Design-theoretic analogies between codes, lattices, and vertex operator algebras

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Abstract
There are many analogies between codes, lattices, and vertex operator algebras. For example, extremal objects are good examples of combinatorial, spherical, and conformal designs. In this study, we investigated these objects from the aspect of design theory.

Keywords Codes · Lattices · Vertex operator algebras · Combinatorial designs · Spherical designs · Conformal designs

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1 Introduction
In this study, we investigated the analogy between codes, lattices, and vertex operator algebras (VOAs), with regard to design theory. To explain our results, we review some of the previous studies conducted on codes, lattices, and VOAs.

First, we review codes and their combinatorial designs. Let $C$ be a doubly even self-dual code of length $n = 24m + 8r$. Then, its minimum weight would satisfy the following equation:

$$\min(C) \leq 4 \left\lfloor \frac{n}{24} \right\rfloor + 4.$$

(1)

We say that $C$ meeting the bound (1) with equality is extremal. Let $C$ be an extremal code of length $n = 24m + 8r$, and let us set

$$C_\ell := \{ c \in C \mid \text{wt}(c) = \ell \}.$$
Then, any $C_\ell$ forms a combinatorial $t$-design, where

$$t = \begin{cases} 
5 & \text{if } n \equiv 0 \pmod{24}, \\
3 & \text{if } n \equiv 8 \pmod{24}, \\
1 & \text{if } n \equiv 16 \pmod{24} 
\end{cases}$$

[2].

Let $C$ be a doubly even self-dual code of length $n = 24m$ with $\min(C) = 4m$. $C$ has the largest minimum weight except for the extremal cases. Any shell of such code $C$ ($C_\ell$) forms a combinatorial 1-design [24].

Then, we review lattices and their spherical designs. Let $L$ be an even unimodular lattice of rank $n = 24m + 8r$. It was shown in [19] that its minimum norm satisfies the following equation:

$$\min(L) \leq 2 \left\lfloor \frac{n}{24} \right\rfloor + 2. \quad (2)$$

We say that $L$ meeting the bound (2) with equality is extremal. Let $L$ be an extremal lattice of rank $n = 24m + 8r$, and let us set

$$L_\ell := \{ x \in L \mid (x, x) = \ell \}.$$ 

Then, any $L_\ell$ forms a spherical $t$-design, where

$$t = \begin{cases} 
11 & \text{if } n \equiv 0 \pmod{24}, \\
7 & \text{if } n \equiv 8 \pmod{24}, \\
3 & \text{if } n \equiv 16 \pmod{24} 
\end{cases}$$

[30] (see also [26]).

Let $L$ be an even unimodular lattice of rank $n = 24m$ with $\min(L) = 2m$. $L$ has the largest minimum norm except for the extremal cases. It is known that any shell of such lattice $L$ ($L_\ell$) forms a spherical 3-design [24].

Finally, we review VOAs and their conformal designs. Let $V$ be a holomorphic VOA of central charge $n = 24m + 8r$. It has been shown in [16] that its minimum weight satisfies the following equation:

$$\min(V) \leq \left\lfloor \frac{n}{24} \right\rfloor + 1. \quad (3)$$

We say that $V$ meeting the bound (3) with equality is extremal. Let $V$ be an extremal VOA of central charge $n = 24m + 8r$. $V$ is graded by $L(0)$-eigenvalues as follows:

$$V = \bigoplus_{n \in \mathbb{Z}} V_n.$$ 

Then, any $V_\ell$ forms a conformal $t$-design, where

$$t = \begin{cases} 
11 & \text{if } n \equiv 0 \pmod{24}, \\
7 & \text{if } n \equiv 8 \pmod{24}, \\
3 & \text{if } n \equiv 16 \pmod{24} 
\end{cases}$$

[17]. (For the detailed expressions of the minimum weights and the conformal $t$-designs, see [16] and [17].)

Let $V$ be a holomorphic VOA of central charge $n = 24m$ with $\min(V) = m$. $V$ has the largest minimum weight except for the extremal cases. Considering this, the question that arises is whether any shell of such VOA $V$ ($V_\ell$) forms a conformal 3-design.

The first major finding of this study is as follows:
Let \( T \) be an even unimodular lattice of rank \( 8 \), \( \ell \) is, \( T_2 = \{ 1, 3, 5, \ldots \} \). (See Sect. 2.3 for the definition of spherical \( T \)-designs.) The second major finding is as follows:

**Theorem 1.2** (1) Let \( C \) be a doubly even self-dual code of length \( n = 24m + 8r \). Then, any \( C_\ell \cup C_{n-\ell} \) forms a combinatorial \( T_2 \)-design with 2-weight. Further, any \( C_\ell \cup C_{n-\ell} \) forms a combinatorial 1-design with 2-weight.

(2) Let \( T_2 \) be the set of positive odd numbers, That is, \( T_2 = \{ 1, 3, 5, \ldots \} \). Then, any non-empty homogeneous space of a holomorphic VOA forms a conformal \( T_2 \)-design.

The third purpose of the present paper is as follows: Let \( L \) be an even unimodular lattice of rank 8, 16, 24. Then the following holds: \( L_\ell \) is a spherical \( T \)-design with

\[
T = \begin{cases} 
\{1, 2, 3, 4, 5, 6, 7, 9, 10, 11\} \cup T_2 & \text{if } \ell = 1 \\
\{1, 2, 3, 5, 6, 7\} \cup T_2 & \text{if } \ell = 2 \\
\{1, 2, 3\} \cup T_2 & \text{if } \ell = 3. 
\end{cases}
\]

The third major finding is as follows:

**Theorem 1.3** Let \( V \) be a holomorphic VOA of central charge \( c = 8\ell \), with \( \ell = 1, 2, 3 \). Then, any non-empty homogeneous space of \( V \) forms a conformal \( T \)-design, with

\[
T = \begin{cases} 
\{1, 2, 3, 4, 5, 6, 7, 9, 10, 11\} \cup T_2 & \text{if } \ell = 1 \\
\{1, 2, 3, 5, 6, 7\} \cup T_2 & \text{if } \ell = 2 \\
\{1, 2, 3\} \cup T_2 & \text{if } \ell = 3. 
\end{cases}
\]

All the homogeneous spaces are conformal 3-designs if \( c \leq 24 \). Moreover, all the homogeneous spaces are conformal 7-designs if \( c = 8 \).

**Remark 1.1** The case \( \ell = 1 \) and \( \ell = 2 \) in Theorem 1.3 have also been mentioned in a remark after Theorem 3.1 of [17]. The proof is essentially the same as [17] with a minimal modification.

The fourth purpose of this study slightly differs from the above three findings. A homogeneous space of VOA \( V_\ell \) has a strength \( t \) if \( V_\ell \) is a conformal \( t \)-design but is not a conformal \( (t + 1) \)-design. We define the concept of strength \( t \) for the spherical \( t \)-designs and the combinatorial \( t \)-designs.

The fourth purpose of this study is to provide other examples for which the strength can be determined (Theorem 1.4 (1)).

We also present other interesting examples of conformal designs. All the known examples of conformal designs \( V_\ell \) have the same strength for each \( \ell \). This leads to the question of whether there are conformal designs \( V_\ell \) for which the strengths are different for each \( \ell \). In the final part of this paper, we give examples for this (Theorem 1.4 (2)).

**Theorem 1.4** (1) Let \( L \) be an even unimodular lattice of rank 24. Then, all the homogeneous spaces \( (V_\ell)_L \) have strength 3.

(2) Let \( L \) be an even unimodular lattice of rank 16. We use \( \text{ord}_p(\ell) \) to denote the number of times that a prime \( p \) occurs in the prime factorization of a non-zero integer \( \ell \). If
ord_\(p(3\ell - 2)\) is odd for some prime \(p \equiv 2 \pmod{3}\), then all the homogeneous spaces \((V_L)_\ell\) have strength 3. Otherwise, the homogeneous spaces \((V_L)_\ell\) are conformal 7-designs.

**Remark 1.2** It is generally difficult to determine the strength of \(C_\ell\), \(L_\ell\), and \(V_\ell\). For example, in [22,30], the following theorem was shown:

**Theorem 1.5** ([22, Theorem 1.2]) Let \(E_8\) be the \(E_8\)-lattice and \(V^\natural\) be the moonshine VOA. Let \(\tau(i)\) be Ramanujan’s \(\tau\)-function:

\[
\Delta(z) = \eta(z)^{24} = \left( q^{1/24} \prod_{i \geq 1} (1 - q^i) \right)^{24} = \sum_{i} \tau(i) q^i,
\]

where \(q = e^{2\pi i z}\). Then, the followings are equivalent:

1. \(\tau(\ell) = 0\).
2. \((E_8)_{2\ell}\) is a spherical 8-design.
3. \((V^\natural)_{\ell+1}\) is a conformal 12-design.

Lehmer’s conjecture gives \(\tau(i) \neq 0\) [18]. Thus, Theorem 1.5 is a reformulation of Lehmer’s conjecture.

We have not yet been able to determine the strength of \((V^\natural)_{\ell}\) for general \(\ell\); hence, Lehmer’s conjecture is still open. This demonstrates the difficulty of determining the strength of \(V_\ell\) for general \(V\). However, in [22,23], There are examples for which the strength \(t\) can be determined. It has been shown that the shells of \(Z^2\)-lattice and \(A_2\)-lattice have strength 3 [22]. It has also been shown that the homogeneous spaces in a \(d\)-free boson VOA have strength 3 [22].

**Remark 1.3** Let \(C\) be a doubly even self-dual code of length \(n\). Let \(\rho : \mathbb{Z}^n \to \mathbb{F}_2^n; x \mapsto x \pmod{2}\).

Then, the construction \(A\) of \(C_L\)

\[
L_C := \frac{1}{\sqrt{2}} \{ x \in \mathbb{Z}^n \mid \rho(x) \in C \}
\]

is an even unimodular lattice. Similarly, let \(L\) be an even unimodular lattice of rank \(n\). Then, we obtain a holomorphic VOA \(V_L\). This leads to the question of whether there is any analogy in design between “\(C\) and \(L_L\)” and “\(L\) and \(V_L\)”.

Summalizing our results, we have the followings. Let \(C\) be a doubly even self-dual code of length 24, 8, 16. Then the following holds: the \(C_\ell\) is a combinatorial \(t\)-design with

\[
t = \begin{cases} 
3 & \text{if } n = 8, \\
1 & \text{if } n = 16, \\
1 \text{ or } 5 & \text{if } n = 24.
\end{cases}
\]

On the other hand, The \((L_C)_\ell\) is a spherical \(T\)-design with

\[
T = \begin{cases} 
\{1, 2, 3, 4, 5, 6, 7, 9, 10, 11\} \cup T_2 & \text{if } n = 8, \\
\{1, 2, 3, 5, 6, 7\} \cup T_2 & \text{if } n = 16, \\
\{1, 2, 3\} \cup T_2 & \text{if } n = 24.
\end{cases}
\]

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Let $L$ be an even unimodular lattice of rank 24, 8, 16. Then, the following holds: the $L_\ell$ is a spherical $t$-designs with

$$t = \begin{cases} 
7 & \text{if } n = 8, \\
3 & \text{if } n = 16, \\
3 \text{ or } 11 & \text{if } n = 24.
\end{cases}$$

On the other hand, The $(V_L)_\ell$ is a spherical $T$-design with

$$T = \begin{cases} 
\{1, 2, 3, 4, 5, 6, 7, 9, 10, 11\} \cup T_2 & \text{if } n = 8, \\
\{1, 2, 3, 5, 6, 7\} \cup T_2 & \text{if } n = 16, \\
\{1, 2, 3\} \cup T_2 & \text{if } n = 24.
\end{cases}$$

Therefore, there exists an analogy in design between “$C$ and $L_C$” and “$L$ and $V_L$”.

This paper is organized as follows: In Sect. 2, we give the definitions of combinatorial, spherical, and conformal $t$-designs; In Sect. 3, we give a proof of Theorem 1.1; In Sect. 4, we give a proof of Theorem 1.2; In Sect. 5, we give a proof of Theorems 1.3 and 1.4; Finally, in Sect. 6, we provide some concluding remarks.

## 2 Preliminaries

### 2.1 Codes and combinatorial $t$-designs

Let $C$ be a subspace of $F_2^n$, where $F_2$ is the binary finite field. $C$ is called a (binary) linear code of length $n$. For $x = (x_1, \ldots, x_n) \in F_2^n$, we put

$$\text{wt}(x) = \sharp\{i \mid x_i = 1\}.$$

The minimum weight of non-zero elements of $C$ is called the minimum weight $\text{min}(C)$ of $C$. In this section, we set $(x, y) = \sum_{i=1}^n x_i y_i$ throughout, for $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n) \in F_2^n$.

Let $C$ be a linear code. We say $C$ is a doubly even self-dual code if it is doubly even (i.e., $\text{wt}(x) \in 4\mathbb{Z}$ for all $x \in C$) and is self-dual (i.e., $C = C^\perp := \{x \in F_2^n \mid (x, y) = 0 \text{ for all } y \in C\}$). It is well known that if there exists a doubly even self-dual code, then $n$ must be a multiple of 8. [7,27] provides the definition of and basic information about codes.

We review the concept of combinatorial $t$-design.

**Definition 2.1** Let $\Omega = \{1, 2, \ldots, v\}$ be a finite set, $\Omega^{(k)}$ be the set of all $k$-element subsets of $\Omega$, and $X$ be a subset of $\Omega^{(k)}$. We say $X$ is a combinatorial $t$-design or $t$-$(v, k, \lambda)$ design if, for any $T \in \Omega^{(t)}$,

$$\sharp\{W \in X \mid T \subset W\} = \lambda.$$ 

We consider the idea of a combinatorial $t$-design with 2-weight.

**Definition 2.2** Let $X$ be a subset of $\Omega^{(k)} \cup \Omega^{(\ell)}$ ($k \neq \ell$). We say $X$ is a combinatorial $t$-design with 2-weight or a $t$-$(v, k, \lambda)$ design with 2-weight if, for any $T \in \Omega^{(t)}$,

$$\sharp\{W \in X \mid T \subset W\} = \lambda.$$ 


Codes provide examples of combinatorial designs and combinatorial designs with 2-weight. The support of a non-zero vector \( \mathbf{x} := (x_1, \ldots, x_n), x_i \in \mathbb{F}_q = \{0, 1, \ldots, q-1\} \) is the set of indices of its non-zero coordinates: \( \text{supp}(\mathbf{x}) = \{i \mid x_i \neq 0\} \). Let \( X := \{1, \ldots, n\} \) and \( B(C_{\ell}) := \{\text{supp}(\mathbf{x}) \mid \mathbf{x} \in C_{\ell}\} \). Then, for a code \( C \) of length \( n \), we say that \( C_{\ell} \) is a combinatorial \( t \)-design (with 2-weight) if \( (X, B(C_{\ell})) \) is a combinatorial \( t \)-design (with 2-weight).

### 2.2 Harmonic weight enumerators

Here, we discuss some definitions and properties of discrete harmonic functions and harmonic weight enumerators [3,8]. Let \( \Omega_1 = \{1, 2, \ldots, n\} \) be a finite set (which will be the set of coordinates of the code), \( \Omega_1^{(k)} \) be the set of its \( k \)-subsets. We denote the free real vector spaces by \( \mathbb{R}\Omega_1, \mathbb{R}\Omega_1^{(k)} \), spanned by the elements of \( \Omega_1, \Omega_1^{(k)} \), respectively. An element of \( \mathbb{R}\Omega_1^{(k)} \) is denoted by \( f = \sum_{z \in \Omega_1^{(k)}, z \subset u} f(z)z \) and is identified with the real-valued function on \( \Omega_1^{(k)} \) given by \( z \mapsto f(z) \). Such an element \( f \in \mathbb{R}\Omega_1^{(k)} \) can be extended to \( \tilde{f} \in \mathbb{R}\Omega_1 \) by setting, for all \( u \in \Omega_1 \),

\[
\tilde{f}(u) = \sum_{z \in \Omega_1^{(k)}, z \subset u} f(z).
\]

If an element \( g \in \mathbb{R}\Omega_1 \) is equal to some \( \tilde{f} \), for \( f \in \mathbb{R}\Omega_1^{(k)} \), we say that \( g \) has degree \( k \). The linear differential operator \( \gamma \) is defined by

\[
\gamma(z) := \sum_{y \in \Omega_1^{(k-1)}, y \subset z} y
\]

for all \( z \in \Omega_1^{(k)} \) and for all \( k = 0, 1, \ldots, n \), and \( \text{Harm}_k \) is the kernel of \( \gamma \):

\[
\text{Harm}_k := \ker(\gamma|_{\mathbb{R}\Omega_1^{(k)}}).
\]

The following theorem is known:

**Theorem 2.1** ([8]) A set \( X \subset \Omega_1^{(k)} \) of blocks is a \( t \)-design if and only if

\[
\sum_{x \in X} \tilde{f}(x) = 0
\]

for all \( f \in \text{Harm}_k, 1 \leq k \leq t \).

Here, we refer to the concept of the combinatorial \( T \)-design and define the concept of the combinatorial \( T \)-design with 2-weight.

**Definition 2.3** ([3,8]) Let \( X \) be a subset of \( \Omega_1^{(k)} \). \( X \) is a combinatorial \( T \)-design if the condition \( \sum_{x \in X} \tilde{f}(x) = 0 \) holds for all \( f \in \text{Harm}_j, j \in T \).

**Definition 2.4** Let \( X \) be a subset of \( \Omega_1^{(k)} \cup \Omega_1^{(\ell)} (k \neq \ell) \). \( X \) is a combinatorial \( T \)-design with 2-weight if the condition \( \sum_{x \in X} \tilde{f}(x) = 0 \) holds for all \( f \in \text{Harm}_j, j \in T \).

To show Theorem 1.2 (1), we review the theory of the harmonic weight enumerator developed in [3].
Definition 2.5 ([3]) Let $C$ be a binary code of length $n$, and let $f \in \text{Harm}_k$. The harmonic weight enumerator associated with $C$ and $f$ is

$$w_{C,f}(x, y) = \sum_{c \in C} f(c)x^{n-\text{wt}(c)}y^{\text{wt}(c)}.$$  

Lemma 2.1 ([3]) Let $C$ be a doubly even self-dual code. Then, for $m > 0$, the non-empty shell $C_m$ is a combinatorial $t$-design if and only if

$$a_m^f = 0 \text{ for every } f \in \text{Harm}_j, \ 1 \leq j \leq t$$

where $a_m^f$ is the coefficient of the harmonic theta series

$$w_{C,f}(x, y) = \sum_{m=0}^{n} a_m^f x^{n-m} y^m.$$  

Let

$$\begin{align*}
P_8(x, y) &= x^8 + 14x^4y^4 + y^8, \\
P_{12}(x, y) &= x^2y^2(x^4 - y^4)^2, \\
P_{18}(x, y) &= xy(x^8 - y^8)(x^8 - 34x^4y^4 + y^8), \\
P_{24}(x, y) &= x^4y^4(x^4 - y^4)^4, \\
P_{30}(x, y) &= P_{12}(x, y)P_{18}(x, y).
\end{align*}$$

We set

$$I_{G, x^k} = \begin{cases} 
\langle P_8(x, y), P_{24}(x, y) \rangle & \text{if } k \equiv 0 \pmod{4} \\
\langle P_{12}(x, y), P_8(x, y), P_{24}(x, y) \rangle & \text{if } k \equiv 2 \pmod{4} \\
\langle P_{18}(x, y), P_8(x, y), P_{24}(x, y) \rangle & \text{if } k \equiv 3 \pmod{4} \\
\langle P_{30}(x, y), P_8(x, y), P_{24}(x, y) \rangle & \text{if } k \equiv 1 \pmod{4}.
\end{cases}$$

The space, which includes $w_{C,f}(x, y)$, is characterized in [3]:

Theorem 2.2 ([3]) Let $C$ be a doubly even self-dual code of length $n$, and let $f \in \text{Harm}_k$. Then, we have $w_{C,f}(x, y) = (xy)^k Z_{C,f}(x, y)$. Moreover, the polynomial $Z_{C,f}(x, y)$ is of degree $n - 2k$ and is in $I_{G, x^k}$.  

2.3 Lattices and spherical $t$-designs

The Euclidean lattice provides an example of a spherical design. A lattice $L \subset \mathbb{R}^n$ of dimension $n$ is unimodular if $L = L^\perp$, where the dual lattice $L^\perp$ of $L$ is defined as $\{x \in \mathbb{R}^n \mid (x, y) \in \mathbb{Z} \text{ for all } y \in L\}$ under the standard inner product $(x, y)$. The norm of a vector $x$ is defined as $(x, x)$. The minimum norm $\min(L)$ of a unimodular lattice $L$ is the smallest norm among all non-zero vectors of $L$.

A unimodular lattice with even norms is said to be even. An even unimodular lattice of dimension $n$ exists if and only if $n \equiv 0 \pmod{8}$.

The concept of a spherical $t$-design has been explained by Delsarte et al. [9].

Definition 2.6 ([9]) Let

$$S^{n-1}(r) = \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_1^2 + \cdots + x_n^2 = r^2\}.$$  

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For a positive integer \( t \), a finite non-empty set \( X \) in the unit sphere \( S^{n-1} \) is called a spherical \( t \)-design in \( S^{n-1} \) if the following condition is satisfied:

\[
\frac{1}{|X|} \sum_{x \in X} f(x) = \frac{1}{|S^{n-1}(1)|} \int_{S^{n-1}(1)} f(x) d\sigma(x)
\]

for all polynomials \( f(x) = f(x_1, \ldots, x_n) \) of degree not exceeding \( t \).

Here, the right-hand side of the equation is the surface integral over the sphere, and \( |S^{n-1}(1)| \) denotes the area of the sphere \( S^{n-1}(1) \). A finite subset \( X \) in \( S^{n-1}(r) \) is also called a spherical \( t \)-design if \( \frac{1}{r} X \) is a spherical \( t \)-design on the unit sphere \( S^{n-1}(1) \). If \( X \) is a spherical \( t \)-design but not a spherical \( (t + 1) \)-design, we can say that \( X \) has strength \( t \).

Lattices provide examples of spherical \( t \)-designs. We say that \( L_\ell \) is a spherical \( t \)-design if \( \frac{1}{\sqrt{\ell}} L_\ell \) is a spherical \( t \)-design.

### 2.4 Spherical theta series

We denote by \( \text{Harm}_j(\mathbb{R}^n) \) as the set of homogeneous harmonic polynomials of degree \( j \) on \( \mathbb{R}^n \). The following theorem is known:

**Theorem 2.3** ([9]) \( X \subset S^{n-1}(1) \) is a spherical \( t \)-design if and only if the condition

\[
\sum_{x \in X} P(x) = 0
\]

holds for all \( P(x) \in \text{Harm}_j(\mathbb{R}^n) \) with \( 1 \leq j \leq t \). If \( X \) is antipodal (i.e., \( x \in X \Rightarrow -x \in X \)), then \( X \) is a spherical \( t \)-design if and only if the condition

\[
\sum_{x \in L_{2m}} P(x) = 0
\]

holds for all \( P \in \text{Harm}_{2j}(\mathbb{R}^n) \) with \( 1 \leq 2j \leq t \).

Let \( T \) be a subset of the natural numbers \( \mathbb{N} = \{1, 2, \ldots\} \). Then, we define the concept of spherical \( T \)-design as follows:

**Definition 2.7** ([23]) \( X \) is a spherical \( T \)-design if the condition

\[
\sum_{x \in X} P(x) = 0
\]

holds for all \( P(x) \in \text{Harm}_j(\mathbb{R}^n) \) with \( j \in T \).

**Remark 2.1** We remark that a spherical \( t \)-design is actually a spherical \( \{1, 2, \ldots, t\} \)-design. Therefore, the concept of a spherical \( T \)-design generalizes that of a spherical \( t \)-design.

Let \( \mathbb{H} := \{ z \in \mathbb{C} \mid \text{Im}(z) > 0 \} \) be the upper half-plane.

**Definition 2.8** Let \( L \) be the lattice of \( \mathbb{R}^n \). Then, for a polynomial \( P \), the function

\[
\vartheta_{L, P}(z) := \sum_{x \in L} P(x) e^{i \pi z(x,x)}
\]

is called the theta series of \( L \) weighted by \( P \).

**Lemma 2.2** ([26,30,31]) Let \( L \) be an integral lattice in \( \mathbb{R}^n \). Then, for \( m > 0 \), the non-empty shell \( L_m \) is a spherical \( t \)-design if and only if

\[
a_m^{(P)} = 0 \text{ for every } P \in \text{Harm}_{2j}(\mathbb{R}^n), \ 1 \leq 2j \leq t,
\]
where \( a_m^{(P)} \) are the Fourier coefficients of the weighted theta-series
\[
\vartheta_{L,P}(z) = \sum_{m \geq 0} a_m^{(P)} q^m,
\]
where \( q = e^{\pi i z} \).

For example, we consider an even unimodular lattice \( L \). Then, the theta series of \( L \) weighted by the harmonic polynomial \( P \), \( \vartheta_{L,P}(z) \), is in a modular form with respect to \( SL_2(\mathbb{Z}) \). In general, we have the following:

**Lemma 2.3** ([26]) Let \( L \subset \mathbb{R}^n \) be an even unimodular lattice of of rank \( n = 8N \) and of minimum norm \( 2M \). Let
\[
E_4(z) := 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^n,
\]
\[
E_6(z) := 1 - 504 \sum_{n=1}^{\infty} \sigma_5(n) q^n,
\]
\[
\Delta(z) := \frac{E_4(z)^3 - E_6(z)^2}{1728} = q - 24q^2 + \cdots,
\]
where \( \sigma_{k-1}(n) := \sum_{d|n} d^{k-1} \). Then we have for \( P \in \text{Harm}_2(\mathbb{R}_n) \),
\[
\vartheta_{L,P} \in \begin{cases} 
\mathbb{C}[E_4, \Delta] & \text{if } j \text{ is even}, \\
E_6 \mathbb{C}[E_4, \Delta] & \text{if } j \text{ is odd}.
\end{cases}
\]
More precisely, there exist \( c_i \in \mathbb{C} \) such that
\[
\vartheta_{L,P} = \begin{cases} 
\sum_{i=M}^{((N+j)/2)/3} c_i \Delta^i E_4^{N+j/2-3i} & \text{if } j \text{ is even}, \\
\sum_{i=M}^{((N+j)/2)/3} c_i E_6 \Delta^i E_4^{N+(j-3)/2-3i} & \text{if } j \text{ is odd}.
\end{cases}
\]
In particular, \( \vartheta_{L,P} = 0 \) if \( j \) is even and \( 3M > N+j/2 \), or \( j \) is odd and \( 3M > N+(j-3)/2 \).

### 2.5 VOAs and conformal \( t \)-designs

First, we review some information about VOAs that will be presented later in this paper. See [6], [13], and [14] for definitions and elementary information about VOAs and their modules.

A VOA \( V \) over the field \( \mathbb{C} \) of complex numbers is a complex vector space equipped with a linear map \( Y : V \to \text{End}(V)[[z, z^{-1}]] \) and two non-zero vectors \( 1 \) and \( \omega \) in \( V \), satisfying certain axioms (cf. [13,14]). We denote a VOA \( V \) by \( (V, Y, 1, \omega) \). For \( v \in V \), we write
\[
Y(v, z) = \sum_{n \in \mathbb{Z}} v(n) z^{-n-1}.
\]
In particular, for \( \omega \in V \), we write
\[
Y(\omega, z) = \sum_{n \in \mathbb{Z}} L(n) z^{-n-2},
\]
and $V$ is graded by $L(0)$-eigenvalues: $V = \bigoplus_{n \in \mathbb{Z}} V_n$. We note that $\{L(n) \mid n \in \mathbb{Z}\} \cup \{id_V\}$ forms a Virasoro algebra. For $V_n$, $n$ is called the weight. In this study, we assume that $V_n = 0$ for $n < 0$, and $V_0 = \mathbb{C}1$. For $v \in V_n$, the operator $v(n - 1)$ is homogeneous and is of degree 0. We set $o(v) = v(n - 1)$. We also assume that the VOA $V$ is isomorphic to a direct sum of the highest weight modules for the Virasoro algebra, i.e.,

$$V = \bigoplus_{n \geq 0} V(n),$$

(4)

where each $V(n)$ is a sum of the highest weight $V_\omega$ modules of the highest weight $n$ and $V(0) = V_\omega$.

In particular, the decomposition (4) yields the natural projection map

$$\pi : V \to V_\omega$$

with the kernel $\bigoplus_{n > 0} V(n)$. Next, we give the definition of a conformal $t$-design, which is based on Matsuo’s study [20].

**Definition 2.9** ([17]) Let $V$ be a VOA of central charge $c$, and let $X$ be an $h$-weight subspace of a module of $V$. For a positive integer $t$, $X$ is referred to as a conformal $t$-design if, for all $v \in V_n$ (where $0 \leq n \leq t$), we have

$$\text{tr}_{|X} o(v) = \text{tr}_{|X} o(\pi(v)).$$

Then, it is easy to prove the following theorem:

**Theorem 2.4** ([17]) Let $X$ be the homogeneous subspace of a module of a VOA $V$. $X$ is a conformal $t$-design if and only if the condition $\text{tr}_{|X} o(\pi) = 0$ holds for all homogeneous $v \in \ker \pi = \bigoplus_{n > 0} V(n)$ of weight $n \leq t$.

**Theorem 2.5** ([17]) Let $V$ be a VOA and let $N$ be a $V$-module graded by $\mathbb{Z} + h$. The following conditions are equivalent:

1. The homogeneous subspaces $N_n$ of $N$ are conformal $t$-designs based on $V$ for $n \leq h$.
2. For all Virasoro highest weight vectors $v \in V_s$ with $0 < s \leq t$ and all $n \leq h$ we obtain

$$\text{tr}_{|N} o(v) = 0.$$  

Let $T$ be a subset of the natural numbers. As an analogue of the concept of spherical $T$-designs, we define the concept of a conformal $T$-design as follows:

**Definition 2.10** $X$ is a conformal $T$-design if the condition $\text{tr}_{|X} o(\pi) = 0$ holds for all homogeneous $v \in \ker \pi = \bigoplus_{n > 0} V(n)$ of weight $j \in T$.

**Remark 2.2** We remark that a conformal $t$-design is actually a conformal $\{1, 2, \ldots, t\}$-design. Therefore, the concept of a conformal $T$-design generalizes that of a conformal $t$-design.

$V_m$ can be considered to have large symmetry if a homogeneous space of VOA $V_m$ is a conformal $t$-design for higher $t$ [20]. A conformal $t$-design is also a conformal $s$-design for all integers $1 \leq s \leq t$. Therefore, it is of interest to investigate the conformal $t$-design for higher $t$.

For the notion of admissible, we refer to [11]. A VOA is called rational if every admissible module is completely reducible. A rational VOA $V$ is called holomorphic if the only irreducible module of $V$ up to isomorphism is $V$ itself. The smallest $h > 0$ for which $V(h) \neq 0$ is called the minimal weight of $V$ and is denoted by $\mu(V)$.

A holomorphic VOA of central charge $c$ exists if and only if $c \equiv 0 \pmod{8}$ [16].
2.6 Graded traces

In this section, we review the concept of the graded trace. As stated earlier, $V$ is a VOA with standard $L(0)$-grading

$$V = \bigoplus_{n \geq 0} V_n.$$ 

Then, for $v \in V_k$, we define the graded trace $Z_V(v, z)$ as follows:

$$Z_V(v, z) = \text{tr} | v o(v) q^{L(0) - c/24} = q^{-c/24} \sum_{n=0}^{\infty} (\text{tr} | v_n o(v)) q^n,$$

where $c$ is the central charge of $V$. If $v = 1$, then

$$Z_V(1, z) = \text{tr} | q^{L(0) - c/24} = q^{-c/24} \sum_{n=0}^{\infty} (\text{dim} V_n) q^n.$$

**Theorem 2.6 ([32])** Let $V$ be a holomorphic VOA of central charge $c$. Let $v \in V_s$ be a Virasoro highest weight vector of conformal weight $s$. Let $S = \pm \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $T = \pm \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. 

Then

$$Z_V(v, q) = q^{-c/24} \sum_{n=0}^{\infty} \text{tr} | v_n o(v) q^n$$

is a meromorphic modular form of weight $s$ for $\text{PSL}_2(\mathbb{Z})$ with character $\rho$

$$\rho(S) = 1 \text{ and } \rho(T) = e^{-2\pi i c/24}.$$ 

**Theorem 2.7 ([12])** Let $L$ be a even unimodular lattice of rank $n$ Then, for every element $v$ in $V_L$, we have

$$Z(v, z) = f(v, z) / \eta(z)^n,$$

where $f(v, z)$ is a sum of modular form of $\text{SL}(2, \mathbb{Z})$.

**Theorem 2.8 ([12])** Let $P$ be a homogeneous spherical harmonic polynomial and let $V_L$ be the lattice vertex operator algebra associated with an even integral lattice $L$ of rank $k$. Then, there exists a Virasoro highest weight vector $v_P$ with the property

$$Z_{V_L}(v_P, q) = \vartheta_L, P(z) / \eta(z)^k,$$

where

$$\eta(z) := q^{1/24} \prod_{i=1}^{\infty} (1 - q^i).$$
2.7 Graded traces for lattice vertex operator algebras

In this section, to prove the Theorem 1.3, we investigate the graded trace of lattice VOAs.

Let $L$ be an even unimodular lattice of rank $n = 24m + 8r$. Then, $V_L$ is a holomorphic VOA of central charge $c = 24m + 8r$. Let $v \in V_\ell$ be a Virasoro highest weight vector of weight $\ell$. It follows from Theorems 2.6 and 2.7 that $\eta(z)^c Z_{V_L}(v, z)$ is a modular form of weight $c/2 + \ell = 12m + 4r + \ell$ for $SL_2(\mathbb{Z})$.

Let $c = 8$, and let $v \in (V_L)_8$. Then

$$\eta(z)^c Z_{V_L}(v, z) = c_1(v)(q + \cdots)$$

is a modular form of weight 12. Therefore, we have

$$\eta(z)^c Z_{V_L}(v, z) = c_1(v)\Delta(z)$$

and

$$Z_{V_L}(v, z) = \frac{c_1(v)\Delta(z)}{\eta(z)^8} = c_1(v)\eta(z)^{16} = c_1(v)q^{-1/3} \sum_{i=1}^{\infty} a(i)q^i \text{ (say),}$$

(5)

where $c_1(v)$ is a constant that depends on $v$. Let $c = 16$ and $v \in (V_L)_4$. Then

$$\eta(z)^c Z_{V_L}(v, z) = c_2(v)(q + \cdots)$$

is a modular form of weight 12. Therefore, we have

$$\eta(z)^c Z_{V_L}(v, z) = c_2(v)\Delta(z)$$

and

$$Z_{V_L}(v, z) = \frac{c_2(v)\Delta(z)}{\eta(z)^8} = c_2(v)\eta(z)^{16} = c_2(v)q^{-2/3} \sum_{i=1}^{\infty} b(i)q^i \text{ (say),}$$

(6)

where $c_2(v)$ is a constant that depends on $v$. Let $c = 24$, and let $v \in (V_L)_4$ be a Virasoro highest weight vector of weight 4. Then

$$\eta(z)^c Z_{V_L}(v, z) = c_3(v)(q + \cdots)$$

is a modular form of weight 16. Therefore, we have

$$\eta(z)^c Z_{V_L}(v, z) = c_3(v)E_4(z)\Delta(z)$$

and

$$Z_{V_L}(v, z) = \frac{c_3(v)E_4(z)\Delta(z)}{\eta(z)^{24}} = c_3(v)E_4(z) = c_3(v)q^{-1} \sum_{i=1}^{\infty} c(i)q^i \text{ (say),}$$

(7)

where $c_3(v)$ is a constant that depends on $v$.

Then, using an argument similar to that presented in the proof of [22, Theorem 1.2], we have the following proposition:

**Proposition 2.1** Let the notation be the same as before. Then, the following (i) and (ii) are equivalent for all $c \in \{8, 16, 24\}$:

1. **Case** $c = 8$:
   (i) $a(\ell) = 0$;
(ii) $(V_L)_t$ is a conformal 8-design.

(2) Case $c = 16$:

(i) $b(t) = 0$;
(ii) $(V_L)_t$ is a conformal 4-design.

(3) Case $c = 24$:

(i) $c(t) = 0$;
(ii) $(V_L)_c$ is a conformal 4-design.

**Proof** (1) Let $c = 8$. Note that $L \cong E_8$. First, note that for $v \in (V_L)_8$, by (5), we have
\[ \eta(z)^8 Z_{V_L}(v, z) = c_1(v) \Delta(z) \in M_{12}(SL_2(\mathbb{Z})). \]
Assume that $a(t) = 0$. Then, for any $v \in (V_L)_8$, we have $\text{tr}|_{(V_L)_t} o(v) = 0$. Therefore, $(V_L)_t$ is a conformal 8-design. Next, we assume the contrary, that is, $a(t) \neq 0$. Since $(V_{E_8})_1$ is not a conformal 8-design (cf [17, Theorem 4.2 (i)]), by (3), there exists $v \in (V_{E_8})_8$ of weight 8 such that
\[ Z_{V_{E_8}}(v, z) = c_1(v)q^{-1/3} \sum_{i=1}^{\infty} a(i)q^i, \]
where $c_1(v) \neq 0$. Hence, we have
\[ \text{tr}|_{(V_{E_8})_t} o(v) = c_1(v) \times a(t) \neq 0, \]
which implies that $(V_{E_8})_t$ is not a conformal 8-design. This completes the proof of Case 1.

(2) Let $c = 16$. For $v \in (V_L)_4$, by (6), we have
\[ \eta(z)^{16} Z_{V_L}(v, z) = c_2(v) \Delta(z) \in M_{12}(SL_2(\mathbb{Z})), \]
where $c_2(v)$ is a constant that depends on $v$. Assume that $b(t) = 0$. Then, for any $v \in (V_L)_4$, we have $\text{tr}|_{(V_L)_t} o(v) = 0$. Therefore, $(V_L)_c$ is a conformal 4-design. On the other hand, based on [26, Lemma 31], there exists $P \in \text{Harm}_4(\mathbb{R}^{16})$ such that $\vartheta_{L, P}(z) = d_1(P) \Delta(z)$, where $d_1(P)$ is a non-zero constant. Therefore, based on Theorem 2.8, there exists $v_P \in (V_L)_4$ such that
\[ Z_{V_L}(v_P, z) = d_1(P) \Delta(z)/\eta(z)^{16} = d_1(P)q^{-2/3} \sum_{i=1}^{\infty} b(i)q^i. \]
We have $c(P) \times b(1) \neq 0$, that is, $(V_L)_1$ is not a conformal 4-design. Then, the rest of the proof is similar to that of Case 1.

(3) Let $c = 24$. For $v \in (V_L)_4$, by (7), we have
\[ \eta(z)^{24} Z_{V_L}(v, z) = c_3(v) E_4(z) \Delta(z) \in M_{16}(SL_2(\mathbb{Z})), \]
where $c_3(v)$ is a constant that depends on $v$. Assume that $c(t) = 0$. Then, for any $v \in (V_L)_4$, we have $\text{tr}|_{(V_L)_t} o(v) = 0$. Therefore, $(V_L)_c$ is a conformal 4-design. Let $L$ be a lattice that is not a Leech lattice. Then, based on [26, Lemma 31], there exists $P \in \text{Harm}_4(\mathbb{R}^{24})$ such that $\vartheta_{L, P}(z) = d_2(P) E_4(z) \Delta(z)$, where $d_2(P)$ is a non-zero constant. Therefore, based on Theorem 2.8, there exists $v_P \in (V_L)_4$ such that
\[ Z_{V_L}(v_P, z) = d_2(P) E_4(z) \Delta(z)/\eta(z)^{24} = q^{-1} \sum_{i=1}^{\infty} c(i)q^i. \]
We have $d_2(P) \times c(1) \neq 0$, that is, $(V_L)_1$ is not a conformal 4-design. The rest of the proof is similar to that of Case 1.
Let $L$ be a Leech lattice. Then, $(V_L)_1 = \langle h_1(-1)\mathbf{1}, \ldots, h_{24}(-1)\mathbf{1} \rangle$, where $\{h_i\}_{i=1}^{24}$ are the orthonormal basis of $\mathfrak{h}$. Let

$$v_4 = h_1(-1)^4 \mathbf{1} - 2h_1(-3)h_1(-1)\mathbf{1} + \frac{3}{2} h_1(-2)^2 \mathbf{1}.$$ 

Then, $v_4$ is the highest weight vector in $(V_L)_4$ (see [22, Proposition 3.2]). Then, we have

$$\text{tr}|_{(V_L)_1} \circ (v_4) \neq 0,$$

that is, $(V_L)_1$ is not a conformal 4-design. The rest of the proof is similar to that of Case 1.

\[\square\]

3 Proof of Theorem 1.1

In this section, we show Theorem 1.1.

Proof of Theorem 1.1 Let $V$ be a holomorphic VOA of central charge $c = 24m$. Let $v \in V_\ell$ be a Virasoro highest weight vector of weight $\ell$. It follows from Theorem from Theorems 2.6 and 2.7 that $\eta(z)^c Z_V(v, z)$ is a modular form of weight $c/2 + \ell = 12m + \ell$ for $\text{SL}_2(\mathbb{Z})$.

Assume that $\ell = 1$ or $\ell = 3$. Then, there is no non-zero holomorphic modular form of weight $12m + \ell$.

Assume that $\ell = 2$. Then

$$\eta(z)^c Z_V(v, z) = q^{\frac{c}{24}} (1 + \cdots)^{\frac{c}{24}} q^{-\frac{c}{24}} (c_m q^m + \cdots) = c_m q^m + \cdots.$$ 

Then, there is no non-zero holomorphic modular form of weight $12m + 2$ such that the leading term is $c_m q^m + \cdots$, that is, $Z_V(v, q) = 0$. Therefore, we can say that any homogeneous spaces of $V$ are conformal 3-design, by Theorem 2.5.

\[\square\]

4 Proof of Theorem 1.2

4.1 Proof of Theorem 1.2 (1)

In this section, we show Theorem 1.2 (1).

Proof of Theorem 1.2 (1) Let $f \in \text{Harm}_\ell$ with $\ell \in T_2$. Let

$$w_{C, f}(x, y) = \sum_{i=0}^{n} c_{C, f}(i)x^{n-i}y^i.$$ 

It is sufficient to show that $c_{C, f}(\ell) + c_{C, f}(n - \ell) = 0$. Let $\ell \equiv 3 \pmod{4}$. Then, from Theorem 2.2,

$$w_{C, f}(x, y) = P_{18}(x, y) \times (\text{a polynomial of } P_{8}(x, y) \text{ and } P_{24}(x, y)).$$

Note that $P_{8}(x, y) = P_{8}(y, x)$, $P_{24}(x, y) = P_{24}(y, x)$ and $P_{18}(x, y) = -P_{18}(y, x)$. These imply that $w_{C, f}(x, y) = -w_{C, f}(y, x)$ and $c_{C, f}(\ell) = -c_{C, f}(n - \ell)$. This completes the
proof for the case $\ell \equiv 3 \pmod{4}$. The case $\ell \equiv 1 \pmod{4}$ can be proved in a similar manner. □

The following corollary is obtained in [1,25]. Theorem 1.2 gives a new proof.

**Corollary 4.1** ([1,25]) Let $C$ be a doubly even self-dual code of length $n = 24m + 8r$. Then, any $C_{n/2}$ forms a combinatorial $T_2$-design. In particular, any $C_{n/2}$ forms a combinatorial 1-design.

**Proof** From Theorem 1.2, $C_k \cup C_{n-k}$ forms a combinatorial $T_2$-design with 2-weight. If $k = n/2$, then $C_{n/2} \cup C_{n-n/2} = C_{n/2}$. This completes the proof of Corollary 4.1. □

### 4.2 Proof of Theorem 1.2 (2)

In this section, we show Theorem 1.2 (2).

**Proof of Theorem 1.2 (2).** Let $V$ be a holomorphic VOA of central charge $c = 24n + 8r$. Let $v \in V_\ell$ be a Virasoro highest weight vector of weight $\ell$. It follows from Theorems 2.6 and 2.7 that $\eta(z)^c Z_V(v, z)$ is a modular form of weight $c/2 + \ell = 12n + 4r + \ell$ for $SL_2(\mathbb{Z})$. Assume that $\ell \equiv 1 \pmod{2}$. Then, there is no non-zero holomorphic modular form of odd weight $c/2 + \ell$, that is, $Z_V(v, q) = 0$. Therefore, we have that any homogeneous spaces of $V$ are conformal $T_2$-design, by Theorem 2.5. □

### 5 Proof of Theorem 1.3 and 1.4

In this section, we give the proof of Theorems 1.3 and 1.4.

#### 5.1 Proof of Theorem 1.3

**Proof of Theorem 1.3** Let $V$ be a holomorphic VOA of central charge $c = 8$. Let $v \in V_\ell$ be a Virasoro highest weight vector of weight $\ell$. Then, $\eta(z)^c Z_V(v, z)$ is a modular form of weight $4 + \ell$ for $SL_2(\mathbb{Z})$. However, there is no non-zero holomorphic modular form of weight 4 + $\ell$ with

$$\ell \in \{1, 2, 3, 4, 5, 6, 7, 9, 10, 11\},$$

that is, $Z_V(v, z) = 0$. Therefore, we conclude that any homogeneous spaces of $V$ are conformal $\{1, 2, 3, 4, 5, 6, 7, 9, 10, 11\} \cup T_2$—design,

from Theorems 2.5 and 1.2. In particular, any homogeneous spaces of $V$ are conformal 7-designs. The proof for the cases $c = 16, 24$ are similar. This completes the proof of Theorem 1.2. □
5.2 Proof of Theorem 1.4

**Proof of Theorem 1.4 (1)** By Proposition 2.1, it is sufficient to show that if \( \text{ord}_p(3\ell - 2) \) is odd, then \( b(\ell) = 0 \); otherwise \( b(\ell) \neq 0 \), where \( b(\ell) \) is defined by (6). Recall that

\[
Z_{V_L}(v, z) = \frac{c_2(v) \Delta(z)}{\eta(z)^{16}} = c_2(v)\eta(z)^8
\]

\[
= c_2(v)q^{-2/3} \sum_{i=1}^{\infty} b(i)q^i
\]

\[
= c_2(v)q^{-16/24}(q - 8q^2 + 20q^3 - 70q^5 + 64q^6 + 56q^7 - 125q^9 + \cdots).
\]

Set

\[
\eta(3z)^8 = \sum_{i=1}^{\infty} b'(i)q^i
\]

\[
= q - 8q^4 + 20q^7 - 70q^{13} + \cdots.
\]

The exponents of the power series of (8) are 1 modulo 3. By [15, Theorem 2.1, Corollary 2.2], for \( p \equiv 2 \pmod{3} \), if \( \text{ord}_p(\ell) \) is odd, then \( b'(\ell) = 0 \); otherwise \( b'(\ell) \neq 0 \), and if \( (\ell, n) = 1 \), then

\[
b'(\ell n) = b'(\ell)b'(n).
\]

Using these properties of \( b'(\ell) \), if \( \text{ord}_p(3\ell - 2) \) is odd for some prime \( p \equiv 2 \pmod{3} \), then \( (V_L)_\ell \) is a conformal 4-design; otherwise, the homogeneous spaces \( (V_L)_v \) are not conformal 4-designs.

Finally, we show that if \( \text{ord}_p(3\ell - 2) \) is odd for some prime \( p \equiv 2 \pmod{3} \), then \( (V_L)_\ell \) is a conformal 7-design. From Theorem 1.2, we show that \( (V_L)_\ell \) is a conformal \( \{1, 2, 3, 5, 6, 7\} \cup T_2 \)-design. As shown above, for \( v \in (V_L)_\ell \) (5 \( \leq k \leq 7 \)), we conclude that \( (V_L)_\ell \) is a conformal 4-design, that is, it is a conformal \( \{1, 2, 3, 4, 5, 6, 7\} \cup T_2 \)-design. Hence, \( (V_L)_\ell \) is a conformal 7-design. Thus, the proof is complete. \( \square \)

**Proof of Theorem 1.4 (2)** Based on Proposition 2.1, it is sufficient to show that for \( \ell \geq 1 \), \( c(\ell) \neq 0 \), where \( c(\ell) \) is defined by (7). Based on (7), we have \( c(\ell) = \sigma_3(\ell) \), where \( \sigma_3(j) \) is a divisor function \( \sigma_3(j) = \sum_{d|j} d^3 \). Then, for \( \ell \geq 1 \), we have \( c(\ell) = \sigma_3(\ell) \neq 0 \). \( \square \)

6 Concluding remarks

(1) Let \( L \) be an even unimodular lattice of rank 16. We showed in Theorem 1.4 that, if \( \text{ord}_p(3\ell - 2) \) is odd for some prime \( p \equiv 2 \pmod{3} \), then \( (V_L)_\ell \) is a conformal 7-design. It is an interesting, unsolved problem to determine whether \( (V_L)_\ell \) is a conformal 8-design. Let \( E_4(z)\eta(z)^8 = q^{-2/3} \sum_{i=1}^{\infty} d(i)q^i \). Using the same method as in the proof of Proposition 2.1, we can say that \( (V_L)_\ell \) is a conformal 8-design if and only if \( d(\ell) = 0 \).

(2) Let \( L \) be an even unimodular lattice of rank 8 (i.e., \( L = E_8 \)-lattice). Then, based on Proposition 2.1, the homogeneous space \( (V_L)_\ell \) is a conformal 8-design if and only if \( a(\ell) = 0 \), where \( a(\ell) \) is defined as follows: \( \eta(z)^{16} = q^{-1/3} \sum_{i=1}^{\infty} a(i)q^i \). It is conjectured in [28] that \( a(\ell) \neq 0 \) for all \( \ell \). Using the same argument as in [4,5,18], we can say that, if \( a(p) \neq 0 \) for all prime numbers \( p \), then \( a(\ell) \neq 0 \) for all \( \ell \).
(3) Note that there is no known combinatorial 6-design among the $C_\ell$ of code $C$. Also note that there are no known spherical or conformal 12-designs among the shell and the homogeneous spaces of any lattices or VOAs, except for the trivial case $V_{A_1}$ [17, Example 2.6]. It is an interesting, unsolved problem to show whether there exists a combinatorial 6-design obtainable from codes, a spherical 12-design obtainable from lattices, or a conformal 12-design obtainable from VOAs.

(4) Let $L = A_1$-lattice (namely, $L = \sqrt{2}\mathbb{Z} = \langle \alpha \rangle \mathbb{Z}$). Then, all homogeneous spaces of the lattice VOA $V_L$ are conformal $t$-designs for all $t$ (cf. [17]). This is because $(V_L)^{\text{Aut}(V_L)} = V_\omega$. Here, let $\theta$ be an element in $\text{Aut}(V_L)$ of order 2, which is a lift of $-1 \in \text{Aut}(L)$, and let $V_L^+$ be the fixed points of the VOA $V_L$ associated with $\theta$. Then, all the homogeneous spaces of $V_L^+$ are conformal 3-designs because $((V_L)^{\text{Aut}(V_L)})_{\leq 3} = (V_\omega)_{\leq 3}$ and because of [17, Theorem 2.5]. On the other hand, let $v_4 = \alpha(-1)^4 1 - 2\alpha(-3)\alpha(-1)1 + \frac{3}{2}\alpha(-2)^2 1 \in (V_L^+)_4$. Then, we calculate the graded trace as follows [10]:

$$Z_{V_L^+}(v_4, z) = q^{1/24} \frac{\eta(2z)^{15}}{\eta(z)^7}.$$ 

Therefore, if the Fourier coefficients of $Z_{V_L^+}(v_4, z)$ do not vanish, then none of the homogeneous spaces of $V_L^+$ are conformal 4-designs. We have checked numerically that the coefficients do not vanish up to the exponent 1000.

(5) Using [26] and [4], we show the following theorem:

**Theorem 6.1** (cf. [26], [4]) The shells in the $\mathbb{Z}^2$-lattice are spherical 3-designs and are not spherical 4-designs. The shells in the $A_2$-lattice are spherical 5-designs and are not spherical 6-designs.

In [21], they showed that the homogeneous spaces of $V_{A_2}$ are conformal 5-designs. Therefore, it is natural to ask whether the corresponding results hold for the lattice VOAs $V_{\sqrt{2}\mathbb{Z}^2}$ and $V_{A_2}$. Specifically,

(a) Are the homogeneous spaces of $V_{\sqrt{2}\mathbb{Z}^2}$ conformal 3-designs and not conformal 4-designs?

(b) Are the homogeneous spaces of $V_{A_2}$ not conformal 6-designs?

As interesting, unsolved problems, these questions remain to be answered in the affirmative or negative.

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1 Prof. Shimakura pointed us to the following: It can be proved that the homogeneous spaces of $V_{\sqrt{2}\mathbb{Z}^2}$ are conformal 3-designs and not conformal 4-designs. Moreover, the homogeneous spaces of $V_{A_2}$ are not conformal 6-designs [29].
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