Robust stability of a class of differential systems with state after-effect dynamics

M. De la Sen
Institute of Research and Development of Processes, P.O. Box 644 of Bilbao, ,Spain
E-mail: manuel.delasen@ehu.eus

Abstract. This paper is concerned with the investigation of the global stability and global asymptotic stability of the error with respect to its nominal version of a non-linear time-varying perturbed functional differential system which is influenced by point, finite-distributed and Volterra-type distributed delayed dynamics. The boundedness of the error and its asymptotic convergence to the zero equilibrium are investigated and some formal “ad-hoc” results are proved.

1. Introduction

The significant research existing nowadays concerned with the problems of global stability of time-delay dynamic systems and their robustness extended issues is nowadays extensive and a variety different problems have been solved with it under the basis that there is a perturbation of a nominal equation and that a norm upper-bounding function of the error is obtained, [1-4]. This paper is concerned with the study of solutions of perturbed time-delay differential systems and their comparison and asymptotic properties of convergence to those of the corresponding unperturbed ones. The differential systems dealt with involve in a combined fashion linear dynamics of point-delays, finitely distributed time-delays and infinitely-distributed Volterra-type delays [5] as well as the presence of perturbations involving nonlinear dynamics depending on delays, in general. In the stability robustness problem focused on through this paper, the number of delays of the perturbed equation and that of its nominal version might be distinct and the matrices describing the linear delayed and delay-free dynamics of both differential systems might be also distinct. Note that the stability of dynamic systems under the presence of nonlinearities and delayed dynamics is a very important issue in theory and applications of the field of control systems [1-8]. The basic problem focused on in this manuscript is the asymptotic convergence of the error between both solutions to zero irrespective of the stability properties of the nominal differential system. The system with delayed dynamics is of a very general structure since it potentially includes in a combined way delay-free dynamics, dynamics under point delays, and finitely and infinitely distributed delayed dynamics.

2. Problem framework

1 To whom any correspondence should be addressed.
Let us consider the following functional \( n \)-th order differential system with point and, in general, both infinite-type Volterra-type as well as finite distributed delays, \([5]\) and includes the contributions of both structured and unstructured stately delayed dynamics:

\[
\dot{x}(t) - Lx_t - g(t) = 0
\]

where:

\[
Lx_t = L_0 x_t + f(t, x_{t-\theta})
\]

\[
L_0 x_t = A_0 x(t) + \sum_{i=1}^{m} A_i(x(t-h_i) + \sum_{i=0}^{m_0} \int_{t-h_i}^{t} d\alpha_i(\tau) A_{\alpha_i}(\tau) x(\tau-h_i) + \sum_{i=m+1}^{m_0+m'} \int_{t-h_i}^{t} d\alpha_i(t-\tau) A_{\alpha_i}(\tau) x(\tau-\theta_0) + g_0(t)
\]

\[
f(t, x_{t-\theta}) = f_0(t, x_{t-\theta}) + f_1(t, x_{t-\theta}) ; \quad g(t) = g_0(t) + \tilde{g}(t)
\]

\[
f_1(t, x_{t-\theta}) = \sum_{i=1}^{m} \tilde{A}_i(t) x(t-h_i) + \sum_{i=0}^{m'} \int_{0}^{t} d\alpha_i(\tau) \tilde{A}_{\alpha_i}(\tau) x(t-h_i) + \sum_{i=m+1}^{m'+m''} \int_{t-h_i}^{t} d\alpha_i(t-\tau) \tilde{A}_{\alpha_i}(\tau) x(\tau-\theta_3)
\]

\( x(t) = \phi(t); t \in [-\theta, 0]\) for any given initial condition vector function \( \phi \in C_\theta (-\theta) \), is the Banach space of continuous functions from \([-\theta, \infty)\) into \( C^n \) endowed with the supremum norm, where \( \theta, \theta_0, \theta_1, \theta_2, \theta_3 \in R_+ \), \( \theta_\alpha = \max(\theta_1, \theta_2) \), \( \theta_\beta = \max(\max(\theta_h), \max(\theta_h')) \) and

\[
\theta = \max \left( \max_{1 \leq i \leq m} \theta_h, \max_{1 \leq i \leq m + m''} \theta_h', \theta_0, \theta_1, \theta_2, \theta_3 \right).
\]

The objective of this research is the comparison of the solution to that of its nominal version \( \dot{x}(t) - L_0 x_t - f_0(t, x_{t-\theta}) - g_0(t) = 0 \). The following hypothesis are made:

(1) \( L : C_\theta (R_{-\theta}) \to C^n \) is a bounded linear functional defined by the right hand-side of (1)

(2) \( h_k \) and \( h'_k \) (\( k=1,2,\ldots,m; \ell=0,1,\ldots,m'+m'' \)) are nonnegative real point delays, infinite-time distributed Volterra-type delays (i.e. the first \( m \) distributed delays) and finite time-interval distributed delays with \( h_0 = h'_0 = 0 \) and \( h = \max \left( \max_{1 \leq i \leq m} h_i, \max_{1 \leq i \leq m + m''} h'_i \right) \leq \theta \) such that \( m \geq m_0 \), \( m' \geq m_0' \) and \( m'' \geq m_0'' \), where the 0-subscripts stand for the nominal equation.

(3) \( g_0 : R_0 \to C^n \) is piecewise-continuous, \( f_0 : R_0 \times C(R_{-\theta}) \to C^n \) describes a perturbed linear dynamics, and \( x_{\theta} : C(R_{-\theta}) \to x(t) = \phi(t), \forall t \in [-\theta,0], \) is a string of the solution of (1).
3. Example of solution construction for a single-delayed linear time-invariant system

Consider the single-input single-output linear time-invariant system

\[ S_h : \dot{x}(t) = A x(t) + A_0 x(t-h) + bu(t) \]

where \( x(t) \) is the state n-vector and \( u(t) \) are, respectively, the state vector and the scalar control input signal and \( h \geq 0 \) is the point delay. If \( h > 0 \), the above system possesses an state delayed dynamics. The real square n-matrices \( A \) and \( A_0 \) are associated with the delay-free and delayed dynamics, respectively and \( b \) is a n dimensioned real control vector. Particular delay-free systems which lie in the above class are:

\[ S_0 : \dot{x}(t) = (A + A_0) x(t) + bu(t) \]
\[ S_\infty : \dot{x}(t) = A x(t) + bu(t) \]

Note that the auxiliary delay-free systems \( S_0 \) and \( S_\infty \) are obtained from (1) for \( h = 0 \) and \( h = \infty \), respectively. The following result holds:

**Proposition 1:** For any absolutely continuous function of initial conditions \( \varphi : [-h, 0] \rightarrow \mathbb{R}^n \) with \( \varphi(0) = x(0) = x_0 \) and any piecewise-continuous control input, the state-trajectory solution is unique and given by any of the equivalent expressions below:

\[
x(t, \varphi, u) = e^{At} x_0 + \int_{-h}^{0} e^{A(t-h-\tau)} A_0 \varphi(\tau) d\tau + \int_{0}^{t-h} e^{A(t-h-\tau)} A_0 I (\tau-h) x(\tau) d\tau + \int_{0}^{t} e^{A(t-\tau)} b u(\tau) d\tau
\]

\[
= \Psi_h(t)x_0 + \int_{-h}^{0} \Psi_h(t-h-\tau) A_0 \varphi(\tau) d\tau + \int_{0}^{t} \Psi_h(t-\tau) b u(\tau) d\tau
\]

where \( I(t) \) is the unity Heaviside function with discontinuity at \( t = 0 \), \( e^{At} \) is a \( C_0 \)-semigroup (popularly known as the state transition matrix) associated with the infinitesimal generator \( A \) of \( S_\infty \), which satisfies \( d(e^{At})/dt = Ae^{At} \), and \( \Psi_h(t) \) is the evolution operator of \( S_h \) for any \( h \geq 0 \) which satisfies

\[ \Psi_h(t) = A \Psi_h(t) + A_0 \Psi_h(t-h) \]

and is explicitly defined by

\[
\Psi_h(t) = e^{At} \left[ I + \int_{0}^{t} e^{-A\tau} A_0 \Psi_h(\tau-h) I (\tau-h) d\tau \right] = \begin{cases} 0 & \text{for } t < 0 \\ e^{At} & \text{for } t \in [0, h] \\ e^{At} \left( I + \int_{0}^{t} e^{-A\tau} A_0 \Psi_h(\tau-h) I (\tau-h) d\tau \right) & \text{for } t \geq h 
\end{cases}
\]

where \( I \) is the n-identity matrix. The proof of this result is made by direct verification.

The subsequent global stability result for the general differential system under consideration can be directly proved after some manipulations:
Theorem 1. Assume that \( \bar{g} \equiv 0 \) and \( f_0 : R_{0+} \times C^n \rightarrow C^n \) is sub-additive with
\[
\left\| f_0(t, x - \bar{g}) \right\| \leq K_0 \sup_{0 \leq \delta \leq h} \left\| x - \bar{g} \right\| + K_0 \quad \text{for some } K_0, K_0 \in R_{0+}; \forall t \in R_{0+}.
\]
Let \( \mu(t_n) = \{ \mu_1, \mu_2, \ldots, \mu_m \} \)
be any \( m_0 \)-tuple defined from a piece-wise constant \( m_0 \)-binary vector function
\( \mu : \{ t_n, t_{n+1} \} \times Z_{0+} \rightarrow \{ \mu_1(t_n), \ldots, \mu_m(t_n) \} \)
for any combination of values of the set of binary variables \( \mu_j(t_n) \in [0, 1]; \forall i \in \bar{m}_0 = \{ 1, 2, \ldots, m_0 \} \).
Define a fundamental matrix of the nominal unforced differential system \( x(t) = L_0 x_t \) of the form:
\[
\psi_{\mu(t_n)}(t, t_n) = e_0^{A_0(h - t_n)} \left( I + \sum_{i=1}^{m_0} \mu_i(t_n) \right) e_0^{A_0(h + r(t))} A_0(r) \psi_{\mu(t_n)}(r - h) U(\tau - h) d\tau \quad \forall t \in \left[ t_n, t_{n+1} \right)
\]
with initial conditions \( \psi_{\mu(t_n)}(t, t_n) = I \); \( \forall t \in R_0 \) and \( \psi_{\mu(t_n)}(t, t_n) = 0 \) for \( t < t_n \),
where \( I \) is the n-identity matrix and \( U(t) \) is the Heaviside function, which satisfies the differential system:
\[
\psi_{\mu(t_n)}(t, t_n) = A_0 \psi_{\mu(t_n)}(t, t_n) + \sum_{i=1}^{m_0} \mu_i(t_n) A_0(t) U(t - h) \psi_{\mu(t_n)}(t - h, t_n)
\]
\( \forall t \in \left[ t_n, t_{n+1} \right), \forall n \in Z_{0+} \).

Then, the subsequent properties hold:

(i) The error norm in-between the current solution and the nominal one on \( \left[ t_n, \infty \right); \forall n \in Z_{0+} \), is upper-bounded by a prescribed positive norm bound \( E \) if \( \| e(t_0) \| \leq E \), the fundamental matrix defined from a binary vector function \( \mu : \{ t_n, t_{n+1} \} \times Z_{0+} \rightarrow \{ \mu_1(t_n), \ldots, \mu_m(t_n) \} \) fulfills for a matrix induced vector norm \( \| \psi_{\mu(t_n)}(t, t_n) \| \leq \psi_{\mu(t_n)}(t, t_n) < 1 \); \( \forall t \in \left[ t_n, t_{n+1} \right), \forall n \in Z_{0+} \), satisfied function \( \psi_{\mu(t)}(\cdot) : Z_{0+} \times R_{0+} \rightarrow R_{0+} \) satisfies \( \psi_{\mu(t)}(t, t_n) = 1 \); \( \forall n \in Z_{0+} \), and the subsequent constraint holds for some sufficiently small real constant \( T^* \in R_+ \):

\[
t - t_n \leq \max_{t_n \leq \tau \leq t_{n+1}} \left( 1 - \psi_{\mu(t_n)}(t, t_n) \right) \left( \sum_{i=1}^{m_0} (1 - \mu_i(t_n)) A_0(\tau) + \sum_{i=0}^{m_0} \int_{t_n}^{t_{n+1}} d\sigma \alpha_i(\tau - \sigma) A_0(\sigma) + \sum_{i=m_0}^{m_0 + m_1} \int_{t_n}^{t_{n+1}} d\sigma (\tau - \sigma) A_0(\sigma) + f(\tau, e, t_n) \right)
\]
\( \forall t \in \left[ t_n, t_{n+1} \right), \forall n \in Z_{0+} \) (1)

for some strictly increasing sequence \( \{ t_0, t_1, \ldots \} \) satisfying:
\[
T^* \leq T_n = t_{n+1} - t_n \leq \min \{ h^*(t_n), \max\{ t_n \in R_+ : r_n = z_n \} \} \quad \forall n \in Z_{0+}
\]

where:
\[ z_n \leq \frac{1 - \psi_{\mu(t_n)}(t_n, t_n + z_n)}{\psi_{\mu(t_n)}(t_n, t_n + z_n)} \max_{t_n \leq t < t_{n+1}} \left[ \sum_{i=1}^{m_0} (1 - \mu_i(t_n)) A_i(\tau) + \sum_{i=0}^{m_0 + m_0} \int_{t=m_0+1}^{\tau-h_i} d\alpha_i(\tau - \sigma) A_{\alpha_i}(\sigma) + f(\tau, x_{t-\theta}) \right] \]

; \forall n \in \mathbb{Z}_{0^+} \quad (2)

\[ h^*(t_n) := \min \left( \min_{1 \leq i \leq m_0} (1 - \mu_i(t_n)) h_i, \min_{m_0+1 \leq i \leq m} h_i, h'_i \right) \; ; \forall n \in \mathbb{Z}_{0^+} \]

\[ \forall \phi \in \mathcal{C}_c \left(-\theta, t_0 \right) := \mathcal{C} \left(-\theta, t_0 \right) \; \mathbb{C}^n \right) \quad \text{with} \quad \theta = \max \left( \max_{1 \leq i \leq m} \left( h_i, \max_{1 \leq i \leq m + m'} \left( h_i', \theta_0, \theta_1, \theta_2, \theta_3 \right) \right) \right), \]

with \( x(t_0) = \phi(t_0) \) for any given \( t_0 \in \mathbb{R}_{0^+} \). If \( \sup_{0 \leq t \leq \theta} \| e(t) \| \leq E \) for any given absolutely continuous vector function of initial conditions \( \varphi : \left[-\theta, 0 \right] \rightarrow \mathbb{C}^n \) then \( \sup_{0 \leq t < \infty} \| e(t) \| \leq E \).

Theorem 1(i) holds if the upper-bound of (8) is replaced with the subsequent one:

\[ 1 - K_0 - \psi_{\mu(t_n)}(t_n, t) \]

\[ \max_{t_n \leq t < t_{n+1}} \left[ \psi_{\mu(t_n)}(t_n, t) \right] \left( K_1 + \sum_{i=1}^{m_0} (1 - \mu_i(t_n)) A_i(\tau) + \sum_{i=0}^{m_0 + m_0} \int_{t=m_0+1}^{\tau-h_i} d\alpha_i(\tau - \sigma) A_{\alpha_i}(\sigma) \right) \]

; \forall t \in \left[ t_n, t_{n+1} \right), \forall n \in \mathbb{Z}_{0^+} \quad (3)

provided that \( \psi_{\mu(t_n)}(t_n, t) < 1 - K_0 \) with \( K_0, K_1 \in \mathbb{R}_+ \) being such that \( \| f(t, x_{t-\theta}) \| \leq K_0 \sup_{0 \leq t \leq \theta} \| e(t) \| + K_1 ; \forall t \in \mathbb{R}_{0^+} \).

(ii) Assume that Property (i) holds by

a) replacing the constraint \( \| \psi_{\mu(t_n)}(t_n, t) \| \leq \psi_{\mu(t_n)}(t_n, t) < 1 \) with \( \| \psi_{\mu(t_n)}(t_n, t) \| \leq \psi_{\mu(t_n)}(t_n, t) < K_n ; \forall t \in \mathbb{R}_{0^+} \).

b) replacing \( E \) by a nonnegative real sequence \( \left\{ E_n \right\} \) satisfying \( E_{n+1} = K_n E_n \) for each \( n \in \mathbb{Z}_+ \) and some decreasing sequence of nonnegative real numbers \( \left\{ K_n \right\} \) such that \( \lim_{n \rightarrow \infty} K_n = K \) for some real constant \( K \in [0, 1] \).

c) replacing “1” in the numerator of (1) and (2) by \( K_n ; \forall n \in \mathbb{Z}_+ \).

Then, \( \psi_{\mu(t_n)}(t_n, t) \rightarrow 0 \) exponentially as \( t \rightarrow \infty \).

A particular stability result for the perturbed system under the stability of the nominal one follows:
**Theorem 2.** Assume that the nominal differential system is globally exponentially stable and \( \tilde{g} : R_{0+} \rightarrow R^n \) is either bounded or integrable or square-integrable. Then, a fundamental matrix \( \Psi_{\mu(t)}(t, t) \) exists such that Theorem still holds for some \( E > \left\| \int_{t_n}^{t} \Psi_{\mu(t)}(t, \tau) \tilde{g}(\tau) d\tau \right\| \) if the needed denominator of (3) is, in each case, corrected with the additive term \( \left( \left\| \int_{t_n}^{t} \Psi_{\mu(t)}(t, \tau) \tilde{g}(\tau) d\tau \right\| \right)^{1/2} \) or with any of its upper-bounds:

\[
\max_{t_n \leq \tau \leq t} \left\| \int_{t_n}^{t} \tilde{g}(\tau) d\tau \right\| ; \quad \max_{t_n \leq \tau \leq t} \left\| g(\tau) \right\| \left\| \int_{t_n}^{t} \Psi_{\mu(t)}(t, \tau) d\tau \right\| ; \quad \left( \left\| \int_{t_n}^{t} \Psi_{\mu(t)}(t, \tau) d\tau \right\| \right)^{1/2} \left( \left\| \int_{t_n}^{t} \Psi_{\mu(t)}(t, \tau) d\tau \right\| \right)^{1/2} .
\]

The so-called Roesser two-dimensional models are difficult to discuss since their horizontal and vertical substates evolve with two discrete arguments but without following recursiveness rules. A two-dimensional discrete linear Roesser model with internal point delays \( d_1, d_2 \in Z_{0+} \) for the states \( x_h(\ldots) \) and \( x_v(\ldots) \), respectively, with \( \max(d_1, d_2) \geq 1 \) generalized from (1)-(3) is the following one:

\[
x_h(n_1 + 1, n_2) = A_{hh} x_h(n_1, n_2) + A_{hv} x_v(n_1, n_2) + A_{hd} x_h(n_1, n_2 - d_2) + B_h u(n_1, n_2)
\]
\[
x_v(n_1, n_2 + 1) = A_{vh} x_h(n_1, n_2) + A_{vv} x_v(n_1, n_2) + A_{vd} x_v(n_1, n_2 - d_2) + B_v u(n_1, n_2)
\]
\[
y(n_1, n_2) = C_h x_h(n_1, n_2) + C_v x_v(n_1, n_2) + D u(n_1, n_2)
\]

where \( n_1 \) and \( n_2 \) are non-negative integers taking account the discretization evolution progress. The above equations in the given order refer to the horizontal and vertical coupled sub-states and the output and \( u(.) \) is the control input with appropriate dimensionalities of all the matrices of parameters taking account of the delay contributions. The whole state dimension \( n = n_h + n_v \) is the sum of the dimensions of the two sub-states. The discrete arguments \( n_1 \) and \( n_2 \) are not necessarily related to a discrete model got from a continuous-time one. The following stability result can be proved.

**Theorem 3.** The following properties hold:

(i) Assume that \( A = \begin{bmatrix} A_{hh} & A_{hv} \\ A_{vh} & A_{vv} \end{bmatrix} \in R^{n \times n} \) is a convergent matrix with convergence abscissa \( \rho \) and that \( \rho + \mu_2(A_d) < 1 \), where \( A_d = \begin{bmatrix} A_{hh} & A_{hd} \\ A_{vd} & A_{vv} \end{bmatrix} \in R^{n \times n} \) and \( \mu_2(A_d) \) is its \( \ell_2 \) measure matrix. Assume also that the initial conditions \( \text{Inc} = \{x_h(0, n_1 + d_1), x_v(n_2 + d_2, 0); n_1, n_2 \in Z_{0+}\} \) of (4)-(6) are absolutely summable for any given \( n_1, n_2 \in Z_{0+} \). Then, the state and output sequences of the unforced system (i.e. zero input) (1)-(3) are bounded and \( \left[ x_h(n_1 + 1, n_2) \right] \rightarrow 0 \), \( \left[ x_v(n_1, n_2 + 1) \right] \rightarrow 0 \), \( y(n_1, n_2) \rightarrow 0 \) as \( \max(n_1, n_2) \rightarrow \infty \). In particular, the result holds if the initial set \( \text{Inc} \) has only a finite number of nonzero members.
(ii) Assume that the system is delay-free. Then, Property (i) holds under the condition $\rho < 1$.

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