Off-policy Confidence Sequences

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Abstract

We develop confidence bounds that hold uniformly over time for off-policy evaluation in the contextual bandit setting. These confidence sequences are based on recent ideas from martingale analysis and are non-asymptotic, non-parametric, and valid at arbitrary stopping times. We provide algorithms for computing these confidence sequences that strike a good balance between computational and statistical efficiency. We empirically demonstrate the tightness of our approach in terms of failure probability and width and apply it to the “gated deployment” problem of safely upgrading a production contextual bandit system.

1 Introduction

Reasoning about the reward that a new policy \(\pi\) would have achieved if it had been deployed, a task known as Off-Policy Evaluation (OPE), is one of the key challenges in modern Contextual Bandits (CBs) Langford and Zhang [2007] and Reinforcement Learning (RL). A typical OPE use case is the validation of new modeling ideas by data scientists. If OPE suggests that \(\pi\) is better, this can then be validated online by deploying the new policy to the real world.

The classic way to answer whether \(\pi\) has better reward than the current policy \(h\) is via a confidence interval (CI). Unfortunately, CIs take a very static view of the world. Suppose that \(\pi\) is better than \(h\) and our OPE shows a higher but not significantly better estimated reward. What should we do? We could collect more data, but since a CI holds for a particular (fixed) sample size and is not designed to handle interactive/adaptive data collection, simply recalculating the CI at a larger sample size invalidates its coverage guarantee.

While there are ways to fix this, such as a crude union bound, the proper statistical tool for such cases is called a Confidence Sequence (CS). A CS is a sequence of CIs such that the probability that they ever exclude the true value
is bounded by a prespecified quantity. In other words, they retain validity under optional (early) stopping and optional continuation (collecting more data).

In this work we develop CSs for OPE using recent insights from martingale analysis (for simpler problems). Besides the aforementioned high probability uniformly over time guarantee, these CSs make no parametric assumptions and are easy to compute. We use them to create a “gated deployment” primitive: instead of deploying \( \pi \) directly we keep it in a staging area where we compute its off-policy CS as \( h \) is collecting data. Then \( \pi \) can replace \( h \) as soon as (if ever) we can reject the hypothesis that \( h \) is better than \( \pi \).

We now introduce some notation to give context to our contributions. We have iid CB data of the form \((x,a,r)\) collected by a historical policy \( h \) in the following way. First a context \( x \) was sampled from an unknown distribution \( D \). Then \( h \) assigns a probability to each action. An action \( a \) is sampled with probability \( h(a;x) \) and performed. A reward \( r \) associated with performing \( a \) in situation \( x \) is sampled from an unknown distribution \( R(x,a) \). Afterwards, we wish to estimate the reward of another policy \( \pi \ll h \). We have

\[
V(\pi) = \mathbb{E}_{x \sim D} \left[ \frac{\pi(a;x)}{h(a;x)} \right] \tag{1}
\]

where the last quantity can be estimated from data. Letting \( w = \frac{\pi(a;x)}{h(a;x)} \) we see that \( \mathbb{E}_{x \sim D, a \sim h} [w] = 1 \), where we write \( w \) instead of \( w(x,a) \) to reduce notation clutter. More generally for any function \( q(x,a) \) — which is typically a predictor of the reward of \( a \) at \( x \) — we have

\[
\mathbb{E}_{x \sim D, a \sim h} [wq(x,a)] = \sum_{a'} \pi(a';x)q(x,a'), \tag{2}
\]

which reduces to \( \mathbb{E}[w] = 1 \) when \( q(x,a) = 1 \) always. Eq. (1) and (2) are the building blocks of OPE estimators. The IPS estimator Horvitz and Thompson [1952] estimates (1) via Monte Carlo: \( \hat{V}^{\text{IPS}}(\pi) = 1/n \sum_{i=1}^{n} \frac{w_i r_i}{h(a_i;x)} \). A plethora of other OPE estimators are discussed in Section 6. In general there is a tension between the desirability of an unbiased estimator like \( \hat{V}^{\text{IPS}} \) and the difficulty of working with it in finite samples due to its excessive variance.

Recently, Kallus and Uehara [2019] proposed an OPE estimator based on Empirical Likelihood Owen [2001] with several desirable properties. Empirical Likelihood (EL) has also been used to derive CIs for OPE in CBs Karampatziakis et al. [2020] and RL Dai et al. [2020]. Our CSs can be thought of as a natural extension to the online setting of the CIs for OPE in the batch setting; its advantages include

- Our CSs hold non-asymptotically, unlike most existing CIs mentioned above which are either asymptotically valid (or nonasymptotic but overly conservative).
- Our CSs are not unnecessarily conservative due to naive union bounds or peeling techniques.
• We do not make any assumptions, either parametric or about the support of \(w\) and \(r\), beyond boundedness.
• Our validity guarantees are time-uniform, meaning that they remain valid under optional continuation (collecting more data) and/or at stopping times, both of which are not true for all aforementioned CIs.

2 Background: OPE Confidence Intervals

We start by reviewing OPE CIs from the perspective of Karampatziakis et al. [2020]. Their CI is constructed by considering plausible distributions from a non-parametric family \(Q\) of distributions \(Q\) for random vectors \((w, r) \in [0, w_{\text{max}}] \times [0, 1]\) under the constraint \(E_Q[w] = 1\). Let \(Q_{wr}\) be the probability that \(Q\) assigns to the event where the importance weight is \(w\) and the reward is \(r\). Then there exists \(Q^* \in Q\) such that

\[
Q^*_{wr} = \mathbb{E}_{x \sim D, a \sim h, \rho \sim R(x,a)} \left[ \mathbb{I} \left( \pi(a; x) \cdot h(a; x) = w \right) \cdot \mathbb{I}[\rho = r] \right]
\]

and \(V(\pi) = \mathbb{E}_{Q^*}[wr]\). To estimate of \(V(\pi)\) we can find \(Q_{\text{mle}} \in Q\) that maximizes the data likelihood. To find a CI we minimize/maximize \(E_{Q}[wr]\) over plausible \(Q \in Q\) so the data likelihood is not far off from that of \(Q_{\text{mle}}\).

Using convex duality the MLE is \(Q_{\text{mle}}^*_{wr} = \frac{1}{n(1+\lambda^*_1(w-1))}\) where \(\lambda^*_1\) is a dual variable solving

\[
\lambda^*_1 = \arg\max_{\lambda_1} \sum_{i=1}^{n} \log(1 + \lambda_1(w_i - 1))
\]

subject to \(1 + \lambda_1(w_{\text{max}} - 1) \geq 0, 1 - \lambda_1 \geq 0\). The profile likelihood \(L(v) = \sup_{Q;E_Q[w]=1, E_Q[wr]=v} \prod_{i=1}^{n} Q_{w_i,r_i}\) is used for CIs. From EL theory, an asymptotic \(1 - \alpha\)-CI is

\[
\left\{ v : -2 \ln \left( \prod_{i=1}^{n} Q_{wr_i}^* \right) \leq \chi^2_{1-\alpha} \right\}
\]

where \(\chi^2_{1-\alpha}\) is the \(1-\alpha\) quantile of a \(\chi^2\) distribution with one degree of freedom. Using convex duality the CI is

\[
\left\{ v : B(v) - \sum_{i=1}^{n} \log(1 + \lambda^*_1(w_i - 1)) \leq \chi^2_{1-\alpha} \right\}
\]

where the dual profile log likelihood \(B(v)\) is

\[
B(v) = \sup_{\lambda_1, \lambda_2} \sum_{i=1}^{n} \log(1 + \lambda_1(w_i - 1) + \lambda_2(w_i r_i - v))
\]
subject to \((\lambda_1, \lambda_2) \in D_v^0\) where
\[
D_v^m = \{(\lambda_1, \lambda_2) : 1 + \lambda_1(w - 1) + \lambda_2(wr - v) \geq m \ \forall (w, r) \in \{0, w_{\max}\} \times \{0, 1\}\}.
\]

The CI endpoints can be found via bisection on \(v\).

### 3 Off-policy Confidence Sequences

We now move from the batch setting and asymptotics to online procedures and finite sample, time-uniform results. We adapt and extend ideas from Waudby-Smith and Ramdas [2020] which constructs CSs for the means of random variables in \([0, 1]\). Our key insight is to combine their construction with an interpretation of (3) as the log wealth accrued by a skeptic who is betting against the hypotheses
\[
E_{Q^*}[w] = 1 \text{ and } E_{Q^*}[wr] = v.
\]

In particular, the skeptic starts with a wealth of 1 and wants to maximize her wealth. Her bet on the outcome \(w - 1\) is captured by \(\lambda_1\), while \(\lambda_2\) represents the bet on the outcome of \(wr - v\) so that the wealth after the \(i\)-th sample is multiplied by \(1 + \lambda_1(w_i - 1) + \lambda_2(w_ir_i - v)\). If the outcomes had been in \([-1, 1]\) then \(|\lambda_1|\) and \(|\lambda_2|\) would have an interpretation as the fraction of the skeptic’s wealth that is being risked on each step. The bets can be positive or negative, and their signs represent the directions of the bet. For example, \(\lambda_2 < 0\) means the skeptic will make money if \(w_ir_i - v < 0\). Enforcing the constraints (4) from the batch setting here means that the resulting wealth cannot be negative.

The first benefit of this framing is that we have mapped the abstract concepts of dual likehood, dual variables, and dual constraints to more familiar concepts of wealth, bets, and avoiding bankruptcy. We now formalize our constructions and show how they lead to always valid, finite sample, CSs. We introduce a family of processes
\[
K_t(v) = \prod_{i=1}^{t}(1 + \lambda_{1,i}(w_i - 1) + \lambda_{2,i}(w_ir_i - v))
\]
where \(\lambda_{1,i}\) and \(\lambda_{2,i}\) are predictable, i.e. based on past data (formally, measurable with respect to the sigma field \(\sigma(\{(w_j, r_j)\}_{j=1}^{i-1})\)). We also formalize CIs and CSs below.

**Definition 1.** Given data \(S_n = \{(x_i, a_i, r_i)\}_{i=1}^{n}\), where \(x_i \sim D\), \(a_i \sim h(\cdot; x_i)\), \(r_i \sim R(x_i, a_i)\), a \((1 - \alpha)\)-confidence interval for \(V(\pi)\) is a set \(C_n = C(h, \pi, S_n)\) such that
\[
\sup_{D, R} \Pr(V(\pi) \notin C_n) \leq \alpha.
\]

In contrast, a \((1 - \alpha)\)-confidence sequence for \(V(\pi)\) is a sequence of confidence intervals \((C_t)_{t \in \mathbb{N}}\) such that
\[
\sup_{D, R} \Pr(\exists t \in \mathbb{N} : V(\pi) \notin C_t) \leq \alpha.
\]
We now have the setup to state our first theoretical result.

**Theorem 1.** $K_t(V(\pi))$ is a nonnegative martingale. Moreover, the sequences $C_t = \{ v : K_t(v) \leq \frac{1}{\alpha} \}$ and $C_t = \bigcap_{i=1}^{t} C_i$ are $(1 - \alpha)$-confidence sequences for $V(\pi)$.

All proofs are in the appendix. The process $K_t(v)$ tracks the wealth of a skeptic betting against $V(\pi) = v$. The process $K_t(V(\pi))$ is a nonnegative martingale so it has a small probability of attaining large values (formally, Ville’s inequality states that the probability of ever exceeding $1/\alpha$ is at most $\alpha$). Of course, we don’t know $V(\pi)$, but if we retain all values of $v$ where the wealth is below $1/\alpha$, and reject the values of $v$ for which it has crossed $1/\alpha$ at some point, this set will always contain $V(\pi)$ with high probability; this is the basis of our construction. The strength of our approach comes from this result, as it guarantees always-valid bounds for $V(\pi)$ using only martingale arguments crucially avoiding parametric or other assumptions.

What about $v \neq V(\pi)$? Can we be sure that $C_t$ does not contain values $v$ very far from $V(\pi)$? That’s where the betting strategy, quantified by the predictable sequences $(\lambda_1, i)$ and $(\lambda_2, i)$, enters. The hope is the skeptic can eventually force $K_t(v)$ to be large via a series of good bets. Importantly, Theorem 1 holds regardless of how the bets are set, but good bets will lead to “small” $C_t$. How to smartly bet is the subject of what follows.

## 4 Main Betting Strategy: MOPE

We develop our main betting strategy, MOPE (Martingale OPE) in steps starting with a slow but effective algorithm and making changes to trade off a small amount of statistical efficiency for large gains in computational efficiency.

### 4.1 Follow The Leader

We begin with a Follow-The-Leader (FTL) strategy that is known to work very well for iid problems [De Rooij et al. 2014]. We define $\ell_i^v(\lambda) = \ln (1 + \lambda_1 (w_i - 1) + \lambda_2 (w_i r_i - v))$ and set $\lambda = [\lambda_1, \lambda_2]$ to maximize wealth in hindsight

$$\lambda_{\text{ftl}}^i(v) = \arg\max_{\lambda} \sum_{i=1}^{t-1} \ell_i^v(\lambda)$$

for every step of betting in $K_i(v)$. The problem (5) is convex and can be solved in polynomial time leading to an overall polynomial time algorithm. However, this approach has three undesirable properties. First, the algorithm needs to store the whole history of $(w, r)$ samples. Second the overall algorithm is tractable but slow. Finally, we need to solve (5) for all values of $v$ that we have not yet rejected.
4.2 Maximizing a lower bound on wealth

We can avoid having to store all history by optimizing an easy-to-maintain lower bound of (5).

**Lemma 1.** For all $x \geq -\frac{1}{2}$ and $\psi = 2 - 4 \ln(2)$, we have

$$\ln(1 + x) \geq x + \psi x^2.$$ 

Observe that if we restrict our bets to lie in the convex set $D_{v}^{1/2}$ (cf. eq. (4)) then for all $\lambda \in D_{v}^{1/2}$

$$\sum_{i=1}^{t-1} \ell_i(\lambda) \geq \lambda^\top \sum_{i=1}^{t-1} b_i(v) + \psi \lambda^\top \left( \sum_{i=1}^{t-1} A_i(v) \right) \lambda$$

where $b_i(v) = \begin{bmatrix} w_i - 1 \\ w_i r_i - v \end{bmatrix}$ and $A_i(v) = b_i(v) b_i(v)^\top$. The first step towards a more efficient algorithm is to set our bets at time $t$ as

$$\lambda_t(v) = \arg\max_{\lambda \in D_{v}^{1/2}} \psi \lambda^\top \left( \sum_{i=1}^{t-1} A_i(v) \right) \lambda + \lambda^\top \sum_{i=1}^{t-1} b_i(v) \quad (6)$$

The restriction $\lambda \in D_{v}^{1/2}$ is very mild: it does not allow the skeptic to lose more than half of her wealth from any single outcome. The first advantage of this formulation is that $\sum_i A_i(v)$ and $\sum_i b_i(v)$ are low degree polynomials of $v$ and can share the coefficients

$$\sum_{i=1}^{t-1} A_i(v) = A_t^{(0)} + v A_t^{(1)} + v^2 A_t^{(2)}$$

$$\sum_{i=1}^{t-1} b_i(v) = b_t^{(0)} + v b_t^{(1)}.$$ 

Secondly, the coefficients can be updated incrementally

$$A_t^{(0)} = \sum_{i=1}^{t-1} \begin{bmatrix} (w_i - 1)^2 \\ (w_i - 1) w_i r_i \\ w_i^2 r_i^2 \end{bmatrix}, \quad (7)$$

$$A_t^{(1)} = \sum_{i=1}^{t-1} \begin{bmatrix} 0 \\ -(w_i - 1) \\ -2 w_i r_i \end{bmatrix}, \quad (8)$$

$$A_t^{(2)} = \sum_{i=1}^{t-1} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad (9)$$

$$b_t^{(0)} = \sum_{i=1}^{t-1} \begin{bmatrix} w_i - 1 \\ w_i r_i \end{bmatrix}. \quad (10)$$
\[ b_t^{(1)} = \sum_{i=1}^{t-1} \begin{bmatrix} 0 \\ -1 \end{bmatrix}. \]  

(11)

Finally, we can solve (6) exactly in \( O(1) \) time. Section 4.4 will elaborate on this using a slight variation of eq. (6).

4.3 Common Bets and Hedging

The most competitive betting sequences for the process \( K(v) \) will take advantage of the knowledge of \( v \). However, placing different bets for different values of \( v \) creates two problems: First, the resulting confidence set need not be an interval and second makes it hard to implement Theorem 1 in a computationally efficient way. Indeed, even in the simpler setup of Waudby-Smith and Ramdas [2020] the authors maintain a grid of test values for the quantity of interest (here \( v \)) and at least keep track of the wealth separately. This is because tracking the wealth for each value in the grid is not straightforward when the bets are different.

To make wealth tracking easy and obtain algorithms that do not require the discretization of the domain of \( v \), a natural proposal would be to use a common bet for all \( v \) in each timestep. Unfortunately, this is not adequate because we do need \( \lambda_2 > 0 \) for \( v < E_Q[w_r] \) and \( \lambda_2 < 0 \) for \( v > E_Q[w_r] \). A simple fix is to use a hedged strategy as in Waudby-Smith and Ramdas [2020]. First, we split the initial wealth equally. The first half is used to bet against low \( v \)'s via the process

\[ K^+(v) = \prod_{i=1}^{t} \left( 1 + \lambda_{1,i}^+(w_i - 1) + \lambda_{2,i}^+(w_i r_i - v) \right) \]

and the second half to bet against high \( v \)'s via a separate process \( K^-(v) \) which for symmetry we parametrize as

\[ K^-(v) = \prod_{i=1}^{t} \left( 1 + \lambda_{1,i}^-(w_i - 1) + \lambda_{2,i}^-(w_i r'_i - v') \right). \]

where \( r'_i = 1 - r_i \) and \( v' = 1 - v \). This can be seen as the wealth process for betting against \( 1 - v \) in an world where \( r \) has been remapped to \( 1 - r \). Thus betting against high values of \( v \) reduces to betting against low values of \( v \) in a modified process. The total wealth of the hedged process is

\[ K^\pm (v) = \frac{1}{2}(K^+(v) + K^-(v)), \]

and it can be used for CSs in the same way as \( K_t(v) \):

**Theorem 2.** The sequence \( C^\pm_t = \{v : K^\pm_t(v) \leq \frac{1}{\alpha} \} \) and its running intersection \( \bigcap_{t=1}^{\infty} C^\pm_t \) are \( 1 - \alpha \) CSs for \( V(\pi) \).

It remains to design a common bet for \( K^+(v) \). Betting against any fixed \( v_0 \) will not work well when \( V(\pi) = v_0 \) since the optimal bet for \( V(\pi) \) is 0 but such a
Algorithm 1 Solve $\lambda^* = \arg\max_{\lambda \in C} \psi \lambda^T A \lambda + \lambda^T b$

Input: $A, b$

$\lambda = -(2\psi A)^{-1} b$

if $\lambda \in C$ then
  Return $\lambda$
end if

$\Lambda = \{ \left[ \frac{1}{2(1-w_{\text{max}})}, 0 \right], \left[ \frac{1}{2}, 0 \right], \left[ 0, \frac{1}{2} \right] \} \ \{ \text{vertices of } C \}$

for $c, d \in \{(0, 1), 0\}, (1, 1), \left( 1 - w_{\text{max}}, 1, \frac{1}{2} \right) \}$ do
  $\mu = -c^T (2\psi A)^{-1} b + d \ \{ \text{Lagrange multiplier} \}$
  $\lambda = -(2\psi A)^{-1} (b + \mu c)$
  if $\lambda \in C$ then
    $\Lambda = \Lambda \cup \{ \lambda \} \ \{ \text{Add feasible solutions on faces} \}$
  end if
end for

Return $\arg\max_{\lambda \in \Lambda} \psi \lambda^T A \lambda + \lambda^T b$

bet cannot help us reject those $v$ that are far from $V(\pi)$. Therefore we propose to adaptively choose the bets against the smallest $v$ that has not been rejected.

As we construct the CS, we have access to the values of $v$ that constitute the endpoints of the CS at the last time step. These values are on the cusp of plausibility given the available data and confidence level which means the bets are neither too conservative nor too detached from what can be estimated.

4.4 Avoiding grid search

Once we have determined $v$ for the current step we could choose $\lambda$ via (6). For reasons that will become apparent shortly, we can also consider

$$
\lambda_t = \arg\max_{\lambda \in C} \psi \lambda^T \left( \sum_{i=1}^{t-1} A_i(v) \right) \lambda + \lambda^T \sum_{i=1}^{t-1} b_i(v),
$$

(13)

where $C = \{ \lambda : \lambda_2 \geq 0 \} \cap \bigcap_{v \in [0,1]} D_v^{1/2}$ or more succinctly

$$
C = \left\{ \lambda : \lambda_2 \geq 0, \lambda_1 + \lambda_2 \leq \frac{1}{2}, \lambda_1 (1 - w_{\text{max}}) + \lambda_2 \leq \frac{1}{2} \right\}.
$$

The constraint $\lambda_2 \geq 0$ is expected for good bets in $K^+_t(v)$ (and by reduction in $K^-_t(v)$) since we are eliminating $v$’s with $E[w - v] > 0$. Since there are only three constraints and two variables we can exactly solve (13) very efficiently. Our implementation first tries to return the unconstrained maximizer, if feasible. If not, we evaluate the objective on up to 6 candidates: up to one candidate per face of $C$ (obtained via maximizing the objective subject to one equality constraint) and its 3 vertices. Algorithm 1 summarizes this.
Algorithm 2 MOPE: Martingale Off-Policy Evaluation

**Input:** process $Z = (w_i, r_i)_{i=1}^\infty, w_{\text{max}}, \alpha$

Let $Z' = (w_i, 1 - r_i)$ for $(w_i, r_i)$ in $Z$

for $v_i, v'_i$ in zip(LCS($Z$), LCS($Z'$)) do

Output($v_i, 1 - v'_i$)

end for

function LCS($Z$)

$\lambda_1 = [0, 0]^{\top}, v = 0$

for $i = 1, \ldots$ do

Observe $(w_i, r_i)$ from $Z$

Update statistics via (7)-(11) and (18)-(22).

if (14) has real roots then

$v = \max(v, \text{largest root of (14)})$

end if

yield $v$ \{execution suspends/resumes here\}

$A = A^{(0)}_i + v A^{(1)}_i + v^2 A^{(2)}_i$

$b = b^{(0)}_i + vb^{(1)}_i$

$\lambda_{i+1} = \arg\max_{\lambda \in C} \psi \lambda^{\top} A \lambda + b^{\top} \lambda$

end for

end function

Given $\lambda_1, \ldots, \lambda_{t-1}$, from (13) we get from Lemma 1

$$\sum_{i=1}^{t-1} \ell_i^*(\lambda_i) \geq \psi \sum_{i=1}^{t-1} \lambda_i^{\top} A_i(v) \lambda_i + \sum_{i=1}^{t-1} \lambda_i^{\top} b_i(v)$$

for all $v \in [0,1]$. Thus, if the lower bound exceeds $\ln(1/\alpha)$ for a particular $v$, the log wealth will also exceed it. Furthermore, the lower bound is quadratic in $v$ so we can easily find those values $v \in [0,1]$ such that

$$\psi \sum_{i=1}^{t-1} \lambda_i^{\top} A_i(v) \lambda_i + \sum_{i=1}^{t-1} \lambda_i^{\top} b_i(v) = \ln \left( \frac{2}{\alpha} \right).$$

(14)

The extra 2 is due to the hedged process. Appendix B explains this and the details of how to incrementally maintain statistics for solving (14) via eqs. (18)-(22). The advantage of (13) over (6) is that the latter cannot ensure that old bets will produce values in $D_v^{1/2}$ for future values of $v$ while the former always does because $C \subseteq D_v^{1/2}, \forall v \in [0,1]$.

The whole process of updating the statistics, tightening the lower bound $v$ via (14) and computing the new bets via (13) is summarized in Algorithm 2.

**Confidence Intervals.** If one only desires a single CI using a fixed batch of data, then a CI can be formed by returning the last set from the CS on any permutation of the data. To reduce variance, we can average the wealth of several independent permutations without violating validity.
Alternative Betting Algorithms  An obvious question is why develop this strategy and not just feed the convex functions \(-\ell_v^i(\lambda)\) to an online learning algorithm? The Online Newton Step (ONS) Hazan et al. [2007] is particularly well-suited as \(-\ell_v^i(\lambda)\) is exp-concave. While ONS does not require storing all history and needs small per-step computation, we could not find an efficient way to efficiently reason about \(K_t(v)\) for every \(v \in [0,1]\). While the ONS bounds the log wealth in terms of the gradients observed at each bet, these gradients depend on \(v\) in a way that makes it hard to efficiently reuse for different values of \(v\). Our approach on the other hand maintains a lower bound on the wealth as a second degree polynomial in \(v\), enabling us to reason about all values of \(v\) in constant time.

5 Extensions

5.1 Adding a Reward Predictor  So far we have only used the special case \(E[w] = 1\) of eq. (2). However, it is common to have access to a reward predictor \(q(x,a)\) mapping context and action to an estimated reward. Here we will just assume that \(q(x,a)\) is any measurable function of \((x,a)\) with codomain \([0,1]\). We use eq. (2) to define the zero mean quantity  \[
  c_i = w_i q(x_i, a_i) - \sum_{a'} \pi(a'; x_i) q(x_i, a').
\]  Thus \(E[w_r - c] = E[w_r]\) but when \(q(x_i, a_i)\) is a good reward predictor, the variance of \(w_r - c\) will be much smaller than that of \(w_r\). We propose the wealth process  \[
  K_t^q(v) = \prod_{i=1}^t (1 + \lambda_1,i (w - 1) + \lambda_2,i (w_r - c - v_i))
\]  for predictable sequences \((\lambda_1,i, \lambda_2,i) \in \mathcal{E}_v^0\), where  \[
  \mathcal{E}_v^m = \{(\lambda_1, \lambda_2) : 1 + \lambda_1 (w - 1) + \lambda_2 (w_r - c - v) \geq m \forall (x, a, r, q) \in \text{supp}(D) \times A \times \{0,1\} \times [0,1]|A|\}.\]  Note that \(w = w(x,a)\) and \(c = c(x,a,q)\) so all quantities are well defined. This set looks daunting but without loss of generality it suffices to only consider two actions: \(a\), which is sampled by \(h\), and an alternative one \(a', h(a) \in \{1/w_{\max}, 1\}\), and \(\pi(a), q(x,a), q(x,a') \in \{0,1\}\). Considering all these combinations and removing redundant constraints leads to the equivalent description for \(\mathcal{E}_v^m\) as  \[
  \left\{ \lambda : \begin{bmatrix} -1 & -1 & w & w \vspace{1mm} -v & -w & w & w \end{bmatrix}^\top \lambda \geq m - 1 \right\},
\]  where \(W = w_{\max} - 1\) and \(v' = 1 - v\).

For an efficient procedure we introduce the set \(\mathcal{C}^q = \bigcap_{v \in [0,1]} \mathcal{E}_v^{1/2}\) to enable the use of our lower bound and common bets for all \(v\). This set can be shown
to be the same as (16) but with \( v = 1 \) and \( v' = 1 \). For predictable sequences of bets \( \lambda_i^{+q}, \lambda_i^{-q} \in C_q \) define the processes

\[
K_i^{+q}(v) = \prod_{i=1}^{t} \left( 1 + \lambda_{i,1}^{+q}(w_i - 1) + \lambda_{i,2}^{+q}(w_ir_i - c_i - v) \right),
\]

\[
K_i^{-q}(v) = \prod_{i=1}^{t} \left( 1 + \lambda_{i,1}^{-q}(w_i - 1) + \lambda_{i,2}^{-q}(w_ir'_i - c'_i - v') \right),
\]

where \( r'_i = 1 - r_i, \; v' = 1 - v \). For the definition of \( c'_i \) we reason as follows: If \( q(x, a) \) is a good reward predictor for \( r_i \) then \( q'(x, a) = 1 - q(x, a) \) is a good reward predictor for \( r'_i \). Plugging \( q'(x, a) \) in place of \( q(x, a) \) in (15) leads to \( c'_i = w_i - 1 - c_i \). Finally, the hedged process is just \( K_i^{+q,v}(v) = \frac{1}{2}(K_i^{+q}(v) + K_i^{-q}(v)) \) and we have

**Theorem 3.** The sequences \( C_i^q = \{v : K_i^q(v) \leq \frac{1}{2} \} \) and \( C_i^{+q} = \{v : K_i^{+q}(v) \leq \frac{1}{2} \} \) as well as their running intersections \( \bigcap_{i=1}^{t} C_i^q \) and \( \bigcap_{i=1}^{t} C_i^{+q} \) are \( 1 - \alpha \) CSs for \( V(\pi) \).

Appendix D contains the details on how to bet. We close this section with two remarks. First, a “bad” \( q(x, a) \) can make \( wr - c \) have larger variance than \( wr \). To protect against this case we can run a *doubly hedged* process:

\[
K_i^{\pm q}(v) = \frac{1}{2}(K_i^{+q}(v) + K_i^{-q}(v)) \]

which will accrue wealth almost as well as the best of its two components. Second our framework allows for \( q(x, a) \) to be updated in every step as long as the updates are predictable.

### 5.2 Scalar Betting

Since \( \mathbb{E}[w] = 1 \), it would seem that the \( \lambda_1 \) bet cannot have any long term benefits. While this will be shown to be false in our experiments we nevertheless develop a betting strategy that only bets on \( w_ir_i - v \). The advantages of this strategy are computational and conceptual simplicity. Similarly to Section 4.3 we use a hedged process \( K_i^{\pm q}(v) = \frac{1}{2}(K_i^{+q}(v) + K_i^{-q}(v)) \) where

\[
K_i^{+q}(v) = \prod_{i=1}^{t} \left( 1 + \lambda_{i,1}^{2,q}(w_ir_i - v) \right),
\]

\[
K_i^{-q}(v) = \prod_{i=1}^{t} \left( 1 + \lambda_{i,2}^{2,q}(w_i(1 - r_i) - (1 - v)) \right).
\]

Appendix C.1 provides an alternative justification via a worst case argument. The upshot is that \( \lambda_1 = \max(0, -\lambda_2) \) is a reasonable choice and it leads to the above processes.

We explain betting for \( K_i^{+q}(v) \), since betting for \( K_i^{-q}(v) \) reduces to that. We use a result by Fan et al. [2015]:

\[
\ln(1 + \lambda \xi) \geq \lambda \xi + (\ln(1 - \lambda) + \lambda) \cdot \xi^2
\]
for all $\xi \geq -1$ and $\lambda \in [0,1)$, which we reproduce in Appendix C.2. We apply it in our case with $\xi_i = w_i r_i - v \geq -1$ and consider the log wealth lower bound for a fixed $\lambda_2$

$$\ln(K^t_\lambda(v)) \geq \lambda_2 \sum_{i=1}^{t-1} \xi_i + (\ln(1-\lambda_2) + \lambda_2) \sum_{i=1}^{t-1} \xi_i^2.$$ 

When $\sum_{i=1}^{t-1} \xi_i^2 > 0$ the lower bound is concave and can be maximized in $\lambda_2$ by setting its derivative to 0. This gives

$$\lambda_{2,i}^\geq(v) = \frac{\sum_{i=1}^{t-1}(w_i r_i - v) - \sum_{i=1}^{t-1}(w_i r_i - v)}{\sum_{i=1}^{t-1}(w_i r_i - v) + \sum_{i=1}^{t-1}(w_i r_i - v)^2}.$$ 

When $\sum_{i=1}^{t-1} \xi_i^2 = 0$ we can set $\lambda_{2,i}^\geq(v) = 0$. Finally, employing the same ideas as Section 4.4 we can adaptively choose the $v$ to bet against and avoid maintaining a grid of values for $v$. Details are in Appendix C.3

### 5.3 Gated Deployment

A common OPE use case is to estimate the difference $V(\pi) - V(h)$. If we can reject all negative values (i.e. the lower CS crosses 0) then $\pi$ should be deployed. Conversely, rejecting all positive values (i.e. the upper CS crosses 0) means $\pi$ should be discarded. Since $h$ is the policy collecting the data we have $V(h) = \mathbb{E}[r]$. Thus we can form a CS around $V(\pi) - V(h)$ by considering the process:

$$K^t_{\lambda}(v) = \prod_{i=1}^{t-1} (1 + \lambda_1 (w_i - 1) + \lambda_2 (w_i r_i - r_i - v))$$

for predictable $\lambda_{1,i}, \lambda_{2,i}$ subject to $(\lambda_{1,i}, \lambda_{2,i}) \in G^0_v$, where

$$G^m_v = \{(\lambda_1, \lambda_2) : 1 + \lambda_1 (w - 1) + \lambda_2 (wr - r) \geq m \quad \forall (w, r) \in \{0, w_{\text{max}}\} \times \{0, 1\}\}.$$ 

As before, we can form a hedged process and restrict bets to a set that enables the use of our lower bound. We defer these details to appendix E. We can then show

**Theorem 4.** The sequences $C^d_i = \{v : K^d_i(v) \leq \frac{1}{\alpha}\}$ and $\bigcap_{i=1} C^d_i$ are $1 - \alpha$ CSs for $V(\pi) - V(h)$.

This CS has two advantages over a classical A/B test. First, we don't have to choose a stopping time in advance. The CS can run for as little or as long as necessary. Second, if $\pi$ were worse than $h$ the A/B test would have an adverse effect on the quality of the overall system, while here we can reason about this degradation without deploying $\pi$. 

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6 Related Work

Apart from IPS, other popular OPE estimators include Doubly Robust Robins and Rotnitzky [1995], Dudík et al. [2011] which incorporates (2) as an additive control variate and SNIPS Swaminathan and Joachims [2015] which incorporates $E[w] = 1$ as a multiplicative control variate. The quest to balance the bias-variance tradeoff in OPE has led to many different proposals Wang et al. [2017], Vlassis et al. [2019]. EL-based estimators are proposed in Kallus and Uehara [2019] and Karampatziakis et al. [2020].

CIs for OPE include both finite-sample Thomas et al. [2015] and asymptotic Li et al. [2015], Karampatziakis et al. [2020] ones. Some works that propose both types are Bottou et al. [2013] and Dai et al. [2020]. The latter obtains CIs without knowledge of $w$, a much more challenging scenario that requires additional assumptions.

We are not aware of any CSs for OPE. For on-policy setups, the most competitive CSs all rely on exploiting (super)martingales, and in some sense all admissible CSs have to Ramdas et al. [2020]. Examples include Robbins’ mixture martingale Robbins [1970] and the techniques of Howard et al. [2020]. The recent work of Waudby-Smith and Ramdas [2020] which leverages a betting view substantially increases the scope of these techniques, while simplifying and tightening the constructions. Similar betting ideas have been recently used in the development of parameter-free online algorithms Orabona and Pál [2016].

7 Experiments

![Figure 1: Empirical coverage for two proposed CSs. The CS that bets on both $w - 1$ and $wr - v$ converges to nominal coverage while the CS that does not bet on $w - 1$ overcovers.](image-url)
Figure 2: The width of 95% CS produced by MOPE and its three ablations. The pointwise asymptotic curve is not a CS.

Figure 3: Three 99.9% CSs with/without a reward predictor and a doubly hedged one that achieves the best of both.
Figure 4: CS for gated deployment and A/B test. $\pi$ can be deployed as soon as the lower CS crosses 0 (dotted line at $t = 657$).

7.1 Coverage

While any predictable betting sequence guarantees correct coverage, some will overcover more than others. Here we investigate the coverage properties of MOPE and the strategy of Section 5.2. We generate 1000 sequences of 100000 $(w, r)$ pairs each from a different distribution. All distributions are maximum entropy distributions subject to $(w, r) \in \{0, 0.5, 2, 100\} \times \{0, 1\}$, $E[w] = 1$, $E[w^2] = 10$ and $V(\pi)$ sampled uniformly in $[0, 1]$. In Figure 1 we show the empirical mean coverage of the two CSs for $\alpha = 0.05$. MOPE approaches nominal coverage from above, a property rarely seen with standard confidence bounds.

7.2 Computational vs. Statistical Efficiency

We run an ablation study for the three ingredients of MOPE, where $-\text{Vector}$ is the scalar betting technique of section 5.2; $-\text{Common}$ solves (6) over a grid of 200 $v$ values at each timestep; and $-\text{Bound}$ optimizes the log wealth exactly rather than the bound of Lemma 1, i.e., Algorithm 2 with equation (5) in lieu of equation (13).

We use four synthetic environments which are distributions over $(w, r)$ generated in the same way as section 7.1 but with $(V(\pi), E[w^2]) \in \{0.05, 0.5\} \times \{10, 50\}$. Table 1 shows the running times for each method in the environment with the largest variance. We see that directly maximizing wealth and individual betting per $v$ are very slow. MOPE and $-\text{Vector}$ are computationally efficient. In Figure 2 we show the average CS width over 10 repetitions for 500000 time steps for MOPE and its ablations as well as the asymptotic CI from Karampatziakis et al. [2020] which is only valid pointwise and provides a lower bound for all CSs in the figure. MOPE is better than $-\text{Vector}$ and as
Table 1: Timings for MOPE and its ablations on 500000 samples

| Method   | MOPE - Vector | - Common | - Bound |
|----------|---------------|----------|---------|
| Time (sec) | 32            | 14.5     | 10440   | 15882   |

good or better than − Bound. While MOPE is not as tight as the (much more computationally demanding) − Common, the gap is small in all but the most challenging environment.

7.3 Effect of a Reward Predictor

We now investigate the use of reward predictors in our CSs using the processes $K_i^{\pm q}(v)$ and $K_i^{\pm t}(v)$ of Section 5.1. We use the first 1 million samples from the mnist8m dataset which has 10 classes and train the following functions: $h$ using linear multinomial logistic regression (MLR), $\pi$ again using MLR but now on 1000 random Fourier features (RFF) Rahimi and Recht [2007] that approximate a Gaussian kernel machine, and finally $q$ which uses the same RFF representation as $\pi$ but instead its $i$-th output is independently trained to predict whether the input is the $i$-th class using 10 binary logistic regressions. We used the rest of the data with the following protocol: for each input/label pair $(x_i, y_i)$, we sample action $a_i$ with probability $0.9h(a_i; x_i) + 0.01$ (so that we can safely set $w_{\text{max}} = 100$), we set $r_i = 1$ if $a_i = y_i$, otherwise $r_i = 0$, and record $w_i$ and $c_i$.

We estimated $V(\pi) \approx 0.9385$ using the next million samples. In Figure 3 we show the CS for $V(\pi)$ averaged over 5 runs each with 10000 different samples using the processes $K_i^{\pm q}(v)$, $K_i^{\pm t}(v)$, and $K_i^{\pm 2t}(v)$. We see that including a reward predictor dramatically improves the lower bound and somewhat hurts the upper bound. The doubly hedged process on the other hand attains the best of both worlds.

7.4 CSs for Gated Deployment

Here we investigate the use of CSs for gated deployment. We use the same $h$ and $\pi$ and the same data as in Section 7.3 but now we are using the process $K_i^{\text{gdl}}(v)$ (or rather a computationally efficient version of this process based on a hedged process with common bets and optimizing a quadratic lower bound c.f. Appendix E). Figure 4 shows the average CS over 5 runs each with 10000 different samples. We see that the CS contains the true difference (about 0.17) and quickly decides that $\pi$ is better than $h$ at $t = 657$ samples. We also include an on-policy CS (from Waudby-Smith and Ramdas [2020]) which can only be computed if $\pi$ is deployed e.g. in an A/B test. While this is riskier when $\pi$ is inferior to $h$, the on-policy rewards typically have lower variance. Thus the on policy CS can conclude that $\pi$ is better than $h$ at the same $\alpha = 0.01$ using 440 samples (220 for each policy). If the roles of $\pi$ and $h$ were swapped, i.e, $\pi$ was the behavior policy and $h$ was a proposed alternative, the on policy CS would
still need to collect 220 samples from \( h \). In contrast, a system using the off-policy CS would never have to experience any regret when the behavior policy is superior.

8 Conclusions

We presented a generic way to construct confidence sequences for OPE in the Contextual Bandit setting. The construction leaves a lot of freedom in designing betting strategies and we mostly explored options with an eye towards computational efficiency. Theoretically we achieve finite sample coverage and validity at any time with minimal assumptions. Empirically the resulting sequences are tight and not too far away from asymptotic and pointwise valid existing work. Theoretical results on the width of our CSs remain elusive and are both an interesting area for future work and a key to unlock much stronger analyses of various algorithms in Bandits and RL.
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A Proofs

A.1 Main Lemma

The following lemma will be helpful in the proofs of all our Theorems.

**Lemma 2.** Suppose we have a family of stochastic processes \((M_t(m))_{t=0}^\infty\) indexed by \(m \in [0, 1]\) and further assume the process \((M_t(\mu))_{t=0}^\infty\) for some \(\mu \in [0, 1]\) is a non-negative martingale with respect to a filtration \(\mathcal{F}_t\) (i.e. \(\mathbb{E}[M_t|\mathcal{F}_{t-1}] = M_{t-1}\) for \(t \geq 1\)) with initial value \(M_0 = 1\). Then for any given \(\alpha \in [0, 1]\) the sequence of sets \(C_t = \{m : M_t(m) \leq \frac{1}{\alpha}\}\) is a \((1 - \alpha)\) confidence sequence for \(\mu\) and so is its running intersection \(\bigcap_{i=1}^t C_i\).

**Proof.** For the first part, by the definition of a CS it suffices to show that \(\Pr(\exists t \in \mathbb{N} : \mu \not\in C_t) \leq \alpha\) or

\[
\Pr\left(\exists t \in \mathbb{N} : \mu \not\in \left\{m : M_t(m) \leq \frac{1}{\alpha}\right\}\right) \leq \alpha.
\]

An error occurs only if \(M_t(\mu)\) exceeds \(1/\alpha\) at any point. This means that it suffices to show that

\[
\Pr\left(\exists t \in \mathbb{N} : M_t(\mu) \geq \frac{1}{\alpha}\right) \leq \alpha,
\]

which is true by Ville’s inequality [1939] since \(M_t(\mu)\) is a non-negative martingale with initial value 1.

For the second part, we need to show that

\[
\Pr\left(\exists t \in \mathbb{N} : \mu \not\in \bigcap_{s=1}^t \left\{m : M_s(m) \leq \frac{1}{\alpha}\right\}\right) \leq \alpha.
\]

This reduces to showing

\[
\Pr\left(\exists t \in \mathbb{N} : \exists s \in \{1, \ldots, t\} : M_s(\mu) \geq \frac{1}{\alpha}\right) \leq \alpha,
\]

which further simplifies to

\[
\Pr\left(\exists t \in \mathbb{N} : M_t(\mu) \geq \frac{1}{\alpha}\right) \leq \alpha,
\]

and this is again implied by Ville’s inequality. \(\square\)

A.2 Proof of Theorem 1

**Proof.** Consider the filtration \((\mathcal{F}_t)_{t=0}^\infty\) generated by the sequence of sigma-fields \(\mathcal{F}_0 \subset \mathcal{F}_1 \subset \ldots\) with \(\mathcal{F}_0\) the trivial sigma-field and \(\mathcal{F}_t = \sigma((w_0, r_0), (w_1, r_1), \ldots, (w_t, r_t))\).
It suffices to show that our betting ensures that $K_t(V(\pi))$ is a non-negative martingale with initial value 1 as we can then apply lemma 2. $K_0(v) = 1$ is by the definition of the process (we start with a wealth of 1), and $K_t(v) \geq 0$ for all $v \in [0, 1]$ because our bets are in the set $\mathcal{D}_u^\omega$ (c.f. eq (4)). Thus it remains to show $\mathbb{E} [K_t(V(\pi)) | \mathcal{F}_{t-1}] = K_{t-1}(V(\pi))$. We have the following chain of equalities

$$
\mathbb{E} [K_t(V(\pi)) | \mathcal{F}_{t-1}] = \mathbb{E} [K_{t-1} (1 + \lambda_{1,t}(w_t - 1) + \lambda_{2,t}(w_tr_t - V(\pi))) | \mathcal{F}_{t-1}]
= K_{t-1} (1 + \mathbb{E} [\lambda_{1,t}(w_t - 1)] | \mathcal{F}_{t-1} + \mathbb{E} [\lambda_{2,t}(w_tr_t - V(\pi))] | \mathcal{F}_{t-1})
= K_{t-1} (1 + \lambda_{1,t} \cdot 0 + \lambda_{2,t} \cdot 0) = K_{t-1}
$$

where we have used that $K_{t-1}$, $\lambda_{1,t}$, $\lambda_{2,t}$ are measurable with respect to $\mathcal{F}_{t-1}$ and that $\mathbb{E}[w] = 1$ and $\mathbb{E}[wr] = V(\pi)$.

A.3 Proof of Lemma 1

Proof. Consider the function $f(x) = \ln(1 + x) - x - \psi x^2$ with domain $[-\frac{1}{2}, \infty)$. Note that $f(-\frac{1}{2}) = 0$ and $\lim_{x \to \infty} f(x) = \infty$. Furthermore $f$ has two critical points: 0 and $-\frac{2\psi+1}{2\psi}$. But $f(0) = 0$ and $f \left(-\frac{2\psi+1}{2\psi}\right) > 0$ so we conclude that $f(x) \geq 0$ for all $x \geq -\frac{1}{2}$. □

A.4 Proof of Theorem 2

Proof. We will first show that $K_t^+(V(\pi))$ is a non-negative martingale with initial value 1. Consider the same filtration as for Theorem 1. Note that $K_0^+(v) = 1$ is by the definition of the process (we start with a wealth of 1). We analyze $K_t^+(v)$ and $K_t^-(v)$ separately. Note that $K_t^+(v) \geq 0$ for all $v \in [0, 1]$ because our bets are in the set $\mathcal{C} \subset \mathcal{D}_u^\omega$ (c.f. eq (4)). For $K_t^-(v)$ we note that the process is isomorphic to a process similar to $K_t^+(v)$ but with the reward and $v$ redefined. Thus our bets keep $K_t^- (v) \geq 0$. We now show the equality

$$
\mathbb{E} \left[ K_t^- \left( V(\pi) \right) | \mathcal{F}_{t-1} \right] = K_{t-1}^-(V(\pi))
$$

as the equality $\mathbb{E} \left[ K_t^+ \left( V(\pi) \right) | \mathcal{F}_{t-1} \right] = K_{t-1}^+(V(\pi))$ is exactly what was shown in Theorem 1. We have

$$
\mathbb{E} \left[ K_t^- \left( V(\pi) \right) | \mathcal{F}_{t-1} \right] = \mathbb{E} \left[ K_{t-1}^- \left( 1 + \lambda_{1,t}^{-}(w_t - 1) + \lambda_{2,t}^{-}(w_t(1-r_t)(1-V(\pi))) \right) | \mathcal{F}_{t-1} \right]
= K_{t-1}^- \mathbb{E} \left[ 1 + \lambda_{1,t}^{-}(w_t - 1) + \lambda_{2,t}^{-}(w_t(1 - V(\pi)) - w_tr_t) | \mathcal{F}_{t-1} \right]
= K_{t-1}^- \left( 1 + \left( \lambda_{1,t}^{-} + \lambda_{2,t}^{-} \right) \mathbb{E} \left[ (w_t - 1) | \mathcal{F}_{t-1} \right] + \lambda_{2,t}^{-} \mathbb{E} \left[ (V(\pi) - w_tr_t) | \mathcal{F}_{t-1} \right] \right)
= K_{t-1}^-(1 + (\lambda_{1,t}^{-} + \lambda_{2,t}^{-}) \cdot 0 + \lambda_{2,t}^{-} \cdot 0) = K_{t-1}^-.
$$

Therefore $\frac{1}{2} \left( K_t^+ \left( V(\pi) \right) + K_t^- \left( V(\pi) \right) \right)$ is also a non-negative martingale with initial value 1. Applying Lemma 2 finishes the proof of the theorem. □
A.5 Proof of Theorem 3

Proof. We note that the proof below works for a sequence of predictable functions \( q_t(x, a) \) but to reduce notation we use \( q(x, a) \). Consider the filtration \((\mathcal{F}_t)_{t=0}^\infty \) generated by the sequence of sigma-fields \( \mathcal{F}_0 \subset \mathcal{F}_1 \subset \ldots \) with \( \mathcal{F}_0 \) the trivial sigma-field and \( \mathcal{F}_t = \sigma((x_1, a_1, r_1), \ldots, (x_t, a_t, r_t)) \). Note that \( K^q_0(v) = 1 \) and \( K^q_t(v) \geq 0 \) for all \( v \in [0, 1] \) because our bets are in the set \( \mathcal{C}^q \). Thus it remains to show \( E[K^q_t(V(\pi)) | \mathcal{F}_{t-1}] = K_{t-1}(V(\pi)) \). We have the following chain of equalities

\[
E[K^q_t(V(\pi)) | \mathcal{F}_{t-1}] = E[K^q_{t-1}(1 + \lambda_{1,t}(w_t - 1) + \lambda_{2,t}(w_t r_t - c_t - V(\pi))) | \mathcal{F}_{t-1}]
\]

\[
= K^q_{t-1}(1 + \lambda_{1,t}E[w_t - 1|\mathcal{F}_{t-1}] + \lambda_{2,t}E[w_t r_t - V(\pi)|\mathcal{F}_{t-1}] - \lambda_{2,t}E[c_t|\mathcal{F}_{t-1}])
\]

\[
= K^q_{t-1} \left( 1 - \lambda_{2,t}E \left[ w_t q(x_t, a_t) - \sum_{a'} \pi(a'; x_t) q(x_t, a') | \mathcal{F}_{t-1} \right] \right) = K^q_{t-1},
\]

where we have used that \( K_{t-1}, \lambda_{1,t}, \lambda_{2,t} \) are measurable with respect to \( \mathcal{F}_{t-1} \) and that \( E[w] = 1 \) and \( E[w r] = V(\pi) \) as well as \( E_{x_t, \sim D,a_t, \sim h} [w_t q(x_t, a_t)] = E_{x_t} \sum \pi(a'; x_t) q(x_t, a') \). Thus the claim for \( C^q_t \) and its running intersection can be shown by applying lemma 2. The claim for \( C^{<q}_t \) is completely analogous using the ideas here and in the proof of Theorem 2. \( \square \)

A.6 Proof of Theorem 4

Proof. Consider the same filtration as for Theorem 1. Note that \( K_0^{gd}(v) = 1 \) is by the definition of the process and that \( K^q_t(v) \geq 0 \) for all \( v \in [0, 1] \) because our bets are in the set \( \mathcal{G}^q_0 \) (c.f. eq (17)). Finally, we have

\[
E \left[ K^{gd}_t(V(\pi) - V(h)) | \mathcal{F}_{t-1} \right] = E \left[ K^{gd}_{t-1}(1 + \lambda_{1,t}(w_t - 1) + \lambda_{2,t}(w_t r_t - r_t - (V(\pi) - V(h))) \right) | \mathcal{F}_{t-1}]
\]

\[
= K^{gd}_{t-1}(1 + \lambda_{1,t}E[w_t - 1|\mathcal{F}_{t-1}] + \lambda_{2,t}E[w_t r_t - V(\pi)|\mathcal{F}_{t-1}] - \lambda_{2,t}E[r_t - V(h)|\mathcal{F}_{t-1}])
\]

\[
= K^{gd}_{t-1} \left( 1 + \lambda_{1,t} \cdot 0 + \lambda_{2,t} \cdot 0 - \lambda_{2,t} \cdot 0 \right) = K^{gd}_{t-1}.
\]

Therefore \( K_t^{gd}(V(\pi) - V(h)) \) is a non-negative martingale with initial value 1. Applying lemma 2 finishes the proof of the theorem. \( \square \)

B Avoiding grid search

We first lower bound each process separately, then lower bound the hedged process. We denote the bets for \( K^+ \) (respectively \( K^- \)) as \( \lambda^+ \), (resp. \( \lambda^- \)). From lemma 1 we have

\[
\ln(K^+_t(v)) \geq \sum_{i=1}^{t-1} \lambda^+_i b_i(v) + \psi \sum_i \lambda^+_i A_i(v) \lambda^+_i,
\]

and

\[
\ln(K^-_t(v)) \geq \sum_{i=1}^{t-1} \lambda^-_i b'_i(v') + \psi \sum_i \lambda^-_i A'_i(v') \lambda^-_i,
\]

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where \( v' = 1 - v \), \( b'_i(v) = \left[ \begin{array}{c} w_i - 1 \\ w_i(1 - r_i) - v \end{array} \right] \) and \( A'_i(v) = b'_i(v)b'_i(v)^t \). For the Hedged process, using that for any \( a, b \)

\[
\ln(\exp(a) + \exp(b)) \geq \max(a, b)
\]
to first establish

\[
\ln(K^\pm(v)) \geq \max(\ln(K^+(v)) - \ln(2), \ln(K^-(v)) - \ln(2))
\]
and further bound each term in the maximum by the respective quadratic lower bound. We conclude that if a \( v \) achieves

\[
\sum_{t=1}^{T-1} \lambda^+_t b_i(v) + \psi \sum_{t=1}^{T-1} \lambda^+_t A_i(v) \lambda^+_t = \ln \left( \frac{2}{\alpha} \right),
\]
or a \( v' = 1 - v \) achieves

\[
\sum_{t=1}^{T-1} \lambda^-_t b'_i(v') + \psi \sum_{t=1}^{T-1} \lambda^-_t A'_i(v') \lambda^-_t = \ln \left( \frac{2}{\alpha} \right),
\]
then we also achieve \( K^\pm_t(v) \geq \frac{1}{\alpha} \). In terms of \( v \) and \( v' \) these expressions are second degree equations and thus their real roots in \( [0, 1] \) (if any) provide a safe bracketing of the confidence region \( \{ v : K^\pm_t(v) \leq 1/\alpha \} \). For \( K^+ \) let

\[
C_t = \sum_{t=1}^{T-1} \lambda^+_t \begin{bmatrix} w_i - 1 \\ w_i r_i \end{bmatrix},
\]

\[
S_t = \sum_{t=1}^{T-1} \lambda^+_t \begin{bmatrix} 0 \\ 1 \end{bmatrix},
\]

\[
Q_t = \sum_{t=1}^{T-1} \psi \lambda^+_t \begin{bmatrix} (w_i - 1)^2 \\ (w_i - 1)w_i r_i \end{bmatrix} \begin{bmatrix} (w_i - 1)w_i r_i \\ w_i^2 r_i^2 \end{bmatrix} \lambda^+_t,
\]

\[
T_t = \sum_{t=1}^{T-1} \psi \lambda^+_t \begin{bmatrix} 0 \\ -(w_i - 1) \end{bmatrix} \begin{bmatrix} -(w_i - 1) \\ -2w_i r_i \end{bmatrix} \lambda^+_t,
\]

\[
U_t = \sum_{t=1}^{T-1} \psi \lambda^+_t \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \lambda^+_t,
\]

and define \( C_t', S_t', Q_t', T_t', U_t' \) similarly by using \( \lambda^-_t \) instead of \( \lambda^+_t \) and \( 1 - r_i \) instead of \( r_i \). Then the largest real root \( v^+ \) of

\[
C_t - S_t v + Q_t + T_t v + U_t v^2 - \ln \left( \frac{2}{\alpha} \right) = 0,
\]
if it exists, satisfies \( K^+_t(v^+) \geq \frac{1}{\alpha} \). Similarly we can obtain \( v' \) as the largest real root of the quadratic with \( C_t', S_t', Q_t', T_t', U_t' \) in place of \( C_t, S_t, Q_t, T_t, U_t \), if it exists. Then \( v^- = 1 - v' \) satisfies \( K^+_t(v^-) \geq \frac{1}{\alpha} \).
C Details of the Scalar Betting Strategy

C.1 Elimination of one bet

Since in the long term $\lambda_1$ should be 0 its purpose can only be as a hedge in the short-term. We formulate this by considering the worst case wealth reduction among three outcomes: $(w, r) = (w_{\text{max}}, 1)$, $(w, r) = (w_{\text{max}}, 0)$ and $w = 0$ with any reward. We choose $\lambda_1$ to maximize the wealth in the worst of these outcomes. Thus we set up a family of Linear Programs (LPs) parametrized by $\lambda_2$ and $v$ and with optimization variables $\alpha$ and $\lambda_1$:

\[
\begin{align*}
\text{maximize} & \quad \alpha \\
\text{subject to} & \quad \alpha \leq 1 + \lambda_1(w_{\text{max}} - 1) + \lambda_2(w_{\text{max}} - v) \quad (z_1) \\
& \quad \alpha \leq 1 + \lambda_1(w_{\text{max}} - 1) - \lambda_2 v \quad (z_2) \\
& \quad \alpha \leq 1 - \lambda_1 - \lambda_2 v \quad (z_3),
\end{align*}
\]

where the variable $z_i$ in parentheses next to each constraint is the corresponding dual variable.

**Theorem 5.** For any $v \in [0,1]$ and any $\lambda_2 \in \mathbb{R}$, the optimal value of $\lambda_1$ in the above LP is $\lambda_1^* = \max(-\lambda_2, 0)$.

**Proof.** The dual program is

\[
\begin{array}{ll}
\text{minimize} & (1 + \lambda_2(w_{\text{max}} - v))z_1 + (1 - \lambda_2 v)z_2 + (1 - \lambda_2 v)z_3 \\
\text{subject to} & z_i \geq 0 \quad (i = 1, 2, 3) \\
& -(w_{\text{max}} - 1)(z_1 + z_2) + z_3 = 0 \\
& z_1 + z_2 + z_3 = 1.
\end{array}
\]

Consider the following two dual feasible settings:

\[
z_1 = 0, \quad z_2 = \frac{1}{w_{\text{max}}}, \quad z_3 = \frac{w_{\text{max}} - 1}{w_{\text{max}}},
\]

and

\[
z_1 = \frac{1}{w_{\text{max}}}, \quad z_2 = 0, \quad z_3 = \frac{w_{\text{max}} - 1}{w_{\text{max}}},
\]

with corresponding dual objectives: $1 - \lambda_2 v$ and $1 - \lambda_2 v + \lambda_2$. From here we see that if $\lambda_2 > 0$ the former attains a better dual objective and is thus a better bound for the primal objective. When $\lambda_2 < 0$ the latter is better.

When $\lambda_2 > 0$, a primal feasible setting is $\alpha = 1 - \lambda_2 v, \lambda_1 = 0$. Furthermore this setting achieves the same objective as the first dual feasible setting so we conclude that these are the optimal primal and dual solutions when $\lambda_2 > 0$.

When $\lambda_2 < 0$, a primal feasible setting is $\alpha = 1 - \lambda_2 v + \lambda_2, \lambda_1 = -\lambda_2$. Furthermore this setting achieves the same objective as the second dual feasible setting so we conclude that these are the optimal primal and dual solutions when $\lambda_2 < 0$.

Finally when $\lambda_2 = 0$ the two cases give the same value for $\lambda_1$ so we conclude $\lambda_1 = \max(-\lambda_2, 0)$ for all $\lambda_2 \in \mathbb{R}$ (and $v \geq 0$). □
The theorem suggests that in a hedged strategy the wealth process eliminating low values of $V(\pi)$ should set $\lambda^+ > 0$ because $E[wr - v] > 0$ and thus $\lambda^+ > 0$. The wealth process that eliminates high values of $V(\pi)$ on the other hand should have $\lambda_1 = -\lambda_2$ because $E[wr - v] < 0$ and thus $\lambda_2 < 0$. Thus the two processes look like

$$K^+_{\pi}(v) = \prod_{i=1}^n \left(1 + \lambda^+_{\pi,i}(w_ir_i - v)\right),$$

$$K^\prec_{\pi}(v) = \prod_{i=1}^n \left(1 - \lambda^\prec_{\pi,i}(w_i - 1) + \lambda^\prec_{\pi,i}(w_ir_i - v)\right) = \prod_{i=1}^n \left(1 - \lambda^\prec_{\pi,i}(w_i(1 - r_i) - (1 - v))\right).$$

In the main text we have redefined $\lambda^\prec_{\pi,i} := -\lambda^\prec_{\pi,i}$ for symmetry.

**C.2 A Technical Lemma**

The following result can be extracted from the proof of Proposition 4.1 in Fan et al. [2015].

**Lemma 3.** For $\xi \geq -1$ and $\lambda \in [0, 1)$ we have

$$\ln(1 + \lambda \xi) \geq \lambda \xi + (\ln(1 - \lambda) + \lambda) \cdot \xi^2. \quad (23)$$

**Proof.** Note that $\lambda \xi \geq -\lambda > -1$. For $x > -1$ the function $f(x) = \frac{\ln(1+x) - x}{x^2}$ is increasing in $x$, therefore $f(\lambda \xi) \geq f(-\lambda)$. Rearranging leads to the statement of the lemma. \qed

We will be using this lemma with bets $\lambda \in [0, 1)$ and $\xi_i = w_ir_i - v$ or $\xi_i = w_i(1 - r_i) - (1 - v_i)$. In either case $\xi_i \geq -1$. This lemma provides a stronger lower bound than that of Lemma 1. The reason we use the latter for vector bets is that the natural extension of (23) to the vector case does not lead to a convex problem.

**C.3 Avoiding grid Search**

Suppose that our bets $\lambda^+_{\pi,i}$ and $\lambda^-_{\pi,i}$ do not depend on $v$. We have the individual lower bounds

$$\ln(K^+(v)) \geq \sum_i \lambda^+_{\pi,i}(w_ir_i - v) + \sum_i (\ln(1 - \lambda^+_{\pi,i}) + \lambda^+_{\pi,i})(w_ir_i - v)^2$$

and

$$\ln(K^-(v)) \geq \sum_i \lambda^-_{\pi,i}(w_ir'_i - v') + \sum_i (\ln(1 - \lambda^-_{\pi,i}) + \lambda^-_{\pi,i})(w_ir'_i - v')^2,$$

where $r' = 1 - r$, $v' = 1 - v$. For the Hedged process, using that for any $a, b$

$$\ln(\exp(a) + \exp(b)) \geq \max(a, b)$$
to first establish

$$\ln(K^+(v)) \geq \max(\ln(K^+(v)) - \ln(2), \ln(K^-(v)) - \ln(2))$$

and further bound each term in the maximum by the respective quadratic lower bound. We conclude that if a $v$ achieves

$$\sum_i \lambda^+_{2,i} (w_i r_i - v) + \sum_i (\ln(1 - \lambda^+_{2,i}) + \lambda^+_{2,i}) (w_i r_i - v)^2 = \ln \left( \frac{2}{\alpha} \right)$$

or a $v' = 1 - v$ achieves

$$\sum_i \lambda^-_{2,i} (w_i r'_i - v') + \sum_i (\ln(1 - \lambda^-_{2,i}) + \lambda^-_{2,i}) (w_i r'_i - v')^2 = \ln \left( \frac{2}{\alpha} \right)$$

then we also achieve $K^+(v) > \frac{1}{\alpha}$. Thus, a valid confidence interval can be obtained by considering the roots of these quadratics. Let

$$C = \sum_i \lambda^+_{2,i} w_i r_i$$
$$C' = \sum_i \lambda^-_{2,i} w_i r'_i$$
$$S = \sum_i \lambda^+_{2,i}$$
$$S' = \sum_i \lambda^-_{2,i}$$
$$Q = \sum_i (\ln(1 - \lambda^+_{2,i}) + \lambda^+_{2,i}) w^2_i r^2_i$$
$$Q' = \sum_i (\ln(1 - \lambda^-_{2,i}) + \lambda^-_{2,i}) w^2_i r'^2_i$$
$$T = \sum_i (\ln(1 - \lambda^+_{2,i}) + \lambda^+_{2,i}) w_i r_i$$
$$T' = \sum_i (\ln(1 - \lambda^-_{2,i}) + \lambda^-_{2,i}) w_i r'_i$$
$$U = \sum_i (\ln(1 - \lambda^+_{2,i}) + \lambda^+_{2,i})$$
$$U' = \sum_i (\ln(1 - \lambda^-_{2,i}) + \lambda^-_{2,i})$$

We obtain:

$$v_{\min} = \frac{2T + S - \sqrt{(2T + S)^2 - 4U(Q + C - \ln(2/\alpha))}}{2U}$$

or $v_{\min} = 0$ if the discriminant is negative, and

$$v_{\max} = 1 - v' = 1 - \frac{2T' + S' - \sqrt{(2T' + S')^2 - 4U'(Q' + C' - \ln(2/\alpha))}}{2U'}$$

or $v_{\max} = 1$ if the discriminant is negative.

### D Reward Predictors

#### D.1 Betting

We describe betting for $K^+_i(v)$. Betting for $K^-_i(v)$ is analogous. We overload the log wealth at step $i$ when betting against $v$ as $\ell^+_i(\lambda) = \ln(1 + \lambda_{1,i}(w_i - 1) + \ldots$
We use lemma 1 to obtain that for any \( \lambda, i \in \mathcal{E}^{1/2}_\alpha \), we have
\[
\ln(K_{t+q}^i(v) = \sum_{i=1}^{t-1} \ell_i^i(\lambda) \geq \lambda^\top \sum_{i=1}^{t-1} b_i(v) + \psi \lambda^\top \left( \sum_{i=1}^{t-1} A_i(v) \right) \lambda,
\]
where now \( b_i(v) = \begin{bmatrix} w_i - 1 & w_i r_i - c_i - v \end{bmatrix} \) and \( A_i(v) = b_i(v) b_i(v)^\top \). As in the case without reward predictor we have that the wealth lower bound is a polynomial in \( v \) with
\[
\sum_{i=1}^{t-1} A_i(v) = A_i^{(0)} + v A_i^{(1)} + v^2 A_i^{(2)},
\]
and the coefficients can be maintained as
\[
A_i^{(0)} = \sum_{i=1}^{t-1} \begin{bmatrix} (w_i - 1)^2 & (w_i - 1)^2 & (w_i - 1)^2 \end{bmatrix} \begin{bmatrix} (w_i - 1)(w_i r_i - c_i) & (w_i r_i - c_i) \end{bmatrix},
\]
\[
A_i^{(1)} = \sum_{i=1}^{t-1} \begin{bmatrix} 0 & -(w_i - 1) & 0 \end{bmatrix} \begin{bmatrix} -(w_i - 1) & -2(w_i r_i - c_i) \end{bmatrix},
\]
\[
A_i^{(2)} = \sum_{i=1}^{t-1} \begin{bmatrix} 0 & 0 & 0 \end{bmatrix},
\]
\[
b_i^{(0)} = \sum_{i=1}^{t-1} \begin{bmatrix} w_i - 1 \end{bmatrix},
\]
\[
b_i^{(1)} = \sum_{i=1}^{t-1} \begin{bmatrix} 0 & 0 \end{bmatrix}.
\]
Given a \( v \) we compute concrete values for these coefficients and then solve
\[
\lambda_t = \arg \max_{\lambda \in \mathcal{E}^{1/2}} \lambda^\top \sum_{i=1}^{t-1} b_i(v) + \psi \lambda^\top \left( \sum_{i=1}^{t-1} A_i(v) \right) \lambda.
\]
A similar procedure like the one in Algorithm 1 can then be used for solving this problem.

### D.2 Avoiding Grid Search

To find the value of \( v \) that we can plug in to the above optimization problem we proceed as in section 4.4, and further explained in Appendix B. To find a \( v \) such that \( K_{t+q}^i(v) \geq \frac{1}{\alpha} \) it suffices to solve
\[
\sum_{i=1}^{t-1} \lambda_i^\top b_i(v) + \psi \sum_{i=1}^{t-1} \lambda_i^\top A_i(v) \lambda_i = \ln \left( \frac{2}{\alpha} \right),
\]
given the previous bets $\lambda_1, \ldots, \lambda_{t-1}$. This is a second degree equation which can be solved by maintaining the quantities

\[
\begin{align*}
C_t &= \sum_{i=1}^{t-1} \lambda_i^\top \begin{bmatrix} w_i - 1 \\ w_i - 1 \end{bmatrix}, \\
S_t &= \sum_{i=1}^{t-1} \lambda_i^\top \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \\
Q_t &= \sum_{i=1}^{t-1} \psi \lambda_i^\top \begin{bmatrix} (w_i - 1)^2 & (w_i - 1)(w_i r_i - c_i) \\ (w_i r_i - c_i) & (w_i r_i - c_i)^2 \end{bmatrix} \lambda_i, \\
T_t &= \sum_{i=1}^{t-1} \psi \lambda_i^\top \begin{bmatrix} 0 & - (w_i - 1) \\ -(w_i - 1) & -2(w_i r_i - c_i) \end{bmatrix} \lambda_i, \\
U_t &= \sum_{i=1}^{t-1} \psi \lambda_i^\top \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \lambda_i,
\end{align*}
\]

and finding the largest real root $v$ of

\[
C_t - S_t v + Q_t + T_t v + U_t v^2 - \ln \left( \frac{2}{\alpha} \right) = 0,
\]

if it exists, otherwise setting $v = 0$.

### D.3 Double Hedging

Double Hedging boils down to running four processes: $K_t^{+q}, K_t^{-q}, K_t^+, \text{ and } K_t^-$. Note that the wealth is split in 4 so anywhere we used $\ln \left( \frac{2}{\alpha} \right)$ in a hedged process now we need to use $\ln \left( \frac{4}{\alpha} \right)$. Note that both $K_t^{+q}(v)$ and $K_t^+(v)$ are trying to establish bounds for the same random variable and in principle they could communicate about values that have been eliminated. However we keep things simple and just run the four processes without sharing any information. The wealth of the doubly hedged process can then be lower bounded by the wealth of the most successful betting strategy starting from a wealth of $\frac{1}{4}$.

### E Gated Deployment

#### E.1 Hedging

Since we don’t typically know whether $\pi$ is better or worse that $h$ we can hedge our bets via the process

\[
K_t^{\pm gd}(v) = \frac{1}{2}(K_t^{+gd}(v) + K_t^{-gd}(v)),
\]
where

\[
K^{+gd}_t(v) = \prod_{i=1}^t \left( 1 + \lambda_{1,i}^+(w_i - 1) + \lambda_{2,i}^+(w_ir_i - r_i - v) \right),
\]

\[
K^{-gd}_t(v) = \prod_{i=1}^t \left( 1 + \lambda_{1,i}^-(w_i - 1) + \lambda_{2,i}^-(w_ir_i' - r_i' - v') \right),
\]

for predictable \( \lambda_{1,i}^+, \lambda_{2,i}^+, \lambda_{1,i}^-, \lambda_{2,i}^- \) subject to \( \lambda_i^+, \lambda_i^- \in \mathcal{G}_v^0 \). As before, \( r_i' = 1 - r_i \) and \( v' = 1 - v \).

### E.2 Betting and Avoiding Grid Search

Betting and avoiding grid search can be obtained using the same equations as for reward predictors but replacing all occurrences of \( c_i \) with \( r_i \).

A key difference we spell out is the feasible region. In order to use common bets and to be able to use the quadratic lower bound of the log wealth we need to specify the set \( \bigcap_{v \in [0,1]} \mathcal{G}_v^m \). This set is equivalent to

\[
\mathcal{G}_v^m = \left\{ \lambda : \begin{bmatrix} -1 & -2 \\ -1 & 0 \\ W & -1 \\ W & W \end{bmatrix} \begin{bmatrix} \lambda \\ m-1 \end{bmatrix} \right\},
\]

where \( W = w_{\text{max}} - 1 \). If we further restrict \( \lambda_2 \geq 0 \) for each of the subprocesses because we expect each to eliminate \( v \) such that \( E[wr - v] > 0 \) and \( v' \) such that \( E[wr' - v'] > 0 \) then the feasible region further simplifies to

\[
\mathcal{G} = \{ \lambda : \lambda_2 \geq 0, W\lambda_1 - \lambda_2 \geq m - 1, -\lambda_1 - 2\lambda_2 \geq m - 1 \}.
\]

Placing bets in this region can be done using the same ideas as Algorithm 1.

### F Reproducibility Checklist

Assumptions: The contextual bandit data is iid. The policy \( \pi \) is absolutely continuous with respect to behavior policy \( h \).

Complexity: MOPE and the scalar Betting Strategy are streaming algorithms. They require constant time per sample and constant memory independent of number of samples. The exact wealth ablation requires memory that scales linearly with the number of samples and time per step that scales at least linearly with the number of samples. The ablation that solves a QP per value \( v \) requires at least \( \frac{1}{\epsilon} \) times more memory and computation that MOPE and provides results that are accurate up to \( \epsilon \). We used \( \epsilon = 0.005 \) in the experiments.

Code: included with the supplementary material and will be released publicly upon acceptance.
Data: synthetic environments are part of the code. Instructions for getting the mnist8m data are in the “Mnist-Policies” notebook.

Hyperparameters: There are no hyperparameters. The confidence level is an input and is stated in each experiment description or the corresponding figure.

Computing infrastructure: Off-the-shelf workstation running Linux (Code works on a Windows laptop as well).