Local numerical range for a class of $2 \otimes d$ hermitian operators

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Abstract
A local numerical range is analyzed for a family of circulant observables and states of composite $2 \otimes d$ systems. It is shown that for any $2 \otimes d$ circulant operator $O$ there exists a basis giving rise to the matrix representation with real non-negative off-diagonal elements. In this basis the problem of finding extremum of $O$ on product vectors $|x\rangle \otimes |y\rangle \in \mathbb{C}^2 \otimes \mathbb{C}^d$ reduces to the corresponding problem in $\mathbb{R}^2 \otimes \mathbb{R}^d$. The final analytical result for $d = 2$ is presented.

1 Introduction

For any linear operator $O$ acting in the Hilbert space $\mathcal{H}$ one defines its numerical range \([1]\)

$$\text{NR}(O) := \{ \langle \psi | O | \psi \rangle : | \psi \rangle \in \mathcal{H}, \| \psi \| = 1 \} .$$

(1)

Clearly, $\text{NR}(O)$ defines a subset of the complex plane. Now, if $O$ is hermitian then $\text{NR}(O) = [\lambda_{\text{min}}, \lambda_{\text{max}}]$, where $\lambda_{\text{min}}$ and $\lambda_{\text{max}}$ denote the minimal and maximal eigenvalue of $O$. Recently, more specific characterization of the hermitian operator called restricted numerical range has been introduced in order to describe the interval of expectation values for some specific sets of vectors in $\mathcal{H}$ \([3]\). In particular, if $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ one introduces the notion of local (product) numerical range \([4]\)

$$L\text{NR}(O) = \{ \langle x \otimes y | O | x \otimes y \rangle : \| x \| = \| y \| = 1 , | x \rangle \in \mathcal{H}_1 , | y \rangle \in \mathcal{H}_2 \} .$$

(2)

It is clear that if $O$ is hermitian then

$$L\text{NR}(O) = [\gamma_{\text{min}}, \gamma_{\text{max}}] \subseteq \text{NR}(O) = [\lambda_{\text{min}}, \lambda_{\text{max}}] .$$

It turns out that the notions of various restricted numerical ranges are useful in many branches of quantum information theory (see \([3, 5, 6]\) for details). For example any entanglement witness $W$ can be written in the following form \([7, 8, 9, 10]\)

$$W = \chi \mathbb{I} - O ,$$

for some hermitian operator $O$ and a positive number $\chi$. Now, the necessary condition for $W$ to be an entanglement witness is $\chi > \gamma_{\text{max}}$. In practice, it is very hard to determine $L\text{NR}$ for a given hermitian operator. In this paper we limit ourselves to the case when $O$ acting on $\mathbb{C}^2 \otimes \mathbb{C}^d$ belongs to a class of circulant operators \([11]\) (see also \([12, 13]\)).
The paper is organized as follows. Sect. 2 is devoted to some basic definitions and properties of circulant bipartite operators. In Sect. 3, we emphasize that it is always possible to bring a matrix representing the circulant operator to the so-called real form using a local unitary transformation. In Sect. 4 we show how to carry out calculations of the local numerical range for circulant operators. The final analytical result for $d = 2$ is presented in Sect. 5 together with some instructive examples.

2 Circulant operators in $\mathbb{C}^2 \otimes \mathbb{C}^d$

Let $\mathcal{H} = \mathbb{C}^2 \otimes \mathbb{C}^d$ and let $\{|g_i \rangle \otimes |f_k \rangle \}$ ($i = 1, 2, k = 1, \ldots, d$) be an orthonormal product basis in $\mathcal{H}$. One defines the family of 2-dimensional subspaces $\Sigma_k$ in $\mathcal{H}$:

$$\Sigma_1 = \text{span}\{|g_1 \rangle \otimes |f_1 \rangle, |g_2 \rangle \otimes |f_2 \rangle\},$$

$$\Sigma_2 = \text{span}\{|g_1 \rangle \otimes |f_2 \rangle, |g_2 \rangle \otimes |f_3 \rangle\},$$

$$\vdots$$

$$\Sigma_d = \text{span}\{|g_1 \rangle \otimes |f_d \rangle, |g_2 \rangle \otimes |f_1 \rangle\}.$$

It is clear that $\Sigma_k$ give rise to the direct sum decomposition $\mathcal{C}^2 \otimes \mathbb{C}^d = \bigoplus_{k=1}^{d} \Sigma_k$. (3)

We shall call (3) a circulant decomposition. Now, we call a linear operator $\mathcal{O} \in B(\mathcal{H})$ to be circulant operator with respect to a circulant decomposition (3) iff

$$\mathcal{O} = \mathcal{O}_1 \oplus \ldots \oplus \mathcal{O}_d,$$

where $\mathcal{O}_k$ is supported on $\Sigma_k$, that is,

$$\mathcal{O}_k = \sum_{i,j=1}^{2} a^{(k)}_{ij} |g_i \rangle \langle f_{i+k} | \otimes |f_j \rangle \langle f_{j+k} |,$$

and $||a^{(k)}_{ij}||$ is a $2 \times 2$ complex matrix. In particular, for $d = 2$ and $d = 3$ we obtain the following matrix representations of the circulant operators (in the basis $|g_i \rangle \otimes |f_k \rangle$)

$$\begin{pmatrix}
    a^{(2)}_{11} & \cdot & \cdot & a^{(2)}_{12} \\
    \cdot & a^{(1)}_{11} & a^{(1)}_{12} & \cdot \\
    \cdot & a^{(1)}_{21} & a^{(1)}_{22} & \cdot \\
    a^{(2)}_{21} & \cdot & \cdot & a^{(2)}_{22}
\end{pmatrix},$$

$$\begin{pmatrix}
    a^{(3)}_{11} & \cdot & \cdot & a^{(3)}_{12} \\
    \cdot & a^{(1)}_{11} & \cdot & \cdot & a^{(1)}_{12} \\
    \cdot & \cdot & a^{(2)}_{11} & a^{(2)}_{12} & \cdot \\
    a^{(3)}_{21} & \cdot & \cdot & a^{(3)}_{22} \\
    \cdot & a^{(1)}_{21} & \cdot & \cdot & a^{(1)}_{22}
\end{pmatrix},$$

where to make the picture more transparent we replaced all zeros by dots. Interestingly for $d = 2$ the circulant matrix displays characteristic X-shape. Such 2-qubit states have been recently investigated in [14, 15, 16, 17, 18]. In the following we limit ourselves to circulant states and
Proposition 1

There exists an orthonormal product basis

\[ w_{ik} = e^{i(k-i)}, \mod d, \]

for \( i = 1, 2, k = 1, \ldots, d, \) and

\[ a_{12}^{(k+2)} = u_k e^{i\alpha_k}, \]

where \( u_k = |a_{12}^{(k+2)}| \geq 0, \) and \( \alpha_k \in (-\pi, \pi]. \) As a consequence, the general circulant observable reads

\[ O = \sum_{i=1}^{2} \sum_{k=1}^{d} w_{ik} |g_i\rangle \langle g_i| \otimes |f_k\rangle \langle f_k| + \left( \sum_{k=1}^{d} u_k e^{i\alpha_k} |g_i\rangle \langle g_i| \otimes |f_k\rangle \langle f_k| + \text{h.c.} \right), \]

where as usual h.c. stands for hermitian conjugation.

3 Real representation of circulant operators

Let \( O \) be an hermitian circulant operator living in \( \mathbb{C}^2 \otimes \mathbb{C}^d. \) One has the following

**Proposition 1** There exists an orthonormal product basis \( \{|g_i\rangle \otimes |f_k\rangle\} \) such that

1. \( O \) is circulant with respect to the circulant decomposition constructed out of \( \{|g_i\rangle \otimes |f_k\rangle\} \).
2. matrix elements of \( O \) with respect to \( \{|g_i\rangle \otimes |f_k\rangle\} \) satisfy:

\[ w_{ik}' = w_{ik}, \quad a_{12}^{(k+2)'} = |a_{12}^{(k+2)}| = u_k. \]

**Proof.** Let \( |g_i\rangle = U_1 |g_i\rangle \) and \( |f_k\rangle = U_2 |f_k\rangle, \) where \( U_1 \) and \( U_2 \) are unitary operators with the following matrix representations in the original basis \( |g_i\rangle \) and \( |f_k\rangle: \)

\[ U_1 = D[1, e^{i\mu_1}], \quad U_2 = D[1, e^{i\mu_2}, \ldots, e^{i\mu_k}], \]

where \( D[a_1, \ldots, a_k] \) denotes diagonal \( k \times k \) matrix with diagonal entries \( a_1, \ldots, a_k. \) One has

\[ O = \sum_{i=1}^{2} \sum_{k=1}^{d} w_{ik}^f |g_i\rangle \langle g_i| \otimes |f_k\rangle \langle f_k| + \left( \sum_{k=1}^{d} u_k e^{i\theta_k} |g_i\rangle \langle g_i| \otimes |f_k\rangle \langle f_k| + \text{h.c.} \right), \]

where the phases \( \theta_k \) satisfying the following relations (mod(2\pi))

\[ \theta_1 = \alpha_1 - \mu_1 - \mu_2, \]
\[ \theta_k = \alpha_k - \mu_1 + \mu_k - \mu_{k+1}, \quad k = 2, \ldots, d - 1 \]
\[ \theta_d = \alpha_d - \mu_1 + \mu_d. \]

Formulas (10) proves that \( O \) is circulant with respect the circulant decomposition constructed out of \( \{|g_i\rangle \otimes |f_k\rangle\} \). Now, we show that one can remove all the phases \( \theta_k \) by the appropriate choice of \( \mu_k. \) Note, that (11) may be rewritten as a matrix equation \( \alpha - \theta = W \mu, \) where the matrix \( W \) is defined by

\[ W_{k1} = 1, \]
\[ W_{kk} = -W_{k,k+1}, \quad k > 1, \]


and the remaining elements vanish. Note that taking $d$-vector $\mu = (\mu_1, \ldots, \mu_d)$ which satisfies the matrix equation
\[ \alpha = W \mu, \] (13)
one finds $\vartheta = 0$. It can be done due to the fact that $\det W = d(-1)^{d+1} \neq 0$ which ends the proof.

We will call the corresponding matrix representation of $O$ with respect to $\{|g'_i\otimes |f'_k\rangle\}$ real representation.

4 Local Numerical Range for a Circulant Operator

Let $O$ be an hermitian circulant operator with respect to a fixed basis $|g_i\rangle \otimes |f_k\rangle$ in $\mathbb{C}^2 \otimes \mathbb{C}^d$, and let us define
\[ F(x, y) = \frac{\langle x \otimes y |O| x \otimes y \rangle}{\langle x \otimes y | x \otimes y \rangle}. \] (14)

Now to provide $\text{LNM}(O)$ one has to find $\gamma_{\text{min}} = \inf F(x, y)$ and $\gamma_{\text{max}} = \sup F(x, y)$. Let
\[ \gamma_{\text{min}} = F(x^-, y^-), \quad \gamma_{\text{max}} = F(x^+, y^+). \] (15)

One has the following

**Proposition 2** The corresponding vectors $|x^\pm\rangle \in \mathbb{C}^2$ and $|y^\pm\rangle \in \mathbb{C}^d$ have the following components with respect to basis $|g'_i\rangle$ and $|f'_k\rangle$ provided in Proposition 1
\[ |x^\pm\rangle = (x^\pm_1, x^\pm_2 e^{i\beta_1}), \quad |y^\pm\rangle = (y^\pm_1, \ldots, y^\pm_d e^{i\beta_d}), \] (16)

where
\[ x^\pm_1 \geq 0, \quad y^\pm_k \geq 0. \] (17)

**Proof.** Consider e.g. $\gamma_{\text{min}}$ and to simplify notation let us write simply $|x\rangle$ instead of $|x^+\rangle$ and $|y\rangle$ instead of $|y^-\rangle$, respectively. Moreover, let us introduce the following parametrization of vectors $|x\rangle \in \mathbb{C}^2$ and $|y\rangle \in \mathbb{C}^d$ in the original basis $|g_i\rangle \otimes |f_k\rangle$:
\[ |x\rangle = (x_1, x_2 e^{i\beta_1}), \quad |y\rangle = (y_1, y_2 e^{i\beta_2}, \ldots, y_d e^{i\beta_d}), \quad x_1, x_2 \geq 0, \quad y_1, \ldots, y_d \geq 0. \] (18)

Using (7) one obtains
\[ \langle x \otimes y |O| x \otimes y \rangle = \sum_{i=1}^{2} \sum_{k=1}^{d} w_{ik} x_i^2 y_k^2 + 2 x_1 x_2 \sum_{k=1}^{d} y_k y_{k+1} u_k \cos \varphi_k, \] (19)

where
\[ \varphi_1 = \alpha_1 + \beta_1 + \beta_2, \quad \varphi_k = \alpha_k + \beta_1 - \beta_k + \beta_{k+1}, \quad k = 2, \ldots, d-1 \] \[ \varphi_d = \alpha_d + \beta_1 - \beta_d. \] (20)

The extremalization procedure leads to the set of equations for real positive variables $x_1, x_2, y_1, \ldots, y_d$ and for the phases $\beta_1, \ldots, \beta_d$ (see Appendix for details). In particular, phases $\beta_k$ can be easily obtained in the generic case, i.e., for $x_i \neq 0$, and $y_k \neq 0$, as shown in (54). Using simple algebra (see the Appendix) one finds
\[ \beta_k = -\mu_k, \quad k = 1, \ldots, d, \] (21)
where \( \mu_k \) are solutions of (13). Hence in the new basis \(|g'_1) \otimes |f'_k)\) the phases \( \beta_k \) are completely removed and the components of \(|x\rangle \) and \(|y\rangle \) are non-negative.

Hence, essentially LNR(\( \mathcal{O} \)) calculations can be done in \( \mathbb{R}^2 \otimes \mathbb{R}^d \) instead of \( \mathbb{C}^2 \otimes \mathbb{C}^d \). Unfortunately, solving the set of \( d + 2 \) polynomial equations (14), (15) is in general very hard. Keeping in mind that in the basis \(|g'_1) \otimes |f'_k)\) all \( \varphi_k = 0 \), we can rewrite (14) as

\[
\begin{aligned}
& (A_1(y) - \lambda_1)x_1 + B(y)x_2 = 0, \\
& B(y)x_1 + (A_2(y) - \lambda_1)x_2 = 0,
\end{aligned}
\]

with

\[ A_\epsilon(y) = \sum_{k=1}^d w_{1k}y_k^2, \quad B(y) = \sum_{k=1}^d u_k y_k y_{k+1}. \]

Now, we obtain the nonzero solution for \( x_1, x_2 \) from a linear set of equations (22) if

\[ \lambda^{\pm}_1 = \frac{1}{2} \left( A_1(y) + A_2(y) \pm \sqrt{(A_1(y) - A_2(y))^2 + 4B(y)^2} \right). \]

Let us write this solution as

\[ x_1 = \frac{1}{\sqrt{1 + C^2_\pm}}, \quad x_2 = C_\pm x_1 = \frac{C_\pm}{\sqrt{1 + C^2_\pm}}, \]

where

\[ C_\pm = \frac{\lambda^{\pm}_1 - A_1(y)}{B(y)} \]

and the normalization of \(|x\rangle\) has been taken into account. Putting (23) into (15) we arrive at the following set of \( d \) nonlinear equations for \( y_1, \ldots, y_d \):

\[ \left[ \frac{1}{C_\pm} w_{1k} + C_\pm w_{2k} - \lambda_2 \frac{1 + C^2_\pm}{C_\pm} y_k + u_k y_{k-1} + u_k y_{k+1} \right] = 0, \quad k = 1, \ldots, d. \]

Clearly, in general the solution of (23) is not feasible. Note however that when \( A_1(y) = A_2(y) \), i.e. \( w_{1k} = w_{2k} \) for \( k = 1, \ldots, d \), one gets \( C_\pm = \pm 1 \) and the set of equations (24) becomes linear.

**Example 1** Let us consider circulant hermitian operator \( \mathcal{O} \) in \( \mathbb{C}^2 \otimes \mathbb{C}^2 \) represented in the standard computational basis by the following real matrix

\[
M_\mathcal{O} = \begin{pmatrix}
2 & 0 & 0 & 1 \\
0 & 1 & -1 & 0 \\
0 & -1 & 1 & 0 \\
1 & 0 & 0 & 2
\end{pmatrix}.
\]

The spectrum of \( M_\mathcal{O} \) is \( \{0, 1, 2, 3\} \). As a consequence, NR(\( \mathcal{O} \)) = \( [0, 3] \), whereas, as we shall see, LNR(\( \mathcal{O} \)) = \( [0.5, 2.5] \). Moreover, the upper bound \( \gamma_{\max} \) is achieved at complex vectors \(|x\rangle = \frac{1}{\sqrt{2}} (1, i)\) and \(|y\rangle = \frac{1}{\sqrt{2}} (1, -i)\) and when calculating expectation values on normalized vectors from \( \mathbb{R}^2 \otimes \mathbb{R}^2 \) we do not go beyond 2.

In order to proof that the upper bound of LNR(\( \mathcal{O} \)) is indeed 2.5, let us bring the observable \( \mathcal{O} \) into the real form by a local unitary transformation (which does not change the ranges but does change the extremal vectors),

\[ U_1 = D[1, -i], \quad U_2 = D[1, i]. \]
Similar proof can be carried out for the lower bound $\gamma$ to $5$ Local Numerical Range for $d$

Consider now 2-qubit case corresponding to $d$ where

\[
\tilde{M} = U_1 \otimes U_2 M \otimes U_1^d \otimes U_2^d = \begin{pmatrix}
2 & 0 & 0 & 1 \\
0 & 1 & 1 & 0 \\
0 & 1 & 1 & 0 \\
1 & 0 & 0 & 2
\end{pmatrix}.
\] (27)

Now, it is easy to show that for $|x\rangle = (x_1, x_2) \in \mathbb{C}^2$ and $|y\rangle = (y_1, y_2) \in \mathbb{C}^2$ we get

\[
\langle x \otimes y | M' | x \otimes y \rangle = 2(|x_1|^2|y_1|^2 + |x_2|^2|y_2|^2) + |x_1|^2|y_2|^2 + |x_2|^2|y_1|^2 + 4\text{Re}(x_1 x_2^*) \text{Re}(y_1 y_2^*)
\leq 2(|x_1|^2|y_1|^2 + |x_2|^2|y_2|^2) + |x_1|^2|y_2|^2 + |x_2|^2|y_1|^2 + 1
\] (28)
due to $\text{Re}(x_1 x_2^*) \leq 1/2$ which follows from the normalization condition $|x_1|^2 + |x_2|^2 = 1$. Equality in (28) is achieved for $|x_1| = |x_2| = \frac{1}{\sqrt{2}}$ and $|y_1| = |y_2| = \frac{1}{\sqrt{2}}$ and therefore $\langle x \otimes y | O | x \otimes y \rangle = 2.5$. Similar proof can be carried out for the lower bound $\gamma_{\text{min}}$.

5 Local Numerical Range for $d = 2$

Consider now 2-qubit case corresponding to $d = 2$. The set of nonlinear equations (24) reduces to

\[
\begin{align*}
\left[\frac{1}{C_{\pm}} w_{11} + C_{\pm} w_{21} - \lambda_2 \frac{1 + C_{\pm}^2}{C_{\pm}}\right] y_1 + (u_1 + u_2) y_2 & = 0, \\
(u_1 + u_2) y_1 + \left[\frac{1}{C_{\pm}} w_{12} + C_{\pm} w_{22} - \lambda_2 \frac{1 + C_{\pm}^2}{C_{\pm}}\right] y_2 & = 0.
\end{align*}
\] (29, 30)

Consider normalized vector $|q\rangle = (q_1, q_2, q_3, q_4) \in \mathbb{R}^4$. It is separable iff $q_1 q_4 = q_2 q_3$. Hence, we define

\[
\tilde{G}(q) = \langle q | M' | q \rangle - \lambda_1 \left(\sum_{j=1}^4 q_j^2 - 1\right) - 2\lambda_2 (q_1 q_4 - q_2 q_3),
\]
where $M' \tilde{G}$ represents matrix of $O$ in the basis $|q\rangle \otimes |f\rangle$, that is,

\[
M' = \begin{pmatrix}
w_{11} & 0 & 0 & u_1 \\
0 & w_{12} & u_2 & 0 \\
u_1 & u_2 & w_{21} & 0 \\
u_1 & 0 & 0 & w_{22}
\end{pmatrix}, \quad u_1, u_2 \geq 0, \quad w_{ij} \in \mathbb{R}.
\] (31)

Now, $d\tilde{G} = 0$ leads to a linear matrix equation

\[
\mathcal{M}|q\rangle = |0\rangle,
\] (32)

where

\[
\mathcal{M} = \begin{pmatrix}
-\lambda_1 + w_{11} & 0 & 0 & u_1 - \lambda_2 \\
0 & w_{12} - \lambda_1 & u_2 + \lambda_2 & 0 \\
u_1 - \lambda_2 & 0 & -\lambda_1 + w_{12} & 0 \\
0 & 0 & 0 & w_{22} - \lambda_1
\end{pmatrix}.
\]

Obviously, this way we arrive at two separate two-dimensional linear problems. In order to obtain nonzero solutions the following condition should be fulfilled:

\[
\det\mathcal{M} = d_1(\lambda_1, \lambda_2) \cdot d_2(\lambda_1, \lambda_2) = 0,
\]
where
\[
\begin{align*}
d_1(\lambda_1, \lambda_2) &= u_2^2 - w_{12}w_{21} + (w_{12} + w_{21})\lambda_1 - \lambda_1^2 + 2u_2\lambda_2 + \lambda_2^2, \\
d_2(\lambda_1, \lambda_2) &= u_1^2 - w_{11}w_{22} + (w_{11} + w_{22})\lambda_1 - \lambda_1^2 - 2u_1\lambda_2 + \lambda_2^2.
\end{align*}
\]

(33)

(34)

Now, assuming
\[
\begin{align*}
d_1(\lambda_1, \lambda_2) &= 0 \\
d_2(\lambda_1, \lambda_2) &\neq 0
\end{align*}
\]

or
\[
\begin{align*}
d_1(\lambda_1, \lambda_2) &\neq 0 \\
d_2(\lambda_1, \lambda_2) &= 0
\end{align*}
\]

and using the separability condition, we get four possible product vectors \(|g_i\rangle \otimes |f_j\rangle|\):
\[
\begin{align*}
\{ &\left( \begin{array}{c} 0 \\ 1 \end{array} \right) \otimes \left( \begin{array}{c} 1 \\ 0 \end{array} \right), \left( \begin{array}{c} 0 \\ 1 \end{array} \right) \otimes \left( \begin{array}{c} 0 \\ 1 \end{array} \right), \left( \begin{array}{c} 0 \\ 1 \end{array} \right) \otimes \left( \begin{array}{c} 1 \\ 0 \end{array} \right) \}
\end{align*}
\]

(35)

whereas solving
\[
\begin{align*}
d_1(\lambda_1, \lambda_2) &= 0 \\
d_2(\lambda_1, \lambda_2) &= 0
\end{align*}
\]

we obtain two solutions \((\lambda_1^+, \lambda_2^+)\) and \((\lambda_1^-, \lambda_2^-)\) which inserted into \((32)\) imply the following conditions:

\[
\begin{align*}
g_1 &= a_\pm q_4 \\
g_2 &= b_\pm q_3 \\
g_1^2 + g_2^2 + g_3^2 + g_4^2 &= 1 \\
g_1q_4 &= q_2q_3.
\end{align*}
\]

(36)

(37)

Solving \((37)\) and factorizing \(|q\rangle = |x\rangle \otimes |y\rangle\) we arrive at
\[
|q\rangle = \left( \frac{\sqrt{\kappa_\pm}}{\sqrt{1+\xi_\pm}} \right) \otimes \left( \frac{\sqrt{\xi_\pm}}{\sqrt{1+\kappa_\pm}} \right),
\]

(38)

where
\[
a_\pm = \frac{l_\pm}{m_\pm}, \quad b_\pm = \frac{g_\pm}{h_\pm}, \quad \kappa_\pm = \frac{a_\pm}{b_\pm}, \quad \xi_\pm = a_\pm \cdot b_\pm,
\]

and
\[
l_\pm = 2u_2^4 + 8u_1^3u_2 + 2u_2^4 \pm (u_1 + u_2)(-w_{11} + w_{12} + w_{21} - w_{22}) \sqrt{\Delta} \\
+ u_2^2(-w_{11}^2 - 2w_{12}w_{21} + w_{11}(w_{12} + w_{21}) + (w_{12} + w_{21})w_{22} - w_{22}^2) \\
+ 2u_1u_2(4u_2^2 - w_{11}^2 - 2w_{12}w_{21} + w_{11}(w_{12} + w_{21}) + (w_{12} + w_{21})w_{22} - w_{22}^2) \\
+ u_1^2(12u_2^2 - w_{11}^2 - 2w_{12}w_{21} + w_{11}(w_{12} + w_{21}) + (w_{12} + w_{21})w_{22} - w_{22}^2),
\]

(34)

\[
m_\pm = (u_1 + u_2)\left( \pm 2(u_1 + u_2) \sqrt{\Delta} + (w_{11} - w_{12})(w_{11} - w_{21})(w_{11} - w_{12} - w_{21} + w_{22}) \\
+ 2u_1u_2(-3w_{11} + w_{12} + w_{21} + w_{22}) + 2u_1u_2(-3w_{11} + w_{12} + w_{21} + w_{22}) + u_2^2(-3w_{11} + w_{12} + w_{21} + w_{22}) \right),
\]

\[
g_\pm = 2u_1^4 + 8u_1^3u_2 + 2u_1^4 \pm (u_1 + u_2)(w_{11} - w_{12} + w_{21} - w_{22}) \sqrt{\Delta} \\
+ u_1^2(-w_{12}^2 - w_{21}^2 + w_{11}(w_{12} + w_{21} - 2w_{22}) + (w_{12} + w_{21})w_{22}) \\
+ 2u_1u_2(4u_1^2 - w_{12}^2 - w_{21}^2 + w_{11}(w_{12} + w_{21} - 2w_{22}) + (w_{12} + w_{21})w_{22}) \\
+ u_1^2(12u_2^2 - w_{12}^2 - w_{21}^2 + w_{11}(w_{12} + w_{21} - 2w_{22}) + (w_{12} + w_{21})w_{22}),
\]

(34)

\[
h_\pm = (u_1 + u_2)\left( \pm 2(u_1 + u_2) \sqrt{\Delta} + (w_{11} - w_{12})(w_{12} - w_{22})(w_{11} - w_{12} - w_{21} + w_{22}) \\
+ u_2^2(w_{11} - w_{12} + w_{21} - w_{22}) + 2u_2u_2(w_{11} - 3w_{12} + w_{21} + w_{22}) + u_2^2(w_{11} - 3w_{12} + w_{21} + w_{22}) \right),
\]

(34)
with
\[ \Delta = (u_1 + u_2)^2 + (w_{11} - w_{21})(w_{12} - w_{22}) \left( (u_1 + u_2)^2 + (w_{11} - w_{12})(w_{21} - w_{22}) \right). \]

Note that, in order to have real components of \(|q\rangle\),
\[ \xi_\pm \geq 0, \quad \kappa_\pm \geq 0, \]
should be fulfilled. As a consequence, either both \(a_\pm, b_\pm\) are nonnegative or both are non-positive.

Finally, for \(|q\rangle\) given by (38) we obtain
\[ \langle q|M'_O|q \rangle \equiv F_\pm = 2(u_1 + u_2)\sqrt{\xi_\pm \kappa_\pm + \xi_\pm \kappa_\pm w_{11} + \xi_\pm w_{12} + \kappa_\pm w_{21} + w_{22}} \]
\[ \frac{(1 + \kappa_\pm)(1 + \xi_\pm)}{1 + \xi_\pm + \kappa_\pm + a_\pm^2}. \]

Taking into account vectors (35) one obtains
\[ \langle g_i \otimes f_j|M'_O|g_i \otimes f_j \rangle = w_{ij}. \]

Hence, LNR of the circulant observable \(O\) is given by \([\gamma_{\min}, \gamma_{\max}]\), where
\[ \gamma_{\min} = \min \left\{ w_{ij}, F_\pm \right\} \]
\[ \gamma_{\max} = \max \left\{ w_{ij}, F_\pm \right\} \]

To summarize, in order to calculate LNR for a given \(C^2 \otimes C^2\) circulant operator, we propose the following procedure

1. if in a given basis a matrix representation of an operator \(M_O\) has complex or negative off-diagonal entries then change the basis due to Proposition 1 and bring the matrix to the real form,
2. determine real vectors \(|x\rangle\) and \(|y\rangle\) (see (35)) together with \(F_\pm\) and compare these values with diagonal elements of \(M'_O\). Then \(\gamma_{\min}, \gamma_{\max} = [\gamma_{\min}, \gamma_{\max}]\), where \(\gamma_{\min}\) and \(\gamma_{\max}\) are defined in (42) and (43), respectively.

Example 2 As an illustration let us consider a two-parameter family of matrices \(Q_{t,s}, t, s \geq 0\), analyzed in [3].

\[ Q_{t,s} = \begin{pmatrix} 2 & 0 & 0 & t \\ 0 & 1 & s & 0 \\ 0 & s & -1 & 0 \\ t & 0 & 0 & -2 \end{pmatrix}. \]

Denoting by \(p = t + s \geq 0\) one obtains
\[ \Delta = (1 + p^2)(9 + p^2) \]
\[ a_\pm = \frac{4p \pm \sqrt{\Delta}}{p^2 - 3} \]
\[ b_\pm = \frac{2p \pm \sqrt{\Delta}}{p^2 + 3} \]
\[ \kappa_\pm = \frac{p^4 + 2p^2 + 9 \pm 2\sqrt{\Delta}}{(p^2 - 3)(p^2 + 3)} \]
\[ \xi_\pm = \frac{p^4 + 18p^2 + 9 \pm 6\sqrt{\Delta}}{(p^2 - 3)(p^2 + 3)}. \]
Note that $b_+ \geq 0$ and $b_- \leq 0$, hence $a_+ \geq 0$ and $a_- \leq 0$. Finally, $|x|$ and $|y|$ are real under the condition $p \geq \sqrt{3}$ (see (39)) and using (40) we arrive at

$$F_\pm = \pm \frac{\sqrt{\Delta}}{2p}.$$ 

Because the maximal and minimal values of $w_{ij}$ are equal to 2 and $-2$, respectively, due to (42) and (43) we get

$$\gamma_{\text{max}} = \begin{cases} 2 & \text{for } 0 \leq p < \sqrt{3} \\ \frac{1}{2p} \sqrt{\Delta} & \text{for } p \geq \sqrt{3} \end{cases}$$

and $\gamma_{\text{min}} = -\gamma_{\text{max}}$ in complete agreement with the result of [3].

**Appendix**

We are going to carry out an extremalization procedure of (19) with two constraints $|x| = 1$, $|y| = 1$ using a Lagrange function $G = F - \lambda_1(|x|^2 - 1) - \lambda_2(|y|^2 - 1)$. As a result we get the following equations:

$$\frac{\partial G}{\partial x_i} = \left[ \sum_{k=1}^{d} w_{ik} y_k^2 - \lambda_1 \right] + x_{i+1} \sum_{k=1}^{d} y_k y_{k+1} u_k \cos \varphi_k = 0, \quad i = 1, 2 \quad (44)$$

$$\frac{\partial G}{\partial y_k} = \left[ \sum_{i=1}^{2} w_{ik} x_i^2 - \lambda_2 \right] + x_1 x_2 \left( y_{k-1} y_{k-1} \cos \varphi_{k-1} + y_{k+1} u_k \cos \varphi_k \right) = 0, \quad k = 1, \ldots, d \quad (45)$$

$$\frac{\partial G}{\partial \beta_1} = x_1 x_2 \sum_{k=1}^{d} y_k y_{k+1} u_k \sin \varphi_k = 0, \quad (46)$$

$$\frac{\partial G}{\partial \beta_k} = x_1 x_2 \left( y_k y_{k+1} u_k \sin \varphi_k - y_{k-1} y_k u_{k-1} \sin \varphi_{k-1} \right) = 0, \quad k = 2, \ldots, d. \quad (47)$$

From the last two equations one obtains in a generic case, i.e., when $x_i \neq 0$, and $y_k \neq 0$, the following set of equations

$$\begin{cases} \sum_{k=1}^{d} z_k = 0 \\ z_{k-1} - z_k = 0, \quad k = 2, \ldots, d. \end{cases} \quad (48)$$

with $z_k = y_k y_{k+1} u_k \sin \varphi_k$ or in a matrix notation $W^T z = 0$, where $W^T$ is a transposition of the matrix given by (12). Now, according to $|\det W^T = d(-1)^{d+1} \neq 0$, the set of homogeneous equations (48) has only zero solution, hence in a generic case, $\sin \varphi_k = 0$ for $k = 1, \ldots, d$. The angles $\beta_k$ can now be easily obtained. It results from $\sin \varphi_k = 0$ that

$$\alpha_1 + \beta_1 + \beta_2 = 0, \quad (49)$$

$$\alpha_k + \beta_1 - \beta_k + \beta_{k+1} = 0, \quad k = 2, \ldots, d - 1 \quad (50)$$

$$\alpha_d + \beta_1 - \beta_d = 0. \quad (51)$$

or in a matrix form

$$\alpha = -W \beta \quad (53)$$
with exactly the same $W$ as in (13). Hence solutions for $\beta_1, \ldots, \beta_d$ differ only by a sign from solutions for $\mu_1, \ldots, \mu_d$ (see (13)) and one can easily find that

$$\begin{align*}
\beta_1 &= -\frac{1}{d} \sum_{k=1}^{d} \alpha_k = -\mu_1, \\
\beta_2 &= -\alpha_1 - \beta_1 = -\mu_2, \\
\beta_{k+1} &= -\alpha_k - \beta_1 + \beta_k = -\mu_{k+1}, \quad k = 2, \ldots, d - 1.
\end{align*}$$

(54)

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