Empirical Likelihood Covariate Adjustment for Regression Discontinuity Designs

Jun Ma†  Zhengfei Yu‡

Abstract

This paper proposes a versatile covariate adjustment method that directly incorporates covariate balance in regression discontinuity (RD) designs. The new empirical entropy balancing method reweights the standard local polynomial RD estimator by using the entropy balancing weights that minimize the Kullback–Leibler divergence from the uniform weights while satisfying the covariate balance constraints. Our estimator can be formulated as an empirical likelihood estimator that efficiently incorporates the information from the covariate balance condition as correctly specified over-identifying moment restrictions, and thus has an asymptotic variance no larger than that of the standard estimator without covariates. We demystify the asymptotic efficiency gain of Calonico, Cattaneo, Farrell, and Titiunik (2019)’s regression-based covariate-adjusted estimator, as their estimator has the same asymptotic variance as ours. Further efficiency improvement from balancing over sieve spaces is possible if our entropy balancing weights are computed using stronger covariate balance constraints that are imposed on functions of covariates. We then show that our method enjoys favorable second-order properties from empirical likelihood estimation and inference: the estimator has a small (bounded) nonlinearity bias, and the likelihood ratio based confidence set admits a simple analytical correction that can be used to improve coverage accuracy. The coverage accuracy of our confidence set is robust against slight perturbation to the covariate balance condition, which may happen in cases such as data contamination and misspecified “unaffected” outcomes used as covariates. The proposed entropy balancing approach for covariate adjustment is applicable to other RD-related settings. For example, we derive a covariate-adjusted estimator of the treatment effect derivative of Dong and Lewbel (2015) and show that it incorporates the covariate information in a more transparent and flexible way than the regression-based adjustment. We conduct Monte Carlo simulations to assess our method’s finite-sample performance and also apply it to a real dataset.

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†School of Economics, Renmin University of China
‡Faculty of Humanities and Social Sciences, University of Tsukuba
1 Introduction

The RD design resembles a randomized experiment conducted near the cut-off of the score (forcing variable) and exploits the discontinuous variation in the probability of treatment to nonparametrically identify the local average treatment effect (LATE) at the cut-off under mild continuity assumptions on the latent variables. The transparent close-form identification (Hahn et al., 2001) of the RD LATE calls for nonparametric estimation and inference methods as they avoid functional form assumptions. See Cattaneo et al. (2019) for a recent review of RD. In practical implementations, information from pre-treatment covariates (i.e., variables that have already been determined before the assignment of the treatment) is incorporated to enhance efficiency and compensate for the low accuracy of nonparametric methods. A widely used procedure is augmented local polynomial (LP) regression, where the covariates enter linearly. Calonico et al. (2019, CCFT, hereafter) formalize this augmented regression approach and derive its (first-order) asymptotic properties. CCFT shows that the augmented LP regression estimator consistently estimates the RD LATE under the covariate balance condition, i.e., the expectations of covariates coincide at both sides of the cut-off. Apart from CCFT, covariate adjustment for RD has received much attention in recent literature. See Frölich and Huber (2019) for an alternative approach that requires smoothing over covariates but allows for the potential failure of covariate balance. Arai et al. (2021) and Kreiß and Rothe (2022) extend CCFT’s approach to control for a high-dimensional covariate vector by regularization. Noack et al. (2021) extend CCFT’s linear regression adjustment to nonparametric adjustment with machine learning methods. See Cattaneo et al. (2021) for a recent review of covariate adjustment for RD.

This paper studies a novel and versatile approach based on (generalized) entropy balancing (EB) to incorporate covariates for RD. The recent literature on the estimation of the average treatment effect (ATE) under the unconfoundedness assumption and also broader causal inference literature (e.g., Doudchenko and Imbens, 2016) flourishes with methods based on balancing. See Ben-Michael et al. (2021) for a review of this strand of literature. To the best of our knowledge, the balancing approach has not been investigated in the RD literature. In this paper, we follow CCFT to consider a potential outcome and covariate framework. Here, the covariate balance condition, which is a restriction on the population feature of the observed covariates, is directly implied by the predeterminedness (zero RD LATE on covariates) assumption and standard smoothness assumptions. Our balancing approach adjusts for covariates by using weights that achieve exact local covariate balance and have the least Kullback–Leibler (KL) divergence from the uniform weights. The EB estimator can be constructed in two intuitive steps: the first step computes the EB weights from a minimum relative entropy problem subject to the covariate balance constraints, and the second step replaces the uniform weights in the standard local polynomial RD estimator (without covariates) with the EB weights. The EB estimator can also be formulated as an empirical likelihood (EL) estimator, for

\[ 1 \text{In a recent study, Hyytinen et al. (2018) confirmed that RD produces estimates that are in line with the results from a comparable experiment if inference is implemented with the method of Calonico et al. (2014).} \]
which covariate balance translates to a set of over-identifying LP moment conditions and is used as “side information.” Therefore, our approach explicitly incorporates the covariate balance condition, which is treated as a maintained assumption in CCFT, into the estimation and inference procedure. We show in Theorem 1 that the EB (EL) estimator is first-order equivalent to the regression adjustment estimator of CCFT. Although CCFT doubted whether covariate adjustment can always lead to asymptotic efficiency gain in RD estimation, it has been pointed out by Kreiß and Rothe (2022) that this is true. This paper provides an explanation of the asymptotic efficiency gain brought by covariate adjustment from the perspective of the generalized method of moments (GMM): the efficiency gain can be attributed to the efficient inclusion of covariate balance as side information (Remark 3). We also offer another explanation of the efficiency gain from the perspective of local randomization (Remark 4). Under CCFT’s stronger version of covariate balance (see Page 446 of CCFT), incorporating functions of baseline covariates can further improve efficiency. Theorem 3 shows that the asymptotic variance of the EB estimator incorporating basis functions of baseline covariates attains the lower bound derived in Noack et al. (2021), if the number of basis functions (i.e., the dimension of the corresponding linear sieve spaces) grows with the sample size.

Since the EB estimator can be formulated as an EL estimator, we expect that the favorable second-order properties (Newey and Smith, 2004) may also be shared by the EB estimator. Theorem 2 shows that the EB estimator has a small (bounded) “nonlinearity bias”. Such a property is analogous to Newey and Smith (2004, Theorem 4.5). Then, we study covariate-adjusted EL inference for RD. A common advantage of EL inference is that it does not require calculating standard errors and explicit studentization. Theorem 4 shows that the EL confidence set is a finite interval with probability approaching one. Theorem 5 shows a new uniform-in-bandwidth extension of the standard Wilks theorem (i.e., the EL ratio is asymptotically $\chi^2$). Our uniform-in-bandwidth version adjusts for specification search over multiple bandwidths, known as bandwidth snooping (Armstrong and Kolesár, 2018b, AK, hereafter), and takes into account the effects from data-dependent bandwidths in a robust manner (Remarks 12 and 15). It also provides a useful tool for sensitivity analysis in the sense of AK (Remark 14). By deriving distributional expansions, we investigate the second-order properties of our EL inference method and show that it enjoys a couple of nice properties in this setting. Theorem 6 characterizes the leading coverage error term (i.e., the discrepancy between the nominal and finite-sample coverage probabilities; see, e.g., Calonico et al., 2020 for Wald-type inference). We consider two choices of the LP order: one less than the assumed smoothness ($p$-th order) and exhausting the smoothness ($(p + 1)$-th order). In the first case, the coverage optimal (CO) bandwidth, which is defined as the minimizer of this leading coverage error, has a simple closed form (Remark 16), which, to the best of our knowledge, cannot be obtained for Wald-type inference (Calonico et al., 2020). In both cases, the simple coverage expansion for the EL confidence sets makes analytical correction possible. The correction aims to remove the leading term in the coverage error and does not require resampling. The correction factor has a very simple form and thus can be estimated with good accuracy in finite samples. Remark 18 proposes Analytically corrected likelihood ratio statistics and confidence sets for conducting covariate-adjusted RD inference.
Remark 19 combines the analytical correction and AK-type correction (Remark 14) and provides a more accurate uniform confidence band that is useful for sensitivity analysis and robust inference.

Theorem 7 considers possible deviations from covariate balance and shows that the coverage accuracy of our proposed EL confidence set is highly insensitive to mild deviations (Remark 20), which we refer to as local imbalance in this paper. Failure of the covariate balance assumption may happen in a realistic situation when the balance condition holds for pre-treatment covariates in theory, but our sample observations on these covariates are contaminated (possibly due to measurement errors that occur after treatment) so that they are drawn from a perturbed population (Kitamura et al., 2013) that slightly violates the balance condition. When covariate balance does not hold exactly, the coverage accuracy of the EL confidence set stays relatively unaffected, while other inference methods may exhibit severe undercoverage (Remark 20). To the best of our knowledge, such a robustness property is novel in the literature.

Our balancing approach is versatile in dealing with covariate-adjustment estimation/inference for parameters and/or models beyond the standard RD, such as the treatment effect derivative (TED) of Dong and Lewbel (2015) and nonlinear estimators for RD with limited outcome variables (e.g., Xu, 2017, 2018). An algorithmic extension of CCFT’s regression adjustment may not be straightforward in these scenarios. Indeed, applying our EB approach is about reweighting a sample-analogue-type estimator (without covariate) in the RD-related context using the EB weights that are fully determined by the covariate balance condition. It does not matter if the initial estimator (without covariate) involves derivative or nonlinear transformation. For this reason, our balancing approach serves as a useful complement to the regression adjustment. We consider the following example in this paper. In addition to the standard RD LATE parameter in the standard RD model, one may be interested in estimating other parameters that have important causal interpretations, such as the TED as a measurement of the external validity of RD. Theorem 8 shows the efficiency gain of the simple TED estimator using our EB weights in place of uniform weights, for which the only assumption needed for consistency is covariate balance. Another class of problems that our approach can tackle is nonlinear estimators with limited outcome variables (e.g., Xu, 2017, 2018). Estimators of Xu (2017, 2018) using the EB weights achieve desired properties (consistency and potential efficiency gain) under covariate balance. Lastly, various extensions to the standard RD model and estimation of the relevant causal parameters have been considered in the recent literature. Our approach has the potential to provide easy-to-implement covariate adjustment with clear causal interpretation. Further investigation is needed in a case-by-case manner.

Related literature. Our EB estimator resembles the method of Hainmueller (2012); Chan et al. (2016) in the literature on balancing methods for estimating ATE under unconfoundedness. See Wong and Chan (2017); Kallus (2020); Hirshberg and Wager (2021) for more recent development of this strand of literature. Graham et al. (2012) show that their balancing-type estimator enjoys a similar small nonlinearity bias property. EL and generalized EL (Newey and Smith, 2004) are popular alternatives to GMM, and they do not require first-step estimation of the efficient weighting matrix. See, e.g., Kitamura (2006) for a comprehensive review of EL and generalized EL. See,
e.g., Chen and Qin (2000); Otsu et al. (2013, 2015); Ma et al. (2019) for EL inference in the context of non-parametric curves. It was shown that EL has favorable properties relative to GMM. See, e.g., Chen and Cui (2007); Kitamura (2001); Matsushita and Otsu (2013); Newey and Smith (2004); Otsu (2010); Ma (2017) among many others. In relation to the literature, Otsu et al. (2015) proposed EL inference for RD without covariates. Their method was based on first-order conditions from standard local linear regression. This paper focuses on covariate adjustment and uses different moment conditions. In another related paper, Ma et al. (2019) studied EL inference for the parameter of interest in the density discontinuity design (Jales and Yu, 2016). Our paper uses a similar approach to covariate adjustment as Wu and Ying (2011); Zhang (2018) who formulated covariate balance in randomized experiments as moment conditions and proposed EL-type methods. We formulate local imbalance and study its impact on coverage accuracy by using standard local asymptotic analysis (e.g., the Pitman approach to local power analysis). Local imbalance can also be viewed as a special case of local misspecification in the GMM framework (see, e.g., Armstrong and Kolesár, 2021 and references therein). However, the approach we take differs from those employed by papers in this strand of literature. Our approach follows Bravo (2003) and is based on the second-order asymptotic expansion of the coverage probability under drifting data-generating processes (i.e., local imbalance).

**Organization.** Section 2 quickly reviews the RD design. Section 3 introduces our EB method for RD with covariates. Section 4 provides results on the asymptotic properties of the EB estimator, including asymptotic normality with a discussion on the efficiency gain (Section 4.1), calculation of the nonlinearity bias (Section 4.2) and extension to balancing over sieve spaces (Section 4.3). In Section 5, we consider inference using the likelihood ratio and show several properties, including a uniform-in-bandwidth Wilks theorem (Section 5.1), derivation of a simple analytical correction (Section 5.2), and sensitivity of the coverage probability to the covariate balance condition (Section 5.3). Section 6 proposes a covariate-adjusted estimator of the TED and provides an asymptotic normality result that shows the efficiency gain. Sections 7 and 8 present results from simulation and empirical exercises. Section 9 concludes. Proofs are collected in the online appendix (available at [ruc-econ.github.io/supplement_Rev_V12.pdf](https://ruc-econ.github.io/supplement_Rev_V12.pdf)).

**Notation.** $\sum_i$ is understood as $\sum_{i=1}^n$. “$a := b$” means that $a$ is defined by $b$ and “$a =: b$” means that $b$ is defined by $a$. For any $k$-times differentiable univariate function $f$, let $f^{(k)}$ denote the $k$-th order derivative. Let $1(\cdot)$ denote the indicator function. For a $d$-dimensional vector $x$, let $x^{(j)}$ denote its $j$-th coordinate, $x^\top$ denote its transpose, $x^{\otimes k}$ denote a vector of the distinct entries of $k$-th Kronecker power for $k = 2, 3, 4$ ($x^{\otimes 2} := \text{vech} \left(xx^\top\right)$, where $\text{vech} \left(xx^\top\right)$ denotes the half vectorization of $xx^\top$, $x^{\otimes 3}$ is the vector obtained by stacking $\left\{x^{(j)}\text{vech} \left(x_jx_j^\top\right) : j = 1, \ldots, d\right\}$, where $x_j := (x^{(j)}, \ldots, x^{(d)})^\top$, and $x^{\otimes 4}$ can be defined similarly) and $\|x\|$ denote its Euclidean norm. Let $1_K$ denote the $K$-dimensional identity matrix. Let $0_{J \times K}$ denote the $J \times K$ matrix in which all elements are zeros. Let $0_J$ denote the $J$-dimensional vector in which all elements are zeros. $A^{(jk)}$ denotes the $jk$-th element of a matrix $A$. For a square matrix $A$, let $\text{tr} (A)$ denote its trace and $\text{mineig} (A)$ and $\maxeig (A)$ denote the smallest and the largest eigenvalues, respectively. For a real-valued function $f : \mathcal{X} \to \mathbb{R}$, let $\|f\|_\infty := \sup_{x \in \mathcal{X}} |f(x)|$ denote the sup-norm. We write $a_n \asymp b_n$, if $a_n = O (b_n)$ and
2 Regression discontinuity designs

Let $X \in \mathbb{R}$ be a continuous score supported on $[\underline{x}, \overline{x}]$. Let $f_X$ denote its density function. We normalize the cutoff point to zero (so that $0 \in [\underline{x}, \overline{x}]$ without loss of generality) for notational brevity. In this paper, we assume that $f_X$ is continuous at the cutoff. Denote $\varphi := f_X(0)$ for simplicity. For a random vector (or matrix) $V$, denote $g_V(x) := E[V \mid X = x]$, $m_V(x) := g_V(x) f_X(x)$ and $g_{V|Z}(z,x) := E[V \mid Z = z, X = x]$. Denote $\mu_V^{(k)} := \lim_{x \to 0} g_V^{(k)}(x)$ and $\psi_V^{(k)} := \lim_{x \to 0} m_V^{(k)}(x)$. $(\mu_V^{(k)}, \psi_V^{(k)})$ are defined similarly with $\lim_{x \to 0}$ replaced by $\lim_{x \to a}$. For simplicity, also denote $\mu_{V,s} := \mu_{V,s}^{(0)}$, $\psi_{V,s} := \psi_{V,s}^{(0)} (s \in \{-, +\})$, $\mu_{V,s} := \mu_{V,s}^{(+)} + \mu_{V,s}^{(-)}$, $\mu_{V,s} := \mu_{V,s}^{(+) - \mu_{V,s}^{(-)}}, \psi_{V,s} := \psi_{V,s}^{(+)} + \psi_{V,s}^{(-)}$. Let $\mu_V$ (or $\psi_V$) denote the common value if $\mu_{V,+} = \mu_{V,-}$ (or $\psi_{V,+} = \psi_{V,-}$). For random vectors $V$ and $U$, $\text{Var}_{[0]}[U]$ is understood as $\lim_{x \downarrow 0} \text{Var}[U \mid X = x] = \mu_{UU\uparrow,+} - \mu_{UU\uparrow,-}$ and $\text{Cov}_{[0]}[V, U]$ is understood as $\lim_{x \downarrow 0} \text{Cov}[V, U \mid X = x] = \mu_{VV\uparrow,+} - \mu_{VV\uparrow,-}$. Similarly, $\text{Var}_{[0]}[U] := \lim_{x \downarrow 0} \text{Var}[U \mid X = x]$ and $\text{Cov}_{[0]}[V, U] := \lim_{x \downarrow 0} \text{Cov}[V, U \mid X = x]$. Also for notational simplicity, let $\text{Var}_{[0]}[U] := \text{Var}_{[0]}[U] + \text{Var}_{[0]}[U]$ and $\text{Cov}_{[0]}[V, U] := \text{Cov}_{[0]}[V, U] + \text{Cov}_{[0]}[V, U]$. $\text{Var}_{[0]}$ and $\text{Cov}_{[0]}$ are understood as $\text{Var} \cdot \mid X = 0$ and $\text{Cov} \cdot \mid X = 0$.

Let $Y \in \mathbb{R}$ denote the outcome variable, $D \in \{0, 1\}$ be the binary treatment and $Z$ be pretreatment covariates. Variables in $Z$ can be continuous, discrete, or mixed. We observe $(Y, D, Z)$ and the score $X$. In an RD model, an incentive is assigned if $X \geq 0$. In the sharp RD case $D = I := 1(X \geq 0)$ (i.e., perfect compliance). The more general fuzzy RD model assumes $D \neq I$ but $g_D$ has a jump discontinuity at $x = 0$ ($\mu_{D,+} \neq \mu_{D,-}$) due to the incentive. This is known as limited compliance in the literature. The RD model can be embedded in the potential outcome and treatment framework. Let $(Y(1), Y(0))$ be the potential outcomes with or without treatment. Let $(D(1), D(0))$ denote the potential treatments with or without incentives. The observed outcome $Y$ and treatment $D$ are determined by $Y = D \cdot Y(1) + (1 - D) Y(0)$ and $D = I \cdot D(1) + (1 - I) D(0)$ respectively. The complier group is defined to be individuals with $D(1) > D(0)$ (i.e., $(D(1), D(0)) = (1, 0)$). We use “co” to denote this event. Following CCFT, we let $(Z(1), Z(0))$ denote potential covariates and then $Z = D \cdot Z(1) + (1 - D) Z(0)$.

Let $B(d) := (Y(d), Z(d))$, for $d \in \{0, 1\}$. Denote $g_{\text{def}}(x) := \text{Pr}[D(1) = d, D(0) = d' \mid X = x]$ and $g_{B(j) \text{def}}(x) := E[B(j) \mid D(1) = d, D(0) = d', X = x]$ for $(j, d, d') \in \{0, 1\}$. The RD LATE (the average treatment effect for individuals with zero score in the complier group) is given by $E[Y(1) - Y(0) \mid X = 0, \text{co}]$. The following assumption is sufficient for the identification in RD and is also imposed in CCFT.

Assumption 1. (a) $(g_{Y(1)|11}, g_{Y(0)|00}, g_{Y(1)|10}, g_{Y(0)|10})$ are all continuous at the threshold 0; (b) $b_n = O(a_n)$. Let $e_{k,s}$ denote the $s$-th unit vector in $\mathbb{R}^k$. 

\footnote{The RD design can be represented by a structural model. See Dong (2018). $(Y, D, Z)$ are assumed to be generated by the structural model $Y = g(D, X, Z, \epsilon), D = h(I, X, \eta)$ and $Z = m(D, X, \xi)$, where $(g, h, m)$ are unknown functions and $(\epsilon, \eta, \xi)$ are (potentially correlated) unobserved disturbances of unrestricted dimensionality. Then the potential outcomes, covariates and treatments are given by $Y(d) = g(d, X, Z, \epsilon), D(d) = h(d, X, \eta)$ and $Z(d) = m(d, X, \xi)$.}
$g_{dd'}$ is continuous at the threshold 0 for all $(d,d') \in \{0,1\}^2$; (c) $\Pr[D(1) \geq D(0) \mid X = 0] = 1$; (d) $\Pr[co \mid X = 0] > 0$; (e) $(gZ(1)|11, gZ(0)|00, gZ(1)|10, gZ(0)|10)$ are all continuous at the threshold 0; (f) $gZ(1)|10(0) = gZ(0)|10(0)$.

It can be shown that under (a,b,c,d), the RD LATE is identified by the standard RD estimand $\vartheta := \mu_{Y,d\dagger}/\mu_{D,d\dagger}$ (i.e., $E[Y(1) - Y(0) \mid X = 0, co] = \vartheta$), see Hahn et al., 2001; Dong, 2018 and Arai et al., 2021 for testable implications of these assumptions), where $\vartheta$ is a population feature of the observed variables. As in Frölich and Huber (2019), the continuity assumption (a) can be viewed as an exclusion restriction. Intuitively, continuity of $g_{Y(j)|dd'}$ essentially requires that $Y(j)$ cannot depend on $I$ or (observed or unobserved) variables related to $I$ (so that their distributions change discontinuously at the cutoff). Since $Y(j)$ often depends on $Z$, continuity of $g_{Y(j)|dd'}$ also implicitly requires that the conditional distributions of $Z$ given $(D(1), D(0), X) = (d,d', x)$ change smoothly at $x = 0$. Since the distribution of $Z$ coincides with that of $Z(d) (Z(d'))$, given $(D(1), D(0), X) = (d,d', x)$ with $x \geq 0 (x < 0)$, continuity of the conditional distribution of $Z$ given $(D(1), D(0), X) = (d,d', x)$, for $(d,d') \in \{(1,1), (0,0), (1,0)\}$, holds if the conditional distributions of the potential covariates change smoothly at $x = 0$ and the distribution of $Z(1)$ given $co$ and $X = 0$ is the same as that of $Z(0)$ given $co$ and $X = 0$. Following CCFT, we consider using weaker versions of these assumptions in (e,f). We consider using the strong versions in Section 4.3. (e) essentially requires that the covariates satisfy the same exclusion restriction (not affected by $I$). It is clear from $g_Z(x) = \sum_{d,d'} g_{Z|dd'}(x) g_{dd'}(x)$, where “$\sum_{d,d'}$” is understood as “$\sum_{(d,d') \in \{0,1\}^2}$”, that covariate balance $\mu_{Z,+} = \mu_{Z,-}$ holds as a testable implication for the population of the observed variables.

3 Empirical entropy balancing

This section introduces the EB method. We quickly review the idea of entropy balancing and reweighting in the literature on ATE estimation under unconfoundedness (i.e., conditional independence of the potential outcomes and the treatment given the covariates). Then, we utilize the idea of EB to propose a new balancing-based method for covariate adjustment for RD.

In observational studies, because of the selection bias, the difference in the sample means corresponding to the treatment and control groups does not consistently estimate the ATE. The balancing weights satisfy the requirement that the weighted control (treatment) group sample moments of the covariates match the unweighted sample moments of the covariates of all units. Within all balancing weights, Hainmueller (2012) defines the EB weights as those being as close as possible to the uniform

\[3\text{In the sharp RD model (} \mu_{D,+} = 1 \text{ and } \mu_{D,-} = 0 \text{ in this case) or under a stronger conditional independence assumption (Hahn et al., 2001), a causal parameter that corresponds to a broader subpopulation (conditional average treatment effect) is identified by the same ratio: } E[Y(1) - Y(0) \mid X = 0] = \vartheta .

\[4\text{Indeed, } \mu_{Z,+} = \mu_{Z,-} \text{ is the null hypothesis of a popular falsification or placebo test for the RD model. See, e.g., Lee (2008); Canay and Kamat (2017). Evidence against } \mu_{Z,+} = \mu_{Z,-} \text{ in the data (so that a hypothesis test of } \mu_{Z,+} = \mu_{Z,-} \text{ is rejected) casts doubts on the validity of the key identifying assumption of the RD design (i.e., Assumption 1(a)). While most empirical works conduct the balance test separately for each covariate, some researchers have noted that the problem of multiple testing may generate statistical imbalance of some covariates by chance. See, e.g., Hyttinen et al. (2018).} \]
weights in the sense of minimal relative entropy (KL divergence). Hainmueller (2012) replaces the uniform weights used by the simple sample means with the EB weights. Chan et al. (2016) construct EB weights that equalize weighted and unweighted sample means of transformations of the covariates via basis functions. Chan et al. (2016) show that the estimator using these EB weights overcomes the selection bias under the unconfoundedness assumption if the number of basis functions of the covariates increases with the sample size.

From a GMM/EL perspective, in Hainmueller (2012); Chan et al. (2016), the entropy balancing and reweighting approach uses weights under which some intentionally misspecified (biased) moment restrictions are satisfied to correct for the selection bias.\(^5\) In our RD case, the moment restrictions (balancing constraints) are correctly specified and entropy balancing and reweighting aim at enhancing efficiency (Section 3.1). In our case, the EB estimator can be formulated as a standard EL estimator (Section 3.3).

### 3.1 Entropy balancing for covariate adjustment in RD

Now we elaborate on the entropy balancing and reweighting approach to covariate adjustment in the RD context. Firstly, we introduce some notations. Let \( K \) denote the kernel function and let \( h \) denote the bandwidth. We assume that \( h = h_n \) decreases with the sample size \( n \). For notational simplicity, we suppress the dependence of \( h \) on \( n \). Let the data \( \{(Y_i, D_i, X_i, Z_i)\}_{i=1}^n \) be i.i.d. copies of \((Y, D, X, Z)\). We drop the subscript \( i \) when we refer to population-level estimands. Let \( p \geq 1 \) be the integer-valued LP order. Let \( r_p(t) := (1, t, \ldots, t^p)^\top \) and let \( H \) be the \((p+1) \times (p+1)\) diagonal matrix with \((1, h, \ldots, h^p)\) being on the diagonal. Denote

\[
\hat{\Pi}_{p,-} := \frac{1}{nh} \sum_i r_p \left( \frac{X_i}{h} \right) r_p^\top \left( \frac{X_i}{h} \right) K \left( \frac{X_i}{h} \right) 1 (X_i < 0). \tag{1}
\]

Let \( \hat{\Pi}_{p,+} \) be defined similarly by the right-hand side of (1) with \( 1 (X_i < 0) \) replaced by \( 1 (X_i > 0) \). Let

\[
\hat{W}_{p,-,i} := e_p r_p(1) \hat{\Pi}_{p,-}^{-1} r_p \left( \frac{X_i}{h} \right) K \left( \frac{X_i}{h} \right) 1 (X_i < 0). \tag{2}
\]

Let \( \hat{W}_{p,+} \) be defined similarly by the right-hand side of (1) with \( 1 (X_i < 0) \) and \( \hat{\Pi}_{p,-} \) replaced by \( 1 (X_i > 0) \) and \( \hat{\Pi}_{p,+} \).\(^6\)

Let \( \hat{W}_{p,i} := \hat{W}_{p,+} - \hat{W}_{p,-} \). The standard LP regression estimator of \( \vartheta \) is

\[
\hat{\vartheta}_p^\vartheta := \frac{1}{nh} \sum_i \hat{W}_{p,i} Y_i, \tag{3}
\]

\[^5\text{In observational data, the population moments of covariates in the control or treatment group may not be the same as the unconditional population moments, since the treatment status is not independent from the covariates.}\]

\[^6\text{We restrict the bandwidths on the left and the right of the cut-off to be the same. It is possible to extend all of the theorems in this paper to accommodate different bandwidths on different sides.}\]
where the numerator \((nh)^{-1} \sum_i \widehat{W}_{p,i} Y_i\) is the LP regression estimator of \(\mu_{Y,t}\) and the denominator is the LP regression estimator of \(\mu_{D,t}\).

Now we incorporate the covariate information to the standard LP estimator \(\hat{\theta}_p\) by reweighting its numerator and denominator using the EB weights computed from the covariate balance constraints. Denote \(Z_i := (1, Z_i^\top)^\top\). We define EB weights \((w_1^{eb}, ..., w_n^{eb})\) as the solution to the following minimum relative entropy problem:

\[
\min_{w_1, ..., w_n} KL \left( \frac{1}{n}, ..., \frac{1}{n} \right) \quad \text{subject to} \quad \sum_i w_i \widehat{W}_{p,i} Z_i = 0_{d_z+1}, \quad \sum_i w_i = 1, \tag{4}
\]

where \(KL(w_1, ..., w_n \| 1/n, ..., 1/n) := -\sum_i \log(n \cdot w_i) / n\) is the KL divergence from \((w_1, ..., w_n)\) to the uniform weights \((1/n, ..., 1/n)\). The construction of these balancing weights is similar to those in Hainmueller (2012). By solving the minimization problem (4), we find the set of weights with the least KL divergence from the uniform weights among these balancing weights. The uniform weights satisfy the important finite-sample property of \(\sum_i \widehat{W}_{p,i} = 0\). By requiring \(\sum_i w_i \widehat{W}_{p,i} = 0\) in the constraint of (4), we require that the balancing weights satisfy the same property. The balancing weights should also satisfy \(\sum_i w_i \widehat{W}_{p+i,i} Z_i = \sum_i w_i \widehat{W}_{p-i,i} Z_i\). This requires that the (kernel-weighted) local averages of the covariates on both sides of the thresholds coincide in finite samples under the new weights for the data points. To solve for the optimal weights, we use strong duality and concentrate out \((w_1, ..., w_n)\) to obtain the following dual characterization

\[
w_i^{eb} = \frac{1}{n} \cdot \frac{1}{1 + (\lambda_p^{eb})^\top \left( \widehat{W}_{p,i} Z_i \right)}, \tag{5}
\]

where

\[
\lambda_p^{eb} := \argmax_{\lambda} \sum_i \log \left( 1 + \lambda^\top \left( \widehat{W}_{p,i} Z_i \right) \right) \tag{6}
\]

is the Lagrangian multiplier. Computing the EB weights requires dealing with a well-understood convex optimization problem (6) that can be solved by the Newton algorithm. The domain of its objective function is the convex set \(\left\{ \lambda : 1 + \lambda^\top \left( \widehat{W}_{p,i} Z_i \right) > 0 \text{ for all } i \right\}\). The algorithm should either take these constraints into account or use a modified objective function defined for all \(\lambda \in \mathbb{R}^{d_z+1}\).

(6) has no solution if the origin is not an interior point of the convex hull of \(\{ \widehat{W}_{p,1} Z_1, ..., \widehat{W}_{p,n} Z_n \}\). See Kitamura (2006) and Owen (2001) for more algorithmic details.\(^7\)

\(^7\)By arguments similar to those in Owen (2001, Chapter 11.2), we can show that if covariate balance holds, the origin lies in the convex hull with probability approaching one.

\(^8\)In the “no solution” scenario, the Newton algorithm trying to solve (6) returns a sequence of vectors with diverging lengths. In this scenario, after the algorithm terminates (either the gradient is sufficiently small or the maximal number of iterations is reached), we would get weights not summing up to one in the former case (Owen, 2001, Chapter 3.14) or a large gradient in the latter case. In our simulation studies and computation for the empirical application, we use the Matlab code written by Kirill Evdokimov and Yuichi Kitamura (https://kitamura.sites.yale.edu/matlabstata-codes-el) and hardly see any “no solution” case.
We propose the following empirical entropy balancing estimator by reweighting the numerator and denominator of $\hat{\vartheta}_p$ in (3) using the EB weights $w_i^{eb}$ defined by (5):

$$
\hat{\vartheta}_p^{eb} := \frac{\sum_i w_i^{eb} \widehat{W}_{p,i} Y_i}{\sum_i w_i^{eb} \widehat{W}_{p,i} D_i}.
$$

(7)

The reweighting form of our EB estimator $\hat{\vartheta}_p^{eb}$ has a clear causal interpretation. The continuity and predeterminedness assumptions in Assumption 1 imply a restriction $\mu_{Z,+} = \mu_{Z,-}$ on the population distribution of the observed covariates. $\hat{\vartheta}_p^{eb}$ directly uses weights that explicitly exploit such information from the covariates.

### 3.2 Generalized entropy balancing

The entropy balancing approach looks for balancing weights closest to the uniform weights, where “closeness” is measured by the KL divergence. It is useful to consider the following extension. Let $(p_1, \ldots, p_n)$ and $(p'_1, \ldots, p'_n)$ be two sets of probability masses. For any $\varrho \in \mathbb{R}$, let

$$
D_{\varrho} (p_1, \ldots, p_n \parallel p'_1, \ldots, p'_n) := \frac{1}{\varrho (1 + \varrho)} \sum_i \left\{ \left( \frac{p_i}{p'_i} \right)^{-\varrho} - 1 \right\} p'_i
$$

be the Cressie-Read divergence from $(p_1, \ldots, p_n)$ to $(p'_1, \ldots, p'_n)$. Taking $\varrho = 0$ gives the KL divergence from $(p_1, \ldots, p_n)$ to $(p'_1, \ldots, p'_n)$. Taking $\varrho = -1$ gives the KL divergence from $(p'_1, \ldots, p'_n)$ to $(p_1, \ldots, p_n)$.\(^9\) The generalized balancing estimator is based on the weights $(w_{gb,1}^{eb}, \ldots, w_{gb,n}^{eb})$ that solve

$$
\min_{w_1, \ldots, w_n} D_{\varrho} \left( w_1, \ldots, w_n \parallel \frac{1}{n}, \ldots, \frac{1}{n} \right)
$$

subject to $\sum_i w_i \widehat{W}_{p,i} Z_i = 0_{d_z+1}$, $\sum_i w_i = 1$.

(9)

We define the generalized balancing estimator $\hat{\vartheta}^{gb}_{\varrho,p}$ by the right hand side of (7) with $w_i^{eb}$ replaced by $w_i^{gb,\varrho}$.

Interestingly, although CCFT takes an augmented regression approach to incorporate pre-treatment covariates, a slight modification of CCFT’s estimator can be written as a generalized balancing estimator with $\varrho = -2$. It is easy to see that $D_{-2} (w_1, \ldots, w_n \parallel 1/n, \ldots, 1/n)$ is proportional to the square of the Euclidean distance between $(w_1, \ldots, w_n)$ and the uniform weights. CCFT’s covariate adjusted estimator $\hat{\vartheta}_p^{CCFT}$ for $\mu_{Y,i}$ is given by the regression coefficient of $I_i := 1 (X_i \geq 0)$ in

$$
\hat{\vartheta}_p^{CCFT} := e_2^{\top \varrho+1+d_z+2} \arg\min_{b_0, b_1, b_2} \sum_i K \left( \frac{X_i}{h} \right) \left\{ Y_i - \varrho_p (X_i) b_0 - I_i \cdot \varrho_p (X_i) b_1 - Z_i b_2 \right\}^2.
$$

(10)

\(^9\) $D_0 (p_1, \ldots, p_n \parallel p'_1, \ldots, p'_n)$ (or $D_{-1} (p_1, \ldots, p_n \parallel p'_1, \ldots, p'_n)$) is defined as the limit of the right hand side of (8) as $\varrho \to 0$ (or $\varrho \to -1$).
Similarly, CCFT’s estimator \( \hat{\vartheta}_{D,p}^{CCFT} \) for \( \mu_{D,\dagger} \) is defined by the right hand side of the above equation with \( Y_i \) replaced by \( D_i \). Then CCFT’s estimator of \( \vartheta \) is \( \hat{\vartheta}_{Y,p}^{CCFT} = \hat{\vartheta}_{Y,p}^{CCFT} / \hat{\vartheta}_{D,p}^{CCFT} \). CCFT shows that by the partitioned regression argument, the numerator \( \hat{\vartheta}_{Y,p}^{CCFT} \) can be written as

\[
\hat{\vartheta}_{Y,p}^{CCFT} = \frac{1}{nh} \sum_i \hat{W}_{p,i} \left( Y_i - Z_i^\top \hat{\gamma}_{Y}^{CCFT} \right),
\]

where \( \hat{\gamma}_{Y}^{CCFT} \) is a consistent estimator of \( \gamma_Y := \left( \text{Var}_{\mid Z} \left[ Z \right] \right)^{-1} \text{Cov}_{\mid Z} \left[ Z, Y \right] \). A similar result holds for \( \hat{\vartheta}_{D,p}^{CCFT} \). To find the optimal weights that solve (9) with \( \varrho = -2 \), we again apply the Lagrangian multiplier method. What differs from EB is that in this case the Lagrangian multiplier has an explicit form. Then we can see that \( \sum_i w_{gb,2} \hat{W}_{p,i}Y_i/h \) (or \( \sum_i w_{gb,2} \hat{W}_{p,i}D_i/h \)) can be written in the form of (11) with a slightly different estimator of \( \gamma_Y \) (or \( \gamma_D \)). More details can be found in Section S9 of our online supplement. Therefore, our formulation provides information-theoretic and balancing interpretation of CCFT’s estimator. The generalized balancing estimators are all first-order equivalent to the EB estimator in the sense that the conclusion of Theorem 1 also holds for them.

### 3.3 Connection to empirical likelihood

We now show that the EB estimator can be formulated as an EL estimator which incorporates covariate balance as side information. The RD estimand \( \vartheta \), which has causal interpretation under the identifying assumptions of the RD model, can be approximately identified by a moment condition. Note that

\[
\lim_{x \downarrow 0} E \left[ Y - \theta D \mid X = x \right] = \lim_{x \uparrow 0} E \left[ Y - \theta D \mid X = x \right] \text{ if and only if } \theta = \vartheta.
\]

By the standard LP regression theory, we have

\[
\frac{1}{nh} \sum_i \hat{W}_{p,i} (Y_i - \theta D_i) \to_p \lim_{x \downarrow 0} E \left[ Y - \theta D \mid X = x \right] - \lim_{x \uparrow 0} E \left[ Y - \theta D \mid X = x \right],
\]

under standard assumptions. Under covariate balance, \( (nh)^{-1} \sum_i \hat{W}_{p,i} Z_i \to_p 0_{d_z+1} \). We consider the following EL-type criterion function:

\[
\ell_p^\text{el} (\theta \mid h) := \min_{w_1, \ldots, w_n} KL \left( w_1, \ldots, w_n \mid \frac{1}{n}, \ldots, \frac{1}{n} \right)
\]

subject to \( \sum_i w_i \hat{W}_{p,i} \left( Y_i - \theta D_i \right) / \hat{Z}_i = 0_{d_z+2}, \sum_i w_i = 1. \) (12)

Note that we have \( 2 + d_z \) LP moment conditions that approximately identify one parameter of interest \( \vartheta \). Note that the covariate balance condition provides a set of over-identifying moment restrictions. We can easily see that the EB estimator is also an EL estimator, defined as a minimizer of the EL criterion function. It is clear that \( \ell_p^\text{el} (\theta \mid h) \geq -n^{-1} \sum_i \log (n \cdot w_i^\text{eb}) \) for all \( \theta \), since
\(-n^{-1} \sum_i \log (n \cdot w_i^\text{eb})\) is the minimum corresponding to a larger constraint set. Since the constraint set of (12) with \( \theta = \hat{\vartheta}_p^\text{el} \) contains the EB weights, we have \( \ell_p^\text{el} (\hat{\vartheta}_p^\text{el} \mid h) \leq -n^{-1} \sum_i \log (n \cdot w_i^\text{eb}) \). Therefore, \( \hat{\vartheta}_p^\text{el} \) is a minimizer of \( \ell_p^\text{el} (\cdot \mid h) \).

We consider replacing the kernel-dependent weight \( \hat{W}_{p,i} \) in (12) by the weight from population-level LP fitting (or solving a minimum contrast (MC) problem) and define the MC-EL estimator as the minimizer. See Bickel and Doksum (2015, Chapter 11.3) for more details about the construction of the population-level LP fitting. Denote \( V_{p,-} := \int_{-1}^{0} r_p(t) r_p^\top(t) K(t) \, dt \) and \( \mathcal{K}_{p,-} (t) := e_{p+1,1}^\top V_{p,-}^{-1} r_p(t) K(t) \). Let \( (V_{p,+}, \mathcal{K}_{p,+}) \) be defined by the same equations with the integral range \([-1,0] \) replaced by \([0,1] \). \( (\mathcal{K}_{p,+}, \mathcal{K}_{p,-}) \) coincide with the “equivalent kernel” associated with the LP regression. See, e.g., Section S2.1 of AK. Let \( W_{p,-,i} := \mathbb{I} (X_i < 0) \mathcal{K}_{p,-} (X_i/h) \), \( W_{p,+;i} := \mathbb{I} (X_i > 0) \mathcal{K}_{p,+} (X_i/h) \) and \( W_{p,i} := W_{p,+;i} - W_{p,-;i} \). By Taylor expansion (see Jiang and Doksum, 2003), \( E [W_p (Y - \theta D)] = O (h^{p+2}) \) if and only if \( \theta = \vartheta \) and \( E [W_p \cdot \bar{Z}] = O (h^{p+2}) \) under suitable smoothness assumptions. Let the MC-EL criterion function \( \ell_p^\text{mc} \) be defined by the right hand side of (12) with \( \hat{W}_{p,i} \) replaced by \( W_{p,i} \). The \( p \)-th order MC-EL estimator is given by \( \hat{\vartheta}_p^\text{mc} := \argmin_{\vartheta} \ell_p^\text{mc} (\vartheta \mid h) \). Similar derivations show that the estimator can be written as a balancing-type estimator. \( \hat{\vartheta}_p^\text{mc} \) is equal to the right hand side of (7) with \( (\hat{W}_{p,i}, w_i^\text{eb}) \) replaced by \( (W_{p,i}, w_i^\text{mc}) \), where \( (w_i^\text{mc}, \lambda_i^\text{mc}) \) are defined by the right hand sides of (5) and (6) with \( \hat{W}_{p,i} \) replaced by \( W_{p,i} \). The MC-EL estimator has similar asymptotic properties as the EB estimator (see Theorems 1 and 2 ahead). The MC-EL criterion function is useful for constructing confidence sets for \( \vartheta \) with favorable second-order properties (see Section 5 ahead).

4 Properties of the empirical likelihood (balancing) estimator

In this section, we show several large-sample properties of the proposed empirical balancing (likelihood) estimator. Section 4.1 gives the asymptotic normality result of the estimators proposed in the preceding section. We compare our result with that of CCFT’s estimator and discuss the efficiency gain brought by the covariates. Section 4.2 gives a result on the “nonlinearity bias” of the proposed estimators. We argue that our estimators have small nonlinearity biases, especially in the situation when a relatively large number of valid covariates satisfying the balance condition are available. In Section 4.3, we consider the situation when strong continuity and predeterminedness assumptions hold. The main result in Section 4.3 shows that an extension of our EB estimator whose weights balance functions of covariates in a sequence of linear sieve spaces achieves the variance lower bound derived in Noack et al. (2021).

4.1 Efficiency gain

This section shows asymptotic normality of the EB and MC-EL estimators, and gives the expression for the asymptotic mean square error (AMSE). We then compare our results with the asymptotic
result from CCFT. Let $B := (Y, Z)^\top$, $\bar{B} := (Y, D, Z)^\top$ and $M := Y - \vartheta D$. The following assumptions are imposed on the population distribution of the observed variables. Let $\mathcal{B} \subseteq [\underline{x}, \overline{x}]$ denote a neighborhood around 0.

**Assumption 2.** (a) $g_B$ is $(p+1)$-times continuously differentiable on $\mathcal{B} \setminus \{0\}$ and $g_B^{(p+1)}$ is Hölder continuous with unknown exponent $\eta \in (0, 1]$; (b) $g_B^{(\otimes 2)}$ is uniformly continuous on $\mathcal{B} \setminus \{0\}$; (c) $f_X$ is $(p+1)$-times continuously differentiable on $\mathcal{B}$ and $f_X^{(p+1)}$ is Hölder continuous with unknown exponent $\eta \in (0, 1]$; (d) $\Var_{|0^+} \left[ (M, Z)^\top \right]$ and $\Var_{|0^-} \left[ (M, Z)^\top \right]$ are positive definite.

Assumption 2 parallels Assumption SA-5 of CCFT. We will invoke it directly in the proofs. These assumptions are satisfied under suitable conditions imposed on the population distribution of the latent variables as in Assumption 1. Since $B = D (1) (1) B (0)$ if $X \geq 0$ and $B = D (0) B (1) + (1 - D (0)) B (0)$ if $X < 0$, by the law of iterated expectations (LIE), for any function $\varphi(\cdot, \cdot)$, we have

$$g_{\varphi(B,D)}(x) = \begin{cases} \sum_{d,d'} g_{dd'}(x) g_{\varphi(B(d),d')} |dd'|(x) & \text{if } x \geq 0 \\ \sum_{d,d'} g_{dd'}(x) g_{\varphi(B(d'),d')} |dd'|(x) & \text{if } x < 0. \end{cases} \tag{13}$$

(a) is satisfied if $(g_B(d)|dd', g_B(d')|dd', g_{dd'})$ are $(p+1)$-times continuously differentiable on $\mathcal{B}$ with uniformly continuous derivatives for all $(d, d') \in \{0, 1\}^2$. The smoothness level in (a) is similar to that commonly assumed in the literature, i.e., the minimal smoothness level $(p+1)$ such that that the leading smoothing bias term of the estimator (using $p$-th order LP) can be explicitly characterized. (b) is satisfied if we impose the additional condition that for all $(d, d') \in \{0, 1\}^2$, $(g_B(d)|dd', g_B(d')|dd', g_{dd'})$ are uniformly continuous on $\mathcal{B}$. (a,c) also guarantee that $m_B = g_B f_X$ have uniformly continuous derivatives up to $(p+1)$-th order on the left and right neighborhoods of 0.

Existence of $\Var_{|0^+} \left[ (M, Z)^\top \right]$ and $\Var_{|0^-} \left[ (M, Z)^\top \right]$ is guaranteed by (b). By the law of total variance (writing $\Var \left[ (M, Z)^\top \right]$ as the sum of $\E \left[ \Var \left[ (M, Z)^\top \mid D (1), D (0), X \right] \right] \mid X$ and $\Var \left[ \E \left[ (M, Z)^\top \mid D (1), D (0), X \right] \right] \mid X$) and (b), $\Var_{|0^+} \left[ (M, Z)^\top \right]$ (or $\Var_{|0^-} \left[ (M, Z)^\top \right]$) is guaranteed to be positive definite if $\Var \left[ B (1) \mid X = 0, \sigma_0 \right]$ (or $\Var \left[ B (0) \mid X = 0, \sigma_0 \right]$) is positive definite.

**Assumption 3.** (a) $K$ is a symmetric continuous probability density function (PDF) supported on $[\overline{x}, \overline{y}]$; (b) $K_{p+}$ is differentiable with bounded first-order derivatives on $(-1, 0)$ and $(0, 1)$.

(a) is standard and also imposed in CCFT. (b) is also found in AK. (a) implies that $K_{p+}(t) = K_{p-}(-t)$ and therefore (b) also holds for $K_{p-}$. Denote $\omega_{p+}^0 := \int_0^1 t \nu \K_k \nu \K_k(0) dt$ and $\omega_{p-}^0 := \int_{-1}^0 t \nu \K_k \nu \K_k(0) dt$. It is easy to see that $\omega_{p+}^k = \omega_{p-}^k =: \omega_{p}^k$. Let $\gamma_M := \left( \Var_{|0^+} [Z] \right)^{-1} \Cov_{|0^+} [Z, M]$, $\epsilon := M - Z^\top \gamma_M$ and $\sigma^2 := \Var_{|0^+} [M] - \Cov_{|0^+} [M, Z] \cdot \gamma_M = \Var_{|0^+} [\epsilon]$. Existence of these quantities is guaranteed by Assumption 2(b). Under Assumption 2(d), $\sigma^2$ is strictly positive. Under Assumption 1, $\mu_{\epsilon, +} = \mu_{\epsilon, -} =: \mu_{\epsilon}$. Assumption 2(a) guarantees that $g_\epsilon$ and $m_\epsilon$ admit continuous derivatives up to $(p + 1)$-th order on the left and right neighborhoods of 0 so that the leading bias terms can
be characterized. For any $j \in \mathbb{N}$, $g_{\|B\|^j}$ is bounded on $\mathbb{B} \setminus \{0\}$ if $(g_{\|B(d)\|^j|dd^*}, g_{\|B(d')\|^j|dd^*}, g_{dd^*})$ are bounded on $\mathbb{B}$, for all $(d, d') \in \{0, 1\}^2$. The following result shows the asymptotic normality of the EB and MC-EL estimators.

**Theorem 1.** Suppose that Assumptions 1, 2 and 3 hold. Assume that $g_{\|B\|^4}$ is bounded on $\mathbb{B} \setminus \{0\}$. Assume that the bandwidth satisfies $nh^{2p+3} = O(1)$ and $nh \rightarrow \infty$. Then,

$$\sqrt{nh} \left( \hat{\theta}_{p}^{eb} - \theta - \mathcal{B}_{p}^{eb} h^{p+1} \right) \rightarrow_{d} \mathcal{N} (0, \mathcal{V}_{p}),$$

where 

$$\mathcal{B}_{p}^{eb} := \frac{\mu_{e,+}^{(p+1)} \omega_{p,+}^{(p+1)} - \mu_{e,-}^{(p+1)} \omega_{p,-}^{(p+1)}}{\mu_{D,\uparrow} (p + 1)!} \quad \text{and} \quad \mathcal{V}_{p} := \frac{\omega_{p,\downarrow}^{2} \sigma^{2}}{\varphi \mu_{D,\uparrow}^{2}}.$$

And,

$$\sqrt{nh} \left( \hat{\theta}_{p}^{mc} - \theta - \mathcal{B}_{p}^{mc} h^{p+1} \right) \rightarrow_{d} \mathcal{N} (0, \mathcal{V}_{p}),$$

where 

$$\mathcal{B}_{p}^{mc} := \frac{\left( \psi_{e,+}^{(p+1)} - \mu_{e} \psi_{e}^{(p+1)} \right) \omega_{p,+}^{(p+1)} - \left( \psi_{e,-}^{(p+1)} - \mu_{e} \psi_{e}^{(p+1)} \right) \omega_{p,-}^{(p+1)}}{\psi_{D,\uparrow} (p + 1)!}.$$

**Remark 1.** The asymptotic smoothing bias $\mathcal{B}_{p}^{eb}$ of the EB estimator is exactly the same as that of CCFT’s estimator. The standard LP regression theory (see, e.g., Imbens and Kalyanaraman, 2011) shows that for the standard estimator $\hat{\theta}_{p}$ defined by (3) without using covariates, we have

$$\sqrt{nh} \left( \hat{\theta}_{p} - \theta - \mathcal{B}_{p}^{lp} h^{p+1} \right) \rightarrow_{d} \mathcal{N} (0, \mathcal{V}_{p}),$$

where 

$$\mathcal{B}_{p}^{lp} := \frac{\mu_{M,+}^{(p+1)} \omega_{p,+}^{(p+1)} - \mu_{M,-}^{(p+1)} \omega_{p,-}^{(p+1)}}{\mu_{D,\uparrow} (p + 1)!} \quad \text{and} \quad \mathcal{V}_{p} := \frac{\omega_{p,\downarrow}^{2} \text{Var} \left[ \mu_{D,\uparrow} [M] \right]}{\varphi \mu_{D,\uparrow}^{2}}.$$

Without further assumptions, the ranking of $|\mathcal{B}_{p}^{eb}|$ versus $|\mathcal{B}_{p}^{lp}|$ is undetermined, in general. Consider the case of $p = 1$, which is the usual choice of LP order for point estimation. It is easy to see that $\omega_{1,+} = \omega_{1,-} =: \omega_{1}$ in this case. By linearity of the conditional expectation, we have $\mu_{e,s}^{(p+1)} = \mu_{e,s}^{(p+1)} - \gamma_{M}$ for $s \in \{-, +\}$. If $g_Z$ is twice continuously differentiable on $\mathbb{B}$ so that $\mu_{M,+}^{(2)} = \mu_{M,-}^{(2)}$, the constant part $\mathcal{B}_{1}^{eb}$ of the leading smoothing bias term coincides with $\mathcal{B}_{1}^{lp}$. The MC-EL estimator has a different asymptotic bias term. It can be seen that $\mathcal{B}_{p}^{mc}$ can be written as the sum of $\mathcal{B}_{p}^{eb}$ and additional terms. However, the ranking of $|\mathcal{B}_{p}^{eb}|$ versus $|\mathcal{B}_{p}^{mc}|$ is undetermined, since the additional terms may have signs opposite to that of $\mathcal{B}_{p}^{eb}$ and cancellation may happen. When $p = 1$, under the additional assumption $\mu_{Z,+}^{(2)} = \mu_{Z,-}^{(2)}$, we have $\mathcal{B}_{1}^{mc} = \left( \psi_{M,+}^{(2)} - \psi_{M,-}^{(2)} \right) \omega_{1}^{2} / (2 \psi_{D,\uparrow})$.

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10 If $(g_{Z(z);d,dd^*}, g_{Z(d');dd^*}, g_{dd^*})$ are smooth, it is clear from (13) that $g_Z$ is twice continuously differentiable on $\mathbb{B}$ if and only if $(d/dx)^j g_{Z(1);0}(x)|_{x=0} = (d/dx)^j g_{Z(0);j}(x)|_{x=0} = 0$ for $j = 0, 1, 2$. A causal interpretation of this condition is that the TED’s up to the second order of the treatment on covariates are zero (i.e., $(d/dx)^j E \{Z(1) - Z(0) \mid X = x, Z \} |_{x=0} = 0$ for $j = 0, 1, 2$, see Section 6 ahead).
Remark 2. The asymptotic variance $\nu_p$ of the EB and MC-EL estimators is also the same as that of CCFT’s estimator.\footnote{Indeed, it can be shown that the EB and CCFT’s estimators are first-order equivalent in a stronger sense: $\hat{\nu}_p^{\text{CCFT}} - \hat{\nu}_p^{\text{EB}} = o_p\left((nh)^{-1/2}\right)$.} Kreiś and Rothe (2022) show that CCFT’s estimator achieves efficiency gain $\nu_p \leq \nu_p^{\text{EB}}$ by using $\text{Var}_{0}[M - Z^\top \gamma_M] = \min_{\nu} \text{Var}_{0}[M - Z^\top \gamma] \leq \text{Var}_{0}[M]$. Consider the case of $p = 1$ and assume that $\mu_{Z+}^{(2)} = \mu_{Z-}^{(2)}$ holds. Since we have $\mathcal{B}_1^{\text{EB}} = \mathcal{B}_1^{\text{EB}}$ in this case, the AMSE of $\hat{\nu}_p^{\text{EB}}$, which equals $\left(\mathcal{B}_1^{\text{EB}}\right)^2 h^4 + \nu_p/(nh)$, is always less than or equal to the AMSE of $\hat{\nu}_p^{\text{EB}}$, which equals $\left(\mathcal{B}_1^{\text{EB}}\right)^2 h^4 + \nu_p^{\text{EB}}/(nh)$. It is noted in Noack et al. (2021, Section 4) that the AMSE-minimizing bandwidth for $\hat{\nu}_p^{\text{EB}}$ is also always less than or equal to that of $\hat{\nu}_p^{\text{EB}}$. As a result, the smoothing bias of $\hat{\nu}_p^{\text{EB}}$ is also smaller than that of $\hat{\nu}_p^{\text{EB}}$, when the AMSE-minimizing bandwidths are used for both estimators.

Remark 3. Theorem 1 and the first-order equivalence between the EL and CCFT estimators explain the asymptotic efficiency ranking from a GMM perspective: CCFT’s estimator can be interpreted as efficiently incorporating the side information from the covariate balance condition, which will typically reduce the asymptotic variance, and in the worst scenario, will yield the same asymptotic variance if the side information is irrelevant. Such an argument is analogous to that of Hirano et al. (2003), which explains the puzzling phenomenon that the inverse probability weighting estimator using the nonparametrically estimated propensity score has a smaller asymptotic variance relative to that uses the true propensity score. Hirano et al. (2003) show that the former is equivalent to an EL estimator that incorporates the side information from knowing the true propensity score efficiently.

Remark 4. When CCFT claim no definite ranking between their estimator and the standard LP estimator without covariates, they interpret such an indeterminacy as “in perfect agreement with those in the literature on analysis of experiments,..., where it is also found that incorporating covariates in randomized controlled trials using linear regression leads to efficiency gains only under particular assumptions”. As the RD design is often viewed as local randomization, let us reconcile the asymptotic efficiency gain and CCFT’s comment from the perspective of randomized experiments. In RD designs, the continuity of the density of the score $X$ implies that the shares of units with $X$ being in small neighborhoods to the left and right of the cutoff are equal (Noack et al., 2021, Section 5.2). Therefore, the RD design is analogous to a randomized experiment with equal probabilities of being in treatment and control groups. In the literature of randomized experiments, Negi and Wooldridge (2014, Theorem 5.2(iv)) show that when the assignment probability is equal to 1/2, the pooled regression adjustment (see Negi and Wooldridge, 2014 for its definition), whose algorithm is analogous to that of the CCFT estimator, always leads to a smaller or equal asymptotic variance. The assignment probability assumption is automatically fulfilled in RD designs.

Remark 5. Theorem 1 also implies that including a covariate will not change the asymptotic variance if and only if the corresponding element in $\gamma_M$ is zero. Note that the (true) projection coefficients $\gamma_M$ are the probabilistic limits of the regression coefficients of $Z_1$ in the “long” regression (10) including all covariates. Consider the partition $Z = (Z_1^\top, Z_2^\top)^\top$ of $Z$ and let $\gamma_M^\top = (\gamma_1^\top, \gamma_2^\top)^\top$.
be the conformable partition of $\gamma_M$ such that the dimension of $\gamma_j^\top$ coincides with that of $Z_j$, $j = 1, 2$. Using Theorem 1 and $\sigma^2 = \text{Var}_{[0]} [M] - \text{Cov}_{[0]} [M, Z] \left( \text{Var}_{[0]} [Z] \right)^{-1} \text{Cov}_{[0]} [Z, M]$, then writing $\text{Var}_{[0]} [Z]$ as a block matrix and inverting it, we can easily show that $\gamma_p$ is equal to the asymptotic variance of the covariate-adjusted estimator using only $Z_1$ if and only if $\gamma_2 = 0$. In this case, $Z_2$ is irrelevant in the sense that dropping $Z_2$ has no first-order impact: it neither leads to efficiency loss nor changes the asymptotic smoothing bias. In conclusion, if we say that an estimator achieves efficiency gain when its asymptotic variance is smaller than that of the standard estimator without covariates, then EB, MC-EL, and CCFT estimators achieve efficiency gain as long as the coefficients of some covariates are nonzero.

### 4.2 Nonlinearity bias

This section carries out a higher-order analysis of the MC-EL and EB estimators $\hat{\vartheta}_{mc}^p$ and $\hat{\vartheta}_{eb}^p$. We apply the quadratic stochastic expansion (Newey and Smith, 2004, Section 3) to the estimator and write it as the sum of a quadratic function of centered sample averages and a remainder term of a smaller order of magnitude. E.g., for CCFT’s estimator, using the expression (11), we simply write

$$\hat{\vartheta}_{CCFT}^{Y,p} = \frac{1}{nh} \sum_i \tilde{W}_{p,i} \left( Y_i - Z_i^\top \gamma_Y \right) - \left( \frac{1}{nh} \sum_i \tilde{W}_{p,i} Z_i \right)^\top \left( \hat{\gamma}_{CCFT}^Y - \gamma_Y \right).$$

The first-order asymptotic analysis is based on the linear term $(nh)^{-1} \sum_i \tilde{W}_{p,i} \left( Y_i - Z_i^\top \gamma_Y \right)$. Using the second term on the right hand side of above equation and replacing $\hat{\gamma}_{CCFT}^Y - \gamma_Y$ by its linearization, we extract the quadratic terms. For $\hat{\vartheta}_{mc}^p$ and $\hat{\vartheta}_{eb}^p$, more complicated derivations are needed.

In our nonparametric context, we write the leading (linear and quadratic) terms as the sum of the first-order stochastic variability term, the first-order smoothing bias term, the second-order stochastic variability term, and a (smoothing) bias-variability interaction term. The first-order (second-order) stochastic variability term is a linear (quadratic) function of centralized sample averages. The first-order stochastic variability term is approximately distributed as $N(0, \gamma_p/(nh))$. The first-order smoothing bias has a leading term given by $\mathcal{B}_{p}^{eb} h^{p+1}$ (or $\mathcal{B}_{p}^{mc} h^{p+1}$). The expectation of the second-order stochastic variability term is referred to as the nonlinearity bias. The following theorem provides an asymptotic representation for the nonlinearity bias.

**Theorem 2.** Suppose that Assumptions 1, 2 and 3 hold. Assume that $g_{[B]}^\varphi$ is bounded on $\mathbb{B} \setminus \{0\}$. The nonlinearity bias of $\hat{\vartheta}_{eb}^p$ is given by

$$\frac{1}{nh} \cdot \left\{ \omega_p^2 \cdot \frac{\text{Cov}_{[0]} [\epsilon, D]}{\varphi h^{2p+1}} + o(1) \right\}.$$

---

$^{12}$Such a bias is referred to as “higher-order bias” by Newey and Smith (2004) and Graham et al. (2012). We use terminology similar to Cattaneo et al. (2013) to distinguish such a bias incurred by (second-order) stochastic variability from smoothing bias in our nonparametric context.
Remark 6. We consider the situation when a relatively large number of valid covariates that satisfy the covariate balance condition are available. The first-order asymptotic theory (Theorem 1 and Remark 2) shows that the covariate-adjusted estimator using more covariates should have a smaller asymptotic variance. Since the covariate adjustment methods can be viewed as effectively incorporating covariate balance as overidentifying moment restrictions, second-order asymptotic analysis (Newey and Smith, 2004) reveals that using more covariates could be costly in terms of increased nonlinearity bias. In our case, we can see that the leading term in the nonlinearity bias admits an upper bound independent of the number of covariates since it follows easily from Cauchy-Schwarz inequality that \( |\text{Cov}_{0\pm} [\varepsilon, D]| \leq \sqrt{2} \cdot \sqrt{\text{Var}_{0\pm} [\varepsilon]} \leq \sqrt{2} \cdot \sqrt{\text{Var}_{0\pm} [M]} \). Such a property is analogous to the small bias properties given by Newey and Smith (2004, Theorem 4.5) and Graham et al. (2012, Theorem 4.1).

Remark 7. Let \( T := (\varepsilon - \mu_\varepsilon) (Z - \mu_Z) ^\top \text{Var}_{0\pm} [Z] ^{-1} (Z - \mu_Z) \). By adapting the proof arguments, we can show that for the generalized balancing estimator defined in Section 3.2, the nonlinearity bias is

\[
\frac{1}{nh} \cdot \left\{ \omega_{0,2} \cdot \text{Cov}_{0\pm} [\varepsilon, D] \varphi_{\mu_D,\varepsilon}^2 - \frac{\varrho}{2} \cdot \omega_{0,3} \cdot \frac{\mu_{T,\varepsilon}}{\varphi_{\mu_D,\varepsilon}} + o(1) \right\},
\]

where \( \varrho \in \mathbb{R} \) is the parameter in the definition of Cressie-Read divergence in (8). It is possible to construct examples where the absolute value of the extra term increases linearly with the number of covariates.13 With a relatively large number of valid covariates, a generalized balancing estimator could have a large nonlinearity bias, while the leading term in the nonlinearity bias of the EB and MC-EL estimators are guaranteed to be bounded. Since CCFT’s estimator is a slight modification of the generalized balancing estimator with \( \varrho = -2 \), we expect that its nonlinearity bias should admit an asymptotic expansion in a form similar to (14) with an extra unbounded term.

4.3 Balancing over functions in linear sieve spaces

The EB approach can incorporate information from not only the covariate balance conditions imposed on \( Z \) but also on those imposed on functions of \( Z \). This improves efficiency relative to CCFT and can achieve the best attainable asymptotic variance derived in Noack et al. (2021, Theorem 3). Let \( Z \subseteq \mathbb{R} ^{d_z} \) denote the support of \( Z \). We assume that \( Z \) is compact. Let \((b_1, \ldots, b_k, \ldots)\) be approximating basis functions defined on \( Z \) (typically, \( b_1 = 1 \)). Denote \( \rho := (b_1, \ldots, b_k) ^\top \). We assume that \( k = k_n \) increases with the sample size \( n \). For notational simplicity, we suppress the dependence of \( k \) on \( n \) and also the dependence of \( \rho \) on \( k \). Examples of such basis functions commonly

13E.g., we consider a modification of the simulation design in Section 7. The outcome and the \( l \) covariates are generated by \( Y = \mathbb{1} (X \geq 0) \left( \mu_{y_1} (X) + 0.28 \cdot \sum_{j=1}^{l} Z_j ^{(i)} \right) + \mathbb{1} (X < 0) \left( \mu_{y_0} (X) + 0.22 \cdot \sum_{j=1}^{l} Z_j ^{(i)} \right) + \varepsilon _{y} \) and \( Z_j ^{(i)} = \mathbb{1} (X \geq 0) \mu_{z_1} (X) + \mathbb{1} (X < 0) \mu_{z_0} (X) + \varepsilon _{z_j} ^{(i)} \) for all \( j = 1, \ldots, l \), where \( \varepsilon _{y} ^{(i)}, \ldots, \varepsilon _{z_j} ^{(i)} \) are i.i.d and \( \varepsilon _{z_j} ^{(i)} \sim \chi ^2 _1 - 1 \) for all \( j = 1, \ldots, l \). Then we get \( \mu_{T,\varepsilon} = 1.44 \cdot l \) by straightforward calculation.
used in econometrics include algebraic polynomials (and their transformations), trigonometric polynomials, and B-spline functions, among others. See, e.g., Chen (2007) and Belloni et al. (2015) for more details. Consider the following problem of balancing functions in the linear sieve space \( \mathcal{M}_k := \{ \rho^\top \gamma : \gamma \in \mathbb{R}^k \} \) and define the EB weights \((w_1^{\text{sieve}}, \ldots, w_n^{\text{sieve}})\) as the solution to

\[
\min_{w_1, \ldots, w_n} \text{KL} \left( w_1, \ldots, w_n \| \frac{1}{n}, \ldots, \frac{1}{n} \right)
\]

subject to \( \sum_i w_i \hat{W}_{p,i} \rho(Z_i) = 0_k, \sum_i w_i = 1. \)

(15)

The constraint in (15) imposes the balancing constraint that \( \sum_i w_i \hat{W}_{p,i} f(Z_i) = 0 \) for all \( f \in \mathcal{M}_k \) for the balancing weights. Then, by the Lagrangian multiplier method, we get the optimal weights and the associated Lagrangian multiplier \((w_1^{\text{sieve}}, \ldots, w_n^{\text{sieve}})\) defined by the right-hand sides of (5) and (6) with \( \bar{Z}_i \) replaced by \( \rho(Z_i) \). Define the sieve EB estimator by

\[
\hat{\theta}_p^{\text{sieve}} := \frac{\sum_i w_i^{\text{sieve}} \hat{W}_{p,i} Y_i}{\sum_i w_i^{\text{sieve}} \hat{W}_{p,i} D_i}.
\]

(16)

The weak continuity and predeterminedness assumptions (Assumption 1(e,f)) allow us to use only the margins in constructing the balancing weights. Now, in order to use functions in broader classes for further efficiency gain, we essentially need CCFT’s strong predeterminedness assumption \( F_{Z(1)|0} (\cdot | 0) = F_{Z(0)|0} (\cdot | 0) \) (see Section III of CCFT for discussion), where \( F_{Z(j)|d|d'} (\cdot | x) \) denotes the conditional cumulative distribution function (CDF) of \( Z(j) \) given \( (D(1), D(0), X) = (d, d', x) \) and \( F_{Z|X} \) denotes the conditional CDF of \( Z \) given \( X \). In addition, we need to replace Assumption 1(e) with the stronger assumption that the conditional distributions of \( (Z(d), Z(d')) \) given \( (D(1) = d, D(0) = d', X = x) \) change smoothly around the threshold \( x = 0 \). Under the strong continuity and predeterminedness assumptions, \( F_{Z|X} \) changes smoothly (\( \lim_{x \downarrow 0} F_{Z|X} (z | x) = \lim_{x \uparrow 0} F_{Z|X} (z | x) \) for all \( z \in \mathbb{Z} \)). Then \( \left( g_{f(Z(d))|d|d'}, g_{f(Z(d'))|d|d'} \right) \) are continuous at 0 for all \( (d, d') \in \{0, 1\}^2 \) if \( f \in \mathcal{M}_k \) satisfies some mild conditions.\(^{14}\) And the strong predeterminedness assumption implies that \( g_{f(Z(1))|0} (0) = g_{f(Z(0))|0} (0) \). It is clear from \( g_f(Z) = \sum_{d,d'} g_{f(Z)|d|d'} (x) g_{d|d'} (x) \) that the covariate balance condition for \( f(Z) \) (i.e., \( \mu_{f(Z),+} = \mu_{f(Z),-} \)) is fulfilled.

Theorem 3 below shows that the sieve EB estimator given in (16) achieves further efficiency gain relative to CCFT’s estimator and our EB estimator in (7). Interestingly, the asymptotic variance of the sieve EB estimator coincides with Noack et al. (2021)’s best attainable asymptotic variance of their LP estimator in which a flexible function is subtracted from the dependent variable. It is clear from (11) that one can write CCFT’s estimator for \( \mu_{Y|\tilde{T}_i} \) as a standard LP regression estimator using \( Y_i - Z_i^{\top \gamma}_Y \) as the dependent variable. Noack et al. (2021) consider

\[^{14}\]If for all \( (j, d, d', x) \in \{0, 1\}^2 \times \mathbb{Z} \), the conditional distribution of \( Z(j) \) given \( (D(1), D(0), X) = (d, d', x) \) admits a density \( f_{Z(j)|d|d'} (\cdot | x) \) with respect to the \( \sigma \)-finite dominating measure \( \nu \) with \( \nu (Z) < \infty \) such that \( f_{Z(j)|d|d'} (z | \cdot) \) is continuous at 0 for all \( z \in \mathbb{Z} \), we can write \( g_{f(Z(j))|d|d'} (x) = \int f (z) f_{Z(j)|d|d'} (z | x) \nu (dz) \). If for all \( x \) in an open neighborhood of 0, \( f_{Z(j)|d|d'} (\cdot | x) \) is uniformly bounded, this condition is satisfied if \( \int f (z) \nu (dz) < \infty \).
replacing the linear adjustment $Z_i \overset{\gamma}{\in} \mathbb{C}^{\text{FFT}}$ in (11) with a nonlinear transformation of the baseline covariates $Z_i$. Let $(\eta_Y, \eta_D)$ be a real-valued adjustment functions defined on $Z$ and we consider the standard LP regression estimator $\hat{\theta}_p(\eta_Y, \eta_D)$ using $Y_i - \eta_Y(Z_i)$ (or $D_i - \eta_D(Z_i)$) as the dependent variable in (3). Such an estimator is consistent if $\mu_{\eta_Y}(Z,+)=\mu_{\eta_Y}(Z,-)$ and $\mu_{\eta_D}(Z,+)=\mu_{\eta_D}(Z,-)$. See Noack et al. (2021, Footnote 6) for more discussion. Denote $\mu_{\eta_Y}(Z,+):=\lim_{x \rightarrow 0} g_M|ZX(z,x)$, $\mu_{\eta_Y}(Z,-):=\lim_{x \rightarrow 0} g_M|ZX(z,x)$, $\eta^*(Z):=(\mu^*(z) + \mu^*(z))/2$, and $\epsilon^*: = M - \eta^*(Z)$. Noack et al. (2021, Theorem 3) show that under some mild conditions on $(\eta_Y, \eta_D)$, the asymptotic variance of $\hat{\theta}_p(\eta_Y, \eta_D)$ cannot be smaller than the best attainable asymptotic variance $\gamma_p^{\text{opt}} := \omega_p^2 \sigma^{2}_{\text{opt}}/(\varphi_{D,1}^2)$, where $\sigma^{2}_{\text{opt}} := \text{Var}_{0+}[\epsilon^*]$.

To show that (16) is asymptotically normally distributed with the asymptotic variance $\gamma_p^{\text{opt}}$, we impose the following assumption on the distribution of the observed variables, which we invoke directly in the proof of Theorem 3. $B$ is defined in Assumption 2. Let $B_\eta := (Y, D, \eta^*(Z))^\top$ and let $f_{X|Z}$ be the conditional PDF of $X$ given $Z$.

**Assumption 4.** (a) $\mu_{\eta_Y}(Z,+)=\mu_{\eta_Y}(Z,-)$; (b) Let $\{\varepsilon_n\}_{n=1}^{\infty}$ denote a sequence of real-valued functions defined on $Z$ such that $\|\varepsilon_n\|_{\infty} \downarrow 0$ as $n \uparrow \infty$ and $g_{\varepsilon_n}(Z)$ is $(p+1)$-times continuously differentiable on $\mathbb{B} \setminus \{0\}$, then, $\sup_{x \in (-h,0) \cup (0,h)} |g^{(p+1)}_{\varepsilon_n}(Z)(x)| \downarrow 0$ as $n \uparrow \infty$; (c) $g_{B_\eta}$ has uniformly continuous derivatives up to the $(p+1)$-th order on $\mathbb{B} \setminus \{0\}$; (d) $\gamma_{B_{\eta}}^{-2}$ is uniformly continuous on $\mathbb{B} \setminus \{0\}$; (e) For all $Z \in Z$, $f_{X|Z}(\cdot \mid z)$ and $g_M|ZX(z,\cdot)$ are Lipschitz continuous on $\mathbb{B} \setminus \{0\}$ with Lipschitz constants $L_f, L_g > 0$ respectively; (f) $\text{Var}_{0+}[\epsilon^*]>0$ and $\text{Var}_{0-}[\epsilon^*]>0$.

Sufficient and easy-to-interpret conditions can be imposed on the population distribution of the latent variables to guarantee that Assumption 4 holds. Under the strong predeterminedness assumption $F_{Z(1)|0}(\cdot \mid 0) = F_{Z(0)|0}(\cdot \mid 0)$, (a) is satisfied if the assumptions on the densities in Footnote 10 hold and $\eta^*$ satisfies the integrability condition in Footnote 14. (b) is a mild regularity condition similar to Noack et al. (2021, Assumption 2). (c,d) are similar to Assumption 2(a,b). By (13), these are satisfied under suitable smoothness assumptions on $(g_{B_{\eta}(\cdot)|dd'}, g_{B_{\eta}(\cdot)|dd'}, g_{dd'})$ and $(g_{B_{\eta}(\cdot)|dd'}, g_{B_{\eta}(\cdot)|dd'}, g_{dd'})$ for $(d, d') \in \{0, 1\}^2$, where $B_\eta(d) := (Y(d), \eta^*(Z(d)))^\top$. The first

15 Let $\eta^* := \eta_Y - \theta_{\eta_D}$. $\gamma_p^{\text{opt}}$ is an asymptotic variance lower bound for all $\hat{\theta}_p(\eta_Y, \eta_D)$ with adjustment functions $(\eta_Y, \eta_D)$ fulfilling the condition $\text{Cov}_{0+}[\eta^*(Z), \mu^*_s(Z)] = \text{Cov}_{0-}[\eta^*(Z), \mu^*_s(Z)]$ for $s \in \{-, +\}$ and $\text{Var}_{0+}[\eta^*(Z) - \eta^*(Z)]$ and $\text{Var}_{0-}[\eta^*(Z) - \eta^*(Z)]$ exist. Under the conditions imposed on the densities in Footnote 14, this assumption is satisfied, if $\int_{\mathbb{B}} (\eta^*)^2 \, d\nu < \infty$ and $\int_{\mathbb{B}} (\mu^*)^2 \, d\nu < \infty$ for $s \in \{-, +\}$. Noack et al. (2021) show how to construct estimators that attain the optimal asymptotic variance.

16Under the existence of the densities of the latent variables defined in Footnote 14, the conditional distribution of $Z$ given $X = x$ admits a density $f_{Z|X}(\cdot \mid x)$ with respect to $\nu$ as a mixture:

$$f_{Z|X}(z \mid x) = \begin{cases} \sum_{d, d'} g_{dd'}(z) f_{Z(d)|dd'}(z \mid x) & \text{if } x \geq 0 \\ \sum_{d, d'} g_{dd'}(z) f_{Z(d')|dd'}(z \mid x) & \text{if } x < 0. \end{cases}$$

(17)

The assumption that $f_{Z|X}(z \mid \cdot)$ is $(p+1)$-times continuously differentiable on $\mathbb{B} \setminus \{0\}$ for all $z \in Z$ and $(\partial/\partial x)^{p+1} f_{Z|X}(z \mid x)$ is uniformly bounded for all $(x, j) \in (\mathbb{B} \setminus \{0\}) \times \{0, 1, ..., p + 1\}$ is satisfied if for all $(d, d') \in \{0, 1\}^2$, $(f_{Z(d)|dd'}(z \mid \cdot), f_{Z(d')|dd'}(z \mid \cdot))$ are $(p+1)$-times continuously differentiable on $\mathbb{B}$ with uniformly (in $z \in Z$) bounded derivatives; (2) $g_{dd'}$ is $(p+1)$-times continuously differentiable on $\mathbb{B}$ with bounded derivatives. Then under these assumptions, we have $g_{\varepsilon_n}(Z) = \int \varepsilon_n(z) ((\partial/\partial x)^{p+1} f_{Z|X}(z \mid x)) \nu(dz)$ and Part (b) holds.
part of Assumption 4(e) is satisfied if \( f_{X|Z} (\cdot | z) \) are differentiable on \( \mathbb{B} \setminus \{0\} \) with uniformly (in \( z \in \mathcal{Z} \)) bounded derivatives.\(^{17}\) The second part of Assumption 4(e) is satisfied if the conditional PDF \( f_{M|Z|X} (y \mid z, \cdot) \) of \( M \) given \((Z, X)\) is differentiable on \( \mathbb{B} \setminus \{0\} \) with derivatives that satisfy some dominance and integrability condition.\(^{18}\) Assumption 4(f) is similar to Assumption 2(d). Under Assumption 1(c), it is satisfied as long as \( \text{Var} [Y(d) - \eta^*(Z(d)) \mid X = 0, \mathcal{C}_0] > 0 \) for \( d \in \{0, 1\} \).

We also impose the following assumption on the basis functions. For notational simplicity, let \( P := \rho(Z) \).

**Assumption 5.**

(a) \( \mu_{b_j(Z)} = \mu_{b_j(Z)}^{-} \) and \( \mu_{b_j(Z)} = \mu_{b_j(Z)}^{-} \) for all \((j, s) \in \mathbb{N} \times \{-, +\}; (b) There exists constants \( 0 < \alpha < \beta < \infty \) independent of \( k \) such that for all \( x \in \mathbb{B} \setminus \{0\} \) and uniformly over all \( k \), mineig \((E[PP^\top | X = x]) > \alpha \), maxeig \((E[PP^\top | X = x]) < \beta \), mineig \((E[PP^\top]) > \alpha \) and maxeig \((E[PP^\top]) < \beta \); (c) \( g_{b_j(Z)} \) has uniformly continuous derivatives up to the \((p + 1)\)-th order on \( \mathbb{B} \setminus \{0\} \), for all \( j \in \mathbb{N} \); (d) There exists a constant \( c_p > 0 \) such that \( \sup_{z \in \mathbb{B} \setminus \{0\}} \left\| g_{P}^{(p+1)} (x) \right\| \leq c_p \sqrt{k} \).

\( \) is satisfied by all commonly used basis functions, as long as the assumption in Footnote 14 is fulfilled and \((\mu_0^+, \mu_0^-)\) satisfy the integrability condition in Footnote 15. (b) is a standard regularity condition imposing restrictions on the collinearity of the basis functions for which mild sufficient conditions are available (see Belloni et al., 2015, Proposition 2.1).\(^{19}\) (c) is analogous to Assumption 4(c). (d) is satisfied under (b) and some other mild regularity conditions.\(^{20}\)

Let \( \alpha_k := \inf_{\gamma \in \mathbb{R}^k} \| \eta^* - \rho^\top \gamma \|_\infty \) and \( \beta_k := \sup_{z \in \mathcal{Z}} \| \rho(z) \| \). Bounds for \( \alpha_k \) under commonly used basis functions are available from the approximation theory. E.g., if we take \((b_1, ..., b_k, ...)\) to be the algebraic polynomials and \( \eta^* \) is \( s \)-smooth (see, e.g., Chen, 2007, Section 2.3.1 for its definition),

\(^{17}\)By the Bayes theorem, we can show that the first part is satisfied if the assumptions discussed in Footnote 16 hold and \( f_X \) is continuously differentiable on \( \mathbb{B} \) with uniformly continuous derivatives.

\(^{18}\)Assume for simplicity that the support \( \mathcal{Y} \) of \( Y \) is bounded. Let \( f_{Y(\cdot)|Z(\cdot)d\mathcal{A}} (\cdot \mid x) \) denote the conditional joint density of \((Y(j), Z(j))\) given \((D(1), D(0), X) = (d, d', x)\), for \((j, d, d', x) \in \{0, 1\}^3 \times \mathcal{Y} \). Then we can write the conditional joint density \( f_{M|Z|X} \) of \((M, Z)\) given \( X \) as a mixture similar to (17) and write \( f_{M|Z|X} = f_{M|Z|X}/f_{Z|X} \). It is clear that the second part is satisfied, if for all \((d, d') \in \{0, 1\}^2 \) \((f_{Y(\cdot)|Z(\cdot)d\mathcal{A}} (y, z \mid \cdot), f_{Y(\cdot)|Z(\cdot)d\mathcal{A}} (y, z \mid \cdot))\) are differentiable on \( \mathbb{B} \) with uniformly (in \((y, z) \in \mathcal{Y} \times \mathcal{Z} \)) bounded derivatives and similar assumptions hold for \((f_{Z(\cdot)|d\mathcal{A}} (z \mid \cdot), f_{Z(\cdot)|d\mathcal{A}} (z \mid \cdot))\) and \( g_{d\mathcal{A}} (2) \left( f_{Z(\cdot)|d\mathcal{A}} (\cdot \mid \cdot), f_{Z(\cdot)|d\mathcal{A}} (\cdot \mid \cdot) \right) \) are bounded away from zero on \( \mathcal{Z} \times \mathbb{B} \) and a similar assumption holds for \( g_{d\mathcal{A}} \).

\(^{19}\)Part (b) is satisfied if (1) the conditional distribution of \( Z \) given \( X = x \) admits a Lebesgue density that is uniformly (for all \( x \in \mathbb{B} \setminus \{0\} \)) bounded away from zero, (2) the marginal distribution of \( Z \) admits a Lebesgue density that is bounded above and away from zero, (3) the basis functions are orthonormal with respect to the Lebesgue measure. By (17), Condition (1) is satisfied if for all \((d, d') \in \{0, 1\}^2 \), \((f_{Z(\cdot)|d\mathcal{A}} (\cdot \mid \cdot), f_{Z(\cdot)|d\mathcal{A}} (\cdot \mid \cdot))\) are bounded above and away from zero on \( \mathcal{Z} \times \mathbb{B} \) and a similar assumption holds for \( g_{d\mathcal{A}} \).

\(^{20}\)Under conditions in Footnote 16, \( g_{P}^{(p+1)} (x) = \int \rho (z) \left( \frac{\partial (\partial / \partial x)^p + f_{Z|X} (z \mid x)}{f_{Z|X} (z \mid x)} \right) \nu (dz) \). Then, by Jensen’s and Cauchy-Schwarz inequalities,

\[ \left\| g_{P}^{(p+1)} (x) \right\|^2 \leq E \left[ \left( \frac{\partial (\partial / \partial x)^p + f_{Z|X} (Z \mid x)}{f_{Z|X} (Z \mid x)} \right)^2 \right] \cdot E \left[ \left\| P \right\|^2 \mid X = x \right]. \]

Under Part (b), \( E \left[ \left\| P \right\|^2 \mid X = x \right] = \text{tr} \left( E \left[ PP^\top \mid X = x \right] \right) \leq k \cdot \mathcal{F} \). Part (d) holds if the first term is bounded. This holds if (1) \( f_{Z|X} \) is bounded away from zero on \( \mathcal{Z} \times (\mathbb{B} \setminus \{0\}) \); (2) \( \partial (\partial / \partial x)^p + f_{Z|X} (\cdot \mid x) \) is uniformly bounded for all \( x \in \mathbb{B} \setminus \{0\} \). Sufficient conditions for these assumptions are discussed in Footnotes 16 and 19.
then \( \alpha_k \) is bounded by \( k^{-s/d_x} \), up to a constant. Bounds for \( \beta_k \) are also available in the literature for commonly used basis functions. For the algebraic polynomials, \( \beta_k \) is bounded by \( k \) up to a constant. If \((b_1, ..., b_k, ...)\) are B-splines, then an upper bound is \( \sqrt{k} \). See Chen (2007) and Belloni et al. (2015) for results for other basis functions. In the statement of the following theorem, we impose Assumptions 4 and 5 in place of Assumption 2.

**Theorem 3.** Suppose that Assumptions 1, 3, 4 and 5 hold. Assume \( \text{E} \left[ (\mu_k^s)^2 (Z) \right] < \infty \) for \( s \in \{-, +\} \), for some \( r \geq 4 \) and \( \zeta \in (0, 1) \), \( g_0 ||w ||_r \) and \( g_{\mu_k^s} (Z)^{2+s} \) \( (s \in \{-, +\}) \) are bounded on \( \mathbb{B} \setminus \{0\} \).

Assume that the tuning parameters \((h, k)\) satisfy \( nh^{2\zeta+3} = O(1) \), \( nh \to \infty \), \( (\alpha_k + h) \beta_k \downarrow 0 \) and \( \left( \beta_k + (nh)^{1/r} \right) k/\sqrt{nh} \downarrow 0 \). Then,

\[
\sqrt{nh} \left( \hat{\alpha}_p \text{sieve} - \bar{\varphi} - \mathcal{B}^{\text{opt}}_p h^{p+1} \right) \to_d N \left( 0, \varphi^{\text{opt}}_p \right),
\]

where

\[
\mathcal{B}_p^{\text{opt}} := \frac{\mu_{k^+}^{p+1, 1} \omega_{p^{1+1}}^{+} - \mu_{k^-}^{p+1, 1} \omega_{p^{1+1}}^{-}}{\mu_{p^{1+1}}^{d, +} (p+1)!}.
\]

**Remark 8.** Theorem 3 is analogous to Noack et al. (2021, Theorem 2). Consider the case of \( p = 1 \) as in Remark 1. If we assume that \( g_{\eta^r(Z)} \) is twice continuously differentiable on the neighborhood \( \mathbb{B} \) of 0 so that \( \mu_{\eta^r(Z)}^{2} = \mu_{\eta^r(Z)}^{2} \) as in Noack et al. (2021, Assumption 1), \( \mathcal{B}^{\text{opt}}_1 \) coincides with \( \mathcal{B}^{d_p}_1 \) (i.e., the constant part of the asymptotic smoothing bias of the standard LP regression estimator without covariates).\(^{21}\)

**Remark 9.** Theorem 3 is analogous to Donald et al. (2003, Theorem 5.6). Viewed as an EL estimator based on a set of over-identified moment restrictions whose dimension grows with the sample size, \( \hat{\alpha}_p \text{sieve} \) attains the variance lower bound derived by Noack et al. (2021) asymptotically.

Theorem 3 also parallels the main result of Chan et al. (2016), which shows that the sieve-based generalized EB estimator for the ATE under unconfoundedness attains the semiparametric efficiency bound. As discussed in the remark following Newey and Smith (2004, Theorem 4.5) (also see Donald et al., 2009), the calculation and conclusion in Theorem 2 and Remark 7 are still valid if the number of effective covariates is allowed to grow with the sample size. The calculation implies that the nonlinearity bias of the sieve EB estimator is of order \( O \left( (nh)^{-1} \right) \), while other sieve-based estimators can have nonlinearity bias of order \( O (k/ (nh)) \).

**Remark 10.** As in Donald et al. (2003), we can consider a generalization using the Cressie-Read divergence defined by (8). The conclusion of Theorem 3 holds for any sieve-based generalized balancing estimator. If \( \varphi = -2 \), the condition \( \left( \beta_k + (nh)^{1/r} \right) k/\sqrt{nh} \downarrow 0 \) can be weakened to

\(^{21}\)By arguments similar to those in Footnote 10, \( g_{\eta^r(Z)} \) is twice continuously differentiable on \( \mathbb{B} \) if and only if \( (d/ dx)^j g_{\eta^r(Z)} (Z(1)) (x) \bigg|_{x=0} = (d/ dx)^j g_{\eta^r(Z(0))} (x) \bigg|_{x=0} \) for \( j = 0, 1, 2 \). A causal interpretation is that the TED’s up to the second order of the treatment on \( \eta^r(Z) \) are zero. This condition holds under \( \frac{\theta}{\theta x} f_{Z(j)} (\cdot | x) \bigg|_{x=0} = \frac{\theta}{\theta x} f_{Z(j)} (\cdot | x) \bigg|_{x=0} \) for \( j = 0, 1, 2 \), where \( f_{Z(j)} (\cdot | x) \) is the density defined in Footnote 14.
\( \beta_k \sqrt{\log (k)} \frac{k}{\sqrt{n}} h \downarrow 0. \) Since the generalized balancing estimator with \( q = -2 \) is a slight modification of CCFT’s estimator. We expect that a “LP-series” regression extension (i.e., replacing \( Z_i \) by \( \rho (Z_i) \) in (10)) of CCFT’s estimator has the same asymptotic distribution under the weaker conditions imposed on the pair of tuning parameters.

5 Likelihood ratio based inference

In this section, we consider inference using the likelihood ratio statistics. Denote \( M_i (\theta) := Y_i - \theta D_i, \)
\( U_i (\theta) := (M_i (\theta), Z_i^\top) \) and \( U_i := U_i (\theta) \) for notational simplicity. Let \( \tau \in (0,1) \) be the significance level. Let \( F_{\chi_1^2} \) and \( f_{\chi_1^2} \) denote the CDF and the PDF of a \( \chi_1^2 \) (\( \chi^2 \) with one degree of freedom) random variable respectively. Let \( c_\tau := F_{\chi_1^2}^{-1} (1 - \tau) \) be the \((1 - \tau)\) quantile of the \( \chi_1^2 \) distribution. The standard EL ratio statistic is given by
\[ LR_p (\theta \mid h) := 2n \left( \ell_{p,\tau}^m (\theta \mid h) - \ell_{p,\tau}^m (\hat{\theta}_{p,\tau} \mid h) \right), \]
which is a function of \( \theta \). An EL confidence set for \( \theta \) with nominal coverage probability \( 1 - \tau \) is
\[ CS_{p,\tau} (h) := \{ \theta : LR_p (\theta \mid h) \leq c_\tau \}. \]
When \( p = 2 \) is taken, our smoothness assumption and construction of \( CS_{p,\tau} (h) \) parallel CCFT in that \( CS_{p,\tau} (h) \) uses the same LP order as CCFT’s inference method and Assumption 2(a) assumes the same (three-times differentiability) smoothness as CCFT’s SRD. \(^{23}\) By the Lagrangian multiplier method and strong duality, for fixed \( \theta \),
\[ \ell_{p,\tau}^m (\theta \mid h) = \sup_{\lambda} \frac{1}{n} \sum_i \log \left( 1 + \lambda^\top (W_{p,i}U_i (\theta)) \right) \] \( (18) \)
By similar derivations as those in Section 3,
\[ \ell_{p,\tau}^m (\hat{\theta}_{p,\tau} \mid h) = -\frac{1}{n} \sum_i \log (n \cdot w_{p,i}^{mc}) = \sup_{\lambda} \frac{1}{n} \sum_i \log \left( 1 + \lambda^\top (W_{p,i}Z_i) \right) \] \( (19) \)
Computation of \( LR_p (\theta \mid h) \) only requires solving convex optimization problems. The right hand side of the second equality in (19) can be \( \infty \) in the “no solution” scenario discussed in Section 3. If our algorithm finds a solution for the maximization problem in (19), then we proceed to compute (18) for fixed \( \theta \) using a similar algorithm. The right hand side of (18) is \( \infty \) if the origin is not in the interior of the convex hull of \{ \( W_{p,1} U_1 (\theta), \ldots, W_{p,n} U_n (\theta) \} \}. In this case, the Newton algorithm would return a very large value for \( \ell_{p,\tau}^m (\theta \mid h) \) and \( \theta \) is excluded from the confidence set. We have the following result on the shape of \( CS_{p,\tau} (h) \).

Theorem 4. Suppose that Assumptions 1, 2 and 3 hold. Assume that \( g_{\|B\|^4} \) is bounded on \( B \setminus \{0\} \). Assume that the bandwidth satisfies \( nh^{2p+3} = o(1) \) and \( nh \to \infty \). Then, \( CS_{p,\tau} (h) \) is a finite interval

\(^{22}\) For fuzzy RD, as Noack and Rothe (2019)”s method, the EL confidence set avoids a “delta method” argument used by the Wald-type inference of CCFT.

\(^{23}\) CCFT proposes Wald-type inference using their local linear estimator with bias correction and standard errors that take into account estimation of the bias. Calonico et al. (2014, Remark 7) show that subtracting the \( p \)-th order LP estimator by the nonparametric estimator for the leading bias term with the same bandwidth is the same as a \( (p+1) \)-th order LP estimator. CCFT’s bias-corrected local linear estimator (with common bandwidths) is the same as a local quadratic regression estimator.
with probability approaching one.

**Remark 11.** Theorem 4 is an extension of Hall and La Scala (1990, Theorem 2.2). It shows that when the sample size is large, with high probability, \( CS_{p,\tau}(h) \) must be a finite interval. In general, EL confidence sets may not satisfy such a property in finite samples. See Otsu et al. (2015, Section 3) for more discussion.\(^{24}\)

In the rest of this section, we give several large-sample properties of the EL inference method. Section 5.1 establishes uniform-in-bandwidth (first-order) validity of the EL confidence set. Sections 5.2 and 5.3 are devoted to second-order properties. Section 5.2 shows the distributional expansion for the likelihood ratio and proposes a simple analytical correction to improve coverage accuracy. Section 5.3 considers a scenario in which covariate balance fails to hold and analyzes the sensitivity of the coverage accuracy to this assumption. We derive the distributional expansion under local perturbation to the covariate balance condition.

### 5.1 Uniform-in-bandwidth Wilks theorem

The following theorem parallels the main result of AK and is a substantial extension of the standard Wilks theorem, which states that \( LR_p(\theta \mid h) \to_d \chi^2_1 \). Our result incorporates covariates and accommodates unbounded outcomes. The proof techniques we use differ from those employed by AK. Let \( \ell^\infty([1, \bar{h}/h]) \) denote the space of all bounded real-valued functions defined on \([1, \bar{h}/h]\) endowed with the sup-norm. Let \( \mathbb{H} := [\bar{h}, \bar{h}] \) be a compact bandwidth set where \( \bar{h} > 0 \) and \( \bar{h} > 0 \) (\( h < \bar{h} \)) are bandwidths that depend on the sample size.\(^{25}\)

**Theorem 5.** Suppose that Assumptions 1, 2 and 3 hold. Suppose that \((h, \bar{h})\) satisfy \( \log(n)\bar{h} = o(1) \), \( nh^{2p+3} = o(1) \) and \( n^{1/r} (nh)^{1/2} + (nh)^{-1/6} = o(\log(n)^{-3}) \). Assume that \( g_{\bar{h}, h} \) is Lipschitz continuous and \( g_{\|B\|^r} \) is bounded for some \( r \geq 4 \). There exists a zero-mean Gaussian process \( \{g_G(s) : s \in [1, \bar{h}/h] \} \) which is a tight random element in \( \ell^\infty([1, \bar{h}/h]) \) with the covariance structure given by

\[
E[ G_G(s) G_G(t) ] = \sqrt{\frac{s}{t}} \cdot \frac{\int_0^\infty K_{p,+}(z) K_{p,+}((s/t)z) \, dz}{\int_0^\infty K_{p,+}(z)^2 \, dz}. \tag{20}
\]

Then, \( \Pr \left[ LR_p(\theta \mid h) \leq z_{\tau}(\bar{h}/h)^2, \forall h \in \mathbb{H} \right] \to 1 - \tau, \) as \( n \uparrow \infty \), where \( z_{\tau}(\bar{h}/h) \) denotes the \( 1 - \tau \) quantile of \( \|G_G\|_{[1, \bar{h}/h]} \).

**Remark 12.** Theorem 5 generalizes the standard Wilks theorem with a single bandwidth. It implies that when \( \bar{h} = \bar{h} = \bar{h}, \) \( \Pr[\theta \in CS_{p,\tau}(h)] \to 1 - \tau \). The standard EL confidence set \( CS_{p,\tau}(h) \) may

\(^{24}\)In the proof of Theorem 4, we show that in finite samples, \( CS_ {p,\tau}(h) \) is unbounded if and only if some covariate-adjusted EL confidence set for \( \psi_{p,1} \) contains zero. In our case, we have the same observation as Otsu et al. (2015, Section 3). Unboundedness of \( CS_{p,\tau}(h) \) is indicative of weak identification in the sense of Feir et al. (2016).

\(^{25}\)As the main result of AK, Theorem 5 assumes deterministic upper and lower bounds. Let \((\bar{h}, \bar{h})\) denote some deterministic bounds that some data-dependent bounds \((h, \bar{h})\) capture. As argued by AK, the conclusion of Theorem 5 still holds under data-dependent bounds if the orders of \( \bar{h}/\bar{h} - 1 \) and \( h/\bar{h} - 1 \) are sufficiently small and \((\bar{h}, \bar{h})\) satisfy the assumptions of Theorem 5.
undercover if the bandwidth is selected after specification search over $\mathbb{H}$. As an example, suppose that $\hat{h} := \arg\max_{h \in \mathbb{H}} LR_p(0 \mid h)$ is selected to maximize the $p$-value for the two-sided hypothesis test of $\theta = 0$. AK shows that $z_{\tau}(\hat{h}/h)^2 > c_\tau$ when $\hat{h}/h > 1$ but $z_{\tau}(\hat{h}/h)$ grows at a logarithmic speed as $\hat{h}/h \uparrow \infty$. It is clear from Theorem 5 that under $\theta = 0$, $\Pr\left[\theta \in CS_{p,\tau}(\hat{h})\right] \rightarrow 1 - \tau$, where $\tau > \tau$ solves $\tau^2(\hat{h}/h)^2 = c_\tau$ and the test does not have asymptotically correct size. Theorem 5 justifies a simple correction for bandwidth snooping as AK by replacing the critical value $\tau$ used by $CS_{p,\tau}(h)$ with $z_{\tau}(\hat{h}/h)^2$. Let $CS_{p,\tau}^{sc}(h \mid \hat{h}/h) := \left\{ \theta : LR_p(\theta \mid h) \leq z_{\tau}(\hat{h}/h)^2 \right\}$ be the snooping corrected confidence set. Then, $CS_{p,\tau}^{sc}(h \mid \hat{h}/h)$ has asymptotically correct coverage no matter how $h$ is selected from $\mathbb{H}$, i.e., $\liminf_{n \uparrow \infty} \Pr\left[\theta \in CS_{p,\tau}^{sc}(h \mid \hat{h}/h)\right] \geq 1 - \tau$, for all $h \in \mathbb{H}$. The critical value $z_{\tau}(\hat{h}/h)$ can be easily simulated.\(^{26}\)

Remark 13. Note that Theorem 5 uses undersmoothing to guarantee that the bias term is asymptotically negligible, so it requires the rate of $\hat{h}$ to be smaller than that optimally trades off bias and variance. The bias-aware inference approaches (Armstrong and Kolesár, 2018a, 2020; Imbens and Wager, 2019) that explicitly characterize the worst-case bias can give shorter confidence intervals. This paper considers a different criterion in bandwidth selection and proposes in Remark 16 a bandwidth that minimizes the coverage error of $CS_{p,\tau}(h)$ and satisfies the rate requirement for $\hat{h}$.

Remark 14. Theorem 7 shows that $\{ CS_{p,\tau}^{sc}(h \mid \hat{h}/h) : h \in \mathbb{H} \}$ is an asymptotically valid confidence band for the constant $\theta$, which uses multiple bandwidth choices. Therefore, such an inference procedure is more robust and less sensitive to bandwidth choice. The uniform confidence band can also be used for sensitivity analysis of the result from the confidence set to bandwidth choice. Let $h_{\text{rf}}$ denote a reference bandwidth, and one computes $CS_{p,\tau}(h_{\text{rf}})$. In case of a statistically insignificant result (i.e., $0 \in CS_{p,\tau}(h_{\text{rf}})$), it can be argued that using a smaller (larger) bandwidth is necessary due to high bias (variance) incurred by $h_{\text{rf}}$. However, the specification search or multiple testing issue undermines the validity of a significant result ($CS_{p,\tau}(h) \subseteq (0, \infty)$ or $CS_{p,\tau}(h) \subseteq (-\infty, 0)$) corresponding to some $h \neq h_{\text{rf}}$. In such a case, with suitable lower and upper bounds $(\underline{h}, \bar{h})$ such that $\underline{h} < h_{\text{rf}} < \bar{h}$, one may follow AK’s approach and use the band $\{ CS_{p,\tau}^{sc}(h \mid \hat{h}/h) : h \in \mathbb{H} \}$. If there exists $h \in \mathbb{H}$ such that $CS_{p,\tau}^{sc}(h \mid \hat{h}/h) \subseteq (0, \infty)$ or $CS_{p,\tau}^{sc}(h \mid \hat{h}/h) \subseteq (-\infty, 0)$, one may conclude that the RD LATE is different from zero, and the validity of such a result is guaranteed by Theorem 5. On the other hand, if $0 \in CS_{p,\tau}^{sc}(h \mid \hat{h}/h)$ for all $h \in \mathbb{H}$, we conclude that the insignificant result is insensitive to bandwidth choice. In the case of $0 \not\in CS_{p,\tau}(h_{\text{rf}})$, it is still necessary to examine the sensitivity of such a significant result to bandwidth choice (Imbens and Lemieux, 2008). With suitable $(\underline{h}, \bar{h})$, one may conclude that $\theta > 0$ in a robust sense if there exists $h \in \mathbb{H}$ such that $CS_{p,\tau}^{sc}(h \mid \hat{h}/h) \subseteq (0, \infty)$ and for all $h \in \mathbb{H}$, $CS_{p,\tau}^{sc}(h \mid \hat{h}/h) \cap (0, \infty) \neq \emptyset$. Compared with AK, our confidence band incorporates information from covariates, so the robust inference based on it is more powerful.

Remark 15. Let $\bar{h}$ be the minimizer of some data-dependent criterion function defined on $[\underline{h}, \bar{h}]$.

\(^{26}\)See the R package BWSnooping from github.com/kolesarm/BWSnooping. If $\hat{h}/h \uparrow \infty$ as $n \uparrow \infty$, then $z_{\tau}(\hat{h}/h)$ can be replaced by its asymptotic counterpart. See AK for more detailed discussion on the critical values.
By Theorem 5, the asymptotic validity of the confidence set \( CS_{p,\tau}^{sc} \left( \hat{h} \mid h/h \right) \) is guaranteed without assuming that \( \hat{h} \) fulfills any property, such as the stochastic order of \( \hat{h}/h - 1 \) is sufficiently small so that the noise in \( \hat{h} \) is negligible, where \( h \) is some deterministic bandwidth that \( \hat{h} \) tries to capture.

5.2 Analytical correction

This section provides coverage expansions of the EL confidence sets. Similar to Calonico et al. (2020, Theorem 3.1(a)), Theorem 6 below considers two scenarios under the given smoothness assumption (Assumption 2(a)). The first scenario uses the LP order \( p \) so that the leading bias term in the coverage error of the confidence set \( CS_{p,\tau} (h) \) can be characterized. The second scenario exhausts the smoothness by setting LP order to \( p + 1 \). The smoothing bias, in this case, is of a smaller order \( O \left( h^{p+1-b} \right) \) but its leading term can not be explicitly characterized. The following mild assumption on the kernel function is used when establishing the validity of the Edgeworth expansions in the proofs of Theorems 6 and 7.

Assumption 6. \( (1, K_{p,+}, K_{p,+}^2, K_{p,+}^3) \) are linearly independent as elements in the vector space of continuous functions on \( (0, 1) \).

Since \( K (\cdot) \) is assumed to be symmetric, an analogous property holds for \( (1, K_{p,-}, K_{p,-}^2, K_{p,-}^3) \) as functions on \( (-1, 0) \) under this assumption. It is clear that the assumption is satisfied if \( K_{p,+} \) is a non-constant polynomial on \([-1, 1]\). If \( p \geq 1 \), this condition is satisfied if \( K (\cdot) \) is any of the commonly used kernel functions (triangular, biweight, trivariate, etc.) including the uniform kernel.\(^{27}\) Denote \( \Xi := \mu^{-1}_{U U^\top, \pm} \), \( \Psi^k_1 := \text{tr} \left( \Xi : \mu_{U (0) U^\top, \pm} \right) \) and \( \Psi^k_2 := \text{tr} \left( \Xi : \mu_{U (0) U^\top, \pm} \Xi : \mu_{U (0) U^\top, \pm} \right) \). Let

\[
\Upsilon^p_1 := \sum_{k,l=1,...,d_z+2} \left( \omega^2 \varphi \right)^{-1} \left\{ \frac{1}{2} \frac{\omega^4}{\omega^2} \Xi^{(k)} \Psi^k_1 - \frac{1}{3} \frac{\omega^6}{\omega^2} \Xi^{(k)} \Psi^k_2 \right\}.
\]

Let \( \Upsilon^p_4 \) be defined by the same formula with \( U \) replaced by \( \bar{Z} \) and the range changed to \( 1, ..., d_z + 1 \) accordingly. Let \( \Upsilon^{LR}_p := \Upsilon^p_4 - \Upsilon^p_1 \) and \( \mathcal{B}^{LR}_{p} := \left( \mathcal{B}^{mc}_{p} \right)^2 / \Upsilon^{p}_p \).

Now we provide distributional expansions for both \( LR_p (\vartheta \mid h) \) and \( LR_{p+1} (\vartheta \mid h) \). Asymptotic expansions of the coverage probabilities follow from these results (e.g., \( \Pr [\vartheta \in CS_{p,\tau} (h)] = \Pr [LR_p (\vartheta \mid h) \leq c_{\tau}] \)). The proof uses the method of Calonico et al. (2022) and calculations in Chen and Cui (2007).

Theorem 6. Suppose that Assumptions 1, 2, 3 and 6 hold. Assume that \( g_{\hat{B}_{\vartheta}} \) is Lipschitz continuous on \( \mathbb{B} \setminus \{0\} \) for \( j = 2, 3, 4 \) and \( g_{\|B\|^{20}} \) is bounded on \( \mathbb{B} \setminus \{0\} \). Suppose that \( h \) satisfies \( nh^{2p+3} = o (1) \)\(^{27}\) Suppose that \( p = 1 \) and \( K \) is the uniform kernel, i.e., \( K (t) = 1 (|t| \leq 1) / 2 \). Then, by simple calculation, \( K_{p,+} (t) = (4 - 6t) 1 (|t| \leq 1) \) and \( K_{p,-} (t) = (4 + 6t) 1 (|t| \leq 1) \).
and \( nh \to \infty \). Then,
\[
\Pr[LR_p(\theta \mid h) \leq x] = F_{\chi_1^2}(x) - \left(nh^{2p+3}B_p^{LR} + \frac{\gamma_p^{LR}}{nh}\right) x f_{\chi_1^2}(x) + O(v_{p,n})  
\]
(22)
and
\[
\Pr[LR_{p+1}(\theta \mid h) \leq x] = F_{\chi_1^2}(x) - \frac{\gamma_{p+1}^{LR}}{nh} x f_{\chi_1^2}(x) + O(v_{p+1,n}),
\]
where \( v_{p,n} := h^{p+1}/\sqrt{nh + (\log(n))^{5/2}/(nh)^{3/2} + h^{p+2} + n^{-1} + (nh)^{2}(h^{p+1})^4 + nh^{2p+3+2h}} \) and \( v_{p+1,n} := nh^{2p+3+2h} + h^{p+1+h}/\sqrt{nh + (\log(n))^{5/2}/(nh)^{3/2} + h^{p+2+h} + n^{-1}} \).

**Remark 16.** In (22), \( nh^{2p+3}B_p^{LR} \) is the “bias” term that is brought by the smoothing bias and \( (nh)^{-1} \gamma_p^{LR} \) is the “variability term” that stems from the stochastic variability. Since \( h \gg n^{-1/(p+2)} \) gives the best coverage error decay rate, following CCFT we restrict our attention to bandwidths that satisfy \( h = H \cdot n^{-1/(p+2)} \) for some constant \( H > 0 \). The leading coverage error is proportional to \( -n^{-1/(p+2)}(B_p^{LR} H^{2p+3} + \gamma_p^{LR} H^{-1}) \).\(^{28}\) Parallel to Calonico et al. (2018), we define \( H_{co} := \text{argmin}_{H > 0} |B_p^{LR} H^{2p+3} + \gamma_p^{LR} H^{-1}| \) to be the optimal constant. Note that \( H_{co} \) is independent of the nominal coverage probability \( 1 - \tau \) and has a simple closed form.\(^{29}\) These properties are not shared by the CO bandwidths for the Wald-type inference methods. In practical implementation, \( H_{co} \) has to be estimated. A simple plug-in estimator \( \hat{\gamma}_p^{LR} \) of \( \gamma_p^{LR} \) that is based on local linear regression with standard rule-of-thumb (ROT) bandwidths (Hansen, 2021, Chapter 21.6) has a relatively fast \( O_p(n^{-2/5}) \) rate of convergence. On the other hand, since \( B_p^{LR} \) involves higher-order derivatives up to the \((p + 1)\)-th order, estimation of derivatives using a working parametric model is recommended for bandwidth selection (see, e.g., Hansen, 2021, Chapter 21.6).\(^{30}\)

**Remark 17.** The confidence set \( CS_{p,\tau}(h) \) uses the same bandwidth \( h \) for all values of \( \theta \) and can be considered as being obtained by inversion of a test of \( H_0 : \theta = \theta \) using the test statistic \( LR_p(\theta \mid h) \). We can consider a bandwidth dependent on the hypothesized value \( \theta \) under \( H_0 \). Let \((B_p^{LR}(\theta), \gamma_p^{LR}(\theta))\) be defined by the formulae of \((B_p^{LR}, \gamma_p^{LR})\) with \( M \) replaced by \( M(\theta) \). By Theorem 6 and similar arguments as those in Remark 16, the size-distortion-minimizing bandwidth is given by \( H_{co}(\theta) \cdot n^{-1/(p+2)} \), where \( H_{co}(\theta) := \text{argmin}_{H > 0} |B_p^{LR}(\theta) H^{2p+3} + \gamma_p^{LR}(\theta) H^{-1}| \).\(^{31}\) Note that the constant \( H_{co} \) defined in Remark 16 is just \( H_{co}(\theta) \). Clearly, the coverage expansion of

\(^{28}\)Note that typically the distributional expansion corresponding to a nonparametric kernel-based Wald-type statistic (e.g., Calonico et al., 2020, Theorem 3.1) is more complicated and involves another “bias-variability” interaction term of order \( h^{p+1} \).

\(^{29}\)If \( \hat{\gamma}_p^{LR} > 0 \), the unique minimizer \( H_{co} \) satisfies the first-order condition. An explicit solution is available from solving it: \( H_{co} = (\gamma_p^{LR} / (2p + 3) B_p^{LR})^{1/(2p+4)} \). If \( \gamma_p^{LR} < 0 \), it is easy to see that \( H_{co} = (\gamma_p^{LR} / B_p^{LR})^{1/(2p+4)} \) and \( B_p^{LR} H_{co}^{2p+3} + \gamma_p^{LR} H_{co}^{-1} = 0 \). In this case, the first-order coverage error vanishes at the optimal bandwidth.

\(^{30}\)If \( h \) is known and the bandwidth is chosen to guarantee the fastest rate of convergence, the rate of a fully nonparametric estimator of \( B_p^{LR} \) based on the \((p + 1)\)-th order LP regression is \( O_p\left(n^{-h/(2p+3+2h)}\right) \) under our smoothness assumption.

\(^{31}\)In the inference part of this paper, we mainly focus on improving the coverage accuracy. If the size of the confidence set is concerned, one may consider local alternatives for a given hypothesized value \( \theta \) under \( H_0 \) and choose the \( \theta \)-dependent power-optimal constant under a criterion from the distributional expansion of the test statistic \( LR_p(\theta \mid h) \) under the local alternatives.
the confidence set \( \hat{CS}_{p,\tau} := \{ \theta : LR_p(\theta \mid H_\Theta(\theta) \cdot n^{1/(p+2)}) \leq c_r \} \) has the same second-order term as \( CS_{p,\tau}(H_\Theta \cdot n^{1/(p+2)}) \). A preliminary estimator of \( \vartheta \) is required for estimation of \( H_\Theta \) but not for estimation of \( H_\Theta(\theta) \). However, in light of Theorem 4, \( CS_{p,\tau}(H_\Theta \cdot n^{1/(p+2)}) \) has a more interpretable form, while \( \hat{CS}_{p,\tau} \) can be disconnected.

**Remark 18.** The simple expression on the right hand side of (22) suggests that analytical correction can be implemented to improve coverage accuracy. E.g., it follows from (22) and Taylor expansion that the distribution of \( LR_p(\vartheta \mid h) / \big( 1 + nh^{2p+3}cLR_p + (nh)^{-1}cLR_p \big) \) is \( F_{\chi^2}^1(x) + O(v_{p,n}) \) (i.e., rescaling completely removes the leading terms). Feasible correction uses nonparametric estimators \( (\hat{\mathcal{B}}_p, \hat{\mathcal{F}}_p) \) of \( (\mathcal{B}_p, \mathcal{F}_p) \). Let \( LR_{bc}^p(\vartheta \mid h) := LR_p(\vartheta \mid h) / \big( 1 + nh^{2p+3}\hat{\mathcal{B}}_p + (nh)^{-1}\hat{\mathcal{F}}_p \big) \) be the likelihood ratio with analytical (Bartlett) correction and let \( CS_{bc}^{\vartheta}(h) := \{ \theta : LR_{bc}^p(\theta \mid h) \leq c_r \} \) be the corrected confidence set. This correction approach removes the leading bias term in (22) by using an estimator \( \hat{\mathcal{B}}_p \). We also consider implicit bias removal based on the idea of Calonico et al. (2014) by increasing the LP order by one. Let \( LR_{bc}^{p+1}(\vartheta \mid h) := LR_{p+1}(\vartheta \mid h) / \big( 1 + (nh)^{-1}\hat{\mathcal{F}}_{p+1} \big) \) be the likelihood ratio with analytical (partial Bartlett) correction (Chen, 1996) and let \( CS_{bc}^{\vartheta}(h) := \{ \theta : LR_{bc}^{p+1}(\theta \mid h) \leq c_r \} \) be the corrected confidence set. This approach essentially trades bias for variability, as the latter can be estimated with good accuracy. By the second part of Theorem 6, under the assumption that \( \hat{\mathcal{F}}_{p+1} - \hat{\mathcal{F}}_{p} = O_p\left(n^{-2/5}\right) \) and \( (nh^3)^{-1} = O(1) \), \( \text{Pr}[LR_{bc}^{p+1}(\vartheta \mid h) \leq x] = F_{\chi^2}^1(x) + O(v_{p+1,n}) \). The confidence set \( CS_{bc}^{\vartheta}(h) \) has a faster coverage error decay rate than \( CS_{bc}^{\vartheta}(h) \) for all \( h \). Viewed differently, \( CS_{bc}^{\vartheta}(h) \) with \( h \propto n^{1/(p+2)} \) follows the idea of partial Bartlett correction of Chen (1996) in that upon removal of the leading variability term, under-smoothing relative to its CO rate \( n^{-1/(p+2)+h} \) reduces the effects from the smoothing bias on the coverage accuracy and gives a faster coverage error decay rate.

**Remark 19.** By using the AK-type correction proposed in Theorem 5, we can also construct a confidence band that uses a continuous range of bandwidths to analyze the sensitivity of the result from \( CS_{p,\tau}^{\vartheta}(h) \) or \( CS_{p+1,\tau}^{\vartheta}(h) \) to bandwidth choice. The conclusion of Theorem 5 still holds for \( LR_{p+1}(\vartheta \mid h) \) and also for \( LR_{bc}^p(\vartheta \mid h) \) and \( LR_{bc}^{p+1}(\vartheta \mid h) \) since they are first-order equivalent to \( LR_p(\vartheta \mid h) \) and \( LR_{p+1}(\vartheta \mid h) \), uniformly in \( h \in \mathbb{H} \). We can take the lower and upper bounds in \( \mathbb{H} \) to be proportional to some commonly used reference bandwidths. The “doubly corrected” confidence sets can be constructed by following the procedure in Remark 12. We also expect a small coverage error for the corrected EL confidence band.\(^{34}\)

\(^{32}\)For this reason, \( CS_{p,\tau} \) with estimated \( H_\Theta(\theta) \) is likely to have better coverage accuracy in finite samples since the estimator of \( H_\Theta(\theta) \) is less variable than that of \( H_\Theta \). Noise in the selection of bandwidth will translate into coverage error of the confidence set (see Ma et al., 2023, Theorem 4 and Remark 5). Also see Hansen (2021, Chapter 21.6).

\(^{33}\)If \( \hat{\mathcal{B}}_p \) is the fully nonparametric estimator in Footnote 30, the coverage error of \( \{ \theta : LR_p^b(\vartheta \mid h) \leq c_r \} \) is of order \( n^{-h/(2p+3+2b)}h^{p+1}+v_{p,n} \), which converges to zero at a rate slower than \( v_{p+1,n} \). It is easy to check that \( v_{p+1,n} = O\left(n^{-1}\right) \) under \( h \approx n^{-1/(p+2)} \) if \( p \geq 1 \) and \( h \geq 1/2 \). However, we note that this does not imply that the finite-sample coverage accuracy of \( CS_{p+1,\tau}^{\vartheta}(h) \) is always better than that of \( CS_{p,\tau}^{\vartheta}(h) \), since the constant terms in the coverage errors are different.

\(^{34}\)In the proof of the asymptotic validity of the confidence band, we show that the distribution of \( \sup_{h \in \mathbb{H}}LR_{bc}^p(\vartheta \mid h) \) is approximated by the distribution of \( \| \Gamma_G(h/h) \|^2 \) with a vanishing error, where \( \Gamma_G(h/h) \) follows the \( \chi^2 \) distribution for all \( h \in \mathbb{H} \). We expect that the distributional approximation of \( \sup_{h \in \mathbb{H}}\Gamma_G(h/h) \) to
5.3 Local imbalance

This section shows that the coverage performance of the EL confidence set is maintained even if the covariate balance assumption is slightly violated, a scenario we call “local imbalance”. Specially, we assume that the observed covariates $Z$ are subject to data contamination (measurement errors) that occurs after treatment. The contaminated covariates may not satisfy the predeterminedness assumption and can be drawn from some perturbed probability law that generates a slight imbalance (Kitamura et al., 2013). On the other hand, the genuine but unobserved predetermined covariates $Z^* \in \mathbb{R}^d$, which typically affects $Y(d)$, still satisfy the balance condition. The continuity of $(g_Y(d)|d^d \cdot g_Y(d')|d'')$ remains to hold. In other words, the imbalance is caused by measurement errors that are known to be excluded from the data-generating process of $Y(d)$. In this case, the standard RD estimand $\vartheta$, which confidence sets try to cover, remains to identify a causal parameter of interest.$^{35}$

Formally, let $\zeta \in \mathbb{R}^d$ denote the measurement errors realized after treatment. The measurement error $\zeta$ is nonclassical in the sense that it relates to $(D, X, Z^*)$. Let $(Z^*(1), Z^*(0), \zeta(1), \zeta(0))$ be potential covariates and measurement errors.$^{36}$ The contaminated potential covariates $(Z(1), Z(0))$ are generated by $Z(d) = Z^*(d) + \zeta(d)$ for $d = 0, 1$. And the observed contaminated covariates are $Z = D \cdot Z(1) + (1 - D) \cdot Z(0)$. We assume that the true covariates satisfy the “predeterminedness” assumption $g_{Z^*(1)|10}(0) = g_{Z^*(0)|10}(0)$, but the measurement errors fail to satisfy it. As a result, local imbalance in essence assumes that $g_{Z(1)|10}(0) - g_{Z(0)|10}(0)$ approaches 0 at the rate of $(nh)^{-1/2}$. We are interested in the coverage probability $\Pr[\vartheta \in CS_{p+1, \tau}(h)]$, which is expected to have a limit in $(0, 1 - \tau)$ and thus captures the phenomenon that covariate imbalance results in undercoverage. We set the bandwidth to $h \asymp n^{-(p+1)/(2p+4)}$ as discussed in Remark 18. Let $l_n := n^{-(p+1)/(2p+4)} \asymp (nh)^{-1/2}$. The following assumption formalizes local imbalance.

**Assumption 7.** (a) $(g_{Z^*(d)|dd'}, g_{\zeta(d)|dd'})$ and $(g_{Z^*(d')|dd'}, g_{\zeta(d')|dd'})$ are all continuous at the threshold 0 for all $(d, d') \in \{0, 1\}^2$; (b) $g_{Z^*(1)|10}(0) = g_{Z^*(0)|10}(0)$; (c) $g_{\zeta(1)|10}(0) - g_{\zeta(0)|10}(0) = \delta \cdot l_n$ for some localizing parameter $\delta \in \mathbb{R}^d$.

Part (a) essentially assumes that no other variables depending on $I$ affect $(Z^*(d), \zeta(d))$. Under Assumption 1(a,b,c,d), the standard RD estimand still identifies the RD LATE. (a,b) imply that the true covariates that may affect $Y(d)$ still satisfy the balance condition $\mu_{Z^*+,} = \mu_{Z^*-,}$. (c) assumes that the RD LATE on $\zeta$ is $\delta \cdot l_n$, which generates the local imbalance in the observed covariate: $\mu_{Z^*+,} - \mu_{Z^*-,} \asymp (nh)^{-1/2}$. By using local asymptotic analysis, we analyze the performance of our EL
confidece set under such a local imbalance condition, which is similar to using locally misspecified moment conditions in the sense of Armstrong and Kolesár (2021). Our result differs from Armstrong and Kolesár (2021) and focuses on the coverage performance of the confidence set when \( \delta \) is close to 0. \(^{37}\)

Let \( N := Z - (\delta \cdot l_n) D = D \cdot N (1) + (1 - D) N (0) \), where \( N (1) := Z^* (1) + \zeta (1) - \delta \cdot l_n \) and \( N (0) := Z^* (0) + \zeta (0) \). It now follows that \( g_{N(1)|10} (0) = g_{N(0)|10} (0) \) and \( \mu_{N,+} = \mu_{N,-} \). Let \( \gamma_N := \left( \text{Var}_{0\pm} [N] \right)^{-1} \text{Cov}_{0\pm} [N, M] \) and \( \mathcal{V}_N := \left( \omega_{p+1}^{0.2} \text{Var}_{0\pm} [M - N^\top \gamma_N] \right) / \left( \varphi_{p+1}^2 \right) \). For simplicity, we assume that the distribution of \( N \) does not vary with \( n \). \(^{38}\) CCFT shows that the covariate-adjusted estimator is inconsistent and the confidence interval fails to have asymptotically correct coverage probability under “global imbalance” \( \mu_{Z,+} \neq \mu_{Z,-} \). Under local imbalance in Assumption 7, CCFT’s estimator and the generalized EB estimators are still consistent. \(^{39}\) Inference suffers from the undercoverage problem, since the coverage probabilities of the confidence sets (CCFT’s or the EL) converge to a limit in \((0, 1 - \tau)\).

We now consider \( \Pr \left[ \vartheta \in C_{p+1, \tau}^{\text{bc}} (h) \right] \) as a function of \( \delta \) under local imbalance. A measure of sensitivity of the coverage accuracy to local imbalance (i.e., how the coverage probability drops relative to that under \( \delta = 0 \)) is given by the slope of \( \Pr \left[ \vartheta \in C_{p+1, \tau}^{\text{bc}} (h) \right] \) as a function of \( \delta \) at \( \delta = 0 \). We extend Theorem 6 and derive a two-term asymptotic expansion \( \Pr \left[ \vartheta \in C_{p+1, \tau}^{\text{bc}} (h) \right] = R (\delta) + o (l_n) \), where \( R (\delta) \) is the sum of the leading terms as an approximation to \( \Pr \left[ \vartheta \in C_{p+1, \tau}^{\text{bc}} (h) \right] \) in finite samples. We show that \( R (0) = 1 - \tau \) and the gradient \( \nabla R (\delta) := (\partial / \partial \delta) R (\delta) \) at \( \delta = 0 \) is equal to 0, so that \( R (\delta) \) is locally constant around \( \delta = 0 \).

Let \( F (\cdot | \iota) \) denote the CDF of a \( \chi^2 (\iota) \) (non-central \( \chi^2 \) with one degree of freedom and non-centrality parameter \( \iota \geq 0 \)) random variable. Let \( F^{(k)} (x | \iota) := (\partial / \partial \iota)^k F (x | \iota) \) be the \( k \)-times partial derivative of \( F (x | \iota) \) with respect to \( \iota \).

**Theorem 7.** Suppose that Assumptions 1(a,b,c,d), 2, 3, 6 and 7 hold. Suppose that \( h \) satisfies \( h = H \cdot n^{-1/(p+2)} \) for some constant \( H > 0 \). Then,

\[
\Pr \left[ \vartheta \in C_{p+1, \tau}^{\text{bc}} (h) \right] = F \left( c_\tau | H \cdot \left( \frac{\gamma_\tau}{\mathcal{V}_N} \right)^2 \right) + \left\{ \mathcal{P}_1 (\delta) F^{(1)} \left( c_\tau | H \cdot \left( \frac{\gamma_\tau}{\mathcal{V}_N} \right)^2 \right) \right\}
\]

\(^{37}\)The approach of Armstrong and Kolesár (2021) specifies a set in which \( \delta \) possibly lies and then adjusts the critical value to take into account the maximal misspecification bias. We take a very different approach in this paper.

\(^{38}\)E.g., this holds if the measurement errors are the following form: \( \zeta (1) = \delta \cdot l_n + \zeta \) and \( \zeta (0) = \zeta \), for some zero-mean \( (\zeta, \zeta) \) that are independent of other variables in the model. Relaxation of this assumption requires more complicated arguments and suitable modification of the assumptions.

\(^{39}\)We can show that \( \hat{\gamma}_Y^{\text{CCFT}} \) (see Section 6 of the online supplement of CCFT for its expression) in the representation (11) converges in probability to \( \left( \text{Var}_{0\pm} [N] \right)^{-1} \text{Cov}_{0\pm} [N, Y] \). Then, since \( \mu_{N,+} = \mu_{N,-} \), we have

\[
\hat{\gamma}_Y^{\text{CCFT}} = \frac{1}{nh} \sum_i \hat{W}_{pi} \left( Y_i - N_i^\top \hat{\gamma}_Y \right) - (\delta \cdot l_n) \left( \frac{1}{nh} \sum_i \hat{W}_{pi} D_i \right) = \mu_{Y,1} + o_p (1).
\]

Under global imbalance, the EB estimator has a probabilistic limit different from that of CCFT’s estimator (see Lemma 1 of CCFT). Neither of them is equal to \( \vartheta \).
\[ + \mathcal{P}_2(\delta) F^{(2)} \left( c_r \mid H \left( \frac{(\gamma_N^T \delta)^2}{\gamma_N^T} \right) \right) l_n + o(l_n), \]

where \( \gamma_N = \gamma_N + o(1) \) and \( \gamma_N = \gamma_N + o(1) \) and \((\mathcal{P}_1, \mathcal{P}_2)\) are homogeneous cubic polynomials with constant coefficients. The expressions of \( (\gamma_N, \gamma_N, \mathcal{P}_1, \mathcal{P}_2)\) are in the supplement.

**Remark 20.** The first-order term \( F \left( c_r \mid H \left( \frac{(\gamma_N^T \delta)^2}{\gamma_N^T} \right) \right) \) is an even function of \( \delta \), and the second-order term is an odd function of \( \delta \). Clearly, we have \( \nabla R(0) = 0 \) and therefore \( R(\cdot) \) is locally constant around the origin.\(^{40}\) We expect that the coverage accuracy of the \( CS_{F,\tau}^{bc+} \) is highly insensitive to local imbalance in finite samples. If \( \|\nabla R(0)\| \) is large in magnitude, a slight perturbation will incur severe undercoverage. To see that the slope is a measure of sensitivity to local imbalance, we consider the approximate minimal coverage \( \min_{\delta \in S_\tau} R(\delta) \) on \( S_\tau \), where \( \tau \) is a positive constant and \( S_\tau := \{ \delta \in \mathbb{R}^d_+ : \|\delta\| = \tau \} \) represents perturbations with equal magnitude \( \tau \) in all directions. \( \delta_R^* := \arg\min_{\delta \in S_\tau} R(\delta) \) corresponds to the direction in which the perturbation results in the most severe undercoverage. Clearly, \( R(\delta_R^*) < 1 - \tau \), and we have the approximation \( R(\delta_R^*) = (1 - \tau) - \|\nabla R(0)\| \tau + o(\tau) \) when \( \tau \) is small.\(^{41}\)

**Remark 21.** In some real applications (see, e.g., Cattaneo et al., 2019 and Cattaneo and Titunik, 2022, Section 4.1 for discussion and examples), the researcher may have access to observations on outcomes \( \tilde{Y} \) determined after treatment but considered unaffected by the treatment and to have no effect on the outcome of interest \( Y \). Cattaneo and Titunik (2022) note that “the principle of covariate balance can be extended beyond pre-determined covariates to variables that are determined after the treatment is assigned but are known to be unaffected by the treatment...”. The balance condition \( \mu_{\tilde{Y},+} = \mu_{\tilde{Y},-} \) should also hold for “unaffected” outcomes. We can also augment the list of covariates in (4) to include unaffected outcomes. While expanding the set of covariates may improve the efficiency, it bears the risk that the prior belief \( \mu_{\tilde{Y},+} = \mu_{\tilde{Y},-} \) is wrong. Imbalance for \( \tilde{Y} \) does not falsify the RD design (the continuity assumption for \( Y \)), since \( \tilde{Y} \) does not affect \( Y \) by assumption. Theorem 7 with \( Z \) replaced by \( \tilde{Y} \) still holds, under the assumption that the potential unaffected outcomes satisfy the continuity assumption and our prior belief is imperfect so that the balance condition is just slightly violated (RD LATE on \( \tilde{Y} \) is \( \delta \cdot l_n \)).

\(^{40}\)Let \( LR_{p+1}(\theta \mid h) \) denote the likelihood ratio with KL divergence replaced by the Cressie-Read divergence (8). Under the same assumptions as in Theorem 7, we can show that \( \Pr \left[ LR_{p+1}(\theta \mid h) \leq c_r \right] \) admits a similar two-term asymptotic expansion with the same first-order term \( F \left( c_r \mid H \left( \frac{(\gamma_N^T \delta)^2}{\gamma_N^T} \right) \right) \) and a second-order term with a non-zero gradient at 0 if \( \vartheta \neq 0 \).

\(^{41}\)By using the Lagrange multiplier method to solve the constrained minimization problem \( \min_{\delta \in S_\tau} R(\delta) \) and mean value expansion, \( \delta_R = -\left( \nabla R(\delta_R) / \|\delta_R\| \right) \tau \) and therefore, \( R(\delta_R) = (1 - \tau) - \left( \nabla R(\delta_R)^T \nabla R(\delta_R) / \|\nabla R(\delta_R)\| \right) \tau \), where \( \delta_R \) is the mean value that lies between \( \delta_R^* \) and 0. Clearly, \( \nabla R(\delta_R)^T \nabla R(\delta_R) / \|\nabla R(\delta_R)\| \rightarrow \|\nabla R(0)\| = 0 \) as \( \tau \downarrow 0 \).
6 Covariate-adjusted estimation of the treatment effect derivative

The EB approach for covariate adjustment applies to parameters of interest other than the standard RD LATE parameter. This section applies EB to covariate-adjusted estimation of the treatment effect derivative (TED). To focus on the main ideas, we consider the sharp design first. Dong and Lewbel (2015) propose using the TED defined as \((d/dx) E[Y(1) - Y(0) | X = x]|_{x=0}\) for evaluating the external validity of RD. A large TED suggests that the LATE would be quite different if the score changes slightly, raising more concern about external validity. The researcher can check whether the RD LATE is likely to have external validity by testing for zero TED. Under the assumption that \(g_Y(d)\) is continuously differentiable on a neighborhood of 0 (Dong and Lewbel, 2015, Assumption A2), the TED is identified: \( (d/dx) E[Y(1) - Y(0) | X = x]|_{x=0} = \pi_{srd} := \mu_{Y,+}^{(1)} - \mu_{Y,-}^{(1)} \). This section proposes an EB estimator for the TED. An inferential procedure and standard errors can be found in Section S10 of the online supplement.

Let \(\tilde{W}_{p,-,i}\) be defined by the right-hand side of (2) with \(e_{p+1,1}^{\top}\) replaced by \(e_{p+1,2}^{\top}\). Similarly, we define \(\tilde{W}_{p,+i}\) and in addition, let \(\tilde{W}_{p,i} = \tilde{W}_{p,+i} - \tilde{W}_{p,-i}\). In our notation, the standard LP estimator proposed in Dong and Lewbel (2015) for the TED is given by \(\hat{\pi}_p^{lp} := (nh^2)^{-1} \sum_i \tilde{W}_{p,i} Y_i \) \((p \geq 2)\). As the EB estimator \(\hat{\pi}_p^{eb}\) for the RD LATE proposed in Section 3, the EB-based TED estimator \(\hat{\pi}_p^{eb}\) with covariate adjustment can also be obtained by replacing the uniform weights with the EB weights \((w_1^{eb}, ..., w_n^{eb})\) defined by (5):

\[
\hat{\pi}_p^{eb} := \frac{1}{nh^2} \sum_i w_i^{eb} \tilde{W}_{p,i} Y_i.
\]

The above construction illustrates the convenience of EB-based covariate adjustment: One can start with the standard estimator (without covariates) for a parameter of interest in an RD-related context and then replace its standard uniform weights with the EB weights. The EB weights are computed using the covariates only, and are independent of the standard estimator. Such an adjustment strategy also works straightforwardly in other RD-related settings, such as the (nonlinear) estimators of Xu (2017, 2018) in the scenarios with limited outcome variables.

Let \(\gamma_{ted} := (\text{Var}_{0+}[Z])^{-1} \left( \text{Cov}_{0+}[Z,Y] - \text{Cov}_{0-}[Z,Y] \right) \) and \(\tilde{K}_{p,s}(t) := e_{p+1,2}^{\top} V_{ps}^{-1} r_p(t) K(t)\), for \(s \in \{-, +\}\). One can easily verify that \(\tilde{K}_{p,+}(t) = -\tilde{K}_{p,-}(-t)\) and \(\int_{-1}^1 \tilde{K}_{p,-}(t) \tilde{K}_{p,+}(t) dt = -\varpi_p\), where \(\varpi_p := \int_0^1 \tilde{K}_{p,+}(t) \tilde{K}_{p,+}(t) dt\). Also denote \(\dot{\omega}_{p,+}^{j,k} := \int_0^1 t^j \tilde{K}_{p,+}(t) dt\) and \(\dot{\omega}_{p,-}^{j,k} := \int_0^1 t^j \tilde{K}_{p,-}(t) dt\). It can be checked that \(\dot{\omega}_{p,+}^{0,2} = \dot{\omega}_{p,-}^{0,2} =: \dot{\omega}_p^{0,2}\). The following theorem shows the asymptotic normality of the EB estimator.

**Theorem 8.** Suppose that Assumptions 1, 2 and 3 hold. Assume that \(g_{\|B\|^4}\) is bounded on \(\mathbb{B} \setminus \{0\}\). Assume that the bandwidth satisfies \(nh^{2p+3} = O(1)\) and \(nh^3 \to \infty\). Then,

\[
\sqrt{nh^3} \left( \hat{\pi}_p^{eb} - \pi_{srd} - \beta_p^{ted} h_p \right) \to_d N \left( 0, \gamma_{ted} \right),
\]

31
where
\[
\hat{\mathcal{R}}_p^{\text{ted}} := \left( \hat{\omega}_p^{(p+1)} \frac{\mu_{Y,+}^{(p+1)}}{(p+1)!} - \hat{\omega}_p^{(p+1)} \frac{\mu_{Y,-}^{(p+1)}}{(p+1)!} \right) - \left( \frac{\omega_p}{\omega_p^{(2)}} \right) \hat{\gamma}_{\text{ted}} \left( \omega_p \frac{\mu_{Z,+}^{(p+1)}}{(p+1)!} - \omega_p \frac{\mu_{Z,-}^{(p+1)}}{(p+1)!} \right) \cdot \hat{\omega}_p^{\frac{2}{2}} \text{Var}_{|x|} \left[ Y \right] \left( \frac{\omega_p}{\omega_p^{(2)}} \right) \hat{\gamma}_{\text{ted}} \left( \text{Var}_{|x|} \left[ Z \right] \right) \hat{\gamma}_{\text{ted}} \frac{1}{\varphi}.
\]

**Remark 22.** The standard LP regression theory shows \( \sqrt{n} h^3 \left( \hat{\pi}_p^{\text{lp}} - \pi_{\text{sd}} \right) \rightarrow_d \mathcal{N} \left( 0, \hat{\gamma}_p^{\text{lp}} \right) \), where \( \hat{\mathcal{R}}_p^{\text{lp}} := \left( \hat{\omega}_p^{(p+1)} \frac{\mu_{Y,+}^{(p+1)}}{(p+1)!} - \hat{\omega}_p^{(p+1)} \frac{\mu_{Y,-}^{(p+1)}}{(p+1)!} \right) / (p+1)! \) and \( \hat{\gamma}_p^{\text{lp}} := \omega_p^{\frac{2}{2}} \text{Var}_{|x|} \left[ Y \right] / \varphi \). The asymptotic variance of \( \hat{\pi}_p^{\text{lp}} \) is larger than \( \hat{\gamma}_p^{\text{ted}} \) provided that \( \gamma_{\text{ted}} \neq 0 \). Therefore, the EB method leads to efficiency gain in the case of estimating TED. Consider the simulation design (the case with one covariate) in Section 7. We get \( \gamma_{\text{ted}} = 1.5 \) and \( \sqrt{\hat{\gamma}_p^{\text{ted}}/2} = 63.5 \) by straightforward calculation, while the asymptotic standard deviation \( \sqrt{\hat{\gamma}_p^{\text{lp}}/2} \) without covariate adjustment is 74.4.

**Remark 23.** CCFT’s regression-based method can also be applied to obtain a covariate-adjusted estimator of the TED, i.e., the regression coefficient of \( I_i \cdot X_i \) in (10). Let \( \hat{\pi}_p^{\text{CCFT}} \) be defined by the right-hand side of (10) with \( \hat{e}_{2(p+1)+d,p+2} \) replaced by \( \hat{e}_{2(p+1)+d,p+3} \). Consistency of \( \hat{\pi}_p^{\text{CCFT}} \) requires covariate balance in the first derivative \( \mu_{Z,+}^{(1)} = \mu_{Z,-}^{(1)} \). Indeed, an extension of Theorem 1 shows that \( \hat{\pi}_p^{\text{CCFT}} \) is first-order equivalent to an EB estimator using weights defined by the right-hand side of (5) with \( \hat{W}_{p,i} \) replaced by \( \hat{W}_{p,i} \). Under continuous differentiability of \( g_Z(1) \) and \( g_Z(0) \), \( \mu_{Z,+}^{(1)} = \mu_{Z,-}^{(1)} \) is equivalent to the predeterminedness-type assumption \( (\text{d}/\text{d}x) E \left[ Z(1) \mid X = x \right] \big|_{x=0} = (\text{d}/\text{d}x) E \left[ Z(0) \mid X = x \right] \big|_{x=0} \) (i.e., zero TED on covariates). In comparison, consistency and efficiency gain of \( \hat{\pi}_p^{\text{lp}} \) require the same predeterminedness condition \( E \left[ Z(1) \mid X = 0 \right] = E \left[ Z(0) \mid X = 0 \right] \) as the covariate-adjusted estimators for the RD LATE do. As the TED estimator is often used to evaluate the external validity of RD LATE, it is more natural to impose the same assumptions as those underlying estimation of the RD LATE in an RD design with covariates.

**Remark 24.** Suppose that a researcher believes that both of the predeterminedness assumption \( E \left[ Z(1) \mid X = 0 \right] = E \left[ Z(0) \mid X = 0 \right] \) (the usual covariate balance condition \( \mu_{Z,+} = \mu_{Z,-} \)) and zero TED \( (\text{d}/\text{d}x) E \left[ Z(1) - Z(0) \mid X = x \right] \big|_{x=0} = 0 \) (the derivative version \( \mu_{Z,+}^{(1)} = \mu_{Z,-}^{(1)} \) of covariate balance) are likely to hold. In this case, CCFT’s estimator \( \hat{\pi}_p^{\text{CCFT}} \) does not fully exploit the information in the covariates. An estimator linearly combining \( \hat{\pi}_p^{\text{lp}} \) and \( \hat{\pi}_p^{\text{CCFT}} \) in the form of \( \zeta \cdot \hat{\pi}_p^{\text{lp}} + (1 - \zeta) \cdot \hat{\pi}_p^{\text{CCFT}} \) achieves further efficiency improvement. We can show the following joint asymptotic normality re-
suit:

\[
\sqrt{n} h^2 \left( \frac{\hat{\pi}_{eb}^p - \pi_{sd} - h^p \hat{\pi}_{CCFT}^p}{\hat{\pi}_{CCFT}^p - \pi_{sd} - h^p \hat{\pi}_{CCFT}} \right) \xrightarrow{d} N \left( \gamma^p_{ted}, \gamma^p_{p} \right),
\]

where \( \gamma^p_{CCFT} = \omega^p_{0,2} \sigma^2 / \varphi \) and

\[
\hat{\rho}^p_{CCFT} := \frac{\mu_{Y-Z}^{(p+1)} p_{i+} - \mu_{Y-Z}^{(p+1)} p_{i-}}{(p+1)!} \]

\[
\varrho^p := \gamma^p_{CCFT} - \frac{\omega^2_p}{\omega^2_{p}} \left( \frac{\text{Cov}_{[0]}{Z|Y]} - \text{Cov}_{[0]}{Z|Y} \right) \gamma_{ted} + \frac{\omega^2_p}{\omega^2_{p}} \gamma_{ted} \left( \mu_{ZZ}^{(1)} \right) \gamma_{Y}.
\]

Therefore, the optimal linear combination that has the smallest asymptotic variance will assign to \( \hat{\pi}_{eb}^p \) the following optimal weight \( \varsigma^* := \left( \gamma^p_{CCFT} - \varrho^p \right) / \left( \gamma^p_{ted} + \gamma^p_{CCFT} - 2 \varrho^p \right) \). Once again, consider the simulation design in Section 7 with slight modification to ensure \( \mu_{Z,+}^{(1)} = \mu_{Z,-}^{(1)} = 1.06 \).\(^{44}\) The optimal weight \( \varsigma^* = 0.35 \) and the resulting asymptotic standard deviation is 47.5, which is smaller than \( \sqrt{\gamma^2_{ted}} = 63.5 \) calculated in Remark 22. Another approach to exploiting the information in both balance conditions is based on the EB weights with a new set of constraints \( \sum_i w_i \hat{W}_{p,1}^i \hat{Z}_i = 0 \) being added to (4). We can show that this estimator is first-order equivalent to the optimal combination. However, such a method is more computationally costly.

Remark 25. In the fuzzy RD, Dong and Lewbel (2015) show that the TED is identified:

\[
\frac{d}{dx} \mathbb{E} [Y(1) - Y(0) | X = x, co] \bigg|_{x=0} = \frac{\mu_{y,+}^{(1)} - \mu_{y,-}^{(1)}}{\mu_{D,+}^{(1)} - \mu_{D,-}^{(1)}} \cdot \frac{\mu_{y,+}^{(1)}}{\mu_{D,+}^{(1)}} \cdot \frac{\mu_{y,-}^{(1)}}{\mu_{D,-}^{(1)}}.
\]

The same equality with \( Y \) replaced by \( Z \) also holds. Covariate-adjusted estimation of TED based on (24) and our EB approach is straightforward. Under covariate balance, \( \mu_{Z,+}^{(1)} - \mu_{Z,-}^{(1)} = 0 \) is implied by \( (d/dx) \mathbb{E} [Z(1) - Z(0) | X = x, co] |_{x=0} = 0 \). EB-based estimation exploiting both predeterminedness and zero TED conditions is also straightforward.

7 Monte Carlo simulations

We conduct simulations to evaluate the finite sample performance of the proposed EL-based inference methods for sharp RD designs with covariates. The data-generating process (DGP) of the outcome variable \( Y_i \), the score \( X_i \) and the first covariate \( Z_i^{(1)} \) is based on the simulation design of CCFT. The incorporation of additional covariates \( Z_i^{(2)}, ..., Z_i^{(l)} \) follows that of Arai et al. (2021). Let

\[
\mu_{y0}(x) := 0.36 + 0.96x + 5.47x^2 + 15.28x^3 + 15.87x^4 + 5.14x^5
\]

\[
\mu_{y1}(x) := 0.38 + 0.62x - 2.84x^2 + 8.42x^3 - 10.24x^4 + 4.31x^5
\]

\(^{44}\)In this numerical example, letting \( \mu_{Z,+}^{(1)} = \mu_{Z,-}^{(1)} = 1.06 \) does not change the asymptotic variance and covariance \( \gamma_{ted}, \varrho_2 \) and \( \gamma_{CCFT}^p \).
\[ \mu_{z0}(x) := 0.49 + \theta_1 x + 5.74 x^2 + 17.14 x^3 + 19.75 x^4 + 7.47 x^5 \]
\[ \mu_{z1}(x) := 0.49 + \theta_1 x - 0.23 x^2 - 3.46 x^3 + 6.43 x^4 - 3.48 x^5 \]

and

\[
\begin{align*}
\mu_y(x,z_1) & := \\
& \begin{cases} 
\mu_{y0}(x) + \gamma_l z_1 & \text{if } x < 0 \\
\mu_{y1}(x) + \gamma_r z_1 & \text{if } x \geq 0 
\end{cases} \\
\mu_z(x) & := \\
& \begin{cases} 
\mu_{z0}(x) & \text{if } x < 0 \\
\mu_{z1}(x) & \text{if } x \geq 0, 
\end{cases}
\end{align*}
\]

(25)

(26)

where with the coefficients \( \gamma_l = 0.22, \gamma_r = 0.28, \theta_l = 1.06 \) and \( \theta_r = 0.61 \), all following CCFT. Then, \( Y_i = \mu_y(X_i, Z_i^{(1)}) + \sum_{j=2}^{l} \pi^{j-1} Z_i^{(j)} + \varepsilon_{y,i} \) and \( Z_i^{(1)} = \mu_z(X_i) + \varepsilon_{z,i} \). Error terms \( (\varepsilon_{y,i}, \varepsilon_{z,i}) \) are bivariate normal with mean 0, standard deviation 1 and correlation coefficient \( \rho = 0.269 \). Additional covariates \( (Z_i^{(2)}, ..., Z_i^{(l)}) \) have a multivariate normal distribution with mean zero and covariance matrix given by \( \text{Cov} [Z_i^{(j)}, Z_i^{(k)}] = 0.5^{\lfloor j-k \rfloor} \), for all \( j, k \geq 2 \). We take \( \pi = 0.2 \). We consider three scenarios with \( l = 0, 2, 4 \), corresponding to the total number of covariates \( d_z = l + 1 \) being 1, 3, 5. CCFT uses local linear regression with bias correction, equivalent to local quadratic regression. Our EL approach parallels CCFT in that the degree of the LP is set to be \( p = 2 \). The sample sizes are \( n = 1000, 2000 \). The number of Monte Carlo replications is 5000.

Table 1 presents the bias, root mean square error (RMSE) of the MC-EL estimator \( \hat{\theta}^{mc}_{p} \) defined in Section 3.3, as well as the empirical coverage probability and the average length of the EL confidence sets \( CS_{p}\tau (h) \) and \( CS_{p+1}\tau (h) \) defined in Remark 18. Following Remark 16, we select a bandwidth of the form \( h = H \cdot n^{-1/(p+2)} \), replace \( H \) with a consistent estimator \( \hat{H} \), and use the bandwidth \( \hat{h} := \hat{H} \cdot n^{-1/(p+2)} \). Calonico et al. (2020, Section 5.3) propose an approach that takes the estimated AMSE optimal bandwidth and rescales it to make it obey the coverage optimal rate (see Section IV(C) of CCFT). One choice of bandwidth \( \hat{h} \) is to follow this approach and use CCFT’s bandwidth, denoted as CCFT in Table 1. CCFT’s bandwidth is computed from R function \texttt{rdrobust} with the options \( p = 1 \), \( \text{rho} = 1 \), and \texttt{bwselect} =“cerrd”.\(^{45}\) Another simpler choice is a rescaled rule of thumb (ROT) bandwidth that uses the constant part \( \hat{H} \) computed according to Hansen (2021, Chapter 21.6)’s ROT bandwidth. For comparison, Table 1 also includes results from CCFT’s method that uses the CCFT bandwidth and restricts \( \rho = h/b = 1 \), where \( b \) stands for the pilot bandwidth used for bias estimation. Table 1 shows that both EL and CCFT approaches perform well for estimation and inference. A closer look reveals that EL with \( p = 2 \) and using the rescaled ROT bandwidth yields similar bias and RMSE compared with CCFT, but slightly better coverage (especially for \( d_z = 3 \) and 5), and shorter confidence intervals. On the other hand, EL that uses CCFT’s bandwidth, which amounts to half of the ROT bandwidth, yields smaller bias but larger RMSE and longer confidence intervals. In particular, the length of \( CS_{p+1}\tau (\hat{h}) \) is longer than

\(^{45}\)The rate of CCFT’s bandwidth is \( n^{-1/4} \), which matches the rate of \( \hat{h} = \hat{H} \cdot n^{-1/(p+2)} \) with \( p = 2 \).
those of other confidence sets. In sum, all the methods we consider deliver satisfactory finite-sample performances.\footnote{The EL confidence intervals are also well-centered. E.g., the average center (across all 5,000 simulation replications) of the EL intervals with ROT bandwidth (EL$_p$ in Table 1), $d_z = 5$ and $n = 1,000$ is only 0.0015 away from the true treatment parameter (0.0494).} Computing the EB weights (for the point estimator) and the EL likelihood ratio statistic (for the confidence set) only requires solving convex optimization problems (corresponding to the “inner loop” in the standard EL computation) and thus is very fast. E.g., computing the row EL$_p$ in Table 1 with $d_z = 5$, $n = 1,000$ and CCFT bandwidth costs 0.06 to 1.08 seconds for one replication, with the average computation time per replication about 0.23 second on an Intel Core i7 processor with 32 GB of RAM.

Table 1: Performance of EL and Wald-type confidence sets in sharp RD with covariates: EL$_p$ corresponding to the point estimator $\hat{\varphi}^{mc}_p$ and the confidence set $CS_{p,\tau}^{bc}(\hat{h})$, EL$_{p+1}$ corresponding to the point estimator $\hat{\varphi}^{mc}_{p+1}$ and the confidence set $CS_{p+1,\tau}^{bc}(\hat{h})$; $\tau = 0.05$ $p = 2$, the bandwidth $\hat{h}$ = rescaled rule of thumb (ROT) or CCFT’s (CCFT) bandwidth with the average bandwidth length for $n = 1,000$ reported in the parenthesis. CCFT’s Wald-type inference uses the CCFT bandwidth, CP = the coverage probability, CIL = the average length of the confidence intervals, $n$ = sample size, $d_z$ = the number of covariates.

| $d_z$ | Methods | $\hat{h}$ | Bias | RMSE | 0.95 CP | 0.95 CIL |
|------|---------|----------|------|------|---------|----------|
| 1    | EL$_p$  | ROT (0.301) | 0.011 | 0.014 | 0.336   | 0.246    | 0.960    | 0.964    | 1.472    | 1.044    |
|      |         | CCFT (0.147) | 0.007 | 0.008 | 0.420   | 0.331    | 0.946    | 0.949    | 1.931    | 1.326    |
|      | EL$_{p+1}$ | ROT      | 0.011 | 0.012 | 0.461   | 0.326    | 0.948    | 0.949    | 1.790    | 1.282    |
|      |         | CCFT     | 0.003 | 0.010 | 0.481   | 0.400    | 0.940    | 0.938    | 2.503    | 1.809    |
|      | CCFT    | CCFT     | 0.013 | 0.014 | 0.334   | 0.238    | 0.945    | 0.951    | 1.822    | 1.285    |
| 3    | EL$_p$  | ROT (0.303) | 0.014 | 0.011 | 0.349   | 0.248    | 0.957    | 0.954    | 1.460    | 1.040    |
|      |         | CCFT (0.145) | 0.007 | 0.004 | 0.421   | 0.334    | 0.933    | 0.946    | 1.917    | 1.345    |
|      | EL$_{p+1}$ | ROT      | 0.007 | 0.002 | 0.499   | 0.335    | 0.934    | 0.940    | 1.784    | 1.279    |
|      |         | CCFT     | 0.001 | -0.000 | 0.471   | 0.414    | 0.932    | 0.937    | 2.484    | 1.822    |
|      | CCFT    | CCFT     | 0.012 | 0.010 | 0.347   | 0.241    | 0.936    | 0.949    | 1.806    | 1.280    |
| 5    | EL$_p$  | ROT (0.302) | -0.001 | -0.001 | 0.358   | 0.253    | 0.946    | 0.957    | 1.452    | 1.035    |
|      |         | CCFT (0.143) | 0.001 | -0.006 | 0.438   | 0.346    | 0.917    | 0.929    | 1.956    | 1.336    |
|      | EL$_{p+1}$ | ROT      | -0.004 | -0.007 | 0.507   | 0.346    | 0.916    | 0.933    | 1.793    | 1.274    |
|      |         | CCFT     | 0.000 | -0.004 | 0.486   | 0.426    | 0.915    | 0.915    | 2.509    | 1.826    |
|      | CCFT    | CCFT     | 0.000 | 0.004 | 0.354   | 0.242    | 0.924    | 0.938    | 1.787    | 1.270    |

We also examine how the coverage performance of EL and CCFT confidence sets changes when the covariate balance condition is slightly violated. We consider the case with one covariate ($d_z = 1$). The data-generating process for $(Y_i, X_i, Z_i(1))$ remains the same but the incorporated covariate is given by $\tilde{Z}_i(1) := Z_i(1) + 1(X_i < 0)\delta$, so that the local covariate imbalance is measured by the
perturbation $\delta$. Figure 1 plots the simulated coverage probabilities of the EL and CCFT confidence sets as a function of $\delta \in [-0.3, 0.3]$. We observe that the coverage probability of $C_{p+1, \tau}^{\text{bc}}(\hat{h})$ is less sensitive to the change of $\delta$, which parallels the discussion in Remark 20.

Figure 1: Sensitivity of coverage probabilities of EL-based confidence sets $C_{p, \tau}^{\text{bc}}(\hat{h})$, $C_{p+1, \tau}^{\text{bc}}(\hat{h})$, and the CCFT confidence set with respect to a local imbalance of magnitude $\delta$, $n = 2000$, $p = 2$, bandwidth $\hat{h} = \text{CCFT’s bandwidth}$.

We then investigate the performance of the EB approach in the covariate-adjusted estimation of TED. We consider the case with one covariate ($d_z = 1$) and modify the coefficients $\gamma_l$, $\gamma_r$, $\theta_l$, and $\theta_r$ in the design in order to highlight two features of covariate-adjusted estimation of TED. First, the magnitude of efficiency gain from incorporating the single covariate is determined by $|\gamma_l - \gamma_r|$. We choose $\gamma_l = 3$ and $\gamma_r = 0$ to highlight the efficiency contribution of the covariate adjustment. Second, note that $\theta_r - \theta_l$ corresponds to $\mu_{Z,+}^{(1)} - \mu_{Z,-}^{(1)}$, which is required to be zero for CCFT’s augmented regression estimator of TED (Remark 23). More specifically, the asymptotic bias of CCFT’s estimator is proportional to $|\theta_l - \theta_r|$. We set $\theta_l = 3$ and $\theta_r = 0$ to highlight such a bias. Table 2 reports the finite-sample performances of three TED estimators: the EB estimator with $p = 2$ given by (23) and its confidence interval constructed following the procedure in Section S10 in our online supplement, CCFT’s TED estimator with $p = 2$ in Remark 23, and the standard local quadratic (LQ) TED estimator (Dong and Lewbel, 2015) without using covariate information, all three methods using Calonico et al. (2014, CCT, hereafter)’s bandwidth for the first derivative computed from the R function rdrobust.\footnote{As expected, Table 2 shows that the CCFT’s TED estimator leads to a substantial bias and undercoverage for TED inference, given that $\mu_{Z,+}^{(1)} - \mu_{Z,-}^{(1)}$ is away from zero. This problem can be solved by the EB estimator, which incorporates the correctly specified covariate balance condition $\mu_{Z,+} = \mu_{Z,-}$ rather than the misspecified condition $\mu_{Z,+}^{(1)} = \mu_{Z,-}^{(1)}$. The standard LQ TED estimator without covariates remains valid but has a larger RMSE and $\text{RMSE}$.}

\textit{We note that the rate of CCT’s bandwidth here is still $n^{-1/4}$, which is also the CO rate for Wald-type inference on the TED. See Calonico et al. (2020, Theorem 3.1).}
yields a longer confidence interval than those from the EB estimator, which reflects the efficiency gain of EB from covariate adjustment.

Table 2: Treatment effect derivative (TED) inference with covariates: the EB method compared with the standard LQ or CCFT, COV indicates whether a method adjusts for covariates; CP = the coverage probability, CIL = the average length of the confidence intervals, n = sample size, d_z = the number of covariates. The true TED = −9.34. All rows use CCT’s bandwidth. The average bandwidth length equals 0.151 for n = 1,000 and 0.123 for n = 2,000.

| Methods | COV | Bias   | RMSE | 0.95 CP | 0.95 CIL |
|---------|-----|--------|------|---------|---------|
|         |     | n = 1,000 | 2,000 | n = 1,000 | 2,000 | n = 1,000 | 2,000 |
| EB      | YES | 0.961   | 0.710 | 11.542  | 8.266  | 0.934   | 0.942  | 40.034 | 30.357 |
| CCFT    | YES | 5.894   | 5.830 | 11.046  | 9.094  | 0.839   | 0.791  | 32.444 | 25.042 |
| Standard| NO  | 0.221   | 0.362 | 12.359  | 9.254  | 0.936   | 0.946  | 43.325 | 33.762 |

8 Empirical illustration: Finnish municipal election data

We apply our estimation/inference method to analyze the individual incumbent advantage in Finnish municipal elections, which was first studied by Hyytinen et al. (2018). The outcome variable Y indicates whether the candidate is elected in an election, and the score X is the vote share margin in the previous election. Table 3 presents the RD LATE point estimate \( \hat{\vartheta} \), the p-value for testing the null hypothesis \( \vartheta = 0 \), the 95% confidence intervals (CI), and the CI length. The first row of Table 3 presents the standard LQ regression estimator that ignores the covariates. Then, we incorporate four covariates \( Z \): candidates’ age, gender, age squared, and age \( \times \) gender. EL estimation and inference use CCT’s bandwidth \( (h_{CCT} = 0.396) \) and the rescaled ROT bandwidth (equal to 2.917). The last row of Table 3 reproduces the “experiment benchmark” reported originally by Hyytinen et al. (2018)(see their Table 2, Column 4, the p-value is imputed by us). Apparently, all RD estimates, with or without covariates, are small in magnitude and statistically insignificant, which agrees with the finding in the experiment benchmark. By comparing the covariate-adjusted estimates (EL and CCFT) with the standard LQ regression without covariates, we see that incorporating covariates helps to reduce the CI length for four out of five confidence intervals, except for \( CS^{bc}_{p+1,\tau} (h_{CCT}) \). Among them, the EL confidence set \( CS^{bc}_{p,\tau} (h_{CCT}) \) that uses the same bandwidth as the standard LQ regression is 7.2% shorter than the standard method and is 5.5% shorter than CCFT. Here, the efficiency improvement is moderate, probably because the election outcome is only weakly correlated with age and gender.

We then conduct a sensitivity analysis of the EL-based covariate-adjusted inference with respect to the bandwidth choice by plotting the confidence band (Remarks 14 and 19). We consider the dataset includes 1351 candidates “for whom the (previous) electoral outcome was determined via random seat assignment due to ties in vote counts” (Hyytinen et al., 2018, Page 1020), which constitutes an experiment benchmark to evaluate the credibility of the RD treatment effect estimated from the non-experimental data (candidates with previous electoral ties are excluded from the RD sample).
continuous range of bandwidths $h \in [\underline{h}, \overline{h}]$ with the lower bound $\underline{h} = h_{\text{CCT}}/3 \approx 0.13$ and the upper bound $\overline{h} = h_{\text{CCT}} \times 2 \approx 0.78$. The rate of $h_{\text{CCT}}$ is $n^{-1/4}$, which satisfies the conditions for $\underline{h}$ and $\overline{h}$ in Theorem 5. Using the R package \texttt{BWSnooping}, we calculate the snooping corrected critical value $2.413^2$ for the triangular kernel and bandwidth ratio $\overline{h}/\underline{h} = 6$. In Figure 2, the solid (or dotted) lines correspond to a 95% uniform (or pointwise) confidence band. For small bandwidth (say, less than 0.2), the uniform confidence band is wide. However, as long as the bandwidth is not so small, the confidence band appears stable. Moreover, the confidence band includes zero over the entire bandwidth range, demonstrating the robustness of the finding of no incumbency advantage with respect to the bandwidth choice.

Lastly, we evaluate the external validity by testing the null hypothesis that the TED is zero. It will tell us whether the RD estimate, which by design only applies to the “local” incumbents whose previous vote share margin resides at the 0 cutoff, can be applied to incumbents whose previous vote share margins are slightly higher than 0. When estimating the TED, we maintain the usual covariate balance condition $\mu_{Z,+} = \mu_{Z,-}$ but do not impose the balance condition $\mu^{(1)}_{Z,+} = \mu^{(1)}_{Z,-}$ for the derivatives, so the CCFT’s augmented regression estimator for TED is not a proper choice, as discussed in Remark 23. Our EB method gives a point estimate of TED equal to $-0.631$, and a $p$-value for testing a zero TED equal to 0.064. In comparison, the standard estimate (without covariates) of TED is $-0.634$ with the $p$-value equal to 0.023.\textsuperscript{49} Therefore, both methods raise the concern of external validity of applying the RD estimate to incumbents with share margins above 0, as the treatment effect is likely to significantly decrease in response to a marginal increase in the score.

Table 3: Incumbency Advantage in Finnish Municipal Election: $\hat{\vartheta} = \text{RD LATE estimate}$, COV: NO = without covariate; YES = with covariate, bandwidth selector being CCT’s bandwidth ($h_{\text{CCT}} = 0.396$) or the ROT bandwidth = 2.917, the $p$-value for testing $\vartheta = 0$. The sample size $n = 154,543$ for all RD methods, and $n = 1,351$ for the experimental data in the last row.

| Methods | COV | $\hat{h}$ | $\hat{\vartheta}$ | $p$-value | 95% CI | CI length |
|---------|-----|---------|-----------------|----------|--------|----------|
| Standard | NO  | CCT    | 0.012           | 0.675    | [-0.067,0.044] | 0.111    |
| EL$_p$   | YES | ROT    | 0.031           | 0.187    | [-0.014,0.075]  | 0.089    |
|          |     | CCT    | -0.009          | 0.741    | [-0.060,0.043]  | 0.103    |
| EL$_{p+1}$ | YES | ROT    | 0.009           | 0.447    | [-0.014,0.033]  | 0.047    |
|          |     | CCT    | -0.049          | 0.294    | [-0.141,0.042]  | 0.183    |
| CCFT     | YES | CCT    | 0.014           | 0.621    | [-0.068,0.041]  | 0.109    |
| Experimental data | NO  | -0.010 | 0.516           | [-0.060,0.040] | 0.100 |

\textsuperscript{49}Both EB and the standard estimates use the CCT bandwidth for the first derivative, which equals 0.462.
Figure 2: A sensitivity analysis of the EL-based covariate-adjusted inference using the Finnish municipal election data: uniform (solid) and pointwise (dotted) confidence bands as functions of the bandwidth $h$ with the $h = h_{CCT}/3 \approx 0.13$ and $h = h_{CCT} \times 2 \approx 0.78$, $\tau = 0.05$. Bandwidth snooping corrected critical value $= 2.413^2$. Vertical line indicates the CCT’s bandwidth $h_{CCT} = 0.396$.

9 Conclusion and further discussion

This paper proposes a balancing approach to covariate adjustment for RD. The covariate balance condition can be viewed as over-identifying restrictions, which the EB estimator incorporates when formulated as an EL estimator. By establishing the first-order equivalence between the EB estimator and CCFT’s regression estimator, we show that the efficiency gain can be attributed to incorporating covariate balance as side information.

The EB problem (4) can be cast in a more general framework under which several extensions can be considered. The construction follows Ben-Michael et al. (2021). Consider the following imbalance measure $\text{imbalance}_M(w_1, ..., w_n) := \sup_{f \in M} \left| \sum_i w_i \hat{W}_p, f(Z_i) \right|$ with respect to a function space $M$. Let $m : \mathbb{R}_+ \rightarrow \mathbb{R}_+ \cup \{\infty\}$ be an increasing and convex function. Let complexity $(w_1, ..., w_n)$ denote some complexity (or dispersion) measure of the weights. Consider the following risk minimization
problem similar to Ben-Michael et al. (2021, Equation (12)):

\[
\min_{w_1 + \cdots + w_n = 1} m \left( \text{imbalance}_M (w_1, ..., w_n) \right) + \zeta \cdot \text{complexity} (w_1, ..., w_n), \tag{27}
\]

for some tuning parameter $\zeta > 0$. Denote $M_0 := \left\{ \mathbb{R}^d_z \ni z \mapsto a + z^T b : |a| + \sum_{j=1}^d |b(j)| \leq 1 \right\}$. We may also take the Cressie-Read divergence $D_\phi (w_1, ..., w_n \parallel 1/n, ..., 1/n)$ defined by (8) as a complexity measure. It is clear that under complexity $(w_1, ..., w_n) = D_\phi (w_1, ..., w_n \parallel 1/n, ..., 1/n)$ the generalized balancing problem (9) can be written in the form (27) with $M = M_0$ (see Ben-Michael et al., 2021, Equation (13)) and $m (\cdot)$ taken to be $\mathbb{R}_+ \ni x \mapsto \infty \cdot 1 (x > 0)$, so that exact balance is required.\(^{50}\) The sieve balancing problem (15) is also of the form (27) with complexity $(w_1, ..., w_n) = KL (w_1, ..., w_n \parallel 1/n, ..., 1/n)$ and $M$ taken to be the broader sieve space.

An alternative balancing scheme similar to Hirshberg and Wager (2021) is based on solving (27) with $M$ taken to be the sieve space, $m (\cdot)$ taken to be $x \mapsto x^2$ and complexity $(w_1, ..., w_n)$ taken to be the “square Euclidean” divergence given by $D_{-2} (w_1, ..., w_n \parallel 1/n, ..., 1/n)$. Then, we expect to find a dual characterization of the optimal weights by using results from Hirshberg and Wager (2021). An asymptotic normality result similar to Theorem 3 is expected to hold under a suitable choice of tuning parameters $(h, k, \zeta)$. With $M$ taken to be a ball in a Reproducing Kernel Hilbert Space (RKHS), we get a balancing scheme similar to Kallus (2020); Wong and Chan (2017) (see Wong and Chan, 2017, Equation (5)). We also expect an asymptotic normality result similar to Theorem (3) holds under a suitable choice of the three tuning parameters ($(h, \zeta)$ and the radius of the ball) and the assumption that the “optimal adjustment function” $\eta^*$ (see Section 4.3) lies in the RKHS.

Our EB approach avoids the selection of the additional tuning parameter. Another advantage is the favorable second-order properties developed in the EL literature carry over to our proposed method. These include a small nonlinearity bias of the point estimator and a simple analytical correction to improve coverage accuracy for the confidence set. We also show a uniform-in-bandwidth Wilks theorem, which can be used for sensitivity analysis and robust inference along the lines of AK. We also derive the distributional expansion for the EL ratio statistics under the local imbalance condition and analyze the sensitivity of the coverage performance to the balance assumption. Lastly, we demonstrate that our approach can address previously unsolved covariate adjustment problems in RD by deriving an EB-based covariate-adjusted estimator for the TED. We also expect the large-deviation optimality results for EL (e.g., Otsu, 2010) to carry over. In the presence of high-dimensional covariates (Arai et al., 2021; Kreiß and Rothe, 2022), resorting to the dual characterization (5), we apply appropriate penalization in (6) (Chang et al., 2018) to reduce the effective

\[^{50}\text{The risk minimization problem can now be written as}
\]

\[
\min_{w_1, ..., w_n} \zeta \cdot \text{complexity} (w_1, ..., w_n) \\
\text{subject to} \quad \text{imbalance}_M (w_1, ..., w_n) = 0.
\]

Clearly, the optimal weights do not depend on the choice of $\zeta$ in this case. Relaxation of the “exact balance” constraint by using a strictly positive threshold (see Ben-Michael et al., 2021, Section 9.1.2) is also straightforward.
number of covariates. Properties of the penalized EB are left for future investigation.

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