D-brane Description of New Open String Solutions in $AdS_5$

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ABSTRACT: In this paper we find D-brane descriptions of some of new open string solutions that were found in 0804.3438[hep-th]. These D5-brane and D3-brane configurations give gravitational dual descriptions of Wilson loops in some particular representations.

KEYWORDS: D-branes, AdS/CFT correspondence
1. Introduction and Summary

Wilson loop operators are non-local gauge invariant operators in gauge theory in which the theory can be formulated. Mathematically we define a Wilson loop as the trace in an arbitrary representation $R$ of the gauge group $G$ of the holonomy matrix associated with parallel transport along a closed curve $C$ in spacetime. Further, as it is well known from the times of birth of AdS/CFT correspondence [1] that Wilson loops in $N = 4$ SYM theory can be calculated in dual description using macroscopic strings [2, 3]. This prescription is based on a picture of the fundamental string ending on the boundary of AdS along the path $C$ specified by the Wilson loop operator. The description of the Wilson loop in terms of a fundamental string is a well established part of the AdS/CFT dictionary. Remarkably it was argued in very interesting paper [4] that Wilson loops have a gravitational dual description in terms of D5-branes or alternatively in terms of D3-branes in $AdS_5 \times S^5$. More precisely, Drukker and Fiol [18] argued that a Wilson loop with matter in the rank $k$ symmetric representation is better described as a D3-brane embedded in $AdS_5$ with $k$ units of electric flux. Further, it was argued in [13, 20, 21] that a Wilson loop with matter in the rank $k$ antisymmetric representation is better described by a D5-brane whose world-volume is a minimal surface in the AdS part of the geometry times an $S^4$ inside the $S^5$ and that supports $k$-units of world-volume electric flux.

In recent interesting paper [22] new open string solutions in $AdS_5$ were found. Further, it was shown that these string solutions end at the boundary on various Wilson lines. Since these solutions correspond to the solutions of the fundamental string equations of motion these configurations describe Wilson loops in fundamental representations. Then it is natural to ask the question whether we can find similar solutions for D3 and D5-brane that should correspond to the Wilson loops in antisymmetric and symmetric representations. We will argue that it is really possible to find such configurations.

We begin with the study of D5-brane that wraps $S^4$ in $S^5$ and that carries electric flux $\Pi$ along its embedding into $AdS_5$. Following [20] we consider D5-brane with Euclidean

\footnote{For related works, see [5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 20, 16, 17, 19].}
world-volume. Then we shown that the solutions of the equations of motion correspond to the solution found in the first part of the paper [22]. Further, when we evaluate the action on this solution we obtain regularized value of the action that corresponds to the Wilson line in level II anti-symmetric representation. This result is in nice agreement with exception.

As the second example we study Euclidean D3-brane that wraps \( S^2 \) subspace of \( S^5 \). Even if this D3-brane configuration does not correspond to the Wilson line in symmetric representation it is still instructive to find the solutions of equation of motion and evaluate corresponding on-shell action.

Finally we consider D3-brane that carries electric flux \( \Pi \) and that is embedded in \( AdS_5 \) completely. We study this action in Minkowski signature in order to include light-cone world-volume coordinates. We find D3-brane configuration that gives gravitation dual description of Wilson loop in II level symmetric representation. On the other hand we also find that the on-shell action vanishes corresponding to the expectation value of the corresponding Wilson line \( < W > = 1 \). This is in complete agreement with the result presented in [22].

Let us outline our paper. We explicitly found D3 and D5-brane configurations that are analogue of fundamental string solutions given in [22] and that correspond to dual Wilson loop in level II anti-symmetric and symmetric representations. We mean that this agreement further supports the claim that D3-branes and D5-branes configurations that end on the boundary of \( AdS_5 \) correspond to expectation values of some non-local operators in dual \( N = 4 \) SYM theory.

This paper can be extended in several ways. First of all it would be certainly interesting to find D3-brane configurations that correspond to the generalization of the circular Wilson line as in [22]. This is very interesting problem since description of circular Wilson loops in terms of D3-branes is rather non-trivial as was shown in [18]. Further it would be certainly interesting to find D3-brane and D5-brane analogue of the fundamental string solutions that were studied in second part of paper [22]. We hope to return to these problems in future.

The organization of this paper is as follows. In the next section 2 we introduce basic notation and study D5-brane and D3-brane configurations that wrap \( S^4 \) subspace and \( S^2 \) subspace of \( S^5 \) respectively. Then in section 3 we study D3-brane configuration where D3-brane is embedded in \( AdS_5 \) only. Finally for reader’s convenience we list in appendix 3 the form of equations of motion for Dp-brane in general background.

2. D5-brane on \( S^4 \) and D3-brane on \( S^2 \)

In this section we find D5-brane and D3-brane configurations that are related to fundamental string solutions studied in [22].

To begin with we write the metric of \( AdS_5 \) and \( S^5 \) in the form

\[
ds^2 = \frac{R^2}{z^2} (\eta_{\mu\nu} dx^\mu dx^\nu + dz^2) + R^2 (d\theta^2 + \sin^2 \theta d\Omega^4), \tag{2.1}
\]
where
\[ \eta_{\mu\nu}dx^\mu dx^\nu = 2dx^+ dx^- + (dx^1)^2 + (dx^2)^2, \]
and where we also introduced two light-cone variables \( x^\pm \) as
\[ x^\pm = \frac{1}{\sqrt{2}}(x^3 \pm x^0). \]

Let us now study dynamics of D5-brane in given background. Note that D5-brane action in Euclidean signature takes the form
\[ S = \tau_5 \int d^6\xi \sqrt{\det A} - i\tau_5 \int (2\pi\alpha') F \wedge C^{(4)}, \]
where \( N \) are units of flux of the Ramond-Ramond five-form, and where we introduced Ramond-Ramond four-form \( C^{(4)} \)
\[ C^{(4)} = R^4 \frac{3}{2}(\theta - \pi) - \sin^3 \theta \cos \theta - \frac{3}{2} \cos \theta \sin \theta) \text{Vol}S^4 \equiv R^4 C(\theta) \text{Vol}S^4. \]

Now we propose following ansatz
\[ x^+ = u(\tau, \sigma), \quad x^1 = \xi^0 \equiv \tau, \quad z = \xi^1 \equiv \sigma, \]
\[ x^2 = a, \quad \theta = \theta_{\Pi}, \quad \phi^a = \xi^a, \quad a = 1, 2, 3, 4, \]
\[ \text{where } \phi^a \text{ label coordinates on } S^4 \text{ and where } \theta_{\Pi} \text{ is a constant that depends on the electric flux in } \tau, \sigma \text{ direction. It turns out that it is necessary to consider complex gauge field so that } (2\pi\alpha')F \rightarrow i(2\pi\alpha')F. \]

Then the matrix \( A \) takes the form
\[ A_{\tau\tau} = \frac{R^2}{\sigma^2}, \quad A_{\sigma\sigma} = \frac{R^2}{\sigma^2}, \]
\[ A_{\tau\sigma} = i2\pi\alpha' F_{\tau\sigma}, \quad A_{\sigma\tau} = i2\pi\alpha' F_{\sigma\tau}, \]
\[ \det A_{ij} = R^8 \sin^8 \theta \det \tilde{g}_{ij}, \quad \det A = R^{12} \left( \frac{1}{\sigma^4} - \frac{(2\pi\alpha')^2}{R^4} F^2 \right) \sin^8 \theta \det \tilde{g}_{ij}, \quad F \equiv F_{\tau\sigma}, \]
\[ \text{where } \tilde{g}_{ij} \text{ is the metric on } S^4 \text{ sphere. Then the equation for } x^- \text{ implies} \]
\[ \partial_\tau \left( \frac{\partial_\tau u}{\sigma^2} \sqrt{\frac{R^4 - \sigma^4 (2\pi\alpha')^2 F^2}{R^4}} \right) + \partial_\sigma \left( \frac{\partial_\sigma u}{\sigma^2} \sqrt{\frac{1}{R^4 - \sigma^4 (2\pi\alpha')^2 F^2}} \right) = 0. \]
Further, the equation of motion for $A_\alpha$ takes the form
\[ \partial_\beta[(A^{-1})^{\alpha\beta}_A \sqrt{\text{det} A}] + \partial_\beta[\epsilon^{3\alpha} C(\theta)] = 0 . \]  
(2.9)

This equation implies an existence of the electric flux $\Pi$
\[ \frac{(2\pi \alpha')\sin^4 \theta}{\sqrt{R^2 - (2\pi \alpha' F)^2}} + C(\theta) = \Pi \]
and consequently
\[ \frac{(2\pi \alpha')F}{\sqrt{R^2 - (2\pi \alpha' F^2)}} = \frac{(\Pi - C(\theta))R^2}{\sigma^2 \sqrt{\sin^8 \theta + (\Pi - C(\theta))^2}} . \]
(2.10)

Using these results we obtain that the equation (2.8) takes the form
\[ \partial^2 u - 2\frac{1}{u} \partial u + \partial^2 u = 0 . \]  
(2.12)

This has the same form as the equation derived in the first part of the paper [22]. Further it is easy to see that the equation of motion for $x^+$ is automatically satisfied. Finally, using (2.11) we obtain that the equation of motion for $\theta$ takes the form
\[ 4 \sin^7 \theta \cos \theta - 4 \sin^4 \theta(\Pi - C(\theta)) = 0 \]
and consequently we find that $\theta$ depends on the value of electric flux $\Pi$ as
\[ \Pi = \frac{3}{4} \sin 2\theta_{\Pi} - \frac{3}{2}(\theta_{\Pi} - \pi) \]
(2.14)
or alternatively
\[ (2\pi \alpha' F) = - \cos \theta_{\Pi} \frac{R^2}{\sigma^2} . \]
(2.15)

Then the action evaluated on this solution takes the form
\[ S = \tau_5 R^6 \int d\text{Vol}S(4) d\sigma d\tau \frac{1}{\sigma^2} [\sin^5 \theta - \cos \theta C(\theta)] = \]
\[ = \frac{N \sqrt{\lambda}}{3\pi^2} \int d\tau d\sigma \frac{1}{\sigma^2} [\sin^5 \theta - \cos \theta C(\theta)] . \]
(2.16)

\[ ^2\text{Using the fact that the volume of unit-$S^4$ sphere is equal to} \]
\[ \text{Vol}S^4 = \frac{8}{3\pi^2} . \]
The D5-brane solution extends to the boundary of $AdS_5$ and ends there along a one-dimensional curve. This opens up the possibility of adding boundary terms to the action. These boundary terms do not change the equations of motion, so the solution is still the same, but the value of the action when evaluated at this solution will in general depend on the boundary terms. Careful discussion of these boundary terms was given in [18]. However for our purposes it is only sufficient to demand that the resulting action together with boundary term is gauge invariant. In fact, as was shown in [18] in order to achieve gauge invariance we have to add to the action the term

$$S_{\text{gauge}} = -\tau_5 \int d\tau d\sigma \sqrt{\text{Vol}} S^{(4)} \Pi(2\pi\alpha') F = -\frac{\sqrt{\lambda} N}{3\pi^2 R^2} \int d\tau d\sigma \Pi(2\pi\alpha') F =$$

$$= \frac{\sqrt{\lambda} N}{3\pi^2} \int d\tau d\sigma \frac{1}{\sigma^2} \Pi \cos \theta_\Pi = \frac{\sqrt{\lambda} N}{3\pi^2} \int d\tau d\sigma \frac{1}{\sigma^2} (\sin^3 \theta_\Pi \cos^2 \theta_\Pi + C(\theta_\Pi) \cos \theta_\Pi)$$

(2.17)

and hence

$$S_{\text{bulk}} + S_{\text{gauge}} = \frac{N \sqrt{\lambda}}{3\pi^2} \sin^3 \theta_\Pi \int d\tau d\sigma \frac{1}{\sigma^2} = \frac{2N}{3\pi} \sin^3 \theta_\Pi S_{\text{funstring}},$$

(2.18)

where $S_{\text{funstring}}$ is an on-shell action that has the same form as in [22]. In other words we have again find that the Wilson loop in the $\Pi$-th antisymmetric representation is given in terms of the loop in the fundamental representation as follows from (2.18). Further, $\theta$ is related to the electric flux through (2.14).

Let us now turn to the study of D3-brane solutions that are related to the solution found in [22] and that are characterized by the property that the embedding of D3-brane corresponds to the embedding of string world-sheet in $AdS_5$ and to the embedding $S^2 \hookrightarrow S^5$. We use the convention given in [21] and write the relevant metric in the form

$$ds^2 = R^2 \frac{\sigma^2}{\sigma^2}[2dx^+ dx^- + dx^i dx^i + d\sigma^2 + \sin^2 \alpha d\Omega^{[2]} + \cos^2 \alpha d\Omega^{[2]}].$$

(2.19)

The Euclidean DBI action for D3-brane takes the form

$$S = \tau_3 \int d^4 \xi \sqrt{\text{det} A},$$

(2.20)

where $\tau_3 = \frac{N}{2\pi^2 R^3}$. Since we are interested in solution where D3-brane has two directions in $AdS_5$ and wraps $S^2 \hookrightarrow S^5$ we do not need to consider WZ term. More precisely, let us consider generalization of the solution given in [22] and propose following ansatz

$$x_+ = u(\tau, \sigma), \quad x^1 = \xi^0 \equiv \tau, \quad z = \xi^1 \equiv \sigma, \quad x^2 = a, \quad \phi^a = \xi^a, \quad a = 2, 3,$$

(2.21)

where $\phi^a$ label coordinates on the first subspace $S^2$ of $S^4$ given in (2.19). Then, as in D5-brane case we consider imaginary $F_{\tau\sigma}$ so that the matrix $A$ takes the form

$$A_{\tau\tau} = \frac{R^2}{\sigma^2}, \quad A_{\sigma\sigma} = \frac{R^2}{\sigma^2},$$
\[
\begin{align*}
A_{\tau\sigma} &= i2\pi\alpha'F_{\tau\sigma}, \quad A_{\sigma\tau} = i2\pi\alpha'F_{\sigma\tau}, \\
\det A_{ab} &= R^4\sin^4\alpha\det \tilde{g}_{ab}, \\
\det A &= -R^8\left(\frac{1}{\sigma^4} - \frac{(2\pi\alpha')^2F^2}{R^4}\right)\sin^4\alpha\det \tilde{g}_{ab}, \\
F &\equiv F_{\tau\sigma},
\end{align*}
\] (2.22)

where \(\tilde{g}_{ab}\) is the metric on \(S^2\) sphere. Let us now study the equations of motion for given ansatz. Firstly, the equation of motion for \(A_\alpha\) implies

\[
\frac{(2\pi\alpha')F\sin^2\alpha}{\sqrt{\frac{R^4}{\sigma^4} - (2\pi\alpha')^2F^2}} = \Pi
\] (2.23)

and consequently

\[
2\pi\alpha'F = -\frac{R^2\Pi}{\sigma^2\sqrt{\sin^4\alpha + \Pi^2}},
\]

\[
\sqrt{\frac{R^4}{\sigma^4} - (2\pi\alpha')^2F^2} = \frac{R^2\sin^2\alpha}{\sigma^2\sqrt{\sin^4\alpha + \Pi^2}}.
\] (2.24)

Then in the similar way as in D5-brane case it is possible to show that the equation of motion for \(x^-\) implies

\[
\partial^2_\tau u - \frac{2}{\sigma}\partial_\sigma u + \partial^2_\sigma u = 0
\] (2.25)

that has again the same form as equation given in [22]. Further, using the linearity of this equation it is possible for given profile of Wilson loop on the boundary specified by curve \(u(0, \tau)\) find corresponding function \(u(\sigma, \tau)\).

As the next step we consider remaining equations of motion. In fact the equation of motion for \(\alpha\) has two solutions, one where \(\alpha_0 = 0\) that corresponds to D3-brane collapsed to a point and the second one \(\alpha_0 = \frac{\pi}{2}\) corresponding D3-brane wrapped \(S^2\). In fact this is solution we are interested in. Finally it is easy to see that remaining equations of motion are satisfied.

Finally we evaluate the action on given solution and we obtain

\[
S_{\text{bulk}} = \frac{2N}{\pi} \int d\tau d\sigma \frac{\sin^4\alpha_0}{\sigma^2\sqrt{\sin^4\alpha_0 + \Pi^2}}.
\] (2.26)

Again as in case of D5-brane we have to include boundary terms to achieve gauge invariance. To do this we include to the action the boundary expression

\[
S_{\text{gauge}} = -\tau_3 \int d\tau d\sigma \text{Vol}S^2\Pi(2\pi\alpha')F = \frac{2N}{\pi} \int d\tau d\sigma \frac{\Pi^2}{\sigma^2\sqrt{\sin^4\alpha_0 + \Pi^2}}.
\] (2.27)
Then the on-shell action is equal to

\[ S = S_{\text{bulk}} + S_{\text{gauge}} = \frac{2N}{\pi} \int d\tau d\sigma \frac{1}{\sigma^2} \sqrt{\sin^4 \alpha_0 + \Pi^2} = \]

\[ = \frac{2N}{\pi} \sqrt{\sin^4 \alpha_0 + \Pi^2} \int d\tau d\sigma \frac{1}{\sigma^2} = \frac{4N}{\sqrt{\lambda}} \sqrt{1 + \Pi^2 S_{\text{funstring}}} . \]

(2.28)

We again see that the resulting action is proportional to the fundamental string action. On the other hand an interpretation of this D3-brane configuration in dual SYM is not complete clear as was argued in [20]. Then we should rather consider this solution as an interesting example of non-trivial D3-brane configuration that ends on the boundary of AdS_5.

3. D3-brane on AdS_5

In this section we consider D3-brane that is embedded in AdS_5 only. This D3-brane with electric flux Π should correspond to the Wilson line in rank Π symmetric representation. As opposite to the examples studied in previous section we start with the D3-brane action where the world-volume has Minkowski signature so that the action takes the form

\[ S_{D3} = -\tau_3 \int d^4 \xi \sqrt{-\text{det} A} + \frac{\tau_3}{4!} \int C_{MNPQ} dX^M \wedge dX^N \wedge dX^P \wedge dX^Q , \]

where

\[ C^{(4)} = \frac{R^2}{z^4} dx^+ \wedge dx^- \wedge dx^1 \wedge dx^2 . \]

(3.1)

To begin with we introduce light-cone variables \( z^\pm = \frac{1}{\sqrt{2}} (\xi^1 \pm \xi^0) \) and consider following ansatz

\[ z = z(\xi^3) \equiv z(\sigma) , \quad X^1 = \xi^2 \equiv \tau , \quad X^+ = \xi^+ , \quad X^- = \xi^- , \quad X^3 = \xi^3 \]

(3.3)

with non-trivial gauge field strengths \( F_{\tau \sigma} = -F_{\sigma \tau} = F , F_{+\sigma} = -F_{\sigma +} = \mathcal{F} \). For this ansatz the matrix \( A_{\alpha \beta} \) takes the form

\[ A_{++} = 0 , \quad A_{--} = 0 , \]

\[ A_{+-} = \frac{R^2}{z^2} , \quad A_{-+} = \frac{R^2}{z^2} , \]

\[ A_{23} = -A_{32} = (2\pi \alpha') F , \]

\[ A_{+3} = -A_{3+} = (2\pi \alpha') \mathcal{F} , \]

\[ A_{22} = \frac{R^2}{z^2} , \quad A_{33} = \frac{R^2}{z^2} (1 + z'^2) , \]

(3.4)

where \( z' = \partial_\sigma z \). Then we easily obtain

\[ \text{det} A = -\frac{R^8}{z^8} (1 + z'^2) - \frac{R^4}{z^4} (2\pi \alpha')^2 F^2 . \]

(3.5)
In order to solve the equations of motion we have to find components of the inverse matrix \( (A^{-1}) \). Using (3.4) we obtain

\[
(A^{-1})^{++} = 0, \quad (A^{-1})^{--} = \frac{R^2(2\pi\alpha'F)^2}{z^2(\frac{R^2}{z^4}(1 + z'^2) + \frac{R^4}{z^6}(2\pi\alpha')^2 F^2)},
\]
\[
(A^{-1})^{+-} = (A^{-1})^{-+} = \frac{R^6(1 + z'^2)}{z^6[\frac{R^2}{z^4}(1 + z'^2) + \frac{R^4}{z^6}(2\pi\alpha')^2 F^2]},
\]
\[
(A^{-1})^{+-} = (A^{-1})^{+2} = 0 = (A^{-1})^{-3} = (A^{-1})^{3+} = 0,
\]
\[
(A^{-1})^{-2} = (A^{-1})^{2-} = -\frac{R^2(2\pi\alpha')^2 F F}{z^2[\frac{R^2}{z^4}(1 + z'^2) + \frac{R^4}{z^6}(2\pi\alpha')^2 F^2]},
\]
\[
(A^{-1})^{3-} = (A^{-1})^{-3} = -\frac{R^4(2\pi\alpha') F}{z^4[\frac{R^2}{z^4}(1 + z'^2) + \frac{R^4}{z^6}(2\pi\alpha')^2 F^2]},
\]
\[
(A^{-1})^{22} = -\frac{R^6(1 + z'^2)}{z^6[\frac{R^2}{z^4}(1 + z'^2) + \frac{R^4}{z^6}(2\pi\alpha')^2 F^2]},
\]
\[
(A^{-1})^{33} = \frac{R^6}{z^6[\frac{R^2}{z^4}(1 + z'^2) + \frac{R^4}{z^6}(2\pi\alpha')^2 F^2]},
\]
\[
(A^{-1})^{23} = -(A^{-1})^{32} = -\frac{R^4(2\pi\alpha') F}{z^4[\frac{R^2}{z^4}(1 + z'^2) + \frac{R^4}{z^6}(2\pi\alpha')^2 F^2]}.
\]

(3.6)

Then using (3.6) it is easy to see that the equations of motion for \( A_+, A_- \) are obeyed automatically. On the other hand the equation of motion for \( A_2, A_3 \) implies an existence of conserved electric flux \( \Pi \)

\[
\frac{(2\pi\alpha') F}{\sqrt{(1 + z'^2) + \frac{z^4}{R^4}(2\pi\alpha')^2 F^2}} = \Pi
\]

(3.7)

and consequently

\[
\sqrt{1 + z'^2 + \frac{z^4}{R^4}(2\pi\alpha')^2 F^2} = \sqrt{\frac{1 + z'^2}{1 - \frac{z^4}{R^4}\Pi^2}}.
\]

(3.8)

Using these relations we obtain that the equation of motion for \( X^+ \) takes the form

\[
\partial_3 \left[ \frac{R^2(2\pi\alpha') F}{z^2 \sqrt{(1 + z'^2) + \frac{z^4}{R^4}(2\pi\alpha')^2 F^2}} \right] + \partial_2 \left[ \frac{(2\pi\alpha')^2 F F}{\sqrt{(1 + z'^2) + \frac{z^4}{R^4}(2\pi\alpha')^2 F^2}} \right] =
\]
\[
= \partial_\sigma \left[ \frac{R^2 \sqrt{1 - \frac{z^4}{R^4 F^2} F^2}}{z^2 \sqrt{1 + z'^2}} \right] + \Pi \partial_7 F = 0.
\]

(3.9)
We return to implications of this equation below. As the next step we consider the equation of motion for $X^3$ and we obtain

$$\partial_3 [g_{33} \left( A^{-1} \right)_{33} \sqrt{-\det A}] - \partial_3 [C_{+23}] =$$

$$= \partial_\sigma \left[ R^4 \left( \frac{\sqrt{1 - \frac{z^4}{R^4} \Pi^2}}{\sqrt{1 + z^2}} - 1 \right) \right] = 0$$

(3.10)

and consequently

$$\frac{\sqrt{1 - \frac{z^4}{R^4} \Pi^2}}{\sqrt{1 + z^2}} = 1 + \frac{z^4}{R^4} B ,$$

(3.11)

where $B$ is an integration constant. In what follows we consider the case when $B = 0$ and introduce complex electric flux as $\Pi = i \tilde{\Pi}$. Then the equation (3.11) takes simple form

$$z' = -\frac{z^2}{R^2} \tilde{\Pi}$$

(3.12)

that can be easily integrated with the result

$$\frac{1}{z} = \frac{\tilde{\Pi}}{R^2} \sigma$$

(3.13)

with appropriate chosen integration constant.

Now we return to the equation (3.9). Using (3.12) and (3.13) this equation simplifies considerably

$$\frac{\tilde{\Pi}^2}{R^2} \partial_\sigma (\sigma^2 \mathcal{F}) + i \tilde{\Pi} \partial_\tau \mathcal{F} = 0$$

(3.14)

Since $\mathcal{F}_{+\sigma} = \mathcal{F} = -\partial_\sigma u_+$ the equation above implies

$$\frac{\tilde{\Pi}}{R^2} \sigma^2 \partial_\sigma u_+ + i \partial_\tau u_+ = 0$$

(3.15)

and we solve this equation with the ansatz

$$u_+(\sigma, \tau) = \int_0^\infty d\omega [e^{i\omega \tau} u_+ (\sigma, \omega) + e^{-i\omega \tau} u_+^*(\sigma, \omega)]$$

(3.16)

so that $u_+^* = u_+$. If we insert (3.16) into (3.15) we obtain

$$\frac{\tilde{\Pi}}{R^2} \sigma^2 u_+ - \omega u_+ = 0$$

(3.17)

with the solution

$$u_+ = C_\omega e^{-\omega \frac{\sigma^2}{\tilde{\Pi}}} = C_\omega e^{-\omega z} .$$

(3.18)

Then the general solution (3.16) takes the form

$$u_+(z, \tau) = \int_0^\infty d\omega e^{-\omega z} [e^{i\omega \tau} C_\omega + e^{-i\omega \tau} C_\omega^*] ,$$

(3.19)
where we can determine $C_\omega$ using the known profile $u_+(z = 0, \tau)$ of Wilson line on the boundary $z = 0$. As final remark note that can be easily shown that the equations of motion for $X^2$ and $Z$ are obeyed for the ansatz (3.3).

Let us finally evaluate the action on solution found above. If we calculate the action for this solution we find that the WZ term exactly cancels the DBI part and consequently $S = 0$. Of course this is expected final answer but we should take the boundary terms into account as well. Careful analysis of this problem was presented in [18] with the result that the boundary terms for this particular D3-brane configurations do not contribute to the action.

Let us outline our results. We found solutions of D3-brane equations of motion that is an analogue of the fundamental string solution presented in [22] and that should be related to the expectation values of Wilson loops evaluated in II-symmetric representation in dual $N = 4$ SYM theory. We again mean that this fact nicely shows an efficiently of the D3-brane description of configurations with large number of fundamental strings.

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A. Dp-brane in general background

In this appendix we review the form of Dp-brane action in general background. We also review corresponding equations of motion.

As is well known the Dp-brane action in general background consists two parts, Dirac-Born-Infeld part (DBI) and Wess-Zumino part (WZ) so that

$$S = S_{DBI} + S_{WZ},$$

$$S_{DBI} = -\tau_p \int d^{p+1} \xi e^{-\Phi} \sqrt{-\det A},$$

$$A_{\alpha \beta} = \partial_\alpha X^M \partial_\beta X^N G_{MN} + (2\pi \alpha') F_{\alpha \beta},$$

$$F_{\alpha \beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha - (2\pi \alpha')^{-1} B_{MN} \partial_\alpha X^M \partial_\beta X^N,$$

$$S_{WZ} = \tau_p \int e^{(2\pi \alpha') F} \wedge C = \tau_p \sum_{n \leq 0} \frac{(2\pi \alpha')^n}{n!(2!)^n q!} \int d^{p+1} \xi e^{i_{1...i_{p+1}}(F^n)_{1...2n}} C_{i_{2n+1}...i_{p+1}},$$

(A.1)

where $\tau_p$ is Dp-brane tension, $\xi^\alpha, \alpha = 0, \ldots, p$ are world-volume coordinates and where $A_\alpha$ is gauge field living on the world-volume of Dp-brane. Note also that $C$ in the last line in (A.1) means collection of Ramond-Ramond forms where $q = p + 1 - 2n$.

If we now perform the variation of (A.1) with respect to $X^M$ we obtain following equations of motion for $X^M$

$$-\tau_p \partial_M [e^{-\Phi} \sqrt{-\det A}$$

$$- \frac{\tau_p}{2} e^{-\Phi} (\partial_M g_{KL} \partial_\alpha X^K \partial_\beta X^L - \partial_K \partial_\alpha X^K \partial_L \partial_\beta X^L) (A^{-1})^{\beta \alpha} \sqrt{-\det A} +$$

$$(A.2)
\[ + \tau_p \partial_{\alpha} \left[ e^{-\Phi} g_{MN} \partial_{\beta} X^N \left( A^{-1} \right)^{\beta \alpha}_S \sqrt{-\det A} \right] - \tau_p \partial_{\alpha} \left[ e^{-\Phi} b_{MN} \partial_{\beta} X^N \left( A^{-1} \right)^{\beta \alpha}_A \sqrt{-\det A} \right] + J_M = 0 , \] 

(A.2)

where

\[ J_M = \frac{\delta S_{WZ}}{\delta X^M} = \tau_p \sum_{n \leq 0} \frac{(2\pi \alpha')^n}{n!(2)!^n q!} e^{i \varepsilon_{1 \ldots i_{p+1}}^{i_1 \ldots i_{2n+1}}} \partial_{i_{2n+2}} X^{i_{2n+1} M_{2n+1}} \ldots \partial_{i_{p+1}} X^{i_{p+1} M_{p+1}} F_M M_{2n+1} \ldots M_{p+1} . \] 

(A.3)

In the same way the variation of (A.1) with respect to \( A_{\alpha} \) implies following equation of motion

\[ 2\pi \alpha' \tau_p \partial_{\alpha} \left[ e^{-\Phi} \left( A^{-1} \right)^{\beta \alpha}_A \sqrt{-\det A} \right] + J_\alpha = 0 , \] 

(A.4)

where

\[ J_\alpha = \frac{\delta S_{WZ}}{\delta A_\alpha} = \epsilon^{1 \ldots i_{p+1}} \sum_{n \geq 0} \frac{(2\pi \alpha')^n}{n!(2)!^n (q-1)!} (F)^n_{i_1 \ldots i_{2n+1}} \partial_{i_{2n+2}} X^{i_{2n+1} M_{2n+2}} \ldots \partial_{i_{p+1}} X^{i_{p+1} M_{p+1}} F_M M_{2n+2} \ldots M_{p+1} . \] 

(A.5)

and where we have also defined

\[ (A^{-1})^{\alpha \beta}_S = \frac{1}{2} \left( (A^{-1})^{\alpha \beta} + (A^{-1})^{\beta \alpha} \right) , \quad (A^{-1})^{\alpha \beta}_A = \frac{1}{2} \left( (A^{-1})^{\alpha \beta} - (A^{-1})^{\beta \alpha} \right) . \] 

(A.6)

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