A GENERALIZATION OF THE CAPELLI IDENTITY

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Abstract. We prove a generalization of the Capelli identity. As an application we obtain an isomorphism of the Bethe subalgebras actions under the \((\mathfrak{gl}_N, \mathfrak{gl}_M)\) duality.

1. Introduction

Let \(A\) be an associative algebra over complex numbers. Let \(A = (a_{ij})_{i,j=1}^n\) be an \(n \times n\) matrix with entries in \(A\). The row determinant of \(A\) is defined by the formula:

\[
\text{rdet}(A) := \sum_{\sigma \in S_n} \text{sgn}(\sigma)a_{1\sigma_1} \ldots a_{n\sigma_n}.
\]

Let \(x_{ij}, i, j = 1, \ldots, M\), be commuting variables. Let \(\partial_{ij} = \partial/\partial x_{ij}\),

\[
E_{ij} = \sum_{a=1}^M x_{ia}\partial_{ja}. \tag{1.1}
\]

Let \(X = (x_{ij})_{i,j=1}^M\) and \(D = (\partial_{ij})_{i,j=1}^M\) be \(M \times M\) matrices.

The classical Capelli identity \([C1]\) asserts the following equality of differential operators:

\[
\text{rdet} \left( E_{ji} + (M - i)\delta_{ij} \right)_{i,j=1}^M = \det(X)\det(D). \tag{1.2}
\]

This identity is a “quantization” of the identity

\[
\det(AB) = \det(A)\det(B)
\]

for any matrices \(A, B\) with commuting entries.

The Capelli identity has the following meaning in the representation theory. Let \(\mathbb{C}[X]\) be the algebra of complex polynomials in variables \(x_{ij}\). There are two natural actions of the Lie algebra \(\mathfrak{gl}_M\) on \(\mathbb{C}[X]\). The first action is given by operators from (1.1) and the second action is given by operators \(\tilde{E}_{ij} = \sum_{a=1}^M x_{ai}\partial_{aj}\). The two actions commute and the corresponding \(\mathfrak{gl}_M \oplus \mathfrak{gl}_M\) action is multiplicity free.

It is not difficult to see that the right hand side of (1.2), considered as a differential operator on \(\mathbb{C}[X]\), commutes with both actions of \(\mathfrak{gl}_M\) and therefore lies in the image of the center of the universal enveloping algebra \(U\mathfrak{gl}_M\) with respect to the first action. Then the

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left hand side of the Capelli identity expresses the corresponding central element in terms of $U\mathfrak{gl}_M$ generators.

Many generalizations of the Capelli identity are known. One group of generalizations considers other elements of the center of $U\mathfrak{gl}_M$, called quantum immanants, and then expresses them in terms of $\mathfrak{gl}_M$ generators, see [C2], [N1], [O]. Another group of generalizations considers other pairs of Lie algebras in place of $(\mathfrak{gl}_M, \mathfrak{gl}_M)$, e.g. $(\mathfrak{gl}_M, \mathfrak{gl}_N)$, $(\mathfrak{sp}_{2M}, \mathfrak{gl}_2)$, $(\mathfrak{sp}_{2M}, \mathfrak{so}_N)$, etc, see [MN], [HU]. The third group of generalizations produces identities corresponding not to pairs of Lie algebras, but to pairs of quantum groups [NUW] or superalgebras [N2].

In this paper we prove a generalization of the Capelli identity which seemingly does not fit the above classification.

Let $z = (z_1, \ldots, z_N)$, $\lambda = (\lambda_1, \ldots, \lambda_M)$ be sequences of complex numbers. Let $Z = (z_i\delta_{ij})_{ij=1}^N$, $\Lambda = (\lambda_i\delta_{ij})_{ij=1}^M$ be the corresponding diagonal matrices. Let $X$ and $D$ be the $M \times N$ matrices with entries $x_{ij}$ and $\partial_{ij}$, $i = 1, \ldots, M$, $j = 1, \ldots, N$, respectively. Let $\mathbb{C}[X]$ be the algebra of complex polynomials in variables $x_{ij}$, $i = 1, \ldots, M$, $j = 1, \ldots, N$. Let $E^{(a)}_{ij} = x_{ia}\partial_{ja}$, where $i, j = 1, \ldots, M$, $a = 1, \ldots, N$.

In this paper we prove that

$$\prod_{a=1}^N (u - z_a) \text{ rdet} \left( (\partial_u - \lambda_i)\delta_{ij} - \sum_{a=1}^N E^{(a)}_{ji} \right)^M = \text{ rdet} \left( u - Z X^t D - \partial_u - \Lambda \right).$$

(1.3)

The left hand side of (1.3) is an $M \times M$ matrix while the right hand side is an $(M + N) \times (M + N)$ matrix.

Identity (1.3) is a “quantization” of the identity

$$\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(A) \det(D - CA^{-1}B)$$

which holds for any matrices $A, B, C, D$ with commuting entries, for the case when $A$ and $D$ are diagonal matrices.

By setting all $z_i, \lambda_j$ and $u$ to zero, and $N = M$ in (1.3), we obtain the classical Capelli identity (1.2), see Section 2.4.

Our proof of (1.3) is combinatorial and reduces to the case of $2 \times 2$ matrices. In particular, it gives a proof of the classical Capelli identity, which may be new.

We invented identity (1.3) to prove Theorem 3.1 below, and Theorem 3.1 in its turn was motivated by results of [MTV2]. In Theorem 3.1 we compare actions of two Bethe subalgebras.

Namely, consider $\mathbb{C}[X]$ as a tensor product of evaluation modules over the current Lie algebras $\mathfrak{gl}_M[t]$ and $\mathfrak{gl}_N[t]$ with evaluation parameters $z$ and $\lambda$, respectively. The action of the algebra $\mathfrak{gl}_M[t]$ on $\mathbb{C}[X]$ is given by the formula

$$E_{ij} \otimes t^n = \sum_{a=1}^N x_{ia}\partial_{ja}z_i^n,$$
and the action of the algebra $\mathfrak{gl}_N[t]$ on $\mathbb{C}[X]$ is given by the formula

$$E_{ij} \otimes t^n = \sum_{a=1}^{M} x_{ai} \partial_{aj} \lambda^n_i.$$ 

In contrast to the previous situation, these two actions do not commute.

The algebra $U\mathfrak{gl}_M[t]$ has a family of commutative subalgebras $\mathcal{G}(M, \lambda)$ depending on parameters $\lambda$ and called the Bethe subalgebras. For a given $\lambda$, the Bethe subalgebra $\mathcal{G}(M, \lambda)$ is generated by the coefficients of the expansion of the expression

$$\text{rdet} \left( (\partial_u - \lambda_i) \delta_{ij} - \sum_{a=1}^{N} \sum_{s=1}^{\infty} (E^{(a)}_{ji} \otimes t^s) u^{-s-1} \right)_{i,j=1}^{M}$$

with respect to powers of $u$ and $\partial_u$, cf. Section 3. For different versions of definitions of Bethe subalgebras and relations between them, see [FFR], [T], [R], [MTV 1].

Similarly, there is a family of Bethe subalgebras $\mathcal{G}(N, z)$ in $U\mathfrak{gl}_N[t]$ depending on parameters $z$.

For fixed $\lambda$ and $z$, consider the action of the Bethe subalgebras $\mathcal{G}(M, \lambda)$ and $\mathcal{G}(N, z)$ on $\mathbb{C}[X]$ as defined above. In Theorem 3.1 we show that the actions of the Bethe subalgebras on $\mathbb{C}[X]$ induce the same subalgebras of endomorphisms of $\mathbb{C}[X]$.

The paper is organized as follows. In Section 2 we describe and prove formal Capelli-type identities and in Section 3 we discuss the relations of the identities to the Bethe subalgebras.

## 2. Identities

### 2.1. The main identity

We work over the field of complex numbers, however all results of this paper hold over any field of characteristic zero.

Let $\mathcal{A}$ be an associative algebra. Let $A = (a_{ij})_{i,j=1}^{n}$ be an $n \times n$ matrix with entries in $\mathcal{A}$. Define the row determinant of $A$ by the formula:

$$\text{rdet}(A) := \sum_{\sigma \in S_n} \text{sgn}(\sigma)a_{1\sigma_1} \ldots a_{n\sigma_n},$$

where $S_n$ is the symmetric group on $n$ elements.

Fix two natural numbers $M$ and $N$ and a complex number $h \in \mathbb{C}$. Consider noncommuting variables $u, p_u, x_{ij}, p_{ij}$, where $i = 1, \ldots, M, j = 1, \ldots, N$, such that the commutator of two variables equals zero except

$$[p_u, u] = h, \quad [p_{ij}, x_{ij}] = h, \quad i = 1, \ldots, M, j = 1, \ldots, N.$$

Let $X, P$ be two $M \times N$ matrices given by

$$X := (x_{ij})_{i=1}^{M}^{j=1}^{N}, \quad P := (p_{ij})_{i=1}^{M}^{j=1}^{N}.$$

Let $\mathcal{A}^{(MN)}_h$ be the associative algebra whose elements are polynomials in $p_u, x_{ij}, p_{ij}$, $i = 1, \ldots, M, j = 1, \ldots, N$, with coefficients that are rational functions in $u$. 


Let $\mathcal{A}^{(MN)}$ be the associative algebra of linear differential operators in $u, x_{ij}, i = 1, \ldots, M, j = 1, \ldots, N$, with coefficients in $\mathbb{C}(u) \otimes \mathbb{C}[X]$.

We often drop the dependence on $M, N$ and write $\mathcal{A}_h, \mathcal{A}$ for $\mathcal{A}_h^{(MN)}$ and $\mathcal{A}^{(MN)}$, respectively. For $h \neq 0$, we have the isomorphism of algebras

$$\iota_h : \mathcal{A}_h \to \mathcal{A},$$

$$u, x_{ij} \mapsto u, x_{ij},$$

$$p_u, p_{ij} \mapsto h \frac{\partial}{\partial u}, h \frac{\partial}{\partial x_{ij}}.$$

Fix two sequences of complex numbers $z = (z_1, \ldots, z_N)$ and $\lambda = (\lambda_1, \ldots, \lambda_M)$. Define the $M \times M$ matrix $G_h = G_h(M, N, u, p_u, z, \lambda, X, P)$ by the formula

$$G_h := \left( (p_u - \lambda_i) \delta_{ij} - \sum_{a=1}^{N} x_{ja}p_{ia} \right)_{i,j=1}^M.$$

**Theorem 2.1.** We have

$$\prod_{a=1}^{N} (u - z_a) \ rdet(G_h) = \sum_{A,B,|A|=|B|} (-1)^{|A|} \prod_{a \notin B} (u - z_a) \prod_{b \in B} (p_u - \lambda_b) \ det(x_{ab})_{a \in A}^{b \in B} \ det(p_{ab})_{a \in A}^{b \in B},$$

where the sum is over all pairs of subsets $A \subset \{1, \ldots, M\}, B \subset \{1, \ldots, N\}$ such that $A$ and $B$ have the same cardinality, $|A| = |B|$. Here the sets $A, B$ inherit the natural ordering from the sets $\{1, \ldots, M\}, \{1, \ldots, N\}$. This ordering determines the determinants in the formula.

Theorem 2.1 is proved in Section 2.5.

2.2. A presentation as a row determinant of size $M + N$. Theorem 2.1 implies that the row determinant of $G$ can be written as the row determinant of a matrix of size $M + N$.

Namely, let $Z$ be the diagonal $N \times N$ matrix with diagonal entries $z_1, \ldots, z_N$. Let $\Lambda$ be the diagonal $M \times M$ matrix with diagonal entries $\lambda_1, \ldots, \lambda_M$:

$$Z := (z_i \delta_{ij})_{i,j=1}^N, \quad \Lambda := (\lambda_i \delta_{ij})_{i,j=1}^M.$$

**Corollary 2.2.** We have

$$\prod_{a=1}^{N} (u - z_a) \ rdet \ G = \ rdet \left( u - Z \begin{pmatrix} X^t & X \\ P & P_u - \Lambda \end{pmatrix} \right),$$

where $X^t$ denotes the transpose of the matrix $X$.

**Proof.** Denote

$$W := \left( u - Z \begin{pmatrix} X^t & X \\ P & P_u - \Lambda \end{pmatrix} \right),$$

The entries of the first $N$ rows of $W$ commute. The entries of the last $M$ rows of $W$ also commute. Write the Laplace decomposition of $\text{rdet}(W)$ with respect to the first $N$ rows.
Each term in this decomposition corresponds to a choice of $N$ columns in the $N \times (N + M)$ matrix $(u - Z, X^T)$. We label such a choice by a pair of subsets $A \subset \{1, \ldots, M\}$ and $B \subset \{1, \ldots, N\}$ of the same cardinality. Namely, the elements of $A$ correspond to the chosen columns in $X^T$ and the elements of the complement to $B$ correspond to the chosen columns in $u - Z$. Then the term in the Laplace decomposition corresponding to $A$ and $B$ is exactly the term labeled by $A$ and $B$ in the right hand side of the formula in Theorem 2.1. Therefore, the corollary follows from Theorem 2.1. □

Let $A, B, C, D$ be any matrices with commuting entries of sizes $N \times N, N \times M, M \times N$ and $M \times M$, respectively. Let $A$ be invertible. Then we have the equality of matrices of sizes $(M + N) \times (M + N)$:

$$
\begin{pmatrix}
A & B \\
C & D
\end{pmatrix} = \begin{pmatrix}
A & 0 \\
C & D - CA^{-1}B
\end{pmatrix} \begin{pmatrix}
1 & A^{-1}B \\
0 & 1
\end{pmatrix}
$$

and therefore

$$
\det \begin{pmatrix}
A & B \\
C & D
\end{pmatrix} = \det(A) \det(D - CA^{-1}B) .
$$

The identity of Corollary 2.2 for $h = 0$ turns into identity (2.3) with diagonal matrices $A$ and $D$. Therefore, the identity of Corollary 2.2 may be thought of as a “quantization” of identity (2.3) with diagonal $A$ and $D$.

### 2.3. A relation between determinants of sizes $M$ and $N$.

Introduce new variables $v, p_v$ such that $[p_v, v] = h$.

Let $\tilde{A}_h$ be the associative algebra whose elements are polynomials in $p_u, p_v, x_{ij}, p_{ij}, i = 1, \ldots, M, j = 1, \ldots, N$, with coefficients in $\mathbb{C}(u) \otimes \mathbb{C}(v)$.

Let $e : \tilde{A}_h \to \tilde{A}_h$ be the unique linear map which is the identity map on the subalgebra of $\tilde{A}_h$ generated by all monomials which do not contain $p_u$ and $p_v$, and which satisfy

$$
e(a p_u) = e(a) v, \quad e(a p_v) = e(a) u,
$$

for any $a \in \tilde{A}_h$.

Let $\tilde{A}$ be the associative algebra of linear differential operators in $u, v, x_{ij}, i = 1, \ldots, M, j = 1, \ldots, N$, with coefficients in $\mathbb{C}(u) \otimes \mathbb{C}(v) \otimes \mathbb{C}[x_{ij}]$. Then for $h \neq 0$, we have the isomorphism of algebras extending the isomorphism (2.1):

$$
\tilde{\iota}_h : \tilde{A}_h \to \tilde{A},
$$

$$
u, v, x_{ij} \mapsto u, v, x_{ij},
$$

$$
p_u, p_v, p_{ij} \mapsto h \frac{\partial}{\partial u}, h \frac{\partial}{\partial v}, h \frac{\partial}{\partial x_{ij}} .
$$

For $a \in \tilde{A}$ and a function $f(u, v)$ let $a \cdot f(u, v)$ denotes the function obtained by the action of $a$ considered as a differential operator in $u$ and $v$ on the function $f(u, v)$.

We have

$$
\tilde{\iota}_h(e(a)) = \exp(-uv/h)\tilde{\iota}_h(a) \cdot \exp(uv/h)
$$
for any \( a \in \bar{A}_h \) such that \( a \) does not depend on either \( p_u \) or \( p_v \).

Define the \( N \times N \) matrix \( H_h = H_h(M, N, v, p_v, \lambda, X, P) \) by

\[
H_h := \left( (p_v - z_i) \delta_{ij} - \sum_{b=1}^{M} \frac{x_{bji}p_b}{v - \lambda_h} \right)_{i,j=1}^N, \tag{2.4}
\]
cf. formula (2.2).

**Corollary 2.3.** We have

\[
e \left( \prod_{a=1}^{N} (u - z_a) \text{rdet}(G_h) \right) = e \left( \prod_{b=1}^{M} (v - \lambda_h) \text{rdet}(H_h) \right).
\]

**Proof.** Write the dependence on parameters of the matrix \( G_h \):

\[G_h = G_h(M, N, u, p_u, z, \lambda, X, P)\]

Then

\[H_h = G_h(N, M, v, p_v, \lambda, z, X^T, P^T)\]

The corollary now follows from Theorem 2.1. \( \square \)

2.4. **A relation to the Capelli identity.** In this section we show how to deduce the Capelli identity from Theorem 2.1.

Let \( s \) be a complex number. Let \( \alpha_s : A_h \to A_h \) be the unique linear map which is the identity map on the subalgebra of \( A_h \) generated by all monomials which do not contain \( p_u \), and which satisfies

\[
\alpha_s(a p_u) = s \alpha_s(a)
\]

for any \( a \in \bar{A}_h \).

We have

\[
i_h(\alpha_s(a)) = u^{-s/h} i_h(a) \cdot u^{s/h}
\]

for any \( a \in \bar{A}_h \).

Consider the case \( z_1 = \cdots = z_N = 0 \) and \( \lambda_1 = \cdots = \lambda_M = 0 \) in Theorem 2.1.

Then it is easy to see that the row determinant \( \text{rdet}(G) \) can be rewritten in the following form

\[
u^M \text{rdet}(G_h) = \text{rdet} \left( h(u p_u - M + i) \delta_{ij} - \sum_{a=1}^{N} x_{ja} p_{ia} \right)_{i,j=1}^M.
\]

Applying the map \( \alpha_s \), we get

\[
\alpha_s(u^M \text{rdet}(G_h)) = \text{rdet} \left( h(s - M + i) \delta_{ij} - \sum_{a=1}^{N} x_{ja} p_{ia} \right)_{i,j=1}^M.
\]

Therefore applying Theorem 2.1 we obtain the identity

\[
\text{rdet} \left( h(s - M + i) \delta_{ij} - \sum_{a=1}^{N} x_{ja} p_{ia} \right)_{i,j=1}^M = \sum_{A,B} (-1)^{|A|} \prod_{b=0}^{M-|A|-1} (s - bh) \det(x_{ab})_{a \in A} \det(p_{ab})_{a \in A}.
\]
In particular, if \( M = N \), and \( s = 0 \), we obtain the famous Capelli identity:

\[
\text{rdet} \left( \sum_{a=1}^{M} x_{ja} p_{ia} + h(M-i) \delta_{ij} \right)_{i,j=1}^{M} = \det X \det P.
\]

If \( h = 0 \) then all entries of \( X \) and \( P \) commute and the Capelli identity reads \( \det(XP) = \det(X)\det(P) \). Therefore, the Capelli identity can be thought of as a “quantization” of the identity \( \det(AB) = \det(A)\det(B) \) for square matrices \( A, B \) with commuting entries.

2.5. Proof of Theorem 2.1. We denote

\[
E_{ij,a} := x_{ja} p_{ia} / (u - z_a).
\]

We obviously have

\[
[E_{ij,a}, E_{kl,b}] = \delta_{ab} (\delta_{kj} (E_{il,a})' - \delta_{il} (E_{kj,a})'),
\]

where the prime denotes the formal differentiation with respect to \( u \).

Denote also \( F_{jk,a}^1 = -E_{jk,a} \) and \( F_{jj,0}^0 = (p_u - \lambda_j) \).

Expand \( \text{rdet}(G) \). We get an alternating sum of terms,

\[
\text{rdet}(G_h) = \sum_{\sigma,a,c} (-1)^{\text{sgn}(\sigma)} F_{1\sigma(1),a(1)} F_{2\sigma(2),a(2)} \cdots F_{M\sigma(M),a(M)},
\]

where the summation is over all triples \( \sigma, a, c \) such that \( \sigma \) is a permutation of \( \{1, \ldots, M\} \) and \( a, c \) are maps \( a : \{1, \ldots, M\} \to \{0,1, \ldots, N\}, c : \{1, \ldots, M\} \to \{0,1\} \) satisfying: \( c(i) = 1 \) if \( \sigma(i) \neq i \), \( a(i) = 0 \) if and only if \( c(i) = 0 \).

Let \( m \) be a product whose factors are of the form \( f(u), p_u, p_{ij}, x_{ij} \) where \( f(u) \) are some rational functions in \( u \). Then the product \( m \) will be called normally ordered if all factors of the form \( p_u, p_{ij} \) are on the right from all factors of the form \( f(u), x_{ij} \). For example, \((u - 1)^{-2} x_{11} p_{u} p_{11} \) is normally ordered and \( p_u (u - 1)^{-2} x_{11} p_{11} \) is not.

Given a product \( m \) as above, define a new normally ordered product \( : m : \) as the product of all factors of \( m \) in which all factors of the form \( p_u, p_{ij} \) are placed on the right from all factors of the form \( f(u), x_{ij} \). For example, \( : p_u (u - 1)^{-2} x_{11} p_{11} : = (u - 1)^{-2} x_{11} p_u p_{11} \).

If all variables \( p_u, p_{ij} \) are moved to the right in the expansion of \( \text{rdet}(G) \) then we get terms obtained by normal ordering from the terms in \((2.5)\) plus new terms created by the non-trivial commutators. We show that in fact all new terms cancel in pairs.

Lemma 2.4. For \( i = 1, \ldots, M \), we have

\[
\text{rdet}(G_h) = \sum_{\sigma,a,c} (-1)^{\text{sgn}(\sigma)} F_{1\sigma(1),a(1)}^{c(1)} \cdots F_{(i-1)\sigma(i-1),a(i-1)}^{c(i-1)} \left( F_{\sigma(i),a(i)}^{c(i)} \cdots F_{M\sigma(M),a(M)}^{c(M)} \right),
\]

where the sum is over the same triples \( \sigma, a, c \) as in \((2.5)\).

Proof. We prove the lemma by induction on \( i \). For \( i = M \) the lemma is a tautology. Assume it is proved for \( i = M, M - 1, \ldots, j \), let us prove it for \( i = j - 1 \).
We have
\[ F_{(j-1)r,a}^1 : \mathcal{F}^{c(j)}_{j\sigma(j),a(j)} \cdots \mathcal{F}^{c(M)}_{M\sigma(M),a(M)} := \]
\[ : F_{(j-1)r,a}^1 : \mathcal{F}^{c(j)}_{j\sigma(j),a(j)} \cdots \mathcal{F}^{c(M)}_{M\sigma(M),a(M)} : + \sum_k : \mathcal{F}^{c(j)}_{j\sigma(j),a(j)} \cdots (-E_{kr,a})' \cdots \mathcal{F}^{c(M)}_{M\sigma(M),a(M)} : , \]
where the sum is over \( k \in \{ j, \ldots, M \} \) such that \( a(k) = a, \sigma(k) = j - 1 \) and \( c(k) = 1 \).

We also have
\[ F_{(j-1)(j-1),0}^0 : \mathcal{F}^{c(j)}_{j\sigma(j),a(j)} \cdots \mathcal{F}^{c(M)}_{M\sigma(M),a(M)} := \]
\[ : F_{(j-1)(j-1),0}^0 : \mathcal{F}^{c(j)}_{j\sigma(j),a(j)} \cdots \mathcal{F}^{c(M)}_{M\sigma(M),a(M)} : + \sum_k : \mathcal{F}^{c(j)}_{j\sigma(j),a(j)} \cdots (-E_{k\sigma(k),a(k)})' \cdots \mathcal{F}^{c(M)}_{M\sigma(M),a(M)} : , \]
where the sum is over \( k \in \{ j, \ldots, M \} \) such that \( c(k) = 1 \).

Using (2.7), (2.8), rewrite each term in (2.6) with \( i = j \). Then the \( k \)-th term obtained by using (2.8) applied to the term labeled by \( \sigma, c, a \) with \( c(j - 1) = 0 \) cancels with the \( k \)-th obtained by using (2.8) applied to the term labeled by \( \tilde{\sigma}, \tilde{c}, \tilde{a} \) defined by the following rules:
\[ \tilde{\sigma}(i) = \sigma(i) \ (i \neq j - 1, k), \quad \tilde{\sigma}(j - 1) = j - 1, \quad \tilde{\sigma}(k) = \sigma(j - 1), \]
\[ \tilde{c}(i) = c(i) \ (i \neq j - 1), \quad \tilde{c}(j - 1) = 0, \]
\[ \tilde{a}(i) = a(i) \ (i \neq j - 1), \quad \tilde{a}(j - 1) = 0. \]

After this cancellation we obtain the statement of the lemma for \( i = j - 1 \). \( \square \)

**Remark 2.5.** The proof of Lemma 2.4 implies that if the matrix \( \sigma G_h \) is obtained from \( G_h \) by permuting the rows of \( G_h \) by a permutation \( \sigma \) then \( \text{rdet}(G_h) = (-1)^{\text{sgn}(\sigma)} \text{rdet}(G_h) \).

Consider the linear isomorphism \( \phi_h : A_h \to A_0 \) which sends any normally ordered monomial \( m \) in \( A_h \) to the same monomial \( m \) in \( A_0 \).

By (2.6) with \( i = 1 \), the image \( \phi_h(\text{rdet}(G_h)) \) does not depend on \( h \) and therefore can be computed at \( h = 0 \). Therefore Theorem 2.4 for all \( h \) follows from Theorem 2.4 for \( h = 0 \). Theorem 2.4 for \( h = 0 \) follows from formula (2.3).

### 3. The \((\mathfrak{gl}_M, \mathfrak{gl}_N)\) duality and the Bethe subalgebras

#### 3.1. Bethe subalgebra

Let \( E_{ij}, i, j = 1, \ldots, M, \) be the standard generators of \( \mathfrak{gl}_M \). Let \( \mathfrak{h} \) be the Cartan subalgebra of \( \mathfrak{gl}_M \):
\[ \mathfrak{h} = \bigoplus_{i=1}^M \mathbb{C} \cdot E_{ii}. \]

We denote \( U\mathfrak{gl}_M \) the universal enveloping algebra of \( \mathfrak{gl}_M \).

For \( \mu \in \mathfrak{h}^* \), and a \( \mathfrak{gl}_M \) module \( L \) denote by \( L[\mu] \) the vector subspace of \( L \) of vectors of weight \( \mu \),
\[ L[\mu] = \{ v \in L \mid hv = \langle \mu, h \rangle v \ \text{for any} \ h \in \mathfrak{h} \}. \]

We always assume that \( L = \bigoplus_{\mu} L[\mu] \).
For any integral dominant weight $\Lambda \in \mathfrak{h}^*$, denote by $L_\Lambda$ the finite-dimensional irreducible $\mathfrak{gl}_M$-module with highest weight $\Lambda$.

Recall that we fixed sequences of complex numbers $z = (z_1, \ldots, z_N)$ and $\lambda = (\lambda_1, \ldots, \lambda_M)$. From now on we will assume that $z_i \neq z_j$ and $\lambda_i \neq \lambda_j$ if $i \neq j$.

For $i, j = 1, \ldots, M$, $a = 1, \ldots, N$, let $E_{ji}^{(a)} = 1^{\otimes (a-1)} \otimes E_{ji} \otimes 1^{\otimes (N-a)} \in (U \mathfrak{gl}_M)^{\otimes N}$.

Define the $M \times M$ matrix $\tilde{G} = \tilde{G}(M, N, z, \lambda, u)$ by

$$\tilde{G}(M, N, z, \lambda, u) := \left( \left( \frac{\partial}{\partial u} - \lambda_i \right) \delta_{ij} - \sum_{a=1}^{N} \frac{E_{ji}^{(a)}}{u - z_a} \right)_{i,j=1}^M.$$ 

The entries of $\tilde{G}$ are differential operators in $u$ whose coefficients are rational functions in $u$ with values in $(U \mathfrak{gl}_M)^{\otimes N}$.

Write

$$\text{rdet}(\tilde{G}(M, N, z, \lambda, u)) = \frac{\partial^M}{\partial u^M} + \tilde{G}_1(M, N, z, \lambda, u) \frac{\partial^{M-1}}{\partial u^{M-1}} + \cdots + \tilde{G}_M(M, N, z, \lambda, u).$$

The coefficients $\tilde{G}_i(M, N, z, \lambda, u)$, $i = 1, \ldots, M$, are called the transfer matrices of the Gaudin model. The transfer matrices are rational functions in $u$ with values in $(U \mathfrak{gl}_M)^{\otimes N}$.

The transfer matrices commute:

$$[\tilde{G}_i(M, N, z, \lambda, u), \tilde{G}_j(M, N, z, \lambda, v)] = 0,$$

for all $i, j, u, v$, see [1] and Proposition 7.2 in [MTV].

The transfer matrices clearly commute with the diagonal action of $\mathfrak{h}$ on $(U \mathfrak{gl}_M)^{\otimes N}$.

For $i = 1, \ldots, M$, it is clear that $\tilde{G}_i(M, N, z, \lambda, u) \prod_{a=1}^{N} (u - z_a)^i$ is a polynomial in $u$ whose coefficients are pairwise commuting elements of $(U \mathfrak{gl}_M)^{\otimes N}$. Let $\mathcal{G}(M, N, z, \lambda) \subset (U \mathfrak{gl}_M)^{\otimes N}$ be the commutative subalgebra generated by the coefficients of polynomials $\tilde{G}_i(M, N, z, \lambda, u) \prod_{a=1}^{N} (u - z_a)^i$, $i = 1, \ldots, M$. We call the subalgebra $\mathcal{G}(M, N, z, \lambda)$ the Bethe subalgebra.

Let $\mathcal{G}(M, \lambda) \subset U \mathfrak{gl}_M[t]$ be the subalgebra considered in the introduction. Let $U \mathfrak{gl}_M[t] \to (U \mathfrak{gl}_M)^{\otimes N}$ be the algebra homomorphism defined by $E_{ij} \otimes t^n \mapsto \sum_{a=1}^{N} E_{ij}^{(a)} z_a^n$. Then the subalgebra $\mathcal{G}(M, N, z, \lambda)$ is the image of the subalgebra $\mathcal{G}(M, \lambda)$ under that homomorphism.

The Bethe subalgebra clearly acts on any $N$-fold tensor products of $\mathfrak{gl}_M$ representations.

Define the Gaudin Hamiltonians, $H_a(M, N, z, \lambda) \subset (U \mathfrak{gl}_M)^{\otimes N}$, $a = 1, \ldots, N$, by the formula

$$H_a(M, N, z, \lambda) = \sum_{b=1, b \neq a}^{N} \frac{\Omega_{ab}^{(ab)}}{z_a - z_b} + \sum_{b=1}^{M} \lambda_b E_{bb}^{(a)},$$

where $\Omega_{ab}^{(ab)} := \sum_{i,j=1}^{M} E_{ij}^{(a)} E_{ji}^{(b)}$. 


Define the dynamical Hamiltonians \( H^\vee_a(M, N, z, \lambda) \subset (U\mathfrak{gl}_M)^{\otimes N} \), \( a = 1, \ldots, M \), by the formula

\[
H^\vee_a(M, N, z, \lambda) = \sum_{b=1, b \neq a}^M \frac{(\sum_{i=1}^N E^{(i)}_{ab})(\sum_{i=1}^N E^{(i)}_{ba}) - \sum_{i=1}^N E^{(i)}_{aa}}{\lambda_a - \lambda_b} + \sum_{b=1}^N z_b E^{(b)}_{aa}.
\]

It is known that the Gaudin Hamiltonians and the dynamical Hamiltonians are in the Bethe subalgebra, see e.g. Appendix B in [MTV1]:

\[
H_a(M, N, z, \lambda) \in \mathcal{G}(M, N, z, \lambda), \quad H^\vee_a(M, N, z, \lambda) \in \mathcal{G}(M, N, z, \lambda),
\]

\( a = 1, \ldots, N \), \( b = 1, \ldots, M \).

3.2. The \( (\mathfrak{gl}_M, \mathfrak{gl}_N) \) duality. Let \( L^{(M)} = \mathbb{C}[x_1, \ldots, x_M] \) be the space of polynomials of \( M \) variables. We define the \( \mathfrak{gl}_M \)-action on \( L^{(M)} \) by the formula

\[
E_{ij} \mapsto x_i \frac{\partial}{\partial x_j}.
\]

Then we have an isomorphism of \( \mathfrak{gl}_M \) modules

\[
L^{(M)} = \bigoplus_{m=0}^{\infty} L^{(M)}_m
\]

the submodule \( L^{(M)}_m \) being spanned by homogeneous polynomials of degree \( m \). The submodule \( L^{(M)}_m \) is the irreducible \( \mathfrak{gl}_M \) module with highest weight \( (m, 0, \ldots, 0) \) and highest weight vector \( x^m_1 \).

Let \( L^{(M,N)} = \mathbb{C}[x_{11}, \ldots, x_{1N}, \ldots, x_{M1}, \ldots, x_{MN}] \) be the space of polynomials of \( MN \) commuting variables. Let \( \pi^{(M)} : (U\mathfrak{gl}_M)^{\otimes N} \to \text{End}(L^{(M,N)}) \) be the algebra homomorphism defined by

\[
E_{ij}^{(a)} \mapsto x_{ia} \frac{\partial}{\partial x_{ja}}.
\]

In particular, we define the \( \mathfrak{gl}_M \) action on \( L^{(M,N)} \) by the formula

\[
E_{ij} \mapsto \sum_{a=1}^N x_{ia} \frac{\partial}{\partial x_{ja}}.
\]

Let \( \pi^{(N)} : (U\mathfrak{gl}_N)^{\otimes M} \to \text{End}(L^{(M,N)}) \) be the algebra homomorphism defined by

\[
E_{ij}^{(a)} \mapsto x_{ai} \frac{\partial}{\partial x_{aj}}.
\]

In particular, we define the \( \mathfrak{gl}_N \) action on \( L^{(M,N)} \) by the formula

\[
E_{ij} \mapsto \sum_{a=1}^M x_{ai} \frac{\partial}{\partial x_{aj}}.
\]
We have isomorphisms of algebras,
\[
\begin{align*}
(C[x_1, \ldots, x_M])^\otimes N & \to L^{(M,N)}_\bullet, \\
(C[x_1, \ldots, x_N])^\otimes M & \to L^{(M,N)}_\bullet,
\end{align*}
\]
where
\[A \otimes 1 \otimes x_i \otimes 1^\otimes (N-j) \mapsto x_{ij},\]
\[1^\otimes (j-1) \otimes x_i \otimes 1^\otimes (N-j) \mapsto x_{ij}.
\]
(3.1)

Under these isomorphisms the space \(L^{(M,N)}_\bullet\) is isomorphic to \((L^{(M)}_\bullet)^\otimes N\) as a \(gl_M\) module and to \((L^{(N)}_\bullet)^\otimes M\) as a \(gl_N\) module.

Fix \(n = (n_1, \ldots, n_N) \in \mathbb{Z}_{\geq 0}^N\) and \(m = (m_1, \ldots, m_M) \in \mathbb{Z}_{\geq 0}^M\) with \(\sum_{i=1}^N n_i = \sum_{a=1}^M m_a\).

The sequences \(n\) and \(m\) naturally correspond to integral \(gl_N\) and \(gl_M\) weights, respectively.

Let \(L_m\) and \(L_n\) be \(gl_N\) and \(gl_M\) modules, respectively, defined by the formulas
\[
L_m = \otimes_{a=1}^M L^{(N)}_{n_a}, \quad L_n = \otimes_{b=1}^N L^{(M)}_{m_b}.
\]
The isomorphisms (3.1) induce an isomorphism of the weight subspaces,
\[
L_n[m] \simeq L_m[n].
\]
(3.2)

Under the isomorphism (3.2) the Gaudin and dynamical Hamiltonians interchange,
\[
\pi^{(M)} H_a(M, N, z, \lambda) = \pi^{(N)} H^\vee_a(N, M, \lambda, z), \quad \pi^{(M)} H^\vee_b(M, N, z, \lambda) = \pi^{(N)} H_b(N, M, \lambda, z),
\]
for \(a = 1, \ldots, N, b = 1, \ldots, M\), see [TV].

We prove a stronger statement that the images of \(gl_M\) and \(gl_N\) Bethe subalgebras in \(\text{End}(L^{(M,N)}_\bullet)\) are the same.

**Theorem 3.1.** We have
\[
\pi^{(M)}(\mathcal{G}(M, N, z, \lambda)) = \pi^{(N)}(\mathcal{G}(N, M, \lambda, z)).
\]
Moreover, we have
\[
\begin{align*}
\prod_{a=1}^N (u - z_a) \pi^{(M)} \text{rdet}(\widetilde{G}(M, N, z, \lambda, u)) &= \sum_{a=1}^N \sum_{b=1}^M A^{(M)}_{ab} u^a \frac{\partial^b}{\partial u^b}, \\
\prod_{b=1}^M (v - \lambda_b) \pi^{(N)} \text{rdet}(\widetilde{G}(N, M, \lambda, z, v)) &= \sum_{a=1}^N \sum_{b=1}^M A^{(N)}_{ab} v^b \frac{\partial^a}{\partial v^a},
\end{align*}
\]
where \(A^{(M)}_{ab}, A^{(N)}_{ab}\) are linear operators independent on \(u, v, \partial/\partial u, \partial/\partial v\) and
\[
A^{(M)}_{ab} = A^{(N)}_{ab}.
\]

**Proof.** We obviously have
\[
\pi^{(M)}(\widetilde{G}(M, N, z, \lambda, u)) = \tilde{i}_{h=1}(G_{h=1}),
\]
\[
\pi^{(N)}(\widetilde{G}(N, M, \lambda, z, v)) = \tilde{i}_{h=1}(H_{h=1}),
\]
where \(G_{h=1}\) and \(H_{h=1}\) are matrices defined in (2.20) and (2.21).

Then the coefficients of the differential operators \(\prod_{a=1}^N (u - z_a) \pi^{(M)} \text{rdet}(\widetilde{G}(M, N, z, \lambda, u))\) and \(\prod_{b=1}^M (v - \lambda_b) \pi^{(N)} \text{rdet}(\widetilde{G}(N, M, \lambda, z, v))\) are polynomials in \(u\) and \(v\) of degrees \(N\) and
by Theorem 2.1. The rest of the theorem follows directly from Corollary 2.3.

3.3. Scalar differential operators. Let \( w \in L_n[m] \) be a common eigenvector of the Bethe subalgebra \( \mathcal{G}(M, N, z, \lambda) \). Then the operator \( \text{rdet}(\tilde{G}(M, N, z, \lambda, u)) \) acting on \( w \) defines a monic scalar differential operator of order \( M \) with rational coefficients in variable \( u \). Namely, let \( D_w(M, N, \lambda, z) \) be the differential operator given by

\[
D_w(M, N, z, \lambda, u) = \frac{\partial^M}{\partial u^M} + \tilde{G}_1^w(M, N, z, \lambda, u) \frac{\partial^{M-1}}{\partial u^{M-1}} + \cdots + \tilde{G}_M^w(M, N, z, \lambda, u),
\]

where \( \tilde{G}_i^w(M, N, z, \lambda, u) \) is the eigenvalue of the \( i \)th transfer matrix acting on the vector \( w \):

\[
\tilde{G}_i(M, N, z, \lambda, u)w = \tilde{G}_i^w(M, N, z, \lambda, u)w.
\]

Using isomorphism (3.2), consider \( w \) as a vector in \( L_m[n] \). Then by Theorem 3.1, \( w \) is also a common eigenvector for algebra \( \mathcal{G}(N, M, \lambda, z) \). Thus, similarly, the operator \( \text{rdet}(\tilde{G}(N, M, \lambda, z, v)) \) acting on \( w \) defines a monic scalar differential operator of order \( N \), \( D_w(N, M, \lambda, z, v) \).

Corollary 3.2. We have

\[
\prod_{a=1}^{N}(u - z_a)D_w(M, N, z, \lambda, u) = \sum_{a=1}^{N} \sum_{b=1}^{M} A_{ab,w}^{(M)} u^a \frac{\partial^b}{\partial u^b},
\]

\[
\prod_{b=1}^{M}(v - \lambda_b)D_w(N, M, \lambda, z, v) = \sum_{a=1}^{N} \sum_{b=1}^{M} A_{ab,w}^{(N)} v^b \frac{\partial^a}{\partial v^a},
\]

where \( A_{ab,w}^{(M)}, A_{ab,w}^{(N)} \) are numbers independent on \( u, v, \partial/\partial u, \partial/\partial v \). Moreover,

\[
A_{ab,w}^{(M)} = A_{ab,w}^{(N)}.
\]

Proof. The corollary follows directly from Theorem 3.1.

Corollary 3.2 was essentially conjectured in Conjecture 5.1 in [MTV2].

Remark 3.3. The operators \( D_w(M, N, z, \lambda) \) are useful objects, see [MV1], [MTV2], [MTV3]. They have the following three properties.

(i) The kernel of \( D_w(M, N, z, \lambda) \) is spanned by the functions \( p_i^w(u)e^{\lambda_i u}, \) \( i = 1, \ldots, M \), where \( p_i^w(u) \) is a polynomial in \( u \) of degree \( m_i \).
(ii) All finite singular points of \( D_w(M, N, z, \lambda) \) are \( z_1, \ldots, z_N \).
(iii) Each singular point \( z_i \) is regular and the exponents of \( D_w(M, N, z, \lambda) \) at \( z_i \) are \( 0, n_i + 1, n_i + 2, \ldots, n_i + M - 1 \).
A converse statement is also true. Namely, if a linear differential operator of order $M$ has properties (i-iii), then the operator has the form $D_w(M, N, z, \lambda)$ for a suitable eigenvector $w$ of the Bethe subalgebra. This statement may be deduced from Proposition 3.4 below.

We will discuss the properties of such differential operators in [MTV4], cf. also [MTV2] and Appendix A in [MTV3].

3.4. The simple joint spectrum of the Bethe subalgebra. It is proved in [R], that for any tensor product of irreducible $\mathfrak{gl}_M$ modules and for generic $z, \lambda$ the Bethe subalgebra has a simple joint spectrum. We give here a proof of this fact in the special case of the tensor product $L_n$.

**Proposition 3.4.** For generic values of $\lambda$, the joint spectrum of the Bethe subalgebra $\mathcal{G}(M, N, z, \lambda)$ acting in $L_n[m]$ is simple.

**Proof.** We claim that for generic values of $\lambda$, the joint spectrum of the Gaudin Hamiltonians $H_a(M, N, z, \lambda)$, $a = 1, \ldots, N$, acting in $L_n[m]$ is simple. Indeed fix $z$ and consider $\lambda$ such that $\lambda_1 \gg \lambda_2 \gg \cdots \gg \lambda_M \gg 0$. Then the eigenvectors of the Gaudin Hamiltonians in $L_n[m]$ will have the form $v_1 \otimes \cdots \otimes v_N + o(1)$, where $v_i \in L_n[m^{(i)}]$ and $m = \sum_{i=1}^{N} m^{(i)}$. The corresponding eigenvalue of $H_a(M, N, z, \lambda)$ will be $\sum_{j=1}^{M} \lambda_j m_j^{(a)} + O(1)$.

The weight spaces $L_n^{(M)}[m_i]$ all have dimension at most 1 and therefore the joint spectrum is simple in this asymptotic zone of parameters. Therefore it is simple for generic values of $\lambda$. \hfill \Box

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