The balanced Voronoi formulas for $GL(n)$

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Abstract

In this paper we show how the $GL(N)$ Voronoi summation formula of [MiSc2] can be rewritten to incorporate hyper-Kloosterman sums of various dimensions on both sides. This generalizes a formula for $GL(4)$ with ordinary Kloosterman sums on both sides that was used in [BLM] to prove nonvanishing of $GL(4)$ $L$-functions by $GL(2)$-twists, and later by the second-named author in [Zho].

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1 Introduction

The Voronoi summation formula for $GL(2)$ has long been a standard tool for studying analytic properties of automorphic forms and their $L$-functions. More recently, the Voronoi formula for $GL(3)$ of the first-named author and Schmid in [MiSc1] has found applications in the study of automorphic forms on $GL(3)$ and their $L$-functions, such as [Mil], [Li], [Mun], and [FoGa]. The Voronoi formula was generalized to $GL(N)$ in [MiSc2], with other proofs later found by [GoLi1], [GoLi2], [IcTe], and [KiZh].

The existing Voronoi formula for $GL(N)$, $N \geq 3$, (e.g., Theorem 2.1) is a Poisson-style summation formula with Fourier coefficients of an automorphic form twisted by additive characters on one side, and those of a contragredient form twisted by (hyper-)Kloosterman sums of dimension $N - 2$ on the other side. The appearance of the (hyper-)Kloosterman sums was already suggested by finite harmonic analysis with Dirichlet characters and Gauss sums, e.g., in [DuIw].

In 2011, the first-named author and Xiaqing Li discovered a different (so called “balanced”) Voronoi-type formula on $GL(4)$, with both sides twisted by ordinary Kloosterman sums (see [BLM] and [Zho, Theorem 1.2]). This formula was first derived by modifying the automorphic-distributional proof in [MiSc2]. The second-named author later generalized that formula to $GL(N)$ under certain hypotheses ([Zho, Theorem 1.1]). In this paper, we complete the general balanced Voronoi formulas for cusp forms on $GL(N,\mathbb{Z}) \setminus GL(N,\mathbb{R})$. These balanced formulas are derived from the original Voronoi formula of [MiSc2], and equate a sum of Fourier coefficients twisted by hyper-Kloosterman sums of dimension $L$ with a contragredient sum twisted by hyper-Kloosterman sums of dimension $M$, where $N = L + M + 2$. The original formula of Miller and Schmid corresponds to the case of $L = 0$ and $M = N - 2$, while the balanced formula of Li and Miller on $GL(4)$

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corresponds to the case of \( L = 1 \) and \( M = 1 \). The latter formula on \( \text{GL}(4) \) is a key ingredient in the recent nonvanishing theorem for \( \text{GL}(2) \)-twists of \( \text{GL}(4) \) \( L \)-functions in [BLM]. This is because the Kloosterman sums in the balanced Voronoi formula on \( \text{GL}(4) \) mesh well with the Kloosterman sums appearing in the Kuznetsov trace formula on \( \text{GL}(2) \). The match between them is used in [BLM] to create a spectral reciprocity formula, from which mean value estimates and the nonvanishing result are deduced.

The proof of our balanced formulas (Theorem 3.1) in this paper is different from the automorphic-distributional method used to prove Li and Miller’s balanced formula on \( \text{GL}(4) \). Our proof is also different from that of [Zho, Theorem 1.1], which instead uses functional equations of twisted automorphic \( L \)-functions.

Before stating the formulas, we need define the hyper-Kloosterman sums which already appear in the Voronoi formula of [MiSc2] for \( \text{GL}(N) \) for \( N \geq 4 \) (restated in Theorem 2.1 below). Denote \( e(x) := \exp(2\pi ix) \). Let \( a, n \in \mathbb{Z}, c \in \mathbb{N} \), and let

\[
q = (q_1, q_2, \ldots, q_N) \quad \text{and} \quad d = (d_1, d_2, \ldots, d_N)
\]

be \( N \)-tuples of positive integers satisfying the divisibility conditions

\[
d_1|q_1c, \quad d_2 \mid \frac{q_1q_2c}{d_1}, \quad \cdots, \quad d_N \mid \frac{q_1\cdots q_Nc}{d_1\cdots d_{N-1}}.
\]

Define the \( N \)-dimensional hyper-Kloosterman sum as

\[
\text{Kl}_N(a, n, c; q, d) := \sum_{x_1 \ (dq_1c \mod d_1), \ x_2 \ (dq_1q_2c \mod d_1d_2), \ \cdots, \ x_N \ (dq_1\cdots q_Nc \mod d_1\cdots d_N)}^* e \left( \frac{d_1x_1a}{c} + \frac{d_2x_2a}{q_1c} + \cdots + \frac{d_Nx_Na}{q_1\cdots q_{N-1}c} + \frac{n\omega x_N}{d_1\cdots d_N} \right),
\]

where \( \sum^* \) indicates that the summations are restricted to coprime residue classes and \( \pi_i \) denotes the multiplicative inverse of \( x_i \) modulo \( \frac{q_iq_{i+1}c}{d_i} \). In the degenerate case of \( N = 0 \), we define \( \text{Kl}_0(a, n, c; \cdot) := e \left( \frac{a\omega}{c} \right) \); when \( N = 1 \) the hyper-Kloosterman sum \( \text{Kl}_1(a, n, c; q_1, d_1) \) reduces to the ordinary Kloosterman sum \( S(\omega q_1, n; q_1c/d_1) \).

Let \( F \) be a cuspidal automorphic form for \( \text{GL}(N, \mathbb{Z}) \). As is customary, we assume that \( F \) generates an irreducible subrepresentation \( \pi \) of \( L^2(\mathbb{Z}_R \text{GL}(N, \mathbb{Z}) \backslash \text{GL}(N, \mathbb{R})) \) under the right regular representation of \( \text{GL}(N, \mathbb{R}) \), where \( \mathbb{Z}_R \) denotes the center of \( \text{GL}(N, \mathbb{R}) \) and \( \xi \) is a central character. Note this does not imply that \( F \) is a Hecke eigenform, which is a stronger assumption that is unnecessary using our methods. Let \( A(\ast, \cdots, \ast) \) denote its abelian Fourier coefficients (see [MiSc2] (2.9) and [Bum] (2.1.5)), which are the Hecke eigenvalues of \( F \) when \( F \) is a normalized Hecke eigenform. The Voronoi summation formula in [MiSc2] is a Poisson-sum style identity relating sums of the abelian Fourier coefficients weighted against test functions \( \omega \) and \( \Omega \), which are related by an integral transform completely determined by \( \pi \). Further background on Voronoi summation and this integral transform (which our new formula shares as well) is given in Section 2.

There are various ways to describe allowable choices of test functions \( \omega \) in the Voronoi summation formula. The simplest approach (which we follow here) is to demand that \( \omega \) be a smooth function on \( \mathbb{R} \) which has compact support contained in \( \mathbb{R}_{>0} = (0, \infty) \); this is natural since \( \omega(x) \) is never evaluated at \( x = 0 \) in Theorem 1.1. However, for some applications (e.g., to \( L \)-functions) it
is important to allow different behavior at the origin, such as fractional powers of the form $|x|^s$ or $|x|^s \text{sgn}(x)$ for $s \in \mathbb{C}$. We shall not pursue this here, other than noting that any admissible function used in the usual Voronoi formula on $\text{GL}(N)$ (see [MiSc2 (1.8)]) can be used in the balanced Voronoi formulas (with only minor modifications to account for parities); this is because our proof constructs the balanced formula as a finite average of formulas of the type given in Theorem 2.1.

At a formal level, the integral transform has the form

$$\Omega(y) = \frac{1}{|y|} \int_{\mathbb{R}^N} \omega \left( \frac{x_1 \cdots x_N}{y} \right) \prod_{1 \leq j \leq N} \left( e(-x_j)|x_j|^{-\lambda_j} \text{sgn}(x_j)^{\delta_j} \right) \, dx_j,$$

where the $\lambda_j$ and $\delta_j$ are the representation parameters of $\pi$ (this notion as well as a reformulation of (2) in terms of Mellin inversion is given in Section 2; see also [MiSc2 §1]).

**Theorem 1.1.** Let $F$ be a cuspidal automorphic form on $\text{GL}(N, \mathbb{Z}) \backslash \text{GL}(N, \mathbb{R})$ for $N \geq 3$ with abelian Fourier coefficients $A(*, \ldots, *)$, and which generates an irreducible representation of $\text{GL}(N, \mathbb{R})$. Let $\omega \in C_c^\infty(\mathbb{R}_{>0})$ and let $L$ and $M$ be two non-negative integers with $L+M+2 = N$. Let $c > 0$ be an integer and let $a$ be any integer with $(a, c) = 1$. Denote by $\overline{a}$ the multiplicative inverse of $a$ modulo $c$. Let $q = (q_1, q_2, \ldots, q_L)$ be an $L$-tuple of positive integers and let $Q = (Q_1, Q_2, \ldots, Q_M)$ be an $M$-tuple of positive integers. Let $\sum_{D|q} \pi_{D|q}^\ast$ stand for $\sum_{D_1|Q_1} \sum_{D_2|Q_2^c} \cdots \sum_{D_M|Q_M^c}$ and let $\sum_{d|q}$ stand for $\sum_{d|q_1} \sum_{d_2|q_2^c} \cdots \sum_{d_L|q_L^c}$. Then

$$\sum_{D|q} \sum_{n=1}^\infty A(q_L, \ldots, q_1, D_1, \ldots, D_M, n) K_{M}(\overline{a}, n, c; Q, D) D_1^{M} D_2^{M-1} \cdots D_M \omega \left( \frac{n D_1^{M+1} D_2^{M} \cdots D_M^{2}}{q_1^{2} q_2^{2} \cdots q_L^{2}} \right)$$

$$= \sum_{d|q} d_1^{L} d_2^{L-1} \cdots d_L \sum_{n=1}^\infty A(n, d_L, \ldots, d_1, Q_1, \ldots, Q_M) K_{L}(\overline{a}, n, c; q, d) \Omega \left( \frac{(-1)^{M+1} n d_1^{L+1} d_2^{L} \cdots d_L^{2}}{c^{N} Q_1^{M} Q_2^{M-1} \cdots Q_M} \right)$$

$$+ \sum_{d|q} d_1^{L} d_2^{L-1} \cdots d_L \sum_{n=1}^\infty A(n, d_L, \ldots, d_1, Q_1, \ldots, Q_M) K_{L}(\overline{a}, -n, c; q, d) \Omega \left( \frac{(-1)^{M} n d_1^{L+1} d_2^{L} \cdots d_L^{2}}{c^{N} Q_1^{M} Q_2^{M-1} \cdots Q_M} \right),$$

where $\Omega$ is the integral transform from (2) (which is rigorously defined as a convergent integral in (2)-(7)).

**Remark 1.2.** As we mentioned earlier, Theorem 1.1 is proved by averaging over a finite number of instances of the original Voronoi formula of [MiSc2] (Theorem 2.1). Consequently, any analysis of test functions for that formula automatically transfers over to our present setting. The construction by finite average also shows that any coefficients $A(*, \ldots, *)$ satisfying the summation formula in [MiSc2] must also satisfy the summation formula in Theorem 3.1. This extends the range of applicability of $F$ to cases where functoriality has not yet been shown. For example, Kiral and the second-named author have shown in [KiZh] that the Voronoi summation formula of [MiSc2] also holds when $F$ is a Rankin-Selberg convolution of two full-level cuspidal automorphic representations (see [KiZh] Examples 1.8-1.9). Therefore Theorem 1.1 and Theorem 3.1 hold for such $F$ as well, despite it not yet being known to be automorphic.

**Remark 1.3.** We have stated the summation formula in Theorem 1.1 so that it only involves a sum over positive integers $n$ on the lefthand side. This is somewhat unnatural from the point of view of automorphic distributions, through which one obtains summation formulas via integration.
against distributions that involve terms for both positive and negative $n$. Also, including both positive and negative $n$ on the lefthand side results in simplifying the righthand side, as well as the analytic assumptions on the behavior of $\omega$ near the origin. Nevertheless, since Voronoi summation formulas are typically applied to sums indexed by positive integers $n$, we have chosen to sacrifice aesthetics for practicality and state our formula as above.

2 Voronoi formulas as Dirichlet series identities

Let $F$ be a cuspidal automorphic form on $GL(N,\mathbb{Z})\backslash GL(N,\mathbb{R})$ and let $\pi$ denote the archimedean representation attached to $F$, which we assume is irreducible. We say that $(\lambda, \delta) \in \mathbb{C}^N \times (\mathbb{Z}/2\mathbb{Z})^N$ is a representation parameter of $F$ if $\pi$ embeds into a subspace of the principal series representation

$$V_{\lambda, \delta} = \left\{ f : GL(N, \mathbb{R}) \rightarrow \mathbb{C} \mid f \left( g \left( \begin{array}{cc} a_1 & 0 \\ \ast & \ddots \\ \ast & \ast & a_n \end{array} \right) \right) = f(g) \prod_{j \leq N} \left( |a_j|^{-s_j - j - \lambda_j} \operatorname{sgn}(a_j)^{\delta_j} \right) \right\},$$

which is a representation space for $GL(N, \mathbb{R})$ under the left translation action $[\pi_{\lambda, \delta}(g)f]h = f(g^{-1}h)$. When $\pi$ is spherical, any simultaneous permutation of the entries of $(\lambda, \delta)$ is also a representation parameter; in this case $\lambda$ coincides with the notion of Langlands parameter, though it does not in general (see [MiSc3, A.1-A.2] for a complete description of all allowable representation parameters of cuspidal automorphic representations of $GL(N, \mathbb{R})$). The $GL(N, \mathbb{Z})$-invariance forces $\delta_1 + \cdots + \delta_n \equiv 0 \pmod{2}$ [MiSc2 (2.2)].

Define the Gamma factor

$$G_\delta(s) := \begin{cases} 2(2\pi)^{-s} \Gamma(s) \cos(\pi s/2), & \text{if } \delta \in 2\mathbb{Z}, \\ 2i(2\pi)^{-s} \Gamma(s) \sin(\pi s/2), & \text{if } \delta \in 2\mathbb{Z} + 1. \end{cases}$$

Alternatively, if $\Gamma_{\mathbb{R}}(s) = \pi^{-s/2} \Gamma(s/2)$ denotes the usual Artin Gamma factor appearing in the functional equation of the Riemann $\zeta$-function, then we have equivalently

$$G_\delta(s) = \begin{cases} \frac{\Gamma_{\mathbb{R}}(s)}{\Gamma_{\mathbb{R}}(1-s)}, & \delta \in 2\mathbb{Z}, \\ i^{\delta} \frac{\Gamma_{\mathbb{R}}(s+1)}{\Gamma_{\mathbb{R}}(2-s)}, & \delta \in 2\mathbb{Z} + 1. \end{cases}$$

(3)

Define

$$G_+(s) = \prod_{j=1}^{N} G_{\delta_j}(s + \lambda_j)^{-1} \quad \text{and} \quad G_-(s) = \prod_{j=1}^{N} G_{1+\delta_j}(s + \lambda_j)^{-1}. \quad (4)$$

The ratio of Gamma factors $G_+(s)$ appears naturally in the functional equations of the standard $L$-function of $F$ and its twists by Dirichlet characters. For this reason we can alternatively write

$$G_+(s) = \frac{L(1-s, \bar{\pi}) \epsilon(s, \pi)}{L(s, \pi)} \quad \text{and} \quad G_-(s) = \frac{L(1-s, \bar{\pi} \otimes \operatorname{sgn}) \epsilon(s, \pi \otimes \operatorname{sgn})}{L(s, \pi \otimes \operatorname{sgn})}, \quad (5)$$

where the local factors are as defined in [Jac, Appendix].

Let $\omega \in C_c^\infty(\mathbb{R}_{>0})$ and let $\tilde{\omega}(s)$ denote its Mellin transform. We shall now clarify the relationship between $\omega$ and its Voronoi transform $\Omega$ from [2]. Decompose $\Omega$ into its even and odd parts for $y > 0$

$$\Omega_+(y) = \frac{1}{2} (\Omega(y) + \Omega(-y))$$

$$\Omega_-(y) = \frac{1}{2} (\Omega(y) - \Omega(-y)). \quad (6)$$
It then follows from \[\text{MiSc2} \ (1.5)\] that  
\[
\Omega_\pm(x) = \frac{1}{2\pi i} \int_{\Re(s) = -\sigma} \tilde{\omega}(s) x^{s-1} G_\pm(s) \, ds
\]  
for \(x > 0\) and some \(\sigma > 0\). Please note that we take some \(\sigma > 0\) to avoid the poles of \(G_\pm(s)\), which are on some right half plane. Also, \(\Omega\) is defined over \(\mathbb{R} \setminus \{0\}\) and \(\Omega_\pm\) over \(\mathbb{R}_{>0}\).

The original Voronoi formula for \(\text{GL}(N)\), \(N \geq 3\), in \[\text{MiSc1}\] \[\text{MiSc2}\] was proven using automorphic distributions. The methods of \[\text{MiSc2}\] §4 can be used to prove it using a reformulation in terms of Dirichlet series, which we state in Theorem 3.1. This reformulation will itself be proved by taking a finite average of a similarly-reformulated version of the \(\text{GL}(N)\) Voronoi summation formula in terms of Dirichlet series, which can be found in \[\text{KiZh}\] (and \[\text{MiSc2}\] (1.12)):

**Theorem 2.1** (Voronoi formula on \(\text{GL}(N)\) of Miller-Schmid \[\text{MiSc2}\]). Let \(F\) be a cuspidal automorphic form on \(\text{GL}(N, \mathbb{Z}) \setminus \text{GL}(N, \mathbb{R})\) with abelian Fourier coefficients \(A(*, \ldots, *)\). Assume that \(F\) generates an irreducible representation \(\pi\) of \(\text{GL}(N, \mathbb{R})\) and let \(G_\pm\) be the ratio of Gamma factors from (4)-(5). Let \(c > 0\) be an integer and let \(a\) be any integer with \((a, c) = 1\). Denote by \(\overline{a}\) the multiplicative inverse of a modulo \(c\). Let \(q = (q_1, q_2, \cdots, q_{N-2})\) be an \((N-2)\)-tuple of positive integers. Then the additively-twisted Dirichlet series

\[
\mathcal{L}_q(s, F, \overline{a}/c) = q_1^{(N-2)s} q_2^{(N-3)s} \cdots q_{N-2}^{s} \sum_{n=1}^{\infty} \frac{A(q_{N-2}, \ldots, q_1, n)}{n^s} e\left(\frac{\overline{a}n}{c}\right),
\]

which is initially convergent for \(\Re s \gg 1\), has an analytic continuation to an entire function of \(s \in \mathbb{C}\) satisfying the functional equation

\[
\mathcal{L}_q(s, F, \overline{a}/c) = 
\]

\[
\frac{G_+(s) - G_-(s)}{2} \sum_{d_1 \mid q_1, c} \frac{\mu(d_1)}{d_1^{s+1}} \sum_{m_1=1}^{q_1} \cdots \sum_{d_{N-2} \mid q_{N-2}, c} \sum_{m_{N-2}=1}^{q_{N-2}} \frac{A(n, d_{N-2}, \ldots, d_2, d_1) \text{Kl}_{N-2}(a, n, c; q, d)}{n^s c^{N-2} d_1^{1-(N-1)s} d_2^{1-(N-2)s} \cdots d_{N-2}^{1-2s}}
\]

\[
+ \frac{G_+(s) + G_-(s)}{2} \sum_{d_1 \mid q_1, c} \frac{\mu(d_1)}{d_1^{s+1}} \sum_{m_1=1}^{q_1} \cdots \sum_{d_{N-2} \mid q_{N-2}, c} \sum_{m_{N-2}=1}^{q_{N-2}} \frac{A(n, d_{N-2}, \ldots, d_2, d_1) \text{Kl}_{N-2}(a, -n, c; q, d)}{n^s c^{N-2} d_1^{1-(N-1)s} d_2^{1-(N-2)s} \cdots d_{N-2}^{1-2s}},
\]

where \(d = (d_1, \ldots, d_{N-2})\) (both terms on the righthand side converge for \(\Re s \ll -1\) and have entire continuations to \(s \in \mathbb{C}\)).

### 3 Proof

We begin by restating Theorem 1.1 in the language of Dirichlet series, analogously to Theorem 2.1.

**Theorem 3.1.** Let \(F\) be a cuspidal automorphic form on \(\text{GL}(N, \mathbb{Z}) \setminus \text{GL}(N, \mathbb{R})\), \(N \geq 3\), with abelian Fourier coefficients \(A(*, \ldots, *)\). Assume that \(F\) generates an irreducible representation \(\pi\) of \(\text{GL}(N, \mathbb{R})\) and let \(G_\pm\) be the ratio of Gamma factors from (4)-(5). Let \(L\) and \(M\) be two non-negative integers whose sum \(L + M = N - 2\). Let \(c > 0\) be an integer and let \(a\) be any integer with

\footnote{The formula stated here corrects a misprint propagating from \[\text{MiSc2}\] (1.9), where the first two arguments in the definition of the Kloosterman sum were mistakenly switched.}
\( (a, c) = 1 \). Denote by \( \overline{a} \) the multiplicative inverse of \( a \) modulo \( c \). Let \( \mathbf{q} = (q_1, q_2, \ldots, q_L) \) be an \( L \)-tuple of positive integers and \( \mathbf{Q} = (Q_1, Q_2, \ldots, Q_M) \) an \( M \)-tuple of positive integers. Define the Dirichlet series

\[
L_{\mathbf{q}, \mathbf{Q}}(s, F, \overline{a}/c) = \sum_{D | Q} \sum_{n=1}^{\infty} \frac{A(q_L, \ldots, q_1, D_1, \ldots, D_M, n)}{n^s} \text{Kl}_M(\overline{a}, n, c; \mathbf{Q}, D) \frac{q_1^{L_s} \cdots q_L^s}{D_1^{(M+1)s-M} \cdots D_M^{2s-1}},
\]

where \( \sum_{D | Q} \) stands for \( \sum_{D_1 | Q_1} \sum_{D_2 | D_1^{Q_2/c}} \cdots \sum_{D_M | D_1^{Q_3/c} \cdots D_{M-1}^{Q_{M-1}/c}} \). This Dirichlet series is convergent for \( \Re s \gg 1 \), and has an analytic continuation to an entire function in \( s \in \mathbb{C} \) which satisfies the functional equation

\[
L_{\mathbf{q}, \mathbf{Q}}(s, F, \overline{a}/c) = c^{M+1-N_s} \left[ \frac{G_+((s)+(-1)^{M+1})G_-(s)L_{\mathbf{q}, \mathbf{Q}}(1-s, \overline{F}, a/c)}{2} + \frac{G_+((s)+(-1)^{M+1})G_-(s)L_{\mathbf{q}, \mathbf{Q}}(1-s, \overline{F}, -a/c)}{2} \right],
\]

where \( L_{\mathbf{q}, \mathbf{Q}}(\ldots, \overline{F}, \ldots) \) is defined using the contragredient coefficients \( \tilde{A}(m_1, \ldots, m_{n-1}) = A(m_{n-1}, \ldots, m_1) \).

The following two lemmas are used in the proof of Theorem 3.1.

**Lemma 3.2.** Let \( C, Q, \) and \( b \) be positive integers and \( y, a \) integers. Assuming \( b | QC, (y, QC/b) = 1 \) and \( (a, C) = 1 \), we have

\[
\sum_{D | QC} \frac{1}{x \left( \mod \frac{QC}{D} \right)} e \left( \frac{Dxa}{C} + \frac{byx}{QC} \right) = \begin{cases} QC, & \text{if } b = Q \text{ and } y \equiv -a \pmod{C}, \\ 0, & \text{otherwise}. \end{cases}
\]

**Proof.** The sum \( \sum_{z \left( \mod QC \right)} e \left( \frac{z(Qa + by)}{QC} \right) \) equals \( QC \) when \( QC | Qa + by \), and vanishes otherwise. Factoring each \( z \) as \( z = Dx \) with \( D = \gcd(z, QC) \) and \( x \in (\mathbb{Z}/\mathbb{Z})^* \), we see this sum equals the lefthand side of (12). It thus suffices to show that the nonvanishing conditions are equivalent. Clearly \( QC | Qa + by \) if \( b = Q \) and \( y \equiv -a \pmod{C} \). Conversely, suppose \( QC | Qa + by \). Thus \( Q | by \), which implies that \( Q \) divides \( \gcd(by, QC) = b \); also, since we have assumed that \( b | QC \), we must have \( b | Qa \) and hence \( b \) divides \( \gcd(Qa, QC) = Q \). Being divisors of each other, \( b \) and \( Q \) are equal; this forces \( C | (a + y) \).

**Proof of Theorem 3.1** We open up the hyper-Kloosterman sums on the lefthand side of (10) com-
pletely, which results in the formal identity

\[
\sum_{D|Q} \sum_{n=1}^{\infty} \frac{A(q, q_1, D_1, \ldots, D_M, n) K_{M}(\tilde{a}, n, D) q_1^{L_1} \ldots q_L^{L_M}}{D_1^{(M+1)s - M} \ldots D_M^{2s - 1}}
\]

\[
= \sum_{D|Q} \sum_{n=1}^{\infty} \frac{A(q, q_1, D_1, \ldots, D_M, n) q_1^{L_1} \ldots q_L^{L_M}}{D_1^{(M+1)s - M} \ldots D_M^{2s - 1}}
\]

\[
\sum_{x_1 \mod \frac{Q_1}{D_1}} ^* \sum_{x_2 \mod \frac{Q_2}{D_1D_2}} ^* \ldots \sum_{x_M \mod \frac{Q_M}{D_1 \ldots D_M}} ^* e \left( \frac{D_1x_1\tilde{a}}{c} + \frac{D_2x_2\tilde{a}}{c} + \ldots + \frac{D_Mx_M\tilde{a}}{c} \right)
\]

\[
= \sum_{D|Q} \sum_{n=1}^{\infty} \frac{A(q, q_1, D_1, \ldots, D_M, n)}{n^s} D_1^{(L+1)s} \ldots D_M^{(L+M)s} q_1^{L_1} \ldots q_L^{L_M} e \left( \frac{n\tilde{a}}{Q_1 \ldots Q_M c \tilde{a}} \right)
\]

By Theorem 2.1, the \(n\)-sum part is absolutely convergent for \(Rs \gg 1\) and has analytic continuation to \(C\), hence the same assertions are true of (10). Applying (9) to the \(n\)-sum, we get

\[
\sum_{D|Q} \sum_{n=1}^{\infty} \frac{A(q, q_1, D_1, \ldots, D_M, n)}{n^s} D_1^{(L+1)s} \ldots D_M^{(L+M)s} q_1^{L_1} \ldots q_L^{L_M} e \left( \frac{n\tilde{a}}{Q_1 \ldots Q_M c \tilde{a}} \right)
\]

which is absolutely convergent for \(Rs \ll -1\). We open up the hyper-Kloosterman sum partially, obtaining

\[
K_{1-L} \left( x_M, n, \frac{Q_1 \ldots Q_M c}{D_1 \ldots D_M}, (D_M, \ldots, D_1, q_1, \ldots, q_L), (b_M, \ldots, b_1, d_1, \ldots, d_L) \right)
\]

\[
= \sum_{y_M \mod \frac{D_1 \ldots D_M}{b_M}} ^* \sum_{y_{M-1} \mod \frac{D_1 \ldots D_{M-1}}{b_M b_{M-1}}} ^* \ldots \sum_{y_1 \mod \frac{D_1 \ldots D_M}{b_M b_{M-1} \ldots b_1}} ^* e \left( \frac{b_M y_M x_M}{D_1 \ldots D_M} + \frac{b_{M-1} y_{M-1} x_{M-1}}{D_1 \ldots D_M} + \ldots + \frac{b_1 y_1 x_1}{D_1 \ldots D_M} \right) K_{1-L} \left( y_1, n, \frac{Q_1 \ldots Q_M c}{b_1 \ldots b_M}, (q_1, \ldots, q_L), (d_1, \ldots, d_L) \right)
\]
After reordering the summations, $L_q(Q(s, F, \tilde{a}/c)$ equals

$$
\sum_{D_1|Q_1c} x_1(\text{mod } \frac{Q_1c}{D_1}) \cdot \ldots \cdot \sum_{D_{M-1}|Q_{M-1}c} x_{M-1}(\text{mod } \frac{Q_{M-1}c}{D_{M-1}}) \cdot \sum_{D_M|Q_{M}c} x_M(\text{mod } \frac{Q_{M}c}{D_M})
$$

$$
\sum_{b_1|Q_1c} y_{1}(\text{mod } \frac{Q_1c}{b_1}) \cdot \ldots \cdot \sum_{b_{M-1}|Q_{M-1}c} y_{M-1}(\text{mod } \frac{Q_{M-1}c}{b_{M-1}}) \cdot \sum_{b_M|Q_{M}c} y_{M}(\text{mod } \frac{Q_{M}c}{b_M})
$$

$$
\sum_{M-1} D_M \cdot D_{M-2} \cdot \ldots \cdot D_1
$$

$$
e \left( \frac{D_1 x_1 \tilde{a}}{c} + \frac{D_2 x_2 \tilde{a}}{c} + \ldots + \frac{D_M x_M \tilde{a}}{c} \right) + \frac{b_1 y_{1}Q_1c}{D_1} + \frac{b_{M-1} y_{M-1}Q_{M-1}c}{D_{M-1}} + \frac{b_M y_{M}Q_{M}c}{D_M} + \ldots + \frac{b_1 y_{2}Q_1c}{D_M}
$$

$$
\sum_{n=1}^{\infty} A(n, d_L, \ldots, d_1, b_1, \ldots, b_M)
$$

$$\cdot \sum_{n=1}^{\infty} A(n, d_L, \ldots, d_1, b_1, \ldots, b_M)
$$

$$= \left[ \frac{G_1(s) - G_2(s)}{2} \right] K_L \left( \tilde{y}_1, n, \frac{Q_{1} \cdot Q_{M}c}{b_1 \cdot b_M}; (q_1, \ldots, q_L), (d_1, \ldots, d_L) \right)
$$

$$+ \left[ \frac{G_1(s) - G_2(s)}{2} \right] K_L \left( \tilde{y}_1, n, \frac{Q_{1} \cdot Q_{M}c}{b_1 \cdot b_M}; (q_1, \ldots, q_L), (d_1, \ldots, d_L) \right)
$$

Observe that $D_M$ is not present in the summations in lines (14b) and (14c). Thus consider the $D_M$- and $x_M$-summations,

$$
\sum_{D_M|Q_{M}c} x_M(\text{mod } \frac{Q_{M}c}{D_M})
$$

$$\sum_{D_M|Q_{M}c} x_M(\text{mod } \frac{Q_{M}c}{D_M})
$$

$$e \left( \frac{D_M x_M \tilde{a}}{c} + \frac{b_M y_{M}Q_{M}c}{D_M} \right)
$$

is precisely the modulus of $y_M$. The overall expression is multiplied by $QC = \frac{Q_1 \cdot Q_{M}c}{D_1 \cdot D_{M-1}}$, so the portion of the summand in (14c) becomes $D_M \cdot D_{M-2} \cdot D_{M-3} \cdot D_{M-2}$.

Now that $b_M$ and $y_M$ have been removed, we see that the remaining indices of summation in (14b) and (14c) do not involve $D_{M-1}$. We continue to apply Lemma 3.2 to $D_{M-1}$- and $x_{M-1}$-summations,

$$
\sum_{D_{M-1}|Q_{M-1}c} x_{M-1}(\text{mod } \frac{Q_{M-1}c}{D_{M-1}})
$$

$$\sum_{D_{M-1}|Q_{M-1}c} x_{M-1}(\text{mod } \frac{Q_{M-1}c}{D_{M-1}})
$$

$$e \left( \frac{D_{M-1} x_{M-1} \tilde{a}_{M-1}}{c} + \frac{b_{M-1} y_{M-1}Q_{M-1}c}{D_{M-1}} \right)
$$

This forces $b_{M-1} = Q_{M-1}$ and $y_{M-1} \equiv (-1)^2 \tilde{a}_{M-2} \left( \text{mod } \frac{Q_1 \cdot Q_{M-2}c}{D_1 \cdot D_{M-2}} \right)$. In turn, we consecutively apply Lemma 3.2 to $D_j$- and $x_j$-summations for $j = M-2, \ldots, 2$,

$$
\sum_{D_j|Q_jc} x_j(\text{mod } \frac{Q_jc}{D_j})
$$

$$\sum_{D_j|Q_jc} x_j(\text{mod } \frac{Q_jc}{D_j})
$$

$$e \left( \frac{D_j x_j \tilde{a}_j}{c} + \frac{(-1)^{M-j} b_j y_j x_j}{D_1 \cdot D_j} \right)
$$

8
forcing \( b_j = Q_j \) and \( y_j \equiv (-1)^{M-j+1}x_{j-1} \pmod{\frac{Q_1-\cdots-Q_{j-1}}{D_1-D_{j-1}}} \). At the final stage, we apply Lemma 3.2 once more to the remaining sum

\[
\sum_{D_1|Q_1c} \sum_{x_1 \pmod{\frac{Q_1c}{D_1}}} \sum_{c \equiv \frac{y_1}{D_1}} e \left( \frac{D_1x_1\bar{a}}{c} + \frac{(-1)^{M-1}b_1x_1}{Q_1c} \right)
\]

to force \( b_1 = Q_1 \) and \( y_1 \equiv (-1)^M\bar{a} \pmod{c} \). At this point the summations in (14a) and (14b) have all disappeared, and we find (13) is equal to

\[
\sum_{d_1|q,c} \cdots \sum_{d_L|q,c} \sum_{n=1}^{\infty} A(n, d_L, \ldots, d_1, Q_1, \ldots, Q_M)c^M Q_1^M \cdots Q_M^M \frac{1}{Q_1^{L+1}s} \cdots Q_M^{L+1}s \frac{1}{d_1^{L+1}s} \cdots d_L^{L+1}s
\]

\[
\times \left[ G_+(s) - G_-(s) KL \left( (-1)^M a, n, c; (q_1, \ldots, q_L), (d_1, \ldots, d_L) \right) \right.
\]

\[
+ \left. \frac{G_+(s) + G_-(s)}{2} KL \left( (-1)^M a, -n, c; (q_1, \ldots, q_L), (d_1, \ldots, d_L) \right) \right]
\]

\[
= \frac{G_+(s) + (-1)^{M+1}G_-(s)}{2} \sum_{d|q} \sum_{n=1}^{\infty} A(n, d_L, \ldots, d_1, Q_1, \ldots, Q_M) KL(a, n, c; q, d) Q_1(M-1-s) \cdots Q_M(M-1-s) \frac{1}{n^{s-1}c^{Ns-1-M}} \frac{1}{d_1^{L+1}s} \cdots d_L^{L+1}s
\]

\[
+ \frac{G_+(s) + (-1)^{M}G_-(s)}{2} \sum_{d|q} \sum_{n=1}^{\infty} A(n, d_L, \ldots, d_1, Q_1, \ldots, Q_M) KL(a, -n, c; q, d) Q_1(M-1-s) \cdots Q_M(M-1-s) \frac{1}{n^{s-1}c^{Ns-1-M}} \frac{1}{d_1^{L+1}s} \cdots d_L^{L+1}s,
\]

which is equivalent to (11).

\[\square\]

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**References**

[BLM] Blomer, Valentin, Xiaqing Li, and Stephen D. Miller. “A spectral reciprocity formula and non-vanishing for L-functions on GL(4) × GL(2).” *arXiv preprint arXiv:1705.04344* (2017).

[Bum] Bump, Daniel. “The Rankin-Selberg method: a survey.” *Number theory, trace formulas and discrete groups (Oslo, 1987)* (1989): 49-109.

[DuIw] Duke, William, and Henryk Iwaniec. “Estimates for coefficients of L-functions. I.” *Automorphic forms and analytic number theory* (1990): 43-47.

[FoGa] Fouvry, Étienne, and Satadal Ganguly. “Strong orthogonality between the Möbius function, additive characters and Fourier coefficients of cusp forms.” *Compos. Math.* 150, no. 05 (2014): 763-797.

[GoLi1] Goldfeld, Dorian, and Xiaqing Li. “Voronoi formulas on GL(n).” *Int. Math. Res. Not. IMRN* 2006 (2006): 86295.
[GoLi2] Goldfeld, Dorian, and Xiaoqing Li. “The Voronoi formula for $GL(n, \mathbb{R})$.” *Int. Math. Res. Not. IMRN* 2008 (2008): rnm144.

[IcTe] Ichino, Atsushi, and Nicolas Templier. “On the Voronoi formula for $GL(n)$.” *Amer. J. Math.* 135, no. 1 (2013): 65-101.

[Jac] Jacquet, Hervé. “Archimedean Rankin-Selberg integrals.” *Automorphic forms and $L$-functions II. Local aspects*, Contemp. Math. 489 (2009): 57-172.

[KiZh] Kıral, Eren Mehmet, and Fan Zhou. “The Voronoi formula and double Dirichlet series.” *Algebra Number Theory* 10-10 (2016): 2267–2286.

[Li] Li, Xiaoqing. “Bounds for $GL(3) \times GL(2)$ $L$-functions and $GL(3)$ $L$-functions.” *Ann. of Math.* (2) 173 (2011): 301-336.

[Mil] Miller, Stephen D. “Cancellation in additively twisted sums on $GL(n)$.” *Amer. J. Math.* 128, no. 3 (2006): 699-729.

[MiSc1] Miller, Stephen D., and Wilfried Schmid. “Automorphic distributions, $L$-functions, and Voronoi summation for $GL(3)$.” *Ann. of Math.* (2) 2006: 423-488.

[MiSc2] Miller, Stephen D., and Wilfried Schmid. “A general Voronoi summation formula for $GL(n, \mathbb{Z})$.” In *Geometry and analysis*. No. 2, volume 18 of Adv. Lect. Math. (ALM), pages 173-224. Int. Press, Somerville, MA, 2011.

[MiSc3] Miller, Stephen D., and Wilfried Schmid. “Adelization of automorphic distributions and mirabolic Eisenstein series.” *Representation theory and mathematical physics*, Contemp. Math. 557 (2011): 289-334.

[Mun] Munshi, Ritabrata. “The circle method and bounds for $L$-functions - IV: Subconvexity for twists of $GL(3)$ $L$-functions.” *Ann. of Math.* (2) 182, no. 2 (2015): 617-672.

[Zho] Zhou, Fan. “Voronoi summation formulae on $GL(n)$.” *J. Number Theory* 162 (2016): 483-495.

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