M/G/1 FAULT-TOLERANT MACHINING SYSTEM WITH IMPERFECTION

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Abstract. The internet of things (IoT) is an emerging archetype of technology for the guaranteed quality of services (QoS). The availability of the uninterrupted power supply (UPS) is one of the most challenging criteria in the successful implementation of the service system of IoT. In this paper, we consider a fault-tolerant power generation system of finite operating machines along with warm standby machine provisioning. The time-to-failure for each of the operating and standby machines are assumed to be exponentially distributed. The time-to-repair by the single service facility for the failed machine follows the arbitrary distribution. For modeling purpose, we have also incorporated realistic machining behaviors like imperfect coverage of the failure of machines, switching failure of standby machine, reboot delay, switch over delay, etc. For the evaluation of the explicit expression for steady-state probabilities of the system, the only required input is the Laplace-Stieltjes transform (LST) of the repair time distribution. The step-wise recursive procedure, illustrative examples, and numerical results have been presented for the following different type of repair time distribution: exponential (M), n-stage Erlang (Ern), deterministic (D), uniform (U(a, b)), n-stage generalized Erlang (GErn) and hyperexponential (He^n). Concluding remarks and future scopes have also been included.

1. Introduction. With the advent of latest technologies, the internet of things (IoT) becomes the great boon for human civilization. It requires various independent working systems like computing system, communication system, power supply system, etc. Power supply system is more prone to failure in nature among these and necessary to be fault-tolerant. The power generation failure causes power interruption which affects adversely on the efficiency, quality, and output of the system. Critical fault-tolerant system recurrently must use redundancy if it is required to...
meet extremely high availability requirement levels. The fault-tolerant system can be modeled mathematically as a machine repair problem (MRP). The major tasks of the monitoring system are to detect the fault in the redundant machining system, isolation, reconfiguration so as to prevent a failed machine from adversely affecting the system’s ongoing performance. The notable research works on Markovian modeling of machine repair problem have been done in the past [cf. [4], [21]]. Redundancy reveals an important role in increasing the availability of the machining systems. Standby redundancy in repairable systems has been examined extensively in the past [cf. [11], [6]]. However, almost all researches in machine repair problem consider perfect coverage and perfect switching of available standby machine on failure where operating machines are prone to break down.

When the breakdown of the operating machine is observed, there exists the possibility to replace it with the available standby machine so that the system can operate in spite of the unexpected failure of the operating machine. The switching failure during the transition of the standby machine to the operational mode in case of replacement of a failed operating machine in a machining system cannot be ignored. Few researchers studied machine repair problem with switching failure of standby in the past. [cf. [13], [8], [14]]. In [12], authors computed steady-state availability of a repairable system with standby switching failure. Using Probabilistic Global Search Lausanne (PGSL) method in cost analysis, in [10], an optimal number of standby machines for machine repair system with standby switching failure was determined. In [20], the authors used the concept of switching failure and geometric reneging in machine interference problem with standby provisioning for performance modeling and reliability analysis. [24] considered the repairable system with multiple working vacations and retrial orbit and dealt with various reliability characteristics.

The likelihood that monitoring system’s tasks are accomplished correctly is called fault coverage or simply perfect coverage. These tasks can seldom be done with perfect certainty i.e. there is some chance that at least one of the tasks, identification, isolation or reconfiguration, is not done correctly. The machining system that is subject to some level of uncertainty in the monitoring process is framed as MRP with imperfect fault coverage [cf. [17], [16]]. In machining system, faults may numerous in type and system analysts sometimes unable to identify them or to locate them i.e. the faults are not covered properly and hence unable to recover them. When operating machine fails and covers perfectly, it is replaced with an available standby machine with some switchover time delay. Otherwise, if it is not covered perfectly, it is cleared from the system through the reboot process. Reboot is very instantaneous so that the occurrence of any other event is rare. Reboot delay and switching delay hinder the functionality of the machining system for a moment [cf. [22], [7]]. Researchers evaluated the MTTF and availability of redundant machine repair problem with switching failure and reboot delay [18]. Recently, in [19], authors dealt with the reliability analysis of the fault-tolerant multi-component machining system having multi-warm spares and reboot provisioning. In [15], the authors discussed the coverage factor of various types of failures in the multi-component repairable system.

Stochastic processes with discrete states in continuous time can be transformed into Markov processes by either the use of regeneration point, or Erlang’s method with division into fictitious stages, or the inclusion of the sufficient supplementary variable. Among these, the well-known method of including supplementary
variables, the remaining or elapsed service time, for determining the queue size distribution explicitly is more justifiable. It is observed that the resulting equations simplify considerably when some arbitrary distributions associated with the process have rational Laplace transforms [cf. [1], [5]]. ([2], [3]) used a supplementary variable, the remaining service time, and a recursive method firstly to obtain the steady-state probability distribution of the number of down machines at arbitrary time epoch of a machine interference problem with spares. [23] used a recursive method and supplementary variable technique to develop steady-state analytic solutions of an $M/G/1$ machine repair problem with multiple imperfect coverages. Using the supplementary variable technique, in [9], authors obtained the distribution for the stationary queue at an arbitrary epoch for the $M/G/1$ queue in a multi-phase random environment with disasters.

In this paper, we study a machine repair problem with two operating machines and the provisioning of the standby machine. The practical example of the studied machining system can be observed in the power supply system having two identical generators with the facility of one standby generator of the same capacity. All generators are under the monitor of an automatic switching system which manages the switchover process, the reboot process, activation of generation by self. It also alarms the repair facility on perfect coverage of the fault. The objective of this research article is to provide an alternative solution technique, using the remaining repair time as the supplementary variable, to compute an explicit expression for steady-state probabilities for the machine repair problem that are used to obtain the various reliability measures of the machining system. The method is recursive and can be used for any repair time distribution such as Markovian $(M)$, Erlangian $(Er_n)$, deterministic $(D)$, uniform $(U(a,b))$, generalized Erlang $(GE_n)$ or hyperexponential $(HE_n)$, etc. These distributions have a different coefficient of variation and represent a different type of repair architects like single or multi-phase repair, different type of repair to different type of machine as per its requirement, etc. The mere input required for computation of state probabilities is the Laplace-Stieltjes transform (LST) of the repair time distribution.

There are many research studies on stochastic models of machine repair problem with realistic environments in recent literature base. Only a few of these studies focus on repairable systems with a supplementary variable technique which used the remaining repair time as a supplementary variable. On the survey base, we find that no research article considers the different types of failures, breakdown, imperfect coverage, switching failure simultaneously in two machines system with standby provisioning. Uncertainty in the failure is one of the important issues in the decision criterion. At present, works on the Markovian machine repairing system with imperfect coverage, switching failure, and reboot delay mainly focus on the exponential repair time distribution. However, from a hands-on perspective, the repair time can be an arbitrary random variable. This research gap gives us the motivation to investigate a machine repairing system with imperfect coverage and switching failure and arbitrary distributed repair times. The purpose of this research is threefold and outlined as follows:

(i) we examine the machine repairing system with imperfect coverage of failure, reboot & switchover delay, and switching failure, in which the repair time follows the general distribution

(ii) we propose a step-wise recursive solution procedure to compute the steady-state probability distribution of the number of failed machines in the system.
which are used to compute various system performance measures and reliability measures.

(iii) we perform a comparative analysis of the different repair time distributions and present numerical simulation in detail.

In this paper, we discuss the types of failure which may hinder the performance of the power supply system in terms of availability. The rest of the paper is organized as follows: Section 2 presents a detailed model description with assumptions and notations. In section 3, we present the recursive method for the computation of the steady-state probabilities and availability of the repairable system. Different types of repair time distribution are used in section 4 and an explicit expression for state probabilities are derived. Parametric sensitivity has been discussed in section 5. Conclusions are drawn and future scope is given in section 6.

2. Model description. For the purpose of the modeling, we consider the requirement of 10 MW power supply in the service system and assume that available generators have 5 MW electricity generation capacity. We consider that there are two primary active generators (operating machines) in the system with the provision of one standby generator (standby machine) to produce reliable and uninterrupted power supply for the service system. All primary active generators and standby generator are under the care of one repairman and automatic monitoring device to detect failure or to provide immediate repair. We visualize studied model as machine repair problem (MRP) with two operating machines, one standby machine, and a single repairman. For developing the mathematical model, we consider the following assumptions and notations:

- The running times of the machines between breakdown have an exponential distribution with mean $1/\lambda$.
- The warm standby machine is also prone to failure in an inactive state too with an assumption that its lifetime has an exponential distribution with mean $1/\nu (0 < \nu < \lambda)$.
- Automatic monitoring device detects the failure of the machine(s) with coverage probability $c$. On successful coverage, the failed machine is promptly supplanted with an available standby machine with exponentially distributed switchover time with mean time $1/\sigma$. The switchover may be unsuccessful with failure probability $p$.
- Whenever failure of the machine(s) is not covered successfully, an unsafe failure state of the system, the failed machine is cleared from the system by a reboot process. Reboot delay is assumed to exponentially distributed with parameter $\beta (\beta \gg \mu, \lambda)$. Reboot process is so fast so that the possibility of the occurrence of any other event is very rare.
- In automatic monitoring device, for uninterrupted in the functioning of the system, available standby machine immediately switches in place of the failed operating machine with switching failure probability $q$.
- The failed machine is repairable and is immediately sent to a single repairman for repair on the basis of first come, first served (FCFS) discipline. The repair times are identically and independently distributed random variables (iidrvs) having a probability density function $b(v)$, distribution function $B(v)$ and mean repair time $b_1$. Once a machine is repaired, it is as good as new.
3. Steady-state probabilities and availability. For the modeling and computational purpose, we use the following supplementary variable: \( V \equiv \text{remaining repair time for the failed machines under repair}. \) The state of the system at time \( t \) is given by:

- \( M(t) \equiv \text{number of active operating machines in the system at time} \ t \),
- \( N(t) \equiv \text{number of active standby machine in the system at time} \ t \),
- \( V(t) \equiv \text{remaining repair time for the failed machines under repair at time} \ t \),

and

\( I(t) \equiv \text{state of the system at time} \ t \)

where

\[
I(t) = \begin{cases} 
0, & \text{if the system is in a safe failure state} \\
1, & \text{if the system is in an unsafe failure state} 
\end{cases}
\]

For \( v \geq 0, \ t \geq 0 \), let us define the following state probabilities:

- \( P_{2,1}(v, t) = \text{Prob} \{ M(t) = 2, N(t) = 1, I(t) = 0, v \leq V(t) \leq v + dv \} \)
- \( P_{2,0}(v, t) = \text{Prob} \{ M(t) = 2, N(t) = 0, I(t) = 0, v \leq V(t) \leq v + dv \} \)
- \( P_{1,0}(v, t) = \text{Prob} \{ M(t) = 1, N(t) = 0, I(t) = 0, v \leq V(t) \leq v + dv \} \)
- \( Q_{1,1}(v, t) = \text{Prob} \{ M(t) = 1, N(t) = 1, I(t) = 0, v \leq V(t) \leq v + dv \} \)
- \( R_{O}(v, t) = \text{Prob} \{ \text{imperfect coverage of the failed operating machine}, \ I(t) = 1, v \leq V(t) \leq v + dv \} \)
- \( R_{S}(v, t) = \text{Prob} \{ \text{imperfect coverage of the failed standby machine}, \ I(t) = 1, v \leq V(t) \leq v + dv \} \)

Hence,

\[
P_{2,1}(t) = \int_{0}^{\infty} P_{2,1}(v, t) \, dv; \quad P_{2,0}(t) = \int_{0}^{\infty} P_{2,0}(v, t) \, dv; \quad P_{1,0}(t) = \int_{0}^{\infty} P_{1,0}(v, t) \, dv
\]

\[
Q_{1,1}(t) = \int_{0}^{\infty} Q_{1,1}(v, t) \, dv; \quad R_{O}(t) = \int_{0}^{\infty} R_{O}(v, t) \, dv \quad \text{and} \quad R_{S}(t) = \int_{0}^{\infty} R_{S}(v, t) \, dv
\]

Using above mentioned assumptions and notations, we develop the following state transition diagram in Fig. 1.

Now, relating the states of the \( M/G/1 \) fault-tolerant system with imperfect coverage, reboot delay, switchover delay, switching failure and standby provisioning at time \( t \) and \( t + dt \) in Fig. 1, we obtain following forward differential-difference equations on balancing the inflow and outflow of the rates:

\[
\frac{dP_{2,1}(t)}{dt} = -(2\lambda + \nu) P_{2,1}(t) + P_{2,0}(0, t)
\]

(1)

\[
\left( \frac{\partial}{\partial t} - \frac{\partial}{\partial v} \right) P_{2,0}(v, t) = -2\lambda P_{2,0}(v, t) + \nu c P_{2,1}(t) + b(v) P_{1,0}(0, t) + \beta R_{S}(t) + (1 - p) \sigma Q_{1,1}(t)
\]

(2)

\[
\left( \frac{\partial}{\partial t} - \frac{\partial}{\partial v} \right) P_{1,0}(v, t) = 2\lambda P_{2,0}(v, t) + 2\lambda q P_{2,1}(t) + P \sigma Q_{1,1}(t)
\]

(3)
In steady-state (as $t \to \infty$), we define

$$P_{2,1} = \lim_{t \to \infty} P_{2,1}(t), \quad P_{2,0} = \lim_{t \to \infty} P_{2,0}(t), \quad P_{1,0} = \lim_{t \to \infty} P_{1,0}(t),$$

$$Q_{1,1} = \lim_{t \to \infty} Q_{1,1}(t), \quad R_O = \lim_{t \to \infty} R_O(t), \quad R_S = \lim_{t \to \infty} R_S(t),$$

$$P_{2,1}(v) = \lim_{t \to \infty} P_{2,1}(v,t), \quad P_{2,0}(v) = \lim_{t \to \infty} P_{2,0}(v,t), \quad P_{1,1}(v) = \lim_{t \to \infty} P_{1,1}(v,t),$$

$$Q_{1,1}(v) = \lim_{t \to \infty} Q_{1,1}(v,t), \quad R_O(v) = \lim_{t \to \infty} R_O(v,t), \quad R_S(v) = \lim_{t \to \infty} R_S(v,t)$$

Further, we also define

$$P_{2,1}(v) = b(v)P_{2,1}$$

$$Q_{1,1}(v) = b(v)Q_{1,1}$$

$$R_O(v) = b(v)R_O$$

$$R_S(v) = b(v)R_S$$

From Eq's (1)-(10) and defined steady-state probabilities, we get the following steady-state equation:

$$- (2\lambda + \nu)P_{2,1} + P_{2,0}(0) = 0$$

$$- \frac{d}{dv}P_{2,0}(v) = -2\lambda P_{2,0}(v) + \nu c b(v)P_{1,0}(0) + \beta b(v)R_S + (1 - p)\sigma b(v)Q_{1,1}$$

$$- \frac{d}{dv}P_{1,0}(v) = 2\lambda P_{2,0}(v) + 2\lambda q b(v)P_{2,1} + p\sigma b(v)Q_{1,1}$$

$$- \sigma Q_{1,1} + 2\lambda(1 - q)cP_{2,1} + \beta R_O = 0$$
- $\beta R_O + 2\lambda (1 - q)(1 - c)P_{2,1} = 0$
\hline
$- \beta R_S + \nu(1 - c)P_{2,1} = 0$
\hline
From Eq\textsuperscript{n}s (11), (15), and (16), we have

\begin{align*}
P_{2,0}(0) &= (2\lambda + \nu)P_{2,1} \quad \text{(17)} \\
R_O &= \frac{2\lambda (1 - q)(1 - c)}{\beta} P_{2,1} \quad \text{(18)} \\
R_S &= \frac{\nu(1 - c)}{\beta} P_{2,1} \quad \text{(19)}
\end{align*}

Hence, from Eq\textsuperscript{n} (14) we get

\begin{align*}
Q_{1,1} &= \frac{2\lambda (1 - q)cP_{2,1} + \beta R_O}{\sigma} \\
&= \frac{2\lambda (1 - q)}{\sigma} P_{2,1} \quad \text{(20)}
\end{align*}

Further, define Laplace transform in terms of Laplace variable $s$ for probability density function $b(v)$ of repair times and state probabilities as follows:

\begin{align*}
\tilde{B}(s) &= \int_0^{\infty} e^{-sv} dB(v) = \int_0^{\infty} e^{-sv} b(v)dv, \\
\tilde{P}_{2,0}(s) &= \int_0^{\infty} e^{-sv} P_{2,0}(v)dv,
\end{align*}

\begin{align*}
P_{2,0} &= \tilde{P}_{2,0}(0) = \int_0^{\infty} P_{2,0}(v)dv, \\
&= \int_0^{\infty} e^{-sv} P_{2,0}(v)dv = s\tilde{P}_{2,0}(s) - P_{2,0}(0)
\end{align*}

and

\begin{align*}
\tilde{P}_{1,0}(s) &= \int_0^{\infty} e^{-sv} P_{1,0}(v)dv, \\
&= \int_0^{\infty} e^{-sv} P_{1,0}(v)dv = \int_0^{\infty} P_{1,0}(v)dv,
\end{align*}

\begin{align*}
&= s\tilde{P}_{1,0}(s) - P_{1,0}(0)
\end{align*}

Taking the Laplace-Stieltjes (LST) on both sides of Eq\textsuperscript{n}s (12) & (13), on simplification, we obtain

\begin{align*}
(2\lambda - s) \tilde{P}_{2,0}(s) &= (\nu cP_{2,1} + P_{1,0}(0) + \beta R_S + (1 - p)\sigma Q_{1,1}) \tilde{B}(s) - P_{2,0}(0) \\
- s\tilde{P}_{1,0}(s) &= 2\lambda \tilde{P}_{2,0}(s) + 2\lambda q \tilde{B}(s) P_{2,1} + p\sigma \tilde{B}(s) Q_{1,1} - P_{1,0}(0)
\end{align*}

Now, we develop a recursive method to get the explicit expression $\tilde{P}_{2,0}(0)$ and $\tilde{P}_{1,0}(0)$ in terms of $P_{2,1}$ as follows:

Substituting Eq\textsuperscript{n}s (17), (19), and (20) into Eq\textsuperscript{n} (21), and setting $s = 2\lambda$ in Eq\textsuperscript{n} (21), it follows

\begin{align*}
P_{1,0}(0) &= \frac{2\lambda + \nu - (2\lambda (1 - p)(1 - q) + \nu) \tilde{B}(2\lambda)}{\tilde{B}(2\lambda)} P_{2,1} \quad \text{(23)}
\end{align*}

Substituting Eq\textsuperscript{n}s (20) and (23) into Eq\textsuperscript{n} (22), and setting $s = 0$ in Eq\textsuperscript{n} (22), we have

\begin{align*}
\tilde{P}_{2,0}(0) &= \frac{(2\lambda + \nu)(1 - \tilde{B}(2\lambda))}{2\lambda B(2\lambda)} P_{2,1} \quad \text{(24)}
\end{align*}
Similarly, on differentiating Eq. (21) with respect to $s$ and set $s = 0$ and put $b_1 = -\tilde{B}(1)(0) = -\left(\frac{\partial \tilde{B}(s)}{\partial s}\right)_{s=0}$ in result. From the resulting expression, we derived the value of $P_{2,0}(0)$ in term of $P_{2,1}$ using the expressions in Eqn’s (19), (20), (23), and (24).

\[
\frac{P_{2,0}(0)}{P_{2,1}} = \frac{(2\lambda + \nu)\left(1 - 2\lambda b_1 - \tilde{B}(2\lambda)\right)}{4\lambda^2 B(2\lambda)} P_{2,1}
\]  

(25)

Similarly, on differentiating Eqn. (22) with respect to $s$ and setting $s = 0$ in result and using Eqn’s (20), (23) and (25) we have

\[
\frac{\tilde{P}_{2,0}(0)}{P_{2,1}} = \frac{(2\lambda + \nu)\left(\tilde{B}(2\lambda) - 1 - 2\lambda b_1\right) - 4\lambda^2 b_1 (p(1-q) + q) \tilde{B}(2\lambda)}{2\lambda B(2\lambda)}
\]  

(26)

Since $\tilde{P}_{2,0}(0)$, $\tilde{P}_{1,0}(0)$, $Q_{1,1}$, $R_O$ and $R_S$ are in term of $P_{2,1}$, on substituting Eqn’s (18), (19), (20), (24) and (26) into the probability normalizing condition

\[
P_{2,1} + \tilde{P}_{2,0}(0) + \tilde{P}_{1,0}(0) + Q_{1,1} + R_O + R_S = 1
\]  

(27)

we obtain

\[
P_{2,1} = \frac{\beta \sigma \tilde{B}(2\lambda)}{D_1}
\]  

(28)

where

\[
D_1 = (q (2\lambda (\beta + \bar{c} \sigma) - 2b_1 \beta \lambda \sigma \bar{p}) + \sigma (\beta (2b_1 \lambda + 1) + \nu \bar{c}) \tilde{B}(2\lambda) + \bar{b}_1 \beta \sigma (2\lambda + \nu)
\]

After obtaining the explicit expression for state probability $P_{2,1}$, we can obtain the explicit expression for remaining state probabilities as follow:

\[
\tilde{P}_{2,0}(0) = \frac{\beta \sigma (2\lambda + \nu)(1 - \tilde{B}(2\lambda))}{2\lambda D_1}
\]  

(29)

\[
\tilde{P}_{1,0}(0) = \frac{4\lambda^2 b_1 (q + (1-q)p) \tilde{B}(2\lambda) + (2\lambda + \nu) \left(2b_1 \lambda + \bar{B}(2\lambda) - 1\right) \beta \sigma}{2\lambda D_1}
\]  

(30)

\[
Q_{1,1} = \frac{2\lambda (1-q) \beta \tilde{B}(2\lambda)}{D_1}
\]  

(31)

\[
R_O = \frac{2\lambda \sigma (1-c) (1-q) \tilde{B}(2\lambda)}{D_1}
\]  

(32)

\[
R_S = \frac{\sigma (1-c) \nu \tilde{B}(2\lambda)}{D_1}
\]  

(33)

where $\bar{c}$, $\bar{p}$, and $\bar{q}$ are the complementary probability of $c$, $p$, and $q$ respectively. Since there is a demand of 10 MW power supply, at least two operating machines should function properly. Hence, the availability of the system is defined as

\[
Av = P_{2,1} + \tilde{P}_{2,0}(0)
\]

and its explicit expression can be derived as follows:

\[
Av = \frac{2\lambda + \nu \left(1 - \tilde{B}(2\lambda)\right)}{2\lambda D_1} \beta \sigma
\]  

(34)
For the arbitrary distribution of the repair time, we derive the explicit expression of state probabilities in Eq's (28)-(33) and availability in Eq (34). For standard repair time distribution, there is no need to repeat the recursive procedure completely for the derivation of expression of state probabilities and corresponding availability of the system. We mere require the Laplace-Stieltjes transform (LST) of the repair time distribution.

A recursive algorithm for computing steady-state probabilities is given as follows:

**Recursive algorithm**

1. Initialize $P_{2,1} = 1$.
2. Compute $P_{2,0}(0)$ using the Eq (17).
3. Compute $R_O$ and $R_S$ using the Eq's (18) and (19).
4. Compute $Q_{1,1}$ using the result obtained in step 3 and the Eq (20).
5. Compute $F_{1,0}(0)$ from the Eq (23).
6. Evaluate $\tilde{P}_{2,0}(0)$ using the Eq's (24).
7. Determine the explicit expression for $\tilde{P}_{2,0}(0)$ from Eq (25).
8. Compute $\tilde{F}_{1,0}(0)$ using Eq (26).
9. Using the normalizing condition in Eq(27) and Eq's (28), compute the value $P_{2,1}$.
10. Using the value of $P_{2,1}$ from step 9 and Eq's (29)-(33), evaluate all state probabilities $\tilde{P}_{2,0}(0), \tilde{F}_{1,0}(0), Q_{1,1}, R_O$, and $R_S$.
11. Compute the availability of the system ($Av$) from the Eq (34).

4. Special cases. Based on the flow of the solution procedure described in the previous section 3, the explicit expression for state probabilities and the availability of the system can easily be derived for different continuous distributions of service times. For the described recursive method, we merely require a Laplace-Stieltjes transform (LST) of the repair time distribution. We present the explicit expression of state probability $P_{2,1}$ and the availability of the system $Av$ for exponential $(M)$, n-stage Erlangian $(Er_n)$ and Deterministic $(D)$ distribution.

**Case 1: Exponential distribution.** The repair times follow an exponential distribution with mean rate $\mu$. It is member of gamma distribution. The Laplace-Stieltjes transform of probability density function $b(v) = e^{-\mu v}$ is given by

$$\tilde{B}(s) = \frac{\mu}{s+\mu}$$

Hence, we have $b_1 = -\tilde{B}'(0) = \frac{1}{\mu}$. Succeeding the recursive algorithm discussed in previous section, we have following expression step wise:

1. $P_{2,1} = 1$
2. $P_{2,0}(0) = 2\lambda + \nu$
3. $R_O = \frac{2(1-q)(1-c)\lambda}{\beta}, \quad R_S = \frac{\nu(1-c)}{\beta}$
4. $Q_{1,1} = \frac{2\lambda(1-q)}{\sigma}$
5. $F_{1,0}(0) = \frac{2\lambda[(1-q)p+q]\mu+\nu+2\lambda}{\mu}$
6. $\tilde{P}_{2,0}(0) = \frac{2\lambda+\nu}{\mu}$
Step 7. $\tilde{P}_{2,0}(0) = \frac{-2\lambda - \nu}{\mu^2}$

Step 8. $\tilde{P}_{1,0}(0) = \frac{2\lambda}{\mu^2} \left[ \left( (1-q)p + q \right) \mu + \nu + 2\lambda \right]$

Step 9.

$P_{2,1} = \left[ \frac{\beta \mu^2 \sigma}{2q\lambda \mu ((\hat{c}\mu - \beta \hat{p}) \sigma + \beta \mu) + \sigma ((\mu + 2 \lambda) (\mu + \nu + 2 \lambda) \beta + \mu^2 \nu \tilde{c})} \right]$ (35)

Step 10.

$\tilde{P}_{2,0}(0) = \left[ \frac{\beta \mu \sigma (2\lambda + \nu)}{2q\lambda \mu ((\hat{c}\mu - \beta \hat{p}) \sigma + \beta \mu) + \sigma ((\mu + 2 \lambda) (\mu + \nu + 2 \lambda) \beta + \mu^2 \nu \tilde{c})} \right]$

$\tilde{P}_{1,0}(0) = \left[ \frac{2\beta \sigma \beta \sigma [((1-p)q + p) \mu + \nu + 2\lambda]}{2q\lambda \mu ((\hat{c}\mu - \beta \hat{p}) \sigma + \beta \mu) + \sigma ((\mu + 2 \lambda) (\mu + \nu + 2 \lambda) \beta + \mu^2 \nu \tilde{c})} \right]$

$Q_{1,1} = \left[ \frac{2\mu^2 \beta \lambda (1-q)}{2q\lambda \mu ((\hat{c}\mu - \beta \hat{p}) \sigma + \beta \mu) + \sigma ((\mu + 2 \lambda) (\mu + \nu + 2 \lambda) \beta + \mu^2 \nu \tilde{c})} \right]$

$R_O = \left[ \frac{2\mu^2 \sigma \lambda (1-q)(1-c)}{2q\lambda \mu ((\hat{c}\mu - \beta \hat{p}) \sigma + \beta \mu) + \sigma ((\mu + 2 \lambda) (\mu + \nu + 2 \lambda) \beta + \mu^2 \nu \tilde{c})} \right]$

$R_S = \left[ \frac{2\mu^2 \sigma \nu (1-c)}{2q\lambda \mu ((\hat{c}\mu - \beta \hat{p}) \sigma + \beta \mu) + \sigma ((\mu + 2 \lambda) (\mu + \nu + 2 \lambda) \beta + \mu^2 \nu \tilde{c})} \right]$

Step 11.

$Av = \left[ \frac{\sigma \beta \mu (2\lambda + \mu + \nu)}{2q\lambda \mu ((\hat{c}\mu - \beta \hat{p}) \sigma + \beta \mu) + \sigma ((\mu + 2 \lambda) (\mu + \nu + 2 \lambda) \beta + \mu^2 \nu \tilde{c})} \right]$ (36)

Case 2: n-stage Erlangian distribution. The repair time of failed machines has a n-stage Erlang distribution with shape parameter $n$ and rate $\mu$. It is also a member of gamma distribution and the sum of $n$ independent exponential variables with mean $1/\mu$ each i.e. repair is done in $n$ stages with mean repair rate $\mu$. In this case, we have

$\tilde{B}(s) = \left( \frac{n\mu}{s + n\mu} \right)^n$

and $b_1 = -\tilde{B}^{(1)}(0) = \frac{1}{\mu}$. For the explicit expression for state probability $P_{21}$ and the availability of the system $Av$, we have obtained following step wise results:

Step 1. $P_{2,1} = 1$

Step 2. $P_{2,0}(0) = 2\lambda + \nu$

Step 3. $R_O = \frac{2(1-q)(1-c)\lambda}{\beta}$, $R_S = \frac{\nu(1-c)}{\beta}$

Step 4. $Q_{1,1} = \frac{2\lambda(1-q)}{\beta}$

Step 5. $P_{1,0}(0) = \frac{\beta \sigma \lambda (1-q)(1-c)}{2\mu^2 \beta \lambda (1-q)} + 2(1-q)(p-1)\lambda - \nu$

Step 6. $\tilde{P}_{2,0}(0) = \frac{1}{2} \left( \frac{2\lambda + \nu}{\lambda} \right) \left( 1 - \left( \frac{r\mu}{\mu r + 2\lambda} \right)^r \right)$
Step 7. \( \tilde{P}_{2,0}(0) = \frac{1}{4} \left( 2\lambda + \nu \right) \left( 1 - \left( \frac{r_{\mu}}{\mu r + 2\lambda} \right)^r - \frac{2\lambda}{\mu} \right) \lambda^2 \left( \frac{r_{\mu}}{\mu r + 2\lambda} \right)^r \)

Step 8. \( \tilde{P}_{1,0}(0) = \frac{1}{2} \left( 2\lambda + \nu \right)(\mu - 2\lambda) - \left[ 4\left( 1 - q \right)p + q \right] \lambda^2 + 2\lambda \mu + \mu \nu \left( \frac{r_{\mu}}{\mu r + 2\lambda} \right)^r \lambda \mu \left( \frac{r_{\mu}}{\mu r + 2\lambda} \right)^r \)

Step 9.

\[ P_{2,1} = \frac{\left( \frac{\mu n}{\mu n + 2\lambda} \right)^n}{D_2} \beta \mu \sigma \] (37)

where

\[ D_2 = 2q \left( \frac{\mu n}{\mu n + 2\lambda} \right)^n \lambda \left\{ (\bar{c} - \beta \bar{p}) \sigma + \beta \mu \right\} + \sigma \left( \left( \mu + 2\lambda \right) \beta + \bar{c} \mu \nu \right) \left( \frac{\mu n}{\mu n + 2\lambda} \right)^n + \beta \left( 2\lambda + \nu \right) \]

Step 10.

\[ \tilde{P}_{2,0}(0) = \frac{1}{4} \mu \beta \sigma (2\lambda + \nu) \left( 1 - \left( \frac{r_{\mu}}{\mu r + 2\lambda} \right)^r \right) \]

\[ \tilde{P}_{1,0}(0) = \frac{\left[ \sigma \beta \left\{ (1 - p)q + p \right\} \lambda^2 + \frac{1}{2} \lambda \mu + \frac{1}{2} \mu \nu \right] \left( \frac{r_{\mu}}{\mu r + 2\lambda} \right)^r - \frac{1}{4} (2\lambda + \nu)(\mu - 2\lambda) }{D_2} \]

\[ Q_{1,1} = \frac{2\lambda \beta \mu (1 - q) \left( \frac{r_{\mu}}{\mu r + 2\lambda} \right)^r}{D_2}, \quad R_O = \frac{2\lambda \sigma \mu (1 - q)(1 - c)}{D_2} \]

\[ R_S = \frac{\nu \sigma \mu (1 - c) \left( \frac{r_{\mu}}{\mu r + 2\lambda} \right)^r}{D_2} \]

Step 11.

\[ A_v = \frac{\sigma \beta \left( 2\lambda + \left( 1 - \left( \frac{\mu n}{\mu n + 2\lambda} \right)^n \right) \nu \right) \mu}{2\lambda D_2} \] (38)

Case 3: Deterministic distribution. Also known as degenerate distribution and takes only single value. The repair time has a deterministic distribution with mean rate \( \mu \). In this case, we have LST of the corresponding PDF as follows:

\[ \tilde{B}(s) = e^{-\left( \frac{\lambda}{\mu} \right)} \]

and \( b_1 = -\tilde{B}^{(1)}(0) = \frac{1}{\mu} \). The step wise recursive expression as per algorithm discussed in previous section are as follows:

Step 1. \( P_{2,1} = 1 \).

Step 2. \( P_{2,0}(0) = 2\lambda + \nu \)

Step 3. \( R_O = \frac{2(1 - q)(1 - c)\lambda}{\beta}, \quad R_S = \frac{\mu(1 - c)}{\beta} \)

Step 4. \( Q_{1,1} = \frac{2\lambda (1 - q)}{\sigma} \)
Step 5. \[ P_{1,0} = \left( \frac{2(\lambda + \nu)}{e^{-\frac{2\lambda}{\nu}}} - 2(1-q)(1-p)\lambda - \nu \right) \]

Step 6. \[ \tilde{P}_{2,0}(0) = \frac{2\lambda e^{-\frac{2\lambda}{\nu}}}{2\lambda e^{-\frac{2\lambda}{\nu}} \left( 1 - e^{-\frac{2\lambda}{\nu}} \right)} \]

Step 7. \[ \tilde{P}_{2,0}^{(1)}(0) = \frac{4\lambda^2 e^{-\frac{2\lambda}{\nu}}}{(2\lambda + \nu) \left( 1 - e^{-\frac{2\lambda}{\nu}} \right) - e^{-\frac{2\lambda}{\nu}} - (2\lambda + \nu)(2\lambda + \mu)} \]

Step 8. \[ \tilde{P}_{1,0}^{(1)}(0) = \frac{(4(1-q)p + 4q)\lambda^2 + 2\lambda\mu + \mu\nu}{2\lambda\mu e^{-\frac{2\lambda}{\nu}}} \]

Step 9. \[ P_{2,1} = \frac{e^{-\frac{2\lambda}{\nu}} \beta \mu \sigma}{D_3} \quad (39) \]

where \[ D_3 = 2q e^{-\frac{2\lambda}{\nu}} \left( (\epsilon \mu - \beta \bar{p}) \sigma + \beta \mu \right) \]

\[ + \sigma \left( (\mu + 2\lambda) \beta + \epsilon \mu \nu \right) e^{-\frac{2\lambda}{\nu}} + \beta (2\lambda + \nu) \]

Step 10. \[ \tilde{P}_{2,0}(0) = \frac{\frac{1}{4} \left( \mu \beta \sigma (2\lambda + \nu) \left( 1 - e^{-\frac{2\lambda}{\nu}} \right) \right)}{D_3} \]

\[ \tilde{P}_{1,0}(0) = \frac{\left( \frac{1}{4} \left( (1-p)q + p \right) \lambda^2 + \frac{\lambda \mu}{2} + \frac{\mu \nu}{4} \right) e^{-\frac{2\lambda}{\nu}} - \frac{1}{4} (2\lambda + \nu)(\mu - 2\lambda) \} \sigma \beta}{D_3} \]

\[ Q_{1,1} = \frac{2\lambda(1-q)\beta \mu e^{-\frac{2\lambda}{\nu}}}{D_3}, \quad R_O = \frac{2\lambda(1-q)(1-c)\mu^2 \sigma}{D_3}, \quad R_S = \frac{\nu(1-c)\sigma \mu e^{-\frac{2\lambda}{\nu}}}{D_3} \]

Step 11. \[ A_v = \frac{\sigma \beta \left( 2\lambda + \left( 1 - e^{-\frac{2\lambda}{\nu}} \right) \nu \right) \mu}{2\lambda D_3} \quad (40) \]

Table 1. State probabilities and availability of the system

| Distribution | $M$ | $Er_3$ | $D$ | $U(a,b)$ | $GE_4$ | $HE_2$ |
|--------------|-----|--------|-----|----------|--------|--------|
| Parameter(s) | $\mu = 25$ | $\mu = 25$ | $\mu = 25$ | $a = 0.02$ | $\mu_1 = 60$ | $\alpha_1 = 0.2$ |
|              | $\mu = 25$ | $\mu = 25$ | $\mu = 25$ | $b = 0.06$ | $\mu_2 = 100$ | $\alpha_2 = 0.8$ |
|              | $\mu = 120$ | $\mu = 120$ | $\mu = 120$ | $\mu_3 = 120$ | $\mu_1 = 15$ | $\mu_4 = 200$ |
|              | $\mu = 120$ | $\mu = 120$ | $\mu = 120$ | $\mu_4 = 200$ | $\mu_2 = 30$ |        |
| $P_{21}$     | 0.9123644  | 0.9123430 | 0.9123320 | 0.9123347 | 0.9123417 | 0.9123711 |
| $P_{20}$     | 0.0437935  | 0.0443790 | 0.0446796 | 0.0446037 | 0.0444134 | 0.0436102 |
| $P_{10}$     | 0.0371515  | 0.0365876 | 0.0362980 | 0.0363711 | 0.0365544 | 0.0373280 |
| $Q_{1,1}$    | 0.0054742  | 0.0054741 | 0.0054740 | 0.0054740 | 0.0054740 | 0.0054742 |
| $R_O$        | 0.0007299  | 0.0007299 | 0.0007299 | 0.0007299 | 0.0007299 | 0.0007299 |
| $R_S$        | 0.0004866  | 0.0004866 | 0.0004866 | 0.0004866 | 0.0004866 | 0.0004866 |
| $A_v$        | 0.9561579  | 0.9567219 | 0.9570115 | 0.9569385 | 0.9567551 | 0.9559813 |
M/G/1 FAULT-TOLERANT MACHINING SYSTEM WITH IMPERFECTION

Besides the state probability and availability for these standard repair time distribution, exponential distribution ($M$), $n$-stage Erlangian distribution ($Er_n$) and Deterministic ($D$), we have identified some more standard continuous distribution numerically in next section 5 also since algebraic expressions are very complex and tedious to express.

5. Numerical result. In this section, the availability of the two-operating machine power system with standby machine provisioning has been dealt extensively with different repair time distribution numerically. For numerical simulation purpose, we fix the values of the governing parameters as follows: $\lambda = 0.5, \mu = 25, \beta = 75, \sigma = 50, q = 0.7, c = 0.8, p = 0.9$.

For the comparison and illustration purpose, we consider six different repair time distributions from different distribution families. For study persistence, we set

$\mu$

| $P_{2,1}$ | $Av$ |
|---|---|
| $M$ | 0.9118, 0.9120, 0.9121, 0.9122, 0.9123, 0.9124, 0.9124, 0.9125, 0.9125, 0.9125 |
| $Er_3$ | 0.9118, 0.9119, 0.9121, 0.9122, 0.9123, 0.9123, 0.9123, 0.9124, 0.9125, 0.9125 |
| $D$ | 0.9118, 0.9119, 0.9121, 0.9122, 0.9123, 0.9123, 0.9123, 0.9124, 0.9125, 0.9125 |

$\mu$

| $P_{2,1}$ | $Av$ |
|---|---|
| $M$ | 0.9556, 0.9557, 0.9559, 0.9560, 0.9561, 0.9562, 0.9562, 0.9563, 0.9563, 0.9564 |
| $Er_3$ | 0.9561, 0.9563, 0.9564, 0.9565, 0.9566, 0.9567, 0.9568, 0.9569, 0.9569, 0.9569 |
| $D$ | 0.9564, 0.9566, 0.9567, 0.9568, 0.9569, 0.9570, 0.9571, 0.9571, 0.9572, 0.9572 |

**Figure 2.** State probability $P_{2,1}$ and availability of the system ($Av$) wrt $\mu$ for different repair time distribution

**Table 2.** Performance indices corresponding to Fig. 2
$b_1 = 0.04$ and fix the corresponding parameter(s) as follows:

- Exponential distribution ($M$): $\mu = 25$
- $n$-stage Erlang distribution ($Er_n$): $n = 3$, $\mu = 25$,
- Deterministic distribution ($D$): $\mu = 25$
- Uniform distribution ($U(a,b)$): $a = 0.02$, $b = 0.06$
- Generalized Erlangian distribution ($GE_n$): $n = 4$, $\mu_1 = 60$, $\mu_2 = 100$, $\mu_3 = 120$, $\mu_4 = 200$
- Hyperexponential distribution ($HE_n$): $n = 2$, $\mu_1 = 15$, $\mu_2 = 30$, $\alpha_1 = 0.2$, $\alpha_2 = 0.8$

For the illustration of the proposed methodology using a supplementary variable, Table 1 comprises the results of steady-state probabilities and availability of the system for six different repair time distributions and their fix parameter(s) value.

From Table 1, it is clearly inferred the order of the availability as $Av_D > Av_{U(a,b)} > Av_{GE_4} > Av_{Er_3} > Av_M > Av_{HE_2}$. As per requirement of the service facility, a system designer can choose appropriate repair time distribution.

Figs. 2-9 or Tables 2-9 summarize the variation in the value of state probability $P_{2,1}$ and availability $Av$ of the system on varying the value of governing parameter $\mu$, $\lambda$, $\nu$, $\beta$, $\sigma$, $p$, $c$, and $q$ respectively. It is clearly noticeable from Fig. 2 or Table 2 that $P_{2,1}$ and $Av$ of the system increase on increasing the repair rate $\mu$ and the order of the availability is $Av_D > Av_{Er_3} > Av_M$. Deterministic repair time distribution is recommended for designing the service system but practically service in phases is more suitable.

Fig. 3 or Table 3 depicts the variation of the $P_{2,1}$ and $Av$ with respect to the failure rate of an operating machine $\lambda$. Both performance indices are decreasing on increasing the value of $\lambda$ which is the expected trend and validate our modeling.
Table 3. Performance indices corresponding to Fig. 3

| Indices | Distribution | \( \lambda \) |
|---------|--------------|---------------|
| \( P_{2,1} \) | \( M \) | 0.9749 0.9667 0.9586 0.9507 0.9428 0.9351 0.9274 0.9198 0.9124 |
| | \( E_{r3} \) | 0.9749 0.9667 0.9586 0.9507 0.9428 0.9350 0.9274 0.9198 0.9123 |
| | \( D \) | 0.9749 0.9667 0.9586 0.9507 0.9428 0.9350 0.9274 0.9198 0.9123 |
| | \( U(a,b) \) | 0.9749 0.9667 0.9586 0.9507 0.9428 0.9350 0.9274 0.9198 0.9123 |
| | \( GE_4 \) | 0.9749 0.9667 0.9586 0.9507 0.9428 0.9350 0.9274 0.9198 0.9123 |
| | \( HE_2 \) | 0.9749 0.9667 0.9586 0.9507 0.9428 0.9350 0.9274 0.9198 0.9124 |

\( \text{Av} \)

| Indices | Distribution | \( \lambda \) |
|---------|--------------|---------------|
| \( M \) | 0.9905 0.9860 0.9816 0.9773 0.9730 0.9687 0.9645 0.9603 0.9562 |
| \( E_{r3} \) | 0.9905 0.9861 0.9818 0.9775 0.9732 0.9690 0.9649 0.9608 0.9567 |
| \( D \) | 0.9905 0.9861 0.9818 0.9775 0.9733 0.9691 0.9650 0.9610 0.9569 |
| \( U(a,b) \) | 0.9905 0.9861 0.9818 0.9775 0.9733 0.9691 0.9650 0.9610 0.9569 |
| \( GE_4 \) | 0.9905 0.9861 0.9818 0.9775 0.9732 0.9690 0.9649 0.9608 0.9568 |
| \( HE_2 \) | 0.9905 0.9860 0.9816 0.9772 0.9729 0.9686 0.9644 0.9602 0.9560 |

**Figure 4.** State probability \( P_{2,1} \) and availability of the system \( (\text{Av}) \) wrt \( \nu \) for different repair time distribution and methodology. Significant discrimination of repair time distribution is observed for the higher value of \( \lambda \). Similar observations are depicted in Fig. 4 or Table 4 in which we plot the variation of \( P_{2,1} \) and \( \text{Av} \) with respect to the failure rate of the standby machine \( \nu \). For the high order of availability and the initial state of the system, it is necessary to take measure to avoid the failure of the standby machine in the inactive state.

Fig. 5 or Table 5 summarizes the variability of the \( P_{2,1} \) and \( \text{Av} \) with respect to reboot rate \( \beta \). Slight improvement in the value of \( \text{Av} \) is observed by increasing the rate of the reboot process. It prompts that the reboot process is a necessary action to avoid the hindrance due to the failure of the machine. Fig. 6 or Table 6 depicts the changeability of the state probability and availability with respect to switchover
Figure 5. State probability $P_{2,1}$ and availability of the system $(Av)$ wrt $\beta$ for different repair time distribution

Figure 6. State probability $P_{2,1}$ and availability of the system $(Av)$ wrt $\sigma$ for different repair time distribution
Table 4. Performance indices corresponding to Fig. 4

| Indices | Distribution | $\nu$ | 0.050 | 0.075 | 0.100 | 0.125 | 0.150 | 0.175 | 0.200 | 0.225 | 0.250 |
|---------|--------------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| $P_{2,1}$ | $M$ | 0.9179 | 0.9170 | 0.9161 | 0.9151 | 0.9142 | 0.9133 | 0.9124 | 0.9114 | 0.9105 |
|          | $E_{r3}$ | 0.9179 | 0.9170 | 0.9160 | 0.9151 | 0.9142 | 0.9133 | 0.9123 | 0.9114 | 0.9105 |
|          | $D$ | 0.9179 | 0.9170 | 0.9160 | 0.9151 | 0.9142 | 0.9133 | 0.9123 | 0.9114 | 0.9105 |
|          | $U(a, b)$ | 0.9179 | 0.9170 | 0.9160 | 0.9151 | 0.9142 | 0.9133 | 0.9123 | 0.9114 | 0.9105 |
|          | $G_{E4}$ | 0.9179 | 0.9170 | 0.9160 | 0.9151 | 0.9142 | 0.9133 | 0.9123 | 0.9114 | 0.9105 |
|          | $H_{E2}$ | 0.9179 | 0.9170 | 0.9161 | 0.9151 | 0.9142 | 0.9133 | 0.9123 | 0.9114 | 0.9115 | 0.9105 |

| Indices | Distribution | $\beta$ | 50 | 55 | 60 | 65 | 70 | 75 | 80 | 85 | 90 |
|---------|--------------|---------|---|---|---|---|---|---|---|---|---|
| $P_{2,1}$ | $M$ | 0.9565 | 0.9564 | 0.9564 | 0.9563 | 0.9563 | 0.9562 | 0.9562 | 0.9561 | 0.9561 |
|          | $E_{r3}$ | 0.9570 | 0.9569 | 0.9569 | 0.9568 | 0.9568 | 0.9568 | 0.9567 | 0.9567 | 0.9567 |
|          | $D$ | 0.9572 | 0.9572 | 0.9572 | 0.9571 | 0.9571 | 0.9570 | 0.9570 | 0.9570 | 0.9570 |
|          | $U(a, b)$ | 0.9572 | 0.9571 | 0.9571 | 0.9570 | 0.9570 | 0.9570 | 0.9570 | 0.9570 | 0.9570 |
|          | $G_{E4}$ | 0.9570 | 0.9570 | 0.9569 | 0.9569 | 0.9568 | 0.9568 | 0.9568 | 0.9567 | 0.9567 |
|          | $H_{E2}$ | 0.9563 | 0.9563 | 0.9562 | 0.9562 | 0.9561 | 0.9560 | 0.9560 | 0.9559 | 0.9559 |

Table 5. Performance indices corresponding to Fig. 5

| Indices | Distribution | $\sigma$ | 30 | 35 | 40 | 45 | 50 | 55 | 60 | 65 | 70 |
|---------|--------------|---------|---|---|---|---|---|---|---|---|---|
| $P_{2,1}$ | $M$ | 0.9090 | 0.9102 | 0.9111 | 0.9118 | 0.9124 | 0.9128 | 0.9132 | 0.9135 | 0.9138 |
|          | $E_{r3}$ | 0.9090 | 0.9102 | 0.9111 | 0.9118 | 0.9123 | 0.9128 | 0.9132 | 0.9135 | 0.9138 |
|          | $D$ | 0.9090 | 0.9102 | 0.9111 | 0.9118 | 0.9123 | 0.9128 | 0.9132 | 0.9135 | 0.9138 |
|          | $U(a, b)$ | 0.9090 | 0.9102 | 0.9111 | 0.9118 | 0.9123 | 0.9128 | 0.9132 | 0.9135 | 0.9138 |
|          | $G_{E4}$ | 0.9090 | 0.9102 | 0.9111 | 0.9118 | 0.9123 | 0.9128 | 0.9132 | 0.9135 | 0.9138 |
|          | $H_{E2}$ | 0.9091 | 0.9102 | 0.9111 | 0.9118 | 0.9123 | 0.9128 | 0.9132 | 0.9135 | 0.9138 |

| Indices | Distribution | $\sigma$ | 30 | 35 | 40 | 45 | 50 | 55 | 60 | 65 | 70 |
|---------|--------------|---------|---|---|---|---|---|---|---|---|---|
| $P_{2,1}$ | $M$ | 0.9527 | 0.9539 | 0.9549 | 0.9556 | 0.9562 | 0.9566 | 0.9570 | 0.9574 | 0.9577 |
|          | $E_{r3}$ | 0.9532 | 0.9545 | 0.9554 | 0.9561 | 0.9567 | 0.9572 | 0.9576 | 0.9579 | 0.9582 |
|          | $D$ | 0.9535 | 0.9548 | 0.9557 | 0.9564 | 0.9570 | 0.9575 | 0.9579 | 0.9582 | 0.9585 |
|          | $U(a, b)$ | 0.9535 | 0.9547 | 0.9556 | 0.9564 | 0.9569 | 0.9574 | 0.9578 | 0.9581 | 0.9584 |
|          | $G_{E4}$ | 0.9533 | 0.9545 | 0.9554 | 0.9562 | 0.9568 | 0.9572 | 0.9576 | 0.9580 | 0.9583 |
|          | $H_{E2}$ | 0.9525 | 0.9537 | 0.9547 | 0.9554 | 0.9560 | 0.9565 | 0.9569 | 0.9572 | 0.9575 |

rate $\sigma$. For a high value of $\sigma$, higher availability and a higher probability of the initial state is observed. This implies that the automatic switchover of the standby machine in place of the failed machine should be prompt.

Figs. 7-9 or Tables 7-9 demonstrate the variation in the value of state probability and availability of the system with respect to probabilities $p$, $c$ and $q$. From these results, we judge how these probabilities are significant in the analysis of the service
Figure 7. State probability $P_{2,1}$ and availability of the system ($Av$) wrt $p$ for different repair time distribution.

Figure 8. State probability $P_{2,1}$ and availability of the system ($Av$) wrt $c$ for different repair time distribution.
system and prompt the action to maintain the high grade of service. The overall observations justify that for the better service facility, the preventive measures like provision of standby machine is a necessary and working condition of operating machines and standby machine should be proper to avoid frequent failure. Corrective measures like prompt and perfect switchover, prompt and fast repair, etc should opt. System analyst may opt for any repair policy as per convenient of the design of the system and analyze the benefits in term of availability. The suitable corrective and preventive action (CAPA) eliminate causes of non-conformities or other undesirable situations.
6. **Conclusion.** In this paper, we analyze six different repair time distributions for a standby provisioning machine repair problem with imperfect coverage, reboot delay, switchover delay, and switching failure. We develop a recursive method for deriving steady-state probabilities of the system using supplementary variable $V$, a remaining repair time of the failed machines. We also derive the explicit expression for the steady-state probabilities and the availability of the system for different repair time distribution algebraically and numerically as a special case. The extensive numerical discussion has also been included to give a quick insight into the characteristics of different repair time distributions of various distribution families. We rank the repair time distribution based on the availability of the system having the same mean repair time. We implicate the present studied model and methodology for a power supply system. The recursive method developed in this research article works efficiently for the machine repair problem having a large number of machines or any other queueing problems with any kind of repair time distribution. Using this technique, we can explore more general queueing problems with state-dependent arrival and service process in the future. We can extend the present model for pressure coefficient in service, the catastrophic effect for the system, mixed standby provisioning, degraded failure, etc. Our study would be very useful to the system manager in making a decision due to the readily available explicit expression for state probabilities and concrete recursive method for determining them even having no knowledge of mathematical modeling. Our study is pivotal for deciding effective corrective and preventive action by investigating the root cause of failure and quality management system (QMS).
Conflict of interest. The authors declare that there is no conflict of interests regarding the publication of this paper.

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![Figure 10](image_url)

Figure 10. State probability $P_{2,1}$ and availability of the system ($Av$) wrt $\mu$ for different repair time distribution

Table 10. State probabilities and availability of the system

| Distribution | $M$ | $Er_3$ | $D$ | $U(a,b)$ | $GE_4$ | $HE_2$ |
|--------------|-----|--------|-----|----------|--------|--------|
| Parameter(s) | $\mu = 25$ | $\mu = 25$ | $\mu = 25$ | $\alpha = 0.02$ | $\mu_1 = 60$ | $\alpha_1 = 0.2$
|              | $b = 0.06$ | $\mu_2 = 100$ | $\alpha_2 = 0.8$ | $\mu_3 = 120$ | $\mu_1 = 15$
|              | $\mu_4 = 200$ | $\mu_2 = 30$ | $\mu_4 = 200$ | $\mu_2 = 30$
| $P_{21}$     | 0.9123644  | 0.9123430 | 0.9123320 | 0.9123347 | 0.9123417 | 0.9123711 |
| $P_{30}$     | 0.0437935  | 0.0443790 | 0.0446796 | 0.0446037 | 0.0444134 | 0.0436102 |
| $P_{40}$     | 0.0371515  | 0.0365876 | 0.0362980 | 0.0363711 | 0.0365544 | 0.0373280 |
| $Q_{21}$     | 0.0054742  | 0.0054741 | 0.0054740 | 0.0054740 | 0.0054741 | 0.0054742 |
| $R_{11}$     | 0.0007299  | 0.0007299 | 0.0007299 | 0.0007299 | 0.0007299 | 0.0007299 |
| $R_{21}$     | 0.0004866  | 0.0004866 | 0.0004866 | 0.0004866 | 0.0004866 | 0.0004866 |
| $Av$         | 0.9561579  | 0.9567219 | 0.9570115 | 0.9569385 | 0.9567551 | 0.9559813 |
Figure 11. State probability $P_{2,1}$ and availability of the system ($Av$) wrt $\lambda$ for different repair time distribution.

Table 11. Performance indices corresponding to Fig. 2

| Indices | Distribution | $\mu$ |
|---------|--------------|------|
| $P_{2,1}$ | $M$ | 0.9118 0.9120 0.9121 0.9122 0.9123 0.9124 0.9125 |
|         | $Er_3$ | 0.9118 0.9119 0.9121 0.9122 0.9123 0.9124 0.9125 |
|         | $D$ | 0.9118 0.9119 0.9121 0.9122 0.9123 0.9124 0.9125 |
| $Av$ | $M$ | 0.9556 0.9557 0.9559 0.9560 0.9561 0.9562 0.9563 |
|         | $Er_3$ | 0.9561 0.9563 0.9564 0.9565 0.9566 0.9567 0.9568 |
|         | $D$ | 0.9564 0.9566 0.9567 0.9568 0.9569 0.9570 0.9571 |

Table 12. Performance indices corresponding to Fig. 3

| Indices | Distribution | $\lambda$ |
|---------|--------------|------|
| $P_{2,1}$ | $M$ | 0.9749 0.9667 0.9586 0.9507 0.9428 0.9351 0.9274 |
|         | $Er_3$ | 0.9749 0.9667 0.9586 0.9507 0.9428 0.9351 0.9274 |
|         | $D$ | 0.9749 0.9667 0.9586 0.9507 0.9428 0.9351 0.9274 |
| $Av$ | $M$ | 0.9905 0.9860 0.9816 0.9773 0.9730 0.9687 0.9645 |
|         | $Er_3$ | 0.9905 0.9861 0.9818 0.9775 0.9732 0.9690 0.9649 |
|         | $D$ | 0.9905 0.9861 0.9818 0.9775 0.9733 0.9691 0.9650 |
Figure 12. State probability $P_{2,1}$ and availability of the system ($Av$) wrt $\nu$ for different repair time distribution.

Figure 13. State probability $P_{2,1}$ and availability of the system ($Av$) wrt $\beta$ for different repair time distribution.
Figure 14. State probability $P_{2,1}$ and availability of the system ($Av$) wrt $\sigma$ for different repair time distribution

Figure 15. State probability $P_{2,1}$ and availability of the system ($Av$) wrt $p$ for different repair time distribution
Figure 16. State probability $P_{2,1}$ and availability of the system $(Av)$ wrt $c$ for different repair time distribution

Figure 17. State probability $P_{2,1}$ and availability of the system $(Av)$ wrt $q$ for different repair time distribution
Table 13. Performance indices corresponding to Fig. 4

| Indices | Distribution | \( \nu \) |
|---------|--------------|---------|
| \( P_{2,1} \) | \( M \) | 0.9179 0.9170 0.9161 0.9151 0.9142 0.9133 0.9124 0.9114 0.9105 |
| | \( E_3 \) | 0.9179 0.9170 0.9160 0.9151 0.9142 0.9133 0.9124 0.9114 0.9105 |
| | \( D \) | 0.9179 0.9170 0.9160 0.9151 0.9142 0.9133 0.9124 0.9114 0.9105 |
| | \( U(a,b) \) | 0.9179 0.9170 0.9160 0.9151 0.9142 0.9133 0.9124 0.9114 0.9105 |
| | \( G_E \) | 0.9179 0.9170 0.9160 0.9151 0.9142 0.9133 0.9124 0.9114 0.9105 |
| | \( H_E \) | 0.9179 0.9170 0.9161 0.9151 0.9142 0.9133 0.9124 0.9114 0.9105 |
| \( Av \) | \( M \) | 0.9565 0.9564 0.9563 0.9562 0.9562 0.9562 0.9562 0.9562 0.9562 |
| | \( E_3 \) | 0.9570 0.9569 0.9568 0.9568 0.9568 0.9568 0.9567 0.9567 0.9566 |
| | \( D \) | 0.9572 0.9572 0.9572 0.9571 0.9571 0.9571 0.9570 0.9570 0.9569 |
| | \( U(a,b) \) | 0.9572 0.9571 0.9571 0.9570 0.9570 0.9569 0.9569 0.9569 0.9569 |
| | \( G_E \) | 0.9570 0.9570 0.9569 0.9568 0.9568 0.9568 0.9567 0.9567 0.9567 |
| | \( H_E \) | 0.9563 0.9563 0.9562 0.9562 0.9562 0.9562 0.9561 0.9560 0.9559 |

Table 14. Performance indices corresponding to Fig. 5

| Indices | Distribution | \( \beta \) |
|---------|--------------|---------|
| \( P_{2,1} \) | \( M \) | 0.9118 0.9120 0.9121 0.9122 0.9123 0.9123 0.9124 0.9124 0.9125 |
| | \( E_3 \) | 0.9118 0.9119 0.9121 0.9122 0.9123 0.9123 0.9124 0.9124 0.9125 |
| | \( D \) | 0.9118 0.9119 0.9121 0.9122 0.9123 0.9123 0.9124 0.9124 0.9125 |
| | \( U(a,b) \) | 0.9118 0.9120 0.9121 0.9122 0.9123 0.9123 0.9124 0.9124 0.9125 |
| | \( G_E \) | 0.9118 0.9119 0.9121 0.9122 0.9123 0.9123 0.9124 0.9124 0.9125 |
| | \( H_E \) | 0.9118 0.9120 0.9121 0.9122 0.9123 0.9123 0.9124 0.9124 0.9125 |
| \( Av \) | \( M \) | 0.9556 0.9557 0.9559 0.9560 0.9561 0.9562 0.9562 0.9562 0.9562 |
| | \( E_3 \) | 0.9561 0.9563 0.9564 0.9564 0.9566 0.9566 0.9567 0.9567 0.9567 |
| | \( D \) | 0.9564 0.9566 0.9567 0.9568 0.9568 0.9569 0.9570 0.9571 0.9572 |
| | \( U(a,b) \) | 0.9564 0.9565 0.9566 0.9568 0.9569 0.9569 0.9570 0.9571 0.9571 |
| | \( G_E \) | 0.9562 0.9563 0.9564 0.9566 0.9566 0.9568 0.9568 0.9569 0.9569 |
| | \( H_E \) | 0.9554 0.9556 0.9557 0.9558 0.9559 0.9560 0.9561 0.9561 0.9562 |

Table 15. Performance indices corresponding to Fig. 6

| Indices | Distribution | \( \sigma \) |
|---------|--------------|---------|
| \( P_{2,1} \) | \( M \) | 0.9090 0.9102 0.9111 0.9118 0.9124 0.9128 0.9132 0.9135 0.9138 |
| | \( E_3 \) | 0.9090 0.9102 0.9111 0.9118 0.9123 0.9128 0.9132 0.9135 0.9138 |
| | \( D \) | 0.9090 0.9102 0.9111 0.9118 0.9123 0.9128 0.9132 0.9135 0.9138 |
| | \( U(a,b) \) | 0.9090 0.9102 0.9111 0.9118 0.9123 0.9128 0.9132 0.9135 0.9138 |
| | \( G_E \) | 0.9090 0.9102 0.9111 0.9118 0.9123 0.9128 0.9132 0.9135 0.9138 |
| | \( H_E \) | 0.9091 0.9102 0.9111 0.9118 0.9124 0.9128 0.9132 0.9135 0.9138 |
| \( Av \) | \( M \) | 0.9527 0.9539 0.9549 0.9556 0.9562 0.9566 0.9570 0.9574 0.9577 |
| | \( E_3 \) | 0.9532 0.9545 0.9554 0.9561 0.9567 0.9572 0.9576 0.9579 0.9582 |
| | \( D \) | 0.9535 0.9548 0.9557 0.9564 0.9570 0.9575 0.9579 0.9582 0.9585 |
| | \( U(a,b) \) | 0.9535 0.9547 0.9556 0.9564 0.9569 0.9574 0.9578 0.9581 0.9584 |
| | \( G_E \) | 0.9533 0.9545 0.9554 0.9562 0.9568 0.9572 0.9576 0.9580 0.9583 |
| | \( H_E \) | 0.9525 0.9537 0.9547 0.9554 0.9560 0.9565 0.9569 0.9572 0.9575 |
### Table 16. Performance indices corresponding to Fig. 7

| Indices | P<sub>2,1</sub> | \( \rho \) | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 |
|--------|-----------------|-----------|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| M      | 0.9204          | 0.9194    | 0.9184 | 0.9174 | 0.9164 | 0.9154 | 0.9144 | 0.9134 | 0.9124 |
| Er<sub>3</sub> | 0.9204 | 0.9194 | 0.9184 | 0.9174 | 0.9164 | 0.9154 | 0.9144 | 0.9134 | 0.9124 |
| D      | 0.9204          | 0.9194    | 0.9184 | 0.9174 | 0.9164 | 0.9154 | 0.9144 | 0.9134 | 0.9124 |
| U(a, b) | 0.9204 | 0.9194 | 0.9184 | 0.9174 | 0.9164 | 0.9154 | 0.9144 | 0.9134 | 0.9124 |
| GE<sub>4</sub> | 0.9204 | 0.9194 | 0.9184 | 0.9174 | 0.9164 | 0.9154 | 0.9144 | 0.9134 | 0.9124 |
| H<sub>E</sub> | 0.9204 | 0.9194 | 0.9184 | 0.9174 | 0.9164 | 0.9154 | 0.9144 | 0.9134 | 0.9124 |

### Table 17. Performance indices corresponding to Fig. 8

| Indices | Av | \( \rho \) | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 |
|--------|-----------------|-----------|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| M      | 0.9646 | 0.9635 | 0.9625 | 0.9614 | 0.9604 | 0.9593 | 0.9583 | 0.9572 | 0.9562 |
| Er<sub>3</sub> | 0.9655 | 0.9644 | 0.9633 | 0.9623 | 0.9612 | 0.9602 | 0.9592 | 0.9581 | 0.9570 |
| D      | 0.9654 | 0.9643 | 0.9633 | 0.9622 | 0.9611 | 0.9601 | 0.9590 | 0.9580 | 0.9570 |
| U(a, b) | 0.9652 | 0.9641 | 0.9631 | 0.9620 | 0.9610 | 0.9599 | 0.9589 | 0.9578 | 0.9568 |
| GE<sub>4</sub> | 0.9644 | 0.9634 | 0.9623 | 0.9612 | 0.9602 | 0.9591 | 0.9581 | 0.9570 | 0.9560 |

### Table 18. Performance indices corresponding to Fig. 9

| Indices | Av | \( \rho \) | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 |
|--------|-----------------|-----------|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| M      | 0.9521 | 0.9527 | 0.9533 | 0.9538 | 0.9544 | 0.9550 | 0.9556 | 0.9562 | 0.9567 |
| Er<sub>3</sub> | 0.9527 | 0.9532 | 0.9538 | 0.9544 | 0.9550 | 0.9556 | 0.9561 | 0.9567 | 0.9573 |
| D      | 0.9530 | 0.9535 | 0.9541 | 0.9547 | 0.9553 | 0.9558 | 0.9564 | 0.9570 | 0.9576 |
| U(a, b) | 0.9529 | 0.9535 | 0.9540 | 0.9546 | 0.9552 | 0.9558 | 0.9564 | 0.9569 | 0.9575 |
| GE<sub>4</sub> | 0.9527 | 0.9533 | 0.9539 | 0.9544 | 0.9550 | 0.9556 | 0.9562 | 0.9568 | 0.9573 |
| H<sub>E</sub> | 0.9519 | 0.9525 | 0.9531 | 0.9537 | 0.9542 | 0.9548 | 0.9554 | 0.9560 | 0.9566 |

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