Kinetics of Mobile Impurities and Correlation Functions in One-Dimensional Superfluids at Finite Temperature

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We examine the hydrodynamic approach to dynamical correlations in one-dimensional superfluids near integrability and calculate the characteristic time scale, \( \tau \), beyond which this approach is valid. For time scales shorter than \( \tau \), hydrodynamics fails, and we develop an approach based on kinetics of fermionic quasiparticles described as mobile impurities. New universal results for the dynamical structure factor relevant to experiments in ultracold atomic gases are obtained.

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The understanding of many-body quantum dynamics is a central topic of current research in ultracold atomic gases [1]. Of special interest are one-dimensional (1D) systems where interactions play a crucial role in their dynamics as was demonstrated in recent experiments [2, 3].

These effects are accessible by measuring the dynamical correlations of 1D bosons, which in contrast to the static ones [4], largely remain an open theoretical problem [5]. One such correlation function is the dynamical structure factor (DSF) defined as the Fourier transform of the density-density correlation function (we use the units such that \( \hbar = 1 \), \( k_B = 1 \) throughout the paper)

\[
S(q, \omega) = \int dx dt e^{-i(qx - \omega t)} \langle \rho(x, t)\rho(0, 0) \rangle. \tag{1}
\]

This quantity is readily accessible by Bragg spectroscopy [6], and its measurement was recently reported for arrays of one-dimensional Bose gases [7].

As the calculation of the correlation functions, especially dynamical ones, is still a formidable task even for integrable models, the problem is often solved by employing a hydrodynamic description valid in the low-energy limit [8, 9], which treats a system as a collection of weakly interacting phononic modes. In the case of weakly interacting 1D bosons at low enough temperatures \( T < m c^2 \), where \( c \) is the sound velocity and \( m \) is the mass of particles, and for small momenta \( 0 < q < T/c \), the DSF is a narrowly peaked function around the line \( \omega = cq \) [10], suggesting that the main contribution into \( S(q, \omega) \) comes from phononic excitations. The smearing of the phononic peak was attributed to the phononic nonlinearity by Andreev [11] who used the hydrodynamic approach to obtain the typical width

\[
\delta \omega_q^{(B)} = 0.394 \left( 1 + \frac{n}{c} \frac{\partial c}{\partial n} \right) \sqrt{\frac{T q^3}{m n}}, \tag{2}
\]

where \( n \) is 1D density. The width in Eq. (2) has a 3/2 power law dependence on momentum, suggesting a relation to the Kardar-Parisi-Zhang (KPZ) model of interface growth [12], as was elucidated and recently confirmed numerically by Kulkarni and Lamacraft [13].

Formally, Eq. (2) (which was obtained under assumptions of the hydrodynamic description, i.e., in the universal long-wavelength limit) is valid for all values of the interaction [14], including the case of impenetrable Tonks-Girardeau (TG) limit. Conversely, in this regime, one can use the Bose-Fermi mapping [15] representing excitations in terms of free fermions. This leads to a different estimate for the width of DSF,

\[
\delta \omega_q^{(F)} \sim \frac{q T}{mc}. \tag{3}
\]

The physics behind Eq. (3) is that the density fluctuations are dominated by effective fermionic particle-hole excitations rather than phonons. The fermions move ballistically with velocity close to \( c \) and the width (3) is just the uncertainty of the energy of a particle-hole pair \( \delta \omega_q^{(F)} = \delta cq \) resulting from the thermal uncertainty in the Fermi velocity \( \delta c \sim T/mc \) [5]. For small momenta \( q < T/c \), the energy uncertainty of a pair exceeds considerably the one predicted by Andreev, \( \delta \omega_q^{(F)} > \delta \omega_q^{(B)} \).
Away from the TG regime, the Bose-Fermi mapping can be generalized to any value of interactions \[^{16}\]. For this effective system of interacting fermions, the linear momentum law, Eq. (3), still holds with a proper redefinition of the Fermi velocity and mass \[^{17}\].

We now examine the hydrodynamic approach of Refs. \[^{11,13}\] and argue that it can be applied only on time scales longer than a certain microscopic time \(\tau\) restricting the hydrodynamic behavior of \(S(q, \omega)\) to very small frequencies i.e., the standard condition \(\omega \tau < 1\). The time \(\tau\) and corresponding length \(l = c \tau\) can be identified with the lifetime and the mean free path, respectively, of fermionic excitations, which can be defined away from the TG limit \[^{19}\] and which represent fast degrees of freedom. So, the apparent contradiction between the results in Eqs. (2) and (3) is naturally resolved: for times shorter than \(\tau\), the fermionic quasiparticles move ballistically, and the spread in their energies is controlled by the thermal uncertainty in their velocities leading to Eq. (3). The DSF for \(\omega > 1/\tau\) is then given by the standard fermionic expression with renormalized mass and quasiparticle residue; see Eq. (22) below. For long times, \(t > \tau\), or small frequencies, \(\omega < 1/\tau\), the fermionic quasiparticles thermalize, and the system enters the hydrodynamic regime. The DSF acquires the form given by Eq. (12) with width given by Eq. (2). There is also an intermediate regime, where the width of \(S(q, \omega)\) is controlled by collisions between excitations; see Eq. (23). These regimes of DSF are shown schematically in Fig. 1.

The exact microscopic calculations of \(\tau\) can only be envisaged for integrable models, but the result \(1/\tau = 0\) is expected due to infinitely many integrals of motion preventing the system from thermalization. We consider a small perturbation of the Lieb-Liniger model and obtain the main result of this Letter, the expression for thermalization time scale,

\[
\frac{1}{\tau} = \frac{\pi^5}{128} \left[ \Gamma_p' \right]^2 \frac{T^7}{m^* c},
\]

where the effective mass \(m^*\) is given by Eq. (10). The time scale \(\tau\) depends strongly on the temperature \(T\) \(\propto T^{-7}\). In addition, it is proportional to the square of the parameter \(\sqrt{\Gamma_p} = \partial \Gamma_p / \partial \rho\). Here, \(\Gamma_p\) is the amplitude of backscattering of phonons by a fermionic quasiparticle with momentum \(p\) introduced in Refs. \[^{20,22}\] so that \(\Gamma_p mc^2\) is the dimensionless small parameter of our theory. It depends on the fine details of the interactions between particles; in particular, it vanishes identically for the Lieb-Liniger model \[^{23}\]. For weakly interacting bosons with weak three-particle interactions, we find \(\Gamma_p' = -\alpha n/(48m^2c^4)\), where \(\alpha = \partial (mc^2/n) / \partial n\). Below, we discuss the derivation of the above results \[^{24}\].

The hydrodynamic description of one-dimensional superfluids \[^{8,9}\] is based on considering smooth configurations of the displacement \(\vartheta\) and phase \(\varphi\) fields related to deviations \(\rho = \partial_\vartheta / \pi\) of the density from its thermodynamic average \(n\) and superfluid velocity \(u = \partial_x \varphi / m\). Using these variables, the low-temperature dynamics is then governed by the Lagrangian density

\[
L_{hyd} = -\rho \dot{\varphi} - \frac{n + \rho}{2m} \left(\partial_x \varphi\right)^2 - \left[ e_0(n + \rho) - e_0(n) - \mu \rho \right],
\]

where \(e_0(n)\) is the ground state energy density and \(\mu = \partial e_0 / \partial n\) is the chemical potential. Expanding Eq. (5) for small \(\rho, u\), one obtains the quadratic phononic Lagrangian density

\[
L_{ph} = -\frac{1}{2} \partial_x \varphi \partial_t \varphi - \frac{c}{2\pi} \left(\partial_x \varphi\right)^2 - \frac{cK}{2\pi} \left(\partial_x \varphi\right)^2
\]

depending on the Luttinger parameter \(K = \pi n mc^2\). For bosons with short-range interactions, \(K \geq 1\). In the weakly interacting regime, \(K \to \infty\), while \(K = 1\) corresponds to the TG gas of hard-core bosons. It is customary to define the right and left chiral fields

\[
\psi_\pm = e^{\pm \varphi/\sqrt{K}}
\]

in terms of which the phononic Lagrangian separates

\[
L_{ph} = \sum_{\nu = \pm} \frac{1}{4\pi} \alpha(\nu \partial_t \varphi + c\partial_x^2 \varphi) \chi_\nu.
\]

The next (cubic) order the expansion of Eq. (5) contains a nonlinear coupling between phonons,

\[
L_{ph}' = \frac{\alpha}{6} \rho^3 + \frac{1}{2m} \partial_x \varphi_0^2
\]

\[
= \frac{1}{12\pi m^*} \left[ (\partial_x \chi_+)^3 + (\partial_x \chi_-)^3 \right] + \ldots.
\]

The omitted terms describe the interactions between phonons of different chirality. They can be safely neglected for low energies, as the interaction time between phonons moving in opposite directions is small. In Eq. (9), we have introduced the effective mass

\[
\frac{1}{m^*} = \frac{1}{2m\sqrt{K}} \left( 1 + \frac{n}{\theta} \frac{\partial c}{\partial n} \right)
\]

(c.f. Ref. \[^{30}\]). In the TG regime, \(m^* = 1\), whereas in the weakly interacting regime, \(m^* = 3/(4\sqrt{K})\). \[^{3}\].

The total Lagrangian density \(L_{ph} + L_{ph}'\) generates equations of motion for the right-moving fields

\[
(\partial_t + c \partial_x) \chi_+ = \frac{1}{2m^*} (\partial_x \chi_+)^2 + D \partial_x^2 \chi_+ + \xi,
\]

and similarly for the left-moving ones. In Eq. (11), we have also added terms describing dissipation and thermal white noise with zero mean and \(\xi(x,t)\xi(x',t') = 4\pi DT/c \delta(x-x')\delta(t-t')\) originating from coupling to a yet unspecified thermal bath with temperature \(T\). Equation (11) can be identified with the celebrated KPZ equation for the dynamics of interface growth \[^{12}\]. Its DSF was obtained in terms of the universal function \(f(s)\) describing the scaling limit of the polynuclear growth model \[^{31}\]. The function \(f(y)\) can only be determined numerically and has a peaked form with height and width of...
order unity. Expressing the density in terms of the chiral field using Eq. 7, the result can be written as

\[ S(q, \omega) = \frac{T}{c} \frac{K}{2\pi^2} \frac{1}{\delta \omega_q(B)} \int \left( \frac{\omega - cq}{\delta \omega_q(B)} \right). \]  (12)

Its typical width is given by

\[ \delta \omega_q(B) = \sqrt{\frac{2Tq^3}{m^2c}} \]  (13)

in full agreement with Andreev’s results [32]. The numerical calculations in Ref. [33] based on the Gross-Pitaevskii equation confirm results [12] and [13].

While the calculations leading to Eqs. (12) and (13) are purely classical, based on the hydrodynamic Lagrangian [5], the calculations of Ref. [19] leading to Eq. (3) are essentially quantum, based on the quantization of bosonic fields using the commutation relations \[ [\hat{\phi}(x), \hat{\rho}(x')] = i\delta(x - x') \] and the introduction of the fermionic operator

\[ \psi_+(x) \sim e^{-i\chi(x)} \]  (14)

and similarly for \[ \psi_- [5]. \] The density can be expressed in terms of these operators as \[ \hat{\rho} = \sqrt{K}(\psi_+^\dagger \psi_+ + \psi_+ \psi_-), \] though only right-moving fermions contribute in our case. The fermionic operators [14] create “kinks” of magnitude \[ \sqrt{K} \] and \[ 1/\sqrt{K} \] in the fields \[ \theta \] and \[ \varphi \], correspondingly. Such kink configurations can be described semiclassically as “mobile impurities” (even in the absence of foreign particles) and were recently studied in Ref. [22].

In the integrable case of the Lieb-Liniger model, one can associate the impuritylike excitations with two excitation modes, Lieb I and Lieb II. \[ E_{I,I}(p) \] predicted by the Bethe ansatz solution [33]. For small momenta \[ E_{I,II}(p) = cp \pm p^2/2m^* [5] \] allowing the interpretation of \[ E_I \] as a particle and \[ E_{II} \] as a hole, the excitations created on top of the Fermi ground state filled for \[ p < 0 \] using the standard prescription \[ E_{I,II}(p) = \pm \varepsilon_+(\pm p) \].

The dispersion law of the right-moving fermions [34],

\[ \varepsilon_+(p) = cp + \frac{p^2}{2m^*} \]  (15)

contains the parameters \( c \) and \( m^* \), the same as the ones entering the hydrodynamic Lagrangian [5] and [3], as both phononic and fermionic descriptions must provide the same equilibrium properties of the system [5] [30].

In the TG regime, the fermions are exact excitations, and Eq. (15) is valid for all momenta. It remains valid even in the weakly interacting regime but for restricted momenta \[ |p| < p_G = (m/m^*)mc \sim mc/\sqrt{K} [5] \]. This puts a further limitation on the temperatures \[ T < T_G = cp_G \sim mc^2/\sqrt{K} \] for the validity of our approach. For \[ |p| > p_G \], the excitations can be treated semiclassically: the Lieb I and II modes cross over into the Bogoliubov mode and gray soliton mode respectively [35].

In the integrable case, the fermionic excitations are exact eigenstates of the system, so they have an infinite lifetime. Away from integrability, they experience backscattering from phononic excitations providing the mechanism for their decay. In the low-temperature limit, the backscattered phonons have very small momenta to transfer to the impurity, and the corresponding amplitude reduces to a function \[ \Gamma_{p} \] of the impurity momentum \( p \) only and can be calculated phenomenologically from the dependence of the dispersion [15] on the background density \( n [22] \). If, in addition, the momentum of the impurity is small, the amplitude \( \Gamma_p \approx \Gamma_0 \), so the backscattering is characterized by only one parameter, \( \Gamma_0 \).

To derive the DSF, we assume the system to be only slightly nonintegrable, so we can rely on perturbation theory in \( \Gamma_0 \) and employ the picture of the near integrable Bose liquid as a collection of phonons described by Eq. [8] and fermionic quasiparticles treated as a dilute gas of mobile impurities with a Lagrangian density

\[ \mathcal{L}_+^t = \psi_+^\dagger \left[ i\partial_t - \varepsilon_+ (-i\partial_x) \right] \psi_+. \]  (16)

The term describing low-energy interactions between fermions and phonons can be written as

\[ \mathcal{L}_{ph-F}^t = \frac{-i\Gamma_0}{2\pi n^*} \partial_x \chi_+ \partial_x \chi_+ \left( \psi_+^\dagger \partial_x^2 \psi_+ \right), \]  (17)

where we have introduced the symmetrized operator \[ \partial_x^2 = (1/4) (\partial_x^2 - 2\partial_x \partial_+ \partial_- + \partial_+^2) \] with arrows denoting the derivatives acting on the operators to the left or right. Operator hats are omitted for clarity.

In the lowest (second) order, the interactions, corresponding to Eq. (17) modify fermionic and phononic propagators. The corresponding self-energies \[ \Sigma_+(k, \varepsilon) \] and \[ \Pi_+(k, \varepsilon) \] are shown in Fig. 2. Their imaginary parts determine the rate of dissipative processes. As expected, these processes consist of collisions of right-moving fermions with left-moving thermal phonons during which the phonons change their direction.

For short time scales, phonons do not participate in the dynamics and only provide a damping mechanism for fermionic quasiparticles. The dynamics at these time scales is described by a kinetic equation [18] for small deviations \[ \delta f_+(x, p; t) = f_+(x, p; t) - f_+(p) \] of the fermionic distribution function from its equilibrium form \[ f_+(p) = 1 - 2n_F(E_+(p)) \approx \tanh(cp/2T). \] Making the scattering time approximation for the linearized collision integral we have

\[ \left[ \partial_t + \left( c + \frac{p}{m^*} \right) \partial_x \right] \delta f_+ = -\delta f_+ / \tau(p). \]  (18)

The time scale \( \tau(p) \) is given by the imaginary part of the retarded component of the fermionic self-energy matrix in the Keldysh space [36]

\[ \frac{1}{\tau(p)} = -2\Im \chi_+(p, E_+(p)) = \frac{\left[ \Gamma_0 \right]^2 T}{2\pi n^*c^3} \Im \left( \frac{cp}{2T} \right) \]  (19)

obtained from the corresponding diagram in Fig. 2. The dimensionless function \( I(y) = \int_{-\infty}^{+\infty} dx x(x +...
For long times, $t > \tau$, the fermionic quasiparticles decay, $\delta f_+ \to 0$, and the dynamics of phonons is governed by Eq. (11). The sole role of the fermionic quasiparticles is now to provide a thermal bath for the phonons. By calculating the diagram on the right in Fig. 2 we get the dissipation and noise terms in Eq. (11) from the imaginary part of the retarded phononic self-energy $\Pi^R(q, \omega)$ [38]. The dissipation is proportional to the diffusivity constant

$$D = -\frac{\pi}{q^2} \Im \Pi^R(q, cq) = \frac{224}{15\pi^2} \left( \frac{c}{T} \right)^2 \frac{1}{\tau} \quad (24)$$

confirming our statement that $\tau$ is the shortest time scale for the applicability of the hydrodynamic approach. The distribution of noise in Eq. (11) follows from the fluctuation-dissipation theorem.

In conclusion, we have shown that the hydrodynamic description for 1D bosons is only valid for times longer than $\tau$. The latter diverges as one approaches the integrable point, and a non-hydrodynamic behavior due to fermionic quasiparticles prevails. Using $\alpha = 12\ln(4/3)\hbar a^2\omega_\perp$ [37], where $a$ is the scattering length and $\omega_\perp$ is the frequency of transverse confinement used to create a one-dimensional system, we can recast Eq. (4) into dimensionless form (reintroducing the Planck and Boltzmann constants)

$$\frac{\hbar}{\tau m^* c^2} = A \left( \frac{m^* c^2}{\hbar \omega_\perp} \right)^2 \left( \frac{k_B T}{m^* c^2} \right)^2,$$  

where the numerical factor $A = 9\pi^2 (\ln(4/3))^2/2^{15} \approx 0.07$. For experiments in Ref. [7], $K \sim 10$, $m^*/\hbar \sim 5.5\text{kHz}$ and $\omega_\perp \sim 66\text{kHz}$, which results in $\tau$ of the order of tens of seconds even for $k_B T/m^* c^2 = 1$. Because of the $T^7$ dependence, decreasing the temperature will result in even longer relaxation times. This makes the fermionic ballistic result Eq. (22) the only one likely to be observed [35].

In contrast to the purely hydrodynamic approach of Ref. [11] and the purely fermionic approach of Ref. [19], our treatment is based on considering both phononic and fermionic excitations of the liquid. This is justified a posteriori as fermionic and phononic fields fluctuate on very different time scales.

For the important special case of a linear spectrum, $1/m^* = 0$, the scenario described above is not correct as $1/\tau = 0$ in this case. Nevertheless, the linearized (free) hydrodynamic description holds on all time scales and is at the origin of standard bosonization approach [39]. However, in this case, the fermionic description is valid as well. The equivalence between the two is due to the simple fact that fermionic wave packets do not disperse, and their form can be parametrized by bosonic fields at all times.

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SUPPLEMENTAL MATERIAL

Kinetics of mobile impurities and correlation functions in one-dimensional superfluids at finite temperature

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I. DERIVATION OF FERMION-PHONON COUPLING

In a quantum liquid, a single impurity interacting with a phononic bath was studied in Ref. [1]. It was shown that it can be described by the universal Lagrangian

$$L = P \dot{X} - H(P, \Lambda) + \Lambda^j \frac{d}{dt} \chi(X, t) .$$

(I.1)

The Hamiltonian $H(P, \Lambda)$ describes the dynamics of the impurity with momentum $P$ together with the depletion of the quantum liquid in its closest vicinity. In addition to the momentum $P$ and coordinate $X$, the state of such a system is described by two parameters $\Lambda^j = (\Lambda_+, \Lambda_-)$. Rewriting $2\Lambda_{\pm} = N/\sqrt{K} \pm (\Phi/\pi)\sqrt{K}$ one can associate the variable $N$ with the number of particles expelled from the vicinity of the impurity and $\Phi$ with the superfluid phase drop. Here the dynamics of the phononic field $\chi(x, t)$ is governed by the Lagrangian $\mathcal{L}_{\text{ph}} + \mathcal{L}'_{\text{ph}}$, Eqs. (8) and (9) of main text. This picture applies equally to a foreign particle, for example an atom in a different hyperfine state considered in Refs. [1, 2] as well as to localized excitations in the superfluid itself, like grey solitons in BEC [3, 4]. The last term in Eq. (I.1) represents universal interactions of the impurity with the phononic fields $\chi^\dagger = (\chi_+, \chi_-)$.

In the absence of phonons the depletion cloud variables take their equilibrium values $\Lambda = \Lambda^*(P)$ obtained from the conditions $\nabla_{\Lambda} H = 0$. Equivalently

$$P = n\Phi^* - mN^* V(P) ,$$

$$\frac{\partial E(P)}{\partial n} = \frac{mc^2}{n} N^* - V(P) \Phi^* ,$$

which must be solved at constant $P$. Here $n$ is linear density of the background, $V(P) = \partial E/\partial P$ is the velocity and $E(P) = H(P, \Lambda^*(P)) = cP + P^2/2m^*$ is the dispersion law of the impurity.

In the presence of long wavelength phonons the depletion cloud parameters change. We can, however, assuming that the change due to phononic background is small, expand $H(P, \Lambda)$ around $\Lambda^*$,

$$H(P, \Lambda^* + \delta \Lambda) \simeq E(P) + \delta \Lambda^1 (2\pi \Gamma^{-1}_P) \delta \Lambda ,$$

(I.2)

and integrate out the fluctuations $\delta \Lambda$. This results in effective Lagrangian of an impurity coupled to the bath of phonons:

$$L = P \dot{X} - E(P) + \Lambda^1 \frac{d}{dt} \chi(X, t) + \frac{1}{8\pi} \left( \frac{d}{dt} \chi^\dagger(X, t) \right) \Gamma_P \left( \frac{d}{dt} \chi(X, t) \right) .$$

(I.3)

The matrix $\Gamma^{-1}_P$ is the Hessian,

$$\Gamma^{-1}_P = \begin{pmatrix} \Gamma_{P,++} & \Gamma_{P,+-} \\ \Gamma_{P,-+} & \Gamma_{P,-} \end{pmatrix} = \frac{1}{4\pi} \begin{pmatrix} H_{\Lambda_+\Lambda_+} & H_{\Lambda_+\Lambda_-} \\ H_{\Lambda_-\Lambda_+} & H_{\Lambda_-\Lambda_-} \end{pmatrix} \Lambda = \Lambda^* ,$$

evaluated at constant $P$ (see also Ref. [1]). It characterizes the scattering rate of phonons by a moving impurity. It will be shown below that at small momenta $\Lambda^*(P)$ is independent of $P$, so that the third term on the right of Eq. (I.3) is a total derivative and can be omitted. The full time derivative of phonon fields can be evaluated with the help of the equation of motion of the impurity, $\dot{X} = V(P)$, leading to $d\chi(X, t)/dt \simeq (\partial_t + V(P)\partial_x) \chi$. Then the Lagrangian becomes

$$L = P \dot{X} - E(P) + \frac{1}{8\pi} \left( \partial_t + V(P)\partial_x \right) \chi^\dagger \Gamma_P \left( \partial_t + V(P)\partial_x \right) \chi ,$$

(I.4)
Evaluating the last term of Eq. (I.4) with the help of the equations of motion of phonons, $\partial_t \chi_{\pm} = \mp c \partial_x \chi_{\pm}$, we find
\[
\frac{d}{dt} \chi_+ \Gamma_{p,+} \frac{d}{dt} \chi_+ = \left( \frac{P}{m^*} \right)^2 \partial_x \chi_+ \Gamma_{p,+} \partial_x \chi_+ ,
\]
\[
\frac{d}{dt} \chi_- \Gamma_{p,-} \frac{d}{dt} \chi_- = \frac{d}{dt} \chi_+ \Gamma_{p,+} \frac{d}{dt} \chi_- = \left( \frac{P}{m^*} \right) \left( \frac{2 c + \frac{P}{m^*}}{2} \right) \partial_x \chi_+ \Gamma_{p,+} \partial_x \chi_+ ,
\]
\[
\frac{d}{dt} \chi_- \Gamma_{p,-} \frac{d}{dt} \chi_- = \left( 2 c + \frac{P}{m^*} \right)^2 \partial_x \chi_- \Gamma_{p,-} \partial_x \chi_- \simeq 4 c^2 \partial_x \chi_- \Gamma_{p,-} \partial_x \chi_- ,
\]
where we have simplified the expression leaving only the leading terms in $P/m^* c \ll 1$. The last term, being the largest, does not contribute, however to the physical quantities of interest (like the dissipation) since the corresponding scattering process is suppressed by the conservation of energy and momentum. We also neglect the first term as being small in the low energy limit. Moreover, since only vertices proportional to $\Gamma_{p,+}$ and $\Gamma_{p,-}$ are left, we denote $\Gamma_{p,+} = \Gamma_{p,-} \equiv \Gamma_{p}$. As it is shown in Sec. VI, for small momenta $\Gamma_{p} \simeq \Gamma_{p,0}^p$.

So far only right-moving impurities have been considered. We can easily extend the discussion to left-moving impurities by means of the substitutions $E(P) \rightarrow E_1(\pm P) = \pm c P + P^2/2m^*$ and $V(P) \rightarrow V_1(\pm P) = \partial E_{\pm}/\partial P$ in the previous results, where $+$ and $-$ labeled right- and left-moving impurities. It can also be shown that $\Gamma_{p,0}^{+} = \Gamma_{p,0}^{-}$ and $\Gamma_{p,0}^{\pm} = \Gamma_{p,0}^{\mp}$. It follows that in second quantization the Lagrangian of right/left chiral fermions, Eq. (I.4), corresponds to the Lagrangian density $\mathcal{L}_{F}^{\pm} + \mathcal{L}_{ph-F}^{\pm}$ with
\[
\mathcal{L}_{F}^{\pm} = \bar{\psi}_{\pm} \left[ \psi_{\pm} - \epsilon^\pm \left( -i \partial_x \psi \right) \right] \psi_{\pm} ,
\]
\[
\mathcal{L}_{ph-F}^{\pm} = \pm \frac{1}{2 \pi m^*} \Gamma_{p,0}^{\pm} \partial_x \psi^\alpha \partial_x \psi^\beta \left( \bar{\psi}_a \left( \mathcal{T}P \right)^2 \psi_b \right) .
\]

Here $\epsilon_{\pm}(P)$ is defined in Eq. (15) of the main text and we defined $\mathcal{T}P$ as the symmetric version of the momentum operator, $\bar{\psi} \mathcal{T}P \psi = \frac{1}{2} \left[ \left( \partial_x \bar{\psi} \right) \psi - \bar{\psi} \left( \partial_x \psi \right) \right]$, in order for it to be Hermitian. The respective term in Eq. (I.6) is explicitly given by $\bar{\psi} \left( \mathcal{T}P \right)^2 \psi = -\frac{1}{2} \left[ \left( \partial_x^2 \bar{\psi} \right) \psi - 2 \left( \partial_x \bar{\psi} \right) \left( \partial_x \psi \right) + \bar{\psi} \left( \partial_x^2 \psi \right) \right]$.

\section*{II. RULES FOR FEYNMAN DIAGRAMS}

In this section we derive the Feynman rules in the Keldysh formalism for the action
\[
S = \int \mathcal{C} dt \! \! \! \int dx \left( \mathcal{L}_{F}^{+} + \mathcal{L}_{ph} + \mathcal{L}_{ph-F}^{+} + \mathcal{L}_{ph-F}^{-} \right) ,
\]
\[\text{(II.1)}\]
where $\mathcal{C}$ is the Keldysh contour. We perform Keldysh rotation
\[
\left( \begin{array}{c} \chi_+ \\ \chi_- \end{array} \right) = \left( \begin{array}{cc} 1 & 1 \\ 1 & -1 \end{array} \right) \left( \chi^e \right) ,
\]
\[\text{(II.2)}\]
for bosonic fields and
\[
\left( \begin{array}{c} \psi_+ \\ \psi_- \end{array} \right) = \frac{1}{\sqrt{2}} \left( \begin{array}{cc} 1 & 1 \\ 1 & -1 \end{array} \right) \left( \begin{array}{c} \psi_1 \\ \psi_2 \end{array} \right) \), \quad \left( \psi^+, \psi^- \right) = \left( \psi_1, \psi_2 \right) \times \frac{1}{\sqrt{2}} \left( \begin{array}{cc} 1 & -1 \\ 1 & 1 \end{array} \right) \)
\[\text{(II.3)}\]
for fermionic ones. In these expressions $\pm$ superscripts denote the forward/backward parts of the Keldysh contour and we have suppressed the chirality indices as the expressions are identical for right/left moving fermions and bosons. As interactions between left and right fermions do not appear in the lowest order of perturbation theory, we concentrate here on right-moving fermions only and often omit the chirality index for clarity. In contrast, as both right and left moving phonons are excited via interaction with right-moving fermions, we indicate their chirality indices explicitly. We will also use the shorthand notation $k$ for $(k, \xi)$ and $q$ for $(q, \omega)$ and omit the cut-off on both momenta. The propagators of fermions become matrices in Keldysh space
\[
\begin{pmatrix} \pm \\ k \end{pmatrix} = G_{\pm}(k) = \begin{pmatrix} \mathcal{G}_{\pm}^R(k, \xi) & \mathcal{G}_{\pm}^K(k, \xi) \\ 0 & \mathcal{G}_{\pm}^A(k, \xi) \end{pmatrix} .
\]
\[\text{(II.4)}\]
For phonons we have similarly
\[ \begin{pmatrix} + \end{pmatrix} = D_{+}(q) = \begin{pmatrix} D^{K}(q, \omega) & D^{R}(q, \omega) \\ D^{R}(q, \omega) & 0 \end{pmatrix} \] (II.5)
and
\[ \begin{pmatrix} - \end{pmatrix} = D_{-}(q) = \begin{pmatrix} D^{K}(q, \omega) & D^{R}(q, \omega) \\ D^{R}(q, \omega) & 0 \end{pmatrix} \] (II.6)

The retarded and advanced propagators are given by
\[ G^{R}_{\pm}(k, \varepsilon) = [G^{A}_{\pm}(k, \varepsilon)]^{\dagger} = \frac{1}{\varepsilon - \varepsilon_{\pm}(k) + i0} , \]
\[ D^{R}_{\pm}(q, \omega) = [D^{A}_{\pm}(q, \omega)]^{\dagger} = \frac{\pi}{\pm(q(\omega \mp eq + i0))} . \] (II.7)

In equilibrium the Keldysh propagators are given by
\[ G^{K}_{\pm}(k, \varepsilon) = F_{\pm}(k)\Delta_{f\pm}(k, \varepsilon) , \]
\[ D^{K}_{\pm}(q, \omega) = F_{\pm}(q)\Delta_{b\pm}(q, \omega) , \] (II.8)
where the density functions are \( F_{\pm}(k) = \tanh(\frac{\varepsilon_{\pm}(k)}{2T}) \) and \( F_{\pm}(q) = \coth(\frac{\varepsilon_{\pm}(q)}{2T}) \) and we have defined
\[ \Delta_{f\pm}(k, \varepsilon) = G^{R}_{\pm}(k, \varepsilon) - G^{A}_{\pm}(k, \varepsilon) = -2\pi i \delta(\varepsilon - \varepsilon_{\pm}(k)) , \]
\[ \Delta_{b\pm}(q, \omega) = D^{R}_{\pm}(q, \omega) - D^{A}_{\pm}(q, \omega) = T i \delta(\omega - \omega_{\pm}(q)) . \] (II.9)

The vertex for the interaction between right chiral fermions and phonons in Eq. (I.6) is given by
\[ L_{ph-F}^{+} = \frac{1}{2\pi m^{*}} \frac{e}{m} \Gamma_{0}^{\prime} \bar{\psi}^{a}_{\alpha}(\bar{F}^{2})^{2}\psi_{\beta}(k) = \Gamma^{ac}_{\beta\gamma}(k; q', q) = \]
\[ = \frac{i}{8\pi m^{*}} \Gamma_{0}^{\prime} \gamma^{ab}_{\alpha\beta} q q'(k + k')^{2} (2\pi)^{2} \delta^{2}(k + q - k' - q') . \]

Here we have introduced Keldysh tensors \( \hat{\gamma}^{ab}_{\alpha\beta} , \hat{\gamma}^{ab}_{\alpha\beta} ; a, b = 1, 2, \alpha, \beta = cl, q \) with components
\[ \hat{\gamma}_{cl} = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} , \quad \hat{\gamma}_{q} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \quad \hat{\gamma}_{clq} = \hat{\gamma}_{qq} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} , \quad \hat{\gamma}_{clq} = \hat{\gamma}_{qcl} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} . \] (II.10)

## III. Calculation of Self-Energies

In this section we will calculate the self-energy of fermions and phonons at the lowest order in \( \Gamma_{0}^{\prime} \). In the Keldysh formalism they are given respectively by
\[ \Sigma_{\pm}(k) = \begin{pmatrix} \Sigma^{R}_{\pm}(k, \varepsilon) & \Sigma^{K}_{\pm}(k, \varepsilon) \\ \Sigma^{R}_{\pm}(k, \varepsilon) & \Sigma^{A}_{\pm}(k, \varepsilon) \end{pmatrix} \]
and
\[ \Pi_{\pm}(q) = \begin{pmatrix} \Pi^{R}_{\pm}(q, \omega) & \Pi^{K}_{\pm}(q, \omega) \\ \Pi^{R}_{\pm}(q, \omega) & \Pi^{A}_{\pm}(q, \omega) \end{pmatrix} \]
and the corresponding diagrams are shown in Fig. III.1.
Figure III.1: Fermion and boson self-energies in the 2nd order perturbation theory.

### A. Fermion self-energy and life-time

The fermion self-energy is depicted in Fig. III.1(a) and reads

\[
-i\Sigma_{\pm}^{ab}(k) = \int \frac{d^2q}{(2\pi)^2} \frac{d^2q'}{(2\pi)^2} \Gamma^{\alpha\beta}_\pm(k+q-q', k; q', q) i\delta^{ac}(k+q-q') iD^{\beta\delta}_\pm(q') \\
\times \Gamma^{\delta\alpha}_\pm(k, k+q-q'; q, q') iD^{\alpha\beta}_\pm(q) \quad (III.1)
\]

where the factor 1/2 coming from the second order perturbation theory is cancelled by the factor 2 due to the symmetry by exchange of the vertices. The retarded component is given by

\[
\Sigma_{\pm}^{R}(k) = \Sigma_{\pm}^{1}(k) = -\frac{1}{(8\pi)^2} \frac{\Gamma^{\alpha\beta}_0}{(m^*)^2} \int \frac{d^2q}{(2\pi)^2} \frac{d^2q'}{(2\pi)^2} (qq')^2 (2k + q - q')^4 \\
\times \left[ G^{\alpha\beta}_R(k + q - q') \left( D^{\beta\delta}_R(q') D^{\delta\alpha}_R(q) - \left( D^{\beta\delta}_R(q') - D^{\delta\alpha}_R(q') \right) \left( D^{\beta\delta}_R(q) - D^{\delta\alpha}_R(q) \right) \right] + \\
+ D^{\alpha\beta}_R(k + q - q') \left( D^{\beta\delta}_R(q') D^{\delta\alpha}_R(q) + D^{\beta\delta}_R(q') D^{\delta\alpha}_R(q) \right) \right] .
\]

The Keldysh component is given by

\[
\Sigma_{\pm}^{K}(k) = \Sigma_{\pm}^{2}(k) = -\frac{1}{(8\pi)^2} \frac{\Gamma^{\alpha\beta}_0}{(m^*)^2} \int \frac{d^2q}{(2\pi)^2} \frac{d^2q'}{(2\pi)^2} (qq')^2 (2k + q - q')^4 \\
\times \left[ G^{\alpha\beta}_K(k + q - q') \left( D^{\beta\delta}_K(q') D^{\delta\alpha}_K(q) - \left( D^{\beta\delta}_K(q') - D^{\delta\alpha}_K(q') \right) \left( D^{\beta\delta}_K(q) - D^{\delta\alpha}_K(q) \right) \right] + \\
- D^{\alpha\beta}_K(q') \left( G^{\beta\delta}_K(k + q - q') - G^{\alpha\beta}_K(k + q - q') \right) \left( D^{\beta\delta}_K(q) - D^{\delta\alpha}_K(q) \right) + \\
+ D^{\alpha\beta}_K(q) \left( G^{\beta\delta}_K(k + q - q') - G^{\alpha\beta}_K(k + q - q') \right) \left( D^{\beta\delta}_K(q') - D^{\delta\alpha}_K(q') \right) \right] .
\]

In order to calculate the life-time and the kinetic equation we need the quantities

\[
\Sigma_{\pm}^{R}(k) = O_f[K_{f\pm}](k), \quad K_{f\pm}(k, k'; q, q') = F_{f\pm}(k') (F_{b\mp}(q') F_{b\pm}(q) - 1) - F_{b\mp}(q') + F_{b\pm}(q) , \\
2i\delta\Sigma_{\pm}^{R}(k) = O_f[T_{f\pm}](k), \quad T_{f\pm}(k, k', q, q') = F_{b\mp}(q') F_{b\pm}(q) - 1 + F_{f\pm}(k') (F_{b\pm}(q) - F_{b\mp}(q')) ,
\]

where, employing Eq. (II.8), we defined

\[
O_f[\mathcal{F}](k) = -\frac{1}{(8\pi)^2} \frac{\Gamma^{\alpha\beta}_0}{(m^*)^2} \int \frac{d^2q}{(2\pi)^2} \frac{d^2q'}{(2\pi)^2} \frac{d^2k'}{(2\pi)^2} (qq')^2 (k + k')^4 \delta_{\pm}(k + q - q' - k') \\
\times \Delta_{f\pm}(k') \Delta_{b\mp}(q') \Delta_{b\pm}(q) \mathcal{F}_{f\pm}(k, k'; q, q') .
\]

(III.2)
Here we imposed the mass shell condition on the external energy as well since the quantities $O_f$’s that we will consider will always appear multiplying $\Delta$. The subscript $\pm$ of the delta function in $O_f$ refers to the energy conservation for right/left chiral fermions. Substituting the explicit values $\Delta_{f,\pm}$, $\Delta_{k,\pm}$, Eq. (II.9), and integrating over $d\omega$, $d\omega'$, $d\varepsilon'$ we find

$$O_{f,\pm} [F](k) = \frac{i}{128\pi} \frac{\Gamma_0^2 e^{2}}{(m^*)^2} \int dq dq' dk' \delta(\varepsilon_{\pm}(k) + \omega_{\pm}(q) - \omega_{\pm}(k') - \varepsilon_{\pm}(k')) \delta(k + q - q' - k') q q' (k + k')^4 F_{f,\pm}(k, k'; q, q').$$

By integrating over $dk'$, making the shift $q' = q - \tilde{q}$ and integrating $dq$ out, we find

$$O_{f,\pm} [F](k) = \pm \frac{i}{128\pi} \frac{\Gamma_0^2}{(2m^*)^3} \int dq q^2 (2k + q)^5 \left[ 1 \pm \frac{2k + q}{4m^* c} \right] F_{f,\pm}(k, k + q; q \pm q \frac{2k + q}{4m^* c}),$$

where in the last line we renamed the dummy variable $\tilde{q}$ to $q$. At the lowest order in $k, q \ll m^* c$ (i.e., non-linearity of the spectrum) we have

$$O_{f,\pm} [F](k) \simeq \pm \frac{i}{128\pi} \frac{\Gamma_0^2}{(2m^*)^3} \int dq q^2 (2k + q)^5 F_{f,\pm}(k, k + q; q \pm q \frac{2k + q}{4m^* c}). \quad (III.3)$$

The last argument of $F_{f,\pm}$ is the momentum of the left-moving (right-moving) phonon in the scattering process which is small due to the conservation of energy and momentum. Indeed it would be zero in the case of a linear dispersion ($1/m^* = 0$), and the addition of a weak non-linearity ($q, k \ll m^* c$) makes it parametrically small (see Fig. III.2a).

The lifetime of fermions is given by

$$\frac{1}{\tau(k)} = -2\Sigma^R(k) = iO_f[T_f,\pm](k),$$

where we omitted the chiral indices as the result will be independent of them. At equilibrium

$$T_{f,\pm}(k, k'; q, q') = \left[ \coth \left( \frac{cq'}{2T} \right) + \coth \left( \frac{cq}{2T} \right) \right] \left[ \tanh \left( \frac{ck'}{2T} \right) - \coth \left( \frac{c(q + q')}{2T} \right) \right].$$

In the limit $k, q \ll m^* c$, the terms with the divergent factor $\coth (cq'/2T)$ dominate as a consequence of the conservation of energy and momentum and, substituting into Eq. (III.3), the expression simplifies to

$$\frac{1}{\tau(k)} = \frac{1}{2\pi} \frac{T^2 \Gamma_0^2}{m^* c^3} I \left( \frac{ck}{2T} \right) = \frac{\pi^5}{128} \frac{T^2 \Gamma_0^2}{(m^* c)^2 c^2} + \left( \frac{103\pi^3}{1920} \frac{T^2 \Gamma_0^2}{(m^* c)^2} \right) k^2 + o(k^4),$$

$$I(y) = \int_{-\infty}^{+\infty} dx x (x + 2y)^4 \left[ \coth(x) - \tanh(x + y) \right]. \quad (III.4)$$
In the leading order in $k$ it results in Eq. (4) of the main text.

The life-time of excitations in the system of nonlinear interacting fermions was investigated in [5, 6]. Their result for a non-thermal particle scales as $k^6$ where $k$ is the momentum of the quasiparticle above Fermi surface. Substituting signum functions for distribution functions,

$$\tanh \left( \frac{E}{2T} \right) \rightarrow \text{sgn}(E),$$

$$\coth \left( \frac{\omega}{2T} \right) \rightarrow \text{sgn}(\omega),$$

we can verify this zero temperature result.

$$\frac{1}{\tau(k)} = -\frac{\Gamma^2 c^2}{256\pi m^2} \int dq dq' \delta \left( q' - \frac{q(2k + q)}{4m^2 c} \right) \times \times q q' (2k + q)^4 \left[ \text{sgn}(q) - \text{sgn}(ck + cq) \right] + \left[ \text{sgn}(ck + cq) \text{sgn}(cq) - 1 \right],$$

where we have neglected the curvature of the spectrum, as well as $q'$, in the distribution functions, but retained it inside the energy delta–function. We have also conveniently rearranged the distribution functions such that the expressions in curly brackets are non–zero only when $-k < q < 0$. Since $\text{sgn}(-q') = \text{sgn}( -(2k + q)/q ) = 1$ in that region, the value of the expression in square brackets gives a factor of $-4$ and the decay rate is given by

$$\tau^{-1}(k) = \frac{\Gamma^2 c^2}{128\pi m^2} \int dq \frac{q^2 (2k + q)^5}{2m^2 c^3} = \frac{73\Gamma^2 c^2 k^8}{14336\pi m^3}.$$

In the case of thermal quasiparticles the temperature dependence $T^3$, see Eq. (III.4), originates from the fact that the energy of the left moving phonon $q'q$ is controlled by first power of temperature $T$, in contrast to second power of the incoming momentum in Refs. [5, 6].

### B. Boson self-energy and diffusivity

The boson self-energy is depicted in Fig. III.1(b) and reads

$$-i\Pi_{\pm,\alpha\beta}(q) = -\int \frac{dk}{(2\pi)^2} \frac{dk'}{(2\pi)^2} \Gamma^0_{\alpha\beta}(k', k; q + k - k', q) iG_+^{cf}(k') iD_\pm^\delta(q + k - k')$$

$$\times \times i\Gamma^0_{\delta\alpha}(k, k'; q, q + k - k') iG_+^{cf}(k)$$

$$= \frac{i}{(8\pi)^2} \Gamma^0_{\alpha\beta} \frac{c^2}{(m^*)^2} \int \frac{dk}{(2\pi)^2} \frac{dk'}{(2\pi)^2} q^2 (q + k - k')^2 (k + k')^4 \left[ \gamma_{\alpha\beta} G_+^{cf}(k') D_\pm^\delta(q + k - k') \gamma_{\delta\alpha} G_\pm^{cd}(k) \right],$$

(III.5)

where the minus sign in the first line is given by the fermionic loop and, again, the factor 1/2 coming from the second order perturbation theory is cancelled by the factor 2 due to the symmetry by exchange of the vertices. The retarded component is given by

$$\Pi^R_{\pm}(q) = \Pi_{\pm,qq'}(q) = \frac{1}{(8\pi)^2} \Gamma^0_{\alpha\beta} \frac{c^2}{(m^*)^2} \int \frac{dk}{(2\pi)^2} \frac{dk'}{(2\pi)^2} q^2 (q + k - k')^2 (k + k')^4$$

$$\times \left[ \left( D_+^\delta(q + k - k') - G_+^{ck}(k') \right) G_\pm^{R}(k) + G_\pm^{ck}(k') \right] (\gamma_{\alpha\beta} G_+^{cf}(k') D_\pm^\delta(q + k - k') \gamma_{\delta\alpha} G_\pm^{cd}(k) \right).$$

The Keldysh component is given by

$$\Pi^K_{\pm}(q) = \Pi_{\pm,qq'}(q) = \frac{1}{(8\pi)^2} \Gamma^0_{\alpha\beta} \frac{c^2}{(m^*)^2} \int \frac{dk}{(2\pi)^2} \frac{dk'}{(2\pi)^2} q^2 (q + k - k')^2 (k + k')^4$$

$$\times \left[ \left( D_+^\delta(q + k - k') \right) \left( G_\pm^{ck}(k') \right) - G_\pm^{ck}(k') \right] \left( G_+^{R}(k) - G_\pm^{ck}(k') \right)$$

$$\gamma_{\alpha\beta} G_+^{cf}(k') D_\pm^\delta(q + k - k') \gamma_{\delta\alpha} G_\pm^{cd}(k) \right].$$
where we used the causality structure of the Green’s functions. In order to calculate the diffusion coefficient and the effective action of bosons we need the quantities

\[ \Pi^K_\pm(q) = O_b[K_{b\pm}](q), \quad K_{b\pm}(k,k';q,q') = \frac{F_{\mp}(q')}{F_{f\pm}(k')} F_{f\pm}(k) - 1 - F_{f\pm}(k') + F_{f\pm}(k), \]

\[ 2i\Im \Pi^R_\pm(q) = O_b[T_{b\pm}](q), \quad T_{f\pm}(k,k';q,q') = \frac{F_{\mp}(q')}{F_{f\pm}(k')} (F_{f\pm}(k) - F_{f\pm}(k')) + F_{f\pm}(k') F_{f\pm}(k) - 1, \]

where we used Eq. (II.8) and defined

\[ O_{b\pm}[\mathcal{F}](q) = \frac{1}{(8\pi)^2} \frac{\Gamma^2_0 c^2}{(m^*)^2} \int \frac{d^2k}{(2\pi)^2} \frac{d^2k'}{(2\pi)^2} d^3q d^3q' \delta^2(q + k + k' - q') \]

\[ \times \Delta_{\mp}(q') \Delta_{f\pm}(k') \Delta_{f\pm}(k) \mathcal{F}_{b\pm}(q,q';k,k'). \]

Substituting the explicit values of \( \Delta_{f\pm}(k) \) and \( \Delta_{b\pm}(q) \), Eq. (II.9), and integrating over \( d\epsilon, d\epsilon', d\omega' \) we find

\[ O_{b\pm}[\mathcal{F}](q) = -\frac{i}{2(8\pi)^2} \frac{\Gamma_0^2 c^2}{(m^*)^2} \int dk dk' dq dq' (k + k')^4 \delta(\omega_{\pm}(q) + \epsilon_{\pm}(k) - \epsilon_{\pm}(k') - \omega_{\mp}(q')) \delta(q + k + k' - q') \mathcal{F}_{b\pm}(q,q';k,k'). \]

Integrating over \( dq' \), making the change of variables \( k \to (k + k')/2, k' \to (k - k')/2 \) and integrating over the new \( dk' \), the integral reduces to

\[ O_{b\pm}[\mathcal{F}](q) = -\frac{i}{256\pi^2} \frac{\Gamma_0^2 c^2}{(2m^*)^2} q^3 \int dk k^5 \mathcal{F}_{b\pm} \left( q, \pm \frac{qk}{4m^*c} \left( \frac{1}{1 \pm \frac{k}{4m^*c}} \right) \right). \]

At the lowest order in \( k, q \ll m^*c \), we have

\[ O_{b\pm}[\mathcal{F}](q) = -\frac{i}{256\pi^2} \frac{\Gamma_0^2 c^2}{(2m^*)^2} q^3 \int dk k^5 \mathcal{F}_{b\pm} \left( q, \pm \frac{qk}{4m^*c} \frac{k - q}{2} \right). \]  

(III.6)

Here the second argument of \( \mathcal{F}_{b\pm} \) represents the momentum of the left-moving (right-moving) boson in the scattering process. The momentum is parametrically small for the same reason explained in the fermionic case (see Fig. III.2b).

The diffusivity of phonons is given by

\[ D(q) = -\left( \frac{\pi}{q} \right)^2 \Im \Pi^R(q) = \pm \frac{\pi}{q} O_b[T_{b\pm}](k), \]

where we omitted the chiral indices as the result will be independent of them. At equilibrium

\[ \mathcal{T}_{b\pm}(q,q';k,k') = \left[ \coth \left( \frac{cq'}{2T} \right) - \coth \left( \frac{c(k' - k)}{2T} \right) \right] \left[ \tanh \left( \frac{ck'}{2T} \right) - \tanh \left( \frac{ck}{2T} \right) \right]. \]

In the limit \( k, q \ll m^*c \), the terms with the divergent factor \( \coth \left( \frac{cq}{2T} \right) \) dominate as a consequence of the conservation of energy and momentum and, substituting into Eq. (III.6), the expression simplifies to

\[ D(q) = -\frac{1}{2\pi} \frac{T\Gamma^2_0}{256(m^*)^2} \int \frac{dk k^4}{q^3} \left[ \tanh \left( \frac{ck + q}{2T} \right) - \tanh \left( \frac{ck - q}{2T} \right) \right]. \]

Using Eq. (A.3) for the second integral, we find [11]

\[ D(q) = \frac{7\pi^3}{120} \frac{T^5\Gamma^2_0}{(m^*c^2)^4} + o\left( q^2 \right). \]  

(III.7)

In the leading order in \( q \) it results in Eq. (24) of the main text.
IV. PHONON EFFECTIVE ACTION

Including the self energies, the Lagrangian of right phonons reads

\[ L[\chi] = \frac{1}{4} \int \frac{d^3q}{(2\pi)^3} \left\{ 2\chi_+^q \left( [D_++^{-1}]^R - i\beta^R \right) \chi_+^q - \chi_+^q \Pi^K_q \chi_+^q \right\}. \]

By means of the FDT we can obtain the Keldysh component of the self-energy,

\[ \Pi^K(q) = \coth \left( \frac{eq}{2T} \right) \Pi^R(q) - \Pi^A(q) = \coth \left( \frac{eq}{2T} \right) \left[ -i \left( \frac{q}{\tau} \right) \tau^{-1}(q) \right] = -\frac{2}{\pi} iDq^3 \coth \left( \frac{eq}{2T} \right), \]

where \( D \) is the constant part of \( D(q) \). The Lagrangian becomes

\[ L[\chi] = \frac{1}{2\pi} \int \frac{d^3q}{(2\pi)^3} \left\{ \chi_+^q \left( q(\omega - cq) + iDq^3 \right) \chi_+^q + \chi_+^q \left( iDq^3 \coth \left( \frac{eq}{2T} \right) \right) \chi_+^q \right\}. \]

The Fourier transform gives [7]

\[ L[\chi] = \frac{1}{2\pi} \int dx \left\{ -\partial_x \chi_+^q \left\{ (\partial_t + c\partial_x) - D\partial_x^2 \right\} \chi_+^q + i \frac{2T}{c} \left( \partial_x \chi_+^q \right)^2 + \frac{\pi T^2}{2} \int dx \partial_x^2 \chi_+^q(x) - \partial_x \chi_+^q(x') \right\}. \]

In the semiclassical limit the last term is zero and the remaining quadratic term in the quantum field can be split by means of the Hubbard-Stratonovich transformation and the Lagrangian becomes

\[ L[\chi] = -\frac{1}{2\pi} \int dx \left\{ \partial_x \chi_+^q \left\{ (\partial_t + c\partial_x) - D\partial_x^2 \right\} \chi_+^q - \partial_x \chi_+^q \xi \right\}, \]

where \( \xi \) is a Gaussian random force with second momentum given by \( \langle \xi(x)\xi(x') \rangle = 4\pi \frac{DT}{e} \delta(x - x') \). Finally, reintroducing the chiral indices, the equation of motion is

\[ (\partial_t \pm c\partial_x) \chi_\pm^q = \frac{\left( \partial_x \chi_\pm^q \right)^2}{2m^*} + D\partial_x^2 \chi_\pm^q + \xi, \]

where we introduced the non-linear term and obtain Eq. (11) of the main text.

V. KINETIC EQUATIONS FOR FERMIONS

The kinetic equations for fermions in the slightly inhomogeneous case is given by [7]

\[ [\partial_t + (\pm c + k/m) \partial_x] F_{f\pm}(x, k) = iI_{f\pm}(x, k), \]

where \( I \) is the collisional integral, given by

\[ I_{f\pm}(x, k) = \Sigma^R_{f\pm}(x, k) - 2iF_{f\pm}(x, k)\Sigma^A_{f\pm}(x, k), \]

and the functional dependence on \( F_{f\pm} \) and \( F_{b\pm} \) has been omitted. Since the system is slightly inhomogeneous, we can neglect the spatial derivatives in the collision integrals, and use the results of Sec. III. In terms of Eq. (III.2), \( I_{f\pm} = O_{f\pm}|T_{f\pm}| \), with

\[ T_{f\pm} = K_{f\pm} - F_{f\pm} |T_{f\pm}|. \]

To study the system in a state out of equilibrium we substitute \( F_{f\pm}(x, k) = \tanh \left( \frac{\epsilon(x,k)}{2T} \right) + f_{f\pm}(x, k) \), where \( f_{f\pm}(x, k) \) is the displacement from the equilibrium distribution, and consider only the linear approximation in the variations. Keeping only the linear terms we find the linearized kinetic equation

\[ (\partial_t + v_{f\pm}(k)\partial_x) f_{f\pm}(x, k) = -\frac{f_{f\pm}(x, k)}{\tau(k)} + \int dqG(k,q) f_{f\pm}(x, k + q), \quad (V.1) \]
where
\[ G(k, q) = \frac{1}{128\pi} \frac{T}{m^2} q(2k + q)^4 \left( \text{coth} \left( \frac{cq}{2T} \right) + \text{tanh} \left( \frac{ck}{2T} \right) \right). \]

The kinetic equation (18) in the main text is considered in the scattering time approximation, that is, only the first term in the right hand side of Eq. (V.1) is kept, with the term containing \( G(k, q) \) being negligible.

VI. CALCULATION OF THE BACK-SCATTERING AMPLITUDE

The backscattering rate can be calculated as the power series expansion in \( P \), up to the first order for low momenta
\[ \Gamma_P = \Gamma_{+-} = \Gamma_{-+} = -\frac{1}{c} \left( \Phi \frac{\partial N}{\partial P} - N \frac{\partial \Phi}{\partial P} + \frac{\partial N}{\partial n} \right) \approx \Gamma_0 + \Gamma'_0 P. \]

Expanding \( N(P) \) and \( \Phi(P) \) in Taylor series
\[ N(P) = N_0 + N_1 P + N_2 P^2 + \ldots, \]
\[ \Phi(P) = \Phi_0 + \Phi_1 P + \Phi_2 P^2 + \ldots, \]
and substituting in the expression for \( \Gamma_P \) one obtains
\[ \Gamma_0 = -\frac{1}{c} \left( \Phi_0 N_1 - N_0 \Phi_1 + \frac{\partial N_0}{\partial n} \right) \]
and
\[ \Gamma'_0 = -\frac{1}{c} \left( 2\Phi_0 N_2 - 2N_0 \Phi_2 + \frac{\partial N_1}{\partial n} \right). \]

The equilibrium values of \( N^*, \Phi^* \) (we drop asterisks from now on) can be obtained by inverting the equations for \( N \) and \( \Phi \):
\[ N(P) = \frac{1}{n} \frac{V(P)P + n\partial E(P)/\partial n}{c^2 - V^2(P)} \]
\[ \Phi(P) = \frac{P}{n} + \frac{mV(P)}{n} N(P). \]

Using the quadratic dispersion, one has
\[ V(P) = c + \frac{P}{m^*}, \]
\[ \frac{\partial E(P)}{\partial n} = \frac{\partial c}{\partial n} P - \frac{P^2}{2(m^*)^2} \frac{\partial m^*}{\partial n}. \]

Therefore,
\[ N_0 = -\frac{m^*}{2n} \left( 1 + \frac{n}{c} \frac{\partial c}{\partial n} \right) = -\sqrt{K} \]
\[ \Phi_0 = -\frac{m^* c}{2n} \left( 1 + \frac{n}{c} \frac{\partial c}{\partial n} \right) = -\frac{\pi}{\sqrt{K}}. \]
Here we have used the expression for effective mass \( m^\ast \) in the Luttinger liquid cited in [8, 9],
\[
\frac{1}{m^\ast} = \frac{c}{K} \frac{\partial}{\partial \mu} (c \sqrt{K}) = \frac{1}{2m} \frac{\pi}{mc} \left( 1 + \frac{n}{c} \frac{\partial c}{\partial n} \right).
\]

The first order coefficients are:
\[
N_1 = \frac{1}{2mc} \left( \frac{n}{2m^\ast} \frac{\partial m^\ast}{\partial n} + \frac{n}{c} \frac{\partial c}{\partial n} - \frac{1}{2} \right)
\]
\[
\Phi_1 = \frac{1}{2n} \left( \frac{n}{2m^\ast} \frac{\partial m^\ast}{\partial n} - \frac{n}{2c} \frac{\partial c}{\partial n} + \frac{1}{2} \right).
\]

The expression for \( \Gamma_0 \) then simplifies to
\[
\Gamma_0 = -\frac{m^\ast}{4mc} \left( 1 - \left( \frac{n}{c} \frac{\partial c}{\partial n} \right)^2 \right) - \frac{1}{c} \frac{\partial N_0}{\partial n}.
\]

Substituting the expression for \( m^\ast \) and using
\[
- \frac{\partial N_0}{\partial n} = \frac{\partial}{\partial n} \sqrt{\frac{\pi n}{mc}} = \sqrt{\frac{\pi}{mnc}} \left( 1 - \frac{n}{c} \frac{\partial c}{\partial n} \right),
\]
on one obtains that \( \Gamma_0 \) vanishes identically,
\[
\Gamma_0 = 0.
\]

\( \Gamma'_0 \) takes the form
\[
\Gamma'_0 = -\frac{1}{c} \left( 1 + \frac{n}{c} \frac{\partial c}{\partial n} \right) \frac{N_1}{n} + \frac{\partial N_1}{\partial n},
\]
where the second order coefficients cancel because of the relation between \( N(P) \) and \( \Phi(P) \):
\[
\Phi_2 = \frac{mc}{n} N_2 + \frac{m}{m^\ast n} N_1.
\]

For
\[
\mu(n) = gn, \quad c(n) = \sqrt{\frac{gn}{m}},
\]
the following expression for \( \Gamma'_0 \) is obtained:
\[
\Gamma'_0 = \frac{1}{16mc^n}. \quad (VI.2)
\]

However, we expect \( \Gamma_p \) to vanish identically for integrable systems, such as Lieb–Liniger model. The above non-zero result is an artefact of the mean-field approximation used for \( \mu(n) \). On the other hand, the contribution of the three-body collisions, for which
\[
\mu(n, \alpha) = gn + \frac{\alpha}{2} n^2, \quad c(n, \alpha) = \sqrt{\frac{gn + \alpha n^2}{m}} \approx \sqrt{\frac{gn}{m}} \left( 1 + \frac{\alpha n}{2g} \right),
\]
is not expected to vanish. Substituting the expression for sound velocity into Eqs. (VI.1), and subtracting Eq. (VI.2), we finally obtain
\[
\Gamma'_0 \approx -\frac{1}{48 m^a c^4}
\]
up to the first order in \( \alpha \).
Appendix A: Identities

Hyperbolic tangent and cotangent identities

\[ 1 - \tanh(a) \tanh(b) = \coth(a - b) [\tanh(a) - \tanh(b)] \]  
(A.1)

\[ 1 - \coth(a) \coth(b) = \coth(a - b) [\coth(a) - \coth(b)] \]  
(A.2)

Integral of difference of hyperbolic tangents

\[ I(y) \equiv \int_{-\infty}^{\infty} dxx^{n-1} [\tanh(x + y) - \tanh(x - y)] = \]

\[ = \frac{4}{n} \left[ y^n + \sum_{m=1}^{(n-1)/2} \left( \frac{n}{2m} \right) \pi^{2m} \left( 1 - 2^{1-2m} \right) |B_{2m}| y^{-2m} \right], \ \text{n odd} \]

Here \( B_n \) are the Bernoulli numbers.

Proof. Integrating by parts we obtain

\[ I(y) = \frac{1}{n} \int_{-\infty}^{\infty} dxx^n \left[ \frac{1}{\cosh^2(x - y)} - \frac{1}{\cosh^2(x + y)} \right] \]

where the boundary term is zero as

\[ \lim_{\lambda \to \infty} \left\{ \frac{x^n}{n} [\tanh(x + y) - \tanh(x - y)] \right\}^\Lambda_{-\Lambda} = \lim_{\lambda \to \infty} \left[ \frac{\sinh(2y)}{1 + 2 \cosh(2y) + e^{-2\pi y} e^{-2\Lambda}} \right]_{-\Lambda} = 0 \]

By shifting the variable as \( x \to x \pm y \) the integral reads

\[ I(y) = \frac{1}{n} \int_{-\infty}^{\infty} dx [(x + y)^n - (x - y)^n] \frac{1}{\cosh^2(x)} \]

Since the integrand is even we can write twice the integral from zero to infinity. Expressing the binomials with the help of the binomial theorem we find

\[ I(y) = \frac{2}{n} \sum_{k=0}^{n} \left( \frac{n}{k} \right) (y^{n-k} - (-y)^{n-k}) \int_{0}^{\infty} dx \frac{x^k}{\cosh^2(x)} \]

The term in parenthesis is zero for \( k \) odd and equal to \( 2y^{n-k} \) for \( k \) even. Therefore, putting \( k = 2m \)

\[ I(y) = \frac{4}{n} \sum_{m=0}^{(n-1)/2} \left( \frac{n}{2m} \right) y^{n-2m} \int_{0}^{\infty} dx \frac{x^{2m}}{\cosh^2(x)} \]

Using formulas 3.511.8 and 3.527.5 of Ref. [10] we find the result.

In the case in which a positive constant \( A \) multiplies the argument of the hyperbolic tangent, the integral becomes

\[ I(y) \equiv \int_{-\infty}^{\infty} dxx^{n-1} [\tanh(A(x + y)) - \tanh(A(x - y))] = \]

\[ = \frac{4}{n} y^n \left[ 1 + \sum_{m=1}^{(n-1)/2} \left( \frac{n}{2m} \right) (\frac{\pi}{A})^{2m} (1 - 2^{1-2m}) |B_{2m}| y^{-2m} \right], \ \text{n odd} \]  
(A.3)

[1] M. Schecter, D. Gangardt, and A. Kamenev, Ann. Phys. 327, 639 (2012).
Both domains of integration are in the range $|k|, |q| < T \Lambda/c$, where $\Lambda$ is the momentum cut-off, but, since the integrand converges to zero exponentially for $|k|, |q| > T \Lambda/c$, the integrals can be well approximated by extending the domain of integration to $\mathbb{R}$.