A Bubble-breaking Phenomenon in the Variation of a Swarm Communication Network

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Abstract

We discuss a specific circumstance in which the topology of the communication network of a robotic swarm has to change during the movement. The variation is caused by a topological obstruction which emerges from certain geometric restrictions on both the environment and the swarm.

1 Introduction

1.1 Networks and system dynamics

Communication among individuals in a group of agents (fish/birds/robots/people) is the foundation for forming system behaviours. The “communication” here can be any exchange/flow of information in any form. In flocking/schooling of birds/fish it may refer to the sensing (by the group members) of changes in positions/velocities of nearby companions [1]. In a multirobot system it can mean the transmission of signals among the robots. In epidemic spreading it is then the transfer of viruses/bacteria.

Representing with vertices the agents and with edges the established communications yields a communication network of the group. These networks serve as basic mathematical structures upon which system dynamics are built [2, 3, 4, 5]. In nature as well as in practise, the ability of an agent to set up communication with other individuals is usually limited. For the case of interest in this note, it is the maximal distance to set up message channels between two robots. As a consequence, the motion of the group may in turn cause changes to the topology of the network.

1.2 A bubble-breaking phenomenon

Put an iron ring (rigid) into a balloon (not inflated) and seal the valve. No matter how hard you try, it is impossible to stretch and attach the entire balloon onto the ring without breaking it. In other words, if you
force as large a part of the balloon as possible to be attached onto the ring, then the rest part of the balloon will be terribly stretched and the continuum of the material may eventually break down. We call this issue a bubble-breaking phenomenon, and its mathematical essence can have implications on various problems from different contexts. For example, in stability theory it implies the fact that, on any sphere enclosing an asymptotically stable limit cycle $P$ of a (smooth) flow $\varphi$ in $\mathbb{R}^3$, there always exists some point $p$ such that the trajectory $t \mapsto \varphi'(p)$ does not converge to $P$. We encountered this fact in [6] when studying path-following control. In this note, we demonstrate a case in which the topology of the communication network of a robotic swarm has to change during its movement. As we will see, the mathematics behind also reflects the essence of the bubble-breaking phenomenon.

The description of the case is detailed in Subsection 2.1 where the central problem of this study is stated as Question 1 and then Proposition 2 is proved in Subsection 2.2 as answer to the question. The analysis in Section 2 assumes a special structure from the initial positions of the robots. In Section 3 we show a natural condition on the initial positions under which the analysis can be applied, and the main result in this part is demonstrated as Proposition 7. In the end, by combining Propositions 2 and 7 we draw the final conclusion as Theorem 11.

2 A Case for the Communication Network of a Robotic Swarm

2.1 Description of the question

Imagine that there is a facility $\mathcal{P}$ (floating in the space) which occupies an area of a solid torus

$$\mathcal{P} = \{(r \cos \theta, r \sin \theta, z) | \theta \in [0, 2\pi], z^2 + (r - \frac{1}{10})^2 \leq \epsilon^2 \}$$

with $\epsilon < \frac{1}{100}$. Note that the circle

$$\mathcal{P} = \{x^2 + y^2 = \frac{1}{10}, z = 0\}$$

is the central axis of the solid torus $\mathcal{P}$. Suppose that the robots can only move in the area out of $\mathcal{P}$, and each robot can only communicate with those within the distance $\delta < \epsilon$.

A movement of a group of $n$ robots can be represented as a continuous map

$$\mathcal{R} : \mathbb{R} \ni t \mapsto (r_1(t), ..., r_n(t)) \in (\mathbb{R}^3)^n$$

with $r_i(t)$ being the position of the $i$th robot at the moment $t$. The communication network among the individuals of the group can be represented by an $n \times n$ “0-1” matrix $[p_{ij}]$. To be precise, if there is a message channel built up between the $i$th and the $j$th robots with $i < j$, then we set $p_{ij} = 1$ and otherwise $p_{ij} = 0$. 

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Since (the structure of) the network may change over time, we use the symbol $[p_{ij}]_t := [p^t_{ij}]$ to denote
the network at the moment $t$.

Denote by $o_i$ the initial position of the $i$th robot, i.e. $o_i = r_i(0)$. For simplicity, we assume that the initial communication network $[p_{ij}]_0$ induces a triangulation of $S^2$. To be precise, let $a_{ij}$ be the minimal geodesic connecting $o_i$ and $o_j$ on $S^2$. If $p^0_{ik} = p^0_{ij} = p^0_{jk} = 1$, the arcs $a_{ij}, a_{jk}, a_{ik}$ enclose a geodesic triangle $\sigma_{ijk}$ (including the interior). If two different triangles $\sigma$ and $\sigma'$ have nonempty intersection, their overlap is either an edge or a vertex. Suppose that at the moment $\bar{t} > 0$, the robots land on $\mathcal{P}$, i.e. $r_i(\bar{t}) \in \mathcal{P}$. We consider the question that whether the topology of the network changes during this movement.

**Question 1.** $[p_{ij}]_t \equiv [p_{ij}]_0$ for $t \in [0, \bar{t}]$?

The answer is not surprising: it has to change. An explanation is given in the next subsection.

### 2.2 Answer to the question

Associated to the matrix $[p_{ij}]_0$ there is a 2-dimensional simplicial complex in $\mathbb{R}^3$. For $i, j, k$ such that $p^0_{ik} = p^0_{ij} = p^0_{jk} = 1$, let $\Delta_{ijk}$ be the triangle in $\mathbb{R}^3$ with vertices $o_i, o_j, o_k$. $\Delta_{ijk}$ can be seen as a linear approximation to the curved triangle $\sigma_{ijk}$ on $S^2$, and

$$K = \bigcup \Delta_{ijk}$$

is a polytope with a triangulation given by $\{\Delta_{ijk}\}$. Moreover, $\Delta_{ijk}$ is homeomorphic to $\sigma_{ijk}$ via the radial projection. That is, for each $p$ on $\Delta_{ijk}$, there exists a unique $b_p \geq 1$ such that $b_p \cdot p \in \sigma_{ijk}$, and the map

$$\Delta_{ijk} \ni p \mapsto \frac{\sigma_{ijk}}{b_p} \cdot p \in \sigma_{ijk}$$

is a homeomorphism. Piecing together the maps $\tau_{ijk}$ we get a homeomorphism $\tau$ from $K$ to $S^2$ with $\tau|_{\Delta_{ijk}} = \tau_{ijk}$. With small $\delta$, $K$ also encloses $\mathcal{P}$. Then

$$h_t(w) = (1 - t) \cdot w + t \cdot \tau^{-1}(w)$$

is a continuous map from $S^2 \times [0,1]$ to $\mathbb{R}^3 - \mathcal{P}$ and it is a homotopy between the inclusion $\iota_{S^2} = h_0$ and $\tau^{-1} = h_1$.

The movement $\mathcal{R}$ induces a homotopy

$$\mathcal{K} : K \times [0,1] \to \mathbb{R}^3$$

by sending each point $p = s_i o_i + s_j o_j + s_k o_k$ in $\Delta_{ijk}$ to the point $s_i r_i(t) + s_j r_j(t) + s_k r_k(t)$. Here $s_i, s_j, s_k \geq 0$ are the weights with $s_i + s_j + s_k = 1$, and $\mathcal{K}$ is well-defined since $\{\Delta_{ijk}\}$ is a triangulation of $K$. Note that
\( \Delta_{ijk}^t = K_t(\Delta_{ijk}) \) is also a “triangle”: it is the convex hull of the points \( r_i(t), r_j(t) \) and \( r_k(t) \) in \( \mathbb{R}^3 \), and its diameter equals to the largest distance between these points.

We conclude the discussion with the following proposition.

**Proposition 2.** There exists \( t \in [0, \bar{t}] \) such that \([p_{ij}]_t \neq [p_{ij}]_0\).

**Proof.** We argue by contradiction. Assume that \([p_{ij}]_t = [p_{ij}]_0\) for all \( t \in [0, \bar{t}] \). It means that, if \( p^0_{ij} = 1 \), then the distance between \( r_i(t) \) and \( r_j(t) \) is always no larger than \( \delta \). As a consequence, the diameter of the “triangle” \( \Delta_{ijk}^t = K_t(\Delta_{ijk}) \) is no more than \( \delta \). Since the robots are moving outside \( \Psi \) and \( \delta \) is smaller than \( \epsilon \) (the radius of \( \Psi \)), we know that \( \Delta_{ijk}^t \) has no intersection with \( \mathcal{P} \). Since

\[
K_t(K) = \bigcup K_t(\Delta_{ijk}),
\]

this means that \( K \) is a continuous map from \( K \times [0, 1] \) to \( \mathbb{R}^3 - \mathcal{P} \). Note that it follows directly from Eq.(2) that \( K_0 \) is the inclusion of \( K \) into \( \mathbb{R}^3 - \mathcal{P} \). Then we get a homotopy \( h \) between \( \iota_{S^2} \) and \( K_t \circ \tau^{-1} \) by letting \( \tilde{h}_t = h_t \) for \( t \in [0, 1] \) and \( \tilde{h}_t = K_{t-1} \circ h_1 \) for \( t \in [1, \bar{t} + 1] \).

Since the diameter of \( \Delta_{ijk}^t \) is no larger than \( \delta \) and \( r_i(\bar{t}) \in \Psi \) for all \( i \in \{1, \ldots, n\} \), each \( \Delta_{ijk}^t \) lies in the \( \delta \)-neighbourhood \( \mathcal{U} \) of \( \Psi \) and then the image

\[
\tilde{h}_{t+1}(S^2) = \bigcup K_t(\Delta_{ijk}) = \bigcup \Delta_{ijk}^t
\]

also lies in \( \mathcal{U} \). Considering the radii of \( \mathcal{P} \) and \( \Psi \), \( \mathcal{U} \) is a thickened 2-torus lying in \( \mathbb{R}^3 - \mathcal{P} \), i.e. \( \mathcal{U} \cong \Psi \times (-\delta, \delta) \) and \( \mathcal{U} \cap \mathcal{P} = \emptyset \). Since \( S^2 \) is simply connected, \( \tilde{h}_{t+1} \) factors through the universal covering \( \mathbb{R}^3 \) (contractible) of \( \mathcal{U} \). The induced homomorphism \( \tilde{h}_{t+1,*} \) from \( H_2(S^2) \) to \( H_2(\mathbb{R}^3 - \mathcal{P}) \) then factors as

\[
\tilde{h}_{(t+1),*} : H_2(S^2) \to H_2(\mathbb{R}^3) \to H_2(\mathcal{U}) \xrightarrow{\text{inclusion}} H_2(\mathbb{R}^3 - \mathcal{P})
\]

and is therefore a null morphism (since \( H_2(\mathbb{R}^3) \cong \{0\} \)), and hence so is \( \iota_{S^2,*} \) due to the homotopy. However, for any \( p_0 \in \mathcal{P} \), \( S^2 \) is a deformation retract of \( \mathbb{R}^3 - \{p_0\} \) and hence

\[
H_2(S^2) \xrightarrow{\iota_{S^2,*}} H_2(\mathbb{R}^3 - \mathcal{P}) \xrightarrow{\text{inclusion}} H_2(\mathbb{R}^3 - \{p_0\})
\]

is an isomorphism, meaning \( \iota_{S^2,*} \) should not be trivial, yielding a contradiction. \( \square \)
3 Further Discussion

In Section 2 we assume that the communication network of the n robots at $t = 0$ together with the initial positions $(o_1, ..., o_n)$ induces a triangulation of $S^2$. In this section we replace this assumption with a more natural condition. This is formalized as Proposition 7, which is proposed in Subsection 3.1 and (eventually) proved in Subsection 3.3. Subsection 3.2 is devoted for a technical preparation.

Remark 3. A position of the swarm (n robots) on $S^2$ is an element $(a_1, ..., a_n)$ in $(S^2)^n$. We say a network $[p_{ij}]$ with a position $(a_1, ..., a_n)$ induces a triangulation of $S^2$ if and only if connecting all those points $a_i$ and $a_j$ by geodesics whenever $p_{ij} = 1$ gives a triangulation.

Remark 4. A sub-network/-graph of the network $[p_{ij}]$ can be represented as a matrix $[p'_{ij}]$ with the same dimension satisfying the relation

$$p'_{ij} = 1 \implies p_{ij} = 1. \quad (3)$$

3.1 A natural condition

If a subgraph fails to keep its structure then so does the whole network. Therefore, Proposition 2 applies as long as there is a sub-graph/sub-network inducing a triangulation on $S^2$. In fact, Proposition 2 still holds even if such a sub-network induces a triangulation only after an admissible perturbation on the initial position $(o_1, ..., o_n)$. Here, an admissible perturbation refers to a position $(o'_1, ..., o'_n) \in (S^2)^n$ from which the swarm can move to the actual initial position $(o_1, ..., o_n)$ while keeping the topology of the network unchanged. To be precise,

Definition 5. $(o'_1, ..., o'_n)$ is a perturbation of $(o_1, ..., o_n)$ admissible to a sub-network $[p'_{ij}]$ if there is a movement

$$\mathcal{R} : [0, 1] \xrightarrow{\mathcal{R}_0 = (o'_1, ..., o'_n)} (\mathbb{R}^3)^n - \mathcal{P} \quad (4)$$

with $\mathcal{R}_0 = (o'_1, ..., o'_n)$ and $\mathcal{R}_1 = (o_1, ..., o_n)$ such that whenever $p'_{ij} = 1$, $|\tilde{r}_i(t) - \tilde{r}_j(t)| < \delta$ holds for all $t \in [0, 1]$.

If a triangulation of $S^2$ is induced by a sub-network $[p'_{ij}]$ with the position $(o'_1, ..., o'_n)$ and the movement $\mathcal{R}$ is admissible to $[p'_{ij}]$, then applying Proposition 2 to the “composed” movement in which the swarm first takes the movement $\mathcal{R}$ from $(o'_1, ..., o'_n)$ to $(o_1, ..., o_n)$ and continues with the movement $\mathcal{R}$ will then prove that the structure of $[p'_{ij}]$ has to change during the whole process. Since it is unchanged in the first movement $\mathcal{R}$, we again shows that the structure of the network has to change in the movement $\mathcal{R}$.

Remark 6. For convenience, we will call $o_i$ and $o'_i$ respectively the actual and the virtual (initial) positions of the $i$th robot. Similarly, $(o_1, ..., o_n)$ and $(o'_1, ..., o'_n)$ are respectively the actual and the virtual (initial) positions of the swarm.
Based on the discussion above, we will look for a condition which allows the initial network \([p_{ij}]_0\) to have a sub-graph inducing a triangulation of \(S^2\) under an admissible perturbation on \((o_1,\ldots,o_n)\). We formalize it as the following proposition.

**Proposition 7.** Suppose that the initial positions \(\{o_1,\ldots,o_n\}\) of the robots constitute a \(\frac{\delta}{6}\)-net on \(S^2\), and any pair of robots will set up a message channel if the distance between them is smaller than \(\delta\). Then a sub-graph of the communication network \([p_{ij}]_0\) induces a triangulation of \(S^2\) after a perturbation on \((o_1,\ldots,o_n)\) which is admissible to the sub-graph.

The condition of the set \(\{o_1,\ldots,o_n\}\) being a \(\frac{\delta}{6}\)-net on \(S^2\) means that every point on \(S^2\) is at a distance less than \(\frac{\delta}{6}\) from some point (robot) \(o_i\). Here we choose the distance to be the (restriction of the) Euclidean metric from \(\mathbb{R}^3\) (on \(S^2\)). That is, the distance between any two points \(o\) and \(o'\) on \(S^2\) is measured by the length of the vector \(o - o'\) in \(\mathbb{R}^3\). With this metric, the \(\frac{\delta}{6}\)-neighbourhood \(D(o,\frac{\delta}{6})\) of \(o\) on \(S^2\) is simply the intersection of \(S^2\) with the 3-dimension ball \(B(o,\frac{\delta}{6})\) in \(\mathbb{R}^3\) (with center \(o\) and radius \(\frac{\delta}{6}\)). Here both \(D(o,\frac{\delta}{6})\) and \(B(o,\frac{\delta}{6})\) are taken as open sets in \(S^2\) and \(\mathbb{R}^3\), respectively. The condition is equivalent to saying the neighbourhoods \(D(o_i,\frac{\delta}{6})\) constitute an open cover of \(S^2\), and we consider it to be natural since it merely gives a description on the density of the robots on \(S^2\). Note that this condition may be coarse in the sense that we could have taken a (much) larger radius than \(\frac{\delta}{6}\), or, say, a (much) smaller density of the robots. However, giving a finer estimation on how sparse the robots can be (for inducing a triangulation) is beyond the scope of this note.

### 3.2 General positions of the swarm on \(S^2\)

For any \(1 \leq i, j, k \leq n\), define a function \(f_{ijk}(a_1,\ldots,a_n)\) with

\[
 f_{ijk}(a_1,\ldots,a_n) := \det[a_i; a_j; a_k].
\]

By saying a general position (of size \(n\)) on \(S^2\) we mean an element \((a_1,\ldots,a_n)\) in \((S^2)^n\) such that \(f_{ijk}(a_1,\ldots,a_n) \neq 0\) for all the triples \((i, j, k)\) with \(i < j < k\). Note that when the swarm is in a general position \((a_1,\ldots,a_n)\) on \(S^2\), the convex hull of any three robots is a triangle in \(\mathbb{R}^3\), and its radial projection on \(S^2\) is a geodesic triangle. In this subsection we show that given any (initial) position \((o_1,\ldots,o_n)\), with an arbitrarily small perturbation it yields a general position \((o'_1,\ldots,o'_n)\). More precisely,

**Lemma 8.** The set \(\mathcal{G}\) of all general positions is open and dense in \((S^2)^n\).

**Proof.** Note that \(\mathcal{G} = (S^2)^n - \mathcal{C}\) with

\[
\mathcal{C} = \bigcup_{1 \leq i < j < k \leq n} f_{ijk}^{-1}(0).
\]

We only need to show that for all triple \((i, j, k)\) with \(i < j < k\), the sets \(\mathcal{G}_{ijk} := \mathcal{G} - f_{ijk}^{-1}(0)\) are open and dense in \((S^2)^n\), and then as their finite intersection \(\mathcal{G}\) is also dense and open in \((S^2)^n\).
Since $f_{ijk}^{-1}(0)$ is closed in $(S^2)^n$, $\mathfrak{S}_{ijk}$ is dense. To see that $\mathfrak{S}_{ijk}$ is dense, we first look at the set $S^2_{ijk}$ of $f_{ijk}^{-1}(0)$ defined by containing all the points $(a_1, ..., a_n)$ with $a_i = a_j = a_k$. It is straightforward to check that $S^2_{ijk}$ is an embedding of $(S^2)^{n-2}$ in $(S^2)^n$, and therefore its complement $\tilde{\mathfrak{S}}_{ijk} := \mathfrak{S} - S^2_{ijk}$ is an open and dense subset of $(S^2)^n$ which contains $\mathfrak{S}_{ijk}$.

It remains to show that $\mathfrak{S}_{ijk} = \tilde{\mathfrak{S}}_{ijk} - f_{ijk}^{-1}(0)$ is dense in $\tilde{\mathfrak{S}}_{ijk}$. For doing this, we will verify that 0 is a regular value of the (restricted) function $f_{ijk} := f_{ijk}|_{\tilde{\mathfrak{S}}_{ijk}}$ on $\tilde{\mathfrak{S}}_{ijk}$. This will imply that

$$f_{ijk}^{-1}(0) \cap \tilde{\mathfrak{S}}_{ijk} = \tilde{f}_{ijk}^{-1}(0)$$

is an embedded submanifold with codimension 1 in $\tilde{\mathfrak{S}}_{ijk}$, and then as its complement $\mathfrak{S}_{ijk}$ is dense in $\tilde{\mathfrak{S}}_{ijk}$.

Suppose that $(a_1, ..., a_n)$ is from the set $\tilde{f}_{ijk}^{-1}(0)$. Without loss of generality, we can assume that $a_i \neq a_j$. From \ref{1} it holds $\det[a_i; a_j; a_k] = 0$, which means that the vectors $a_i$, $a_j$ and $a_k$ locate on a 2 dimensional vector subspace $V$ of $\mathbb{R}^3$. Let $u$ be a unit vector perpendicular to $V$. Since $a_i, a_j, a_k \in S^2 \cap V$, we know that $u$ is vertical to these three vectors, and then $(a_k, u)$ is a tangent vector of $S^2$ at $a_k$. Check that

$$\frac{d}{dt} f_{ijk}(a_1, ..., a_k + tu, ..., a_n) = \frac{d}{dt} \det[a_i; a_j; tu] = \det[a_i; a_j; u] \neq 0,$$

which implies that $(a_1, ..., a_n)$ is a regular point of $f_{ijk}$. \hfill \Box

As a consequence of Lemma \ref{1} we get a general position $(o_1', ..., o_n')$ from any given $\delta$-net $\{o_1, ..., o_n\}$ on $S^2$ via an arbitrarily small perturbation. Furthermore, when $(o_1', ..., o_n')$ is sufficiently closed to $(o_1, ..., o_n)$, i.e. the displacement

$$\delta o := \max_{1 \leq i \leq n} \{|o_i - o_i'|\} \tag{6}$$

is small enough, $(o_1', ..., o_n')$ is an admissible perturbation from $(o_1, ..., o_n)$, and $\{o_1', ..., o_n'\}$ is also a $\delta$-net on $S^2$. We give this an explanation in the following.

Since there are only finitely many pairs of robots with initial conditions $|o_i - o_j| < \delta$, the number

$$l_o := \max \{|o_i - o_j| \mid |o_i - o_j| < \delta\}$$

is strictly smaller than $\delta$. If $\delta o < l_o$, then the “straightline” movement of the swarm (out of $\Psi$) defined by

$$\tilde{r}_i(t) := (1 - t) \cdot o_i' + t \cdot o_i$$

gives an admissible perturbation.

Since the neighbourhoods $D(o_i, \delta/6)$ form a finite cover of $S^2$, for a number $\delta'$ slightly smaller than $\delta$, the
neighbourhoods $D(o_i, \frac{\delta'}{6})$ also cover $S^2$. To see this, we first take a look at the compact set

$$A_1 = S^2 - \bigcup_{j \neq 1} D(o_j, \frac{\delta}{6}).$$

Since $A_1$ has no intersection with $D(o_j, \frac{\delta}{6})$ for all $j > 1$, it is contained in $D(o_1, \frac{\delta}{6}).$ Due to the compactness of $A_1,$ with $\delta'_1$ slightly but strictly smaller than $\delta,$ $D(o_1, \frac{\delta'_1}{6})$ also contains $A_1.$ Therefore, the sets $D(o_1, \frac{\delta}{6})$ and $D(o_j, \frac{\delta}{6})$ for $j > 1$ constitute an open cover of $S^2.$ Now apply this process inductively to the rest $D(o_1, \frac{\delta}{6}).$ Suppose that we already get an open cover consisting of the sets $U_i = D(o_1, \frac{\delta'_i}{6}), \text{ for } i \leq k$ and $U_i = D(o_i, \frac{\delta}{6})$ for $i \geq k + 1.$ The compact set

$$A_{k+1} = S^2 - \bigcup_{j \neq k+1} U_j$$

is then contained in $U_{k+1} = D(o_{k+1}, \frac{\delta}{6}).$ Replace it with $D(o_{k+1}, \frac{\delta'_k}{6})$ and continue until we eventually get an open cover $D(o_i, \frac{\delta'_i}{6})$ with $\delta'_i < \delta$ for $i = 1, ..., n.$ Take

$$\delta' = \max\{\delta'_i\}$$

and then the neighbourhoods $D(o_i, \frac{\delta'_i}{6})$ again cover $S^2$ with $\delta' < \delta,$ i.e. $\{o_1, ..., o_n\}$ is also a $\frac{\delta'}{6}$-net.

Now we take $\delta o < \min\{\frac{\delta - \delta'}{6}, l_o\}.$ Since $\{o_1, ..., o_n\}$ is a $\frac{\delta'}{6}$-net, each $p \in S^2$ is contained in some $D(o_i, \frac{\delta'}{6}),$ i.e. $|p - o_i| < \frac{\delta'}{6}.$ Therefore, it holds that

$$|p - o'_i| < \frac{\delta'}{6} + \delta o < \frac{\delta}{6},$$

and hence we conclude that:

**Conclusion 9.** Suppose that the initial positions $o_1, ..., o_n$ on $S^2$ form a $\frac{\delta}{6}$-net on $S^2.$ Then by admissible perturbation the swarm is in such a general position $(o'_1, ..., o'_n)$ that $\{o'_1, ..., o'_n\}$ is also a $\frac{\delta}{6}$-net on $S^2.$

### 3.3 Sub-networks inducing triangulations

As a $\frac{\delta}{6}$-net, $\{o_1, ..., o_n\}$ is also a $\frac{\delta'}{6}$-net on $S^2$ for some $\delta' < \delta.$ According to Conclusion 9 we can perturb $(o_1, ..., o_n)$ to a general position $(o'_1, ..., o'_n)$ such that $\{o'_1, ..., o'_n\}$ is a $\frac{\delta'}{6}$-net, and, the displacement $\delta o$ defined by (9) is smaller than $\frac{\delta - \delta'}{2}.$ By the triangle inequality, if a subgraph with vertices from $\{o'_1, ..., o'_n\}$ has all the edges shorter than $\delta'$, these edges will then remain shorter than $\delta$ during the swarm moves back to the actual position $(o_1, ..., o_n)$ along the straightlines $\overline{o'_i o_i}.$

Instead of directly triangulate $S^2$ with $\{o'_1, ..., o'_n\}$ as the vertices, we will build a triangulation with the vertices from another set $V,$ and show that the points in $V$ are sufficiently close to $\{o'_1, ..., o'_n\}.$ More specifically, each point $q$ in $V$ is obtained from a point $o'$ in $\{o'_1, ..., o'_n\}$ with $|o' - q| < \frac{\delta'}{6},$ and the mapping $q \mapsto o'$ is injective.
3.3.1 The set of vertices $\mathcal{V}$

By rotating $S^2$ if necessary, we assume that $o'_n$ is the north pole $(0,0,1)$. For $h \in [-1,1]$, denote by $L_h$ the latitude

$$L_h = S^2 \cap \{(x,y,z) | z = h\}.$$  

Note that $L_0$ is the equator. For each $e^{i\theta} \in S^1$, denote by $R_{e^{i\theta}}$ the longitude line intersecting with $L_0$ at the point $(e^{i\theta},0)$. For convenience, we will refer to $h$ and $e^{i\theta}$ the latitude and the longitude, respectively. Since $(o'_1, ..., o'_n)$ is a general position, when $i, j \neq n$, $o'_i$ and $o'_j$ have different longitude. Choose a sequence from $(-1, 1)$

$$h_k < h_{k+1} < ... < h_0 = 0 < h_1 < ... < h_k$$

such that the distance between any two adjacent latitude lines $L_j = L_{h_j}$ and $L_{j+1} = L_{h_{j+1}}$ is $\frac{\delta'}{3}$. The integer $k$ is taken in such a way that the distance between $L_k$ and $o'_n = (0, 0, 1)$ is smaller than $\frac{2\delta'}{3}$ but no less than $\frac{\delta'}{3}$. Note that by symmetry this means that the distance between $L_{-k}$ and $(0, 0, -1)$ is also between $\frac{\delta'}{3}$ and $\frac{2\delta'}{3}$.

For each $j \in \{-k, ..., k\}$, let $C_j$ be the subset of $O' = \{o'_1, ..., o'_n\}$ containing all those $o'_i$ such that $L_j$ intersects with $D(o'_i, \frac{\delta'}{6})$, or equivalently, with $B(o'_i, \frac{\delta'}{6})$. That is,

$$C_j := \{o'_i \in O' | L_j \cap B(o'_i, \frac{\delta'}{6}) \neq \emptyset\}.  \tag{7}$$

Re-label the points in $C_j$ as $o'_{(j,1)}, ..., o'_{(j, n_j)}$ in such a way that the corresponding longitude $w_{(j,1)}, ..., w_{(j, n_j)}$ are counter clockwise on $S^1$. By the way the circles $L_j$ are chosen, their diameters are strictly larger than $\frac{\delta'}{4}$. Therefore, the intersection of $L_j$ with each neighbourhood $D(o'_{(j,l)}, \frac{\delta'}{6})$ (or equivalently, $B(o'_{(j,l)}, \frac{\delta'}{6})$) is an arc/interval $I_{(j,l)}$ centred at a point $q_{j,l}$. Here, $q_{j,l}$ is the intersection of the longitude $R_{w_{(j,l)}}$ with the latitude.

**Conclusion** 10. The distance between $o'_{(j,l)}$ and $q_{j,l}$ is less than $\frac{\delta'}{6}$. Since $\{I_{(j,l)}\}$ covers $L_j$, the distance between each consecutive points $q_{j,l}$ and $q_{j,l+1}$ is less than $\frac{\delta'}{4}$. Here, $l \in \{1, ..., n_j\}$ and $q_{j,l+1}$ refer to $q_{j,1}$ when $l = n_j$.

For each $j = -k, ..., k$, we set

$$\mathcal{V}_j := \{q_{j,l}, l = 1, ..., n_j\}.  \tag{8}$$

For $j < k - 1$, we will triangulate the annulus $A_{j,j+1}$ between $L_j$ and $L_{j+1}$ with the points in $\mathcal{V}_j$ and $\mathcal{V}_{j+1}$ as the vertices. T triangulate the discs $D_k$ and $D_{-k}$ enclosed by respectively $L_k$ and $L_{-k}$, we still need two more points lying in the discs. For $D_k$ we pick $o'_n$. For $D_{-k}$, we choose a point $o'_s$ such that $(0, 0, -1) \in D(o'_{s}, \frac{\delta'}{6})$. Since the distance from $(0, 0, -1)$ to the circle $L_{-k}$ is no less than $\frac{\delta'}{3}$, we know that $o'_s$ lies in disc $D_{-k}$. Moreover, the neighbourhoods $D(o'_{(s,k,j)}, \frac{\delta'}{6})$ do not cover the south pole $(0, 0, -1)$ and then $o'_s$ is not in $C_{-k}$.
(or any other \(C_j\)). We take the set of vertices to be
\[
\mathcal{V} := \left( \bigcup_j \mathcal{V}_j \right) \bigcup \{o_n', o_n''\}.
\]

3.3.2 Connecting the vertices and constructing the triangulation

Now we describe the construction of the triangulation. The latitude lines decompose the sphere into areas \(A_{j,j+1}\) between \(L_j\) and \(L_{j+1}\) for \(j \leq k-1\) and discs \(D_k\) and \(D_{-k}\) on \(S^2\) respectively enclosed by \(L_k\) and \(L_{-k}\). The triangulation of \(S^2\) to be constructed will also triangulate each of these areas.

First of all, we connect \(q_{j,l}\) with \(q_{j,l+1}\) for each \(j\) and \(l\). When \(l = n_j\), \(q_{j,l+1}\) refers to \(q_{j,1}\). We can simply take the closed arcs \([q_{j,l}, q_{j,l+1}]\) to be the edges. Note that the distance between \(q_{j,l}\) and \(q_{j,l+1}\) is less than \(\frac{\delta}{3}\), while the diameter of \(L_j\) is larger than \(\sqrt{2} \cdot \frac{\delta}{3}\). Therefore, \([q_{j,l}, q_{j,l+1}]\) is a minor arc on \(L_j\).

Second, we connect \(o_n\) to each \(q_{k,l}\) in \(\mathcal{V}_k\) and \(o_n'\) to each \(q_{k,l}\) in \(\mathcal{V}_{-k}\). It indeed gives triangulations of \(D_k\) and \(D_{-k}\) since each of the sets \(\mathcal{V}_k\) and \(\mathcal{V}_{-k}\) contains at least three points. We show this for \(\mathcal{V}_k\) and it works for \(\mathcal{V}_{-k}\) in the same way. The radius of the circle \(L_k\) (in \(\mathbb{R}^3\)) is larger than \(\frac{\sqrt{2}}{3} \cdot \frac{\delta}{3} > \frac{\delta}{5}\). Meanwhile, the intersection of each \(B(o_i(k,l), \frac{\delta}{5})\) with the plane \(\{z = h_k\}\) is a disc \(D_{(k,l)}\), with radius no more than \(\frac{\delta}{5}\). If \(\mathcal{V}_k\) has only two points, it means that \(D_{(k,1)}\) and \(D_{(k,2)}\) should cover \(L_k\). Then by symmetry, at least one of the discs should cover both the ends of a diameter of \(L_k\), which is impossible due to their sizes. Moreover, the distance from \(o_n\) to \(L_k\) is less than \(\frac{2\delta}{5}\), and by the triangle inequality \(o_n'\) is no farther than \(\frac{2\delta}{5} + \frac{\delta}{6}\) from each \(q_{k,l}\) since \((0, 0, -1) \in D(o_n', \frac{\delta}{6})\).

Last, we complete the triangulation of the area \(A_{j,j+1}\) between \(L_j\) and \(L_{j+1}\) by connecting the points from \(\mathcal{V}_j\) with those \(\mathcal{V}_{j+1}\) in a proper way. For \(q \in R_w\), \(q' \in R_w\) and \(q'' \in R_w\), we say that \(q'\) is before \(q''\) from the right-hand side of \(q\) if and only if the relations \(w' = e^{i\theta'} w\) and \(w'' = e^{i\theta''} w\) hold for \(0 < \theta' < \theta'' < 2\pi\). For \(j = -k, ..., k-1\), we connect each point \(q_{j,l}\) to the first point in \(\mathcal{V}_{j+1}\) from the right-hand side of \(q_{j,l}\), and \(q_{j+1,l'}\) to the first point in \(\mathcal{V}_j\) from the right-hand side of \(q_{j+1,l'}\). We explain in detail for \(j \geq 0\) that this gives a triangulation of the area \(A_{j,j+1}\) and the distance between each pair of connected points is less than \(\frac{2\delta}{5}\). The explanation also works for the other cases in the same way.

For \(j \geq 0\), the radius of \(L_j\) is larger then \(L_{j+1}\). Let \(q_{j,l}'\) be the point on \(L_{j+1}\) with the same longitude with \(q_{(j,l)}\). Observe that the distance between \(q_{j,l}\) and \(q_{j,l}'\) is smaller than that between \(q_{j,l}\) and \(q_{j,l+1}\), and hence is smaller than \(\frac{\delta}{5}\). Now we have two families of points on \(L_{j+1}\): \(\{q_{j+1,l'}\}\) and \(\{q_{j,l}'\}\). For convenience, we color \(q_{j+1,l'}\) in grey and \(q_{j,l}'\) in red. The circle \(L_{j+1}\) can be decomposed into intervals each of which contains points in a single color, and each point from \(\{q_{j+1,l'}\}\) and \(\{q_{j,l}'\}\) belongs to one of these intervals. Suppose that \(I^r\) is an interval contains points in red. Then it holds \(I^r \subset (q_{j+1,l'}, q_{j+1,l'+1})\) for some \(l' \in \{1, ..., n_{j+1}\}\). Since \(\{q_{j+1,l'}, q_{j+1,l'+1}\}\) is a minor arc on \(L_{j+1}\), each points \(q_{j,l}'\) in \(I^r\) is at a distance less than \(\frac{\delta}{5}\) from \(q_{j+1,l'+1}\), which is exactly the first point in \(\mathcal{V}_{j+1}\) from the right-hand side of \(q_{j,l}\). Similarly, an interval \(I^g\)
the robots land on \( R \)

two robots will set up a message channel if the distance between them is smaller than \( q_{j,l'} \) in \( I^g \) is no farther than \( \frac{\delta'}{3} \) from \( q_{j,l+1}^l \), and \( q_{j,l+1}^l \) is the first point in \( V_j \) from the right-hand side of \( q_{j+1,l'} \).

Since the distance between \( L_j \) and \( L_{j+1} \) is \( \frac{\delta'}{3} \), by the triangle inequality we know that the distances between the connected pairs are indeed less than \( \frac{3\delta'}{3} \).

\[ 3.4 \quad \text{Conclusion} \]

We describe the sub-network \( |p'_{j,l}|_0 \) obtained from the triangulation constructed above. Take the subset of \( \{o'_1, ..., o'_n\} \)

\[ \mathcal{C} := (\bigcup_j \mathcal{C}_j) \cup \{o'_n, o'_s\}, \]

and then the points in \( \mathcal{C} \) are in one-one correspondence with the points in \( V \). (Recall that each point in \( \mathcal{C}_j \) is re-labelled as \( o'_{(j,l)} \) and corresponds to \( q_{j,l} \) in \( V_j \), and the points \( o'_n, o'_s \) simply correspond to themselves). For \( i, i' \in \{1, ..., n\} \), if \( o'_i \) and \( o'_{i'} \) are in \( \mathcal{C} \) and the corresponding points in \( V \) are connected in the triangulation, we set \( p'_{i,i'} = 1 \), otherwise, set \( p'_{i,i'} = 0 \).

We still need to verify that when \( p'_{i,i'} = 1 \), it holds \( |o'_i - o'_{i'}| < \delta' \). Suppose that \( o'_i, o'_{i'} \in (\bigcup_j \mathcal{C}_j) \) and correspond to \( q_{i,l} \) and \( q_{i',l'} \) in \( (\bigcup_j V_j) \), respectively. Then when they are connected, we have \( |q_{i,l} - q_{i',l'}| < \frac{2\delta'}{3} \). The distances between \( q_{i,l} \) and \( o'_{(i,l)} \) are less than \( \frac{2\delta'}{3} \) for all \( (j,l) \), and hence it holds

\[ |o'_i - o'_{i'}| < |o'_{(i,l)} - q_{i,l}| + |q_{i,l} - q_{i',l'}| + |q_{i',l'} - o'_{i',l'}| < \delta'. \]

If \( o'_i = o'_n \) and \( o'_{i'} = o'_{(k,l)} \), it holds \( |o'_i - q_{k,l}| < \frac{2\delta'}{3} + \frac{\delta'}{3} \) and then again we get \( |o'_i - o'_{i'}| < \delta' \) from the triangle inequality. The same works for the case with \( o'_i = o'_s \) and \( o'_{i'} = o'_{(-k,l)} \).

Combining Propositions 2 and 7 we conclude that

\[ \textbf{Theorem 11.} \quad \text{Suppose that the initial positions} \ \{o_1, ..., o_n\} \ \text{of the robots constitute a} \ \frac{\delta}{3} \text{-net on} \ S^2, \ \text{and any two robots will set up a message channel if the distance between them is smaller than} \ \delta. \ \text{If after a movement} \ \mathcal{R} \ \text{the robots land on} \ \Psi \ \text{at the moment} \ t > 0, \ \text{then} \ |p_{i,j}|_t \neq |p_{i,j}|_0 \ \text{for some} \ t \in [0, t]. \]

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