On the volumes of complex hyperbolic manifolds with cusps

Jun-Muk Hwang

Abstract

We study the problem of bounding the number of cusps of a complex hyperbolic manifold in terms of its volume. Applying algebro-geometric methods using Mumford’s work on toroidal compactifications and its generalization due to N. Mok and W.-K. To, we get a bound which is considerably better than those obtained previously by methods of geometric topology.

MSC Number: 32Q45

1 Introduction

There have been some interests on the problem of bounding the number of cusps of a complex hyperbolic manifold in terms of its volume, as a generalization of the corresponding problem for a real hyperbolic manifold. We refer the readers to [3], [5] and the references therein for the historical background and the motivation for studying problems of this type from the viewpoint of geometric topology.

It seems that the following bounds of John R. Parker’s are the best published results on this problem.

Theorem 1 [5, Theorem D and Theorem F] Let $X$ be an $n$-dimensional complex hyperbolic manifold of finite volume. Let $k$ be the number of cusps of $X$ and let $\text{Vol}(X)$ be the volume of $X$ with respect to the Bergmann metric with holomorphic sectional curvature $-1$. Then

$$\frac{\text{Vol}(X)}{k} \geq \frac{2^{n-1}}{n(6\pi)^{2n^2-3n+1}}.$$  

When $n = 2$,

$$\frac{\text{Vol}(X)}{k} \geq \frac{2}{3}.$$  

The method used in [5], based on the earlier work of [3], is motivated by the corresponding method in the study of real hyperbolic manifolds. More precisely, these authors constructed certain disjoint neighborhoods of the cusps whose volumes can be estimated.

1Supported by the Korea Research Foundation Grant (KRF-2002-070-C00003).
The goal of this paper is to explain a completely different approach to the problem, using techniques of algebraic geometry. To state our result, let

\[ P(\ell) := \frac{(n\ell + n + \ell)!}{n!(n\ell + \ell)!}. \]

**Theorem 2** *In the notation of Theorem 1, for \( n \geq 2 \),

\[ \frac{\text{Vol}(X)}{k} \geq \frac{(4\pi)^n}{n!(P(4) - P(2))}(1 - \frac{n + 1}{P(4) - P(2)}). \]

Note that the right hand side is at least \( \frac{2^{2n-1} \pi^n}{(5n+4)n} \) which is considerably better than Theorem 1. For \( n = 2 \), \( P(4) - P(2) = 63 \) and the right hand side is

\[ \frac{(4\pi)^2}{2 \cdot 63}(1 - \frac{3}{63}) = \frac{160}{1323}\pi^2 \geq 1.19..., \]

which is better than Theorem 1. Note that our argument is uniform in all dimensions \( \geq 2 \), while the case \( n = 2 \) in Parker’s work was obtained by a special argument which did not apply in higher dimensions.

Theorem 2 is obtained by examining the dimensions of the spaces of certain cusp forms. The proof depends essentially on the existence of a toroidal compactification of \( X \) and its metric property which was established by Mumford [4] for \( X \) defined by an arithmetic group and generalized to arbitrary \( X \) by N. Mok and W.-K. To [7]. Excepting these results, we only need standard methods of algebraic geometry.

Yum-Tong Siu told us that one may be able to get a bound of the above type also by the differential geometric method used in [6]. It is not clear however whether the resulting bound would be as good as ours.

## 2 Results from toroidal compactifications

In this section, we will recall some basic facts about toroidal compactifications which we need for the proof of Theorem 2.

Throughout, \( X \) denotes a complex hyperbolic manifold of dimension \( n \geq 2 \) with finite volume. Denote by \( X^* \) the minimal compactification of \( X \), which was constructed by Baily-Borel [2] for \( X \) defined by an arithmetic group and by Siu-Yau [6] for arbitrary \( X \). The complement \( X^* \setminus X \) consists of \( k \) cusp points, which we denote by

\[ X^* \setminus X = \{Q_1, \ldots, Q_k\}. \]

\( X^* \) is a normal projective variety and there exists an ample line bundle \( K_{X^*} \) extending the canonical bundle of \( X \).
Denote by $\bar{X}$ a toroidal compactification of $X$, which was constructed by Mumford et al. [1] for $X$ defined by an arithmetic group and by Mok for arbitrary $X$ as explained in [7, p.61]. $\bar{X}$ is a smooth projective variety and the complement $\bar{X} \setminus X$ is a smooth divisor $E$ with $k$ components, which we denote by

$$\bar{X} \setminus X = E = E_1 \cup \cdots \cup E_k.$$  

Each component $E_i$ is an abelian variety of dimension $n - 1$ whose normal bundle in $\bar{X}$ is a negative line bundle, as described in [7, pp.61-62]. There is a canonical morphism

$$\psi : \bar{X} \to X^*$$

which contracts each $E_i$ to a cusp point $Q_i$. Let us denote by $L$ the nef and big line bundle $\psi^* K_{X^*}$. Then by [4, Proposition 3.4 (b)],

$$L = K_{\bar{X}} + E.$$  

The key property of $L$ is that the Bergman metric on $X$ induces a singular metric on $L$ which is good in the sense of [4, Section 1]. This was proved by [4, Main Theorem 3.1 and Proposition 3.4 (b)] for $X$ defined by an arithmetic group and by [7, Section 2] for arbitrary $X$. This implies Hirzebruch proportionality [4, Theorem 3.2]. One special case we need is the following.

**Proposition 1** [4, Theorem 3.2]

$$\text{Vol}(X) = \frac{(4\pi^n)^n}{n!(n+1)^n} L^n.$$  

This is not exactly [4, Theorem 3.2] because Mumford uses different normalization of the metric from ours. One can check that the volume of $X$ in [4] corresponds to $\frac{n!}{(4\pi^n)^n} \text{Vol}(X)$ in our notation.

One consequence of Hirzebruch proportionality is a formula for the dimension of the space $V_\ell$ of cusp forms of weight $\ell$. By definition, $V_\ell$ is the space of sections of $L^{\otimes \ell}$ which vanish on $E$. In other words,

$$V_\ell := H^0(\bar{X}, \mathcal{O}(\ell L - E)).$$

Mumford showed that the formula for the dimension of $V_\ell$ in the case of compact $X$ continues to hold for non-compact $X$ with an error term of degree bounded by the dimension of $X^* \setminus X$. More precisely,

**Proposition 2** [4, Corollary 3.5] Let

$$P(\ell) := h^0(P_n, \mathcal{O}(\ell(n+1))) = \frac{(n\ell + n + \ell)!}{n!(n\ell + \ell)!}.$$  

Then there exists a constant $P_0$ such that for all $\ell \geq 2$,

$$\dim V_\ell = \frac{n!}{(4\pi)^n} \text{Vol}(X) P(\ell - 1) + P_0.$$  

An immediate consequence is

**Corollary 1** For any $\ell \geq 2$, $\dim V_{\ell+1} > \dim V_\ell$. In particular, $V_3 \neq 0$. 

3
3 Proof of Theorem 2

To prove Theorem 2, we need the following two lemmas.

**Lemma 1** Recall that $E_1, \ldots, E_k$ are the components of $E = \bar{X} \setminus X$. For each $1 \leq i \leq k$, there exists $\sigma_i \in H^0(\bar{X}, \mathcal{O}(2L))$ such that

$$\sigma_i|_{E_i} \neq 0, \text{ but } \sigma_i|_{E_j} = 0 \text{ for each } j \neq i.$$

**Proof of Lemma 1.** Consider the short exact sequence on $\bar{X}$,

$$0 \to \mathcal{O}(2L - E) \to \mathcal{O}(2L) \to \mathcal{O}(2L)|_E \to 0.$$

Since $L = K_{\bar{X}} + E$ is nef and big, Kawamata-Viehweg vanishing gives

$$H^1(\bar{X}, \mathcal{O}(2L - E)) = H^1(\bar{X}, \mathcal{O}(K_{\bar{X}} + L)) = 0.$$

Thus we have the surjectivity of the restriction map

$$H^0(\bar{X}, \mathcal{O}(2L)) \to H^0(E, \mathcal{O}(2L)|_E).$$

Since $E_i$ is contracted by $\psi : \bar{X} \to X^*$, the line bundle $L|_{E_i}$ is trivial. So we have the surjectivity of

$$H^0(\bar{X}, \mathcal{O}(2L)) \to \bigoplus_{i=1}^k H^0(E_i, \mathcal{O}_{E_i})$$

from which Lemma 1 follows. \(\square\)

**Lemma 2** Suppose $V_\ell \neq 0$. Then $\dim V_{\ell+2} - \dim V_\ell \geq k - 1$.

**Proof of Lemma 2.** Recall that elements of $V_\ell$ are sections of $L^\otimes \ell$ which vanish on $E$. Choose $v \in V_\ell$ such that the vanishing order of $v$ along $E_1$ is the highest among all non-zero elements of $V_\ell$. Fix a basis $\{v_1, \ldots, v_m\}$ of $V_\ell$ with $m = \dim V_\ell$. Consider the following $(m + k - 1)$ elements of $V_{\ell+2}$.

$$\sigma_2 \cdot v, \ldots, \sigma_k \cdot v, \sigma_1 \cdot v_1, \ldots, \sigma_1 \cdot v_m$$

where $\sigma_1, \ldots, \sigma_k$ are as in Lemma 1. We claim that they are linearly independent. Suppose

$$\sum_{j=2}^k a_j(\sigma_j \cdot v) + \sum_{i=1}^m b_i(\sigma_1 \cdot v_i) = 0$$

for some complex numbers $a_j, b_i$. Then

$$\left(\sum_{j=2}^k a_j \sigma_j\right) \cdot v = -\sigma_1 \cdot w$$

for $w = \sum_{i=1}^m b_i v_i \in V_\ell$. The left hand side has vanishing order along $E_1$ strictly higher than that of $v$. Since the vanishing order of non-zero $w$ along $E_1$ can’t be bigger than that of $v$, we see
that \( w = 0 \). This yields \( a_j = b_i = 0 \) for all \( 2 \leq j \leq k \) and \( 1 \leq i \leq m \). This proves the claim. Lemma 2 follows immediately from the claim. □

**Proof of Theorem 2.** From Corollary 1 and Lemma 2, we see that

\[
\dim V_5 - \dim V_3 \geq k - 1.
\]

By Proposition 2,

\[
\dim V_5 - \dim V_3 = \frac{n!}{(4\pi)^n} \text{Vol}(X)(P(4) - P(2)) > 0.
\]

Thus

\[
\text{Vol}(X) \geq \frac{(4\pi)^n}{n!(P(4) - P(2))}(k - 1).
\]

As quoted in [3, p.179], Gromov’s generalization of Gauss-Bonnet says

\[
\text{Vol}(X) = \frac{(-4\pi)^n}{(n + 1)!}e(X)
\]

where \( e(X) \) denotes the topological Euler number of \( X \). This implies

\[
\text{Vol}(X) \geq \frac{(4\pi)^n}{(n + 1)!}.
\]

Thus when \( k \leq \frac{P(4) - P(2)}{n + 1} \),

\[
\text{Vol}(X) \geq \frac{(4\pi)^n}{n!(P(4) - P(2))}k
\]

and the statement of Theorem 2 holds automatically.

When \( k \geq \frac{P(4) - P(2)}{n + 1} \),

\[
k - 1 \geq (1 - \frac{n + 1}{P(4) - P(2)})k.
\]

Thus

\[
\text{Vol}(X) \geq \frac{(4\pi)^n}{n!(P(4) - P(2))}(k - 1) \geq \frac{(4\pi)^n}{n!(P(4) - P(2))}(1 - \frac{n + 1}{P(4) - P(2)})k
\]

which proves the theorem. □.

**References**

[1] A. Ash, D. Mumford, M. Rapoport and Y. Tai, *Smooth compactification of locally symmetric varieties*, Math. Sci. Press, 1975

[2] W. L. Baily and A. Borel, Compactification of arithmetic quotients of bounded symmetric domains, *Annals of Math.* 84 (1966), 442-528

[3] S. Hersonsky and F. Paulin, On the volumes of complex hyperbolic manifolds, *Duke Math. J.* 84 (1996), 719-737

[4] D. Mumford, Hirzebruch’s proportionality theorem in the non-compact case, *Invent. math.* 42 (1977), 239-272
[5] J. R. Parker, On the volumes of cusped, complex hyperbolic manifolds and orbifolds, *Duke Math. J.* **94** (1998), 433-464

[6] Y.-T. Siu and S.-T. Yau, Compactifications of negatively curved complete Kähler manifolds of finite volume, *Ann. Math. Stud.* **102** (1980), 363-380

[7] W.-K. To, Total geodesy of proper holomorphic immersions between complex hyperbolic space forms of finite volume, *Math. Ann.* **297** (1993), 59-84

Korea Institute for Advanced Study  
207-43 Cheongryangri-dong  
Seoul, 130-722, Korea  
jmhwang@kias.re.kr