Complex space-time and the classification of elementary particles

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Abstract

It is shown that the Dirac theory implies complex space-time and complex space-time can lead to the Dirac equation. It is suggested that fermions are grouped into doublets, those doublets are then divided into color singlets (leptons) and color triplets (quarks), then they are further divided into generation singlets and generation triplets. If the exclusion principle for fermions works in the internal space, then the non-observation of free quarks can be explained. However, it is suggested that a possible new free quark may exist with a color triplet and generation singlet state which is antisymmetric in the internal space.

Key words: Complex space-time, classification of elementary particles, explanation for non-existing free quarks, possible existence of new free quarks.
PACS: 03.65.P, 11.30.Ly, 14.65.-q, 14.80.-j
I. INTRODUCTION

In his great discovery of the Dirac equation [1,2], Dirac introduced $\gamma_\mu$ into the theory and describes naturally internal spin and antiparticles. However, there are many more internal variables such as isospin, hypercharge, color, ... etc. To have a theory that contains all of these variables, new degrees of freedom for the internal motion may be needed. The purpose of this article is to show that Dirac theory implies the existence of a complex space-time, that complex space-time can account for the fact that a free particle is a free wave as well, and also can reproduce the Dirac equation. It is suggested that all fermions can be classified into fermion doublets, color singlets and triplets, and generation singlets and triplets. Finally, if the principle of anti-symmetrization for fermion state, can be generalized to be valid in the internal space, then the non-observation of a free quark is a result of the present treatment. However, it is also suggested that there is a possibility that a new free quark may exist with an internally anti-symmetric state function.

II. DIRAC THEORY AND COMPLEX SPACE-TIME

Consider the Dirac Hamiltonian of a free electron

$$H = \vec{\alpha} \cdot \vec{P} + \beta m, \quad (c = \hbar = 1).$$

The $x_1$-component of the velocity and the $x_1$-component of coordinate are given as (see Chapter XI of ref.[2])

$$\dot{x}_1 = \alpha_1 = p_1 H^{-1} + \frac{1}{2} \tilde{a}_1^0 \exp(-2iHt)H^{-1} \equiv \dot{x}_1(\text{classical}) + \eta_1,$$

$$x_1 = p_1 H^{-1}t + a_1 - \frac{1}{4} \tilde{a}_1^0 \exp(-2iHt)H \equiv x_1(\text{classical}) + \eta_1,$$

with $x_1(\text{classical}) = a_1 + p_1 H^{-1}t, \dot{x}_1(\text{classical}) = p_1 H^{-1}$. We can consider the system of a particle with $(x_\mu, p_\mu; \mu = 1, \cdots 4)$ in the physical space coupled to its internal variables
(αᵢ, βᵢ; i = 1, 2, 3) in the internal space. Due to the coupling, the generalized normal coordinates become mixtures of both parts (the physical part and the internal part), and similarly for the velocity components. The motion of the ηᵢ is known as Zitterbewegung which has never been observed experimentally, so it cannot exist in our physical space, and can only stay in the internal space. In fact, ηᵢ is related to the αᵢ and βᵢ. Since the total angular momentum is conserved

\[ \frac{d}{dt}(\vec{P}(\text{classical}) \times \vec{P} + \frac{1}{2} \vec{σ}) = \frac{d}{dt}(\vec{η} \times \vec{P} + \frac{1}{2} \vec{σ}) = 0. \]  

\( \vec{η} \) is responsible for the conservation of total angular momentum (orbital plus spin). From equation (2), \( \dot{η}_1 \) is a part of \( α_1(\dot{η}_i = α_i \text{ when } \vec{P} \text{ is zero}) \). This is because \( \vec{α} \) is responsible for the Lorentz covariant 4-currents which are physically very important in describing interactions between quantum systems. Therefore \( \vec{η} \) is physically important. A good physical quantity must be Lorentz covariant, so there must exist an \( η_4 \) so as to make \( η_μ \) Lorentz covariant. Consider the case when \( P_i = 0 \), and \( η_i = (−i/2m)βα_i \), so \( η_μ = (−i/2m)γ^μ \). \( η_μ \) is a component of a dynamical quantity in the internal space implying that there are 4 axes, \( y_μ \) in that space. The \( y_μ \) is complex. In our physical space-time, \( x_i \) is real and \( x_4 \) is imaginary. There is no a priori reason why the \( y_i \) or \( y_4 \) should be real or imaginary when the universe was created. However, nature has preference for simplicity and unification. Therefore, the physical space-time and the internal space should be created together as a whole by taking all possible combinations of \( x_i \) and \( x_4 \) with equal probability to make a greater universe as

\[ GU = P_1(x^R_i, x^R_4) + P_2(x^R_i, x^I_4) + P_3(x^I_i, x^R_4) + P_4(x^I_i, x^I_4), \]

where R=real, I=imaginary. The \( P_2 \) part is the physical space-time and the other three parts are the internal space. A 4-vector in this GU is in general complex and we conclude that the Dirac theory implies a complex space-time. (In 1998, the author [3] suggested a complex space-time, however, the derivation was not so satisfactory).

To describe a free particle with complex space-time, there must exist a well-behaved physical function \( Φ \). Suppose we try a harmonic function that has continuous partial derivatives
of the second order and that satisfies the Laplace equation. For a stationary state, the spatial distribution must be independent of time, so $\Phi$ can be factorized and we have

$$
\Phi(X, T) = \Phi(X)\Phi(T), \quad X = x + i\bar{x}, \quad T = T + i\bar{t},
$$

(6)

$$
\frac{\partial^2 \Phi(X)}{\partial x^2} + \frac{\partial^2 \Phi(X)}{\partial \bar{x}^2} = 0, \quad \frac{\partial^2 \Phi(T)}{\partial t^2} + \frac{\partial^2 \Phi(T)}{\partial \bar{t}^2} = 0.
$$

(7)

Since $x$ and $\bar{x}$ are basically independent, we can write $\Phi(X) = \Phi(x)\Phi(\bar{x})$, and similarly for $\Phi(T)$. The solution can be written as

$$
\Phi = A \exp(\pm i\omega t \pm ik_x) \cdot \exp(\pm \omega \bar{t} \pm k\bar{x}).
$$

(8)

We can choose the direction of $k$ to be in the $x \pm i\bar{x}$ axes. Note that there is a non-physical solution where $\omega$ and $k$ are changed into $i\omega$ and $ik$ in equation (8). Setting $\bar{t} = \bar{t} = 0$, we get a projected solution with only real space-time variables. The result is a free wave with $\omega$ and $k$ as the frequency and wave number. That is, a free particle is a free wave as well.

Consider $\Phi(T)$ of equation (8) in the rest frame with the wave number $k = 0$. The magnitude of $\omega$ is uniquely determined by the rest mass of the particle and its value can assume to be $\pm \omega$ and $\pm i\omega$. If we require $\Phi$ to remain finite for all values of $t$ and $\bar{t}$, there are four independent wave functions in the complex $T$-plane given as

$$
\phi_1 = e^{-i\omega t - \omega \bar{t}}, \quad \phi_2 = e^{-\omega t - i\omega \bar{t}}, \quad \phi_3 = e^{i\omega t - \omega \bar{t}}, \quad \phi_4 = e^{-\omega t + i\omega \bar{t}},
$$

(9)

(set $t = \bar{t} = 0$ at the moment of the creation of the universe). If $t$ and $\bar{t}$ are exchanged, $\phi_1$ and $\phi_2$ would be exchanged and also for $\phi_3$ and $\phi_4$. Therefor, there is a real-imaginary symmetry in the system. Define the generalized energy operator as

$$
\hat{E} = \hat{E}_R + \hat{E}_1 = i \left( \frac{\partial}{\partial t} + \frac{\partial}{\partial \bar{t}} \right).
$$

(10)

It is easy to see that $\phi_1$ and $\phi_2$ have the same generalized energy. This can be viewed as the system having an internal degree of freedom with two allowed states with the same generalized energy in the complex $T$-plane. We therefore have an SU(2) internal symmetry. Consider an SU(2) transformation $U$
The generalized energy of \((\varphi_1, \varphi_2)\) does not change. That is, the generalized energy of \((\varphi_1, \varphi_2)\) is invariant under an SU(2) transformation, and \((\varphi_1, \varphi_2)\) forms a basis representation of an SU(2) group which has the three \(\sigma_i\) as generators. Similarly, it is also true for the negative frequency solutions \((\phi_3, \phi_4)\). By combining the four functions together we can write

\[
\begin{pmatrix}
\varphi_1 \\
\varphi_2
\end{pmatrix} = U \begin{pmatrix}
\phi_1 \\
\phi_2
\end{pmatrix} = \begin{pmatrix}
a_1\phi_1 + a_2\phi_2 \\
b_1\phi_1 + b_2\phi_2
\end{pmatrix},
\]

\(11\)

with \(\varphi_1 = a_1\phi_1 + a_2\phi_2, \varphi_2 = b_1\phi_1 + b_2\phi_2, \varphi_3 = a'_1\phi_3 + a'_2\phi_4, \varphi_4 = b'_1\phi_3 + b'_2\phi_4\). To get physical solutions in the rest frame with only real time \(t\), we project out the imaginary time by putting \(\mathbf{t} = 0\) in equations (9) and (12). For a stationary state, we must also project out the unphysical part by setting \(a_2 = b_2 = a'_2 = b'_2 = 0\), otherwise the state is a mixture of \(e^{-i\omega t}\) and \(e^{-\omega t}\). The four resulting functions form a complete set of basis functions of the time \(t\) with internal symmetry. Any stationary state function of the particle can be written as

\[
\psi(\mathbf{r}, t) = \psi(\mathbf{r}) \left[ \sum A_j \psi_j(\text{projected}) \right].
\]  

If \(\psi(\mathbf{r}, t)\) of equation (13) is a proper state function of the system, there must exist a conserved current which is Lorentz covariant and satisfies the continuity condition. Noticing that each \(\psi_i\) is a spinor function, it is natural to construct a spinor current \(j^\mu\) as

\[
j^\mu = \bar{\psi} \gamma^\mu \psi, \quad \bar{\psi} = \psi^\dagger \gamma^0,
\]

\(14\)

and require the continuity condition. We then get

\[
\bar{\psi} [i\gamma^\mu \partial_\mu - m] \psi + [i\gamma_\mu \bar{\psi} \gamma^\mu + m] \psi = 0.
\]  

\(15\)
Since $\psi$ and $\bar{\psi}$ are linearly independent, the argument in each of the square brackets must vanish separately, and we get back the Dirac equation and its adjoint equation with $m$ being the rest mass of the particle. That is, complex space-time can reproduce the Dirac equation.

III. CLASSIFICATION OF FERMION ELEMENTARY PARTICLES

If no interaction exists between $P_i$ and $P_j$ in equation (5), the Lagrangian of the system in GU is a sum of the $L_j$, $j = 1, \cdots, 4$, where $L_j$ is the Lagrangian of $P_j$. It is effectively a 4-body problem, and the total state function $\Psi$ is a product function $\Pi \Psi_j$, where $\Psi_j$ is a free particle Dirac solution for $P_j$ with a spin part and an $x_\mu$-part. The $x_\mu$-part of $\Psi_j$ must satisfy equation (7). The solution of equation (8) represents the $x_\mu$-parts of $\Psi_2 \Psi_3$. Since $P_2$ is the physical space-time, $\Psi_2$ must have good momentum, so there is only one momentum solution for $\Psi_2$, while $\Psi_3$ can take two momentum solutions like those in equation (8). By exchanging the $x_4^I$ in $P_2$ with $x_4^R$ in $P_3$, we get $P_1$ and $P_4$. Therefore, the pair $P_1P_4$ is just the same system as the pair $P_2P_3$ with the exchange of $x_4^I$ and $x_4^R$ and the solution is obtained by the same way. Thus there are $2 \otimes 2$ momentum states in $\Psi$. On the other hand, there are 4 spin functions in $\Psi$. Since the spin in the physical space-time is an eigen state in the rest frame and cannot mix with other internal states. Therefore, the total internal states for a Dirac particle can be obtained as $2 \otimes 2 \otimes 2 \otimes 2 = 2 \otimes (1 \oplus 3) \otimes (1 \oplus 3)$. That is, fermions are grouped into doublets, then all the doublets are divided into color singlets (leptons) and color triplets (quarks), then they are further divided into generation singlets and generation triplets. There are two possibilities:

(A) generation singlet \( (\nu^*_\tau^* \tau^*) \) N.F. \( (t^*b^*) \) N.F.

\[
\begin{align*}
\text{generation triplet} & \\
(\nu_\tau \tau) & (tb) \\
(\nu_\mu \mu) & (cs) \\
(\nu_e e) & (ud)
\end{align*}
\]
(B) generation triplet

\[
\begin{pmatrix}
(\nu^* \tau^*) & \text{N.F.} & (t^* b^*) & \text{N.F.} \\
(\nu_\tau) & (t b) \\
(\nu_{\mu \mu}) & (c s)
\end{pmatrix}
\]

generation singlet \((\nu_e e)(u d)\).

N. F.=not found yet. Take the fermion isospin \(T\) to be 1/2 and \(T_3\) as 1/2 for the left-hand side of the doublets and -1/2 for the other. Define the fermion hypercharge as \(SG/N\), where \(N\) is the multiplicity of the color multiplet (1 for leptons and 3 for quarks), and \(SG\) is -1 for leptons and 1 for quarks (opposite sign for antiparticles). Then the electric charge of the particle is given as \(Q = T_3 + Y/2\).

It may be worthwhile to mention that when the momentum of the particle is zero, states in GU are degenerate. However, since \(\pm k/|k| = \pm 1 = \lim_{k \to 0} (\pm k/|k|)\), so the degeneracy is removed.

**IV. THE NON-OBSERVATION OF FREE QUARKS AND A POSSIBLE NEW FREE QUARK**

In the GU of equation (5), the real space \(x_i\) is the overlap of \(P_1\) and \(P_2\). (Note that the momentum \(p_i\) in \(P_2\) is \(\partial L_2/\partial (dx_i/dt)\), while that in \(P_1\) is \(\partial L_1/\partial (dx_i/dt)\) which is unphysical). The states in \(P_2\) do not mix with other states. This can be regarded as saying that, in the real space \(x_i\), there is a physical particle (from \(P_2\)), and a non-physical particle (from \(P_1\)), which are not identical particles. The imaginary space \(\pi_i\) is the overlap of \(P_3\) and \(P_4\). There are two identical particles which are both non-physical and each has two momentum states and two spin states, and can couple to singlets and triplets. If the principle of anti-symmetrization for fermions can be generalized to be valid in the GU, then in the above list of (A), the leptons of color singlets and generation triplets are anti-symmetric states and can exist in free particle states, while the quarks of color triplets and generation triplets are symmetric states and cannot exist in free particle states (as expected by experiments). However, if there is no other condition such as “electric charge has to be quantized in units
of electron charge e”. then there is a possibility that the free quarks of color triplets and generation singlet in (A) can exist in nature. On the contrary, free leptons with a color singlet and a generation singlet can not be observed. Therefore, if the anti-symmetrization principle works in the greater universe, the above possibility (B) should be ruled out due to the existence of free electrons.

The author wishes to thank Professors B. Rosenstein and E. Yen for valuable discussions, and J. Nester for reading the manuscript.

V. REFERENCES

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