Remarks on the factorization and monotonicity method for inverse acoustic scatterings

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Abstract

We study the factorization and monotonicity method for inverse acoustic scattering problems. Firstly, we give a new general functional analysis theorem for the monotonicity method. Comparing with the factorization method, the general theorem of the monotonicity generates reconstruction schemes under weaker a priori assumptions for unknown targets, and can directly deal with mixed problems so that the unknown targets have several different boundary conditions. Using the general theorem, we give the reconstruction scheme for the mixed crack that the Dirichlet boundary condition is imposed on one side of the crack and the Neumann boundary condition on the other side, which is a new extension of monotonicity method.

1 Introduction

In this paper, we study the factorization and monotonicity method for inverse acoustic scattering problems. The factorization method has first been introduced by Kirsch ([17]) for the inverse acoustic obstacle scattering. It has been studied so far by many authors (see e.g., [1, 3, 4, 7, 12, 18, 19, 20, 21, 23, 25]), and it is well-known as one of classical qualitative methods, which includes the linear sampling method of Colton and Kirsch ([5]), the singular sources method of Potthast ([24]), the probe method of Ikehata ([16]), etc. The monotonicity method, on the other hand, has been recently introduced by Harrach in [15] for the electrical impedance tomography. Very recently, it was extended to the Helmholtz equation in the bounded domain [13, 14], the inverse acoustic obstacle [2], Dirichlet crack [9], and medium scatterings [11, 22]. It was found from these works that the monotonicity method has advantage over the factorization method that we can give the reconstruction scheme under weaker a priori assumptions for unknown targets.

The contributions of this paper are the following.

[A] We give a general functional analysis theorem for the monotonicity method (Theorem 4.2) including previous works [2, 9, 11] for inverse acoustic scatterings.

[B] Using the general theorem, we give the reconstruction scheme for the inverse mixed crack scattering, which is a new extension of the monotonicity method (Theorems 5.13 and 5.14).
A characteristic of the factorization method is to prepare the general functional analysis theorem which generates reconstruction schemes by spectrums of the far-field operator (see Theorem 2.15 of [19]). Based on this idea, the contribution [A] will be discussed as the version of the monotonicity method.

The general theorem of the factorization method assumes that the real part of the middle operator of the far-field operator has a decomposition into a positive coercive operator and a compact operator, while the imaginary part of the middle operator becomes strictly positive (see the assumptions (b) and (c) in Theorem 3.1). This two assumptions cause a priori assumptions for the unknown target, in particular, the positivity of the imaginary part corresponds to restrictions for the wave number. For example, it is necessary for the inverse medium scattering that the wave number is not a transmission eigenvalue with respect to the unknown medium (see e.g., Theorem 4.10 of [19]), and for the inverse obstacle scattering that the wave number is not an eigenvalue with respect to the boundary condition of the unknown obstacle (see e.g., Corollary 2.16 of [19]). However, the general theorem of the monotonicity method does not assume the positivity of the imaginary part (see Theorem 4.2), which means that the monotonicity can essentially avoid restrictions for the wave number. In fact, monotonicity reconstructions for inverse obstacle (Theorem 5.3 of [2]) and medium (Theorems 5.1–5.3 of [11]) have been successful without restrictions for the wave number.

The advantage of the monotonicity over the factorization is not only weaker a priori assumptions, but also to directly deal with mixed problems that the unknown target consists of two separate components with different boundary conditions. The real part of middle operator of the far-field operator for mixed problems is not decomposed into a positive coercive operator and a compact operator (see e.g., Theorem 3.4 of [19], Theorem 3.2 of [20]). In order to make a decomposition of the coercivity and the compactness, the factorization method for mixed problems needs masking approaches that one component is covered by an artificial domain disjoint with the other component we want to reconstruct (see e.g., Lemma 3.5 of [19], (3.26) of [20]). However, the general theorem of the monotonicity method (see (3) of Theorem 4.2) does not assume such a real part decomposition, which means that the monotonicity essentially do not need masking approaches. In fact, monotonicity reconstructions for inverse mixed obstacle (Theorem 5.5 of [2]) have been successful without the masking approach.

This paper studies not only previous works from the viewpoint of the general theorem, but also a new extension of the monotonicity to the inverse acoustic mixed crack scattering, which corresponds to the contribution [B]. The mixed crack consists of only one component, but imposes the Dirichlet boundary condition on one side of the unknown crack and the Neumann boundary condition on the other side (see the beginning of Section 5.5). The factorization method for the mixed crack has been studied in [25], but an extensive closed curve of the unknown crack should be known, which is a very restrictive assumption (see Theorem 3.3 of [25]). Using the general theorem for the monotonicity method, we give a reconstruction scheme without assuming such an extensive curve (see Theorems 5.13 and 5.14).

This paper is organized as follows. In Section 2, we define the inverse acoustic scattering problem. In Section 3, we recall the functional analysis theorem for the fac-
torization method. In Section 4, we give a new general functional analysis theorem (Theorem 4.2) for the monotonicity method. In Section 5, we study several applications of the general theorem. The results discussed in Sections 5.1–5.4 are the same in previous works [2, 11, 9], while the one in Section 5.5 is new in the monotonicity for the inverse mixed crack scattering. Finally in Section 6, we give numerical examples for our theoretical results.

2 Inverse acoustic scattering

First of all, we define the inverse acoustic scattering problem. Let $k > 0$ be the wave number, and let $\theta \in \mathbb{S}^{d-1}$ ($d = 2, 3$) be the incident direction. We denote the incident field $u^{inc}(x, \theta)$ with incident direction $\theta$ by the plane wave of the form

$$u^{inc}(x, \theta) := e^{ikx \cdot \theta}, \ x \in \mathbb{R}^d. \quad (2.1)$$

Let $\Omega \subset \mathbb{R}^d$ be a bounded open set with the smooth boundary $\partial \Omega$ such that the exterior $\mathbb{R}^d \setminus \overline{\Omega}$ is connected. Here, we denote by $\overline{\Omega} = \Omega \cup \partial \Omega$. In particular, we discuss the following two cases. The first case is that the scatterer $\Omega$ is a penetrable medium, and determine the total field $u = u^{sca} + u^{inc}$ such that

$$\Delta u + k^2 (1 + q) u = 0 \text{ in } \mathbb{R}^d, \quad (2.2)$$

$$\lim_{r := |x| \to \infty} r^{\frac{d-1}{2}} \left( \frac{\partial u^{sca}}{\partial r} - iku^{sca} \right) = 0, \quad (2.3)$$

where $q \in L^\infty(\mathbb{R}^d)$ has a compact support such that $\Omega = \text{supp} \ q$, and $\Delta$ is the Laplace operator. The Sommerfeld radiation condition (2.3) holds uniformly in all directions $\hat{x} := \frac{x}{|x|}$. The second case is that $\Omega$ is an impenetrable obstacle, and determine the total field $u = u^{sca} + u^{inc}$ such that

$$\Delta u + k^2 u = 0 \text{ in } \mathbb{R}^d \setminus \overline{\Omega}, \quad (2.4)$$

$$B u = 0 \text{ on } \partial \Omega, \quad (2.5)$$

and $u^{sca}$ satisfies the Sommerfeld radiation condition (2.3) where (2.5) means the boundary conditions, for example, the Dirichlet boundary condition $Bu = u$, the Neumann boundary condition $Bu = \frac{\partial u}{\partial \nu}$, etc. In both cases, it is well-known that there exists a unique solution $u^{sca}$ that has the following asymptotic behaviour (see e.g., [6])

$$u^{sca}(x) = \frac{e^{ikr}}{r^\frac{d-1}{2}} \left\{ u^\infty(\hat{x}, \theta) + O(1/r) \right\}, \ r \to \infty. \quad (2.6)$$

The function $u^\infty$ is called the far-field pattern of the scattered field $u^{sca}$. With the far-field pattern $u^\infty$, we define the far-field operator $F : L^2(\mathbb{S}^{d-1}) \to L^2(\mathbb{S}^{d-1})$ by

$$F g(\hat{x}) := \int_{\mathbb{S}^{d-1}} u^\infty(\hat{x}, \theta) g(\theta) d\sigma(\theta), \ \hat{x} \in \mathbb{S}^{d-1}. \quad (2.7)$$

In the inverse acoustic scattering problem, we reconstruct $\Omega$ from the far-field pattern $u^\infty(\hat{x}, \theta)$ for all $\hat{x}, \theta \in \mathbb{S}^{d-1}$, and fixed $k > 0$. In other words, given the far-field operator $F$, we reconstruct $\Omega$.
3 The factorization method

Here, we recall the general functional analysis theorem for the factorization method. The following functional analytic theorem is proved by Theorem 2.15 of [19] and Theorem 3.1 of [8].

**Theorem 3.1** (Theorem 2.15 of [19] and Theorem 3.1 of [8]). Let $X \subset U \subset X^*$ be a Gelfand triple with a Hilbert space $U$ and a reflexive Banach space $X$ such that the embedding is dense. Furthermore, let $Y$ be a Hilbert space and let $F : Y \rightarrow Y$, $G : X \rightarrow Y$, $T : X^* \rightarrow X$ be linear bounded operators such that

$$F = GTG^*.$$  \hfill (3.1)

We make the following assumptions:

(a) $G$ is compact with dense range in $Y$.

(b) $\text{Re}T$ has the form $\text{Re}T = C + K$ where $K : X^* \rightarrow X$ is some self-adjoint compact operator and $C : X^* \rightarrow X$ is some positive coercive operator, i.e., there exists a constant $c > 0$ such that

$$\langle \varphi, C\varphi \rangle \geq c\|\varphi\|^2_{X^*}, \text{ for all } \varphi \in X^*, \hfill (3.2)$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $X^*$ and $X$.

(c) $\text{Im}\langle \varphi, T\varphi \rangle > 0$ for all $\varphi \in \overline{\text{Ran}(G^*)}$ with $\varphi \neq 0$.

Then, the operator $F_\# := \left| \text{Re}F \right| + \text{Im}F$ is non-negative, and the ranges of $G : X \rightarrow Y$ and $F^{1/2}_\# : Y \rightarrow Y$ coincide with each other, that is, we have the following range identity:

$$\text{Ran}(G) = \text{Ran}(F^{1/2}_\#).$$  \hfill (3.3)

Here, the real part and the imaginary part of an operator $A$ are self-adjoint operators given by

$$\text{Re}A = \frac{A + A^*}{2} \text{ and } \text{Im}A = \frac{A - A^*}{2i}.$$  \hfill (3.4)

**Remark 3.2.** It has been well-known that in Theorem 2.1 of [23] the assumption (c) can be replaced by the injectivity of $T$, and it has been mainly used especially for the relaxation of the assumption that the wave number $k > 0$ is not a transmission eigenvalue in inverse medium scatterings (see e.g., [7, 19, 20, 23]). However, it was found that this replacement is not correct (see Remark 3.2 of [8]), thus the factorization method for inverse medium scatterings essentially needs the assumption of transmission eigenvalues.
4 General theorems for the monotonicity method

In this section, we give a new general functional analysis theorem for the monotonicity method.

**Definition 4.1.** Let $A, B : H \to H$ be self-adjoint compact linear operators on a Hilbert space $H$. We write

$$A \leq_{\text{fin}} B, \tag{4.1}$$

if $B - A$ has only finitely many negative eigenvalues.

**Theorem 4.2.** Let $X \subset U \subset X^*$, $	ilde{X} \subset \tilde{U} \subset \tilde{X}^*$ be Gelfand triples with Hilbert spaces $U$, $	ilde{U}$ and reflexive Banach spaces $X$, $	ilde{X}$ such that the embeddings are dense. Furthermore, let $Y$ be a Hilbert space and let $F : Y \to Y$, $\tilde{F} : Y \to Y$, $G : X \to Y$, $\tilde{G} : \tilde{X} \to Y$, $T : X^* \to X$, $\tilde{T} : \tilde{X}^* \to \tilde{X}$ be linear bounded operators such that

$$F = GTG^*, \quad \tilde{F} = \tilde{G}\tilde{T}\tilde{G}^*. \tag{4.2}$$

(1) Assume that

(a) $\text{Re} T$ has the form $\text{Re} T = C + K$ where $C : X^* \to X$ is some positive coercive operator and $K : X^* \to X$ is some self-adjoint compact operator.

(b) There exists a compact operator $R : \tilde{X} \to X$ such that $\tilde{G} = GR$.

Then,

$$\text{Re} \tilde{F} \leq_{\text{fin}} \text{Re} F. \tag{4.3}$$

(2) Assume that

(a) $\text{Re} \tilde{T}$ has the form $\text{Re} \tilde{T} = \tilde{C} + \tilde{K}$ where $\tilde{C} : \tilde{X}^* \to \tilde{X}$ is some positive coercive operator and $\tilde{K} : \tilde{X}^* \to \tilde{X}$ is some self-adjoint compact operator.

(b) There exists an infinite dimensional subspace $W$ in $\text{Ran}(\tilde{G})$ such that $W \cap \text{Ran}(G) = \{0\}$.

Then,

$$\text{Re} \tilde{F} \not\leq_{\text{fin}} \text{Re} F. \tag{4.4}$$

(3) Let $X_j \subset U_j \subset X_j^*$ $(j = 1, 2)$ be a Gelfand triple with a Hilbert space $U_j$ and a reflexive Banach space $X_j$ such that the embedding is dense. Let $F^{\text{Mix}} : Y \to Y$, $G^{\text{Mix}} : X_1 \times X_2^* \to Y$, $T^{\text{Mix}} : X_1^* \times X_2 \to X_1 \times X_2^*$ be linear bounded operators such that

$$F^{\text{Mix}} = G^{\text{Mix}} T^{\text{Mix}} G^{\text{Mix}}^*. \tag{4.5}$$

Assume that
\( \text{(3a)} \) \( \text{Re} T^{\text{Mix}} \) has the form \( \text{Re} T^{\text{Mix}} = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} + K^{\text{Mix}} \) where \( C_{11} : X_1^* \to X_1 \) is some positive coercive operator, \( C_{12} : X_2 \to X_1 \) and \( C_{21} : X_1^* \to X_2^* \) are some linear bounded operators, \( C_{22} : X_2 \to X_2^* \) is some negative coercive operator (that is, \(-C_{22}\) is positive coercive), and \( K^{\text{Mix}} : X_1^* \times X_2 \to X_1 \times X_2^* \) is some self-adjoint compact operator.

\( \text{(3b)} \) There exists an infinite dimensional subspace \( W_1 \) in \( \text{Ran}(G^{\text{Mix}} R_1^*) \) such that \( W_1 \cap \text{Ran}(\begin{pmatrix} \tilde{G}, G^{\text{Mix}} R_2^* \end{pmatrix}) = \{0\} \) where \( R_1 : X_1^* \times X_2 \to X_1^* \) is defined by
\[
R_1 \begin{pmatrix} f \\ g \end{pmatrix} := f, \quad \text{and its adjoint operator } R_1^* : X_1 \to X_1 \times X_2^* \text{ is given by } R_1^* \phi := \begin{pmatrix} \phi \\ 0 \end{pmatrix}.
\]

Then,
\[
\text{Re} F^{\text{Mix}} \not\leq_{\text{fin}} \text{Re} \tilde{F}.
\]

**Remark 4.3.** If the assumption (3b) is replaced by

\( \text{(3b)'} \) There exists a finite dimensional subspace \( W_2 \) in \( \text{Ran}(G^{\text{Mix}} R_2^*) \) such that \( W_2 \cap \text{Ran}(\begin{pmatrix} \tilde{G}, G^{\text{Mix}} R_1^* \end{pmatrix}) = \{0\} \) where \( R_2 : X_1^* \times X_2 \to X_2^* \) is defined by
\[
R_2 \begin{pmatrix} f \\ g \end{pmatrix} := g,
\]
and its adjoint operator \( R_2^* : X_1^* \times X_2 \to X_1 \times X_2^* \) is given by
\[
R_2^* \psi := \begin{pmatrix} 0 \\ \psi \end{pmatrix},
\]
then, we can show that \(-\text{Re} F^{\text{Mix}} \not\leq_{\text{fin}} \text{Re} \tilde{F}\) by the same argument.

We recall the following technical lemmas which will be useful to prove Theorem 4.2.

**Lemma 4.4** (Corollary 3.3 of [14]). Let \( A,B : H \to H \) be self-adjoint compact linear operators on a Hilbert space \( H \) with an inner product \((\cdot,\cdot)_H\). Then, the following statements are equivalent:

1. \( A \leq_{\text{fin}} B \)
2. There exists a finite dimensional subspace \( V \) in \( H \) such that
\[
((B - A)v,v)_H \geq 0,
\]
for all \( v \in V^\perp \).

**Lemma 4.5** (Lemma 4.6 in [14]). Let \( X, Y, \) and \( Z \) be Hilbert spaces, and let \( A : X \to Y \) and \( B : X \to Z \) be bounded linear operators. Then,
\[
\exists C > 0 : \|Ax\|_Y^2 \leq C \|Bx\|_Z^2 \text{ for all } x \in X \iff \text{Ran}(A^*) \subset \text{Ran}(B^*). \]

**Lemma 4.6** (Lemma 4.7 in [14]). Let \( X, Y, V \subset Z \) be subspaces of a vector space \( Z \). If
\[
X \cap Y = \{0\}, \quad \text{and } X \subset Y + V,
\]
then, \( \dim(X) \leq \dim(V) \).
Proof of Theorem 4.2. (1) Since the restriction $C|_U : U \to U$ is positive, there exists a positive square root $\hat{W} : U \to U$, i.e., $C|_U = \hat{W}^2$. Since we have,

$$\|\hat{W}\varphi\|_U^2 = (\varphi, \hat{W}^2\varphi)_U = (\varphi, C|_U\varphi)_U = (\varphi, C\varphi) \leq \|C\|\|\varphi\|_{X^*}^2,$$

for all $\varphi \in U, \quad (4.10)$

and the embedding $U \subset X^*$ is dense, $\hat{W}$ has a bounded extension $W : X^* \to U$ of $\hat{W}$. By the positive coercivity of $C$, there exists a constant $c > 0$ such that for all $\varphi \in U$,

$$c\|\varphi\|_{X^*}^2 \leq (\varphi, C\varphi) = (\varphi, C|_U\varphi)_U = \|\hat{W}\varphi\|_U^2.$$ \quad (4.11)

Hence, by the dense embedding $U \subset X^*$, we have $c\|\varphi\|_{X^*}^2 \leq \|W\varphi\|_U^2$ for all $\varphi \in X^*$, which implies that the extension $W : X^* \to U$ of $\hat{W}$ is bounded invertible. It is easy to check that $C = W^*W$. By this, the factorization (4.2) of operators $F$ and $\tilde{F}$, assumptions (1a) and (1b), we have

$$\text{Re}F - \text{Re}\tilde{F} = G \left[ C + \text{Re}K - R(\text{Re}\tilde{T})R^* \right] G^*$$

$$= [GW^*] \left[ W^{*-1}CW^{-1} + W^{*-1} \left\{ \text{Re}K - R(\text{Re}\tilde{T})R^* \right\} W^{-1} \right] [GW^*]^*$$

$$=: \tilde{G}[I_U + K]\tilde{G}^*.$$ \quad (4.12)

Let $\{\mu_j, \phi_j\}$ be an eigensystem of the self-adjoint compact operator $\tilde{K} : U \to U$, and let

$$V := \text{span} \left\{ \phi_j : \mu_j \leq -\frac{1}{2} \right\}. \quad (4.13)$$

Then, $V$ is a finite dimensional subspace of $U$, and for all $\varphi \in \left[ \text{Ran} \left( \tilde{G}|_V \right) \right]_+$, which is equivalent to $\tilde{G}^*\varphi \in V_+ = \text{span} \left\{ \phi_j : \mu_j > -\frac{1}{2} \right\}$,

$$((\text{Re}F - \text{Re}\tilde{F})\varphi, \varphi)_Y = ((I_U + \tilde{K})\tilde{G}^*\varphi, \tilde{G}^*\varphi)_U \geq \frac{1}{2} \|\tilde{G}^*\varphi\|_U^2 \geq 0. \quad (4.14)$$

From $\text{dim} \left[ \text{Ran} \left( \tilde{G}|_V \right) \right] < \infty$ and Lemma 4.4, we conclude (4.3).

(2) Assume on the contrary that $\text{Re}\tilde{F} \leq_{\text{fin}} \text{Re}F$. Then by Lemma 4.4, there exists a finite dimensional subspace $V_1$ in $Y$ such that

$$(\text{Re}\tilde{F}\varphi, \varphi)_Y \leq (\text{Re}F\varphi, \varphi)_Y, \quad (4.15)$$

for all $\varphi \in V_1^\perp$. By the same argument in the beginning of (1), there exists a bounded invertible operator $\tilde{W} : \tilde{X}^* \to \tilde{U}$ and a self-adjoint compact operator $K$ such that

$$\text{Re}\tilde{F} = [\tilde{G}\tilde{W}^*][I_{\tilde{U}} + K][\tilde{G}\tilde{W}^*]^*,$$ \quad (4.16)

which implies that by the same argument in (4.13)-(4.14) there exists a finite dimensional subspace $V_2$ in $Y$ and a constant $c > 0$ such that

$$(\text{Re}\tilde{F}\varphi, \varphi)_Y \geq \frac{1}{2} \|\tilde{W}\tilde{G}^*\varphi\|_{\tilde{U}}^2 \geq c\|\tilde{G}^*\varphi\|_{X^*}^2,$$ \quad (4.17)
for all $\varphi \in V^\perp$. Setting $V := V_1 \cup V_2$, $V$ is a finite dimensional subspace in $Y$. Then by (4.15) and (4.17) we have

\[ c\|G^*\varphi\|_{X^*}^2 \leq (\text{Re} F \varphi, \varphi)_Y \leq \|\text{Re} T\|\|G^*\varphi\|_{X^*}^2, \tag{4.18} \]

for all $\varphi \in V^\perp$.

On the other hand, by the assumption (2b) and Lemma 4.6, we have $W \not\subset \text{Ran}(G) + V$, which implies that by $W \subset \text{Ran}(\tilde{G})$

\[ \text{Ran}(\tilde{G}) \not\subset \text{Ran}(G) + V = \text{Ran} ([G, P_V]), \tag{4.19} \]

where $P_V$ denotes the orthogonal projection on $V$. By this and Lemma 4.5, there exists a sequence $(\varphi_n)_{n \in \mathbb{N}} \subset Y$ such that

\[ \|\tilde{G}^*\varphi_n\|_{X^*}^2 > n^2 \left(\|G^*\varphi_n\|_{X^*}^2 + \|P_V \varphi_n\|_{Y}^2\right), \tag{4.20} \]

for all $n \in \mathbb{N}$. Setting $\psi_n := \frac{1}{n^2}\varphi_n$, we obtain

\[ \|G^*\psi_n\|_{X^*}^2 + \|P_V \psi_n\|_{Y}^2 = \frac{n}{\|G^* \varphi_n\|_{X^*}^2} \|G^* \varphi_n\|_{X^*}^2 + \|P_V \varphi_n\|_{Y}^2 \leq \frac{1}{n}, \quad \|G^* \psi_n\|_{X^*} = n. \tag{4.21} \]

Setting $\tilde{\psi}_n := (I - P_V) \psi_n \in V^\perp$, we finally obtain

\[ \|\tilde{G}^*\tilde{\psi}_n\|_{X^*} \geq \|G^* \psi_n\|_{X^*} - \|G^*\|\|P_V \psi_n\|_{Y} \to \infty, \quad \text{as } n \to \infty, \tag{4.22} \]

\[ \|G^* \tilde{\psi}_n\|_{X^*} \leq \|G^* \psi_n\|_{X^*} + \|G^*\|\|P_V \psi_n\|_{Y} \to 0, \quad \text{as } n \to \infty, \tag{4.23} \]

which contradicts (4.18). Therefore, we conclude (4.4).

(3) Assume on the contrary that $\text{Re} F^{\text{Mix}} \leq \text{fin} \ \text{Re} \tilde{F}$. Then by Lemma 4.4, there exists a finite dimensional subspace $V_1$ in $Y$ such that

\[ (\text{Re} F^{\text{Mix}} \varphi, \varphi)_Y \leq (\text{Re} \tilde{F} \varphi, \varphi)_Y, \tag{4.24} \]

for all $\varphi \in V^\perp_1$. Since $C_{11} : X^*_1 \to X_1$ and $-C_{22}^{-1} : X^*_2 \to X_2$ are positive coercive, by the same argument in the beginning of (1) there exists a bounded invertible operator $W_{jj} : X^*_j \to U_j$ such that

\[ C_{11} = W^*_1 W_{11}, \quad -C_{22}^{-1} = W^*_2 W_{22}, \tag{4.25} \]

We denote by the operator $W := \begin{pmatrix} W^*_1 & 0 \\ 0 & W^*_2 \end{pmatrix} : U_1 \times U_2 \to X_1 \times X_2^*$, hence, we have

\[ \text{Re} F^{\text{Mix}} = [G^{\text{Mix}} W] \begin{pmatrix} W^{-1} \left( \begin{array}{cc} C_{11} & C_{12} \\ C_{21} & C_{22} \end{array} \right) + K^{\text{Mix}} \end{pmatrix} W^{-1} ] [G^{\text{Mix}} W]^*, \tag{4.26} \]
where \( \tilde{C}_{12} := W_{11}^{-1}C_{12}W_{22}^* : U_2 \to U_1 \), \( \tilde{C}_{21} := W_{22}C_{21}W_{11}^* : U_1 \to U_2 \), and \( \tilde{K}_{Mix} = W^{-1}K_{Mix}W^{-1}^* : U_1 \times U_2 \to U_1 \times U_2 \). Since \( \tilde{K}_{Mix} \) is a self-adjoint compact operator, by the same argument in \((4.13)-(4.14)\), there exists a finite dimensional subspace \( V_2 \) in \( Y \), and constants \( c_1, c_2, c_3 > 0 \) such that

\[
\begin{align*}
(\Re F_{Mix}^{Mix}, \varphi, \varphi)_Y \\
= & \left( \begin{pmatrix} \tilde{C}_{12} & \tilde{C}_{21} \\ \tilde{C}_{21} & -I_{U_2} \end{pmatrix} \right) + \tilde{K}_{Mix}^{Mix} \cdot \varphi, \tilde{K}_{Mix}^{Mix} \cdot \varphi)_{U_1 \times U_2} \\
\geq & \left\| W_{11}R_1G_{Mix}^{Mix} \cdot \varphi \right\|_{U_1}^2 - \left\| W_{22}^{-1}R_2G_{Mix}^{Mix} \cdot \varphi \right\|_{U_2}^2 \\
& - \frac{1}{2} \left\{ \left\| W_{11}R_1G_{Mix}^{Mix} \cdot \varphi \right\|_{U_1}^2 + \left\| W_{22}^{-1}R_2G_{Mix}^{Mix} \cdot \varphi \right\|_{U_2}^2 \right\} \\
& + \left( \begin{pmatrix} 0 & C_{12} \\ C_{21} & 0 \end{pmatrix} \right) G_{Mix}^{Mix} \cdot \varphi, G_{Mix}^{Mix} \cdot \varphi)_{U_1 \times U_2} \\
\geq & c_1\left\| R_1G_{Mix}^{Mix} \cdot \varphi \right\|_{X_1}^2 - c_2\left\| R_2G_{Mix}^{Mix} \cdot \varphi \right\|_{X_2}^2 \\
& - c_3\left\| R_1G_{Mix}^{Mix} \cdot \varphi \right\|_{X_1} \left\| R_2G_{Mix}^{Mix} \cdot \varphi \right\|_{X_2}, \quad (4.27)
\end{align*}
\]

for all \( \varphi \in V_2^\perp \). Setting \( V := V_1 \cup V_2 \), \( V \) is finite dimensional subspace in \( Y \). Then by \((4.24)\) and \((4.27)\) we have

\[
\begin{align*}
c_1\left\| R_1G_{Mix}^{Mix} \cdot \varphi \right\|_{X_1}^2 \leq & c_2\left\| R_2G_{Mix}^{Mix} \cdot \varphi \right\|_{X_2}^2 \\
& + c_3\left\| R_1G_{Mix}^{Mix} \cdot \varphi \right\|_{X_1} \left\| R_2G_{Mix}^{Mix} \cdot \varphi \right\|_{X_2} + \left\| \Re \tilde{T} \right\| \left\| \tilde{G}^* \varphi \right\|_{X_2}^2, \quad (4.28)
\end{align*}
\]

for all \( \varphi \in V^\perp \).

On the other hand, by the assumption \((3b)\) and Lemma \(4.6\), we have \( W_1 \not\subset \text{Ran} \left( \tilde{G}, G_{Mix}^{Mix}R_2^* \right) + V \), which implies that by \( W_1 \subset \text{Ran} (G_{Mix}^{Mix}R_1^*) \)

\[
\text{Ran} (G_{Mix}^{Mix}R_1^*) \not\subset \text{Ran} \left( \tilde{G}, G_{Mix}^{Mix}R_2^* \right) + V = \text{Ran} \left( \tilde{G}, G_{Mix}^{Mix}R_2^*, P_V \right). \quad (4.29)
\]

By this and Lemma \(4.5\), there exists sequence \( (\varphi_n)_{n \in \mathbb{N}} \subset Y \) such that

\[
\left\| R_1G_{Mix}^{Mix} \cdot \varphi_n \right\|_{X_1}^2 > n^2 \left( \left\| \tilde{G}^* \varphi_n \right\|_{X_2}^2, + \left\| R_2G_{Mix}^{Mix} \cdot \varphi_n \right\|_{X_2}^2 + \left\| P_V \varphi_n \right\|_{Y}^2 \right), \quad (4.30)
\]

for all \( n \in \mathbb{N} \). Setting \( \psi_n := \frac{n^\frac{3}{2} \varphi_n}{\left\| R_1G_{Mix}^{Mix} \cdot \varphi_n \right\|_{X_1}^2} \), we obtain

\[
\left\| \tilde{G}^* \psi_n \right\|_{X_2}^2 + \left\| R_2G_{Mix}^{Mix} \cdot \psi_n \right\|_{X_2}^2 + \left\| P_V \psi_n \right\|_{Y}^2 \leq \frac{1}{n}, \quad \left\| R_1G_{Mix}^{Mix} \cdot \varphi_n \right\|_{X_1}^2 = n. \quad (4.31)
\]

Setting \( \tilde{\psi}_n := (I - P_V)\psi_n \in V^\perp \), we finally obtain as \( n \to \infty \)

\[
\left\| R_1G_{Mix}^{Mix} \cdot \tilde{\psi}_n \right\|_{X_1} \geq \left\| R_1G_{Mix}^{Mix} \cdot \psi_n \right\|_{X_1} - \left\| R_1G_{Mix}^{Mix} \cdot \left\| P_V \psi_n \right\|_{Y}, \quad (4.32)
\]

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\[ \| \tilde{G}^* \tilde{\psi}_n \|_{X^*} + \| R_2 G^{Mix} \tilde{\psi}_n \|_{X_2} \leq \| \tilde{G}^* \tilde{\psi}_n \|_{X^*} + \| \tilde{G}^* \| \| P_V \psi_n \|_{Y} \]
\[ + \| R_2 G^{Mix} \tilde{\psi}_n \|_{X_2} + \| G^* \| \| P_V \psi_n \|_{Y} \to 0, \quad (4.33) \]
\[ \| R_1 G^{Mix} \tilde{\psi}_n \|_{X^*_1} \| R_2 G^{Mix} \tilde{\psi}_n \|_{X_2} \leq (\| R_1 G^{Mix} \psi_n \| + \| R_1 G^{Mix} \| \| P_V \psi_n \|) \]
\[ \times (\| R_2 G^{Mix} \psi_n \| + \| R_2 G^{Mix} \| \| P_V \psi_n \|) \to 1 + \| R_2 G^{Mix} \|, \quad (4.34) \]
which contradicts (4.28). Therefore, we conclude (4.6).

5 Applications of the general theorem

In the following, we study many applications of Theorem 4.2 to inverse acoustic scatterings.

5.1 Dirichlet obstacle

Let \( F^{Dir}_{\Omega} \) be the far-field operator for a Dirichlet obstacle \( \Omega \), that is, \( F^{Dir}_{\Omega} \) is the far-field operator defined by (2.7) corresponding to the solution of (2.4)–(2.5) where \( B u = u \).

\( F^{Dir}_{\Omega} \) has the factorization (see Theorem 1.15 of [19])

\[ F^{Dir}_{\Omega} = -G^{Dir}_{\Omega} S^\ast_{\Omega} G^{Dir}_{\Omega} \ast, \quad (5.1) \]

where \( G^{Dir}_{\Omega} : H^{1/2}(\partial \Omega) \to L^2(\mathbb{S}^{d-1}) \) is the data-to-pattern operator defined by \( G^{Dir}_{\Omega} f := v^\infty \) where \( v^\infty \) is the far-field pattern of a radiating solution \( v \) (that is, \( v \) satisfies the Sommerfeld radiation condition (2.3)) such that

\[ \Delta v + k^2 v = 0 \text{ in } \mathbb{R}^d \setminus \Omega, \quad v = f \text{ on } \partial \Omega, \quad (5.2) \]

and \( S_{\Omega} : H^{-1/2}(\partial \Omega) \to H^{1/2}(\partial \Omega) \) is the single layer boundary operator defined by

\[ S_{\Omega} \varphi(x) := \int_{\partial \Omega} \varphi(y) \Phi(x,y) ds(y), \quad x \in \partial \Omega, \quad (5.3) \]

where \( \Phi(x,y) \) denotes the fundamental solution for the Helmholtz equation in \( \mathbb{R}^d \), i.e.,

\[ \Phi(x,y) := \begin{cases} 
\frac{i}{4} H^{(1)}_0(k|x-y|), & d = 2, \\
\frac{e^{ik|x-y|}}{4\pi|x-y|}, & d = 3,
\end{cases} \quad (5.4) \]

where \( H^{(1)}_0 \) is the Hankel function of the first kind of order one. The single layer boundary operator \( S_{\Omega} \) is of the form (see Lemma 1.14 of [19])

\[ S_{\Omega} = C_{\Omega} + K_{\Omega}, \quad (5.5) \]
where \( C_\Omega : H^{-1/2}(\partial \Omega) \to H^{1/2}(\partial \Omega) \) is some positive coercive operator and \( K_\Omega : H^{-1/2}(\partial \Omega) \to H^{1/2}(\partial \Omega) \) is some compact operator.

For a bounded domain \( B \subset \mathbb{R}^d \) with the smooth boundary, we define the Herglotz operator \( H_{\partial B} : L^2(S^{d-1}) \to L^2(\partial B) \) by
\[
H_{\partial B}g(x) := \int_{S^{d-1}} e^{ik\theta \cdot x} g(\theta) ds(\theta), \ x \in \partial B,
\] (5.6)
If the range is restricted to the space \( H^{1/2}(\partial B) \), we denote its Herglotz operator by \( \hat{H}_{\partial B} : L^2(S^{d-1}) \to H^{1/2}(\partial B) \). From the definition, we have \( \hat{H}^*_{\partial B} = G^D_B S_B \) (see e.g., Theorem 1.15 of [19]). Let \( J_{\partial B} : H^{1/2}(\partial B) \to H^{-1/2}(\partial B) \) be a self-adjoint compact embedding. Then, we have
\[
H^*_{\partial B} H_{\partial B} = \hat{H}^*_{\partial B} J_{\partial B} \hat{H}_{\partial B} = G^D_B S_B J_{\partial B} S^*_B G^D_B^* \nonumber
\]
\[
= G^D_B \left[ C_B J_{\partial B} C_B + \hat{K}_B \right] G^D_B^*, \quad (5.7)
\]
where \( C_B J_{\partial B} C_B \) is a positive coercive operator and \( \hat{K}_B \) is some self-adjoint compact operator.

Assume that \( B \subset \Omega \). Then, we can define \( R : H^{1/2}(\partial B) \to H^{1/2}(\partial \Omega) \) by \( Rf := v|_{\partial \Omega} \) where \( v \) is a radiating solution \( v \) of (5.2) replacing \( \Omega \) with \( B \). Since \( v|_{\partial \Omega} \in C^\infty(\partial \Omega) \), \( R \) is a compact operator, and by the definition we have
\[
G^D_B = G^D_{\Omega}, \quad (5.8)
\]
which corresponds to the assumption (1a) of Theorem 4.2. Applying (1) of Theorem 4.2 as
\[
F = -F^D_{\Omega} = G^D_{\Omega}(C_\Omega + K_\Omega^*) G^D_{\Omega}^*, \nonumber
\]
\[
\tilde{F} = H^*_{\partial B} H_{\partial B} = G^D_B \left[ C_B J_{\partial B} C_B + \hat{K}_B \right] G^D_B^*, \nonumber
\]
we have
\[
H^*_{\partial B} H_{\partial B} \leqslant \text{fin} - \text{Re} F^D_{\Omega}. \quad (5.9)
\]
Assume that \( B \not\subset \Omega \). Then, there exists a bounded domain \( B_0 \subset B \) such that \( B_0 \cap \Omega = \emptyset \). We set \( W := \text{Ran}(G^D_{B_0}) \subset \text{Ran}(G^D_B) \), then, \( W \) is an infinite dimensional subspace of \( L^2(S^{d-1}) \) because \( G^D_{B_0} \) is injective (see e.g., Lemma 1.13 of [19]). From \( B_0 \cap \Omega = \emptyset \), we obtain
\[
W \cap \text{Ran}(G^D_{\Omega}) = \{0\}, \quad (5.10)
\]
(see e.g., Lemma 4.2 of [2]) which corresponds to the assumption (2b) of Theorem 4.2. Applying (2) of Theorem 4.2 as
\[
F = -F^D_{\Omega} = G^D_{\Omega}(C_\Omega + K_\Omega^*) G^D_{\Omega}^*, \nonumber
\]
\[
\tilde{F} = H^*_{\partial B} H_{\partial B} = G^D_B \left[ C_B J_{\partial B} C_B + \hat{K}_B \right] G^D_B^*, \nonumber
\]
we have
\[
H^*_{\partial B} H_{\partial B} \not\leqslant \text{fin} - \text{Re} F^D_{\Omega}. \quad (5.11)
\]
From the above discussion, we conclude the following theorem, which is the same result as Theorem 5.3 of [2].
Theorem 5.1 (Theorem 5.3 of [2]). Let \( B \subset \mathbb{R}^d \) be a bounded domain with the smooth boundary. Then,
\[
B \subset \Omega \iff H_{\partial B}^* H_{\partial B} \leq \Re F_{\Omega}^{\text{Dir}}, \quad (5.12)
\]
By the same argument in Theorem 5.1, one can apply (1) and (2) of Theorem 4.2 as
\[
F = H_{\partial B}^* H_{\partial B} = G_{B}^{\text{Dir}} \left[ C_B J_{\partial B} C_B + \hat{K}_B \right] G_{B}^{\text{Dir}}^*,
\]
\[
\tilde{F} = -F_{\Omega}^{\text{Dir}} = G_{\Omega}^{\text{Dir}} (C_{\Omega} + K_{\Omega}^*) G_{\Omega}^{\text{Dir}}^*.
\]
Then, we also conclude the following theorem.

Theorem 5.2. Let \( B \subset \mathbb{R}^d \) be a bounded domain with the smooth boundary. Then,
\[
\Omega \subset B \iff -\Re F_{\Omega}^{\text{Dir}} \leq \Re F_{\Omega}^{\text{Dir}}, \quad (5.13)
\]

Remark 5.3. We remark that the factorization reconstruction for the inverse obstacle scattering needs to assume that \( k^2 \) is not a Dirichlet eigenvalue of \( -\Delta \) in \( \Omega \) (see e.g., Corollary 2.16 of [19]). While, the monotonicity reconstruction (Theorems 5.1 and 5.2) does not require the assumption of Dirichlet eigenvalues.

5.2 Inhomogeneous medium

Let \( F_{\Omega}^{\text{Med}} \) be the far-field operator for an inhomogeneous medium \( \Omega \) with the function \( q \in L^\infty(\Omega) \), that is, \( F_{\Omega}^{\text{Med}} \) is the far-field operator defined by (2.7) corresponding to the solution of (2.2)–(2.3). Throughout this section, we assume that there exists a constant \( q_0 > 0 \) such that \( q \geq q_0 \) in \( \Omega \). \( F_{\Omega}^{\text{Med}} \) has the following factorization by the same argument in Theorem 4.5 of [19]
\[
F_{\Omega}^{\text{Med}} = H_{\Omega}^* T_{\Omega} H_{\Omega}, \quad (5.14)
\]
where \( H_{\Omega} : L^2(\mathbb{S}^{d-1}) \rightarrow L^2(\Omega) \) is the Herglotz operator defined by
\[
H_{\Omega} g(x) := \int_{\mathbb{S}^{d-1}} e^{ik\theta \cdot x} g(\theta) d\sigma(\theta), \quad x \in \Omega, \quad (5.15)
\]
and some operator \( T_{\Omega} : L^2(\Omega) \rightarrow L^2(\Omega) \) is of the form
\[
T_{\Omega} = k^2 q I_{L^2(\Omega)} + K_{\Omega}, \quad (5.16)
\]
where \( K_{\Omega} \) is some compact operator.

Let \( B \subset \mathbb{R}^d \) be a bounded domain with the smooth boundary. Assume that \( B \subset \Omega \). We define the restriction operator \( R : L^2(\Omega) \rightarrow L^2(B) \) by \( Rf := f|_B \). Then by the definition we have \( H_{B} = R H_{\Omega} \), and for \( \alpha \in (0, k^2 q_0) \) we have
\[
F_{\Omega}^{\text{Med}} - \alpha H_{B}^* H_{B} = H_{\Omega}^* \left[ k^2 q I_{L^2(\Omega)} - \alpha R^* R + K_{\Omega} \right] H_{\Omega}, \quad (5.17)
\]
where the operator \(k^2 q I_{L^2(\Omega)} - \alpha R^* R\) is positive coercive when \(\alpha \in (0, k^2 q_0)\). Applying (1) of Theorem 4.2 as

\[
F = F_\Omega^{Med} - \alpha H_B^* H_B = H_\Omega^*[k^2 q I_{L^2(\Omega)} - \alpha R^* R + K_\Omega] H_\Omega,
\]

\[\tilde{F} = 0,\]

we have for \(\alpha \in (0, k^2 q_0)\)

\[\alpha H_B^* H_B \preceq_{\text{fin}} \text{Re} F_\Omega^{Med}.\]  

(5.18)

Assume that \(B \not\subset \Omega\). Then, there exists a bounded domain \(B_0 \Subset B\) such that \(B_0 \cap \Omega = \emptyset\). We set \(W := \text{Ran}(H_{B_0}^*) \subset \text{Ran}(H_B^*)\), then, \(W\) is an infinite dimensional subspace of \(L^2(S^1)\) because \(H_{B_0}^*\) is injective. From \(B_0 \cap \Omega = \emptyset\), we obtain

\[W \cap \text{Ran}(H_\Omega^*) = \{0\}.\]  

(5.19)

(see e.g., Lemma 4.3 of [11]). Applying (2) of Theorem 4.2 as

\[
F = F_\Omega^{Med} = H_\Omega^* T_\Omega H_\Omega, \quad \tilde{F} = \alpha H_B^* H_B,
\]

we have for \(\alpha \in (0, k^2 q_0)\)

\[\alpha H_B^* H_B \preceq_{\text{fin}} \text{Re} F_\Omega^{Med}.\]  

(5.20)

From the above discussion, we conclude the following theorem, which is the same result as Theorem 5.1 of [11].

**Theorem 5.4** (Theorem 5.1 of [11]). Let \(B \subset \mathbb{R}^d\) be a bounded domain with the smooth boundary. Then, for \(\alpha \in (0, k^2 q_0)\)

\[B \subset \Omega \iff \alpha H_B^* H_B \preceq_{\text{fin}} \text{Re} F_\Omega^{Med}.\]  

(5.21)

Assume that \(\Omega \subset B\). Then, we can define the compact operator \(R : L^2(\Omega) \to H^{1/2}(\partial B)\) by \(Rg := w|_{\partial B}\) where \(w\) is a radiating solution \(w\) of

\[\Delta w + k^2 (1 + q)w = -k^2 qg\text{ in }\mathbb{R}^d,\]  

(5.22)

From the definition we obtain \(G_\Omega^{Med} = G_{B}^{Dir} R\) where the data-to-pattern operator \(G_\Omega^{Med}\) is defined by \(G_\Omega^{Med} g := w_\infty\) and \(G_B^{Dir}\) is defined by (5.2) replacing \(\Omega\) by \(B\). Since \(G_\Omega^{Med} = H_\Omega^* T_\Omega\) and \(T_\Omega\) is bounded invertible (see e.g., the arguments of Theorem 4.5 of [19]), we have

\[H_\Omega^* = G_\Omega^{Med} T_\Omega^{-1} = G_B^{Dir} R T_\Omega^{-1}.\]  

(5.23)

Applying (1) of Theorem 4.2 as

\[
F = H_{\partial B}^* H_{\partial B} = G_B^{Dir} \left[ C_B J_{\partial B} C_B + \bar{K}_B \right] G_B^{Dir} *\,
\]

\[\tilde{F} = F_\Omega^{Med} = H_\Omega^* T_\Omega H_\Omega = H_\Omega^* [k^2 q I_{L^2(\Omega)} + K_\Omega] H_\Omega,\]  

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we have
\[ \text{Re} F_{\Omega}^{Med} \preceq_{\text{fin}} H_{\partial B}^* H_{\partial B}. \] (5.24)

Assume that \( \Omega \not\subset B \). Then, there exists a bounded domain \( \Omega_0 \subset \Omega \) such that \( \Omega_0 \cap B = \emptyset \). We set \( W := \text{Ran}(G_{\Omega_0}^{Med}) \subset \text{Ran}(G_{\Omega}^{Med}) = \text{Ran}(H_{\Omega}^*), \) then, \( W \) is an infinite dimensional subspace of \( L^2(\mathbb{S}^1) \) because \( G_{\Omega_0}^{Med} \) is injective. From \( \Omega_0 \cap B = \emptyset \), we obtain
\[ W \cap \text{Ran}(G_{B}^{\text{Dir}}) = \{0\}, \] (5.25)
(see e.g., Lemma 4.3 of [11]). Applying (2) of Theorem 4.2 as
\[ F = H_{\partial B}^* H_{\partial B} = G_{B}^{\text{Dir}} \left[ C_B J_{\partial B} C_B + \hat{K}_\Omega \right] G_{B}^{\text{Dir}}^*, \]
\[ \tilde{F} = F_{\Omega}^{Med} = H_{\Omega}^* T_\Omega H_{\Omega} = H_{\Omega}^* \left[ k^2 q I_{L^2(\Omega)} + K_\Omega \right] H_{\Omega}, \]
we have
\[ \text{Re} F_{\Omega}^{Med} \preceq_{\text{fin}} H_{\partial B}^* H_{\partial B}. \] (5.26)

From the above discussion, we conclude the following theorem.

**Theorem 5.5.** Let \( B \subset \mathbb{R}^d \) be a bounded domain with the smooth boundary. Then,
\[ \Omega \subset B \iff \text{Re} F_{\Omega}^{Med} \preceq_{\text{fin}} H_{\partial B}^* H_{\partial B}, \] (5.27)

**Remark 5.6.** We remark that the factorization reconstruction for the inverse medium scattering needs to assume that \( k^2 \) is not a transmission eigenvalue in \( \Omega \) (see e.g., Theorem 4.10 of [19]). While, the monotonicity reconstruction (Theorems 5.4 and 5.5) does not require the assumption of transmission eigenvalues.

### 5.3 Dirichlet crack

Let \( F_{\Gamma}^{\text{Dir}} \) be the far-field operator for a Dirichlet crack \( \Gamma \) where \( \Gamma \subset \mathbb{R}^d \) is a smooth non-intersecting open arc \((d = 2)\) or surface \((d = 3)\), and we assume that \( \Gamma \) can be extended to some smooth, connected, closed curve \((d = 2)\) or surface \((d = 3)\) \( \partial \Omega \) enclosing a bounded domain \( \Omega \) in \( \mathbb{R}^d \). The corresponding far-field pattern is defined by solving the scattering problem (2.4)–(2.5) where \( \Omega \) in (2.4) is replaced by \( \Gamma \) and the boundary condition (2.3) is replaced by
\[ u_- = 0 \text{ on } \Gamma, \quad u_+ = 0 \text{ on } \Gamma, \] (5.28)
where we denote by \( u_\pm \) the limit of \( u \) approaching the boundary from exterior (+) and interior (-) of an extensive domain \( \Omega \) (see Figure 1). \( F_{\Gamma}^{\text{Dir}} \) has the factorization (see Lemma 3.4 of [21])
\[ F_{\Gamma}^{\text{Dir}} = -G_{\Gamma}^{\text{Dir}} S_\Gamma^* G_{\Gamma}^{\text{Dir}}^*, \] (5.29)
where \( G_{\Gamma}^{\text{Dir}} : H^{1/2}(\Gamma) \to L^2(\mathbb{S}^{d-1}) \) is the data-to-pattern operator corresponding to the crack \( \Gamma \), and \( S_\Gamma : \tilde{H}^{-1/2}(\Gamma) \to H^{1/2}(\Gamma) \) is the single layer boundary operator corresponding to the crack \( \Gamma \) where we denote by
\[ H^{1/2}(\Gamma) := \{ u|_{\Gamma}; u \in H^{1/2}(\partial \Omega) \}, \] (5.30)

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\[ \tilde{H}^{1/2}(\Gamma) := \{ u|_\Gamma; u \in H^{1/2}(\partial \Omega), \text{supp}(u) \subset \Gamma \}, \]  
(5.31)

and \( H^{-1/2}(\Gamma) \) and \( \tilde{H}^{-1/2}(\Gamma) \) the dual spaces of \( \tilde{H}^{1/2}(\Gamma) \) and \( H^{1/2}(\Gamma) \), respectively. We have the following inclusion relation

\[ \tilde{H}^{1/2}(\Gamma) \subset H^{1/2}(\Gamma) \subset L^2(\Gamma) \subset \tilde{H}^{-1/2}(\Gamma) \subset H^{-1/2}(\Gamma). \]  
(5.32)

The single layer boundary operator \( S_\Gamma \) is of the form (see Lemma 3.2 of [21])

\[ S_\Gamma = C_\Gamma + K_\Gamma, \]  
(5.33)

where \( C_\Gamma : \tilde{H}^{-1/2}(\Gamma) \to H^{1/2}(\Gamma) \) is some positive coercive operator and \( K_\Gamma : \tilde{H}^{-1/2}(\Gamma) \to H^{1/2}(\Gamma) \) is some compact operator.

Let \( \sigma \subset \mathbb{R}^d \) be a smooth arc \((d = 2)\) or surface \((d = 3)\). Assume that \( \sigma \subset \Gamma \). Then, we define \( R : L^2(\Gamma) \to L^2(\sigma) \) by \( Rf := f|_\sigma \). Let \( J : H^{1/2}(\Gamma) \to L^2(\Gamma) \) be the compact embedding. Since \( \tilde{H}^*_\Gamma = G^{\text{Dir}}_\Gamma \) (see e.g., Lemma 3.4 of [21]), we have

\[ H^*_\sigma = RH_\Gamma = RJ\tilde{H}_\Gamma = RJ\tilde{S}^*_\Gamma G^{\text{Dir}}_\Gamma, \]  
(5.34)

where \( H_\Gamma : L^2(S^{d-1}) \to L^2(\Gamma) \) is the Herglotz operator corresponding to \( \Gamma \) and \( \tilde{H}_\Gamma : L^2(S^{d-1}) \to H^{1/2}(\Gamma) \) is the Herglotz operator that its range is restricted to the space \( H^{1/2}(\Gamma) \). Applying (1) of Theorem 4.2 as

\[ F = -F^{\text{Dir}}_\Gamma = G^{\text{Dir}}_\Gamma (C_\Gamma + K^*_\Gamma)G^{\text{Dir}}_\Gamma, \]  
(5.35)

we have

\[ H^*_\sigma H_\sigma \leq_{\text{fin}} -\text{Re}F^{\text{Dir}}_\Gamma. \]  
(5.36)

Assume that \( \sigma \nsubseteq \Gamma \). Then, there exists \( \sigma_0 \subset \sigma \) such that \( \sigma_0 \cap \Gamma = \emptyset \). We set \( W := \text{Ran}(H^*_\sigma) \subset \text{Ran}(H^*_\sigma) \), then, \( W \) is an infinite dimensional subspace of \( L^2(S^1) \) because \( H^*_\sigma \) is injective. From \( \sigma_0 \cap \Gamma = \emptyset \), we obtain

\[ W \cap \text{Ran}(G^{\text{Dir}}_\Gamma) = \{0\}, \]  
(5.37)

(see e.g., Lemma 4.1 of [9]). Applying (2) of Theorem 4.2 as

\[ F = -F^{\text{Dir}}_\Gamma = G^{\text{Dir}}_\Gamma (C_\Gamma + K^*_\Gamma)G^{\text{Dir}}_\Gamma, \]  
(5.38)
\( \tilde{F} = H_\sigma^* H_\sigma, \)

we have

\[ H_\sigma^* H_\sigma \leq_{\text{fin}} -\text{Re} F_\Gamma^{\text{Dir}}. \]  

(5.37)

From the above discussion, we conclude the following theorem, which is the same result as Theorem 1.1 of [9].

**Theorem 5.7** (Theorem 1.1 of [9]). Let \( \sigma \subset \mathbb{R}^d \) be a smooth arc (\( d = 2 \)) or surface (\( d = 3 \)). Then,

\[ \sigma \subset \Gamma \iff H_\sigma^* H_\sigma \leq_{\text{fin}} -\text{Re} F_\Gamma^{\text{Dir}}. \]  

(5.38)

By the same argument in Theorem 5.7, one can apply (1) and (2) of Theorem 4.2 as

\[ F = H_{\beta_B}^* H_{\beta_B} = G_B^{\text{Dir}} \left[ C_B J_{\beta_B} C_B + \hat{K}_B \right] G_B^{\text{Dir}} \ast, \]

\[ \tilde{F} = -F_\Gamma^{\text{Dir}} = G_\Gamma^{\text{Dir}} (C_\Gamma + K_\Gamma^*) G_\Gamma^{\text{Dir}} \ast. \]

Then, we also conclude the following theorem, which is the same result as Theorem 1.2 of [9].

**Theorem 5.8** (Theorem 1.2 of [9]). Let \( B \subset \mathbb{R}^d \) be a bounded domain with the smooth boundary. Then,

\[ \Gamma \subset B \iff -\text{Re} F_\Gamma^{\text{Dir}} \leq_{\text{fin}} H_{\beta_B}^* H_{\beta_B}. \]  

(5.39)

**Remark 5.9.** We remark that the factorization reconstruction for the inverse crack scattering also does not restrict the wave number. (see Theorem 3.9 of [21]). One of the advantage of the monotonicity over the factorization is to have not only inside tests (Theorem 5.7), but also outside tests (Theorem 5.8).

**5.4 Mixed obstacle**

Let \( F_{\Omega_1, \Omega_2}^{\text{Mix}} \) be the far-field operator for the mixed obstacle \( \Omega = \Omega_1 \cup \Omega_2 \) with the Dirichlet part \( \Omega_1 \) and the Neumann part \( \Omega_2 \) where \( \Omega_1, \Omega_2 \) are bounded domains with the smooth boundary such that \( \Omega_1 \cup \Omega_2 = \emptyset \). The corresponding far-field pattern is defined by solving the scattering problem (2.4)–(2.5) where the boundary condition (2.5) is replaced by

\[ u = 0 \text{ on } \partial \Omega_1, \quad \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial \Omega_2. \]  

(5.40)

\( F_{\Omega_1, \Omega_2}^{\text{Mix}} \) has the factorization (see Theorem 3.4 of [19])

\[ F_{\Omega_1, \Omega_2}^{\text{Mix}} = -G_{\Omega_1, \Omega_2}^{\text{Mix}} T_{\Omega_1, \Omega_2}^{\text{Mix}} G_{\Omega_1, \Omega_2}^{\text{Mix}} \ast, \]  

(5.41)

where \( G_{\Omega_1, \Omega_2}^{\text{Mix}} : H^{1/2}(\partial \Omega_1) \times H^{-1/2}(\partial \Omega_2) \rightarrow L^2(S^{d-1}) \) is the data-to-pattern operator for the mixed obstacle \( \Omega \), i.e., defined by \( G_{\Omega_1, \Omega_2}^{\text{Mix}} \left( \begin{array}{c} f_1 \\ f_2 \end{array} \right) := v^\infty \) where \( v^\infty \) is the far-field pattern of a radiating solution \( v \) such that

\[ \Delta v + k^2 v = 0 \text{ in } \mathbb{R}^d \setminus \overline{\Omega}, \quad v = f_1 \text{ on } \partial \Omega_1, \quad \frac{\partial v}{\partial \nu} = f_2 \text{ on } \partial \Omega_2, \]  

(5.42)
and some operator $T_{\Omega_1, \Omega_2}^{Mix}: H^{-1/2}(\partial \Omega_1) \times H^{1/2}(\partial \Omega_2) \to H^{1/2}(\partial \Omega_1) \times H^{-1/2}(\partial \Omega_2)$ has the form (Theorem 3.4 of [19])

$$T_{\Omega_1, \Omega_2}^{Mix} = \begin{pmatrix} C_{\Omega_1}^+ & 0 \\ 0 & C_{\Omega_2}^- \end{pmatrix} + K_{\Omega_1, \Omega_2}^{Mix}, \quad (5.43)$$

where $C_{\Omega_1}^+: H^{-1/2}(\partial \Omega_1) \to H^{1/2}(\partial \Omega_1)$ is some positive coercive operator, $C_{\Omega_2}^-$ : $H^{1/2}(\partial \Omega_2) \to H^{-1/2}(\partial \Omega_2)$ is some negative coercive operator, and $K_{\Omega_1, \Omega_2}^{Mix}: H^{-1/2}(\partial \Omega_1) \times H^{1/2}(\partial \Omega_1) \times H^{-1/2}(\partial \Omega_2)$ is some compact operator.

Assume that $\Omega_1 \subset \tilde{B}$. Then, there exists bounded domain $\tilde{B} \Subset B$ such that $\tilde{B} \cap \Omega_2 = \emptyset$ and $\tilde{B} \subset \tilde{B}$. Define $R : H^{1/2}(\partial \Omega_1) \times H^{-1/2}(\partial \Omega_2) \to H^{1/2}(\partial \tilde{B}) \times H^{-1/2}(\partial \Omega_2)$ by

$$R \left( \begin{array}{c} f_1 \\ f_2 \end{array} \right) := \left( \begin{array}{c} v|_{\partial \tilde{B}} \\ f_2 \end{array} \right).$$

Then, $R$ has the form $R = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}$, compact, and

$$G_{\Omega_1, \Omega_2}^{Mix} = G_{\Omega, B, \Omega_2}^{Mix} R, \quad (5.44)$$

where $G_{\tilde{B}, \Omega_2}^{Mix}: H^{1/2}(\partial \tilde{B}) \times H^{-1/2}(\partial \Omega_2) \to L^2(S^{d-1})$ is the data-to-pattern operator corresponding to $\tilde{B} \cup \Omega_2$ with the Dirichlet part $\tilde{B}$ and the Neumann part $\Omega_2$. We also define $R : H^{1/2}(\partial \tilde{B}) \to H^{1/2}(\partial \tilde{B}) \times H^{-1/2}(\partial \Omega_2)$ by

$$\tilde{R} g := \left( \frac{\partial w}{\partial \nu}|_{\partial \Omega_2} \right)$$

where $w$ is a radiating solution $w$ of

$$\Delta w + k^2 w = 0 \text{ in } \mathbb{R}^d \setminus \overline{\tilde{B}}, \quad w = g \text{ on } \partial \tilde{B}, \quad (5.45)$$

Then, $\tilde{R}$ has the form $\tilde{R} = \begin{pmatrix} I \\ 0 \end{pmatrix}$, compact, and

$$G_{\tilde{B}}^{Dir} = G_{\tilde{B}, \Omega_2}^{Mix} \tilde{R}. \quad (5.46)$$

By these and (5.7), we have

$$F^{Mix} + H^*_{\partial \tilde{B}} H_{\partial \tilde{B}} = G_{\tilde{B}, \Omega_2}^{Mix} \begin{pmatrix} R(-T_{\Omega_1, \Omega_2}^{Mix}) R^* + \tilde{R} S_{\tilde{B}} J_{\partial \tilde{B}} S^*_{\tilde{B}} \tilde{R}^* \\ C_{\tilde{B}}^J_{\partial \tilde{B}} C_{\tilde{B}} 0 \\ 0 -C_{\Omega_2}^- \end{pmatrix} G_{\tilde{B}, \Omega_2}^{Mix}, \quad (5.47)$$

where

$$\begin{pmatrix} C_{\tilde{B}}^J_{\partial \tilde{B}} C_{\tilde{B}} 0 \\ 0 -C_{\Omega_2}^- \end{pmatrix}$$

is a positive coercive operator and $G_{\tilde{B}, \Omega_2}^{Mix}$ is some compact operator. Applying (1) of Theorem 4.2 as

$$F = F_{\Omega_1, \Omega_2}^{Mix} + H^*_{\partial \tilde{B}} H_{\partial \tilde{B}} = G_{\tilde{B}, \Omega_2}^{Mix} \begin{pmatrix} C_{\tilde{B}}^J_{\partial \tilde{B}} C_{\tilde{B}} 0 \\ 0 -C_{\Omega_2}^- \end{pmatrix} G_{\tilde{B}, \Omega_2}^{Mix},$$

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\[ \tilde{F} = 0, \]

we have

\[ -\text{Re} F^{\text{Mix}}_{\Omega_1, \Omega_2} \leq \text{fin} H^*_\partial B H^\partial B. \quad (5.48) \]

Since we have \( \tilde{B} \subset B \), one can show by the same argument in (5.8) that there exists a compact operator \( R_B : H^{1/2}(\partial \tilde{B}) \to H^{1/2}(\partial B) \) such that

\[ G_B^{\text{Dir}} = G_B^{\text{Div}} R_B. \quad (5.49) \]

Then, applying (1) of Theorem 4.2 as \( F = H^*_\partial B H^\partial B, \tilde{F} = H^*_\partial \tilde{B} H^\partial \tilde{B} \) (remark (5.7)), it follows that

\[ H^*_\partial \tilde{B} H^\partial \tilde{B} \leq \text{fin} H^*_\partial B H^\partial B, \quad (5.50) \]

which implies that with (5.48) we conclude that

\[ -\text{Re} F^{\text{Mix}}_{\Omega_1, \Omega_2} \leq \text{fin} H^*_\partial B H^\partial B. \quad (5.51) \]

Assume that \( \Omega_1 \not\subset B \) and \( \mathbb{R}^d \setminus (B \cup \Omega) \) is connected. Then, there exists \( \Gamma \subset \partial \Omega_1 \) such that \( \Gamma \cap B = \emptyset \). Define \( E_\Gamma : H^{1/2}(\Gamma) \to H^{1/2}(\partial \Omega_1) \) by \( E_\Gamma f = f \) on \( \Gamma \), otherwise zero. Denote by \( X_\Gamma \subset H^{1/2}(\Gamma) \) the subspace of piecewise linear continuous functions on \( \Gamma \) that vanish on \( \partial \Gamma \). Set \( W := \text{Ran} \left( G^{\text{Mix}} R_1^* E_\Gamma \big|_{X_\Gamma} \right) \subset \text{Ran}(G^{\text{Mix}} R_1^*), \) it is infinite dimensional because \( X_\Gamma \) is infinite dimensional and the operator \( G^{\text{Mix}} R_1^* E_\Gamma \) is injective. By Lemma 4.6 of [2], we have

\[ W \cap \text{Ran}(G^{\text{Dir}} B, G^{\text{Mix}} R_2^*) = \{0\}. \quad (5.52) \]

Applying (3) of Theorem 4.2 as

\[ F^{\text{Mix}} = -F^{\text{Mix}}_{\Omega_1, \Omega_2} = C^{\text{Mix}}_{\Omega_1, \Omega_2} \left( \begin{pmatrix} C^+_{\Omega_1} & 0 \\ 0 & C^-_{\Omega_2} \end{pmatrix} + K^{\text{Mix}}_{\Omega_1, \Omega_2} \right) G^{\text{Mix}}_{\Omega_1, \Omega_2}^*, \]

\[ \tilde{F} = H^*_\partial \tilde{B} H^\partial \tilde{B} = G_B^{\text{Div}} \left[ C_B J_{\partial \tilde{B}} C_B + \tilde{K}_B \right] G_B^{\text{Div}}^*, \]

we have

\[ -\text{Re} F^{\text{Mix}}_{\Omega_1, \Omega_2} \leq \text{fin} H^*_\partial \tilde{B} H^\partial \tilde{B}. \quad (5.53) \]

From the above discussion, we conclude the following theorem, which is the same result as Theorem 5.5 of [2]. We remark that the monotonicity reconstruction discussed here was succeeded without assuming that the wavenumber \( k^2 \) is neither a Dirichlet eigenvalue of \( -\Delta \) in \( \Omega_1 \) and \( B \), nor a Neumann eigenvalue in \( \Omega_2 \), although Theorem 5.5 of [2] assumed it.

**Theorem 5.10 (Theorem 5.5 of [2]).** Let \( B \subset \mathbb{R}^d \) be a bounded domain with the smooth boundary such that \( \mathbb{R}^d \setminus (B \cup \Omega) \) is connected. Then,

\[ \Omega_1 \subset B \iff -\text{Re} F^{\text{Mix}}_{\Omega_1, \Omega_2} \leq \text{fin} H^*_\partial B H^\partial B. \quad (5.54) \]
By the same argument in Theorem 5.10, one can apply (1) and (3) replacing (3b) by (3b) of Theorem 4.2 as

\[ F_{\Omega_1, \Omega_2}^{\text{Mix}} = -G_{\Omega_1, \Omega_2}^{\text{Mix}} \left( \begin{array}{cc} C_{\Omega_1}^+ & 0 \\ 0 & C_{\Omega_2}^- \end{array} \right) + K_{\Omega_1, \Omega_2}^{\text{Mix}} \] \[ \tilde{F} = H_{\partial B}^* H_{\partial B} = G_{B}^{\text{Dir}} \left[ C_B J_{\partial B} C_B + \hat{K}_B \right] G_{B}^{\text{Dir}} \]

for the reconstruction of the Neumann part \( \Omega_2 \). Then, we also conclude the following theorem, which is the same result as Theorem 5.5 of [2] without the restriction of the wave number \( k > 0 \) as well as Theorem 5.10.

**Theorem 5.11** (Theorem 5.5 of [2]). Let \( B \subset \mathbb{R}^d \) be a bounded domain with the smooth boundary such that \( \mathbb{R}^d \setminus (\bar{B} \cup \Omega) \) is connected. Then,

\[ \Omega_2 \subset B \iff \Re F_{\Omega_1, \Omega_2}^{\text{Mix}} \leq \text{fin} \ H_{\partial B}^* H_{\partial B}. \] (5.55)

**Remark 5.12.** We remark that the factorization reconstruction for the mixed obstacle has to assume that one component is covered by some artificial domain which is disjoint with the other one we want to reconstruct, and furthermore assume that \( k^2 \) is neither a Dirichlet eigenvalue of \( -\Delta \) in \( \Omega_1 \) and artificial covering domain, nor a Neumann eigenvalue in \( \Omega_2 \) (see Lemma 3.5 of [19]). However, the monotonicity reconstruction (Theorems 5.10 and 5.11) does not require both of them.

### 5.5 Mixed crack

Let \( F_{\Gamma}^{\text{Mix}} \) be the far-field operator for the mixed crack \( \Gamma \) (The assumption for \( \Gamma \) is the same as Section 5.3). The corresponding far-field pattern is defined by solving the scattering problem (2.4)–(2.5) where \( \Omega \) in (2.4) is replaced by \( \Gamma \) and the boundary condition (2.3) is replaced by

\[ u_- = 0 \text{ on } \Gamma, \quad \frac{\partial u_+}{\partial \nu} = 0 \text{ on } \Gamma. \] (5.56)

\( F_{\Gamma}^{\text{Mix}} \) has the factorization (see (3.6) and (3.13) of [25])

\[ F_{\Gamma}^{\text{Mix}} = -G_{\Gamma}^{\text{Mix}} M_{\Gamma}^{\text{Mix}} G_{\Gamma}^{\text{Mix}} \]

where \( G_{\Gamma}^{\text{Mix}} : H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma) \rightarrow L^2(\mathbb{S}^{d-1}) \) is the data-to-pattern operator for the mixed crack \( \Gamma \), i.e., defined by \( G_{\Gamma}^{\text{Mix}} \left( \begin{array}{c} f_1 \\ f_2 \end{array} \right) := v^\infty \) where \( v^\infty \) is the far-field pattern of a radiating solution \( v \) such that

\[ \Delta v + k^2 v = 0 \text{ in } \mathbb{R}^d \setminus \Gamma, \quad v_- = f_1 \text{ on } \Gamma, \quad \frac{\partial v_+}{\partial \nu} = f_2 \text{ on } \Gamma, \] (5.58)

and \( M_{\Gamma}^{\text{Mix}} : \tilde{H}^{-1/2}(\Gamma) \times \tilde{H}^{1/2}(\Gamma) \rightarrow H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma) \) has the form (see (3.12) of [25])

\[ M_{\Gamma}^{\text{Mix}} = \left( \begin{array}{cc} C_{\Gamma} & -I \\ -I & C_{\Gamma}^- \end{array} \right) + K_{\Gamma}^{\text{Mix}}, \] (5.59)
where $C_1^+ : \tilde{H}^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$ is some positive coercive operator, $C_1^- : \tilde{H}^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ is some negative coercive operator, and $K_{\Gamma}^{Mix} : \tilde{H}^{-1/2}(\Gamma) \times \tilde{H}^{1/2}(\Gamma) \to H^1(\Gamma) \times H^{-1/2}(\Gamma)$ is some compact operator.

Assume that $\Gamma \subset B$. Define $R : H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma) \to H^{1/2}(\partial B)$ by $R \left( \begin{array}{c} f_1 \\ f_2 \end{array} \right) := v|_{\partial B}$, then, $R$ is a compact operator and

\[ G_{\Gamma}^{Mix} = G_B^{Dir} R. \]  \hfill (5.60)

Applying (1) of Theorem 4.2 as $F = H_B^* H_{\partial B} = G_B^{Dir} \left[ C_B J_{\partial B} C_B + \hat{K}_B \right] G_B^{Dir} \ast$, we have

\[ -\text{Re} F_{\Gamma}^{Mix} \leq_{\text{fin}} H_{\partial B}^* H_{\partial B}. \]  \hfill (5.61)

Assume that $\Gamma \not\subset B$. Then, there exists $\hat{\Gamma} \in \Gamma$ such that $\hat{\Gamma} \cap B = \emptyset$. Setting $W := \text{Ran} \left( G_{\Gamma}^{Mix} R_{\Gamma}^1 E_{\Gamma} \right) \subset \text{Ran}(G_B^{Mix} R_B^*)$, it is infinite dimensional because $X_{\Gamma}$ is infinite dimensional and the operator $G_{\Gamma}^{Mix} R_{\Gamma}^1 E_{\Gamma}$ is injective. By the same argument in Lemma 4.6 of [2], we have

\[ W \cap \text{Ran}(G_B^{Dir}, G_{\Gamma}^{Mix} R_2^*) = \{0\}. \]  \hfill (5.62)

Applying (3) of Theorem 4.2 as

\[ F^{Mix} = -F_{\Gamma}^{Mix} = G_{\Gamma}^{Mix} \left( \left( C_{\Gamma}^+ \begin{array}{c} -I \\ -I \end{array} \right) + K_{\Gamma}^{Mix} \right) G_{\Gamma}^{Mix} \ast, \]

we have

\[ -\text{Re} F_{\Gamma}^{Mix} \leq_{\text{fin}} H_{\partial B}^* H_{\partial B}. \]  \hfill (5.63)

From the above discussion, we conclude the following theorem.

**Theorem 5.13.** Let $B \subset \mathbb{R}^d$ be a bounded domain. Then,

\[ \Gamma \subset B \iff -\text{Re} F_{\Gamma}^{Mix} \leq_{\text{fin}} H_{\partial B}^* H_{\partial B}. \]  \hfill (5.64)

By the same argument in Theorem 5.13 one can apply (1) and (3) replacing (3b) by (3b)' of Theorem 4.2 as

\[ F^{Mix} = F_{\Gamma}^{Mix} = -G_{\Gamma}^{Mix} \left( \left( C_{\Gamma}^+ \begin{array}{c} -I \\ -I \end{array} \right) + K_{\Gamma}^{Mix} \right) G_{\Gamma}^{Mix} \ast, \]

we have

\[ -\text{Re} F_{\Gamma}^{Mix} \leq_{\text{fin}} H_{\partial B}^* H_{\partial B}. \]  \hfill (5.63)

From the above discussion, we conclude the following theorem.

**Theorem 5.13.** Let $B \subset \mathbb{R}^d$ be a bounded domain. Then,

\[ \Gamma \subset B \iff -\text{Re} F_{\Gamma}^{Mix} \leq_{\text{fin}} H_{\partial B}^* H_{\partial B}. \]  \hfill (5.64)

By the same argument in Theorem 5.13 one can apply (1) and (3) replacing (3b) by (3b)' of Theorem 4.2 as

\[ F^{Mix} = F_{\Gamma}^{Mix} = -G_{\Gamma}^{Mix} \left( \left( C_{\Gamma}^+ \begin{array}{c} -I \\ -I \end{array} \right) + K_{\Gamma}^{Mix} \right) G_{\Gamma}^{Mix} \ast, \]

we have

\[ -\text{Re} F_{\Gamma}^{Mix} \leq_{\text{fin}} H_{\partial B}^* H_{\partial B}. \]  \hfill (5.63)

From the above discussion, we conclude the following theorem.
Theorem 5.14. Let $B \subset \mathbb{R}^d$ be a bounded domain. Then,
\[
\Gamma \subset B \iff \text{Re} F^\text{Mix}_\Gamma \leq \text{fin} H^*_{\partial B} H_{\partial B}.
\] (5.65)

Remark 5.15. The factorization reconstruction for the mixed crack has been studied in [25], but an extensive closed curve of the unknown crack should be known, which is a very restrictive assumption (see Theorem 3.3 of [25]). However, the monotonicity reconstruction (Theorems 5.13 and 5.14) does not require it. We also remark that our works in this section would be a new extension of the monotonicity to the inverse acoustic mixed crack scattering.

6 Numerical examples

In this section, we study numerical examples of our theoretical results in 2 dimensions. The far-field operator $F$ is approximated by the matrix
\[
F \approx \frac{2\pi}{N} \left( \int_B e^{iky \cdot (\hat{x}_l - \hat{x}_m)} dy \right)_{1 \leq l, m \leq N} \in \mathbb{C}^{N \times N},
\] (6.1)
where $\hat{x}_l = (\cos(\frac{2\pi l}{N}), \sin(\frac{2\pi l}{N}))$ and $\theta_m = (\cos(\frac{2\pi m}{N}), \sin(\frac{2\pi m}{N}))$. For the far-field pattern $u^\infty(\hat{x}, \theta)$ of the Dirichlet obstacle and the Dirichlet crack, we numerically compute the integral
\[
u^\infty(\hat{x}, \theta) = \frac{e^{\frac{i\pi}{4}}}{\sqrt{8\pi k}} \int_{\Gamma} e^{-ik\hat{x}_l \phi_\theta(y)} ds(y),
\] (6.2)
where $\phi_\theta$ is given by solving
\[-e^{ikx \cdot \theta} = \int_{\Gamma} \Phi(x, y) \phi_\theta(y) ds(y), \ x \in \Gamma.
\] (6.3)

For the inhomogeneous medium, we numerically compute the integral
\[
u^\infty(\hat{x}, \theta) = \frac{e^{\frac{i\pi}{4}}}{\sqrt{8\pi k}} \int_{\Omega} e^{-ik\hat{x}_l q(y)} (u_\theta(y) + e^{iky \cdot \theta}) dy,
\] (6.4)
where $u_\theta$ is given by solving
\[
u(x) = k^2 \int_{\Omega} \Phi(x, y) q(y) (u_\theta(y) + e^{iky \cdot \theta}) dy, \ x \in \Omega.
\] (6.5)

For a bounded domain (or a smooth curve) $B$, the operator $H^*_{\partial B} H_{\partial B}$ is approximated by
\[
H^*_{\partial B} H_{\partial B} \approx \frac{2\pi}{N} \left( \int_B e^{iky \cdot (\theta_m - \hat{x}_l)} dy \right)_{1 \leq l, m \leq N} \in \mathbb{C}^{N \times N}.
\] (6.6)

We denote by the sampling square region $[-R, R]^2$, which includes the unknown target. We also denote by a grid point $z_{i,j} := \left( \frac{R_i}{M}, \frac{R_j}{M} \right)$ $(i, j = -M, -M + 1, ..., M)$ in the sampling square region $[-R, R]^2$ (see Figure 2). Throughout our examples, we fix parameters $R = 1.5$, $M = 100$, and $N = 20$. 21
6.1 Dirichlet obstacle and inhomogeneous medium

We consider the Dirichlet obstacle and the inhomogeneous medium detections discussed in Sections 5.1 and 5.2. The following shapes \( \partial \Omega_j \) \((j = 1, 2)\) are considered (see Figure 6):

\[
\partial \Omega_1 = \{(0.7 \cos(\pi s), 0.7 \sin(\pi s)) \mid -1 \leq s \leq 1\}.
\]

\[
\partial \Omega_2 = \{(0.3 \cos(\pi s) - 0.7, 0.3 \sin(\pi s)) \mid -1 \leq s \leq 1\}
\cup \{(0.3 \cos(\pi s) + 0.7, 0.3 \sin(\pi s)) \mid -1 \leq s \leq 1\}.
\]

Based on Theorems 5.1 and 5.4 the indicator functions for the Dirichlet obstacle and inhomogeneous medium are given by

\[
I_{MM}^{\text{dir}}(B) := \# \{\text{negative eigenvalues of } -\text{Re}F_{\Omega_j}^{\text{Dir}} - H_{\partial B}^* H_{\partial B}\},
\]

\[
I_{MM}^{\text{med}}(B) := \# \{\text{negative eigenvalues of } \text{Re}F_{\Omega_j}^{\text{Med}} - \alpha H_{\partial B}^* H_{\partial B}\},
\]

for a bounded domain \( B \), respectively. For the medium, we always consider \( q = 1 \) in \( \Omega_j \), and \( \alpha = 1 \). Here, \( B \) is chosen as a square, i.e., \( B = B_{i,j}(r) := z_{i,j} + [-\frac{r}{2}, \frac{r}{2}] \times [-\frac{r}{2}, \frac{r}{2}] \)
where a grid point \( z_{i,j} \) is the center of a square (see Figure 3).

\[
\int_{B_{i,j}(r)} e^{iky \cdot (\theta_m - \hat{x}_l)} dy = r^2 e^{ik(\theta_m - \hat{x}_l) \cdot z_{i,j}} \text{sinc} \left( \frac{kr}{2} (\theta_m - \hat{x}_l)_1 \right) \text{sinc} \left( \frac{kr}{2} (\theta_m - \hat{x}_l)_2 \right).
\]
Figures 7 and 8 are given by plotting the values of the indicator functions

\[ I_{\text{MM dir}}(z_{i,j}) := I_{\text{MM dir}}(B_{i,j}(r)), \quad \text{for each } i,j = -100, -99, ..., 100, \tag{6.12} \]

\[ I_{\text{MM med}}(z_{i,j}) := I_{\text{MM med}}(B_{i,j}(r)), \quad \text{for each } i,j = -100, -99, ..., 100, \tag{6.13} \]

respectively, for different lengths \( r = 0.1, 0.5 \), wavenumbers \( k = 1, 5 \), and shapes \( \Omega_1, \Omega_2 \).

We also plot the values of the indicator function for the factorization method

\[ I_{\text{FM}}(z_{i,j}) := \left( \sum_{n=1}^{\infty} \frac{\langle \phi_{z_{i,j}}, \varphi_n \rangle_{L^2(S^1)}^2}{\mu_n} \right)^{-1}, \quad \text{for each } i,j = -100, -99, ..., 100, \tag{6.14} \]

with \( \{\mu_n, \varphi_n\} \) an eigensystem of the self-adjoint compact operator \( |\Re F| + |\Im F| \) where \( F \) is some far-field operator, and \( \phi_{z_{i,j}} \) is defined by

\[ \phi_{z_{i,j}}(\hat{x}) := e^{-ik\hat{x} \cdot z_{i,j}}, \quad \hat{x} \in S^1, \tag{6.15} \]

Figures 9 and 10 are corresponding to \( F = F^\text{Dir}_\Omega \) for the Dirichlet obstacle and \( F = F^\text{Med}_\Omega \) for the inhomogeneous medium, respectively. For details of introductions of the indicator function \( I_{\text{FM}} \), we refer to Corollary 2.16 and Theorem 4.9 of [19].

6.2 Dirichlet crack

We consider the Dirichlet crack detection discussed in Section 5.3. The following shapes \( \Gamma_j \ (j = 1, 2) \) are considered (see Figure 11):

\[ \Gamma_1 = \{(\cos(2s), \sin(2s)) | -1 \leq s \leq 1\}. \tag{6.16} \]

\[ \Gamma_2 = \{(-0.4\cos(2s) - 0.7, 0.4\sin(2s)) | -1 \leq s \leq 1\} \]

\[ \cup \{(0.4\cos(2s) + 0.7, 0.4\sin(2s)) | -1 \leq s \leq 1\}. \tag{6.17} \]

Based on Theorem 5.7, the indicator function is given by

\[ I_{\text{MM dir}}(\sigma) := \# \{ \text{negative eigenvalues of } - \Re F^\text{Dir}_\Gamma - H^\sigma H_\sigma \}. \tag{6.18} \]

for a smooth arc \( \sigma \). Here, \( \sigma \) is chosen as a line segment, i.e., \( \sigma = \sigma_{i,j}(\eta, r) := z_{i,j} + L(\eta, r) \)

where a grid point \( z_{i,j} \) is the center of line segments, and \( L(\eta, r) \) is defined by

\[ L(\eta, r) := \left\{ (s, \tan(\eta)) \big| -\frac{r}{2}\cos(\eta) \leq s \leq \frac{r}{2}\cos(\eta) \right\}, \tag{6.19} \]

where \( \eta \in [0, \pi] \) is the angle and \( r > 0 \) is the length of the line segment (see Figure 4).

Then, we can compute the integral

\[ \int_{\sigma_{i,j}(\eta, r)} e^{iky \cdot (\theta_m - \hat{x}^1)} ds(y) = re^{ik(\theta_m - \hat{x}^1) \cdot z_{i,j}} \text{sinc} \left( \frac{rk}{2} \left( \cos(\eta)(\theta_m - \hat{x}^1)_1 + \sin(\eta)(\theta_m - \hat{x}^1)_2 \right) \right). \tag{6.20} \]
Figures 12 and 13 are given by plotting the values of the indicator function for $\Gamma_1$ and $\Gamma_2$

$$I_{\text{dir}}^{\text{MM}}(z_{i,j}) := I_{\text{dir}}^{\text{MM}}(\sigma_{i,j}(\eta, r)),$$

for each $i, j = -100, -99, \ldots, 100$, respectively, for different angles $\eta = 0, \pi/2$, lengths $r = 0.01, 0.1$, and wavenumbers $k = 1, 5$.

In Figure 14 we also plot the values of the indicator function for the factorization method

$$I_{\text{dir}}^{\text{FM}}(z_{i,j}) := \left( \sum_{n=1}^{\infty} \frac{|\langle \phi_{z_{i,j}}, \varphi_n \rangle_{L^2(S^1)}|^2}{\mu_n} \right)^{-1},$$

for each $i, j = -100, -99, \ldots, 100$, (6.22)

with $\{\mu_n, \phi_n\}$ an eigensystem of the self-adjoint compact operator $|\text{Re}F_{\Gamma}^{\text{Dir}}| + |\text{Im}F_{\Gamma}^{\text{Dir}}|$ and $\phi_{z_{i,j}}$ is defined in (6.15). For details of introductions of the indicator function $I_{\text{dir}}^{\text{FM}}$, we refer to Theorem 3.9 of [21].

### 6.3 Mixed crack

We consider the mixed crack detection discussed in Section 5.5. The following shape $\Gamma_3$ is considered (see Figure 15):

$$\Gamma_3 = \{(0.5\cos(2s), 0.5\sin(2s)) \mid -1 \leq s \leq 1\}. \quad (6.23)$$

Based on Theorem 5.13, the indicator function is given by

$$I_{\text{mix}}^{\text{MM}}(B) := \# \{ \text{negative eigenvalues of } \text{Re}F_{\Gamma}^{\text{Mix}} + H_{\partial B}^* H_{\partial B} \}. \quad (6.24)$$

for a bounded domain $B$. Here, $B$ is chosen as a circle, i.e., $B = B_r(z)$ is an open circle with center $z \in \mathbb{R}^2$ and radius $r > 0$. Then, we can compute the integral

$$\int_{\partial B_r(z)} e^{iky \cdot (\theta_m - \bar{z}_i)} ds(y) = 2\pi r e^{i(kr|\theta_m - \bar{z}_i|)} J_0(kr), \quad (6.25)$$
where \( J_0 \) is the Bessel function of the first kind for the zero order. Figures 16 and 17 are given by plotting the values of two types (see Figure 5) of indicator functions

\[
I_{mix,r}(z_{i,j}) := I_{MM}^{mix}(B_r(z_{i,j})), \quad \text{for each } i,j = -100, -99, ..., 100, \tag{6.26}
\]

\[
I_{mix,p}(z_{i,j}) := I_{MM}^{mix}(B_{|z_{i,j} - p|}(p)), \quad \text{for each } i,j = -100, -99, ..., 100, \tag{6.27}
\]

respectively, for different radiuses \( r = 0.25, 1 \), points \( p = (0,0), (1,0) \), wavenumbers \( k = 1, 5 \).

Figure 5: The shifting circle test(left) and the shrinking circle test(right).

In Figures 18 we also plot the values of the indicator function for the factorization method

\[
I_{mix}(z_{i,j}) := \left( \sum_{n=1}^{\infty} \frac{|\langle \phi_{z_{i,j}}, \varphi_n \rangle_{L^2(S_1)}|^2}{\mu_n} \right)^{-1}, \quad \text{for each } i,j = -100, -99, ..., 100, \tag{6.28}
\]

with \( \{ \mu_n, \varphi_n \} \) an eigensystem of the self-adjoint compact operator \( |\text{Re} \tilde{F}| + |\text{Im} \tilde{F}| \). The operator \( \tilde{F} \) is defined by adding the Herglotz operator corresponding to the auxiliary \( \partial \Omega_3 \) to the original far-field operator

\[
\tilde{F} := F_{mix}^\Gamma - pH_{\partial \Omega_3}^\ast H_{\partial \Omega_3}, \tag{6.29}
\]

for \( p \in \mathbb{C} \setminus \{0\} \). We choose \( p = 0.01 + 0.01i \) or \( 0.1 + 0.1i \) in our numerical examples. Here, the auxiliary \( \partial \Omega_3 \) is defined by

\[
\partial \Omega_3 = \{ (0.5\cos(\pi s), 0.5\sin(\pi s)) | -1 \leq s \leq 1 \}, \tag{6.30}
\]

which is an extension of \( \Gamma_3 \), that is, \( \Gamma_3 \subset \partial \Omega_3 \). For details of introductions of the indicator function \( I_{mix}^{FM} \), we refer to Theorem 3.4 of [25].

Conclusions

In this paper, we studied the factorization and monotonicity method for inverse acoustic scattering problems. The main contribution was to give a new general functional analysis theorem (Theorem 4.2) for the monotonicity method, which can provide reconstruction schemes under weaker a priori assumptions rather than the factorization
method (see the assumptions in Theorems 3.1 and 4.2). Furthermore, we observed that the factorization method needs the real and imaginary parts of the far-field operator (see (3.3)), while the monotonicity needs only the real part (see (4.3), (4.4), and (4.6)), which is also the advantage over the factorization in terms of less data. After proving the general theorem, we also showed how the general theorem is applied to three typical inverse scattering problems (obstacle in Sections 5.1 and 5.4, medium in Section 5.2, and crack in Sections 5.3 and 5.5). However, it can be applied to other inverse problems as well (especially for inverse problems that the factorization method already studied, e.g., inverse scatterings by a layered medium [3], a mixed-type scatterer of a obstacle and a medium [20], and an obstacle in an homogeneous half-space [10]).

We also provided several numerical examples to compare the factorization method with the monotonicity method. The factorization method is a point test which checks whether a point \( z \) is included in the unknown target or not, while the monotonicity is a domain test which checks whether a domain \( B \) is included in the unknown target or not. For the domain test, we have to find the appropriate choice of \( B \). By testing the monotonicity for many different shapes and sizes of \( B \), we obtained as accurate reconstructions as the factorization (see Figures 7, 8, 9, 10, 12, 13, and 14).

In our numerical examples for the mixture crack, we observed that the factorization method is more accurate than the monotonicity (see Figures 16, 17, 18). This comes from that the factorization method used the Herglotz operator corresponding to the auxiliary closed curve that is an extension of the unknown crack (see (6.29)), which is a very restrictive assumption. On the other hand, the monotonicity can give the reconstruction scheme without using the auxiliary closed curve. Its scheme is the outside domain test that checks whether a domain \( B \) contains \( \Gamma \) or not. However in our numerical examples, the outside domain test can not obtain the exact shape of the unknown crack \( \Gamma \) although the location and size can be done (see Figures 16, 17). As we see numerical examples for the mixed obstacle discussed in [2], it also seems to be difficult to obtain the shape of unknown obstacle from outside domain tests. The numerical tests for the mixed problem in the monotonicity has to be developed in the future.

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Figure 6: The original domains Ω₁ (left) and Ω₂ (right).
Figure 7: Reconstruction for the Dirichlet obstacle by the monotonicity method for different lengths $r = 0.1, 0.5$, wavenumbers $k = 1, 5$, and shapes $\Omega_1, \Omega_2$. 
Figure 8: Reconstruction for the inhomogeneous medium by the monotonicity method for different lengths $r = 0.1, 0.5$, wavenumbers $k = 1, 5$, and shapes $\Omega_1, \Omega_2$
Figure 9: Reconstruction for the Dirichlet obstacle by the factorization method for different wavenumbers $k = 1, 5$ and shapes $\Omega_1, \Omega_2$. 
Figure 10: Reconstruction for the inhomogeneous medium by the factorization method for different wavenumbers \( k = 1, 5 \) and shapes \( \Omega_1, \Omega_2 \).

Figure 11: The original open arcs \( \Gamma_1 \) (left) and \( \Gamma_2 \) (right).
Figure 12: Reconstruction for the Dirichlet crack \( \Gamma_1 \) by the monotonicity method for different angles \( \eta = 0, \pi/2, \) lengths \( r = 0.01, 0.1, \) and wavenumbers \( k = 1, 5. \)
Figure 13: Reconstruction for the Dirichlet crack $\Gamma_2$ by the monotonicity method for different angles $\eta = 0, \pi/2$, lengths $r = 0.01, 0.1$, and wavenumbers $k = 1, 5$. 
Figure 14: Reconstruction for the Dirichlet crack by the factorization method for different wavenumbers $k = 1, 5$ and shapes $\Gamma_1, \Gamma_2$. 
Figure 15: The original open arc $\Gamma_3$.

Figure 16: Reconstruction for the mixed crack $\Gamma_3$ by the shifting circle test of monotonicity method for different radiuses $r = 0.25, 1$ and wavenumbers $k = 1, 5$. 
Figure 17: Reconstruction for the mixed crack $\Gamma_3$ by the shrinking circle test of monotonicity method for different points $p = (0, 0), (1, 0)$ and wavenumbers $k = 1, 5$.

Figure 18: Reconstruction for the mixed crack $\Gamma_3$ by the factorization method for different wavenumbers $k = 1, 5$ and complex numbers $p = 0.01 + 0.01i, 0.1 + 0.1i$. 

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