Nonexistence of local conservation laws for generalized
Swift–Hohenberg equation

Pavel Holba

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Abstract
We prove that the generalized Swift–Hohenberg equation with nonlinear right-hand side, a natural generalization of the Swift–Hohenberg equation arising in physics, chemistry and biology and describing inter alia pattern formation, has no nontrivial local conservation laws.

Keywords Conservation laws · Generalized Swift–Hohenberg equation · Nonlinear PDEs · Evolution equations

Mathematics Subject Classification 37K05

1 Introduction
Conservation laws are important for natural sciences for many reasons [1–3, 5, 6, 9–11, 13–16, 18]. While the presence of an infinite series of conservation laws is usually a sign of integrability in the sense of soliton theory, cf. e.g. [2, 5, 9–11, 13–16] and references therein, even existence of a finite number of conservation laws can be quite helpful in establishing the qualitative behavior of solutions, like e.g. preservation of the solution norm in a certain functional space or of some important physical characteristics like energy or momentum, in the course of time evolution, see for example [1, 2, 9, 11]. Notice that the search for conservation laws is a highly nontrivial task whose complexity grows significantly with the increase of number of independent variables and/or the order of the equation under study, cf. e.g. [2, 3, 5, 6, 9–11, 14, 16, 18].

Of course, an immediate consequence of the above is that it is also quite important to know that a certain equation has no (nontrivial) conservation laws at all, or, say, of order higher than a certain number, and below we prove just such a result,
establishing nonexistence of nontrivial local conservation laws, for the generalized Swift–Hohenberg equation with any number \( n \) of space variables, that is,

\[
u_t = A(\Delta)u + N(u),
\]

where \( A(\Delta) = \sum_{i=0}^{k} a_i \Delta^i \), \( k \geq 1 \), \( a_i \) are constants, \( \Delta = \sum_{i=1}^{n} \partial^2 / \partial x_i^2 \) is the Laplace operator, and \( N(u) \) is a smooth function of \( u; t, x_1, \ldots, x_n \) are independent variables and \( u \) is the dependent variable. All variables \( t, x_1, \ldots, x_n, u \), as well as the coefficients \( a_0, \ldots, a_k \) are assumed real, and the function \( N(u) \) is assumed real-valued.

In what follows we make a (natural) blanket assumption that \( n \) is a nonnegative nonzero integer.

The choice of name for Eq. (1) is motivated by the fact that it is a natural generalization of the original Swift–Hohenberg equation [17], which corresponds to the case when

\[
A(\Delta) = a(\Delta + b)^2 + c,
\]

where \( a, b, c \) are real constants, or, even more specifically, \( a = b = 1 \), and has a number of important applications in physics, chemistry and biology, cf. e.g. [4, 8]. In particular, (1) with \( A \) given by (2) serves as a model for the study of various issues in pattern formation and is a subject of intense research, see e.g. [4, 7, 8, 12] and references therein.

We prove that (1) admits no local conservation laws if \( \partial^2 N / \partial u^2 \neq 0 \) and \( A(\Delta) \) is nonconstant, see Theorem 1 below for details. Note that establishing this result in full generality by direct search for conservation laws is apparently impossible, in particular because (1) can, in view of freedom in choosing \( A \), be of arbitrarily high order.

2 Preliminaries

In spirit of [11], we shall say that a \emph{differential function} is a smooth function of \( x_1, \ldots, x_n, t, u \) and finitely many \( x \)-derivatives of \( u \). Here and below \( D_t \) and \( D_{x_i} \) will stand for total derivatives and \( \delta / \delta u \) for the variational derivative with respect to \( u \), see e.g. [2, 6, 9, 11] for further details on those.

A \emph{local conservation law} for (1) is, cf. e.g. [2, 9, 11] and references therein, a differential expression

\[
D_t(\rho) + \sum_{i=1}^{n} D_{x_i} \sigma_i
\]

that vanishes modulo (1) and its differential consequences; here \( \rho, \sigma_1, \ldots, \sigma_n \) are differential functions.

We shall refer to the quantity \( \delta \rho / \delta u \) as to the \emph{characteristic} of a conservation law (3). It is readily seen that for the case of (1) this definition is equivalent to the standard one [2, 6, 9, 11].
A local conservation law (3) is said to be trivial if its characteristic is identically zero or, equivalently, if (3) vanishes identically, without the need of invoking (1) or its differential consequences, cf. e.g. [2, 6, 9, 11] for details. It can be shown that trivial conservation laws are pretty much of no interest for applications [2, 9, 11].

3 Main result

Theorem 1  Equation (1) with \( \partial^2 N/\partial u^2 \neq 0 \) and nonconstant polynomial \( A(\Delta) \) has no nontrivial local conservation laws.

Before proceeding to the proof note that the condition that the polynomial \( A(\Delta) \) is nonconstant ensures that (1) is necessarily a partial, rather than ordinary, differential equation for \( u \).

Proof  The necessary condition for a differential function, say \( Q \), to be a characteristic of a local conservation law for (1) is readily seen, cf. e.g. [2, 6, 9, 11], to take the form

\[
D_t(Q) + \frac{\partial N}{\partial u} Q + \sum_{i=0}^{k} a_i \tilde{\Delta}(Q) = 0, \quad \text{where} \quad \tilde{\Delta} = \sum_{i=1}^{n} D_{x_i}^2.
\] (4)

Equation (4) is a linear equation in total derivatives for \( Q \) and it is immediate that the coefficients of this equation depend at most on \( u \). In view of this fact and since the polynomial \( A(\Delta) \) is nonconstant by assumption, (1) satisfies the conditions of Theorem 6 from [6], and therefore, for any local conservation law of (1) its characteristic \( Q \) depends at most on \( t, x_1, \ldots, x_n \) but not on \( u \) and its derivatives.

With this in mind upon applying \( \partial / \partial u \) to both sides of (4) we get

\[
\frac{\partial^2 N}{\partial u^2} Q = 0,
\] (5)

which implies that if \( \partial^2 N/\partial u^2 \neq 0 \) then \( Q = 0 \), so under the assumptions of Theorem 1 Eq. (1) can have only trivial local conservation laws, and the result follows.

4 Conclusions and discussion

We have shown, see Theorem 1 above, that under very mild assumptions, which essentially boil down to being nonlinear and partial (rather than ordinary) differential equation, the generalized Swift–Hohenberg equation (1) has no nontrivial local conservation laws.
In particular, it follows from Theorem 1 that the original Swift–Hohenberg equation, i.e., (1) with \( A \) given by (2), \( a \neq 0 \), and \( \partial^2 N/\partial u^2 \neq 0 \), has no nontrivial local conservation laws. Let us reiterate, however, that Theorem 1 establishes nonexistence of nontrivial local conservation laws for the entire class of generalized Swift–Hohenberg equations of the form (1) with nonconstant polynomial \( A(\Delta) \) and nonlinear \( N(u) \), which is a far stronger result.

Note that the absence of nontrivial local conservation laws is a strong indication that generalized Swift–Hohenberg equations of the form (1) with nonconstant polynomial \( A(\Delta) \) and nonlinear \( N(u) \) are not integrable in the sense of soliton theory, cf. e.g. [9, 11, 15, 16] and references therein on the relation among existence of (infinitely many) conservation laws and integrability.

Another important consequence of absence of nontrivial local conservation laws for the equations under study satisfying the assumptions of Theorem 1 is that the choice of appropriate discretization for solving such equations numerically is not constrained by the necessity of taking into account the conservation laws of the equations in question, cf. e.g. [1] and references therein.

It is an interesting open problem to find out whether (1) admits nontrivial differential coverings (see e.g. [9] and references therein on general theory of those) and, if yes, whether (1) could have nontrivial nonlocal conservation laws associated with these coverings.

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