Quaternary dichotomous voting rules

Annick Laruelle · Federico Valenciano

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Abstract In this article, we provide a general model of “quaternary” dichotomous voting rules (QVRs), namely, voting rules for making collective dichotomous decisions (to accept or reject a proposal), based on vote profiles in which four options are available to each voter: voting (“yes”, “no”, or “abstaining”) or staying home and not turning out. The model covers most of actual real-world dichotomous rules, where quorums are often required, and some of the extensions considered in the literature. In particular, we address and solve the question of the representability of QVRs by means of weighted rules and extend the notion of “dimension” of a rule.

1 Introduction

This article deals with dichotomous decision-making that specify collective acceptance or collective rejection of a proposal. Most of the social choice literature on dichotomous voting rules considers only binary rules, where the passage or rejection of a proposal is decided on the basis of the votes cast by those who vote “yes.” This implicitly assumes either that voting “yes” and “not” are the only feasible options,
or that abstention and not showing up are counted as “noes.” In the real world, rules often distinguish between these options. A well-known example is the UN Security Council. A proposal can be passed if at least 9 of the 15 members are in favor and no permanent member is against (i.e., all permanent members approve or abstain). In many Parliaments, the requirement for passing a bill is often based on the number of members (MPs) present, not on the total number of MPs. Other examples include all rules with a quorum, that is, those that require the presence of a minimum number of voters for a vote to take place.

Still ternary dichotomous voting rules have been studied. Felsenthal and Machover (1997) study rules where the three actions are voting “yes,” voting “no,” and abstention. They deal with the measurement of power in this context. Uleri (2002), Côrte-Real and Pereira (2004), Herrera and Mattozzi (2010), and Maniquet and Morelli (2010) study ternary rules where the third option is not participating. They study the strategic aspect induced by the quorum.

Dougherty and Edward (2010) compare the simple majority and the absolute majority in a context where all four options (voting “yes,” voting “no,” abstention, and non participation) are possible.

In this article, we extensively study quaternary dichotomous voting rules (QVRs). The four possible options are those mentioned above, and the outcome is dichotomous, i.e., either acceptance or rejection of a proposal. This article is complementary to Freixas and Zwicker (2003), Freixas and Zwicker (2009), and Zwicker (2009). On the one hand, they consider $j$ options and $k$ outcomes, while we focus on 4 inputs and 2 outputs. On the other hand, they study rules with any number of ordered options or levels of support in the input, and any number of ordered levels of approval in the output. We consider the case where levels of support are not necessarily ordered. In particular, we study rules with quorum, where the “not participating” option and the “no” option cannot be ranked: depending on the vote profile, the “no” option may be more favorable or less favorable to the rejection of the proposal than the “not participating” option. We extend Freixas and Zwicker’s notion of weighted rules and also define the notion of dimension in this context and prove its well definedness.

The article is organized as follows. In Sect. 2, the general model of QVR is introduced. In Sect. 3, a lattice of “monotonic” classes of QVRs (i.e., specified by an admissible notion of monotonicity) is introduced and its basic properties are studied. In Sect. 4, some real world examples, their monotonicities, and minimal monotonic classes containing each of them are examined. In Sect. 5, the notions of weighted majority rule and dimension are extended for the class of rules introduced here, and the well-definedness of the notion of dimension is proved. Finally, we briefly summarize our conclusions and point out some lines of further research.

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1 The existence of real world examples has been noticed in the literature, as, for instance, in Freixas and Zwicker (2003).
2 Quaternary dichotomous rules

A dichotomous voting rule specifies a collective decision, acceptance, or rejection, for each possible vote profile. In the binary case usually considered in the literature (see, for instance, Laruelle and Valenciano 2008) voters can only cast either a positive or a negative vote (or any action other than voting “yes” counts as voting “no”).

If \( n \) is the number of seats on the committee, we label them by \( 1, 2, \ldots, n \), and \( \mathcal{N} = \{1, 2, \ldots, n\} \). The same labels are also used to designate the voters that occupy the corresponding seats. A (binary) vote configuration is a 2-partition of \( \mathcal{N}, (S^Y, S^N) \), where \( S^Y \) is the set of yes voters. Then a binary dichotomous voting rule specifies a set \( \mathcal{V} \subseteq 2^\mathcal{N} \) of “winning configurations” (i.e., those which lead to the acceptance of the proposal)

\[
\mathcal{V} = \left\{ S = (S^Y, S^N) \in 2^\mathcal{N} : S \text{ leads to acceptance} \right\}
\]

that satisfies the following conditions. First, if all voters vote “yes,” the proposal should be adopted (full-support condition): \( (\mathcal{N}, \emptyset) \in \mathcal{V} \); second, if all voters vote “no” (or none votes “yes”), the proposal should be rejected (null-support condition): \( (\emptyset, \mathcal{N}) \notin \mathcal{V} \); and third, if a vote configuration is winning, then any other configuration with a larger set of “yes”-voters is also winning (monotonicity condition): if \( S \in \mathcal{V} \) and \( S^Y \subseteq T^Y \) then \( T \in \mathcal{V} \). A fourth condition is usually added. The possibility of a proposal and its negation both being accepted should be prevented. If a proposal leads to a configuration \( S \), its opposite should lead to the configuration \( T \) where \( T^Y = S^N \). Both configurations should not be winning: if \( S \in \mathcal{V} \) then \( T \notin \mathcal{V} \). In this case, the rule is called proper. A binary dichotomous rule that does not satisfy this condition is said to be improper. Most binary rules in the real world are proper. Nevertheless, it proves convenient to widen the class so as to include improper rules within the domain of voting rules.

In general, more options than “yes” or “no” are offered to voters and the final outcome may be sensitive to these options. Here we are interested in this more general type of voting rule. We consider the quaternary case where four different options are offered to each voter: a voter may not show up at the vote, may turn up but abstain, may come and vote yes, or may come and vote no.2 The precise account of a particular vote is specified by a vote configuration or vote profile \( S = (S^Y, S^A, S^H, S^N) \) that keeps track of the action taken by the voter occupying each seat, where \( S^Y \) is the set of “yes”-voters, \( S^N \) is the set of “no”-voters, \( S^A \) is the set of those who participate and abstain, and \( S^H \) that of those who do not participate or “stay at home.” A (quaternary) vote configuration is thus in general a 4-partition of \( \mathcal{N} \), i.e., any two of these subsets are disjoint and \( \mathcal{N} = S^Y \cup S^A \cup S^N \cup S^H \). The number of “yes”-voters in the configuration \( S \) is denoted by \( s^Y \), \( s^N \) denotes the number of “no”-voters, etc. We denote by \( 4^\mathcal{N} \) the set\(^3 \) of 4-partitions of \( \mathcal{N} \).

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2 In fact generality can be pushed a bit further by including the possibility of spoilt votes as a separate option, but they are most often identified with abstention or absence.

3 This is an abbreviated notation for the set of maps from \( \mathcal{N} \) to the set of actions \( \{Y, A, H, N\} \), whose usual set-theoretic notation is \( \{Y, A, H, N\}^\mathcal{N} \).
A QVR based on this input should specify a set \( \mathcal{W} \subseteq 4^N \) of “winning configurations”\(^4\)

\[
\mathcal{W} = \left\{ S = (S^Y, S^A, S^H, S^N) \in 4^N : S \text{ leads to acceptance} \right\}.
\]

Now let us consider what conditions may reasonably be imposed on such a set of winning configurations for a sound and general enough notion of voting rule. Note that monotonicity for actual rules with a quorum is different from that for binary dichotomous rules, in that a configuration can be winning, while a configuration with a larger set of “yes”-voters (and a smaller set of “no”-voters) may be losing, as the following example shows:

**Example 1** In the Belgian Parliament, a bill must receive more votes in favor than votes against in order to be passed, and a quorum of 76 (out of 150 MPs) is required. If 50 MPs go and vote “yes,” 30 go and vote “no” and 70 are absent, the proposal is accepted, while if 60 MPs vote “yes” and 90 are absent, the proposal is rejected.

In the example the set of “yes”-voters is not extended exclusively at the expense of the set of no-voters, the set of voters who stay at home also becomes larger. If the set of “yes”-voters is extended exclusively at the expense of the “no”-voters, a winning configuration should not become losing. We refer to this condition as \(NY\)-monotonicity. In similar terms we define and assume \(AY\)-monotonicity (extension of the set of “yes”-voters solely at the expense of the abstainers), and \(HY\)-monotonicity (extension of the set of “yes”-voters solely at the expense of the voters who stay at home). We thus have three monotonicity conditions:

- **NY-monotonicity**: If \( S \in \mathcal{W} \), then \( T \in \mathcal{W} \) for any \( T \) such that \( S^Y \subseteq T^Y \), \( S^A = T^A \) and \( S^H = T^H \).
- **AY-monotonicity**: If \( S \in \mathcal{W} \), then \( T \in \mathcal{W} \) for any \( T \) such that \( S^Y \subseteq T^Y \), \( S^N = T^N \) and \( S^H = T^H \).
- **HY-monotonicity**: If \( S \in \mathcal{W} \), then \( T \in \mathcal{W} \) for any \( T \) such that \( S^Y \subseteq T^Y \), \( S^A = T^A \) and \( S^N = T^N \).

The following diagram represents these three monotonicities, that is, the transitions of votes that keep a winning vote configuration winning (“No” \( \to \) “Yes,” etc.) that we assume for any dichotomous rule based on such voting profiles:

\[
\begin{array}{ccc}
\text{Yes} & \uparrow & \leftarrow \\
\text{No} & \text{Abstain} & \text{Home}
\end{array} \quad \text{or just} \quad
\begin{array}{ccc}
\text{Y} & \uparrow & \leftarrow \\
\text{N} & \text{A} & \text{H}
\end{array}
\]

Most real world voting rules satisfy further monotonicities (see the next section). Nevertheless we include only these basic ones in the basic definition of voting rule in order to have a sufficiently general notion.

\(^4\)A more suitable term in this setting, where a vote profile cannot be summarized by the set of “yes”-voters as it must include at least those that have chosen three different actions, would be “accepting configuration” or “yes-winning configuration.” Nevertheless, for the sake of simplicity, we maintain the term “winning configuration.”
Now let us consider the extension of the other two properties satisfied by binary rules for a set $\mathcal{W} \subseteq 4^N$. Full support of a proposal should imply its acceptance, thus we impose:

**Full-support:** A set $\mathcal{W} \subseteq 4^N$ is said to satisfy the full-support condition if a unanimous “yes” leads to the acceptance of the proposal: If $S^Y = N$, then $S \in \mathcal{W}$.

As to the extension of “null-support” condition the situation is more delicate. An obvious extension is this:

**Null-support:** A set $\mathcal{W} \subseteq 4^N$ is said to satisfy the null-support condition if the proposal is rejected in case of null support: If $S^Y = \emptyset$, then $S \notin \mathcal{W}$.

Nevertheless, if we want to avoid clashes with some configurations of monotonicities that are to be found in the specification of some real world voting rules, then this condition is too strong. For instance, a quorum requirement considers as equivalent the options of voting “yes,” “no,” and abstaining. In the corresponding rule, a configuration where $S^Y = \emptyset$ (but $S^A \neq \emptyset$) could be winning. Thus, we weaken the “null-support” condition in order to avoid ruling out such rules. To that end, we need a previous notion. Let $X, Z$ be two options, i.e., $X, Z \in \{Y, A, H, N\}$ and $\mathcal{W} \subseteq 4^N$. We say that $X$ and $Z$ are “equivalent in $\mathcal{W}$” and write $X \equiv_{\mathcal{W}} Z$, if for all $S \in 4^N$

$$S \in \mathcal{W} \Rightarrow T \in \mathcal{W}, \text{ for all } T \in 4^N \text{ s.t. }$$

$$S^X \cup S^Z = T^X \cup T^Z, \text{ and } S^V = T^V \text{ for all action } V \in \{Y, A, H, N\}\{Z, X\}.$$ 

Now we can formulate a weak version of “null-support” for this type of rule:

**Weak null-support:** A set $\mathcal{W} \subseteq 4^N$ is said to satisfy the weak null-support condition if for all $S \in \mathcal{W}$, either $S^Y \neq \emptyset$ or there exists $X \in \{A, H, N\}$ s.t. $X \equiv_{\mathcal{W}} Y$ and $S^X \neq \emptyset$.

This condition is obviously weaker than “null-support” and equivalent when no action is equivalent to voting “yes.” Adding this condition and that of “full-support” to the above monotonicities, we define what in the sequel we refer to as a QVR.

**Definition 1** An $n$-voter “quaternary dichotomous voting rule” (QVR) is a set $\mathcal{W}$ of 4-partitions of $N$ that satisfies full-support, weak null-support, NY-monotonicity, AY-monotonicity, and HY-monotonicity.

**Remark** Given the monotonicities assumed, the “full-support” condition can be replaced by this:

**Nonemptiness:** $\mathcal{W} \neq \emptyset$.

---

5 In real world rules, a participation quorum is usually associated with other requirements that break the equivalence between the “yes” option and the others.
The following preorders (i.e., binary reflexive and transitive relations) on the set of vote configurations can be naturally associated with each of the three basic monotonicities assumed:

\[ S \preceq_{NY} T \iff (S^Y \subseteq T^Y, S^A = T^A \text{ and } S^H = T^H), \quad (1) \]

\[ S \preceq_{AY} T \iff (S^Y \subseteq T^Y, S^N = T^N \text{ and } S^H = T^H), \quad (2) \]

\[ S \preceq_{HY} T \iff (S^Y \subseteq T^Y, S^A = T^A \text{ and } S^N = T^N). \quad (3) \]

By means of these relations, by just replacing "XY" by the desired monotonicity (i.e., NY, AY, or HY), the three monotonicities assumed can be expressed in the form: A rule \( \mathcal{W} \) verifies \( XY \)-monotonicity if

\[ (S \in \mathcal{W} \text{ and } S \preceq_{XY} T) \Rightarrow T \in \mathcal{W}, \]

entirely analogous to the monotonicity condition for binary rules: preorder "\( \preceq_{XY} \)" merely replaces "\( SY \subseteq TY \)."

As has been done for each of these monotonicities separately, when all three of them are assumed it is possible to formulate all their implications by means of a single preorder \( \preceq_{QVR} \), given by the transitive closure of the union of the preorders associated with each of the three monotonicities, as stated explicitly by the following:

**Definition 2** Given two vote configurations, \( S \) and \( T \), \( S \preceq_{QVR} T \) if and only if there exists a finite sequence of vote configurations \( S_1, S_2, \ldots, S_k \) such that \( S_1 = S, S_k = T \) and for all \( j = 1, 2, \ldots, k - 1 \): \( S_j \preceq_{XY} S_{j+1} \), where "\( \preceq_{XY} \)" is any of the relations defined by (1), (2), or (3).

Now it is possible to express the three monotonicities by a single implication which yields the following alternative definition of QVR:

**Definition 3** An \( n \)-voter quaternary voting rule is a nonempty set \( \mathcal{W} \) of 4-partitions of \( N \) that satisfies weak null-support and such that

\[ (S \in \mathcal{W} \text{ and } S \preceq_{QVR} T) \Rightarrow T \in \mathcal{W}, \quad (4) \]

where \( \preceq_{QVR} \) is the relation given by Definition 2.

Preorder \( \preceq_{QVR} \) can also be formulated exclusively in terms of configurations \( S \) and \( T \) as the following proposition shows:

**Proposition 1** For any two vote configurations \( S \) and \( T \):

\[ S \preceq_{QVR} T \iff \begin{cases} S^N \supseteq T^N \\ S^H \supseteq T^H \\ S^A \supseteq T^A. \end{cases} \quad (5) \]

**Proof** \((\Rightarrow)\) Assume \( S_1, S_2, \ldots, S_k \) are such that \( S_1 = S, S_k = T \) and for all \( j = 1, 2, \ldots, k - 1 \): \( S_j \preceq_{XY} S_{j+1} \). Then, from (1), (2), and (3), it can thus immediately
be concluded that for any \( j = 1, 2, \ldots, k - 1 \), we have: \( S_j^N \supseteq S_{j+1}^N \), \( S_j^H \supseteq S_{j+1}^H \), and \( S_j^A \supseteq S_{j+1}^A \). Thus the three inclusions hold for \( S \) and \( T \).

(\( \iff \)) Now, reciprocally, assume that the three inclusions hold for \( S \) and \( T \). Then let \( S_1, S_2, S_3, \) and \( S_4 \) be the following configurations: \( S_1 = S \); \( S_2 \) such that

\[
S_2^Y = S_1^Y \cup (S_1^A \setminus T^A), \quad S_2^H = S^H, \quad S_2^A = T^A, \quad S_2^N = S^N;
\]

\( S_3 \) such that

\[
S_3^Y = S_2^Y \cup (S^H \setminus T^H), \quad S_3^H = T^H, \quad S_3^A = T^A, \quad S_3^N = S^N;
\]

and \( S_4 = T \). Then we have

\[
S = S_1 \preceq_A Y S_2 \preceq_H Y S_3 \preceq_N Y S_4 = T.
\]

The latter because

\[
S_4^Y = S_3^Y \cup (S^N \setminus T^N), \quad S_3^H = T^H, \quad S_3^A = T^A, \quad S_3^N = T^N,
\]

that is, \( S_3 \preceq_N Y S_4 = T \). Therefore \( S \preceq_{QVR} T \).

The preorder \( \preceq_{QVR} \) allows us to formulate the notion of “minimal” vote configuration (relative to the monotonicities summarized by \( \preceq_{QVR} \)), i.e., those winning configurations whose winning character cannot be inferred from that of other configurations and the three monotonicity conditions.

**Definition 4** A vote configuration \( S \in 4^N \) is *minimal winning* in rule \( W \) w.r.t. \( \preceq_{QVR} \) if \( S \in W \) and there is no other winning configuration \( T \) such that \( T \prec_{QVR} S \) (i.e., such that \( T \preceq_{QVR} S \) but \( S \not\prec_{QVR} T \)); and \( S \) is *maximal losing* w.r.t. \( \preceq_{QVR} \) if \( S \not\in W \) and there is no other losing configuration \( T \) such that \( S \prec_{QVR} T \).

A rule is *anonymous* if a vote configuration is winning or not dependent solely on the number of voters of each type.

**Definition 5** A quaternary dichotomous voting rule is “anonymous” if for all \( S \in W \) and all \( T \) such that \( t^Y = s^Y, t^N = s^N, t^A = s^A \) and \( t^H = s^H \), we have \( T \in W \).

When the rule is anonymous the inclusions (5) that define the binary relation \( \preceq_{QVR} \) can be replaced by a set of inequalities involving the cardinalities of the sets of different types of vote. It is enough to replace each set by its cardinality (e.g., \( S^Y \) by \( s^Y \), etc.), and replacing “\( \subseteq \)” by “\( \leq \)”.

Thus, we have the following self-contained definition:

**Definition 6** An anonymous quaternary voting rule (AnQVR) is a nonempty set \( W \) of 4-partitions of \( N \) that satisfies weak null-support and such that

\[
(S \in W \text{ and } S \preceq_{\text{AnQVR}} T) \Rightarrow T \in W,
\]

where \( \preceq_{\text{AnQVR}} \) is the relation given by
\[
S \preceq_{\text{AnQVR}} T \iff \begin{cases} 
s^N \geq t^N \\
s^H \geq t^H \\
s^A \geq t^A.
\end{cases}
\]

Finally, the notion of “proper” rule remains to be extended to this wider class of rules, but we postpone this to the next section.

3 The lattice of classes of monotonic QVRs

All real world dichotomous voting rules based on the four options satisfy the three monotonicities considered so far and consequently fit into the above general definition. Nevertheless, actual dichotomous voting rules often satisfy further monotonicities.

Definition 7 Given any two options \(X, Z \in \{Y, A, H, N\}\), a QVR \(\mathcal{W}\) is \(XZ\)-monotonic if

\[
S \in \mathcal{W} \Rightarrow T \in \mathcal{W} \text{ for all } T \text{ s.t. } S \preceq_{XZ} T,
\]

where

\[
S \preceq_{XZ} T \iff (S^Z \subseteq T^Z, \text{ and } S^V = T^V \text{ for } V \in \{Y, A, H, N\} \setminus \{Z, X\}).
\]

By assuming different combinations of monotonicities (in addition to the three basic ones), a complete lattice of subclasses of QVRs related by inclusion arises. In the next section, we constrain our attention to some combinations of monotonicities to be found in real world examples. Nevertheless it is convenient first to establish a few basic facts about the general lattice that will be useful later.

In what follows we refer to subclasses of \(n\)-voter QVRs (denoted by \(C, C_1, C_2, \ldots\)) monotonic in the following sense:

Definition 8 A class of “monotonic” QVRs is a class of QVRs that contains all QVRs that satisfy a specific set of \(XZ\)-monotonicities that includes the three basic ones.

Note that as a class \(C\) of monotonic QVRs is characterized by a set of monotonicities, all the monotonicities within the class can be summarized in the form:

\[(S \in \mathcal{W} \text{ and } S \preceq_C T) \Rightarrow T \in \mathcal{W},\]

where \(\preceq_C\) is the preorder determined by the monotonicities that characterize \(C\), as has been done for the whole class of QVRs. As in the case of \(\preceq_{\text{QVR}}\), it is given by the transitive closure of the union of the preorders associated with each of those monotonicities. In the next section, some monotonic classes and their associated preorders are explicitly specified, but here we provide a list of all possible “constellations” of monotonicities. In order to simplify the list, we use the following notation 
Quaternary dichotomous voting rules

\[ \{V, X, Z\} = \{A, H, N\}, \] and “\(X \rightarrow Z\)” means that \(XZ\)-monotonicity holds, while “\(X \equiv Z\)” means that these options are interchangeable as both \(X \rightarrow Z\) and \(Z \rightarrow X\) hold. In the diagrams we omit those monotonicities that are implied by those that are stated explicitly: for instance, in \(V \equiv X \rightarrow Z \rightarrow Y\) the arrow \(X \rightarrow Y\) (and another 2) is omitted as implied by \(X \rightarrow Z\) and \(Z \rightarrow Y\). In this way, for instance, the pattern of monotonicities \(V \equiv X \rightarrow Z \rightarrow Y\) represents three possible variations depending on whether \(Z = A\) or \(Z = H\) or \(Z = N\). These are the possible configurations of monotonicities specifying a monotonic class:

1. With (essentially) only one option there is a unique possibility: the trivial degenerated rule where

\[ Y \equiv A \equiv H \equiv N. \]

2. With up to two (essentially) different options there are three possible configurations:

\[
\begin{array}{ccc}
Y & Y \equiv Z & Y \equiv Z \equiv X \\
\uparrow & \uparrow & \uparrow \\
A \equiv H \equiv N & X \equiv V & V
\end{array}
\]

3. With up to three (essentially) different options there are five possibilities:

\[
\begin{array}{cccccc}
Y & Y & Y \equiv Z & Y & Y \equiv Z \\
\uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\
Z & Z \equiv X & X & \rightarrow \leftarrow & \rightarrow \leftarrow \\
\uparrow & \uparrow & \uparrow & Z \equiv X & V & V \\
X \equiv V & V & V
\end{array}
\]

4. With up to four different options there are five possibilities:

\[
\begin{array}{ccccc}
Y & Y & Y & Y & Y \\
\uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\
Z & Z & X & X & \rightarrow \leftarrow \\
\uparrow & \uparrow & \uparrow & \uparrow & \uparrow \leftarrow \rightarrow \\
X & X & V & V & V \\
\uparrow & \uparrow & \uparrow & \uparrow & A \ H \ N \\
V & V & V
\end{array}
\]

Observe that each monotonic class \(C\) is specified by a combination of monotonicities that fits one of the above diagrams, which can be seen as a preorder \(\leq_C\) over the set of options \(\{Y, A, H, N\}\) (that determines a preorder \(\preceq_C\) over vote configurations). In some cases, the preorder over the set of options is linear (i.e., it is complete and transitive). This is the case for all monotonicity configurations with only one or two options, but also for the first three of those with 3 options and the first one with 4 options. In what follows we refer to such monotonic classes of QVRs as the linear...
classes in reference to this linear preorder, and we refer as linear rules to those that belong to any of these classes.

Then we have the following definitions and facts that state some general conclusions about these classes and these QVRs, some of which are used later (their proof is given in an Appendix). The reader less interested in technical details may skip the rest of this section and go directly to Sect. 4.

First note that the notions of minimal winning and maximal losing configuration are relative to the preorder over vote configurations that summarizes the monotonicities that characterize the monotonic class within which we are working. That is, if \( \preceq \) is the preorder associated with class \( C \), we denote

\[
\text{Min}_w^C(W) := \{ S \in W : (T \prec^C S \Rightarrow T \not\in W) \}
\]

\[
\text{Max}_l^C(W) := \{ S / \in W : (S \prec^C T \Rightarrow T \in W) \}
\]

Then we have the following relations.

**Proposition 2** Let \( C_1 \) and \( C_2 \) be two classes of monotonic QVRs, then: (i) \( C_1 \subseteq C_2 \) if and only if \( \preceq \subseteq \preceq \) (or, equivalently, \( \preceq \subseteq \preceq \)); (ii) If \( W \in C_1 \subseteq C_2 \), then \( \text{Min}_w^{C_1}(W) \subseteq \text{Min}_w^{C_2}(W) \) and \( \text{Max}_l^{C_1}(W) \subseteq \text{Max}_l^{C_2}(W) \).

Given two classes of monotonic QVRs, \( C_1 \) and \( C_2 \), we call their meet, and denote by \( C_1 \wedge C_2 \), to the class of monotonic QVRs that satisfy all the monotonicities in \( C_1 \) and all the monotonicities in \( C_2 \), and call their join, and denote by \( C_1 \vee C_2 \), to the class of monotonic QVRs that satisfy all the monotonicities that hold both in \( C_1 \) and in \( C_2 \). In fact “\( \wedge \)” (“\( \vee \)””) gives the greatest (least) lower (upper) bound of any two classes in the complete lattice of all monotonic classes partially ordered by inclusion. The minimal element in the lattice is the class where all monotonicities hold, which contains only the degenerated rule where all vote configurations are winning (first in the list of classes given above). Note that this rule satisfies “full-support” and “weak null-support.” The maximal class is that of all QVRs (last in the list given above). Meet and join are related with union and intersection by the following proposition, whose simple proof is omitted.

**Proposition 3** Let \( C_1 \) and \( C_2 \) be two monotonic classes of QVRs, then: (i) \( C_1 \wedge C_2 = C_1 \cap C_2 \) and \( \prec \leq \) is the transitive closure of \( \leq \); and (ii) \( C_1 \vee C_2 \supseteq C_1 \cup C_2 \) and \( \leq \leq \) is the transitive closure of \( \leq \).

The union and the intersection are means of defining new rules from preexisting ones, and we are interested in how properties and these operations interact. Let us examine first the basic conditions and then the monotonicities.

**Proposition 4** Given \( W_1, W_2 \subseteq 4^N \), then:

(i) If both \( W_1 \) and \( W_2 \) satisfy “full-support” (null-support), then \( W_1 \cup W_2 \) and \( W_1 \cap W_2 \) also satisfy it.

(ii) Even if only one of them satisfies “full-support” (null-support) then \( W_1 \cup W_2 \) (\( W_1 \cap W_2 \)) also satisfies it.
(iii) If both satisfy “weak null-support” \( \mathcal{W}_1 \cup \mathcal{W}_2 \) also satisfies it, but \( \mathcal{W}_1 \cap \mathcal{W}_2 \) may fail to satisfy it.

As to the monotonicities we have:

**Proposition 5** Let \( C_1 \) and \( C_2 \) be two classes of QVRs specified by sets of monotonicities with associated preorders \( \preceq_{C_1} \) and \( \preceq_{C_2} \). If \( \mathcal{W}_1 \in C_1 \), and \( \mathcal{W}_2 \in C_2 \), then \( \mathcal{W}_1 \cup \mathcal{W}_2 \in C_1 \land C_2 \), and, if \( \mathcal{W}_1 \cap \mathcal{W}_2 \) satisfies “weak null-support,” \( \mathcal{W}_1 \cap \mathcal{W}_2 \in C_1 \lor C_2 \).

In Sect. 5 we address the representability of QVRs by means of weighted rules. The following result will be useful there:

**Proposition 6** Let \( C_1 \) and \( C_2 \) be two monotonic classes of QVRs and \( C = C_1 \lor C_2 \), then for all \( \mathcal{W} \in C \) there exist \( \mathcal{W}_1 \in C_1 \) and \( \mathcal{W}_2 \in C_2 \) such that \( \mathcal{W} = \mathcal{W}_1 \cap \mathcal{W}_2 \).

We now extend the notion of “proper” rule to the class of QVRs. The point of this condition, satisfied by all binary rules by means of which issues of substance are decided upon, is to prevent two disjoint groups of voters with opposed preferences from both being winning when supporting opposite proposals. The difficulty of extending this condition to QVRs is that for real world rules that can be expressed as QVRs it is often the case that two disjoint groups of voters can win a vote if a sufficient number of voters abstain and/or do not turn out. For instance, if only more “yes” than “no” voters is required in addition to a certain quorum (i.e., a certain maximal number of staying-at-home voters) to pass a decision, this may happen. But no real problem arises, as it is the result of admitting abstention and staying at home as legitimate options for possibly indifferent voters. The problem arises if this may also happen in cases where all voters have strict preferences either for approval or rejection of a proposal. This motivates the following

**Definition 9** A set of voters \( R \subseteq \mathcal{N} \) is “strong winning” for a QVR \( \mathcal{W} \subseteq 4^\mathcal{N} \) if for all \( S \in 4^\mathcal{N} \) such that \( SY = R \) we have \( S \in \mathcal{W} \).

This notion allows for associating the following underlying binary rule to each QVR. Let \( \mathcal{W} \subseteq 4^\mathcal{N} \) be a QVR, the core binary rule associated with \( \mathcal{W} \) is the binary rule (that we denote by \( V_{\mathcal{W}}^* \))

\[
V_{\mathcal{W}}^* := \{ T \in 2^\mathcal{N} : T^Y \text{ is strong winning in } \mathcal{W} \}.
\]

Observe that \( V_{\mathcal{W}}^* \) is actually a binary rule: Full-support of \( \mathcal{W} \) ensures that \( (\mathcal{N}, \emptyset) \in V_{\mathcal{W}}^* \). Now assume that \( (\emptyset, \mathcal{N}) \in V_{\mathcal{W}}^* \). In this case, by the three basic monotonicities, it is immediate to check that for all \( S \in 4^\mathcal{N}, S \subseteq \mathcal{W} \). In other words, \( \mathcal{W} \) is the degenerated rule. Thus, \( V_{\mathcal{W}}^* \) satisfies full support and null support. Finally, if \( T, Q \in 2^\mathcal{N} \), with \( T \in V_{\mathcal{W}}^* \) and \( T^Y \subseteq Q^Y \), then the basic monotonicities of \( \mathcal{W} \) imply that \( Q \in V_{\mathcal{W}}^* \). Thus \( V_{\mathcal{W}}^* \) is a binary voting rule.

Alternatively, the notion of the associated core rule can also be formulated as a QVR:
**Definition 10** The “core QVR” associated with a QVR $\mathcal{W}$ is the QVR

$$\mathcal{W}^* := \{ S \in 4^N : S^Y \text{ is strong winning in } \mathcal{W} \}.$$ 

Observe that in this formulation $\mathcal{W}^* \subseteq \mathcal{W}$. In fact, the core rule $\mathcal{W}^*$ is the maximal QVR in the class

$$Y \uparrow \quad A \equiv H \equiv N$$

contained in $\mathcal{W}$, which motivates the term “core” rule. Now we can formulate a sensible notion of properness for QVRs consistent with the usual notion for binary rules.

**Definition 11** A quaternary voting rule $\mathcal{W}$ is “proper” if the associated core binary rule is proper.

The following straightforward fact is interesting when defining rules by intersection of others.

**Proposition 7** The intersection of two QVRs is proper if at least one of them is proper.

### 4 Some examples and classes of QVRs rules

The preceding section adopted a very general point of view for a general study the lattice of classes of monotonic QVRs. In this section, based on actual majority rules used in Parliaments, we examine systematically real world examples in the light of the model adopted here. For each example, we check the combination of monotonicities that identify the minimal monotonic class containing each of them. Finally, we identify the minimal monotonic class containing all these classes and examples.

1. **(C1)** A simple majority with a quorum

$$\mathcal{W} = \left\{ S \in 4^N : (s^Y > s^N) \& \left( s^Y + s^N + s^A > \frac{n}{2} \right) \right\}$$

is used in Parliaments such as those of Belgium and Italy. The monotonicity diagram of the minimal monotonic class containing it and the associated preorder are:

\[
\begin{array}{ccc}
\downarrow & \downarrow & \downarrow \\
Y & A & S \leq_{C1} T \iff \left\{ \begin{array}{c}
S^N \supseteq T^N \\
S^H \supseteq T^H \\
S^Y \subseteq T^Y.
\end{array} \right.
\end{array}
\]
(C2) A majority of present voters with a quorum

\[ \mathcal{W} = \left\{ S \in 4^N : \left( s_Y > \frac{s_Y + s_N + s^A}{2} \right) \& \left( s^H < \frac{n}{2} \right) \right\} \]

is used in the Spanish and German Parliaments. The monotonicity diagram of the minimal monotonic class containing it and the associated preorder are:

\[ Y \xrightarrow{A \equiv N} H \quad S \leq_{C2} T \Leftrightarrow \left\{ \begin{array}{l} S^H \supseteq T^H \\ S_Y \cup S^H \subseteq T^Y \cup T^H \end{array} \right. \]

(C3) A simple majority (with no quorum):

\[ \mathcal{W} = \{ S \in 4^N : s_Y > s_N \} \]

is used in the Swedish Parliament. The monotonicity diagram and the associated preorder are:

\[ Y \xrightarrow{A \equiv H} \quad S \leq_{C3} T \Leftrightarrow \left\{ S^N \supseteq T^N \\ S_Y \subseteq T^Y \right. \]

(C4) A majority of members present (used in the Finnish Parliament)

\[ \mathcal{W} = \left\{ S \in 4^N : s_Y > \frac{(s^Y + s^N + s^A)}{2} \right\} \]

and a majority of those present with an approval threshold (used in the Greek Parliament)

\[ \mathcal{W} = \left\{ S \in 4^N : \left( s_Y > \frac{s_Y + s_N + s^A}{2} \right) \& \left( s_Y > \frac{n}{4} \right) \right\} \]

have the same monotonicity diagram and associated preorder, which are:

\[ Y \xrightarrow{H \equiv A} \quad S \leq_{C4} T \Leftrightarrow \left\{ \begin{array}{l} S^Y \subseteq T^Y \\ S^Y \cup S^H \subseteq T^Y \cup T^H \end{array} \right. \]
(C5) An absolute 3/5-majority

$$\mathcal{W} = \left\{ S \in 4^N : s^Y \geq \frac{3}{5} n \right\}$$

is used in some Parliaments such as that of Estonia (in order to amend the Constitution) or that of Poland (to overrule the veto of the President). The monotonicity diagram of the minimal monotonic class containing it and the associated preorder are:

$$\begin{align*}
Y & \uparrow \\
A & \equiv H \equiv N
\end{align*}$$

$$S \leq_{C5} T \iff s^Y \subseteq T^Y.$$  

Observe that all preceding examples verify $NA$-monotonicity (for a precise definition just replace “$XZ$”- by “$NA$” in Definition 7). If we add this condition to the three basic ones, we specify the class of “$NA$-monotonic” rules, whose associated diagram and preorder are

$$\begin{align*}
Y & \uparrow \\
A & \leftarrow H \leftarrow N
\end{align*}$$

$$S \leq_{NA-mon} T \iff \begin{cases} 
S^N \supseteq T^N \\
S^H \supseteq T^H \\
S^Y \cup S^H \subseteq T^Y \cup T^H.
\end{cases}$$

In fact, as the reader may easily check, this class is the minimal monotonic class containing all the preceding classes and examples. In fact, we know no actual rule that fits in the QVR model provided here and does no belong to this class, with the sole exception of the voting rule that was used in the US Congress till 1890. ⁶ According to this rule abstaining could be more effective for rejection than voting “no,” a sort of simple majority with a “votes-cast” quorum:

$$\mathcal{W} = \left\{ S \in 4^N : (s^Y > s^N) \& (s^Y + s^N > \frac{n}{2}) \right\}.$$  

The rule was abandoned when Speaker Thomas Reed replaced the “votes-cast” quorum by a traditional quorum⁷ is a monotonic class of QVRs according to general Definition 1.

It is worth remarking that even purely “auxiliary” rules that specify quorum requirements belong to the class of $NA$-monotonic rules. For instance, if $\frac{n}{2} \leq q \leq n$, then the rule that specifies $q$-quorum requirement,

$$\mathcal{W} = \{ S \in 4^N : s^Y + A^S + s^N > q \},$$

⁶ Source: Vermeule (2007).
⁷ Nevertheless, observe that this rule satisfies the three basic monotonicities and therefore is a QVRs according to general Definition 1.

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displays $YA$-monotonicity and $YH$-monotonicity. Thus the monotonicity diagram of the minimal monotonic class containing it and the associated preorder are$^8$:

$$Y \equiv A \equiv N \uparrow H \quad S \subseteq_{\text{quorum}} T \iff T^H \subseteq S^H.$$ 

This class is obviously contained in that of $NA$-monotonic rules.

**Remarks**

(i) Classes $C1$ and $C2$ are not covered by previous models in the literature. In both cases, $NH$-monotonicity does not hold and as a result it may be preferable for a voter against the proposal to stay at home rather than come and vote “no.”$^9$

(ii) At the other extreme, class $C5$ is isomorphic to that of classical binary voting rules. In other words, the general model of QVR includes the binary model as a particular case.

(iii) In classes $C2$ and $C3$ and $C4$ two options collapse by identification into one ($A$ and $N$ in $C2$ and $C4$, $H$ and $A$ in $C3$), thus leaving only three really different options. Felsenthal and Machover (1997) “ternary” voting rules correspond to class $C3$ as they consider “yes,” “no,” and “abstention” as separate options and the monotonicities that they assume are precisely the ones for this class. Observe that the rules in classes $C2$ and $C4$ could also be called “ternary” (as there are actually three options: $Y$, $N$ and $A \equiv H$) but they are not covered by the model considered by Felsenthal and Machover.

(iv) Classes $C3$ and $C4$ can be seen as $(3, 2)$-rules and $C5$ as $(2, 2)$-rules in the terms of Freixas and Zwicker (2003).

(v) As with the general class of QVRs, for each of the monotonic classes considered above the preorder associated with its subclass of anonymous rules can be specified by replacing each SET by its cardinality (e.g., $S^Y$ by $s^Y$, etc.), “$\subseteq$” by “$\leq$” and “$\cup$” by “$+$” in the different definitions. For instance, the binary relation associated with anonymous rules in the monotonic class $C4$ is defined by

$$S \leq_{C4} T \iff \begin{cases} s^Y \leq t^Y \\ s^Y + s^H \geq t^Y + t^H. \end{cases}$$

Then, condition

$$(S \in \mathcal{W} \text{ and } S \leq_{\text{AnC4}} T) \Rightarrow T \in \mathcal{W},$$

along with nonemptiness and “weak null-support” specifies the anonymous subclass of $C4$. All the the other classes can be similarly “anonymized.”

---

$^8$ Note that although such rules conflict with the “null-support” condition (as this implies that some configuration where nobody votes “yes” is winning) they satisfy “weak null-support.”

$^9$ This is what Côrte-Real and Pereira (2004) refer to as the ‘no-show’ paradox.
5 Weighted QVRs and dimension

The simplest and best known binary dichotomous voting rules are weighted \( q \)-majority voting rules. In fact, most binary rules to be found in real world collective decision bodies are either of this type or such that their sets of winning vote configurations are the intersections of the sets of winning configurations of two or more such rules. In this section, based on Freixas and Zwicker (2003), we extend the notion of weighted \( q \)-majority voting rule to the wider domain of QVR and address the question of the representability of QVR as (or by means of) such weighted quaternary \( q \)-majority voting rules.

A binary weighted majority rule is specified by a system of weights \( w = (w_1, \ldots, w_n) \), and a quota \( Q > 0 \), so that the final result is “yes” if the sum of the weights in favor of the proposal is larger than the quota. Denoting this rule by \( B(Q, w) \), we have:

\[
B(Q, w) = \left\{ S \in 2^\mathcal{N} : \sum_{i \in S} w_i > Q \right\}. \tag{6}
\]

As is well known, not all binary voting rules can be represented in this way: some can only be represented by a double weighted majority, triple majority, etc. A \( k \)-multiple binary weighted majority rule is specified by \( k \) systems of weights \( w_r = (w_{r1}, \ldots, w_{rn}) \), where \( w_{ri} \) represents the weight of voter \( i \) in rule \( r \), and \( k \) quotas \( Q_r (r = 1, 2, \ldots, k) \), each quota \( Q_r \) corresponding to \( w_r \) system of weights. The final result is “yes” if for all systems the sum of the weights in favor of the proposal is larger than the corresponding quota. The resulting rule is

\[
\bigcap_{r=1}^{k} B(Q_r, w_r) = \left\{ S \in 2^\mathcal{N} : \sum_{i \in S} w_{ri} > Q_r, \text{ for all } r = 1, 2, \ldots, k \right\}. \tag{7}
\]

Taylor and Zwicker (1992, 1993, 1999) give necessary and sufficient conditions for a binary voting rule to be representable as a weighted majority rule, and introduce the notion of the dimension of a binary voting rule as the minimum number of weighted majority rules necessary to represent the voting rule (i.e., the minimum \( k \) for which the rule can be expressed in the form (7). This notion is well defined as they prove that any binary voting rule can be represented in this way.

We now extend the notion of weighted majority rule to the class of QVR dealt with in this article. We use the following notation:

**Definition 12** For any two vectors in \( \mathbb{R}^m, x = (x_1, \ldots, x_m), y = (y_1, \ldots, y_m) \),

\[
x \leq y \iff x_i \leq y_i, \text{ for all } i = 1, \ldots, m.
\]

\[
x < y \iff x_i < y_i, \text{ for all } i = 1, \ldots, m.
\]
We follow Freixas and Zwicker (2003) general definition of weighted (4, 2)-rules.

**Definition 13** An *n*-voter quaternary dichotomous weighted rule is specified by a system of weights \(w^Y = (w^Y_1, \ldots, w^Y_n), w^A = (w^A_1, \ldots, w^A_n), w^H = (w^H_1, \ldots, w^H_n), w^N = (w^N_1, \ldots, w^N_n)\) such that for all \(i\)
\[
    w^Y_i \geq \max\{w^A_i, w^H_i, w^N_i\},
\]
and \(w^A, w^H, \text{ and } w^N\) are linearly (pre)ordered (i.e., such that \(w^X \leq w^Z\) or \(w^Z \leq w^X\) for any \(X, Z \in \{A, H, N\}\)), and by a quota \(Q > 0\), so that a vote configuration \(S \in 4^N\) is winning if and only if
\[
    \sum_{i \in S^Y} w^Y_i + \sum_{i \in S^A} w^A_i + \sum_{i \in S^H} w^H_i + \sum_{i \in S^N} w^N_i > Q.
\]

Such a rule is denoted by \(Q(Q; w^Y, w^A, w^H, w^N)\).

This definition constrains the weights by (8) so as to make the rule consistent with the basic monotonicities assumed for all QVRs. Moreover, the ranking of the weights corresponds to the monotonicities. For instance, if \(w^Y > w^A > w^H > w^N\) the rule belongs to the monotonic class whose monotonicities are \(N \rightarrow H \rightarrow A \rightarrow Y\); while if \(w^Y > w^A = w^H > w^N\) it belongs to the class where \(N \rightarrow H \equiv A \rightarrow Y\), etc. In other words, all weighted rules belong to the linear classes introduced in Sect. 3.

As we prove later, the class of weighted rules is rich enough to express any QVR as a “dichotomous quaternary k-multiple weighted rule.” That is, as the intersection of \(k\) weighted rules \(\bigcap_{r=1}^{k} Q(r; w^Y_r, w^A_r, w^H_r, w^N_r)\). More precisely, we have the following definition.

**Definition 14** An *n*-voter dichotomous quaternary k-multiple weighted rule is specified by \(k\) systems of weights \(w^Y_r = (w^Y_{r1}, \ldots, w^Y_{rn}), w^A_r = (w^A_{r1}, \ldots, w^A_{rn}), w^H_r = (w^H_{r1}, \ldots, w^H_{rn}), w^N_r = (w^N_{r1}, \ldots, w^N_{rn})\) (where \(r = 1, \ldots, k\)), and \(k\) quotas \(Q_r\), such that for each \(r\)
\[
    Q(Q_r; w^Y_r, w^A_r, w^H_r, w^N_r)
\]
is a quaternary dichotomous weighted rule according to Definition 13, so that a vote configuration \(S \in 4^N\) is winning if and only if

\[
    \sum_{i \in S^Y} w^Y_i + \sum_{j \in S^A} w^A_j + \sum_{k \in S^H} w^H_k + \sum_{l \in S^N} w^N_l \leq Q
\]

for any \(S\) such that \(S^Y = \emptyset\), or a somewhat more complicated condition to ensure “weak null-support.”

---

\(^{10}\) In order to grant that a QVR consistent with Definition 4 results, two conditions should be added: first, to ensure “full-support,” \(\sum_{i \in N} w^Y_i > Q\), and, second, either to ensure “null-support,”
\[
    \sum_{j \in S^A} w^A_j + \sum_{k \in S^H} w^H_k + \sum_{l \in S^N} w^N_l \leq Q
\]
for any \(S\) such that \(S^Y = \emptyset\), or a somewhat more complicated condition to ensure “weak null-support.”
\[ \sum_{i \in SY} w_{Yi} + \sum_{i \in SA} w_{Ai} + \sum_{i \in SH} w_{Hi} + \sum_{i \in SN} w_{Ni} > Q_r, \text{ for all } r = 1, \ldots, k. \]

Now the point is the representability of all QVRs by means of weighted rules. Let us first address the representability of rules belonging to the linear classes.

For the case of the linear classes, given the fact that these classes correspond to the possible orders between weights consistent with the constraints in Definition 13, the result of Freixas and Zwicker (2003) applies straightforwardly. That is, their characterization theorem answers the question of the necessary and sufficient conditions for the representability of a linear QVR as a (single) weighted rule. But there is the question of the well-definedness of the notion of “dimension” for general quaternary rules. The following result extends the proof of Taylor and Zwicker (1999) to the case of linear classes of monotonic rules, showing that the notion of dimension is sound for quaternary rules in the linear classes.

**Theorem 1** Let \( C \) be a linear class of monotonic rules, then for any rule \( \mathcal{W} \in C \) there exists a dichotomous quaternary \( k \)-multiple weighted rule \( \mathcal{Q} \) such that \( \mathcal{W} = \mathcal{Q} \).

**Proof** We provide the proof for class \( C4 \), whose monotonocities are

\[ N \rightarrow H \rightarrow A \rightarrow Y, \]

but the proof is entirely analogous for any linear class. With the notation introduced in Sect. 3, we have \( N <_C4 H <_C4 A <_C4 Y \). Given a vote configuration \( S = (SY, SA, SH, SN) \), for each \( i \in N \) and \( X \in \{Y, A, H, N\} \), we denote \( S^{-1}(i) = X \) if \( i \in SX \). Let \( \mathcal{W} \) be a rule in \( C4 \). The proof consists of constructing a weighted rule for each maximal (w.r.t. \( \leq_{C4} \)) losing vote configuration, and showing that \( \mathcal{W} \) is the intersection of all of them. Let \( L = (LY, LA, LH, LN) \) be a maximal losing vote configuration in \( \mathcal{W} \) relative to \( \leq_{C4} \), that is, \( L \in \text{Max}_C4(\mathcal{W}) \). Consider the weighted rule \( \mathcal{Q}(QL; w_{Yi}^L, w_{Ai}^L, w_{Hi}^L, w_{Ni}^L) \), such that \( Q_L = 1/2 \), and its weight system is given by:

|   | \( i \in SY \) | \( i \in SA \) | \( i \in SH \) | \( i \in SN \) |
|---|---|---|---|---|
| \( w_{Yi}^L \) | 0 | 1 | 1 | 1 |
| \( w_{Ai}^L \) | 0 | 0 | 1 | 1 |
| \( w_{Hi}^L \) | 0 | 0 | 0 | 1 |
| \( w_{Ni}^L \) | 0 | 0 | 0 | 0 |

Obviously, \( w_Y \geq w_A \geq w_H \geq w_N \), and \( \text{Max}_C4(\mathcal{Q}(QL; w_Y^L, w_A^L, w_H^L, w_N^L)) = \{L\} \) (to see this, note that the cells in the diagonal contain the weights to be added corresponding to vote configuration \( L \), all of them 0, and any transfer of votes from this configuration in the sense of the monotonocities in \( C4 \) would yield a winning configuration). Then we have
\[ S = (S^Y, S^A, S^H, S^N) \in \mathcal{W} \iff \forall L \in \text{Max}_C(W) : S \not\leq_C L \]

\[ \iff \forall L \in \text{Max}_C(W) : \exists i \text{ s.t. } L^{-1}(i) <_C S^{-1}(i) \]

\[ \iff \forall L \in \text{Max}_C(W) : \exists i \text{ s.t. } w_{Li}^{S^{-1}(i)} = 1 \]

\[ \iff \forall L \in \text{Max}_C(W) : \sum_{i \in N} w_{Li}^{S^{-1}(i)} \geq 1 > 1/2 \]

\[ \iff \forall L \in \text{Max}_C(W) : S \in \mathcal{Q}(Q_L; w_L^Y, w_L^A, w_L^H, w_L^N). \]

Therefore

\[ \mathcal{W} = \bigcap_{L \in \text{Max}_C(W)} \mathcal{Q}(Q_L; w_L^Y, w_L^A, w_L^H, w_L^N). \]

In other words, \( \mathcal{W} \) is a quaternary \( k \)-multiple weighted rule with \( k = \#\text{Max}_C(W) \).

\[ \square \]

**Remark** Observe that \( \mathcal{Q}(Q_L; w_L^Y, w_L^A, w_L^H, w_L^N) \) satisfies full support and null support, but, as is the case for binary rules (see Taylor and Zwicker 1999), it is not proper in the sense of Def. 11.

Therefore there only remains the question of the representability of QVRs in the “non linear” classes. In view of Theorem 1, it is enough to prove that any rule in any class that is not linear can be represented as the intersection of a finite set of linear rules. In fact, this can be derived as a consequence of Szpilrajn (1930) extension theorem, given that any partial order can be extended to a linear order and is the intersection of all such extensions. Nevertheless, we provide a direct proof showing in addition that only two linear rules are needed.

**Proposition 8** Any quaternary voting rule \( \mathcal{W} \) either is linear or is the intersection of two linear rules.

**Proof** Assume \( \mathcal{W} \) is not linear. In view of Proposition 6, it is enough to see that, whatever the monotonic class \( C \) belongs to, there exist two linear classes, \( C_1 \) and \( C_2 \) such that \( C = C_1 \lor C_2 \). We prove that this is so for the six types of class corresponding to the six essentially different non linear configurations of monotonicities.

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(as discussed in Sect. 3). Assume $\mathcal{W} \in C$. For the six possible cases for $C$ we give two linear classes whose join is $C$. We specify the classes in terms of their diagrams of monotonicities. 11

1. \[
\begin{bmatrix}
Y \\
\downarrow \leftarrow \\
Z \equiv X & V
\end{bmatrix} =
\begin{bmatrix}
Y \\
\uparrow \\
Z \equiv X
\end{bmatrix}
\cap
\begin{bmatrix}
Y \\
\uparrow \\
V
\end{bmatrix}
\cap
\begin{bmatrix}
Y \\
\uparrow \\
Z \equiv X
\end{bmatrix}
\]

2. \[
\begin{bmatrix}
Y \equiv Z \\
\uparrow \leftarrow \\
X & V
\end{bmatrix} =
\begin{bmatrix}
Y \equiv Z \\
\uparrow \\
V
\end{bmatrix}
\cap
\begin{bmatrix}
Y \equiv Z \\
\uparrow \\
X
\end{bmatrix}
\]

3. \[
\begin{bmatrix}
Y \\
\uparrow \\
Z \\
\downarrow \leftarrow \\
X & V
\end{bmatrix} =
\begin{bmatrix}
Y \\
\uparrow \\
X
\end{bmatrix}
\cap
\begin{bmatrix}
Y \\
\uparrow \\
V
\end{bmatrix}
\cap
\begin{bmatrix}
Y \\
\uparrow \\
X
\end{bmatrix}
\]

4. \[
\begin{bmatrix}
Y \\
\downarrow \leftarrow \\
Z & X
\end{bmatrix} =
\begin{bmatrix}
Y \\
\uparrow \\
Z
\end{bmatrix}
\cap
\begin{bmatrix}
Y \\
\uparrow \\
X
\end{bmatrix}
\cap
\begin{bmatrix}
Y \\
\uparrow \\
V
\end{bmatrix}
\]

5. \[
\begin{bmatrix}
Y \\
\uparrow \\
Z \\
\downarrow \leftarrow \\
V & X
\end{bmatrix} =
\begin{bmatrix}
Y \\
\uparrow \\
X
\end{bmatrix}
\cap
\begin{bmatrix}
Y \\
\uparrow \\
V
\end{bmatrix}
\cap
\begin{bmatrix}
Y \\
\uparrow \\
X
\end{bmatrix}
\]

6. \[
\begin{bmatrix}
Y \\
\uparrow \leftarrow \\
Z & V \\
\downarrow \leftarrow \\
X
\end{bmatrix} =
\begin{bmatrix}
Y \\
\uparrow \\
Z
\end{bmatrix}
\cap
\begin{bmatrix}
Y \\
\uparrow \\
V
\end{bmatrix}
\cap
\begin{bmatrix}
Y \\
\uparrow \\
X
\end{bmatrix}
\]

It is easy to check that in all cases $\leq_C \leq C_1 \cap \leq C_2$. \hfill \Box

11 In other terms, we show explicitly how for each class $C$, the associated partial order $\leq_C$ has Dushnik–Miller's (1941) dimension 2.
The decompositions given in the proof are not unique. For instance, case 5 can alternatively be expressed in the following way:

\[
\begin{bmatrix}
Y \\
Z \\
X
\end{bmatrix}
\begin{bmatrix}
\uparrow \\
\equiv \\
\equiv
\end{bmatrix}
\begin{bmatrix}
V \\
Z \\
X
\end{bmatrix}
= \begin{bmatrix}
Y \\
Z \\
X
\end{bmatrix}
\begin{bmatrix}
\uparrow \\
\equiv \\
\equiv
\end{bmatrix}
\begin{bmatrix}
V
\end{bmatrix}
\]

As a consequence we have the following

**Corollary 1** All quaternary voting rules can be represented as the intersection of a finite number of weighted QVRs. In particular, non linear QVRs can only be represented as the intersection of at least two weighted QVRs.

**Proof** By Theorem 1, all linear rules are representable in this way. By Proposition 8, non linear QVRs can be represented as the intersection of two linear ones, but never as a single linear rule. This is because a weighted QVR is always linear.

Therefore the notion of dimension of a QVR as the minimum number of weighted QVRs by means of which it can be expressed\(^{12}\) is well defined for any QVR, and non linear QVRs have at least dimension two.

### 6 Concluding remarks

We have provided a model of QVR that covers dichotomous rules admitting abstention and “staying home” as legitimate and possibly differentiated options, and covers all real world dichotomous voting rules based on these four options and different types of “quorum” that we know of. The model extends consistently simpler models such as binary and ternary voting rules. In particular, notions such as that of “proper” rule and “minimal winning” and “maximal losing” vote configurations have been naturally extended. We have also extended the notion of weighted rule to this enlarged class of rules and the notion of dimension. We have also proved the representability of all QVR as the intersection of such weighted rules.

As main lines of further research, the following seem worth investigating. For binary dichotomous voting rules actions trivially follow preferences, at least when no indifferences occur. Assuming that voting is not costly, a rational voter in favor of the proposal should vote “yes,” and a rational voter against should vote “no.” Under the same assumptions, for QVR a voter against the proposal may be better off by staying at home than voting “no” and game-theoretic considerations are pertinent. The framework provided suggests a study of different classes of rules and rules to be found in parliaments from this point of view. The framework provided also suggests a systematic study of different anonymous majority rules in parliaments and other decision-making bodies, and perhaps founding reasonable proposals.

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\(^{12}\) Equivalently the minimal \(k\) for which it can be expressed as a \(k\)-multiple weighted QVR.
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Appendix

Proof of Proposition 2

(i) $C_1 \subseteq C_2$ means that any monotonicity in $C_2$ holds also in $C_1$ (i.e., $\leq_{C_2} \leq_{C_1}$), and more monotonicities in $C_1$ means a richer preorder relationship, thus the conclusion follows straightforwardly.

(ii) Assume $\mathcal{W} \in C_1 \subseteq C_2$. Let $S \in \min w_{C_1}(\mathcal{W})$, if $T \prec_{C_2} S$, then, in view of Proposition 2-(i), $T \prec_{C_1} S$, and consequently $T \not\in \mathcal{W}$. Therefore $S \in \min w_{C_2}(\mathcal{W})$. The other inclusion is proved in the same way.

Proof of Proposition 4

Parts (i) and (ii) are straightforward.

(iii) Assume that $\mathcal{W}_1$ and $\mathcal{W}_2$ satisfy “weak null-support.” If $S \in \mathcal{W}_1 \cup \mathcal{W}_2$, then either $S \in \mathcal{W}_1$ or $S \in \mathcal{W}_2$. As both rules satisfy “weak null-support,” we have that either

$$(S \neq \emptyset) \text{ or } (\exists X \equiv_{\mathcal{W}_1} Y \text{ s.t. } S^X \neq \emptyset)$$

or

$$(S \neq \emptyset) \text{ or } (\exists Z \equiv_{\mathcal{W}_2} Y \text{ s.t. } S^Z \neq \emptyset).$$

Therefore, either $S \neq \emptyset$ or

$$(\exists X \equiv_{\mathcal{W}_1} Y \text{ s.t. } S^X \neq \emptyset) \text{ or } (\exists Z \equiv_{\mathcal{W}_2} Y \text{ s.t. } S^Z \neq \emptyset).$$

In the latter case assume, for instance, that $\exists X \equiv_{\mathcal{W}_1} Y \text{ s.t. } S^X \neq \emptyset$, then $X \equiv_{\mathcal{W}_1 \cup \mathcal{W}_2} Y$ and $S^X \neq \emptyset$, so that $\mathcal{W}_1 \cup \mathcal{W}_2$ satisfies “weak null-support”.

The following counter example shows that $\mathcal{W}_1 \cap \mathcal{W}_2$ may fail to satisfy “weak null-support.” Let $\mathcal{W}_1$ and $\mathcal{W}_2$ be the 30-voters rules specified by

$$\mathcal{W}_1 = \left\{ S \in 4^N : (s^Y + s^A + 0.5s^H > s^N) \& (s^Y + s^A > 9) \right\}$$

and

$$\mathcal{W}_2 = \left\{ S \in 4^N : (s^Y + s^H + 0.5s^A > s^N) \& (s^Y + s^H > 9) \right\}.$$ 

Both rules satisfy “weak null-support,” in $\mathcal{W}_1$ we have $A \equiv_{\mathcal{W}_1} Y$, and in $\mathcal{W}_2$ it holds that $H \equiv_{\mathcal{W}_2} Y$, while in $\mathcal{W}_1 \cap \mathcal{W}_2$ we have $\not\exists Z \equiv_{\mathcal{W}_1 \cap \mathcal{W}_2} Y$. Then for any configuration $S = (s^Y, s^A, s^H, s^N)$ s.t. $s^Y = 0$, and $s^A = s^H = s^N = 10$, we have

$$S \not\in \mathcal{W}_1 \cap \mathcal{W}_2,$$
Quaternary dichotomous voting rules

$S^Y = \emptyset$, and $\exists Z \equiv \mathcal{W}_1 \cap \mathcal{W}_2 Y$. Thus, $\mathcal{W}_1 \cap \mathcal{W}_2$ does not satisfy “weak null-support.”

**Proof of Proposition 5** Just observe that $\mathcal{W}_1 \cap \mathcal{W}_2$ satisfies those monotonicities that are satisfied by both $\mathcal{W}_1$ and $\mathcal{W}_2$, while $\mathcal{W}_1 \cup \mathcal{W}_2$ satisfies all those satisfied by $\mathcal{W}_1$ and all those satisfied by $\mathcal{W}_2$. Then by Proposition 3 the conclusion follows, but note that in the light of Proposition 4 “weak null support” is not guaranteed for $\mathcal{W}_1 \cap \mathcal{W}_2$, hence the if clause in the statement about $\mathcal{W}_1 \cap \mathcal{W}_2$.

**Proof of Proposition 6** Let $\preceq_C$ denote the preorder associated with class $C$ and $\mathcal{W} \in C$, so that we have:

$$\mathcal{W} = \{T \in 4^N : S \preceq_C T \text{ for some } S \in \text{Min}_w C(\mathcal{W})\}.$$ 

Let $\preceq_{C_1}$ and $\preceq_{C_2}$ denote the preorders associated with $C_1$ and $C_2$. Let $\mathcal{W}_1$ and $\mathcal{W}_2$ be the rules defined by:

$$\mathcal{W}_1 = \{T \in 4^N : S \preceq_{C_1} T \text{ for some } S \in \text{Min}_w C(\mathcal{W})\}$$

and

$$\mathcal{W}_2 = \{T \in 4^N : S \preceq_{C_2} T \text{ for some } S \in \text{Min}_w C(\mathcal{W})\}.$$ 

Then, as can easily be checked, $\mathcal{W}_1$ and $\mathcal{W}_2$ satisfy “full-support” and “weak null-support.” Thus, obviously, $\mathcal{W}_1 \in C_1$, $\mathcal{W}_2 \in C_2$, and we have

$$\mathcal{W}_1 \cap \mathcal{W}_2 = \{T \in 4^N : S \preceq_{C_1} T, S \preceq_{C_2} T \text{ for some } S \in \text{Min}_w C(\mathcal{W})\} = \{T \in 4^N : S \preceq^* T \text{ for some } S \in \text{Min}_w C(\mathcal{W})\},$$

where $\preceq^* = \preceq_{C_1} \cap \preceq_{C_2}$, and as $C = C_1 \lor C_2$, by Proposition 14 we have, $\preceq_{C_1} \cap \preceq_{C_2} = \preceq_C$. Then we conclude

$$\mathcal{W}_1 \cap \mathcal{W}_2 = \{T \in 4^N : S \preceq C T \text{ and } S \in \text{Min}_w C(\mathcal{W})\} = \mathcal{W}.$$ 

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