BOGOMOLOV MULTIPLIERS OF $p$-GROUPS OF MAXIMAL CLASS

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Abstract. Let $G$ be a $p$-group of maximal class and order $p^n$, where $n \geq 4$. We show that the Bogomolov multiplier $B_0(G)$ is trivial if and only if $[P_1, P_1] = [P_1, \gamma_{n-2}(G)]$, where $P_1 = C_G(\gamma_2(G)/\gamma_4(G))$. On the other hand, if $G$ has positive degree of commutativity and $P_1$ is metabelian, we prove that $B_0(G)$ coincides with the coinvariants of the $G$-module $P_1 \rtimes P_1$. This result covers all $p$-groups of maximal class of large enough order and, furthermore, it allows us to give the first natural examples of $p$-groups with Bogomolov multiplier of arbitrarily high exponent.

1. Introduction

The Bogomolov multiplier is a group-theoretical invariant, originally introduced as an obstruction to the famous rationality problem in algebraic geometry. This problem asks whether a given field extension $E/k$ is rational (purely transcendental). Of particular interest is the situation when a finite group $G$ acts on the function field $L$ of an affine space by permuting indeterminates. The subfield $L^G$ of fixed points of this action represents the function field of the quotient variety. Noether [16] asked whether the extension $L^G/k$ is always purely transcendental. This version of the rationality problem is known as Noether’s problem. Saltman [17] found explicit examples of groups for which the answer to Noether’s problem is in general negative, even when taking $k = \mathbb{C}$. His approach was to use a certain Galois-cohomological invariant associated to the group $G$, namely, the unramified Brauer group. It was Bogomolov [2] who found a purely group-cohomological way of computing this invariant, now known as the Bogomolov multiplier and denoted by $B_0(G)$. This object represents an obstruction to Noether’s problem, which has a negative answer for $G$ provided that $B_0(G)$ is nontrivial.

Recently, Moravec [14] introduced the curly exterior square $G \triangleright G$, which is the group generated by the symbols $x \triangleright y$ for all pairs $x, y \in G$, subject to the relations $\ xy \triangleright z = (x^y \triangleright z^y)(y \triangleright z)$, $\ x \triangleright yz = (x \triangleright z)(x^y \triangleright y^z)$, $\ a \triangleright b = 1$, for all $x, y, z \in G$ and all $a, b \in G$ with $[a, b] = 1$. By [14], there is a canonical epimorphism $G \triangleright G \to [G, G]$, whose kernel is isomorphic to $B_0(G)$. In this sense, the Bogomolov multiplier can be interpreted as a measure of how the commutator relations in a group fail to follow from the so-called universal ones, see [8]. We will

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work exclusively with this combinatorial interpretation of the Bogomolov multiplier in this paper.

Explicitly determining the Bogomolov multiplier of a group $G$ can be quite involved. By general homological arguments, it boils down to studying Bogomolov multipliers of the Sylow subgroups of $G$. For a prime, the smallest examples of $p$-groups with nontrivial Bogomolov multiplier are of order 64 if $p = 2$ [3, 4], and of order $p^5$ if $p$ is odd [2, 15]. These groups are all of maximum nilpotency class given their order.

Motivated by these examples, in this paper we set to inspect Bogomolov multipliers of $p$-groups $G$ of maximal class, that is, of order $p^n$ and nilpotency class $n-1$. We may assume that $n \geq 4$, and we will tacitly do so in the remainder. Then we can consider the chief series $G > P_1 > \cdots > P_n = 1$, with $P_i = \gamma_i(G)$ for $i \geq 2$, and $P_1 = C_G(P_2/P_4)$. We will see that among groups of maximal class, most of them have nontrivial Bogomolov multipliers. More precisely, we prove the following result.

**Theorem 1.1.** Let $G$ be a $p$-group of maximal class and order $p^n$. Then $B_0(G)$ is trivial if and only if $[P_1, P_1] = [P_1, P_{n-2}]$.

By the theory of groups of maximal class, this implies that for $n \geq p + 2$, $G$ has trivial Bogomolov multiplier if and only if the maximal subgroup $P_1$ is abelian.

We also consider the family of all $p$-groups of maximal class with $P_1$ metabelian and positive degree of commutativity (see the next section for the definition). As we will see, both properties are satisfied if $n$ is large enough with respect to $p$. In this case, we show that the structure of $B_0(G)$ can be read from the commutator structure of $P_1$.

**Theorem 1.2.** Let $G$ be a $p$-group of maximal class with positive degree of commutativity and $P_1'$ abelian. Then $B_0(G)$ is isomorphic to the coinvariants $(P_1 \wr P_1)^G$.

Here, $P_1 \wr P_1$ is seen as a $G$-module under the action induced by the rule $(x \wr y)g = x^g \wr y^g$ for all $x, y \in P_1$ and $g \in G$.

Following the proof of Theorem 1.2, we can explicitly determine the Bogomolov multipliers of some specific examples of $p$-groups of maximal class. In particular, we get the following result.

**Theorem 1.3.** Let $p \geq 5$ be a prime, and let $m$ and $n$ be integers such that $m \geq 4$ and $m \leq n \leq 2m - 2$. If we write $n - m + 1 = x(p - 1) + y$ with $x \geq 0$ and $0 \leq y < p - 1$, then there exists a $p$-group of maximal class of order $p^n$ with $P_1' = P_m$ and Bogomolov multiplier isomorphic to $C_{p^{x+1}} \times C_{p^{(p-1-y)/2}}$.

As a consequence of the previous theorem, we obtain the first natural family of $p$-groups in the literature whose Bogomolov multipliers have unbounded exponent.

2. **General results**

In this section, we prove the two general results given in the introduction about Bogomolov multipliers of $p$-groups of maximal class, namely Theorems 1.2 and 1.1.
We start by proving Theorem 1.2, from which we can deduce one of the implications of Theorem 1.1, and then we complete the proof of this theorem.

Before proceeding, we recall some results from the theory of p-groups of maximal class, started by Blackburn in [1] and further developed by Leedham-Green and McKay [10, 11, 12], by Shepherd [18], and by the first author [5]. General accounts of this theory can be found in [6], [7] or [13], and we refer the reader to them for the following facts about p-groups of maximal class. We set \( P_i = \gamma_i(G) \) for all \( i \geq 1 \) and \( P_1 = C_G(P_2/P_4) \). Observe that \( P_i = 1 \) for \( i \geq n \), and that \( P_2, \ldots, P_n \) are the only normal subgroups of \( G \) of index greater than \( p \). If we pick arbitrary elements \( s \in G \setminus P_1 \cup C_G(P_{n-2}) \) and \( s_1 \in P_1 \setminus P_2 \), then \( s \) and \( s_1 \) generate \( G \). Also, we have \( C_G(s) = \langle s \rangle P_{n-1} \), which is of order \( p^2 \). Let us define \( s_i = [s_{i-1}, s] \) for all \( i \geq 2 \). Then \( s_i \in P_1 \setminus P_{i+1} \) for \( 1 \leq i \leq n-1 \), and \( s_i = 1 \) for \( i \geq n \). As a consequence, every \( g \in G \) can be uniquely written in the form

\[
g = s^i s_1^{j_1} \cdots s_{n-1}^{j_{n-1}},
\]

with \( 0 \leq i < p \) for all \( j = 0, \ldots, n-1 \). We refer to this as the normal form of \( g \) with respect to \( s \) and the \( s_i \). We also note that all elements of \( G' \) are commutators of the form \([s, g]\) with \( g \in G \), and then, by [14, Theorem 3.11], we have an epimorphism \( B_0(G) \to B_0(G/P_1) \) for every \( i \geq 2 \).

The degree of commutativity of \( G \) is the largest integer \( \ell \leq n-3 \) such that \( [P_i, P_j] \leq P_{i+j+t} \) for all \( i, j \geq 1 \). Then \( \ell \geq 0 \), and \( \ell = n-3 \) if and only if \( P_1 \) is abelian. We write \( \ell(G) \) when we want to emphasize the group \( G \). A fundamental result of Blackburn is that \( \ell > 0 \) if \( n \geq p+2 \). On the other hand, we have \( \ell(G/N) \geq \ell(G) \) for every \( N \triangleleft G \). Also, \( \ell(G/Z(G)) \) is always positive and, as a consequence, \( \ell(G) = 0 \) if and only if \([P_1, P_{n-2}] = P_{n-1}\).

Our approach to the proof of Theorem 1.2 relies on considering a free presentation of the p-group of maximal class and referring to a Hopf-type formula for the Bogomolov multiplier.

**Proof of Theorem 1.2.** By [13, Exercise 3.3(4)], the group \( G \) may be presented in the following manner. Let \( F \) be the free group on \( t \) and \( t_1 \). Denote \( t_i = [t_{i-1}, t] \) for all \( i \geq 2 \). Every element \( g \) of \( G \) has a normal form in terms of the generating set \( s \) and \( s_i \) for \( 1 \leq i \leq n-1 \). For a word \( w \) of \( F \), let \([w]\) denote the word in \( t \) and \( t_1 \) for \( 1 \leq i \leq n-1 \) obtained by replacing \( s \) with \( t \) and \( s_i \) with \( t_i \) in the normal form of the element of \( G \) that is represented by the word \( w \). Denote by \( \rho(w) = w[w]^{-1} \) the relator associated to \( w \). Set

\[
\mathcal{R}_0 = \{t_1\}, \quad \mathcal{R}_1 = \{\rho(t^p), \rho(t(t_1)^p)\}, \quad \mathcal{R}_2 = \{\rho([s_{2i}, s_1]) | 1 \leq i \leq (p-1)/2\},
\]

and let \( R \) be the normal subgroup of \( F \) generated by \( \mathcal{R}_0 \cup \mathcal{R}_1 \cup \mathcal{R}_2 \). Then \( F/R \) is a presentation of the group \( G \). Finally, let \( M = \langle t_1 \rangle F/R \) be the maximal subgroup of \( F \) that corresponds to \( P_1 \).

Define \( \lambda \) to be the map

\[
\lambda: M' \to \frac{F'}{K(F) \cap R} \cong B_0(G), \quad w \mapsto \rho(w)/\langle K(F) \cap R \rangle.
\]

The rest of the proof is devoted to showing how \( \lambda \) induces the desired isomorphism between \((P_1 \lambda P_1)_G\) and \( B_0(G) \).
I. We first claim that λ is a homomorphism. To see this, first observe that since $P_m$ is assumed to be abelian, we have $[M', \gamma_m(F), M' \gamma_m(F)] \leq R$. Now pick any $x, y \in M'$. Note that $[x], [y] \in \gamma_m(F)$. Hence

$$\lambda(x)\lambda(y) = x[x]^{-1}y[y]^{-1} \equiv xy([x][y])^{-1} \pmod{(K(F) \cap R)}.$$ 

Observe now that since $[\gamma_m(F), \gamma_m(F)] \leq (K(F) \cap R)$, every element of $\langle t_m, \ldots, t_{n-1} \rangle$ may be rewritten as

$$\prod_{i=m}^{n-1} t_i^{m_i} \equiv \left[ \prod_{i=m}^{n-1} t_i^{m_i}, t \right] \pmod{(K(F) \cap R)}.$$ 

Thus $\langle t_m, \ldots, t_{n-1} \rangle \cap R \leq (K(F) \cap R)$. Now, as $[x][y][xy]^{-1} \in R \cap \langle t_m, \ldots, t_{n-1} \rangle$, we conclude

$$\lambda(x)\lambda(y) = x[x]^{-1} = \lambda(xy) \pmod{(K(F) \cap R)}.$$ 

II. Let us now show that λ is surjective. Consider the group $F/\langle R \cap K(F) \rangle$. Its subgroup $R/\langle R \cap K(F) \rangle$ is an abelian group that can be generated by the cosets of the elements of $R_1 \cup R_2$. Observe that $R/\langle R \cap F \rangle \cong RF'/F'$ can be generated by the cosets of elements of $R_2$. Moreover, the elements $t^F$ and $t_i^{F'}$ form a base of the free abelian group $RF'/F'$ of rank 2. Hence we have that the torsion subgroup $(R \cap F')/(R \cap K(F))$ is generated by the cosets of the elements of $R_2$. Now note that $R_2 \leq \rho(M')$. Therefore λ is indeed surjective.

III. The homomorphism λ factors through $(K(M) \cap R)[M', t]$. It is clear that $(K(M) \cap R)$ is contained in the kernel of λ. To see that the same holds for $[M', t]$, consider $\lambda(m')$ for some $m \in M'$. We have that $[m'] \equiv [m]^t$ modulo $R \cap \langle t_m, \ldots, t_{n-1}, t_n \rangle \leq (K(F) \cap R)$. Whence

$$\lambda(m') = m'[m']^{-1} \equiv m'[m]^{-1} \equiv m[m]^{-1}[m][m]^{-1}, t \equiv \lambda(m) \pmod{(K(F) \cap M)}),$$

proving our claim. There is thus an induced homomorphism

$$\tilde{\lambda}: \frac{M'}{(K(M) \cap R)[M', t]} \xrightarrow{\cong} (P_1 \times P_1)_G \rightarrow \frac{F' \cap R}{(K(F) \cap R)} \cong B_0(G).$$

IV. Lastly, let us show that $\tilde{\lambda}$ is injective. To this end, let $w \in M'$ represent an element in ker $\tilde{\lambda}$. So $w[w]^{-1} \in (K(F) \cap R)$. Write $w = [w] \cdot \prod_{i \in I} [x_i, y_i]$ for some $[x_i, y_i] \in R$. Collect all those indices $i$ for which $x_i, y_i \in M$ into a set $I$. Upon replacing $w$ by $w \prod_{i \in I} [x_i, y_i]^{-1}$, we may assume that we have $x_i \not\in M$ and $y_i \in M$ for all indices $i$. Write $x_i = t_i^{a_i}m_i$ for some $1 \leq a_i < p$ and $m_i \in M$. Since $[x_i, y_i] \in R$, it follows that $y_i = t_i^{b_i}r_i$ for some $0 \leq b_i < p$ and $r_i \in R$. So $[x_i, y_i] = [t_i^{a_i}m_i, t_i^{b_i}r_i] \equiv [t_i^{a_i}, t_i^{b_i}r_i][m_i, t_i^{b_i}r_i]$. Now $[m_i, t_i^{b_i}r_i] \in \langle K(M) \cap R \rangle$, so that $[x_i, y_i] \equiv [t_i, t_i^{b_i}r_i]^{a_i}$. We may thus write $w \equiv [w] \cdot [t_i, t_i^{b_i}r_i]^{a_i} \pmod{t_i^p}$ for some $0 \leq a_i < p$ and $r_i \in R$. As $R$ is generated by $R_1 \cup R_2 \cup R_3$ modulo $R' \leq M'$ and we are in a setting where $[M', t] \equiv 1$, it follows that we may write $w \equiv \prod_{i \geq m} t_i^{a_i}$ for some integers $a_i$.

Note that the image of the group $\langle t_i \mid i \geq m \rangle$ in $M/(K(M) \cap R)$ is abelian, and so it is a quotient of the free abelian group generated by the elements $t_i$ for $i \geq m$. Moreover, the group $M/M'$ is the quotient of the free abelian group generated by the elements $t^F$ and $t_i$ for $i \geq 1$ subject only to the relations

$$\prod_{i=1}^{p} t_i^{c_i} \equiv 1 \pmod{M'}$$

where $c_i$ are integers.
for all $j \geq 1$. These arise from expanding $[t_j, t^p] \in M'$. As the element $w \equiv \prod_{i \geq m} \ell_{i}$ belongs to $M'$, it can therefore be written as a product of some powers of elements of the form $\prod_{i=1}^{p} t_{j+i}^{(i)}$ for some $j \geq m$. Now, observe that 

$$\prod_{i=1}^{p} t_{j+i}^{(i)} \equiv [t_j, t^p] \equiv 1 \mod (K(M) \cap R)$$

for all $j \geq m - 1$. Therefore $w \equiv 1$ in the domain of $\bar{\lambda}$ and the proof is complete. \hfill $\square$

Our main result covers all but finitely many $p$-groups of maximal class for every prime $p$, as the next corollary shows.

**Corollary 2.1.** Let $G$ be a $p$-group of maximal of order $p^n$. If 

$$n \geq \max\{p+2, 6p-29\},$$

then $B_0(G) \cong (P_1 \rtimes P_1)_G$.

**Proof.** The condition $n \geq p+2$ ensures that $G$ has positive degree of commutativity. Hence, according to Theorem 1.2 it suffices to check that $P_1$ is metabelian. If $p = 2, 3$ or 5, then the degree of commutativity $\ell$ of $G$ is at least $n-3$, $n-4$ or $(n-6)/2$, respectively, and the result readily follows. On the other hand, if $p \geq 7$ then $2\ell \geq n-2p+5$ by [5]. Since $P'_1 = [P_1, P_2] \leq P_{t+3}$, we have

$$[P'_1, P'_1] \leq [P_{t+3}, P_{t+3}] = [P_{t+3}, P_{t+4}] \leq P_{3\ell+7},$$

where

$$3\ell + 7 \geq \frac{3}{2}(n-2p+5) + 7 = n + \frac{1}{2}(n-6p+29) \geq n,$$

since $n \geq 6p-29$ by hypothesis. Thus $P_1$ is metabelian, as desired. \hfill $\square$

Now we continue by proving Theorem 1.1.

**Proof of Theorem 1.1.** We first assume that $[P_1, P_1] = [P_1, P_{n-2}]$ is strictly larger than $[P_1, P_{n-1}]$, and prove that $B_0(G)$ is nontrivial. If $\ell(G) = 0$ then $[P_1, P_{n-2}] = P_{n-1}$, and so $G/P_{n+1}$ is a proper quotient of $G$. We conclude that $\ell(G/P_{n+1}) > 0$ in every case. Then we may apply Theorem 1.2 to $G/P_{m+1}$ to get $B_0(G/P_{m+1}) \cong (P_1/P_{m+1} \rtimes P_1/P_{m+1})_G$. Since $B_0(G)$ surjects onto $B_0(G/P_{m+1})$ and $(P_1/P_{m+1} \rtimes P_1/P_{m+1})_G$ surjects onto $[P_1/P_{m+1}, P_1/P_{m+1}]_G = (P_m/P_{m+1})_G \cong C_p$, we conclude that $B_0(G) \neq 1$.

Now we prove the converse, namely that the condition $[P_1, P_1] = [P_1, P_{n-2}]$ implies that $B_0(G) = 1$. If $P_1$ is abelian, then $G$ is abelian-by-cyclic and hence $B_0(G)$ is trivial by [2]. So assume that $P_1$ is not abelian. The restriction $[P_1, P_1] = [P_1, P_{n-2}]$ gives $[P_1, P_1] = P_{n-1}$. Note that $P_{n-1}$ is generated by the element $s_{n-1}$ of order $p$. Moreover, since $[P_1, P_{n-2}] = P_{n-1}$ and $[P_2, P_{n-2}] = 1$, there exists a $\lambda \not \equiv 0 \mod p$ with $[s_1, s_{n-2}] = s_{n-1}^\lambda$. The latter equality may be rewritten as $[s^\lambda s_1, s_{n-2}] = 1$. Expanding $1 = s^\lambda s_1 \rtimes s_{n-2}$ in the curvy exterior square $G \rtimes G$ gives

$$(s^\lambda \rtimes s_{n-2})^\alpha (s_1 \rtimes s_{n-2}) = (s^\lambda[s^\lambda, s_1] \rtimes s_{n-2} s_{n-2}^\lambda)(s_1 \rtimes s_{n-2}) = (s \rtimes s_{n-2})^\lambda(s_1 \rtimes s_{n-2}),$$

therefore $(s_{n-2} \rtimes s)^\lambda = (s_1 \rtimes s_{n-2})$. Furthermore, pick any $s_i, s_j, s_k, s_l$ in $P_1$ and assume that both of the elementary wedges $s_i \rtimes s_j$ and $s_k \rtimes s_l$ are nontrivial in $G \rtimes G$. As $[P_1, P_1] = P_{n-1}$, both of the commutators $[s_i, s_j]$ and $[s_k, s_l]$ equal a power of
that in this case, we have \(P_1\) trivial by \([8, \text{Corollary 4.1}]\), there exists an \(m > 0\) such that \((s_1 \cdot s_j)(s_k \cdot s_l)^m \in B_0(P_1)\) trivial. The natural homomorphism \(B_0(P_1) \to B_0(G)\) shows that \((s_1 \cdot s_j)(s_k \cdot s_l)^m\) is also trivial in \(G \cdot G\). Hence all the elementary wedges \(s_1 \cdot s_j\) are equal to a power of the nontrivial one \(s_1 \cdot s_{n-2}\).

Now let \(w\) be an arbitrary element of \(B_0(G)\). For any \(x, y \in P_1\) and \(g, h \in G\), we have \([x \cdot y, g \cdot h] = [x, y] \cdot [g, h] = 1\) in \(G \cdot G\), since \(P_{n-1} = Z(G)\). Note also that \([s_1 \cdot s, s_j \cdot s] = s_{i+1} \cdot s_{j+1}\). The element \(w\) may therefore be written in the form

\[
w = \prod_{i=1}^{n-2} (s_i \cdot s)^{\alpha_i} \cdot (s_1 \cdot s_{n-2})^{\beta}
\]

for some integers \(\alpha_i, \beta\). Observe that

\[
1 = s_1 \cdot s = (s_1 \cdot s^{p-1})(s_1 \cdot s)^{p-1} = (s_1 \cdot s)^{p} \cdot \prod_{j>i}(s_j \cdot s)^{\lambda_j} \cdot (s_1 \cdot s_{n-2})^{\beta}
\]

for some \(\alpha_j, b\). We may thus assume that \(0 \leq \alpha_i < p\), and the same for \(\beta\). Note that \(w\) belongs to \(B_0(G)\) if and only if we have \(\prod_{i=1}^{n-2} (s_i \cdot s_{n-2})^{\lambda_i} \cdot (s_1 \cdot s_{n-2})^{\beta} = 1\) in \(G\). Collecting the left hand side in its normal form and comparing exponents gives \(\alpha_i = 0\) for all \(i \leq n-3\) and \(\alpha_{n-2} + \lambda \beta = 0\). We thus have \(w = ((s \cdot s_{n-2})^\lambda (s_1 \cdot s_{n-2}))^\beta\), and so \(w = 1\) by above. Hence \(B_0(G)\) is trivial, as required.

\[\square\]

**Corollary 2.2.** Let \(G\) be a \(p\)-group of maximal class of positive degree of commutativity. Then \(B_0(G)\) is trivial if and only if \(P_1\) is abelian.

With respect to isoclinism, this amounts to precisely the groups isoclinic to a group on the single infinite line of the tree of all \(p\)-groups of maximal class, where every group \(G\) is linked to \(G/Z(G)\). This infinite line corresponds to the quotients of the only infinite pro-\(p\) group of maximal class. Also, the cases \(p = 2, 3\) are special. When \(p = 2\), all the groups have abelian \(P_1\), and therefore trivial Bogomolov multipliers. When \(p = 3\), all the groups have positive degree of commutativity and \([P_1, P_1] \leq 3\), so we either have that \(P_1\) is abelian, in which case \(B_0(G)\) is trivial, or \([P_1, P_1] = P_{n-1}\) and the degree is positive, in which case \(B_0(G)\) is nontrivial. Moreover, it follows from the proof of Theorem 1.2 that in this case, we have \(B_0(G) = \langle (s \cdot s_{n-1})^\lambda (s_i \cdot s_j) \rangle \cong C_3\) for some \(\lambda, i, j\).

3. **Examples**

In this section, we provide some examples of \(p\)-groups of maximal class, for \(p \geq 5\), for which the structure of their Bogomolov multipliers may be explicitly determined, and as a consequence, we prove Theorem 1.3.

To do this, we recall that the structure of a \(p\)-group \(G\) of maximal class with \(P_1\) of nilpotency class 2 can be given in terms of the ring of integers in the \(p\)-th cyclotomic number field \(\mathcal{O}\). So \(\mathcal{O} = \mathbb{Z}[\theta]/(1 + \theta + \cdots + \theta^{p-1})\), where \(\theta\) is a primitive complex \(p\)-th root of unity. Denote \(\kappa = \theta - 1\) and let \(p = (\kappa)\). There is an action of \(\mathcal{O}\) on \(P_m\) with \(\theta\) acting via conjugation by \(s\). By \([13, \text{Lemma 8.2.1}]\), there is an \(\mathcal{O}\)-module isomorphism between \(P_i/P_{i+j}\) and \(\mathcal{O}/p^j\), induced by the map

\[\mathcal{O} \to P_i/P_{i+j}, \quad \sum_u a_u \kappa^u \rightarrow \prod_u a_{s_i^u} \kappa^{s_i^u}\]

The commutator structure of \(P_1\) can thus be understood in terms of the homomorphism

\[\alpha: \mathcal{O}/p^{m-1} \otimes \mathcal{O}/p^{m-1} \rightarrow \mathcal{O}/p^{n-m} \cong P_m\]
This is in fact a homomorphism of $(\theta) = C_p$ modules. Set
\[ K\alpha = \ker\alpha \cap \{\text{elementary wedges in } \mathcal{O}/p^{m-1} \wedge \mathcal{O}/p^{m-1}\}. \]

Now consider the induced epimorphism
\[ \alpha_{C_p} : (\mathcal{O}/p^{m-1} \wedge \mathcal{O}/p^{m-1})_{C_p} \to (\mathcal{O}/p^{n-m})_{C_p} \cong \mathcal{O}/p \cong P_m/P_{m+1} \cong C_p \]

obtained by factoring out the action of $\theta$. Correspondingly, there is the induced kernel
\[ K\alpha_{C_p} = \ker\alpha_{C_p} \cap \{\text{image of elementary wedges in } (\mathcal{O}/p^{m-1} \wedge \mathcal{O}/p^{m-1})_{C_p}\}. \]

Notice that
\[ (\mathcal{O}/p^{m-1} \wedge \mathcal{O}/p^{m-1})_{C_p} \cong \left( \frac{\mathcal{O}/p^{m-1} \wedge \mathcal{O}/p^{m-1}}{K\alpha} \right)_{C_p} \]

by right-exactness of coinvariants. We make the following identification:

\[ P_i \cdot P_i = \frac{P_i/P_m \wedge P_i/P_m}{(xP_m \wedge yP_m \mid [x, y] = 1)} \cong \frac{\mathcal{O}/p^{m-1} \wedge \mathcal{O}/p^{m-1}}{K\alpha} \]

Now, to provide concrete examples, we show that by carefully selecting the map $\alpha$, which in turn determines the group $G$, one may achieve that the image of the map $K\alpha_{C_p}$ in $(\mathcal{O}/p^{m-1} \wedge \mathcal{O}/p^{m-1})_{C_p}$ is trivial. Based on the above identification, this amounts to constructing groups $G$ with $B_0(G) \cong (\mathcal{O}/p^{m-1} \wedge \mathcal{O}/p^{m-1})_{C_p}$. Such a commutator structure will therefore produce groups whose Bogomolov multipliers will have largest possible rank and exponent for the given values of $n$ and $m$. Furthermore, essentially the same argument will deal with quotients of such extreme groups. The construction we give below covers this more general case.

Fix any $m \geq 4$ and set $\ell = m - 3$. The number $\ell$ will be the degree of commutativity of the constructed group. Now pick any $n$ satisfying $m < n \leq 2m - 2$. Set $\mu = n - m + 2$, so that $2 < \mu \leq m$. Let $g$ be a primitive root modulo $p$ and pick an integer $a$ so that $a \equiv (g + 1)^{-1} \pmod{p}$. It is here that we need $p \geq 5$. In the case when $a > (p - 1)/2$, replace $a$ by $1 - a$, so that in the end, $2 \leq a \leq (p - 1)/2$. Now define $\alpha : \mathcal{O}/p^{m-1} \wedge \mathcal{O}/p^{m-1} \to \mathcal{O}/p^{n-m}$ by the rule
\[ \alpha(x \wedge y) = \kappa^{-1} \cdot (\sigma_a(x)\sigma_{1-a}(y) - \sigma_a(y)\sigma_{1-a}(x)) \]

for $x, y \in \mathcal{O}/p^{m-1}$. Here, $\sigma_a$ is the automorphism of $\mathcal{O}/p^{m-1}$ which maps $\theta^a$ to $\theta^a$. This corresponds to the map induced by $\kappa^{-1}S_a$ in [13, Theorem 8.3.1]. Set $u_a = (\theta^a - 1)/\kappa \in \mathcal{O}^*$. Then
\[ \alpha(k^i \wedge k^j) = \text{sgn}(i + j)k^{i+j-1}(u_{\mathcal{O}^1}u_{\mathcal{O}^1})^{\text{min}(i,j)}(u_{\mathcal{O}^1}^{i-j} - u_{\mathcal{O}^1}^{i-j}) \in p^{i+j-1}. \]

Observe that $u_{\mathcal{O}^1}^{i-j} - u_{\mathcal{O}^1}^{i-j} \equiv a^{i-j} - (1 - a)^{i-j} \pmod{p}$. This element belongs to $p$ if and only if we have $(a^{-1} - 1/2^{i-j}) \equiv 1 \pmod{p}$. By our choice of $a$, this occurs precisely when $i \equiv j \pmod{p - 1}$. The commutator map $\alpha$ therefore satisfies $\alpha(k^i \wedge k^j) \in p^{i+j-1}\backslash p^{i+j}$ whenever $i \neq j \pmod{p - 1}$.

Invoking [13, Theorem 8.2.7], there is a $p$-group $G$ of maximal class of order $p^n$ whose commutator structure is described by the map $\alpha$ given above. In terms of the $P_i$-series of $G$, the above discussion shows that we have $[P_i, P_j] = P_{i+j+\ell}$ for all $i, j \geq 1$ that satisfy $i \neq j \pmod{p - 1}$.

This highly restricted commutator structure enables us to completely understand commuting pairs of $G$. 
Lemma 3.1. Let \( x \in P_i \setminus P_{i+1} \). Then \( C_{P_i}(x) = \langle x, P_{i+j} \rangle \), where \( j = \max\{n-2i-\ell, 1\} \).

Proof. Clearly the right hand side centralizes \( x \). Conversely, suppose that \( y \in P_i \setminus P_{k+1} \) for some \( k > i \) and \( [x,y] = 1 \). Assume that \( y \not\in \langle x \rangle \). If \( k \equiv i \) (mod \( p-1 \)), then \( y = x^r z \) for some \( r > 0 \) and \( z \in P_i \setminus P_{k+1} \) with \( [z,x] = 1 \) and \( k' \not\equiv i \) (mod \( p-1 \)). In this case, replace \( y \) by \( z \) and \( k \) by \( k' \), so that we may assume \( k \not\equiv i \) (mod \( p-1 \)). Now, since \( [P_i, P_k] = P_{i+k+\ell} \) and \( [P_{i+1}, P_k][P_i, P_{k+1}] \leq P_{i+k+\ell+1} \), it follows that \( P_{i+k+\ell} = P_{i+k+\ell+1} \), which is only possible when \( i + k + \ell \geq n \). \( \square \)

In particular, note that \( Z(P_1) \geq P_\mu \geq P_n \) in the group \( G \). Transferring to the \( C_p \)-module \( O/p^\mu \), we thus have that the elementary wedges in \( p^{\mu-1}/p^{\mu-1} \) are all contained in \( K\alpha \). Using Lemma 3.1 more precisely, we now show that wedges that arise from commuting pairs are, modulo the action of \( C_p \), nothing but the latter.

Lemma 3.2. \( K\alpha_{C_p} = (O/p^{m-1} \wedge O/p^{m-1}) + [O/p^{m-1} \wedge O/p^{m-1}, C_p] \).

Proof. Let \( x \land y \in K\alpha \) for some \( x, y \in O/p^{m-1} \). Suppose that \( x \) corresponds to an element in \( P_i \setminus P_{i+1} \) and \( y \) to an element in \( P_j \setminus P_{j+1} \) with \( i \leq j \). We will prove that \( x \land y \) is equivalent to an element of the submodule \( p^{\mu-1}/p^{\mu-1} \wedge O/p^{\mu-1} \) modulo \([O/p^{\mu-1} \wedge O/p^{\mu-1}, C_p] \) by induction on \( i \).

If \( i \equiv j \) (mod \( p-1 \)), then as in the proof of Lemma 3.1, we may write \( y = x^r z \) with \( z \in P_j \setminus P_{j+1} \) and \( j' \not\equiv i \) (mod \( p-1 \)). Then \( x \land y = x \land z \), so we may without loss of generality assume that \( i \not\equiv j \) (mod \( p-1 \)). By the lemma, we then have \( i + j + \ell \geq n \). If \( i = 1 \), this implies that \( j \geq n - \ell - 1 = \mu \), whence \( x \land y \in O/p^{m-1} \wedge p^{\mu-1}/p^{m-1} \). This is the base for the induction. Suppose now that \( i > 1 \). Then \( x = \tilde{x} \land \kappa \) for some \( \tilde{x} \in O/p^{m-1} \) corresponding to a group element in \( P_{i-1} \setminus P_i \). Observe that

\[
\tilde{x} \land \kappa y + x \land y + x \land \kappa y = \kappa(\tilde{x} \land y) \in [O/p^{m-1} \wedge O/p^{m-1}, C_p].
\]

Note that \( \kappa y \) corresponds to a group element in \( P_{i+1} \), and therefore \( \tilde{x} \land \kappa y \) and \( x \land \kappa y \) both belong to \( K\alpha \). Using this reasoning, we show our claim by reverse induction on \( j \). When \( j \geq \mu \), it is clear that \( x \land y \in O/p^{m-1} \wedge p^{\mu-1}/p^{m-1} \). Assume now that \( j < \mu \). By induction, both \( \tilde{x} \land \kappa y \) (since \( \tilde{x} \) belong to a higher level) and \( x \land \kappa y \) (since \( \kappa y \) belongs to a lower level) are contained in \( O/p^{m-1} \wedge p^{\mu-1}/p^{m-1} \) modulo \([O/p^{m-1} \wedge O/p^{m-1}, C_p] \). An application of (3) then implies that the same holds for \( x \land y \), as claimed. \( \square \)

The above gives that

\[
B_0(G) = \frac{(O/p^{m-1} \wedge O/p^{m-1})_{C_p}}{K\alpha_{C_p}} = (O/p^{\mu-1} \wedge O/p^{\mu-1})_{C_p}.
\]

Finally, a structure description of the group \((O/p^{\mu-1} \wedge O/p^{\mu-1})_{C_p}\) may be read off from the explicit \( C_p \)-module decomposition of \( O/p^{\mu-1} \wedge O/p^{\mu-1} \) into a direct sum of cyclic submodules as given in [12, Theorem 8.13]. Then Theorem 1.3 readily follows.

Let us consider some special cases of Theorem 1.3. When \( n \) is chosen so that \( n \equiv m-1 \) (mod \( p-1 \)), we obtain a group \( G \) with \( B_0(G) \) homocyclic of rank
(p - 1)/2 and exponent $p^{(n-m+1)/(p-1)}$. Further selecting $n \approx 2m$, we have the property $\exp B_0(G) \approx \sqrt[p]{\exp G}$. Consider now the option $n = m + 1$. In this case, we obtain groups that are immediate descendants of groups on the main line of the maximal class tree. Their Bogomolov multipliers are $C_p$. In the very special case when $m = 4$, we obtain the known groups of order $p^9$ with nontrivial Bogomolov multipliers. Another extreme option is picking $n = 2m - 2$. In this case, we have $n - m + 1 = m - 1$, so by varying $m$, the groups exhaust all the possibilities for the Bogomolov multiplier, depending on the value $m - 1 \pmod{p-1}$. Finally, consider the option of selecting consecutive values $n = m + 1, m + 2, \ldots, 2m - 2$. In terms of the constructed groups, this corresponds to a path in the maximal class tree, starting from an immediate descendant of a group on the main line (which is of order $p^m$) and going deeper into the branch. In this process, the value $n - m + 1$ grows one by one, so that the corresponding Bogomolov multipliers grow in size by a factor of $p$ on each second step.

The above rank of the Bogomolov multiplier is in fact largest possible for $p$-groups of maximal class. To see this, we first recall the following result.

**Lemma 3.3** ([9, Corollary 5.2]). Let $G$ be a finite group generated by $d(G)$ elements with $r(G)$ defining relations, and let $r_K(G)$ be the number of defining relations which are commutators. Then $\text{rank } B_0(G) \leq r(G) - r_K(G) - d(G)$.

A direct consequence is the following.

**Corollary 3.4.** Let $G$ be a $p$-group of maximal class. Then $\text{rank } B_0(G) \leq (p-1)/2$.

**Proof.** This is immediate from the previous lemma and [13, Exercise 3.3(4)].

The above exponents are also the largest possible for groups in which $P_1$ is of nilpotency class 2. This is so because if $[P_1, P_1] = P_m$, then $B_0(G)$ is a quotient of the group $(G/P_m \cap G/P^{m-1})C_p$, whose exponent is $p^{((n-m+1)/(p-1))} \leq p^{n/(2(p-1))} \approx \sqrt[p]{\exp G}$.

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