The study of periodic oscillations and global stability in the Tal’ model via the Tsypkin method and the LPRS method

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Abstract. Analysis of the phase space of discontinuous systems via classical methods of the theory of oscillations can be quite a difficult task. Moreover, in some cases this analysis is impossible to perform. Due to the emergence of frequency methods for analysis of discontinuous systems and the development of various analytical-numerical methods, it has become possible to give a more accurate description of the phase space of systems with discontinuous right-hand sides. The present report tries to adopt to the Watt’s governor model that takes account of the self-regulation of the object and the derivative action. We compare the results with those obtained by rigorous analytical method.

Introduction
The rapid development of analytical-numerical methods in recent years has made it possible to find the limiting dynamical regimes in classical systems that were not found before. Therefore, it was necessary to revise some well-known results in the theory of oscillations, which was reflected in the new classification [1] as hidden and self-excited. If an oscillation can be visualized numerically by a trajectory released from the vicinity of an unstable equilibrium state then it is called self-excited, otherwise it is called hidden. Thus, it has become necessary to revise the limits of applicability of classical methods for determining stability and the onset of oscillations in nonlinear control systems, as well as to develop special analytical-numerical methods for localizing hidden oscillations in such systems [2–4].

One of the methods for analysis of the phase space of discontinuous systems is the Andronov point-mapping method, which allows to analyze it rigorously. This method can be used to obtain necessary and sufficient conditions for global stability; however, in many cases its application can be problematic, in particular, for systems of higher order. Moreover, the method is applicable only to systems with piecewise-linear nonlinearities. Due to the development of the global stability theory for discontinuous systems, at present, using the frequency criteria developed by A.Kh. Gel’g and G.A. Leonov and their followers, it has become possible to obtain sufficient conditions for global stability in a general form for systems of any order.

Among engineers, one of the most common general methods for searching and analyzing oscillations in nonlinear control systems is the classical harmonic balance method (HBM).
For Lurie systems with relay nonlinearity, the Tsypkin method and the locus of a perturbed relay system (LPRS) method can be applied. These methods can be considered as a further development of the classical harmonic balance method for relay systems and, therefore, may give more accurate results. Thus, by combining frequency methods and methods for analysis of periodic oscillations, it is possible to give a more accurate description of the phase space of the system.

In this report we study the phase space of the nonlinear model of the Watt governor that takes into account dry friction and self-regulation of the object. The mathematical formulation of this problem was proposed by A.A. Tal’ in [5]. We also compare the results obtained with the ones presented in the original paper.

1. Oscillations and stability of relay systems
Consider a Lurie system with one scalar nonlinearity of the relay type

$$\dot{x} = Px + q\varphi(\sigma), \quad \sigma = r^*x,$$

where $x \in \mathbb{R}^n$, $P \in \mathbb{R}^{n \times n}$, $q \in \mathbb{R}^{n \times 1}$, $r \in \mathbb{R}^{n \times 1}$. Here we consider the solution of the system in the Filippov sense [6]. The linear part of the system (1) can be represented by the following transfer function $W(s) = r^*(P - sI)^{-1}q$. Next, we consider several classical and modern methods for localization of periodic oscillations and analysis of global stability in Lurie systems (1).

1.1. Harmonic balance method
The classical harmonic balance method was one of the first approximate analytical methods for analyzing oscillations in nonlinear control systems. It was developed in the first half of the 20th century in the works of B. van der Pol [7] and N.M. Krylov and N.N. Bogolyubov [8].

The main assumption of this method is that a periodic oscillation is harmonic: $a \cos \omega_0 t$. The values of the frequency $\omega_0 > 0$ and the linearization coefficient $k$ can be found from the following equations:

$$\text{Im} \ W(i\omega_0) = 0, \quad k = -(\text{Re} \ W(i\omega_0))^{-1}.$$  \hspace{1cm} (2)

The amplitude $a$ can be found from the harmonic balance equation

$$\frac{2\pi}{\omega_0} \int_0^{2\pi/\omega_0} \text{sign}(a \cos(\omega_0 t)) \cos(\omega_0 t) dt = ak \int_0^{2\pi/\omega_0} (\cos(\omega_0 t))^2 dt.$$  \hspace{1cm} (3)

The classical HBM is an approximate method, but for some systems it can give an exact answer about the presence or absence of oscillations in the system. For example, it is true for the Vyshnegrasky model [9,10] which describes the dynamics of the machine and the motion of the Watt governor taking into account viscous and Coulomb friction. However, there are various examples where application of the classical HBM can lead to incorrect conclusions about the presence or absence of oscillations. For instance, when applied to a two-dimensional model of flutter suppression of aircraft controls by hydraulic damper with dry friction, the method proposed by M.V. Keldysh [11] for some values of the parameters indicates the presence of periodic oscillations in the phase space of the system, but in fact they do not exist [12]. On the other hand, for a four-dimensional model of flutter suppression, the classical HBM cannot find periodic oscillations, but they can be localized using analytical-numerical methods [10].

1 A.A. Tal’ is a famous Soviet scientist in the field of control theory and pneumatic automation equipment; after graduating from the Bauman Moscow State Technical University, he worked in the Institute of Control Sciences of the USSR Academy of Sciences.
1.2. The Tsypkin method and the LPRS method

The Tsypkin method and the LPRS method were developed by Ya.Z. Tsypkin and I.M. Boyko and described in [13] and [14], respectively. As it was mentioned above, they can be considered as further developments of the ideas behind the classical HBM and may give more accurate results. Within these methods, it is necessary to construct special functions $J_{\text{Tsyp}}$ and $J_{\text{LPRS}}$:

$$J_{\text{Tsyp}}(\omega) = \frac{4}{\pi} \left( \sum_{k=1}^{\infty} \text{Re} W(i(2k-1)\omega) \right) + i \sum_{k=1}^{\infty} \frac{1}{2k-1} \text{Im} W(i(2k-1)\omega),$$

$$J_{\text{LPRS}}(\omega) = -0.5C \left( A^{-1} + \frac{2\pi}{\omega} \left( I - e^{\frac{2\pi}{\omega} A} \right)^{-1} e^{\frac{\pi}{\omega} A} \right) B +$$

$$+ i \frac{\pi}{4} C \left( I + e^{\frac{\pi}{\omega} A} \right)^{-1} \left( I - e^{\frac{\pi}{\omega} A} \right) A^{-1} B,$$

and in the case of a three-position relay without hysteresis, the frequency values of the periodic solutions can be found from the following equations

$$\begin{cases}
\text{Im} J_{\text{Tsyp}}(\omega_0) = 0, \\
\text{Re} J_{\text{Tsyp}}(\omega_0) < 0;
\end{cases}$$

$$\text{Im} J_{\text{LPRS}}(\omega_0) = 0.$$ (6)

The Tsypkin method and the LPRS method allow to draw more accurate conclusions on the existence of periodic oscillations in the phase space of relay system. For example, they can be used to localize a symmetric periodic solution in a four-dimensional flutter suppression [10] model, which cannot be found via the classical HBM. However, these methods still have limitations: they cannot be used to find asymmetric periodic solutions. Nevertheless, the ideas behind the LPRS method can be used to find such solutions [15].

2. Tal’ model

The model considered by A.A. Tal’ in [5] is a generalization of Watt governor model described by Vyshnegradsky. A rigorous global analysis of the nonlinear model of the Watt governor and justification of the conditions for the absence of oscillations obtained by Vyshnegradsky became possible largely due to the development of A.A. Andronov’s mathematical theory of oscillations [9]. Following the results obtained by A.A. Andronov and A.G. Mayer [16], Tal’ carried out similar analysis and obtained the global stability conditions for some parameter values.

Let us consider the equations that describe the dynamics of the machine and the motion of the Watt governor taking into account the self-regulation of the object, the first derivative action, viscous and Coulomb friction given in [5]:

$$\ddot{x} + B\dot{x} + (A + D)x = (1 - CD)y - \frac{1}{2} \text{sign}(\dot{x}), \quad \dot{y} + Cy = -x,$$ (7)

where $A$ – regulator unevenness parameter, $B$ – viscous friction parameter, $C$ – object self-regulation parameter, $D$ – first derivative action parameter. The Lurie form (1) of the system (7) is as follows

$$P = \begin{pmatrix} 0 & 1 & 0 \\ -(A + D) & -B & 1 - CD \\ -1 & 0 & -C \end{pmatrix}, \quad q = \begin{pmatrix} 0 \\ -\frac{1}{2} \\ 0 \end{pmatrix}, \quad r = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$ (8)

Let us now consider system (8) with the following values of parameters: $A = 0.5$, $B = 2$, $C = -1$, $D = 2.6$. These values were also considered by Tal’ in [5], and it was shown that the system
(8) has one stable periodic solution. Application of new approaches to analysis of the phase space of the discontinuous systems allows these results to be verified. Visualisation of the phase space of the system with the considered values of parameters can be seen in Fig. 1.

![Figure 1](image_url)

**Figure 1.** Stable self-excited symmetric periodic solution of system (8) with $A = 0.5$, $B = 2$, $C = -1$, $D = 2.6$.

The methods for localization of periodic oscillations developed in recent years allowed us to confirm the existing results and, moreover, to obtain new ones. For instance, we were able to obtain a set of parameter values that were not considered in [5] and for which there is a symmetric periodic solution in the phase space of the corresponding system. For example, according to the Tsypkin method and the LPRS method, system (8) with parameters $A = 1$, $B = 3$, $C = -0.5$, $D = 0.8$ has a stable symmetric periodic solution with the frequency $\omega = 0.429354696$, which can be seen by visualizing corresponding functions (4) and (5) (see Fig. 2 and Fig. 3). Visualization of these periodic solutions is presented in Fig. 4.

To analyse the phase space of the system in a more general case, we use the frequency criterion for discontinuous systems [17, p. 206]. Hence, we obtained sufficient conditions for the global stability of the system (8):

$$\begin{align*}
C &> 0, \\
B &> 0, \\
D &> \frac{1}{C}, \\
A &> -\frac{1}{C}, \\
B &> 0, \\
D &> \frac{1 - BC^2}{C}, \\
A &> -\frac{1}{C}.
\end{align*}$$

(9)

This analytical result gives us sufficient conditions of global stability and makes it possible
Figure 2. The Tsypkin locus of system (8) with $A = 1$, $B = 3$, $C = -0.5$, $D = 0.8$ and $\omega \in [0.3, 0.7]$.

Figure 3. The LPRS of system (8) with $A = 1$, $B = 3$, $C = -0.5$, $D = 0.8$ and $\omega \in [0.3, 0.7]$.

Figure 4. Stable self-excited symmetric periodic solution of system (8) with parameters $A = 1$, $B = 3$, $C = -0.5$, $D = 0.8$.

to estimate the stability region via application of analytical-numerical methods.

Conclusion
In this paper we discussed different approaches to the analysis of the phase space of the systems with discontinuous right-hand sides and applied these approaches to the model of the Watt governor considered by A.A. Tal’. With the help of the developments of the classical HBM, namely the Tsypkin and the LPRS methods, we were able to verify the results obtained by Tal’ and also to consider a new set of parameters that had not been studied before, and carry out
more detailed analysis. However, the Tsypkin method and the LPRS method are known to have certain limitations, so further development of the ideas that are put in the foundation of these methods may be possible. We also showed that stability criterion application combined with the methods for localization of periodic solutions allows to obtain an efficient estimation of the stability region.

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