— Introduction. Recent developments in computational nuclear physics reveals that the microscopic description of nucleon in terms of nucleon degrees of freedom requires three-nucleon interactions. In fact, although the nuclear three-body interaction is weaker than the two-body interaction, the binding energies of light nuclei cannot be understood without taking into account the three-body terms. This is due to a large cancellation of the kinetic energy and two-body attraction. The main component of the three-body interaction is associated with two-pion exchange, such as the Fujita-Miyazawa force. However, in addition to this, a repulsive three-body interaction of short range is required for quantitative description of nuclear systems. The short-range three-nucleon interaction, which is assumed to be spin-isospin independent in many cases, is important for determination of the nuclear equation of state at high density.

The two-body interactions adopted in those many-body calculations are determined by the phase-shift analysis of nucleon-nucleon scattering data. However, much less information is available for the N-body forces. Of course, we know that, in principle, the nuclear properties should be derived from QCD. However, QCD is strongly coupled at the nuclear energy scale, which leads to a huge gap between QCD and nuclear many-body problems.

A recent progress in string theory can bridge this gap, analytically. It is called holographic QCD, an application of gauge/string duality to strongly coupled QCD. We apply the holographic QCD to N-body nuclear force (N ≥ 3).

In holographic QCD, one of the most successful D-brane models is Sakai-Sugimoto model (SS model). The theory, which is a U(N_f) Yang-Mills-Chern-Simons (YM-CS) theory in a warped 5-dimensional space-time, was conjectured to be dual to low energy massless QCD with N_f flavors, in the large N_c and large λ limits (λ ≡ 1/\sqrt{N_c}).

We provide a calculation of N-body (N ≥ 3) nucleon interactions at short distances in holographic QCD. In the Sakai-Sugimoto model of large N_c massless QCD, N baryons are described by N Yang-Mills instantons in 5 spacetime dimensions. We compute a classical short distance interaction hamiltonian for N 'tHooft instantons. This corresponds to N baryons sharing identical classical spins and isospins. We find that genuine N-body nuclear forces turn out to vanish for N ≥ 3, at the leading order. This suggests that classical N-body forces are always suppressed compared with 2-body forces.

N-Body Nuclear Forces at Short Distances in Holographic QCD

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two deviations from the YM theory in the flat space, at the leading order in $1/\lambda$ expansion: (i) the effect of the CS term, and (ii) the effect of the space weakly curved along the $x^4$ direction. The additional hamiltonians are

$$H_{\text{pot}}^{U(1)} = -\frac{a N_c}{2} \int \prod_M dx^M \hat{A}_0 \Box \hat{A}_0,$$  

$$H_{\text{pot}}^{SU(2)} = \frac{a N_c}{6} \int \prod_M dx^M \left( x^4 \right)^2 \text{tr}(F_{MN})^2,$$  

respectively, with $a \equiv 1/(216\pi^3)$. Note that we work in the unit $M_{KK} = 1$, where $M_{KK}$ is the unique scale parameter appearing in the model, and it can be fixed by fitting the $\rho$ meson mass, giving $M_{KK} = 949 \text{[MeV]}$.

In particular, for a single baryon, the leading order solution is a single instanton in the flat 4-dimensional space, that is BPS instanton [13]. The instanton has moduli parameters: the instanton location $X^M$, the size $\rho$ and the orientation in $SU(2)$. These hamiltonians induce potentials in the moduli space of the instanton, and $\rho$ and $X_i$ prefer particular values classically, $(\rho_3)^2 = \frac{1}{\pi^2 a} \sqrt{\frac{6}{g}}, X^4_i = 0$. For multi-instantons, there appears a potential for the moduli representing the distance between the instantons, which is in fact the nuclear force.

In Ref. [14], this 2-body nuclear force was evaluated explicitly. Multi-instanton solutions are available in flat space, while in this particular curved space it is difficult to find them. However, when instantons are close enough to each other, the effect of the curved space can be neglected, and as a leading order solution we can use the multi-instanton solutions in the flat space. Therefore, the distance $r_{ij}$ between the $i$-th and the $j$-th nucleons allowed in this approximation is $|r_{ij}| < M_{KK}^{-1}$ ($|r_{ij}| < \Lambda^{3/2} M_{KK}^{-1}$) in the original (rescaled) coordinates. Thus we probe only the short range for the nuclear force.

The construction of the two-instanton solution owes to the renowned ADHM (Atiyah-Drinfeld-Hitchin-Manin) method [16, 17]. The moduli parameters of generic $N$ instanton solutions are completely encoded in the real $N \times N$ matrix function $L(x; X, \cdots)$. The Osborn’s formula [18] tells us the instanton density

$$\text{tr}(F_{MN})^2 = \Box^2 \log \det L$$  

where $\Box \equiv \partial_M \partial_M$. Using this expression, the equation of motion for the U(1) part of the gauge field which is sourced by the instanton density is solved as [10]

$$\hat{A}_0 = \frac{1}{32\pi^2 a} \Box \log \det L.$$  

With this explicit dependence on the instanton moduli parameters in $L$, one can compute the hamiltonians [2] and [3] as functions of them. Then, the expectation value of the hamiltonians for given baryon states (the wave functions are written by the moduli parameters) gives the nuclear force at short range [14].

There is the third contribution to the additional hamiltonians, $H_{\text{kin}}$, which is present only in multi-instanton case. This comes from the metric of the instanton moduli space. In Ref. [14], it was shown that it is higher order in $1/N_c$ compared to the other two hamiltonians [2] and [3], so we need not compute it in this paper.

--- 3-body Nuclear Force. The 2-body nuclear force computed in Ref. [14] is for generic spin/isospin components. But since explicit generic $N$ instanton solution is not available, we consider a special solution called ‘tHooft instanton which has $5N - 3$ moduli parameters (while generic instanton solution has $8N - 3$ moduli parameters). It is important to notice that once we restrict our moduli space by hand like this, we cannot get the generic expression for the nuclear force for given baryon states. Instead, what we will obtain is a classical analogue of the nuclear force.

The moduli parameters of the ‘tHooft instantons are only the size $\rho_i$ and the location $X^M_i$ of each instanton ($i = 1, 2, \cdots, N$). The missing parameters, the orientations of the instants in $SU(2)$, are responsible for the spin/isospin wave functions of the baryons. Thus our analysis with the ‘tHooft instantons is restricted to “classical” baryons, where all the spin/isospins of the baryons are identical classically.

First, let us show that $H_{\text{pot}}^{SU(2)}$ given in Eq. (3) is irrelevant to the three-body nuclear forces. We can use the generic formula obtained in Appendix C of Ref. [14],

$$\int d^4 x \left( x^4 \right)^2 \text{tr}(F_{MN})^2 = 8\pi^2 \sum_{i=1}^N \left( 2(X^4_i)^2 + \rho_i^2 \right),$$  

for the $N$ ‘tHooft instantons. The expression consists of just a sum of each instanton sector, which means that there is no term involving the inter-nucleon distance, that is, no contribution to the nuclear force. Therefore, we compute the other hamiltonian [2] in this paper. (The contribution from $H_{\text{kin}}$ is suppressed as in the case of the 2 instantons [19].)

In this section, we concentrate on the case for $N = 3$, i.e. the 3-body force.

For three ‘tHooft instantons, which correspond to nucleons sharing classically identical spins/isospins, we have

$$L = \begin{pmatrix} (x - X_1)^2 + \rho_1^2 & \rho_1 \rho_2 & \rho_1 \rho_3 \\ \rho_1 \rho_2 & (x - X_2)^2 + \rho_2^2 & \rho_2 \rho_3 \\ \rho_1 \rho_3 & \rho_2 \rho_3 & (x - X_3)^2 + \rho_3^2 \end{pmatrix},$$  

where we omit the index $M$ and denote $(x^M)^2$ by $x^2$. Then the Osborn’s formula becomes particularly simple, 

$$\log \det L = \sum_i \log(x - X_i)^2 + \log f,$$  

with $f \equiv 1 + \sum_i \frac{\rho_i^2}{(x - X_i)^2}$. This gives the U(1) gauge field

$$\hat{A}_0 = -\frac{1}{32\pi^2 a} \left[ \Box \sum_i \log(x - X_i)^2 + \frac{\Box f}{f} - \left( \partial_M f \right)^2 \right].$$
The first term in Eq. (8) is a self-energy which was already computed, and \( f \) is a harmonic function, i.e., \( \Box f = 0 \). Thus, all we need to evaluate is only the last term in Eq. (9), \( (\partial_M f)^2 / f^2 \).

For three instantons, we can expand the expression for \((x - X_1)^2 \ll (x - X_2)^2, (x - X_3)^2\). In particular, we can approximate \((x - X_1)^2 \approx (X_1 - X_2)^2 \equiv X_{12}^2\), and a similar expression for \(X_{13}\). Furthermore, for simplicity we put \(X_{1i} = 0\). Then, the expansion is

\[
\hat{A}_0 = \frac{1}{8\pi^2 a} \left\{ \frac{1}{x^2} \left( 1 - \frac{\rho_1^4}{(x^2 + \rho_1^2)^2} \right) + \frac{1}{X_{12}^2} \frac{1}{X_{13}^2} \right. \\
+ \frac{2\rho_1^4}{(x^2 + \rho_1^2)^3} \left( \frac{\rho_2^3}{X_{12}^2} + \frac{\rho_3^3}{X_{13}^2} \right) + \frac{-3\rho_1^4 x^2}{(x^2 + \rho_1^2)^4} \left( \frac{\rho_2^3}{X_{12}^2} + \frac{\rho_3^3}{X_{13}^2} \right)^2 \\
+ \frac{-2\rho_1^2}{(x^2 + \rho_1^2)^2} \left( \frac{\rho_2^3 x \cdot X_{12}^2}{X_{12}^2} + \frac{\rho_3^3 x \cdot X_{13}^2}{X_{13}^2} \right) + \text{higher} \right\}
\]  

We like to compute the potential (9). As seen from the expression for \(\hat{A}_0\), the leading term of the nontrivial three-body force has the form \(\frac{1}{X_{12}^2 X_{13}^2}\). This means that, we should expect the nuclear force appearing in \(\text{proper to the 3-body is} at\), the leading order, 

\[ H = \mathcal{O}\left(\frac{N_c}{\chi^2}\right) \frac{1}{X_{12}^2 X_{13}^2}, \]  

in the original coordinates. Let us consider only terms of this leading form (11). We obtain

\[
H_{\text{pot}}^{(1)} \bigg|_{3\text{-body}} = \frac{-a N_c}{2(8\pi^2 a)^2} \times \int d^4x \left\{ \frac{1}{x^2} \left( 1 - \frac{\rho_1^4}{(x^2 + \rho_1^2)^2} \right) \right. \\
- \frac{6\rho_1^4 x^2}{(x^2 + \rho_1^2)^3} \frac{\rho_2^2 \rho_3^2}{X_{12}^2 X_{13}^2} \left( 1 - \frac{\rho_1}{(x^2 + \rho_1^2)^2} \right) \\
+ \frac{1}{X_{12}^2 X_{13}^2} \left( 1 + \frac{2\rho_1^2 \rho_2^3}{(x^2 + \rho_1^2)^3} \right) \left( 1 + \frac{2\rho_1^2 \rho_3^3}{(x^2 + \rho_1^2)^3} \right) \\
+ \left. + (1 \rightarrow 2 \rightarrow 3) + (1 \rightarrow 3 \rightarrow 2) \right\}. \]  

Performing the derivatives, and using the following integration formulas

\[
\int d^4x \frac{x^{2(N-j)}}{(x^2 + \rho_1^2)^{N+j}} = \frac{\pi^2 (j+2)! (N-j+1)!}{\rho_1^{2j+6} (N+4)!}, \]  

we find that the right hand side of Eq. (12) vanishes. Therefore, the leading term of the order \(1/(X_{12}^2 X_{13}^2)\) vanishes. This means that the expansion starts from the next-to-leading order,

\[
H_{\text{pot}}^{(1)} \bigg|_{3\text{-body}} = \frac{-N_c}{128\pi^4 a} \mathcal{O}\left(\frac{(\rho_1^4)^4}{X_{12}^2 X_{13}^2}, \frac{(\rho_1^4)^4}{X_{12}^2 X_{13}^2}, \cdots\right), \]  

where \(\cdots\) represents terms obtained by permutation for the indices 1, 2, 3. Here the dependence on \(\rho_i\) \((i = 1, 2, 3)\) is fixed to be \((\rho_1^4)\) by a dimensional analysis. The expectation value of this \((\rho_1^4)\) at the leading order in large \(N_c\) is given by the classical value given before. Then, rescaling the coordinates back as \(X_{12} \rightarrow \lambda^{1/2} X_{12}\) and write it as the 3-dimensional inter-nucleon distance \(r_{12}\) since we substitute the classical value \(X_{12}^4 = 0\), we obtain, at the leading order in \(1/N_c\),

\[
H_{\text{pot}}^{(1)} \bigg|_{3\text{-body}} = \frac{N_c}{\chi^2} \mathcal{O}\left(\frac{1}{r_{12}^4}, \frac{1}{r_{12}^4}, \cdots\right), \]  

again \(\cdots\) represents term obtained by permutation for the indices 1, 2, 3.

Note that we are working in a regime \(\lambda^{-1/2} \ll X_{12,13} \ll 1\) in the unit \(M_{KK} = 1\). The natural scale for the 2-body force \(14\) is \(\mathcal{O}(N_c/\chi X_{12}^2)\). So, if we consider a natural separation of the nucleons as \(X_{ij} \sim 1/M_{KK}\), the 3-body force is suppressed compared to the 2-body force. We conclude that the 3-body force at short range is small, for baryons carrying classical and equal spin/iso spins.

— N-body Nuclear Force. We can easily extend the analysis in the previous section to \(N\) 'tHooft instantons. The result for the leading term vanishes again, as we explain briefly below.

The quantity necessary for computing \(\hat{A}_0\) is

\[
\left(\frac{\partial f}{\partial \lambda}\right)^2 = \left[ \frac{4\rho_1^4}{x^6} + \sum_{i=2}^N \frac{4\rho_1^2 \rho_i^2 x \cdot X_{1i}}{x^4 X_{1i}^2} + \frac{N}{X_{1i}^2} \right] \left( 1 + \frac{\rho_i^2}{x^2} + \frac{\rho_i^2}{X_{1i}^2} \right)^{-2}. \]  

The expansion analogous to 3-body case implies that the leading order of the short-range nuclear force in proper to \(N\)-body would be

\[
H_{\text{pot}}^{(1)} \bigg|_{N\text{-body}} = \frac{N_c}{\chi^{N-1}} \mathcal{O}\left(\prod_{i=2}^N \frac{1}{X_{1i}^2}, \cdots\right). \]  

Again, we have rescaled back the coordinates to the original coordinates and \(\cdots\) represents permutation terms. In the previous section, we showed that for \(N = 3\) this leading contribution vanishes, for the 'tHooft instantons. In this section, we prove that for any \(N\) this leading contribution vanishes.

First, in the integral (2), it is straightforward to obtain

\[
\left(\frac{\partial f}{\partial \lambda}\right)^2 = (N - 1)! \frac{64\rho_1^4 (-1)^{N-1}}{x^2 (x^2 + \rho_1^2)^{N-1}} \sum_{i=2}^N \frac{1}{X_{1i}^2} \sum_{l=1}^N l(N - l + 1) \times \left[ (N-l)(N-l-1)\rho_i^4 (N-l+1) x^2 \rho_i^2 + 6x^4 \right]. \]  

Using the formula (13), this can be easily integrated with \(f d^4x\) to give 0.
In the Hamiltonian (17), there are additional terms
\begin{equation}
- \int d^4x \left( \frac{4}{x^2} + \sum_{i=2}^{N} \frac{4}{X_{ii}}^{2} \right) \left( \frac{\partial M(f)}{f} \right)^2
\end{equation}
coming from the first term in Eq. (9). In the same manner this is shown to vanish at the order (17). Therefore, we conclude that the leading order $N$-body nuclear force (17) vanishes, for arbitrary $N$. Note that the 2-body nuclear force does not vanish at the leading order, as $N = 2$ computation is exceptional.

— Summary and Discussions. Using the SS model of holographic QCD, we have found that the $N$-body nuclear force at short range ($N \geq 3$) is order of $N_c(\lambda r^2)^{-N}$, for nucleons sharing identical classical spin/isospins. This is small compared to the 2-body force which is $O(N_c(\lambda r^2)^{-1})$ in contrast, and it leads to a hierarchy of the $(N+1)$-body / $N$-body ratio $V^{(N+1)}/V^{(N)} \sim 1/(\lambda r^2) \ll 1$ for $N \geq 3$, in the unit $M_{KK} = 1$. This suppression is consistent with our empirical knowledge.

Effects of the short-range many-body interaction becomes more prominent for higher-density nuclear matter. Therefore, for physics of neutron stars and supernovae, for instance, properties of $N$-body interactions such as what is revealed in this letter are important, even if qualitative.

Our computation is not fully satisfactory since the quantum spin/isospin states of each baryon have not been incorporated. The wave function of the classical spin/isospin is a delta-function of the SU(2) orientational moduli of the instantons, thus it is difficult to relate it with the quantum spin/isospins. Nevertheless, it is quite remarkable that the generic $N$-body nuclear force can be obtained by analytic computations. The successful performance of this computation owes in particular to the simplicity of the SS model, in contrast to other chiral soliton models.

Furthermore, the theory on which our computations of the nuclear force is based is not a phenomenological model but a theory which has been “derived” from large $N_c$ QCD at strong coupling, through the gauge/string duality. Therefore, in principle, we can try to address theoretically what is different from QCD and what is inherited from it. The present computations go beyond the limitations of arguments using universality of chiral symmetry breaking.

Although the model at hand is for large $N_c$ QCD, these two properties of baryons in holographic QCD would be sufficiently strong motivations for studying holographic QCD and its relation to nuclear physics further.

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[19] Let us explain briefly why $H_{\text{kin}} \equiv -\nabla^2/16\pi^2 a_{\text{kin}}$ gives smaller contributions compared to the other two hamiltonians. Here $\nabla^2$ is the metric on the multi-instanton moduli space. We evaluate the expectation value of $\nabla^2$ with the baryon states. The dimension of $\nabla^2$ is $[\text{length}]^{-2}$. The leading 2-body forces are $\propto 1/X_{ij}^{2}$, so the remaining factor should be dimensionless. Thus $\nabla^2$ originating in the metric is written only by using a dimensionless operator $y \cdot \partial^2/\partial y^2$, where $y$ is the moduli $\rho$ and the SU(2) orientations. Then it was found in Ref. [13] that $\langle \nabla^2 \rangle \sim O(N_c)$, so, in total, after rescaling back $|X|^2 \rightarrow x|X|^2$, one obtains $\langle H_{\text{kin}} \rangle = O(\lambda^{-3})$, which is smaller than $\langle H^{(2)} \rangle \sim \langle H^{(1)} \rangle \sim O(N_c/\lambda)$ by the factor of $1/N_c$. This suppression by $1/N_c$ is expected to any instanton number, that is, general $N$-body force, and so in this paper we don’t consider $H_{\text{kin}}$. 