Criteria for system-environment entanglement generation for systems of any size in pure-dephasing evolutions

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An evolution between a system and its environment which leads to pure dephasing of the system may either be a result of entanglement building up between the system and the environment or not (the second option is only possible for initially mixed environmental states). We find a way of distinguishing between an entangling and non-entangling evolutions for systems which are larger than a single qubit. The generalization of the single qubit separability criterion to larger systems is not sufficient to make this distinction (it constitutes a necessary condition of separability). A set of additional conditions for the operators describing the evolution of the environment depending on the state of the system is required. We find that the commutation of these environmental operators with the initial state of the environment does not guarantee separability, products of the operators need to commute among themselves for a pure dephasing evolution not to be accompanied by system-environment entanglement generation. This is a qualitative difference with respect to the single-qubit case, since it allows for a system to entangle with an initially completely mixed environment.

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I. INTRODUCTION

The detection of system-environment entanglement (SEE) is an involved problem both theoretically and experimentally once the size of either the system or the environment exceeds a couple of qubits [1–5]. In general the quantification of entanglement requires the knowledge of the whole system-environment density matrix. For open quantum systems with large environments, this is easier to have in theory than in experiment, but is often also challenging theoretically, since many of the standard approximate methods are inapplicable in this case [6–9], since their aim is to provide the evolution of the system of interest alone. Only then can the numerical calculation of an entanglement measure be performed (one that can be found from a density matrix for larger systems, such as Negativity [10–12]). Hence, the studies of entanglement build-up between a system and its environment are rather limited in literature [13–19], and although they provide insight into the workings of a given open system, they can be used to draw more general conclusions in a limited extent.

If the class of systems under study is reduced to such systems for which the interaction with the environment can only lead to pure dephasing of the system of interest, an effective theoretical tool to distinguish entangling and non-entangling evolutions has recently become available, as long as the system is only a qubit [20]. We first extend the results of Ref. [20] and find that qubit-like criteria for system-environment separability are not sufficient to distinguish between entangling and non-entangling system-environment evolutions. These criteria bind the initial state of the environment to operators which describe the evolution of the environment conditional on the state of the system. We find that, if any of these criteria is not satisfied then entanglement is definitely present, but there exist entangled states for which all of the qubit-like criteria are fulfilled. Furthermore, for larger systems it is possible for an initially fully mixed environment to entangle with the qubit during their joint pure-dephasing evolution, a phenomenon, which is impossible in the case of the qubit.

In fact, for larger systems a completely different set of separability criteria is necessary in addition to the qubit-like conditions, to fully separate entangled states obtained during the evolution from non-entangled ones. We find the second set of criteria, which are qualitatively different from the qubit-like conditions, as they are defined on the conditional evolution operators of the environment alone.

The paper is organized as follows. In Sec. II we describe, what is meant by “pure dephasing evolutions” in terms of the system-environment Hamiltonian and write the operators which govern the joint evolution of the system and environment in this case. In Sec. III we recount the results of Ref. [20] concerning the generation of qubit-environment entanglement, and find a convenient and general explicitly separable form of any system-environment density matrix which fulfills the separability criterion. Sec. IV is devoted to the study of a qutrit interacting with an environment of any size and introduces the additional separability criteria for larger systems. In Sec. V a full set of separability criteria for both a system of any size and an environment of any size is introduced. Sec. VI concludes the paper.
II. PURE DEPHASING EVOLUTIONS

In the study of the generation of SEE (or to be precise, the distinction between entangling and non-entangling evolutions) for systems which are larger than a qubit, we remain in the same framework as in the single qubit study [20], meaning that we study a class of Hamiltonians for which the interaction with the environment leads to pure dephasing of the system of interest (after tracing out the environmental degrees of freedom). For systems of dimension \( N \) and an unspecified, arbitrary size of the environment, the class of Hamiltonians is

\[
\hat{H} = \sum_{k=0}^{N-1} \varepsilon_k |k\rangle\langle k| + \hat{H}_E + \sum_{k=0}^{N-1} |k\rangle\langle k| \otimes \hat{V}_k, \tag{1}
\]

where the first term on the right describes the free Hamiltonian of the system in its eigenbasis \( \{|k\rangle\} \), \( \hat{H}_E \) is an arbitrary free Hamiltonian of the environment, and the last term describes the interaction, which has to be diagonal in the basis \( \{|k\rangle\} \) of the system, while the environmental operators \( \hat{V}_k \) are also arbitrary.

Since the free Hamiltonian of the system commutes with all other terms of the Hamiltonian and local unitary operations do not affect the amount of SEE, in the following we will always implicitly write the evolution of the system (and the corresponding evolution operators) without the unitary oscillations of the system which result from this first term of the Hamiltonian. Hence, the density matrix of the system and environment under consideration is in fact of the form \( \hat{U}_S(t) \hat{\sigma}(t) \hat{U}_S(t)^\dagger \) with

\[
\hat{U}_S(t) = \exp \left( -\frac{i}{\hbar} \sum_{k=0}^{N-1} \varepsilon_k |k\rangle\langle k| t \right) = \sum_{k=0}^{N-1} e^{-i \varepsilon_k t} |k\rangle\langle k|.
\]

This does not affect any of the later conclusions drawn with respect to SEE.

The evolution operator for the system and environment (without the local unitary oscillations of the system) is of the form

\[
\hat{U}(t) = \sum_{k=0}^{N-1} |k\rangle\langle k| \otimes \hat{\omega}_k(t), \tag{2}
\]

with the evolution operator of the environment conditional on the state of the system given by

\[
\hat{\omega}_k(t) = \exp \left( -\frac{i}{\hbar} (\hat{H}_E + \hat{V}_k) t \right). \tag{3}
\]

We assume that the system and environment are initially in a product state, with the system in a pure state \( |\psi\rangle = \sum_{k=0}^{N-1} c_k |k\rangle \), while there are no limitations on the initial state of the environment, \( \hat{R}(0) \). Hence the time evolution of the combined system-environment state is given by

\[
\hat{\sigma}(t) = \hat{U}(t) \left( |\psi\rangle\langle\psi| \otimes \hat{R}(0) \right) \hat{U}_S(t)^\dagger = \sum_{k,l=0}^{N-1} c_k c_l^* |k\rangle\langle l| \hat{R}_{kl}(t), \tag{4}
\]

where

\[
\hat{R}_{kl}(t) = \hat{\omega}_k(t) \hat{R}(0) \hat{\omega}_l^\dagger(t). \tag{5}
\]

III. SEPARABILITY FOR A QUBIT

Let us first look at the situation, when our system is a qubit, \( N = 2 \), following the results of Ref. [20]. In this case the “if and only if” condition for separability at time \( t \) is \( [\hat{R}(0), \hat{\omega}_0(t) \hat{\omega}_1(t)] = 0 \) [20], or, equivalently, \( \hat{R}_{00}(t) = \hat{R}_{11}(t) \). Hence, the density matrix of an evolving system and environment which is separable at time \( t \) can always be written as

\[
\hat{\sigma}(t) = \left( \begin{array}{cc} |c_0|^2 \hat{R}_{00}(t) & c_0 c_1^* \hat{R}_{01}(t) \\ \bar{c}_0 c_1 \hat{R}_{10}(t) & \bar{c}_1^2 \hat{R}_{00}(t) \end{array} \right). \tag{6}
\]

Here we use a notation, in which the matrix is written in terms of system states, while the environmental degrees of freedom are contained in the operators \( \hat{R}_{ij}(t) \), which would be explicitly written in Dirac notation. To see that the form of the density matrix given by eq. (6) actually guarantees qubit-environment separability, it is best to transform the off-diagonal environmental matrices \( \hat{R}_{ij}(t) \), \( i \neq j \), in the following way,

\[
\hat{R}_{01}(t) = \hat{\omega}_0(t) \hat{R}(0) \hat{\omega}_1(t) = \hat{R}_{00}(t) \hat{\omega}_0(t) \hat{\omega}_1(t),
\]

\[
\hat{R}_{10}(t) = \hat{\omega}_1(t) \hat{R}(0) \hat{\omega}_0(t) = \hat{R}_{11}(t) \hat{\omega}_1(t) \hat{\omega}_0(t) = \hat{R}_{00}(t) \hat{\omega}_1(t) \hat{\omega}_0(t). \tag{7}
\]

Note, that this procedure can be performed in such a way that the \( \hat{\omega}_k(t) \) operators are left over on the left side, leading to

\[
\hat{R}_{01}(t) = \hat{\omega}_0(t) \hat{\omega}_1(t) \hat{R}_{00}(t),
\]

\[
\hat{R}_{10}(t) = \hat{\omega}_1(t) \hat{\omega}_0(t) \hat{R}_{00}(t),
\]

which shows that the following commutation relations are true

\[
\left[ \hat{R}_{00}(t), \hat{\omega}_1(t) \hat{\omega}_0(t) \right] = 0, \tag{8a}
\]

\[
\left[ \hat{R}_{00}(t), \hat{\omega}_0(t) \hat{\omega}_1(t) \right] = 0, \tag{8b}
\]

\[
\left[ \hat{\omega}_0(t) \hat{\omega}_1(t), \hat{\omega}_1(t) \hat{\omega}_0(t) \right] = 0. \tag{8c}
\]

The last commutation relation comes from the fact that \( \hat{\omega}_0(t) \hat{\omega}_1(t) \) implies commutation (since they can be diagonalized in the same basis). Note that there is nothing assumed about the commutation of \( \hat{\omega}_0(t) \) and \( \hat{\omega}_1(t) \), or \( \hat{\omega}_0(t) \hat{\omega}_1(t) \) and \( \hat{\omega}_0(t) \hat{\omega}_1(t) \).

With the use of eqs (7), we can write the whole density matrix (6) in the form

\[
\hat{\sigma}(t) = \left( \begin{array}{cc} |c_0|^2 \hat{R}_{00}(t) & c_0 c_1^* \hat{R}_{01}(t) \hat{R}_{00}(t) \hat{\omega}_0(t) \hat{\omega}_1(t) \\ \bar{c}_0 c_1 \hat{R}_{10}(t) \hat{R}_{00}(t) \hat{\omega}_0(t) \hat{\omega}_1(t) & \bar{c}_1^2 \hat{R}_{00}(t) \end{array} \right). \tag{9}
\]
Since the commutation relations imply that there exists an environmental basis \( \{|n(t)\}\) which diagonalizes \( \hat{R}_{00}(t), \hat{w}_0(t)\hat{w}_0^\dagger(t), \) and \( \hat{w}_1(t)\hat{w}_0^\dagger(t) \) at time \( t \), we can write all three operators with the help of this basis,

\[
\hat{R}_{00}(t) = \sum_n p_n(t)|n(t)\rangle\langle n(t)|, \quad \text{(10a)}
\]
\[
\hat{w}_0(t)\hat{w}_0^\dagger(t) = \sum_n e^{i\phi_n(t)}|n(t)\rangle\langle n(t)|, \quad \text{(10b)}
\]
\[
\hat{w}_1(t)\hat{w}_0^\dagger(t) = \sum_n e^{-i\phi_n(t)}|n(t)\rangle\langle n(t)|. \quad \text{(10c)}
\]

Now the density matrix can be written with the help of basis \( \{|n(t)\}\) in an obviously separable form,

\[
\hat{\sigma}(t) = \sum_n p_n(t) \left( \frac{|c_0|^2}{c_0^*c_1} e^{i\phi_n(t)} |c_1|^2 \right) \otimes |n(t)\rangle\langle n(t)|. \quad \text{(11)}
\]

With the density matrix in this form, it is also obvious that it is zero-discordant with respect to the environment, but not necessarily with respect to the qubit [21].

IV. SEPARABILITY OF A QUTRIT

A. The inadequacy of the qubit-like separability conditions alone

The previous section allows us to easily find the only possible separable form of the qubit-environment density matrix for pure dephasing evolutions. Furthermore, it will allow us to extend the reasoning to systems of dimension \( N > 2 \), and check, if a straightforward extension of the qubit separability condition to bigger systems is enough to guarantee separability for \( N > 2 \).

Let us for now restrict ourselves to an \( N = 3 \) system (a qutrit) interacting with the environment as described in Sec. III. The idea is not to repeat the reasoning of Ref. [20] and look at minors; instead, we will look at the time-evolved qutrit-environment density matrix

\[
\hat{\sigma}(t) = \begin{pmatrix}
|c_0|^2\hat{R}_{00}(t) & c_0c_1^*\hat{R}_{01}(t) & c_0c_2^*\hat{R}_{02}(t) \\
c_0^*c_1\hat{R}_{01}(t) & |c_1|^2\hat{R}_{11}(t) & c_1c_2^*\hat{R}_{12}(t) \\
c_0^*c_2\hat{R}_{02}(t) & c_1^*c_2\hat{R}_{12}(t) & |c_2|^2\hat{R}_{22}(t)
\end{pmatrix}, \quad \text{(12)}
\]

and impose the conditions, which are an extension of the separability conditions for a qubit,

\[
\hat{R}(0), \hat{w}_1, \hat{w}_2 = \hat{R}(0), \hat{w}_1, \hat{w}_2 = 0, \quad \text{(13)}
\]

or, equivalently, \( \hat{R}_{00}(t) = \hat{R}_{11}(t) = \hat{R}_{22}(t) \). Under these conditions, we get (as in eq. [11])

\[
\hat{R}_{ij}(t) = \hat{R}_{00}(t)\hat{w}_i(t)\hat{w}_j^\dagger(t) \quad \text{(14)}
\]

and

\[
\begin{align*}
\hat{R}_{00}(t), \hat{w}_i(t)\hat{w}_j^\dagger(t) &= 0, \quad \text{(15a)} \\
\hat{R}_{00}(t), \hat{w}_j(t)\hat{w}_i^\dagger(t) &= 0, \quad \text{(15b)} \\
\hat{w}_j(t)\hat{w}_i^\dagger(t), \hat{w}_k(t)\hat{w}_l^\dagger(t) &= 0, \quad \text{(15c)}
\end{align*}
\]

for all \( i \) and \( j \).

Hence, if the density matrix of a qutrit and its environment fulfills the conditions, it can be written as

\[
\hat{\sigma}(t) = \begin{pmatrix}
|c_0|^2\hat{R}_{00}(t) & c_0c_1^*\hat{R}_{01}(t) & c_0c_2^*\hat{R}_{02}(t) \\
c_0^*c_1\hat{R}_{01}(t) & |c_1|^2\hat{R}_{11}(t) & c_1c_2^*\hat{R}_{12}(t) \\
c_0^*c_2\hat{R}_{02}(t) & c_1^*c_2\hat{R}_{12}(t) & |c_2|^2\hat{R}_{22}(t)
\end{pmatrix}. \quad \text{(16)}
\]

This density matrix is not necessarily always separable, because although for all \( i \neq j \) there exists a basis \( \{|n_{ij}(t)\}\) which diagonalizes \( \hat{R}_{00}(t), \hat{w}_i(t)\hat{w}_j^\dagger(t), \) and \( \hat{w}_j(t)\hat{w}_i^\dagger(t) \), there is no reason for the different bases \( \{|n_{ij}(t)\}\) to be the same. Hence, unless we also have

\[
\left( \hat{w}_i(t)\hat{w}_j^\dagger(t), \hat{w}_k(t)\hat{w}_l^\dagger(t) \right) = 0 \quad \text{(17)}
\]

for all \( i, j, k, l = 0, 1, 2, \) it is impossible to write the density matrix in the simple separable form as done in eq. (11) at the end of Sec. III. The conditions actually come down to one independent, nontrivial equation; it should be obvious that there are three such equations taking into account the commutation relation (15c), since there are three relevant combinations of indexes 01, 02, and 12, while the reduction to one will become apparent in Sec. IV C.

It is fairly simple to show that, if any of the conditions of eq. (13) are not fulfilled, then the system is entangled with its environment following the method used in Ref. [20]. The proof of this fact for a system of any size is provided in Appendix A.

Hence, if condition (13) is broken, it indicates entanglement, while if both condition (13) and conditions (17) are fulfilled, the system and environment are separable. It is the gray area where condition (13) is fulfilled, while any of the conditions (17) are broken, that is interesting. There are three possibilities:
• The difference between the qubit and qutrit situations is only apparent, and the fulfillment of condition {13} is both necessary and sufficient for separability.

• The fulfillment of condition {13} is necessary for separability, while the conditions {17} are necessary and sufficient for separability (so when any of the conditions {17} are not met, there is entanglement in the system).

• The fulfillment of condition {13} is necessary for separability, while the conditions {17} are sufficient (so when the conditions {17} are met, there is no entanglement in the system, but there are also situations, when the conditions {17} are not fulfilled and there is no SEE).

The following example will eliminate the first option, since it will show that there exists a situation when a system is entangled with its environment, even though condition {13} is fulfilled.

B. Example: Entanglement of a qutrit with a initial completely mixed state of the environment

Let us look at an exemplary situation, in which condition {13} is fulfilled, but some of the conditions {17} are not. To make the example as simple as possible we will consider a single-qubit environment (the system is a qutrit). We assume that the initial state of the qutrit is an equal superposition state \(|\psi\rangle = 1/\sqrt{3}(|0\rangle + |1\rangle + |2\rangle)| for simplicity. The initial state of the environment is a completely mixed state \(\hat{R}(0) = 1/2(|0\rangle \langle 0| + |1\rangle \langle 1|)\), which guarantees the fulfillment condition {13}, since the unit matrix commutes with all other matrices.

We are interested in a certain instant of time \(t\), when the conditional evolution operators of the environment are proportional to unity and two of the Pauli matrices,

\[
\hat{w}_0(t) = 1, \quad \hat{w}_1(t) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \hat{w}_2(t) = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix},
\]

so \(\hat{w}_0(t)\) commutes with \(\hat{w}_1(t)\) and \(\hat{w}_2(t)\) (all three are hermitian), but \(\hat{w}_1(t)\) and \(\hat{w}_2(t)\) do not commute with each other, nor do they commute with

\[
\hat{w}_1(t)\hat{w}_2(t) = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix},
\]

which is proportional to the third Pauli matrix. Hence the system-environment density matrix at time \(t\) is given by

\[
\hat{\sigma}(t) = \frac{1}{6} \begin{pmatrix} 1 & \hat{w}_1(t) & \hat{w}_2(t) & \hat{w}_3(t) \\ \hat{w}_1(t) & 1 & \hat{w}_3(t) & \hat{w}_2(t) \\ \hat{w}_2(t) & \hat{w}_3(t) & 1 & \hat{w}_1(t) \\ \hat{w}_3(t) & \hat{w}_2(t) & \hat{w}_1(t) & 1 \end{pmatrix} = \frac{1}{6} \begin{pmatrix} 1 & 0 & i & 0 \\ 0 & 1 & 0 & -i \\ i & 0 & 1 & 0 \\ 0 & i & 0 & 1 \end{pmatrix}.
\]

The matrix in eq. {20b} is written explicitly in the system-environment basis in the following order \({|0\rangle, |1\rangle, |2\rangle, |3\rangle}\), which are not transposed (all three are orthogonal for simplicity. The initial state of the environment is \(|e\rangle\).

To check, if there is SEE in the state described by the density matrix {20b}, it suffices to find the eigenvalues of the matrix after partial transposition with respect to either the system or the environment. If the transposed matrix is not a density matrix (has negative eigenvalues) then there is entanglement in the system (as stated by the Peres-Horodecki criterion {22, 23}). After partial transposition with respect to the environment we get

\[
\hat{\sigma}_E(t) = \frac{1}{6} \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & -i \\ 0 & 1 & 0 & -1 & i & 0 \\ 1 & 0 & 1 & 0 & 0 & i \\ 0 & -1 & 0 & 1 & i & 0 \\ 0 & i & 0 & -i & 1 & 0 \\ i & 0 & -i & 0 & 0 & 1 \end{pmatrix}.
\]

The matrix has two negative eigenvalues equal to \((-\frac{1}{2}\sqrt{2}\). (and four positive eigenvalues equal to \(\frac{1}{2}\)), so it is not a density matrix, and there is entanglement between the system and the environment in the (not transposed) density matrix {20b}.

There are a couple of interesting conclusions, which can be drawn from the above example. The first obviously is that the fulfillment of condition {13} is not sufficient for separability. The second, which should be surprising, is that pure-dephasing evolution can lead to entanglement with the environment even, if the environment is initially in a completely mixed state (hence, fully classical). This is absolutely not possible, if the system is a qubit.

C. The commutation conditions between different environmental evolution operators are necessary and sufficient for separability

In the following we will show that systems for which the qubit-like condition {13} is met, but at least one of the conditions {17} is not, display SEE. To this end, we study the qutrit-environment density matrix {13}, which is already written in a form equivalent to the fulfillment of the qubit-like conditions {13}.
For simplicity, let us denote
\[
\hat{W}_{ij}(t) = \hat{w}_i(t)\hat{w}^*_j(t) = \hat{W}^\dagger_{ij}(t),
\]
(22)
which automatically implies \(\hat{W}_{12}(t) = \hat{W}_{10}(t)\hat{W}^\dagger_{10}(t)\). It is now convenient to write \(\hat{R}_{00}(t), \hat{W}_{10}(t)\) and \(\hat{W}_{20}(t)\) in the common eigenbasis of \(\hat{R}_{00}(t)\) and \(\hat{W}_{10}(t)\), \(\{|n_{01}(t)\}\) (this basis may change with time),
\[
\begin{align*}
\hat{R}_{00}(t) &= \sum_n p_n |n_{01}(t)\rangle \langle n_{01}(t)|, \\
\hat{W}_{10}(t) &= \sum_n e^{i\phi(t)} |n_{01}(t)\rangle \langle n_{01}(t)|, \\
\hat{W}_{20}(t) &= \sum_{nm} x_{nm} |n_{01}(t)\rangle \langle m_{01}(t)|.
\end{align*}
\]
(23a–23c)
Obviously \(\hat{W}_{20}(t)\) is not diagonal in this basis. If that were the case then also \(\hat{W}_{10}(t)\hat{W}^\dagger_{20}(t)\) would be diagonal in the same basis and the state would have to be separable, since the conditions (17) would be fulfilled. Furthermore, it should now be evident that, if one of the conditions (17) is met, it implies the fulfillment of the other two, since from
\[
\begin{bmatrix}
\hat{w}_1(t)\hat{w}^*_1(t), \hat{w}_0(t)\hat{w}^*_2(t)
\end{bmatrix} = \begin{bmatrix} \hat{W}_{10}(t), \hat{W}^\dagger_{20}(t) \end{bmatrix} = 0,
\]
(24)
we get
\[
\begin{bmatrix}
\hat{w}_0(t)\hat{w}^*_1(t), \hat{w}_1(t)\hat{w}^*_2(t)
\end{bmatrix} = \begin{bmatrix} \hat{W}^\dagger_{10}(t), \hat{W}_{10}(t)\hat{W}^\dagger_{20}(t) \end{bmatrix} = 0,
\]
\[
\begin{bmatrix}
\hat{w}_2(t)\hat{w}^*_1(t), \hat{w}_1(t)\hat{w}^*_2(t)
\end{bmatrix} = \begin{bmatrix} \hat{W}_{20}(t), \hat{W}_{10}(t)\hat{W}^\dagger_{20}(t) \end{bmatrix} = 0.
\]
(25)
It is now possible to write the whole density matrix \(\hat{\sigma}\) at time \(t\) using the environmental basis \(\{|n_{01}(t)\}\), which will be denoted as simply \(\{|n(t)\}\) in what follows. This yields
\[
\hat{\sigma}(t) = \begin{pmatrix}
|c_0|^2 \sum_n p_n |n(t)\rangle \langle n(t)| & c_0 c_1^* \sum_n p_n e^{-i\phi(t)} |n(t)\rangle \langle n(t)| & \cdots & c_0 c_M^* \sum_{nm} p_n x_{nm}^* |n(t)\rangle \langle m(t)| \\
\end{pmatrix}.
\]
(26)

Note that in this basis, five out of nine submatrices describing the environment conditional on the element of the density matrix of the system are diagonal.

To check, if the density matrix is entangled, we again use the Peres-Horodecki criterion \([22, 23]\), firstly applying a partial transposition to eq. (25) with respect to the system and then checking, if the resulting matrix has negative eigenvalues (if it does, then there is entanglement between the system and the environment, otherwise the question is unanswered). The partial transposition yields
\[
\hat{\sigma}^{\text{TS}}(t) = \begin{pmatrix}
|c_0|^2 \sum_n p_n |n(t)\rangle \langle n(t)| & c_0 c_1^* \sum_n p_n e^{i\phi(t)} |n(t)\rangle \langle n(t)| & \cdots & c_0 c_M^* \sum_{nm} p_n x_{nm} |n(t)\rangle \langle m(t)| \\
\end{pmatrix}.
\]
(27)
Since the diagonalization of the matrix (26) for an arbitrary size of the environment is impossible, we restrict ourselves to the study of its principal minors, since a matrix has only non-negative eigenvalues, if and only if all of its principal minors are non-negative.

We have identified a class of principal minors, which solve the question of qubit-environment entanglement generation, when the conditions (17) are not fulfilled. These minors are determinants of \(3 \times 3\) matrices obtained by symmetrically crossing out \((3M - 3)\) rows and columns out of the transposed matrix (26), where \(M\) is the dimension of the environment. They are described by two indices \(k, q = 0, 1, \ldots, M - 1\), which correspond to two states of the environment. Note, that the rows and columns of the matrix (26), which is of dimension \(3M \times 3M\) are numbered by both the system and the environmental states and are ordered in such a way, that the row/column number \(r = sM + e\), where \(s = 0, 1, 2\) denote system states, while \(e = 0, 1, \ldots, M - 1\) denote states of the environment. To obtain the \(3 \times 3\) matrix for a given \(k\) and \(q\), we cross out all rows and columns with the exception of the \(k\)-th, \((M + k)\)-th and \((2M + q)\)-th rows and columns. This means that only the \(k\)-th diagonal elements from the parts of the matrix (26) proportional to \(|c_0|^2\) and \(|c_1|^2\) are left over, as well as the \(q\)-th element from the part proportional to \(|c_1|^2\), which yields principal minors of the form...
\[ D_{kq} = \det \begin{pmatrix} |c_0|^2 p_k & c_0 c_1 p_k e^{i \phi_k(t)} & c_0 c_1 p_k e^{-i \phi_k(t)} \\ c_0 c_1 p_k e^{-i \phi_k(t)} & |c_1|^2 p_k & c_2 c_0 p_q x_{kq} \\ c_0 c_1 p_k e^{i \phi_k(t)} x_{kq} & c_2 c_1 p_q e^{-i \phi_k(t)} x_{kq} & |c_2|^2 p_q \end{pmatrix}. \] (27)

Since from the condition (13) we get eq. (15a) and consequently \([\hat{R}_{00}, \hat{W}_{20}] = 0\), this means that either \(p_k = p_q\) or \(x_{kq} = 0\). If \(x_{kq} = 0\), then the principal minor \(D_{kq} = 0\). Otherwise we get

\[ D_{kq} = -2 |c_0 c_1 c_2|^2 p_k |x_{kq}|^2 \left[1 - \cos (\phi_k(t) - \phi_q(t))\right], \] (28)

which cannot be positive, and is equal to zero only in four situations. Two of them are trivial. The first is when the initial occupation of the qubit \(|c_i|^2 = 0\) for any \(i = 0, 1, 2\), which means that the studied system is operationally a qubit, not a qutrit. In the second trivial situation both of the eigenvalues of the matrix \(\hat{R}_{00}(t)\) are zero, \(p_k = p_q = 0\), so regardless of the value of the parameter \(x_{kq}\), there are no transitions between the states \(|k(t)\rangle\) and \(|q(t)\rangle\). In this situation, no transitions will occur to states \(|k(t)\rangle\) or \(|q(t)\rangle\), since for any state \(|n(t)\rangle\), either \(p_n = p_{k/q}\) and must be equal to zero, or \(x_{(k/q)n} = 0\). Hence, the density matrix elements corresponding to the states \(|k(t)\rangle\) and \(|q(t)\rangle\) must be equal to zero during any point of the evolution, and \(|k(t)\rangle\) and \(|q(t)\rangle\) can be eliminated from the subspace of environmental degrees of freedom taken into account. Note that the discussed system-environment state is already restricted by the fulfillment of conditions (13); the above conclusion is by no means general.

The other two situations are relevant, if we wish to distinguish between entangling and non-entangling evolutions. For the determinant \(D_{kq}\) to be zero, we must either have \(x_{kq} = 0\) or \(e^{i \phi_k(t)} = e^{i \phi_q(t)} = 0\), otherwise \(D_{kq} < 0\) which means that there is entanglement in the system. If it is true for all \(k\) and \(q\) that \(x_{kq} = 0\) or \(e^{i \phi_k(t)} - e^{i \phi_q(t)} = 0\), then it is equivalent to write

\[ [\hat{W}_{10}(t), \hat{W}_{20}(t)] = 0, \] (29)

which means that eq. (24) is also fulfilled, and consequently the conditions (17) are met. In this case we can show, using the definition of separability that the qutrit is separable from its environment. Otherwise there is qutrit-environment entanglement in the studied state.

Hence, we have shown that both the condition (13) and the condition (29) must be fulfilled for a qutrit to be separable from its environment. If either of the conditions are not met, then there is entanglement in the system. Note that for a qutrit coupled to a qubit environment (as in the example of Sec. XVIII), the “grey area” where the condition (13) is met, but condition (29) is not, contains only situations with an initially fully classical environment \(p_0 = p_1 = 1/2\), because \(p_0 \neq p_1\) implies \(x_{01} = x_{10} = 0\), for which two-dimensional operators \(\hat{W}_{10}(t)\) and \(\hat{W}_{20}(t)\) must commute. For larger environments, other, more quantum initial states of the environment may lead to the generation of this kind of entanglement.

\section{V. LARGER SYSTEMS}

There is no qualitative difference when studying systems of larger dimensionality, \(N > 3\), compared to the qutrit case, as there is between a qubit and a qutrit. The qubit-like conditions, correspondnig to those given by eq. (13) for the qutrit, now take the form that for all \(i, j = 0, 1, ..., N - 1\) (the indices label system states)

\[ [\hat{R}, w_i^\dagger(t) \hat{w}_j(t)] = 0. \] (30)

Obviously, if the condition (30) is satisfied for \(i\) and \(j\), then it must be satisfied for \(j\) and \(i\) (interchanged indices) and the conditions are automatically fulfilled for \(i = j\). Furthermore, since for a given \(i\) and \(j\) the condition of eq. (30) is equivalent to \(R_{ii}(t) = R_{jj}(t)\), it is evident that if condition (30) is satisfied for all \(i, j\), as well as for \(i, k\) and \(k, j\), then it must be satisfied for \(i, j, k\) as well. Hence, there are \((N - 1)\) non-trivial qubit-like commutation conditions in eq. (30). If any of these conditions is broken, then there is entanglement between the system and its environment in the density matrix given by eq. (1). A proof of this fact is given in Appendix (A).

When all of the conditions (30) are satisfied, the question of separability is still open and another set of conditions needs to be verified. These are similar to the conditions introduced for the qutrit scenario (17), namely that for all \(i, j, k, l\) (where the indices again label system states only)

\[ [\hat{W}_{ij}(t), \hat{W}_{kl}(t)] = 0, \] (31)

where the environmental operators \(\hat{W}_{ij}(t)\) are given by eq. (22).

Note that only \((N - 1)(N - 2)/2\) of these conditions are independent. This is obvious, if we first set two of the system indices to a fixed an equal value, say \(j = k = 0\). It is then straightforward to show that if for all \(i \neq l\), the conditions

\[ [\hat{W}_{il}(t), \hat{W}_{0l}(t)] = 0, \] (32)

are fulfilled, also all conditions (31) must be satisfied. To
this end, we can write
\[
\begin{align*}
\left[ \hat{W}_{ij}(t), \hat{W}_{kl}(t) \right] &= \hat{W}_{ij}(t)\hat{W}_{kl}(t) - \hat{W}_{kl}(t)\hat{W}_{ij}(t) \\
&= \hat{W}_{i0}\hat{W}_{j0}\hat{W}_{k0}\hat{W}_{l0}(t) - \hat{W}_{l0}(t)\hat{W}_{ij}(t)\hat{W}_{kl}(t) \\
&= \hat{W}_{kl}(t)\hat{W}_{i0}\hat{W}_{j0}(t) - \hat{W}_{ij}(t)\hat{W}_{kl}(t) \\
&= \hat{W}_{ij}(t)\hat{W}_{kl}(t) - \hat{W}_{ij}(t)\hat{W}_{kl}(t) = 0,
\end{align*}
\]
since $\hat{W}_{i0}(t) = \hat{W}_{i0}^\dagger(t)$ and the operators are unitary (so the commutation relations of eq. \[2\] hold, if one or both operators undergo hermitian conjugation).

VI. CONCLUSION AND OUTLOOK

We have shown that there is a qualitative difference between system-environment generation in case of larger systems and in case of a qubit. Although a generalization of the qubit separability condition to larger systems does constitute an entanglement witness (if any of the qubit-like conditions is broken, this means that there is entanglement in the system), entanglement can be generated also, if all of such conditions are met.

Furthermore, we have identified an additional set of separability conditions which apply for systems of larger dimensionality than a qubit, when all qubit-like conditions are fulfilled. These conditions act on evolution operators which govern environment behavior depending on different states of the system, and contrarily to the qubit-like conditions, are independent of the actual initial state of the environment (neither set of conditions is dependent on the system state, as long as it is a superposition of pointer states). This set of conditions allows us to make a final distinction between entangling and non-entangling evolutions, since if they are all fulfilled at a given time (additionally to the qubit-like conditions), the system-environment state at this time is separable. Otherwise it is always entangled.

The number of separability conditions expectedly grows with the size of the studied system. There are $N - 1$ conditions of the first type and $(N - 1)(N - 2)/2$ conditions of the second type for a system of size $N$. Hence, the number of qubit-like conditions grows more slowly, but the number of conditions of the other type is smaller for a both a qubit and a qutrit, while for $N = 4$ there are three non-trivial conditions of each type. Still, for reasonably small systems, and/or interactions of large symmetry, checking entanglement generation using the proposed method should be manageable.

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Appendix A: Proof of entanglement generation when any of the qubit-like conditions is broken

Let us consider a system of any size $N$ interacting with an environment of any size $M$ via a pure-dephasing interaction described in Sec. [1]. The time-evolved density matrix of the system and its environment can be written using eq. (1) and after partial transposition with respect to the system, we get

$$\hat{\sigma}^{TS}(t) = \sum_{k,l=0}^{N-1} c_k c_l^* |l\rangle\langle k| \hat{R}_{kl}(t),$$

(A1)

where the states $|l\rangle$ and $|k\rangle$ are interchanged with respect to (1).

To prove that there is entanglement between the system and its environment in all situations when at least one of the qubit-like conditions given by eq. (30) is broken, it is convenient to write the conditions in an equivalent form,

$$\hat{R}_{ii}(t), \hat{W}_{ij}(t) = 0,$$

(A2)

where $\hat{R}_{ii}(t)$ is given by eq. (5) and $\hat{W}_{ij}(t)$ is given by eq. (22).

To prove the equivalence of conditions (30) and (A2), we first provide the derivation of (A2) from (30). Since we may always write $\hat{R}_{ij}(t) = \hat{R}_{ii}(t)\hat{W}_{ij}(t) = \hat{W}_{ij}(t)\hat{R}_{ij}(t)$ and the condition (30) can easily be transformed into $\hat{R}_{ii}(t) = \hat{R}_{jj}(t)$, we get $\hat{R}_{ii}(t)\hat{W}_{ij}(t) = \hat{W}_{ij}(t)\hat{R}_{ii}(t)$. For the derivation of (30) from (A2), we start by writing eq. (A2) explicitly in terms of the $\hat{w}_i(t)$ operators and the initial state of the environment $\hat{R}(0)$,

$$\hat{w}_i(t)\hat{R}(0)\hat{w}_i^+(t)\hat{w}_j(t)\hat{w}_j^+(t) - \hat{w}_i(t)\hat{w}_j^+(t)\hat{w}_i(t)\hat{w}_j^+(t) = 0.$$

(A3)

Since $\hat{w}_i(t)$ are unitary, multiplying on the right by $\hat{w}_i(t)$ yields

$$\hat{w}_i(t) \left( \hat{R}(0)\hat{w}_i^+(t)\hat{w}_i(t) - \hat{w}_i^+(t)\hat{w}_i(t)\hat{R}(0) \right) = 0,$$

(A4)

and multiplying by $\hat{w}_i(t)$ on the left yields the condition (30). Hence, showing that the violation of any of the conditions (A2) guarantees system-environment entanglement is equivalent to showing the same for the violation of any of the conditions (30).

To show that, if for any $i \neq j$ there is $\hat{R}_{ii}(t), \hat{W}_{ij}(t) \neq 0$, then there is system-environment entanglement at time $t$, we use the Peres-Horodecki criterion [22, 23]. Hence, we must show that in this case the system-environment density matrix after partial transposition (A1) has negative eigenvalues. Since a matrix has negative eigenvalues, if any of its principal minors is negative, we study a class of minors which is indicative of this type of entanglement, extending the results of Ref. [20]. For given system states $i$ and $j$ we first eliminate all elements of the matrix (A1) describing other system states, by symmetrically crossing out rows and columns denoted by other system indices than $ii$, $ij$, $ji$, and $jj$. The resulting matrix is of the form

$$\hat{M}_{ij} = \begin{pmatrix} |c_i|^2 \hat{R}_{ii}(t) & c_i^* c_j \hat{W}_{ij}(t) \hat{R}_{ii}(t) \\ c_i^* c_j^* \hat{R}_{ii}(t) \hat{W}_{ij}(t) & |c_j|^2 \hat{W}_{ij}(t) \hat{R}_{ii}(t) \hat{W}_{ij}(t) \end{pmatrix},$$

(A5)

since

$$\hat{R}_{ij}(t) = \hat{W}_{ij}^+(t) \hat{R}_{ii}(t) \hat{W}_{ij}(t),$$

(A6a)

$$\hat{R}_{ji}(t) = \hat{R}_{ii}(t) \hat{W}_{ij}(t),$$

(A6b)

$$\hat{R}_{ij}(t) = \hat{W}_{ij}(t) \hat{R}_{ii}(t).$$

(A6c)

We will not study the minor corresponding to the matrix (A5), since for an arbitrary size of the environment, calculating it is too complex. Instead we write the matrix in the eigenbasis of $\hat{R}_{ii}(t)$, which is specific for time $t$, and which we denote as $\{\langle n(t)\rangle\}$. In this basis, we have

$$\hat{R}_{ii}(t) = \sum_n p_n(t) |n(t)\rangle\langle n(t)|,$$

(A7a)

$$\hat{W}_{ij}(t) = \sum_{nm} y_{nm}(t) |n(t)\rangle\langle m(t)|,$$

(A7b)

where $p_n(t)$ are the eigenvalues of $\hat{R}_{ii}(t)$ and $y_{nm}(t) = \langle n(t)|\hat{W}_{ij}(t)|m(t)\rangle$. The indices $i$ and $j$ are omitted here. Now the class of relevant principal minors is obtained by symmetrically crossing out $M-1$ rows and columns from the matrix (A5) in such a way, that only one diagonal element proportional to $|c_{ij}|^2$ is left (and then finding the determinant). Hence, we get $M$ principal minors in this class labeled by the environmental state index $n = 0, 1, ..., M-1$ (the explicit time-dependence of the parameters has been dropped in the following).
This class of minors is of exactly the same form as in Ref. [20] and their analysis leads to an analogous conclusion. There are three relevant situations:

1. If all of the eigenvalues of the matrix \( \hat{R}_{ii}(t) \), \( p_k \), are non-zero, then

\[
Y_{n}^{ij} = |c_i|^2 M |c_j|^2 \prod_{q \neq r} p_q \sum_k \left[ p_k |y_{kn}|^2 - \frac{p_n^2}{p_k} |y_{nk}|^2 \right].
\]

(8)

If we now choose such \( n' \) that the corresponding \( p_{n'} \) is largest (\( p_{n'} \geq p_k \) for all \( k \)), it is easy to show that \( \sum_k p_k |y_{kn}|^2 \leq p_{n'} \) and \( \sum_k \frac{p_n^2}{p_k} |y_{nk}|^2 \geq p_{n'} \), since \( \sum_k |y_{kn}|^2 = \sum_k |y_{nk}|^2 = 1 \) (because the operators \( \hat{W}_{ij}(t) \) are unitary). Hence, the minor \( Y_{n}^{ij} \) is equal to zero only if, for all \( k \) either \( p_k = p_{n'} \) or \( |y_{kn}| = |y_{nk}| = 0 \) (when \( \sum_k p_k |y_{kn}|^2 = p_{n'} \) and \( \sum_k \frac{p_n^2}{p_k} |y_{nk}|^2 = p_{n'} \)). Otherwise it is negative and there must be SEE present.

In the situation, when \( Y_{n}^{ij} = 0 \), \( n' \) should be studied, for which the eigenvalue \( p_{n''} \) second largest (\( p_{n''} \leq p_{n'} \) and \( p_{n''} \geq p_k \) for all \( k \neq n' \)). Both conditions stemming from \( Y_{n}^{ij} = 0 \) lead to the conclusion that \( \sum_k p_k |y_{kn}|^2 \leq p_{n''} \) and \( \sum_k \frac{p_n^2}{p_k} |y_{nk}|^2 \geq p_{n''} \). Hence, \( Y_{n}^{ij} = 0 \) if and only if either \( p_k = p_{n''} \) or \( |y_{kn''}| = |y_{nk}| = 0 \) for all \( k \), otherwise it is negative.

Repeating this reasoning for all minors in the order of diminishing \( p_n \) leads to the conclusion that there is SEE unless for all \( k \) and \( q \) either \( p_k = p_q \) or \( |y_{kq}| = |y_{qk}| = 0 \) (and all principal minors from the class \( Y_{n}^{ij} = 0 \)). This is equivalent to the statement that entanglement has been generated unless \( \hat{R}_{ii}(t), \hat{W}_{ij}(t) = 0 \).

2. If only one of the eigenvalues of the matrix \( \hat{R}_{ii}(t) \) is equal to zero (let us denote the corresponding eigenstate as \( |r(t)\rangle \), so \( p_r = 0 \), then

\[
Y_{n}^{ij} = -|c_i|^2 M |c_j|^2 \prod_{q \neq r} p_q \frac{p_n^2}{p_{nr}} |y_{nr}|^2, \quad (10)
\]

for \( n \neq r \) and \( Y_{n}^{rr} = 0 \). Hence, if there exists \( |y_{nr}| \neq 0 \) for any \( n \neq r \), then there is SEE. Otherwise the environmental state \( |r(t)\rangle \) does not take part in the system-environment evolution and does not need to be taken into account, so entanglement generation may be probed using the minors of eq. (10) after eliminating the state \( |r(t)\rangle \) from the subspace of the environment.

Note that this situation can still be described using the commutation of \( \hat{R}_{ii}(t) \) and \( \hat{W}_{ij}(t) \). This is because, if there exists \( |y_{nr}| \neq 0 \), with \( p_r = 0 \), but \( p_n \neq 0 \), then \( Y_{n}^{ij} < 0 \) and furthermore the commutation of \( \hat{R}_{ii}(t) \) and \( \hat{W}_{ij}(t) \) is impossible. Otherwise, for \( |y_{nr}| = 0 \) for all \( n \) and there is entanglement in the system unless \( \hat{R}_{ii}(t) \) and \( \hat{W}_{ij}(t) \) can be diagonalized in the same basis. Hence, if the condition \( \hat{R}_{ii}(t), \hat{W}_{ij}(t) = 0 \) is not satisfied, then the system is entangled with its environment.

3. If more than one eigenvalue of of the matrix \( \hat{R}_{ii}(t) \) is equal to zero, then the class of minors given by eq. (10) is not a good class for the study of entanglement generation, since all of the minors in this class are always equal to zero. If for any of the states \( \{ |p\rangle \} \) for which \( p_p = 0 \), we have \( |y_{np}| = 0 \) for all \( n \), then these states can be eliminated from the subsystem of environmental states for the analysis of entanglement generation, since they do not take part in the system-environment evolution. If none, or only one relevant \( |p\rangle \) state is present in the system, then the system should be treated as described above.

Otherwise, if after eliminating the parts of the Hilbert space of the environment, which do not take part in the evolution, there are still \( K \geq 2 \) states from the subspace \( \{ |p\rangle \} \), then the analysis of entanglement generation requires a different class of principal minors. This is obtained by crossing out all but one of the rows and columns corresponding to diagonal elements equal to zero in the matrix given on the left in eq. (10), under the determinant.
where \( \{ p \} \) denotes the set of indices for which \( p_p = 0 \). The minors \( \tilde{Y}_{ij} \) are negative when \( |y_{nr}| \neq 0 \), so in the described situation, a negative minor must exist and entanglement is present. Furthermore, since the states corresponding to the non-zero element of the \( W_{ij}(t) \) operator, \( |y_{nr}| \), have different occupations in the environmental state \( \hat{R}_{ii}(t) \), \( p_n \neq p_c = 0 \), \( W_{ij}(t) \) and \( \hat{R}_{ii}(t) \) do not commute.

Hence, we have shown that there is SEE present in the density matrix (11) of a system of any size \( N \) and environment of any size \( M \). We can now conclude that all the qubit-like conditions (30), which are necessary for system-environment separability (but not sufficient), are satisfied. Hence, the system-environment density matrix can be written as

\[
\hat{\sigma}(t) = \sum_i |c_i|^2 \langle i | \hat{\sigma}_0(t) \rangle |i \rangle + \sum_{i > j} \left( c_i^* c_j \langle i | \hat{\sigma}_0(t) \rangle |j \rangle + H.c. \right),
\]

where (30) implies \( \hat{R}_{ii}(t) = \hat{\sigma}_0(t) \), for all \( i \). After partial transposition with respect to the system, we get

\[
\hat{\sigma}^{T_S}(t) = \sum_i |c_i|^2 \langle i | \hat{\sigma}_0(t) \rangle |i \rangle + \sum_{i > j} \left( c_i^* c_j \langle i | \hat{\sigma}_0(t) \rangle |j \rangle + H.c. \right).
\]

By studying the principal minors for the study of qubit-environment entanglement generation, when the conditions (30) are satisfied is also the same as in Sec. (IV.C). They are determinants of 3 \times 3 matrices obtained by symmetrically crossing out all but one row and column, the diagonal elements of which are proportional to \( |c_i|^2 \), \( |c_j|^2 \), and \( |c_l|^2 \), respectively. Furthermore, the two rows and columns left over with diagonal elements proportional to

\[
\tilde{M}_{ij} = \left( \begin{array}{ccc} |c_i|^2 \sum_n p_n |n \rangle \langle n| & c_i^* c_j \sum_n p_n e^{-i\phi_n} |n \rangle \langle n| & c_i^* \sum_n p_n x_{nm} |n \rangle \langle m| \\ c_j^* \sum_n p_n e^{i\phi_n} |n \rangle \langle n| & |c_j|^2 \sum_n p_n |n \rangle \langle n| & c_j^* \sum_n p_n x_{nm} e^{i\phi_m} |n \rangle \langle m| \\ c_i \sum_n p_n x_{nm} |n \rangle \langle m| & c_j \sum_n p_n x_{mn} e^{-i\phi_m} |n \rangle \langle m| & |c_l|^2 \sum_n p_n |n \rangle \langle n| \end{array} \right).
\]

As before, we will use the Peres-Horodecki criterion and the fact that a matrix has negative eigenvalues, if and only if at least one of its principal minors is negative. Hence, the existence of a negative principal minor of the density matrix after partial transposition (12) means that SEE has been generated at time \( t \) during the evolution.

Since the desired set of separability conditions (31) is qualitatively different than in Appendix A, so will the studied set of principal minors be. We start by choosing three system states \( i, j, \) and \( l \) and symmetrically eliminating all rows and columns from the matrix (12) which describe the system-environment occupations and coherences not confined to the \( \{ |i\rangle, |j\rangle, |l\rangle \} \) subspace of the system. This yields the matrix (the explicit time dependence is omitted further on)

\[
\hat{M}_{ij} = \left( \begin{array}{ccc} c_i^* c_j \sum_n p_n |n \rangle \langle n| & c_i^* c_j \sum_n p_n e^{-i\phi_n} |n \rangle \langle n| & c_i^* \sum_n p_n x_{nm} |n \rangle \langle m| \\ c_j^* \sum_n p_n e^{i\phi_n} |n \rangle \langle n| & |c_j|^2 \sum_n p_n |n \rangle \langle n| & c_j^* \sum_n p_n x_{nm} e^{i\phi_m} |n \rangle \langle m| \\ c_i \sum_n p_n x_{nm} |n \rangle \langle m| & c_j \sum_n p_n x_{mn} e^{-i\phi_m} |n \rangle \langle m| & |c_l|^2 \sum_n p_n |n \rangle \langle n| \end{array} \right).
\]

It is now convenient to write the matrix (13) in terms of eigenstates which diagonalize both \( \hat{\sigma}_0(t) \) and \( \hat{W}_{ij}(t) \), which we denote as \( |n\rangle \). Note, that the basis which diagonalizes \( W_{ij}(t) \) (or \( \tilde{W}_{ij}(t) \)) also diagonalizes \( \hat{\sigma}_0(t) \), but not necessarily \( \hat{W}_{ij}(t) \). In this basis we have

\[
\hat{\sigma}_0(t) = \sum_n p_n |n \rangle \langle n|,
\]

\[
\hat{W}_{ij}(t) = \sum_n e^{i\phi_n} |n \rangle \langle n|,
\]

\[
\hat{W}_{ij}(t) = \sum_n x_{nm} e^{-i\phi_m} |n \rangle \langle m|.
\]

In this basis the matrix \( \hat{M}_{ij} \) can be written in an identical form as the partially transposed qubit-environment density matrix, which satisfies the qubit-like conditions (16), given by eq. (16),

\[
\hat{M}_{ij} = \left( \begin{array}{ccc} |c_i|^2 \sum_n p_n |n \rangle \langle n| & c_i^* c_j \sum_n p_n e^{-i\phi_n} |n \rangle \langle n| & c_i^* \sum_n p_n x_{nm} |n \rangle \langle m| \\ c_j^* \sum_n p_n e^{i\phi_n} |n \rangle \langle n| & |c_j|^2 \sum_n p_n |n \rangle \langle n| & c_j^* \sum_n p_n x_{nm} e^{i\phi_m} |n \rangle \langle m| \\ c_i \sum_n p_n x_{nm} |n \rangle \langle m| & c_j \sum_n p_n x_{mn} e^{-i\phi_m} |n \rangle \langle m| & |c_l|^2 \sum_n p_n |n \rangle \langle n| \end{array} \right).
\]
to \(|c_i|^2\) and \(|c_j|^2\) correspond to the \(k\)-th environmental state (the diagonal element is proportional to \(p_k\)), while the row and column left over with a diagonal element proportional to \(|c_i|^2\) corresponds to the \(q\)-th state of the environment (the diagonal element is proportional to \(p_q\)). The minors of interest are therefore labeled by to environmental indices \(k\) and \(q\), and are given by

\[
X_{kj} = \det \begin{pmatrix}
|c_i|^2 p_k & c_i c_j^* p_k e^{-i \phi_k} x_{kj} & c_i c_j^* p_q x_{kj} \\
0 & |c_j|^2 p_k & c_j c_i^* p_k e^{-i \phi_k} x_{kj} \\
c_j c_i^* p_q & c_j c_i^* p_q e^{-i \phi_q} x_{kj} & |c_i|^2 p_k
\end{pmatrix} = -2 |c_i c_j|^2 p_k^2 |x_{kj}|^2 [1 - \cos (\phi_k - \phi_q)],
\]

(B6)

since \([R_{00}, \hat{W}_{li}] = 0\) implies that for \(x_{kj} \neq 0\), we must have \(p_k = p_q\). No minor \(X_{kj}^{ij}\) can be positive, and they can be equal to zero only in four situations, two of which are trivial. The first trivial one is when at least one of the initial qubit occupations is zero and therefore \(|c_i c_j|^2 = 0\), so that the studied system state is of lower dimension than \(N\) and the system-environment density matrix should be adjusted accordingly. In the second trivial situation \(p_k = p_q = 0\). Because for the studied state the qubit like conditions (B9) are satisfied, it follows that for \(p_k = p_q = 0\), \(x_{kn} = 0\) and \(x_{qn} = 0\) for all \(n\) with \(p_n \neq 0\), and \(|k\rangle\) and \(|q\rangle\) can be eliminated from subspace of environmental degrees of freedom taken into account.

Otherwise, the principal minor \(X_{kj}^{ij}\) is equal to zero (is non-negative) only if either \(x_{kj} = 0\) or \(e^{i \phi_k} - e^{i \phi_q} = 0\). If all of the principal minors of this class, \(X_{kj}^{ij}\), for fixed \(i, j,\) and \(l\), but for every possible value of \(k \neq q\) are non-negative, then it is equivalent to write

\[
[\hat{W}_{ji}(t), \hat{W}_{kl}(t)] = 0.
\]

(B7)

Obviously \(X_{kj}^{ij} < 0\) for any \(k\) or \(q\) means that there is qubit-environment entanglement in the state, so if the condition (B7) is not met, there must be entanglement in the system.

Repeating the procedure detailed above for all \(i \neq j \neq l\) yields a set of conditions of the form of eq. (B7) for system-environment states which already fulfill all of the conditions (B9). If for any system states \(i \neq j \neq l\), the condition (B7) is not satisfied (for at least one pair of environment states \(k\) and \(q\)), then there is entanglement between the system and the environment in the state. Otherwise, for all \(i \neq j \neq l\), operators \(\hat{W}_{ji}(t)\) and \(\hat{W}_{kl}(t)\) commute. This is enough to show that there is no SEE, if the system is a qutrit, since in this case \(\hat{R}_{00}, \hat{W}_{01}(t), \hat{W}_{12}(t),\) and \(\hat{W}_{20}(t)\) have a common eigenbasis and the system-environment density matrix (B9) can be written in a separable form. For larger systems, for there to exist an environmental basis, which diagonalizes \(\hat{R}_{00}(t)\) and all operators \(\hat{W}_{ij}(t)\), also \(\hat{W}_{ij}(t)\) and \(\hat{W}_{kl}(t)\) must commute for all \(i \neq j \neq k \neq l\) (in the case of the qutrit, such four system indices do not exist). Incidentally, no new class of minors is needed in this case, since the commutation of two \(\hat{W}_{ij}(t)\) operators described by four different indices can be derived from commutation of \(\hat{W}_{ij}(t)\) operators with only three different indices. If the condition (B7) is fulfilled for all \(i \neq j \neq l\), we get

\[
[\hat{W}_{ij}(t), \hat{W}_{kl}(t)] = \hat{W}_{ij}(t)\hat{W}_{kl}(t) - \hat{W}_{kl}(t)\hat{W}_{ij}(t) = \hat{W}_{kj}(t)\hat{W}_{jl}(t) - \hat{W}_{jl}(t)\hat{W}_{kj}(t) = \hat{W}_{kl}(t)\hat{W}_{ij}(t) - \hat{W}_{ij}(t)\hat{W}_{kl}(t) = \hat{W}_{kl}(t)\hat{W}_{ij}(t) - \hat{W}_{ij}(t)\hat{W}_{kl}(t) = 0.
\]

Here we used the fact that the condition (B7) also means that

\[
[\hat{W}_{ij}(t), \hat{W}_{kl}(t)] = 0, \\
[\hat{W}_{ij}(t), \hat{W}_{il}(t)] = 0, \\
[\hat{W}_{ij}(t), \hat{W}_{lt}(t)] = 0,
\]

since \(\hat{W}_{ij}(t) = \hat{W}_{ij}^\dagger(t)\) and the operators \(\hat{W}_{ij}(t)\) are unitary, and the fact that \(\hat{W}_{kl}(t) = \hat{W}_{kl}(t)\hat{W}_{ij}(t)\).

Hence, if for all \(i \neq j \neq l\), the condition (B7) is fulfilled, then all operators \(\hat{W}_{ij}(t)\) commute with each other. This means that the system-environment density matrix may be written in a basis \(|\{n(t)\}\rangle\), which is not only the eigenbasis of \(\hat{R}_{00}(t)\), but also diagonalizes all of the operators \(\hat{W}_{ij}(t)\) (and the initial, pointer basis of the system) in the obviously separable form

\[
\hat{\sigma}(t) = \sum_n p_n(t)\hat{\rho}_n(t) \otimes |n(t)\rangle\langle n(t)|,
\]

(B8)

where the density matrices of the system conditional on the state of the environment are given by

\[
\hat{\rho}_n(t) = \sum_{ij} c_i^* c_j^* e^{i \phi_{ij}(t)} |i\rangle\langle j|,
\]

(B9)

with \(|i\rangle\) and \(|j\rangle\) denoting the system states for which the Hamiltonian (B) is diagonal, and the oscillating factors
being a product of the diagonalization of the operators

\[ \hat{W}_{ij}(t) = \sum_n e^{i\phi_{in}(t)} |n(t)\rangle \langle n(t)|, \]

\[ e^{i\phi_{ii}(t)} = 1, \quad \text{and} \quad e^{i\phi_{ji}(t)} = e^{-i\phi_{ij}(t)}. \]

In a last step, let us reduce the number of relevant conditions (B7), by restricting ourselves to the independent ones, which cannot be derived from one another. This involves choosing a fixed index \( i \) in the set of conditions (B7), say \( i = 0 \). The set of all independent conditions is such that for all \( j > l > 0 \),

\[ [\hat{W}_{j0}(t), \hat{W}_{l0}(t)] = 0. \]  

(B11)

All other commutation relations, \( [\hat{W}_{ij}(t), \hat{W}_{kl}(t)] = 0 \), for any \( i, j, k, l \), can be derived from (B11) using \( \hat{W}_{ij}(t) = \hat{W}_{i0}(t)\hat{W}_{0j}(t) \), \( \hat{W}_{ij}(t) = I \), and the fact that the operators \( \hat{W}_{ij}(t) \) are unitary, so commutation is unaffected by hermitian conjugation of one or both operators. Hence, there are \( (N-1)(N-2)/2 \) independent commutation relations for a system of size \( N \), the fulfillment of which guarantees separability of the system from its environment, if system-environment state already satisfies the \( (N-1) \) qubit-like conditions (30).