Fractals and Symbolic Dynamics as Invariant Descriptors of Chaos in General Relativity

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The study of dynamics in general relativity has been hampered by a lack of coordinate independent measures of chaos. Here I review a variety of invariant measures for quantifying chaotic dynamics in relativity that exploit the coordinate independence of fractal dimensions and symbolic entropies.

1 Time and chaos

Historically, chaos theory was developed for Newtonian dynamics where time and space are absolute and the notion of a mechanical phase space is clear. In contrast, both space and time are dynamical and intermixed in general relativity. There is no such thing as the time direction. The fundamentally different role played by time in relativity and Newtonian mechanics manifests itself in the coordinate, or gauge, dependence of chaotic measures such as Lyapunov exponents. Lyapunov exponents quantify a system’s sensitive dependence on initial conditions. If two initially close trajectories separate along a given eigendirection in phase space such that the separation \( \varepsilon(t) \) grows as \( \varepsilon(t) = \varepsilon_0 e^{\lambda t} \), then \( \lambda \) represents the Lyapunov exponent along that direction. If \( \lambda > 0 \) for a set of trajectories with non-zero measure, the system is said to exhibit sensitive dependence on initial conditions with a characteristic chaotic, or Lyapunov, timescale \( T_L = 1/\lambda \). Unfortunately, this nice picture breaks down when applied to general relativity. Consider the allowed coordinate transformation \( t \rightarrow \ln \tau \). In terms of this time variable we find \( \varepsilon(\tau) = \varepsilon_0 \tau^\lambda \), which describes the standard power-law divergence of trajectories found in integrable system. In particular, the Lyapunov exponents in this coordinate system would all be zero. It should be mentioned that the Lyapunov exponents also depend on the choice of distance measure in phase space and are therefore variant under spatial coordinate transformations also. From the above discussion it is clear that standard coordinate dependent measures of chaos have to be either modified, abandoned or augmented in general relativity. This problem has now been solved in a series of papers which introduced and illustrated the effectiveness of fractal dimensions and symbolic codings as invariant descriptors of chaos in general relativity. Central to both of these methods is the concept of a chaotic invariant set of orbits. It is interesting to note that the problem of describing chaos in quantum mechanics was also solved by focusing on periodic orbits. In this talk I review these developments and illustrate the methods by applying them to photon orbits in a binary black hole system.

2 Invariant methods

In order to be precise, I will begin by stating the definitions and theorems that underpin our approach. I will then illustrate the ideas pictorially for photon trajectories.
tories in the field of two fixed extremal black hole\textsuperscript{5}.

**Definition:** A dynamical system will be called chaotic if it contains a *chaotic invariant set* of unstable periodic orbits. This set is characterised by having a non-zero topological entropy and non-integer fractal dimensions. Conversely, the system is integrable if there exist sufficient constants of motion to restrict all trajectories to smooth tori in phase space.

**Definition:** A future invariant set forms the fractal boundary between possible outcomes of a scattering system. The set consists of trajectories that are future asymptotic to unstable periodic orbits.

**Theorem:** (see eg. Barnsley\textsuperscript{8})

Fractal dimensions are invariant under coordinate transformations that are one to one, onto and Lipschitz ($C^1$), *ie.* bounded deformations with no rips, tears, folds or infinite stretching. Thus, fractal dimensions are diffeomorphism invariant.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1.png}
\caption{(i) Outcomes for an incident photon (ii) Outcome basins with fractal boundaries.}
\end{figure}

The procedure we adopt is very simple. First we identify the possible outcomes for the dynamics. Figure (1.i) shows the four outcomes that can occur when a photon is fired into the field of two fixed, extremal black holes. The photon is either captured by one of the black holes, or it escapes and scatters up or down. The outcomes are each assigned a different colour and the initial conditions are colour coded according to their outcome. The result of applying this procedure to a $840 \times 840$ grid of initial conditions is shown in Fig. (1.ii). The boundaries between the various outcomes form a fractal future invariant set with capacity dimension $D_0 = 1.36 \pm 0.02$. Since the set has a non-integer capacity dimension, we can conclude in a coordinate invariant way that photon dynamics in this spacetime is chaotic.

An alternative way of showing that a system is chaotic is to find a symbolic coding for all the periodic orbits and then show that the coding is complex. The philosophy behind this approach being that chaos is a global phenomenon, so it can be studied by reducing the detailed local information required to specify a trajectory down to a discrete set of symbols. This is accomplished by dividing
phase space into a finite number of partitions and labelling each partition with a symbol. Trajectories are then represented by strings of these symbols.

\[ N(k + 1) = 3N(k) \]

A symbolic coding is said to be grammatically complex if the number of orbits, \( N(L) \), with period less than or equal to \( L \) grows exponentially. The growth in the number of orbits is measured by the topological entropy:

\[ H_T = \lim_{L \to \infty} \frac{L}{- \ln N(L)} \]

For photon orbits in the field of two extremal black holes we find \( H_T = \ln 3 \), so the dynamics is chaotic.

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