Atoroidal Manifolds in Small Covers

Lisu Wu

Abstract

We show that a 3-dimensional small cover is atorodal if and only if there is no 4-belt in the corresponding simple polytope.

1 Introduction

Small covers are a class of closed manifolds which admit locally standard $\mathbb{Z}_n^2$-actions, such that the orbit spaces are some simple convex polytope in an Euclidean space. Or defined equivalently, an $n$-dimensional small cover consists of $2^n$ copies of a simple convex polytope whose faces are guled according to the colorings in $\mathbb{Z}_n^2$. In general, the topological properties of a small cover are closely related to the combinatorics of the orbit polytope and the colorings on its facets (codimension-1 faces). For instance, the cohomology ring of a small cover with $\mathbb{Z}_2$ coefficients is isomorphic to $\mathbb{Z}_2/I + J$, where $\mathbb{Z}_2/I$ is the Stanley-Reisner ring of the orbit polytope, and $J$ depends on the colorings on its facets [5, Theorem 4.14]; and a small cover is orientable if and only if the sum of entries of the coloring on each face is odd [11, Theorem 1.7].

Let $P$ be an $n$-dimensional simple convex polytope in $\mathbb{R}^n$, and $\pi : M \rightarrow P$ be a small cover over $P$. We denote the set of all facets of $P$ by $\mathcal{F}(P)$, then there is a characteristic map $\lambda : \mathcal{F}(P) \rightarrow \mathbb{Z}_2^n$ with the coloring of each facet identical with $\lambda(F)$, where $\mathbb{Z}_2^n$ is taken as a multiplicative group. For any proper face $f$ of $P$, we define

$$G_f \triangleq \text{the subgroup of } \mathbb{Z}_2^n \text{ generated by the set}\{\lambda(F) \mid f \subset F\}.$$ and $G_P \triangleq \{1\} \subset \mathbb{Z}_2^n$. Then $M$ is homeomorphic to the quotient space

$$P \times \mathbb{Z}_2^n / \sim$$

where $(p, g) \sim (q, h)$ if and only if $p = q$ and $g^{-1}h \in G_{f(p)}$, and $f(p)$ is the unique face of $P$ that contains $p$ in its relative interior. The right-angled Coxeter group of $P$ is defined by:

$$W_P \triangleq \langle s_{F}, F \in \mathcal{F}(P) \mid s_F^2 = 1, \forall F; (s_F s_{F'})^2 = 1, \forall F \cap F' \neq \emptyset \rangle.$$ By giving the Borel construction $M_{\mathbb{Z}_2^n} = E\mathbb{Z}_2^n \times\mathbb{Z}_2 M$ of $M$, Davis and Januszkiewicz [5, Corollary 4.5] proved that the fundamental group of a small cover, denoted as $\pi_1(M)$, is isomorphic to the kernel of a homomorphism from $W_P$ to $\mathbb{Z}_2^n$, which is induced by the characteristic map over $\mathcal{F}(P)$. Actually, there is a short right split group sequence

$$1 \rightarrow \pi_1(M) \rightarrow W_P \overset{\phi}{\rightarrow} \mathbb{Z}_2^n \rightarrow 1$$
where $\phi(s_F) = \lambda(F)$ for each facet $F$ of $P$. Wu and Yu [14] described this relation more explicitly based on the presentation of fundamental groups they calculated. Furthermore, they showed that a facial submanifold of a small cover is $\pi_1$-injective if and only if there is no 3-belt conclude the corresponding face [14, Theorem 3.3].

Let $P$ be a 3-dimensional simple polytope. A $k$-circuit in $P$ is a simple loop on the boundary of $P$ which intersects transversely the interior of exactly $k$ distinct edges, and a $k$-circuit $c$ is called prismatic if the endpoints of all edges which $c$ intersects are distinct. A $k$-belt in $P$ is a set of $k$ distinct faces $F_1, \ldots, F_k$ of $P$ such that $F_i \cap F_{i+1} \neq \emptyset$ for $1 \leq i \leq k-1$, $F_k \cap F_1 \neq \emptyset$, and any three face in the belt have no common intersection. It is clear that each $k$-belt determines a prismatic $k$-circuit. Furthermore, a prismatic 3-circuit can determines a 3-belt; and if there is no prismatic 3-circuit, then a prismatic 4-circuit determines a 4-belt. It is clearly that the cross section surrounded by a 4-circuit $c$ determined by a 4-belt is a square, we denote this cross section by $F$. If $c$ is prismatic, then $\pi^{-1}(F)$ is torus or Klein bottle in $M$, denoted by $M_{F}$. And a closed 3-manifold is atoroidal if it contains no essential torus, otherwise it is toroidal.

In this paper, we consider the $\pi_1$-injectivity of the sectional submanifold $M_{F}$ which is determined by a 4-belt in simple polytope $P$. According to the following diagram,

$$
\begin{array}{cccccc}
1 & \to & \pi_1(M_{F}) & \xrightarrow{\psi_F} & W_{F} & \xrightarrow{\phi_F} & \mathbb{Z}_2^3 \\
& & i_* & & j_* & & \\
1 & \to & \pi_1(M) & \xrightarrow{\psi} & W_{P} & \xrightarrow{\phi} & \mathbb{Z}_2^3 \\
\end{array}
$$

where $i_*$ and $j_*$ are induced by inclusion map of sectional submanifold $M_{F}$, we show that each 4-belt in $P$ gives a $\pi_1$-injective torus or Klein bottle for $M$, i.e. $M$ is toroidal in this case. Furthermore, we give a topological proof for Andreev’s Theorem in the right-angled case. The main result of this paper is the following theorem.

**Theorem** Let $M$ be a 3-dimensional small cover over a simple polytope $P$, then $M$ is atoroidal if and only if there is no 4-belt in $P$.

The paper is organized as follows. In section 2, we construct a presentation of the fundamental group of the sectional submanifold which is determined by a 4-belt. Using this presentation, we give a group homomorphism from $\pi_1(M_{F})$ to the Coxeter group $W_{F}$. In section 3, we show that sectional submanifold $M_{F}$ is $\pi_1$-injective. In section 4, we prove Andreev’s Theorem in the right-angled case by the result in section 3.
2 Presentations of $\pi_1(M_F)$

Let $P$ be a 3-dimensional simple polytope, and $\pi : M \rightarrow P$ be a small cover over $P$. If $c$ is a prismatic 4-circuit in $P$ determined by a 4-belt $F_1, F_2, F_3, F_4$, where $F_1 \cap F_2 = F_2 \cap F_4 = \emptyset$. Then $c$ intersects exactly 4 edges and 4 faces of $P$. Thus $c$ encloses a square in $P$, denoted by $F$. And four faces bounded $F$ are $F_1, F_2, F_3, F_4$ respectively, the edge of $F$ denoted by $f_i = F_1 \cap F_i$, for $i = 1, 2, 3, 4$. Then the Coxeter group of $F$ is

$$W_F \cong \langle s_1, s_2, s_3, s_4 | s_1^2 = s_2^2 = s_3^2 = s_4^2 = 1; (s_1s_2)^2 = (s_2s_3)^2 = (s_3s_4)^2 = (s_4s_1)^2 = 1 \rangle$$

(5)

There are, essentially, 5 cases of colorings on $F = \{f_1, f_2, f_3, f_4\}$ which is induced by characteristic map $\lambda : F(P) \rightarrow \mathbb{Z}_2^3$. We define the induced characteristic map of $F$:

$$\lambda_F : F \rightarrow \mathbb{Z}_2^3$$

(6)

where $\lambda_F(F) = \lambda_F(f_1, f_2, f_3, f_4) \in \{(e_1, e_2, e_1, e_2), (e_1, e_2, e_1, e_1e_2), (e_1, e_2, e_3, e_2), (e_1, e_2, e_3, e_1e_2), (e_1, e_2, e_3, e_1e_2), (e_1, e_2, e_3, e_1e_2)\}$, and $e_1, e_2, e_3$ are basis of $\mathbb{Z}_2^3$.

Then the sectional submanifold $M_F \cong \pi^{-1}(F)$ determined by $F$ is

$$M_F = F \times \mathbb{Z}_2^3 / \sim$$

(7)

where $(f_i, g) \sim (f_j, h)$ if and only if $i = j$ and $g^{-1}h = \langle \lambda_F(f_i) \rangle$. Thus sectional submanifolds determined by above 5 colorings are double tori, double Klein bottles, torus, Klein bottle, torus, respectively.

According to the construction of $M_F$, we consider the following two cases:

**Case 1:** $\lambda_F(F) = (e_1, e_2, e_1, e_2)$ or $(e_1, e_2, e_1, e_1e_2)$.

In this case, $\text{Im}(\lambda_F)$ generates the subgroup $\mathbb{Z}_2^2$ of $\mathbb{Z}_2^3$. The sectional submanifold is a disjoint union of two tori or two Klein bottles, one of which we denote by $M_F$. Then $M_F$ is glued by 4 copies of $F$, we choose a vertex $p_0 = f_1 \cap f_2$ of $F$ as the base point of $\pi_1(M_F)$, and glue $\{(F, g) | g \in \mathbb{Z}_2^2\}$ along its faces $\{(f_i, g) | i = 1, 2, g \in \mathbb{Z}_2^2\}$. For each face $(f_i, g) \subset (F, g)$, we choose a simple closed circles $\beta_{i,g}$, which crosses $p_0$ and $(f_i, g)$ in $M_F$, as a generator of $\pi_1(M_F)$.

Then

$$\pi_1(M_F, p_0) = \langle \beta_{i,g}, i = 1, 2, 3, 4 | \beta_{i,g} \beta_{j,g} \lambda_F(f_i) = \beta_{j,g} \beta_{i,g} \lambda_F(f_i), \forall f_i \cap f_j \neq \emptyset, \forall g; \beta_{i,g} = 1, i = 1, 2, \forall g \rangle$$

And consider the following short group sequence

$$1 \longrightarrow \pi_1(M_F, p_0) \xrightarrow{\psi_F} W_F \xrightarrow{\phi_F} \mathbb{Z}_2^2 \longrightarrow 1$$

(8)

where $\gamma_F : \mathbb{Z}_2^2 \rightarrow W_F$ is defined by $\gamma_F(e_1) = s_1, \gamma_F(e_2) = s_2$. And define

$$\psi_F : \pi_1(M_F, p_0) \rightarrow W$$

$$\beta_{i,g} \mapsto \gamma_F(g \cdot \lambda_F(f_i))s_i(g) \cong S_i,g$$

It is easy to check that both $\gamma_F$ and $\psi_F$ are well-defined. We have the following lemma.
Lemma 1 The sequence \( \{8\} \) is right split and exact.

Proof. cf. [13, Lemma 2.9]

Case 2: \( \lambda_F(F) \in \{(e_1, e_2, e_3, e_2), (e_1, e_2, e_3, e_1 e_2 e_3), (e_1, e_2, e_3, e_1 e_2 e_3)\} \).

In this case, \( \text{Im}(\lambda_F) \) generates \( \mathbb{Z}_2^3 \). The sectional submanifold \( M_F \) is glued by 8 copies of \( F \). Similarly, we choose a vertex \( p_0 = f_1 \cap f_2 \) of \( F \), and glue \( \{(F, g) \} (g \in \mathbb{Z}_2^3) \) along along its faces \( \{(f_i, g)\} \) for \( i = 1, 2, g \notin \langle e_1, e_2 \rangle \) or \( i = 3, g \notin \langle e_1 \rangle \). We shrink the faces \( \{f_2, g\}, g \notin \langle e_1 \rangle \) to a point, which is also denoted as \( p_0 \) and taken as the base point of \( \pi_1(M) \). And for each pair \( (f_i, g) \subset (F, g) \), we choose a simple closed circles \( \beta_{i,g} \), which crosses \( p_0 \) and \( (f_i, g) \) in \( M_F \), as a generator of \( \pi_1(M_F) \). Then

\[
\pi_1(M_F) = \langle \beta_{i,g}, i = 1, 2, 3, 4; g \in \mathbb{Z}_2^3 | \beta_{i,g} \beta_{i,g} \lambda_F(f_i) = 1, \forall g; \\
\beta_{i,g} \beta_{j,g} \lambda_F(f_j) = \beta_{j,g} \beta_{i,g} \lambda_F(f_j), \forall f_i \cap f_j \neq \emptyset; \forall g; \\
\beta_{i,g} = 1, i = 1, 2; \forall g \text{ or } i = 3, \forall \gamma \in G \rangle
\]

(9)

where \( G = \{e_1, e_1 e_2, e_1 e_3, e_1 e_2 e_3\} \). And consider the following short group sequence

\[
1 \longrightarrow \pi_1(M_F, p_0) \xrightarrow{\psi_F} W_F \xrightarrow{\phi_F \gamma_F} \mathbb{Z}_2^3 \longrightarrow 1 \tag{10}
\]

where \( \gamma_F : \mathbb{Z}_2^3 \longrightarrow W_F \) is defined by \( \gamma_F(e_1) = s_1, \gamma_F(e_2) = s_2, \gamma_F(e_3) = s_1 s_3 s_1 \).

And

\[
\psi_F : \pi_1(M_F, p_0) \longrightarrow W \\
\beta_{i,g} \longmapsto \gamma_F(g \cdot \lambda_F(f_i)) s_i \gamma(g) \triangleq S_{i,g}
\]

And both \( \gamma_F \) and \( \psi_F \) are well-defined. Then the similar lemma follows as above.

Lemma 2 The sequence \( \{10\} \) is right split and exact.

Proof. \( \phi_F \circ \gamma_F = id_{\mathbb{Z}_2^3} \) implies the right splitting of \( \{10\} \). We just need to prove \( \pi_1(M_F, p_0) \cong \text{ker}(\phi_F) \). \( \phi_F \circ \psi_F(\beta_{i,g}) = \phi_F(\gamma_F(g s_i \gamma(g) \lambda_F(f_i))) = \phi_F(\gamma_F(g) \cdot \phi_F(s_i) \cdot \phi_F(\gamma(g \lambda_F(f_i)))) = g \cdot \lambda_F(f_i) \cdot g \lambda_F(f_i) = 1, \) thus \( \text{im}(\psi_F) \subseteq \text{ker}(\phi_F) \).

Since \( \text{im}(\psi_F) \) is a normal group of \( W_F \), and

\[
W_F = \langle s_1, s_2, s_3, s_4 | s_i^2 = s_i^3 = s_i^4 = 1; \\
(s_1 s_2)^2 = (s_2 s_3)^2 = (s_3 s_4)^2 = (s_4 s_1)^2 = 1 \rangle
\]

(11)

thus \( W_F / \text{im}(\psi_F) \cong \mathbb{Z}_2^3 \) implies that \( \text{im}(\psi_F) \cong \text{ker}(\phi_F) \). Hence the sequence \( \{10\} \) is exact.
3 \(\pi_1\)-injectivity of \(M_F\)

In this section, we take \(v = F_0 \cap F_1 \cap F_2\) as the base point, then the fundamental group of \(M\) [14, Proposition 2.1]

\[
\pi_1(M, v) = \langle \alpha_{F,g}, \forall F \in \mathcal{F}(P); \forall g \in \mathbb{Z}_3^2 | \alpha_{F,g} \alpha_{F,g} \lambda(F) = 1, \forall g; \alpha_{F,g} \alpha_{F',g} \lambda(F') = 1, \forall F \cap F' \neq \emptyset; \forall g \rangle
\]

and the Coxeter group of \(P\)

\[
W_P = \langle t_F, F \in \mathcal{F}(P) \mid t_F^2 = 1, \forall F; (t_F t_{F'})^2 = 1, \text{for } F \cap F' \neq \emptyset \rangle
\]

Consider the following diagram

\[
\begin{array}{ccc}
\pi_1(M) & \xrightarrow{\psi} & W_P \\
\downarrow i_* & & \downarrow j_* \\
\pi_1(M_F) & \xrightarrow{\psi_F} & W_F \\
\end{array}
\]

where \(\psi\) is defined at [14, Lemma 2.9], and \(s_0\) represents the generator in \(W_P\) determined by the face \(F_0\). \(W_P \rightarrow W_P/\langle s_0 \rangle\) is a quotient map. \(i_*\) and \(j_*\) are induced by inclusion map.

\[j_* : W_F \rightarrow W_P\]

\[s_i \mapsto t_{F_i}\]

and in Case 1, we define

\[i_* : \pi_1(M_F, p_0) \rightarrow \pi_1(M, v)\]

\[\beta_{i,g} \mapsto \alpha_{F_i,g}, \ g \in \langle e_1, e_2 \rangle\]

in Case 2, we define

\[i_* : \pi_1(M_F, p_0) \rightarrow \pi_1(M, v)\]

\[\beta_{i,g} \mapsto \alpha_{F_i,g}, \ \text{for } i = 1, 2, 4;\]

\[\beta_{3,g} \mapsto \alpha_{F_3,g} \lambda(F_3) \lambda(F_1) \alpha_{F_3,g} = \alpha_{F_3,g} e_1 e_2 \alpha_{F_3,g}\]

where \(g \in \mathbb{Z}_3^2\). It can be check that the diagram [14] is communicate both in Case 1 & 2.

**Lemma 3** Both \(i_*\) and \(j_*\) are injective.

**Proof.** The injectivity of \(j_*\) is showed in [14, Theorem 3.3], and the injectivity of \(i_*\) can be proved by the injectivity of \(j_*\) and the commutativity of diagram [14]. \(\square\)
A simple polytope is flag if any pairwise intersecting faces have a common intersection. Davis [4, Corollary 5.4] proved that a small cover is aspherical if and only if the orbit polytope is flag. And there are at most 3 faces intersecting at one point in a 3-dimensional simple polytope other than $\Delta^3$, So we have the proposition as following.

**Proposition 1** Let $P \neq \Delta^3$ be a simple polytope of dimension 3, then $P$ is flag if and only if there is no prismatic 3-circuit in $P$. 

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**Theorem 1** Let $M$ be a 3-dimensional small cover over a simple polytope $P$, then $M$ is atoroidal if and only if there is no 4-belt in $P$.

**Proof.** The Elliptization and Hyperbolization Theorems together imply that every atoroidal closed 3-manifold is either spherical or hyperbolic, i.e. $\pi_1(M)$ is finite or infinite without subgroup $\cong \mathbb{Z}^2$. If there exists a 4-belt in $P$, then according to Lemma 3, there is a $\pi_1$-injective torus or Klein bottle in $M$, which implies that there is a subgroup $\cong \mathbb{Z}^2$ in $\pi_1(M)$. This is a contradiction.

Conversely, we show that $M$ toroidal implies a 4-belt in $P$. If $M$ toroidal, then there is a subgroup $\mathbb{Z}^2$ in $\pi_1(M)$. And $\pi_1(M)$ embeds in $W_P$, thus the subgroup $\mathbb{Z}^2$ embeds in $W_P$. In other words, there are two free commutable elements in $W_P$, written as $x, y$. We assume that $x = s_1 s_2 \cdots s_m$ and $y = t_1 t_2 \cdots t_n$ are shortest expressions of $x$ and $y$, where $s_i, t_j$ are generators of $W_P$, and $m, n \geq 2$. Then $xy = yx$ implies $s_i t_j = t_j s_i, \forall i, j$, i.e. $F_{s_i} \cap F_{t_j} \neq \emptyset, \forall i, j$. Since $x$ is free, there exist $s, s'$ such that $F_s \cap F_{s'} = \emptyset$. And because $y$ is free, there also exist $t, t'$ such that $F_t \cap F_{t'} = \emptyset$ and $t, t', s, s'$ are four distinct generators. Otherwise, we have $(t, t') = (s, s')$ or $(t, t') = (s', s)$, now there must exist another two generators in $x$ or $y$ such that the intersection of the corresponding two faces is empty set. Otherwise, the only two non-commutable generators in $x$ and $y$ is $s$ and $s'$, which implies that $x^2$ and $y^2$ is one of $(ss')^2$ and $(s's)^2$. Thus $x^2 = y^2$ or $x^2 = y^{-2}$, which contradict to $x, y$ generating $\mathbb{Z}^2$. Hence there exists a 4-belt. □
4 Andreev’s Theorem in the right-angled case

Andreev[1971] (see also [12]) gives a complete characterization of compact hyperbolic polyhedra in dimension 3 with nonobtuse dihedral angles. In the right-angled case, we have

**Theorem 2 (Andreev’s Theorem in the right-angled case)**

A simple polytope $P \neq \Delta^3$ has a geometric realization in $\mathbb{H}^3$ as a right-angled hyperbolic polytope if and only if there is no prismatic 3 or 4-circuit in $P$. Furthermore, such geometric realization is unique up to isometry.

Now, we give a topological proof for the above theorem. For any 3-dimensional simple polytope $P$, there exists a small cover $M$ over $P$ by the 4-Colors Theorem. Conversely, The group $\mathbb{Z}_2^3$ that acts on a hyperbolic small cover of dimension 3 produces a simple orbit polytope with all dihedral angles right-angled. Thus, we just need to prove that $M$ is hyperbolic if and only if there is no prismatic 3 or 4-circuit in $P$.

**Proof of the necessity part.** If $M$ is hyperbolic, then $P$ can not be a 3-simplex. If not, there is a element of order 2 in $\pi_1(M)$, then $\pi_1(M)$ is not torsion-free, which is a contradiction. And other cases can be shown by Gauss-Bonnet theorem.

**Proof of the sufficiency part.** If there is no prismatic 3-circuit, then $M$ is aspherical according to Proposition[1]. Furthermore, If there is no prismatic 3-circuit, then there is no prismatic 4-circuit if and only if there is no 4-belt in $P$. Thus $M$ is atoroidal by Theorem[1] if there is no prismatic 3 and 4-circuit. Hence $M$ is hyperbolic by Thurston’s Hyperbolization Theorem and such hyperbolic structure is unique up to isometry, thus $P$ has a unique geometric realization in $\mathbb{H}^3$, and now its all dihedral angles are right angle. □

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**School of Mathematical Sciences, Fudan University, Shanghai, 200433, P.R.China.**

*E-mail address:* wulisuwulisu@qq.com