Research Article

Construction and Stability of Riesz Bases

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We construct some new Riesz bases and consider the stability of them. The investigation is based on the stability of Riesz bases of cosines and sines in the Hilbert space \( L^2[0,\pi] \).

1. Introduction

As is well known, Riesz basis is not only a base but also a special frame. The research of frame and Riesz basis plays important role in theoretical research of wavelet analysis [1]; because of the redundancy of frame and Riesz basis, they have been extensively applied in signal denoising, feature extraction, robust signal processing, and so on. Therefore, construction of Riesz basis has attracted much attention of the researchers due to their wide applications.

In 1934, Paley and Wiener studied the problem of finding sequences \( \{\lambda_n\} \) for which \( \{\exp(i\lambda_n x)\} \) is a Riesz basis in \( L^2[-\pi,\pi] \) [2]. Since then many results on the Riesz basis have been obtained [3–5]. Also the Riesz basis of the systems of sines and cosines in \( L^2[0,\pi] \) and Riesz basis associated with Sturm-Liouville problems have been studied in many papers [6–12]; moreover, on the problems of expansion of eigenfunctions, we refer to [13–18] and references cited therein.

Motivated by these works, on the one hand, we construct two groups of Riesz bases \( \{1\} \cup \{\cos(2nx)\} \cup \{\sin(2nx)\} \) and \( \{\sin((2n-1)x)\} \cup \{\cos((2n-1)x)\} \) and study the stability of them. On the other, we consider the problem of finding a new sequence associated with eigenfunctions of Sturm-Liouville problem

\[
-y'' + qy = \lambda y, \quad \text{on } [0,\pi];
\]

\[
y(0) = y(\pi) = 0,
\]

such that it forms a Riesz basis.

2. Riesz Bases Generated by Sines and Cosines

Let us first recall some basic concepts. Let \( \{f_n\}, n \in \mathbb{N}, \) be a sequence in a Hilbert space \( H \), where \( \mathbb{N} \) is the set of positive integers. The sequence is called complete if its closed span equals \( H \) [5, P. 154]. We say that \( \{f_n\} \) is a Bessel sequence if \( \sum_{n=1}^{\infty} |\langle f, f_n \rangle|^2 < \infty \) for every element \( f \in H \) and that the sequence \( \{f_n\} \) is a Riesz-Fischer sequence if the moment problem \( \langle f, f_n \rangle = c_n \) (\( n = 1, 2, 3, \ldots \)) admits at least one solution \( f \in H \) whenever \( \{c_n\} \in \ell^2 \) [5, P. 154].

A basis \( \{f_n\} \) of Hilbert space is called a Riesz basis if it is obtained from an orthonormal basis by means of a bounded linear invertible operator. Two sequences of elements \( \{f_n\} \) and \( \{g_n\} \) from Hilbert space \( H \) are called quadratically close if \( \sum_{n=1}^{\infty} \|f_n - g_n\|^2 < \infty \) [5, P. 45]. A sequence \( \{\lambda_n\} \) of real or complex numbers is said to be separated if, for some positive number \( \epsilon \), \( |\lambda_n - \lambda_m| \geq \epsilon \) whenever \( n \neq m \) [5, P. 98]. A sequence \( \{f_n\} \) is called \( \omega \)-linearly independent if the equality \( \sum_{n=1}^{\infty} c_n f_n = 0 \) is possible only for \( c_n = 0 \) (\( n \geq 1 \)) [5, P. 40].

Next we need the following lemmas to get our main results.

Lemma 1 ([5, P. 155]).

(i) The sequence \( \{f_n\} \) is a Bessel sequence with bound \( M \) if and only if the inequality

\[
\left\| \sum_n c_n f_n \right\|^2 \leq M \sum_n |c_n|^2
\]

holds for every finite systems \( \{c_n\} \) of complex numbers.
(i) The sequence \( \{f_n\} \) is a Riesz-Fischer sequence with bound \( m \) if and only if the inequality
\[
m \sum_{n} |c_n|^2 \leq \left\| \sum_{n} c_n f_n \right\|^2
\]
holds for every finite systems \( \{c_n\} \) of complex numbers.

**Lemma 2** ([16, P. 95]). Let two sequences \( \{f_n\} \) and \( \{g_n\} \) be quadratically close and let \( \{f_n\} \) be an Riesz basis in \( H \).

(i) If the sequence \( \{g_n\} \) is \( \omega \)-linearly independent, then \( \{g_n\} \) is a Riesz basis in \( H \).

(ii) If the sequence \( \{g_n\} \) is complete in \( H \), then \( \{g_n\} \) is \( \omega \)-linearly independent.

Using the above lemmas, we obtain the following lemmas.

**Lemma 3.** If \( \{\cos(\lambda_n x)\} \cup \{\sin(\tilde{\lambda}_n x)\} \) is a Riesz-Fischer sequence in \( L^2[0, \pi] \) with real \( \lambda_n \) and \( \tilde{\lambda}_n \), then the sequences \( \{\lambda_n\} \) and \( \{\tilde{\lambda}_n\} \) are separated, respectively.

Proof. Let \( m \) be a lower bound of \( \{\cos(\lambda_n x)\} \cup \{\sin(\tilde{\lambda}_n x)\} \). With \( c_n = 1, c_k = -1 \) and \( c_n = 0, d_n = 0 \), it follows from (3) that
\[
\sqrt{2m} \leq \left\| \cos(\lambda_n x) - \cos(\lambda_k x) \right\|.
\]

On the other hand,
\[
\begin{align*}
\left\| \cos(\lambda_n x) - \cos(\lambda_k x) \right\|^2 & = \int_{0}^{\pi} \left| \cos(\lambda_n x) - \cos(\lambda_k x) \right|^2 dx \\
& \leq \int_{0}^{\pi} |\lambda_n - \lambda_k|^2 x^2 dx = \frac{\pi^3}{3} |\lambda_n - \lambda_k|^2.
\end{align*}
\]

Thus \( \{\lambda_n\} \) is separated by definition.

Similarly, setting \( d_m = 1, d_k = -1 \) and \( c_n = 0, d_n = 0 \) in (3), we also have that \( \{\tilde{\lambda}_n\} \) is separated. \( \square \)

**Lemma 4.** Let \( \{\lambda_n\} \cup \{\tilde{\lambda}_n\} \) and \( \{\mu_n\} \cup \{\tilde{\mu}_n\} \), \( n \in \mathbb{N} \), be two sequences of nonnegative real numbers such that \( \lambda_m \neq \lambda_k \), \( \tilde{\lambda}_m \neq \tilde{\lambda}_k \), \( \mu_m \neq \mu_k \), and \( \tilde{\mu}_m \neq \tilde{\mu}_k \) for all \( m \neq k \) and
\[
\sum_{n=1}^{\infty} (\lambda_n - \mu_n)^2 + \sum_{n=1}^{\infty} (\tilde{\lambda}_n - \tilde{\mu}_n)^2 < \infty.
\]

Then \( \{\cos(\lambda_n x)\} \cup \{\sin(\tilde{\lambda}_n x)\} \) is a Riesz basis in \( L^2[0, \pi] \) if and only if \( \{\cos(\mu_n x)\} \cup \{\sin(\tilde{\mu}_n x)\} \) is a Riesz basis in \( L^2[0, \pi] \).

Proof. Let \( f_n(x) = \cos(\lambda_n x), \tilde{f}_n(x) = \sin(\tilde{\lambda}_n x) \) and \( g_n(x) = \cos(\mu_n x), \tilde{g}_n(x) = \sin(\tilde{\mu}_n x) \). Suppose that \( \{f_n\} \cup \{\tilde{f}_n\} \) is a Riesz basis in \( L^2[0, \pi] \). By Lemma 3, we find that the sequences \( \{\lambda_n\} \) and \( \{\tilde{\lambda}_n\} \) are separated, respectively. Using (6), we get that the sequences \( \{\mu_n\} \) and \( \{\tilde{\mu}_n\} \) are also separated, respectively. Therefore, we can assume
\[
0 \leq \mu_1 < \mu_2 < \mu_3 \ldots,
\]
and there is a positive \( \varepsilon \) such that \( \mu_n \geq \varepsilon n \) and \( \tilde{\mu}_n \geq \varepsilon n \) for all \( n \in \mathbb{N} \). Since
\[
\begin{align*}
\|f_n - g_n\| & \leq \pi|\lambda_n - \mu_n|, \\
\|\tilde{f}_n - \tilde{g}_n\| & \leq \pi|\tilde{\lambda}_n - \tilde{\mu}_n|,
\end{align*}
\]
we obtain that
\[
\begin{align*}
\sum_{n=1}^{\infty} \|f_n - g_n\|^2 & + \sum_{n=1}^{\infty} \|\tilde{f}_n - \tilde{g}_n\|^2 \\
& \leq \pi^2 \left( \sum_{n=1}^{\infty} |\lambda_n - \mu_n|^2 + \sum_{n=1}^{\infty} |\tilde{\lambda}_n - \tilde{\mu}_n|^2 \right) < \infty;
\end{align*}
\]
thus two sequences \( \{f_n\} \cup \{\tilde{f}_n\} \) and \( \{g_n\} \cup \{\tilde{g}_n\} \) are quadratically close. In particular, \( \{f_n - g_n\} \cup \{\tilde{f}_n - \tilde{g}_n\} \) is a Bessel sequence.

We can define a bounded linear operator
\[
T \left( \sum_{n=1}^{\infty} c_n f_n + \sum_{n=1}^{\infty} d_n \tilde{f}_n \right)
\]
\[
= \sum_{n=1}^{\infty} c_n (f_n - g_n) + \sum_{n=1}^{\infty} d_n (\tilde{f}_n - \tilde{g}_n)
\]
on \( L^2[0, \pi] \), as \( \{f_n\} \cup \{\tilde{f}_n\} \) is a Hilbert-Schmidt operator. Furthermore, by Lemma 2, it is sufficient to prove that 1 is a regular point of \( T \) in order to prove that \( \{g_n\} \cup \{\tilde{g}_n\} \) is a Riesz basis.

Assume that 1 is not a regular point of \( T \). By the compactness of \( T \), \( I - T \) is not one to one; i.e., there exists a sequence \( \{c_n\} \cup \{d_n\} \in l^2 \), not identically zero, such that
\[
\sum_{n=1}^{\infty} c_n g_n + \sum_{n=1}^{\infty} d_n \tilde{g}_n = 0.
\]

Let \( \lambda \in C \) such that \( \lambda \neq \pm \mu_n, \pm \tilde{\mu}_n \) for all \( n \in \mathbb{N} \). Then, the series
\[
g(x) = \sum_{n=1}^{\infty} c_n \cos(\lambda_n x) + \sum_{n=1}^{\infty} d_n \sin(\tilde{\lambda}_n x)
\]
is convergent uniformly on \([0, \pi]\). Similarly,
\[
g'(x) = \sum_{n=1}^{\infty} c_n \cos(\mu_n x) + \sum_{n=1}^{\infty} d_n \sin(\tilde{\mu}_n x)
\]
\[
= -\sum_{n=1}^{\infty} c_n \mu_n \cos(\lambda_n x) + \sum_{n=1}^{\infty} d_n \mu_n \sin(\tilde{\lambda}_n x)
\]
\[
+ \sum_{n=1}^{\infty} \frac{d_n}{\mu_n^2 - \lambda^2} \cos(\mu_n x)
\]
also converges uniformly on \([0, \pi]\). Because of
\[
g''(x) = -\frac{c_n^2}{\lambda_n^2} g_n(x),
\]
\[
g''(x) = -\frac{d_n^2}{\tilde{\lambda}_n^2} \tilde{g}_n(x),
\]
we can deduce that
\[
\begin{align*}
\sum_{n=1}^{m} \frac{c_n}{\mu_n^2 - \lambda^2} g_n''(x) + \sum_{n=1}^{m} \frac{d_n}{\mu_n^2 - \lambda^2} \tilde{g}_n''(x) \\
= -\sum_{n=1}^{m} \frac{c_n \mu_n^2}{\mu_n^2 - \lambda^2} g_n(x) - \sum_{n=1}^{m} \frac{c_n \mu_n^2}{\mu_n^2 - \lambda^2} \tilde{g}_n(x) \\
= -\left( \sum_{n=1}^{m} c_n g_n(x) + \sum_{n=1}^{m} d_n \tilde{g}_n(x) \right) - \lambda^2 \left( \sum_{n=1}^{m} \frac{c_n}{\mu_n^2 - \lambda^2} g_n(x) + \sum_{n=1}^{m} \frac{d_n}{\mu_n^2 - \lambda^2} \tilde{g}_n(x) \right).
\end{align*}
\]
(15)

When \( m \to \infty \), the sequence on the right-hand side of (15) converges to \(-\lambda^2 g(x)\) in \( L^2[0, \pi] \). This shows that \( g(x) \) is twice differentiable and \( g''(x) = -\lambda^2 g(x) \) for all \( x \in [0, \pi] \). Due to
\[
g(0) = \sum_{n=1}^{\infty} \frac{c_n}{\mu_n^2 - \lambda^2},
\]
(16)
\[
g'(0) = \sum_{n=1}^{\infty} \frac{d_n}{\mu_n^2 - \lambda^2},
\]
we obtain that
\[
g(x) = u(\lambda) \cos(\lambda x) + v(\lambda) \sin(\lambda x),
\]
(17)
where \( u(\lambda) = g(0) \) and \( v(\lambda) = \lambda^{-1} g'(0) \). The functions \( u(\lambda) \) and \( v(\lambda) \) are meromorphic and not identically zero, respectively. Thus it has at most countably many zeros. If \( u(\lambda) v(\lambda) \neq 0 \), by (12) and (17), we have that \( \{\cos(\lambda x)\} \cup \{\sin(\lambda x)\} \) is in the closed linear span of \( \{\cos(\mu_n x)\} \cup \{\sin(\mu_n x)\} \). Owing to \( \{\cos(\nu_n x)\} \cup \{\sin(\nu_n x)\} \), which is continuous about \( (x, \lambda) \), we get that \( \{\cos(\lambda x)\} \cup \{\sin(\lambda x)\} \) is in the closed linear span of \( \{\cos(\mu_n x)\} \cup \{\sin(\nu_n x)\} \). This follows that \( \{\sin(n \nu)\} \), \( n \in \mathbb{N} \), is in the closed linear span of \( \{g_n(x)\} \cup \{\tilde{g}_n(x)\} \), so \( \{g_n(x)\} \cup \{\tilde{g}_n(x)\} \) is complete in \( L^2[0, \pi] \). Hence the \( (I - T) \) is dense in \( L^2[0, \pi] \). Using the fact that \( T \) is compact, we have that \( (I - T) = L^2[0, \pi] \) and \((-1)^k (I - T)\) is one to one; this contradicts the assumption.

Similarly, assume that \( \{g_n(x)\} \cup \{\tilde{g}_n(x)\} \) is a Riesz basis in \( L^2[0, \pi] \), then \( \{f_n(x)\} \cup \{\tilde{f}_n(x)\} \) is also a Riesz basis in \( L^2[0, \pi] \).

Let \( F(x) = \int_{0}^{x} f(t) dt \); integration by parts yields that
\[
\begin{align*}
\int_{0}^{\pi} f(x) \cos(2nx) dx \\
= F(x) \cos(2nx) \big|_{0}^{\pi} + 2n \int_{0}^{\pi} F(x) \sin(2nx) dx \\
= 0.
\end{align*}
\]
(19)
Thus
\[
\int_{0}^{\pi} F(x) \sin(2nx) dx = 0.
\]
(20)

Setting \( t = 2x - \pi \), we obtain
\[
\int_{0}^{\pi} F(x) \sin(2nx) dx = \frac{(-1)^n}{2} \int_{-\pi}^{\pi} F\left(\frac{t + \pi}{2}\right) \sin(nt) dt = 0,
\]
(21)
\[
\int_{0}^{\pi} f(x) \sin(2nx) dx = \frac{(-1)^n}{2} \int_{-\pi}^{\pi} F'\left(\frac{t + \pi}{2}\right) \sin(nt) dt = 0.
\]
(22)

Combining (18), (21), and (22), we obtain \( f(x) \equiv 0 \). Therefore, \( \{1\} \cup \{\cos(2nx)\} \cup \{\sin(2nx)\}, n \in \mathbb{N} \), is complete in \( L^2[0, \pi] \). The orthogonality of \( \{1\} \cup \{\cos(2nx)\} \cup \{\sin(2nx)\} \), \( n \in \mathbb{N} \), will be proved by establishing that \( \{\cos(2nx)\} \) and \( \{\sin(2nx)\} \) are orthogonal for all \( m, n \in \mathbb{N} \), using the fact that \( \{1\} \cup \{\cos(2nx)\} \) and \( \{1\} \cup \{\sin(2nx)\} \), \( n \in \mathbb{N} \), are the orthogonal sequences in \( L^2[0, \pi] \), respectively.

It follows from
\[
\langle \cos(2nx), \sin(2nx) \rangle = \int_{0}^{\pi} \cos(2nx) \sin(2nx) dx = 0,
\]
(23)
that \( \cos(2nx) \) and \( \sin(2nx) \) are orthogonal for all \( m, n \in \mathbb{N} \). Clearly, it is also a Riesz basis in \( L^2[0, \pi] \). This completes the proof of (i).

(ii) Suppose \( f(x) \in L^2[0, \pi] \), such that
\[
\begin{align*}
\int_{0}^{\pi} f(x) \sin((2n - 1)x) dx &= 0, \\
\int_{0}^{\pi} f(x) \cos((2n - 1)x) dx &= 0.
\end{align*}
\]
(24)

Let \( F(x) = \int_{0}^{x} f(t) dt \). By partial integration,
\[
\int_{0}^{\pi} f(x) \sin((2n - 1)x) dx \]
(25)
\[
\int_{0}^{\pi} F(x) \cos((2n - 1)x) dx = 0.
\]
Hence
\[
\int_{0}^{\pi} F(x) \cos((2n - 1)x) dx = 0.
\]
(26)
Setting \( t = 2x - \pi \), we obtain
\[
\int_0^\pi F(x) \cos((2n - 1)x) \, dx = \frac{(-1)^n}{2} \int_{-\pi}^\pi F\left(\frac{t + \pi}{2}\right) \sin\left((n - \frac{1}{2})t\right) \, dt = 0,
\]
and corresponding normalized eigenfunctions are
\[
y_n(x) = \sqrt{\frac{2}{\pi}} \sin(nx) + O\left(\frac{1}{n}\right)
\]
and
\[
y_n(x) = \sqrt{\frac{2}{\pi}} \sin(nx) + O\left(\frac{1}{n}\right)
\]

Theorem 7. Let \( u_n(x, q) = g_1(x, \lambda_n)g_2(x, \lambda_n) \), \( n \in \mathbb{N} \), where \( g_i(x, \lambda_n), i = 1, 2 \), are the solutions of (30) satisfying the initial conditions
\[
g_1(0, \lambda, q) = g'_1(0, \lambda, q) = 1;
g_2(0, \lambda, q) = g_2(0, \lambda, q) = 0.
\]
This clearly vanishes for \( m = n \). If \( m \neq n \), then \( \lambda_n \neq \lambda_m \), and we can use
\[
[y_m, y_n]' = (\lambda_m - \lambda_n) y_m y_n
\]

3. Riesz Bases Associated with the Eigenfunctions of Strum-Liouville Problems

We consider the Strum-Liouville problem
\[
-\gamma'' + g(x)y = \lambda^2 y, \quad x \in [0, \pi],
\]
where \( \lambda \in \mathbb{C} \) and \( g(x) \in L^2([0, \pi], \mathbb{R}) \).

It is well known that (see, for example, [19]) the eigenvalues of problem (30) are
\[
\lambda_n = n + O\left(\frac{1}{n}\right)
\]
Thus the sequence \( \{1\} \cup \{1 - \pi y \} \) is quadratically close with the Riesz basis \( \{1\} \cup \{\cos(2nx)\} \cup \{\sin(2nx)\} \), respectively. Based on this result, we find that a new sequence associated with eigenfunctions of Sturm-Liouville problem forms a Riesz basis in \( L^2[0, \pi] \).

\[ \|g_1(x, \lambda) \|_0^2 = 0. \]

\[ \|g_2(x, \lambda) \|_0^2 = 0. \]

The authors declare that there are no conflicts of interest.

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### 4. Conclusion

Riesz bases have been extensively applied in signal denoising, feature extraction, robust signal processing, and also the corresponding inverse problems. This paper gives that \( \{1\} \cup \{\cos(2nx)\} \cup \{\sin(2nx)\} \) and \( \{\cos((2n - 1)x)\} \cup \{\cos(2n - 1)x\} \) form a Riesz basis in \( L^2[0, \pi] \), respectively. Based on this result, we find that a new sequence associated with eigenfunctions of Sturm-Liouville problem forms a Riesz basis in \( L^2[0, \pi] \).

### Data Availability

No data were used to support this study.

### Conflicts of Interest

The authors declare that there are no conflicts of interest.

### Authors’ Contributions

All authors contributed equally to the writing of this paper. The authors read and approved the final manuscript.

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