Three dimensional quantum algebras: 
a Cartan-like point of view

A. Ballesteros\textsuperscript{1}, E. Celeghini\textsuperscript{2} and M.A. del Olmo\textsuperscript{3}

\textsuperscript{1}Departamento de Física, Universidad de Burgos, 
E-09006, Burgos, Spain.

\textsuperscript{2}Departimento di Fisica, Universitá di Firenze and INFN–Sezione di Firenze
I50019 Sesto Fiorentino, Firenze, Italy

\textsuperscript{3}Departamento de Física Teórica, Universidad de Valladolid,
E-47011, Valladolid, Spain.

\textsuperscript{1}\textsuperscript{2}\textsuperscript{3}e-mail: angelb@ubu.es, celeghini@fi.infn.it, olmo@fta.uva.es

Abstract

A perturbative quantization procedure for Lie bialgebras is introduced and used to classify all three dimensional complex quantum algebras compatible with a given coproduct. The role of elements of the quantum universal enveloping algebra that, analogously to generators in Lie algebras, have a distinguished type of coproduct is discussed, and the relevance of a symmetrical basis in the universal enveloping algebra stressed. New quantizations of three dimensional solvable algebras, relevant for possible physical applications for their simplicity, are obtained and all already known related results recovered. Our results give a quantization of all existing three dimensional Lie algebras and reproduce, in the classical limit, the most relevant sector of the complete classification for real three dimensional Lie bialgebra structures given in \[1\].

MSC: 81R50, 81R40, 17B37

Keywords: Lie bialgebras, quantization, quantum algebras
1 Introduction

Cartan classification of semisimple Lie algebras has facilitated their applications in Physics. Quantum algebras are not been classified in a similar way and their physical applications are far to be developed or understanding. For these reasons, this paper deals with some facets of the problem of the construction and classification of quantum universal enveloping algebras (hereafter, quantum algebras) \[2\]-\[5\]. It is well-known that any quantum algebra \((\mathcal{U}_z(a), \Delta)\) with deformation parameter \(z\) defines a unique Lie bialgebra structure \((a, \eta)\), a pair determined by the Lie algebra \(a\) and a skew-symmetric linear map (cocommutator) \(\eta : a \rightarrow a \times a\). Such cocommutator \(\eta\) is defined as the first order skew-symmetric part of the coproduct

\[
\eta(X) = \frac{1}{2}(\Delta(X) - \sigma \circ \Delta(X)) \quad \text{mod} \ z^2 \quad \forall X \in a \tag{1.1}
\]

where \(\sigma\) is the flip operator \(\sigma(X \otimes Y) = Y \otimes X\).

Therefore, quantum deformations of a given Lie algebra \(a\) can be classified according to this “semiclassical” limit. Moreover, we recall that Lie bialgebras are in one to one correspondence with Poisson-Lie structures on the group \(\text{Lie}(a)\) \[6\] that arise again as the first order in \(z\) of the quantum groups dual to the quantum algebras \((\mathcal{U}_z(a), \Delta)\). With this in mind, some classifications of Lie bialgebra structures for several physically relevant Lie algebras have been obtained. For the three dimensional (3D) case we will refer to the full classification of Lie bialgebras given in \[1\] and for some higher dimensional Lie bialgebras we refer to \[7\] and references therein.

However, it is clear that the inverse problem has also to be faced: that is, to obtain general recipes for the “quantization” (i.e. the associated quantum algebra) of a given Lie bialgebra \((a, \eta)\). Although the existence of such an object is indeed guaranteed (see \[3\], Chapter 6), only for coboundary triangular bialgebras the Drinfel’d twist operator gives rise to the associated quantum algebra \[4\]. However, quasitriangular and even non-coboundary Lie bialgebras do exist and for them the twist operator approach to quantization is not available. We recall that some attempts have been performed in order to get structural properties of the quantization of arbitrary Lie bialgebras (see \[8, 9\] for a prescription to get the quantum coproduct –but not the deformed commutation rules– for a wide class of examples). Moreover, to our knowledge, a general investigation concerning the uniqueness of this Lie bialgebra quantization process has not been given yet and only restrictive results for certain deformations of simple Lie algebras have been obtained (see \[10\], Chapter 11). As a consequence of the abovementioned facts, a complete classification of quantum algebras in the spirit of the Cartan classification for Lie algebras is still far to be reached.

In this paper we present a “direct approach” to the quantization problem, together with an operational procedure \(\text{à la Cartan}\) (in a sense that will be made more explicit in the sequel) for the classification of quantum algebras. The essential ingredients of the quantization method are described in Section 2. In particular, given a Lie bialgebra we shall firstly obtain a coassociative quantum coproduct with some outstanding symmetry properties and, afterwards, we shall solve order by order the compatibility equations for the deformed commutation rules. Lie-Poisson brackets are immediately recover using a completely symmetrized basis for \(\mathcal{U}_z(a)\) non previously considered in the literature. In Section 3 this approach will be explicitly developed for a relevant type of cocommutator \(\eta\) that is compatible with all the non-isomorphic 3D complex
Lie algebras, as described—for instance—in Jacobson [11]. This cocommutator gives rise to a coproduct very simple and deformed commutators that are not specially complex. In this way we obtain three different families of quantum algebras whose deformed commutation rules are very general and depend on several structure constants.

Another point to stress is that our quantization method has been made using a symmetrized basis instead the usual Poincaré-Birkhoff-Witt (PBW) basis. This procedure has not been previously considered in the literature and can be relevant in quantum physics applications and in the semiclassical limit.

In order to classify such quantizations, in Section 4 we consider the equivalence transformations on the basis of the quantum algebra that are defined through invertible maps in $U_z(\mathfrak{a})$ that leave formally invariant the coproduct for the elements of the basis. This constraint regarding the formal invariance of the coproduct can be understood as a way to identify certain basis elements as some sort of “generators” of a quantum algebra, thus following for quantum algebras the “Cartan” approach to the classification of Lie algebras within $U(\mathfrak{a})$, where only linear transformations in the space of generators leaving invariant their (primitive) coproduct are performed. After a such classification is systematically developed, new quantizations of 3D solvable algebras are obtained, and already known results are recovered. The above mentioned simplicity of the algebraic and coalgebraic structures of this new quantizations increases notably their possible interest in Physics. In order to make more precise the range of 3D deformations that have been covered, a detailed comparison with the complete classification of 3D real Lie bialgebra structures presented in [11] is explicitly given within a final Section which also includes some further comments and conclusions.

2 The quantization method

Let us consider the Lie algebra $\mathfrak{a}$ and its universal enveloping algebra $U(\mathfrak{a})$, an associative algebra that is obtained as the quotient $T(\mathfrak{a})/I$, where $T(\mathfrak{a})$ is the tensor algebra of $\mathfrak{a}$ and $I$ is the ideal generated by the elements $XY - YX - [X, Y]$ ($X, Y \in \mathfrak{a}$). If we define the coproduct, counit and antipode ($\forall X \in \mathfrak{a}$)

\[
\Delta_0(X) = 1 \otimes X + X \otimes 1, \quad \Delta_0(1) = 1 \otimes 1, \quad \epsilon(X) = 0, \quad \epsilon(1) = 1, \quad \gamma(X) = -X,
\]

and we extend by linearity all these (anti)automorphisms to the full $U(\mathfrak{a})$, we shall endow the universal enveloping algebra with a Hopf algebra structure. In general, an element $Y$ of a Hopf algebra is called primitive if

\[
\Delta(Y) = 1 \otimes Y + Y \otimes 1.
\]

Within $U(\mathfrak{a})$, it can be shown that the only primitive generators under the coproduct $\Delta_0$ are the generators of $\mathfrak{a}$ (this result is known as the Friedricks theorem [12]). Note that an essential property of the Hopf algebra $U(\mathfrak{a})$ is its cocommutativity, since $\Delta_0$ is invariant under the action of the flip operator $\sigma$.

Let us now consider the Hopf algebra $U(\mathfrak{a})$ and a deformation parameter $z$. A quantum algebra $(U_z(\mathfrak{a}), \Delta)$ is a Hopf algebra of formal power series in the deformation parameter $z$
with coefficients in $U(a)$ and such that $\[3\]

$$U_z(a)/zU_z(a) \simeq U(a).$$

Therefore, $U(a)$ is obtained (as Hopf algebra) in the limit $z \to 0$, and the first order in $z$ of the coproduct is directly related to the cocommutator of an underlying Lie bialgebra $(a, \eta)$ through (1.1). In this way, Lie bialgebras can be used to characterize quantum deformations. Amongst all the Hopf algebra axioms to be imposed, we recall that the quantum coproduct $\Delta$ has to be a coassociative map, namely:

$$(\Delta \otimes id) \circ \Delta = (id \otimes \Delta) \circ \Delta. \quad (2.1)$$

From now on, we shall refer to the “quantization” of a given Lie bialgebra $(a, \eta)$ as the problem of finding a quantum algebra $(U_z(a), \Delta)$ such that (1.1) is fulfilled (we recall that the uniqueness of such construction cannot be take for granted).

The general quantization procedure that we propose is based on three essential ingredients that can be considered whatever the dimension of the Lie bialgebra is. The first one is a generalized cocommutativity property for the coproduct, the second one is related to the choice of a basis in the universal enveloping algebra and the third one selects a given type of power series expansion for the deformed commutation rules.

1. **Generalized cocommutativity.** In general, we impose the (noncocommutative) quantum coproduct $\Delta$ to be invariant under the composition $\tilde{\sigma} = \sigma \circ T$ of the flip operator $\sigma$ and a change of sign of (all) the deformation parameter(s):

$$\tilde{\sigma} \circ \Delta = \Delta \quad \text{where} \quad \tilde{\sigma} = T \circ \sigma \quad \text{and} \quad T(z) = -z.$$

We point out that the definition (1.1) for the underlying cocommutator implies that the deformation parameter appear explicitly as multiplicative factors within the cocommutator. This symmetry property of the coproduct can be imposed in any dimension and makes much easier the procedures of symmetrization and ‘hermitation’ [13]. In particular, given a certain Lie bialgebra $(a, \eta)$, the above assumption implies that the first order deformation of the coproduct will be just given by the (skewsymmetric) Lie bialgebra cocommutator.

$$\Delta(X) = \Delta_0(X) + \eta(X) + O[z^2], \quad X \in g. \quad (2.2)$$

Moreover, the invariance of $\Delta$ under the transformation $\tilde{\sigma}$ together with the fact that the coproduct is an algebra homomorphism with respect to the deformed commutation rules $[\cdot, \cdot]_z$, i. e. $\Delta([\cdot, \cdot]_z) = [\Delta(\cdot), \Delta(\cdot)]_z$ implies that $[\cdot, \cdot]_z$ has to be an even function in the deformation parameter $z$.

2. **The choice of a basis in $U_z(a)$.** In contradistinction with previous works on this subject in which the PBW basis $X_1^\alpha X_2^\beta \ldots X_l^\gamma$ is considered, we introduce using the operator Sym a basis in $U_z(a)$ given by the completely symmetrized monomials. We, thus, define the linear operator Sym by

$$\text{Sym} \left\{ \sum c_{\alpha\beta\ldots\gamma} X_1^\alpha X_2^\beta \ldots X_l^\gamma \right\} = \sum c_{\alpha\beta\ldots\gamma} \text{Sym} \left\{ X_1^\alpha X_2^\beta \ldots X_l^\gamma \right\}$$
\[ \text{Sym } \{A_1 \ldots A_n\} := \frac{1}{n!} \sum_{p \in S_n} p(A_1 \ldots A_n), \]

with \( S_n \) the group of permutations of \( n \) elements. Note that Sym is the identity for commuting operators.

This symmetrization Ansatz, although very convenient in quantum mechanical terms, has not been previously considered in the literature, and turns out to be very efficient in order to get the explicit form of the deformed commutation rules. We remark that one of the main advantages of the symmetrized basis in the quantum algebra is the fact that if we replace the deformed commutation rules by Poisson brackets, the correspondent Poisson-Hopf algebra is uniquely and immediately defined.

### 3. Deformed commutation rules

We will assume

\[ [X, Y]_z = \frac{1}{z} \text{Sym}\left(f(zX_1, zX_2, \ldots, zX_l)\right), \]

where \( f \) is a meromorphic function at \( z = 0 \) and also odd in \( z \);

We extend the discussion to the multiparametric case considering meromorphic deformations of Lie algebras with \( z_i/z_j \) fixed.

Under all these assumptions, given a (family of) Lie bialgebra \((a, \eta)\), the “direct” quantization procedure that we propose would be sketched as follows:

- Assume that the first order coproduct is of the form (2.2).
- Order by order in the deformation parameter(s), get the relations coming from the coassociativity constraint (2.1) and solve them recursively by taking into account the invariance under \( \tilde{\sigma} \) of the solution, thus obtaining the full quantum coproduct.
- Obtain, again order by order, the deformed commutation rules by solving the compatibility equations coming from the fact that the coproduct has to be an algebra homomorphism.

### 3 Three dimensional quantum algebras

Let \( \{A, B, C\} \) be the generators of an arbitrary complex 3D Lie algebra \( \mathcal{L} \) with commutators

\[
\begin{align*}
[A, B] &= c_1 A + c_2 B + c_3 C \\
[A, C] &= b_1 A + b_2 B + b_3 C \\
[B, C] &= a_1 A + a_2 B + a_3 C,
\end{align*}
\]

(3.1)

where the structure constants are complex numbers subjected to some (nonlinear) relations coming from Jacobi identity.

We recall that the complete classification of the 3D complex Lie algebras is given in [11] (see for instance [14] for the real case). According to the dimension of the derived algebra \( \mathcal{L}' = [\mathcal{L}, \mathcal{L}] \), the nonisomorphic classes of 3D Lie algebras read:
• Type I: $\dim L' = 0$. Then $L$ is abelian. We shall not consider this case from the point of view of quantum deformations, since any coassociative coalgebra with primitive non-deformed limit is compatible with the abelian commutation rules.

• Type II: $\dim L' = 1$. We have two algebras, the Heisenberg-Weyl algebra and a central extension $L = B \oplus C$ of the Borel algebra, where $B$ is a Borel algebra and $C$ commutes with $B$.

• Type III: $\dim L' = 2$. We have the family on non-isomorphic Lie algebras labeled by the nonzero complex number $\alpha$ with commutators

$$[X_1, X_2] = 0, \quad [X_1, X_3] = X_1, \quad [X_2, X_3] = \alpha X_2, \quad (3.2)$$

and the Lie algebra

$$[X_1, X_2] = 0, \quad [X_1, X_3] = X_1 + X_2, \quad [X_2, X_3] = X_2. \quad (3.3)$$

• Type IV: $\dim L' = 3$. The only element in this class is the simple Lie algebra $A_1$.

Note that in the Type III the algebra of dilations and the 2D Euclidean algebra for $\alpha = 1$ and $\alpha = -1$, respectively, are included.

A complete classification of the Lie bialgebra structures of the real 3D Lie algebras has been given in [1], and the Lie bialgebra classification for the complex case can be extracted from there.

Obviously, once a set of values for the structure constants $\{a_i, b_i, c_i\}$ is given, a suitable linear transformation $X_i = X_i(A, B, C)$ with complex coefficients can be found in such a way that the Lie algebra (3.1) is reduced to one of the Jacobson cases. However, while in (3.1) all $X_i$ are primitive and, thus, equivalent, this is not longer true in presence of deformation.

In particular, by looking at the complexified form of the classification of 3D Lie bialgebras given in [1], one can realize that a cocommutator of the form

$$\eta(A) = 0 \quad \eta(B) = z A \land B, \quad \eta(C) = \rho z A \land C, \quad z, \rho \in \mathbb{C} \quad (3.4)$$

defines a Lie bialgebra structure for each of the types of Lie algebras given in Jacobson’s classification provided that different linear transformations $X_i = X_i(A, B, C)$ are defined and that $\rho$ takes certain appropriate values. Since we are interested in obtaining the most general types of deformed commutation rules arising in 3D quantizations, we shall apply the perturbative quantization procedure described in the previous Section in order to get a quantum coproduct coming from the cocommutator (3.4) together with a compatible deformation of the commutation rules (3.1). We remark that the cocommutator (3.4) can be thought of as a two parameter structure in $z$ and $\chi = \rho z$. Moreover, complex values for all the parameters (including the structure constants $\{a_i, b_i, c_i\}$ and $\rho$) will be considered, and the results here presented will also contain the quantizations in which the deformation parameter is a root of unity.
Let us follow step by step the procedure introduced in Section 2. Firstly, we assume that the quantum coproduct will be of the form (2.2), namely
\[
\Delta(A) = 1 \otimes A + A \otimes 1,
\]
\[
\Delta(B) = 1 \otimes B + B \otimes 1 + z A \wedge B + O[z^2],
\]
\[
\Delta(C) = 1 \otimes C + C \otimes 1 + \rho z A \wedge C + O[z^2].
\]
(3.5)

In this case, it is straightforward to prove that the following coassociative coproduct has a first order given by the cocommutators (3.4) and is invariant under the transformation \(\tilde{\sigma}\):
\[
\Delta(A) = 1 \otimes A + A \otimes 1,
\]
\[
\Delta(B) = e^{zA} \otimes B + B \otimes e^{-zA},
\]
\[
\Delta(C) = e^{\rho zA} \otimes C + C \otimes e^{-\rho zA},
\]
(3.6)

Now we have to obtain the deformed commutation rules by solving the compatibility equations coming from the fact that the coproduct has to be an algebra homomorphism. Since the \(\tilde{\sigma}\) invariance of the coproduct implies that the first deformed term in the commutation rules has to be of the order \(z^2\), the first order coproduct (3.5) has to be compatible with the non deformed commutation rules. This condition leads to the following relations between the structure constants and \(\rho\):
\[
b_2(1 - \rho) = 0, \quad c_3(1 - \rho) = 0, \quad b_1 = -\rho a_2, \quad a_3 = \rho c_1, \quad b_3 = -\rho c_2.
\]
(3.7)

Jacobi identity requires the relation
\[
(1 - \rho)[a_2 c_1 (1 + \rho) - a_1 c_2] = 0
\]
(3.8)

Now we can distinguish the following cases:

1. Case \(\rho \neq \pm 1\):
\[
b_2 = 0, \quad c_3 = 0, \quad b_1 = -\rho a_2, \quad a_3 = \rho c_1, \quad b_3 = -\rho c_2;
\]
\[
a_2 c_1 (1 + \rho) - a_1 c_2 = 0.
\]
(3.9)

Note that the space of parameters is 3D

2. Case \(\rho = +1\):
\[
b_1 = -a_2, \quad a_3 = c_1, \quad b_3 = -c_2.
\]
(3.10)

In this case the space of parameters is 6D.

3. Case \(\rho = -1\):
\[
b_2 = 0, \quad c_3 = 0, \quad b_1 = a_2, \quad a_3 = -c_1, \quad b_3 = c_2;
\]
\[
a_1 c_2 = 0.
\]
(3.11)

Here, the space of parameters is 3D.
In all the cases no more conditions are found between the structure constants to higher orders, in spite of the fact that we have assumed no dependence on the deformation parameter of the structure constants. Note that the order by order procedure has to be solved, in general, simultaneously for both the quantum coproduct and the deformed commutation rules.

Now, we are in conditions to obtain the deformed commutators. So, the integration of the above equations to all orders gives the general $q$–algebra of three generators compatible with the deformed coproduct. We find:

1. Case $\rho \neq \pm 1$:

1.1) $c_2 \neq 0$:

\[
\begin{align*}
[A, B] &= c_1 \sinh(zA)/z + c_2 B, \\
[A, C] &= -a_2 \sinh(z \rho A)/z - \rho c_2 C, \\
[B, C] &= \frac{a_2 c_1 \sinh[z(1 + \rho)A]}{c_2} z + a_2 \text{Sym } [B \cosh(z \rho A)] + \rho c_1 \text{Sym } [C \cosh(zA)].
\end{align*}
\] (3.12)

1.2) $c_2 = 0$, $a_2 = 0$:

\[
\begin{align*}
[A, B] &= c_1 \sinh(zA)/z, \\
[A, C] &= 0, \\
[B, C] &= a_1 \sinh[z(1 + \rho)A]/z(1 + \rho) + \rho c_1 C \cosh(zA).
\end{align*}
\] (3.13)

$(c_2 = 0, a_2 = 0)$.

2. Case $\rho = +1$:

2.1) $c_2 = 0$:

\[
\begin{align*}
[A, B] &= c_1 \sinh(zA)/z + c_2 B + c_3 C, \\
[A, C] &= -a_2 \sinh(zA)/z + b_2 B - c_2 C, \\
[B, C] &= a_1 \sinh(2zA)/(2z) + \text{Sym } \{a_2 B + c_1 C \cosh(zA)\}.
\end{align*}
\] (3.14)

3. Case $\rho = -1$:

3.1) $c_2 \neq 0$:

\[
\begin{align*}
[A, B] &= c_1 \sinh(zA)/z + c_2 B, \\
[A, C] &= b_1 \sinh(zA)/z + c_2 C, \\
[B, C] &= \text{Sym } \{(b_1 B - c_1 C) \cosh(zA)\}.
\end{align*}
\] (3.15)
3.2) $c_2 = 0$:

$$[A, B] = c_1 \sinh(zA)/z,$$

$$[A, C] = b_1 \sinh(zA)/z,$$

$$[B, C] = a_1 A + \text{Sym} \{ (b_1 B - c_1 C) \cosh(zA) \}. \tag{3.16}$$

It is worthy to notice that in many of these case there is a form invariance of the commutators related with the interchange of the generators $B$ and $C$.

4 Equivalence and classification

In general, two quantum algebras are said to be equivalent (isomorphic) if there exists an invertible (nonlinear in many cases) map between their corresponding quantum universal enveloping algebras as Hopf algebras. In this way, equivalence classes of quantum deformations can be defined. But it is clear that, due to the infinite number of possibilities given by arbitrary nonlinear maps, such equivalence classes are huge, and presumably some general criteria for the choice of “canonical” representatives of each of them should be helpful for classification purposes. Moreover, the definition of a such canonical representative for the Hopf algebra structure would be given, indeed, in terms of what could be properly called as “generators” of the quantum algebra.

In fact, the standard classification of (non deformed) Lie algebras can be understood as a particular application of the abovementioned procedure to their corresponding universal enveloping algebras as Hopf algebras, since Friedrichs theorem states that the only primitive elements in $U(a)$ under the coproduct $\Delta_0$ are just the generators of the Lie algebra $a$. Therefore, we can define the generators of a Lie algebra $a$ as those elements of $U(a)$ which have a primitive coproduct. In this way, the (Cartan) classification of Lie algebras is performed by obtaining appropriate generators (primitive elements) having the “simplest” commutation rules (minimum number of non-vanishing structure constants). We stress that, in order to find such “irreducible” commutation rules only equivalence transformations leaving the coproduct invariant are allowed (in the Lie case, these are just linear transformations).

From this perspective, we propose a definition of the “canonical” representatives of quantum algebras by following a similar procedure. In this case is essential to realize that a quantum algebra $U_z(a)$ is endowed with a deformed coproduct $\Delta$ which is no longer cocommutative, but $\Delta$ is (through our quantization procedure) invariant under $\tilde{\sigma}$ (generalized cocommutativity). Thus, in order to find the “canonical” generators we shall move within the equivalence subclass defined through the restricted set of Hopf algebra isomorphisms of quantum universal enveloping algebras (always in a symmetrized basis) that leave the coproduct formally invariant. Through such restricted isomorphisms we shall look for representatives with “irreducible” deformed commutation rules having a minimal number of non-zero terms.

By proceeding in this way we have succeeded in classifying all the non-isomorphic quantum algebras that are contained in the three multiparameter families given in the previous section.
Let us explicitly obtain them by eliminating many irrelevant parameters through coproduct-preserving mappings.

4.1 Case 1: $\rho \neq \pm 1$

Let us start with the case $\rho \neq \pm 1$. Let us consider the following transformation [15]:

$$
\mathcal{A} = \alpha A, \quad \mathcal{B} = \beta B + \delta \frac{\sinh(A)}{z}, \quad \mathcal{C} = \nu C + \eta \frac{\sinh(z \rho A)}{z \rho},
$$

$$
\hat{z} = \alpha^{-1}z, \quad \alpha, \beta, \delta, \nu, \eta \in \mathbb{C}.
$$

(4.1)

After this transformation the coproduct (3.6) becomes

$$
\Delta(\mathcal{A}) = 1 \otimes \mathcal{A} + \mathcal{A} \otimes 1,
$$

$$
\Delta(\mathcal{B}) = e^{\hat{z}A} \otimes \mathcal{B} + \mathcal{B} \otimes e^{-\hat{z}A},
$$

$$
\Delta(\mathcal{C}) = e^{\hat{z}\rho A} \otimes \mathcal{C} + \mathcal{C} \otimes e^{-\hat{z}\rho A},
$$

(4.2)

i.e., it remains formally invariant. So, all the elements of the form (4.1) belong of the same class.

1.1) $c_2 \neq 0$

In this case the above mentioned change of basis (4.1) can be reduced to

$$
\mathcal{A} = \mathcal{A}/c_2, \quad \mathcal{B} = c_2 B + c_1 \frac{\sinh(zA)}{z}, \quad \mathcal{C} = c_2 C + c_2 \frac{\sinh(z \rho A)}{z \rho},
$$

$$
\hat{z} = c_2 z.
$$

(4.3)

1.1.1) Under this change of basis we obtain the new Lie commutators

$$
[A, B] = \sinh(\hat{z}A)/\hat{z}, \quad [A, C] = 0, \quad [B, C] = \rho C \cosh(\hat{z}A),
$$

(4.4)

which correspond to a quantization of the Lie algebra (3.2). Its associated bialgebra is non-coboundary, i.e., there is no classical-$r$ matrix.

1.2) $c_2 = 0$

We have two quantum algebras. Following a procedure analogous to (4.3) they can be written as

1.2.1) $a_2 = 0, \ c_1 \neq 0$

$$
[A, B] = \sinh(\hat{z}A)/\hat{z}, \quad [A, C] = 0, \quad [B, C] = \rho C \cosh(\hat{z}A).
$$

(4.5)

The bialgebra is coboundary: the classical $r$-matrix is $r = \hat{z} \mathcal{A} \wedge \mathcal{B}$. It is non-standard, i.e. it verifies the classical Yang-Baxter equation. This is another quantization of the Lie algebra (3.2).
1.2.2) \( a_2 = 0, a_1 \neq 0, c_1 = 0 \)

\[
[\mathcal{A}, \mathcal{B}] = 0, \quad [\mathcal{A}, \mathcal{C}] = 0, \quad [\mathcal{B}, \mathcal{C}] = \frac{\sinh(\hat{z}(1 + \rho)\mathcal{A})}{\hat{z}(1 + \rho)}. \tag{4.6}
\]

This is also a coboundary deformation with standard \( r \)-matrix, i.e. it verifies the modified classical Yang-Baxter equation, \( r = \hat{z}\mathcal{B} \wedge \mathcal{C} \).

Note that for \( \rho = 0 \) we have obtained two deformations of the extended Borel algebra and one deformation of the Heisenberg-Weyl algebra, both of them of Type II in Jacobson.

4.2 Case 2: \( \rho = +1 \)

The equivalence classes are defined by applying to (3.14) the transformation:

\[
\hat{\mathcal{A}} = \alpha \mathcal{A}, \quad \mathcal{B} = \beta \mathcal{B} + \gamma \mathcal{C} + \delta \frac{\sinh(z \mathcal{A})}{z}, \quad \mathcal{C} = \mu \mathcal{B} + \nu \mathcal{C} + \eta \frac{\sinh(z \mathcal{A})}{z},
\]

\( \hat{z} = \alpha^{-1}z \quad \alpha, \beta, \gamma, \delta, \mu, \nu, \eta \in \mathbb{C}. \tag{4.7} \)

This transformation allows us to distinguish the quantum algebras characterized by \( b_2c_3 + c_2^2 \neq 0 \) and those in which \( b_2c_3 + c_2^2 = 0 \).

2.1) \( b_2c_3 + c_2^2 \neq 0 \)

2.1.1) \[
[\mathcal{A}, \mathcal{B}] = \mathcal{B}, \quad [\mathcal{A}, \mathcal{C}] = -\mathcal{C}, \quad [\mathcal{B}, \mathcal{C}] = \frac{\sinh(2\hat{z}\mathcal{A})}{2\hat{z}}. \tag{4.8}
\]

This quantum algebra is just \( A_1(q) \). The classical \( r \)-matrix is \( r = z\mathcal{B} \wedge \mathcal{C} \), and is standard.

2.1.2) \[
[\mathcal{A}, \mathcal{B}] = \mathcal{B}, \quad [\mathcal{A}, \mathcal{C}] = -\mathcal{C}, \quad [\mathcal{B}, \mathcal{C}] = 0. \tag{4.9}
\]

This is the complexification of the first discovered contraction of \( su_q(2) \) deformation of the Euclidean algebra \( \mathcal{E}(2) \) in two dimensions. It is a non-coboundary one.

2.2) \( b_2c_3 + c_2^2 = 0 \)

2.2.1) \( a_2c_2 + b_2c_1 \neq 0 \)

\[
[\mathcal{A}, \mathcal{B}] = -\frac{\sinh(\hat{z}\mathcal{A})}{\hat{z}}, \quad [\mathcal{A}, \mathcal{C}] = \mathcal{B}, \quad [\mathcal{B}, \mathcal{C}] = -\text{Sym}\{\mathcal{C} \cosh(\hat{z}\mathcal{A})\}. \tag{4.10}
\]

This is the symmetrized version of the well-known Jordanian deformation of \( A(1) \) with non-standard classical \( r \)-matrix \( r = \hat{z}\mathcal{A} \wedge \mathcal{B} \). According to the commutation relations (4.10) we obtain that

\[
\text{Sym}(\mathcal{C} \cosh(\hat{z}\mathcal{A})) = \frac{1}{2}(\mathcal{C} \cosh(\hat{z}\mathcal{A}) + \cosh(\hat{z}\mathcal{A})\mathcal{C}) + \frac{1}{12} \hat{z}^2 \sinh \frac{2\hat{z}\mathcal{A}}{2\hat{z}}.
\]

thus, the quantum algebra presented in \([18]\) is a case of (3.14) with \( z \)-dependent parameters.
2.2.2) \(a_2c_2 + b_2c_1 = 0\)

This condition implies that the commutators \([A, B]\) and \([A, C]\) are proportional. We obtain the following algebras:

2.2.2.1) \[
[A, B] = \frac{\sinh(\hat{z}A) - z}{\hat{z}}, \quad [A, C] = 0, \quad [B, C] = C \cosh(\hat{z}A)\]

(4.11)

We recover a non-standard deformation of the Euclidean group in two dimensions \(E(2)\) [19]. The classical \(r\)-matrix is non-standard, \(r = \hat{z}A \wedge B\).

2.2.2.2) \[
[A, B] = 0, \quad [A, C] = -B, \quad [B, C] = \frac{\sinh(2\hat{z}A)}{2\hat{z}}\]

(4.12)

We have the standard deformation of \(E(2)\) with classical \(r\)-matrix \(r = \hat{z}B \wedge C\).

2.2.2.3) \[
[A, B] = 0, \quad [A, C] = 0, \quad [B, C] = \frac{\sinh(2\hat{z}A)}{2\hat{z}}\]

(4.13)

It corresponds to a deformation of the Heisenberg-Weyl algebra with classical \(r\)-matrix is standard, \(r = \hat{z}C \wedge B\).

2.2.2.4) \[
[A, B] = 0, \quad [A, C] = B, \quad [B, C] = 0\]

(4.14)

This is a non-coboundary deformation of the Heisenberg-Weyl algebra.

4.3 Case 3: \(\rho = -1\)

The classification of the quantum algebras corresponding to the case \(\rho = -1\) can be made considering the transformation:

\[
A = \alpha A, \quad B = \beta B + \delta \frac{\sinh(zA)}{z}, \quad C = \nu C + \eta \frac{\sinh(zA)}{z}, \quad \hat{z} = \alpha^{-1}z,
\]

(4.15)

where \(\alpha, \beta, \delta, \nu, \eta\) are complex numbers.

3.1) \(c_2 \neq 0\)

\[
[A, B] = B, \quad [A, C] = C, \quad [B, C] = 0.
\]

(4.16)

We have a deformation of the dilations algebra such that there is not \(r\)-matrix.

3.2) \(c_2 = 0\):  

3.2.1) \[
[A, B] = -\frac{\sinh(\hat{z}A)}{\hat{z}}, \quad [A, C] = \frac{\sinh(\hat{z}A)}{\hat{z}}, \quad [B, C] = A + (B + C) \cosh(\hat{z}A).
\]

(4.17)

Like in the previous case there is not \(r\)-matrix.
Note that in the limit of $\hat{z} \to 0$ we recover

$$[A, B] = -A, \quad [A, C] = A, \quad [B, C] = A + (B + C).$$

that can be rewritten under a change $(B + C \to B)$ like

$$[A, B] = 0, \quad [A, C] = A, \quad [B, C] = A + B. \quad (4.18)$$

Therefore we have obtained a quantum deformation of the Lie algebra (3.3).

3.2.2) Other deformation of (3.3), now non-standard, is

$$[A, B] = 0, \quad [A, C] = \sinh(\hat{z} A)/\hat{z}, \quad [B, C] = A + B \cosh(\hat{z} A). \quad (4.19)$$

The $r$-matrix is $r = \hat{z} A \wedge C$.

3.2.3)

$$[A, B] = -\sinh(\hat{z} A)/\hat{z}, \quad [A, C] = \sinh(\hat{z} A)/\hat{z}, \quad [B, C] = (B + C) \cosh(\hat{z} A). \quad (4.20)$$

This is a deformation of the dilation algebra in two dimensions without $r$-matrix.

3.2.4) \[3.2.4\]

$$[A, B] = 0, \quad [A, C] = 0, \quad [B, C] = A. \quad (4.21)$$

It corresponds to a coboundary deformation of the Heisenberg-Weyl algebra with standard $r$-matrix, $r = \hat{z} B \wedge C$.

3.2.5) \[3.2.5\]

$$[A, B] = -\sinh(\hat{z} A)/\hat{z}, \quad [A, C] = 0, \quad [B, C] = C \cosh(\hat{z} A). \quad (4.22)$$

We have a deformation of the dilatation algebra in two dimensions. In this case the classical $r$-matrix is non-standard, $r = \hat{z} A \wedge B$.

5 Conclusions and remarks

This quantization method can be simultaneously applied and successfully solved for a multiparameter family of Lie bialgebras that share some structural properties.

Throughout the paper we have considered certain 3D complex Lie bialgebras. In particular, the parameter $\rho$ is complex but we do also have isolated solutions for $\rho = \pm 1$. On the other hand, the comparison with the complete classification of 3D real Lie bialgebras given in [1] can be worked out by considering the isomorphisms among the complexified versions of 3D real Lie algebras.

In this way it can be shown that all the quantum algebras that we have obtained in section 4 are quantizations of the complexifications of the dual version of the Lie bialgebras given in [1].
In order to find out the correspondence explicitly, in Table III of [1] we have to identify a given algebra \( a \) with the complex version of the dual Lie algebra \( g^* \) and, consequently, the dual of the cocommutator \( \eta \) will have to be isomorphic to one of the algebras \( g \) in the first row of such Table. By proceeding in this way we find the following correspondences (we write first the Lie bialgebras \((g^*, g \equiv \eta^* )\) as labeled in Table III of [1] and afterwards the corresponding quantum algebra according to our classification):

\[
\begin{align*}
5 & \rightarrow 1.2.2); \ 6 \rightarrow 1.2.1); \ 7 \rightarrow 1.1.1); \\
(1) & \rightarrow 2.1.1); \ (2), (4) \rightarrow 2.1.1); \ (3) \rightarrow 2.2.1); \ 9 \rightarrow 2.1.2); \ 11, 11' \rightarrow 2.2.2.2); \\
10 & \rightarrow 2.2.2.4); 5_{\rho=1} \rightarrow 2.2.2.3); 6_{\rho=1} \rightarrow 2.2.2.1); 7_{\rho=1} \rightarrow 2.1.2)_{\rho=1}; \\
5' & \rightarrow 3.2.4); 8 \rightarrow 3.2.1); (14) \rightarrow 3.2.2); (11) \rightarrow 3.2.3); 6_{\rho=-1} \rightarrow 3.2.5)_{\rho=-1}; \\
7_{\rho=-1} & \rightarrow 3.1)_{\rho=-1}.
\end{align*}
\]

In this way we can realize that our choice (3.4) for the cocommutator implies that we have just obtained the quantizations for the full set of dual Lie bialgebras \((g^*, g \equiv \eta)\) of [1] such that \( \eta^* \equiv \tau_3(\rho) \) for all values of \( \rho \). In fact, the \( \rho \) parameter in (3.4) is identified with the one appearing in Gomez’s classification.

**Acknowledgments**

This work has been partially supported by DGI of the Ministerio de Ciencia y Tecnología (Projects BMF2002-0200 and BFM2000-1055), the FEDER Programme and Junta de Castilla y León (Projects VA085/02 and BU04/03) (Spain). The visit of E.C. to Valladolid have been financed by Universidad de Valladolid, by CICYT-INFN and by Ministerio de Educación y Cultura (Spain).

**References**

[1] X. Gomez, *J. Math. Phys.* **41**. (2000) 4939

[2] V.G. Drinfel’d, *Quantum Groups* in “Proceedings of the International Congress of Mathematicians”, Berkeley, 1986, A.M. Gleason (ed.) (AMS, Providence, 1987)

[3] V. Chari and A. Pressley, *A Guide to Quantum Groups* (Cambridge University Press, Cambridge, 1995)

[4] S. Majid, *Foundations of Quantum Group Theory* (Cambridge University Press, Cambridge, 1995)

[5] T. Tjin, *Int. J. Mod. Phys. A: Math. Gen.* **7** (1992) 6175

[6] V.G. Drinfel’d, *Soviet Math. Dokl.* **27** (1983) 68
[7] A. Ballesteros, E. Celeghini and F.J. Herranz, *J. Phys. A: Math. Gen.* **33** (2000) 3431

[8] V. Lyakhovsky and A.I. Mudrov, *J. Phys. A: Math. Gen.* **25** (1992) L1139

V. Lyakhovsky, *Group-like structures in quantum-Lie algebras and the procedure of quantization* in “Quantum Groups, Formalism and Applications” ed. J Lukierski et al (Warsaw: Polish Scientific Publishers, 1995)

[9] A.I. Mudrov, *J. Math. Phys.* **38** (1997) 476

[10] S. Schneider and S. Sternberg, Quantum Groups from Coalgebras to Drinfeld algebras (Int. Press, Cambridge (Mass.), 1993)

[11] N. Jacobson, Lie algebras (Dover, New York, 1979)

[12] M. Postnikov, Lectures in Geometry: Lie Groups and Lie Algebras (Mir, Moscow, 1982)

[13] E. Celeghini and M.A. del Olmo, *Europhys. Lett.* **61** (2003) 438

[14] J. Patera, R.T. Sharp, P. Winternitz and H. Zassenhaus, *J. Math. Phys.* **17** (1976) 986

[15] B.L. Aneva, D. Arnaudon, A. Chakrabarti, V.K.Dobrev and S.G. Mihov, *J. Math. Phys.* **42** (2001) 1236

[16] M. Jimbo, *Lett. Math. Phys.* **10** (1985) 63

[17] L.L. Vaksman and L.I. Korogodski, *Sov. Math. Dokl.* **39** (1989) 173

[18] Ch. Ohn, *Lett. Math. Phys.* **25** (1992) 85

[19] A. Ballesteros, E. Celeghini, F.J. Herranz, M.A. del Olmo and M. Santander, *J. Phys. A: Math. Gen.* **28** (1995) 3129