ON THE FRAME SET OF THE SECOND-ORDER B-SPLINE

A. GANIOU D. ATINDEHOU, CHRISTINA FREDERICK, YÉBÉNI B. KOUAGOU, AND KASSO A. OKOUDJOU

Abstract. The frame set of a function \( g \in L^2(\mathbb{R}) \) is the set of all parameters \((a, b) \in \mathbb{R}^2_+ \) for which the collection of time-frequency shifts of \( g \) along \( a\mathbb{Z} \times b\mathbb{Z} \) form a Gabor frame for \( L^2(\mathbb{R}) \). Finding the frame set of a given function remains a challenging open problem in time-frequency analysis. In this paper, we establish new regions of the frame set of the second-order B-spline. Our method uses the compact support of this function to partition a subset of the putative frame set and finds an explicit dual window function in each subregion. Numerical evidence indicates the existence of further regions belonging to the frame set.

1. Introduction and main results

Given a window \( g \in L^2(\mathbb{R}) \), and \( a, b > 0 \), the collection of time-frequency shifts

\[
\mathcal{G}(g, a, b) = \{ M_{\ell b}T_{ka}g = e^{2\pi ib \cdot} g(\cdot - ka) : (\ell, k) \in \mathbb{Z}^2 \}
\]

is called a Gabor frame for \( L^2(\mathbb{R}) \) if there exist constants \( A, B > 0 \) such that

\[
A \| f \|_2^2 \leq \sum_{\ell, k \in \mathbb{Z}} |\langle f, M_{\ell b}T_{ka}g \rangle|^2 \leq B \| f \|_2^2,
\]

for all \( f \in L^2(\mathbb{R}) \). When \( \mathcal{G}(g, a, b) \) is a Gabor frame, there exists a dual window \( \gamma \in L^2(\mathbb{R}) \) such that \( \mathcal{G}(\gamma, a, b) \) is also a Gabor frame for \( L^2(\mathbb{R}) \) called the (canonical) dual to \( \mathcal{G}(g, a, b) \). Consequently, for any \( f \in L^2(\mathbb{R}) \) we have the following reconstruction formulas:

\[
f = \sum_{k \in \mathbb{Z}} \langle f, M_{\ell b}T_{ka}\gamma \rangle M_{\ell b}T_{ka}g = \sum_{k \in \mathbb{Z}} \langle f, M_{\ell b}T_{ka}g \rangle M_{\ell b}T_{ka}\gamma.
\]

We refer to [2, 7] for more on Gabor frame theory.

Despite the outstanding advances in the theory and applications of Gabor frames over the last three decades, the problem of characterizing the set of all points \((a, b) \in \mathbb{R}^2_+ \) such that \( \mathcal{G}(g, a, b) \) is a frame for a given \( g \in L^2(\mathbb{R}) \) remains largely unresolved. This set is known as the frame set of \( g \) and will be denoted by \( \mathcal{F}(g) \). The current state-of-the-art result in this direction states that if \( g \) is either:

1. in \( \{ e^{-\pi x^2}, \frac{1}{\cosh x}, \chi_{[0, \infty)} e^{-x}, e^{-|x|}\} \), or
2. a totally positive function of finite type, or
3. a totally positive function of exponential type

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Yébéni B. Kouagou suddenly passed away in 2018, a few weeks after the first version of this work was released. This version is dedicated to his memory.
then $F(g) = \{(a, b) \in \mathbb{R}_+^2, ab < 1\}$ (see [8] [10] [15] [12] [14] [13] [18] [19] [20]). At the same time, the frame set of $g = \chi_{[0,c]}$, known as the Janssen’s tie [14] has a more complex structure that was fully described by Dai and Sun [5] [11].

In this paper we consider the frame set of the $B$-spline of order 2:

$$B_2(x) = \begin{cases} 1 + x & x \in [-1,0] \\ 1 - x & x \in [0,1]. \end{cases}$$

It is known that $F(B_2)$ is an open set in $\mathbb{R}_+^2$ [6], but a full characterization of this set remains an open question. To date, it has been shown that

$$\left\{ (a,b) \in \mathbb{R}_+^2 : ab < 1, 0 < a < 2, 0 < b \leq \max_a \left( \frac{4}{2+3a} \right) \right\} \subset F(B_2)$$

(see Figure 1 for a sketch and [1, 4, 3, 2, 9, 17, 16] for details).

In an earlier work [1], we introduced a framework for determining the frame sets of compactly supported functions, including the $B$-splines of order $N \geq 2$. This framework unified many of the known results on frame sets of $B$-splines [4, 3, 2, 9, 17, 16]. In this paper, we use a similar linear algebra based approach to shed new light on set $F(B_2)$ and prove the following result

**Theorem 1.** The frame set of the second-order $B$-spline contains the set $\Gamma_3 \cup \Lambda$, that is,

$$F(B_2) \supset \Gamma_3 \cup \Lambda,$$

where

$$\Gamma_3 := \left\{ (a,b) \in \mathbb{R}_+^2 : a \in \left( 0, \frac{2}{9} \right] \cup \left( \frac{2}{7}, \frac{1}{2} \right), b \in \left( \frac{4}{2 + 3a}, \frac{2}{1 + a} \right) \right\}$$

$$\Lambda_3 := \left\{ (a,b) \in \mathbb{R}_+^2 : a \in \left[ \frac{1}{2}, \frac{4}{5} \right], b \in \left( \frac{6}{2 + 3a}, \frac{2}{2 + 5a}, b > 1 \right) \right\},$$

and for $m \geq 4$

$$\Lambda_m = \left\{ (a,b) \in \mathbb{R}_+^2 : a \in \left[ \frac{m - 3}{m - 2}, \frac{2(m - 1)}{2m - 1} \right], b \in \left( \frac{2(m - 1)}{2 + (2m - 3)a}, \min_a \left( \frac{2m}{2 + (2m - 1)a}, \frac{2}{1 + a} \right), b > 1 \right) \right\}.$$
Figure 1. Sketch depicting known results on the frame set of $B_2$. The shaded regions in gray were proved to be in $\mathcal{F}(B_2)$ in [1, 4, 8, 2, 9, 17, 16] and the light purple region was established as a subset of $\mathcal{F}(B_2)$ in [1]. Points that are brown are known not to be in $\mathcal{F}(B_2)$.

Theorem 1 asserts that the union of $\Gamma_3$ (dark purple) and $\bigcup_{m=3}^{\infty} \Lambda_m$ (shades of blue) belong to $\mathcal{F}(B_2)$. We show that the additional points marked with purple diamonds are in $\mathcal{F}(B_2)$.

To deal with the regions $\Lambda_m$, let $h$ be a bounded, real-valued function with support on $[-\frac{(2m-1)a}{2}, \frac{(2m-1)a}{2}]$. Then, it is known that the Gabor systems $\mathcal{G}(B_2, a, b)$ and $\mathcal{G}(h, a, b)$ are dual frames for $L^2(\mathbb{R})$ if and only if

$$\sum_{k=1-m}^{-1} B_2(x - \ell/b + ka)h(x + ka) = b\delta_{\ell,0}, \quad |\ell| \leq m-1, \text{ for a.e } x \in [\frac{-a}{2}, \frac{a}{2}].$$

Our main goal is to show the existence of a bounded compactly function $h$ that solves (1) when $(a, b) \in \Lambda_m$, for $m \geq 3$.

To do this, we rewrite (1) as a matrix-vector equation using the compact support of $B_2$,

$$G_m H_m = \begin{pmatrix} F_{m-1} & A_{m-1} \\ 0 & C_{m-1} \end{pmatrix} H_m = V$$

where $G_m$ is a $(2m-1) \times (2m-1)$ matrix, and the column vectors $H_m$ and $V$ are given by

$$H_m(x) = [h(x + ka)]_{|k| \leq m-1}, \quad V = [b\delta_{\ell,0}]_{|\ell| \leq m-1}.$$

Furthermore, $C_{m-1}$ is a $(m-2) \times (m-2)$ upper triangular matrix with strictly positive diagonal entries, and $F_{m-1}$ is the $(m+1) \times (m+1)$ tridiagonal matrix given by

$$F_{m-1}(x) = \left[ B_2(x - \ell/b + ka) \right]_{1-m \leq \ell, k \leq 1}.$$

We observe that the diagonal entries of $F_{m-1}$ are also strictly positive. Therefore, the existence of a dual is guaranteed when $G_m$ has a strictly positive determinant, or equivalently...
Proposition 1. For a dual Gabor frame to $G_m$, let Lemma 1. An induction argument in Lemma 2. establishes the invertibility of the matrices when $(a, b) \in \Lambda$.

To deal with the region $\Gamma_3$, we effectively compute the determinant a $4 \times 4$ matrix that models $[1]$. However, in this case the matrix is no longer tridiagonal. Nonetheless, we again exploit the structure of $B_2$ to show that the determinant of this matrix is also strictly positive which allows us to find a compactly supported and bounded function $h$ which generates a dual Gabor frame to $G(B_2, a, b)$. This is established in Section 3.

Notation. In the sequel, given a $p \times p$ matrix $A$ and $E \subset \{1, 2, 3, \ldots, p\}$, we denote by $A^E$ the $\#E \times \#E$ sub-matrix of $A$ using rows and columns from $E$, and denote by $|A|$ the determinant of the matrix $A$. We also denote by $g_{l,k}$ the function

$$g_{l,k}(x) := B_2(x - \frac{l}{b} + ka).$$

2. $F(B_2)$ contains $A$

In this section we prove the first part of Theorem 1 by establishing the following result.

Theorem 2. For $m \geq 3$, let $(a, b) \in \Lambda_m$. Then, the Gabor system $G(B_2, a, b)$ is a frame for $L^2(\mathbb{R})$, and there is a unique dual window $h \in L^2(\mathbb{R})$ such that $\text{supp}(h) \subseteq \left[-\frac{2m-1}{2}a, \frac{2m-1}{2}a\right]$. Furthermore, for each $(a, b) \in \Lambda$, the Gabor system $G(B_2, a, b)$ is a frame for $L^2(\mathbb{R})$.

To prove Theorem 2 we only need to show that $[1]$ has a solution $h$ that is a bounded and compactly supported function. As mentioned earlier, the determinant of the (block) matrix $G_m$ is

$$|G_m| = |F_{m-1}| |C_{m-1}| = |F_{m-1}| \prod_{1-m}^{2} g_{-k,-k}$$

where we used the fact $C_{m-1}$ is an upper triangular matrix. Because for each $k = 1 - m, \ldots, -2, g_{-k,-k} > 0$ on $[-\frac{a}{2}, 0]$, we turn our attention to establishing that the determinant $|F_{m-1}|$ of this tridiagonal matrix is strictly positive when $(a, b) \in \Lambda_m$.

Proposition 1. For $m \geq 3$, let $(a, b) \in \cup_{k=m}^{\infty} \Lambda_k$. Then $|F_{m-1}| > 0$ on $[-\frac{a}{2}, 0]$.

We will prove the result by showing that $|F_{m-1}|$ never vanishes on $[-\frac{a}{2}, 0]$. Since this matrix is a tri-diagonal, we could rely on standard formulas to find its determinant. However, the challenging part is to establish that the determinant is never 0. We will do this by an induction argument on $m$, relying on the fact that $B_2$ is a compactly supported piecewise linear function. In Lemma 1 we prove the result for the base case $m = 3$, and complete the induction argument in Lemma 2.

Lemma 1. Let $(a, b) \in \cup_{k=3}^{\infty} \Lambda_k$. Then, $|F_2| > 0$ on $[-\frac{a}{2}, 0]$.

Proof. We note that for $(a, b) \in \cup_{k=3}^{\infty} \Lambda_k$
Figure 2. The frame set of $B_2$ established in Lemma 1 by partitioning $\cup_{k=3}^{\infty} \Lambda_k$ into the four regions shown in the left plot. In each region, for each fixed $b$, the interval $[-\frac{a}{2}, 0]$ is subdivided into three intervals (Cases (I) and (III)) or four intervals (Cases (II) and (IV)) as depicted by the shading in the plots on the right.

\begin{equation}
F_2 = \begin{pmatrix}
g_{-2,-2} & g_{-2,-1} & 0 & 0 \\
g_{-1,-2} & g_{-1,-1} & g_{-1,0} & 0 \\
0 & g_{0,-1} & g_{0,0} & g_{0,1} \\
0 & 0 & g_{1,0} & g_{1,1}
\end{pmatrix}.
\end{equation}

In each case below, we write $[-\frac{a}{2}, 0]$ as a union of intervals and show that $|F_2| > 0$ on each of them. Figure 2 shows plots of the four regions of $\cup_{k=3}^{\infty} \Lambda_k$ considered (left) as well as the partitions of the interval $[-\frac{a}{2}, 0]$ (right).

(I) For $a \in \left[\frac{1}{2}, \frac{2}{3}\right)$ and $2 - \frac{3}{b} + a \leq 0$,

$$[-\frac{a}{2}, 0] = \left[\frac{a}{2}, 1 - \frac{2}{b} + a\right] \cup \left[1 - \frac{2}{b} + a, -1 + \frac{1}{b}, 0\right].$$

On the first interval, $g_{1,0} = 0$, and since $g_{-1,-1} > g_{-1,-2}$, $g_{1,-1} \geq g_{-1,0}$ and

$$\begin{vmatrix} g_{-2,-2} & 0 \\ 0 & g_{0,0} \end{vmatrix} > \begin{vmatrix} g_{-2,-2} & g_{-2,-1} \\ g_{0,-1} & 0 \end{vmatrix} + \begin{vmatrix} g_{-2,-1} & 0 \\ g_{0,-1} & g_{0,0} \end{vmatrix},$$

it follows that $|F_2| = g_{1,1}F_2^{(1,2,3)} > 0$. On the second interval, $g_{2,1} = g_{1,0} = 0$, and since $g_{1,-1} > g_{1,0}$ and $g_{0,-1} < g_{0,0}$, it follows that $|F_2| = g_{-2,-2}g_{1,1}F_2^{(2,3)} > 0$. On the third interval, $g_{2,-1} = 0$, and since $g_{0,0} > \max\{g_{0,-1}, g_{0,1}\}$, and

$$\begin{vmatrix} g_{-1,-1} & 0 \\ 0 & g_{1,1} \end{vmatrix} > \begin{vmatrix} g_{-1,-1} & g_{-1,0} \\ g_{0,1} & 0 \end{vmatrix} + \begin{vmatrix} g_{-1,0} & 0 \\ g_{1,0} & g_{1,1} \end{vmatrix},$$

it follows that $|F_2| = g_{-2,-2}F_2^{(2,3,4)} > 0$.

(II) For $a \in \left[\frac{2}{3}, 1\right]$ and $2 - \frac{3}{b} + a \leq 0$,

$$[-\frac{a}{2}, 0] = \left[-\frac{a}{2}, a - 1\right] \cup (a - 1, 1 - \frac{2}{b} + a] \cup \left[1 - \frac{2}{b} + a, \frac{1}{b} - 1\right] \cup (\frac{1}{b} - 1, 0].$$
On the first interval, we have $g_{0,-1} = g_{1,0} = 0$, and since $g_{-2,-2} > g_{-2,-1}, g_{-1,-2} < g_{-1,-1}$, it follows that $|F_2| = \prod_{\ell=0}^1 g_{\ell,\ell}|F_2^{(1,2)}| > 0$. The last three intervals are treated as in the previous case.

(III) For $\left[\frac{1}{2}, \frac{3}{2}\right]$ and $2 - \frac{3}{b} + a > 0$,

$$\left[-\frac{a}{2}, 0\right] = \left[-\frac{a}{2}, -1 + \frac{1}{b}\right] \cup (-1 + \frac{1}{b}, 1 - \frac{2}{b} + a) \cup \left[1 - \frac{2}{b} + a, 0\right].$$

On the first interval, $g_{1,0} = 0$. As in (I), we conclude that $|F_2| = g_{1,1}|F_2^{(1,2,3)}| > 0$.

On the second interval, we have $g_{0,0} > g_{0,-1}, g_{0,0} > g_{0,1}$, $|F_2^{(1,2)}| > 0$ and $|F_2^{(1,2,4)}| > g_{1,0}|F_2^{(1,2)}| + g_{-2,-2}g_{1,0}g_{1,1}$, and therefore

$$|F_2| = \begin{vmatrix} g_{-2,-2} & g_{-2,-1} & 0 & 0 \\ g_{-1,-2} & g_{-1,-1} & g_{-1,0} & 0 \\ 0 & g_{0,-1} & g_{0,0} & g_{0,1} \\ 0 & 0 & g_{1,0} & g_{1,1} \end{vmatrix} > 0.$$  

On the third interval, $g_{-2,-1} = 0$. As in part (I), we conclude $|F_2| = g_{-2,-2}|F_2^{(2,3,4)}| > 0$.

(IV) For $a \in \left[\frac{3}{2}, 1\right]$ and $2 - \frac{3}{b} + a > 0$,

$$\left[-\frac{a}{2}, 0\right] = \left[-\frac{a}{2}, -1 + a\right] \cup (-1 + a, -1 + \frac{1}{b}) \cup (-1 + \frac{1}{b}, 1 - \frac{2}{b} + a) \cup \left[1 - \frac{2}{b} + a, 0\right].$$

On the first interval, we have $g_{0,-1} = g_{1,0} = 0$, and therefore $|F_2| = g_{1,1}g_{0,0}|F_2^{(1,2)}| > 0$.

The remaining three intervals can be treated as above.

□

We organize the induction step in the following result.

**Lemma 2.** Suppose that for some $m \geq 3$, $|F_{m-k}| > 0$ on $\left[-\frac{a}{2}, 0\right]$ for each $(a, b) \in \Lambda_k$, for all $k \geq m$. Then $|F_m| > 0$ on $\left[-\frac{a}{2}, 0\right]$ for $(a, b) \in \Lambda_k$, and all $k \geq m + 1$.

**Proof.** We first prove that $|F_m| > 0$ on $\left[-\frac{a}{2}, 0\right]$ for $(a, b) \in \Lambda_{m+1}$ in the following four cases:

I) For $a \in \left[\frac{m-2}{m+1}, \frac{4m-2}{m+1}\right]$ and $2 - \frac{3}{b} + a \leq 0$, then $-1 - \frac{m-2}{b} + (m-1)a \leq -\frac{a}{2}$. In this case,

$$\left[-\frac{a}{2}, 0\right] = \left[-\frac{a}{2}, -1 - \frac{m-2}{b} + (m-1)a\right] \cup \left[1 - \frac{m}{b} + (m-1)a, 0\right].$$

On the first interval, $g_{-k,-(k+1)} = 0$ for all $k \in \{1, \ldots, m-3\}$. Therefore, since $g_{-m+1,-m} < g_{-m+1,-m+1}, g_{-m+1,-m+1} > g_{-m+1,-m+2}$, and

$$|F_m| = \prod_{\ell=0}^{m-3} g_{\ell,\ell}|F_m^{(1,2,3)}| > 0.$$

On the second interval, $g_{-m,-(m-1)} = 0$.

The induction assumption implies that $|F_{m-k}| = g_{m-k,0}|F_{m-k-1}| > 0$.

II) For $a \in \left(\frac{2m-2}{2m-1}, \frac{2m}{2m+1}\right]$ and $2 - \frac{3}{b} + a \leq 0$, we have $-\frac{a}{2} \leq -1 - \frac{m-2}{b} + (m-1)a$, and

$$\left[-\frac{a}{2}, 0\right] = \left[-\frac{a}{2}, -1 - \frac{m-2}{b} + (m-1)a\right]$$

$$\cup \left(-1 - \frac{m-2}{b} + (m-1)a, 1 - \frac{m}{b} + (m-1)a\right)$$

$$\cup \left[1 - \frac{m}{b} + (m-1)a, 0\right].$$
On the first interval, \( g_{-k, -(k+1)} = 0 \) for all \( k \in \{-1, \ldots, m-3\} \). Since \( F_m^{(1,2)} \) is diagonally dominant, it follows that \(|F_m| = \prod_{\ell=-1}^{m-2} g_{-\ell, -\ell}|F_m^{(1,2)}| > 0\). On the second interval, \( g_{-k, -(k+1)} = 0 \) for all \( k \in \{-1, \ldots, m-3\} \). As in part (I), we conclude \(|F_m| = \prod_{\ell=-1}^{m-3} g_{-\ell, -\ell}|F_m^{(1,2,3)}| > 0\). On \([1 - \frac{m}{b} + (m-1)a, 0]\), \( g_{-m, -(m-1)} = 0 \). Consequently \(|F_m| = g_{m, -m}|F_{m-1}| > 0\) by the induction assumption.

III) For \( a \in (\frac{4m-2}{4m+1}, \frac{2m-2}{2m-1}) \) and \( 2 - \frac{3}{b} + a \leq 0 \), the quantity \(-1 - (m-2)/b + (m-1)a + \frac{a}{2}\) can be either positive or negative, falling into the categories of (I) and (II).

IV) If \( 2 - \frac{3}{b} + a > 0 \), then \( a \in [\frac{m-2}{m-1}, \frac{4m-2}{4m+1}] \) and

\[
\left[ -\frac{a}{2}, 0 \right] = \left[ -\frac{a}{2}, -1 - \frac{m-3}{b} + (m-2)a \right] \\
\hspace{1cm} \cup \left( -1 - \frac{m-3}{b} + (m-2)a, 1 - \frac{m}{b} + (m-1)a \right) \\
\hspace{1cm} \cup \left[ 1 - \frac{m}{b} + (m-1)a, 0 \right].
\]

On the first interval, \( g_{-k, -(k+1)} = 0 \) for all \( k \in \{-1, \ldots, m-3\} \). As in part (I), we have \(|F_m| = \prod_{\ell=-1}^{m-3} g_{-\ell, -\ell}|F_m^{(1,2,3)}| > 0\). On the second interval, \( g_{-k, -(k+1)} = 0 \) for all \( k \in \{-1, \ldots, m-4\} \). We then have \( F_m^{(1,2,3,4)}(x) = F_2(x + \frac{m-2}{b} - (m-2)a) \), where \( F_2 \) is the 4 \( \times \) 4 matrix (4). Hence,

\[
|F_m| = \prod_{\ell=-1}^{m-1} g_{-\ell, -\ell}|F_m^{(1,2,3,4)}| > 0.
\]

On \([1 - \frac{m}{b} + (m-1)a, 0]\), \( g_{-m, -(m-1)} = 0 \). Hence, \(|F_m| = g_{m, -m}|F_{m-1}| > 0\) by the induction assumption.

To establish the result for \((a, b) \in \Lambda_k\), \( k \geq m + 2 \), we prove that \(|F_m| > 0\) on each interval in a partition of \([-\frac{a}{2}, 0]\) and reduce the analysis to the case \((a, b) \in \Lambda_{m+1}\). We omit details of the proof, only indicating the relevant partitions of \([-\frac{a}{2}, 0]\).

For \( 2 - \frac{3}{b} + a \leq 0 \) we have the following partition:

\[
\left[ k - \frac{3}{b}, 2(k-1) \right] = \left[ k - \frac{3}{b}, 4k - 6 \right] \cup \left[ 4k - 6, 2k - 4 \right] \cup \left[ 2k - 4, 2(k-1) \right].
\]

(I) If \( a \in \left[ k - \frac{3}{b}, \frac{4k-6}{4k-3} \right] \), then \(-1 - k \frac{3}{b} + (k-2)a \leq -\frac{a}{2}\), and we write

\[
\left[ -\frac{a}{2}, 0 \right] = \bigcup_{\ell=1}^{k-2} (Q_\ell \cup T_\ell) \cup T_{k-1}
\]

where \( T_1 = \left[ -1 + \frac{1}{b}, 0 \right], T_{k-1} = \left[ -\frac{a}{2}, 1 - \frac{k-1}{b} + (k-2)a \right], \) and for \( \ell = 2, \ldots, k-2 \) we have \( T_\ell = \left[ -1 - \ell \frac{2}{b} + (\ell - 1)a, 1 - \ell \frac{2}{b} + (\ell - 1)a \right] \); while for \( \ell = 1, \ldots, k-2 \) we have \( Q_\ell = \left[ 1 - \ell \frac{2}{b} + \ell a, 1 - (\ell - 1)a \right] \).

(II) If \( a \in \left[ \frac{2k-4}{2k-3}, \frac{2(k-1)}{2k-1} \right] \), then \(-1 - k \frac{3}{b} + (k-2)a \geq -\frac{a}{2}\), and we have the partition

\[
\left[ -\frac{a}{2}, 0 \right] = \bigcup_{\ell=1}^{k-1} (Q_\ell \cup T_\ell)
\]
shown that $h$ with the convention that $Q$, where $H$ (5) $G$ is even and therefore can be defined on the interval $[0, \infty)$. Theorem 3. Let $(a, b) \in \Gamma_3$. Then, the Gabor system $G(B_2, a, b)$ is a frame for $L^2(\mathbb{R})$, and there is a unique dual window $h \in L^2(\mathbb{R})$ such that $\text{supp}(h) \subseteq \left[-\frac{5a}{2}, \frac{5a}{2}\right]$. We observe that when $(a, b) \in \Gamma_3$, the matrix $G_3$ becomes

(6) 
\[ G_3 = \begin{pmatrix} D & v \\ 0 & g_{2,2} \end{pmatrix}. \]
Figure 3. Plots of nonzero values of the dual $h(x)$ of $G(B_2, a, b)$ for the following $(a, b)$: (0.15, 1.70) and (0.35, 1.40) in $\Gamma_3$; (0.55, 1.20) in $\Lambda_3$; and (0.85, 1.05) in $\Lambda_6$.

where $0$ is a $1 \times 4$ matrix of $0$s, $v$ is a column vector in $\mathbb{R}^4$ and $D$ denotes the $4 \times 4$ matrix obtained by deleting the last row and the last column of $G_3$ and given by

$$D = \begin{pmatrix} g_{-2,-2} & g_{-2,1} & 0 & 0 \\ g_{-1,-2} & g_{-1,1} & g_{-1,0} & g_{-1,1} \\ g_{0,-2} & g_{0,1} & g_{0,0} & g_{0,1} \\ g_{1,-2} & g_{1,1} & g_{1,0} & g_{1,1} \end{pmatrix}. $$

Because $g_{2,2} > 0$ on $[-\frac{a}{2}, 0]$, $G_3$ is invertible on $[-\frac{a}{2}, 0]$ if and only if $D$ is invertible. The following proposition shows that the matrix $D$ is invertible for $(a, b) \in \Gamma_3 = \Gamma_3' \cup \Gamma_3''$, where

$$\Gamma_3' := \left\{ (a, b) \in \mathbb{R}_+^2 : a \in \left(0, \frac{2}{9}\right], b \in \left(\frac{4}{2 + 3a}, \frac{2}{1 + a}\right) \right\}, \quad \text{and}$$

$$\Gamma_3'' := \left\{ (a, b) \in \mathbb{R}_+^2 : a \in \left(\frac{4}{7}, \frac{1}{2}\right], b \in \left(\frac{4}{2 + 3a}, \frac{2}{1 + a}\right) \right\}. $$

(7) Proposition 2. Let $(a, b) \in \Gamma_3 = \Gamma_3' \cup \Gamma_3''$. Then $|D| > 0$, and therefore $|G_3| > 0$, on $[-\frac{a}{2}, 0]$. 
Proof. We first consider the case \( a \in (0, \frac{2}{13}] \). Then \( g_{1,-2} > 0 \) and \( -\frac{a}{2} < 1 - \frac{2}{b} + a \leq 0 \). A series of computations shows that \(|D| > 0\), since

\[
|D(x)| = \begin{cases} 
\frac{-4a}{b} (bx - b + 2)(1-b)x + a(1-b) & x \in \left[ -\frac{a}{2}, 1 - \frac{2}{b} + a \right] \\
\frac{a}{b} g_{-2,-2}(x) & x \in \left[ 1 - \frac{2}{b} + a, 0 \right].
\end{cases}
\]

Next, assume that \( a \in \left( \frac{2}{13}, \frac{1}{5} \right) \). Then \( g_{1,-2}(x) > 0 \) if and only if \( x \in (-1 + \frac{1}{b} + 2a, 1 + \frac{1}{b} + 2a) \). We first treat the subcase \(-1 + \frac{1}{b} + 2a \geq 0\), then \( g_{1,-2} = 0 \). Therefore

\[
|D(x)| = \begin{cases} 
I(x) & x \in \left[ -\frac{a}{2}, 1 - \frac{2}{b} + a \right] \\
g_{-2,-2}(x) & x \in \left[ 1 - \frac{2}{b} + a, 0 \right].
\end{cases}
\]

We now prove that \( I > 0 \) on the interval \([-\frac{a}{2}, 1 - \frac{2}{b} + a]\) by showing that \( I' > 0 \) on \([-\frac{a}{2}, 0]\) and \( I(-\frac{a}{2}) > 0 \). It can be proved that \( f(a,b) := I(-\frac{a}{2}) \), as a function of \((a,b)\) has no critical point in the interior of the domain \( \left[ \frac{2}{13}, \frac{1}{5} \right) \times \left[ \frac{2}{2+3a}, \frac{2}{1+a} \right] \). Thus, the minimum value of \( f(a,b) \) is achieved on the boundary of the domain. Furthermore, a series of calculations shows that \( f \) is positive on the boundary. Consequently \( f(a,b) > 0 \) for all \((a,b) \in \left[ \frac{2}{13}, \frac{1}{5} \right) \times \left[ \frac{4}{2+3a}, \frac{2}{1+a} \right] \).

Similarly, we can show \( L(x,a) := I'(x) > 0 \) for all \((x,a,b) \in \left[ \frac{1}{10}, 0 \right] \times \left[ \frac{2}{13}, \frac{1}{5} \right] \times \left[ \frac{20}{13}, \frac{26}{13} \right] \) which contains the compact set \([-\frac{a}{2}, 0] \times \left[ \frac{2}{13}, \frac{1}{5} \right] \times \left[ \frac{4}{2+3a}, \frac{2}{1+a} \right] \).

For the subcase \(-1 + \frac{1}{b} + 2a < 0 \) (with \( a \in \left( \frac{2}{13}, \frac{1}{5} \right) \)) we note that \( g_{1,-2} \geq 0 \). In this case, a series of computations shows that \(|D|\) is given by either (8) or (9). We proceed similarly to establish that the determinant is positive for \( x \in [-\frac{a}{2}, 0] \).

Finally, assume \( a \in \left[ \frac{1}{5}, \frac{2}{7} \right) \). It follows that \( g_{1,-2} = 0 \), and a series of computations shows that \(|D|\) is given by (9), which can be shown to be positive. Consequently, for \((a,b) \in \Gamma'_3\), then \(|D| > 0 \) on \([-\frac{a}{2}, 0]\).

We now consider \((a,b) \in \Gamma''_3\). Assume that \( a \in \left( \frac{2}{7}, \frac{1}{3} \right) \), \( b \in \left( \frac{4}{2+3a}, \frac{3}{2} \right) \). If \(-1 + \frac{1}{b} + a > 0\), then

\[
|D(x)| = \begin{cases} 
 J(x) & x \in \left[ -\frac{a}{2}, 1 - \frac{1}{b} - a \right] \\
g_{-2,-2}(x) |D^{2,3,4}(x)| & x \in \left[ 1 - \frac{1}{b} - a, 0 \right].
\end{cases}
\]

As in the case of \( I \), we prove that \( J \neq 0 \) and \(|D^{2,3,4}| > 0\), since

\[
\begin{vmatrix} g_{-1,0} & g_{-1,1} \\ g_{1,0} & g_{1,1} \end{vmatrix} > 0, \quad \begin{vmatrix} g_{-1,-1} - g_{1,0} \\ 0 \end{vmatrix} > 0, \quad \text{and} \quad \begin{vmatrix} g_{-1,-1} - g_{1,1} \\ 0 \end{vmatrix} > 0.
\]

The same decomposition is obtained for \( b \in \left( \frac{4}{2+3a}, \frac{3}{2} \right) \), \(-1 + \frac{1}{b} + a > 0 \) and \( a \in \left( \frac{2}{7}, \frac{1}{3} \right), \ b \in \left( \frac{3}{2}, \frac{2}{1+a} \right] \).

Let \( a \in \left( \frac{1}{5}, \frac{2}{7} \right] \) and \( 2a - \frac{1}{b} > 0 \). Then,

\[
|D(x)| = \begin{cases} 
 J(x) & x \in \left[ -\frac{a}{2}, 1 - \frac{1}{b} - a \right] \\
M(x) & x \in \left[ 1 - \frac{1}{b} - a, 1 - \frac{2}{b} + a \right] \\
g_{-2,-2}(x) |D^{2,3,4}(x)| & x \in \left[ 1 - \frac{2}{b} + a, 0 \right].
\end{cases}
\]

The previous determinants are obtained in the case \( a \in \left( \frac{1}{5}, \frac{2}{7} \right] \) and \( 2a - \frac{1}{b} \leq 0 \). Let \( a \in \left( \frac{2}{5}, \frac{1}{2} \right] \) and \( 2 - \frac{2}{b} - a \leq 0 \). Then

\[
|D(x)| = \begin{cases} 
 g_{1,1}(x) |D^{1,2,3}(x)| & x \in \left[ -\frac{a}{2}, -1 + \frac{1}{b} \right] \\
N(x) & x \in \left( -1 + \frac{1}{b}, 1 - \frac{2}{b} + a \right) \\
g_{-2,-2}(x) |D^{2,3,4}(x)| & x \in \left[ 1 - \frac{2}{b} + a, 0 \right].
\end{cases}
\]
Bessel sequence. By construction, it also follows that $h$. In particular, on $(13)$.

Remark

P > 0. Then

\begin{equation}
|D(x)| = \begin{cases}
  P(x) & x \in [-\frac{a}{2}, 2a - 1] \\
  J(x) & x \in (2a - 1, 1 - \frac{1}{b} - a) \\
  M(x) & x \in [1 - \frac{1}{b} - a, 1 - \frac{2}{b} + a) \\
  g_{-2, -2}(x)|D^{(2,3,4)}(x)| & x \in [1 - \frac{2}{b} + a, 0]
\end{cases}
\end{equation}

Similar techniques used for $|D^{(2,3,4)}|$ can be use to prove that $|D^{(1,2,3)}| > 0$, $N > 0$ and $P > 0$. We conclude that for $(a, b) \in \Gamma'''_3$, $D$ is invertible on $[-\frac{a}{2}, 0]$. \hfill \square

Remark 1. Using numerical simulations (Figure 4), we observe that for $(a, b) \in \Gamma'''_3 := \{(a, b) \in \mathbb{R}^2 : a \in (\frac{2}{5}, \frac{1}{2}], b \in (\frac{4}{2+4a}, \frac{2}{1+a})\}$, then $|D| > 0$. However, we have not yet been able to prove this.

Proof of Theorem 3. We recall from Propositions 2 that for $(a, b) \in \Gamma_3$, we have $|G_3| \neq 0$ on $[-\frac{a}{2}, 0]$. Since $h$ is compactly supported, we know that $h(x) = 0$ for $|x| > \frac{5a}{2}$. In addition, on $[-\frac{5a}{2}, \frac{5a}{2}]$, $h$ is determined by the entries of $H_3 = b(G_3^{-1})_3$, where $H_3$ is given by (5).

Then, we can determine $h$ on $(-\frac{a}{2} + ka, ka]$ where $k = 0, \pm 1, \pm 2$. This, along with the symmetry of $h$, will define the function $h$ everywhere except possibly at $\frac{ka}{2}$ for $k = \pm 1, \pm 2, \pm 3, \pm 5$. In particular, $h(x + 2a) = 0$ for $x \in (-\frac{a}{2}, 0)$. That is, $h = 0$ on $(-\frac{5a}{2}, -2a)$, and $(2a, \frac{5a}{2})$.

Consequently, $h$ is a compactly supported and bounded function for which $G(h, a, b)$ is a Bessel sequence. By construction, it also follows that $g$ and $h$ are dual windows. \hfill \square

Remark 2. (1) We observe that the dual window $h$ constructed from Theorem 2 and Theorem 3 is discontinuous. This is proved in the same way as in [1, Remark 4].

(2) On the line $b = \frac{3}{2}$, all of the dyadic points in the set $\{(\frac{1}{2}, \frac{3}{2})\}_{j=1}^{\infty} \cup \{(\frac{3}{4}, \frac{3}{2})\}_{j=3}^{\infty}$ belong to $\mathcal{F}(B_2)$, since most of these points belong $\Gamma_3$, except for $\{(\frac{3}{4}, \frac{3}{2}), (\frac{3}{8}, \frac{3}{2})\}$, and $\{(\frac{3}{4}, \frac{3}{2})\}$ for which we omit the proof.

(3) Our results extend to regions beyond the frame set established here and also to higher-order B-splines, as demonstrated in Figures 4 and 5.
Figure 5. The results presented here extend in a straightforward way to higher-order B-splines (left). On the right are plots of the nonzero values of the corresponding dual frames $h$ of $G(B_N, a, b)$ for $(a, b) = (0.85, 1.05)$ and $N = 2, 3, 4, 5$.

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4. Appendix

Listing 1. MATLAB code for plotting the dual of $G(B_N, a, b)$.

```matlab
function plotDual
a = .85;
b = 1.05;
N = 2;
g = @(x) fnval(spmak([-N:N,[0 1 0]]),x);
x=-a/2:.001:a/2;

% Set up and solve the linear systems for h(x)
for j=1:length(x)
    [m, Gm] = G(x(j),g,a,b);
    if m==0
        H = NaN;
    else
        bv = zeros(2*m-1,1);
        bv(m) = b;
        H(:,j) = Gm\bv;
    end
end

% concatenate the components of H
X = []; h = [];
for k = -(m-1): (m-1)
    X = [X x+k*a];
    h = [h, H(k+m,:)];
end

figure;
plot(x(abs(h)>1e-10) , h(abs(h)>1e-10) , '.');
title(sprintf('Dual of $\mathcal{G}(B_N)$ \n N=%d, a =%1.2f, b =%1.2f,' , N, a, b))
end

function [m, Y] = G(x, g, a, b, m_max)
% Returns the (2m-1) x (2m-1) matrix Y = G_3(x) and the integer m between 1 and m_max such
% that (a,b) is in \Lambda_m.  
% If (a,b) is not in \Lambda_m for any m\geq 1, Y= NaN.
% The window g is a function handle,  m_max is a large integer and x, a, and b are real
% numbers.
if nargin<5
    m_max=50;
end

m = NaN;
for mm = 1: m_max
    if (b>2*(mm-1)/(2*(2*mm-3)*a)) && (b<2*mm/(2*(2*mm-1)*a)) && (b<2/(1+a)) && (b>1)
        m = mm;
    end
end
if ~ isnan(m)
    [l,m] = meshgrid(-(m-1): (m-1));
    Y = g(x-l'/b+l*a);
else
    Y = NaN;
end
end
```

Département de Mathématiques/FAST/UAC, 01 BP : 4521, Cotonou 01, Bénin
Email address: ganiou.atindehou@fast.uac.bj

Department of Mathematical Sciences, New Jersey Institute of Technology
Email address: christin@njit.edu

Department of Mathematics, Tufts University, Medford, MA 02155
Email address: kasso.okoudjou@tufts.edu