D=4, N=1 Supersymmetric Henneaux-Knaepen Models

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**Abstract**

We construct \( N = 1 \) supersymmetric versions of four-dimensional Freedman-Townsend models and generalizations thereof found recently by Henneaux and Knaepen, with couplings between 1-form and 2-form gauge potentials. The models are presented both in a superfield formulation with linearly realized supersymmetry and in WZ gauged component form. In the latter formulation the supersymmetry transformations are nonlinear and do not commute with all the gauge transformations. Among others, our construction yields \( N = 1 \) counterparts of recently found \( N = 2 \) supersymmetric gauge theories involving vector-tensor multiplets with gauged central charge.

**1 Introduction**

Four-dimensional Freedman-Townsend models \[1\] involve peculiar gauge invariant self-couplings of 2-form gauge potentials. These couplings are local, but nonpolynomial in the fields and in the coupling constant. Nonpolynomial couplings of a similar type, but between 2-form gauge potentials and ordinary gauge fields, are met in recently constructed \( D = 4, N = 2 \) supersymmetric gauge theories \[2, 3\] involving so-called vector-tensor multiplets \[4, 5\].

In the latter models, the nonpolynomial couplings arise from gauging a nonstandard global symmetry, the so-called central charge of the vector-tensor multiplet. This was illustrated in \[3\] through a nonsupersymmetric toy-model. In contrast, the nonpolynomial couplings appearing in Freedman-Townsend models are not related to a global symmetry that is gauged.

Nevertheless there is a relationship between all these models from which one can also understand the origin and similarity of the peculiar couplings appearing in them. In fact, the Freedman-Townsend models, the toy-model discussed in \[3\], and the purely bosonic part of the \( N = 2 \) supersymmetric gauge theories in \[2, 3\], with scalars set to constants, can all be fit in a larger class of (nonsupersymmetric) gauge theories found recently by Henneaux and Knaepen \[7\].

As Henneaux-Knaepen models arise so naturally in the supersymmetric gauge theories mentioned above, it is tempting to seek supersymmetric versions of these models.
The purpose of the present paper is the construction of $N=1$ globally supersymmetric Henneaux-Knaepen models in four spacetime dimensions.

We shall first review four-dimensional nonsupersymmetric Henneaux-Knaepen models in section 2. In section 3 we construct supersymmetric Henneaux-Knaepen models with linearly realized supersymmetry in terms of appropriate superfields, generalizing earlier work [8] on supersymmetric Freedman-Townsend models. Section 4 provides the component version of these models in an appropriate “Wess-Zumino (WZ) gauge” and is the main part of the paper. In section 5 we illustrate the results for two simple examples, one of which is an $N=1$ counterpart of the aforementioned $N=2$ gauge theories. The paper is ended with some concluding remarks in section 6 and a short appendix containing among others our conventions.

2 D=4 Henneaux-Knaepen models

The models couple sets of 2-form and 1-form gauge potentials. We shall label these gauge potentials by indices $A$ and $a$ respectively, and denote their components by $B_{\mu\nu A} = -B_{\nu\mu A}$ and $A_{\mu a}^A$. The action and gauge transformations can be elegantly written by means of auxiliary vector fields $V_{\mu}^A$. In this first order formulation, they are polynomial. The nonpolynomial form is then obtained upon eliminating the auxiliary fields. In first order form, the Lagrangian reads

$$L = L_{FT} + L_{HK} + L_{CM} + L_{aux}$$

where $c_{ab}$ are arbitrary real constant coefficients, and $V_{\mu}^A$ and $\hat{F}_{\mu\nu}^a$ are given by

$$V_{\mu}^A = \partial_\mu V^A - \partial_\nu V^A + g f_{BC}^A V^B \nabla_\nu C$$

$$\hat{F}_{\mu\nu}^a = \nabla_\mu A^a - \nabla_\nu A^a , \quad \nabla_\mu A^a_\nu = \partial_\mu A^a_\nu + g V^A T^a_{\mu} A^b_{\nu} .$$

Here $g$ is a coupling constant with dimension $-1$, and $f_{BC}^A$ and $T^a_{A b}$ are real constants which satisfy

$$f_{[AB}^D f_{C]D}^E = 0$$

$$T^a_{Ac} T^b_{Bc} - T^a_{Bc} T^b_{Ac} = f_{AB}^C T^a_{C b} .$$

According to (2.8) and (2.9), the $f_{AB}^C$ are the structure constants of a Lie algebra $G$, while the $T^a_{A b}$ are the entries of matrices $T_A$ representing $G$.

$$[T_A , T_B] = f_{AB}^C T_C .$$

Further conditions are not imposed. In particular, $G$ can be any finite dimensional Lie algebra (not necessarily compact), $T_A$ can be any real representation thereof, and $\delta_{ab}$,
$c_{ab}$ and $\delta_{AB}$, which appear in $L_{HK}$, $L_{CM}$ and $L_{aux}$ respectively, need not be $\mathcal{G}$-invariant
tensors (hence, in general $L$ is not globally $\mathcal{G}$-invariant).

We shall refer to $L_{FT}$, $L_{HK}$, $L_{CM}$ as the Freedman-Townsend, Henneaux-Knaepen,
and Chapline-Manton part of the Lagrangian respectively. We note that by combining
all these parts in a single action we have slightly deviated from [7] where such a combi-
nation was not considered (rather, Chapline-Manton type couplings, of a more general
form, were discussed separately from the other two types). The reason is that the
Chapline-Manton part arises naturally in the supersymmetric exten-
sions constructed later on, and therefore we have introduced it already here. $L_{CM}$ gives rise to couplings of
the 2-form gauge potentials (or, more precisely, their field strengths) to Chern-Simons
forms, similar to those appearing in [9] and in the Green-Schwarz anomaly cancella-
tion mechanism [10]. This becomes clear upon elimination of the auxiliary fields (see
below). Note that the $V$-independent part of $L_{CM}$ is a total derivative.

Eq. (2.8) guarantees the invariance of the action under the following gauge transfor-
mations,

$$\delta_C B_{\mu \nu A} = \nabla_\mu C_{\nu A} - \nabla_\nu C_{\mu A} , \quad \delta_C A_\mu^a = 0 , \quad \delta_C V_\mu^A = 0 , \quad (2.10)$$

where the $C_{\mu A}$ are arbitrary fields, and $\nabla_\mu C_{\nu A}$ is given by

$$\nabla_\mu C_{\nu A} = \partial_\mu C_{\nu A} - g V_\mu^B f_{BA}^C C_{\nu C} . \quad (2.11)$$

Indeed, the $\delta_C$-transformation of the Lagrangian is a total derivative,

$$\delta_C L = \delta_C L_{FT} = -\frac{1}{2} \partial_\rho (\varepsilon^{\mu \rho \sigma} V_{\mu \nu}^A C_{\sigma A}) .$$

This holds because the terms in $\delta_C L_{FT}$ without derivatives (i.e., those which are cubic in $V$) cancel thanks to (2.8). One can easily deduce this from the Bianchi identity

$$\varepsilon^{\mu \rho \sigma} \nabla_\rho V_{\mu \nu}^A = \varepsilon^{\mu \rho \sigma} (\partial_\rho V_{\mu \nu}^A + g V_\rho^B f_{BC}^A V_{\mu \nu}^C) = 0 .$$

This Bianchi identity holds thanks to (2.8) for any $\mathcal{G}$, as $V_{\mu \nu}^A$ has precisely the form of
a nonabelian Yang-Mills strength. Note however that $V_{\mu \nu}^A$ cannot be interpreted as a
Yang-Mills gauge field (the action is clearly not invariant under corresponding Yang-
Mills gauge transformations, due to the presence of $L_{aux}$). The gauge transformations $\delta_C$ are reducible because a shift $C_{\mu A} \rightarrow C_{\mu A} + \nabla_\mu Q_A$ modifies $\delta_C B_{\mu \nu A}$ only by the term $[\nabla_\mu , \nabla_\nu] Q_A = -g V_{\mu \nu}^B f_{BA}^C Q_C$ which vanishes on-shell for any fields $Q_A$ (as $V_{\mu \nu}^A$ vanishes by the equations of motion for $B_{\mu \nu A}$).

Eq. (2.9) guarantees that the action is also gauge invariant under

$$\delta_\epsilon A_\mu^a = \partial_\mu \epsilon^a + g V_\mu^A T_{A b}^a \epsilon^b \equiv \nabla_\mu \epsilon^a , \quad \delta_\epsilon V_\mu^A = 0 , \quad (2.12)$$

where the $\epsilon^a$ are arbitrary fields. Indeed, thanks to (2.9) one has

$$\delta_\epsilon \hat{F}_{\mu \nu} = [\nabla_\mu , \nabla_\nu] \epsilon^a = g V_{\mu \nu}^A T_{A b}^a \epsilon^b .$$

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It is now easy to verify that \( \delta (L_{HK} + L_{CM}) \) is precisely canceled by \( \delta L_{FT} \),

\[
\delta (L_{HK} + L_{CM}) = - \delta L_{FT} \quad \Rightarrow \quad \delta c \equiv 0 .
\]

Let us briefly discuss the formulation without auxiliary fields \( V^A \). Up to a total derivative, the Lagrangian takes the form

\[
\frac{1}{2} V^A \epsilon^{\mu \nu} K_{AB} V^B - V^A \mathcal{H}^\mu_A - \frac{1}{4} \delta_{ab} F_{\mu \nu}^a F^{\mu \nu b} ,
\]

where

\[
K_{AB}^{\mu \nu} = \delta_{AB} \eta^{\mu \nu} - \frac{1}{2} g f_{ABC} \epsilon^{\mu \nu \rho \sigma} B_{\rho \sigma C} - g^2 T_a^B T^b_A (\delta_{ab} \eta^{\mu \nu} A^c_{\mu \rho} - \delta_{ab} A_{\mu \rho} A_{\nu c} - c_{ab} \epsilon^{\mu \nu \rho \sigma} A^c_{\mu \rho} A^d_{\nu \sigma})
\]

\[
\mathcal{H}^\mu_A = \frac{1}{2} \epsilon^{\mu \nu \rho \sigma} \partial_{\nu} B_{\rho \sigma A} + g T_a^B A^c_{\rho \sigma} (\delta_{ab} F_{\mu \nu} + \frac{1}{2} c_{ab} \epsilon^{\mu \nu \rho \sigma} F_{\rho \sigma}^b)
\]

\[
F_{\mu \nu}^a = \partial_{\mu} A^a_{\nu} - \partial_{\nu} A^a_{\mu}.
\]

The equations of motion for \( V^A \) give (\( \approx \) denotes equality on-shell)

\[
K_{AB}^{\mu \nu} V^B \approx \mathcal{H}^\mu_A \quad \Rightarrow \quad V^A \approx (K^{-1})^{AB}_\mu \mathcal{H}^\nu_B , \quad (K^{-1})^{AC}_\mu \ K^{\mu \nu}_{CB} = \delta^A_B \delta^\nu_\mu .
\]

The formulation without auxiliary fields is thus obtained by substituting \( (K^{-1})^{AB}_\mu \mathcal{H}^\nu_B \) for \( V^A \) in the above expressions for the action and gauge transformations. For instance, the action turns into

\[
S = - \int d^4 x \left[ \frac{1}{2} \mathcal{H}^\mu_A (K^{-1})^{AB}_\mu \mathcal{H}^\nu_B + \frac{1}{4} \delta_{ab} F_{\mu \nu}^a F^{\mu \nu b} \right] . \tag{2.13}
\]

Note that \( K_{AB}^{\mu \nu} \) depends on the fields, but not on derivatives thereof. Hence, its inverse is nonpolynomial in the fields, but still local. As a consequence, in the formulation without auxiliary fields, the action and gauge transformations are also nonpolynomial but remain local. In fact, the action contains only terms with exactly two spacetime derivatives, while the gauge transformations are linear in derivatives. The gauge transformations commute on-shell, i.e., they are abelian.

To understand the nature of the above models and of their gauge symmetries, it is instructive to view them as deformations of corresponding free theories (in fact, this is how they were derived in [7]). The free action \( (g = 0) \) reads

\[
S_{\text{free}} = - \int d^4 x \left[ \frac{1}{2} H^\mu_A H^\nu_A + \frac{1}{4} F_{\mu \nu}^a F^{\mu \nu a} \right]
\]

where

\[
H^\mu_A = \frac{1}{2} \epsilon^{\mu \nu \rho \sigma} \partial_{\nu} B_{\rho \sigma A} , \quad H^A_\mu = \delta^{AB} H^A_B , \quad F_{\mu \nu}^a = \delta_{ab} F_{\mu \nu}^b .
\]

The free theory has, among others, global symmetries generated by

\[
\Delta_a A^b_\mu = H^A_\mu f^b_a A^c_\nu , \quad \Delta_a B_{\mu \nu} = \frac{1}{2} \epsilon_{\mu \nu \rho \sigma} F_{\rho \sigma}^b T^b_a .
\]
and corresponding Noether currents
\[ j^\mu_a = T^b_{Aa} H^A_{\nu} F^{\mu\nu}_b. \]
Furthermore, it possesses conserved nontrivial currents of second order,
\[ j^{\mu\nu A} = \frac{1}{2} f_{BC} A^{\nu} \varepsilon^{\mu\nu\rho\sigma} H^B_{\rho} H^C_{\sigma}. \]
Expanding the action (2.13) in \( g \), one finds
\[ S = S_{\text{free}} - g \int d^4x \left( B_{\mu\nu A} j^{\mu\nu A} + A^a_{\mu} j^\mu_a + c_{ac} T^c_{A b} \varepsilon^{\mu\nu\rho\sigma} H^A_{\mu} A^a_{\nu} \partial^\rho_A A^b_{\sigma} \right) + O(g^2). \]
Hence, to first order in \( g \) the action couples \( A^a_{\mu} \) and \( B_{\mu\nu A} \) to first and second order currents \( j^\mu_a \) and \( j^{\mu\nu A} \) of the free theory. These couplings arise from \( L_{\text{HK}} \) and \( L_{\text{FT}} \) respectively. In addition there are couplings of \( H^A_{\mu} \) to abelian Chern-Simons forms originating from \( L_{\text{CM}} \).

Analogously one may expand the gauge transformations in \( g \). At zeroth order this reproduces of course the gauge symmetries of the free theory. The first order pieces involve the global symmetries of the free action given above through transformations \( g_{\epsilon} a \Delta_a \). Hence, \( \delta_{\epsilon} \) gauges these global symmetries (\( \delta_{\epsilon} B_{\mu\nu A} \) involves in addition terms related to the Chapline-Manton couplings). This explains why Henneaux-Knaepen models arise when one gauges the central charge of the \( N=2 \) vector tensor multiplet, as this central charge is a global symmetry of the above type.

We remark that \( j^\mu_a \), \( j^{\mu\nu A} \) are conserved for any constants \( T^b_{Aa}, f_{BC} A^{\nu} \), i.e., whether or not these constants fulfill (2.8) and (2.9). The latter conditions arise at order \( g^2 \) from the requirement that the deformed action be invariant under deformed versions of the gauge transformations of the free model [7].

### 3 Superfield formulation

We shall now construct a supersymmetric extension of the Lagrangian (2.1) in terms of superspace integrals. To this end we associate an appropriate superfield with each of the fields \( B_{\mu\nu A}, A^a_{\mu} \) and \( V^A_{\mu} \), and generalize the gauge transformations (2.10) and (2.12) to these superfields. Similarly to the nonsupersymmetric case, the superfield associated with \( V^A_{\mu} \) is auxiliary and may eventually be eliminated algebraically. Our construction applies to general \( \mathcal{G} \) (not necessarily compact), and any real representation \( T_A \) thereof. As a consequence, even in the pure Freedman-Townsend case, in general we cannot use traces over matrix valued fields in order to construct the Lagrangian. Therefore we must spell out indices \( A \) and \( a \) explicitly, where necessary.

The superfields associated with \( A^a_{\mu} \) and \( V^A_{\mu} \) are standard real vector superfields which we denote by \( A^a \) and \( V^A \) respectively. We assign dimension 0 to \( A^a \) and dimension 1 to \( V^A \), as the latter is auxiliary.

The superfield associated with \( B_{\mu\nu A} \) is a spinorial one as in [11] and denoted by \( \Psi^\alpha_A \). It is chiral,
\[ \bar{D}_\alpha \Psi^\alpha_A = 0, \quad (3.1) \]
where $\bar{D}_\alpha$ is a supercovariant derivative mapping superfields into superfields, cf. appendix. We assign dimension $1/2$ to $\Psi^\alpha_A$. Then the independent component fields of $\Psi^\alpha_A$ are two Weyl fermions with dimension $1/2$ and $3/2$ respectively, a complex scalar field with dimension $1$ and a 2-form gauge potential, also with dimension $1$. We remark that $\Psi^\alpha_A$ is the prepotential of a real linear superfield $\Phi_A = D^\alpha \Psi_{\alpha A} + \bar{D}_\alpha \bar{\Psi}^\alpha_A (D^2 \Phi_A = 0)$.

In order to construct the superfield action, we define two chiral superfields, $Y^a_\alpha$ and $W^A_\alpha$, constructed of $A^a$ and $V^A$. $Y^a_\alpha$ is given by

$$Y^a_\alpha = -\frac{i}{4} \bar{D}^2 (e^{-2iV} D_\alpha e^{iV})^a,$$

$$V = g V^A T_A,$$

where $T_A$ are real matrices representing $\mathcal{G}$ as in (2.9), and $\bar{D}^2 = \bar{D}_\alpha \bar{D}^\alpha$. In (3.2), $\bar{D}_\alpha$ and $D_\alpha$ act on everything to their right, and ordinary matrix multiplication is understood, i.e.,

$$(e^{-2iV} D_\alpha e^{iV})^a = (e^{-2iV})^a_b D_\alpha [(e^{iV})^b_c A^c].$$

$W^A_\alpha$ is defined analogously to the spinorial field strength in super-Yang-Mills theory,

$$g W^A_\alpha T_A = -\frac{i}{4} \bar{D}^2 (e^{-2iV} D_\alpha e^{2iV}).$$

We are now prepared to present our superfield Lagrangian. It reads

$$L = L_{FT} + L_{HK} + L_{CM} + L_{FI} + L_{aux}$$

$$L_{FT} = -\frac{1}{2} \int d^2 \theta W^A T_A + c.c.$$

$$L_{HK} + L_{CM} = \int d^2 \theta k_{ab} Y^a Y^b + c.c.$$

$$L_{FI} = \int d^2 \theta d^2 \bar{\theta} \mu_\alpha A^\hat{a}$$

$$L_{aux} = \int d^2 \theta d^2 \bar{\theta} \mathcal{F}(\bar{V}) + c.c.$$

where $k_{ab} = k_{ba}$ are arbitrary complex numbers, and $\mathcal{F}(\bar{V})$ is any function of the $V^A$ that allows eventually to eliminate $V^A$ (e.g., one may take $\mathcal{F}(V) \propto \delta_{AB} V^A V^B$, but more general choices are admissible too). $L_{FI}$ is present only in the special case that all the representation matrices $T_A$ have a vanishing row in common, i.e., only if

$$T^\alpha_{\hat{a} b} = 0 \quad \forall A, b$$

for some $\hat{a}$. In that case one may include $L_{FI}$, with arbitrary real numbers $\mu_\alpha$. $L_{FI}$ is of course a Fayet-Iliopoulos contribution \[12\]. The supersymmetric Henneaux-Knaepen and Chapline-Manton parts of the Lagrangian, $L_{HK}$ and $L_{CM}$, arise from the real and imaginary part of $k_{ab}$ respectively.

Thanks to the use of superspace techniques, the action $\int d^4 x \, L$ is manifestly supersymmetric. We shall now show that it has in addition gauge symmetries corresponding
to (2.10) and (2.12). As in [8], the counterpart of (2.10) is generated on the superfields through

$$\delta_C \Psi \alpha A = i \bar{D}^2 (e^{-2i \bar{V}} D \alpha e^{i \bar{V}} C)_A, \quad \delta_C A^a = 0, \quad \delta_C V^A = 0$$

(3.10)

where the $C_A$ are arbitrary real vector superfields, and

$$\nabla_A^B \equiv -g V^C f_{CA}^B.$$  

(3.11)

In (3.10), matrix multiplication is understood, as in (3.2). In order to verify the invariance of the action, one calculates

$$\delta_C L = \delta_C L_{FT} = -\frac{i}{2} \int d^2 \theta W^\alpha A D^2 (e^{-2i \bar{V}} D \alpha e^{i \bar{V}} C)_A + \text{c.c.}$$

$$\simeq 2i \int d^2 \theta d^2 \bar{\theta} (W^\alpha e^{-2i \bar{V}})^A D \alpha (e^{i \bar{V}} C)_A + \text{c.c.}$$

$$\simeq -2i \int d^2 \theta d^2 \bar{\theta} C_B [(e^{i \bar{V}})_A^B D^\alpha (W \alpha e^{-2i \bar{V}})_A - \text{c.c.}]$$

$$= 0,$$

where $\simeq$ denotes equality up to total derivatives and we have used that $C_A$ is real. The last equality holds thanks to the identity

$$(e^{i \bar{V}})_A^B D^\alpha (W \alpha e^{-2i \bar{V}})_A = (e^{-i \bar{V}})_A^B \bar{D}_\alpha (W \alpha e^{2i \bar{V}})_A$$

(3.12)

where

$$(W \alpha e^{-2i \bar{V}})_A = W_B^\alpha (e^{-2i \bar{V}})_B.$$

(3.13) is nothing but the “super-Bianchi identity” (for any $G$) familiar from super-Yang-Mills theory, cf. appendix.

(3.10) extends indeed the gauge transformation (2.10) to the superfields, as the field $C_{\mu A}$ which appears in (2.10) corresponds just to the vector field contained in $C_A$.

Finally we present the superfield version of the gauge transformations (2.12). It reads

$$\delta_\Lambda A^a = i(e^{i \bar{V}} \Lambda - e^{-i \bar{V}} \bar{\Lambda})^a, \quad \delta_\Lambda V^A = 0$$

$$\delta_\Lambda \Psi_A = 4ig k_{ab} Y^a T_A^b \Lambda^c$$

(3.13)

where the $\Lambda^a$ are arbitrary chiral superfields,

$$\bar{D}_\alpha \Lambda^a = 0.$$  

(3.14)

Using (3.14), one verifies that (3.13) implies

$$\delta_\Lambda Y^a = ig W^A T^a_{A b} \Lambda^b.$$  

(3.15)
It is now easy to check that the superfield Lagrangian is $\delta_\Lambda$-invariant. Indeed, $L_{\text{aux}}$ is evidently invariant, while the transformations of $L_{\text{FT}}$ and $L_{\text{HK}} + L_{\text{CM}}$ cancel,

$$
\delta_\Lambda (L_{\text{FT}} + L_{\text{HK}} + L_{\text{CM}}) = \int d^2\theta \left( -\frac{1}{2} W^A \delta_\Lambda \Psi_A + 2k_{ab} Y^a \delta_\Lambda Y^b \right) + \text{c.c.} = 0 .
$$

Finally, if (3.9) holds, then $\exp(\pm iV)^{\hat{a}}_b = \delta^{\hat{a}}_b$, and (2.12) implies

$$
\delta_\Lambda A^a = i(\Lambda^\hat{a} - \bar{\Lambda}^{\hat{a}})
$$

which in turn guarantees the gauge invariance of (3.7), as $\Lambda^a$ is a chiral superfield. Note that $A^\hat{a}$ transforms exactly as a standard abelian gauge superfield.

The lowest component field of $\Lambda^a + \bar{\Lambda}^a$ corresponds to $\epsilon^a$ in (2.12).

## 4 Models in WZ gauge

The gauge transformations (3.10) and (3.13) act as shift symmetries on some of the component fields of the superfields $\Psi_A$ and $A^a$. As usual, this signals that the action can actually be written in terms of fewer fields, with a correspondingly reduced gauge invariance and modified supersymmetry transformations. In this section we shall construct such a “WZ gauged” version of the models.

(3.10) suggests that, in WZ gauge, the remaining fields originating from $\Psi_A$ will be those of a real linear multiplet, i.e., a real scalar field $\varphi_A$ with dimension 1, the components $B_{\mu\nu}^A$ of a real 2-form gauge potential, also with dimension 1, and a Weyl spinor $\chi_A$ with dimension 3/2. Similarly (3.13) indicates that, in WZ gauge, $A^a$ will give rise only to a real vector field $A^a_\mu$ with dimension 1, a Weyl spinor $\lambda^a$ with dimension 3/2 and a real auxiliary field $D^a$ with dimension 2.

We shall now work in component formalism with such a field content ($B_{\mu\nu}^A$, $\varphi_A$, $\chi_A$, $A^a_\mu$, $\lambda^a$, $D^a$). Again, we complement these fields by all the component fields of the auxiliary superfields $V^A$ in order to work in a convenient first order formulation. This is possible because the latter fields are invariant under the gauge transformations (3.10) and (3.13) and can thus be kept in WZ gauge. As before, the component fields of $V^A$ are auxiliary and may be eliminated algebraically at the end, along with the $D^a$. Hence, in a formulation without auxiliary fields, one is left with the field content $B_{\mu\nu}^A$, $\varphi_A$, $\chi_A$, $A^a_\mu$, $\lambda^a$.

Now, from the experience with other supersymmetric gauge theories, one expects that the supersymmetry algebra holds in WZ gauge only modulo gauge transformations. This is our motivation for using a particular gauge covariant graded commutator algebra of supersymmetry and gauge transformations as the starting point for the construction of WZ gauged models. We shall then use this algebra to construct the Lagrangian, supersymmetry and gauge transformations.

The algebra has an unusual form which is inspired by the models in sections 3 and 3 (see discussion below). On gauge covariant quantities constructed of $A^a_\mu$, $\lambda^a$, $D^a$ and the component fields of $V^A$ it reads

\footnote{One would not expect that (4.1) can be realized also on $B_{\mu\nu}^A$, $\varphi_A$ and $\chi_A$ since it does not contain...}
\[ [D_{\alpha}, D_{\beta\dot{\alpha}}] = -2g \epsilon_{\alpha\beta} \dot{\lambda}_{\dot{\alpha}}^{\gamma} \delta\gamma \]

\[ \{D_{\alpha}, \bar{D}_{\dot{\alpha}}\} = -i D_{\alpha}\dot{\alpha} \]

where \( D_{\alpha} \) and \( \bar{D}_{\dot{\alpha}} \) generate the supersymmetry transformations (on component fields), the \( \delta\alpha \) generate gauge transformations corresponding to (2.12) resp. (3.13), \( \Gamma^{A} \) and \( V^{A}_{\mu} \) will be constructed of the auxiliary fields (see below), and \( D_{\mu} \) are gauge covariant derivatives

\[ D_{\mu} = \partial_{\mu} - gA^{a}_{\mu} \delta\alpha. \]

Note that (4.1) is somewhat similar to the gauge covariant algebra in WZ gauged super-Yang-Mills theories. However there is a remarkable difference to the latter theories (and to other supersymmetric gauge theories as well): the supersymmetry transformations do not commute with all the gauge transformations. In order to explain this unusual feature we remark:

(a) From sections 2 and 3 it is clear that the algebra (4.1) should in the special case \( T_{A} = 0 \) reproduce the supersymmetry algebra of usual abelian gauge theory in WZ gauge. Hence, \( \{D_{\alpha}, \bar{D}_{\dot{\alpha}}\} \) should thus contain the covariant derivative rather than the partial one, reflecting the presence of a gauge transformation in the commutator of two supersymmetry transformations.

(b) We aim at the construction of supersymmetrized Henneaux-Knäepen models. To that end \( [\delta_{\alpha}, D_{\mu}] \) must not vanish because otherwise we would get \( \tilde{F}^{a}_{\mu\nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu} \) rather than an expression like (2.7). This is seen from the following calculation which uses (4.2) in the form \( \partial_{\mu} = \partial_{\mu} + gA^{a}_{\mu} \delta\alpha \):

\[
0 = [\partial_{\mu}, \partial_{\nu}] = \partial_{\mu}(D_{\nu} + gA^{a}_{\nu} \delta\alpha) - (\mu \leftrightarrow \nu)
= (D_{\mu} + gA^{a}_{\mu} \delta\alpha)D_{\nu} + g(\partial_{\mu}A^{a}_{\nu}) \delta\alpha + gA^{b}_{\nu}(D_{\mu} + gA^{b}_{\mu} \delta\beta) \delta\alpha - (\mu \leftrightarrow \nu)
= [D_{\mu}, D_{\nu}] + gA^{a}_{\mu}[\delta\alpha, D_{\nu}] - gA^{b}_{\mu}[\delta\alpha, D_{\mu}]
+ g(\partial_{\mu}A^{a}_{\nu} - \partial_{\nu}A^{a}_{\mu}) \delta\alpha + g^{2}A^{a}_{\nu}A^{b}_{\mu}[\delta\beta, \delta\alpha].
\]

Since \( \{D_{\alpha}, \bar{D}_{\dot{\alpha}}\} = -i D_{\alpha}\dot{\alpha} \) and \( [\delta_{\alpha}, D_{\mu}] = 0 \) would imply \( [\delta_{\alpha}, D_{\mu}] = 0 \), the requirements in (a) and (b) show that \( [\delta_{\alpha}, D_{\mu}] \) must not vanish because otherwise we would not end up with supersymmetric Henneaux-Knäepen models (rather, we would get a

\[ [D_{\mu}, D_{\nu}] = -g\tilde{F}^{a}_{\mu\nu} \delta\alpha \]

and since the gauge transformations (2.12) do not commute off-shell on \( B_{\mu\nu A} \). Indeed, we shall find that the algebra has an accordingly modified form on \( B_{\mu\nu A} \), \( \varphi_{A} \) and \( \chi_{A} \).

\[ \text{Actually the algebra (4.1) alone would still permit the possibility that supersymmetry and gauge transformations commute on-shell. However, this will not be the case, as one expects since the gauge transformations (2.12) do not vanish on-shell (cf. also remarks at the end of this section).} \]
supersymmetric abelian gauge theory of the standard type). Besides, the calculation in (b) also shows that the algebra (4.1) reproduces exactly the curvature (2.7),

\[ \hat{F}^a_{\mu\nu} = \partial_{\mu} A^a_{\nu} - \partial_{\nu} A^a_{\mu} + g T^a_{A b} (V^A_{\mu} A^b_{\nu} - V^A_{\nu} A^b_{\mu}) . \]

Now, an analysis of (4.1) and the Bianchi identities following from it shows that the algebra is realized off-shell by the following supersymmetry and gauge transformations of \( A^a_{\mu} \), \( \lambda^a \), \( D^a \) and the component fields of \( V^A \):

(i) All the component fields of \( V^A \) are gauge invariant and (4.1) reduces thus on these fields to the standard supersymmetry algebra. Hence, the component fields of \( V^A \) form a standard real \( N = 1 \) vector multiplet, as in the superfield formulation. \( \Gamma^A_\alpha \) is defined through

\[ g \Gamma^A_\alpha T_A = (e^{-iV} D_\alpha e^{iV}) | , \quad g \bar{\Gamma}^A_\alpha T_A = (e^{iV} \bar{D}_\alpha e^{-iV}) | \]  (4.3)

where \( | \) denotes the \( \theta \)-independent part of a superfield, and we used a notation as in (3.3). This implies

\[ \mathcal{D}_\alpha \Gamma^A_\beta + \mathcal{D}_\beta \Gamma^A_\alpha + g \Gamma^B_\alpha \Gamma^C_{\beta} f_{BC}^A = 0 , \]

as required by the Bianchi identity

\[ \{ [\delta_\alpha, \mathcal{D}_\alpha], \mathcal{D}_\beta \} + \{ [\mathcal{D}_\alpha, \mathcal{D}_\beta], \delta_\alpha \} - \{ [\mathcal{D}_\beta, \delta_\alpha], \mathcal{D}_\alpha \} = 0 . \]

The analogous Bianchi identity with \( \mathcal{D}_\beta \) replaced by \( \bar{\mathcal{D}}_\alpha \) determines \( V^A_\mu \):

\[ V^A_\alpha = i (\mathcal{D}_\alpha \bar{\Gamma}_\dot{\alpha} + \bar{\mathcal{D}}_\dot{\alpha} \Gamma^A_\alpha + g f_{BC}^A \Gamma^B_\alpha \Gamma^C_\dot{\alpha}) . \]  (4.4)

For later purpose we note that one gets

\[ \mathcal{D}_\alpha V^A_\mu = -(\sigma_{\mu} \bar{\eta}^A)_\alpha + \partial_{\mu} \Gamma^A_\alpha - g \Gamma^B_\alpha V^C_{\mu} f_{BC}^A \]

\[ \mathcal{D}_\alpha \eta^A_\beta = \varepsilon_{\alpha \beta} h^A + \frac{i}{2} \sigma^{\mu \nu} \omega^{A}_{\mu \nu} \]

\[ \bar{\mathcal{D}}_\alpha \eta^A_\beta = -g \bar{\Gamma}^B_\alpha \eta^C_{\beta} f_{BC}^A \]

\[ \mathcal{D}_\alpha h^A = -\frac{i}{2} \partial_{\alpha \dot{\alpha}} \bar{\eta}^{A \dot{\alpha}} - \frac{i}{2} g V^A_{\alpha \dot{\alpha}} \bar{\eta}^{A \dot{\alpha}} f_{BC}^A - g \Gamma^B_\alpha h^C f_{BC}^A \]

\[ \delta_\alpha \eta^A_\alpha = \delta_\alpha V^A_\mu = \delta_\alpha h^A = 0 , \]  (4.5)

where \( \eta^A \), \( h^A \) and \( V^A_{\mu \nu} \) can be obtained from a superfield \( \mathcal{W}^A \) defined in the appendix,

\[ \eta^A_\alpha = \mathcal{W}^A_\alpha | \]

\[ h^A = \frac{1}{4} (D^A \mathcal{W}^A + g f_{BC}^A \Gamma^A_\alpha \mathcal{W}^C_\beta) | + \text{c.c.} , \]

\[ V^A_{\mu \nu} = -i \sigma^{\mu \nu A \beta} (D_\alpha \mathcal{W}^A_\beta + g f_{BC}^A \Gamma^A_\alpha \mathcal{W}^C_\beta) | + \text{c.c.} \]

\[ = \partial_{\mu} V^A_{\nu} - \partial_{\nu} V^A_{\mu} + g V^B_{\mu} V^C_{\nu} f_{BC}^A . \]  (4.6)
(ii) The supersymmetry transformations of $A^\mu$, $\lambda^a$ and $D^a$ are given by
\[
\begin{align*}
\mathcal{D}_\alpha A^\mu &= - (\sigma^\mu \bar{\lambda}^a)_\alpha - g \Gamma^a_{\alpha\beta} A^b \\
\mathcal{D}_\alpha \lambda^a &= \varepsilon_{\alpha\beta} D^a + i \tilde{F}^a_{\alpha\beta} - g \Gamma^a_{\alpha\beta} \lambda^b \\
\bar{\mathcal{D}}_\alpha \lambda^a &= - g \bar{\Gamma}^a_{\alpha\beta} \lambda^b \\
\mathcal{D}_\alpha D^a &= - \frac{i}{2} (\mathcal{D}_\mu + g V^a T_A)_{\alpha}^a (\sigma^\mu \bar{\lambda}^b)_\alpha - g \Gamma^a_{\alpha\beta} D^b.
\end{align*}
\]
\[\delta_a \] is realized on $\lambda^a$, $\tilde{F}^a_{\mu\nu}$ and $D^b$ by
\[
\delta_a \lambda^b = \eta^a_{\alpha \beta} T_A^b, \quad \delta_a \tilde{F}^a_{\mu\nu} = V^a T_A^b \quad \delta_a D^b = h^a T_A^b.
\]
The corresponding gauge transformations of $A^b_{\mu}$, $\lambda^b$ and $D^b$ are
\[
\delta_e A^b_{\mu} = \partial_e e^b + g V^a T_A^b a^a_e, \quad \delta_e \lambda^b = g \eta^a_{\alpha \beta} T_A^b a^a_e, \quad \delta_e D^b = g h^a T_A^b a^a_e.
\]

We are now prepared to construct the WZ gauged Lagrangian, along with the supersymmetry and gauge transformations of $B_{\mu\nu A}$, $\varphi_A$ and $\chi_A$. The Lagrangian is
\[
L = L_{FI} + L_{HK} + L_{CM} + L_{FI} + L_{aux}
\]
\[\text{(4.10)} \]
\[
L_{FI} = - \frac{1}{4} \varepsilon_{\mu\nu\rho\sigma} V^A_{\mu\nu} B_{\rho\sigma A} + h^A \varphi_A + \eta^A \chi_A + \bar{\eta}^A \bar{\chi}_A
\]
\[\text{(4.11)} \]
\[
L_{HK} + L_{CM} = - \frac{1}{4} \mathcal{D}^2 [k_{ab} (e^{-iv} \lambda)^a (e^{-iv} \lambda)^b] + \text{c.c.}
\]
\[\text{(4.12)} \]
\[
L_{FI} = \mu_a D^a
\]
\[\text{(4.13)} \]
with $L_{aux}$ as in \[\text{(3.8)}\] (since the component fields of $V^A$ are gauge invariant and have the same supersymmetry transformations as in the superfield formulation). In \[\text{(4.12)}\] we used the notation $\mathcal{D}^2 = \mathcal{D}^a \mathcal{D}_a$, $k_{ab}$ are arbitrary complex numbers as in \[\text{(3.8)}\], and $v$ is a matrix valued field constructed of the lowest component fields of the $V^A$ and the representation matrices $T_A$,
\[
(e^{-iv} \lambda^a)_\alpha^b = (e^{-iv})_\alpha^b \lambda^a_\alpha, \quad v = g V^A T_A, \quad v^A = V^A
\]
\[\text{(4.14)} \]
\[\text{(4.12)}\] will be spelled out explicitly at the end of this section.

As in the superfield formulation, the Fayet-Iliopoulos part \[\text{(1.13)}\] is present only if all the representation matrices $T_A$ have a vanishing row in common, i.e., if \[\text{(3.9)}\] holds. $L_{aux}$ and $L_{FI}$ are separately supersymmetric (up to total derivatives) and gauge invariant and therefore need not be discussed further (indeed, \[\text{(3.9)}\], \[\text{(4.7)}\] and \[\text{(4.8)}\] imply $\mathcal{D}_{\alpha} D^a = - \frac{1}{2} \partial_{\alpha a} \bar{\lambda}^{\dot{a}a}$ and $\bar{\mathcal{D}}_{\alpha} D^a = 0$). We note that the Freedman-Townsend part \[\text{(1.11)}\] can be directly obtained from \[\text{(1.5)}\] by defining the component fields of $\Psi^A$ appropriately, but we skip the details of these definitions as they do not matter.

The crucial part of the Lagrangian is \[\text{(4.12)}\]. This part is neither gauge invariant nor supersymmetric by itself. However, its gauge and supersymmetry variations can be canceled (up to total derivatives) by choosing the gauge and supersymmetry transformations of $B_{\mu\nu A}$, $\varphi_A$ and $\chi_A$ appropriately, such that the gauge and supersymmetry
The fact that similarly, the supersymmetry transformations of (4.12) are analysed, using (4.1) and the variations of (4.12) are killed by terms in the variations of (4.11) (up to total derivatives). To show this, we introduce the notation

\[ P = k_{ab} (e^{-iv})^a (e^{-iv})^b . \]

Using the algebra (4.1), one obtains straightforwardly

\[ \delta_a D^2 P = \frac{1}{4} \left[ -g^2 \Gamma^A \Gamma^B T_{A c} T_{B a} + 2g \Gamma^A \Gamma^B T_{A a} D_\alpha + g(D^a \Gamma^A) T_{A b} - \delta_b D^2 \right] \delta_b P . \] (4.15)

Similarly, the supersymmetry variations of (4.12) can be written as linear combinations of the variations of (4.11), and from \( \overline{\gamma} \), \( \alpha \) and supersymmetry variations of (4.12) can be written as linear combinations of the eqs. (4.15), (4.17) and (4.18) we can thus infer that, up to total derivatives, the gauge invariance of \( v \) and supersymmetry variations of (4.11) allows us therefore to cancel these linear combinations through appropriately chosen terms in the transformations of \( B_{\mu A}, \varphi_A \) and \( \chi_A \) which are obtained from evaluating (4.11) and (4.17) explicitly.

In order to analyse (4.15) and (4.17), we use that eq. (4.8) gives, due to the gauge invariance of \( v \),

\[ \delta_a P = 2 \Omega_{a A} \eta_A^a . \] (4.18)

where

\[ \Omega_{a A a} = \lambda^b G_{bc}(v) T_{A a}^c , \quad G_{ab}(v) = k_{cd} (e^{-iv})^c_a (e^{-iv})^d_b . \] (4.19)

Eqs. (4.3) show that each term in \( D_a \eta_A^a \) and in \( D^2 \eta_A^a \) contains exactly one of the fields \( \eta_A^a, \bar{\eta}_A^a, h^A \) or \( V_{\mu \nu}^A \), where all these fields appear undifferentiated except for \( \bar{\eta}_A^a \). From eqs. (4.15), (4.17) and (4.18) we can thus infer that, up to total derivatives, the gauge and supersymmetry variations of (4.12) can be written as linear combinations of the undifferentiated fields \( \eta_A^a, \bar{\eta}_A^a, h^A \) and \( V_{\mu \nu}^A \) with field dependent coefficient functions. The particular form of (4.11) allows us therefore to cancel these linear combinations through appropriately chosen terms in the transformations of \( B_{\mu \nu A}, \varphi_A \) and \( \chi_A \) which are obtained from evaluating (4.11) and (4.17) explicitly.

This yields the following gauge transformations of \( B_{\mu \nu A}, \varphi_A \) and \( \chi_A \):

\[ \delta_c \varphi_A = g e^a \delta_a \varphi_A \quad \delta_c B_{\mu \nu A} = g e^a \delta_a B_{\mu \nu A} \quad \delta_c \chi_A = g e^a \delta_a \chi_A \]

\[ \delta_a \varphi_A = - \nabla \Omega_{A a} - \nabla \bar{\Omega}_{A a} \]

\[ \delta_a B_{\mu \nu A} = - \nabla \sigma_{\mu \nu} \Omega_{A a} - \nabla \sigma_{\mu \nu} \bar{\Omega}_{A a} \]

\[ \delta_a \chi_A = \frac{i}{2} \nabla^2 \Omega_{A a} + i \partial_{a a} \bar{\Omega}^a_{A a} + i g V_{a a}^B \bar{\Omega}^a_{C a} f_{B A} C , \] (4.20)

where

\[ \nabla \Omega_{A a} = D_a \Omega_{A a} - g \Gamma^B_a (T_{B a}^b \Omega_{A b} + f_{B A} C \Omega_{B C a} ) \]
\[\nabla^2 \Omega_{\beta A} = D^a \nabla_a \Omega_{\beta A} - g \Gamma^\alpha_{\beta A} \nabla_\alpha \Omega_{\beta a} + f_{BA} C \nabla_a \Omega_{\beta CA} \, . \tag{4.21}\]

Analogously one determines the terms in the supersymmetry transformations of \(B_{\mu A}, \varphi_A\) and \(\chi_A\) that compensate for the supersymmetry variation of (4.12). The supersymmetry transformations of \(B_{\mu A}, \varphi_A\) and \(\chi_A\) still have to be completed by contributions which cancel those terms in the supersymmetry variation of (4.11) originating from the transformations of the auxiliary fields (up to total derivatives). Not surprisingly, the additional contributions contain the standard supersymmetry transformations of a linear multiplet, plus some nonlinear extra terms involving the auxiliary fields. Altogether one finds

\[D_\alpha \varphi_A = \chi_{\alpha A} - i g A^a_{\alpha \dot{\alpha}} \Omega_{\dot{\alpha} A} - g \Gamma^B_{\alpha} f_{AB} C \varphi_C \]

\[D_\alpha B_{\mu A} = (\sigma_{\mu \dot{\alpha}} \chi_A)_\alpha + i g A^a_\rho (\sigma^\rho \sigma_{\mu \dot{\alpha}} \Omega_{\dot{\alpha} A})_\alpha - g \Gamma^B_{\alpha} f_{AB} C B_{\rho C} \]

\[D_\alpha \chi_{\beta A} = -g \Gamma^A_{\beta} f_{AB} \chi_C \]

\[D_{\dot{\alpha}} \chi_{\alpha A} = -\frac{i}{2} (\partial_\alpha \varphi_A + g V^A_{\dot{\beta}} f_{AB} C \varphi_C) - 2 i g \chi^{\dot{a}}_{\dot{\alpha}} \Omega_{\alpha A} - i g A^a_{\beta \dot{\alpha}} \nabla^\beta \Omega_{\alpha A} + \frac{i}{2} \sigma_{\rho \alpha \dot{a}} \varepsilon^{\rho \mu \nu \sigma} (\partial_\mu B_{\sigma \rho A} + g V^B_{\nu} f_{AB} C B_{\rho C}) - g \Gamma^B_{\alpha} f_{AB} C \chi_C \, . \tag{4.22}\]

In addition the Lagrangian is gauge invariant under transformations of \(B_{\mu A}\) as in (2.10), with all other fields invariant under these gauge transformations,

\[\delta_C B_{\mu A} = \nabla_\mu C_{\nu A} - \nabla_\nu C_{\mu A} , \quad \delta_C (\text{all other fields}) = 0 \, . \tag{4.23}\]

Let us now return to the algebra of supersymmetry and gauge transformations. One finds that (1.20) and (1.22) realize the algebra (1.1) on \(B_{\mu A}, \varphi_A\) and \(\chi_A\) only on-shell\(^3\) and up to gauge transformations (4.23). In particular one gets

\[\{D_\alpha, D_\beta\} B_{\mu A} = 0 \]

\[\{D_\alpha, D_{\dot{\alpha}}\} B_{\mu A} = -i D_{\alpha \dot{\alpha}} B_{\mu A} + i (\nabla_\mu Z_{\nu \alpha \dot{\alpha} A} - \nabla_\nu Z_{\mu \alpha \dot{\alpha} A}) \, , \]

where

\[Z_{\mu A} = \frac{1}{2} \eta_{\mu \nu} \varphi_A - B_{\mu A} , \quad \nabla_\mu Z_{\nu \rho A} = \partial_\mu Z_{\nu \rho A} + g V^B_{\mu} f_{AB} C Z_{\nu \rho C} \, . \]

Altogether, we find that the commutator of two supersymmetry transformations involves a translation and gauge transformations \(\delta_\epsilon\) and \(\delta_C\) with field dependent \(\epsilon^a\) and \(C_{\mu A}\). More precisely, denoting a supersymmetry transformation with anticommuting parameters \(\xi\) by

\[\Delta_\xi = \xi^a D_a + \bar{\xi}_{\dot{a}} D_{\dot{a}} \, , \]

one gets on all the fields

\[\{ \Delta_\xi, \Delta_{\xi'} \} = a^\mu \partial_\mu - \delta_\epsilon - \delta_C \]

\[a^\mu \equiv i \xi^\alpha \sigma^\mu \bar{\xi} - i \xi^\alpha \sigma^\mu \bar{\xi}' , \quad \epsilon^a \equiv a^\mu A^a_{\mu} , \quad C_{\mu A} \equiv \frac{1}{2} a^\mu \varphi_A - a^\nu B_{\mu \nu A} \, . \tag{4.24}\]

\(^3\)As in the nonsupersymmetric case, \([\delta_a, \delta_b]\) vanishes only on-shell.
In first order formulation this holds off-shell, in the formulation without auxiliary fields only on-shell.

Finally, we spell out (4.12) explicitly,

\[
\begin{align*}
L_{HK} + L_{CM} &= G_{ab}(v) \left[ D^a D^b - \frac{1}{8} \hat{F}^{a\mu} \hat{F}^{\mu b} + \frac{i}{16} \epsilon^{\mu\nu\rho\sigma} \hat{F}^{a\mu}_{\rho} \hat{F}^{b\nu}_{\sigma} - i \lambda^a \sigma^\mu \partial_\mu \lambda^b \right] \\
&+ i g A^a_{\mu} \Omega A_{\alpha} \sigma^\mu \bar{\eta}^A - i g V^A_{\mu} \Omega A_{\alpha} \sigma^\mu \bar{\lambda}^A - \frac{g}{2} \left( \Omega_{\alpha A} + G_{ab}(v) T^a_{\alpha} c^c \right) \times \\
&\times \left[ 2 \hat{\Gamma}^A D^a + i \sigma^{\mu\nu} \hat{\Gamma}^A \hat{F}^{a\mu}_{\nu} + g T^a_{\alpha} d^d \hat{\Gamma}^A \hat{\Gamma}^B \right] \\
&+ \frac{g}{2} G_{ab}(v) \left[ e^{iv} D^a (e^{-iv} \hat{\Gamma}^A T_{\alpha} e^{iv}) e^{-iv} \lambda^a \right] \lambda^b \\
&+ c.c.
\end{align*}
\]

with the abbreviation

\[
\hat{\Gamma}^A \equiv \Gamma^A + \Gamma^B (e^{-iv}) B^A.
\] (4.26)

**Remarks.**

1. As \( B_{\mu\nu A}, \varphi_A \) and \( \chi_A \) appear only in the Freedman-Townsend part \( L_{FT} \) of the Lagrangian, one immediately concludes that \( V^A_{\mu\nu}, h^A \) and \( \eta^A \) vanish on-shell. It is also easy to infer that \( h^A \) appears only linearly in the action and that its equation of motion yields \( v^A \) as a function of the \( \varphi_A \) (the precise relation between the \( v^A \) and \( \varphi_A \) depends on the choice of the \( T_A \) and the function \( F \) in \( L_{aux} \)).

2. The previous remark implies that the gauge transformations of \( \lambda^a, D^a \) and \( \varphi_A \) vanish on-shell (for \( \lambda^a \) and \( D^a \), this is seen from (4.9) because \( \eta^A \) and \( h^A \) vanish on-shell; for \( \varphi_A \), it follows from the fact that \( \varphi_A \) equals on-shell a function of the \( v^A \)). The algebra (4.1) shows thus that, on these fields, the supersymmetry transformations commute on-shell with all the gauge transformations. The same is however not true for \( A^a_{\mu} \) and \( B_{\mu\nu A} \), as their gauge transformations do not vanish on-shell.

3. As in usual supersymmetric gauge theories, a Fayet-Iliopoulos contribution breaks supersymmetry spontaneously, as is seen from the equation of motion for \( D^a \) and from \( D_a \lambda^A_\beta \) in (4.7). The gauge symmetries remain unbroken, as one can infer from the fact that the gauge transformation of \( \varphi_A \) vanishes on-shell (cf. previous remark).

4. \( g = 0 \) reproduces the usual supersymmetric gauge theories for free real linear multiplets (in first order formulation) and abelian WZ gauged vector multiplets. Hence, the models are deformations of these standard supersymmetric gauge theories. For \( g \neq 0 \) but \( T_A = 0 \), (4.23), (4.7) and (4.9) still reproduce the Lagrangian, supersymmetry and gauge transformations of standard free abelian supersymmetric gauge theory in WZ gauge (as \( G_{ab} \) is constant for \( T_A = 0 \), while the linear and auxiliary multiplets establish supersymmetric pure Freedman-Townsend models in WZ gauge without couplings to the abelian gauge multiplets \( A^a_{\mu}, \lambda^a, D^a \).

5 **Examples**

To illustrate some features of the models constructed in the previous sections, we will now discuss two examples. We begin with the simplest case of one gauge multiplet, one
linear and one auxiliary multiplet. We thus drop the indices $A$ and $a$ in the following, and take $T = 1$. The field dependent coupling $G$ and the spinor $\Omega$ defined in eq. (4.19) reduce to

$$G(v) = ke^{-2ig\nu}, \quad \Omega = ke^{-2ig\nu}\lambda,$$

where we shall further simplify the discussion by considering $k = 1$ only. In this case we have $\Gamma_\alpha = iD_\alpha v$, $V_{\alpha\dot{\alpha}} = [D_\alpha, D_{\dot{\alpha}}]v$, and the field strengths are

$$V_{\mu\nu} = \partial_\mu V_\nu - \partial_\nu V_\mu, \quad \tilde{F}_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + g(V_\mu A_\nu - V_\nu A_\mu).$$

If we take $\mathcal{F}(V)$ to be quadratic, the complete Lagrangian reads

$$L_1 = \frac{1}{2}V_\mu K^{\mu\nu}V_\nu - V_\mu \mathcal{H}^\mu + \frac{1}{4}\partial_\mu v \partial^\mu v - \frac{i}{2}(\sigma^\mu \partial_\mu \phi + h(2v + \varphi) + \eta(\chi - 2i \Gamma - igA_\mu \sigma^\mu \bar{\Omega}) + \bar{\eta}(\bar{\chi} - 2i \bar{\Gamma} - igA_\mu \sigma^\mu \Omega) + \frac{1}{2}(M + 2g \overline{\Omega}\bar{\lambda})(\bar{M} + 2g \Omega\bar{\lambda}) - 2g^2 \Omega \lambda \bar{\Omega} \lambda + 2\cos(2gv)D^2$$

$$- \frac{1}{4} \cos(2gv) F_{\mu\nu} F^{\mu\nu} + \frac{1}{8} \sin(2gv) \varepsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma}$$

$$- iG(v)\lambda\sigma^\mu \partial_\mu \lambda + i\bar{G}(v)\partial_\mu \lambda \sigma^\mu \bar{\lambda} - 4gD(\Omega + \bar{\Omega})$$

$$- 4g^2 \Gamma \lambda \Omega - 4g^2 \Gamma \bar{\lambda} \bar{\Omega} + 2ig F_{\mu\nu}(\Gamma \sigma^{\mu\nu} \Omega + \Omega \sigma^{\mu\nu} \Gamma),$$

where $M = iD^2 v$, and

$$K^{\mu\nu} = \eta^{\mu\nu} + g^2 \cos(2gv)(A^\mu A^\nu - \eta^{\mu\nu} A^\rho A_\rho)$$

$$\mathcal{H}^\mu = \frac{1}{2} \varepsilon^{\mu\nu\rho\sigma} \partial_\nu B_{\rho\sigma} + ig \Omega \sigma^\mu \lambda - ig \lambda \sigma^\mu \bar{\Omega}$$

$$+ g \cos(2gv) F^{\mu\nu} A_\nu - g \sin(2gv) \varepsilon^{\mu\nu\rho\sigma} A_\nu \partial_\rho A_\sigma$$

$$- 4ig^2 A_\nu(\Gamma \sigma^{\mu\nu} \Omega + \Omega \sigma^{\mu\nu} \Gamma)$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.$$

Let us now discuss the formulation without auxiliary fields. By virtue of the equation of motion for $h$ we can replace $v$ with $-\frac{1}{2} \varphi$. To eliminate $V_\mu$, we need to invert the matrix $K^{\mu\nu}$. In the simple case at hand the inverse can be given explicitly,

$$(K^{-1})_{\mu\nu} = \frac{1}{E}(\eta_{\mu\nu} - g^2 \cos(g\varphi)A_\mu A_\nu), \quad E \equiv 1 - g^2 \cos(g\varphi)A^\rho A_\rho.$$

We note that $\mathcal{H}^\mu$ is of the form

$$\mathcal{H}^\mu = H^\mu + L^{\mu\nu}A_\nu, \quad H^\mu \equiv \frac{1}{2} \varepsilon^{\mu\nu\rho\sigma} \partial_\nu B_{\rho\sigma} - 2g \sin(g\varphi)\lambda \sigma^\mu \bar{\lambda}$$

with $L^{\mu\nu}$ antisymmetric. So the equation of motion for $V_\mu$ yields

$$\frac{1}{2}V_\mu K^{\mu\nu}V_\nu - V_\mu \mathcal{H}^\mu \approx - \frac{1}{2E} \mathcal{H}^\mu \mathcal{H}_\mu + \frac{g^2}{2E} \cos(g\varphi) (A_\mu H^\mu)^2.$$
It proves convenient to eliminate $\chi$ in favor of $\Gamma$, which we keep as an independent field instead. Variation with respect to $\eta$ then identifies $\chi$ as the combination

$$\chi \approx 2i \Gamma + i g e^{-ig\varphi} A_\mu \sigma^\mu \tilde{\lambda}.$$  

Elimination of $D$ gives rise to four fermion terms only, as a Fayet-Iliopoulos term is not possible here,

$$D \approx \frac{g}{\cos(g\varphi)} (e^{ig\varphi}\Gamma \lambda + e^{-ig\varphi}\Gamma \tilde{\lambda}).$$

Inserting the above expressions back into the Lagrangian, we finally arrive at

$$L_1 \approx -\frac{1}{2E} \mathcal{H}^\mu \mathcal{H}_\mu + \frac{g^2}{2E} \cos(g\varphi) (A_\mu H^\mu)^2 + \frac{1}{16} \partial_\mu \varphi \partial^\mu \varphi - \frac{i}{2} \Gamma \sigma^\mu \partial_\mu \Gamma$$

$$- \frac{1}{4} \cos(g\varphi) F_{\mu\nu} F^{\mu\nu} - \frac{1}{8} \sin(g\varphi) \varepsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma}$$

$$- ie^{ig\varphi} \lambda \sigma^\mu \partial_\mu \tilde{\lambda} + ie^{-ig\varphi} \partial_\mu \lambda \sigma^\mu \tilde{\lambda} - 2g^2 \lambda \lambda \tilde{\lambda} \tilde{\lambda}$$

$$- \frac{2g^2}{\cos(g\varphi)} (e^{ig\varphi}\Gamma \lambda + e^{-ig\varphi}\Gamma \tilde{\lambda})^2 - 4g^2 e^{ig\varphi}\Gamma \Gamma \lambda \lambda - 4g^2 e^{-ig\varphi}\Gamma \Gamma \tilde{\lambda} \tilde{\lambda}$$

$$+ 2ig F_{\mu\nu}(e^{ig\varphi}\Gamma \sigma^\mu \lambda + e^{-ig\varphi}\Gamma \sigma^\mu \tilde{\lambda}).$$

(5.9)

As a second example, we present an $N = 1$ supersymmetric counterpart of the toy model in [6] and the $N = 2$ supersymmetric models in [2, 3]. In [7] it was observed that these theories correspond to the case

$$T^a_{1b} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

(5.10)

i.e., we now deal with two gauge multiplets, one linear and one auxiliary multiplet. Again, as the index $A$ takes only one value, we drop it in the following. Since we now get

$$e^{-i g v T} = \begin{pmatrix} 1 & 0 \\ -i g v & 1 \end{pmatrix},$$

(5.11)

the field dependent coupling reads

$$G_{ab}(v) = \begin{pmatrix} k_{11} - 2i g v k_{12} - (g v)^2 k_{22} & k_{12} - i g v k_{22} \\ k_{12} - i g v k_{22} & k_{22} \end{pmatrix},$$

(5.12)

with complex numbers $k_{ab}$. As the entries in the second column of $T$ are zero, so is the second component of the doublet $\Omega_a$,

$$\Omega_a = (k_{22} \lambda^2 + (k_{12} - i g v k_{22}) \lambda^1, 0),$$

(5.13)
and it follows from eqs. (1.18), (1.5) and (1.20) together with (1.21) that the gauge transformations with parameter $e^2$ act trivially on all the fields, except for $A^2_\mu (\delta_\epsilon A^2_\mu = \partial_\mu e^2 + gV_\mu e^1)$. The field strengths now are

$$
\hat{F}^1_{\mu \nu} = \partial_\mu A^1_\nu - \partial_\nu A^1_\mu , \quad V_{\mu \nu} = \partial_\mu V_\nu - \partial_\nu V_\mu , \\
\hat{F}^2_{\mu \nu} = \partial_\mu A^2_\nu - \partial_\nu A^2_\mu + g(V_\mu A^1_\nu - V_\nu A^1_\mu) .
$$

(5.14)

We shall again take $\mathcal{F}(V)$ to be quadratic. Due to the increased complexity we give only the bosonic part of the Lagrangian,

$$
L_2 = \frac{1}{2} V_\mu K^{\mu \nu} V_\nu - V_\mu \mathcal{H}^\mu + \frac{1}{4} \partial_\mu v \partial_\nu v + h(2v + \phi) + \frac{1}{2} M M \\
+ \text{Re} \ k_{22} \left( 2D^2 D^2 - \frac{1}{4} F_{\mu \nu}^2 F^{\mu \nu} \right) - \frac{1}{8} \text{Im} \ k_{22} v^2 \partial_\mu \partial_\nu v + \frac{1}{8} \text{Im} \ k_{22} v^2 \partial_\mu \partial_\nu v
$$

(5.15)

where

$$
K^{\mu \nu} = \eta^{\mu \nu} + \frac{1}{2} g^2 \text{Re} \ k_{22} (A^{\mu 1} A^{\nu 1} - \eta^{\mu \nu} A^{1 1}_1 A^{1 1}_1) \\
\mathcal{H}^\mu = \frac{1}{2} \varepsilon^{\mu \rho \sigma \tau} \partial_\nu B_{\rho \sigma \tau}
$$

(5.16)

$$
+ g(\text{Re} \ k_{12} + \text{Im} \ k_{22} \partial_\mu A^{2 1}_\nu + \text{Im} \ k_{22} \varepsilon^{\mu \rho \sigma \tau} A^{2 1}_\nu \partial_\rho A^{2 1}_\sigma \\
+ g(\text{Im} \ k_{12} - \text{Re} \ k_{22} \varepsilon^{\mu \rho \sigma \tau} A^{2 1}_\nu \partial_\rho A^{2 1}_\sigma + \text{Re} \ k_{22} F^{\mu \nu} A^{1 1}_1) .
$$

(5.17)

As in this case the matrix $T$ has a vanishing first row, a Fayet-Iliopoulos term has been added for $D^1$, spontaneously breaking supersymmetry.

Elimination of the auxiliary vector $V_\mu$ works exactly as in the previous example,

$$
(K^{-1})_{\mu \nu} = \frac{1}{E} \left( \eta_{\mu \nu} - \frac{1}{2} g^2 \text{Re} k_{22} A^{1 1}_\mu A^{1 1}_\nu \right) , \quad E \equiv 1 - \frac{1}{2} g^2 \text{Re} k_{22} A^{1 1}_\mu A^{1 1}_\mu
$$

(5.18)

$$
\Rightarrow \frac{1}{2} V_\mu K^{\mu \nu} V_\nu - V_\mu \mathcal{H}^\mu \approx \frac{1}{2E} \mathcal{H}^\mu \mathcal{H}_\mu + \frac{g^2}{4E} \text{Re} k_{22} (A^{1 1}_\mu H^\mu)^2 .
$$

Comparing with the $N = 2$ supersymmetric models $[3, 3]$, $A^2_\mu$ corresponds to the gauge field in the vector-tensor multiplet, while $A^1_\mu$ is the analog of the vector field used to gauge the central charge.

6 Conclusion

We have constructed $N = 1$ supersymmetric versions of all the models presented in section $[3]$. The resulting supersymmetric models are nontrivial deformations of the
standard supersymmetric gauge theories for free linear and vector multiplets. They have several unusual properties as compared to other globally supersymmetric gauge theories. We find particularly remarkable that, in the WZ gauge constructed in section 4, the supersymmetry transformations do not commute with all the gauge transformations, in contrast to the formulation with linearly realized supersymmetry given in section 3. We have presented arguments which suggest that this unusual feature might be an inevitable property of this type of supersymmetric models, but we admit that these arguments rely on our construction and are therefore not completely cogent.

Another unusual feature of the WZ gauged models is that neither the Henneaux-Knaepen nor the Chapline-Manton parts of the action are supersymmetric by themselves but only together with the Freedman-Townsend part, again in contrast to the formulation with linearly realized supersymmetry. This property is less surprising because, as already in the nonsupersymmetric case, the Henneaux-Knaepen and Chapline-Manton parts of the action are not separately gauge invariant, but only together with the Freedman-Townsend part.

Our results suggest several possible generalizations. For instance, one may investigate extensions of the models constructed here by including further fields. Furthermore, one might try to couple these models to supergravity. Another interesting extension of our results would be their generalization to \(N = 2\) supersymmetry. In particular this might streamline and generalize the results of \([2,3]\). A possible starting point for such generalizations could be the algebra (4.1) or suitably modified versions thereof.

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A Appendix

We use \(\eta_{\mu\nu} = \text{diag}(+,-,-,-)\) and \(\{D_\alpha, \bar{D}_{\dot{\alpha}}\} = -i\partial_{\alpha\dot{\alpha}}\) (note the absence of a factor 2 here). Apart from this, our conventions agree with those in \([13]\). Supercovariant derivatives, mapping superfields into superfields, are thus

\[
D_\alpha = \frac{\partial}{\partial \theta^\alpha} + \frac{i}{2}(\sigma^\mu \bar{\theta})_\alpha \partial_\mu, \quad \bar{D}_{\dot{\alpha}} = -\frac{\partial}{\partial \bar{\theta}^{\dot{\alpha}}} - \frac{i}{2}(\theta \sigma^\mu)_{\dot{\alpha}} \partial_\mu.
\]

The supersymmetry transformations of the component fields of a superfield \(\Sigma\) are related to the supercovariant derivatives of \(\Sigma\) through

\[
\mathcal{D}_\alpha \Sigma = D_\alpha \Sigma, \quad \mathcal{D}_{\dot{\alpha}} \Sigma = \bar{D}_{\dot{\alpha}} \Sigma
\]

where \(\mathcal{D}_\alpha\) and \(\mathcal{D}_{\dot{\alpha}}\) act only on the component fields (and anticommute with the \(\theta\)’s).

The matrix valued superfields in (3.3) and (4.3) are related by

\[
\mathcal{W}_A^\alpha T_A = \mathcal{W}_A^\alpha e^{iV} T_A e^{-iV} \equiv \mathcal{W}_\alpha^A.
\]
\( \mathcal{W}_\alpha \) satisfies a Bianchi identity familiar from super Yang-Mills theory,

\[
D^\alpha \mathcal{W}_\alpha + g \{ \tilde{\Gamma}^\alpha , \mathcal{W}_\alpha \} - \text{c.c.} = 0 , \quad g \tilde{\Gamma}_\alpha = e^{-iV} D_\alpha e^{iV} . \tag{A.1}
\]

(3.12) is equivalent to (A.1). This can be derived from the identity

\[
e^{iV} T_A e^{-iV} = (e^{-i\tilde{V}})_A^B T_B , \tag{A.2}
\]

which holds for any matrix representation \( \{ T_A \} \) of \( \mathcal{G} \) because the entries of \( T_A \) are \( \mathcal{G} \)-invariant tensors. (A.2) implies

\[
\mathcal{W}_\alpha^A = W_\alpha^B (e^{-i\tilde{V}})_B^A , \quad \{ \tilde{\Gamma}_\alpha , \mathcal{W}_\alpha \} = - W_\alpha^B (e^{-i\tilde{V}})_B^A [ \tilde{\Gamma}_\alpha , T_A ] .
\]

The commutator in the latter expression can be written as follows

\[
g[ \tilde{\Gamma}^\alpha , T_A ] = e^{-iV} D^\alpha \{ e^{iV} T_A \} - \{ T_A e^{-iV} \} D^\alpha e^{iV} \\
= e^{-iV} D^\alpha [(e^{-i\tilde{V}})_A^B T_B e^{iV}] - (e^{-i\tilde{V}})_A^B e^{-iV} T_B D^\alpha e^{iV} \\
= \{ e^{-iV} T_B e^{iV} \} D^\alpha (e^{-i\tilde{V}})_A^B \\
= (e^{i\tilde{V}})_B^C T_C D^\alpha (e^{-i\tilde{V}})_A^B ,
\]

where expressions \( \{ \ldots \} \) have been rewritten using (A.2). Altogether we get

\[
D^\alpha \mathcal{W}_\alpha + g \{ \tilde{\Gamma}^\alpha , \mathcal{W}_\alpha \} = (e^{i\tilde{V}})_A^B D^\alpha (W_\alpha e^{-2iV})^A T_B ,
\]

which implies the equivalence of (3.12) and (A.1).

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