Abstract

A set $S$ of vertices in a graph $G$ is a dominating set if every vertex not in $S$ is adjacent to a vertex in $S$. If, in addition, $S$ is an independent set, then $S$ is an independent dominating set. The independent domination number $i(G)$ of $G$ is the minimum cardinality of an independent dominating set in $G$. In 2013 Goddard and Henning [Discrete Math 313 (2013), 839–854] conjectured that if $G$ is a connected cubic graph of order $n$, then $i(G) \leq \frac{3}{8}n$, except if $G$ is the complete bipartite graph $K_{3,3}$ or the 5-prism $C_5 \square K_2$. Further they construct two infinite families of connected cubic graphs with independent domination three-eighths their order. They remark that perhaps it is even true that for $n > 10$ these two families are only families for which equality holds. In this paper, we provide a new family of connected cubic graphs $G$ of order $n$ such that $i(G) = \frac{3}{8}n$. We also show that if $G$ is a subcubic graph of order $n$ with no isolated vertex, then $i(G) \leq \frac{1}{2}n$, and we characterize the graphs achieving equality in this bound.

Keywords: Independent domination; Cubic graph; Subcubic graph

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1 Introduction

A set $S$ of vertices in a graph $G$ is a dominating set if every vertex not in $S$ is adjacent to a vertex in $S$. If, in addition, $S$ is an independent set, then $S$ is an independent dominating set, abbreviated ID-set. The independent domination number, denoted $i(G)$, of $G$ is the minimum cardinality of an ID-set in $G$. The concept of independent domination number of graphs is studied extensively in the literature, for example see [1, 2, 3, 4, 5, 7, 8, 9, 10, 11, 13, 14, 15, 16, 17, 18, 19, 20]. A survey on independent domination in graphs can be found in [6].

For notation and graph theory terminology we generally follow [12]. The order of a graph $G$ with vertex set $V(G)$ and edge set $E(G)$ is denoted by $n(G) = |V(G)|$ and its size by $m(G) = |E(G)|$. Two vertices are neighbors in $G$ if they are adjacent. The open neighborhood of a vertex $v$ in $G$ is the set of neighbors of $v$, denoted $N_G(v)$. Thus, $N_G(v) = \{u \in V(G) \mid uv \in E(G)\}$. The closed neighborhood of $v$ is the set $N_G[v] = N_G(v) \cup \{v\}$. The degree of a vertex $v$ in $G$ is denoted $d_G(v) = |N_G(v)|$. We denote the minimum and maximum degrees among the vertices of $G$ by $\delta(G)$ and $\Delta(G)$, respectively. A cubic graph is a graph in which every vertex has degree 3, while a subcubic graph is a graph with maximum degree at most 3.

Further, the subgraph obtained from $G$ by deleting all vertices in $S$ and all edges incident with vertices in $S$ is denoted by $G - S$. If $S = \{v\}$, we simply denote $G - \{v\}$ by $G - v$. A leaf of a graph $G$ is a vertex of degree 1 in $G$, while a support vertex of $G$ is a vertex adjacent to a leaf. A star is the graph $K_{1,k}$, where $k \geq 1$; that is, a star is a tree with at most one vertex that is not a leaf. A double star is a tree with exactly two (adjacent) non-leaf vertices. Further if one of these vertices is adjacent to $r$ leaves and the other to $s$ leaves, then we denote the double star by $S(r,s)$. We denote the path and cycle on $n$ vertices by $P_n$ and $C_n$, respectively, and we denote a complete bipartite with partite sets of cardinalities $n$ and $m$ by $K_{n,m}$. The corona $\text{cor}(G)$ of a graph $G$, also denoted $G \circ P_1$ in the literature, is the graph obtained from $G$ by adding a pendant edge to each vertex of $G$. For $k \geq 1$ an integer, we use the standard notation $[k] = \{1, \ldots, k\}$ and $[k]_0 = \{0, 1, \ldots, k\}$.

2 Motivation and Known Results

As remarked in [6], since every bipartite graph is the union of two independent sets, each of which dominates the other, we have the following well-known bound on the independent domination number of a bipartite graph.

Proposition 1 If $G$ is a bipartite graph with no isolated vertices of order $n$, then $i(G) \leq \frac{1}{2}n$.

As noted in [6], the bound in Proposition 1 is sharp as may be seen by taking $G = K_{k,k}$ for any $k \geq 1$. In particular, if $G = K_{k,k}$ and $k \in [3]$, then $G$ is a connected subcubic graph of order $n = 2k$ such that $i(G) = \frac{1}{2}n$.  

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It remains an open problem to determine best possible upper bounds on the independent domination number of a connected cubic graph in terms of its orders. Lam, Shiu, and Sun [14] proved that if $G$ is a connected cubic graph of order $n$ different from $K_{3,3}$, then $i(G) \leq \frac{2}{5}n$, where the graph $K_{3,3}$ is given in Figure 1(a). This bound is achieved by the 5-prism $C_5 \square K_2$ which is illustrated in Figure 1(b).

(a) $K_{3,3}$
(b) $C_5 \square K_2$

Figure 1: The graphs $K_{3,3}$ and $C_5 \square K_2$.

Goddard and Henning [6] posed the conjecture that the $\frac{2}{5}n$ bound on the independent domination number can be improved if we forbid the exceptional graphs $K_{3,3}$ and $C_5 \square K_2$.

**Conjecture 1** ([6]) If $G \not\in \{K_{3,3}, C_5 \square K_2\}$ is a connected, cubic graph of order $n$, then $i(G) \leq \frac{3}{8}n$.

Dorbec, Henning, Montassier, and Southey [3] proved Conjecture 1 in the case when there is no subgraph isomorphic to $K_{2,3}$. In general, however, Conjecture 1 remains unresolved.

Goddard and Henning [6] constructed two infinite families $G_{cubic}$ and $H_{cubic}$ of connected cubic graphs with independent domination number three-eighths their orders as follows. For $k \geq 1$, a graph in the family $G_{cubic}$ is constructed by taking two copies of the cycle $C_{4k}$ with respective vertex sequences $a_1b_1c_1d_1 \ldots a_kb_kc_kd_k$ and $w_1x_1y_1z_1 \ldots w_kx_ky_kz_k$, and joining $a_i$ to $w_i$, $b_i$ to $x_i$, $c_i$ to $z_i$, and $d_i$ to $y_i$ for each $i \in [k]$. For $\ell \geq 1$, a graph in the family $H_{cubic}$ is constructed by taking a copy of a cycle $C_{3\ell}$ with vertex sequence $a_1b_1c_1 \ldots a_{\ell}b_{\ell}c_{\ell}$, and for each $i \in [\ell]$, adding the vertices $\{w_i, x_i, y_i, z_i^1, z_i^2\}$, and joining $a_i$ to $w_i$, $b_i$ to $x_i$, and $c_i$ to $y_i$, and further for each $j \in [2]$, joining $z_i^j$ to each of the vertices $w_i$, $x_i$, and $y_i$. Graphs in the families $G_{cubic}$ and $H_{cubic}$ are illustrated in Figure 2(a) and 2(b), respectively.

(a) $G$
(b) $H$

Figure 2: Graphs $G \in G_{cubic}$ and $H \in H_{cubic}$ of order $n$ with $i(G) = i(H) = \frac{3}{8}n$. 

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Theorem 1 \([6]\) If \(G \in G_{\text{cubic}} \cup H_{\text{cubic}}\) has order \(n\), then \(i(G) = \frac{3}{8}n\).

It is remarked in \([6]\) that “Perhaps even more than Conjecture 1 is true, in that the only extremal graphs are those in \(G_{\text{cubic}} \cup H_{\text{cubic}}\). We have confirmed by computer search that this is true when \(n \leq 20\).”

We remark that several papers, see for example \([1, 2, 3, 11, 19]\), in which upper bounds are obtained on the independent domination number of cubic graphs present more general results on subcubic graphs.

3 Main Results

In this paper we have two immediate aims. Our first aim is to provide a new family of connected cubic graphs, different from the families \(G_{\text{cubic}}\) and \(H_{\text{cubic}}\), such that every graph \(G\) of order \(n\) in the family satisfies \(i(G) = \frac{3}{8}n\). We shall prove the following result, where \(F_{\text{cubic}}\) is the family of connected cubic graphs constructed in Section 4.

**Theorem 2** If \(G \in F_{\text{cubic}}\) has order \(n\), then \(n \geq 16\) and \(n \equiv 0 \pmod{8}\) and \(i(G) = \frac{3}{8}n\).

Our second aim is to provide a tight upper bound on the independent domination number of a subcubic graph, and to characterize the graphs achieving equality in this bound. Let \(G_1 \cong K_{2,2}\) and \(G_2 \cong K_{3,3}\), and let \(G_3, G_4, G_5\) be the three graphs shown in Figure 3.

![Figure 3: The graphs \(G_3, G_4\) and \(G_5\)](image)

We shall prove the following result, a proof of which is presented in Section 4.

**Theorem 3** If \(G\) is a subcubic graph of order \(n\) with no isolated vertex, then \(i(G) \leq \frac{1}{2}n\), with equality if and only if the following holds.

(a) \(G \in \{G_1, G_2, G_3, G_4, G_5\}\).
(b) \(n = 2k\) for some \(k \geq 1\) and \(G = \text{cor}(P_k)\).
(c) \(n = 2k\) for some \(k \geq 3\) and \(G = \text{cor}(C_k)\).

4 Proof of Theorem 2

Let \(X\) and \(Y\) be the graphs shown in Figure 4(a) and 4(b), respectively.
Let $\mathcal{F}_{\text{cubic}}$ be the family of graphs constructed as follows. A graph $G$ in the family $\mathcal{F}_{\text{cubic}}$ is constructed as follows. We start by taking a cycle $C: v_1v_2 \ldots v_kv_1$ on $k \geq 2$ (for $k = 2$ we mean two vertices adjacent with two different edges) vertices and coloring every vertex on the cycle $C$, red or blue in such a way that the number of red vertices is even. We then replace each red vertex on $C$ with a copy of $X$, and each blue vertex on $C$ with a copy of $Y$. (In the case $k = 2$ we only replace each vertex by a copy of $Y$.) We call each resulting copy of $X$ and $Y$ an $X$-copy and $Y$-copy of $G$, respectively. Let $G_i$ be the $X$-copy or $Y$-copy associated with the vertex $v_i$ on the cycle $C$ for each $i \in [k]$. Thus if $v_i$ is colored red, then $G_i \cong X$, while if $v_i$ is colored blue, then $G_i \cong Y$ for $i \in [k]$. We note that there are an even number of $X$-copies in $G$. Next we partition these $X$-copies into pairs. For each resulting pair $\{X_1, X_2\}$ where $X_i \cong X$ for $i \in [2]$, we add two edges as follows: If $v_{i1}$ and $v_{i2}$ denote the two (adjacent) vertices of degree 2 in $X_i$ for $i \in [2]$, then we add the edges $v_{i1}v_{i2}$ and $v_{i2}v_{i2}$ as illustrated in Figure 5.

![Figure 5: Joining of the pairs $X_1$ and $X_2$](image)

We note that each $X$-copy of $G$ contains a vertex of degree 1 and each $Y$-copy of $G$ contains two vertices of degree 2. We now complete the construction of the graph $G$ as follows. Consider the subgraphs $G_i$ and $G_{i+1}$ where addition is taken modulo $k$ and where $i \in [k]$. If $G_i$ is an $X$-copy, then let $x_i$ denote the vertex of degree 1 in $G_i$, while if $G_i$ is a $Y$-copy, then let $y^1_i$ and $y^2_i$ denote the two vertices of degree 2 in $G_i$. If both $G_i$ and $G_{i+1}$ are $X$-copies, then add the edge $x_1x_2$. If both $G_i$ and $G_{i+1}$ are $Y$-copies, then add the edge $y^2_1y^1_{i+1}$. If $G_i$ is an $X$-copy and $G_{i+1}$ is a $Y$-copy, then add the edge $x_1y^1_{i+1}$. If $G_i$ is a $Y$-copy and $G_{i+1}$ is an $X$-copy, then add the edge $y^2_{i+1}x_{i+1}$. We do this for each $i \in [k]$, and let $G$ denote the resulting graph. An example of a graph $G$ in the family $\mathcal{F}_{\text{cubic}}$ constructed from a colored 7-cycle (here $k = 7$) with four red vertices and three blue vertices is illustrated in Figure 6.

![Figure 6](image)

We are now in a position to prove Theorem 2. Recall its statement.

**Theorem 2** If $G \in \mathcal{F}_{\text{cubic}}$ has order $n$, then $n \geq 16$ and $n \equiv 0 \pmod{8}$ and $i(G) = \frac{3}{8}n$. 

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If \( G \in \mathcal{F}_{\text{cubic}} \) has order \( n \), then by construction \( G \) is obtained from a \( k \)-cycle for some \( k \geq 2 \) by replacing each vertex with a copy of \( X \) or \( Y \) and adding certain edges to produce a connected cubic graph. Since each copy of \( X \) and \( Y \) has order 8, we note that \( n = 8k \). Thus, \( n \geq 16 \) and \( n \equiv 0 \pmod{8} \). Next we show that \( i(G) = \frac{3}{8}n \).

Let \( S \) be an arbitrary ID-set in \( G \). We show that \( S \) contains at least three vertices from every \( X \)-copy and \( Y \)-copy in \( G \). First we consider an \( X \)-copy in \( G \), and let the vertices in this \( X \)-copy be named as in Figure 7. For notational convenience, we simply call this subgraph \( X \).

Figure 6: A graph \( G \) in the family \( \mathcal{F}_{\text{cubic}} \) constructed from a colored 7-cycle

\begin{figure}[h]
\centering
\includegraphics[width=0.5\linewidth]{fig6}
\caption{A graph \( G \) in the family \( \mathcal{F}_{\text{cubic}} \) constructed from a colored 7-cycle}
\end{figure}

If \( \{a_1, a_2\} \cap S = \emptyset \), then in order to dominate the vertex \( b_i \), we note that either \( b_i \in S \) or \( c_i \in S \). Thus, \( |\{b_i, c_i\} \cap S| = 1 \) for all \( i \in [3] \), implying that \( |S \cap V(X)| \geq 3 \), as desired. Hence we may assume that \( |\{a_1, a_2\} \cap S| \geq 1 \), for otherwise the desired result follows. Renaming \( a_1 \) and \( a_2 \) if necessary, we may further assume that \( a_1 \in S \). Since \( S \) is an independent set, we note that \( \{b_1, b_2, b_3\} \cap S = \emptyset \), implying that \( a_2 \in S \). We show that the set \( S \) contains at least one vertex from the set \( \{c_1, c_2, c_3\} \). Suppose, to the contrary, that \( \{c_1, c_2, c_3\} \cap S = \emptyset \). By the construction, \( c_i \) has exactly one neighbor \( c'_i \) in \( G \setminus V(X) \) for \( i \in [2] \). In order to dominate the vertices \( c_1 \) and \( c_2 \), our earlier observations imply that \( c'_1, c'_2 \in S \). However by construction, the vertices \( c'_1 \) and \( c'_2 \) are adjacent, implying that the set \( S \) contains two adjacent vertices, contradicting the fact that \( S \) is an independent set. Hence, \( \{c_1, c_2, c_3\} \cap S \neq \emptyset \), implying that \( |S \cap V(X)| \geq 3 \), as desired.

\begin{figure}[h]
\centering
\includegraphics[width=0.2\linewidth]{fig7}
\caption{An \( X \)-copy in \( G \)}
\end{figure}

If \( \{a_1, a_2\} \cap S = \emptyset \), then in order to dominate the vertex \( b_i \), we note that either \( b_i \in S \) or \( c_i \in S \). Thus, \( |\{b_i, c_i\} \cap S| = 1 \) for all \( i \in [3] \), implying that \( |S \cap V(X)| \geq 3 \), as desired. Hence we may assume that \( |\{a_1, a_2\} \cap S| \geq 1 \), for otherwise the desired result follows. Renaming \( a_1 \) and \( a_2 \) if necessary, we may further assume that \( a_1 \in S \). Since \( S \) is an independent set, we note that \( \{b_1, b_2, b_3\} \cap S = \emptyset \), implying that \( a_2 \in S \). We show that the set \( S \) contains at least one vertex from the set \( \{c_1, c_2, c_3\} \). Suppose, to the contrary, that \( \{c_1, c_2, c_3\} \cap S = \emptyset \). By the construction, \( c_i \) has exactly one neighbor \( c'_i \) in \( G \setminus V(X) \) for \( i \in [2] \). In order to dominate the vertices \( c_1 \) and \( c_2 \), our earlier observations imply that \( c'_1, c'_2 \in S \). However by construction, the vertices \( c'_1 \) and \( c'_2 \) are adjacent, implying that the set \( S \) contains two adjacent vertices, contradicting the fact that \( S \) is an independent set. Hence, \( \{c_1, c_2, c_3\} \cap S \neq \emptyset \), implying that \( |S \cap V(X)| \geq 3 \), as desired.
Next we consider a $Y$-copy in $G$, and let the vertices in this $Y$-copy be named as in Figure 8. For notational convenience, we simply call this subgraph $Y$. We show that $|S \cap V(Y)| \geq 3$.

If $\{a_1, a_2\} \cap S = \emptyset$, then as observed earlier, $|\{b_i, c_i\} \cap S| = 1$ for all $i \in [3]$, implying that $|S \cap V(X)| \geq 3$, as desired. Hence we may assume that $|\{a_1, a_2\} \cap S| \geq 1$, for otherwise the desired result follows. Further we may assume that $a_1 \in S$. As observed earlier, this implies that $\{b_1, b_2, b_3\} \cap S = \emptyset$ and $a_2 \in S$. In order to dominate the vertex $c_2$, the set $S$ contains at least one of the vertices $c_1$, $c_2$ and $c_3$. Thus, $|\{c_1, c_2, c_3\} \cap S| \geq 1$, and so $|S \cap V(Y)| \geq 3$, as desired. This completes the proof of Theorem 2.

5 Proof of Theorem 3

In this section, we present a proof of Theorem 3. First we prove that the independent domination number of a subcubic graph with no isolated vertex is at most one-half the order of the graph.

Theorem 4 If $G$ is a subcubic graph of order $n$ with no isolated vertex, then $i(G) \leq \frac{1}{2}n$.

Proof. By linearity, the independent domination number of a graph is the sum of the independent domination numbers of its components. Hence it suffices for us to prove the bound for connected graphs; that is, we prove that if $G$ is a connected subcubic graph of order $n \geq 2$, then $i(G) \leq \frac{1}{2}n$. We proceed by induction on the order $n \geq 2$. If $n = 2$, then $G = K_2$ and $i(G) = 1 = \frac{1}{2} \times 2$. If $n = 3$, then $G = K_3$ or $G = P_3$, and in both cases, $i(G) = 1 < \frac{1}{2} \times 3$. This establishes the base cases. Let $n \geq 4$ and assume that if $G'$ is a connected subcubic graph of order $n'$ where $2 \leq n' < n$, then $i(G') \leq \frac{1}{2}n'$. Let $G$ be a connected subcubic graph of order $n$. If $G$ is a bipartite graph, then the desired bound follows from Proposition 1. Hence we may assume that $G$ contains an odd cycle $C$.

First, assume that there exists a vertex $u$ on the cycle $C$ with a leaf neighbor, say $w$, and consider the graph $H = G - \{u, w\}$. Since the two neighbors of $u$ on the cycle $C$ are connected in $H$ by the path $C - u$, the graph $H$ is a connected subcubic graph. Since $n \geq 4$, we note that $|V(H)| = n - 2 \geq 2$. Applying the induction to $H$, we have $i(H) \leq \frac{1}{2}|V(H)| = \frac{1}{2}(n - 2)$. Every minimum ID-set in $H$ can be extended to an ID-set of $G$ by adding to it the vertex $w$, implying that $i(G) \leq i(H) + 1 \leq \frac{1}{2}n$. Hence we may assume that no vertex on the cycle $C$ has a leaf neighbor.
Next, assume there are two consecutive vertices, say \( u \) and \( w \), on the cycle \( C \) both of degree 2 in \( G \). We now consider the graph \( H = G - \{u, w\} \). We note that the graph \( H \) is a connected subcubic graph. Further since \( n \geq 4 \), we note that \( |V(H)| = n - 2 \geq 2 \). Applying the induction to \( H \), we have \( i(H) \leq \frac{1}{2}|V(H)| = \frac{1}{2}(n - 2) \). Let \( S' \) be a minimum ID-set of \( H \), and let \( u' \) be the neighbor of \( u \) different from \( w \) and let \( w' \) be the neighbor of \( w \) different from \( u \). (Possibly, \( u' = w' \).) If \( u' \notin S' \), then let \( S = S' \cup \{u\} \). If \( u' \in S \) and \( u' \notin S \), then let \( S = S' \cup \{v\} \). If \( u' \in S \) and \( u' \notin S \), then let \( S = S' \). In all cases, \( S \) is an ID-set of \( G \), and so \( i(G) \leq |S| \leq |S'| + 1 = i(H) + 1 \leq \frac{1}{2}n \). Hence we may assume that no two consecutive vertices on the cycle \( C \) both have degree 2 in \( G \).

Let \( u \) and \( v \) be two arbitrary consecutive (adjacent) vertices on the cycle \( C \). Suppose that there exists a vertex \( w \) of degree 2 adjacent to both \( u \) and \( v \). In this case, we consider the connected subcubic graph \( H = G - \{u, v, w\} \). If \( |V(H)| = 1 \), then \( n = 4 \) and \( G \cong K_4 - e \) where \( e \) is the missing edge of the complete graph \( K_4 \). In this case, \( i(G) = 1 < \frac{1}{2} \times 4 \). Hence we may assume that \( |V(H)| \geq 2 \). Applying the induction to \( H \) we have \( i(H) \leq \frac{1}{2}|V(H)| = \frac{1}{2}(n - 3) \). Every minimum ID-set of \( H \) can be extended to an ID-set of \( G \) by adding to it the vertex \( w \), implying that \( i(G) \leq i(H) + 1 < \frac{1}{2}n \). Hence we may assume that there is no vertex of degree 2 adjacent to both \( u \) and \( v \).

We now consider the subcubic graph \( H = G - \{u, v\} \). With our assumptions, we note that \( H \) has at most three components, each of which has order at least 2. Let \( H_1, \ldots, H_t \) be the components of \( H \), and so \( t \leq 3 \). Let \( S_i \) be a minimum ID-set of \( H_i \) for \( i \in [t] \). By the inductive hypothesis, \( |S_i| \leq \frac{1}{2}|V(H_i)| \) for \( i \in [t] \). Let
\[
S' = \bigcup_{i=1}^{t} S_i.
\]
If \( u \) has no neighbor in \( S' \), then let \( S = S' \cup \{u\} \). If \( u \) has a neighbor in \( S' \) and \( v \) has no neighbor in \( S' \), then let \( S = S' \cup \{v\} \). If both \( u \) and \( v \) have a neighbor in \( S' \), then let \( S = S' \). In all three cases, the set \( S \) is an ID-set of \( G \), and so \( i(G) \leq |S| \leq |S'| + 1 \leq \frac{1}{2}|V(H)| + 1 = \frac{1}{2}n \). This completes the proof of Theorem 3.

We are now in a position to present a proof of Theorem 3. Recall its statement.

**Theorem 3.** If \( G \) is a subcubic graph of order \( n \) with no isolated vertex, then \( i(G) \leq \frac{1}{2}n \). Further, if \( G \) is connected, then equality holds if and only if the following holds.

- (a) \( G \in \{G_1, G_2, G_3, G_4, G_5\} \).
- (b) \( n = 2k \) for some \( k \geq 1 \) and \( G = \text{cor}(P_k) \).
- (c) \( n = 2k \) for some \( k \geq 3 \) and \( G = \text{cor}(C_k) \).

**Proof.** The upper bound \( i(G) \leq \frac{1}{2}n \) is a restatement of Theorem 4. If \( G \) is a connected subcubic graph of order \( n \) that satisfies (a), (b) or (c) in the statement of the theorem, then it is a simple exercise to check that \( i(G) = \frac{1}{2}n \). Hence it suffices to prove that if \( G \) is a connected subcubic graph of order \( n \geq 2 \) satisfying \( i(G) = \frac{1}{2}n \), then (a), (b) or (c) in the statement of the theorem hold. We proceed by induction on the order \( n \geq 2 \). We note that \( n = 2i(G) \) is even since \( i(G) \) is an integer. If \( n = 2 \), then \( G = K_2 = \text{cor}(P_1) \). Suppose that
n = 4. If Δ(G) = 3, then \(i(G) = 1 < \frac{1}{2} \times 4\), a contradiction. Hence, Δ(G) = 2, and so either \(G = P_4 = \text{cor}(P_2)\) or \(G = K_{2,2} = G_1\). This establishes the base cases. Let \(n \geq 6\) be even and assume that if \(G'\) is a connected subcubic graph of even order \(n'\) where \(2 \leq n' < n\) satisfying \(i(G') = \frac{1}{2}n'\), then (a), (b) or (c) in the statement of the theorem hold. Let \(G\) be a connected subcubic graph of order \(n\) satisfying \(i(G) = \frac{1}{2}n\). We proceed further with two claims. Recall that \(G_2 = K_{3,3}\).

Claim 1 If the graph \(G\) contains no support vertex, then \(G = G_2\).

Proof. Assume that the graph \(G\) contains no support vertex. Thus, every vertex of \(G\) has degree at least 2 and degree at most 3. If Δ(G) = 2, then \(G\) is a cycle \(C_n\) where \(n \geq 6\), and so \(i(G) = i(C_n) = \lceil \frac{1}{3}n \rceil < \frac{1}{2}n\), a contradiction. Hence, Δ(G) = 3. Let \(v\) be an arbitrary vertex of degree 3 in \(G\) and let \(N_G(v) = \{x,y,z\}\). We now consider the graph \(H = G \setminus N_G[v]\). Suppose that \(H\) has no isolated vertex. Let \(H_1,\ldots,H_t\) be the components of \(H\) and let \(S_i\) be a minimum ID-set of \(H_i\) for \(i \in [t]\). By Theorem 4, \(|S_i| = i(H_i) \leq \frac{1}{2}|V(H_i)|\) for \(i \in [t]\). Let
\[
S' = \bigcup_{i=1}^{t} S_i \quad \text{and} \quad S = S' \cup \{v\}.
\]
The set \(S\) is an ID-set of \(G\), implying that
\[
\begin{align*}
i(G) \leq |S| &= 1 + |S'| = 1 + \sum_{i=1}^{t} |S_i| \\
&< 1 + \sum_{i=1}^{t} \frac{1}{2}|V(H_i)| \\
&= 1 + \frac{1}{2}|V(H)| \\
&= 1 + \frac{1}{2}(n - 4) \\
&< \frac{1}{2}n,
\end{align*}
\]
a contradiction. Hence, \(H\) contains at least one isolated vertex. If \(H\) contains at least four isolated vertices, then since \(\delta(G) \geq 2\) each such isolated vertex in \(H\) has at least two neighbors in \(G\) belonging to the set \(\{x,y,z\}\), implying by the Pigeonhole Principle that at least one of the vertices \(x, y\) and \(z\) has degree at least 4 in \(G\), a contradiction. Therefore, \(H\) contains at most three isolated vertices.

We show that \(H\) contains at most two isolated vertices. Suppose, to the contrary, that \(H\) contains three isolated vertices. Since \(\delta(G) \geq 2\) and \(\Delta(G) = 3\), the graph \(G\) is now determined. In this case, \(n = 7\) and this contradicts the fact that \(n\) is even. Hence, \(H\) contains at most two isolated vertices.

Next we show that \(H\) contains exactly two isolated vertices. Suppose, to the contrary, that \(H\) contains exactly one isolated vertex, say \(u\). In this case, we consider the graph
$H' = H \setminus \{u\}$. Since $n \geq 6$ is even and $H'$ contains no isolated vertex, every component of $H'$ has order at least 2. Applying Theorem 4 to $H'$, we have $i(H') \leq \frac{1}{2}|V(H')| = \frac{1}{2}(n-5)$. Since $n$ is even, this implies that $i(H') \leq \frac{1}{2}(n-6)$. A minimum ID-set of $H'$ can be extended to an ID-set of $G$ by adding to it the vertices $u$ and $v$, implying that $i(G) \leq i(H') + 2 < \frac{1}{2}n$, a contradiction. Hence, $H$ contains exactly two isolated vertices.

Let $u$ and $w$ be the two isolated vertices of $H$. Each of $u$ and $w$ has either two or three neighbors in $G$ that belong to the set $\{x, y, z\}$, implying that $u$ and $w$ have at least one common neighbor.

Suppose that $u$ and $w$ have exactly one common neighbor. Renaming the neighbors of $v$ if necessary, we may assume that $N_G(u) = \{x, y\}$ and $N_G(w) = \{y, z\}$. In particular, $y$ is the common neighbor of $u$ and $w$. Let $H' = H - \{u, w\}$. We note that $H'$ has no isolated vertex. Applying Theorem 4 to $H'$, we have $i(H') \leq \frac{1}{2}|V(H')| = \frac{1}{2}(n-6)$. Let $S$ be a minimum ID-set of $H'$. If $N(z) \cap S = \emptyset$, then $S \cup \{u, z\}$ is an ID-set of $G$, a contradiction. If $N(x) \cap S = \emptyset$, then similarly we get a contradiction. Now, assume that $N(z) \cap S \neq \emptyset$ and $N(x) \cap S \neq \emptyset$. In this case, $S \cup y$ is an ID-set of $G$, a contradiction. Hence, $u$ and $w$ have at least two common neighbors.

Suppose that $u$ and $w$ have exactly two common neighbors. Renaming neighbors of $v$ if necessary, we may assume in this case that $\{x, y\} = N_G(u) \cap N_G(w)$. By assumption, $z$ is adjacent to at most one of $u$ and $w$. Renaming $u$ and $w$, we may assume that $z$ is not adjacent to $w$. If $n = 6$, then $\{w, z\}$ is an ID-set of $G$, implying that $i(G) = 2 < \frac{1}{2} \times 6$, a contradiction. Hence, $n \geq 8$. We now consider the connected subcubic graph $H' = G - \{u, v, w, x, y\}$. Applying Theorem 4 to $H'$, we have $i(H') \leq \frac{1}{2}|V(H')| = \frac{1}{2}(n-5)$. Since $n$ is even, this implies that $i(H') \leq \frac{1}{2}(n-6)$. A minimum ID-set of $H'$ can be extended to an ID-set of $G$ by adding to it the vertices $x$ and $y$, implying that $i(G) \leq i(H') + 2 < \frac{1}{2}n$, a contradiction.

Hence, the vertices $u$ and $w$ have three common neighbors. The graph $G$ is now determined, and $G = K_{3,3} = G_2$. This completes the proof of the claim. (c)

By Claim 1 we may assume that the graph $G$ contains at least one support vertex, for otherwise $G = G_2$ and the desired result follows. Since $n \geq 6$, we note that every support vertex of $G$ has at most two leaf neighbors. Recall that $G_4$ is the double star $S(2, 2)$ shown in Figure 3(b).

Claim 2 If the graph $G$ contains a support vertex with two leaf neighbors, then $G = G_4$.

Proof. Suppose that $G$ contains a support vertex $v$ with two leaf neighbors, say $u$ and $w$. Let $x$ be the third neighbor of $v$. Since $n \geq 6$, we note that $d_G(x) \geq 2$. We show that $x$ is a support vertex. Suppose, to the contrary, that $x$ is not a support vertex. In this case, we consider the subcubic graph $H = G - N_G[v] = G - \{u, v, w, x\}$. Since $x$ is not a support vertex in $G$, every component of $H$ has order at least 2. Applying Theorem 4 to $H$, we have $i(H) \leq \frac{1}{2}|V(H)| = \frac{1}{2}(n-4)$. A minimum ID-set of $H$ can be extended to an ID-set of $G$ by adding to it the vertex $v$, implying that $i(G) \leq i(H) + 1 < \frac{1}{2}n$, a contradiction. Hence, $x$ is a support vertex.
Next, we show that \( x \) has two leaf neighbors. Suppose, to the contrary, that \( x \) has exactly one leaf neighbor, say \( y \). Since \( n \geq 6 \), we note that in this case the vertex \( x \) has degree 3. We consider the connected subcubic graph \( H = G - \{u, v, w, x, y\} \). We note that \( H \) has order at least 2. Applying Theorem 2 to \( H \), we have \( i(H) \leq \frac{1}{2}|V(H)| = \frac{1}{2}(n-5) \). Since \( n \) is even, this implies that \( i(H) \leq \frac{1}{2}(n-6) \). A minimum ID-set of \( H \) can be extended to an ID-set of \( G \) by adding to it the vertices \( v \) and \( y \), implying that \( i(G) \leq i(H) + 2 < \frac{1}{2}n \), a contradiction. Hence, \( x \) has exactly two leaf neighbors; that is, \( G = G_4 \). (c)

By Claim 2, we may assume that every support of \( G \) has exactly one leaf neighbor, for otherwise \( G = G_4 \) and the desired result follows. Among all support vertices of \( G \), let \( v \) be chosen so that the following holds, where \( u \) is the leaf neighbor of \( v \).

1. The degree, \( d_G(v) \), of \( v \) is a minimum.
2. Subject to (1), the number of components of \( G - \{u, v\} \) is a minimum.

We note that either \( d_G(v) = 2 \) or \( d_G(v) = 3 \). Further, we note that either \( G - \{u, v\} \) is connected or has two components. Let \( H = G - \{u, v\} \). Each component of \( H \) contains a neighbor of \( v \). Since \( u \) is the only leaf neighbor of \( v \), the graph \( H \) has no isolated vertex, and so each component of \( H \) has order at least 2. Applying Theorem 3 to \( H \), we have \( i(H) \leq \frac{1}{2}|V(H)| = \frac{1}{2}(n-2) \). A minimum ID-set of \( H \) can be extended to an ID-set of \( G \) by adding to it the vertex \( u \), implying that \( \frac{1}{2}n = i(G) \leq i(H) + 1 \leq \frac{1}{2}n \). Hence we must have equality throughout this inequality chain, implying that \( i(H) = \frac{1}{2}|V(H)| \) and that every component \( H' \) of \( H \) satisfies \( i(H') = \frac{1}{2}|V(H')| \). Applying the inductive hypothesis to each component \( H' \) of \( H \), the component \( H' \) satisfies (a), (b) or (c) in the statement of the theorem. Recall that \( G_3 \) and \( G_5 \) are the graphs shown in Figure 3(a) and 3(b), respectively.

Claim 3 The graph \( H \) is connected.

Proof. Suppose, to the contrary, that \( H \) is disconnected. Thus, \( H \) has two components, say \( H_1 \) and \( H_2 \). In particular, this implies that \( d_G(v) = 3 \). Let \( v_i \) be the neighbor of \( v \) that belongs to \( H_i \) for \( i \in [2] \). Let \( H_i \) have order \( n_i \) for \( i \in [2] \). As observed earlier, \( n_i \geq 2 \) and \( i(H_i) = \frac{1}{2}n_i \) for \( i \in [2] \). Further, \( H_i \) satisfies (a), (b) or (c) in the statement of the theorem for \( i \in [2] \).

Claim 3.1 \( H_i \notin \{G_1, G_2, G_3, G_4, G_5\} \) for \( i \in [2] \).

Proof. Suppose, to the contrary, that \( H_1 \in \{G_1, G_2, G_3, G_4, G_5\} \). We note that \( H_1 \neq G_2 = K_{3,3} \) since the vertex \( v_1 \) has degree at most 2 in \( H_1 \). Thus, \( H_1 = G_1 \), in which case \( n_1 = 4 \), or \( H_1 \in \{G_3, G_4, G_5\} \), in which case \( n_1 = 6 \). Further we note that \( H_1 - v_1 \) is a connected graph of odd order \( n_1 - 1 \geq 3 \), implying by Theorem 2 that \( i(H_1 - v_1) \leq \frac{1}{2}(n_1 - 2) \). Let \( S_1 \) be a minimum ID-set of \( H_1 \). We note that the set \( S_1 \) contains no neighbor of \( v_1 \). We now consider the connected subcubic graph \( G' = G - (V(H_1) \setminus \{v_1\}) \). Let \( G' \) have order \( n' \). Since \( n' = n - n_1 + 1 \) is odd, Theorem 3 implies that \( i(G') \leq \frac{1}{2}(n' - 1) = \frac{1}{2}(n - n_1) \). If \( S' \) is an ID-set of \( G' \) of minimum cardinality, then \( S' \cup S_1 \) is an ID-set of \( G \), implying that

\[
i(G) \leq |S_1| + |S'| \leq \frac{1}{2}(n_1 - 2) + \frac{1}{2}(n - n_1) < \frac{1}{2}n,
\]
a contradiction. (c)

By Claim 3.1 and our earlier observations, \( H_i \) satisfies (b) or (c) in the statement of the theorem for \( i \in [2] \). Thus, \( n_i = 2k_i \) and \( H_i = \text{cor}(P_{k_i}) \) for some \( k_i \geq 3 \) and \( i \in [2] \). Recall that among all support vertices of \( G \), the vertex \( v \) was chosen to have minimum degree. This implies that if \( H_i = \text{cor}(P_{k_i}) \), then \( k_i \geq 2 \) for \( i \in [2] \), for otherwise if \( H_i = \text{cor}(P_1) = P_2 \), then the vertex \( v_i \) would be a support vertex of \( G \) of degree 2, a contradiction. In particular, we note that \( n_i \geq 4 \) for \( i \in [2] \). If \( H_1 = \text{cor}(P_{k_1}) \) for some \( k_1 \geq 2 \), then at least one of the two support vertices of degree 2 in \( H \) is a support vertex of degree 2 in \( G \), contradicting our choice of the support vertex \( v \). If \( H_1 = \text{cor}(C_{k_1}) \) for some \( k_1 \geq 3 \), then at least one support vertex (of degree 3) in \( H \) is a support vertex in \( G \). However, the removal of such a support vertex and its leaf neighbor in \( G \) produces a connected graph, once again contradicting our choice of the support vertex \( v \). This completes the proof of Claim 3. (c)

By Claim 3 the graph \( H \) is connected. Let \( H \) have order \( n' \), and so \( n' = n - 2 \). As observed earlier, \( H \) satisfies (a), (b) or (c) in the statement of the theorem. Thus, \( H \in \{G_1, G_2, G_3, G_4, G_5\} \) or \( n' = 2k' \) for some \( k' \geq 1 \) and \( H = \text{cor}(P_{k'}) \) or \( n' = 2k' \) for some \( k' \geq 3 \) and \( H = \text{cor}(C_{k'}) \).

Claim 4 If \( H \in \{G_1, G_2, G_3, G_4, G_5\} \), then \( G = G_3 \).

Proof. Suppose that \( H \in \{G_1, G_2, G_3, G_4, G_5\} \). We consider each possibility in turn. Suppose that \( H = G_1 = C_4 \), and so \( n = 6 \). Let \( H \) be the cycle \( C: w_1w_2w_3w_4w_1 \), where \( vw_1 \) is an edge of \( G \). If \( vw_3 \) is not an edge of \( G \), then \( \{v, w_3\} \) is an ID-set of \( G \), and so \( i(G) = 2 < \frac{1}{2} \times 6 \), a contradiction. Hence, \( vw_3 \) is an edge of \( G \), implying that \( G = G_3 \).

We note that the vertex \( v \) has one or two neighbors in \( H \), and each neighbor of \( v \) in \( H \) has degree at most 2 in \( H \), implying that \( H \neq G_2 \cong K_{3,3} \).

Suppose that \( H = G_3 \), and so \( n = 8 \). Let \( a_1 \) and \( a_2 \) be the two vertices of \( H \) with three common neighbors, say \( b_1 \), \( b_2 \) and \( b_3 \), where \( b_3 \) has degree 3 in \( H \). Let \( w \) be the leaf neighbor of \( b_3 \) in \( H \). If \( vw \in E(G) \), then let \( S = \{a_1, a_2, v\} \). If \( vw \notin E(G) \) and \( v \) is adjacent to both \( b_1 \) and \( b_2 \), then let \( S = \{b_3, v\} \). If \( vw \notin E(G) \) and \( v \) is adjacent to exactly one of \( b_1 \) and \( b_2 \), say to \( b_1 \), then let \( S = \{b_2, b_3, v\} \). In all three cases, the set \( S \) is an ID-set of \( G \) and \( |S| \leq 3 \). Thus, \( i(G) \leq 3 < \frac{1}{2} \times 8 \), a contradiction.

Suppose that \( H = G_4 \), and so \( n = 8 \). Let \( x \) and \( y \) be the two central vertices of the double star \( H \). By our earlier assumptions, every support vertex of \( G \) has exactly one leaf neighbor. Hence, the vertex \( v \) is adjacent in \( G \) to a leaf neighbor in \( H \) of \( x \) and a leaf neighbor in \( H \) of \( y \). Thus if \( x' \) be the leaf neighbor of \( x \) in \( H \) that is not adjacent to \( v \) in \( G \), then the set \( \{v, x', y\} \) is an ID-set of \( G \), and so \( i(G) \leq 3 < \frac{1}{2} \times 8 \), a contradiction.

Suppose that \( H = G_5 \), and so \( n = 8 \). Thus, \( H \) is obtained from a path \( a_1a_2a_3a_4a_5a_6 \) by adding the edge \( a_2a_5 \). Suppose that \( v \) is adjacent to \( a_1 \) or \( a_6 \), say to \( a_1 \). If \( v \) is not adjacent to \( a_3 \), then let \( S = \{v, a_3, a_5\} \). If \( v \) is adjacent to \( a_3 \), then let \( S = \{v, a_5\} \). In both cases,
the set $S$ is an ID-set of $G$ and $|S| \leq 3$. Thus, $i(w1G) \leq 3 < \frac{1}{2} \times 8$, a contradiction. Hence, $v$ is adjacent to neither $a_1$ nor $a_6$. Thus, the only possible neighbors of $v$ in $H$ are $a_3$ or $a_4$. By symmetry, we may assume that $va_3 \in E(G)$. Thus, $\{v, a_1, a_5\}$ is an ID-set of $G$, and so $i(G) \leq 3 < \frac{1}{2} \times 8$, a contradiction. This completes the proof of Claim 4. ($\diamond$

Let $n' = |V(H)|$, and so $n' = n - 2$. Recall that $n \geq 6$, and so $n' \geq 4$. By Claim 4, we may assume that $H \notin \{G_1, G_2, G_3, G_4, G_5\}$, for otherwise $G = G_3$, and the desired result follows. Hence $n' = 2k'$ and $H = \text{cor}(P_{k'})$ for some $k' \geq 2$.

Claim 5 $H = \text{cor}(P_{k'})$ for some $k' \geq 2$.

**Proof.** Suppose that $H = \text{cor}(C_{k'})$ for some $k' \geq 3$. Thus, $n' = 2k'$ and $n = 2k' + 2$. Let $H$ be the corona of the cycle $C: v_1v_2 \ldots v_{k'}v_1$, and let $u_i$ be the resulting leaf neighbor of $v_i$ in $H$ for $i \in [k']$. Since $G$ is a subcubic graph, we note that the only possible neighbors of $v$ that belong to $H$ are the leaves of $H$. We now consider the connected subcubic graph $G' = G - N_G[v]$ of order at least 2. Applying Theorem 4 to the graph $G'$, we have $i(G') \leq \frac{1}{2}|V(G')| \leq \frac{1}{2}(n - 3)$. Since $n$ is even, this implies that $i(G') \leq \frac{1}{2}(n - 4)$. A minimum ID-set of $G'$ can be extended to an ID-set of $G$ by adding to it the vertex $v$, implying that $i(G) \leq i(G') + 1 < \frac{1}{2}n$, a contradiction. ($\diamond$

By Claim 5, $H = \text{cor}(P_{k'})$ for some $k' \geq 2$. Thus, $n' = 2k'$ and $n = 2k' + 2$. Let $H$ be the corona of the path $P: v_1v_2 \ldots v_{k'}$, and let $u_i$ be the resulting leaf neighbor of $v_i$ in $H$ for $i \in [k']$. Since $G$ is a subcubic graph, we note that the only possible neighbors of $v$ that belong to $H$ are the vertices $u_i$ for $i \in [k']$ or the vertices $v_1$ and $v_{k'}$ of degree 2 in $H$.

Claim 6 If $d_G(v) = 2$, then $G = \text{cor}(P_k)$ where $k = k' + 1$.

**Proof.** Suppose that $d_G(v) = 2$. Let $w$ be the neighbor of $v$ different from $u$. If $w = u_i$ for some $i \in [k']$, then we consider the connected subcubic graph $G' = G - \{u, v, w\}$ of order at least 3. Applying Theorem 4 to the graph $G'$, we have $i(G') \leq \frac{1}{2}|V(G')| = \frac{1}{2}(n - 3)$. Since $n$ is even, this implies that $i(G') \leq \frac{1}{2}(n - 4)$. A minimum ID-set of $G'$ can be extended to an ID-set of $G$ by adding to it the vertex $v$, implying that $i(G) \leq i(G') + 1 < \frac{1}{2}n$, a contradiction. Hence, either $w = v_1$ or $w = v_{k'}$. In both cases, $G = \text{cor}(P_k)$ where $k = k' + 1$, as desired. ($\diamond$

By Claim 6, we may assume that $d_G(v) = 3$. Hence, the vertex $v$ has two neighbors in $H$, say $w$ and $x$. If both neighbors $w$ and $x$ are leaves in $H$, then we consider the connected subcubic graph $G' = G - \{u, v, w, x\}$ of order at least 2. Applying Theorem 4 to the graph $G'$, we have $i(G') \leq \frac{1}{2}|V(G')| = \frac{1}{2}(n - 4)$. A minimum ID-set of $G'$ can be extended to an ID-set of $G$ by adding to it the vertex $v$, implying that $i(G) \leq i(G') + 1 < \frac{1}{2}n$, a contradiction. Hence, renaming $w$ and $x$ if necessary, we may assume that $w = v_1$.

If $x = v_{k'}$, then $G = \text{cor}(C_k)$ where $k = k' + 1$, and the desired result follows. Hence, we may assume that $x$ is a leaf of $H$. 

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If $x = u_1$, then again we consider the connected subcubic graph $G' = G - \{u, v, w, x\}$, and as before obtain the contradiction $i(G) \leq i(G') + |\{v\}| < \frac{1}{2}n$. Hence, $x = u_i$ for some $i \in [k'] \setminus \{1\}$. Suppose that $k' \geq 3$. In this case, we consider the connected subcubic graph $G' = G - \{u, v, w, x, u_1\}$ of order at least 3. Applying Theorem 4 to the graph $G'$, we have $i(G') \leq \frac{1}{2}|V(G')| = \frac{1}{2}(n - 5)$. Since $n$ is even, this implies that $i(G') \leq \frac{1}{2}(n - 6)$. A minimum ID-set of $G'$ can be extended to an ID-set of $G$ by adding to it the vertices $u_1$ and $v$, implying that $i(G) \leq i(G') + 2 < \frac{1}{2}n$, a contradiction. Hence, $k' = 2$, implying that $G = G_5$, and the desired result follows. This completes the proof of Theorem 3.}

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