FREE SUBALGEBRAS OF THE SKEW POLYNOMIAL
RINGS \( k[x, y][t; \sigma] \) AND \( k[x^{\pm 1}, y^{\pm 1}][t; \sigma] \)

S. PAUL SMITH

ABSTRACT. Let \( k \) be a field, \( R \) a commutative \( k \)-algebra, and \( \sigma \) a \( k \)-algebra automorphism of \( R \). The skew polynomial ring \( R[t; \sigma] \) is generated by \( R \) and an indeterminate \( t \) subject to the relations \( ta = \sigma(a)t \) for all \( a \in R \). This paper shows that for certain \( R \) and appropriate \( \sigma \) there are elements \( a, b \in R \) such that the subalgebra of \( R[t; \sigma] \) generated by \( at \) and \( bt \) is a free algebra. For example, if \( \sigma \) is an automorphism of the polynomial ring \( k[x, y] \), then the subalgebra of \( k[x, y][t; \sigma] \) generated by \( xt \) and \( yt \) is free if and only if \( \sigma \) is not conjugate to an automorphism of the form \( x \mapsto ax + p(y) \), \( y \mapsto by + c \), for any \( a, b, c \in k \) and \( p(y) \in k[y] \). Similarly, if \( \sigma \) is an automorphism of \( k[x^{\pm 1}, y^{\pm 1}] \) of the form \( \sigma(x) = x^a y^b \) and \( \sigma(y) = x^c y^d \), then the subalgebra of \( k[x^{\pm 1}, y^{\pm 1}][t; \sigma] \) generated by \( xt \) and \( yt \) is free if the spectral radius of \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) is \( > 2 \); indeed, \( k[x^{\pm 1}, y^{\pm 1}][t; \sigma] \) contains a free subalgebra if and only if the spectral radius of \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) is \( > 1 \).

1. INTRODUCTION

1.1. The main results. Let \( k \) be a field. Let \( R \) denote the commutative polynomial ring \( k[x, y] \) or its localization \( k[x^{\pm 1}, y^{\pm 1}] \). Let \( \sigma \) be a \( k \)-algebra automorphism of \( R \). The skew polynomial ring \( R[t; \sigma] \) is the vector space \( R \otimes_k k[t] \) with multiplication defined so that \( R \otimes 1 \) and \( 1 \otimes k[t] \) are subalgebras and \( t^n r = \sigma^n(r) t^n \) for all \( r \in R \) and \( n \geq 0 \).

We prove the following results.

Theorem 1.1. Let \( \sigma \) be an automorphism of \( k[x, y] \). Then \( k[x, y][t; \sigma] \) contains a free subalgebra if and only if \( \sigma \) is not conjugate to an automorphism of the form \( x \mapsto ax + p(y) \), \( y \mapsto by + c \), for any \( a, b, c \in k \) and \( p(y) \in k[y] \).

Theorem 1.2. Let \( M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}(2, \mathbb{Z}) \) and let \( \sigma \) be the automorphism of \( k[x^{\pm 1}, y^{\pm 1}] \) defined by \( \sigma(x) = x^a y^b \) and \( \sigma(y) = x^c y^d \). Then \( k[x^{\pm 1}, y^{\pm 1}][t; \sigma] \) contains a free subalgebra if and only if the spectral radius of \( M \) is \( > 1 \). In that case, \( k\{xt^{2n}, yt^{2n}\} \) is a free subalgebra of \( k[x^{\pm 1}, y^{\pm 1}][t; \sigma] \) for all \( n \) such that the spectral radius of \( M^{2n} \) is \( \geq 2 \).

Some of our results apply to skew polynomial rings over other commutative rings.

1991 Mathematics Subject Classification. 16S35, 16W20, 16S10.

Key words and phrases. Free algebras, automorphisms, skew polynomial rings, skew Laurent extensions.
1.2. **Skew Laurent extensions.** It is natural to view a skew polynomial ring $R[t;\sigma]$ as a subalgebra of a slightly larger algebra. Let $R$ be a commutative $k$-algebra and $\sigma \in \text{Aut}_k(R)$. The skew Laurent extension

$$R[t^{\pm 1};\sigma] := \bigoplus_{n \in \mathbb{Z}} R t^n$$

is defined by declaring that it is the free left $R$-module with basis $\{t^i \mid i \in \mathbb{Z}\}$ and multiplication defined to be the $k$-linear extension of $(ft^i)(gt^j) := f\sigma^i(g)t^{i+j}$ for all $f, g \in R$ and all $i, j \in \mathbb{Z}$. When $\sigma$ is the identity this is the ordinary ring of Laurent polynomials $R[t^{\pm 1}]$.

We make $R[t^{\pm 1};\sigma]$ a graded ring by setting $\deg(t) = 1$ and $\deg(R) = 0$.

1.3. We consider graded subalgebras

$$k\{at, bt\} := \bigoplus_{n=0}^{\infty} (Vt)^n \subset R[t;\sigma]$$

where $V := ka + kb$ is a 2-dimensional subspace of $R$. The degree-$(n + 1)$ component of $k\{at, bt\}$ is

$$V\sigma(V)\ldots\sigma^n(V)t^{n+1}$$

so $k\{at, bt\}$ is a free algebra if and only if $\dim_k (V\sigma(V)\ldots\sigma^n(V)) = 2^{n+1}$ for all $n \geq 0$.

1.4. **Conventions/Notation.** Throughout this paper $k$ is a field and $R$ a commutative $k$-algebra. We always assume that $R$ is an integral domain and write $K$ for its field of fractions. Whenever we say “free (sub)algebra” we will mean “free (sub)algebra on $\geq 2$ variables”.

We write $k[x, y]$ for the polynomial ring on two variables and $k[x^{\pm 1}, y^{\pm 1}]$ for its localization obtained by inverting $x$ and $y$.

If $(G, +)$ is an abelian group we write $G^\times$ for $G - \{0\}$.

We write $\rho(M)$ for the spectral radius of a matrix $M$.

We write $|S|$ for the cardinality of a set $S$.

If $u$ and $v$ are elements in a $k$-algebra $A$ we will write $k\{u, v\}$ for the $k$-subalgebra they generate.

1.5. **Relation to other work.**

1.5.1. **Free subalgebras of division algebras.** In 1983, Makar-Limanov discovered that the division ring of fractions of the ring of differential operators $\mathbb{C}[x, \partial/\partial x]$ contains a free subalgebra [8]. Since then the question of which division algebras contain free subalgebras has been of considerable interest.

A good account of recent progress on the question of which division algebras contain free subalgebras can be found in Bell and Rogalski’s paper [2]. They make further progress on this question in [3].
1.5.2. Despite the many results on free subalgebras of division algebras, there seem to be no known examples of free subalgebras of $R[t; \sigma]$ when $R$ is a finitely generated commutative $k$-algebra. Theorems 1.1 and 1.2 provide a host of such examples. Furthermore, since $R$ is a domain in those cases, $R[t; \sigma]$ has a division ring of fractions which then has a free subalgebra. Because the division algebra of fractions of an algebra $A$ is much larger than $A$ it is much easier to find free subalgebras of division algebras. For example, if $\sigma$ is the automorphism of the polynomial ring $C[z]$ defined by $\sigma(z) = z + 1$, then $C[z][t; \sigma]$ does not contain a free algebra\footnote{The ring $C[z][t; \sigma]$ is isomorphic to the enveloping algebra of the 2-dimensional non-abelian Lie algebra.} but its the division ring of fractions, Fract $(C[z][t; \sigma])$, is isomorphic to Fract $(C[x, \partial/\partial x])$ so Fract $(C[z][t; \sigma])$ has a free subalgebra.

1.5.3. Gelfand-Kirillov dimension. Gelfand-Kirillov dimension is a function $GKdim : \{k\text{-algebras}\} \to \mathbb{R}_{\geq 0} \cup \{\infty\}$. We refer the reader to [7] for its definition. We will use two properties:

(a) If $B$ is a free algebra on $\geq 2$ variables, then $GKdim(B) = \infty$.

(1) If $B$ is a subalgebra of $A$, then $GKdim(B) \leq GKdim(A)$.

Thus, an algebra of finite GK-dimension does not contain a free subalgebra.

1.5.4. Exponential growth. Let $A = A_0 \oplus A_1 \oplus \cdots$ be an $\mathbb{N}$-graded $k$-algebra. We say $A$ is connected if $A_0 = k$; locally finite if $\dim_k(A_n) < \infty$ for all $n$; finitely graded if it is locally finite and finitely generated as a $k$-algebra \cite{10}.

Such an $A$ has exponential growth if

$$\limsup_{n \to \infty} \left( \frac{\dim_k(A_n)}{n} \right)^{1/n} > 1.$$

The ur-example of an algebra with exponential growth is the free algebra, $k\langle x_1, \ldots, x_r \rangle$ on $r \geq 2$ variables each having degree 1: the dimension of its degree $n$ component is $r^n$.

1.5.5. It is natural to ask if a locally finite graded algebra having exponential growth has a graded free subalgebra on $\geq 2$ variables.

Golod and Shafarevich [5] showed the answer to that question is “no” by constructing a finitely generated, connected, graded $k$-algebra $A$ of exponential growth such that every element in $A_{\geq 1}$ is nilpotent. Their algebra is not finitely presented. The question of whether every finitely presented locally finite graded algebra having exponential growth contains a graded free subalgebra on $\geq 2$ variables remains open.

1.5.6. Given a specific algebra of exponential growth one can ask if it contains a free subalgebra on $\geq 2$ variables. This paper shows the answer is “yes” for various algebras of the form $R[t; \sigma]$. The question can be sharpened: given $R$ and $\sigma$, and particular $a, b \in R$, and an integer $n$, when is the subalgebra $k\{at^n, bt^n\}$ free? Most of our results that prove the existence
of a free subalgebra give explicit $a, b$ and $n$, such that $k\{at^n, bt^n\}$ is a free algebra.

1.5.7. Relation to the results in [12]. The results in this paper complement those in [12] where the following result is proved.

**Theorem 1.3.** Let $K/k$ be a finitely generated extension field of transcendence degree two. Let $X$ be a smooth projective surface such that $k(X) \cong K$.

1. If $\sigma$ is an automorphism of $X$ and also denotes the induced automorphism of $k(X)$, then $k(X)[t^\pm 1; \sigma]$ contains a free subalgebra if and only if the spectral radius of the automorphism of the Néron-Severi group induced by $\sigma$ is $> 1$.

2. If $\sigma$ is an automorphism of $K$, then $K[t^\pm 1; \sigma]$ contains a free subalgebra if and only if the dynamical degree of $\sigma$ is $> 1$.

The methods in this paper are more accessible to algebraists than those in [12]. Because the results in [12] are more general they are not as sharp as those in this paper. For example, suppose $\sigma$ is such that $k(x, y)[t; \sigma]$ contains a free subalgebra of the form $k\{at^n, bt^n\}$ for suitable $a, b \in k(x, y)$; the results in this paper generally produce a smaller $n$ than those in [12]. All the free algebras in [12] are of the form $k\{at^n, bt^n\}$ where the divisor of zeroes of $a^{-1}b$ is very ample. In this paper, some of the free algebras are of the form $k\{at^n, \sigma(a)t^n\}$ and the question of whether the divisor of zeroes of $a^{-1}\sigma(a)$ is very ample does not enter into the argument.

1.6. Acknowledgements. Dan Rogalski’s paper [9] prompted my interest in free subalgebras. He told me that a test case was the subalgebra $k\{xt, yt\} \subset k[x^{\pm 1}, y^{\pm 1}][t; \sigma]$ where $\sigma(x) = xy$ and $\sigma(y) = xy^2$. By Theorem 1.2 that subalgebra is free. I thank George Bergman for showing me that the argument I used to answer Rogalski’s question could be simplified, and improved to show that the subalgebra of $k[x^{\pm 1}, y^{\pm 1}][t; \sigma]$ generated by $xt^n$ and $yt^n$ is free for all $n \geq 1$. The argument used to prove Theorem 1.3 is, in part, based on Bergman’s ideas.

2. Observations

2.1. The question of whether $R[t; \sigma]$ contains a free subalgebra depends on the conjugacy class of $\sigma$ because if $\sigma, \tau \in \text{Aut}_k(R)$, there is a $k$-algebra isomorphism $\Phi : R[t; \sigma] \to R[s; \tau \sigma \tau^{-1}]$ given by $\Phi(ft^j) := \tau(f)s^j$ for $f \in R$ and $j \in \mathbb{N}$.

2.2. Let $a$ and $b$ be non-zero elements of $K$. We will show that $k\{at, bt\} \cong k\{t, a^{-1}bt\}$.

Define $a_0 = 1$. For each integer $m \geq 1$ define $a_m := a\sigma(a) \cdots \sigma^{m-1}(a)$ and $a_{-m} := \sigma^{-m}(a^{-1}) \cdots \sigma^{-1}(a^{-1}) = \sigma^{-m}(a_m)^{-1}$. The fact that $a_m \sigma^n(a_n) = a_{m-n}$ for all $m, n \in \mathbb{Z}$ ensures that the $k$-linear extension of the map $\Psi : K[t^\pm 1; \sigma] \to K[t^\pm 1; \sigma]$ defined by $\Psi(gt^m) := a_mgt^m$ for $g \in K$ is a graded
Lemma 3.1. Let \( k \)-algebra automorphism. Since \( \Psi(t) = at \) and \( \Psi(a^{-1}bt) = bt \), \( \Psi \) restricts to a graded \( k \)-algebra isomorphism \( k\{t, a^{-1}bt\} \to k\{at, bt\} \).

Thus, \( K[t^{\pm 1}; \sigma] \) contains a free subalgebra of the form \( k\{at, bt\} \) for some \( a, b \in K^\times \) if and only if it contains a free subalgebra of the form \( k\{t, ct\} \) for some \( c \in K^\times \).

2.3. If \( \sigma = \text{id}_R \), then \( R[t; \sigma] = R[t] \), the polynomial ring of with coefficients in \( R \). If \( R \) is a finitely generated \( k \)-algebra, then the Gelfand-Kirillov dimension of \( R[t] \) is \( 1 + \text{GKdim}(R) \).

2.4. If \( \sigma^n = \text{id}_R \), then the subalgebra of \( R[t; \sigma] \) generated by \( t^n \) is the commutative polynomial ring over \( R \) on the indeterminate \( t^n \) and \( R[t; \sigma] \) is a free \( R[t^n] \)-module with basis \( \{1, t, \ldots, t^{n-1}\} \) whence \( \text{GKdim}(R[t; \sigma]) = 1 + \text{GKdim}(R) \). Furthermore, \( R[t; \sigma] \) embeds in the ring of \( n \times n \) matrices over \( R[t^n] \). A matrix ring over a commutative ring never contains a free subalgebra on \( \geq 2 \) variables.

2.5. Let \( \sigma \) be the automorphism of \( k[x^{\pm 1}, y^{\pm 1}] \) given by \( \sigma(x) = x \) and \( \sigma(y) = xy \). Then \( k[x^{\pm 1}, y^{\pm 1}]|_{t^{\pm 1}; \sigma} \) is isomorphic to the group algebra of the discrete Heisenberg group, the subgroup of \( \text{GL}(3, \mathbb{Z}) \) consisting of upper triangular matrices with 1’s on the diagonal. As is well-known, the growth rate of this group is 4 so \( \text{GKdim}(k[x^{\pm 1}, y^{\pm 1}]|_{t^{\pm 1}; \sigma}) = 4 \).

2.6. Let \( E \) be an elliptic curve and \( K = k(E)(z) \), the field of rational functions over the function field of \( E \). Artin and Van den Bergh [1, Ex. 5.19] showed there is an automorphism \( \sigma \) of \( K \) and a finitely graded subalgebra \( B \subset K[t; \sigma] \) such that \( \text{GKdim}(B) = 5 \).

2.7. By (the proof of) [1, Cor. 5.17], if \( \sigma \) is an automorphism of a smooth projective surface \( X \) such that the induced automorphism of the Neron-Severi group has an eigenvalue that is not a root of unity, then \( k(X)[t^{\pm 1}; \sigma] \) has exponential growth.

2.8. In [9], Rogalski proved that when \( K/k \) is a finitely generated field extension of transcendence degree the Gelfand-Kirillov dimension of a finitely generated \( k \)-subalgebra of \( K[t^{\pm 1}; \sigma] \) is either 3, 4, 5, or \( \infty \). Rogalski’s proof uses ideas and results from complex dynamics and algebraic geometry.

2.9. A noetherian locally finite \( \mathbb{N} \)-graded \( k \)-algebra never has exponential growth [13, Thm. 0.1] so can’t contain a free algebra.

3. Valuations

Let \( \nu : K \to \mathbb{R} \sqcup \{\infty\} \) be a valuation such that \( \nu(a) = 0 \) for all \( a \in K^\times \).

We note that if \( \nu(x) \neq \nu(y) \), then \( \nu(x + y) = \min\{\nu(x), \nu(y)\} \).

If \( S \) is a subset of \( K \) we write \( \nu(S) := \{\nu(x) \mid x \in S\} \).

Lemma 3.1. Let \( x_1, \ldots, x_n \in K^\times \).

1. If \( \nu(x_i) > \nu(x_1) \) for all \( i \geq 2 \), then \( x_1 + x_2 + \cdots + x_n \neq 0 \).
(2) If $|\{\nu(x_1), \ldots, \nu(x_n)\}| = n$, then $\{x_1, \ldots, x_n\}$ is linearly independent over $k$.

(3) If $U$ is a $k$-subspace of $K$, then $\dim_k(U) \geq |\nu(U^\times)|$.

**Proof.** (1) This is a small variation on [11, Lemma 1, p.8] where the result is proved for a discrete valuation. Multiplying the $x_i$s by $x_1^{-1}$ we can assume that $x_1 = 1$. Thus, $\nu(x_i) > 0$ for all $i \geq 2$. Hence each $x_i$, $i \geq 2$, belongs to the maximal ideal of the valuation ring associated to $\nu$. The result now follows from the fact that 1 is not in this maximal ideal.

(2) Suppose the statement is false. Then there is a non-empty subset $I \subset \{1, \ldots, n\}$ and $\lambda_i \in k^\times$ such that $\sum_{i \in I} \lambda_i x_i = 0$. Since $\nu(\lambda_i x_i) = \nu(x_i)$, there is $j \in I$ such that $\nu(\lambda_j x_j) > \nu(\lambda_i x_i)$ for all $i \in I - \{j\}$. It follows from (1) that $\sum_{i \in I} \lambda_i x_i \neq 0$. This is a contradiction so (2) must be true.

(3) This follows at once from (2). \qed

Lemma 3.1(3) will be used to obtain lower bounds on the dimensions of $k$-subspaces of $K$.

**Theorem 3.2.** Let $K/k$ be a field extension and $\sigma \in \text{Aut}_k(K)$. Let $a, b \in K$. Let $L$ be the smallest $\sigma$-stable subfield of $K$ that contains $a$ and $b$. Suppose $\nu : L \to \mathbb{R} \cup \{\infty\}$ is a valuation on $L/k$ such that $\infty \neq \nu(a) \neq \nu(b)$.

If there is a number $\beta \in \mathbb{R}$ such that $\nu(\sigma(z)) = \beta \nu(z)$ for all $z \in L^\times$, then $k\{at^n, bt^n\}$ is a free subalgebra of $K[t; \sigma]$ for all $n$ such that $|\beta^n| \geq 2$.

**Proof.** We note that $\nu(\sigma^n(z)) = \beta^n \nu(z)$ for all $z \in K^\times$. The subalgebra of $K[t; \sigma]$ generated by $at^n$ and $bt^n$ is isomorphic to the subalgebra of $K[t; \sigma^n]$ generated by $at$ and $bt$ so, after replacing $\sigma$ by $\sigma^n$ and $\beta$ by $\beta^n$, we can, and will, assume $|\beta| \geq 2$ and $\nu(\sigma(z)) = \beta \nu(z)$ for all $z \in K$.

Define $\Delta := \{\nu(a), \nu(b)\}$, $V := ka + kb$, and

$$V_n := \nu(V) a^2(V) \cdots a^{n-1}(V).$$

Because $\nu(a) \neq \nu(b)$, $\dim_k(V) = 2$. The degree-$n$ component of $k\{at, bt\}$ is $V_n k^n$. To prove this theorem it suffices to show that $\dim_k(V_n) = 2^n$.

We will do this by showing that $|\nu(V_n^\times)| = 2^n$ for all $n \geq 1$ and then invoking Lemma 3.1(2) to conclude that $\dim_k(V_n) \geq |\nu(V_n^\times)| = 2^n$.

An induction argument on $n$ shows that $\{\nu(\sigma^n(V^\times))\} = \beta^n \Delta$. Hence $\nu(V_n^\times) = \Delta + \beta \Delta + \cdots + \beta^{n-1} \Delta$. By hypothesis, $|\Delta| = 2$.

Suppose $|\nu(V_n^\times)| = 2^n$ but $|\nu(V_{n+1}^\times)| < 2^{n+1}$. Then

$$e_0' + e_1' \beta + \cdots + e_n' \beta^n + \nu(a) \beta^{n+1} = e_0 + e_1 \beta + \cdots + e_n \beta^n + \nu(b) \beta^{n+1}$$

for some $e_0', e_1', \ldots, e_n' \in \Delta$. Hence

$$(3-1) \quad (\nu(a) - \nu(b)) \beta^{n+1} = (e_0' - e_0) + (e_1' - e_1) \beta + \cdots + (e_n' - e_n) \beta^n.$$ 

The absolute value of the left-hand side of (3-1) is $|\nu(a) - \nu(b)| \beta^{n+1}$ and the absolute value of the right-hand side is

$$\leq |\nu(a) - \nu(b)| \frac{|\beta|^{n+1} - 1}{|\beta| - 1}.$$
The equality in (3-1) therefore implies

\[ |\beta|^{n+1} \leq \frac{|\beta|^{n+1} - 1}{|\beta| - 1} \]

which is false because \(|\beta| \geq 2\). We deduce that \(\nu(V_{n+1}^\times) = 2^{n+1}\) and therefore \(k\{at, bt\}\) is free. \(\Box\)

4. Monomial automorphisms of \(k[x^{\pm 1}, y^{\pm 1}]\)

Let \(R = k[x^{\pm 1}, y^{\pm 1}]\). Let \(M = (\begin{array}{cc} a & b \\ c & d \end{array}) \in GL(2, \mathbb{Z})\). The automorphism \(\sigma : R \rightarrow R\) defined by

\[(4-1) \quad \sigma(x) := x^a y^b \quad \text{and} \quad \sigma(y) := x^c y^d\]

is called a monomial automorphism of \(K/k\).

4.1. The finite-order case. As remarked in [2.4], if \(\sigma\) has finite order, then \(R[t^{\pm 1}; \sigma]\) is a finite module over its center for every commutative ring \(R\) so neither \(R[t^{\pm 1}; \sigma]\), nor its division ring of fractions in the case when \(R\) is a domain, contains a free subalgebra on \(\geq 2\) variables.

The order of a monomial automorphism \(\sigma\) is equal to the order of \(M\). Since we will obtain results showing that \(k\{xt, yt\}\) is a free subalgebra of \(k[x^{\pm 1}, y^{\pm 1}][t; \sigma]\) for suitable \(\sigma\) we will briefly note some relations satisfied by \(xt\) and \(yt\) when \(\sigma\) has finite order.

Lemma 4.1. If \(n \geq 1\) and \(M^n = I\), then \(k\{xt, yt\}\) is not free because \((xt)^n(yt)^n = (yt)^n(xt)^n\).

Proof. The hypothesis implies that \(\sigma^n\) is the identity map. Therefore

\[ x\sigma(x) \ldots \sigma^{n-1}(x)y\sigma(y) \ldots \sigma^{n-1}(y) = y\sigma(y) \ldots \sigma^{n-1}(y)x\sigma(x) \ldots \sigma^{n-1}(x). \]

Hence \((xt)^n(yt)^n = (yt)^n(xt)^n\). \(\Box\)

Lemma 4.2. The subalgebra \(k\{xt, yt\} \subset k[x^{\pm 1}, y^{\pm 1}][t; \sigma]\) is not free if

1. \(\text{Tr}(M) = 0\), or
2. \(\text{Tr}(M) = 1\) and \(\det(M) = -1\), or
3. \(\text{Tr}(M) = \det(M) = -1\), or
4. \(\text{Tr}(M) \in \{\pm 1, \pm 2\}\) and \(\det(M) = 1\).

Proof. (1) If \(\text{Tr}(M) = 0\), then \(M^4 = I\) so \((xt)^4(yt)^4 = (yt)^4(xt)^4\).

(2) A calculation shows that \((xt)^2(yt) = (yt)^2(xt)\).

(3) A calculation shows that \((xt)(yt)^2 = (yt)(xt)^2\).

(4) One can easily verify the following claims:

(a) if \(\text{Tr}(M) = 2\), then \((xt)(yt)^2(xt) = (yt)(xt)^2(yt)\);

(b) if \(\text{Tr}(M) = -2\), then \((xt)^2(yt)^2 = (yt)^2(xt)^2\);

(c) if \(\text{Tr}(M) = 1\), then \((xt)(yt)(xt) = (yt)(xt)(yt)\);

(d) if \(\text{Tr}(M) = -1\), then \((xt)^3 = (yt)^3\).

In case (c), we also note that \(M^6 = I\) so \((xt)^6(yt)^6 = (yt)^6(xt)^6\). \(\Box\)
4.2. A matrix \( M \in \text{GL}(2, \mathbb{Z}) \) has finite order if and only if \( \rho(M) = 1 \), i.e., if and only if condition (1) or (4) in Lemma 4.2 holds. If condition (2) or (3) in Lemma 4.2 is satisfied, then \( \rho(M) = \frac{1}{2}(1 + \sqrt{5}) \) and, conversely, if \( \rho(M) = \frac{1}{2}(1 + \sqrt{5}) \), either (2) or (3) holds. If \( \rho(M) \neq 1 \), then \( \rho(M) \) is either \( \frac{1}{2}(1 + \sqrt{5}) \) or \( > 2 \).

The trace of \( M \) is the sum of its eigenvalues so \( \rho(M) > 1 \) if \( |\text{Tr}(M)| > 2 \).

**Theorem 4.3.** Let \( M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}(2, \mathbb{Z}) \) and define \( \sigma \in \text{Aut}_k(k[x^{\pm 1}, y^{\pm 1}]) \) by \( \sigma(x) = x^a y^b \) and \( \sigma(y) = x^c y^d \).

(1) If \( \rho(M) = 1 \), then \( k[x^{\pm 1}, y^{\pm 1}][t; \sigma] \) is a finite module over its center so does not contain a free algebra on \( \geq 2 \) variables.

(2) If \( \rho(M) = \frac{1}{2}(1 + \sqrt{5}) \), then \( k\{xt, yt\} \) is a free algebra but \( k\{xt, yt\} \) is not.

(3) If \( \rho(M) > \frac{1}{2}(1 + \sqrt{5}) \), then \( k\{xt, yt\} \) is a free algebra.

Thus, \( k[x^{\pm 1}, y^{\pm 1}][t; \sigma] \) contains a free subalgebra if and only if \( \rho(M) > 1 \).

**Proof.** (1) Suppose \( M \) has a non-real eigenvalue. Since the eigenvalues are the zeroes of the characteristic polynomial \( x^2 - \text{Tr}(M)x + \det(M) \), \( \text{Tr}(M)^2 - 4\det(M) < 0 \). Hence \( \det(M) = 1 \) and \( \text{Tr}(M) \in \{0, \pm 1\} \). By Lemma 4.2, \( k\{xt, yt\} \) is not a free algebra.

Suppose \( M \) has a single real eigenvalue. Then \( \text{Tr}(M)^2 - 4\det(M) = 0 \) whence \( \text{Tr}(M) = \pm 2 \) and \( \det(M) = 1 \); Lemma 4.2(4) tells us that \( k\{xt, yt\} \) is not free.

Suppose \( M \) has two distinct real eigenvalues. Those eigenvalues must be \( \pm 1 \) and \( -1 \) so \( \text{Tr}(M)^2 - 4\det(M) = 4 \) which implies \( \text{Tr}(M) = 0 \) and \( \det(M) = -1 \). Lemma 4.2(1) tells us that \( k\{xt, yt\} \) is not free.

(2) The fact that \( k\{xt, yt\} \) is not a free algebra is proved by parts (2) and (3) of Lemma 4.2. The fact that \( k\{xt^2, yt^2\} \) is free follows from part (3) of the present theorem because \( k\{xt^2, yt^2\} \) is isomorphic to the subalgebra of \( k[x, y][t; \sigma^2] \) generated by \( xt \) and \( yt \).

(3) Let \( \beta \) be an eigenvalue for \( M \) such that \( |\beta| \geq 2 \).

If \( bc = 0 \), then either \( a \) or \( d \) is equal to \( \beta \); but \( ad = \pm 1 \) so that cannot be the case. Hence \( bc \neq 0 \).

Both \( (b \alpha) \) and \( (d - \beta) \) are \( \beta \)-eigenvectors for \( M \). Thus \( M \) has a \( \beta \)-eigenvector of the form \( (a \alpha) \). Because \( M(1) = \beta(1) \),

\[
a + ba = \beta \quad \text{and} \quad c + da = \alpha \beta.
\]

If \( \alpha = 1 \), then \( a + b = c + d = \beta \) whence \( \det(M) = (a - c)\beta = \pm 1 \), contradicting the fact that \( |\beta| \geq 2 \). Therefore \( \alpha \neq 1 \).

Let \( \nu \) be the valuation on \( k[x^{\pm 1}, y^{\pm 1}] \) defined by

\[
\nu \left( \sum a_{ij} x^i y^j \right) := \min \{i + j\alpha \mid a_{ij} \neq 0\}.
\]

Let \( \Delta := \{\nu(x), \nu(y)\} = \{1, \alpha\} \). A simple calculation shows that \( \nu(\sigma(x^i y^j)) = \beta \nu(x^i y^j) \) whence \( \nu(\sigma^n(x)), \nu(\sigma^n(y)) = \beta^n \Delta \).
Let $A$ be the algebra generated by $xt$ and $yt$. Then $A_{n+1} = V_{n+1}t^{n+1}$ where $V_{n+1}$ is the linear span of
$$\{x_0\sigma(x_1)\ldots\sigma^n(x_n) \mid x_i \in \{x, y\}\}.$$ To prove the theorem we must show that $\dim_k(V_{n+1}) = 2^{n+1}$. Obviously, $\dim_k(V_{n+1}) \leq 2^{n+1}$.

It is clear that
$$\{\nu(x_0\sigma(x_1)\ldots\sigma^n(x_n)) \mid x_i \in \{x, y\}\} = \left\{\sum_{i=0}^n \delta_i \mid \delta_i \in \beta^i \Delta\right\}.$$ Since $\alpha \neq 1$, $\beta^i \Delta$ has exactly two elements. Since $\nu$ is a valuation, $\dim_k(V_{n+1})$ is at least the number of elements in the right-hand side of (4-2). To complete the proof we show that the set on the right-hand side of (4-2) contains $2^{n+1}$ elements. To do that it suffices to prove the following claim.

Claim: If $\delta_i, \delta'_i \in \beta^i \Delta$ and $\delta_0 + \cdots + \delta_n = \delta'_0 + \cdots + \delta'_n$, then $\delta_i = \delta'_i$ for all $i$.

Proof: We argue by induction on $n$. The claim is true for $n = 0$. Suppose the claim is true for $n$ but false for $n+1$. Then there are elements $\delta_i, \delta'_i \in \beta^i \Delta$, $0 \leq i \leq n + 1$, such that
$$\delta_0 + \cdots + \delta_n + \delta_{n+1} = \delta'_0 + \cdots + \delta'_n + \delta'_{n+1}$$
and some $\delta_i \neq \delta'_i$. If $\delta_{n+1} = \delta'_{n+1}$ the induction hypothesis implies that $\delta_i = \delta'_i$ for all $0 \leq i \leq n$. That is not the case so $\delta_{n+1} \neq \delta'_{n+1}$.

Since $|\delta'_j - \delta_j| < |\beta| ||1 - \alpha||$, the absolute value of the right-hand side is
$$\sum_{j=0}^n (\delta'_j - \delta_j) \leq |1 - \alpha| \sum_{j=0}^n |\beta|^j < |1 - \alpha||\beta^{n+1}| = |\delta_{n+1} - \delta'_{n+1}|$$
where the strict inequality follows from the hypothesis that $|\beta| \geq 2$. Therefore
$$\delta_{n+1} - \delta'_{n+1} \neq \sum_{j=0}^n (\delta'_j - \delta_j).$$
This contradicts (4-3) so we conclude that the claim must be true for $n+1$. The validity of the claim completes the proof of the theorem.

4.3. The automorphism of $k[x^\pm 1, y^\pm 1]$ corresponding to the matrix $M = \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ \end{pmatrix}$ is given by $\sigma(x) = xy$, $\sigma(y) = xy^2$. The spectral radius of $M$ is $\frac{1}{2}(3 + \sqrt{5})$ so $k\{xt, yt\}$ is a free subalgebra of $k[x^\pm 1, y^\pm 1][t; \sigma]$. Thus, the answer to the question Rogalski asked—see 1.4—is “yes”.

G. Bergman noticed that $\sigma$ is the square of the automorphism $\tau(x) = y$ and $\tau(y) = xy$ and showed that the subalgebra $k\{xt, yt\} \subset k[x^\pm 1, y^\pm 1][t; \tau]$ is not free because $(xt)^2(yt) = (yt)^2(xt)$. Although $k\{xt, yt\}$ is not free it has exponential growth. The automorphism $\tau$ corresponds to $M' = \begin{pmatrix} 0 & 1 \\ 1 & 1 \\ \end{pmatrix}$, $M = (M')^2$, and $\rho(M') = \frac{1}{2}(1 + \sqrt{5}) > 1$. The subalgebra $k\{xt^2, yt^2\} \subset$
5. Automorphisms of $k[x, y]$

5.1. An elementary automorphism of $k[x, y]$ is an automorphism $\tau$ of the form $x \mapsto ax + p(y)$, $y \mapsto by + c$, for some $a, b, c \in k$, $p(y) \in k[y]$.

A Hénon automorphism of $k[x, y]$ is an automorphism $\tau$ of the form $\tau(x, y) = (p(x) - ay, x)$ with deg$(p(x)) \geq 2$ and $a \in k^\times$. A composition of Hénon automorphisms is called a Hénon map.

**Theorem 5.1** (Friedland-Milnor). [4] Thm.2.6 An automorphism of $\mathbb{C}[x, y]$ is conjugate to either an elementary automorphism or a Hénon map.

In this section we show that $k[x, y][t; \sigma]$ contains a free subalgebra if and only if $\sigma$ is conjugate to a Hénon map.

The dynamical system $(\mathbb{C}^2, \sigma : \mathbb{C}^2 \to \mathbb{C}^2)$ where $\sigma$ is a Hénon map of the form $(x, y) \mapsto (1 + y - ax^2, bx)$ has been intensively studied in the context of complex dynamics. There is a belief that the “chaotic behavior” of any complex dynamical system is already exhibited by a Hénon map for suitable $a$ and $b$. Guedj and Sibony say “it is clear that only the [Hénon maps] are dynamically interesting” [6].

**Theorem 5.2.** Let $\nu$ be a valuation on a field extension $K/k$. Let $\sigma \in \text{Aut}_k(K)$. Suppose there is a real number $\beta > 1$ and an element $g \in K$ such that either

(1) $\nu(\sigma^{m+1}(g)) \geq \beta \nu(\sigma^m(g)) > 0$ for all $m \geq 0$ or

(2) $\nu(\sigma^{m+1}(g)) \leq \beta \nu(\sigma^m(g)) < 0$ for all $m \geq 0$.

If $\beta^n \geq 2$, then $gt^n$ and $\sigma^n(g)t^n$ generate a free subalgebra of $K[t; \sigma]$.

**Proof.** Define $\tau = \sigma^n$. If case (1) holds, then $\nu(\tau^{m+1}(g)) \geq 2\nu(\tau^m(g)) > 0$ for all $m \geq 0$. If case (2) holds, then $\nu(\tau^{m+1}(g)) \leq 2\nu(\tau^m(g)) < 0$ for all $m \geq 0$.

The subalgebra of $K[t; \sigma]$ generated by $gt^n$ and $\sigma^n(g)t^n$ is isomorphic to the subalgebra of $K[t; \tau]$ generated by $gt$ and $\tau(g)t$ so, after replacing $\sigma$ by $\tau$ and $\beta$ by $\beta^n$, we can, and will, assume $\beta = 2$ and either (1) or (2) holds.

The first part of the proof applies to both cases (1) and (2).

Let $A$ be the algebra generated by $gt$ and $\sigma(g)t$. Then $A_{n+1} = V_{n+1}t^{n+1}$ where $V_{n+1}$ is the linear span of

$$\{x_0\sigma(x_1)\ldots\sigma^n(x_n) \mid x_i \in \{g, \sigma(g)\}\}.$$

To prove the theorem we must show that $\dim_k(V_{n+1}) = 2^{n+1}$. Obviously, $\dim_k(V_{n+1}) \leq 2^{n+1}$.

Define $\Delta_{n+1} := \{\nu(x_0\sigma(x_1)\ldots\sigma^n(x_n)) \mid x_i \in \{g, \sigma(g)\}\}$. Then

$$\Delta_{n+1} := \left\{ \sum_{i=0}^n \nu(\sigma^i(x_i)) \right\} \quad \nu(x_j) \in \{g, \sigma(g)\}.$$

$k[x^{\pm 1}, y^{\pm 1}][t; \tau]$ is free because it is isomorphic to the subalgebra $k\{xt, yt\}$ in the previous paragraph.
Since $\nu$ is a valuation, $\dim_k(V_{n+1}) \geq |\Delta_{n+1}|$. We will complete the proof by showing that $|\Delta_{n+1}| = 2^{n+1}$. This is true for $n = 0$ because $\Delta_1 = \{\nu(g), \nu(\sigma(g))\}$. Suppose the result is true for $\Delta_n$. Since

$$\Delta_{n+1} = \left(\Delta_n + \nu(\sigma^n(g))\right) \cup \left(\Delta_n + \nu(\sigma^{n+1}(g))\right)$$

it suffices to show that

$$\left(\Delta_n + \nu(\sigma^n(g))\right) \cap \left(\Delta_n + \nu(\sigma^{n+1}(g))\right) = \emptyset.$$

Let $\delta_i = \nu(\sigma^i(g))$. To prove the intersection is empty it suffices to show that $\delta_{n+1} - \delta_n \neq \delta - \delta'$ for all $\delta, \delta' \in \Delta_n$.

Now we split the proof into two separate parts according to the two cases in the statement of the theorem.

1. In this case $0 < \delta_0 \leq \frac{1}{2}\delta_1 \leq \cdots \leq \frac{1}{2^n}\delta_n$. The largest element in $\Delta_n$ is $\delta_1 + \cdots + \delta_n = \nu(\sigma(g) \cdots \sigma^n(g))$ and the smallest is $\delta_0 + \cdots + \delta_{n-1} = \nu(\sigma(g) \cdots \sigma^{n-1}(g))$. If $\delta, \delta' \in \Delta_n$, then

$$\delta - \delta' \leq (\delta_1 + \cdots + \delta_n) - (\delta_0 + \cdots + \delta_{n-1}) = \delta_n - \delta_0$$

which is strictly smaller than $\delta_{n+1} - \delta_n$.

2. In this case $0 > \delta_0 \geq \frac{1}{2}\delta_1 \geq \cdots \geq \frac{1}{2^n}\delta_n$. The smallest element in $\Delta_n$ is $\delta_1 + \cdots + \delta_n = \nu(\sigma(g) \cdots \sigma^n(g))$ and the largest is $\delta_0 + \cdots + \delta_{n-1} = \nu(\sigma(g) \cdots \sigma^{n-1}(g))$. If $\delta, \delta' \in \Delta_n$, then

$$\delta - \delta' \geq (\delta_1 + \cdots + \delta_n) - (\delta_0 + \cdots + \delta_{n-1}) = \delta_n - \delta_0$$

which is strictly larger that $\delta_{n+1} - \delta_n$.

\[ \square \]

**Corollary 5.3.** Suppose $B$ be a commutative $\mathbb{N}$-graded $k$-algebra and an integral domain. Let $\sigma \in \text{Aut}_k(B)$ and $g \in B$. If

$$\deg(\sigma^{m+1}(g)) \geq 2 \deg(\sigma^m(g)) > 0$$

for all $m \geq 0$, then $k\{gt, \sigma(g)t\}$ is a free subalgebra of $B[t; \sigma]$.

**Proof.** Write $K$ for the field of fractions of $B$. Let $\nu$ be the unique valuation on $K/k$ such that $\nu(ab^{-1}) = \deg(b) - \deg(a)$ whenever $a, b \in B$. The hypothesis in the statement of the corollary implies that $g$ satisfies condition (2) in Theorem 5.2. \[ \square \]

**Corollary 5.4.** Let $\sigma$ be a $k$-algebra automorphism of $k[x, y]$.

1. $k[x, y][t; \sigma]$ contains a free subalgebra if and only if $\sigma$ is not conjugate to an elementary automorphism.

2. If $\sigma$ is not conjugate to an elementary automorphism, then $k\{\sigma^n(x)t, \sigma^{n+1}(x)t\}$ is a free subalgebra of $k[x, y][t; \sigma]$ for all $n \geq 0$.

3. If $\sigma$ is conjugate to an elementary automorphism, then $\text{GKdim}(k[x, y][t; \sigma]) = 3$.
The claim is true for $k$.

An induction argument shows that $GKdim_k kt$.

**Proof.** (3) This is surely well-known but we could not find an argument in the literature so give one here.

Suppose $\sigma$ is conjugate to a elementary automorphism. As noted in [21], the isomorphism class of $k[x,y][t;\sigma]$ as a graded $k$-algebra depends only on the conjugacy class of $\sigma$ so we can, and will, assume that $\sigma(x) = ax + p(y)$ and $\sigma(y) = by + c$, for some $a, b, c \in k, p(y) \in k[y]$.

Suppose $\deg(p) = d$. Let $V = k + ky + \cdots + ky^d + kx$ and let $W = V + kt$. Since $1, x, y, t \in W$, we can measure the GK-dimension of $k[x,y][t;\sigma]$ by measuring the rate at which $\dim_k(W^n)$ grows. Since $\sigma(V) = V, tV = Vt$. An induction argument shows that

$$W^n = V^n + V^{n-1}t + V^{n-2}t^2 + \cdots + Vt^{n-1} + kt^n.$$ 

Therefore $\dim_k(W^n) = \dim_k(D^n)$ where $D$ is the subspace of the commutative polynomial ring $k[X,Y,T]$ spanned by $\{1, X, T, Y, Y^2, \ldots, Y^d\}$. Thus, $\text{GKdim}(k[x,y][t;\sigma]) = \text{GKdim}(k[X,Y,T]) = 3$.

(1) If $\sigma$ is conjugate to an elementary automorphism $k[x,y][t;\sigma]$ does not contain a free algebra on $\geq 2$ variables because its GK-dimension is 3.

Suppose $\sigma$ is not conjugate to an elementary automorphism. By [13 Cor. 9], the degree of $\sigma^{n+1}(x)$ is at least twice the degree of $\sigma^n(x)$. By Corollary 5.3, $k\{\sigma^n(x)t, \sigma^{n+1}(x)t\}$ is a free subalgebra of $k[x,y][t;\sigma]$ for all $n \geq 0$. This completes the proof of (1) and also proves (2). \hfill $\square$

5.2. Let $a, b \in \mathbb{C}$ with $ab \neq 0$. Let $\sigma$ be the automorphism of $\mathbb{C}[x,y]$ defined by

(5-1) \hspace{1cm} \sigma(x) = 1 + y - ax^2 \quad \text{and} \quad \sigma(y) = bx.$$

By Corollary 5.4, $\mathbb{C}\{xt, \sigma(x)t\}$ is a free subalgebra of $\mathbb{C}[x,y][t;\sigma]$.

**Proposition 5.5.** If $\sigma$ is the automorphism of $\mathbb{C}[x,y]$ given by (5-1), then $\mathbb{C}\{xt, yt\}$ is a free subalgebra of $\mathbb{C}[x,y][t;\sigma]$.

**Proof.** For the duration of this proof we give $\mathbb{C}[x,y]$ the grading determined by $\deg(x) = 2$ and $\deg(y) = 1$. Since

$$\deg(\sigma(x^iy^j)) = i \deg(\sigma(x)) + j \deg(\sigma(y)) = 4i + 2j = 2 \deg(x^iy^j),$$

$$\deg(\sigma(f)) \leq 2 \deg(f)$$

for all $f \in \mathbb{C}[x,y]$. Since $\deg(x) = 2$, an induction argument shows that $\deg(\sigma^n(x)) \leq 2^{n+1}$. (As we will shortly show, $\deg(\sigma^n(x)) = 2^{n+1}$.)

**Claim:** the degree-2 component of $\sigma^n(x)$ is a non-zero scalar multiple of $x^{2^n+1}$. **Proof:** The claim is true for $n = 0$ and $n = 1$. Suppose the claim is true for $n$, i.e., there is $\lambda_n \in \mathbb{C}^\times$ such that $\sigma^n(x) = \lambda_n x^{2^n} + \text{l.d.t}$ where l.d.t stands for lower-degree terms, a term being a non-zero scalar multiple of some $x^iy^j$. Hence

$$\sigma^{n+1}(x) = \lambda_n \sigma(x)^{2^n} + \sigma(\text{l.d.t}) = \lambda_n(1 + y - ax^2)^{2^n} + \sigma(\text{l.d.t}).$$
6. Big subalgebras

6.1. The definition. (Rogalski and Zhang [10] p.435, [9] Defn. 6.1). Let \( R \) be a commutative \( k \)-algebra and \( \sigma \in \text{Aut}_k(R) \). A locally finite \( \mathbb{N} \)-graded subalgebra \( \bigoplus_{n=0}^{\infty} V_n t^n \subset R[t; \sigma] \), where each \( V_n \subset R \), is a big subalgebra of \( R[t; \sigma] \) if some \( V_n \) contains a unit of \( R \), \( u \) say, such that \( \text{Fract}(R) = \text{Fract}(k[V_n u^{-1}]) \).

Proposition 6.1. [10] Cor. 2.4] Let \( R \) be a commutative \( k \)-algebra. If a single finitely graded subalgebra of \( R[t^{\pm 1}; \sigma] \) has exponential growth so does every finitely graded big subalgebra of \( R[t^{\pm 1}; \sigma] \).

Let \( \sigma \in \text{Aut}_k(k[x^{\pm 1}, y^{\pm 1}]) \). The subalgebra of \( k[x^{\pm 1}, y^{\pm 1}][t; \sigma] \) generated by \( \{t, xt, yt\} \) is a big subalgebra for all \( \sigma \). The next result shows that the subalgebra of \( k[x^{\pm 1}, y^{\pm 1}][t; \sigma] \) generated by \( xt \) and \( yt \) can be free without being a big subalgebra.

Lemma 6.2. Let \( M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}(2, \mathbb{Z}) \) and let \( \sigma \) be the automorphism of \( k[x^{\pm 1}, y^{\pm 1}] \) given by \( \sigma(x) = x^a y^b \) and \( \sigma(y) = x^c y^d \). Assume \( a + b \equiv c + d \pmod{2} \). If \( \rho(M) > 2 \), then \( k\{xt, yt\} \) is free but is not a big subalgebra of \( k[x^{\pm 1}, y^{\pm 1}][t; \sigma] \).

Proof. Since \( \rho(M) > 2 \), \( k\{xt, yt\} \) is a free algebra by Theorem 4.3.

Let \( A = k\{xt, yt\} \) and write \( A_n = V_n t^n \) where \( V_n = V \sigma(V) \cdots \sigma^{n-1}(V) \) and \( V = kx + ky \).

Claim: If \( v, v' \in \sigma^n(V^\times) \), then \( \deg(v) \equiv \deg(v') \pmod{2} \). Proof: We will prove this by induction on \( n \). All non-zero elements in \( V \) have odd degree so the claim is true for \( n = 1 \). Suppose the claim is true for \( n \). Let \( x^i y^j \in \sigma^n(V) \). Since

\[
\deg(\sigma(x^i y^j)) = (a + b)i + (c + d)j \equiv (a + b)(i + j) \pmod{2}
\]

\( \deg(\sigma(x^i y^j)) \pmod{2} \) is the same for all \( x^i y^j \in \sigma^n(V) \). Hence the claim is true for \( n + 1 \).

It follows from the claim that \( \deg(u) \equiv \deg(u') \pmod{2} \) for all \( u, u' \in V_n^\times \). Therefore \( V_n u^{-1} \subset k(x^2, xy, y^2) \) for all \( u \in V_n^\times \) and all \( n \geq 0 \). Thus, \( A \) is not a big subalgebra of \( k[x^{\pm 1}, y^{\pm 1}][t; \sigma] \).

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Department of Mathematics, Box 354350, Univ. Washington, Seattle, WA 98195

E-mail address: smith@math.washington.edu