On Little Groups and Boosts of $\kappa$-deformed Poincaré Group

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Abstract

We show how Wigner’s little group approach to the representation theory of Poincaré group may be generalized to the case of $\kappa$-deformed Poincaré group. We also derive the deformed Lorentz transformations of energy and momentum. We find that if the $\kappa$-deformed Poincaré group is adopted as the fundamental symmetry of nature, it results in deviations from predictions of the Poincaré symmetry at large energies, which may be experimentally observable.
1 Introduction

Since the discovery of quantum groups as generalizations of ordinary groups, there is a tendency in theoretical physics to apply these objects in physical theories. In high energy physics groups of symmetries appear at various levels. First, there is the group of symmetries of the space-time manifold. In the standard model this is the Poincaré group. Second, there are the gauge groups: $SU(3), SU(2) \times U(1)$, or other groups in the grand unified theories. There also appear some groups related to flavor. Any of these groups may be deformed, here we only consider the deformation of the Poincaré group. We shall not go into the details of the idea that lies behind the quantization of groups. We only mention that although quantum groups were first introduced in the context of integrable models, nowadays people try to use them in other problems, hoping that a quantum group may be a better tool to formulate the symmetry of a theory. Our interest is the structure of space-time, therefore we consider the Poincaré group. This group is related to the structure of space-time and we know, from Einstein’s general relativity, that gravity affects this structure.

Being the semidirect product of Lorentz and translation groups, the Poincaré group is not semisimple, therefore, its quantization is not unique. Up to now there have been discovered three different deformations of Poincaré group \cite{1, 2, 3}. In this paper we consider $\kappa$-deformations of the Poincaré group as introduced by Lukierski et al. \cite{1}. We prefer this deformation because it is minimal in the sense that only two of the commutation relations are deformed and there is a very clear correspondence between the generators of this quantum group and the generators of the un-deformed Poincaré group. Furthermore, there are some physical reasons in choosing this deformation \cite{1}.

Our aim in this paper is to get some insight in the consequences of this deformation from a physical point of view and obtain some of the deviations that these deformations imply up to first order in observable quantities such as energy and momenta. Whether or not these deviations may be tested experimentally requires more careful study. Specifically, one has to take into account any other perturbative effect that has the same order.

In section 2 we briefly review the Poincaré group and its $\kappa$-deformed version. In section 3 we apply Wigner’s little group approach to the $\kappa$-Poincaré group. In section 4 we see that the non-trivial co-product of $\kappa$-Poincaré leads to a feature that is not present in the Poincaré group. In section 5 we consider boosts, infinitesimal and finite. We end this paper by conclusion in section 6.
2 The Poincaré group and its \( \kappa \)-deformations

The Poincaré group is generated by 10 generators: \( P_0, P_i, M_i, L_i \) for \( i = 1, 2, 3 \). The commutation relations are:

\[
\begin{align*}
[P_0, P_i] &= 0, & [P_i, P_j] &= 0, & [M_i, M_j] &= i\epsilon_{ijk}M_k, \\
[M_i, P_0] &= 0, & [M_i, P_j] &= i\epsilon_{ijk}P_k, & [M_i, L_j] &= i\epsilon_{ijk}L_k, \\
[L_i, P_0] &= iP_i, & [L_i, P_j] &= i\delta_{ij}P_0, & [L_i, L_j] &= -i\epsilon_{ijk}M_k.
\end{align*}
\] (2.1)

The \( \kappa \)-deformation is obtained by a contraction of \( U_q(SO(2,3)) \) which is the deformed anti de Sitter algebra \( \mathbb{H} \). The commutation relations for the \( \kappa \)-Poincaré algebra are exactly the same as those for the Poincaré algebra except for the following two relations:

\[
\begin{align*}
[L_i, P_j] &= i\delta_{ij}\kappa \sinh\left(\frac{P_0}{\kappa}\right), \\
[L_i, L_j] &= -i\epsilon_{ijk}(M_k \cosh\left(\frac{P_0}{\kappa}\right) - \frac{1}{\kappa^2}P_kP_lM_l).
\end{align*}
\] (2.2)

Here \( \kappa \) is a constant with the dimensions of energy. Usually \( \kappa \) is regarded as real, however, a pure imaginary one also leads to a Hopf algebra.

The deformed Pauli–Lubanski four-vector is defined as:

\[
W_0 = P_iM_i \quad W_i = \kappa \sinh\left(\frac{P_0}{\kappa}\right)M_i + \epsilon_{ijk}P_jL_k.
\] (2.3)

The two Casimir operators are

\[
\begin{align*}
c_1 &= 4\kappa^2 \sinh^2\left(\frac{P_0}{2\kappa}\right) - P_iP_i, \\
c_2 &= (\cosh(\frac{P_0}{\kappa}) - \frac{1}{4\kappa^2}P_iP_i)W_0^2 - W_iW_i.
\end{align*}
\] (2.4)

We note that the \( \kappa \to \infty \) limit of the above relations are the familiar ones for the Poincaré algebra. Therefore, if \( \kappa \)-Poincaré is the correct symmetry algebra of nature \( \kappa \) must be large.

If we want to have real energy \( P_0 \) and momenta \( P_i \) then \( \kappa \) must be real or pure imaginary.

Imaginary \( \kappa \) has the odd property that the total energy and total momenta of a system composed of two subsystems will not be real: the co-product of the corresponding Hopf algebra contains \( \exp(i\frac{P_0}{\kappa}) \), cf. [3]. For completeness we write the co-product and antipode.

\[
\begin{align*}
\Delta(P_0) &= P_0 \otimes 1 + 1 \otimes P_0, \\
\Delta(M_i) &= M_i \otimes 1 + 1 \otimes M_i, \\
\Delta(P_i) &= P_i \otimes \exp(i\frac{P_0}{2\kappa}) + \exp(-i\frac{P_0}{2\kappa}) \otimes P_i, \\
\Delta(L_i) &= L_i \otimes \exp(i\frac{P_0}{2\kappa}) + \exp(-i\frac{P_0}{2\kappa}) \otimes L_i, \\
&\quad + \frac{i}{2\kappa}\epsilon_{ikl}(\exp(i\frac{P_0}{2\kappa})M_k \otimes P_l + P_k \otimes M_l \exp(i\frac{P_0}{2\kappa})), \\
S(P_\mu) &= -P_\mu \quad S(M_i) = -M_i \quad S(L_i) = L_i - \frac{3}{2\kappa}P_i.
\end{align*}
\] (2.5)
3 Mass-shells and little groups

The idea of using little groups to obtain information about the representations of the Poincaré group is due to Wigner [6]. For a complete accounting of this we refer the reader to [7]. Here we briefly review this idea and try to apply it to the $\kappa$-deformed Poincaré group.

Because of (2.1), we can diagonalize energy and momenta simultaneously and consider physical states with definite energy and momenta $|P_0, P_i>$. Next we look at the orbits of these states and the little groups. By an orbit we mean a $c_1 = \text{constant}$ surface in the $\mathbb{R}^4 \sim \{(P_0, P_1, P_2, P_3)\}$ for the undeformed Poincaré group these orbits are: light-cone for a massless particle, two time-like hyperbolas for a massive particle and a hyperboloid of revolution for a tachyon. A general (finite) action of the Poincaré group can change the four-momentum of a particle but only on its mass-shell. We know also that the action of the Lorentz group on these orbits is transitive\footnote{We omit the origin $P_\mu = 0$ from the light-cone.}. Although for the $\kappa$–deformations the finite action is not well-understood, we know that the orbits; i.e. the mass shells; are stable. This is because $c_1$ is the Casimir operator. However we don’t know about the transitivity of this action.

For a given $P_\mu$ the little group is defined as the subgroup of the Lorentz group that leaves the given $P_\mu$ intact. The Lie algebra of this little group is taken as the subset of the algebra defined by eq (2.1) which is closed and leaves a physical state invariant. This Lie algebra is the Lie algebra of $SO(3)$, $E(2)$ or $SO(2,1)$ for time-like, light-like and space-like four-momenta respectively. The topology of these Lie groups may be quite different from $SO(3)$, $E(2)$ and $SO(2,1)$, therefore we call them $SO(3)$—like, $E(2)$—like and $SO(2,1)$—like\footnote{For example, the little group of electrons is $SU(2)$ which is the double cover of $SO(3)$.

Let’s take a look at the orbits. For the undeformed Poincaré group these are the well-known hyperboloids $E^2 - P^2 = c_1$ where $E$ is for $P_0$, $P$ is for $P_3$ and $P_1 = P_2 = 0$. For the real $\kappa$-deformation the mass-shells are topologically like the ones in special relativity. The defining equation in this case is $4\kappa^2 \sinh(\frac{E}{2\kappa}) - P^2 = c_1$. For the imaginary $\kappa$ this equation becomes $c_1 = 4\chi^2 \sin^2(\frac{E}{2\chi}) - P^2$ where $\kappa = i\chi$. For this $\chi$-deformation we note that: (i) Because of the periodicity of $\sin^2(\frac{E}{2\chi})$ the levels $E = 2n\pi\chi$ are identified, therefore, all orbits are closed. (ii) For a particle with definite $c_1$ there is a bound for momentum. If $c_1 < 0$; i.e. for tachyons; there is a lower and an upper bound for momentum and no bound for energy except the
periodicity of energy. If \( c_1 \geq 0 \) there is only an upper bound for the momentum and a lower bound for the energy: the rest mass. For these massive particles there is an upper bound for \( c_1 \) itself \( c_1 < 4 \chi^2 \). A particle with the critical value \( c_1 = 4 \chi^2 \) has zero momentum in any frame. Particles with mass near this critical value have almost zero momenta in all frames. This has no correspondence in special relativity; i.e. in Poincaré group; or in the real \( \kappa \)–deformation. In special relativity massless particles in all frames have the same velocity but their energy and momenta differ. Now in a \( \chi \)–deformed special relativity we face with the possibility of the existence of particles with the property that in all frames they have the same energy and zero momenta. It is interesting to study the velocity of these particles in different frames.

Wigner’s approach to the representation theory of the Poincaré group is based on the idea of little groups. Since for the deformed Poincaré group the momenta commute; \([P_\mu, P_\nu] = 0\); this idea may be applied to them also. The key idea is that:

\[
[W_\mu, P_\nu] = 0 \tag{3.1}
\]

so that the action of the \( W_\mu \) leaves \( P_\nu \) intact. The remaining point is whether \( \{W_\mu\}_{\mu=0}^3 \) generate a subgroup of the Lorentz group. After a little calculations one can prove that:

\[
[W_0, W_1] = i(W_2 P_3 - W_3 P_2)
\]
\[
[W_1, W_2] = i(W_1 \alpha P_1 P_3 + W_2 \alpha P_2 P_3 + W_3 (\alpha P_3^2 + \xi)) \tag{3.2}
\]

and relations obtained from these by cyclic permutations of the indices. Here we have used the notations:

\[
\xi = \kappa \sinh(P_0 \kappa) \quad \alpha \xi = \frac{1}{\kappa^2} P_1 P_i - \cosh(P_0 \kappa) \tag{3.3}
\]

Note that in all these commutation relations momenta have gone to the right. This means that acting on eigenvectors of the momenta they are \( c \)-numbers, therefore, they can be moved to the left and all the commutation relations are of the following form:

\[
[W_\mu, W_\nu] = C^\sigma_{\mu\nu} W_\sigma \tag{3.4}
\]

where \( C^\sigma_{\mu\nu} \) s are the structural constants depending on the given four-momentum. So, the little groups of the \( \kappa \)– and \( \chi \)–deformations of the Poincaré group are ordinary Lie groups. Because there is a linear relation among the \( W \)s,

\[
\xi W_0 - P_i W_i = 0 \tag{3.5}
\]
this Lie algebra is three dimensional. To reveal its structure we consider cases such that:

\[ P_0 = E \quad P_1 = 0 \quad P_2 = 0 \quad P_3 = P. \]  

We get:

\[
W_0 = P M_3 \quad W_1 = \xi M_1 - P L_2 \quad W_2 = \xi M_2 + P L_1 \quad W_3 = \xi M_3
\]  

(3.7)

where \( \xi = \kappa \sinh \left( \frac{E}{\kappa} \right) \). Note that \( W_0 \) and \( W_3 \) are proportional and for \( E \to 0 \), \( W_0 \) is well-defined and \( W_3 = 0 \). For the Lie algebra we get:

\[
[W_1, W_2] = i \xi f(E, P) W_3 \quad [W_2, W_3] = i \xi W_1 \quad [W_3, W_1] = i \xi W_2
\]  

(3.8)

where

\[
f(E, P) = \kappa^2 \sinh^2 \left( \frac{E}{\kappa} \right) - P^2 \left( \cosh \left( \frac{E}{\kappa} \right) - \frac{P^2}{4\kappa^2} \right) = c_1 \left( 1 + \frac{c_1}{4\kappa^2} \right).
\]  

(3.9)

From these we conclude that the little groups for the \( \kappa \)-deformation are:

\begin{align*}
\text{a} & \quad SO(3)\text{-like} & \text{for } c_1 > 0 & \text{massive} \\
\text{b} & \quad E(2)\text{-like} & \text{for } c_1 = 0 & \text{massless} \\
\text{c} & \quad SO(2, 1)\text{-like} & \text{for } -4\kappa^2 < c_1 < 0 & \text{tachyonic A} \\
\text{d} & \quad E(2)\text{-like} & \text{for } c_1 = -4\kappa^2 & \text{tachyonic B} \\
\text{e} & \quad SO(3)\text{-like} & \text{for } c_1 < -4\kappa^2 & \text{tachyonic C}
\end{align*}

(3.10)

Cases \( d \) and \( e \) have no counterpart in the Poincaré group. For the \( \chi \)-deformation the little groups are:

\begin{align*}
\text{a'} & \quad SO(3)\text{-like} & \text{for } 0 < c_1 < 4\chi^2 & \text{massive} \\
\text{b'} & \quad E(2)\text{-like} & \text{for } c_1 = 0 & \text{massless} \\
\text{c'} & \quad SO(2, 1)\text{-like} & \text{for } c_1 < 0 & \text{tachyonic} \\
\text{d'} & \quad E(2)\text{-like} & \text{for } c_1 = 4\chi^2 & \text{massive}
\end{align*}

(3.11)

Case \( d' \) has no counterpart in the Poincaré group. Therefore, depending on the values for \( P_0, P_1, P_2, \text{ and } P_3 \), we get different little groups. In the \( \kappa \)-deformation there is a region which is not present in the undeformed case where for a space-like four-momentum, i.e. a tachyon, the little group is always \( SO(2, 1)\text{-like} \). In the \( \kappa \)-deformed Poincaré group tachyons divide into three classes. In one class, which we name type \( \text{A} \), near the light cone, the little group
is still $SO(2,1)$–like. For the special value $c_1 = -4\kappa^2$ the little group is $E(2)$–like and we name such a tachyon of type B. For type C we have $c_1 < -4\kappa^2$, the little group is $SO(3)$–like. In the imaginary $\kappa$–deformed Poincaré group there is a time like orbit consisting of just one point: $E = \pi \chi$, $P_1 = P_2 = P_3 = 0$. For this orbit the little group is $E(2)$–like. The little group for the vacuum, i.e. $P_\mu = 0$, is $SO(3,1)$–like in all three cases.

It is interesting that the function $f$ in (3.9) is the Casimir operator found recently by Ruegg et al. in [8]. According to their results this is the “covariant” length squared of the four-momentum: $P_\mu P^\mu$.

4 Little group of a composite system

Ordinarily one can consider a system composed of two non-interacting particles. This composite system must carry a representation of the Poincaré group. One can therefore speak about its little group. In the (non-deformed) Poincaré group, if the two particles have identical energy and momenta; i.e. they have identical rest mass and are at rest with respect to each other; then, the composite system has the same little group type. For example, the little group of two electrons is $SO(3)$–like. In the $\chi$– and $\kappa$–deformations there are some differences. In the $\kappa$–deformation the composite system made up of two tachyons of type A may be of type B or C; and a system composed of two tachyons of type B is of type C. To see this we use the co-product to obtain the structural constants of the composite system. Since the two particles have the same energy and momenta, and because of the form of the co-product for $\kappa$–Poincaré group, we have

$$E_{\text{tot}} = 2E \quad P_{\text{tot}} = 2P \cosh\left(\frac{E}{2\kappa}\right)$$

(4.1)

Therefore, we obtain:

$$f_{\text{tot}} = f\left(2E, 2P \cosh\left(\frac{E}{2\kappa}\right)\right) = 4\cosh^2\left(\frac{E}{2\kappa}\right) c_1 \left(\kappa^2 + \cosh^2\left(\frac{E}{2\kappa}\right) c_1\right).$$

(4.2)

From this last equation we see that if $c_1$ is negative then the sign of $f_{\text{tot}}$ may be different from the sign of $f$. Setting $\chi = -i\kappa$ we see that this may happen for positive $c_1$ of $\chi$–deformation, where the little group of two ordinary particles may be $E(2)$–like.
5 Deformation of the boost operators

The Poincaré group is the symmetry group of particles in the relativistic quantum mechanics. There must be a representation of the Poincaré group acting on the Hilbert space of the particle. In the rest frame of a particle there is a non-trivial part of the Poincaré group that leaves the particle at rest, viz. the little group. However, since the particle is at rest, the angular momentum generators must act on the spin degrees of freedom. So we have

\[ J_i = M_i + S_i \]

\[ B_i = L_i + K_i \]  \hspace{1cm} (5.1)

where \( J \) is the total angular momentum, \( M \) is the orbital angular momentum, \( S \) is the spin, \( B \) is the total boost operator, \( L \) is the operator that boosts the space-time coordinates and \( K \) is the operator that boosts the spin degrees of freedom. This form of total angular momentum and boost come from the trivial co-product of the Poincaré group:

\[ \Delta(J_i) = J_i \otimes 1 + 1 \otimes J_i = M_i + S_i \] etc.  \hspace{1cm} (5.2)

For the \( \kappa \)-deformation we use the non-trivial co-product to obtain:

\[ J_i = M_i + S_i \quad B_i = L_i + \exp\left(-\frac{P_0}{2\kappa}\right)K_i + \frac{1}{2\kappa}\epsilon_{ijk}P_jS_k. \] \hspace{1cm} (5.3)

\( \{S_i, K_i, i = 1, 2, 3 \} \) generates a non-deformed Lorentz group. The new feature of the \( \kappa \)-deformed Poincaré group is \( \text{(5.3)} \) which states that the effect of a boost depends on the energy-momentum four-vector. Expanding \( \text{(5.3)} \) to first order in \( \frac{1}{\kappa} \) we get:

\[ B_i = L_i + (1 - \frac{P_0}{2\kappa})K_i - \frac{1}{2\kappa}\epsilon_{ijk}P_jS_k. \] \hspace{1cm} (5.4)

For our standard case, i.e. \( P_0 = E, P_1 = P_2 = 0 \) and \( P_3 = P \) we get:

\[ B_1 = L_1 + (1 - \frac{E}{2\kappa})K_1 - \frac{1}{2\kappa}pS_2 \]

\[ B_2 = L_2 + (1 - \frac{E}{2\kappa})K_2 + \frac{1}{2\kappa}pS_1 \]

\[ B_1 = L_3 + (1 - \frac{E}{2\kappa})K_3 \] \hspace{1cm} (5.5)

Now we turn to finite Lorentz (boost) transformations. A glance at the commutation relations shows that the finite action of the rotation subgroup is not deformed. So is the action of translations on the angular momenta. The deformation appear only in boosts. Let’s consider the effects of a finite boost in the \( z \) direction on \( E = P_0 \) and \( P = P_3 \).

\[ E \rightarrow E' = \exp(-i\eta L)E \exp(i\eta L) = \sum_{n=0}^{\infty} \frac{(i\eta)^n}{n!}L^n(E) \] \hspace{1cm} (5.6)
\[ P \rightarrow P' = \exp(-i\eta L) P \exp(i\eta L) = \sum_{n=0}^{\infty} \frac{(i\eta)^n}{n!} L^n(P) \]  

(5.7)

where \( \eta \) is the rapidity; \( L = L_3 \) and

\[ L^0(x) = x \quad L^{n+1}(x) = [L, L^n(x)]. \]  

(5.8)

In writing (5.6) and (5.7) we have assumed that the action of a generator \( \Omega \) is given by

\[ \delta(\Omega) = i[X, \Omega] \]  

(5.9)

as usual.

For the Poincaré group (5.6) and (5.7) lead to the familiar Lorentz transformations of energy and momentum. We want to see the effect of deformations on these transformations. Using commutation relations one can calculate \( E' \) and \( P' \) to any order \( n \) in \( \eta \). The result will be two polynomials in \( \sinh(\frac{\eta}{\kappa}) \), \( \cosh(\frac{\eta}{\kappa}) \) and \( P \). In this form the approximation is in ignoring \( O(\eta^{n+1}) \) and everything is exact in \( \kappa \). Let’s try to calculate everything to first non-zero order in \( \frac{1}{\kappa} \) but to all orders in \( \eta \). In this form the deformation is seen more transparently. To this end we write everything up to first order in \( \frac{1}{6\kappa^2} \).

\[
\begin{align*}
L(E) &= iP \\
L(P) &= i(E + \frac{1}{6\kappa^2}E^3) \\
L(\frac{1}{6\kappa^2}E^3) &= i\frac{3}{6\kappa^2}E^2P \\
L(\frac{1}{6\kappa^2}E^2P) &= i\frac{1}{6\kappa^2}(2EP^2 + E^3) \\
L(\frac{1}{6\kappa^2}EP^2) &= i\frac{1}{6\kappa^2}(P^3 + 2E^2P) \\
L(\frac{1}{6\kappa^2}P^3) &= i\frac{3}{6\kappa^2}EP^2 \\
\end{align*}
\]

(5.10)

In matrix form this can be written as

\[
\begin{pmatrix}
E \\
P \\
\frac{1}{6\kappa^2}E^3 \\
\frac{1}{6\kappa^2}E^2P \\
\frac{1}{6\kappa^2}EP^2 \\
\frac{1}{6\kappa^2}P^3 \\
\end{pmatrix}
\rightarrow
\begin{pmatrix}
E' \\
P' \\
\frac{1}{6\kappa^2}E^{3'} \\
\frac{1}{6\kappa^2}E^{2'P} \\
\frac{1}{6\kappa^2}E'P^{2'} \\
\frac{1}{6\kappa^2}P^{3'} \\
\end{pmatrix}
= \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 3 & 0 & 0 \\
0 & 0 & 1 & 0 & 2 & 0 \\
0 & 0 & 0 & 2 & 0 & 1 \\
0 & 0 & 0 & 0 & 3 & 0 \\
\end{pmatrix} 
\begin{pmatrix}
E \\
P \\
\frac{1}{6\kappa^2}E^3 \\
\frac{1}{6\kappa^2}E^2P \\
\frac{1}{6\kappa^2}EP^2 \\
\frac{1}{6\kappa^2}P^3 \\
\end{pmatrix}
\]  

(5.11)

Now to calculate the effect of a finite boost with rapidity \( \eta \) to order \( \frac{1}{6\kappa^2} \) one has to compute \( \exp(-\eta\Lambda) \) where \( \Lambda \) is the \( 6 \times 6 \) matrix in (5.11). The resulting matrix gives the transformed
energy $E'$ and momentum $P'$. Because of the form of $\Lambda$ the transformation has the following form:

$$E' = \cosh(\eta)E - \sinh(\eta)P + \frac{1}{6\kappa^2} \left( a_E(\eta)E^3 + b_E(\eta)E^2P + c_E(\eta)EP^2 + d_E(\eta)P^3 \right)$$  \hspace{1cm} (5.12)

$$P' = -\sinh(\eta)E + \cosh(\eta)P + \frac{1}{6\kappa^2} \left( a_P(\eta)E^3 + b_P(\eta)E^2P + c_P(\eta)EP^2 + d_P(\eta)P^3 \right).$$  \hspace{1cm} (5.13)

For the special case $E = m_0$ and $P = 0$ we get:

$$E' = \left( \gamma + \frac{a_E(\eta)}{6}(\frac{m_0}{\kappa})^2 \right) m_0 \hspace{1cm} P' = \left( \gamma + \frac{a_P(\eta)}{6v}(\frac{m_0}{\kappa})^2 \right) m_0 v$$  \hspace{1cm} (5.14)

where

$$v = \tanh(\eta) \hspace{1cm} \gamma = \cosh(\eta).$$  \hspace{1cm} (5.15)

The functions $a_E, b_E, \ldots d_P$ may be computed to any desired order.

6 Conclusion

At the energies near the Plank scale the structure of space-time is not well-understood. Even at energies far below the Plank scale the validity of Lorentz invariance is not clear. If the structure of space-time at high energies is altered such that the symmetry group of nature is affected, it may be replaced with a deformed Poincaré group. This means that some deviations from Lorentz transformations may be observed. For the $\kappa$-deformed Poincaré group we find two classes of deviations. Firstly, those which concern tachyonic states. As these do not exist in nature, these deviations will not be verifiable. Secondly, there are deviations for particles with real mass but at very high energies, or in very fast moving frames. Here we mention two of such deviations.

Equations (5.5) mean that the spin degrees of freedom lag the space-time degrees of freedom when boosting to high energies. This is because the coefficient of $K_i$ is less than the coefficient of $L_i$, and $K_i$ is the operator that boosts the spin degrees of freedom. Also because of $\frac{1}{2\kappa}pS_i$ term in (5.5), there is a rotation in spin degrees of freedom in such boosts. This may results in an observable discrepancy in the spectra of atoms as seen from a fast moving frame, or the spin flip of protons or electrons in an accelerator.

Referring to (5.14) for an electron with energy one TeV we obtain:

$$P' \simeq E' = 10^6(1 + 2.5 \times 10^7(\frac{m_0}{\kappa})^2)m_0$$  \hspace{1cm} (6.1)
where $m_0$ is electron’s rest mass. The discrepancy from Einstein’s formula is $2.5 \times 10^7 (\frac{m_0}{\kappa})^2$. For a proton with the same energy the discrepancy is of the order $10^5 (\frac{M_0}{\kappa})^2$ where $M_0$ is the proton’s rest mass. Although proton is not a point particle, the trivial co-product of the $\kappa$–deformed Poincaré group for energy $P_0$ justifies this reasoning.

**Acknowledgments**

This research was supported in part by the Vice Chancellor for Research, Sharif University of Technology and in part by Institute for Studies in Theoretical Physics and Mathematics. The authors would like to thank A. Aghamohammadi, F. Ardalan, H. Arfaei, M. Khorrami and V. Karimipour for fruitful discussions.

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