Morse functions on the moduli space
of $G_2$ structures

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Abstract

Let $\mathcal{M}$ be the moduli space of torsion free $G_2$ structures on a compact 7-manifold $M$, and let $\mathcal{M}_1 \subset \mathcal{M}$ be the $G_2$ structures with $\text{volume}(M) = 1$. The cohomology map $\pi^3 : \mathcal{M} \to H^3(M, R)$ is known to be a local diffeomorphism. It is proved that every nonzero element of $H^4(M, R) = H^3(M, R)^*$ is a Morse function on $\mathcal{M}_1$ when composed with $\pi^3$. When $\dim H^3(M, R) = 2$, the result in particular implies $\pi^3$ is one to one on each connected component of $\mathcal{M}$. Considering the first Pontryagin class $p_1(M) \in H^4(M, R)$, we formulate a compactness conjecture on the set of $G_2$ structures of $\text{volume}(M) = 1$ with bounded $L^2$ norm of curvature, which would imply that every connected component of $\mathcal{M}$ is contractible. We also observe the locus $\pi^3(\mathcal{M}_1) \subset H^3(M, R)$ is a hyperbolic affine sphere if the volume of the torus $H^3(M, R)/H^3(M, Z)$ is constant on $\mathcal{M}_1$.

Key words: $G_2$ structure, moduli space, Morse function

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1 Introduction

The moduli space of complex structures on a compact Riemann surface of genus 1 or $\geq 2$ can be identified with the deformation space of Riemannian metrics of constant curvature 0 (up to scale) or $-1$ respectively, while the latter definition naturally gives rise to the Weil-Petersson metric.

Let $M$ be a compact, oriented, and spin manifold of dimension 7. Then $M$ admits a differential 3-form $\phi$ of generic type called a definite (positive) 3-form (Section 2) [Br], and such $\phi$ determines a unique Riemannian metric $g_\phi$ and an orientation on $M$. $\phi$ is called a $G_2$ form if $d\phi = 0$, $d^*\phi = 0$, and the orientation determined by $\phi$ agrees with the given orientation of $M$, where $^*\phi$ is the Hodge star operator with respect to $g_\phi$. The stabilizer of a definite 3-form in the Euclidean space $R^7$ is isomorphic to compact simple Lie group $G_2$, and thus the existence of a $G_2$ form is equivalent to a torsion free $G_2$ structure on $M$. Throughout this paper, a $G_2$ structure would mean a torsion free $G_2$ structure and a $G_2$ manifold would mean a manifold with a $G_2$ structure. Note the holonomy of the underlying metric of a $G_2$ manifold is necessarily a subgroup of $G_2$.

It is known that the moduli space of $G_2$ structures, denoted by $\mathfrak{M}$, is a smooth manifold of dimension $b^3 = \dim H^3(M, R)$. When $M$ has full holonomy $G_2$, or equivalently
$b^1 = \dim H^1(M, R) = 0$, a connected component of $\mathcal{M}$ coincides with the (Ricci flat) Einstein deformation space of the underlying $G_2$ metric (the property for a Ricci flat metric to support a parallel spinor is preserved under Einstein deformation). In this perspective, one of the motivation for the present work is to examine the analogy or the difference between classical Teichmüller theory and the deformation of $G_2$ structures [Tr].

Another motivation comes from the question: Can one find the best $G_2$ form (structure) on a given $G_2$ manifold? A natural condition would be to require a $G_2$ form $\phi$ to satisfy $[\ast \phi, \phi] = -p_1(M)$, where $p_1(M) \in H^4(M, R)$ is (nonzero) Pontryagin class. Such $\phi$ is best in the sense that it locally minimizes the $L^2$ norm of the curvature of the associated metric $g_\phi$ among the set of $G_2$ forms with fixed volume $\text{Vol}_\phi(M)$.

The main result of this paper is that any nonzero element $\beta \in H^4(M, R) = H^3(M, R)^*$ composed with the cohomology map $\mathcal{M} \to H^3(M, R)$ is a Morse function on $\mathcal{M}_1$, where $\mathcal{M}_1 \subset \mathcal{M}$ is the moduli space of $G_2$ structures of volume 1. In particular, when $b^1 = 0$ and $\beta = -p_1(M) \neq 0$, every critical point is a positive local minimum. As a corollary, $p_1(M) \neq 0$ implies each connected component of $\mathcal{M}_1$ is noncompact. We also prove Torelli theorem in case $b^1 = 0$ and $b^3 = 2$: the cohomology map $\mathcal{M} \to H^3(M, R)$ is one to one on each connected component.

For further application of this result toward the question raised above, an analogue of Mumford compactness theorem for Riemann surfaces would play a role. Based on the geometry of the Torelli map (to be defined below) and the main results above, we formulate a conjecture on the compactness of the set of $G_2$ structures of volume 1 with uniformly bounded height with respect to $-p_1(M)$. Specifically, let $\{\phi_n\}$ be a sequence of $G_2$ forms of volume 1 in a connected component of the moduli space of $G_2$ forms on a compact manifold $M$ with $b^1 = 0$. Suppose $\{-p_1(M)([\phi_n])\}$ is bounded from above. Does there exists a subsequence $\{\phi_{n_k}\}$, a sequence of diffeomorphisms $f_k$ isotopic to the identity, and a $G_2$ form $\phi$ of volume 1 such that $f_k^* \phi_{n_k} \to \phi$ in $C^1$? If this
compactness theorem is true, a simple Morse theory argument then implies that each
connected component \( \mathcal{M} \) is contractible, and that a \( G_2 \) manifold \( M \) supports a \( G_2 \) form
\( \phi \) unique up to diffeomorphism such that \( [\ast \phi \phi] = -p_1(M) \).

Torelli theorem for compact Riemann surfaces states that the period map of a Rie-
mann surface determines its complex structure [Gr]. An essential characterization of the
image of the period map, inversion problem, is also known [AM]. In the last section, we
consider the Torelli map \( \pi : \mathcal{M} \to H^3(M, R) \oplus H^4(M, R) \) defined by
\[
\pi(\langle \phi \rangle) = (\pi^3(\langle \phi \rangle), \pi^4(\langle \phi \rangle)) = ([\phi], [\ast \phi \phi]) \in H^3(M, R) \oplus H^4(M, R)
\]
where \( \langle \phi \rangle \) is the equivalence class of \( G_2 \) forms represented by \( \phi \). It is proved that \( \pi \) is a
Lagrangian immersion, and that it is an isometric immersion (up to sign) along \( \mathcal{M}_1 \) when
\( b^1 = 0 \), where \( H^3(M, R) \oplus H^4(M, R) \) is equipped with the canonical symplectic form
and the metric of signature \( (b^3, b^4) \). If the ratio of Jacobian of the projections \( \pi^3 \) and \( \pi^4 \)
is constant along \( \mathcal{M}_1 \), each hypersurface \( \pi^3(\mathcal{M}_1) \subset H^3(M, R) \) and \( \pi^4(\mathcal{M}_1) \subset H^4(M, R) \)
describes a hyperbolic affine sphere centered at the origin. This is also equivalent to the
volume of the torus \( H^3(M, R)/H^3(M, Z) \) being constant along \( \mathcal{M}_1 \) [Hi].

2 Moduli space of \( G_2 \) structures

Consider an exterior 3-form on \( R^7 \)
\[
\varphi = (dx^{12} + dx^{34} + dx^{56}) \wedge dx^7 + dx^{135} - dx^{146} - dx^{362} - dx^{524},
\]
where \( x^i \)'s are coordinate functions and \( dx^{12} = dx^1 \wedge dx^2 \), etc. The stabilizer of \( \varphi \) is a
compact simple simply connected Lie group \( G_2 \subset GL(7, R) \), and it also preserves the
7-form \( dx^{12..7} \) and a metric uniquely determined by
\[
\langle u, v \rangle dx^{12..7} = \frac{1}{6} (u \hook\varphi) \wedge (v \hook\varphi) \wedge \varphi
\]
for \( u, v \in R^7 \) [Br]. The orbit of \( \varphi \) is one of the two open orbits in \( \bigwedge^3 (R^7)^* \) under \( GL(7, R) \) action.

Let \( M \) be a compact oriented 7-manifold. Let \( PM \subset \bigwedge^3 T^*M \) be the open subset whose fiber at each point \( p \in M \) consists of \( \phi_p \in \bigwedge^3 T_p^*M \) that can be identified with (1) under an oriented isomorphism between \( T_pM \) and \( R^7 \). An element \( \phi \in C^\infty (PM) \) is called a positive (definite) 3-form. \( M \) admits a positive 3-form if and only if it is spin, and a positive 3-form \( \phi \) in turn determines a metric \( g_\phi \) and a volume form \( dvol_\phi \) by (2) [Br]. We denote by \( *_\phi \) the Hodge star operator on differential forms defined by \( g_\phi \). Since \( PM \subset \bigwedge^3 T^*M \) is an open subset, the tangent space \( T_\phi C^\infty (PM) \) can be identified with the space of differential 3-forms

\[
T_\phi C^\infty (PM) \cong C^\infty (\bigwedge^3 T^*M) = \Omega^3,
\]

and \( C^\infty (PM) \) becomes a Riemannian manifold with respect to the \( L^2 \) metric on \( \Omega^3 \) determined by \( g_\phi \). Note the diffeomorphism group \( \text{Diff}(M) \) acts on \( C^\infty (PM) \) as a group of isometry.

**Definition 1** A positive differential 3-form \( \phi \) is a \( G_2 \) form if \( d\phi = 0 \) and \( d*_\phi \phi = 0 \). A compact oriented manifold \( M \) is a \( G_2 \) manifold if it admits a \( G_2 \) form.

A \( G_2 \) metric \( g_\phi \) is necessarily Ricci-flat. The holonomy of the \( G_2 \) metric on a \( G_2 \) manifold is isomorphic to a subgroup of \( G_2 \), and it is full \( G_2 \) whenever the fundamental group \( \pi_1 (M) \) is finite, or equivalently the first de Rham cohomology \( H^1 (M, R) = 0 \).

We denote the space of \( G_2 \) forms by \( \mathfrak{M} \subset C^\infty (PM) \). Let \( \mathcal{D} \subset \text{Diff}(M) \) be the group of diffeomorphisms of \( M \) isotopic to the identity.

**Definition 2** Let \( M \) be a \( G_2 \) manifold. The moduli space of \( G_2 \) structures (forms) is the quotient space \( \mathfrak{M} = \mathfrak{M}/\mathcal{D} \).

Given a \( G_2 \) form \( \phi \), we denote its equivalence class by \( \langle \phi \rangle \).
It is known that $\mathcal{M}$ is a smooth manifold of dimension $b^3 = \dim H^3(M, R)$ [Br][Jo]. From the remark above, there exists a unique Riemannian metric on $\mathcal{M}$ for which $\widehat{\mathcal{M}} \rightarrow \mathcal{M}$ is a Riemannian submersion. When $b^1 = \dim H^1(M, R) = 0$, a connected component of $\mathcal{M}$ coincides with the Einstein deformation space of the underlying $G_2$ metric, which has a real analytic structure [Ko].

From the definition of $\mathcal{D}$, the projection $\pi^3 : \mathcal{M} \rightarrow H^3(M, R)$ is well defined. The image of a class is denoted by $\pi^3(\langle \phi \rangle) = [\phi]$ for simplicity.

Remark. Let $f$ be a diffeomorphism of $M$. Then

$$\ast f \ast f \phi = f^*(\ast \phi),$$

and the cohomology map $\pi^4 : \mathcal{M} \rightarrow H^4(M, R)$ by $\pi^4(\langle \phi \rangle) = [\ast \phi]$ is also well defined. In fact, we may take the dual definition of the moduli space of $G_2$ structures as the equivalence class of definite 4-forms $\psi$ that satisfies $d\psi = 0$ and $d \ast \psi \psi = 0$. (3) shows these two definitions "commute" with the projection maps $\pi^3$ and $\pi^4$. Note however a definite 4-form does not determine an orientation.

Let $\phi \in \widehat{\mathcal{M}}$. Then $\bigwedge^* T^*M$ admits a $G_2$ invariant decomposition

$$\bigwedge^2 T^*M = \bigwedge^2_7 \oplus \bigwedge^2_{14},$$

$$\bigwedge^3 T^*M = \bigwedge^3_1 \oplus \bigwedge^3_7 \oplus \bigwedge^3_{27},$$

where, for instance,

$$\ast \phi \phi \wedge (\bigwedge^3_7 \oplus \bigwedge^3_{27}) = 0.$$

In particular, for any $X \in \Omega^3$, there exists a unique quadratic form $h_X$ and a vector field $v_X$ such that

$$X = h_X \cdot \phi + v_X \mathcal{J} \ast \phi$$

where $h_X \cdot \phi$ means the action of $h_X$ as a derivation.

We denote $\Omega^p_k = C^\infty(\bigwedge^p_k)$ and write $X = X_1 + X_7 + X_{27}$ for a given $X \in \Omega^3$ for
its irreducible components. Since Hodge Laplacian commutes with this decomposition, let $H_k^p$ be the corresponding decomposition of the cohomology $H^p(M, R)$.

$\Omega^3 \cong T_\phi C^\infty(PM)$ admits a $L^2$ decomposition along $\widehat{M}$ as follows [Jo].

$$T_\phi \widehat{M} = \{ X \in \Omega^3 \mid dX = 0, \quad d*_{\phi} \left( \frac{4}{3}X_1 + X_7 - X_{27} \right) = 0 \}$$

$$V_\phi = \{ d\Omega_7^2 \}$$

$$H_\phi = V_\phi^\perp \cap T_\phi \widehat{M}$$

$$N_\phi = (T_\phi \widehat{M})^\perp.$$  

Here $*_{\phi} \left( \frac{4}{3}X_1 + X_7 - X_{27} \right)$ is the derivative of the map $\phi \to *_{\phi} \phi$. $H_\phi$ and $V_\phi$ represent the horizontal and vertical subspaces of $T_\phi \widehat{M}$ with respect to the submersion $\widehat{M} \to M$. The orthogonal projection map from $T_\phi C^\infty(PM)$ to these subspaces will be denoted by $\Pi^V_\phi$, $\Pi^H_\phi$, and $\Pi^N_\phi$ respectively.

### 3 Horizontal geodesics on $\widehat{M} \to M$

Let $\{w^1, w^2, \ldots, w^7\}$ be a local coframe on $M$, and

$$X = X_{ijk} w^i \wedge w^j \wedge w^k \in T_\phi C^\infty(PM) \cong \Omega^3,$$

where $X_{ijk}$ is skew symmetric in all of its indices. Fix $\phi \in \widehat{M}$ so that $\langle w^i, w^j \rangle_{g_\phi} = g_\phi^{ij}$.

The $L^2$ inner product on $T_\phi C^\infty(PM)$ is defined by

$$\langle X, Y \rangle_\phi = 6 \int_M X_{ijk} Y_{i'j'k'} g_\phi^{i'i'} g_\phi^{j'j} g_\phi^{kk'} \, dvol_{g_\phi}.$$  

(5)

Note $\langle \phi, \phi \rangle_\phi = 7 Vol_\phi(M)$.

Let $\nabla$ be the Levi-Civita connection on $C^\infty(PM)$. Since tangent vectors on $C^\infty(PM)$ can be identified with $\Omega^3$ valued functions,

$$\nabla_X Y = X(Y) + D_X Y$$

(6)
where $X(Y)$ is the directional derivative of $Y$ as an $\Omega^3$ valued function, and $D_X Y$ is the covariant derivative of $Y$ considered as a translation invariant vector field.

$D_X Y$ can be computed by the following lemma [Br].

**Lemma 1** Let $Z = h_Z \cdot \phi + v_Z \ast c_\phi \phi \in \Omega^3$ and consider a curve $\phi_t = \phi + t Z + O(t^2) \in \hat{\mathbb{M}}$. Then $g_{\phi_t} = g_\phi + t h_Z + O(t^2)$.

**Proposition 1** Let $X, Y, Z \in \Omega^3$ be translation invariant vector fields on $C^\infty(PM)$.

Then

$$2 \ll D_X Y, Z \gg_\phi = -2 \ll h_X \cdot Y + h_Y \cdot X, Z \gg_\phi + 2 \ll Y, h_Z \cdot X \gg_\phi$$

$$+ \int_M \text{tr}_{g_\phi}(h_X)(Y, Z)_{g_\phi} \, dvol_{g_\phi} + \int_M \text{tr}_{g_\phi}(h_Y)(X, Z)_{g_\phi} \, dvol_{g_\phi}$$

$$- \int_M \text{tr}_{g_\phi}(h_Z)(X, Y)_{g_\phi} \, dvol_{g_\phi}.$$  

Proof. Differentiate (5) and use the fact $[X, Y] = [Y, Z] = [Z, X] = 0$. □

Let $\{\gamma_t\} \subset \mathcal{M}$ be a geodesic and $\phi_t \in \hat{\mathcal{M}}$ be one of its horizontal lifts, which is also a geodesic in $\hat{\mathcal{M}}$. As a curve in $C^\infty(PM)$,

$$\Pi_t^N(\phi'_t) = 0 \quad (7)$$

$$\Pi_t^N(\nabla_{\phi'_t} \phi'_t) = \nabla_{\phi'_t} \phi'_t, \quad (8)$$

where $\Pi_t^N = \Pi_{\phi_t}^N$, and

$$\nabla_{\phi'_t} \phi'_t = \phi''_t + D_{\phi'_t} \phi'_t. \quad (9)$$

From (8),

$$\phi''_t = \Pi_0^N(\phi''_0) - \Pi_0^{V+H}(D_{\phi'_0} \phi'_0). \quad (10)$$

Differentiating (7),

$$\frac{d}{dt} \Pi_t^N(\phi'_0)\big|_{t=0} + \Pi_0^N(\phi''_0) = 0, \quad (11)$$

and we obtain

$$\phi''_0 = -\frac{d}{dt} \Pi_t^N(\phi'_0)\big|_{t=0} - \Pi_0^{V+H}(D_{\phi'_0} \phi'_0). \quad (12)$$
We record the following for later application.

**Lemma 2** Let \( \phi_t \in \hat{\mathcal{M}} \) be a curve and put \( \psi_t = *_{\phi_t} \phi_t \). Let \( X \in V_{\phi_0} \oplus H_{\phi_0} \). Then

\[
\int_{\hat{\mathcal{M}}} \psi_0 \wedge \frac{d}{dt} \Pi_t^N(X)|_{t=0} = 0,
\]

where \( X \) is considered as a translation invariant vector field along \( \phi_t \).

**Proof.** Since \( X \) is closed, \( \Pi_t^N(X) \) is closed all \( t \), and form the decomposition (4), \( \Pi_t^N(X) \) is in fact exact. \( \square \)

## 4 Morse functions on \( \mathcal{M}_1 \)

Let \( \hat{\mathcal{M}}_1 \subset \hat{\mathcal{M}} \) be the set of \( G_2 \) forms of volume 1, and \( \mathcal{M}_1 \subset \mathcal{M} \) its image under the projection \( \hat{\mathcal{M}} \to \mathcal{M} \). Then \( \mathcal{M} \cong \mathcal{M}_1 \times R^+ \), and \( \mathcal{M}_1 \) is an embedded hypersurface of \( \mathcal{M} \).

In this section, we show each nonzero element in \( H^4(M, R) \) becomes a Morse function when composed with the cohomology map \( \mathcal{M}_1 \to H^3(M, R) \).

Let \( \phi \in \hat{\mathcal{M}}_1 \). Then the gradient of the volume function on \( \hat{\mathcal{M}} \) at \( \phi \) is \( \frac{1}{3} \phi \in H_\phi \), (19). Let \( \nu_\phi = \frac{1}{\sqrt{7}} \phi \in H_\phi \) be the unit normal to \( \hat{\mathcal{M}}_1 \subset \hat{\mathcal{M}} \). The second fundamental form of \( \hat{\mathcal{M}}_1 \) is

\[
\Pi = - \ll d\nu, d\phi \gg = - \frac{1}{\sqrt{7}} \ll d\phi, d\phi \gg,
\]

and \( \hat{\mathcal{M}}_1 \subset \hat{\mathcal{M}} \) is a umbilic hypersurface. Note that \( d\phi \) in the expression above is not the exterior derivative, but it is a tautological 1-form that represents the infinitesimal displacement of \( \phi \) in \( \hat{\mathcal{M}}_1 \).

We wish to introduce a class of functions on \( \mathcal{M} \). Let \( \beta \in H^4(M, R) \) be a nonzero class, and define

\[
F^\beta(\langle \phi \rangle) = \beta(\langle [\phi] \rangle)
\]

where \( \beta \) is regarded as an element of \( H^3(M, R)^* \). Let \( F^\beta_1 \) denote its restriction to \( \mathcal{M}_1 \), and \( \text{Crit}(F^\beta_1) \) its critical point set.
Remark. From the Remark below Definition 2, we may also define
\[ G^\alpha(\langle \phi \rangle) = \alpha([*\phi]), \]
for \( \alpha \in H^3(M, R) \).

Proposition 2
\[ \text{Crit}(F^\beta_1) = \{ \langle \phi \rangle | \beta = c^\beta_{\langle \phi \rangle} [*_\phi] \} \]
for some constant \( c^\beta_{\langle \phi \rangle} \).

Note 7 \( c^\beta_{\langle \phi \rangle} = F^\beta_1(\langle \phi \rangle) \neq 0 \).

Proof. From the description of the unit normal \( \nu \) above, \( \phi \) is a critical point whenever \( \beta \) annihilates \( H^3_\phi \oplus H^3_{27} \). The proposition follows for \( H^3(M, R)^* = H^4(M, R) \). \( \square \)

Let \( \langle \phi_0 \rangle \in \text{Crit}(F^\beta_1) \) and put \( \psi_0 = *_{\phi_0} \phi_0 \). Then one finds
\[
\nabla^2 F^\beta_1 |_{\langle \phi_0 \rangle} = \nabla^2 F^\beta_{\langle \phi_0 \rangle} + \frac{\partial}{\partial \nu} F^\beta_{|_{\langle \phi_0 \rangle \Pi}}
= \nabla^2 F^\beta_{|_{\langle \phi_0 \rangle}} - c^\beta_{\langle \phi_0 \rangle} \ll d\phi, d\phi \gg_{\langle \phi_0 \rangle} .
\]
(14)

Now, let \( X \in H_{\phi_0} \) be a horizontal lift of a tangent vector \( x \in T_{\langle \phi_0 \rangle} \mathfrak{M}_1 \) and let \( \phi_t \subset \hat{\mathfrak{M}} \) be a horizontal geodesic with \( \phi'_0 = X \). From (12) and Lemma 2,
\[
\nabla^2 F^\beta(x, x) = c^\beta_{\langle \phi_0 \rangle} \int_M \psi_0 \wedge \phi_0''
= -c^\beta_{\langle \phi_0 \rangle} \ll \phi_0, D_X X \gg_{\phi_0} ,
\]
(15)
for \( \phi_0 \in H_{\phi_0} \). Proposition 1 then gives,
\[
\ll \phi_0, D_X X \gg_{\phi_0} = -2 \ll h_X \cdot X, \phi_0 \gg_{\phi_0} + \ll X, X \gg_{\phi_0}
= -\frac{7}{6} \ll X, X \gg_{\phi_0} \text{ since } h_{\phi_0} = \frac{1}{3} g_{\phi_0} \text{ and } X_1 = 0
= -\frac{1}{6} \ll X, X \gg_{\phi_0} - 2 \ll h_X \cdot X, \phi_0 \gg_{\phi_0}
= -\frac{1}{6} \ll X, X \gg_{\phi_0} - 2 \ll X_{27}, X_{27} \gg_{\phi_0} .
\]
(16)
Theorem 1 Let $F^β_1$ be a function on $\mathcal{M}_1$ defined in (13) with $β \in H^4(M,R)$. Then at a critical point $⟨φ_0⟩ \in \mathcal{M}_1$,

$$\nabla^2 F^β_1|_{⟨φ_0⟩}(x,x) = -c^β_{⟨φ_0⟩}(\frac{5}{6} ≪ X_7, X_7 \gg_{φ_0} - \frac{7}{6} ≪ X_27, X_27 \gg_{φ_0})$$

where $X \in H_{φ_0}$ is the horizontal lift of $x \in T_{⟨φ_0⟩} \mathcal{M}_1$, and $β = c^β_{⟨φ_0⟩} [∗_{φ_0} φ_0]$ with $7c^β_{⟨φ_0⟩} = F^β_1(⟨φ_0⟩)$. $F^β_1$ is a Morse function on $\mathcal{M}_1$ for any nonzero $β \in H^4(M,R)$.

Note that in case $H^1(M,R) = 0$, every critical point is either a positive local minimum or a negative local maximum.

It is known that $H^1_i$ component of the Pontryagin class $p_1(M)$ is $p_1(M)_1 = -\frac{1}{56π^2} ∥R_{g_φ}∥^2 [∗_{φ} φ]$ for any $G_2$ form $φ \in \mathcal{M}_1$, where $∥R_{g_φ}∥^2$ is the $L^2$ norm of the curvature tensor of the associated metric $g_φ$.

Corollary 1 Let $M$ be a compact $G_2$ manifold with $p_1(M) \neq 0$. Let $\mathcal{M}_1$ be the moduli space of $G_2$ structures of volume 1. Then each connected component of $\mathcal{M}_1$ is noncompact.

Proof. $p_1(M) \neq 0$ implies $F^{p_1(M)}_1 < 0$ on $\mathcal{M}_1$. The corollary follows from maximum principle. □.

Corollary 2 Let $M$ be a compact $G_2$ manifold with $b^1 = 0$ and $b^3 = 2$. Suppose $φ_1$ and $φ_2$ are isotopic $G_2$ forms, i.e., they can be connected through $G_2$ forms, and that $[φ_2] = λ[φ_1]$ or $[∗_{φ_2} φ_2] = λ^4[∗_{φ_1} φ_1]$ for some constant $λ > 0$. Then there exits a diffeomorphism $f$ of $M$ isotopic to the identity such that $φ_2 = λ f^*(φ_1)$. In particular, each $π^3 : \mathcal{M} \rightarrow H^3(M,R)$ and $π^4 : \mathcal{M} \rightarrow H^4(M,R)$ is one to one on every connected components of $\mathcal{M}$.

Proof. Assume $φ_1, φ_2 \in \mathcal{M}_1$. Take $β = [∗_{φ_1} φ_1]$. Since $b^1 = 0$ and $b^3 = 2$, $F^β_1$ is a positive function on the connected component of $\mathcal{M}_1$ containing $⟨φ_1⟩$ with a unique critical point $⟨φ_1⟩$. 
If $[\ast_2 \phi_2] = \lambda^2 [\ast_1 \phi_1]$, it follows from considering $G_1^\alpha$ with $\alpha = [\phi_1]$. □

In case $b_1 = 0$, the image $[\mathcal{M}_1] \subset H^3(M, R)$ displays the following geometric properties.

1. $[\mathcal{M}_1] \subset H^3(M, R)$ is an immersed orientable locally convex hypersurface that is transversal to the radial direction.

2. Restriction to $[\mathcal{M}_1]$ of every linear(height) function in $H^4(M, R)$ is a Morse function with only positive local minima or negative local maxima.

3. Take $\beta = p_1(M) \neq 0$, the first Pontryagin class. Then $F_1^{\beta}(F^\beta)$ is negative on $\mathcal{M}_1(\mathcal{M})$ [Br].

4. Similarly, take $\beta = \sigma^2$ for any nonzero $\sigma \in H^2(M, R)$. Then $F_1^{\beta}(F^\beta)$ is negative on $\mathcal{M}_1(\mathcal{M})$ [Br].

In view of Theorem 1 and the properties listed above, we propose the following conjecture on the compactness of $G_2$ structures of volume 1 with bounded $L^2$ norm of curvature.

**Conjecture.** Let $\{\phi_n\}$ be a sequence of $G_2$ forms of volume 1 in a connected component of the moduli space $\mathcal{M}_1$ of $G_2$ forms on a compact manifold $M$ with $b_1 = 0$. Suppose $\{-p_1(M)([\phi_n])\} = \{\frac{1}{60\pi^2} \| R_{g_{\phi_n}} \|^2 \}$ is bounded from above. Then there exists a subsequence $\{\phi_{n_k}\}$, a sequence of diffeomorphisms $f_k$ isotopic to the identity, and a $G_2$ form $\phi$ of volume 1 such that $f_k^* \phi_{n_k} \rightarrow \phi$ in $C^1$.

Suppose this conjecture is true, and consider the function $F_1^{\beta}$ with $\beta = -p_1(M)$. From the $C^1$ convergence,

$$[f_k^* \phi_{n_k}] \rightarrow [\phi]$$

$$[f_k^* \ast_{\phi_{n_k}} \phi_{n_k}] \rightarrow [\ast_\phi \phi],$$
and the conjecture implies the set

\[(F_1^3)^c = \{ (\phi) \in \mathcal{M}_1 | F_1^3(\langle \phi \rangle) \leq c \}\]

is compact for any constant \(c\). Since every critical points of \(F_1^3\) is a positive local minimum, a result form Morse theory shows that there exists a unique critical point \(\langle \phi \rangle\) in each connected component of \(\mathcal{M}_1\), and that every connected component of \(\mathcal{M}_1\) is contractible. By **Theorem 1**, \([*\phi\phi]\) must be a constant multiple of \(-p_1(M)\). Since \(\mathcal{M} \cong \mathcal{M}_1 \times \mathbb{R}^+\), this implies every connected component of \(\mathcal{M}\) is contractible and contains a unique element \(\langle \phi \rangle\) such that \([*\phi\phi] = -p_1(M)\). Such \(\langle \phi \rangle\) is in fact unique up to diffeomorphism, for Pontryagin class is a diffeomorphism invariant.

## 5 Torelli map

**Definition 3** Let \(M\) be a \(G_2\) manifold, and let \(\mathcal{M}\) be the moduli space of \(G_2\) structures on \(M\). Torelli map \(\pi : \mathcal{M} \to H^3(M, R) \oplus H^1(M, R)\) is defined by

\[\pi(\langle \phi \rangle) = (\pi^3(\langle \phi \rangle), \pi^4(\langle \phi \rangle)) = ([\phi], [*\phi\phi]).\]

For the period map of Riemann surfaces of fixed genus, the image is known to be an open set of an irreducible analytic subset [Gr]. An essential characterization of the image of the period map, inversion problem, is also known via automorphic forms [AM].

The purpose of this section is to describe the image of the Torelli map \(\pi(\mathcal{M})\), or the projections \(\pi^3(\mathcal{M}_1)\) and \(\pi^4(\mathcal{M}_1)\) in more detail. A general idea is that any invariant of \(\pi(\mathcal{M})\) as a Lagrangian submanifold, or the invariants of \(\pi^3(\mathcal{M}_1)\) as an affine hypersurface, is an invariant of \(\mathcal{M}\) or \(\mathcal{M}_1\).

Set \(l = b^3\), and let \(\{\alpha_1, \alpha_2, .. \alpha_l\}\) be a basis of \(H^3(M, R)\) and \(\{\beta^1, \beta^2, .. \beta^l\}\) be the dual basis of \(H^4(M, R)\) so that \(\beta^A(\alpha_B) = \delta^A_B\). Let \(x^t = (x^1, x^2, .. x^t)\) and \(y^t = (y_1, y_2, .. y_t)\) be the coordinate functions on \(H^3(M, R)\) and \(H^4(M, R)\) with respect to these bases. Then
\[ \sum_A dA^A \wedge dA^A \] is the canonical symplectic form, and \( G_0 = \sum_A dA^A dA^A \) is the canonical metric of signature \((l, l)\) on \( H^3(M, R) \oplus H^4(M, R) \).

From the definition,

\[
\pi^3(\phi) = \sum_A x^A(\langle \phi \rangle) \alpha_A
\]
\[
\pi^4(\phi) = \sum_A y_A(\langle \phi \rangle) \beta^A,
\]

where

\[
x^A(\langle \phi \rangle) = \beta^A([\phi]),
\]
\[
y_A(\langle \phi \rangle) = \alpha_A([\ast_\phi \phi]).
\]

Each \( x \) and \( y \) is then a local diffeomorphism from \( \mathfrak{M} \) to \( R^l \) [Jo].

Define a function \( U : \mathfrak{M} \to R \) by

\[
U(\langle \phi \rangle) = \sum_A x^A y_A = 7 Vol_\phi(M) = \int_M \phi \wedge \ast_\phi \phi.
\]

Then,

\[
dU = \sum_A x^A dy^A + y_A dx^A = \int_M \phi \wedge d\pi^4 + d\pi^3 \wedge *_\phi \phi.
\]
\[
= \frac{7}{3} \int_M d\pi^3 \wedge *_\phi \phi.
\]
\[
= \frac{7}{3} \sum_A y_A dx^A,
\]

where the third equality follows from (4). Hence

\[
3 \sum_A x^A dy^A = 4 \sum_A y_A dx^A,
\]

and the Torelli map \( \pi \) is a Lagrangian immersion [Jo].

Let \( p = (p_{AB}) = (p_{BA}) \) be the unique \( Gl(l, R) \) valued function on \( \mathfrak{M} \) such that

\[
dy_A = \sum_B p_{AB} dx^B.
\]
From (20), we have

$$4y = 3px.$$  \hspace{1cm} (21)

Thus the invertible symmetric matrix function \(p\) transforms one period function to the other.

Upon a change of basis \(x^* = A^{-1}x, y^* = A^t y\) for \(A \in GL(l,R)\), \(p\) becomes \(p^* = A^t p A\), and \(dx^t p dx = dx^t dy\) is a well defined quadratic form on \(\mathcal{M}\), which is the metric induced from \(G_0\) via Torelli map \(\pi\).

**Proposition 3** The signature of \(\pi^*G_0\) is \((1 + b^1, b^3 - b^1 - 1)\). If \(b^1 = 0\), \(\pi^*G_0\) is the negative of the \(L^2\) metric when restricted to \(\mathcal{M}_1 \subset \mathcal{M}\).

**Proof.** Let \(X \in T_\phi \hat{\mathcal{M}}\). Then, with an abuse of notation,

$$\pi^*G_0(X,X) = dx^t(X) dy(X)$$

$$= \frac{4}{3} \ll X_1, X_1 \gg_\phi + \ll X_7, X_7 \gg_\phi - \ll X_{27}, X_{27} \gg_\phi$$

by (4). \(\square\)

Let \(\{\xi_A(\phi)\}_1^l\) be a basis of \(H^3(M,R)\) where \([\xi_A(\phi)] = \alpha_A\) and \(\xi_A(\phi)\) is harmonic with respect to \(g_\phi\). Define \(m_{AB}(\langle \phi \rangle)\) by

$$[*_\phi \xi_A(\phi)] = \sum_B m_{AB} \beta^B,$$

or equivalently

$$m_{AB}(\langle \phi \rangle) = \int_M *_\phi \xi_A(\phi) \wedge \xi_B(\phi),$$  \hspace{1cm} (22)

and

$$m_{AB} dx^A dx^B$$

is the \(L^2\) metric on \(\mathcal{M}\). Suppose \(b^1 = 0\). From **Proposition 3**, we get

$$p_{AB} = -m_{AB} + \frac{7}{3U} y_A y_B$$

$$p^{AB} = -m^{AB} + \frac{7}{4U} x^A x^B,$$  \hspace{1cm} (23)
where \((p^{AB})\) and \((m^{AB})\) are the inverse matrices of \((p_{AB})\) and \((m_{AB})\) respectively.

We now turn our attention to the hypersurface \(\Sigma = [\mathcal{M}_V] \subset H^3(M, R)\), where \(\mathcal{M}_V \subset \mathcal{M}\) is the set of \(G_2\) structures of volume \(V\). For definiteness, let us assume \(\{\alpha_1, \alpha_2, .. \alpha_l\}\) is a basis of \(H^3(M, Z)\) modulo torsion, choose an orientation for \(H^3(M, R)\), and consider the properties of \(\Sigma\) that are invariant under the linear change of basis by \(Sl(l, R)\) [Ch2][NS]. We agree on the index range \(1 \leq i, j \leq l - 1\) and \(1 \leq A, B \leq l\).

Let
\[
\vec{x} = \sum_A x^A \alpha_A
\]
\[
\vec{y} = \sum_A y_A \beta^A
\]
be the immersions defined by (17), (18). Then since \(\mathcal{M}_V\) is defined by the equation \(U = 7V\),
\[
\sum_A y_A dx^A = \sum_A x^A dy_A = 0
\]
on \(\mathcal{M}_V\), and we may write
\[
d\vec{x} = \sum_i dx^i \alpha_i + dx^l \alpha_l
\]
\[
= \sum_i dx^i (\alpha_i - \frac{y_i}{y_l} \alpha_l) + (dx^l + \frac{1}{y_l} \sum_i y_i dx^i) \alpha_l
\]
\[
= \sum_i \omega^i e_i + \omega^l e_l
\]
where
\[
e_i = \alpha_i - \frac{y_i}{y_l} \alpha_l
\]
\[
e_l = \alpha_l
\]
and
\[
\omega^i = dx^i
\]
\[
\omega^l = dx^l + \frac{1}{y_l} \sum_i y_i dx^i = 0.
\]
Following the general theory of moving frames, put
\[ de_A = \sum_B \omega_A^B e_B. \]

Differentiating (27), we get
\[ de_i \equiv \omega_i^j e_j \mod e_1, e_2, .. e_{l-1} \]
\[ \equiv -d\left(\frac{y_i}{y_l}\right)e_l, \]
and
\[ \omega_i^l = -d\left(\frac{y_i}{y_l}\right) \]
\[ = \sum_j h_{ij} \omega^j \]
where
\[ h_{ij} = h_{ji} \]
\[ = -\frac{p_{ij}}{y_l} - \frac{p_{il}}{y_l} y_i y_j + \frac{y_i y_j + y_j y_i}{y_l^2}. \]

By (28), the second fundamental form \( II \) of \( \Sigma \)
\[ II = \sum_{ij} h_{ij} \omega^i \omega^j \]
\[ = -\frac{1}{y_l} \sum_{AB} p_{AB} dx^A dx^B, \]
and it is definite whenever \( b^1 = 0 \) by Proposition 3.

Set \( H = \det(h_{ij}) \neq 0 \). Then the normalized second fundamental form
\[ \hat{II} = |H|^{-\frac{1}{1+m}} II \]
is an affine invariant called Blaschke metric.

Remark. If \( b^1 = 0 \), one may assume \( H > 0 \) on \( \Sigma \).

Since
\[ \omega_1^l \wedge \omega_2^l \ldots \wedge \omega_{l-1}^l = H \omega_1 \wedge \omega_2 \wedge \ldots \wedge \omega_{l-1}, \]
a computation using (30) gives,
\[
\omega_1 \wedge \omega_2 \ldots \wedge \omega_{l-1} = \frac{1}{y_l} \left( \sum_A y_A \frac{\partial}{\partial y_A} \right) dy_1 \wedge dy_2 \wedge \ldots \wedge dy_l
\]
(33)
\[
= \frac{1}{y_l} \left( \sum_A x^A \frac{\partial}{\partial x^A} \right) dx_1 \wedge dx_2 \wedge \ldots \wedge dx_l
\]
(34)
for \( \sum_A y_A \frac{\partial}{\partial y_A} = \frac{3}{4} \sum_A x^A \frac{\partial}{\partial x^A} \) by (21). By expanding (34) and using \( U = 7V \),
\[
H = \frac{21}{4} V \det(p)(-1)^{l-1} y_l^{-(l+1)},
\]
(35)
and
\[
\hat{H} = -\left( \frac{21}{4} V \right)^{\frac{l}{l+1}} |\det(p)|^{\frac{1}{l+1}} \sum_{AB} P_{AB} dx^A dx^B.
\]
(36)
Here we assume \( \{ \alpha_1, \alpha_2, \ldots \alpha_l \} \) is the right orientation for \( H^3(M, R) \).

Another affine invariant called affine normal \( \xi^x \) is defined by
\[
\xi^x = |H|^{\frac{1}{l+1}} (e_i + \sum_i t^i e_i),
\]
(37)
where \( t^i \)'s are uniquely determined by the equation
\[
\frac{1}{l+1} d \log |H| + \sum_i t^i \omega_i^l = 0 = \lambda \sum_A x^A dy_A,
\]
(38)
for some nonzero \( \lambda \).

Set
\[
q^A = \frac{1}{(l+1)\det(p)} \frac{\partial \det(p)}{\partial y_A}.
\]
Then from (35) and (30), \( \lambda = \frac{1}{7V} (-1 + \sum_A y_A q^A) \), and we get
\[
t^i = y_l (q^i - x^i \lambda).
\]
(39)

The affine normal \( \xi^x \) is now determined to be
\[
\xi^x = |H|^{\frac{1}{l+1}} (e_i + \sum_i t^i e_i),
\]
(40)
\[
= |H|^{\frac{1}{l+1}} \frac{y_l}{7V} \left( (\sum_A x^A \alpha_A) + (7V q^A - x^A (\sum_B y_B q^B)) \alpha_A \right)
\]
\[
= \frac{1}{7V} \left( \frac{21}{4} V \right)^{\frac{l}{l+1}} |\det(p)|^{\frac{1}{l+1}} \left( (\sum_A x^A \alpha_A) + (7V q^A - x^A (\sum_B y_B q^B)) \alpha_A \right).
\]
A hypersurface in an affine space is an affine sphere if all the normal lines pass through a fixed point (finite or infinite) called the center. In case it is locally convex, it is called elliptic or hyperbolic depending on whether the center is on the convex or concave side.

**Theorem 2** The locus of projection \( \pi^3(\mathcal{M}_1) \subset H^3(M, R) \) is an affine sphere centered at the origin if \( \det(p) \) is constant along \( \mathcal{M}_1 \). In case \( b^1 = 0 \), this is equivalent to \( \det(m) \) being constant on \( \mathcal{M}_1 \), and \( \pi^3(\mathcal{M}_1) \) is a (locally convex) hyperbolic affine sphere.

**Remark.** \( \det(m) \) is constant on \( \mathcal{M}_1 \) if and only if the volume of the torus \( H^3(M, R)/H^3(M, Z) \) is constant on \( \mathcal{M}_1 \).

**Proof.** \( \xi^x \) is proportional to \( \vec{x} \) if \( q^A = \mu x^A \) for all \( A \) for some function \( \mu \) on \( \mathcal{M}_1 \). By (40) and (25), this implies \( d \det(p) = 0 \) along \( \mathcal{M}_1 \).

In case \( b^1 = 0 \),

\[
\sum_{AB} p^{AB} dp_{AB} \equiv \sum_{AB} m^{AB} dm_{AB} \mod dU
\]

from (23). \( \pi^3(\mathcal{M}_1) \) is locally convex by **Proposition 3** and hyperbolic by **Theorem 1**.

\( \square \)

Similar computation for the conormal hypersurface \( \Sigma' = \pi^4(\mathcal{M}_1) \subset H^4(M, R) \) defined by \( \tilde{y} \) in (24) gives the affine normal

\[
\xi_y = \frac{1}{7V} \left( \frac{3}{28V} \right) \sum_A \frac{\partial \det(p)}{\partial x^A} \left( \sum_A y_A \beta^A \right) + \left( -7Vq_A + y_A(\sum_B x^B q_B) \right) \beta^A,
\]

with

\[
q_A = \frac{1}{(l + 1)\det(p)} \frac{\partial \det(p)}{\partial x^A} = \sum_B p_{AB} q^B.
\]
Note the functions
\[
\xi_y(\bar{x}) = \left(\frac{3}{28V}\right)^{\frac{l+1}{l+1}}|\det(p^{-1})|^{\frac{1}{l+1}}
\]
\[
\xi^x(\bar{y}) = \left(\frac{21V}{4}\right)^{\frac{l-1}{l+1}}|\det(p)|^{\frac{1}{l+1}}
\]
\[
\xi_y(\xi^x) = \frac{1}{7V}\left(\frac{9}{16}\right)^{\frac{l}{l+1}}\left(1 + \left(\sum_A x^A q_A\right)\left(\sum_B y_B q_B\right) - 7V \sum_A q^A q_A\right).
\]

are independent of the choice of the orientation for \(H^3(M, R)\). From (19),
\[
\sum_A A \frac{\partial \det(p)}{\partial x^A} = \frac{l}{3} \det(p)
\]
\[
\sum_A y_A \frac{\partial \det(p)}{\partial y_A} = \frac{l}{4} \det(p).
\]

Hence
\[
\xi_y(\xi^x) = \frac{1}{7V}\left(\frac{9}{16}\right)^{\frac{l}{l+1}}\left(1 + \frac{l^2}{12(l+1)^2} - 7V \sum_A q^A q_A\right).
\]
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