Abstract. In this paper, we extend the idea of comprehensive Gröbner bases given by Weispfenning (1992) to border bases for zero dimensional parametric polynomial ideals. For this, we introduce a notion of comprehensive border bases and border system, and prove their existence even in the cases where they do not correspond to any term order. We further present algorithms to compute comprehensive border bases and border system. Finally, we study the relation between comprehensive Gröbner bases and comprehensive border bases w.r.t. a term order and give an algorithm to compute such comprehensive border bases from comprehensive Gröbner bases.

1. Introduction

Since Weispfenning (1992) introduced and proved the existence of comprehensive Gröbner bases (CGB) for parametric ideals, there has been a lot of work done in this direction. Stability of Gröbner bases under specialization was studied in (Kalkbrener, 1997) and based on this, an improved algorithm for computing the CGB was given in (Montes, 1999). An algorithm for computing CGB from the Gröbner bases of the initial ideal using the computation in the polynomial ring over ground field was given in (Suzuki & Sato, 2006). Recently an efficient method for computing CGB was given by Kapur et al. (2013).

Border bases, an alternative to Gröbner bases, is known to have more numerical stability as compared to Gröbner bases (Stetter, 2004). Recently there have been much interest in the theory of border bases though they are restricted to zero dimensional ideals. Characterization of border bases was given in (Kehrein & Kreuzer, 2005) along with some parallel results from Gröbner bases theory. Complexity of border bases detection was studied in (Ananth & Dukkipati, 2011, 2012). In this paper, we define and study the border system and comprehensive border bases similar to Gröbner system and CGB respectively.

Contributions. We define comprehensive border bases (CBB) and border system (BS) on the similar lines of comprehensive Gröbner bases (CGB) and Gröbner system (GS). We show the existence of the comprehensive border bases, even in the case where they do not correspond to any term order. We then provide algorithms for computing border system and comprehensive border bases. We show that for a given term order the border form ideal (Kehrein & Kreuzer, 2005), is same as leading term ideal and, using this
fact, we present a relation between CGB and comprehensive border bases for a given term order.

**Organization.** The rest of the paper is organized as follows. In Section 2, we present preliminaries on CGB and border bases. In Section 3, we introduce border system and comprehensive border bases and prove the existence of the same. In Section 4, we present an algorithm for computing border system and comprehensive border bases. A relation between comprehensive border bases and CGB is presented in Section 5. A brief discussion of CBB over von Neumann regular rings is presented in Section 6. Finally a detailed example is given in Section 7 and we give concluding remarks in Section 8.

## 2. Background & Preliminaries

### 2.1. Notations.
Throughout this paper, $k$ denotes a field, $\bar{k}$ denotes the field extension (algebraic closure) of $k$ and $\mathbb{N}$ the set of natural numbers. A polynomial ring in indeterminates $x_1, \ldots, x_n, u_1, \ldots, u_m$ over $k$ is denoted by $k[x_1, \ldots, x_n, u_1, \ldots, u_m](= k[X,U])$, where $x_1, \ldots, x_n$ are the main variables and $u_1, \ldots, u_m$ are the parameters. A polynomial ring in the main variables $x_1, \ldots, x_n$ over $k$ is denoted by $k[x_1, \ldots, x_n](= k[X])$ and a polynomial ring in the parameters $u_1, \ldots, u_m$ over $k$ is denoted by $k[u_1, \ldots, u_m](= k[U])$. A monomial $x_1^{\alpha_1} \cdots x_n^{\alpha_n} \in \mathbb{T}_n$ is denoted by $x^\alpha$, with the understanding that $x = (x_1, \ldots, x_n)$ and $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$. We denote the set of all monomials in variables $x_1, \ldots, x_n$ by $\mathbb{T}_n$. $\mathbb{T}_n^\ell$ denotes the set of monomials of degree $\ell$. When we need to deal with more than one monomials, say $\ell$, in variables $x_1, \ldots, x_n$, we index these monomials as $x^{\alpha^{(1)}}, x^{\alpha^{(2)}}, \ldots, x^{\alpha^{( \ell)}}$.

With respect to a term order $\prec$, we have the leading monomial $(\text{LM}_\prec)$, leading coefficient $(\text{LC}_\prec)$, leading term $(\text{LT}_\prec)$ and the degree of a polynomial $(\text{deg}_\prec)$, where $\text{LT}_\prec(f) = \text{LC}_\prec(f) \text{LM}_\prec(f)$ and $\text{deg}_\prec(f) = \text{deg}_\prec(\text{LM}_\prec(f))$. If it is clear from the context we will drop the subscript for term order.

A substitution or specialization is a ring homomorphism $$\sigma : \bar{k}[x_1, \ldots, x_n, u_1, \ldots, u_m] \rightarrow \bar{k}[x_1, \ldots, x_n]$$ specified by $x_i \mapsto x_i$, $i = 1, \ldots, n$, $u_j \mapsto c_j$, $j = 1, \ldots, m$, where $c_1, \ldots, c_m \in \bar{k}$. By this the specialization $\sigma$ can be uniquely identified by $\sigma = (c_1, \ldots, c_m) \in \bar{k}^m$. A set of specializations is termed as a condition in parameters $u_1, \ldots, u_m$ and is denoted by $\gamma$. For an ideal $a \subseteq k[U]$, $\gamma(a) \subseteq \bar{k}^m$ denotes the algebraic set of $a$, i.e., $$\gamma(a) = \{(c_1, \ldots, c_m) \in \bar{k}^m \mid f \in a, f(c_1, \ldots, c_m) = 0\}.$$

### 2.2. Comprehensive Gröbner bases.

**Definition 2.1.** Let $F$ be a finite set of generators of an ideal $a \subseteq k[U][X]$ and $S$ be a subset of $\bar{k}^m$. A finite subset $G$ in $k[U][X]$ is called a comprehensive Gröbner basis on $S$ for $F$, if $\sigma(G)$ is a Gröbner basis for the ideal $\sigma(F)$ in $\bar{k}[X]$ for any specialization $\sigma \in S$. If $S = \bar{k}^m$, then $G$ is called a comprehensive Gröbner basis (CGB) for $F$.
Let $G_{=0} = \{g_1, \ldots, g_s\}$ and $G \neq 0 = \{g'_1, \ldots, g'_t\}$ be a finite set of polynomial equations and polynomial inequalities in $k[U]$ respectively. Then a condition, $\gamma \subseteq \overline{k}^n$, can also be represented as $\mathcal{V}(G_{=0}) \setminus \mathcal{V}(G \neq 0)$.

**Definition 2.2.** Let $F$ be a subset of $k[U][X]$, $\gamma_1, \ldots, \gamma_t$ be conditions in $\overline{k}^m$, $G_1, \ldots, G_t$ be subsets of $k[U][X]$, and $S$ be a subset of $\overline{k}^m$ such that $S \subseteq \gamma_1 \cup \ldots \cup \gamma_t$. A finite set $\mathcal{G} = \{(G_1, \gamma_1), \ldots, (G_t, \gamma_t)\}$ is called a Gröbner system on $S$ for $F$, if $\sigma(G_i)$ is a Gröbner basis for the ideal $\sigma(F)$ in $k[X]$ for any specialization $\sigma \in \gamma_i$, $i = 1, \ldots, t$. If $S = \overline{k}^m$, then $\mathcal{G}$ is called a Gröbner system (GS) for $F$.

For any distinct pairs $(G_i, \gamma_i), (G_j, \gamma_j) \in \mathcal{G}$ such that $\gamma_i \cap \gamma_j = \gamma_t$, we can replace $(G_i, \gamma_i), (G_j, \gamma_j) \in \mathcal{G}$ with $(G_i, \gamma_i \setminus \gamma_t), (G_j, \gamma_j \setminus \gamma_t), (G_t, \gamma_t)$, where $G_t$ is either $G_i$ or $G_j$. So without any loss of generality we will assume that the conditions, $\gamma_1, \ldots, \gamma_t$, are pairwise disjoint.

A CGB $G$ for $F$ is called faithful, if $G \subseteq \langle F \rangle$. A Gröbner system $\mathcal{G} = \{(G_1, \gamma_1), \ldots, (G_t, \gamma_t)\}$ for $F$ is called faithful if every element of $G_i$, is also in $\langle F \rangle$, for $i = 1, \ldots, t$.

The first algorithm to compute CGB was given by Weispfenning [1992]. The central idea in this approach is to generate a finite partition (case distinction) of $\text{spec}(k[U])$ such that, inside each set of the partition the leading term ideal remains same for all specialization satisfying the condition of the set. The polynomials are ‘conditionally colored’ to calculate the conditional Gröbner bases under a condition. Under a given partition (condition) each term is colored according to the value of its coefficient as follows: the terms with coefficients that vanish under the condition are colored as ‘green’ and terms with coefficients that do not vanish under the condition are colored ‘red’. When the condition is not sufficient to color the term as red or green, we color the term as ‘white’. A polynomial with all terms colored is called a colored polynomial. The Gröbner bases computations are then done with respect to conditional leading term of a colored polynomial, to make sure we generate a conditional Gröbner bases that is not outside the original ideal. The conditions are further refined in case the Gröbner bases computation generate a white leading term polynomial. Gröbner bases along with each partition is called Gröbner system and the union of all the Gröbner bases of Gröbner system gives CGB. For a detailed exposition one can refer to Dunn III [1995].

### 2.3. Border bases in $k[x_1, \ldots, x_n]$.

**Definition 2.3.** A finite set of terms $O \subset \mathbb{T}^n$ is called an order ideal if it is closed under forming divisors i.e. for $x^\alpha \in \mathbb{T}^n$ if, $x^\beta \in O$ and $x^\alpha | x^\beta$ then it implies $x^\alpha \in O$.

**Definition 2.4.** Let $O \subset \mathbb{T}^n$ be an order ideal. The border of $O$ is the set $\partial O = (\mathbb{T}_n^o) \setminus O = (x_1 O \cup \ldots \cup x_n O) \setminus O$. The first border closure of $O$ is defined as the set $O \cup \partial O$ and it is denoted by $\overline{O}$. Note that $\overline{O}$ is also an order ideal.

By convention for $O = \emptyset$ we set $\partial O = 1$. 

Definition 2.5. Let \( \mathcal{O} = \{x^{\alpha(1)}, \ldots, x^{\alpha(s)}\} \subset \mathbb{T}^n \) be an order ideal, and let \( \partial \mathcal{O} = \{x^{\beta(1)}, \ldots, x^{\beta(t)}\} \) be its border. A set of polynomials \( \mathcal{B} = \{b_1, \ldots, b_t\} \subset \mathbb{k}[X] \) is called an \( \mathcal{O} \)-border prebasis if the polynomials have the form \( b_j = x^{\beta(j)} - \sum_{i=1}^{s} c_{ij} x^{\alpha(i)} \), where \( c_{ij} \in \mathbb{k} \) for \( 1 \leq i \leq s \) and \( 1 \leq j \leq t \).

Definition 2.6. Let \( \mathcal{O} = \{x^{\alpha(1)}, \ldots, x^{\alpha(s)}\} \subset \mathbb{T}^n \) be an order ideal and \( \mathcal{B} = \{b_1, \ldots, b_t\} \) be an \( \mathcal{O} \)-border prebasis consisting of polynomials in \( \mathfrak{a} \subseteq \mathbb{k}[X] \). We say that the set \( \mathcal{B} \) is an \( \mathcal{O} \)-border bases of \( \mathfrak{a} \) if the residue classes of \( x^{\alpha(1)}, \ldots, x^{\alpha(s)} \) form a \( \mathbb{k} \)-vector space basis of \( \mathbb{k}[x_1, \ldots, x_n]/\mathfrak{a} \).

Now we give a brief account of characterization of border bases given in [Kehrein & Kreuzer, 2005].

Index of \( x^\beta \in \mathbb{T}^n \) is defined as \( \text{ind}_{\mathcal{O}}(x^\beta) = \min\{r \geq 0 | x^\beta \in \partial \mathcal{O}\} \). Given a non-zero polynomial \( f = c_1 x^{\alpha(1)} + \ldots + c_s x^{\alpha(s)} \in \mathfrak{a} \subseteq \mathbb{k}[X] \), where \( c_1, \ldots, c_s \in \mathbb{k} \setminus \{0\} \) and \( x^{\alpha(1)}, \ldots, x^{\alpha(s)} \in \mathbb{T}^n \), we order the terms in the support of \( f \) such that \( \text{ind}_{\mathcal{O}}(x^{\alpha(1)}) \geq \text{ind}_{\mathcal{O}}(x^{\alpha(2)}) \geq \ldots \geq \text{ind}_{\mathcal{O}}(x^{\alpha(s)}) \). Then we call \( \text{ind}_\mathcal{O}(f) = \text{ind}_{\mathcal{O}}(x^{\alpha(1)}) \) the index of \( f \).

Definition 2.7. Given a polynomial \( f \in \mathfrak{a} \subseteq \mathbb{k}[X] \), there is a representation of \( f = c_1 x^{\alpha(1)} + \ldots + c_s x^{\alpha(s)} \) with \( c_1, \ldots, c_s \in \mathbb{k} \setminus \{0\} \) and \( x^{\alpha(1)}, \ldots, x^{\alpha(s)} \in \mathbb{T}^n \) such that \( \text{ind}_{\mathcal{O}}(x^{\alpha(1)}) \geq \text{ind}_{\mathcal{O}}(x^{\alpha(2)}) \geq \ldots \geq \text{ind}_{\mathcal{O}}(x^{\alpha(s)}) \).

\begin{enumerate}
\item The polynomial \( \text{BF}_\mathcal{O}(f) = \sum_{\text{ind}(x^{\alpha(i)}) = \text{ind}(f)}^{s} c_i x^{\alpha(i)} \) is called the border form of \( f \) with respect to \( \mathcal{O} \). For \( f = 0 \), we let \( \text{BF}_\mathcal{O}(f) = 0 \).
\item Given an ideal \( \mathfrak{a} \), the ideal \( \text{BF}_\mathcal{O}(\mathfrak{a}) = \{\text{BF}_\mathcal{O}(f) \mid f \in \mathfrak{a}\} \) is called the border form ideal of \( \mathfrak{a} \) with respect to \( \mathcal{O} \).
\end{enumerate}

Definition 2.8. \( F \) is \( \mathcal{O} \)-border bases of \( \mathfrak{a} \subseteq \mathbb{k}[X] \) if and only if one of the following equivalent conditions is satisfied:

\begin{enumerate}
\item For all \( f \in \mathfrak{a} \), \( \text{supp}(\text{BF}_\mathcal{O}(f)) \) is contained in \( \mathbb{T}^n \setminus \mathcal{O} \).
\item \( \text{BF}_\mathcal{O}(\mathfrak{a}) = (x^{\beta(1)}, \ldots, x^{\beta(t)}) \), where \( x^{\beta(i)} \in \{\text{BF}_\mathcal{O}(f_i) \mid f_i \in F\} \).
\end{enumerate}

Reduced Gröbner bases is unique for an ideal with respect to a given term order. While the monomial ordering in the Gröbner bases theory is given by term ordering, in the border bases theory it can be given by the shortest distance from the order ideal.

Proposition 2.9. For a given term order and an ideal there exists a unique reduced Gröbner bases and a unique border bases.

The proof of above proposition follows from the following results.

Theorem 2.10 (Macaulay-Buchberger Basis Theorem [Buchberger, 1965]). Let \( G = \{g_1, \ldots, g_t\} \) be a Gröbner basis for an ideal \( \mathfrak{a} \subseteq \mathbb{k}[x_1, \ldots, x_n] \). A basis for the vector space \( \mathbb{k}[x_1, \ldots, x_n]/\langle \mathfrak{a} \rangle \) is given by \( S = \{x^{\alpha} + \mathfrak{a} : \text{LM}(g_i) \mid x^{\alpha}, i = 1, \ldots, t\} \).
Theorem 2.11. ([Kehrein et al., 2004]) Let $\mathcal{O} = \{x^{\alpha(1)}, \ldots, x^{\alpha(n)}\} \subset \mathbb{T}^n$ be an order ideal, let $a$ be a zero-dimensional ideal in $k[X]$, and assume that the residue classes of the elements in $\mathcal{O}$ form a $k$-vector space basis of $k[X]/a$:

1. There exists a unique $\mathcal{O}$-border basis $G$ of $a$.
2. Let $G'$ be an $\mathcal{O}$-border prebasis whose elements are in $a$. Then $G'$ is the $\mathcal{O}$-border basis of $a$.
3. Let $k$ be the field of definition of $a$. Then we have $G \subseteq k[X]$.

As we have mentioned earlier, given an ideal there exists border bases that do not correspond to Gröbner bases for any term ordering. An example of such a border basis is given in [Kehrein & Kreuzer, 2006].

3. Comprehensive Border bases

First we define the notion of ‘scalar’ border bases.

Definition 3.1. Let $a \subseteq k[X]$ be an ideal, for $\{b_1, \ldots, b_l\} \subseteq a$, $\mathcal{O} \subset \mathbb{T}^n$ an order ideal and $c_1, \ldots, c_l \in k$, $\{c_1b_1, \ldots, c_lb_l\}$ is called scalar $\mathcal{O}$-border bases of $a$ if $\{b_1, \ldots, b_l\}$ is an $\mathcal{O}$-border basis of $a$.

We need the notion of scalar border bases because our aim is to construct border bases for the ideals in $k[X]$ that result from specialization of the ideals in $k[U][X]$. As the coefficient space changes from the ring $k[U]$ to the field $k$ under a specialization, we will not be able to get the monic border form terms for the border bases in $k[X]$ from the polynomials in $k[U][X]$. Now we define the notion of border system and comprehensive border bases.

Definition 3.2. Consider a zero dimensional ideal $a \subseteq k[X, U]$ and $S \subseteq \tilde{k}^m$. Let $\gamma_\ell$ is a condition, $\mathcal{O}_\ell$ is a order ideal and $B_\ell \subset k[X, U]$, for $\ell = 1, \ldots, L$. Then the set $\mathcal{B} = \{(\gamma_\ell, \mathcal{O}_\ell, B_\ell) : \ell = 1, \ldots, L\}$ is said to be a border system on $S$ for $a$ if,

1. $\gamma_i \cap \gamma_j = \emptyset$ for all $i, j$ such that $1 \leq i, j \leq L$, $i \neq j$.
2. $S \subseteq \bigcup_{i=1}^{L} \gamma_i$, and
3. for every specialization $\sigma \in \gamma_\ell$, $\sigma(B_\ell)$ is a scalar $\mathcal{O}$-border bases of $\sigma(a)$ in $\tilde{k}[X]$.

If $S = \tilde{k}^m$, then $\mathcal{B}$ is called border system for $a$.

Definition 3.3. A finite set $B \subset k[X, U]$ is said to be a comprehensive border basis on $S \subseteq \tilde{k}^m$ for a zero dimensional ideal $a = \langle B \rangle$ if for all specializations $\sigma \in S$, there exists an order ideal $\mathcal{O} \subset \mathbb{T}^n$ such that, $\sigma(B)$ is a scalar $\mathcal{O}$-border basis of $\sigma(a) \subseteq \tilde{k}[X]$. If $S = \tilde{k}^m$, then $B$ is called comprehensive border basis for $a$.

A border system $\mathcal{B} = \{(\gamma_\ell, \mathcal{O}_\ell, B_\ell) : \ell = 1, \ldots, L\}$ for an ideal $a$ is called faithful, if in addition, every element of $B_\ell$, $\ell = 1, \ldots, L$ is also in $a$. A comprehensive border basis $B$ for $a$ is called faithful, if in addition, every element of $B$ is also in $a$. 

Definition 3.4. Let $\sigma$ be a specialization and $a \subseteq k[X]$ be an ideal, then $B \subset a$ is called a conditional border basis of $a$ under $\sigma$ if, $\sigma(B)$ is an $O$-border basis of $\sigma(a) \subseteq k[X]$ for an order ideal $O \subset \mathbb{T}^n$. In particular, we say, $B$ is a $\sigma$-conditional $O$-border basis for $\sigma(a) \subseteq k[X]$.

Let $a \subseteq k[x_1, \ldots, x_n, u_1, \ldots, u_m]$ be an ideal and $a' = a \cap k[u_1, \ldots, u_m]$. For each specialization $\sigma \notin \mathcal{V}(a')$, we have $\sigma(a) = \langle 1 \rangle = k[x_1, \ldots, x_n]$ (Suzuki & Satd, 2000). We use this fact along with Lemma 3.5.1 for zero dimensional ideals to show the existence of comprehensive border bases.

Lemma 3.5. If $a \subseteq k[x_1, \ldots, x_n, u_1, \ldots, u_m]$ is zero-dimensional ideal, then $a' = a \cap k[u_1, \ldots, u_m]$ is also a zero dimensional ideal.

Proof. Let $G$ be a Gröbner basis w.r.t. an elimination (term) order $\leq$ with $\{x_1, \ldots, x_n\} \leq \{u_1, \ldots, u_m\}$ and a lex term order within the main variables and parameters. If $a$ is a zero-dimensional ideal then for each $x_i$ (or $u_i$), $\exists g_i \in G$ such that $\text{LT}(g_i) = x_i^\ell$ (or $u_i^\ell$ respectively), for some $\ell \in \mathbb{N}$ (Adams & Loustaunau, 1994). Also for each $u_i \exists g_i \in G$ such that $\text{LT}(g_i) = u_i^\ell$ and $\text{supp}(g_i) \subseteq k[u_1, \ldots, u_m]$. The Gröbner bases of $a'$ is $G \cap k[u_1, \ldots, u_m]$ (Adams & Loustaunau, 1994) and we denote it by $G'$. Now for each $u_i \exists g_i \in G'$ such that $\text{LT}(g_i) = u_i^\ell$ implying $a'$ is a zero dimensional ideal. 

It is easy to see that if $a \subseteq k[X, U]$ is a zero dimensional ideal, then $\sigma(a) \subset k[X]$ is also a zero dimensional ideal. Hence there exists a border basis for $\sigma(a)$.

Proposition 3.6. For a zero dimensional ideal $a \subseteq k[X, U]$, border system and hence comprehensive border basis always exists.

Proof. Let $F$ be the given set of generators of the ideal $a$ in $k[X, U]$ and $\sigma \in \mathbb{K}^m$ be a specialization. Let $a' = a \cap k[u_1, \ldots, u_m]$ then by Lemma 3.5.1 we have $\mathcal{V}(a')$ is finite. Now, either $\sigma \in \mathcal{V}(a')$ or $\sigma \notin \mathcal{V}(a')$. In the first case, for each $\sigma \in \mathcal{V}(a')$ we can compute conditional border basis for the ideal generated by $F$ as we can exactly determine which coefficients vanishes and which does not under any specialization $\sigma$. In the second case, where $\sigma \notin \mathcal{V}(a')$, for all $\sigma \in \mathbb{K}^m \setminus \mathcal{V}(a')$, $F$ can be considered as a conditional border basis as $\sigma(F) = \sigma(a) = \langle 1 \rangle$.

There exists a border system $B$ that is constructed as, for each $\sigma_i \in \mathcal{V}(a')$ with $B_i$ calculated as conditional $O_i$-border basis we add $(\sigma_i, O_i, B_i)$ to $B$. For all $\sigma \in \mathbb{K} \setminus \mathcal{V}(a')$ we add the tuple $(\sigma, \emptyset, F')$, where $F' = F$ (or $F'$ can also be the finite set of generators of the ideal $a'$).

Consider $\Phi$ as a function which maps an element in $\mathbb{K}^m$ to the corresponding order ideal in the border system computed above. Now CBB can be computed as the union of all the conditional border bases from the border system along with $\Phi$. 

In Section 5, we give a method to construct CBB from CGB, by which one can establish the existence of CBB. But the significance of the Proposition 3.6 arises from the fact that it establishes the existence of border
system and CBB that need not correspond to any term order. Before we study the relation between CGB and CBB we present the algorithms to construct border system and CBB.

4. Algorithm

Here we briefly recall the ‘coloring terminology’ used by Weispfenning (1992). Let \( t = ax^\alpha \in k(U)[X] \) be a term, where \( k(U) \) is the field of fractions of \( k[U] \), \( a \in k(U) \) and \( x^\alpha \in \mathbb{T}^n \) is a monomial in main variables \( x_1, \ldots, x_n \). Then color of \( t \) is green if \( \sigma(a) \) is zero, otherwise it is red.

The basic idea for coloring as given by Weispfenning (1992) was to do the computations for an ideal \( \sigma(a) \subset k[X] \) and still be able to generate and deal with the polynomials in \( a \subset k(U)[X] \). As we know the exact specialization for which we need to compute border bases, we will be able to color the every term of the polynomials in \( a \subset k(U)[X] \) as either red or green. Now we can use any border bases algorithm and compute the conditional border bases for \( \sigma(a) \subset k[X] \) with respect to the red (non zero) term of the colored polynomials making sure that the modified polynomials belongs to \( a \).

The proof of correctness of the colored border bases algorithm is implied by the proof of correctness for the main border bases algorithm. As the proof holds for the red terms of the polynomials and the green terms vanishes under substitution.

We list a procedure to compute border system in Algorithm 1.

**Algorithm 1** Border System

**Input:** \( F \subset k[X, U] \), such that \( \langle F \rangle \) is a zero dimensional ideal.

**Output:** A border system \( B \) for \( F \).

1. Compute \( V(a') \in \bar{k}^m \), where \( a' = \langle F \rangle \cap k[U] \).
2. \( F' = F' \setminus k[U] \subset k(U)[X] \).
3. for each \( \sigma \in V(a') \) do
4. Color \( F' \) with \( \sigma \), remove the polynomials with all green terms, call \( F'' \).
5. Compute conditional \( O \)-border bases \( B \) for \( \langle F'' \rangle \) by the colored version of the border bases algorithm in \( k(U)[X] \).
6. Convert the conditional \( O \)-border bases \( B \) generated above to conditional scalar \( O \)-border bases \( B' \) in \( k[U][X] \).
7. Update \( B = B \cup \{(\sigma, O, B')\} \)
8. end for
9. Update \( B = B \cup \{ (\bar{k}^m \setminus V(a'), \emptyset, F) \} \)
10. Compress \( B \)
11. Return \( B \)

The termination of the algorithm is obvious by Lemma 3.5. Note that Step 10 combines two elements of the border system \( B \), when they differ only in the specialization of the parameters. This optional step is shown in the example we present in Section 7. The correctness of the border system algorithm follows from Proposition 3.6.
We must note that the nature of border system generated by Algorithm 1 depends on the nature of the border bases algorithm used in Step 5. Using a border bases (colored version) algorithm we can generate border system and hence the CBB that do not correspond to any term order.

We need the following definition of vanishing polynomial with respect to a specialization \( \sigma \) and a finite condition, \( \gamma \).

**Definition 4.1.** Let \( \sigma = (c_1, \ldots, c_m) \in \overline{k}^m \) be a specialization of parameters \((u_1, \ldots, u_m)\), then the vanishing polynomial at \( \sigma \) is defined as

\[
f_{\sigma} = (u_1 - c_1)^2 + \ldots + (u_m - c_m)^2.
\]

For a finite condition \( \gamma = \{\sigma_1, \sigma_2, \ldots, \sigma_\ell\} \) we define vanishing polynomial w.r.t. \( \gamma \) as,

\[
f_{\gamma} = f_{\sigma_1}f_{\sigma_2}\ldots f_{\sigma_\ell}.
\]

We list a method to compute CBB in Algorithm 2.

**Algorithm 2 Comprehensive border bases**

**Input:** The border system \( B = \{ (\gamma_\ell, \mathcal{O}_\ell, B_\ell) : \ell = 1, \ldots, L \} \) of the ideal \( a \subseteq k[X, U] \). We represent \( a' = a \cap k[u_1, \ldots, u_m] \).

**Output:** A comprehensive border basis \( B \) of \( a \)

1: \( B_1 = \{ (\gamma, \mathcal{O}_i, B_i) \} \) such that \( \gamma \) is a condition belonging to \( \overline{k}^m - \mathcal{V}(a') \).
2: \( B' = B \setminus B_1 \)
3: Let \( f_{a'} = \prod_{\gamma \in \mathcal{V}(a')} f_{\gamma} \)
4: Replace each \( B_1 \) belonging to \( (\gamma_i, \mathcal{O}_i, B_i) \in B' \) by \( B_i^{mark} \), where \( B_i^{mark} \) is marked version of \( B_i \) such that, border form of each polynomial in \( B_i \) is marked with respect to \( \mathcal{O}_i \).
5: Replace each \( B_i^{mark} \) belonging to \( (\gamma_i, \mathcal{O}_i, B_i^{mark}) \in B' \) by \( B_i^{CBB} \)

\[
B_i^{CBB} = \{ (f_{a'}/f_{\gamma_i}) (f) | f \in B_i^{mark} \}
\]

6: \( B = \bigcup_{B_i^{CBB} \in B'} B_i^{CBB} \cup \{ B_i | B_i \in B_1 \} \)
7: Return \( B \) (CBB)

Correctness of the algorithm can be seen from the nature of the vanishing polynomial under a specialization. For a specialization \( \gamma_i \in \mathcal{V}(a') \) such that \( (\gamma_i, \mathcal{O}_i, B_i) \in B' \) the vanishing polynomial \( f_{\gamma_i} \) is multiplied to all the conditional border bases in \( B' \) except for \( B_i \) (Step 5). This implies for all \( B_j \in B', j \neq i, B_j^{CBB} \) vanishes under \( \gamma_i \) and also \( B_i \) corresponding to \( B_i \) vanishes under \( \gamma_i \). For \( \gamma_\ell \notin \mathcal{V}(a') \) such that \( (\gamma_\ell, \mathcal{O}_\ell, B_\ell) \in B_1 \) we have \( \langle \sigma(B_\ell) \rangle = \langle 1 \rangle \) for all \( \sigma \in \gamma_\ell \).

One must note here that the border system and hence the CBB constructed in Algorithm 1 and Algorithm 2 are not faithful. As the CBB for an ideal is computed from the border system, if the border system is faithful, the CBB constructed also be faithful. We can obtain the faithful border system by restricting polynomial operations (i.e. avoiding division or multiplication by inverses) in \( k[U][X] \) in the colored border bases algorithm (Algorithm 1).
Step 5). This is similar to the technique followed in (Weispfenning, 1992), where a ‘modified’ definition of reduction and S-polynomial is used to avoid divisions.

For construction of border bases under a specialization from the CBB computed in Algorithm 2, we have to calculate the order ideal \( O \) as \( T^n \langle \langle M \rangle \rangle \), where \( M \) is the set of all marked monomials (Algorithm 2, Step 4) that do not vanish on specialization. The correctness of \( O \) is implied by the construction of CBB shown above.

A note on identifying whether a given CBB correspond to any term order or not. If a border basis correspond to a term order if and only if it contains a reduced Gröbner bases as the subset. These border bases are hence also the Gröbner bases for the corresponding term order.

Let \( B \in \mathbb{k}[U][X] \) be a CBB and \( \sigma \in \bar{\mathbb{k}}^m \) be a specialization. One can verify whether \( \sigma(B) \subseteq \bar{\mathbb{k}}[X] \) is a Gröbner bases or not using algorithm given in (Sturmfels, 1996). If it is a Gröbner basis then it implies that \( B \) correspond to a term order else \( B \) is a CBB that do not correspond to any term order. We must note that the nature of CBB generated depends on the border bases algorithm (colored version) used in Algorithm 1. If we already know the nature of border bases algorithm used then CBB generated will also be of the same nature.

5. Relation between CGB and CBB

We establish the following result that is crucial for studying the relation between CGB and CBB.

**Proposition 5.1.** For a given term order and an ideal, border form ideal is same as leading term ideal.

**Proof.** For an ideal \( \mathfrak{a} \subseteq \mathbb{k}[X] \) and a given term order \( \leq \), assume \( G \subset \mathfrak{a} \) be a reduced Gröbner basis. Let \( O \) be the \( \mathbb{k} \)-vector space basis of \( \mathbb{k}[X]/\mathfrak{a} \) corresponding to \( G \). Then there exists a unique \( O \)-border basis \( B \) for \( \mathfrak{a} \) corresponding to \( \leq \). Now from Gröbner basis \( G \), \( O \) is given by \( T^n \setminus \langle \text{LT}(G) \rangle = T^n \setminus \langle \text{LT}_{\leq}(\mathfrak{a}) \rangle \). And from \( O \)-border bases \( B, O \) is given by \( T^n \setminus \langle BF_O(\mathfrak{a}) \rangle \). Hence, for a given term order \( \leq \) we have a unique \( \mathbb{k} \)-vector space basis \( O \) such that \( \text{LT}_{\leq}(\mathfrak{a}) = BF_O(\mathfrak{a}) \). \( \square \)

With the above proposition, one must note that the existential proof of CBB is intuitive from CGB only in the case, where the border bases correspond to the Gröbner bases (term order) and not otherwise.

\( \bar{\mathbb{k}}. \)

Let \( \mathfrak{a} \subseteq \mathbb{k}[X,U] \) be an ideal and a term ordering \( \leq \) with an elimination order satisfying \( X \geq U \). \( \text{LM}_\gamma(g) \) denotes conditional leading red term of the polynomial \( g \) under the condition \( \gamma \), and \( \text{LM}_\gamma(G) \) is the set of conditional leading red terms of the polynomials in \( G \) under \( \gamma \). From the property of Gröbner system and hence reduced Gröbner system, for each tuple \( \{(G, \gamma)\} \in \mathcal{G} \), conditional leading red terms of the polynomials in \( G \) under the condition \( \gamma \) is uniquely determined. The process of computation
of CBB from CGB is described below. CGB \( \xrightarrow{\text{Weispfenning(1992)}} \) Gröbner system, \( \mathcal{G} \xrightarrow{\text{Weispfenning(1992)}} \) Reduced Gröbner system \( \xrightarrow{\text{Algorithm 2}} \) border system, \( \mathcal{B} \xrightarrow{\text{Algorithm 2}} \) CBB

**Algorithm 3** Reduced Gröbner system to border system

**Input:** Reduced Gröbner system \( \mathcal{G} = \{(G, \gamma)\} \) w.r.t \( \leq \) term order.

**Output:** Border system \( \mathcal{B} \)

1. \( \mathcal{B} = \emptyset \)
2. \( \text{while } \mathcal{G} \neq \emptyset \text{ do} \)
3. \( \text{Select any } (G_i, \gamma_i) \in \mathcal{G} \)
4. \( \mathcal{G} = \mathcal{G} \setminus \{(G_i, \gamma_i)\} \)
5. Compute \( \mathcal{O}_i = \mathbb{T}^n \setminus \{LM_{\gamma_i}(G_i)\} \)
6. Compute \( \partial \mathcal{O}_i \)
7. \( \partial \mathcal{O}_i' = \partial \mathcal{O}_i \setminus \{LM_{\gamma_i}(G_i)\} \)
8. \( \text{while } \partial \mathcal{O}_i' \text{ is not empty do} \)
9. \( \text{Select a monomial } x^\alpha \in \partial \mathcal{O}_i' \)
10. \( \partial \mathcal{O}_i' = \partial \mathcal{O}_i' \setminus x^\alpha \)
11. \( \text{Get the variable } x_t, \text{ such that } x^\alpha = x_tLM_{\gamma_i}(g_j) \text{ for some } g_j \in G_i. \)
12. \( \text{Consider } b = x_tg_j = cx^\alpha + p \) \( \triangleright c \in \mathbb{k} \)
13. \( \text{Reduce } p \text{ to } p', \text{ such that } p' \text{ is irreducible with } G_i \text{ relative to } \gamma_i \)
14. \( b = x^\alpha + p' \)
15. \( B_i = B_i \cup b \)
16. \( \text{end while} \)
17. \( \mathcal{B} = \mathcal{B} \cup \{(\mathcal{V}(\gamma_i), \mathcal{O}_i, B_i)\} \)
18. \( \text{end while} \)
19. \( \text{Return } \mathcal{B} \)

As for each tuple \( \{(G, \gamma)\} \in \mathcal{G} \), for any specialization \( \sigma \) corresponding to \( \gamma \), \( \sigma(G) \) is a zero dimensional ideal and so we have a finite order ideal for \( \langle \sigma(G) \rangle \) corresponding to each tuple. The finiteness of the order ideal shows the termination of the inner while loop. The reduced Gröbner system calculated from Gröbner system is a finite set of tuples, which shows the termination of the outer while loop. Hence, Algorithm 3 terminates.

It is easy to see that for each tuple \( (G, \gamma) \in \mathcal{G} \) (reduced Gröbner system) the conditional leading term ideal is same for all specializations \( \sigma \) corresponding to the condition \( \gamma \), and so the border form ideal will remain same for the ideal formed by \( \mathcal{G} \) under any such specialization. Due to this fact the conditional reduced Gröbner bases \( G \) under \( \gamma \) maps to conditional scalar border bases \( \mathcal{B} \) under \( \gamma \) (Proposition 5.1). So an element in \( \mathcal{G} \) maps to an element in \( \mathcal{B} \). Each tuple \( (G, \gamma) \in \mathcal{G} \) under a specialization \( \sigma \) corresponding to the condition \( \gamma \) forms a scalar reduced Gröbner bases (reduced Gröbner bases except for the monic leading coefficient). We have to convert each of this conditional scalar reduced Gröbner bases to conditional scalar border bases. Step 5 to Step 16 is the colored version of the standard Gröbner bases to border bases algorithm. It therefore computes conditional scalar border
bases from scalar reduced Gröbner bases. Rest of the algorithm uses Step 5 to Step 16 to map each element in $G$ to an element in $B$.

One must note that comprehensive border bases calculated w.r.t to some term order, will also be the CGB. This is obvious by the fact that for a given term order, border bases is a superset of reduced Gröbner bases.

6. Comprehensive Border bases and regular rings

Regular rings (commutative von Neumann regular rings) can be viewed as a certain subdirect products of the fields. The close relation between comprehensive Gröbner bases over fields and non-parametric Gröbner bases over von Neumann regular rings was shown in Weispfenning (2002, 2006).

Any algorithm for Gröbner bases over fields can be modified to compute Gröbner bases over von Neumann regular rings, these Gröbner bases over von Neumann regular rings also give us comprehensive Gröbner bases.

Computation of the comprehensive border bases can also be done through computing border bases over von Neumann regular rings. All the operations and the ideas come exactly from the computation of Gröbner bases over regular rings. We must make sure that the final output of border bases over regular rings have all the boolean closed polynomials. The difference comes in the structure of the order ideal over regular rings. We know that the order ideal contains all the monic terms (monomials), but the definition of monicness changes over regular rings.

Let $R$ be a regular ring and $\alpha \in R$, then the element $\alpha^*$, such that $\alpha \cdot \alpha^* = \alpha$, is called the idempotent element of $\alpha$. Let $f \in R[X]$ be a polynomial over regular ring, $R$.

Definition 6.1 (Sato (1998)). A polynomial $f$ is called monic if it satisfies $LC(f) = (LC(f))^*$

So the order ideal will have monic coefficients coming from $R$. This can be seen as the coordinate wise order ideal for border bases over regular rings (coordinate wise, order ideal and the border bases over the field).

Also, the polynomial coefficients from the polynomial ring $k[U]$ can be extended to computable ring of terraces (Suzuki & Sato, 2003) so that we get the closure under addition, multiplication and inverses. Ring of terraces becomes von Neumann regular rings and the terrace arithmetic is well studied in (Suzuki & Sato, 2003). Output of the Gröbner bases computed with terraces as the coefficients is similar to the output of the Gröbner system, in the sense that, we know the corresponding Gröbner bases for each possible substitution of the parameters. So for border bases we can associate an order ideal with each possible border bases output.

7. Example

For the colored version of a border bases algorithm, we modify the border basis algorithm given in Kehrein & Kreuzer (2006) and use the deglex term ordering. Consider a simple example for the algorithms given above.
Example: \( F = \{ x^2 - z^2 - 6x + 4z + 5, 3y^2 + z^2 - 12y - 4z + 12, z^3 - 8z^2 + 19z - 12, x^2z^3 - 8x^2z^2 + 19x^2z + xz^2 - 12x^2 - 4xz - z^2 + 3x + 4z - 3, x^2z^3 - 8x^2z^2 + 19x^2z + yz^2 - 12x^2 - 4yz - 2z^2 + 3y + 8z - 6 \} \)

First, we compute a border system \( B \), using Algorithm [1]

(1) \( \mathcal{V}(a') = \{ 4, 3, 1 \} \), where \( a' = (z^3 - 8z^2 + 19z - 12) \)

(2) \( F' = \{ x^2 - 6x - (z^2 - 4z - 5), 3y^2 - 12y + (z^2 - 4z + 12), (z^3 - 8z^2 + 19z - 12)x^2 + (z^2 - 4z + 3)x - (z^2 - 4z + 3), (z^3 - 8z^2 + 19z - 12)x^2 + (z^2 - 4z + 3)y - 2(z^2 - 4z + 3) \} \)

(3) **pass 1 for specialization** \( z = 4 \)

(4) \( F'_{\sigma} = \{ x^2_R - 6x_R - (z^2 - 4z - 5)R, 3y^2_R - 12y_R + (z^2 - 4z + 12)R, (z^3 - 8z^2 + 19z - 12)x^2_G + (z^2 - 4z + 3)x_R - (z^2 - 4z + 3)R, (z^3 - 8z^2 + 19z - 12)x^2_G + (z^2 - 4z + 3)y_R - 2(z^2 - 4z + 3)R \} \)

(5) \( \mathcal{O} = \{ 1 \}, \partial \mathcal{O} = \{ x, y \} \)

\( \mathcal{O} \)-BB\(_{z=4} = \{ \langle (z - 4)x^2_G + x_R - 1, (z - 4)x^2_G + y_R - 2R \rangle \} \)

(6) \( \mathcal{O} = \{ 1 \}, \partial \mathcal{O} = \{ x, y \} \)

\( \mathcal{O} \)-BB\(_{z=4} = \{ \langle (z - 4)x^2 + x - 1, (z - 4)x^2 + y - 2 \rangle \} \)

(7) \( B = \{ \langle 4 \}, \{ x, y \}, \langle (z - 4)x^2 + x - 1, (z - 4)x^2 + y - 2 \rangle \} \)

(3) **pass 2 for specialization** \( z = 3 \)

(4) \( F'_{\sigma} = \{ x^2_R - 6x_R - (z^2 - 4z - 5)R, 3y^2_R - 12y_R + (z^2 - 4z + 12)R \} \)

(5) \( \mathcal{O} = \{ 1, x, y, xy \}, \partial \mathcal{O} = \{ x^2, y^2, x^2y, xy^2 \} \)

\( \mathcal{O} \)-BB\(_{z=3} = \{ \langle x^2_R - 6x_R - (z^2 - 4z - 5)R, y^2_R - 4y_R + 1/3(z^2 - 4z + 12)R, x^2y_R - 6xy_R - y(z^2 - 4z - 5)R, xy^2_R - 4xy_R + 1/3(x(z^2 - 4z + 12)) \rangle \} \)

(6) \( \mathcal{O} = \{ 1, x, y, xy \}, \partial \mathcal{O} = \{ x^2, y^2, x^2y, xy^2 \} \)

\( \mathcal{O} \)-BB\(_{z=3} = \{ \langle x^2 - 6x - (z^2 - 4z - 5), y^2 - 4y + 1/3(z^2 - 4z + 12), x^2y - 6xy - y(z^2 - 4z - 5), xy^2 - 4xy + 1/3(x(z^2 - 4z + 12)) \rangle \} \)

(7) \( B = \{ \langle 4 \}, \{ 1 \}, \langle (z - 4)x^2 + x - 1, (z - 4)x^2 + y - 2 \rangle, \langle 3 \}, \{ x, y, xy \}, \langle x^2 - 6x - (z^2 - 4z - 5), y^2 - 4y + 1/3(z^2 - 4z + 12), x^2y - 6xy - y(z^2 - 4z - 5), xy^2 - 4xy + 1/3(x(z^2 - 4z + 12)) \rangle \} \)

(3) **pass 3 for specialization** \( z = 1 \)

(4) \( F'_{\sigma} = \{ x^2_R - 6x_R - (z^2 - 4z - 5)R, 3y^2_R - 12y_R + (z^2 - 4z + 12)R \} \)

(5) \( \mathcal{O} = \{ 1, x, y, xy \}, \partial \mathcal{O} = \{ x^2, y^2, x^2y, xy^2 \} \)

\( \mathcal{O} \)-BB\(_{z=3} = \{ \langle x^2_R - 6x_R - (z^2 - 4z - 5)R, y^2_R - 4y_R + 1/3(z^2 - 4z + 12)R, x^2y_R - 6xy_R - y(z^2 - 4z - 5)R, xy^2_R - 4xy_R + 1/3(x(z^2 - 4z + 12)) \rangle \} \)

(6) \( \mathcal{O} = \{ 1, x, y, xy \}, \partial \mathcal{O} = \{ x^2, y^2, x^2y, xy^2 \} \)
We should note that the colored version of any border bases algorithm is just used as a plugin for Algorithm \textbf{4} in step 5. As we have used the border bases algorithm which uses a degree compatible term ordering, we can also verify the output above with the CGB output. We can compare our output here because of Theorem \textbf{2.11} and Section \textbf{5}.

\textbf{Note:} The border system output depends on the colored version of the border bases algorithm we use. If we use the border bases algorithm which does not correspond to any Gröbner bases then we can generate comprehensive border bases for which we can’t verify the output with CGB output.

Let us see the output of comprehensive border bases algorithm. We will show the marked term of a polynomial by subscript \( m \).

(1) \( B_1 = \{(\{\overline{k} \setminus \{4, 3, 1\}\}, \emptyset, \{z^3 - 8z^2 + 19z - 12\}\} \)

(2) \( B' = \{(\{4\}, \{1\}, \{(z - 4)x^2 + x - 1, (z - 4)x^2 + y - 2\}, \{(3, 1) \}, \{1, x, y, xy\}, \{x^2 - 6x - (z^2 - 4z - 5), y^2 - 4y + 1/3(z^2 - 4z + 12)\}, \{x^2y - 6xy - y(z^2 - 4z + 12)\}\}\}

(3) Let \( f_3 = (z - 1)(z - 3)(z - 4) \)

(4) \( B' = \{(\{4\}, \{1\}, \{(z - 4)x^2 + x - 1, (z - 4)x^2 + y - 2\}, \{(3, 1) \}, \{1, x, y, xy\}, \{x^2 - 6x - (z^2 - 4z - 5), y^2 - 4y + 1/3(z^2 - 4z + 12)\}, \{x^2y - 6xy - y(z^2 - 4z + 12)\}\}\}

(5) \( B' = \{(\{4\}, \{1\}, \{(z - 1)(z - 3)\}(z - 4)x^2 + x - 1, (z - 1)(z - 3)\}(z - 4)x^2 + x - 1, (z - 3)(z - 4)x^2 + y - 2)\), \{(3, 1) \}, \{1, x, y, xy\}, \{x^2 - 6x - (z^2 - 4z - 5), y^2 - 4y + 1/3(z^2 - 4z + 12)\}, \{x^2y - 6xy - y(z^2 - 4z + 12)\}\}\}

(6) \( B = \{(z - 1)(z - 3)\}(z - 4)x^2 + x - 1, (z - 4)x^2 + y - 2)\), \{(3, 1) \}, \{1, x, y, xy\}, \{x^2 - 6x - (z^2 - 4z - 5), y^2 - 4y + 1/3(z^2 - 4z + 12)\}, \{x^2y - 6xy - y(z^2 - 4z + 12)\}\}\}

\( \mathcal{O} \)-BB$_{e=3} : \{x^2 - 6x - (z^2 - 4z - 5), y^2 - 4y + 1/3(z^2 - 4z + 12)\}, x^2y - 6xy - y(z^2 - 4z - 12)\}} \)

(7) \( B = \{(\{4\}, \{1\}, \{(z - 4)x^2 + x - 1, (z - 4)x^2 + y - 2\}, \{(3, 1) \}, \{1, x, y, xy\}, \{x^2 - 6x - (z^2 - 4z - 5), y^2 - 4y + 1/3(z^2 - 4z + 12)\}, x^2y - 6xy - y(z^2 - 4z - 12)\}} \)

(8) end for loop

(9) \( B = \{(\{4\}, \{1\}, \{(z - 4)x^2 + x - 1, (z - 4)x^2 + y - 2\}, \{(3, 1) \}, \{1, x, y, xy\}, \{x^2 - 6x - (z^2 - 4z - 5), y^2 - 4y + 1/3(z^2 - 4z + 12)\}, x^2y - 6xy - y(z^2 - 4z - 12)\}, \{(\overline{k} \setminus \{4, 3, 1\}\}, \emptyset, \{z^3 - 8z^2 + 19z - 12\}\} \)

(10) \( B = \{(\{4\}, \{1\}, \{(z - 4)x^2 + x - 1, (z - 4)x^2 + y - 2\}, \{(3, 1) \}, \{1, x, y, xy\}, \{x^2 - 6x - (z^2 - 4z - 5), y^2 - 4y + 1/3(z^2 - 4z + 12)\}, x^2y - 6xy - y(z^2 - 4z - 12)\}, \{(\overline{k} \setminus \{4, 3, 1\}\}, \emptyset, \{z^3 - 8z^2 + 19z - 12\}\} \)
\[(z - 4)(x^2y_m - 6xy - y(z^2 - 4z - 5)),\]
\[(z - 4)(xy^2_m - 4xy + 1/3z(z^2 - 4z + 12)),\]
\[z^3 - 8z^2 + 19z - 12}\]

Let's calculate \(O\)-BB for the specialization \(z = 4\) and \(z = 7\).

(1) \(z = 4\) i.e. \(\sigma_4\)

\[
\sigma_4(B) = \{x_m - 1, y_m - 2\} = \{x - 1, y - 1\}
\]
\[
O = T^n \setminus \langle x, y \rangle = \{1\}
\]
We can verify that the \(\sigma_4(B)\) is \(O\)-BB of \(\sigma_4(a)\).

(2) \(z = 7\) i.e. \(\sigma_7\)

\[
\sigma_7(B) = \{3x^2 + x_m - 1, \]
\[3x^2 + y_m - 2, \]
\[x^2_m - 6x - (z^2 - 4z - 5), \]
\[y^2_m - 4y + 1/3(z^2 - 4z + 12), \]
\[x^2y_m - 6xy - y(z^2 - 4z - 5), \]
\[xy^2_m - 4xy + 1/3x(z^2 - 4z + 12), 1\}
\[= \{1\}\]
\[
O = T^n \setminus \langle 1 \rangle = \emptyset
\]
We can verify that the \(\sigma_7(B)\) is \(O\)-BB of \(\sigma_7(a)\).

8. Concluding remarks

The theory of comprehensive Gröbner bases is an important tool for the studying parametric ideals. Applications of comprehensive Gröbner bases include ideal membership depending upon parameters, elimination of quantifier-blocks in algebraically closed fields, deformation of residue algebras, geometric theorem proving and many more.

In this paper we proposed the notion of comprehensive border bases for zero dimensional ideals. We established the existence of comprehensive border bases that need not correspond to any term order and hence to any comprehensive Gröbner bases. We also propose an algorithm to compute comprehensive border bases and study its relation with comprehensive Gröbner bases. The main aim of the proposed comprehensive border bases is to bring the features of border bases computation in the studies of zero-dimensional parametric ideals.

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References

Adams, W. & Loustaunau, P. (1994). An Introduction to Gröbner Bases. American Mathematical Society.

Ananth, P. V. & Dukkipati, A. (2011). Border basis detection is np-complete. In: Proceedings of the 36th international symposium on Symbolic and algebraic computation. ACM.

Ananth, P. V. & Dukkipati, A. (2012). Complexity of Gröbner basis detection and border basis detection. Theoretical Computer Science 459, 1–15.
Buchberger, B. (1965). An Algorithm for Finding a Basis for the Residue Class Ring of a Zero-Dimensional Polynomial Ideal (in German). Ph.D. thesis, University of Innsbruck, Austria. (reprinted in Buchberger (2006)).

Buchberger, B. (2006). Bruno Buchberger’s PhD thesis 1965: An algorithm for finding the basis elements of the residue class ring of a zero dimensional polynomial ideal. Journal of Symbolic Computation 41, 475–511.

Dunn III, W.-M. (1995). Algorithms and applications of comprehensive groebner bases. The University of Arizona.

Kalkbrener, M. (1997). On the stability of groebner bases under specializations. Journal of Symbolic Computation 24(1), 51–58.

Kapur, D., Sun, Y. & Wang, D. (2013). An efficient algorithm for computing a comprehensive gröbner system of a parametric polynomial system. Journal of Symbolic Computation 49, 27–44.

Kehrein, A. & Kreuzer, M. (2005). Characterizations of border bases. Journal of Pure and Applied Algebra 196(2), 251–270.

Kehrein, A. & Kreuzer, M. (2006). Computing border bases. Journal of Pure and Applied Algebra 205(2), 279–295.

Kehrein, A., Kreuzer, M. & Robbiano, L. (2005). An algebraists view on border bases. In: Solving polynomial equations. Springer, pp. 169–202.

Montes, A. (1999). Basic algorithms for specialization in gröbner bases. SIGSAM Bulletin (ACM Special Interest Group on Symbolic and Algebraic Manipulation) 33(3), 18.

Sato, Y. (1998). A new type of canonical gröbner bases in polynomial rings over von neumann regular rings. In: Proceedings of the 1998 international symposium on Symbolic and algebraic computation. ACM.

Stetter, H. J. (2004). Numerical polynomial algebra. SIAM.

Sturmfels, B. (1996). Grörbner bases and convex polytopes, vol. 8. AMS Bookstore.

Suzuki, A. & Sato, Y. (2003). An alternative approach to comprehensive grörbner bases. Journal of Symbolic Computation 36(3), 649–667.

Suzuki, A. & Sato, Y. (2006). A simple algorithm to compute comprehensive grörbner bases using grörbner bases. In: Proceedings of the 2006 international symposium on Symbolic and algebraic computation. ACM.

Weispfenning, V. (1992). Comprehensive grörbner bases. Journal of Symbolic Computation 14(1), 1–29.

Weispfenning, V. (2002). Comprehensive grörbner bases and regular rings. In: Symposium in Honor of Bruno Buchbergers 60th Birthday.

Weispfenning, V. (2006). Comprehensive grörbner bases and regular rings. Journal of Symbolic Computation 41(3), 285–296.

E-mail address: dubey.abhishek@csa.iisc.ernet.in, ad@csa.iisc.ernet.in

Dept. of Computer Science & Automation, Indian Institute of Science, Bangalore - 560012