A Marginal Analysis Framework to Incorporate the Externality Effect of Ordering Perishables

Katsunobu Sasanuma\textsuperscript{a,*}, Mohammad Delasay\textsuperscript{a}, Christine Pitocco\textsuperscript{a}, Alan Scheller-Wolf\textsuperscript{b}, Thomas Sexton\textsuperscript{a}

\textsuperscript{a}College of Business, Stony Brook University, Stony Brook, NY 11794, USA
\textsuperscript{b}Tepper School of Business, Carnegie Mellon University, Pittsburgh, PA 15213, USA

Abstract
Finding the optimal policy for multi-period perishable inventory systems requires solving computationally-expensive stochastic dynamic programs (DP). To avoid the difficulty of solving DP models, we propose a framework that uses an externality term to capture the long-term impact of ordering decisions on the average cost over an infinite horizon. By approximating the externality term, we yield a tractable approximate optimality condition, which is solved through standard marginal analysis. The resulted policy is near-optimal in long-run average cost and ordering decisions.

Keywords: Perishable inventory, marginal analysis, externality, constant base-stock policy

1. Introduction
The exact analysis of perishable inventory models using stochastic dynamic programming (DP) is computationally expensive, rendering it intractable for models with large state spaces. As noted in Karaesmen et al. [1], “The policy structures outlined in Fries [2] and Nahmias [3] are quite complex; perishability destroys the simple base-stock structure of optimal policies for discrete review models without fixed order costs in the absence of perishability.” Furthermore, the DP approach does not provide any insight into the form of the inventory-dependent (which we refer to as state-dependent) optimal policy. Many researchers have thus sought effective heuristic methods [for comprehensive reviews, see, e.g., 1, 4]. Among these heuristics, the constant base-stock (CBS) policy—despite its simplicity—has been shown to be an excellent alternative to the optimal state-dependent policies [5, 6, 7]. Many state-dependent policies have also been proposed, among which two approaches have received increased attention, namely $L^2$-convexity [e.g., 8, 9, 10] and the marginal cost accounting

\textsuperscript{*}Corresponding author.
Email addresses: katsunobu.sasanuma@stonybrook.edu (Katsunobu Sasanuma), mohammad.delasay@stonybrook.edu (Mohammad Delasay), christine.pitocco@stonybrook.edu (Christine Pitocco), awolf@andrew.cmu.edu (Alan Scheller-Wolf), thomas.sexton@stonybrook.edu (Thomas Sexton)

Preprint submitted to Elsevier July 28, 2021
scheme [e.g., 11, 12, 13, 14]. The marginal cost accounting scheme utilizes marginal analysis, which provides an efficient algorithm for perishable inventory models. The key of the marginal cost accounting scheme is to develop an effective cost-balancing technique for the specific model under consideration, which is often not straightforward to identify.

We develop a marginal analysis framework that incorporates the externality effect—the indirect long-term impact of ordering decisions—on the average cost of a perishable inventory system. To our knowledge, the inclusion of the externality effect in a marginal analysis framework has not been employed in the inventory management literature, though it has been widely implemented to study economic concepts, including congestion pricing [e.g., 15, 16]. Using this framework, we derive an approximate optimality condition for the general state-dependent policy. This optimality condition is a recursive equation, which unfortunately is difficult to solve. However, by utilizing the properties of the CBS policy, we can reduce the externality effect into a fixed cost or benefit, representing the marginal external cost. Thus, we convert the original complex exact optimality condition into a simple approximate optimality condition in which only a single order amount for a given initial inventory level is involved. This single-decision condition is almost identical to the optimality condition for the newsvendor model, and hence, is easy to solve, for example, in a spreadsheet. This approach provides near-optimal solutions both with respect to the average cost and the individual order amounts. In addition, our approach provides insight into the state-dependent characteristics of near-optimal ordering policies.

There is an abundance of near-optimal heuristics for perishable inventory systems in the literature. In this paper, our primary contribution is not to add one more element to this list, but rather to provide a general framework to convert numerically intractable multi-decision stochastic dynamic inventory models to tractable single-decision models. Our framework is motivated by density functional theory [17] and its local density approximation [18]—the most popular and successful methods in computational physics and chemistry to convert multi-body problems to single-body problems. We showcase the application of our framework on one of the classic perishable inventory models, but we believe it has the potential to be applied to other inventory models as well.

2. General Formulation

In this section we describe our marginal analysis framework we use to derive the optimality condition of the state-dependent policy for a general infinite-horizon inventory system with a single perishable product. (In §3, we illustrate how to apply it to the model introduced in [3].) In an infinite-horizon single-product perishable inventory system, the initial inventory is reviewed in each period, a new order is placed, demand is fulfilled, and perished products are discarded. Let \( m \geq 1 \) be the product lifetime and \( x_i \) be the number of units with \( i \in [1, m] \) remaining periods of lifetime. Then, the initial inventory and order amount are represented by \( x_1, \ldots, x_{m-1} \) and \( x_m \), respectively. For ease of representation, let \( x^i = \sum_{j=1}^i x_j \) be the inventory level with remaining lifetime of at most \( i \) periods and \( x = x^{m-1} \) be the total initial inventory. Similarly, let \( \mathbf{x}^i = (x_1, \ldots, x_i) \) be the inventory vector with the remaining lifetime of at most \( i \) periods and \( \mathbf{x} = \mathbf{x}^{m-1} \) be the total initial
inventory vector. For notational convenience, let \( x^0 = x^0 = 0 \). Let \( \Omega = \mathbb{R}^{m-1}_+ \), a set of non-negative real numbers in an \((m - 1)\)-dimensional vector space; then \( x \in \Omega, \forall x \).

A stationary ordering policy may be characterized by its order-up-to level \( q(x) \), which is a scalar valued function of the initial inventory vector \( x \). With a slight abuse of notation, we denote \( q \) to represent either the order-up-to level \( q(x) \) for a particular \( x \) or the policy \( q(\cdot) \), a function of \( x \), distinguishing between the two when necessary. When implementing the policy \( q \), the order amount at the beginning of each period becomes \( x_m = \max\{q(x) - x, 0\} \), or simply \( x_m = [q - x]^+ \).

We propose a stationary model of this infinite-horizon problem based on the ensemble-average cost (taken over the initial inventory distribution), instead of its stochastic dynamic program (DP) model, which is known to be computationally difficult to solve due to the curse of dimensionality and dependence of decisions among different periods. In our stationary model, the complexity of tracking inventory levels in infinite time periods is incorporated into the initial inventory distribution. The infinite-horizon DP model and the stationary model represent the same average total cost; DP calculates the time-average cost, and the stationary model calculates the ensemble-average cost.

When demand is independent and identically distributed (i.i.d.), we can define each period’s initial inventory \( X \) as a non-negative random vector following a stationary distribution \( f_X(\cdot) \) given the policy \( q \). Let \( L(q,x) \) be the one-period cost associated with the single ordering decision \( q \) when the initial inventory \( x \) is observed at the beginning of the period. Then, the average total cost of the stationary model follows:

\[
L(q) = \mathbb{E}_X[L(q,X)] = \int_{\Omega} L(q(k), k) f_X(k) dk,
\]

which is a functional of the policy \( q = q(x), \forall x \in \Omega \). Let \( q^* \) be the minimizer of \( L(q) \) (i.e, \( q^* \) is the optimal inventory policy). When \( L(q) \) is convex (which is the case for many inventory models including the perishable inventory model in §3), \( q^* \) satisfies the following optimal functional derivative condition:

\[
\frac{\delta L(q)}{\delta q(x)} = \frac{\partial L(q(x), x)}{\partial q(x)} f_X(x) + \int_{\Omega} L(q(k), k) \frac{\partial f_X(k)}{\partial q(x)} dk = 0, \quad \forall x \in \Omega.
\]

The derivation of (2) is motivated by the Kohn-Sham approach to reduce the dimensionality of multi-body problems in Physics [13]; the details of the derivations and proofs appear in the online appendix. The optimality condition (2) for the stationary model has two components. When the order-up-to level (i.e., the policy) changes from \( q(x) \) to \( q(x) + \delta q(x) \) for the initial inventory \( x \): (i) The first term is the contribution of this policy change to the average total cost \( L(q) \), assuming the initial inventory distribution remains the same; and (ii) The second term is the contribution of the policy change to the average total cost \( L(q) \) due to the change in the initial inventory distribution. This second term, which we refer to as externality, captures the long-term impact of ordering decisions, since an equilibrium inventory distribution is reached only after infinitely many periods.

The externality term is the main source of complexity in the exact optimality condition (2); specifically, it is difficult to evaluate the function \( \partial f_X(k)/\partial q(x) \) representing the impact of the policy change at \( x \) on the distribution at all \( k \in \Omega \). To resolve this complexity,
we can approximate the externality term using any simple and reasonably good policy $\tilde{q}$: Specifically, we replace $\frac{\partial f_X(q)}{\partial q(x)}$ in the externality term with $\frac{\partial f_X(k)}{\partial q(x)}$ along with some necessary modifications due to normalization. Adopting the idea of the local density approximation [18], we utilize the CBS policy, which is simple to optimize and known to be a reasonably good policy for many inventory models. By using the optimal CBS policy $q^*_e$ instead of the optimal policy $q^*$, we simplify the externality term in (2) to (3), in which $V_{ex}(q_e)$ follows [4]. We expect this to be a good approximation because: (i) The expected change in the one-period cost originating from the change of the initial inventory distribution is conserved: $L(q^*(x), x)\delta f_X(q^*) \approx L(q_e^*, x)\delta f_X^e(q_e^*)$; (ii) The expected change in the order amount is conserved: $f_X^e(q_e^*)(\delta q^*(x)) \approx \delta q^*_e$ (or equivalently, $\partial q_e / \partial q(x)|_{q=q^*} \approx f_X^e(q_e^*)$). Therefore

$$\int_\Omega L(q^*(k), k) \frac{\partial f_X(q)}{\partial q(x)} \bigg|_{q=q^*} dk \approx \int_\Omega L(q_e^*(k), k) \frac{\partial f_X^e(k)}{\partial q_e} \bigg|_{q=q^*} \frac{\partial q_e}{\partial q(x)} \bigg|_{q=q^*} dk \approx V_{ex}(q_e^*) f_X^e(q_e^*),$$

$$V_{ex}(q_e) = \int_\Omega L(q_e, k) \frac{\partial f_X^e(k)}{\partial q_e} dk. \tag{3}$$

Combining (2) and (3), we derive the approximate optimality condition (5) for the stationary model, conditioned on $x$ being recurrent (i.e., $f_X^e(x) > 0$):

$$\left( \frac{\partial L(q, x)}{\partial q(x)} + V_{ex}(q_e^*) \right) f_X^e(x) = 0, \forall x \in \Omega \implies \frac{\partial L(q, x)}{\partial q(x)} + V_{ex}(q_e^*) = 0. \tag{5}$$

Similar to (2), the optimality condition (5) has two components. We refer to the first term $\partial L(q, x) / \partial q(x)$ as the marginal internal cost (MIC) and the second term $V_{ex}(q_e^*)$ as the marginal external cost (MEC), which is a constant since it is independent of $x$ under the CBS approximation. Without the MEC term, (5) reduces to the optimality condition of a standard single-decision inventory model, which is easy to solve. But the MEC term does not increase the computational complexity of solving (5) as it simply is a constant. Nevertheless, it plays an important role in minimizing the average cost. By Solving (5), we obtain the approximate optimal policy $q_e^*$, which is a state-dependent policy (due to the MIC term) like the optimal policy.

3. Applying the Framework for a Perishable Inventory Model

In this section we showcase how our framework, described in [2], can be used to analyze the classic model by Nahmias [3]: A periodic-review perishable inventory model with lost sales, fixed product lifetime, no lead time, i.i.d. demand (denoted by $D$), and a FIFO issuance policy. The four cost parameters include: purchase ($c$ per unit), holding ($h \geq 0$ per period per unit), shortage ($r$ per unit), and wastage ($\theta$ per unit). The optimal policy for this model is obtained using DP in [3]. In this section, we show how to derive the approximate optimality condition (5) for this model, and then, we evaluate its accuracy.

To derive the one-period cost $L(q, x)$, we evaluate the costs associated with a single ordering decision at the beginning of period 1, considering that the holding and shortage costs are incurred in period 1 and wastage cost is incurred in period $m$. Let the random
variable $D^i(x^{i-1})$ represent the $i$-period effective demand, i.e., the total outflow through demand and wastage from periods 1 to $i$ (excluding the wastage in period $i$). Denoting the random variables for demand and wastage in period $i$ by $D_i$ and $R_i$, respectively, we can define $D^i(x^{i-1})$ recursively:

$$D^i(x^{i-1}) = \begin{cases} D^{i-1}(x^{i-2}) + R_{i-1} + D_i & \text{if } i = 2, \ldots, m, \\ D_1 & \text{if } i = 1, \end{cases}$$

$$R_{i-1} = [x^{i-1} - D^{i-1}(x^{i-2})]_+ \quad \text{if } i = 2, \ldots, m, \quad 0 \quad \text{if } i = 1. \tag{6}$$

Note that $D^m(x^{m-1}) = D^m(x)$. We denote the number of units being held, in shortage, and wasted under policy $q$ as $[q - D]^+$, $[D - q]^+$, and $[q - D^m(x)]^+$, and we express their corresponding expectations by $n_h(q)$, $n_s(q)$, and $n_w(q)$, respectively. Then, we can represent the expected one-period cost $L(q, x)$ and the average total cost $L(q)$ as:

$$L(q, x) = h\mathbb{E}_D[q - D]^+ + (r - c)\mathbb{E}_D[D - q]^+ + (\theta + c)\mathbb{E}_{D^m(x)}[q - D^m(x)]^+. \tag{8}$$

$$L(q) = h\mathbb{E}_X\mathbb{E}_D[q(X) - D]^+ + (r - c)\mathbb{E}_X\mathbb{E}_D[D - q(X)]^+ + (\theta + c)\mathbb{E}_{D^m(x)}[q(X) - D^m(x)]^+. \tag{9}$$

**Marginal Internal Cost (MIC).** To evaluate the MIC term, we need $L(q, x)$, which in turn depends on the $m$-period effective demand $D^m(x)$. Proposition 1 shows how to obtain its cumulative distribution function (CDF).

**Proposition 1.** The CDF of $D^m(x)$ is obtained by applying the following recursively:

$$F_{D^{i+1}(x^i)}(z) = \begin{cases} \int_{z=x^i}^\infty F_{D^i(x^{i-1})}(z - \xi)f_{D^{i+1}}(\xi)d\xi & \text{if } z > x^i, i = 1, \ldots, m - 1, \\ 0 & \text{if } z \leq x^i, i = 1, \ldots, m - 1, \\ F_{D}(z) & \text{if } i = 0. \end{cases} \tag{10}$$

We can represent $L(q, x)$ and its partial derivative (i.e., MIC) as $[11]$ and $[12]$, respectively. We can confirm that $L(q, x)$ is strictly convex as $\partial^2 L(q, x)/\partial q_i^2 > 0$, $\forall q \in [0, \infty)$. Let $X^q$ denote

$$L(q, x) = h\int_{q}^\infty (q - z)f_D(z)dz + (r - c)\int_{q}^\infty (z - q)f_D(z)dz + (\theta + c)\int_{0}^{q}(q - z)f_{D^m(x)}(z)dz. \tag{11}$$

$$MIC: \frac{\partial L(q, x)}{\partial q(x)} = -(h + r - c)\bar{F}_D(q) + (\theta + c)F_{D^m(x)}(q) + h, \tag{12}$$

where $\bar{F}_D(q)$ is the complementary CDF of the demand distribution.

**Marginal External Cost (MEC).** To evaluate the MEC term $V_{ex}(q)$, we substitute the expression for $L(q, x)$ from $[11]$ into $[4]$. The first and second terms in $[11]$ do not contribute to the MEC term as they do not depend on $x^i$. As a result, we derive $[13]$. Let $X^q$ denote

---

1 For ease of exposition, we incorporate the purchase cost when the unit is either sold or perished; i.e., $r - c > 0$ and $\theta + c > 0$ represent the shortage (understocking) and the wastage (overstocking) costs.
2 This is because $\int_{q}^\infty \frac{\partial}{\partial q_c}f_{X^q}(x)dx = \frac{\partial}{\partial q_c}\int_{q}^\infty f_{X^q}(x)dx|_{q=q} = \frac{\partial}{\partial q_c} = 0$, where we use Leibniz’s rule.
the initial inventory random vector under the CBS policy $q_c$. We can compute $w_{ex}(q_c)$ following (14) by discretizing $q_c$ with step size $\Delta$ and evaluating the difference between two expectations. According to Proposition 2, $w_{ex}(q_c)$ is bounded.

$$MEC : V_{ex}(q_c) = (\theta + c) \int_{\Omega} \int_0^{q_c} (q_c - z) f_{D^n(x)}(z)dz \frac{\partial f_R^c(h)}{\partial q_c} dh = (\theta + c) w_{ex}(q_c), \quad (13)$$

$$w_{ex}(q_c) = \frac{1}{\Delta} \left( \frac{\mathbb{E}[q_c - D^n(X^{q_c + \Delta})]}{n_{\Delta}(q_c)} - \frac{\mathbb{E}[q_c - D^n(X^{q_c})]}{n_{\Delta}(q_c)} \right). \quad (14)$$

Proposition 2. The externality is negative and bounded; i.e. $-1 < w_{ex}(q_c) \leq 0, \forall q_c \geq 0$.

By replacing the MIC term (12) and the MEC term (13) in the approximate optimality condition (5), we obtain the approximate optimality condition as follows:

$$(\theta + c) F_{D^n(x)}(q) = (h + r - c) \hat{F}_D(q) - h - (\theta + c) w_{ex}(q^*_h). \quad (15)$$

The solution to (15) is unique (Proposition 3). We can thus find the approximate optimal policy $q^*_m(x), \forall x \in \Omega$ numerically. Based on (13), the approximate optimal policy approaches CBS as $h$ or $r$ grow large; the same patterns hold for the actual optimal policy [5]. Also, the optimal policy is asymptotically CBS when demand variability decreases or $m$ increases [7]; the same patterns hold for our approximate optimal policy, following Proposition 4.

Proposition 3. There exists a unique finite order-up-to level (approximate optimal policy) $q^*_m(x)$ satisfying the optimality condition (15) for any initial inventory vector $x \in \Omega$.

Proposition 4. The approximate policy approaches CBS if and only if $F_{D^n(0)}(q^*_m(0)) \rightarrow 0$.

We compare the average total costs under the approximate optimal policy (using the algorithm laid out in Table 1) and the optimal policy (using the DP algorithm described in [3]), for $c = 0$ and other parameters as specified in Table 2 under exponential and Poisson demands with mean 10. Note that the optimal policy is the same for any combination of the parameters that result in the same $r - c$ and $\theta + c$. As $m$ grows, the optimal policy approaches CBS; therefore, the relative gap $G$ between the optimal and our approximate policies is going to be more stark (if such a gap exists) for small $m$ values, where the optimal policy is highly state-dependent. According to our numerical experiments, the average and maximum cost deviations between the approximate and optimal policies are very small—around 0.05% and 0.34%, respectively.

We also examine the accuracy of the approximate optimal policy with respect to individual order amounts. Figs. 1a and 1b show examples of this comparison: We observe that the order amounts following policy $q^*_h$ closely match those from the optimal policy in recurrent regions; the discrepancies in non-recurrent regions do not impact average costs. In Table 2 we report the mean absolute deviation (MAD) between the order amounts of the optimal and approximate order policies for the recurrent initial inventory levels.

\footnote{The threshold for the recurrent region is specified by the maximum possible initial inventory (i.e., $q^*_h(0)$), as the initial inventory level cannot exceed the order amount at $x = 0$.}

\footnote{For $m = 3$, we compare the order amounts when $x_1 = 0$ and $x_2 \geq 0$.}
Table 1: Algorithm

Pre-processing (performed for each combination of \( m \) and demand distribution):

- Derive \( F_D^{(m)}(q) \) (eq. (10)).
- Discretize \( q_c \) and \( x \) for continuous distributions. For each \( q_c \), simulate a system with CBS policy and evaluate \((n_h(q_c), n_s(q_c), n_w(q_c), n_w^*(q_c))\) (eqs. (9) and (14)).

Marginal analysis (performed for each combination of \( c, h, r, \) and \( \theta \)):

- For each value of \( q_c \), evaluate \( L(q_c) \) (eq. (9)).
- Find \( q^*_c = \arg\min_{q_c} L(q_c) \) and evaluate \( w_{ex}(q^*_c) \) (eq. (14)).
- For each \( x \in \Omega \), conduct marginal analysis to determine \( q^*_h(x) \) (eq. (15)).
- Find the order amount \( x_m = [q^*_h(x) - x]^+ \).

Table 2: The comparison between the DP and approximate policies

| \( h, r, \theta \) | \( m = 2 \) | \( m = 3 \) | \( m = 2 \) | \( m = 3 \) |
|------------------|------------------|------------------|------------------|------------------|
|                  | Exponential Demand | Poisson Demand | Exponential Demand | Poisson Demand |
| \( 0, 5, 5 \)     | 19.84 0.04 0.28   | 12.14 0.34 0.62  | 1.47 0.14 0.13   | 0.13 0.27 0.00  |
| \( 0, 5, 10 \)    | 25.40 0.06 0.26   | 16.05 0.07 0.26  | 2.09 0.11 0.20   | 0.19 0.04 0.11  |
| \( 0, 5, 20 \)    | 30.74 0.02 0.14   | 20.24 0.02 0.33  | 2.92 0.08 0.07   | 0.26 0.00 0.00  |
| \( 0, 8, 7 \)     | 30.06 0.05 0.29   | 18.31 0.20 0.48  | 2.16 0.00 0.06   | 0.19 0.00 0.00  |
| \( 0, 10, 5 \)    | 29.19 0.09 0.38   | 17.49 0.17 0.46  | 1.95 0.00 0.06   | 0.17 0.00 0.00  |
| \( 1, 5, 5 \)     | 25.39 0.01 0.14   | 20.88 0.00 0.05  | 5.26 0.27 0.43   | 4.93 0.00 0.00  |
| \( 1, 5, 10 \)    | 28.93 0.01 0.14   | 22.69 0.00 0.04  | 5.52 0.00 0.00   | 4.93 0.00 0.00  |
| \( 1, 5, 20 \)    | 32.81 0.00 0.05   | 25.03 0.01 0.10  | 5.88 0.00 0.00   | 4.94 0.00 0.00  |
| \( 1, 8, 7 \)     | 36.51 0.02 0.17   | 28.38 0.02 0.16  | 6.36 0.00 0.00   | 5.68 0.00 0.00  |
| \( 1, 10, 5 \)    | 38.25 0.02 0.18   | 30.24 0.04 0.25  | 6.63 0.00 0.20   | 6.05 0.00 0.00  |

Average 0.033 0.202 0.088 0.274 0.060 0.116 0.027 0.011
Maximum 0.09 0.38 0.34 0.62 0.27 0.20 0.27 0.11

Notes: Results are based on \( 10^6 \)-period Monte Carlo simulations (\( 10^4 \) burn-in periods). We discretize the exponential demand with a step size of 0.1. The underlined values specify that the optimal policy is CBS.

4. Concluding Remarks

We develop a framework to convert a stochastic dynamic inventory model to a single-decision model, by capturing the complex interactions of multi-period decisions in a single externality term. The resulting single-decision model is simpler to solve and yields not only near-optimal average cost, but also close-to-optimal initial inventory-dependent ordering amounts, implying our method captures the fundamental properties of the optimal policy. Our framework has the potential to be applied to various perishable inventory models, including the more advanced models (like those considered in [19] and [20], for example), with appropriate modifications.

References

[1] I. Z. Karaesmen, A. Scheller-Wolf, B. Deniz, Managing perishable and aging inventories: review and future research directions, in: Planning production and inventories in the extended enterprise, Springer, 2011, pp. 393–436.
[2] B. E. Fries, Optimal ordering policy for a perishable commodity with fixed lifetime, Operations Research 23 (1975) 46–61.
Figure 1: Order amounts; $r = 10, \theta = 5$. $\tau$ separates the recurrent and non-recurrent inventory levels.

[3] S. Nahmias, Optimal ordering policies for perishable inventory—II, Operations Research 23 (1975) 735–749.
[4] O. Baron, Managing perishable inventory, in: Wiley Encyclopedia of Operations Research and Management Science, Wiley Online Library, 2011.
[5] W. T. Huh, G. Janakiraman, J. A. Muckstadt, P. Rusmevichientong, Asymptotic optimality of order-up-to policies in lost sales inventory systems, Management Science 55 (2009) 404–420.
[6] M. Bijvank, W. T. Huh, G. Janakiraman, W. Kang, Robustness of order-up-to policies in lost-sales inventory systems, Operations Research 62 (2014) 1040–1047.
[7] J. Bu, X. Gong, X. Chao, Asymptotic optimality of base-stock policies for perishable inventory systems, Available at SSRN (2020).
[8] P. Zipkin, On the structure of lost-sales inventory models, Operations Research 56 (2008) 937–944.
[9] W. T. Huh, G. Janakiraman, On the optimal policy structure in serial inventory systems with lost sales, Operations Research 58 (2010) 486–491.
[10] X. Chen, Z. Pang, L. Pan, Coordinating inventory control and pricing strategies for perishable products, Operations Research 62 (2014) 284–300.
[11] R. Levi, G. Janakiraman, M. Nagarajan, A 2-approximation algorithm for stochastic inventory control models with lost sales, Mathematics of Operations Research 33 (2008) 351–374.
[12] V.-A. Truong, Approximation algorithm for the stochastic multi-period inventory problem via a look-ahead optimization approach, Mathematics of Operations Research 39 (2014) 1039–1056.
[13] X. Chao, X. Gong, C. Shi, H. Zhang, Approximation algorithms for perishable inventory systems, Operations Research 63 (2015) 585–601.
[14] H. Zhang, C. Shi, X. Chao, Approximation algorithms for perishable inventory systems with setup costs, Operations Research 64 (2016) 432–440.
[15] W. S. Vickrey, Congestion theory and transport investment, The American Economic Review 59 (1969) 251–260.
[16] R. C. Larson, K. Sasanuma, Congestion pricing: A parking queue model, Journal of Industrial and Systems Engineering 4 (2010) 1–17.
[17] P. Hohenberg, W. Kohn, Inhomogeneous electron gas, Physical Review 136 (1964) B864.
[18] W. Kohn, L. J. Sham, Self-consistent equations including exchange and correlation effects, Physical Review 140 (1965) A1133.
[19] H. Abouee-Mehrizi, O. Baron, O. Berman, D. Chen, Managing perishable inventory systems with multiple priority classes, Production and Operations Management 28 (2019) 2305–2322.
[20] C. Kouki, B. Legros, M. Z. Babai, O. Jouini, Analysis of base-stock perishable inventory systems with general lifetime and lead-time, European Journal of Operational Research 287 (2020) 901–915.
Online Appendix for “A Marginal Analysis Framework to Incorporate the Externality Effect of Ordering Perishables”

Appendix A  Deriving Eq. (2)

We apply variational principle for the functional $L(q)$.

\[
\frac{\delta L(q)}{\delta q(x)} = \int_\Omega \left[ \frac{\partial L(q(k), k)}{\partial q(k)} \delta q(k) \delta_X(k) + L(q(k), k) \frac{\partial f^q_X(k)}{\partial q(x)} \right] dk
\]

\[
= \int_\Omega \left[ \frac{\partial L(q(k), k)}{\partial q(k)} \delta(k - x)f^q_X(k) + L(q(k), k) \frac{\partial f^q_X(k)}{\partial q(x)} \right] dk
\]

\[
= \int \frac{\partial L(q(x), x)}{\partial q(x)} f^q_X(x) + \int L(q(k), k) \frac{\partial f^q_X(k)}{\partial q(x)} dk = 0, \quad \forall x \in \Omega,
\]

where we apply chain rule first and then product rule of the functional derivative [Appendix A of [1]] to obtain the first line, replace $\delta q(k)/\delta q(x)$ with the Dirac delta function $\delta(k - x)$ to obtain the second line, and apply its sifting property $[g(x) = \int g(k)\delta(k - x)dk]$ for every continuous function $g(\cdot)$; see, e.g., [2] to obtain the fourth line.

Appendix B  Proof of Proposition [1]

We can rewrite (7) as

\[
R_i = \max\{x^i - D^i(x^{i-1}), 0\}, \quad i \geq 1,
\]

which is equivalent to

\[
D^i(x^{i-1}) + R_i = \max\{D^i(x^{i-1}), x^i\}, \quad i \geq 1.
\]

Therefore, for $i \geq 1$,

\[
F_{D^i(x^{i-1})+R_i}(\zeta) = \begin{cases} F_{D^i(x^{i-1})}(\zeta) & \text{if } \zeta \geq x^i, \\ 0 & \text{if } \zeta < x^i. \end{cases}
\]

Combining this result with [6], we obtain

\[
F_{D^{i+1}(x^{i})}(z) = Pr\{D^{i+1}(x^i) \leq z\} = Pr\{D^i(x^{i-1}) + R_i + D_{i+1} \leq z\}
\]

\[
= Pr\{D^i(x^{i-1}) + R_i \leq z - D_{i+1}\} = \int_{\xi=-\infty}^{\infty} F_{D^i(x^{i-1})+R_i}(z - \xi)f_{D_{i+1}}(\xi) d\xi
\]

\[
= \int_{\xi=0}^{z-x^i} F_{D^i(x^{i-1})}(z - \xi)f_{D_{i+1}}(\xi) d\xi, \quad \text{if } z > x^i, i \geq 1,
\]

and $F_{D^{i+1}(x^{i})}(z) = 0$, if $z \leq x^i, i \geq 1$. \qed
Appendix C Deriving Eq. (14)

We discretize the continuous order-up-to level \( q_c \) with a step size of \( \Delta \).

\[
\begin{align*}
    w_{ex}(q_c) &= \int_{\Omega} \int_{0}^{q_c} (q_c - z) f_{D^m(k)}(z) dz \frac{\partial f_X(k)}{\partial q_c} dk \\
    &= \int_{\Omega} \int_{0}^{q_c} (q_c - z) f_{D^m(k)}(z) dz \frac{f_{q_c+\Delta}(k) - f_{q_c}(k)}{\Delta} dk \\
    &= \int_{\Omega} \int_{0}^{q_c} (q_c - z) f_{D^m(k)}(z) dz f_X(k) \Delta - \int_{\Omega} \int_{0}^{q_c} (q_c - z) f_{D^m(k)}(z) dz f_X(k) dk \\
    &= \mathbb{E}[q_c - D^m(X^{q_c+\Delta})] + \mathbb{E}[q_c - D^m(X^{q_c})].
\end{align*}
\]

\[
\frac{n_{w}^\Delta(q_c) - n_{w}(q_c)}{\Delta}.
\]

Appendix D Proof of Proposition 2

We assume (as in §3) that \( h \geq 0, r - c > 0, \theta + c > 0 \), and \( f_D(d) > 0, \forall d \geq 0 \). Since we discuss the properties of random initial inventory vectors, it is convenient to use the concept of the first-order stochastic dominance (FSD), which is defined as follows:

**Definition 1.** A random variable \( X \) first-order stochastically dominates another random variable \( Y \) (\( X \succeq_{FSD} Y \)) if and only if \( F_X(x) \leq F_Y(x), \forall x \in \mathbb{R} \).

For notational convenience, we write \( X \succeq_{FSD} Y \) for random vectors \( X \) and \( Y \) if the FSD property holds componentwise: \( X_i \succeq_{FSD} Y_i \) for all \( i \)th elements of \( X \) and \( Y \). To prove the FSD property for random variables, the following property is convenient and well-known [see, e.g., 3].

**Property 1.** \( X \succeq_{FSD} Y \iff \mathbb{E}[f(X)] \geq \mathbb{E}[f(Y)] \) for any non-decreasing function \( f(\cdot) \).

Next, we present two useful FSD relationships for \( D^m(\cdot) \). Let \( x \) and \( y \) be initial inventory vectors in \( \Omega = \mathbb{R}_{\geq 0}^m \), and \( X \) and \( Y \) be the corresponding random vectors.

**Lemma 1.** \( x \geq y \) component-wise \( \iff \) \( D^m(x) \succeq_{FSD} D^m(y) \).

**Proof of Lemma 1** The proof follows from Proposition 1 and Definition 1.

**Lemma 2.** \( X \succeq_{FSD} Y \implies D^m(X) \succeq_{FSD} D^m(Y) \).

**Proof of Lemma 2**. Combining Property 1 and Lemma 1, we have \( x \geq y \) componentwise \( \implies \mathbb{E}_{D^m(x)}[f(D^m(x))] \geq \mathbb{E}_{D^m(y)}[f(D^m(y))] \) for any non-decreasing function \( f(\cdot) \). This result indicates that \( g(x) = \mathbb{E}_{D^m(x)}[f(D^m(x))] = \mathbb{E}_{D^m(x)|X=x}[f(D^m(x))|X=x] \) is a non-decreasing function in \( x \) componentwise (because \( g(x) \geq g(y) \) whenever \( x \geq y \) componentwise). Using Property 1 once again this time with \( g(x) \) we define above and the law of total expectation, \( X \succeq_{FSD} Y \implies \mathbb{E}_X[g(X)] \geq \mathbb{E}_Y[g(Y)] \iff \mathbb{E}_{D^m(X)}[f(D^m(X))] \geq \mathbb{E}_{D^m(Y)}[f(D^m(Y))] \).
\[ \mathbb{E}_{D^m(Y)}[f(D^m(Y))] \] for any non-decreasing function \( f(\cdot) \), which indicates \( D^m(X) \succeq_{FSD} D^m(Y) \).

Let \( X^c \) and \( X^c_m \) be the initial inventory random vector and the new order under the CBS policy \( q_c \), respectively. Let \( X^c_0 = (X^c, X^c_m) \in \mathbb{R}_{\geq 0}^m \). Since the entire inventory follows the CBS policy \( q_c \), \( \sum_{i=1}^m X^c_i = q_c \) must hold. Consider increasing the order-up-to level \( q_c \) by a positive infinitesimal \( \delta q_c \). Then the stationary distribution of the entire inventory (including the new order) shifts from \( X^c \) to \( X^c + \delta q_c \). The following relationship holds:

**Lemma 3.** \( X^c + \delta q_c \succeq_{FSD} X^c \).

**Proof of Lemma 3:** Define a discrete time stochastic process \( \{\tilde{X}^c(t), t = 0, 1, 2, \ldots\} \) to represent the entire inventory at time period \( t \in \mathbb{Z}_{\geq 0} \). Consider a sample path \( \tilde{X}^c(t; \omega) \).

Without loss of generality, we assume \( \tilde{X}^c(0; \omega) = 0 \) and \( X^c_m(0; \omega) = q_c \), which repeatedly appear one period after we encounter a shortage of inventory (note: \( x = 0 \) is recurrent). Suppose the CBS policy is modified from \( q_c \) to \( q_c + \delta q_c \), where \( \delta q_c \) is a positive infinitesimal that is non-divisible. Then the sample path at \( t = 0 \) shifts from \( \tilde{X}^c(0; \omega) = (0, q_c) \) to \( \tilde{X}^c + \delta q_c(0; \omega) = (0, q_c + \delta q_c) \). Assuming that this \( \delta q_c \) is used last in each age category, either one of the two occurs every period: (1) \( \delta q_c \) is not used, in which case \( \delta q_c \) becomes older (or wasted) and shows up in the older age category (or the new order category) in the next period, or (2) \( \delta q_c \) is used, in which case \( \delta q_c \) shows up in the same or newer age category in the next period. Hence, the revised sample path is represented as \( \tilde{X}^c + \delta q_c(t; \omega) = \tilde{X}^c(t; \omega) + \delta q_c \mathbf{I}(t; \omega) \), where \( \mathbf{I} \) is a random unit vector (one of the age categories is 1 and all others are 0) and \( \mathbf{I}(t; \omega) \) is its sample path. It follows that, for each age category \( i \), \( F_{X^c + \delta q_c}(x) = Pr\{X^c_i + \delta q_c \leq x\} \leq Pr\{X^c_i \leq x\} = F_{X^c_i}(x), \forall x [0, \infty) \). Hence, from Definition \( 1 \) we obtain \( \tilde{X}^c + \delta q_c \succeq_{FSD} \tilde{X}^c \), and therefore, \( X^c + \delta q_c \succeq_{FSD} X^c \). (Note: \( \tilde{X} \) and \( \mathbf{I} \) are not independent, but the dependency does not affect the conclusion.)

**Lemma 4.** \( D^m(X^c + \delta q_c) \succeq_{FSD} D^m(X^c) \).

**Proof of Lemma 4:** The result is immediately obtained from Lemmas 2 and 3.

Using Property 1 and Lemma 4, we can bound the externality term and obtain Proposition 2.

**Proof of Proposition 2:** To prove this property, we rewrite the partial derivative with the expression using a positive infinitesimal change \( \delta q_c \): \( \partial f_{D^m(X^c)}(z)/\partial q_c = [f_{D^m(X^c + \delta q_c)}(z) - f_{D^m(X^c)}(z)]/\delta q_c \).

First part \( w_{c_0}(q_c) \leq 0 \): Changing the order of two integrations and the partial derivative
Appendix E Proof of Proposition 3

We continue to assume (as in [3]) that \(h \geq 0, r - c > 0, \theta + c > 0, \) and \(f_D(d) > 0, \forall d \geq 0.\) Proposition 3 follows from Proposition 2.

Let \(g_X(q) = (\theta + c) F_{D^m(x)}(q) - (h + r - c) F_D(q) + h + (\theta + c) w_{\text{ex}}(q_c),\) where \(q_c\) is independent of \(q.\) Observe that \(g_X(q)\) is an increasing function with respect to \(q\) as \(\partial g_X(q)/\partial q = (\theta + c) F_{D^m(x)}(q) + (h + r - c) F_D(q) > 0.\) Also, using Proposition 2, \(g_X(0) = -(r - c) + (\theta + c) w_{\text{ex}}(q_c) < 0.\) Finally there always exists a finite \(\hat{q}\) satisfying \(F_{D^m(x)}(\hat{q}) > 1 - \hat{\delta},\) where \(\hat{\delta} = \frac{(\theta + c)(1 + w_{\text{ex}}(q_c))}{h + r + \theta} \in (0, 1).\) Such \(\hat{q}\) also satisfies \(F_D(\hat{q}) > 1 - \hat{\delta}\) (or equivalently, \(\hat{F}_D(\hat{q}) < \hat{\delta}\)) since \(D^m(x) \geq F_{\text{SD}}\) (see [6] and Definition 1). Using Proposition 2 and \(\hat{\delta}\) defined above, this \(\hat{q}\) satisfies

\[
g_X(\hat{q}) = (\theta + c) F_{D^m(x)}(\hat{q}) - (h + r - c) F_D(\hat{q}) + h + (\theta + c) w_{\text{ex}}(q_c) \\
> (\theta + c)(1 - \hat{\delta}) - (h + r - c) \hat{\delta} + h + (\theta + c) w_{\text{ex}}(q_c) \\
= (\theta + c)(1 + w_{\text{ex}}(q_c)) - (h + \theta + r) \hat{\delta} + h = h \geq 0.
\]
Since $g_x(q)$ is monotonic, we can conclude that there exists a unique and finite solution $q_h^*(x) \in (0,1)$ that satisfies $g_x(q) = 0$ (and hence the optimality condition (15)) for any initial inventory vector $x \in \Omega$. \hfill \Box

Finally, we present a corollary of Proposition 3 which is utilized in the proof of Proposition 4.

**Corollary 1.** Consider two initial inventory vectors $x_1, x_2 \in \Omega$, where $x_1 \neq x_2$. Then

$$|q_h^*(x_1) - q_h^*(x_2)| \to 0 \iff |F_{D^m(x_1)}(q_h^*(x_1)) - F_{D^m(x_2)}(q_h^*(x_2))| \to 0.$$

**Proof of Corollary 1:** Since the solution to (15) is unique and finite (Proposition 3), we can write two optimality equations corresponding to initial inventory vectors $x_1$ and $x_2$. Note that $w_{ex}(q_c^*)$ in (15) does not depend on $x$. By subtracting one from the other, we obtain

$$(\theta + c)(F_{D^m(x_1)}(q_h^*(x_1)) - F_{D^m(x_2)}(q_h^*(x_2))) = (h + r - c)(\bar{F}_D(q_h^*(x_1)) - \bar{F}_D(q_h^*(x_2))).$$

The result follows from the assumptions $\theta + c > 0$, $h + r - c > 0$, and continuous $F_D(d)$ for $d \in [0, +\infty)$.

**Appendix F  Proof of Proposition 4**

We first show the relationship between two solutions with different initial inventory levels (Lemma 5), from which we can determine the upper and lower bounds of the solution (Lemma 6). If the gap between the upper and lower bounds shrinks, a state-dependent policy should approach CBS. The condition to make the gap shrink is provided in Proposition 4.

**Lemma 5.** $q_h^*(x_1) \geq q_h^*(x_2)$ and $F_{D^m(x_1)}(q_h^*(x_1)) \leq F_{D^m(x_2)}(q_h^*(x_2))$ if $x_1 \geq x_2$ component-wise.

**Proof of Lemma 5:** As in the proof of Proposition 3, we define $g_x(q) \equiv (\theta + c)F_{D^m(x)}(q) - (h + r - c)\bar{F}_D(q) + h + (\theta + c)w_{ex}(q_c^*)$. This $g_x(q)$ is an increasing function with respect to $q$. Now, let $q_h^*(x_1)$ and $q_h^*(x_2)$ be the unique, finite solutions to $g_{x_1}(q) = 0$ and $g_{x_2}(q) = 0$, respectively. Since $x_1 \geq x_2$ componentwise $\implies D^m(x_1) \succeq_{FS\text{D}} D^m(x_2)$ (Lemma 1) $\iff F_{D^m(x_1)}(q) \leq F_{D^m(x_2)}(q), \forall q \in \mathbb{R}$ (Definition 1), it follows that $g_{x_1}(q) \leq g_{x_2}(q), \forall q \in \mathbb{R}$. In particular, at $q = q_h^*(x_2)$, we obtain $g_{x_1}(q_h^*(x_2)) \leq g_{x_2}(q_h^*(x_2)) = 0$, which implies $q_h^*(x_1) \geq q_h^*(x_2)$. Furthermore, $q_h^*(x_1) \geq q_h^*(x_2)$ implies $\bar{F}_D(q_h^*(x_1)) \leq \bar{F}_D(q_h^*(x_2))$ because $\bar{F}_D(q)$ is a decreasing function of $q$. Combining this result with (15), we obtain $F_{D^m(x_1)}(q_h^*(x_1)) \leq F_{D^m(x_2)}(q_h^*(x_2))$. \hfill \Box

**Lemma 6.** $q^l \leq q_h^*(x) \leq q^\dagger$ and $F_{D^m}(q^\dagger) \geq F_{D^m(x)}(q_h^*(x)), \forall x \in \Omega$.

**Proof of Lemma 6:** From Lemma 5, we have $q_h^*(0) \leq q_h^*(x) \leq q_h^*(y) \leq q_h^*(0)$ and $F_{D^m(0)}(q_h^*(0)) \geq F_{D^m(x)}(q_h^*(x)), \forall y \geq x \in \Omega$ componentwise. We obtain the result by taking the limit of a large initial inventory $y$ and denoting $D^m = D^m(0), q^l = q_h^*(0)$, and $q^\dagger = \lim_{v \to \infty} q_h^*(y)$, where $v$ is the smallest component in the initial inventory vector $y$. \hfill \Box
Proof of Proposition 4: We split the proof in three parts:

(First part: $F_{D^m}(q^\dagger) \to 0 \implies |q^\dagger - q^\dagger| \to 0$) Using Lemma 6, $F_{D^m}(q^\dagger) \to 0 \implies F_{D^m}(x)(q^\dagger_h(x)) \to 0, \forall x \in \Omega$. Since $F_{D^m}(x)(q^\dagger_h(x))$ converges to the same value (0) for any initial inventory vector $x$, using Corollary 1 we can conclude $|q^\dagger - q^\dagger| \to 0$.

(Second part: $|q^\dagger - q^\dagger| \to 0 \implies q^\dagger_c(x) \to q^\dagger_c, \forall x \in \Omega_r$) This part is trivial because $\Omega_r \subseteq \Omega$.

(Third part: $q^\dagger_h(x) \to q^\dagger, \forall x \in \Omega_r \implies F_{D^m}(q^\dagger) \to 0$) Consider two initial inventory vectors: $x_1 = 0$ and $x_2 = (q^\dagger, 0, \ldots, 0)$; $x_2$ represents $q^\dagger (= q^\dagger_h(0) = q^\dagger_h(x_1))$ units of initial inventory with remaining lifetime of $m - 1$ periods. Note that $x_1 \neq x_2$ and both $x_1, x_2 \in \Omega_r$ because we assume that $D$ can take 0 and any large amount. Note also that from Proposition 1 we know $F_{D^m(x_2)}(q^\dagger) = 0$. (This is intuitively obvious: $D^m(x_2)$ is the total outflow (through demand and wastage) from periods 1 to $m$ (excluding the wastage in period $m$) when the initial inventory is $x_2$. Hence, the support of its CDF is bounded below by $q^\dagger$.) Now, suppose $q^\dagger_h(x) \to q^\dagger, \forall x \in \Omega_r$, then $q^\dagger_h(x_2) \to q^\dagger_h(x_1) = q^\dagger$. Therefore, using Corollary 1 and replacing $q^\dagger_h(x_2)$ with $q^\dagger$, we obtain $F_{D^m}(q^\dagger) = F_{D^m(x_1)}(q^\dagger_h(x_1)) \to F_{D^m(x_2)}(q^\dagger_h(x_2)) \to F_{D^m(x_2)}(q^\dagger) = 0$. □

References

[1] R.G. Parr, W. Yang. 1989. Density-Functional Theory of Atoms and Molecules. Oxford University Press.
[2] R.N. Bracewell. 1986. The Fourier Transform and Its Application. McGraw-Hill New York.
[3] E. Wollstetter. 1999. Topics in Microeconomics: Industrial Organization, Auctions, and Incentives. Cambridge University Press.