A Boolean-valued model approach to conditional risk*

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Abstract

Based on Boolean-valued models we provide a method to interpret a theorem of representation of convex risk measures as a theorem for conditional risk measures which also holds thanks to transfer principle of Boolean-valued models. In particular, we establish a general robust representation theorem for conditional risk measures.

Keywords: Boolean-valued models; conditional risk measures; robust representation.

Introduction

Boolean-valued models are a tool in mathematical logic that was developed as a way to formalize the method of forcing that Paul Cohen created to solve the first problem in the famous Hilbert’s list: it is impossible neither to prove nor to disprove that every infinite set of reals can be bijected either with the natural numbers or with the whole real line [16]. The theory was first formulated by Scott [40] based on some ideas of Solovay, while Vopěnka created independently a similar theory. Very loosely speaking, the idea behind Boolean-valued models is, for a given complete boolean algebra \(A\), to define a model \(V^A\) for the Zermelo-Fraenkel set theory with the axiom of choice, ZFC for short, so that the truth values of propositions are not limited to “true” (1) and “false” (0), but instead they can take any value \(a\) in \(A\).

Over the past two decades and having its origins in the seminal paper [3], the theory of dual representation of risk measures has been an active and prolific area of research cf. [1, 2, 8, 15, 17, 21, 27, 33, 34, 36]. The simplest situation is the case in which only two instants of time matter: today 0 and tomorrow \(T > 0\). In this case, the market information that will be available at time \(T\) is described by a \(\sigma\)-algebra \(E\). A risk measure is a function (satisfying some financial conditions) that assigns to any \(E\)-measurable random variable \(x\), which models a final pay-off, a real number \(\rho(x)\) which quantifies the riskiness of \(x\).

A more intricate situation is when we have a dynamic configuration of time, in which the arrival of new information at an intermediate date \(0 < t < T\) is taken into account, cf. [4, 7, 9, 10, 14, 15, 19, 25, 26, 28]. In this case, the information available at \(t\) is encoded in a sub-\(\sigma\)-algebra \(F\) of the \(\sigma\)-algebra \(E\) of the information at time \(T\). Then the conditional risk measure is a map (satisfying some financial conditions) that assigns to any final payoff, modelled by an \(E\)-measurable random variable \(x\), a \(F\)-measurable random variable \(\rho(x)\) which quantifies the risk arisen from \(x\).

As explained in [22], classical convex analysis—which perfectly applies to the one-period case—seems to have rather delicate application to the multi-period model, requiring sometimes of heavy measurable selection criteria. These difficulties have motivated some new developments in functional analysis. For instance, Filipovic et al. [22] proposed to consider modules over...
$L^0(F)$, the space of (equivalence classes of) $F$-measurable random variables (also see [23, 13]). A more sophisticated machinery is provided in [19], where the so-called conditional set theory is introduced and developed. Other related approaches are given in [11, 12, 20, 30].

In [5] it was proven that some of these developments provided can be seen as particular instances of Boolean-valued models. One of the advantages of this approach is the so-called transfer principle. Namely, the transfer principle asserts that any known result of set theory has a reformulation in the Boolean-valued setting, which is also true. This is a machine to produce a theorem from another theorem. For instance, it is shown in [5] that the main results of [22] are just reformulations in a suitable boolean-valued universe of known results of convex analysis and they automatically follow by means of the transfer principle without the necessity of a step-by-step proof.

This paper goes further in the boolean-valued approach. Let $A$ denote the measure algebra associated to the $\sigma$-algebra $F$ of information, which is a complete boolean algebra, and consider the boolean-valued universe $V(A)$. We provide a method to produce from known theorems on (one-period) risk measures a statement for conditional risk measures which also holds — it is also a theorem due to the transfer principle of $V(A)$. Filipovic et al. [23] proposed the so-called $L^p$-type modules as a model space for conditional risk measures. Namely, for a given $1 \leq p \leq \infty$, the $L^p$-type module, denoted by $L^p_F(E)$, is the $L^0(F)$-module generated by the classical function space $L^p(E)$. Thereby, we will see that $L^p_F(E)$ is essentially the interpretation of some space $L^p(\Sigma)$ of random variables within the universe $V(A)$ and that a conditional risk measure $\rho : L^p_F(E) \rightarrow L^0(F)$ is nothing else but an interpretation of a risk measure $\rho : L^p(\Sigma) \rightarrow \mathbb{R}$ within $V(A)$. By the transfer principle, any dual representation theorem on a risk measure $\rho : L^p(\Sigma) \rightarrow \mathbb{R}$ is also true within $V(A)$, thus they can be interpreted as a statement for conditional risk measure on $\rho : L^p_F(E) \rightarrow L^0(F)$ which gives rise to a new dual representation theorem. We do this not only for $L^p$-type modules, but also for modular versions of Orlicz and Orlicz-heart spaces. Moreover, we will use this method to provide a general representation result for conditional risk measures defined on a solid $L^0(F)$-submodule of $L^0(E)$.

In Section 1, we present some representation results for conditional risk measure, these are just examples as this method can be applied to many other theorems on risk measures. These results cannot be shown by scalarization techniques. Also, a proof based on conditional set theory of them would require the development of a conditional theory of Riesz spaces which has not been established so far. Besides, the boolean-valued approach provides a general method which can be applied to obtain more and more results, without the necessity of providing step-by-step proofs for conditional or modular versions of any single theorem needed.

The paper is structured as follows: In Section 1, we present different theorems for conditional risk measures which will be derived later by means of the transfer principle; Section 2 is devoted to some foundations of boolean-valued models, providing a short review of the construction of the boolean-valued universe and explaining the main elements and principles of this theory; in Section 3, we develop the boolean-valued approach to $L^p$-type modules, Orlicz-type modules and Orlicz-heart-type modules, and provide the machinery that allows to derive theorems for conditional risk measures.

1 Representation of conditional risk measures

In this section we present some theorems of representation of conditional risk measures which follows from the theory developed in Section 3 and the transfer principle of boolean-valued models. They are just examples of what can be derived as the method introduced could be applied to any other result on convex risk measures.

Let us introduce some notation and notions. Let $(\Omega, E, P)$ be a probability space and let
$\mathcal{F} \subset \mathcal{E}$ a sub-$\sigma$-algebra. We denote by $L^0(\mathcal{E})$ the space of $\mathcal{E}$-measurable real-valued random variables on $\Omega$ identified whenever their difference is $\mathbb{P}$-negligible. Given $x, y \in L^0(\mathcal{E})$ we understand $x \leq y$ and $x < y$ in the almost surely sense. We say that $\lim_n x_n = x$ in $L^0(\mathcal{E})$ whenever $(x_n)$ converges almost surely to $x \in L^0(\mathcal{E})$ (or equivalently, $x_n$ order converges to $x$). We define the following sets: $L^0_+(\mathcal{F}) := \{ \eta \in L^0(\mathcal{F}) : \eta \geq 0 \}$, $L^0_{++}(\mathcal{F}) := \{ \eta \in L^0(\mathcal{F}) : \eta > 0 \}$ and $L^0(\mathcal{F}, \mathbb{N}) := \{ \eta \in L^0(\mathcal{F}) : \eta(\Omega) \subset \mathbb{N} \}$. Also, $L^0(\mathcal{F})$ stands for the space of all (classes of equivalence of) $\mathcal{F}$-measurable random variables which take values in $\mathbb{R} \cup \{ \pm \infty \}$. Let $\mathcal{A} := \mathcal{A}_{\mathcal{F}}$ be the measure algebra associated to $\mathcal{F}$ defined by identifying two elements of $\mathcal{F}$ whenever their symmetric difference has probability 0. The measure algebra $\mathcal{A}$ is a complete boolean algebra which satisfies the countable chain condition (ccc), i.e. every family of positive pairwise disjoint elements in $\mathcal{A}$ is at most countable. The 0 of $\mathcal{A}$ is represented by the empty set $\emptyset$ and the I of $\mathcal{A}$ is represented by $\Omega$. We denote by $p(I)$ the sets of partitions of $I$ to $\mathcal{A}$, which are at most countable due to the ccc. Given $a \in \mathcal{A}$, $1_a$ stands for the class in $L^0(\mathcal{F})$ of the characteristic function of some representative of $a$.

Let introduce some nomenclature:

- A non-empty subset $S$ of $L^0(\mathcal{E})$ is said to be stable if for any partition $(a_k) \in p(I)$ and any sequence $(x_k) \subset S$, it is satisfied that $\sum_{k \in \mathbb{N}} 1_{a_k} x_k \in S$;
- A non-empty collection $\mathcal{C}$ of subsets of $E$ is called stable if any $S \in \mathcal{C}$ is stable and for any partition $(a_k) \in p(I)$ and countable family $(S_k) \subset \mathcal{C}$ it is satisfied that $\sum_{k \in \mathbb{N}} 1_{a_k} S_k := \{ \sum_k 1_{a_k} x_k : x_k \in S_k \} \in \mathcal{C}$.

Throughout, $\mathcal{X}$ denotes a stable $L^0(\mathcal{F})$-submodule of $L^0(\mathcal{E})$ which is solid (i.e. $y \in \mathcal{X}$ and $|x| \leq |y|$ implies that $x \in \mathcal{X}$) and such that $L^0(\mathcal{F}) \subset \mathcal{X}$.

**Example 1.1.**

1. $L^p$-type modules (see [22]): Given $1 \leq p \leq \infty$ we define
   $$\| \cdot |\mathcal{F}|_p : L^0(\mathcal{E}) \to \bar{L}^0(\mathcal{F})$$
   by
   $$\| x |\mathcal{F}|_p := \begin{cases} \lim_n \mathbb{E}_{\mathcal{F}}[|x|^{p} \wedge n |\mathcal{F}|]^{1/p} & \text{if } p < \infty \\ \text{ess. inf} \{ \eta \in L^0(\mathcal{F}) : \eta \geq |x| \} & \text{if } p = \infty \end{cases}$$
   and set $L^p_+(\mathcal{E}) := \{ x \in L^0(\mathcal{E}) : \| x |\mathcal{F}|_p \in L^0(\mathcal{F}) \}$. It follows by inspection that $L^p(\mathcal{E})$ is an $L^0(\mathcal{F})$-submodule of $L^0(\mathcal{E})$ which is stable and solid. Further, $L^p_+(\mathcal{E}) \subset L^p_+(\mathcal{E})$.

2. Orlicz-type modules (see [13]) and Orlicz-heart-type modules: Let $\phi : [0, \infty) \to [0, \infty]$ be a Young function (i.e. an increasing left-continuous convex function finite on a neighborhood of 0 with $\phi(0) = 0$ and $\lim_{x \to \infty} \phi(x) = \infty$) and let
   $$L^\phi(\mathcal{E}) := \{ x \in L^0(\mathcal{E}) : \exists \eta \in L^0_{++}(\mathcal{F}), \lim_n \mathbb{E}_{\mathcal{F}}[\phi(\eta|x|) \wedge n |\mathcal{F}] \in L^0(\mathcal{F}) \}$$
   (Orlicz-type module),
   $$H^\phi(\mathcal{E}) := \{ x \in L^0(\mathcal{E}) : \forall \eta \in L^0_{++}(\mathcal{F}), \lim_n \mathbb{E}_{\mathcal{F}}[\phi(\eta|x|) \wedge n |\mathcal{F}] \in L^0(\mathcal{F}) \}$$
   (Orlicz-heart-type module).

   It is not difficult to verify that $L^\phi(\mathcal{E})$ and $H^\phi(\mathcal{E})$ are $L^0(\mathcal{F})$-submodules of $L^0(\mathcal{E})$ which are stable and solid with $H^\phi(\mathcal{E}) \subset L^\phi(\mathcal{E}) \subset L^1_+(\mathcal{E})$. 

Hereafter, we will additionally suppose that $\mathcal{X} \subset L^1_F(\mathcal{E})$. Also, for $x \in L^1_F(\mathcal{E})$, its extended expectation is given by $\mathbb{E}_p[x|\mathcal{F}] := \lim_n \mathbb{E}_p[x^+ \wedge n|\mathcal{F}] - \lim_n \mathbb{E}_p[x^- \wedge n|\mathcal{F}] \in L^0(\mathcal{F})$.

We define the Köthe dual $L^0$-module of $\mathcal{X}$ to be

$$\mathcal{X}^\ast := \{y \in L^0(\mathcal{E}) : xy \in L^1_F(\mathcal{E}) \text{ for all } x \in \mathcal{X}\}.$$  

It simple to verify that $\mathcal{X}^\ast$ enjoys the same properties as $\mathcal{X}$. Namely, $\mathcal{X}^\ast$ is a solid and stable $L^0(\mathcal{F})$-submodule with $L^0(\mathcal{F}) \subset \mathcal{X}^\ast \subset L^1_F(\mathcal{E})$.

We can define the so-called stable filter base and stable compactness introduced in $[19, Definition 3.12]$ and the notion of the conditional compactness introduced in $[19, Definition 3.5$ and $3.11$ of $[19]$ (the relation between both settings is expressed in the language of categories in $[37, Theorem 1.2]$ and $[31]$).

Let us introduce more notions:

- A stable filter base is a filter base $\mathcal{B}$ on $L^0(\mathcal{E})$ which is also a stable collection.
- A stable subset $S \subset \mathcal{X}^\ast$ is a stable filter base $\mathcal{B}$ on $S$ has a cluster point $x \in S$ w.r.t. the topology $\sigma(\mathcal{X}^\ast, \mathcal{X})$ (resp. $\sigma(\mathcal{X}^\ast, \mathcal{X})$-stably compact).

The notion of stable filter base and stable compactness were established in $[31]$ as a transcription in the framework of $L^0$-modules of the notion of conditional initial topologies introduced in $[19, Definition 3.12]$ and the notion of the conditional compactness introduced in $[19, Definition 3.24]$, respectively.

**Definition 1.1.** $[22]$ A function $f : \mathcal{X} \to \hat{L}^0(\mathcal{F})$ is said to be $\sigma(\mathcal{X}^\ast, \mathcal{X})$-lower semi-continuous if for any $\eta \in \hat{L}^0(\mathcal{F})$ it is satisfied that the sublevel set

$$V_\eta(\rho) := \{x \in \mathcal{X} : \rho(x) \leq \eta\}$$

is close w.r.t. $\sigma(\mathcal{X}^\ast, \mathcal{X})$.

Let us turn to our financial problem. We assume that the probability space $(\Omega, \mathcal{E}, \mathbb{P})$ models the information of a financial market that will be available at some time horizon $T > 0$, and the available market information at some future time $t$, $0 < t < T$, is encoded in the sub-$\sigma$-algebra $\mathcal{F} \subset \mathcal{E}$. Our model module $\mathcal{X}$ is going to describe all possible final payoffs $x : \Omega \to \mathbb{R}$ where $x(\omega)$ is the discounted net worth of the position at time $T$.

The risk at time $t$ arisen from a financial position $x \in \mathcal{X}$ is uncertain at time $t$ and contingent to the information encoded in $\mathcal{F}$. Thus this risk is quantified by a function $\rho : \mathcal{X} \to L^0(\mathcal{F})$ which satisfies:

1. **Monotonicity**: if $x \leq y$ in $\mathcal{X}$, then $\rho(x) \geq \rho(y)$;
2. **$L^0(\mathcal{F})$-cash invariance**: $\rho(x + \eta) = \rho(x) - \eta$ whenever $\eta \in L^0(\mathcal{F})$, $x \in \mathcal{X}$;
Then, $ho(x) = \rho(y) + (1-\eta)\rho(z)$ whenever $\eta \in L^0(\mathcal{F})$ with $0 \leq \eta \leq 1$ and $x, y \in \mathcal{F}$.

Such a function is called a conditional risk measure. This type of risk measure was introduced in [23] when $\mathcal{F} = L^p_\infty(\mathcal{E})$ with $1 \leq p \leq \infty$.

A conditional risk measure $\rho : \mathcal{F} \to L^0(\mathcal{F})$ is said to have:

- **The Fatou property** if for any sequence $(x_n) \subset \mathcal{F}$
  
  \[ y \in \mathcal{F}, |x_n| \leq y \text{ for all } n \in \mathbb{N}, \lim_n x_n = x \text{ in } \mathcal{F} \implies \liminf_n \rho(x_n) \geq \rho(x); \]

- **The Lebesgue property** if for any sequence $(x_n) \subset \mathcal{F}$
  
  \[ y \in \mathcal{F}, |x_n| \leq y \text{ for all } n \in \mathbb{N}, \lim_n x_n = x \text{ in } \mathcal{F} \implies \lim_n \rho(x_n) = \rho(x). \]

Now, we are ready to present the results:

**Theorem 1.1.** Let $\rho : \mathcal{F} \to L^0(\mathcal{F})$ a conditional risk measure and set

\[ \rho^*(y) := \text{ess. sup} \{ \mathbb{E}[\rho(y) - \rho(x)] : x \in \mathcal{F} \} \quad \text{for } y \in \mathcal{F}^\vee. \]

Then, $\rho$ is $\sigma(\mathcal{F}^\vee, \mathcal{F})$-lower semi-continuous if and only if $\rho$ admits a representation

\[ \rho(x) = \text{ess. sup} \{ \mathbb{E}[\rho(y) - \rho^*(y)] : y \leq 0, \mathbb{E}[y] = -1 \} \quad \forall x \in \mathcal{F}. \quad (1) \]

In this case, the following are equivalent:

1. The representation (1) is attained for each $x \in \mathcal{F}$, i.e. for every $x \in \mathcal{F}$ there exists $y \in \mathcal{F}^\vee$ with $y \leq 0$ and $\mathbb{E}[y] = -1$ such that

   \[ \rho(x) = \mathbb{E}[y] - \rho^*(y); \]

2. $\rho$ has the Lebesgue property;

3. For each $\eta \in L^0_\infty(\mathcal{F})$, \[ V_\eta(\rho^*) := \{ y \in \mathcal{F}^\vee : \rho^*(y) \leq \eta \} \]

   is $\sigma(\mathcal{F}^\vee, \mathcal{F})$-stably compact.

If $\mathcal{F} = L^1_\infty(\mathcal{E})$, we have the following:

**Theorem 1.2.** Let $\rho : L^1_\infty(\mathcal{E}) \to L^0(\mathcal{F})$ a conditional risk measure and set

\[ \rho^*(y) := \text{ess. sup} \{ \mathbb{E}[\rho(y) - \rho(x)] : x \in L^1_\infty(\mathcal{E}) \} \quad \text{for } y \in L^1_\infty(\mathcal{E}). \]

Then, $\rho$ has the Fatou property if and only if $\rho$ admits a representation

\[ \rho(x) = \text{ess. sup} \{ \mathbb{E}[\rho(y) - \rho^*(y)] : y \leq 0, \mathbb{E}[y] = -1 \} \quad \forall x \in \mathcal{F}. \quad (2) \]

In this case, the following are equivalent:

1. The representation (2) is attained for each $x \in L^1_\infty(\mathcal{E})$, i.e. for every $x \in L^1_\infty(\mathcal{E})$ there exists $y \in L^1_\infty(\mathcal{E})$ with $y \leq 0$ and $\mathbb{E}[y] = -1$ such that

   \[ \rho(x) = \mathbb{E}[y] - \rho^*(y); \]
2. \( \rho \) has the Lebesgue property;

3. For each \( \eta \in L^0_{++}(\mathcal{F}) \),
   \[
   V_0(\rho^*):=\{y \in L^1_\mathcal{F}(\mathcal{E}): \rho^*(y) \leq \eta\}
   \]
   is \( \sigma(\mathcal{L}^1_\mathcal{F}(\mathcal{E}), \mathcal{L}^\infty_\mathcal{F}(\mathcal{E})) \)-stably compact.

Briefly, let us explain the results for \( L^p \)-type modules \( (1 \leq p < \infty) \), Orlicz-type and Orlicz-heart-type modules. Suppose that \( \phi \) is a Young function and \( \psi(r):=\sup_{s \geq 0}\{rs-\phi(s)\} \) is its conjugate Young function. If \( \mathcal{X}=L^\psi_\mathcal{F}(\mathcal{E}) \), we will see that, as in the classical case, \( \mathcal{X}^z=L^\psi_\mathcal{F}(\mathcal{E}) \). Then Theorem 1.1 applies to the pairing \( \langle L^\phi_\mathcal{F}(\mathcal{E}), L^\psi_\mathcal{F}(\mathcal{E}) \rangle \).

If \( (p,q) \) are Hölder conjugates and \( \mathcal{X}=L^p_\mathcal{F}(\mathcal{E}) \), we will show that \( \mathcal{X}^z=L^q_\mathcal{F}(\mathcal{E}) \) and any conditional risk measure \( \rho:L^p_\mathcal{F} \to L^0(\mathcal{F}) \) admits a representation which is attained for any \( x \in L^q_\mathcal{F}(\mathcal{E}) \).

If \( \phi \) is finite-valued and \( \mathcal{X}=H^\phi_\mathcal{F}(\mathcal{E}) \), then we will see that \( \mathcal{X}^z=H^\phi_\mathcal{F}(\mathcal{E}) \). In this case, we will show that any conditional risk measure defined on \( H^\phi_\mathcal{F}(\mathcal{E}) \) admits a representation which is attained for any \( x \in H^\phi_\mathcal{F}(\mathcal{E}) \).

\[ \text{Remark 1.1.} \] \( \text{The topologies } \sigma(\mathcal{X}, \mathcal{X}^z) \text{ and } \sigma(\mathcal{X}^z, \mathcal{X}) \text{ are stable locally } L^0 \text{-convex topologies in the sense of } [31, \text{Definition 4.8}]. \text{ It is proven in } [31, \text{Proposition 5.2}] \text{ that this type of topologies are anti-compact (i.e. any compact set is finite) if the underlying probability space } (\Omega, \mathcal{F}, \mathbb{P}) \text{ does not have atoms. This shows that it is not possible to establish a robust representation theorem for conditional risk measures with full generality based on the classical notion of compactness, and the stably weak topologies is a suitable substitute for the classical compactness.} \]

2 \ Foundations of Boolean-valued models

The precise formulation of Boolean-valued analysis requires some familiarity with the basics of set theory and logic, and in particular with first order logic, ordinals and transfinite induction. For a detailed description we can refer the reader to \([6, 32, \text{Chapter 14}], \text{or } [35, \text{Chapter 2}]\). We make now a quick review.

Let us consider a universe of sets \( V \) satisfying the axioms of the Zermelo-Fraenkel set theory with the axiom of choice (ZFC), and a first-order language \( \mathcal{L} \) which allows the formulation of statements about the elements of \( V \). In the universe \( V \) we have all possible mathematical objects (real numbers, topological spaces, and so on.). The language \( \mathcal{L} \) consists of nouns for the elements of \( V \) plus a finite list of symbols for logic symbols (\( \forall, \land, \neg, \) and parenthesis), variables (with the symbol \( x \) we can express any variables we need as \( x, xx, xxx, \ldots \)) and the verbs = and \( \in \). Though we usually use a much richer language by introducing more and more intricate definitions, in the end any usual mathematical statement can be written using only those mentioned. The elements of the universe \( V \) are classified into a transfinite hierarchy:

\[
V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_\omega \subset V_{\omega+1} \subset \cdots,
\]

where \( V_0 = \emptyset \), \( V_{\alpha+1} = \mathcal{P}(V_\alpha) \) is the family of all sets whose elements come from \( V_\alpha \), and \( V_\beta = \bigcup_{\alpha < \beta} V_\alpha \) for limit ordinal \( \beta \).

Now consider our measure algebra \( \mathcal{A} \) which encodes the future market information. We will construct now \( V^{(\mathcal{A})} \), the \textit{Boolean-valued model} of \( \mathcal{A} \), whose elements are called \( \mathcal{A} \)-\textit{valued sets}, that we interpret as objects which we can talk about in the future time \( t \). We proceed by induction over the class Ord of ordinals of the universe \( V \). We start by defining \( V^{(\mathcal{A})}_0 := \emptyset \). If \( \alpha + 1 \) is the successor of the ordinal \( \alpha \), we define

\[
V^{(\mathcal{A})}_{\alpha+1} := \{u : u \text{ is an } \mathcal{A} \text{-valued function with } \text{dom}(u) \subset V^{(\mathcal{A})}_\alpha\}. 
\]
The idea is that for \( v \in \text{dom}(u) \), \( v \) will become an element of \( u \) in the future time \( t \) if \( u(v) \) happens. If \( \alpha \) is a limit ordinal \( V^{(A)}_\alpha := \bigcup_{\xi < \alpha} V^{(A)}_\xi \). Finally, let \( V^{(A)} := \bigcup_{\alpha \in \text{Ord}} V^{(A)}_\alpha \).

Given an element \( u \) in \( V^{(A)} \) we define its rank as the least ordinal \( \alpha \) such that \( u \) is in \( V^{(A)}_{\alpha+1} \).

We consider a first-order language which allows to produce statements about \( V^{(A)} \). Namely, let \( L^{(A)} \) be the first-order language which is the extension of \( L \) by adding names for each element of \( V^{(A)} \). Throughout, we will not distinguish between an \( A \)-valued set and its name.

Suppose that \( \varphi \) is any formula of the language \( L^{(A)} \), its Boolean truth value \( \llbracket \varphi \rrbracket \) is defined by induction in the length of \( \varphi \). If one got the right intuition, all the formulas that follow should look natural. We start by defining the Boolean truth value of the atomic formulas \( u \in v \) and \( u = v \) for \( u \) and \( v \) in \( V^{(A)} \). Namely, proceeding by transfinite recursion we define

\[
\llbracket u \in v \rrbracket = \bigvee_{t \in \text{dom}(v)} v(t) \land t = u,
\]

\[
\llbracket u = v \rrbracket = \bigwedge_{t \in \text{dom}(u)} (u(t) \Rightarrow \llbracket t \in v \rrbracket) \land \bigwedge_{t \in \text{dom}(v)} (v(t) \Rightarrow \llbracket t \in u \rrbracket),
\]

where, for \( a, b \in A \), we denote \( a \Rightarrow b := a^c \lor b \). For non-atomic formulas we have

\[
\llbracket \exists x \varphi(x) \rrbracket := \bigvee_{u \in V^{(A)}} \llbracket \varphi(u) \rrbracket \quad \text{and} \quad \llbracket \forall x \varphi(x) \rrbracket := \bigwedge_{u \in V^{(A)}} \llbracket \varphi(u) \rrbracket;
\]

\[
\llbracket \varphi \land \psi \rrbracket := \llbracket \varphi \rrbracket \land \llbracket \psi \rrbracket \quad \text{and} \quad \llbracket \neg \varphi \rrbracket := \llbracket \varphi \rrbracket^c.
\]

We say that two names \( u, v \) are equivalent when \( \llbracket u = v \rrbracket = I \). It is not difficult to verify that the truth value of a formula is not affected when we change a name by an equivalent one. However, the relation \( \llbracket u = v \rrbracket = I \) does not mean that the functions \( u \) and \( v \) (considered as elements of \( V \)) coincide. For example, the empty function \( u := \emptyset \) and the function \( v : \{\emptyset\} \to A \) with \( v(\emptyset) := 0 \) are different as functions; however, \( \llbracket u = v \rrbracket = I \). In order to avoid technical difficulties, we will construct the so-called separated universe. Namely, let \( \overline{V}^{(A)} \) be the subclass of \( V^{(A)} \) defined by choosing a representative of the least rank in each class of the equivalence relation \( \{(u, v) : \llbracket u = v \rrbracket = I \}\).

The universe \( V \) can be embedded into \( V^{(A)} \). Given a set \( x \) in \( V \), we define its canonical name \( \check{x} \) in \( V^{(A)} \) by transfinite induction. Namely, we put \( \emptyset := \emptyset \) and for \( x \) in \( V \) we define \( \check{x} \) to be the unique representative in \( \overline{V}^{(A)} \) of the name given by the function

\[
\{ y : y \in x \} \to \check{A} \quad y \mapsto \check{I}.
\]

Given a name \( u \) with \( \llbracket u \neq \emptyset \rrbracket = I \) we define its descent by

\[
u \downarrow := \{ v \in \overline{V}^{(A)} : \llbracket v \in u \rrbracket = I \}.
\]

Notice that, if \( u \in V^{(A)}_\alpha \), then any element of the class \( u \downarrow \) is also in \( V^{(A)}_\alpha \). Therefore, we have that \( u \downarrow \) is a set in \( V \).

If \( u \) is a name with \( \llbracket u \neq \emptyset \rrbracket = I \) and such that \( \llbracket v \neq \emptyset \rrbracket = I \) for any \( v \in u \downarrow \), we define its second descent by

\[
u \downarrow := \{ v \downarrow : v \in u \downarrow \}.
\]

It is well-known that every theorem of ZFC is true in \( V^{(A)} \) with the Boolean true value:

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1The construction can be done by transfinite induction. We put \( \overline{V}^{(A)}_{\alpha+1} = \{ u \in V^{(A)}_{\alpha+1} : \llbracket u = v \rrbracket < 1 \text{ for all } y \in V^{(A)}_\alpha \} \) and for a limit ordinal \( \alpha \) we put \( V^{(A)}_\alpha := \bigcup_{\xi < \alpha} \overline{V}^{(A)}_\xi \). The class \( \overline{V}^{(A)} \) is frequently defined in literature and is called the separated universe, see eg [35, 32].
Theorem 2.1. (Transfer Principle) If \( \varphi \) is a theorem of ZFC, then \( \llbracket \varphi \rrbracket = I \).

Also, it will be important to keep in mind the following results, which will allow to manipulate \( V^{(A)} \) and are well-known within Boolean-valued models theory:

**Theorem 2.2. (Maximum Principle)** Let \( \varphi(x) \) be a formula with one free variable \( x \). Then there exists an element \( u \) of \( V^{(A)} \) such that \( \llbracket \varphi(u) \rrbracket = \llbracket \exists x \varphi(x) \rrbracket \).

**Theorem 2.3. (Mixing Principle)** Let \( (a_k) \in p(I) \) and let \( (u_k) \) be a sequence of elements of \( V^{(A)} \). Then there exists a name \( u \) such that \( \llbracket u = u_k \rrbracket \geq a_k \) for all \( k \in \mathbb{N} \). Moreover, if \( v \) is another element of \( V^{(A)} \) satisfying \( \llbracket v = u_k \rrbracket \geq a_k \) for all \( k \in \mathbb{N} \), then \( \llbracket u = v \rrbracket = I \).

Given a partition \( (a_k) \in p(I) \) and a sequence \( (u_k) \) of elements of \( V^{(A)} \), we denote by \( \sum_k u_k a_k \), the unique name \( u \) in \( V^{(A)} \) satisfying \( \llbracket u = u_k \rrbracket \geq a_k \) for all \( k \in \mathbb{N} \).

If \( u \) is a name which satisfies \( \llbracket u \text{ “is a function”} \rrbracket = I \), that is, \( u \) satisfies the definition of function in the language \( L^{(F)} \), then we say that \( u \) is a name for a function. Of course, this can be done for any mathematical concept, thus we will talk about names for a topology, names for a vector space, names for a topology, and so on.

We will use the following notational convention. Suppose that \( F(x_1, \ldots, x_n) \) is a function symbol which could be introduced in the language \( L^{(A)} \) to give a definition based on \( x_1, \ldots, x_n \). For instance, if one wants to introduce in \( L^{(A)} \) the symbol “\( \{x_1, x_2\} \)” to denote the doubleton of \( x_1 \) and \( x_2 \), we can consider the function symbol \( F(x_1, x_2) \) := \( \{x_1, x_2\} \). Let \( u_1, \ldots, u_n \in V^{(A)} \).

By applying first the transfer principle and then the maximum principle, there is a unique name \( u \) in \( V^{(A)} \) such that \( \llbracket u = F(u_1, \ldots, u_n) \rrbracket = I \). We will denote the canonical representative \( u \) in \( V^{(A)} \) by \( F(u_1, \ldots, u_n)_A \), which is called the interpretation of \( F(u_1, \ldots, u_n) \). Thus, if \( F(x_1, x_2) = \{x_1, x_2\} \), given to names \( u_1, u_2 \) in \( V^{(A)} \), we denote by \( \{u_1, u_2\}_A \) the interpretation of the doubleton of \( u_1 \) and \( u_2 \).

Suppose that \( u, v \) are two names with \( \llbracket (u \neq \emptyset) \land (v \neq \emptyset) \rrbracket = I \). A function \( f : u \downarrow \rightarrow v \downarrow \) such that

\[
\llbracket w = t \rrbracket \leq \llbracket f(w) = f(t) \rrbracket \quad \text{for all} \quad w, t \in u \downarrow
\]

is called extensional.

The following result will be useful later:

**Theorem 2.4.** [39, Theorem 2.5.6] Let \( u, v \) be names with \( \llbracket (u \neq \emptyset) \land (v \neq \emptyset) \rrbracket = I \) and suppose that \( f : u \downarrow \rightarrow v \downarrow \) is an extensional function. Then there exists \( w \) in \( V^{(A)} \), which is a name for a function from \( u \) to \( v \), such that \( \llbracket f(t) = w(t) \rrbracket = I \) for all \( t \in u \downarrow \).

Let us see a variation of the result above.

Suppose that \( u, v, w \) are names with \( \llbracket (u \neq \emptyset) \land (v \neq \emptyset) \land (w \neq \emptyset) \rrbracket = I \). A function \( f : u \downarrow \times v \downarrow \rightarrow w \downarrow \) is called extensional if

\[
\llbracket w_1 = t_1 \rrbracket \land \llbracket w_2 = t_2 \rrbracket \leq \llbracket f(w_1, w_2) = f(t_1, t_2) \rrbracket \quad \text{for all} \quad w_1, t_1 \in u \downarrow, w_2, t_2 \in v \downarrow.
\]

We have the following:

**Theorem 2.5.** [39, Theorem 3.7] Let \( u, v, w \) be names with \( \llbracket (u \neq \emptyset) \land (v \neq \emptyset) \land (w \neq \emptyset) \rrbracket = I \) and suppose that \( f : u \downarrow \times v \downarrow \rightarrow w \downarrow \) is an extensional function. Then there exists \( g \) in \( V^{(A)} \), which is a name for a function from \((u \times v)_A\) to \( w \), such that \( \llbracket f(t_1, t_2) = g(t_1, t_2) \rrbracket = I \) for all \( t_1 \in u \downarrow, t_2 \in v \downarrow \).
The set of real numbers is a definable notion of ZFC. We will denote by $\mathbb{R}_A$ the unique name in $\mathcal{V}^{(A)}$ which satisfies the definition of real numbers. Likewise, we will denote by $\mathbb{N}_A$ the unique name in $\mathcal{V}^{(A)}$ which satisfies the definition of natural numbers.

It is well-known that $L^0(\mathcal{F})$ can be identified with $\mathbb{R}_A \downarrow$, see eg [41, Chapter 2, Section 3]. More precisely speaking, there is a bijection $\Phi : L^0(\mathcal{F}) \rightarrow \mathbb{R}_A \downarrow, \eta \mapsto \eta^\star$, such that:

(i) $\Phi(L^0(\mathcal{F}, \mathbb{N})) = \mathbb{N}_A \downarrow$;
(ii) $[0^\star = 0] = I, [1^\star = 1] = I, [\eta^\star + \xi^\star = (\eta + \xi)^\star] = I$ and $[\eta^\star \xi^\star = (\eta \xi)^\star] = I$ for all $\eta, \xi \in L^0(\mathcal{F})$;
(iii) $[\eta^\star = \xi^\star] = \bigvee\{a \in A : 1_a \eta = 1_a \xi\}$ and $[\eta^\star \leq \xi^\star] = \bigvee\{a \in A : 1_a \eta \leq 1_a \xi\}$ for all $\eta, \xi \in L^0(\mathcal{F})$.

Given $u \in \mathbb{R}_A \downarrow$, we denote by $u^\circ$, the unique $\eta \in L^0(\mathcal{F})$ such that $\eta^\star = u$.

The next results allow to connect sequences of $L^0(\mathcal{F})$ and names for sequences of real numbers.

**Proposition 2.1.** ([41] Proposition 2.2.1) Suppose that $u$ is a name for a sequence of real numbers, i.e. $[u : \mathbb{N} \rightarrow \mathbb{R}] = I$. Then $\lim_n u(n)^\circ = \eta$ in $L^0(\mathcal{F})$ if and only if $[\lim_n u(n) = \eta^\star] = I$.

If $(x_n) \subset L^0(\mathcal{E})$, we can define $(x_h)_{h \in L^0(\mathcal{F}, \mathbb{N})}$ where $h = \sum_{k \in \mathbb{N}} 1_{\{h=k\}} x_k$ and the function $\mathbb{N}_A \downarrow \rightarrow \mathbb{R}_A \downarrow, u \mapsto x_u$, is extensional. Due to Theorem 2.4 we can find a name $e$ for a sequence in $\mathbb{N}$ (i.e. $[v : \mathbb{N} \rightarrow \mathbb{N}] = I$) with $[\forall n \in \mathbb{N}(x_e^\star = v(n))] = I$.

Then, as a consequence of Proposition 2.1, we have:

**Proposition 2.2.** If $(x_n)$ is a sequence in $L^0(\mathcal{F})$, then $\lim_n x_n = x$ in $L^0(\mathcal{F})$ if and only if $[\lim_n x_n^\star = x^\star] = I$.

### 3 The boolean-valued approach

Let us go back to our model probability space $(\Omega, \mathcal{E}, \mathbb{P})$ with $\mathcal{F} \subset \mathcal{E}$ and let $\mathcal{A}_E$ denote the measure algebra associated to $\mathcal{E}$, all in the universe $\mathcal{V}$ of sets.

We shall define some names in the universe $\mathcal{V}^{(A)}$.

- For any $e \in \mathcal{A}_E$ we define $\tilde{e}$ to be the representative in $\mathcal{V}^{(A)}$ of the name given by the function $\{f : f \in \mathcal{A}_E \} \rightarrow \mathcal{A}, f \mapsto \bigvee\{a \in A : a \land e = a \land f\}$.
- Let $\mathcal{A}_E \uparrow$ denote the representative in $\mathcal{V}^{(A)}$ of the name given by the function $\{e : e \in \mathcal{A}_E \} \rightarrow \mathcal{A}, e \mapsto I$.

**Proposition 3.1.** The map $\iota_E : \mathcal{A}_E \rightarrow \mathcal{A}_E \uparrow, e \mapsto \tilde{e}$ is a bijection such that for any partition $(a_k) \in p(\mathcal{I})$ and countable family $(e_k) \subset \mathcal{A}_E$ it is satisfied that $\iota_E(\bigvee_k a_k \land e_k) = \sum_k \iota_E(e_k) a_k$.

\[\text{Given } (x_n), \text{ throughout, we will use } (x_h) \text{ without further explanations.}\]
Proof. First, we will prove that $\lceil \hat{e} = \hat{f} \rceil = a_{e,f}$ where $a_{e,f} := \vee \{ a \in \mathcal{A} : a \land e = a \land f \}$. Indeed, we have

$$\lceil \hat{e} = \hat{f} \rceil = \bigwedge_{g \in \mathcal{A}_e} (a_{e,g}^c \lor \lceil \hat{g} \rceil \cap \bigwedge_{h \in \mathcal{A}_e} (a_{f,h}^c \lor \lceil \hat{h} \rceil)).$$

(3)

In addition, if $g, h \in \mathcal{A}_e$, it is not difficult to compute that $\lceil \hat{g} \rceil = a_{g,f}$ and $\lceil \hat{h} \rceil = a_{h,e}$. Then, replacing in (3), we obtain

$$\lceil \hat{e} = \hat{f} \rceil = \bigwedge_{g \in \mathcal{A}_e} (a_{e,g}^c \lor a_{g,f}) \cap \bigwedge_{h \in \mathcal{A}_e} (a_{f,h}^c \lor a_{h,e}).$$

By taking above $g = e$ and $h = f$, it follows that $\lceil \hat{e} = \hat{f} \rceil \leq a_{e,f}$. Now, if we prove that $a_{e,f} \leq a_{e,g}^c \lor a_{g,f}$ for all $g \in \mathcal{A}_e$, we obtain the desired (by symmetry we also have $a_{e,f} \leq a_{f,h}^c \lor a_{h,e}$ for all $h$). To reach a contradiction, suppose that $0 < a := a_{e,f} \land (a_{e,g}^c \lor a_{g,f})^c$. Since $a_{e,f} \land e = a_{e,f} \land f$ and $a \leq a_{e,f} \land a_{e,g}$, one has $a = a \land e = a \land f = a \land g$. But $a \leq a_{e,g}^c$, hence $a \land e \neq a \land g$, which is a contradiction.

Let us show that $i_e$ is injective. If $\hat{e} = \hat{f}$, then $a_{e,f} = \lceil \hat{e} = \hat{f} \rceil = I$. This means that $e = f$.

Let us prove the surjectivity. If $u \in \mathcal{A}_e \uparrow$, then $\Omega = \lceil u \rceil \subseteq \mathcal{A}_e \uparrow$. Then we can find $(a_k) \in p(I)$ and $(e_k) \subset \mathcal{A}_e$ so that $\lceil u = \hat{e}_k \rceil \geq a_k$ for all $k \in \mathbb{N}$. Set $e := \bigvee_k e_k \land a_k$. One has $\lceil \hat{e} = \hat{e}_k \rceil = a_{e,e_k} \geq a_k$ for each $k$. By the mixing principle, we have that $\lceil u = \hat{e} \rceil = I$ and taking representatives we obtain $u = \hat{e}$. Thus $e$ is the desired preimage of $u$.

A identical argument shows that for any partition $(a_k) \in p(I)$ and countable family $(e_k) \subset \mathcal{A}_e$ it is satisfied that $i_e(\bigvee_k a_k \land e_k) = \sum i_e(e_k)a_k$. □

**Proposition 3.2.** $\mathcal{A}_e \uparrow$ is a name for a Dedekind $\sigma$-complete boolean algebra. Moreover, for any $e, f \in \mathcal{A}_e$, one has $\lceil (e \lor f)^\sim = \hat{e} \lor \hat{f} \rceil = I$, $\lceil (e \land f)^\sim = \hat{e} \land \hat{f} \rceil = I$, $\lceil (e^c)^\sim = \hat{e}^\sim \rceil = I$, $\lceil [0 = 0] \rceil = I$ and $\lceil [0 = 1] \rceil = I$. Further, if $u$ is a name for a non-empty countable family in $\mathcal{A}_e \uparrow$, then there exists a countable family $(e_n) \subset \mathcal{A}_e$ such that $\lceil [\bigvee_k e_k]^\sim = \bigvee u \rceil = I$ and $u \subseteq \{ \hat{e}_h : h \in L^0(\mathcal{F}, \mathbb{N}) \}$ where $e_h := \bigvee_{k \in \mathbb{N}} \{ h = k \} \land e_k$.

**Proof.** We consider the functions

$$\lambda_\lor : \mathcal{A}_e \uparrow \times \mathcal{A}_e \uparrow \to \mathcal{A}_e \uparrow \downarrow \quad \lambda_\lor(\hat{e}, \hat{f}) := (e \lor f)^\sim;$$

$$\lambda_\land : \mathcal{A}_e \uparrow \times \mathcal{A}_e \uparrow \to \mathcal{A}_e \uparrow \downarrow \quad \lambda_\land(\hat{e}, \hat{f}) := (e \land f)^\sim;$$

$$\lambda_\ast : \mathcal{A}_e \uparrow \to \mathcal{A}_e \uparrow \downarrow \quad \lambda_\ast(\hat{e}) := (e^c)^\sim.$$
For $e \in \mathcal{A}_E$, we define the conditional probability of $e$ by $P(e|\mathcal{F}) := \mathbb{E}_P[1_e|\mathcal{F}]$.

Since $e \mapsto \tilde{e}$ is a bijection the map

$$\mathcal{A}_\mathcal{E} \uparrow \downarrow \mapsto \mathbb{R}_A \downarrow, \quad \tilde{e} \mapsto P(e|\mathcal{F})$$

is well-defined. This function is also extensional, thus Theorem 2.4 yields a name $\tilde{P}$ for a function from $\mathcal{A}_\mathcal{E} \uparrow$ to $\mathbb{R}_A$ such that

$$[P(e|\mathcal{F})] = \tilde{P}(\tilde{e}) = I \quad \text{for all } e \in \mathcal{E}.$$

The following result is simple to verify, so we omit the proof.

**Proposition 3.3.** $(\mathcal{A}_\mathcal{E} \uparrow, \tilde{P})_A$ is a name for a probability algebra.

The proposition above asserts that $(\mathcal{A}_\mathcal{E} \uparrow, \tilde{P})_A$ is a name for a probability algebra. However, it is not clear if $(\mathcal{A}_\mathcal{E} \uparrow, \tilde{P})_A$ is a name for a measure algebra of some probability space.

We have the following variation of the Stone theorem:

**Proposition 3.4.** [26, Theorem 321J] Any measure algebra is isomorphic, as measure algebra, to the measure algebra of some measure space. In particular, any probability algebra is isomorphic, as probability algebra, to the measure algebra of some probability space.

The transfer principle applied to the proposition above yields the following:

**Proposition 3.5.** There exists a name $(\mathcal{A}_\Sigma, Q)_A$ for the measure algebra of a probability space (where $\Sigma$ is a name for a $\sigma$-algebra of sets) such that $(\mathcal{A}_\mathcal{E} \uparrow, \tilde{P})_A$ and $(\mathcal{A}_\Sigma, Q)_A$ are names for isomorphic probability algebras.

For given $e \in \mathcal{A}_E$, let $e^*$ be in $\mathcal{A}_\Sigma \downarrow$ the image of $\tilde{e}$ via the name of the isomorphism provided in the proposition above. Then $e \mapsto e^*$ is a bijection from $\mathcal{A}_E$ to $(\mathcal{A}_\Sigma)_A \downarrow$, which preserves the boolean operations. We will denote by $u \mapsto u^*$ the inverse map.

Let $(L^0(\Sigma))_A$ be the name for the space of (classes of equivalence) of $\Sigma$-measurable random variables. For given $x \in L^0(\mathcal{E})$, we have that $\bigvee_{\eta \in L^0(\mathcal{F})} \{x \geq \eta\} = I$, $\bigwedge_{\eta \in L^0(\mathcal{F})} \{x \geq \eta\} = 0$ and for any $\xi \in L^0(\mathcal{F})$ we have $\{x \geq \xi\} = \bigvee_{\eta \geq \xi} \{x \geq \eta\}$. Thus $[\bigvee_{r \in \mathbb{R}} \{x \geq r^+\}] = I$, $[\bigwedge_{r \in \mathbb{R}} \{x \geq r^+\}] = 0$ and $[\forall s \in \mathbb{R} \{x \geq s^+\}] = \bigvee_{r \geq x} \{x \geq r^+\} = I$. By means of the transfer principle, we derive the existence of a unique element $x^* \in (L^0(\Sigma))_A \downarrow$ satisfying

$$\left[ x^* = \text{ess. sup}_{r \in \mathbb{R}} \{r1_{\{x \geq r^+\} - \infty 1_{\{x < r^+\}}\} \right] = I.$$

We have that $x \mapsto x^*$ is a canonical bijection from $L^0(\mathcal{E})$ to $(L^0(\Sigma))_A \downarrow$ which extends the canonical bijection $\eta \mapsto \eta^*$ from $L^0(\mathcal{F})$ to $\mathbb{R}_A \downarrow$.

**Proposition 3.6.** The map $x \mapsto x^*$ is an isomorphism of lattice rings between $L^0(\mathcal{E})$ and $(L^0(\Sigma))_A \downarrow$ which sends $L^0(\mathcal{F})$ to $\mathbb{R}_A \downarrow$.

**Proof.** By means of maximum principle, $(L^0(\Sigma))_A \downarrow$ inherits the lattice rings operations of $(L^0(\Sigma))_A$.

Given $x \in L^0(\mathcal{E})$, it can be verified that $\{x \geq \eta\} = \{x^* \geq \eta^*\}_A$ for all $\eta \in L^0(\mathcal{F})$. In particular, if $x^* = y^*$, then we have that $\{x \geq \eta\} = \{y \geq \eta\}$ for all $\eta \in L^0(\mathcal{F})$. Thus, $x = y$. This proves the injectivity.

If $u \in (L^0(\Sigma))_A \downarrow$, then $[u = \text{ess. sup}_{r \in \mathbb{R}} \{r1_{\{u \geq r\} - \infty 1_{\{u < r\}}\} \right] = I$. We have that

$$x := \text{ess. sup}_{\eta \in L^0(\mathcal{F})} \{\eta1_{\{u \geq \eta^*\} - \infty 1_{\{u < \eta^*\}}^+} \}
$$

is a preimage of $u$. The rest of properties follow by inspection.
Lemma 3.1. Let \((x_n) \subset L^0(\mathcal{E})\). Then \(\lim_n x_n = x\) in \(L^0(\mathcal{E})\) if and only if \(\lim_n x_n^* = x^*\) if.

Proof. For any \(h \in L^0(\mathcal{F},\mathbb{N})\), define \(x_h := \sum_{k\in\mathbb{N}}1\{h=k\}x_k\). The map \(\mathbb{N}_A \rightarrow (L^0(\Sigma))_A \downarrow\) \(u \mapsto u^*\) is extensional. We can find a name \(u\) with \(\|u : \mathbb{N} \rightarrow L^0(\Sigma)\| = I\) and so that \(\|x_h^* = u(h^*)\| = I\) for all \(h \in L^0(\mathcal{F},\mathbb{N})\).

Let \(h, l\) denote elements of \(L^0(\mathcal{F},\mathbb{N})\) and \(n, m\) denote natural numbers.

On one hand, we have that

\[
\limsup_n x_n^* = \left(\inf_i \sup_{h \geq l} x_h^*\right) = I.
\]

On the other, we have that \(\inf_i \sup_{h \geq l} x_h^* = \sup_{m \geq m} x_n^*\).

Thus

\[
\limsup_n x_n^* = \left(\inf_n x_n^*\right) = I.
\]

Similarly, we have \(\liminf_n x_n^* = \left(\inf_n x_n^*\right) = I\). Thus, we obtain the assertion. \qed

Let \((L^1(\Sigma))_A\) be the interpretation of a name for the space of (classes of equivalence) of \(\Sigma\)-measurable random variables with finite expectation. We have the following:

Proposition 3.7. The map \(x \mapsto x^*\) is a bijection between \(L^1(F)\) and \((L^1(\Sigma))_A \downarrow\). Moreover, \([E_p[x|F]^* = E_Q[x^*]] = I\) for all \(x \in L^1_F(\mathcal{E})\).

Proof. Given \(e \in A_{\mathcal{E}}, \) since \(E_p[1_e|F] = P(e|F), \) \([P(e|F)^* = Q(e^*)]|F = I\), \([Q(e^*)]_A = E_Q[1_e|F] = I\) and \(1_e = 1_e^*\), we have that \([E_p[1_e|F]^* = E_Q[1_e^*]|F = I]\).

If \(s \in L^0(\mathcal{E})^*\) is a simple function, by linearity it follows \([E_p[s|F]^* = E_Q[s]|F = I]\).

Now suppose that \(x \in L^1_F(\mathcal{E})\). We can find a sequence \((s_n)\) of simple functions such that \(0 \leq s_n \not\rightarrow x\).

For each \(h \in L^0(\mathcal{F},\mathbb{N})\), define \(s_h = \sum_k1\{h=k\}s_k\). Lemma 3.1 proves that \([E_p[s_n|F]^* = E_Q[x^*]|F = I]\).

Let \(\eta \in L^0(F), \) \((\eta_n) \subset L^0(F)\) with \([\eta^* = E_Q[x^*]|F = I\) and \([\eta_n^* = E_Q[s_n]|F = E_Q[x|F]|F\) for all \(n \in \mathbb{N}\). Notice that \(\eta_n = E_p[s_n|F]\). Then, by Lemma 3.1, we have that \([E_p[s_n|F] = \eta_n]\)

and by monotone convergence \(\lim_n E_p[s_n|F] = \eta, \) thereby, we conclude that \([E_p[x|F] = E_Q[x^*]|F = I]\).

By linearity, the equality extends to any \(x \in L^1_F(\mathcal{E})\). This also proves that \(x^* \in (L^1(\Sigma))_A \downarrow\) whenever \(x \in L^1_F(\mathcal{E})\).

Now, if \(u \in (L^1(\Sigma))_A \downarrow, \) take \(x := u^0 \in L^0(\mathcal{E})\). Then, for any \(h \in L^0(\mathcal{F},\mathbb{N}), \) it follows that \(|x| \land h \in L^1_F(\mathcal{E})\). Therefore, we have \([E_p[|x| \land h|F]^* = E_Q[|x| \land h]|F = I]\). By the transfer principle applied to the monotone convergence theorem we obtain that \([\lim_n E_p[|x| \land n^0|F]^* = E_Q[|x^0|]|F = I]\). By Lemma 2.1, we conclude that \([\lim_n E_p[|x| \land n|F] = \text{finite}\) and thus \(x \in L^1_F(\mathcal{E})\). \qed

The next result is an immediate consequence of Proposition 3.7.

Corollary 3.1. If \(1 \leq p < \infty\), then \(x \mapsto x^*\) is bijection between \(L^p_F(\mathcal{E})\) and \((L^p(\Sigma))_A \downarrow\). If \(\phi\) is a Young function, then \(x \mapsto x^\phi\) is a bijection between \(L^p_F(\mathcal{E})\) and \((L^p(\Sigma))_A \downarrow, \) and between \(H^p_F(\mathcal{E})\) and \((H^p(\Sigma))_A \downarrow\).

Proposition 3.7 and Corollary 3.1 show that the \(L^p\)-type modules, the Orlicz-type modules and the Orlicz-heart-type modules can be seen as interpretations of \(L^p\) spaces, Orlicz spaces and Orlicz-heart spaces of random variables, respectively, within the universe \(\mathcal{V}(A)\). Thus, the transfer principle will allow us to interpret theorems on \(L^p\) spaces, Orlicz spaces and Orlicz-heart spaces of random variables as theorem for \(L^p\)-type modules, the Orlicz-type modules and the Orlicz-heart-type modules.

Now, let us define some relevant names:
• If $S$ is a stable subset of $L^0(\mathcal{E})$, let $S \uparrow$ denote the unique name in $\overline{V}(A)$ equivalent to

$$\{ x^\bullet : x \in S \} \rightarrow A \quad x^\bullet \mapsto I.$$  

• If $\mathcal{C}$ is a stable collection of subsets of $L^0(\mathcal{E})$, we denote by $\mathcal{C} \uparrow$ the unique name in $\overline{V}(A)$ equivalent to

$$\{ S \uparrow : S \in \mathcal{C} \} \rightarrow A \quad S \mapsto I.$$  

A similar argument to Proposition 3.1 allows to prove the following:

**Proposition 3.8.** Let $S \subset L^0(\mathcal{E})$ be stable, then $[\emptyset \neq S \uparrow \subset L^0(\Sigma)] = I$. Moreover, the map $i_S : S \rightarrow S \uparrow \downarrow$, $x \mapsto x^\bullet$, is a bijection such that $i_S(\sum_k 1_{a_k} x_k) = \sum i_S(x_k) a_k$ whenever $(a_k) \in p(I)$ and $(x_k) \subset S$.

Let $\mathcal{C}$ be a stable collection of subsets of $L^0(\mathcal{E})$, then $[\emptyset \neq \mathcal{C} \uparrow \subset P(L^0(\Sigma))] = I$. Moreover, the map $i_\mathcal{C} : \mathcal{C} \rightarrow \mathcal{C} \uparrow \downarrow$, $S \mapsto S \uparrow \downarrow$, is a bijection such that $i_\mathcal{C}(\sum_k 1_{a_k} S_k) = \sum i_\mathcal{C}(S_k) a_k$ whenever $(a_k) \in p(I)$ and $(S_k) \subset \mathcal{C}$.

Let us go back to our model solid stable $L^0(\mathcal{F})$-submodule $\mathcal{X}$. It can be verified that $\mathcal{X} \uparrow$ is a name for a solid subspace of $(L^0(\Sigma), A)$. Further, since $\mathcal{X} \subset L^0(\mathcal{E})$, it is fulfilled that $[\emptyset \subset \mathcal{X} \uparrow \subset L^0(\Sigma)] = I$.

Notice that the collection $\mathcal{U}_{\sigma, (\mathcal{X}, \mathcal{X}^\pi)}$ is stable. Therefore, we can consider the name $\mathcal{U}_{\sigma, (\mathcal{X}, \mathcal{X}^\pi)} \uparrow$. The proof of the next result is easy to find and therefore omitted.

**Proposition 3.9.** $\mathcal{U}_{\sigma, (\mathcal{X}, \mathcal{X}^\pi)} \uparrow$ (resp. $\mathcal{U}_{\sigma, (\mathcal{X}^\pi, \mathcal{X})} \uparrow$) is a name for a base for the weak topology induced by the pairing $(\mathcal{X} \uparrow, \mathcal{X}^\pi \uparrow)_A$ (resp. $(\mathcal{X}^\pi \uparrow, \mathcal{X} \uparrow)^*_A$).

Moreover, we have:

**Proposition 3.10.** Let $S \subset \mathcal{X}$ (resp. $S \subset \mathcal{X}^\pi$) be stable. Then $S$ is $\sigma_s(\mathcal{X}, \mathcal{X}^\pi)$-closed (resp. $\sigma_s(\mathcal{X}^\pi, \mathcal{X})$-closed) if and only if $S \uparrow$ is a name for a $\sigma(\mathcal{X}, \mathcal{X}^\pi)$-closed (resp. $\sigma(\mathcal{X}^\pi, \mathcal{X})$-closed) set.

**Proof.** Let $\mathcal{U} := \mathcal{U}_{\sigma, (\mathcal{X}, \mathcal{X}^\pi)}$. Note that $[\bigcap_{U \in \mathcal{U}} (S \uparrow + U) = S \uparrow] = I$ if and only if $S = \bigcap_{U \in \mathcal{U}} (S + U)$. Then the result follows. □

**Proposition 3.11.** Let $S \subset \mathcal{X}$ (resp. $S \subset \mathcal{X}^\pi$) be stable. Then $S$ is $\sigma_s(\mathcal{X}, \mathcal{X}^\pi)$-stably compact (resp. $\sigma_s(\mathcal{X}^\pi, \mathcal{X})$-stably compact) if and only if $S \uparrow$ is a name for a $\sigma(\mathcal{X}, \mathcal{X}^\pi)$-compact (resp. $\sigma(\mathcal{X}^\pi, \mathcal{X})$-compact) set.

**Proof.** Suppose that $[S \uparrow \text{“is compact”}] = I$. If $\mathcal{B}$ is a stable filter base on $S$, it can be verified that $\mathcal{B} \uparrow$ is a name for a filter base on $S \uparrow$. By the maximum principle, there exists a name $u \in S \downarrow \uparrow$ for a cluster point of $\mathcal{B} \uparrow$. Then $u^\circ$ is a cluster point of $\mathcal{B}$.

Conversely, suppose that $S \subset \mathcal{X}$ is stably compact. Let $u$ be a name for a filter base on $S \subset \mathcal{X}$. Set $\mathcal{B} := \{ i_{\wp}^{-1}(w) : w \in u \downarrow \}$. Then inspection shows that $\mathcal{B}$ is a stable filter base on $S$. Since $S$ is stably compact, we have that $\mathcal{B}$ has a cluster point $x \in S$. It can be verified that $x^\bullet$ is a name for a cluster point of $u$. □

Proposition 3.8 allows to prove the following:

**Lemma 3.2.** Let $S \subset \mathcal{X}$ be stable, let $f : \mathcal{X} \rightarrow L^0(\mathcal{F})$ be a function with the local property and suppose that $f$ is a name for a function from $\mathcal{X} \uparrow$ to $\mathbb{R}_A$ such that $[f(x^\bullet) = (f(x))^\bullet] = I$ for all $x \in S$. Then

$$[\sup_{u \in S \uparrow} f(u) = (\text{ess} \sup_{x \in S} f(x))^\bullet] = I.$$
At this point we are ready to apply the transfer principle to verify the statements presented in Section 1. With this aim, we will recall some known facts on (one-period) risk measures. Let $L^0$ be the space of (classes of equivalence of) random variables of an arbitrary probability space which we will not specify for simplicity in the exposition. Let $L^1$ be the space of integrable random variables of this probability space and let $E[\cdot]$ denote the expectation with respect its probability. Let us fix a solid subspace $X$ of $L^1$ (i.e. an order ideal) with $\mathbb{R} \subset X$. Recall that a **convex risk measure** is a convex function $\rho : X \to \mathbb{R}$ satisfying:

1. **monotonicity**: i.e. $\rho(x) \leq \rho(y)$ whenever $x \geq y$ in $X$;
2. **cash invariance**: i.e. $\rho(x + r) = \rho(x) - r$ for all $r \in \mathbb{R}$, $x \in X$.

In addition, $\rho$ has the Fatou (resp. Lebesgue) property if for every order bounded sequence $(x_n)$ with $\lim_n x_n = x$ it is fulfilled that $\liminf_n \rho(x_n) \geq x$ (resp. $\lim_n \rho(x_n) = x$).

Suppose that $\rho : X \to \mathbb{R}$ is a convex risk measure. The classical Köthe dual space of $X$ is defined by

$$Y := \{x \in L^0 : xy \in L^1, \forall x \in X\}.$$ 

As a consequence of the Fenchel-Moreau theorem (see [34] Theorem 2.1) applied to the weak topology $\sigma(X, Y)$, $\rho$ is $\sigma$-lower semi-continuous if and only if

$$\rho(x) = \sup_{y \in Y} \{E[xy] - \rho^*(y)\} \quad \text{for } x \in X,$$

with $\rho^*(y) := \sup_{x \in X} \{E[xy] - \rho(x)\}$ for $y \in Y$.

In this case, it is satisfied that $\rho^*(y) < \infty$ only if $y \leq 0$ and $E[y] = -1$. Thus the lower semi-continuity of $\rho$ is equivalent to

$$\rho(x) = \sup \{E[xy] - \rho^*(y) : E[y] = -1, y \leq 0\} \quad \text{for all } x \in X.$$

Now, let us go back to the initial setting. Suppose that $\rho : \mathcal{X} \to L^0(\mathcal{F})$ is a conditional risk measure. Since $\rho$ is $L^0(\mathcal{F})$-convex, we know from Theorem [22] Theorem 3.2 that $\rho$ has the local property, i.e. $1_a \rho(1_a x) = 1_a \rho(1_a) x$ for all $a \in \mathcal{A}$. This allows to show that the map $\mathcal{X} \uparrow \to \mathbb{R}_+, u \mapsto \rho(u^* \cdot)$ is extensional. We have a name $\tilde{\rho}$ for a function from $\mathcal{X}$ to $\mathbb{R}_+$ such that $\tilde{\rho}(x^*) = \rho(x^*)^* = I$ for all $x \in \mathcal{X}$. Moreover, $\tilde{\rho}$ is a name for a convex risk measure. We also have that $\rho^*$ has the local property; thus, we can find a name $\hat{\rho}^*$ with $\hat{\rho}^*: \mathcal{X}^* \uparrow \to \mathbb{R}_+$ such that $\hat{\rho}^*(y^*) = \rho^*(y^*)^* = I$ for all $y \in \mathcal{X}$. Then Lemma [3.2] yields that $\hat{\rho}^*(\rho^*)^* = I$.

Using that $\hat{\rho}^*(\rho^*)^* = \hat{\rho}^*$ we can prove:

**Proposition 3.12.** For any $x \in \mathcal{X}$,

$$[\sup \{E_Q[x^* v] - \hat{\rho}^*(v) : v \in \mathcal{X} \uparrow, E_Q[v] = -1, v \leq 0\} = (\esssup_M \{E_E[x y | \mathcal{F}] - \rho^*(y)\})^* \in I,$$

where $M := \{y \in \mathcal{X}_- : E_E[y | \mathcal{F}] = -1, y \leq 0\}$.

Then, due to Proposition [3.10] $\rho$ is $\sigma_0(\mathcal{X}, \mathcal{X}_-)$-l.s.c. if and if

$$\hat{\rho}^* \text{ is } \sigma(\mathcal{X}_-, \mathcal{X}_-)$-l.s.c.
Due to the transfer principle, the latter is equivalent to

$$\|\rho(x)\| = \sup \{ E_Q[x^*v] - \tilde{\rho}^*(v) : v \in \mathcal{H}, E_Q[v] = -1, v \leq 0 \} = I \quad \text{for all } x \in \mathcal{H},$$

and, in view of Proposition 3.12, this is equivalent to

$$\rho(x) = \text{ess. sup}_M \{ E_P[xy|\mathcal{F}] - \rho^*(y) \}.$$
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