WHAT ARE SYMMETRIES OF NONLINEAR PDES AND WHAT ARE THEY THEMSELVES?

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Abstract. The general theory of (nonlinear) partial differential equations originated by S. Lie had a significant development in the past 30-40 years. Now this theory has solid foundations, a proper language, proper techniques and problems, and a wide area of applications to physics, mechanics, to say nothing about traditional mathematics. However, the results of this development are not yet sufficiently known to a wide public. An informal introduction in a historical perspective to this subject presented in this paper aims to give to the reader an idea about this new area of mathematics and, possibly, to attract new researchers to this, in our opinion, very promising area of modern mathematics.

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This is neither a research nor a review paper but some reflections about general theory of (nonlinear) partial differential equations (N)PDEs and its strange marginal status in the realm of modern mathematical sciences.

For a long time a “zoological-botanical” approach was and still continues to dominate in the study of PDEs, and, especially, of NPDEs. Namely, single equations coming from geometry, physics, mechanics, etc, were “tamed and cultivated” like single animals/plants of a practical or theoretical interest. As a rule, for each of these equations were found some prescriptions for the treatment motivated by some concrete external, i.e., physical, etc reasons, but not based on the knowledge of their intrinsic mathematical nature. Mainly, these prescriptions are focused on how to construct the solutions rather than to answer numerous questions concerning global properties of the PDE itself.

Modern genetics explains what are living things, their variety and how to treat them to get the desired result. Obviously, a similar theory is indispensable for PDEs, i.e., a solid, well established general theory. The recent spectacular progress in genetics became possible only on the basis of not less spectacular developments in chemistry and physics in the last century. Similarly, the general state of the art in mathematics 50-60 years ago was not sufficiently mature to think about the general theory of PDEs. For instance, the fact that an advanced homological algebra will become an inherent feature of this theory could have been hard to imagine at that time.

Recent developments in general theory of PDEs are revealing more and more its intimate relations with quantum mechanics, quantum field theory and related areas of contemporary theoretical physics, which, also, could be hardly expected a priori. Even more, now we can be certain that the difficulties and shortcomings of current physical theories are largely due to this historically explainable ignorance.

In this paper we informally present in a historical perspective problems, ideas and results that had led to the renaissance of a general theory of PDEs after the long dead season that followed the pioneering S. Lie opera. Our guide was a modern interpretation of the Erlangen program in the form of the principle : look for the symmetries and you will find the right way. Also, one of our goals was to show that this theory is not less noble part of pure mathematics than algebraic geometry, which may be viewed as its zero-dimensional subcase. The paradox is that the
number of mathematicians who worked on this theory does not exceed the number of those who studied Kummer surfaces.

**Warnings.** The modern general theory of PDEs is written in a new, not commonly known mathematical language, which was formed in the past 30-40 years and was used by a very narrow circle of experts in this field. This makes it impossible to present this theory to a wide mathematical audience, to which this paper is addressed, in its native language. This is why the author was forced to be sometimes rather generic and to refer to some “common places” instead. His apology is in a maxim attributed to Confucius: “An ordinary man wonders marvelous things, a wise man wonders common places”.

**Notation.** Throughout the paper we use $\Lambda^k(M)$ (resp., $D_k(M)$) for the $C^\infty(M)$–module of $k$–th order differential forms (resp., $k$–vector fields) on a smooth manifold $M$. For the rest the notation is standard.

1. A BRIEF HISTORY OF NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS.

Sophus Lie was the pioneer who sought for an order in the primordial chaos reigning in the world of NPDE at the end of the XIX century. The driving force of his approach was the idea to use symmetry considerations in the context of PDEs in the same manner as they were used by E. Galois in the context of algebraic equations. In the initial phase of realization of this program, Lie was guided by the principle “chercher la symétrie” and he discovered that behind numerous particular tricks found by hand in order to solve various concrete differential equations there are groups of transformations preserving these equations, i.e., their symmetries. Then, based on these “experimental data”, he developed the machinery of transformation groups, which allows one to systematically compute what is now called point or classical symmetries of differential equations. Central in Lie theory is the concept of an infinitesimal transformation and hence of an infinitesimal symmetry. Infinitesimal symmetries of a differential equation or, more generally, of an object in differential geometry, form a Lie algebra. This Lie’s invention is among the most important in the history of mathematics.

Computation of classical symmetries of a system of differential equations leads to another nonlinear system, which is much more complicate than the original one. Lie resolved this seemingly insuperable difficulty by passing to infinitesimal symmetries. In order to find them one has to solve an overdetermined system of linear differential equations, which is a much easier task and it non infrequently allows a complete solution. Moreover, by exponentiating infinitesimal symmetries one can find almost all finite symmetries. A particular case of this mechanism is the famous relation between Lie algebras and Lie groups.

Initial expectations that groups of classical symmetries are analogues of Galois group for PDEs had led to a deep delusion. Indeed, computations show that this group for a generic PDE is trivial. This was one of the reasons why systematic applications of Lie theory to differential equations be frozen for a long time and the original intimate relations of this theory with differential equations were lost. Only much later in 1960-70 L. V. Ovsiannikov and his collaborators resumed these relations (see [43, 20]) and now they are extending in various directions.

Contact geometry was another important contribution of S. Lie to the general theory of NPDE’s. Namely, he discovered that symmetries of a first order NPDE
imposed on one unknown function are contact transformations. These transformations not only mix dependent and independent variables but their derivatives as well. For this reason they are much more general than the above-mentioned point transformations, which mix only dependent and independent variables. Moreover, it turned out that contact symmetries are sufficient to build a complete theory for this class of equations, which includes an elegant geometrical method of construction of their solutions. In this sense contact symmetries play the role of a Galois group for this class of equations. On the other hand, the success of contact geometry in the theory of first order NPDEs led to the suspicion that classical symmetries form just a small part of all true symmetries of NPDEs. But the question of what are these symmetries remained unanswered for a long time up to the discovery of integrable systems (see below). But some signs of an implicit use of such symmetries in some concrete situation can already be found in works of A. V. Bächml, E. Noether.

A courageous attempt to build a general theory of PDEs was undertaken by Charles Riquier at the very end of the XIX.th century. “Courageous” because at that time the only way to deal with general PDEs was to manipulate their coordinate descriptions. His results then gathered in a handsome book [46] of more than 600 pages present, from the modern point of view, the first systematically developed general theory of formal integrability. This book is full of cumbersome computations, and the obtained results are mostly of a descriptive nature and do not reveal structural units of the theory. Nevertheless, it demonstrated that a general theory of PDE, even on a formal level, is not impossible. Moreover, Riquier showed that the formal theory duly combined with the Cauchy-Kowalewski theorem lead to various existence results in the class of analytic functions such as the famous Cartan-Kähler theorem (see [7, 24]). In its turn Riquier’s work motivated Elie Cartan to look for a coordinate-free language for the formal/analytical theory and it led him to the theory of differential systems based on the calculus of differential forms (see [7]). Cartan’s theory was later developed and extended by E. Kähler [24], P. K. Rashevsky [45], Kuranishi [32] and others. The reader will find its latest version in [6].

In the middle of the XX.th century the theory of differential systems circulated in a narrow group of geometers as the most general theory of PDEs. However, this was an exaggeration. For instance, there were no relations between this theory and the theory of linear PDEs, which was in a booming growth at that time. Moreover, this theory did not produce any, worth to be mentioned, application to the study of concrete PDEs. We can say that it is even hardly possible to imagine that the study of the Einstein or Navier-Stokes equations will become easier after being converted into differential systems. So, the apparatus of differential forms did not confirm the expectations to become a natural base language for the general theory of PDEs, but it became one of the basic instruments in modern differential geometry and in many areas of its applications.

The original Riquier approach was improved and developed by Janet ([23]). But, unfortunately, his works were for a long period shadowed by E. Cartan works. Their vitality was confirmed much later at the beginning of the new era for NPDEs (see [44]). This era implicitly starts with the concept of a jet bundle launched by Ch. Ehresmann (see [12]). Ehresmann himself did not develop applications of jet bundles to PDEs. But, fortunately, this term became a matter of fashion and
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later was successfully used in various areas of differential geometry. In particular, D. C. Spencer and H. Goldschmidt essentially used jet bundles in their new theory of formal integrability by inventing a new powerful instrument, namely, the Spencer cohomology (see [50, 44]). In this way were discovered the first structural blocks of the general theory of PDEs, and this new theory demonstrated some important advantages in comparison with the theory of differential systems.

The Ehresmann concept of jet bundle is, however, too restrictive to be applied to general NPDEs and for this reason should be extended to that of jet space (or manifold). Namely, the $k$–th order jet bundle $J^k(\pi)$ associated with a smooth bundle $\pi : E \to M$ consists of $k$–th order jets of sections of $\pi$, while the $k$–th order jet space $J^k(E, n)$ consists of $k$–th order jets of $n$–dimensional submanifolds of the manifold $E$. Jet spaces are naturally supplied with a structure, called the Cartan distribution, which allows an interpretation of functions on them as nonlinear differential operators. Differential equations in the standard but coordinate independent meaning of this term are naturally interpreted as submanifolds of jet spaces. The first systematic study of geometry of jet spaces was done by A. M. Vinogradov and participants of his Moscow seminar in the seventieth of the passed century (see [60, 70, 30]). Later, on this basis, it was understood that various natural differential operators and constructions that are necessary for the study of a system of PDEs of order $k$ do not live necessarily on the $k$–th order jet space but involve jet spaces of any order. This is equivalent to say that a conceptually complete theory of PDEs is possible only on infinite order jet spaces. A logical consequence of this fact is that objects of the category of partial differential equations are diffieties, which duly formalize the vague idea of the “space of all solutions” of a PDE. Diffieties are a kind of infinite dimensional manifolds, and the specific differential calculus on them, called secondary calculus, is a native language to deal with PDEs and especially with NPDEs (see [66, 27, 29]).

Below we shall show how to come to secondary calculus and hence to the general theory of (nonlinear) partial differential equations by trying to answer the question “what are symmetries of a PDE”. It is worth stressing that Klein’s Erlangen program was a good guide in this expedition, which was decisive in finding the right way in some crucial moments.

2. EVOLUTION OF THE NOTION OF SYMMETRY FOR DIFFERENTIAL EQUATIONS.

A retrospective view on how the answer to the question “what are symmetries of a PDE” evolved historically will be instructive for our further discussion. From the very beginning this question was more implicitly than explicitly related with the answer to question “what is a PDE”. It seems that the apparent absurdity of this question prevented its exact formulation and hence slowed the development of the general theory.

Below we shall use the following notation. If $\sigma = (\sigma_1, \ldots, \sigma_n)$ is a multiindex, then $|\sigma| = \sigma_1 + \cdots + \sigma_n$, and

\[
\frac{\partial^{|\sigma|} f(x)}{\partial x^\sigma} = \frac{\partial^s f(x)}{\partial x_1^{\sigma_1} \cdots x_n^{\sigma_n}}, \quad |\sigma| = s, \quad f(x) = f(x_1, \ldots, x_n).
\]

We assume that $\frac{\partial^{|\sigma|} f}{\partial x^\sigma} = f$ if $\sigma = (0, \ldots, 0)$. 
**Standard (“classical”) definition.** According to the commonly accepted point of view a system of PDEs is a set of expressions

\[ F_i(x, u, u_{[1]}, \ldots, u_{[k]}) = 0, \quad i = 1, 2, \ldots, l. \]  

(1)

where \( x = (x_1, \ldots, x_n) \) are independent variables, \( u = (u^1, \ldots, u^m) \) dependent ones, and \( u_{[\sigma]} \) stands for the totality of symbols \( u_i^\sigma \), \( 1 \leq i \leq m \), with \( |\sigma| = s \). Further we shall use short “PDE” for “system of PDEs” and, accordingly, write

\[ F(x, u, u_{[1]}, \ldots, u_{[k]}) = 0 \]  

assuming that \( F = (F_1, \ldots, F_l) \).

**Solutions.** A system of functions \( f_1(x), \ldots, f_m(x) \) is a solution of the PDE (1) if the substitutions \( \frac{\partial^{|\sigma|} f_i}{\partial x^\sigma} \to u_i^\sigma \) transform the expressions (1) to identically equal to zero functions of \( x \).

This traditional view on PDEs is presented in all, modern and classical, textbooks. For instance, in Wikipedia one may read that a PDE is “an equation that contains unknown multivariable functions and their partial derivatives” or “une équation aux dérives partielles (EDP) est une équation dont les solutions sont les fonctions inconnues vérifiant certaines conditions concernant leurs dérivés partielles.”

**Symmetries: the first idea.** The “common sense” coherent with this point of view suggests to call a symmetry of PDE (1) transformations

\[ x_i = \phi_i(\bar{x}_1, \ldots, \bar{x}_n), \quad i = 1, \ldots, n, \quad u^j = \psi^j(\bar{u}^1, \ldots, \bar{u}^m), \quad j = 1, \ldots, m, \]

(2)

of dependent and independent variables that “preserve the form” of relations (1). More exactly, this means that the so-obtained functions \( \bar{F}_i = \bar{F}_i(\bar{x}, \bar{u}, \bar{u}_{[1]}, \ldots, \bar{u}_{[k]}) \)s are linear combinations of functions \( F_i \)s with functions of \( x, u, u_{[1]}, \ldots, u_{[k]} \) as coefficients. Here we used the confusing classical notation where \( (x, u) \) stands for coordinates of the image of the point \( (\bar{x}, \bar{u}) \). Also, it is assumed that transformations of symbols \( u_i^\sigma \) are naturally induced by those of \( x \) and \( u \).

Many fundamental equations in physics and mechanics inherits space-time symmetries, and these are “first idea” symmetries. Very popular in mechanics of continua dimensional analysis is also based on the so-understood concept of symmetry (see [4, 5, 43]).

**Example 2.1.** The Burgers equation \( u_t = u_{xx} + uu_x \) is invariant, i.e., symmetric, with respect to space shifts \( (x = \bar{x} + c, t = \bar{t}, u = \bar{u}) \), time shifts \( (x = \bar{x}, t = \bar{t} + c, u = \bar{u}) \) and the passage to another Galilean inertial frame moving with the velocity \( v = (x = \bar{x} + vt, t = \bar{t}, u = \bar{u}) \). This equation possesses also scale symmetries: \( x = \lambda \bar{x}, t = \lambda^2 \bar{t}, u = \lambda^{-1} \bar{u}, \lambda \in \mathbb{R} \).

The above definition of a symmetry is based on the a priori premise that the division of variables into dependent and independent ones is an indispensable part of the definition of a PDE. However, many arguments show that this point of view is too restrictive. In particular, numerous tricks that were found by hands to resolve various concrete PDEs involves transformations which do not respect this division. For instance, transformations

\[ x = \frac{\bar{x}}{\tau t + 1}, \quad t = \frac{\bar{t}}{\tau t + 1}, \quad u = \bar{u} + \tau(\bar{u} \bar{u} - \bar{x}) \]

depending on a parameter \( \tau \in \mathbb{R} \) leave the Burgers equations invariant. They, however, do not respect sovereignty of of the dependent variable \( u \).
What are symmetries

In this connection more obvious and important argument is that

what is called functions in the traditional definition of a PDE are not, generally, functions but elements of coordinate-wise descriptions of certain objects, like tensors, submanifolds, etc.

Indeed, if dependent variables \( u \) are components of a tensor, then a transformation of independent variables induces automatically a transformation of independent ones. So, the division of variables into dependent and independent ones cannot, in principle, be respected in such cases. Moreover, this, as banal as well-known observation, which is nevertheless commonly ignored, poses a question

what are “independent variables”, i.e., what are mathematical objects that are subjected by PDEs?

The “obvious” answer that these are “objects that are described coordinate-wisely by means of functions” is purely descriptive and hence not very satisfactory. In fact, this question is neither trivial, nor stupid, and, in particular, its analysis directly leads to the conception of jets (see below).

Symmetries: the second idea. Under the pressure of the the above arguments it seems natural to call a symmetry of PDE (1) a transformation of independent and dependent, which respect the status of independent variables only, i.e., a transformation of the form

\[
\begin{align*}
x_i &= \phi_i(\bar{x}_1, \ldots, \bar{x}_n), & i &= 1, \ldots, n, \\
u^j &= \psi^j(\bar{x}_1, \ldots, \bar{x}_n, \bar{u}^1, \ldots, \bar{u}^m), & j &= 1, \ldots, m.
\end{align*}
\]

This idea is consistent with many situations in physics and mechanics where space-time coordinates play role of independent variables, while “internal” characteristics of the considered continua, fields, etc, refer to dependent ones. Mathematically, these quantities are represented as sections of suitable fiber bundles, and transformations that preserve the bundle structure are exactly of the form (5). Gauge transformations in modern physics are of this kind.

On the other hand, since the second half of 18th century, the development of differential geometry put in light various problems related with surfaces and, later, with manifolds and their maps (see [41, 15, 10]) formulated in terms of PDEs. A surface in the 3-dimensional Euclidean space \( E^3 \) is not, generally, the graph of a function. So, the phrase that the equation

\[
(1 + u^2_y)u_{yy} - 2u_xu_yu_{xy} + (1 + u^2_y)u_{xx} = 0
\]

is the equation of minimal surfaces is not, rigorously speaking, true. More exactly, it is true only locally for surfaces of the form \( z = u(x, y) \) with \((x, y, z)\) being standard Cartesian coordinates in \( E^3 \). So, the question “what is the true (global) equation of minimal surfaces” should be clarified. This question, which was historically ignored, becomes, however, rather relevant if one thinks about global topological properties of minimal surfaces. Also, isometries of \( E^3 \) preserve the class of minimal surfaces and hence they must be considered as symmetries of the “true” equation of minimal surfaces in any reasonable sense of this term. But, generally, these transformations do not respect the status of both independent and dependent variables. This and many other similar examples show that the second idea is still too restrictive.
**Symmetries: the third idea.** The next obvious step is to consider transformations
\[ x_i = \phi_i(\bar{x}_1, \ldots, \bar{x}_n, \bar{u}^1, \ldots, \bar{u}^m), \quad u^j = \psi^j(\bar{x}_1, \ldots, \bar{x}_n, \bar{u}^1, \ldots, \bar{u}^m), \]
\[ i = 1, \ldots, n, \quad j = 1, \ldots, m, \] (5)
as eventual symmetries of PDEs. In this form the idea of a symmetry of a PDE was formulated by S. Lie at the end of 19th century and was commonly accepted for a long time up to the discovery of integrable systems at the late 1960s, when some strong doubts about it arose. Symmetries of form (5) are called *point symmetries* in order to distinguish them from *contact symmetries* (see below) and more general ones that recently emerged.

**Symmetries: the fourth incomplete idea.** But S. Lie himself created the ground for such doubts by developing the theory of one first order PDEs in the form of contact geometry. From this point of view natural candidates for symmetries of such a PDE are contact transformations, which mix independent and dependent variables and their first order derivatives in an almost arbitrary manner. In particular, this means that, generally, transformations of dependent and independent variables involve also first derivatives, i.e.,
\[ \left\{ \begin{array}{l}
x_i = \phi_i(\bar{x}_1, \ldots, \bar{x}_n, \bar{u}, \bar{u}_{x_1}, \ldots, \bar{u}_{x_m}), \\
u = \psi^j(\bar{x}_1, \ldots, \bar{x}_n, \bar{u}, \bar{u}_{x_1}, \ldots, \bar{u}_{x_m}).
\end{array} \right. \] (6)
Transformations (6) are to be completed by transformations of first derivatives
\[ x_i = \phi_i(\bar{x}_1, \ldots, \bar{x}_n, \bar{u}, \bar{u}_{x_1}, \ldots, \bar{u}_{x_m}), \quad i = 1, \ldots, n, \]
in a way that respects the “contact condition” \( d\bar{u} - \sum_{i=1}^n \bar{u}_{x_i} dx_i = 0. \)

Contact transformations can be naturally prolonged to transformations of higher order derivatives and, therefore, considered as candidates for true symmetries of PDEs with one dependent variable. For instance, as such they are very useful in the study of Monge-Ampère equations (see [33]). In other words, the third idea becomes too restrictive, at least, for equations with one dependent variable.

The above discussion leads to a series of questions:

**QUESTION 1.** What are analogues of contact transformations for PDEs with more than one dependent variable?

**QUESTION 2.** Are there higher order analogues of contact transformations, i.e., transformations mixing dependent and independent variables with derivatives of order higher than one?

To answer this questions we, first, need to bring the traditional approach to PDEs to a more conceptual form. In particular, a coordinate-free definition of a PDE equivalent to the standard one is needed. This is done in the next section.

3. Jets and PDEs.

Various objects (functions, tensors, submanifolds, smooth maps, geometrical structures, etc.) that are subjected to PDEs may be interpreted as submanifolds of a suitable manifold. For instance, functions, sections of fiber bundles, in particular, tensors, and smooth maps may be geometrically viewed as the corresponding graphs. So, we assume this unifying point of view and interpret PDEs as *differential restrictions* imposed on submanifolds of a given manifolds.
3.1. **Jet spaces.** So, objects of our further considerations will be \(n\)-dimensional submanifolds of an \((n + m)\)-dimensional submanifold \(E\). Let \(L \subset E\) be such one. In order to locally describe it in a local chart \((y_1, \ldots, y_{n + k})\) we must choose among these coordinates \(n\) independent on \(L\) ones, say, \((y_1, \ldots, y_n)\) and declare the remaining \(y_i\)s to be dependent ones. The notation \(x_1 = y_{i_1}, \ldots, x_n = y_{i_n}, u^1 = y_{j_1}, \ldots, u^m = y_{j_m}\) with \(\{j_1, \ldots, j_m\} = \{1, \ldots, n + m\} \setminus \{i_1, \ldots, i_n\}\) stresses this artificial division of local coordinates into dependent and independent ones. We shall refer to \((x, u)\) as a divided chart. By construction \(L\) is locally described in this divided chart by equations of the form \(u^i = f^i(x), i = 1, \ldots, n\). The next step is to understand what is the manifold in which \((x, u, u[1], \ldots, u[k])\) is a local chart. The answer is as follows.

Let \(M\) be a manifold, \(z \in M\) and \(\mu_z = \{f \in C^\infty(M) \mid f(z) = 0\}\) the ideal of the point \(z\). Elements of the quotient algebra \(J_z(M) = C^\infty(M)/\mu_z^{k + 1}\) (\(\mu_z^k\) stands for the \(k\)-th power of the ideal \(\mu_z\)) are called \(k\)-th order jets of functions at the point \(z \in M\). The \(k\)-th jet of \(f\) at \(z\) denoted by \([f]_k^z\) is the image of \(f\) under the factorization homomorphism \(C^\infty(M) \rightarrow J_z(M)\). This definition also holds for \(k = \infty\) if we put \(\mu_z^\infty = \bigcap_{k \in \mathbb{N}} \mu_z^k\). It is easy to see that \([f]_k^z = [g]_k^z\) if and only if in a local chart all the derivatives of the functions \(f\) and \(g\) of order \(\leq k\) at the point \(z\) are equal.

Two \(n\)-dimensional submanifolds \(L_1, L_2 \subset E\) are called **tangent with the order** \(k\) **at a common point** \(z\) if for any \(f \in C^\infty(M)\) \([f|_{L_1}]^k_z = 0\) implies \([f|_{L_2}]^k_z = 0\) and vice versa. Obviously, \(k\)-th order tangency is an equivalence relation.

**Definition 3.1.** The equivalence class of \(n\)-dimensional submanifolds of \(E\), which are \(k\)-th order tangent to \(L\) at \(z \in L\), is called the \(k\)-th order jet of \(L\) at \(z\) and is denoted by \([L]^k_z\).

The set of all \(k\)-jets of \(n\)-dimensional submanifolds \(L\) of \(E\) is naturally supplied with the structure of a smooth manifold, which will be denoted by \(J^k(E, n)\). Namely, associate with an \(n\)-dimensional submanifold \(L\) of \(E\) the map

\[ j_k(L) : L \rightarrow J^k(E, n), \quad L \ni z \mapsto [L]^k_z. \]

and call a function \(\phi\) on \(J^k(E, n)\) smooth if \(j_k(L)\) \(\in C^\infty(L)\). The so-defined smooth function algebra will be denoted by \(\mathcal{F}_k(E, n)\), i.e., \(C^\infty(J^k(E, n)) = \mathcal{F}_k(E, n)\).

**Remark 3.1.** If \(k < \infty\) the above definition of the smooth structure on \(J^k(E, n)\) is equivalent to the standard one, which use charts and atlases (see below). But it becomes essential for \(k = \infty\), since the standard “cartographical” approach in this case creates some boring inconveniences.

If \(L\) is given by equations \(u^i = f^i(x), i = 1, \ldots, n\), in a divided chart and \((x_1^0, \ldots, x_n^0, u_1^0, \ldots, u_m^0)\) are coordinates of \(z\) in this chart, then, as it is easy to see, \([L]^k_z\) is uniquely defined by the derivatives

\[ u^i_{\sigma, 0} = \frac{\partial^{|\sigma|} f^i}{\partial x_\sigma}(x_1^0, \ldots, x_n^0), \quad 1 \leq i \leq m, \quad |\sigma| \leq k, \tag{7} \]

and vice versa. So, the numbers \(x_j^0\) together with the numbers \(u^i_{\sigma, 0}\) may be taken for local coordinates of the point \(\theta = [L]^k_z \in J^k(E, n)\). By observing that

\[ u^i_{\sigma, 0} = j_k(L)^* \left( \frac{\partial^{|\sigma|} f^i}{\partial x_\sigma} \right)(\theta) \]
we see that the functions $u^j_\sigma \overset{\text{def}}{=} j_\sigma(L)^* \left( \frac{\partial^m f}{\partial x^j} \right)$, $1 \leq i \leq m$, $|\sigma| \leq k,m$ together with the functions $x_j$ form a smooth local chart on $J^k(E,n)$.

Thus we see that $(x,u,u[1],\ldots,u[k])$ is a local chart on $J^k(E,n)$ and hence (1) is the equation of a submanifold in $J^k(E,n)$. This allows us to interpret the standard definition of PDEs in an invariant coordinate-free manner.

**Definition 3.2.** A system of PDEs of order $k$ imposed on $n$-dimensional submanifolds of a manifold $E$ is a submanifold $\mathcal{E}$ of $J^k(E,n)$.

**Remark 3.2.** $\mathcal{E}$ as a submanifold of $J^k(E,n)$ may have singularities.

3.2. **Jet tower.** Note that $E$ is naturally identified with $J^0(E,n) : z \mapsto [L]^0_z$, and natural projections

$$\pi_{k,1} : J^k(E,n) \to J^l(E,n), \ [L]^k_z \mapsto [L]^l_z, \ l \leq k,$$

relate jet spaces of various orders in a unique structure

$$E = J^0(E,n) \overset{\pi_{1,0}}{\leftarrow} J^1(E,n) \overset{\pi_{2,1}}{\leftarrow} \cdots \overset{\pi_{k-1,k}}{\leftarrow} J^k(E,n) \overset{\pi_{k+1,k}}{\leftarrow} J^{k+1}(E,n) \cdots J^{\infty}(E,n). \ (8)$$

It is easy to see that $J^{\infty}(E,n)$ is the inverse limit of the system of maps $\{\pi_{k,1}\}$. Also note that $\pi_{k,1} : J^k(E,n) \to J^l(E,n)$ is a fiber bundle. Moreover, $\pi_{k,k-1} : J^k(E,n) \to J^{k-1}(E,n)$ is an affine bundle if $k \geq 2$ and $m > 1$ or if $k \geq 3$ and $m = 1$ (see [30, 44]).

Dually to (8), smooth function algebras on jet spaces form a telescopic system of inclusions

$$C^\infty(E) = \mathcal{F} \overset{\pi_{1,0}}{\leftarrow} \mathcal{F}_1 \overset{\pi_{2,1}}{\leftarrow} \cdots \overset{\pi_{k,k-1}}{\leftarrow} \mathcal{F}_k \overset{\pi_{k+1,k}}{\leftarrow} \cdots \mathcal{F}_\infty. \ (9)$$

So, $\mathcal{F}_\infty$ may be viewed as the direct limit of (9). By identifying $\mathcal{F}_k$ with $\pi_{\infty,k}(\mathcal{F}_k)$ we get the filtered algebra $\mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_k \subset \cdots$, $\mathcal{F}_\infty = \bigcup_{k=0}^\infty \mathcal{F}_k$.

Since two submanifolds of the same dimension are first order tangent at a point $z$ if and only if they have the common tangent space at $z$, $[L]^1_z$ is naturally identified with $T_zL$. In this way $J^1(E,n)$ is identified with the Grassmann bundle $Gr_n(E)$ of $n$-dimensional subspaces tangent to $E$, and the canonical projection $Gr_n(E) \to E$ is identified with $\pi_{1,0}$. In particular, the standard fiber of $\pi_{1,0}$ is the Grassmann manifold $Gr_{n,m}$ so that $\pi_{1,0}$ is not an affine bundle. If $m = 1$, then the fiber of $\pi_{2,1}$ is the Lagrangian Grassmanian.

**Example 3.1.** The equation $\mathcal{E}$ of minimal surfaces in the 3-dimensional Euclidean space $E^3$ is a hypersurface in $J^2(E^3,2)$. The projection $\pi_{2,1} : \mathcal{E} \to J^1(E^3,2)$ is a nontrivial bundle whose fiber is the 2-dimensional torus. So, rigorously speaking, (4) is not the equation of minimal surfaces but a local piece of it.

This and many other similar examples show that, generally, (1) is just a local coordinate-wise description of a PDE.

3.3. **Classical symmetries of PDEs.** The language of jet spaces not only gives a due conceptual rigor to the traditional theory of PDEs but it also simplifies many technical aspects of it and makes transparent and better workable various basic constructions. This will be shown in the course of the subsequent exposition. But now we shall illustrate this point by explaining how “point transformations” acts on PDEs.
First, we observe that (5) is just a local coordinate-wise description of a diffeomorphism \( F : E \to E \). Now the question we are interested in is: how does \( F \) act on jets. The answer is obvious: \( F \) induces the diffeomorphism

\[
F_k : J^k(E, n) \to J^k(E, n), \quad [L]^k \mapsto [F(L)]^k_{F(L)}
\]

called the \( k \)-lift of \( F \). This immediately leads to formulate the definition of a “classical” (= “point”) symmetry of a PDE.

**Definition 3.3.** A classical/point symmetry of a PDE \( E \subset J^k(E, n) \) is a diffeomorphism \( F : E \to E \) such that \( F_k(E) = E \).

Similarly one can define lifts of “infinitesimal point transformations”, i.e., vector fields on \( E \). Recall that if \( X \) is a vector field on \( E \) and \( F_t : E \to E \) is the flow generated by it, then

\[
X = \frac{d(F_t^\ast)}{dt} \bigg|_{t=0} \text{ with } F_t^\ast : C^\infty(E) \to C^\infty(E).
\]

Then the lift \( X_k \) of \( X \) to \( J^k(E, n) \) is defined as

\[
X_k = \frac{d((F_t^\ast(k))}{dt} \bigg|_{t=0}.
\]

**Definition 3.4.** A vector field \( X \) on \( E \) is an infinitesimal classical/point symmetry of a PDE \( E \subset J^k(E, n) \) if \( X_k \) is tangent to \( E \).

Nonlinear partial differential operators are also easily defined in terms of jets.

**Definition 3.5.** A (nonlinear) partial differential operator \( \square \) of order \( k \) sending \( n \)-dimensional submanifolds of \( E \) to \( E' \) is defined as the composition \( \Phi \circ j_k \) with \( \Phi : J^k(E, n) \to E' \) being a (smooth) map, i.e., \( \square(L) = \Phi(j_k(L)) \).

For instance, functions on \( J^1(E, n) \) are naturally interpreted as (nonlinear) differential operators.

The above definitions and constructions are easily specified to fiber bundles. Namely, if \( \pi : E \to M \), \( \dim M = n \), is a fiber bundle, then the \( k \)-th order jet \([s]^k\) of an local section of it \( s : U \to E \) (\( U \) is an open in \( M \)) at a point \( x \in M \) is defined as \([s]^k_x = [s(U)]^k_{j_k(x)}\). These specific jets form an everywhere dense open subset in \( J^k(E, n) \) denoted by \( J^k(\pi) \) and called the \( k \)-order jet bundle of \( \pi \). The substitute of maps \( j(L) \) in this context are maps \( j_k(s) \equiv j_k(s(U)) \). Additionally, we have natural projections \( \pi_k = \pi \circ \pi_{k,0} : J^k(\pi) \to M \). If \( \pi \) is a vector bundle, then \( \pi_k \) is a vector bundle too, and the equation \( E \subset J^k(\pi) \) is linear if \( E \) is a linear sub-bundle of \( \pi_k \), etc. For further details concerning the “fibered” case, see [29, 70, 44].

4. **Higher order contact structures and generalized solutions of NPDEs.**

4.1. **Higher order contact structures.** Now we are going to reformulate the standard definition of a solution of a PDE in a coordinate-free manner. Put \( L_{(k)} = \text{Im} j_k(L) \) for an \( n \)-dimensional submanifold of \( E \). Obviously, \( L_{(k)} \) is an \( n \)-dimensional submanifold \( L \) of \( J^k(E, n) \), which is projected diffeomorphically onto \( L \) via \( \pi_{k,0} \).
Definition 4.1. Let $L$ be a solution in the standard sense of a PDE $\mathcal{E} \subset J^k(E,n)$ if $L_{(k)} \subset \mathcal{E}$.

If $u^i = F^i(x), i = 1, \ldots, n$ are local equations of $L$, then

$$u^i_\sigma = \partial^{\sigma i} f^i(x), \quad i = 1, \ldots, n, \quad |\sigma| \leq n,$$

are local equations of $L_{(k)}$ in $J^k(E,n)$. This shows that the coordinate-free definition 4.1 coincides with the standard one. Also, we see that the $L_{(k)}$’s form a very special class of $n$-dimensional submanifolds in $J^k(E,n)$. This class is not intrinsically defined, and hence Definition 4.1 is not intrinsic. For this reason it is necessary to supply $J^k(E,n)$ with an additional structure, which allows to distinguish the submanifolds $L_{(k)}$ from others. Such a structure is a distribution on $J^k(E,n)$ defined as follows.

Definition 4.2. The minimal distribution $C^k : J^k(E,n) \ni \theta \mapsto C^k_\theta \subset T_\theta(J^k(E,n))$ on $J^k(E,n)$ such that all the $L_{(k)}$ are integral submanifolds of it, i.e., $T_\theta(L_{(k)}) \subset C^k_\theta, \forall \theta \in L_{(k)}$, is called the $k$-th order contact structure or the Cartan distribution on $J^k(E,n)$.

It directly follows from the definition that

$$C^k_\theta = \text{span}\{T_\theta(L_{(k)})\} \text{ for all } L \text{ such that } L_{(k)} \ni \theta \}. \tag{11}$$

Due to the importance of the subspaces $T_\theta(L_{(k)}) \subset T_\theta(J^k(E,n))$ we shall call them $R$-planes (at $\theta$). By construction any $R$-plane at $\theta$ belongs to $C^k_\theta$. The following simple fact is very important and will be used in various constructions further on.

Lemma 4.1. Let $\theta = [L]^k_z = [N]^k_z$ and $\theta' = \pi_{k,k-1}(\theta)$. Then $T_{\theta'}(L_{(k-1)}) = T_{\theta'}(N_{(k-1)})$ and hence the $R$-plane $R_{\theta} = T_{\theta'}(L_{(k-1)})$ is uniquely defined by $\theta$. Moreover, the correspondence $\theta \mapsto R_{\theta}$ between points of $J^k(E,n)$ and $R$-planes at points of $J^{k-1}(E,n)$ is bimune.

This lemma allows to identify the fiber $\pi_{k,k-1}^{-1}(\theta')$ with the variety of all $R$-planes at $\theta$ and hence $J^k(E,n)$ with the variety of $R$-planes at points of $J^{k-1}(E,n)$.

Below we list some basic facts concerning of the Cartan distribution and $R$-planes (see [60, 30, 70]).

Proposition 4.1. (1) $C^k_{\theta} = (d_{\theta} \pi_{k,k-1})^{-1}(R_{\theta})$ with $d_{\theta} \pi_{k,k-1} : T_\theta(J^k(E,n)) \to T_{\theta}(J^{k-1}(E,n))$ being the differential of $\pi_{k,k-1}$ at $\theta$. In particular,

$$d_{\theta} \pi_{k,k-1}(C^k_{\theta}) \subset C^{k-1}_{\theta}. \tag{2}$$

(2) In local coordinates the Cartan distribution is given by the equations

$$\omega^i = d u^i_\sigma - \sum_j u^i_{\sigma+1,j} dx_j = 0, \quad |\sigma| < k, \text{ where } (\sigma + 1)_j = \sigma + \delta_{ij},$$

or, dually, is generated by the vector fields

$$D^i_k = \frac{\partial}{\partial x_i} + \sum_{j,|\sigma| < k} u^i_{\sigma+1,j} \frac{\partial}{\partial u^j_\sigma}, \quad i = 1, \ldots, n, \text{ and } \frac{\partial}{\partial u^i_\sigma}, \quad |\sigma| = k. \tag{3}$$

$$\dim C^k_{\theta} = m \left( \frac{n + k - 1}{k} \right) + n, \text{ if } 0 \leq k < \infty; \quad \dim C^\infty_{\theta} = n.$$
(4) Tautologically, a point $\theta = [L]_\infty \in J^\infty(E, n)$ is the inverse limit of $\theta_k = \pi_{\infty, k}(\theta) = [L]_{k+1}^k$. Then $C^\infty_\theta$ is the inverse limit of the chain

\[ \ldots \to \cdots \to \pi_{k, k+1} \to \pi_{k, k+2} \to \cdots \]

\[ \theta_k \to \theta_{k+1} \to \theta_{k+2} \to \cdots \]

(5) Distributions $C^k$, $k < \infty$, are, in a sense, “completely non-integrable”, while their inverse limit $C^\infty$ is completely (Frobenius) integrable and locally generated by commuting total derivatives

\[ D_i = \frac{\partial}{\partial x_i} + \sum_{j, \sigma} u_{\sigma+1,i} \frac{\partial}{\partial u_{\sigma}^i}, \quad i = 1, \ldots, n. \]

(6) If an $n$-dimensional integral submanifold $N$ of $C^k$, $k < \infty$, is transversal to fibers of $\pi_{k, k-1}$, then, locally, $N$ is of the form $L_{(k)}$ and, therefore, $\pi_{k, 0}(N)$ is an immersed $n$-dimensional submanifold of $E$.

Cartan’s forms $\omega^i_\sigma$ figuring in assertion (2) of the above proposition were systematically used by E. Cartan in his reduction of PDEs to exterior differential systems. Hence the term “Cartan distribution”.

Note that if $m = 1$, then the manifold $J^1(E, n)$ supplied with the Cartan distribution $C^1$ is a contact manifold. The contact distribution $C^1$ is locally given by the classical contact form $du - \sum_{i=1}^n u^i dx_i = 0$. So, $C^k$ whose construction word for word mimics the classical construction of contact geometry may be viewed as its higher order analogue, i.e., the $k$-th order contact structures.

Recall now how the theory of one 1-st order PDE with one independent variable is formulated in terms of contact geometry. Let $K, \dim K = r + 1$, be a manifold supplied with an $r$-dimensional distribution $\mathcal{C} : K \ni x \mapsto \mathcal{C}_x \subset T_x K$. The fiber at $x \in K$ of the normal to the $\mathcal{C}$ vector bundle $\nu_\mathcal{C} : N_\mathcal{C} \to K$ is $T_x K/\mathcal{C}_x$, and $\dim \nu_\mathcal{C} = 1$. We shall write $X \in \mathcal{C}$ if the vector field $X$ belongs to $\mathcal{C}$, i.e., $X_x \in \mathcal{C}_x, \forall x \in K$. By abusing language we shall denote also by $\mathcal{C}$ the $C^\infty(K)$–module of vector fields belonging to $\mathcal{C}$ and put $N_\mathcal{C} = \Gamma(\nu_\mathcal{C})$. The curvature of $\mathcal{C}$ is the following $C^\infty(K)$–bilinear skew-symmetric form $\Omega$ with values in $N_\mathcal{C}$:

\[ \Omega_\mathcal{C}(X, Y) = [X, Y] \mod \mathcal{C}, \quad X, Y \in \mathcal{C}. \]

$\Omega_\mathcal{C}$ is nondegenerate if the map

\[ \mathcal{C} \ni X \mapsto \Omega_\mathcal{C}(X, \cdot) \in \Lambda^1(K) \otimes_{C^\infty(K)} N_\mathcal{C} \]

is an isomorphism of $C^\infty(K)$–modules. The pair $(K, \mathcal{C})$ is a contact manifold if the 2-form $\Omega_\mathcal{C}$ is nondegenerate. In such a case $r$ is odd, say, $r = 2n + 1$.

This definition of contact manifolds is not standard (see [2, 33]) but is more convenient for our purposes. By the classical Darboux lemma a contact manifold locally possesses canonical coordinates $(x_1, \ldots, x_n, u, p_1, \ldots, p_n)$ in which $\mathcal{C}$ is given by the 1-form $\omega^{\text{def}} = du - \sum_{i=1}^n p_i dx_i = 0$. Then $e = [\partial/\partial u \mod \mathcal{C}]$ is a local base of $N_\mathcal{C}$ and $\Omega_\mathcal{C} = -d\omega \otimes e = (\sum_{i=1}^n dp_i \wedge dx_i) \otimes e$.

If a hypersurface $\mathcal{E} \subset K$ is interpreted as a 1-st order PDE, then a (generalized) solution of $\mathcal{E}$ is a Legendrian submanifold $L$ in $K$ belonging to $\mathcal{E}$. Recall that a Legendrian submanifold $L$ is an $n$-dimensional integral submanifold of $\mathcal{C}$, or, more conceptually, a locally maximal integral submanifold of $\mathcal{C}$. “Locally maximal” means that even locally $L$ does not belong to an integral submanifold of greater dimension.

These considerations lead to conjecture that
Proposition 4.2. Let \( \mathcal{C}^k \) be the Cartan distribution. All maximal integral submanifolds are Legendrian and hence are locally exceptional cases (ii) and (iii). On the contrary, in the case (i) (classical contact geometry), the fibers of the projection \( \pi \) are regular while the remaining ones are not. This assertion shows that in the regular case \( \dim W = s \), then

\[
\dim \mathcal{L}(W) = s + m \left( \frac{n + k - s - 1}{n - s - 1} \right).
\]

(3) If \( N \subseteq J^k(E, n) \) is a locally maximal integral submanifold, then there is an open and everywhere dense subset \( N_0 \) in \( N \) such that

\[
N_0 = \bigcup_{\alpha} U_\alpha \quad \text{with} \quad U_\alpha \quad \text{being an open domain in} \quad \mathcal{L}(W_\alpha)
\]

(4) If \( \dim W_1 < \dim W_2 \), then \( \dim \mathcal{L}(W_1) > \dim \mathcal{L}(W_2) \) except in the cases (i) \( n = m = 1 \), (ii) \( k = m = 1 \) and (iii) \( m = 1 \), \( \dim W_1 + 1 = \dim W_2 = n \).

An important consequence of Proposition 4.2 is that it disproves the above conjecture. Existence of locally maximal integral submanifolds of different dimensions is what makes a substantial difference between higher order contact structures and the classical original. In particular, this creates a problem in definition of solutions of PDEs in an intrinsic manner. To resolve it we need some additional arguments.

Situations (i)-(iii) in assertion (4) of Proposition 4.2 will be called exceptional, while the remaining ones regular. This assertion shows that in the regular case integral submanifolds \( W_\alpha \) figuring in assertion (3) must have the same dimension. This dimension will be called the type of the maximal integral submanifold \( N \). For some other reasons, which we shall skip, the notion of type can be defined also in exceptional cases (ii) and (iii). On the contrary, in the case (i) (classical contact geometry!) all maximal integral submanifolds are Legendrian and hence are locally equivalent.

Now we may notice that, except for the case \( k = m = 1 \) (classical contact geometry), the fibers of the projection \( \pi_{k,k-1} \), \( k > 1 \) are intrinsically characterized as locally maximal integral submanifolds of zero-th type. Therefore, the manifold \( J^{k-1}(E, n) \) may be interpreted as the variety of such submanifolds and, similarly, the distribution \( \mathcal{C}^{k-1} \) can be recovered from \( \mathcal{C}^k \). So, the obvious induction arguments show that by starting from the \( k \)-th order contact manifold \( (J^k(E, n), \mathcal{C}^k) \) we can intrinsically recover the whole tower

\[
J^k(E, n) \xrightarrow{\pi_{k,k-1}} J^{k-1}(E, n) \xrightarrow{\pi_{k-1,k-2}} \cdots \xrightarrow{\pi_{1,0}} J'(E, n)
\]
where $\epsilon = 0$ if $m > 1$ and $\epsilon = 1$ if $m = 1$. In particular, the projections $\pi_{k,0}$ (resp., $\pi_{k,1}$) can be intrinsically characterized in terms of the $k$–th order contact structure if $m > 1$ (resp., if $m = 1$ and $k > 1$). So, if $m > 1$, submanifolds $L_{(k)}$ are characterized in these terms as locally maximal integral submanifolds of type $n$ that diffeomorphically project on their images via $\pi_{k,0}$. If $m = 1$, then only contact manifold $(J^1(E, n), C^1)$ can be intrinsically described in terms of a $k$–th order contact structure as the image of the intrinsically defined projection $\pi_{k,1}$. So, in this case in order to characterize the submanifolds $L_{(k)}$ we additionally need to supply the image of $\pi_{k,1}$ with a fiber structure, which mimics $\pi_{k,0}$.

4.3. Generalized solutions of NPDEs. The above considerations lead us to the following definition.

**Definition 4.3.**

1. A locally maximal integral submanifold of type $n$ will be called $R$-manifold. In particular, submanifolds $L_{(k)}$ are $R$-manifolds.

2. Generalized (resp., “usual”) solutions of a PDE $E \subset J^k(E, n)$ are $R$-manifolds (resp., manifolds $L_{(k)}$) belonging to $E$.

With this definition we gain

> the concept of generalized solutions for nonlinear PDE’s, which, principally, cannot be formulated in terms of functional analysis as in the case of linear PDEs.

(see [51, 48, 17]). This is one of many instances where a geometrical approach to PDEs can be in no way substituted by methods of functional analysis or by other analytical methods.

Definition 4.3 may be viewed as an extension of the concept of a generalized solution of a linear PDE in the sense of Sobolev-Schwartz to general NPDEs. We have no sufficient “space-time” to discuss this very interesting question here. A very rough idea about this relation is that a generalized solution in the sense of Definition 4.3 may be viewed as a multivalued one. If the equation is linear, then it is possible to construct a 1-valued one just by summing up various branches of a multivalued one. The result of this summation is, generally, no longer a smooth function but a “generalized” one. A rigorous formalization of this idea requires, of course, a more delicate procedure of summation and the Maslov index (see [40]) naturally appears in this context.

4.4. PDEs versus differential systems. According to E. Cartan, a PDE $E \subset J^k(E, n)$ can be converted into a differential system by restricting the distribution $C^k$ to $E$. The restricted distribution denoted by $C^k_E$ is defined as

$$C^k_E : E \ni \theta \mapsto C^k \cap T_{\theta}E.$$ 

Originally, E. Cartan used the Pfaff (exterior) system $\omega^i_\sigma = 0, i = 1, \ldots, m, |\sigma| < k$, in order to describe $C^k_E$, and this explains the term *exterior differential system*.

The passage from the equation $E$ understood as a submanifold of $J^k(E, n)$ to the differential system $(E, C^k_E)$ means, in essence, that we forget that $E$ is a submanifold of $J^k(E, n)$ and consider it as an abstract manifolds equipped with a distribution. Cartan was motivated by the idea of replacing non-invariant, i.e., depending on the choice of local coordinate, language of partial derivatives by the invariant calculus of differentials and hence of differential forms. The idea that the general theory of PDEs requires an invariant and adequate language is of fundamental importance,
and E. Cartan was probably the first who raised it explicitly. On the other hand, it turned out later that the language of differential forms is not sufficient in this sense. For instance, Proposition 4.2 illustrates the fact that the concept of a solution for a generic differential system is not well-defined because of the existence of integral submanifolds of different types. The rigidity theory (see [60, 30, 70]) sketched below makes this point more precise.

First, note that locally maximal integral submanifolds of the restricted distribution $\mathcal{C}_k^E$ are intersections of such submanifolds for $\mathcal{C}^k$ with $\mathcal{E}$. So, if $\mathcal{E}$ is not very overdetermined, i.e., if the codimension of $\mathcal{E}$ in $J^k(E, n)$ is not too big, then the difference between locally maximal integral submanifolds of $\mathcal{C}^k$ of different types survives the restriction to $\mathcal{E}$. So, the information about this difference in an explicit form gets lost when passing to the differential system $(\mathcal{E}, \mathcal{C}_k^E)$. The problem to recover it becomes rather difficult especially if the 1–forms $\omega_i \in \Lambda^1(\mathcal{E})$ of the Pfaff system $\omega_i = 0$ describing the distribution $\mathcal{C}_k^E$ are arbitrary, say, not Cartan ones. Moreover, if we have a generic differential system $(M, \mathcal{D})$ with $\mathcal{D} = \{\rho_i = 0\}$, $\rho_i \in \Lambda^1(M)$, then it is not even clear which class of its integral submanifolds should be called solutions. To avoid this inconvenience, E. Cartan proposed to formulate the problem associated with a differential system as the problem of finding its integral submanifolds (locally maximal or not) of a prescribed dimension. But numerous examples show that a differential system may possess integral submanifolds of an absolutely different nature, which have the same dimension.

One of the simplest examples of this kind is the differential system $(J^k(E, 1), \mathcal{C}^k)$ with $\dim E = 2$, $k > 1$, for which locally maximal integral submanifolds of types 0 (fibers or the projection $\pi_{k,k-1}$) and 1 (R-manifolds) are all 1-dimensional. Moreover, integral submanifolds of type 0 are irrelevant/“parasitic” in the context of the theory of differential equations.

Secondly, an equation $\mathcal{E} \subset J^k(E, n)$ is called rigid if the $k$-th order contact manifold $(J^k(E, n), \mathcal{C}^k)$ can be recovered if $(\mathcal{E}, \mathcal{C}_k^E)$ as an abstract differential system is only known. For instance, if the codimension of $\mathcal{E}$ in $J^k(E, n)$ is less than the difference of dimensions of locally maximal integral submanifolds of types 0 and 1, then $\mathcal{E}$ is, as a rule, rigid. Indeed, in this case integral submanifolds of $(\mathcal{E}, \mathcal{C}_k^E)$ of absolutely maximal dimension are intersections of fibers of $\pi_{k,k-1}$ with $\mathcal{E}$. In other words, these are fibers of the projection $\pi_{k,k-1}|_{\mathcal{E}} : \mathcal{E} \to J^{k-1}(E, n)$. If, additionally, this projection is surjective, then $J^{k-1}(E, n)$ is recovered as the variety of integral submanifolds of $(\mathcal{E}, \mathcal{C}_k^E)$ of maximal dimension. Next, under some weak condition projections of spaces $\mathcal{C}_k^E, \theta \in \mathcal{E}$, on the so-interpreted jet space $J^{k-1}(E, n)$ span the distribution $\mathcal{C}^{k-1}$. In this way $(J^{k-1}(E, n), \mathcal{C}^{k-1})$ is recovered from $(\mathcal{E}, \mathcal{C}_k^E)$ and, finally, $(J^k(E, n), \mathcal{C}^k)$ is recovered from $(J^{k-1}(E, n), \mathcal{C}_k^{k-1})$ as the variety of $R$–planes on $J^{k-1}(E, n)$ according to Proposition 4.1, (1). Thus converting rigid equations into differential systems is counterproductive, since this procedure create non-necessary additional problems. In this connection it is worth mentioning that the most important PDEs in geometry, mechanics and physics we deal with are determined or slightly overdetermined systems of PDE’s, like Maxwell or Einstein equations, and hence are rigid.

Even more important arguments, which do not speak in favor of differential systems, come from the fact that the calculus of differential forms is a small part of a much richer structure formed by natural functors of differential calculus and objects representing them. For instance, indispensable for formal integrability theory diff-
and jet-Spencer complexes are examples of this kind (see [50, 54, 59, 44, 30, 70]).
Finally, our distrust of differential systems is supported by the fact that practical
computations of symmetries, conservation laws and other quantities characterizing
PDEs become much more complicated in terms of differential systems.

4.5. **Singularities of generalized solutions.** The concept of generalized solutions
for NPDEs, which is important in itself, naturally leads to an important
part of a general theory of PDEs, namely, the theory of singularities of generalized
solutions. Below we shall outline some key points of this theory.

Let \( N \subset J^k(E,n) \) be an \( R \)–manifold. A point \( \theta \in N \) is called *singular of type s*
if the kernel of the differential \( d\theta \pi_{k,k-1} \) restricted to \( T_\theta N \) is of dimension \( s > 0 \).
Otherwise, \( \theta \) is called *regular*. It should be stressed here that “singular” refers to
singularities of the map \( \pi_{k,k-1} | _N \), while, by definition, \( N \) is a smooth submanifold.

According to Proposition 4.1, (6), \( N \) is of the form \( L(k) \) in a neighborhood of any
regular point.

Put \( F_\theta = (\pi_{k,k-1})^{-1}(\pi_{k,k-1}(\theta)) \) (the fiber of \( \pi_{k,k-1} \) passing through \( \theta \)) and
\( VC^k_\theta = C^k_\theta \cap T_\theta (F_\theta) \). The bend of \( N \) at a point \( \theta \in N \) is

\[
B_\theta N \overset{\text{def}}{=} \ker d\theta (\pi_{k,k-1} | _N) = T_\theta N \cap T_\theta (F_\theta) \subset VC^k_\theta.
\]

Also, we shall call an *s–bend* (at \( \theta \in J^k(E,n) \)) an *s–dimensional subspace of \( VC^k_\theta \),
which is of the form \( B_\theta N \) for some \( R \)–manifold \( N \). Bends are very special subspaces in
\( VC^k_\theta \). A remarkable fact is that \( s \)–dimensional bends are classified by
\( s \)–dimensional Jordan algebras of a certain class over \( R \), which contains all unitary
algebras (see [64, 68]).

PDEs differ from each other by the types of singularities which their generalized solutions admit.

For instance, 2-dimensional Jordan algebras associated with 2-dimensional bends
are 2-dimensional unitary algebras and hence are isomorphic to one of the following
three algebras

\[
C_\epsilon = \{ a + b\zeta \mid a, b \in \mathbb{R}, \zeta^2 = \epsilon 1 \} \quad \text{with} \quad \epsilon = \pm \text{ or } 0.
\]

Obviously, \( C_- = C \) and \( C_+ = \mathbb{R} \oplus \mathbb{R} \) (as algebras). An equation in two independent variables is *elliptic* (resp., *parabolic* or *hyperbolic*) if its generalized solutions
possess singularities of type \( C_- \) (resp., \( C_0 \), \( C_+ \)) only. Geometrically, singularities corresponding to algebra \( C \) are Riemann ramifications, while bicharacteristics of
hyperbolic equations reflect the fact that \( C_+ \) splits into the direct sum \( \mathbb{R} \oplus \mathbb{R} \).

Obviously, the simplest singularities correspond to the algebra \( \mathbb{R} \). They present
a kind of folding and can be *analytically detected* in terms of non-uniqueness of
Cauchy data. A similar analytic approach is hardly possible for more complicated
algebras. This explains why analogues of the classical subdivision of PDEs in two
independent variables into elliptic, parabolic and hyperbolic ones are not yet known.
This fact emphasizes once again that only analytical methods for PDEs, even linear
ones, are not sufficient and the geometrical approach is indispensable.

Description of singularities that solutions of a given PDE admit is naturally
settled as follows. Let \( \Sigma \) be a type of \( s \)–bends, which may be identified with the
corresponding Jordan algebra. If \( N \) is an \( R \)–manifold, then

\[
N_\Sigma = \{ \theta \in N \mid B_\theta N \text{ is of the type } \Sigma \}
\]
is the locus of its singular points of type $\Sigma$. Generally, $\dim N_\Sigma = n - s$. If $N$ is a solution of a PDE $\mathcal{E}$, then $N_\Sigma$ must satisfy an auxiliary system of PDEs, which we denote by $\mathcal{E}_\Sigma$. For "good" equations $\mathcal{E}_\Sigma$ is, generally, a nonlinear, undetermined system of PDEs in $n - s$ independent variables.

4.6. **The reconstruction problem.** So, any PDE is not a single but is surrounded by an "aura" of subsidiary equations, which put in evidence the internal structure of its solutions. The importance of these equations becomes especially clear in the light of the reconstruction problem:

*Whether the behavior of singularities of solutions of a PDE $\mathcal{E}$ uniquely determines the equation itself or, equivalently, whether is possible to reconstruct $\mathcal{E}$ assuming that the $\mathcal{E}_\Sigma$’s are known?*

In a physical context this question sounds as

*Whether the behavior of singularities of a field (medium, etc.) completely determines the behavior of the field (medium, etc.) itself?*

A remarkable example of this kind is the deduction of Maxwell’s equations from elementary laws of electricity and magnetism (Coulomb, . . . , Faraday) (see [35]).

The reconstruction problem resolves positively for hyperbolic NPDEs on the basis of equations $\mathcal{E}_{\text{FOLD}}$ that describes singularities corresponding to the algebra $\mathbb{R}$. The equations describing wave fronts of solutions of a linear hyperbolic PDE $\mathcal{E}$ are part of the system $\mathcal{E}_{\text{FOLD}}$.

**Example 4.1.** Fold–type singularities for the equation $u_{xx} - \frac{1}{c^2}u_{tt} - mu^2 = 0$.

Consider wave fronts of the form $x = \varphi(t)$ and put

$$ g = u|_{\text{wave front}}, \quad h = u_x|_{\text{wave front}}. $$

Then we have

$$ \begin{cases} \ddot{g} + (cm)^2 g = \pm 2ch \\ 1 - \frac{1}{c^2} \dot{\varphi}^2 = 0 \iff \dot{\varphi} = \pm c \end{cases} \iff \text{Equations describing the behavior of fold–type singularities}$$

The second of these equations is of eikonal type and describe the space-time shapes of singularities. On the contrary, the first equation describes a “particle” in the “field” $h$. If this field is constant $\iff h = 0$, then the first equation represents a harmonic oscillator of frequency $\nu = mc$.

**Example 4.2.** Fold–type singularities for the Klein–Gordon equation

$$(\partial_t^2 - \vec{\nabla}^2 + m^2)u = 0.$$

Consider wave fronts of the form $t = \varphi(x_1, x_2, x_3)$ and $g$ and $h$ as in example 4.1

$$ \mathcal{E}_{\text{FOLD}} = \begin{cases} (\vec{\nabla}\varphi)^2 = 1 \iff \text{eikonal type equation} \\ \nabla^2 h + m^2 h - g - (\nabla^2 \varphi)g = 2\vec{\nabla}\varphi \cdot \vec{\nabla}g \iff \text{??} \end{cases}$$

The physical meaning of the second of these equations is unclear.

**Example 4.3.** Classical Monge-Ampère equations are defined as equations of the form

$$ S(u_{xx}u_{yy} - u_{xy}^2) + Au_{xx} + Bu_{xy} + Cu_{yy} + D = 0 $$

with $S, A, B, C, D$ being functions of $x, y, u, u_x, u_y$ (see [33]). As it was already observed by S. Lie this class of equations is invariant with respect to contact transformations. This fact forces to think that Monge-Ampère equations are distinguished
by some “internal” property. This is the case, and Monge-Ampère equations are completely characterized by the fact that the reconstruction problem for these equations is equivalent to a problem in contact geometry (see [8, 39]).

The reader will find in [36] further details and examples concerning the auxiliary singularities equations. Some exact generalized solutions of Einstein equations (the “square root” of the Schwarzschild solution, etc) are described in [49].

4.7. Quantization as a reconstruction problem. Let $\mathcal{E}$ be a PDE, whose solutions admit fold-type singularities. Then we have the following series of interconnected equations:

$$
\mathcal{E} \implies \mathcal{E}_{\text{FOLD}} \implies \mathcal{E}_{\text{eikonal}} \implies \mathcal{E}_{\text{char}}.
$$

(12)

Here $\mathcal{E}_{\text{eikonal}}$ is the equations from the system $\mathcal{E}_{\text{FOLD}}$ that describes space-time shape ("wave front") of fold-type singularities. It is a Hamilton-Jacobi equation (see Examples 4.1 and 4.2). In its turn $\mathcal{E}_{\text{char}}$ is the system of ODEs that describes characteristics of $\mathcal{E}_{\text{eikonal}}$. In the context where space-time coordinates are independent variables $\mathcal{E}_{\text{eikonal}}$ is a Hamiltonian system whose Hamiltonian is the main symbol of $\mathcal{E}$. Now we see that the correspondence

$$
\text{CHAR} : \mathcal{E} (\text{PDE}) \implies \mathcal{E}_{\text{char}} (\text{Hamiltonian system of ODE’s})
$$

(13)

is parallel to the correspondence between quantum and classical mechanics

$$
\text{BOHR} : (\text{Schroedinger’s PDE}) \implies (\text{Hamiltonian ODEs}).
$$

(14)

Moreover, the correspondence (13) is at the root of the famous “optics-mechanics analogy”, which guided E. Schrödinger in his discovery of the “Schrödinger equation” (see Schrödinger’s Nobel lecture [47]).

It is remarkable that in “Cauchy data” terms correspondence (13) was known already to T. Levi-Civita and he tried to put it at the foundations of quantum mechanics (see [34]). From what is known today this attempt was doomed to failure. However, the idea that quantization is something like the reconstruction problem explains well why numerous quantization procedures proposed up to now form a kind of recipe book not based on some universal principles. Indeed, from this point of view the quantization looks like an attempt to restore the whole system $\mathcal{E}_{\text{FOLD}}$ on the basis of knowledge of $\mathcal{E}_{\text{char}}$ only. This is manifestly impossible, since $\mathcal{E}_{\text{char}}$ depends only on the main symbol of $\mathcal{E}$. On the other hand, the above outlined solution singularity theory admits some interesting generalizations and refinements, which not only keep alive the Levi-Civita idea but even make it more attractive.

4.8. Higher order contact transformations and the Erlangen program. The above interpretation of PDEs as submanifolds of higher order contact manifolds is the first step toward a “conceptualization” of the standard approach to PDE’s. It is time now to test its validity through the philosophy of the Erlangen program. First of all, this means that we have to describe the symmetric group of higher contact geometries, i.e., the group of higher contact transformations.

**Definition 4.4.** A diffeomorphism/transformation $\Phi : J^k(E,n) \to J^k(E,n)$ is called a $k$-order contact if for any $X \in C^k$, $\Phi(X) \in C^k$ or, equivalently, $d_\theta \Phi(C^k_\theta) = C^k_{\Phi(\theta)}, \forall \theta \in \Phi$.

If $\Phi$ is a $k$-th order contact, then, obviously, it preserves the class of locally maximal integral submanifolds of type $s$. In particular, it preserves fibers of the
projection $\pi_{k,k-1}$ and hence locally maximal integral submanifolds of type $n$ that are transversal to these fibers. But the latter are locally of the form $L(k)$ (Proposition 4.1, (6)). This proves that the differential of $\Phi$ sends $R$-planes into $R$-planes. By identifying these $R$-planes with points of $J^{k+1}(E,n)$ we see that $\Phi$ induces a diffeomorphism $\Phi_{(1)}$ of $J^{k+1}(E,n)$. More exactly, if $\theta \in J^{k+1}(E,n)$ and $\theta' = \pi_{k+1,k}(\theta)$, then $(d\theta_{(1)}\Phi)(R_\theta)$ is an $R$-plane and hence is of the form $R_\theta$ for a $\theta \in J^{k+1}(E,n)$. Then we put $\Phi(\theta) = \theta$. Moreover, it directly follows from Proposition 4.1, (1), that $\Phi_{(1)}$ is a $(k+1)$-order contact and the diagram

$$
\begin{array}{ccc}
J^{k+1}(E,n) & \xrightarrow{\Phi_{(1)}} & J^{k+1}(E,n) \\
\downarrow \pi_{k+1,k} & & \downarrow \pi_{k+1,k} \\
J^{k}(E,n) & \xrightarrow{\Phi} & J^{k}(E,n)
\end{array}
$$

commutes. By continuing this process we, step by step, construct contact transformations $\Phi_{(l)} : J^{k+l}(E,n) \xrightarrow{\Phi_{(l)}} J^{k+l}(E,n), \Phi_{(l)} \equiv (\Phi_{(l-1)})_{(1)}$.

**Theorem 4.1.** Let $\Phi : J^{k}(E,n) \rightarrow J^{k}(E,n)$, $k > 0$, be a $k$-order contact transformation. Then $\Phi = \Psi_{(l)}$ (resp., $\Phi = \Psi_{(l-1)}$) where $\Psi$ is a diffeomorphism of $E$ if $m > 1$ (resp., a contact transformation of $J^{l}(E,n)$ if $m = 1$).

A proof of this fundamental result for the classical symmetry theory can be easily deduced from the fact explained above that a $k$-th order contact transformation preserves fibers of $\pi_{k,k-1}$ and hence induces a $(k-1)$-th order contact transformation of $J^{k-1}(E,n)$. For $m = 1$ it was proven by Lie and Bäcklund (see [60, 29]).

If one takes Definition 3.2 for a true definition of PDEs, then the definition of a symmetry of a PDE should be

**Definition 4.5.** A symmetry of a PDE $\mathcal{E} \subset J^{k}(E,n)$ is

1. a $k$-th order contact transformation $\Phi : J^{k}(E,n) \rightarrow J^{k}(E,n)$ such that $\Phi(\mathcal{E}) = \mathcal{E}$ (à la S. Lie);
2. a diffeomorphism $\Psi : \mathcal{E} \rightarrow \mathcal{E}$ preserving the distribution $\mathcal{C}_{E}^{k}$ (à la E. Cartan).

The rigidity theory shows that Definitions (1) and (2) are equivalent for rigid PDEs, i.e., for almost all PDEs of practical interest. Moreover, by Theorem 4.1, Definitions 4.5 and 3.3 are equivalent in this case too.

**Remark 4.1.** There are analogues of theorem 4.1 and definition 4.5 for infinitesimal $k$-order contact transformations and symmetries. They do not add anything new to our discussion, and we shall skip them.

In the light of the “Erlangen philosophy” the result of Theorem 4.1 looks disappointing. Indeed, it tells us that the group of $k$-order contact transformations coincides with the group of first order transformations. So, higher order contact geometries are governed by the same group as the classical one. This does not meet a natural expectation that transformations of higher order geometries should form some larger groups. Hence, by giving credit to this philosophy, we are forced to conclude that

Definition 3.2 or what is commonly meant by a differential equation is not a conceptual definition but should be considered just as a description of an object, whose nature must be still discovered.
So, the question of what object is hidden under this description is to be investigated. One rather evident hint is to examine the remaining case $k = \infty$. This is psychologically difficult, since $J^\infty(E, n)$ being an infinite-dimensional manifold of a certain kind does not possess any “good” topology or norm, etc. which seem indispensable for the existence of a “good” differential calculus on it. Another hint comes from the principle “chercher la symétrie”. For instance, if $\mathcal{E}$ (resp., $\Box$) is a linear equation (resp., a linear differential operator) with constant coefficients, then $\Box$ sends solutions of $\mathcal{E}$ to the solutions. For this reason $\Box$ may be considered as a symmetry of $\mathcal{E}$, finite or infinitesimal. Symmetries of this kind are not, generally, classical and their analytical description involves partial derivatives of any order. Hence one may expect that something similar takes place for general PDEs, and we are going to show that this is the case.

5. From integrable systems to diffieties and higher symmetries.

5.1. New experimental data: integrable systems. The discovery in the late 1960s of some remarkable properties of the now famous Korteweg-de Vries equation and later of other integrable systems brought to light various new facts, which had no conceptual explanation in terms of the classical symmetry theory. In particular, any such equation is included in an infinite series of similar equations, the hierarchy, which are interpreted as commuting Hamiltonian flows with respect to an, in a sense, infinite-dimensional Poisson structure. For this reason equations of this hierarchy may be considered as infinitesimal symmetries of each other. Moreover, they involve derivatives of any order and hence are outside the classical theory (see [71]). So, attempts to include these non-classical symmetries into common with classical symmetries frames directly leads to infinite jets.

5.2. Infinite jets and infinite order contact transformations. Recall that the Cartan distribution $\mathcal{C}^\infty$ on $J^\infty(E, n)$ is (paradoxically!) $n$-dimensional and completely integrable (Proposition 4.1, (5)). A consequence of this fact is that locally maximal integral submanifolds of $\mathcal{C}^\infty$ are of the same type in sharp contrast with finite-order contact geometries (Proposition 4.2). This is a weighty argument in favor of infinite jets. After that we have to respond to the question of whether the group of infinite-order contact transformations is broader than the group of classical ones. More exactly, we ask whether there are infinite-order contact transformations that are not of the form $\Phi(\infty)$ where $\Phi$ is a finite-order contact transformation (see Theorem 4.1). Here $\Phi(\infty)$ stands for the direct limit of $\Phi(\ell)$’s. The answer is positive: this (local) group consists of all invertible differential operators (in the generalized sense outlined above) acting on $n$-dimensional submanifolds of $E$. These operators involve partial derivatives of arbitrary orders and in this sense they justify the credit given to infinite jets. We shall skip the details (see [61]), since the same question about infinite order infinitesimal symmetries is much more interesting from the practical point of view and at the same time it reveals some unexpected a priori details, which become essential for the further discussion.

Recall that an infinitesimal symmetry of a distribution $\mathcal{C}$ on a manifold $M$ is a vector field $X \in D(M)$ such that $[X, Y] \in \mathcal{C}$ if $Y \in \mathcal{C}$ (symbolically, $[X, \mathcal{C}] \subseteq \mathcal{C}$). Infinitesimal symmetries form a subalgebra in $D(M)$ denoted $D_{\mathcal{C}}(M)$. The flow generated by a field $X \in D_{\mathcal{C}}(M)$ moves, if it is globally defined, (maximal) integral submanifolds of $\mathcal{C}$ into themselves. If it not globally defined this flow moves only
sufficiently small pieces of integral submanifolds. In this sense we can speak of a local flow in the “space of (maximal) integral submanifolds of $C$”.

If the distribution $C$ is integrable/Frobenius, then it may be interpreted as a foliation whose leaves are its locally maximal integral submanifold. In this case $C$ is an ideal in $D_C(M)$. If $N \subset M$ is a leaf of $C$, then any $Y \in D_C(M) \setminus C$. Hence the flow generated by $Y$ in the “space of all leaves of $C$” is uniquely defined by the coset $[Y \ mod \ C]$, and the quotient Lie algebra

$$\text{Sym} C \overset{\text{def}}{=} \frac{D_C(M)}{C},$$

called the symmetry algebra of $C$, is naturally interpreted as the algebra of vector fields on the “space of leaves” of $C$. It should be stressed that it would be rather counterproductive to try to give a rigorous meaning to the “space of leaves”. On the contrary, the above interpretation of the quotient algebra (15) is very productive and may be interpreted as the smile of the Cheshire Cat.

Now we shall apply the above construction to the distribution $C^\infty$ and introduce for this special case the following notation:

$$CD(J^\infty(E,n)) = C^\infty, \ D_C(J^\infty(E,n)) = D_C(J^\infty(E,n)), \ \kappa = \text{Sym} C^\infty.$$ (16)

The Lie algebra $\kappa$ will play a prominent role in our subsequent investigation. At the moment we know that it is the “algebra of vector fields on the space of all locally maximal integral submanifolds of $J^\infty(E,n)$”. As a first step we have to describe $\kappa$ in coordinates. However, in order to do that with a due rigor we have to clarify before what is differential calculus on infinite-dimensional manifolds of the kind. It is rather obvious that the usual approaches based on “limits”, “norms”, etc, cannot be applied to this situation. So, we need the following digression.

5.3. On differential calculus over commutative algebras. Let $A$ be a unitary, i.e., commutative and with unit, algebra over a field $k$ and $P$ and $Q$ be some $A$–modules.

**Definition 5.1.** $\Delta : P \rightarrow Q$ is a linear differential operator (DO) of order $\leq m$ if $\Delta$ is $k$–linear and $[a_0, [a_1, \ldots, [a_m, \Delta] \ldots]] = 0, \forall a_0, a_1, \ldots, a_m \in A$.

Elements $a_i \in A$ figuring in the above multiple commutator are understood as the multiplication by $a_i$ operators.

If $A = C^\infty(M), P = \Gamma(\pi), Q = \Gamma(\eta)$ with $\pi, \eta$ being some vector bundles, then Definition 5.1 is equivalent to the standard one. The “logic” of differential calculus is formed by functors of differential calculus together with their natural transformations and representing them objects in a differentially closed category of $A$–modules [54, 60, 59]. In particular, this allows one to construct analogues of all known structures in differential geometry, say, tensors, connections, de Rham and Spencer cohomology, an so on, over an arbitrary unitary algebra. The reader will find in [42] an elementary introduction to this subject based on a physical motivation.

By applying this approach to the filtered algebra $\mathcal{F}_\infty = \{F_i\}$ (see (9)) we shall get all necessary instruments to develop differential calculus on spaces $J^\infty(E,n)$ and,
more generally, on diffieties (see below). The informal interpretation of the filtered algebra $F_\infty$ as the smooth function algebra on the “cofiltered manifold” $J^{\infty}(E,n)$ helps to keep the analogy with the calculus on smooth manifolds under due control.

In coordinates an element of $F_\infty$ looks as a function of a finite number of variables $x_i$ and $u^j_\sigma$. This reflects the fact that any “smooth function” on $J^{\infty}(E,n)$ is, by definition, a smooth function on a certain $J^k(E,n)$, $k < \infty$, pulled back onto $J^{\infty}(E,n)$ via $\pi_{\infty,k}$. So, the filtered structure of $F_\infty$ is essential and differential operators $\Delta : F_\infty \rightarrow F_\infty$ (in the sense of definition 5.1) must respect it. This means that $\Delta(F_k) \subset F_{k+s}$ for some $s$. In particular, a vector field on $J^{\infty}(E,n)$ is defined as a derivation of $F_\infty$, which respects, in this sense, the filtration. In coordinates such a vector field looks as an infinite series

$$X = \sum_i \alpha_i \frac{\partial}{\partial x_i} + \sum_{j,\sigma} \beta^j_\sigma \frac{\partial}{\partial u^j_\sigma}, \quad \phi_i, \psi^j_\sigma \in F_\infty. \quad (17)$$

The $F_\infty$–module of vector fields on $J^{\infty}(E,n)$ will be denoted by $D(J^{\infty}(E,n))$.

### 5.4. Algebra $\mathcal{X}$ in coordinates.

Since $C^\infty$ is an $F_\infty$–module generated by the vector fields $D_i$’s (Proposition 4.1, (5)), it is convenient to represent a vector field $X \in D(J^{\infty}(E,n))$ in the form

$$X = \sum_i \psi_i D_i + \sum_{j,\sigma} \varphi^j_\sigma \frac{\partial}{\partial u^j_\sigma}, \quad \psi_i, \varphi^j_\sigma \in F_\infty. \quad (18)$$

where the first summation, which belongs to $C^\infty$, is the horizontal part of $X$, while the second one is its vertical part. This splitting of a vector field into horizontal and vertical parts is unique but depends on the choice of coordinates. Obviously, the coset $[X \mod C^\infty]$ is uniquely characterized by the vertical part of $X$.

Below we use the notation $D_\sigma \overset{\text{def}}{=} D^{\sigma_1}_1 \cdots D^{\sigma_n}_n$ for a multiindex $\sigma = (\sigma_1, \ldots, \sigma_n)$.

**Proposition 5.1.**

1. $\mathcal{X}$ is a $F_\infty$–module and $\partial/\partial u^1, \ldots, \partial/\partial u^m$ is its local basis in the chart $U$ with coordinates $(\ldots, x_i, \ldots, u^j_\sigma, \ldots)$;

2. the correspondence

$$(F_\infty)_U \ni \varphi = (\varphi^1, \ldots, \varphi^m) \leftrightarrow \mathcal{E}_\varphi = \sum D_\sigma(\varphi^i) \frac{\partial}{\partial u^j_\sigma} \in \mathcal{X}_U$$

is an isomorphism of $F_\infty$–modules localized to the chart $U$;

3. the Lie algebra structure $\{\cdot, \cdot\}$ in $\mathcal{X}_U$ is given by the formula

$$\{\varphi, \psi\} = \mathcal{E}_\varphi(\psi) - \mathcal{E}_\psi(\varphi), \quad [\mathcal{E}_\varphi, \mathcal{E}_\psi] = \mathcal{E}_{\{\varphi, \psi\}};$$

4. $(f, \mathcal{E}_\varphi) \mapsto \mathcal{E}_{f\varphi}$ is the $(F_\infty)_U$–module product in $\mathcal{X}_U$.

The vector fields $\mathcal{E}_\varphi$’s locally representing elements of the module $\mathcal{X}$ are called evolutionary derivations, and $\varphi$ is called the generating function of $\mathcal{E}_\varphi$. The bracket $\{\cdot, \cdot\}$ introduced for the first time in [58] (see also [61, 29]) is a generalization of both the Poisson and the contact brackets. Indeed, these are particular cases where $m = 1$ and the generating functions depend only on the $x_i$s and on the first derivatives and in the contact case also of $u$. If $Y$ is a vector field on $E$ ($m > 1$) or a contact vector field on $J^1(E,n)$ ($m = 1$) and $Y_{(\infty)}$ is its lift to $J^{\infty}(E,n)$, then $Y_{(\infty)} \in D_C(J^{\infty}(E,n))$ and the composition $Y \mapsto Y_{(\infty)} \mapsto [Y_{(\infty)} \mod CD(J^{\infty}(E,n))] \in \mathcal{X}$ is injective. So, infinitesimal point and contact transformations are naturally included in $\mathcal{X}$. Their generating functions depends only on $x$, $u$ and first derivatives, and we see that the Lie algebra $\mathcal{X}$ is much larger than the algebras of infinitesimal
point and contact transformations. Hence the passage to infinite jets is in fairly good accordance with the “Erlangen philosophy”. But in order to benefit from this richness of infinite order contact transformations we must bring PDEs in the context of infinite order contact geometry. But in that case we cannot mimic Definition 3.2, since, in sharp contrast with finite order jet spaces, an arbitrary submanifold of \( S \subset J^\infty(E, n) \) cannot be interpreted as a PDE. Indeed, the restriction of \( C^\infty \) to \( S \) is, generally, not \( n \)-dimensional, while we need \( n \)-dimensional integral submanifolds to define the solutions. So, we must concentrate on those submanifolds \( S \) to which \( C^\infty \) is tangent, i.e., such that \( C^\infty_\theta \subset T_\theta S, \forall \theta \in S \). These are obtained by means of the prolongation procedure.

5.5. Prolongations of PDEs and diffieties. Let \( \mathcal{E} \subset J^k(E, n) \) be a PDE in the sense of Definition 3.2 and \( N \subset \mathcal{E} \) be its solution (Definition 4.3). Then, obviously, \( T_\theta N \subset \mathcal{E}_\theta, \forall \theta \in N \). So, if \( \mathcal{E} \) admits a solution passing through a point \( \theta \in \mathcal{E} \), then there is at least one \( R \)-plane at \( \theta \), which is tangent to \( \mathcal{E} \). Since any \( R \)-plane is of the form \( R_\vartheta, \pi_{k+1}^{k+1}(E) = \theta \), the variety of all \( R \)-planes tangent to \( \mathcal{E} \) is identified with the submanifold (probably, with singularities)

\[
\mathcal{E}^{(1)} \overset{\text{def}}{=} \{ \vartheta \in J_{k+1}(E, n) | R_\vartheta \text{ is tangent to } \mathcal{E} \} \subset J^{k+1}(E, n).
\]

So, tautologically, a solution of \( \mathcal{E} \) passes only through points of \( \pi_{k+1}^{k+1}(\mathcal{E}^{(1)}) \subset \mathcal{E} \). In other words, a solution of \( \mathcal{E} \) is automatically a solution of \( \pi_{k+1}^{k+1}(\mathcal{E}^{(1)}) \). Hence by substituting \( \pi_{k+1}^{k+1}(\mathcal{E}^{(1)}) \) for \( \mathcal{E} \) we eliminate “parasitic” points. Moreover, by construction, if \( L_{(k)} \subset \mathcal{E} \), then \( L_{(k+1)} \subset \mathcal{E}^{(1)} \) and vice versa. Hence \( \mathcal{E} \) and \( \mathcal{E}^{(1)} \) have common “usual” solutions but \( \mathcal{E}^{(1)} \) is without “parasitic” points of \( \mathcal{E} \). By continuing this process of elimination of “parasitic” points we inductively construct successive prolongations \( \mathcal{E}^{(r)} \overset{\text{def}}{=} (\mathcal{E}^{(r-1)})^{(1)} \) of \( \mathcal{E} \). In this way we get an infinite series of equations, which have common “usual” solutions:

\[
\mathcal{E} = \mathcal{E}^{(0)} \overset{\pi_{k+1}^{k+1}}{\leftarrow} \mathcal{E}^{(1)} \overset{\pi_{k+2}^{k+2}}{\leftarrow} \mathcal{E}^{(2)} \overset{\pi_{k+3}^{k+3}}{\leftarrow} \ldots, \text{ with } \mathcal{E}^{(r)} \subset J^{k+r}.
\]

The inverse limit \( \mathcal{E}_\infty \) of the sequence (19) called the infinite prolongation of \( \mathcal{E} \) is a submanifold of \( J^\infty(E, n) \) (in the same sense as the latter) and one of the results of the formal theory of PDE’s tells:

**Proposition 5.2.** If the distribution \( C^\infty \) is tangent to a submanifold \( S \subset J^\infty(E, n) \), then \( S = \mathcal{E}_\infty \) for a PDE \( \mathcal{E} \).

(see [23, 50, 44, 30]).

In coordinates, prolongations of \( \mathcal{E} \) are described as follows

\[
\mathcal{E}^{(2)} = \left\{ \begin{array}{l}
\mathcal{E}^{(1)} = \left\{ \begin{array}{l}
\mathcal{E} = \{ F_s(x, u, \ldots, u^i_\sigma, \ldots) = 0, \ s = 1, \ldots, l \} \\
D_i F_s = 0 \\
D_i D_j F_s = 0
\end{array} \right.
\end{array} \right. \downarrow
\mathcal{E}_\infty = \{ D_\sigma F_s = 0, \forall s, \sigma \}
\]

**Remark 5.1.** \( \mathcal{E}_\infty \) may be empty.
The algebra $F_{∞}(E) \overset{\text{def}}{=} F|_{E_{∞}}$ plays the role of the smooth function algebra on $E_{∞}$. It is a filtered algebra $F_{0}(E) \subset \ldots \subset F_{s}(E) \subset \ldots F_{∞}(E)$ with $F_{s}(E) = \text{Im}(C^{∞}(E_{s}) \overset{π_{∞,k−s}}{\rightarrow} F_{∞}(E))$.

As in the case of infinite jets differential calculus on $E_{∞}$ is understood as differential calculus over the filtered algebra $F_{∞}(E)$.

Thus we have constructed the central object of general theory of PDEs.

**Definition 5.2.** The pair $(E_{∞}, C^{∞}_{E})$ with $C^{∞}_{E} \overset{\text{def}}{=} C^{∞}|_{E_{∞}}$ is called the *diffiety* associated with $E$.

The distribution $C^{∞}_{E}$ is $n$–dimensional, since $C^{∞}$ is tangent to $E_{∞}$. The projection $π_{∞,k}$ establishes a one-to-one correspondence between integral submanifolds of $C^{∞}_{E}$ and those of $C^{k}|_{E}$, which are transversal to fibers of $π_{k,k−1}$. So, $n$–dimensional integral submanifolds of $C^{∞}_{E}$ are identified with non-singular solutions of $E$.

The following interpretation, even though absolutely informal, is a very good guide in the task of deciphering the native language that NPDEs speak and, therefore, in terms of which they can be only understood adequately:

_The diffiety associated with a PDE $E$ (in the standard sense of this term) is the space of all solutions of $E._*

**Remark 5.2.** The reader may have already observed that nontrivial generalized solutions of $E$ cannot be interpreted as integral submanifolds of $C^{∞}_{E}$ and hence the diffiety $(E, C^{∞}_{E})$ is not the “space of all solutions of $E$”. However, this is not a conceptual defect, since this diffiety can be suitably completed.

As a rule, diffieties are infinite-dimensional. Diffieties of finite dimension are foliations, probably, with singularities. Diffieties associated with determined and overdetermined systems of ordinary differential equations (ODEs) are 1–dimensional foliations on finite-dimensional manifolds. On the contrary, diffieties associated with underdetermined systems of ODEs are infinite-dimensional. A good part of control theory is naturally interpreted as structural theory of this kind of diffieties (see [13]).

5.6. **Higher infinitesimal symmetries of PDEs.** Now having in hands the concept of diffiety we can extend the classical symmetry theory described above by including in it the new already mentioned “experimental data” that come from the theory of integrable systems. To this end it is sufficient to apply the same approach we have used to understand what are infinite-order infinitesimal contact transformations.

As before, by abusing the language, we shall denote the $F_{∞}(E)$–module of vector fields on $E_{∞}$ belonging to $C^{∞}_{E}$ by the same symbol $C^{∞}_{E}$. Since the distribution $C^{∞}$ is tangent to $E_{∞}$, vector fields $D_{i}$s are also tangent to $E_{∞}$. For this reason restrictions of the $D_{i}$ to $E_{∞}$ are well-defined. Denote them by $\bar{D}_{i}$. The Lie algebra of infinitesimal transformations preserving the distribution $C^{∞}_{E}$ is

$$D_{C}(E_{∞}) \overset{\text{def}}{=} \{X \in D(E_{∞}) \mid [X, Y] \in C^{∞}_{E}, \forall Y \in C^{∞}_{E}\}. \tag{22}$$

Now the Lie algebra of *infinitesimal higher symmetries* of a PDE $E$ is defined as

$$\text{Sym} \ E = \frac{D_{C}(E_{∞})}{C^{∞}_{E}} \tag{23}$$
This definition merits some comments. First, we use the adjective “higher” to stress the fact that generating functions of elements of the algebra $\text{Sym} E$ may depend, contrary to the classical symmetries, on arbitrary order derivatives. Next, in conformity with the above interpretation of the diffiety $(E, C_\infty^\infty)$, the informal interpretation of Definition (23) is:

\[
\text{Elements of the Lie algebra Sym} E \text{ are vector fields on the “space of all solutions of} \ E\text{”}.
\]

The importance of this interpretation is that it forces the question:

\[
\text{What are tensors, differential operators, PDEs, etc. on the “space of all solutions of} \ E\text{”}.
\]

Later we shall give some examples and indications on how to define and use this kind of objects. These objects form the thesaurus of secondary calculus, which is a natural language of the general theory of PDE’s (see [66, 27, 29]).

Finally, note that higher symmetries are not genuine vector fields as in the classical theory but just some cosets of them modulo $C_\infty^\infty$. For this reason their action on functions on $E_\infty$ is not even defined. This at first glance discouraging fact leads to the bifurcation point: either to give up or to understand what are functions on the “space of solutions of $E$”. Since, as we shall see, Definition (23), works well, the first alternative should be discarded, while the second one will lead us to discover differential forms on the “space of solutions of $E$”.

5.7. Computation of higher symmetries. Though elements of $\mathcal{X}$ are cosets of vector fields modulo $E_\infty$ we can say that $\chi = [X] \in \mathcal{X}$ is tangent to $E_\infty$ if the vector field $X$ is tangent to $E_\infty$. Since $C_\infty$ is tangent to $E_\infty$, this definition is correct. If $E_\infty$ is locally given by equations (19) and $\chi$ by the evolutionary derivation $\Theta_\varphi$, then $\chi$ is tangent to $E_\infty$ if and only if $\Theta_\varphi(D_\sigma(F_i))|_{E_\infty} = 0$, $\forall \sigma, s$. Since $\Theta_\varphi$ and the $D_i$ commute these conditions are equivalent to $\Theta_\varphi(F_s)|_{E_\infty} = 0$, $\forall s$, or, shortly, to $\Theta_\varphi(F)|_{E_\infty} = 0$ with $F = (F_1, \ldots, F_r)$. The bidifferential operator $(\varphi, F) \mapsto \Theta_\varphi(F)$ may be rewritten in the form $\Theta_\varphi(F) = \ell_F(\varphi)$ with

\[
\ell_F = \left( \begin{array}{ccc}
\sum_{\sigma} \frac{\partial \ell_F}{\partial u^{\sigma}} D_\sigma & \ldots & \sum_{\sigma} \frac{\partial \ell_F}{\partial u^{s}} D_\sigma \\
\vdots & \ddots & \vdots \\
\sum_{\sigma} \frac{\partial \ell_F}{\partial u^{1}} D_\sigma & \ldots & \sum_{\sigma} \frac{\partial \ell_F}{\partial u^{s}} D_\sigma
\end{array} \right)
\]

and $\ell_F$ is called the universal linearization operator. Being tangent to $E_\infty$ the fields $D_i$’s can be restricted to $E_\infty$. It follows from (24) that $\ell_F$ can also be restricted to $E_\infty$. This restriction will be denoted by $\bar{\ell}_F$. So, by definition, $\bar{\ell}_F(G|_{E_\infty}) = \ell_F(G)|_{E_\infty}$, $\forall G$. In these terms the condition of tangency of $\chi$ to $E_\infty$ reads

\[
\bar{\ell}_F(\bar{\varphi}) = 0, \quad \bar{\varphi} = \varphi|_{E_\infty} \quad \Rightarrow \quad \text{Sym} E = \ker \bar{\ell}_F
\]

Hence the problem of the computation of the infinitesimal symmetries of a PDE $E$ is reduced to resolution of equation (25). This equation is not a usual PDE, since it is imposed on functions depending on unlimited number of variables. Nevertheless, it is not infrequent that it can be exactly solved. For instance, this method allows not only to easily rediscover “classical” hierarchies associated with well known integrable systems but also to find various new ones (see [29, 25]).

The interpretation of higher symmetries as vector fields on the “space of solutions of $E$” leads to the question: What are the trajectories of this field? The equation
of trajectories of $\chi$ is very natural:

$$u_t = \varphi(x, u, \ldots, u^i, \ldots) \quad \text{with} \quad \varphi = (\varphi_1, \ldots, \varphi_m). \quad (26)$$

Equation (26) is the exact analogue of the classical equations

$$x_i = a_i(x), \quad i = 1, \ldots, m, \quad x = (x_1, \ldots, x_m), \quad (27)$$

which describe trajectories of the vector field $X = \sum_{i=1}^m a_i \partial/\partial x_i$. An essential difference between equations (26) and (27) is that the initial data uniquely determine solutions of (27), while it is not longer so for (26). Indeed, the uniqueness for the partial evolution equation is guaranteed by some additional to the initial conditions, for instance, the boundary ones. For this reason a “vector field” $\chi \in \kappa$ does not generate a flow on the “space of solutions of $\mathcal{E}$”.

A very important consequence of this fact is that in this new context the classical relation between Lie algebras and Lie groups breaks down. Consequently, the absolute priority should be given to infinitesimal symmetries, not to the finite ones.

One of the most popular applications of symmetry theory takes an especially simple form if expressed in terms of generating functions. Namely, imagine for a while that the flow generated by $\chi \in \kappa$ exists. Then, according to (26), “stable points” of this flow are solutions of the equation $\varphi = 0$. In other words, these “stable points” are solutions of the last equation. If $\varphi^1, \ldots, \varphi^l$ are generating functions of some symmetries of $\mathcal{E}$, then solutions of the system

$$\begin{cases}
F = 0 \\
\varphi^1 = 0 \\
\vdots \\
\varphi^l = 0
\end{cases} \quad (28)$$

represent those solutions of $\mathcal{E}$ that are stable in the above sense with respect to “flows” generated by $\varphi^1, \ldots, \varphi^l$. System (28) is well overdetermined and by this reason can be exactly solved in many cases. For instance, famous multi-soliton solutions of the KdV equation are solutions of this kind.

5.8. What are partial differential equations? The fact that we have built a self-consistent and well working theory of symmetries for PDEs based on diffieties gives a considerable reason to recognize diffieties as objects of category of PDEs. Another argument supporting this idea is as follows.

Take any PDE, say,

$$u_{xx}u_{tt}^2 + u_{tx}^2 + (u_x^2 - u_t)u = 0. \quad (29)$$

This is a hypersurface $\mathcal{E} \subset J^2(E, 2), \dim E = 3$. The equivalent system of first order PDEs is

$$\begin{cases}
u_x = v \\
u_t = w \\
v_xu_t^2 + v_tw_x + (v^2 - w)u = 0.
\end{cases} \quad (30)$$

This is a submanifold $\mathcal{E}' \subset J^1(E', 2), \dim E' = 5$, of codimension 3. $\mathcal{E}$ and $\mathcal{E}'$ live in different jet spaces and have different dimensions. For this reason their classical symmetries cannot even be compared. On the other hand, associated with $\mathcal{E}$ and $\mathcal{E}'$ diffieties are naturally identified and hence have the same (higher) symmetries. So, this fact may be interpreted by saying that (29) and (30) are different descriptions of the same object, namely, of the associated diffiety.
Another example illustrating priority of diffieties is the factorization problem. Namely, if \( G \) is a Lie algebra of classical symmetries of an equation \( \mathcal{E} \), then the question is: Can \( \mathcal{E} \) be factorized by the action of \( G \) and what is the resulting “quotient equation”? In terms of diffieties the answer is almost obvious: this is the equation \( \mathcal{E}' \) such that \( \mathcal{E}_\infty \setminus G = \mathcal{E}'_\infty \). On the contrary, it is not very clear how to answer this question in terms of the usual approach.

Example 5.1. Let \( G \) be the group of translations of the Euclidean plane. Obviously, these translations are symmetries of the Laplace equation \( u_{xx} + u_{yy} = 0 \). Then the corresponding quotient equation is again the Laplace equation.

There are many other examples manifesting that a PDE as a mathematical object is a diffiety, while what is usually called a PDE is just one of many possible “identity cards” of it.

It should be stressed that the diffiety associated with a system of PDEs (in the usual sense of this word) is the exact analogue of the algebraic variety associated to a system of algebraic equations. Indeed, if a system of algebraic equations is \( f_1 = 0, \ldots, f_r = 0 \), then the ideal defining the corresponding variety is algebraically generated by polynomials \( f_i \). In the case of a PDE \( \mathcal{E} = \{ F_i = 0 \} \) the ideal defining \( \mathcal{E}_\infty \) is algebraically generated not only by functions \( F_i \) but also by all their differential consequences \( D_\sigma (F_i) \) (see (20)). Viewed from this side algebraic geometry is seen as the zero-dimensional case of the general theory of PDEs.

6. ON THE INTERNAL STRUCTURE OF DIFFITIES.

On the surface, a diffiety \( \mathcal{O} = (\mathcal{E}, \mathcal{C}_\infty) \) looks as a simple enough object like a foliation. All foliations of given finite dimension and codimension are locally equivalent. On the contrary, the situation drastically changes when the codimension becomes infinite. So, the problem of how to extract all the information on the equation \( \mathcal{E} \), which is encoded in the “poor” Frobenius distribution \( \mathcal{C}_\infty \), naturally arises and becomes central. To gain a first insight into the problem we consider as a simple model a Frobenius distribution \( \mathcal{D} \), or, equivalently, a foliation, on a finite-dimensional manifold \( M \).

6.1. The normal complex of a Frobenius distribution. Let \( \mathcal{D} \) be an \( r \)-dimensional Frobenius distribution on a manifold \( M \). The quotient \( C^\infty(M) \)-module \( \mathcal{N} = D(M)/\mathcal{D} \) is canonically isomorphic to \( \Gamma(\nu) \) where \( \nu \) is the normal to \( \mathcal{D} \) bundle, i.e., the bundle whose fiber over \( x \in M \) is \( T_xM/\mathcal{D}_x \). Put \( \hat{Y} = [Y \mod \mathcal{D}] \in \mathcal{N} \) for \( Y \in D(M) \) and \( \nabla_X(\hat{Y}) = [X,Y] \) for \( X \in \mathcal{D} \).

It is easy to see that \( \nabla_Yf = f\nabla_X, \nabla_X(f\hat{Y}) = X(f)\hat{Y} + f\nabla_X(\hat{Y}) \) if \( f \in C^\infty(M) \) and \( [\nabla_X, \nabla_Y] = \nabla_{[X,Y]} \). These formulas tell that the correspondence \( \nabla : X \mapsto \nabla_X \) is a flat \( \mathcal{D} \)-connection. This means that this construction can be restricted to a leaf \( \mathcal{L} \) of the foliation associated with \( \mathcal{D} \) and this restriction is a flat connection \( \nabla^\mathcal{L} \) in the normal to \( \mathcal{D} \) bundle \( \nu \) restricted to \( \mathcal{L} \). Recall that with a flat connection \( \nabla^\mathcal{L} \) associated to \( \mathcal{L} \) there is a de Rham-like complex (see [11]), which for \( \nabla^\mathcal{L} \) is

\[
0 \rightarrow \mathcal{N}_\mathcal{L} \xrightarrow{\nabla^\mathcal{L}} \Lambda^1(\mathcal{L}) \otimes_{C^\infty(\mathcal{L})} \mathcal{N}_\mathcal{L} \xrightarrow{\nabla^\mathcal{L}} \cdots \xrightarrow{\nabla^\mathcal{L}} \Lambda^r(\mathcal{L}) \otimes_{C^\infty(\mathcal{L})} \mathcal{N}_\mathcal{L} \rightarrow 0
\]
where the covariant differential is abusively denoted also by $\nabla^\mathcal{L}$ and $N_\mathcal{L} = \Gamma(\nu|_\mathcal{L})$. This complex is, in fact, the restriction of the complex to $\mathcal{L}$.

$$0 \rightarrow N \xrightarrow{\nabla} \Lambda^1_P \otimes_{C^\infty(M)} N \xrightarrow{\nabla} \cdots \xrightarrow{\nabla} \Lambda^k_P \otimes_{C^\infty(M)} N \rightarrow 0 \quad (32)$$

where $\Lambda^i_P = \Lambda^i(M)/\mathcal{D}\Lambda^i(M)$ with

$$\mathcal{D}\Lambda^i(M) = \{ \omega \in \Lambda^i(M) \mid \omega(X_1, \ldots, X_i) = 0, \forall X_1, \ldots, X_i \in \mathcal{D} \}.$$  

The terms of the complex (32) are $N$–valued differential forms on $\mathcal{D}$, i.e., $\rho(X_1, \ldots, X_s) \in N$ if $X_1, \ldots, X_s \in \mathcal{D}$. The covariant differential $\nabla$ is defined as

$$\nabla(\rho)(X_1, \ldots, X_{s+1}) = \sum_{i=1}^{s+1} (-1)^{i-1}\nabla_{X_i}(\rho(X_1, \ldots, \tilde{X_i}, \ldots, X_{s+1})) +$$

$$\sum_{i<j}(-1)^{i+j}\rho([X_i, X_j], X_1, \ldots, \tilde{X_i}, \ldots, X_{j-1}, X_{j+1}, \ldots, X_{s+1}).$$

Pictorially, this situation may be seen as a “foliation” of the complex (32) by complexes (31). The $i$-th cohomology of complexes (31) and (32) will be denoted by $H^i(\nabla^\mathcal{L})$ and $H^i(\nabla)$, respectively. We also have a natural restriction map $H^i(\nabla) \rightarrow H^i(\nabla^\mathcal{L})$ in cohomology.

Formally, the above construction remains valid for any Frobenius distribution and hence can be applied to diffieties. In order to duly specify complex (32) to this particular case we need a new construction from differential calculus over commutative algebras.

6.2. Modules of jets. Let $A$ be an unitary algebra and let $P, Q$ be $A$–modules. Denote by $\text{Diff}_k(P, Q)$ the totality of DO’s of order $\leq k$ considered as a left $A$–module, i.e., $(a, \square) \mapsto a\square$, $a \in A$, $\square \in \text{Diff}_k(P, Q)$. Consider a subcategory $\mathcal{K}$ of the category of $A$-modules such that $\text{Diff}_k(P, Q) \in \text{Ob} \mathcal{K}$ if $P, Q \in \text{Ob} \mathcal{K}$. For a fixed $P$ we have the functor $Q \mapsto \text{Diff}_k(P, Q)$. We say that a pair composed of an $A$–module $\mathcal{J}_k^\mathcal{K}(P)$ and a $k$-th order DO $j_k = j_k^{\mathcal{K}, P, Q} : P \rightarrow \mathcal{J}_k^\mathcal{K}$ represents this functor in the category $\mathcal{K}$ if the map $\text{Hom}_A(\mathcal{J}_k^\mathcal{K}(P), Q) \ni h \mapsto h \circ j_k \in \text{Diff}_k(P, Q)$ is an isomorphism of $A$–modules. Under some weak condition on $\mathcal{K}$, which we skip, the representing object $(\mathcal{J}_k^\mathcal{K}(P), j_k)$ exists and is unique up to isomorphism. $\mathcal{J}_k^\mathcal{K}(P)$ is called the module of $k$-th order jets of $P$ in $\mathcal{K}$. Thus for a DO $\square \in \text{Diff}_k(P, Q)$ there is a unique $A$–module homomorphism $h_\square : \mathcal{J}_k^\mathcal{K}(P) \rightarrow Q$ such that $\square = h_\square \circ j_k$.

As an $A$–module $\mathcal{J}_k^\mathcal{K}(P)$ is generated by elements $j_k(p), p \in P$. A natural transformation of functors $\text{Diff}_l(P, \cdot) \rightarrow \text{Diff}_k(P, \cdot), l \leq k$, induces a homomorphism $\pi_{k, l} = \pi_{k, l}^{\mathcal{K}, P, Q} : \mathcal{J}_k^\mathcal{K}(P) \rightarrow \mathcal{J}_l^\mathcal{K}(P)$ of $A$–modules such that $j_l = \pi_{k, l} \circ j_k$. This allows to define the inverse limit of pairs $(\mathcal{J}_k^\mathcal{K}(P), j_k)$ called the module of infinite jets of $P$ and denoted by $(\mathcal{J}_\infty^\mathcal{K}, j_\infty) = \mathcal{J}_\infty^{\mathcal{K}, P, Q}$. Natural projections $\pi_{k, \infty} : \mathcal{J}_\infty^\mathcal{K}(P) \rightarrow \mathcal{J}_k^\mathcal{K}(P)$ come from the definition. These maps supply $\mathcal{J}_\infty^\mathcal{K}(P)$ with a decreasing filtration

$$\mathcal{J}_\infty^\mathcal{K}(P) \supset ker(\pi_{\infty, 0}) \supset ker(\pi_{\infty, 1}) \supset \cdots \supset ker(\pi_{\infty, k}) \supset \cdots \quad (33)$$

Finally, we stress that $\mathcal{J}_k^\mathcal{K}(P)$ and all related constructions essentially depend on $\mathcal{K}$.

Any operator $\square \in \text{Diff}_r(P, Q)$ induces a homomorphism

$$h_\square^r : \mathcal{J}_k^{k+r}(P) \rightarrow \mathcal{J}_k^\mathcal{K}(Q), \quad r \geq 0.$$  

Namely, the composition $P \xrightarrow{\square} Q \xrightarrow{\mathcal{J}_k^\mathcal{K}} \mathcal{J}_k^\mathcal{K}(Q))$ is a DO of order $\leq k + r$. So, it can be presented in the form $h_{j_r \circ \square} \circ j_{k+r}$, and we put

$$\def\hbar{\hat{h}}\hbar^r \circ \square : \mathcal{J}_k^{k+r}(P) \rightarrow \mathcal{J}_k^\mathcal{K}(Q). \quad (34)$$
The inverse limit of homomorphisms $h^k$'s defines a homomorphism of filtered modules

$$ h^k \colon J^\infty_k(P) \to J^\infty_k(Q) $$

which shifts filtration (33) by $-k$.

If $Q = J^k_s(P)$ and $\square = j_k$, then the above construction gives natural inclusions

$$ t_k : J^k_s(\square) \to J^k_j(P). $$

Their inverse limit of these inclusions is

$$ t_\infty : J^\infty_k(P) \to J^\infty_k(J^\infty_k(P)). $$

Now we shall describe constructively the above conceptually defined modules of jets for geometrical modules over the algebra $A = C^\infty(M)$. Recall that an $A$-module $P$ is geometrical if all its elements $p$ such that $p \in \mu_z \cdot P, \forall z \in M$, are equal to zero (see [42]). Here $\mu_z = \{ f \in C^\infty(M) \mid f(z) = 0 \}$. The category of geometrical $A$-modules will be denoted by $\mathcal{G}$ and we shall write simply $J^k(P)$ for $J^k_0(P)$.

Put $J^k = J^k(A)$ and note that $J^k$ is a unitary algebra with the product $(f_1 j_k(g_1)) \cdot (f_2 j_k(g_2)) = f_1 f_2 j_k(g_1 g_2), f_i, g_i \in A$. In particular, $J^k$ is a bimodule. Namely, left (standard) and right multiplications by $f \in A$ are defined as $(f, \theta) \mapsto f \theta$ and $(f, \theta) \mapsto \theta j_k(f)$, respectively. $J^k$ supplied with the right $A$-module structure will be denoted by $J^k_s$.

We have (see [42]).

**Proposition 6.1.** (1) Let $\alpha_k$ be the vector bundle whose fiber over $z \in M$ is $J^1_0(z)$ (see section 2.1). Then $J^k = \Gamma(\alpha_k)$.

(2) $J^k(P) = J^k_s \otimes_A P$.

**6.3. Jet-Spencer complexes.** The $k$-th jet-Spencer complex of $P$ denoted by $S_k(P)$ is defined as

$$ 0 \to J^k(P) \to J^{k-1}(P) \oplus A \Lambda^1(M) \to \cdots \to J^k_n(P) \oplus A \Lambda^n(M) \to 0 $$

with $S_k(j_{k-s}(p) \otimes \omega) = j_{k-s-1}(p) \otimes d\omega, \omega \in \Lambda^s(M)$. Here $n = \dim M$ and we assume that $J^s(P) = 0$ if $s < 0$.

Differentials of Spencer complexes are 1-st order DOs. For $k \geq l$, the homomorphisms

$$ \pi_{k-s,l-s} \otimes \text{id}_{\Lambda^s(M)} : J^{k-s}(P) \otimes_A \Lambda^s(M) \to J^{l-s}(P) \otimes_A \Lambda^s(M), s = 0, 1, \ldots, n, $$

define a cochain map $\sigma_{k,l} : S_k(P) \to S_l(P)$. In particular, we have the following sequence of Spencer complexes

$$ 0 \to S_0(P) \xleftarrow{\sigma_{1,0}} S_1(P) \xleftarrow{\sigma_{2,1}} \cdots \xleftarrow{\sigma_{k-1,1}} S_k(P) \xleftarrow{\sigma_{k+1,1}} \cdots $$

The infinite jet-Spencer complex $S_\infty(P)$ is defined as the inverse limit of (36) together with natural cochain maps $\sigma_{\infty,k}$. As in the case of jets the complex $S_\infty(P)$ is filtered by subcomplexes $\ker(\sigma_{\infty,k})$.

An operator $\square \in \text{Diff}_r(P, Q)$ induces a cochain map of Spencer complexes

$$ \sigma^\square_{k} : S_k(P) \to S_{k-r}(Q), $$

which acts on the $s$-th term of $S_k(P)$ as

$$ h_{\square}^{k-s} \otimes \text{id}_{\Lambda^s(M)} : J^{k-s}(P) \otimes_A \Lambda^s(M) \to J^{k-s-r}(Q) \otimes_A \Lambda^s(M) $$

(38)
used leaves of D in terms of the distribution D hence d to DS dial. Similarly, respectively. In this way we get the horizontal de Rham factorize to \(\bar{\Lambda}^i\) of all differential forms (resp., jets) whose restrictions to all leaves of L single leaves. In other words, if viewed as a family of differential forms (resp., jet, Spencer complex) defined on \(C^\infty(N)\)-modules can also be considered as \(C^\infty(M)\)-modules, the composition\(\square\)

\[ P \xrightarrow{\text{restriction}} P_N \xrightarrow{j_N^k} J^k(P_N) \]

is a \(k\)-th order DO over \(C^\infty(M)\). The homomorphism of \(C^\infty(M)\)-modules \(h_\square : J^k(P) \to J^k(P_N)\) associated with \(\square\) is, by definition, the restriction operator. Now, by tensoring this restriction operator with the well-known restriction operator for differential forms we get the restriction operator for terms of \(S_k(P)\). Finally, by passing to the inverse limit we get the restriction operator for \(S_\infty(P)\).

6.4. Foliation of Spencer complexes by a Frobenius distribution. Take the notation of Subsection 6.1 and denote by \(D\Lambda^i(M)\) (resp., \(D\mathcal{J}^k(M)\)) the totality of all differential forms (resp., jets) whose restrictions to all leaves of \(D\) are trivial. Similarly, \(DS_k(P)\) stands for the maximal subcomplex of \(S_k(P)\) such that its restrictions to leaves of \(D\) are trivial. Horizontal (with respect to \(D\)) differential forms and jets are elements of the quotient modules

\[ \bar{\Lambda}^i_D(M) \overset{\text{def}}{=} \Lambda^i(M)/D\Lambda^i(M), \quad \mathcal{J}^k(P) \overset{\text{def}}{=} J^k(P)/D\mathcal{J}^k(P). \]

Similarly, the horizontal jet-Spencer complex is

\[ \bar{S}_k(P) \overset{\text{def}}{=} S_k(P)/D S_k(P). \]

Restrictions of all the above horizontal objects to leaves of \(D\) are naturally defined. Conversely, a horizontal differential form (resp., jet, Spencer complex) may be viewed as a family of differential forms (resp., jet, Spencer complex) defined on single leaves. In other words, if \(L\) runs the all leaves of \(D\), then the \(\Lambda^i(L)\) foliate \(\bar{\Lambda}^i_D(M)\), and similarly for jets and Spencer complexes.

Accordingly, the exterior differential \(d\) as well as the Spencer differential \(S_k\) factorize to \(\Lambda^i(M)\) and \(\bar{S}_k(P)\), since \(D\Lambda^i(M)\) and \(DS_k(P)\) are stable with respect to \(d\) and \(S_k\), respectively. These quotient differentials will be denoted by \(d\) and \(\bar{S}_k\), respectively. In this way we get the horizontal de Rham and Spencer complexes and hence horizontal de Rham and Spencer cohomology.

Remark 6.1. For simplicity, in the above definition of horizontal objects we have used leaves of \(D\). It fact, with a longer formal procedure this can be done explicitly in terms of the distribution \(D\).
6.5. The normal complex of a diffiety and more symmetries. First, we shall describe the normal complex for $J^\infty(E, n)$. Denote the normal bundle to $C_k$ by $\nu_k$, $1 \leq k \leq \infty$, and put $\kappa_k \overset{\text{def}}{=} \Gamma(\nu_k) = D(J^k(E, n))/C_k$, $\kappa_{l,k} \overset{\text{def}}{=} \Gamma(\pi^*_{k,l}(\nu_l)) = \pi^*_{k,l}(\kappa_l)$, $l \leq k$.

**Proposition 6.3.**

1. $\kappa = \kappa_1, \infty$, $\kappa_k, \infty = \tilde{J}^k(\kappa)$ and $\kappa_\infty = \tilde{J}^\infty(\kappa)$;
2. the normal to $C_\infty$ complex is isomorphic to $\tilde{S}_\infty(\kappa)$;
3. $H^0(\tilde{S}_\infty) = \kappa$, $H^i(\tilde{S}_\infty) = 0$, $i \neq 0$.

Assertion (3) in this proposition is a pro-finite consequence of Proposition 6.2. The following important interpretation is a consequence of this and the second assertions of the Proposition 6.3:

$\kappa$ is the zero-th cohomology space of the complex normal to the infinite contact structure $C_\infty$.

Now we can describe the normal complex to $C_\infty E$ by restricting, in a sense, Proposition 6.3 to $E_\infty$. First, to this end, we need a conceptually satisfactory definition of the universal linearization operator (24), which was defined coordinate-wisely in Subsection 5.7.

The equation $E \subset J^k(E, n)$ may be presented in a coordinate-free form as $\Phi = 0$, $\Phi \in \Gamma(\xi)$ with $\xi$ being a suitable vector bundle over $J^k(E, n)$. If $P = \Gamma(\pi^*_\infty, k(\xi))$, then $E_\infty = \{\tilde{j}_\infty(\Phi_\infty) = 0\}$. Additionally, assume that $P$ is supplied with a connection $\nabla$. If $Y \in C_\infty$, then, as it is easy to see, $\nabla_Y(\Phi)|E_\infty = 0$.

For this reason the following definition is correct.

$$\ell_E(\chi) \overset{\text{def}}{=} \nabla_X(\Phi)|E_\infty \quad \text{with} \quad \chi = [X \mod C_\infty].$$

(39)

The operator $\ell_E : \kappa \rightarrow P$ does not depend on the choices of $\Phi$ and $\nabla$. It defines the cochain map of jet-Spencer complexes

$$\sigma^*_{\ell_E} : S_\infty(\kappa) \rightarrow S_\infty(P)$$

(40)

(see (37)).

**Proposition 6.4.** Let $N_E$ be the normal to the distribution $C_\infty E$ complex on $E_\infty$. Then

1. $N_E$ is isomorphic to the complex ker $\sigma^*_{\ell_E}$;
2. The cohomology $H^i(N_E)$ of the complex $N_E$ is trivial if $i > n$.
3. $\text{Sym}_E = H^0(N_E) = \ker \ell_E$.

Assertions (2) and (3) of this proposition are consequences of the first one, which allows to compute the cohomology of $N_E$. Moreover, assertion (3) is one of many other arguments that motivate the following definition.

**Definition 6.1.** The Lie algebra of (higher) infinitesimal symmetries of a PDE $E$ is the cohomology of the normal complex $N_E$.

Accordingly, denote by $\text{Sym}_i E$ the $i$-th cohomology space of $N_E$. So, the whole Lie algebra of infinitesimal symmetries of $E$ is graded:

$$\text{Sym}_* E = \sum_{i=0}^{n} \text{Sym}_i E, \quad \text{Sym}_i E = H^i(N_E)$$

In particular, $\text{Sym} E = \text{Sym}_0 E$ (see Subsection 5.6).
Remark 6.2. The description of the Lie product in \( \text{Sym} \ast \mathcal{E} \) is not immediate and requires some new instruments of differential calculus over commutative algebras. For this reason we shall skip it.

The following proposition illustrates what the algebra \( \text{Sym} \mathcal{E} \) looks like.

Proposition 6.5. If \( \mathcal{E} \) is not an overdetermined system of PDEs, then

\[
\text{Sym}_0 \mathcal{E} = \ker \ell \mathcal{E}, \quad \text{Sym}_1 \mathcal{E} = \text{coker} \ell \mathcal{E} \quad \text{and} \quad \text{Sym}_i \mathcal{E} = 0 \quad \text{if} \quad i \neq 0, 1.
\]

In this connection we note that a great majority of the PDEs of current interest in geometry, physics and mechanics are not overdetermined. As an exception we mention the system of Yang-Mills equations, which is sightly overdetermined, and for these equations \( \text{Sym}_2 \mathcal{E} \neq 0 \).

To conclude this section we would like to emphasize the role of the structure of differential calculus over commutative algebras in the above discussion. While we have used the “experimental data” coming from the theory of integrable systems to discover “by hands” the conceptually simplest part of infinitesimal symmetries of PDEs, i.e., the Lie algebra \( \text{Sym} \mathcal{E} \), a familiarity with the structures of differential calculus over commutative algebras is indispensable to discover that it is just the zeroth component of the full symmetry algebra \( \text{Sym} \ast \mathcal{E} \), which in its turn is the cohomology of a certain complex.

7. Nonlocal symmetries and once again : what are PDEs ?

In the previous section we have constructed a self-consistent symmetry theory, which, from one side, resolves shortcomings of the classical theory discussed in Sections 2-5 and, from another side, incorporates “experimental data” that emerged in the theory of integrable systems. However, one important element of this theory was taken into account. Namely, we have in mind nonlocal symmetries. Roughly speaking, these are symmetries whose generating function depends on variables of the form \( D^{-1} u \). Fortunately, these unusual symmetries can be tamed by introducing only one new notion we are going to describe.

7.1. Coverings of a diffiety. Schematically, a diffiety \( \mathcal{O} \) is a pro-finite manifold \( \mathfrak{M} \) supplied with a finite-dimensional pro-finite Frobenius distribution \( \mathcal{D} = D_\mathcal{O} : \mathcal{O} = (\mathfrak{M}, \mathcal{D}) \). We omit technical details that these data must satisfy.

Recall that a pro-finite manifold is the inverse limit of a sequence of smooth maps

\[
M_0 \xleftarrow{\mu_1} M_1 \xleftarrow{\mu_2} \cdots \xleftarrow{\mu_k} M_k \xleftarrow{\mu_{k+1}} \cdots \subseteq \mathfrak{M}. \tag{41}
\]

A pro-finite distribution on \( \mathfrak{M} \) is the inverse limit via \( \mu_i \)'s of distributions \( \mathcal{D}_i \)'s on \( M_i \)'s. The associated sequence of homomorphisms of smooth function algebras

\[
C^\infty(M_0) \xrightarrow{\mu_1^*} C^\infty(M_1) \xrightarrow{\mu_2^*} \cdots \xrightarrow{\mu_k^*} C^\infty(M_k) \xrightarrow{\mu_{k+1}^*} \cdots \Rightarrow \mathcal{F}_\mathfrak{M} \tag{42}
\]

with \( \mathcal{F}_\mathfrak{M} \) being the direct limit of homomorphisms \( \mu_k^* \) is filtered by subalgebras

\[
\mathcal{F}_\mathfrak{M} \overset{\text{def}}{=} \mu_{\infty,k}^* C^\infty(M_k) \quad \text{where} \quad \mu_{\infty,k} : \mathfrak{M} \to M_k \quad \text{is a natural projection. Differential calculus on} \ \mathfrak{M} \quad \text{is interpreted as the calculus over the filtered algebra} \ \mathcal{F}_\mathfrak{M} \quad \text{(see Subsection 5.3). The dimension of} \ \mathcal{D} \quad \text{is interpreted as the “number of independent variables”}.
\]
A morphism $F : O \to O'$ of a diffiety $O = (\mathcal{M}, \mathcal{D})$ to a diffiety $O' = (\mathcal{M}', \mathcal{D}')$ is, abusing the notation, a map $F : \mathcal{M} \to \mathcal{M}'$ such that $F^*(\mathcal{D}') \subset \mathcal{D}$, $F^*$ is compatible with filtrations and $d_\theta F(\mathcal{D}_\theta) \subset \mathcal{D}'_{F(\theta)}$, $\forall \theta \in \mathcal{M}$.

**Definition 7.1.** A surjective morphism $F : O \to O'$ of diffieties is called a covering if $\dim \mathcal{D} = \dim \mathcal{D}'$ and $d_\theta F$ isomorphically sends $\mathcal{D}_\theta$ to $\mathcal{D}'_{F(\theta)}$, $\forall \theta \in \mathcal{M}$.

This terminology emphasizes the analogy with the standard notion of a covering in the category of manifolds. Namely, fibers of a covering are zero-dimensional diffieties in the sense that their structure distributions $\mathcal{D}'$'s are zero-dimensional. If these fibers are finite-dimensional in the usual sense, then the covering is called finite-dimensional.

A covering $F : \mathcal{E}_\infty \to \mathcal{E}'_\infty$ may be interpreted as a (nonlinear) DO, which sends solutions of $\mathcal{E}$ to solutions of $\mathcal{E}'$. More exactly, it associates with a solution of $\mathcal{E}'$ a families of solutions of $\mathcal{E}$. For instance, the famous Cole-Hopf substitution $v = 2u_x/u$ that sends solutions of the heat equation $\mathcal{E} = \{u_t = u_{xx}\}$ to solutions of the Burgers equation $\mathcal{E}' = \{v_t = v_{xx} + vv_x\}$ comes from a 1-dimensional covering of $\mathcal{E}'$. Equivalently, the passage from a PDE to a covering equation is the inversion of a (nonlinear) DO on solutions of this PDE. For instance, by inverting the operator $v \mapsto 2v_x/v$ on solutions of the Burgers equation one gets the heat equation.

7.2. **Where coverings appear.** The notion of a covering of a diffiety was introduced by the author (see [62]) as a common basis for various constructions that appeared in PDE’s. Below we list and briefly discuss some of them.

1) In the language of diffieties the passage from Lagrange’s description of a continuum media to that of Euler is interpreted as a covering. This interpretation allows to apply instruments of secondary calculus to this situation and, as a result, to derive from this fact some important consequences for mechanics of continua.

2) **Factorization of PDE’s.** If $G$ is a symmetry group of a diffiety $O$, then under some natural conditions the quotient diffiety $O\setminus G$ is well-defined and $O \to O\setminus G$ is a covering. In particular, if $O = \mathcal{E}_\infty$, then $O\setminus G = \mathcal{E}'_\infty$. In such a case $\mathcal{E}'$ is the quotient equation of $\mathcal{E}$ by $G$. A remarkable fact is that the group $G$ in this construction may be an “infinite-dimensional” Lie group like the group $\text{Diffeo}(M)$ of diffeomorphisms of a manifold $M$, or the group of contact transformations, etc.

3) **Differential invariants and characteristic classes** Let $\pi : E \to M$ be a fiber bundle of geometrical structures of a type $\mathcal{G}$ on $M$ (see [1]). Then $\text{Char} \mathcal{G} = J^\infty(\pi) \setminus \text{Diffeo}(M)$ is the characteristic diffiety for $\mathcal{G}$– structures. This diffiety is with singularities, which are in turn diffieties with a smaller numbers of independent variables. Functions on $\text{Char} \mathcal{G}$ are scalar differential invariants of $\mathcal{G}$– structures, horizontal de Rham cohomology is composed of their characteristic classes, etc.

Similarly one can define differential invariants and characteristic classes for solutions of natural PDEs, i.e., those that are invariant with respect to the group $\text{Diffeo}(M)$ or some more specific subgroups of this group. For instance, Einstein equations and many other equations of mathematical physics are natural. Gel’fand-Fuks characteristic classes are quantities of this kind. The reader will find more details and examples in [65, 39, 53].
4) **Bäcklund transformations.** The notion of covering allows to rigorously define Bäcklund transformations. Namely, the diagram

![Diagram](image)

where $F'$ and $F''$ are coverings presents the Bäcklund transformation $F'' \circ (F')^{-1}$ from $\mathcal{E}'$ to $\mathcal{E}''$ and its inverse $F' \circ (F'')^{-1}$. The importance of this definition lies in the fact that it suggests an efficient and regular method for finding Bäcklund transformations for a given PDE (see [28, 29, 22]). Previously this was a kind of handcraft art. Moreover, it turned out to be possible to prove for the first time nonexistence of Bäcklund transformations connecting two given equations (see [21]). This seems to be an impossible task by using only the standard techniques of the theory of integrable systems.

5) **Poisson structures and the Darboux lemma in field theory.** The efficiency and elegance of the Hamiltonian approach to the mechanics of systems with a finite number of degrees of freedom motivates to look for its extension to the mechanics of continua and field theory. Obviously, this presupposes a due formalization of the idea of a Poisson structure in the corresponding infinite-dimensional context. Over the past 70-80 years various concrete constructions of the Poisson bracket in field theory were proposed, mainly, by physicists. But the first attempts to build a systematic general theory can be traced back only to late 1970’s. Here we mention B. A. Kuperschmidt’s paper [31] where he constructs an analogue of the Poisson structure on the cotangent bundle on infinite jets, and the paper by I. M. Gel’fand and I. Dorfman [16] in the context of “formal differential geometry”. A general definition of a Poisson structure on infinite jets was proposed by the author in [57] but its extension to general diffieties appeared to be a not very trivial task.

More precisely, while the necessary definition of multivectors in secondary calculus, sometimes also called variational multivectors, is a natural generalization of Definition 6.1, some technical aspects of the related Schouten bracket mechanism are to be still elaborated. See [66, 26, 18] for further results.

On the other hand, in the context of integrable systems numerous concrete Poisson structures were revealed. Among them the bi-hamiltonian ones deserves a special mention (see [38]). So, arises the question of their classification. In the finite-dimensional case the famous Darboux lemma tells that symplectic manifolds or, equivalently, nondegenerate Poisson structures of the same dimension are locally equivalent. “What is its analogue in field theory?” is a good question, which, at first glance, seems to be out of place as many known examples show. Nevertheless, by substituting “coverings” for “diffeomorphisms” in the formulation of this lemma and observing that these two notions are locally identical for finite-dimensional manifolds we get some satisfactory results. Namely, all Poisson structures explicitly described up to now on infinite jets are obtained from a few models by passing to suitable coverings. See [3] for more details.

7.3. **Nonlocal symmetries.** The first idea about nonlocal symmetries its takes origin at a seemingly technical fact. It was observed that the PDEs forming the KdV
hierarchy are obtained from the original KdV equation $\mathcal{E} = \{u_t = uu_x + u + u_{xxx}\}$ by applying the so-called recursion operator. This operator

$$R = D_x^2 + \frac{2}{3}u + \frac{1}{3}u_xD^{-1}_x$$

is not defined rigorously. By applying it to generating functions of symmetries of $\mathcal{E}$ one gets new ones that may depend on $D^{-1}_x u = \int u\,dx$. A due rigor to this formal trick can be given by passing to a 1-dimensional covering $\mathcal{E}_c \rightarrow \mathcal{E}_\infty$ by adding to standard coordinates on $\mathcal{E}_\infty$ a new one $w$ such that $D_x w = u$ and $D_t w = u_{xx} + \frac{1}{2}u^2$.

In this setting the above symmetries of $\mathcal{E}$ depending on $\int dx$, i.e., nonlocal ones, become symmetries of $\mathcal{E}_c$ in the sense of Definition 6.1, i.e., local ones. This and other similar arguments motivate the following definition.

**Definition 7.2.** A nonlocal symmetry (finite or infinitesimal) of an equation $\mathcal{E}$ is a local symmetry of a diffiety $\mathcal{O}$, which covers $\mathcal{E}_\infty$. If $\tau : \mathcal{O} \rightarrow \mathcal{E}_\infty$ is a covering, then symmetries of $\mathcal{O}$ are called $\tau$–symmetries of $\mathcal{E}$.

Similarly are defined nonlocal quantities of any kind. For instance, Poisson structures in field theory discussed in Subsection 7.2 are nonlocal with respect to the original PDE/diffiety.

Let $\tau_i : \mathcal{O}_i \rightarrow \mathcal{E}_\infty$, $i = 1, 2$, be two coverings of $\mathcal{E}_\infty$. A remarkable fact, which is due to I. S. Krasil’shchik, is that the Lie bracket of a $\tau_1$–symmetry and a $\tau_2$–symmetry can be defined as a $\tau$–symmetry for a suitable covering $\tau : \mathcal{O} \rightarrow \mathcal{E}_\infty$ together with coverings $\tau'_i : \mathcal{O} \rightarrow \mathcal{O}_i$, $i = 1, 2$, such that $\tau = \tau_i \circ \tau'_i$. The covering $\tau$ is not defined uniquely. Nevertheless, this non-uniqueness can be resolved by passing to a common covering for “all parties in question”. The Jacobi identity as well as other ingredients of Lie algebra theory can be settled in a similar manner (see [28, 29]). So, nonlocal symmetries of a PDE $\mathcal{E}$ form this strange Lie algebra, and this fact in turn confirms the validity of Definition 7.2.

Thus this definition incorporates all theoretically or experimentally known candidates for symmetries of a PDE. Moreover, it brings us to a new challenging question:

*Symmetries of which object are the elements of the above “strange” Lie algebra?*

Indeed, this algebra can be considered not only as the algebra of nonlocal symmetries of the equation $\mathcal{E}$ but also as the symmetry algebra of any equation that covers $\mathcal{E}$. In other words, the question: What are partial differential equations? arises again in this new context. But before we shall take a necessary look at the related problem of construction of coverings.

### 7.4. Finding of coverings

The problem of how to find coverings of a given equation is key from both practical and theoretical points of view. At present we are rather far from its complete solutions. So, below we shall illustrate the situation by sketching a direct method, which works well for PDEs in two independent variables and also supplies us with an interesting experimental material.

Let $\mathcal{E} \subset J^k(E, n), \mathcal{O} = (\mathfrak{M}, D)$ and $\tau : \mathcal{O} \rightarrow \mathcal{E}_\infty$ be a covering. A $\tau$–projectable vector field $X \in D$ is of the form $X = \tilde{X} + V$ where $\tilde{X} \in \mathcal{C}_\mathcal{E}$ and $V$ is $\tau$–vertical, i.e., tangent to the fibers of $\tau$. Locally $\tau$ can be represented as the projection $U \times W \rightarrow \mathcal{E}_\infty$ with $W$ being a pro-finite manifold and $U$ a domain in $\mathcal{E}_\infty$. If $U$ is sufficiently small, then the restrictions $\tilde{D}_i, i = 1, \ldots, n$, of the total derivatives $D_i$'s
to $U$ span the distribution $\mathcal{C}_E|_U$. The vector fields $\hat{D}_i \in \mathcal{D}$ that project onto the $\hat{D}_i$ span $\mathcal{D}|_{\tau^{-1}(U)}$ and $\hat{D}_i = \hat{D}_i + V_i$ where $V_i$ is $\tau$–vertical. The Frobenius property of $\mathcal{D}$ is equivalent to

$$0 = [\hat{D}_i, \hat{D}_j] \Leftrightarrow [\hat{D}_i, V_j] - [\hat{D}_j, V_i] + [V_i, V_j] = 0, \quad 1 \leq i < j \leq n. \quad (43)$$

By inverting this procedure we get a method to search for coverings of $\mathcal{C}_E$. Namely, take a pro–finite manifold $W$ with coordinates $w_1, w_2, \ldots$ and vector fields $V_i = \sum_r a_r \partial/\partial w_s$ on $U \times W$ with indeterminate coefficients $a_r \in C^\infty(U \times W)$. Any choice of these coefficients satisfying relations (43) defines a Frobenius distribution $\text{span}\{\hat{D}_1, \ldots, \hat{D}_n\}$, which covers $\mathcal{C}_E$. So, by resolving equations (43) with respect to the $a_r$, we get local coverings of $\mathcal{C}_E$. Many exact solutions of these equations for concrete PDEs of interest can be found for $n = 2$ and they reveal a very interesting structure, which we illustrate with the following example.

**Example 7.1.** For the KdV equation $\mathcal{E} = \{u_t = uu_x + u_{xxxx}\}$ we may take $t, x, u, u_x, u_{xx}, \ldots$ for coordinates on $\mathcal{C}_E$. Then the vector fields

$$D_x \overset{def}{=} \hat{D}_1 = \frac{\partial}{\partial x} + \sum_{s=0}^\infty u_{s+1} \frac{\partial}{\partial u_s}, \quad D_t \overset{def}{=} \hat{D}_2 = \frac{\partial}{\partial t} + \sum_{s=0}^\infty D_x^s(u_3 + uu_1) \frac{\partial}{\partial u_s},$$

with $u_s = u_{s\times s}$ ($s$–times) span $\mathcal{C}_E$. Put $V_x = V_1, V_t = V_2$ and look for solutions of (43) assuming that $a_r = a_r(u, u_1, u_2, w_1, w_2, \ldots)$ for simplicity. The result is worth to be reported in details. We have

$$V_x = u^2 A + uB + C,$$

$$V_t = 2u w_2 A + u_2 B - u^2 A + u_1 [B, C] + \frac{2}{9} u^3 A + \frac{1}{2} (B + [B, [C, B]]) + u[C, [C, B]] + D$$

with $A, B, C, D$ being some fields on $W$ such that

$$[A, B] = [A, C] = [C, D] = 0, \quad [B, D] + [C, [C, [C, B]]] = 0,$$

$$[B, [B, [C, B]]] = 0, \quad [A, D] + \frac{3}{2} [B, [C, [C, B]]] = 0. \quad (45)$$

This results tells that if we consider the Lie algebra generated by four elements $A, B, C, D$, which are subject to the relations (45), then any representation of this algebra by vector fields on a manifold $W$ gives a covering of $\mathcal{C}_E$ associated with the vector fields (44).

**Remark 7.1.** The Lie algebra defined by relations (45) “mystically” appeared for the first time in the paper by H. D. Wahlquist and F. B. Estabrook [72], in which they introduced the so-called prolongation structures. The fact that it is, as explained above, a necessary ingredient in the construction of coverings is due to the author.

The reader will find many other examples of this kind together with related nonlocal symmetries, conservation laws, recursion operators, Bäcklund transformations, etc. in [28, 29, 25].

7.5. **But what really are PDEs?** Now we can turn back to the question posed at the end of Subsection 7.3. Recall that the possibility to commute nonlocal symmetries of a PDE $\mathcal{E}$ living in different coverings of $\tau_i : \mathcal{O}_i \to \mathcal{E}_\infty$, $i = 1, \ldots, m$, is ensured by the existence of a common covering diffiety $\mathcal{O}$, i.e., a system of coverings $\tau'_i : \mathcal{O} \to \mathcal{O}_i$, such that $\tau = \tau_i \circ \tau'_i$. So, in order to include into consideration all nonlocal symmetries we must consider “all” coverings $\tau_\alpha : \mathcal{O}_\alpha \to \mathcal{E}_\infty$ of $\mathcal{E}_\infty$ as
well as coverings $O_\alpha \to O_\beta$. In this way we come to the category \textbf{Cobweb} $E$ of coverings of $E_\infty$. Then it is natural to call the \textit{universal covering of $E$} the \textit{terminal object} of \textbf{Cobweb} $E$. Denote this hypothetical universal covering by $\tau_E : O_E \to E_\infty$, $O_E = (M_E, D_E)$. Now it is easy to see that $\textbf{Cobweb} E = \textbf{Cobweb} E'$ if and only if there is a common covering diffiety $O$, $E_\infty \leftarrow O \to E'_\infty$. In other words, $E$ and $E'$ are related by a Bäcklund transformation (see Subsection 7.2). Recalling that coverings present inversions of differential operators we can trace the following analogy with algebraic geometry:

\begin{itemize}
  \item \textit{Affine algebraic variety associated with an algebraic equation} $\Rightarrow E_\infty$.
  \item \textit{Birational transformations connecting two affine varieties} $\Rightarrow$ Bäcklund transformations.
  \item \textit{The field of rational functions on an affine variety} $\Rightarrow O_E$.
\end{itemize}

This analogy becomes a tautology if one considers algebraic varieties as PDEs in \textit{zero independent variables}. Indeed, any DO in this case is of zero order, i.e., multiplications by a function, and hence the inversion of such a DO is the division by this function.

Unfortunately, the universal covering understood as a terminal object of a category is not sufficiently constructive to work with. However, we have some indication of how to proceed. From the theoretical side, the indication is to look for an analogue of the fundamental group in the category of diffieties in order to construct the universal covering. By taking into account that we deal with infinitesimal symmetries it would be more adequate to look for the infinitesimal fundamental group, i.e., for the \textit{fundamental Lie algebra} of the diffiety $E_\infty$. On the other side, this idea is on an "experimental" ground. Namely, the Lie algebra associated with a Wahlquist-Estabrook prolongation structure (see Example 7.1) is naturally interpreted as the universal algebra for a special class of coverings.

In this connection a very interesting result by S. Igonin should be mentioned. In [21] he constructed an object which possesses basic properties of the fundamental algebra for a class of PDEs in two independent variables. Moreover, on this basis he succeeded to prove the non-existence of Bäcklund transformations connecting some integrable PDE’s, for instance, the KdV equation and the Krichever-Novikov equation.

Thus the question: What are PDE’s? continues to resist well, and the reader may see that this is a highly nontrivial conceptual problem. Yet though universal coverings of diffieties (if they exist!) point at a plausible answer, a good bulk of work should be done in order to put these ideas on a firm ground.

8. A couple of words about secondary calculus.

In these pages we, first, tried to attract attention to two intimately related questions: “what are symmetries of an object?” and “what is the object itself?”. They form something like an electro-magnetic wave when one of them induces the other and vice versa. Probably, this dynamical form is the most adequate adaptation of the background ideas of the Erlangen program to realities of present-day mathematics. The launch of such a wave in the area of nonlinear partial differential equations was the inestimable contribution of S. Lie to modern mathematics as it is now clearly seen in the hundred-years retrospective.
In the above picture of the post-Lie phase of propagation of this wave we did not touch such fundamental questions as what are general tensor fields, connections, differential operators, etc, on the “space of all solutions” of a given PDE, i.e., on the corresponding diffiety. They all together form what we call secondary calculus. It turns out that any natural notion or construction of the standard “differential mathematics” has an analogue in secondary calculus, which is referred to by adding the adjective “secondary”. In these terms (higher) symmetries of a PDE are nothing but secondary vector fields on \( \mathcal{E}_\infty \). Surprisingly, all secondary notions are cohomology classes of suitable natural complexes of differential operators, one of which, the jet-Spencer complex, was discussed in Section 6. For the whole picture see [66].

To illustrate this point we shall give some details on secondary differential forms. They constitute the first term of the \( C \)-spectral sequence, which is defined as follows. Let \( \mathcal{O} = (\mathfrak{M}, \mathcal{D}) \) be a diffiety and \( \mathcal{D} \Lambda(\mathcal{O}) = \oplus_{i \geq 0} \mathcal{D} \Lambda^i(\mathcal{O}) \) the ideal of differential forms on \( \mathfrak{M} \) vanishing on the distribution \( \mathcal{D} \). This ideal is differentially closed and its powers \( \mathcal{D}^k \Lambda(\mathcal{O}) \) form a decreasing filtration of \( \Lambda(\mathcal{O}) \). The \( C \)-spectral sequence \( \{ E^p,q_\mathcal{O} \}, d^{p,q} \mathcal{O} \} \) is the spectral sequence associated with this filtration. By definition, the space of secondary differential forms of degree \( p \) is the graded object \( \oplus_{q=0}^n E^{p,q}_\mathcal{O} \) and \( d_1 \) is the secondary exterior differential. Note that a smooth fiber bundle may be naturally viewed as a diffiety and the corresponding \( C \)-spectral sequence is identical to the Leray-Serre spectral sequence of this bundle.

Nontrivial terms of the \( C \)-spectral sequence are all in the strip \( 0 \leq q \leq n, p \geq 0 \) with \( n = \dim \mathcal{D} \), and \( E_1^{0,q}(\mathcal{O}) = H^q(\mathcal{H}(\mathfrak{M}, \mathcal{D})) \) (horizontal de Rham cohomology of \( \mathcal{O} \), see Subsection 6.4). Below we write simply \( E^{p,q}_\mathcal{O} \) for \( E^{p,q}_\mathcal{O} \) if the context does not allow a confusion. Also recall that \( C \)-differential DOs are those that admit restrictions to integral submanifolds of \( \mathcal{D} \).

The following proposition illustrates the fact that the calculus of variations is just an element of the calculus of secondary differential forms.

**Proposition 8.1.** Let \( \mathcal{O} = J^\infty(E, n) \). Then

1. If \( E_1^{p,q} \) is nontrivial, then either \( p = 0 \) or \( q = n \) (“one line theorem”).
2. \( E_1^{0,q} = H^q(J^1(E, n)) \), if \( q < n \), and \( E_1^{0,n} \) is composed of variational functionals \( \int \omega dx_1 \wedge \cdots \wedge dx_n \).
3. \( d_1^{0,n} \) is the Euler operator of the calculus of variations:

\[
E_1^{0,n} = H^n(J^\infty(E, n)) \ni \int \omega dx_1 \wedge \cdots \wedge dx_n \xrightarrow{d_1^{0,n}} \ell^*_\omega(1) \in \hat{\mathcal{R}}
\]

where \( \hat{\mathcal{R}} \equiv \text{Hom}_F(x, \Lambda^n(J^\infty(E, n))) \) and \( \ell^*_\omega \) stands for the adjoint to \( \ell_\omega \) \( C \)-differential operator.
4. \( E_1^{2,n} = C \text{Diff}^{alt}(x, \hat{\mathcal{R}}) \equiv \{ \text{skew-self-adjoint } C \text{-differential operators from } x \text{ to } \hat{\mathcal{R}} \} \), and

\[
d_1^{1,n} : \hat{\mathcal{R}} \ni \Psi \mapsto \ell^*_\Psi - \ell^*_\Psi \in C \text{Diff}^{alt}(x, \hat{\mathcal{R}}).
\]
5. \( E_2^{p,n} = H^{p+n}(J^1(E, n)) \) and, in particular, the complex \( \{ E_1^{p,n}, d_1^{p,n} \}_{p \geq 0} \) is locally acyclic.

The reader will find a similar description of the terms \( E_1^{p,n} \) and the differentials \( d_1^{p,n} \) for \( p > 2 \) in [63, 66].
If \( \mathcal{O} = \mathcal{E}_\infty \), then the terms \( E_1^{0,q}(\mathcal{O}) \) present various conserved quantities of the equation \( \mathcal{E} \). For instance, the Gauss electricity conservation law is an element of \( E_1^{0,2}(\mathcal{E}_\infty) \) for the system of Maxwell equations \( \mathcal{E} \). The term \( E_1^{0,n-1}(\mathcal{E}_\infty) \) is composed of standard conservation laws of a PDE \( \mathcal{E} \), which are associated with conserved densities. In this connection we have

**Proposition 8.2.** Let \( \mathcal{E} \) be a determined system of PDEs and \( \mathcal{CL}(\mathcal{E}) \) the vector space of conservation laws for \( \mathcal{E} \). Then

1. If \( E_1^{p,q} \) is nontrivial, then either \( p = 0 \) or \( q = n-1 \) ("two lines theorem").
2. \( \ker d_1^{0,n-1} = H^{n-1}(\mathcal{E}) \) (trivial conservation laws).
3. \( E_1^{1,n-1} = \ker \ell^*_\mathcal{E} \) and \( E_1^{1,n} = \coker \ell^*_\mathcal{E} \).

\( \Upsilon = d_1^{0,n-1}(\Omega) \) is called the generating function of a conservation law \( \Omega \in \mathcal{CL}(\mathcal{E}) \). Assertion (2) of Proposition 8.2 tells that a conservation law is uniquely defined by its generating function up to a trivial one. Moreover, by assertion (3) of this proposition, generating functions are solutions of the equation \( \ell^*_\mathcal{E} \Upsilon = 0 \), and this is the most efficient known method for finding conservation laws (see [67, 29, 27]).

Propositions 8.1 and 8.2 unveil the nature of the classical Noether theorem. Namely, by assertions (3) and (4) of Proposition 8.1, the Euler-Lagrange equation \( \mathcal{E} \) corresponding to the Lagrangian \( \int \omega dx_1 \wedge \cdots \wedge dx_n \) is \( \Psi = 0 \) with \( \Psi = \ell^*_\mathcal{E} (1) \) and \( \ell^*_\mathcal{E} = \ell^*_\mathcal{E} \). In other words, Euler-Lagrange equations are self-adjoint. So, in this case the equation \( \ell^*_\mathcal{E} = 0 \) whose solutions are generating functions of conservation laws of \( \mathcal{E} \) (assertion (3) of Proposition 8.2) coincides with the equations \( \ell^*_\mathcal{E} = 0 \) whose solutions are generating functions of symmetries of \( \mathcal{E} \) (formula (25)). Moreover, we see that this relation between symmetries and conservation laws takes place for a much larger than the Euler-Lagrange class of PDEs, namely, the class of conformally self-adjoint equations: \( \ell^*_\mathcal{E} = \lambda \ell^*_\mathcal{E} \), \( \lambda \in \mathcal{F}_\mathcal{E} \).

All natural relations between vector fields and differential forms such as Lie derivatives, insertion operators, etc survive at the level of secondary calculus in the form of some relations between the horizontal jet-Spencer cohomology and the first term of the \( \mathcal{C} \)-spectral sequence. Also, a morphism of diffieties induces a pullback homomorphism of \( \mathcal{C} \)-spectral sequences. In particular, this allows to define nonlocal conservation laws of a PDE \( \mathcal{E} \) as conservation laws of diffieties that cover \( \mathcal{E}_\infty \). These are just a few of numerous facts that show high self-consistence of secondary calculus and its adequacy for needs of physics and mechanics.

**Remark 8.1.** The \( \mathcal{C} \)-spectral sequence was introduced by the author in [56]. It was preceded by some works by various authors on the inverse problem of calculus of variations and the resolvent of the Euler operator (or the Lagrange complex). These works may now be seen as results about the \( \mathcal{C} \)-spectral sequence for \( \mathcal{O} = J^\infty(\pi) \) (see, for instance, [31, 52]). If \( \mathcal{E} \subset J^k(\pi) \), then the first term of the \( \mathcal{C} \)-spectral sequence for \( \mathcal{E}_\infty \subset J^\infty(\pi) \) acquires the second differential coming from the spectral sequence of the fiber bundle \( \pi_\infty : J^\infty(\pi) \rightarrow M \) and it becomes the variational bicomplex associated with \( \mathcal{E} \). This local interpretation of the \( \mathcal{C} \)-spectral sequence is due to T. Tsujishita [53], who described these two differentials in a semi-coordinate manner.

9. NEW LANGUAGE AND NEW BARRIERS.

In the preceding pages we were trying to show that a pithy general theory of PDEs exists and to give an idea about the new mathematics that comes into light.
When developing this theory in a systematic way. Even now this young theory provides many new instruments allowing to discover new features and facts about well-known and for long time studied PDE's in geometry, mechanics and mathematical physics. The theory of singularities of solutions of PDEs sketched in Section 4 is an example of this to say nothing about symmetries, conservation laws, hamiltonian structures and other more traditional aspects. Moreover, numerous possibilities, which are within one arm’s reach, are still waiting to be duly elaborated simply because of a lack of workmen in this new area. This situation is to a great extent due to a language barrier, since

the specificity of the general theory of PDEs is that it cannot be systematically developed in all its aspects on the basis of the traditionally understood differential calculus.

Indeed, one very soon loses the way by performing exclusively direct manipulations with coordinate-wise descriptive definitions of objects of differential calculus, especially if working on such infinite-dimensional objects as diffieties. By their nature, these descriptive definitions cannot be applied to various situations when some kind of singularities or other nonstandard situations occur naturally. Not less important is that descriptive definitions give no idea about natural relations between objects of differential calculus. Typical questions that can in no way be neglected when dealing with foundations of the theory of PDE’s are: “What are tensor fields on manifolds with singularities, or on pro-finite manifolds, or what are tensor fields respecting a specific structure on a smooth manifolds”, etc. This kind of questions becomes much more delicate when working with diffieties.

All these questions can be answered by analyzing why and how the traditional differential calculus of Newton and Leibniz became a natural language of classical physics (including geometry and mechanics). Since the fundamental paradigm of classical physics states that existence means observability and vice versa, the first step in this analysis must be a due mathematical formalization of the observability mechanism in classical physics.

We do that by assuming that from a mathematical point of view a classical physical laboratory is the unitary algebra \( A \) over \( \mathbb{R} \) generated by measurement instruments installed in this laboratory and called the algebra of observables. A state of an observed object is interpreted as a homomorphism \( h : A \to \mathbb{R} \) of \( \mathbb{R} \)-algebras (≡ “readings of all instruments”). Hence the variety of all states of the system is identified with the real spectrum \( \text{Spec}_\mathbb{R} A \) of \( A \). The validity of this formalization of the classical observation mechanism is confirmed by the fact that all aspects of classical physics are naturally and, even more, elegantly expressed in terms of this language. Say, one of the simplest necessary concepts, namely, that of velocity of an object at a state \( h \in \text{Spec}_\mathbb{R} A \) is defined as a tangent vector to \( \text{Spec}_\mathbb{R} A \) at the “point” \( h \), i.e., as an \( \mathbb{R} \)-linear map \( \xi : A \to \mathbb{R} \) such that \( \xi(ab) = h(a)\xi(b) + h(b)\xi(a) \). So, velocity is a particular first order DO over the algebra \( A \) of observables in the sense of Definition 5.1. In this case \( P = A \) and \( Q = \mathbb{R} \) as an \( \mathbb{R} \)-vector space with the \( \mathbb{R} \)-module product \( a \star r \overset{\text{def}}{=} h(a)r, \quad a \in A, \quad r \in \mathbb{R} \). The reader will find other simple examples of this kind in an elementary introduction to the subject [42].

Thus, by formalizing the concept of a classical physical laboratory as a commutative algebra, we rediscover differential calculus in a new and much more general form. The next question is: “What is the structure of this new language and what
are its informative capacities?” In the standard approach the zoo of various structures and constructions in modern differential and algebraic geometry, mechanics, field theory, etc. that are based on differential calculus seems not to manifest any regularity. Moreover, numerous questions like “why do skew symmetric covariant tensors, i.e., differential forms, possess a natural differential $d$, while the symmetric ones do not” cannot be answered within this approach. On the contrary, in the framework of differential calculus over commutative algebras all these “experimental materials” are nicely organized within a scheme composed of functors of differential calculus connected by natural transformations and the objects that represent them in various categories of modules over the ground algebra.

The reader will find in a series of notes [69] various examples illustrating what one can discover by analyzing the question “what is the conceptual definition of covariant tensors”. From the last three notes of this series he can also get an idea on the complexity of the theory of iterated differential forms and, in particular, tensors, in secondary calculus.

It should be especially mentioned that new views, instruments and facts coming from the general theory of PDEs and related mathematics offer not only new perspectives for many branches of contemporary mathematics and physics but at the same time put in question some popular current approaches and expectations ranging from algebraic geometry to QFT. Unfortunately, there is too much to say in order to present the necessary reasons in a satisfactory manner.

We conclude by stressing that

*The complexity and the dimension of problems in general theory of PDEs are so high that a new organization of mathematical research similar to that in experimental physics is absolutely indispensable.*

Unfortunately, the dominating mentality and the “social organization” of the modern mathematical community seems not to be sufficiently adequate to face this challenge.

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