GALOIS LINES FOR NORMAL ELLIPTIC SPACE CURVES, II

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Abstract. For each linearly normal elliptic curve $C$ in $\mathbb{P}^3$, we determine Galois lines and their arrangement. The results are as follows: the curve $C$ has just six $V_4$-lines and in case $j(C) = 1$, it has eight $Z_4$-lines in addition. The $V_4$-lines form the edges of a tetrahedron, in case $j(C) = 1$, for each vertex of the tetrahedron, there exist just two $Z_4$-lines passing through it. We obtain as a corollary that each plane quartic curve of genus one does not have more than one Galois point.

key words and phrases : Galois line, space elliptic curve, Galois covering

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1. Introduction

This is a continuation of [1], where we found three $V_4$-lines for each linearly normal elliptic curve $C$ in $\mathbb{P}^3$, and four $Z_4$-lines for such curve $C$ with $j(C) = 1$. However, those lines are not all the ones. In this article we determine all Galois lines and describe their arrangement. First let us recall the definition of Galois lines briefly.

Let $k$ be the ground field of our discussion, we assume it to be algebraically closed, later we assume it the field $\mathbb{C}$ of complex numbers. Let $C$ be a smooth irreducible non-degenerate curve of degree $d$ in the projective three space $\mathbb{P}^3$ and $\ell$ a line in $\mathbb{P}^3$ not meeting $C$. Let $\pi_\ell : \mathbb{P}^3 \to \ell_0$ be the projection with center $\ell$, where $\ell_0$ is a line not meeting $\ell$. Restricting $\pi_\ell$ to $C$, we get a surjective morphism $\pi_\ell|_C : C \to \ell_0$ and hence an extension of fields $(\pi_\ell|_C)^* : k(\ell_0) \hookrightarrow k(C)$, where $[k(C) : k(\ell_0)] = d$.

Note that the extension of fields does not depend on $\ell_0$, but on $\ell$.

Definition 1. The line $\ell$ is said to be a Galois line for $C$ if the extension $k(C)/k(\ell_0)$ is Galois, or equivalently, if $\pi_\ell|_C$ is a Galois covering. In this case $\text{Gal}(k(C)/k(\ell_0))$ is said to be the Galois group for $\ell$ and denoted by $G_\ell$.

If $\ell$ is the Galois line, then each element $\sigma \in G_\ell$ induces an automorphism of $C$ over $\ell_0$. We denote it by the same letter $\sigma$. Hereafter, assume $C$ is linearly normal, i.e., the hyperplanes cut out the complete linear series $|\mathcal{O}_C(1)|$. Then, the automorphism $\sigma$ can be extended to a projective transformation of $\mathbb{P}^3$, which will be also denoted by the same letter $\sigma$.

We use the following notation and convention:

- $V_4$ : the Klein 4-group
- $Z_4$ : the cyclic group of order four
- $\sim$ : the linear equivalence of divisors
- $\text{Aut}(C)$ : the automorphism group of $C$
\[ L(D) := \{ f \in k(C) \setminus \{0\} \mid \text{div}(f) + D \geq 0 \} \cup \{0\}, \text{ where div}(f) \text{ is the divisor of } f \text{ and } D \text{ is a divisor on } C. \]

\[ \langle \cdots \rangle : \text{the group generated by the set } \{ \cdots \} \text{ or the linear subvariety spanned by the set } \{ \cdots \} \]

\[ V(F) : \text{the variety defined by } F = 0 \]

\[ C \cdot H : \text{the intersection divisor of } C \text{ and } H \text{ on } C, \text{ where } H \text{ is a plane.} \]

\[ \ell_{PQ} : \text{the line passing through } P \text{ and } Q \]

2. Statement of Results

We assume \( k = \mathbb{C} \) and use the same notation as in [1].

**Definition 2.** When \( \ell \) is a Galois line for \( C \) and \( G_\ell \cong V_4 \) (resp. \( Z_4 \)), we call \( \ell \) a \( V_4 \)-line.

There exist \( V_4 \)-lines for the curve which is given by an intersection of hypersurfaces as follows.

**Lemma 1.** Suppose \( S_1 \) and \( S_2 \) are irreducible quadratic surfaces in \( \mathbb{P}^3 \) satisfying the following conditions:

1. \( S_i \) (\( i = 1, 2 \)) has a singular point \( Q_i \) and \( Q_1 \neq Q_2 \).
2. \( S_1 \cap S_2 \) is a smooth curve \( \Delta \).
3. The line \( \ell \) passing through \( Q_1 \) and \( Q_2 \) does not meet \( \Delta \).

Then, \( \Delta \) is a linearly normal elliptic curve and \( \ell \) is a \( V_4 \)-line for \( \Delta \).

Let \( C \) be a linearly normal elliptic curve in \( \mathbb{P}^3 \). Then, there exists a divisor \( D \) of degree four on an elliptic curve \( E \) such that \( C \) is given by an embedding of \( E \) associated with the complete linear system \( |D| \). Note that \( C \) can be expressed as an intersection of two quadratic surfaces.

**Lemma 2.** There exist just four irreducible quadratic surfaces \( S_i \) (\( 0 \leq i \leq 3 \)) such that each \( S_i \) has a singular point and contains \( C \). Let \( Q_i \) be the unique singular point of \( S_i \). Then the four points are not coplanar.

**Remark 3.** Let \( \pi_Q : \mathbb{P}^3 \to \mathbb{P}^2 \) be the projection with center \( Q \in \mathbb{P}^3 \setminus C \). If \( \pi_Q \) induces a 2 to 1 morphism from \( C \) onto its image in Lemma 2 then \( Q \) coincides with one of \( Q_i \).

The main theorem is stated as follows:

**Theorem 1.** For each linearly normal elliptic curve in \( \mathbb{P}^3 \), there exist four non-coplanar points \( Q_i \) (\( 0 \leq i \leq 3 \)) such that the lines passing through each two of them are \( V_4 \)-lines for \( C \). Namely, all the \( V_4 \)-lines form the six edges of a tetrahedron. Further, if the Weierstrass normal form of \( E \) is given by \( y^2 = 4(x-e_1)(x-e_2)(x-e_3) \), then we can present explicitly the coordinates of \( Q_i \) (by taking a suitable coordinates of \( \mathbb{P}^3 \)) as follows:

\[ Q_0 = (0 : 0 : 0 : 1) \text{ and } Q_i = (1 : -c_i : e_i : 0), \quad (i = 1, 2, 3), \]

where \( c_i = e_i^2 + e_j e_k \) such that \( \{i, j, k\} = \{1, 2, 3\} \).
Remark 4. In the case of an elliptic curve $E$ in $\mathbb{P}^2$, it has a Galois point if and only if $j(E) = 0$, and then it has just three $\mathbb{Z}_3$-points.

In the case where the $j$-invariant $j(C) = 1$, there exists an automorphism of order four with a fixed point. This curve has the other Galois lines as follows.

**Theorem 2.** Under the same assumption as in Theorem 1, if $j(C) = 1$, then there exist eight $\mathbb{Z}_4$-lines (in addition to the $V_4$-lines). To state in more detail, for each vertex $Q_i$ ($0 \leq i \leq 3$) of the tetrahedron in Theorem 1, there exist two $\mathbb{Z}_4$-lines passing through it. Therefore, for each vertex, there exist three $V_4$-lines and two $Z_4$-lines passing through it and the total number of Galois lines is fourteen. Two $Z_4$-lines do not meet except at one of the vertices.

Let $\Sigma$ be the set of six $V_4$-lines in Theorem 1. In the case where $j(C) = 1$ let $\Sigma'$ be the set of eight $Z_4$-lines in Theorem 2. The following corollary is an answer to the question for the case of outer Galois point [3, Theorem 2].

**Corollary 5.** For a plane quartic curve $\Gamma$ with genus one, the number of (outer) Galois points is at most one. If $\Gamma$ has the Galois point, then the Galois group $G$ is isomorphic to $V_4$ or $\mathbb{Z}_4$. Further, if $G \cong V_4$ (resp. $Z_4$), then $\Gamma$ is obtained by a projection $\pi_Q : \mathbb{P}^3 \dashrightarrow \mathbb{P}^2$ with center $Q$, where $Q \in \Sigma$ (resp. $Q \in \Sigma'$) such that $Q \neq Q_i$ ($0 \leq i \leq 3$).

Remark 6. Different from the case of the space quartic curve, a plane quartic curve of genus one does not necessarily have a Galois point.

Remark 7. Since $C$ is given by the embedding associated with a complete linear system and has a Galois line, the embedding is called a Galois embedding, which has been defined in [6].

3. Proofs

First we prove Lemma 1. It is easy to see that $\Delta$ has genus one and $\dim H^0(\Delta, O_\Delta(1)) = 4$. Hence $\Delta$ is a linearly normal elliptic curve. Let $\pi_{Q_i}$ be the projection $\mathbb{P}^3 \dashrightarrow \mathbb{P}^2$ with center $Q_i$ ($i = 1, 2$) and put $\Delta_i = \pi_{Q_i}(\Delta) \subset \mathbb{P}^2$ and $R_i = \pi_{Q_i}(\ell \setminus \{Q_i\})$. Then $\Delta_i$ is a conic and $R_i$ is a point not on $\Delta_i$. Let $\varpi_{R_i}$ be the projection $\mathbb{P}^2 \dashrightarrow \mathbb{P}^1$ with center $R_i$. Restricting $\varpi_{R_i}$ to $\Delta_i$, we get a surjective morphism $\varpi_{R_i}|_{\Delta_i} : C_i \dashrightarrow \mathbb{P}^1$. Therefore we have two morphisms

$$\pi_i = \varpi_{Q_i} \circ \pi_{Q_i} : \Delta \dashrightarrow \mathbb{P}^1$$
of degree four. They coincide with the restriction of the projection \( \pi_\ell : \mathbb{P}^3 \to \mathbb{P}^1 \). Note that \( k(\Delta_1) \) and \( k(\Delta_2) \) are distinct subfields of \( k(\Delta) \) and \( [k(\Delta) : k(\Delta_i)] = [k(\Delta_i) : k(\mathbb{P}^1)] = 2 \). We infer that \( k(\Delta) \) is a \( V_4 \)-extension of \( k(\mathbb{P}^1) \), hence \( \pi_\ell|_\Delta \) is a \( V_4 \)-Galois covering. This proves Lemma 1.

Fix a universal covering \( \pi : C \to C/\mathcal{L} \), where \( \mathcal{L} \) is the lattice in \( C \) defining a complex torus. We assume \( \mathcal{L} = \mathbb{Z} + \mathbb{Z} \omega \), where \( \Im \omega > 0 \). Let \( \wp(z) \) be the Weierstrass \( \wp \)-function with respect to \( \mathcal{L} \). Then, the map \( \phi : C \to E \) defined by \( \phi(z) = (\wp(z) : \wp'(z) : 1) \), induces an isomorphism \( \bar{\phi} : C/\mathcal{L} \to E \). The defining equation of the elliptic curve \( E \) is the Weierstrass normal form \( y^2 = 4x^3 + px + q \).

We assume it to be factored as \( 4(x - e_1)(x - e_2)(x - e_3) \). Put \( P_\alpha = \phi(\alpha) \) for \( \alpha \in \mathbb{C} \).

Denote by \( + \) the sum of divisors on \( E \) and, at the same time, the sum of complex numbers. For example, \( P_\alpha + P_\beta \) and \( \alpha + \beta \) denote the sum of divisors and complex numbers respectively.

\textbf{Lemma 8.} We have the linear equivalence of divisors on \( E \):
\[ P_\alpha + P_\beta \sim P_{\alpha + \beta} + P_0. \]

\textit{Proof.} This may be well-known. See, for example, [2, Ch. IV, Theorem 4.13B]. \( \Box \)

\textbf{Lemma 9.} Let \( D \) be the divisor of degree four on \( E \). By taking a suitable translation \( \tau \) on \( E \), we have \( \tau^*(D) \sim 4P_0 \).

\textit{Proof.} Suppose \( D = \sum_{i=1}^4 P_{\alpha_i} \). Then, take \( \beta = -\sum_{i=1}^4 \alpha_i/4 \). Let \( \tau \) be the translation on \( E \) induced from the one \( z \mapsto z + \beta \) on \( \mathbb{C} \). Then we have \( \tau^*(D) = \sum_{i=1}^4 P_{\alpha_i + \beta} \).

Using Lemma 8 we get \( \tau^*(D) \sim 4P_0 \). \( \square \)

Let \( D \) be a hypeplane section of \( C \). Applying Lemma 9, we see that there exists an elliptic curve \( C_0 \) in \( \mathbb{P}^3 \) given by the embedding associated with \( |4P_0| \) and an isomorphism \( \psi : \mathbb{P}^3 \to \mathbb{P}^3 \) satisfying that \( \psi(C_0) = C \) and \( 4P_0 \sim \psi^*(D) \). So that we have the following lemma.

\textbf{Lemma 10.} We can assume \( C \) is given by the embedding associated with \( |4P_0| \).

Therefore it is sufficient for our purpose to consider the curve embedded by \( |4P_0| \). Let \( \phi : E \to C \subset \mathbb{P}^3 \) be the embedding of \( E \) associated with \( |4P_0| \).

\begin{center}
\begin{tikzpicture}
  \node (C) at (0,0) {$C$};
  \node (pi) at (-2,-2) {$\pi$};
  \node (C-L) at (-4,-4) {$C/\mathcal{L}$};
  \node (varphi) at (-2,-3) {$\varphi$};
  \node (varphi-L) at (-4,-5) {$\bar{\varphi}$};
  \node (phi) at (-2,-4) {$\phi$};
  \node (E) at (0,-4) {$E$};
  \node (C-C) at (0,-2) {$C \subset \mathbb{P}^3$};

  \draw[->] (C) -- (pi);
  \draw[->] (pi) -- (C-L);
  \draw[->] (C-L) -- (varphi-L);
  \draw[->] (varphi-L) -- (varphi);
  \draw[->] (varphi) -- (phi);
  \draw[->] (phi) -- (E);
  \draw[->] (E) -- (C-C);
\end{tikzpicture}
\end{center}

In order to study the number and arrangement of Galois lines, we provide some lemmas. Let \( \mathcal{S} \) and \( \mathcal{G} \) be the set of Galois lines for \( C \) and the set of subgroups of
Proof. Since a Galois line $\ell$ determine the Galois group $G_{\ell}$ in $\text{Aut}(C)$ uniquely, we can define the following map.

**Definition 3.** We define an arrangement-map $\rho : S \rightarrow G$ by $\rho(\ell) = G_{\ell}$.

We study the map $\rho$ in detail. Note that each element of $G_{\ell}$ can be extended to a projective transformation. That is, we have a faithful representation $r : G_{\ell} \rightarrow \text{PGL}(3, \mathbb{C})$.

**Lemma 11.** The map $\rho$ is injective.

*Proof.* For two elements $\ell_i$ of $S$ ($i = 1, 2$), suppose $\rho(\ell_1) = \rho(\ell_2)$ and $\ell_1 \neq \ell_2$. Then, the following two cases take place:

(i) $\ell_1 \cap \ell_2$ consists of one point $P$.
(ii) $\ell_1 \cap \ell_2 = \emptyset$.

In the case (i), for a general point $Q \in C$, put $H_iQ = \langle \ell_i, Q \rangle$ ($i = 1, 2$) : the plane spanned by $\ell_i$ and $Q$. Since $G_{\ell_1} = G_{\ell_2}$, we have $H_iQ \cap \ell_0 = H_2Q \cap \ell_0 = \{R\}$, where $\ell_0$ is the line defined in Introduction. Further, since $\pi_{\ell_i}(H_1Q \cap C) = \pi_{\ell_2}(H_2Q \cap C) = R$, the set of four points $H_1Q \cap C$ is equal to that of $H_2Q \cap C$ and they lie on the line $H_1Q \cap H_2Q$, which passes through $P$. This implies $C$ is contained in the plane spanned by $\ell_0$ and $P$. Since $C$ is assumed to be non-degenerate, this is a contradiction. Next we treat the case (ii). Similarly, for a general point $Q \in C$, put $H_iQ = \langle \ell_i, Q \rangle$. Then, by the same argument as above, the four points $H_1Q \cap C$ and $H_2Q \cap C$ lie on the line $H_1Q \cap H_2Q$. Thus $C$ is contained in a rational normal scroll $\Sigma$. However, $H_iQ \cap \Sigma$ is a line, so that $\Sigma$ must be a plane. This is a contradiction. \qed

We present a criterion when $G \subset \text{Aut}(C)$ can be the image of an element of $S$. See [3] Theorem 2.2 for a similar one. Hereafter we use the notation $P_\alpha' = \phi(P_\alpha) = (\phi\varphi)_{(\alpha)} \in C$ for brevity.

**Lemma 12.** A subgroup $G = \{\sigma_1, \ldots, \sigma_4\}$ of $\text{Aut}(C)$ is an image of $\rho$ if and only if $G$ satisfies the following condition $(\odot)$:

$(\odot)$ For each point $Q \in C$ the divisor $\sum_{i=1}^4 \sigma_i(Q)$ is linearly equivalent to $4P_0'$ and $C/G$ is a rational curve.

*Proof.* If $G = \rho(\ell)$, then clearly $C/G \cong \mathbb{P}^1$. Take a plane $H$ satisfying that $H \ni \ell$ and $H \ni Q$. By definition the point $\sigma_i(Q)$ ($1 \leq i \leq 4$) lies on $H$, hence the divisor is linearly equivalent to $4P_0'$. Conversely, for a point $Q \in C$, put $D = \sum_{i=1}^4 \sigma_i(Q)$. By assumption we have $D \sim 4P_0'$, hence $G$ acts on $H^0(C, \mathcal{O}_C(1))$. Therefore each element of $G$ can be extended to a projective transformation. Letting $\pi : C \rightarrow C/G \cong \mathbb{P}^1$, we take independent sections $s_0$ and $s_1$ of $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1))$ and put $\tilde{s}_i = \pi^*(s_i)$ ($i = 1, 2$). Then we have $\sigma^*(\tilde{s}_i) = \tilde{s}_i$. Taking a basis of $H^0(C, \mathcal{O}_C(1))$ containing $\tilde{s}_1$ and $\tilde{s}_2$, we obtain a Galois line $\ell$ such that $\rho(\ell) = G$. \qed

We study whether $\ell_1 \cap \ell_2 = \emptyset$ or $\neq \emptyset$ by observing $G_{\ell_1} \cap G_{\ell_2}$ in $\text{Aut}(C)$.

**Lemma 13.** Suppose $\ell_1$ and $\ell_2$ are distinct Galois lines. Then, the following two cases take place.

1. If $\ell_1 \cap \ell_2 = \emptyset$, then $G_{\ell_1} \cap G_{\ell_2} = \{\text{id}\}$ in $\text{Aut}(C)$.
2. If $\ell_1 \cap \ell_2$ is a point $P$, then it is a singular point of some quadratic surface containing $C$. Further, we have $G_{\ell_1} \cap G_{\ell_2} = \langle \sigma \rangle$, where $\sigma$ has order two and has a fixed point as an automorphism of $C$. 


**Proof.** Take an element \( \sigma \in G_{\ell_1} \cap G_{\ell_2} \). It can be extended to a projective transformation. Since every plane \( H_i \) containing \( \ell_i \) is invariant by \( \sigma \), we infer \( \sigma|_{\ell_i} = \ell_i \) (\( i = 1, 2 \)). Therefore, for each hyperplane \( H_1 \supset \ell_1 \), if \( H_1 \cap \ell_2 = \{ Q \} \), then \( \sigma|Q = Q \), i.e., \( \sigma|_{\ell_2} = \text{id} \). By the same argument we also have \( \sigma|_{\ell_1} = \text{id} \). Since \( \ell_1 \cap \ell_2 = \emptyset \), \( \sigma \) is identity on \( \mathbb{P}^3 \). Next we treat the second case. Suppose \( \ell_1 \cap \ell_2 \) consists of one point \( P \). Then, for each point \( Q \in C \), put \( H_{iQ} = \langle \ell_i, Q \rangle \) and \( \ell_Q = H_{1Q} \cap H_{2Q} \). Since \( H_{iQ} \supseteq \ell_Q \) for \( i = 1 \) and \( 2 \), we have \( \sigma|Q \in \ell_Q \). Therefore \( C \) is contained in the cone passing through \( P \). Clearly the order of \( \sigma \) is two. Since the quotient curve \( C/\langle \sigma \rangle \) is isomorphic to \( \mathbb{P}^1 \), the \( \sigma \) has a fixed point in \( C \).

From Lemma [13] we infer the following remark.

**Remark 14.** Let \( \ell \) be a Galois line and take a point \( P \in \ell \). Let \( \pi_P : \mathbb{P}^3 \rightarrow \mathbb{P}^2 \) be a projection with center \( P \). If \( P \) is not the vertex of the tetrahedron, then \( \pi_P(\ell \setminus \{ P \}) \) is a Galois point for the quartic curve \( \pi_P(C) \). However, if \( P \) is the one, then \( \pi_P|_C \) turns out to be a 2 to 1 morphism onto its image and \( \pi_P(C) \) is a conic in \( \mathbb{P}^2 \).

Hereafter we denote by \( \sigma_i \) (\( 0 \leq i \leq 3 \)) an automorphism of \( E \) such that the representation on \( \mathbb{C} \) is

\[
\sigma_0(z) = -z, \quad \sigma_1(z) = -z + \frac{1}{2}, \quad \sigma_2(z) = -z + \frac{\omega}{2}, \quad \sigma_3(z) = -z + \frac{1+\omega}{2}.
\]

**Lemma 15.** The number of \( V_4 \)-lines is at most six.

**Proof.** Suppose \( C \) has a \( V_4 \)-line \( \ell \). Then, let \( H \) be a plane containing \( \ell \) and \( P_0' \). Since \( \pi_{\ell|C} : C \rightarrow \mathbb{P}^1 \) is a \( V_4 \)-covering, the intersection divisor \( H \cdot C \) on \( C \) can be expressed in one of the following two types:

\[
\begin{align*}
\text{(i)} & \quad H \cdot C = 2P_0' + 2P_1' \\
\text{(ii)} & \quad H \cdot C = P_0' + P_1' + P_2' + P_3'.
\end{align*}
\]

Suppose \( G = \langle \sigma, \tau \rangle \), where

\[
\sigma(z) = -z + \alpha \quad \text{and} \quad \tau(z) = z + \beta
\]

on the universal covering \( \mathbb{C} \), where \( 2\beta \equiv 0 \pmod{\mathbb{L}} \) and \( \beta \not\equiv 0 \pmod{\mathbb{L}} \). The case (i) (resp. (ii)) occurs when \( \alpha \equiv 0 \pmod{\mathbb{L}} \) (resp. \( \alpha \not\equiv 0 \pmod{\mathbb{L}} \)) in \( \mathbb{L} \). We consider the possibility of \( \alpha \not\equiv 0 \), i.e., we treat the case (ii). Since \( H \cdot C \) is invariant by the action of \( G \), it can be expressed as \( P_0' + P_1' + P_2' + P_3' \). Since this is linearly equivalent to \( 4P_0' \), we infer

\[
\begin{align*}
\text{(2)} & \quad P_{\alpha} + P_{\beta} + P_{\alpha+\beta} \sim 3P_0
\end{align*}
\]

on \( E \). The left hand side of (2) is linearly equivalent to \( P_{2(\alpha+\beta)} + 2P_0 \) by Lemma [8]. Therefore we have \( P_{2(\alpha+\beta)} \sim P_0 \). This implies \( 2(\alpha + \beta) \equiv 0 \pmod{\mathbb{L}} \), i.e., \( 2\alpha \equiv 0 \pmod{\mathbb{L}} \). Then, let us find the distinct subgroups \( G \) of \( \text{Aut}(C) \) such that \( G \) is generated by order two elements. By taking two from \( \sigma_i \) (\( 0 \leq i \leq 3 \)), we have six subgroups \( G_{ij} = \langle \sigma_i, \sigma_j \rangle \), where \( 0 \leq i < j \leq 3 \). Clearly \( G_{ij} \cong V_4 \). For example, \( G_{12} = \{ \text{id}, \sigma_1, \sigma_2, \sigma_1\sigma_2 \} \), where \( (\sigma_1\sigma_2)(z) = z + (1 + \omega)/2 \).

**Lemma 16.** Putting \( a_i = (e_i - e_j)(e_i - e_k) \), we have

\[
\sigma_0^*(x) = x, \quad \sigma_0^*(y) = -y
\]
Lemma 17. Using the same notation $K$ formulas of $\mathcal{P}$ the point $Q$ coordinates on $P$ $\sigma$, $\varphi$, $\rho$ bedding $\pi$ $\mathcal{P}$ $\mathcal{L}$, $\mathcal{L}$ $\mathcal{P}$ $\sigma_i$, $\sigma_j$ $F_i = XY - Z^2$ and $F_2 = 4YZ + pXZ + qX^2 - W^2$.  

Lemma 17. Using the same notation $G_{ij} = \langle \sigma_i, \sigma_j \rangle$ as in the proof of Lemma 15, we denote by $K_{ij} = k(x, y)G_{ij}$ the fixed subfield of $k(x, y)$ by $G_{ij}$. Then we have $K_{0i} = k \left( \frac{x^2 + c_i}{x - c_i} \right)$, where $1 \leq i \leq 3$ and $1 \leq i < j \leq 3$ and $(k - i)(k - j) \neq 0$. In particular, the Galois lines which correspond to $G_{0i}$ and $G_{ij}$ by the arrangement-map $\rho$ are $Y + c_iX = Z - c_iX = 0$ and $c_kX - Y + 2c_kZ = W = 0$ respectively.

Proof. By making use of Lemma 16, we can check the assertions by direct calculations. □

Now we proceed with the proof of Lemma 17. Let $S = V(F)$ be a surface containing $C$. Then $F$ can be expressed as $\lambda_1 F_1 + \lambda_2 F_2$, where $(\lambda_1 : \lambda_2) \in \mathbb{P}^1$. In case $\lambda_2 = 0$, the point $Q_0 = (0 : 0 : 0 : 1)$ is the singular point of $V(F_1)$. On the other hand, in case $\lambda_0 \neq 0$, put $b = \lambda_1/\lambda_0$. So we assume $F = bF_1 + F_2$. Consider the condition that $V(F)$ has a singular point, i.e., consider the simultaneous linear equations

(3) \[ F_X = F_Y = F_Z = F_W = 0. \]

This is equivalent to consider the rank of the matrix

(4) \[ M_b = \begin{pmatrix} 2q & b & p & 0 \\ b & 0 & 4 & 0 \\ p & 4 & -2b & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix}. \]
The equations \( Q \) have a non-trivial solution if and only if
\[
(5) \quad b^3 + 4pb - 16q = 0.
\]
It easy to see that the left hand side of \( (5) \) can be factored into
\[
(b + 4e_1)(b + 4e_2)(b + 4e_3). \quad \text{Thus, there exist three distinct solutions of \( (3) \).}
\]
Since the rank of \( M_b \) is three for each solution of \( (3) \), each surface \( S_i = V(b_{i1}F_1 + F_2) \) is irreducible, where \( b_i = -4e_i \). Let \( Q_i \) be the unique singular point of \( S_i \). By simple calculations we obtain \( Q_i = (8 : -2p - b_i^2 : -2b_i : 0) = (1 : -c_i : e_i : 0) \), where
\[
c_i = e_i^2 + e_j e_k \text{ such that } \{i, j, k\} = \{1, 2, 3\}. \quad \text{Since}
\]
\[
\det \begin{pmatrix} 1 & -c_1 & e_1 \\ 1 & -c_2 & e_2 \\ 1 & -c_3 & e_3 \end{pmatrix} = 2(e_1 - e_2)(e_2 - e_3)(e_3 - e_1) \neq 0,
\]
the four points are not coplanar. This completes the proof.

The proof of Remark 3 is as follows. Let \( \Sigma_Q \) be the set \( \{ \ell_{QR} \mid R \in C \} \). Then there exists a cone \( S_Q \) with the singularity at \( Q \) such that \( S_Q \supset C \) and \( S_Q \supset \Sigma \). Therefore, by Lemma 2, we have \( Q = Q_i \) for some \( i \).

Combining Lemmas 1, 2 and 15, we infer readily Theorem 1.

Remark 18. By using the condition \( (\Diamond) \) in Lemma 12, we can prove that the number of \( V_4 \)-lines is just six. However, Lemmas 1 and 2 give the more detailed structure of the arrangement of \( V_4 \)-lines.

Now we go to the proof of Theorem 2. Since \( j(C) = 1 \), we can assume \( \omega = \sqrt{-1} \). Hereafter, for simplicity we use \( i \) instead of \( \sqrt{-1} \), so \( \mathcal{L} = \mathbb{Z} + i\mathbb{Z} \).

Lemma 19. The number of \( \mathbb{Z}_4 \)-lines is at most eight.

Proof. Suppose \( C \) has a \( \mathbb{Z}_4 \)-line \( \ell \). Then, let \( H \) be a plane containing \( \ell \) and \( P_0' \). Since \( \pi_{|C} : C \longrightarrow \mathbb{P}^1 \) is a \( \mathbb{Z}_4 \)-covering, one of the following three cases take place:

(i) \( H \cdot C = 4P_0' \).
(ii) \( H \cdot C = 2P_0' + 2P_\gamma' \).
(iii) \( H \cdot C = P_0' + P_\gamma' + P_{\gamma_2'} + P_{\gamma_3'} \).

Suppose \( G = \langle \sigma \rangle \), where
\[
(6) \quad \sigma(z) = iz + \alpha
\]
on the universal covering \( \mathbb{C} \). The case (i) occurs if and only if \( P_0' \) is a fixed point for \( \sigma \), i.e., \( \alpha \equiv 0 \pmod{\mathcal{L}} \) in \( (3) \). The case (ii) occurs if and only if \( P_0' \) is a fixed point for \( \sigma^2 \), i.e., \( 2\alpha \equiv 0 \pmod{\mathcal{L}} \) in \( (3) \). Concerning the last case (iii), since \( H \cdot C \) is invariant by the action of \( G \), it can be expressed as \( P_0' + P_\alpha' + P_{\alpha'} + P_{(1+i)\alpha}' \).

Since this is linearly equivalent to \( 4P_0' \), we infer
\[
(7) \quad P_\alpha + P_{i\alpha} + P_{(1+i)\alpha} \sim 3P_0
\]
on the curve \( E \). Moreover the left hand side of \( (7) \) is linearly equivalent to \( P_{2(1+i)\alpha} + 2P_0 \) by Lemma \( Q \). Therefore we have \( P_{2(1+i)\alpha} \sim P_0 \). This implies \( 2(1 + i)\alpha \equiv 0 \pmod{\mathcal{L}} \). To find the possibility of \( \alpha \), it is sufficient to solve the equation \( 2(1 + i)\alpha \equiv 0 \pmod{\mathcal{L}} \). By a simple calculation we have \( \alpha = (m + ni)/4 \), where
\[
(m, n) = (0, 0), (2, 2), (2, 0), (0, 2), (3, 1), (1, 3), (1, 1), (3, 3).
\]
Thus we get eight subroups, which might be the images of $\rho$ of Definition 8. □

Checking the condition $(\diamondsuit)$ of Lemma 12 we now prove Theorem 2. As we see from the proof of Lemma 19 we have $G = \langle \sigma \rangle$, where $\sigma(z) = iz + \alpha$. Since $\sigma$ has fixed points, the curve $C/G$ is rational. For each point $Q \in C$ there exists $\gamma \in \mathbb{C}$ satisfying that $Q = P_\gamma$. So it is sufficient to prove that $P_\gamma + P_{\sigma(\gamma)} + P_{\sigma^2(\gamma)} + P_{\sigma^3(\gamma)} \sim 4P'_0$. Since $2(1+i)\alpha \equiv 0 \pmod{L}$ as in the proof of Lemma 19 this holds true by Lemma 8. Since $j_1(i) = 1/2$, $e_2 = 1/2$, $e_3 = 0$. Thus we have $Q_0 = (0 : 0 : 0 : 1)$, $Q_1 = (4 : -1 : 2 : 0)$, $Q_2 = (4 : -1 : -2 : 0)$ and $Q_3 = (4 : 1 : 0 : 0)$. Let $\ell_1$ and $\ell_2$ are $Z_4$-lines and $G_{\ell_1} = \langle \tau_1 \rangle$ and $G_{\ell_2} = \langle \tau_2 \rangle$. If $\ell_1$ and $\ell_2$ meet, then we have $\tau_1^2 = \tau_2^2$ by Lemma 13. Letting $\tau_1(z) = iz + \alpha_1$ and $\tau_2(z) = iz + \alpha_2$, we have $(1+i)(\alpha_1 - \alpha_2) \in L$. Denote by $\ell(m, n)$ the line corresponding to the group $(\tau)$ by the arrangement-map $\rho$, where $\tau(z) = iz + (m + ni)/4$. The following assertion is easy to see.

Claim 1. Putting $\sigma_{mn}(z) = iz + (m + ni)/4$ and $G_{mn} = \langle \sigma_{mn} \rangle$, we have $G_0 \cap G_{22} = \langle \sigma_0 \rangle$, $G_20 \cap G_{02} = \langle \sigma_3 \rangle$, $G_{11} \cap G_{33} = \langle \sigma_2 \rangle$ and $G_{31} \cap G_{13} = \langle \sigma_1 \rangle$.

Claim 2. The intersections of the eight $Z_4$-lines are $\ell(0, 0) \cap \ell(2, 2) = Q_0$, $\ell(2, 0) \cap \ell(0, 2) = Q_3$, $\ell(1, 1) \cap \ell(3, 3) = Q_2$ and $\ell(3, 1) \cap \ell(1, 3) = Q_1$.

Proof. The intersection points are found by Lemma 17. For example, the point $\ell(1, 1) \cap \ell(3, 3)$ is found as follows: Since $G_{11} \cap G_{33} = \langle \sigma_2 \rangle$, the point is the intersection of two lines

\[ c_3X - Y + 2e_1Z = W = 0 \quad \text{and} \quad c_1X - Y + 2e_1 = W = 0, \]

where $e_1 = 1/2$, $e_3 = 0$ and $c_1 = 1/4$, $c_3 = -1/4$. So it is $Q_2$. □

Now, we prove Corollary 1. Let $E$ be the Weierstrass normal form of the normalization of $\Gamma$ and let $\mu : E \to \Gamma \subset \mathbb{P}^3$ be the normalization morphism. Put $D = \mu^*(L)$ for a line $L$ in $\mathbb{P}^2$. By Lemma 10 we can assume $C$ is given by the embedding by $|4P_0|$. Therefore, $\Gamma$ is regained as $\pi_P(C)$, where $\pi_P : \mathbb{P}^3 \dashrightarrow \mathbb{P}^2$ is the projection with center $P$. Suppose $\Gamma$ has two Galois points $Q_1$ and $Q_2$. Then, letting $\ell_1 = \pi_P^*(Q_1)$ and $\ell_2 = \pi_P^*(Q_2)$, they are Galois lines for $C$ and $\ell_1 \cap \ell_2 = \{P\}$. However, as we have seen Remark 14 the projection $\pi_P$ induces a 2 to 1 morphism from $C$ to $\Gamma$ and $\pi_P(C)$ is a rational curve, this is a contradiction. On the other hand, if $P$ lies in one of the Galois lines, i.e., $P \in \ell$ and is not the vertex, then $\pi_P$ induces a birational transformation on $C$ by Remark 3 and $\pi_P(\ell \setminus \{P\})$ is a Galois point for $\Gamma = \pi_P(C)$.

Finally, we mention Remark 6. Take a point $Q \in \mathbb{P}^3$ which does not lie on the Galois lines. Then, the curve $\Gamma = \pi_Q(C)$ is a quartic curve with no Galois point. Because, by Remark 3 it is birational to $C$. Suppose it has a Galois point. Then, there exists a smooth quartic curve $C'$ in $\mathbb{P}^3$ and a Galois line $\ell'$ and a point $P' \in \mathbb{P}^3$ satisfying that $\pi_{P'}(C') = \Gamma$. Moreover, there exists an isomorphism $\varphi : \mathbb{P}^3 \dashrightarrow \mathbb{P}^3$ such that $\varphi(C') = C$ and $\varphi(\ell')$ coincides with some Galois line for $C$. Since $\ell' \ni P'$, we have $\varphi(\ell') \nRightarrow P$, which is a contradiction.

Thus we complete all proofs.
Problem. We ask the following questions concerning Galois embedding of elliptic curves.

(a) In case $\ell$ is not a Galois line, consider the Galois group $G$ of the Galois closure curve [5, Definition 1.3]. If $\ell$ is general, then the Galois group is a full symmetric group [5, Theorem 2.2], see also [4]. So we ask if $\ell$ is neither general (i.e., $G \not\sim S_4$) nor Galois, then what group can appear. For the group which appears, how are the arrangements of the lines with the group?

(b) Let $D$ be a divisor of degree $d \geq 5$ on $E$. Then, study the Galois embedding by $|D|$. In particular, consider the Galois group and the arrangement of Galois subspaces ([6]).

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