A CLASS OF PRESERVERS ON HILBERT SPACE EFFECTS
INCLUDING ORTHO-ORDER AUTOMORPHISMS
AND SEQUENTIAL AUTOMORPHISMS

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ABSTRACT

In this paper we study a new class of transformations on the set of all Hilbert space effects. This consists of the bijective maps which preserve the order and zero product in both directions. The main result of the paper gives a complete description of the structure of those transformations. As applications we obtain additional new results and some former ones as easy corollaries. In particular, we obtain the form of the ortho-order automorphisms as well as that of the sequential automorphisms. In the last paragraph of the paper we show that these two kinds of automorphisms belong to our class of transformations even when their domain is the set of all effects in a general von Neumann algebra.
I. Introduction and Statements of the Results

Effects play a very important role in the quantum theory of measurement (see, for example, [3]). In the Hilbert space formalism of the theory, the so-called Hilbert space effects are the positive bounded linear operators on a Hilbert space $H$ which are bounded by the identity $I$. The set of all effects on $H$ is denoted by $E(H)$.

There are several operations and relations defined on $E(H)$ which are important from different aspects of the theory. What concerns the present paper, here we are interested in the ortho-order structure and in the sequential structure on $E(H)$. The first one is obtained as follows. The usual ordering $\leq$ among self-adjoint bounded linear operators gives rise to a partial order on $E(H)$ and the operation $': A \mapsto I - A$ defines a kind of orthocomplementation on $E(H)$. This relation and operation together give the ortho-order structure on $E(H)$ [10]. As for the second structure, it comes from the sequential product which is defined as follows. If $A, B \in E(H)$, then their sequential product is $A \circ B = \sqrt{AB}\sqrt{A}$ [8].

Supposing dim $H \geq 3$, the automorphisms of $E(H)$ with respect to the ortho-order structure (called ortho-order automorphisms) as well as the ones with respect to the sequential product (called sequential automorphisms) are known to be implemented by unitary or antiunitary operators. This means that all those automorphisms are of the form

$$\phi(A) = UAU^* \quad (A \in E(H))$$

where $U$ is a unitary or antiunitary operator. The result concerning ortho-order automorphisms was obtained by Ludwig in [10, Section V.5] (the proof was later clarified in [4]) while the corresponding result
on sequential automorphisms was given by Gudder and Greechie in [6]. In the paper [12] we showed that Ludwig’s theorem holds also in the 2-dimensional case.

In our recent paper [11] we presented some characterizations of the ortho-order automorphisms of $E(H)$ by means of their preserver properties. This investigation was motivated by the extensive study of preserver problems in matrix theory and in operator theory. Roughly speaking, preserver problems are concerned with the description of all transformations (called preservers) on a given set (preferably equipped with an algebraic structure) which preserve a certain relation between the elements, or a given subset of elements, or a quantity attached to the elements of the underlying set. In the present paper we continue the investigation started in [11] by studying the bijective maps $\phi : E(H) \to E(H)$ which preserve the order and zero product in both directions, i.e., which satisfy

(a) \[ A \leq B \iff \phi(A) \leq \phi(B) \quad (A, B \in E(H)) \]

and

(b) \[ AB = 0 \iff \phi(A)\phi(B) = 0 \quad (A, B \in E(H)). \]

In what follows we present a complete description of the structure of those maps and give several applications of the corresponding result. Among others, we obtain the form of all bijective transformations of $E(H)$ which

(i) preserve the order in both directions and

(ii) map one single nontrivial scalar operator $\lambda I$ ($\lambda \neq 0, 1$) to an operator of the same type.
At the first glance, it might be rather surprising that such maps are of a nice form but it turns out that they in fact satisfy (a) and (b). Next, we easily recover one of the main results in [11] on maps preserving the order and coexistence in both directions (the definition of coexistence is given below). What is probably more important, we also show that the main result of the present paper readily implies the former results on the structure of ortho-order automorphisms and sequential automorphisms of $E(H)$ given in [10] and in [6], respectively. In fact, we prove that those automorphisms also satisfy (a) and (b).

It should be emphasized that, due to the proof of our main result, all statements presented in the paper are valid in 2-dimensional case as well. In particular, this holds for the result on the form of sequential automorphisms of $E(H)$ which is a new result.

Finally, in the last paragraph of the paper we demonstrate that even on the set $E(A)$ of all effects belonging to a general von Neumann algebra, the ortho-order automorphisms and the sequential automorphisms (the definitions should be self-explanatory) belong to our new class of preservers, i.e., they preserve the order and zero product in both directions. This observation is worth mentioning since in that generality it is quite hard to see any connection between those two kinds of automorphisms. Therefore, we believe that our preservers deserve attention and it has sense to study them in other contexts as well.

Now we turn to the precise formulations of the results of the paper. In what follows we assume that $H$ is a (complex) Hilbert space with $\dim H \geq 2$. Our main result reads as follows.

**Theorem 1.** Let $\phi : E(H) \to E(H)$ be a bijective map which preserves the order and zero product in both directions. Then there is an either unitary or antiunitary operator $U$ on $H$ and a real number $p < 1$ such
that with the function \( f_p(x) = \frac{x}{x^{p+1} (1-p)} \) \((x \in [0,1])\) we have
\[
\phi(A) = U f_p(A) U^* \quad (A \in E(H)).
\]
Here, \( f_p(A) \) denotes the image of the function \( f_p \) under the continuous function calculus belonging to the operator \( A \).

To get the form of all bijective maps on \( E(H) \) with the properties (i), (ii), we need the following proposition which might be interesting on its own right.

**Proposition 2.** Let \( \phi : E(H) \to E(H) \) be a bijective map which preserves the order in both directions and suppose that there is a single pair of scalars \( \lambda, \mu \in ]0,1[ \) such that \( \phi(\lambda I) = \mu I \). Then \( \phi \) preserves zero product in both directions.

The above results have the following immediate consequences. To the second statement in the corollary below observe that scalar operators in \( E(H) \) can be characterized as effects which commute with every other effect.

**Corollary 3.** Let \( \phi : E(H) \to E(H) \) be a bijective map which preserves the order in both directions and suppose that there is a single pair of scalars \( \lambda, \mu \in ]0,1[ \) such that \( \phi(\lambda I) = \mu I \). Then there exist an either unitary or antiunitary operator \( U \) on \( H \) and a real number \( p < 1 \) such that with the function \( f_p(x) = \frac{x}{x^{p+1} (1-p)} \) \((x \in [0,1])\) we have
\[
\phi(A) = U f_p(A) U^* \quad (A \in E(H)).
\]
In particular, we obtain the same form for any bijection of \( E(H) \) which preserves the order and commutativity in both directions.
It is easy to see that if a function $f_p$ appearing above satisfies $f_p(\lambda) = \lambda$ for some $\lambda \in ]0,1[$, then $p = 0$ and hence $f_p$ is the identity. This gives us the following corollary stating that if an order preserving bijection $\phi$ of $E(H)$ fixes one single nontrivial scalar operator, then $\phi$ is implemented by an either unitary or antiunitary operator.

**Corollary 4.** Let $\phi : E(H) \to E(H)$ be a bijective map which preserves the order in both directions and suppose that there is a scalar $\lambda \in ]0,1[$ such that $\phi(\lambda I) = \lambda I$. Then there exists an either unitary or antiunitary operator $U$ on $H$ such that
\[
\phi(A) = UAU^* \quad (A \in E(H)).
\]

The following corollary in the case when $\dim H \geq 3$ appears in [11] as Theorem 1. Here we shall present a short proof of it based on our main result which works also in the 2-dimensional case. Recall that two effects $A, B \in E(H)$ are called coexistent if they are in the range of a positive operator valued measure or, equivalently, if there are effects $E, F, G \in E(H)$ such that
\[
A = E + G, \quad B = F + G, \quad E + F + G \in E(H).
\]

**Corollary 5.** Let $\phi : E(H) \to E(H)$ be a bijective map which preserves the order and coexistence in both directions. Then there is an either unitary or antiunitary operator $U$ on $H$ such that
\[
\phi(A) = UAU^* \quad (A \in E(H)).
\]

As for the last two corollaries below, we shall see that the structure of the ortho-order automorphisms and that of the sequential automorphisms of $E(H)$ can be deduced from our main result even in the 2-dimensional case. The proofs are based on the observation that both
kinds of automorphisms preserve the order and zero product in both
directions. (For a similar observation concerning effects in general von
Neumann algebras, see the last paragraph of the paper.) Recall that a
bijective map $\phi : E(H) \to E(H)$ is called an ortho-order automorphism
if for any $A, B \in E(H)$ we have

$$A \leq B \iff \phi(A) \leq \phi(B),$$

(i.e., $\phi$ preserves the order in both directions) and

$$\phi(A') = \phi(A)'$$

(i.e., $\phi$ preserves the orthocomplements). Moreover, a bijective map
$\phi : E(H) \to E(H)$ is called a sequential automorphism if

$$\phi(A \circ B) = \phi(A) \circ \phi(B)$$

holds for every $A, B \in E(H)$.

**Corollary 6.** Let $\phi : E(H) \to E(H)$ be an ortho-order automorphism.
Then $\phi$ preserves the order and zero product in both directions. In fact,
there is an either unitary or antiunitary operator $U$ on $H$ such that

$$\phi(A) = UAU^* \quad (A \in E(H)).$$

**Corollary 7.** Let $\phi : E(H) \to E(H)$ be a sequential automorphism.
Then $\phi$ preserves the order and zero product in both directions. In fact,
there is an either unitary or antiunitary operator $U$ on $H$ such that

$$\phi(A) = UAU^* \quad (A \in E(H)).$$

Finally, we remark that the converse statements in all of the above
results excluding Proposition 2 are also true. We mean that if a trans-
formation is of the form which appears in the formulation of the corre-
sponding result, then it has the properties which were required there.
Since the verification of this observation needs only elementary computations, hence we omit them.

II. PROOFS

First we emphasize that the proof of our main result has many common points with the proof of [12, Theorem]. In fact, the argument to be presented below can be considered as an adaptation of the proof given in [12] for another situation. So, it would have been a possibility just to point out where and how the proof in [12] should be modified to get the statement of the main result of this paper but we think that such a proof is quite hard to follow and hence it does not meet the most elementary requirements. Hence, we decided to present a complete, self-contained proof.

We begin with some notation and useful facts that we shall apply in our arguments.

First, we note that the concept of the strength of an effect along a ray plays very important role in what follows. This notion was introduced by Busch and Gudder in [2]. If $A$ is an effect on $H$, $\varphi$ is a unit vector in $H$ and $P_\varphi$ is the rank-one projection onto the subspace generated by $\varphi$, then the quantity

$$
\lambda(A, P_\varphi) = \sup \{ \lambda \in [0, 1] : \lambda P_\varphi \leq A \}
$$

is called the strength of $A$ along the ray represented by $\varphi$. Due to [2, Theorem 4] there is a very useful formula to compute the strength. In fact, we have

$$
\lambda(A, P_\varphi) = \begin{cases} 
\|A^{-1/2} \varphi\|^{-2}, & \text{if } \varphi \in \text{rng}(A^{1/2}); \\
0, & \text{else.}
\end{cases}
$$
(Here, rng denotes the range of an operator and $A^{-1/2}$ denotes the inverse of $A^{1/2}$ on its range.)

Let $\phi : E(H) \to E(H)$ be a bijective map which preserves the order in both directions. It was proved by Ludwig in [10, Theorem 5.8., p. 219] that $\phi$ necessarily preserves the projections in both directions. It is then trivial to see that $\phi$ also preserves the rank of the projections (cf. the proof of [11, Theorem 1]).

An easy fact follows what we shall use several times. Namely, if $A, B$ are effects, $B$ is of rank one and $A \leq B$, then $A$ is a scalar multiple of $B$. This observation and the previous one have, among others, the following corollary. Let $\phi$ be as above, i.e., suppose that it is a bijection of $E(H)$ which preserves the order in both directions. Then for every rank-one projection $P$, there is a function $f_P : [0, 1] \to [0, 1]$ such that

$$\phi(tP) = f_P(t)\phi(P) \quad (t \in [0, 1]).$$

By the order preserving property of $\phi$ and $\phi^{-1}$ we see that $f_P$ is strictly increasing and bijective. In fact, we have

$$\phi^{-1}(t\phi(P)) = f_P^{-1}(t)P \quad (t \in [0, 1]).$$

Now, we turn to the proofs. In the proof of our main result Theorem 1 we need the following proposition which presents the solution of a functional equation.

**Proposition 8.** Let $f, g : [0, 1] \to [0, 1]$ be functions and suppose that $f$ is a strictly monotone increasing bijection. Let

$$f\left(\frac{x}{x + (1 - x)y}\right) = \frac{f(x)}{f(x) + (1 - f(x))g(y)} \quad (x, y \in [0, 1]).$$

Then there are positive real numbers $a, b, c$ such that

$$f(x) = \frac{x^c}{x^c + a(1 - x)^c} \quad (x \in [0, 1]).$$
and
\[ g(y) = by^c \quad (y \in ]0,1[). \]

**Proof.** First note that since the function \( f \) is continuous, equation (4) implies the continuity of \( g \).

Next observe that with the notation
\[
\alpha(t) = \frac{1}{1 + e^t} \quad (t \in \mathbb{R}),
\beta(x) = \ln \frac{1 - x}{x} \quad (x \in ]0,1[),
\gamma(y) = \ln y \quad (y \in ]0,1[),
\]
we have the identity
\[
\frac{x}{x + (1 - x)y} = \frac{1}{1 + \exp(\ln \frac{1 - x}{x} + \ln y)} = \alpha(\beta(x) + \gamma(y))
\]
for all \( x, y \in ]0,1[. \) Therefore, equation (4) can be rewritten as
\[
(5) \quad f \circ \alpha(\beta(x) + \gamma(y)) = \alpha(\beta \circ f(x) + \gamma \circ g(y)) \quad (x, y \in ]0,1[).
\]

Substituting \( x = \beta^{-1}(u) \) and \( y = \gamma^{-1}(v) \) into (5) and applying the inverse function of \( \alpha \) to both sides of (5), we get
\[
(6) \quad \alpha^{-1} \circ f \circ \alpha(u + v) = \beta \circ f \circ \beta^{-1}(u) + \gamma \circ g \circ \gamma^{-1}(v)
\]
for all \( u \in \mathbb{R} \) and \( v \in ]-\infty,0[. \) This means that the functions
\[
F = \alpha^{-1} \circ f \circ \alpha, \quad G = \beta \circ f \circ \beta^{-1}, \quad \text{and} \quad H = \gamma \circ g \circ \gamma^{-1}
\]
satisfy the following so-called Pexider equation
\[
F(u + v) = G(u) + H(v) \quad (u \in \mathbb{R}, v \in ]-\infty,0[).
\]

Then, by known results of the theory of functional equations (cf. [1], or [9]) and by the continuity of \( F, G, H \), it follows that there exist
constants \(a, b, c \in \mathbb{R}\) such that

\begin{align*}
F(w) &= cw + a + b \quad (w \in \mathbb{R}), \\
G(u) &= cu + a \quad (u \in \mathbb{R}), \\
H(v) &= cv + b \quad (v \in ]-\infty, 0[).
\end{align*}

Using (7) and the definition of \(G\), we get that \(\beta \circ f(x) = c\beta(x) + a\).

Easy computation yields that

\[ f(x) = \frac{x^c}{x^c + e^a(1-x)^c} \quad (x \in ]0, 1[). \]

Similarly, the definition of \(H\) and \(\gamma\), and equation (8) give

\[ g(y) = e^b y^c \quad (y \in ]0, 1[). \]

The function \(f\) being strictly increasing, \(G\) is also increasing and hence we get \(c > 0\). \(\square\)

Now, we are in a position to prove our main result.

**Proof of Theorem 1.** The clue of the proof is to show that the functions \(f_P\) (see the first part of this section) do not depend on the rank-one projections \(P\). This will be done in what follows.

Let \(P, Q\) be arbitrary mutually orthogonal rank-one projections. By the order, rank and orthogonality preserving properties of \(\phi\) on set of all projections we clearly have

\[ \phi(P + Q) = \phi(P) + \phi(Q). \]

Let \(\lambda \in [0, 1]\). From the inequality

\[ \phi(Q) \leq \phi(\lambda P + Q) \leq \phi(P + Q) = \phi(P) + \phi(Q) \]

we obtain

\[ 0 \leq \phi(\lambda P + Q) - \phi(Q) \leq \phi(P). \]
As \( \phi(P) \) is of rank-one, according to the introduction of the present section, this implies that there is a scalar \( h_P(\lambda) \in [0,1] \) such that

\[
\phi(\lambda P + Q) - \phi(Q) = h_P(\lambda) \phi(P)
\]

or, equivalently, that

\[
\phi(\lambda P + Q) = h_P(\lambda) \phi(P) + \phi(Q).
\]

Since \( \phi^{-1} \) has the same properties as \( \phi \), it can be seen that the function \( h_P : [0, 1] \to [0, 1] \) is a strictly monotone increasing bijection. In fact, we have

\[
(9) \quad \phi^{-1}(\lambda \phi(P) + \phi(Q)) = h_P^{-1}(\lambda)P + Q.
\]

We assert that \( h_P = f_P \). Indeed, since

\[
f_P(\lambda) \phi(P) = \phi(\lambda P) \leq \phi(\lambda P + Q) = h_P(\lambda) \phi(P) + \phi(Q),
\]

it follows that \( f_P \leq h_P \). By (3) and (9), if one considers \( \phi^{-1} \), it follows that \( f_P^{-1} \leq h_P^{-1} \). Since the functions \( f_P, h_P : [0, 1] \to [0, 1] \) are monotone increasing we then conclude that \( f_P = h_P \). From the inequality

\[
f_P(\lambda) \phi(P) = \phi(\lambda P) \leq \phi(\lambda(P + Q)) \leq \\
\phi(\lambda P + Q) = h_P(\lambda) \phi(P) + \phi(Q) = f_P(\lambda) \phi(P) + \phi(Q)
\]

we infer that

\[
0 \leq \phi(\lambda(P + Q)) - f_P(\lambda) \phi(P) \leq \phi(Q).
\]

As \( \phi(Q) \) is of rank one, this implies that

\[
(10) \quad \phi(\lambda(P + Q)) = f_P(\lambda) \phi(P) + k_P(\lambda) \phi(Q)
\]

holds for some scalar \( k_P(\lambda) \in [0, 1] \).
With the notation $F = P + Q$ it follows from the equality (10) that the operator $\phi(\lambda F)$ is diagonalizable with respect to any orthonormal basis in the range of the projection $\phi(F) = \phi(P) + \phi(Q)$. We obtain that $\phi(\lambda F)$ is a constant multiple of $\phi(F)$. This gives us that we have $f_P = kP$ and hence

$$\phi(\lambda(P + Q)) = f_P(\lambda)\phi(P) + f_P(\lambda)\phi(Q).$$

Let $R$ be a rank-one projection whose range is included in the subspace generated by $\text{rng } P$ and $\text{rng } Q$. Then we have

$$f_R(\lambda)\phi(R) = \phi(\lambda R) \leq \phi(\lambda F) = f_P(\lambda)(\phi(P) + \phi(Q)) = f_P(\lambda)\phi(F).$$

This gives us that

$$(11) \quad f_R \leq f_P$$

whenever $P, R$ are rank-one projections. It is then trivial that there is in fact equality in (11) and this proves that $f_P$ does not depend on $P$. Denote by $f$ this common function.

Our next claim is to show that $f$ satisfies a functional equation of the form (4). Fix mutually orthogonal rank-one projections $P, Q$ on $H$. Pick $\mu \in [0, 1]$ and let $E = \mu P + Q$. Take any rank-one projection $R$ on $H$ whose range is contained in the subspace generated by the ranges of $P$ and $Q$ and which is neither equal nor orthogonal to $P$. Using the formula (2), one can easily verify that

$$\lambda(E, R) = \frac{\mu}{\mu + (1 - \mu) \text{tr } PR}.$$

By the definition of $\lambda(E, R)$ and the order preserving property of $\phi$ it is clear that

$$f(\lambda(E, R)) = \sup \{f(\lambda) : \lambda R \leq E\} = \sup \{f(\lambda) : \phi(\lambda R) \leq \phi(E)\} = \sup \{f(\lambda) : f(\lambda)\phi(R) \leq \phi(E)\} = \lambda(\phi(E), \phi(R)).$$
Since $\phi(E) = \phi(\mu P + Q) = f(\mu)\phi(P) + \phi(Q)$, we obtain the equality

$$f\left(\frac{\mu}{\mu + (1 - \mu) \text{tr} PR}\right) = \frac{f(\mu)}{f(\mu) + (1 - f(\mu)) \text{tr} \phi(P)\phi(R)}.$$ 

As the quantities $\text{tr} PR$ and $\text{tr} \phi(P)\phi(R)$ do not depend on $\mu$, it follows from this equality that $\text{tr} \phi(P)\phi(R)$ can be uniquely expressed as a function of $\text{tr} PR$. Denoting $g(\text{tr} PR) = \text{tr} \phi(P)\phi(R)$, we get a bijective function $g : [0,1] \rightarrow [0,1]$ for which

$$f\left(\frac{\mu}{\mu + (1 - \mu) \nu}\right) = \frac{f(\mu)}{f(\mu) + (1 - f(\mu)) g(\nu)} \quad (\mu, \nu \in ]0,1[).$$ 

This gives us the desired functional equation for $f$ and $g$. By Proposition 8 we obtain that there are positive real numbers $a, b, c$ such that

$$f(x) = \frac{x^c}{x^c + a(1 - x)^c} \quad (x \in ]0,1[)$$

and

$$g(y) = by^c \quad (y \in ]0,1[).$$

But our function $g$ has the additional property that $g(1 - x) = 1 - g(x)$ ($x \in ]0,1[)$. In fact, this follows from the equality

$$g(\text{tr} PR) + g(1 - \text{tr} PR) = g(\text{tr} PR) + g(\text{tr} QR) =$$

$$\text{tr} \phi(P)\phi(R) + \text{tr} \phi(Q)\phi(R) = 1.$$ 

One can easily deduce that we necessarily have $b = 1, c = 1$, i.e., $g$ is the identity on $]0,1[$. This further implies that our function $f$ is of the form

$$f(x) = \frac{x}{x + a(1 - x)} = \frac{x}{x(1 - a) + a} \quad (x \in ]0,1[).$$

Because of continuity, the above equality holds also on the whole interval $[0,1]$. Hence, we have that $f$ is of the form $f = f_p$ where $p = 1 - a$.

Since the function $g$ above is the identity, we have

$$\text{tr} PQ = \text{tr} \phi(P)\phi(Q)$$
for all rank-one projections $P, Q$ on $H$. Hence, using Wigner’s celebrated theorem on symmetry transformations (sometimes called unitary-antiunitary theorem) we obtain that there exists an either unitary or antiunitary operator $U$ on $H$ such that

$$\phi(P) = UPU^*$$

holds for every rank-one projection $P$ on $H$. Consider the transformation

$$\psi : A \mapsto f^{-1}(U^*\phi(A)U)$$

on $E(H)$. It is not hard to see that this map is a bijection of $E(H)$ which preserves the order (as well as zero product) in both directions and it has the additional property that it fixes the so-called weak atoms, that is, the effects of the form $\lambda P$ where $\lambda \in [0, 1]$ and $P$ is a rank-one projection. As, according to [2, Corollary 3], every effect is equal to the supremum of the set of all weak atoms it majorizes, we have that $\psi$ is the identity on $E(H)$. Transforming back, we see that

$$\phi(A) = Uf(A)U^* \quad (A \in E(H)).$$

The proof is complete. \hfill \Box

Proof of Proposition 2. Let $\phi : E(H) \to E(H)$ be an order preserving bijection and $\lambda, \mu$ be a pair of nontrivial scalars for which we have $\phi(\lambda I) = \mu I$. Keeping the notation introduced in the first part of this section, we claim that

$$f_P(\lambda) = \mu,$$

i.e., that $\phi(\lambda P) = \mu \phi(P)$ holds for every rank-one projection $P$. Indeed, from

$$f_P(\lambda)\phi(P) = \phi(\lambda P) \leq \phi(\lambda I) = \mu I$$
we deduce that \( f_P(\lambda) \leq \mu \). Now, considering \( \phi^{-1} \) and \( \phi(P) \) in the place of \( \phi \) and \( P \), respectively, we also have \( f_P^{-1}(\mu) \leq \lambda \). Since \( f_P \) is increasing, this implies \( \mu \leq f_P(\lambda) \) and hence we get (12).

We next assert that \( \phi \) preserves the orthogonality between rank-one projections. To see this, let \( P, Q \) be mutually orthogonal rank-one projections. Denote by \( P' = I - P \) the orthogonal complement of \( P \). Consider the effect \( E = \lambda P + P' \). Clearly, we have \( \lambda I \leq E \leq I \), the strength of \( E \) along \( P \) is \( \lambda \) and along \( Q \) (which is a subprojection of \( P' \)) is 1. It follows from the order preserving property of \( \phi \) and from (12) that

\begin{itemize}
  \item \( \mu I \leq \phi(E) \leq I \),
  \item the strength of \( \phi(E) \) along \( \phi(P) \) is \( \mu \),
  \item the strength of \( \phi(E) \) along \( \phi(Q) \) is 1.
\end{itemize}

Now, Lemma 3 in [11] tells us that in this case the ranges of \( \phi(P) \) and \( \phi(Q) \) are subspaces of the eigenspaces of \( \phi(E) \) corresponding to the eigenvalues \( \mu \) and 1, respectively. This yields that the ranges of \( \phi(P) \) and \( \phi(Q) \) are orthogonal to each other.

As every projection is equal to the supremum of the set of all rank-one projections it majorizes, it follows that \( \phi \) preserves the orthogonality of projections of any rank. It is also easy to verify that \( \phi \) preserves the range projections of effects. This means that if \( R \) is the range projection of \( A \) (i.e., the projection onto \( \text{rng} \ A \)), then \( \phi(R) \) is the range projection of \( \phi(A) \). Indeed, this preserver property follows from the simple fact that the range projection of the effect \( A \) is equal to the infimum of the set of all projections which are greater than or equal to \( A \). It is clear that for any \( A, B \in E(H) \), we have \( AB = 0 \) if and only if the range projections of \( A \) and \( B \) are orthogonal. Using these
observations we can infer that
\[ AB = 0 \iff \phi(A)\phi(B) = 0. \]
This completes the proof. \hfill \square

**Proof of Corollary 5.** By [11, Lemma 2] an effect is coexistent with every other effect if and only if it is a scalar multiple of the identity. This implies that our transformation \( \phi \) maps scalar operators to scalar operators. Hence, by Corollary 3 we infer that up to unitary-antiunitary equivalence, \( \phi \) is of the form
\[ \phi(A) = f_p(A) \quad (A \in E(H)) \]
for some \( p < 1 \). We claim that \( p = 0 \), i.e., \( f_p \) is the identity. To see this, let \( P, Q \) be different rank-one projections. It was proved in [11, Lemma 2] that two rank-one effects with different ranges are coexistent if and only if their sum is an effect. Since \( \lambda P \) and \( (1 - \lambda)Q \) are always coexistent (indeed, their sum is an effect), we obtain that \( \phi(\lambda P) \) and \( \phi((1 - \lambda)Q) \) must be also coexistent. By the just mentioned characterization we obtain that for any \( \lambda \in ]0, 1[ \) we have
\[ f_p(\lambda)P + f_p(1 - \lambda)Q = \phi(\lambda P) + \phi((1 - \lambda)Q) \leq I. \]
If we let \( Q \) tend to \( P \), we infer from this inequality that
\[ f_p(\lambda) + f_p(1 - \lambda) \leq 1 \quad (\lambda \in [0, 1]). \]
Since \( \phi^{-1} \) has the same properties as \( \phi \), it also follows that
\[ f_p^{-1}(f_p(\lambda)) + f_p^{-1}(1 - f_p(\lambda)) \leq 1. \]
This further implies
\[ f_p^{-1}(1 - f_p(\lambda)) \leq 1 - \lambda \]
and by the monotonicity of \( f_p \) we obtain

\[
1 - f_p(\lambda) \leq f_p(1 - \lambda).
\]

Comparing this with (13), we see that

\[
f_p(1 - \lambda) = 1 - f_p(\lambda)
\]

holds for every \( \lambda \in [0,1] \). It is easy to show that this implies \( p = 0 \) which completes the proof. \( \Box \)

**Proof of Corollary 6.** We first show that \( \phi \) preserves zero product in both directions. As \( \phi \) preserves the order in both directions, we know that \( \phi \) preserves the projections in both direction. Since \( \phi \) is an ortho-order automorphism, we obtain that it also preserves the orthogonality between projections. Now, one can use the same argument as in the last paragraph of the proof of Proposition 2 to verify that \( \phi \) preserves zero product in both directions. Since \( \phi \) preserves the orthocomplements, we can compute

\[
\phi\left(\frac{1}{2}I\right) = \phi\left(\left(\frac{1}{2}I\right)'ight) = \phi\left(\frac{1}{2}I\right)'
\]

which implies that \( \phi\left(\frac{1}{2}I\right) = \frac{1}{2}I \). One can apply Corollary 4 to complete the proof. \( \Box \)

**Proof of Corollary 7.** It was proved in [7] that the order on \( E(H) \) is completely determined by the sequential product. More precisely, [7, Theorem 5.1] tells us that for any \( A, B \in E(H) \) we have

\[
A \leq B \iff \exists C \in E(H) : A = B \circ C.
\]

As \( \phi \) is a sequential automorphism, using this characterization it follows that \( \phi \) preserves the order in both directions. It is easy to see that

\[
A \circ B = (\sqrt{B} \sqrt{A})^* \sqrt{B} \sqrt{A} = 0 \iff AB = 0.
\]
Therefore, $\phi$ also preserves zero product in both directions. By Theorem 1 we obtain that up to unitary-antiunitary equivalence, $\phi$ is of the form

$$\phi(A) = f_p(A) \quad (A \in E(H))$$

for some $p < 1$. We assert that $p = 0$, i.e., $f_p$ is the identity. In fact, as $\phi$ is a sequential automorphism, we obtain from (14) that $f_p$ is a multiplicative function on the unit interval. It is then obvious that $p = 0$ and the proof is complete. \qed

In conclusion, according to our promise given in the introduction, we make some remarks on the relation between the class of our preservers and the collections of ortho-order automorphisms, resp. sequential automorphisms in the general setting of von Neumann algebras. So, let $A$ be a von Neumann algebra of operators acting on the Hilbert space $H$. Denote by $E(A)$ the set of all effects which belong to $A$, i.e., let $E(A) = E(H) \cap A$. The definitions of the order, orthocomplementation, and sequential product are straightforward and so are the definitions of ortho-order automorphisms and sequential automorphisms.

First, let $\phi$ be an ortho-order automorphism of $E(A)$. It is easy to see that the sharp elements in $E(A)$ (i.e., the elements $A$ for which the infimum of $A$ and $A' = I - A$ is 0) are exactly the projections. Hence we obtain that $\phi$ preserves the projections in $E(A)$ as well as their orthogonality in both directions. Since, as it is well-known, the range projection of any element of a von Neumann algebra also belongs to the algebra, we see that $\phi$ preserves the range projections of the elements of $E(A)$ in the same sense as it was mentioned in the proof of Proposition 2. Then one can argue as in that proof to verify that $\phi$ preserves zero product in both directions. So, we obtain that every
ortho-order automorphism of $E(\mathcal{A})$ belongs to our class of preservers, that is, those automorphisms preserve the order and zero product in both directions.

Now, let $\phi : E(\mathcal{A}) \to E(\mathcal{A})$ be a sequential automorphism. It is not hard to prove that the above mentioned characterization of the order by means of the sequential product due to Gudder and Greechie holds true also in the setting of von Neumann algebras. This means that for any $A, B \in E(\mathcal{A})$ we have

\begin{equation}
A \leq B \iff \exists C \in E(\mathcal{A}) : A = B \circ C.
\end{equation}

In fact, if $A, B \in E(\mathcal{A})$ and $A \leq B$, then by [7, Theorem 5.1] there is an operator $C \in E(H)$ such that $A = B \circ C$. Following the proof in [7], one can see that the existence of this operator $C$ is a consequence of a well-known result of Douglas [5]. In fact, in the corresponding part of the proof of Douglas’ result this $C$ was constructed. Examining the construction, it is not hard to verify that $C$ belongs to the von Neumann algebra $\mathcal{A}$, i.e., we have $C \in E(\mathcal{A})$. This gives one implication from the asserted equivalence in (15). The other implication is trivial. Then, just as in the proof of Corollary 7, one can show that $\phi$ preserves the order and zero product in both directions.

To sum up, it has turned out that the ortho-order automorphisms and the sequential automorphisms all belong to our new class of preservers even in the setting of von Neumann algebras. This seems to be a worthwhile observation as in that generality it is quite hard to see any connection between those two kinds of automorphisms.
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