A Numerical Scheme for BSVIEs

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Abstract

In this paper, we consider the Euler method for backward stochastic Volterra integral equations. First, we approximate the original equation by a family of backward stochastic equations (BSDEs, for short). Then we solve the BSDEs by the Euler method. Finally, by virtue of the numerical solutions to BSDEs, we get the numerical solution to original equation and obtain the global $1/2$ order convergence speed in $L^2$ norm.

Keywords: Backward stochastic Volterra integral equation, the Euler method, backward stochastic differential equation, Malliavin analysis.

AMS subject classification: 60H20, 65C30

1 Introduction

Throughout this paper, we let $T \in (0, +\infty)$, $(\Omega, \mathcal{F}, \mathbb{F}, P)$ be a complete probability space and $\mathbb{F} = \{\mathcal{F}_t, t \in [0, T]\}$ be the natural filtration generalized by a 1-dimensional Wiener process $\{W(t) : t \in [0, T]\}$ satisfying the usual conditions. The purpose of this work is to present a numerical scheme for solving the following backward stochastic Volterra integral equation (BSVIE, for short):

\begin{equation}
Y(t) = g(t, x(T)) + \int_t^T f(t, s, x(s), Y(s), Z(t, s))ds - \int_t^T Z(t, s)dW(s), \ t \in [0, T],
\end{equation}

where $f : \Delta^c \times \mathbb{R}^d \times \mathbb{R}^n \to \mathbb{R}^n$, $g : [0, T] \times \mathbb{R}^d \to \mathbb{R}^n$ are given maps with $\Delta^c = \{(t, s) \in [0, T]^2 : t < s\}$, and $x(\cdot)$ satisfies the following stochastic Volterra integral equation (SVIE, for short):

\begin{equation}
x(t) = \varphi(t) + \int_0^t b(t, s, x(s))ds + \int_0^t \sigma(t, s, x(s))dW(s), \ t \in [0, T].
\end{equation}

Here $\varphi : [0, T] \times \Omega \to \mathbb{R}^d$, $b, \sigma : [0, T]^2 \times \mathbb{R}^d \to \mathbb{R}^d$.

BSVIEs are natural and nontrivial extensions of backward stochastic differential equations (BSDEs, for short), and the general BSVIEs can not be reduced to BSDEs (see [17]). The main feature of SVIEs/BSVIEs is that these equations contain memories, which is closer to reality. We refer

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to [3], [10] for the pioneering work on SVIEs. Nonlinear BSVIEs was first introduced in 2002 ([11]). Later, Yong ([17]) studied the well-posedness of solutions to generalized BSVIEs. Thereafter, BSVIEs turned out to be an extremely useful tool in the study of stochastic control problems for SVIEs, time-inconsistent stochastic differential utility and risk management (see, e.g., [6, 16]).

Generally, it is impossible to obtain the true solutions to BSDEs/BSIVEs. Hence, the study of numerical solutions becomes necessary and interesting. In recent period, the study of numerical solutions to stochastic differential equations (SDEs, for short) becomes an active topic. So far, the following numerical schemes for BSDEs have been presented: the four step scheme, the Euler method, the random walk approach, the Wiener chaos expansion method, the finite transposition method and so on (see, e.g., [1, 4, 5, 7, 12, 13, 14, 19]). But for BSVIE, the numerical method is quiet limited. Here we mention [2]. In [2], the numerical method for the following BSVIE is considered:

\[
Y(t) = g(t, W) + \int_t^T f(s, Y(s))ds - \int_t^T Z(t, s)dW(s), t \in [0, T],
\]

which is approximated by a family of discrete BSVIEs driven by a binary random walk with solutions \((Y^{(n)}, Z^{(n)})\). Under suitable conditions, \(Y^{(n)}\) converges weakly to \(Y\) in the Skorokhod topology. That result relies on a representation for BSVIEs by systems of quasilinear PDEs of parabolic type.

In this paper, we employ the Euler method to present the numerical solution to BSVIE (1.1). To be specific, suppose a partition \(\pi : 0 = t_0 < t_1 < \cdots < t_N = T\) of \([0, T]\) with the mesh size \(|\pi| = \max_{0 \leq i \leq N} |t_{i+1} - t_i|\). Then we denote \(\Delta_i = t_{i+1} - t_i\) \(\Delta_iW = W(t_{i+1}) - W(t_i)\), for \(i = 0, 1, \cdots, N - 1\).

For \(0 \leq k \leq N - 1\), we present the Euler method for BSVIE (1.1) as follows:

\[
\begin{align*}
Y^{k, \pi}(t_N) &= g(t_k, x^\pi(T)), \\
Y^{k, \pi}(t_l) &= \mathbb{E}\left( Y^{k, \pi}(t_{l+1}) + f(t_l, t_{l+1}, x^\pi(t_l), Y^{l, \pi}(t_{l+1}), Z^{k, \pi}(t_l))\Delta_l | \mathcal{F}_t \right), \\
Z^{k, \pi}(t_l) &= \mathbb{E}\left( \frac{\Delta_l W}{\Delta_l} (Y^{k, \pi}(t_{l+1}) + f(t_l, t_{l+1}, x^\pi(t_l), Y^{l, \pi}(t_{l+1}), Z^{k, \pi}(t_l))) | \mathcal{F}_t \right),
\end{align*}
\]

\(k \leq l \leq N - 1\).

Here \(x^\pi(\cdot)\) is the numerical solution to SVIE (1.2) stated as

\[
\begin{align*}
x^\pi(0) &= x^\pi(t_0) = \varphi(0), \\
x^\pi(t_{i+1}) &= \varphi(t_{i+1}) + \sum_{k=0}^{i} \left[ b(t_{i+1}, t_k, x^\pi(t_k))\Delta_k + \sigma(t_{i+1}, t_k, x^\pi(t_k))\Delta_k W \right],
\end{align*}
\]

\(i = 0, 1, \cdots, N - 1\).

Under suitable conditions on \(f, g, \varphi, b\) and \(\sigma\) (assumptions (A1)-(A4) below), in the cases: (I) \(f = f(t, s, x, y)\); (II) \(f = f(t, s, x, z)\), we can prove that (Theorem 4.3)

\[
\max_{0 \leq k \leq N} \mathbb{E}|Y(t_k) - Y^{k, \pi}(t_k)|^2 + \sum_{k=0}^{N-1} \mathbb{E} \int_{t_k}^{t_{k+1}} \int_{t}^{T} |Z(t, s) - Z^{k, \pi}(\tau(s))|^2 ds \leq K|\pi|,
\]
where $\tau(\cdot)$ is a map on $[0,T]$ defined by $\tau(s) = t_i, s \in [t_i, t_{i+1}), i = 0, 1, \cdots, N - 1$ and $K$ is a constant.

The rest of the paper is organized as follows: In Section 2, we review some of the standard results on SDEs and BSDEs, introduce our general setting and show the well-posedness of SVIE (1.2) and BSVIE (1.1). In Section 3, we present the Euler method to obtain the numerical solution to SVIE (1.2) and get the convergence speed. In Section 4, we adopt the Euler method for BSVIE (1.1), and the convergence and error analysis are also provided. A numerical example is presented in Section 5.

## 2 Preliminaries

Recall that $\mathbb{R}^n$ is the $n$-dimensional Euclidean space with the standard Euclidean norm $|\cdot|$ induced by the standard Euclidean inner product $\langle \cdot, \cdot \rangle$. Hereafter, the superscript $^\top$ denotes the transpose of a vector or a matrix. We now introduce some spaces: for $p, q \geq 1$,

- $L^p_{\mathcal{F}_T}(\Omega; \mathbb{R}^n)$ is the space of all $\mathcal{F}_T$-measurable random variances $\xi$ valued in $\mathbb{R}^n$ such that
  $$\|\xi\|_{L^p_{\mathcal{F}_T}(\Omega; \mathbb{R}^n)} = (\mathbb{E}[|\xi|^p])^{\frac{1}{p}} < \infty.$$  
- $L^p_F(\Omega; L^q(0,T; \mathbb{R}^n))$ is the space of all $\mathbb{F}$-progressively measurable processes $\varphi(\cdot)$ valued in $\mathbb{R}^n$ such that
  $$\|\varphi(\cdot)\|_{L^p_F(\Omega; L^q(0,T; \mathbb{R}^n))} = \left[\mathbb{E}\left(\int_0^T |\varphi(t)|^q dt\right)^{\frac{1}{q}}\right]^{\frac{1}{p}} < \infty.$$  
  When $p = q$, we write $L^p_F(\Omega \times (0,T); \mathbb{R}^n)$ for simplicity.
- $L^{1,2}_a(\mathbb{R}^n)$ is the space of all $\mathbb{F}$-progressively measurable processes $u(\cdot)$ valued in $\mathbb{R}^n$ satisfying
  
  (i) For almost all $t \in [0,T]$, $u(t) \in D^{1,2}(\mathbb{R}^n)$;
  
  (ii) $\mathbb{E}\left(\int_0^T |u(t)|^2 dt + \int_0^T \int_0^T |D_\theta u(t)|^2 d\theta dt\right) < \infty$.

The following lemma collects some standard results in SDE and BSDE literature. We only list them.

**Lemma 2.1.** Suppose that $b_o, \sigma_0 : \Omega \times [0,T] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ and $f_0 : \Omega \times [0,T] \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ are $\mathbb{F}$-adapted random fields, satisfying:

(a) they are uniformly Lipschitz continuous with respect to $x \in \mathbb{R}^d$, $y \in \mathbb{R}^n$ and $z \in \mathbb{R}^n$,

(b) $b_o(\cdot, 0), \sigma_0(\cdot, 0) \in L^2_F(\Omega \times (0,T); \mathbb{R}^d)$, $f_0(\cdot, 0, 0) \in L^2_F(\Omega \times (0,T); \mathbb{R}^n)$.

For any $x \in \mathbb{R}^d$ and $\xi \in L^2_{\mathcal{F}_T}(\Omega; \mathbb{R}^n)$, $X(\cdot)$ is the solution to the following SDE:

$$X(t) = x + \int_0^t b_o(s, X(s)) ds + \int_0^t \sigma_0(s, X(s)) dW(s), \ t \in [0,T],$$

and $(Y(\cdot), Z(\cdot))$ solves the BSDE:

$$Y(t) = \xi + \int_t^T f_0(s, Y(s), Z(s)) ds - \int_t^T Z(s) dW(s), \ t \in [0,T].$$
Then, for any $p \geq 2$, we have the following estimates:

$$E\left( \sup_{0 \leq t \leq T} |X(t)|^p \right) \leq C \left\{ |x|^p + E\left( \int_0^T |b_0(t, 0)| dt \right)^p + E\left( \frac{1}{2} |\sigma_0(t, 0)|^2 dt \right)^{p/2} \right\},$$

$$E\left( \sup_{0 \leq t \leq T} |Y(t)|^p \right) + E\left( \int_0^T |Z(s)|^2 ds \right)^{p/2} \leq C \left\{ E|\xi|^p + E\left( \int_0^T |f_0(s, 0)| ds \right)^p \right\},$$

where $C$ is a constant.

Throughout the paper, we will make use of the following assumptions.

(A1) $f : [0, T]^2 \times \mathbb{R}^d \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^d$, and there exists a constant $L$ such that

$$|f(t_1, s_1, x, y, z) - f(t_2, s_2, x, y, z)| \leq L(|t_1 - t_2|^{1/2} + |s_1 - s_2|^{1/2}),$$

and $f$ has continuous and uniformly bounded first and second partial derivatives with respect to $x$, $y$ and $z$ (boundary is $L$).

(A2) $g : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^n$, and there exists a constant $L$ such that

$$|g(t_1, x) - g(t_2, x)| \leq L|t_1 - t_2|^{1/2}, \quad t_1, t_2 \in [0, T], \quad x \in \mathbb{R}^d,$$

$$|g(\cdot, 0)| \leq L,$$

and $g$ has continuous and uniformly bounded first and second partial derivatives with respect to $x$ (boundary is $L$).

(A3) $b, \sigma : [0, T]^2 \times \mathbb{R}^d \rightarrow \mathbb{R}^d$, and there exists a constant $L$ such that

$$|b(t_1, s_1, x) - b(t_2, s_2, x)| + |\sigma(t_1, s_1, x) - \sigma(t_2, s_2, x)| \leq L(|t_1 - t_2|^{1/2} + |s_1 - s_2|^{1/2}),$$

and $b, \sigma$ has continuous and uniformly bounded first and second partial derivatives with respect to $x$ (boundary is $L$).

(A4) $\varphi(\cdot)$ is $\mathbb{F}$-adapted continuous process and there exists a constant $p_0 > 2$ and $L$ such that

$$E|\varphi(t) - \varphi(s)|^2 \leq L|t - s|, \quad t, s \in [0, T],$$

$$E|D_{\theta_1} \varphi(t) - D_{\theta_2} \varphi(t)|^2 \leq L|\theta_1 - \theta_2|, \quad 0 \leq \theta_1, \theta_2 \leq t \leq T,$$

$$\sup_{0 \leq \theta_1, \theta_2 \leq t \leq T} E\left[ |\varphi(t)|^{2p_0} + |D_{\theta_1} \varphi(t)|^{2p_0} + |D_{\theta_1} D_{\theta_2} \varphi(t)|^{p_0} \right] \leq L^{2p_0}.$$

In what follows, $K$ and $C$ are positive constants, depending only on $L$ and $T$, and may be different from line to line.

### 2.1 Regularity of $x(\cdot)$

In this part, we review the wellposedness of SVIE (1.2). Under assumptions (A3)–(A4), the wellposedness of SVIEs can be proved by a routine successive approximation argument ([10]). The following properties on $x(\cdot)$ are need later.
Lemma 2.2. Under assumptions (A3)–(A4), for any $0 \leq t_0 \leq t \leq T$, $0 \leq \theta_1, \theta_2 \leq t$, it holds that

$$
\sup_{0 \leq \theta_1, \theta_2 \leq t \leq T} \mathbb{E}\left(|x(t)|^{2p_0} + |D_{\theta_1} x(t)|^{2p_0} + |D_{\theta_2} x(t)|^{2p_0}\right) < K,
$$

(2.5)

$$
\sup_{0 \leq \theta_1, \theta_2 \leq t \leq T} \mathbb{E}|x(t) - x(t_0)|^2 \leq K(t - t_0),
$$

$$
\sup_{0 \leq \theta_1, \theta_2 \leq t \leq T} \mathbb{E}|D_{\theta_1} x(t) - D_{\theta_2} x(t)|^2 \leq K|\theta_1 - \theta_2|,
$$

where $K$ is a constant depending only on $p_0$, $L$ and $T$.

Proof. Suppose that $0 \leq t_0 \leq t \leq T$. Then by SVIE (1.2), one obtains

$$
\mathbb{E}|x(t)|^{2p_0} \leq 3^{2p_0-1} \mathbb{E}|\varphi(t)|^{2p_0} + (3T)^{2p_0-1} \mathbb{E} \int_0^t |b(t, s, x(s))|^{2p_0} ds
$$

$$
+ 3^{2p_0-1} T^{2p_0-1} \mathbb{E} \int_0^t |\sigma(t, s, x(s))|^{2p_0} ds
$$

$$
\leq 3^{2p_0-1} L^{2p_0} + (6T)^{2p_0-1} \mathbb{E} \int_0^t |b(t, s, x(s)) - b(t, s, 0)|^{2p_0} + |b(t, s, 0)|^{2p_0} ds
$$

$$
+ 6^{2p_0-1} T^{2p_0-1} \mathbb{E} \int_0^t |\sigma(t, s, x(s)) - \sigma(t, s, 0)|^{2p_0} + |\sigma(t, s, 0)|^{2p_0} ds
$$

$$
\leq K + K \mathbb{E} \int_0^t |x(s)|^{2p_0} ds.
$$

(2.6)

Consequently, by virtue of Gronwall’s inequality, we can get $\sup_{0 \leq t \leq T} \mathbb{E}|x(t)|^{2p_0} < K$. Also by the routine successive approximation argument, $D_\theta x(\cdot)$, the Malliavin derivative of $x(\cdot)$, satisfies the following SVIE: for any $0 \leq \theta \leq t \leq T$

$$
D_\theta x(t) = D_\theta \varphi(t) + \sigma(t, \theta, x(\theta)) + \int_\theta^t b_x(t, s, x(s)) D_\theta x(s) ds + \int_\theta^t \sigma_x(t, s, x(s)) D_\theta x(s) dW(s).
$$

Similarly, we also can obtain $\sup_{0 \leq t \leq T, 0 \leq \theta_1, \theta_2 \leq t} \mathbb{E}\left(|D_{\theta_1} x(t)|^{2p_0} + |D_{\theta_2} x(t)|^{2p_0}\right) < K$, which is the first inequality of (2.5).

Now, making use of the first inequality of (2.5), with the similar estimate to that of (2.6), we can obtain

$$
\mathbb{E}|x(t) - x(t_0)|^2 \leq 5 \mathbb{E}|\varphi(t) - \varphi(t_0)|^2 + 5T \mathbb{E} \int_0^{t_0} |b(t, s, x(s)) - b(t_0, s, x(s))|^2 ds
$$

$$
+ 5 \mathbb{E} \int_0^{t_0} |\sigma(t, s, x(s)) - \sigma(t_0, s, x(s))|^2 ds
$$

$$
+ 5 \mathbb{E} \left| \int_0^t b(t, s, x(s)) ds \right|^2 + 5 \mathbb{E} \int_0^t |\sigma(t, s, x(s))|^2 ds
$$

$$
\leq K|t - t_0| + K \mathbb{E} \int_{t_0}^t |x(s)|^2 ds
$$

$$
\leq K|t - t_0|,
$$

which is the second inequality of (2.5).
For the third inequality of (2.5), suppose that \( \theta_2 \leq \theta_1 \leq t \). Since
\[
D_{\theta_1}x(t) - D_{\theta_2}x(t) = (D_{\theta_1}\varphi(t) - D_{\theta_2}\varphi(t)) + (\sigma(t, \theta_1, x(\theta_1)) - \sigma(t, \theta_2, x(\theta_2)))
\]
\[
+ \int_{\theta_1}^{t} b_x(D_{\theta_1}x(t) - D_{\theta_2}x(t)) \, ds + \int_{\theta_1}^{t} \sigma_x(D_{\theta_1}x(t) - D_{\theta_2}x(t)) \, dW(s)
\]
\[
- \int_{\theta_2}^{\theta_1} b_x D_{\theta_2}x(t) \, ds - \int_{\theta_2}^{\theta_1} \sigma_x D_{\theta_2}x(t) \, dW(s),
\]
it easy to calculate that
\[
\mathbb{E}|D_{\theta_1}x(t) - D_{\theta_2}x(t)|^2
\]
\[
= 6\mathbb{E}|D_{\theta_1}\varphi(t) - D_{\theta_2}\varphi(t)|^2 + 6L^2(\theta_1 - \theta_2) + \mathbb{E}|x(\theta_1) - x(\theta_2)|^2
\]
\[
+ 6(T + 1)L^2 \mathbb{E} \int_{\theta_1}^{t} |D_{\theta_1}x(s) - D_{\theta_2}x(s)|^2 \, ds + 6(T + 1)L^2 \mathbb{E} \int_{\theta_2}^{\theta_1} |D_{\theta_2}x(s)|^2 \, ds
\]
\[
\leq K|\theta_2 - \theta_1| + 6(T + 1)L^2 \mathbb{E} \int_{\theta_1}^{t} |D_{\theta_1}x(s) - D_{\theta_2}x(s)|^2 \, ds.
\]
Hence, by Gronwall’s inequality, we have
\[
\sup_{\theta_1, \theta_2 \leq t \leq T} \mathbb{E}|D_{\theta_1}x(t) - D_{\theta_2}x(t)|^2 \leq K|\theta_1 - \theta_2|,
\]
completing the proof.

2.2 Regularity of \((Y(\cdot), Z(\cdot, \cdot))\)

The following result on wellposedness of BSVIE (1.1) comes from [17, Theorem 3.7 and 4.1].

**Theorem 2.3.** Under assumptions (A1)–(A4), BSVIE (1.1) admits a unique solution \((Y(\cdot), Z(\cdot, \cdot))\).

Moreover, the following estimates hold:
\[
\mathbb{E} \int_{S}^{T} |Y(t)|^2 \, dt + \mathbb{E} \int_{S}^{T} \int_{t}^{T} |Z(t, s)|^2 \, ds \, dt
\]
\[
\leq C \left\{ \mathbb{E} \int_{S}^{T} |g(t, x(T))|^2 \, dt + \mathbb{E} \int_{S}^{T} \left( \int_{t}^{T} |f(t, s, 0, 0, 0)| \, ds \right)^2 \, dt \right\}, \text{ for any } S \in [0, T],
\]
\[
\sum_{i=1}^{n} \mathbb{E} \left\{ \int_{S}^{T} |D_{i}Y(t)|^2 \, dt + \int_{S}^{T} \int_{t}^{T} |D_{i}Z(t, s)|^2 \, ds \, dt \right\}
\]
\[
\leq C \mathbb{E} \left\{ \int_{S}^{T} |g(t, x(T))|^2 \, dt + \sum_{i=1}^{n} \int_{S}^{T} |D_{i}g(t, x(t)|^2 \, dt
\]
\[
+ \int_{S}^{T} \left( \int_{t}^{T} |f(t, s, 0, 0, 0)| \, ds \right)^2 \, dt \right\}, \text{ for any } r, S \in [0, T].
\]

Moreover, \((D_{i}Y(\cdot), D_{i}Z(\cdot, \cdot))\) is the adapted solution to the following BSVIE:
\[
D_{r}Y(t) = D_{r}g(t, x(T)) + \int_{t}^{T} \left( g_x(t, s, x(s), Y(s), Z(t, s))D_{r}x(s)
\]
\[
+ g_y(t, s, x(s), Y(s), Z(t, s))D_{r}Y(s)
\]
\[
+ g_z(t, s, x(s), Y(s), Z(t, s))D_{r}Z(t, s) \right) \, ds
\]
\[
- \int_{r}^{T} D_{r}Z(t, s) \, dW(s), \ t \in [r, T].
\]
In addition, for any $0 \leq t < u \leq T$, $1 \leq i \leq n$,

\begin{align}
Z_i(t, u) &= D^i_u g(t, x(T)) + \int_u^T \left( f_x(t, s, x(s), Y(s), Z(t, s)) D^i_u x(s) \\
&\quad \quad \quad + f_y(t, s, x(s), Y(s), Z(t, s)) D^i_u Y(s) \\
&\quad \quad \quad + f_z(t, s, x(s), Y(s), Z(t, s)) D^i_u Z(t, s) \right) ds \\
&\quad \quad \quad - \int_u^T D^i_u Z(t, s) dW(s).
\end{align}

(2.10)

The following result is used to deduce the convergence speed in the Euler method for BSVIE (1.1).

**Lemma 2.4.** Under assumptions (A1)–(A4), for any $t, t_0 \in [0, T]$, it holds that

\begin{align}
\mathbb{E}|Y(t) - Y(t_0)|^2 + \mathbb{E} \int_{t \vee t_0}^T |Z(t, s) - Z(t_0, s)|^2 ds \leq C|t - t_0|,
\end{align}

(2.11)

where $C$ is a constant.

**Proof.** Suppose that $t_0 < t$. By [17, Corrolary 3.6], under assumptions (A1)–(A4), we have

\begin{align}
\mathbb{E}|Y(t) - Y(t_0)|^2 + \mathbb{E} \int_t^T |Z(t, s) - Z(t_0, s)|^2 ds &\leq C \left\{ \mathbb{E}|g(t, x(T)) - g(t_0, x(T))|^2 + \mathbb{E} \left( \int_t^t |f(t, s, x(s), Y(s), Z(t, s))| ds \right)^2 \right. \\
&\quad \quad \quad \quad \quad \quad + \mathbb{E}\left( \int_t^t |f(t, s, x(s), Y(s), Z(t, s)) - f(t_0, s, x(s), Y(s), Z(t, s))| ds \right)^2 \\
&\quad \quad \quad \quad \quad \quad + \mathbb{E} \left. \int_t^t |Z(t_0, s)|^2 ds \right\} \\
&\leq C|t - t_0| + C \mathbb{E} \int_t^t (|Y(s)|^2 + |Z(t_0, s)|^2) ds.
\end{align}

(2.12)

For $\mathbb{E}|Y(\cdot)|^2$, also by [17, Corrolary 3.6], one has

\begin{align}
\mathbb{E}|Y(t)|^2 + \mathbb{E} \int_t^T |Z(t, s)|^2 ds &\leq C \left\{ \mathbb{E}|g(t, x(T))|^2 + \mathbb{E} \left( \int_t^T |f(t, s, x(s), Y(s), 0)| ds \right)^2 \right\} \\
&\leq C + C \mathbb{E} \int_t^T |Y(s)|^2 ds.
\end{align}

(2.13)

By Gronwall’s inequality, one gets that $\sup_{0 \leq t \leq T} \mathbb{E}|Y(t)|^2 \leq C$. Thus

$\mathbb{E} \int_{t_0}^t |Y(s)|^2 ds \leq C|t - t_0|$. 

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Setting $t = t_0$ in (2.10), by [17, Corollary 3.6], Lemma 2.2 and (2.8), we can obtain
\[
\mathbb{E}|Z(t_0, u)|^2 + \mathbb{E} \int_t^T |D_u Z(t_0, s)|^2 ds \\
\leq C \left\{ \mathbb{E}|D_u g(t_0, x(T))|^2 + \mathbb{E} \left( \int_t^T \left( |f_x(t_0, s, x(s), Y(s), Z(t_0, x)) D_u x(s)| + |f_y(t_0, s, x(s), Y(s), Z(t_0, x)) D_u Y(s)| \right)^2 ds \right) \right\} \\
\leq C \left\{ \mathbb{E}|D_u x(T)|^2 + \mathbb{E} \int_t^T \left( |D_u x(s)|^2 + |D_u Y(s)|^2 \right) ds \right\} \\
\leq C |t - t_0|.
\] (2.14)

Now, (2.12), together with (2.13) and (2.14), yields that
\[
\mathbb{E}|Y(t) - Y(t_0)|^2 + \mathbb{E} \int_t^T |Z(t, s) - Z(t_0, s)|^2 ds \leq C|t - t_0|,
\]
which is (2.11). 

\section{The Euler method for SVIEs}

The aim of this section is to review the Euler method for SVIE (1.2) under assumptions (A3)–(A4). For numerical solutions to general SVIEs with singular kernels, one can refer to [20].

For simplicity, throughout this paper, we assume that $\Delta_i = |\pi_i| = \frac{T}{N} \leq 1$, for each $i = 0, 1, \cdots, N - 1$. Our numerical scheme still works for general uniform partition of $[0, T]$ (i.e., there exists a constant $K$, such that $K|\pi_i| \leq \Delta_j$, for any $j = 0, 1, \cdots, N - 1$). We also need the following two functions $\tau(\cdot)$ and $\pi(\cdot)$ defined on $[0, T]$ by
\[
\tau(t) = t_i, \quad \pi(t) = i, \quad t \in [t_i, t_{i+1}), i = 0, 1, \cdots, N - 1.
\]

The Euler method for SVIE (1.2) is as follows:
\[
x^\pi(0) = x^\pi(t_0) = \varphi(0), \\
\left\{
\begin{array}{l}
x^\pi(t_{i+1}) = \varphi(t_{i+1}) + \sum_{k=0}^i \left( b(t_{i+1}, t_k, x^\pi(t_k)) \Delta_k + \sigma(t_{i+1}, t_k, x^\pi(t_k)) \Delta_k W \right), \quad i = 0, 1, \cdots, N - 1.
\end{array}
\right.
\]
\quad (3.2)

In order to obtain the convergent speed, we introduce the following SVIE related to (3.2):
\[
x^\pi(t) = \varphi(t) + \int_0^t b(t, \tau(s), x^\pi(\tau(s))) ds + \int_0^t \sigma(t, \tau(s), x^\pi(\tau(s))) dW(s), \quad t \in [0, T].
\]
\quad (3.3)

Now, we are in the step to obtain the convergent speed for the Euler method (3.2). By SVIEs (1.2) and (3.3), one has, for any $t \in [0, T],
\[
x(t) - x^\pi(t) = \int_0^t \left( b(t, s, x(s)) - b(t, \tau(s), x^\pi(t(s))) \right) ds \\
+ \int_0^t \left( \sigma(t, s, x(s)) - \sigma(t, \tau(s), x^\pi(\tau(s))) \right) dW(s).
\]

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A direct calculation leads to
\[
\mathbb{E}|x(t) - x^\pi(t)|^2 \\
\leq 2T\mathbb{E} \int_0^t \left| b(t, s, x(s)) - b(t, \tau(s), x^\pi(\tau(s))) \right|^2 ds \\
+ 2\mathbb{E} \int_0^t \left| \sigma(t, s, x(s)) - \sigma(t, \tau(s), x^\pi(\tau(s))) \right|^2 ds \\
\leq 2(T + 1)L^2\mathbb{E} \int_0^t \left( |x(s) - x^\pi(s)|^2 + |x^\pi(s) - x^\pi(\tau(s))|^2 \right) ds.
\] (3.4)

By the definition of \( \tau \) in (3.1) (suppose that \( t \in [t_i, t_{i+1}] \), \( i = 0, 1, \cdots, N - 1 \),
\[
\mathbb{E} \int_0^t (s - \tau(s)) ds = \sum_{k=0}^{i-1} \int_{t_k}^{t_{k+1}} (s - t_k) ds + \int_{t_i}^t (s - t_i) ds = \sum_{k=0}^{i-1} \frac{\Delta_k^2}{2} + \frac{(t - t_i)^2}{2} \leq T|\pi|.
\] (3.5)

Now, supposing that \( t \in [t_i, t_{i+1}] \), we estimate \( \mathbb{E}|x^\pi(t) - x^\pi(\tau(t))|^2 \). By SVIE (3.3), one has
\[
x^\pi(t) - x^\pi(\tau(t)) = x^\pi(t) - x^\pi(t_i) \\
= (\varphi(t) - \varphi(t_i)) + \int_{t_i}^t \left( b(t, \tau(s), x^\pi(\tau(s))) - b(t_i, \tau(s_i), x^\pi(\tau(s_i))) \right) ds \\
+ \int_{t_i}^t \left( \sigma(t, \tau(s), x^\pi(\tau(s))) - \sigma(t_i, \tau(s_i), x^\pi(\tau(s_i))) \right) dW(s) \\
+ \int_{t_i}^t b(t, t_i, x^\pi(t_i)) ds + \int_{t_i}^t \sigma(t, t_i, x^\pi(t_i)) dW(s).
\]

Then, under assumptions (A3)–(A4), it is easy to check that
\[
\mathbb{E}|x^\pi(t) - x^\pi(\tau(t))|^2 \leq 5\mathbb{E}|\varphi(t) - \varphi(t_i)|^2 + 5\int_{t_i}^t L\sqrt{t - t_i} ds + 5\int_{t_i}^t L^2(t - t_i) ds \\
+ 5T\mathbb{E} \int_{t_i}^{t_{i+1}} |b(t, t_i, x^\pi(t_i))|^2 ds + 5\mathbb{E} \int_{t_i}^{t_{i+1}} |\sigma(t, t_i, x^\pi(t_i))|^2 ds \\
\leq C|\pi| + C\mathbb{E}|x^\pi(t_i)|^2|\pi|.
\] (3.6)

For \( \mathbb{E}|x^\pi(t_i)|^2 \), also by SVIE (3.3),
\[
\mathbb{E}|x^\pi(t)|^2 \leq 3\mathbb{E}|\varphi(t)|^2 + 6T\mathbb{E} \int_0^t |b(t, \pi(s), 0)|^2 ds + 6TL^2 \int_0^t |x^\pi(\pi(s))|^2 ds \\
+ 6\mathbb{E} \int_0^t |\sigma(t, \pi(s), 0)|^2 ds + 6L^2 \int_0^t |x^\pi(\pi(s))|^2 ds \\
\leq C + 6L^2(T + 1) \int_0^t \mathbb{E}|x^\pi(\pi(s))|^2 ds.
\]

Setting \( g(t) = \sup_{s \in [0,t]} \mathbb{E}|x^\pi(s)|^2 \), by Gronwall’s inequality, one obtains
\[
g(t) \leq Ce^{6L^2(T+1)}, \text{ for all } t \in [0, T].
\] (3.7)

(3.6), together with (3.7), yields that
\[
\mathbb{E}|x^\pi(t) - x^\pi(\tau(t))|^2 \leq C|\pi|, \forall t \in [0, T].
\] (3.8)
By (3.4), (3.5) and (3.8), we have

\[ \mathbb{E}|x(t) - x^\pi(t)|^2 \leq C|\pi| + 2L^2(T + 1) \int_0^t \mathbb{E}|x(s) - x^\pi(s)|^2ds, \]

which, by Gronwall’s inequality, deduces that

\[ \sup_{t \in [0,T]} \mathbb{E}|x(t) - x^\pi(t)|^2 \leq e^{2L^2(T+1)}C|\pi|. \]

By the above analysis, we get the following convergence speed of the Euler method (3.2) for SVIE (1.2).

**Theorem 3.1.** Let (A3)–(A4) hold. Then for \( x(\cdot) \) and \( x^\pi(\cdot) \) defined as in (1.2) and (3.2), respectively, there exists a constant \( C \), depending only on \( L \) and \( T \), such that

\[ \max_{0 \leq i \leq N} \mathbb{E}|x(t_i) - x^\pi(t_i)|^2 \leq C|\pi|. \]  

### 4 The Euler method for BSVIEs

In this section, we mainly present the Euler method to calculate the numerical solution to BSVIE (1.1), and prove the convergence speed of that method for (1.1). For \( 1 \leq k \leq N - 1 \), we present the Euler method for BSVIE (1.1) as follows:

\[
\begin{align*}
Y^{k,\pi}(t_N) &= g(t_k, x^\pi(T)), \\
Y^{k,\pi}(t_l) &= \mathbb{E}\left(Y^{k,\pi}(t_{l+1}) + f(t_k, t_l, x^\pi(t_l), Y^{l,\pi}(t_{l+1}), Z^{k,\pi}(t_l))\Delta_l\big|\mathcal{F}_t\right), \\
Z^{k,\pi}(t_l) &= \mathbb{E}\left(\frac{\Delta l}{\Delta t}W_l(Y^{k,\pi}(t_{l+1}) + f(t_k, t_l, x^\pi(t_l), Y^{l,\pi}(t_{l+1}), Z^{k,\pi}(t_l))\Delta_l)\big|\mathcal{F}_t\right),
\end{align*}
\]

\( k \leq l \leq N - 1 \).

Here \( x^\pi(\cdot) (k \leq l \leq N - 1) \) is defined by (3.2).

In order to obtain the convergence speed, we introduce \( (Y^{k,\cdot}, Z^{k,\cdot}) (k = 0, 1, \cdots, N - 1) \) solving the following BSDE:

\[
\begin{align*}
dY^{k}(s) &= -f(t_k, s, x(s), Y^{l}(s), Z^{k}(s))ds + Z^{k}(s)dW(s), s \in (t_l, t_{l+1}), \ k + 1 \leq l \leq N - 1, \\
dY^{k}(s) &= -f(t_k, s, x(s), Y^{k}(s), Z^{k}(s))ds + Z^{k}(s)dW(s), s \in [t_k, t_{k+1}], \\
Y^{k}(T) &= g(t_k, x(T)), Y^{k}(t_l) = Y^{k}(t_l + 0), \ k + 1 \leq l \leq N - 1,
\end{align*}
\]

and \( (Y^{k,\pi,\cdot}, \hat{Z}^{k,\pi,\cdot}) (k = 0, 1, \cdots, N - 1) \) solving the following BSDE:

\[
\begin{align*}
Y^{k,\pi}(t_{l+1}) - Y^{k,\pi}(t) &= -f(t_k, t_l, x^\pi(t_l), Y^{l,\pi}(t_{l+1}), \hat{Z}^{k,\pi}_{0}(t_l))\Delta_l + \int_{t_l}^{t_{l+1}} \hat{Z}^{k,\pi}(s)dW(s), \quad t \in (t_l, t_{l+1}], \ k + 1 \leq l \leq N - 1, \\
Y^{k,\pi}(t_{k+1}) - Y^{k,\pi}(t) &= -f(t_k, t_k, x^\pi(t_k), Y^{k,\pi}(t_{k+1}), \hat{Z}^{k,\pi}_{0}(t_k))\Delta_k + \int_{t_k}^{t_{k+1}} \hat{Z}^{k,\pi}(s)dW(s), \quad t \in [t_k, t_{k+1}], \\
Y^{k,\pi}(T) &= g(t_k, x^\pi(T)), Y^{k,\pi}(t_l) = Y^{k,\pi}(t_l + 0), \ k + 1 \leq l \leq N - 1, \\
\hat{Z}^{k,\pi}_{0}(t_N) &= 0, \hat{Z}^{k,\pi}_{0}(t_l) = \frac{1}{\Delta_l}\mathbb{E}\left(\int_{t_l}^{t_{l+1}} \hat{Z}^{k,\pi}(u)dW(u)\big|\mathcal{F}_t\right), \ k \leq l \leq N - 1.
\end{align*}
\]
Remark 4.1. (i) When $|\pi| < L^2$, BSDE (4.3) admits a unique solution.
(ii) By (4.1) and BSDE (4.3), in the cases: (I) $f = f(t, s, x, y)$; (II) $f = f(t, s, x, z)$, we can easily check that, for any $k = 0, 1, \cdots, N - 1$, $k + 1 \leq j \leq N - 1$,
\begin{equation}
Z^k,\pi(t_j) = \frac{1}{\Delta_j} \mathbb{E}\left( \int_{t_j}^{t_{j+1}} \tilde{Z}^k,\pi(u)du | \mathcal{F}_{t_j} \right) = \tilde{Z}^k,\pi(t_j).
\end{equation}

By the definition of $\tau(\cdot)$ and $\pi(\cdot)$ in (3.1), we can define $Y^\pi(t),\pi(t) = Y^k(t), Z^\pi(s) = Z^k(s)$, and $Y^\pi(t),\pi(\tau(t)) = Y^k,\pi(t_k), Z^\pi(\pi(s)) = Z^k,\pi(t_j)$, for $t \in [t_k, t_{k+1})$, $k = 0, 1, \cdots, N - 1$, and $s \geq t$, $s \in [t_j, t_{j+1})$, $k \leq j \leq N - 1$. The following result comes from [18].

**Theorem 4.2.** Let (A1)–(A4) hold. Then, BSDE (4.2) admits a unique solution $(Y^\pi(\cdot), Z^\pi(\cdot))$, and
\[ \mathbb{E} \int_0^T |Y(t) - Y^\pi(t)|^2 dt + \mathbb{E} \int_0^T \int_t^T |Z(t, s) - Z^\pi(s)|^2 ds dt \leq K|\pi|, \]
where $K$ is a constant only depending on $L$ and $T$.

Now we state our main result on convergence speed of the Euler method (4.1) for BSVE (1.1).

**Theorem 4.3.** Suppose that $f = f(t, s, x, y)$ or $f = f(t, s, x, z)$ in BSDE (4.2), and let (A1)–(A4) hold. Then
\begin{equation}
\sup_{0 \leq t \leq T} \mathbb{E}|Y(\tau(t)) - Y^\pi(\tau(t))|^2 + \mathbb{E} \int_0^T \int_t^T |Z(t, s) - Z^\pi(\pi(s))|^2 ds \leq K|\pi|,
\end{equation}
where $K$ is a constant depending only on $L$ and $T$.

The proof of Theorem 4.3 is lengthy, we split it into several lemmas.

### 4.1 Regularity of $(Y^\pi(\cdot), Z^\pi(\cdot))$

In this part, we mainly study the regularity of $Y^\pi(\cdot)$ and $Z^\pi(\cdot)$, which is crucial in proving Theorem 4.3. First, we need the next lemma.

**Lemma 4.4.** (1) Suppose that $a_i \geq 0$, $b_i \geq 0$, $c_0 > 0$ ($i = 0, 1, \cdots, N - 1$), and
\[ a_i \leq b_i + c_0 \sum_{k=i+1}^{N-1} a_k. \]
Then
\begin{equation}
a_i \leq b_i + c_0 \sum_{k=i+1}^{N-1} (1 + c_0)^{k-i-1} b_k.
\end{equation}

(2) Suppose that $b$, $K$ are positive constants, $\gamma = 1$ or $2$, and for any $k = 0, 1, \cdots, N - 1$, $k+1 \leq j \leq N - 1$
\begin{align}
a_{k,j} \leq ba_{k,j+1} + b|\pi|a_{j,j} + bK|\pi|^{\gamma}, \\
a_{k,k} \leq ba_{k,k+1} + bK|\pi|^{\gamma}.
\end{align}
Then, the following holds true:
\[
\begin{align*}
    a_{k,j} &\leq b^{N-k-j}a_{k,N-1} + b^{N-k-j} b^N \left| \sum_{l=0}^{N-2} (1 + b|\pi|)^l a_{k+l,t,N-1} + bK|\pi|^7 \sum_{l=0}^{N-2} b^l (1 + b|\pi|)^l, \\
    \quad \text{for any } N \geq k, \quad j \geq 1.
\end{align*}
\]

Proof. We prove (4.6) by induction,
\[
\begin{align*}
    a_{N-1} &\leq b^{N-1}b_{N-1}; \\
    a_{N-2} &\leq b^{N-2}b_{N-2} + c_0b_{N-1}; \\
    a_{N-3} &\leq b^{N-3}b_{N-3} + c_0b_{N-2} + c_0(c_0 + 1)b_{N-1}; \\
    a_{N-4} &\leq b_{N-4} + c_0b_{N-3} + c_0(c_0 + 1)b_{N-2} + c_0(c_0 + 1)^2b_{N-1}; \\
    \vdots
    \end{align*}
\]
\[
\begin{align*}
    a_i &\leq b_i + c_0 \sum_{k=i+1}^{N-1} (c_0 + 1)^{k-i-1}b_k.
\end{align*}
\]

Hence we obtain (4.6). (8.8) can also proved by induction.

The following Lemma is about the regularity of $Y^{\pi(t)}$.

Lemma 4.5. Suppose that (A1)–(A4) hold true. Then, for any $k = 0, 1, \cdots, N-1$, $k \leq j \leq N-1$ and $t \in [t_j, t_{j+1}]$, there exists a constant $C$, depending only on $L$ and $T$, such that
\[
    \mathbb{E}|Y^k(t) - Y^k(t_j)|^2 + |Y^k(t) - Y^k(t_{j+1})|^2 \leq C|\pi|.
\]

Proof. For any $t \in [t_j, t_{j+1}]$, by ESDE (4.2), it is easy to see that
\[
\begin{align*}
    \mathbb{E}|Y^k(t) - Y^k(t_j)|^2 &\leq 2\mathbb{E}\int_t^{t_j} |f(t_k, s, x(s), Y^j(s), Z^k(s))]^2 ds(t-t_j) + 2\mathbb{E}\int_t^{t_j} |Z^k(s)|^2 ds \\
    \leq &8L^2\mathbb{E}\int_t^{t_j} (|f(t_k, s, 0, 0, 0)|^2 + |x(s)|^2 + |Y^j(s)|^2 + |Z^k(s)|^2)ds(t-t_j) + 2\mathbb{E}\int_t^{t_j} |Z^k(s)|^2 ds \\
    \leq &K(t-t_j) + 8L^2\mathbb{E}\int_t^{t_j} |Y^j(s)|^2 ds(t-t_j) + (8L^2(t-t_j) + 2)\mathbb{E}\int_t^{t_j} |Z^k(s)|^2 ds.
\end{align*}
\]

We now estimate $\mathbb{E}|Y^k(t)|^2$, for $k = 0, 1, \cdots, N-1$ and $t \in [t_k, t_{k+1}]$, which appears on the right side of (4.10). By Itô’s formula,
\[
\begin{align*}
    \mathbb{E}|Y^k(t)|^2 &\leq \mathbb{E}\int_t^{t_{k+1}} |Z^k(s)|^2 ds \\
    \leq &\mathbb{E}|Y^k(t_{k+1})|^2 + \mathbb{E}\int_t^{t_{k+1}} (2L + 3L^2 + 1)|Y^k(s)|^2 ds + |f(t_k, s, 0, 0, 0)|^2 + |x(s)|^2 + \frac{1}{2}|Z^k(s)|^2 ds \\
    \leq &\mathbb{E}|Y^k(t_{k+1})|^2 + (2L + 3L^2 + 1) \mathbb{E}\int_t^{t_{k+1}} |Y^k(s)|^2 ds + (L^2 + K)(t_{k+1} - t) + \frac{1}{2} \mathbb{E}\int_t^{t_{k+1}} |Z^k(s)|^2 ds.
\end{align*}
\]

Consequently, by Gronwall’s inequality,
\[
(4.11) \quad \mathbb{E}|Y^k(t)|^2 \leq e^{(2L + 3L^2 + 1)(t_{k+1} - t)} \left( \mathbb{E}|Y^k(t_{k+1})|^2 + (L^2 + K)|\pi| \right).
\]
Now, we estimate $\mathbb{E}|Y^k(t)|^2$, for any $t \in [t_j, t_{j+1})$ $(k + 1 \leq j \leq N - 1)$, which appears in (4.11). With the similar calculus to that of (4.11), one can obtain

$$
\mathbb{E}|Y^k(t)|^2 + \mathbb{E} \int_t^{t_{j+1}} |Z^k(s)|^2 ds
\leq \mathbb{E}|Y^k(t_{j+1})|^2 + 2\mathbb{E} \int_t^{t_{j+1}} |Y^k(s)|(|f(t_k, s, 0, 0)| + L|x(s)| + L|Y^j(s)| + L|Z^k(s)|) ds
\leq \mathbb{E}|Y^k(t_{j+1})|^2 + (4L^2 + 1)\mathbb{E} \int_t^{t_{j+1}} |Y^k(s)|^2 ds
+ \mathbb{E} \int_t^{t_{j+1}} (|f(t_k, s, 0, 0)|^2 + |x(s)|^2 + |Y^j(s)|^2 + \frac{1}{2}|Z^k(s)|^2) ds
\leq \mathbb{E}|Y^k(t_{j+1})|^2 + (4L^2 + 1)\mathbb{E} \int_t^{t_{j+1}} |Y^k(s)|^2 ds
+ (L^2 + K)(t_{j+1} - t) + \mathbb{E} \int_t^{t_{j+1}} |Y^j(s)|^2 ds + \frac{1}{2}\mathbb{E} \int_t^{t_{j+1}} |Z^k(s)|^2 ds.
$$

Also, by Gronwall’s inequality, one has,

$$
(4.12) \quad \mathbb{E}|Y^k(t)|^2 \leq e^{(4L^2+1)(t_{j+1} - t)} \left( \mathbb{E}|Y^k(t_{j+1})|^2 + (L^2 + K)|\pi| + \mathbb{E} \int_t^{t_{j+1}} |Y^j(s)|^2 ds \right).
$$

Set $\bar{L} = \max\{2L + 3L^2 + 1, 4L^2 + 1\}$, $\alpha = e^{\bar{L}|\pi|}$, $\bar{K} = L^2 + K$, and $J_{k,j} = \sup_{t_j \leq t < t_{j+1}} \mathbb{E}|Y^k(t)|^2$, $j \geq k + 1$. Then, by (4.11), (4.12) and Lemma 4.4, it comes out

$$
(4.13) \quad J_{k,j} \leq \alpha^{N-1-j}J_{k,N-1} + \alpha^{N-j}|\pi| \sum_{l=0}^{N-2-j} (1 + \alpha|\pi|)^lJ_{j+l,N-1} + \bar{K}|\pi| \sum_{l=1}^{N-1-j} \alpha^l(1 + \alpha|\pi|)^l.
$$

Since for all $k \leq N - 2$,

$$
J_{k,N-1} = \sup_{t_{N-1} \leq t \leq T} \mathbb{E}|Y^k(t)|^2
\leq e^{\bar{L}|\pi|} \left( \mathbb{E}|g(t_k, x(T))|^2 + \bar{K}|\pi| + \int_{t_{N-1}}^{T} |Y^{N-1}(s)|^2 ds \right)
\leq e^{\bar{L}|\pi|} \left( L^2T + L^2\mathbb{E}|x(T)|^2 + \bar{K}|\pi| + |\pi|J_{N-1,N-1} \right),
$$

and

$$
J_{N-1,N-1} = \sup_{t_{N-1} \leq t \leq T} \mathbb{E}|Y^{N-1}(t)|^2 \leq e^{\bar{L}|\pi|} \left( \mathbb{E}|g(t_{N-1}, x(T))|^2 + \bar{K}|\pi| \right)
\leq e^{\bar{L}|\pi|} \left( L^2T + L^2\mathbb{E}|x(T)|^2 + \bar{K}|\pi| \right) \leq C < \infty,
$$

(4.14), together with (4.15), yields that, for all $k = 0, 1, \cdots, N - 1$,

$$
(4.16) \quad J_{k,N-1} \leq C < \infty.
$$

Now, we estimate the right side of (4.13) term by term.

$$
(4.17) \quad \alpha^{N-i-j} \leq \alpha^N = e^{\bar{L}|\pi|N} = e^{LT} < \infty,
$$

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\[
\alpha^{N-j}|\pi| \sum_{l=0}^{N-2-j} (1 + \alpha|\pi|)^l J_{j+l,N-1} \leq C e^{LT} \frac{(1 + \alpha|\pi|)^N}{\alpha} \\
= C\left(1 + \frac{T e^{LT}}{N}\right)^N \leq C e^{LT} < \infty,
\]
and
\[
\bar{K}|\pi| \sum_{l=1}^{N-1-j} \alpha^l (1 + \alpha|\pi|)^l \leq \bar{K}|\pi| \frac{\alpha^N (1 + \alpha|\pi|)^N}{\alpha + \alpha^2|\pi| - 1} \\
\leq \bar{K}|\pi| \frac{\alpha^N (1 + \alpha|\pi|)^N}{L|\pi|} \leq \frac{\bar{K}}{L} e^{LT} e^{LT} < \infty.
\]
Hence, (4.13), together with (4.17)–(4.19), leads to, for any \(k = 0, 1, \cdots, N - 1\) and \(j \geq k + 1\),
\[
J_{k,j} = \sup_{t_j \leq t \leq t_{j+1}} \mathbb{E}|Y^k(t)|^2 < \infty.
\]
Thereafter
\[
J_{k,k} \leq \alpha J_{k,k+1} + \alpha \bar{K}|\pi| < \infty.
\]
Furthermore, the second term of the right side in (4.10) turns into
\[
\mathbb{E} \int_{t_j}^t |Y^j(s)|^2 ds (t - t_j) \leq C|t - t_j|^2 \leq C|t - t_j|.
\]

Now, we need to estimate \(\mathbb{E}|Z^k(t)|^2\), for any \(t \in [t_j, t_{j+1}]\), which appears on the third term of the right side in (4.10). Since \((D_\theta Y^\pi(\cdot), D_\theta Z^\pi(\cdot))\), the Malliavin derivative of \((Y^\pi(\cdot), Z^\pi(\cdot))\), satisfies the following BSDE ([8, Proposition 5.3]):
\[
\begin{cases}
D_\theta Y^k(t_{j+1}) - D_\theta Y^k(t) \\
= \int_t^{t_{j+1}} \left(-f_x(t_k, s, x(s), Y^j(s), Z^k(s))D_\theta x(s) - f_y(t_k, s, x(s), Y^j(s), Z^k(s))D_\theta Y^j(s) \\
- f_z(t_k, s, x(s), Y^j(s), Z^k(s))D_\theta Z^k(s)\right) ds + \int_t^{t_{j+1}} D_\theta Z^k(s) dW(s), \\
\begin{array}{l}
t \in [t_j, t_{j+1}], \\
\theta \in [0, t], \\
Z^k(t) = D_t Y^k(t), t \in [t_k, T].
\end{array}
\end{cases}
\]
By Itô’s formula, for any \(t \in [t_k, t_{k+1}]\), one has
\[
\mathbb{E}|D_\theta Y^k(t)|^2 + \mathbb{E} \int_t^{t_{k+1}} |D_\theta Z^k(s)|^2 ds \\
\leq \mathbb{E}\left\{|D_\theta Y^k(t_{k+1})|^2 + (2L + 3L^2) \int_t^{t_{k+1}} |D_\theta Y^k(s)|^2 ds \\
+ \int_t^{t_{k+1}} |D_\theta x(s)|^2 ds + \frac{1}{2} \int_t^{t_{k+1}} |D_\theta Z^k(s)|^2 ds\right\},
\]
by Gronwall’s inequality, which deduces that,
\[
\mathbb{E}|D_\theta Y^k(t)|^2 \leq e^{(2L + 3L^2)(t_{k+1} - t)} \left(\mathbb{E}|D_\theta Y^k(t_{k+1})|^2 + K|\pi|\right).
\]
Similarly, for any $t \in [t_j, t_{j+1}]$,

$$
\mathbb{E}|D_0Y^k(t)|^2 + \mathbb{E} \int_t^{t_{j+1}} |D_0Z^k(s)|^2 ds 
$$

(4.23)

$$
\leq \mathbb{E} \left\{ |D_0Y^k(t_{j+1})|^2 + 4L^2 \int_t^{t_{j+1}} |D_0Y^k(s)|^2 ds + \int_t^{t_{j+1}} (|D_0x(s)|^2 + |D_0Y^j(s)|^2) ds + \frac{1}{2} \int_t^{t_{j+1}} |D_0Z^k(s)|^2 ds \right\}
$$

Therefore, similar to (4.20), by virtue of Lemma 4.4, we have

(4.24)

$$
\sup_{t_j \leq t \leq t_{j+1}} \mathbb{E}|D_0Y^k(t)|^2 < \infty.
$$

By setting $\theta = t$ in (4.24), one gets, for any $t \in [t_k, T]$,

(4.25)

$$
\mathbb{E}|Z^k(t)|^2 = \mathbb{E}|D_1Y^k(t)|^2 < \infty.
$$

(4.10), together with (4.21) and (4.25), yields that

(4.26)

$$
\mathbb{E}|Y^k(t) - Y^k(t_j)|^2 \leq C|t - t_j| \leq C|\pi|.
$$

Similarly, we can get

$$
\mathbb{E}|Y^k(t) - Y^k(t_{j+1})|^2 \leq C|t - t_{j+1}| \leq C|\pi|.
$$

That completes the proof.

With this result at hand we can conclude:

**Proposition 4.6.** Suppose that (A1)–(A4) hold true. Then, there exists a constant $K$, such that, for any $k = 0, 1, \cdots, N - 1$,

$$
\mathbb{E}|Y(t_k) - Y^k(t_k)|^2 + \mathbb{E} \int_{t_k}^{T} |Z(t_k, s) - Z^k(s)|^2 ds \leq K|\pi|.
$$

**Proof.** Setting $h(t, s, z) = f(t, s, x(s), Y(s), z)$ and $\bar{h}(t, s, z) = f(\tau(t), s, x(s), Y^\pi(s)(s), z)$, by [17, Corrolary 3.6], we have

$$
\mathbb{E}|Y(t_k) - Y^k(t_k)|^2 + \mathbb{E} \int_{t_k}^{T} |Z(t_k, s) - Z^k(s)|^2 ds 
$$

(4.27)

$$
\leq K \mathbb{E} \left( \int_{t_k}^{T} |f(t_k, s, x(s), Y(s), Z(t_k, s)) - f(\tau(t_k), s, x(s), Y^\pi(s)(s), Z(t_k, s))| ds \right)^2 
$$

$$
\leq K \sum_{j=k}^{N-1} \mathbb{E} \int_{t_j}^{t_{j+1}} \left( |Y(s) - Y(t_j)|^2 + |Y^j(s) - Y^j(t_j)|^2 + |Y(t_j) - Y^j(t_j)|^2 \right) ds 
$$

$$
\leq K|\pi| + K|\pi| \sum_{j=k+1}^{N-1} \mathbb{E}|Y(t_j) - Y^j(t_j)|^2 + K|\pi| \mathbb{E}|Y(t_k) - Y^k(t_k)|^2.
$$

Here we apply Lemma 2.4 and Lemma 4.5. Taking $N > 2KT$ ($N \in \mathbb{N}$), then $K|\pi| \leq \frac{1}{2}$, and

(4.28)

$$
\frac{1}{2} \mathbb{E}|Y(t_k) - Y^k(t_k)|^2 \leq K|\pi| + K|\pi| \sum_{j=k+1}^{N-1} \mathbb{E}|Y(t_j) - Y^j(t_j)|^2.
$$
Denote \( a_k = \mathbb{E}[Y(t_k) - Y^k(t_k)]^2 \), \( b_k = 2K|\pi| \) and \( c_0 = 2K|\pi| \). Therefore, by Lemma 4.4, for any \( k = 0, 1, \ldots, N - 1 \),

\[
\mathbb{E}|Y(t_k) - Y^k(t_k)|^2 = a_k \leq b_k + c_0 \sum_{i=k+1}^{N-1} (1 + c_0)^{i-k-1}b_i
\]

\[
\leq 2K|\pi| + 2K|\pi|(c_0 + 1)^N \leq 2K|\pi| + 2K|\pi|e^{2KT} \leq K|\pi|.
\]

Combining (4.27) with (4.29), we can have \( \mathbb{E} \int_{t_k}^T |Z(t_k, s) - Z^k(s)|^2 ds \leq K|\pi| \). That completes the proof.

In the following part, we mainly provide the regularity of \( Z^\pi(\cdot) \). Such a regularity, combining with that for \( x(\cdot) \) and \( Y^\pi(\cdot) \), can derive the rate of convergence of the Euler method (4.1). We present that regularity in two different cases: (I) \( f = f(t, s, x, y) \); (II) \( f = f(t, s, x, z) \). Here, we borrow some idea from [9].

**Lemma 4.7.** Suppose that \( f = f(t, s, x, y) \) in BSDE (4.2), and (A1)–(A4) hold true. Then, for any \( k = 0, 1, \ldots, N - 1 \), \( k \leq j \leq N - 1 \) and \( s \in [t_j, t_{j+1}] \), there exists a constant \( C \), such that

\[
\mathbb{E}|Z^k(s) - Z^k(t_j)|^2 \leq K|\pi|.
\]

**Proof.** We divide the proof into two steps.

**Step 1.** For any \( k = 0, 1, \ldots, N - 1 \), \( k \leq j \leq N - 1 \) and \( s \in [t_j, t_{j+1}] \), by (4.22), one gets

\[
Z^k(s) - Z^k(t_j) = D_sY^k(s) - D_{t_j}Y^k(t_j)
\]

\[
= (D_sY^k(s) - D_{t_j}Y^k(s)) + (D_{t_j}Y^k(s) - D_{t_j}Y^k(t_j)).
\]

We claim that

\[
\mathbb{E}|D_sY^k(s) - D_{t_j}Y^k(s)|^2 \leq K|\pi|.
\]

Indeed, by (4.22), for \( \theta_1, \theta_2 \in [t_j, t_{j+1}] \), \( \theta_2 \leq \theta_1 \leq s \),

\[
\mathbb{E}|D_{\theta_1}Y^k(s) - D_{\theta_2}Y^k(s)|^2 + \mathbb{E} \int_{\theta_1}^{t_{j+1}} |D_{\theta_1}Z^k(t) - D_{\theta_2}Z^k(t)|^2 dt
\]

\[
= \mathbb{E}|D_{\theta_1}Y^k(t_{j+1}) - D_{\theta_2}Y^k(t_{j+1})|^2
\]

\[
+ 2 \mathbb{E} \int_{\theta_1}^{t_{j+1}} \left\langle D_{\theta_1}Y^k(t) - D_{\theta_2}Y^k(t), f_x(D_{\theta_1}x(t) - D_{\theta_2}x(t)) + f_y(D_{\theta_1}Y^j(t) - D_{\theta_2}Y^j(t)) \right\rangle dt
\]

\[
\leq \mathbb{E}|D_{\theta_1}Y^k(t_{j+1}) - D_{\theta_2}Y^k(t_{j+1})|^2 + 2L^2 \mathbb{E} \int_{\theta_1}^{t_{j+1}} |D_{\theta_1}Y^k(t) - D_{\theta_2}Y^k(t)|^2 dt
\]

\[
+ \mathbb{E} \int_{\theta_1}^{t_{j+1}} |D_{\theta_1}x(t) - D_{\theta_2}x(t)|^2 dt + \mathbb{E} \int_{\theta_1}^{t_{j+1}} |D_{\theta_1}Y^j(t) - D_{\theta_2}Y^j(t)|^2 dt.
\]

Hence, by Lemma 2.2 and Gronwall’s inequality, one has, for any \( s \in [t_j, t_{j+1}] \),

\[
\mathbb{E}|D_{\theta_1}Y^k(s) - D_{\theta_2}Y^k(s)|^2
\]

\[
\leq C.
\]
\[ s \in E \]

\[ \theta \]

**Setting**

By Lemma 4.4, with the similar procedure used in the proof of Lemma 4.5, one can get, for any \( s \in [t_j, t_{j+1}] \),

\[
\mathbb{E}[D_{\theta_1} Y^k(s) - D_{\theta_2} Y^k(s)]^2 \leq e^{2L^2(t_{k+1} - s)} \left( \mathbb{E}[D_{\theta_1} Y^k(t_{k+1}) - D_{\theta_2} Y^k(t_{k+1})]^2 + K|\pi|^2 \right).
\]

By Lemma 4.4, with the similar procedure used in the proof of Lemma 4.5, one can get, for any \( s \in [t_j, t_{j+1}] \),

\[
\sup_{t_j \leq s \leq t_{j+1}} \mathbb{E}[D_{\theta_1} Y^k(s) - D_{\theta_2} Y^k(s)]^2 \leq K|\pi|.
\]

Setting \( \theta_1 = s, \theta_2 = t_j \), one easily obtains (4.32).

**Step 2.** We claim that, for any \( s \in [t_j, t_{j+1}] \),

\[
\mathbb{E}[D_{t_j} Y^k(s)]^2 \leq K|s - t_j|.
\]

For any \( \theta \leq t_j, t \in [t_j, t_{j+1}] \), by virtue of Eq. (4.2),

\[
D_{\theta} Y^k(t) = \mathbb{E}\left(D_{\theta} Y^k(T) + \int_t^T F(s) ds | \mathcal{F}_t\right),
\]

where

\[
\int_t^T F(s) ds = \int_t^{t_{j+1}} f_x(t_k, s, x(s), Y^j(s)) D_{\theta} x(s) + f_y(t_k, s, x(s), Y^j(s)) D_{\theta} Y^j(s) ds + \sum_{l=1}^{N-1} \int_{t_l}^{t_{l+1}} f_x(t_k, s, x(s), Y^l(s)) D_{\theta} x(s) + f_y(t_k, s, x(s), Y^l(s)) D_{\theta} Y^l(s) ds.
\]

Then

\[
D_{\theta} Y^k(t) - D_{\theta} Y^k(t_j)
\]

(4.35) = \left\{ \mathbb{E}(D_{\theta} Y^k(T)|\mathcal{F}_t) - \mathbb{E}(D_{\theta} Y^k(T)|\mathcal{F}_{t_j}) \right\} + \left\{ \mathbb{E}\left( \int_t^T F(s) ds | \mathcal{F}_t \right) - \mathbb{E}\left( \int_t^T F(s) ds | \mathcal{F}_{t_j} \right) \right\}

:= I_1 + I_2.

For \( I_1 \), since

\[
D_{\theta} Y^k(T) = \mathbb{E}D_{\theta} Y^k(T) + \int_0^T \mathbb{E}(D_{\theta} D_{\theta} Y^k(T)|\mathcal{F}_s) dW(s),
\]

by Lemma 2.2, one can have

\[
\mathbb{E} I_1^2 = \mathbb{E}\int_{t_j}^t \left| \mathbb{E}(D_{\theta} D_{\theta} Y^k(T)|\mathcal{F}_s) \right|^2 ds \leq \mathbb{E}\int_{t_j}^t \left| D_{\theta} D_{\theta} Y^k(T) \right|^2 ds
\]

(4.36) = \mathbb{E}\int_{t_j}^t \left| D_{\theta} g(t_k, x(T)) \right|^2 ds = \mathbb{E}\int_{t_j}^t \left| g_{x}\theta D_{\theta} x(T) D_{\theta} x(T) + g_{x} D_{\theta} x(T) \right|^2 ds \leq K|t - t_j|.
For $I_2$,
\[
I_2 = \left\{ \mathbb{E}\left( \int_t^T F(s)ds \right)_{\mathcal{F}_t} - \mathbb{E}\left( \int_t^T F(s)ds \right)_{\mathcal{F}_t} \right\} \\
\quad + \left\{ \mathbb{E}\left( \int_t^T F(s)ds \right)_{\mathcal{F}_t} - \mathbb{E}\left( \int_t^T F(s)ds \right)_{\mathcal{F}_t} \right\} \\
:\Rightarrow I_{21} + I_{22}.
\]

(4.37)

By Lemma 2.2 and (4.24), it is easy to check that
\[
\mathbb{E}I_{21}^2 = \mathbb{E}\left| \int_t^{t_{j+1}} \left( f_x(t_k, s, x(s), Y^j(s))D_\theta x(s) + f_y(t_k, s, x(s), Y^j(s))D_\theta Y^j(s) \right) ds \right|^2 \\
\leq K(t - t_j)\mathbb{E}\int_t^{t_{j+1}} \left( |D_\theta x(s)|^2 + |D_\theta Y^j(s)|^2 \right) ds \leq K|t - t_j|.
\]

(4.38)

For the $I_{22}$ part, by Clark-Ocone representation formula,
\[
\int_t^T F(s)ds = \mathbb{E}\left( \int_t^T F(s)ds \right) + \int_0^T \mathbb{E}\left( D_u \int_t^T F(s)ds \right)_{\mathcal{F}_u} dW(u),
\]

it admits the following representation:
\[
I_{22} = \int_t^T \mathbb{E}\left( D_u \int_t^T F(s)ds \right)_{\mathcal{F}_u} dW(u).
\]

It is easy to check that
\[
D_u \int_t^T F(s)ds = \int_t^T D_u F(s)ds \\
= \sum_{l=j}^{N-1} \int_{t_l}^{t_{l+1}} \left( f_{xx}D_\theta x(s)D_u x(s) + f_{xy}D_\theta x(s)D_u Y^l(s) + f_x D_u D_\theta x(s) \\
+ f_{yx}D_\theta y(l(s)D_u x(s) + f_{yy}D_\theta Y^l(s)D_u Y^l(s) + f_y D_u D_\theta Y^l(s) \right) ds.
\]

Therefore,
\[
\mathbb{E}\left| D_u \int_t^T F(s)ds \right|^2 \leq T\mathbb{E}\int_t^T |D_u F(s)|^2 ds \\
\leq K \sum_{l=j}^{N-1} \mathbb{E}\int_{t_l}^{t_{l+1}} \left( |D_\theta x(s)D_u x(s)|^2 + |D_\theta x(s)D_u Y^l(s)|^2 + |D_u D_\theta x(s)|^2 \\
+ |D_\theta Y^l(s)D_u x(s)|^2 + |D_\theta Y^l(s)D_u Y^l(s)|^2 + |D_u D_\theta Y^l(s)|^2 \right) ds \\
\leq K \sum_{l=j}^{N-1} \mathbb{E}\int_{t_l}^{t_{l+1}} \left( |D_\theta x(s)|^4 + |D_u x(s)|^4 + |D_u Y^l(s)|^4 + |D_\theta Y^l(s)|^4 \\
+ |D_u D_\theta x(s)|^2 + |D_u D_\theta Y^l(s)|^2 \right) ds.
\]

(4.39)
Now, we estimate each term on the right side of the above inequality. By Itô’s formula,
\[
\mathbb{E}|D_0 Y^k(t)|^4 + 6 \mathbb{E} \int_t^{t_{j+1}} |D_0 Y^k(s)|^2 |D_0 Z^k(s)|^2 ds
\]
\[
= \mathbb{E}|D_0 Y^k(t_{j+1})|^4 + 4 \mathbb{E} \int_t^{t_{j+1}} |D_0 Y^k(s)|^2 \langle D_0 Y^k(s), f_x D_0 x(s) + f_y D_0 Y^j(s) \rangle ds
\]
\[
\leq \mathbb{E}|D_0 Y^k(t_{j+1})|^4 + 4 \mathbb{E} \int_t^{t_{j+1}} \left( \frac{3}{2} |D_0 Y^k(s)|^4 + 4 |D_0 x(s)|^4 + 4 |D_0 Y^j(s)|^4 \right) ds
\]
\[
\leq \mathbb{E}|D_0 Y^k(t_{j+1})|^4 + 6 \mathbb{E} \int_t^{t_{j+1}} |D_0 Y^k(s)|^4 ds + K |\pi| + KE \int_t^{t_{j+1}} |D_0 Y^j(s)|^4 ds.
\]
Thus, by Lemma 4.4, one get
\[
(4.40) \quad \sup_{t_j \leq t \leq t_{j+1}} \mathbb{E}|D_0 Y^k(t)|^4 < \infty.
\]
For any \( u \leq t_j \leq t \leq t_{j+1} \),
\[
D_u D_0 Y^k(t_{j+1}) - D_u D_0 Y^k(t)
= \int_t^{t_{j+1}} \left( f_{xx} D_0 x(s) D_u x(s) + f_{xy} D_0 x(s) D_u Y^j(s) + f_x D_u D_0 x(s)
\right. \\
\left. + f_{yx} D_0 Y^j(s) D_u x(s) + f_{yy} D_0 Y^j(s) D_u Y^j(s) + f_y D_u D_0 Y^j(s) \right) ds
\]
\[
+ \int_t^{t_{j+1}} D_u D_0 Z^k(s) dW(s).
\]
Hence, by Itô’s formula,
\[
\mathbb{E}|D_u D_0 Y^k(t)|^2 + \mathbb{E} \int_t^{t_{j+1}} |D_u D_0 Z^k(s)|^2 ds
\leq \mathbb{E}|D_u D_0 Y^k(t_{j+1})|^2 + \mathbb{E} \int_t^{t_{j+1}} \left( 6L^2 |D_u D_0 Y^k(s)|^2 + |D_0 x(s) D_u x(s)|^2 + |D_0 Y^j(s) D_u Y^j(s)|^2 \\
+ |D_u D_0 x(s)|^2 + |D_0 Y^j(s) D_u x(s)|^2 + |D_0 Y^j(s) D_u Y^j(s)|^2 + |D_0 D_0 Y^j(s)|^2 \right) ds
\]
\[
\leq \mathbb{E}|D_u D_0 Y^k(t_{j+1})|^2 + 6L^2 \mathbb{E} \int_t^{t_{j+1}} |D_u D_0 Y^k(s)|^2 ds + K |\pi| + \mathbb{E} \int_t^{t_{j+1}} |D_u D_0 Y^j(s)|^2 ds
\]
Also, by Lemma 4.4, we have
\[
(4.42) \quad \sup_{t_j \leq t \leq t_{j+1}} \mathbb{E}|D_u D_0 Y^k(t)|^2 < \infty.
\]
Therefore, (4.39), together with (4.40) and (4.42), yields that
\[
\mathbb{E} \left| D_u \int_{t_j}^T F(s) ds \right|^2 < \infty.
\]
Furthermore,
\[
(4.43) \quad \mathbb{E}|I_{22}|^2 = \mathbb{E} \int_{t_j}^t \left| \mathbb{E} \left( D_u \int_{t_j}^T F(s) ds \bigg| F_u \right) \right|^2 du \leq \mathbb{E} \int_{t_j}^t |D_u \int_{t_j}^T F(s) ds|^2 du \leq K |t - t_j|.
\]
Finally, by (4.35)–(4.38) and (4.43), one gets
\[ \mathbb{E}|D_y Y_k(t) - D_y Y_k(t_j)|^2 \leq K|t - t_j|, \]
which deduces (4.34) by setting \( \theta = t_j \). Now combining (4.31) with (4.32) and (4.34), we have the regularity of \( Z \) (4.30).

The following regularity of \( Z^{\pi_c}() \) is in the case: \( f = f(t, s, x, z) \).

**Lemma 4.8.** Suppose that \( f = f(t, s, x, z) \) in BSDE (4.2), and (A1)–(A4) hold true. Then, for any \( k = 0, 1, \cdots, N - 1 \), \( k \leq j \leq N - 1 \) and \( s \in [t_j, t_{j + 1}] \), there exists a constant \( C \), such that
\begin{equation}
\mathbb{E}|Z^k(s) - Z^k(t_j)|^2 \leq C|t|.
\end{equation}

We need the following lemma to prove the above result.

**Lemma 4.9.** Let (A1) hold, and for any \( k = 0, 1, \cdots, N - 1 \), \( \Psi_k() \) and \( \Phi_k() \) solve the following SDEs
\begin{equation}
d\Psi_k(t) = \Psi_k(t)f_z(t_k, t, x(t), Z^k(t))dW(t), \quad t \in [0, T),
\end{equation}
\begin{equation}
\Psi(0) = I_n
\end{equation}
and
\begin{equation}
d\Phi_k(t) = (f_z(t_k, t, x(t), Z^k(t)))^2 \Phi(t)dt
\end{equation}
\begin{equation}
- f_z(t_k, t, x(t), Z^k(t)) \Phi_k(t)dW(t), \quad t \in [0, T),
\end{equation}
\begin{equation}
\Phi(0) = I_n,
\end{equation}
respectively. Then, for any \( p \geq 2 \),
\begin{equation}
\mathbb{E}\left( \sup_{0 \leq t \leq T} |\Psi_k(t)|^p \right) + \mathbb{E}\left( \sup_{0 \leq t \leq T} |\Phi_k(t)|^p \right) \leq C,
\end{equation}
\begin{equation}
\mathbb{E}\left( \sup_{s \leq t \leq T} |\Phi_k(s)\Psi_k(t)|^p \right) + \mathbb{E}\left( \sup_{s \leq t \leq T} |\Phi_k(t)\Psi_k(s)|^p \right) \leq C,
\end{equation}
\begin{equation}
\mathbb{E}(|\Phi_k(t) - \Phi_k(s)|\Psi_k(T_0)|^p \leq C|t - s|^\frac{p}{2}, \quad t, s \leq T_0 \leq T;
\end{equation}
and for any \( p \in [2, 2p_0] \),
\begin{equation}
\mathbb{E}\left( \sup_{\theta, s \leq t \leq T} |D_y(\Phi_k(s)\Psi_k(t))|^p \right) \leq C,
\end{equation}
where \( C \) depends only on \( p, L \) and \( T \).

**Proof.** First of all, for any \( x_0 \in \mathbb{R}^n \), set \( x(\cdot) = \Psi^\top_k(\cdot)x_0 \). Then \( x(\cdot) \) solves the following SDE:
\begin{equation}
\begin{cases}
 dx(t) = f_z^\top(t)x(t)dW(t), \quad t \in [0, T),
 x(0) = x_0.
\end{cases}
\end{equation}
Then, by Lemma 2.1,
\[ \mathbb{E}\left( \sup_{0 \leq t \leq T} |x(t)|^p \right) \leq C|x_0|^p. \]
Consequently,
\[ \mathbb{E}\left( \sup_{0 \leq t \leq T} |\Psi(t)|^p \right) = \sup_{x_0 \in \mathbb{R}^n} \mathbb{E}\left( \sup_{0 \leq t \leq T} |x(t)|^p \right) \leq C. \]
Here C depends only on p, L and T. Similarly, one can prove \( \mathbb{E}\left( \sup_{0 \leq t \leq T} |\Phi_k(t)|^p \right) \leq C \), and then (4.47) is proved.

Next, we only prove the second part \( \mathbb{E}\left( \sup_{s \leq t \leq T} |\Phi_k(t)\Psi_k(s)|^p \right) \leq C \) of (4.48). The first one can be proved with the similar procedure. For any \( x_0 \in \mathbb{R}^n \), set \( x_s(t) = \Phi_k(t)\Psi_k(s)x_0 \). Then \( x_s(t) \) solves the following SDE:
\[ \begin{cases} 
  dx_s(t) = (f_z)^2 x_s(t)dt - f_z x_s(t)dW(t), & t \in [s, T), \\
  x_s(s) = x_0.
\end{cases} \]
Then, also by Lemma 2.1,
\[ \mathbb{E}\left( \sup_{s \leq t \leq T} |\Phi_k(t)\Psi_k(s)x_0|^p \right) = \mathbb{E}\left( \sup_{s \leq t \leq T} |x_s(t)|^p \right) \leq C|x_0|^p, \]
where C depends only on p, L and T.

Now, by Eq. (4.46), one has
\begin{align}
\mathbb{E}\left( (\Phi_k(t) - \Phi_k(s))\Psi_k(T_0) \right)^p &= \mathbb{E}\left( \left( \int_s^t (f_z(\tau))^2 \Phi_k(\tau)d\tau \Psi_k(T_0) + \int_s^t f_z(\tau)\Phi_k(\tau)dW(\tau)\Psi_k(T_0) \right)^p \right) \\
&\leq C\mathbb{E}\left( \int_s^t |\Phi_k(\tau)\Psi_k(T_0)|^p d\tau \right)^{\frac{p}{2}} + C\mathbb{E}\left( \int_s^t f_z(\tau)\Phi_k(\tau)dW(\tau)\Psi_k(T_0) \right)^p \\
&:= CJ_1 + CJ_2.
\end{align}
For \( J_1 \), by (4.48), we have
\begin{align}
J_1 &\leq \mathbb{E}\int_s^t |\Phi_k(\tau)\Psi_k(T_0)|^p d\tau \left( \int_s^t 1d\tau \right)^{p-1} \\
&= \int_s^t \mathbb{E}|\Phi_k(\tau)\Psi_k(T_0)|^p d\tau(t-s)^{p-1} \leq C(t-s)^p,
\end{align}
where C depends only on p, L and T. For \( J_2 \), by (4.48), Hölder’s inequality and Burkholder-Davis-Gundy inequality, we also can obtain
\begin{align}
J_2 &= \mathbb{E}\left( \int_s^t f_z(\tau)\Phi_k(\tau)dW(\tau)\Psi_k(T_0) \right)^p = \mathbb{E}\left( \int_s^t f_z(\tau)\Phi_k(\tau)\Psi_k(s)dW(\tau)\Phi_k(s)\Psi_k(T_0) \right)^p \\
&\leq \left( \mathbb{E}\left( \int_s^t f_z(\tau)\Phi_k(\tau)\Psi_k(s)dW(\tau) \right)^{2p} \right)^{1/2} \left( \mathbb{E}|\Phi_k(s)\Psi_k(T_0)|^{2p} \right)^{1/2} \\
&\leq C\left( \left( \int_s^t 1d\tau \right)^{p-1} \left( \int_s^t |\Phi_k(\tau)\Psi_k(s)|^2d\tau \right)^{\frac{p}{2}} \right)^{1/2} \\
&\leq C\left\{ \left( \int_s^t 1d\tau \right)^{\frac{p-1}{p}} \left( \int_s^t |\Phi_k(\tau)\Psi_k(s)|^{2p}d\tau \right)^{\frac{1}{2}} \right\}^{1/2} \\
&\leq C(t-s)^{\frac{p}{2}},
\end{align}
where \( C \) depends only on \( p, L \). Combining (4.51)–(4.53), we have (4.49).

Finally, we prove (4.50). Indeed, for any \( 0 \leq \theta, s \leq t \leq T \), \( D_\theta(\Phi_k(s)\Psi_k(\cdot)) \) satisfies the following SDE:

\[
\begin{cases}
    dD_\theta(\Phi_k(s)\Psi_k(t)) = \left(D_\theta(\Phi_k(s)\Psi_k(t)) f_z(t_k, t, x(t), Z^k(t))
    
    + (\Phi_k(s)\Psi_k(t))(f_{zz}D_\theta x(t) + f_{zz}D_\theta Z^k(t))\right) dW(t), \quad \theta \leq t \leq T,
    
    D_\theta(\Phi_k(s)\Psi_k(t)) = 0, \quad 0 \leq t < \theta.
\end{cases}
\]

For any \( x_0 \in \mathbb{R}^n \), set \( x_{\theta,s}(\cdot) = D_\theta(\Psi_k^T(\cdot)\Phi_k^T(s))x_0 \) and \( y_{\theta}(\cdot) = \Psi_k^T(\cdot)\Phi_k^T(s)x_0 \). Then \( x_{\theta,s}(\cdot) \) satisfies the following SDE:

\[
\begin{cases}
    dx_{\theta,s}(t) = (f_z^T(t)x_{\theta,s}(t) + (D_\theta x^T(t)f_{zz}^T(t) + D_\theta Z^k(t)f_{zz}^T(t))y_{\theta}(t))dW(t), \quad \theta \leq t \leq T,
    
    x_{\theta,s}(\theta) = 0, 
    
    x_{\theta,s}(t) = 0, \quad 0 \leq t < \theta \leq T.
\end{cases}
\]

For any \( p \in [2, 2p_0) \), by Lemma 2.1, we have

\[
\begin{aligned}
E\left(\sup_{\theta \leq t \leq T} |D_\theta(\Psi_k^T(t)\Phi_k^T(s))x_0|^p\right)
\leq & C E\left(\int_\theta^T |(D_\theta x^T(t)f_{zz}^T(t) + D_\theta Z^k(t)f_{zz}^T(t))y_{\theta}(t)|^2 dt\right)^{\frac{p}{2}}
\leq & C\left\{E\left(\int_\theta^T |D_\theta Z^k(t)|^2 dt\right)^{\frac{2p_0}{2}} + C\right\}^{\frac{p}{2p_0}}
\left\{E\left(\sup_{s \leq t \leq T} |y_{\theta}(t)|^{2p_0}\right)\right\}^{\frac{p}{2p_0}},
\end{aligned}
\]

and

\[
\begin{aligned}
E\left(\int_\theta^T |D_\theta x(t)|^2 |y_{\theta}(t)|^2 dt\right)^{\frac{p}{2}} & \leq \left\{E\left(\int_\theta^T |D_\theta x(t)|^{2p_0} dt\left(\int_\theta^T |y_{\theta}(t)|^2 dt\right)^{\frac{p_0-2}{2}}\right)\right\}^{\frac{p}{2p_0}}
\leq C\left\{E\left(\int_\theta^T |D_\theta x(t)|^{2p_0} dt\right)^{\frac{p}{2p_0}}\right\}^{\frac{p}{2p_0}}
\leq C\left\{E\left(\sup_{\theta \leq t \leq T} |D_\theta x(t)|^{2p_0}\right)\right\}^{\frac{p}{2p_0}}
\leq C\left\{E\left(\sup_{\theta \leq t \leq T} |y_{\theta}(t)|^{2p_0}\right)\right\}^{\frac{p}{2p_0}}
\end{aligned}
\]

\[
\begin{aligned}
E\left(\int_\theta^T |D_\theta Z^k(t)|^2 dt\right)^{\frac{2p_0}{p}} & \leq C\left\{E|D_\theta x(T)|^{2p} + E\left(\int_\theta^T |D_\theta x(t)| dt\right)^{2p_0}\right\},
\end{aligned}
\]

where \( C \) depends only on \( p_0, L \) and \( T \). (4.54), together with (4.55), (4.56) and (4.48), yields (4.50). That completes the proof.

Now, we can prove Lemma 4.8.
Proof of Lemma 4.8. We split the proof into two steps.

Step 1. Similar to (4.31), we also obtain: for any $k = 0, 1, \cdots, N - 1$, $k \leq j \leq N - 1$ and $s \in [t_j, t_{j+1}]$,
\begin{equation}
Z^k(s) - Z^k(t_j) = D_s Y^k(s) - D_{t_j} Y^k(t_j)
\end{equation}
(4.57)
\begin{equation}
= (D_s Y^k(s) - D_{t_j} Y^k(s)) + (D_{t_j} Y^k(s) - D_{t_j} Y^k(t_j)),
\end{equation}
and
\begin{equation}
\mathbb{E} |D_s Y^k(s) - D_{t_j} Y^k(t_j)|^2 \leq K|\pi|.
\end{equation}
(4.58)

Step 2. We claim that, for any $s \in [t_j, t_{j+1}]$, there exists a constant $K$, such that
\begin{equation}
\mathbb{E} |D_{t_j} Y^k(s) - D_{t_j} Y^k(t_j)| \leq K|s - t_j|.
\end{equation}
(4.59)
In order to do this, for any $\theta \leq t_j$, applying Itô’s formula to $\Psi_k(\cdot)D_\theta Y^k(\cdot)$, we obtain
\begin{equation}
\Psi_k(t)D_\theta Y^k(t) = \Psi_k(T)D_\theta Y^k(T) + \int_t^T \Psi_k(s)f_s D_\theta x(s)ds
\end{equation}
\begin{equation}
+ \int_t^T \Psi_k(s)(f_s D_\theta Y^k(s) + D_\theta Z^k(s))dW(s),
\end{equation}
(4.60)
t \in [t_j, t_{j+1}], j \geq k.
Since $\Psi_k(\cdot)\Phi_k(\cdot) = I_n$, one can get
\begin{equation}
D_\theta Y^k(t) = \mathbb{E}
\left(\Phi_k(t)\Psi_k(T)D_\theta Y^k(T) + \Phi_k(t)\int_t^T \Psi_k(s)f_s D_\theta x(s)ds|\mathcal{F}_t\right).
\end{equation}

Then
\begin{equation}
D_\theta Y^k(t) - D_\theta Y^k(t_j)
\end{equation}
\begin{equation}
= \mathbb{E}(\Phi_k(t)\Psi_k(T)D_\theta Y^k(T)|\mathcal{F}_t) - \mathbb{E}(\Phi_k(t_j)\Psi_k(T)D_\theta Y^k(T)|\mathcal{F}_t)
\end{equation}
\begin{equation}
+ \mathbb{E}(\Phi_k(t)\int_t^T \Psi_k(s)f_s D_\theta x(s)ds|\mathcal{F}_t) - \mathbb{E}(\Phi_k(t_j)\int_{t_j}^T \Psi_k(s)f_s D_\theta x(s)ds|\mathcal{F}_{t_j})
\end{equation}
\begin{equation}
:= I_1 + I_2.
\end{equation}

Now, we estimate $I_1$ and $I_2$, respectively. $I_1$ can be written as
\begin{equation}
I_1 = \mathbb{E}
\left((\Phi_k(t) - \Phi(t_j))\Psi_k(T)D_\theta Y^k(T)|\mathcal{F}_t\right)
\end{equation}
\begin{equation}
+ \mathbb{E}(\Phi(t_j)\Psi_k(T)D_\theta Y^k(T)|\mathcal{F}_t) - \mathbb{E}(\Phi(t_j)\Psi_k(T)D_\theta Y^k(T)|\mathcal{F}_{t_j})
\end{equation}
\begin{equation}
:= I_{11} + I_{12}.
\end{equation}
By (4.49), a direct calculate leads to
\begin{equation}
\mathbb{E}|I_{11}|^2 \leq \left(\mathbb{E}|(\Phi_k(t) - \Phi(t_j))\Psi_k(T)|^4\mathbb{E}|D_\theta Y^k(T)|^4\right)^{1/2} \leq C|t - t_j|.
\end{equation}
(4.62)
Meanwhile, by Clark-Ocone representation formula,
\begin{equation}
\Phi(t_j)\Psi_k(T)D_\theta Y^k(T) = \mathbb{E}(\Phi(t_j)\Psi_k(T)D_\theta Y^k(T)) + \int_0^T u_\theta(s)dW(s),
\end{equation}
(4.63)
Thus, it is easy to check that
\[
\mathbb{E}|u_\theta(s)|^2 \leq C \left\{ \left( \mathbb{E}|D_\theta \Psi_k(T)|^4 \mathbb{E}|D_\theta Y^k(T)|^4 \right)^{1/2} + \left( \mathbb{E}|\Phi_k(T)|^4 \mathbb{E}|D_\theta Y^k(T)|^4 \right)^{1/2} \right\}
\leq C < \infty.
\]

Thus,
\[
\mathbb{E}|I_{12}|^2 = \mathbb{E} \left| \int_{t_j}^{t} u_\theta(s) dW(s) \right|^2 = \mathbb{E} \int_{t_j}^{t} |u_\theta(s)|^2 ds \leq C|t - t_j|.
\]

For \( I_2 \), we can rewrite it as follows:
\[
I_2 = \mathbb{E} \left( \int_{t_j}^{t} (\Phi_k(t) - \Phi_k(t_j)) \Psi_k(s) f_x D_\theta x(s) ds \big| \mathcal{F}_t \right) + \mathbb{E} \left( \Phi_k(t_j) \left( \int_{t_j}^{T} \Psi_k(s) f_x D_\theta x(s) ds - \int_{t_j}^{T} \Psi_k(s) f_x D_\theta x(s) ds \right) \big| \mathcal{F}_t \right)
+ \mathbb{E} \left( \int_{t_j}^{T} \Phi_k(t_j) \Psi_k(s) f_x D_\theta x(s) ds \big| \mathcal{F}_t \right) - \mathbb{E} \left( \int_{t_j}^{T} \Phi_k(t_j) \Psi_k(s) f_x D_\theta x(s) ds \big| \mathcal{F}_t \right)
:= I_{21} + I_{22} + I_{23}.
\]

It is easy to check that
\[
\mathbb{E}I_{21}^2 \leq \left( \mathbb{E} \int_{t_j}^{T} |(\Phi_k(t) - \Phi_k(t_j)) \Psi_k(s)|^4 ds \right)^{1/2} \left( \mathbb{E} \int_{t_j}^{T} |f_x D_\theta x(s)|^4 ds \right)^{1/2}
\leq K(t - t_j) \left( \sup_{0 \leq s \leq T} \mathbb{E}|D_\theta x(s)|^4 \right)^{1/2}
\leq K|t - t_j|,
\]
and
\[
\mathbb{E}I_{22}^2 \leq |t - t_j| \mathbb{E} \int_{t_j}^{T} |\Phi_k(t_j) \Psi_k(s)|^2 |f_x D_\theta x(s)|^2 ds
\leq K(t - t_j) \left( \mathbb{E} \int_{t_j}^{T} |\Phi_k(t_j) \Psi_k(s)|^4 ds \right)^{1/2} \left( \mathbb{E} \int_{t_j}^{T} |f_x D_\theta x(s)|^4 ds \right)^{1/2}
\leq K|t - t_j|.
\]

Now, we are in the step to estimate \( I_{23} \). By Clark-Ocone representation formula,
\[
\int_{t_j}^{T} \Phi_k(t_j) \Psi_k(s) f_x D_\theta x(s) ds = \mathbb{E} \int_{t_j}^{T} \Phi_k(t_j) \Psi_k(s) f_x D_\theta x(s) ds + \int_{0}^{T} v_\theta(u) dW(u),
\]
where
\[
v_\theta(u) = \mathbb{E} \left( D_u \int_{t_j}^{T} \Phi_k(t_j) \Psi_k(s) f_x D_\theta x(s) ds \big| \mathcal{F}_u \right)
\]
\[
= \mathbb{E} \left( \int_{t_j}^{T} \Phi_k(t_j) D_u \Psi_k(s) f_x D_\theta x(s) ds \big| \mathcal{F}_u \right) + \mathbb{E} \left( \int_{t_j}^{T} \Phi_k(t_j) D_u \Psi_k(s) D_u (f_x D_\theta x(s)) ds \big| \mathcal{F}_u \right)
:= V_1 + V_2.
\]
For $V_1$, by (4.50), it is easy to check that

$$ (4.68) \quad \mathbb{E}|V_1|^2 = \left( \mathbb{E} \int_{t_j}^{T} |\Phi_k(t_j)D_u\Psi_k(s)|^4 ds \mathbb{E} \int_{t_j}^{T} |f_xD_\theta x(s)|^4 ds \right)^{1/2} \leq K. $$

For $V_2$,

$$ (4.69) \quad \mathbb{E}|V_2|^2 \leq \mathbb{E} \left| \int_{t_j}^{T} \Phi_k(t_j)\Psi_k(s)D_u(f_xD_\theta x(s)) ds \right|^2 \leq K \mathbb{E} \left( \int_{t_j}^{T} |\Phi_k(t_j)\Psi_k(s)| \times \left( |D_\theta x(s)D_u x(s)| + |D_\lambda x(s)D_u z_k(s)| + |D_u D_\lambda x(s)| \right) ds \right)^2. $$

We estimate the right side of (4.69) term by term. By Lemma 2.2 and Hölder’s inequality,

$$ (4.70) \quad \mathbb{E} \left( \int_{t_j}^{T} |\Phi_k(t_j)\Psi_k(s)| |D_\lambda x(s)D_u x(s)| ds \right)^2 \leq \mathbb{E} \left\{ \sup_{0 \leq s \leq T} |\Phi_k(t_j)\Psi_k(s)|^2 \left( \int_{t_j}^{T} (|D_\lambda x(s)|^2 + |D_u x(s)|^2) ds \right)^2 \right\} \leq K \left( \mathbb{E} \sup_{0 \leq s \leq T} |\Phi_k(t_j)\Psi_k(s)| \frac{2 \rho_0}{\rho_0 - 1} \left( \mathbb{E} \int_{t_j}^{T} |D_\lambda x(s)|^{2 \rho_0} + |D_u x(s)|^{2 \rho_0} ds \right)^{\frac{1}{\rho_0}} \right) \leq K < \infty. $$

Similarly,

$$ (4.71) \quad \mathbb{E} \left( \int_{t_j}^{T} |\Phi_k(t_j)\Psi_k(s)||D_u D_\lambda x(s)| ds \right)^2 \leq \mathbb{E} \left\{ \sup_{0 \leq s \leq T} |\Phi_k(t_j)\Psi_k(s)|^2 \left( \int_{t_j}^{T} |D_u D_\lambda x(s)| ds \right)^2 \right\} \leq \left( \mathbb{E} \sup_{0 \leq s \leq T} |\Phi_k(t_j)\Psi_k(s)| \frac{2 \rho_0}{\rho_0 - 2} \left( \mathbb{E} \int_{t_j}^{T} |D_u D_\lambda x(s)|^{\rho_0} ds \right)^{\frac{1}{\rho_0}} \right) \leq K < \infty. $$

Now, we estimate the left terms in the right side of (4.69). For any $k = 0, 1, \cdots, N - 1$, applying Lemma 2.1, one can get

$$ \mathbb{E} \left( \int_{t_k}^{T} |D_\theta z_k(s)|^2 ds \right)^{\rho_0} \leq K \left\{ \mathbb{E} |D_\theta Y^k(T)|^{2 \rho_0} + \mathbb{E} \int_{t}^{t+1} |f_xD_\theta x(s)|^{2 \rho_0} ds \right\} \leq K < \infty. $$

Therefore,

$$ (4.72) \quad \mathbb{E} \left( \int_{t_j}^{T} |\Phi_k(t_j)\Psi_k(s)||D_\theta x(s)D_u z_k(s)| ds \right)^2 \leq \mathbb{E} \left\{ \sup_{0 \leq s \leq T} |\Phi_k(t_j)\Psi_k(s)|^2 \left( \sup_{0 \leq s \leq T} |D_\theta x(s)|^4 + \left( \int_{t_j}^{T} |D_u z_k(s)|^2 ds \right)^2 \right) \right\}. $$

25
\[ \leq \left( \mathbb{E} \sup_{0 \leq s \leq T} |\Phi_k(t_j)\Psi_k(s)|^{\frac{2\rho_0}{\rho_0-2}} \right)^{\frac{\rho_0-2}{\rho_0}} \times \left\{ \left( \mathbb{E} \sup_{0 \leq s \leq T} |D_\theta x(s)|^{2\rho_0} \right)^{\frac{2}{\rho_0}} + \left[ \mathbb{E} \left( \int_{t_j}^T |D_u Z^k(s)|^2 ds \right)^{\frac{2}{p_0}} \right]^{\frac{2}{p_0}} \right\} \]

\[ \leq K < \infty. \]

Hence, (4.67), together with (4.68)–(4.72), yields that

\[ (4.73) \quad \mathbb{E} |I_{23}|^2 = \mathbb{E} \int_{t_j}^t v_\theta(u) dW(u) \leq \mathbb{E} \int_{t_j}^t |v_\theta(u)|^2 du \leq K |t - t_j|. \]

Finally, by (4.60)–(4.66) and (4.73), one gets

\[ \mathbb{E} |D_\theta Y_k(t) - D_\theta Y_k(t_j)|^2 \leq K |t - t_j|, \]

which deduces (4.59) by setting \( \theta = t_j \). Now combining (4.57) with (4.58) and (4.59), we have the regularity of \( Z \) (4.44).

**Remark 4.10.** From the proof of Lemma 4.7 and Lemma 4.8, we can see that when \( f = f(t, s, x, y) \) in BSDE (4.2), we only need \( p_0 = 2 \) in assumption (A4); but when \( f = f(t, s, x, z) \), \( p_0 > 2 \) is needed.

### 4.2 Proof of Theorem 4.3

In this part, we prove our main result Theorem 4.3. Firstly, we need the following lemma on conditional expectation. One can refer to [15] for proof.

**Lemma 4.11.** For any \( \varphi(\cdot) \in L^2_{F_t}(\Omega \times (0, T); \mathbb{R}^n) \) and \( 0 \leq s < t \leq T \), write

\[ \varphi_0 = \frac{1}{t - s} \mathbb{E} \left( \int_s^t \varphi(\tau) d\tau \bigg| F_s \right). \]

Then for any \( \xi \in L^2_{F_s}(\Omega; \mathbb{R}^n) \), it holds that

\[ \mathbb{E} \int_s^t |\varphi(\tau) - \varphi_0|^2 d\tau \leq \mathbb{E} \int_s^t |\varphi(\tau) - \xi|^2 d\tau. \]

The following lemma is on the relation between \( (Y^{\pi(\cdot)}(\cdot), Z^{\pi(\cdot)}(\cdot)) \) and \( (Y^{\pi(\cdot), \pi(\cdot)}, Z^{\pi(\cdot), \pi(\cdot)}) \).

**Lemma 4.12.** Let (A1)–(A4) hold. Then, for any \( k = 0, 1, \cdots, N - 1 \),

\[ (4.74) \quad \sup_{k \leq j \leq N} \mathbb{E} |Y^k(t_j) - Y^{k,\pi}(t_j)|^2 + \mathbb{E} \int_{t_k}^T |Z^k(s) - Z^{k,\pi}(\tau(s))|^2 ds \leq K |\pi|, \]

where \( K \) is a constant depending only on \( L \) and \( T \).
Proof. We split the proof into three steps.

**Step 1.** For any \( k = 0, 1, \ldots, N - 1 \) and \( k \leq j \leq N - 1 \), denote

\[
I_{k,j} = \sup_{t_j \leq s \leq t_{j+1}} \mathbb{E}[Y^k(t) - Y^{k,\pi}(t)]^2 + \frac{1}{2} \mathbb{E} \int_{t_j}^{t_{j+1}} |Z^k(s) - \hat{Z}^{k,\pi}(s)|^2 ds;
\]

\[
I_{k,N} = \mathbb{E}[g(t_k, x(T)) - g(t_k, x^\pi(T))]^2.
\]

By Eq. (4.2) and (4.3), for \( j \geq k \), we have

\[
\begin{align*}
(Y^k(t_j) - Y^{k,\pi}(t_j)) + \int_{t_j}^{t_{j+1}} (Z^k(s) - \hat{Z}^{k,\pi}(s))dW(s) \\
= & (Y^k(t_{j+1}) - Y^{k,\pi}(t_{j+1})) \\
& + \int_{t_j}^{t_{j+1}} (f(t_k, s, x(s), Y^j(s), Z^k(s)) - f(t_k, t_j, x^\pi(t_j), Y^{j,\pi}(t_{j+1}), Z_0^{k,\pi}(t_j)))ds.
\end{align*}
\]

Squaring and taking expectation on both sides of the above equation, we obtain

\[
\begin{align*}
\mathbb{E}[Y^k(t_j) - Y^{k,\pi}(t_j)]^2 + & \mathbb{E} \int_{t_j}^{t_{j+1}} |Z^k(s) - \hat{Z}^{k,\pi}(s)|^2 ds \\
\leq & \left(1 + \frac{8\Delta}{\varepsilon}\right) \mathbb{E}[Y^k(t_{j+1}) - Y^{k,\pi}(t_{j+1})]^2 \\
& + \left(8 + \frac{\varepsilon}{\Delta}\right)L^2 \bigg\{ \mathbb{E} \left[ \int_{t_j}^{t_{j+1}} \sqrt{s - t_j} ds \right]^2 \\
& + \mathbb{E} \left[ \int_{t_j}^{t_{j+1}} x(s) - x(t_j) ds \right]^2 \\
& + \mathbb{E} \left[ \int_{t_j}^{t_{j+1}} Y^j(s) - Y^{j,\pi}(t_{j+1}) ds \right]^2 \\
& + \mathbb{E} \left[ \int_{t_j}^{t_{j+1}} \frac{1}{\Delta} \mathbb{E} \left[ \int_{t_j}^{t_{j+1}} (Z^k(t_j) - Z^k(\tau)) d\tau | \mathcal{F}_{t_j} \right] ds \right]^2 \\
& + \mathbb{E} \left[ \int_{t_j}^{t_{j+1}} \frac{1}{\Delta} \mathbb{E} \left[ \int_{t_j}^{t_{j+1}} (Z^k(\tau) - \hat{Z}^{k,\pi}(\tau)) d\tau | \mathcal{F}_{t_j} \right] ds \right]^2 \bigg\} \\
\leq & \left(1 + \frac{8\Delta}{\varepsilon}\right) \mathbb{E}[Y^k(t_{j+1}) - Y^{k,\pi}(t_{j+1})]^2 \\
& + \left(8 + \frac{\varepsilon}{\Delta}\right)L^2 \bigg\{ K|\pi|^3 + |\pi|^2 \mathbb{E}[Y^j(t_{j+1}) - Y^{j,\pi}(t_{j+1})]^2 \\
& + |\pi| \mathbb{E} \int_{t_j}^{t_{j+1}} |Z^k(s) - \hat{Z}^{k,\pi}(s)|^2 ds \bigg\}.
\end{align*}
\]

Now, choosing \( \varepsilon = \frac{1}{2L^2} \), then for \( |\pi| \leq \frac{1}{16L^2} \), one gets

\[
\begin{align*}
\mathbb{E}[Y^k(t_j) - Y^{k,\pi}(t_j)]^2 + & \frac{1}{2} \mathbb{E} \int_{t_j}^{t_{j+1}} |Z^k(s) - \hat{Z}^{k,\pi}(s)|^2 ds \\
\leq & \left(1 + 16L^2|\pi|\right) \mathbb{E}[Y^k(t_{j+1}) - Y^{k,\pi}(t_{j+1})]^2 \\
& + \left(8|\pi| + \frac{1}{2L^2}\right)L^2 \bigg\{ K|\pi|^3 + |\pi| \mathbb{E}[Y^j(t_{j+1}) - Y^{j,\pi}(t_{j+1})]^2 \bigg\}.
\end{align*}
\]
In the above inequality, we use Lemma 2.2, Theorem 3.1 and Lemma 4.5. For simplicity, denote \( b = 1 + 16L^2|\pi| \), \( c = b|\pi| \). Then, by induction, we can get

\[
I_{k,j} \leq bI_{k,j+1} + cI_{j,j+1} + cK|\pi|
\]

(4.78)

\[
\leq b^{N-j}I_{k,N} + \sum_{l=0}^{N-j-1} b^{N-j-l-1}c(b+c)^l I_{j+l,N} + cK|\pi| \sum_{l=0}^{N-j-1} (b+c)^l.
\]

For any \( k \), by (A2) and Theorem 3.1,

\[
I_{k,N} = E|g(t_k, x(T)) - g(t_k, x^\pi(T))|^2 \leq K E|x(T) - x^\pi(T)|^2 \leq K|\pi|.
\]

(4.79)

Also, it is easy to check that,

\[
\sum_{l=0}^{N-j-1} b^{N-j-l-1}c(b+c)^l = b^{N-j-1}c \sum_{l=0}^{N-j-1} \frac{(b+c)^l}{b^l} \leq b^N \left( 1 + \frac{c}{b} \right)^N \leq e^{16L^2T}cT,
\]

(4.80)

and

\[
cK|\pi| \sum_{l=0}^{N-j-1} (b+c)^l = cK|\pi| \frac{(b+c)^{N-j} - 1}{b+c - 1} = Kb|\pi|^2 \frac{(b+b|\pi|)^{N-j} - 1}{b+b|\pi| - 1}.
\]

(4.81)

Since

\[
b + b|\pi| = 1 + (1 + 16L^2)|\pi| + 16L^2|\pi|^2 \leq 1 + (2 + 32L^2)|\pi|,
\]

and \( (b + b|\pi|) - 1 \geq 16L^2|\pi| \), (4.81) turns into

\[
cK|\pi| \sum_{l=0}^{N-j-1} (b+c)^l \leq K(1 + 16L^2|\pi|)|\pi|^2 \frac{(1 + (2 + 32L^2)|\pi|)^N}{16L^2|\pi|}
\]

\[
\leq \frac{K}{16L^2|\pi|}(1 + 16L^2|\pi|)e^{(2+32L^2)T} \leq K|\pi|.
\]

Hence, (4.78), together with (4.79), (4.80) and (4.82), yields that

\[
E|Y^k(t_j) - Y^{k,\pi}(t_j)|^2 \leq K|\pi|.
\]

(4.83)

That is the first part of (4.74).

**Step 2.** Now, we estimate \( E\int_{t_k}^{T} |Z^k(s) - Z^{k,\pi}(\tau(s))|^2 ds \). By (4.77), summing from \( j = k \) to \( N - 1 \) leads to

\[
\sum_{j=k}^{N-1} E|Y^k(t_j) - Y^{k,\pi}(t_j)|^2 + \frac{1}{2} E\int_{t_k}^{t_N} |Z^k(s) - \hat{Z}^k,\pi(s)|^2 ds
\]

\[
\leq (1 + 16L^2|\pi|) \sum_{j=k}^{N-1} E|Y^k(t_j+1) - Y^{k,\pi}(t_{j+1})|^2
\]

\[
+ (1 + 16L^2|\pi|)|\pi| \sum_{j=k}^{N-1} \{ K|\pi| + E|Y^j(t_{j+1}) - Y^{j,\pi}(t_{j+1})|^2 \}.
\]
Hence, by (4.83),
\[
\mathbb{E} \int_{t_k}^{T} |Z^k(s) - \hat{Z}^{k,\pi}(s)|^2 ds \\
\leq 32L^2|\pi| \sum_{j=k}^{N-1} \mathbb{E}|Y^k(t_{j+1}) - Y^{k,\pi}(t_{j+1})|^2 + 2\mathbb{E}|Y^k(t_N) - Y^{k,\pi}(t_N)|^2 \\
- 2\mathbb{E}|Y^k(t_k) - Y^{k,\pi}(t_k)|^2 + (2 + 32L^2|\pi|) \sum_{j=k}^{N-1} \{ K|\pi| + \mathbb{E}|Y^j(t_{j+1}) - Y^{j,\pi}(t_{j+1})|^2 \}
\]
(4.84)
\[
\leq K|\pi|.
\]

**Step 3.** For any \(k = 0, 1, \cdots, N - 1\), and \(k \leq j \leq N - 1\), denote
\[
\hat{Z}^k(t_j) = \frac{1}{\Delta_j} \mathbb{E}\left( \int_{t_j}^{t_{j+1}} Z^k(s) ds | F_{t_j} \right).
\]
Then, by Lemma 4.7, Lemma 4.8 and Lemma 4.11 and (4.84), a direct calculation leads to
\[
\mathbb{E} \int_{t_k}^{T} |Z^k(s) - Z^{k,\pi}(\tau(s))|^2 ds = \sum_{j=k}^{N-1} \mathbb{E} \int_{t_j}^{t_{j+1}} |Z^k(s) - Z^{k,\pi}(\tau(s))|^2 ds
\]
\[
\leq 2 \sum_{j=k}^{N-1} \mathbb{E} \int_{t_j}^{t_{j+1}} (|Z^k(s) - \hat{Z}^k(t_j)|^2 + |\hat{Z}^k(t_j) - Z^{k,\pi}(t_j)|^2) ds
\]
\[
\leq 2 \sum_{j=k}^{N-1} \mathbb{E} \int_{t_j}^{t_{j+1}} \left( |Z^k(s) - Z^k(t_j)|^2 + \left| \frac{1}{\Delta_j} \mathbb{E}\left( \int_{t_j}^{t_{j+1}} Z^k(u) - \hat{Z}^{k,\pi}(u) du | F_{t_j} \right) \right|^2 \right) ds
\]
\[
\leq K|\pi| + 2 \sum_{j=k}^{N-1} \mathbb{E} \int_{t_j}^{t_{j+1}} |Z^k(u) - \hat{Z}^{k,\pi}(u)|^2 du
\]
\[
\leq K|\pi|.
\]
That completes the proof.

Now, we are in the step to prove Theorem 4.3.

**Proof of Theorem 4.3.** By Lemma 4.12, we can see that \(\sup_{0 \leq t \leq T} \mathbb{E}|Y(\tau(t)) - Y^{\pi(t),\pi}(\tau(t))|^2 \leq K|\pi|\) is true.

For the second term, \(\mathbb{E} \int_0^T \int_t^T |Z(t,s) - Z^{\pi(t),\pi}(\tau(s))|^2 ds dt\), on the left side of (4.5), It is easy to check that
\[
\mathbb{E} \int_0^T \int_t^T |Z(t,s) - Z^{\pi(t),\pi}(\tau(s))|^2 ds dt
\]
(4.85)
\[
\leq \mathbb{E} \int_0^T \int_t^T |(Z(t,s) - Z(\tau(t),s)) + (Z(\tau(t),s) - Z^{\pi(t),\pi}(s)) + (Z^{\pi(t),\pi}(s) - Z^{\pi(t),\pi}(\tau(s)))|^2 ds dt
\]
\begin{align*}
\leq & 3E \int_0^T \int_t^T |Z(t, s) - Z(\tau(t), s)|^2 ds dt + 3E \int_0^T \int_{\tau(t)}^T |Z(\tau(t), s) - \hat{Z}^{\pi(t), \pi(s)}|^2 ds dt \\
& + 3E \int_0^T \int_{\tau(t)}^T |\hat{Z}^{\pi(t), \pi(s)} - Z^{\pi(t), \pi(\tau(s))}|^2 ds dt.
\end{align*}

By Lemma 2.4, one has
\begin{equation}
E \int_0^T \int_t^T |Z(t, s) - Z(\tau(t), s)|^2 ds dt \leq C \int_0^T (t - \tau(t)) dt \leq C|\pi|.
\end{equation}

For the third term on the right side of (4.85), by (4.84) and (4.2), one has
\begin{equation}
\begin{align*}
& \leq 2E \int_0^T \int_{\tau(t)}^T |\hat{Z}^{\pi(t), \pi(s)} - Z^{\pi(t), \pi(\tau(s))}|^2 ds dt \\
& \leq 2E \int_0^T \int_{\tau(t)}^T |\hat{Z}^{\pi(t), \pi(s)} - Z^{\pi(t), \pi(\tau(s))}|^2 ds dt + 2E \int_0^T \int_{\tau(t)}^T |Z^{\pi(t), \pi(\tau(s))}|^2 ds dt \\
& \leq K|\pi|.
\end{align*}
\end{equation}

Now, we estimate \( E \int_0^T \int_{\tau(t)}^T |Z(\tau(t), s) - \hat{Z}^{\pi(t), \pi(s)}|^2 ds dt \). By Eq. (1.1) and (4.3), for any \( k = 0, 1, \ldots, N - 1 \), one can easily calculate
\begin{equation}
\begin{align*}
& E|Y(t_k) - Y^{k, \pi(t_k)}|^2 + E \int_{t_k}^T |Z(t_k, s) - \hat{Z}^{k, \pi(s)}|^2 ds \\
& \leq \left| g(t_k, x(T)) - g(t_k, x^\pi(T)) \right|^2 \\
& + \sum_{l=k}^{N-1} \left( f(t_k, s, x(s), Y(s), Z(t_k, s)) - f(t_k, t_l, x^\pi(t_l), Y^l, \hat{Z}^{k, \pi(t_l)}) \right)^2 ds dt \\
& \leq 2L^2 E|x(T) - x^\pi(T)|^2 \\
& + 2N|\pi| \sum_{l=k}^{N-1} E \int_{t_l}^{t_{l+1}} |f(t_k, s, x(s), Y(s), Z(t_k, s)) - f(t_k, t_l, x^\pi(t_l), Y^l, \hat{Z}^{k, \pi(t_l)})|^2 ds dt \\
& \leq K|\pi| + K \sum_{l=k}^{N-1} E \int_{t_l}^{t_{l+1}} \left( |s - t_l| + |x(s) - x(t_l)|^2 + |x(t_l) - x^\pi(t_l)|^2 \\
& + |Y(s) - Y^l(s)|^2 + |Y^l(s) - Y^l(t_{l+1})|^2 + |Y^l(t_{l+1}) - Y^l, \hat{Z}^{k, \pi(t_l)}|^2 \\
& + |Z(t_k, s) - Z^k(s)|^2 + |Z^k(s) - Z^k, \hat{Z}^{k, \pi(t_l)}|^2 \right) ds dt \\
& \leq K|\pi|.
\end{align*}
\end{equation}

Here, we use Theorem 4.2, Lemma 4.5, Lemma 4.12, Proposition 4.6 and (4.2). Now, (4.85), together with (4.86)–(4.88), yields that
\begin{equation}
E \int_0^T \int_{t_l}^{t_{l+1}} |Z(t, s) - Z^{\pi(t), \pi(\tau(s))}|^2 ds dt \leq K|\pi|.
\end{equation}

That completes the proof of the convergent speed of the Euler method for BSVIE (1.1).
Figure 1: \((Y(t), Z(t, s))\) and its approximation \((Y^{\pi(t)}, Z^{\pi(t)})\). For \(Z(t, s)\) and its approximation \(Z^{\pi(t)}\), we choose \(t = 0.1, 0.2, 0.3\) for one sample path \(\omega \in \Omega\).

5 A Numerical example

In this section, we mainly present a numerical example. Consider the following BSVIE:

\[(5.1) \quad Y(t) = t \sin(W(1)) \int_{t}^{1} \frac{t}{2} \sin(W(s))ds - \int_{t}^{1} Z(t, s)dW(s), \; t \in [0, 1],\]

with \(T = d = n = 1\), which admits a unique solution \((t \sin(W(t)), t \cos(W(s)))\).

In Figure 1, choosing \(N = 100\) (i.e. \(|\pi| = 0.01\)), we simulate true solution \((Y(t), Z(t, s))\) (in red) and its approximation \((Y^{\pi(t)}, Z^{\pi(t)})\) (in blue). For the \(Z\) part, we take three cases: \(t = 0.1, 0.2, 0.3\) and one sample path \(\omega \in \Omega\).

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