Relativistic stars in scalar-tensor theories with disformal coupling

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We present a general formulation to analyze the structure of slowly rotating relativistic stars in a broad class of scalar-tensor theories with disformal coupling to matter. Our approach includes theories with generalized kinetic terms, generic scalar field potentials and contains theories with conformal coupling as particular limits. In order to investigate how the disformal coupling affects the structure of relativistic stars, we propose a minimal model of a massless scalar-tensor theory and investigate in detail how the disformal coupling affects the spontaneous scalarization of slowly rotating neutron stars. We show that for negative values of the disformal coupling parameter between the scalar field and matter, scalarization can be suppressed, while for large positive values of the disformal coupling parameter stellar models cannot be obtained. This allows us to put a mild upper bound on this parameter. We also show that these properties can be qualitatively understood by linearizing the scalar field equation of motion in the background of a general-relativistic incompressible star. To address the intrinsic degeneracy between uncertainties in the equation-of-state of neutron stars and gravitational theory, we also show the existence of universal equation-of-state-independent relations between the moment of inertia and compactness of neutron stars in this theory. We show that in a certain range of the theory’s parameter space the universal relation largely deviates from that of general relativity, allowing, in principle, to probe the existence of spontaneous scalarization with future observations.

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\section{Introduction}

Although Einstein’s general relativity (GR) has passed all the current weak-field tests of gravity in the weak-field/slow-motion regimes with flying colors [1], it remains fairly unconstrained in the strong-gravity regime [2] and on the cosmological scales [3]. The recent observation of gravitational waves generated during the merger of two black holes (BHs) by the LIGO/Virgo Collaboration, in accordance with general-relativistic predictions [4, 5], has offered us a first glimpse of gravity in a fully nonlinear and highly dynamical regime whose theoretical implications are still being explored [6]. Nevertheless, the pressing issues on understanding the nature of dark matter and dark energy, the inflationary evolution of the early Universe and the quest for an ultraviolet completion of GR have served as driving forces in the exploration of modifications to GR [2, 3].

In general modifications of GR introduce new gravitational degree(s) of freedom in addition to the metric tensor and can be described by a scalar-tensor theory of gravity [7]. On the theoretical side, scalar-tensor theories should not contain Ostrogradski ghosts [8], i.e. the equations of motion should be written in terms of the second-order differential equations despite the possible existence of the higher-order derivative interactions at the action level. On the experimental/observational side, any extension of GR must pass all the current weak-field tests which GR has successfully passed. Therefore realistic modifications of gravity should contain a mechanism to suppress scalar interactions at small scales [9, 10] or (to be interesting) satisfy weak-field tests, but deviate from GR at some energy scale. Some models satisfying these requirements belongs to the so-called Horndeski theory [11–14], the most general scalar-tensor theory with second-order equations of motion.

In scalar-tensor theories, the scalar field may directly couple to matter, and hence matter does not follow geodesics as in the Einstein and Jordan frames described by $g_{\mu\nu}$ and $\tilde{g}_{\mu\nu}$, respectively. The two frames described by $g_{\mu\nu}$ and $\tilde{g}_{\mu\nu}$ are often referred to as the Einstein and Jordan frames, respectively.

\subsection{Spontaneous scalarization}

For relativistic stars, such as neutron stars (NSs), the conformal coupling to matter can trigger a tachyonic instability (due to a negative effective mass) of the scalar field when the star has a compactness above a certain threshold. This instability spontaneously scalarizes the NS, whereupon it harbors a nontrivial scalar field configuration which smoothly decays outside the star. In its simplest realization, scalarization occurs when the conformal factor in Eq. (1) is chosen as $A(\varphi) = \exp(\beta_1 \varphi^2 / 2)$, where $\beta_1$ is a free parameter of the theory and $\varphi$ is a massless scalar field. This theory passes all weak-field tests, but the presence of the scalar field can significantly modify the bulk properties of NSs, such as masses and radii, in comparison with GR. This effect was first analyzed for isolated NSs by Damour and Esposito-Farèse [15, 16]. The properties and observational consequences of this phe-
nomenon were studied in a number of situations, including stability [17, 18], asteroseismology [19–22], slow (and rapidly) rotating NS solutions [16, 23–26], its influence on geodesic motion of particles around NSs [27, 28], tidal interactions [26] and the multipolar structure of the spacetime [29, 30]. Moreover, the dynamical process of scalarization was studied in Ref. [31] and stellar collapse (including the associated process of scalar radiation emission) was investigated in Refs. [32–34]. We refer the reader to Ref. [35] for an extensive literature review.

Additionally, a semiclassical version of this effect [36] (cf. also [37–40] and [41] for a connection with the Damour-Esposito-Farèse model [15, 16]) has been shown to awaken the vacuum state of a quantum field leading to an exponential growth of its vacuum energy density in the background of a relativistic star.

These nontrivial excitations of scalar fields induced by relativistic stars are consequences of the absence of a “no-hair theorem” for these objects (see Refs. [42–44] for counterexamples), in contrast to the case of BHs, and can potentially be an important source for signatures of the presence of fundamental gravitational scalar degrees of freedom through astronomical observations [2, 45], including the measurements of gravitational and scalar radiation signals [46].

The phenomenological implications of spontaneous scalarization have also been explored in binary NS mergers [47–50] and in BHs surrounded by matter [51, 52]. In the former situation, a dynamical scalarization allows binary members to scalarize under conditions where this would not happen if they were isolated. This effect can dramatically change the dynamics of the system in the final cycles before the merger with potentially observable consequences. In the latter case, the presence of matter can cause the appearance of a nontrivial scalar field configuration, growing “hair” on the BH.

On the experimental side, binary-pulsar observations [53] have set stringent bounds on $\beta_1$, whose value is presently constrained to be $\beta_1 \geq -4.5$. This tightly constrains the effects of spontaneous scalarization in isolated NSs, for it has been shown that independently of the choice of the equation of state (EOS) scalarization can occur only if $\beta_1 \leq -4.35$ for NSs modeled by a perfect fluid [31, 54, 55]. These two results confine $\beta_1$ to a very limited range, in which, even if it exists in nature, the effect of scalarization on isolated NSs are bound to be small; see Refs. [24, 55] for examples where the threshold value of $\beta_1$ can be increased and Refs. [56–58] for recent work exploring the large positive $\beta_1$ region of the theory.

B. Disformal coupling

It was recently understood that modern scalar-tensor theories of gravity, under the umbrella of Horndeski gravity [11, 59], offer a more general class of coupling [60, 61] between the scalar field and matter through the so-called disformal coupling [62]

\[
\tilde{g}_{\mu\nu} = A^2(\varphi) \left[ g_{\mu\nu} + \Lambda B^2(\varphi) \varphi_\mu \varphi_\nu \right],
\]

where $\varphi_\mu = \nabla_\mu \varphi$ is the covariant derivative of the scalar field associated with the gravity frame metric $g_{\mu\nu}$, and $\Lambda$ is a constant with dimensions of $(\text{length})^2$. For $\Lambda = 0$ we recover the purely conformal case of Eq. (1). Disformal transformations were originally introduced by Bekenstein and consist of the most general coupling constructed from the metric $g_{\mu\nu}$ and the scalar field $\varphi$ that respects causality and the weak equivalence principle [62]. Disformal couplings have been investigated so far mainly in the context of cosmology [63–65]. They also arise in higher-dimensional gravitational theories with moving branes [66, 67] in relativistic extensions of modified Newtonian theories, the tensor-vector-scalar theories [68, 69], and in the decoupling limit of the nonlinear massive gravity [70–73]. Moreover, in Ref. [60] it was shown that the mathematical structure of Horndeski theory is preserved under the transformation (2), namely if the scalar-tensor theory written in terms of $g_{\mu\nu}$ belongs to a class of the Horndeski theory the same theory rewritten in terms of $\tilde{g}_{\mu\nu}$ belongs to another class of the Horndeski theory. Thus disformal transformations provide a natural generalization of conformal transformations.

Disformal coupling was also considered in models of a varying speed of light [74] and inflation [75, 76]. The invariance of cosmological observables in the frames related by the disformal relation (2) was verified in Refs. [77–82]. Although applications to early Universe models are still limited, disformal couplings have been extensively applied to late-time cosmology [63, 65, 66, 83–89]. A new screening mechanism of the scalar force in the high-density region was proposed in Ref. [86], where in the presence of disformal coupling the nonrelativistic limit of the scalar field equation seemed to be independent of the local energy density. However, a reanalysis suggested that no new screening mechanism from disformal coupling could work [83, 90]. It was also argued that disformal coupling could not contribute to a chameleon screening mechanism around a nonrelativistic source [91]. Experimental and observational constraints on disformal coupling to particular matter sectors have also been investigated. Disformal couplings to baryons and photons have been severely constrained in terms of the nondetection of new physics in collider experiments [75, 92–96], the absence of spectral distortion of the cosmic microwave background and the violation of distance reciprocal relations [94, 97–99], respectively. On the other hand, disformal coupling to the dark sector has been proposed in [84, 100] and is presently less constrained in comparison with coupling to visible matter sectors.

When conformal and disformal couplings are universal to all the matter species, they can only be constrained through experimental tests of gravity. A detailed study of scalar-tensor theory with the pure disformal coupling $A(\varphi) = 1$ and $B(\varphi) = 1$ in the weak-field limit was presented in [83] and the post-Newtonian (PN) corrections due to the presence of pure disformal coupling were computed [90]. In these papers [83, 90], in contrast to the claim of Refs. [66, 86], it was shown that no screening mechanism which could suppress the scalar force in the vicinity of the source exists and the difference of the parametrized post-Newtonian (PPN) parameters from GR are of order $|\Lambda| H_0^2$, where $H_0 \sim (10^{-28} \text{ cm}^{-1})$ is the present-day Hubble scale. The strongest bound on $|\Lambda|$ comes
from the constraints on the PPN preferred frame parameter \( \alpha_2 \). The near perfect alignment between the Sun’s spin axis and the orbital angular momenta of the planets provides the constraint \( \alpha_2 < 4 \times 10^{-5} \) (see Ref. [101] for a discussion), which implies that \( |A| \lesssim 10^{-6} H_0^2 \sim 10^{40} \text{km}^2 \). With the inclusion of the conformal factor, i.e. \( A(\varphi) \neq 1 \), the authors of Ref. [90] argued that the Cassini bound \( |\gamma - 1| < 2.1 \times 10^{-5} \) [102] imposes a constraint on \( \alpha(\varphi_0) \), where \( \varphi_0 \) is the cosmological background value of the scalar field and

\[
\alpha(\varphi) := \frac{d \log A(\varphi)}{d \varphi}, \quad \beta(\varphi) := \frac{d \log B(\varphi)}{d \varphi}.
\]

On the other hand the disformal part of the coupling \( \beta(\varphi_0) \) remains unconstrained, because corrections to the PPN parameters which include \( \beta(\varphi) \) are subdominant compared to the conformal part. These weaker constraints on the disformal coupling parameters are due to the fact that in the nonrelativistic regime with negligible pressure and a slowly evolving scalar field the disformal coupling becomes negligible. We also point out that in the weak-field regime such as in the Solar System, typical densities are small therefore preventing the appearance of ghosts in the theory for negative values of \( \Lambda \).

In the strong-gravity regime such as that found in the interior of NSs, the pressure cannot be neglected and the disformal coupling is expected to be as important as the conformal one. This would affect the spontaneous scalarization mechanism and consequently influence the structure (and stability) of relativistic stars, or have significant impact on gravitational-wave astronomy [2]. The influence of disformal coupling on the stability of matter configurations around BHs was analyzed in Ref. [103]. The authors of Ref. [103] derived the stability conditions of the system by generalizing the case of pure conformal coupling [51, 52]. They also generalized these works to scalar-tensor theories with noncanonical kinetic terms and disformal coupling, finding that the disformal coupling could make matter configurations more unstable, triggering spontaneous scalarization. In the present work within the same class of scalar-tensor theory considered in Ref. [103], we will study relativistic stars and investigate the influence of disformal coupling on the scalarization of NSs.

C. Organization of this work

This paper is organized as follows. In Sec. II we review the fundamentals of scalar-tensor theories with generalized kinetic term and disformal coupling. In Sec. III we present a general formulation to analyze the structure of slowly rotating stars in theories with disformal coupling. In Sec. IV, as a case study, we consider a canonical scalar field with a generic scalar field potential. We particularize the stellar structure equations to this model and discuss how to solve them numerically. In Sec. V we explore the consequences of the disformal coupling by studying small scalar perturbations to an incompressible relativistic star in GR. In particular we investigate the conditions for which spontaneous scalarization happens. In Sec. VI we present our numerical studies about the influence of disformal coupling on the spontaneous scalarization by solving the full stellar structure equations. In Sec. VII as an application of our numerical integrations, we examine the EOS independence between the moment of inertia and compactness of NSs in scalar-tensor theory comparing it against the results obtained in GR. Finally, in Sec. VIII we summarize our main findings and point out possible future avenues of research.

II. SCALAR-TENSOR THEORY WITH THE DISFORMAL COUPLING

We consider scalar-tensor theories in which matter is disformally coupled to the scalar field. The action in the Einstein frame reads

\[
S = \frac{1}{2\kappa} \int d^4x \sqrt{-g} \left[ R + 2P(X, \varphi) \right] + \int d^4x \sqrt{-\tilde{g}(\varphi, \varphi_\mu) \mathcal{L}_m[\tilde{g}_{\mu\nu}(\varphi, \varphi_\mu), \Psi]},
\]

where \( x^\mu (\mu = 0, 1, 2, 3) \) represents the coordinate system of the spacetime, \( g_{\mu\nu} \) and \( \tilde{g}_{\mu\nu} \) are respectively the Einstein and Jordan frame metrics disformally related by (2), \( g := \det(g_{\mu\nu}) \) and \( \tilde{g} := \det(\tilde{g}_{\mu\nu}) \), \( R \) is the Ricci scalar curvature associated with \( g_{\mu\nu} \), \( \kappa := (8\pi G)/c^4 \), where \( G \) is the gravitational constant defined in the Einstein frame and \( c \) is the speed of light in vacuum. \( P(X, \varphi) \) is an arbitrary function of the scalar field \( \varphi \) and \( X := \frac{2}{c^4}g^{\mu\nu}\varphi_\mu\varphi_\nu \), and \( \mathcal{L}_m \) represents the Lagrangian density of matter fields \( \Psi \). We note that the canonical scalar field corresponds to the case of \( P(X, \varphi) = 2X - V(\varphi) \), but we will not restrict the form of \( P(X, \varphi) \) at this stage. In this paper we will not omit \( G \) and \( c \).

Varying the action (4) with respect to the Einstein frame metric \( g_{\mu\nu} \), we obtain the Einstein field equations

\[
G^{\mu\nu} = \kappa \left( T_{(m)}^{\mu\nu} + T_{(\varphi)}^{\mu\nu} \right),
\]

where the energy-momentum tensors of the matter fields \( \Psi \) and scalar field \( \varphi \) are given by

\[
T_{(m)}^{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta \left( \sqrt{-g}\mathcal{L}_m[\tilde{g}(\varphi, \Psi)] \right)}{\delta g_{\mu\nu}},
\]

and

\[
T_{(\varphi)}^{\mu\nu} = \frac{1}{\kappa} \frac{2}{\sqrt{-g}} \frac{\delta \left( \sqrt{-g}P(X, \varphi) \right)}{\delta g_{\mu\nu}} = \frac{1}{\kappa} \left( P_X \varphi^{\mu} \varphi_\nu + P g^{\mu\nu} \right),
\]

respectively, where \( P_X := \partial_X P \) and \( \varphi^{\mu} := g^{\mu\nu}\varphi_\nu \). From Eq. (2), the inverse Jordan frame metric \( \tilde{g}^{\mu\nu} \) is related to the inverse Einstein frame metric \( g^{\mu\nu} \) by

\[
\tilde{g}^{\mu\nu} = A^{-2}(\varphi) \left[ g^{\mu\nu} - \frac{\Lambda B^2(\varphi)}{\chi(X, \varphi)} \chi^{\mu\nu} \right],
\]

where we have defined

\[
\chi(X, \varphi) := 1 - 2\Lambda B^2(\varphi)X.
\]
The volume element in the Jordan frame $\sqrt{-g}$ is given by $\sqrt{-g} = A^4(\varphi)\sqrt{-\tilde{g}}\sqrt{\chi(X, \varphi)}$. In order to keep the Lorentzian signature of the Jordan frame metric $\tilde{g}_{\mu\nu}$, $\chi$ must be non-negative. We note that in the purely conformal coupling limit $\Lambda = 0$ and $\chi = 1$.

The contravariant energy-momentum tensor in the Jordan frame $T_{(m)}^{\mu\nu}$ is related to that in the Einstein frame by

$$
\tilde{T}_{(m)}^{\mu\nu} := \frac{2}{\sqrt{-g}} \frac{\delta}{\delta \tilde{g}_{\mu\nu}} \left[ -\frac{\sqrt{-g}L_m [\tilde{g}, \Psi]}{\sqrt{\chi(X, \varphi)}} \right],
$$

$$
= \sqrt{\tilde{g}} \frac{\delta g_{\alpha\beta}}{\tilde{g} \delta \tilde{g}_{\mu\nu}} T_{(m)}^{\alpha\beta} = \frac{A_0^2(\varphi)}{\sqrt{\chi(X, \varphi)}} T_{(m)}^{\mu\nu},
$$

(10)

The mixed and covariant energy-momentum tensors in the Jordan frame are respectively given by

$$
\tilde{T}_{(m)}^{(m)\mu} = \frac{A^{-4}(\varphi)}{\sqrt{\chi(X, \varphi)}} \left( \delta_\mu^\alpha + \Lambda B^2(\varphi) \varphi_\mu \varphi^\alpha \right) T_{(m)\alpha},
$$

(11a)

$$
\tilde{T}_{(m)\mu\nu} = \frac{A^{-2}(\varphi)}{\sqrt{\chi(X, \varphi)}} \left( \delta_\mu^\alpha + \Lambda B^2(\varphi) \varphi_\mu \varphi^\alpha \right)
\times \left( \delta_\nu^\beta + \Lambda B^2(\varphi) \varphi_\nu \varphi^\beta \right) T_{(m)\alpha\beta},
$$

(11b)

and

$$
T_{(m)}^{\mu\nu} = A^6(\varphi) \sqrt{\chi(X, \varphi)} \tilde{T}_{(m)}^{\mu\nu},
$$

(12a)

$$
T_{(m)\nu} = A^4(\varphi) \sqrt{\chi(X, \varphi)} \left( \delta_\nu^\alpha - \Lambda B^2(\varphi) \varphi_\nu \varphi^\alpha \right) \tilde{T}_{(m)\rho},
$$

(12b)

$$
T_{(m)\mu\nu} = A^2(\varphi) \sqrt{\chi(X, \varphi)} \left( \delta_\mu^\alpha - \Lambda B^2(\varphi) \varphi_\mu \varphi^\alpha \right)
\times \left( \delta_\nu^\beta - \Lambda B^2(\varphi) \varphi_\nu \varphi^\beta \right) \tilde{T}_{(m)\rho\sigma},
$$

(12c)

In terms of the covariant tensors, the Einstein equations in the Einstein frame (5) can be recast as

$$
G_{\mu\nu} = \kappa A^2(\varphi) \sqrt{\chi(X, \varphi)} \left( \delta_\mu^\alpha - \frac{\Lambda B^2(\varphi) \varphi_\mu \varphi^\alpha}{\chi(X, \varphi)} \right)
\times \left( \delta_\nu^\beta - \frac{\Lambda B^2(\varphi) \varphi_\nu \varphi^\beta}{\chi(X, \varphi)} \right) \tilde{T}_{(m)\rho\sigma} + P_X \varphi_\mu \varphi_\nu + g_{\mu\nu} P,
$$

(13)

Varying the action (4) with respect to the scalar field $\varphi$, we obtain the scalar field equation of motion

$$
P_X \square \varphi + P_\varphi - P_{XX} \varphi_\rho \varphi^\rho \varphi_\sigma - 2X P_{X\varphi} = \kappa Q,
$$

(14)

where the function $Q$ characterizes the strength of the coupling of matter to the scalar field

$$
Q := \Lambda \nabla_\rho \left( B^2(\varphi) T_{(m)\rho\sigma} \varphi^\sigma - \alpha(\varphi) T_{(m)} \right)
- \Lambda B^2(\varphi) \left[ \alpha(\varphi) + \beta(\varphi) \right] T_{(m)\rho\sigma} \varphi^\rho \varphi_\sigma,
$$

(15)

where $T_{(m)} := g^{\rho\sigma} T_{(m)\rho\sigma}$ is the trace of $T_{(m)\rho\sigma}$, and $\alpha(\varphi)$ and $\beta(\varphi)$ were defined in Eq. (3). Taking the divergence of Eq. (5), employing the contracted Bianchi identity $\nabla_\rho G^{\rho\sigma} = 0$, and using the scalar field equation of motion (14), we obtain

$$
\nabla_\rho T_{(m)\rho\sigma} = -\nabla_\rho \tilde{T}_{(m)\rho\sigma} = -Q \varphi^\sigma,
$$

(16)

and the coupling strength $Q$ can be rewritten as

$$
Q = \Lambda B^2(\varphi) \left( \nabla_\rho T_{(m)\rho\sigma} \varphi_\sigma + \nabla_\rho Y \right),
$$

(17)

where we have introduced

$$
Y := \Lambda B^2(\varphi) \left\{ \left[ \beta(\varphi) - \alpha(\varphi) \right] T_{(m)\rho\sigma} \varphi_\rho \varphi_\sigma + T_{(m)\rho\sigma} \varphi_\rho \varphi_\sigma \right\}
- \alpha(\varphi) T_{(m)}.
$$

(18)

Multiplying Eq. (16) by $\varphi_\sigma$ and solving it with respect to $\left( \nabla_\rho T_{(m)\rho\sigma} \right) \varphi_\sigma$, we obtain

$$
\chi \left( \nabla_\rho T_{(m)\rho\sigma} \right) \varphi_\sigma = 2XY.
$$

(19)

Then, substituting it in Eq. (17), using $Q = Y/\chi$, and finally eliminating $Q$ from Eq. (14), we obtain the reduced scalar field equation of motion

$$
P_X \square \varphi + P_\varphi - P_{XX} \varphi_\rho \varphi^\rho \varphi_\sigma - 2X P_{X\varphi} = \kappa \chi \left( \nabla_\rho T_{(m)\rho\sigma} \right) \varphi_\sigma
\left\{ \Lambda B^2(\varphi) \left[ \left( \beta(\varphi) - \alpha(\varphi) \right) T_{(m)\rho\sigma} \varphi_\rho \varphi_\sigma + T_{(m)\rho\sigma} \varphi_\rho \varphi_\sigma \right]
- \alpha(\varphi) T_{(m)} \right\}.
$$

(20)

### III. THE EQUATIONS OF STELLAR STRUCTURE

#### A. Equations of motion

In this section, we consider a static and spherically symmetric spacetime with line element

$$
ds^2 = g_{\mu\nu} dx^\mu dx^\nu
= -e^\nu(r) e^\rho(r) dr^2 + e^\lambda(r) d\sigma^2 + r^2 \gamma_{ij} d\theta^i d\theta^j,
$$

(21)

where $\nu(r)$ and $\lambda(r)$ are functions of the radial coordinate $r$ only, $\gamma_{ij}$ is the metric of the unit 2-sphere, and the coordinates $\theta^i$ ($i = 1, 2$) run over the directions of the unit 2-sphere, such that $\gamma_{ij} d\theta^i d\theta^j = d\theta^2 + \sin^2 \theta d\phi^2$. We also assume by symmetry that the scalar field is only a function of $r$, $\varphi = \varphi(r)$. Hence the coupling functions $A(\varphi)$ and $B(\varphi)$ are also only functions of $r$ through $\varphi(r)$.

We assume that in the Jordan frame only diagonal components of the energy-momentum tensor of matter are nonvanishing

$$
\tilde{T}_{(m)\rho\sigma}^{\rho\sigma} = -\tilde{\rho} e^2, \quad \tilde{T}_{(m)\rho\rho}^{\rho\rho} = \tilde{p}_r, \quad \tilde{T}_{(m)\rho\theta}^{\rho\theta} = \tilde{p}_\theta \delta^j_l,
$$

(22)

where $\tilde{\rho}, \tilde{p}_r$, and $\tilde{p}_\theta$ are respectively the energy density, radial and tangential pressures of an anisotropic fluid in the Jordan
frame [104]. Using Eq. (12b), they are related to the components of the energy-momentum tensor of matter in the Einstein frame, which are represented by the quantities without a \( \tilde{\text{a}} \), by

\[
\rho = A^4(\varphi) \sqrt{\chi} \tilde{\varphi}, \quad p_r = \frac{A^4(\varphi)}{\sqrt{\chi}} \tilde{\varphi}_r, \quad p_t = A^4(\varphi) \sqrt{\chi} \tilde{\varphi}_t,
\]

where in the background given by Eq. (21), the quantity \( \chi \) defined in Eq. (9) reduces to

\[
\chi = 1 + e^{-\lambda} A B^2(\varphi)(\varphi')^2.
\]

We note that even if the fluid in the Jordan frame has an isotropic pressure, \( \tilde{p}_r = \tilde{p}_t \), it is transformed into an anisotropic one in the Einstein frame i.e. \( p_r \neq p_t \) in the presence of disformal coupling \( \chi \neq 1 \).

The nontrivial radial component of the energy-momentum conservation law in the Einstein frame (16) gives us

\[
\frac{d\tilde{\varphi}_r}{dr} = - \left[ \frac{\nu'}{2} + \alpha(\varphi) \varphi' \right] (\tilde{p}_c^2 + \tilde{p}_r) - 2 \left[ \frac{1}{r} + \alpha(\varphi) \varphi' \right] \tilde{\sigma},
\]

where we have defined \( \tilde{\sigma} := \tilde{\varphi}_r - \tilde{\varphi}_t \), which measures the degree of anisotropy of the fluid [104]. The same result can be obtained from the conservation law in the Jordan frame \( \nabla_{\mu} T^{\mu \nu}_{(m)} = 0 \), where \( \nabla_{\mu} \) represents the covariant derivative associated with the Jordan frame metric \( \tilde{g}_{\mu \nu} \). The conservation law (29) depends implicitly on \( B(\varphi) \) and its derivative through \( \nu' \) [cf. Eq. (26)].

\subsection{The reduced equations of motion}

We then reduce the set of equations (25)-(27), (28) and (29) into a form more convenient for a numerical integration. We introduce the mass function \( \mu(r) \) through

\[
e^{-\lambda(r)} := 1 - \frac{2\mu(r)}{r},
\]

and replace all \( \lambda(r) \) dependence with \( \mu(r) \). We also introduce the first-order derivative of the scalar field \( \psi(r) \), i.e.

\[
\psi := \frac{d\varphi}{dr}.
\]

We can write the kinetic energy as

\[
X = - \frac{r - 2\mu}{2r} \psi^2.
\]

and \( \chi \) can then be expressed as

\[
\chi = 1 + \frac{r}{2r} \frac{-2\mu}{\lambda B^2(\varphi)\psi^2}.
\]

The \((t, t)\) component of the Einstein equations [cf. Eq (25)] determines the gradient of \( \mu \)

\[
\frac{d\mu}{dr} = \frac{r^2}{2} \left[ A^4(\varphi) \sqrt{\chi} \tilde{\varphi}^2 - P \right].
\]

Similarly, the \((r, r)\) component of the Einstein equations (26) reduces to

\[
\frac{d\nu}{dr} = \frac{2\mu}{r(r - 2\mu)}
\]

\[
+ r \left\{ \psi^2 P_X + \frac{r}{r - 2\mu} \left[ P + \frac{A^4(\varphi)}{\sqrt{\chi}} (\kappa \tilde{p}_r) \right] \right\}.
\]
The conservation law (29) combined with Eq. (35) leads to
\[
\frac{d\tilde{p}_r}{dr} = -\left\{\alpha(\varphi)\psi + \frac{\mu}{r(r - 2\mu)} + \frac{r}{2} \left[\psi^2 P_X + \frac{r}{r - 2\mu} \left(P + \frac{A^4(\varphi)}{\sqrt{X}}(\kappa\tilde{p}_r)\right)\right] \right\} \left(\tilde{p}_r \psi^2 + \tilde{p}_r\right) - 2 \left[\frac{1}{r} + \alpha(\varphi)\psi\right] \tilde{\sigma}. \tag{36}
\]

Finally, the scalar field equation of motion (28) reduces to
\[
\begin{align*}
\chi \left(P_X - e^{-\lambda}\psi^2 P_{XX} \right) - \kappa \Delta A^4(\varphi) B^2(\varphi) \frac{\tilde{p}_r}{\sqrt{X}} \psi' &+ \left\{\chi \left[\left(\nu' - \frac{\chi}{2} \right) \lambda + \frac{2}{r} \right] P_X + \frac{\chi}{2} e^{-\lambda}\psi^2 P_{XX} + P_{X\varphi}\right\} \psi \\
- \kappa \Delta A^4(\varphi) B^2(\varphi) \left(\frac{\sqrt{X}}{2} \psi^2 c^2 - 2\tilde{p}_t\right) &= - e^\lambda \chi P_{\varphi} + \kappa A^4(\varphi) \alpha(\varphi) c^2 \left[\frac{\tilde{p}_r}{\sqrt{X}} + \sqrt{X}(\tilde{p}_c^2 - 2\tilde{p}_t)\right]. \tag{37}
\end{align*}
\]
Eliminating \(\nu'\) and \(\nu'\) from Eq. (37), and using Eqs. (25)-(26), the scalar field equation of motion (37) can be rewritten as
\[
C_2 \frac{d\psi}{dr} = -C_1 \psi + \frac{r}{r - 2\mu} \left\{-\chi P_{\varphi} + \kappa A^4(\varphi) \alpha(\varphi) \left[-\frac{\tilde{p}_r}{\sqrt{X}} - \sqrt{X}(\tilde{p}_c^2 + 2\tilde{p}_t)\right]\right\}, \tag{38}
\]
where we introduced
\[
\begin{align*}
C_2 &= \chi \left[P_X - \left(\frac{2\mu}{r}\right) \psi^2 P_{XX}\right] - \kappa \Delta A^4(\varphi) B^2(\varphi) \frac{\tilde{p}_r}{\sqrt{X}}, \\
C_1 &= \chi \left\{P_X \left[\frac{2(r - \mu)}{r(r - 2\mu)} + \frac{r^2}{2} \psi^2 P_{XX} + \frac{r^2}{r - 2\mu} \left(P - \frac{\kappa}{2} A^4(\varphi) \left(\sqrt{X}\tilde{p}_c^2 - \tilde{p}_r\right)\right)\right]\right\} + \frac{1}{2} \left[-\frac{2\mu}{r^2} + r \left(P + A^4(\varphi) \sqrt{X}(\kappa\tilde{p}_c^2)\right)\right] \psi^2 P_{XX} + P_{X\varphi} \\
&- \kappa \Delta A^4(\varphi) B^2(\varphi) \left\{-\frac{1}{r - 2\mu} \left(\frac{\mu}{r} + \frac{r^2 P}{2}\right) \left(\sqrt{X}\tilde{p}_c^2 - \tilde{p}_r\right) - \frac{\tilde{p}_c^2}{2} \sqrt{X} \psi^2 P_X r \right\} - \frac{\kappa}{r - 2\mu} \tilde{p}_c^2 A^4(\varphi) + \frac{2}{\sqrt{X}} \tilde{p}_t + \psi \left(\beta(\varphi) - \alpha(\varphi)\right) \tilde{p}_r. \tag{39}
\end{align*}
\]

The set of Eqs. (31), (34), (35), (36) and (38) together with a given EOS
\[
\tilde{p}_r = \tilde{p}_r(\tilde{p}), \quad \tilde{p}_t = \tilde{p}_t(\tilde{p}), \tag{40}
\]
form a closed system of equations to analyze the structure of relativistic stars in the scalar-tensor theory (4).

\section*{C. Slowly rotating stars}

In this subsection, we extend our calculation to the case of slowly rotating stars. Once the set of the equations of motion for a static and spherically symmetric star is given, it is simple to take first-order corrections due to rotation into consideration using the Hartle-Thorne scheme [105, 106]. At first order in the Hartle-Thorne perturbative expansion, we derive our results in a manner as general as possible, similarly to the previous section.

In the Einstein frame, the line element including the first-order correction due to rotation is given by
\[
ds^2 = -e^{\nu(r)} c^2 dt^2 + e^{\lambda(r)} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) + 2 (\omega - \Omega) r^2 \sin^2 \theta dtd\phi, \tag{41}
\]
where \(\omega(r)\) is a function of \(r\), which is of the same order as the star’s angular velocity \(\Omega\). We can construct the Jordan frame line element using Eqs. (2) and (8). The construction of the energy-momentum tensor for the anisotropic fluid in the Jordan frame is similar to what was done before, except that now, the normalization of the four-velocity, demands that
\[
\begin{align*}
\tilde{u}^t &= \left[- \left(\tilde{g}_{tt} + 2\tilde{\Omega}_t \tilde{g}_{t\phi} + \tilde{\Omega}^2 \tilde{g}_{\phi\phi}\right)\right]^{-1/2} \tag{42a}, \\
\tilde{u}^r &= \tilde{u}^\theta = \tilde{u}^\phi = 0, \tag{42b}
\end{align*}
\]
where \(\tilde{\Omega}\) is the star’s angular velocity in the Jordan frame [measured in the coordinates of \(x^\mu = (t, r, \theta, \phi)\)]
\[
\begin{align*}
\tilde{g}_{tt} &= A^2 g_{tt}, \quad \tilde{g}_{rr} = A^2 \left[g_{rr} + \Lambda B^2 (\varphi)^2\right], \tag{43a} \\
\tilde{g}_{ij} &= A^2 g_{ij}, \quad (i, j = \theta, \phi) \tag{43b} \\
\tilde{g}_{t\phi} &= A^2 g_{t\phi}, \tag{43c}
\end{align*}
\]
and we must expand all expressions, keeping only terms of order \( O(\Omega) \). As shown in the Appendix the star’s angular velocity is disformally invariant, \( \tilde{\Omega} = \Omega \). We also note that rotation can induce a dependence of the scalar field on \( \theta \), which appears however only at more than second order in rotation, \( O(\Omega^2) \) \cite{26}. Thus in our case, the scalar field configuration remains the same as in the nonrotating situation.

At the first order in rotation, the diagonal components of the Einstein equations and the scalar field equation of motion remain the same as Eqs. (31), (34), (35), (36) and (38). A new equation comes however from the disformal coupling: 

\[
\frac{d^2 \omega}{dr^2} - \left( \frac{4}{r} - \frac{\lambda' + \nu'}{2} \right) \frac{d\omega}{dr} + 2\kappa A^4(\varphi) r \sqrt{\chi} \left( \frac{\hat{\rho} \beta^2 + \hat{p}_r - \hat{\sigma}}{(r - 2\mu)} \right) \omega(r) = 0. \tag{44}
\]

By eliminating \( \nu' \) and \( \lambda' \) with the use of Eqs. (34) and (35), we obtain the frame-dragging equation

\[
\frac{d^2 \omega}{dr^2} + \left[ \frac{1}{2} r P_X \omega^2 + \frac{k r^2 A^4(\varphi)}{2 \sqrt{\chi}(r - 2\mu)} \left( \hat{p}_r + \chi \hat{\rho} \beta^2 \right) - \frac{4}{r} \right] \frac{d\omega}{dr} + 2\kappa A^4(\varphi) r \sqrt{\chi} \left( \frac{\hat{\rho} \beta^2 + \hat{p}_r - \hat{\sigma}}{(r - 2\mu)} \right) \omega(r) = 0. \tag{45}
\]

Equation (45) can be solved together with Eqs. (31), (34), (35), (36) and (38). Together these equations fully describe a slowly rotating anisotropic relativistic star in the theory described by the action (4).

D. Particular limits

The equations obtained in the previous section represent the most general set of stellar structure equations for a broad class of scalar-tensor theories with a single scalar degree of freedom with a disformal coupling between the scalar field and a spherically symmetric slowly rotating anisotropic fluid distribution. Because of its generality, we can recover many particular cases previously studied in the literature:

1. In the limit of the pure conformal coupling, \( \Lambda \to 0 \) (thus \( \chi \to 1 \)), we recover the case studied in Ref. [55].

2. If we additionally assume isotropic pressure \( \hat{p}_r = \hat{p}_t = \hat{p} \), we recover the standard equations given in Refs. [15, 16].

3. If we assume a kinetic term of the form \( P(X, \varphi) = 2X - V(\varphi) \), where \( V(\varphi) \) is a mass term \( m^2 \varphi^2 \), isotropic pressure and purely conformal coupling we recover the massive scalar-tensor theory studied in Refs. [107, 108] and the asymteron scenario proposed in Ref. [109] by appropriately choosing \( A(\varphi) \).

IV. SCALAR-TENSOR THEORY WITH A CANONICAL SCALAR FIELD

A. Stellar structure equations

Now let us apply the general formulation developed in the previous section to the canonical scalar field with the potential \( V(\varphi) \), i.e. \( P = 2X - V(\varphi) \). The stellar structure equations (31), (34), (35), (36) and (38) reduce to

\[
\frac{d\mu}{dr} = \frac{r(r - 2\mu)}{2} \psi^2 + \frac{r^2}{2} V(\varphi) + A^4(\varphi) \sqrt{\chi} \left( \frac{\kappa}{2} \hat{\rho} \beta^2 + \frac{\hat{p}_r}{\sqrt{\chi}} \right), \tag{46a}
\]

\[
\frac{d\nu}{dr} = \frac{r(2\mu - r)}{2} \psi^2 - \frac{r^2}{r - 2\mu} V(\varphi) + \frac{r^2}{r - 2\mu} \sqrt{\chi} \left( \frac{\kappa}{2} \hat{p}_r \right) - 2 \left( \frac{1}{r} + \alpha(\varphi) \right) \tilde{\sigma}, \tag{46b}
\]

\[
\frac{d\hat{p}_r}{dr} = - \left[ \alpha(\varphi) - \frac{\mu}{r - 2\mu} \right] \tilde{\sigma} + \frac{r^2}{r - 2\mu} \sqrt{\chi} \left( \frac{\kappa}{2} \hat{p}_r \right), \tag{46c}
\]

\[
\frac{d\varphi}{dr} = \psi, \tag{46d}
\]

\[
C_2 \frac{d\psi}{dr} = -C_1 \psi + \frac{r \chi V_\varphi(\varphi)}{r - 2\mu} + \frac{k r}{r - 2\mu} A^4(\varphi) \alpha(\varphi) \times \left[ -\frac{\hat{p}_r}{\sqrt{\chi}} + \sqrt{\chi} (\hat{\rho} \beta^2 - 2\hat{p}_r) + 2 \sqrt{\chi} \tilde{\sigma} \right], \tag{46e}
\]

where

\[
C_1 = \frac{2\chi}{r - 2\mu} \left[ \frac{2(r - \mu)}{r} - \frac{r^2}{2} V(\varphi) - \frac{\kappa}{4} A^4(\varphi) r^2 \right] \times \left( \sqrt{\chi} \hat{\rho} \beta^2 - \frac{\hat{p}_r}{\sqrt{\chi}} \right) - \kappa A^4(\varphi) B^2(\varphi) \times \left[ -\frac{\mu}{r - 2\mu} \right] \left( \sqrt{\chi} \hat{\rho} \beta^2 - \frac{\hat{p}_r}{\sqrt{\chi}} \right) - \frac{r^2}{(r - 2\mu)} \left( \sqrt{\chi} \hat{\rho} \beta^2 - \frac{\hat{p}_r}{\sqrt{\chi}} \right) + \frac{2}{r} \left( \frac{\hat{p}_r}{\sqrt{\chi}} \right) + \frac{\psi}{\sqrt{\chi}} (\beta(\varphi) - \alpha(\varphi) \hat{p}_r) \tag{47}
\]

and

\[
C_2 = 2\chi - \kappa A^4(\varphi) B^2(\varphi) \left( \frac{\hat{p}_r}{\sqrt{\chi}} \right) \tag{48}
\]

In the case of a slowly rotating star, the frame-dragging
When integrating the scalar field equation \((\phi)\), the Scalar field approaches a given cosmological value outside the star, the scalar field becomes comparable to the standard ones in the scalar-tensor theory as \(R_D = \sqrt{\Lambda}B(\phi)\). If \(R_D > R\), where \(r = R\) is the star’s radius, the contributions of disformal coupling to the gradient terms become important throughout the star, while if \(R_D < R\) they could be important only in a portion of the star’s interior \(r < R_D\). When \(B \to 1\), \(R_D \approx \sqrt{\Lambda}A\) and therefore \(\sqrt{\Lambda}\) characterizes the length scale for which the disformal coupling effects become apparent. As the radius of a typical NS is about 10 km, the effects of disformal coupling of the star become apparent when \(\Lambda > O(100\text{ km}^2)\).

We note that in the presence of the disformal coupling, when integrating the scalar field equation \((\phi)\), the coefficient \(C_2\) in the \(d\psi/ dr\) equation may vanish at some \(r = R_+\), i.e.

\[
\mu(r) = \frac{1}{6} \left[ \kappa\tilde{\rho}_c c^2 A^4(\varphi_c) + V(\varphi_c) \right] r^3 + O(r^5),
\]

\[
\nu(r) = \frac{1}{6} \left[ \kappa (\tilde{\rho}_c c^2 + 3\tilde{\rho}_c) A^4(\varphi_c) - 2V(\varphi_c) \right] r^2 + O(r^4),
\]

\[
\varphi(r) = \varphi_c + \frac{\kappa A^4(\varphi_c)\alpha(\varphi_c) (\tilde{\rho}_c c^2 - 3\tilde{\rho}_c) + V_\varphi(\varphi_c)}{12 [2 - \kappa\Lambda\tilde{\rho}_c A^4(\varphi_c)B^2(\varphi_c)]} r^2 + O(r^4),
\]

\[
\tilde{p}(r) = \tilde{p}_c - \frac{1}{12} \tilde{\rho}_c c^2 + \tilde{\rho}_c \left\{ \kappa A^4(\varphi_c) \left[ \tilde{\rho}_c c^2 + 3\tilde{\rho}_c + \alpha(\varphi_c) \left[ 1 - \frac{\tilde{\rho}_c c^2 - 3\tilde{\rho}_c}{2\Lambda\tilde{\rho}_c A^4(\varphi_c)B^2(\varphi_c)} \right] \right] \right\} r^2 + O(r^4),
\]

where \(\tilde{\rho}_c\) is fixed by \(\tilde{\rho}_c\) through the EOS, i.e., \(\tilde{\rho}_c = \tilde{\rho}(\tilde{\rho}_c)\). The central value of the scalar field \(\varphi_c\) is fixed by demanding that outside the star the scalar field approaches a given cosmological value \(\varphi_0\) as \(r \to \infty\), which is consistent with observational constraints. We will come back to this in Sec. IV C.

As a well-behaved stellar model requires \(\tilde{\rho}''(0) < 0\), we impose

\[
\kappa A^4(\varphi_c) \left[ \tilde{\rho}_c c^2 + 3\tilde{\rho}_c + \alpha(\varphi_c)^2 \left[ 1 - \frac{\tilde{\rho}_c c^2 - 3\tilde{\rho}_c}{2\Lambda\tilde{\rho}_c A^4(\varphi_c)B^2(\varphi_c)} \right] \right] - 2V(\varphi_c) \left[ 1 - \frac{\alpha(\varphi_c)V_\varphi(\varphi_c)}{2V(\varphi_c)} - \frac{\tilde{\rho}_c c^2}{2\Lambda\tilde{\rho}_c A^4(\varphi_c)B^2(\varphi_c)} \right] > 0.
\]

For a large positive disformal coupling parameter \(\Lambda > 0\) and a large pressure at the center \(\tilde{\rho}_c\) such that \(C_2(R_+) = 0\). This could happen when both \(\Lambda > 0\) and the pressure at the center of the star is large enough such that \(C_2 < 0\) in the vicinity of \(r = 0\). In such a case, we integrate the equations outwards, since the radial pressure \(\tilde{p}_c\) decreases and vanishes at the surface of the star, there must be a point \(R_+\) where \(C_2\) vanishes. This point represents a singularity of our equations and a regular stellar model cannot be constructed.

The nonexistence of a regular relativistic star for a large positive \(\Lambda\) is one of the most important consequences due to the disformal coupling. The appearance of the singularity is due to the fact that the gradient term in the scalar field equation of motion \((\phi)\) picks a wrong sign (i.e., negative speed of sound) and is an illustration of the gradient instability pointed out in Refs. [72, 86, 110].

### B. Interior solutions

From this section onwards, we focus on the case of isotropic pressure \(\tilde{p} = \tilde{p}_r = \tilde{p}_\mu\). We then derive the boundary conditions at the center of the star, \(r = 0\), which have to be specified when integrating Eqs. \((\phi)\) and \((\phi)\). We assume that at \(r = 0\), \(\tilde{\rho}(0) = \tilde{\rho}_c\). The remaining metric and matter variables can be expanded as

\[
1 - \frac{m^2}{4\pi\tilde{\rho}_c A^4(\varphi_c)B^2(\varphi_c)} \ll 1,\quad \text{the } r^2\text{ terms of the scalar field and pressure diverge and the Taylor series solution (50) breaks down. Such a property is a direct consequence of the appearance of the singularity inside the star which was mentioned in the previous subsection. Assuming that } A(\varphi_c) \approx 1\text{ and } B(\varphi_c) \approx 1,\text{ the maximal positive value of } \Lambda_{\text{max}}\text{ can be roughly estimated as}
\]

\[
\Lambda_{\text{max}} \approx \frac{c^4}{4\pi G\tilde{\rho}_c} \approx 10^2 \text{ km}^2\]

for \(\tilde{\rho}_c = 10^{36} \text{ dyne/cm}^2\), which agrees with the numerical analysis done in Sec. VI. On the other hand, for a large negative value of the disformal coupling \(\Lambda < 0\), no singularity appears, from Eq. \((\phi)\) the \(r^2\) correction to the scalar field amplitude is suppressed, and \(\varphi(r) \to \varphi_c\) everywhere inside the star. This indicates that \(A(\varphi_c) \approx \text{constant}\), and for a
vanishing potential $V(\varphi) = 0$ the stellar configuration approaches that in GR.

In the case of slowly rotating stars, the boundary condition for $\omega$ near the origin reads

$$\omega = \omega_0 \left[ 1 + \frac{\kappa}{5} A^4(\varphi_c) \left( \tilde{p}_c c^2 + 3 \tilde{p}_c \right) r^2 \right] + \mathcal{O}(r^4).$$ (53)

1. Stellar models in purely disformal theories

It is interesting to analyze the stellar structure equations in the purely disformal coupling limit, when $A(\varphi) = 1$. In this case we find that the expansions near the origin are

$$\mu(r) = \frac{1}{6} \left[ \kappa \tilde{p}_c c^2 + V(\varphi_c) \right] r^3 + \mathcal{O}(r^5),$$

$$\nu(r) = \frac{1}{6} \left[ \kappa \left( \tilde{p}_c c^2 + 3 \tilde{p}_c \right) - 2 V(\varphi_c) \right] r^2 + \mathcal{O}(r^4),$$

$$\varphi(r) = \varphi_c + \frac{V_\varphi(\varphi_c)}{12 \left[ 1 - \frac{2}{5} \Lambda \tilde{\rho}_c B^2(\varphi_c) \right]} r^2 + \mathcal{O}(r^4),$$

$$\tilde{p}(r) = \tilde{p}_c - \frac{1}{12} \left( \tilde{p}_c + 3 \tilde{p}_c c^2 \right) \left[ \kappa \left( \tilde{p}_c c^2 + 3 \tilde{p}_c \right) - 2 V(\varphi_c) \right] r^2 + \mathcal{O}(r^4).$$ (54)

Thus for $V(\varphi) = 0$, $\varphi = \varphi_c$ everywhere, and the disformal coupling term does not modify the stellar structure with respect to GR. Only with a nontrivial potential $V(\varphi)$, the disformal coupling can modify the profile of the scalar field inside the NS. It was argued in Ref. [83] that for a simple mass term potential $V_\varphi \sim m^2 \varphi$, where $m$ is the mass of the scalar field, disformal contributions can be neglected and the NS solution is the same as in GR.

2. Metric functions in the Jordan frame

Finally, we mention the behaviors of the metric functions in the Jordan frame. In the Appendix we derive the relationship of the physical quantities defined in the two frames. The boundary conditions (50) indicate that in the singular stellar solution of the Einstein frame the metric functions $\mu$ and $\nu$ remain regular. Using Eqs. (A3) and (A9), the metric functions in the Jordan frame behave as

$$\bar{\nu}(r) = \ln A(\varphi_c)^2 + \frac{1}{6} \left[ \left( \tilde{p}_c c^2 + 3 \tilde{p}_c \right) - 2 V(\varphi_c) + \alpha(\varphi_c) \frac{\kappa A^4(\varphi_c) \alpha \left( \tilde{p}_c c^2 - 3 \tilde{p}_c \right) + V'(\varphi_c)}{1 - \frac{\kappa}{2} A^4(\varphi_c) B^2(\varphi_c) \tilde{p}_c} \right] r^2 + \mathcal{O}(r^4),$$ (55)

$$\bar{\mu}(r) = \frac{A(\varphi_c)}{18} \left[ 3 \left( A^4(\varphi_c) \tilde{p}_c c^2 + V(\varphi_c) \right) + 3 \alpha(\varphi_c) \frac{\kappa A^4(\varphi_c) \alpha \left( \tilde{p}_c c^2 - 3 \tilde{p}_c \right) + V'(\varphi_c)}{1 - \frac{\kappa}{2} A^4(\varphi_c) B^2(\varphi_c) \tilde{p}_c} \right] r^3 + \mathcal{O}(r^5).$$ (56)

Therefore, for $\left| 1 - \frac{\kappa}{2} A^4(\varphi_c) B^2(\varphi_c) \tilde{p}_c \right| \ll 1$, the Taylor series solutions for $\bar{\mu}(r)$ and $\bar{\nu}(r)$ break down, which indicates that the metric functions in the Jordan frame $\bar{\mu}$ and $\bar{\nu}$ diverge at some finite radius and a curvature singularity appears there.

C. Exterior solution

In the vacuum region outside the star $r > R$, the fluid variables $\tilde{p}$, $\tilde{p}_r$, and $\tilde{p}_i$ vanish. The exterior solution should be the vacuum solution of GR coupled to the massless canonical scalar field. The following exact solution can be obtained [15, 111]

$$ds^2 = -e^{-\nu(\varphi)} c^2 dt^2 + e^{-\nu(\varphi)} \left[ d\rho^2 + \left( \rho^2 - \frac{2 G s}{c^2 \rho} \right) \gamma_{ij} d\theta^i d\theta^j \right],$$ (57)

$$\nu(\varphi) = \nu_0 + \ln \left( 1 - \frac{2 G s}{c^2 \rho} \right) \frac{\varphi}{2 M},$$ (58)

$$\varphi(\rho) = \varphi_0 - \frac{Q}{2 M} \ln \left( 1 - \frac{2 G s}{c^2 \rho} \right) \frac{\varphi}{2 M},$$ (59)
Thus the integration constants \( J \) and \( Q \) correspond to the Arnowitt-Deser-Misner (ADM) mass and the scalar charge in the Einstein frame, respectively. For later convenience we also define the fractional binding energy

\[
E_b := \frac{M_b}{M} - 1,
\]

which is positive for bound (but not necessarily stable) configurations. We note that for the vanishing scalar field at asymptotic infinity the ADM mass is disformally invariant, \( \tilde{M} = M \) [see Eq. (A10)].

In the slowly rotating case, the integration of Eq. (44) in vacuum \( \tilde{p} = \tilde{p}_t = 0 \) gives

\[
\omega' = \frac{6G}{c^2r^4} \tilde{\omega} J,
\]

where \( J \) is the integration constant. In the vacuum case, we can find the exact exterior solution at the first order in rotation [16]. Expanding it in the vicinity of \( r \to \infty \) gives

\[
\omega = \Omega - 2GJ \frac{\omega}{c^2r^3} + O\left(\frac{1}{r^4}\right).
\]

Thus \( J \) corresponds to the angular momentum in the exterior spacetime.

### D. Matching

At the surface of the star, the interior solution is matched to the exterior solution (57). Then the cosmological value of the scalar field \( \varphi_0 \), the ADM mass \( M \) and the scalar charge \( Q \) are evaluated as

\[
\varphi_0 = \varphi_s + \ln \left(\frac{x_1 + x_2}{x_1 - x_2}\right) \frac{\psi_s}{x_2},
\]

\[
M = \frac{c^2 R^2 \nu'}{2G} \left(1 - \frac{2\mu_s}{R}\right) \frac{1}{(x_1 + x_2)} \frac{\nu'}{x_1 - x_2} - \frac{\nu^2}{x_1 - x_2},
\]

\[
q := \frac{Q}{M} = -\frac{2\psi_s}{\nu_s},
\]

where we introduced \( x_1 := \nu'_s + 2/r \) and \( x_2 := \sqrt{\nu_s^2 + \psi_s^2} \). We also defined \( \mu_s := \mu(R) \) and \( \nu_s := \nu(R) \).

In the case of a slowly rotating star, the angular velocity and angular momentum of the star, \( \Omega \) and \( J \), are evaluated as

\[
\Omega = \omega_s - \frac{3c^2J}{4G^2M^3(3 - \alpha(\phi_s)^2)} \left[\frac{4}{x_1^2 - x_2^2} \left(\frac{x_1 - x_2}{x_1 + x_2}\right)^{2\nu_s'/x_2} \times \left(\frac{3\nu_s'}{R} + \frac{1}{R^2} + 3\nu_s^2 - \psi_s^2\right) - 1\right],
\]

\[
J = \frac{c^2R^4}{6G} \sqrt{1 - \frac{2\mu_s}{R}} e^{-4\omega_s R} \tilde{\omega}_s.
\]

The moment of inertia can be obtained by

\[
I := \frac{J}{\Omega},
\]

or equivalently by integrating Eq. (44), using Eqs. (34) and (35)

\[
I = \frac{8\pi}{3c^2} \int_0^R dr A^4(\varphi) \sqrt{x} r^4 e^{-\frac{\omega}{x^2}} (\tilde{\rho} e^2 + \tilde{p}_b) \left(\frac{\omega}{\Omega}\right).
\]

We observe that this relation for the moment of inertia holds for any choice of \( P(X, \varphi), A(\varphi) \) and \( B(\varphi) \). In the purely conformal theory we obtain the result of Ref. [55].

For a given EOS the equations of motion (46) and (49) are numerically integrated from \( r = 0 \) up to the surface of the star \( r = \tilde{R} \), where the pressure vanishes \( \tilde{p}(R) = 0 \). With the values of various variables at the surface at hand, we can compute \( \varphi_0, M, q \) and \( I \) using the matching conditions.

From the Einstein frame radius \( R \), we can calculate the physical Jordan frame radius \( \tilde{R} \) through [cf. Eq. (2)]

\[
\tilde{R} := \sqrt{A^2(\varphi) [R^2 + \Lambda B^2(\varphi) \psi_s^2]}.
\]

where we introduced \( \varphi_s := \varphi(R) \) and \( \psi_s := \psi(R) \). For a vanishing scalar field we have \( \tilde{R} = R \).

The total baryonic mass of the star \( M_b \) can be obtained by integrating

\[
M_b = \int_0^R dr A^3(\varphi) \sqrt{\frac{4\pi \tilde{m}_b r^2}{\sqrt{1 - \frac{2\tilde{m}_b}{r}}} \tilde{n}(r)},
\]

where \( \tilde{m}_b = 1.66 \times 10^{-24} \text{ g} \) is the atomic mass unit and \( \tilde{n} \) is the baryonic number density.

In the Appendix we show that the physical quantities related to the rotation of fluid and spacetime, namely \( I \) and \( J \) as well as \( \omega \) and \( \Omega \), are invariant under the disformal transformation (2).
V. A TOY MODEL OF SPONTANEOUS SCALARIZATION
WITH AN INCOMPRRESSIBLE FLUID

Before carrying out the full numerical integrations of the stellar structure equations it is illuminating to study under which conditions scalarization can occur in our model. This can be accomplished by studying a simple toy model where a scalar field lives on the background of an incompressible fluid star. The results obtained in this section will be validated in Sec. VI.

Let us start by assuming that the star has a constant density \( \rho \) (incompressible) and an isotropic pressure \( p = p_r = p_t \). The scalar field \( \varphi \) is massless, and has a canonical kinetic term and small amplitude, such that we can linearize the equations of motion. The conformal and disformal coupling functions can be expanded as

\[
A(\varphi) = 1 + \frac{1}{2} \beta_1 \varphi^2 + O(\varphi^3),
\]

\[
B(\varphi) = 1 + \frac{1}{2} \beta_2 \varphi^2 + O(\varphi^3),
\]

where we have defined \( \beta_1 = A_{\varphi\varphi}(0) \) and \( \beta_2 = B_{\varphi\varphi}(0) \). As at the background level the scalar field is trivial \( \varphi = 0 \), the Jordan and Einstein frames coincide, and \( \bar{\rho} = \rho \) and \( \bar{\rho} = \bar{p} = p \).

For an incompressible star, the Einstein field equations admit an exact solution of the form (21) given by [112]

\[
e^\lambda(r) = \left(1 - \frac{2GMr^2}{c^2R^3}\right)^{-1},
\]

\[
e^\nu(r) = \left[\frac{3}{2} \left(1 - \frac{2GM}{c^2R}\right)^{1/2} - \frac{1}{2} \left(1 - \frac{2GM^2}{c^2R^3}\right)^{1/2}\right]^2,
\]

\[
p(r) = \rho c^2 \left(1 - \frac{2GM^2}{c^2R^3}\right)^{1/2} - \frac{1}{2} \left(1 - \frac{2GM}{c^2R}\right)^{1/2}
\]

\[
3 \left(1 - \frac{2GM}{c^2R}\right)^{1/2} - \frac{1}{2} \left(1 - \frac{2GM^2}{c^2R^3}\right)^{1/2},
\]

where \( r = R \) is the surface of the star, at which \( p(R) = 0 \). Here, \( M \) and \( C \) are the total mass and compactness of the star:

\[
M = \frac{4\pi R^3}{3} \rho, \quad C = \frac{GM}{c^2R}.
\]

We then consider the perturbations to the background (74) induced by the fluctuations of \( \varphi \). Since the corrections to the Einstein equations appear in \( O(\varphi^2, \varphi_\mu^2) \), at the leading order of \( \varphi \) only the scalar field equation of motion becomes non-trivial. In the linearized approximation, \( \chi = 1 + O(\varphi^2) \), \( \alpha = \beta_1 \varphi + O(\varphi^2) \) and \( \beta = \beta_2 \varphi + O(\varphi^2) \), and the scalar field equation of motion (20) for the massless and minimally coupled scalar field \( P = 2X \) reduces to

\[
\left( g^{\rho\sigma} - \frac{\kappa \Lambda}{2} T^{\rho\sigma}_{(m)} \right) \varphi_{\rho\sigma} = -\frac{\kappa \beta_1}{2} T^{\rho}_{(m)} \rho \varphi + O(\varphi^2, \varphi_\mu^2).
\]

Thus, as expected, in the Einstein frame the corrections from disformal coupling appear as the modification of the kinetic term via the coupling to the energy-momentum tensor.

Taking the \( s \)-wave configuration for a stationary field, \( \dot{\varphi} = \ddot{\varphi} = 0 \), we get

\[
\varphi'' + \frac{\varphi - \chi'}{2} + \frac{2}{r} - \frac{\kappa \Lambda}{2} \left( -\frac{\varphi'}{2} + \frac{2}{r} \right) p(r) \varphi' + \frac{1 - \frac{\kappa \beta_1}{2} \varphi}{1 - \frac{\kappa \beta_1}{2} \varphi} p(r) \varphi + \mathcal{O}(\varphi^2, \varphi^2) = 0.
\]

Inside the star, the scalar field equation of motion in the stationary background (77) can be expanded as

\[
\varphi'' + \frac{2}{r} \left[1 + \mathcal{O} \left( \frac{C r^2}{R^2} \right) \right] \varphi' + u \left[1 + \mathcal{O} \left( \frac{C r^2}{R^2} \right) \right] \varphi = 0,
\]

where we have defined

\[
u := \frac{6(3\sqrt{1 - 2C} - 2) C}{(3\sqrt{1 - 2C} - 1) R^2 + 3C (\sqrt{1 - 2C} - 1) A} \left| \beta_1 \right|.
\]

By neglecting the correction terms of order \( O \left( \frac{C r^2}{R^2} \right) \) in Eq. (78), the approximated solution inside the star satisfying the regularity boundary condition at the center, \( \varphi(0) = \varphi_c \) and \( \varphi'(0) = 0 \), is given by

\[
\varphi(r) \approx \varphi_c \sin \left( \frac{\sqrt{ur}}{\sqrt{ur}} \right).
\]

We note that at the surface of the star, \( r = R \), the corrections to this approximate solution (80) would be of \( O(\mathcal{C}) \), which is negligible for \( \mathcal{C} \approx 1 \) and gives at most a 10% error even for \( \mathcal{C} \approx 0.1 \). Thus the solution (80) provides a good approximation to the precise interior solution of Eq. (77), up to corrections of \( O(10\%) \) for typical NSs.

Outside the star, where \( \bar{\rho} = \bar{p} = 0 \), the scalar field equation of motion (77) reduces to

\[
\varphi'' + \left(\frac{1}{r} + \frac{1}{r - \frac{2GM}{c^2r}}\right) \varphi' = 0.
\]

The exterior solution of the scalar field is given by

\[
\varphi(r) = \varphi_0 + \frac{Q}{2M} \ln \left(1 - \frac{2GM}{c^2r} \right),
\]

which can be expanded as

\[
\varphi(r) = \varphi_0 - \frac{GQ}{c^2r} + \mathcal{O} \left( \frac{1}{r^2} \right),
\]

where \( Q \) denotes scalar charge. Matching at the surface \( r = R \) gives

\[
\frac{GQ}{c^2R \varphi_0} = -\frac{2C (1 - 2C) (\sqrt{u}R - \tan(\sqrt{u}R))}{\Xi},
\]

\[
\frac{\varphi_c}{\varphi_0} = -\frac{2C \sqrt{u} R}{\cos(\sqrt{u}R)} \frac{1}{\Xi},
\]
where we introduced
\[ \Xi = (1 - 2C) \sqrt{uR} \ln (1 - 2C) - [2C + (1 - 2C) \ln (1 - 2C)] \tan (R \sqrt{u}) \] (86)

The scalar charge \( Q \) and the central value of the scalar field \( \varphi_c \) blow up when
\[ \tan \left( \sqrt{\frac{uR}{2}} \right) = \frac{(1 - 2C) \ln (1 - 2C)}{2C + (1 - 2C) \ln (1 - 2C)}. \] (87)

Thus, inside the star, the scalar field can be enhanced and the scalarization takes place when
\[ \sqrt{uR} \approx \frac{\pi}{2} \left( 1 + \frac{4}{\pi^2} C \right). \] (88)

The condition (88) can be rewritten as
\[ |\beta_1^{\text{crit}}| \approx \frac{\pi^2}{24C} \frac{3 \sqrt{1 - 2C} - 1 + 3C \left( \sqrt{1 - 2C} - 1 \right)}{3 \sqrt{1 - 2C} - 2} \times \left( 1 + \frac{4}{\pi^2} C \right)^2, \] (89)

where \( \beta_1^{\text{crit}} \) is the critical value of \( \beta_1 \) for which scalarization can be triggered.

For small compactness \( C \ll 1 \), we find at leading order
\[ |\beta_1^{\text{crit}}| \approx \frac{\pi^2}{12C} \left( 1 - \frac{3C^2}{2R^2} \Lambda \right). \] (90)

For a typical NS, the compactness parameter \( C \approx 0.2 \), and if \( \Lambda \) is negligibly small \( |\beta_1^{\text{crit}}| \approx \pi^2/(12C) \approx 4.1 \), which agrees with the ordinary scalarization threshold [15, 17]. On the other hand, disformal coupling becomes important when \( \Lambda \approx (R/C)^2 \), which for \( R \sim 10 \text{ km and } C \approx 0.2 \), corresponds to \( \Lambda \approx 2500 \text{ km}^2 \).

In the other limit, for sufficiently large negative disformal coupling parameters \( |\Lambda| \gg (R/C)^2 \), as \( uR^2 \approx 2R^2/(|\Lambda|C^2) \ll 1 \), from Eqs. (84) and (85) we have
\[ \frac{GQ}{c^2 R \varphi_0} \approx - \frac{1 - 2C}{3} uR^2 \ll 1 \quad \text{and} \quad \varphi_c \approx \varphi_0, \] (91)

and the scalar field excitation is suppressed inside the star; the stellar configuration is that of GR.

In the next section, we will show explicit examples of the numerical integrations of the stellar structure and scalar field equations [(46) and (49)], and explore how the disformal coupling affects the standard scalarization mechanism in the models proposed in Refs. [15, 16]. We will confirm our main conclusions from the perturbative calculations presented here.

VI. NUMERICAL RESULTS

Having gained analytical insight into the effect of the disformal coupling on spontaneous scalarization, we now will perform full numerical integrations of the stellar structure equations.

For simplicity, we will focus on the simple case of a canonical scalar field without a potential, \( V(\varphi) = 0 \), and we will assume the special form of the coupling functions that enter Eq. (2)
\[ A(\varphi) = e^{\beta_1 \varphi^2}, \quad B(\varphi) = e^{\beta_2 \varphi^2}, \] (92)
as a minimal model to include the disformal coupling in our problem. In the absence of the disformal coupling function \( (\Lambda = 0) \), this model reduces to that studied originally by Damour and Esposito-Farèse [15, 16]. Another input from the theory is the cosmological value of the scalar field \( \varphi_0 \), which for simplicity we take to be zero throughout this section. We also studied the case \( \varphi_0 = 10^{-3} \), which does not alter our conclusions.

Under these assumptions our model is invariant under the transformation \( \varphi \to -\varphi \) (reflection symmetry). Therefore for each scalarized NS with scalar field configuration \( \varphi \), there exists a reflection-symmetric counterpart with \( \varphi \to -\varphi \). For both families of solutions the bulk properties (such as masses, radii and moment of inertia) are the same, while the scalar charges \( Q \) have opposite sign, but the same magnitudes. Moreover, \( \varphi = 0 \) is a trivial solution of the stellar structure equations. These solutions are equivalent to NSs in GR.

In this section we sample the \( (\beta_1, \beta_2, \Lambda) \) parameter space of the theory, analyzing each parameter’s influence on NS models and on spontaneous scalarization. As mentioned in Sec. I, binary-pulsar observations have set a constraint of \( \beta_1 \gtrsim -4.5 \) in what corresponds to the purely conformal coupling \( (\Lambda = 0) \) limit of our model. This lower bound on \( \beta_1 \) is not expected to apply for our more general model and therefore, so far, the set of parameters \( (\beta_1, \beta_2, \Lambda) \) are largely unconstrained.

A. Equation of state

To numerically integrate the stellar structure equations we must complement them with a choice of EOS. Here we consider three realistic EOSs, namely APR [113], SLy4 [114] and FPS [115], in decreasing order of stiffness. The first two support NSs with masses larger than the \( M = 2.01 \pm 0.04M_\odot \) lower bound from the pulsar PSR J0348+0432 in GR [116]. On the other hand, EOS FPS has a maximum mass of \( \sim 1.8M_\odot \) in GR and is in principle ruled out by Ref. [116]. Nevertheless, as we will see this EOS can support NSs with \( M \gtrsim 2M_\odot \), albeit scalarized, for certain values of the theory’s parameters.

With this set of EOSs we validated our numerical code by reproducing the results of Refs. [28, 55] in the purely conformal coupling limit. Our results including the presence of the disformal coupling are presented next.

B. Stellar models in the minimal scalar-tensor theory with disformal coupling

In Sec. V we found that \( \beta_1 \) always needs to be sufficiently negative for scalarization to be triggered. For this reason, let
us first analyze how $\Lambda$ and $\beta_2$ affect scalarized nonrotating NSs assuming a fixed value of $\beta_1$.

In Fig. 1, we consider what happens when we change the value of $\Lambda$ while maintaining $\beta_1$ and $\beta_2$ fixed. We observe that for sufficiently negative values of $\Lambda$ the effects of scalarization become suppressed. This can be qualitatively understood from Eq. (90): as $\Lambda/R^2 \to -\infty$ we need $|\beta_1^\text{con}| \to \infty$ for scalarization to happen. For fixed values of $\beta_1$ and $\mathcal{C}$, there will be a sufficiently negative value of $\Lambda$, for which $\beta_1^\text{con} > \beta_1$ and scalarization ceases to occur. Although in Fig. 1 we show $\Lambda = -3000$ km$^2$, we have confirmed this by constructing stellar models for even smaller values of $\Lambda$. Also, in agreement with Sec. V, we see that $\Lambda$ alters the threshold for scalarization.

This is most clearly seen in the right panel of Fig. 1, where for different values of $\Lambda$ scalarization starts (evidenced by a nonzero scalar charge $q$) when different values of compactness $\mathcal{C}$ are reached.\footnote{In the preceding section, because of the weak (scalar) field approximation the Jordan and Einstein frame radii are approximately the same, i.e $R = R'$. This is not the case in this section and hereafter the compactness uses the Jordan frame radius, i.e. $C = GM/(c^2 R)$.} In particular, for $\Lambda > 0$, because of the minus sign in the disformal term in Eq. (90), NSs can scalarize for smaller values of $\mathcal{C}$, while the opposite happens when $\Lambda < 0$. We remark that for large positive $\Lambda$ the structure becomes singular at the origin as discussed in Sec. IV. This prevents nonrelativistic stars, for which $C \to 0$, from scalarizing.

In Fig. 2, we consider what happens when we change the value of $\beta_2$ while maintaining $\beta_1$ and $\Lambda$ fixed. We see that in agreement with Eq. (90), the parameter $\beta_2$ does not affect the threshold for scalarization. Moreover, we observe that $\beta_2 < 0$ ($\beta_2 > 0$) makes scalarization more (less) evident with respect to $\beta_2 = 0$. In fact, in Eqs. (48) and (47), we see that $\beta_1$ and $\beta_2$ contribute to the scalar field equation through the factors $\Lambda A^2 B^2$ and $\beta - \alpha$, which have competing effects in sourcing the scalar field for $\beta_1 < 0$ and $\beta_2 \neq 0$. Our numerical integrations indicate that the former is dominant and that $\beta_2 \neq 0$ affects only very compact NSs ($C \gtrsim 0.15$ in the example of Fig. 2).

It is also of interest to see how scalarization affects the interior of NSs. In Fig. 3, we show the normalized pressure profile $p/p_c$ (top left), the dimensionless mass function $\mu/M_\odot$ (top right), the scalar field $\varphi$ (bottom left) and the disformal factor $\chi$ (bottom right) in the stellar interior. The radial coordinate was normalized by the Einstein frame radius $R$. The quantities correspond to three stellar configurations using SLy4 EOS with fixed baryonic mass $M_b/M_\odot = 1.5$, which in GR yields a canonical NS with mass $M \approx 1.4 M_\odot$, for the sample values of $(\beta_1, \beta_2, \Lambda)$ indicated in Table I. In agreement with our previous discussion we see that NSs with $\Lambda > 0$ ($\Lambda < 0$) support a larger (smaller) value of $\varphi_c$, which translates to a larger (smaller) value of $q$. It is particularly important to observe that $\chi$ is non-negative for all NS models, guaranteeing the Lorentzian signature of spacetime [cf. Eq. (9)].

In Fig. 4 we show the mass-radius curves (top panels) and moment of inertia-mass (lower panels) for increasing values of $\beta_1$ (from left to right), for three realistic EOSs, keeping $\beta_2 = 0$, but using different values of $\Lambda$. As we anticipated in Fig. 1, negative values of $\Lambda$ reduce the effects of scalarization, while positive values increase them. The case $\Lambda = 0$ corresponds to the purely conformal theory of Ref. [15]. We observe that scalarized NS models branch from the GR family at different points for different values of $\Lambda$ (when $\beta_1$ is fixed). In agreement with our previous discussion, sufficiently negative values of $\Lambda$ can completely suppress scalarization. Indeed for $\beta = -4.5$ the solutions with $\Lambda = -1000$ km$^2$ are identical to GR, while scalarized solutions exist when $\Lambda = 0$.

Additionally, we observe degeneracy between families of solutions in theories with different parameters. For instance, the maximum mass for a NS assuming EOS APR is approximately the same, $M/M_\odot \approx 2.38$, for both $\beta_1 = -5.5, \Lambda = 0$ and $\beta_1 = -6.0, \Lambda = -1000$ km$^2$. We also point out the degeneracy between the choice of EOS and of the parameters of the theory. For instance, the maximum mass predicted by

![Figure 1](image1.png)

**Fig. 1.** We show the role of $\Lambda$ in spontaneous scalarization. In both panels we consider stellar models using EOS SLy4 with $\beta_1 = -6.0, \beta_2 = 0$ and for $\Lambda = (-500, -3000, 50)$ km$^2$. For reference the solid line corresponds to GR. Left panel: The mass-radius relation. Right panel: The dimensionless scalar charge $q := -Q/M$ [15] as a function of the compactness $\mathcal{C}$. We see that $\Lambda > 0$ slightly increases scalarization with respect to the purely conformal theory (cf. Fig. 2).

![Table I](image2.png)

**TABLE I.** The properties of NSs in GR and scalar-tensor theory using EOS SLy4 and fixed baryonic mass $M_b/M_\odot = 1.5$. The radial profiles of some of the physical variables involved in the integration of the stellar model are shown in Fig. 3.
We observe that independently of the choice of EOS, solutions obtained in this section. In general, for a given set of the mass $M$ and scalar field $\varphi_0$, we have more than one stellar configuration with different values of the mass $M$. Following the arguments of Refs. [15, 17, 35], we take the solution of smallest mass $M$, i.e., larger fractional binding energy $E_b$, defined in Eq. (63), to be the one which is energetically favorable to be realized in nature. In Fig. 6, we show $E_b$, as a function of $M_b$, for the two families of solutions in a theory with $(\beta_1, \beta_2, \Lambda) = (-6, 0, 50)$ and $\varphi_0 = 0$. The dashed line corresponds to solutions which are indistinguishable from the ones obtained in GR, while the solid line (which branches off from the former around $M_b/M_\odot \approx 1.1$) corresponds to scalarized solutions. We see that scalarized stellar configurations in our model are energetically favorable, as happens in the case of purely conformal coupling theory [15, 17, 35].

C. Stability of the solutions

Let us briefly comment on the stability of the scalarized solutions obtained in this section. In general, for a given set of parameters $(\beta_1, \beta_2, \Lambda)$ and fixed values of $M_b$ and $\varphi_0$, we have more than one stellar configuration with different values of the mass $M$. Following the arguments of Refs. [15, 17, 35], we take the solution of smallest mass $M$, i.e., larger fractional binding energy $E_b$, defined in Eq. (63), to be the one which is energetically favorable to be realized in nature. In Fig. 6, we show $E_b$, as a function of $M_b$, for the two families of solutions in a theory with $(\beta_1, \beta_2, \Lambda) = (-6, 0, 50)$ and $\varphi_0 = 0$. The dashed line corresponds to solutions which are indistinguishable from the ones obtained in GR, while the solid line (which branches off from the former around $M_b/M_\odot \approx 1.1$) corresponds to scalarized solutions. We see that scalarized stellar configurations in our model are energetically favorable, as happens in the case of purely conformal coupling theory [15, 17, 35].

VII. AN APPLICATION: EOS-INDEPENDENT I-C RELATIONS

As we have seen in the previous sections the presence of the disformal coupling modifies the structure of NSs making scalar-tensor theories generically predict different bulk properties with respect to GR. However, as we discussed based on Figs. 4 and 5, modifications caused by scalarization are usually degenerate with the choice of EOS, severely limiting our ability to constrain the parameters of the theory using current
FIG. 4. We show NS models in scalar-tensor theories with disformal coupling for three choices of realistic EOSs, namely APR, SLy4 and FPS, in decreasing order of stiffness. We illustrate the effect of varying the values of $\beta_2$ and $\Lambda$, while keeping $\beta_1$ fixed ($\beta_2 = 0$) for simplicity. The curves corresponding to $\Lambda = 0$, represent stellar models in purely conformal theory [15, 16]. Top panels: Mass-radius relations. Bottom panels: Moment of inertia versus mass. As seen in Fig. 1 already, while $\Lambda < 0$ weakens scalarization, $\Lambda > 0$ strengthens the effect. For $\beta_2 = 0$, this latter effect is very mild, being more evident by $\beta_2 < 0$ (cf. Fig. 5).

NS observations (see e.g. Ref. [118]). Moreover, different theory parameters can yield similar stellar models for a fixed EOS.

An interesting possibility to circumvent these problems is to search for EOS-independent (or at least weakly EOS-dependent) properties of NSs. Accumulating evidence favoring the existence of such EOS independence between certain properties of NSs, culminated with the discovery of the $I$-Love-$Q$ relations [119, 120] connecting the moment of inertia, the tidal Love number and the rotational quadrupole moment (all made dimensionless by certain multiplicative factors) of NSs in GR.

If such relations hold in modified theories of gravitation they can potentially be combined with future NS measurements to constrain competing theories of gravity. This attractive idea was explored in the context of dynamical Chern-Simons theory [120], Eddington-inspired Born-Infeld gravity [121], Einstein-dilaton-Gauss-Bonnet (EdGB) gravity [122, 123], $f(R)$ theories [124] and the Damour-Esposito-Farèse model of scalar-tensor gravity [24, 26].

Within the present framework we cannot compute the $I$-Love-$Q$ relations, since while on one hand we can compute $I$, the tidal Love number requires an analysis of tidal interactions, and the rotational quadrupole moment $Q$ requires pushing the Hartle-Thorne perturbative expansion up to order $O(\Omega^2)$. Nevertheless, we can investigate whether the recently proposed $I$-$C$ relations [125] between the moment of inertia $I$ and the compactness $C$ remain valid in our theory. For a
FIG. 5. In comparison to Fig. 4, here we show the influence of $\beta_2$ in spontaneous scalarization while keeping $\Lambda = -1000$ km$^2$. As we have seen in Fig. 1 (and by the analytic treatment of Sec. V), negative values of $\Lambda$ suppress scalarization. This effect is such that for $\beta_1 = -4.5$, scalarization is suppressed altogether (top left panel). For smaller values of $\beta_1$, this value of $\Lambda$ weakens scalarization and we clearly see that $\beta_2$ affects the most scalarized stellar models in the conformal-coupling theory. Note that the range covered by the axis here and in Fig. 4 is the same, making it clear that scalarization is less strong for the values of $\beta_2$ adopted.

recent study in the Damour-Esposito-Farèse and $R^2$ theories, see Ref. [126]. This relation was also studied for EdGB and the subclass of Horndeski gravity with nonminimal coupling between the scalar field and the Einstein tensor in Ref. [127]. The relation proposed in Ref. [125] for the moment of inertia $\bar{I} := I/M^2$ and the compactness $\mathcal{C}$ is

$$
\bar{I} = a_1 \mathcal{C}^{-1} + a_2 \mathcal{C}^{-2} + a_3 \mathcal{C}^{-3} + a_4 \mathcal{C}^{-4},
$$

(93)

where the coefficients $a_i$ (i = 1, ..., 4) are given by $a_1 = 8.134 \times 10^{-1}$, $a_2 = 2.101 \times 10^{-1}$, $a_3 = 3.175 \times 10^{-3}$ and $a_4 = -2.717 \times 10^{-4}$. This result is valid for slowly rotating NSs in GR, although it can easily be adapted for rapidly rotating NSs [125]. The coefficients in Eq. (93) are obtained by fitting the equation to a large sample of EOSs. For earlier work considering a different normalization for $\bar{I}$, namely $\bar{I}/(MR^2)$, see e.g. Refs. [128–132].

We confront this fit against stellar models in two scalar-tensor theories with the parameters $(\beta_1, \beta_2, \Lambda)$ having the values $(-6, -20, -500)$ and $(-7, -20, -500)$ that support highly scalarized solutions. As seen in Fig. 7, the deviations from GR can be quite large, up to 40% for the theory with $\beta_1 = -7$ in the range of compactness for which spontaneous scalarization happens (cf. Fig. 7, bottom panel). Nevertheless, the EOS independence between $\bar{I}$ and $\mathcal{C}$ remains even when scalarization occurs (cf. Fig. 7, top panel).

Since our model is largely unconstrained observationally, measurements of the moment of inertia and compactness of NSs could in principle be used to constrain it or, more opti-
so tightly constrained by binary pulsar observations of the Esposito-Farèse model, for which the theory’s parameters are relativistically, indicate the occurrence of spontaneous scalarization in NSs. This is in contrast with the standard Damour-Esposito-Farèse model in scalar-tensor theories, especially, conformal and disformal couplings to matter, nonstandard scalar kinetic terms and coupling due to slow rotation. The disformal coupling is negligibly favor over the general relativistic ones (dashed line). The turning point at the solid curve corresponds to the maximum in the $M/R$ relation, cf. Fig. 1.

**FIG. 6.** We show the fractional binding energy $E_b$ as a function of the baryonic mass for stellar models using EOS SLy4 and for theory with $(\beta_1, \beta_2, \Lambda) = (-6, 0, 50)$. Solutions in this theory branch around $M_0/M_0 \approx 1.1$ with scalarized solutions (solid line) being energetically favorable over the general relativistic ones (dashed line). The turning point at the solid curve corresponds to the maximum in the $M/R$ relation, cf. Fig. 1.

mistically, indicate the occurrence of spontaneous scalarization in NSs. This is in contrast with the standard Damour-Esposito-Farèse model, for which the theory’s parameters are tightly constrained by binary pulsar observations [133], that spontaneous scalarization (if it exists) is bound to have a negligible influence on the $I-C$ relation [126]. We stress however that in general it will be difficult to constrain the parameter space $(\beta_1, \beta_2, \Lambda)$ only through the $I-C$ relation. The reason is in the degeneracy of stellar models for different values of the parameters; see the discussion in Sec. VI.B.

**FIG. 7.** We consider the $I-C$ relation in scalar-tensor gravity. Top panel: The fit (93) obtained in the context of GR (thick solid line) is confronted against stellar models obtained in GR (solid line); and scalar-tensor theories with parameters $(\beta_2, \Lambda) = (-20, -50)$, but with $\beta_1 = -6$ (dashed lines) and $\beta_2 = -7$ (dashed-dotted lines), using EOSs APR, SLy4 and FPS. Middle panel: Relative error between the fit for GR against scalar-tensor theory with $\beta_1 = -6$. Bottom panel: Similarly, but for $\beta_1 = -7$. In all panels the shaded regions correspond to approximately the domain of compactness for which spontaneous scalarization occurs in each theory. While for GR, the errors are typically below 6%, scalarized models can deviate from GR by 20% (for $\beta_1 = -6$) and up to 40% (for $\beta_1 = -7$).

**VIII. CONCLUSIONS AND OUTLOOK**

In this paper we have presented a general formulation to analyze the structure of relativistic stars in scalar-tensor theories with disformal coupling, including the leading-order corrections due to slow rotation. The disformal coupling is negligibly small in comparison with conformal coupling in the weak-gravity or slow-motion regimes, where the scalar field is slowly evolving and typical pressures are much smaller than the energy density scales, but it may be comparable to the ordinary conformal coupling in the strong-gravity regime found inside relativistic stars. Our calculation covers a variety of scalar-tensor models, especially, conformal and disformal couplings to matter, nonstandard scalar kinetic terms and generic scalar potential terms.

After obtaining the stellar structure equations, we have particularly focused on the case of a canonical scalar field with a generic scalar potential. We showed that in the absence of both conformal coupling and a scalar potential, the disformal coupling does not modify the stellar structure with respect to GR. On the other hand, this result shows us that inside relativistic stars the effects of disformal coupling always appear only when there is conformal coupling to matter and/or a non-trivial potential term. The strength of disformal coupling crucially depends on the coupling strength $\Lambda$ in Eq. (2) with dimensions of (length)$^2$. For a canonical scalar field, $\Lambda$ has to be of $\mathcal{O}(10^3)$ km$^2$ to significantly influence the structure of NSs.

In our numerical analyses, we have investigated the effects of the disformal coupling on the spontaneous scalarization of NSs in the scalar-tensor theory with purely conformal coupling. We found that the effects of disformal coupling depend on the sign of $\Lambda$. We showed that for negative values of $\Lambda$ the mass and moment of inertia of NSs decrease, approaching the values in GR for sufficiently large negative values of $\Lambda$. We speculate that this is the consequence of a mechanism similar to the disformal screening proposed in Ref. [86] where in a high density or a large disformal coupling limit the response of the scalar field becomes insensitive to the local matter density, exemplified here by studying relativistic stars. On the other hand, for positive values of $\Lambda$, we showed that the mass and moment of inertia increase but for too large positive values of $\Lambda$ the stellar structure equation becomes singular and a regular NS solution cannot be found. This allowed us to derive a mild upper bound of $\Lambda \lesssim 100$ km$^2$, that does not depend on the choice of the EOS.

We have also tested the applicability of a recently proposed EOS-independent relation between the dimensionless
moment of inertia $I/M^3$ and the compactness $C$ for NSs in GR. We found that for a certain domain of the theory’s parameter space, the deviations from GR can be as large as $\sim 40\%$, suggesting that future measurements of NS moment of inertia might be used to test scalar-tensor theories with disformal coupling. Because of the large dimensionality of the parameter space, modifications with respect to GR are generically degenerate between different choices of $\beta_1$, $\beta_2$ and $\Lambda$. Thereby, even though deviations from GR can be larger, it seems unlikely that constraints can be put on the theory’s parameters using exclusively the $I-C$ relation. In this regard, it would be worth extending our work and studying how the $I-Love-Q$ relations are affected by the disformal coupling, generalizing the works of Refs. [24–26] for scalar-tensor theories with disformal coupling.

Still in this direction, one could investigate whether the “three-hair” relations – EOS-independent relations connecting higher-order multipole moments of rotating NSs in terms of the first three multipole moments in GR [134–136] - remain valid in scalar-tensor theory, including those with disformal coupling. This could be accomplished by combining the formalism developed in [29] with numerical solutions for rotating NSs such as those obtained in Ref. [24].

Although the main subject of this paper was to investigate the hydrostatic equilibrium configurations in scalar-tensor theories with disformal coupling, let us briefly comment on the gravitational (core) collapse resulting in the formation of a NS (see e.g. Ref. [34]). A fully numerical analysis of dynamical collapse in this theory is beyond the scope of our paper, but an important issue in this dynamical process may be the possible appearance of ghost instabilities for negative values of $\Lambda$ [72, 75, 86, 110]. During collapse, matter density at a given position increases, and if at some instant it reaches the threshold value where the effective kinetic term in the scalar field equation vanishes, the time evolution afterwards cannot be determined. For a canonical scalar field $P = 2X$, in a linearized approximation where $\chi \simeq 1$ and $B(\varphi) \simeq 1$, the effective kinetic term of the equation of motion (20) is roughly given by

$$-\left(1 - \frac{\kappa A|\varphi|}{2\bar{\rho}c^2}\right)\ddot{\varphi},$$

where a dot represents a time derivative. The sign of the kinetic term may change in the region of a critical density higher than $\bar{\rho}_{\text{crit}} = 2/\left(\kappa c^2|\Lambda|\right)$. The choice of $\Lambda = -100$ km$^2$ gives $\bar{\rho}_{\text{crit}} \simeq 10^{15}$ g/cm$^3$, which is a typical central density of NSs. Thus for $|\Lambda| \lesssim 100$ km$^2$ a NS is not expected to suffer an instability while for other values it might occur in the interior of the star. Of course, for a more precise estimation, nonlinear interactions between the dynamical scalar field, spacetime and matter must be taken into consideration. A detailed study of time-dependent processes in our theory is definitely important, but is left for future work.

Another interesting prospect for future work would be to study compact binaries within our model. The most stringent test of scalar-tensor gravity comes from the measurement of the orbital decay of binaries with asymmetric masses, which constrains the emission of dipolar scalar radiation by the system [53]. We expect that the disformal coupling parameters $\beta_2$ and $\Lambda$ should play a role in the orbital evolution of a binary system by influencing the emission of scalar radiation from the system. In fact, both parameters are expected to modify the so-called sensitivities [137, 138] that enter at the lowest PN orders sourcing the emission of dipolar scalar radiation. An investigation of compact binaries within our model could, combined with current observational data, yield tight constraints on disformal coupling. Moreover one could study NS solutions for other classes of scalar-tensor theories not considered here. This task is facilitated by the generality of our calculations presented in Sec. III. Work in this direction is currently underway and we hope to report it soon.

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**Appendix A: Disformal invariance**

In this appendix, we study how the physical quantities associated with the stellar properties transform under the disformal transformation (2). We write the metrics with slow rotation of spacetimes in the Einstein and Jordan frames as

$$ds^2 = -e^{\nu(r)}c^2 dt^2 + e^{\lambda(r)}dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) + 2(\omega - \Omega) r^2 \sin^2 \theta dt d\phi,$$  \hspace{1cm} (A1)

and

$$d\tilde{s}^2 = -e^{\tilde{\nu}(\bar{r})}c^2 d\bar{t}^2 + e^{\bar{\lambda}(\bar{r})}d\bar{r}^2 + \bar{r}^2 (d\bar{\theta}^2 + \sin^2 \bar{\theta} d\bar{\phi}^2) + 2(\bar{\omega} - \bar{\Omega}) \bar{r}^2 \sin^2 \bar{\theta} d\bar{t} d\bar{\phi}.$$  \hspace{1cm} (A2)

We can relate Eqs. (A1) and (A2) using the disformal relation (2) as

$$e^{\tilde{\nu}} = A^2(\varphi)e^{\nu},$$ \hspace{1cm} (A3)

$$e^{\bar{\lambda}} d\bar{r} = A(\varphi)\sqrt{c} e^{\lambda} dr,$$ \hspace{1cm} (A4)

$$\bar{r} = r A(\varphi),$$ \hspace{1cm} (A5)

$$\bar{\omega} - \bar{\Omega} = \omega - \Omega,$$ \hspace{1cm} (A6)

where we recall that due the symmetries of the problem $\varphi = \varphi(r)$. From Eqs. (A4) and (A5) we get

$$e^{\bar{\lambda}} = \frac{\chi}{(1 + r A'(\varphi))^2} e^{\lambda}.$$ \hspace{1cm} (A7)
Introducing $\mu$ and $\bar{\mu}$ in the Einstein and Jordan frames by
\[ e^{-\lambda} = 1 - \frac{2\mu}{r}, \quad e^{-\bar{\lambda}} = 1 - \frac{2\bar{\mu}}{r}, \quad (A8) \]
and using Eqs. (A5) and (A7) we find
\[ \bar{\mu} = -\frac{rA(\varphi)}{2} \left[ \left( 1 - \frac{2\mu}{r} \right) \left( 1 - \frac{r\alpha(\varphi)\varphi'}{\chi} \right)^2 - 1 \right]. \quad (A9) \]
As it is reasonable to set $\varphi_0 = 0$ and $\varphi'_0 = 0$ at asymptotic infinity, in the class of models considered in the text [Eq. (73)], $A(\varphi_0) = 1$, $\alpha(\varphi_0) = 0$ and $\chi(\varphi_0, \varphi'_0) = 1$, we find that the ADM mass obtained from the leading-order values of $\mu$ and $\bar{\mu}$ at asymptotic infinity is disformally invariant
\[ \bar{M} = M. \quad (A10) \]

The energy-momentum tensors of the matter fields in the Einstein and Jordan frames are defined by
\[ T_{(m)\mu\nu} = \rho c^2 u_{\mu} u_{\nu} + p_{r} \kappa_\mu \kappa_\nu + p_{t} (\dot{g}_{\mu\nu} + u_{\mu} u_{\nu} - \kappa_\mu \kappa_\nu), \]
\[ \bar{T}_{(m)\mu\nu} = \bar{\rho} c^2 \bar{u}_{\mu} \bar{u}_{\nu} + \bar{p}_{r} \bar{\kappa}_\mu \bar{\kappa}_\nu + \bar{p}_{t} (\dot{\bar{g}}_{\mu\nu} + \bar{u}_{\mu} \bar{u}_{\nu} - \bar{\kappa}_\mu \bar{\kappa}_\nu), \quad (A11) \]
where $u^\mu$ ($\bar{u}^\mu$) and $\kappa^\mu$ ($\bar{\kappa}^\mu$) are the four-velocity and unit radial vectors in the Einstein (Jordan) frame, respectively [55]. Within the first order of Hartle-Thorne’s slow-rotation approximation [106], in the Einstein frame
\[ u^\mu = \left( \frac{1}{\sqrt{-g_{tt}}}, 0, 0, \frac{\Omega}{\sqrt{-g_{tt}}} \right), \quad \bar{u}^\mu = \left( 0, \frac{1}{\sqrt{-g_{rr}}}, 0, 0 \right), \quad (A12) \]
and in the Jordan frame $\bar{u}^\mu$ and $\bar{\kappa}^\mu$ are defined in the same way as Eq. (A12) with an overbar. The nonvanishing components of the energy-momentum tensors in both frames are then given by
\[ T_{(m)t} = -\rho c^2, \quad T_{(m)r} = p_r, \quad T_{(m)\theta} = T_{(m)\phi} = p_t, \quad (A13a) \]
\[ \bar{T}_{(m)t} = -\bar{\rho} c^2, \quad \bar{T}_{(m)r} = \bar{p}_r, \quad \bar{T}_{(m)\theta} = \bar{T}_{(m)\phi} = \bar{p}_t, \quad (A13b) \]
and
\[ T_{(m)\phi} = \left( \frac{\rho}{c^2} \right) e^{-\nu} \omega^2 r^2 \sin^2 \theta, \quad (A14a) \]
\[ \bar{T}_{(m)\phi} = \left( \frac{\bar{\rho}}{c^2} \right) e^{-\nu} \omega^2 r^2 \sin^2 \theta. \quad (A14b) \]

In the Jordan frame, we then make a coordinate transformation from $\bar{x}^\mu = (t, \bar{r}, \theta, \phi)$ to $x^\mu = (t, r, \theta, \phi)$, such that
\[ \bar{T}_{(m)\mu}^\nu := \frac{\partial \bar{x}^\rho}{\partial x^\mu} \frac{\partial x^\sigma}{\partial \bar{x}^\nu} \bar{T}_{(m)\rho}^\sigma. \quad (A15) \]
Introducing the components of the energy-momentum tensor $T_{(m)\mu\nu}$ as (A13a)-(A13b) with a tilde, we find
\[ \bar{\rho} = \tilde{\rho}, \quad \bar{p}_r = \tilde{p}_r, \quad \bar{p}_t = \tilde{p}_t, \quad (A16) \]
and consequently
\[ \bar{T}_{(m)\phi} = \tilde{T}_{(m)\phi}. \quad (A17) \]
The components of the energy-momentum tensor in the Einstein and Jordan frames are related by (23) and
\[ T_{(m)\phi} = A^4(\varphi)\sqrt{\bar{T}_{(m)\phi}}. \quad (A18) \]
Substituting (23), (A3), (A5), (A14a), (A14b), (A16) and (A17) into Eq. (A18) we find
\[ \bar{\omega} = \omega. \quad (A19) \]
Thus from (A6),
\[ \bar{\Omega} = \Omega. \quad (A20) \]

The angular momenta in the Einstein and Jordan frames are given by
\[ J = \int dr d\theta d\phi \, r^2 \sin \theta \, e^{\nu + \lambda} T_{(m)\phi}, \quad (A21) \]
\[ \bar{J} = \int d\bar{r} d\theta d\phi \, \bar{r}^2 \sin \theta \, e^{\bar{\nu} + \bar{\lambda}} \bar{T}_{(m)\phi}. \quad (A22) \]
Using again (A3), (A4), (A5), (A17) and (A18), we find that the angular momentum is disformally invariant
\[ \bar{J} = J. \quad (A23) \]
From Eqs. (A20) and (A23) we find that the moments of inertia in the Einstein and Jordan frames, $I = \bar{J}/\bar{\Omega}$ and $\bar{I} = J/\Omega$, are also disformally invariant
\[ \bar{I} = I. \quad (A24) \]
Thus all quantities associated with rotation are disformally invariant. Our arguments in this appendix can be applied to a generic class of the Horndeski theory connected by the disformal transformation [60].

[1] C. M. Will, The Confrontation between General Relativity and Experiment, Living Rev. Rel. 17, 4 (2014), arXiv:1403.7377 [gr-qc].
[2] E. Berti et al., Testing General Relativity with Present and Future Astrophysical Observations, Class. Quant. Grav. 32, 243001 (2015), arXiv:1501.07274 [gr-qc].
[3] T. Clifton, P. G. Ferreira, A. Padilla, and C. Skordis, Modified Gravity and Cosmology, Phys. Rept. 513, 1–189 (2012), arXiv:1106.2476 [astro-ph.CO].
[4] B. . Abbott et al. (Virgo, LIGO Scientific), Observation of Gravitational Waves from a Binary Black Hole Merger, Phys. Rev. Lett. 116, 061102 (2016), arXiv:1602.03837 [gr-qc].
[5] B. P. Abbott et al. (Virgo, LIGO Scientific), Tests of general relativity with GW150914, (2016), arXiv:1602.03841 [gr-qc].
[6] N. Yunes, K. Yagi, and F. Pretorius, Theoretical Physics Implications of the Binary Black-Hole Merger GW150914, (2016), arXiv:1603.08955 [gr-qc].
[7] Y. Fujii and K.-I. Maeda, The Scalar-Tensor Theory of Gravitation, by Yasunori Fujii and Kei-ichi Maeda, pp. 256. ISBN 0521811597. Cambridge, UK: Cambridge University Press, March 2003. (Cambridge University Press, 2003).
[8] R. P. Woodard, Ostrowgradsky’s theorem on Hamiltonian instability, Scholarpedia 10, 32243 (2015), arXiv:1506.02210 [hep-th].
[9] A. Vainshtein, To the problem of nonvanishing gravitation mass, Phys. Lett. B39, 393–394 (1972).
[10] P. Brax, C. van de Bruck, A.-C. Davis, J. Khoury, and A. Weltman, Detecting dark energy in orbit - The Cosmological chameleon, Phys. Rev. D70, 123518 (2004), arXiv:astro-ph/0408415 [astro-ph].
[11] G. W. Horndeski, Second-order scalar-tensor field equations in a four-dimensional space, Int. J. Theor. Phys. 10, 363–384 (1974).
[12] C. Deffayet, S. Deser, and G. Esposito-Farèse, Generalized Galileons: All scalar models whose curved background extensions maintain second-order field equations and stress-tensors, Phys. Rev. D80, 064015 (2009), arXiv:0906.1967 [gr-qc].
[13] C. Deffayet, X. Gao, D. Steer, and G. Zahariade, From k-essence to generalised Galileons, Phys. Rev. D84, 064039 (2011), arXiv:1103.3260 [hep-th].
[14] T. Kobayashi, M. Yamaguchi, and J. Yokoyama, Generalized G-inflation: Inflation with the most general second-order field equations, Prog. Theor. Phys. 126, 511–529 (2011), arXiv:1105.5723 [hep-th].
[15] T. Damour and G. Esposito-Farèse, Nonperturbative strong field effects in tensor - scalar theories of gravitation, Phys. Rev. Lett. 70, 2220–2223 (1993).
[16] T. Damour and G. Esposito-Farèse, Tensor-scalar gravity and binary pulsar experiments, Phys. Rev. D54, 1474–1491 (1996), arXiv:gr-qc/9602056 [gr-qc].
[17] T. Harada, Stability analysis of spherically symmetric star in scalar-tensor theories of gravity, Prog. Theor. Phys. 98, 359–379 (1997), arXiv:gr-qc/9706014 [gr-qc].
[18] T. Chiba, T. Harada, and K.-i. Nakao, Gravitational physics in scalar tensor theories: Tests of strong field gravity, Prog. Theor. Phys. Suppl. 128, 335–372 (1997).
[19] H. Sotani and K. D. Kokkotas, Probing strong-field scalar-tensor gravity with gravitational wave astrophysics, Phys. Rev. D70, 084026 (2004), arXiv:gr-qc/0409066 [gr-qc].
[20] H. Sotani and K. D. Kokkotas, Stellars oscillations in scalar-tensor theory of gravity, Phys. Rev. D71, 124038 (2005), arXiv:gr-qc/0506060 [gr-qc].
[21] H. Sotani, Scalar gravitational waves from relativistic stars in scalar-tensor gravity, Phys. Rev. D89, 064031 (2014), arXiv:1402.5699 [astro-ph.HE].
[22] H. O. Silva, H. Sotani, E. Berti, and M. Horbatsch, Torsional oscillations of neutron stars in scalar-tensor theory of gravity, Phys. Rev. D90, 124044 (2014), arXiv:1410.2511 [gr-qc].
[23] H. Sotani, Slowly Rotating Relativistic Stars in Scalar-Tensor Gravity, Phys. Rev. D86, 124036 (2012), arXiv:1211.6986 [astro-ph.HE].
[24] D. D. Doneva, S. S. Yazadjiev, N. Stergioulas, and K. D. Kokkotas, Rapidly rotating neutron stars in scalar-tensor theories of gravity, Phys. Rev. D88, 084060 (2013), arXiv:1309.0605 [gr-qc].
[25] D. D. Doneva, S. S. Yazadjiev, K. V. Staykov, and K. D. Kokkotas, Universal I-Q relations for rapidly rotating neutron and strange stars in scalar-tensor theories, Phys. Rev. D90, 104021 (2014), arXiv:1408.1641 [gr-qc].
[26] P. Pani and E. Berti, Slowly rotating neutron stars in scalar-tensor theories, Phys. Rev. D90, 024025 (2014), arXiv:1405.4547 [gr-qc].
[27] S. DeDeo and D. Psaltis, Testing strong-field gravity with quasiperiodic oscillations, (2004), arXiv:astro-ph/0405067 [astro-ph].
[28] D. D. Doneva, S. S. Yazadjiev, N. Stergioulas, K. D. Kokkotas, and T. M. Athanasiasis, Orbital and epicyclic frequencies around rapidly rotating compact stars in scalar-tensor theories of gravity, Phys. Rev. D90, 044004 (2014), arXiv:1405.6976 [astro-ph.HE].
[29] G. Pappas and T. P. Sotiriou, Multipole moments in scalar-tensor theory of gravity, Phys. Rev. D91, 044011 (2015), arXiv:1412.3494 [gr-qc].
[30] G. Pappas and T. P. Sotiriou, Geodesic properties in terms of multipole moments in scalar-tensor theories of gravity, Mon. Not. Roy. Astron. Soc. 453, 2862–2876 (2015), arXiv:1505.02882 [gr-qc].
[31] J. Novak, Neutron star transition to strong scalar field state in tensor scalar gravity, Phys. Rev. D58, 064019 (1998), arXiv:gr-qc/9806022 [gr-qc].
[32] T. Harada, T. Chiba, K.-i. Nakao, and T. Nakamura, Scalar gravitational wave from Oppenheimer-Snyder collapse in scalar - tensor theories of gravity, Phys. Rev. D55, 2024–2037 (1997), arXiv:gr-qc/9611031 [gr-qc].
[33] J. Novak, Spherical neutron star collapse in tensor scalar theory of gravity, Phys. Rev. D57, 4789–4801 (1998), arXiv:gr-qc/9707041 [gr-qc].
[34] D. Gerosa, U. Sperhake, and C. D. Ott, Numerical simulations of stellar collapse in scalar-tensor theories of gravity, (2016), arXiv:1602.06952 [gr-qc].
[35] M. Horbatsch and C. Burgess, Semi-Analytic Stellar Structure in Scalar-Tensor Gravity, JCAP 1108, 027 (2011), arXiv:1006.4411 [gr-qc].
[36] W. C. Lima, G. E. Matsas, and D. A. Vanzella, Awakening the vacuum in relativistic stars, Phys. Rev. Lett. 105, 151102 (2010), arXiv:1009.1771 [gr-qc].
[37] A. G. S. Landulfo, W. C. C. Lima, G. E. A. Matsas, and D. A. T. Vanzella, Particle creation due to tachyonic instability in relativistic stars, Phys. Rev. D86, 104025 (2012), arXiv:1204.3654 [gr-qc].
[38] R. F. Mendes, G. E. Matsas, and D. A. Vanzella, Quantum versus classical instability of scalar fields in curved backgrounds, Phys. Rev. D89, 047503 (2014), arXiv:1310.2185 [gr-qc].
[39] A. G. S. Landulfo, W. C. C. Lima, G. E. A. Matsas, and D. A. T. Vanzella, From quantum to classical instability in relativistic stars, Phys. Rev. D91, 024011 (2015), arXiv:1410.2274 [gr-qc].
[40] R. F. P Mendes, G. E. A. Matsas, and D. A. T Vanzella, Instability of nonminimally coupled scalar fields in the spacetime of slowly rotating compact objects, Phys. Rev. D90, 044053 (2014), arXiv:1407.6405 [gr-qc].
[41] P. Pani, V. Cardoso, E. Berti, J. Read, and M. Sagdoo, The vacuum revealed: the final state of vacuum instabilities in compact stars, Phys. Rev. D83, 081501 (2011), arXiv:1012.1343 [gr-qc].
[42] K. Yagi, L. C. Stein, N. Yunes, and T. Tanaka, Post-Newtonian, Quasi-Circular Binary Inspirals in Quadratic Modified Gravity, Phys. Rev. D85, 064022 (2012), arXiv:1110.5950 [gr-qc].
[43] K. Yagi, L. C. Stein, and N. Yunes, Challenging the Presence
arXiv:1601.06083 [gr-qc].

[126] K. V. Staykov, D. D. Doneva, and S. S. Yazadjiev, Moment of inertia-compactness universal relations in scalar-tensor theories and $R^2$ gravity, (2016), arXiv:1602.00504 [gr-qc].

[127] A. Maselli, H. O. Silva, M. Minamitsuji, and E. Berti, Fab Four Neutron Stars, (2016), arXiv:1603.04876 [gr-qc].

[128] D. G. Ravenhall and C. J. Pethick, Neutron star moments of inertia, Astrophys. J. 424, 846–851 (1994).

[129] J. Lattimer and M. Prakash, Neutron star structure and the equation of state, Astrophys. J. 550, 426 (2001), arXiv:astro-ph/0002232 [astro-ph].

[130] M. Bejger and P. Haensel, Moments of inertia for neutron and strange stars: Limits derived for the Crab pulsar, Astron. Astrophys. 396, 917 (2002), arXiv:astro-ph/0209151 [astro-ph].

[131] J. M. Lattimer and B. F. Schutz, Constraining the equation of state with moment of inertia measurements, Astrophys. J. 629, 979–984 (2005), arXiv:astro-ph/0411470 [astro-ph].

[132] M. Urbanec, J. C. Miller, and Z. Stuchlik, Quadrupole moments of rotating neutron stars and strange stars, Mon. Not. Roy. Astron. Soc. 433, 1903 (2013), arXiv:1301.5925 [astro-ph.SR].

[133] J. Antoniadis, M. van Kerkwijk, D. Koester, P. Freire, N. Wex, et al., The relativistic pulsar-white dwarf binary PSR J1738+0333 I. Mass determination and evolutionary history, Mon.Not.Roy.Astron.Soc. 423, 3316 (2012), arXiv:1204.3948 [astro-ph.HE].

[134] G. Pappas and T. A. Apostolatos, Effectively universal behavior of rotating neutron stars in general relativity makes them even simpler than their Newtonian counterparts, Phys.Rev.Lett. 112, 121101 (2014), arXiv:1311.5508 [gr-qc].

[135] K. Yagi, K. Kyutoku, G. Pappas, N. Yunes, and T. A. Apostolatos, Effective No-Hair Relations for Neutron Stars and Quark Stars: Relativistic Results, Phys.Rev. D89, 124013 (2014), arXiv:1403.6243 [gr-qc].

[136] B. Majumder, K. Yagi, and N. Yunes, Improved Universality in the Neutron Star Three-Hair Relations, Phys. Rev. D92, 024020 (2015), arXiv:1504.02506 [gr-qc].

[137] C. M. Will and H. W. Zaglauer, Gravitational Radiation, Close Binary Systems, and the Brans-dicke Theory of Gravity, Astrophys. J. 346, 366 (1989).

[138] H. Zaglauer, Neutron stars and gravitational scalars, Astrophys.J. 393, 685–696 (1992).