ON THE REAL AND COMPLEX ZEROS OF THE QUADRILATERAL ZETA FUNCTION

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Abstract. Let $0 < a \leq 1/2$ and define the quadrilateral zeta function by $2Q(s,a) := \zeta(s,a) + \zeta(s,1-a) + \text{Li}_s(e^{2\pi i a}) + \text{Li}_s(e^{2\pi i (1-a)})$, where $\zeta(s,a)$ is the Hurwitz zeta function and $\text{Li}_s(e^{2\pi i a})$ is the periodic zeta function.

In the present paper, we show that there exists a unique real number $a_0 \in (0,1/2)$ such that $Q(\sigma,a_0)$ has a unique double real zero at $\sigma = 1/2$ when $\sigma \in (0,1)$, for any $a \in (a_0,1/2]$, the function $Q(\sigma,a)$ has no zero in the open interval $\sigma \in (0,1)$ and for any $a \in (0,a_0)$, the function $Q(\sigma,a)$ has at least two real zeros in $\sigma \in (0,1)$.

Moreover, we prove that $Q(s,a)$ has infinitely complex zeros in the region of absolute convergence and the critical strip when $a \in \mathbb{Q} \cap (0,1/2) \setminus \{1/6,1/4,1/3\}$. The Riemann-von Mangoldt formula for $Q(s,a)$ is also shown.

1. Main results and Some remarks

1.1. Introduction and Main results. For $0 < a \leq 1$, we define the Hurwitz zeta function $\zeta(s,a)$ by

$$\zeta(s,a) := \sum_{n=0}^{\infty} \frac{1}{(n+a)^s}, \quad \sigma > 1,$$

and the periodic zeta function $\text{Li}_s(e^{2\pi i a})$ by

$$\text{Li}_s(e^{2\pi i a}) := \sum_{n=1}^{\infty} \frac{e^{2\pi i a n}}{n^s}, \quad \sigma > 1.$$

Note that the both Dirichlet series of $\zeta(s,a)$ and $\text{Li}_s(e^{2\pi i a})$ converge absolutely in the half-plane $\sigma > 1$ and uniformly in each compact subset of this region. The Hurwitz zeta function $\zeta(s,a)$ can be extended for all $s \in \mathbb{C}$ except $s = 1$, where there is a simple pole with residue 1 (see for instance [11, Section 12]). On the other hand, the Dirichlet series of the function $\text{Li}_s(e^{2\pi i a})$ with $0 < a < 1$ converges uniformly in each compact subset of the half-plane $\sigma > 0$ (see for example [9, p. 20]). The function $\text{Li}_s(e^{2\pi i a})$ with $0 < a < 1$ is analytically continuable to the whole complex plane (see for instance [9, Section 2.2]).

Obviously, one has $\zeta(s,1) = \text{Li}_s(1) = \zeta(s)$, where $\zeta(s)$ is the Riemann zeta function. We can easily see that $\zeta(1) > 0$ when $\sigma > 1$ form the series expression of $\zeta(s)$. Since (see for instance [17, (2.12.4)])

$$\left(1 - 2^{1-s}\right)\zeta(s) = 1 - \frac{1}{2^s} + \frac{1}{3^s} - \frac{1}{4^s} + \cdots > 0, \quad 0 < \sigma < 1,$$

one has $\zeta(\sigma) < 0$ if $0 < \sigma < 1$. Moreover, we have $\zeta(0) = -1/2 < 0$ (see for example [17, (2.4.3)]). Therefore, from the functional equation of $\zeta(s)$, we have the following (see for example [7, Theorem 1.6.1] or [17, Section 2.12]).

Theorem A. All real zeros of $\zeta(s)$ are simple and at only the negative even integers.

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For $0 < a \leq 1/2$, we define the quadrilateral zeta function $Q(s, a)$ by

$$2Q(s, a) := \zeta(s, a) + \zeta(s, 1 - a) + \text{Li}_s(e^{2\pi i a}) + \text{Li}_s(e^{2\pi i(1-a)}).$$

The function $Q(s, a)$ can be continued analytically to the whole complex plane except $s = 1$. In [11, Theorem 1.1], the author proved the functional equation

$$Q(1 - s, a) = \frac{2\Gamma(s)}{(2\pi)^s} \cos \left( \frac{\pi s}{2} \right) Q(s, a).$$

(1.1)

It should be emphasised that we have $Q(0, a) = -1/2 = \zeta(0)$ by (2.4) and the equation (1.1) completely coincides with the functional equation of $\zeta(s)$ if we replace $Q(s, a)$ and $Q(1 - s, a)$ with $\zeta(s)$ and $\zeta(1 - s)$ (see also [12, Section 1.3]). In [11, Theorem 1.2], he showed the following.

Theorem B. For any $0 < a \leq 1/2$, there exist positive constants $A(a)$ and $T_0(a)$ such that the numbers of zeros of the quadrilateral zeta function $Q(s, a)$ on the line segment from $1/2$ to $1/2 + iT$ is greater than $A(a)T$ whenever $T \geq T_0(a)$.

In the present paper, we show the following statements. Note that by Mathematica 11.3, the real number $a_0 \in (0, 1/2)$ appeared in the next theorems is

$$a_0 = 0.11837513961527293582719034552119129714717699990531455491427859384268411483278906208314018589873082...$$

(1.2)

### Theorem 1.1.
There exists the unique real number $a_0 \in (0, 1/2)$ such that

1. the function $Q(\sigma, a_0)$ has a unique double real zero at $\sigma = 1/2$ when $\sigma \in (0, 1)$,
2. for any $a \in (a_0, 1/2]$, the function $Q(\sigma, a)$ has no real zero in $\sigma \in (0, 1)$,
3. for any $a \in (0, a_0)$, the function $Q(\sigma, a)$ has at least two real zeros in $\sigma \in (0, 1)$.

### Corollary 1.2.
All real zeros of the quadrilateral zeta function $Q(s, a)$ are simple and at only the negative even integers just like $\zeta(s)$ if and only if $a_0 < a \leq 1/2$.

### Proposition 1.3.
Suppose that $a = 1/6, 1/4, 1/3$ or $1/2$. Then the Riemann hypothesis is true if and only if all non-real zeros of $Q(s, a)$ are on the critical line $\sigma = 1/2$.

### Proposition 1.4.
Let $a \in \mathbb{Q} \cap (0, 1/2) \setminus \{1/6, 1/4, 1/3\}$. Then for any $\delta > 0$, there exist positive constants $C_a^\theta(\delta)$ and $C_a^\varphi(\delta)$ such that the function $Q(s, a)$ has more than $C_a^\theta(\delta)T$ and less than $C_a^\varphi(\delta)T$ complex zeros in the rectangles $1 < \sigma < 1 + \delta$ and $0 < t < T$, and $-\delta < \sigma < 0$ and $0 < t < T$ if $T$ is sufficiently large.

Furthermore, for any $1/2 < \sigma_1 < \sigma_2 < 1$, there are positive numbers $C_a^\theta(\sigma_1, \sigma_2)$ and $C_a^\varphi(\sigma_1, \sigma_2)$ such that the function $Q(s, a)$ has more than $C_a^\theta(\sigma_1, \sigma_2)T$ and less than $C_a^\varphi(\sigma_1, \sigma_2)T$ non-trivial zeros in the rectangles $\sigma_1 < \sigma < \sigma_2$ and $0 < t < T$, $1 - \sigma_2 < \sigma < 1 - \sigma_1$ and $0 < t < T$ when $T$ is sufficiently large.

Let $N(T, F)$ count the number of non-real zeros of a function $F(s)$ having $|\Im(s)| < T$. Then we have the following Proposition which implies that for any $0 < a \leq 1/2$,

$$N(T, \zeta(s)) - N(T, Q(s, a)) = O_a(T).$$

### Proposition 1.5.
For $0 < a \leq 1/2$ and $T > 2$, we have

$$N(T, Q(s, a)) = \frac{T}{\pi} \log T - \frac{T}{\pi} \log(2\pi a^2) + O_a(\log T).$$

Some remarks for Theorem 1.1, Propositions 1.4 and 1.5 and the Epstein zeta function are given in Section 1.2. In Section 2.1, we prove Theorem 1.1 and Corollary 1.2. The proofs of Propositions 1.3 and 1.4 are given in Section 1.2. We give a proof of Proposition 1.5 and related propositions and their proofs in Section 2.3.
1.2. The Epstein zeta function. Let $B(x, y) = ax^2 + bxy + cy^2$ be a positive definite integral binary quadratic form, and denote by $r_B(n)$ the number of solutions of the equation $B(x, y) = n$ in integers $x$ and $y$. Then the Epstein zeta function for the binary quadratic form $B$ is defined by the series

$$
\zeta_B(s) := \sum_{(x, y) \in \mathbb{Z}^2 \setminus (0, 0)} \frac{1}{B(x, y)^s} = \sum_{n=1}^{\infty} \frac{r_B(n)}{n^s}
$$

for $\sigma > 1$. It is widely known that the function $\zeta_B(s)$ admits analytic continuation into the entire complex plane except for a simple pole at $s = 1$ with residue $2\pi(-D)^{-1/2}$, where $D := b^2 - 4ac < 0$. Moreover, the function $\zeta_B(s)$ fulfills the functional equation

$$
\left(\frac{\sqrt{-D}}{2\pi}\right)^s \Gamma(s) \zeta_B(s) = \left(\frac{\sqrt{-D}}{2\pi}\right)^{1-s} \Gamma(1-s) \zeta_B(1-s).
$$

Denoted by $N_{\text{Ep}}^{\text{CL}}(T)$ the number of the zeros of the Epstein zeta function $\zeta_B(s)$ on the critical line and whose imaginary part is smaller than $T > 0$. Potter and Tichmarsh \[13\] showed $N_{\text{Ep}}^{\text{CL}}(T) \gg T^{1/2-\varepsilon}$. The current (January, 2020) best result is $N_{\text{Ep}}^{\text{CL}}(T) \gg T^{4/7-\varepsilon}$, shown by Baier, Srinivas and Sangale \[2\].

The distribution of zeros of $\zeta_B(s)$ off the critical line depends on the value of the class number $h(D)$ of the imaginary quadratic field $\mathbb{Q}(\sqrt{D})$. If $h(D) = 1$, it is expected that $\zeta_B(s)$ satisfies an analogue of the Riemann hypothesis since $\zeta_B(s)$ has an Euler product. When $h(D) \geq 2$, Davenport and Heilbronn \[5\] proved that $\zeta_B(s)$ has infinitely many zeros in the region of absolute convergence $\Re(s) > 1$. Voronin \[16\] proved that if $h(D) \geq 2$, then there exists a positive constant $C_B^0(\sigma_1, \sigma_2)$ such that the function $\zeta_B(s)$ has more than $C_B^0(\sigma_1, \sigma_2)T$ non-trivial zeros in the rectangle $\sigma_1 < \sigma < \sigma_2$ and $0 < t < T$, when $T$ is sufficiently large (see also \[7\] Theorem 7.4.3)). Recently, for example, Lee \[10\] and Lamzouri \[8\] improve Voronin’s theorem (more precisely, their theorems imply $C_B^0(\sigma_1, \sigma_2) = C_B^0(\sigma_1, \sigma_2)$ in Theorem C below). By Davenport and Heilbronn’s Theorem, Voronin’s theorem, the functional equation and the almost periodicity of $\zeta_B(s)$ in the region of absolute convergence, we have the following.

**Theorem C.** Let $h(D) \geq 2$. Then for any $\delta > 0$, there are positive numbers $C_B^0(\delta)$ and $C_B^4(\delta)$ such that the Epstein zeta function $\zeta_B(s)$ has more than $C_B^0(\delta)T$ and less than $C_B^4(\delta)T$ complex zeros in the rectangles $1 < \sigma < 1 + \delta$ and $0 < t < T$, and $-\delta < \sigma < 0$ and $0 < t < T$ if $T$ is sufficiently large.

Furthermore, for any $1/2 < \sigma_1 < \sigma_2 < 1$, there are positive constants $C_B^0(\sigma_1, \sigma_2)$ and $C_B^4(\sigma_1, \sigma_2)$ such that the zeta function $\zeta_B(s)$ has more than $C_B^0(\sigma_1, \sigma_2)T$ and less than $C_B^4(\sigma_1, \sigma_2)T$ non-trivial zeros in the rectangles $\sigma_1 < \sigma < \sigma_2$ and $0 < t < T$, $1 - \sigma_2 < \sigma < 1 - \sigma_1$ and $0 < t < T$ when $T$ is sufficiently large.

Hereafter, let $B(x, y) = ax^2 + bxy + cy^2$ be a positive definite binary quadratic form, namely, $a, b, c \in \mathbb{R}$, $a > 0$ and $d := b^2 - 4ac < 0$. And define $\kappa > 0$ by putting

$$
\kappa^2 := \frac{|d|}{4a^2} = \frac{4ac - b^2}{4a^2} = \frac{c}{a} - \left(\frac{b}{2a}\right)^2.
$$

Bateman and Grosswald \[3\] showed the following theorem. It should be noted that this result was announced by Chowla and Selberg \[4\] without a proof.

**Theorem D.** Let $\kappa \geq \sqrt{3}/2$. Then one has

$$
\zeta_B(1/2) > 0 \quad \text{if} \quad \kappa \geq 7.00556
$$

(or if $4\kappa^2 = |d|/a^2 \geq 199.2$) but,

$$
\zeta_B(1/2) < 0 \quad \text{if} \quad \sqrt{3}/2 \leq \kappa \leq 7.00554
$$
(or if \(3 \leq 4\,\kappa^2 = |d|/a^2 \leq 199.1\)).

It should be mentioned that \(\zeta_B(s)\) vanishes in the interval \((1/2, 1)\) if \(\kappa \geq 7.00556\) by the theorem above and \(\lim_{s \to 1^-} \zeta_B(s) = -\infty\). Moreover, it is probable that \(\zeta_B(s) < 0\) for all \(\sigma \in (0, 1)\) if \(\sqrt{3}/2 \leq \kappa \leq 7.00554\).

Furthermore, we have the following Riemann-von Mangoldt formula for \(\zeta_B(s)\) (see for example [15, p. 692]).

**Theorem E.** Denote by \(m(B)\) the minimum of the values of the quadratic form \(B(x, y)\) for \((x, y) \neq (0, 0)\). Then we have

\[
N(T, \zeta_B(s)) = \frac{2T}{\pi} \log \frac{|d|^{1/2}T}{2\pi e m(B)} + O(\log T).
\]

**Remark.** Proposition 1.4 can be regarded as an analogue of Theorem C. Furthermore, we can regard Theorem 1.1 as an analogue of Theorem D since Theorem 1.1 implies that \(Q(1/2, a) > 0\) if and only if \(0 < a < a_0\).

Corollary 1.2 should be compared with the facts that all real zeros of \(\zeta(s, a) + \zeta(s, 1 - a)\) are simple and on only the non-positive even integers if and only if \(1/4 \leq a \leq 1/2\) (see [11, Theorem 1.3]), and all real zeros of \(\text{Li}_s(e^{2\pi i a}) + \text{Li}_s(e^{2\pi i(1-a)})\) are simple and on only the negative even integers if and only if \(1/4 \leq a \leq 1/2\) (see [11, Theorem 1.4]). Proposition 1.5 is an analogue of Theorem E.

## 2. Proofs

### 2.1. Proof of Theorem 1.1

For \(0 < a \leq 1/2\), put

\[
Z(s, a) := \zeta(s, a) + \zeta(s, 1 - a), \quad P(s, a) := \text{Li}_s(e^{2\pi i a}) + \text{Li}_s(e^{2\pi i(1-a)}).
\]

Note that one has \(2Q(s, a) = Z(s, a) + P(s, a)\). By [12, Lemma 4.1], we have

\[
Z(1 - s, a) = \frac{2\Gamma(s)}{(2\pi)^s} \cos\left(\frac{\pi s}{2}\right) P(s, a), \quad P(1 - s, a) = \frac{2\Gamma(s)}{(2\pi)^s} \cos\left(\frac{\pi s}{2}\right) Z(s, a). \tag{2.1}
\]

In [12, Lemma 4.4], it is shown that

\[
\frac{\partial}{\partial a} Z(\sigma, a) < 0, \quad \frac{\partial}{\partial a} P(\sigma, a) < 0, \quad 0 < \sigma < 1.
\]

Hence we have the following inequality.

**Lemma 2.1.** Let \(0 < a < 1/2\). Then it holds that

\[
\frac{\partial}{\partial a} Q(\sigma, a) < 0, \quad 0 < \sigma < 1. \tag{2.2}
\]

From [12, Theorem 1.3 and Proposition 4.5], we have the following.

**Lemma 2.2.** One has

1. Let \(1/6 \leq a \leq 1/2\). Then one has \(Z(\sigma, a) < 0\) for \(0 < \sigma < 1\).
2. When \(0 < a < 1/6\), the function \(Z(\sigma, a)\) has precisely one simple zero in \((0, 1)\). Let \(\beta_Z(a)\) denote the unique zero of \(Z(\sigma, a)\) in \((0, 1)\). Then the function \(\beta_Z(a) : (0, 1/6) \to (0, 1)\) is a strictly decreasing \(C^\infty\)-diffeomorphism.

**Proof of Theorem 1.1.** From Lemma 2.2 there is the unique \(0 < a_0 < 1/6\) such that \(Z(1/2, a_0) = 0\). By the functional equations (2.1), we have \(P(1/2, a_0) = 0\). Hence, from (2.2), the exists the unique \(0 < a_0 < 1/6\) such that

\[
2Q(1/2, a_0) = Z(1/2, a_0) + P(1/2, a_0) = 0.
\]
It should be noted that the $0 < a_0 < 1/6$ above is given by (1.2) numerically. By the functional equation (1.1), we have

$$Q(1/2 - \varepsilon, a_0)Q(1/2 + \varepsilon, a_0) \geq 0$$

if $\varepsilon > 0$ is sufficiently small. The inequality above implies that

$$Q'(1/2, a_0) = 0, \quad Q'(\sigma, a_0) := \frac{\partial}{\partial \sigma} Q(\sigma, a_0).$$

(2.3)

Hereafter, we only consider the open interval $(0, 1/2)$ in virtue of (1.1). We have

$$2Q(0, a) = Z(0, a) + P(0, a) = 0 - 1 = -1, \quad 0 < a \leq 1/2$$

(2.4)

from [12, (4.11) and (4.12)]. Put

$$a_\flat := 0.11837513961 < a_0 < 0.11837513962 =: a_\sharp.$$ 

Then we have the following ten figures by Mathematica 11.3.

F. 1. $\{Z(1/2, a) : a_\flat \leq a \leq a_\sharp\}$

F. 2. $\{Q(1/2, a) : a_\flat \leq a \leq a_\sharp\}$

F. 3. $\{Q(\sigma, a_\flat) : 0 \leq \sigma \leq 1/2\}$

F. 4. $\{Q(\sigma, a_\sharp) : 0 \leq \sigma \leq 1/2\}$

We can see that $Q''(\sigma, a_0) < -2 < 0$ for all $1/3 \leq \sigma \leq 1/2$ by the seventh and eighth figures. Therefore, by (2.3), the functional equation (1.1) and ten figures we have

$$Q(1/2, a_0) = Q'(1/2, a_0) = 0, \quad Q(\sigma, a_0) < 0, \quad \sigma \in (0, 1/2) \cup (1/2, 1)$$

which implies the first statement of Theorem 1.1. Moreover, from (1.1) and (2.2), one has

$$Q(\sigma, a) < 0, \quad \sigma \in (0, 1), \quad a_0 < a \leq 1/2.$$ 

(2.5)

which implies the second statement of Theorem 1.1.

Next suppose $0 < a < a_0$. Then we have $Q(1/2, a) > 0$ from (2.2). On the other hand, one has $Q(0, a) = -1/2$ for all $0 < a \leq 1/2$. Thus we have the third statement of Theorem 1.1 according to the intermediate value theorem. □
Proof of Corollary 1.2. When $\sigma > 1$ and $0 < a \leq 1/2$, we have

$$2Q(\sigma, a) = Z(\sigma, a) + P(\sigma, a) > \sum_{n=0}^{\infty} \left( \frac{1}{(n+a)^{\sigma}} + \frac{1}{(n+1-a)^{\sigma}} - \frac{2}{(n+1)^{\sigma}} \right) > 0. \quad (2.6)$$

Moreover, we have $Q(\sigma, a) < 0$ if $\sigma \in (0, 1)$ and $a_0 < a \leq 1/2$ from [2.5]. Recall that one has $Q(0, a) = -1/2$ by [2.4] or [12] (4.11) and (4.12). Hence, all real zeros of $Q(1-s, a)$ with $a_0 < a \leq 1/2$ and $\sigma > 1$ come from $\cos(\pi s/2) = 0$ with $\sigma > 1$ from [11.1] and the fact that $\Gamma(\sigma) > 0$ when $0 < \sigma < 1$. Hence every real zero of $Q(s, a)$ with $a_0 < a \leq 1/2$ and $\sigma < 1$ is caused by

$$\cos\left( \frac{\pi(1-s)}{2} \right) = 0, \quad \sigma < 0,$$

which is equivalent to that $s$ is a negative even integer. \qed
2.2. Proofs of Propositions. From \[12\] (4.1), (4.2), (4.3), (4.4), (4.5), (4.6), (4.7) and (4.8), it holds that
\[
\begin{align*}
Z(s, 1/2) &= 2(2^s - 1)\zeta(s), \quad P(s, 1/2) = 2(2^{1-s} - 1)\zeta(s), \quad (2.7) \\
Z(s, 1/3) &= (3^s - 1)\zeta(s), \quad P(s, 1/3) = (3^{1-s} - 1)\zeta(s), \quad (2.8) \\
Z(s, 1/4) &= 2^s(2^s - 1)\zeta(s), \quad P(s, 1/4) = 2^{1-s}(2^{1-s} - 1)\zeta(s), \quad (2.9) \\
Z(s, 1/6) &= (2^s - 1)(3^s - 1)\zeta(s), \quad P(s, 1/6) = (2^{1-s} - 1)(3^{1-s} - 1)\zeta(s). \quad (2.10)
\end{align*}
\]

Proof of Proposition \[12\] When \(a = 1/2\), we have
\[
2Q(s, 1/2) = 2(2^s - 1)\zeta(s) + 2(2^{1-s} - 1)\zeta(s) = 2(X - 2 + 2X^{-1})\zeta(s),
\]
where \(0 \neq X := 2^s\), form (2.4). The solutions of \(X^2 - 2X + 2 = 0\) are \(X = 1 \pm i\). Obviously, the all solutions of \(2^s = 1 \pm i\) are on the line \(\sigma = 1/2\).

Assume that \(a = 1/3\). By using (2.8), we have
\[
2Q(s, 1/3) = (3^s - 1)\zeta(s) + (3^{1-s} - 1)\zeta(s) = (Y - 2 + 3Y^{-1})\zeta(s),
\]
where \(0 \neq Y := 3^s\). We can easily see that \(Y = 1 \pm i\sqrt{2}\) are the roots of \(Y^2 - 2Y + 3 = 0\) and the all solutions of \(3^s = 1 \pm i\sqrt{2}\) are on the line \(\sigma = 1/2\).

Let \(a = 1/4\). By (2.9), it holds that
\[
2Q(s, 1/4) = 2^s(2^s - 1)\zeta(s) + 2^{1-s}(2^{1-s} - 1)\zeta(s) = (X^2 - X + 4X^{-2} - 2X^{-1})\zeta(s).
\]
The solutions of \(X^2 - X + 4X^{-2} - 2X^{-1} = 0\) are
\[
X = \frac{1}{4} \left(1 + \sqrt{17} \pm i\sqrt{2(7 - \sqrt{17})}\right), \quad \frac{1}{4} \left(1 - \sqrt{17} \pm i\sqrt{2(7 + \sqrt{17})}\right).
\]
The absolute value of the numbers above are \(\sqrt{2}\). Therefore, the all roots of \(X^2 - X + 4X^{-2} - 2X^{-1} = 0\), where \(X := 2^s\), are on the line \(\sigma = 1/2\).

Finally, suppose that \(a = 1/6\). From (2.10), one has
\[
2Q(s, 1/6) = \left((2^s - 1)(3^s - 1) + (2^{1-s} - 1)(3^{1-s} - 1)\right)\zeta(s)
= (3^s - 1)(2^{1-s} - 1)(g_2(s) + g_3(1 - s))\zeta(s),
\]
where the function \(g_p(s)\) is defined as
\[
g_p(s) = \frac{p^s - 1}{p^{1-s} - 1}, \quad p = 2, 3.
\]
Obviously, we have \(g_p(1 - s) = 1/g_p(s)\) from the definition. By modifying the argument appeared at the end of [12] Section 3.1], we can prove that
\[
|g_p(s)| = 1, \quad \sigma = 1/2, \quad |g_p(s)| > 1/2, \quad \sigma > 1/2, \quad |g_p(s)| < 1/2, \quad \sigma < 1/2.
\]
Hence \((2^s - 1)(3^s - 1) + (2^{1-s} - 1)(3^{1-s} - 1)\) does not vanish when \(\sigma \neq 1/2\). \(\Box\)

Let \(\chi\) be the Euler totient function, \(\chi\) be a primitive Dirichlet character of conductor of \(q \in \mathbb{N}\) and \(L(s, \chi)\) be the Dirichlet \(L\)-function. Let \(G(\overline{\chi})\) denote the Gauss sum \(G(\overline{\chi}) := \sum_{n=1}^{q} \chi(n)e^{2\pi i n/q}\) associated to a Dirichlet character \(\overline{\chi}\). When \(0 < r/q \leq 1/2\), where \(q\) and \(r\) are relatively prime integers, we have
\[
Q(s, r/q) = \frac{1}{2\varphi(q)} \sum_{\chi \mod q} \left(1 + \chi(-1)\right)\left(\overline{\chi}(r)q^s + G(\overline{\chi})\right)L(s, \chi) \quad (2.11)
\]
from \([12\ (2.3)]\). It is well-known that \(\varphi(q) \leq 2\) if and only if \(q = 1, 2, 3, 4, 6\).
Theorem 3.1.2 and 3.1.3), it holds that

Proof. By the functional equation (1.1), it suffices to estimate

According to the inequalities

ξ is a function of at most 1. The order is exactly 1 since one has

For ζ, proved by the Bohr-Landau method (see for instance [17, Theorem 9.15 (A)]), the mean square of ζ(s, a) and Li_s(e^{2πia}) (see [9, Theorem 4.2.1]) and the inequality

The lower bounds for the number of zeros of ζ(s, a) in the half-planes 1 < σ < 1 + δ and −δ < σ < 0 are proved by [1, (2.11)] and [1, Corollary]. Furthermore, the lower bounds for the number of zeros of ζ(s, a) in the half-planes σ_1 < σ < σ_2 and 1 − σ_2 < σ < 1 − σ_1 are shown by [1, (2.11)] and [6, Theorem 2] and the definition of ζ(s, a).

2.3. Proof of Proposition 1.4. First we show the Hadamard product formula for the quadrilateral zeta function.

Lemma 2.3. For any 0 < a < 1/2 and η > 0, there exists σ_0(η) \geq 3/2 such that |ζ(s, a)| > η for all \Re(s) \geq σ_0(η).

Proof. Fix 0 < a < 1/2. When σ \geq 3/2 we have

by the series expression of ζ(s, a). Note that the function a^{-σ} − (1−a)^{-σ} is monotonically increasing with respect to σ > 1. Hence the inequality above implies Lemma 2.3 with 0 < a < 1/2. When a = 1/2 and σ \geq 3/2, one has

Hence we have this lemma for 0 < a \leq 1/2.

Lemma 2.4. For 0 < a \leq 1/2, define the function ξ_Q(s, a) by

2ξ_Q(s, a) := s(s−1)\pi^{-s/2}Γ(s/2)ζ(s, a).

Then ξ_Q(s, a) is an entire function of order 1.

Proof. By the functional equation (1.1), it suffices to estimate |ξ_Q(s, a)| on the half-plane \Re(s) \geq 1/2. From the approximate functional equations of ζ(s, a) and Li_s(e^{2πia}) (see [9, Theorems 3.1.2 and 3.1.3]), it holds that

According to the inequalities

where c_1 and c_2 are some positive constants (see for instance [7, p. 20]), ξ_Q(s, a) is a function of at most 1. The order is exactly 1 since one has

4Q(σ, a) \geq a^{-σ}, \quad \log Γ(σ) \geq (σ − 1/2) log σ − 2σ,
where $\sigma > 0$ is sufficiently large, by the general Dirichlet series expression of $Q(s,a)$ and Stirling’s formula.

Proposition 2.5. Let $0 < a \leq 1/2$ and $\rho_a$ be the zeros of $\xi_Q(s,a)$. Then we have
\[
\xi_Q(s,a) = e^{A+B(a)s} \prod_{\rho_a} \left(1 - \frac{s}{\rho_a}\right) e^{s/\rho_a}, \quad e^A = \frac{1}{2}, \quad B(a) = \frac{Q'(0,a)}{Q(0,a)} - 1 - \frac{\gamma_E + \log \pi}{2},
\]
where $\gamma_E$ is the Euler constant.

Proof. By Lemma 2.4 and Hadamard’s factorization theorem, we only determine the constants $A$ and $B(a)$. From (2.4), one has
\[
\xi_Q(0,a) = -Q(0,a) = 1/2 = e^A.
\]
Hence it holds that
\[
Q(s,a) = \frac{e^{B(a)s} \pi^{s/2}}{s(s-1)\Gamma(s/2)} \prod_{\rho_a} \left(1 - \frac{s}{\rho_a}\right) e^{s/\rho_a} = \frac{e^{(B(a)+(\log \pi)/2)s}}{2(s-1)\Gamma(s/2+1)} \prod_{\rho_a} \left(1 - \frac{s}{\rho_a}\right) e^{s/\rho_a}.
\]
By taking the logarithmic derivative of the formula above, we obtain
\[
\frac{Q'(s,a)}{Q(s,a)} = B(a) + \frac{\log \pi}{2} - \frac{1}{s-1} - \frac{1}{2} \frac{\Gamma'(s/2+1)}{\Gamma(s/2+1)} + \sum_{\rho_a} \left(\frac{1}{s-\rho_a} + \frac{1}{\rho_a}\right)
\]
By making $s \to 0$, we have
\[
\frac{Q'(0,a)}{Q(0,a)} = B(a) + 1 + \frac{\gamma_E + \log \pi}{2}
\]
which implies Proposition 2.5.

Proposition 2.6. Let $0 < a \leq 1/2$, $\sigma_a := \sigma'_a + 1$, where $\sigma'_a := \sigma'_a(\eta) \geq 3/2$ is given in Lemma 2.3, and $\gamma_a$ be the imaginary part of the zeros of $\xi_Q(s,a)$. Then for $1-\sigma_a \leq \sigma \leq \sigma_a$ and $s = \sigma + it$, it holds that
\[
\frac{Q'(s,a)}{Q(s,a)} = -\frac{1}{s-1} + \sum_{|t-\gamma_a| \leq 1} \frac{1}{s-\rho_a} + O_a(|t|+2)).
\]

Proof. From (2.12) and the Stirling formula
\[
\frac{\Gamma'(s/2)}{\Gamma(s/2)} = \log(s/2) + O_a(|s|^{-1}),
\]
it holds that
\[
\frac{Q'(s,a)}{Q(s,a)} = -\frac{1}{s-1} + \sum_{\rho_a} \left(\frac{1}{s-\rho_a} + \frac{1}{\rho_a}\right) + O_a(|t|+2)).
\]
By putting $s = \sigma_a + it$, we obtain
\[
\sum_{\rho_a} \left(\frac{1}{\sigma_a + it - \rho_a} + \frac{1}{\rho_a}\right) = O_a(|t|+2))
\]
since the general Dirichlet series of $Q(s,a)$ converges absolutely and does not vanish when $\sigma \geq \sigma_a = \sigma'_a + 1$ from Lemma 2.3. Let $\rho_a := \beta_a + i\gamma_a$. Then, the real part of the each term in the infinite summation above can be expressed as
\[
\frac{\sigma_a - \beta_a}{(\sigma_a - \beta_a)^2 + (t - \gamma_a)^2} + \frac{\beta_a}{\beta_a^2 + \gamma_a^2} \geq \frac{\sigma_a - \beta_a}{(\sigma_a - \beta_a)^2 + (t - \gamma_a)^2} \gg \frac{1}{1 + (t - \gamma_a)^2}.
\]
Hence the infinite summation in (2.18) with $|t| > |t - \gamma|_1$ is $O_a(\log(|t| + 2))$ by (2.16). Moreover, it holds that

\[
\sum_{|t - \gamma|_1 \leq 1} \frac{1}{\sigma_a + it - \rho_a} = O_a(\log(|t| + 2))
\]

from (2.17) and the inequality $|\sigma_a + it - \rho_a| \geq 1$ which is proved by Lemma 2.3 and the definitions of $\sigma_a'$ and $\sigma_a$. Therefore, we have (2.13) by (2.18).

**Proof of Proposition 1.5** The proof method is using the argument principle (see for instance [2, Section 1.8] and [17, Section 9.3]). Fix $0 < a \leq 1/2$ and assume that $Q(s, a)$ does not vanish on the line $\Im(s) = 0$. Let $C$ be the rectangular contour with vertices $s = \sigma_a \pm T$, $s = 1 - \sigma_a \pm T$, where $\sigma_a > 0$ is given in Proposition 2.6. By Theorem 1.1 and Littlewood’s Lemma, it holds that

\[
N(T, Q(s, a)) = \frac{1}{2\pi i} \int_C \frac{\xi_Q'(s, a)}{\xi_Q(s, a)} ds + O_a(1).
\]  

From the definition of $\xi_Q(s, a)$, one has

\[
\frac{\xi_Q'(s, a)}{\xi_Q(s, a)} = \frac{1}{s} + \frac{1}{s - 1} - \frac{\log \pi}{2} + \frac{1}{2} \frac{\Gamma'(s/2)}{\Gamma(s/2)} + \frac{Q'(s, a)}{Q(s, a)}.
\]  

The equation (2.19) can be expressed as

\[
N(T, Q(s, a)) + O_a(1) = \frac{1}{2\pi} \int_{-T}^{T} \left( \frac{\xi_Q'(\sigma_a + it, a)}{\xi_Q(\sigma_a + it, a)} - \frac{\xi_Q'(1 - \sigma_a + it, a)}{\xi_Q(1 - \sigma_a + it, a)} \right) dt
\]

\[
+ \frac{1}{2\pi i} \int_{\sigma_a}^{1 - \sigma_a} \left( \frac{\xi_Q'(\sigma - iT, a)}{\xi_Q(\sigma - iT, a)} - \frac{\xi_Q'(\sigma + iT, a)}{\xi_Q(\sigma + iT, a)} \right) d\sigma
\]

\[
=: I_1 + I_2,
\]

where $I_1$ and $I_2$ are the first and second integrals in the last formula.

First we find an upper bound of for $|I_2|$. By (2.14) and the definition of $I_2$, we have

\[
I_2 = \frac{1}{2\pi i} \int_{1 - \sigma_a}^{\sigma_a} \left( \frac{Q'(\sigma - iT, a)}{Q(\sigma - iT, a)} - \frac{Q'(\sigma + iT, a)}{Q(\sigma + iT, a)} \right) d\sigma + O_a(\log T).
\]

From Proposition 2.6 one has

\[
\frac{Q'(s, a)}{Q(s, a)} = \sum_{|T - \gamma|_1 \leq 1} \frac{1}{\sigma + iT - \rho_a} + O_a(\log T),
\]  

(2.21)

(2.22)
where \( \rho_a \) are the zeros of \( \xi_Q(s, a) \). Now let \( C' \) be the rectangular contour with vertices \( s = 1 - \sigma_a + iT, \ s = \sigma_a + iT, \ s = \sigma_a + i(T - 2), \ s = 1 - \sigma_a + i(T - 2) \). Note that the number of \( \rho_a \) satisfying \( T - 2 \leq \Im(\rho_a) \leq T \) is \( O_a(\log T) \) from (2.17). Hence, we obtain

\[
\int_{C'} \left( \sum_{|T - \gamma_a| \leq 1} \frac{1}{s - \rho_a} \right) ds = O_a(\log T).
\]

This integral over \( C' \) can also be written as

\[
\int_{C'} \left( \sum_{|T - \gamma_a| \leq 1} \frac{1}{s - \rho_a} \right) ds = \sum_{T - 1 \leq \gamma_a \leq T + 1} \int_{C'} \frac{ds}{s - \rho_a} = \sum_{T - 1 \leq \gamma_a \leq T + 1} \int_{1 - \sigma_a}^{\sigma_a} \frac{d\sigma}{\sigma + iT - \rho_a} + \sum_{T - 1 \leq \gamma_a \leq T + 1} \int_{1 - \sigma_a}^{\sigma_a} \frac{dt}{\sigma + i(T - 2) - \rho_a} - i \sum_{T - 1 \leq \gamma_a \leq T + 1} \int_{T - 2}^{T} \frac{dt}{1 - \sigma_a + iT - \rho_a} + \sum_{T - 1 \leq \gamma_a \leq T + 1} \int_{1 - \sigma_a}^{\sigma_a} \frac{d\sigma}{\sigma + i(T - 2) - \rho_a}
\]

We can easily see that the last three sums are \( O_a(\log T) \). Hence the first sum is also \( O_a(\log T) \). From this fact and (2.22), we can conclude that

\[
I_2 = O_a(\log T).
\]

Next we estimate \( I_1 \), making use of (2.20) and the relation

\[
\frac{\xi'_Q(s, a)}{\xi_Q(s, a)} = -\frac{\xi'_Q(1 - s, a)}{\xi_Q(1 - s, a)}.
\]

From (2.14), (2.20) and the formula above, we have

\[
\frac{\xi_Q(\sigma_a + it, a)}{\xi_Q(\sigma_a + it, a)} = \frac{\xi_Q(1 - \sigma_a + it, a)}{\xi_Q(1 - \sigma_a + it, a)} = \frac{\xi'_Q(\sigma_a + it, a)}{\xi_Q(\sigma_a + it, a)} + \frac{\xi'_Q(\sigma_a - it, a)}{\xi_Q(\sigma_a - it, a)} \frac{\sigma_a^2 + t^2}{\sigma_a^2 + (1 - \sigma_a)^2 + t^2} - \log \pi + \frac{1}{2} \log \frac{\sigma_a^2 + t^2}{\sigma_a^2} + O_a((\sigma_a^2 + t^2)^{-1/2})
\]

\[
+ \frac{Q'(\sigma_a + it, a)}{Q(\sigma_a + it, a)} + \frac{Q'(\sigma_a - it, a)}{Q(\sigma_a - it, a)}.
\]

Obviously, one has

\[
\int_{-T}^{T} \frac{Q'(\sigma_a + it, a)}{Q(\sigma_a + it, a)} dt = \int_{-T}^{T} \frac{(a^{-\sigma_a - it})'}{a^{-\sigma_a - it}} dt + \int_{-T}^{T} \frac{(a^{\sigma_a + it}Q(\sigma_a + it, a))'}{a^{\sigma_a + it}Q(\sigma_a + it, a)} dt.
\]

For the first integral, we have

\[
\int_{-T}^{T} \frac{(a^{-\sigma_a - it})'}{a^{-\sigma_a - it}} dt = -i \left[ \log a^{\sigma_a - it} \right]_{-T}^{T} = -2T \log a.
\]

For some \( \theta > 0 \), without loss of generality we can assume that \( \sigma_a \) satisfies

\[
2\Re(a^{\sigma_a + it}Q(\sigma_a + it, a)) \geq 1 - \left( \frac{a}{1 - a} \right)^{\sigma_a} - 4a^{\sigma_a} \zeta(\sigma_a) > \theta
\]

when \( 0 < a < 1/2 \) by modifying the proof of Lemma 2.3. One can assume similarly when \( a = 1/2 \) (see the proof of Lemma 2.3). Hence we obtain

\[
\int_{-T}^{T} \frac{(a^{\sigma_a + it}Q(\sigma_a + it, a))'}{a^{\sigma_a + it}Q(\sigma_a + it, a)} dt = -i \left[ (\log a^{\sigma_a + it}Q(\sigma_a + it, a)) \right]_{-T}^{T} = O_a(1).
\]
Therefore, it holds that
\[
I_1 = \frac{1}{2\pi} \int_{-T}^{T} \left( \frac{\xi_Q(\sigma_a + it, a)}{\xi_Q(\sigma_a + it, a)} + \frac{\xi_Q'(\sigma_a - it, a)}{\xi_Q'(\sigma_a - it, a)} \right) dt
\]
\[
= -\frac{T}{\pi} \log \pi - \frac{T}{\pi} \log 2 + \frac{T}{\pi} \log T - \frac{T}{\pi} - \frac{2T}{\pi} \log a + O_a(\log T)
\]
\[
= \frac{T}{\pi} \log T - \frac{T}{\pi} \log(2e\pi a^2) + O_a(\log T).
\]
The theorem in this case follows from the formula above and the bound for $I_2$. If we suppose that $Q(s, a)$ has zeros on the line $\Im(s) = T$, then the theorem follows from the case above of the theorem along with [2.17]. □

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References

[1] T. M. Apostol, *Introduction to Analytic Number Theory*. Undergraduate Texts in Mathematics, Springer, New York, 1976.

[2] S. Baier, K. Srinivas and U. K. Sanga, *A note on the gaps between zeros of Epstein's zeta-functions on the critical line*. Funct. Approx. Comment. Math. 57 (2017), no. 2, 235–253.

[3] P. T. Bateman and E. Grosswald, *On Epstein's zeta function*, Acta Arith. 9 (1964), 365–373.

[4] S. Chowla and A. Selberg, *On Epstein's zeta function. I*, Proc. Nat. Acad. Sci. U. S. A. 35 (1949), 371–374.

[5] H. Davenport and H. Heilbronn, *On the zeros of certain Dirichlet series*. J. Lond. Math. Soc. 11 (1936), 181–185, 307–312.

[6] J. Kaczorowski and M. Kulas, *On the non-trivial zeros off line for $L$-functions from extended Selberg class*, Monatshefte Math. 150 (2007), no. 3, 217–232.

[7] A. A. Karatsuba and S. M. Voronin, *The Riemann zeta-function*. Translated from the Russian by Neal Koblitz. De Gruyter Expositions in Mathematics, 5. Walter de Gruyter & Co., Berlin, 1992.

[8] Y. Lamzouri, *Zeros of the Epstein zeta function to the right of the critical line*. (arXiv:1907.06387).

[9] A. Laurinčikas and R. Garunkštis, *The Lerch zeta-function*. Kluwer Academic Publishers, Dordrecht, 2002.

[10] J. Steuding, *On the zero-distribution of Epstein zeta-functions*. Math. Ann. 333 (2005), no. 3, 689–697.

[11] S. M. Voronin, *The zeros of zeta-functions of quadratic forms* (in Russian), Tr. Mat. Inst. Steklova 142 (1976), 135–147.

[12] E. C. Titchmarsh, *The theory of the Riemann zeta-function*, Second edition. Edited and with a preface by D. R. Heath-Brown. The Clarendon Press, Oxford University Press, New York, 1986.