On the space of harmonic 2-spheres in $\mathbb{C}P^2$

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Abstract

Carrying further work of T.A. Crawford, we show that each component of the space of harmonic maps from the 2-sphere to complex projective 2-space of degree $d$ and energy $4\pi E$ is a smooth closed submanifold of the space of all $C^j$ maps ($j \geq 2$). We achieve this by showing that the Gauss transform which relates them to spaces of holomorphic maps of given degree and ramification index is smooth and has injective differential.

1 Introduction

In [4], T.A. Crawford showed that if $E \leq 5|d| + 10$, the subset $\operatorname{Harm}_{d,E}(\mathbb{C}P^2)$ of the space of harmonic maps from $S^2$ to $\mathbb{C}P^2$ consisting of those maps of degree $d$ and energy $4\pi E$ can be given the structure of a complex manifold and that this manifold is connected. This he did by showing that the space $\operatorname{Hol}^*_{k,r}(\mathbb{C}P^2)$ of full holomorphic maps from $S^2$ to $\mathbb{C}P^2$ of degree $k$ and ramification index $r$ is a complex manifold if $r \leq (k + 1)/2$, and that the “Gauss transform” $G'_{k,r}$ which maps $\operatorname{Hol}^*_{k,r}(\mathbb{C}P^2)$ to $\operatorname{Harm}_{k-r-2,3k-r-2}(\mathbb{C}P^2)$ bijectively is a homeomorphism, so that the manifold structure of $\operatorname{Hol}^*_{k,r}(\mathbb{C}P^2)$ can be transported to the topological space $\operatorname{Harm}_{d,E}(\mathbb{C}P^2)$. This does not prove that the transported structure is the one induced by the natural inclusion of $\operatorname{Harm}_{d,E}(\mathbb{C}P^2)$ in the space of maps from $S^2$ to $\mathbb{C}P^2$.

In this paper, after giving a treatment of Crawford’s result adapted to our needs, we show that $G'_{k,r}$ is a smooth map from $\operatorname{Hol}^*_{k,r}(\mathbb{C}P^2)$ to $C^j(S^2, \mathbb{C}P^2)$ (for $j \geq 2$) and has injective differential. From this we obtain:

**Theorem 1.1** For $0 \leq r \leq \frac{k + 1}{2}$ and $\frac{4k - 11}{3} \leq r \leq \frac{3}{2}k - 3$ the map

$$G'_{k,r} : \operatorname{Hol}^*_{k,r}(\mathbb{C}P^2) \to C^j(S^2, \mathbb{C}P^2)$$

is a smooth embedding onto $\operatorname{Harm}_{k-r-2,3k-r-2}(\mathbb{C}P^2)$ for any $j \geq 2$.

Each component $\operatorname{Harm}_{d,E}(\mathbb{C}P^2)$ of $\operatorname{Harm}(\mathbb{C}P^2)$ with $E \leq 5|d| + 10$ is a closed smooth submanifold of $C^j(S^2, \mathbb{C}P^2)$ of dimension $6E + 4$ if $E = |d|$ (in which case it consists of holomorphic or antiholomorphic maps) and of dimension $2E + 8$ (otherwise).

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Added in proof: In a revised version of [4], Crawford has shown that $\text{Hol}^*_{k,r}(\mathbb{C}P^2)$ is a manifold for all $k, r$. It follows that all our results (Theorems 1.1, 1.3, Proposition 3.1, 4.1, 5.2 and Lemma 5.1) are valid for this range and the restriction $E \leq 5|d| + 10$ can be removed from Theorem 1.1. Proofs are unchanged except for those in Sec. 3, see below.

**Remark 1.2** The space $\text{Hol}^*_{k,r}(\mathbb{C}P^2)$ is non-empty precisely for $k \geq 2$, $0 \leq r \leq \frac{2}{3}k - 3$ (see Proposition 2.7 below).

We shall first of all prove that $\text{Hol}^*_{k,r}(\mathbb{C}P^2)$ is a complex manifold for the range $k \geq 2$, $0 \leq r \leq (k + 1)/2$. The passage from this range to the second $k \geq 3$, $(4k - 11)/3 \leq r \leq 3k/2 - 3$ is achieved by the conjugate polar (see Definition 2.3 below).

In fact, we have:

**Theorem 1.3** For $0 \leq r \leq \frac{k + 1}{2}$ and for $\frac{4k - 11}{3} \leq r \leq \frac{3}{2}k - 3$, $\text{Hol}^*_{k,r}(\mathbb{C}P^2)$ is a complex submanifold of the complex manifold $\text{Hol}^*_k(\mathbb{C}P^2)$ of dimension $3k - r + 2$.

Added in proof: Note that Lemma 3.3 in the proof is false outside the above range; we give a counter example for $k = 6, r = 4$. Hence the description of the manifold structure on $\text{Hol}^*_{k,r}(\mathbb{C}P^2)$ in Sec. 3 is special to the range $r \leq (k + 1)/2$ — see the revised version of [4] for a description valid for any $k, r$.

The contents of the subsequent sections are as follows.

In Section 2, we recall the construction of J. Eells and the second author of the harmonic maps of $S^2$ to $\mathbb{C}P^2$, stressing the limits of the values of the parameters involved, and we give some examples illustrating the behaviour of this construction.

In Section 3, we present a proof of the result of T.A. Crawford on the structure of $\text{Hol}^*_{k,r}(\mathbb{C}P^2)$, adapted to our needs. We use ideas from the paper of Crawford, and a construction suggested to us by M. Guest.

In Sections 4 and 5, we show successively that the Gauss transform is smooth and that it is an embedding. The main ingredient is a property of smooth dependence of the common factor of smooth families of polynomials (Lemma 4.5). We note at this point that the methods of the present paper will not generalize easily to $\mathbb{C}P^n$ for $n > 2$, however M. Guest and Y. Ohnita [10] show that some topological questions on $\mathbb{C}P^n$ reduce to $\mathbb{C}P^2$.

Recall finally that the harmonic maps from $S^2$ to any manifold are precisely the minimal branched immersions of $S^2$, in particular, they are conformal (see e.g. [3] (5.15))

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2 Gauss transform and polars

Following work of others \cite{3, 4}, see also \cite{5}, J. Eells and the second author classified the harmonic maps from the 2-sphere $S^2$ to a complex projective space $\mathbb{C}P^n$, as follows:

**Theorem 2.1** \cite{3}. There is a bijective correspondence between the set of pairs $(f, s)$ where $f : S^2 \to \mathbb{C}P^n$ is a full holomorphic map and $s$ an integer, $0 \leq s \leq n$, and the set of full harmonic maps $\varphi : S^2 \to \mathbb{C}P^n$.

Here “full” means “not having image in any proper projective subspace of $\mathbb{C}P^n$”. For $s = 0$, $\varphi = f$ is holomorphic, for $s \neq 0, n$, $\varphi$ is neither holomorphic nor antiholomorphic, and for $s = n$, $\varphi$ is antiholomorphic and is called the **polar** of $f$.

We now restrict to the case of $\mathbb{C}P^2$, where the description of this construction is simpler.

Let $f : S^2 \to \mathbb{C}P^2$ be a holomorphic map. Identifying $S^2$ with $\mathbb{C} \cup \{\infty\}$ by stereographic projection, $f$ can be represented on $\mathbb{C}$ by a map $p : \mathbb{C} \to \mathbb{C}^3 \setminus \{0\}$ where $p(z) = (p_0(z), p_1(z), p_2(z))$ is a triple of coprime polynomials with max (degree $p_0$, degree $p_1$, degree $p_2$) = degree of $f$. We shall write $f = [p_0, p_1, p_2]$.

A map $S^2 \to \mathbb{C}P^2$ is called **full** if its image lies in no complex projective line. Note that if a harmonic map is not full, its image lies in a $\mathbb{C}P^1$ and it is then $\pm$-holomorphic, since it is a conformal map between surfaces. Thus all harmonic non $\pm$-holomorphic maps $f : S^2 \to \mathbb{C}P^2$ are full.

We denote by $\text{Hol}_k^*(\mathbb{C}P^2)$ the space of full holomorphic maps of degree $k$. All values of $k \geq 2$ occur and $\text{Hol}_k^*(\mathbb{C}P^2)$ is a complex manifold of dimension $3k + 2$ with coordinate charts given by the coefficients.

Recall that a holomorphic map $f : S^2 \to \mathbb{C}P^2$ is said to be **ramified** at a point $z \in S^2$ if $df(z) = 0$. The ramification index of $f$ at $z$ is the order of the zero of $df(z)$ and the **total ramification index** $r$ of $f$ is the sum of ramification indices.

Consider a full holomorphic map $f$. The harmonic map $\varphi$ associated to the pair $(f, 1)$ in Theorem 2.1 is obtained by means of the $\partial$-**Gauss transform** (in the terminology of \cite{3} - in \cite{12} it is called the $\partial$-**transform**) which is defined as follows \cite{7}:

Let $\pi : \mathbb{C}^3 \setminus \{0\} \to \mathbb{C}P^2$ be the canonical projection sending $(z_0, z_1, z_2)$ to the point of $\mathbb{C}P^2$ with homogeneous coordinates $[z_0, z_1, z_2]$. For a map $f : S^2 \to \mathbb{C}P^2$, say that $F : U \to \mathbb{C}^3 \setminus \{0\}$ represents $f$ on the open set $U$ if $f|_U = \pi \circ F$, in which case we write $f = [F]$. For $f$ holomorphic and full, the **first associated curve** $f_{(1)} : S^2 \to G_2(\mathbb{C}^3)$ is the holomorphic map defined as follows. Let $F : U \to \mathbb{C}^3 \setminus \{0\}$ represent $f$ on a domain $(U, z)$ of $\mathbb{C}$. Consider the map $F \wedge F' : U \to \wedge^2 \mathbb{C}^3$ where $F' = \frac{dF}{dz}$. At a point $x$ where $f$ is not ramified, $F \wedge F'$ is non zero and so defines a complex two-dimensional subspace $f_{(1)}(x)$. If, on the other hand, $f$ is ramified at $x$ with ramification index $k$, then $F \wedge F' = (z - x)^k \cdot \Psi$ for some smooth nonzero map $\Psi : U' \to \wedge^2 \mathbb{C}^3$ on an open neighbourhood of $x$. Since $\Psi(y)$ is decomposable for all $y \in U'$, $y \neq x$, it remains decomposable for $y = x$ and we can define $f_{(1)}(x)$ as the complex two-dimensional subspace defined by $\Psi(x)$. The resulting map $f_{(1)} : S^2 \to G_2(\mathbb{C}^3)$ is well-defined and smooth. This leads us to two maps associated to $f$:
Definition 2.2 The ∂'-Gauss transform \( \varphi = G'(f) : S^2 \to \mathbb{C}P^2 \) is defined by the formula
\[
\varphi(x) = f(x) \perp f_{(1)}(x).
\] (1)

By Theorem 2.1, it is a smooth and full harmonic map.

Definition 2.3 The polar of the holomorphic map \( f \) is the antiholomorphic map
\[
g(x) = f_{(1)}(x) \perp.
\]

Note that for any \( x \in S^2 \), \( f(x), \varphi(x) \) and \( g(x) \) are Hermitian orthogonal complex lines.

For convenience, we consider the conjugate polar \( h \) of \( f \) defined by taking in \( \mathbb{C}^3 \) the complex conjugate of the values of \( g(x) : h(x) = \overline{f_{(1)}(x)} \).

More explicitly, represent \( f \in \text{Hol}^*_k(\mathbb{C}P^2) \) by \([p_0, p_1, p_2]\) as above. Identifying \( \wedge^2 \mathbb{C}^3 \) with \( \mathbb{C}^3 \), the first associated curve, or equivalently the conjugate polar \( h \) of \( f \), is represented by the polynomials
\[
h = [p_{12}, p_{20}, p_{01}]
= [p_1p_2' - p_1'p_2, p_2p_0' - p_0'p_2, p_0p_1' - p_1'p_0].
\]

Once they have been divided by their common factor. Explicitly, if \( f \) has finite ramification points \( z_I \) with multiplicities \( r_I \) (\( I = 1, \ldots, R \)), then the ramification divisor \( R(f) \) is the monic polynomial
\[
R(f) = \prod(z - z_I)^{r_I}.
\]

If \( f \) is not ramified at \( \infty \) this has degree equal to the ramification index, otherwise it has lower degree.

In either case, \( R(f) \) is the highest common factor of the polynomials \( p_{ij} \), and
\[
h = \left[ \frac{p_{12}}{R(f)}, \frac{p_{20}}{R(f)}, \frac{p_{01}}{R(f)} \right].
\]

One checks easily that \( h \) has degree \( 2k - 2 - r \) (indeed, the terms of degree \( 2k - 1 \) in \( p_{ij} \) cancel).

Note that formula (1) has a counterpart:
\[
\overline{\varphi(x)} = h(x) \perp h_{(1)}(x).
\]

Theorem 2.1 can be rephrased in \( \mathbb{C}P^2 \) by saying that the Gauss transform defines a bijection between the space of full holomorphic maps and the space of harmonic non \( \pm \)-holomorphic maps, and that the passage to the polar defines a bijection between full holomorphic and antiholomorphic maps.

However, this does not immediately provide a simple description of the space of harmonic maps. Indeed, the Gauss transform \( G' : \text{Hol}^*_k(\mathbb{C}P^2) \to \text{Harm}(\mathbb{C}P^2) \) is not a continuous map, when the spaces are equipped with their \( C^0 \)-topology. This appears for instance in the following example (brought to our attention by F. Burstall).
Example 2.4 Let \( f_t : S^2 \to \mathbb{C}P^2 \) be defined by \( f_t(z) = [F_t(z)] \), where
\[
F_t(z) = (1, tz + z^3, z^2) \quad (z \in \mathbb{C}, t \in \mathbb{R})
\]
(so that \( f_t(\infty) = [0, 1, 0] \)). Note that \( f_t(0) = [1, 0, 0] \) for all \( t \) and that \( f_t \) is a smooth family of full holomorphic maps. Then \( F_t'(z) = (0, t + 3z^2, 2z) \) and \( F_t \wedge F_t'(z) = (tz^2 - z^4, -2z, t + 3z^2) \). If \( t \neq 0 \), at \( z = 0 \) this equals \( (0, 0, t) \neq t(1, 0, 0) \wedge (0, 1, 0) \) so that the first associated curve has the value \( f_t(1)(0) = \text{span}\{(1, 0, 0), (0, 1, 0)\} \) and \( f_t'(0)(0) = [0, 1, 0] \). However, if \( t = 0 \), then \( F_t \wedge F_t'(z) = (-z^4, -2z, 3z^2) = z\psi(z) \) where \( \psi(z) = (-z^3, -2, 3z) \). In particular \( \psi(0) = (0, -2, 0) = (1, 0, 0) \wedge (0, 0, 1) \) so that \( f_t(1)(0) = \text{span}\{(1, 0, 0), (0, 0, 1)\} \) and \( f_t'(0)(0) = [0, 0, 1] \). This shows that \( f_t(1) \) and \( f_t'(0) \) do not vary continuously with \( t \). The reason for this is that \( f_t \) is unramified when \( t \neq 0 \) but ramified with ramification index 1 at \( z = 0 \) when \( t = 0 \), and that in the presence of ramification, both \( g' \) and \( g \) involve division of polynomials by their common factor, a discontinuous process when the degree of the factor changes.

To proceed, define \( \text{Hol}_{k,r}^*(\mathbb{C}P^2) \) as the space of full holomorphic maps of degree \( k \) and total ramification index \( r \), and \( \text{Harm}_{d,E}(\mathbb{C}P^2) \) the space of all harmonic maps of degree \( d \) and energy \( 4\pi E \). By results of [3], Theorem 2.3 specializes to

**Proposition 2.5** For each pair of integers \( k \geq 2 \), \( 0 \leq r \leq \frac{3}{2}k - 3 \), there is a bijective correspondence
\[
G_{k,r} : \text{Hol}_{k,r}^*(\mathbb{C}P^2) \to \text{Harm}_{d,E}(\mathbb{C}P^2)
\]
given by the restriction of the Gauss transform, where \( d = k - r - 2 \) and \( E = 3k - r - 2 \).

**Proposition 2.6** For each pair of integers \( k \geq 2 \), \( 0 \leq r \leq \frac{3}{2}k - 3 \), the map \( f \mapsto \text{conjugate polar} \) of \( f \) restricts to a bijection
\[
\text{Hol}_{k,r}^*(\mathbb{C}P^2) \to \text{Hol}_{k',r'}^*(\mathbb{C}P^2)
\]
where \( k' = 2k - r - 2 \), \( r' = 3k - 2r - 6 \).

This allows us to specify the values of \( k \) and \( r \) as follows:

**Proposition 2.7** The space \( \text{Hol}_{k,r}^*(\mathbb{C}P^2) \) is non-empty precisely for the range \( k \geq 2 \), \( 0 \leq r \leq \frac{3}{2}k - 3 \).

**Proof.** The well known examples (see [3])
\[
f(z) = [1, (z+1)^{k-r+1}, z^k]
\]
provide maps \( f \) in \( \text{Hol}_{k,r}^*(\mathbb{C}P^2) \) for all \( k \geq 2 \), \( 0 \leq r \leq k - 2 \).

The Plücker formulae (see e.g. [4]) show that the involutive map \( f \mapsto \text{conjugate polar} \) of \( f \) restricts to bijections
\[
\text{Hol}_{k,r}^*(\mathbb{C}P^2) \to \text{Hol}_{k',r'}^*(\mathbb{C}P^2)
\]
with $k', r'$ as in Proposition 2.6. Thus to get $\text{Hol}_{k,r}^*(C P^2)$ (and $\text{Hol}_{k',r'}^*(C P^2)$) non empty, we need $r' \geq 0$, i.e. $r \leq \frac{3}{2}k - 3$.

On the other hand, the conjugate polars of the above examples provide maps in $\text{Hol}_{k,r}^*(C P^2)$ for all

$$ k - 2 \leq r \leq \frac{3}{2}k - 3. $$

As in [3], we note that the above shows that $\text{Harm}_{d,E}(C P^2)$ is non-empty precisely for pairs $(d, E)$ of integers with either $E = |d|$ (in which case it consists of $\pm$-holomorphic maps) or $E = 3|d| + 4 + 2r$ for some $r \geq 0$ (otherwise).

Indeed, all such values of $E$ with $d \geq 0$ are achieved with $0 \leq r \leq k - 2$, the range $k - 2 < r \leq 3k/2 - 3$ giving $d < 0$.

The main contribution of the present paper is to show that $G'_{k,r} : \text{Hol}_{k,r}^*(C P^2) \to \text{Harm}_{d,E}(C P^2) \subset C^j(S^2, C P^2)$ is a smooth embedding and that the map $f \mapsto$ conjugate polar of $f$ is a complex analytic equivalence from $\text{Hol}_{k,r}^*(C P^2)$ to $\text{Hol}_{k',r'}^*(C P^2)$.

We conclude this section by an example showing that in $\text{Hol}_{k,r}^*(C P^2)$, the ramification divisor of a smooth family of polynomials can vary smoothly, even when individual common roots of the $p_{ij}$'s vary only continuously.

**Example 2.8** Identifying $S^2$ with $C \cup \{0\}$ by stereographic projection, let $f_t : S^2 \to C P^2$ be defined by

$$ F_t(z) = (z^4 + 1, (1 - 3t^2)z^3 + (-3t + t^3)z, 2tz^2 + (1 - t^2)) \quad (z \in C, t \in R) $$

(so that $f_t(\infty) = [1, 0, 0]$).

Identifying $\Lambda^2 C^3$ with $C^3$ we have $(F_t \wedge F'_t)(z) = (z^2 - t)\psi(z)$ where

$$ \psi(z) = ((-2t + 6t^3)z^2 + (-3 + t^2)(1 - t^2), 4z(tz^2 + 1), (-1 + 3t^2)z^4 + 8tz^2 + 3 - t^2). $$

which shows that, if $t \neq 0$, $f_t$ is ramified at $z = \pm \sqrt{t}$ with index 1, but if $t = 0$, these ramification points coalesce into a ramification point at $z = 0$ of index 2. Further $f_{t(1)}(z) = [\psi(z)]$. We see from this that $f_t \in \text{Hol}_{4,2}^*(C P^2)$ for all $t$ and that $f_{t(1)}$ and so $G'(f_t)$ vary smoothly with $t$, even though each root does not.

## 3 Spaces of holomorphic maps

In this section, we give a proof of the

**Proposition 3.1** For any $k \geq 2$ and $0 \leq r \leq \frac{k+1}{2}$, $\text{Hol}_{k,r}^*(C P^2)$ is a complex submanifold of $\text{Hol}_k^*(C P^2)$ of dimension $3k - r + 2$. 

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Proof. Let
\[ \text{Hol}_k'(\mathbb{C}P^2) = \{ f \in \text{Hol}_k^* (\mathbb{C}P^2) : f = [p_0, p_1, p_2], p_0 \text{ is monic of degree } k \text{ with distinct roots, } f \text{ is not ramified at } \infty \}. \]

Here, \( p_0, p_1 \) and \( p_2 \) are always assumed to be coprime. \( \text{Hol}_k'(\mathbb{C}P^2) \) is an open subset of \( \text{Hol}_k^* (\mathbb{C}P^2) \) and so a complex manifold. It can be embedded as an open subset in \( \mathbb{C}^{3k+2} \) by sending the polynomials \( (p_0, p_1, p_2) \) to their coefficients (omitting the leading coefficient of \( p_0 \), which is equal to 1).

The group \( G = PGL_2(\mathbb{C}) \times PGL_3(\mathbb{C}) \) acts on \( \text{Hol}_k^* (\mathbb{C}P^2) \) in a natural way preserving the subsets \( \text{Hol}_{k,r}^* (\mathbb{C}P^2) \). Given \( f \in \text{Hol}_k'(\mathbb{C}P^2) \), a variation of the proof of Lemma 4.3 below shows that there exist \( g \in G \) and a neighbourhood \( U \) of \( f \) in \( \text{Hol}_k'(\mathbb{C}P^2) \) such that \( g(U) \subset \text{Hol}_k'(\mathbb{C}P^2) \). Setting \( \text{Hol}_{k,r}^* (\mathbb{C}P^2) = \text{Hol}_{k,r}^* (\mathbb{C}P^2) \cap \text{Hol}_k'(\mathbb{C}P^2) \), it suffices therefore to show that \( \text{Hol}_{k,r}^* (\mathbb{C}P^2) \) is a complex submanifold of \( \text{Hol}_k'(\mathbb{C}P^2) \subset \mathbb{C}^{3k+2} \).

To do this, we use a construction suggested by M. Guest following ideas of T.A. Crawford. Set
\[ X_{k,r}'' = \{ (a, f) = (a, (p_0, p_1, p_2)) \in \mathbb{C}^r \times \mathbb{C}^{3k+2} : f \in \text{Hol}_k'(\mathbb{C}P^2), a \text{ is a monic polynomial of degree } r, (a, p_0) \text{ are coprime and } a \text{ divides } R(f) \}. \]

Note that \( X_{k,r}'' \) is an algebraic subvariety of \( \mathbb{C}^r \times \mathbb{C}^{3k+2} \). We shall prove that \( X_{k,r}'' \) is a complex submanifold provided \( r \leq (k+1)/2 \).

There is an injective map
\[ i : \text{Hol}_{k,r}^* (\mathbb{C}P^2) \to X_{k,r}'' \]
given by \( i(f) = (R(f), (p_0, p_1, p_2)) \) where \( f = [p_0, p_1, p_2] \) as above, with image \( X_{k,r}'' = \{ (a, f) \in X_{k,r}' : \deg f = k, \text{ ramification index } (f) = r \} \).

To check this, we need only show that in \( (R(f), f) = (a, (p_0, p_1, p_2)), a \) and \( p_0 \) are coprime.

Suppose, to the contrary, that there is an \( x \) with \( a(x) = p_0(x) = 0 \). Then \( p_0(z) = (z-x)p(z) \), and since \( a \) divides \( p_0p_i - p_ip_0 \), it follows that \( p_i(x)p(x) = 0 \) for \( i = 1, 2 \). Since \( p_0 \) has distinct roots, \( p(x) \neq 0 \) so that \( p_i(x) = 0 \) for \( i = 1, 2 \), contradicting the fact that \( p_0, p_1 \) and \( p_2 \) are coprime.

By Lemma 4.3 below, the injective map \( i \) is complex analytic, since the map \( f \mapsto R(f) \) is.

Note that since \( f \in \text{Hol}_{k,r}^* (\mathbb{C}P^2) \) is not ramified at infinity, \( R(f) \) is a polynomial of degree \( r \). The complement of \( X_{k,r}' \) in \( X_{k,r}'' \) is a proper subvariety of \( X_{k,r}'' \), so that if \( X_{k,r}'' \) is a complex submanifold of \( \mathbb{C}^r \times \mathbb{C}^{3k+2} \), so is \( X_{k,r}' \).

To study \( X_{k,r}'' \), we embed it in the trivial holomorphic vector bundle \( \pi : E \to A \), where \( A \) is the open set in \( \mathbb{C}^{r+k} \) given by
\[ A = \{ (a, p_0) \in \mathbb{C}^r \times \mathbb{C}^k : a \text{ and } p_0 \text{ are monic coprime polynomials of degrees } r \text{ and } k \text{ respectively and } p_0 \text{ has no repeated root} \}; \]
\[ E = \{ (a, p_0, p_1, p_2) : (a, p_0) \in A, (p_1, p_2) \in \mathbb{C}^{k+1} \times \mathbb{C}^{k+1} \} \]
and \( \pi \) is the natural projection.
For \((a, p_0) \in A\), let \(T_{(a,p_0)} : \mathbf{C}^{k+1} \to \mathbf{C}^r\) be the linear map which sends a polynomial \(p\) of degree \(\leq k\) (represented by its coefficients) to the remainder of the division of \(p_0p' - p'_0p\) by \(a\). We have:

**Lemma 3.2** Let \((a, p_0, p_1, p_2) \in E\) and \(f = [p_0, p_1, p_2]\). Then \(a|R(f)\) if and only if \(p_1\) and \(p_2\) lie in \(\ker T_{(a,p_0)}\).

**Proof.** With the notation \(p_{ij} = p_ip'_j - p_jp'_i\), we have immediately

\[p_1p_{02} - p_2p_{01} = p_0p_{12}.\]

If \(p_1, p_2 \in \ker T_{(a,p_0)}\), then \(a\) divides the left hand side, and since \(a\) and \(p_0\) are coprime, \(a\) must divide \(p_{12}\). Therefore \(a\) divides \(R(f)\).

The converse is immediate.

Now, note that \(X''_{k,r}\) is the kernel of the morphism of holomorphic vector bundles

\[E = A \times (\mathbf{C}^{k+1})^2 \to A \times (\mathbf{C}^r)^2\]

defined by

\[((a, p_0), (p_1, p_2)) \mapsto ((a, p_0), T_{(a,p_0)}(p_1), T_{(a,p_0)}(p_2))\].

Hence \(X''_{k,r}\) is a complex submanifold of \(\mathbf{C}^r \times \mathbf{C}^{k+2}\) if \(\dim \ker T_{(a,p_0)}\) is independent of \((a, p_0) \in A\).

**Lemma 3.3** If \(r \leq \frac{k + 1}{2}\), \(\dim \ker T_{(a,p_0)} = k + 1 - r\) \(\forall (a, p_0) \in A\).

**Proof.** Let the zeros of \(a\) be \(\alpha_1, \ldots, \alpha_R\) with multiplicities \(m_1, \ldots, m_R\), so that \(\sum_{J=1}^R m_J = r\).

For any \(p\), set \(h(p) = p_0p' - p'_0p\). Then \(p \in \ker T_{(a,p_0)}\) iff

\[(h(p))^{(I)}(\alpha_J) = 0 \quad \forall J = 1, \ldots, R, \ I = 0, \ldots, m_J - 1\]  \hspace{1cm} (2)

where \((h(p))^{(I)}\) denotes the \(I^{\text{th}}\) derivative of the polynomial \(h(p)\).

Now (2) is a system of \(r\) linear equations in \(k + 1\) unknowns. Indeed, we can replace \(T_{(a,p_0)}\) by the linear map \(\mathbf{C}^{k+1} \to \mathbf{C}^r\) which sends \(p \in \mathbf{C}^{k+1}\) to the vector

\[
((h(p))^{(I)}(\alpha_J), J = 1, \ldots, R, I = 0, \ldots, m_J - 1) \in \mathbf{C}^r.
\]

We shall show that this map has rank \(r\) by finding \(r\) polynomials \(P_{K,L} \in \mathbf{C}^{k+1}\) \((L = 1, \ldots, R, K = 1, \ldots, m_L)\) whose images are linearly independent.

To do this, choose for \(P_{K,L}\) a polynomial of degree \(\leq k\) with roots \(\alpha_L\) of multiplicity \(K\) and \(\alpha_J\) (for \(J \neq L\)) of multiplicity \(m_J + 1\).

This is possible since

\[m_L + \sum_{J \neq L} (m_J + 1) = r + R - 1 \leq 2r - 1 \leq k\]

\[m_L + \sum_{J \neq L} (m_J + 1) = r + R - 1 \leq 2r - 1 \leq k\]
by the hypothesis \( r \leq (k + 1)/2 \).

Then
\[
(h(P_{K,L}))^{(I)}(\alpha_J) = \begin{cases} 
0 & \text{if } J \neq L \\
\text{non zero} & \text{or } J = L \text{ and } I < K - 1 \\
& \text{if } J = L, I = K - 1
\end{cases}
\]

If we order the components of the vector \( (h(p))^{(I)}(\alpha_J) \) in lexicographical order, viz.
\[(J, I) = (1, 0), (1, 1), \ldots, (1, m_1 - 1), (2, 0), \ldots, (R, m_R - 1),\]
we observe that the matrix
\[(h(P_{K,L}))^{(I)}(\alpha_J)\]
is in echelon form.

Thus, the images of the \( P_{K,L} \) are linearly independent, which shows that the rank of \( T_{(a,p_0)} \) is \( r \), and so its kernel has dimension \( k + 1 - r \).

**Remark 3.4** An example, obtained in conjunction with M. Guest, shows this lemma to be false for \( k = 6, r = 4 \). Namely, when
\[p_0 = 4z^6 - 12z^5 + 10z^4 + 2z^2 - 4z + 4\]
and
\[a = z(z - 1)(z + 1)(z - 2),\]
the kernel of \( T_{(a,p_0)} \) is of dimension 4, instead of 3.

We deduce from the lemma that \( X''_{k,r} \), and so \( X'_{k,r} \), is a complex submanifold of dimension \( 3k - r + 2 \).

Now the restriction of the projection \( X'_{k,r} \to \text{Hol}'_{k,r}(CP^2) \) which forgets \( a \) is complex analytic and has image \( \text{Hol}'_{k,r}(CP^2) \) (indeed, it is the inverse of the map \( i : \text{Hol}'_{k,r}(CP^2) \to X''_{k,r}, f \mapsto (R(f), f)) \).

Thus \( \text{Hol}'_{k,r}(CP^2) \) is a complex analytic submanifold of \( C^{3k+r+2} \), which concludes the proof of Proposition 3.1.

### 4 The smooth nature of \( G'_{k,r} \)

In this section we shall prove, with notation as in §1,

**Proposition 4.1** For \( 0 \leq r \leq \frac{k + 1}{2} \) and for \( \frac{4k - 11}{3} \leq r \leq \frac{3}{2}k - 3 \), \( G'_{k,r} : \text{Hol}^*_{k,r}(CP^2) \to C^j(S^2, CP^2) \) is a \( C^\infty \) map between \( C^\infty \) manifolds, for any \( 2 \leq j < \infty \).

**Lemma 4.2** Let \( g_t \) and \( h_t \) be two families of polynomials in a single (complex) variable which depend smoothly (resp. complex analytically) on a parameter \( t \in U \subseteq \mathbb{R}^N \) (resp. \( \mathbb{C}^N \)), where \( U \) is open. Suppose that the degrees of \( g_t, h_t \) and of their highest common factor \( l_t \) are all constant, i.e. do not vary with \( t \). Then the polynomial \( l_t \) depends smoothly (resp. complex analytically) on \( t \).
Remark 4.3 We can take the polynomial $l_t$ to be monic; then the statement of the lemma means that the remaining coefficients depend smoothly (or complex analytically) on $t$.

Remark 4.4 In the situation of Lemma 4.2, each root of the polynomials depends continuously on $t$, but not smoothly, in general, when the multiplicity changes. So the linear factors of the polynomials do not always vary smoothly.

Proof of Lemma 4.2. It is sufficient to prove the lemma for $t$ close to a fixed $t_0 \in U$. We can suppose $\deg l_t > 0$, or there is nothing to prove.

First case: Suppose that for one value of $t$, $g_t$ divides $h_t$ or $h_t$ divides $g_t$. If for instance $g_t$ divides $h_t$, we have $l_t = a(t)g_t$, where $a(t)$ is a scalar function. Since all degrees are constant, we see that, for all $s$, $\deg l_s = \deg g_s$, so that again $l_s = a(s)g_s$, and $l_s$ is smooth.

Second case: For each $t$, $g_t$ (resp. $h_t$) does not divide $h_t$ (resp. $g_t$). In particular, $g_t$ and $h_t$ are not proportional.

For brevity of notation we shall now omit the parameter $t$ from the notation.

Claim 1. There exist unique polynomials $\lambda$ and $\mu$ with

$$\deg \lambda < \deg \frac{h}{l} \quad \text{and} \quad \deg \mu < \deg \frac{g}{l} \quad (3)$$

such that $\lambda g + \mu h = l$.

The Euclidean algorithm ensures the existence of $\lambda$ and $\mu$ such that $\lambda g + \mu h = l$. Suppose $\deg \lambda \geq \deg \frac{h}{l}$ and let $\tilde{\lambda}$ be the unique polynomial such that

$$\lambda = q \cdot \frac{h}{l} + \tilde{\lambda},$$

with $\tilde{\lambda} = 0$ or $\deg \tilde{\lambda} < \deg \frac{h}{l}$.

If $\tilde{\lambda} = 0$, we have $\lambda = q \cdot \frac{h}{l}$ and $\lambda g + \mu h = 1$ becomes $(qg/l + \mu)h = l$, which is impossible with $l \neq 0$ and $\deg l < \deg h$.

If, instead, $\tilde{\lambda} \neq 0$, and $\deg \tilde{\lambda} < \deg \frac{h}{l}$, we have $\tilde{\lambda} = \lambda - q \cdot \frac{h}{l}$.

Setting $\tilde{\mu} = \mu + q \cdot g/l$, we see that

$$\tilde{\lambda} g + \tilde{\mu} h = l$$

and

$$\deg \tilde{\mu} + \deg h = \deg \tilde{\lambda} + \deg g.$$

This implies

$$\deg \tilde{\mu} < \deg h - \deg l + \deg g - \deg h = \deg \frac{g}{l}.$$

Thus we have existence of $\tilde{\lambda}$ and $\tilde{\mu}$ satisfying (3). Unicity is easily checked.

Claim 2. Let $\lambda$ and $\mu$ be polynomials satisfying (3) and such that $\deg(\lambda g + \mu h) \leq \deg l$. Then $\lambda g + \mu h = a \cdot l$, with $a(t)$ a scalar function.
Since \( l \) divides \( q \) and \( h \), we have \( \text{deg}(\lambda g + \mu h) \geq \text{deg} \, l \) or \( \lambda g + \mu h = 0 \).

In the second case, we have \( \lambda g/l = -\mu h/l \), and by (3) \( g/l \) must have a common factor with \( h/l \), a contradiction.

Therefore, \( \text{deg}(\lambda g + \mu h) = \text{deg} \, l \). With the hypotheses of the claim, we have \( \text{deg}(\lambda g + \mu h) = \text{deg} \, l \) and \( l \) divides \( \lambda g + \mu h \) so that \( \lambda g + \mu h = a \cdot l \).

We conclude from the two claims that for \( \lambda \) and \( \mu \) satisfying (3), \( l \) is characterized up to a non-zero scalar factor as the unique polynomial of the form \( \lambda g + \mu h \) such that \( \text{deg}(\lambda g + \mu h) \leq \text{deg} \, l \equiv L \).

Writing the parameter \( t \) back in, this is equivalent to \( \frac{d^{L+1}}{dt^{L+1}}(\lambda t g_t + \mu_t h_t) = 0 \), a system of homogeneous linear equations in the unknown coefficients of \( \lambda_t \) and \( \mu_t \), with coefficients smooth in \( t \).

At \( t = t_0 \), consider any non-zero coefficient of \( \lambda_t \) or \( \mu_t \) and scale the solution by setting the coefficient equal to 1 for \( t \) close to \( t_0 \). The system becomes inhomogeneous, and can be solved by Cramer’s rule, so that the solutions are smooth, which proves Lemma 4.2.

**Lemma 4.5** Let \( g_t, h_t \) and \( k_t \) be three families of polynomials in a single complex variable which depend smoothly (resp. complex analytically) on a parameter \( t \in U \subseteq \mathbb{R}^N \) (resp. \( \mathbb{C}^N \)). Suppose that the degrees of \( g_t \) and of the highest common factor \( l_t \) of \( g_t, h_t, k_t \) are constant and that \( \text{deg} \, h_t \leq \text{deg} \, g_t \) and \( \text{deg} \, k_t \leq \text{deg} \, g_t \) for all \( t \). Then \( l_t \) depends smoothly (resp. complex analytically) on \( t \).

**Proof.** The idea of the proof is to replace \( g_t, h_t \) and \( k_t \) by linear combinations \( \tilde{g}_t, \tilde{h}_t \) and \( \tilde{k}_t \), so that the common factor remains the same, but any two of the three polynomials have no further common factor.

First, replace \( h_t \) by \( h_t + a \cdot g_t \) and \( k_t \) by \( k_t + b \cdot g_t \) so that the three polynomials (still denoted by \( g_t, h_t \) and \( k_t \)) all have the same constant degree.

Consider now a fixed value of the parameter — say \( t = 0 \). We shall show that there exists \( \epsilon > 0 \) such that \( l_t \) is smooth for \( ||t|| < \epsilon \).

Let \( A \) be a 3 by 3 matrix, which we shall choose close to the identity matrix \( I \), and in particular invertible. Set

\[
\begin{pmatrix}
\tilde{g}_t \\
\tilde{h}_t \\
\tilde{k}_t
\end{pmatrix} = A \begin{pmatrix}
g_t \\
h_t \\
k_t
\end{pmatrix}.
\]

(4)

For \( ||t|| < \epsilon_1 \), all roots of the three polynomials move in a compact set of \( \mathbb{C} \), since the degrees are constant.

If \((a'_1, \ldots, a'_r)\) are the roots common to \( g_t, h_t \) and \( k_t \), they are also the roots common to \( \tilde{g}_t, \tilde{h}_t \) and \( \tilde{k}_t \). We shall now study the common roots of two (but not three) of these polynomials.

Suppose that \( \alpha \) is a root of \( g_0 \) and \( \beta \) a root of \( h_0 \), with \( \alpha \neq \beta \). By continuity of the roots of a family of polynomials, for \( A - I \) small enough, the corresponding roots of \( \tilde{g}_0 \) and \( \tilde{h}_0 \) remain distinct. Applying this remark a finite number of times, we see that any pair of distinct roots of the polynomials remain distinct after transformation by \( A \), when \( A - I \) is small enough.
Consider now a complex number $b$ which is a root of $g_t$ and $h_t$, but not $k_t$, so that $g_t(b) = h_t(b) = 0$ and $k_t(b) = B \neq 0$.

Then for $\theta \in \mathbb{C}$ small enough, replace $g_0$ by $g_0 + \theta k_0$. We see that $g_0 + \theta k_0$ and $h_0$ do not any more have the common root $b$. By the preceding remark no new common root has been created.

Note that the same applies if $b$ is one of the $a_0^r$, by which we mean that $b$ is a root of the three polynomials, of order $m + n$ for $g_0$ and $h_0$ and of order $m$ for $k_0$, with $m \geq 1$, $n \geq 1$. Indeed, in this case,

$$g_0(z) = (z - b)^m(z - b)^n \tilde{g}(z),$$

$$k_0(z) = (z - b)^m \tilde{k}(z)$$

and

$$(g_0 + \theta k_0)(z) = (z - b)^m((z - b)^n \tilde{g}(z) + \tilde{k}(z)),$$

the last factor being non zero at $b$.

Thus $b$ is not any more a common root of $g_0 + \theta k_0$ and $h_0$, except for the multiplicity of the root in all three polynomials.

Repeating this argument a finite number of times, we can replace $g_0, h_0$ and $k_0$ by three new polynomials given by

$$\begin{pmatrix} \tilde{g}_0 \\ \tilde{h}_0 \\ \tilde{k}_0 \end{pmatrix} = A \begin{pmatrix} g_0 \\ h_0 \\ k_0 \end{pmatrix}$$

in such a way that the only roots common to two of them are in fact common to all three.

Defining $\tilde{g}_t, \tilde{h}_t$ and $\tilde{k}_t$ by $[\mathbb{P}]$ for the same matrix $A$, we see that for $||t||$ small enough the roots have the same property.

Thus, the common factor $l_t$ of $g_t, h_t$ and $k_t$ is also the common factor of, for example, $g_t$ and $h_t$. By Lemma 4.2, it varies smoothly (resp. complex analytically) with $t$.

**Proof of Proposition 4.1.** Let $f \in \text{Hol}_{k,r}^*(\mathbb{C}P^2)$. Then identifying $S^2$ with $\mathbb{C} \cup \{\infty\}$ by stereographic projection, $f$ can be represented (at least on $\mathbb{C} \subset S^2$) by a map $p : \mathbb{C} \to \mathbb{C}^3 \setminus \{0\}$ with $p(z) = (p_0(z), p_1(z), p_2(z))$, $(z \in \mathbb{C})$, a triple of polynomials with no common zeros and with max (degree $p_0$, degree $p_1$, degree $p_2$) = $k$. Now since $f$ has a finite number of ramification points (in fact no more than $r$) we can choose the pole of the stereographic projection such that none of them is at $\infty$, then let the ramification points be $z_1, \ldots, z_t \in \mathbb{C}$ with ramification indices $k_1, \ldots, k_t$. Note that $\sum_{i=1}^t k_i = r$.

Next, the first associated curve (see §2) $f_{(1)}$ is represented by $q = p \wedge p' : \mathbb{C} \to \Lambda^2 \mathbb{C}^3 \equiv \mathbb{C}^3$. This is a triple of polynomials $q = (g, h, k)$ and it is easily seen that, since $f$ is not ramified at infinity, the maximum of the degrees of $f, g, k$ is equal to $2k - 2$. Further $(g, h, k)$ have a common zero at $z_i$ of order $k_i$ if and only if $z_i$ is a ramification point of $f$ of ramification index $k_i$ so that the highest common factor of $(g, h, k)$ is the ramification divisor of $f$ given by $R(z) = \prod_{i=1}^t (z - z_i)^{k_i}$, a polynomial of degree $r$. Then $q/R$ is a triple of polynomials with no common roots and, for $z \in \mathbb{C}$, $f_{(1)}(z)$ is the 2-plane spanned by $q(z)/R(z)$.
Now suppose that \( f_t \in \text{Hol}^*_k, r(CP^2) \) is a family of holomorphic maps depending smoothly on a parameter \( t \in U \subset \mathbb{R}^N \). Then we can choose a family of polynomial maps \( p_t : C \to \mathbb{C}^3 \setminus \{0\} \) as above representing \( f_t \) and depending smoothly on \( t \) with no ramification point at infinity.

Here we use the fact that the ramification points, being roots of polynomials, vary continuously with \( t \).

Since the total ramification stays constant (= \( r \)), the ramification divisor \( R_t \) of \( f_t \) has constant degree and we can apply Lemma 4.5 to see that \( R_t \) depends smoothly on \( t \). Hence the corresponding \( q_t/R_t \) depends smoothly on \( t \) and so does \( (f_t)^{(1)} \). Since \( \varphi_t = f_t^{\perp} \cap f_t^{(1)} \) it is clear that this too varies smoothly with \( t \) and the proposition is proven.

**Proof of Theorem 1.3.** Similarly, Lemma 4.5 allows us to conclude that the passage from \( f \) to its conjugate polar is a complex analytic map from \( \text{Hol}^*_k, r(CP^2) \to \text{Hol}^*_k', r'(CP^2) \), and that the same applies to its inverse.

## 5 The diffeomorphic nature of \( G'_{k,r} \)

In this section, we complete the proof of Theorem 1.1. First we show

**Lemma 5.1** For \( k \geq 2 \), \( 0 \leq r \leq \frac{k+1}{2} \) and \( k \geq 3 \), \( \frac{4k-11}{3} \leq r \leq \frac{3}{2}k-3 \) \( G'_{k,r} : \text{Hol}^*_k, r(CP^2) \to \mathcal{C}^j(S^2, CP^2) \) has injective differential at all points \( f_0 \in \text{Hol}^*_k, r(CP^2) \), for any \( j \geq 2 \).

**Proof.** Let \( f_t \in \text{Hol}^*_k, r(CP^2) \) be a family of holomorphic maps depending smoothly on a real parameter \( t \in (-\epsilon, \epsilon) \), \( \epsilon > 0 \), i.e. a smooth curve in \( \text{Hol}^*_k, r(CP^2) \). Then, by Proposition 1.1 and the proof of Theorem 1.3, \( \varphi_t = G'_{k,r}(f_t) \) and \( g_t \) (the polar of \( f_t \)) are smooth curves in \( \mathcal{C}^j(S^2, CP^2) \). Working on a coordinate domain \((U,z)\), let \( F_t, \Phi_t, G_t : U \to \mathbb{C}^3 \setminus \{0\} \) be families of smooth maps representing \( f_t, \phi_t, g_t \) respectively with \( F_t \) holomorphic and \( G_t \) antiholomorphic. Then for each \( z \in U \), \( t \in (-\epsilon, \epsilon) \), \( \{F_t(z), \Phi_t(z), G_t(z)\} \) is a Hermitian orthogonal basis of \( \mathbb{C}^3 \). Now \( \frac{d\varphi_t}{dt} = \pi \left( \frac{d\Phi_t}{dt} \right) \) (where, as before, \( \pi : \mathbb{C}^3 \setminus \{0\} \to CP^2 \) is the canonical projection). Suppose that \( \frac{d\varphi_t}{dt} = 0 \) for some value of \( t \). Then \( \frac{d\Phi_t}{dt} \) must be in direction \( \Phi_t \), so that, in particular,

\[ < \frac{d\Phi_t}{dt}, F_t > = 0. \]

(Here \( <,> \) denotes the standard Hermitian inner product on \( \mathbb{C}^3 \)). This last equation is equivalent to

\[ < \frac{dF_t}{dt}, \Phi_t > = 0, \]

hence

\[ \frac{dF_t}{dt} = \alpha F_t + \beta G_t \]  \hspace{1cm} (5)
for some smooth functions $\alpha, \beta$ on $U \times (-\epsilon, \epsilon)$. Differentiating with respect to $\zeta$, since $F_t$ is holomorphic we obtain
\[
0 = \frac{\partial \alpha}{\partial \zeta} F_t + \frac{\partial \beta}{\partial \zeta} G_t + \beta \frac{\partial G_t}{\partial \zeta}.
\]
Now the triple $\{F_t, \frac{\partial G_t}{\partial \zeta}, G_t\}$ is linearly independent except at the isolated points where $h_t = \bar{g}_t$ is ramified. Hence $\beta \equiv 0$ and so from (3)
\[
\frac{dF_t}{dt} = \alpha F_t
\]
which implies that $df_t/dt = 0$. The lemma follows.

The proof of Theorem 1.1 is completed by

**Proposition 5.2** For $k \geq 2, 0 \leq r \leq \frac{k+1}{2}$ and $k \geq 3, \frac{4k-11}{3} \leq r \leq \frac{3}{2}k-3$
\[G'_{k,r} : \text{Hol}_{k,r}^*(CP^2) \to C^j(S^2, CP^2)\]

is an embedding with image the closed submanifold $\text{Harm}_{k-2-r, 3k-2-r}(CP^2)$ of $C^j(S^2, CP^2)$ for any $j \geq 2$.

**Proof.** Since $\text{Hol}_{k,r}^*(CP^2)$ is finite dimensional, the differential of $G'_{k,r}$ splits at each point and so by Proposition 2.5 it is injective and has image $\text{Harm}_{k-2-r, 3k-2-r}(CP^2)$, and we show below that it is a homeomorphism onto its image. Thus it is an embedding and its image is a closed submanifold of $C^j(S^2, CP^2)$. The dimension of the space $\text{Harm}_{k-2-r, 3k-2-r}(CP^2)$ is thus equal to the (real) dimension of $\text{Hol}_{k,r}^*(CP^2)$ which is $6k - 2r + 4$; the stated dimensions follow easily.

To show that $G'_{k,r}$ is a homeomorphism it suffices to show that it is proper. To do this, following [1, Lemma 3.3] consider a sequence $(\phi_n)$ which converges to $\phi$ in $G'_{k,r}(\text{Hol}_{k, r}^*(CP^2)) = \text{Harm}_{k-2-r, 3k-2-r}(CP^2)$. Let $f_n = (G'_{k,r})^{-1}(\phi_n) \in \text{Hol}_{k,r}^*(CP^2)$. It suffices to prove that a subsequence converges in $\text{Hol}_{k,r}^*(CP^2)$. Note first that $\text{Hol}_{k,r}^*(CP^2)$ can be injected into the projective space $P(C^{3k-3})$ by the map $i : f \mapsto$ projective class of the coefficients of the polynomials describing $f$. Since that space is compact, a subsequence of $(i(f_n))$ converges to $[p] = [p_0, p_1, p_2]$. (Note that $(p_0, p_1, p_2)$ need not be coprime and that their maximal degree need not be $k$.) We retain the notation $i(f_n)$ for the subsequence. We can then write $[p] = [b_{q_0}, b_{q_1}, b_{q_2}]$ where the $q_i$'s are coprime and $[q] = [q_0, q_1, q_2]$ lies in $\text{Hol}_{k-m}(CP^2)$, the space of not necessarily full holomorphic maps from $S^2$ to $CP^2$ of degree $k - m$, for some $m \geq 0$. For each $n$, let $h_n$ be the conjugate polar of $f_n$ belonging to $\text{Hol}_{2k-2-r}^*(CP^2)$, and consider its image by $i$ in the appropriate projective space of coefficients $P(C^{6k-3})$. Again, a subsequence converges to $[t] = [t_0, t_1, t_2]$, and we have $[t_0, t_1, t_2] = [a_{s_0}, a_{s_1}, a_{s_2}]$ with the $s_i$'s coprime and of possibly lower degree than $2k - 2 - r$. Now $f_n \perp \overline{g_n}$ for all $n$ so that $q \perp \overline{s}$ and $\psi = (q \oplus \overline{s})$ is well-defined. Further on $S^2 \setminus \{\text{zeros of } a \cup \text{zeros of } b \cup \{\infty\}$, $G'(f_n)(x)$ converges to $\psi(x)$ so that $\phi$ coincides with $\psi$ on a dense set, and therefore everywhere. Then $3k - 2 - r = E(\phi) = E(\psi) = \deg q + \deg s$ with $\deg q \leq k$ and $\deg s \leq 2k - 2 - r$ so that both these inequalities must be equalities and there can be no loss of degree above. Thus $q$ must have degree $k$ and ramification index $r$. Further, since $r \leq 3k/2 - 3$, $q$ is full, otherwise by the Riemann-Hurwitz formula we would have $r = 2k - 2 \geq 3k/2 - 3$. Hence $q \in \text{Hol}_{k,r}^*(CP^2)$ as required.
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