Thermodynamics and critical behaviors of topological dS black holes with nonlinear source

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We discuss black hole solutions of Einstein gravity in presence of nonlinear electrodynamics in dS spacetime. Considering prescribed entropy, thermodynamic volume of dS spacetime, We investigate properties of the effective thermodynamic quantities under influence of nonlinearity parameter $\alpha$. They show a similar phase transition and criticality properties with that of black holes in AdS spacetime. And the nonlinearity parameter $\alpha$ combined with electric charge is found to have effects on the phase structure. By the Ehrenfest equations we prove the critical phase transition is a second order equilibrium transition.

Keywords: de Sitter spacetime; phase transition; nonlinear source; effective thermodynamic quantities; stability

PACS numbers: 04.70.-s, 05.70.Ce

I. INTRODUCTION

Since the four laws of black hole mechanics was discovered, black holes are widely believed to be thermodynamic and they possess standard thermodynamic quantities such as temperature and entropy\[1, 2]. The particular thermodynamic quantities of black hole and its holographic property are quantum essentially, so black hole is a macroscopical quantum system. Therefore, the studies of black hole thermodynamic properties provide an important window to investigate quantum gravity\[1, 3]. Phase transition is one of interesting subjects in studying a thermodynamic system. It is found that black holes may go through some interesting phase transition, for example the Hawking-Page phase transition\[4], and have

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some critical phenomena similar to that of usual thermodynamic systems.

Theoretically the cosmological constant term is expected to arise from the vacuum expectation value of a quantum field and hence can vary. Therefore, it may be considered in the first law of thermodynamics with its conjugate \[\text{[5–7]}\]. By this generalization, the cosmological constant \(\Lambda\) and its conjugate can be interpreted as thermodynamic pressure and volume of a black object system respectively. For \(d\)-dimensional spacetime the thermodynamic pressure \(P\) and its conjugate thermodynamic volume \(V\) are defined as \[\text{[8–11]}\]

\[
P = -\frac{1}{8\pi} \Lambda = \frac{(d - 1)(d - 2)}{16\pi l^2} \tag{1.1}
\]

\[
V = \left(\frac{\partial M}{\partial P}\right)_{S,Q,l,J_k} \tag{1.2}
\]

Recently critical behaviors and phase transitions of black holes have been extensively investigated by considering the cosmological constant as thermodynamic pressure \[\text{[12–28]}\]. It is interesting that the studies on the charged black holes show they may have an analogous phase transition with that of van der Waals-Maxwell’s liquid-gas. People have also been trying to construct a complete liquid-gas analogue system for black holes \[\text{[29, 30]}\].

Nonlinear field theories are of interest to different branches of mathematical physics because most physical systems are inherently nonlinear in the nature. The nonlinear electrodynamics (NLED) theories are considerably richer than the Maxwell field and in special case they reduce to the linear Maxwell theory. Recently, It was shown that NLED objects can remove both the big bang and black hole singularities \[\text{[31–36]}\]. The first attempt to couple the NLED with gravity was made by Hoffmann \[\text{[37]}\]. After that the effects of Born-Infeld (BI) NLED coupled to the gravitational field have been extensively studied. In the paper of Hendi \[\text{[38–41]}\], thermodynamic properties of AdS black holes with nonlinear source have been discussed. We take into account a NLED source and investigate the effects of nonlinearity on the phase transition properties of black holes in dS spacetime in this paper.

In the era of inflation, the Universe is in a quasi-dS space. The cosmological constant corresponds to vacuum energy and is usually considered as a candidate for dark energy. The accelerating Universe will evolve into another dS phase. In order to construct the entire evolution history of our Universe, we should have a clear perspective on the classical and quantum properties of dS space. It is known that with appropriate parameters dS spacetimes possess not only black hole horizon but cosmological horizon. Moreover both horizons have
thermal radiation but they are of different temperatures, and the two sets of thermodynamic quantities for the both horizons respectively satisfy the first law of thermodynamics \[42, 43\]. Recently, the study on the physical properties of dS spacetime has aroused great interest \[44–47\]. Take into account the thermodynamic relevance of the two horizons of dS spacetime and analyze the first law of thermodynamics satisfied by the two horizons, the Ref\[48–51\] obtained the effective temperature, effective pressure and effective electric potential. Moreover the thermodynamic behaviors of some dS spacetimes were discussed by the effective thermodynamics quantities. In this paper, we discuss the critical properties of charged dS spacetime with nonlinear source(RN-dS-N system). Based on the effective thermodynamic quantities of the RN-dS-N system, we put emphasis on discussing the effects of nonlinear electrodynamics disturbance on the critical behaviors of dS spacetime.

Outline of the paper is as follow: In the next section, we introduce the topological black holes with nonlinear source and the two sets of thermodynamic quantities corresponding to the two horizons of dS spacetime. In the third section, we analyze the relations of the thermodynamic quantities and consider the dS spacetime as a whole, then get the effective thermodynamic quantities. The critical behaviors of the dS spacetime as a thermodynamic system and the effect the nonlinearity parameters on critical behaviors are analysed in the fourth section. And in fifth section, by Ehrenfest scheme we testify the critical behaviors belong to the second order phase transition. In the last section, we make some discussions and conclusions. (We use the units \(G = \hbar = k_B = c = 1\))

\section{II. TOPOLOGICAL BLACK HOLES WITH NONLINEAR SOURCE}

The \((n+1)\)-dimensional action of Einstein gravity of nonlinear electrodynamics is \[52, 53\]:

\[
I_G = -\frac{1}{16\pi} \int_M d^{n+1}x \sqrt{-g} \left[ R - 2\Lambda + L(F) \right] - \frac{1}{8\pi} \int_{\partial M} d^n x \sqrt{-\gamma} \Theta(\gamma),
\]

where \(R\) is the scalar curvature, \(\Lambda\) is the cosmological constant. In this action,

\[
L(F) = -F + \alpha F^2 + O(\alpha^2),
\]

is the Lagrangian of nonlinear electrodynamics. \(F = F^\mu_\nu F^\nu_\mu\) is the Maxwell invariant, in which \(F^\mu_\nu = \partial_\mu A_\nu - \partial_\nu A_\mu\) is the electromagnetic field tensor and \(A_\mu\) is the gauge potential. In addition, \(\alpha\) denotes nonlinearity parameter which is small, so the effects of nonlinearity should be considered as a perturbation.
The \((n+1)\)-dimensional topological black hole solutions can take the form of

\[ ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2d\Omega_{n-1}^2, \quad (2.3) \]

where

\[ f(r) = k - \frac{m}{r^{n-2}} - \frac{2\Lambda r^2}{n(n-1)} + \frac{2q^2}{(n-1)(n-2)r^{2n-4}} - \frac{4q^4\alpha}{[2(n-2)(n+2) + (n-3)(n-4)]r^{4n-6}}. \quad (2.4) \]

\(m\) is an integration constant which is related to the mass of the black hole and the last term in Eq. (2.4) indicates the effect of nonlinearity. The asymptotical behavior of the solution is AdS or dS provided \(\Lambda < 0\) or \(\Lambda > 0\) and the case of asymptotically flat solution is permitted for \(\Lambda = 0\) and \(k = 1\).

When \(\Lambda > 0\), the black holes have black hole horizon and cosmological horizon, and \(f(r_{+,c}) = 0\). The temperatures at the two horizons respectively are

\[ T_+ = \frac{f'(r_+)}{4\pi} = \frac{1}{2\pi(n-1)} \left( \frac{(n-1)(n-2)k}{2r_+} - \Lambda r_+ - \frac{q^2}{r_+^{2n-3}} + \frac{2q^4\alpha}{r_+^{4n-5}} \right), \quad (2.5) \]

\[ T_c = -\frac{f'(r_c)}{4\pi} = -\frac{1}{2\pi(n-1)} \left( \frac{(n-1)(n-2)k}{2r_c} - \Lambda r_c - \frac{q^2}{r_c^{2n-3}} + \frac{2q^4\alpha}{r_c^{4n-5}} \right). \quad (2.6) \]

The ADM (Arnowitt-Deser-Misner) mass and electric charge parameter \(Q\) per unit volume \(V_{n-1}\) of the black hole are

\[ M = \frac{V_{n-1}(n-1)m}{16\pi}, \quad (2.7) \]

\[ Q = \frac{q}{4\pi}V_{n-1}, \quad (2.8) \]

with \(V_{n-1} = \frac{2\pi^{n/2}}{\Gamma(n/2)}\).

The mass of the black hole can be expressed as

\[ M = \frac{V_{n-1}(n-1)}{16\pi} \left( kr_{+,c}^{n-2} - \frac{2\Lambda r_{+,c}^{n}}{n(n-1)} + \frac{2q^2}{(n-1)(n-2)r_{+,c}^{2n-2}} \right. \]

\[ - \left. \frac{4q^4\alpha}{[2(n-2)(n+2) + (n-3)(n-4)]r_{+,c}^{4n-4}} \right). \quad (2.9) \]

The entropy and thermodynamic volume of the black hole corresponding to black hole horizon and cosmology horizon are
\[
S_+ = \frac{V_n-1r_+^{n-1}}{4}, \quad S_c = \frac{V_n-1r_c^{n-1}}{4}, \quad V_+ = \frac{V_n-1r_+^n}{n}, \quad V_c = \frac{V_n-1r_c^n}{n}. \tag{2.10}
\]

Recently, in the view of that cosmological constant is considered as thermodynamic pressure of black holes\cite{8–11}, substituting eq. (2.10) into eqs. (2.5) and (2.6) two thermodynamic systems can be obtained corresponding to black hole horizon and cosmological horizon respectively. On account of the two thermodynamic systems are independent of each other, their thermodynamic properties have been researched and achieved some meaningful results\cite{9, 43}.

### III. EFFECTIVE THERMODYNAMIC QUANTITIES OF DS SPACETIME

The thermodynamics volume and entropy of spherically symmetric dS spacetime satisfy\cite{9, 43}
\[
V = V_c - V_+ = \frac{V_n-1r_c^n}{n} (1 - x^n), \quad S = S_c + S_+ = \frac{V_n-1r_c^{n-1}}{4} (1 + x^{n-1}). \tag{3.1}
\]

in which, \(x = \frac{r_+}{r_c} \leq 1\). From Eq. (2.9), one can see that for the dS spacetime with black hole horizon and cosmological horizon both the positions of the two horizons \(r_+\) and \(r_c\) are the functions of the spacetime energy (mass)\(M\), electric charge\(Q\), and cosmological constant\(\Lambda\). So \(r_+\) and \(r_c\) are not independent of each other. Therefore if the two horizons are considered as two thermodynamic systems, they are not independent. We should take into account the relevance in studying the thermodynamic properties of dS spacetime.

Using the Eqs. (3.1) and (2.9), the energy (mass)\(M\) can be expressed as
\[
M = \frac{V_n-1(n-1)}{16\pi(1 - x^n)} r_c^{n-2} \left[ k(x^n - x^n) + \frac{2q^2}{(n-1)(n-2)r_c^{2(n-2)}} \left(1 - x^{2n-2} \right) \right] - \frac{4q^4\alpha}{[2(n-2)(n+2) + (n-3)(n-4)]r_c^{2(2n-3)}x^{3n-4}} (1 - x^{4n-4}) \right]. \tag{3.2}
\]

Combining Eqs. (3.1) and (3.2), as \(\alpha\) is a constant, one can see that the energy (mass) of the system is a function of entropy, thermodynamic volume, and electric charge, that is
\[
M = M(S, V, Q). \tag{3.3}
\]

It is well known that when the energy of the spacetime acts as a function of the thermodynamic quantities corresponding to black hole horizon or those corresponding to cosmological
horizon, the functions meet the first law of thermodynamics

\[ \delta M = T_+ \delta S_+ + \Phi_+ \delta Q + V_+ \delta P_0, \quad \delta M = -T_c \delta S_c + \Phi_c \delta Q + V_c P_0, \quad (3.4) \]
in which \( \Phi_+ \) and \( \Phi_c \) are the charge potential at black hole horizon and at cosmological horizon respectively and

\[ P_0 = -\frac{\Lambda}{8\pi}. \quad (3.5) \]

There exist two baffling problems considering the two thermodynamic systems presented in the Eq. (3.4): Firstly, the energy \( M \), electric charge \( Q \) and cosmological constant \( \Lambda \) are the common state parameters of the two systems. Thus the thermodynamic quantities of the two systems are dependent. Secondly, the black hole horizon radiation temperature \( T_+ \) and the cosmological radiation temperature \( T_c \) present in Eq. (3.4) are usually different. So dS spacetime is in nonequilibrium. At present, a mature theory has not been found to analyze the nonequilibrium thermodynamic system. The characteristic that the two thermodynamic systems own the common state parameters reminds us to build an effective thermodynamic system to reflect the thermodynamics properties of the dS spacetime.

\[ dM = TdS + \Phi dQ - PdV. \quad (3.6) \]

The effective temperature \( T \), effective electric potential \( \Phi \), and effective pressure \( P \) respectively are

\[ T = \left( \frac{\partial M}{\partial S} \right)_{Q,V} = \left( \frac{\partial M}{\partial x} \right)_{r_c,q} \left( \frac{\partial V}{\partial r} \right)_{x,q} - \left( \frac{\partial V}{\partial x} \right)_{r_c,q} \left( \frac{\partial M}{\partial r} \right)_{x,q}, \quad (3.7) \]

\[ \Phi = \left( \frac{\partial M}{\partial Q} \right)_{S,V} = \left( \frac{\partial M}{\partial q} \frac{\partial q}{\partial Q} \right)_{x,q} = \frac{(n-1)(1-x^{2n-2})q}{(1-x^n)r_c^{n-2}x^{n-2}} \left[ \frac{1}{(n-1)(n-2)} \right], \quad (3.8) \]

\[ P = -\left( \frac{\partial M}{\partial V} \right)_{Q,S} = \left( \frac{\partial M}{\partial x} \right)_{r_c,q} \left( \frac{\partial S}{\partial r} \right)_{x,q} - \left( \frac{\partial S}{\partial x} \right)_{r_c,q} \left( \frac{\partial M}{\partial r} \right)_{x,q} \quad (3.9) \]

The Eq. (3.6) reflects thermodynamic properties of the whole dS spacetime rather than that of a horizon, therefore it gives more comprehensive view of dS spacetime.
IV. PHASE TRANSITION IN TOPOLOGICAL BLACK HOLES WITH NONLINEAR SOURCE SPACETIME

On the basis of the previous section, we study the phase transition and critical behaviors of the RN-dS-N system. We analyze the effective thermodynamic quantities by Van der Waals equation, and investigate the relation of the effective pressure and thermodynamic volume when the temperature is kept constant. Using the Gibbs free energy criterion, we analyze the phase transition of the system.

When the electric charge $q$ and the nonlinearity parameter $\alpha$ are kept as constant, the critical point can be obtained by

$$\left(\frac{\partial P}{\partial V}\right)_T = 0, \quad \left(\frac{\partial^2 P}{\partial V^2}\right)_T = 0. \quad (4.1)$$

$$\left(\frac{\partial P}{\partial V}\right)_T = \frac{\partial (P, T)}{\partial (x, r_c)} = f(x, r_c) = 0, \quad (4.2)$$

$$\left(\frac{\partial^2 P}{\partial V^2}\right)_T = \left(\frac{\partial f}{\partial V}\right)_T = \frac{\partial (f, T)}{\partial (x, r_c)} = 0. \quad (4.3)$$

By the Eqs. (3.1), (3.7), (3.9) and (4.1) the critical quantities can be derived when $Q$, $\alpha$ and dimensionality $n$ are given certain values. Table 1 below shows some critical values at dimensionality $n = 3$.

Under the conditions $0 < x < 1$ and $T > 0$, we derive the relation of $P$ and $V$ numerically as $T$ take some certain values nearby the critical temperature $T_c$. And we depict these effective isotherms of the RN-dS-N system in the Fig.1.

It can be seen in Table 1 that with increasing $\alpha$ all of the critical values $x_c$, the critical temperature $T_c$ and the critical pressure $P_c$ are increasing as $Q$ is given. And when $Q$ is greater the increases are more obvious. The other critical quantities in the table go just the opposite. Similarly, as $\alpha = 0.1$, all of $x_c$, $T_c$ and $P_c$ are increasing with increasing $Q$, the other critical quantities in the table go the opposite.

Both Table 1 and Fig. 1 show that the nonlinear parameters $\alpha$ and the electric charge $Q$ influence the critical state together. Fig.1 shows that as either of the values of $\alpha$ and $Q$ is greater or both of them are greater the more acceptable physical states exist within the smaller volume range, that is, the phase structure is more completed and it is more like
### TABLE I: Critical values of the effective thermodynamic system for different $Q$ and $\alpha$ as $n = 3$

| $n=3$ | $Q=1$ | $Q=2$ | $Q=3$ |
|-------|-------|-------|-------|
|       | $\alpha=0.001$ | $\alpha=0.01$ | $\alpha=0.1$ | $\alpha=0.3$ | $\alpha=0.6$ | $\alpha=0.1$ | $\alpha=0.3$ | $\alpha=0.6$ |
| $x^c$ | 0.73222 | 0.73230 | 0.73316 | 0.73559 | 0.74274 | 0.13443 | 0.08697 | 0.15771 | 0.23708 |
| $r^c$ | 3.5058 | 3.5025 | 3.4675 | 3.3760 | 7.5636 | 13.915 | 10.554 | 8.8944 |
| $T^c$ | 0.008015 | 0.008018 | 0.008139 | 0.008312 | 4.103E-5 | 6.541E-6 | 4.562E-5 | 1.601E-4 |
| $P^c$ | 6.055E-4 | 6.061E-4 | 6.128E-4 | 6.298E-4 | 6.666E-4 | 1.775E-6 | 1.550E-7 | 1.409E-6 |
| $V^c$ | 109.64 | 109.30 | 105.82 | 97.019 | 78.215 | 1807.9 | 11278.4 | 4904.4 |
| $M^c$ | 1.2086 | 1.2080 | 1.2007 | 1.1821 | 1.1404 | 2.1764 | 3.6970 | 3.1454 |
| $S^c$ | 59.315 | 59.207 | 58.077 | 55.179 | 48.758 | 182.96 | 612.89 | 358.61 |
| $G^c$ | 0.7996 | 0.7995 | 0.7979 | 0.7942 | 0.7873 | 2.172 | 3.695 | 3.136 |

**FIG. 1:** Effective isotherms in $P - V$ diagrams of $n+1$ dimensional RN-dS-N system as $n = 3$.

The dashed lines match $T < T^c$, the thick solid lines match $T = T^c$, and the thin solid lines match $T > T^c$.

that in AdS spacetime. It can not be seen in the Fig.1 whether there exist stable phase transition, especially in Fig.1(a) and Fig.1(b). We discuss the phase transition near the
effective critical temperature $T^c$ by Gibbs free energy criterion.

Gibbs free energy defined [54] as

$$G = M - TS + PV$$  \hspace{1cm} (4.4) $$

Under the conditions $0 < x < 1$ and $T > 0$, we depict the $G - P$ diagrams with some effective isotherms nearby the critical temperature $T^c$ for the RN-dS-N system in Fig.2.

![Graphs showing effective isotherms](image)

(a) $Q = 1, \alpha = 0.1$  \hspace{1cm} (b) $Q = 3, \alpha = 0.1$  \hspace{1cm} (c) $Q = 3, \alpha = 0.3$

FIG. 2: Effective isotherms in $G - P$ diagrams of $n + 1$ dimensional RN-dS-N system as $n = 3$. The dashed lines, the thick solid lines and the thin solid lines mean the same as that in Fig.1.

In Fig.2, Gibbs free energy $G$ versus effective pressure $P$ presents a swallow tail form below the critical temperature. According to Gibbs free energy criterion, below the critical temperature there exist phase transitions but above the critical temperature there not. And the larger $Q$ and $\alpha$, the phase transition is more obvious. Near the effective critical temperature $T^c$, the phase structure of the RN-dS-N system is similar that of van der Waals-Maxwell’s liquid-gas system.

V. ANALYTICAL CHECK OF THE CLASSICAL EHRENFEST EQUATIONS
AT CRITICAL POINT

According to Ehrenfest scheme, at the phase transition point, that the chemical potential (molar Gibbs free energy) and its first partial derivatives are continuous but its second partial derivatives are mutational indicate that the phase transition belongs to a second order one. We derive the specific heat $C_P$, expansion coefficient $\beta$, and the isothermal compressibility $\kappa$ of the RN-dS-N system. They are given below as the dimension $n$, electric charge $q$, and
α are regarded as constants.

\[ C_P = T \left( \frac{\partial S}{\partial T} \right)_P = -T \frac{\partial^2 G}{\partial T^2} = T \left( \frac{\partial S}{\partial x} \right)_{rc} \left( \frac{\partial P}{\partial x} \right)_{rc} - \left( \frac{\partial S}{\partial x} \right)_{x} \left( \frac{\partial P}{\partial x} \right)_{rc}, \]  
(5.1)

\[ \beta = \frac{1}{V} \left( \frac{\partial V}{\partial T} \right)_P = \frac{1}{V} \frac{\partial^2 \mu}{\partial T \partial P} = \frac{1}{V} \left( \frac{\partial V}{\partial x} \right)_{rc} \left( \frac{\partial P}{\partial x} \right)_{rc} - \left( \frac{\partial V}{\partial x} \right)_{x} \left( \frac{\partial P}{\partial x} \right)_{rc}, \]  
(5.2)

\[ \kappa = -\frac{1}{V} \left( \frac{\partial V}{\partial P} \right)_T = -\frac{1}{V} \frac{\partial^2 \mu}{\partial P^2} = \frac{1}{V} \left( \frac{\partial V}{\partial x} \right)_{rc} \left( \frac{\partial T}{\partial x} \right)_{rc} - \left( \frac{\partial V}{\partial x} \right)_{x} \left( \frac{\partial T}{\partial x} \right)_{rc}, \]  
(5.3)

In Fig.3, \( C_P - P, \beta - P, \) and \( \kappa - P \) curves for given values of \( Q \) and \( \alpha \) are shown. According to a preliminary inspection the phase transitions at critical temperature belong to a second order one. The phase transitions below critical temperature are more complex.

In Fig. 3, \( C_P - P, \beta - P, \kappa - P \) diagrams of \( n + 1 \) dimensional RN-dS-N system as \( n = 3 \). The dashed lines, thick solid lines, and thin solid lines represent the same as that in Fig. 1.

The Ehrenfest Equations

\[ \left( \frac{\partial P}{\partial T} \right)_S = \frac{C_P^2 - C_P^1}{TcVc(\beta^2_q - \beta^1_q)} = \frac{\Delta C_P}{TcVc\Delta \beta_q}, \]  
(5.4)
\[
\left( \frac{\partial P}{\partial T} \right)_V = \frac{\beta^2_q - \beta^1_q}{\kappa^2_q - \kappa^1_q} = \frac{\Delta \beta_q}{\Delta \kappa_q},
\] (5.5)

The superscripts 1 and 2 represent phases 1 and 2 respectively, and the subscript \( q \) represents \( q \) remains unchanged.

According to Maxwell relations
\[
\left( \frac{\partial V}{\partial S} \right)_P = \left( \frac{\partial T}{\partial P} \right)_S, \quad \left( \frac{\partial V}{\partial S} \right)_T = \left( \frac{\partial T}{\partial P} \right)_V,
\] (5.6)

we can get
\[
\left[ \left( \frac{\partial P}{\partial T} \right)_S \right]^c = \left[ \left( \frac{\partial S}{\partial V} \right)_P \right]^c, \quad \left[ \left( \frac{\partial S}{\partial V} \right)_T \right]^c = \left[ \left( \frac{\partial P}{\partial T} \right)_V \right]^c.
\] (5.7)

Note that the footnote 'c' denotes the values of physical quantities at the critical point.

Substitute Eq.(5.7) into Eqs. (5.4) and (5.5),
\[
\Delta C_P^T = \left[ \left( \frac{\partial S}{\partial V} \right)_P \right]^c, \quad \Delta \beta_q \Delta \kappa_q = \left[ \left( \frac{\partial S}{\partial V} \right)_T \right]^c.
\] (5.8)

From Eq.(4.2), at the critical point, there is
\[
\frac{\partial (P, T)}{\partial (r_c, x)} = 0.
\] (5.9)

Moreover
\[
\left( \frac{\partial S}{\partial V} \right)_P = \frac{\partial (S, P)}{\partial (r_c, x)}, \quad \left( \frac{\partial S}{\partial V} \right)_T = \frac{\partial (S, T)}{\partial (r_c, x)}.
\] (5.10)

Substitute (5.9) into (5.10), one can get
\[
\left( \frac{\partial S}{\partial V} \right)_P = \left( \frac{\partial S}{\partial V} \right)_T^c.
\] (5.11)

So far, we have proved the validity of both Ehrenfest equations at the critical point. Utilizing Eqs. (5.11) and (5.8), the Prigogine-Defay (PD) ratio can be calculated,
\[
\Pi = \frac{\Delta C_P \Delta \kappa_q}{T_c V_c \Delta \beta_q^2} = 1.
\] (5.12)

Hence the phase transition occurring at \( T = T_c \) is a second order equilibrium transition. The conclusion is similar to that in AdS spacetime \[54\].
VI. DISCUSSIONS AND CONCLUSIONS

In this paper, considering the correlation of the thermodynamic quantities corresponding to the two horizons respectively in dS spacetime, we discuss thermodynamic properties of RN-dS spacetime with nonlinear source by a set of effective thermodynamic quantities, which reflect thermodynamic property of the two horizons and the whole dS spacetime as a thermodynamic system.

we investigate the phase transition of the RN-dS-N system, and found that the nonlinearity parameters $\alpha$ along with electric charge influence the phase structure of the system, which can be seen in Table 1, Fig.1 and Fig.2. When the effective temperature is below critical temperature, a phase transition can happen, which can be seen in Fig. 2 and inferred from Gibbs free energy criterion.

We carry out an analytical check of Ehrenfest equations and derive the speical heat, expansion coefficient, and the isothermal compressibility of the effective RN-dS-N system. And find that the RN-dS-N system undergoes a second order equilibrium phase transition at the critical point. This result is similar to the nature of Van der Waala liquid-gas phase transition at the critical point.

Acknowledgments

This work is supported by NSFC under Grant No.11475108, by the doctoral Sustentation Fund of Shanxi Datong University (2015-B-10), and by the Natural Science Foundation for Young Scientists of Shanxi Province,China (Grant No.2012021003-4).

[1] J. D. Bekenstein, Phys. Rev. D 7, 949 (1973).
[2] J. M. Bardeen, B. Carter, and S. W. Hawking, Commun. Math. Phys. 31, 161 (1973).
[3] S. W. Hawking, Nature 248, 30 (1974).
[4] S. Hawking and D. N. Page, Commun. Math. Phys. 87, 577 (1983).
[5] G. Gibbons, R. Kallosh, and B. Kol, Phys. Rev. Lett. 77, 4992 (1996).
[6] J. D. Brown and C. Teitelboim, Phys. Lett. B 195, 177 (1987).
[7] M. M. Caldarelli, G. Cognola, and D. Klemm, Classical Quantum Gravity 17, 399 (2000).
[8] D. Kubiznak and Robert B. Mann, J. High Energy Phys. 07 (2012) 033, arXiv:1205.0559
[9] B. P. Dolan, David Kastor, David Kubiznak, Robert B. Mann, and Jennie Traschen, Phys. Rev. D. 87, 104017 (2013), arXiv:1301.5926
[10] S. Gunasekaran, D. Kubiznak, and R. B. Mann, JHEP 1211, 110 (2012), arXiv:1208.6251
[11] M. Cvetic, G.W. Gibbons, D. Kubiznak, and C.N. Pope, Phys. Rev. D 84, 024037 (2011), arXiv:1012.2888
[12] D. Kastor, S. Ray, and J. Traschen, Classical Quantum Gravity 26, 195011 (2009).
[13] R. Banerjee, S. K. Modak, and S. Samanta, Phys. Rev. D 84, 064024 (2011), arXiv:1005.4832
[14] R. Banerjee and D. Roychowdhury, J. High Energy Phys. 11 (2011) 004, arXiv:1109.2433
[15] R. Banerjee, S. Ghosh, and D. Roychowdhury, Phys. Lett. B. 696, 156 (2011), arXiv:1008.2644
[16] R.-G. Cai, L.-M. Cao, L. Li, and R.-Q. Yang, J. High Energy Phys. 09 (2013) 005., arXiv:1306.6233[gr-qc].
[17] M.-S. Ma, F. Liu, and R. Zhao, Classical Quantum Gravity 31, 095001 (2014), arXiv:1403.0449
[18] R. Zhao, M. Ma, H. Li and L. Zhang, Adv. High Energy Phys. 2013, 371084 (2013).
[19] R. Zhao, H.-H. Zhao, M.-S. Ma, and L.-C. Zhang, Eur. Phys. J. C 73, 2645 (2013), arXiv:1305.3725.
[20] M.-S. Ma and R. Zhao, Phys. Rev. D 89, 044005 (2014).
[21] M.-S. Ma, Phys. Lett. B 735, 45 (2014).
[22] D.-C. Zou, S.-J. Zhang, and B. Wang, Phys. Rev. D 89, 044002 (2014), arXiv:1311.7299
[23] D.-C. Zou, Y. Liu, and B. Wang, Phys. Rev. D 90, 044063 (2014), arXiv: 1404.5194.
[24] S.-W. Wei and Y.-X. Liu, Phys. Rev. D 87, 044014 (2013), arXiv:1209.1707[gr-qc].
[25] S.-Wen Wei and Y.-X. Liu, Phys. Rev. Lett. 115, 111302 (2015), arXiv:1502.00386
[26] M. Kord Zangeneh, A. Dehyadegari, A. Sheykhi, arXiv:1602.03711[hep-th].
[27] Y. Liu, D.-C. Zou, and B. Wang, J. High Energy Phys. 09 (2014) 179, arXiv:1405.2644
[28] H.-H. Zhao, L.-C. Zhang, M.-S. Ma, and R. Zhao, Classical Quantum Gravity 32, 145007 (2015).
[29] S.H. Hendi, R. B. Mann, S. Panahiyan, and B. Eslam Panah, Phys. Rev. D 95, 021501 (2017).
[30] A. Dehyadegaria, A. Sheykha, and A. Montakhaba, Phys. Lett. B 768, 235 (2017).
[31] E. Ayon-Beato and A. Garcia, Gen. Relativ. Gravit. 31, 629 (1999).
[32] E. Ayon-Beato and A. Garcia, Phys. Lett. B 464, 25 (1999).
[33] V. A. De Lorenci, R. Klippert, M. Novello and J. M. Salim, Phys. Rev. D 65, 063501 (2002).
[34] I. Dymnikova, Classical Quantum Gravity 21, 4417 (2004).
[35] C. Corda and H. J. Mosquera Cuesta, Mod. Phys. Lett. A 25, 2423 (2010).
[36] C. Corda and H. J. Mosquera Cuesta, Astropart. Phys. 34, 587 (2011).
[37] B. Hoffmann, Phys. Rev. 47, 877 (1935).
[38] S. H. Hendi and M. H. Vahidinia, Phys. Rev. D 88, 084045 (2013), arXiv: 1510.06269.
[39] S. H. Hendi, B. Eslam Panah, M. Momennia and S. Panahiyan, Eur. Phys. J. C 75, 457 (2015), arXiv: 1509.03081.
[40] S. H. Hendi, S. Panahiyan and M. Momennia, Int. J. Mod. Phys. D 25, 1650063 (2016), arXiv: 1503.03340.
[41] S. H. Hendi, S. Panahiyan, M. Momennia and B. Eslam Panah, Int. J. Mod. Phys. D 26, 1750026 (2017).
[42] R. G. Cai, Nucl. Phys. B 628, 375 (2002).
[43] Y. Sekiwa, Phys. Rev. D 73, 084009 (2006), arXiv:hep-th/0602269.
[44] E. T. Akhmedov, Int. J. Mod. Phys. D 23, 1430001 (2014).
[45] X. Chen, Int. J. Mod. Phys. A 27, 1250166 (2012).
[46] I. Arraut, Mod. Phys. Lett. A 28, 1350019 (2013).
[47] R. Zhao, M. Ma, H. Zhao, and L. Zhang, Adv. High Energy Phys. 2014, 124854 (2014).
[48] M.-S. Ma, H.-H. Zhao, L.-C. Zhang, and R. Zhao, Int. J. Mod. Phys. A 29, 1450050 (2014), arXiv: 1312.0731.
[49] H.-H. Zhao, L.-C. Zhang, M.-S. Ma, and R. Zhao, Phys.Rev. D 90, 064018 (2014).
[50] M.-S. Ma, L.-C. Zhang, H.-H. Zhao, and R. Zhao, Adv. High Energy Phys. 2015, 134815 (2015), arXiv:1410.5950.
[51] X. Guo, H. Li, L. Zhang, and R. Zhao, Phys. Rev. D 91, 084009 (2015).
[52] S. H. Hendi, and M. Momennia, Eur. Phys. J. C 75, 54 (2015), arXiv:1501.04863.
[53] S. H. Hendi and R. Naderi, Phys. Rev. D 91, 024007 (2015), arXiv:1510.06269.
[54] J.-X. Mo, G.-Q. Li, W.-B. Liu, Phys. Lett. B 730, 111 (2014).