FRW Cosmology From Five Dimensional Vacuum Brans–Dicke Theory

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Abstract

We follow the approach of induced–matter theory for a five–dimensional (5D) vacuum Brans–Dicke theory and introduce induced–matter and induced potential in four dimensional (4D) hypersurfaces, and then employ a generalized FRW type solution. We confine ourselves to the scalar field and scale factors be functions of the cosmic time. This makes the induced potential, by its definition, vanishes, but the model is capable to expose variety of states for the universe. In general situations, in which the scale factor of the fifth dimension and scalar field are not constants, the 5D equations, for any kind of geometry, admit a power–law relation between the scalar field and scale factor of the fifth dimension. Hence, the procedure exhibits that 5D vacuum FRW–like equations are equivalent, in general, to the corresponding 4D vacuum ones with the same spatial scale factor but a new scalar field and a new coupling constant, \( \tilde{\omega} \). We show that the 5D vacuum FRW–like equations, or its equivalent 4D vacuum ones, admit accelerated solutions. For a constant scalar field, the equations reduce to the usual FRW equations with a typical radiation dominated universe. For this situation, we obtain dynamics of scale factors of the ordinary and extra dimensions for any kind of geometry without any priori assumption among them. For non–constant scalar fields and spatially flat geometries, solutions are found to be in the form of power–law and exponential ones. We also employ the weak energy condition for the induced–matter, that gives two constraints with negative or positive pressures. All types of solutions fulfill the weak energy condition in different ranges. The power–law solutions with either negative or positive pressures admit both decelerating and accelerating ones. We illustrate that the accelerating power–law solutions, which satisfy the weak energy condition and have non–ghost scalar fields, are compatible with the recent observations in ranges \(-4/3 < \omega < -1.3151\) for the coupling constant and \(1.5208 < n < 1.9583\) for dependence of the fifth dimension scale factor with the usual scale factor. These ranges also fulfill the condition \(\tilde{\omega} > -3/2\) which prevents ghost scalar fields in the equivalent 4D vacuum Brans–Dicke equations. The results are presented in a few tables and figures.

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1 Introduction

Attempts to geometrical unification of gravity with other interactions, using higher dimensions other than our conventional 4D space–time, began shortly after invention of the special relativity (SR). Nordstrom was the first who built a unified theory on the base of extra dimensions [1]. Tight
connection between SR and electrodynamics, namely the Lorentz transformation, led Kaluza [2] and Klein [3] to establish 5D versions of general relativity (GR) in which electrodynamics rises from the extra fifth dimension. Since then, considerable amount of works have been focused on this idea either using different mechanism for compactification of extra dimension or generalizing it to non–compact scenarios (see e.g. Ref. [4]) such as Brane–World theories [5], space–time–matter or induced–matter (IM) theories [6] and references therein. The latter theories are based on the Campbell–Magaard theorem which asserts that any analytical $N$–dimensional Riemannian manifold can locally be embedded in an $(N + 1)$–dimensional Ricci–flat Riemannian manifold [7]. This theorem is of great importance for establishing 4D field equations with matter sources locally to be embedded in 5D field equations without priori introducing matter sources. Indeed, the matter sources of 4D space–times can be viewed as a manifestation of extra dimensions. This is actually the core of IM theory which employs GR as the underlying theory.

On the other hand, Jordan [8] attempted to embed a curved 4D space–time in a flat 5D space–time and introduced a new kind of gravitational theory, known as the scalar–tensor theory. Following his idea, Brans and Dicke [9] invented an attractive version of the scalar–tensor theory, an alternative to GR, in which the weak equivalence principle is saved and a non–minimally scalar field couples to curvature. The advantage of this theory is that it is more Machian than GR, though mismatching with the solar system observations is claimed as its weakness [10]. However, the solar system constraint is a generic difficulty in the context of the scalar–tensor theories [11], and it does not necessarily denote that the evolution of the universe, at all scales, should be close to GR, in which there are some debates on its tests on cosmic scales [12].

Although it is sometimes desirable to have a higher dimensional energy–momentum tensor or a scalar field, for example in compactification of extra curved dimensions [13], but the most preference of higher dimensional theories is to obtain macroscopic 4D matter from pure geometry. In this approach, some features of a 5D vacuum Brans–Dicke (BD) theory based on the idea of IM theory have recently been demonstrated [14], in where the role of GR as fundamental underlying theory has been replaced by the BD theory of gravitation. Actually, it has been shown that 5D vacuum BD equations, when reduced to four dimensions, lead to a modified version of the 4D Brans–Dicke theory which includes an induced potential. Whereas in the literature, in order to obtain accelerating universes, inclusion of such potentials has been considered in priori by hand. A few applications and a $D$–dimensional version of this approach have been performed [15, 16]. Though, in Refs. [15], it has also been claimed that their procedure provides explicit definitions for the effective matter and induced potential. Besides, some misleading statements and equations have been asserted in Ref. [14], and hence we have re–derived the procedure in Section 2. Actually, the reduction procedure of a 5D analogue of the BD theory, with matter content, on every hypersurface orthogonal to an extra cyclic dimension (recovering a modified BD theory described by a 4–metric coupled to two scalar fields) has previously been performed in the literature [17]. However, the key point of IM theories are based on not introducing matter sources in 5D space–times.

In addition, recent measurements of anisotropies in the microwave background suggest that our ordinary 4D universe should be spatially flat [18], and the observations of Type Ia–supernovas indicate that the universe is in an accelerating expansion phase [19]. Hence, the universe should mainly be filled with a dark energy or a quintessence which makes it to expand with acceleration [20]. Then after an intensive amount of work has been performed in the literature to explain the acceleration of the universe.

In this work, we explore the Friedmann–Robertson–Walker (FRW) type cosmology of a 5D vacuum BD theory and obtain solutions and related conditions. This model has extra terms, such as a scalar field and scale factor of fifth dimension, which make it capable to present accelerated universes beside decelerated ones. In the next section, we give a brief review of the induced modified BD theory from a 5D vacuum space–time to rederive the induced energy–momentum tensor, as has been introduced in Ref. [14], for our purpose to employ the energy density and pressure. In Section 3, we consider a generalized FRW metric in the 5D space–time and specify FRW cosmological equations.
and employ the weak energy condition (WEC) to obtain the energy density and pressure conditions. Then, we probe two special cases of a constant scale factor of the fifth dimension and a constant scalar field. In Section 4, we proceed to exhibit that 5D vacuum BD equations, employing the generalized FRW metric, are equivalent, in general, to the corresponding vacuum 4D ones. This equivalency can be viewed as the main point within this work which distinguishes it from Refs. [14, 15]. In Section 5, we find exact solutions for flat geometries and proceed to get solutions fulfilling the WEC while being compatible with the recent observational measurements. We also provide a few tables and figures for a better view of acceptable range of parameters. Finally, conclusions are presented in the last section.

2 Modified Brans–Dicke Theory From Five–Dimensional Vacuum

Following the idea of IM theories [6], one can replace GR by the BD theory of gravitation as the underlying theory [14, 15, 17]. For this purpose, the action of 5D Brans–Dicke theory can analogously be written in the Jordan frame as

\[ S [g_{AB}, \phi] = \int \sqrt{|(5)g|} \left( \phi^{(5)} R - \frac{\omega}{\phi} g^{AB} \phi_{;A} \phi_{;B} + 16\pi L_m \right) d^5x, \]  

(1)

where \( c = 1 \), the capital Latin indices run from zero to four, \( \phi \) is a positive scalar field that describes gravitational coupling in five dimensions, \( (5)R \) is 5D Ricci scalar, \( (5)g \) is the determinant of 5D metric \( g_{AB} \), \( L_m \) represents the matter Lagrangian and \( \omega \) is a dimensionless coupling constant. The field equations obtained from action (1) are

\[ (5)G_{AB} = \frac{8\pi}{\phi} (5)T_{AB} + \frac{\omega}{\phi^2} \left( \phi_{;A} \phi_{;B} - \frac{1}{2} g_{AB} \phi_{;C} \phi_{;C} \right) + \frac{1}{\phi} \left( \phi_{;AB} - g_{AB} (5)\Box \phi \right), \]  

(2)

and

\[ (5)\Box \phi = \frac{8\pi}{4 + 3\omega} (5)T, \]  

(3)

where \( (5)\Box \equiv \partial^A \partial_A \), \( (5)G_{AB} \) is 5D Einstein tensor, \( (5)T_{AB} \) is 5D energy–momentum tensor, \( (5)T \equiv (5)T^c_c \). Also, in order to have a non–ghost scalar field in the conformally related Einstein frame, i.e. a field with a positive kinetic energy term in that frame, the BD coupling constant must be \( \omega > -4/3 \) [21, 22].

As explained in the introduction, we propose to consider a 5D vacuum state, i.e. \( (5)T_{AB} = 0 = (5)T \), where equations (2) and (3) read

\[ (5)G_{AB} = \frac{\omega}{\phi^2} \left( \phi_{;A} \phi_{;B} - \frac{1}{2} g_{AB} \phi_{;C} \phi_{;C} \right) + \frac{1}{\phi} \left( \phi_{;AB} - g_{AB} (5)\Box \phi \right), \]  

(4)

and\(^1\)

\[ (5)\Box \phi = 0. \]  

(5)

For cosmological purposes one usually restricts attention to 5D metrics of the form, in local coordinates \( x^A = (x^\mu, y) \),

\[ dS^2 = g_{AB} (x^C) dx^A dx^B = (5)g_{\mu\nu} (x^C) dx^\mu dx^\nu + g_{44} (x^C) dy^2 \equiv (5)g_{\mu\nu} (x^C) dx^\mu dx^\nu + \epsilon^2 (x^C) dy^2, \]  

(6)

where \( y \) represents the fifth coordinate, the Greek indices run from zero to three and \( \epsilon^2 = 1 \). It should be noted that this ansatz is restrictive, but one limits oneself to it for reasons of simplicity. Assuming the 5D space–time is foliated by a family of hypersurfaces, \( \Sigma \), defined by fixed values of the fifth coordinate, then the metric intrinsic to every generic hypersurface, e.g. \( \Sigma_\phi (y = y_0) \), can be obtained

\(^1\)We have purposely kept the null term in equation (4) for later on convenient.
when restricting the line element (6) to displacements confined to it. Thus, the induced metric on the hypersurface \( \Sigma \) can have the form

\[
\begin{align*}
\text{ds}^2 &= (5) g_{\mu\nu}(x^\alpha, y_o)dx^\mu dx^\nu \\
&= g_{\mu\nu}dx^\mu dx^\nu,
\end{align*}
\]

in such a way that the usual 4D space–time metric, \( g_{\mu\nu} \), can be recovered.

Hence, equation (4) on the hypersurface \( \Sigma \) can be written as

\[
G_{\alpha\beta} = \frac{8\pi}{\phi} T^{(BD)}_{\alpha\beta} + \frac{\omega}{\phi^2} \left( \phi \phi,_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} \phi,_{\sigma} \phi,^\sigma \right) + \frac{1}{\phi} \left[ \phi_{,\alpha\beta} - g_{\alpha\beta} \left( \Box \phi - \frac{1}{2} V(\phi) \right) \right],
\]

where \( T^{(BD)}_{\alpha\beta} \) is an induced energy–momentum tensor of the effective 4D modified BD theory, which is defined as

\[
T^{(BD)}_{\alpha\beta} \equiv T^{(IM)}_{\alpha\beta} + T^{(\phi)}_{\alpha\beta},
\]

with

\[
T^{(IM)}_{\alpha\beta} \equiv \frac{\phi}{8\pi} \left\{ b_{\alpha\beta} \frac{b'}{b} - \frac{b'}{b} g_{\alpha\beta} - \frac{\epsilon}{2b^2} \left[ b' g_{\alpha\beta} - g_{\alpha\beta} + g_{\mu\nu} g^\mu_{,\alpha} g^\nu_{,\beta} - \frac{1}{2} g_{\mu\nu} g^\mu_{,\alpha} g^\nu_{,\beta} \right] - g_{\alpha\beta} \left( b' g_{\mu\nu} g^\mu_{,\alpha} g^\nu_{,\beta} - \frac{1}{2} g_{\mu\nu} g^\mu_{,\alpha} g^\nu_{,\beta} \right) \right\}
\]

and

\[
T^{(\phi)}_{\alpha\beta} \equiv -\frac{\epsilon}{8\pi b^2} \left\{ g_{\alpha\beta} \left[ \phi'' + \left( \frac{1}{2} g_{,\mu\nu} g^\mu_{,\alpha} g^\nu_{,\beta} - \frac{b'}{b} \right) \phi' + \epsilon b b_{,\mu\phi}^\mu \right] - \frac{1}{2} g_{\alpha\beta} \phi' \right\}.
\]

Also, the induced potential has been defined in the formal identification as [14]

\[
V[\phi] \equiv -\frac{\omega}{b^2} \phi^2 \bigg|_{\Sigma},
\]

where the prime denotes derivative with respect to the fifth coordinate. Such an identification has been claimed [23] to be valid depending on metric background and considering separable scalar fields. However, this definition is different from what has been used in Ref. [15].

Reduction of equation (5) on the hypersurface \( \Sigma \) gives

\[
\Box \phi = -\frac{\epsilon}{b^2} \left[ \phi'' + \phi' \left( \frac{g_{,\mu\nu} g^\mu_{,\alpha} g^\nu_{,\beta}}{2} - \frac{b'}{b} \right) \right] - \frac{b_{,\mu}}{b} \phi^\mu,
\]

which after manipulation resembles the other field equation of a modified BD theory in four dimensions with induced potential. The definition \( T^{(BD)}_{\alpha\beta} \) and equation (13) are all we need for our purpose in this work and an interested reader can consult Refs. [14, 15] for further details.

In the next section we assume a generalized FRW metric in a vacuum 5D universe to find its cosmological implications.

3 Generalized FRW Cosmology

For a 5D universe with an extra space–like dimension in addition to the three usual spatially homogenous and isotropic ones, metric (6) can be written as

\[
dS^2 = -dt^2 + a^2(t, y) \left[ \frac{dr^2}{1 - kr^2} + r^2(d\theta^2 + \sin^2 \theta d\phi^2) \right] + b^2(t, y)dy^2,
\]

We have corrected miscalculations mentioned in the Introduction.
that can be considered as a generalized FRW solution. The scalar field $\phi$ and the scale factors $a$ and $b$, in general, are functions of $t$ and $y$. However, for simplicity and physical plausibility, we assume the extra dimension is cyclic, i.e. the hypersurface–orthogonal space–like is a Killing vector field in the underlying 5D space–time [17]. Hence, all fields are functions of the cosmic time only, and definition (12) makes the induced potential vanishes. In this case, we will show that such a universe can have accelerating and decelerating solutions. Note that, the functionality of the scale factor $b$ on $y$, either can be eliminated by transforming to a new extra coordinate if $b$ is a separable function, and or makes no changes in the following equations if $b$ is the only field that depends on $y$. Besides, in the compactified extra dimension scenarios, all fields are Fourier–expanded around $y_o$, and henceforth one can have terms independent of $y$ to be observable, i.e. physics would thus be effectively independent of compactified fifth dimension [4].

Considering metric (14), equations (4) and (5) result in cosmological equations

$$H^2 - \frac{\omega}{6} F^2 + HF + \frac{k}{a^2} = -\left( HB + \frac{1}{3} BF \right),$$  
$$2 \dot{H} + \dot{F} + 3H^2 + \left( \frac{\omega}{2} + 1 \right) F^2 + 2HF + \frac{k}{a^2} = -\left( \dot{B} + B^2 + 2HB + BF \right),$$  
$$2 \dot{H} + 4H^2 + \frac{\omega}{3} F^2 + 2\frac{k}{a^2} = \frac{2}{3} BF,$$

and

$$\dot{F} + F^2 + 3HF = -BF,$$

which are not independent equations and where $H \equiv \dot{a}/a$, $B \equiv \dot{b}/b$ and $F \equiv \dot{\phi}/\phi$. By employing relation (9), one can interpret the right hand side of equations (15) and (16) as energy density and pressure of the induced effective perfect fluid, i.e.

$$\rho_{BD} \equiv -T^{(BD)}_{tt} = -\frac{\phi}{8\pi} (3HB + BF)$$

and

$$p_{BD} \equiv T^{(BD)}_{ii} = \frac{\phi}{8\pi} (\dot{B} + B^2 + 2HB + BF) = -\frac{\phi}{8\pi} HB,$$

where $i = 1$ or 2 or 3 without summation on it. The latter equality in (20) comes from equation (31) which will be derived in the next section. Therefore, the equation of state is

$$p_{BD} = w_{\text{eff}} \rho_{BD} \quad \text{with} \quad w_{\text{eff}} = \frac{1}{F/H + 3}.$$  

The usual matter in our universe has a positive energy density, this basically has been demanded by the WEC, in which time–like observers must obtain positive energy densities. Actually, the complete WEC is [24]

$$\left\{ \begin{array}{l} \rho_{BD} \geq 0 \\ \rho_{BD} + p_{BD} \geq 0. \end{array} \right.$$  

Now, let us consider that the scale factor of the fifth dimension and the scalar field are not constant values, i.e. $B \neq 0$ and $F \neq 0$. Then, by applying conditions (22) into relations (19) and (20), one gets

$$\left\{ \begin{array}{l} B > 0 \\ F \leq -4H \end{array} \right. \quad \text{or} \quad \left\{ \begin{array}{l} B < 0 \\ F \geq -3H \end{array} \right.$$  

5
where we also have assumed expanding universes, i.e. \( H > 0 \). Using conditions (23) and (24) in relation (21) gives
\[-1 \leq w_{\text{eff}} \leq 0 \quad (25)\]
or
\[w_{\text{eff}} \geq 0, \quad (26)\]
in where the effective dust matter can be achieved when \( F/H \) goes to negative or positive infinity, respectively.

In Section 5, we explore characteristic of the corresponding universes for the above results. Meanwhile, in the following, we consider two special cases of a constant scale factor of the fifth dimension and a constant scalar field.

**Constant Scale Factor of Fifth Dimension**

When \( b \) is a constant, equations (15)–(18) reduce to
\[H^2 - \frac{\omega}{6} F^2 + HF + \frac{k}{a^2} = 0, \quad 2 \dot{H} + 4H^2 + \frac{\omega}{3} F^2 + 2 \frac{k}{a^2} = 0 \quad \text{and} \quad \dot{F} + F^2 + 3HF = 0. \quad (27)\]
These are exactly the ordinary vacuum BD equations in 4\( D \) space–time, with \( \rho_{\text{BD}} = 0 = p_{\text{BD}} \), as expected.

**Constant Scalar Field**

When \( \phi \) is a constant, action (1) reduces to a 5\( D \) Einstein gravitational theory that has been considered in Ref. [25] in general situation (i.e. the extra dimension is not cyclic). In this case, equations (15)–(18) become
\[H^2 + \frac{k}{a^2} = -HB, \quad \dot{H} + 2H^2 + \frac{k}{a^2} = 0 \quad \text{and} \quad \dot{B} + B^2 + 3HB = 0. \quad (28)\]
And, the usual FRW equations are equipped with \( p_{\text{BD}} = \rho_{\text{BD}} / 3 \equiv -HB/8\pi G \), which refers to a radiation–like dominated universe for any kind of geometry without a priori assumption that the scale factor of the fifth dimension is proportional to the inverse of the usual scale factor, i.e. \( b \propto a^{-1} \). Actually, the radiation–like result is expected. For where there is no dependency on the extra dimension, the usual four dimensional part of metric (14) and the third equation (28) give a wave equation for the scale factor of fifth dimension. Hence, definitions (10) and (11) yield a traceless induced energy–momentum tensor, as mentioned in Ref.[25].

Exact solution of the second equation of (28) is
\[a = \sqrt{-kt^2 + \alpha t}. \quad (29)\]
Substituting solution (29) into the first or third equation of (28) gives
\[b = |\beta \dot{a}| = \left| \beta \frac{-2kt + \alpha}{2\sqrt{-kt^2 + \alpha t}} \right| = \left| \beta \frac{\sqrt{\alpha^2 - 4ka^2}}{2a} \right|, \quad (30)\]
where \( \alpha \) and \( \beta \) are constants of integration, and we have assumed that 4\( D \) space–time has originated from a big bang.

For a closed geometry, solution (29) admits \( \alpha > 0 \) and predicts a big crunch at \( t = \alpha \) for the usual spatial coordinates while the fifth dimension tends to infinite size and is always real, for the maximum value of the usual scale factor is \( \alpha/2 \). But, a flat geometry expands for ever and accepts \( \alpha > 0 \). An open geometry also expands for ever and admits \( \alpha \geq 0 \). In this case, \( \alpha = 0 \) results in \( a = t \) and \( b = |\beta| \). Time evolution of scale factors correspond to closed, flat and open geometries have been illustrated in Fig. 0 with constant values of \( \alpha = 1 \) and \( \beta = 1 \) as an example.

In the next two sections, we again consider a more general situation in which the scale factor of the fifth dimension and the scalar field are not constants.
4 Correspondence Between 5D Equations and 4D Ones

Let us explore an equation (if any) similar to equation (18) – which is an integrable equation – for when the rules of $F$ and $B$ are replaced. For this purpose, adding equations (15) and (16), then subtracting equations (17) and (18) from it, yields

$$\dot{B} + B^2 + 3HB + BF = 0. \tag{31}$$

Comparing equations (18) and (31) shows that they are equivalent to each other if one replaces $B$ by $F$. Indeed, integrating equations (18) and (31) gives

$$\dot{a}^3b = m_1 \quad \text{and} \quad \dot{b}a^3\phi = m_2, \tag{32}$$

where $m_1 \neq 0$ and $m_2 \neq 0$ are constants of integration in general situations when $\phi$ and $b$ are not constants. Actually, vanishing $m_1$ or $m_2$ gives $\phi$ or $b$ to be a constant value, respectively, which have been discussed in the previous section. Dividing equations (32) by each other leads to

$$B = m'F, \tag{33}$$

where $m' \equiv m_2/m_1$. Relation (33) obviously gives

$$b = b_o \left( \frac{\phi}{\phi_o} \right)^{m'}, \tag{34}$$

where $b_o$ and $\phi_o$ are initial values.

Now, considering relation (33), equations (15)–(18) lead to three independent equations

$$H^2 - \frac{\tilde{\omega}}{6} \tilde{F}^2 + H\tilde{F} + \frac{k}{a^2} = 0, \quad 2\dot{H} + 4H^2 + \frac{\tilde{\omega}}{3} \tilde{F}^2 + 2\frac{k}{a^2} = 0 \quad \text{and} \quad \dot{\tilde{F}} + \tilde{F}^2 + 3H\tilde{F} = 0, \tag{35}$$

with

$$\tilde{\phi} \equiv \phi^{m'+1} \quad \text{and} \quad \tilde{\omega} \equiv \frac{\omega - 2m'}{(m'+1)^2}, \tag{36}$$

where $m' \neq -1$ and $\tilde{F} = (m'+1)F$. Equations (35) are exactly the FRW equations of 4D vacuum BD theory. However, one also needs to check if this new scalar field is a wave function in 4D vacuum as well. For this purpose, it is easy to show that

$$\Box\tilde{\phi} = (m' + 1) \left( \phi^{m'} \phi'' + \phi^{m'} \Box \phi \right) = 0, \tag{37}$$
where equations (13) and (14) for a cyclic extra dimension have been employed to get the second equality.

Hence, this procedure exhibits that 5D vacuum FRW–like equations, equations (15)–(18), are equivalent to the corresponding 4D vacuum ones, equations (35), with the same spatial scale factor but a new (or modified) scalar field and a new coupling constant, \( \tilde{\phi} \) and \( \tilde{\omega} \), in which to have a non–ghost scalar field one must have \( \tilde{\omega} > -3/2 \) [21, 22].

For the special case of \( m' = -1 \), i.e. when \( b \propto \phi^{-1} \), equations (15)–(18) reduce to

\[
H^2 - \left( \frac{\omega}{6} + \frac{1}{3} \right) F^2 + \frac{k}{a^2} = 0, \quad 2\dot{H} + 3H^2 + \left( \frac{\omega}{2} + 1 \right) F^2 + \frac{k}{a^2} = 0 \quad \text{and} \quad \dot{F} + 3HF = 0. \tag{38}
\]

From the third equation of (38) one gets

\[
F = F_o \left( \frac{a_o}{a} \right)^3. \tag{39}
\]

Using relation (39) into the first and second equations of (38) yields

\[
\ddot{a}a^5 = 2A \quad \text{and} \quad (a^2 + k)a^4 = -A, \tag{40}
\]

for \( \dot{a} \neq 0 \) and \( \omega \neq -2 \) and where \( A \equiv -(\omega + 2)F_o^2a_o^6/6 \). These equations, or actually their division i.e. \( \ddot{a}a + 2(\dot{a}^2 + k) = 0 \), can be solved by non–algebraic procedures, and their solutions include the inverse–elliptic functions, although we do not perform it further. For a static universe, i.e. \( \dot{a} = 0 \), equations (38) lead to a flat universe with \( \omega = -2 \). On the other hand, if \( \omega = -2 \), then equations (38) give \( \ddot{a} = 0 \) and \( \dot{a}^2 = -k \) which restrict the geometry either to be flat or open. For \( k = 0 \), one again gets a static universe with \( b = b_o \exp(-F_o t) \) and \( \phi = \phi_o \exp(F_o t) \). In the case \( k = -1 \), it leads to a uniform expanding universe with \( a = t \), and the evolution of scale factor of the fifth dimension is \( b = b_o \exp(F_o a_o^3/2t^2) \).

In the next section we continue our investigations for cosmological implications of equations (15)–(18) for a flat universe compatible with the recent observations.

## 5 Exact Solutions for Flat Universe Compatible with Observations

Measurements of anisotropies in the cosmic microwave background radiation indicate that the universe is spatially flat [18], so we concentrate on solutions with flat 3–spaces. Therefore, equations (15)–(17) yield

\[
\dot{H} + 3H^2 + (B + F)H = 0, \tag{41}
\]

that gives

\[
\ddot{a}a^2b\phi = m_3, \tag{42}
\]

where \( m_3 \) is an integration constant. The case of vanishing \( m_3 \) gives a static universe which is not compatible with observations. In general, relations (32) and (42) lead to

\[
b = b_o \left( \frac{a}{a_o} \right)^n \quad \text{and} \quad \phi = \phi_o \left( \frac{a}{a_o} \right)^m, \tag{43}
\]

for \( m_3 \neq 0 \) and where \( m \equiv m_1/m_3 \) and \( n \equiv m_2/m_3 = mn' \), also for general situations \( B \neq 0 \) and \( F \neq 0 \), we have \( n \neq 0 \) and \( m \neq 0 \).

Indeed, if in \textit{priori}, one had assumed \( b \propto a^n \) (or \( \phi \propto a^m \)), then equations (15)–(18) would restrict the geometry to be spatially flat, and automatically would give \( \phi \propto a^m \) (or \( b \propto a^n \)). Therefore, the power–law relation between the scale factor of the fifth dimension and the scalar field with the usual scale factor is a characteristic of the spatially flat universe.

Substituting solutions (43) into equation (41) gives

\[
\frac{\ddot{a}}{a} + (m + n + 2)\frac{\dot{a}}{a} = 0. \tag{44}
\]
For $m + n \neq -3$, equation (44) has a power–law solution
\[ a(t) = a_o \left( \frac{t}{t_o} \right)^s \quad \text{with} \quad H = \frac{s}{t}, \quad (45) \]
where $s \equiv (m + n + 3)^{-1}$, and assumption expanding universes makes $s > 0$. Hence, solutions (43) lead to
\[ b(t) = b_o \left( \frac{t}{t_o} \right)^{ns} \quad \text{with} \quad B = \frac{ns}{t} \quad (46) \]
and
\[ \phi(t) = \phi_o \left( \frac{t}{t_o} \right)^{ms} \quad \text{with} \quad F = \frac{ms}{t}. \quad (47) \]
There is also a constraint relation among the initial values, namely $a_o^3 b_o \phi_o = (m_1 + m_2 + 3) t_o$. Incidentally, the effective energy density and pressure, equations (19) and (20) become
\[ \rho_{BD} = -\frac{\phi_o s^2}{8\pi t_o^{ms}} n(m + 3)t^{ms-2} \quad \text{and} \quad p_{BD} = -\frac{\phi_o s^2}{8\pi t_o^{ms}} n t^{ms-2}. \quad (48) \]

In the case $m + n = -3$, equations (43) and (44) give exponential solutions
\[ a(t) = a_o e^{\lambda(t-t_o)} \quad \text{with} \quad H = \lambda, \quad (49) \]
\[ b(t) = b_o e^{n\lambda(t-t_o)} \quad \text{with} \quad B = n\lambda \quad (50) \]
and
\[ \phi(t) = \phi_o e^{m\lambda(t-t_o)} \quad \text{with} \quad F = m\lambda, \quad (51) \]
where $\lambda$ is a constant and its positive values give expanding universes, thus we assume $\lambda > 0$. Incidentally, the constraint relation among the initial values is $a_o^3 b_o \phi_o = m_3 / \lambda$. In this case, the energy density and pressure are
\[ \rho_{BD} = -\frac{\phi_o \lambda^2}{8\pi e^{m\lambda t_o}} n(m + 3) e^{m\lambda t} \quad \text{and} \quad p_{BD} = -\frac{\phi_o \lambda^2}{8\pi e^{m\lambda t_o}} n e^{m\lambda t}. \quad (52) \]

Note that, for both groups of solutions, the power law and exponential ones, one has $w_{eff} = 1/(3+m)$. We should emphasize that all solutions of this section have been obtained without a priori ansatz for functionality of the scale factor and the scalar field.

In the next two subsections, we discuss properties of these solutions. We should also remind that our vanishing induced potential case is not consistent with zero potential case of Ref. [15] (where there, it requires $\omega = -1$ only).

### 5.1 Power–Law Solutions

Solutions are generally confined within some constraints that are originated from mathematical or physical reasons. First of all, due to equations (15)–(17), the parameters $n$ and $m$ are not independent. Substituting solutions (45)–(47) into either of equations (15)–(17) gives
\[ m_\pm = \frac{n + 3 \pm \sqrt{(n + 3)^2 + 6\omega(n + 1)}}{\omega} \quad (53) \]
and hence
\[ s_\pm = \frac{\omega}{(\omega + 1)(n + 3) \pm \sqrt{(n + 3)^2 + 6\omega(n + 1)}}. \quad (54) \]

Besides, our constraints are as follows. We have assumed $s > 0$, $m_\pm \neq 0$, $n \neq 0$ and $m_\pm + n \neq -3$ for power–law solutions. Real solutions of relation (53) dictate that $(n + 3)^2 + 6\omega(n + 1) \geq 0$. By substituting solutions (45)–(47) in the WEC (23) or (24), we get
\[ \begin{cases} 
    n > 0 \\
    m \leq -4
\end{cases} \quad (55) \]
Case Ia: Decelerated Universe

It is supposed that the universe for a long time, when it was in the radiation or dust dominated phases, was in a decelerating regime. In our model, decelerating solutions can be obtained when $0 < s_{\pm} < 1$. Acceptable domains of $n$ and $\omega$ for such a range, without considering the WEC, is given in Table 1 and Fig. 1. Note that, in Fig. 1, the part (ii) completely covers the part (i).

In the following, we employ these constraints for when they lead to cases of decelerated and especially accelerated universes. Meanwhile, we should also remind that the deceleration parameter, $q = -\ddot{a}a/a^2$, in our model for the power–law solutions is $q = 1 - s$.

| $n$         | $\omega$ for $s_+$ values | $\omega$ for $s_-$ values |
|-------------|---------------------------|---------------------------|
| $n \leq -3$ | No solution               | $\frac{2(n^2 + 2n + 3)}{(n+2)^2} < \omega < 0$ |
| $-3 < n \leq -2$ | $0 < \omega \leq \frac{(n+3)^2}{6(n+1)}$ | $-\frac{2(n^2 + 2n + 3)}{(n+2)^2} < \omega < 0$ or $0 < \omega \leq \frac{(n+3)^2}{6(n+1)}$ |
| $-2 < n \leq -1$ | $\omega < \frac{2(n^2 + 2n + 3)}{(n+2)^2}$ or $0 < \omega \leq \frac{(n+3)^2}{6(n+1)}$ | $\omega < 0$ or $0 < \omega \leq \frac{(n+3)^2}{6(n+1)}$ |
| $n = -1$ | $\omega < -4$ or $\omega > 0$ | No solution |
| $-1 < n < 0$ | $\frac{(n+3)^2}{6(n+1)} \leq \omega < \frac{2(n^2 + 2n + 3)}{(n+2)^2}$ or $\omega > 0$ | $-\frac{2(n+3)^2}{6(n+1)} \leq \omega < 0$ or $\omega > 0$ |
| $0 < n \leq 1$ | $\omega > 0$ | $-\frac{(n+3)^2}{6(n+1)} \leq \omega < 0$ or $\omega > 0$ |
| $n > 1$ | $\frac{(n+3)^2}{6(n+1)} \leq \omega < \frac{2(n^2 + 2n + 3)}{(n+2)^2}$ or $\omega > 0$ | $-\frac{(n+3)^2}{6(n+1)} \leq \omega < 0$ or $\omega > 0$ |

Table 1: Ranges of $n$ and $\omega$ for decelerating power–law solutions.

Figure 1: Domains of $n$ and $\omega$ correspond to Table 1, (i) $s_+$ and (ii) $s_-$ values. Note that, the line $n = -1$ is excluded in part (ii).

or

$$\left\{ \begin{array}{l} n < 0 \\ m \geq -3 \end{array} \right.$$

respectively. Note that, conditions (55) and (56) are compatible with conditions (25) and (26), as expected.

In the following, we employ these constraints for when they lead to cases of decelerated and especially accelerated universes. Meanwhile, we should also remind that the deceleration parameter, $q = -\ddot{a}a/a^2$, in our model for the power–law solutions is $q = 1 - s$.
\[
\begin{array}{|c|c|c|}
\hline
n & \omega \text{ for } s_+ \text{ values} & \omega \text{ for } s_- \text{ values} \\
\hline
2 < n < 3 & -\frac{n+9}{8} \leq \omega < -\frac{2(n^2+2n+3)}{(n+2)^2} & \text{No solution} \\
\hline
n \geq 3 & \frac{(n+3)^2}{6(n+1)} \leq \omega < -\frac{2(n^2+2n+3)}{(n+2)^2} & -\frac{(n+3)^2}{6(n+1)} \leq \omega < -\frac{n+9}{8} \\
\hline
\end{array}
\]

Table 2: Ranges of \( n \) and \( \omega \) for decelerating power–law solutions which adapt the WEC (55).

\[\begin{array}{|c|c|c|}
\hline
n & \omega \text{ for } s_+ \text{ values} & \omega \text{ for } s_- \text{ values} \\
\hline
n \leq -3 & \text{No solution} & -\frac{2(n^2+2n+3)}{(n+2)^2} < \omega < 0 \\
-3 < n \leq -2 & 0 < \omega \leq -\frac{(n+3)^2}{6(n+1)} & -\frac{2(n^2+2n+3)}{(n+2)^2} < \omega < 0 \text{ or } 0 < \omega \leq -\frac{(n+3)^2}{6(n+1)} \\
-2 < n < -1 & \omega < -\frac{2(n^2+2n+3)}{(n+2)^2} \text{ or } 0 < \omega \leq -\frac{(n+3)^2}{6(n+1)} & \omega < 0 \text{ or } 0 < \omega \leq -\frac{(n+3)^2}{6(n+1)} \\
n = -1 & \omega < -4 \text{ or } \omega > 0 & \text{No solution} \\
-1 < n < 0 & -\frac{(n+3)^2}{6(n+1)} \leq \omega < -\frac{2(n^2+2n+3)}{(n+2)^2} \text{ or } \omega > 0 & -\frac{(n+3)^2}{6(n+1)} \leq \omega < 0 \text{ or } \omega > 0 \\
\hline
\end{array}\]

Table 3: Ranges of \( n \) and \( \omega \) for decelerating power–law solutions which adapt the WEC (56).

Figure 2: Domains of \( n \) and \( \omega \) correspond to Table 2, (i) \( s_+ \) and (ii) \( s_- \) values.

Figure 3: Domains of \( n \) and \( \omega \) correspond to Table 3, (i) \( s_+ \) and (ii) \( s_- \) values. Note that, the line \( n = -1 \) is excluded in part (ii).
### Table 4: Ranges of $n$ and $\omega$ for accelerating power–law solutions.

| $n$          | $\omega$ for $s_+$ values | $\omega$ for $s_-$ values |
|--------------|----------------------------|-----------------------------|
| $n \leq -3$  | No solution                | $-\frac{2(n^2+2n+3)}{(n+2)^2} < \omega < -\frac{2(n^2+3n+6)}{(n+3)^2}$ |
| $-3 < n < -2$| $-\frac{2(n^2+3n+6)}{(n+3)^2} < \omega < -8$ | No solution |
| $n = -2$     | $\omega < -\frac{2(n^2+3n+6)}{(n+3)^2}$ | No solution |
| $-2 < n < 0$ | $\omega < -\frac{2(n^2+3n+6)}{(n+3)^2}$ | No solution |
| $0 < n < 1$  | $\omega < -\frac{2(n^2+3n+6)}{(n+3)^2}$ | No solution |
| $n \geq 1$   | $\omega < -\frac{2(n^2+3n+6)}{(n+3)^2}$ | No solution |

Table 4: Ranges of $n$ and $\omega$ for accelerating power–law solutions.

### Figure 4: Domains of $n$ and $\omega$ correspond to Table 4, (i) $s_+$ and (ii) $s_-$ values.

**Case IIa: Accelerated Universe**

Recent observations show that the universe is in an accelerating regime at the present epoch [19]. This makes $s_+ > 1$, and acceptable values of $n$ and $\omega$ corresponding to this condition, without considering the WEC, are given in Table 4 and Fig. 4. In Fig. 4, the maximum value of $\omega$ for $s_+$ values tends to $-5/4$ when $n = 1$, and for $s_-$ values tends to $-4/3$ when $n \to 1^-$. Corresponding cases with the WECs (55) and (56) are illustrated in Table 5 with Fig. 5 and Table 6 with Fig. 6, respectively.

It should be emphasized that though astronomical tests in the solar system requires a positive large value for $\omega$, but still in the large cosmological scale, one cannot definitely rule out small or even negative values of the BD coupling constant. Indeed, these values of $\omega$ have achieved considerable interests in the literature.

By considering the WEC (55) in relations (48), one gets positive energy densities, as expected, but with negative pressures, where both $\rho_{BD}$ and $|p_{BD}|$ decrease with the time. In this case, even
Figure 5: Domains of \( n \) and \( \omega \) correspond to Table 5 for \( s_+ \) values.

| \( n \)     | \( \omega \) for \( s_+ \) values                                   | \( \omega \) for \( s_- \) values                                   |
|------------|--------------------------------------------------------------------|--------------------------------------------------------------------|
| \( n \leq -3 \) | No solution                                                        | \( -\frac{2(n^2+3n+6)}{(n+3)^2} < \omega < -\frac{2(n^2+2n+3)}{(n+2)^2} \) |
| \(-3 < n < -2 \) | \( \omega < -\frac{2(n^2+3n+6)}{(n+3)^2} \)                      | \( \omega < -\frac{2(n^2+3n+6)}{(n+3)^2} \)                      |
| \( n = -2 \)   | \( \omega < -8 \)                                                  | No solution                                                        |
| \(-2 < n < 0 \) | \( -\frac{2(n^2+2n+3)}{(n+2)^2} < \omega < -\frac{2(n^2+3n+6)}{(n+3)^2} \) | No solution                                                        |

Table 6: Ranges of \( n \) and \( \omega \) for accelerating power–law solutions which adapt the WEC (56).

Figure 6: Domains of \( n \) and \( \omega \) correspond to Table 6, (i) \( s_+ \) and (ii) \( s_- \) values.
Table 7: Ranges of $n$ and $\omega$ for decelerating power-law solutions with non-ghost scalar fields which adapt the WEC (56).

| $n$       | $\omega$ for $s_+$ values | $\omega$ for $s_-$ values |
|-----------|---------------------------|---------------------------|
| $n \leq -3$ | No solution               | $-\frac{4}{3} < \omega < 0$ |
| $-3 < n < -1$ | $0 < \omega \leq -\frac{(n+3)^2}{6(n+1)}$ | $-\frac{4}{3} < \omega < 0$ or $0 < \omega \leq -\frac{(n+3)^2}{6(n+1)}$ |
| $n = -1$   | $\omega > 0$              | No solution                |
| $-1 < n < 0$ | $\omega > 0$              | $-\frac{4}{3} < \omega < 0$ or $\omega > 0$ |

Figure 7: Domains of $n$ and $\omega$ correspond to Table 7, (i) $s_+$ and (ii) $s_-$ values. Note that, the line $n = -1$ is excluded in part (ii).

though the pressure is negative, but Figs. 2 and 5 illustrate that one has decelerating and accelerating solutions. On the other hand, using condition (56) into relations (48) gives positive energy densities and pressures. Although, in this situation the pressure is positive, but still Figs. 3 and 6 indicate that one again has decelerating and accelerating solutions. In this situation, for decreasing energy density and pressure with the time, one has to restrict $ms < 2$, which most of the solutions fulfill it.

Yet we have one more condition, namely non-ghost scalar fields with $\omega > -4/3$, to be imposed. With this situation, acceptable solutions are as follows.

Case Ib: Decelerated Universe

Acceptable values of $n$ and $\omega$ for the range $\omega > -4/3$ restrict Table 3 and Fig. 3, and the results are shown in Table 7 and Fig. 7. Hence, this model admits a typical decelerated universe with non-ghost scalar fields, positive induced energy density and pressure, fulfilling the WEC (56), where the scale factor of fifth dimension shrinks with the time. Incidentally, Fig. 2 illustrates that there is not any decelerated solution with non-ghost scalar fields which complies with the WEC (55).

Case Iib: Accelerated Universe

Table 8 and Fig. 8, which are the reductions of Table 5 and Fig. 5, illustrate the corresponding domains of $n$ and $\omega$ for $\omega > -4/3$. Therefore, the model also admits a typical accelerated universe with non-ghost scalar fields, positive induced energy density and negative pressure, fulfilling the WEC (55), where the scale factor of fifth dimension grows with the time. This situation restricts $1 < n < 3$, contrary to the assumption of $n = 1$ in Ref. [14]. Also, Fig. 6 indicates that accelerated solutions do not exist for non-ghost scalar fields which fulfill the WEC (56).

By employing the recent observational measurements of $q$, namely $-0.92 \leq q_0 \leq -0.42$ [26], we obtain $1.42 \leq s \leq 1.92$. This range of $s$ is only compatible with the accelerating Case II, but
Table 8: Ranges of $n$ and $\omega$ for accelerating power–law solutions with non–ghost scalar fields which fulfill the WEC (55).

| $n$ | $\omega$ for $s_+$ values |
|-----|---------------------------|
| $1 < n < \frac{5}{3}$ | $-\frac{n+9}{8} \leq \omega < -\frac{2(n^2+3n+6)}{(n+3)^2}$ |
| $\frac{5}{3} \leq n < 3$ | $-\frac{4}{3} < \omega < -\frac{2(n^2+3n+6)}{(n+3)^2}$ |

Figure 8: Domains of $n$ and $\omega$ correspond to Table 8. The area below the dashed line corresponds to Table 9, and the upper border curve corresponds to Table 10 with non–ghost scalar fields.

imposes more restrictions on domains of $n$ and $\omega$. Indeed, Table 8 and Fig. 8 of Case IIb with these values of $s$ lead to Table 9 and the area below the dashed line in Fig. 8. Thus, one obtains ranges $1.5208 \leq n < 1.9583$ and $-4/3 < \omega \leq -1.3151$. These are the best values of $n$ and $\omega$, in this model, that are compatible with the recent observations. These values also fulfill the condition $\tilde{\omega} > -3/2$, which is required, in Section 4, for having non–ghost scalar fields in the equivalent 4D vacuum BD equations.

It is interesting to note that, one can infer from relations (53) and (54) that $m_+$ and $s_+$ do not allow $\omega = 0$. However, the other constraints does not obviously show that the zero value of $\omega$ is prevented, but the mathematical procedure for all tables and figures indicates that $\omega \neq 0$.

5.2 Exponential Solutions

Relation (49) represents an exponential growth of the scale factor, i.e. an inflationary universe. However, no such a rapid expansion has been indicated at present or throughout almost the whole history of the universe except at the very early universe stage. Nevertheless, let us probe some properties of these solutions.

Table 9: Ranges of $n$ and $\omega$ for accelerating power–law solutions with non–ghost scalar fields which fulfill the WEC (55) for $s_+$ values, and are compatible with the recent observations.

| $n$ | $\omega$ for $s_+$ values |
|-----|---------------------------|
| $n = 1.5208$ | $\omega = -1.3151$ |
| $1.5208 \leq n < \frac{5}{3}$ | $-\frac{n+9}{8} \leq \omega \leq -\frac{32(144n^2+357n+639)}{2304n^2+11424n+14161}$ |
| $\frac{5}{3} \leq n < 1.9583$ | $-\frac{4}{3} < \omega \leq -\frac{32(144n^2+357n+639)}{2304n^2+11424n+14161}$ |
\[ n \quad \omega \text{ for } m_+ \text{ values} \quad \omega \text{ for } m_- \text{ values} \\
\hline n = 1 \quad \omega = \frac{-2}{3} \quad \omega = \frac{-1}{4} \\
n > 1 \quad \text{No solution} \quad \omega = \frac{-2(n^2 + 3n + 6)}{(n+3)^2} \\
\]  

Table 10: Ranges of \( n \) and \( \omega \) for exponential solutions which fulfill the WEC (55).

\[ n \quad \omega \text{ for } m_+ \text{ values} \quad \omega \text{ for } m_- \text{ values} \\
\hline n < -3 \quad \text{No solution} \quad \omega = \frac{-2(n^2 + 3n + 6)}{(n+3)^2} \\
-3 < n < -1 \quad \omega = \frac{-2(n^2 + 3n + 6)}{(n+3)^2} \quad \text{No solution} \\
-1 < n < 0 \quad \omega = \frac{-2(n^2 + 3n + 6)}{(n+3)^2} \quad \text{No solution} \\
\]  

Table 11: Ranges of \( n \) and \( \omega \) for exponential solutions which fulfill the WEC (56).

First of all, we have \( m + n = -3 \) that, with assumptions \( m \neq 0 \) and \( n \neq 0 \), restricts \( m \neq -3 \), \( n \neq -3 \) and \( m \neq -n \). By employing solutions (49)–(51) into equations (15)–(17), one gets

\[ m_\pm = \frac{-3 \pm \sqrt{(12\omega + 15)}}{\omega + 2}. \tag{57} \]

Substituting condition \( m + n = -3 \) into relation (57) gives

\[ \omega = \frac{-2(n^2 + 3n + 6)}{(n+3)^2}, \tag{58} \]

for \( n \neq 0, -3 \). Real solutions of relation (57) impose \( \omega \leq -5/4 \). Solutions (49)–(51) satisfy conditions (55) and (56) with more restrictions. Using the WEC (55), in relations (52) gives positive energy densities and negative pressures, where both \( \rho_{BD} \) and \( |p_{BD}| \) decrease rapidly with the time. But, considering condition (56) in relations (52), gives positive energy densities and pressures. If one takes \( m < 0 \) in condition (56), the energy density and pressure again will decrease with the time.

Acceptable ranges of \( n \) and \( \omega \) for exponential solutions fulfilling the WECs (55) and (56) are shown in Tables 10 and 11. These tables indicate that, only for the range \( 1 \leq n < 3 \), solutions do avoid ghost scalar fields. In Fig. 8, the upper border curve illustrates the acceptable values of \( n \) and \( \omega \) for an inflationary universe fulfilling the WEC (55) with non–ghost scalar fields.

6 Conclusions

Analogous to the approach of IM theories, one can consider the BD gravity as the underlying theory. Hence, extra geometrical terms, coming from the fifth dimension, are regarded as an induced–matter and induced potential. We have followed, with some corrections, the procedure of Ref. [14] for introducing the induced potential and have employed a generalized FRW type solution for a 5D vacuum BD theory. Hence, the scalar field and scale factors of the 5D metric can, in general, be functions of the cosmic time and the extra dimension. However, for simplicity, we have assumed the scalar field and scale factors to be only functions of the cosmic time, where this makes the induced potential, by its definition, vanishes.

We then have revealed that in general situations, in which the scale factor of the fifth dimension and scalar field are not constants, the 5D equations, for any kind of geometry, admit a power–law relation between the scalar field and scale factor of the fifth dimension. Hence, the procedure exhibits that 5D vacuum FRW–like equations are equivalent, in general, to the corresponding 4D vacuum ones.
with the same spatial scale factor but a new (or modified) scalar field and a new coupling constant. This equivalency can be viewed as the distinguished point of this work from Refs. [14, 15]. Indeed, through investigating the 5D vacuum FRW–like equations, we have shown that its equivalent 4D vacuum equations admit accelerated scale factors, contrary to what one may have expected from a vacuum space–time. Conclusions of the complete investigation of the induced 4D equations are as follows.

Following our investigations for cosmological implications, we have shown that for the special case of a constant scale factor of the fifth dimension, the 5D vacuum FRW–like equations reduce to the corresponding equations of the usual 4D vacuum BD theory, as expected. In the special case of a constant scalar field, the action reduces to a 5D Einstein gravitational theory and the equations reduce to the usual FRW equations with a typical radiation dominated universe. For this situation, we also have obtained dynamics of scale factors of the ordinary and extra dimensions for any kind of geometry without any priori assumption among them. Solutions predict a limited life time for closed geometries and unlimited one for flat and open geometries. A typical time evolutions of scale factors correspond to closed, flat and open geometries have been illustrated in Fig. 0.

Then, we have focused on spatially flat geometries and have obtained exact solutions of scale factors and scalar field. Solutions are found to be in the form of power–law and exponential ones in the cosmic time. We also have employed the WEC for the induced–matter of the 4D modified BD gravity, that gives two conditions (55) and (56). We then have pursued properties of these solutions and have indicated mathematically and physically acceptable ranges of them, and the results have been presented in a few tables and figures.

All types of solutions fulfill the WECs in different ranges, where the exponential solutions are more restricted. The solutions fulfilling the WEC (55) have negative pressures, but the figures illustrate that for the power–law results there are decelerating solutions beside accelerating ones. For this condition, both \( \rho_{BD} \) and \( |p_{BD}| \) decrease with the cosmic time, but the extra dimension grows. On the other hand, the solutions satisfying the WEC (56) have positive pressures, where the power–law results accept accelerating solutions in addition to decelerating ones. For this condition, again decreasing energy density and pressure with the time can occur for some solutions, however all with shrinking extra dimension. The homogeneity between the extra dimension and the usual spatial dimensions, i.e. \( b \propto a \), can take place in the solutions, but for the power–law ones the WECs exclude it.

By considering non–ghost scalar fields and appealing the recent observational measurements, the solutions have been more restricted. Actually, we have illustrated that the accelerating power–law solutions, which satisfy the WEC and have non–ghost scalar fields, are compatible with the recent observations in ranges \(-4/3 < \omega \leq -1.3151\) for the BD coupling constant and \(1.5208 \leq n < 1.9583\) for dependence of the fifth dimension scale factor with the usual scale factor. These ranges also fulfill the condition \( \tilde{\omega} > -3/2 \) which prevents ghost scalar fields in the equivalent 4D vacuum BD equations. Incidentally, this range is more restricted than the one obtained in Ref. [15], i.e. \(-1.5 < \omega < -1\), where the difference may have been caused by the distinct definition of the induced potential in two approaches of Ref. [14] and Ref. [15]. However, we should remind that it has also been shown [21] that the WEC, for 5D space–times, requires \(-4/3 \leq \omega\), in which no other experimental evidences have been considered.

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