SCISSORS CONGRUENCE GROUPS AND THE THIRD HOMOLOGY OF SL₂ OF LOCAL RINGS AND FIELDS

KEVIN HUTCHINSON

Abstract. We describe the third homology of \( SL_2 \) of local rings, over \( \mathbb{Z}[\frac{1}{2}] \), in terms of a refined Bloch group. We use this to derive a localization sequence for the third homology of \( SL_2 \) of certain discrete valuation rings and to calculate the \( H_3 \) of \( SL_2 \) of higher-dimensional local fields and their associated discrete valuation rings in terms of indecomposable \( K_3 \) and scissors congruence groups of intermediate residue fields.

1. Introduction

The study of the homology of special linear groups divides naturally into the stable and unstable cases. For a field \( F \) and for given \( n \in \mathbb{N} \) when \( m \) is sufficiently large (see \cite{25, 10}) the groups \( H_n(SL_m(F), \mathbb{Z}) \) stabilize (in the sense that they become independent of \( m \)). These stable homology groups embed as natural direct summands in the corresponding homology of the general linear group \( H_n(GL_m(F), \mathbb{Z}) \) and their calculation is therefore closely tied to the calculation of the homology of the general linear group and to algebraic K-theory. When \( m \) is small, by contrast, it usually happens that neither the stabilization maps nor the map to the homology of the general linear group is injective and the kernels of these maps are interesting invariants of the field.

For a field or local ring \( A \), the structure of \( H_2(SL_m(A), \mathbb{Z}) \) is well-understood, thanks to the Theorem of Matsumoto-Moore (\cite{13, 17}), and the results of W. van der Kallen (\cite{24} and A. Suslin (\cite{22}): Stability begins at \( m = 3 \) and \( H_2(SL_m(A), \mathbb{Z}) \cong K_2(A) = K_2^M(A) \) for all \( m \geq 3 \). For a field \( F \), the map \( H_2(SL_2(F), \mathbb{Z}) \to H_2(SL_3(F), \mathbb{Z}) \) is surjective and the kernel is naturally \( I^3(F) \), the third power of the fundamental ideal of the Witt ring of the field \( F \).

For the groups \( H_3(SL_m(F), \mathbb{Z}) \), stability also begins at \( m = 3 \) and

\[
H_3(SL_m(F), \mathbb{Z}) \cong K_3(F)/\{−1\} \cdot K_2(F)
\]

for all \( m \geq 3 \) (\cite{23, 9}). The map \( H_3(SL_2(F), \mathbb{Z}) \to H_3(SL_3(F), \mathbb{Z}) \) has cokernel isomorphic to \( 2K_3^M(F) \) and image isomorphic - up to 2-torsion - to \( K_3^{\text{ind}}(F) \) (\cite{9}). The group \( H_3(SL_2(F), \mathbb{Z}) \) has been much studied because of its connections with K-theory, the dilogarithm function, hyperbolic geometry and other topics (\cite{11, 20, 23, 27, 5, 2, 18}). However, its structure for general fields or rings is far from being understood. For instance the structure of \( H_3(SL_2(\mathbb{Q}), \mathbb{Z}) \) is not yet known (but see \cite{8}).

For any field \( F \) there is a natural surjective homomorphism

\[
H_3(SL_2(F), \mathbb{Z}) \longrightarrow K_3^{\text{ind}}(F)
\]
from the third homology of \( SL_2(F) \) to the indecomposable \( K_3 \) of the field (see \([9]\) for infinite fields and \([6]\) for finite fields). This map is an isomorphism when \( F \) is algebraically, or even quadratically, closed (\([20, 14]\)).

In fact the map \((1)\) is naturally a map of \( R_F \)-modules where \( R_F \) is the group ring \( \mathbb{Z}[F^x/(F^x)^2] \). The action of \( R_F \) on \( K_3^{\text{ind}}(F) \) is trivial, while the square class \( \langle a \rangle \) of the element \( a \in F^x \) acts on the left via conjugation by the matrix \( \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \) (or more generally by any \( 2 \times 2 \) matrix with determinant \( a \)). It turns out – see \([4, 14]\) – that essentially (i.e. up to some possible 2-torsion) the only obstruction to the injectivity of the map \((1)\) is the nontriviality or otherwise of the action of \( R_F \) on \( H_3(SL_2(F), \mathbb{Z}); \) i.e. there is an induced isomorphism
\[
H_3(SL_2(F), \mathbb{Z}[\frac{1}{2}])_{F^x} \cong K_3^{\text{ind}}(F)[\frac{1}{2}] .
\]

Equivalently we have
\[
\text{Ker}(H_3(SL_2(F), \mathbb{Z}[\frac{1}{2}]) \to K_3^{\text{ind}}(F)[\frac{1}{2}]) = \text{I}_F H_3(SL_2(F), \mathbb{Z}[\frac{1}{2}])
\]
where \( \text{I}_F \) is the augmentation ideal in the group algebra \( R_F \).

Now for a field \( F \), \( K_3^{\text{ind}}(F) \) is closely related to \( \mathcal{B}(F) \), the Bloch group of \( F \) (\([3, 23]\)). This is a natural subgroup of the scissors congruence group \( \mathcal{P}(F) \) (or pre-Bloch group in the terminology of \([23]\)), which is described by a presentation where the relations are derived from the 5-term functional equation of dilogarithm function.

In \([6]\) we introduced an \( R_F \)-module \( \mathcal{RB}(F) \), the refined Bloch group of the field \( F \), which is closely related to \( H_3(SL_2(F), \mathbb{Z}) \) – for details see section 2.4 below – and which can be explicitly calculated in some interesting cases (\([8]\)). \( \mathcal{RB}(F) \) surjects naturally onto \( \mathcal{B}(F) \) with kernel \( \mathcal{RB}_0(F) \) and we have for any field \( F \)
\[
\mathcal{RB}_0(F)[\frac{1}{2}] = \text{I}_F \mathcal{RB}(F)[\frac{1}{2}] = \text{I}_F H_3(SL_2(F), \mathbb{Z}[\frac{1}{2}]).
\]

The main purpose of the present article is firstly to generalize the results of \([6]\) and \([8]\) from fields to local rings – including the case of finite residue fields – and then to apply these results to compare the third homology of \( SL_2 \) of local rings with that of their fields of fractions and with related invariants of their residue fields. This leads to what we call a localization sequence for the third homology of \( SL_2 \), with coefficients in \( \mathbb{Z}[\frac{1}{2}] \), of a certain (quite restricted) class of discrete valuation rings. This takes the form of a short exact sequence
\[
0 \to H_3(SL_2(A), \mathbb{Z}[\frac{1}{2}]) \to H_3(SL_2(K), \mathbb{Z}[\frac{1}{2}]) \to \overline{\mathcal{RP}}_1(F)[\frac{1}{2}] \to 0
\]
where \( A \) is a discrete valuation ring with field of fractions \( K \) and residue field \( F \). In fact, the sequence is defined for any discrete valuation ring and we would expect it to be exact for a larger class of rings than those to which our methods of proof apply.

In this exact sequence, the functor \( \overline{\mathcal{RP}}_1(F) \) is essentially a refined version of the classical scissors congruence group of the field \( F \). In particular, we show that for a large class of higher-dimensional local fields it can be expressed as a direct sum of the scissors congruence group of the field and of copies of the scissors congruence groups of the intermediate residue fields.

We refer to the sequence \((2)\) as a localization sequence because of an analogy with \( K \)-theory and related functors. For example, in a forthcoming article (\([7]\)) we show that for an infinite field \( F \) there is a natural (split) short exact sequence
\[
0 \to H_3(SL_2(F), \mathbb{Z}[\frac{1}{2}]) \to H_3(SL_2(F[T, T^{-1}]), \mathbb{Z}[\frac{1}{2}]) \to \overline{\mathcal{RP}}_1(F)[\frac{1}{2}] \to 0.
\]
(There is also a corresponding statement for \( H_3(SL_2(F), \mathbb{Z}) \) where the associated functor is Milnor-Witt \( K_1 \).)
The article is laid out as follows:

In section 2 we review the definitions of (refined) scissors congruence groups and Bloch groups of commutative rings and recall the required results from [6] and [8].

In section 3 we prove (Theorem 3.22) that for local rings $A$ with sufficiently large residue field there is a natural short exact sequence

$$0 \to \text{tor}(\mu_A, \mu_A) \left[ \frac{1}{2} \right] \to H_3(\text{SL}_2(A), \mathbb{Z} \left[ \frac{1}{2} \right]) \to \mathcal{R}\mathcal{B}(A) \left[ \frac{1}{2} \right] \to 0.$$ 

The proof follows the same route as the proof of the corresponding theorem for fields in [6], but we supply all the necessary details for the convenience of the reader.

In section 4, we consider certain submodules and quotient modules of the refined scissors congruence group $\mathcal{R}\mathcal{P}(A)$ which play an important role in our calculations. Again, we are here following a route already covered in the case of fields. Only small adaptations are needed to extend the results for fields to the more general case of local rings, but we include (most) details for the reader’s convenience. In particular, the key identity $\langle\langle x \rangle\rangle_D = \psi_1(x) - \psi_2(x)$ in $\mathcal{R}\mathcal{P}(A)$ (see Theorem 4.19) is crucial to our later calculations.

In section 5 we re-visit the specialization homomorphism for the refined Bloch group of a field with valuation which was introduced in [8] and used to calculate the third homology of $S\text{L}_2$ of local and global fields. The small technical improvement here is that we re-prove the specialization theorem replacing the original target of this map, the module $\mathcal{R}\mathcal{P}(k)_F$, with the larger module $\mathcal{R}\mathcal{P}(k)_F$ (Theorem 5.2). (The improvement is not hugely significant: there is a surjective map $\mathcal{R}\mathcal{P}(k)_F \to \mathcal{R}\mathcal{P}(k)_F$ whose kernel is annihilated by 3.)

In section 6 we use the specialization homomorphism to compare the refined Bloch group of a field with discrete valuation and finitely many square classes with the refined scissors congruence group of the residue field. In this section, we also compare the refined scissors congruence group of a local ring to that of its residue field. We apply these results and Theorem 3.22 to calculate the third homology of $S\text{L}_2$ of some higher-dimensional local fields, greatly generalizing the main result of [8]. The restriction to fields with finitely many square classes here is primarily an artefact of our method of proof: All of the objects of study are modules over the group of square classes of the field or local ring, and we compare related modules by comparing the associated eigenspaces for the characters of the group of square classes.

In section 7 we use the results of the previous section to prove the localization theorem (Theorem 7.2). Although we would expect the theorem to be much more widely valid, our methods of proof restrict us to a very small class of discrete valuation rings; those whose residue field either has at most two square classes or has finitely many square classes and a discrete valuation satisfying a number of conditions.

Finally, in section 8 we detail a range of particular cases of the results of sections 6 and 7.

2. Scissors congruence groups and Bloch Groups

2.1. Preliminaries and Notation. For a commutative ring $A$, we let $U_A$ denote the group of units of $A$ and we let $G_A$ denote the multiplicative group, $U_A/U_A^2$, of square classes of $U_A$. For $x \in U_A$, we will let $\langle x \rangle \in G_A$ denote the corresponding square class. Let $R_A$ denote the integral group ring $\mathbb{Z}[G_A]$ of the group $G_A$. We will use the notation $\langle\langle x \rangle\rangle$ for the basis elements, $\langle x \rangle - 1$, of the augmentation ideal $I_A$ of $R_A$.

For any $a \in U_A$, we will let $p^a$ and $p^{-a}$ denote respectively the elements $1 + \langle a \rangle$ and $1 - \langle a \rangle$ in $R_A$. 

For any abelian group $G$ we will let $G\left[\frac{1}{2}\right]$ denote $G \otimes \mathbb{Z}\left[\frac{1}{2}\right]$. For an integer $n$, we will let $n'$ denote the odd part of $n$. Thus if $G$ is a finite abelian group of order $n$, then $G\left[\frac{1}{2}\right]$ is a finite abelian group of order $n'$.

We let $e^a_+$ and $e^a_-$ denote respectively the mutually orthogonal idempotents

$$e^a_+ := \frac{p^a_+}{2} = \frac{1 + \langle a \rangle}{2}, \quad e^a_- := \frac{p^a_-}{2} = \frac{1 - \langle a \rangle}{2} \in \mathbb{R}[\frac{1}{2}].$$

(Of course, these operators depend only on the class of $a$ in $\mathcal{G}_A$.)

For an abelian group $G$ and $n \in \mathbb{N}$, $G[n]$ will denote the subgroup $\{g \in G \mid ng = 0\}$.

### 2.2. Indecomposable $K_3$.

Let $A$ be a either a local ring or a field. Let $K_*(A)$ denote the Quillen $K$-theory of $A$ and let $K^M_*(A)$ be the Milnor $K$-theory. There is a natural homomorphism of graded rings $K^M_*(A) \to K_*(A)$. Indecomposable $K_3$ of $A$ is the group $K_3^{\text{ind}}(A) := \text{Coker}(K^M_3(\to)K_3(A))$. We will require the following theorem from $K$-theory:

**Theorem 2.1.** Let $A$ be discrete valuation ring with field of fractions $K$ and residue field field $F$. Suppose that either $\text{char}(K) = \text{char}(F)$ or that $F$ is finite. Then the inclusion $A \to K$ induces an isomorphism $K_3^{\text{ind}}(A) \cong K_3^{\text{ind}}(K)$.

**Proof.** Let $F$ be the residue field of $A$ and let $\pi$ be a uniformizer. There is a commutative diagram with exact columns

$$
\begin{array}{ccccccccc}
0 & & & & & & & & \\
\downarrow & & & & & & & & \\
K^M_3(A) & \longrightarrow & K^M_3(K) & \delta_3 & K^M_2(F) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
K_3(A) & \longrightarrow & K_3(K) & \delta_3 & K_2(F) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
K_3^{\text{ind}}(A) & \longrightarrow & K_3^{\text{ind}}(K) & & & & \\
\downarrow & & & & & & & & \\
0 & & 0 & & & & & & \\
\end{array}
$$

The exactness of the top row is well-known (see [26 Corollary 6.6.2] for example). The second row is exact by Gersten’s conjecture, which is known for the case of equicharacteristic discrete valuation rings or discrete valuation rings with finite residue fields (see, for example, [26 Corollary 6.7.2, Corollary 6.9.2]). The result follows by the snake lemma. \hfill \square

### 2.3. Scissors congruence groups and Bloch groups.

For a commutative ring $A$, let $\mathcal{W}_A$ denote the set $\{u \in U_A : 1-u \in U_A\}$. The scissors congruence group or pre-Bloch group, $\mathcal{P}(A)$, is the group generated by the elements $[x], x \in \mathcal{W}_A$, subject to the relations

$$R_{xy} : \quad [x] - [y] + \left[\frac{y}{x}\right] - \left[\frac{1 - x^{-1}}{1 - y^{-1}}\right] = \left[\frac{1 - x}{1 - y}\right] \quad \text{for all } x, y, x/y \in \mathcal{W}_A.$$

Given an abelian group $G$ we let $S^2_\geq (G)$ denote the group

$$\frac{G \otimes \geq G}{< x \otimes y + y \otimes x | x, y \in G >}.$$
and, for \(x, y \in G\), we denote by \(x \circ y\) the image of \(x \otimes y\) in \(S^2_\mathbb{Z}(G)\).

For a commutative ring \(A\), the map
\[
\lambda : \mathcal{P}(A) \to S^2_\mathbb{Z}(U_A), \quad [x] \mapsto (1 - x) \circ x
\]
is well-defined, and the Bloch group of \(A\), \(\mathcal{B}(A) \subset \mathcal{P}(A)\), is defined to be the kernel of \(\lambda\).

For a field \(F\), the Bloch group is known to be closely related to the indecomposable \(K_3\): There is a natural exact sequence
\[
0 \to \text{Tor}_1^F(\mu_F, \mu_F) \to K_3^{\text{ind}}(F) \to \mathcal{B}(F) \to 0
\]
where \(\text{Tor}_1^F(\mu_F, \mu_F)\) is the unique nontrivial extension of \(\text{Tor}_1^F(\mu_F, \mu_F)\) by \(\mathbb{Z}/2\). (See Suslin [23] for infinite fields and [6] for finite fields.)

2.4. The refined Bloch group and \(H_3(\text{SL}_2(F), \mathbb{Z})\). For any field \(F\) there is a natural surjective homomorphism
\[
H_3(\text{SL}_2(F), \mathbb{Z}) \twoheadrightarrow K_3^{\text{ind}}(F).
\]
When \(F\) is quadratically closed (i.e. when \(G_F = 1\)) this map is an isomorphism.

However, for any commutative ring \(A\), the group extension
\[
1 \to \text{SL}_2(A) \to \text{GL}_2(A) \to U_A \to 1
\]
induces an action – by conjugation – of \(U_A\) on \(H_*(\text{SL}_2(A), \mathbb{Z})\) which factors through \(G_A\).

For a field \(F\), it can be shown that the map (3) induces an isomorphism
\[
H_3(\text{SL}_2(F), \mathbb{Z}) \bigl[\frac{1}{2}\bigr]_{G_F} \cong K_3^{\text{ind}}(F) \bigl[\frac{1}{2}\bigr]
\]
(see [14]), but – as our calculations, in [8] and below, show – the action of \(G_F\) on \(H_3(\text{SL}_2(F), \mathbb{Z})\) is in general non-trivial.

Thus \(H_3(\text{SL}_2(A), \mathbb{Z})\) is naturally an \(R_A\)-module, and for general fields or rings, in order to give a Bloch-type description of it, we must incorporate the \(R_A\)-module structure at each stage of the process.

The refined scissors congruence group or refined pre-Bloch group, \(\mathcal{RP}(A)\), of a commutative ring \(A\) which has at least 4 elements, is the \(R_A\)-module with generators \([x], x \in \mathcal{W}_A\) subject to the relations
\[
S_{x,y} : 0 = [x] - [y] + \langle x \rangle \biggl[\frac{y}{x} - x^{-1} - 1\biggr] \biggl[\frac{1 - x^{-1}}{1 - y^{-1}}\biggr] + \langle 1 - x \rangle \biggl[\frac{1 - x}{1 - y}\biggr], \quad \text{for all } x, y, x/y \in \mathcal{W}_A
\]

Of course, from the definition it follows immediately that \(\mathcal{P}(A) = (\mathcal{RP}(A))_{G_A} = H_0(G_A, \mathcal{RP}(A))\).

Let \(\Lambda = (\lambda_1, \lambda_2)\) be the \(R_A\)-module homomorphism
\[
\mathcal{RP}(A) \to I_A^2 \oplus S^2_\mathbb{Z}(U_A)
\]
where \(\lambda_1 : \mathcal{RP}(A) \to I_A^2\) is the map \([x] \mapsto \langle 1 - x \rangle \langle x \rangle\), and \(\lambda_2\) is the composite
\[
\mathcal{RP}(A) \xrightarrow{\Lambda} \mathcal{P}(A) \xrightarrow{\lambda_1} S^2_\mathbb{Z}(U_A).
\]

It can be shown that \(\Lambda\) is well-defined.

The refined Bloch group of the commutative ring \(A\) (with at least 4 elements) to be the \(R_A\)-module
\[
\mathcal{RB}(A) := \ker(\Lambda : \mathcal{RP}(A) \to I_A^2 \oplus S^2_\mathbb{Z}(U_A)).
\]

Remark 2.2. The functor \(\mathcal{RP}_1(A) := \ker(\lambda_1)\) will also play an essential role in what follows.
We recall some results from [6]: The main result there is

**Theorem 2.3.** Let $F$ be any field.

1. If $F$ is infinite, there is a natural complex
   \[ 0 \to \text{Tor}_1^R(\mu_F, \mu_F) \to H_3(\text{SL}_2(F), \mathbb{Z}) \to \mathcal{RB}(F) \to 0. \]
   which is exact everywhere except possibly at the middle term. The middle homology is annihilated by 4.
   In particular, for any infinite field there is a natural short exact sequence
   \[ 0 \to \text{Tor}_1^R(\mu_F, \mu_F) \left[ \frac{1}{2} \right] \to H_3(\text{SL}_2(F), \mathbb{Z} \left[ \frac{1}{2} \right]) \to \mathcal{RB}(F) \left[ \frac{1}{2} \right] \to 0. \]
2. If $F = \mathbb{F}_q$ is finite of characteristic $p$, there is a natural short exact sequence
   \[ 0 \to \text{Tor}_1^R(\mu_F, \mu_F) \to H_3(\text{SL}_2(F), \mathbb{Z} \left[ \frac{1}{p} \right]) \to \mathcal{B}(F) \to 0. \]
   and furthermore $H_3(\text{SL}_2(F), \mathbb{Z}) = H_3(\text{SL}_2(F), \mathbb{Z} \left[ \frac{1}{p} \right])$ for $q > 27$.

Now for any field $F$, let
\[ H_3(\text{SL}_2(F), \mathbb{Z})_0 := \text{Ker}(H_3(\text{SL}_2(F), \mathbb{Z}) \to K_3^{\text{ind}}(F)) \]
and
\[ \mathcal{RB}_0(F) := \text{Ker}(\mathcal{RB}(F) \to \mathcal{B}(F)) \]

The following is Lemma 5.2 in [6].

**Lemma 2.4.** Let $F$ be an infinite field. Then

1. $H_3(\text{SL}_2(F), \mathbb{Z} \left[ \frac{1}{2} \right])_0 = \mathcal{RB}_0(F) \left[ \frac{1}{2} \right]$
2. $H_3(\text{SL}_2(F), \mathbb{Z} \left[ \frac{1}{2} \right])_0 = I_F H_3(\text{SL}_2(F), \mathbb{Z} \left[ \frac{1}{2} \right])$ and $\mathcal{RB}_0(F) \left[ \frac{1}{2} \right] = I_F \mathcal{RB}(F) \left[ \frac{1}{2} \right]$.
3. $H_3(\text{SL}_2(F), \mathbb{Z} \left[ \frac{1}{2} \right])_0 = H_3(\text{SL}_2(F), \mathbb{Z} \left[ \frac{1}{2} \right]) \to H_3(\text{SL}_3(F), \mathbb{Z} \left[ \frac{1}{2} \right]) \to H_3(\text{GL}_2(F), \mathbb{Z} \left[ \frac{1}{2} \right])$

**Lemma 2.5.** If the field $F$ is quadratically closed, real-closed or finite then $I_F \mathcal{RB}(F) = 0$, and hence $\mathcal{RB}_0(F) \left[ \frac{1}{2} \right] = 0$ also.

**Proof.** Since $H_3(\text{SL}_2(F), \mathbb{Z})$ maps onto $\mathcal{RB}(F)$ as an $R_F$-module, this follows from the fact that $G_F$ acts trivially on $H_3(\text{SL}_2(F), \mathbb{Z})$ in each of these cases.

For a quadratically closed field this is vacuously true.

This result is proved by Parry and Sah in [19] for the field $\mathbb{R}$, but their argument extends easily to any real-closed field.

For finite fields, the relevant result is Lemma 3.8 of [6].

Furthermore, the calculations in [6], sections 5 and 7, show that

**Lemma 2.6.** Let $q$ be a power of a prime. Then
\[ \mathcal{RB}(\mathbb{F}_q) = \mathcal{B}(\mathbb{F}_q) \cong \begin{cases} \mathbb{Z}/(q + 1)/2, & q \text{ odd} \\ \mathbb{Z}/(q + 1), & q \text{ even} \end{cases} \]
and $\mathcal{B}(\mathbb{F}_q) \left[ \frac{1}{2} \right] = \mathcal{P}(\mathbb{F}_q) \left[ \frac{1}{2} \right]$. 

3. The third homology of $SL_2$ of local rings

In this section, we generalize Theorem 2.3 to local rings with sufficiently large residue field.

3.1. Homological interpretation of $R^n(A)$. Let $A$ be a commutative ring. A row vector $u = (u_1, u_2) \in A^2$ is said to be unimodular if $Au_1 + Au_2 = A$. Equivalently, $u$ is unimodular if there exists $v \in A^2$ such that

$$\begin{bmatrix} u \\ v \end{bmatrix} \in GL_2(A).$$

We let $U_2 = U_2(A)$ denote the set of 2-dimensional unimodular row vectors of $A$. $U_2$ is a right $GL_2(A)$-set. In particular this induces an action of $U_A = Z(GL_2(A))$ acting as multiplication by scalars.

Let

$$U_n^\text{gen} = U_n^\text{gen}(A) := \left\{ (u_1, \ldots, u_n) \in U_2^n : \begin{bmatrix} u_i \\ u_j \end{bmatrix} \in GL_2(A) \text{ for all } i \neq j \right\}.$$

$U_n^A$ acts entry-wise on $U_n^\text{gen}$ and we let $X_n = X_n(A) = U_n^\text{gen}/U_n^A$. Observe that $U_n^\text{gen}$ and $X_n$ are right $GL_2(A)$-sets (with the natural diagonal action).

In particular, $X_1 = U_2/U_A$ and $X_n \subset X_1^n$. If $u = (u_1, u_2) \in U_2$ we will denote the corresponding class in $X_1$ by $\tilde{u}$ or $[u_1, u_2]$.

We have two natural injective maps from $A$ to $X_1$:

$$\iota_+ : A \to X_1, \ a \mapsto a_+ := [a, 1] \text{ and } \iota_- : A \to X_1, \ a \mapsto a_- := [1, a].$$

Clearly, $a_+ = b_-$ in $X_1$ if and only if $a, b \in U_A$ and $b = a^{-1}$. We will identify $U_A$ with its image in $X_1$ under the map $\iota_+$.

If $(u, v) \in U_2^\text{gen}$, we set

$$d(u, v) := \det\begin{bmatrix} u \\ v \end{bmatrix} \in U_A$$

and

$$T_{u,v} := \begin{bmatrix} u \\ v \end{bmatrix}^{-1} \cdot \begin{bmatrix} 0 & -1 \\ d(u, v) & 0 \end{bmatrix} \in SL_2(A).$$

Thus, for $(u, v) \in U_2^\text{gen}$ with $d = d(u, v)$ we have

$$u \cdot T_{u,v} = (0, -1) \text{ and } v \cdot T_{u,v} = (d, 0) \text{ in } U_2$$

and hence

$$\tilde{u} \cdot T_{u,v} = 0_+, \ \tilde{v} \cdot T_{u,v} = 0_- \text{ in } X_1.$$

Let $\phi : U_3^\text{gen} \to X_1$ be the map defined by

$$\phi(u, v, w) := \tilde{w} \cdot T_{u,v} \in X_1.$$

Lemma 3.1. Let $A$ be a commutative ring and let $(u, v, w) \in U_3^\text{gen}$. Then

$$\phi(u, v, w) = \left( \frac{d(u, w) \cdot d(u, v)}{d(v, w)} \right)_+ \in U_A \subset X_1.$$

Proof. A straightforward direct calculation gives

$$w \cdot T_{u,v} = \left( d(u, w), \frac{d(v, w)}{d(u, v)} \right) \in U_2,$$

$\square$
Corollary 3.2. Let $A$ be a commutative ring and let $(u, v, w) \in U^\text{gen}_3$.

1. For all $a, b, c \in U_A$ we have
   $$\phi(u \cdot a, v \cdot b, w \cdot c) = \phi(u, v, w) \cdot a^2.$$

2. For all $X \in \text{GL}_2(A)$ we have
   $$\phi((u, v, w) \cdot X) = \phi(u, v, w) \cdot \det(X).$$

Now, for $n \geq 1$, let
$$Y_n = Y_n(A) := \{(y_1, \ldots, y_n) \in U^n_A : y_i - y_j \in U_A \text{ for all } i \neq j\}.$$ We will consider $Y_n$ as a right $U_A$-set via $(y_1, \ldots, y_n) \cdot a := (y_1a, \ldots, y_na)$.

For $n \geq 3$, let $\Phi_n : U^\text{gen}_n \to U^n_A$ be the map
$$\Phi_n(u_1, \ldots, u_n) := (\phi(u_1, u_2, u_3), \ldots, \phi(u_1, u_2, u_n)).$$

Lemma 3.3. For all $(u_1, \ldots, u_n) \in U^\text{gen}_n$, we have $\Phi_n(u_1, \ldots, u_n) \in Y_{n-2}$.

Proof. For $3 \leq i \leq n$, let $y_i := \phi(u_1, u_2, u_i)$. Let $3 \leq i < j \leq n$. Then
$$\begin{bmatrix} u_i \\ u_j \end{bmatrix} \in \text{GL}_2(A) \implies \begin{bmatrix} u_i \\ u_j \end{bmatrix} \cdot T_{u_1,u_2} \in \text{GL}_2(A).$$

But
$$\begin{bmatrix} u_i \\ u_j \end{bmatrix} \cdot T_{u_1,u_2} = \begin{bmatrix} y_i & 1 \\ y_j & 1 \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$$

for some $a, b \in U_A$. On taking the determinant, it follows that $y_i - y_j \in U_A$. \qed

From Corollary [3.2] we immediately deduce:

Lemma 3.4. Let $(u_1, \ldots, u_n) \in U^\text{gen}_n$.

1. For all $a_1, \ldots, a_n \in U_A$ we have
   $$\Phi_n(u_1 \cdot a_1, \ldots, u_n \cdot a_n) = \Phi_n(u_1, \ldots, u_n) \cdot a_1^2.$$

2. For all $X \in \text{GL}_2(A)$ we have
   $$\Phi_n((u_1, \ldots, u_n) \cdot X) = \Phi_n(u_1, \ldots, u_n) \cdot \det(X).$$

It follows that $\Phi_n$ induces a well-defined map of orbit sets (which we will continue to denote $\Phi_n$)
$$X_n/\text{SL}_2(A) \to Y_{n-2}/U^2_A.$$ Furthermore, this is a map of right $G_A$-sets (noting that the matrix $X \in \text{GL}_2(A)$ acts via the square class of $\det(X)$ on the left).

Proposition 3.5. For all $n \geq 3$, $\Phi_n$ induces a bijection of $G_A$-sets
$$X_n/\text{SL}_2(A) \leftrightarrow Y_{n-2}/U^2_A.$$

Proof. Let $\Psi_n : Y_{n-2} \to U^\text{gen}_n$ be the map
$$(y_3, \ldots, y_n) \mapsto ((0, -1), (1, 0), (y_3, 1), \ldots, (y_n, 1)).$$

Then $\Phi_n \circ \Psi_n = \text{Id}_{Y_{n-2}}$ since $T_{(0,-1),(1,0)}$ is the identity matrix.

Now let $\tilde{\Psi}_n$ be the induced map from $Y_{n-2}$ to $X_n$, given by the formula
$$(y_3, \ldots, y_n) \mapsto (0, 0, (y_3)_+, \ldots, (y_n)_+).$$
For any \( y \in A \) and \( a \in U_A \) we have

\[
[y, 1] \begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix} = [ya, a^{-1}] = [ya^2, 1] = (ya^2)_+ \text{ in } X_1.
\]

It follows that \( \bar{\Psi}_n \) induces a well-defined map \( Y_{n-2}/U_A^2 \to X_n/\text{SL}_2(A) \) satisfying \( \Phi_n \circ \bar{\Psi}_n = \text{Id}_{Y_{n-2}/U_A^2} \).

It remains to show that \( \bar{\Psi}_n : Y_{n-2}/U_A^2 \to X_n/\text{SL}_2(A) \) is surjective: If \( (u_1, \ldots, u_n) \in U_n^\text{gen} \) then in \( X_n/\text{SL}_2(A) \) we have

\[
(\bar{u}_1, \ldots, \bar{u}_n) = (\bar{u}_1, \ldots, \bar{u}_n) \cdot T_{u_1, u_2} = (0_+, 0_-, (y_3)_+, \ldots, (y_n)_+) = \bar{\Psi}_n(y_3, \ldots, y_n)
\]

where \( y_i = \phi(u_1, u_2, u_i) \) for \( i \geq 3 \).

Taking the quotient set for the action of \( G_A \) on both sides, we deduce:

**Corollary 3.6.** For \( n \geq 3 \), \( \Phi_n \) induces a natural bijection

\[
X_n/\text{GL}_2(A) \leftrightarrow Y_{n-2}/U_A.
\]

For \( n \geq 1 \), we let

\[
Z_n = Z_n(A) := \{(z_1, \ldots, z_n) \in W^m_{A} : z_i/z_j \in W_{A} \text{ for all } i \neq j\}
\]

and we let \( Z_0 = Z_0(A) := \{1\} \).

We observe that for \( n \geq 1 \) there is a natural bijection of \( U_A \)-sets

\[
Y_n \leftrightarrow U_A \times Z_{n-1}, \quad (y_1, \ldots, y_n) \leftrightarrow \left( y_1, \frac{y_2}{y_1}, \ldots, \frac{y_n}{y_1} \right)
\]

(where \( U_A \) acts on the first factor of the right-hand side). Thus, taking quotient sets for the action of \( U_A^\text{gen} \) and \( U_A \), we obtain:

**Corollary 3.7.** For all \( n \geq 3 \) we have natural bijections

\[
X_n/\text{SL}_2(A) \leftrightarrow G_A \times Z_{n-3}
\]

\[
X_n/\text{GL}_2(A) \leftrightarrow Z_{n-3}
\]

**Remark 3.8.** Retracing our steps above, an explicit formula for the first bijection is

\[
(\bar{u}_1, \ldots, \bar{u}_n) \mapsto \begin{pmatrix} \phi(u_1, u_2, u_3) \\ \phi(u_1, u_2, u_4) \\ \vdots \\ \phi(u_1, u_2, u_n) \end{pmatrix} = \begin{pmatrix} d(u_1, u_3)d(u_1, u_2) \\ d(u_1, u_4)d(u_1, u_3) \\ \vdots \\ d(u_1, u_n)d(u_1, u_3) \end{pmatrix} \times \begin{pmatrix} d(u_2, u_3)d(u_2, u_1) \\ d(u_2, u_4)d(u_1, u_3) \\ \vdots \\ d(u_2, u_n)d(u_1, u_3) \end{pmatrix}.
\]

and hence the formula for the second is

\[
(\bar{u}_1, \ldots, \bar{u}_n) \mapsto \begin{pmatrix} d(u_1, u_4)d(u_2, u_3) \\ \vdots \\ d(u_1, u_n)d(u_2, u_3) \end{pmatrix}.
\]

**Remark 3.9.** If \( A = F \) is a field, then clearly \( X_1 = (F^2 \setminus \{0\})/F^\times = \mathbb{P}^1(F) \) and more generally \( X_n \) is naturally the set of \( n \)-tuples of distinct points of \( \mathbb{P}^1(F) \). On the other hand, \( W_F = F^\times \setminus \{1\} = \mathbb{P}^1(F) \setminus \{\infty, 0, 1\} \) and \( Z_n \) consists of \( n \)-tuples of distinct points of \( F^\times \setminus \{1\} \).

The point \( x \in F \) is identified with the point of \( \mathbb{P}^1(F) \) represented by \( (x, 1) \). Since \( d((x, 1), (y, 1)) = x - y \), the bijection \( X_n/\text{GL}_2(F) \leftrightarrow Z_{n-3} \) is thus given by the formula

\[
(x_1, \ldots, x_n) \mapsto ([x_1 : x_2 : x_3 : x_4], \ldots, [x_1 : x_2 : x_3 : x_n])
\]
where
\[
\{x_1 : x_2 : x_3 : x_4\} = \frac{(x_1 - x_4)(x_2 - x_3)}{(x_1 - x_2)(x_3 - x_4)}
\]
is the classic cross ratio.

**Corollary 3.10.** For all \(n \geq 3\) there are natural isomorphisms of \(R_A\)-modules
\[
\mathbb{Z}[X_n]_{SL_2(A)} \cong R_A[Z_{n-3}]
\]
and natural isomorphisms of \(\mathbb{Z}\)-modules
\[
\mathbb{Z}[X_n]_{GL_2(A)} \cong \mathbb{Z}[Z_{n-3}].
\]

**Proof.** If \(G\) is a group and if \(X\) is a right \(G\)-set, then for any ring \(R\) there is a natural isomorphism
\[
R[X]_G \cong R[X/G], \; \bar{x} \mapsto \bar{x}.
\]
Thus, for \(n \geq 3\),
\[
\mathbb{Z}[X_n]_{SL_2(A)} \cong \mathbb{Z}[X_n/SL_2(A)] \cong \mathbb{Z}[G_A \times Z_{n-3}] \cong \mathbb{Z}[G_A][Z_{n-3}] = R_A[Z_{n-3}].
\]
\(\square\)

For \(n \geq 1\), let \(\delta_n : \mathbb{Z}[X_{n+1}] \to \mathbb{Z}[X_n]\) be the simplicial boundary map
\[
(\bar{u}_1, \ldots, \bar{u}_{n+1}) \mapsto \sum_{i=1}^{n+1} (-1)^{i+1}(\bar{u}_1, \ldots, \widehat{\bar{u}_i}, \ldots, \bar{u}_{n+1})
\]
and let \(\mathcal{A}(A) : = \text{Coker}(\delta_4)\). Note that \((\mathbb{Z}[X_4], \delta_n)\) is a complex of \(GL_2(A)\)-modules and that \(\mathcal{A}(A)\) is thus also a \(GL_2(A)\)-module.

**Proposition 3.11.** For any commutative ring \(A\), \(\mathcal{R}\mathcal{P}(A) \cong \mathcal{A}(A)_{SL_2(A)}\) as \(R_A\)-modules, and \(\mathcal{P}(A) \cong \mathcal{A}(A)_{GL_2(A)}\) as \(\mathbb{Z}\)-modules.

**Proof.** By right exactness of coinvariants, \(\mathcal{A}(A)_{SL_2(A)}\) is naturally identified with the cokernel of the map \(\tilde{\delta}_4 : \mathbb{Z}[X_5]_{SL_2(A)} \to \mathbb{Z}[X_4]_{SL_2(A)}\) of \(R_A\)-modules induced by \(\delta_4\). Now
\[
\mathbb{Z}[X_5]_{SL_2(A)} \cong R_A[Z_3] \text{ and } \mathbb{Z}[X_4]_{SL_2(A)} \cong R_A[Z_1],
\]
and, under these identifications, the map \(\tilde{\delta}_4\) is described as follows: \((z_1, z_2) \in R_A[Z_2]\) corresponds to \((1, z_1, z_2) \in Y_3/U_1^2\) and this in turn corresponds to the element \((0_+, 0_-, 1_+, (z_1)_+, (z_2)_+)\) in \(\mathbb{Z}[X_5]_{SL_2(A)}\). The image of this under \(\tilde{\delta}_4\) is
\[
(0_-, 1_+, (z_1)_+, (z_2)_+)-(0_+, 1_+, (z_1)_+, (z_2)_+)+(0_+, 0_-, (z_1)_+, (z_2)_+)-(0_+, 0_-, 1_+, (z_1)_+, (z_2)_+)+0_+, 0_-, 1_+, (z_1)_+, (z_2)_+)
\]
in \(\mathbb{Z}[X_4]_{SL_2(A)}\). Recalling that \((\bar{u}_1, \ldots, \bar{u}_4) \in \mathbb{Z}[X_4]_{SL_2(A)}\) corresponds to
\[
\left(\frac{d(u_1, u_3)d(u_1, u_2)}{d(u_2, u_3)}\right)\left(\frac{d(u_1, u_4)d(u_2, u_3)}{d(u_2, u_4)d(u_1, u_3)}\right) \in R_A[Z_1]
\]
and observing that \(d(a_+, b_+) = a - b\) and \(d(0_-, a_+) = 1\) for all \(a \neq b \in A\), we see that
\[
\tilde{\delta}_4(z_1, z_2) = (1 - z_1) \left(\frac{1 - z_1}{1 - z_2}\right) - \left(z_1^{-1} - 1\right) \left(\frac{1 - z_1}{1 - z_2}\right) + \left(z_1\right) \left(\frac{z_2}{z_1}\right) - (z_2) + (z_1) \in R_A[Z_1].
\]
Thus the map \(R_A[Z_1] \to \mathcal{R}\mathcal{P}(A), (z) \mapsto [z]\) induces an isomorphism
\[
\text{Coker}(\tilde{\delta}_4) \cong \mathcal{R}\mathcal{P}(A).
\]
\(\square\)
Remark 3.12. We will call the (composite) map
\[ \mathbb{Z}[X_4] \to R_A[Z_1] \to R\mathcal{P}(A), \quad (\tilde{u}_1, \ldots, \tilde{u}_4) \mapsto \left( \frac{d(u_1, u_3)d(u_1, u_2)}{d(u_2, u_3)} \right) \left( \frac{d(u_1, u_4)d(u_2, u_3)}{d(u_2, u_4)d(u_1, u_3)} \right) \]
the refined cross ratio map, and will denote it by \( \text{cr} \). In the special case where \( u_i = \iota_r(x_i) \) for \( x_i \in A \), it takes the form
\[ (x_1, x_2, x_3, x_4) \mapsto \left( \frac{(x_1 - x_3)(x_1 - x_2)}{x_2 - x_3} \right) \left( \frac{(x_1 - x_4)(x_1 - x_3)}{x_2 - x_3} \right). \]

3.2. The isomorphism \( H_n(T, \mathbb{Z}) \cong H_n(B, \mathbb{Z}) \). In order to prove Proposition 3.19 below, we follow the strategy of Suslin’s proof of Theorem 1.8 in [21].

Lemma 3.13. ([21, Lemma 1.1]) Suppose that \( \phi_1, \ldots, \phi_m : k \to F \) are field embeddings such that for any \( x \in k^c \) we have \( \prod_{i=1}^m \phi_i(x) = 1 \). Then \( k \) is a finite field of order \( p^f \) with \( m \geq (p-1) \cdot f \).

Remark 3.14. This simple but useful result can be extended in many directions. For example: (See [11, Lemma 2.2.4]) Let \( A \) be a ring with many units. Let \( B \) be any ring. For any \( m \geq 1 \), there do not exist ring homomorphisms \( \phi_1, \ldots, \phi_m : A \to B \) satisfying \( \prod_{i=1}^m \phi_i(x) = 1 \) for all \( x \in U_A \).

Local rings with infinite residue fields are rings with many units, but we will want to include the case of local rings with finite residue field below.

Corollary 3.15. Suppose that \( \phi_1, \ldots, \phi_m : k \to F \) are field embeddings such that for any \( x \in k^c \) we have \( \prod_{i=1}^m \phi_i(x') = 1 \). Then \( k \) is a finite field of order \( p^f \) and \( mr = (p-1) \cdot t \) for some \( t \geq f \).

Proof. We have \( 1 = \prod_{i=1}^m \phi_i(x') = \prod_{i=1}^m \phi_i(x)^r := \prod_{i=1}^m \psi_i(x) \), and thus \( k \) is finite of characteristic \( p > 0 \) and \( mr \geq (p-1)f \) by Lemma 3.13.

On the other hand, if \( a \in \mathbb{F}_p \subset k \) is a primitive root modulo \( p \), then \( 1 = \prod_i \phi_i(a') = a^{mr} \) and thus \( p - 1 | mr \).

Corollary 3.16. Let \( A \) be a local ring with maximal ideal \( M \) and residue field \( k \). Suppose that \( r \geq 1 \) and \( \phi_1, \ldots, \phi_m : A \to F \) are homomorphisms from \( A \) to the field \( F \) satisfying \( \prod_{i=1}^m \phi_i(u') = 1 \) for all \( u \in U_A \). Then \( k \) is a finite field with \( p^f \) elements and \( mr = (p-1)t \) where \( t \geq f \). In particular, \( (p-1)f \leq mr \).

Proof. \( F \) must have positive characteristic, for otherwise we can choose \( 1 < n \in U_A \cap \mathbb{Z} \), and the hypothesis gives \( n^{mr} = 1 \) in \( F \).

Let \( \text{char}(F) = p > 0 \). Replacing \( A \) by \( A/pA \) if necessary, we can assume that \( A \) is an \( \mathbb{F}_p \)-algebra. We complete the proof by showing that \( M \subset \text{Ker}(\phi_i) \) for all \( i \) (and hence that the \( \phi_i \) factor through \( k \)).

Let \( x \in M \). For \( i = 1, \ldots, m \), let \( x_i = \phi_i(x) \in F \). If \( f(T) \in \mathbb{F}_p[T] \) satisfies \( f(0) \neq 0 \), then \( f(x) \in U_A \). In this case we have
\[ 1 = \prod_{i=1}^m \phi_i(f(x)) = \prod_{i=1}^m f(x_i). \]

Thus, let \( I \) be the ideal of \( \mathbb{F}_p[T_1, \ldots, T_m] \) generated by the set
\[ \left\{ \left( \prod_{i=1}^m f(T_i) \right) - 1 \mid f(T) \in \mathbb{F}_p[T] \text{ with } f(0) \neq 0 \right\}. \]

Let \( V \) be the corresponding variety. Then \( (x_1, \ldots, x_m) \in V(F) \).
We observe that $(0, \ldots, 0) \in V$ if and only if $p - 1|mr$.

On the other hand, suppose that $(a_1, \ldots, a_m) \in \mathbb{F}_p^m$ is algebraic and that $a_j \neq 0$ for some $j$. Then there exists $f(T) \in \mathbb{F}_p[T]$ with $f(0) \neq 0$ and $f(a_j) = 0$. It follows that $\prod_i f(a_i) = 0$ and hence $(a_1, \ldots, a_m) \notin V(\mathbb{F}_p)$. Thus

$$V(\mathbb{F}_p) = \begin{cases} \{0\}, & p - 1|mr \\ \emptyset, & \text{otherwise.} \end{cases}$$

It follows from the Nullstellensatz that the ideal, $J$, of $V$ in $\mathbb{F}_p[T_1, \ldots, T_m]$ is given by

$$J = \begin{cases} \langle T_1, \ldots, T_m \rangle, & p - 1|mr \\ \mathbb{F}_p[T_1, \ldots, T_m], & \text{otherwise.} \end{cases}$$

and hence, for any field $K$ we have

$$V(K) = \begin{cases} \{0\}, & p - 1|mr \\ \emptyset, & \text{otherwise.} \end{cases}$$

Since $(x_1, \ldots, x_m) \in V(F)$, it follows that $p - 1|mr$ and $x_i = \phi_i(x) = 0$ for all $i$. \hfill \Box

For $r \geq 1$, we denote by $A(r)$ the $\mathbb{Z}[U_A]$-module obtained by making $u \in U_A$ act on $A$ as multiplication by $u^r$.

**Lemma 3.17.** Let $m, r \geq 1$. Let $n_1, \ldots, n_k$ satisfy $n_1 + \cdots + n_k = m$ and $n_i \geq 1$. Let $A$ be a local ring with residue field $k$. If $k$ is finite of order $p^f$ we suppose that $mr < (p - 1)f$.

Let $T^n(A(r))$ denote either $\wedge^n_{\mathbb{Z}}(A(r))$ or $\operatorname{Sym}^n_{\mathbb{Z}}(A(r))$, considered as $U_A$ modules with the diagonal action.

Then

$$H_i(U_A, T^n(A(r)) \otimes \cdots \otimes T^n(A(r))) = 0$$

for all $i \geq 0$.

**Proof.** This follows from Corollary 3.16 by the same argument verbatim as that by which Suslin proves Corollary 1.6 from Lemma 1.1 in [21]. \hfill \Box

**Lemma 3.18.** Let $m, r \geq 1$. Let $A$ be a local integral domain with residue field $k$. If $k$ is finite of order $p^f$ we suppose that $mr < (p - 1)f$.

For all $i \geq 0$ we have

$$H_i(U_A, H_m(A(r), \mathbb{Z})) = 0.$$

**Proof.** If $\operatorname{char}(A) = 0$, then $H_m(A(r), \mathbb{Z}) = \wedge^n_{\mathbb{Z}}(A(r))$ and the statement follows at once from Lemma 4.6.

Otherwise $A$ is an $\mathbb{F}_p$-algebra for some $p > 0$. Then $H_m(A(r), \mathbb{F}_p)$ is a direct sum of modules of the form $\wedge^n_{\mathbb{Z}}(A(r)) \otimes \operatorname{Sym}^t_{\mathbb{Z}}(A(r))$ with $s + t \leq m$. It follows from Lemma 4.6 that $H_i(U_A, H_m(A(r), \mathbb{F}_p)) = 0$.

On the other hand, the short exact sequence

$$0 \longrightarrow \mathbb{Z} \overset{p}{\longrightarrow} \mathbb{Z} \longrightarrow \mathbb{Z}/p\mathbb{Z} = \mathbb{F}_p \longrightarrow 0$$

induces a long exact homology sequence for $H_*(A(r),)$, which decomposes into short exact sequences

$$0 \rightarrow H_k(A(r), \mathbb{Z}) \rightarrow H_k(A(r), \mathbb{F}_p) \rightarrow H_{k-1}(A(r), \mathbb{Z}) \rightarrow 0 \ (k \geq 2)$$

and an isomorphism

$$H_1(A(r), \mathbb{Z}) \cong A(r) \cong H_1(A(r), \mathbb{F}_p).$$
The vanishing of \(H_i(U_A, H_n(A(Γ), Z))\) for all \(i \geq 0\) then follows from a straightforward induction on \(m\).

We let \(T = T(A)\) denote the subgroup of \(SL_2(\mathbb{A})\) consisting of diagonal matrices:

\[
T(A) := \left\{ \begin{bmatrix} u & 0 \\ 0 & u^{-1} \end{bmatrix} \middle| u \in U_A \right\}.
\]

Thus \(T(A) \cong U_A\). We let \(B = B(A)\) denote the subgroup consisting of lower triangular matrices:

\[
B(A) := \left\{ \begin{bmatrix} u & a \\ 0 & u^{-1} \end{bmatrix} \middle| u \in U_A, a \in A \right\}.
\]

Thus there is natural (split) group extension

\[
1 \to V \to B \to T \to 1
\]

where

\[
V = T(A) := \left\{ \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \middle| a \in A \right\} \cong A.
\]

Here \(T \cong U_A\) acts on \(V \cong A\) by conjugation. With the given identifications, \(u \in U_A\) acts on \(a \in A\) as multiplication by \(u^2\). Thus \(V \cong A(2)\) as a \(\mathbb{Z}[U_A]\)-module.

**Proposition 3.19.** Let \(n \geq 1\). Let \(A\) be a local integral domain with residue field \(k\). If \(k\) is finite of order \(p^f\) we suppose that \((p - 1)f > 2n\).

The natural maps \(B \to T\) and \(T \to B\) induce isomorphisms on homology

\[
H_n(T, Z) \cong H_n(B, Z)
\]

**Proof.** The Hochschild-Serre spectral sequence associated to the extension \(\xrightarrow{4}\) takes the form

\[
E^{2}_{ij} = H_i(T, H_j(V, Z)) = H_i(U_A, H_j(A(2), Z)) \Rightarrow H_{i+j}(B, Z).
\]

By Lemma \([3.18]\) it follows that \(E^{2}_{ij} = 0\) if \(0 < j \leq n\), and \(E^{2}_{i,0} = H_i(T, Z)\) for all \(i\).

Hence \(H_n(B, Z) = H_n(T, Z)\).

**Remark 3.20.** In particular, \(H_n(T, Z) \cong H_n(B, Z)\) for all \(n \leq 3\) provided \(|k| \notin \{2, 3, 4, 5, 7, 8, 9, 16, 27, 32, 64\}\).

3.3. **The complex \(L_\bullet\).** For a commutative ring \(A\), we let \(L_n = L_n(A) := \mathbb{Z}[X_{n+1}]\). Equipped with the boundary \(\delta_n : L_n \to L_{n-1}\) this yields a complex, \(L_\bullet\), of right \(GL_2(A)\)-modules. Restricting the group action, this is also a complex of \(SL_2(A)\)-modules.

We now restrict attention to the case where \(A\) is a commutative local ring with residue field \(k\). We let \(\pi : A \to k\) denote the canonical surjective quotient map. So \(U_A = \pi^{-1}(k^\times)\). More generally, if \(X \in M_n(A)\) is an \(n \times n\) matrix with coefficients in \(A\), we let \(\pi(X) \in M_n(k)\) denote the matrix obtained by applying \(\pi\) to each entry of \(X\). Since \(A\) is a local ring \(X \in GL_n(A)\) if and only if \(\pi(X) \in GL_n(k)\).

Similarly, \(u = (u_1, u_2) \in U_2(\mathbb{A})\) if and only if \(\pi(u) \in U_2(k) = k^2 \setminus \{0\}\) and \(\bar{u} \in X_1(\mathbb{A})\) if and only if \(\overline{\pi(u)} \in X_1(k) = \mathbb{P}^1(k)\). Furthermore,

\[
(u_1, \ldots, u_n) \in U_n^\text{gen}(A) \iff (\pi(u_1), \ldots, \pi(u_n)) \in U_n^\text{gen}(k)
\]

and hence

\[
(\bar{u}_1, \ldots, \bar{u}_n) \in X_n(\mathbb{A}) \iff (\overline{\pi(u_1)}, \ldots, \overline{\pi(u_n)}) \in X_n(k).
\]

**Lemma 3.21.** \(H_n(L_\bullet) = 0\) for \(1 \leq n < |k|\).
Proof. When \( A = k \) is a field, the argument is given in \([6]\), Lemma 4.4. This argument is easily adapted to the current situation as follows:

For any subset \( S \) of \( \mathbb{P}^1(k) \), let \( D_n(S) \) denote the subgroup of \( L_n(A) \) generated by those \((n+1)\)-tuples \((\bar{u}_1, \ldots, \bar{u}_{n+1}) \in X_{n+1}(A)\) which satisfy \( S \subseteq \{ \pi(u_1), \ldots, \pi(u_{n+1}) \} \). Thus \( D_n(S) = 0 \) if \(|S| > n+1\). Furthermore, \( D_n(S_1 \cup S_2) = D_n(S_1) \cap D_n(S_2) \) for any \( S_1, S_2 \subseteq \mathbb{P}^1(k) \).

Now for each \( x \in \mathbb{P}^1(k) \), choose \( u_\bullet \in U_2(A) \) satisfying \( \pi(u_\bullet) = x \) and for \( n \geq 0 \) define a homomorphism \( S_x : L_n \to L_{n+1} \) by

\[
S_x(\bar{u}_1, \ldots, \bar{u}_{n+1}) = \begin{cases} 
(\bar{u}_x, \bar{u}_1, \ldots, \bar{u}_{n+1}), & x \notin \{ \pi(u_1), \ldots, \pi(u_{n+1}) \} \\
0, & \text{otherwise}
\end{cases}
\]

Thus if \((\bar{u}_1, \ldots, \bar{u}_{n+1}) \in X_{n+1}(A)\) and if \( x \notin \{ \pi(u_1), \ldots, \pi(u_{n+1}) \} \) then

\[
dS_x(\bar{u}_1, \ldots, \bar{u}_{n+1}) = (\bar{u}_1, \ldots, \bar{u}_{n+1}) - S_x\delta(\bar{u}_1, \ldots, \bar{u}_{n+1}).
\]

On the other hand, if \( x = \pi(u_j) \) for some \( j \) then

\[
S_x\delta(\bar{u}_1, \ldots, \bar{u}_{n+1}) = (-1)^{j+1}(\bar{u}_x, \bar{u}_1, \ldots, \bar{u}_j, \ldots, \bar{u}_{n+1}).
\]

and hence

\[
0 = dS_x(\bar{u}_1, \ldots, \bar{u}_{n+1})
\]

\[
= (\bar{u}_1, \ldots, \bar{u}_{n+1}) - S_x\delta(\bar{u}_1, \ldots, \bar{u}_{n+1}) - \left\{ S_x\delta(\bar{u}_1, \ldots, \bar{u}_{n+1}) - (-1)^j(\bar{u}_x, \bar{u}_1, \ldots, \bar{u}_j, \ldots, \bar{u}_{n+1}) \right\}.
\]

In either case we have

\[
dS_x(\bar{u}_1, \ldots, \bar{u}_{n+1}) = (\bar{u}_1, \ldots, \bar{u}_{n+1}) - S_x\delta(\bar{u}_1, \ldots, \bar{u}_{n+1}) + w
\]

where \( w \in D_n(\{x\}) \). Furthermore, if \((\bar{u}_1, \ldots, \bar{u}_{n+1}) \in D_n(S)\) for some subset \( S \) of \( \mathbb{P}^1(k) \) then \( w \in D_n(S \cup \{x\}) \).

Suppose now that \( 1 \leq n < |k| \) and that \( x_1, \ldots, x_{n+2} \) are \( n + 2 \) distinct points of \( \mathbb{P}^1(k) \). Let \( z \in L_n(A) \) be a cycle. Then

\[
(\delta S_{x_1} - \text{Id})z = S_{x_1}\delta(z) + z_1 = z_1
\]

where \( z_1 \in D_n(\{x_1\}) \) and \( z_1 \) is again a cycle.

Thus \( (\delta S_{x_2} - \text{Id})z_1 = z_2 \) where \( z_2 \) is a cycle belonging to \( D_n(\{x_1, x_2\}) \). Repeating the process we get

\[
(\delta S_{x_{n+2}} - \text{Id})(\delta S_{x_{n+1}} - \text{Id}) \cdots (\delta S_{x_1} - \text{Id})z \in D_n(\{x_1, \ldots, x_{n+2}\}) = 0.
\]

This equation has the form \( \delta(y) + (-1)^{n+2}z = 0 \) and hence \( z = \delta((-1)^{n+1}y) \) is a boundary as required.

\[
\square
\]

3.4. \( H_3(\text{SL}_2(A), \mathbb{Z}[\frac{1}{2}]) \).

**Theorem 3.22.** Let \( A \) be a local integral domain with residue field \( k \) satisfying \( |k| \not\in \{2, 3, 4, 5, 7, 8, 9, 16, 27, 32, 64\} \). Then there is a natural short exact sequence of \( R_A[\frac{1}{2}] \)-modules

\[
0 \to \text{tor}(\mu_A, \mu_A) \left[ \frac{1}{2} \right] \to H_3(\text{SL}_2(A), \mathbb{Z} \left[ \frac{1}{2} \right] ) \to \mathcal{R} \mathcal{B}(A) \left[ \frac{1}{2} \right] \to 0.
\]

**Proof.** In the case where \( A \) is a field, the proof can be found in \([6]\) section 4. We indicate here the adaptations needed to extend that proof to the current context:

Associated to the complex \( L_\bullet = L_\bullet(A) \) there is hyperhomology spectral sequence of the form

\[
E_{p,q}^{1} = H_{p}(\text{SL}_2(A), L_{q} \left[ \frac{1}{2} \right] ) \Rightarrow H_{p+q}(\text{SL}_2(A), L_{\bullet} \left[ \frac{1}{2} \right] ).
\]
and furthermore the augmentation $L_0 \to \mathbb{Z}$ induces an isomorphism
\[ H_n(\text{SL}_2(A), L_\bullet \left[ \frac{1}{2} \right]) \cong H_n(\text{SL}_2(A), \mathbb{Z} \left[ \frac{1}{2} \right]) \]
for $n \leq 3$ by Remark 3.20.

The $\text{SL}_2(A)$-modules $L_\bullet$ are permutation modules, so the $E^1$-terms are calculated using Shapiro’s Lemma:

$\text{SL}_2(A)$ acts transitively on $X_1$ and the stabilizer of $0_+ \in X_1$ is $B = B_\Lambda$. Thus
\[ L_0 = \mathbb{Z}[X_1] \cong \mathbb{Z}[B/\text{SL}_2(A)] \cong \text{Ind}^\mathbb{Z}[\text{SL}_2(A)]_{\mathbb{Z}} \]
and hence
\[ E^1_{p,0} = H_p(\text{SL}_2(A), L_0 \left[ \frac{1}{2} \right]) \cong H_p(B, \mathbb{Z} \left[ \frac{1}{2} \right]). \]

Similarly, $\text{SL}_2(A)$ acts transitively on $X_2$ and the stabilizer of $(0_+, 0_-) = T = T_A$, so that
\[ E^1_{p,1} = H_p(\text{SL}_2(A), L_1 \left[ \frac{1}{2} \right]) \cong H_p(T, \mathbb{Z} \left[ \frac{1}{2} \right]). \]

For $n \geq 3$, the stabilizer in $\text{SL}_2(A)$ of $(\bar{u}_1, \ldots, \bar{u}_n) \in X_n$ is $Z(\text{SL}_2(A)) \cong \mu_2(A)$. By Corollary 3.10 it follows that for $q \geq 2$ we have
\[ E^1_{p,q} = R_A \left[ \frac{1}{2} \right][Z_{q-2}] \otimes H_p(\mu_2(A), \mathbb{Z}) = \begin{cases} \text{R}_A \left[ \frac{1}{2} \right][Z_{q-2}], & p = 0 \\ 0, & p > 0 \end{cases} \]
where $Z_n = Z_n(A)$ as above.

Thus our $E^1$-page has the form
\[ \begin{array}{cccc}
R_A \left[ \frac{1}{2} \right][Z_2] & 0 & \vdots & \vdots & \ldots \\
\downarrow d^1 & & & & & \\
R_A \left[ \frac{1}{2} \right][Z_1] & 0 & 0 & 0 & \ldots \\
\downarrow d^1 & & & & & \\
R_A \left[ \frac{1}{2} \right] & 0 & 0 & 0 & \ldots \\
\downarrow d^1 & & & & & \\
\mathbb{Z} \left[ \frac{1}{2} \right] & H_1(T, \mathbb{Z} \left[ \frac{1}{2} \right]) & H_2(T, \mathbb{Z} \left[ \frac{1}{2} \right]) & H_3(T, \mathbb{Z} \left[ \frac{1}{2} \right]) & \ldots \\
\downarrow d^1 & & & & & \\
\mathbb{Z} \left[ \frac{1}{2} \right] & H_1(T, \mathbb{Z} \left[ \frac{1}{2} \right]) & H_2(T, \mathbb{Z} \left[ \frac{1}{2} \right]) & H_3(T, \mathbb{Z} \left[ \frac{1}{2} \right]) & \ldots \\
\end{array} \]

Now $T \cong U_A$. Thus $E^1_{p,q} \cong H_p(U_A, \mathbb{Z} \left[ \frac{1}{2} \right])$ for $p \leq 3$ and $q \in \{0, 1\}$.

Now let
\[ w := \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \in \text{SL}_2(A). \]

Then $w(0_+) = 0_-$ and $w(0_-) = 0_+$. It follows easily that the differential
\[ d^1 : E^1_{p,1} = H_p(T, \mathbb{Z} \left[ \frac{1}{2} \right]) \to H_p(T, \mathbb{Z} \left[ \frac{1}{2} \right]) = E^1_{p,0} \]
is the map

\[ H_p(T, \mathbb{Z}[\frac{1}{2}]) \xrightarrow{w_\rho^{-1}} H_p(T, \mathbb{Z}[\frac{1}{2}]) \]

where \( w_\rho : H_p(T, \mathbb{Z}[\frac{1}{2}]) \rightarrow H_p(T, \mathbb{Z}[\frac{1}{2}]) \) is the map induced by conjugation by \( w \). However, conjugating by \( w \) is just the inversion map on \( U_A \cong T \). Thus \( d^1 : \mathbb{Z}[\frac{1}{2}] = E_{0,1}^1 \rightarrow E_{0,0}^1 = \mathbb{Z}[\frac{1}{2}] \) is the zero map. \( d^1 : U_A[\frac{1}{2}] = E_{1,1}^1 \rightarrow E_{1,0}^1 = U_A[\frac{1}{2}] \) is the map \( u \mapsto u^{-1} \) and hence is an isomorphism. \( d^1 : \wedge_2^2(U_A[\frac{1}{2}]) = E_{2,1}^1 \rightarrow E_{2,0}^1 = \wedge_2^2(U_A[\frac{1}{2}]) \) is the zero map.

Finally, \( E_{3,1}^1 = E_{3,0}^1 = H_3(U_A, \mathbb{Z}[\frac{1}{2}]) \cong \wedge_2^2(U_A[\frac{1}{2}]) \oplus \text{tor}(\mu_A, \mu_A)[\frac{1}{2}] \). The map \( d^1 : E_{3,1}^1 \rightarrow E_{3,0}^1 \) is an isomorphism of the first factor and the zero map on the second factor.

The differential

\[ d^1 : \mathbb{R}_A[\frac{1}{2}] \cong H_0(\text{SL}_2(A), L_2) = E_{0,2}^1 \rightarrow E_{0,1}^1 = H_0(\text{SL}_2(A), L_1) \cong \mathbb{Z}[\frac{1}{2}] \]

is the natural augmentation sending \( \langle u \rangle \) to 1 for any \( u \in U_A \).

As in the proof of [6, Theorem 4.3], the differential

\[ d^1 : \mathbb{R}_A[\frac{1}{2}][Z_1] \cong H_0(\text{SL}_2(A), L_3) = E_{0,3}^1 \rightarrow E_{0,2}^1 = H_0(\text{SL}_2(A), L_2) \cong \mathbb{R}_A[\frac{1}{2}] \]

is the \( \mathbb{R}_A \)-homomorphism sending \( (z) \) to \( \langle z \rangle \langle 1 - z \rangle \in I_A^2 \) for any \( z \in W_A \).

By the proof of Proposition 3.11 above, the differential

\[ d^1 : \mathbb{R}_A[\frac{1}{2}][Z_2] \cong H_0(\text{SL}_2(A), L_4) = E_{0,4}^1 \rightarrow E_{0,3}^1 = H_0(\text{SL}_2(A), L_3) \cong \mathbb{R}_A[\frac{1}{2}][Z_1] \]

is the map

\[ (z_1, z_2) \mapsto \langle 1 - z_1 \rangle \left( \frac{1 - z_1}{1 - z_2} \right) - \langle z_1^{-1} - 1 \rangle \left( \frac{1 - z_1^{-1}}{1 - z_2^{-1}} \right) + \langle z_1 \rangle \left( \frac{z_2}{z_1} \right) - (z_2) + (z_1) \]

Thus the \( E^2 \)-page of our spectral sequence has the form

\[ \mathbb{R}P_1(A)[\frac{1}{2}] \begin{array}{ccc} 0 & 0 & \vdots \\
I_A[\frac{1}{2}]/J_A[\frac{1}{2}] & 0 & 0 \\
0 & 0 & \mathbb{R}_A[\frac{1}{2}] \\
0 & 0 & \wedge_2^2(U_A[\frac{1}{2}]) \\
\mathbb{Z}[\frac{1}{2}] & 0 & \wedge_2^2(U_A[\frac{1}{2}]) \oplus \text{tor}(\mu_A, \mu_A)[\frac{1}{2}] \end{array} \]

where \( J_A \subset R_A \) is the ideal generated by the Steinberg elements \( \langle u \rangle \langle 1 - u \rangle \).
Clearly there are no nonzero $d^2$-differentials. So the $E^3$-page has the form

\[
\begin{array}{ccc}
E^3_{0,4} & 0 & 0 \\
\mathcal{RP}_1(A) \left[ \frac{1}{2} \right] & 0 & 0 \\
I_A \left[ \frac{1}{2} \right] / \mathcal{F}_A \left[ \frac{1}{2} \right] & 0 & 0 \\
\mathbb{Z} \left[ \frac{1}{2} \right] & 0 & \Lambda^2_Z(U_A \left[ \frac{1}{2} \right]) \quad \text{tor}(\mu_A, \mu_A) \left[ \frac{1}{2} \right] \\
\end{array}
\]

The argument now concludes exactly as in [6, section 4]: The cokernel of the differential $d^3 : E^3_{0,4} \to \Lambda^2_Z(U_A \left[ \frac{1}{2} \right])$ is annihilated by 2 and hence $E^4_{2,1} = E^\infty_{2,1} = 0$. There is a commutative diagram

\[
\begin{array}{ccc}
\mathcal{RP}_1(A) \left[ \frac{1}{2} \right] & \xrightarrow{d^3} & \Lambda^2_Z(U_A \left[ \frac{1}{2} \right]) \\
\downarrow & & \downarrow \cong \\
\mathcal{P}(A) \left[ \frac{1}{2} \right] & \xrightarrow{\lambda} & S^2_Z(U_A \left[ \frac{1}{2} \right]) \\
\end{array}
\]

It follows that $E^3_{0,3} = E^\infty_{0,3} = \text{Ker}(\lambda_2 : \mathcal{RP}_1(A) \left[ \frac{1}{2} \right] \to S^2_Z(U_A \left[ \frac{1}{2} \right]) = \mathcal{RB}(A) \left[ \frac{1}{2} \right]$. This completes the proof of the theorem. □

3.5. Local integral domains with infinite residue field. We will require the following two results of B. Mirzaii:

**Proposition 3.23.** ([15 Corollary 5.4],[16 Theorem 3.7]) Let $R$ be a ring with many units.

1. If $R$ is an integral domain, there is a natural short exact sequence

   \[0 \to \text{tor}(\mu_R, \mu_R) \left[ \frac{1}{2} \right] \to H_3(\text{SL}_2(R), \mathbb{Z} \left[ \frac{1}{2} \right]) \to \mathcal{B}(R) \left[ \frac{1}{2} \right] \to 0.\]

2. There is a natural isomorphism

   \[H_3(\text{SL}_2(R), \mathbb{Z} \left[ \frac{1}{2} \right])_{U_A} = H_3(\text{SL}_2(R), \mathbb{Z} \left[ \frac{1}{2} \right])_{\mathfrak{g}_R} \cong K_3^{\text{ind}}(R) \left[ \frac{1}{2} \right].\]

**Remark 3.24.** A local ring with infinite residue field is an example of a ring with many units.

We note the following immediate corollary to Mirzaii’s results:

**Corollary 3.25.** Let $A$ be a discrete valuation ring with field of fractions $K$ and infinite residue field $F$. Suppose that $\text{char}(F) = \text{char}(K)$. Then the functorial map $\mathcal{B}(A) \left[ \frac{1}{2} \right] \to \mathcal{B}(K) \left[ \frac{1}{2} \right]$ is an isomorphism.
Proof. By Theorem \[2.1\] the natural map $K_3^{\text{ind}}(A) \to K_3^{\text{ind}}(K)$ is an isomorphism. Thus there is a commutative diagram with exact rows

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \text{tor}(\mu_A, \mu_A) \left[ \frac{1}{2} \right] & \longrightarrow & H_3(\text{SL}_2(A), \mathbb{Z} \left[ \frac{1}{2} \right]) & \longrightarrow & \mathcal{B}(A) \left[ \frac{1}{2} \right] & \longrightarrow & 0 \\
\downarrow & & & & & & & & \\
0 & \longrightarrow & \text{tor}(\mu_K, \mu_K) \left[ \frac{1}{2} \right] & \longrightarrow & K_3^{\text{ind}}(K) \left[ \frac{1}{2} \right] & \longrightarrow & \mathcal{B}(K) \left[ \frac{1}{2} \right] & \longrightarrow & 0
\end{array}
$$

from which the result immediately follows. □

Combining Mirzaii’s results with Theorem \[3.22\] gives:

**Corollary 3.26.** Let $A$ be a local integral domain with infinite residue field. Then there is a natural short exact sequence

$$
0 \to \mathcal{R}\mathcal{B}_0(A) \left[ \frac{1}{2} \right] \to H_3(\text{SL}_2(R), \mathbb{Z} \left[ \frac{1}{2} \right]) \to K_3^{\text{ind}}(A) \left[ \frac{1}{2} \right] \to 0
$$

and

$$
\mathcal{R}\mathcal{B}_0(A) \left[ \frac{1}{2} \right] \cong I_A \mathcal{R}\mathcal{B}(A) \left[ \frac{1}{2} \right] \cong I_A H_3(\text{SL}_2(R), \mathbb{Z} \left[ \frac{1}{2} \right]).
$$

Proof. Combining Theorem \[3.22\] with the statements in Proposition \[3.23\] we obtain a commutative diagram with exact rows and columns

$$
\begin{array}{ccccccccc}
0 & & & & & & & & \\
\downarrow & & & & & & & & \\
\quad I_A H_3(\text{SL}_2(A), \mathbb{Z} \left[ \frac{1}{2} \right]) & \longrightarrow & \mathcal{R}\mathcal{B}_0(A) \left[ \frac{1}{2} \right] & \longrightarrow & 0 \\
\downarrow & & & & & & & & \\
0 & \longrightarrow & \text{tor}(\mu_A, \mu_A) \left[ \frac{1}{2} \right] & \longrightarrow & H_3(\text{SL}_2(A), \mathbb{Z} \left[ \frac{1}{2} \right]) & \longrightarrow & \mathcal{B}(A) \left[ \frac{1}{2} \right] & \longrightarrow & 0 \\
\downarrow & & & & & & & & \\
0 & \longrightarrow & \text{tor}(\mu_A, \mu_A) \left[ \frac{1}{2} \right] & \longrightarrow & K_3^{\text{ind}}(A) \left[ \frac{1}{2} \right] & \longrightarrow & \mathcal{B}(A) \left[ \frac{1}{2} \right] & \longrightarrow & 0 \\
\downarrow & & & & & & & & \\
0 & & & & & & & & \\
\end{array}
$$

from which the statements of this corollary immediately follow. □

**Corollary 3.27.** Let $A$ be a local integral domain with infinite residue field. Suppose that $U_A = U_A^2$. Then

$$
\mathcal{R}\mathcal{B}_0(A) \left[ \frac{1}{2} \right] = 0 \text{ and } \mathcal{R}\mathcal{B}(A) \left[ \frac{1}{2} \right] \cong \mathcal{B}(A) \left[ \frac{1}{2} \right].
$$

4. Submodules of $\mathcal{R}\mathcal{P}(A)$

In this section, $A$ will denote a commutative local ring, with maximal ideal $\mathcal{M}$ and residue field $k = A / \mathcal{M}$. Furthermore, we will suppose that $k$ has at least four elements.

In this case, we have $U_A = A \setminus \mathcal{M}$ and $\mathcal{W}_A = U_A \setminus U_1$, where $U_1 = U_{1,A} = 1 + \mathcal{M}$. In particular, if $x \in U_A$, then $x \in \mathcal{W}_A \iff x^{-1} \in \mathcal{W}_A$. 
4.1. **The modules** $\mathcal{K}_A^{(i)}$. As in [8], we define two families of elements of $RP(A)$.

Given $x \in W_A$ we define

$$\psi_1(x) := [x] + (-1)[x^{-1}]$$

and

$$\psi_2(x) := (1-x) \left( [x] + [x^{-1}] \right) = \left( x^{-1} - 1 \right) [x] + (1-x)[x^{-1}].$$

Observe, from the definitions, that $\psi_i(x^{-1}) = (-1)\psi_i(x)$ for all $x \in W_A$.

**Lemma 4.1.** For $i = 1, 2$ we have

1. $\psi_i(xy) = \langle x \rangle \psi_i(y) + \psi_i(x)$ whenever $x, y, xy \in W_A$.
2. $\langle x \rangle \psi_i(x^{-1}) + \psi_i(x) = 0$ for all $x \in W_A$.

**Proof.**

(1) The proof of Lemma 3.1 in [8] adapts without alteration.

(2) Let $x \in W_A$. Choose $y \in W_A$ such that $xy \in W_A$ also. (Note that this is possible because of the hypothesis that $|k| \geq 4$.)

Then by part (1) we have

$$\psi_i(xy) = \langle y \rangle \psi_i(x) + \psi_i(y) \implies \langle y \rangle \psi_i(x) = \psi_i(xy) - \psi_i(y)$$

and

$$\psi_i(y) = \psi_i(xy \cdot x^{-1}) = \langle xy \rangle \psi_i(x^{-1}) + \psi_i(xy) \implies \langle xy \rangle \psi_i(x^{-1}) = \psi_i(y) - \psi_i(xy).$$

Form these we deduce that

$$\langle y \rangle \psi_i(x) = -\langle xy \rangle \psi_i(x^{-1}).$$

Multiplying both sides of this equation by $\langle y \rangle$ gives the result.

**Lemma 4.2.** Let $u \in U_1$. For any $w_1, w_2 \in W_A$, we have

$$\psi_i(w_1u) - \langle u \rangle \psi_i(w_1) = \psi_i(w_2u) - \langle u \rangle \psi_i(w_2)$$

for $i = 1, 2$.

**Proof.** First suppose that $w_1 \equiv w_2 \pmod{U_1}$. Then $w_1w_2^{-1} \in W_A$ and hence

$$\psi_i(w_1u) = \psi_i \left( (w_1^{-1}w_1) \cdot (w_2u) \right) = \langle w_2u \rangle \psi_i \left( w_1^{-1}w_2 \right) + \psi_i(w_2u) = \langle u \rangle \left( \psi_i(w_2) - \psi_i(w_1) \right) + \psi_i(w_2u)$$

giving the result in this case.

On the other hand, if $w_1 \equiv w_2 \pmod{U_1}$ choose $w_3 \in W_A$ with $w_3 \neq w_1 \pmod{U_1}$. Then

$$\psi_i(w_1u) - \langle u \rangle \psi_i(w_1) = \psi_i(w_3u) - \langle u \rangle \psi_i(w_3) \psi_i(w_2u) - \langle u \rangle \psi_i(w_2).$$

We now extend the definition of $\psi_i(x)$ to allow $x \in U_1$. For $u \in U_1$, we define

$$\psi_i(u) := \psi_i(uw) - \langle u \rangle \psi_i(w)$$

for any $w \in W_A$. 

Proposition 4.3. For \( i = 1, 2 \) the maps \( U_A \to \mathcal{RP}(A) \), \( x \mapsto \psi_i(x) \) define 1-cocycles; i.e. we have
\[
\psi_i(xy) = \langle x \rangle \psi_i(y) + \psi_i(x)
\]
for all \( x, y \in U_A \).

Proof. If \( x, y, xy \in \mathcal{W}_A \), this is part (1) of Lemma 4.1.
If \( x, y \in \mathcal{W}_A \), but \( xy \in U_1 \), then
\[
\psi_i(xy) = \psi_i(xy \cdot y^{-1}) - \langle xy \rangle \psi_i(y^{-1})
= \psi_i(x) - \langle x \rangle \left( \langle y \rangle \psi_i(y^{-1}) \right)
= \psi_i(x) + \langle x \rangle \psi_i(y)
\]
using Lemma 4.1(2) in the last step.
If \( x \in U_1 \) and \( y \in \mathcal{W}_A \), the identity is just the definition of \( \psi_i(x) \).
On the other hand, if \( x \in \mathcal{W}_A \) and \( y \in U_1 \), then choose \( w \in \mathcal{W}_A \) such that \( xw \in \mathcal{W}_A \). We have
\[
\langle x \rangle \psi_i(y) + \psi_i(x) = \langle x \rangle (\psi_i(wy) - \langle y \rangle \psi_i(w)) + \psi_i(x)
= \langle x \rangle \psi_i(wy) + \psi_i(x) - \langle xy \rangle \psi_i(w)
= \psi_i(xyw) - \langle x \rangle \psi_i(yw)
= \psi_i(xy).
\]
Finally, suppose that \( x, y \in U_1 \). Let \( w \in \mathcal{W}_A \). Then
\[
\psi_i(xy) = \psi_i(xyw) - \langle xy \rangle \psi_i(w)
= \langle x \rangle \psi_i(yw) + \psi_i(x) - \langle xy \rangle \psi_i(w)
= \langle x \rangle (\psi_i(yw) - \langle y \rangle \psi_i(w)) + \psi_i(x)
= \langle x \rangle \psi_i(y) + \psi_i(x)
\]
as required. \( \square \)

We recall here, from [8] some of the basic algebraic properties of the \( \psi_i(x) \). (The proofs given in [8] for the case of fields adapt without change to the case of local rings.)

Proposition 4.4. For \( i \in \{1, 2\} \) we have:

1. \( \langle x \rangle \psi_i(y) = \langle y \rangle \psi_i(x) \) for all \( x, y \)
2. \( \psi_i(x^2) = \psi_i(x) + \psi_i(y^2) \) for all \( x, y \)
3. \( \langle x \rangle \psi_i(y^2) = 0 \) for all \( x, y \)
4. \( 2 \cdot \psi_i(-1) = 0 \) for all \( i \)
5. \( \psi_i(x^2) = -\langle x \rangle \psi_i(-1) \) for all \( x \)
6. \( 2 \cdot \psi_i(x^2) = 0 \) for all \( x \) and if \( -1 \) is a square in \( U_A \) then \( \psi_i(x^2) = 0 \) for all \( x \).
7. \( \langle x \rangle \langle y \rangle \psi_i(-1) = 0 \) for all \( x, y \)
8. \( \langle -1 \rangle \langle x \rangle \psi_i(y) = \langle x \rangle \psi_i(y) \) for all \( x, y \)
9. Let
\[
\epsilon(A) := \begin{cases} 
1, & -1 \in U_A^2 \\
2, & -1 \notin U_A^2
\end{cases}
\]
The map \( U_A/U_A^2 \to \mathcal{RP}(A), \langle x \rangle \mapsto \epsilon(A) \psi_i(x) \) is a well-defined 1-cocycle.

Corollary 4.5. For \( i = 1, 2 \) and \( a \in U_A \)
\[
\psi_i(a) - \psi_i(-a^{-1}) = \psi_i(-1).
\]
Proof. For \(a \in U_A\), since \((-1) \psi_1(a^{-1}) = \psi_i(a)\) we have
\[
\psi_i(-a^{-1}) = \psi_i(-1 \cdot a^{-1}) = (-1) \psi_i(a^{-1}) + \psi_i(-1) = \psi_i(a) + \psi_i(-1).
\]

\[\square\]

**Lemma 4.6.** For \(i = 1, 2\) and for all \(x \in U_A\)
\[
\begin{align*}
(1) \quad & \lambda_1(\psi_i(x)) = -p_+^{-1}\langle x \rangle = \langle -x \rangle \langle x \rangle \in I_A^2 \\
(2) \quad & \lambda_2(\psi_i(x)) = (-x) \circ x \in S^2_2(U_A).
\end{align*}
\]

Proof. (1) For \(x \in \mathcal{W}_A\), this is a straightforward calculation given in [8, Lemma 3.3]. For \(x \in U_1\) it follows from the identity \(\langle xw \rangle - \langle x \rangle \langle w \rangle = \langle x \rangle \) in \(R_A\).

(2) Since \(U_A\) acts trivially on \(S^2_2(U_A)\), we have for any \(x \in \mathcal{W}_A\)
\[
\begin{align*}
\lambda_2(\psi_1(x)) &= \lambda_2(\psi_2(x)) \\
&= \lambda_2([x]) + \lambda_2([x^{-1}]) \\
&= (1 - x) \circ x + (1 - x^{-1}) \circ x^{-1} \\
&= (1 - x) \circ x - \frac{1 - x}{-x} \circ x \\
&= (-x) \circ x.
\end{align*}
\]

On the other hand, if \(x \in U_1\), the result follows from the identity
\[
(-xw) \circ (xw) = (-w) \circ w = (-x) \circ x
\]
in \(S^2_2(U_A)\).

\[\square\]

Let \(K_A^{(i)}\) denote the \(R_A\)-submodule of \(\mathcal{RP}(A)\) generated by the set \(\{\psi_i(x) \mid x \in U_A\}\).

**Lemma 4.7.** Then for \(i \in \{1, 2\}\)
\[
\lambda_1(K_A^{(i)}) = p_+^{-1}(I_A) \subset I_A^2
\]
and \(\ker(\lambda_1|_{K_A^{(i)}})\) is annihilated by 4.

Proof. The first statement follows from Lemma 4.6.

For the second, the proof of Lemma 3.3 in [8] applies without change.

\[\square\]

Let
\[
\mathcal{RP}(A) := \mathcal{RP}(A)/K_A^{(1)}
\]
Then the \(R_A\)-homomorphism
\[
\Lambda : \mathcal{RP}(A) \rightarrow I_A^2 \oplus S^2_2(U_A)/\lambda(K_A^{(1)})
\]
is well-defined and we set \(\overline{RB}(A) := \ker(\Lambda)\).

**Corollary 4.8.** The natural map \(RB(A) \rightarrow \overline{RB}(A)\) is surjective with kernel annihilated by 4. In particular,
\[
RB(A) \left[ \frac{1}{2} \right] = \overline{RB}(A) \left[ \frac{1}{2} \right].
\]
Proof. From the definitions,

\[ \overline{\mathcal{RB}}(A) \cong \frac{\mathcal{RB}(A)}{\mathcal{RB}(A) \cap \mathcal{K}^{(1)}_A} \]

and

\[ \mathcal{RB}(A) \cap \mathcal{K}^{(1)}_A = \text{Ker}(\Lambda|_{\mathcal{K}^{(1)}_A}) \subset \text{Ker}(\Lambda|_{\mathcal{K}^{(1)}}) \]

which is annihilated by 4. \qed

4.2. The constant $D_A$. In \cite{23}, Suslin shows that, for an infinite field $F$, the elements $[x] + [1 - x] \in \mathcal{B}(F) \subset \mathcal{P}(F)$ are independent of $x$ and that the resulting constant, $\tilde{C}_F$, has order dividing 6. Furthermore Suslin shows that $\tilde{C}_\mathbb{R}$ has exact order 6.

In \cite{3} Lemma 3.5] it is shown that the elements

\[ C(x) = [x] + (-1) [1 - x] + \langle 1 - x \rangle \psi_1 (x) \in \overline{\mathcal{RB}}(F) \]

are constant (for a field with at least 4 elements) and have order dividing 6. In fact the proof given there extends without alteration to the case of local rings:

**Lemma 4.9.** Let $A$ be a local ring whose residue field has at least 4 elements. For all $a, b \in \mathcal{W}_A$

\[ C(a) := [a] + (-1) [1 - a] + \langle 1 - a \rangle \psi_1 (a) \in \overline{\mathcal{RB}}(A) \text{ and } C(a) = C(b). \]

We denote this constant by $C_A$ and we set $D_A := 2C_A$. Of course, $C_A$ maps to $\tilde{C}_A$ under the natural map $\overline{\mathcal{P}}(A) \to \mathcal{P}(A)$. Similarly, we denote the image of $D_A$ in $\mathcal{P}(A)$ by $\tilde{D}_A$. Thus, of course, $\tilde{D}_A = 2\tilde{C}_A$ in $\mathcal{P}(A)$. In fact, these elements lie in $\mathcal{B}(A)$ by Lemma 4.9.

Let $\Phi(X)$ denote the polynomial $X^2 - X + 1 \in A[X]$.

**Lemma 4.10.** Let $A$ be a local ring whose residue field has at least 4 elements.

1. $3C_A = \psi_1 (-1)$ and $6C_A = 0$.
2. If $\Phi(X)$ has a root in $A$ then $D_A = 0$ and $C_A = \psi_1 (-1)$.

**Proof.**

1. Let $a \in \mathcal{W}_A$. Then

\[
3C_A = C(a) + C(1 - a^{-1}) + C\left(\frac{1}{1 - a}\right)
\]

\[
= [a] + (-1) [1 - a] + \left\langle \frac{1}{1 - a} \right\rangle \psi_1 (a)
\]

\[
+ [1 - a^{-1}] + (-1) \left[ a^{-1} \right] + \langle a \rangle \psi_1 \left(1 - a^{-1}\right)
\]

\[
+ \left[\frac{1}{1 - a}\right] + (-1) \left[\frac{1}{1 - a^{-1}}\right] + \langle 1 - a^{-1}\rangle \psi_1 \left(\frac{1}{1 - a}\right)
\]

\[
= \psi_1 (a) + \psi_1 \left(1 - a^{-1}\right) + \psi_1 \left(\frac{1}{1 - a}\right)
\]

\[
+ \left\langle \frac{1}{1 - a} \right\rangle \psi_1 (a) + \langle a \rangle \psi_1 \left(1 - a^{-1}\right) + \left\langle 1 - a^{-1} \right\rangle \psi_1 \left(\frac{1}{1 - a}\right)
\]

\[
= \left\langle \frac{1}{1 - a} \right\rangle \psi_1 (a) + \langle a \rangle \psi_1 \left(1 - a^{-1}\right) + \left\langle 1 - a^{-1} \right\rangle \psi_1 \left(\frac{1}{1 - a}\right)
\]

\[
= \psi_1 \left(\frac{1}{a^{-1} - 1}\right) - \psi_1 \left(\frac{1}{1 - a}\right) + \psi_1 (a - 1) - \psi_1 (a) + \psi_1 \left(\frac{1}{a}\right) - \psi_1 \left(1 - a^{-1}\right)
\]

\[
= 3\psi_1 (-1) = \psi_1 (-1)
\]
Lemma 4.11. For any $a$ commutative diagram with exact rows

Corollary 4.13. For any field $F$, $D$ induces an isomorphism

Let $A$ be a local ring with residue field $k$. Suppose either that $U \not\equiv -1 \pmod{3}$ and that $\Phi$ is a primitive cube root of unity, $\zeta_3$. Let $k$ be the prime subfield of $F$. If $k = \mathbb{Q}$, then $\hat{D}_k = \hat{D}_Q \in B(\mathbb{Q})$ has order $3$ (since $\mathbb{Q}$ embeds in $\mathbb{R}$). On the other hand, if $k = \mathbb{F}_p$, then $\hat{D}_k$ has order $3$ if and only if $p \equiv -1 \pmod{3}$ by Lemma 7.11. Thus, in all cases, $\hat{D}_k = 0$ in $B(k)$ if and only if $\zeta_3 \in k$.

By Suslin’s Theorem (Theorem 5.2) – or by Corollary 7.5] when $k$ is finite – we have a commutative diagram with exact rows

where the middle vertical arrow is injective (Corollary 4.6). If $\hat{D}_k \neq 0$, then it lies in the kernel of $B(k) \to B(F)$ only if $3|\mu_F$. Thus $\hat{D}_F = 0$ only if $\zeta_3 \in F$.

Corollary 4.14. Let $A$ be a local ring with residue field $k$. Suppose either that $U_{1,A} = U_{1,A}^2$ and that $\mathrm{char}(k) \neq 2, 3$ or that $\mathrm{char}(k) = 2$ and $A$ is henselian. Then the functorial map $\mathcal{R}B(A) \to \mathcal{R}B(k)$ induces an isomorphism $\mathbb{Z} \cdot D_A \cong \mathbb{Z} \cdot D_k$.

Proof. Clearly $D_A = 0 \implies D_k = 0$. Conversely, if $D_k = 0$ then $\Phi(X)$ has a root in $k$. The hypotheses then imply that $\Phi(X)$ has a root in $A$ and thus $D_A = 0$ also.

Remark 4.15. When $\mathrm{char}(k) = 3$, the result may fail. For example, $\Phi(X)$ has no root in $\mathbb{Q}_3$, so that $D_{\mathbb{Q}_3} 
eq 0$, and hence $D_{\mathbb{Q}_3} 
eq 0$ also. But $D_{\mathbb{F}_3} = 0$.
4.3. A key identity. The purpose of this section is to prove the important identity \( \langle a \rangle D_A = \psi_1(a) - \psi_2(a) \) in \( \mathcal{R}\mathcal{P}(A) \) for all \( a \in U_A \) (Theorem 4.19 below). This identity was proved in the case when \( A \) is a field in [8].

Let \( t \) denote the matrix of order 3

\[
\begin{pmatrix}
-1 & -1 \\
1 & 0
\end{pmatrix} \in \text{SL}_2(\mathbb{Z}).
\]

It can be shown that \( H_3(\text{SL}_2(\mathbb{Z}), \mathbb{Z}) \) is cyclic of order 12 and that the inclusion \( G := \langle t \rangle \to \text{SL}_2(\mathbb{Z}) \) induces an isomorphism \( H_3(G, \mathbb{Z}) \cong H_3(\text{SL}_2(\mathbb{Z}), \mathbb{Z})[3] \).

We will identify \( A \) with \( A_+ = \iota_*(A) \subset X_1 \).

For any subset \( S \) of \( A \), we have \( S^n \subset X^n_1 \) and we let

\[
\Delta(n, S) := S^n \cap X_n = \{(s_1, \ldots, s_n) \in S^n | s_i - s_j \in U_A \text{ for } i \neq j\}.
\]

Thus there is a natural inclusion of additive groups \( \mathbb{Z}[\Delta(n, S)] \to \mathbb{Z}[X_n] \).

We note that \( \mathcal{W}_A \) is a right \( G \)-set since for any \( a \in \mathcal{W}_A \)

\[
a \cdot t = [a, 1] \cdot \begin{pmatrix}
-1 & -1 \\
1 & 0
\end{pmatrix} = [1-a, -a] = [1-a^{-1}, 1] = 1 - a^{-1} \in \mathcal{W}_A \subset X_1.
\]

For a local ring \( A \), let \( \tilde{\mathcal{W}}_A := \mathcal{W}_A \setminus \{a \in A | \Phi(\pi(a)) = 0 \text{ in } k\} \).

**Lemma 4.16.** If \( x \in \tilde{\mathcal{W}}_A \), then \( x \cdot t^i - x \cdot t^j \in U_A \) whenever \( i \neq j \) (mod 3).

**Proof.** If \( x \in \tilde{\mathcal{W}}_A \), then \( x, 1 - x, \Phi(x) \in U_A \). The statement thus follows from the identities

\[
\begin{align*}
x - x \cdot t &= x - 1 + \frac{1}{x} = \frac{\Phi(x)}{x} \\
x - x \cdot t^2 &= x - \frac{1}{1-x} = \frac{\Phi(x)}{x-1} \\
x \cdot t - x \cdot t^2 &= 1 - \frac{1}{x} - \frac{1}{1-x} = \frac{\Phi(x)}{x(x-1)}.
\end{align*}
\]

**Lemma 4.17.** Let \( A \) be a local ring whose residue field, \( k \), has at least 10 elements. Let \( L_n = L_n(A) = \mathbb{Z}[X_{n+1}] \) as above. Let \( F_\bullet \) be the (right) homogeneous standard resolution of \( \mathbb{Z} \) over \( \mathbb{Z}[G] \). Then an augmentation-preserving chain map of right \( \mathbb{Z}[G] \)-modules \( F_\bullet \to L_\bullet \) in dimensions three and below can be constructed as follows:

Let \( x \in \tilde{\mathcal{W}}_A \) and let \( y \in \tilde{\mathcal{W}}_A \) with

\[
\pi(y) \notin \{\pi(x), \pi(x) \cdot t, \pi(x) \cdot t^2\} \subset k.
\]

Then define \( \beta_n^{x,y} : F_n \to L_n \) as follows:

Given \( g \in G \), let \( \beta_n^{x,y}(g) = x \cdot g \in A \subset X_1 \).

Given \( g_0, g_1 \in G \) let

\[
\beta_1^{x,y}(g_0, g_1) = \begin{cases}
(x \cdot g_0, x \cdot g_1), & g_0 \neq g_1 \\
0, & g_0 = g_1
\end{cases}
\]
Given \( g_0, g_1, g_2 \in G \), let

\[
\beta_2^{x,y}(g_0, g_1, g_2) = \begin{cases} 
(x \cdot g_0, x \cdot g_1, x \cdot g_2), & \text{if } g_0, g_1, g_2 \text{ are distinct} \\
0, & g_0 = g_1 \text{ or } g_1 = g_2 \\
(y \cdot g_0, x \cdot g_0, x \cdot g_1) & \text{if } g_0 = g_2 \neq g_1 \\
+ (y \cdot g_0, x \cdot g_1, x \cdot g_0), & \text{if } g_0 = g_2 = g_1 \text{ and } \text{g_1 = g_2 are distinct} \\
\end{cases}
\]

Given \( g_0, g_1, g_2, g_3 \in G \), let

\[
\beta_3^{x,y}(g_0, g_1, g_2, g_3) = \begin{cases} 
0, & g_i = g_{i+1} \text{ for some } 0 \leq i \leq 2 \\
(y \cdot g_0, y \cdot g_1, x \cdot g_0, x \cdot g_1) & \text{if } g_0 = g_2 \neq g_1 \text{ and } g_1 = g_3 \\
(x \cdot g_0, y \cdot g_0, x \cdot g_1, x \cdot g_2) & \text{if } g_1 = g_3 \text{ and } \text{g_1 = g_3 are distinct} \\
+(x \cdot g_0, y \cdot g_1, x \cdot g_2, x \cdot g_1) & \text{if } g_0 = g_2 \text{ and } \text{g_0 = g_1, g_2 are distinct} \\
(y \cdot g_0, x \cdot g_0, x \cdot g_1, x \cdot g_3) & \text{if } g_0 = g_2 \text{ and } \text{g_0 = g_1, g_2 are distinct} \\
-(y \cdot g_0, x \cdot g_1, x \cdot g_0, x \cdot g_3) & \text{if } g_0 = g_2 \text{ and } \text{g_0 = g_1, g_2 are distinct} \\
(y \cdot g_0, x \cdot g_1, x \cdot g_2, x \cdot g_0) & \text{if } g_0 = g_2 \text{ and } \text{g_0 = g_1, g_2 are distinct} \\
-(y \cdot g_0, x \cdot g_0, x \cdot g_1, x \cdot g_2) & \text{if } g_0 = g_2 \text{ and } \text{g_0 = g_1, g_2 are distinct} \\
\end{cases}
\]

Proof. By Lemma \([4,16]\), the image of \( \beta_2^{x,y} \) lies in \( \mathbb{Z}[\Delta(n,A)] \subset \mathbb{Z}[X_{n+1}] \). It is a straightforward computation to verify that this map is an augmentation-preserving chain map.

Corollary 4.18. Let \( x, y \) be chosen as in Lemma \([4,17]\). Let
\[
C := H_3(C, K) = \text{cr}(y, x \cdot t, x \cdot t^2, x) - \text{cr}(y, x, x \cdot t, x \cdot t^2) + \text{cr}(y, y \cdot t, x, x \cdot t) + \text{cr}(y, y \cdot t, x \cdot t, x) \in \mathcal{RP}(A).
\]
Then \( C \) is independent of the choice of \( x, y, C \in \mathcal{RB}(A) \) and \( 3C = 0 \).

Proof. Let \( K \to \text{SL}_2(A) \) be any group homomorphism. Then there is a hypercohomology spectral sequence

\[
E_{i,j}^1(K) = H_n(K, L_q) \Rightarrow H_{p+q}(K, L_*)
\]
and \( H_{p+q}(K, L_*) = H_{p+q}(K, \mathbb{Z}) \) when \( p + q \) is not too large. There are associated edge homomorphisms giving a commutative diagram

\[
\begin{CD}
H_0(K, \mathbb{Z}) @>>> E_{0,0}^2(K) = H_0(L_*) \subset H_0(\text{SL}_2(A)) \\
@. @VV\alpha_0(K) V @VH_0(\text{SL}_2(A)) VV \\
H_0(\text{SL}_2(A), \mathbb{Z}) @>>> E_{0,0}^2(\text{SL}_2(A)) = H_0(L_*)_{\text{SL}_2(A)}
\end{CD}
\]

The map \( \alpha_0(K) \) can be constructed as follows: Let \( F_0(K) \) be a projective right \( \mathbb{Z}[K] \)-resolution of \( \mathbb{Z} \) and let \( \beta_* : F_0(K) \to L_* \) be any augmentation-preserving homomorphism of right \( \mathbb{Z}[k] \)-complexes. Then \( \alpha_0(K) \) is the induced map on \( n \)-th homology groups associated to the map of complexes \( F_0(K) \otimes_{\mathbb{Z}[k]} \mathbb{Z} \to L_* \otimes_{\mathbb{Z}[\text{SL}_2(A)]} \mathbb{Z} \). The map \( \alpha_0(K) \) is independent of the choices of resolution \( F_0(K) \) and chain map \( \beta_* \).

Applying this to \( G, F_* \) and \( \beta_3^{x,y} \) as in Lemma \([4,17]\) and observing that the cycle \((1, t, t, t^2, 1) + (1, t, t, t^2, 1) + (1, t, t, t) \in F_3 \) represents a generator of \( H_3(G, \mathbb{Z}) \cong \mathbb{Z}/3 \), we see that the map \( \alpha_3 : \mathbb{Z}/3 = H_3(G, \mathbb{Z}) \to H_3(L_*, \mathbb{Z})_{\text{SL}_2(A)} \) sends 1 to the class of

\[
(y, x \cdot t, x \cdot t^2, x) - (y, x, x \cdot t, x \cdot t^2) + (y, y \cdot t, x, x \cdot t) + (y, y \cdot t, x \cdot t, x) \in L_3
\]

But the proof of Theorem \([3,22]\) shows that \( H_3(L_*)_{\text{SL}_2(A)} \cong \mathcal{RP}_1(A) \) and that this isomorphism is induced by the refined cross ratio map.

Thus \( 1 \in \mathbb{Z}/3 \) maps to \( C = C(x, y) \in \mathcal{RP}_1(A) \) under \( \alpha_3 \). It follows that \( C \) is independent of \( x, y \) and that \( 3C = 0 \).
Finally, the proof of Theorem 3.22 shows that, at least after tensoring with \( \mathbb{Z} \left[ \frac{1}{2} \right] \), the image of the edge homomorphism lies in \( E^\infty_{0,3} = \mathcal{R}B(A) \left[ \frac{1}{2} \right] \). Since 3C = 0, it follows that \( C \in \mathcal{R}B(A) \).

**Theorem 4.19.** Let \( A \) be a local ring whose residue field has at least 10 elements. Then

1. For all \( x \in U_A \), \( \langle x \rangle D_A = \psi_1(x) - \psi_2(x) \).
2. \( \langle x \rangle D_A = 0 \) if \( x \in U_A \) is of the form \( \pm \Phi(a)u^2 \) for some \( a, u \in U_A \).

**Proof.** Choose \( x, y \in \tilde{W}_A \) as in Lemma 4.17. Then, the calculations in the proof of Theorem 3.12 of [8] show that

\[
C = C(x, y) = \langle -\Phi(x)r \rangle \psi_2(r) - \psi_1(r) - D_A
\]

where

\[
r = r(x, y) = \frac{x - y}{x - 1 - xy} \in \mathcal{W}_A.
\]

By the Corollary, \( C \) has order 3 and is independent of \( x \) and \( y \). Now, by choice of \( x \) and \( y \), \( r \) can assume any value in \( \mathcal{W}_A \). In particular, we can arrange for \( r \) to have the form \( -u^2 \) for some unit \( u \). Since \( 4C = 4D_A = 0 \) and \( 2\psi_1(-u^2) = 0 \) for \( i = 1, 2 \), multiplying by 4 gives

\[
C = -\langle \Phi(x) \rangle D_A
\]

for any \( x \in \tilde{W}_A \).

Given \( x \in \mathcal{W}_A \) we can find \( x' \in \tilde{W}_A \) such that \( xx' - 1, x + x' - 1 \in U_A \) and

\[
x'' = \frac{xx' - 1}{x + x' - 1} \in \mathcal{W}_A.
\]

Since \( \Phi(x)\Phi(x') = \Phi(x'') \cdot (x + x' - 1)^2 \), we have

\[
C = -\langle \Phi(x'') \rangle D_A = -\langle \Phi(x) \rangle \langle \Phi(x') \rangle D_A = \langle \Phi(x) \rangle C
\]

for any \( x \in \tilde{W}_A \), and hence \( C = -D_A \).

It follows that \( \langle \Phi(x) \rangle D_A = D_A \) for any \( x \in \tilde{W}_A \); i.e. \( \langle \Phi(x) \rangle D_A = 0 \) for all \( x \in \tilde{W}_A \). Since \( \langle -1 \rangle D_A = D_A \) also, it follows that

\[
\left\langle \pm \Phi(x)u^2 \right\rangle D_A = 0
\]

for all \( x \in \tilde{W}_A \) and \( u \in U_A \).

We now have that

\[
-D_A = C = \langle -\Phi(x)r \rangle \psi_2(r) - \psi_1(r) - D_A
\]

for all \( r \in \mathcal{W}_A \) and some \( x \in \tilde{W}_A \). Thus

\[
-\langle \Phi(x)r \rangle D_A = \psi_2(r) - \psi_1(r) - D_A \implies \langle r \rangle D_A = \psi_1(r) - \psi_2(r)
\]

since \( \langle \Phi(x) \rangle D_A = D_A \). It follows that

\[
\langle r \rangle D_A = \psi_1(r) - \psi_2(r)
\]

for all \( r \in \mathcal{W}_A \).

Finally, let \( x \in U_{1,A} = U_A \setminus \mathcal{W}_A \). Fix \( r \in \mathcal{W}_A \). Then \( \psi_i(x) = \psi_i(rx) - \langle x \rangle \psi_i(r) \) for \( i = 1, 2 \). Furthermore, \( \langle x \rangle = \langle rx \rangle - \langle x \rangle \langle r \rangle \). It follows that

\[
\langle r \rangle D_A = \psi_1(rx) - \psi_2(rx) - \langle x \rangle \psi_1(r) + \langle x \rangle \psi_2(r)
\]

as required. \( \square \)
By Theorem 4.19 (2), we have

$$\langle a \rangle C_A = \psi_2 (a) - \psi_1 (a^{-1}).$$

Of course, since $3 \neq 4.4.$ The module $D_A$ will be required in our computations below.

**Corollary 4.20.** Let $A$ be a local ring whose residue field has at least $10$ elements. For all $a \in U_A$

$$\langle a \rangle C_A = \psi_2 (a) - \psi_1 (a^{-1}).$$

**Proof.** Since $3C_A = \psi_1 (-1)$ and $2C_A = D_A$, we have $C_A = \psi_1 (-1) - D_A$.

Thus, if $a \in U_A$ we have

$$\langle a \rangle C_A = \langle a \rangle \psi_1 (-1) - \langle a \rangle D_A = \langle a \rangle \psi_1 (-1) - \langle -a \rangle D_A$$

since $\langle -1 \rangle D_A = 0$. Thus, by Theorem 4.19

$$\langle a \rangle C_A = \langle a \rangle \psi_1 (-1) - \langle a \rangle \psi_2 (-a) = \langle a \rangle \psi_1 (-1) - \langle a \rangle \psi_1 (-1) - \psi_1 (a) + \psi_2 (-a) = -\langle \psi_1 (-1) + \psi_1 (a) \rangle + \psi_2 (-a) = -\langle \psi_1 (-1) + \langle -1 \rangle \psi_1 (a^{-1}) \rangle + \psi_2 (-a) = \psi_2 (-a) - \psi_1 (a^{-1})$$

However, $\langle -1 \rangle C_A = 0$. Thus $\langle a \rangle C_A = \langle -a \rangle C_A = \psi_2 (a) - \psi_1 (a^{-1}).$ \hfill \qed

**4.4. The module $D_A$.** We let $D_A$ denote the cyclic $\hat{R}_A$-submodule of $\hat{RB}(A)$ generated by $D_A$. Of course, since $3D_A = 0$ always, in fact $D_F$ is a module over the group ring $\mathbb{F}_3 [G_A]$.

Let $N_A$ be the subgroup of $U_A$ generated by elements of the form $\pm \Phi(x)u^2$, $x \in \hat{W}_A, u \in U_A$. By Theorem 4.19 (2), we have $\langle x \rangle D_A = D_A$ if $a \in N_A$. Let $\bar{G}_A = U_A/N_F$. Thus the action of $G_A$ on $D_A$ factors through the quotient $\bar{G}_A$, and hence $D_A$ is a cyclic module over the ring $\hat{R}_A := \mathbb{F}_3 [\bar{G}_A]$.

**Remark 4.21.** The results of section 6 below (see Corollary 6.45) show that if $F$ is a higher-dimensional local field, satisfying certain conditions, then $D_F$ is a free of rank one over $\hat{R}_F$. In this case it follows that we have a converse to (2) of Theorem 4.19: $\langle x \rangle D_F = 0$ if and only if $x \in N_F$. (And hence, over such fields, $\psi_1 (x) = \psi_2 (x)$ if and only if $x \in N_F$.)

**Lemma 4.22.** Let $A$ be a local ring with residue field $F$. Then the natural map $D_A \rightarrow D_F$ is surjective.

Furthermore, if $U_{1,A} = U_{1,A}^2$ and if $D_F$ is free (of rank $1$) as a $\hat{R}_F$-module, then $D_A \cong D_F$.

**Proof.** Since $G_A$ maps onto $G_F$ and hence $R_A$ maps onto $R_F$ the first statement is clear.

For the second statement, note that the conditions ensure that $G_A \cong G_F$ and under this isomorphism, $N_A$ corresponds to $N_F$. Thus $\hat{R}_A \cong \hat{R}_F$ and the composite map

$$\hat{R}_A \xrightarrow{\cong} D_A \xrightarrow{\cong} D_F$$

is an isomorphism. \hfill \qed

**4.5. Reduced Bloch Groups.** We introduce some quotient groups of (pre-)Bloch groups which will be required in our computations below.

First recall that

$$\hat{R}P(A) = \frac{\hat{R}P(A)}{K_A^{(1)}} \quad \text{and} \quad \hat{R}B(A) = \text{Ker}(\hat{\Lambda}) \subset \hat{R}P(A).$$
Furthermore, the map $\mathcal{RB}(A) \rightarrow \overline{\mathcal{RB}}(A)$ is surjective with kernel annihilated by 4. Since $D_A$ is annihilated by 3, it follows that the composite

$$D_A \rightarrow \mathcal{RB}(A) \rightarrow \overline{\mathcal{RB}}(A)$$

is injective and we will identify $D_A$ with its image in $\overline{\mathcal{RB}}(A)$.

We further define and

$$\overline{\mathcal{RP}}(A) := \frac{\mathcal{RP}(A)}{\mathcal{K}^{(1)}_A + D_A} \quad \text{and} \quad \overline{\mathcal{RB}}(A) := \frac{\mathcal{RB}(A)}{D_A}. $$

Corollary 4.8 implies:

**Lemma 4.23.** For any field or local ring $A$ there is a short exact sequence

$$0 \rightarrow D_A \rightarrow \mathcal{RB}(A) \left[ \frac{1}{2} \right] \rightarrow \overline{\mathcal{RB}}(A) \left[ \frac{1}{2} \right] \rightarrow 0. $$

Observe also that $\mathcal{K}_A^{(2)} \subset \mathcal{K}_A^{(1)} + D_A$ by Theorem 4.19 (2). It follows that for $i = 1, 2$ and all $x \in U_A$ we have $\psi_i(x) = 0$ in $\overline{\mathcal{RP}}(A)$.

**Lemma 4.24.** Let $A$ be a local ring.

1. For all $x \in W_A$, $[x^{-1}] = -\langle -1 \rangle [x] = -\langle x \rangle [x]$ in $\overline{\mathcal{RP}}(A)$.
2. For all $x \in W_A$, $\langle y \rangle [x] = 0$ in $\overline{\mathcal{RP}}(A)$ whenever $y \equiv -x \pmod{U_A^2}$.
3. For all $x \in W_A$, $[1 - x] = \langle -1 \rangle [x] = [x^{-1}]$ in $\overline{\mathcal{RP}}(A)$.

**Proof.**

1. The first equality follows from $\psi_1(x) = 0$, the second from $\psi_2(x) = 0$.
2. From $\langle x \rangle [x] = \langle -1 \rangle [x]$ it follows that $\langle y \rangle [x] = \langle -x \rangle [x] = [x]$.
3. Since $2C_A = D_A$ and $3C_A = \psi_1(-1) \in \mathcal{K}_A^{(1)}$, it follows that $C_A = 0$ in $\overline{\mathcal{RP}}(A)$, and thus we have (from the definition of $C_A$) that $0 = [x] + \langle -1 \rangle [1 - x]$ in $\overline{\mathcal{RP}}(A)$. □

4.6. **The module** $\overline{\mathcal{RP}}_1(A)$. Let $A$ be a commutative local ring or a field.

Recall that, for any $x \in U_A$, $\lambda_1(\psi_1(x)) = -p^{-1}_+ \langle x \rangle = \langle -x \rangle \langle x \rangle \in I_A^2 \subset R_A$. Let $\overline{R}_A := R_A/p^{-1}_+ R_A$. If $\epsilon$ denotes the augmentation homomorphism $R_A \rightarrow \mathbb{Z}$, then $\epsilon(p^{-1}_+) = 2$ and thus

$$p^{-1}_+ R_A \cap I_A^2 = p^{-1}_+ R_A \cap I_A = p^{-1}_+ I_A.$$ 

Therefore:

**Lemma 4.25.** The inclusion map $I_A^2 \rightarrow R_A$ induces an injective map of $R_A$-modules $I_A^2/p^{-1}_+ I_A \rightarrow \overline{R}_A$.

Let $\lambda_1$ denote the composite map

$$\overline{\mathcal{RP}}(A) \rightarrow \frac{I_A^2}{p^{-1}_+ I_A} \rightarrow R_A, \quad [x] \mapsto \langle 1 - x \rangle \langle x \rangle$$

and let $\overline{\mathcal{RP}}_1(A) := \text{Ker}(\lambda_1)$.

Let $S^2_Z(U_A)$ denote the quotient of the abelian group $S^2_Z(U_A)$ modulo the subgroup, $Q_A$ say, generated by the elements $x \circ -x$, $x \in U_A$. Observe that $Q_A = A_2(\mathcal{K}_A^{(1)})$. 


\textbf{Lemma 4.26.} \( \overline{RB}(A) = \text{Ker}(\overline{\lambda}_2) \) where \( \overline{\lambda}_2 \) is the restriction of the map
\[ \overline{RP}(A) \to \overline{S}_2^2(U_A), \quad [x] \mapsto (1 - x) \circ x \]
to \( \overline{RP}_1(A) \)

\textit{Proof.} Let \( \overline{\alpha} = \alpha + \mathcal{K}_A^{(1)} \in \overline{RP}(A) \). Then, by definition, \( \overline{\alpha} \in \overline{RB}(A) \) if and only if there exists \( \beta \in \mathcal{K}_A^{(1)} \) such that \( \lambda_i(\alpha) = \lambda_i(\beta) \) for \( i = 1, 2 \). On the other hand, \( \overline{\alpha} \in \text{Ker}(\overline{\lambda}_2) \) if and only if there exist \( \beta_1, \beta_2 \in \mathcal{K}_A^{(1)} \) with \( \lambda_i(\alpha) = \lambda_i(\beta_i) \) for \( i = 1, 2 \).

Thus \( \overline{RB}(A) \subset \text{Ker}(\overline{\lambda}_2) \), and to prove the converse inclusion, it’s enough to prove that if \( \alpha \in \overline{RP}(A) \) and \( \beta_i \in \mathcal{K}_A^{(1)} \) for \( i = 1, 2 \) such that \( \lambda_i(\beta_i) = \lambda_i(\alpha) \) for \( i = 1, 2 \), then \( \lambda_2(\alpha) = \lambda_2(\beta_1) \) also.

To see this we recall that there are natural homomorphisms of abelian groups
\[ p_1 : I_A^2 \to \text{Sym}_2^2(\mathcal{G}_A), \quad \langle a \rangle \langle b \rangle \mapsto \langle a \rangle * \langle b \rangle \]
and
\[ p_2 : S_2^2(U_A) \to \text{Sym}_2^2(\mathcal{G}_A), \quad a \circ b \mapsto \langle a \rangle * \langle b \rangle \]
satisfying \( p_1 \circ \lambda_1 = p_2 \circ \lambda_2 \) (see \cite{6} for more details).

Furthermore, let \( \tau : U_A \to Q_A \) be the map \( x \mapsto (-x) \circ x \). It is easily verified that \( \tau \) is a group homomorphism containing \(-1\) and \((U_A)^2\) in its kernel. Thus \( \tau \) induces a group homomorphism
\[ \overline{\tau} : U_A/\pm(U_A)^2 \to Q_A, \quad x \mapsto (-x) \circ x. \]

Composing this with \( p_2 \) gives the map
\[ U_A/\pm(U_A)^2 \to \text{Sym}_2^2(\mathcal{G}_A) = \text{Sym}_2^2(\mathcal{G}_A), \quad x \mapsto \langle -1 \rangle * \langle x \rangle + \langle x \rangle * \langle x \rangle \]
which is clearly injective. It follows that \( \overline{\tau} \) is an isomorphism and that the restriction of \( p_2 \) to \( Q_A \) is injective.

But \( \lambda_2(\alpha) \in \lambda_2(\mathcal{K}_A^{(1)}) = Q_A \) and
\[ p_2(\lambda_2(\alpha)) = p_1(\lambda_1(\beta_1)) = p_2(\lambda_2(\beta_1)) \]
so that \( \lambda_2(\alpha) = \lambda_2(\beta_1) \) as required. \( \square \)

Let \( \overline{P}(A) \) denote \( P(A) \) modulo the subgroup generated by the elements \([x] + [x^{-1}]\) (of order 2).

Let \( \overline{B}(A) \) denote the kernel of the map \( \overline{P}(A) \to \overline{S}_2^2(U_A) \), \([x] \mapsto (1 - x) \circ x\).

Recall that \( \overline{RB}_0(A) \) denotes the kernel of the natural map \( \overline{RB}(A) \to B(A) \).

We let \( \overline{RB}_0(A) \) denote the kernel of the map from \( \overline{RB}(A) \) to \( \overline{B}(A) \). Equivalently, \( \overline{RB}_0(A) \) is the image of \( \overline{RB}_0(A) \) in \( \overline{RB}(A) \).

\textbf{Corollary 4.27.} There is a natural exact sequence of \( R_A \)-modules
\[ 0 \to \overline{RB}_0(A) \to \overline{RP}_1(A) \to \overline{P}(A). \]

Furthermore, the cokernel of the right-most map is annihilated by 2.

\textit{Proof.} The exactness of the sequence follows from the commutative diagram with exact rows
\[
\begin{array}{cccccc}
0 & \longrightarrow & \overline{RB}(A) & \longrightarrow & \overline{RP}_1(A) & \longrightarrow & \overline{S}_2^2(U_A) \\
& & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \overline{B}(A) & \longrightarrow & \overline{P}(A) & \longrightarrow & \overline{S}_2^2(U_A)
\end{array}
\]
For the second statement, let $x \in W_A$. Then $p_{+}^{-1} [x] \in \overline{RP}_1 (A)$, since
\[ \lambda_1(p_{+}^{-1} [x]) = p_{+}^{-1}(\langle \langle 1 - x \rangle \rangle) = 0 \text{ in } \overline{R}_A \]
and $p_{+}^{-1} [x]$ maps to $2 [x] \in \overline{P}(A)$. \hfill \Box

We will also need the following corollary below.

**Corollary 4.28.** Let $A$ be a field or a local integral domain with infinite residue field. There are natural isomorphisms of $R_A \left[ \frac{1}{2} \right]$-modules
\[ \overline{RB}_0 (A) \left[ \frac{1}{2} \right] \cong \overline{RB}_0 (A) \left[ \frac{1}{2} \right] \cong I_A \overline{RB} (A) \left[ \frac{1}{2} \right] \cong I_A \overline{RP}_1 (A) \left[ \frac{1}{2} \right]. \]

**Proof.** By Corollary 4.8, the kernel of the surjective map $\overline{RB}_0 (A) \rightarrow \overline{RB}_0 (A)$ is annihilated by 4, and the first equality follows.

The second equality is part of the statement of Lemma 2.4 when $A$ is a field and is part of the statement of Corollary 3.26 when $A$ is a local integral domain.

For the third equality, we begin by observing that since $G_A$ acts trivially on $\overline{P}(A)$, we have
\[ I_A \overline{RP}_1 (A) \left[ \frac{1}{2} \right] \subset \overline{RB}_0 (A) \left[ \frac{1}{2} \right] \]
by Corollary 4.27. On the other hand we have
\[ \overline{RB}_0 (A) \left[ \frac{1}{2} \right] = I_A \overline{RB} (A) \left[ \frac{1}{2} \right] = I_A \overline{RB} (A) \left[ \frac{1}{2} \right] \subset I_A \overline{RP}_1 (A) \left[ \frac{1}{2} \right]. \]
\hfill \Box

5. **Valuations**

5.1. **The specialization homomorphism revisited.** Given a field $F$ with a valuation $v : F^\times \rightarrow \Gamma$ and corresponding residue field $k$, we let $U = U_F$ denote the corresponding group of units and $U_1 = U_1 (F)$ the units mapping to 1 in $k^\times$. The residue map $U \rightarrow k^\times$ will be denoted $a \mapsto \bar{a}$. Given any $R_k$-module $M$, we define the induced $R_F$-module
\[ M_F := R_F \otimes_{Z[U]} M = R_F \otimes_{Z[U/U]} M. \]

Observe that when $U_1 = U_1^2$ then $U/U^2 \cong G_k$ so that $M_F = R_F \otimes_{R_k} M$ in this case, and there is a natural (split) short exact sequence of groups
\[ 1 \rightarrow G_k \rightarrow G_F \rightarrow \Gamma/2 \rightarrow 0 \]
and $R_k \subset R_F$ naturally.

In [8] we proved the existence of a natural surjective specialization homomorphism
\[ S_v : \overline{RP}(F) \rightarrow \overline{RP}(k)_F \]
where $\overline{RP}(k) := \overline{RP}(k)/(K_k^{(1)} + I_k D_k) = \overline{RP}(k)/(K_k^{(1)} + K_k^{(2)})$.

In this subsection, we revisit this specialization map and prove that it can be defined with target $\overline{RP}(k)_F$, rather than $\overline{RP}(k)_F$. (It can be shown that there is no reasonable specialization homomorphism with target $\overline{RP}(k)_F$, however.)

**Lemma 5.1.** Let $F$ be a field and let $a \in F^\times$.

1. $\langle a \rangle C_F = C_F + \psi_2 (a)$ in $\overline{RP}(F)$.
2. $\psi_2 (a) = \psi_2 (a^{-1}) = \psi_2 (-a)$ in $\overline{RP}(F)$.

**Proof.** (1) This is an immediate consequence of the formula in Corollary 4.20 above.
(2) This follows immediately from (1) since \( \langle a \rangle = \langle a^{-1} \rangle \) and \( -1 \) \( C_F = C_F \).

\[ \square \]

Note also, that in \( \overline{RP}(F) \), we have, by definition, that \( C_F = [x] + \langle -1 \rangle [1 - x] \) for any \( x \neq 0, 1 \).

**Theorem 5.2.** There is a surjective \( R_F \)-module homomorphism

\[
S_v : \overline{RP}(F) \rightarrow \overline{RP}(k)_F
\]

\[ [a] \mapsto \begin{cases} 
1 \otimes [\bar{a}], & v(a) = 0 \\
1 \otimes C_k, & v(a) > 0 \\
-(1 \otimes C_k), & v(a) < 0
\end{cases}
\]

**Remark 5.3.** The proof we give here follows closely the proof of Theorem 4.9 in [8]. Only the following cases differ: Case (v) (a) and (b), Case (vii). These cases use Lemma 5.1, which in turn relies on Theorem 4.19

**Proof.** Let \( Z_1 \) denote the set of symbols of the form \( [x], x \neq 1 \) and let \( T : R_F[Z_1] \rightarrow \overline{RP}(k)_F \) be the unique \( R_F \)-homomorphism given by

\[ [a] \mapsto \begin{cases} 
1 \otimes [\bar{a}], & v(a) = 0 \\
1 \otimes C_k, & v(a) > 0 \\
-(1 \otimes C_k), & v(a) < 0
\end{cases}
\]

We must prove that \( T(S_{x,y}) = 0 \) for all \( x, y \in F^\times \setminus \{1\} \).

Through the remainder of this proof we will adopt the following notation: Given \( x, y \in F^\times \setminus \{1\} \), we let

\[
u = \frac{y}{x} \quad \text{and} \quad w = \frac{1 - x}{1 - y}.
\]

Note that

\[
\frac{1 - x^{-1}}{1 - y^{-1}} = \frac{y}{x} \cdot \frac{x - 1}{y - 1} = uw.
\]

Thus, with this notation, \( S_{x,y} \) becomes \( [x] - [y] + \langle x \rangle [u] - \langle x^{-1} - 1 \rangle [uw] + \langle 1 - x \rangle [w] \).

We divide the proof into several cases:

**Case (i):** \( v(x), v(y) \neq 0 \)

**Subcase (a):** \( v(x) = v(y) > 0 \).

Then \( 1 - x, 1 - y \in U_1 \) and hence \( w \in U_1 \), so that \( \bar{w} = 1 \) and \( \bar{w}w = \bar{u} \). Thus

\[
T(S_{x,y}) = 1 \otimes C_k - 1 \otimes C_k + \langle x \rangle \otimes [\bar{u}] = (x^{-1} - 1) \otimes [\bar{u}]
\]

However, \( x^{-1} - 1 = x^{-1}(1 - x) \), so that \( (x^{-1} - 1) \otimes [\bar{u}] = \langle x \rangle \otimes (1 - \bar{x}) [\bar{u}] = \langle x \rangle \otimes [\bar{u}] \), and thus \( T(S_{x,y}) = 0 \) as required.

**Subcase (b):** \( v(x) = v(y) < 0 \).

Then \( u \in U \) and \( uw \in U_1 \) so that \( \bar{w} = \bar{u}^{-1} \). Thus

\[
T(S_{x,y}) = -1 \otimes C_k + 1 \otimes C_k + \langle x \rangle \otimes [\bar{u}] = \langle 1 - x \rangle \otimes [\bar{u}^{-1}]
\]

But \( 1 - x = -x(1 - x^{-1}) \) and \( 1 - x^{-1} \in U_1 \), so that the last term is \( \langle -x \rangle \otimes [\bar{u}^{-1}] \) and hence \( T(S_{x,y}) = \langle x \rangle \otimes \psi_1 (\bar{u}) = 0 \) in \( \overline{RP}(k)_F \).

**Subcase (c):** \( v(x) > v(y) > 0 \).

Then \( w \in U_1 \) and \( v(u), v(uw) < 0 \). So

\[
T(S_{x,y}) = 1 \otimes C_k - 1 \otimes C_k - \langle x \rangle \otimes C_k + \langle x^{-1} - 1 \rangle \otimes C_k.
\]
But since \( x^{-1} - 1 = x^{-1}(1-x) \) and \( 1-x \in U_1 \) it follows that \( \langle x^{-1} - 1 \rangle \otimes C_k = \langle x \rangle \otimes C_k \) and hence \( T(S_{x,y}) = 0 \).

Subcase (d): \( v(x) > 0 > v(y) \).
Then \( v(u) = v(y) - v(x) < 0 \), \( v(w) = -(1-y) = -(v(y)-1) = -v(y) > 0 \) and \( v(uw) = v(u) + v(w) = -v(x) < 0 \).
So \( T(S_{x,y}) = 1 \otimes C_k + 1 \otimes C_k - \langle x \rangle \otimes C_k + \langle x^{-1} - 1 \rangle \otimes C_k + \langle 1-x \rangle \otimes C_k \).

But \( 1-x \in U_1 \) and \( \langle x^{-1} - 1 \rangle = \langle x \rangle(1-x) \). So this gives \( T(S_{x,y}) = 1 \otimes 3C_k \).

However, \( 3C_k = \psi_1(-1) \) in \( \mathcal{RP}(k) \).

Subcase (e): \( 0 > v(x) > v(y) \).
Then \( v(u) = v(y) - v(x) < 0 \) and \( uw \in U_1 \), so that \( v(w) = -v(u) > 0 \) and \( \tilde{w} = \tilde{u}^{-1} \).
Thus \( \langle x \rangle \otimes \langle 1-x \rangle \).

Now \( 1-x = -(1-x^{-1}) \) and \( 1-x^{-1} \in U_1 \), so the last term is \( \langle x \rangle \otimes (-1)C_k = \langle x \rangle \otimes C_k \).
This gives \( T(S_{x,y}) = 0 \) as required.

Subcases (f),(g),(h): The corresponding calculations when \( v(y) > v(x) \) are almost identical.

Case (ii): \( x, y \in U_1 \).

Subcase (a): \( v(1-x) \neq v(1-y) \).
Then \( u \in U_1 \) and \( v(w) = v(uw) \neq 0 \). So
\[
T(S_{x,y}) = \pm \left( \langle x^{-1} - 1 \rangle \otimes C_k - \langle x \rangle \otimes C_k \right) = 0
\]
since \( \langle x^{-1} - 1 \rangle = \langle x^{-1} \rangle \langle 1-x \rangle \) and \( x^{-1} \in U_1 \).

Subcase (b): \( v(1-x) = v(1-y) \).
Then \( u \in U_1 \) and \( w, uw \in U \) with \( uw = \tilde{w} \). So
\[
T(S_{x,y}) = -\langle x^{-1} - 1 \rangle \otimes \tilde{w} + \langle 1-x \rangle \otimes \tilde{w}
\]
which is 0 by the same argument as the previous (sub)case.

Case (iii): \( x \in U_1, v(y) \neq 0 \).
Then \( u = v(y) \). Observe that \( v(1-y) = \min(v(1), v(y)) = \min(0, v(y)) \leq 0 \). Of course, \( v(1-x) > 0 \).
Thus \( v(w) = v(1-x) - v(1-y) > 0 \) and \( v(uw) = v(u) + v(w) = v(1-x) + v(y) - v(1-y) > 0 \)
since \( v(y) - v(1-y) \geq 0 \).
So
\[
T(S_{x,y}) = \pm(1 \otimes C_k - 1 \otimes C_k - \langle x^{-1} - 1 \rangle \otimes C_k + \langle 1-x \rangle \otimes C_k = 0
\]
since \( x \in U_1 \) and thus \( \langle x^{-1} - 1 \rangle \otimes C_k \langle 1-x \rangle \otimes C_k \).

Case (iv): \( v(x) \neq 0, y \in U_1 \)
Arguing as in the last case, \( v(w), v(uw) < 0 \) in this case.

Subcase (a): \( v(x) > 0 \)
Then \( v(u) = -v(x) < 0 \). So
\[
T(S_{x,y}) = 1 \otimes C_k - \langle x \rangle \otimes C_k + \langle x^{-1} - 1 \rangle \otimes C_k + \langle 1-x \rangle \otimes C_k = 0
\]
since \( 1-x \in U_1 \).

Subcase (b): \( v(x) < 0 \).
Then \( v(u) > 0 \) and
\[
T(S_{x,y}) = -1 \otimes C_k + \langle x \rangle \otimes C_k + \langle x^{-1} - 1 \rangle \otimes C_k - \langle 1-x \rangle \otimes C_k
\]
since \( 1-x^{-1} \in U_1 \).
Case (v): $x \in U \setminus U_1$ and $v(y) \neq 0$.

Subcase (a): $v(y) > 0$.

Then $v(u) = v(y) > 0$. Since $1 - x \in U$ and $1 - y \in U_1$, it follows that $v(w) = 0$, $\tilde{w} = 1 - \tilde{x}$ in $k$ and $v(uw) > 0$. Thus

$$T(S_{x,y}) = 1 \otimes [\bar{x}] - 1 \otimes C_k + \langle x \rangle \otimes C_k - \langle x^{-1} - 1 \rangle \otimes C_k + (1 - x) \otimes [1 - \bar{x}] = 1 \otimes X.$$  

Using Lemma 5.1 (1) we have

$$X = [\bar{x}] + \langle 1 - \bar{x} \rangle [1 - \bar{x}] - C_k + \psi_2(\bar{x}) - \psi_2(\bar{x}^{-1} - 1) \in \mathcal{RP}(k).$$

But

$$\psi_2(\bar{x}^{-1} - 1) = \psi_2(\bar{x}^{-1} \cdot (1 - \bar{x})) = \langle \bar{x} \rangle \psi_2(1 - \bar{x}) + \psi_2(\bar{x}^{-1})$$

and thus

$$X = [\bar{x}] + \langle 1 - \bar{x} \rangle [1 - \bar{x}] - C_k - \langle \bar{x} \rangle \psi_2(1 - \bar{x})$$

(using Lemma 5.1 (2)). However,

$$C_k = [\bar{x}] + \langle -1 \rangle [1 - \bar{x}] \text{ and } \langle \bar{x} \rangle \psi_2(1 - \bar{x}) = \langle 1 - \bar{x} \rangle [1 - \bar{x}] + \left[ \frac{1}{1 - \bar{x}} \right].$$

Thus

$$X = -\langle -1 \rangle [1 - \bar{x}] - \left[ \frac{1}{1 - \bar{x}} \right] = -\psi_1 \left( \frac{1}{1 - \bar{x}} \right) = 0 \text{ in } \mathcal{RP}(k).$$

Subcase (b): $v(y) < 0$.

We have $v(u) = v(y) < 0$ and $v(w) = -v(1 - y) = -v(y) > 0$. Thus $v(uw) = v(u) + v(w) = v(y) - v(y) = 0$. Furthermore, since $1 - y^{-1} \in U_1$, $\tilde{u}\tilde{w} = 1 - \tilde{x}^{-1}$.

Thus $T(S_{x,y}) = 1 \otimes X$ where

$$X = [\bar{x}] + C_k - \langle \bar{x} \rangle C_k - \langle \bar{x}^{-1} - 1 \rangle \left[ 1 - \bar{x}^{-1} \right] + \langle 1 - \bar{x} \rangle C_k \in \mathcal{RP}(k).$$

Using Lemma 5.1 (1) again, and the identity $C_k = \left[ 1 - \bar{x}^{-1} \right] + \langle -1 \rangle \left[ \bar{x}^{-1} \right]$ we deduce

$$x = \psi_1(\bar{x}) + [1 - \bar{x}] - \langle \bar{x}^{-1} - 1 \rangle \left[ 1 - \bar{x}^{-1} \right] + \psi_2(1 - \bar{x}) - \psi_2(\bar{x})$$

$$= [1 - \bar{x}] - \langle \bar{x}^{-1} - 1 \rangle \left[ 1 - \bar{x}^{-1} \right] + \psi_2(1 - \bar{x}) - \psi_2(\bar{x})$$

$$= [1 - \bar{x}] + \langle 1 - \bar{x}^{-1} \rangle \left[ \frac{1}{1 - \bar{x}^{-1}} \right] + \psi_2(1 - \bar{x}) - \psi_2(\bar{x})$$

(using $\psi_1(1 - \bar{x}^{-1}) = 0$ in $\mathcal{RP}(k)$ in the last step).

However

$$[1 - \bar{x}] + \langle 1 - \bar{x}^{-1} \rangle \left[ \frac{1}{1 - \bar{x}^{-1}} \right] = \langle 1 - \bar{x} \rangle \psi_2(\left[ \frac{1}{1 - \bar{x}^{-1}} \right])$$

$$= \psi_2 \left( \frac{1 - \bar{x}}{1 - \bar{x}^{-1}} \right) - \psi_2(1 - \bar{x})$$

$$= \psi_2(-\bar{x}) - \psi_2(1 - \bar{x})$$

and so $X = 0$ by Lemma 5.1 (2) again.

Case (vi): $y \in U \setminus U_1$ and $v(x) \neq 0$
Subcase (a): $v(x) > 0$
we have $v(u) = -v(x) < 0$, $v(w) = 0$ and $1 - x \in U_1$ so that $\bar{w} = (1 - \bar{y})^{-1}$. Finally $v(\bar{u}w) = v(u) < 0$. Thus

$$T(S_{x,y}) = 1 \otimes C_k - 1 \otimes [\bar{y}] - \langle x \rangle \otimes C_k + \left(x^{-1} - 1\right) \otimes C_k + \langle 1 - x \rangle \otimes \left[\frac{1}{1 - \bar{y}}\right].$$

However, $\left(x^{-1} - 1\right) \otimes C_k = \langle x \rangle \langle 1 - x \rangle \otimes C_k = \langle x \rangle \otimes \langle 1 \rangle C_k = 0$. Thus $T(S_{x,y}) = 1 \otimes Y$ where

$$Y = C_k - [\bar{y}] + \left[\frac{1}{1 - \bar{y}}\right] \in \underaccent{h}\mathcal{P}(k).$$

Since $C_k = [\bar{y}] + \langle -1 \rangle [1 - \bar{y}]$ in $\underaccent{h}\mathcal{P}(k)$ we thus have

$$Y = \langle -1 \rangle [1 - \bar{y}] + \left[\frac{1}{1 - \bar{y}}\right] = \psi_1 \left(\frac{1}{1 - \bar{y}}\right) = 0 \text{ in $\underaccent{h}\mathcal{P}(k)$.}$$

Subcase (b): $v(x) < 0$
We have $v(u) = -v(x) > 0$ and $v(w) = v(1-x) = v(x) < 0$ and $v(\bar{u}w) = v(u) + v(w) = 0$. Furthermore, since $1 - x^{-1} \in U_1$, $\bar{u}w = 1/(1 - \bar{y})^{-1}$. Thus

$$T(S_{x,y}) = -(1 \otimes C_k) - 1 \otimes [\bar{y}] + \langle x \rangle \otimes C_k - \left(x^{-1} - 1\right) \otimes \left[\frac{1}{1 - \bar{y}^{-1}}\right] - \langle 1 - x \rangle \otimes C_k$$

$$= 1 \otimes Y$$

where

$$Y = -C_k - [\bar{y}] - \langle -1 \rangle \left[\frac{1}{1 - \bar{y}^{-1}}\right]$$

since $\langle x \rangle - \langle 1 - x \rangle \otimes C_k = -\langle x \rangle \langle x^{-1} - 1 \rangle \otimes C_k = \langle x \rangle \otimes \langle -1 \rangle C_k = 0$.

But

$$-C_k - [\bar{y}] - \langle -1 \rangle \left[\frac{1}{1 - \bar{y}^{-1}}\right] = -C_k - [\bar{y}] - \langle -1 \rangle \left[\frac{1}{1 - \bar{y}^{-1}}\right] + \psi_1 \left(1 - \bar{y}^{-1}\right)$$

$$= -C_k - [\bar{y}] + [1 - \bar{y}^{-1}]$$

$$= -C_k - [\bar{y}] + [1 - \bar{y}^{-1}] + \psi_1 (y)$$

$$= -C_k + \langle -1 \rangle [\bar{y}] + [1 - \bar{y}^{-1}] = 0.$$
Case (viii): \( x \in U_1 \) and \( y \in U \setminus U_1 \)
We have \( u \in U \) and \( \bar{u} = \bar{y} \), \( v(w) = v(1 - x) > 0 \) and \( v(\bar{u}w) = v(w) > 0 \). So
\[
T(S_{x,y}) = -(1 \otimes [\bar{y}]) + \langle x \rangle \otimes [\bar{y}] - \langle x^{-1} - 1 \rangle \otimes C_k + \langle 1 - x \rangle \otimes C_k
= -\langle 1 - x \rangle \langle C \bar{y} \rangle C_k = 0 \text{ since } \bar{x} = 1.
\]

Case (ix): \( x, y \in U \setminus U_1 \)
In this case \( \bar{x}, \bar{y} \in k^\times \setminus \{1\} \) and
\[
T(S_{x,y}) = 1 \otimes S_{\bar{x},\bar{y}} = 1 \otimes 0.
\]
\[\square\]

Of course, the specialization map \( S_\nu \) induces a well-defined map, which will also be denoted \( S_\nu \), from \( \overline{RP}(F) \) to \( \overline{RP}(k) \).

5.2. The specialization map and \( \overline{RP}_1(F) \).

**Lemma 5.4.** Let \( F \) be a field with valuation \( \nu \) and corresponding residue field \( k \). The following diagram commutes:

\[
\begin{array}{ccc}
\overline{RP}(F) & \xrightarrow{\lambda_1} & \overline{R}_F \\
\downarrow{S_\nu} & & \downarrow{}
\end{array}
\]

\[
\overline{RP}(k)_F \xrightarrow{(\lambda_1)_F} \overline{(R)_k}_F
\]

where the right-hand vertical map sends \( x \) to \( x \otimes 1 \in (\overline{R}_k)_F = R_F \otimes_{\mathbb{Z}[U]} \overline{R}_k = \overline{R}_F \otimes_{\mathbb{Z}[U]} \overline{R}_k \).

**Proof.** Let \( a \in F^\times \). We must show that
\[
\lambda_1([a]) \otimes 1 = 1 \otimes \lambda_1(S_\nu([a])) \text{ in } \overline{R}_F \otimes_{\mathbb{Z}[U]} \overline{R}_k.
\]

Case (i): \( a \in U \setminus U_1 \).
Then \( a, 1 - a \in U \) and \( S_\nu([a]) = [\bar{a}] \). Thus
\[
\lambda_1([a]) \otimes 1 = \langle 1 - a \rangle \langle a \rangle \otimes 1 = 1 \otimes \langle 1 - a \rangle = \langle 1 - a \rangle = \langle 1 - a \rangle \otimes 0 = 0 = 0.
\]

Case (ii): \( a \in U_1 \).
Then \( \bar{a} = 1 \) and hence \( S_\nu([a]) = [\bar{a}] = 0 \) while
\[
\lambda_1([a]) \otimes 1 = \langle 1 - a \rangle \langle a \rangle \otimes 1 = \langle 1 - a \rangle \langle a \rangle = \langle 1 - a \rangle \otimes 0 = 0 = 0.
\]

Case (iii): \( \nu(a) > 0 \).
Then \( S_\nu(a) = 1 \otimes C_k \) and \( \lambda_1(C_k) = 0 \). On the other hand,
\[
\lambda_1([a]) \otimes 1 = \langle 1 - a \rangle \langle a \rangle \otimes 1 = \langle a \rangle \langle 1 - a \rangle = \langle a \rangle \otimes 0 = 0 = 0.
\]

Case (iv): \( \nu(a) < 0 \).
Then \( S_\nu([a]) = -(1 \otimes C_k) \) and hence \( 1 \otimes \lambda_1(S_\nu([a])) = 0 \). Thus we must show that
\[
\langle 1 - a \rangle \langle a \rangle \otimes 1 = 0 \text{ in } \overline{R}_F \otimes \overline{R}_k.
\]
Let \( b = 1/a \). Then \( \langle a \rangle = \langle b \rangle \) and
\[
1 - a = 1 - \frac{1}{b} = -\frac{1 - b}{b} = -b(1 - b) \pmod{(F^\times)^2}.
\]
Thus \(\langle 1-a \rangle = \langle -b(1-b) \rangle = \langle -b \rangle \langle 1-b \rangle + \langle -b \rangle + \langle 1-b \rangle \) and hence
\[
\langle 1-a \rangle \langle a \rangle \otimes 1 = (\langle -b \rangle \langle 1-b \rangle + \langle -b \rangle + \langle 1-b \rangle) \langle b \rangle \otimes 1
\]
\[
= (\langle -b \rangle \langle b \rangle \langle 1-b \rangle \langle 1-b \rangle + \langle -b \rangle \langle b \rangle + \langle 1-b \rangle \langle b \rangle) \otimes 1
\]
\[
= (\langle -b \rangle \langle b \rangle + (\langle -b \rangle + 1) \otimes 1 + \langle 1-b \rangle \langle b \rangle) \otimes 1
\]
\[
= -p_+^{-1} \langle b \rangle ((\langle 1-b \rangle + 1) \otimes 1 + \langle b \rangle \otimes \langle 1-b \rangle)
\]
\[
= - (\langle b \rangle ((\langle 1-b \rangle + 1) \otimes p_+^{-1} + \langle b \rangle \otimes 0 = 0.
\]

\[\square\]

**Corollary 5.5.** Let \(F\) be a field with valuation \(v\) and corresponding residue field \(k\). Then the map \(S_v : \overline{\mathcal{RP}}(F) \to \overline{\mathcal{RP}}(k)_F\) restricts to a homomorphism of \(R_F\)-modules \(S_v : \overline{\mathcal{RP}}_1(F) \to \overline{\mathcal{RP}}_1(k)_F\).

Furthermore, if \(U_1 = U_1^2\), this restricted homomorphism is surjective.

**Proof.** If \(M\) is a \(R_k\)-module, then
\[
M_F = R_F \otimes \mathbb{Z}[U] M = R_F \otimes \mathbb{Z}[U/U^2] M.
\]
Since \(R_F\) is a free \(\mathbb{Z}[U/U^2]\)-module, it follows that the functor \(M \mapsto M_F\), from \(R_k\)-modules to \(R_F\)-modules, is exact. Thus \(\overline{\mathcal{RP}}_1(k)_F = \text{Ker}((\bar{\lambda})_F)\) and the first statement follows from Lemma 5.4.

If \(U_1 = U_1^2\), then \(\mathbb{Z}[U] = R_k\) and the map \(\overline{\mathcal{RP}} \to (\bar{\mathcal{R}}k)_F\) is an isomorphism. The second statement then follows by diagram-chasing. \[\square\]

**Remark 5.6.** On the other hand, \(S_v\) does not restrict to a homomorphism \(\overline{\mathcal{RB}}(F) \to \overline{\mathcal{RB}}(k)_F\), as our calculations below repeatedly demonstrate. See also [\text{[R]} section 4] for further discussion of this point.

5.3. **The module \(L_v.** Recall that \(S_v\) denotes the \(R_F\)-module homomorphism \(\overline{\mathcal{RP}}(F) \to \overline{\mathcal{RP}}(k)_F\) of Theorem 5.2.

We will also let \(S_v\) denote the induced (surjective) specialization homomorphism on fully reduced pre-Bloch groups
\[
S_v : \overline{\mathcal{RP}}(F) \to \overline{\mathcal{RP}}(k)_F, \quad [a] \mapsto \begin{cases} \ 1 \otimes [\bar{a}], & v(a) = 0 \\ \ 0, & v(a) \neq 0 \end{cases}
\]
Given a triple \((F, v, k)\), we set
\[
L_v := \{ [a] \in \overline{\mathcal{RP}}(F) | v(a) \neq 0 \}_{R_F}.
\]

**Lemma 5.7.** As submodules of \(\overline{\mathcal{RP}}(F)\) we have
\[
L_v = \langle [au] - [a] \ | \ a \in U \setminus U_1, u \in U_1 \rangle_{R_F} = \langle [u] \ | \ u \in U_1 \rangle_{R_F}
\]

**Proof.** Let
\[
L'_v = \langle [au] - [a] \ | \ a \in U \setminus U_1, u \in U_1 \rangle_{R_F} \text{ and } L''_v = \langle [u] \ | \ u \in U_1 \rangle_{R_F}.
\]
Let \(u \in U_1\). Then \[u\] = \(-1\) \((-1) \{ 1-u \} \in L_v \) since \(v(1-u) > 0\). Thus \(L'_v \subset L_v\).

Conversely, suppose that \(a \in F^\times \) with \(v(a) \neq 0\). Since \(a^{-1} = -(-1) [a] \) in \(\overline{\mathcal{RP}}(F)\), we can suppose that \(v(a) > 0\). Then \(1 - a \in U_1\) and hence \([a] \in L''_v\). Hence \(L_v \subset L''_v\).

Now suppose that \(a \in U \setminus U_1\) and \(u \in U_1\). Then
\[
0 = [a] - [au] + [a] [u] - (a^{-1} - 1) \left[ u \cdot \frac{1-a}{1- au} \right] + (1-a) \left[ \frac{1-a}{1- au} \right].
\]
Since $1 - a \equiv 1 - au \pmod{U_1}$, it follows that $[au] - [a] \in L''_v$. Thus $L'_v \subset L''_v$.

Finally, suppose that $u \in U_1$ and choose $b \in U$ such that $a = b^2 \notin U_1$. Then

$$0 = [a] - [au] + \langle a \rangle [u] - \langle a^{-1} - 1 \rangle \left[ u \cdot \frac{1 - a}{1 - au} \right] + \langle 1 - a \rangle \left[ \frac{1}{1 - au} \right] \equiv \langle a \rangle [u] \quad \text{(mod } L'_v)$$

since $(1 - a)/(1 - au) \in U_1$ and $\langle a^{-1} - 1 \rangle = \langle a^{-1} \rangle (1 - a) = \langle 1 - a \rangle$. It follows that $L''_v \subset L'_v$. \[\square\]

**Lemma 5.8.** There is a short exact sequence of $R_F$-modules

$$0 \longrightarrow L_v \longrightarrow \overline{RP}(F) \xrightarrow{S_v} \overline{RP}(k)_F \longrightarrow 0$$

**Proof.** By definition of $L_v$, $L_v \subset \text{Ker}(S_v)$ and hence $S_v$ induces a surjective homomorphism

$$\overline{RP}(F)_v := \frac{\overline{RP}(F)}{L_v} \twoheadrightarrow \overline{RP}(k)_F.$$ 

Since $L_v = L'_v$ by Lemma 5.7, for any $a \in U$ the element $[a]$ in $\overline{RP}(F)_v$ depends only on image, $\bar{a}$, of $a$ in $k^\times$. 

Furthermore, the action of $\mathbb{Z}[U]$ on $\overline{RP}(F)_v$ descends to an $R_k$-module structure: Let $a \in U$ and $u \in U_1$. Then $\langle -a \rangle [a] = [a]$ by Lemma 4.24. However, $[a] = [au]$ in $\overline{RP}(F)_v$. Thus, we also have $\langle -au \rangle [a] = [a]$ in $\overline{RP}(F)_v$, from which it follows that

$$\langle u \rangle [a] = \langle -a \rangle [a] = [a] \text{ in } \overline{RP}(F)_v.$$ 

It therefore follows that there is a well-defined $R_k$-module homomorphism

$$T_v : \overline{RP}(k) \rightarrow \overline{RP}(F)_v, \quad [\bar{a}] \mapsto [a]$$

which extends to an $R_F$-module homomorphism

$$T_{v,F} : \overline{RP}(k)_F \rightarrow \overline{RP}(F)_v.$$ 

Since $T_{v,F}$ is surjective by the work above, and since clearly $S_v \circ T_{v,F} = \text{Id}_{\overline{RP}(k)_F}$, it follows that $S_v : \overline{RP}(F)_v \rightarrow \overline{RP}(k)_F$ is an isomorphism with inverse $T_{v,F}$. \[\square\]

Recall that for any field there is a natural homomorphism of $R_F$-modules

$$\tilde{\lambda}_1 = \tilde{\lambda}_{1,F} : \overline{RP}(F) \rightarrow \tilde{R}_F$$

with kernel $\overline{RP}_1(F)$.

**Corollary 5.9.** Suppose that $U_1 = U_1^2$. Then the homomorphism $S_v : \overline{RP}(F) \rightarrow \overline{RP}(k)_F$ induces a short exact sequence

$$0 \rightarrow L_v \left[ \frac{1}{2} \right] \rightarrow \overline{RP}_1(F) \left[ \frac{1}{2} \right] \rightarrow \overline{RP}_1(k)_F \left[ \frac{1}{2} \right] \rightarrow 0.$$ 

**Proof.** By Corollary 5.5 there is a commutative diagram with exact rows

$$\begin{array}{cccc}
0 & \longrightarrow & \overline{RP}_1(F) & \longrightarrow & \overline{RP}(F) & \longrightarrow & \overline{R}_F \\
& & \downarrow & & \downarrow s_v & & \\
0 & \longrightarrow & \overline{RP}_1(k)_F & \longrightarrow & \overline{RP}(k)_F & \longrightarrow & (\overline{R}_k)_F
\end{array}$$

However, since $U_1 = U_1^2$ we have $R_k \cong \mathbb{Z}[U/U^2] \subset R_F$ and thus

$$(\overline{R}_k)_F = R_F \otimes_{R_k} \frac{R_k}{p^3_1R_k} = \frac{R_F}{p^3_1R_F} = \tilde{R}_F.$$
The result now follows from Lemma \ref{lem:5.8}. \hfill \Box

For a commutative local ring \( A \) with residue field \( k \) we let
\[
\mathcal{L}_A := \langle [au] - [u] \in \overline{RP}(A) \mid a \in \mathcal{W}_A, u \in U_1 \rangle_{R_A}.
\]

**Lemma 5.10.** There is a short exact sequence of \( R_A \)-modules
\[
0 \to \mathcal{L}_A \to \overline{RP}(A) \to \overline{RP}(k) \to 0.
\]

**Proof.** The functorial map \( \overline{RP}(A) \to \overline{RP}(k) \) is clearly surjective and \( \mathcal{L}_A \) is contained in its kernel. Now the argument in the proof of Lemma \ref{lem:5.8} shows that the action of \( R_A \) on \( \overline{RP}(A)/\mathcal{L}_A \) descends to an action of \( R_k \); i.e. if \( u \in U_1 \), then \( \langle u \rangle \) acts trivially on this quotient. This allows us to construct a well-defined inverse map \( \overline{RP}(k) \to \overline{RP}(A)/\mathcal{L}_A, [\overline{a}] \mapsto [\overline{a}] + \mathcal{L}_A \). \hfill \Box

6. **Bloch groups of local rings and scissors congruence groups of residue fields**

6.1. **Some preliminaries.** Let \( A \) be a commutative local ring or a field.

For a character \( \chi \in \widehat{G}_A = \text{Hom}(G_A, \mu_2) \), let \( I^\chi \) be the ideal of \( R_A \) generated by the elements \( \{ \langle a \rangle - \chi(a) \mid \langle a \rangle \in G_A \} \). In other words \( I^\chi \) is the kernel of the ring homomorphism \( \rho(\chi) : R_A \to \mathbb{Z} \)

If \( M \) is an \( R_A \)-module, we let \( M^\chi = I^\chi M \) and we let
\[
M^\chi := M/I^\chi M = (R_A/I^\chi) \otimes_{R_A} M = (R_A)_\chi \otimes_{R_A} M.
\]

In particular, if \( \chi = 1 \) then \( I^1 = I_A \) and \( M^1 = M_\widehat{G}_A \).

Given \( \langle a \rangle \in G_A \) and \( \chi \in \widehat{G}_A \), let \( e^a_{\pm \chi} \in R_A \left[ \frac{1}{2} \right] \) denote the idempotent
\[
\frac{1 \pm \chi(a) \langle a \rangle}{2} = \begin{cases} 
0, & \chi(a) = 1 \\
1, & \chi(a) = -1
\end{cases}
\]

so that \( \langle a \rangle e^a_{\pm \chi} = \pm \chi(a) e^a_{\pm \chi} \) for all \( a \). Of course, we have
\[
e^a_{\chi} + e^a_{-\chi} = 1 \quad \text{and} \quad e^a_{\chi} \cdot e^a_{-\chi} = 0
\]

for all \( a, \chi \).

More generally, for any finite subset \( S \) of \( G_A \) and any \( \chi \in \widehat{G}_A \), let
\[
e^S_{\chi} := \prod_{\langle a \rangle \in S} e^a_{\pm \chi} = \prod_{\langle a \rangle \in S} \left( \frac{1 \pm \chi(a) \langle a \rangle}{2} \right) \in R_A \left[ \frac{1}{2} \right].
\]

**Lemma 6.1.** For any character \( \chi \) and finite \( S \subset G_A \), \( 1 - e^S_{\chi} \in I^\chi \subset R_A \left[ \frac{1}{2} \right] \).

**Proof.** This follows from \( \rho(\chi)(e^a_{\chi}) = 1 \) for any \( a \in U_A \). \hfill \Box

**Corollary 6.2.** For any \( \chi \in \widehat{G}_A \) and \( R_A \left[ \frac{1}{2} \right] \)-module \( M \), we have
\[
M^\chi = \{ m \in M \mid e^S_{\chi} m = 0 \text{ for some finite } S \subset G_A \}.
\]

**Proof.** Let \( m \in M^\chi \). Then there exist \( a_1, \ldots, a_r \in U_A \) such that
\[
m = \sum_{i=1}^r e^{a_i}_{\chi} m_i
\]

from which it follows that \( e^S_{\chi} m = 0 \) where \( S = \{ \langle a_1 \rangle, \ldots, \langle a_r \rangle \} \).
Conversely, suppose that \( e^S \chi m = 0 \) for some finite set \( S \). Then
\[
m = (1 - e^S \chi) m \in M^\chi
\]
by Lemma 6.1.

**Corollary 6.3.** The functor \( M \mapsto M^\chi \) is an exact functor on the category of \( R_A \left[ \frac{1}{2} \right] \)-modules.

**Proof.** Since this is the functor \((R_A)_\chi \otimes_{R_F} (-)\) it is right-exact. On the other hand, if \( N \) is an \( R_A \left[ \frac{1}{2} \right] \)-submodule of \( M \) then Corollary 6.2 implies immediately that \( N^\chi = M^\chi \cap N \), from which left-exactness follows. \( \square \)

Note that, in practice, we may assume, when required, that \( S \) is a subgroup of \( \mathcal{G}_A \):

**Lemma 6.4.** Let \( S \subset \mathcal{G}_A \) be a finite set and let \( \chi \in \widehat{\mathcal{G}_A} \). Let \( T \) be the subgroup of \( \mathcal{G}_A \) generated by \( S \). Then \( e^S_\chi = e^T_\chi \).

**Proof.** Let \( \langle b \rangle \in T \). Then there exist \( \langle a_1 \rangle, \ldots, \langle a_r \rangle \in S \) with \( b = \prod_i \langle a_i \rangle \). Now \( e^S_\chi a_i = \chi(a_i) e^S_\chi \) for each \( i \). It follows that \( \langle b \rangle e^S_\chi = \chi(b) e^S_\chi \) and hence that \( e^S_\chi = e^T_\chi e^S_\chi \). \( \square \)

When \( \mathcal{G}_A \) itself is finite, we let \( e_\chi := e^{\mathcal{G}_A}_\chi \).

**Lemma 6.5.** Suppose that \( |\mathcal{G}_A| < \infty \) and \( M \) is an \( R_A \left[ \frac{1}{2} \right] \)-module. For any character \( \chi \) the composite
\[
e^\chi \xymatrix{ M \ar[r] & M^\chi }\]
is an isomorphism of \( R_A \left[ \frac{1}{2} \right] \)-modules.

**Proof.** Since \( 1 - e_\chi \in I^\chi \) by 6.1, while \( e_\chi I^\chi = 0 \), it follows that \( I^\chi = (1 - e_\chi) R_A \left[ \frac{1}{2} \right] \) and more generally that \( M^\chi = (1 - e_\chi) M \).

Thus \( M = e_\chi M \oplus (1 - e_\chi) M = e_\chi M \oplus M^\chi \) as an \( R_A \left[ \frac{1}{2} \right] \)-module. \( \square \)

**Lemma 6.6.** Suppose that \( \mathcal{G}_A \) is finite and \( M \) is a \( R_A \left[ \frac{1}{2} \right] \)-module. Let \( \chi_0 \in \widehat{\mathcal{G}_A} \). Then
\[
M \cong \bigoplus_{\chi \in \mathcal{G}_A} M_\chi
\]
and
\[
I_F M \cong \bigoplus_{\chi \neq \chi_0} M_\chi.
\]

**Proof.** We have \( 1 = \sum_{\chi} e_\chi \) in \( R_F \left[ \frac{1}{2} \right] \), and hence, for any \( m \in M \), \( m = \sum_{\chi} e_\chi m \).

For the second statement observe that \( e_{\chi_0} M \cong M_{\chi_0} \cong M \otimes_{R_F} \mathbb{Z} \left[ \frac{1}{2} \right] \). \( \square \)

For any \( \chi \in \widehat{\mathcal{G}_A} \) and \( a \in \mathcal{W}_A \), we will let \([a]_\chi \) denote the image of \([a] \) in \( \widehat{R\mathcal{P}(A)}_\chi \).

**Lemma 6.7.** Let \( \chi \in \widehat{\mathcal{G}_A} \). Then \( 2 [a]_\chi = 0 \) in \( \widehat{R\mathcal{P}(A)}_\chi \) if \( \chi((-a)) = -1 \).

**Proof.** By definition of \( M_\chi \), \( \langle(-a) + 1 \rangle x = 0 \) for all \( x \in \widehat{R\mathcal{P}(A)}_\chi \). On the other hand, \( \langle(-a) - 1 \rangle [a] = 0 \) in \( \widehat{R\mathcal{P}(A)} \) by Lemma 4.24. Thus \( 0 = (1 + \langle -a \rangle) [a]_\chi + (1 - \langle -a \rangle) [a]_\chi = 2 [a]_\chi \). \( \square \)
6.2. **Local rings with two square classes.**

**Proposition 6.8.** Let $A$ be a local ring with residue field $F$. Suppose that $U_{1,A} = U_{2,A}^2$ and that $|\mathcal{G}_A| = |\mathcal{G}_F| = 2$. Let $\chi$ be the nontrivial character on $\mathcal{G}_A$. Then $\mathcal{L}_A \left[ \frac{1}{2} \right] = 0$.

**Proof.** We distinguish two cases:

Case (i): $-1 \notin (F^\times)^2$

In this case, $\mathcal{G}_A = \mathcal{G}_F$ is generated by the square class of $-1$. Thus $\chi((-1)) = -1$ and hence by Lemma 6.7, $\{a\}_\chi = 0$ in $\mathcal{R}(A) \left[ \frac{1}{2} \right]$ for all $a \in \mathcal{W}_A \cap U_{2,A}$.

Since $\{a\} + \{-1\} \cdot \{1 - a\} = 0$ in $\mathcal{R}(A)$, it follows that $\{a\}_\chi = 0$ whenever $a \in \mathcal{W}_A$ with $1 - a \in U_{2,A}$.

Now let $\mathcal{N} := \{a \in \mathcal{W}_A \mid a, 1 - a \in U_{2,A}^\times\}$. (Note that if $F$ is real-closed then $\mathcal{N} = \emptyset$.) We have shown that $\{a\}_\chi = 0$ whenever $a \notin \mathcal{N}$.

Let $a, b \in \mathcal{N}$ with $b/a \in \mathcal{W}_A$. Then $b/a, (1-a)/(1-b), (1-a^{-1})/(1-b^{-1}) \in U_{2,A}$.

Thus

$$0 = (S_{a,b})_\chi = \{a\}_\chi - \{b\}_\chi = \frac{b}{a}_\chi \pm \left( \frac{1 - a^{-1}}{1 - b^{-1}} \right) \chi \pm \left( \frac{1 - a}{1 - b} \right) \chi = \{a\}_\chi - \{b\}_\chi.$$

Thus if $a \in \mathcal{W}_A$ and if $1 \not\in u \in U_{1,A}$ then either $a \notin \mathcal{N}$ and $\{a\}_\chi = 0 = \{au\}_\chi$ or $a \in \mathcal{N}$ and $\{a\}_\chi = \{au\}_\chi$. Either way, $\{a\}_\chi - \{au\}_\chi = 0$ for all $a \in \mathcal{W}_A$, $u \in U_{1,A}$ and hence $\mathcal{L}_A \left[ \frac{1}{2} \right] = 0$.

Case (i): $-1 \in (F^\times)^2$

In this case, $\chi((-a)) = -1$ if and only if $a \notin U_{2,A}$ and hence, by Lemma 6.7, and the argument of case (i), $\{a\}_\chi = 0$ in $\mathcal{R}(A) \left[ \frac{1}{2} \right]$ if either $a \notin U_{2,A}^\times$ or $1 - a \notin U_{2,A}$.

Let $\mathcal{R} := \{a \in \mathcal{W}_A \mid a, 1 - a \in U_{2,A}^\times\}$ and let $\mathcal{R}' := \{a \in \mathcal{W}_A \mid a \in U_{2,A}^\times \text{ and } 1 - a \notin U_{2,A}^\times\}$. Note that $\{a\}_\chi = 0$ if $a \in \mathcal{R}'$.

We begin by noting that $\mathcal{R}' \neq \emptyset$: Let $b \notin U_{2,A}$. If $1 - b \in U_{2,A}$ then $1 - b \in \mathcal{R}'$ by definition. Otherwise $1 - b \notin U_{2,A}$. In this latter case, $b^{-1} \notin U_{2,A}$ but

$$1 - b^{-1} = (-1) \cdot \frac{1 - b}{b} \in U_{2,A}^2$$

and hence $1 - b^{-1} \in \mathcal{R}'$.

Now let $a \in \mathcal{R}$ and $s \in \mathcal{R}'$.

If $t = as \in \mathcal{R}'$ then $1 - a \in U_{2,A}^\times$, $1 - t \notin U_{2,A}^\times$ so that $(1-a)/(1-t), (1-a^{-1})/(1-t^{-1}) \notin U_{2,A}$ and hence

$$0 = (S_{a,s})_\chi = \{a\}_\chi - \{t\}_\chi = \left( \frac{1 - a^{-1}}{1 - t^{-1}} \right) \chi \pm \frac{1 - a}{1 - t} \chi = \{a\}_\chi.$$ 

On the other hand, if $t = as \notin \mathcal{R}'$, then $t \in \mathcal{R}$, $1 - a, 1 - t \in U_{2,A}$. On the other hand,

$$1 - s = 1 - \frac{a - t}{t} \notin U_{2,A}^\times \iff a - t \notin U_{2,A}^\times.$$

It follows, in this case, that $(1-a)/(1-t) \in U_{2,A}^\times$ but

$$1 - \frac{1 - a}{1 - t} = \frac{a - t}{1 - t} \notin U_{2,A}^\times$$

so that $(1-a)/(1-t) \in \mathcal{R}'$. Similarly, replacing $a$ with $a^{-1}$ and $t$ with $t^{-1}$, $(1-a^{-1})/(1-t^{-1}) \in \mathcal{R}'$. Thus, if $a, as \in \mathcal{R}$ then

$$0 = (S_{a,s})_\chi = \{a\}_\chi - \{t\}_\chi = \{a\}_\chi - \{as\}_\chi.$$
From all of this we conclude that for any \( a \in \mathcal{W}_A \), either \([a]_x = 0\) or \( a \in \mathcal{R}\) and \( a, s \in \mathcal{R}'\), \([a]_x = [as]_x\) for all \( s \in \mathcal{R}'\).

Finally, let \( a \in \mathcal{W}_A \), \( 1 \neq u \in U_{1,A} \). Fix \( s \in \mathcal{R}'\). Then \( s^{-1}u \in \mathcal{R}'\) also. Thus either \([a]_x = [au]_x = 0\) or \( a \in \mathcal{R}\) and
\[
0 = [a]_x = [as]_x = [(as) \cdot (s^{-1}u)]_x = [au]_x.
\]
As in case (i), it follows that \( L_A \left( \frac{1}{2} \right) \chi = 0 \).

\[\square\]

**Corollary 6.9.** Let \( A \) be a local integral domain with infinite residue field \( F \). Suppose that \( U_{1,A} = U_{1,A}^2 \) and that \( |G_A| = |G_F| = 2 \). Then the homomorphism \( A \to F \) induces an isomorphism
\[
\overline{RB}_0(A) \left[ \frac{1}{2} \right] \cong \overline{RB}_0(F) \left[ \frac{1}{2} \right].
\]
Furthermore, if \( D_F \) is free of rank one as an \( \overline{R}_F \)-module then
\[
\overline{RB}_0(A) \left[ \frac{1}{2} \right] = \overline{RB}_0(F) \left[ \frac{1}{2} \right].
\]

**Proof.** By Corollary 5.9, there is an induced isomorphism \( \overline{RP}_1(A) \left[ \frac{1}{2} \right] \cong \overline{RP}_1(F) \left[ \frac{1}{2} \right] \).

However, since \( |G_A| = |G_F| = 2 \), for any \( R_A \left[ \frac{1}{2} \right] \)-module \( M \), we have \( I_A M = I_F M = M_A \). Thus
\[
I_A \overline{RP}_1(A) \left[ \frac{1}{2} \right] \cong I_F \overline{RP}_1(F) \left[ \frac{1}{2} \right].
\]

By Corollary 4.28 \( \overline{RB}_0(A) \left[ \frac{1}{2} \right] = I_A \overline{RP}_1(A) \left[ \frac{1}{2} \right] \) and \( \overline{RB}_0(F) \left[ \frac{1}{2} \right] = I_F \overline{RP}_1(F) \left[ \frac{1}{2} \right] \). This proves the first statement.

For the second statement, \( D_A \cong D_F \) by Lemma 4.22.

\[\square\]

**Remark 6.10.** Note that if \( F \) is real-closed, we have \( \overline{RB}_0(F) = 0 \). Furthermore, since the group of square classes is generated by \(-1\), we have \( N_F = F^\times \) and hence \( \overline{R}_F = \mathbb{R} \). Thus \( D_F = \mathbb{R}_3 \cdot D_F \cong \mathbb{Z}/3 \) is a free module of rank 1 over \( \overline{R}_F \), and hence the second statement of Corollary applies to real-closed fields.

**6.3. Local rings with finite residue field.** To prove a corresponding result in the case of finite residue fields requires a separate argument, since Mirzaii’s results (Proposition 3.23) require the residue fields to be infinite.

We let \( \Sigma_0 \) denote the set \( \{2, 3, 4, 5, 7, 8, 9, 16, 27, 32, 64\} \) of prime powers \( p^f \) satisfying \( (p - 1)f \leq 6 \) (see Remark 3.20).

**Lemma 6.11.** Let \( A \) be a local ring with finite residue field \( F \) of odd characteristic. Suppose that \( U_{1,A} = U_{1,A}^2 \). Then
\[
I_A \overline{RB}(A) \left[ \frac{1}{2} \right] = \overline{RB}_0(A) \left[ \frac{1}{2} \right] = \overline{RB}_0(F) \left[ \frac{1}{2} \right] = 0.
\]

**Proof.** By Proposition 6.8 we have
\[
I_A \overline{RP}_1(A) \left[ \frac{1}{2} \right] \cong I_F \overline{RP}_1(F) \left[ \frac{1}{2} \right].
\]
However, by Lemma 4.22 \( D_A \cong D_F \) also and hence this isomorphism lifts to an isomorphism
\[
I_A \overline{RP}_1(A) \left[ \frac{1}{2} \right] \cong I_F \overline{RP}_1(F) \left[ \frac{1}{2} \right].
\]
However, for a finite field \( F \)
\[
I_F \overline{RP}_1(F) \left[ \frac{1}{2} \right] = \overline{RB}_0(F) \left[ \frac{1}{2} \right] = 0.
\]
Thus \( I_A \overline{\mathbb{P}}_1(A)[\frac{1}{2}] = 0 \) and hence \( \overline{\mathbb{P}}_1(A)[\frac{1}{2}] = \overline{\mathbb{P}}_1(A)[\frac{1}{2}]_{G_A} \).

Let \( M \) be the \( R_A[\frac{1}{2}] \)-module \( \text{Im}(\lambda_1)[\frac{1}{2}] \subset \overline{R}_A[\frac{1}{2}] \). Consider the short exact sequence of \( R_A[\frac{1}{2}] \)-modules

\[
0 \to \overline{\mathbb{P}}_1(A)[\frac{1}{2}] \to \overline{\mathbb{P}}(A)[\frac{1}{2}] \to M \to 0.
\]

Taking coinvariants for the action of the cyclic group \( G_A \) of order 2 is exact here and \( M_{G_A} = 0 \) since the square class of \(-1 \) acts as multiplication by \(-1 \) on \( M \). Thus there is an induced isomorphism

\[
\overline{\mathbb{P}}_1(A)[\frac{1}{2}] \cong \overline{\mathbb{P}}(A)[\frac{1}{2}]_{G_f} = \mathcal{P}(A)[\frac{1}{2}].
\]

By Corollary 4.27 it follows that \( \mathcal{R}B_0(A)[\frac{1}{2}] = 0 \). Since it is always the case that \( I_A \mathcal{R}B(A) \subset \mathcal{R}B_0(A) \), the result follows. \( \square \)

**Corollary 6.12.** Let \( A \) be a local integral domain with finite residue field \( k \) with \(|k| \notin \Sigma_0 \). Suppose that \( U_{1,A} = U_{1,A}^2 \). Then there is a natural short exact sequence

\[
0 \to \text{tor}(\mu_A, \mu_A)[\frac{1}{2}] \to H_3(SL_2(A), \mathbb{Z}[\frac{1}{2}]) \to \mathcal{B}(A)[\frac{1}{2}] \to 0
\]

and the natural action of \( G_A = U_A/U_A^2 \) on \( H_3(SL_2(A), \mathbb{Z}[\frac{1}{2}]) \) is trivial.

**Proof.** We combine Theorem [6.47] with Lemma [6.11] \( \square \)

**Lemma 6.13.** Let \( K \) be a field with discrete valuation \( v \) and corresponding discrete valuation ring \( A \). Then the natural functorial map \( \mathcal{B}(A)[\frac{1}{2}] \to \mathcal{B}(K)[\frac{1}{2}] \) is surjective.

**Proof.** Fix \( u \in \mathcal{W}_A \). Let \( a \in K^\times \) with \( v(a) > 0 \). Then

\[
[a] + [1 - a] = [u] + [1 - u] \quad \text{in} \quad \mathcal{P}(K)
\]

and hence

\[
[a] = -[1 - a] + [u] + [1 - u] \in \text{Im}(\mathcal{P}(A) \to \mathcal{P}(K)).
\]

On the other hand, if \( a \in K^\times \) with \( v(a) < 0 \) then \( 2([a] + [a^{-1}]) = 0 \) in \( \mathcal{P}(K) \). So \( [a] = -[a^{-1}] \) in \( \mathcal{P}(K)[\frac{1}{2}] \), and thus \( [a] \in \text{Im}(\mathcal{P}(A)[\frac{1}{2}] \to \mathcal{P}(K)[\frac{1}{2}]) \) since \( v(a^{-1}) > 0 \).

It follows that the map \( \mathcal{P}(A)[\frac{1}{2}] \to \mathcal{P}(K)[\frac{1}{2}] \) is surjective.

Since the map \( U_A \to K^\times \cong \mathbb{Z} \times U_A \) is a split injection, the induced map \( \mathcal{S}_2^2(U_A) \to \mathcal{S}_2^2(K^\times) \) is split injective. The statement thus follows from the commutative diagram with exact rows

\[
\begin{array}{c}
0 \to \mathcal{B}(A)[\frac{1}{2}] \to \mathcal{P}(A)[\frac{1}{2}] \to \mathcal{S}_2^2(U_A)[\frac{1}{2}] \\
\downarrow \quad \downarrow \quad \downarrow \\
0 \to \mathcal{B}(K)[\frac{1}{2}] \to \mathcal{P}(K)[\frac{1}{2}] \to \mathcal{S}_2^2(K^\times)[\frac{1}{2}].
\end{array}
\]

\( \square \)

For a local domain \( A \) with field of fractions \( K \), let \( \mathcal{B}(A; K) \) denote \( \text{Ker}(\mathcal{B}(A) \to \mathcal{B}(K)) \).

**Remark 6.14.** When \( A \) is a discrete valuation ring, the injectivity of the map \( \mathcal{S}_2^2(U_A) \to \mathcal{S}_2^2(K^\times) \) implies that \( \mathcal{B}(A; K) = \text{Ker}(\mathcal{P}(A) \to \mathcal{P}(K)) \) also.

**Remark 6.15.** If \( A \) is a discrete valuation ring with infinite residue field \( F \) and field of fractions \( K \) and if \( \text{char}(F) = \text{char}(K) \), then \( \mathcal{B}(A; K)[\frac{1}{2}] = 0 \) by Corollary 3.25 above.
Corollary 6.16. Let $K$ be a field with discrete valuation $v$ and corresponding discrete valuation ring $A$. Suppose that the residue field $k$ is finite, that $|k| \notin \Sigma_0$ and that $U_{1,A} = U_{1,A}^2$. Then there is a short exact sequence

$$0 \to \mathcal{B}(A; K) \left[ \frac{1}{2} \right] \to \mathrm{H}_3(\mathrm{SL}_2(A), \mathbb{Z} \left[ \frac{1}{2} \right]) \to K^\text{ind}_3(A) \left[ \frac{1}{2} \right] \to 0.$$ 

**Proof.** $K^\text{ind}_3(A) \cong K^\text{ind}_3(K)$ by Theorem 2.1 and thus by Corollary 6.12 and Lemma 6.13 there is a commutative diagram with exact rows and columns

\[
\begin{array}{ccccccccc}
0 & \to & \mathcal{B}(A; K) \left[ \frac{1}{2} \right] & \to & \cdots & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & \text{tor}(\mu_A, \mu_A) \left[ \frac{1}{2} \right] & \to & \mathrm{H}_3(\mathrm{SL}_2(A), \mathbb{Z} \left[ \frac{1}{2} \right]) & \to & \mathcal{B}(A) \left[ \frac{1}{2} \right] & \to & 0 \\
\downarrow & \cong & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & \text{tor}(\mu_K, \mu_K) \left[ \frac{1}{2} \right] & \to & K^\text{ind}_3(K) \left[ \frac{1}{2} \right] & \to & \mathcal{B}(K) \left[ \frac{1}{2} \right] & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & & & & & & & & \\
\end{array}
\]

6.4. Fields with valuation and finitely many square classes. For this section we will suppose that $A$ is a local ring with residue field $F$. We interpret this to include the case $A = F$. If $a \in A$, its image in $F$ will be denoted $\bar{a}$. We suppose further that $U_{1,A} = U_{1,A}^2$ and hence there are natural identifications $\mathcal{G}_A = \mathcal{G}_F$ and $R_A = R_F$.

Suppose that $v : F^\times \to \Gamma$ is a valuation with residue field $k$. Suppose further that $U_{1,v} = U_{1,v}^2$. Then $U/U^2 \cong \mathcal{G}_k$ and with this identification $\mathcal{G}_k$ is a subgroup of $\mathcal{G}_F$ and there is a short exact sequence of $\mathbb{Z}/2$-modules

$$1 \to \mathcal{G}_k \to \mathcal{G}_F \to \Gamma/2 \to 0.$$ 

Let $j : \Gamma/2 \to \mathcal{G}_F$ be a choice of splitting of this sequence. Then there is an induced isomorphism

$$\hat{\mathcal{G}}_k \times \hat{\Gamma}/2 \cong \hat{\mathcal{G}}_F, \quad (\chi, \psi) \mapsto \chi \times_j \psi$$

where

$$(\chi \times_j \psi)(\langle u \cdot j(\gamma) \rangle) := \chi(\langle \bar{u} \rangle) \cdot \psi(\gamma).$$

The inverse map is given by

$$\hat{\mathcal{G}}_F \to \hat{\mathcal{G}}_k \times \hat{\Gamma}/2, \quad \rho \mapsto (\rho|_{\mathcal{G}_k}, \rho \circ j).$$

Recall that given a splitting $j$ and a character $\psi \in \hat{\Gamma}/2$ there is a ring homomorphism $\rho_{j,\psi} : R_F \to R_k$ sending $\langle j(\gamma) \cdot u \rangle$ to $\psi(\gamma) \langle \bar{u} \rangle$, and an associated ‘restriction’ functor giving any $R_k$-module $M$ the structure of an $R_F$-module via

$$\langle j(\gamma) \cdot u \rangle m := \psi(\gamma) \langle \bar{u} \rangle \cdot m$$

for any $\gamma \in \Gamma$, $u \in U$, and $m \in M$. We will denote this $R_F$-module structure by $M(j, \psi)$. Furthermore, given any $R_F$-module we let $M(j, \psi)$ denote the $R_F$-module $R_k \otimes_{R_F} M$ obtained from the map $\rho_{j,\psi}$. We record the following observations for future use:
Remark 6.17. In the special case, of most interest in this article, where $\Gamma = \mathbb{Z}$, we denote the characters of $\mathbb{Z}/2$ by $\varepsilon \in \{-1, 1\}$ (according to where $1 + 2\mathbb{Z}$ is sent). In this case, a splitting $j$ amounts to the choice of a (square class of a) uniformizer $\pi$. So we will also denote the pair $(j, \psi)$ as $(\pi, \varepsilon)$.

Thus $\rho_{\pi, \varepsilon} : R_F \to R_k$ is the ring homomorphism sending $\langle \pi \cdot u \rangle$ to $\varepsilon \cdot \langle \bar{u} \rangle$.

Furthermore, observe that if $M$ is any $R_F$ module, then $M(\pi, 1) \cong \pi M$ and $M(\pi, -1) \cong \pi M$ as $R_F$-modules. In particular, $M \cong M(\pi, 1) \oplus M(\pi, -1)$.

Lemma 6.18. Suppose given a splitting $j$ and $\psi \in \Gamma/2$.

1. For any $R_k$-module $M$ we have $(M_F)(j, \psi) \cong M(j, \psi)$ as $R_F$-modules.
2. For any $R_F$-module $M$ and for any $\chi \in \hat{G}_k$ we have $M_{\chi \times j, \psi} = (M(j, \psi))_{\chi}$ as $R_F$-modules.

Proof. (1) This is immediate from the definitions.

(2) The diagram

$$
\begin{array}{ccc}
R_F & \xrightarrow{\rho_{j, \psi}} & R_k \\
\downarrow{\rho_{\chi \times j, \psi}} & & \downarrow{\rho(\chi)} \\
\mathbb{Z} & & \\
\end{array}
$$

commutes; i.e. $\rho(\chi \times j, \psi) = \rho(\chi) \circ \rho_{j, \psi}$.

\[\square\]

Corollary 6.19. The specialization map $S_\psi : \overline{RP}(F) \to \overline{RP}(k)_F$ induces a surjective homomorphism

$$
S_{(j, \psi)} : \overline{RP}(F)(j, \psi) \to \left(\overline{RP}(k)_F\right)(j, \psi) = \overline{RP}(j, \psi).
$$

For any $\chi \in \hat{G}_k$ it induces a homomorphism

$$
S_{\chi \times j, \psi} : \overline{RP}(F)_{\chi \times j, \psi} \to \overline{RP}(k)_{\chi}.
$$

We let $W_{v, A} := \{a \in U_A | \bar{a} \in U_{1, v}\}$. Thus $U_{1, A} \subset W_{v, A}$ and our conditions on $A$ and $F$ ensure that $W_{v, A} = W_{v, A}^2$.

Lemma 6.20. Let $\chi \in \hat{G}_F$. Suppose $a \in W_A$ and $[b]_\chi = 0$ in $\overline{RP}(A)$ for all $b \in aU_A^2$ satisfying $v(b) \neq 0$. Then $[b]_\chi = 0$ for all $b \in -aU_A^2$ satisfying $v(b) \neq 0$.

Proof. Let $b \in -aU_A^2$ with $v(b) \neq 0$. Since $\left[\begin{bmatrix} b^{-1} \end{bmatrix} = -\langle -1 \rangle [b] \right.$ in $\overline{RP}(F)$, we can suppose that $v(b) < 0$. But then $1 - b = -b(1 - b^{-1}) = -bw$ with $w \in W_{v, A}$. Since $W_{v, A} = W_{v, A}^2$, $w \in U_A^2$ and hence $1 - b = -bU_A^2 = aU_A^2$. Since $v(1 - b) = v(b)$, it follows that $[1 - b]_\chi = 0$ in $\overline{RP}(A)_\chi$. Thus $[b]_\chi = 0$ since $[b] = -\langle -1 \rangle [1 - b]$ in $\overline{RP}(A)$. \[\square\]

Let $\mathcal{L}_{v, A} := \langle [a] \in \overline{RP}(A) | a \in W_A, v(\bar{a}) \neq 0 \rangle_{R_F}$.

Analogously to $\mathcal{L}_v$, we have

Lemma 6.21.

$$
\mathcal{L}_{v, A} = \langle [au] - [a] | a \in W_A, u \in W_{v, A}\rangle_{R_F} = \langle [u] | u \in W_A \cap W_{v, A}\rangle_{R_F}
$$
Corollary 6.22. \( \mathcal{L}_{v,A} \) is the inverse image of \( \mathcal{L}_v \) under the surjective functorial homomorphism \( \overline{\mathcal{R}}(A) \to \overline{\mathcal{R}}(F) \) and 
\[
\overline{\mathcal{R}}(A)/\mathcal{L}_{v,A} \cong \overline{\mathcal{R}}(F)/\mathcal{L}_v \cong \overline{\mathcal{R}}(k)_F.
\]

Proposition 6.23. Suppose that \( F \) is a field with valuation \( v : F^\times \to \Gamma \) with residue field \( k \) satisfying

1. \( U_{1,v} = U_{1,v}^2 \)
2. \( \Gamma \) has a minimal positive element \( \gamma_0 \)
3. \( |k| \geq 4 \) and \( k \) is either perfect or of characteristic not equal to 2.

Suppose given a character \( \psi \in \widehat{\Gamma}/2 \) satisfying \( \psi(\gamma_0) = -1 \). Then for any \( \chi \in \widehat{G}_k \) and any splitting \( j : \Gamma/2 \to \mathcal{G}_F \) we have 
\[
(\mathcal{L}_{v,A}[\frac{1}{2}])_{\chi \times j}^0 = 0.
\]

Proof. Let \( \rho := \chi \times j \psi \). We must prove that \( [a]_\rho = 0 \) in \( \overline{\mathcal{R}}(A)[\frac{1}{2}]_\rho \), whenever \( v(\bar{a}) \neq 0 \).

Since \( [a^{-1}] = -\langle -1 \rangle [a] \), it is enough to prove that \( [a]_\rho = 0 \) whenever \( v(\bar{a}) > 0 \).

Furthermore, if \( \rho(\langle -a \rangle) = -1 \), then \( [a]_\rho = 0 \) by Lemma 6.7. Thus we can suppose for the remainder of the proof that \( \rho(\langle -a \rangle) = 1 \) and \( v(\bar{a}) > 0 \).

Suppose that \( \chi(\langle -1 \rangle) = -1 \). Then \( \rho(\langle -1 \rangle) = -1 \) and \( \rho(\langle a \rangle) = -1 \). Thus \( [b]_\rho = 0 \) for all \( b \in \langle a \rangle \) by Lemma 6.7 again. It follows that \( [b]_\rho = 0 \) for all \( b \in \langle a \rangle \) with \( v(\bar{b}) \neq 0 \) by Lemma 6.20. In particular, it follows that \( [a]_\rho = 0 \) as required.

Thus we can assume that \( \rho(\langle -1 \rangle) = \langle -1 \rangle = 1 \).

Let \( \gamma = v(a) > 0 \). Then \( a = p \cdot u \) where \( \langle p \rangle = j(\langle \gamma \rangle) \) and \( v(\bar{u}) = 0 \). We consider two cases:

Case (i) \( \psi(\gamma) = 1 \).

Thus, in this case, \( 1 = \rho(a) = \chi(u)\psi(\gamma) = \chi(u) \).

Let \( \pi \in U_A \) satisfy \( v(\bar{\pi}) = \gamma_0 \) and \( \pi \in j(\langle \gamma_0 \rangle) \). Thus \( \rho(\pi) = \psi(\gamma_0) = -1 \), by hypothesis.

Observe that for any \( x \in U_A \), if \( \rho(\pm x) = 1 \) then \( \rho(\pm \pi x) = -1 \) and thus \( [\pm \pi x]_\rho = 0 \) in \( \overline{\mathcal{R}}(A)[\frac{1}{2}]_\rho \) by Lemma 6.7.

In \( \overline{\mathcal{R}}(A)[\frac{1}{2}]_\rho \) we have
\[
0 = S_{\frac{1}{\pi}, \varphi} = \left[ \frac{1}{\pi} \right]_\rho - \left[ \frac{a}{\pi} \right]_\rho \pm [a]_\rho \pm [az]_\rho \pm [z]_\rho,
\]
where
\[
\bar{z} := \frac{1 - \frac{1}{\pi}}{1 - \frac{1}{\pi}}.
\]

But
\[
v(\bar{\pi}) = \gamma - \gamma_0 > 0
\]
since \( \gamma \neq \gamma_0 \) (because \( \psi(\gamma) = 1 \) while \( \psi(\gamma_0) = -1 \)). Thus
\[
1 - \frac{a}{\pi} \in W_{v,A} \subset U_A^2.
\]

It follows that
\[
\langle z \rangle = \left\langle 1 - \frac{1}{\pi} \right\rangle = \langle -\pi \rangle
\]
since \( 1 - \pi \in W_{v,A} \subset U_A^2 \) also. Thus \( [z]_\rho = 0 = [1/\pi]_\rho \) by the remarks just above.

Similarly \( \langle az \rangle = \langle -a\pi \rangle \) and thus \( [az]_\rho = 0 = [a/\pi]_\rho \).
It follows that \([a]_\rho = 0\) as required. 

Case (ii) \(\psi(y) = -1\).

In this case, we have \(1 = \rho(a) = \psi(y)\chi(u) = -\chi(u)\), so that \(\chi(u) = -1\).

By the hypotheses on the field \(k\), given any \(x \in k^\times\) there exist \(y, z \in k^\times\) satisfying \(x = y^2 - z^2\). Furthermore, the conditions \(U_{1,v} = U_{1,v}'\) and \(U_{1,A} = U_{1,A}'\) ensure that squares in \(k^\times\) can be lifted to squares in \(U_A\). Thus there exist \(r, s \in U_A\) with \(v(\bar{r}) = v(\bar{s}) = 0\) and \(u = r^2 - s^2\). Then

\[
1 - \frac{u}{r^2} = \left(\frac{s}{r}\right)^2 \in U_A^2.
\]

In \(\widetilde{\mathcal{R}}F(A)\), we have

\[
0 = S \frac{1}{r^2p} \cdot u = \left[\frac{1}{r^2p}\right]_\rho - \left[\frac{u}{r^2}\right]_\rho \pm [a]_\rho \pm [az]_\rho \pm [z]_\rho
\]

where

\[
z := \frac{1 - \frac{r^2}{p}}{1 - \frac{u}{r^2}} \in \left\langle 1 - \frac{r^2}{p}\right\rangle = \left\langle -r^2p\right\rangle = \left\langle -p\right\rangle
\]

since \(1 - r^2p \in W_{v,A} \subset U_A^2\).

Since \(\rho(-p) = \rho(p) = -1\) it follows, as Case (i), that

\[
[z]_\rho = [az]_\rho = \left[\frac{1}{r^2p}\right]_\rho = 0.
\]

Furthermore,

\[
\left[\frac{u}{r^2}\right]_\rho = 0
\]

by Lemma 6.7 since \(\rho((\pm u)) = -1\). This completes the proof of the theorem.

\(\square\)

Combining this proposition with Lemma 5.8 and Corollary 6.19 above we immediately deduce:

**Corollary 6.24.** Under the hypotheses of Proposition 6.23 the map \(S \chi \times j \psi\) induces an isomorphism

\[
\widetilde{\mathcal{R}}F(A) \left[\frac{1}{2}\right]_{\chi \times j \psi} \cong \widetilde{\mathcal{R}}F(F) \left[\frac{1}{2}\right]_{\chi \times j \psi} \cong \widetilde{\mathcal{R}}F(k) \left[\frac{1}{2}\right]_{\chi}
\]

for any \(\chi \in \hat{G}_k\).

**Remark 6.25.** Observe that the isomorphism in this result is naturally an isomorphism of \(R_F\)-modules (or, equivalently, of \(R_A\)-modules). The right-hand term has an \(R_F\)-module structure which depends on the the pair \((\psi, j)\). Namely

\[
(j(\langle y \rangle) \cdot \langle u \rangle) \cdot x := \psi(\langle y \rangle) \langle \bar{u} \rangle \cdot x \text{ for all } x \in \widetilde{\mathcal{R}}F(k) \left[\frac{1}{2}\right]_{\chi}.
\]

**Corollary 6.26.** Under the hypotheses of Proposition 6.23 there are isomorphisms

\[
\widetilde{\mathcal{R}}F(A) \left[\frac{1}{2}\right]_{\chi \times j \psi} \cong \widetilde{\mathcal{R}}F(F) \left[\frac{1}{2}\right]_{\chi \times j \psi} \cong \widetilde{\mathcal{R}}F(k) \left[\frac{1}{2}\right]_{\chi}
\]

**Proof.** This follows from Corollary 5.9 together with the identity \(\widetilde{R}_A = \widetilde{R}_F\). \(\square\)
Corollary 6.27. Hypotheses as in Proposition 6.23. Suppose further that $\Gamma = \mathbb{Z}$ and that $\lambda \in \hat{G}_F = \hat{G}_A$ is any nontrivial character. Then $S_v$ induces a natural isomorphism

$$\overline{RP}_1(A) \left[\frac{1}{2}\right]_A \cong \overline{RP}_1(F) \left[\frac{1}{2}\right]_A \cong \overline{RP}_1(k) \left[\frac{1}{2}\right]_{A_k}$$

where

$$\lambda_k := \lambda|_{\hat{G}_k}.$$

Furthermore,

1. If $\lambda_k \neq \chi_0$, then there is an induced isomorphism

$$\overline{RB}(F) \left[\frac{1}{2}\right]_A \cong \overline{RB}(k) \left[\frac{1}{2}\right]_{A_k}.$$

2. If $\lambda_k = \chi_0$, we have

$$\overline{RB}(F) \left[\frac{1}{2}\right]_A = \overline{RP}(F) \left[\frac{1}{2}\right]_A \cong \overline{RP}(k) \left[\frac{1}{2}\right]_{A_k} \equiv \mathcal{P}(k) \left[\frac{1}{2}\right]$$

(the last isomorphism being induced by the projection $\pi : \mathcal{P}(k) \to \mathcal{P}(k)$).

Proof. We begin by observing that, since $\lambda \neq 1$, there is a uniformizer $\pi$ such that $\lambda(\pi) = -1$. This can be seen as follows:

Let $a \in F^\times$ with $\lambda(\langle a \rangle) = -1$. If $v(a)$ is odd, then $\langle a \rangle$ is represented by some uniformizer $\pi$. If $v(a)$ is even, then the class $\langle a \rangle$ is represented by a unit $u$. Let $\pi'$ be any uniformizer. If $\lambda(\pi') = -1$, then $\pi'$ will serve. Otherwise $\lambda(\pi) = -1$ where $\pi = \pi'u$.

Now, let $j : \Gamma/2 = \mathbb{Z}/2 \to G_F$ be the associated splitting (sending $\bar{1}$ to $\langle \pi \rangle$). Then

$$\lambda = (-1) \times j \lambda_k$$

and the first statement follows from Corollary 6.24.

By Corollary 5.5, $S_v$ induces an isomorphism

$$\overline{RP}_1(A) \left[\frac{1}{2}\right]_A \cong \overline{RP}_1(F) \left[\frac{1}{2}\right]_A \cong \overline{RP}_1(k) \left[\frac{1}{2}\right]_{A_k}.$$

Thus

1. If $\lambda_k \neq \chi_0$, then

$$\overline{RP}_1(k) \left[\frac{1}{2}\right]_{A_k} = I \mathcal{P} \overline{RP}_1(k) \left[\frac{1}{2}\right]_{A_k} = \overline{RB}(k) \left[\frac{1}{2}\right]_{A_k}$$

by Corollary 4.28.

2. On the other hand, if $\lambda_k = \chi_0$ then

$$\overline{RP}_1(k) \left[\frac{1}{2}\right]_{A_k} = \overline{RP}_1(k) \left[\frac{1}{2}\right]_{\chi_0} = \mathcal{P}(k) \left[\frac{1}{2}\right]$$

by Corollaries 4.27 and 4.28.

Recall that $\overline{RB}(F) = \overline{RB}(F)/\mathcal{D}_F$.

We define

$$\overline{RB}_0(F) := \text{Ker}(\overline{RB}(F) \to \overline{B}(F))$$

Lemma 6.28. For any field $F$

1. $\overline{RB}_0(F) \equiv \overline{RB}_0(F)/I_F \mathcal{D}_F$.
2. $\overline{RB}_0(F) \left[\frac{1}{2}\right] = I_F \overline{RP}_1(F) \left[\frac{1}{2}\right]$. 
Proof. (1) From the definitions, we have a commutative diagram with exact rows

\[
\begin{array}{c}
\begin{array}{c}
0 \\
\downarrow \\
0 \longrightarrow I_F \mathcal{D}_F \longrightarrow \mathcal{D}_F \longrightarrow \mathbb{Z} \cdot D_F \longrightarrow 0 \\
\downarrow \\
0 \\
\downarrow \\
0 \longrightarrow \overline{RB}_0(F) \longrightarrow \overline{RB}(F) \longrightarrow \overline{B}(F) \\
\downarrow \\
\overline{RB}_0(F) \longrightarrow \overline{RB}(F) \longrightarrow \overline{B}(F) \\
\downarrow \\
0 \\
\downarrow \\
0 \\
\downarrow \\
0 \\
\end{array}
\end{array}
\]

where \( \overline{B}(F) := \overline{B}(F)/\mathbb{Z} \cdot D_F \).

(2) Recall that \( I_F \overline{RP}_1(F) \left[ \frac{1}{2} \right] = I_F \overline{RP}_1(F) \left[ \frac{1}{2} \right] \) by Corollary 6.28. Furthermore \( I_F^2 \mathcal{D}_F = I_F \mathcal{D}_F \) since \( 3 \cdot D_F = 0 \). Thus, using (1), we have

\[
I_F \overline{RP}_1(F) \left[ \frac{1}{2} \right] = \frac{I_F \overline{RP}_1(F) \left[ \frac{1}{2} \right]}{I_F \mathcal{D}_F \left[ \frac{1}{2} \right]} = \frac{\overline{RB}_0(F) \left[ \frac{1}{2} \right]}{I_F \mathcal{D}_F \left[ \frac{1}{2} \right]} = \overline{RB}_0(F) \left[ \frac{1}{2} \right].
\]

\( \square \)

**Corollary 6.29.** Hypotheses as is Proposition 6.23. Suppose further that \(|G_F| < \infty \). Then \( S_v \) induces an isomorphism

\[
\overline{RB}_0(F) \left[ \frac{1}{2} \right] \cong I_F \left( \overline{RP}_1(k) F \left[ \frac{1}{2} \right] \right).
\]

Proof. We have \( \overline{RB}_0(F) \left[ \frac{1}{2} \right] = I_F \overline{RP}_1(F) \left[ \frac{1}{2} \right] \).

Now, if \( M \) is an \( R_F \left[ \frac{1}{2} \right] \)-module, then \( I_F M = \bigoplus_{\lambda \neq \chi_0} M_\lambda \). Thus if \( \lambda \neq \chi_0 \), then for any \( R_F \left[ \frac{1}{2} \right] \)-module \( M \) we have \( I_F M_\lambda = M_\lambda \) and if \( f : M \rightarrow N \) is a map of \( R_F \left[ \frac{1}{2} \right] \)-modules and if \( f \) induces isomorphisms \( M_\lambda \cong N_\lambda \) for all \( \lambda \neq \chi_0 \), then \( f \) induces an isomorphism \( I_F M \cong I_F N \).

The result now follows from Corollary 6.27. \( \square \)

Suppose that \( v : F^\times \rightarrow \mathbb{Z} \) is a discrete valuation with residue field \( k \), and that \( U_1 = U^1_1 \). For any \( R_k \)-module \( M \), we have \( M_F = R_F \otimes_{R_k} M \). Now \( R_F \) is a free \( R_k \)-module of rank 2. If \( \pi \) is a uniformizer for \( v \), \( R_k \) is the group ring over \( R_k \) of the cyclic group of order 2 with generator \( \langle \pi \rangle \).

Thus there is a split short exact of \( R_k \)-modules

\[
0 \longrightarrow R_k \longrightarrow R_F \longrightarrow \rho \longrightarrow 0
\]

where \( \rho(x + y \langle \pi \rangle) = y \).

It follows that if \( M \) is any \( R_k \)-module there is an induced split short exact sequence of \( R_k \)-modules

\[
0 \longrightarrow M \longrightarrow M_F \longrightarrow \rho \otimes \text{id} \longrightarrow M \longrightarrow 0
\]

In this situation we denote by \( \delta_\pi : \overline{RB}(F) \rightarrow \overline{RP}_1(k) \) the composite \( R_k \)-homomorphism

\[
\overline{RB}(F) \overset{S_v}{\longrightarrow} \overline{RP}_1(k) F \overset{\rho \otimes \text{id}}{\longrightarrow} \overline{RP}_1(k).
\]
Lemma 6.30. Let \( v \) be a discrete valuation on \( F \) with residue field \( k \) and suppose that \( U_1 = U_1^2 \).
Let \( \pi \) be a choice of uniformizer.

1. There is a natural split short exact sequence of \( R_k \)-modules

\[
0 \rightarrow I_k M \rightarrow I_F(M) \xrightarrow{\rho_\pi \otimes \text{id}} M \rightarrow 0
\]

2. If \( M \) is an \( R_k \left[ \frac{1}{2} \right] \)-module, then there is an isomorphism of \( R_F \left[ \frac{1}{2} \right] \)-modules

\[
I_F(M) \cong (I_k M) \oplus M
\]

where \( \langle \pi \rangle \) acts as multiplication by \( 1 \) on the first factor and as multiplication by \( -1 \) on the second.

Proof. (1) There is a commutative diagram of \( R_k \)-modules with exact row and columns:

\[
\begin{array}{ccccccc}
0 & \rightarrow & 0 & \rightarrow & I_k M & \rightarrow & I_F(M) \xrightarrow{\rho_\pi \otimes \text{id}} M & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
I_k M & \rightarrow & I_F M_F & \rightarrow & M & \rightarrow & M_F \xrightarrow{\rho_\pi \otimes \text{id}} M & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & M & \rightarrow & M_F & \xrightarrow{\rho_\pi \otimes \text{id}} M & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & M_{\bar{\theta}_k} & \rightarrow & M_{\bar{\theta}_k} & \rightarrow & 0 & \rightarrow & 0
\end{array}
\]

(2) There is a decomposition of the \( R_F \left[ \frac{1}{2} \right] \)-module \( M_F \) for the action of \( \langle \pi \rangle \) (see Remark 6.17 above):

\[
M_F = (M_F)_{(\pi, 1)} \oplus (M_F)_{(\pi, -1)} = M(\pi, 1) \oplus M(\pi, -1).
\]

But \( I_F M(\pi, -1) = M(\pi, -1) \) and \( I_F M(\pi, 1) = (I_k M)(\pi, 1) \).

\( \square \)

Corollary 6.31. Let \( v \) be a discrete valuation on \( F \) with residue field \( k \) and suppose that \( U_1 = U_1^2 \). Let \( \pi \) be a choice of uniformizer. Suppose that \( |G_F| < \infty \). Then

1. There is a split short exact sequence of \( R_k \)-modules

\[
0 \rightarrow R\mathcal{B}_0(k) \left[ \frac{1}{2} \right] \rightarrow R\mathcal{B}_0(F) \left[ \frac{1}{2} \right] \xrightarrow{\delta_Z} R\mathcal{P}_1(k) \left[ \frac{1}{2} \right] \rightarrow 0.
\]

2. There is an isomorphism of \( R_F \left[ \frac{1}{2} \right] \)-modules

\[
R\mathcal{B}_0(F) \left[ \frac{1}{2} \right] \cong R\mathcal{B}_0(k) \left[ \frac{1}{2} \right] \oplus R\mathcal{P}_1(k) \left[ \frac{1}{2} \right]
\]

where \( \pi \) acts as (multiplication by) \( 1 \) on the first factor and \( -1 \) on the second factor.

Proof. By Corollary 6.29

\[
R\mathcal{B}_0(F) \left[ \frac{1}{2} \right] \cong I_F \left( R\mathcal{P}_1(k)_F \left[ \frac{1}{2} \right] \right).
\]

The result follows from Lemma 6.30, together with the identification \( I_k R\mathcal{P}_1(k) \left[ \frac{1}{2} \right] \cong \overline{R\mathcal{B}_0(k)} \left[ \frac{1}{2} \right] \).

\( \square \)
Corollary 6.32. Let $F$ be a field with a discrete value $v$ and residue field $k$. Suppose that $U_1 = U_1^2$ and that either the characteristic of $k$ is not 2 or $k$ is perfect. Suppose further that $\overline{G}_k < \infty$. Then
\[
\overline{RB}_2(F) \left[ \frac{1}{2} \right] \cong \overline{P}(k) \left[ \frac{1}{2} \right] \oplus \left( \overline{RB}_2(k) \left[ \frac{1}{2} \right] \right)^{\oplus 2}.
\]

Proof. This follows from Corollary 6.31 together with the isomorphisms
\[
\overline{P}_1(k) \left[ \frac{1}{2} \right] \cong \bigoplus_i \overline{P}_1(k) \left[ \frac{1}{2} \right] \oplus \overline{P}_1(k)_{x_0} \cong \overline{RB}_2(k) \left[ \frac{1}{2} \right] \oplus \overline{P}(k) \left[ \frac{1}{2} \right].
\]

Remark 6.33. By the proof of Corollary 6.27 if we fix a choice of uniformizer $\pi$, the isomorphism is an isomorphism of $R F \left[ \frac{1}{2} \right]$-modules, where $\pi$ acts as $-1$ on $\overline{P}(k) \left[ \frac{1}{2} \right]$ and on one of the factors of $\overline{RB}_2(k) \left[ \frac{1}{2} \right]$, while it acts as 1 on the other factor.

An argument by induction on $n$ then gives the following version for $n$-dimensional local fields:

Corollary 6.34. Let $F_0, F_1, \ldots, F_n$ be fields with the following properties: For each $i \in \{1, \ldots, n\}$ there is a discrete valuation $v_i$ on $F_i$ whose residue field is $F_{i-1}$. Suppose furthermore that

1. $U_1(F_i) = U_1(F_i)$ for $i = 1, \ldots, n$
2. Either $\text{char}(F_0) \neq 2$ or $F_0$ is perfect
3. $\overline{G}_{F_0} < \infty$.

Then
\[
\overline{RB}_0(F) \left[ \frac{1}{2} \right] \cong \left( \bigoplus_{i=0}^{n-1} \frac{\overline{P}(F_i) \left[ \frac{1}{2} \right]}{2^{n-i-1}} \right) \bigoplus \overline{RB}_0(F_0) \left[ \frac{1}{2} \right]^{\oplus 2^n}.
\]

Corollary 6.35. Under the same hypotheses as Corollary 6.34, suppose also that $F_0$ is quadratically closed or real or finite. Then
\[
\overline{RB}_0(F) \left[ \frac{1}{2} \right] \cong \bigoplus_{i=0}^{n-1} \frac{\overline{P}(F_i) \left[ \frac{1}{2} \right]}{2^{n-i-1}}.
\]

Proof. By Lemma 2.5, $\overline{RB}_0(F_0) = 0$ in these cases. 

Remark 6.36. From the observation of Remark 6.33 and induction, the direct sum decomposition on the right can be understood as an eigenspace decomposition of $\overline{RB}(F) \left[ \frac{1}{2} \right]$ (or of $\overline{P}_1(F) \left[ \frac{1}{2} \right]$) as follows:

There are natural injective maps $\overline{G}_{F_{i-1}} \rightarrow \overline{G}_{F_i}$ ($1 \leq i \leq n$) which induce surjective homomorphisms $\overline{G}_{F_i} \rightarrow \overline{G}_{F_{i-1}}$. For each $i \leq n$, let
\[
W_i := \text{Ker}(\overline{G}_F \rightarrow \overline{G}_{F_i}) = \{ \chi \in \overline{G}_F \mid \chi|_{\overline{G}_{F_i}} = 1 \}.
\]

For each $i < n$ and for each $\chi \in W_i \setminus W_{i+1}$, the $\chi$-eigenspace of $\overline{RB}(F) \left[ \frac{1}{2} \right]$ is isomorphic to the module $\overline{P}(F_i) \left[ \frac{1}{2} \right]$ (on which the square class of $a$ acts as multiplication by $\chi(a)$).
6.5. **Lifting back to** $H_3(\text{SL}_2(F), \mathbb{Z}\left[\frac{1}{2}\right])$. In this section we discuss some circumstances under which the preceding results can be lifted from $\overline{RB}(F)$ to $RB(F)$.

We begin with the following observation:

**Lemma 6.37.** Let $F$ be a field with discrete value $v$ and residue field $k$. Suppose that either $\text{char}(k) \neq 2, 3$ or $\text{char}(k) = 2$ and $v$ is complete. Then $D_F = 0$ if and only if $D_k = 0$ and hence there is an isomorphism of groups $\mathbb{Z} \cdot D_F \cong \mathbb{Z} \cdot D_k$.

**Proof.** The hypotheses ensure that $\Phi(x)$ has a root in $k$ if and only if it has a root in $F$. \hfill $\Box$

**Remark 6.38.** When $F = \mathbb{Q}_3$ and $k = \mathbb{F}_3$ we have $D_F \neq 0$ but $D_k = 0$.

**Lemma 6.39.** The specialization map $S_v$ induces a surjective homomorphism of $R_F$-modules $D_F \longrightarrow (D_k)_F$.

**Proof.** The map $S_v$ from $R_F \cdot D_F = D_F$ to $R_F \otimes_{\mathbb{Z}[U]} R_k \cdot D_k = (D_k)_F$ sends $\alpha \cdot D_F$ to $\alpha \otimes D_k$. \hfill $\Box$

We let $D_F(v) := \text{Ker}(D_F \rightarrow (D_k)_F)$.

**Lemma 6.40.** Let $F$ be a field with discrete value $v$ and residue field $k$. Suppose that either $\text{char}(k) \neq 2, 3$ or $\text{char}(k) = 2$ and $v$ is complete. Then $D_F(v) \cong \text{Ker}(I_F D_F \rightarrow I_F(D_k)_F)$

**Proof.** For any $R_F$-module $M$ there is a natural short exact sequence $0 \rightarrow I_FM \rightarrow M \rightarrow M \otimes_{R_F} \mathbb{Z} = M_{\mathbb{Z}_F} \rightarrow 0$.

In the case of the module $D_F = R_F \cdot D_F$, this sequence takes the form $0 \rightarrow I_F D_F \rightarrow D_F \rightarrow \mathbb{Z} \cdot D_F \rightarrow 0$.

On the other hand, if $M$ is a $R_k$-module, then $(M_F)_{\mathbb{Z}_F} = (M \otimes_{\mathbb{Z}[U]} R_F) \otimes_{R_F} \mathbb{Z} \cong M \otimes_{\mathbb{Z}[U]} \mathbb{Z} = M \otimes_{R_k} \mathbb{Z} = M_{\mathbb{Z}_F}$.

Thus we have a map of short exact sequences

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & I_F D_F & \longrightarrow & D_F & \longrightarrow & \mathbb{Z} \cdot D_F & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \cong & & \\
0 & \longrightarrow & I_F(D_k)_F & \longrightarrow & (D_k)_F & \longrightarrow & \mathbb{Z} \cdot D_k & \longrightarrow & 0 \\
\end{array}
$$

\hfill $\Box$

**Remark 6.41.** Of course, when $\text{char}(F) = 0$ and $\text{char}(k) = 3$ it may happen that $D_F \neq 0$ but $D_k = 0$. In this case, $D_F(v) = D_F$.

**Corollary 6.42.** Let $F$ be a field with discrete value $v$ and residue field $k$. Suppose that either $\text{char}(k) \neq 2, 3$ or $\text{char}(k) = 2$, $k$ is perfect and $v$ is complete. Suppose further that $|G_F| < \infty$. Then there is an exact sequence of $R_F$-modules

$$
0 \rightarrow D_F(v) \rightarrow R_B(F) \left[\frac{1}{2}\right] \rightarrow I_F \left(\overline{RP}_1(k)_F \left[\frac{1}{2}\right]\right) \rightarrow 0
$$

and

$$
I_F \left(\overline{RP}_1(k)_F \left[\frac{1}{2}\right]\right) \cong R_B^0(k) \left[\frac{1}{2}\right] \oplus \overline{RP}_1(k) \left[\frac{1}{2}\right]
$$
Proof. In the commutative diagram of $R_F$-modules with

\[
\begin{array}{ccccccc}
0 & 
\longrightarrow & I_F((D_k)_F) & 
\longrightarrow & I_F(\mathcal{R} \mathcal{P}_1(k)_F \left\{ \frac{1}{2} \right\}) & 
\longrightarrow & I_F(\mathcal{R} \mathcal{P}_1(k)_F) & 
\longrightarrow & 0 \\
0 & 
\longrightarrow & (D_k)_F & 
\longrightarrow & \mathcal{R} \mathcal{P}_1(k)_F \left\{ \frac{1}{2} \right\} & 
\longrightarrow & \mathcal{R} \mathcal{P}_1(k)_F \left\{ \frac{1}{2} \right\} & 
\longrightarrow & 0 \\
0 & 
\longrightarrow & \mathbb{Z} \cdot D_k & 
\longrightarrow & \mathbb{P}(k) \left\{ \frac{1}{2} \right\} & 
\longrightarrow & \mathbb{P}(k) \left\{ \frac{1}{2} \right\} & 
\longrightarrow & 0 \\
0 & 
\longrightarrow & 0 & 
\longrightarrow & 0 & 
\longrightarrow & 0 & 
\longrightarrow & 0
\end{array}
\]

the columns are exact and the second and third rows are exact. It follows that the top row is exact also, and hence there is a natural map of short exact sequences

\[
\begin{array}{ccccccc}
0 & 
\longrightarrow & I_F(D_F) & 
\longrightarrow & \mathcal{R} \mathcal{B}_0(F) \left\{ \frac{1}{2} \right\} & 
\longrightarrow & \mathcal{R} \mathcal{B}_0(F) \left\{ \frac{1}{2} \right\} & 
\longrightarrow & 0 \\
0 & 
\longrightarrow & I_F((D_k)_F) & 
\longrightarrow & I_F(\mathcal{R} \mathcal{P}_1(k)_F \left\{ \frac{1}{2} \right\}) & 
\longrightarrow & I_F(\mathcal{R} \mathcal{P}_1(k)_F) & 
\longrightarrow & 0
\end{array}
\]

The last statement follows by the argument given above in the case of $I_F(\mathcal{R} \mathcal{P}_1(k)_F \left\{ \frac{1}{2} \right\})$.

Corollary 6.43. Let $F$ be a field with discrete value $v$ and residue field $k$. Suppose that either $\text{char}(k) \neq 2, 3$ or $\text{char}(k) = 2, k$ is perfect and $v$ is complete. Suppose further that $|G_F| < \infty$. Let $\pi$ be a choice of uniformizer. Then there is an exact sequence of $R_F$-modules

\[
0 \to D_F(v) \to H_3(\text{SL}_2(F), \mathbb{Z} \left\{ \frac{1}{2} \right\}) \to K_3^\text{ind}(F) \left\{ \frac{1}{2} \right\} \oplus \mathbb{P}(k) \left\{ \frac{1}{2} \right\} \oplus (\mathcal{R} \mathcal{B}_0(k) \left\{ \frac{1}{2} \right\})^\otimes 2 \to 0.
\]

(where $\pi$ acts as $-1$ on $\mathbb{P}(k) \left\{ \frac{1}{2} \right\}$ and on one of the factors of $\mathcal{R} \mathcal{B}_0(k) \left\{ \frac{1}{2} \right\}$ and as $1$ on the other factor).

Proof. There is a decomposition of $R_F \left\{ \frac{1}{2} \right\}$-modules

\[
H_3(\text{SL}_2(F), \mathbb{Z} \left\{ \frac{1}{2} \right\}) \cong H_3(\text{SL}_2(F), \mathbb{Z} \left\{ \frac{1}{2} \right\}) \otimes I_F H_3(\text{SL}_2(F), \mathbb{Z} \left\{ \frac{1}{2} \right\}) \cong K_3^\text{ind}(F) \left\{ \frac{1}{2} \right\} \oplus \mathcal{R} \mathcal{B}_0(F) \left\{ \frac{1}{2} \right\}.
\]

The result then follows from Corollary 6.42 together with the isomorphisms

\[
I_F(\mathcal{R} \mathcal{P}_1(K)_F \left\{ \frac{1}{2} \right\}) \cong \mathcal{R} \mathcal{B}_0(k) \left\{ \frac{1}{2} \right\} \oplus \mathcal{R} \mathcal{P}_1(k) \left\{ \frac{1}{2} \right\} \cong \mathcal{R} \mathcal{B}_0(k) \left\{ \frac{1}{2} \right\}^\otimes 2 \oplus \mathbb{P}(k) \left\{ \frac{1}{2} \right\}.
\]

Lemma 6.44. Let $F$ be a field with discrete valuation $v$ and residue field $k$. Suppose that $U_1 = U_1^2$, that $\text{char}(k) \neq 3$ and that $\zeta_3 \notin k$. Let $\pi \in F$ be a uniformizer and let $C_2$ denote the cyclic subgroup of $G_F$ generated by the square-class of $\pi$. Suppose that $|\overline{G}_k| < \infty$. Then $\overline{G}_F \cong \overline{G}_k \times C_2$ and $\overline{R}_F$ is isomorphic to the group algebra $\overline{R}_k[C_2]$.‌
Proof. Let $\tilde{N}_k = \{u \in U | \tilde{u} \in N_k\}$. It is enough to show that
\[
N_F = \mathbb{Z}^2 \cdot \tilde{N}_k.
\]
Let $u \in \tilde{N}_k$. Thus $\pm \tilde{u} = s^2 - \tilde{t} + \tilde{c}^2$ for some $s, t \in U$. Thus $\pm u = (s^2 - st + t^2)w^2$ for some $w \in U_1 = U_1^2$, and hence $u \in N_F$.

It remains to show that $\pi \not\in N_F$: Suppose that $a, b \in F^\times$. Then $v(a^2 - ab + b^2) \equiv v(1 - c + c^2) \pmod{2}$ where $c = b/a$. If $v(c) \neq 0$ then $v(1 - c + c^2) \equiv 0 \pmod{2}$. On the other hand if $v(c) = 0$ then $v(1 - c + c^2) = 0$ since $1 - \bar{c} + \bar{c}^2 \neq 0$ in $k$ by hypothesis.

It follows that if $x \in N_F$ then $v(x)$ is even, and thus $\pi \not\in N_F$ in this case. □

Corollary 6.45. Let $F$ be a field with discrete valuation $v$ and residue field $k$. Suppose that $U_1 = U_1^2$ that $\text{char}(k) \neq 3$ and that $\zeta_3 \not\in k$. Suppose further that $D_k$ is a free rank one $\hat{R}_k$-module.

Then $D_F$ is a free rank one $\hat{R}_F$-module and $D_F(v) = 0$.

Proof. By Lemma 6.44, the natural ring homomorphism $\hat{R}_k \rightarrow \hat{R}_F$ induces an isomorphism of rings $(\hat{R}_k)_F \cong \hat{R}_F$. It follows that the composition of surjective maps
\[
\hat{R}_F \twoheadrightarrow D_F \xrightarrow{S_k} (D_k)_F \xrightarrow{=} (\hat{R}_k)_F
\]
is an isomorphism, and thus each of the maps appearing in this sequence is an isomorphism. □

Corollary 6.46. Let $F$ be a field with discrete valuation $v$ and residue field $k$ satisfying:

1. Either
   a. $\text{char}(k) \neq 2, 3$ and $U_1 = U_1^2$
   b. $\text{char}(k) = \text{char}(F) = 3$
   c. $\text{char}(k) = 2$ and $v$ is complete
2. $|\mathcal{G}_k| < \infty$ and either $D_k = 0$ or $D_k$ is free on rank one as an $\hat{R}_k$-module.

Then there is an isomorphism of $R_F \left[\frac{1}{2}\right]$-modules
\[
H_3(SL_2(F), \mathbb{Z} \left[\frac{1}{2}\right]) \cong K_3^{\text{ind}}(F) \left[\frac{1}{2}\right] \oplus \mathcal{P}(k) \left[\frac{1}{2}\right] \oplus (\mathcal{R}B_0(k) \left[\frac{1}{2}\right])^{\oplus 2}
\]
As noted above, if $k$ is either quadratically closed or real-closed or finite, then $\mathcal{R}B_0(k) = 0$.
Furthermore, $\mathcal{G}_k$ is either trivial or generated by the class of $-1$. It follows that $\bar{\mathcal{G}}_k = \{1\}$ and hence $\hat{R}_k = \mathbb{F}_3$. Hence, in this case, either $D_k = 0$ or $D_k = \mathbb{F}_3$. $D_k$ is free of rank one as a $\hat{R}_k$-module. Using this observation together with induction, we deduce:

Theorem 6.47. Let $F_0, F_1, \ldots, F_n = F$ be fields with the following properties: For each $i \in \{1, \ldots, n\}$ there is a discrete valuation $v_i$ on $F_i$ whose residue field is $F_{i-1}$. Suppose that $F_0$ is quadratically closed or real-closed or finite. Suppose furthermore that

1. $U_1(F_i)^2 = U_1(F_i)$ for $i = 1, \ldots, n$
2. If $\text{char}(F_i) = 2$ for some $i < n$ then $F_i$ is perfect and $v_{i+1}$ is complete
3. Either $\text{char}(F) = 3$ or $\text{char}(F_0) \neq 3$ or $\zeta_3 \not\in F$

Then
\[
H_3(SL_2(F), \mathbb{Z} \left[\frac{1}{2}\right]) \cong K_3^{\text{ind}}(F) \left[\frac{1}{2}\right] \oplus \left( \bigoplus_{i=0}^{n-1} \mathcal{P}(F_i) \left[\frac{1}{2}\right] \right)^{\oplus 2^{n-1}}.
\]
7. The localization sequence for discrete valuation rings

7.1. Discrete valuation rings with infinite residue field.

**Lemma 7.1.** Let $A$ be a local integral domain with infinite residue field $F$. Suppose that $U_{1,A} = U_{1,F}$ and that either

1. $|\mathcal{G}_F| \leq 2$ or
2. $|\mathcal{G}_F| < \infty$ and $F$ has a discrete valuation $v$ for which $U_{1,v} = U_{1,v}^2$.

Suppose further that $D_A$ is free of rank one as an $\hat{R}_F$-module.

Then the quotient map $A \to F$ induces an isomorphism $\mathcal{R}_B_0(A)[\frac{1}{2}] \cong \mathcal{R}_B_0(F)[\frac{1}{2}]$.

**Proof.** We have

$$\mathcal{R}_B_0(A)[\frac{1}{2}] \cong \mathcal{R}_B_0(F)[\frac{1}{2}]$$

by Corollaries 6.10 and 6.27. But $D_A \cong D_F$ by Lemma 4.22, and hence this lifts to an isomorphism $\mathcal{R}_B_0(A)[\frac{1}{2}] \cong \mathcal{R}_B_0(F)[\frac{1}{2}]$.

Let $\delta_\pi$ denote the map $H_3(SL_2(K), \mathbb{Z}[\frac{1}{2}]) \to \mathcal{RP}_1(F)[\frac{1}{2}]$ obtained by tensoring the following composite map with $\mathbb{Z}[\frac{1}{2}]$:

$$H_3(SL_2(K), \mathbb{Z}) \to \mathcal{R}_B(K) \xrightarrow{S_w} \mathcal{RP}_1(F)_K \xrightarrow{\rho_v \otimes \text{id}} \mathcal{RP}_1(F).$$

**Theorem 7.2.** Let $K$ be a field with discrete valuation $w$ and infinite residue field $F$ satisfying $\text{char}(F) = \text{char}(K)$. Let $O_w$ be the associated valuation ring. Suppose that $U_{1,w} = U_{1,w}^2$ and that either

1. $|\mathcal{G}_F| \leq 2$ or
2. $|\mathcal{G}_F| < \infty$ and $F$ has a discrete valuation $v$ for which $U_{1,v} = U_{1,v}^2$.

Suppose further that $D_F$ is free of rank one as an $\hat{R}_F$-module.

Let $\pi \in O_w$ be a uniformizer. Then there is a natural split short exact sequence of $\mathcal{R}_F[\frac{1}{2}]$-modules

$$0 \to H_3(SL_2(O_w), \mathbb{Z}[\frac{1}{2}]) \to H_3(SL_2(K), \mathbb{Z}[\frac{1}{2}]) \xrightarrow{\delta_\pi} \mathcal{RP}_1(F)[\frac{1}{2}] \to 0.$$

**Proof.** We have

$$\mathcal{R}_B_0(O_w)[\frac{1}{2}] \cong \mathcal{R}_B_0(F)[\frac{1}{2}] \cong I_F \mathcal{RP}_1(F)[\frac{1}{2}]$$

by Lemma 7.1 and

$$\mathcal{R}_B_0(K)[\frac{1}{2}] \cong I_K \left(\mathcal{RP}_1(F)_K[\frac{1}{2}]\right)$$

by Corollary 6.42.

It follows that there is a natural short exact sequence

$$0 \to \mathcal{R}_B_0(O_w)[\frac{1}{2}] \to \mathcal{R}_B_0(K)[\frac{1}{2}] \xrightarrow{\delta_\pi} \mathcal{RP}_1(F)[\frac{1}{2}] \to 0.$$
by Lemma 6.30, which fits into the commutative diagram with exact columns

$$\begin{array}{ccccccccc}
0 & \rightarrow & \mathcal{R}B_0(O_w) & \left\{\frac{1}{2}\right\} & \rightarrow & \mathcal{R}B_0(K) & \left\{\frac{1}{2}\right\} & \xrightarrow{\delta_x} & \mathcal{R}P_1(F) & \left\{\frac{1}{2}\right\} & \rightarrow & 0 \\
| & & | & & | & & | & & | & & | \\
0 & \rightarrow & \mathcal{H}_3(\text{SL}_2(O_w), \mathbb{Z}) & \left\{\frac{3}{2}\right\} & \rightarrow & \mathcal{H}_3(\text{SL}_2(K), \mathbb{Z}) & \left\{\frac{3}{2}\right\} & \xrightarrow{\delta_x} & \mathcal{P}(F) & \left\{\frac{1}{2}\right\} & \rightarrow & 0 \\
| & & | & & | & & | & & | & & | \\
K_3^{\text{ind}}(O_w) & \left\{\frac{1}{2}\right\} & \xrightarrow{=} & K_3^{\text{ind}}(K) & \left\{\frac{1}{2}\right\} \\
| & & | & & | & & | & & | & & | \\
0 & & 0 & & 0 & & 0 & & 0 & & 0
\end{array}$$

\[\square\]

7.2. **Discrete valuation rings with finite residue field.** For the case of finite residue fields, we are only in the position to assert the following.

**Lemma 7.3.** Let \( K \) be a field with discrete valuation \( w \). Let \( O_w \) be the corresponding discrete valuation ring and \( F \) the residue field. Let \( \pi \) be a uniformizing parameter. Suppose that \( F \) is finite, \( |F| \notin \Sigma_0 \) and that \( U_{1,w} = U_{2,w}^2 \). Then there is a natural exact sequence

$$0 \rightarrow \mathcal{B}(O_w; K) \left\{\frac{1}{2}\right\} \rightarrow \mathcal{H}_3(\text{SL}_2(O_w), \mathbb{Z}) \left\{\frac{1}{2}\right\} \rightarrow \mathcal{H}_3(\text{SL}_2(K), \mathbb{Z}) \left\{\frac{1}{2}\right\} \xrightarrow{\delta_x} \mathcal{P}(F) \left\{\frac{1}{2}\right\} \rightarrow 0.$$

**Proof.** Note that \( \mathcal{R}P_1(F) \left\{\frac{1}{2}\right\} \cong \mathcal{P}(F) \left\{\frac{1}{2}\right\} \) since \( \mathcal{R}B_0(F) = 0 \). Thus there is an isomorphism of \( R_F \left\{\frac{1}{2}\right\} \)-modules

$$\mathcal{R}B_0(K) \left\{\frac{1}{2}\right\} \xrightarrow{\cong} I_K \left( \mathcal{R}P_1(F)_K \left\{\frac{1}{2}\right\} \right) \xrightarrow{\delta_x} \mathcal{P}(F) \left\{\frac{1}{2}\right\}.$$

From this, using Corollary 6.16, we obtain a commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \rightarrow & \mathcal{B}(O_w; K) & \left\{\frac{1}{2}\right\} & \rightarrow & \mathcal{H}_3(\text{SL}_2(O_w), \mathbb{Z}) & \left\{\frac{1}{2}\right\} & \rightarrow & K_3^{\text{ind}}(A) & \left\{\frac{1}{2}\right\} & \rightarrow & 0 \\
| & & | & & | & & | & & | & & | \\
0 & \rightarrow & \mathcal{R}B_0(K) & \left\{\frac{1}{2}\right\} & \rightarrow & \mathcal{H}_3(\text{SL}_2(K), \mathbb{Z}) & \left\{\frac{1}{2}\right\} & \rightarrow & K_3^{\text{ind}}(K) & \left\{\frac{1}{2}\right\} & \rightarrow & 0 \\
| & & | & & | & & | & & | & & | \\
0 & \rightarrow & \mathcal{P}(F) & \left\{\frac{1}{2}\right\} & \rightarrow & 0 & & 0 & & 0 & & 0
\end{array}$$

from which the statement follows. \[\square\]

**Remark 7.4.** Of course, by Corollary 6.16 above \( \mathcal{B}(O_w; K) \left\{\frac{1}{2}\right\} = 0 \) if and only if \( \mathcal{H}_3(\text{SL}_2(O_w), \mathbb{Z}) \left\{\frac{1}{2}\right\} \cong K_3^{\text{ind}}(O_w) \left\{\frac{1}{2}\right\} \) and we would expect this to be the case (at least when the residue field is sufficiently large).
Theorem 6.19 in \[8\] says that if \( F \) is a field complete with respect to a discrete valuation with finite residue field of odd characteristic (if \( \mathbb{Q}_3 \subset F \), we require that \( [F : \mathbb{Q}_3] \) is odd), then \( \mathcal{R}\mathcal{B}_0(F) \left[ \frac{1}{2} \right] \cong \mathcal{P}(k) \left[ \frac{1}{2} \right] \cong \mathcal{B}(k) \left[ \frac{1}{2} \right] \) and hence
\[
\mathcal{H}_3(\text{SL}_2(F), \mathbb{Z} \left[ \frac{1}{2} \right]) \cong K_3^{\text{ind}}(F) \left[ \frac{1}{2} \right] \oplus \mathcal{P}(k) \left[ \frac{1}{2} \right].
\]
In particular, \( \mathcal{R}\mathcal{B}_0(\mathbb{Q}_p) \left[ \frac{1}{2} \right] \cong \mathcal{P}(\mathbb{F}_p) \left[ \frac{1}{2} \right] \) is \( p \geq 5 \). This is a cyclic group of order \((p + 1)^{\prime}\).

**Example 8.1.** Letting \( K = \mathbb{C} \langle x \rangle \) in Theorem 7.2 gives a (split) short exact sequence of \( \mathbb{Z} \left[ \frac{1}{2} \right] \)-modules
\[
0 \longrightarrow \mathcal{H}_3(\text{SL}_2(\mathbb{C} \langle x \rangle), \mathbb{Z} \left[ \frac{1}{2} \right]) \longrightarrow \mathcal{H}_3(\text{SL}_2(\mathbb{C} \langle x \rangle), \mathbb{Z} \left[ \frac{1}{2} \right]) \longrightarrow \mathcal{P}(\mathbb{C}) \longrightarrow 0
\]
since \( \mathcal{R}\mathcal{P}_1(\mathbb{C}) = \mathcal{P}(\mathbb{C}) \) is a \( \mathbb{Q} \)-vector space. Furthermore, we have
\[
\mathcal{H}_3(\text{SL}_2(\mathbb{C} \langle x \rangle), \mathbb{Z} \left[ \frac{1}{2} \right]) \cong K_3^{\text{ind}}(\mathbb{C} \langle x \rangle) \left[ \frac{1}{2} \right] \cong K_3^{\text{ind}}(\mathbb{C} \langle x \rangle) \left[ \frac{1}{2} \right]
\]
by Corollary 3.27 and Theorem 2.1. (More generally, \( \mathbb{C} \) can be replaced with any infinite quadratically closed field in this example.)

**Example 8.2.** Letting \( K = \mathbb{R} \langle x \rangle \) in Theorem 7.2 gives a (split) short exact sequence of \( \mathbb{Z} \left[ \frac{1}{2} \right] \)-modules
\[
0 \longrightarrow \mathcal{H}_3(\text{SL}_2(\mathbb{R} \langle x \rangle), \mathbb{Z} \left[ \frac{1}{2} \right]) \longrightarrow \mathcal{H}_3(\text{SL}_2(\mathbb{R} \langle x \rangle), \mathbb{Z} \left[ \frac{1}{2} \right]) \longrightarrow \mathcal{P}(\mathbb{R}) \left[ \frac{1}{2} \right] \longrightarrow 0
\]
since \( \mathcal{R}\mathcal{B}_0(\mathbb{R}) = 0 \) and hence \( \mathcal{R}\mathcal{P}_1(\mathbb{R}) \left[ \frac{1}{2} \right] = \mathcal{P}(\mathbb{R}) \left[ \frac{1}{2} \right] \).
Furthermore, we have
\[
\mathcal{H}_3(\text{SL}_2(\mathbb{R} \langle x \rangle), \mathbb{Z} \left[ \frac{1}{2} \right]) \cong K_3^{\text{ind}}(\mathbb{R} \langle x \rangle) \left[ \frac{1}{2} \right] \cong K_3^{\text{ind}}(\mathbb{R} \langle x \rangle) \left[ \frac{1}{2} \right]
\]
by Corollaries 3.26 and 6.10 and Theorem 2.1. (More generally, \( \mathbb{R} \) can be replaced with any real-closed field in this example.)

**Example 8.3.** Let \( p \) be a prime. For simplicity, we suppose that \( p \geq 5 \). Letting \( K = \mathbb{Q}_p \langle x \rangle \) in Theorem 7.2 gives a (split) short exact sequence of \( \mathbb{Z} \left[ \frac{1}{2} \right] \)-modules
\[
0 \longrightarrow \mathcal{H}_3(\text{SL}_2(\mathbb{Q}_p \langle x \rangle), \mathbb{Z} \left[ \frac{1}{2} \right]) \longrightarrow \mathcal{H}_3(\text{SL}_2(\mathbb{Q}_p \langle x \rangle), \mathbb{Z} \left[ \frac{1}{2} \right]) \longrightarrow \mathcal{P}(\mathbb{Q}_p) \left[ \frac{1}{2} \right] \longrightarrow 0.
\]
Furthermore, we have
\[
\mathcal{H}_3(\text{SL}_2(\mathbb{Q}_p \langle x \rangle), \mathbb{Z} \left[ \frac{1}{2} \right]) \cong K_3^{\text{ind}}(\mathbb{Q}_p \langle x \rangle) \left[ \frac{1}{2} \right] \oplus \mathcal{R}\mathcal{B}_0(\mathbb{Q}_p \langle x \rangle) \left[ \frac{1}{2} \right] \text{ by Corollary 3.26}
\]
\[
\cong K_3^{\text{ind}}(\mathbb{Q}_p \langle x \rangle) \left[ \frac{1}{2} \right] \oplus \mathcal{R}\mathcal{B}_0(\mathbb{Q}_p) \left[ \frac{1}{2} \right] \text{ by Lemma 7.1}
\]
\[
\cong K_3^{\text{ind}}(\mathbb{Q}_p \langle x \rangle) \left[ \frac{1}{2} \right] \oplus \mathcal{P}(\mathbb{F}_p) \left[ \frac{1}{2} \right] \text{ by Corollary 6.42}
\]
and
\[
\mathcal{R}\mathcal{P}_1(\mathbb{Q}_p) \left[ \frac{1}{2} \right] \cong \mathcal{R}\mathcal{B}_0(\mathbb{Q}_p) \left[ \frac{1}{2} \right] \oplus \mathcal{P}(\mathbb{Q}_p) \left[ \frac{1}{2} \right] \cong \mathcal{P}(\mathbb{F}_p) \left[ \frac{1}{2} \right] \oplus \mathcal{P}(\mathbb{Q}_p) \left[ \frac{1}{2} \right].
\]
Alternatively, for the calculation of $H_3(SL_2(\mathbb{Q}_p ((x)), \mathbb{Z}[\frac{1}{2}])$, we can use Theorem 6.47 to conclude that there is an $R_K[\frac{1}{2}]$-module decomposition

$$H_3(SL_2(\mathbb{Q}_p ((x)), \mathbb{Z}[\frac{1}{2}]) \cong K_3^{\text{ind}}(\mathbb{Q}_p ((x)))[\frac{1}{2}] \oplus \mathcal{P}(\mathbb{Q}_p)[\frac{1}{2}] \oplus \mathcal{P}(\mathbb{F}_p)[\frac{1}{2}]^\oplus 2.$$

More generally, $\mathbb{Q}_p$ can be replaced with any local field of residue characteristic at least 5 in this example.

If we replace $\mathbb{Q}_p$ with a local field of residue characteristic 3, our calculations may be missing a small amount of 3-torsion, but certainly remain valid if we replace $\mathbb{Z}[\frac{1}{2}]$ with $\mathbb{Z}[\frac{1}{6}]$.

If we replace $\mathbb{Q}_p$ with a local field of residue characteristic 2 our arguments generally do not apply, since the condition $U_1 = U_2^2$ no longer holds. (Nevertheless, we would conjecture that the results remain valid in this case.)

**Example 8.4.** Replacing $\mathbb{Q}_p$ with $F = \mathbb{C}((x))$ in Example 8.3, we obtain again a (split) short exact sequence of $R_F[\frac{1}{2}]$-modules

$$0 \longrightarrow H_3(SL_2(\mathbb{C}((x)) [\mathfrak{m}]), \mathbb{Z}[\frac{1}{2}]) \longrightarrow H_3(SL_2(\mathbb{C}((x)) [\mathfrak{m}]), \mathbb{Z}[\frac{1}{2}]) \xrightarrow{\delta_3} R\mathcal{P}_1(\mathbb{C}((x)))[\frac{1}{2}] \longrightarrow 0.$$

Furthermore, we have

$$H_3(SL_2(\mathbb{C}((y)) [\mathfrak{m}]), \mathbb{Z}[\frac{1}{2}]) \cong K_3^{\text{ind}}(\mathbb{C}((y)) [\mathfrak{m}])[\frac{1}{2}] \oplus \mathcal{R}\mathcal{B}_0(\mathbb{C}((y)) [\mathfrak{m}])[\frac{1}{2}] \oplus \mathcal{P}(\mathbb{C}((y)))[\frac{1}{2}] \cong \mathcal{P}(\mathbb{C})[\frac{1}{2}] \oplus \mathcal{P}(\mathbb{C}((y)))[\frac{1}{2}].$$

and

$$\mathcal{R}\mathcal{P}_1(\mathbb{C}((y)))[\frac{1}{2}] \cong \mathcal{R}\mathcal{B}_0(\mathbb{C}((y)))[\frac{1}{2}] \oplus \mathcal{P}(\mathbb{C}((y)))[\frac{1}{2}] \cong \mathcal{P}(\mathbb{C})[\frac{1}{2}] \oplus \mathcal{P}(\mathbb{C}((y)))[\frac{1}{2}].$$

Alternatively, for the calculation of $H_3(SL_2(\mathbb{C}((y)) [\mathfrak{m}]), \mathbb{Z}[\frac{1}{2}])$, we can use Theorem 6.47 to conclude that there is an $R_K[\frac{1}{2}]$-module decomposition

$$H_3(SL_2(\mathbb{C}((y)) [\mathfrak{m}]), \mathbb{Z}[\frac{1}{2}]) \cong K_3^{\text{ind}}(\mathbb{C}((y)) [\mathfrak{m}])[\frac{1}{2}] \oplus \mathcal{P}(\mathbb{C}((y)))[\frac{1}{2}] \oplus \mathcal{P}(\mathbb{C}((y)))[\frac{1}{2}]^\oplus 2.$$

Of course, we can replace $\mathbb{C}$ by any quadratically closed field in this example. We can even replace $\mathbb{C}$ by any real-closed field in this example. We can even replace $\mathbb{C}$ by any real-closed field, and the results remain valid, since for such fields $k$ we have $\mathcal{R}\mathcal{B}_0(k) = 0$.

**Example 8.5.** Let $K = \mathbb{Q}_p ((x_1)) \cdots ((x_n))$ with $p \geq 5$ for simplicity. Then, by Theorem 6.47, we have $R_K[\frac{1}{2}]$-module isomorphisms

$$H_3(SL_2(K), \mathbb{Z}[\frac{1}{2}]) \cong K_3^{\text{ind}}(K)[\frac{1}{2}] \oplus \bigoplus_{i=1}^{n-1} \left( \mathcal{P}(\mathbb{Q}_p ((x_1)) \cdots ((x_i)), \mathbb{Z}[\frac{1}{2}]) \oplus \mathcal{P}(\mathbb{F}_p)[\frac{1}{2}]^\oplus 2 \right)$$

and

$$H_3(SL_2(K [\mathfrak{m}]), \mathbb{Z}[\frac{1}{2}]) \cong K_3^{\text{ind}}(K)[\frac{1}{2}] \oplus \mathcal{R}\mathcal{B}_0(K)[\frac{1}{2}] \cong K_3^{\text{ind}}(K)[\frac{1}{2}] \oplus \bigoplus_{i=1}^{n-1} \left( \mathcal{P}(\mathbb{Q}_p ((x_1)) \cdots ((x_i)), \mathbb{Z}[\frac{1}{2}]) \oplus \mathcal{P}(\mathbb{F}_p)[\frac{1}{2}]^\oplus 2 \right).$$

Thus, we have shown the following:

**Theorem 8.7.** Let $K = \mathbb{Q}_p ((x_1)) \cdots ((x_n))$ with $p \geq 5$ for simplicity. Then, by Theorem 6.47, we have $R_K[\frac{1}{2}]$-module isomorphisms

$$H_3(SL_2(K), \mathbb{Z}[\frac{1}{2}]) \cong K_3^{\text{ind}}(K)[\frac{1}{2}] \oplus \bigoplus_{i=1}^{n-1} \left( \mathcal{P}(\mathbb{Q}_p ((x_1)) \cdots ((x_i)), \mathbb{Z}[\frac{1}{2}]) \oplus \mathcal{P}(\mathbb{F}_p)[\frac{1}{2}]^\oplus 2 \right)$$

and

$$H_3(SL_2(K [\mathfrak{m}]), \mathbb{Z}[\frac{1}{2}]) \cong K_3^{\text{ind}}(K)[\frac{1}{2}] \oplus \mathcal{R}\mathcal{B}_0(K)[\frac{1}{2}] \cong K_3^{\text{ind}}(K)[\frac{1}{2}] \oplus \bigoplus_{i=1}^{n-1} \left( \mathcal{P}(\mathbb{Q}_p ((x_1)) \cdots ((x_i)), \mathbb{Z}[\frac{1}{2}]) \oplus \mathcal{P}(\mathbb{F}_p)[\frac{1}{2}]^\oplus 2 \right).$$
Example 8.6. Let \( p > 7 \) be a prime number. Then
\[
H_3(\text{SL}_2(\mathbb{Q}_p), \mathbb{Z}[\frac{1}{2}]) \cong K_3^{\text{ind}}(\mathbb{Q}_p)[\frac{1}{2}] \oplus \mathcal{P}(\mathbb{F}_p)[\frac{1}{2}]
\]
and, by Lemma[7.3] there is an exact sequence
\[
0 \to \mathcal{B}(\mathbb{Z}_p; \mathbb{Q}_p)[\frac{1}{2}] \to H_3(\text{SL}_2(\mathbb{Z}_p), \mathbb{Z}[\frac{1}{2}]) \to H_3(\text{SL}_2(\mathbb{Q}_p), \mathbb{Z}[\frac{1}{2}]) \to \mathcal{P}(\mathbb{F}_p)[\frac{1}{2}] \to 0
\]
where \( \mathcal{B}(\mathbb{Z}_p; \mathbb{Q}_p) = \text{Ker}(\mathcal{B}(\mathbb{Z}_p) \to \mathcal{B}(\mathbb{Q}_p)) \) is conjecturally 0 and
\[
\text{Im}(H_3(\text{SL}_2(\mathbb{Z}_p), \mathbb{Z}[\frac{1}{2}]) \to H_3(\text{SL}_2(\mathbb{Q}_p), \mathbb{Z}[\frac{1}{2}])) \cong K_3^{\text{ind}}(\mathbb{Q}_p)[\frac{1}{2}] .
\]

Acknowledgement I wish to thank Behrooz Mirzaii for answering some questions concerning his results on the third homology of \( SL_2 \).

References
[1] Spencer J. Bloch. Higher regulators, algebraic K-theory, and zeta functions of elliptic curves, volume 11 of \textit{CRM Monograph Series}. American Mathematical Society, Providence, RI, 2000.
[2] Jean-Louis Cathelineau. Homologie du groupe linéaire et polylogarithmes (d’après A. B. Goncharov et d’autres). \textit{Astérisque}, (216):Exp. No. 772, 5, 311–341, 1993. Séminaire Bourbaki, Vol. 1992/93.
[3] Johan L. Dupont and Chih Han Sah. Scissors congruences. II. \textit{J. Pure Appl. Algebra}, 25(2):159–195, 1982.
[4] Philippe Elbaz-Vincent. The indecomposable \( K \)-groups of rings and homology of \( SL_2 \). \textit{J. Pure Appl. Algebra}, 132(1):27–71, 1998.
[5] Alexander Goncharov. Volumes of hyperbolic manifolds and mixed Tate motives. \textit{J. Amer. Math. Soc.}, 12(2):569–618, 1999.
[6] Kevin Hutchinson. A Bloch-Wigner complex for \( SL_2 \). \textit{J. K-Theory}, To appear.
[7] Kevin Hutchinson. Low-dimensional homology of \( SL_2 \) of Laurent polynomials. \textit{In preparation}.
[8] Kevin Hutchinson. A refined Bloch group and the third homology of \( SL_2 \) of a field. \textit{J. Pure Appl. Algebra}, 217:2003–2035, 2013.
[9] Kevin Hutchinson and Liqun Tao. The third homology of the special linear group of a field. \textit{J. Pure Appl. Algebra}, 213:1665–1680, 2009.
[10] Kevin Hutchinson and Liqun Tao. Homology stability for the special linear group of a field and Milnor-Witt \( K \)-theory. \textit{Doc. Math.}, (Extra Vol.):267–315, 2010.
[11] Kevin P. Knudson. \textit{Homology of linear groups}, volume 193 of \textit{Progress in Mathematics}. Birkhäuser Verlag, Basel, 2001.
[12] Marc Levine. The indecomposable \( K \) of fields. \textit{Ann. Sci. École Norm. Sup. (4)}, 22(2):255–344, 1989.
[13] Hideya Matsumoto. Sur les sous-groupes arithmétiques des groupes semi-simples déployés. \textit{Ann. Sci. École Norm. Sup. (4)}, 2:1–62, 1969.
[14] Behrooz Mirzaii. Third homology of general linear groups. \textit{J. Algebra}, 320(5):1851–1877, 2008.
[15] Behrooz Mirzaii. Bloch-Wigner theorem over rings with many units. \textit{Math. Z.}, 268(1-2):329–346, 2011.
[16] Behrooz Mirzaii. Third homology of general linear groups over rings with many units. \textit{J. Algebra}, 350:374–385, 2012.
[17] Calvin C. Moore. Group extensions of \( p \)-adic and adelic linear groups. \textit{Inst. Hautes Études Sci. Publ. Math.}, (35):157–222, 1968.
[18] Walter D. Neumann and Jun Yang. Bloch invariants of hyperbolic 3-manifolds. \textit{Duke Math. J.}, 96(1):29–59, 1999.
[19] Walter Parry and Chih-Han Sah. Third homology of \( SL(2, \mathbb{R}) \) made discrete. \textit{J. Pure Appl. Algebra}, 30(2):181–209, 1983.
[20] Chih-Han Sah. Homology of classical Lie groups made discrete. III. \textit{J. Pure Appl. Algebra}, 56(3):269–312, 1989.
[21] A. A. Suslin. Homology of \( GL_n \), characteristic classes and Milnor \( K \)-theory. In \textit{Algebraic K-theory, number theory, geometry and analysis (Bielefeld, 1982)}, volume 1046 of \textit{Lecture Notes in Math.}, pages 357–375. Springer, Berlin, 1984.
[22] A. A. Suslin. Torsion in \( K \) of fields. \textit{K-Theory}, 1(1):5–29, 1987.
[23] A. A. Suslin. \( K \) of a field, and the Bloch group. \textit{Trudy Mat. Inst. Steklov.}, 183:180–199, 229, 1990. Translated in Proc. Steklov Inst. Math. \textbf{1991}, no. 4, 217–239, Galois theory, rings, algebraic groups and their applications (Russian).
[24] Wilberd van der Kallen. The $K_2$ of rings with many units. *Ann. Sci. École Norm. Sup. (4)*, 10(4):473–515, 1977.

[25] Wilberd van der Kallen. Homology stability for linear groups. *Invent. Math.*, 60(3):269–295, 1980.

[26] Charles A. Weibel. *The K-book*, volume 145 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2013. An introduction to algebraic $K$-theory.

[27] Don Zagier. The dilogarithm function. In *Frontiers in number theory, physics, and geometry. II*, pages 3–65. Springer, Berlin, 2007.

School of Mathematical Sciences, University College Dublin

E-mail address: kevin.hutchinson@ucd.ie