \) satisfying \( S_\epsilon(\omega_{\epsilon,t} + \phi_{\epsilon,t}) \in C_{\epsilon,t} \). Furthermore, we have the estimate

\[
||\phi_{\epsilon,t}||_{H^2(\Omega)} \leq C \epsilon.
\]

6. The reduction problem. In this section, we solve the reduced problem and prove our main existence result, Theorem 2.5.

By lemma 5.2, for any \( t \in B_\epsilon(t^0) \), there exists a unique solution \( \phi_{\epsilon,t} \in K^{-1}_{\epsilon,t} \) such that

\[
S_{\epsilon,t}[\omega_{\epsilon,t} + \phi_{\epsilon,t}] = v_{\epsilon,t} \in C_{\epsilon,t}.
\]

Our ideal is to find \( t^\epsilon = (t_1^\epsilon, \ldots, t_N^\epsilon) \in B_\epsilon(t^0) \) such that also

\[
S_{\epsilon,t}[\omega_{\epsilon,t} + \phi_{\epsilon,t}] \perp C_{\epsilon,t}.
\]

Therefore, \( S_{\epsilon,t}[\omega_{\epsilon,t} + \phi_{\epsilon,t}] = 0 \).

To this end, we define

\[
W_{\epsilon,i}(t) := \epsilon^{-1} \int_{\Omega} S_{\epsilon,t}[\omega_{\epsilon,t} + \phi_{\epsilon,t}] \frac{d\hat{\omega}_i}{dx} \, dx,
\]

\[
W_\epsilon(t) := (W_{\epsilon,1}(t), \ldots, W_{\epsilon,N}(t)) : B_\epsilon(t^0) \to R^N.
\]

Then \( W_\epsilon(t) \) is a map which is continuous of \( t \) and our problem is reduced to find a zero of the vector field \( W_\epsilon(t) \).

Let us now calculate \( W_\epsilon(t) \). By (4.16) and (4.18), we have

\[
\epsilon^{-1} \int_{\Omega} S_{\epsilon,t}[\omega_{\epsilon,t} + \phi_{\epsilon,t}] \frac{d\hat{\omega}_i}{dx} \, dx = \epsilon^{-1} \int_{\Omega} S_{\epsilon,t}[\omega_{\epsilon,t}] \frac{d\hat{\omega}_i}{dx} \, dx + \epsilon^{-1} \int_{\Omega} N_{\epsilon,t}[\phi_{\epsilon,t}] \frac{d\hat{\omega}_i}{dx} \, dx
\]

\[
= I_1 + I_2 + I_3
\]

where \( I_1, I_2 \) and \( I_3 \) are defined by the last equality.
The computation of $I_3$ is as follows: note that by Taylor’s expansion for (5.8), the first term in the expansion of $N_{x,t}$ is quadratic in $\phi_{x,t}$. So
\[ I_3 = O(\varepsilon). \] (6.3)

We now compute $I_1$ and $I_2$.

For $I_1$, we have
\[ I_1 = \int \frac{d\omega_i}{dx} dx + O(\varepsilon) = I_{11} + I_{12} + O(\varepsilon), \]
where $E_1$ and $E_2$ have been given by (4.14) and (4.15), respectively. Using (4.16), we obtain
\[ I_{11} = -\mu'(t_i)\xi^{-1}_i \int y \omega_i(y)dy + O(\varepsilon) = \mu'(t_i)\xi^{-1}_i \int (\omega_i(y))dy + O(\varepsilon) = 3\mu'(t_i)\xi^{-1}_i \mu_i^{3/2} + O(\varepsilon) \quad \text{(by 2.5)}. \]

Next, we calculate $I_{12}$
\[ I_{12} = -\varepsilon^{-1} \int \frac{d\omega_i}{dx} dx \]
\[ = -\varepsilon^{-1} \int R \left\{ \sum_{k=1}^N b_k \xi^2_k \omega^2_k \xi^{-1}_i p_k(y) + 6y(\xi^{-1}_i \mu^{3/2}_k \nabla_x H_D(x,t_k)|_{x=t_k} \right. \]
\[ + \sum_{j \neq i} \xi^{-1}_j \mu^{3/2}_j \nabla_x G_D(x,t_j)|_{x=t_k} \right\} \omega_i dy + O(\varepsilon) \quad \text{(By 4.18)} \]
\[ = -\frac{b}{3} \xi^{-2}_i \mu^{5/2}_i \int R \omega^3(y)dy \int R \omega^2(y)dy \left\{ \sum_{j \neq i} \xi^{-1}_j \mu^{3/2}_j \nabla_x H_D(x,t_j)|_{x=t_i} \right\} + O(\varepsilon) \quad \text{(using (2.5) and (4.3))} \]
\[ = -14.4b\mu^{5/2}_i \xi^{-2}_i \sum_{k=1}^N \left\{ \xi^{-1}_k \mu^{3/2}_k \nabla_x G_D(x,t_k)|_{x=t_i} (1 - \delta_{ik}) \right\} + O(\varepsilon), \]

since $P_i(y)$ is an even function.

For $I_2$, by (4.12), (4.9) and (4.4), using the following results
\[ |\mu(t_i) - \mu(x)| = O(\varepsilon |y|), \]
\[ ||\phi_{x,t}||_{H^2(\Omega)} = O(\varepsilon), \]
\[ |T'[\omega_{x,t}](\phi_{x,t})(t_i)| = O(\varepsilon), \]
\[ |T'[\omega_{x,t}](\phi_{x,t})(\varepsilon y + t_i) - T'[\omega_{x,t}](\phi_{x,t})(t_i)| = O(\varepsilon^2 |y|), \]
one has
\[ I_2 = \varepsilon^{-1} \int \Omega S'_{x,t}[\omega_{x,t}] \phi_{x,t} \frac{d\omega_i}{dx} dx \]
Therefore the vector field $W$ mapping degree of $\omega$ properties required in Theorem 2.5. The proof is finished.

**Proof of Theorem 2.5.** By proposition 6.1, there exists a $t$ for any $\varepsilon > 0$ and $\varepsilon > 0$, $\mu_2 > 0$. Thus for $t = 0$ we have $F(t) = O(\varepsilon)$ as $\varepsilon \to 0$.

Combining the estimates for $I_1, I_2$ and $I_3$, we have

$$W_{\varepsilon,i}(t) = \mu_i^{3/2} \left( c_i \mu_i'(t_i) + d_{ij} \nabla t_i H_1(t_i, t_i) + \sum_{j \neq i} d_{ij} \nabla t_i G_1(t_i, t_j) \right) + O(\varepsilon)$$

where

$$c_i = 3 \xi_i^{-1}, \quad d_{ij} = -14.4 \mu_i \xi_i^{-2} \xi_j^{3/2}.$$

Recall from (2.15) that

$$F(t) = (F_1(t), \ldots, F_N(t)),$$

thus

$$W_{\varepsilon,i}(t) = 2.4 \mu_i^{5/2} F_i(t) + O(\varepsilon), \quad i = 1, \ldots, N.$$

By assumption (H3), we have $F(t^0) = 0$ and

$$\det(D_{\varepsilon} F(t^0)) \neq 0.$$

Therefore the vector field $W_{\varepsilon}(t) = (W_{\varepsilon,1}(t), \ldots, W_{\varepsilon,N}(t))$ satisfies

$$W_{\varepsilon}(t) = D_{\varepsilon} F(t^0)(t - t^0) + O(\varepsilon).$$

Thus for $\varepsilon$ small enough $F(t)$ has exactly one zero in $B_{\varepsilon}(t^0)$ and we compute the mapping degree of $W_{\varepsilon,t}(t)$ for the set $B_{\varepsilon}(t^0)$ and the value 0 as follows:

$$\deg(W_{\varepsilon}, 0, B_{\varepsilon}(t^0)) = \text{sign} \det(D_{\varepsilon} F(t^0)) \neq 0.$$

Therefore, standard degree theory implies that, for $\varepsilon$ small enough, there exists a $t^\varepsilon \in B_{\varepsilon}(t^0)$ such that $t^\varepsilon \to t^0$ as $\varepsilon \to 0$.

Thus we have proved the following proposition.

**Proposition 6.1.** For any $\varepsilon$ sufficiently small there exists a point $t^\varepsilon \in B_{\varepsilon}(t^0)$ with $t^\varepsilon \to t^0$ such that $W_{\varepsilon}(t^\varepsilon) = 0$.

*Proof of Theorem 2.5.* By proposition 6.1, there exists a $t^\varepsilon \in B_{\varepsilon}(t^0)$ such that $t^\varepsilon \to t^0$ and $W_{\varepsilon,t} = 0$. In other words, $S_{\varepsilon} \omega_{\varepsilon,t} + \phi_{\varepsilon,t} = 0$. Let $\omega_{\varepsilon,t} = \omega_{\varepsilon,t} + \phi_{\varepsilon,t}$. By the maximum principle, $\omega_{\varepsilon,t} > 0$. Moreover, by construction, $\omega_{\varepsilon,t}$ has all the properties required in Theorem 2.5. The proof is finished. 

### 7. Stability analysis I: Large eigenvalue

In this section, we consider the large eigenvalues of the associated linearized eigenvalue problem.

Let $(u_\varepsilon, v_\varepsilon)$ be the $N$ peak solution constructed in previous section. We have

$$u_\varepsilon = \sum_{i=1}^{N} \xi_i^{-1} \tilde{\omega}_i + \phi_{\varepsilon,t^\varepsilon}, \quad v_\varepsilon(t^\varepsilon_i) = \xi_i, \quad i = 1, \ldots, N, \quad (7.1)$$
We linear (1.6) at \((u_\varepsilon, v_\varepsilon)\). The eigenvalue problem becomes
\[
\begin{cases}
\varepsilon^2 \phi''_\varepsilon - \mu(x) \phi_\varepsilon + 2u_\varepsilon v_\varepsilon \phi_\varepsilon + u^2_\varepsilon \psi_\varepsilon = \lambda_\varepsilon \phi_\varepsilon, \\
D\psi''_\varepsilon - b \psi_\varepsilon u^2_\varepsilon - 2b \psi_\varepsilon u_\varepsilon \phi_\varepsilon = \varepsilon \lambda_\varepsilon \psi_\varepsilon.
\end{cases}
\tag{7.2}
\]
Here \(\lambda_\varepsilon\) is some complex number and
\[
\phi'_\varepsilon(\pm 1) = \psi'_\varepsilon(\pm 1) = 0.
\]

We consider two cases: The large eigenvalue case with \(\lambda_\varepsilon \to \lambda_0 \neq 0\) and the small eigenvalue \(\lambda_\varepsilon \to 0\). The second case will be considered in the next section.

We now analysis the large eigenvalues there exists some small \(c > 0\) such that \(|\lambda_\varepsilon| \geq -c\) for \(\varepsilon\) sufficiently small. We are going looking for a condition under which \(\text{Re}(\lambda_\varepsilon) < 0\) for all eigenvalues \(\lambda_\varepsilon\) of (7.2) if \(\varepsilon\) is sufficiently small. If \(\text{Re}(\lambda_\varepsilon) < c\), then \(\lambda_\varepsilon\) is a stable large eigenvalue. Therefore for the rest of this section we assume that \(\text{Re}(\lambda_\varepsilon) \geq -c\) and study the stability properties of such eigenvalues.

Let us assume that
\[
||\phi_\varepsilon||_{H^2(\Omega)} = 1.
\tag{7.3}
\]

We cut off \(\phi_\varepsilon\) as follows:
\[
\phi_{\varepsilon,j}(y) := \phi_\varepsilon(y) \chi(\varepsilon y - t^j_j), \quad j = 1, \ldots, N,
\tag{7.4}
\]
where \(\chi\) has been given by (4.2). Thus, we have
\[
\phi_\varepsilon(y) = \sum_{j=1}^{N} \phi_{\varepsilon,j}(y) + \text{e.s.t.} \quad \text{in } H^2(\Omega).
\tag{7.5}
\]

Then by the standard procedure (for example see [4]), we extend \(\phi_{\varepsilon,j}\) to a function defined in \(R\) such that
\[
||\phi_{\varepsilon,j}||_{H^2(R)} \leq C ||\phi_{\varepsilon,j}||_{H^2(\Omega)}, \quad j = 1, \ldots, N.
\tag{7.6}
\]

Since \(||\phi_{\varepsilon,j}||_{H^2(\Omega)} = 1, ||\phi_{\varepsilon,j}||_{H^2(R)} \leq C.\)

By taking subsequence of \(\varepsilon\), we may assume that \(\phi_{\varepsilon,j} \to \phi_j\).

Now, using (7.1) and the equation of \(\psi_\varepsilon\), we have as \(\varepsilon \to 0\), \(\psi_\varepsilon \to \psi_0\), and \(\psi_0\) satisfies
\[
D\psi''_0 - b \sum_{j=1}^{N} \xi_j^{-2} \int_R \omega_j^2(x) dx \delta(y - t^j_j) \psi_0 - 2b \sum_{j=1}^{N} \int_R \omega_j \phi_j dy = 0.
\tag{7.7}
\]

Sending \(\varepsilon \to 0\) with \(\lambda_\varepsilon \to \lambda_0\), the equation of \(\psi_\varepsilon\) (in (7.2)) for \(x \in B_\varepsilon(t^0_j)\) can be written as:
\[
\Delta \phi_i - \mu \phi_i + 2\omega_i \phi_i + \psi_0(t^0_i) \hat{\omega}_i^2 = \lambda_0 \phi_i.
\tag{7.8}
\]

We first need to solve \(\psi_0\). In Appendix (9.34), we shall show the following relations hold:
\[
(D + C)\Psi = \eta,
\tag{7.9}
\]
where \(\Psi\) and \(\eta\) are defined by
\[
\Psi = \begin{pmatrix} \psi_0(t^0_1) \\ \vdots \\ \psi_0(t^0_N) \end{pmatrix}, \quad \eta = \begin{pmatrix} \eta_1 \\ \vdots \\ \eta_N \end{pmatrix}, \quad \eta_j = -\frac{2b}{D} \int_R \omega_j \phi_j dy, \quad j = 1, \ldots, N,
\tag{7.10}
\]

\(D, C\) have been given by (2.10) (2.11), respectively.

Since \(D + C\) is invertible, We obtain that
\[
\Psi = (D + C)^{-1}\eta.
\tag{7.11}
Substituting (7.11) into (7.8) and using the transformation $\tilde{y}_j = \sqrt{\mu_j} y_j$, this implies that (after dropping the tilde)

$$L\Phi = \Delta \Phi - \Phi + 2\omega \Phi - 2\sigma \left( \int_R \omega \Phi dy \right) \left( \int R \omega^2 dy \right)^{-1} \omega^2 = \lambda_0 \Phi,$$

(7.12)

where

$$\Phi = \begin{pmatrix} \phi_1 \\ \vdots \\ \phi_N \end{pmatrix} \in (H^2(R))^N \text{ and } B = D(D + C)^{-1}.$$

Then we have the following theorem:

**Theorem 7.1.** Let $\lambda_\varepsilon$ be an eigenvalue of (7.2) such that $Re(\lambda_\varepsilon) > -c$ for some $c > 0$.

1. Suppose that (for suitable sequences $\varepsilon_n \to 0$) we have $\lambda_{\varepsilon_n} \to \lambda_0 \neq 0$. Then $\lambda_0$ is an eigenvalue of the problem given by in (7.12).

2. Let $\lambda_0 \neq 0$ with $Re(\lambda_0) > 0$ be an eigenvalue of the problem given in (7.12). Then for $\varepsilon$ sufficiently small, there is an eigenvalue $\lambda_\varepsilon$ of (7.2) with $\lambda_\varepsilon \to \lambda_0$ as $\varepsilon \to 0$.

This proof is similar to Theorem 7.1 [21], more details can see Theorem 7.1 [21] or Theorem 8.1 [20].

Now, we study the stability of (7.2) for large eigenvalues explicitly and prove (2.25) and (2.27) of Theorem 2.6.

Let $\sigma_i, i = 1, \ldots, N$ be the eigenvalues of the matrix $B$, These eigenvalues are real. Then the system (7.12) can be re-written as

$$L\phi_i = \Delta \phi_i - \phi_i + 2\omega \phi_i - 2\sigma_i \left( \int R \omega \phi_i dy \right) \left( \int R \omega^2 dy \right)^{-1} \omega^2 = \lambda_0 \phi_i \quad i = 1, \ldots, N,$$

(7.13)

where

$$\phi_i \in H^2(R) \quad i = 1, \ldots, N.$$

Suppose that we have

$$\min_{\sigma \in \sigma(B)} \sigma < \frac{1}{2},$$

(7.14)

by Theorem 3.1(1), there exists a positive eigenvalue of (7.13) and also of (7.12).

By Theorem 7.1(2), for $\varepsilon$ sufficiently small, there exists an eigenvalue $\lambda_\varepsilon$ of (7.2) such that $Re(\lambda_\varepsilon) > c_0$ for some positive number $c_0 > 0$. This implies that $u_\varepsilon = \omega_{\varepsilon,t} + \phi_{\varepsilon,t}$ is (linearly) unstable.

Suppose now that

$$\min_{\sigma \in \sigma(B)} \sigma > \frac{1}{2},$$

(7.15)

is satisfied, then by Theorem 3.1(2), we know that for any nonzero eigenvalue $\lambda_0$ in (7.13) and so also in (7.12), we have

$$Re(\lambda_0) \leq c_0 < 0 \quad \text{for some } c_0 > 0.$$  

So by Theorem 7.1(1), for $\varepsilon$ sufficiently small, all nonzero large eigenvalues of (7.2) all have strictly negative real parts. We conclude that in this case all eigenvalues $\lambda_\varepsilon$ of (7.2), for which $|\lambda_\varepsilon| \geq c > 0$ holds, satisfy $Re(\lambda_\varepsilon) \leq -c < 0$ for $\varepsilon$ sufficiently small. This implies that $u_\varepsilon = \omega_{\varepsilon,t} + \phi_{\varepsilon,t}$ is (linearly) stable. $\square$
In conclusion, we have finished the study of large eigenvalues and derived results on their stability properties. It remains to study small eigenvalues which will be done in the next section.

8. **Stability analysis II: Small eigenvalue.** Now we study (7.2) for small eigenvalue. Namely, we assume that $\lambda_\varepsilon \to 0$ as $\varepsilon \to 0$.

Let

$$\tilde{\omega}_\varepsilon = \omega_{\varepsilon,t^*} + \phi_{\varepsilon,t^*}, \quad \tilde{H}_\varepsilon = T[\omega_{\varepsilon,t^*} + \phi_{\varepsilon,t^*}],$$

where $t^* = (t_1^*, \ldots, t_N^*)$.

Recall the eigenvalue problem (7.2):

$$\begin{cases}
\Delta \varepsilon^2 \phi_\varepsilon - \mu(x) \phi_\varepsilon + 2\tilde{\omega}_\varepsilon \tilde{H}_\varepsilon \phi_\varepsilon + \tilde{\omega}_\varepsilon^2 \psi_\varepsilon = \lambda_\varepsilon \phi_\varepsilon, \\
D \Delta \psi_\varepsilon - \frac{b}{\varepsilon} \psi_\varepsilon \tilde{\omega}_\varepsilon^2 - \frac{2b}{\varepsilon} \tilde{H}_\varepsilon \tilde{\omega}_\varepsilon \phi_\varepsilon = \varepsilon \lambda_\varepsilon \psi_\varepsilon.
\end{cases}$$

(8.2)

Our basic ideal is the following: the eigenfunction $\phi_\varepsilon$ can be expanded as

$$\sum_{j=1}^N \alpha_j \frac{\partial \omega_{\varepsilon,t}}{\partial t_j}.$$  

(8.1)

Note that $\omega_{\varepsilon,t} \sim \sum_{i=1}^N \varepsilon^{-1}_i \omega_i(x)$. So when we differentiate $\omega_{\varepsilon,t}$ with respect to $t_j$, we also need to differentiate $\xi_j$ and $\mu(t_j)$ with respect to $t_j$. Therefore, we need to expand $\phi_\varepsilon$ up to $O(\varepsilon^2)$.

Let us define

$$\tilde{\omega}_{\varepsilon,j}(x) = \tilde{\omega}_\varepsilon \chi(x - t_j^*), \quad j = 1, \ldots, N,$$

(8.3)

where $\chi(x)$ was defined by (4.2). Similarly as in section 5, we define

$$\mathcal{K}_{\varepsilon,t^*}^{\text{new}} := \text{span}\{\tilde{\omega}_{\varepsilon,j}^* \mid j = 1, \ldots, N\} \subset H^2(\Omega_\varepsilon),$$

$$\mathcal{C}_{\varepsilon,t^*}^{\text{new}} := \text{span}\{\tilde{\omega}_{\varepsilon,j}^* \mid j = 1, \ldots, N\} \subset L^2(\Omega_\varepsilon).$$

Then it is easy to see that

$$\tilde{\omega}_\varepsilon(x) = \sum_{j=1}^N \tilde{\omega}_{\varepsilon,j}(x) + e.s.t. \quad \text{in } H^2(\Omega).$$

(8.4)

Note that

$$\tilde{\omega}_{\varepsilon,j}(x) = \xi_j^{-1} \omega_j \left(\frac{x - t_j^*}{\varepsilon}\right) \quad \text{in } H^2_{\text{loc}}(\Omega),$$

and $\tilde{\omega}_{\varepsilon,j}$ satisfies

$$\varepsilon^2 \Delta \tilde{\omega}_{\varepsilon,j} - \mu(x) \tilde{\omega}_{\varepsilon,j} + \tilde{H}_\varepsilon \tilde{\omega}_{\varepsilon,j}^2 + e.s.t. = 0.$$ 

Thus $\tilde{\omega}_{\varepsilon,j}^\prime := \frac{d \tilde{\omega}_{\varepsilon,j}}{dx}$ satisfies

$$\varepsilon^2 \Delta \tilde{\omega}_{\varepsilon,j}^\prime - \mu(x) \tilde{\omega}_{\varepsilon,j}^\prime + \tilde{H}_\varepsilon \tilde{\omega}_{\varepsilon,j}^2 + \tilde{\omega}_{\varepsilon,j}^2 \tilde{H}_\varepsilon^\prime - \mu(x) \tilde{\omega}_{\varepsilon,j}^\prime + e.s.t. = 0.$$ 

(8.5)

Let us decompose

$$\phi_\varepsilon = \varepsilon \sum_{j=1}^N a_j^\varepsilon \tilde{\omega}_{\varepsilon,j} + \phi_\varepsilon^\perp$$

(8.6)

with complex numbers $a_j^\varepsilon$, (the factor $\varepsilon$ is for scaling), where $\phi_\varepsilon^\perp \perp \mathcal{K}_{\varepsilon,t^*}^{\text{new}}$.

Suppose that

$$\|\phi_\varepsilon\|_{H^2(\Omega_\varepsilon)} = 1.$$ 

Since $\|\varepsilon \tilde{\omega}_{\varepsilon,j}\|_{H^2(\Omega_\varepsilon)} \geq C > 0$, $|a_j^\varepsilon| \leq C$. 

Similarly, we can decompose
\[ \psi_{\varepsilon,j} = \varepsilon \sum_{j=1}^{N} a^\varepsilon_j \psi_{\varepsilon,j} + \psi_{\varepsilon,j}^\perp, \]
where \( \psi_{\varepsilon,j} \) satisfies
\[ \begin{cases} \Delta \psi_{\varepsilon,j} - \frac{b}{\varepsilon} \psi_{\varepsilon,j} \bar{\omega}_\varepsilon^2 - 2 \frac{b}{\varepsilon} \bar{\omega}_\varepsilon \bar{\psi}_{\varepsilon,j} = \varepsilon \lambda_{\varepsilon,1} \\ \psi_{\varepsilon,j}(\pm1) = 0, \end{cases} \quad (8.7) \]
and \( \psi_{\varepsilon,j}^\perp \) satisfies
\[ \begin{cases} \Delta \psi_{\varepsilon,j}^\perp - \frac{b}{\varepsilon} \psi_{\varepsilon,j}^\perp \bar{\omega}_\varepsilon^2 - 2 \frac{b}{\varepsilon} \bar{\omega}_\varepsilon \phi_{\varepsilon,j}^\perp = \varepsilon \lambda_{\varepsilon,1} \\ (\psi_{\varepsilon,j}^\perp)'(\pm1) = 0. \end{cases} \quad (8.8) \]

Throughout this section, we denote
\[ \mu_j = \mu(t_j^\varepsilon), \quad \mu_j' = \mu'(t_j^\varepsilon), \quad \mu_j'' = \mu''(t_j^\varepsilon), \]
\[ \mu^{-1} := (\mu_j^{-1}), \quad \mu^{3/2} := (\mu_j^{3/2}), \quad \mu' := (\mu_j'), \quad \mu^{1/2} := (\mu_j^{1/2}). \]
\[ \mathcal{H}_1 := (\xi^{-1}), \quad \mathcal{H}_2 := (\xi^{-2}), \quad \mathcal{H}^{-1} := (\xi). \]
Substituting the decompositions of \( \phi_{\varepsilon} \) and \( \psi_{\varepsilon} \) into (8.2) we have, using (8.5)
\[ \varepsilon \sum_{j=1}^{N} a_j^\varepsilon [\bar{\omega}_\varepsilon^2 \psi_{\varepsilon,j} - (\bar{\omega}_{\varepsilon,j})^2 \bar{H}_\varepsilon] + \varepsilon \mu'(x) \sum_{j=1}^{N} a_j^\varepsilon \bar{\omega}_{\varepsilon,j} \]
\[ + \varepsilon^2 \Delta \phi_{\varepsilon,j} - \mu(x) \phi_{\varepsilon,j} + 2 \bar{H}_\varepsilon \omega_{\varepsilon,j} \phi_{\varepsilon,j} - \bar{\omega}_{\varepsilon,j} \phi_{\varepsilon,j} + e.s.t. = \lambda_{\varepsilon} \varepsilon \sum_{j=1}^{N} a_j^\varepsilon \bar{\omega}_{\varepsilon,j}. \]

Let us first compute
\[ \varepsilon \sum_{j=1}^{N} a_j^\varepsilon [\bar{\omega}_\varepsilon^2 \psi_{\varepsilon,j} - (\bar{\omega}_{\varepsilon,j})^2 \bar{H}_\varepsilon] \]
\[ = \varepsilon \sum_{j=1}^{N} a_j^\varepsilon [\bar{\omega}_{\varepsilon,j}]^2 [\psi_{\varepsilon,j} - \bar{H}_\varepsilon] + \varepsilon \sum_{j=1}^{N} a_j^\varepsilon \psi_{\varepsilon,j} \sum_{k \neq j} (\bar{\omega}_{\varepsilon,k})^2 + e.s.t. \]
\[ = \varepsilon \sum_{j=1}^{N} a_j^\varepsilon [\bar{\omega}_{\varepsilon,j}]^2 [\psi_{\varepsilon,j} - \bar{H}_\varepsilon] + \varepsilon \sum_{j=1}^{N} \sum_{k \neq j} a_j^\varepsilon \psi_{\varepsilon,j} (\bar{\omega}_{\varepsilon,j})^2 + e.s.t. \]
\[ = \varepsilon \sum_{j=1}^{N} \sum_{k=1}^{N} a_j^\varepsilon [\bar{\omega}_{\varepsilon,j}]^2 [\psi_{\varepsilon,j} - \delta_{jk} \bar{H}_\varepsilon] + e.s.t. \quad (8.11) \]

Let us also put
\[ \bar{L}_{\varepsilon} \phi := \varepsilon^2 \Delta \phi_{\varepsilon} - \mu(x) \phi_{\varepsilon} + 2 \bar{H}_\varepsilon \omega_{\varepsilon} \phi_{\varepsilon} + \omega_{\varepsilon}^2 \phi_{\varepsilon} \]
\[ + \sum_{j=1}^{N} \int_{-1}^{1} \bar{\omega}_{\varepsilon,j} \bar{\omega}_{\varepsilon,j}' dx \]
\[ = \sum_{j=1}^{N} \int_{-1}^{1} \bar{\omega}_{\varepsilon,j} \bar{\omega}_{\varepsilon,j}' dx \]
\[ = \varepsilon \lambda_{\varepsilon} \sum_{j=1}^{N} a_j^\varepsilon \int_{-1}^{1} \bar{\omega}_{\varepsilon,j} \bar{\omega}_{\varepsilon,j}' dx \]

Multiplying both sides of (8.10) by \( \bar{\omega}_{\varepsilon,j}' \) and integrating over \((-1,1)\), we have, using (2.5),
\[ \text{r.h.s.} = \varepsilon \lambda_{\varepsilon} \sum_{j=1}^{N} a_j^\varepsilon \int_{-1}^{1} \bar{\omega}_{\varepsilon,j} \bar{\omega}_{\varepsilon,j}' dx \]
\[ = \varepsilon \lambda_{\varepsilon} \sum_{j=1}^{N} a_j^\varepsilon \int_{-1}^{1} \bar{\omega}_{\varepsilon,j} \bar{\omega}_{\varepsilon,j}' dx \]
\[ = \varepsilon \lambda_{\varepsilon} \sum_{j=1}^{N} a_j^\varepsilon \int_{-1}^{1} \bar{\omega}_{\varepsilon,j} \bar{\omega}_{\varepsilon,j}' dx \]
Lemma 8.1.

and, using (8.11),

\[ I.h.s. = \varepsilon N \sum_{j=1}^{N} \sum_{k=1}^{N} a_k \int_{-1}^{1} (\hat{\omega}_{\varepsilon,j})^2 |\psi_{\varepsilon,k} - \delta_{jk} \hat{H}_j| \hat{\omega}'_{\varepsilon,l} dx + \varepsilon \sum_{j=1}^{N} a_j \int_{-1}^{1} \mu'(x) \hat{\omega}_{\varepsilon,j} \hat{\omega}'_{\varepsilon,l} dx \]

\[ + \int_{-1}^{1} \hat{\omega}^2_\varepsilon \psi_\varepsilon^j \hat{\omega}'_{\varepsilon,l} dx + \int_{-1}^{1} \mu'(x) \phi_\varepsilon^j \hat{\omega}_{\varepsilon,l} dx - \int_{-1}^{1} (\hat{\omega}_{\varepsilon,l})^2 \hat{H}'_j \phi_\varepsilon^j dx + o(\varepsilon^2) \]

\[ = J_{i,1} + J_{i,2} + J_{i,3} + J_{i,4} + J_{i,5} + o(\varepsilon^2). \]

where \( J_{i,l} \), \( l = 1, \ldots, 5 \) are defined by the last equality.

For \( J_{i,2} \), integrating by parts gives

\[ J_{i,2} = a_i \int_{-1}^{1} \mu'(x) \hat{\omega}_{\varepsilon,l} \hat{\omega}'_{\varepsilon,l} dx \]

\[ = -\frac{\varepsilon}{2} a_i \int_{-1}^{1} \nabla G \nabla \xi^l \hat{\omega}_{\varepsilon,l} \hat{\omega}'_{\varepsilon,l} dx + o(\varepsilon^2) \]

\[ = -\frac{\varepsilon^2}{2} a_i \xi^l - 2 \mu_{l} \mu_{l}^{3/2} \int_{-1}^{1} \omega^2 dx + o(\varepsilon^2). \]

For \( J_{i,3} \), we have

\[ J_{i,3} = \int_{-1}^{1} \hat{\omega}^2_\varepsilon \psi_\varepsilon^l \hat{\omega}'_{\varepsilon,l} dx \]

\[ = \int_{-1}^{1} \hat{\omega}^2_\varepsilon \psi_\varepsilon^l (t_1^l) \hat{\omega}'_{\varepsilon,l} dx + \int_{-1}^{1} \hat{\omega}^2_\varepsilon [\psi_\varepsilon^l (x) - \psi_\varepsilon^l (t_1^l)] \hat{\omega}'_{\varepsilon,l} dx \]

\[ = \int_{-1}^{1} \hat{\omega}^2_\varepsilon [\psi_\varepsilon^l (x) - \psi_\varepsilon^l (t_1^l)] \hat{\omega}'_{\varepsilon,l} dx. \]

We define the vectors

\[ J_i = (J_{i,1}, \ldots, J_{i,N})^T, \quad i = 1, \ldots, 5. \]

We have the following lemma:

**Lemma 8.1.**

\[ J_1 = -\varepsilon^2 \left( 2b \int_{R} \omega^3 \right) \mu^{5/2} H^2 Q a^0 + \varepsilon^2 \left( 2b \int_{R} \omega^3 \right) \mu^{5/2} H^2 \nabla \hat{G}_D \mu^{3/2} H^2 \]

\[ \times \left( I - \frac{1}{N} E - 6b \hat{G}_D \mu^{3/2} H^2 \right)^{-1} \left[ \left( I - \frac{1}{N} E + 6b \hat{G}_D \mu^{3/2} H^2 \right) \nabla \xi - 9b \hat{G}_D \mu^{1/2} \right] a^0 + O(\varepsilon^3), \]

\[ J_2 = -3\varepsilon^2 \mu'' \mu^{3/2} H^2 a^0 + O(\varepsilon^3), \]

\[ J_3 = -\varepsilon^2 \left( 4b \int_{R} \omega^3 \right) \mu^{5/2} H^2 \nabla \hat{G}_D \mu^{3/2} H^2 (I - \frac{1}{N} E - 6b \hat{G}_D \mu^{3/2} H^2)^{-1} \]

\[ \times \left( I - \frac{1}{N} E \right) \left\{ \nabla \xi - \frac{3}{4} \mu'^{-1} \right\} a^0 + O(\varepsilon^3), \]
\[ J_4 = \varepsilon^2 \mu \mu_1^{3/2} H^3 \left( \nabla \xi - \frac{3}{4} H^{-1} \mu_1^{-1} \right) a_0 + O(\varepsilon^3), \]
\[ J_5 = -9 \varepsilon^2 \mu \mu_1^{3/2} H^3 \left( \nabla \xi - \frac{5}{6} H^{-1} \mu_1^{-1} \right) a_0 + O(\varepsilon^3). \]

where \( G_D, Q \) and \( H \) are defined by (2.6), (2.20) and (2.14), respectively, \( a_0 \) is given by (8.13) and

\[ a_0 = \lim_{\varepsilon \to 0} \tilde{a}^\varepsilon. \]  

(8.16)

The proof of lemma 8.1 is delayed to the next section.

Proof of Theorem 2.6. By the previous lemma, we obtain

l.h.s. = \( J_1 + J_2 + J_3 + J_4 + J_5 \)

\[ = -\varepsilon^2 \left( 2b \int_R \omega_1 \mu \mu_1^{3/2} H^3 \nabla G_D \mu \mu_1^{3/2} H^2 \left( \nabla \xi - \frac{3}{2} H^{-1} \mu_1^{-1} \right) a_0 \right) \]
\[ -\varepsilon^2 \left( 2b \int_R \omega_1 \mu \mu_1^{3/2} H^3 Q a_0 - 3\varepsilon^2 \mu \mu_1^{3/2} H^3 \left( \nabla \xi - \frac{1}{2} H^{-1} \mu_1^{-1} \right) a_0 \right) \]
\[ -3\varepsilon^2 \mu \mu_3^{3/2} H^3 a_0 + O(\varepsilon^3). \]

Combining (8.14) and (2.21), we have

\[ \lambda \varepsilon^2 \mu_1^{5/2} H^2 a^\varepsilon \int_R (\omega_1(y)) dy(1 + O(\varepsilon)) = -2.4\varepsilon^3 \mu_1^{5/2} H^3 M(t^\varepsilon) a^\varepsilon + O(\varepsilon^3), \]

using (1.8). Above equation shows that the small eigenvalue \( \lambda_\varepsilon \) of (8.2) are

\[ \lambda_\varepsilon \sim -2\sigma(M(t^0)). \]

Arguing as in Theorem 7.1, this shows that if all the eigenvalues of \( M(t^0) \) are positive, then the small eigenvalues are stable. On other hand, if \( M(t^0) \) has a negative eigenvalue, then we can construct eigenfunctions and eigenvalues to make the system unstable.

This proves Theorem 2.6. \( \square \)

9. Computation of the small eigenvalues II: Proof of Lemma 8.1. In this section, we prove lemma 8.1. First note that

\[ \varepsilon \sum_{j=1}^N \sum_{k=1}^N a_k^\varepsilon \int_{-1}^1 (\omega_{\varepsilon,j})^2 [\psi_{\varepsilon,k} - \delta_{jk} \tilde{H}^\varepsilon \omega'_{\varepsilon,j}] dz = \varepsilon \sum_{k=1}^N a_k^\varepsilon \int_{-1}^1 (\omega_{\varepsilon,k})^2 [\psi_{\varepsilon,k} - \delta_{jk} \tilde{H}^\varepsilon \omega'_{\varepsilon,k}] dz + o(\varepsilon) \]

(9.1)

So we need to study the asymptotic behavior of \( \psi_{\varepsilon,j} \) near \( t_\varepsilon \). Since \( \psi_{\varepsilon,j} \) satisfies (8.7), we have that

\[ \psi_{\varepsilon,j}(x) - \overline{\psi_{\varepsilon,j}} = \frac{b}{\varepsilon} \int_{-1}^1 G_D(x,z) [\psi_{\varepsilon,j} \omega_z^2 + 2\omega_z \tilde{H} \omega'_{\varepsilon,j}] dz + o(\varepsilon) \]

where \( \overline{\psi_{\varepsilon,j}} = \frac{1}{2} \int_{-1}^1 \psi_{\varepsilon,j} \).

Hence we have

\[ \psi_{\varepsilon,j}(t_\varepsilon^\varepsilon) - \overline{\psi_{\varepsilon,j}} = \frac{b}{\varepsilon} \int_{-1}^1 G_D(t_\varepsilon^\varepsilon,z) \psi_{\varepsilon,j} \omega_z^2 dz + 2\frac{b}{\varepsilon} \int_{-1}^1 G_D(t_\varepsilon^\varepsilon,z) \omega_z \tilde{H} \omega'_{\varepsilon,j} dz + o(\varepsilon) \]
\[ = I_1 + I_2 + o(\varepsilon) \]

(9.2)

where \( I_1, I_2 \) are defined by the last equality.
For $I_1$, using (8.4) and (1.10)

$$I_1 = \frac{b}{\varepsilon} \sum_{m=1}^{N} \int_{-1}^{1} G_D(t_k^\varepsilon, z) \hat{\omega}_{\varepsilon,m}^2 \psi_{\varepsilon,j}(t_{m}^\varepsilon) d\bar{z} + o(\varepsilon)$$

$$= b \sum_{m \neq k} \int_{R} G_D(t_k^\varepsilon, x_m + \varepsilon \bar{z}) \hat{\omega}_{\varepsilon,m}^2 (x_m + \varepsilon \bar{z}) \psi_{\varepsilon,j}(x_m + \varepsilon \bar{z}) d\bar{z}$$

$$+ b \int_{R} H_D(t_k^\varepsilon, x_m + \varepsilon \bar{z}) \hat{\omega}_{\varepsilon,m}^2 (x_m + \varepsilon \bar{z}) \psi_{\varepsilon,j}(x_m + \varepsilon \bar{z}) d\bar{z} + o(\varepsilon)$$

$$= b \sum_{m=1}^{N} G_D(t_k^\varepsilon, t_m^\varepsilon) \psi_{\varepsilon,j}(t_m^\varepsilon \mu_{m_k}^{3/2} \xi_k \int_{R} \omega^2 + O(\varepsilon).$$

For $I_2$, we have

$$I_2 = \frac{2b}{\varepsilon} \int_{-1}^{1} G_D(t_k^\varepsilon, z) \hat{\omega}_{\varepsilon,j} \hat{H}_{\varepsilon,j}^\prime d\bar{z}$$

$$= \frac{2b}{\varepsilon} \int_{-1}^{1} G_D(t_k^\varepsilon, z) \hat{\omega}_{\varepsilon,j} \hat{H}_{\varepsilon,j}^\prime d\bar{z} + O(\varepsilon) \quad \text{(integrating by parts)}$$

$$= -b \int_{-1}^{1} \nabla \epsilon G_D(t_k^\varepsilon, z) \hat{\omega}_{\varepsilon,j} \hat{H}_{\varepsilon,j}^\prime d\bar{z} - \frac{b}{\varepsilon} \int_{-1}^{1} G_D(t_k^\varepsilon, z) \hat{\omega}_{\varepsilon,j} \hat{H}_{\varepsilon,j}^\prime d\bar{z} + O(\varepsilon)$$

$$= -b \nabla \epsilon G_D(t_k^\varepsilon, t_j^\varepsilon) \frac{\mu_k^{3/2}}{\xi_k} \int_{R} \omega^2 - b G_D(t_k^\varepsilon, t_j^\varepsilon) \hat{H}_{\varepsilon,j}^\prime(t_k^\varepsilon) \frac{\mu_k^{3/2}}{\xi_k} \int_{R} \omega^2 + O(\varepsilon).$$

Hence, we obtain

$$\psi_{\varepsilon,j}(t_k^\varepsilon) - \psi_{\varepsilon,j} = b \sum_{m=1}^{N} G_D(t_k^\varepsilon, t_m^\varepsilon) \psi_{\varepsilon,j}(t_m^\varepsilon) \frac{\mu_m^{3/2}}{\xi_m} \int_{R} \omega^2 - b \nabla \epsilon G_D(t_k^\varepsilon, t_j^\varepsilon) \frac{\mu_k^{3/2}}{\xi_k} \int_{R} \omega^2$$

$$- b G_D(t_k^\varepsilon, t_j^\varepsilon) \hat{H}_{\varepsilon,j}^\prime(t_k^\varepsilon) \frac{\mu_k^{3/2}}{\xi_k} \int_{R} \omega^2 + O(\varepsilon)$$

(9.3)

On the other hand, integrating (8.7), we obtain

$$\sum_{m=1}^{N} \psi_{\varepsilon,j}(t_m^\varepsilon) \frac{\mu_m^{3/2}}{\xi_m} \int_{R} \omega^2 - H_j(t_j^\varepsilon) \frac{\mu_j^{3/2}}{\xi_j} \int_{R} \omega^2 = O(\varepsilon).$$

(9.4)

Note that by appendix (9.42), we have

$$\sum_{k=1}^{N} \nabla \epsilon G_D(t_k^0, t_j^0) = 0, \quad \sum_{k=1}^{N} G_D(t_k^0, t_j^0) = \lambda_1,$$

(9.5)

where $\lambda_1$ is a constant independent of $m$. By using (9.3),(9.4),(9.5) and the fact $t^\varepsilon \in B_\varepsilon(t^0)$, we have

$$\psi_{\varepsilon,j} = \frac{1}{N} \sum_{k=1}^{N} \psi_{\varepsilon,j}(t_k^\varepsilon) + O(\varepsilon).$$

(9.6)

Hence,

$$\Psi_\varepsilon = -6 b (I - \frac{1}{N} E - 6 b G_D \mu^{3/2} \mathcal{H}^2)^{-1} [(\nabla G_D)^\top \mu^{3/2} \mathcal{H} + G_D \hat{H}^\prime \mu^{3/2} \mathcal{H}^2] + O(\varepsilon)$$

(9.7)
where $\bar{H}'$ and $\Phi_\varepsilon$ are given by the following:

$$\bar{H}' := (\bar{H}'_ε(t'_k) \delta_{ij}), \quad \Psi_\varepsilon := (\Psi_{\varepsilon,1}, \ldots, \Psi_{\varepsilon,N}), \quad \Psi_{\varepsilon,j} := \begin{pmatrix} \psi_{\varepsilon,j}(t'_{k}) \\ \vdots \\ \psi_{\varepsilon,j}(t'_N) \end{pmatrix}. \quad (9.8)$$

From (9.1), we also see that for $j \neq k$

$$\psi_{\varepsilon,j}(t'_k + \varepsilon y) - \psi_{\varepsilon,j}(t'_k) = \frac{b}{\varepsilon} \int_{-1}^{1} \left| G_D(t'_k + \varepsilon y, z) - G_D(t'_k, z) \right| \psi_{\varepsilon,j} \omega^2_d dz + \frac{2b}{\varepsilon} \times \int_{-1}^{1} \left[ G_D(t'_k + \varepsilon y, z) - G_D(t'_k, z) \right] \tilde{\omega}_z \tilde{H}'_ε \tilde{\omega}'_z dz + O(\varepsilon^2)$$

$$= I_{11} + I_{12} + O(\varepsilon^2),$$

where $I_{11}$ and $I_{12}$ are defined by the last equality.

For $I_{11}$, using (8.4) and (1.10)

$$I_{11} = \frac{b}{\varepsilon} \sum_{m \neq k} \int_{-1}^{1} \left| G_D(t'_k + \varepsilon y, z) - G_D(t'_k, z) \right| \psi_{\varepsilon,j} \omega^2_{d,m} dz$$

$$+ \frac{b}{\varepsilon} \int_{-1}^{1} \frac{\left| t'_k + \varepsilon y - z \right| - \left| t'_k - z \right|}{2D} \psi_{\varepsilon,j} \omega^2_{d,k} dz$$

$$+ \frac{b}{\varepsilon} \int_{-1}^{1} \left[ H_D(t'_k + \varepsilon y, z) - H_D(t'_k, z) \right] \psi_{\varepsilon,j} \omega^2_{d,k} dz + O(\varepsilon^2)$$

$$= 6b \sum_{m=1}^{N} \varepsilon y \nabla \varepsilon_{t'_j} G_D(t'_k, t'_m) \frac{\mu_{m2}^{3/2}}{\xi_m^2} \psi_{\varepsilon,j}(t'_m) + b \varepsilon \frac{\psi_{\varepsilon,j}(t'_k)}{\xi_k^2} \times \int_{R} \frac{|y - \tilde{z}| - |\tilde{z}|}{2D} \omega^2_k (\varepsilon \tilde{z} + t'_k) d\tilde{z} + O(\varepsilon^2 y).$$

For $I_{12}$, using $\nabla \varepsilon_{t'_j} G_D(t'_k, t'_j) = 0$, we can obtain

$$I_{12} = -6b \nabla \varepsilon_{t'_k} G_D(t'_k, t'_j) \tilde{H}'_ε(t'_k) \frac{\mu_{j2}^{3/2}}{\xi_j^2} \varepsilon y + O(\varepsilon^2 y).$$

Hence, for $j \neq k$

$$\psi_{\varepsilon,j}(x_k + \varepsilon y) - \psi_{\varepsilon,j}(x_k) = 6b \sum_{m=1}^{N} \varepsilon y \nabla \varepsilon_{t'_j} G_D(t'_k, t'_m) \frac{\mu_{m2}^{3/2}}{\xi_m^2} \psi_{\varepsilon,j}(t'_m)$$

$$+ b \varepsilon \frac{\psi_{\varepsilon,j}(t'_k)}{\xi_k^2} \times \int_{R} \frac{|y - \tilde{z}| - |\tilde{z}|}{2D} \omega^2_k (\varepsilon \tilde{z} + t'_k) d\tilde{z} + O(\varepsilon^2 y). \quad (9.9)$$

Similarly, for $j = k$, we can obtain

$$\psi_{\varepsilon,j}(x_k + \varepsilon y) - \psi_{\varepsilon,j}(x_k) = 6b \varepsilon y \sum_{m=1}^{N} \nabla \varepsilon_{t'_k} G_D(t'_k, t'_m) \frac{\mu_{m2}^{3/2}}{\xi_m^2} \psi_{\varepsilon,j}(t'_m) + b \varepsilon \frac{\psi_{\varepsilon,j}(t'_k)}{\xi_k^2} \times \int_{R} \frac{|y - \tilde{z}| - |\tilde{z}|}{2D} \omega^2_k (\varepsilon \tilde{z} + t'_k) d\tilde{z} - 6b \varepsilon y \frac{\mu_{k2}^{3/2}}{\xi_k^2} \tilde{H}'_ε(t'_k)$$
Using the Green’s function for $Du'$, we have

$$\bar{H}_t^x = \frac{b}{D\xi_k} \int_R \frac{1}{2D} |y - \bar{z}| (\omega_k \omega'_k)(\varepsilon \bar{z} + t_k^x)d\bar{z}$$

(9.10)

$\bar{H}_t^x$ satisfies

$$\begin{cases}
D\Delta \bar{H}_t^x - \frac{b}{\varepsilon} \omega_k^2 \bar{H}_t^x - 2 \frac{b}{\varepsilon} \omega'_k \omega_{z} \bar{H}_z = 0 & \text{in } (-1, 1), \\
\bar{H}_t^x(\pm 1) = 0.
\end{cases}$$

Using the Green’s function for $Du'$, $-1 < x < 1$, $u(1) = u(-1) = 0$ is

$$\bar{H}_t^x(x) = \int_{-1}^{1} \left[ \frac{1}{2D} |x - z| + \frac{1}{2D} (xz - 1) \right] \frac{b}{\varepsilon} \omega_k^2 \bar{H}_t^x + 2 \frac{b}{\varepsilon} \omega'_k \omega_{z} \bar{H}_z dz.$$  

Then it is easy to see that

$$\bar{H}_t^x(t_k^x + \varepsilon y) - \bar{H}_t^x(t_k^x) = \frac{b}{D\xi_k} \int_R |y - \bar{z}| (\omega_k \omega'_k)(\varepsilon \bar{z} + t_k^x)d\bar{z}$$

$$+ \frac{b}{\varepsilon} \omega'_k \int_R \frac{|y - \bar{z}| - |\bar{z}|}{2D} \omega_k^2 (\varepsilon \bar{z} + t_k^x) d\bar{z} - \frac{3b}{D} \frac{y}{2} \sum_{m=1}^{N} \frac{\mu_m^{3/2}}{\xi_m}$$

$$+ 2b \omega'_k \frac{b}{D} \int_R \frac{|y - \bar{z}| - |\bar{z}|}{2D} (\omega_k \omega'_k)(\varepsilon \bar{z} + t_k^x) d\bar{z} + O(\varepsilon^2 y),$$

(9.11)

and

$$\bar{H}_t^x(t_k^x) = 6b \sum_{m=1}^{N} \nabla t_k^x G_D(t_k^x, t_m^x) \frac{\mu_m^{3/2}}{\xi_m} + O(\varepsilon^2).$$

(9.12)

Combining (9.9), (9.10) and (9.11), we have

$$\psi_{\varepsilon,j} - \bar{H}_t^x(x) = \psi_{\varepsilon,j} - \bar{H}_t^x(t_k^x)$$

$$= 6b\varepsilon y \sum_{m=1}^{N} \nabla t_k^x G_D(t_k^x, t_m^x) \frac{\mu_m^{3/2}}{\xi_m} - \psi_{\varepsilon,j}(t_m^x) + \delta_{jk} \frac{3b}{D} \frac{y}{2} \sum_{m=1}^{N} \frac{\mu_m^{3/2}}{\xi_m}$$

$$- 6b \nabla t_k^x G_D(t_k^x, t_j^x) \bar{H}_t^x(t_j^x) \frac{\mu_j^{3/2}}{\xi_j^2} \varepsilon y - \delta_{jk} \frac{b}{D} \frac{y}{2} \sum_{m=1}^{N} \frac{\mu_m^{3/2}}{\xi_m}$$

$$+ 6b \nabla t_k^x G_D(t_k^x, t_j^x) \bar{H}_t^x(t_j^x) \frac{\mu_j^{3/2}}{\xi_j^2} \varepsilon y - \delta_{jk} \frac{b}{D} \frac{y}{2} \sum_{m=1}^{N} \frac{\mu_m^{3/2}}{\xi_m}$$

$$+ b \varepsilon \frac{\mu_j^{3/2}}{\xi_j^2} \int_R \frac{|y - \bar{z}| - |\bar{z}|}{2D} \omega_k^2 (\varepsilon \bar{z} + t_k^x) d\bar{z} + \delta_{jk} \frac{b}{D} \int_R \bar{H}_t^x(t_j^x)$$

$$\times \int_R |y - \bar{z}| (\omega_k \omega'_k)(\varepsilon \bar{z} + t_k^x) d\bar{z} + O(\varepsilon^2 y).$$  

(9.13)

Next we study the asymptotic expansion of $\phi_{\varepsilon,j}$. Let us denote

$$\phi_{\varepsilon,j}^1(x) = -\nabla t_j^x \sum_{i=1}^{N} \xi_i^{-1} \xi_i \omega_{z,i}$$

(9.14)

$$= \sum_{i=1}^{N} \xi_i^{-2} \omega_{z,i} \nabla t_j^x \xi_i - \xi_i^{-1} \mu_j' \omega_j(\mu_j x) + \frac{1}{2} \mu_j^{-1/2} \mu_j' \omega' \left( \sqrt{\mu_j x} \right)$$
\[ \phi_\epsilon^1 = \epsilon \sum_{j=1}^{N} a_j^\epsilon \phi_{\epsilon,j}^1. \]  

**Lemma 9.1.** For \( \epsilon \) sufficiently small, we have

\[ \| \phi_\epsilon^1 - \epsilon \phi_\epsilon^1 \|_{H^2(\Omega_\epsilon)} = O(\epsilon^2). \]  

**Proof.** As the first step in the proof of Lemma 9.1, we obtain a relation between \( \psi_\epsilon^1 \) and \( \phi_\epsilon^1 \). Note that similar to the proof of proposition 5.1, \( \tilde{L}_\epsilon \) is invertible from \((K_{new})^\perp\) and \((C_{new})^\perp\) with uniformly bounded invertible for \( \epsilon \) small enough. By (8.10), (8.11), (9.13) and the fact that \( \tilde{L}_\epsilon \) is invertible, we deduce that

\[ \| \phi_\epsilon^1 \|_{H^2(\Omega_\epsilon)} = O(\epsilon). \]  

Let us decompose

\[ \tilde{\phi}_{\epsilon,j}^1 = \frac{1}{\epsilon} \phi_{\epsilon,j}^1 \chi(x-t_j^\epsilon), \]

then

\[ \phi_\epsilon^1 = \epsilon \sum_{j=1}^{N} \tilde{\phi}_{\epsilon,j}^1 + O(\epsilon^2) \quad \text{in } H^2(\Omega_\epsilon). \]

Let us also define

\[ \tilde{\phi}_{\epsilon,j}^1(y) = \mu_j \phi_{\epsilon,j}^1(\sqrt{\mu_j}y). \]

Suppose that

\[ \phi_{\epsilon,j}^1 \to \phi_j \quad \text{in } H^1(\Omega_\epsilon). \]

Set

\[ \Phi_\epsilon = (\phi_{\epsilon,1}^1, \ldots, \phi_{\epsilon,N}^1)^T \quad \text{and } \Phi_0 = (\phi_1, \ldots, \phi_N)^T. \]

By the equation for \( \psi_\epsilon^1 \):

\[ \begin{cases} 
D \Delta \psi_\epsilon^1 - \frac{b}{2} \psi_\epsilon^1 \omega_\epsilon^2 - 2 \frac{b}{\epsilon} \omega_\epsilon \tilde{H}_\epsilon \phi_\epsilon^1 = \epsilon \lambda_\epsilon \psi_\epsilon^1, \\
(\psi_\epsilon^1)'(\pm 1) = 0.
\end{cases} \]

We see that

\[ \psi_\epsilon^1(t_k^\epsilon) - \overline{\psi_\epsilon^1} = \frac{b}{\epsilon} \int_{-1}^{1} G_D(t_k^\epsilon, z)[\psi_\epsilon^1 \omega_\epsilon^2 + \omega_\epsilon \tilde{H}_\epsilon \phi_\epsilon^1]dz \]

\[ = 6b \sum_{m=1}^{N} G_D(t_k^\epsilon, t_m^\epsilon) \frac{\mu_m^{3/2}}{\xi_m} \psi_\epsilon^1(t_m^\epsilon) + 2b \lambda_1 \epsilon \sum_{m=1}^{N} \int_{\Omega_\epsilon} \omega \phi_{\epsilon,m} + O(\epsilon^2). \]

Thus,

\[ \sum_{m=1}^{N} \psi_\epsilon^1(t_m^\epsilon) - N \overline{\psi_\epsilon^1} = \lambda_1 6b \sum_{m=1}^{N} \int_{\Omega_\epsilon} \omega \phi_{\epsilon,m} + O(\epsilon^2). \]  

(9.17)

For the equation of \( \psi_\epsilon^1 \), integrating over \((-1, 1), \)

\[ 6b \sum_{m=1}^{N} \frac{\mu_m^{3/2}}{\xi_m} \psi_\epsilon^1(t_m^\epsilon) + 2b \lambda_1 \epsilon \sum_{m=1}^{N} \int_{\Omega_\epsilon} \omega \phi_{\epsilon,m} = O(\epsilon^2). \]

(9.18)
Combining (9.17) and (9.18), we have

$$\psi_\perp^\perp = \frac{1}{N} \sum_{m=1}^{N} \psi_\perp^\perp (t_m^\perp) + o(\varepsilon).$$

Hence

$$\Psi_\perp^\perp = (\psi_\perp^\perp (t_1^\perp), \ldots, \psi_\perp^\perp (t_N^\perp))^T = 2b\varepsilon(I - \frac{1}{N}E - 6bG_D\mu^{3/2}\mathcal{H}^2)^{-1}G_D\mu^{3/2} \int \omega\Phi_0 + O(\varepsilon^2).$$

This relation between $\psi_\perp^\perp$ and $\phi_\perp^\perp$ which will be very important for the rest of the proof.

Now we substitute (9.19) into (8.10) and using (9.13), we have $\Phi_0$ satisfies

$$\Delta\Phi_0 - \Phi_0 + 2\omega\Phi_0 - 2\int \omega\beta_0\Phi_0 \omega^2 - \frac{5}{4}\mathcal{H}^{-1}\mu^{-1}\mu'\alpha^0\omega^2 + \mathcal{H}\mu^{-1}\mu'\alpha^0\omega$$

$$- 6b(I - \frac{1}{N}E - 6bG_D\mu^{3/2}\mathcal{H}^2)^{-1}(\nabla G_D)^T \mu^{3/2}\mathcal{H} + \frac{5}{4}G_D\mathcal{H}\mu^{1/2}\mu'\alpha^0\omega^2 = 0.$$

Where

$$\beta_0 = -6b\mathcal{H}^2(I - \frac{1}{N}E - 6bG_D\mu^{3/2}\mathcal{H}^2)^{-1}G_D\mu^{3/2}.$$

Recall $L_0\phi = \Delta\phi - \phi + 2\phi\omega$, using the relations

$$L_0^{-1}\omega^2 = \omega, \quad L_0^{-1}\omega = \frac{1}{2}y\omega' + \omega,$$

by (2.18), (2.19) we have

$$\Phi_0 = \mathcal{H}^2\nabla \xi^0\omega - \mathcal{H}\mu'\mu^{-1}\alpha^0\left(\frac{1}{2}y\omega' + \omega\right). \tag{9.20}$$

Now we compare $\phi_\perp$ with $\phi_\perp^\perp$. By definition

$$\phi_\perp^\perp = \varepsilon \sum_{j=1}^{N} \tilde{\phi}_{\perp, j}^\perp = \varepsilon \sum_{j=1}^{N} \alpha_j^\perp \xi_j^2\tilde{\omega}_{\varepsilon, j}\nabla t_j^\perp\xi_j - \varepsilon \sum_{j=1}^{N} \alpha_j^\perp \xi_j^{-1}[\mu_j'\omega_j(\mu_jx) + \frac{1}{2}\mu_j^{1/2}\mu_j'\omega'((\sqrt{\mu_j}x))]. \tag{9.21}$$

On the other hand

$$\phi_\perp = \varepsilon \sum_{j=1}^{N} \phi_{\varepsilon, j} + O(\varepsilon^2) = \varepsilon \sum_{j=1}^{N} \phi_j\left(\frac{x - t_j^\perp}{\varepsilon}\right) + O(\varepsilon^2). \tag{9.22}$$

Using (9.20) and comparing (9.22) and (9.21), we can obtain (9.16).

From Lemma 9.1 and (9.19), we have that

$$\Psi_\perp^\perp = (\psi_\perp^\perp (t_1^\perp), \ldots, \psi_\perp^\perp (t_N^\perp))^T$$

$$= 12b\varepsilon(I - \frac{1}{N}E - 6bG_D\mu^{3/2}\mathcal{H}^2)^{-1}G_D\mu^{3/2} \times [\mathcal{H}^2\nabla \xi - \frac{3}{4}\mu'\mathcal{H}^{-1}\mu^{-1}]\alpha^0 + O(\varepsilon^2). \tag{9.23}$$
Furthermore,
\[
\psi^\perp_\varepsilon(t_k^\varepsilon + \varepsilon y) - \psi^\perp_\varepsilon(t_k^\varepsilon) = 6b\varepsilon y \sum_{m=1}^{N} \nabla t_f G_D(t_k^\varepsilon, t_m^\varepsilon) \frac{3/2}{\xi_m^2} \psi^\perp_\varepsilon(t_m^\varepsilon) \\
+ 2b\varepsilon^2 y \frac{\xi}{N} \sum_{m=1}^{N} \nabla t_f G_D(t_k^\varepsilon, t_m^\varepsilon) \mu_m^{3/2} \int_{\Omega_\varepsilon} \omega\phi^\perp_{\varepsilon,m} + O(\varepsilon^3). \tag{9.24}
\]

According to (9.20), one has
\[
(\psi^\perp_\varepsilon(t_1^\varepsilon + \varepsilon y) - \psi^\perp_\varepsilon(t_1^\varepsilon), \ldots, \psi^\perp_\varepsilon(t_N^\varepsilon + \varepsilon y) - \psi^\perp_\varepsilon(t_N^\varepsilon)))^T \\
= 2b\varepsilon^2 y \nabla G_D \mu^{3/2} H^2 (I - \frac{1}{N} E - 6bG_D \mu^{3/2} H^2)^{-1} \\
\times (I - \frac{1}{N} E) H^{-2} (\int_R \omega \Phi_0) a^0 + O(\varepsilon^3). \tag{9.25}
\]

**Proof of lemma 8.1.** Substituting (9.13) into the computation of J_{1,l}, we obtain that
\[
J_{1,l} = \varepsilon \sum_{j=1}^{N} \sum_{k=1}^{N} a_{k_j} \int_{-1}^{1} (\tilde{\omega}_{\varepsilon,j})^2 [\psi_{\varepsilon,k} - \delta_{jk} \tilde{H}_\varepsilon'] \tilde{\omega}_{\varepsilon,l} \, dx \\
= \varepsilon^2 \sum_{k=1}^{N} a_{k} \int_R (\tilde{\omega}_{\varepsilon,l})^2 [\tilde{\omega}_{\varepsilon,l}] (t_f^\varepsilon + \varepsilon y) \left\{ [\psi_{\varepsilon,k} - \delta_{jk} \tilde{H}_\varepsilon'](t_f^\varepsilon + \varepsilon y) \\
- [\psi_{\varepsilon,k} - \delta_{jk} \tilde{H}_\varepsilon'](t_f^\varepsilon) \right\} \, dy \\
= - \varepsilon^2 \left( 2b \int_R \omega^3 \right) \frac{\mu_l^3}{\xi_l^2} \sum_{k=1}^{N} a_{k} \left\{ \sum_{m=1}^{N} \nabla t_f G_D(t_f^\varepsilon, t_m^\varepsilon) \frac{3/2}{\xi_m^2} \psi_{\varepsilon,j}(t_m^\varepsilon) \\
- \nabla t_f G_D(t_f^\varepsilon, t_k^\varepsilon) \frac{3/2}{\xi_k^2} \tilde{H}_\varepsilon'(t_f^\varepsilon) + \gamma \delta_{kl} + O(\varepsilon^3) \right\},
\]
where \( \gamma = \frac{1}{2D} \sum_{m=1}^{N} \frac{\mu_m^{3/2}}{\xi_m^2} \).

Therefore, combining (9.7) and (9.12), we have
\[
J_1 = - \varepsilon^2 \left( 2b \int_R \omega^3 \right) G_D \mu^{5/2} H^3 Q a^0 + \varepsilon^2 \left( 2b \int_R \omega^3 \right) \mu^{5/2} H^3 \nabla G_D \mu^{3/2} H^2 \\
\times (I - \frac{1}{N} E - 6bG_D \mu^{3/2} H^2)^{-1} (I - \frac{1}{N} E + 6bG_D \mu^{3/2} H^2) \nabla \xi \\
- 9bG_D \mu^{1/2} H a^0 + O(\varepsilon^3),
\]
where Q was given by (2.20).

By (8.15), we have
\[
J_2 = -3\varepsilon^2 \mu'' \mu^{3/2} H^3 a^0 + O(\varepsilon^3).
\]

By (9.24), we have
\[
J_{3,l} = \int_{-1}^{1} \tilde{\omega}_{\varepsilon,l}(x) \psi_{\varepsilon}^\perp(x) - \psi_{\varepsilon}^\perp(t_f^\varepsilon) \, dx
\]
\[
\begin{align*}
&= \frac{1}{\xi} \int_R (\omega_r^2 \omega_l^2) (x + t_i^j)(y + t_i^j) [\psi^+ (y + t_i^j) - \psi^+ (t_i^j)] dy \\
&= \frac{1}{\xi} \int_R (\omega_r^2 \omega_l^2) (x + t_i^j) dy \left\{ 6b \xi \sum_{m=1}^N \nabla \epsilon \phi_i G_D(t_k^m, t_m^k) \mu_m^{3/2} \psi^+ (t_m^k) \\
&\quad + 2b \xi^2 \sum_{m=1}^N \nabla \epsilon \phi_i G_D(t_k^m, t_m^k) \mu_m^{3/2} \int_{\Omega_x} \omega \psi^+ (t_m^k) \right\} + O(\varepsilon^3).
\end{align*}
\]
Hence, by (9.25), we have
\[
J_3 = - \varepsilon^2 \left( 4b \int_R \omega^3 \mu^{5/2} \nabla \phi_i G_D \mu^{3/2} \omega^2 (I - \frac{1}{N} E - 6b G_D \mu^{3/2} \omega^2)^{-1} (I - \frac{1}{N} E) \int_R (\phi^+ \omega^2) dy + O(\varepsilon^3).
\]
For \( J_{4,l} \) and \( J_{5,l} \),
\[
J_{4,l} = \int_{-1}^1 \mu'(x) \phi^+ \dot{\psi}_{x,l} dx
\]
\[
= \varepsilon^2 \mu'(t_i^j) \mu_l^{3/2} \int_R (\phi^+ \omega) (y) dy + O(\varepsilon^3),
\]
\[
J_{5,l} = - \int_{-1}^1 (\dot{\psi}_{x,l})^2 \dot{\psi}_l dx
\]
\[
= - \varepsilon^2 \int_R (\phi^+ \omega^2) dy + O(\varepsilon^3).
\]
By (9.20), we obtain
\[
J_4 = 6\varepsilon^2 \mu'^2 \mu^{3/2} \int_R (\phi^+ \omega^2) dy + O(\varepsilon^3),
\]
\[
J_5 = - 9\varepsilon^2 \mu'^2 \mu^{3/2} \int_R (\phi^+ \omega^2) dy + O(\varepsilon^3).
\]
This proof is completed. \(\square\)

Appendix. We first analyze problem (7.7) in this section. The problem (7.7) is equivalent to
\[
D \psi''_j = 0 \quad -1 < x < 1, \quad \psi_j(\pm 1) = 0; \quad [\psi_j] = 0; \quad (9.26)
\]
\[
[D \psi'_j] = \frac{6b \mu_j^{3/2}}{\xi_j^2} \psi_j(x_j) + 2b \int_R \omega \psi_j, \quad j = 1, \ldots, N. \quad (9.27)
\]
Here we use \([f]_j\) denotes the jump of \(f\) at \(x_j\).
On the one hand, by (9.26), for \(-1 < x < t_0^j\), \(\psi''_j = \psi'_j = 0\), we have
\[
\psi_j(x) = \psi_j(x_1) = \eta_1. \quad (9.28)
\]
Similarly, for \(t_i^{0-1} < x < t_i^0\), \(i = 2, \ldots, N,\)
\[
\psi_j(x) = \eta_{i-1} \frac{t_i^0 - x}{t_i^0 - t_i^{0-1}} + \eta_i \frac{x - t_i^{0-1}}{t_i^0 - t_i^{0-1}}.
\]
Hence
\[
\psi'_j(x) = \frac{N}{2} (\eta_i - \eta_{i-1}) \quad \text{for} \quad t_i^{0-1} < x < t_i^0. \quad (9.29)
\]
Finally, for $x_N < x < 1$, we have
\begin{equation}
\psi'_0(x) = 0, \quad \psi_0(x) = \eta_N. \tag{9.30}
\end{equation}

On the other hand, by (9.27) and (9.28), at $t^0_1$,
\begin{equation}
\frac{DN}{2} [\eta_2 - \eta_1] = \frac{6b\mu_1^{3/2}}{\xi^2_1} \eta_1 + 2b \int_R \omega_1 \phi_1. \tag{9.31}
\end{equation}

At $t^0_i$, $i = 2, \ldots, N - 1$, using (9.29), we have
\begin{equation}
\frac{DN}{2} [\eta_{i+1} - 2\eta_i + \eta_{i-1}] = \frac{6b\mu_i^{3/2}}{\xi^2_i} \eta_i + 2b \int_R \omega_i \phi_i. \tag{9.32}
\end{equation}

At $t^0_N$, using (9.30), we have
\begin{equation}
\frac{DN}{2} [0 - (\eta_N - \eta_{N-1})] = \frac{6b\mu_N^{3/2}}{\xi^2_N} \eta_N + 2b \int_R \omega_N \phi_N. \tag{9.33}
\end{equation}

From (9.31), (9.32) and (9.33), we obtain
\begin{equation}
(D + C)\Psi = \eta, \tag{9.34}
\end{equation}

where those matrices $D, C, \Psi$ and $\eta$ have been given by (7.10).

Next, let us study Green’s function $G_D(x, z)$:
\begin{equation}
\begin{cases}
DG''_D(x, z) + \frac{1}{2} - \delta_z = 0 & \text{in } (-1, 1), \\
\int_{-1}^1 G_D(x, z) dx = 0, & \\
G_D(-1, z) = G_D(1, z) = 0.
\end{cases}
\end{equation}

We easily calculate
\begin{equation}
G_D(x, z) = \begin{cases}
\frac{1}{2D} [\frac{1}{3} - \frac{(x+1)^2}{4} - \frac{(1-z)^2}{4}], & -1 < x \leq z, \\
\frac{1}{2D} [\frac{1}{3} - \frac{(z+1)^2}{4} - \frac{(1-x)^2}{4}], & z < x < 1.
\end{cases} \tag{9.35}
\end{equation}

We decompose
\begin{equation}
G_D(x, z) = \frac{1}{2D} |x - z| + H_D(x, z). \tag{9.36}
\end{equation}

By simple computation,
\begin{equation}
H_D(x, z) = \frac{1}{2D} \left[ -\frac{1}{3} - \frac{x^2}{2} - \frac{z^2}{2} \right]. \tag{9.37}
\end{equation}

For $x \neq z$ we calculate
\begin{equation}
\nabla_x \nabla_z G_D(x, z) = 0, \quad \nabla_x G_D(x, z) = \begin{cases}
-\frac{x+1}{2D} & \text{if } x \leq z, \\
-\frac{x}{2D} & \text{if } z \leq x.
\end{cases} \tag{9.38}
\end{equation}

We further have
\begin{equation}
\nabla_x G_D(x, z) \big|_{x=\zeta} = \nabla_x H_D(x, z) \big|_{x=\zeta} = -\frac{z}{2D}. \tag{9.39}
\end{equation}

Let $x_j = -1 + \frac{2j-1}{N}$. So we obtain
\begin{equation}
\nabla G_D = (c_{ij})(-\frac{1}{2D}), \tag{9.40}
\end{equation}

where
\begin{equation}
c_{ij} = \begin{cases}
x_i + 1, & \text{if } i < j, \\
x_i - 1, & \text{if } j > i, \\
x_i, & \text{if } j = i.
\end{cases} \tag{9.41}
\end{equation}
We claim that
\[ \sum_{i=1}^{N} G_D(x_i, x_j) = \lambda_1 = - \frac{1}{6DN}, \]
\[ \sum_{j=1}^{N} \nabla_{x_i} G_D(x_i, x_j) = \sum_{i=1}^{N} \nabla_{x_j} G_D(x_i, x_j) = 0, \] (9.42)
where \( x_j = -1 + \frac{2(j-1)}{N}, \ i, j = 1, \ldots, N. \)

**Proof.** By direct computation,
\[ 2D \sum_{j=1}^{N} G_D(x_i, x_j) = N \left( -\frac{1}{3} - \frac{x_i^2}{2} - \frac{1}{2} \right) + \frac{2}{N} \sum_{j=1}^{N} (j - \frac{1}{2}) \]
\[ - \frac{2}{N^2} \sum_{j=1}^{N} (j - \frac{1}{2})^2 + \frac{2}{N} \sum_{j=1}^{N} |j - i| \]
\[ = (-\frac{1}{3} - \frac{x_i^2}{2})N + \frac{2}{N} \sum_{j=1}^{N} |j - i| - \frac{1}{2} \sum_{j=1}^{N} x_j^2 \]
\[ = -\frac{1}{3}N = 2D\lambda_1. \]
\[ \sum_{j=1}^{N} \nabla_{x_i} G_D(x_i, x_j) = \sum_{j=1}^{N} c_{ij} = \sum_{j=1}^{N} x_i + (N - i) = x_iN + N - 2i - 1 = 0. \]

By \( G_D(x_i, x_j) = G_D(x_i, x_j) \), we know (9.42) is true. \( \square \)

**REFERENCES**

[1] W. W. Ao, M. Musso and J. C. Wei, On spikes concentrating on line-segments to a semilinear Neumann problem, *Journal of Differential Equations*, 251 (2011), 881–901.
[2] D. Benson, P. Maini and J. Sherratt, Unravelling the Turing bifurcation using spatially varying diffusion coefficients, *J. Math. Biol.*, 37 (1998), 381–417.
[3] A. Floer and A. Weinstein, Nonspraying wave packets for the cubic Schrödinger equations with a bounded potential, *J. Functional Analysis*, 69 (1986), 397–408.
[4] D. Gilbarg and N. Turdinger, *Elliptic Partial Differential Equations of Second Order*, Springer, Berlin, 1983.
[5] C. F. Gui and J. C. Wei, Multiple interior peak solutions for some singularly perturbation problems, *J. Differential Equations*, 158 (1999), 1–27.
[6] C. F. Gui, J. C. Wei and M. Winter, Multiple boundary peak solutions for some singularly perturbed Neumann problems, *Ann. Inst. H. Poincaré Anal.*, 17 (2000), 47–82.
[7] D. Iron, J. C. Wei and M. Winter, Stability analysis of Turing patterns generated by the Schnakenberg model, *J. Math. Biol.*, 49 (2004), 358–390.
[8] Y. G. Oh, Existence of semi-classical bound states of nonlinear Schrödinger equations with potentials of the class \((V)\), *Comm. Partial Differential Equations*, 13 (1990), 1499–1519.
[9] Y. G. Oh, On positive multi-bump bound states of nonlinear Schrödinger equations under multiple-well potentials, *Comm. Math. Phys.*, 131 (1990), 223–253.
[10] J. Schnakenberg, Simple chemical reaction systems with limit cycle behavior, *J. Theoret. Biol.*, 81 (1979), 389–400.
[11] A. Turing, The chemical basis of morphogenesis, *Phil. Trans. Roy.*, 237 (1952), 37–72.
[12] M. Ward and J. C. Wei, Asymmetric spike patterns for the one-dimensional Gierer-Meinhardt model: equilibria and stability, *J. Appl. Math.*, 13 (2002), 283–320.
[13] M. Ward and J. C. Wei, The existence and stability of asymmetric spike patterns for the Schnakenberg model, *Michigan Math. J.*, 109 (2002), 229–264.
[14] J. C. Wei, On single interior spike solutions of Gierer-Meinhardt system: Uniqueness and spectrum estimates, *European. J. Appl. Mth.*, 10 (1999), 353–378.
[15] J. C. Wei and M. Winter, Stationary solutions for the Cahn-Hilliard equation, *Ann. Inst.H.Poincaré Anal.* 348 (1996), 975–995.
[16] J. C. Wei and M. Winter, On the Cahn-Hilliard equations: Interior spike layer solutions, *J. Differential Equations*, 148 (1998), 231–267.
[17] J. C. Wei and M. Winter, Spikes for the two-dimensional Gierer-Meinhardt system: The weak coupling case, *J.Nonlinear Science*, 11 (2001), 415–458.
[18] J. C. Wei and M. Winter, Spikes for the Gierer-Meinhardt system in the two dimensions: The weak coupling case, *J.Differential Equations*, 178 (2002), 478–518.
[19] J. C. Wei and M. Winter, Existence and stability analysis of asymmetric patterns for the Gierer-Meinhardt system, *J. Math.Pures.Appl.*, 83 (2004), 433–476.
[20] J. C. Wei and M. Winter, Existence, Classification and Stability analysis of multiple-peaked solution for the Gierer-Meinhardt system in $\mathbb{R}^1$, *Methods and Applications of Analysis*, 14 (2007), 119–163.
[21] J. C. Wei and M. Winter, On the Gierer-Meinhardt system with precursors, *Discrete Contin. Dyn. Syst.*, 25 (2009), 303–398.
[22] J. C. Wei and M. Winter, Flow-distributed spikes for Schnakenberg Kinetics, *Mathematical Biology*, 64 (2012), 211–254.

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