THE MINIMAL HIDDEN COMPUTER NEEDED TO IMPLEMENT A VISIBLE COMPUTATION

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Conditional distribution of state at $t = 1$ given state at $t = 0$

- Visible states arrayed along black lines
Visible states arrayed along black lines
Hidden states arrayed along dotted purple lines
Visible states arrayed along black lines
Hidden states arrayed along dotted purple lines
Each “column” of arrows = one timestep
More hidden states allows fewer timesteps
ROADMAP

Overview

How to erase a bit: continually-embeddable matrices

How to flip a bit - hidden states

How to flip a bit - hidden timesteps

Tradeoff of numbers of hidden states and timesteps
Consider a physical system implementing a conditional distribution \( \pi(x_1 | x_0) \) over "visible" state space \( X \) (\( X \) finite, time units arbitrary)

- \( \pi \) observed to govern some naturally occurring system

- \( \pi \) constructed by human engineers

**Example:** Update function of a (re-usable) gate in a digital circuit (a single-valued conditional distribution)

**Example:** Update function of an entire (re-usable) digital computer

Will mostly focus here on \( \pi \) that are functions (like gates)
Consider a physical system implementing a conditional distribution \( \pi(x_1 \mid x_0) \) over a visible state space \( X \).

“Implement” here means the dynamics for \( t \in [0, 1] \) is described by a (time-inhomogeneous) continuous-time Markov chain (CTMC).

- This encompasses stochastic thermodynamics
- However our results are even more general

Example: Stochastic thermodynamic analysis of flipping a bit stored as state of a quantum dot.
Consider a system governed by a CTMC that implements a conditional distribution $\pi(x_1 \mid x_0)$ over a “visible” state space $X$.

We prove that for many $\pi$’s, the CTMC must actually evolve over a space including “hidden” states $Z$, in addition to the visible states $X$.

More precisely:
- For many $\pi(x_1 \mid x_0)$, any CTMC implementing $\pi$ over $X$ must actually evolve across some space $X \cup Z$.
- $\pi(x_1 \mid x_0)$ is the restriction to $X$ of the CTMC over $X \cup Z$. 

**HIDDEN STATES**
For many π’s, the CTMC must actually evolve over a space including hidden states Z, in addition to the visible states X.

**Bit flip example:**
- \( X = \{0, 1\} \)
- Start in either state 0 or 1
For many $\pi$’s, the CTMC must actually evolve over a space including hidden states $Z$, in addition to the visible states $X$.

- **Bit flip example:**
  - $X = \{0, 1\}$
  - Start in either state 0 or 1

![Diagram showing bit flip example with states 0 and 1 and transitions: 1 → 0 and 0 → 1]
 Bit flip example:
- \( X = \{0, 1\}, Z = \{2\} \)
- Start in either state 0 or 1

For many \( \pi \)'s, the CTMC must actually evolve over a space including hidden states \( Z \), in addition to the visible states \( X \).
HIDDEN STATES

For many $\pi$'s, the CTMC must actually evolve over a space including hidden states $Z$, in addition to the visible states $X$

- Bit flip example:
  - $X = \{0, 1\}, Z = \{2\}$
  - Start in either state 0 or 1
• Consider a system governed by a CTMC that implements a conditional distribution $\pi(x_1 \mid x_0)$ over a visible state space $X$.

There is a natural way to view any CTMC as dividing $t \in [0, 1]$ into a countable number of contiguous intervals.

• I.e., any CTMC taking $x_0$ to $\pi(x_1 \mid x_0)$ runs through a sequence of “hidden timesteps” within $[0, 1]$.

• Often there is a cost to any engineer constructing a CTMC to implement $\pi$, which increases with the number of hidden timesteps.
**Example:** To implement a bit flip requires at least *three hidden timesteps*

**Bit flip example:**
- $X = \{0, 1\}$, $Z = \{2\}$
- Start in either state 0 or 1

```
$\begin{array}{c}
1 \rightarrow 0 \\
0 \\
1 \\
2 \\
\end{array}$
```

```
$\begin{array}{c}
0 \rightarrow 1 \\
0 \\
1 \\
2 \\
\end{array}$
```
Consider a system governed by a CTMC that implements a conditional distribution $\pi(x_1 \mid x_0)$ over a visible state space $X$.

There is a natural way to view any CTMC as dividing $t \in [0, 1]$ into a countable number of contiguos intervals.

In general, for any $\pi(x_1 \mid x_0)$, the more hidden states a CTMC can use, the fewer hidden timesteps it needs to implement $\pi$.

I.e., a tradeoff between number of hidden states and minimal number of hidden timesteps needed to implement $\pi$ with a CTMC.

This tradeoff depends on the details of $\pi$. 
ROADMAP

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*How to erase a bit: continually-embeddable matrices*

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Tradeoff of numbers of hidden states and timesteps
**PAST WORK ON EMBEDDING**

- Given any $\pi(x_1 \mid x_0)$, the **embedding problem** is to determine if there is a CTMC with rate matrix $R_{x,x'}(t)$ such that $\text{OE}(R)[1] = \pi(x_1 \mid x_0)$

- First studied by Kingman (1962) who derived necessary and sufficient conditions for any $\pi(x_1 \mid x_0)$ to be embeddable **for binary $X$**, by a time-homogeneous CTMC

- We still do not know necessary and sufficient conditions for larger $X$, even for time-homogeneous CTMCs
PAST WORK ON EMBEDDING

• Given any $\pi(x_1 \mid x_0)$, the embedding problem is to determine if there is a CTMC with rate matrix $R_{x,x'}(t)$ such that $\text{OE}(R)[1] = \pi(x_1 \mid x_0)$.

• Goodman (1970) derived necessary conditions for $\pi$ to be embeddable by a (time-\textit{inhomogeneous}) CTMC for arbitrary finite $X$.

• In particular,

**If $\det \pi \leq 0$, $\pi$ cannot be embedded by any CTMC**

• Intuition: For any time-varying $R_{x,x'}(t)$,

$$\det \pi = e^{\int_0^1 dt \text{ Tr} R(t)} > 0$$

• Lencastre et al. (2016) is a nice review.
CONTINUALLY-EMBEDDABLE $\pi$

If $\det \pi \leq 0$, $\pi$ cannot be embedded by any CTMC

- But ... bit erasure is the stochastic matrix $\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$

- This has determinant 0, and yet many physical systems erase bits.
  
  $\text{??}$

- **Intuitive Solution**: A “quasi-static” CTMC, that is arbitrarily close to bit erasure.
  
  (So the determinant of the matrix $\pi$ that it implements is infinitesimal - but positive)
CONTINUALLY-EMBEDDABLE $\pi$

If $\det \pi \leq 0$, $\pi$ cannot be embedded by any CTMC

- Intuitive Solution: A “quasi-static” CTMC, that is arbitrarily close to bit erasure (so determinant is infinitesimal – but positive)

- Formally: $\pi$ is \textit{continually-embeddable} if $\exists$ sequence of CTMCs with transition matrices $\{T^{(n)}(t, t'): n = 1,2,...\}$ such that
  
  1) $T(t, t')$ is continuous in $t$ and $t'$ for all $t, t' \in [0, 1]: t < t'$
  2) $\pi = T(0, 1)$

  where for all $t, t' \in [0, 1]$, $T(t, t') = \lim_{n \to \infty} T^{(n)}(t, t')$
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Tradeoff of numbers of hidden states and timesteps
HIDDEN STATES

Theorem: Any noninvertible function (like bit erasure) is continually embeddable

Theorem: No invertible function (except the identity) is continually-embeddable

- Intuition: bit-flip is the matrix \[
\begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix}
\]
  with determinant -1.
  - This is not infinitesimally close to a positive determinant

- So bit-flip is not continually-embeddable

- But many physical systems flip bits (not to mention perform more complicated invertible maps)
Illustration of flipping a bit:
- Visible states $X = \{0, 1\}$, hidden states $Z = \{2\}$
- Start in either (visible) state 0 or 1

Each step is noninvertible, and so continually-embeddable - but not continually-embeddable over $X \cup Z$.

Not continually-embeddable over $X$
Overview

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Example of flipping a bit:

1 → 0

0 → 1

If construct a CTMC with transition matrix $T(t, t')$ to do this, you find that at transitions from column 1 to 2 and from column 2 to 3, a term in $T(t, t')$ changes from 0 to nonzero or vice-versa.
Example of flipping a bit:

1 → 0
0
1
2

0 → 1
0
1
2

In conventional single heat-bath stochastic thermodynamics, such changes typically correspond to changing energy gaps from being \textit{infinite} to being \textit{finite} or vice-versa.
Example of flipping a bit:

1 → 0

Often difficult for an engineer to construct a system whose transition matrix has terms that change from zero to nonzero or vice-versa. 

*Treat number of such changes as a cost*
Example of flipping a bit:
- Visible states $X = \{0, 1\}$, hidden states $Z = \{2\}$
- Start in either (visible) state 0 or 1

3 successive idempotent functions
- Is that fewest possible? I.e., does \textit{timestep cost} = 3?
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Tradeoff of numbers of hidden states and timesteps
**Theorem:** Timestep cost of noninvertible $\pi$ is minimal number of idempotent functions whose product is $\pi$

- Analyzing a formally identical semigroup theory question, Saito (1989) showed that this cost is either

$$\left\lfloor \frac{|X| + \text{cycl} \, \pi - \text{fix} \, \pi}{|X| - |\text{img} \, \pi|} \right\rfloor$$

or 1 more than this, where:

- $\text{cycl}(\pi)$ is number of (invertible) cyclic orbits of $\pi$
- $\text{fix}(\pi)$ is number of fixed points of $\pi$
- $\text{img}(\pi)$ is size of image of $X$ under $\pi$
Theorem: Timestep cost of noninvertible $\pi$ is minimal number of idempotent functions whose product is $\pi$

- Simple extension of this result to allow $k$ hidden states, and include invertible $\pi$: Timestep cost is either

$$\left[ \frac{k + |X| + \max (\text{cycl}(\pi) - k, 0) - \text{fix}(\pi)}{k + |X| - |\text{img}(\pi)|} \right]$$

or 1 more than this
EXAMPLE: BIT FLIP

- Timestep cost of $\pi$ with $k$ hidden states:

$$\left\lceil \frac{k + \lvert X \rvert + \max (\text{cycl} (\pi) - k, 0) - \text{fix} (\pi)}{k + \lvert X \rvert - \lvert \text{img} (\pi) \rvert} \right\rceil$$

- $|X| = 2$, $\text{cycl}(\pi) = 1$, $\text{fix}(\pi) = 0$, $\text{img}(\pi) = 2$, $k = 1$

- So timestep cost = 3 – three successive steps is smallest possible.
EXAMPLE – Maps over 4 bits

- Space/time trade-off for two functions over $X = \{0, \ldots, 15\}$:
  - ‘Cycle’ is $x \rightarrow x + 1 \mod 16$.
  - ‘Complement’ represents each element of $X$ as a four-bit string and then applies bitwise NOT.
REALISTIC SETS OF IDEMPOTENT FUNCTIONS

• Analysis so far assumes can use arbitrary idempotent functions

• In real world, severe constraints on set of idempotent functions we can build into our devices

• Ex: X is all bit strings of length 128
  - Number of possible idempotent functions lower-bounded by the number of partitions of X, i.e., the Bell number of $2^{128}$
  - This is huge — so results above, which assume we can use all those functions are not appropriate for such an X

How does analysis change with realistic constraints on set of idempotent functions we can use?
REALISTIC SETS OF IDEMPOTENT FUNCTIONS

Ex: X is all bit strings of length 128

- Suppose only *two types of idempotent function* we can use:
  - Functions that work on one spin (bit) at a time
  - Functions that work on two spins (bits) at a time

- The set of all such functions includes all logical ANDs, NOTs or ORs of individual bits

- So can implement any Boolean function of x by a sequence of such idempotent functions

- Calculating timestep cost with k hidden states similar to circuit complexity, *but different*
CONCLUSIONS

• Derived a novel “hidden” space/time trade-off applicable to all continuous-time Markov chains

• **Physical meaningful** as minimal “costs” of any stochastic thermodynamic process that implements a given function

• Unlike traditional costs in thermodynamics of computation, these new ones involve *state-space resources* and *timestep resources*

• Can extend to non-single-valued (“stochastic”) $\pi$ (another talk).

• Space / time tradeoffs of **a single gate** within an overall circuit of many gates... that is *itself* subject to space / time tradeoffs...

• Lots of future work!