Length problems for Bazilevič functions

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Abstract: Let $C(r)$ denote the curve which is image of the circle $|z| = r < 1$ under the mapping $f$. Let $L(r)$ be the length of $C(r)$ and $A(r)$ the area enclosed by the curve $C(r)$. Furthermore $M(r) = \max_{|z|=r} |f(z)|$. We present some relations between these notions for Bazilevič functions.

Keywords: Bazilevič function, close-to-convex functions, convex functions, starlike function, convex function

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1 Preliminaries

Let $\mathcal{H}$ denote the class of functions $f$ which are analytic in the unit disk $\mathbb{D} = \{ z \in \mathbb{C} : |z| < 1 \}$, and $\mathcal{A}$ be the subclass of $\mathcal{H}$ consisting of functions normalized by $f(0) = 0 = f'(0) - 1$. Let $S \subset \mathcal{A}$ be the class of functions univalent (i.e. one-to-one) in $\mathbb{D}$. Denote by $S^*$ the subclass of $S$ of starlike functions, i.e. the class of functions $f \in \mathcal{A}$ such that $f(\mathbb{D})$ is starlike with respect to the origin. It is well-known, since the work of [1], that $f \in S^*$ if, and only if, $f \in \mathcal{A}$ and

$$\Re \left\{ \frac{zf''(z)}{f'(z)} \right\} > 0, \ z \in \mathbb{D}.$$ \hfill (1.1)

Recall that a set $E \subset \mathbb{C}$ is said to be starlike with respect to to the origin if, and only if, the linear segment joining 0 to every other point $w \in E$ lies entirely in $E$. By $\mathcal{P}$ we denote the class of Carathéodory functions $p$ which are analytic in $\mathbb{D}$, satisfying the condition $\Re \left\{ p(z) \right\} > 0$ for $z \in \mathbb{D}$, with $p(0) = 1$.

Suppose now that $f \in \mathcal{A}$, then $f$ is close-to-convex if, and only if, there exists $a \in (-\pi/2, \pi/2)$, and a function $g \in S^*$ such that

$$\Re \left\{ e^{ia} \frac{zf''(z)}{g(z)} \right\} > 0, \ z \in \mathbb{D}.$$ \hfill (1.1)

This class of close-to-convex functions was introduced in [2]. Functions defined by (1.1) with $a = 0$ were considered earlier by Ozaki [3], see also Umezawa [4, 5]. Moreover, Lewandowski [6, 7] defined the class of functions $f \in \mathcal{A}$ for which the complement of $f(\mathbb{D})$ with respect to the complex plane is a linearly accessible domain in the large sense. The Lewandowski class is identical with the class of close-to-convex functions. Here, we denote this class by $\mathcal{K}$, and note that $S^* \subset \mathcal{K} \subset S$. The class of close-to-convex functions forms an important subclass of $S$. Length problems for close-to-convex functions were recently considered in [8]. A proper subset of $\mathcal{K}$ is the class of bounded boundary rotation of $f$ such that $f'(z) \neq 0$ in the unit disc and

$$4\pi \leq \lim_{r \to 1} \int_{0}^{2\pi} \left| \Re \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} \right| \, d\theta, \ z = re^{i\theta}.$$
Another even larger subset of $S$ is formed by the Bazilevič functions. Bazilevič [9] introduced a class of functions $f \in A$ which are defined by the following

$$f(z) = \left\{ \frac{\beta}{1 + \alpha^2} \int_0^z \left( h(z) - i \alpha \right) \chi^{(-\alpha \beta/(1 + \alpha^2)) - 1} g^{\beta/(1 + \alpha^2)}(\zeta) d\zeta \right\}^{(1+i)\alpha/\beta},$$

where $h \in P$ and $g \in S^*$, $\alpha$ is any real number and $\beta > 0$. Bazilevič showed that all such functions are univalent in $D$. Putting $\alpha = 0$ in (1) and differentiating it, we have

$$zf'(z) = (f(z))^{1-\beta} (g(z))^\beta h(z)$$

and

$$\Re \{ h(z) \} = \Re \left\{ \frac{zf'(z)}{f^{1-\beta}(z)g^\beta(z)} \right\} > 0, \quad z \in D. \quad (1.2)$$

Thomas [10] called a function satisfying condition (1.2) a Bazilevič function of type $\beta$. For further works on Bazilevič functions we refer to [11]-[15]. It is easy to see that Bazilevič functions of type $\beta = 1$ are close-to-convex functions, univalent in $D$. Furthermore, the set of starlike functions is contained in the set of Bazilevič functions of type $\beta$.

Let $C(r)$ denote the curve which is image of the circle $|z| = r < 1$ under the mapping $f$. Let $L(r)$ be the length of $C(r)$ and $A(r)$ the area enclosed by the curve $C(r)$. Furthermore $M(r) = \max_{|z|=r} |f(z)|$. In [16], Thomas has shown the following:

**Theorem 1.1.** [16, Th.1] If $g \in S^*$, then

$$L(r) \leq 2 \sqrt{\pi A(r)} \left( 1 + \log \frac{1 + r}{1 - r} \right) \quad \text{as} \quad r \to 1.$$

Note that in [17], Thomas considered $L(r)$ for the class of bounded close-to-convex functions and asked the following question.

Does there exist a starlike function for which

$$\liminf_{r \to 1} \frac{L(r)}{M(r) \log \frac{1}{1-r}} > 0$$

or

$$\liminf_{r \to 1} \frac{L(r)}{\sqrt{A(r)} \log \frac{1}{1-r}} > 0? \quad (1.3)$$

Applying the result of [18], we give a negative partial result of the above open problem (1.3). Some related problems were considered in [19, 20].

## 2 On Bazilevič functions of bounded rotation

The following lemma is due to Pommerenke [21].

**Lemma 2.1.** [21] Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be analytic and univalent in $D$. Then we have

$$M(r) = \frac{A}{\sqrt{\pi}} \left( A(r) \log \frac{3}{1-r} \right)^{1/2} \quad \text{as} \quad r \to 1.$$

**Lemma 2.2.** Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be analytic in $D$. Then we have

$$M(r) \leq C \left( S(\sqrt{\tau}) \log \frac{1}{1-r} \right)^{1/2} \quad \text{as} \quad r \to 1,$$

(2.1)
where $\Omega$ means the Landau's symbol and

$$S(r) = \frac{1}{2\pi} \int_0^r \int_0^{2\pi} |f'(pe^{i\theta})|^2 d\theta dp.$$  

Proof. Assume that $0 < r_1 < r$, $\zeta = \sqrt[p]{pe^{i\theta}}$, $0 \leq |t| \leq r$, $0 < p < r$ and throughout $C$ will denote an absolute constant not necessarily the same each time. We have

$$|f(z)| = \left| \int_0^r f'(t) dt \right|.$$  

Now, by using the substitution

$$t = pe^{i\theta}, \ dt = e^{i\theta} dp, \ \zeta = \sqrt[p]{pe^{i\theta}},$$
this becomes

$$|f(z)| = \left| \int_0^r f'(pe^{i\theta}) pe^{i\theta} dp \right|$$

$$\leq \int_0^r |f'(pe^{i\theta})| pe^{i\theta} dp$$

$$\leq \frac{1}{2\pi} \int_0^r \int_\zeta^{e^{i\theta}} \frac{|f'( \zeta )|}{|\zeta - pe^{i\theta}|} d\zeta dp$$

$$\leq \frac{1}{2\pi} \int_0^r \int_0^{2\pi} \sqrt[p]{|f'( \zeta )|} d\phi dp + \frac{1}{2\pi} \int_0^r \int_0^{2\pi} \sqrt[p]{|f'( \zeta )|} d\phi dp$$

$$\leq C + \frac{1}{2\pi \sqrt{r_1}} \int_0^r \int_0^{2\pi} \frac{|f'( \zeta )|}{|\zeta - pe^{i\theta}|} d\phi dp.$$  

Further, because

$$\left( \int_D \int |f(x,y)g(x,y)| dxdy \right)^2 \leq \left( \int_D \int |f(x,y)|^2 dxdy \right) \left( \int_D \int |g(x,y)|^2 dxdy \right),$$
we have

$$C + \frac{1}{2\pi \sqrt{r_1}} \int_0^r \int_0^{2\pi} \frac{|f'( \zeta )|}{|\zeta - pe^{i\theta}|} d\phi dp$$

$$\leq C + \frac{1}{\sqrt{r_1}} \left( \frac{1}{2\pi} \int_0^r \int_0^{2\pi} \sqrt[p]{|f'( \sqrt[p]{pe^{i\theta}})|^2} d\phi dp \right)^{1/2} \left( \frac{1}{2\pi} \int_0^r \int_0^{2\pi} \frac{\sqrt[p]{p}}{|\sqrt[p]{pe^{i\theta}} - pe^{i\theta}|^2} d\phi dp \right)^{1/2}$$

$$= C + \frac{1}{\sqrt{r_1}} \left( \frac{1}{2\pi} \int_0^r \int_0^{2\pi} \sqrt[p]{|f'( \sqrt[p]{pe^{i\theta}})|^2} d\phi dp \right)^{1/2} \left( \frac{1}{2\pi} \int_0^r \int_0^{2\pi} \frac{\sqrt[p]{p}}{p - \rho^2} d\phi dp \right)^{1/2}$$

$$\leq C + \frac{1}{\sqrt{r_1}} \left( \frac{1}{2\pi} \int_0^r \int_0^{2\pi} \sqrt[p]{|f'( \sqrt[p]{pe^{i\theta}})|^2} d\phi dp \right)^{1/2} \left( \frac{1}{2\pi \sqrt{r_1}} \int_0^r \frac{1}{1 - \rho} d\rho \right)^{1/2}$$
\[ L(r) \leq C + \frac{1}{\sqrt{r_1}} \sqrt{S(\sqrt{r}) \left( \frac{1}{2\pi\sqrt{r_1}} \int_0^r \frac{1}{1-\rho} \, d\rho \right)^{1/2}} = O \left( \sqrt{S(\sqrt{r}) \sqrt{\log \frac{1}{1-r}}} \right) \] as \( r \to 1, \)

where \( 0 < r_1 < r < 1 \). Because \( M(r) = \max_{|z|=r} |f(z)| \), we finally obtain (2.1).

**Remark 1.** If \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \) is analytic and univalent in \( \mathbb{D} \), then it is trivial that \( S(r) = A(r) \) for \( 0 < r < 1 \)

so in this case (2.1) becomes

\[ M(r) \leq O \left( A(\sqrt{r}) \log \frac{1}{1-r} \right)^{1/2} \text{ as } r \to 1. \]

**Theorem 2.3.** Let \( f \) be a Bazilevič function of type \( \beta \) and let \( f \) be a function of bounded rotation on \( 0 < |z| = r < 1 \), and suppose that

\[ M(r) = O \left\{ (1-r)^{-\alpha} p(r) \right\} \text{ as } r \to 1 \] (2.2)

for all \( \alpha \), where \( 0 < \alpha \leq 2 \), while \( p(r) \) is monotone increasing function of \( r \) in a wider sense, and \( O \) in (2.2) cannot be replaced by \( o \). We then have

\[ L(r) = O \left( A(r) \log \frac{1}{1-r} \right)^{1/2} \text{ as } r \to 1. \]

**Proof.** From (2.2) and applying the same method as in the proof of Theorem 1 in [22], we have

\[ L(r) = \int_0^{2\pi} |zf'(z)| \, d\theta = \int_0^{2\pi} |(f(z))^{1-\beta} (g(z))^\beta h(z)| \, d\theta \]

\[ \leq \int_0^r \int_0^{2\pi} |(1-\beta)f'(z)(g(z))^\beta h(z)| \, d\theta \, d\rho + \int_0^r \int_0^{2\pi} |f^1(\beta)(g(z))^\beta h'(z)| \, d\theta \, d\rho \]

\[ = I_1 + I_2 + I_3, \text{ say.} \]

Then, from [18, p.338], and from Lemma 2.1, we have the following

\[ I_1 \leq 2\pi(1-\beta)M(r) = O \left( A(r) \log \frac{1}{1-r} \right)^{1/2} \text{ as } r \to 1. \]

Next, we have (2.3) below from [22, p.277] and Lemma 2.1:

\[ I_2 = C \int_0^r \frac{M(\rho)}{1-\rho} \, d\rho + C \]

\[ \leq C \int_0^r \frac{p(\rho)}{(1-\rho)^{1+\alpha}} \, d\rho + C \]

\[ \leq \frac{Cp(r)}{\alpha} (1-r)^{-\alpha} + C \]
= \mathcal{O}(M(r)) = \mathcal{O}\left(A(r) \log \frac{1}{1-r}\right)^{1/2} \text{ as } r \to 1,

where \( \delta \) is fixed \( 0 < \delta < \rho \leq r < 1 \). Applying the result of [22, p.277] and the same method as in the calculation (2.3), we have

\[
I_3 = 2\pi \left\{ |1-\beta| C + |\beta| \right\} \int_0^r \frac{M(\rho)}{1-\rho} d\rho
\]

= \mathcal{O}\left(A(r) \log \frac{1}{1-r}\right)^{1/2} \text{ as } r \to 1.

This completes the proof of Theorem 2.3. \( \square \)

From Theorem 2.3, we easily have the following corollary.

**Corollary 2.4.** Let \( f \) be a Bazilevič function of type \( \beta \) and let \( f \) be a function of bounded rotation on \( 0 < |z| = r < 1 \) and suppose that

\[
M(r) = \mathcal{O}\left\{ (1-r)^{-\alpha} \left( \log \frac{1}{1-r}\right)^{1/2} \right\} \text{ as } r \to 1.
\]

Then there is no Bazilevič function of type \( \beta \) satisfying the condition (1.3).

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