Soliton generation by local resonance interaction

Glebov S.G.,† Kiselev O.M., ‡ Lazarev V.A.§

1 Introduction

A wide class of the mathematical physics problems is connected with forced nonlinear Schrödinger equation (NLSE). Usually authors consider either the small amplitude solutions [3, 4] or the asymptotics for perturbed soliton solutions [5, 6, 7]. The problems on soliton generation usually are not discussed or solved numerically [8, 9].

In this paper we show a mechanism of the soliton generation in nonlinear Schrödinger equation due to a small fast oscillating driving force:

\[ i\partial_{\theta}\Psi + \partial_{\xi}\Psi + |\Psi|^2\Psi = \varepsilon^2 f e^{iS/\varepsilon^2}, \quad 0 < \varepsilon \ll 1. \quad (1) \]

We investigate formal asymptotic solutions with a small amplitude of the order of \(O(\varepsilon^2)\). The leading-order term is the solution of linear nonhomogeneous equation. We consider the one phase asymptotic solution with the phase function \(S/\varepsilon^2\). Other terms of the order of \(O(\varepsilon^2)\) do not play any role in investigated phenomenon.

The specific of the problem consists in existence of a local resonance. The resonance curve is defined by equation:

\[ \partial_{\theta}S + (\partial_{\xi}S)^2 = 0, \quad (2) \]

where \(S\) is the phase function of the driving force. The amplitude of forced oscillations increases from \(O(\varepsilon^2)\) up to \(O(\varepsilon)\) in the neighborhood of this curve.

*This work was supported by RFBR grant (00-15-96038)
†Ufa State Petroleum Technical University; gloomy69@ufanet.ru
‡Institute of Mathematics Ufa Centre RAS; ok@ufanet.ru
§Ufa State Petroleum Technical University; lazva@mail.ru
The increase of the amplitude is similar to a phenomenon of a local resonance in the linear nonhomogeneous ordinary differential equation of the second order with the constant coefficients [10]. The amplitude increases linearly. The domain of this increase usually is called a resonance layer. The additional term of the order of the layer width will appear in the asymptotic solution after passage of the resonance layer. The order $O(\varepsilon^2)$ of residual part of the solution does not change.

After resonance the leading-order term of the asymptotic solution has the order of $O(\varepsilon)$. The spatial variable $\varepsilon \xi$ is defined by amplitude of the driving force. The relation between the order of the solution amplitude and typical scale of the independent variables is such that the leading-order term of the asymptotics is defined from NLSE. So specific relations usually appear due to specific asymptotic representation of the solution as was shown in well known works on asymptotic transfer to NLSE [4], [13]-[15]. In this work we show that such kind relation can be obtained due to resonance with driving force.

Thus after resonance layer the leading-order term of the asymptotics has the order $O(\varepsilon)$ and is defined from the Cauchy problem. The initial data are equal to the value of the amplitude $f$ of driving force on the resonance curve. The further evolution of the leading term of the asymptotics does not depend on driving force.

The effect of the solution increase after local resonance is well known in the ordinary differential equations [10] and partial differential equations [11, 12]. Also it is well known the effect of appearance of NLSE in asymptotic constructions for the problems with strong dispersion, cubic nonlinearity and the appropriate relation between scales. This problem contains both of these effects. The passage through the local resonance leads to the increase of the amplitude. The modulation of the amplitude of the driving force and scales of the variables lead to NLSE behind the resonance curve. The similar results can be obtained for other problems on the resonance passage by driving force in systems with the cubic nonlinearity and strong dispersion.

The considered problem is the model problem for the wide class problems on the local resonance passage by driving force. In order to show essence of this phenomenon we assume that the amplitude $f$ of perturbation is a smooth function with respect to variable $\varepsilon \xi$ and phase function is $S = (\varepsilon^2 \theta)^2 / 2$.

The result of this work is the following proposition

**Proposition.** One phase asymptotic solution of the order of $O(\varepsilon^2)$ in the domain $\theta \ll -\varepsilon^{-1}$ will have the order of $O(\varepsilon)$ in the domain $\varepsilon^{-1} \ll \theta \leq K \varepsilon^{-2}$, $K = \text{const} > 0$. The leading-order term of the asymptotics

$$\Psi = \varepsilon^0 u + O(\varepsilon^2)$$
is determined from the Cauchy problem for NLSE

\[ i \, \frac{0}{\partial t} + \frac{0}{\partial x^2} + |0|^2 \, 0 = 0, \]
\[ 0 \, 0 \big|_{t=0} = (1 - i)\sqrt{\pi} f(x). \]

In particular, if the amplitude of the perturbation force has the form

\[ f(x) = \frac{1}{1 - i} \frac{2\eta \exp(-i\mu(x + y_0))}{\sqrt{\pi} \cosh(2\eta(x - x_0))}, \]

then the leading-order term \( \hat{u} \equiv \frac{0}{\partial t} \) of postresonance expansion is a soliton of NLSE, where \( x_0, y_0, \mu, \eta \) are soliton parameters.

## 2 Forced oscillations

In this section we construct the formal asymptotic solution for equation (1) in preresonance domain.

**Lemma 1.** In the domain \( \theta \ll -\varepsilon^{-1} \) the formal asymptotic solution for equation (1) with respect to base \( \mathcal{O}(\varepsilon^8) \) has a form

\[ \Psi(\xi, \theta, \varepsilon) = \left[ \varepsilon^2 \frac{1}{A}(\varepsilon \xi, \varepsilon^2 \theta) + \varepsilon^4 \frac{2}{A}(\varepsilon \xi, \varepsilon^2 \theta) + \varepsilon^6 \frac{3}{A}(\varepsilon \xi, \varepsilon^2 \theta) \right] \exp(iS/\varepsilon^2), \quad \varepsilon \to 0, \]

the coefficients \( A \) of the asymptotics are determined from algebraic equations (4).

Equation (1) is written in terms of fast variables \( \xi, \theta \) but further for convenience we also use slow variables \( x = \varepsilon \xi \) and \( t = \varepsilon^2 \theta \).

The substitution (3) into equation (1) gives a recurrent sequence of equations for coefficients of asymptotics:

\[ -S' \frac{1}{A} = f, \]
\[ -S' \frac{2}{A} = -i \frac{1}{A_t} - \partial_{xx} \frac{1}{A}, \]
\[ -S' \frac{3}{A} = -i \frac{2}{A_t} - \partial_{xx} \frac{2}{A} - \frac{1}{A} \frac{1}{|A|} \frac{1}{A}. \]

The solutions of this systems have singularities at \( t = 0 \). The constructed asymptotic solution can be represented as \( t \to 0 \) in the form:

\[ \psi(\varepsilon \xi, t, \varepsilon) = \left[ \varepsilon^2 \left( \frac{f}{t} \right) + \varepsilon^4 \left( \frac{if}{t^3} - \frac{f_{xx}}{t^2} \right) + \right. \]

\[ \left. \varepsilon^6 \left( \frac{3i}{t^5} - \frac{3f_{xx}}{t^4} + \frac{f_{xxx}}{t^3} \right) + \right. \]
\[ \varepsilon^6 \left( \frac{3i_f}{t^4} - \frac{f_{xxxx}}{t^3} - \frac{|f|^2 f}{t^4} + 3f \right) \exp(iS/\varepsilon^2). \]

The structure of singularities as \( t \to -0 \) of coefficients in asymptotic representation (3) allows us to determine the domain of validity for external asymptotics:

\[ \theta \ll -\varepsilon^{-1}. \]

Lemma 1 is proved.

\section{Internal solution}

In this section we construct the formal asymptotic solution for equation (1) into resonance layer. The asymptotics of these type are usually called by internal asymptotics \[16\].

**Lemma 2.** In the domain \(-\varepsilon^{-2} \ll \theta \ll \varepsilon^{-2}\) the formal asymptotic solution for equation (7) with respect to base \( \mathcal{O}(\varepsilon^4) \) has a form

\[ \Psi(\xi, \tau, \varepsilon) = \varepsilon^0 \tilde{w}(\varepsilon \xi, \tau) + \varepsilon^2 \varepsilon \varepsilon^2 \tilde{w}(\varepsilon \xi, \tau) + \varepsilon^3 \tilde{w}(\varepsilon \xi, \tau) \quad \varepsilon \to 0, \]

where \( \tau = \varepsilon \theta \). The coefficients \( \tilde{w} \) of the asymptotics are determined from ordinary differential equations (9), (11), (12).

The structure of singularities for coefficients in (3) as \( t \to -0 \) allows us to determine internal scaled variable \( \tau = t/\varepsilon = \varepsilon \theta \).

The internal asymptotic solution is constructed in form (5). The behaviour of coefficients can be obtained by matching:

\[ \tilde{w}(\varepsilon \xi, \tau) \sim \left[ -\frac{f}{\tau} + \frac{i f}{\tau^2} + \frac{3f}{\tau^5} \right] \exp\{i\tau^2/2\}, \quad \tau \to -\infty. \]

\[ \tilde{w}(\varepsilon \xi, \tau) \sim \left[ -\frac{f_{xx}}{\tau^2} + \frac{3i f_{xx}}{\tau^4} - \frac{|f|^2 f}{\tau^4} \right] \exp\{i\tau^2/2\}, \quad \tau \to -\infty. \]

\[ \tilde{w}(\varepsilon \xi, \tau) \sim \left[ -\frac{f_{xxxx}}{\tau^3} \right] \exp\{i\tau^2/2\}, \quad \tau \to -\infty. \]

The leading-order term of (5) is determined from equation:

\[ i \tilde{w}_\tau = f \exp(i\tau^2/2). \]

The solution of this equation can be written out in terms of Fresnel integral:

\[ \tilde{w} = -if(\varepsilon \xi) \int_{-\infty}^{\tau} \exp(i\theta^2/2) d\theta. \]
The first order correction $\frac{1}{\tau} w$ is determined by equation:
\[ i \frac{1}{\tau} w + \frac{0}{\tau} w + | \frac{0}{\tau} w |^2 \tau = 0, \]  
(11)
the second order correction is determined by equation:
\[ i \frac{2}{\tau} w + \frac{1}{\tau} w + 2 | \frac{0}{\tau} w |^2 \tau + \frac{0}{\tau} w + \frac{1}{\tau} w = 0. \]  
(12)

In order to determine the behaviour of the asymptotic solution after resonance passage we need to know the asymptotics of the coefficient $s$ of (5) as $\tau \rightarrow +\infty$. Direct calculations lead us to formulas
\[ \frac{0}{\tau} w (\varepsilon, \tau) = -if(\varepsilon) \left[ ic_1 + \frac{\exp(i\tau^2/2)}{i\tau} + O(\tau^{-3}) \right], \]
where $c_1 = (1 - i) \sqrt{\tau};$
\[ \frac{1}{\tau} w (\varepsilon, \tau) = \tau \frac{1}{\tau} w (\varepsilon) + \frac{1}{\tau} w_0 (\varepsilon) + g_1(\varepsilon) \frac{\exp(i\tau^2/2)}{i\tau} + O(\tau^{-4}), \]
where
\[ \frac{1}{\tau} w (\varepsilon, \tau) = \lim \int_{-\infty}^{\tau} \left[ \frac{1}{\tau} w (\varepsilon, \theta) + | \frac{0}{\tau} w (\varepsilon, \theta) |^2 \frac{0}{\tau} w (\varepsilon, \theta) \right] d\theta - (c_1 f_x + | c_1 f |^2 c_1 f), \]
$g_1(\varepsilon) = k_1 f_x + k_2 | f |^2 f$, $k_1$ and $k_2$ are constants;
\[ \frac{2}{\tau} w (\varepsilon, \tau) = \tau^2 \frac{2}{\tau} w (\varepsilon) + \tau \frac{2}{\tau} w_1 (\varepsilon) + \frac{2}{\tau} w_0 (\varepsilon) + g_2(\varepsilon) \frac{\exp(i\tau^2/2)}{i\tau} + O(\tau^{-2}), \]
where
\[ \frac{2}{\tau} w_2 (\varepsilon) = i \left( (\frac{1}{\tau} w_{1x} + 2 | \frac{0}{\tau} w_0 |^2 \frac{1}{\tau} w_1 + \frac{0}{\tau} w_2 \frac{1}{\tau} w_1) \right), \]
\[ \frac{2}{\tau} w_1 (\varepsilon) = i \left( (\frac{1}{\tau} w_{0x} + 2 | \frac{0}{\tau} w_0 |^2 \frac{0}{\tau} w_0 + 2 \frac{0}{\tau} w_0 \frac{0}{\tau} w_{-1} \frac{1}{\tau} w_1 - \frac{2}{\tau} w_{-1} \frac{0}{\tau} w_0 \frac{1}{\tau} w_1 + \frac{0}{\tau} w_0 \frac{1}{\tau} w_{-1} \frac{1}{\tau} w_1 + \frac{0}{\tau} w_0 \frac{1}{\tau} w_0) \right), \]
\[ \frac{2}{\tau} w_0 = \lim \int_{-\infty}^{\tau} \left[ (\frac{1}{\tau} w_{xx} (\varepsilon, \theta) + 2 | \frac{0}{\tau} w (\varepsilon, \theta) |^2 \frac{1}{\tau} w (\varepsilon, \theta) + \frac{0}{\tau} w \frac{2}{\tau} w_1) \frac{d\theta}{\tau^2 \frac{2}{\tau} w_2 - \tau \frac{2}{\tau} w_1} \right], \]
and the function $g_2(\varepsilon)$ is smooth and expressed by $f(\varepsilon)$. The asymptotic behaviour of the coefficients in (5) gives us the domain of validity for this solution:
\[ | \tau | \ll \varepsilon^{-1} \quad \text{or} \quad | \theta | \ll \varepsilon^{-2} \]
Lemma 2 is proved.
4 Postresonance solution

In this section we construct the formal asymptotic solution for equation (1) in postresonance domain.

Lemma 3. In the domain $\varepsilon^{-1} \ll \theta \leq K\varepsilon^{-2}$ the formal asymptotic solution for equation (11) with respect to base $O(\varepsilon^4)$ has a form

$$\Psi(\xi, \theta, \varepsilon) = \varepsilon^0 u(\varepsilon^2(\xi, \varepsilon^2\theta) + \varepsilon^2(\frac{1}{2}u(\varepsilon^2\xi) + \frac{1}{2}B(\varepsilon^2\xi, \varepsilon^2\theta) \exp(iS/\varepsilon^2)) +$$

$$\varepsilon^3(\frac{1}{2}u(\varepsilon^2\xi) + \frac{1}{2}B_1(\varepsilon^2\xi, \varepsilon^2\theta) \exp(iS/\varepsilon^2) + \frac{1}{2}B_{-1}(\varepsilon^2\xi, \varepsilon^2\theta) \exp(-iS/\varepsilon^2)) + (13)$$

$$+ \varepsilon^4(\frac{3}{2}B_1(\varepsilon^2\xi, \varepsilon^2\theta) \exp(iS/\varepsilon^2) + \frac{3}{2}B_2(\varepsilon^2\xi, \varepsilon^2\theta) \exp(2iS/\varepsilon^2) +$$

$$+ \frac{3}{2}B_{-1}(\varepsilon^2\xi, \varepsilon^2\theta) \exp(-iS/\varepsilon^2)),$$

the coefficients $B_k$ of the asymptotics are determined from algebraic equations (14) and the coefficients $\bar{u}$ are solutions of the Cauchy problems for either NLSE or linearized NLSE (15)-(17).

The substitution of (13) in equation (1) gives:

$$\varepsilon \left( - S' \frac{1}{2} B - f \right) \exp(iS/\varepsilon^2) + \varepsilon^2 \left( \frac{1}{2} u_t + \frac{1}{2} u_{xx} + |\frac{1}{2} u|^2 \frac{1}{2} u \right)$$

$$+ \varepsilon^3 \left( \frac{1}{2} u_t + \frac{1}{2} u_{xx} + 2|\frac{1}{2} u|^2 \frac{1}{2} u + \frac{1}{2} u^2 \frac{1}{2} u + \frac{1}{2} B \exp(iS/\varepsilon^2) + \frac{1}{2} B_{xx} \exp(iS/\varepsilon^2) +$$

$$+ 2|\frac{1}{2} u|^2 \frac{1}{2} B \exp(iS/\varepsilon^2) + \frac{1}{2} u^2 \frac{1}{2} B \exp(-iS/\varepsilon^2) -$$

$$- S' \frac{1}{2} B_1 \exp(iS/\varepsilon^2) + S' \frac{1}{2} B_{-1} \exp(-iS/\varepsilon^2) \right) +$$

$$+ \varepsilon^4 \left( \frac{1}{2} u_t + \frac{1}{2} u_{xx} + 2 |\frac{1}{2} u|^2 \frac{1}{2} u + \frac{1}{2} u^2 \frac{1}{2} u + \frac{1}{2} u^2 \frac{1}{2} u + 2 |\frac{1}{2} u|^2 \frac{1}{2} u \right)$$

$$+ \frac{1}{2} u^2 \frac{1}{2} B \exp(2iS/\varepsilon^2) + 2 \frac{1}{2} u^2 \frac{1}{2} B \exp(iS/\varepsilon^2) +$$

$$+ 2 \frac{1}{2} uu^* \frac{1}{2} B \exp(iS/\varepsilon^2) + 2 \frac{1}{2} uu^* \frac{1}{2} B \exp(-iS/\varepsilon^2) +$$

$$+ 2 |\frac{1}{2} u|^2 \frac{1}{2} B_1 \exp(iS/\varepsilon^2) + 2 |\frac{1}{2} u|^2 \frac{1}{2} B_{-1} \exp(-iS/\varepsilon^2) +$$

$$+ \frac{1}{2} u^2 \frac{1}{2} B_1 \exp(-iS/\varepsilon^2) + \frac{1}{2} u^2 \frac{1}{2} B_{-1} \exp(iS/\varepsilon^2) +$$

$$+ i \frac{1}{2} B_1 \exp(iS/\varepsilon^2) + \frac{1}{2} B_{xx} \exp(iS/\varepsilon^2) + \frac{1}{2} B_{xx} \exp(-iS/\varepsilon^2)$$

$$- i \frac{1}{2} B_{-1} \exp(-iS/\varepsilon^2) + \frac{1}{2} B_{-1} \exp(iS/\varepsilon^2) +$$

$$- \frac{1}{2} B_1 \exp(iS/\varepsilon^2) + \frac{1}{2} B_{-1} \exp(-iS/\varepsilon^2) +$$

$$- S' \frac{3}{2} B_1 \exp(iS/\varepsilon^2) + S' \frac{3}{2} B_{-1} \exp(-iS/\varepsilon^2) + 2S' \frac{3}{2} B_2 \exp(2iS/\varepsilon^2) \right) = O(\varepsilon^5)$
Functions $B_1, B_{\pm}$ are determined from algebraic equation:

$$
-S' B_1 = f,
$$

$$
S' B_{1} = i B_t + B_{xx} + |u|^2 B,
$$

$$
S' B_{-1} = -u^2 B^*,
$$

$$
S' B_{1} = i B_1 t + B_{1x} + |u|^2 B + 2 \frac{u^*}{u} \frac{1}{B} + u^* B + u^* B^* - u^* B_{-1},
$$

$$
2S' B_{1} = u^* \frac{1}{B}^2,
$$

$$
S' B_{1} = i B_1 t + B_{1x} + |u|^2 B + 2 \frac{u^*}{u} \frac{1}{B} + u^* B + u^* B^* - u^* B_{-1},
$$

$$
S' B_{-1} = i B_{-1} t - B_{-1x} - \frac{u^*}{B} B^* - u^* B_{1},
$$

$$
2S' B_{1} = u^* \frac{1}{B}^2.
$$

The matching with internal solutions give us the recurrent system of the Cauchy problems for coefficients $u, u^*$:

$$
i u_t + u_{xx} + |u|^2 u = 0,
$$

$$
|u|_{t=0} = c_1 f(x); \quad (15)
$$

$$
i u_t + u_{xx} + 2\frac{u^*}{u} \frac{1}{B} + u^* B + u^* B^* = 0,
$$

$$
|u|_{t=0} = \frac{1}{B} w_0 (x); \quad (16)
$$

$$
i \frac{u}{2} + u_{xx} + 2\frac{u}{2} \frac{1}{u} + u^* = -|B|^2 u - |u|^2 u - u^* u^*,
$$

$$
\frac{u}{2} |_{t=0} = \frac{2}{u} w_0 (x). \quad (17)
$$

Coefficients $B_k$ have singularities at $t = 0$. The domain of validity for solution (13) is determined by inequalities:

$$
t \gg \varepsilon \quad \text{or} \quad \theta \gg \varepsilon^{-1}.
$$

Lemma 3 is proved.

In order to determine the behaviour of solution (13) for large $t > 0$ it is necessary to know the structure of the leading-order term $u$. It is easy to see that the behaviour of $u$ is determined by amplitude of the perturbation force in original equation (1). In particular, if the function $f(x)$ has the form:

$$
f(x) = \frac{1}{(1-i)\sqrt{\pi}} \frac{2\eta \exp(-i \mu(x + y_0))}{\cosh(2\eta(x - \xi_0))}, \quad \text{where} \quad \xi_0, y_0, \mu, \eta = \text{const},
$$

then to determine the leading-order term of asymptotics (13) it is necessary to solve the Cauchy problem for NLSE with initial data of the soliton type.

We are grateful to L.A. Kalyakin and B.I. Suleimanov for stimulating discussions and for help in improving of the mathematical presentation of the results.
References

[1] D.J. Kaup, A.C. Newell, Phys. Rev. B, 18, 1978, 5162.
[2] J.C. Eilbeck, P.S. Lomdahl, A.C. Newell Phys. Lett. A 87, 1, 1981.
[3] Ahmanov S.A., Hohlov R.V., Problemy nelineinoi optiki, M.-Nauka, 1964.
[4] Kalyakin L.A., Dlinnovolnovye asimptotiki. Integriruemye uravneniya kak asimptoticheskii predel nelineinyh sistem, //Uspehi mat. nauk, t.44, vyp.1, 1989, s. 5-34.
[5] Kaup D.J. A perturbation expansion for the Zakharov-Shabat inverse scattering transform. //SIAM J.on Appl.Math., 1976, v. 31, p. 121–133.
[6] V.I.Karpman, E.I. Maslov. Teoriya vozruschenii dlya solitonov //ZhETF, 1977, t..73, s. 537–559.
[7] L.A. Kalyakin The first order corrections and justification of soliton perturbation./ Preprint of Inst. Math. of Ural branch of RAS, Ufa, 1992, p.13.
[8] I.V. Barashenkov, Yu. S. Smirnov, Bifurcation to multisoliton complexes in the ac-driven, damped nonlinear Schrödinger equation, / Research Reports RR 3/97, University of Cape Town, 1997, p.56.
[9] L. Friedland, A.G. Shagalov, Excitation of solitons by adiabatic multiresonant forcing, // Phys. Rev. Lett., v. 81, n. 20 , 1998, p.4357-4360.
[10] Kevorkian J. Passage through resonance for a one-dimensional oscillator with slowly varying frequency // SIAM J. Appl. Math.–1971.– v. 20. – P.364-373.
[11] Kalyakin L.A. Lokal’nyi rezonans v slabonelineinoi zadache. // Mat. zametki. – 1988. – T.44, 5, C. 697 – 699.
[12] Glebov S.G., Ob odnoi slabo nelineinoi zadache s lokal’nymi rezonansami // Diff. uravneniya. – 1995. –31(8). – C. 1402-1408.
[13] Zaharov V.E., Ustoichivost’ periodicheskikh voln konechnoi amplitudy na poverhnosti glubokoi zhidkosti, // PMTF, 1968, N.2, s. 86-94.
[14] Talanov V.I. O samofokusirovke malych puchkov v nelineynyh sredah, // Pis’ma v ZhETF, 1965, N.2, s. 218–222.
[15] Kelley P.L. Self-focusing of optical beams. // Phys. Rev. Lett. 1965, v.15, pp. 1005-1008.

[16] Il’in A.M., Soglasovanie asimptoticheskikh razlozhenii reshenii kraevyh zadach. M.: Nauka. – 1988.