Strong Decoherence*

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Abstract

We introduce a condition for the strong decoherence of a set of alternative histories of a closed quantum-mechanical system such as the universe. The condition applies, for a pure initial state, to sets of homogeneous histories that are chains of projections, generally branch-dependent. Strong decoherence implies the consistency of probability sum rules but not every set of consistent or even medium decoherent histories is strongly decoherent. Two conditions characterize a strongly decoherent set of histories: (1) At any time the operators that effectively commute with generalized records of history up to that moment provide the pool from which — with suitable adjustment for elapsed time — the chains of projections extending history to the future may be drawn. (2) Under the adjustment process, generalized record operators acting on

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the initial state of the universe are approximately unchanged. This expresses the permanence of generalized records. The strong decoherence conditions (1) and (2) guarantee what we call “permanence of the past” — in particular the continued decoherence of past alternatives as the chains of projections are extended into the future. Strong decoherence is an idealization capturing in a general way this and other aspects of realistic physical mechanisms that destroy interference, as we illustrate in a simple model. We discuss the connection between the reduced density matrices that have often been used to characterize mechanisms of decoherence and the more general notion of strong decoherence. The relation between strong decoherence and a measure of classicality is briefly described.
I. INTRODUCTION

In this article we continue our efforts to explore the quantum mechanics of closed systems, most generally and realistically the universe as a whole, and within that framework to understand the significance of the quasiclassical realm\footnote{In previous work we called a decoherent set of alternative coarse-grained histories a “domain”. However, that term can be confusing because of its other uses in physics. We do not want to call such a set a “world” because that word connotes a single history and not a set of alternative ones. Hence, we now call a decoherent set of alternative coarse-grained histories a “realm”.} that includes familiar experience.

We introduce a strong realistic principle of decoherence for sets of alternative coarse-grained histories of a closed system and discuss the relationships between this and other principles that have been put forward. We also examine the concept of classicality, including the role played by the realistic principle of decoherence in characterizing it.

The most general predictions of quantum mechanics are the probabilities of individual members of a set of alternative coarse-grained histories of the universe\footnote{These are \textit{a priori} probabilities. They can also be thought of as the statistical probabilities for an ensemble of universes, but in that case we have access only to one member of this ensemble.} A set of coarse-grained histories is a partition of one of the sets of fine-grained histories (which are the most refined possible descriptions of the closed system) into mutually exclusive classes. The classes are the individual coarse-grained histories, which are thus “bundles” of fine-grained histories.

The absence of quantum-mechanical interference between the individual histories in a set is necessary, at the very least, for quantum theory to assign consistent probabilities to the alternative possibilities. Such sets of histories for which interference is absent are said to \textit{decohere}. Except for pathological cases, coarse-graining is necessary for decoherence.

Various conditions for the decoherence of sets of histories have been proposed. Some authors have tried to weaken the condition as much as possible to get the minimum condition necessary for probabilities to be defined. Our point of view has always been to try and describe a realistic principle of decoherence that characterizes in a general way the physical processes by which the dissipation of interference occurs. We have therefore been led to investigate conditions of decoherence that are as strong as possible compatible with the physical mechanisms that destroy interference. We shall investigate such a strong condition for decoherence in this paper.

Implementing a strong form of decoherence is part of a program to understand how the quasiclassical realm that includes everyday experience arises in quantum mechanics from the Hamiltonian of the elementary particles and the initial condition of the universe\footnote{A set of coarse-grained histories is a partition of one of the sets of fine-grained histories (which are the most refined possible descriptions of the closed system) into mutually exclusive classes. The classes are the individual coarse-grained histories, which are thus “bundles” of fine-grained histories.} By a quasiclassical realm we mean an exhaustive set of mutually exclusive coarse-grained alternative histories that obey a realistic principle of decoherence, that consist largely of similar but branch-dependent alternatives at a succession of times, with individual histories exhibiting patterns of correlations implied by effective classical equations of motion subject to frequent small fluctuations and occasional major ones, the whole set being maximally refined given these properties. The theory may exhibit essentially inequivalent quasiclassical realms, but there is certainly at least one that includes familiar experience. This is the \textit{usual} quasiclas-
tical realm, described, at least in part, by alternative values of hydrodynamic operators that are integrals, over suitable volumes, of densities of conserved or nearly conserved quantities. Examples are densities of energy, momentum, baryon number, and, in late epochs of the universe, of nuclei and even chemical species. The sizes of the volumes and the spacing of the alternatives in time are limited above by maximality. The size and spacing are limited below by decoherence and the requirement that the volumes have sufficient “inertia” to enable them to resist deviations from predictability caused by quantum spreading and by the noise that typical mechanisms of decoherence produce [1,2].

A key property of the usual quasiclassical realm is the persistence of the past. Histories of quasiclassical alternatives up to a given time can be extended into the future to give further such histories without endangering the decoherence of the past alternatives. This persistence of the past is not guaranteed by quantum mechanics alone. Extending a set of histories into the future is a kind of fine graining and this carries the risk of losing decoherence. However, the persistence of the past is critical to the utility of the quasiclassical realm. It is the reason that we do not need to do an elaborate calculation verifying the preservation of past decoherence on every occasion when we want to predict the probability of a quasiclassical alternative in the future conditioned on our experience of the past. We proceed, secure in the understanding that in a quasiclassical realm the past (including the decoherence of past alternatives) will continue to persist.

In this article we discuss a strong form of decoherence that guarantees the persistence of the past. The idea is closely related to the notion of “generalized” records that we treated in our earlier work [4]. The physical picture is that, at every branching of the coarse-grained histories of the universe, each of the exhaustive and mutually exclusive possibilities is correlated with a different state of something like a photon or neutrino going off to infinity and unaffected by subsequent alternatives. The orthogonality of those states is the realistic mechanism underlying decoherence. For each of the alternative coarse-grained histories up to some time, a projection operator $R$ describes the information of that kind that has been stored up. The projections constitute the “generalized records” associated with the different histories. They are all orthogonal to one another and that orthogonality gives rise to the decoherence of histories.

The present work assumes a pure state for the universe, that is, a density matrix $\rho$ of the form $|\Psi\rangle\langle\Psi|$. In some earlier articles [4] we allowed $\rho$ to be more general. We then defined another kind of “strong decoherence” which was equivalent, for a pure state, to medium decoherence. We now suggest restricting the term “strong decoherence” to what we are discussing here and abandoning it as a name for the earlier concept, which may be too restrictive when the state is not pure and is redundant otherwise [2]. For the rest of the article we assume that $\rho = |\Psi\rangle\langle\Psi|$.

The present strong decoherence condition is stronger than our earlier “medium decoherence”, which in turn is stronger than our “weak decoherence” condition. Other authors have discussed still weaker conditions, for example, the “consistent histories” condition of Griffiths [5] and Omnèes [6] and the linearly positive histories of Goldstein and Page [7]. A

‡Indeed, Dowker and Kent [3] have given examples with special final conditions where a quasiclassical realm cannot be extended at all.
simple and instructive case of the present strong decoherence was discussed in an insightful paper by Finkelstein [8], who called it “PT-decoherence” and showed how it is related to the “decoherence of density matrices” that has been discussed by many (e.g. [9–12]). We shall consider this relationship in a more general context and show how a variety of reduced density matrices can be constructed for individual histories up to a given time that are diagonal in appropriate alternatives at the next time as a consequence of strong decoherence.

In Section II we shall review the various decoherence conditions after introducing some necessary notation. Section III introduces strong decoherence and describes the connection with reduced density matrices. Section IV explores these ideas in simplified models in which the coarse grainings are restricted to those that follow one set of fundamental coordinates while ignoring all others. In Section V we review our program to provide a measure of classicality and discuss the role that strong decoherence might play in such a program.

II. VARIETIES OF DECOHERENCE

The ideas of the quantum mechanics of closed systems, including the (medium) decoherence of sets of alternative coarse-grained histories, can be formulated in perfect generality for quantum field theory. One can include the effects of a quantized spacetime metric, as in a field theory (Lagrangian) version of superstring theory, by using the principles of generalized quantum theory [13–15]. However, it is convenient, as well as an excellent approximation for many accessible coarse grainings, to consider a fixed spacetime geometry with well defined timelike directions. In the following brief review of the quantum mechanism of closed systems we shall adopt this approximation, using a time variable $t$ and the associated Hamiltonian $H$.

One way of specifying a set of alternative histories is to give sets of alternative projection operators as a sequence of times $t_1 < t_2 \cdots < t_n$. At each time $t_k$, we have a set of Heisenberg picture projection operators $\{P_k^\alpha_{\alpha_k\cdots\alpha_1}(t_k; t_{k-1}, \ldots, t_1)\}$ where $\alpha_k = 1, 2, 3 \cdots$ denotes the particular alternative in the set. The notation is designed to indicate the branch dependence that is characteristic of useful sets of alternative coarse-grained histories of the universe [3]. The different alternatives in the set at time $t_k$ correspond to different $\alpha_k$. However, in a branch-dependent set of histories, the set of alternatives at a given time will depend on previous history. Useful sets of alternative histories of the universe (such as those constituting a quasiclassical realm) will be branch-dependent because the efficacy of physical mechanisms of decoherence depends on particular present circumstances and past history. Branch dependence is indicated explicitly by the extended subscript $\alpha_k\alpha_{k-1}\cdots\alpha_1$ and the dependence on previous times $t_{k-1}\cdots t_1$. The projection operators are mutually exclusive and exhaustive as expressed by the relations:

$$P_k^\alpha_{\alpha_k\cdots\alpha_1}P_k^{\alpha'_k\alpha_{k-1}\cdots\alpha_1} = \delta_{\alpha_k\alpha'_k}P_k^\alpha_{\alpha_k\cdots\alpha_1}, \quad \Sigma_{\alpha_k}P_k^\alpha_{\alpha_k\cdots\alpha_1} = I,$$

(2.1)

where, as will often be convenient, we have suppressed the time labels for the sake of compactness. The same physical set of alternatives can be expressed at later times $t > t_k$ by way of the Heisenberg equations of motion

$$P_k^\alpha_{\alpha_k\cdots\alpha_1}(t; t_{k-1}\cdots t_1) = e^{iH(t-t_k)}P_k^\alpha_{\alpha_k\cdots\alpha_1}(t_k; t_{k-1}\cdots t_1)e^{-iH(t-t_k)}.$$

(2.2)
Each history is then a particular sequence of alternatives $\alpha = (\alpha_1, \cdots, \alpha_k)$ and is represented by the corresponding chain of projections:

$$H_{\alpha_k \cdots \alpha_1} = P^k_{\alpha_k \cdots \alpha_1}(t_k; t_{k-1}, \cdots, t_1)P^{k-1}_{\alpha_{k-1} \cdots \alpha_1}(t_{k-1}; t_{k-2}, \cdots, t_1) \cdots P^1_{\alpha_1}(t_1). \quad (2.3)$$

As mentioned above, we shall not always indicate the various times. Indeed, since any time label may be altered (preserving the order) by reexpressing the corresponding projections in terms of field operators at another time using the equations of motion, we shall generally suppress these labels.

Sets of histories consisting of chains of projections like (2.3) are not the only sets potentially assigned probabilities by quantum mechanics. As we have mentioned, the general notion of coarse-graining is a partition of a fine-grained set into classes $c_\beta$, $\beta = 1, 2, \cdots$. Such histories may consist of sums of chains

$$C_\beta = \sum_{(\alpha_1, \cdots, \alpha_k) \in \beta} H_{\alpha_k \cdots \alpha_1}. \quad (2.4)$$

Evidently,

$$\sum_\beta C_\beta = I. \quad (2.5)$$

Various authors have discussed different conditions for when a set of histories $\{c_\beta\}$ decoheres and can be assigned probabilities $p(\beta)$ in quantum theory. Below we list them in increasing order of strength — first for an initial condition described by a density matrix $\rho$ and then for the special case that $\rho$ is pure, $\rho = |\Psi\rangle \langle \Psi|$.

- The “linearly positive” condition of Goldstein and Page [7]:

$$p(\alpha) = Re\ Tr(C_\alpha \rho) \geq 0, \quad (2.6a)$$
$$p(\alpha) = Re\langle \Psi| C_\alpha |\Psi\rangle \geq 0. \quad (2.6b)$$

- The “consistent histories” condition of Griffiths [3] and Omnès [8] for sets of histories that are chains of the form (2.3) (homogeneous sets):

$$Re\ Tr(C_{\alpha'} \rho C^\dagger_{\alpha}) = \delta_{\alpha'\alpha} p(\alpha), \quad (2.7a)$$
$$Re\langle \Psi| C^\dagger_{\alpha} C_{\alpha'} |\Psi\rangle = \delta_{\alpha'\alpha} p(\alpha), \quad (2.7b)$$

provided $C_\alpha + C'_{\alpha}$ is a chain as well.

\footnote{These are called \textit{homogeneous} histories by Isham [13]. We used a slightly different notation in [2] with $P^k_{\alpha_k}(t_k; \alpha_{k-1}, t_{k-1}, \cdots, \alpha_1)$ instead of $P^k_{\alpha_k \cdots \alpha_1}(t_k; t_{k-1} \cdots t_1)$.}
• Weak decoherence:

\[
\begin{align*}
\text{Re} \, \text{Tr}(C_{\alpha'} \rho C_{\alpha}^\dagger) &= \delta_{\alpha',\alpha} p(\alpha), \\
\text{Re} \, \langle \Psi | C_{\alpha}^\dagger C_{\alpha'} | \Psi \rangle &= \delta_{\alpha',\alpha} p(\alpha),
\end{align*}
\]

with no restriction to chains or on the sums of \(C\)'s.

• Medium decoherence:

\[
\begin{align*}
\text{Tr}(C_{\alpha'} \rho C_{\alpha}^\dagger) &= \delta_{\alpha',\alpha} p(\alpha), \\
\langle \Psi | C_{\alpha}^\dagger C_{\alpha'} | \Psi \rangle &= \delta_{\alpha',\alpha} p(\alpha),
\end{align*}
\]

again with no restrictions on the \(C\)'s.

In this paper we shall discuss a yet stronger condition of decoherence, which for a pure state has the form

\[
\langle \Psi | C_{\alpha}^\dagger M^\dagger M' C_{\alpha'} | \Psi \rangle = 0, \quad \alpha \neq \alpha',
\]

for any operators \(M\) included in a set \(\{M\}_\alpha\) and \(M'\) included in a set \(\{M\}_{\alpha'}\), both sets including the identity, \(I\). We shall write this as

\[
\langle \Psi | C_{\alpha}^\dagger \{M\}_\alpha \{M\}_{\alpha'} C_{\alpha'} | \Psi \rangle = 0 \quad \alpha' \neq \alpha.
\]

where the occurrence of \(\{M\}\) in an equation means that it holds for each \(M \in \{M\}\). The probabilities for a set satisfying this condition are

\[
p(\alpha) = \langle \Psi | C_{\alpha}^\dagger C_{\alpha} | \Psi \rangle.
\]

The properties of the sets \(\{M\}_\alpha\) that make this a condition of strong decoherence are discussed in the next section.

III. STRONG DECOHERENCE

We shall now introduce a notion of strong decoherence applying to a set of histories that are chains of projections. This strong decoherence is a special form of medium decoherence and thus permits the assignment of probabilities to any set of histories that is a coarse graining of the set of chains, whether or not the sums of chains involved are themselves chains. Such coarse grainings can also be considered to be strongly decoherent.

The definition of strong decoherence is connected with the properties of generalized records of histories that we have described in earlier work. When a set of histories obeys medium decoherence and the initial condition is a pure state \(|\Psi\rangle\), then the branch states \(C_\alpha |\Psi\rangle\) corresponding to the individual histories are mutually orthogonal. There is, therefore, a set of orthogonal projection operators \(\{R_\alpha\}\) on the branches. Indeed, unless the branches constitute a basis in Hilbert space (a full set of histories), there will be many different possible choices of the \(\{R_\alpha\}\). Thus, medium decoherence implies
\[ C_\alpha |\Psi \rangle = R_\alpha |\Psi \rangle \]  
(3.1)

for various sets of projections satisfying
\[ R_\alpha R_{\alpha'} = \delta_{\alpha \alpha'} R_\alpha . \]  
(3.2)

The \( R_\alpha \) are generalized records of the histories \( C_\alpha \). We call them “generalized” records because it does not follow from this definition alone that these operators have any of the further properties that might normally be required of records — being quasiclassical operators, persisting over a period of time, being accessible to an observer, etc.

Medium decoherence and a pure initial state imply the existence of the generalized records as we have just described, but conversely the existence of generalized records yields medium decoherence. To see this, merely note that eqs. (3.1) and (3.2) imply the orthogonality of the branches \( C_\alpha |\Psi \rangle \) and that is what medium decoherence means. In this paper we are going to discuss generalized records of a special kind.

We are going to introduce at any stage in history a pool of operators from which future history may be drawn so as to ensure the permanence of the past. This pool will contain some operators that are very different from the record operators, as in the realistic mechanisms we have described, but correlated with those record operators through the state \( |\Psi \rangle \). In general, the histories will consist of chains of projections that have this character (or sums of such chains). To that end, it is useful, for a given choice of generalized records, to consider the set of operators \( \{ M \} \) that commute with the \( R_\alpha \) when acting on the branch state vector, that is
\[ [M,R_\alpha]C_\alpha |\Psi \rangle = 0, \]  
(3.3a)

or
\[ [M,R_\alpha]R_\alpha |\Psi \rangle = 0. \]  
(3.3b)

We denote this linear space of operators by \( \{ M \}_\alpha \). Writing out (3.3) and using \( (R_\alpha)^2 = R_\alpha \), one finds the equivalent condition
\[ R_\alpha M R_\alpha |\Psi \rangle = M R_\alpha |\Psi \rangle . \]  
(3.4)

This says that \( M R_\alpha |\Psi \rangle \) lies in the subspace designated by \( R_\alpha \). Thus for \( M \in \{ M \}_\alpha \) and \( M' \in \{ M \}_{\alpha'} \), we have that \( M R_\alpha |\Psi \rangle \) is orthogonal to \( M' R_{\alpha'} |\Psi \rangle \), or, what is the same thing, \( M C_\alpha |\Psi \rangle \) is orthogonal to \( M' C_{\alpha'} |\Psi \rangle \).

The sets \( \{ M \}_\alpha \) that effectively commute with the generalized records \( R_\alpha \) are thus operators for which
\[ \langle \Psi | C_\alpha^\dagger \{ M' \}^\dagger \{ M \}_\alpha C_{\alpha'} |\Psi \rangle = 0, \quad \alpha' \neq \alpha , \]  
(3.5)
as advertised in (2.11).

Each \( R_\alpha \) can be expressed in terms of the operators in \( \{ M \}_\alpha \) and the branch states \( C_\alpha |\Psi \rangle \). In fact,
\[ R_\alpha = \text{Proj}(\{ M \}_\alpha C_\alpha |\Psi \rangle) . \]  
(3.6)
by which we mean that \( R_\alpha \) projects onto the subspace spanned by the vectors of the form \( M C_\alpha |\Psi\rangle \) for \( M \in \{ M \}_\alpha \). In the future we shall use the notation \( R_\alpha \) both for the operator and the subspace onto which it projects. To derive (3.6), one simply has to note that, if \( |v\rangle \) is a vector in \( R_\alpha \), then it necessarily has the form \( MR_\alpha |\Psi\rangle \) for some operator \( M \) that effective commutes with \( R_\alpha \) as in (3.3). (The operator \( M = |v\rangle \langle \Psi| R_\alpha \) is a simple example.) Eq. (3.4) then follows.

The existence of generalized records of some kind and their concomitant \( M \)'s follows from medium decoherence and the assumption of a pure state. The decoherence condition (3.5) is therefore no stronger than medium decoherence unless the generalized records or the concomitant \( M \)'s are restricted in some way. We shall shortly define strong decoherence by conditions on the records that guarantee the permanence of the past. Before doing so, however, it is useful to spell out how the above discussion would go if the \( M \)'s were specified first rather than the \( R \)'s.

Suppose we are given sets \( \{ M \}_\alpha \) that satisfy (3.5). We require all the sets \( \{ M \}_\alpha \) to contain \( I \), and so such decoherent sets of histories will be medium decoherent and generalized records satisfying (3.4) will exist. There are many choices for these records, but an attractive one is (3.6). These \( R \)'s are orthogonal [cf. (3.2)] as a consequence of (3.5). They are generalized records satisfying (3.4) because, with \( M = I \) contained in \( \{ M \}_\alpha \), the branch state vector \( C_\alpha |\Psi\rangle \) is in the subspace \( R_\alpha \). Thus

\[
R_\alpha C_\beta |\Psi\rangle = \delta_{\alpha\beta} C_\beta |\Psi\rangle.
\]

Summing over \( \beta \) gives (3.1). It is then an easy computation to show that, as in (3.3), the \( M \)'s effectively commute with the \( R \)'s defined by (3.6). Eq. (3.6) is not the most general identification of records with this commutation property. One could add to the \( \{ R_\alpha \} \) any mutually orthogonal projectors \( V_\alpha \) that are also orthogonal to all the \( R_\alpha \) given by (3.6). If that is done with non-vanishing \( V_\alpha \), then the class of operators effectively commuting with \( R_\alpha \) is wider than the \( \{ M \}_\alpha \), containing also all operators leading from \( C_\alpha |\Psi\rangle \) to the space \( V_\alpha \).

From now on, we shall adopt the point of view that the \( V \)'s are zero and (3.6) holds. That way either the \( R \)'s or the \( M \)'s could be regarded as primary. We shall also deal with explicitly decoherent sets of histories composed of chains of projectors, bearing in mind that we could also treat any coarse grainings of such sets.

We are now in a position to define a notion of strong decoherence that will provide a systematic, step by step procedure for extending sets of histories into the future so as to guarantee the permanence of the past. We restrict attention to sets of homogeneous histories represented by branch-dependent chains of the form (2.3) at \( n \) times \( t_1, \ldots, t_n \). Partial histories up to time \( t_\ell \) are represented by chains

\[
H_{\alpha_\ell \cdots \alpha_1}^\ell = P_{\alpha_\ell \cdots \alpha_1}^\ell P_{\alpha_{\ell-1} \cdots \alpha_1}^{\ell-1} \cdots P_{\alpha_1}^1,
\]

where we have introduced a superscript \( \ell \) on \( H_{\alpha_\ell \cdots \alpha_1}^\ell \) to indicate that it refers to the first \( \ell \) times. The definition of a strongly decoherent set of histories concerns the relation between the histories at a time \( t_\ell \) and those for all future times up to \( t_n \), for each \( t_\ell \).

Strong decoherence is a special case of medium decoherence, so that all the sets \( \{ H_{\alpha_\ell \cdots \alpha_1}^\ell \} \), \( \ell = 1, \ldots, n \) satisfy the condition (2.4). For each \( \ell \) there is a set of generalized records \( \{ R_{\alpha_\ell \cdots \alpha_1}^\ell \} \). From these, the sets of operators \( \{ M \}^\ell_{\alpha_\ell \cdots \alpha_1} \) that effectively commute with the \( R_{\alpha_\ell \cdots \alpha_1}^\ell \) [cf. (3.3)] may be constructed.
Two conditions define a strongly decoherent set. The first is that at each \( \ell \) and for each history \( \alpha_1 \cdots \alpha_1 \), the set of operators \( \{ M \}_{\alpha_1 \cdots \alpha_1}^\ell \), suitably adjusted for evolution, forms the pool from which future histories are drawn. Specifically, we assume for each \( \ell < k \leq n \)

\[
P_{\alpha_k \cdots \alpha_1}^k \cdots P_{\alpha_{k+1} \cdots \alpha_1}^{\ell+1} \in U_{kl} \{ M \}_{\alpha_1 \cdots \alpha_1}^\ell ,
\]

where \( U_{kl} \) is a unitary operator that adjusts the \( M \)'s in the set \( \{ M \}_{\alpha_1 \cdots \alpha_1}^\ell \) to those appropriate to a later time. The range of possibilities for the \( U_{kl} \), as well as the need for this kind of adjustment, will become clearer when we discuss particular models in the next section. We note that if histories are extended by choosing the future alternatives from among the present \( M \)'s by (3.9), the extended set will continue to be medium decoherent. That follows from (3.5) and the unitary character of the \( U \)'s. The permanence of the past is thus guaranteed in these extensions to the future. In particular, if we construct strongly decoherent sets step by step by choosing the \( P \)'s at the next time from the pool of \( M \)'s at this time, the permanence of the past is assured.

Note that the condition (3.9) incorporates branch dependence. The pool of operators for extending each history depends on that history. Note also, that the condition (3.9) is already more restrictive than just medium decoherence. We showed that the operators that effectively commute with \( R_{\alpha_k \cdots \alpha_1}^k \) satisfy (3.5), but we did not show that every possible medium decoherent extension commutes with the \( R_{\alpha_k \cdots \alpha_1}^k \).

To enforce the condition (3.9) for each pair of times \( \ell \) and \( k \) such that \( \ell < k \leq n \) requires a consistency condition between the sets of \( M \)'s at different times, relations between the adjustment operators \( U \), and restrictions on the concomitant records. The consistency condition for the \( U \)'s is the the elementary requirement of composition:

\[
U_{kl} U_{lj} = U_{kj}.
\]

To discuss the requirements on the \( M \)'s, it is useful to introduce the following notation for \( \ell < k \leq n \):

\[
\{ M \}_{\alpha_k \cdots \alpha_1}^k \equiv U_{kl} \{ M \}_{\alpha_1 \cdots \alpha_1}^\ell ,
\]

\[
\{ R \}_{\alpha_k \cdots \alpha_1}^k \equiv U_{kl} \{ R \}_{\alpha_1 \cdots \alpha_1}^\ell U_{kl}^{\dagger} .
\]

The defining condition (3.9) may then be restated

\[
P_{\alpha_k \cdots \alpha_1}^k \cdots P_{\alpha_{k+1} \cdots \alpha_1}^{\ell+1} \in \{ M \}_{\alpha_1 \cdots \alpha_1}^k .
\]

Consistency requires that not only the future histories be drawn from the pool of \( M \)'s, but also the future \( M \)'s as well. At the weakest this requires that

\[
\text{Proj} \left[ \{ M \}_{\alpha_k \cdots \alpha_1}^k P_{\alpha_k \cdots \alpha_1}^k \cdots P_{\alpha_{k+1} \cdots \alpha_1}^{\ell+1} H_{\alpha_1 \cdots \alpha_1}^\ell |\Psi \right] \subseteq \text{Proj} \left[ \{ M \}_{\alpha_1 \cdots \alpha_1}^k H_{\alpha_k \cdots \alpha_1}^k |\Psi \right].
\]

According to (3.6) the record operators \( R_{\alpha_k \cdots \alpha_1}^k \) are the projections onto the subspaces spanned by the vectors \( \{ M \}_{\alpha_k \cdots \alpha_1}^k H_{\alpha_k \cdots \alpha_1}^k |\Psi \). A consequence of the consistency condition (3.13) is therefore
\[ R_{\alpha_k \ldots \alpha_1}^k \subseteq R_{\alpha_\ell \ldots \alpha_1}^k, \quad \ell < k \leq n. \tag{3.15} \]

Thus the records nest. Eq. (3.15) means that, up to unitary adjustment, the subspaces corresponding to the records narrow with each extension into the future. In this sense the records of the earlier parts of history are preserved in records of the later parts.

Nesting of records (3.15) implies the operator form of the consistency relation (3.14). To see this suppose that \( M \in \{M\}_{\alpha_k \ldots \alpha_1}^k \). This means that \( M \) effectively commutes with \( R_{\alpha_k \ldots \alpha_1}^k \), or explicitly [cf. (3.4)]

\[ R_{\alpha_k \ldots \alpha_1}^k M H_{\alpha_\ell \ldots \alpha_1}^\ell \Psi = M H_{\alpha_k \ldots \alpha_1}^k \Psi. \tag{3.16} \]

This can be rewritten

\[ R_{\alpha_k \ldots \alpha_1}^k \left( M P_{\alpha_k \ldots \alpha_1}^k \ldots P_{\alpha_\ell+1 \ldots \alpha_1}^{\ell+1} \right) H_{\alpha_\ell \ldots \alpha_1}^\ell \Psi = \left( M P_{\alpha_k \ldots \alpha_1}^k \ldots P_{\alpha_\ell+1 \ldots \alpha_1}^{\ell+1} \right) H_{\alpha_\ell \ldots \alpha_1}^\ell \Psi. \tag{3.17} \]

But, since \( R_{\alpha_k \ldots \alpha_1}^k \) is contained within \( R_{\alpha_\ell \ldots \alpha_1}^\ell \), this means that \( M P_{\alpha_k \ldots \alpha_1}^k \ldots P_{\alpha_\ell+1 \ldots \alpha_1}^{\ell+1} \) must effectively commute with \( R_{\alpha_k \ldots \alpha_1}^k \). Thus, \( M P_{\alpha_k \ldots \alpha_1}^k \ldots P_{\alpha_\ell+1 \ldots \alpha_1}^{\ell+1} \in \{M\}_{\alpha_\ell \ldots \alpha_1}^k \). This is the operator form of the consistency relation (3.14)

\[ \{M\}_{\alpha_k \ldots \alpha_1}^k P_{\alpha_k \ldots \alpha_1}^k \ldots P_{\alpha_\ell+1 \ldots \alpha_1}^{\ell+1} \in \{M\}_{\alpha_\ell \ldots \alpha_1}^k. \tag{3.18} \]

Effective consistency (3.14) thus implies the necessary operator form (3.18).

The nesting of generalized records (3.15) can be regarded as a modern form of a very old idea. If the generalized records were to consist of genuinely independent “registrations” in commuting variables (“memory slots”), then we would have

\[ R_{\alpha_k \ldots \alpha_1}^k = \prod_{j=1}^{k} A_{\alpha_j}^{(j)}, \tag{3.19} \]

where the \( \{A_{\alpha_j}^{(j)}\} \) are a set of orthogonal projection operators for each \( j \), all the operators in each set commuting with all those in other sets with different \( j \). In that case proceeding to the next time would result in the registration of a new record in a further exhaustive set of alternatives \( \{A_{\alpha_{k+1}}^{(k+1)}\} \) commuting with all the previous ones. The nesting condition (3.15) would then follow immediately. In the jargon of the early exponents of quantum mechanics, the \( A \)’s, and hence the \( R \)’s, could be regarded as “c-numbers”. The non-commuting operators \( P_{\alpha_k \ldots \alpha_1}^k \ldots P_{\alpha_1}^1 \) (“q-numbers”) are registered in commuting “c-numbers” just as in the old theory of measurement** (See, e.g. [16]).

The nesting inclusion relation (3.13) implies

\[ \sum_{\alpha_{k \ldots \alpha_\ell+1}} R_{\alpha_k \ldots \alpha_\ell+1}^k \subseteq R_{\alpha_\ell \ldots \alpha_1}^k, \tag{3.20} \]

**By means of generalized records the universe could be said to be measuring itself. However, we do not recommend this terminology because, as we have stressed before, these records need not be quasiclassical operators or accessible to an observer as they were in some formulations of the old theory of measurement.
since the $R$'s are a set of orthogonal projection operators. It is implausible that in realistic situations this inclusion relation becomes an equality. Equality in the simplified model just described would require that the $\{A_{\alpha}^{(j)}\}$ be an exhaustive set of projections summing to unity. But there is no particular reason that the alternative configurations of the “memory slots” that record history should exhaust all possible configurations.

However, effective equality is plausible. That is because it follows, by summing both sides of (3.1), that

$$\sum_{\alpha_k \cdots \alpha_{\ell+1}} R_{\alpha_k \cdots \alpha_{\ell}}^k |\Psi\rangle = R_{\alpha_k \cdots \alpha_{\ell}}^\ell |\Psi\rangle.$$  

(3.21)

The effective equality corresponding to the inclusion relation (3.20) would thus mean

$$R_{\alpha_k \cdots \alpha_{\ell}}^\ell |\Psi\rangle \approx R_{\alpha_k \cdots \alpha_{\ell}}^k |\Psi\rangle, \quad k > \ell.$$  

(3.22)

This relation expresses the permanence of the record of the history $(\alpha_1, \cdots, \alpha_\ell)$ that we expect in physically realistic mechanisms of decoherence. We take it to be the second defining property of strong decoherence.

Eq. (3.22) cannot be exactly satisfied because it holds only for times $t_k$ later than $t_\ell$. The vanishing of the difference between left and right hand sides of (3.22) just for $t_k > t_\ell$ cannot be strictly true in quantum mechanics, where evolution is continuous and amplitudes analytic, but it can be true to an excellent approximation, which improves rapidly as the interval increases. Indeed, physically one cannot expect it to hold until a time after $t_\ell$ when the record is created.

Many authors have investigated the properties of reduced density matrices in connection with models of the mechanisms of decoherence (e.g. [9–12]). In the framework we have been discussing, a variety of “reduced” density matrices can be constructed for each history up to a given time. They are generalizations of the kinds considered in those works. Strong decoherence guarantees that these “reduced” density matrices are diagonal with respect to the alternatives at the next time in the history. A very useful example of such a construction is as follows:

To be definite, fix attention on the history $(\alpha_1, \alpha_2)$ and consider the sets $\{M\}_{\alpha_3, \alpha_2, \alpha_1}^3$ for the various possible values of $\alpha_3$. (These are the sets of operators from which the alternatives in the chains of histories at time $t_3$ are drawn.) Take any common subset of all these sets that is closed under Hermitian conjugation and multiplication as well as addition and multiplication by complex numbers. More succinctly, take a common subset that is an algebra closed under Hermitian conjugation. Denote this algebra by $\{F\}_{\alpha_2, \alpha_1}^3$ and let $\{A_{\alpha_2, \alpha_1}^i\}$ be a Hermitian operator basis for it conveniently normalized by $Tr(A_{\alpha_2, \alpha_1}^i A_{\alpha_2, \alpha_1}^j) = \delta_{ij}$. A reduced density matrix may be specified by applying the Jaynes construction to the coarse-graining defined by the operators $\{F\}_{\alpha_2, \alpha_1}^3$. That is, we define $\tilde{\rho}_{\alpha_2, \alpha_1}^i$ as the density matrix

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††We have previously discussed how the creation of permanent records of alternative histories for subsystems can produce decoherence in connection with ideal measurement models. (See [13], Section II.10). These models show how the usual “Copenhagen” quantum mechanics of measured subsystems is an approximation to the more general quantum mechanics of closed systems under discussion here. To the extent that these idealized models reflect typical experimental situations, strong decoherence may be said to be assumed in the “Copenhagen” approximation.
that maximizes the entropy functional $-\text{Tr}(\hat{\rho} \log \hat{\rho})$ subject to the constraint of reproducing the correct expectation values of the $F$’s. Specifically, $\tilde{\rho}^{\alpha_3\alpha_1}$ maximizes entropy subject to the constraints

$$\text{Tr}(A_i^{\alpha_3\alpha_1} \tilde{\rho}) = \langle \Psi_{\alpha_2\alpha_1}^2 | A_i^{\alpha_3\alpha_1} | \Psi_{\alpha_2\alpha_1}^2 \rangle \ .$$

(3.23)

Here $|\Psi_{\alpha_2\alpha_1}^2\rangle$ is the normalized branch state vector corresponding to the history $(\alpha_1, \alpha_2)$:

$$|\Psi_{\alpha_2\alpha_1}^2\rangle \propto H_{\alpha_2\alpha_1}^2 |\Psi\rangle = R_{\alpha_2\alpha_1}^2 |\Psi\rangle \ .$$

(3.24)

Since the set $\{F\}_{\alpha_2\alpha_1}^3$ is closed under multiplication, $\tilde{\rho}^{\alpha_3\alpha_1}$ may be computed explicitly (see, e.g. [22]), and comes out

$$\tilde{\rho}^{\alpha_3\alpha_1} = \sum_i \langle \Psi_{\alpha_2\alpha_1} | A_i^{\alpha_3\alpha_1} | \Psi_{\alpha_2\alpha_1} \rangle A_i^{\alpha_3\alpha_1} \ .$$

(3.25)

Strong decoherence implies that the density matrix defined by eq. (3.25) is diagonal in the context of the model discussed in the next Section.

The important observation in the above argument is the vanishing of the matrix elements (3.26). We could replace the whole notion of a reduced density matrix by eq. (3.26). Indeed, because $\{F\}_{\alpha_2\alpha_1}^3$ is closed under multiplication, a much stronger relation holds

$$\langle \Psi_{\alpha_i'\alpha_2\alpha_1}^3 | \{F\}_{\alpha_2\alpha_1}^3 \{F\}_{\alpha_2\alpha_1}^3 | \Psi_{\alpha_i'\alpha_2\alpha_1}^3 \rangle = 0, \quad \alpha_i' \neq \alpha_3 \ .$$

(3.26)

Thus, the reduced density matrix defined by (3.26) is diagonal in $\alpha_3$. We shall make the connection with the usual construction of reduced density matrices more concrete in the context of the the model discussed in the next Section.

The context of the next alternatives in the chain — those labeled by $\alpha_3$. To see this note that because $\{F\}_{\alpha_2\alpha_1}^3$ is a subset of $\{M\}_{\alpha_3\alpha_2\alpha_1}^3$ for any $\alpha_3$,

$$\langle \Psi_{\alpha_i'\alpha_2\alpha_1}^3 | \{F\}_{\alpha_2\alpha_1}^3 | \Psi_{\alpha_i'\alpha_2\alpha_1}^3 \rangle = 0, \quad \alpha_i' \neq \alpha_3 \ .$$

(3.26)

The $\{A_i^{\alpha_3\alpha_1}\}$ are contained in the $\{F\}_{\alpha_2\alpha_1}^3$, so that

$$\langle \Psi_{\alpha_i'\alpha_2\alpha_1}^3 | \tilde{\rho}^{\alpha_3\alpha_1} | \Psi_{\alpha_i'\alpha_2\alpha_1}^3 \rangle = 0, \quad \alpha_i' \neq \alpha_3 \ .$$

(3.27)

Thus, for any two different values of $\alpha_3$, the matrix elements of the $F$’s do not simply vanish between two branch states $|\Psi_{\alpha_i'\alpha_2\alpha_1}^3\rangle$, and $|\Psi_{\alpha_3\alpha_2\alpha_1}^3\rangle$ for distinct $\alpha_i'$ and $\alpha_3$, but between any vectors in the two subspaces $\{F\}_{\alpha_2\alpha_1}^3 | \Psi_{\alpha_i'\alpha_2\alpha_1}^3 \rangle$ and $\{F\}_{\alpha_2\alpha_1}^3 | \Psi_{\alpha_3\alpha_2\alpha_1}^3 \rangle$. Such sets of operators $\{F\}_{\alpha_2\alpha_1}^3$ are not difficult to find. One example is the set of operators commuting with the $R_{\alpha_3\alpha_2\alpha_1}^3$ for all values of $\alpha_3$. As the $R$’s in physically interesting examples are typically very large subspaces, there are many subalgebras that also give possible sets of $F$’s. We will illustrate all this in the context of a concrete model in the next Section.

The closure of the set of $F$’s under Hermitian conjugation and multiplication gives the relation (3.26), an elegant connection with the method of Jaynes, and, as we shall see, a close connection with the models frequently discussed. But these assumptions are not necessary if all we want is to define a reduced density matrix for the history $(\alpha_1, \alpha_2)$. Consider, for instance, products of the form $M'M$ for $M \in \{M\}_{\alpha_3\alpha_2\alpha_1}^3$, and $M' \in \{M\}_{\alpha_i'\alpha_2\alpha_1}^3$ for various values of $\alpha_i'$ and $\alpha_3$. Suppose all these sets have in common a closed linear set of
operators (depending on \((\alpha_1, \alpha_2)\)) but not necessarily closed under Hermitian conjugation and multiplication. There is still an operator basis and, if we denote that basis by \(\{A_1^{\alpha_2 \alpha_1}\}\), the construction \((3.23)\) still yields a reduced density matrix and it is still diagonal in \(\alpha_3\). The construction with the \(F\)'s is a special case of this. Other examples of “reduced” density matrices could be defined by using different linear subsets of the set of \(M \dagger M\)’s.

Strong decoherence is an idealization and generalization of the ideas emerging from simple models that posit coarse grainings following one set of fundamental coördinates (the “system”) while ignoring the “environment” consisting of the remaining coördinates. In those models the decoherence of histories of alternatives of the “system” is effected by the creation of approximately correlated, approximately permanent records in the “environment”. The generality afforded by the notion of strong decoherence is important because the coarse grainings characterizing the realistic quasiclassical domain (using values of hydrodynamic variables, for instance) do not correspond to a distinction between two kinds of coördinates. In the following, however, we shall make more explicit contact with these earlier ideas by investigating simple models.

**IV. A SIMPLIFIED MODEL**

As we mentioned in the last Section, strong decoherence is a generalization and idealization of the idea that a physical mechanism for decoherence is the correlation of the strings of non-commuting operators constituting alternative histories with commuting operators that are records of those histories. In this Section we shall make this concrete and explicit in a simple model. The model is a specialization of a class that we have already discussed (\[2\], Section IV) and we shall rely on that discussion here.

The kind of model we have in mind divides the fundamental coördinates into two groups, the \(x\)'s and \(Q\)'s, with a corresponding factorization of the Hilbert space \(\mathcal{H} = \mathcal{H}^x \otimes \mathcal{H}^{Q_1} \otimes \mathcal{H}^{Q_2} \otimes \cdots\). We assume that the total Hamiltonian describing this system can be written

\[
H = H_x(x, p) + H_0(Q, P) + H_I(x, Q),
\]

where \(H_x\) acts on the \(x\)'s alone, \(H_0\) is a sum of terms each acting on independently on the Hilbert space \(\mathcal{H}^{Q_i}\) of a single coördinate \(Q_i\), and \(H_I\) is the interaction between all the coördinates.

The coarse grainings studied consist of chains of projections at times \(t_1, t_2, \cdots\) that in the Schrödinger picture act only on \(\mathcal{H}^x\) and represent partitions of some complete set of states in \(\mathcal{H}^x\) at each time. We denote the wave functions of these complete orthogonal sets by \(\{\phi_{r_1}^1(x)\}\) at time \(t_1\), \(\{\phi_{r_2}^2(x)\}\) at time \(t_2\), etc., later augmenting this notation to indicate the branch dependence of the possible sets. If, for instance, the set \(\{\phi_{r_1}^1(x)\}\) is partitioned into classes \(\alpha_1, \alpha_2, \cdots\), the Schrödinger picture projection operator onto the class \(\alpha_k\) is

\[
\langle x''|\hat{P}_{\alpha_k}^1|x'\rangle = \sum_{r_1 \in \alpha_k} \phi_{r_1}^1(x'') \phi_{r_1}^1(x')
\]

We use a hat to distinguish Schrödinger picture operators from their Heisenberg counterparts. The coarse grainings therefore refer only to alternatives in \(\mathcal{H}^x\) which, in the usual terminology, is the “system” while the Hilbert space of the \(Q\)'s is the “environment”. 

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The basic assumptions of the model are most easily stated in the Schrödinger picture. We assume a pure wave function $\Psi(x, Q, t_0)$ at the initial time and prescribe its evolution to later times. The idea is that alternatives in $x$ become correlated with record slots in successive coordinates $Q^1, Q^2, \cdots$ at times $t_1, t_2, \cdots$. Specifically, we assume that, at these times the wave function has the form

$$\Psi(x, Q, t_1) = \sum_{\alpha_1} \chi^{1\alpha_1}(Q^1) \sum_{r_1 \in \alpha_1} \phi^1_{r_1}(x) \bar{\chi}^1_{r_1} \left( Q^2, Q^3, \cdots \right) \quad (4.3a)$$

$$\Psi(x, Q, t_2) = \sum_{\alpha_2 \alpha_1} \chi^{2\alpha_1}(Q^1) \chi^{2\alpha_2 \alpha_1}(Q^2) \sum_{r_2 \in \alpha_2} \phi^{2\alpha_1}_{r_2}(x) \bar{\chi}^{2\alpha_1}_{r_2} \left( Q^3, Q^4, \cdots \right) \quad (4.3b)$$

$$\Psi(x, Q, t_3) = \sum_{\alpha_3 \alpha_2 \alpha_1} \chi^{3\alpha_1}(Q^1) \chi^{3\alpha_2 \alpha_1}(Q^2) \chi^{3\alpha_3 \alpha_2 \alpha_1}(Q^3) \sum_{r_3 \in \alpha_3} \phi^{3\alpha_2 \alpha_1}_{r_3}(x) \bar{\chi}^{3\alpha_2 \alpha_1}_{r_3} \left( Q^4, Q^5, \cdots \right), \quad (4.3c)$$

etc. (We are endeavoring here and elsewhere to avoid a débâuche d’indices by quoting specific forms rather than the general case, the form of which should be immediately apparent). Here the $\chi^{k\alpha_3 \cdots \alpha_1}(Q^j)$ are orthogonal functions of a single coordinate $Q^j$ for time $k$, that is

$$\left( \chi^{k\alpha'_{j-1} \cdots \alpha_1}, \chi^{k\alpha_j \alpha_{j-1} \cdots \alpha_1} \right)_{\mathcal{H}^j} = \delta^{\alpha'_j \alpha_j}. \quad (4.4)$$

The evolution described above is assumed to hold for each branch. That is, the individual terms in the sum over $\alpha_1$ in (4.3a) evolve into the corresponding individual terms in the sums over $\alpha_1$, in (4.3b) and (4.3c) and similarly for the sum over $\alpha_2$ in (4.3d).

The orthogonal functions $\chi^{k\alpha_3 \alpha_2 \alpha_1}(Q^3)$ represent the memory slots in which the alternatives represented by the collection $\{\phi^{3\alpha_2 \alpha_1}_{r_3}\}$, $r_3 \in \alpha_3$ are stored at times $k \geq 3$. The central assumption of the model is that once this registration is accomplished the variable $Q^3$ effectively no longer interacts with the rest of the system. Specifically, we assume that as factors of the complete wave function (4.3), the $\chi^{k\alpha_3 \cdots}$ evolve independently after the time of registration, viz.

$$\chi^{k\alpha_j \cdots \alpha_1}(Q^j) = e^{-iH_0(t_k-t_0)}\chi^{\ell\alpha_j \cdots \alpha_1}(Q^j), \quad k \geq \ell. \quad (4.5)$$

Were $H_0$ effectively zero when acting on these states, we could say that the alternatives $\alpha_j$ were registered in unchanging marks. However, more generally, the marks themselves will evolve according to the Hamiltonian $H_0$.

This model is close in spirit to that used by Finkelstein [8] in his discussion of “PT-decoherence”. Finkelstein also used special coarse grainings that distinguished “system” from “environment”. The assumption (4.5) for the evolution of the records accomplishes the same purpose as his more drastic assumption that the interaction Hamiltonian vanishes between coordinate $Q^j$ and the rest for times $t_k > t_j$.

The alert reader will not have failed to see that the above model is a souped-up version of the usual ideal measurement model expressed in the language of the quantum mechanics of closed systems [1]. The coordinates $x$ are those of the “measured subsystem” while the coordinates $Q$ include the apparatus. Our model is therefore subject to all the caveats

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[1] See, e.g., [16], or almost any current text in quantum mechanics. For an exposition from the point of view of the quantum mechanics of closed systems see [13], Section II.10.
associated with the ideal measurement model. In particular, as was shown by Wigner \[17\] and Araki and Yanase \[18\], there are no realistic Hamiltonians \( H \) that will effect the evolution (4.3) exactly. It also follows from analyticity that a relation (4.5) cannot hold exactly after one time and not before it. The evolution (4.3) and (4.5) must therefore be understood as holding approximately.

The history operators corresponding to alternatives defined by the successive collections of \( \phi \)'s are, in the Schrödinger picture,

\[
\hat{H}^\ell_{\alpha_\ell, \cdots, \alpha_1} = \hat{P}_{\alpha_\ell, \cdots, \alpha_1} e^{-iH(t_\ell - t_{\ell-1})} \hat{P}_{\alpha_{\ell-1}, \cdots, \alpha_1} \cdots e^{-iH(t_2 - t_1)} \hat{P}_{\alpha_1},
\]

(4.6)

where we have augmented the notation of (4.2) to include the branch dependence of the \( \phi \)'s. Acting with \( \hat{H}^\ell_{\alpha_\ell, \cdots, \alpha_1} \) on \( \Psi(x, Q, t_0) \) just gives the branch wave function for the history \((\alpha_\ell, \cdots, \alpha_1)\) at time \(t_\ell\), e.g.

\[
\hat{H}^2_{\alpha_2, \alpha_1} \Psi(x, Q, t_0) = \Psi(x, Q, t_2) = \chi^{\alpha_2}(Q^2) \chi^\alpha_1(Q^3, \cdots) \sum_{r^2 \in \alpha_2} \phi^r_{\alpha_2, \alpha_1}(x, Q, t_2)
\]

(4.7)

and similarly for other chains.

From the evolution prescribed in (4.3) it is easy to identify candidate record operators of these histories as projections on the \( \chi \) functions that are correlated with them. Specifically,

\[
\hat{R}^1_{\alpha_1} = \text{Proj} \left( \chi^{\alpha_1}(Q^1) \right),
\]

(4.8a)

\[
\hat{R}^2_{\alpha_2, \alpha_1} = \text{Proj} \left( \chi^{\alpha_1}(Q^1) \chi^{\alpha_2}(Q^2) \right),
\]

(4.8b)

\[
\hat{R}^3_{\alpha_3, \alpha_2, \alpha_1} = \text{Proj} \left( \chi^{\alpha_1}(Q^1) \chi^{\alpha_2}(Q^2) \chi^{\alpha_3}(Q^3) \right),
\]

(4.8c)

etc. Evidently we have

\[
\hat{H}^\ell_{\alpha_\ell, \cdots, \alpha_1} \Psi(x, Q, t_0) = \hat{R}^\ell_{\alpha_\ell, \cdots, \alpha_1} \Psi(x, Q, t_\ell),
\]

(4.9)

which is the Schrödinger picture representative of the relation (3.1) defining records.

Taking the evolution of the individual \( \chi \)'s given by (4.5) into account, we can define

\[
\hat{R}^k_{\alpha_\ell, \cdots, \alpha_1} = \hat{U}_{k\ell} \hat{R}^\ell_{\alpha_\ell, \cdots, \alpha_1} \hat{U}^\dagger_{k\ell},
\]

(4.10)

where

\[
\hat{U}_{k\ell} = \exp \left[ -iH_0(t_k - t_\ell) \right].
\]

(4.11)

Because of the standard relation between Schrödinger and Heisenberg pictures

\[
O = e^{iHt} \hat{O} e^{-iHt},
\]

(4.12)
eq. (4.10) is immediately seen to be the Schrödinger picture representative of the definition (3.12) with
\[ U_{k\ell} = e^{iH_k t} e^{-iH_0 (t_k - t_\ell)} e^{-iH_\ell t}. \] (4.13)

Thus we identify the \( U_{k\ell} \) entering into the definition of strong decoherence for this choice of records.

The following nesting is immediate from the definition (4.8)
\[ \hat{R}_{\alpha_k \cdots \alpha_1}^k \subseteq \hat{R}_{\alpha_\ell \cdots \alpha_1}^\ell, \; k < \ell. \] (4.14)

From (4.12) and (4.13) this is easily seen to be the Schrödinger picture representative of the general nesting relation (3.15). From (4.8) we also see that the nesting relation is not generally an equality because the functions \( \chi^{\alpha_j \cdots \alpha_1}(Q^j), \alpha_j = 1, 2, \cdots \) are not generally a complete basis in \( H_\ell \).

However, the equality in (4.14) can be seen to be satisfied effectively, i.e. when the operator relation is acting on the state. That is a simple consequence of the fact that the evolution prescribed in (4.3) was assumed to hold for each branch. Therefore projecting on a branch \((\alpha_\ell, \cdots, \alpha_1)\) at time \( t_\ell \) and evolving to time \( t_k \) is the same as evolving to time \( t_k \) and projecting on the branch. At time \( t_k \) the projection on the branch \((\alpha_\ell, \cdots, \alpha_1)\) is the projection on the evolved functions \( \chi \), e.g.
\[ \hat{R}_{\alpha_{k+1} \alpha_1}^k \equiv e^{-iH_0 (t_k - t_\ell)} \hat{R}_{\alpha_2 \alpha_1}^2 e^{iH_0 (t_k - t_\ell)} \] (4.15a)
\[ = \text{Proj} \left( \chi^{\alpha_1}(Q^1) \chi^{\alpha_2 \alpha_1}(Q^2) \right). \] (4.15b)

Specifically, therefore, from the form of (4.8) and (4.10) we have
\[ \hat{R}_{\alpha_\ell \cdots \alpha_1}^\ell \Psi(x, Q, t_k) = e^{-iH(t_k - t_\ell)} \hat{R}_{\alpha_\ell \cdots \alpha_1}^\ell \Psi(x, Q, t_\ell). \] (4.16)

This relation expresses the permanence of the records of the history \((\alpha_\ell \cdots \alpha_1)\) in the independently-evolving functions \( \chi^{\alpha_1} \chi^{\alpha_2 \alpha_1} \cdots \) at times later than \( t_\ell \). Translating to the Heisenberg picture, we see that (4.16) is the general relation (3.22). Thus one of the two defining conditions of strong decoherence is satisfied in this particular model. In the discussion of (3.22) we mentioned that it could only be expected to be satisfied approximately, while here it is satisfied exactly as a consequence (4.8). However, as we also mentioned above the evolution, (4.3) can be expected to hold only approximately, so there is in fact no conflict with the result (4.16).

To complete the demonstration that the idealized model under discussion in this Section provides an example of strong decoherence, it remains only to show that the first of the two defining conditions is satisfied, namely, the condition (3.9) or (3.13) that at time \( t_k \) future extensions of history to time \( t_k \) are contained in the set of operators \( M \) effectively commuting with the records at \( t_k \). Specifically, we seek to show
\[ \left[ U_{k\ell}^\dagger H_{\alpha_\ell \cdots \alpha_1}^{(k, \ell + 1)}, \hat{R}_{\alpha_\ell \cdots \alpha_1}^\ell \right] \hat{R}_{\alpha_\ell \cdots \alpha_1}^\ell |\Psi\rangle = 0, \] (4.17)
where the \( \hat{R}_{\alpha_\ell \cdots \alpha_1}^\ell \) are specified in the Schrödinger picture by (4.8), \( U_{k\ell} \) is given by (4.13), and we have introduced a convenient notation for the extensions of histories:
$$H^{(k,\ell+1)}_{\alpha k\cdots\alpha_1} \equiv P^k_{\alpha k\cdots\alpha_1} P^{k-1}_{\alpha k-1\cdots\alpha_1} \cdots P^{\ell+1}_{\alpha \ell+1\cdots\alpha_1}$$

$$= e^{iH_t} \hat{p}^{(k,\ell)}_{\alpha k\cdots\alpha_1} e^{-iH(t_{k-1} - t_{k-2})} \hat{p}^{(k-1)}_{\alpha k-1\cdots\alpha_1} \cdots \hat{p}^{(\ell+1)}_{\alpha \ell+1\cdots\alpha_1} e^{-iH_{t+1}}$$

$$= e^{iH_t} \hat{\Psi}^{(k,\ell+1)}_{\alpha k\cdots\alpha_1} e^{-iH_{t+1}}.$$  

(4.18a)  

(4.18b)  

(4.18c)

With a little care, (4.17) may be transformed back to the Schrödinger picture where it becomes the condition

$$\hat{R}^\ell_{\alpha k\cdots\alpha_1} e^{iH_0(t_k - t_\ell)} \hat{R}^{(k,\ell)}_{\alpha k\cdots\alpha_1} \Psi^{(k,\ell)}_{\alpha k\cdots\alpha_1}(x, Q, t_\ell) = e^{iH_0(t_k - t_\ell)} \hat{H}^{(k,\ell)}_{\alpha k\cdots\alpha_1} \Psi^{(k,\ell)}_{\alpha k\cdots\alpha_1}(x, Q, t_\ell),$$  

(4.19)

where $$\Psi^{(k,\ell)}_{\alpha k\cdots\alpha_1}(x, Q, t_\ell)$$ is the branch wave function corresponding to history $$(\alpha_\ell, \cdots, \alpha_1)$$. Explicitly [cf. (3.24)],

$$\Psi^{(k,\ell)}_{\alpha k\cdots\alpha_1}(x, Q, t_\ell) = \hat{R}^\ell_{\alpha k\cdots\alpha_1} \Psi(x, Q, t_\ell) = \hat{H}^\ell_{\alpha k\cdots\alpha_1} \Psi(x, Q, t_0)$$  

(4.20)

as illustrated in (4.7). (These branch wave functions are not normalized as were those in (3.24).) The operator $$\hat{H}^{(k,\ell+1)}_{\alpha k\cdots\alpha_1}$$ simply extends the branch forward to yield the following condition:

$$\hat{R}^\ell_{\alpha k\cdots\alpha_1} e^{iH_0(t_k - t_\ell)} \Psi^{(k,\ell+1)}_{\alpha k\cdots\alpha_1}(x, Q, t_k) = \Psi^{(k,\ell+1)}_{\alpha k\cdots\alpha_1}(x, Q, t).$$  

(4.21)

This can be seen to be satisfied as follows: We have first

$$\Psi^{(k,\ell+1)}_{\alpha k\cdots\alpha_1}(x, Q, t_k) = \chi^{k_1}(Q_1) \chi^{k_2 a_1}(Q_2) \cdots \chi^{k_\ell a_{\ell+1}}(Q_{\ell+1}) \chi^{a_{\ell+1} a_1}(Q) \times \left( \text{functions of } x \text{ and } Q_{\ell+1} \cdots Q_n \right).$$  

(4.22)

The crucial point is that the Hamiltonian $$H_0$$ was assumed to act independently on all the coordinates and the coordinates $$Q_1 \cdots Q_\ell$$ were assumed to effectively decouple from the rest after time $$t_\ell$$. Thus the operator $$\exp(iH_0(t_k - t_\ell))$$ acts to evolve the individual $$\chi$$‘s back to time $$t_\ell$$, viz.

$$e^{iH_0(t_k - t_\ell)} \Psi^{(k,\ell+1)}_{\alpha k\cdots\alpha_1}(x, Q, t_k) = \chi^{\ell a_1}(Q_1) \chi^{\ell a_2 a_1}(Q_2) \cdots \chi^{a_{\ell+1} a_1}(Q_\ell) \times \left( \text{functions of } x \text{ and } Q_{\ell+1} \cdots Q_n \right).$$  

(4.23)

With this result, it is immediate that (4.21) is satisfied and therefore also (4.17). Thus both conditions for strong decoherence are satisfied in this model.

A consistency condition (3.18) is implicit in the assumption that future histories could be drawn from the pool $$\{M\}_{\alpha \ell \cdots a_1}$$ of operators that effectively commute with the records at time $$t_\ell$$. That does not have to be checked separately because we showed in the discussion following (3.17) that the condition of nesting of records implies the necessary consistency condition, and nesting of the records was verified explicitly for this model in eq. (1.14).

The extensions of histories $$\hat{H}^{(k,\ell)}_{\alpha k\cdots\alpha_1}$$ do not exhaust the set of operators $$\{M\}_{\alpha \ell \cdots a_1}$$ that effectively commute with $$\hat{R}^\ell_{\alpha k\cdots\alpha_1}$$. The $$\hat{R}^\ell_{\alpha k\cdots\alpha_1}$$ for any $$(\alpha_\ell, \cdots, \alpha_1)$$ provide a simple example of a different operator. More generally, it is not difficult to see that the set $$\{M\}_{\alpha \ell \cdots a_1}$$ includes any operator acting on $$\mathcal{H}^x \otimes \hat{R}^\ell_{\alpha k\cdots\alpha_1} \otimes \mathcal{H}^{Q_{\ell+1}} \otimes \cdots \otimes \mathcal{H}^{Q_n}$$, where $$\hat{R}^\ell_{\alpha k\cdots\alpha_1}$$ is the subspace of $$\mathcal{H}^{Q_1} \otimes \cdots \otimes \mathcal{H}^{Q_n}$$ orthogonal to the subspace defined by $$\hat{R}^\ell_{\alpha k\cdots\alpha_1}$$. There may be other operators belonging to the set $$\{M\}_{\alpha \ell \cdots a_1}$$.  

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This model provides a concrete illustration of the connection, described in the previous Section, that can be made between the notion of strong decoherence and the work of many authors who investigate the evolution of a reduced density matrix for a system in the presence of an environment (e.g. \[3\] \[2\]). We discussed this kind of connection in the context of similar models in \(4\) (Section, IV) and related ideas were discussed by Finkelstein \(8\) in connection with his “PT-decoherence”.

Consider for definiteness the history \((\alpha_1, \alpha_2)\). Among the operators of \(\{\hat{M}\}_3^{3,\alpha_2\alpha_3}\) for any value of \(\alpha_3\) are those that act only on the Hilbert space \(\mathcal{H}^x\). These form a linear set closed under Hermitian conjugation and multiplication. The construction of a “reduced” matrix (3.23) may therefore be applied, with \(\{\hat{F}\}_3^{3,\alpha_2\alpha_3}\) being the set of all operators of the form \(\hat{O}_x \otimes I^Q\), where \(\hat{O}_x\) is any operator in \(\mathcal{H}^x\). The \(\hat{A}^{\alpha_2\alpha_3}_i\) are of the form \(\hat{A}_i^x \otimes I^Q\) where \(\hat{A}_i^x\) is an operator basis in \(\mathcal{H}^x\) (e.g. \(|x^i\rangle \langle x^i|\)). The “reduced” density matrix so defined is thus nothing more than the usual reduced density matrix for the “system” described by the \(x^i\)’s. It is constructed by tracing out the \(Q^i\)’s from the full density matrix. For example, assuming the evolution described by eq. (4.3), the reduced density matrix for the branch corresponding to the history \((\alpha_1, \alpha_2)\) is

\[
\hat{\rho}^{\alpha_2\alpha_3}(x', x) = \left(\chi_{3}^{\alpha_1}(Q^1), \chi_{3}^{\alpha_1}(Q^1)\right) \left(\chi_{3}^{\alpha_2\alpha_3}(Q^2), \chi_{3}^{\alpha_2\alpha_3}(Q^2)\right) \times \left(\chi_{3}^{\alpha_2\alpha_3}(Q^3), \chi_{3}^{\alpha_2\alpha_3}(Q^3)\right) \sum_{\alpha_3} \sum_{r_3} \sum_{r_3} \phi_{r_3}^{\alpha_2\alpha_3}(x') \phi_{r_3}^{\alpha_2\alpha_3}(x) \\
\left(\chi_{r_3}^{\alpha_2\alpha_3}(Q^4, Q^5, \ldots), \chi_{r_3}^{\alpha_2\alpha_3}(Q^4, Q^5, \ldots)\right). \tag{4.24}
\]

(A hat should properly be on the \(\hat{\rho}\) but is omitted for typographical reasons.) This reduced density matrix is easily seen to be diagonal in \(\alpha_3\) because of the orthogonality of the record states \(\chi_{3}^{\alpha_3\alpha_2\alpha_3}(Q^3)\) in \(\alpha_3\) [cf. (4.3)]. Thus orthogonality of generalized records guarantees the diagonality of the reduced density matrix in the quantities correlated with those records.

The operators of \(\mathcal{H}^x\) are not the only possible sets of \(F\’s\) in this model. They could be augmented by including, for example, operators acting on \(\mathcal{H}^Q^1 \otimes \mathcal{H}^Q^2 \otimes \ldots\) or on the subspace of \(\mathcal{H}^Q^1\) orthogonal to \(\chi_{3}^{\alpha_1}(Q^1)\) and similarly for the other record states. These will result in different, less “reduced”, density matrices, but strong decoherence guarantees the diagonality of them all in \(\alpha_3\).

The orthogonality of the record states implies much more than just the diagonality of the reduced density matrices that we have constructed. For example, we have

\[
\int dQ \Psi_{\alpha_3}^{3} \Psi^{3,\alpha_2\alpha_3}(x', Q) \Psi^{3,\alpha_3\alpha_2\alpha_3}(x, Q) = 0, \quad \alpha_3' \neq \alpha_3, \tag{4.25}
\]

and many other relations besides. Eq. (4.24) is the general relation (3.28) that follows in this model because operators on \(\mathcal{H}^x\) are closed under multiplication. The general content of the diagonalization of the “reduced” density matrices that can be constructed is perhaps best exhibited by relations such as this.

Thus strong decoherence implies the diagonalization of a variety of reduced density matrices in this model but generalizes that idea to more realistic cases where the assumptions of the model do not apply.

The model we have been discussing expresses concretely the idea of a physical mechanism of decoherence in which phases are dispersed irretrievably into an “environment”, as in the
picture elaborated with increasing precision by the pioneers of this subject (e.g. [20], [21], [9], [10]). However, this model is too restrictive for realistic application. First, it is highly idealized, as we have pointed out in several places in the discussion. More importantly, the realistic hydrodynamic variables that characterize the coarse grainings of the usual quasiclassical realm do not correspond to a distinction between system coördinates and environmental coördinates. The general notion of strong decoherence described in Section III is an idealization that captures the physical ideas without such strong and unrealistic assumptions as are made in this model. In the next Section we shall discuss the role that strong decoherence can play in defining classicality.

V. CLASSICALITY

Quantum mechanics, along with the correct theory of the elementary particles (represented by the Hamiltonian $H$) and the correct initial condition in the universe (represented by the state vector $|\Psi\rangle$), presumably exhibits a great many essentially different strongly decohering realms, but only some of those are quasiclassical. For the quasiclassical realms to be viewed as an emergent feature of $H$, $|\Psi\rangle$, and quantum mechanics, a good technical definition of classicality is required. (One can then go on to investigate whether the theory exhibits many essentially inequivalent quasiclassical realms or whether the usual one is nearly unique.)

In earlier papers, [1,2,4] we have made a number of suggestions about the definition of classicality and it is appropriate to continue that discussion here. It is clear that from those earlier discussions that classicality must be related in some way to a kind of entropy for alternative coarse-grained histories. We must therefore begin with an abstract characterization of entropy and then investigate the application to histories. An entropy $S$ is always associated with a coarse graining, since a perfectly fine-grained version of entropy in statistical mechanics would be conserved instead of tending to increase with time. Classically, if all fine-grained alternatives are designated by $\{r\}$, with probabilities $p_r$ summing to one, that fine-grained version of entropy would be

$$S_{f-g} = -\sum_r p_r \log p_r ,$$  \hspace{1cm} (5.1)

where log means $\log_2$ and where, for convenience, we have put Boltzmann’s constant $k$ times $\log_e 2$ equal to unity. A true, coarse-grained entropy has the form

$$S \equiv -\sum_r \tilde{p}_r \log \tilde{p}_r ,$$  \hspace{1cm} (5.2)

where the probabilities $\tilde{p}_r$ are coarse-grained averages of the $\{p_r\}$. A coarse graining $p_r \to \tilde{p}_r$ must have certain properties (see [22,23] for more details):

1) the $\{\tilde{p}_r\}$ are probabilities , \hspace{1cm} (5.3a)
2) $\tilde{\tilde{p}}_r = \tilde{p}_r$ , \hspace{1cm} (5.3b)
3) $-\sum_r p_r \log \tilde{p}_r = -\sum_r \tilde{p}_r \log \tilde{p}_r$ . \hspace{1cm} (5.3c)
These properties are not surprising for an averaging procedure. The significance of the last one is easily seen if we make use of the well known fact that for any two sets of probabilities \( \{ p_r \} \) and \( \{ p'_r \} \) we have

\[- \sum_r p_r \log p_r \leq - \sum_r p_r \log p'_r . \quad (5.4)\]

Putting \( p'_r = \tilde{p}_r \) for each \( r \) and using (5.1), (5.2), (5.3), and (5.4), we obtain

\[ S_{f-g} = - \sum_r p_r \log p_r \leq - \sum_r p_r \log \tilde{p}_r = - \sum_r \tilde{p}_r \log \tilde{p}_r = S , \quad (5.5) \]

so that \( S_{f-g} \) provides a lower bound for the entropy \( S \). If the initial condition and the coarse graining are related in such away that \( S \) is initially near its lower bound, then it will tend to increase for a period of time. That is the way the second law of thermodynamics comes to hold.

In order to know what nearness to the lower bound means, we should examine the upper bound on \( S \). That upper bound is achieved when all fine-grained alternatives have equal coarse-grained probabilities \( \tilde{p}_r \), corresponding in statistical mechanics to something like “equilibrium” or infinite temperature. Each \( \tilde{p}_r \) is then equal to \( N^{-1} \), where \( N \) (assumed finite) is the number of fine-grained alternatives, and the maximum entropy is thus

\[ S_{\text{max}} = \log N . \quad (5.6) \]

The simplest example of coarse graining utilizes a grouping of the fine-grained alternatives \( \{ r \} \) into exhaustive and mutually exclusive classes \( \{ \alpha \} \), where a class \( \alpha \) contains \( N_\alpha \) elements and has lumped probability

\[ p_\alpha \equiv \sum_{r \in \alpha} p_r . \quad (5.7) \]

Of course we have

\[ \sum_\alpha N_\alpha = N, \quad \sum_\alpha p_\alpha = 1 . \quad (5.8) \]

The coarse-grained probabilities \( \tilde{p}_r \) in this example are the class averages

\[ \tilde{p}_r = p_\alpha / N_\alpha , \quad r \in \alpha , \quad (5.9) \]

and they clearly have the properties (5.3). The entropy comes out

\[ S = - \sum_\alpha p_\alpha \log p_\alpha + \sum_\alpha p_\alpha \log N_\alpha , \quad (5.10) \]

where the second term contains the familiar logarithm of the number of fine-grained alternatives (or microstates) in a coarse-grained alternative (or macrostate), averaged over all the coarse-grained alternatives.

Besides entropy, it is useful to introduce the concept of algorithmic information content (AIC) as defined some thirty years ago by Kolmogorov, Chaitin, and Solomonoff (all working
For a string of bits $s$ and a particular universal computer $U$, the AIC of the string, written $K_U(s)$, is the length of the shortest program that will cause $U$ to print out the string and then halt. The string can be used as the description of some entity $e$, down to a given level of detail, in a given language, assuming a given amount of knowledge and understanding of the world, encoded in a given way into bits [23]. The AIC of the string can then be regarded as $K_U(e)$, the AIC of the entity so described.

We now discuss a way of approaching classicality that utilizes AIC as well as entropy. Some authors have tried to identify AIC in a straightforward way with complexity, and in fact AIC is often called algorithmic complexity. However, AIC is greatest for a “random” string of bits with no regularity and that hardly corresponds to what is usually meant by complexity in ordinary parlance or in scientific discourse. To illustrate the connections among AIC, entropy or information, and an effective notion of complexity, take the ensemble $\tilde{E}$ consisting of a set of fine-grained alternatives $\{r\}$ together with their coarse-grained probabilities $\tilde{p}_r$. We can then consider both $K_U(\tilde{E})$, the AIC of the ensemble, and $K_U(r|\tilde{E})$, which is the AIC of a particular alternative $r$ given the ensemble. For the latter we have the well known inequality (see, for example [26]):

$$\sum_r \tilde{p}_r K_U(r|\tilde{E}) \geq -\sum_r \tilde{p}_r \log \tilde{p}_r = S.$$  \hfill (5.11)

Moreover, it has been shown by R. Schack [27] that, for any $U$, a slight modification $U \rightarrow U'$ permits $K_U(r|\tilde{E})$ to be bounded on both sides as follows:

$$S + 1 \geq \sum_r \tilde{p}_r K_{U'}(r|\tilde{E}) \geq S,$$  \hfill (5.12)

so that we have

$$\sum_r \tilde{p}_r K_{U'}(r|\tilde{E}) \approx S.$$  \hfill (5.13)

(Previous upper bounds had $O(1)$ in place of 1, but there was nothing to prevent $O(1)$ from being millions or trillions of bits!)

Looking at the entropy $S$ as a close approximation to $\sum_r \tilde{p}_r K_{U'}(r|\tilde{E})$, we see that it is natural to complete it by adding to it the quantity $K_U(\tilde{E})$ — the AIC of the ensemble with respect to the same universal computer $U'$. This last quantity can be connected with the idea of effective complexity — the length of the most concise description of the perceived regularities of an entity $e$. Any particular set of regularities can be expressed by describing $e$ as a member of an ensemble $\tilde{E}$ of possible entities sharing those regularities. Then $K_{U'}(\tilde{E})$ may be identified with the effective complexity of $e$ or of the ensemble $\tilde{E}$ [23,24]. Adding this effective complexity to $S$, we have:

$$\Sigma \equiv K_{U'}(\tilde{E}) + S.$$  \hfill (5.14)

This sum of the effective complexity and the entropy (or Shannon information) may be labeled either “augmented entropy” or “total information”. If the coarse graining is the

***For a discussion of the original papers see [24].
simple one obtained by partitioning the set of fine-grained alternatives \{r\} into classes \{\alpha\} with cardinal numbers \(N_\alpha\), then the total information becomes

\[ \Sigma = K_U(\tilde{E}) - \sum_\alpha p_\alpha \log p_\alpha + \sum_\alpha p_\alpha \log N_\alpha . \]  

(5.15)

In (5.14), the first term becomes smaller as the set of perceived regularities becomes simpler, while the second term becomes smaller as the spread of possible entities sharing those regularities is reduced. Minimizing \(\Sigma\) corresponds to optimizing the choice of regularities and the resulting effective complexity thereby becomes less subjective. Thus, the total information or augmented entropy is useful in a wide variety of contexts [23,25]. We apply it here to sets of alternative decohering coarse-grained histories in quantum mechanics.

The general idea of augmenting entropy with a term referring to algorithmic information content was proposed in a different context by Zurek [28]. However, as far as we know, the emphasis on the utility of the quantity \(\Sigma\) in (5.14) and (5.15) is new. We discussed the general idea of an entropy for histories in [1]. Earlier, Lloyd and Pagels [29] introduced a quantity they called thermodynamic depth, applicable to alternative coarse-grained classical histories \(\alpha\). They defined it as

\[ D = \sum_\alpha p_\alpha \log(p_\alpha/q_\alpha) , \]  

(5.16)

where \(q_\alpha\) is an “equilibrium probability”, which in our notation would be \(N_\alpha/N\) for the simple coarse graining we have discussed. We clearly have

\[ D = \log N + \sum_\alpha p_\alpha \log p_\alpha - \sum_\alpha p_\alpha \log N_\alpha \]  

(5.17)

or

\[ D = S_{\text{max}} - S \]  

(5.18)

for the set of alternative coarse-grained histories. We see that thermodynamic depth is intimately related to the notion of an entropy for histories.

In applying augmented entropy to sets of coarse-grained histories in quantum mechanics, one must take into account that there are infinitely many different sets of fine-grained histories and that these sets do not generally have probabilities because they fail to decohere. The quantities \(N_\alpha\) may therefore conceivably depart from their obvious definition as the numbers of fine-grained histories in the coarse-grained classes \{\alpha\}. In fact, there may be some latitude in the precise definition of the complexity and entropy terms in the total information (5.14). For example, one could consider instead of \(\tilde{E}\) an ensemble \(\hat{E}\) consisting of the coarse-grained histories \(\alpha = \{\alpha_1, \alpha_2, \cdots, \alpha_n\}\), their probabilities \(p_\alpha\), and the numbers \(N_\alpha\). A more general definition of the entropy \(S\) of histories may help to define these numbers. The generalized Jaynes construction for coarse-grained histories provides one framework for doing this [22]. In the most general situation, such a construction defines the entropy \(S\) as the maximum of \(-Tr(\hat{\rho}\log \hat{\rho})\) over all density matrices \(\hat{\rho}\) that preserve the decoherence and probabilities of a given ensemble \(E\) of coarse-grained histories. Other Jaynes-like constructions may also be useful, for example ones that define entropy by proceeding step by step through the histories. We are investigating these various possibilities.
In any case, our augmented entropy (5.15) for coarse-grained decohering histories in quantum mechanics is a negative measure of classicality: the smaller the quantity, the closer the set of alternative histories is to a quasiclassical realm. Reducing the first term in (5.15) favors making the description of the sequences of projections simple in terms of the field variables of the theory and the Hamiltonian \( H \). It favors sets of projections at different times that are related to one another by time translations, as are many sequences of projections on quasiclassical alternatives at different times in the usual quasiclassical realm.

Reducing the second term favors more nearly deterministic situations in which the spread of probabilities is small. Approximate determinism is, of course, a property of a quasiclassical realm. Reducing the last term corresponds roughly to approaching “maximality”, allowing the finest graining that still permits decoherence and nearly classical behavior. A quasiclassical realm must be maximal in order for it to be a feature exhibited by the initial condition and Hamiltonian and not a matter of choice by an observer.

Any proposed measure of closeness to a quasiclassical realm must be tested by searching for pathological cases of alternative decohering histories that make the quantity small without resembling quasiclassical realm of everyday experience. The worst pathology occurs for a set of histories in which the \( P \)’s at every time are projections on \( |\Psi\rangle \) and on states orthogonal to \( |\Psi\rangle \). We see that in this pathological case the description of the histories and their probabilities is simple because the description of the initial state is simple, so that \( K_{U'}(E) \) is small. The term \( -\sum_\alpha p_\alpha \log p_\alpha \) is zero and the third term is also zero since the only \( \alpha \) with \( p_\alpha \neq 0 \) corresponds to projecting onto the pure state \( |\Psi\rangle \), so that \( N_\alpha \) is one and \( \log N_\alpha \) vanishes.

Evidently the smallness of \( \Sigma \) is not by itself a sufficient criterion for characterizing a quasiclassical realm. Further criteria can be introduced if we require that quasiclassical realms be strongly decohering with suitable restrictions on the sets \( \{M\}_\alpha \) of operators from which the future histories are constructed. Requiring strong decoherence ensures a physical mechanism of decoherence and guarantees the permanence of the past. The sets \( \{M\}_\alpha \) must be restricted so as to rule out pathologies such as discussed above. Presumably they must all belong to a huge set with certain straightforward properties. Those properties might be connected with locality, since quantum field theory is perfectly local. (Even superstring theory is local — although the string is an extended object, interaction among strings is always local in spacetime.) It would be in this way that strong decoherence enters a definition of classicality.

A quasiclassical realm would then be characterized in quantum mechanics as a strongly decoherent set of histories, with suitable restrictions on the sets \( \{M\}_\alpha \), that minimizes the augmented entropy given by (5.15). Quasiclassical realms so defined would be an emergent feature of \( H \), \( |\Psi\rangle \), and quantum mechanics — a feature of the universe independent of human choice. In principle, given \( H \) and \( |\Psi\rangle \), we could compute the quasiclassical realm that these theories exhibit. We could then investigate the important question of whether the usual quasiclassical realm is essentially unique or whether the quantum mechanics of the universe exhibits essentially inequivalent quasiclassical realms. Either conclusion would be of central importance for understanding quantum mechanics.

We are continuing our efforts to complete, refine, and investigate the consequences of this definition of classicality.
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