Relationship between (2+1) and (3+1)–Friedmann–Robertson–Walker cosmologies

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In this work we establish the correspondence between solutions to the Friedmann–Robertson–Walker cosmologies for perfect fluid and scalar field sources, where both ones fulfill state equations of the form \( p + \rho = \gamma f(\rho) \), not necessarily linear ones. Such state equations are of common use in the case of matter–fluids, nevertheless, for a scalar field, they introduce relationships on the potential and kinetic scalar field energies which restrict the set of solutions. A theorem on this respect is demonstrated: From any given (3+1)–cosmological solution, obeying the quoted state equations, one can derive its (2+1)–cosmological counterpart or vice-versa. Some applications are given.

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I. INTRODUCTION

Scalar fields play a crucial role in describing cosmological models. In the standard big-bang theory such fields are included for solving most of the problems found at very early times in the evolution of the universe, and are called “inflaton” scalar field \([1, 2, 3]\). This scalar field is characterized by its scalar potential. In fact, different inflationary universe models differ from each other by its potential form. For instance, the scalar potential related to new inflation is very different from the potential of chaotic inflation. These scalar fields not only are appropriated for describing the evolution of the universe, but also they give the necessary initial condition for the formation of the large scale structure observed in the universe.

At the same time, the measurements of the luminosity–redshift relations observed for the fifty newly discovered type Ia supernovae with redshift \( z > 0.35 \) \([4, 5]\), indicate that at present the universe is expanding with an accelerated fashion suggesting a net negative pressure for the universe. One of the plausible explanation of this astronomical observation is based on the introduction of a scalar field, which is called “quintessence” or “dark energy” scalar field. This field has an associated effective scalar potential, which on its turn plays a crucial role in describing tracker solutions.

Although these scalar fields are quite different in nature, there are authors who think that the “inflaton” and the “quintessence” fields might be of the same nature, in which a very specific scalar potential form is used \([6]\). In these cases, the scalar field emerges in a kinetic–dominated regime at energy densities above the tracker solution.

On the other hand, during the past decade the three–dimensional gravity has been received much attention \([7, 8]\). The reasons for this are many and varied; however the principal one is the existence of black holes solutions in (2+1)–anti de Sitter spacetimes, which possess certain features inherent to the (3+1)–black holes \([9]\). Often it is useful to consider a physical system in lower dimensions, as for instance is done in quantum field theory. Thus it is reasonable to extend this procedure to gravity. It is believed that (2+1)–gravity will provide new insights towards a better understanding of the physically relevant (3+1)–gravity.

Most of the studies on this respect are related to the black hole spacetimes. The literature on three–dimensional cosmological models is rather scarce. Some Friedmann–Robertson–Walker cosmologies (FRW) models had been analyzed in Ref. \([10, 11, 12]\). It is noteworthy to point out that some of the non–trivial features of the (2+1)–gravity are apparent in the behavior of cosmic strings and domain walls in (3+1)–dimensions (see \([11]\) and the references therein). Among the works on cosmology, one can cite \([12]\), in which Cornish and Frankel consider the three–dimensional Einstein gravity and give the solutions for the isotropic dust–filled and radiation–dominated universes for \( k = -1, 0, 1 \). They also study FRW models in alternative relativistic theories of gravity. Cruz and Martínez \([13]\) have examined (2+1)–FRW models with a perfect fluid and a homogeneous scalar field minimally coupled to gravity.

The purpose of this contribution is to provide a new insight towards the relation between (2+1) and (3+1)–standard cosmologies coupled to a perfect fluid and a scalar field.

The outline of the present paper is as follows: In Sec.
II we briefly review the well known Einstein equations for the FRW metrics in four and three–dimensional gravities coupled to a perfect fluid and a scalar field. A theorem concerning the correspondence of cosmological solutions of a certain kind is formulated and demonstrated. In Sec III some applications are given.

II. FRW COSMOLOGY WITH A PERFECT FLUID AND A SCALAR FIELD

In this section we shall consider homogeneous FRW models filled with a perfect fluid and homogeneous scalar field $\phi$ minimally coupled to gravity with a self–interacting potential $V(\phi)$.

In (3+1)–dimensions the FRW model is given by the metric

$$ds^2 = dt^2 - a(t)^2 \left( \frac{dr^2}{1 - kr^2} + r^2 d\Omega^2 \right),$$

where as usual $d\Omega^2 := d\theta^2 + \sin^2 \theta d\phi^2$, $a(t)$ is the scale factor, and $k = -1,0,1$. The scale factor $a(t)$ of the metric is governed by equations, commonly modelled in terms of perfect fluids and a cosmological constant, if present, in which participate the matter energy density $\rho$, the matter isotropic pressure $p$, the conventional scalar field pressure $p_{\phi}$, the scalar field energy density $\rho_{\phi}$, defined through the scalar field $\phi$, and self–interacting potential $V(\phi)$ by

$$\rho_{\phi} = \frac{1}{2} \dot{\phi}^2 + V_4, \quad p_{\phi} = \frac{1}{2} \dot{\phi}^2 - V_4. \quad (2)$$

The four–dimensional field equations are:

$$3 \frac{k + \dot{a}^2}{a^2} = \kappa_4 (\rho + \rho_{\phi}), \quad (3a)$$

$$\dot{\rho}_s + 3 \frac{\dot{a}}{a} (\rho + p_s) = 0, \quad (3b)$$

$$\dot{\rho}_{\phi} + 3 \frac{\dot{a}}{a} (\rho_{\phi} + p_{\phi}) = 0. \quad (3c)$$

The equation (3b) represents the conservation of matter content, while Eq. (3c) corresponds to the energy conservation associated to the scalar field. In this case the scalar field interacts only with the gravitational field and hence, each energy–momentum tensor is conserved independently from one to another.

Similarly, in the (2+1)–dimensional case, the metric has the form

$$ds^2 = dt^2 - a(t)^2 \left( \frac{dr^2}{1 - kr^2} + r^2 d\theta d\phi \right),$$

where now $d\Omega^2 := d\theta^2$, and the conventional scalar field pressure $p_{\phi}$ and the scalar field energy density $\rho_{\phi}$ are defined similarly to the Eq. (2)

$$\rho_{\phi} = \frac{1}{2} \dot{\phi}_4^2 + V_3, \quad p_{\phi} = \frac{1}{2} \dot{\phi}_3^2 - V_3. \quad (4)$$

The following set of three–dimensional field equations has to be fulfilled:

$$\frac{k + \dot{a}^2}{a^2} = \kappa_3 (\rho + \rho_{\phi}), \quad (5a)$$

$$\dot{\rho}_s + 2 \frac{\dot{a}}{a} (\rho + p_s) = 0, \quad (5b)$$

$$\dot{\rho}_{\phi} + 2 \frac{\dot{a}}{a} (\rho_{\phi} + p_{\phi}) = 0. \quad (5c)$$

The main result of this contribution can be stated in the form of a theorem

**Theorem:** Assuming invariance of the time–coordinate, as well as the structural form invariance of the scale factor $a(t)$ in both $(2+1)$ and $(3+1)$–dimensional cosmologies, coupled to a single scalar field and perfect fluid subjected to state equations $p_s + p_{\phi} = \gamma F(\rho_s)$ and $p + \rho = \gamma f(\rho)$ respectively, where $F(\rho_{\phi})$ and $f(\rho)$ are dimensional invariant structural functions, then the gravitational constant $\kappa$ and the state parameters $\gamma$ and $\Gamma$ obey the following rules:

$$\frac{\kappa_4}{3} \equiv \kappa_3, \quad 3 \gamma_4 \equiv 2 \gamma_3, \quad 3 \Gamma_4 \equiv 2 \Gamma_3, \quad \text{(6)}$$

while the structural functions fulfill:

$$\rho_s \equiv \rho_3, \quad \rho_{\phi} \equiv \rho_\phi, \quad V_4 = \frac{1}{4} \dot{\phi}_4^2 \rightarrow V_3, \quad V_3 = \frac{1}{6} \dot{\phi}_3^2 \rightarrow V_4. \quad \text{(7)}$$

**Proof:** Considering that the time coordinate $t$ as well the scale factor $a(t)$ remain unchanged, then from Eqs. (3b) and (5a), assuming the densities $\rho$ and $\rho_{\phi}$ independent, one has

$$\frac{\dot{\rho}_s + 3 \frac{\dot{a}}{a} (\rho + p_s)}{\kappa_4} = \frac{\kappa_3}{3} (\rho_s + \rho_{\phi}) \Rightarrow \rho_s \equiv \rho_3, \quad \rho_{\phi} \equiv \rho_{\phi}, \quad \text{(8)}$$

At this point, it is in order to explain what we mean by a dimensional invariant structural function; as such we define a function whose dependence on the $t$–coordinate or scalar function $\phi$–variable, is the same in both dimensions and, if gravitational and state parameters are present in it, under (6) the form of the function remains unchanged.

Further, assuming that the state equation for matter is of the form $p + \rho = \gamma f(\rho)$, where $f(\rho)$ is a structurally invariant function, matter conservation equations yield

$$\frac{da}{a} = - \frac{1}{3 \gamma_4} \frac{d\rho_s}{f(\rho_s)} = - \frac{1}{2 \gamma_3} \frac{d\rho_3}{f(\rho_3)}, \quad \text{(9)}$$

hence, because of by assumption $f(\rho_s) \equiv f(\rho_3)$, one has

$$\ln \frac{a}{a_0} = - \frac{1}{3 \gamma_4} \int^{\rho_s} d\rho f(\rho) = - \frac{1}{2 \gamma_3} \int^{\rho_3} d\rho f(\rho), \quad \text{(10)}$$
therefore
\[ 3\gamma_4 \equiv 2\gamma_3. \] (11)

Next, assuming that the state equation for the scalar field is of the form \( p_a + \rho_a = \Gamma F(\rho_a) \), where \( F(\rho_a) \) is a structurally invariant function, the scalar field conservation equations yield
\[ \frac{da}{a} = -\frac{1}{3\Gamma_4} \frac{d\rho_{a4}}{F(\rho_{a4})} = -\frac{1}{2\Gamma_3} \frac{d\rho_{a3}}{F(\rho_{a3})}, \] (12)

hence,
\[ \ln \frac{a}{a_0} = -\frac{1}{3\Gamma_4} \int^{\rho_{a4}} \frac{d\rho_a}{F(\rho_a)} = -\frac{1}{2\Gamma_3} \int^{\rho_{a3}} \frac{d\rho_a}{F(\rho_a)}. \] (13)

therefore
\[ 3\Gamma_4 \equiv 2\Gamma_3. \] (14)

From the equations \( \dot{\phi}^2 = \rho_a + p_a \), one obtains
\[ \dot{\phi}_4^2 = \Gamma_4 F(\rho_{a4}), \quad \dot{\phi}_3^2 = \Gamma_3 F(\rho_{a3}), \] (15)

thus
\[ \frac{\dot{\phi}_4}{\sqrt{\Gamma_4}} = \frac{\dot{\phi}_3}{\sqrt{\Gamma_3}} = \sqrt{F(\rho_a)}. \] (16)

consequently, taking into account Eqs. (14), up to additive constants one gets
\[ \dot{\phi}_4 \equiv \sqrt{\frac{2}{3}} \dot{\phi}_3 \equiv \phi_4 \equiv \sqrt{\frac{3}{2}} \phi_3. \] (17)

Finally, from equations \( 2V(\phi) = \rho_a - p_a = 2\rho_a - \Gamma F(\rho_a) \), namely
\[ V(\phi_a) = \rho_{a4} - \frac{\Gamma_4}{2} F(\rho_{a4}), \quad V(\phi_3) = \rho_{a3} - \frac{\Gamma_3}{2} F(\rho_{a3}), \] (18)

one arrives at
\[ V_4 - \frac{1}{4} \phi_4^2 \rightarrow V_3, \quad V_3 + \frac{1}{6} \phi_3^2 \rightarrow V_4. \] (19)

It is noteworthy to point out that this theorem is based mainly on the equality of the dynamical structure of the field equations in different dimensions. The physical content of solutions in the presence of perfect fluids changes as viewed from different dimensional spacetimes; for instance, starting in (3+1)–cosmology with dust, the (2+1)–counterpart will be a fluid with \( \gamma_3 = 1/2 \), therefore is no way within this treatment to relate dust with dust in the considered dimensions. On the other hand, one has to recall the constraints that energy conditions impose on the matter.

**Corollary:** Cosmologies in (2+1) and (3+1)–dimensions coupled to a single scalar field, in which is assumed the time-coordinate as well as the scale factors to be the same, up to the parametrization of the gravitational and state constants given below, for both (3+1) and (2+1)–spaces, are related according to the following rules:
\[ \frac{\kappa_4}{3} \equiv \kappa_3, \quad 3\Gamma_4 \equiv 2\Gamma_3, \]
\[ \rho_{a4} \equiv \rho_{a3}, \quad \sqrt{\frac{3}{2}} \phi_4 \equiv \phi_3, \quad V_4 - \frac{1}{4} \phi_4^2 \rightarrow V_3, \quad V_3 + \frac{1}{6} \phi_3^2 \rightarrow V_4. \] (20)

The proof follows immediately from the theorem above.

Remark: It is noteworthy to point out that state equations of the form \( p + \rho = \gamma(\rho) \) although they work well for matter-perfect fluids, in the case of a scalar field they introduce a relation between the scalar potential \( V(\phi) \) and the kinetic energy \( \dot{\phi}^2/2 \), restricting in this way the set of scalar field solutions. It would be of interest to search for wider classes of scalar field solutions and establish an algorithm to relate FRW cosmological models in different dimensions, this kind of research is in progress.

## III. Generating Solutions Via Transformations

From any given solution in (2+1)–cosmology with a single scalar field, the above relations (16) or (20) allow one to construct solutions of the similar kind in (3+1)–spacetime, and conversely. In particular, in what follows we shall restrict our study to flat FRW cosmologies.

### A. Barrow–Burd–Lancaster and Madsen inflationary solutions

Barrow, Burd and Lancaster [11] determined two exact solutions exhibiting the evolution of cosmological models containing self-interacting scalar fields with physically interesting potentials in the zero-curvature FRW model. In this case the equation of state, a non–linear one, of the scalar field is given by \( p_{a4} + \rho_{a4} = \alpha \rho_{a4}^{1/2} \). One of the solutions is given by
\[ a(t) = a_0 e^{-\frac{1}{2}((\kappa_3 A^2 e^{\sqrt{\kappa_3}})^{1/3}),} \] (21a)
\[ \Phi_4(t) = A e^{\sqrt{\kappa_3} t/2}, \] (21b)
\[ V(\phi_3) = \frac{1}{2} \kappa_3 \phi_3^2 - \phi_3^2, \] (21c)

where \( a_0, \mu \) and \( A \) are constants.

Thus using the relations (20), we can obtain for the (3+1)–counterpart the following zero–curvature scalar FRW cosmology:
\[ a(t) = a_0 e^{-\frac{1}{2}((\kappa_3 A^2 e^{\sqrt{\kappa_3}})^{1/3})}, \] (22a)
\[ \phi_4(t) = A \sqrt{\frac{2}{3}} e^{\pm \sqrt{\kappa_4} t/2}. \quad (22b) \]

\[ V(\phi_4) = \mu \left( \frac{3}{8} \kappa_4 \phi_4^4 - \phi_4^2 \right), \quad (22c) \]

This four-dimensional cosmological model has been previously found by Madsen \[14\]. In this case the inflationary solution corresponds to the negative sign in the exponents and admits symmetry breaking.

A second four-dimensional inflationary solution may be obtained from the (2+1)-expanding universe given by

\[ a(t) = t^2 \sqrt{1 + \frac{A}{t^3}}, \quad (23a) \]

\[ \phi_3 = \sqrt{2 \rho_0} t \ln \left[ C_0 t^2 \left( 1 + \frac{A}{t^3} \right) \right], \quad (23b) \]

\[ V(\phi_3) = 12 \rho_0 \frac{t}{A + t^3} = 12 \rho_0 C_0 e^{-\phi_3/\sqrt{2 \rho_0}}, \quad (23c) \]

where \( A, \rho_0 \) and \( C_0 \) are constants. For writing the above solution in the Barrow’s form one has to choose \( \rho_0 = \frac{1}{4 \kappa_4} \) and \( C_0 = \frac{\kappa_4 A}{3} \).

Thus, using the relations \[24\] we can obtain for the (3+1)–counterpart of the following zero-curvature scalar FRW cosmology:

\[ a(t) = t^2 \sqrt{1 + \frac{A}{t^3}}, \quad (24a) \]

\[ \phi_4 = \sqrt{\frac{4 \rho_0}{3}} \ln \left[ C_0 t^2 \left( 1 + \frac{A}{t^3} \right) \right], \quad (24b) \]

\[ V(\phi_4) = \frac{\rho_0}{3} \frac{40 t^6 + 32 A t^3 + A^2}{t^2(A + t^3)^2}. \quad (24c) \]

This solution, as far as we know, has not been reported before in the literature. Then it is of certain interest to study this four-dimensional inflationary cosmological model.

B. Four-dimensional version of the Cruz–Martínez solution

Cruz and Martínez \[15\] have obtained a solution which describes a (2+1)–flat FRW cosmology with a homogeneous scalar field with a self–interacting potential whose energy density redshifts as \( a^{-2 \gamma_4} \), where \( a(t) \) is the scale factor. The solution with state equation \( p_{\phi_4} + \rho_{\phi_4} = \gamma_4 \rho_{\phi_4} \) may be written as:

\[ a(t) = (t_0 + \epsilon_4 \sqrt{\kappa_4} t)^{1/\gamma_4}, \quad (25a) \]

\[ \phi_4(t) = \frac{1}{\sqrt{\kappa_4} \gamma_4} \ln \left( t_0 + \epsilon_4 \sqrt{\kappa_4} t \right) + \phi_0, \quad (25b) \]

\[ V(\phi_4) = \frac{2 - \gamma_4}{2} e^{-2 \sqrt{\kappa_4} t \phi_4} - \phi_0), \quad (25c) \]

where \( \epsilon_4 = \pm 1, \rho_{\phi_0}, \phi_0 \) and \( a_0 \) are constants of integration.

Using the relations \[26\] we derive the (3+1)–scalar FRW cosmology counterpart:

\[ a(t) = \left( t_0 + \frac{3 \gamma_4 \epsilon_4}{2} \sqrt{\kappa_4} t \right)^{3/4}, \quad (26a) \]

\[ \phi_4(t) = \frac{2 - \gamma_4}{2} \ln \left( t_0 + \frac{3 \gamma_4 \epsilon_4}{2} \sqrt{\kappa_4} t \right) + \phi_0, \quad (26b) \]

\[ V(\phi_4) = \frac{2 - \gamma_4}{2} e^{-3 \kappa_4 \gamma_4 \gamma_4 (\phi_4 - \phi_0)}, \quad (26c) \]

where \( \epsilon_4 = \pm 1, \rho_{\phi_0}, \phi_0 \) and \( a_0 \) are the same constants of integration as for \[24\]. Notice that for \( 3 \gamma_4 < 2 \) this solution describes an accelerating universe.

C. Barrow-Saich solution

The previously treated solutions contain only a single scalar field as a source. We shall now assume the source of a (3+1)–universe to be a perfect fluid, with a equation of state \( p_\gamma + \rho_\gamma = \gamma_4 \rho_\gamma \), and a scalar field with equation of state \( p_\phi + \rho_\phi = \frac{1}{2} \gamma_4 \rho_\phi \). The conservation equations of the perfect fluid and scalar field give \( p_\gamma = A_\gamma a^{-2(3 \gamma_4)} \) and \( p_\phi = -A_\phi a^{-\gamma_4} \), respectively. In this case the obtained solution is defined through \[16\]:

\[ a(t) = \left( \frac{3 \kappa_4}{16} A_\phi (t - t_0)^2 - A/A_\phi \right)^{2/(3 \gamma_4)}, \quad (27a) \]

\[ \phi_4(t) = \phi_0 + \sqrt{\frac{2}{3}} \frac{1}{\gamma_4 \kappa_4} \ln \left[ \frac{3 \gamma_4}{4} A_\phi \frac{\kappa_4}{3} (t - t_0) \right] + \sqrt{\frac{3 \gamma_4^2}{16} \kappa_4} A_\phi^2 (t - t_0)^2 - A, \quad (27b) \]

\[ V(\phi_4) = \frac{1}{2} \left( 4 - \gamma_4 \right) A_\phi^2 \frac{e^{\sqrt{3 \kappa_4 \gamma_4} t \phi_4} (\phi_4 - \phi_0)}{e^{\sqrt{3 \kappa_4 \gamma_4} t \phi_4} (\phi_4 - \phi_0) - A} \left( \phi_4 - \phi_0 \right) \]

where \( t_0, A_\phi, \phi_0 \) and \( A \) are arbitrary constants. Again, in order to have an accelerating universe we should restrict \( \gamma_4 < 4/3 \).

Now using the relations \[6\] and \[7\] we can obtain the following (2+1)–flat FRW cosmology:

\[ a(t) = \left( \frac{\kappa_4}{4} A_\phi (t - t_0)^2 - A/A_\phi \right)^{1/\gamma_4}, \quad (28a) \]
\[ \phi_3(t) = \phi_0 + \sqrt{\frac{2}{\gamma_3 \kappa_3}} \ln \left[ \frac{\gamma_3}{2} A_s \sqrt{\kappa_3} (t - t_0) \right] + \sqrt{\frac{\gamma_3^2 A_s^2}{4} (t - t_0)^2 - A} \] (28b)

\[ V(\phi_3) = (4 - \gamma_3) A_s^2 \frac{e^{\sqrt{2 \kappa_3 \gamma_3} (\phi_3 - \phi_0)}}{[e^{\sqrt{2 \kappa_3 \gamma_3} (\phi_3 - \phi_0)} - A]^2} \] (28c)

where \( t_0, A_s, \phi_0 \) and \( A \) are constants.

\section*{IV. CONCLUDING REMARKS}

In this report it is established a theorem which allows one to put in correspondence (2+1) and (3+1)–FRW cosmologies. The established relationship holds for solutions modelled through conventional perfect fluids–matter and scalar potential fluids–obeying state equations of the form \( p + \rho = \gamma f(\rho) \), notice that they are not necessarily linear state equations, which in the case of the scalar field impose conditions on the energies and consequently restrict the set of scalar field solutions. A procedure to derive from a given (2+1)–FRW solution a (3+1)–FRW solution and vice-versa was exhibited. For instance, we have shown that the Barrow et al (2+1)–metric becomes the (3+1)–Madsen cosmological solution. Moreover, from the (3+1)–Barrow–Saich metric structure one derive its (2+1)–counterpart. Information about the physical interpretation of the above considered solutions can be found in the quoted references. One of the important features of this approach resides on the possibility of interpreting the related cosmologies from dimensionally different points of view; state equations in different dimensions reveal dissimilar physical content. For instance, (3+1)–radiation possesses as counterpart the (2+1)–stiff matter. In spite of the generality of this theorem, based mainly on the equality of the dynamical structure in different dimensions, there are classes of state equations which remain uncovered by this theorem, for example the polytropic law; a work on this line is in progress and will be published elsewhere.

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