SL(2,R) INVARIANCE OF NON-LINEAR ELECTRODYNAMICS COUPLED TO AN AXION AND A DILATON

G W GIBBONS
&
D A RASHEED*

D.A.M.T.P.
University of Cambridge
Silver Street
Cambridge CB3 9EW
U.K.

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Abstract

The most general Lagrangian for non-linear electrodynamics coupled to an axion $a$ and a dilaton $\phi$ with $SL(2,\mathbb{R})$ invariant equations of motion is

$$-\frac{1}{2} (\nabla \phi)^2 - \frac{1}{2} e^{2\phi} (\nabla a)^2 + \frac{1}{4} a F_{\mu\nu} \star F^{\mu\nu} + L_{\text{inv}}(g_{\mu\nu}, e^{-\frac{1}{2}\phi}F_{\rho\sigma})$$

where $L_{\text{inv}}(g_{\mu\nu}, F_{\rho\sigma})$ is a Lagrangian whose equations of motion are invariant under electric-magnetic duality rotations. In particular there is a unique generalization of Born-Infeld theory admitting $SL(2,\mathbb{R})$ invariant equations of motion.

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1 Introduction

In a recent paper [1] we found the condition on the Lagrangian function $L_{\text{inv}}(g_{\mu\nu}, F_{\rho\sigma})$ for a non-linear electrodynamic theory coupled to gravity that the equations of motion, including the Einstein equations, are invariant under the action of an $SO(2)$ group of generalized electric-magnetic duality rotations. One such Lagrangian is the Born-Infeld Lagrangian [2]

$$L_{\text{BI}} = 1 - \sqrt{1 + \frac{1}{2} F^2 - \frac{1}{16} (F \star F)^2}. \quad (1.1)$$

Note that $L_{\text{inv}}(g_{\mu\nu}, F_{\rho\sigma})$ itself is not invariant under duality rotations.

In this letter we shall extend these results by including a coupling to a scalar dilaton field $\phi$ and a pseudo-scalar axion field $a$. They contribute to the action the following kinetic terms

$$L_{\text{ax-dil}} = -\frac{1}{2} (\nabla \phi)^2 - \frac{1}{2} e^{2\phi} (\nabla a)^2 \quad (1.2)$$

which are $SL(2,\mathbb{R})$ invariant. We shall show that this $SL(2,\mathbb{R})$ invariance may be extended to the equations of motion (but not the action) if and only if the action takes the form

$$\int d^4x \sqrt{g} \left\{ R - 2\Lambda - \frac{1}{2} (\nabla \phi)^2 - \frac{1}{2} e^{2\phi} (\nabla a)^2 + \frac{1}{4} a F \star F + L_{\text{inv}}(g_{\mu\nu}, e^{-\frac{\phi}{2}} F_{\rho\sigma}) \right\} \quad (1.3)$$

where $L_{\text{inv}}(g_{\mu\nu}, F_{\rho\sigma})$ is a Lagrangian with $SO(2)$ invariant equations of motion. In particular there is just one generalization of the Born-Infeld Lagrangian admitting $SL(2,\mathbb{R})$ invariant equations of motion. This $SL(2,\mathbb{R})$ invariant generalization of the Born-Infeld Lagrangian does not coincide with that discussed in [1] in connection with string theory. The relation of our new results to string theory is currently under investigation.

2 SL(2,R) Duality at Lowest Order

In 4 dimensions, the bosonic sector of N=4 supergravity and string theory compactified on a torus, at lowest order, may be described by the following Lagrangian [3]

$$L = R - \frac{1}{2} (\nabla \phi)^2 - \frac{1}{2} e^{2\phi} (\nabla a)^2 + \frac{1}{4} a F_{\mu\nu} \star F^{\mu\nu} + \frac{1}{4} e^{-\phi} F_{\mu\nu} F^{\mu\nu} \quad (2.1)$$
where for simplicity we consider only a single $U(1)$ gauge field. The resulting theory admits an $SL(2, \mathbb{R})$ electric-magnetic duality which mixes the electromagnetic field equations with the Bianchi identities and also transforms the axion and dilaton.

In a local orthonormal frame, the electric intensity $E$ and magnetic induction $B$ may be defined by $E_i = F_{i0}$ and $B_i = \frac{1}{2} \epsilon_{ijk} F_{jk}$. The Bianchi identities $dF = 0$ or $\partial_{[\alpha} F_{\beta\gamma]} = 0$ are then equivalent to

\[ \nabla \cdot B = 0 \]  
\[ \nabla \times E = -\frac{\partial B}{\partial t}. \]  
\[ \nabla \cdot D = 0 \]  
\[ \nabla \times H = +\frac{\partial D}{\partial t}, \]  

where the electric induction $D$ and magnetic intensity $H$ are defined by $D_i = G_{i0}$ and $H_i = \frac{1}{2} \epsilon_{ijk} G_{jk}$. For the Lagrangian (2.1), $D$ and $H$ are given by

\[ D = +\frac{\partial L}{\partial E} = e^{-\phi} E + aB \]  
\[ H = -\frac{\partial L}{\partial B} = e^{-\phi} B - aE. \]  

\[ ^{1} \text{In this paper we use units in which } \mu_0 = \varepsilon_0 = \hbar = c = 16\pi G = 1. \]  
\[ ^{2} \text{There is some ambiguity in the definition of this partial derivative depending on whether or not one takes into account the antisymmetry of } F_{\mu \nu}. \text{ Here we treat } F_{\mu \nu} \text{ and } F_{\nu \mu} \text{ as independent variables, hence the factor of } 2. \]
These are the constitutive relations and may be rewritten as

\[
\begin{pmatrix}
\mathbf{E} \\
\mathbf{H}
\end{pmatrix} = \underbrace{\begin{pmatrix}
e^\phi & -ae^\phi \\
-ae^\phi & e^{-\phi} + a^2e^\phi
\end{pmatrix}}_{\mathcal{M}} \begin{pmatrix}
\mathbf{D} \\
\mathbf{B}
\end{pmatrix}.
\]

(2.6)

It is convenient to define a complex scalar field \( \lambda \) by

\[
\lambda = a + ie^{-\phi}
\]

(2.7)

and a complex 2-component vector \( \psi \) by

\[
\psi = \begin{pmatrix} 1 \\ -\lambda \end{pmatrix}.
\]

(2.8)

Then the matrix \( \mathcal{M} \) may be written as

\[
\mathcal{M} = \frac{\psi\psi^\dagger + \text{c.c.}}{\sqrt{\det(\psi\psi^\dagger + \text{c.c.})}}.
\]

(2.9)

Chosing the first component of \( \psi \) to be 1 fixes the representation of \( \mathcal{M} \). The \( SL(2,\mathbb{R}) \) duality transformation may then be constructed so that it automatically leaves the constitutive relations invariant:

\[
\psi \rightarrow \psi' \propto (S^T)^{-1}\psi \quad \Rightarrow \quad \mathcal{M} \rightarrow (S^T)^{-1}\mathcal{M}S^{-1}
\]

(2.10)

\[
\begin{pmatrix}
\mathbf{D} \\
\mathbf{B}
\end{pmatrix} \rightarrow S \begin{pmatrix}
\mathbf{D} \\
\mathbf{B}
\end{pmatrix}, \quad \begin{pmatrix}
\mathbf{E} \\
\mathbf{H}
\end{pmatrix} \rightarrow (S^T)^{-1}\begin{pmatrix}
\mathbf{E} \\
\mathbf{H}
\end{pmatrix},
\]

where \( S \in SL(2,\mathbb{R}) \). If

\[
S = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \quad \text{where } ps - qr = 1,
\]

(2.11)

then the induced transformations of the axion and dilaton fields are given by a Mobius transformation of \( \lambda \):

\[
\lambda \rightarrow \frac{p\lambda + q}{r\lambda + s}.
\]

(2.12)
It is easy to check that the axion and dilaton equations of motion are invariant under these transformations and also so is the energy momentum tensor, so it is consistent to assume that the metric is unchanged under the action of this duality.

In the covariant notation, the transformations of $F$ and $G$ are given by

$$
\begin{align*}
F_{\mu\nu} &\rightarrow sF_{\mu\nu} + r*G_{\mu\nu} \\
G_{\mu\nu} &\rightarrow pG_{\mu\nu} - q*F_{\mu\nu}.
\end{align*}
$$

(2.13)

Defining the complex 2-forms $\mathcal{F} = F + i*F$ and $\mathcal{G} = *G - iG$, the duality may be written more compactly as:

$$
\begin{pmatrix} \mathcal{F} \\ \mathcal{G} \end{pmatrix} \rightarrow \begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix} \begin{pmatrix} \mathcal{F} \\ \mathcal{G} \end{pmatrix}, \quad \lambda \rightarrow \frac{p\lambda + q}{r\lambda + s}, \quad g_{\mu\nu} \rightarrow g_{\mu\nu}.
$$

(2.14)

3 SL(2,R) Duality in Non-Linear Electrodynamics

Since in both string theory and in supergravity theories, higher order terms in the electromagnetic field arise, causing the electrodynamic equations of motion to become non-linear, it is natural to ask under what circumstances the $SL(2,\mathbb{R})$ duality above continues to hold. It has been shown that, in the case of pure non-linear electrodynamics with no axion or dilaton, the equations of motion will admit an $SO(2)$ duality provided the Lagrangian satisfies a simple differential constraint: $G_{\mu\nu} * G^{\mu\nu} = F_{\mu\nu} * F^{\mu\nu}$, or equivalently $E \cdot B = D \cdot H$. Moreover there are, roughly speaking, as many Lagrangians satisfying this constraint as there are functions of a single real variable. Amongst this class of Lagrangians is the Born-Infeld Lagrangian:

$$
\sqrt{g}L = \sqrt{g} - \sqrt{\det(g_{\mu\nu} + F_{\mu\nu})},
$$

(3.1)

which, in 4 dimensions gives

$$
L = 1 - \sqrt{1 + \frac{1}{2}F^2 - \frac{1}{16}(F*F)^2}.
$$

(3.2)

We will consider here a 4 dimensional Lagrangian $L(g, F, a, \phi)$ which has the same axion and dilaton kinetic terms as (2.1) but we will allow an arbitrary
dependence on $a$, $\phi$ and $F_{\mu\nu}$. We will not consider the higher order derivative terms in $F$ which also occur in string theory. There is some evidence that the singularities present in solutions of the Einstein-Maxwell theory are absent in string theory due to higher order corrections and consequently the higher order derivative terms in $F$ may be neglected \cite{4}.

Infinitesimally, the $SL(2,\mathbb{R})$ transformation above may be described by the matrix

$$S = \begin{pmatrix} p & q \\ r & s \end{pmatrix} = \begin{pmatrix} 1 + \alpha & \beta \\ \gamma & 1 - \alpha \end{pmatrix}. \quad (3.3)$$

The fields then transform according to

$$\begin{align*}
\delta a &= 2\alpha a + \beta - \gamma(a^2 - e^{-2\phi}) \\
\delta \phi &= 2(a\gamma - \alpha) \\
\delta F_{\mu\nu} &= \gamma \ast G_{\mu\nu} - \alpha F_{\mu\nu} \\
\delta G_{\mu\nu} &= \alpha G_{\mu\nu} - \beta \ast F_{\mu\nu} \\
\delta g_{\mu\nu} &= 0.
\end{align*} \quad (3.4)$$

### 3.1 Invariance of constitutive relations

Invariance of the constitutive relation under these transformations requires that

$$\delta G^{\mu\nu} = -2 \frac{\partial L}{\partial F_{\rho\sigma}} \left( \frac{\partial L}{\partial F_{\mu\nu}} \right) \delta F^{\rho\sigma} - 2 \frac{\partial}{\partial a} \left( \frac{\partial L}{\partial F_{\mu\nu}} \right) \delta a - 2 \frac{\partial}{\partial \phi} \left( \frac{\partial L}{\partial F_{\mu\nu}} \right) \delta \phi. \quad (3.5)$$

Comparing the coefficients of $\alpha$, $\beta$ and $\gamma$ in this equation gives 3 differential constraints on the Lagrangian. Firstly, the $\beta$ equation reads

$$\frac{\partial^2 L}{\partial a \partial F_{\mu\nu}} = \frac{1}{2} \ast F^{\mu\nu}. \quad (3.6)$$

Integrating this implies that $L$ must be of the form

$$L = R - \frac{1}{2} (\nabla \phi)^2 - \frac{1}{2} e^{2\phi} (\nabla a)^2 + \frac{1}{4} a F_{\mu\nu} \ast F^{\mu\nu} + \tilde{L}(F, \phi) + f(a, \phi). \quad (3.7)$$

The coefficients of $\alpha$ in (3.3) then give the following constraint on $\tilde{L}$:

$$\frac{\partial}{\partial F_{\rho\sigma}} \left( F_{\rho\sigma} \frac{\partial \tilde{L}}{\partial F_{\mu\nu}} \right) + 2 \frac{\partial^2 \tilde{L}}{\partial \phi \partial F_{\mu\nu}} = 0. \quad (3.8)$$
Defining a new 2-form field \( F_{\mu\nu} = e^{-\frac{1}{2}\phi}F_{\mu\nu} \) and changing variables to \( \tilde{F} \) and \( \phi \) in \( \tilde{L} \), this last constraint reads

\[
\frac{\partial^2 \tilde{L}}{\partial \phi \partial F_{\mu\nu}} = 0 \tag{3.9}
\]

which implies that \( \tilde{L} = \tilde{L}(e^{-\frac{1}{2}\phi}F) \) plus an arbitrary function of \( \phi \) which we are free to absorb into \( f(a, \phi) \). The coefficients of \( \gamma \) in (3.5) then give another constraint on \( \tilde{L} \):

\[
\frac{\partial}{\partial F_{\mu\nu}} \left( \frac{\partial \tilde{L}}{\partial F_{\rho\sigma}} \frac{\partial \tilde{L}}{\partial F_{\lambda\tau}} \right) \eta_{\rho\sigma\lambda\tau} = \frac{1}{2} \eta^{\mu\nu\rho\sigma} \tilde{F}_{\rho\sigma}, \tag{3.10}
\]

where \( \eta_{\mu\nu\rho\sigma} \) is the completely antisymmetric tensor of the 4 dimensional spacetime. It is natural at this stage to define

\[
\tilde{G}^{\mu\nu} = -2 \frac{\partial \tilde{L}}{\partial F_{\mu\nu}} = e^{\frac{1}{2}\phi}(G_{\mu\nu} + a \ast F^{\mu\nu}). \tag{3.11}
\]

Then the last constraint can be integrated to give

\[
\tilde{G}_{\mu\nu} \ast \tilde{G}^{\mu\nu} = \tilde{F}_{\mu\nu} \ast \tilde{F}^{\mu\nu} + 4C \tag{3.12}
\]

where \( C \) is an arbitrary constant.

### 3.2 Invariance of axion and dilaton equations

The dilaton and axion equations of motion are respectively

\[
-\nabla^2 \phi = -e^{2\phi}(\nabla a)^2 + \frac{1}{4} F_{\mu\nu}(G_{\mu\nu} + a \ast F_{\mu\nu}) + \frac{\partial f}{\partial \phi} \tag{3.13}
\]

and

\[
-\nabla^2 a = 2(\nabla a)(\nabla \phi) + \frac{1}{4} e^{-2\phi} F_{\mu\nu} \ast F^{\mu\nu} + \frac{\partial f}{\partial a}. \tag{3.14}
\]

These equations are also required to be invariant under the \( SL(2, \mathbb{R}) \) transformations and (3.3) implies that they must transform into one another according to

\[
\delta(\text{Eq. (3.13)}) = 2\gamma \text{ Eq. (3.14)} \tag{3.15}
\]

\[
\delta(\text{Eq. (3.14)}) = 2(\alpha - a\gamma) \text{ Eq. (3.14)} - 2\gamma e^{-2\phi} \text{ Eq. (3.13)}
\]
Comparing terms involving $F$ and $G$ and using equation (3.12) implies that the constant $C$ must vanish. Comparing the terms involving $f(a, \phi)$ implies that $f$ is at most a constant. The remaining terms then balance. Combining the results so far, we have narrowed down the choice of possible Lagrangians to those of the form

$$L = R - 2\Lambda - \frac{1}{2} (\nabla \phi)^2 - \frac{1}{2} e^{2\phi} (\nabla a)^2 + \frac{1}{4} a F_{\mu\nu} \star F^{\mu\nu} + \bar{L}(e^{-\frac{2}{\phi}F})$$  \hspace{1cm} (3.16)

where $\bar{L}(\bar{F})$ is required to satisfy

$$\bar{G}_{\mu\nu} \star \bar{G}^{\mu\nu} = \bar{F}_{\mu\nu} \star \bar{F}^{\mu\nu}$$  \hspace{1cm} (3.17)

or equivalently

$$(G_{\mu\nu} + a \star F_{\mu\nu})(\star G^{\mu\nu} - a F^{\mu\nu}) = e^{-2\phi} F_{\mu\nu} \star F^{\mu\nu}.$$  \hspace{1cm} (3.18)

In terms of $E$, $B$, $D$ and $H$ this condition reads

$$(D - aB) \cdot (H + aE) = e^{-2\phi} E \cdot B$$  \hspace{1cm} (3.19)

which is clearly satisfied by the $D$ and $H$ fields defined in (2.5) for the Lagrangian (2.1).

### 3.3 Invariance of energy-momentum tensor

The final point to be checked is that the energy-momentum tensor is invariant under this action of $SL(2, \mathbb{R})$, otherwise it would not be consistent to assume that the metric is invariant, which we have already implicitly done in a number of the steps above. The energy-momentum tensor is most conveniently defined as

$$T^{\mu\nu} = g^{\mu\nu} L - \frac{\partial L}{\partial (\partial_{\mu} A_{\lambda})} (\partial^{\nu} A_{\lambda}) - \frac{\partial L}{\partial (\partial_{\mu} \phi)} (\partial^{\nu} \phi) - \frac{\partial L}{\partial (\partial_{\mu} a)} (\partial^{\nu} a)$$  \hspace{1cm} (3.20)

which gives\(^3\)

$$T_{\mu\nu} = g_{\mu\nu} L + G_{\mu}^{\lambda} F_{\nu\lambda} + (\partial_{\mu} \phi)(\partial_{\nu} \phi) + e^{2\phi}(\partial_{\mu} a)(\partial_{\nu} a).$$  \hspace{1cm} (3.21)

\(^3\)Note that this is indeed symmetric: $L$ depends on $F$ only via the two invariants $F_{\mu\nu} F^{\mu\nu}$ and $F_{\mu\nu} \star F^{\mu\nu}$. Therefore $G_{\mu\nu}$ will contain only terms proportional to $F_{\mu\nu}$ and $\star F_{\mu\nu}$, so $G_{\mu}^{\lambda} F_{\nu\lambda}$ will contain only terms proportional to $F_{\mu}^{\lambda} F_{\nu\lambda}$ and $F_{\mu}^{\lambda} \star F_{\nu\lambda} = \frac{1}{4} g_{\mu\nu} F^2$ which are both symmetric in the indices $\mu, \nu$. 

7
All the terms involving derivatives of the axion and dilaton are invariant, since they are the same terms as those that come from the Lagrangian (2.1). The Lagrangian is not invariant but transforms according to

$$\delta L = \frac{1}{4} a F_{\mu \nu} \star \delta F^{\mu \nu} + \frac{1}{4} \delta a F_{\mu \nu} \star F^{\mu \nu} + \frac{1}{4} \delta a F_{\mu \nu} \star F_{\rho \sigma} + \frac{\partial L}{\partial F_{\mu \nu}} e^{-\frac{1}{2} \phi} \left( \delta F_{\mu \nu} - \frac{1}{2} \delta \phi F_{\mu \nu} \right).$$

(3.22)

Using (3.4), (3.11) and (3.18) this gives

$$\delta L = -\frac{1}{2} a F_{\mu \nu} G^{\mu \nu} + \frac{1}{4} (\beta - \gamma a^2 - \gamma e^{-2\phi}) F_{\mu \nu} \star F^{\mu \nu}. \quad (3.23)$$

So, using (3.4), the transformation of the energy-momentum tensor is

$$\delta T_{\mu \nu} = \gamma G^{\mu \lambda} \star G_{\nu \lambda} - \beta \star F_{\mu \lambda} F_{\nu \lambda} + g_{\mu \nu} \delta L. \quad (3.24)$$

Using (3.18) and the fact that in 4 dimensions, the components of any 2-form satisfy $F_{\mu \nu} \star F^{\mu \nu} = \frac{1}{4} g_{\mu \nu} F_{\rho \sigma} \star F^{\rho \sigma}$, this variation of $T_{\mu \nu}$ vanishes as required.

### 4 Conclusions

We have shown that Lagrangians of the form (2.1) but with higher order $F$-terms may retain $SL(2, \mathbb{R})$ invariance provided they are of the form (3.16), (3.17). The condition (3.17) may be recognized as the same condition (in rescaled variables) that a theory of pure electrodynamics has to satisfy in order that it admit an $SO(2)$ duality. Thus for every theory of non-linear electrodynamics described by a Lagrangian $L_{\text{inv}}(g, F)$ which admits an $SO(2)$ duality, we may construct a new theory with Lagrangian

$$L = R - 2 \Lambda - \frac{1}{2} (\nabla \phi)^2 - \frac{1}{2} e^{2\phi} (\nabla a)^2 + \frac{1}{4} a F_{\mu \nu} \star F^{\mu \nu} + L_{\text{inv}}(g, e^{-\frac{1}{2} \phi} F) + \text{const.} \quad (4.1)$$

and the new theory will admit an $SL(2, \mathbb{R})$ duality. Thus, as in [1], there will be as many such Lagrangians as there are functions of a single real variable. One such example is the generalization of the Born-Infeld Lagrangian to include axion and dilaton fields:

$$L = R - 2 \Lambda - \frac{1}{2} (\nabla \phi)^2 - \frac{1}{2} e^{2\phi} (\nabla a)^2 + \frac{1}{4} a F_{\mu \nu} \star F^{\mu \nu} + 1 - \sqrt{1 + \frac{1}{2} e^{-\phi} F^2 - \frac{1}{16} e^{-2\phi} (F \star F)^2} \quad (4.2)$$

and this will be the only generalization of the Born-Infeld theory with the $SL(2, \mathbb{R})$ duality (2.14).
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