A PRODUCT FORMULA FOR GROMOV-WITTEN INVARIANTS.

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Abstract. We establish a product formula for Gromov-Witten invariants for closed relatively semi-positive Hamiltonian fibrations, with connected fiber, and over any connected symplectic base. Furthermore, we show that the fibration projection induces a locally trivial (orbi-)fibration map from the moduli space of pseudo-holomorphic maps with marked points in the total space of the Hamiltonian fibration to the corresponding moduli space of pseudo-holomorphic maps with marked points in the base. We use this induced map to recover the product formula by means of integration. Finally, we give applications to $c$-splitting and symplectic uniruledness.

1. Introduction

We consider rational Gromov-Witten invariants ($GW$-invariants) of closed Hamiltonian fibrations with connected fiber over any connected symplectic base. In a symplectic manifold $(X,\omega)$ with $\omega$-tame almost complex structure $J$, $GW$-invariants are given by counting the (algebraic) number of unparametrized $J$-(pseudo)-holomorphic genus 0 maps with $l$ distinct marked points, representing a fixed spherical homology class $A \in H_2(X,\mathbb{Z})$, and intersecting transversally $l$ given cycles of $X$ at the marked points. Roughly speaking, if $\mathcal{M}_{0,l}(X,A,J)$ denotes the moduli space of unparametrized (genus 0) $J$-holomorphic maps with $l$ markings, $(u,x_1,\ldots,x_l)$, representing the class $A$, and if $M_1,\ldots,M_l$ are cycles in $X$ representing given classes $c_1,\ldots,c_l \in H_*(X,\mathbb{Q})$, then these invariants can be seen as the values of the multilinear homomorphism:

$$\langle \bigwedge_{0,l}^{X} : (H_*(X,\mathbb{Q}))^{\otimes l} \to \mathbb{Q}$$

$$c_1 \otimes \ldots \otimes c_l \mapsto ev_l^X(M_1 \times \ldots \times M_l)$$

the intersection pairing $\cdot$ being taken with respect to the evaluation at the marked points:

$$ev_l^X : \mathcal{M}_{0,l}(X,A,J) \to X^l, \ (u,x_1,\ldots,x_l) \mapsto (u(x_1),\ldots,u(x_l)).$$

It is well known [17], [18], [22], that when the symplectic manifold is semi-positive the $GW$-invariants are generically well-defined $\mathbb{Z}$-valued invariants of the symplectic manifold defined on $H_*(X)$, the singular homology of $X$ modulo torsion.

In the case where the Hamiltonian fibration is a product of two symplectic manifolds, Ruan and Tian [22], and shortly after Kontsevich and Manin [9], showed that when both the base and the fiber are semi-positive, the $GW$-invariants of the total space are given by products of related $GW$-invariants in the base and in the fiber, giving rise to splitting of the quantum product. A priori, we cannot expect for such a splitting to hold in the non-trivial case, as even the cup product may not split already. Nevertheless, one may still ask about the algebraic relations that can be established out of the invariants of the base, the fiber, and the total space. We give such a relation under some assumptions, in particular when the reference fiber of the Hamiltonian fibration is semi-positive relative to the total space. We now explain
Hamiltonian fibrations. By definition, a symplectic fibration is a smooth locally trivial fibration \( \pi : P \to B \) with symplectic reference fiber \((F, \omega)\), and which structure group lies in the group of symplectic diffeomorphisms of the fiber, denoted \( \text{Symp}(F, \omega) \). It follows that each fiber \( F_b := \pi^{-1}(b) \) is naturally equipped with a symplectic form \( \omega_b \). A symplectic fibration is Hamiltonian if the structure group can be reduced to the group \( \text{Ham}(F, \omega) \) of Hamiltonian diffeomorphisms. By a result of Guillemin-Lerman-Sternberg [5], this is equivalent to satisfying the following two conditions:

\((H_1)\): \( P \) is symplectically trivial over the 1-skeleton, \( B_1 \), of \( B \);
\((H_2)\): there exists a connection \( \text{Hor}_\tau \subset TP \) with holonomy in \( \text{Ham}(F, \omega) \), induced by a (canonical) closed 2-form \( \tau \in \Omega^2(P) \) extending the family \( \{\omega_b\}_{b \in B} \).

The closed 2-form is usually referred to as the coupling form. Since \( B \) is assumed to be closed and symplectic, the existence of such a form is sufficient to give \( P \) a symplectic structure compatible with the family \( \{\omega_b\}_{b \in B} \) by considering the form

\[ \omega_{P, \kappa} := \tau + \kappa \pi^* \omega_B, \]

where \( \kappa > 0 \) is a real number chosen large enough so that \( \omega_{P, \kappa} \) is non-degenerate.

A less trivial class of examples is that of Hamiltonian fibrations over \( S^2 \). These latter fibrations correspond (up to isomorphism) to homotopy classes of loops in \( \text{Ham}(F, \omega) \). This direct connection with the fundamental group of \( \text{Ham}(F, \omega) \) makes these objects particularly interesting from the point of view of symplectic topology as pointed out by Seidel [23].

Product formula. When the minimal Chern number of the fiber, \( N_F \), satisfies the following semi-positivity relative to \( P \),

\[ N_F \geq \frac{1}{2} \dim P - 2, \]

we give a product formula relating the rational \( GW \)-invariants of the base with the \( GW \)-invariants of the total space, as suggested in [10]. In order to do so we equip the fibration \( P \) with an \( (almost) \) complex structure \( J_P \), chosen compatibly with the fibration structure and a Hamiltonian connection. Concretely, \( J_P \) is uniquely given by the choice of coupling form \( \tau \), an \( \omega_B \)-tame complex structure \( J_B \) on \( B \) and a family of \( \omega_b \)-tame almost complex structures \( J_b \) in \( F_b \). Such structures are said to be compatible with \( \pi \) and \( \tau \) or just fibered. In the present formula we consider classes \( c_i^P \in H_*(P) \), \( i = 1, ..., l \), that are given by product classes \( c_i^B \otimes c_i^F \), with \( c_i^B \in H_*(B) \) and \( c_i^F \in H_*(F) \), such that the following condition is verified for some integer \( 0 \leq m \leq l \):

\[ \begin{cases} 
  c_i^B = pt, & \text{for } i = 1, ..., m \\
  c_i^F = [F] & \text{for } i = m + 1, ..., l. 
\end{cases} \]

Concretely, the classes in \( P \) are such that (up to multiplication by well chosen integers) their Poincaré duals can be represented either by a submanifold of the fiber or by the preimage under \( \pi \) of a submanifold in \( B \). Let \( P_C \) denote the restriction of \( P \) along the image \( C \) of a smooth map from \( S^2 \) to \( B \). This defines a Hamiltonian fibration over \( S^2 \) with coupling form given by the pull-back of \( \tau \) under the natural
inclusion $i_P^P_C : P_C \hookrightarrow P$. Set,

$$B_\sigma := \{ \sigma' \in H_2(P_C,\mathbb{Z}) | i_P^P_C(\sigma') = \sigma \},$$

where $i_P^P_C$ is understood as the induced map in homology, and let $i_F^P_C : F \hookrightarrow P_C$ denote both the natural inclusion and its induced map in homology. The product formula is as follows:

**Theorem A.** Let $\pi : P \rightarrow B$ be a Hamiltonian fibration with semi-positive fiber $(F,\omega)$ relatively to $P$. Let $\sigma \in H_2(P,\mathbb{Z})$ and suppose $\sigma_B := \pi_*(\sigma) \neq 0$ only admits irreducible effective decompositions for some $J_B$. Define $c^P_c, c^B_c, c^F_c$ as in (22). Then for a generic fibered complex structure the following equation holds:

$$\langle c^P_1, ..., c^P_l \rangle_{0,l,\sigma} = \langle c^B_1, ..., c^B_l |_{\sigma_B} \rangle \cdot \sum_{\sigma' \in B_\sigma} \langle i_F^P_C(c^F_1), ..., i_F^P_C(c^F_l) \rangle_{0,l,\sigma'},$$

where $C$ is a curve counted in $\langle c^B_1, ..., c^B_l \rangle_{0,l,\sigma_B}$.\[\text{(1.1)}\]

This formula states that the number $\langle \quad \rangle_{0,l,\sigma}$ of curves above $C$ is independent of the chosen $C$, which turns out to be essentially a consequence of $(H_1)$. Note that by Gromov’s compactness the above sum is finite. It is even possible to simplify the expression of the formula by considering equivalence classes on the preimage $\pi_\ast^{-1}(\sigma_B) \subset H_2(P,\mathbb{Z})$. More precisely, we say that $\sigma_1, \sigma_2 \in \pi_\ast^{-1}(\sigma_B)$ are equivalent if and only if

$$\tau(\sigma_1 - \sigma_2) = 0 = c^v(\sigma_1 - \sigma_2),$$

where $c^v$ denotes the first Chern class of the vertical subbundle $\ker d\pi \subset TP$. Let $[\sigma]_{\sigma_B}$ denote the equivalence class of $\sigma$ in the product formula. Note that, under the pull-back by $i_P^P_C$, any element in $\pi_\ast^{-1}(\sigma_B)$ defines a section class in $P_C$ (i.e a class projecting on $[S^2]$ under $\pi_\ast$), and it is not hard to see that the preimage of $[\sigma]_{\sigma_B}$ under $i_P^P_C$ yields a well-defined equivalence class of section classes in $P_C$. If $C$ denotes this equivalence class, then the sum in the product formula disappears:

$$\langle c^P_1, ..., c^P_l \rangle_{0,l,[\sigma]_{\sigma_B}} = \langle c^B_1, ..., c^B_l |_{\sigma_B} \rangle \cdot \langle i_F^P_C(c^F_1), ..., i_F^P_C(c^F_l) \rangle_{0,l,\sigma_C}. \text{(1.1)}$$

Finally, we remark that the case $\sigma_B = 0$ leads to the Parametric Gromov-Witten invariants which are well known [11, 12].

An important issue in the proof of this theorem is establishing that the GW-invariants involved are generically and simultaneously well-defined. The irreducibility hypothesis on $\sigma_B$, which is realized by primitive classes, arises for transversality purposes. We make this more precise. In the present context, the projection $\pi$ naturally induces a map

$$\pi : \mathcal{M}_{0,l}(P,\sigma, J_P) \rightarrow \mathcal{M}_{0,l}(B,\sigma_B, J_B), \quad \sigma_B := \pi_\ast \sigma.$$

The issue is to realize generically both moduli spaces as smooth oriented manifolds together with preserving $\pi$. In the general context of a symplectic manifold $(X,\omega)$, it is well-known that the irreducible component, $\mathcal{M}^s_{0,l}(X, A, J)$ of $\mathcal{M}_{0,l}(X, A, J)$, consisting of simple maps, i.e maps $u$ that are somewhere injective (there exists a point $z_0 \in S^2$ such that $du(z_0)$ is injective and $u^{-1}(u(z_0)) = z_0$), is actually an oriented open manifold of finite dimension for a generic choice of $\omega$-tame structure $J$ (see [17, 20]). Here we cannot directly apply this result since tame complex structures of the total space do not coincide with fibered complex structures. Nevertheless, generalizing a result of McDuff and Salamon in the case of Hamiltonian fibrations over Riemann surfaces [17] we prove that:
Theorem B. Suppose $\sigma_B \neq 0$. There exists a second category subset, $J_{P,\text{reg}}$, of fibered almost complex structures such that for every $J_P \in J_{P,\text{reg}}$:

1) the subset $M_{0,l}^*(P,\sigma,J_P)$ of $M_{0,l}(P,\sigma,J_P)$ consisting of simple maps that project to simple maps under $\pi$, and the moduli space $M_{0,l}^*(B,\sigma_B,J_B)$, are open oriented manifolds.

2) for any countable set $Z$ of elements in $M_{0,l}^*(B,\sigma_B,J_B)$, for every $u \in Z$, the preimage $\pi^{-1}(u)$ is an open oriented manifold.

The dimensions of the manifolds in 1) are respectively given by the indices of the linearizations, $D^P$ and $D^B$, of the Cauchy Riemann operators $\overline{\partial}_{J_P}$ and $\overline{\partial}_{J_B}$, while the dimension of the moduli space in 2) is given by the index of $D^v$, the restriction of $D^P$ to vector fields along the curves that are vertically valued. The proof of Theorem B is based on the relation $\pi_* D^P = D^B \circ \pi_*$. Consequently, $\pi$ induces a submersion of Fredholm systems (see Section 2) between the Fredholm systems relative to the operators $\overline{\partial}_{J_P}$ and $\overline{\partial}_{J_B}$, as defined in [2]. From this we derive an exact sequence

$$0 \to \ker D^v \to \ker D^P \to \ker D^B \to \text{coker } D^v \to \text{coker } D^P \to \text{coker } D^B \to 0.$$ 

The result then follows by ensuring that the cokernels vanish, at least at the level of the universal moduli spaces, for example the irreducibility hypothesis on $\sigma_B$ ensures that the last term of the sequence vanishes. It is worth pointing out that without this assumption standard transversality may fail a priori, due to multiple coverings as shown in [18]. The remaining obstructions are dealt with by perturbing the Hamiltonian connection as in [17]. More generally, $\pi$ extends to a map, still denoted $\pi$, between the compactifications $\overline{M}_{0,l}(P,\sigma,J_P)$ and $\overline{M}_{0,l}(B,\sigma_B,J_B)$ of the moduli spaces. These compactifications are stratified spaces for which the strata can be represented by stable stratum data $S$, as pointed out by Kontsevitch, that is, by connected trees with tails together with an effective decomposition of the represented second homology class. We can repeat the arguments above for each stratum $M_{S_P}(P)$ mapped to a stratum $M_{S_B}(B)$ under $\pi$, in order to show that transversality is generically realized for the irreducible elements in $M_{S_P}(P)$ whenever $M_{S_B}(B)$ does not contain reducible elements. Then condition (i) ensures that the “boundary” of the compactified moduli spaces above, given by lower strata, have codimension at least 2 with respect to the top stratum consisting of simple maps.

Remark 1.1. At this point, it is worth mentioning that the restriction to the genus 0 case is not essential. Although we have not treated the case of higher genus curves, all the results should still go through with minor modifications, except regarding the applications to $c$-splitting and symplectic uniruledness below.

Another noteworthy observation, is that the restrictions on $\sigma_B$ and the relative semipositivity conditions are only of technical order. It is believed that those ad hoc hypothesis can be avoided by using virtual perturbations (see [2], [13], [21], [25]), which have been developed in order to deal with transversality issues for general symplectic manifolds. Removing these assumptions, is part of a joint work in progress with Shengda Hu.

Fibration structure. It is natural to ask about the structure of the map

$$\overline{\pi} : \overline{M}_{0,l}(P,\sigma,J_P) \to \overline{M}_{0,l}(B,\sigma_B,J_B).$$
In particular, it would be interesting to understand when \( \pi \) is a fibration, at least above the top stratum of the target space. When this is the case, we can recover the product formula using integration over the fibers of \( \pi \) (see Section 5). Assuming the linearized operators involved in the exact sequence above are all surjective, it follows that the restriction of \( \pi \) to \( \mathcal{M}_{0,l}(P, \sigma, J_P) \) is a smooth submersion onto \( \mathcal{M}_{0,l}(B, \sigma_B, J_B) \). However, this map is not proper. This latter condition is important, as one can easily construct a smooth submersion which is not proper and which does not induce a fibration structure. To solve this problem, we consider the fiberwise compactification of \( \pi \). The properness issue then “disappears” but at the cost of losing the obvious smooth structure. Nevertheless, Chen and Li recently showed in the general case of a symplectic manifold \((X, \omega)\), that one can define a smooth orbifold atlas on \( \mathcal{M}_{0,l}(X, A, J) \), where the charts are given by gluing maps [2]. There are many variants in the gluing of pseudo-holomorphic spheres procedure (see [2], [14], [17], [21], [24], among others), which appears naturally in Gromov-Witten theory, as well as in Floer theory. The approach followed in [2] is to use balanced curves in order to define a natural slice for the action of the group of reparametrizations of \( S^2 \), \( PSL_2(\mathbb{C}) \), reducing the action of this latter non-compact group to that of \( S^1 \). As a consequence, they obtain gluing maps which are well-defined after quotient by the reparametrizations. Adapting their ideas to the Hamiltonian fibration case, we construct gluing maps \( Gl_P \) and \( Gl_B \) satisfying
\[
\pi \circ Gl_P = Gl_B \circ \pi.
\]
Under some transversality assumptions for a given fibered almost complex structure \( J_P \) on \( P \), realized by projective fibrations over projective space (as described in (6.2)), we obtain the following:

**Theorem C.** The moduli spaces \( \overline{\mathcal{M}}_{0,l}(P, \sigma, J_P) \) and \( \overline{\mathcal{M}}_{0,l}(B, \sigma_B, J_B) \) are smooth orbifolds, and the map \( \pi \) restricts to a smooth locally trivial fibration (of orbifolds) above each stratum of \( \overline{\mathcal{M}}_{0,l}(B, \sigma_B, J_B) \). Moreover, the product formula can be recovered using integration over the fibers of \( \pi \) above the top stratum of \( \overline{\mathcal{M}}_{0,l}(B, \sigma_B, J_B) \).

**Applications.** In 1997, Seidel defined in [23], a representation of the space of Hamiltonian loops of a given symplectic manifold in the automorphism group of the corresponding quantum homology. Lalonde, McDuff and Polterovich have shown, under the relative semi-positivity assumption, that the rational cohomology of the total space splits as modules for any Hamiltonian fibration over \( S^2 \) [11]. McDuff removed the semi-positivity assumption using virtual techniques [15]. In general, we say that a fibration is *rationally c-split* if
\[
H^*(P; \mathbb{Q}) \cong H^*(B; \mathbb{Q}) \otimes H^*(F; \mathbb{Q})
\]
as modules. This splitting is realized when the inclusion \( \iota^P_F : F \hookrightarrow P \) induces an injection in rational homology, and if, in addition, the second page of the Leray-Serre spectral sequence splits. More generally, Lalonde and McDuff conjectured that every Hamiltonian fibration verifies the c-splitting property [10]. They showed that this splitting property holds for a large panel of Hamiltonian fibrations, in particular for Hamiltonian fibrations over \( \mathbb{C}P^n \). The proof they give can be seen as a simple consequence of the product formula (when \( \mathcal{L} \) is satisfied), and of the invertibility of Seidel’s morphism [11], [23].

Another consequence is the symplectic uniruledness of Hamiltonian fibrations over rationally connected bases. As defined in [6], a symplectic manifold \((X, \omega)\) is *(symplectically) uniruled* if there is a non vanishing \( GW \)-invariant with at least one
point as a constraint. In other words if there exists \( A \in H_2(X, \mathbb{Z}) \) and homology classes \( c_2, ..., c_l \in H_s(X) \) such that:

\[
\langle pt, c_2, c_3, ..., c_l \rangle_X^{0, l, A} \neq 0.
\]

A symplectic manifold is \textit{rationally connected} if there is a non-zero \( GW \)-invariant involving 2 point insertions \([6]\), i.e the equation above is still true with \( c_2 = pt \). In summary:

**Corollary.** Let \( \pi : P \to B \) be a Hamiltonian fibration. Assume \( B \) is rationally connected with respect to a class, \( \sigma_B \in H_2(B; \mathbb{Z}) \), verifying the hypothesis of Theorem \( B \). Then, \( P \) is \( c \)-split and symplectically uniruled.

**Proof:** Let \( C \) be the image of a map counted in \( \langle pt, pt, c_3^B, ..., c_l^B \rangle_{0, l, \sigma_B} \neq 0 \). As already mentioned, \( P_C \) is a Hamiltonian fibration over \( S^2 \), and by a result of Lalonde, McDuff and Polterovich. \([11]\), for every \( a \in H_4(F) \) there is an equivalence class \( \sigma' \) of section classes in \( P_C \), as well as an element \( b \in H_4(F) \), such that:

\[
0 \neq \langle \iota^P_F(a), \iota^P_F(b) \rangle^{P_C}_{0, 2, \sigma'} = \langle \iota^P_{P_C}(a), \iota^P_{P_C}(b), \iota^P_{P_C}([F]), ..., \iota^P_{P_C}([F]) \rangle^{P_C}_{0, 1, \sigma'},
\]

where the last equality is a consequence of the Divisor axiom. Applying the product formula, as given in \((1.1)\), we conclude that:

\[
(1.2) \quad \langle \iota^P_{P_C}(a), \pi^{-1}(\iota^P_{P_C}([F]))_{0, l, \sigma} \rangle \neq 0.
\]

Hence, by taking \( a = pt \) we obtain that \( P \) is uniruled. Now, suppose \( \pi \) is not \( c \)-split. Then there exists \( a \in H_s(F; \mathbb{Z}) \) in the kernel of \( \iota^P_F \). Therefore, the \( GW \)-invariants having \( \iota^P_F(a) \) as an entry, must vanish. But this would contradict \((1.2)\). \( \square \)

The proof of this corollary indicates, that unless we have a good knowledge of Seidel’s morphism, the number of point insertions should a priori decrease. Still, as a result, every Hamiltonian fibration over \( (\mathbb{C}P^n, \omega_{FS}) \) is \( c \)-split and uniruled, where \( \omega_{FS} \) is the standard Kähler form on the complex projective space. The same applies for Hamiltonian fibrations over \( (S^2 \times S^2, \omega \oplus \omega) \) since in that case there is only one curve representing the diagonal and passing through two points.

The paper is organized as follows. Section 1, introduces the basic ingredients needed. We also define a particular affine connection on \( P \), whose torsion is given by the symplectic curvature associated to the coupling form. In Section 2, we describe the linearization of the Cauchy-Riemann problem associated to the fibered almost complex structures. It is shown that the linearization is compatible with the projection \( \pi \). Then we prove the \textit{structure theorems} in Section 4, ensuring the \( GW \)-invariants are well-defined. In Section 5, we give the proof of the product formula. The Section 6, is devoted to showing the locally trivial smooth (orbi-)fibration structure of \( \overline{\pi} \). We then recover the product formula using integration over the fibers of \( \overline{\pi} \). We conclude by giving an example of a non-trivial induced fibration of moduli spaces.

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2. The framework

In this section we set the basic notions that will be needed in the rest of the paper. We recall that symplectic and Hamiltonian fibrations are classified by \( \text{BSymp}(F,\omega_F) \) and \( \text{BHAm}(F,\omega_F) \), respectively. Again, for \( b \in B \), let \( \omega_b \) be the induced symplectic form on the fiber \( F_b := \pi^{-1}(b) \). For the sake of clarity, we begin by recalling the notions of Hamiltonian connections and coupling form. This exposition follows \([5]\) and \([17]\), where the proofs of all the claims can be found.

2.1. Hamiltonian connections and coupling form. Consider the vertical sub-bundle, \( \text{Vert} \subset TP \), over \( P \), whose fiber at each point, \( p \in P \), is given by the subspace \( \text{Vert}_p := \ker d\pi(p) \). A connection on the fibration, \( \pi : P \to B \), is defined by a splitting of \( TP \) for each \( p \in P \):

\[
T_pP = \text{Hor}(p) \oplus \text{Vert}_p,
\]

where \( \text{Hor}(p) \) is called the horizontal plane at \( p \). The notations, \( X^h \) and \( X^v \), will refer to the horizontal and vertical parts of a vector field \( X \) on \( P \), with respect to the above splitting. Also, given a vector field \( X \) on \( B \), we will denote by \( \overline{X} \) its horizontal lift to \( TP \). Now, let \( \mathbf{h} \) denote the the symplectic curvature associated to the connection. This is the 2-form on \( B \) with values in \( \text{Vert} \), such that for \( v \) and \( w \) two vector fields on \( B \):

\[
R(v,w)(p) := [\overline{v},\overline{w}]^v(p), \quad p \in P.
\]

Any closed extension 2-form \( \tau \) of \( \omega_F \) defines a connection with horizontal planes:

\[
\text{Hor}_\tau(p) := \{ v \in T_pP \mid \tau(v,w) = 0 \ \forall w \in \text{Vert}_p \}.
\]

In fact, any form \( \tau' \in \Omega^2(P) \) such that \( \ker(\tau' - \tau) \) is in \( \text{Vert} \), defines the same horizontal plane field. Nevertheless, a specific choice can be made by requiring that:

\[
\pi_*\tau^{n+1} = \int_F \tau^{n+1} = 0.
\]

Such \( \tau \) is called coupling form associated to the connection, and its values on pairs of horizontal vectors is determined by the curvature of the connection:

\[
-d(\tau(p)(v,w)) := \iota(R(d\pi(p)v,d\pi(p)w)(p))\omega_\pi(p)(p).
\]

This results from the fact that the holonomy of the connection is Hamiltonian.

For transversality purposes we will need to allow the connection to vary. For \( b \in B \), let \( C^\infty_0(F_b) \) denote the space of smooth functions on \( F_b \) having zero mean value. We will consider Hamiltonian deformations of the coupling form \( \tau \). By this we mean exact deformations,

\[
\tau_H := \tau - d\mathbf{H},
\]

where \( \mathbf{H} \) is a section in \( C^\infty(\pi^*TB) \), i.e \( \mathbf{H}_p \) is a cotangent vector in \( T_{\pi(p)}B \), and it satisfies the property that for fixed \( b \in B \) and \( X \in T_bB \) the function \( \mathbf{H}_{b,X}(p) := \mathbf{H}_p(X) \) belongs to \( C^\infty_0(F_b) \). The subset of \( C^\infty(\pi^*TB) \) having this property will be denoted by \( \mathcal{H} \). By definition, \( \tau_H \) is a closed extension 2-form of \( \omega_F \), and one verifies that its associated horizontal distribution is given by:

\[
\text{Hor}_{\tau_H}(p) = \{ v - X_{\mathbf{H}_p(v)}(p) \mid v \in \text{Hor}_\tau(p) \}, \quad p \in P,
\]

where \( X_{\mathbf{H}_p(v)} \) denotes the (unique) Hamiltonian vector field on \( F_{\pi(p)} \) induced by the function \( \mathbf{H}_p(v) \).

Lemma 2.2. The space $\mathcal{J}_P$ is parametrized by the product $\mathcal{J}_B \times \mathcal{J}^V \times \mathcal{H}$.

The above isomorphism is given by the choice of $\tau$. In fact, the dependance is on the factor $\mathcal{H}$ corresponding to affine space of Hamiltonian connections. The choice of $\tau$ simply fixes the origin.

Remark 2.3. Note that, for any family $\{J_b\}_{b \in B} \in \mathcal{J}^V$ and any given $J_B$, we can find a positive $\kappa \in \mathbb{R}$ such that $J_P$ is $\omega_P$-tame for $\omega_P = \tau + \kappa \pi^* \omega_B$.

2.3. A specific affine connection. Fix a coupling form $\tau$. We define an affine connection on $TP$, extending the vertical Levi-Civita (L-C) connection introduced in [17] (Chapter 8), and lifting the L-C connection on $TB$. This construction will be needed in order to relate the linearization of the Cauchy-Riemann associated to $J_P = (J_B, J, H) \in \mathcal{J}_P$, to the linearization of the Cauchy-Riemann operator associated to $J_B$ (see Section 2). First, let $g_{J_P}$ be the hermitian metric on $P$ defined as 

$$g_{J_P} := gJ \oplus \pi^* g_{J_B},$$
relatively to the splitting $TP = \text{Vert} \oplus \text{Hor}_{\tau_B}$, where
\[
g_{\text{Vert}} := \frac{1}{2}(\omega_B(.,J_B.) - \omega_B(J_B., .)),
\]
and $g_J := \{g_{J_b}\}_{b \in B}$ is the analogous family of Hermitian metrics on $Vert$. Let $\nabla^B$ denote the L-C connection on $TB$ relatively to $g_{J_B}$. Also, set $\nabla^F$ to be the L-C connection on $TF_b$, with $b \in B$, relatively to $g_{J_b}$. For any vector fields $X$ and $Y$ on $P$, set
\[
\nabla_X Y := [X^h, Y^v]^u + \nabla^F_{X^h} Y^v + [X^v, Y^v]^h + (\nabla^B_{\pi_* X^h} \pi_* Y^h).
\]
This operation is clearly bilinear in $X$ and $Y$. In fact, the sum of the first two terms corresponds to the \textit{vertical L-C connection}, $\nabla^v$, which is the unique connection on $Vert$ induced by the Hamiltonian connection and which restricts to the L-C connection on $F$. The remaining part is what is needed to extend this vertical connection to an affine connection on $P$ lifting the L-C connection on $B$, which torsion $T$ is given by the symplectic curvature:
\[
T(X, Y) = -R(X, Y) = [X^h, Y^h]^v, \quad X, Y \in \mathcal{X}(TP).
\]
We show below that $\nabla$ is indeed a connection. Since $\nabla^v$ is a connection, it suffices to show that for $f \in C^\infty(P)$ and $\xi \in \text{Hor}_p$, and any $w \in TP$:
\[
\nabla_w f \xi = (w(f))\xi + f \nabla_w \xi, \quad \nabla f w \xi = f \nabla_w \xi,
\]
Suppose that $w$ is vertical. Then, by definition:
\[
\nabla_w f \xi = [w, f \xi]^h = f [w, \xi]^h + w(f) \xi.
\]
Analogously, we have $[f w, \xi] = f [w, \xi] - \xi(f) w$, implying that
\[
[f w, \xi]^h = f [w, \xi]^h
\]
since $w$ is vertical. Now, suppose that $w$ is horizontal. Let $\alpha_t, t \in (\epsilon, \epsilon)$ be the flow of $w$ starting at $p$, and let $P_t^B(\pi_* w_p)$ denote the parallel transport along the projected curve $\pi(\alpha_t)(p)$. Then,
\[
(\nabla_w f \xi)_p = \left. \frac{d}{dt} \right|_{t=0} (f(\alpha_t(p)) P_t^B(\pi_* w_p) \pi_{\alpha_t(p)} \xi)_p^h = df(p) w \xi + f(p)(\nabla_w \xi)_p,
\]
where the first equality is given by linearity of the parallel transport in $B$. We further have that
\[
\nabla f w = \left. \frac{d}{dt} \right|_{t=0} (P_t^B(\pi_* f(p) w_p) \pi_{\alpha_t(p)} \xi)_p^h = \left. \frac{d}{dt} \right|_{t=0} (P_t^B(\pi_* w_p) \pi_{\psi(t)(p)} \xi)_p^h
\]
for some reparametrization $\psi(t)$ of the interval, so that finally we have $f \nabla_w \xi$.

\textbf{Remark 2.4.} Note that $\nabla^v$ may not preserve the metric $g_J$ along horizontal directions, whereas $\tilde{\nabla}^v = \nabla^v - \frac{1}{2} J(\nabla^v J)$ does \cite[Lemma 8.3.6]{17}. Furthermore, given any vector fields $w, \xi_1$ and $\xi_2$ in $TP$, one can show that $(\nabla_w g)(\xi_1, \xi_2)$ coincides with $(\nabla_{w^h} g)(\xi_1^h, \xi_2^h)$. It follows that the $J_P$-preserving connection $\tilde{\nabla} := \nabla - \frac{1}{2} J_P(\nabla J_P)$ preserves $g_{J_P}$.

Let exp stand for both the exponential maps with respect to $\nabla$ and $\nabla^B$. The following straightforward identities will be useful in the gluing section:

\textbf{Lemma 2.5.} For $p \in P$, $X \in T_p P$ and $q \in P$ in the injective radius of $\exp_p$, we have
\[
(2.1) \quad \pi(\exp_p X) = \exp_{\pi(p)} \pi_* X, \quad \pi_* \exp^{-1}_p(q) = \exp^{-1}_{\pi(p)}(\pi(q)).
\]
2.4. Curve independance. From $(H_1)$ in the characterization of Hamiltonian fibrations we deduce the following lemma, which plays a crucial part in the proof of the product formula:

**Lemma 2.6.** Let $\pi : P \to B$ be a Hamiltonian fibration as above. Assume we have $u_1, u_2 \in C^\infty(S^2, B)$ such that $[u_1(S^2)] = [u_2(S^2)]$. Then the restricted bundles $P|_{u_1}$ and $P|_{u_2}$ are isomorphic as Hamiltonian bundles.

**Proof:** If $B$ is simply connected then it follows directly from Hurwicz isomorphism between $\pi_2(B)$ and $H_2(B, \mathbb{Z})$. Assume $B$ is not simply-connected. As Ham$(F, \omega)$ is connected, any classifying map for $\pi$ connected, any classifying map for $\pi$, say $f$, factorizes up to homotopy through a map $f' : B/B_1 \to B\text{Ham}(F, \omega)$. In other words, if $\pi_{B_1}$ denotes the projection from $B$ to $B/B_1$, the maps $f$ and $f' \circ \pi_{B_1}$ are homotopic. Let $P' := (f')^*E\text{Ham}(F, \omega)$ and consider

$$u'_1 := \pi_{B_1} \circ u_1, \quad u'_2 := \pi_{B_1} \circ u_2.$$ 

These two maps represent the same homology class $\pi_{B_1}^*(\sigma_B) \in H_2(B/B_1; \mathbb{Z})$, hence, $u'_1$ and $u'_2$ are homotopic so that:

$$P|_{u_1} \cong (u'_1)^*P' \cong (u'_2)^*P' \cong P|_{u_2}.$$

\hfill \boxrule}

3. The Cauchy-Riemann Fredholm problem in Hamiltonian fibrations

Let $j_0$ denote the complex structure on $S^2 = \mathbb{C} \cup \{\infty\}$ inherited from the multiplication by $i := \sqrt{-1}$. In a general symplectic manifold $(X, \omega)$ with $\omega$-tame almost complex structure $J$, a rational $J$-holomorphic sphere is a smooth map, $u : S^2 \to X$, satisfying the Cauchy-Riemann equation,

$$\overline{\partial}_J u := \frac{1}{2}(du + J \circ du \circ j_0) = 0.$$ 

The set of all such solutions representing a given class $A \in H_2(X, \mathbb{Z})$,

$$\tilde{\mathcal{M}}(X, A, J) := \{ u \in C^\infty(S^2, X) | \overline{\partial}_J u = 0, \ [u(S^2)] = A \},$$

is the moduli space of parametrized $J$-holomorphic maps representing $A$. Consider the Fréchet space,

$$\mathcal{B}_X(A) := \{ u \in C^\infty(S^2, X) | [u(S^2)] = A \},$$

and the space

$$\mathcal{E}_X(A, J) = \bigsqcup_{u \in \mathcal{B}_X(A)} \mathcal{E}_{X,u}(J) := \bigsqcup_{u \in \mathcal{B}_X(A)} C^\infty(A_{0,1}^J(S^2, u^*TX)).$$

The obvious projection from $\mathcal{E}_X(A, J)$ to $\mathcal{B}_X(A)$, defines a locally trivial bundle between Fréchet spaces. Then, $\overline{\partial}_J$ is a section of this bundle, and $\tilde{\mathcal{M}}(X, A, J)$ is the corresponding zero set. The linearized operator $D_{\overline{\partial}_J}$ of $\overline{\partial}_J$ at $u \in \tilde{\mathcal{M}}(X, A, J)$ is defined as the differential of $\overline{\partial}_J$ at $u$ composed with the projection on the fiber $\mathcal{E}_{X,u}(J)$. To give a meaning to this vertical projection outside of the zero section, we consider the Hermitian connection on $\mathcal{E}_X(A, J)$,

$$\tilde{\nabla}^X := \nabla^X - 1/2(J(\nabla^X J)),$$
where $\nabla^X$ is the L-C connection on $X$ with respect to the metric $g_J$. Then,
\[
D^X_u : \mathcal{X}_{X,u} := C^\infty(S^2, u^*TX) \to \mathcal{E}_{X,u}(J) \\
\xi \mapsto \nabla^X u \cdot J(u) = (\nabla^X u)^{0,1}(u) \xi - \frac{1}{2} J(u)(\nabla^X J)(\partial J u),
\]
where $(\nabla^X)^{0,1}$ is the $J$ anti-linear part of $\nabla^X$. It is well-known that $D^X_u$ is Fredholm for $u \in \mathcal{M}(X,A,J)$. Moreover, if $\mathcal{J}_u$ is transversal to the zero section, then the moduli space is a smooth finite dimensional oriented manifold $[17], [22]$.

Now, let $\pi : P \to B$ be a Hamiltonian fibration with coupling form $\tau$, let $J_P$ be a fibered complex structure relatively to $\tau$, and let $\sigma \in H_2(P, \mathbb{Z})$. The connection $\nabla$ we constructed induces a splitting of the tangent space of $\mathcal{E}_{P}(\sigma, J_P)$ at all points, and projection to the fiber direction can again be defined. Hence, the linearization $D^P_u$ of $\mathcal{J}_u$ can be defined for all $u \in B_{P}(\sigma)$. Note that $\nabla$ is not Levi-Civita since its torsion is given by the symplectic curvature. This is what gives rise to the extra curvature term in the following expression for $D^P_u$:

**Lemma 3.1.** Let $u \in B_{P}(\sigma)$ and $\xi \in C^\infty(S^2, u^*TP)$. Then,
\[
D^P_u \xi = (\nabla^P u)^{0,1}(u) \xi - \frac{1}{2} J_P(u)(\nabla^P u)(\partial J_P u) + R^{0,1}(du^h, \xi^h),
\]
where $\nabla^{0,1}$ and $R^{0,1}$, respectively stand for the $J_P$ anti-linear parts of $\nabla$ and $R$.

**Remark 3.2.** When $\xi$ is vertically valued the curvature term disappears and we recover the vertical linearized operator introduced by McDuff and Salamon ([17], Chapter 8). In the rest of the paper we will designate by $D^v$ the restriction of $D^P$ to vertically valued vector fields.

### 3.1. Splitting of Fredholm systems.

Since we consider fibered almost complex structures on $P$, the projection $\pi$ naturally induces a map,
\[
\pi : \mathcal{M}(P, \sigma, J_P) \to \mathcal{M}(B, \sigma_B, J_B), \quad u \mapsto \pi(u),
\]
where $\sigma_B := \pi_* \sigma$. As we will see, $\pi$ induces a submersion between the Fredholm system $(B_{P}(\sigma), \mathcal{E}_{P}(\sigma, J_P, \mathcal{J}_P))$ and $(B_{B}(\sigma_B), \mathcal{E}_{B}(\sigma_B, J_B, \mathcal{J}_B))$. We recall the notion of Fredholm system as given in [2].

**Definition 3.1.** A Fredholm system of index $d$ is a triple $(B, \mathcal{E}, s)$ such that:

$(F_1)$ $\mathcal{E}$ is the total space of a Banach fibration over a Banach manifold $B$, with projection $p$,

$(F_2)$ $s : B \to \mathcal{E}$ is a section such that, for all $x \in s^{-1}(0)$, the linearization $L_x : T_x B \to \mathcal{E}_x := \pi^{-1}(x)$ of $s$ at $x$, is Fredholm of index $d$.

When, in addition, $s$ is proper, we say that the system is compact.

We call the set, $s^{-1}(0)$, the moduli space of the system.

**Example 3.3.** Let $(X, \omega)$ be a general symplectic manifold with $A \in H_2(X, \mathbb{Z})$ and $\omega$-tame almost complex structure $J$. Fix an integer $p > 2$. For $u \in \mathcal{B}_X(A)$ we respectively equip $\mathcal{X}_{X,u}$ and $\mathcal{E}_{X,u}(J)$ with $W^{1,p}$ (Sobolev) and $L^p$ norms (relatively to the metric $g_J$ and a fixed volume form on $S^2$). Explicitly, given $\xi \in \mathcal{X}_{X,u}$ and $\eta \in \mathcal{E}_{X,u}(J)$:
\[
\|\xi\|_{1,p} = \left(\int_{S^2} (|\xi|^p_{g_J} + |\nabla^{1}\xi|^p_{g_J}) dvol_{S^2}\right)^{\frac{1}{p}}, \quad \|\eta\|_{p} = \left(\int_{S^2} |\xi|^p_{g_J} dvol_{S^2}\right)^{\frac{1}{p}}.
\]
We denote by $X^{1,p}_{X,u}$ and $E^p_{X,u}(J)$ the completed vector spaces with respect to these norms, and by $B^1_{X}(A)$ and $E^p_{X}(A,J)$ the corresponding completions of $B_X(A)$ and $E_X(A,J)$. Under these completions, $D^X_u$ is Fredholm so that the triple,

$$(B^1_{X}(A), E^p_{X}(A,J), \overline{\sigma}),$$

satisfies conditions $(F_1)$ and $(F_2)$. This Fredholm system will in general not be compact.

Next we define the following natural notion of morphisms between two Fredholm systems $(B, E, s)$ and $(B', E', s')$.

**Definition 3.2.** A map $\Pi := (\pi, \overline{\pi}) : (B, E, s) \to (B', E', s')$ between Fredholm systems is a Banach orbifold vector bundle morphism, i.e

$$\pi : B \to B', \quad \overline{\pi} : E \to E',$$

such that

i) $s' \circ \pi = \overline{\pi} \circ s$

ii) $L''_{\pi(x)} \circ d\pi(x) = d\overline{\pi}(x,0) \circ L_x$, for every $x \in s^{-1}(0)$.

We say that $\Pi$ is a submersion if, furthermore, $d\pi$ and $d\overline{\pi}$ are surjective.

When $\Pi$ is a submersion, we directly extract the following exact sequence:

$$(3.1) \quad \ker L''_x \rightarrow \ker L_x \rightarrow \ker L''_{\pi(x)} \rightarrow \operatorname{coker} L''_x \rightarrow \operatorname{coker} L_x \rightarrow \operatorname{coker} L''_{\pi(x)},$$

where $L''_x$ stands for the operator $L_x$ restricted to $\ker d\pi(x)$. From exactness of this sequence we deduce splitting at the level of operator indices, i.e for every $x \in s^{-1}(0)$:

$$\operatorname{Ind}(L_x) = \operatorname{Ind}(L''_{\pi(x)}) + \operatorname{Ind}(L'_{\pi(x)}).$$

Obviously, if both $L''_x$ and $L'_{\pi(x)}$ are unobstructed (i.e. their cokernels vanish), then $L_x$ is also unobstructed. As a result, $\ker L_x$ is isomorphic to $\ker L''_x \oplus \ker L'_{\pi(x)}$ (upto a choice of a section from $\ker L'_{\pi(x)}$ to $\ker L_x$).

**Definition 3.3.** A submersion $\Pi$ between Fredholm systems is a splitting, if the sequence $(3.1)$ is obstruction free, i.e if for all $x \in s^{-1}(0)$:

$$\ker L''_x = \ker L_x = \ker L'_{\pi(x)} = 0.$$
Proof: For the left handside diagram, commutativity follows from the fact that $J_P$ is fibered. Thus, $\pi$ maps $(\overline{\partial}J_p)^{-1}(0)$ into $(\overline{\partial}B)^{-1}(0)$. We prove that the second diagram commutes. Namely, we show that for every $\xi \in X(u^\ast TP)$:

$$\pi_\ast D_u \xi = D^B_{\pi \circ u} \pi_\ast \xi.$$ 

This is a consequence of the following three identities: let $\xi$ and $X$ be vector fields on $P$, and let $p \in P$, then

(i) $\pi_\ast [\xi^v, J_P X^h] = J_B \pi_\ast [\xi^v, X^h],$

(ii) $\pi_\ast (\nabla_{\xi} J_P)_{\pi(p)} X = (\nabla^B_{\pi_\ast \xi} J_B)_{\pi(p)} (\pi_\ast X),$

(iii) $\pi_\ast (\nabla^B_{du})^0_1 \pi_\ast \xi - \pi_\ast ([\overline{\partial}J_p(u)]^h, \xi^v)^h.$

Assuming these are verified we obtain that:

$$\pi_\ast D_u \xi = (\nabla^B_{d(\pi \circ u)})^0_1 \pi_\ast \xi - \pi_\ast ([\overline{\partial}J_p(u)]^h, \xi^v)^h - \frac{1}{2} J_B (\pi \circ u)(\nabla^B_{\pi_\ast \xi} J_B)(\pi_\ast \partial J_p u),$$

since $\nabla^B_{du}$ and $R^0_1 (du, \xi)$ are vertically valued. Moreover, since $\pi_\ast \partial J_p u = \partial J_B u_B$ it only remains to show that:

$$[[\overline{\partial}J_p(u)]^h, \xi^v]^h_{u(z)} = 0 \ \forall z \in S^2.$$

But, if $(\overline{\partial}J_p u)(z)^h \neq 0$ for some $z$, consider the horizontal lift $X$ of $(\overline{\partial}J_p (\pi(u)))(z)$, which is defined on $T(\pi^{-1}(u(z)))$, and agrees with $(\overline{\partial}J_p u)(z)^h$ at $u(z)$. Then,

$$[[\overline{\partial}J_p(u)]^h, \xi^v]^h_{u(z)} = [X, \xi^v]_{u(z)},$$

where the right handside vanishes since $X$ is constant along vertical directions.

Now, equality (i) follows by definition of the bracket, by holomorphicity of the projection, and since the flow of a vertical vector field starting at a point $p$ remains in the fiber above $\pi(p)$. Equality (ii) is just a consequence of (i), since this latter is equivalent to

$$[\xi^v, J_P X^h]^h = J_P [\xi^v, X^h]^h,$$

which by definition of $\nabla$ is the same as:

$$\nabla_{\xi^v} (J_P X^h) = J_P \nabla_{\xi^v} (X^h).$$

Hence, $(\nabla_{\xi^v} J_P)(X^h) = 0$, which, combined with the fact that the connection is vertically valued when its two entries are in $\text{Vert}$, gives us the second equation.

For (iii) we have that:

$$\nabla^B_{du}^0_1 \xi = \frac{1}{2}(\nabla_{\xi} du + J_P(u)\nabla_{\xi}(du \circ j_0)) - R^0_1 (du, \xi).$$

Hence, by definition of the connection and because the curvature is vertically valued, we get

$$2\pi_\ast (\nabla^B_{du}^0_1 \xi) = \pi_\ast \nabla_{\xi} du + J_B (\pi(u))\pi_\ast \nabla_{\xi}(du \circ j_0)$$

$$= \nabla^B_{\pi_\ast \xi} d(\pi \circ u) + \pi_\ast [\xi^v, (du)^h]^h + J_B (\pi(u))^0_1 \pi_\ast \xi^v, (du \circ j_0)^h]^h$$

$$+ J_B (\pi(u))\pi_\ast [\xi^v, (du \circ j_0)^h]^h$$

$$= \nabla^B_{d(\pi \circ u)} \pi_\ast \xi + J_B (\pi(u))^0_1 \nabla^B_{d(\pi \circ u) \circ j_0} \pi_\ast \xi + \pi_\ast [\xi^v, (du)^h + J_P(u)(du \circ j_0)^h]^h$$

$$= 2(\nabla^B_{d(\pi \circ u)})^0_1 \pi_\ast \xi + \pi_\ast [\xi^v, 2(\overline{\partial}J_p(u))^h]^h,$$

where the third equality is due to (i) and the fact that $\nabla^B$ is torsion free. The last follows since $J_P$ preserves the horizontal distribution and the vertical subbundle. \qed
The symplectic connection on $P$ induces the splittings:

$$\mathcal{E}_{P,u} = \Gamma(\Lambda_0^1(S^2, u^*TP^h)) \oplus \Gamma(\Lambda_0^{0,1}(S^2, u^*TP^v)) =: \mathcal{E}^h_{P,u}(J_P) \oplus \mathcal{E}^v_{P,u}(J_P),$$

and

$$\mathcal{X}_{P,u} = \Gamma(S^2, u^*TP^h) \oplus \Gamma(S^2, u^*TP^v) = \mathcal{X}^h_{P,u} \oplus \mathcal{X}^v_{P,u}.$$ 

In this splitting, $D^P_u$ takes the following matrix form:

$$\begin{pmatrix} (DB^{(u)}(u))^h & 0 \\ L_u & D^v_u \end{pmatrix}$$

where $L_u$ is linear and given by:

$$L_u : \mathcal{X}^h_{P,u} \longrightarrow \mathcal{X}^v_{P,u}, \quad \xi \mapsto -\frac{1}{2} J(u)(\nabla \xi J)(\partial J_P u) + R^{0,1}(du^h, \xi).$$

Thus, applying the diagram (3.1), we obtain the exact sequence:

$$\ker D^v_u \xrightarrow{\sim} \ker D^P_u \longrightarrow \ker DB^{\pi(u)} \longrightarrow \coker D^v_u \longrightarrow \coker D^P_u \longrightarrow \coker DB^{\pi(u)}$$

where the connectant is given by the restriction of $L_u$ to the kernel of $DB^{\pi(u)}$.

**Lemma 3.5.** A Hamiltonian fibration induces, up to transversality, a splitting of Fredholm system for $(B^1_P(\sigma), \mathcal{E}^p_P(\sigma, J_P), \overline{J}_P)$.

The issues regarding transversality are considered in the next section.

### 4. Structure theorems

Let $\widetilde{\mathcal{M}}_{0,l}(P, \sigma, J_P)$ be the moduli space of parametrized $J_P$-holomorphic maps with $l$ marked points, representing $\sigma$. This space consists of tuples, 

$$(u, x_1, \ldots, x_l) \in \mathcal{M}(P, \sigma, J_P) \times (S^2)^l,$$

where the points $x_1, \ldots, x_l \in S^2$ are pairwise distinct. The group $G := PSL_2(\mathbb{C})$ of reparametrizations of $S^2$ acts (diagonally) on this moduli space. The quotient of $\mathcal{M}_{0,l}(P, \sigma, J_P)$ under this action is usually not compact. Still, its “Gromov’s compactification”, $\overline{\mathcal{M}}_{0,l}(P, \sigma, J_P)$, is a stratified space consisting of stable holomorphic maps $[7, 17, 22]$. Concretely, the stratification is given by the combinatorial-type of the labeled connected trees with tails modeling the stable maps. When these strata are automorphism free they can be given a manifold structure for a generic choice of fibered almost complex structure. After giving the description of the stable holomorphic maps in Hamiltonian fibrations, we show that the latter structure theorem holds compatibly with the projection $\pi$. This is the content of Theorem B’, which is a slight extension of Theorem B. We conclude by observing that changing of generic almost complex structure induces a cobordism between the corresponding moduli spaces.

#### 4.1. Stable holomorphic maps and Hamiltonian fibrations.

In order to fix notations and terminology, we begin by introducing, first, the combinatorics needed to describe stable holomorphic maps, and second, the moduli space of stable pseudo-holomorphic maps.
4.1.1. *Labeled graphs.* This exposition is mostly taken from [3]. Nevertheless, emphasis is put on how the combinatorics change when given a homomorphism of semi-groups.

Let $S = (V, Fl; \text{pr}, \varrho)$ be a graph. Here, $V$ denotes the set of vertices, $Fl$ denotes the set of flags. The map $\text{pr} : Fl \to V$ assigns to a given flag, its associated vertex, i.e. for $v \in V$, the set $\text{pr}^{-1}(v)$ gives the valences at $v$. The notation $|v|$ denotes the number of valences at $v$. The map, $\varrho : Fl \to Fl$ is an involution. The set $E$ of edges in $S$, is identified to the 2-elements orbits of $\varrho$, while the fixed points of $\varrho$ gives the set, $T$, of tails of $S$. In the rest of the paper, we use the notation, $vEw \ (v, w \in V)$, to indicate that there is an edge between the $v$ and the $w$ vertices of $S$.

Denote by $C$ the set of connected components of $S$. There is an induced map, $\text{pr}_C : T \to C$, assigning a tail to the connected component it is attached to. The genus, $g(S)$, of the graph is the Euler number of $S$, i.e

$$g(S) = |V| - |E| + 1.$$  

When all the components of $S$ have genus 0, we say that it is a forest; if, furthermore, $S$ is connected, we say that it is a tree. In the rest of the paper we will only consider forests.

The subgraph, $(v, Fl_v := \text{pr}^{-1}(v); \text{pr}|_{Fl_v}, id)$, is called the star of $S$ at $v$. Given a subset, $V' \subset V$, the complement $S\setminus V'$ of $V'$ in $S$, is defined as the subgraph given by $S$ from which we removed all the stars at points coming from $V'$.

A map, $\beta : V \to B$, where $B$ is an Abelian semi-group, defines a $B$-labeling of $S$. We denote by $(S, \beta)$ the corresponding labeled graph. A $B$-labeling $\beta$ induces a $B$-labeling of $C$:

$$\beta_C : C \to B, \quad \beta_C(c) := \sum_{v \in c} \beta(v).$$

An example of labeling, is the *genus-labeling* given by a map, $h : V \to N$, assigning to each vertex the corresponding genus. In what follows, the genus labeling will always be trivial.

**Composition of graphs.** The composition, $S'' := S' \Rightarrow S$, of two graphs $S'$ and $S$, is the graph obtained by replacing the vertices in $S$ by the components of $S'$. More precisely, $S''$ is determined by an isomorphism of maps between finite sets:

$$[\text{pr}'_C : T' \to C'] \cong [\text{pr} : Fl \to V],$$

that is, we have isomorphisms $T' \cong Fl$, and $C' \cong V$, commuting with $\text{pr}'_C$, and $\text{pr}$. If such isomorphisms exist, we say that $S'$ and $S$ are *composable*. In that case, $S''$ is given as follows:

$$V'' = V', \quad Fl'' = Fl' \quad \text{pr}'' = \text{pr}', \quad \text{and} \quad \varrho'' = \begin{cases} g' & \text{on elements not fixed by } \varrho \\ \varrho & \text{otherwise.} \end{cases}$$

This notion of composition defines a partial order $\prec$ on graphs: we set $S'' \prec S$. More generally, labeled graphs can be composed. Let $(S, \beta)$ and $(S', \beta')$ be two labeled graphs such that $S$ and $S'$ are composable, and let $\pi : B' \to B$ be a homomorphism of semi-groups. We say that the graphs are $\pi$-composable, if $\beta$ and $\beta'_C$ commute with respect to $\pi$. The $\pi$-composition is the labeled graph, $(S'', \beta'')$, where $\beta'' := \pi \circ \beta'$. This defines a $B$-labeling. The *labeled composition* of $(S, \beta)$ with $(S', \beta')$, is the $B'$-labeled graph $(S'', \beta')$.

Note that composition of graphs induces a surjective map on the sets of vertices, $\gamma : V'' \to V$, and of flags, $\gamma_{Fl} : Fl'' \to Fl$. A *section* of the composition is a pair of right inverses, $\iota : V \to V''$ and $\iota_{Fl} : Fl \to Fl''$, such that $(\text{pr}''\iota)^{-1}(\iota(v)) = \iota_{Fl}(\text{pr}^{-1}(v))$. 

which further verify $\beta'' \circ t = \pi \circ \beta' \circ t = \beta$ when we have labelings. In this latter case, we say that $(i, t_{\mathbb{F}1})$ is a labeled section. Note that a section always exists while a labeled one may not. Let $(S'', \beta')$ be a labeled composition of $(S', \beta')$ with $(S, \beta)$, and let $(i, t_{\mathbb{F}1})$ be a labeled section. The components of $S'' \setminus I(V)$, together with the restriction of the labeling $\beta'$, are called contracted components. Note that $(S, \beta)$ is obtained by replacing the contracted components with either tails (if the contracted component had some) or edges between two vertices of $V$. We say that $(S, \beta)$ is the contraction of $(S'', \beta')$. As an example, consider the case where we contract a vertex $v \in V''$ with $|v| \leq 2$. First, delete the star at $v$, thus creating either one or two tails. Assume the graph is connected. If it has only one vertex, everything disappears. Suppose it has more than one vertex. If $v$ had a tail, then you don’t delete the new tail created on its neighbour. If $v$ had two edges, we replace the deleted vertex by an edge obtained by gluing together the two new tails. In the case it only had one edge and no tail, we also delete the new tail. This operation is called elementary contraction.

Stability of graphs. Let $S$ be a forest. A vertex $v \in V$ is said to be stable if $|v| \geq 3$. The forest $S$ is said to be stable, if all its vertices are stable. If, furthermore, we are given a $B$-labeling $\beta$ on $S$, we say that a vertex $v \in V$ is $(B)$-stable if either $\beta(v) \neq 0$ or $v$ is stable. The labeled graph $(S, \beta)$ is $B$-stable when all its vertices are $(B)$-stable. The following lemma, showing how the combinatorics change under a homomorphism $\pi : B' \to B$ of semi-groups, is standard.

Lemma 4.1. Suppose $(S'', \beta')$ is a stable labeled tree, then we have one of the following: (1) there exist stable tree $(S, \beta)$ and a stable forest $(S', \beta')$ such that $(S'', \pi \circ \beta')$ is their $\pi$-composition; (2) $|V''| = 2$. Moreover, in case (1), there is a labeled section of the composition.

Proof: Suppose $(S'', \pi \circ \beta')$ is not stable. This implies that there is at least one vertex of $S''$ which is not stable. Let $v$ be such a vertex, then $|v| \leq 2$ and $\pi(\beta'(v)) = 0$. In the case $|v| = 0$, we directly have (2). If $|v| > 0$, then contract the unstable vertices inductively until we get a stable forest, or we get (2). These elementary contractions don’t change the number of tails and preserve connectedness of the graph. Suppose that the resulting tree, $(S, \beta)$, is stable. By construction, we have natural inclusions $i : V \hookrightarrow V''$ and $i_{\mathbb{F}1} : F I \hookrightarrow F I''$. The forest, $(S', \beta')$, is then given by $S' = (V'', F I'') ; p'' \circ g'$. We define $g'$. By the definition of composition, each edge of $S'$, that is each 2-element orbit $\{f_1, f_2\}$ of $g$, should correspond to two tails in $S''$. So, letting $\{f'_1 := i_{\mathbb{F}1}(f_1), f'_2 := i_{\mathbb{F}1}(f_2)\}$ be the 2-element orbit of $f_1$ under $g''$, we then set $g'(f'_1) = f'_1$ and $g'(f'_2) = f'_2$. Now, for the flags $f \in F I''$ not coming from $S$, we simply set $g'(f) = g''(f)$. \qed

The proof indicates that the tree $(S, \beta)$ is unique while $(S', \beta')$ is not in general. The natural section mentioned in the proof will be called $\pi$-section.

Definition 4.1. Let $\pi : B' \to B$ be a homomorphism of semi-groups. We say that the tree $(S, \beta)$ above is the $\pi$-stabilization of $(S'', \beta')$ and we denote it $S_\pi(S'', \beta')$. The vertices in $S$ are called $\pi$-stable, and the contracted components are called $\pi$-unstable.

Let $(S, \beta) = S_\pi(S'', \beta')$. The vertices in the image of section are said to be $B$-stable. Now, let $c$ be one of the contracted components in $S''$. Then $c$ is a tree with at most 2 tails. A tail of $c$ is called exterior if it is also a tail of $S''$. Otherwise, it is called interior. In the case $c$ has 1 interior tail, we say that $c$ is a $(\pi, \cdot)$-contracted
branch. In the case $c$ has two interior tails, we say that $c$ is a $(\pi\cdot)$-connecting branch.

For a connecting branch there is a unique path of vertices connecting the 2 tails. This path forms a connecting chain.

4.1.2. Moduli space of stable maps. The discussion below is taken from [17] and [22], where the proofs of all the claims can be found. Denote by $\overline{M}_{g,l}$ the Deligne-Mumford moduli space of stable curves with genus $g$ and $l$-marked points. We shall only consider the case $g = 0$. Points in $\overline{M}_{0,l}$ are given by isomorphism classes of elements,

$$j \equiv (\Sigma, j, x := (x_1, \ldots, x_l)),$$

where $(\Sigma, j)$ is a nodal Riemann surface of arithmetic genus 0 with no self-intersection, together with $l$ pairwise disjoint marked points on $\Sigma$ denoted $x$, which are disjoint from the nodes. Furthermore, each component of $\Sigma$ has at least three special points (i.e marked points or nodes). Two pointed nodal curves $(\Sigma, j, x)$ and $(\Sigma', j', x')$ are isomorphic, if there is a diffeomorphism $\varphi : \Sigma \to \Sigma'$ satisfying:

$$\varphi^*j' = j, \quad \text{and} \quad \varphi(x_i) = x'_i.$$

We denote by $\text{Aut}(\Sigma, j, x)$ the automorphisms of $(\Sigma, j, x)$, i.e the subset of diffeomorphisms of nodal surfaces, $\varphi : \Sigma \to \Sigma$, such that $\varphi^*j = j$ and $\varphi(x) = x$. This group is invariant under isomorphism of nodal curves. Also, it is standard that the elements of $\overline{M}_{0,l}$ are automorphism free.

Let $\overline{M}_{0,l}(X, A, J)$ denote the compactified moduli space of stable $J$-(pseudo)-holomorphic maps, from (nodal) curves of genus 0 with $l$ marked points into the symplectic manifold $X$, representing the class $A \in H_2(X, \mathbb{Z})$. Points in $\overline{M}_{0,l}(X, A, J)$ are given by isomorphism classes of parametrized stable pseudo-holomorphic maps $(j, u) \equiv ((\Sigma, j, x), u)$, where $(\Sigma, j, x)$ is a Riemann nodal curve of genus 0 with $l$ marked points (not necessarily stable), and, $u : \Sigma \to X$, is $(j, J)$-holomorphic and such that each component on which $u$ is constant has at least three special points. We say that $(j, u)$ is isomorphic to $(j', u')$, if there is an isomorphism of pointed nodal curve, $\varphi$, between $j$ and $j'$, such that $\varphi^*u = u'$. Let $\text{Aut}(j, u)$ denote the corresponding automorphism group. It is well-known that stability implies finiteness of the automorphism groups. Moreover, if the map $u$ is reduced, or simple in the sense that $(j, u)$ has no ramified component or any two component having the same non-constant image in $X$, then $\text{Aut}(j, u) = id$ (when the map has one only one component, this notion of simple map coincides with the notion of somewhere injective map). It is well known that any stable pseudo-holomorphic map can be reduced to a simple stable map. The reduction process however changes the homology class of the map (see [17]).

The moduli spaces $\overline{M}_{0,l}$ and $\overline{M}_{0,l}(X, A, J)$ are stratified, with strata labeled by stable labeled trees of genus 0 called stable stratum data of the strata, or combinatorial type. For a tree $S = (V, F; l; pr, g)$, the set $V$ corresponds to the components of $(\Sigma, j)$, while $F$ corresponds to the set of special points on the curve. For $\overline{M}_{0,l}(X, A, J)$ we have a $H_2(X, \mathbb{Z})$-labeling giving the homology class represented by the image of each component in $X$. Note that $\overline{M}_{0,l}$ and $\overline{M}_{0,l}(X, A, J)$ coincide when $X = pt$ and $A = 0$. A strata with stratum data $S$ will be denoted $M_S$ or $M_S(X, J)$, and $M_S(X, J)$ will denote the subset of $M_S(X, J)$ consisting of simple stable maps. Furthermore, the partial order $\prec$ on labeled graphs induces inclusions of strata:

$$M_{S'} \subset M_S \iff S' \prec S.$$
and similarly for $\mathcal{M}_S(X,J)$. Note that there are finitely many strata in the compactified moduli space, since, given $l$ and $A$, the set $\mathcal{D}^A_{0,l}$ of possible combinatorial types for genus 0 stable maps, with $l$ markings representing $A$, is finite.

As we are considering only genus 0 stable maps, we can in fact fix the complex structure on each component of the nodal surface to be the standard one. Let $\mathcal{M}_S(X,J)$ denote the moduli space of parametrized stable $J$-holomorphic maps $(\Sigma, u, x)$ representing $S$, and let $G_S$ be the reparametrization group of the domain $(\Sigma, x)$. The stratum $\mathcal{M}_S(X,J)$ is then identified to the quotient of $\mathcal{M}_S(X,J)$ under the (proper) action of $G_S$. For the sake of the reader, we describe it. First, note that a stable map, $(\Sigma, u, x)$, is determined by a triple, $(u, y, x)$, where $y := \{y_{v,v'}\}_{v,v' \in V}$ for $v, v' \in V$, is the data given by the nodal points in $\Sigma$. Let $\Sigma_v$ denote the component of $\Sigma$ corresponding to $v \in V$, and let $u_v$ be the restriction of $u$ to $\Sigma_v$. The group $G_S$ consists of pairs $((\{\varphi_v\}_{v \in V}, \gamma)$, where $\gamma \in \text{Aut}(S)$ is a tree-with-tails automorphism, and $\varphi_v : \Sigma_v \to \Sigma_{\gamma(v)}$ is an element of $\text{PSL}_2(\mathbb{C})$. Then,

$$((\{\varphi_v\}, \gamma) \cdot (u, y, x) := ((u_v \circ \varphi_v^{-1}), \{\varphi_v(y_{v,v'})\}_{v,v' \in V},\{\varphi_p(x_k)(x_k)\}_{k \in \{1,...,l\}}),$$

is the considered action of $G_S$ on $\mathcal{M}_S(X,J)$. Before carrying out the description of stable $J_P$ holomorphic maps in a Hamiltonian fibration $\pi : P \to B$, we first make sure that we have the appropriate energy bounds in order to apply Gromov’s compactness ([17, 22]).

4.1.3. Energy identities. Suppose $J_P$ is a fibered structure obtained from a connection $\tau$, an element $J_B \in \mathcal{J}_B$ and a family $J \in \mathcal{J}^V$, and let $g_{J_P}$ be the corresponding split metric on $P$. For a smooth map $u : S^2 \to P$, we define its total energy to be its Dirichlet norm with respect to $g_{J_P}$:

$$E(u) := \frac{1}{2} \int_{S^2} \|du\|^2_{g_{J_P}} d\text{vol}_{S^2}.$$ 

Since $g_{J_P}$ is split, $E(u)$ can be written as the sum,

$$\frac{1}{2} \int_{S^2} \|d(\pi(u))\|^2_{g_B} d\text{vol}_{S^2} + \frac{1}{2} \int_{S^2} \|(du)^\nu\|^2_{g_J} d\text{vol}_{S^2} := E_B(\pi(u)) + E^\text{vert}(u),$$

where $E_B(\pi(u))$ is the energy of $u_B := \pi(u)$ (with respect to $J_B$), and $E^\text{vert}(u)$ is the vertical energy. When $u$ is $J_P$-holomorphic, it turns out that:

$$E_B(u_B) = \int_{S^2} u_B^* \omega_B, \quad \text{and} \quad E^\text{vert}(u) = \int_{S^2} u^* \tau + \int_{S^2} R(u) d\text{vol}_{S^2},$$

where the second identity is obtained since:

$$\tau(du, J_P du) = \omega(du^\nu, J du^\nu) - R(du^b, J_P du^b).$$

Consider the Hofer norm of the symplectic curvature,

$$\|R\|_H := \int_B \left(\max_{p \in F_h} R(p) - \min_{p \in F_h} R(p)\right) \omega_B^H,$$

which is bounded by compactness of $P$. Then, we obtain the following upper bound:

**Lemma 4.2.** For every $J_P$-holomorphic map $u$:

$$E(u) \leq \int_{S^2} u^* \tau + \|R\|_H + \int_{S^2} u_B^* \omega_B.$$ (4.1)
Applying Gromov’s compactness, we conclude that any sequence of simple \( J_P \)-holomorphic map representing \( \sigma \), must converge (up to taking a sub-sequence) to a stable \( J_P \)-holomorphic map. From the upper bound [14] and the stability condition for stable maps we deduce the following standard result [17, 22]:

**Lemma 4.3.** We have \( |D_{0,1}^\sigma| < \infty \).

4.1.4. **Forgetful maps and Hamiltonian fibrations.** The natural map \( \pi_{pt} : X \to pt \), induces the map: \( \pi_{pt,*} : H_2(X, \mathbb{Z}) \to \{0\} \) on the labeling groups. We then obtain the standard *forgetful* map:

\[
\pi_{pt} : M_{0,1}(X, A) \to \overline{M}_{0,1}, \quad \pi_{pt}(j, u) = j^{st},
\]

where \( j^{st} \) denotes the stabilization of \( j \), that is \( j^{st} \) is obtained by contracting the unstable components recursively. It is well-known that, when restricted to strata, the forgetful map defines maps:

\[
\pi^S_{pt} : M_S(X) \to M_{S^{pt}^*}(S).
\]

This procedure can be generalized to any Hamiltonian fibration \( \pi : P \to B \) with coupling form \( \tau \). Again, the moduli spaces of stable pseudo-holomorphic maps in \( P \) and \( B \) are stratified, with strata labeled by stable labeled trees. Note that \( \pi \)

induces a map \( \pi_* : H_2(P, \mathbb{Z}) \to H_2(B, \mathbb{Z}) \) between the labeling groups. Let \( J_P \) be a \((\pi, \tau)\) compatible almost complex structure on \( P \) projecting on \( J_B \). Given a stable stratum data \( S_P \) for \( J_P \)-holomorphic maps in \( P \), representing a class \( \sigma \in H_2(P, \mathbb{Z}) \), we have that \( S_B := S_{\pi_*}(S_P) \) is a stable stratum data for \( \overline{M}_{0,1}(B, \pi_*^*(\sigma), J_B) \), and we have a \( \pi \)-*forgetful* map:

\[
\pi_{S_P} : M_{S_P}(P, J_P) \to M_{S_B}(B, J_B), \quad (j, u) \mapsto (j^{st,\pi}, u_B := \pi(u)),
\]

where \( j^{st,\pi} \) is the Riemann nodal surface consisting of the components determined by the \( \pi_* \)-section of the \( \pi_* \)-stabilization of labeled graphs (which has combinatorial type \( S_{\pi_*}(S_P) \)), and \( u_B \) restricts to the \( \pi_* \)-stable components. We further have a reparametrization-group-equivariant map:

\[
\pi_{S_P} : \widetilde{M}_{S_P}(P, J_P) \to \widetilde{M}_{S_B}(B, J_B), \quad (u, y, x) \mapsto (u_B, y_B := y^{st,\pi}, x)
\]

lifting \( \pi_{S_P} \). Note that, a priori, \( \pi_{S_P} \) may not respect simplicity of pseudo-holomorphic stable maps (e.g. by sending a simple \( J_P \)-holomorphic map to a multiply covered \( J_B \)-holomorphic map). This has dramatic effects regarding transversality within the range of fibered almost complex structures. For each stable stratum data \( S_P \) we will consequently restrict our attention to the subset, \( M^*_{S_P}(P, J_P) \), of simple stable elements in \( M_{S_P}(P, J_P) \) lying in the preimage of \( M^*_B(B, J_B) \) under \( \pi_{S_P} \). In particular, we denote by \( (\pi_{S_P}^{-1}(\overline{B}))^* \) the set of simple parametrized stable pseudo-holomorphic maps lying in the fiber of \( \pi_{S_P} \) above \( \overline{B} \in \widetilde{M}^*_B(B, J_B) \), and we use the notation \( (\overline{\pi}_{S_P}^{-1}(\overline{B}))^* \) to denote the corresponding quotient under reparametrizations.

4.2. **Transversality on every strata.** We begin by recalling some standard notations and fact concerning transversality for a symplectic manifold \((X, \omega)\). Then, we apply it to Hamiltonian fibrations. Finally, we formulate the corresponding cobordism invariance.
4.2.1. The non-fibered case. Consider a stable stratum data \( S_X = (V, F; \text{pr}, \varrho) \) for \( \overline{\mathcal{M}}_{0,1}(X, A, J) \), with homological labeling \( \beta \). For every \( v \in V \), let \( \sigma_v := \beta(v) \) and set
\[
\widetilde{\mathcal{M}}^*(X, \beta, J_X) := \left\{ (u := \{ u_v \}_{v \in V}, J) \mid J \in J_X \text{ and } u_v \in \widetilde{\mathcal{M}}^*(X, \sigma_v, J) \right\}.
\]
This defines a subset of
\[
\mathcal{B}^1_{\mathcal{X}}(\beta, J_X) := \prod_{v \in V} \mathcal{B}^1_{\mathcal{X}}(\sigma_v) \times J_X.
\]

We describe \( \widetilde{\mathcal{M}}^*_S(X, J) \) as a subset of \( \widetilde{\mathcal{M}}^*(X, \beta, J_X) \times I(S_X) \), where the incidental subvariety \( I(S_X) \) is the subset of all uples,
\[
(y, x) := \left( \{ y_{v,v'} \}_{v \in E \forall v' \}, x_1, \ldots, x_l \right) \in (S^2)^{|E|} \times (S^2)^{|V|},
\]
such that for every \( v \in V \), the points \( y_{v,v'} \) for \( v \in E \), and \( \sigma_v \) with \( \text{pr}(\sigma_v) = v \), are disjoint. First, to each edge of the graph \( S_X \), we associate a copy of the diagonal \( \Delta_X \subset X^2 \), and set the edge diagonal, \( \Delta_E \subset X^{2|E|} \), to be the product of these diagonals over the set \( E \) of all edges. We have a natural map,
\[
ev_E : \widetilde{\mathcal{M}}^*(X, \beta, J_X) \times I(S_X) \to (X)^{2|E|},
\]
called universal edge evaluation map, assigning to each pair \((u, J, y, x)\) the corresponding “evaluation at the nodes”:
\[
u(y) := \{(u_v(y_{v,v'}), u_{v'}(y_{v,v'}))\}_{v \in E}.
\]

The preimage of \( \Delta_E \) under \( \ev_E \), is the (parametrized) universal moduli space denoted \( \widetilde{\mathcal{M}}^*_S(X, J_X) \). Then, it is easy to see that
\[
\widetilde{\mathcal{M}}^*_S(X, J) = (p^{J_X})^{-1}(J) \cap \widetilde{\mathcal{M}}^*_S(X, J_X),
\]
where \( p^{J_X} \) denotes the projection from \( \widetilde{\mathcal{M}}^*(X, \beta, J_X) \times I(S_X) \) to \( J_X \). The standard transversality theorem asserts the following:

**Theorem 4.4.** ([17, 22]) There exists a subset \( J_X, \text{reg}(S_X) \subset J_X \) of second category, such that for each \( J \in J_X, \text{reg}(S_X) \), the moduli space \( \widetilde{\mathcal{M}}^*_S(X, J) \) is a smooth oriented manifold of dimension:
\[
dim(\mathcal{M}^*_S(X, J)) = 2n + 2 \sum_{v \in V} c_1^TX(\sigma_v) + 2|l| - 2|E| - 6.
\]

The set \( J_X, \text{reg}(S_X) \) of regular \( \omega \)-tame almost complex structure for \( S_X \) is explicitly given by the following conditions:

i) for every \( v \in V \), for every \( u \in \widetilde{\mathcal{M}}^*(X, \sigma_v, J) \), the linearization \( D^X_u \), of \( \overline{\sigma}_J \) at \( u \), is surjective.

ii) the restriction of \( \ev_E \) to \( \widetilde{\mathcal{M}}^*_S(X, J) \) is transversal to \( \Delta_E \).

Concretely, by i), for regular \( J \), the moduli spaces \( \widetilde{\mathcal{M}}^*(X, \sigma_v, J) \), \( (v \in V) \), are naturally oriented manifolds of real dimension
\[
\text{ind} \ (D^X) = 2n + 2c_1^TX(\sigma_v).
\]

Point ii) then implies that \( \widetilde{\mathcal{M}}^*_S(X, J) \) is a smooth oriented manifold. Since the \( 6(|E|+1) \)-dimensional group, \( G_S_X \), acts freely and properly by orientation preserving diffeomorphisms on this latter manifold, it follows that \( \mathcal{M}^*_S(X, J) \) is a smooth oriented manifold of the stated dimension.

We briefly sketch the idea of proof for the genericity of \( J_X, \text{reg}(S_X) \). We refer to [17] for the details. The main idea is to show that the universal moduli space is
a separable Banach manifold, and that $p^T_X$ is a Fredholm map between separable Banach manifolds in order to apply Sard-Smale. Of course, this does not apply straightforwardly here, since, for instance, $J_X$ is not Banach. Instead, we consider, $\mathcal{J}_X^r$, the set of $\omega$-tame almost complex structure of class $C^r$, with $r \geq 2$, and let $\mathcal{E}^p_X(\beta)$ be the disjoint union over $\mathcal{J}_X^r$ of the direct sums:

$$\bigoplus_{v \in V} \mathcal{E}^p_{X,u_v}(J)$$

where $(u, J) \in B^{1,p}_X(\beta, \mathcal{J}_X^r)$.

By a standard argument, $\mathcal{E}^p_X(\beta)$ is a locally trivial $C^{r-1}$ Banach fibration over $B^{1,p}_X(\beta, \mathcal{J}_X^r)$. This Banach fibration admits the section

$$s_X : (u, J) \mapsto \partial J(u)$$

and clearly, $\tilde{\mathcal{M}}^*(X, \beta, \mathcal{J}_X^r)$ is a subset of $s_X^{-1}(0)$. Moreover, the linearization, $\tilde{D}^X_{(u, J)}$, of $s_X$ at $(u, J)$ is given by:

$$\tilde{D}^X_{(u, J)} : \bigoplus_{v \in V} \mathcal{A}^{1,p}_{X,u_v} \oplus T_v \mathcal{J} \longrightarrow \bigoplus_{v \in V} \mathcal{E}^p_{X,u_v}(J)$$

$$(\xi, Y) \mapsto D^X_v \xi + \frac{1}{2} Y(du)_v J_0$$

One can show by a standard argument that this operator is onto for every $(u, J)$. It follows that $\mathcal{M}^*(X, \beta, \mathcal{J}_X^r)$ is a separable Banach manifold. One concludes that $\tilde{\mathcal{M}}^*(X, \beta, \mathcal{J}_X^r)$ is a Banach manifold by showing that the edge evaluation map $ev_E$ is transversal to $\Delta_E$. This is done recursively on the set of all labeled forest, the induction argument being made on the number of edges of the forests.

Finally, a simple computation shows that $p^T_X$ is Fredholm, with the same kernel and cokernel as the linearized operator $\tilde{D}^X$. Thus, by Sard-Smale (when $r$ is big enough) one obtains a generic subset $\mathcal{J}_X^r,Reg \subset J_X^r$, which in fact coincides with the regularizing set defined above, but with $C^r$ elements only. One concludes in the $C^\infty$ case by an argument due to Taubes (see [17], Chapter 3).

4.2.2. The Hamiltonian fibration case. Let $\pi : P \to B$ be a Hamiltonian fibration with coupling form $\tau$. Before reformulating Theorem B and giving its proof, we fix some notations.

Fix a stable stratum data $S_P = (V, Fl; pr, g)$ with homological labeling $\beta_P$, and let $S_B = S_{\pi_*(S_P)} = (V_B, Fl_B; pr_B, g_B)$, with corresponding labeling $\beta_B$. Let $E$ and $E_B$ denote the corresponding set of edges. We will assume below that the homomorphism $\beta'_B := \pi_* \circ \beta_P$ is non-zero, thus forcing $\beta_B$ to be non zero. Set $\sigma_v := \beta_P(v)$.

Let, $\tilde{M}^{**}(P, \beta_P, J_P)$, be the restriction of $\tilde{M}^*(P, \beta_P, J_P)$ to simple maps having simple projection under $\pi$, and let

$$\tilde{M}^{**}_{S_P}(P, J_P) := ev^{-1}_E(\Delta_E)$$

be the corresponding universal moduli space. Similarly to the general case, we say that $J_P \in J_{P,reg}(S_P)$ if and only if: i) $\forall v \in V$, $\forall u \in \tilde{M}^{**}(P, \sigma_v, J_P)$, the operator $D^P_u$ is onto; ii) the restriction of $ev_E$ to $\tilde{M}^{**}_{S_P}(P, J)$ is transversal to $\Delta_E$.

Note that we have a natural map

$$\Pi_{S_P} : \tilde{M}^{**}(P, \beta_P, J_P) \times I(S_P) \to \tilde{M}^*(B, \beta_B, J_B) \times I(S_B)$$

$$(u, J_P, y, x) \mapsto (u_B, J_B, y_B, x),$$
induced from $\pi$ and the projection $p_1 : J_P \to J_B$. In fact, the restriction of $\Pi_{S_p}$ to $\widetilde{\mathcal{M}}_{S_p}^{\ast}(P, J_P)$ induces a map between universal moduli spaces:

$$\widetilde{\pi}_{S_p} := \pi_{S_p} \times p_1 : \widetilde{\mathcal{M}}_{S_p}^{\ast}(P, J_P) \to \widetilde{\mathcal{M}}_{S_B}^{\ast}(B, J_B).$$

Since the edge evaluation maps commute with $\widetilde{\pi}_{S_p}$ and the projection from $P^{2|E|}$ to $B^{2|E|}$, it follows easily that if $J_P$ is regular for $S_p$ then $J_B$ is regular for $S_B$.

Now, fix a regular $J_B$. Note that

$$(\pi_{S_p}^{-1}(\pi_B))^{*} = (p_{23} \circ p^{J_P})^{-1}(J, H) \cap \Pi_{S_p}^{-1}(u_B, J_B) \cap ev_{E}^{-1}(\Delta_E \cap F^{2|E|}),$$

where $p_{23} : J_P \to J^V \times H$ is the obvious projection. The set $J\mathcal{H}_{\text{reg}}(\Pi_B, J_B, S_B)$ of fiber regularizing for $(\pi_{S_p}^{-1}(\pi_B))^{*}$, actually consists of pairs $(J, H) \in J^V \times H$ that turn this intersection into an oriented manifold. More precisely, a pair $(J, H)$ is fiber regularizing for $(\pi_{S_p}^{-1}(\pi_B))^{*}$ if for every $(u \equiv \{u_v\}_{v \in V}, y, x) \in (\pi_{S_p}^{-1}(\pi_B))^{*}$, the following conditions are satisfied:

a) $\forall v \in V, \forall u_v \in \widetilde{\mathcal{M}}^{\ast}(\sigma_v, J_P)$, the operator $D_{(u_v, J_P)}^{v}$ is onto;

b) the restriction of $ev_E$ to $(\pi_{S_p}^{-1}(\pi_B))^{*}$ is transversal to $\Delta_E \cap F^{2|E|}$.

Concretely, it follows from a) and b), and the exact sequence (3.4) that for a regularizing pair $(J, H)$, the set $\pi_{S_p}^{-1}(\pi_B)^{*}$ is a smooth oriented manifold with dimension:

$$\dim(\pi_{S_p}^{-1}(\pi_B)^{*}) = \dim \widetilde{\mathcal{M}}_{S_p}^{\ast}(P, J_P) - \dim \widetilde{\mathcal{M}}_{S_B}^{\ast}(B, J_B).$$

Since we are only considering vertical deformations of the maps, the dimension of the reparametrizations is $6(|E| - |E_B|)$, thus giving after quotient:

$$\dim(\pi_{S_p}^{-1}(\pi_B)^{*}) = 2n_F + 2 \sum_{v \in V} c_1(\sigma_v) - 2|E| + 2|E_B|.$$ 

We give the proof of the following extension of Theorem B to any strata, which can be seen as a mild extension to the Hamiltonian fibration case of Theorem 4.3.

**Theorem B’.** Let $S_p$ and $S_B$ be as above.

1) $J_{P, \text{reg}}(S_p)$ is of second category in $J_P$.

2) For any $J_B \in J_{B, \text{reg}}(S_B)$ and any $\Pi_B \in \mathcal{M}_{S_B}^{\ast}(B, J_B)$, the set of regularizing pairs, $J\mathcal{H}_{\text{reg}}(\Pi_B, J_B, S_B)$, is of second category in $J^V \times H$.

**Proof:** The notations used here are the same as the one introduced in Section 4.2.1. Following the guideline given in the preceding section, we show that

$$(\text{4.3}) \quad \pi_{S_p} : \widetilde{\mathcal{M}}_{S_p}^{\ast}(P, J_P) \to \mathcal{M}_{S_B}^{\ast}(B, J_B) \quad r \geq 2,$$

is a submersion between separated $C^{r-1}$ Banach submanifolds of $\mathcal{B}^{1,P}_{P}(\beta_P, J_P)$ and $\mathcal{B}^{1,P}_{B}(\beta_B, J_B)$. By the discussion in Section 4.2.1 we already know that $\mathcal{M}_{S_B}^{\ast}(B, J_B)$ is a Banach manifold. We will then proceed as follows:

(I) we show that $\widetilde{\mathcal{M}}^{\ast}(P, \beta_P, J_B)$ is a Banach manifold and that the restriction of the natural map,

$$p := \pi_{S_p} \times p_1 : \mathcal{B}^{1,P}_{P}(\beta_P, J_P) \to \mathcal{B}^{1,P}_{B}(\beta_B, J_B), \quad (u, J_P) \mapsto (\pi(u), J_B),$$

is a smooth submersion onto $\mathcal{M}^{\ast}(B, \beta_B, J_B)$;

(II) assuming $ev_E$ is transversal to $\Delta_E$, we show that $\widetilde{\pi}_{S_p}$ is a submersion;
(III) we show that for every labeled forest $S_P$ with $l$-tails, the restriction of $dev_E$ to $\ker d\Pi_{S_P}$ is transversal to the subspace $T\Delta_E|_{TPE}$. This implies that $ev_E$ is transversal to $\Delta_E$ for any labeled forest $S_P$ since $ev_{EB}$ is transversal to $\Delta_{EB}$ for every labeled forest. Hence, $\bar{M}^{*}(\mathcal{F}, \mathcal{J}_{EB})$ is a Banach manifold.

The rest of the proof is verbatim the same as in the non-fibered situation and we omit it.

Proof of (I). First, by a standard argument ([8]) involving the use of both the connection induced by $\tau_H$ and the L-C connection on $TB$, the sets, $\mathcal{E}_{P}^{p,r}(\beta_{P})$ and $\mathcal{E}_{B}^{p,r}(\beta_{B})$, are locally trivial $C^{r-1}$ Banach fibrations, over $\mathcal{B}_{P}^{p}(\beta_{P}, \mathcal{J}_{P})$ and $\mathcal{B}_{B}^{p}(\beta_{B}, \mathcal{J}_{B})$, which can be locally trivialized compatibly. Let $s_{P}$ and $s_{B}$ be the corresponding Cauchy-Riemann sections. The linearization $\bar{D}_{P}^{P}(u_{B}, J_{B})$ of $s_{B}$ at $(u_{B}, J_{B})$ is given by (4.2), while for the linearization $\bar{D}_{(u_{B}, J_{B})}^{P}$ of $s_{P}$ at $(u, J_{P})$:

$$
\bar{D}_{(u,J_{P})}^{P} : \bigoplus_{v \in V} \mathcal{X}_{P,u,v}^{P} \oplus T_{P,J_{P}} \rightarrow \bigoplus_{v \in V} \mathcal{E}_{P,u,v}^{P}(J_{P})
$$

$$(\xi, Y^{v}, Y, f) \mapsto D_{u} \xi + \frac{1}{2}(Y . d\pi(u).j_{0})^{h} + \frac{1}{2}Y^{v}.(du)^{v}.j_{0} + X_{f(du)}^{0,1}
$$

Let $\mathcal{P}$ be the fibration map corresponding to the projection $p$:

$$
\mathcal{P} : \mathcal{E}^{p,r}(\beta_{P}) \rightarrow \mathcal{E}_{P}^{p,r}(\beta_{P}), \quad (\eta, J_{P}) \mapsto (d\pi(\eta), J_{B}).
$$

By definition, $\mathcal{P} \circ s_{P} = s_{B} \circ p$. Furthermore, from lemma 3.4 and since $X_{f(du)}^{0,1}$ and $Y^{v} . (du)^{v}.j_{0}$ are vertically valued, we deduce that:

$$
d\mathcal{P} \circ \bar{D}_{(u,J_{P})}^{P} = \bar{D}_{(p(u),J_{B})}^{P} \circ dp.
$$

The maps $dp$ and $d\mathcal{P}$ being both surjective, the pair $(p, \mathcal{P})$ defines a submersion of Fredholm systems, and we end up having the exact sequence:

$$
0 \rightarrow \ker \bar{D}^{v} \rightarrow \ker \bar{D}^{P} \rightarrow \ker \bar{D}^{B} \rightarrow \ker \bar{D}^{v} \rightarrow \ker \bar{D}^{P} \rightarrow \ker \bar{D}^{B} \rightarrow 0,
$$

where $\bar{D}^{v}$ denotes the vertical operator associated to $\bar{D}^{P}$.

To prove the claim, we show that the pair $(p, \mathcal{P})$ defines a splitting when we restrict $s_{P}$ to $\bar{M}^{*}(P, \beta_{P}, \mathcal{J}_{P})$ and $s_{B}$ to $\bar{M}^{*}(B, \beta_{B}, \mathcal{J}_{B})$. It is enough to prove this when the tree structure of $S_{P}$ is preserved under $S_{P}$. Let $V_{0}$ denote the subset of $V$ on which $\beta_{B}$ vanishes, and denote by $V_{+}$ its complement in $V$. Since $\bar{D}^{B}$ is onto for every $(u_{B}, J_{B}) \in \bar{M}^{*}(B, \beta_{B}, \mathcal{J}_{B})$, it suffices to show that the vertical operator $\bar{D}^{v}$ is surjective at every points of $\bar{M}^{*}(P, \beta_{P}, \mathcal{J}_{P})$. Notice that $\bar{D}^{v}$ is closed, and suppose it is not dense. Then, by Hahn-Banach, we would have a non-zero element:

$$
\{\eta_{v}\}_{v \in V} \in \bigoplus_{v \in V} L^{q}((\Lambda^{0,1}(S^{2}, u^{*}T P^{v}))), \quad \frac{1}{p} + \frac{1}{q} = 1
$$

such that each $\eta_{v}$, is of class $W^{1,p}$, is in the cokernel of $D_{u_{v}}^{v}$, and is such that:

$$
0 = \int_{S^{2}} \sum_{v \in V_{+}, v \neq v} \left\langle \frac{1}{2}Y^{v}.(du_{v})^{v}.j_{0} + X_{f(du_{v})}^{0,1}, \eta_{v} \right\rangle + \sum_{v \in V_{0}} \left\langle \frac{1}{2}Y^{v}.(du_{v})^{v}.j_{0}, \eta_{v} \right\rangle dvol_{S^{2}}.
$$

Next, we show that we can find $Y^{v}$ and $f$ such that all the components in the sum must be strictly positive unless all the $\eta_{v}$ are identically zero. Let $Z(u_{v})$ denote the set of non-injective points of $u_{v}$, and consider the subset in $S^{2}$:

$$
X(u_{v}) := Z(u_{v}) \cup \bigcup_{v \in V_{+}, v \neq v} u_{v}^{-1}(u_{v}(S^{2})) \cup \bigcup_{v \in V_{0}, v \neq v} u_{v}^{-1}(u_{v}(S^{2})).
$$

Since we consider simple maps, the complement of this set is open dense in $S^2$. Let $x_v$ be a point of the complement of $X(u_v)$. Then, there is a neighborhood $V$ of $x_v$ which is embedded via $u_v$ into a neighborhood $U_v$ of $u(x_v)$ in $P$, and which does not intersect the image of any other $u_{v'}$. Now, assume $v \in V_p$. From transitivity of the action of Hamiltonian diffeomorphisms on the manifold, we can find a function $f \in T_H^{\mathcal{H}} = \mathcal{H}$ supported in $U_v$, such that:

$$
\int_{S^2} \left< X^{0,1}_{f(du_v)}, \eta_v \right> d\text{vol}_{S^2} > 0.
$$

When $v$'s in $V_0$, we can also find an element $Y^v \in T_I \mathcal{J}^{\text{vert}}$ supported in $U_v$ and such that

$$
\int_{S^2} \left< \frac{1}{2} Y^v(du_v)^v \cdot j_0, \eta_v \right> d\text{vol}_{S^2} > 0.
$$

The neighbourhoods $U_v$ can be chosen small enough so that they are pairwise non intersecting. Set $Y^v|_{u_v} = 0$ for $v \in V_0$, and $f|_{u_v} = 0$ for $v \in V_0$. Then $Y^v$ and $f$ are well-defined on the whole manifold $P$, which implies that all the $\eta_v$'s are vanishing. This ends the proof of the first claim.

**Proof of (II).** Assume $ev_E$ is transversal to $\Delta_E$ when $S_P$ is any labeled forest. Furthermore, suppose that the graph structure of $S_P$ is preserved under $S_{\pi *}$. This is enough since we can always place ourselves in this situation by adding marked points in the fiber components so that they are all equipped with at least three special points. This procedure does not alter the transversality for $ev_E$, as the latter does not depend on the infinitesimal movement of the marked points. Let $S^*_P$ be the stable stratum data resulting from adding the marked points, and consider the map

$$
For^P : \tilde{\mathcal{M}}^{**}_{S^*_P}(P, J_P) \to \tilde{\mathcal{M}}^{**}_{S^*_P}(P, J_P)
$$

that forgets the $k$ added marked points, together with stabilizing the resulting map. Define $For^B$ in a similar way. Then:

$$
For^B \circ \pi_{S^*_P} = \pi_{S^*_P} \circ For^P.
$$

Clearly $For^P$ is a submersion. It is not hard to see that $For^B$ is also a submersion. Moreover, using an adaptation to the fibered case of Lemma 3.4.7 in [17] one can show that $\pi_{S^*_P}$ is also submersion [8].

**Proof of (III).** The proof proceeds by induction on the number of edges of the labeled forests $S_P$. When the forest has no edge, the assertion is vacuous. Suppose it is true for forests with at most $N$ edges, and suppose $S_P$ is a forest with $N + 1$ edges. Pick any edge given by the pair, $(y_{vv'}, y_{v'v})$, cut it out and replace it by the two new marked points, $y_{vv'}$ and $y_{v'v}$. This procedure gives a new forest $S'_P$ with two more tails, which satisfies the induction hypothesis, and such that the sets $I(S_P)$ and $I(S'_P)$ coincide. Let $E'$ denote the set of edges of $S'_P$. Then $ev_{E'}$ is transversal to $\Delta_E'$ so that $\tilde{\mathcal{M}}^{**}_{S'^*_P}(P, J'_P)$ is a Banach manifold. Consider the evaluation

$$
ev_{vv'} : \tilde{\mathcal{M}}^{**}_{S'^*_P}(P, J'_P) \to P \times P, \ (u, y, x, J_P) \mapsto (u(v, y_{vv'}), u_{v'}(y_{v'v})).
$$

We prove that $ev_{vv'}$ is transversal to the diagonal $\Delta_P \subset P \times P$. Assume that $\pi$ preserves the tree structure of $S'_P$ (if not we can add marked points). Let $ev^B_{vv'}$ be the analog of $ev_{vv'}$, but in the case of the base $B$. It is known that $ev^B_{vv'}$ is transversal to $\Delta_B \subset B \times B$ at every point of $\tilde{\mathcal{M}}^{*}_{S_{\pi *}(S'_P)}(B, J'_B)$ (see [17]). Furthermore,

$$
ev^B_{vv'} \circ \pi_{S'^*_P} = (\pi \times \pi) \circ ev_{vv'},
$$
Since both $π × π$ and $\overline{π}$ are submersions it suffices to check that:

$$\forall \bar{u} \in \widetilde{\mathcal{M}}_{\mathcal{S}_p}^*(P, J_F), \text{ coker } \text{dev}_{uv'}(\bar{u})|_{\text{ker } d\overline{\pi}(\bar{u})} = 0.$$ 

By the symmetry arising from quotienting $TF ⊗ TF$ by $TΔ_F$, it suffices to show that the restriction of $\text{dev}_{uv'}(\bar{u})$ to $V$ surjects onto $T_{uv(\Delta F)}F \times \{0\}$. But:

$$W = \left\{ (\{ξ_v\}, 0, 0, 0, Y_v, f) ∈ T_{\bar{u}}\widetilde{\mathcal{M}}_{\mathcal{S}_p}^*(P, J_F^p) \mid ξ_v ∈ W^1 orbit(u^*_v TP^v), \forall v ∈ V \right\}.$$ 

Hence, by definition of $\text{ev}_{uv'}$:

$$\text{dev}_{uv'}(\bar{u})(\{ξ_v\}, 0, 0, 0, Y_v, f) = (ξ_v(\Delta F), ξ_{uv'}(\Delta F)).$$ 

Now let $(v, 0) ∈ T_{uv(\Delta F)}F \times \{0\}$ and suppose the $i$ component is not ghost. Then choose any $ξ_v ∈ X_{\mathcal{P}_u, v}$ such that $ξ_v(\Delta F) = v$. Adapting Lemma 3.4.7 in [17] to the present situation, we can find $Y_v$ or $f$ supported in a small enough neighbourhood in $P$ (such that it does not intersect the image of any other component) and a vector field $ζ ∈ W^1 orbit(u^*_v TP^v)$ such that, $ζ(\Delta F) = 0$ for $vEv'$ and $(ξ_v - ζ, 0, Y_v, 0)$, or $(ξ_v - ζ, 0, 0, f)$, lies in $\ker D(\delta_{uv(\Delta F)})$. Then set

$$ξ_{uv'} = 0 \text{ } \forall v' \neq v.$$ 

If $u_v$ is ghost, then consider $V_{gh}(v)$ the vertices of the largest subtree in $\mathcal{S}'_p$, containing $v ∈ V$ and consisting only of ghost components. For all $k ∈ V_{gh}(v)$ we must have $ξ_k = w$. Consider now all the elements $k ∈ V \setminus V_{gh}(v)$ such that there exists $v' ∈ V_{gh}(v)$ for which $kEv'$ and write this set as $K$. All these components have a point in common in the image of the stable map, namely, $u_v(Σ_v) = u_v(\Delta F)$. Then, for every $m ∈ K$ choose any $ξ_m ∈ X_{\mathcal{P}_u, v}$ such that $ξ_m(\Delta F) = w$. Applying the argument used in the non-ghost case to all the components indexed by $K$, we find vertically valued vector fields $\{ξ_m\}_{m ∈ K}$ such that

$$ζ(\Delta F) = 0 \text{ when } m \leq j, \text{ } m ∈ K \text{ and } v' ∈ V_{gh}(v),$$

and such that, for all $m ∈ K$, either $(ξ_m - ζ, 0, Y_v, 0)$ or $(ξ_m - ζ, 0, 0, f)$ lies in the kernel of $D(\delta_{uv(\Delta F)}).$ Finally, set $ξ_{uv'} ≡ 0$ for every component not indexed by $K \sqcup V_{gh}(v).$}

**Remark 4.5.** In the proof above, it is essential that we allow the connection to vary. In particular such perturbations enable us to avoid horizontal $J_F$-holomorphic maps, i.e maps $u$ such that $\text{Im}(du) ⊆ \text{Hor}$, for which the index of $D^u_v$ is negative (see [17]).

### 4.2.3. Cobordisms

We end this section by stating the invariance of the moduli spaces under changes of the regular structures. Let $\mathcal{S}_P$ be a stable stratum data, and let $\mathcal{S}_B$ be its projection. Given two regular structures $J^0_F$ and $J^1_F$ in $J_{F, \text{reg}}(\mathcal{S}_P)$, we designate by $J_F(J^0_F, J^1_F)$ the set of pairs $\{J^s_F\}$ in $J_F$, $s ∈ [0, 1]$, with endpoints $J^0_F$ and $J^1_F$. Similarly define $J_B(J^0_B, J^1_B)$ for pairs $J^0_B$ and $J^1_B$ in $J_{B, \text{reg}}(\mathcal{S}_B)$. For elements $γ ∈ J_F(J^0_F, J^1_F)$ and $γ_B ∈ J_B(J^0_B, J^1_B)$ we set:

$$\widetilde{W}^*_{\mathcal{S}_P}(P, \{J^s_F\}) := γ^*\widetilde{\mathcal{M}}^*_{\mathcal{S}_P}(P, J_F) \quad \text{and} \quad \widetilde{W}^*_{\mathcal{S}_B}(B, \{J^s_B\}) := γ^*_B\widetilde{\mathcal{M}}^*_{\mathcal{S}_B}(B, J_B).$$

It is not hard to see if $γ$ is transversal to $p^TF$, and respectively $γ_B$ is transversal to $p^TB$, the quotient under $G_{\mathcal{S}_P}$ and $G_{\mathcal{S}_B}$ of the above pullbacks are then oriented manifolds with boundaries, and with dimensions:

$$\dim(\mathcal{M}^*_{\mathcal{S}_P}(P, J^0_F)) + 1 \quad \text{and} \quad \dim(\mathcal{M}^*_{\mathcal{S}_B}(B, J^0_B)) + 1.$$
In such case we say that the paths are regular, and denote by
\[ \mathcal{J}_{P,\text{reg}}(S_P, J_p^0, J_p^1) \quad \text{and} \quad \mathcal{J}_{B,\text{reg}}(S_B, J_B^0, J_B^1) \]
the set of such regular paths. Note that a regular path \( \gamma \) in \( \mathcal{J}_P \) projects to a regular path \( \gamma_B := p_1(\gamma) \) in \( \mathcal{J}_B \).

For fixed \( J_B \in \mathcal{J}_{B,\text{reg}}(S_B) \) and \( u_B \in \widetilde{M}_B^*(B, J_B) \) set \( \mathcal{J}\mathcal{H}((J, H)^0, (J, H)^1) \) to be the set of paths with endpoints \((J, H)^0\) and \((J, H)^1\) in \( \mathcal{J}\mathcal{H}_{\text{reg}}(u_B, J_B, S_P) \), and for any such path \( \gamma \) set:
\[ \tilde{W}^*_{S_P}(\pi^{-1}(u_B), J_B, \{(J, H)^s\}) := \gamma^*(\tilde{\pi})^{-1}(u_B, J_B). \]
If \( \gamma \) is transversal to the restriction of \( p_{23} \circ p^P \) to the fiber \((\tilde{\pi})^{-1}(u_B, J_B)\), then the quotient under \( G_{S_P} \) of the corresponding pullback is an oriented manifold with boundary, of dimension
\[ \dim(\pi^{-1}(u_B) \cap C^{**}(P, J_p^0)) + 1. \]

In such case, we say that \( \gamma \) is regular, and the set of regular paths is denoted by \( \mathcal{J}\mathcal{H}_{\text{reg}}(u_B, J_B, S_P, (J, H)^0, (J, H)^1) \).

**Proposition 4.6.** The regularizing sets: \( \mathcal{J}_{P,\text{reg}}(S_P, J_p^0, J_p^1), \mathcal{J}_{B,\text{reg}}(S_B, J_B^0, J_B^1) \) and \( \mathcal{J}\mathcal{H}_{\text{reg}}(u_B, J_B, S_P, (J, H)^0, (J, H)^1) \) are of second category.

### 5. The Product Formula

In this section we establish the product formula. Before doing so, we recall the definition of Gromov-Witten invariant for a semi-positive symplectic manifold \((X, \omega)\). For a detailed exposition of the following standard facts, we refer to [17] or [22]. Let \( A \in H_2(X, \mathbb{Z}) \), and consider the \( l \)-pointed evaluation map:
\[ ev^X_{l, J} : M^*_{0,l}(X, A, J) \to X^l, \quad (u, x_1, \ldots, x_l) \mapsto (u(x_1), \ldots, u(x_l)). \]
This defines a \( \dim(M^*_{0,l}(X, A, J)) \)-pseudocycle of \( X^l \) for every \( J \in \mathcal{J}_{X,\text{reg}} \subset \mathcal{J}_X \), where
\begin{equation}
\mathcal{J}_{X,\text{reg}} := \bigcap_{S \in \mathcal{D}^A_{0,l}} \mathcal{J}_{X,\text{reg}}(S)
\end{equation}
which is of second category since \( \mathcal{D}^A_{0,l} \) is finite. Note that \( \mathcal{J}_{X,\text{reg}} \) depends on \( \omega \).

We recall that a \( d \)-dimensional pseudo-cycle in a manifold \( X \), is a pair \((M, f)\), where \( M \) is an oriented manifold of dimension \( d \), and \( f : M \to X \) is a smooth map such that the closure, \( \overline{f}(M) \), is compact, and such that its omega-limit \( \Omega_f \) is of codimension at least 2 in \( X \). Given classes, \( c^X_1, \ldots, c^X_l \in H_*(X) \), it is possible to represent them by pseudo-cycles \((M_1, f_1), \ldots, (M_l, f_l)\) in \( X \), of respective dimensions \( \dim M_i := \deg(c^X_i) \). We can further assume that these cycles are in general position, and such that \( ev^X_{l, J} \) is strongly transverse to the product cycle,
\[ \mathcal{C} := (M_1, f_1) \times \ldots \times (M_l, f_l). \]
Then, the corresponding Gromov-Witten invariant is the algebraic number of isolated points in the preimage of \( \mathcal{C} \) under \( ev^X_{l, J} \),
\[ \langle c^X_1, \ldots, c^X_l \rangle^X_{0,l, \sigma} := ev^X_{l, J} \mathcal{C}, \]
which is set to be 0 unless:
\[ 2n(1 - l) + 2c^T_{1} X(\sigma) + 2l - 6 + \sum_{i=0}^{l} \deg(c^X_i) = 0. \]
This number only depends on the bordism class of the pseudo-cycles involved. In particular, it does not depend on the regular almost complex structure, which we will drop from the notations.

Now, let $\pi : P \to B$ be a Hamiltonian fibration with coupling form $\tau$, and let $\iota_P^B$ denote the inclusion of $F$ in $P$. Consider $\sigma \in H_2(P, \mathbb{Z})$ with $\sigma_B = \pi_* \sigma \neq 0$. For $(u_B, x) \in M_{0,l}^*(B, \sigma_B, J_B)$, we have the commutative diagram:

$$
\begin{array}{cc}
\pi^{-1}(u_B, x)/G & M_{0,l}^*(P, \sigma, J_P) \\
\downarrow^{ev_{(u_B, x)}} & \downarrow^{ev_{i, J_P}} \\
F^l & \pi^l \\
\downarrow^{(\iota_F)^l} & \downarrow^{\pi^l} \\
B^l & 
\end{array}
$$

where

$$
ev_{(u_B, x)} : \pi^{-1}(u_B, x)/G \to F^l, \quad u \mapsto u(x) := (u(x_1), \ldots, u(x_l)) \in \bigoplus_{i=1}^l F_{u_B(x_i)}.
$$

The product formula is obtained by considering the (respective) intersections of $ev_{(u_B, x)}$, $ev_{i, J_B}$, and $ev_{i, J_P}$, with the product pseudo-cycles:

$$(C_F, f^F) := \prod_{i=1}^l (M_i^F, f_i^F), \quad (C_B, f^B) := \prod_{i=1}^l (M_i^B, f_i^B), \quad (C_P, f^P) := \prod_{i=1}^l (M_i^P, f_i^P),$$

where, $(M_i^F, f_i^F)$, $(M_i^B, f_i^B)$, and $(M_i^P, f_i^P)$, respectively represent torsion free homology classes, $c_i^F$, $c_i^B$, and $c_i^P$, verifying condition (\[\text{[XXX]}\]). In particular,

$$
\begin{cases}
(M_{B_i}^B, f_{B_i}^B) = (pt, f_i^B) & \text{for } i = 1, \ldots, m \\
(M_{i+1}^F, f_{i+1}^F) = (F, id_F) & \text{for } i = m + 1, \ldots, l.
\end{cases}
$$

Furthermore, a $d$-dimensional pseudo-cycle $(M, f)$ in the fiber $F$ of $P$ defines a $d$-dimensional pseudo-cycle $(M, \iota_P^B \circ f)$ in the total space. Similarly, any $d$-dimensional pseudocycle $(M, f)$ in $B$ defines a $d + \dim F$ pseudocycle $(f^*P, \bar{f})$ in $P$, where $\bar{f}$ stands for the bundle map associated to $f$. These operations actually preserve the bordism classes. We conclude that:

$$(M_i^P, f_i^P) = \begin{cases}
(M_i^F, \iota_P^B(f_i^F)) & \text{if } i = 1, \ldots, m \\
((f_i^B)^*P, f_i^B) & \text{otherwise}.
\end{cases}
$$

Regarding orientations of the product pseudo-cycles, the exact sequence,

$$0 \to df^F(TC_F) \xrightarrow{(\iota_F^l)^l} df^P(TC_P) \xrightarrow{\pi^l} df^B(TCB) \to 0,$$

gives:

$$\det df^P(TC_P) \cong \det df^B(TCB) \otimes \det df^F(TC_F).$$

Therefore, if we choose the cycles $(C_B, f^B)$ and $(C_F, f^F)$ to be positively oriented, the cycle $(C_P, f^P)$ must also be positively oriented. Now, assume the evaluations are pseudo-cycles, and that strong transversality with the product cycles is achieved. Then $(ev_{i, J_P}^B)^{-1}(f^B)$ is a finite set, $\{(u_B, \alpha, x_{\alpha})\}_{\alpha \in A}$, of isolated simple, $l$-pointed, $J_B$-holomorphic maps. For each $\alpha$, let $\iota_{P, \alpha}$ denote the embedding of $F^l$ into $F_{u_B, \alpha}(x_{\alpha})$.

Also, in order to simplify notations, set

$$ev_\alpha := ev_{(u_B, \alpha, x_{\alpha})}, \quad f_i^F := \iota_{P, \alpha} \circ f^F,$$

and write:

$$n_\alpha := ev_\alpha \cdot f_i^F, \quad n_B := ev_i^B \cdot f^B, \quad \text{and} \quad n_P := ev_i^P \cdot f^P.$$
Notice that the numerical conditions under which $n_\alpha$ and $n_B$ are non necessarily zero, provide the condition under which $n_P$ is possibly non vanishing. Now, in the above notations, the product formula now reads:

\[(\text{PF})\quad \forall \alpha \in A, \quad n_P = n_\alpha n_B.\]

We will prove this relation, and then prove the Corollary in the last subsection. Before doing so, we make sure that the evaluation maps in \((5.2)\), are simultaneously pseudo-cycles that remain in the same bordism class under change of regularizing almost complex structure. This will give meaning to all the numbers in \((\text{PF})\).

5.1. Evaluation maps as pseudo-cycles. We begin by fixing some notations. Define $J_{B,\text{reg}} \subset J_B$ and $J_{P,\text{reg}} \subset J_P$ as in \((5.1)\). These sets are of second category. For $J_B \in J_{B,\text{reg}}$ and $u_B \in M^*(B,\sigma_B, J_B)$, we set,

$$
J_H\text{reg}(u_B, J_B) := \bigcap_{\{S_P|S_\alpha(S_P) = S_B^{\text{top}}\}} J_H\text{reg}(u_B, J_B, S_P),
$$

where $S^{\text{top}}_B$ denotes the top stable stratum data for $\overline{M}(B,\sigma_B, J_B)$, i.e. the stratum having only one vertex as a tree structure. By Theorem B’, this set is also of second category. Furthermore, for fixed $J_B$, we say that the class $\sigma_B \in H_2(B,\mathbb{Z})$ only admits irreducible effective decompositions with respect to $J_B$ if every stratum $M_{\sigma_B}(B,J_B)$ is only made of irreducible elements. This condition is in particular realized by primitive classes, for example the class of a line in $\mathbb{CP}^n$, or the diagonal in $S^2 \times S^2$ with the standard product complex structure. Let $J_{\text{irr}}(\sigma_B)$ denote the subset of $J_B$ with respect to which $\sigma_B$ admits only irreducible effective decompositions. Then $J_{\text{irr}}(\sigma_B)$ is open in $J(B,\omega_B)$. Nevertheless, nothing guarantees that it is non-empty. In the theorem below, the restriction to $J_{\text{irr}}(\sigma_B)$ is essential in order to avoid simple stable maps having a reducible projection.

We show that all the evaluation map in the above diagram are pseudo-cycles. Note that condition \((\Box)\) is equivalent to:

\[(5.3)\quad \forall A \in \pi_2(F) : \quad \omega(A) > 0, \quad c^{TF}_2(A) \geq 3 - n_P \quad \implies c^{TF}_2(A) \geq 0.
\]

This is weaker than asking for $P$ to be semi-positive. However, this implies that the fiber is semi-positive.

**Theorem 5.1.** Assume \((5.3)\) and that $J_{\text{irr}}(\sigma_B) \neq \emptyset$. Then:

i) For every $J_P \in J_{P,\text{reg}}$ with $J_B \in J_{\text{irr}}(\sigma_B)$, the evaluation maps, $ev_{l,J_B}^B$ and $ev_{l,J_P}^P$, are pseudo-cycles. Moreover, changing of regular structure along regular path induces a bordism between the relevant evaluation maps, as long as the almost complex structure on $B$ varies in a connected component of $J_{\text{irr}}(\sigma_B)$.

ii) Fix a regular structure $J_B$, and let $(u_B, x) \in M^*_0(B, J_B, \sigma_B)$. Then, for any element in $J_H\text{reg}(u_B, J_B)$, the couple $(\pi^{-1}(u_B, x), ev(a_B, x))$ is a pseudo-cycle that remains in the same bordism class, under change of regularizing pair along regular paths.

**Proof:** We only show the first statement, the proof of the second being similar. Fix $J_P \equiv (J_B, J, H) \in J_{P,\text{reg}}$. Hence $J_B \in J_{B,\text{reg}}$ and $ev_{l,J_B}^B$ is then a pseudo-cycle \((17)\). By Lemma 4.3 there are only finitely many stable stratum datas $S_P$ representing geometric limits of curves in $M_{0,l}^*(P, \sigma, J_P)$. Hence, by Gromov’s compactness,

$$
\Omega_{\text{ev}_l,J_P} \subset \bigcup_{S_P} ev_{l,J_P}(M_{S_P}^*(P, J_P)),
$$
where the union is taken over all reduced stratum datas. Let $\beta_P^{\text{red}}$ denote the homological labeling associated to a reduction $S_P^{\text{red}}$ of $S_P$. Let $V_p^{\text{red}}$ denote the set of $\pi$-stable components in $S_P^{\text{red}}$, and let $V_0^{\text{red}}$ denote the set of $\pi$-unstable components in $S_P^{\text{red}}$. Since, by assumption, $\sigma_B$ only admits irreducible decompositions, only $\pi$-unstable components of $S_P$ are contracted in the reduction process. Therefore, there exist integers $m_v > 0$ for all $v \in V_0^{\text{red}}$ such that:

$$\sigma = \sum_{v \in V_0^{\text{red}}} \beta_P^{\text{red}}(v) + \sum_{v \in V_0^{\text{red}}} m_v \beta_P^{\text{red}}(v).$$

Condition [5.3] further implies that $c^v(\beta_P^{\text{red}}(v)) > 0$, for every $v \in V_0^{\text{red}}$, and we conclude that

$$\dim M_{S_P^{\text{red}}}^{**}(P, J_P) \leq \dim M_{P_0^* l}(P, \sigma, J_P) - 2$$

for every stratum reduction. Hence, $\mathcal{e}v_{l, \sigma}^P$ is a pseudo-cycle. The independance statement is shown as follows. Let $J_P'$ be a regular path of fibered almost complex structures between regular fibered structures, projecting on a path in $J_{B, \text{reg}}(J_P^0, J_B) \cap J_{\text{irr}}(\sigma_B)$ (with $J_B^0, J_B^1$ in the same connected component of $J_{\text{irr}}(\sigma_B)$). Then, any Gromov limit of a sequence in $\mathcal{W}_{\alpha, l}^0(P, \sigma, \{J_P^i\})$, is such that its non-trivial roots are irreducible, while its fiber components may actually be reducible. Argumenting exactly as above, the lower strata in $\mathcal{W}_{\alpha, l}^0(P, \sigma, \{J_P^i\})$ have codimension at least 2 in $P$. Thus,

$$\mathcal{e}v_{l, \sigma}^P(J_P^i) : \mathcal{W}_{\alpha, l}^0(P, \sigma, \{J_P^i\}) \to P^l$$

is a pseudo-cycle inducing a bordism between $\mathcal{e}v_{l, \sigma}^P$ and $\mathcal{e}v_{l, \sigma}^P$.

**Remark 5.2.** Note that in this context we do not need to impose any semi-positivity assumption on $B$ due to the specific decomposability hypothesis imposed on $\sigma_B$.

Note also that if $\sigma_B$ is undecomposable, we can drop the restriction on $J_{\text{irr}}(\sigma_B)$.

**5.2. Proof of theorem A.** We begin by proving that all the terms in $[\mathcal{P}_F]$ are well-defined. From Theorem [5.1] and since $J_{\text{irr}}(\sigma_B) \neq \emptyset$, the evaluation maps $\mathcal{e}v_{l, \sigma}^P$ and $\mathcal{e}v_{l, \sigma}^{P}$ generically define pseudo-cycles. Choose (generically) the cycles $(M_i^B, f_i^B)$ so that $\mathcal{e}v_{l, \sigma}^B$ is strongly transverse to $(B, f_B)$. Then $(\mathcal{e}v_{l, \sigma}^B)^{-1}(f_B^l)$ is finite and, as already mentioned, the corresponding GW-invariant, $n_B$, only depends on the bordism classes of $(M_i^B, f_i^B)$, and on the connected components of $J_{\text{irr}}(\sigma_B) \cap J_{B, \text{reg}}$.

Thus,

$$(\mathcal{e}v_{l, \sigma}^B)^{-1}(f_B^l) = \{(u_{B, \alpha}, x_\alpha)|\alpha \in A\},$$

is finite, and for every $\alpha \in A$, the map $\mathcal{e}v_{l, \sigma}^B$ also defines a pseudocycle for generic fiber regularizing pairs. Consequently, $(F, f_F^l)$ is a pseudo-cycle of $F_{u_{B, \alpha}}^l(x_\alpha)$ for every $\alpha \in A$. Since $A$ is finite, we can furthermore choose the cycles $(M_i^F, f_i^F)$ such that, for every $\alpha \in A$, the evaluation $\mathcal{e}v_{l, \sigma}^B$ is transversal to $f_i^F$. This, together with the fact that $\mathcal{e}v_{l, \sigma}^B$ is transversal to $f_B^l$ implies that $\mathcal{e}v_{l, \sigma}^P$ is transversal to $f_F^l$. The independance of $n_B$ and $n_P$, with respect to the choice of regularizing triple, follows from Theorem [5.1].

Next, we prove $[\mathcal{P}_F]$. Let $C_\alpha \subset B$ denote the image of $u_{B, \alpha}$, let $P_{C_\alpha}$ be the restriction of $P$ to $C_\alpha$, and let, $t_\alpha : P_{C_\alpha} \hookrightarrow P$ and $t_\alpha^{P_{C_\alpha}} : F \hookrightarrow P_{C_\alpha}$, denote the natural inclusions. Consider the subset of section classes in $P_{C_\alpha}$,

$$B_{\sigma}^\alpha = \{\sigma' \in H_2(P_{C_\alpha}, \mathbb{Z})|\langle t_\alpha^{P_{C_\alpha}}, \sigma' \rangle = \sigma\}.$$
Then,
\[
(5.4) \quad n_\alpha = \sum_{\sigma' \in B_2} \langle \iota^*_{PC} (c_1^\alpha), \ldots, \iota^*_{PC} (c_l^\alpha) \rangle_{0,l,\sigma'}.
\]

Indeed, let \(J_{PC_{\alpha}}\) denote the restriction of \(J_P\) to \(PC_{\alpha}\), then \(\iota_\alpha\) naturally induces an identification:
\[
\tau_\alpha : \bigsqcup_{\sigma' \in B_2} \mathcal{M}(PC_{\alpha}, J_{PC_{\alpha}}, \sigma') \to \pi^{-1}(u_{B,\alpha}, x_\alpha),
\]
which is an orientation preserving diffeomorphism when restricted to any stratum. Furthermore, by simplicity of \(u_{B,\alpha}\), the \(l\) marked points are naturally identified to the 2l-dimensional manifold \(M^*_{0,l}(C_{\alpha}, [C_{\alpha}])\). We obtain the following diagram:
\[
\begin{align*}
\pi^{-1}(x_\alpha) &= \bigsqcup_{\sigma' \in B_2} \mathcal{M}^*(PC_{\alpha}, \sigma') \xrightarrow{e_{\alpha_{PC}}} \bigsqcup_{\sigma' \in B_2} \mathcal{M}^*_{0,l}(PC_{\alpha}, \sigma') \xrightarrow{\pi} \mathcal{M}^*_{0,l}(C_{\alpha}, [C_{\alpha}]) \\
F^l \xrightarrow{ev_{\alpha_{PC}}} P^l_{C_{\alpha}} \xrightarrow{ev_{l}} C^l_{\alpha}
\end{align*}
\]

where the complex structures are dropped in order to simplify notations. By definition, \(ev_{\alpha_{PC}}\) is the composition of \(ev_\alpha\) with \(\iota_\alpha\), hence is a pseudo-cycle for generic fiber regularizing parameters. Using the above diagram we conclude that \(ev_{l_{PC_{\alpha}}}\) is generically a pseudo-cycle. Then, equation (5.4) follows since there is only one positively oriented curve in \(\mathcal{M}^*_{0,l}(C_{\alpha}, [C_{\alpha}])\) intersecting transversally \(l\) points at the \(l\)-marked points (giving \(\langle pt, \ldots, pt \rangle_{0,l,[C_{\alpha}]} = 1\)).

Now, consider the sign functions, \(\epsilon_P, \epsilon_B,\) and \(\epsilon_\alpha\) respectively associated to the curves counted in \(n_P, n_B\) and \(n_\alpha\). We have to make sure that the signs of the counted curves are given compatibly with \(\pi^l\), i.e that:
\[
\forall \alpha \in A, \quad \epsilon_P = \epsilon_B \times \epsilon_\alpha.
\]

But this is the case since for every fiber regularizing pair we have the exact sequence:
\[
0 \to T_u (\pi^{-1}(u_{B,\alpha}; x_\alpha)) \to T(\iota_{\alpha}(u); x_\alpha) \mathcal{M}^*_{0,l}(P, \sigma) \to T(n_{B,\alpha}; x_\alpha) \mathcal{M}^*_{0,l}(B, \sigma_B) \to 0.
\]

Consequently:
\[
\begin{align*}
n_P &= \sum_{\alpha \in A} \sum_{\{u \in ev^{-1}_{\alpha}(f^P_{C_{\alpha}})\}} \epsilon_P(\iota_{\alpha}(u)) \\
&= \sum_{\alpha \in A} \left( \epsilon_B(u_{B,\alpha}; x_\alpha) \sum_{\{u \in ev^{-1}_{\alpha}(f^P_{C_{\alpha}})\}} \epsilon_{PC_{\alpha}}(x_\alpha(u)) \right)
\end{align*}
\]
which, by Lemma 2.6 coincides with \(n_\alpha n_B\) for any \(\alpha_0 \in A\).

6. Gluing and fibration of moduli spaces

The aim of this section is to show that, under some circumstances, \(\pi\) defines a locally trivial smooth orbi-fibration above the top stratum of \(\mathcal{M}^*_{0,l}(B, \sigma_B, J_B)\), with \(\sigma_B \neq 0\). We start with the following simple observation. From the transversality theorem, the restriction of \(\pi_{S_P}\) to \(\mathcal{M}^{**}(P, \sigma, J_P)\) is generically a smooth submersion onto \(\mathcal{M}^*(B, \sigma_B, J_B)\) over countably many points. It is natural to ask if there exists a fibered \(J_P\) with respect to which the latter map is everywhere regular. For fixed \(J_B\) and \(J\), we say that \(H \in \mathcal{H}\) is parametric if \(D^u_H\) is surjective for every \(u \in \)
\(M^{**}(P, \sigma, J_P)\). From exactness of \(\pi\) is a smooth submersion for parametric \(H\). As a result, if we assume that \(\sigma\) is an indecomposable effective class projecting onto a non-trivial indecomposable class in \(B\) and that \(H\) is parametric, then \(\pi_{\mathcal{S}_P}\) is a (smooth) locally trivial fibration (Indeed, since \(\sigma\) is indecomposable, \(M^{**}(P, \sigma, J_P)\) is compact so that \(\pi_{\mathcal{S}_P}\) is proper, as desired).

**Remark 6.1.** The set of parametric \(H\)'s may be empty. Consider the \((\mathbb{C}P^1, \omega_{FS})\)-

\[\pi : E := \mathbb{P}(O_{\mathbb{C}P^2}(-2) \oplus \mathbb{C}) \longrightarrow (\mathbb{C}P^2, \omega_{FS}).\]

Denote by \(L\) the homology class of a line in \(\mathbb{C}P^2\), and let \(L_0 \in H_2(E, \mathbb{Z})\) be the class such that \(L = \pi_*(L_0)\) and \(L_0 \cap [F] = [pt]\), where \([F]\) stands for the class of a fiber in \(E\). If \(u\) is a holomorphic curve in \(E\) representing \(L_0\), the fibration \(u^*TE\) is isomorphic to the direct sum of \(O_{\mathbb{C}P^1}(2)\), \(O_{\mathbb{C}P^1}(1)\) and \(O_{\mathbb{C}P^1}(-2)\). A straightforward computation then shows that the index of the vertical linearized operator must be \(-2\), hence there are no parametric \(H\).

We will make the following assumption throughout this section:

**Assumption 6.2.** Let \(\mathcal{S}_P\) be a stable stratum data for pseudo-holomorphic maps in \(P\) representing \(\sigma\). We ask that for every \(u \in \bar{\mathcal{M}}_{\mathcal{S}_P}(P, J_P)\):

- the operators \(D^u\) and \(D^B\) are surjective;
- the edge evaluation maps \(ev^B\) and \(ev_{\pi_{\mathcal{S}_P}(u)}\) are transversal to the corresponding diagonals.

Then for every \(u \in \mathcal{M}_{\mathcal{S}_P}(P, J_P)\) the operator \(D^P_u\) is surjective, and \(ev^P\) is transverse to the associated diagonal (see Section 3). By a standard argument the quotient spaces \(\mathcal{M}_{\mathcal{S}_P}(P, J_P)\), \(\mathcal{M}_{\mathcal{S}_B}(B, J_B)\), and the fiber of \(\pi_{\mathcal{S}_P}\) over \(\pi_{\mathcal{S}_P}(u)\), are smooth orbifolds. Furthermore, the commutativity \(\pi_* \circ D^P = \pi^*_{\pi_{\mathcal{S}_P}(u)} \circ \pi_*\) implies that these orbifold structures can be chosen compatibly with \(\pi_{\mathcal{S}_P}\). We will drop the almost complex structures from the notations since it is understood that we made a choice here.

### 6.1. Gluing in the non-fibered case: \(B = pt\)

We give the gluing procedure in \(\mathcal{M}_{0,1}(X, A)\) with \((X, \omega)\) a general symplectic manifold. This will be our guideline when considering more general base. We start by gluing in \(\mathcal{M}_{0,1}\). We follow standard approaches in the litterature, such as [17], [24], [2] amongst others.

#### 6.1.1. Gluing for nodal curves

Let, \(S := (V, F; \text{pr}, g)\), be a stable stratum data for \(\mathcal{M}_{0,1}\), and let \(j \equiv (\Sigma, j, x) \in \bar{\mathcal{M}}_S\). For \(v \in V\), let \(\Sigma_v\) denote the corresponding (irreducible) component, and for \(f \in F\), denote by \(z_f\) the corresponding special point on \(\Sigma_v\). By definition each \(\Sigma_v\) is stable. Then, upto isometry, there exists a unique isometric action of a Fuchsian group \(\Gamma\) on the hyperbolic half plane \(\mathbb{H}\) with respect to which:

\[\Sigma_v \setminus \{z_f | \text{pr}(f) = v\} \cong \mathbb{H}/\Gamma.\]

The induced metrics belongs to the conformal class given by the complex structure on \(\Sigma_v\). If \(D \subset \mathbb{H}\) denotes a Dirichlet region of \(\Gamma\), then each \(z_f\) corresponds to a vertex at infinity, and we can choose the fundamental region \(D\) such that \(z_f\) corresponds to infinity with edges \(x \equiv 0\) and \(x \equiv 1\). It is well known that given a real \(b > 1\), the horocycle at \(z_f\),

\[\{x + iy \in \mathbb{H} | 2\pi y > b\} / (z \rightarrow z + 1),\]

represents a component of \(\Sigma_v\).
defines a neighbourhood of $z_f$, which can be identified to a punctured disc $D^*(e^{1-b}) \subset \mathbb{C}$ via the map $z \mapsto e^{2\pi iz+1}$. We have such neighbourhoods for each $z_f$, and we denote it $D^*_f(r_f)$, with small $r_f > 0$. Now, $D^*_f(r_f)$ is conformally equivalent to a cylinder with negative end:

$$(-\infty, \ln r_f] \times \mathbb{R}/2\pi\mathbb{Z} \cong D^*_f(r_f) : (s, t) \mapsto e^{s+it}$$

or with positive end:

$$[-\ln r_f, +\infty) \times \mathbb{R}/2\pi\mathbb{Z} \cong D^*_f(r_f) : (s, t) \mapsto e^{-s-it}.$$

Let $e_f, e'_f$ be an edge of $S$ between the vertices $v := \text{pr}(f)$ and $v' := \text{pr}(f')$. For a complex number

$$\rho_{vv'} := e^{-R_{vv'} + i\theta_{vv'}} \equiv r_{vv'} e^{i\theta_{vv'}}, \quad \theta \in [0, 2\pi),$$

such that $|\rho_{vv'}| < \min\{r_f, r_{f'}\}$, we can glue the components $\Sigma_v$ and $\Sigma_{v'}$ at $z_f$ and $z_{f'}$ as follows. Put positive (resp. negative) cylindrical coordinates on $D^*_f(r_f)$ (resp. $D^*_{f'}(r_{f'})$) and identify the annuli:

$$[-R_{vv'}/2 - 1, -R_{vv'}/2 + 1] \times \mathbb{R}/2\pi\mathbb{Z} \xrightarrow{\cong} [R_{vv'}/2 - 1, R_{vv'}/2 + 1] \times \mathbb{R}/2\pi\mathbb{Z},$$

$$(s, t) \mapsto (s + R_{vv'}, t + \theta_{vv'})$$

This gives the patching procedure between the annuli $A_f := D^*_f(r_f) \setminus D^*_f(e^{-R_{vv'}})$ and $A_{f'} := D^*_{f'}(r_{f'}) \setminus D^*_{f'}(e^{-R_{vv'}})$. The resulting curve $\Sigma_{\rho_{vv'}}(f, f')$ has a natural complex structure and is the gluing of $j$ at $e_f, e'_f$ with parameter $\rho_{vv'}$.

Note that $\rho_{vv'}$ can be naturally identified to an element of

$$\mathbb{C}_{vv'} := T_{y_{vv'}} \Sigma_v \otimes T_{y_{vv'}} \Sigma_{v'} \cong \mathbb{C}, \quad vE'v'.$$

Denoting by $C_j$ the direct sum of the $\mathbb{C}_{vv'}$ over all edges in $S$, and by $B_{\epsilon}$ the ball of radius $\epsilon$ at the origin of $C_j$, then for small enough $\epsilon > 0$, the gluing gives a map:

$$gl_j : B_{\epsilon} \subset C_j \rightarrow \overline{M}_{0, l}, \quad \rho \mapsto j_\rho \equiv \Sigma_\rho,$$

which coincides with the identity map when $\rho \equiv 0$. Note that $gl_j$ is Aut($j$)-equivariant (for the linear action of Aut($j$) on $C_j$), and it follows from stability of the curve that $gl_j$ is injective. Taking the union over $M_S$ of the $C_j$ actually defines an orbibundle

$$p_S : L_S \rightarrow M_S.$$

Let $L_{S, \epsilon}$ denote the restriction to an $\epsilon$ neighbourhood of the zero section, and let $L^*_S$ be $L_S$ with the zero section removed. Given a proper open subset $U \subset M_S$, there exists $\epsilon > 0$, depending on $U$, such that the above gluing map extends to a locally diffeomorphic map:

$$gl_S : L^*_S,_{\epsilon, U} := L^*_S|_U \rightarrow \overline{M}_{0, l}, \quad (j, \rho) \mapsto gl_j(\rho).$$

More generally, given two stable stratum datas such that $S \prec S'$, there exists a subbundle $L_{S, S'}$ of $L_S$ with fibers identified to $\mathbb{C}^{|E'|-|E|}$, $|E'|$ being the number of edges in $S'$, as well as gluing maps (defined on proper open subsets)

$$gl_{S, S'} : L^*_{S, S', \epsilon} \rightarrow \overline{M}_S.$$

When $S$ is unstable, one can still define $L_S$ over $M_S$, as well as a gluing map $gl_S$, but $gl_S$ is injective and locally diffeomorphic if and only if $S$ is stable.
6.1.2. Gluing stable components. Let \( S_X := (V, F; pr, \rho) \) be a stable stratum data. Here we assume that the forgetful map \( S := S_{\pi, \rho} (S_X) \) and \( S_X \) have the same tree structure. Then \( \overline{\pi}^S_X \) coincides with the \textit{forgetting-the-map}:

\[
\mathcal{F}_X : \mathcal{M}_{S_X}(X) \to \mathcal{M}_S, \quad (u, j) \mapsto j.
\]

Let \( \mathcal{L}_{S_X} \) denote the orbibundle \( \mathcal{F}^*_X \mathcal{L}_S \) over \( \mathcal{M}_{S_X}(X) \). In local coordinates, an element in \( \mathcal{L}_{S_X} \) is given by a triple \((u, j, \rho)\). Consider the bundle map:

\[
\overline{\pi}^S_{pt} : \mathcal{L}_{S_X} \to \mathcal{L}_S, \quad (u, j, \rho) \mapsto (j, \rho).
\]

The gluing consists in constructing, compatibly with \( \overline{\pi}^S_{pt} \), a \( J \)-holomorphic map with domain \( gl_S(j, \rho) \) out of any \((u, j, \rho)\). More precisely, the following holds:

**Theorem 6.3.** ([2], [17]) For every proper subset \( U_X \subset \mathcal{M}_{S_X}(X) \), there exists a constant, \( \epsilon_X > 0 \), and a locally diffeomorphic map:

\[
Gl_{S_X} : \mathcal{L}^*_S \mathcal{L}_{U_X} := \mathcal{L}^*_{S_X, \epsilon} \big|_{U_X} \to \mathcal{M}_{0, l}(X, A), \quad (u, j, \rho) \mapsto Gl_{S_X}(u, j, \rho),
\]

such that

\[
(6.2) \quad \overline{\pi}^S_{pt} \circ Gl_{S_X} = gl_S \circ \overline{\pi}^S_{pt}.
\]

**Proof:** We give a sketched proof of this theorem which serves as a guiding principle when gluing pseudo-holomorphic curves. This is done in several steps.

**Step 1:** pregluing. Here we construct an approximatively \( J \)-holomorphic smooth map out of \((u, j, \rho) \in \mathcal{L}^*_S \). For \( v \in V \) let \( u_v : \Sigma_v \to X \) denote the corresponding component of \( u \). Also, for an edge \( e_{f, f'} \) of \( S_X \) with \( pr(f) = v \) and \( pr(f') = v' \), let \( y_{v, v'} \) denote the corresponding node on \( \Sigma \). Let \( \rho_{v, v'} \in \mathbb{C}^* \), defined by (6.1), denote the gluing parameter associated to \( y_{v, v'} \). Furthermore, let \( \beta : \mathbb{R} \to [0, 1] \) denote a smooth cut-off function with uniformly bounded derivative, \( |\beta'(r)| \leq 2 \), such that \( \beta(r) = 0 \) if \( r \leq 1 \) and \( \beta(r) = 1 \) for \( r \geq 2 \). The pregluing of \((u, j)\) with parameter \( \rho \), is the smooth map,

\[
u_{\rho} \equiv pg_{g}X(u, j, \rho) : \Sigma_{\rho} \to X,
\]

defined as follows: for each \( v \in V \), for every \( v' \in V \) such that \( vE v' \),

\[
u_{\rho}(z) := \left\{ \begin{array}{ll}
p_{v, v'} := u_v(y_{v, v'}) = u_{v'}(y_{v, v'}) & \text{if } z \in D_f(r_{v, v'}^{1/4})\{D_f(r_{v, v'}^{3/4})}, \\
\exp_{p_{v, v'}}\left(\beta(|z|/r_{v, v'})\{\exp_{p_{v, v'}}(u_{v}(z))\} & \text{if } z \in D_f(2r_{v, v'}^{1/4})\{D_f(r_{v, v'}^{1/4})} \\
\exp_{p_{v, v'}}\left(\beta(r_{v, v'}/|z|)\{\exp_{p_{v, v'}}(u_{v'}(z))\} & \text{if } z \in D_f(r_{v, v'}^{3/4})\{D_f(r_{v, v'}^{3/4})/2}.
\end{array} \right.
\]

and \( u_{\rho}(z) \) coincides with \( u_v(z) \) away from the annuli above. Here we need to assume \( \rho \) small enough, so that the discs \( D_f(4r_{v, v'}^{1/4}) \) are sent under \( u_v \) in a normal neighbourhood of \( p_{v, v'} \). Note that the mapping \( pg_{g}X \) is continuous with respect to \( \rho \).

**Estimates from the pregluing.** The following estimates are all standard and their proofs can be found in [17], [2]. The first estimate tells that \( u_{\rho} \) is approximately \( J \)-holomorphic. The second gives a quadratic estimate ensuring existence and unicity of the gluing map. Finally, the third is needed to derive that the constructed gluing map is locally diffeomorphic.

**Lemma 6.4.** Let \( p > 2 \) an integer, and \( U_X \) a proper open subset of \( \mathcal{M}_{S_X}(X) \). There is a uniform (with respect to \( \rho \)) constant \( c^X \) such that for every \((u, j) \in \mathcal{M}_{S_X}(X)\):

\[
\|du_{\rho}\|_{L^\infty} \leq c^X, \quad \|\overline{\partial}J u_{\rho}\|_{L^p} \leq c^X |\rho|^{1/2p}.
\]


Consequently, there is a uniform constant \( c_1^X \) such that
\[
\|N_u^X(\xi_1) - N_u^X(\xi_2)\|_{L^p} \leq c_1^X (\|\xi_1\|_{W^{1,p}} + \|\xi_2\|_{W^{1,p}}) \|\xi_1 - \xi_2\|_{W^{1,p}},
\]
where \( N_u^X \) denotes the non linear part in the Taylor expansion of \( \overline{\partial}_f \) at \( u \):
\[
N_u^X(\xi) = \overline{\partial}_f \exp_u \xi - \overline{\partial}_f u - D_u^X \xi, \quad \xi \in W^{1,p}(u^*TX).
\]
Finally, let \( u_t := \{(u_{v,t}, u_{v',t})\}_{vEv'} \), \( t \in [0,1) \) be a path in \( B_{\overline{\partial}_f}^{1,p} \), with
\[
\bar{\zeta} \equiv \{(\zeta_v, \zeta_{v'})\}_{vEv'} := \frac{d}{dt} \bigg|_{t=0} u_t.
\]
Let \( u_{v,t} \) denote the corresponding path of preglued curves and set \( \zeta_\rho := \frac{d}{dt} \bigg|_{t=0} u_{\rho,t} \).
There is a uniform constant \( \bar{c}^X \) such that
\[
\|\zeta_\rho\|_{L^{1,p}} \leq \bar{c}^X \|\zeta\|_{L^{1,p}} \quad \text{and} \quad \|D_{u_{v,0}}^X \zeta_\rho\|_{L^p} \leq \|D_{u_0}^X \zeta\| + \|\bar{c}^X\|^{1/2p}.
\]
In particular, if \( u_t \) is a path of holomorphic curves, i.e if \( \zeta \) is in the kernel of \( D_{u_0}^X \), the first term of the right handside of the second inequality vanishes.

**Step 2: Right inverses.** The gluing operation will give a holomorphic map with domain \( \Sigma_\rho \) obtained by perturbing the preglued map in directions that are transverse to the kernel of \( D_{u_0}^X \), i.e lying in the image of a uniformly bounded family \( Q_{u_0}^X \) of right inverses for \( D_{u_0}^X \). Below we sketch the proof of the following (for details we refer to [17]):

**Proposition 6.5.** Let \( p > 2 \) and \( U_X \) as before. There exists a uniformly bounded family \( Q_{u_0}^X \) of right inverses for \( D_{u_0}^X \), i.e there is a uniform constant \( c^X \) such that for every \((j, u) \in U_X\):
\[
\|Q_{u_0}^X u\|_{W^{1,p}} \leq c^X \|\eta\|_{L^p}.
\]

**Proof:** First, one constructs an interpolation \( w_\rho := \{u_{v,\rho, \Sigma_v \to X}\}_{v \in V} \) between \( u \) and \( u_\rho \) as follows: for \( v \in V \),
\[
u_{v,\rho} := \begin{cases} u_\rho(z) & \text{if } z \in \Sigma_v \setminus \bigcup_{v'} \{v \backslash v_{Ev'}\} D_j(r^{1/4}) \\ u_v(y_{Ev'}) = p_{vv'} & \text{otherwise.} \end{cases}
\]
where \( f \) is the flag in \( pr^{-1}(v) \) associated to the edge between \( v \) and \( v' \). As \( \rho \) goes to 0, the “flattened” map \( u_{v,\rho} \) converges to \( u_v \) in \( W^{1,p} \) norm. Thus, \( D_{u_{v,\rho}}^X \) converges to \( D_{u_v}^X \) in the operator norm, and since \( D_{u_v}^X \) is surjective then \( D_{u_{v,\rho}}^X \) also is. As a result, \( D_{u_{v,\rho}}^X \) is surjective. Therefore, \( D_{X_{w_{u_\rho}}}^X \) has right inverse \( Q_{w_{u_\rho}}^X \), which can be chosen uniquely by requiring that its image lies in the \( L^2 \)-orthogonal complement of ker \( D_{w_{u_\rho}}^X \). Moreover, \( Q_{w_{u_\rho}}^X \) is uniformly bounded. Next, we construct a quasi right inverse \( R_{u_\rho}^X \) for \( D_{u_\rho}^X \) out of \( Q_{w_{u_\rho}}^X \), defined by,
\[
R_{u_\rho}^X : \mathcal{E}_{u_{v,\rho}}^X \to \mathcal{A}_{u_{v,\rho}}^X, \quad \eta \mapsto \Gamma \circ Q_{w_{u_\rho}}^X \circ \Lambda(\eta).
\]

We give the definitions of \( \Lambda \) and \( \Gamma \). For \( v_{Ev'} \), with corresponding edge \( e_{j,f'} \), recall that \( \Sigma_\rho(f, f') \) is obtained by patching together the annuli \( A_f \) and \( A_{f'} \). Consider the circles in \( \Sigma_\rho \) corresponding to the circles
\[
\{-R_{w_{u_\rho}}/2\} \times \mathbb{R}/2\pi\mathbb{Z} \cong \{R_{w_{u_\rho}}/2\} \times \mathbb{R}/2\pi\mathbb{Z},
\]
and let \( \mathcal{C} \) be the union of all these circles. Consider the biholomorphism (onto its image),
\[
\pi_\rho : \Sigma_\rho \setminus \mathcal{C} \to \Sigma,
\]
defined as the identity map outside the annuli $A_{f, f'} := A_f \sim A_{f'}$ and given by:

$$\pi_\rho(z_f, z_{f'}) := \begin{cases} z_f & \text{if } |z_f| > |z_{f'}|, \\ z_{f'} & \text{if } |z_{f'}| > |z_f|. \end{cases}$$

Observe that,

$$u_\rho := \begin{cases} w_\rho \circ \pi_\rho & \text{on } \Sigma_\rho \setminus \mathcal{C}, \\ u_\rho(y_{vv'}) = u_\rho(y_{vv'}) = p_{vv'} & \text{on } \mathcal{C}. \end{cases}$$

Now set,

$$\Lambda : L^p(\Lambda^*_f(\Sigma_\rho, u_\rho^*TX)) \to L^p(\Lambda^*_{f'}(\Sigma_\rho, u_\rho^*TX)), \quad \eta \mapsto \begin{cases} (\pi^*_\rho)^{-1}\eta & \text{on } \operatorname{Im}(\pi_\rho), \\ 0 & \text{otherwise.} \end{cases}$$

Next, we define $\Gamma$. Set $\xi_{v,v'} := \xi(y_{v,v'})$ and $\beta_{v,v'}(z) := \beta(4\log |z|/\log |v_{vv'}|)$, where $\beta$ is a cut-off function as before. Then $\Gamma$ is an interpolation between $\xi_v$ and $\xi_{v'}$:

$$\Gamma : W^{1,p}(u_\rho^*TX) \to W^{1,p}(u_\rho^*TX), \quad \xi := \{\xi_v\}_{v \in V} \mapsto \xi_\rho,$$

where

$$\xi_\rho = \begin{cases} \xi_v(z_f) + \beta_{v,v'}(z_f)(\xi_{v'}(z_{f'}) - \xi_v(z_{f})) & \text{when } |z_{f'}| \leq |z_f| \leq |v_{vv'}|^{1/4} \\ \xi_\rho = \xi(z) & \text{otherwise.} \end{cases}$$

It follows from the estimates (6.10) in Lemma 6.6 below that the maps are $D^X_{u_\rho} R^X_{u_\rho}$ is invertible for small enough gluing parameters. Consequently,

$$(6.9) \quad Q^X_{u_\rho} := R^X_{u_\rho}(D^X_{u_\rho} R^X_{u_\rho})^{-1}$$

is the unique right inverse for $D^X_{u_\rho}$ having the same image as $R^X_{u_\rho}$. \hfill \Box

**Right inverses estimates.** The first estimate below gives the invertibility of $D^X_{u_\rho} R^X_{u_\rho}$. The second estimates how the right inverses $Q^X_{u_\rho}$ vary along a path in $\mathcal{B}^p_X$ of preglued maps, which is needed to show that the gluing map is locally diffeomorphic. For detailed proofs we refer to [17].

**Lemma 6.6.** Let $p > 2$. The operator $R^X_{u_\rho}$ depends smoothly on $(j, u, \rho)$ and there are constants $C^X$ and $\bar{C}^X$ independent of $\rho$, such that:

$$(6.10) \quad \|D^X_{u_\rho} R^X_{u_\rho} \eta - \eta\|_p \leq \frac{C^X}{|\log |\rho|^{1-1/p}} \|\eta\|_p, \quad \|R^X_{u_\rho} \eta\|_{W^{1,p}} \leq \frac{\bar{C}^X}{2} \|\eta\|_p.$$

Moreover, let $\{u_t\}_{t \in [0,1]}$ be a path of stable maps, with $\zeta := \frac{d}{dt}_{|t=0} u_t$. Let $\{u_{\rho,t}\}_{t \in [0,1]}$ be the corresponding path of preglued maps, with $u_{\rho,0} := u_\rho$. There is a uniform constant $\bar{C}^X$ such that:

$$(6.11) \quad \left\| \frac{d}{dt}_{|t=0} Q^X_{u_{\rho,t}} \right\| \leq \bar{C}^X \|\zeta\|_{W^{1,p}}.$$

**Proof:** We only give the proof of estimate (6.11).

$$\frac{d}{dt}_{|t=0} Q^X_{u_{\rho,t}} = \left( \frac{d}{dt}_{|t=0} R^X_{u_{\rho,t}} \right) (D^X_{u_\rho} R^X_{u_\rho})^{-1} + R^X_{u_\rho} \frac{d}{dt}_{|t=0} (D^X_{u_{\rho,t}} R^X_{u_{\rho,t}})^{-1}.$$

On one hand,

$$\left\| \frac{d}{dt}_{|t=0} R^X_{u_{\rho,t}} \right\| = \left\| \Gamma \left( \frac{d}{dt}_{|t=0} Q^X_{u_{\rho,t}} \right) \Lambda \right\| \leq M \|\zeta\|_{W^{1,p}}.$$
since, by a standard argument, \( \| \frac{d}{dt}|_{t=0} Q^X_{u_\rho,t} \| \) is bounded above by \( C\| \xi \|_{W^{1,p}} \) for some positive constant \( C \) (see [17]). On the other hand, 
\[
(D^X_{u_\rho,t} R^X_{u_\rho,t}) \frac{d}{dt}|_{t=0} (D^X_{u_\rho,t} R^X_{u_\rho,t})^{-1} = -(D^X_{u_\rho,t} R^X_{u_\rho,t})^{-1} \left( \frac{d}{dt}|_{t=0} D^X_{u_\rho,t} R^X_{u_\rho} + D^X_{u_\rho} \frac{d}{dt}|_{t=0} R^X_{u_\rho,t} \right)
\]
But the norm of derivative of \( D^X_{u_\rho,t} \) is bounded and the estimate follows. \( \square \)

**Step 3: the gluing map \( G\mathcal{S}_X \).** Let \( U_X \) be a proper open subset of \( \mathcal{M}_{S_X}(X) \). Let \( \overline{B}_{\delta_X}(0) \) denote the \( \delta_X \in \mathbb{R}^+ \) neighbourhood around the 0-section in \( \mathcal{E}^p_X(J) \). Also let \( \overline{B}_{\delta_X,u_\rho}(0) \) denote the corresponding ball in \( \mathcal{E}^p_{X,u_\rho}(J) \). Then, for each \( (u,j,\rho) \) we can find a unique element
\[
(6.12) \quad f^X(u,j,\rho) \in \overline{B}_{\delta_X,u_\rho}(0) \subset L^p(\Lambda^{0,1}_J(S^2,u_\rho^*TX))
\]
verifying:
\[
(6.13) \quad \overline{f}_J(\exp_{u_\rho} Q^X_{u_\rho}(f^X(u,j,\rho))) = 0.
\]
The _gluing map_ is then defined by
\[
(6.14) \quad G\mathcal{S}_X : \mathcal{L}_{S_X,\varepsilon_X,u_X} \rightarrow \mathcal{M}_{0,1}(X,A)
\]
for some positive constant \( \varepsilon_X \) given by the Implicit Function Theorem below. Note that we directly have the commutativity (6.2). Also, note that in reality, the gluing map is defined on local uniformizing system for \( \mathcal{L}_{S_X,\varepsilon_X,u_X} \), but since \( S_X \) and \( S \) are stable, the gluing is invariant under \( \text{Aut}(j) \), hence well-defined after quotient by the reparametrizations. Existence and unicity of \( f^X(u,j,\rho) \) are given by the following standard parametric version of the implicit function theorem, see [17].

**Implicit Function Theorem.** Let \( p > 2 \) and let \( c_1^X \) denote the positive constant in (6.1). There is a constant \( \varepsilon_X \) such that, for every \( (u,j,\rho) \in \mathcal{L}_{S_X,\varepsilon_X,u_X} \) we have a uniform positive constants \( \delta_X, \varepsilon_1^X, \) and \( c_2^X \), verifying:
\[
\| \overline{f}_J\rho \|_{L^p} \leq \varepsilon_1^X, \quad \| Q^X_{u_\rho} \| \leq c_2^X, \quad \varepsilon_1^X < \delta_X/4, \quad \varepsilon_1^X < (8c_1^X(c_2^X)^2)^{-1}
\]
and a smooth map,
\[
f^X : \mathcal{L}_{S_X,\varepsilon_X,u_X} \rightarrow \overline{B}_{\delta_X}(0),
\]
such that \( f^X(u,j,\rho) \) is the unique solution to (6.13). Furthermore,
\[
\| f^X(u,j,\rho) \|_{L^p} < 2\varepsilon_1^X.
\]

**Proof:** The constants \( \varepsilon_1^X \) and \( c_1^X \) are given by the estimates in Lemma 6.6 while \( c_2^X \) is given by Proposition 6.6. Consider the Fredholm fibered map:
\[
F^X : (\text{pgl}^X)^*TB_X \rightarrow \mathcal{E}^p_X(J), \quad ((u,j,\rho),\xi) \mapsto F^X_{(u,j,\rho)}(\xi) := \Phi^{-1}_{X,u_\rho}(\xi)\overline{f}_J\rho,\rho(\exp_{u_\rho} \xi),
\]
where \( \Phi_X \) is the parallel transport induced by the hermitian connection induced from the L-C connection \( \nabla^TX \) with respect to \( g_J \). When \( \xi = 0 \), this map coincides with \( \overline{f}_J\rho,\rho \) so that
\[
DF^X_{(u,j,\rho)}(0)(\xi) = D^X_{u_\rho}\xi.
\]
For every \( x := (u,j,\rho) \in \mathcal{L}_{S_X,\varepsilon_X,u_X} \), we want to find \( f^X(x) \) in \( \mathcal{E}^p_X(J) \), such that \( F^X(Q^X_x f^X(x)) = 0 \), in other words such that
\[
0 = F^X_x(0) + f^X(x) + N^X_x(Q^X_x f^X(x)),
\]
where \( N^X_x \) is the non-linear term of the expansion of \( F^X_x \) around \( \xi \equiv 0 \). Consider the family of operators

\[
H_x : \mathcal{E}^p_{X,x}(J) \to \mathcal{E}^p_{X,x}(J), \quad \eta \to -F^X_x(0) - N^X_x(Q^X_x \eta).
\]

We have

\[
\|H_x(\eta)\|_{L^p} \leq \|F^X_x(0)\|_{L^p} + \|N^X_x(Q^X_x \eta)\|_{L^p} \leq \epsilon^X_1 + \epsilon^X_1 (e^X_2)^2 \|\eta\|_{L^p}^2
\]

so that \( H \) maps the ball \( B_{\delta^X_1}(0) \) into itself whenever \( \epsilon^X_1 + \epsilon^X_1 (e^X_2)^2 \delta^X_X \leq \delta^X \). This is realized when \( \epsilon^X_1 < \delta^X/4 \) and \( 4 \epsilon^X_1 (e^X_2)^2 \epsilon^X_1 < 1/2 \). Furthermore,

\[
\|H_x(\eta_1) - H_x(\eta_2)\|_{L^p} = \|N^X_x(Q^X_x \eta_1) - N^X_x(Q^X_x \eta_2)\|_{L^p} \leq 2 \epsilon^X_1 (e^X_2)^2 \delta^X_X \|\eta_1 - \eta_2\|_{L^p} < \|\eta_1 - \eta_2\|_{L^p}.
\]

Thus, \( H \) is a contraction map so that existence and uniqueness follow. One can further show that the map \( H \) defines a contraction from \( B_{2\epsilon^X_1}(0) \) to itself, when \( 4 \epsilon^X_1 (e^X_2)^2 \epsilon^X_1 < 1/2 \). This gives the estimate \( \|f^X(x)\|_{L^p} < 2 \epsilon^X_1 \). Smoothness of \( f^X \) follows from implicit function theorem since \( DF^X_x(Q^X_x f^X(x)) \) is an isomorphism from \( Q^X_x(\mathcal{E}^p_{X,X}(J)) \) to \( \mathcal{E}^p_{X,X}(J) \).

Next, we prove that \( GL_{S_X} \) is locally diffeomorphic: we show that there is a proper open subset \( U_X^p \subset U_X \) and a positive constant \( \delta^X < \delta^X \) with respect to which the map \( GL_{X} |_{U_X^p} \) is a diffeomorphism onto its image.

Fix \((u_0, j, \rho_0)\) in \( \mathcal{L}^p_{S_X, c, U_X} \) and set \( u_{\rho_0} \) to be the corresponding pregluing. Let \( W \) be an open neighbourhood of 0 in \( W^{1,p}(S^2, u_{\rho_0}^*, TX) \). Furthermore, let \( U_1 \) be the image under \( \text{pgl}^X \) of a neighbourhood \( U_0 \) of \((u_0, j, \rho_0)\). We decompose \( GL_{S_X} \) as follows:

\[
U_0 \xrightarrow{\text{pgl}^X} U_1^p \xrightarrow{1 \times f^X} U_1 \xrightarrow{\Phi} W \xrightarrow{\exp_{\rho^X_0}} \mathcal{M}_{0,1}(P)
\]

where \( \Phi(u_{\rho}, \eta) := \xi_{\rho} + Q^X_{u_{\rho}, \eta}, \) and \( \exp_{u_{\rho_0}} \xi_{\rho} = u_{\rho} \). Note that the pregluing map is locally diffeomorphic. Also, for any path \( u_{\rho_t} \) starting at \( u_{\rho_0} \), with derivative \( \xi \) at \( t = 0 \), differentiating the equation

\[
0 = \overline{\partial}f(u_{\rho_t}) + f^X(u_{\rho_t}) + N^X_{u_{\rho_t}}(Q^X_{u_{\rho_t}, f^X(u_{\rho_t})})
\]

and using the estimates proved so far, one obtains the estimate:

\[
\|\frac{df^X}{d\xi}\|_{L^p} < C r^{1/2p} \|\xi\|_{W^{1,p}}.
\]

This ensures that for small enough gluing parameter, the differential of the gluing map is well-defined. Therefore it suffices to check that \( \Phi \) is a diffeomorphism. To prove this we identify \( W^{1,p}(S^2, u_{\rho_0}^*, TX) \) with \( \ker D^X_{u_{\rho_0}} \oplus L^p(S^2, u_{\rho_0}^*, TX) \) via the map

\[
\xi \mapsto ((Id - Q^X_{u_{\rho_0}} D^X_{u_{\rho_0}}) \xi, D^X_{u_{\rho_0}} \xi),
\]

and then rewrite \( \Phi \) as

\[
\Phi(u_{\rho}, \eta) = ((Id - Q^X_{u_{\rho_0}} D^X_{u_{\rho_0}})(\xi_{\rho} + Q^X_{u_{\rho}, \eta}), D^X_{u_{\rho_0}} (\xi_{\rho} + Q^X_{u_{\rho}, \eta})).
\]

Then, \( D^X \Phi(u_{\rho}, \eta) = Id + K(u_{\rho}, \eta) \) where

\[
K(u_{\rho}, \eta)(\xi, \zeta) = \begin{pmatrix}
(Id - Q^X_{u_{\rho_0}} D^X_{u_{\rho_0}})(\xi + \frac{dQ^X_{u_{\rho_0}}}{d\xi} \eta) - \xi & (Id - Q^X_{u_{\rho_0}} D^X_{u_{\rho_0}}) Q^X_{u_{\rho}, \eta} \\
D^X_{u_{\rho_0}} (\xi + \frac{dQ^X_{u_{\rho_0}}}{d\xi} \eta) & (Q^X_{u_{\rho_0}} D^X_{u_{\rho_0}} - Id) \xi
\end{pmatrix}
\]
Hence we deduce from corollary [6.5] and lemma [6.11] that for proper susbet $U' \subset U$ and $\delta_1 < \delta$, the operator $D^X\Phi(u_\rho, \eta)$ is invertible for any $(u_\rho, \eta) \in U' \times B_{\delta_1}(0)$, and that

$$\|D^X\Phi(u_\rho, \eta)\| \leq 2.$$  

For the injectivity of $\Phi$, let $N(\xi_\rho, \eta)$ be the non linear part in the expansion of $\Phi(u_\rho, \eta)$ around $(u_{\rho_0}, 0)$. Then,

$$N^X(\xi_\rho, \eta) = ((Id - Q^X_{u_{\rho_0}} D^X_u \Phi) Q^X_{u_\rho} \eta, D^X_{u_{\rho_0}} Q^X_{u_\rho} \eta - \eta).$$

By standard arguments, for $(\xi_{\rho_1}, \eta_1)$ and $(\xi_{\rho_2}, \eta_2)$, the following estimate holds,

$$\|N^X(\xi_{\rho_1}, \eta_1) - N^X(\xi_{\rho_2}, \eta_2)\|_p \leq C(\|(\xi_{\rho_1}, \eta_1)\|_{1,p} + \|(\xi_{\rho_2}, \eta_2)\|_{1,p})(\|\xi_{\rho_1} - \xi_{\rho_2}, \eta_1 - \eta_2\|_{1,p}).$$

Assume that $\Phi(u_{\rho_1}, \eta_1) = \Phi(u_{\rho_2}, \eta_2)$, for $(u_{\rho_1}, \eta_1) \neq (u_{\rho_2}, \eta_2)$. Then,

$$\|D^X\Phi(u_{\rho_0}, 0)(\xi_{\rho_1} - \xi_{\rho_2}, \eta_1 - \eta_2)\|_{L^p} = \|N^X(\xi_{\rho_1}, \eta_2) - N^X(\xi_{\rho_2}, \eta_2)\|_{L^p}.$$  

Since $\|(\xi_{\rho_1} - \xi_{\rho_2}, \eta_1 - \eta_2)\|_{W^{1,p}} > 0$ by assumption, we have,

$$0 < \|D^X\Phi(u_{\rho_0}, 0)\|_{L^p} \leq C(\|(\xi_{\rho_1}, \eta_1)\|_{1,p} + \|(\xi_{\rho_2}, \eta_2)\|_{1,p}),$$

which is impossible for small enough $(\xi_{\rho_i}, \eta_i)$, $i = 1, 2$. \hfill \Box

As such, the construction above still applies when the domain is not stable, nevertheless:

- the obtained gluing maps are only defined at the level of parametrized maps, i.e before quotienting by $\text{Aut}(j)$ even though these maps are $\text{Aut}(u, j)$ equivariant.
- so far we have parametrized our gluing maps according to the gluing for the domains, but the gluing for nodal surfaces is neither injective, nor locally diffeomorphic in the unstable case. Therefore, we can’t treat the gluing parameters $\rho$ as parameters.

The problem is to find a slice for the action of the group of reparametrizations of the domain.

### 6.1.3. Gluing unstable components

Let $(u, j) \in M_{S^X}(X)$, and let $v \in V$ be a $\pi_{pl,-}$ unstable component. Note that $|v| > 2$ is possible. Following Chen-Li [2], we describe the notion of balanced component when $|v| \leq 2$. For the commodity of the readers, we furnish the details below, as it will be generalized to the case of general base $B$. Let $\text{tr} \cong \mathbb{C} \subset G$ denote the subgroup of translations and let $\mathfrak{m} \cong \mathbb{C}^* \subset G$ denote the subgroup acting on $S^2 = \mathbb{C} \cup \{\infty\}$ by complex multiplication. Then the semi direct product $G := \mathfrak{t} \ltimes \mathfrak{m}$, acts on $\mathbb{C}$ by:

$$(m, t) \in G, \quad (t, m) \cdot z := m(z - t).$$

1) **Balanced maps.** First assume $|v| = 2$. Upto the action of $\mathfrak{m}$, the component $v$ can be parametrized by $\mathbb{CP}^1$ with special points $0 = [1 : 0]$ and $\infty = [0 : 1]$. Identify $\Sigma_v \setminus \{\infty\}$ with $\mathbb{C}$. The parametrization is balanced if:

$$\frac{1}{2} \int_{|z| \leq 1} \|du\|^2 dvols^2 = h$$

where $h$ denotes the minimal energy of a non-constant pseudo-holomorphic map in $X$. Next, when $|v| = 1$, upto the action of $G$, the component $v$ can be parametrized...
by \( \mathbb{C}P^1 \) such that the special point is \( \infty \). Identify \( \Sigma_v \setminus \{ \infty \} \) with \( \mathbb{C} \). The parametrization is balanced if (6.15) holds and if the center of energy of \( u \) is 0:

\[
\int z\|du\|^2dvolS^2 = 0, \quad \text{where} \quad z \in \mathbb{C}.
\]

We will say that \( u \) is centered if it is the case. The \( \pi_{pt^*} \)-unstable component \( \Sigma_v \) with balanced parametrization is called a balanced component. Recall that the reparametrizations of a balanced component is given by (6.15). Here the reparametrizations of a balanced component is given by \( S^1 \). Note that the neighbourhoods of the special points in a balanced parametrization for \( \Sigma_v \) can be put in standard cylindrical coordinates: for \( \infty \),

\[
[0, \infty) \times \mathbb{R}/2\pi \mathbb{Z} \cong D^*(r_{\infty}) : (s, t) \mapsto e^{s+it}
\]

while for 0,

\[
(-\infty, 0] \times \mathbb{R}/2\pi \mathbb{Z} \cong D^*(r_0) : (s, t) \mapsto e^{s+it}.
\]

To each unbalanced map \( u_v : \Sigma_v \to X \) smoothly corresponds a unique element \( \phi_X(u_v) \in \mathcal{G} \) consisting of the pair of translation and real dilation such that: the center of energy of

\[
u^b_v := u_v \circ (\phi_X(u_v))
\]

is zero; half the total energy of \( u^b_v \) lies in the unit disc around zero. Let \( \tilde{\mathcal{M}}^{b}_{0,i}(X, A_v) \), \( i = 1, 2 \), denote the sets of balanced \( J \)-holomorphic maps for \( A_v \) with one and two marked points. The map \( u_v \mapsto u^b_v \) sends an orbit of \( \mathcal{G} \) to an \( S^1 \) orbit where \( S^1 \) acts by rotations around the origin, hence we have the following natural identifications:

\[
\tilde{\mathcal{M}}^{b}_{0,i}(X, A) \cong \tilde{\mathcal{M}}^{b}_{0,i}(X, A)/S^1, \quad i = 1, 2.
\]

More generally, we say that \((u, j) \in \tilde{\mathcal{M}}_{S_X}(X)\) is balanced if each of \( \pi_{pt^*} \)-unstable component \( v \) with \(|v| \leq 2 \) is balanced. Let \( \tilde{\mathcal{M}}_{S_X}(X) \) denote the subset of all balanced stable maps, and call it moduli space of balanced \( J \)-holomorphic maps for \( S_X \). On \( \tilde{\mathcal{M}}^{b}_{S_X}(X) \) the action of the reparametrizations reduces to the action of \( \text{Aut}^X_{red} \):

\[
\text{Aut}^X_{red} \cong (S^1)^{V_b} \times \text{Aut}(S_X).
\]

where \( V_b \subset V \) denotes the subset of \( \pi_{pt^*} \)-unstable components \( v \) with \(|v| \leq 2 \). From the discussion above we have the identification:

\[
\mathcal{M}_{S_X}(X) = \tilde{\mathcal{M}}^{b}_{S_X}(X)/\text{Aut}^X_{red}.
\]

Next we define the gluing maps for balanced curves.

2) Gluing balanced maps. Let \( \mathcal{S} := S_{\pi_{pt^*}}(S_X) \). Set \( \mathcal{S}^u := \mathcal{F}_X(S_X) \) and let \( \tilde{\mathcal{M}}_{S^u} \) denote the set of parametrized nodal curves having \( \mathcal{S}^u \) as stratum data. Let \( \tilde{L}_{S^u} \) be the corresponding fiber bundle of gluing parameters. Also, let \( \tilde{L}_{S_X} \) denote the bundle \( \mathcal{F}_X \tilde{L}_{S^u} \) over \( \tilde{\mathcal{M}}_{S_X}(X) \), and let \( \tilde{L}^b_{S_X} \) denote its restriction to \( \tilde{\mathcal{M}}^{b}_{S_X}(X) \). The forgetful map \( \pi^{S^u}_{pt} \) induces a map:

\[
\tilde{\mathcal{M}}^{S_X}_{pt} : \tilde{L}^b_{S_X} \to \tilde{L}_S, \quad (u, j, \rho) \mapsto (\pi^{S_X}_{pt}(u, j), \rho^{st}),
\]

where \( \rho^{st} \) leaves unchanged the gluing parameters between \( \pi_{pt^*} \)-stable components, forgets about all the gluing parameters of components lying in contracted branch and sends the gluing parameters of a connecting branch to the product of the parameters of the corresponding connecting chain.
Note that the group $\text{Aut}^X_{\text{red}}$ acts naturally by rotations on $\tilde{L}_S^b$. Since we consider balanced parametrizations, $\rho^S$ is invariant under the reparametrizations of $j$ and $\overline{\pi}_{pt}^S$ is well-defined after quotient:

$$\overline{\pi}_{pt}^S : L^b_S := \tilde{L}_S^b / \text{Aut}^X_{\text{red}} \to \mathcal{L}_S.$$ 

Now, by hypothesis (6.2), for any element of $\tilde{M}_S^b(X)$, the linearization of the Cauchy-Riemann operator is surjective. It follows from Theorem 6.3 that for any proper open subset $U_X$ of $\tilde{M}_S^b(X)$, that we may choose to be $\text{Aut}^X_{\text{red}}$-invariant, there exists $\epsilon_X > 0$ and a map

$$\tilde{G}^b_l^S : \tilde{L}^b_{S, \epsilon_X, U_X} \to \tilde{\mathcal{M}}_{0, l}(X, A).$$

This map is $\text{Aut}^X_{\text{red}}$-equivariant, thus the gluing map is well-defined after the quotient by the action:

$$\tilde{G}^b_l^S : \tilde{L}^b_{S, \epsilon_X, U_X} / \text{Aut}^X_{\text{red}} \to \tilde{\mathcal{M}}_{0, l}(X, A).$$

Note that the domain of $\tilde{G}^b_l^S(u, j, \rho)$ is $gl_S^u(j, \rho)$. Since $S^u$ is not stable, $gl_S^u$ is not injective nor it is locally diffeomorphic and $\rho$ cannot be treated as a parameter anymore. Moreover, note that for $l \geq 3$, this gluing map takes value in $\tilde{\mathcal{M}}_{0, l}(X, A)$, while for $l < 3$, one needs to make sure that the image of the gluing gives a slice for the action of the automorphisms of $S^2$ with less than three marked points. 

**Theorem 6.7.** (Chen-Li [2]) $G^b_l^S$ is locally diffeomorphic. Furthermore,

$$\pi^S \circ G^b_l^S = gl_S \circ \overline{\pi}_{pt}^S.$$ 

**Proof:** The gluing among $\pi^*_s$-unstable components is divided into two cases: (1) the gluing between a balanced component and a stable component; (2) the gluing between two balanced components. Let $\Sigma_v$ and $\Sigma_{v'}$ be the two components to be glued at the edge $e_{f, f'}$. We may write

$$\Sigma_v = (S^2, z_f \equiv 0, \{x_k\}_{k=1, \ldots, m}) \quad \text{and} \quad \Sigma_{v'} = (S^2, z_{f'} \equiv \infty, \{x'_k\}_{k=1, \ldots, m'}).$$

Then (1) and (2) can be deduced from: (a) $\Sigma_v$ is $\pi^*_s$-stable while $\Sigma_{v'}$ is not, $m \geq 3$ and $m' = 0$; (b) both components are $\pi^*_s$-unstable and $m = m' = 0$.

**Case (a).** For simplicity we forget about the marked points on $\Sigma_v$. Let $V_{z_f}$ denote a neighbourhood of $z_f$, then $gl_S^u$ sends the neighbourhood $V_{z_f} \times \mathbb{C}^*$ of $j$ to $(S^2, \infty)$. Let $\tilde{N}_{u_0} \times V_{\rho_0}$ denote a neighbourhood of $(u_0, \rho_0)$ in $\tilde{\mathcal{M}}_{S^u}(X) \times \mathbb{C}^*_f$, in local coordinates for $\tilde{L}_S^b$. We want to define a gluing map:

$$Gl : \tilde{N}_{u_0} \times V_{\rho_0} \to \tilde{\mathcal{M}}_{0, l}(X, A).$$

By choosing a proper slice for the action of $\mathcal{G}$ we can construct a well-defined gluing map, namely $G^b_l^S$, locally given by:

$$G^b_l^S : N_{u_0, j} \times V_{z_f} \times V_{\rho_0} \to \tilde{\mathcal{M}}_{0, l}(X, A),$$

where $N_{u_0, j}$ stands for an $S^1$ slice in $\tilde{M}_S^b(X) \cap F^{-1}_X(j)$ around $u_0$. To see that this map is locally diffeomorphic, we compare it to a gluing map already encountered. To do this, use the identification of $m$ with $V_{\rho_0}$ to obtain a new map:

$$Gl^1 : \tilde{N}_{u_0} \times \{\rho_0\} \to \tilde{\mathcal{M}}_{0, l}(X, A).$$
Next, add two marked points on the second component, \( \{0\} \) and \( \{1\} \), in order to stabilize, and let \( \mathcal{S}_X(2) \) denote the corresponding stratum data. Then \( \widetilde{N}_{u_0} \) can be written as a product of \( V_{U_{s_0}} \), with a neighbourhood \( N_{u_0,s_0} \) of \( u_0 \) in \( \mathcal{F}_{X}^{-}(j) \cap \mathcal{M}_{\mathcal{S}_X(2)}(X) \). Then we use the natural identification between \( \mathbf{t} \) and \( V_{U_{s_0}} \) in order to obtain a gluing map \( GL_1^{b} \) defined on \( N_{u_0,s_0} \times \{ z_f \} \times \{ \rho_0 \} \), which we know is diffeomorphic. This map is in fact \( C^1 \)-close to \( GL_1^{b} \), so that \( GL_1^{b} \) is locally diffeomorphic.

**Case (b).** We may assume that \( z_f = 0 \) and \( z_{f'} = \infty \) in the balanced parametrizations. This time \( \Sigma_v = (S^2, 0) \) and \( \Sigma_{v'} = (S^2, \infty) \). Furthermore,

\[
\text{Aut}(\Sigma) = \mathcal{G}_1 \times \mathcal{G}_2 = (t_1 \times m_1) \times (t_2 \times m_2).
\]

Again \( \text{Aut}(\Sigma) \) acts on \( \mathbb{C}^*_\epsilon X \), and we set \( \text{Aut}_{v'v'}(\Sigma) \) to be the normal subgroup that fixes the gluing parameters under this action. Hence, \( \text{Aut}_{v'v'}(\Sigma) \) is isomorphic to \( t_1 \times t_2 \times \mathbb{C}^*_1 \), where \( \mathbb{C}^*_1 := \Delta^{-1}(1) \) and

\[
\Delta : m_1 \times m_2 \to m, \quad (m_1, m_2) \mapsto m_1m_2.
\]

The complementary of \( \mathbb{C}^*_1 \) in \( m_1 \times m_2 \) is denoted by \( \mathbb{C}^*_2 \) and is naturally identified with \( \mathbb{C}^*_\epsilon X \). The map \( GL_1^{b} \mathcal{S}_X \) comes from a map on \( \widetilde{M}_{\mathcal{S}_X}(X) \) locally given by:

\[
\widetilde{GL}_1 \mathcal{S}_X : U_{u_0} \times V_{\rho_0} \to \widetilde{M}_{0,0}(X, A),
\]

where \( U_{u_0} \) denotes a \( \mathcal{G}_1 \times \mathcal{G}_2 \)-invariant neighbourhood of \( u_0 \in \widetilde{M}_{\mathcal{S}_X}(X) \), and \( V_{\rho_0} \) is a neighbourhood of \( \rho_0 \in \mathbb{C}^*_\epsilon X \). The map \( GL_1^{b} \mathcal{S}_X \) is obtained by choosing an appropriate slice for the action of \( \text{Aut}(\Sigma) \) once we restrict ourselves to balanced maps. Quotienting by the automorphism group we get a map

\[
GL_1 \mathcal{S}_X : \frac{U_{u_0} \times V_{\rho_0}}{\mathcal{G}_1 \times \mathcal{G}_2} \to \frac{\widetilde{M}_{0,0}(X, A)}{\text{Aut}(S^2)},
\]

which we would like to take values in \( \frac{\widetilde{M}_{0,0}(X, A)}{\text{Aut}(S^2)} \). Using the identification between \( V_{\rho_0} \) and a neighbourhood of the identity in \( \mathbb{C}^*_2 \) we get a new map:

\[
GL_1 \mathcal{S}_X,1 : \frac{U_{u_0}}{\text{Aut}_{v'v'}(\Sigma)} \times \{ \rho_0 \} \to \frac{\widetilde{M}_{0,0}(X, A)}{\text{Aut}(S^2)},
\]

which is well-defined, locally diffeomorphic, and close to \( GL_1^{b} \mathcal{S}_X \). That this map is indeed well-defined follows since \( \text{Aut}_{v'v'}(\Sigma) \) and \( \text{Aut}(S^2) \) are locally diffeomorphic around the identity.

**Proof of (6.21).** First consider a contracted branch. It is connected to a unique \( \pi_* \)-stable component. The subgraph is a tree with a distinguished root that is attached to the \( \pi_* \)-stable component. We can parametrize each component \( \Sigma_v \) of the branch by \( \mathbb{C}P^1 \), such that the special point closest to the root is given by \( \infty = [0 : 1] \). By gluing from the farthest component to the closest, it suffices to consider the case of only one component attached to a root. But in that case, the glued surface \( J_\rho \) is isomorphic to the domain of the root component for every small enough \( \rho \), so that the forgetful map takes the corresponding glued maps to the same point.

Now consider a connecting branch. Observe that we can treat the components that are not in the connecting chain in the same way as the the components of a contracted branch. Therefore we only consider the case where the connecting branch coincides with the connecting chain. This chain is connected to exactly two \( \pi_* \)-stable components. The subgraph is a tree with two distinguished components, a root and a top that are attached to \( \pi_* \)-stable components. We can parametrize each component \( \Sigma_v \) of the branch by \( \mathbb{C}P^1 \), such that the special point that is the closest to
the root is given by $\infty = [0 : 1]$, and the point that is farthest is given by $0 = [1 : 0]$. Let $k$ denote the number of components of the connecting chain. By adding one marked point, say $\{1\}$, on every component of the chain, the resulting nodal Riemann surface $\Sigma'$ becomes stable. Then for every $\rho$, the gluing $\Sigma'_{\rho}$ is obtained from $\Sigma'_{\rho}$ by forgetting the added marked points. Note that each such choice of point $\{1\}$ fixes one gluing parameter on each component of the chain. For $\rho = (\rho_1, \ldots, \rho_{k+1})$, starting from the top, we can fix all gluing parameters a fixed small $\rho_0$, except for the gluing parameter associated to the root and the $\pi_s$-stable component, which is then given $\bar{\rho}$ the product of all the $\rho_i$’s. Now, $\Sigma'_{\rho}$ and $\Sigma'_{\bar{\rho}}$ are isomorphic since they are both realized by gluing on a cylinder of length $\sum \log |\rho_i|$.

\[ \square \]

6.2. Gluing for general $B$. It is completely parallel to the special case $B = pt$ treated above. We mainly point out the differences. Let $S_P := (V_P, F_l_P; \text{pr}_P, g_P)$ be a stable stratum data for $\overline{M}_{\text{cl}}(P, \sigma)$, and let $S_B := (V_B, F_l_B; \text{pr}_B, g_B)$ be its image under $S_{\pi_s}$. Also let $S$ denote the image of $S_P$ (or $S_B$) under the forgetful map.

6.2.1. Pregluing. Let $(u,j)$ be a $J_P$-holomorphic stable map in $P$, representing the stratum data $S_P$. We show that the pregluing of $(u,j)$ projects under $\pi$ to the pregluing of $(\pi(u),j)$ with same gluing parameter:

**Lemma 6.8.** For every $(u,j)$ stable map, and gluing parameter $\rho$:

$$\pi(u_\rho) \equiv \pi(pg_{\rho}^\pi(u,j)) = pg_{\rho}^\pi((\pi(u),j)) \equiv \pi(u)_\rho$$

**Proof:** Assume for simplicity in the notations that $|V| = 2$ with elements $v$ and $v'$ and let $e_{f,f'}$ be the corresponding edge. Set

$$\xi_v(z) := \exp_{p_{vv'}}^{-1} u_v(z), \quad \xi_{v'}(z) := \exp_{p_{vv'}}^{-1} u_{v'}(\rho_{vv'}/z),$$

and let $\beta^+$ and $\beta^-$ respectively denote the functions $\beta(|z|/r^{1/4}_{vv'})$ and $\beta(r^{3/4}_{vv'}/|z|)$. From (2.1) we deduce that on $\Sigma_v$,

$$\pi(u_\rho) = \begin{cases} 
\pi(u_v) & \text{if } z \in \Sigma_v \setminus D_f(2r^{1/4}_{vv'}), \\
\pi(p_{vv'}) = \pi(u_v(y_{v,v'})) = \pi(u_{v'}(y_{v,v'})) & \text{if } z \in D_f(r^{1/4}_{vv'}) \setminus D_f(r^{3/4}_{vv'}) \\
\exp(\pi(p_{vv'})) \left( \beta^+ \pi_{\rho v v'} \xi_v(z) + \beta^- \pi_{\rho v v'} \xi_{v'}(z) \right) & \text{otherwise}.
\end{cases}$$

We can rewrite this last expression as follows:

$$\exp(\pi(p_{vv'})) \left( \beta^+ \exp_{\pi(p_{vv'})}^{-1}(\pi(u_v(z))) + \beta^- \exp_{\pi(p_{vv'})}^{-1}(\pi(u_{v'}(\rho_{vv'}/z))) \right),$$

so that $\pi(u_\rho)$ coincides with the pregluing $\pi(u)_\rho$. \[ \square \]

**Remark 6.9.** The map obtained is not the pregluing of the stabilized map $\pi(u,j)$. If $u$ is only made of $\pi_s$-stable components, these two pregluings coincide. Furthermore, $u_\rho$ may not necessarily lie in the restriction $P|_{\pi(u)}$, e.g. if $u$ has only one $\pi_s$-stable component $u_0$. Nevertheless, we will see that the glued map projects to $\pi(u_0)$.

Let $p > 2$ and let $(j, u) \in \overline{M}_{x,P}(P)$. Here are some estimates that follow directly from Lemma 6.3 in the $B = pt$ case. From (6.3) there are uniform positive constants $c^B$ and $c^v$ such that:

$$||\overline{J}_P u_\rho||_{L^p} \leq c^v |\rho|^{1/2p}, \quad ||\overline{J}_P \pi(u)_\rho||_{L^p} \leq c^B |\rho|^{1/2p}.$$
Moreover, by definition of $g_{J_P}$:

$$
\|du_P\|_{L^p} \leq \|du_{B,P}\|_{L^p} + \|du_P\|_{L_P},
$$

hence there is a positive uniform constant $c^p$ such that:

$$
(6.22) \quad \|du_P\|_{L^\infty} \leq c^p, \quad \|\overline{\partial}_{J_P} u_P\|_{L_P} \leq c^p |\rho|^{1/2p}.
$$

Also, from (6.4) there is a uniform positive constant $\gamma_1^P$ such that:

$$
(6.23) \quad \|N^P_{u_\rho}(\xi_1) - N^P_{u_\rho}(\xi_2)\|_{L_P} \leq \gamma_1^P (\|\xi_1\|_{W^{1,p}} + \|\xi_2\|_{W^{1,p}})\|\xi_1 - \xi_2\|_{W^{1,p}}.
$$

Furthermore, from (6.5), if $u_t := \{ (u_{xt}, u_{v't}) \}_{t \in [0,v]}$ is a a path in $B_{\rho}^{1,P}$, with $
abla := \frac{d}{dt}|_{t=0} u_{\rho,t}$, and if $u_{\rho,t}$ is the corresponding path of preglued with $\zeta_{\rho} := \frac{d}{dt}|_{t=0} u_{\rho,t}$, there are uniform constants $\gamma_2^P$ and $\gamma_2^B$ such that $\gamma_2^P \leq \gamma_2^P$ and

$$
(6.24) \quad \|\zeta_{\rho}\|_{L^{1,p}} \leq \gamma_2^P \|\zeta\|_{L^{1,p}}, \quad \|D^P_{u_{\rho,0}} \zeta_{\rho}\|_{L_P} \leq \|D_{u_{\rho,0}} \zeta\| + \gamma_2^P |\rho|^{1/2p},
$$

and

$$
(6.25) \quad \|\pi_* \zeta_{\rho}\|_{L^{1,p}} \leq \gamma_2^B \|\pi_* \zeta\|_{L^{1,p}}, \quad \|D^B_{\pi(u_{\rho,0})} \pi_* \zeta_{\rho}\|_{L_P} \leq \|D_{\pi(u_{\rho,0})} \pi_* \zeta\| + \gamma_2^B |\rho|^{1/2p}.
$$

6.2.2. Right inverses. We give the description of right inverses for $D^P_{u}$ which are induced by right inverses for $D^v_{u}$ and right inverses for $D^B_{\pi(u)}$. By assumption $D^B_{\pi(u)}$ and $D^v_{u}$ are surjective and we can therefore consider their unique $L^2$-orthogonal right inverses $Q^B_{\pi(u)}$ and $Q^v_{u}$, with respect to $g_{J_B}$ and $g_J$. Set

$$
(6.26) \quad Q^P_{u} := \begin{pmatrix}
(Q^B_{\pi(u)})^h & 0 \\
L^u & Q^v_u
\end{pmatrix}.
$$

From the matrix expression (6.22) for $D^P_{u}$ we get

$$
(6.27) \quad D^P_{u} \circ Q^P_{u} := D^P_{u} \circ \begin{pmatrix}
(Q^B_{\pi(u)})^h & 0 \\
L^u & Q^v_u
\end{pmatrix} = \begin{pmatrix}
L^u \circ (Q^B_{\pi(u)})^h + D^v_u \circ L^u & 0 \\
0 & 0
\end{pmatrix},
$$

where $L$ is given by (3.3). Thus $Q^P_{u}$ is a right inverse, if $L' : (E_P)^h_u \to (\mathcal{X}_P)^v_u$ verifies $L^u \circ (Q^B_{\pi(u)})^h + D^v_u \circ L^u = 0$.

A natural choice for $L'$ is $L' = -Q^v_u \circ L \circ (Q^B)^h$.

Remark 6.10. By definition, $L_u$ is bounded. Namely, $\|L_u\| \leq C''$ for some constant $C''$ depending on $\|J\|_{C^1}$, $\|du\|_{L^\infty}$, and $\|R\|_{L^H}$.

The following is immediate.

Lemma 6.11. Let $Q^P_{u}$ be a right inverse for $D^P_{u}$ and suppose $Q^B_{\pi(u)}$ and $Q^v_{u}$ are as above. Then $L'$ is uniquely determined by (6.22) and the requirement that $Q^P_{u}$ has for image the $L^2$-orthogonal complement of ker $D^P_{u}$. In this case,

$$
L' = -Q^v_u \circ L \circ (Q^B_{\pi(u)})^h.
$$

In particular if $\|Q^B_{\pi(u)}\| < C^h$ and $\|Q^v_u\| < C^v$, for some positive constants $C^v$ and $C^h$ depending on $\|\rho\|_{L^\infty}$, then $\|L'\| < C^h C'' C^v$.

Remark 6.12. The $L^2$-orthogonal complementarity condition is a commodity assumption. The lemma above still holds for different choices of right inverses as long as we ask that the image of $Q^P_{u}$ is given by the images of $Q^B_{u}$ and $Q^v_{u}$. 

Let $p > 2$, and let $(j, u) \in \tilde{\mathcal{M}}_{S_\rho}(P)$. From Assumption 6.2 Lemma 6.6 and Proposition 6.5 we have uniform constants, $c^B$ and $c^v$, and right inverses $Q^B_{\pi(u)\rho}$ and $Q^v_{u\rho}$ for $D^B_{\pi(u)\rho}$ and $D^v_{u\rho}$, such that:

\[(6.28) \quad \|Q^B_{\pi(u)\rho}\eta\|_{W^{1,p}} \leq c^B\|\eta\|_{L^p}, \quad \|Q^v_{u\rho}\eta\|_{W^{1,p}} \leq c^v\|\eta\|_{L^p}.
\]

Similarly to Proposition 6.5 we have:

**Proposition 6.13.** There exists a constant $c^P$ independant of $\rho$ and right inverse $Q^P$ for $D^P$ such that for $(u, j) \in \mathcal{M}_{S_\rho}(P)$:

\[(6.29) \quad Q^P_{u\rho} := \begin{pmatrix} (Q^B_{\pi(u)\rho})^h & 0 \\ -Q^v_{u\rho} \circ L_{u\rho} \circ (Q^B_{\pi(u)\rho})^h & Q^v_{u\rho} \end{pmatrix}
\]

and such that

\[(6.30) \quad \|Q^P_{u\rho}\eta\|_{W^{1,p}} \leq c^P\|\eta\|_{L^p}.
\]

**Proof:** The right inverses $Q^B_{\pi(u)\rho}$ and $Q^v_{u\rho}$ are obtained from quasi-inverses $R^B_{\pi(u)\rho}$ for $D^B_{\pi(u)\rho}$, and $R^v_{u\rho}$ for $D^v_{u\rho}$, constructed as in (6.38). From these we construct a quasi-inverse for $D^P_{u\rho}$:

\[(6.31) \quad R^P_{u\rho} : \mathcal{E}^B_{P_{u\rho}} \equiv \mathcal{E}^B_{P_{u\rho}} \oplus \mathcal{E}^v_{P_{u\rho}} \rightarrow \mathcal{X}^1_{P_{u\rho}} \equiv \mathcal{X}^1_{P_{u\rho}} \oplus \mathcal{X}^1_{P_{u\rho}}
\]

\[\begin{align*}
\eta^h, \eta^v &\mapsto ((R^B_{\pi(u)\rho})^h \eta^h + L_{u\rho} \eta^h, R^v_{u\rho} \eta^v).
\end{align*}
\]

In fact,

\[(6.32) \quad R^P_{u\rho} := \Gamma \circ Q^P_{w\rho} \circ \Lambda.
\]

where $w\rho$, $\Lambda$ and $\Gamma$ are defined as in the $B = pt$ case. Note that $\Gamma$ and $\Lambda$ preserve the splitting induced by the Hamiltonian connection on $TP$, hence they have the following matrix representation

\[
\Lambda \equiv \begin{pmatrix} \Lambda^h & 0 \\ 0 & \Lambda^v \end{pmatrix} \quad \Gamma \equiv \begin{pmatrix} \Gamma^h & 0 \\ 0 & \Gamma^v \end{pmatrix}.
\]

It follows from the matrix form of $Q^P_{u\rho}$ that

\[
R^P_{u\rho} = \begin{pmatrix} \Gamma^h (Q^B_{\pi(w)\rho})^h \Lambda^h & 0 \\ -\Gamma^v Q^v_{w\rho} L_{w\rho} (Q^B_{\pi(w)\rho})^h \Lambda^h & \Gamma^v Q^v_{w\rho} \Lambda^v \end{pmatrix} \equiv \begin{pmatrix} R^h_{u\rho} & 0 \\ L^v_{u\rho} R^v_{u\rho} \end{pmatrix}
\]

we end up with the desired expression for $R^P_{u\rho}$. Note that $R^B \equiv d\pi \circ R^h$.

We show that $R^P$ is bounded and that $D^P_{u\rho} R^P_{u\rho}$ is invertible for small enough gluing parameters:

\[
\|D^P_{u\rho} R^P_{u\rho}\eta - \eta\|_{L^p} \leq \frac{C^P}{\log |\rho|^{1-1/p}}\|\eta\|_{L^p}, \quad \|R^P_{u\rho}\eta\|_{W^{1,p}} \leq \frac{\tilde{C}^P}{2}\|\eta\|_{L^p},
\]

for uniform constants $C^P$ and $\tilde{C}^P$. But from Lemma 6.6 there are uniform constants, $C^B$, $C^\tilde{B}$, $C^v$ and $\tilde{C}^v$, such that:

\[
\|D^B_{u,B,\rho} R^B_{u,B,\rho}\eta - \eta\|_{L^p} \leq \frac{C^B}{\log |\rho|^{1-1/p}}\|\eta\|_{L^p}, \quad \|R^B_{u,B,\rho}\eta\|_{W^{1,p}} \leq \frac{\tilde{C}^B}{2}\|\eta\|_{L^p}
\]

and

\[
\|D^v_{u,\rho} R^v_{u,\rho}\eta - \eta\|_{L^p} \leq \frac{C^v}{\log |\rho|^{1-1/p}}\|\eta\|_{L^p}, \quad \|R^v_{u,\rho}\eta\|_{W^{1,p}} \leq \frac{\tilde{C}^v}{2}\|\eta\|_{L^p}.
\]
Set $\xi_\rho = R^P_{u_\rho} \eta$ and suppose without loss of generality that $|V| = 2$, and let $V = \{v,v'\}$ with edge $e_{f,f'}$. Outside the patched annuli $D_f(r^{1/4}_v) \setminus D_f(r^{3/4}_v)$, we have that $\xi_\rho = \xi = Q^P_{u_\rho} \eta$ and $u_\rho = w_\rho$ which implies that $D^P_{u_\rho} \xi_\rho = \eta$. Therefore, the desired estimate for $R^P_{u_\rho}$ is trivially realized on this part of the curve. Then, it suffices to understand what happens on $D_f(r^{1/4}_v) \setminus D_f(r^{1/2}_v)$ and suppose without loss of generality that $\eta$.

Now $\rho$ is constant with value $p_{vv'}$. Hence, $D^P_{u_\rho}$, $D^P_{u_{v',\rho}}$ and $D^P_{u_{vv',\rho}}$ coincide with the standard Cauchy-Riemann operator:

$$
\begin{pmatrix}
(\bar{\partial} J_B(\pi(p_{vv'})))^h & 0 \\
0 & -\bar{\partial} J_B(p_{vv'})
\end{pmatrix}.
$$

Furthermore, $L^R_{u_\rho}$ must vanish since $L_{w_\rho}$ vanishes pointwise. Then, the result follows from the estimations of $R^B$ and $R^v$; the first estimate in (6.32) is obtained by choosing $C^P \geq \max(C^B, C^v)$, and the second by taking $C^P \geq \max(C^B, C^v)$.

Hence, $D^P_{u_\rho} R^P_{u_\rho}$ is invertible for small enough gluing parameters and we set:

$$
Q^P_{u_\rho} := R^P_{u_\rho}(D^P_{u_\rho} R^P_{u_\rho})^{-1}.
$$

Now $D^P_{u_\rho} R^P_{u_\rho}$ is of the following form:

$$
DR := \begin{pmatrix}
D^h R^h & 0 \\
L_{D^P R^P} & D^v R^v
\end{pmatrix},
$$

where

$$
L_{D^P R^P, u_\rho} = L_{u_\rho} R^h_{u_\rho} - D^v \Gamma^v Q^v_{w_\rho} L_{w_\rho}(Q^B_{\pi(w)_\rho})^h \Lambda^h.
$$

Furthermore,

$$
(D^P R^P)^{-1} \begin{pmatrix}
(D^h R^h)^{-1} & 0 \\
L_{(D^P R^P)^{-1}} & (D^v R^v)^{-1} \end{pmatrix}.
$$

Since all the operators involved are lower triangular we must have that

$$
Q^P_{u_\rho} := \begin{pmatrix}
(Q^B_{\pi(u)_\rho})^h & 0 \\
L^P_{u_\rho} & Q^v_{u_\rho}
\end{pmatrix}.
$$

We identify $L''_{u_\rho}$. To simplify notations we will omit the $u_\rho$ subscripts and we will set $(D^B)^h = D^h$. Again, $L_{w_\rho}$ vanishes in the region $D_f(r^{1/4}_v) \setminus D_f(r^{3/4}_v)$, so that the image of $L_{w_\rho}$ must lie in the image of $\Lambda^v$. Now this latter map is injective therefore

$$
L^R = -R^v(\Lambda^v)^{-1} L_{w_\rho}(Q^B_{\pi(w)_\rho})^h \Lambda^h.
$$

A simple computation then gives:

$$
L'' = L^R(D^h R^h)^{-1} + R^v L_{(D^P R^P)^{-1}} = -Q^v L R^h(D^h R^h)^{-1} = -Q^v L(Q_B)^h.
$$

Finally, let $\{u_t\}_{t \in [0,v]}$ be a path in $\mathcal{B}_P^{1,p}$, where $\zeta := \frac{d}{dt}|_{t=0} u_t$. Let $\{u_{\rho,t}\}_{t \in [0,v]}$ be the corresponding path of preglued curves, with $u_{\rho,0} =: u_\rho$. There are uniform constants $\tilde{c}^B$ and $\tilde{c}^P$ such that:

$$
\| \frac{d}{dt}|_{t=0} Q^B_{\pi(u_{\rho,t})} \| \leq \tilde{c}^B \| \pi_* \zeta \|_{W^{1,p}}, \quad \| \frac{d}{dt}|_{t=0} Q^P_{u_{\rho,t}} \| \leq \tilde{c}^P \| \zeta \|_{W^{1,p}}.
$$
6.2.3. **Gluing stable components.** We assume here that both \( \Sigma _{\pi ^*} \) and \( \overline{\pi }_{pl}^S \) preserves the tree structure of \( \mathcal{S}_P \). Then \( \overline{\pi }_{pl}^S \equiv \mathcal{F}_P, \overline{\pi }_{pl}^B \equiv \mathcal{F}_B \) and
\[
\overline{\pi }_{S_P} : \mathcal{M}_{S_P}(P) \rightarrow \mathcal{M}_{S_B}(B), \quad (u,j) \mapsto (u_B := \pi (u), j).
\]
This induces a map between orbibundles:
\[
\overline{\pi }_{S_P} : \mathcal{L}_{S_P} := \mathcal{F}_P \mathcal{L}_S \rightarrow \mathcal{L}_{S_B} := \mathcal{F}_B \mathcal{L}_S \quad (u,j,\rho) \mapsto (\pi (u), j, \rho).
\]
Let \( U_P \subset \mathcal{M}_{S_P}(P) \) and \( U_B \subset \mathcal{M}_{S_B}(B) \) be proper open subsets such that \( \overline{\pi }_{S_P}(U_P) = U_B \). To simplify the exposition, we assume that the maps involved do not have automorphism. If not so, the gluing maps \( \text{Gl}_{S_P} \) and \( \text{Gl}_{S_B} \) obtained below, are actually defined on local uniformizing systems for neighbourhoods \( W^P \) and \( W^B \) around the stable (holomorphic) maps \((u,j)\) and \((\pi (u), j)\), which are compatible with the projection \( \overline{\pi }_{S_P} \).

From Theorem \ref{thm:6.3} there exists a positive constant \( \epsilon_B \), and a diffeomorphism
\[
\text{Gl}_{S_B} : \mathcal{L}_{S_B, \epsilon B, U_B}^* \rightarrow \mathcal{M}_{0,1}(B, \sigma_B), \quad (u_B,j,\rho) \mapsto \text{Gl}_{S_P}(u_B,j,\rho),
\]
such that:
\[
(6.34) \quad \overline{\pi } \circ \text{Gl}_{S_P} = \text{Gl}_{S_B} \circ \overline{\pi }_{S_P}.
\]
Proof: First, recall that for \((\pi (u), j, \rho) \in \mathcal{L}_{S_B, \epsilon B, U_B}^*\),
\[
\text{Gl}_{S_B}(\pi (u), j, \rho) = (j_P, \exp _{\pi (u), \rho} Q^B_{\pi (u), \rho}(f^B(\pi (u), j, \rho))),
\]
where \( f^B \) is as in \( (6.12) \):
\[
f^B(\pi (u), j, \rho) \in B_{\delta_B, \pi (u), 0}(0) \subset L^p(\Lambda ^{0,1}_j(S^2, \pi (u)^* T_B))
\]
for a positive (uniform) constant \( \delta_B \) given by the Implicit Function Theorem. From the estimates \( (6.22), (6.23), (6.30) \), Implicit Function Theorem applies here, and there are uniform constants \( \epsilon_P \) and \( \delta_P \), and a smooth map
\[
f^P : \mathcal{L}_{\epsilon P, U_P} \rightarrow B_{\delta_P, 0}(0)
\]
such that
\[
f^P(u,j,\rho) \in B_{\delta_P, u, 0}(0) \subset L^p(\Lambda ^{0,1}_j(S^2, u^* T_P))
\]
is the unique solution to \( \overline{\partial }_{J_P}(\exp _{u, \rho} Q^P_{u, \rho}(f^P(u,j,\rho))) = 0 \). Then the gluing map \( \text{Gl}_{S_P} \) is defined by;
\[
\text{Gl}_{S_P}(u,j,\rho) = (j_P, \exp _{u, \rho} Q^P_{u, \rho}(f^P(u,j,\rho))).
\]
The proof that this is a locally diffeomorphic is verbatim the proof of Theorem \ref{thm:6.3} using the estimates \( (6.24) \) and \( (6.33) \).

Next we show \( (6.34) \). We begin by showing that
\[
d\pi \circ f^P = f^B \circ \overline{\pi }_{S_P}.
\]
By Lemma \ref{lem:6.8} \( \pi (u, \rho) = \pi (u) \rho \). Let \( \xi ^P \) denote \( Q^P_{\pi (u), \rho}(f^P(u,j,\rho)) \). Then,
\[
\pi (\exp _{u, \rho} (\xi ^P)) = \exp _{\pi (u), \rho} d\pi (\xi ^P).
\]
Since \( \xi ^P \) is the unique solution to \( \overline{\partial }_{J_P} \exp _{u, \rho} (\xi ^P) = 0 \), we deduce that
\[
\overline{\partial }_{J_P} \exp _{\pi (u), \rho} d\pi (\xi ^P) = 0.
\]
Moreover,
\[ \xi^B := d\pi(f^P) = d\pi(Q_{u_0}^P f^P(u, j, \rho)) = Q_{\pi(u), \rho}^B d\pi(f^P(u, j, \rho)), \]
iimplying that \( \xi^B \) is in the image of \( Q_{u,\rho}^B \), so that \( d\pi(f^P(u, j, \rho)) \) is in the image of \( f^B \) (by the implicit function theorem and choosing \( \delta_P \) smaller than \( \delta_B \)). Finally,
\[ \pi(GL_{S_P}(u, j, \rho)) = \pi(\exp_{u_0} Q_{u_0}^P f^P(u, j, \rho)) \]
\[ = \exp_{\pi(u_0)}(d\pi(Q_{u_0}^P f^P(u, j, \rho))) \]
\[ = \exp_{\pi(u)}(Q_{\pi(u)}^B f^B(\pi(u), j, \rho)) \]
\[ = GL_{S_B}(\pi(u), j, \rho). \]

6.2.4. Gluing: the unstable case. Let \((u, j) \in M_{S_P}(P)\), and let \(v \in V_P\) be a \( \pi_*\)-unstable component. Again, \(|v| > 2\) is possible. Generalizing the approach in [2], we describe the notion of balanced component when \(|v| \leq 2\).

1) **Balanced maps.** First assume \(|v| = 2\). In this case \(u_0 : \Sigma_v \to P\) can be parametrized by \(\mathbb{C}P^1\) with special points \(0 = [1 : 0]\) and \(\infty = [0 : 1]\). Identify \(\Sigma_v \setminus \{\infty\}\) with \(\mathbb{C}\). If \(\pi(u_0)\) is non-constant, we say that \(u_0\) is balanced if it is horizontally balanced, i.e if \(\pi(u_0)\) is balanced in the sense of (6.35). On the other hand, if \(\pi(u_0)\) is constant, we say that \(u_0\) is balanced if it is vertically balanced, i.e if half the vertical energy \(\langle h^v \rangle\) of \(u_0\) is contained in the unit disc around 0.

Next, when \(|v| = 1\), \(\Sigma_v\) can be parametrized by \(\mathbb{C}P^1\) with special \(\infty = [0 : 1]\). Identify \(\Sigma_v \setminus \{\infty\}\) with \(\mathbb{C}\). If \(\pi(u_0)\) is not constant, \(u_0\) is balanced if it is horizontally balanced as in the \(|v| = 2\) case, and if \(u_0\) is horizontally centered, i.e if the center of energy of \(\pi(u_0)\) is zero. If \(\pi(u_0)\) is constant, we say that \(u_0\) is balanced if it is vertically balanced as above, and if vertically centered, i.e if the mean value for \(\|du_0^v\|_{g_{j,P}}^2\) is zero.

For \(i = 1, 2\), let \(\widetilde{M}_{0,i}^{h^h}(P, \sigma_v)\), resp. \(\widetilde{M}_{0,i}^{h^v}(P, \sigma_v)\), denote the set of horizontally, resp. vertically, balanced \(J_P\)-holomorphic maps with \(i = 1\) or 2 marked point representing \(\sigma_v\). In order to simplify notations, \(\widetilde{M}_{0,i}^{h^v}(P, \sigma_v)\) will designate the set of (vertically and horizontally) balanced \(J_P\)-holomorphic maps with \(i\) marked point. When \(\pi_*\sigma_v \neq 0\), the projection \(\pi\) naturally induces a map:
\[ \pi : \widetilde{M}_{0,1}^{h^h}(P, \sigma_v) = \widetilde{M}_{0,1}^{h^v}(P, \sigma_v) \to \widetilde{M}_{0,i}^{h^h}(B, \pi_*\sigma_v). \]
As in the \(B = pt\) case, there is a natural smooth surjective map between the moduli space of holomorphic maps to the moduli space of balanced maps, \(u_0 \mapsto u_0^b\), where \(u_0^b\) is given by (6.17), and it is not hard to see that
\[ \pi(u_0)^b = \pi(u_0) \text{ if } \pi_*\sigma_v \neq 0. \]

We conclude that
\[ M_{0,i}(P, \sigma_v) \cong \widetilde{M}_{0,i}^{h^h}(P, \sigma_v) / S^1, \quad i = 1, 2, \]
and when \(\pi_*\sigma \neq 0\), the map \(\pi\) in (6.35) descends to the (expected) map:
\[ \pi : M_{0,i}(P, \sigma_v) \to M_{0,i}(B, \pi_*\sigma_v), \quad i = 1, 2. \]

Finally, we say that \((u, j) \in M_{S_P}(P)\) is balanced if each of its \(\pi_*\)-unstable components \(u_v\) with \(|v| \leq 2\) is balanced. The corresponing moduli space of balanced
\(J_P\)-holomorphic maps is denoted \(\tilde{\mathcal{M}}^b_{P}(P)\). Similarly to \((6.19)\), we have a natural identification,
\[
\mathcal{M}_{S_P}(P) = \tilde{\mathcal{M}}^b_{S_P}(P)/\text{Aut}_{\text{red}}^P,
\]
where \(\text{Aut}_{\text{red}}^P\) denotes the group of reparametrizations acting on \(\tilde{\mathcal{M}}^b_{S_P}(P)\) (see \((6.18)\)). Observe that the projection \(\pi_{S_P}\) restricts to an \(\text{Aut}_{\text{red}}^P, \text{Aut}_{\text{red}}^B\) equivariant map
\[
\pi_{S_P} : \tilde{\mathcal{M}}^b_{S_P}(P) \rightarrow \mathcal{M}_{S_B}(B),
\]
and descends to \(\pi_{S_P} : \mathcal{M}_{S_P}(P) \rightarrow \mathcal{M}_{S_B}(B)\) after quotienting by the reparametrizations.

**Remark 6.15.** Let \(P\) denote the bundle \((S^2 \times S^2, \omega_0 + \omega_0)\) over the base \(B := (S^2, \omega_0)\), where \(\omega_0\) denotes the Fubini-Study form on \(S^2\). Here \(\pi\) represents the projection to the first factor, \(J_P\) is the product complex structure. Consider the holomorphic section \(u(z) = (z, z + b)\). A simple computation shows that \(\pi(u)\) is balanced (the map \(z \mapsto az + b\) is balanced if and only if \(|a| = 1\) and \(b = 0\)). In addition, one sees that \(u(z-b)\) is vertically balanced. Note that if we had adopted the definition for balanced maps in \(P\) with respect to the energy density \(\|du\|^2_g J_P\), the mapping \((6.35)\) would not necessarily exist. Indeed, the map \(u(z-b/2)\) is balanced in this sense, but projects to \(z-b/2\) which is not balanced.

Finally, had we defined a balanced map in \(P\) as being both horizontally and vertically balanced, the compatibility \((6.36)\) may not be realized. For example, consider the maps \(u(z) = (z, a\bar{z})\), for \(a \in \mathbb{R}^+\), with \(a \neq 1\). Then the center of energy of \(u\) is 0, so that it is horizontally balanced. However, the map is not vertically balanced, and if \(h^b\) denotes the energy of the projection \(\pi(u)\), then the energy of \(u\) in the unit disc around 0 is given by \(h^b + \pi(1 - \frac{1}{1+a^2})\), which gives \(h\) if and only if \(a = 1\).

2) **Gluing of balanced maps and compatibility.** Let \(S^b_P\) and \(S^u_P\) respectively denote the projections of \(S_P\) under \(\pi_*\) and \(F_P\). These stratum data are not stable here. Note that \(S^b_P\) and \(S_P\) have the same tree structure but the homological data for \(S^b_P\) is the projection of the homological data for \(S_P\) under \(\pi_*\).

Let \(L_{S_P}\) and \(\tilde{L}_{S_B}\) denote the bundles of gluing parameters over the balanced moduli spaces \(\tilde{\mathcal{M}}^b_{S_P}(P)\) and \(\tilde{\mathcal{M}}^b_{S_B}(B)\) obtained by pull-backing \(L_{S^u_P}\) under \(F_P\) and \(F_B\) respectively. Consider the bundle map lifting \(\pi_{S_P}:\)
\[
\overline{\pi}_{S_P} : \tilde{L}_{S_P} \rightarrow \tilde{L}_{S_B}, \quad (u, j, \rho) \mapsto (\pi_{S_P}(u, j), \rho^{st}),
\]
where \(\rho^{st}\) denotes the stabilization of \(\rho\) with respect to \(\pi_*\) this time (see \((6.20)\)). This map descends to a well-defined map:
\[
\overline{\pi}_{S_P} : L_{S_P} := \tilde{L}_{S_P}/\text{Aut}_{\text{red}}^P \rightarrow L_{S_B} := \tilde{L}_{S_B}/\text{Aut}_{\text{red}}^B.
\]

Let \(U_P \subset \mathcal{M}_{S_P}(P)\) and \(U_B \subset \mathcal{M}_{S_B}(B)\) be proper open subsets such that \(\overline{\pi}_{S_P}(U_P) = U_B\). From the discussion in Section 6.1.3 we have a gluing map,
\[
\text{Gl}^b_{S_P} : \tilde{L}_{S^*,S_P,U_P}/\text{Aut}_{\text{red}}^P \rightarrow \mathcal{M}_{0,1}(P, \sigma),
\]
and a gluing map \(\text{Gl}^b_{S_B}\) above \(U_B\) defined similarly. We prove the following generalization of Theorem 6.7
Theorem 6.16. The gluing maps $Gl^{b}_{\mathcal{S}^{u}_B}$ and $Gl^{b}_{\mathcal{S}^{u}_B}$ are locally diffeomorphic and such that
\begin{equation}
\pi \circ Gl^{b}_{\mathcal{S}^{u}_B} = Gl^{b}_{\mathcal{S}^{u}_B} \circ \overline{\pi}_{\mathcal{S}^{u}_B}.
\end{equation}

Proof: The diffeomorphic issue follows directly from Theorem [6.7] The compatibility (6.37) is obtained as follows. Let $\mathcal{M}_{B}^{b}(B)$ denote the image of $\mathcal{M}_{\mathcal{S}^{u}_B}(B)$ under $\pi: (u, j) \mapsto (\pi(u), j)$. Note that the parametrization of the domain $\pi(u)$ comes from the parametrization of the domain of $u$, which is fixed up to an $S^1$-action. Now, let $\mathcal{L}^u_{\mathcal{S}^{u}_B}$ denote the pull-back of $\mathcal{L}^u_{\mathcal{S}^{u}_B}$ under the forgetting-the-map map. Then, by assumption (6.2), we can construct a gluing map $\tilde{Gl}^{b}_{\mathcal{S}^{u}_B}$ with value in $\overline{\mathcal{M}}_{0,1}(B, J_B)$, and domain a neighbourhood of the zero section in $\mathcal{L}^u_{\mathcal{S}^{u}_B}$ restricted to $\pi(U_B)$. This map is equivariant with respect to the balanced maps automorphisms, but since $\mathcal{S}^{u}_B$ is not stable, it is not locally diffeomorphic nor injective. Nevertheless, $\tilde{Gl}^{b}_{\mathcal{S}^{u}_B}$ still verifies the compatibility:
\[\pi \circ \tilde{Gl}^{b}_{\mathcal{S}^{u}_B} = \tilde{Gl}^{b}_{\mathcal{S}^{u}_B} \circ \pi.\]

Again, the problem is localized in the contracted branches and the connecting branches.

Contracted branch. Set $(u_B, j_B) := \pi_{\mathcal{S}^{u}_B}(u, j)$. As in Theorem [6.7], for every $\rho$ the glued surface $j_{\rho}$ is isomorphic to $j_B$ i.e to the domain of the $\pi_*$-stable component. Therefore, $u_B$ and $\tilde{Gl}^{b}_{\mathcal{S}^{u}_B}(u, j, \rho)$ have the same domain. Fix $\rho_0$ with small radius. We show that
\[\tilde{Gl}^{b}_{\mathcal{S}^{u}_B}(\pi(u), j, \rho_0) = (u_B, j_B).\]
By definition, the pregluing of $(\pi(u), j, \rho_0)$ coincides with $\pi(u)$ except on disc with radius determined by $\rho_0$ on which
\[\pi(u)_{\rho_0}(z) \equiv \text{pgl}_{B}(\pi(u), j, \rho_0)(z) = \exp_{\pi(p_{\infty})}(\beta(z/\rho_0^{-1}) \exp_{\pi(p_{\infty})}^{-1}(\pi(u_0(z)))),\]
$p_{\infty}$ being the image under $u$ of the point of the root identified to $\infty$. Set
\[\xi := \exp_{\pi(p_{\infty})}^{-1}(\pi(u_0(z))) \quad \text{and} \quad \beta_{\rho_0} := \beta(z/\rho_0^{-1/4}).\]
Then, $\tilde{Gl}^{b}_{\mathcal{S}^{u}_B}(\pi(u), j, \rho_0)$ coincides with $u_B \equiv \exp_{\pi(u)_{\rho_0}}((1 - \beta_{\rho_0})\xi)$.

Connecting branch. As in Theorem [6.7], we can assume the contracting branch is a connecting chain. By adding $k$-marked point, one for each component of the connecting branch, $\pi(u)$ becomes stable. Let $j'$ denote the conformal structure resulting from this operation. Let $\mathcal{M}_{\mathcal{S}^{u}_B(k)}(B)$ denote the stabilization by adding $k$-marked points. We have a gluing map
\[\tilde{Gl}^{b}_{\mathcal{S}^{u}_B(k)}: \mathcal{L}^{k}_{\mathcal{S}^{u}_B(k)} \equiv \mathcal{F}_{B}(\mathcal{L}_{\mathcal{S}^{u}_B(k)}) \rightarrow \overline{\mathcal{M}}_{0,1+k}(B, \sigma_B).\]
By definition of the gluing, and since the components of $\pi(u)$ coming from the connecting branch are constant, the maps
\[\tilde{Gl}^{b}_{\mathcal{S}^{u}_B(k)}(\pi(u), j', \rho) \quad \text{and} \quad \tilde{Gl}^{b}_{\mathcal{S}^{u}_B}(\pi(u), j, \rho)\]
coincide for every $\rho$, and any choice of balanced parametrization on the domain $\Sigma$ of $j$ (here the $\text{Aut}(j)$-equivariance of $Gl^{b}_{\mathcal{S}^{u}_B}$ is needed). Note that $j_{\rho}$ is obtained from $j'_{\rho}$ by forgetting the added marked points. We compare the gluing of $(\pi(u), j')$ with that of $(u_B, j_B) := \pi_{\mathcal{S}^{u}_B}(u, j)$. Note that $(u_B, j_B)$ is the image of $(\pi(u), j')$ via the map that forgets the added marked points and stabilizes. The fiber over $(u_B, j_B)$
corresponds to the set of all possible reparametrizations for \( \gamma \), since \( u_B \) and \( \pi(u) \) have same image. Thus, it suffices to identify the gluings of the corresponding domains but this follows from Theorem 6.7. Finally,

\[
\pi \circ \widetilde{Gl}_{\mathcal{S}}(u, j, \rho) = \widetilde{Gl}_{\mathcal{S}}(\pi(u), j, \rho) = \widetilde{Gl}_{\mathcal{S}}(\pi(u), j, \rho^*) = \widetilde{Gl}_{\mathcal{S}}(u_B, j_B, \rho^*) = \widetilde{Gl}_{\mathcal{S}} \circ \overline{\pi}_{\mathcal{S}}(u, j, \rho).
\]

From now on, we will always assume that gluing maps are obtained by considering balanced maps, and we will drop the \( b \) indices from the notations for the gluing maps.

6.2.5. **Gluing maps between strata.** One can generalize the preceding discussions and introduce gluing maps between different stable strata. Let \( \mathcal{S}_X \) and \( \mathcal{S}'_X \) be stable stratum data for \( \mathcal{M}_{0, l}(X, \mathcal{A}) \), such that \( \mathcal{S}_X \prec \mathcal{S}'_X \). Let \( \mathcal{S} \) and \( \mathcal{S}' \) denote their projections under \( \mathcal{F}_X \), and consider the bundles \( \mathcal{L}_{\mathcal{S}, \mathcal{S}'} \) and \( \mathcal{L}_{\mathcal{S}_X, \mathcal{S}'_X} := \mathcal{F}_X^{\ast} \mathcal{L}_{\mathcal{S}, \mathcal{S}'} \). For an open proper subset \( U_X \) of \( \mathcal{M}_{\mathcal{S}_X}(X) \) there exists a positive constant \( \epsilon_X \) and locally diffeomorphic map:

\[
\widetilde{Gl}_{\mathcal{S}_X, \mathcal{S}'_X} : \mathcal{L}_{\mathcal{S}_X, \mathcal{S}'_X}^{\ast} \times_{\epsilon_X, U_X} \to \mathcal{M}_{\mathcal{S}'_X}(X),
\]

which coincides with the identity on the zero section. Also, from the definition of \( \mathcal{L}_{\mathcal{S}_X, \mathcal{S}'_X} \), a point of \( \mathcal{L}_{\mathcal{S}_X} \) is locally given by a tuple \((u, j, \rho_1, \rho_2)\) where \((u, j, \rho_1) \in \mathcal{L}_{\mathcal{S}_X, \mathcal{S}'_X} \) and where \( \rho_2 \) accounts for the remaining gluing parameters. Therefore, \( \widetilde{Gl}_{\mathcal{S}_X, \mathcal{S}'_X} \) induces a map:

\[
\mathcal{L}_{\mathcal{S}_X} \to \mathcal{L}_{\mathcal{S}'_X}, \quad (u, j, \rho_1, \rho_2) \mapsto (\widetilde{Gl}_{\mathcal{S}_X, \mathcal{S}'_X}(u, j, \rho_1), \rho_2)
\]

It follows that \( \mathcal{L}_{\mathcal{S}_X} \) coincides with the pullbacks \( \widetilde{Gl}_{\mathcal{S}_X, \mathcal{S}'_X}^{\ast} \mathcal{L}_{\mathcal{S}_X, \mathcal{S}'_X} \). Suppose now we have a third stratum data \( \mathcal{S}_X'' \) such that \( \mathcal{S}_X' \prec \mathcal{S}_X'' \). Since \( \widetilde{Gl}_{\mathcal{S}_X, \mathcal{S}'_X} \) is locally diffeomorphic, we can define a new gluing map:

\[
\widetilde{Gl}_{\mathcal{S}_X, \mathcal{S}'_X} : \mathcal{L}_{\mathcal{S}_X, \mathcal{S}_X''} \times_{\epsilon_X, \epsilon_X} \to \mathcal{M}_{\mathcal{S}'_X}(X),
\]

extending the identity map on the zero section. This new gluing does not necessarily coincide with \( \widetilde{Gl}_{\mathcal{S}_X, \mathcal{S}_X''} \). The equality would mean that the gluing procedure is associative, which is a priori not true due to the numerous choices made along the gluing construction (in particular the independance with respect to the choice of right inverses). Nevertheless we can see that these maps are close, in the \( C^\infty \) sense, which is enough to give the moduli spaces the structure of smooth orbifolds, as we will see in the next Section.

Now consider the Hamiltonian Fibration case \( \pi : P \to B \). Let \( \mathcal{S}_P \) and \( \mathcal{S}'_P \) be stable stratum data for \( \mathcal{M}_{0, l}(P, \sigma) \) such that \( \mathcal{S}_P \prec \mathcal{S}'_P \), and let \( \mathcal{S}_B \) and \( \mathcal{S}'_B \) be their corresponding projections via \( \mathcal{S}_X \). We see that \( \mathcal{S}_B \prec \mathcal{S}'_B \). By the discussion above, for \( U_P \) and \( U_B \) be proper open subsets of \( \mathcal{M}_{\mathcal{S}_P}(P) \) and \( \mathcal{M}_{\mathcal{S}_B}(B) \) such that \( \overline{\pi}_{\mathcal{S}_P}(U_P) = U_B \) we do have gluing maps \( \widetilde{Gl}_{\mathcal{S}_B, \mathcal{S}'_B} \) and \( \widetilde{Gl}_{\mathcal{S}_P, \mathcal{S}'_P} \) such that:

\[
\widetilde{Gl}_{\mathcal{S}_B, \mathcal{S}'_B} \circ \overline{\pi}_{\mathcal{S}_P, \mathcal{S}'_P} = \overline{\pi}_{\mathcal{S}_P, \mathcal{S}'_P} \circ \widetilde{Gl}_{\mathcal{S}_P, \mathcal{S}'_P}.
\]

Suppose now we have a third stratum data \( \mathcal{S}_P'' \) projecting on \( \mathcal{S}_B \) and such that \( \mathcal{S}_P' \prec \mathcal{S}_P'' \). Then we also have the commutativity:

\[
\widetilde{Gl}_{\mathcal{S}_P, \mathcal{S}_P''} \circ \overline{\pi}_{\mathcal{S}_P, \mathcal{S}_P''} = \overline{\pi}_{\mathcal{S}_P, \mathcal{S}_P''} \circ \widetilde{Gl}_{\mathcal{S}_P, \mathcal{S}_P''}.
\]
6.3. **Fibration of moduli spaces.** In this section we prove that a Hamiltonian Fibration structure induces a fibration structure between the appropriate compactified moduli spaces:

**Theorem 6.17.** Under hypothesis \([6.2]\), the moduli spaces \(\overline{\mathcal{M}}_{0,l}(P, \sigma)\) and \(\overline{\mathcal{M}}_{0,l}(B, \sigma_B)\) are smooth orbifolds, and the maps \(\pi_{S_P}\) extend to a map

\[
\pi : \overline{\mathcal{M}}_{0,l}(P, \sigma) \to \overline{\mathcal{M}}_{0,l}(B, \sigma_B),
\]

which restricts to a smooth locally trivial fibration (of orbifolds) above each strata of \(\overline{\mathcal{M}}_{0,l}(B, \sigma_B)\).

Regarding the fibration structure, it suffices to show this above a proper open subset \(U_B\). But each \(\pi_{S_P}\) is a smooth submersion. Moreover, the fibers of \(\pi\) are compact, hence \(\pi\) is proper which ends the proof of the fibration statement.

Following \([2]\), we show below how the compactified moduli spaces can be given the structure of smooth orbifolds compatibly with the \(\pi_{S_P}\).

6.3.1. **Charts data and admissible gluing maps.** Consider a Fredholm system \((B, E, s)\) with moduli space \(M = s^{-1}(0)\), modeled on maps. Assume that the linearization \(L_{x_0}\) at \(x_0 \in M\) is surjective. Then, a standard construction gives a local coordinate chart around \(x_0\). Such a chart is given by a triple \((U, \varphi, f)\) where:

(i) \(U\) is a submanifold of a neighbourhood \(V_{x_0}\) of zero in \(T_{x_0}B\) (which we identify to a neighbourhood \(V_{x_0}'\) of \(x_0\) in \(B\) via the exponential map);

(ii) \(\varphi : U \times B_\delta \to V_{x_0}'\) is a diffeomorphism where \(B_\delta \subset E_{x_0}\) is an open ball, \(V_{x_0}'\) is a neighbourhood of 0 in \(V_{x_0}\);

(iii) \(f\) is a smooth section \(f : U \to B_\delta\);

with the property that

\[
\Psi : U \overset{1 \times f}{\longrightarrow} U \times B_\delta \overset{\phi}{\longrightarrow} V_{x_0}' \cap M,
\]

is a diffeomorphism from \(U\) onto \(V_{x_0}' \cap M\). Here \(x_0\) serves as a reference point.

More generally, fix \(x_0 \in B\), which may not belong to \(M\), and let \(V_{x_0} \subset B\) be a neighbourhood of \(x_0\).

**Definition 6.1.** A triple \((U, \phi, \Psi)\), or \((U, \phi, f)\), verifying conditions i), ii) and iii) above is called a chart data for \(M\).

**Remark 6.18.** It follows immediately from the definition, that the triples \((U_P, f^P, Gl_{S_P})\) and \((U_B, f^B, Gl_{S_B})\) give chart datas for \(\mathcal{M}_{0,l}(P)\) and \(\mathcal{M}_{0,l}(B)\). Moreover, these are compatible with \(\pi\).

In fact, one can construct chart data for \(\mathcal{M}_{S_P}(P)\) from pairs, \(Q_P := (U_P, Q^P)\), where \(U_P\) is a smooth submanifold of \(B_{S_P}^{1,p}\), and where

\[
Q^P := \{Q^P_u | u \in U_P\}
\]

is a smooth \(U_P\)-family of right inverses for \(D^P_u\). In order to do so, we assume the following conditions (that we actually met when constructing the gluing maps):

**Assumption 6.19.** • for every \(u \in U_P\),

\[
\|du\|_{L^p} \leq C_P \quad \text{and} \quad \|J_P u\|_{L^p} \leq \epsilon_P,
\]

• for all \(\xi \in T_u U_P\),

\[
\left\| \frac{d}{d\xi} J_P u \right\|_{L^p} \leq \epsilon_P \|\xi\|_{W^{1,p}},
\]

\[
\frac{d}{d\xi} J_P u \right\|_{L^p} \leq \epsilon_P \|\xi\|_{W^{1,p}},
\]
the family $Q^P_u$ is Lipschitz continuous for the constant $C_P$ and for all $u \in U_P$:

$$\|Q^P_u\| \leq C_P,$$

the constants $C_P$ and $\epsilon_P$ being such that $C_P \epsilon_P << 1$, with $\epsilon_P$ small.

The chart data is then given as follows. Fix $u_0 \in U_P$, and let $W$ be a neighbourhood of 0 in $T_{u_0}B^*_P$. Denote by $U_P$, the lift (around $u_0$) of $U_P$ in $W$ via $\exp_{u_0}^{-1}$. Then set

$$\phi_P : U_P \times \mathcal{E}^p_{U_P,u_0}(S_P) \to \mathcal{X}^1_{U_P,u_0}, \quad (\xi, \eta) \mapsto \xi + Q^P_{u_0} \eta.$$

From the assumptions above, there is a unique smooth map $f^P : U_P \to B_\delta$, around $u_0$, such that

$$\overline{\partial}_{U_P} \exp_{u_0} \phi_P(\xi, f^P(\xi)) = 0.$$

By reducing $\delta$ and the neighbourhoods involved, we further have that $\phi_P$ is diffeomorphic, hence $(U_P, \phi_P, f^P)$ is a chart data for $\mathcal{M}_{S_P}(P)$.

**Remark 6.20.** Regarding compatibility of the coordinate charts one needs to be careful, as pointed out in [21] Section 3. In fact, the $C^\infty$ compatibility is ensured if we restrict our attention to smooth stable maps, which is sufficient to study pseudoholomorphic stable maps (by elliptic regularity).

From the data $Q_P$, we can furthermore define another type of gluing map. We explain this. Suppose that $(U_P, \phi_P, f^P)$ is a chart data for a proper subset $\tilde{U}_P$ of $\mathcal{M}_{S_P}(P)$, i.e $\tilde{U}_P$ is the image of the diffeomorphism: $\Psi_P := \phi_P \circ (1 \times f^P)$. By further reducing $U_P$ if necessary, we can find a pair as above

$$Q' = (U'_P := \text{pgl}(\Psi_P^{-1}(\tilde{U}_P)), Q')$$

where $Q'$ is a family of right inverses for the elements in $U'_P$, which is constructed from the original family of right inverses $Q^P$, giving a gluing map defined by the composition:

$$G_{Q_P} : \mathcal{L}^*_{S_P}|_{\tilde{U}_P} \to \Psi_P^* \mathcal{L}^*_{S_P}|_{U_P} \xrightarrow{Gl} \mathcal{M}_{0,l}(P, \sigma).$$

Following [2], we say that $G_{Q_P}$ is admissible. Moreover, if $U_P \subset \mathcal{M}_{S_P}(P)$, we say that $G_{Q_P}$ is of type-1, otherwise we say that it is of type-2. In particular, the gluing map $Gl_{S_P}$ constructed directly from $\tilde{U}_P$ is admissible and of type-1.

Using gluing maps we introduce a topology basis on $\overline{\mathcal{M}}_{0,l}(P, \sigma)$ as follows: an open neighbourhood of $(u, j) \in \overline{\mathcal{M}}_{0,l}(P, \sigma)$ will be the image of some gluing map $G_{Q_P}$ previously constructed. Hence, a neighbourhood is given by charts data of the type $(\mathcal{L}^*_{S_P,\epsilon_P, U_P}, Gl_{S_P})$.

A standard argument shows that these charts are $C^0$ compatible [2], [17]. This can be proved by comparing any admissible gluing map $Gl_{Q_P}$ arising from a chart data $Q_P$ for a proper subset $\tilde{U}_P \subset \mathcal{M}_{S_P}(P)$, with the type-1 gluing map $Gl_{S_P}$ on $\tilde{U}_P$. Concrely, one shows that for small enough $\rho$, the map $(Gl_{S_P})^{-1}Gl_{Q_P}$ is close to the identity map, hence continuous. Thus, the moduli space $\overline{\mathcal{M}}_{0,l}(P, \sigma)$ has the structure of an orbifold in the topology given by the gluing maps.

The smooth orbifold structure is given by the two lemmas in the next subsection. In these two lemmas we prove more. Namely, we construct smooth atlases on both $\overline{\mathcal{M}}_{0,l}(P, \sigma)$ and $\overline{\mathcal{M}}_{0,l}(B, \sigma_B)$ compatibly with $\pi$. 
6.3.2. Structure of orbi-bundle. Consider now the fibration context. We begin by the following observation. Let $Q_P := (U_P, Q^P)$ projecting onto $Q_B := (U_B, Q^B)$ in the sense that $\pi_{SP}(U_P) = U_B$ and $Q^P$ is of the matrix form $\pi_{SP}$.

Also, suppose that both pairs satisfy the assumption (6.19), and that they generate charts datas, $(U_P, \phi_P, f_P)$ and $(U_B, \phi_B, f_B)$, for some proper open subsets, $U_P \subset M_{SP}(P)$ and $U_B \subset M_{SB}(B)$ such that $\pi_{SP} (\tilde{U}_P) = \tilde{U}_B$. Then, repeating the arguments in the gluing map section, we obtain that

$$\pi \circ Gl_{Q_P} = Gl_{Q_B} \circ \pi_{SP}.$$ 

This implies that the orbifold structures on the compactified moduli spaces are defined compatibly with $\pi$ which is continuous, open, and surjective, in the topology of the gluing maps. We now construct smooth atlases on both $\overline{M}_{0,l}(P, \sigma)$ and $\overline{M}_{0,l}(B, \sigma_B)$ compatibly with $\pi$. In order to do so we introduce stratum-coverings.

**Definition 6.2.** A strata-covering of $\overline{M}_{0,l}(P, \sigma)$ consists in pairs $(U_P, \epsilon_{SP})$ for each stratum data $S_P$, such that:

- $U_P$ is a proper open subset of $M_{SP}(P)$,
- there exists a well-defined gluing map $Gl_{SP}$ with domain $L_{SP, \epsilon_{SP}, U_P}$,
- letting $W_{SP}$ be the image $Gl_{SP}(L_{SP, \epsilon_{SP}, U_P})$, we have that for any two (effective) stratum datas $S_P$ and $S^\prime_P$:

$$W_{SP} \cap W_{S^\prime_P} \neq \emptyset \quad \text{iff} \quad S_P \prec S^\prime_P, \text{ or } S^\prime_P \prec S_P,$$

- the family $\{W_{SP}\}_P^{0,l} \text{ yields an open covering of } \overline{M}_{0,l}(P, \sigma)$.

**Lemma 6.21.** There exists strata-coverings $(U_P, \epsilon_{SP})$ and $(U_B, \epsilon_{SB})$ for $\overline{M}_{0,l}(P, \sigma)$ and $\overline{M}_{0,l}(B, \sigma_B)$, such that $\pi_{SP} (U_P) = U_B$.

**Proof:** The proof is an induction on the stratum datas in $D^B := D_{0,l}^{\sigma_B,JB}$ and $D^P := D_{0,l}^{\sigma,JP}$. Let $S_{B,0}$ be the set of lowest strata in $D^B$. For $S_B$ in $S_{B,0}$ set $U_B = M_{SB}(B)$. Since $S_B$ is minimal $U_B$ is compact and there exists $\epsilon_{SB}$ and a gluing map $Gl_{SB}$ defined on the restriction of $L^*_{SB, \epsilon_{SB}}$, to $U_B$. Furthermore, the minimal strata are isolated and for each $S_B \in S_{B,0}$ we can choose a small enough $\epsilon_B$ so that the resulting gluing neighbourhoods never intersect. Now let $S_{P,0,0,SB}$ be the set of lowest strata in $D^P \cap S^{-1}_{\pi^*}(D^B)$, where $S_B$ is minimal. For any $S_P \in S_{P,0,0,SB}$ set $U_P = M_{SP}(P)$. Argumenting as above, there is $\epsilon_{SP}$ and $Gl_{SP}$, with domain $L^*_{SP, \epsilon_{SP}, U_P}$, such that $\pi \circ Gl_{SP} = Gl_{SB} \circ \pi_{SP}$. Once again, we can choose the $\epsilon_{SP}$ such that $W_{SP} \cap W_{S^\prime_P} = \emptyset$, for any two strata in $S_{P,0,0,SB}$.

Define inductively $S_{B,k}$ as being the set of minimal strata in $D^B \backslash S_{B,k-1}$, and $S_{P,k,m_k}$ as being the set of minimal strata in $D^P \cap S^{-1}_{\pi^*}(S_{B,k}) \backslash S_{P,k,m_k-1}$. Suppose that each pairs $(U_B, \epsilon_{SB})$ and $(U_P, \epsilon_{SP})$, for $S_B \in S_{B,n}$ and $S_P \in S_{P,n,m_n}$ with $n \leq k - 1$ and $m_n \leq m_k - 1$, have been chosen so that the induction holds. Set

$$W_{SB} = Gl_{SB}(L_{SB, \epsilon_{SB}, U_B}).$$

Then, for $S_B \in S_{B,k}$ we can choose a proper open subset $U_B'$ such that $\{W_{SB} \cap |SB \prec S_B'\} \cup U_B'$, is a covering for $M_{SB}(B)$. Furthermore, there is $\epsilon_{SB}'$ and a gluing map

$$Gl_{SB} : L^*_{SB, \epsilon_{SB}, U_B'} \rightarrow M_{0,l}(B, \sigma_B),$$

and we can make sure that for all $S_B' \in S_{B,k}$ and $S_B \in \bigcup_{i=0}^{k} S_{B,i}$, the intersection $W_{SB}' \cap W_{SB}$ is empty unless $S_B \prec S_B'$ (by choosing smaller $\epsilon_{SB}'$ and $\epsilon_{SB}$ if necessary).
Since the gluings commute with the projection, it suffices to fix \( S_B \in S_{B,k} \), and to apply the arguments given for \( k = 0 \) to the elements of \( S_{P,k,m,k}S_B := S_{P,k,m,k} \cap S_{\pi_*}(S_B) \). Set

\[
W_{S_p',S'_p} := Gl_{S_p',S'_p}(L_{S_p',S'_p,\epsilon_{S_p},U_{S_p}}),
\]

where \( S_p' \) projects onto \( S_B \). Note that \( W_{S_p',S'_p} \) also projects onto \( W_{\pi(S_B),\pi(S_B')} \). We can choose \( U_{S_p'} \) such that \( \{ W_{S_p',S'_p} \mid \epsilon_{S_p',S_B} \} \cup U_{S_p}' \) covers \( M_{S_p'}(P) \), and for a well chosen \( \epsilon_{S_p'} \), we have a map \( Gl_{S_p'} \) which image does not intersect the neighbourhoods obtained so far unless it comes from a stratum \( S_p \) such that \( S_p \prec S_p' \).

A strata-covering gives an atlas. If the transition functions were to be smooth we would directly have a smooth orbifold structure on the considered moduli spaces. However, this may be hard to show and even not true in full generality. Instead, we show (cf \[2\]) that for each stratum data there are charts \( Gl_{S_p} \) such that the composition \( Gl_{S_p} \circ Gl_{S_B}^{-1} \) is smooth for every \( S_p' \prec S_B \), which provides, not canonically, a smooth atlas.

**Lemma 6.22.** There are strata-coverings \((U_p,\epsilon_{S_p})\) and \((U_B,\epsilon_{S_B})\), and gluing maps, \( Gl_{S_p} \) and \( Gl_{S_B} \), compatible with \( \pi_{S_p} \), such that for every stratum data \( S_p \) and \( S_B \), the maps \( Gl_{S_p} \) and \( Gl_{S_B} \) coincide with any other gluing maps

\[
Gl'_{S_p} = Gl_{S_p} \circ Gl_{S_B}^{-1}
\]

and

\[
Gl'_{S_B} = Gl_{S_B} \circ Gl_{S_B}^{-1},
\]

where \( S_p \prec S_p \) and \( S_B \prec S_B \).

**Proof:** The proof is again by induction. Let \( S_{B,k} \) and \( S_{P,k,m,k} \) as in lemma \[6.21\]. We see that the result holds for \( S_{B,0} \) and \( S_{P,0,0} \). Suppose it is true for all \( S_B \in S_{B,n} \) and \( S_p \in S_{P,n,m,n} \) such that \( n \leq k - 1 \) and \( m \leq m_k - 1 \). Let \( S_B \in S_{B,k} \) and set \( W_{S_B} := \cup S_B \prec S_B W_{S_B} \). Let \( Gl'_{S_B}(S_B') \) be the gluing map induced by \( S_B' \prec S_B \).

Recall that this map is defined above \( W_{S_B} \). We must show that

\[
(6.38)
\]

on \( W_{S_B} \). But this latter intersection is non-empty if and only if \( S_B'' \prec S_B' \).

But from the induction, \( Gl'_{S_B}(S_B'') = Gl_{S_B} \) on \( W_{S_B''} \). Thus

\[
Gl'_{S_B}(S_B') = Gl_{S_B} \circ Gl_{S_B}^{-1}
\]

\[
= Gl_{S_B} \circ Gl_{S_B}^{-1} \circ Gl_{S_B} \circ Gl_{S_B}^{-1}
\]

\[
= Gl_{S_B} \circ Gl_{S_B}^{-1} \circ Gl_{S_B}^{-1}
\]

giving (6.38). As a result, we obtain a gluing map \( Gl'_{S_B} \) defined on \( W_{S_B} \). Now, given a gluing map \( Gl''_{S_B} \) on \( U_B \), we derive a third map \( Gl_{S_B} \), which is obtained as an interpolation between \( Gl'_{S_B} \) and \( Gl''_{S_B} \) using a cut-off function. This ends the induction for \( \forall \phi(B,\sigma_B) \).

We explain in details how to interpolate \( Gl'_{S_B} \) and \( Gl''_{S_B} \. By definition \( Gl'_{S_B}(S_B') \) is of type-2 with domain:

\[
W_{S_B} := Gl_{S_B}(L_{S_B',S_B',U_B}) \equiv W_{S_B}(\epsilon_{S_B}).
\]

For \( Gl_{S_B} \) admissible, the associated chart data is a triple \((V := pgl(L_{S_B',S_B',U_B}),\phi,F)\) where:

\[
F: V \to M_{S_B}(B), \quad (u_B,\rho,j_B) \mapsto \exp_{u_B,\rho}^B Q_{u_B,\rho} f_{S_B}(u_B,j_B,\rho).
\]
Consider a function
\[ \nu : L_{FB}(S'_B, F_B(S_B), \epsilon S'_B) \to \mathbb{R}, \quad \nu(j, \rho) \mapsto \begin{cases} 0 & \text{if } |\rho| \leq 0.5 \epsilon S'_B \\ 1 & \text{if } |\rho| \geq 0.75 \epsilon S'_B. \end{cases} \]
We now “glue” the domains of the chart datas for \( GL^p_{S_B} \) and \( GL^l_{S_B} (S'_B) \), i.e we glue \( U_B \) and \( V \). The new domain
\[ V' := \text{Im} \left( \exp_{u_B, \rho} \left( \nu(j, \rho) q_{u_B, \rho} \right) s_B(u_B, j, \rho) \right), \]
coincides with \( V \) for \(|\rho| \leq 0.5 \epsilon S'_B \), and with \( U_B' \) when \(|\rho| \geq 0.75 \epsilon S'_B \). We can make sure that the pair, \( (V', \{ Q_{u_B} | u_B \in V' \}) \), satisfies hypothesis \((6.19)\) since \( V' \) is a uniform deformation between \( V \) and \( U_B \) (\( \nu \) does not depend on \( u_B \)). Denote by \( GL_{S_B} \) the gluing map arising from the pair \( (V', \{ Q_{u_B} | u_B \in V' \}) \). Then, \( GL_{S_B} \) coincides with \( GL^l_{S_B} (S'_B) \) on \( V \cap V' = W_{S_B, S_B} (0.5 \epsilon S'_B) \), and with \( GL^p_{S_B} \) on \( U_B \cap V' \). In particular, this map extends to \( U_B \backslash W_{S_B, S_B} (\epsilon S'_B) \). Regarding \( \overline{M}_{0, l}(P, \sigma) \) we proceed similarly. Fixing \( S_B \in S_B, k \), and applying the same arguments as above to the elements \( S_P \in S_{P, k, m, S_B} \), ends the proof. \( \square \)

6.4. The product formula revisited. In what follows, we will assume hypothesis \((6.2)\) is verified for a given fibered structure \( J_P = (J_B, J, H) \) and classes \( \sigma, \sigma_B \neq 0 \) such that \( \pi_* \sigma = \sigma_B \). Hence, \( \overline{\pi} \) is fibration when restricted to the top stratum of \( \overline{M}_{0, l}(B, \sigma_B) \). In addition, the lower strata in \( \overline{M}_{0, l}(B, \sigma_B) \) and \( \overline{M}_{0, l}(P, \sigma) \) are of codimension at least two. The same applies for the lower strata in \( \overline{\pi^{-1}}(u_B, x) \), for every \((u_B, x) \in \overline{M}_{0, l}(B, \sigma_B) \).

Proposition 6.23. Under the assumptions above the product formula is obtained using integration over the fibers of \( \overline{\pi} \).

Proof: Set
\[ M^p_{0, l} := M^p_{0, l}(P, \sigma) \quad \text{and} \quad M^b_{0, l} := M^b_{0, l}(B, \sigma_B), \]
and consider the evaluation maps,
\[ ev^p : M^p_{0, l} \to P^l, \quad ev^b : M^b_{0, l} \to B^l, \quad ev_{(u_B, x)} : \overline{\pi^{-1}}(u_B, x) \to F^l, \]
where \((u_B, x) \in M^b_{0, l}\). These are smooth with respect to the gluing topology.

Now, let \( c_i^p, c_i^b \) and \( c_i^F \) (\( i = 1, \ldots, m \)) be classes as in condition \((6.3)\). We can represent these classes by submanifolds after multiplying them by well chosen integers, if necessary. Now represent the Poincaré duals of \( c_i^p, c_i^b \) and \( c_i^F \), by differential forms, \( \alpha_i^p, \alpha_i^b \), and \( \alpha_i^F \), compactly supported in a small enough tubular neighbourhoods around the submanifolds. Then the pull-backs of these forms along \( ev^p, ev^b \) or \( ev_{(u_B, x)} \), are also compactly supported. By definition,
\[ \alpha_i^p = \pi^*(\alpha_i^b), \quad i = m + 1, \ldots, l. \]
Moreover, since \( c_i^b = pt \):
\[ \alpha_i^b = \text{vol}(B), \quad i = 1, \ldots, m. \]
If \( N_i \) denotes a tubular neighbourhood around the fiber above \( c_i^b = pt \), for \( i = 1, \ldots, m \), and if \( \rho_i : N_i \to F \) is a deformation retract associated to this normal neighbourhood, we obtain that:
\[ \alpha_i^p = \pi^* \rho_i^* \alpha_i^b = \pi^* \text{vol}(B) \wedge \rho_i^* \alpha_i^F. \]
Here we have to make sure that the support of $\pi^* \text{vol}(B)$ does not strictly contain the support of $\rho_1^* \alpha_1^F$, but this can be realized by decreasing the support of $\text{vol}(B)$ if necessary. By definition, (see [20]) we have that:

$$
\langle c_1^P, ..., c_l^P \rangle_{0,l,\sigma} = \int_{\mathcal{M}_{0,l}^P} (ev_P)^* \left( \bigwedge_{i=1}^l \alpha_i^P \right) = \int_{\mathcal{M}_{0,l}^P} \overline{\pi}_*(ev_P)^* \left( \bigwedge_{i=1}^l \alpha_i^P \right),
$$

where $\overline{\pi}_*$ stands for integration along the fibers. Using (6.39), the equation above then equals:

$$
\int_{\mathcal{M}_{0,l}^P} \overline{\pi}_*(ev_P)^* \left( \bigwedge_{i=1}^k (\pi^* \alpha_i^B \wedge \rho_i^* \alpha_i^F) \right) \wedge \left( \bigwedge_{i=k+1}^l \pi^* \alpha_i^B \right).
$$

Let $ev_i^P$ be the projection of $ev_P$ on the $i$th component of $P$, and define $ev_i^B$ and $ev_{(u_B, x), i}$ similarly. For $i = 1, ..., l$, we have $\pi \circ ev_i^P = ev_i^B \circ \overline{\pi}$ so that:

$$
\langle c_1^P, ..., c_l^P \rangle_{0,l,\sigma} = (-1)^{\alpha} \int_{\mathcal{M}_{0,l}^P} \overline{\pi}_* \left( \bigwedge_{i=1}^l \pi^*(ev_i^B)^* \alpha_i^B \wedge \bigwedge_{i=1}^k (ev_i^P)^* \rho_i^* \alpha_i^F \right),
$$

where $\alpha = \sum_{i=k+1}^l \deg \alpha_i^B \sum_{i=1}^k \deg \alpha_i^F$ is odd (by a simple dimension argument). Furthermore, since $\pi_*(a \wedge \pi^* b) = (\pi_*(a) \wedge b)$, for any form $a$ and $b$ we finally have:

$$
\langle c_1^P, ..., c_l^P \rangle_{0,l,\sigma} = \int_{\mathcal{M}_{0,l}^P} \left( (ev_i^B)^* \bigwedge_{i=1}^l \alpha_i^B \right) \wedge \overline{\pi}_* \left( \bigwedge_{i=1}^k (ev_i^P)^* \rho_i^* \alpha_i^F \right),
$$

where the term involving integration over the fibers of $\pi$ must be a function $\psi$ on $\mathcal{M}_{0,l}^P$ given by (cf [5.2]):

$$
\psi(u_B, x) = \int_{\overline{\pi}^{-1}(u_B, x)} ev_s^*(u_B, x) \left( \bigwedge_{i=1}^k \alpha_i^F \wedge \bigwedge_{i=k+1}^l \text{vol}(F) \right) = n_\alpha
$$

where $n_\alpha$ is given in [5.4]. Hence, $\psi$ does not depend on $(u_B, x)$ and we can withdraw it from the integral. This ends the proof. \hfill \Box

6.5. **An example of non-triviality of the induced fibration.** Consider the Hamiltonian fibration $\pi : P \rightarrow \mathbb{CP}^2$ with fiber $(F := \mathbb{CP}^1, \omega_{FS})$, where $P$ is the projectivization of the rank 2 holomorphic vector bundle,

$$
\pi : V := \mathcal{O}_{\mathbb{CP}^2}(1) \oplus 
$$

Let $J_0$ be the standard complex structure on $\mathbb{CP}^2$ which is compatible with $\omega_{FS}$. Let $J_P$ be the integrable structure on $P$ induced by $J_0$, the structure of complex fibration on $V$ and the holomorphic hermitian connection on $V$ (inducing the coupling form here.).

Let $h \in \mathcal{H}^2(P; \mathbb{Z})$ denote the pull-back under $\pi$ of the positive generator in $\mathcal{H}^2(\mathbb{CP}^2, \mathbb{Z})$, Poincaré dual to the class $L \in \mathcal{H}_2(\mathbb{CP}^2, \mathbb{Z})$ of a line in $\mathbb{CP}^2$. Let also $\xi \in \mathcal{H}^2(P; \mathbb{Z})$ be the first Chern class of the dual of the tautological bundle over $P$. It is standard that:

$$
\mathcal{H}^*(P; \mathbb{Z}) \cong \frac{\mathbb{Z}[h, \xi]}{\{ h^3 = 0, \xi^2 + h \xi = 0 \}}.
$$

Then, $\mathcal{H}_2(P; \mathbb{Z})$ is generated by the duals of the classes $h^2$ and $h \xi$. Let $L_0 \in \mathcal{H}_2(P; \mathbb{Z})$ denote the Poincaré dual of $h^2 + h \xi$. If $\pi_*$ represents integration along the fibers of $\pi$, then $\pi_* L_0 = L$. 


The map \( \pi \) induces the projection
\[
\pi : \overline{\mathcal{M}}(P, L_0, J_F) \to \overline{\mathcal{M}}(\mathbb{CP}^2, L, J_0) \cong (\mathbb{CP}^2)^*.
\]
The source moduli space is made of two strata: \( S_0 \) the top stratum of simple maps, and \( S_1 \) the stratum consisting of stable maps having two components, one being a \( \pi_* \)-stable component representing the class of the exceptional divisor, \( PD(h\xi) \), and the other being a \( \pi_* \)-unstable component representing the class \( PD(h^2) = [F] \).
Observe that the second stratum contains only irreducible elements.

**Lemma 6.24.** The fibered complex structure \( J_F \) is regular and parametric for the class \( L_0 \).

**Proof:** First, since \( L \) is \( J_0 \)–indecomposable, \( J_0 \) belongs to \( J_{irr}(L) \). Consider \( u \in S_0 \).
From the exact sequence (3.4), we only have to verify that \( \text{coker } D_u = 0 \), but
\[
\text{coker } D_u = H^{0,1}(\mathbb{CP}^2, u^*TP) = H^{0,1}(\mathbb{CP}^2, O_{\mathbb{CP}^2}(1)) \cong H^0(\mathbb{CP}^2, O(-3)) = 0.
\]
Now consider \((u_1, u_2, y_{12}, y_{21}) \in S_1 \), where \( u_1 \) denotes the \( \pi_* \)-stable component, and \( u_2 \) denotes the \( \pi_* \)-unstable component. Again, we have to check that \( \text{coker } D_{u_1} \) and \( \text{coker } D_{u_2} \) vanish. Moreover, we have to show that for every line \( \ell \in (\mathbb{CP}^2)^* \) the edge evaluation map
\[
ev : \pi^{-1}(\ell) \cap S_1 \to \pi^{-1}(b) \times \pi^{-1}(b), \quad (u_1, u_2, y_{12}, y_{21}) \mapsto \left(u_1, (y_{12}, u_1(y_{21}))\right),
\]
is transverse to the diagonal \( \Delta_{\pi^{-1}(b)} \), where \( b := \pi(\overline{b}) \). Again, for every holomorphic map representing \([F]\), \( \text{coker } D_{u_2} \) can be identified to \( H^{0,1}(\mathbb{CP}^2, T\mathbb{CP}^1) \) which vanishes, and similarly,
\[
\text{coker } D_{u_1} = H^{0,1}(\mathbb{CP}^2, u_1^*TP) = H^{0,1}(\mathbb{CP}^2, O(-1)) = 0.
\]
Finally, transversality of \( ev \) follows since for every \( X_0 \in T_{\pi^{-1}(b)} \) there exists a holomorphic vector field on \( \mathbb{CP}^1 \cong \pi^{-1}(b) \) with value \( X_0 \) at \( \overline{b} \). \( \square \)

Let \( \mathbb{CP}^2 \) denote the blow-up of \( \mathbb{CP}^2 \) at one point. Then the restriction \( P|_\ell \) of \( P \) to a line \( \ell \) in \( \mathbb{CP}^2 \) is identified to \( \mathbb{CP}^2 \). The fibers of \( \pi \) are then given by \( \overline{\mathcal{M}}(\mathbb{CP}^2, L_0, J) \) where \( J \) is the complex structure associated to the Hirzebruch surface \( \mathbb{P}(O_{\mathbb{CP}^1}(1) \oplus \mathbb{C}) \), and \( L_0 \) is the class represented by the zero section. Furthermore,
\[
\mathcal{M}^* \cong H^0(\mathbb{CP}^1, O_{\mathbb{CP}^1}(1)) \cong \mathbb{C}\langle u, v \rangle,
\]
where \([u : v]\) stands for the homogeneous coordinates on \( \mathbb{CP}^1 \). The Gromov closure then corresponds to adding the line at infinity so that
\[
\overline{\mathcal{M}}(\mathbb{CP}^2, L_0, J) = \mathbb{CP}^2.
\]
Next, we show that \( \pi \) is non-trivial, more precisely that it is a \( \mathbb{CP}^2 \)-fibration obtained as the projectivization of a non-trivial rank two holomorphic bundle over \( (\mathbb{CP}^2)^* \). Consider the incidence variety:
\[
W := \left\{(p, \ell) \in \mathbb{CP}^2 \times (\mathbb{CP}^2)^* \mid p \in \ell \right\},
\]
and let \( \pi_1, \pi_2 \) denote the projections on the first and second factors. Note that \((W, \pi_2) \) is the projectivization of \( O_{(\mathbb{CP}^2)^*}(-1) \oplus \mathbb{C} \). Consider the direct image sheaf over \((\mathbb{CP}^2)^* \),
\[
\mathcal{R} := \pi_2^* \pi_1^* O_{\mathbb{CP}^2}(1),
\]
which germ at \( \ell \in (\mathbb{CP}^2)^* \) is given by
\[
H^0(\pi_2^{-1}(\ell), O_{\mathbb{CP}^1}(1)|_{\pi_2^{-1}(\ell)}) \cong H^0(\mathbb{CP}^1, O_{\mathbb{CP}^1}(1)),
\]
Hence \( R = S_0 \). Let \( D \) be a line in \((\mathbb{CP}^2)^*\). Then \( \pi_2^{-1}(D) \) can be identified to \( \widetilde{\mathbb{CP}^2} \), where the blown-up point is given by the intersection of all the lines generated by \( D \). In fact,

\[
\pi_2^{-1}(D) \cong \frac{D^+ \times \mathbb{CP}^1 \cup D^- \times \mathbb{CP}^1}{(\lambda, [u : v]) \sim (\lambda^{-1}, [\lambda u : v]), \ \lambda \neq 0},
\]

where \( D^+ \) and \( D^- \) respectively denote the complements of \([0 : 1]\) and \([1 : 0]\) in \( \mathbb{CP}^1 \). Hence, the restriction of \( R \) to \( D \) is the direct sum of \( \mathcal{O}_{\mathbb{CP}^1}(-1) \) with \( \mathbb{C} \), so that \( \pi \) is non-trivial.

As an example, we compute

\[
\langle pt, pt \rangle^P_{0, 2, L_0}.
\]

Since the homology of the fiber injects in the homology of the total space, the product formula simplifies to give:

\[
\langle pt, pt \rangle^P_{0, 2, L_0} = \langle pt, pt \rangle^{\mathbb{CP}^2}_{0, 2, L} \cdot \langle pt, pt \rangle^{P_C}_{0, 2, B}
\]

where \( C \) is the image of a holomorphic map in \( \mathbb{CP}^2 \) representing a line passing through two points, and \( B \) is the Poincaré dual of the restriction of \( h^2 + h\xi \) to \( P_C \). This latter class corresponds to the sum of the fiber and the exceptional divisor in the blow up. It follows that both members of the above product give 1, hence

\[
\langle pt, pt \rangle^P_{0, 2, L_0} = 1.
\]

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