QUASI-LINEAR SCHRÖDINGER-POISSON SYSTEM
UNDER AN EXPONENTIAL CRITICAL NONLINEARITY:
EXISTENCE AND ASYMPTOTIC BEHAVIOUR OF SOLUTIONS

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Abstract. In this paper we consider the following quasilinear Schrödinger-Poisson system in a bounded domain in $\mathbb{R}^2$:

$$
\begin{cases}
-\Delta u + \phi u = f(u) & \text{in } \Omega, \\
-\Delta \phi - \varepsilon^4 \Delta_4 \phi = u^2 & \text{in } \Omega, \\
u = \phi = 0 & \text{on } \partial \Omega
\end{cases}
$$

depending on the parameter $\varepsilon > 0$. The nonlinearity $f$ is assumed to have critical exponential growth. We first prove existence of nontrivial solutions $(u_\varepsilon, \phi_\varepsilon)$ and then we show that as $\varepsilon \to 0^+$ these solutions converge to a nontrivial solution of the associated Schrödinger-Poisson system, that is by making $\varepsilon = 0$ in the system above.

1. Introduction

In this paper we study the following system

$$(P_\varepsilon)
\begin{cases}
-\Delta u + \phi u = f(u) & \text{in } \Omega, \\
-\Delta \phi - \varepsilon^4 \Delta_4 \phi = u^2 & \text{in } \Omega, \\
u = \phi = 0 & \text{on } \partial \Omega
\end{cases}
$$

where $\Omega \subset \mathbb{R}^2$ is a smooth and bounded domain, $\Delta_4 = \text{div}(|\nabla \phi|^2 \nabla \phi)$ is the 4-Laplacian and $f$ satisfies suitable assumptions, allowing to have critical growth.

Problem $(P_\varepsilon)$ is the planar version of the so called quasilinear Schrödinger-Poisson system which, after the papers $[4, 7]$ has attracted the attention of mathematicians in these recent years. However few papers deal with this kind of system. We cite here $[8]$ where the authors consider the quasilinear Schrödinger-Poisson system in the unitary cube in $\mathbb{R}^3$ under periodic boundary conditions; they show global existence and uniqueness of solutions. In $[10]$ the author proves existence and uniqueness of a global mild solution in the one dimensional case. In the recent paper $[5]$, the problem in $\mathbb{R}^3$ with an asymptotically linear $f$ is considered. The authors prove existence and the behaviour of the ground state solutions as $\varepsilon \to 0^+$. Again the solutions converge to the solution of the “limit” problem with $\varepsilon = 0$. Finally in $[6]$ we studied the problem in $\mathbb{R}^3$ under a critical nonlinearity, showing again that the solutions converge to a solution of the Schrödinger-Poisson system.

As explained in $[1]$ (see also $[4, 7]$) the system appears by studying a quantum physical model of extremely small devices in semi-conductor nanostructures and takes into account the quantum structure and the longitudinal field oscillations during the beam propagation. This is reflected into the fact that the dielectric permittivity depends on the electric field by

$$
c_{\text{die}}(\nabla \phi) = 1 + \varepsilon^4 |\nabla \phi|^2, \quad \varepsilon > 0 \text{ and constant.}
$$

We refere the reader to $[6]$ where the system is deduced in the framework of Abelian Gauge Theories.

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Before to state our results let us introduce some notations. In this paper we fix an arbitrary \( r > 2 \) and consider the auxiliary problem

\[
(A) \quad \begin{cases}
-\Delta u = |u|^{r-2}u & \text{in } \Omega, \\
u = 0 & \text{on } \partial\Omega.
\end{cases}
\]

Let

\[
R(u) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 - \frac{1}{r} \int_{\Omega} |u|^r
\]

be the the functional associated to problem \((A)\) and let

\[
\mathcal{N} = \{ u \in H^1_0(\Omega) \setminus \{0\} : R'(u)[u] = 0 \}.
\]

be the Nehari manifold. Hereafter \( H^1_0(\Omega) \) is the usual Sobolev space endowed with scalar product and (squared) norm given by

\[
\langle u, v \rangle := \int_{\Omega} \nabla u \nabla v, \quad \|u\|^2 = \int_{\Omega} |\nabla u|^2.
\]

We have \( H^1_0(\Omega) \hookrightarrow L^p(\Omega), \) for \( p \geq 1. \) The \( L^p - \)norm will be simply denoted with \( | \cdot |_p. \)

Standard arguments give the existence of a ground state \( u \in H^1_0(\Omega) \) for problem \((A)\) which satisfies

\[
m := R(u) = \min_{\mathcal{N}} R, \quad R'(u) = 0
\]

and

\[
m = \frac{r-2}{2r} \int_{\Omega} |u|^r = \frac{r-2}{2r} \|u\|^2.
\]

(1.1)

Now we can state our assumptions on \( f \) in order to study problem \((P_\varepsilon)\).

Let \( f : \mathbb{R} \rightarrow \mathbb{R} \) be a continuous function such that

\[
(f_0) \quad f(t) = 0 \text{ for } t \leq 0,
\]

\[
(f_1) \quad \lim_{t \to 0} \frac{f(t)}{t} = 0,
\]

\[
(f_2) \quad \text{there exists } \alpha_0 > 0 \text{ such that}
\]

\[
\lim_{t \to \infty} \frac{f(t)}{\exp(\alpha t^2)} = \begin{cases}
0 & \text{for } \alpha > \alpha_0, \\
\infty & \text{for } \alpha < \alpha_0,
\end{cases}
\]

\[
(f_3) \quad \text{there exists } \theta \in (4, +\infty) \text{ such that}
\]

\[
0 < \theta F(t) = \theta \int_0^t f(s) ds \leq tf(t), \quad \text{for all } t > 0,
\]

\[
(f_4) \quad \text{there is } \tau \geq \tau^*(\varepsilon) \text{ such that}
\]

\[
f(t) \geq \tau t^{r-1}, \quad \forall t \geq 0
\]

where

\[
\tau^*(\varepsilon) := \max\left\{ \frac{\theta m(\alpha_0 + 1)}{\pi(\theta - 2)} \left(\frac{r-2}{2}\right)^{\frac{r-2}{2}}, \frac{4\theta m}{(\theta - 2)^2} \left(\frac{r-2}{2}\right)^{\frac{r-2}{2}} \right\},
\]

and where \( \mathcal{T}(\varepsilon) > 0 \) will appear later.

We would like to highlight that the model nonlinearity

\[
f(t) = \begin{cases}
\tau^*(\varepsilon)|t|^{r-2}t \exp(\alpha_0 t^2) & \text{for } t \geq 0 \\
0 & \text{for } t \leq 0.
\end{cases}
\]

satisfies all the assumptions above.

We define

\[
X := H^1_0(\Omega) \cap W^{1,4}_0(\Omega)
\]
which is a Banach space under the norm
\[ \| \phi \|_X := |\nabla \phi|_2 + |\nabla \phi|_4. \]

Note that \( X \hookrightarrow L^{\infty}(\Omega) \).

By a solution of \((P_\varepsilon)\) we mean a pair \((u_\varepsilon, \phi_\varepsilon) \in H^1_0(\Omega) \times X\) such that
\[
\forall v \in H^1_0(\Omega) : \quad \int_\Omega \nabla u_\varepsilon \nabla v + \int_\Omega \phi_\varepsilon u_\varepsilon v = \int_\Omega f(u_\varepsilon) v
\]
\[ (1.2) \]
\[
\forall \xi \in X : \quad \int_\Omega \nabla \phi_\varepsilon \nabla \xi + \varepsilon^4 \int_\Omega |\nabla \phi_\varepsilon|^2 \nabla \phi_\varepsilon \nabla \xi = \int_\Omega \xi u^2_\varepsilon.
\]

The main results of this paper are the following.

**Theorem 1.** Assume that conditions \((f0)-(f4)\) hold. Then, for every \( \varepsilon > 0 \) problem \((P_\varepsilon)\) admit a solution \((u_\varepsilon, \phi_\varepsilon) \in H^1_0(\Omega) \times X\). Moreover \( \phi_\varepsilon, u_\varepsilon \) are nonnegative.

We study also the asymptotic behaviour of the solutions \( u_\varepsilon, \phi_\varepsilon \) as \( \varepsilon \to 0^+ \) obtaining the following

**Theorem 2.** Let \( f : \mathbb{R} \to \mathbb{R} \) be a function satisfying conditions \((f0)-(f2)\) and consider the Schrödinger-Poisson system

\[
(P_0) \quad \begin{cases}
-\Delta u + \phi u = f(u) & \text{in } \Omega, \\
-\Delta \phi = u^2 & \text{in } \Omega, \\
u = \phi = 0 & \text{on } \partial \Omega.
\end{cases}
\]

If \( \{u_\varepsilon, \phi_\varepsilon\}_{\varepsilon > 0} \) are solutions of \((P_\varepsilon)\) satisfying also \( \|u_\varepsilon\|^2 \leq 2\pi/(\alpha_0 + 1) \) then,

1. \( \lim_{\varepsilon \to 0^+} u_\varepsilon = u_0 \) in \( H^1_0(\Omega) \),
2. \( \lim_{\varepsilon \to 0^+} \phi_\varepsilon = \phi_0 \) in \( H^1_0(\Omega) \),

where \((u_0, \phi_0)\) is a nontrivial solution of \((P_0)\).

In particular Theorem 2 gives the existence of a nontrivial solution for \((P_0)\), which we were not able to find in the mathematical literature under our assumption on \( f \).

Our contribution in this paper is to give a better understanding on this quasilinear problem, on which there are just few papers (cited above) in the literature. Moreover, to the best of our knowledge, this is the first paper dealing with the two dimensional case and involving a critical nonlinearity; and in fact the main difficulties are related to (i) the “fourth” order term in the equation (hence in particular any homogeneity property is lost) and (ii) to the critical growth of the nonlinearity.

We find solutions by using variational methods by using Mountain Pass arguments. Indeed the solutions will be critical points of a functional \( J_\varepsilon \). However, to avoid the previous difficulties, we introduce a suitable truncated functional, \( J^T_\varepsilon \), depending on a parameter \( T > 0 \), in such a way that we have compactness at the Mountain Pass level of the truncated functional, and even more, we can recover a critical point of the untruncated functional. Then by using suitable estimates with respect to \( \varepsilon \) we are able to show that the solutions of \((P_\varepsilon)\) tends, as \( \varepsilon \) tends do zero, to a nontrivial solution of the Schrödinger-Poisson system.

The paper is organized as follows.

In Section 2 we introduce the variational framework, by defining a \( C^1 \) functional \( J_\varepsilon \) naturally associated to \((P_\varepsilon)\).

The truncated functional \( J^T_\varepsilon \) is introduced in Section 3, where we prove also a suitable estimate on its Mountain Pass level.

In Section 4 we show that, for a suitable choice of the truncation parameter \( T \), the Mountain Pass level of the \( J^T_\varepsilon \) satisfies an estimate which permits to have compactness and recover a critical point of \( J_\varepsilon \), hence a solution of \((P_\varepsilon)\), proving Theorem 1.

Finally in Section 5 we prove Theorem 2.
As a matter of notation we use for brevity the notation $\int_{\Omega} w$ to mean $\int_{\Omega} w(x)dx$.

2. The variational framework

Let us start by noticing that from \((\text{f1})-(\text{f3})\), for all $\delta > 0$ and for all $\alpha > \alpha_0$, there exist constants $C_\delta, \tilde{C}_\delta > 0$ such that

\[(2.1) \quad \int_{\Omega} f(u)u \leq \delta \int_{\Omega} u^2 + C_\delta \int_{\Omega} |u|^q \exp(\alpha u^2) \]

and

\[(2.2) \quad \int_{\Omega} F(u) \leq \delta \int_{\Omega} u^2 + \tilde{C}_\delta \int_{\Omega} |u|^q \exp(\alpha u^2), \]

for all $u \in H^1_0(\Omega)$ and for all $q \geq 0$. Let us recall the following Trundiger-Moser inequality.

**Proposition 1** ([11]). If $\alpha > 0$ and $u \in H^1_0(\Omega)$, then

\[\int_{\Omega} \exp(\alpha u^2) < \infty.\]

Moreover if $\alpha < 4\pi$ there exists a constant $C = C(\alpha) > 0$ such that

\[\sup_{\|u\| \leq 1} \int_{\Omega} \exp(\alpha u^2) \leq C.\]

It is easy to see that the critical points of the smooth functional

\[J_\varepsilon(u, \phi) = \frac{1}{2}\|u\|^2 + \frac{1}{2} \int_{\Omega} \phi u^2 - \int_{\Omega} F(u) - \frac{1}{4} \int_{\Omega} |\nabla \phi|^2 - \frac{\varepsilon^4}{8} \int_{\Omega} |\nabla \phi|^4\]

on $H^1_0(\Omega) \times X$ are exactly the weak solutions of \((P_\varepsilon)\), according to \((1.2)\) and \((1.3)\). However since this functional $J_\varepsilon$ is strongly indefinite in the product space $H^1_0(\Omega) \times X$ we use a well known by now reduction method which consists in studying a suitable functional of a single variable. We give the details in the next subsection, which basically consists in solving for every $u \in H^1_0(\Omega)$ the second equation of the system and then substituting in the first equation.

The value of $\varepsilon > 0$ has to be considered fixed until the end of Section 4. In Section 5 where we will pass to the limit with respect to $\varepsilon$ to prove Theorem 2.

2.1. Study of the quasilinear Schrödinger-Poisson equation. Let us study here the second equation of the system \((P_\varepsilon)\). Note that, for every $u \in H^1_0(\Omega)$ the map (with abuse of notations)

\[u^2 : \phi \in X \mapsto \int_{\Omega} \phi u^2 \in \mathbb{R}\]

is linear and continuous, hence $u^2 \in X'$. Then the unique solution of

\[(2.3) \quad \begin{cases} -\Delta \phi - \varepsilon^4 \Delta_4 \phi = u^2 & \text{in } \Omega, \\ \phi = 0 & \text{in } \Omega, \end{cases}\]

is the unique critical point (the minimum) of the functional

\[\phi \in X \mapsto \frac{1}{2} \int_{\Omega} |\nabla \phi|^2 + \frac{\varepsilon^4}{4} \int_{\Omega} |\nabla \phi|^4 - \int_{\Omega} \phi u^2 \in \mathbb{R}.\]

Hence it makes sense to consider the map

\[(2.4) \quad \Phi_\varepsilon : u \in H^1_0(\Omega) \mapsto \phi_\varepsilon(u) \in X\]

where $\phi_\varepsilon(u)$ is the unique solution of \((2.3)\). The continuity of $\Phi_\varepsilon$ follows by the next result, whose proof is exactly as in [6, Lemma 1].
Lemma 1. Let \( g_n \to g \) in \( X' \). Then, we have
\[
\int_{\Omega} |\nabla \phi_{\varepsilon}(g_n)|^2 \to \int_{\Omega} |\nabla \phi_{\varepsilon}(g)|^2, \quad \int_{\Omega} |\nabla \phi_{\varepsilon}(g_n)|^4 \to \int_{\Omega} |\nabla \phi_{\varepsilon}(g)|^4.
\]
In particular the operator \( \Phi_{\varepsilon} \) in (2.4) is continuous and \( \phi_{\varepsilon}(g_n) \to \phi_{\varepsilon}(g) \) in \( L^\infty(\Omega) \).

In the remaining of the paper, \( \phi_{\varepsilon}(u) \) will always denote the unique solution of (2.3) with fixed \( u \). Note that it satisfies
\[
(2.5) \quad \int_{\Omega} |\nabla \phi_{\varepsilon}(u)|^2 + \varepsilon^4 \int_{\Omega} |\nabla \phi_{\varepsilon}(u)|^4 = \int_{\Omega} \phi_{\varepsilon}(u)u^2.
\]

The next result will be useful in the following.

Lemma 2. Let \( q \in [1, +\infty) \). If \( \{u_n\} \) converges to some \( w \) in \( L^q(\Omega) \) then,
\[
(a) \quad \lim_{n \to +\infty} \int_{\Omega} |\nabla \phi_{\varepsilon}(u_n)|^2 = \int_{\Omega} |\nabla \phi_{\varepsilon}(w)|^2,

(b) \quad \lim_{n \to +\infty} \int_{\Omega} |\nabla \phi_{\varepsilon}(u_n)|^4 = \int_{\Omega} |\nabla \phi_{\varepsilon}(w)|^4,

(c) \quad \lim_{n \to +\infty} \int_{\Omega} \phi_{\varepsilon}(u_n)u_n^2 = \int_{\Omega} \phi_{\varepsilon}(w)w^2,

(d) \quad \lim_{n \to +\infty} \phi_{\varepsilon}(u_n) = \phi_{\varepsilon}(w) \text{ in } L^\infty(\Omega),

(e) \quad \text{for all } v \in H^1_0(\Omega) : \lim_{n \to +\infty} \int_{\Omega} \phi_{\varepsilon}(u_n)u_nv = \int_{\Omega} \phi_{\varepsilon}(w)wv.
\]

Proof. Under our assumptions we have,
\[
\|u_n^2 - w^2\| = \sup_{\|\phi\|_X = 1} \left| \int_{\Omega} \phi(u_n^2 - w^2) \right| \leq \|\phi\|_q |u_n^2 - w^2|_q \leq C |u_n^2 - w^2|_q \to 0.
\]

Then we can apply Lemma 1 and (2.5) and conclude the proof of (a), (b), (c), (d). The proof of (e) follows by using an Hölder inequality and (d). \( \square \)

Let \( G(\Phi_{\varepsilon}) \) be the graph of the map \( \Phi_{\varepsilon} : u \in H^1_0(\Omega) \mapsto \phi_{\varepsilon}(u) \in X \).

Since the functional \( J_{\varepsilon} \) is \( C^2 \), classical arguments using the Implicit Function Theorem (see e.g. \[3\]) for the Schrödinger-Poisson system) give that
\[
G(\Phi_{\varepsilon}) = \{(u, \phi) \in H^1_0(\Omega) \times X : \partial_\phi J_{\varepsilon}(u, \phi) = 0\}.
\]

and actually \( \Phi_{\varepsilon} \in C^1(\overline{H^1_0(\Omega)}; X) \).

As a consequence, the functional (recall (2.5))
\[
J_{\varepsilon}(u) := J_{\varepsilon}(u, \Phi_{\varepsilon}(u)) = \frac{1}{2} \|u\|^2 + \frac{1}{4} \int_{\Omega} |\nabla \phi_{\varepsilon}(u)|^2 + \frac{3\varepsilon^4}{8} \int_{\Omega} |\nabla \phi_{\varepsilon}(u)|^4 - \int_{\Omega} F(u)
\]
is of class \( C^1 \) and in particular we have
\[
J'_{\varepsilon}(u)[v] = \partial_u J_{\varepsilon}(u, \phi_{\varepsilon}(u))[v] + \partial_{\phi} J_{\varepsilon}(u, \phi_{\varepsilon}(u)) \circ \Phi'_{\varepsilon}(u)[v] = \partial_u J_{\varepsilon}(u, \phi_{\varepsilon}(u))[v].
\]

Then
\[
J'_{\varepsilon}(u)[v] = \int_{\Omega} \nabla u \nabla v + \int_{\Omega} \phi_{\varepsilon}(u)uv - \int_{\Omega} f(u)v
\]
which shows that finding a critical point \( u_{\varepsilon} \) of \( J_{\varepsilon} \) is equivalent to obtain a solution \( (u_{\varepsilon}, \phi_{\varepsilon}) \) of \( (P_{\varepsilon}) \) where \( \phi_{\varepsilon} = \Phi_{\varepsilon}(u_{\varepsilon}) \). We are then reduced to study the problem
\[
\begin{cases}
-\Delta u + \phi_{\varepsilon}(u)u = f(u) & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega.
\end{cases}
\]
For brevity we introduce the functional
\[ I_\varepsilon : u \in H_0^1(\Omega) \mapsto \frac{1}{4} \int_\Omega |\nabla \phi_\varepsilon(u)|^2 + \frac{3\varepsilon^4}{8} \int_\Omega |\nabla \phi_\varepsilon(u)|^4 \in \mathbb{R} \]
so that we can write
\[ J_\varepsilon(u) = \frac{1}{2} \|u\|^2 + I_\varepsilon(u) - \int_\Omega F(u). \]

3. The truncated functional

In order to overcome the “growth” of order 4 in \( J_\varepsilon \), let us define a truncation for the functional \( J_\varepsilon \) in the following way. Consider a smooth cut-off and non-increasing function \( \psi : [0, +\infty) \to [0, +\infty) \) such that
\[
\begin{cases}
\psi(t) = 1, & t \in [0, 1], \\
0 \leq \psi(t) \leq 1, & t \in (1, 2), \\
\psi(t) = 0, & t \in [2, \infty), \\
|\psi'|_\infty \leq 2.
\end{cases}
\]

We define \( h_T(u) := \psi \left( \|u\|^2/T^2 \right) \) and the truncated functional \( J^T_\varepsilon : H_0^1(\Omega) \to \mathbb{R} \) given by
\[
J^T_\varepsilon(u) := \frac{1}{2} \|u\|^2 + h_T(u) \left[ \frac{3}{4} \int_\Omega |\nabla \phi_\varepsilon(u)|^2 + \frac{3\varepsilon^4}{8} \int_\Omega |\nabla \phi_\varepsilon(u)|^4 \right] - \int_\Omega F(u) = \frac{1}{2} \|u\|^2 + h_T(u) I_\varepsilon(u) - \int_\Omega F(u).
\]

The functional \( J^T_\varepsilon \) is \( C^1 \) with differential given, for all \( u, v \in H_0^1(\Omega) \), by
\[
(J^T_\varepsilon)'(u)[v] = \langle u, v \rangle + \frac{2}{T^2} \psi' \left( \frac{\|u\|^2}{T^2} \right) \langle u, v \rangle I_\varepsilon(u) + h_T(u) \int_\Omega \phi_\varepsilon(u) uv - \int_\Omega f(u)v.
\]

3.1. The Mountain Pass Geometry for \( J^T_\varepsilon \). The next two results deal with the Mountain Pass geometry for the functional \( J^T_\varepsilon \) where \( \varepsilon, T > 0 \).

We point out that the Mountain Pass structure of \( J^T_\varepsilon \) does not depend on \( \varepsilon \). In other words,
- \( \beta, \rho \) in Lemma 3 does not depend on \( \varepsilon \), neither on \( T \).
- \( e_T \) in Lemma 4 just depend on \( T \), and not on \( \varepsilon \).

The reason of this independence on \( \varepsilon \) is because the terms involving \( \varepsilon \) (that is \( I_\varepsilon \)) is suitably thrown away.

**Lemma 3.** Assume that conditions (1) and (2) hold. Then, there exists numbers \( \rho, \beta > 0 \) such that,
\[ \forall T > 0, \quad J^T_\varepsilon(u) \geq \beta, \quad \text{whenever } \|u\| = \rho. \]

**Proof.** Let \( \alpha > \alpha_0 \) and use (2.2) with \( q > 2 \): taking \( \delta > 0 \) sufficiently small there exists \( D_1 > 0 \) such that
\[ J^T_\varepsilon(u) \geq D_1 \|u\|^2 - \tilde{C}_\delta \int_\Omega |u|^q \exp(\alpha u^2). \]

Using Hölder’s inequality
\[
J^T_\varepsilon(u) \geq D_1 \|u\|^2 - \tilde{C}_\delta \left( \int_\Omega |u|^{2q} \right)^{1/2} \left( \int_\Omega \exp \left( 2\alpha \|u\|^2 \frac{u^2}{\|u\|^2} \right) \right)^{1/2}
\]
\[ \geq D_1 \|u\|^2 - D_2 \|u\|^q \left( \int_\Omega \exp \left( 2\alpha \|u\|^2 \frac{u^2}{\|u\|^2} \right) \right)^{1/2}. \]

Then we can choose \( \rho_1 = \|u\| > 0 \) small enough such that \( 2\alpha \rho_1^2 < 4\pi \), so that, by Proposition 1 we get
\[ J^T_\varepsilon(u) \geq D_1 \rho_1^2 - D_2 \rho_1^q, \]
for some \( D_2 > 0 \). Thus there exists \( \beta > 0 \) such that \( J^T_\varepsilon(u) \geq \beta > 0 \), for all \( 0 < \rho < \rho_1 \) which proves the Lemma.
Lemma 4. Assume that conditions (f4) hold. Then for every \( T > 0 \), there exists \( e_T \in H^1_0(\Omega) \) such that \( J^T_e(e_T) < 0 \) and \( \|e_T\| > \rho \),

where \( \rho \) is given in Lemma 3.

Proof. Let \( T > 0 \) be fixed. Let now \( v \in C_0^\infty(\Omega) \), positive, with \( \|v\| = 1 \). Using (f4) and considering \( t > 2T \), we get

\[ J^T_e(tv) \leq \frac{1}{2} t^2 - \tau \frac{t^r}{r} \int_\Omega v^r \]

Since \( 2 < r \), the result follows by choosing some \( t_* > 2T \) large enough and setting \( e_T := t_*v \). \( \square \)

4. Proof of Theorem 1

Since for every \( \varepsilon, T > 0 \) the functional \( J^T_\varepsilon \) satisfies the geometric assumptions of Mountain Pass Theorem (see [2]), we know that there exists a \((PS)\) sequence at this level, that is a sequence \( \{u_n\} \subset H^1_0(\Omega) \) satisfying

\[ J^T_\varepsilon(u_n) \to c^T_\varepsilon > 0 \quad \text{and} \quad (J^T_\varepsilon)'(u_n) \to 0, \]

where

\[ c^T_\varepsilon := \inf_{\gamma \in \Gamma^T_\varepsilon} \max_{t \in [0,1]} J^T_\varepsilon(\gamma(t)) > 0 \]

and

\[ \Gamma^T_\varepsilon := \{ \gamma \in C([0,1], H^1_0(\Omega)) : \gamma(0) = 0, \ J^T_\varepsilon(\gamma(1)) < 0 \}. \]

It is clear that this sequence \( \{u_n\} \) should depend also on \( \varepsilon \) and \( T \) but we omit this for simplicity.

Observe that there exists \( k > 0 \) such that \( 0 < k \leq c^T_\varepsilon \) for all \( \varepsilon, T \), by Lemma 3. Moreover since \( e_T \) found in Lemma 4 does not depends on \( \varepsilon \), by setting

\[ \gamma_* : t \in [0,1] \mapsto te_T \in H^1_0(\Omega) \]

we get \( \gamma_* \in \bigcap_{\varepsilon > 0} \Gamma^T_\varepsilon \).

Our next aim is to show that for a suitable choice of \( T > 0 \) (see Lemma 6) the \((PS)\) sequence \( \{u_n\} \) given above for \( J^T_\varepsilon \) at level \( c^T_\varepsilon \) is bounded and is actually a \((PS)\) sequence for the untruncated functional \( J_\varepsilon \) (see Lemma 7).

First few preliminaries are in order. It is well-known that, for every \( \varepsilon, T > 0 \) there is a unique \( t_{\varepsilon,T} > 0 \) such that \( c^T_\varepsilon \leq J^T_\varepsilon(t_{\varepsilon,T}u) \). Recall that \( u \) is the ground state of the auxiliary problem (A).

The important fact now is that there is a bound on \( t_{\varepsilon,T} \) independent on \( T \).

Lemma 5. There exists \( K_\varepsilon > 0 \), such that for every \( T > 0 \)

\[ t_{\varepsilon,T} \leq K_\varepsilon. \]

Proof. Since \((J^T_\varepsilon)'(t_{\varepsilon,T}u)|_{t_{\varepsilon,T}u} = 0 \) and \( h'(t_{\varepsilon,T}u) \leq 0 \), by (3.1) we easily get

\[ \int_\Omega f(t_{\varepsilon,T}u)t_{\varepsilon,T}u \leq t_{\varepsilon,T}^2 \|u\|^2 + h_T(t_{\varepsilon,T}u)I'_\varepsilon(t_{\varepsilon,T}u) \]

and then, by hypothesis (f4),

\[ t_{\varepsilon,T}^r\|u\|^2 \leq \tau t_{\varepsilon,T}^r \int_\Omega |u|^r \leq \int_\Omega f(t_{\varepsilon,T}u)t_{\varepsilon,T}u \leq t_{\varepsilon,T}^2 \|u\|^2 + h_T(t_{\varepsilon,T}u)I'_\varepsilon(t_{\varepsilon,T}u). \]

It follows that, if \( \lim_{T \to +\infty} t_{\varepsilon,T} = +\infty \), then for \( T \) larger and larger \( t_{\varepsilon,T}^r\|u\|^2 \leq t_{\varepsilon,T}^2 \|u\|^2 \) which is not possible, being \( r > 2 \). \( \square \)

Observe that all we have done up to now is true for every \( T > 0 \). Now we will choose a particular value of \( T \).

Lemma 6. Let \( K_\varepsilon \) be the value given in Lemma 5. For \( \overline{T}(\varepsilon) := K_\varepsilon\|u\|/2 \), the Mountain Pass value \( c^T_\varepsilon(\cdot) \) satisfies

\[ 0 < c^T_{\varepsilon(\overline{T})} \leq \frac{\pi(\theta - 2)}{\theta(\alpha_0 + 1)}. \]
Proof. Using (4.1) and once that \(h_\varepsilon(K_\varepsilon u) = \psi \left(\frac{K_\varepsilon^2\|u\|^2}{\mathcal{T}_\varepsilon}\right) = 0\), we obtain

\[c_\varepsilon^{\mathcal{T}(\varepsilon)} \leq J_\varepsilon^{\mathcal{T}(\varepsilon)}(t_\varepsilon, \mathcal{T}_\varepsilon u) = \frac{t_\varepsilon^2}{2}\|\mathcal{T}_\varepsilon u\|^2 - \tau_\varepsilon \mathcal{T}_\varepsilon u \int_\Omega |u|^r = \left(\frac{t_\varepsilon^2}{2} - \tau_\varepsilon \mathcal{T}_\varepsilon u\right) \int_\Omega |u|^r.
\]

Using (1.1), we have

\[c_\varepsilon^{\mathcal{T}(\varepsilon)} \leq \frac{2r\mathcal{T}_\varepsilon}{r-2} \max_{\xi \geq 0} \left\{ \frac{\xi^2}{2} - \frac{\tau_\varepsilon \mathcal{T}_\varepsilon u}{r}\right\} = \frac{m}{\tau^{2/(r-2)}} \leq \frac{\pi(\theta - 2)}{\theta(\alpha_0 + 1)},
\]

finishing the proof. \(\square\)

**Remark 1.** It is worth to point out that the bound on \(c_\varepsilon^{\mathcal{T}(\varepsilon)}\) does not depend on \(\varepsilon\).

From now on we will consider the truncated functional with the value of \(\mathcal{T}_\varepsilon\) given in the above Lemma. The reason is explained in the next

**Lemma 7.** Let \(\varepsilon > 0\) be fixed and let \(\{u_n\} \subset H^1_0(\Omega)\) be the (PS) sequence for the functional \(J_\varepsilon^{\mathcal{T}(\varepsilon)}\) at level \(c_\varepsilon^{\mathcal{T}(\varepsilon)}\) given above. Then,

\[
\limsup_{n \to \infty} \|u_n\|^2 \leq \min \left\{ \frac{2\pi}{\alpha_0 + 1} \frac{\mathcal{T}_\varepsilon}{2}, \frac{\mathcal{T}_\varepsilon}{2} \frac{\mathcal{T}_\varepsilon}{2} \right\}.
\]

As a consequence \(\|u_n\| < \mathcal{T}_\varepsilon\) and then \(\{u_n\}\) is also a (PS) sequence for the untruncated functional \(J_\varepsilon\) at level \(c_\varepsilon^{\mathcal{T}(\varepsilon)} > 0\).

**Proof.** Using the fact that \(\theta > 4\) we get

\[
c_\varepsilon^{\mathcal{T}(\varepsilon)} = J_\varepsilon^{\mathcal{T}(\varepsilon)}(u_n) - \frac{1}{\theta}(J_\varepsilon^{\mathcal{T}(\varepsilon)})'(u_n)|u_n| + o_n(1)
\]

\[\geq \frac{\theta - 2}{2\theta} \|u_n\|^2 + o_n(1).
\]

By Lemma 6 we deduce

\[\|u_n\|^2 \leq \frac{2\theta}{\theta - 2} \frac{m}{r^{2/(r-2)}} + o_n(1).
\]

Since \(\tau \geq \tau^*(\varepsilon)\) in (4.1), then few computations show that

\[\|u_n\|^2 \leq \min \left\{ \frac{2\pi}{\alpha_0 + 1} \frac{\mathcal{T}_\varepsilon}{2}, \frac{\mathcal{T}_\varepsilon}{2} \right\} + o_n(1)
\]

and the first part holds. Since \(\|u_n\| \leq \mathcal{T}_\varepsilon/\sqrt{2} < \mathcal{T}_\varepsilon\), the Lemma is completely proved. \(\square\)

In view of the previous Lemma, there exists \(u_\varepsilon \in H^1_0(\Omega)\) such that \(u_n \rightharpoonup u_\varepsilon\) in \(H^1_0(\Omega)\). In particular we have a bound on \(\|u_\varepsilon\|\) independent of \(\varepsilon\); this fact will be used in Section 5.

The next result deal with the convergence of the nonlinear term \(f\).

**Lemma 8.** The (PS) sequence \(\{u_n\}\) for the functional \(J_\varepsilon\) at the level \(c_\varepsilon^{\mathcal{T}(\varepsilon)}\) is such that

\[
\int_\Omega f(u_n)u_n \to \int_\Omega f(u_\varepsilon)u_\varepsilon, \quad \int_\Omega f(u_n)u_\varepsilon \to \int_\Omega f(u_\varepsilon)u_\varepsilon.
\]

**Proof.** Let us prove just the first limit since the second one is similar.

We can assume that

\[
u_n \rightharpoonup u_\varepsilon \quad \text{in} \quad H^1_0(\Omega),
\]

\[u_n \to u_\varepsilon \quad \text{in} \quad L^p(\Omega), \quad p \geq 1,
\]

\[u_n(x) \to u_\varepsilon(x) \quad \text{a.e. in} \quad \Omega
\]

and \(f(u_n(x))u_n(x) \to f(u_\varepsilon(x))u_\varepsilon(x)\) a.e. in \(\Omega\). We apply inequality (2.1) with \(q = 1\) and \(a = \alpha_0 + 1\): for \(\delta > 0\), there exists \(C_\delta > 0\) such that

\[
f(u_n(x))u_n(x) \leq \delta u_n^2(x) + C_\delta |u_n(x)| \exp \left((\alpha_0 + 1)|u_n(x)|\right).
\]
If we show that
\begin{equation}
\exists h \in L^1(\Omega): |f(u_n(x))u_n(x)| \leq h(x) \quad \text{a.e. } x \in \Omega,
\end{equation}
then by the Dominated Convergece Theorem we obtain the first limit in the Lemma. So let us estimate both terms in the right hand side of (4.1).

Clearly \( \{u_n^2\} \) converges in \( L^1(\Omega) \), then up to subsequences,
\begin{equation}
\exists h_1 \in L^1(\Omega): u_n^2(x) \leq h_1(x) \quad \text{a.e. } x \in \Omega.
\end{equation}

Let us bound now \( g_n := |u_n|\exp((\alpha_0 + 1)u_n^2) \) by some \( h_2 \in L^1(\Omega) \). Of course
\begin{equation}
g_n(x) \to |u_\varepsilon(x)|\exp((\alpha_0 + 1)u_\varepsilon^2(x)) \quad \text{a.e. } x \in \Omega.
\end{equation}

Now by Lemma 7, by choosing \( p \in (1,2) \) we have
\[ \limsup_{n \to \infty} \|u_n\|^2 \leq \frac{2\pi}{\alpha_0 + 1} < \frac{4\pi}{p(\alpha_0 + 1)} \]
and then we conclude, by Proposition 1, that
\[ \int_{\Omega} \exp\left(p(\alpha_0 + 1)u_n^2\right) = \int_{\Omega} \exp\left(p(\alpha_0 + 1)\|u_n\|^2 \frac{u_n^2}{\|u_n\|^2}\right) \leq C \]
where \( C \) does not depend on \( n \). Since \( \exp((\alpha_0 + 1)u_n^2(x)) \to \exp((\alpha_0 + 1)u_\varepsilon^2(x)) \) a.e. in \( \Omega \) we infer
\begin{equation}
\exp((\alpha_0 + 1)u_n^2) \to \exp((\alpha_0 + 1)u_\varepsilon^2) \quad \text{in } L^p(\Omega),
\end{equation}
see e.g. and [9, Lemma 4.8]. Of course it is also
\begin{equation}
|u_n| \to |u_\varepsilon| \quad \text{in } L^p'(\Omega), \quad \text{where } p^{-1} + p'^{-1} = 1
\end{equation}
and then by (4.5) and (4.6)
\[ \int_{\Omega} g_n = \int_{\Omega} |u_n|\exp((\alpha_0 + 1)u_n^2) \to \int_{\Omega} |u_\varepsilon|\exp((\alpha_0 + 1)u_\varepsilon^2). \]

But then by (4.4), we can invoke the the Brezis-Lieb Lemma and deduce that \( g_n \to |u_\varepsilon(x)|\exp((\alpha_0 + 1)u_\varepsilon^2(x)) \) in \( L^1(\Omega) \), so that (possibly passing to a subsequence)
\begin{equation}
\exists h_2 \in L^1(\Omega): g_n(x) = |u_n(x)|\exp((\alpha_0 + 1)u_n^2(x)) \leq h_2(x).
\end{equation}

Then by (4.3) and (4.7) we deduce (4.2). \( \square \)

Now we can conclude the proof of Theorem 1.
Since \( J'_\varepsilon(u_n)[u_n] = o_\varepsilon(1) \) and \( J''_\varepsilon(u_n)[u_n] = o_\varepsilon(1) \), we have
\[ o_\varepsilon(1) = \|u_n\|^2 + \int_{\Omega} \phi_\varepsilon(u_n)u_n^2 - \int_{\Omega} f(u_n)u_n - \langle u_n, u_\varepsilon \rangle - \int_{\Omega} \phi_\varepsilon(u_n)u_n u_\varepsilon + \int_{\Omega} f(u_n)u_\varepsilon. \]
Then by the fact that \( u_n \to u_\varepsilon \) in \( H_0^1(\Omega) \), by Lemma 2 items (c), (e), with \( w := u_\varepsilon \), and Lemma 8 we conclude that
\[ \|u_n\| \to \|u_\varepsilon\| \]
and then \( u_n \to u_\varepsilon \) in \( H_0^1(\Omega) \).

Then we deduce that \( u_\varepsilon \) is a critical point of \( J_\varepsilon \) at level \( \mathcal{T}_\varepsilon(c) \) and then setting \( \phi_\varepsilon := \phi_\varepsilon(u_\varepsilon) \), we have that \( (u_\varepsilon, \phi_\varepsilon) \) is a solution of \( (P_\varepsilon) \).

Moreover is easy to see that \( \phi_\varepsilon(u_\varepsilon) \geq 0 \): this is achieved by multiplying the second equation in \( (P_\varepsilon) \) by \( \phi_\varepsilon(u_\varepsilon)^- \), its negative part, and integrating. Then arguing similarly for the equation
\[ -\Delta u_\varepsilon + \phi_\varepsilon(u_\varepsilon) = f(u_\varepsilon) \]
we see that \( u_\varepsilon \geq 0 \) and the proof of Theorem 1 is concluded.
5. Proof of Theorem 2

All the limits in this section are taken as \( \varepsilon \to 0^+ \). We denote also with \( o_\varepsilon(1) \) a quantity which tends to zero as \( \varepsilon \to 0^+ \).

Hence let \( \{ u_\varepsilon, \phi_\varepsilon \}_{\varepsilon > 0} \) be solutions of

\[
\begin{align*}
-\Delta u + \phi u &= f(u) \quad \text{in } \Omega, \\
-\Delta \phi - \varepsilon^4 \Delta_1 \phi &= u^2 \quad \text{in } \Omega, \\
u &= \phi = 0 \quad \text{on } \partial \Omega
\end{align*}
\]

where \( f \) satisfies just (f0)-(f2). We know that \( u_\varepsilon \) is a critical point of

\[
J_\varepsilon(u) = \frac{1}{2} \|u\|^2 + I_\varepsilon(u) - \int_{\Omega} F(u)
\]

and by assumptions \( \|u_\varepsilon\|^2 < 2\pi/(\alpha_0 + 1) \). Then there exists \( u_0 \) such that \( u_\varepsilon \to u_0 \) in \( H^1_0(\Omega) \) as \( \varepsilon \to 0^+ \).

Let us show this convergence is strong. Denote with \( \phi_0(u_0) \in H^1_0(\Omega) \) the unique solution of

\[
\begin{align*}
-\Delta \phi &= u_0^2 \quad \text{in } \Omega, \\
\phi &= 0 \quad \text{on } \partial \Omega.
\end{align*}
\]

We need now the following

**Lemma 9.** It holds

\[
\lim_{\varepsilon \to 0^+} \phi_\varepsilon(u_\varepsilon) = \phi_0(u_0) \quad \text{in } H^1_0(\Omega) \quad \text{and} \quad \lim_{\varepsilon \to 0^+} \varepsilon \phi_\varepsilon(u_\varepsilon) = 0 \quad \text{in } W^{1,4}_0(\Omega).
\]

**Proof.** It is done exactly as in [4, Lemma 3.2]), observing that we have convergence \( u_\varepsilon \to u_0 \) in \( L^p(\Omega) \) for \( p \geq 1 \). \( \square \)

As at the end of the previous Section, by combining the identities \( J'_\varepsilon(u_\varepsilon)[u_\varepsilon] = 0 \) and \( J'_\varepsilon(u_\varepsilon)[u_0] = 0 \) we deduce

(5.1) \( 0 = \|u_\varepsilon\|^2 + \int_{\Omega} \phi_\varepsilon(u_\varepsilon) u_\varepsilon^2 - \int_{\Omega} f(u_\varepsilon) u_\varepsilon - \langle u_\varepsilon, u_0 \rangle - \int_{\Omega} \phi_\varepsilon(u_\varepsilon) u_\varepsilon u_0 + \int_{\Omega} f(u_\varepsilon) u_0. \)

Observe that

\[
\left| \int_{\Omega} \phi_\varepsilon(u_\varepsilon) u_\varepsilon^2 - \int_{\Omega} \phi_\varepsilon(u_\varepsilon) u_\varepsilon u_0 \right| \leq |\phi_\varepsilon(u_\varepsilon)||u_\varepsilon|^3 |u_\varepsilon - u_0|^3 = o_\varepsilon(1)
\]

and as in Lemma 8, simply using the fact that \( \|u_\varepsilon\| \leq 2\pi/(\alpha_0 + 1) \),

\[
\left| \int_{\Omega} f(u_\varepsilon) u_\varepsilon - \int_{\Omega} f(u_\varepsilon) u_0 \right| = o_\varepsilon(1).
\]

Then from (5.1) we deduce \( \|u_\varepsilon\| \to \|u_0\| \) and so

(5.2) \( \lim_{\varepsilon \to 0^+} u_\varepsilon = u_0 \quad \text{in } H^1_0(\Omega). \)

Moreover

\[
0 = J'_\varepsilon(u_\varepsilon)[u_\varepsilon] \geq \|u_\varepsilon\|^2 - \int_{\Omega} f(u_\varepsilon) u_\varepsilon \geq \|u_\varepsilon\|^2 - \tau \int_{\Omega} |u_\varepsilon| r \geq \|u_\varepsilon\|^2 - C \|u_\varepsilon\|^r,
\]

which implies (being \( r > 2 \)) that there exists a constant \( h > 0 \) such that,

\[
\forall \varepsilon > 0 : \quad 0 < h \leq \|u_\varepsilon\|.
\]

In particular \( u_0 \neq 0 \) and then also \( \phi_0(u_0) \neq 0 \); moreover from Lemma 7, \( \|u_\varepsilon\| < \mathcal{T}(\varepsilon) \) and then \( \mathcal{T}(\varepsilon) \not\to 0 \) as \( \varepsilon \to 0^+ \).

Finally, from \( J'_\varepsilon(u_\varepsilon) = 0 \) we get

(5.3) \( \forall v \in H^1_0(\Omega) : \int_{\Omega} \nabla u_\varepsilon \nabla v + \int_{\Omega} \phi_\varepsilon(u_\varepsilon) u_\varepsilon v = \int_{\Omega} f(u_\varepsilon) v. \)
We want to pass to the limit in $\varepsilon$ in the identity above. Since, by Lemma 9, $\phi_\varepsilon(u_\varepsilon) \to \phi_0(u_0)$ in $L^3(\Omega)$, $u_\varepsilon \to u_0$ in $L^3(\Omega)$ and $v \in L^3(\Omega)$, by the Hölder inequality we have

\begin{equation}
(5.4) \lim_{\varepsilon \to 0^+} \int_\Omega \phi_\varepsilon(u_\varepsilon)u_\varepsilon v = \int_\Omega \phi_0(u_0)u_0 v.
\end{equation}

On the other hand, again as in Lemma 8,

\begin{equation}
(5.5) \lim_{\varepsilon \to 0^+} \int_\Omega f(u_\varepsilon)v = \int_\Omega f(u_0)v.
\end{equation}

Then by (5.3)-(5.5), we get

$$\forall v \in H^1_0(\Omega) : \int_\Omega \nabla u_0 \nabla v + \int_\Omega \phi_0(u_0)u_0 v = \int_\Omega f(u_0)v,$$

showing that the pair $u_0, \phi_0 := \phi_0(u_0)$ solves the Schrödinger-Poisson system. Then in view of the first limit in Lemma 9 and (5.2), Theorem 2 is completely proved.

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