The Calogero Model - Anyonic Representation, Fermionic Extension and Supersymmetry

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ABSTRACT

We discuss several applications and extensions of our previous operator solution of the $N$-body Calogero problem, i.e. $N$ particles in 1 dimension subject to a two-body interaction of the form $\frac{1}{2} \sum_{i,j}[(x_i - x_j)^2 + g/(x_i - x_j)^2]$. Using a complex representation of the deformed Heisenberg algebra underlying the Calogero model, we explicitly establish the equivalence between this system and anyons in the lowest Landau level. A construction based on supersymmetry is used to extend our operator method to include fermions, and we obtain an explicit solution of the supersymmetric Calogero model constructed by Freedman and Mende. We also show how the dynamical $OSp(2;2)$ supersymmetry is realized by bilinears of modified creation and annihilation operators, and how to construct a supersymmetic extension of the deformed Heisenberg algebra.

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1. Introduction and Summary

The Calogero model is a quantum mechanical system of $N$ particles on a line interacting via the two-body potential $\omega^2 x^2 + \nu(\nu - 1)x^{-2}$, where $\omega$ is the harmonic oscillator frequency and $\nu(\nu - 1)$ the coupling constant. As pointed out by Calogero\cite{1, 2} in his 1971 paper, this model has some very remarkable properties. First, in order to be normalizable, all wave functions must, up to a common factor, be either totally symmetric or totally antisymmetric. Secondly, the energy spectrum is that of $N$ bosons or fermions interacting via harmonic forces only, but with a total energy shift proportional to $\nu$. Although the spectrum is known, it has been proved difficult to construct the corresponding wave functions. With rather laborious techniques, the wave functions up to the five-particles one were constructed\cite{2, 3, 4}, but in a recent paper\cite{5} we found all $N$ particle eigenwavefunctions using an operator formulation.

The Calogero model is a prime example of a solvable low-dimensional model, with interesting physical applications as well as deep connections to various branches of mathematics. In fact, it has been shown to be but one in a large family of two dimensional (2d) quantum integrable models (for a review see e.g. \cite{6}). In particular, it is expected to have close links to 2d conformal models. This is indicated by the observations that the model is closely related to the matrix models \cite{7, 8} and that the differential operators which are central in our treatment of the model (see also \cite{9}) appear in the decoupling equations in certain formulations of conformal models \cite{10, 11, 12, 13}. Another intriguing connection is to the higher-spin gauge theories in 3 and 4 space-time dimensions (see \cite{14, 15, 16, 17, 18} and references therein) based on a certain class of infinite-dimensional symmetry algebras, higher-spin algebras, introduced in \cite{19, 20, 21, 22, 23}. In \cite{5} it was observed that a version of higher-spin algebras investigated in \cite{22, 23} is precisely the algebra of observables of the two-body Calogero problem.

The aim of this paper is twofold. First we stress the algebraic aspects of our operator solution of the Calogero model, and show that the pertinent extended Heisenberg algebra can be given a complex, (or Bargmann-Fock type) representation in which the wave raising operators, and thus the wave functions, become very simple. It is in fact this representation that provides the link between the Calogero model and anyons in the lowest Landau level. That these two systems are in fact equivalent, was conjectured in \cite{24} and in this paper we provide the proof. Secondly we extend our operator method to the supersymmetric Calogero model originally constructed by Freedman and Mendel\cite{25}. In fact, we show that this model is only one in a family of (non-super symmetric) extensions of the Calogero model. Another member of this
family coincides with one of the integrable models of Calegero type with internal degrees of freedom, recently found by Minahan and Polychronakos [26]. We also construct the relevant extended super-Heisenberg algebra, and give the Bargman-Fock formulation of the super-Calogero model.

As already mentioned, the central ingredient of our method is an extended Heisenberg algebra which, in the simple case of 2 particles (eliminating the center of mass coordinate), is defined by,

\begin{align*}
[D, x] &= 1 - \nu K \quad (1.1) \\
KD &= -DK, \quad (1.2) \\
Kx &= -xK, \quad (1.3) \\
K^2 &= 1, \quad (1.4)
\end{align*}

where $K$ is the so called Klein operator and $\nu$ is an arbitrary constant.

As emphasized in [22, 23] the higher-spin algebras used in [17, 18] amount to algebras of functions of $x$, $D$, and $K$ (i.e. the algebras with generating elements $x$, $D$, and $K$ subject to (1.1) - (1.4)). It was also shown in [22, 23] that bilinears of $x$ and $D$, i.e. $x^2$, $D^2$, and $\{x, D\}$ span the Lie algebra $sl_2$ with respect to commutators, while its quadratic Casimir operator depends on $\nu$. In the group theoretical formulation of the two-body Calogero model, with $sl_2$ as a spectrum generating algebra [27], $\nu$ is identified with the Calogero coupling constant. The same relation was also found in the two-anyon case and this observation was in fact the starting point for our analysis.

Remarkably enough, (1.1) can be consistently generalized to the $N$-body case [4, 8]

\begin{align*}
[D_i, x_j] &= A_{ij} = \delta_{ij}(1 + \nu \sum_{l=1}^{N} K_{il}) - \nu K_{ij}, \quad (1.5) \\
[x_i, x_j] &= 0, \quad (1.6) \\
[D_i, D_j] &= 0, \quad (1.7)
\end{align*}

where $K_{ij}$ are generating elements of the symmetric group,

\begin{align*}
K_{ij}x_j &= x_i K_{ij} \quad (1.8) \\
K_{ij}D_j &= D_i K_{ij}, \quad (1.9)
\end{align*}

\begin{align*}
K_{ij} &= K_{ji} \quad (1.10) \\
K_{ij}K_{jl} &= K_{jl}K_{ii} = K_{ii}K_{ij} \quad (1.11)
\end{align*}

and all quantities $K_{ij}$, $x_i$, and $D_k$ are mutually commuting when all the labels like $i, j, l$, and $k$ are pairwise noncoinciding. Let us emphasize that the center
of mass coordinates
\[ \tilde{x} = \sum_{i=1}^{N} x_i, \quad \tilde{D} = \sum_{i=1}^{N} D_i \] (1.12)
decouple from \textit{i.e.} commute with) the relative coordinates, and that the two-body relations (1.1) - (1.4) hold for the relative coordinates \( x = x_1 - x_2, \)
\( 2D = D_1 - D_2, K = K_{12}. \)

As will be demonstrated in section 2, the algebra (1.5) - (1.7) is crucial for understanding the \( N \)-body Calogero model. Raising and lowering operators can be constructed as \( a_i^\pm = (x_i \mp D_i)/\sqrt{2}, \) and the generators of the spectrum generating \( sl_2 \) algebra (which gives the complete solution to the two-body problem) can be constructed from bilinears in these step operators.

Several problems of both physical and mathematical nature remain to be investigated. On the physics side, one should ask in particular whether there is any interesting theory like \textit{e.g.} some generalized higher-spin theory or string theory with an underlying infinite-dimensional symmetry generated by the relations (1.5) - (1.7).

An interesting mathematical issue is the relation to the spherical function theory on a Riemannian symmetric spaces, namely to the analysis of hypergeometric functions associated with a root systems, and particularly the connection to the so called "Dinkham shift operators" which were introduced in \([28, 29]\). They turn out to be closely related to the realization of the operators \( D_i \) given in the section 2. These operators are a particular case of differential-difference operators introduced in \([30]\) for arbitrary Coxeter group, which corresponds to the root system \( A_{N-1}. \)

### 2. Operator structure of the Calogero model

In this section we first recapitulate the operator solution to the Calogero problem given in \([3]\). The model is defined by the Hamiltonian
\[ H_{Cal} = \frac{1}{2} \sum_{i=1}^{N} \left[ -d_i^2 + x_i^2 \right] + \sum_{j<i}^{N} \frac{g}{(x_i - x_j)^2}, \] (2.1)
(where \( d_i = \frac{\partial}{\partial x_i} \)) which differs from that in reference \([2]\) by an overall normalization and a harmonic oscillator term for the center of mass coordinate, and we have also put the frequency \( \omega \) to one. This must be borne in mind when explicitly comparing our spectrum with that of \([2]\).

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1 We are grateful to Dr. A. Matsuo for informing us about these works.
To solve $H_{Cal} \Psi = E \Psi$, we make the ansatz:

$$\Psi^\pm = \prod_{i > j} (x_i - x_j)^\nu \Phi^\pm = \beta^\nu \Phi^\pm ,$$

where $x_i > x_j$ for $i > j$, while $+$ and $-$ refers to totally symmetric and antisymmetric wave functions $\Phi^+$ and $\Phi^-$, respectively. Consider now the deformed Heisenberg algebra (1.5) - (1.7). A representation of the "covariant derivative" $D_i$ can be given as

$$D_i = d_i + \nu \sum_{j \neq i} \frac{1}{(x_i - x_j)} (1 - K_{ij}).$$

(2.3)

(Note that essentially the same modified derivatives were introduced previously in [30]. There the operators $K_{ij}$ were not used explicitly but rather their representation on the functions of $x_i$. A slightly different expression for $D_i$ with the factor of $K_{ij}$ instead of $(1 - K_{ij})$ was given in [9]. The definition (2.3) is unambiguously fixed by the requirement that $D_i$ induces no new poles when acting on regular functions of $x_i$. In particular $D_i$ can easily be seen to leave the space of polynomials invariant.)

The square of this operator is then

$$D^2 = \sum_{i=1}^N D_i^2 = \sum_{i=1}^N \left[ d_i^2 + \nu \sum_{i \neq j} \frac{2}{x_i - x_j} d_i - \nu \sum_{i \neq j} \frac{1}{(x_i - x_j)^2} (1 - K_{ij}) \right].$$

(2.4)

Noting that $(1 - K_{ij})$ gives 0 and 2 respectively on the totally symmetric and antisymmetric wave functions, we finally get

$$H_{Cal} \Psi^\pm = \beta^\nu \frac{1}{2} (-D^2 + X^2) \Phi^\pm$$

(2.5)

where $X^2 = \sum_{i=1}^N x_i^2$, and $g = \nu (\nu \mp 1)$, where the upper and lower sign refers to symmetric and antisymmetric wave functions respectively.

Given the complicated form of the $D_i$:s the algebra (1.5) - (1.7) as well as (2.5) is amazingly simple. Note that the $A_{ij}$ in (1.5) is symmetric in $i$ and $j$ so that we can construct creation and annihilation operators via

$$a_i^\mp = \frac{1}{\sqrt{2}} (x_i \pm D_i)$$

(2.6)

obeying the commutation relations

$$[a_i^\pm, a_j^\pm] = 0$$
$$[a_i^-, a_j^+] = A_{ij}.$$

(2.7)
The Hamiltonian can now be expressed as
\[
H = \frac{1}{2}(-D^2 + X^2) = \frac{1}{2} \sum_i \{a_i^+, a_i^-\}, \quad (2.8)
\]
and turns out to obey the standard commutation relations with the creation and annihilation operators,
\[
[H, a_i^\pm] = \pm a_i^\pm. \quad (2.9)
\]
This last relation again follows from a series of nontrivial algebraic manipulations using the properties of the \( K_{ij} \). The eigenfunctions are now obtained via the construction,
\[
\Phi^{\pm}(n_i) = S\{\prod_{i=1}^N (a_i^+)^{n_i}\} \Phi^{\pm}_0, \quad (2.10)
\]
where \( S \) denotes total symmetrization, and the vacuum state \( \Phi^{\pm}_0 \) satisfies
\[
a_i^- \Phi^{\pm}_0 = 0, \quad (2.11)
\]
and
\[
K_{ij} \Phi^{\pm}_0 = \pm \Phi^{\pm}_0. \quad (2.12)
\]
It is well known that every symmetric polynomial of \( a_i^+ \) or \( a_i^- \) can be expressed as a polynomial of elementary symmetric polynomials
\[
C_n^\pm = \sum_{i=1}^N (a_i^\pm)^n, \quad (2.13)
\]
hence (anti)symmetric wave functions have the following basis functions
\[
\Phi^{\pm}_{\{n_i\}} = \prod_{i=1}^N (C_i^+)^{n_i} \Phi^{\pm}_0. \quad (2.14)
\]

Using (2.8), (2.11), (2.12) and the commutation relations (2.7) we find the ground state energy of \( H_{Cal} \) to be \( E_0^+ = \frac{N}{2} \pm \nu_1^2 N(N - 1) \), so the complete spectrum is that of \( N \) bosons or fermions in a harmonic oscillator well, shifted by this constant. This is Calogero’s original result. Solving (2.11) and (2.12) also immediately gives Calogero’s ground state wave function. As advertised, the new result is the explicit expression (2.10) for the wave functions. Needless
to say, the expressions very quickly become very cumbersome because of the sums in the definition of $D_i$.

Let us now discuss the relation between our construction and the algebraic approach of Perelomov\cite{Perelomov1, Perelomov2} and Gambardella\cite{Gambardella}. To this end it is convenient to use the center mass coordinate frame. After separating the center of mass coordinates our construction can be rewritten in terms of the relative derivatives $\tilde{D}_i$:

$$\tilde{D}_i = D_i - \frac{1}{N} \sum_{l=1}^{N} d_l .$$

(2.15)

Note that $\sum d_i = \sum D_i$ and hence $\sum \tilde{D}_i = 0$. The relative parts of creation and annihilation operators can be defined in a similar way:

$$\tilde{a}_i^\pm = a_i^\pm - \frac{1}{N} \sum_{l=1}^{N} a_l^\pm .$$

The characteristic property of these relative operators is that they leave the space of smooth functions vanishing on the hyperplane $\sum_1^N x_i = 0$ invariant.

In \cite{Perelomov1} Perelomov proposed to look for such symmetric (\textit{i.e.} transforming (anti)symmetric wavefunctions into (anti)symmetric ones) operators $B_n^\pm$ that increase (decrease) the energy \textit{i.e.}

$$[H, B_n^\pm] = \pm nB_n^\pm$$

(2.16)

and

$$B_n^- \Phi_0 = 0 .$$

(2.17)

If such operators are mutually commuting and linearly independent the wavefunctions

$$\Phi_{\{n_i\}} = \prod_i (B_i^+)^{n_i} \Phi_0$$

(2.18)

with $\sum_i in_i = m$ form linearly independent states degenerate in energy. The dimension of this subspace for a given $m$ coincides with the number of solutions of the equation $\sum_i in_i = m$ in non-negative integers.

In \cite{Perelomov1} Perelomov has obtained the operators $B_2^+, B_3^+$ and $B_4^+$, and in \cite{Gambardella} Gambardella constructed $B_5^+$. All these operators were shown to be mutually commuting. To construct such operators Perelomov deduced the equations for the coefficients of their expansion in the powers of annihilation and creation...
operators of pure oscillator system - $a_i$, $a_i^\dagger$. These equations follow from (2.16)
and depend on the potential $V = g \sum_{i<j}(x_i - x_j)^{-2}$. The operators $B_n^\pm$, which
because of (2.16), generate classes of states for the general N body problem, provide the full solution for $N \leq 5$.

From the commutation relation (2.9) it immediately follows that

$$[H, (\tilde{a}_i^\pm)^n] = \pm n(\tilde{a}_i^\pm)^n \quad (2.19)$$

Thus the elementary symmetric polynomials $C_n^\pm$ which generate all symmetric polynomials of $\tilde{a}_i^\pm$ satisfy (2.19) like $B_n$ and leave the subspace of symmetric wave functions invariant. All polynomials $C_n^\pm$ commute among themselves and with the operators $K_{ij}$. These polynomials can be expressed in the following form,

$$C_n^+ = \sum_\alpha C_n^\alpha F_n^\alpha (K_{ij}) \quad (2.20)$$

where the operators $C_n^\alpha$ depend on $x_i$ and $\frac{d}{dx_i}$ but not on the permutation operators $K_{ij}$, while the operators $F_n^\alpha$ are some functions of $K_{ij}$ independent of $x_i$ and $\frac{d}{dx_i}$. The point is that the operators

$$S C_n^+ = \sum_\alpha C_n^\alpha F_n^\alpha |_{K_{ij}=1} \quad (2.21)$$

are precisely Perelomov’s operators $B_n^+$ because they obey all requirements imposed in $[3]$.

Actually, it can be shown that the operators $S C_n^+$ are symmetric (i.e. commute with $K_{ij}$) and satisfy the relations $[H, S C_n^+] = n S C_n^+$. Hence the coefficients of expansion of these operators $S C_i^\pm$ in pure oscillator creation and annihilation operators satisfy the equations obtained by Perelomov where in place of potential $V = g \sum_{i<j}(x_i - x_j)^{-2}$ the "potential" $\tilde{V} = -\nu/2 \sum_{i\neq j}(x_i - x_j)^{-1}(d_i - d_j)$ is used. This implies that the operators $S C_i^\pm$ are equivalent to the operators $B_i^\pm$ after performing the similarity transformation (2.2).

To prove that the $S C_n^+$:s are mutually commuting it is convenient to use another representation of these operators,

$$C_n^+ = S C_n^+ + \sum_{i,j} \Phi_{ij} \cdot (1 - K_{ij}) \quad (2.22)$$

where $\Phi_{ij}$ are some functions of $x_i$, $\frac{d}{dx_i}$ and $K_{ij}$. Because the $C_n^+:s$ are commuting and the $S C_n^+$:s are symmetrical it follows from (2.22) that

$$0 = [C_n^+, C_m^+] = [S C_n^+, S C_m^+] + \sum_{i,j} \Psi_{ij}(1 - K_{ij}) \quad (2.23)$$
with some operators $\Psi_{ij}$. This implies that the symmetric operators $C^+_n$ are commuting when restricted to the space of symmetric functions. A slightly more complicated analysis shows that all these commutators vanish identically.

Let us mention that the above construction makes it less mysterious why the construction of Perelomov leads to normalizable wave functions despite the poles in the operators $B^+_n$.

It is also to be noted, that the operators $B^\pm_2$ together with $B^0_2 = \frac{1}{2}H$, satisfy the $sp(2, R)$ algebra,

\[
[B_0, B_\pm] = \pm B_\pm
\]  

(2.24)

\[
[B_+, B_-] = -2B_0
\]  

(2.25)

which is the spectrum generating algebra for the $N = 2$ case. In the following we shall consider other representations of this algebra.

3. The Complex Representation and Anyons

The connection between the Calogero problem and fractional statistics in 1+1 dimension was first noted in [31], and the N-body problem was discussed in [32] and [24]. In particular it was noted that there is a close similarity between the Calogero problem and N anyons in the lowest Landau level [24, 33]. In [24] it was shown that the system of two anyons in the lowest Landau level is in fact equivalent to the 2-body Calogero problem. It was also shown that, after appropriate rescalings, the spectrum of the total angular momentum operator for N anyons is identical to that of the N-body Calogero Hamiltonian. The wave functions for anyons in the lowest Landau level are all explicitly known, and can in fact be constructed with the help of raising and lowering operators. The conjecture in [24] that the systems are in fact equivalent, was strongly supported by our operator construction of the wave functions. We shall now prove this by finding a complex representation of the operator algebra (1.5) - (1.7) and showing that the corresponding states are precisely those of anyons in the lowest Landau level. In the case of two particles, we also explicitly show the connection to the treatment in [24].

We start from the Bargmann-Fock representation for a collection of harmonic oscillators, where the creation and annihilation operators are repre-
sented by

\[ \tilde{a}_i = \frac{\partial}{\partial z_i} = \partial_i \]  

(3.1)

\[ \tilde{a}_i^\dagger = z_i \]  

(3.2)

where \( z_i = \frac{1}{\sqrt{2}} (x_i + ip_i) \). These operators act on holomorphic functions of \( z \), and the scalar product (in the one particle case) is defined by,

\[ \langle \psi_2 | \psi_1 \rangle = \int \frac{dz dz'}{2\pi i} e^{-z\overline{z}} \overline{\psi_2(z)} \psi_1(z) \]  

(3.3)

where bar denotes complex conjugation. In this representation the \( g = 0 \) part of the Calogero Hamiltonian (2.1), i.e. a system of particles interacting only by harmonic forces, can be written

\[ H_{ho} = \sum_{i=1}^{N} (z_i \partial_i + \frac{1}{2}) \]  

(3.4)

An ON basis is given by products of the one-particle states

\[ \psi_{n_i}(z_i) = \frac{1}{\sqrt{n_i!}} z_i^{n_i} \]  

(3.5)

This representation for the harmonic oscillator is connected to the conventional one (where the wave functions involve Hermite polynomials) via a unitary transformation. On the other hand, we can interpret \( H_{ho} \) as the angular momentum operator of \( N \) particles in the lowest Landau level in radial gauge (\( A_r = 0 \)). More precisely \( H_{ho} = \sum_{i=1}^{N} (\hat{L}_i + \frac{1}{2}) \), where \( \hat{L}_i = z_i \partial_i \) is the angular momentum operator of particle \( i \).

We now introduce the operators

\[ \tilde{a}^-_i = \partial_i + \nu \sum_{j \neq i} \frac{1}{(z_i - z_j)(1 - K_{ij})} \]  

\[ \tilde{a}^+_i = z_i \]  

(3.6)

(3.7)

These are direct complex generalizations of the operators \( D_i \) and \( x_i \) in section 1 and satisfy the same algebra (1.5) - (1.7). The Hamiltonian (2.8) generalizes to

\[ \hat{H} = \sum_{i=1}^{N} \frac{1}{2} \{ \tilde{a}^-_i, \tilde{a}^+_i \} = \sum_{i=1}^{N} \left[ z_i \partial_i + \frac{1}{2} \right] + \nu \frac{N(N-1)}{2} \]  

(3.8)
which again describes particles in the lowest Landau level but with the angular momentum shifted by a constant value $\nu$ for each of the $N(N - 1)/2$ pairs. This is precisely a system of $N$ anyons! Note that although the annihilation operators, $\tilde{a}_i^-$ are complicated, the creation operators $\tilde{a}_i^+ = z$ act trivially on the wave functions. In the real (or Calogero) representation both $a_i$ and $a_i^\dagger$ act non-trivially on the wave functions. This is the basic reason for why the wave functions are so simple in the complex (or anyonic) description, and so complicated in the real one.

For the following discussion of the $N = 2$ case, we need the explicit expressions, in the complex representation, for the operators $B^\pm_2$ and $B^0_2$ introduced at the end of section 2. A simple calculation yields

\[
\tilde{B}^+_2 = \frac{1}{2} z^2 \\
\tilde{B}^-_2 = \frac{1}{2} D^2 = \frac{1}{2} \left[ \partial^2 + \frac{2\nu}{z} \partial - \nu (1 - K) \right] \\
\tilde{B}^0_2 = \frac{1}{2} \left[ \hat{L} + \frac{1}{2} \right] = \frac{1}{2} \left[ z \partial + \nu + \frac{1}{2} \right],
\]

(3.9)

where $z = z_1 - z_2$ is the relative coordinate, $\partial$ and $D$ the corresponding derivatives, and $K = K_{12}$. The operators are properly normalized to satisfy the $sp(2,R)$ algebra (2.24), (2.25).

We are now ready to show how the above results relate to those in ref. [24]. There the equivalence between the two-body Calogero system and two anyons in the lowest Landau level was demonstrated by an explicit construction of the generators for the spectrum generating $sp(2,R)$ algebra (2.24), (2.25) in the anyon case. For this, consider the representations in the discrete series defined by,

\[
B_0 |k, \mu \rangle = (k + \mu) |k, \mu \rangle \\
\Gamma |k, \mu \rangle = \mu(\mu - 1) |k, \mu \rangle,
\]

(3.10)

(3.11)

where $\Gamma = B^2_0 - \frac{1}{2} (B_+ B_- + B_- B_+)$ is the quadratic Casimir operator, $\mu > 0$ and $k = 0, 1, 2...$. Different values for the real parameter $\mu$ correspond to inequivalent representations. With appropriate phase conventions the commutation relations (2.24), (2.25) imply

\[
B_+ |k, \mu \rangle = \sqrt{(k + 1)(k + 2\mu)} |k + 1, \mu \rangle \\
B_- |k, \mu \rangle = \sqrt{k(k + 2\mu - 1)} |k - 1, \mu \rangle.
\]

(3.12)

(3.13)
Using these relations, it was shown in [24] that the $B_\pm$ can be represented by the following differential operators,

\[
\begin{align*}
\hat{B}_2^0 &= \frac{1}{2} \left( z \hat{D} + \frac{1}{2} \right), \\
\hat{B}_2^+ &= \frac{1}{2} z^2 M, \\
\hat{B}_2^- &= \frac{1}{2} M \hat{D}^2,
\end{align*}
\]

(3.14)

where $\hat{D} = \partial + \bar{z}/2$ and $M$ is the square root of the operator

\[
M^2 = 1 - \frac{\nu(\nu - 1)}{(L + 1)(L + 2)} \equiv 1 - \nu(\nu - 1) \frac{1}{z^2} \frac{1}{D^2},
\]

(3.15)

which is positive and hermitian, and where $\mu$ in (3.13) and (3.12) is related to the anyonic parameter $\nu$ by

\[
\mu = \frac{\nu}{2} + \frac{1}{4}.
\]

(3.16)

The operators (3.14) act on the normalized ”anyonic” wave functions in the lowest Landau level given by,

\[
\psi_k(z) = N_k z^k e^{-\frac{1}{2} \bar{z} z},
\]

(3.17)

where $N_k = [\pi \Gamma(\ell + 1)]^{-\frac{3}{2}}$, and the integer $k$ is restricted by the requirement that the angular momentum $\ell = 2k + \nu$ fulfills $\ell \geq 0$ in order for the wave function to be regular at the origin.

Thus, the observables (3.14) for the two anyon problem, satisfy the same algebra as the observables in the $N = 2$ Calogero problem, and in [24] it was concluded that the systems are indeed equivalent. This is of course nothing but a special case of the equivalence just proved for general $N$. It remains to understand the connection between the representation (3.14) found in [24] and the representation (3.9) obtained from our creation and annihilation operator formalism. We do this by following the same procedure as in [24], but require that $\tilde{B}_2^+ = \frac{1}{2} z^2$ and $\tilde{B}_2^0 = \frac{1}{2}(z \partial + \frac{1}{2} + \nu)$ and act on the functions

\[
\tilde{\psi}_k(z) = \tilde{N}_k z^{2k},
\]

(3.18)

\footnote{Note that in [24] the exponential factor was included in the wave functions rather than in the measure, hence the occurrence of $\tilde{D}$ rather than $\partial$.}
From (3.12) it now follows
\[ \tilde{N}_{k-1} = 2 \sqrt{k(k - \frac{1}{2} + \nu)} \tilde{N}_k \]  
(3.19)

We then make the ansatz,
\[ \tilde{B}^- = \frac{1}{2} \tilde{M} \partial^2 \]  
(3.20)

and use (3.19) and (3.13) to obtain
\[ \tilde{M} = 1 + \frac{2\nu}{2k - 1} = 1 + \frac{1}{z} \]  
(3.21)

so finally
\[ \tilde{B}^- = \frac{1}{2} \partial^2 + \nu \frac{1}{z} \partial \]  
(3.22)

which is exactly the result (3.9) obtained by squaring our creation operator for the symmetric case where \( K = 1 \). We thus see that there is a large freedom in representing the algebra (2.24), (2.25) on holomorphic functions, and we have explicitly demonstrated how to construct the representation (3.9) both in our approach, and in that used in [24].

Note that \( \tilde{B}^- \) in (3.9) annihilates both the even and the odd ground states (i.e. both 1 and \( z \)), while \( \tilde{B}^- \) in (3.22) only annihilates the even one. This is because in the latter construction we explicitly used the symmetric functions (3.18). It is easy to repeat the construction for wave functions \( \sim z^{2k+1} \) to obtain the extra term \( \sim 1/z^2 \) in (3.9) for the odd case.

### 4. Spinning Models and the Super-Calogero Model

In section 2 we saw that the spectrum of the Calogero model coincides with the one of the ordinary harmonic oscillators, apart from a shift of the vacuum energy. It is also known that the one-dimensional harmonic oscillator has a unique, \( N = 2 \), supersymmetric extension. This is easily seen in a Lagrangian formulation of the one-particle case. Define the superfield
\[ \Phi(\tau, \vartheta) = x(\tau) + i\vartheta^i \theta^i(\tau) + i\vartheta^1 \vartheta^2 F(\tau) \]  
(4.1)

and construct the covariant derivative
\[ \mathcal{D}_i = \frac{\partial}{\partial \vartheta^i} + i\vartheta^i \frac{\partial}{\partial \tau} \quad i = 1, 2. \]  
(4.2)
Then consider the action

\[ S = -\frac{1}{2} \int d\tau d\vartheta d\vartheta^2 [\mathcal{D}_1 \Phi \mathcal{D}_2 \Phi + i \omega \Phi^2] \]  \hspace{1cm} (4.3)

On dimensional grounds we see that this construction is unique. Note that we have reintroduced the frequency \( \omega \) to make the argument about the dimension clearer. It will also be useful to have it explicit for the discussions that will follow. If we perform the \( \vartheta \)-integrations, eliminate the auxiliary fields and go over to a hamiltonian formalism with \( N \) interacting particles and set

\[ a_i = \frac{1}{\sqrt{2}} (\omega x_i + ip_i) \]  \hspace{1cm} (4.4)
\[ a_i^\dagger = \frac{1}{\sqrt{2}} (\omega x_i - ip_i) \]  \hspace{1cm} (4.5)
\[ \theta_i = \frac{1}{\sqrt{2}} (\theta_i^1 + i \theta_i^2) \]  \hspace{1cm} (4.6)
\[ \theta_i^\dagger = \frac{1}{\sqrt{2}} (\theta_i^1 - i \theta_i^2) \]  \hspace{1cm} (4.7)

and construct the supercharges

\[ Q = \sum_i \theta_i^\dagger a_i, \]  \hspace{1cm} (4.8)
\[ Q^\dagger = \sum_i \theta_i a_i^\dagger, \]  \hspace{1cm} (4.9)

then the quantized Hamiltonian takes the following explicitly supersymmetric form

\[ H = \{Q, Q^\dagger\} = \frac{1}{2} \sum_i \left(\{a_i^\dagger, a_i\} + \omega [\theta_i^1, \theta_i]\right) \]  \hspace{1cm} (4.10)

The wave function for the \( N \) particle system is a sum of the form

\[ \Psi(x_m, \theta_i) = \psi(x_m) + \sum_i \psi_i(x_m) \theta_i + \sum_{ij} \psi_{ij}(x_m) \theta_i \theta_j + \ldots \]

To describe a system of identical bosons and fermions, we demand that \( \Psi \) is symmetric or antisymmetric under the combined exchange \((x_i, \theta_i) \leftrightarrow (x_j, \theta_j)\). It is easy to see that this will ensure the correct symmetrization and antisymmetrization of the bosonic and fermionic wavefunctions respectively.

It is now straightforward to supersymmetrize the Calogero model by simply replacing the free raising and lowering operators \( a_i^\dagger \) and \( a_i \) with the corresponding operators \( a_i^+ \) and \( a_i^- \) given in (2.6). Note that since the frequency is reintroduced, the commutation relations (2.7) take the form \([a_i^-, a_j^+] = \omega A_{ij}\). The Hamiltonian corresponding to \( H \) in (2.8) is

\[ H_s = \{Q^+, Q^-\} = \frac{1}{2} \sum_i \{a_i^+, a_i^-\} + \frac{1}{2} \omega \sum_{i,j} [\theta_i^1, \theta_j^1] A_{ij} \]  \hspace{1cm} (4.11)
Here

\[ Q^- = \sum_i \theta_i^+ a_i^- \quad (4.12) \]
\[ Q^+ = \sum_i \theta_i a_i^+ \quad (4.13) \]

By introducing the explicit form (1.5) for the \( A_{ij} \)'s, the last term in (4.11) becomes

\[ \frac{1}{2} \omega \sum_{i,j} [\theta_i^+, \theta_j] A_{ij} = \frac{1}{2} \omega \sum_i [\theta_i^+, \theta_i] + \frac{\nu \omega}{2} \sum_{j \neq i} \frac{1}{2} [\theta_i^+, \theta_j] K_{ij} \quad (4.14) \]

By direct calculation, one can now show that the combination

\[ K_{ij}^\theta = \frac{1}{2} [\theta_i^+, \theta_j] = 1 - (\theta_i - \theta_j)(\theta_i^+ - \theta_j^+) \quad (4.15) \]

occurring in (4.14) is the permutation \( K \) operator for the fermionic variables, i.e. it fulfills

\[ \theta_i K_{ij}^\theta = K_{ij}^\theta \theta_j \quad , \quad (K_{ij}^\theta)^2 = 1 \quad (4.16) \]

and the algebraic relations (1.10) and (1.11). A slightly different realization of the permutation operators acting on inner labels was used in the paper by Minahan and Polychronakos[26]. We see that \( K_{ij}^\theta \) naturally appear in the supersymmetric extension of the Calogero model. Since \( K_{ij} \) and \( K_{ij}^\theta \) commute we can define a total \( K \) operator, again satisfying (1.8) - (1.11), that exchanges both bosonic and fermionic coordinates by

\[ K_{ij}^{tot} = K_{ij} K_{ij}^\theta \quad . \quad (4.17) \]

and rewrite the Hamiltonian (4.11) as

\[ H_s = \frac{1}{2} \sum_i \{a_i^+, a_i^-\} + \frac{1}{2} \omega \sum_i [\theta_i^+, \theta_i] + \frac{1}{2} \omega \nu \sum_{i \neq j} K_{ij}^{tot} = H_B + H_F + H_K . \quad (4.18) \]

Now one observes that when restricted to the subspaces of totally symmetric or totally antisymmetric wavefunctions the operators \( K_{ij}^{tot} \) act as a constant, \( K_{ij}^{tot} = \pm 1 \), and, therefore, for these subspaces the Hamiltonian (4.18) amounts to a particular case of the family of Hamiltonians

\[ H = \frac{1}{2} \sum_i \{a_i^+, a_i^-\} + \frac{1}{2} \omega_F \sum_i [\theta_i^+, \theta_i] + c \quad (4.19) \]

with \( \omega_F = \omega \) and \( c = \pm \frac{\nu \omega}{2} N (N - 1) \).
Evidently the Hamiltonian (4.19) obeys the following commutation relations
\[
[H, a_i^\pm] = \pm \omega a_i^\pm, \quad [H, \theta_i^\dagger] = \omega_F \theta_i^\dagger, \quad [H, \theta_i] = -\omega_F \theta_i
\]
and therefore is exactly solvable for arbitrary values of bosonic and fermionic frequencies. All eigenstates of (4.19) can be obtained by acting with creation operators \(a_i^+\) and \(\theta_i^\dagger\) on the groundstate \(\Phi_0\) which satisfies the equations
\[
a_i \Phi_0 = \theta_i \Phi_0 = 0
\]
Groundstates have the following form
\[
\Phi_0(x, \theta) = \theta_1 \theta_2 \ldots \theta_N \Phi_0(x).
\]
Here \(\Phi_0(x, \theta)\) as well as \(\Phi_0(x)\) has a definite parity
\[
K_{ij}^{\text{tot}} \Phi_0(x, \theta) = \pi_0 \Phi_0(x, \theta), \quad K_{ij}^{\text{tot}} \Phi_0(x) = -\pi_0 \Phi_0(x), \quad (4.20)
\]
where \(\pi_0 = \pm 1\). The ground energy can be easily computed
\[
E_0 = \frac{1}{2} N(\omega - \omega_F) + c - \pi_0 \frac{\nu \omega}{2} N(N - 1) \quad , \quad (4.21)
\]
and as expected \(E_0 = 0\) in the supersymmetric case. The symmetric and antisymmetric excited eigenstates of (4.19) can be obtained by acting on a symmetric or antisymmetric \(\Phi_0\) with polynomials of \(a_i^+\) and \(\theta_i^\dagger\) that are symmetric under the total exchange \(K_{ij}^{\text{tot}}\).

Using (4.15) - (4.17) one can write \(K_{ij}\) as \(K_{ij}^\theta K_{ij}^{\text{tot}}\). When restricted to the subspace of symmetric wavefunctions with \(K_{ij}^{\text{tot}} = 1\) this enables one to replace bosonic permutation operators with the fermionic ones that effectively leads to interactions between bosons and fermions. For the particular case of the supersymmetric Hamiltonian (4.11) we reintroduce the factor \(\beta^\nu\) to get the following supersymmetric generalization of the Hamiltonian in (2.1)
\[
H_{\text{Cal}} = \frac{1}{2} \sum_i \left[ -d_i^2 + \omega^2 x_i^2 \right] + \frac{\nu^2}{4} \sum_{i<j} \frac{1}{x_i - x_j} \left[ -\theta_i^\dagger - \theta_j^\dagger + \theta_i + \theta_j \right] + \frac{\nu \omega}{2} N(N - 1) \quad . \quad (4.22)
\]
This is exactly the supersymmetric extension of the Calogero model originally found by Freedman and Mende[23].

Some comments are now in order. It is clear that supersymmetry is not important for solvability of these models. Moreover, because of the term \(H_K\)
in (4.18) the supersymmetric model turns out to be explicitly solvable only when restricted to the subspace of totally (anti)symmetric wave functions when \( H_K \) amounts to some constant. On the other hand the Hamiltonians (4.19) are explicitly solvable for arbitrary \( \omega_B \) and \( \omega_F \) and all types of symmetry properties of wave functions.

It is also clear that if we introduce an internal symmetry by adding more fermionic variables \( \theta_i^a, \ a = 1, 2, \ldots \) we can generalize (4.19) to other solvable models. For the particular case \( \omega_F = 0 \) these Hamiltonians coincide with a special case of those considered by Minahan and Polychronakos in [26].

It is also amusing to write the supersymmetric version of the "anyonic" representation of the Calogero model, i.e. the supersymmetric version of (3.8)

\[
\hat{H}_{sCal} = \frac{1}{2} \sum_i \left[ \left\{ z_i, \frac{\partial}{\partial z_i} \right\} - \left\{ \theta_i, \frac{\partial}{\partial \theta_i} \right\} \right].
\]

Although written in this way the supersymmetric Calogero model indeed looks rather trivial, the unitary transformation that relates the two representations to each other is not trivial at all.

As shown by Freedman and Mendes[23], the super Calogero model possesses a dynamical \( OSp(2; 2) \) supersymmetry. We will now show how this symmetry can be realized in terms of bilinears of the modified creation and annihilation operators that we already introduced. The result is quite simple since all basic quantities are expressed in terms of the modified creation and annihilation operators essentially in the same way as for the ordinary harmonic oscillator. (To simplify formulae from now on we again set \( \omega = 1 \).)

The basic supercharges have already been introduced in (4.12) and (4.13), and they of course fulfill

\[
(Q^+)^2 = (Q^-)^2 = 0,
\]

We also introduce the notation

\[
\{Q^+, Q^-\} = H = T_3 + J
\]

where

\[
T_3 = \frac{1}{2} \sum_i \{a_i^-, a_i^+\}
\]

\[
J = \frac{1}{2} (\sum_i [\theta_i^+, \theta_i^-] + \nu \sum_{i \neq j} K_{ij}^{\text{tot}})
\]
where $\theta^+ = \theta^\dagger$ and $\theta^- = \theta$. To obtain the full $osp(2; 2)$ algebra we need the additional supercharges
\[ S^- = \sum_i a_i^- \theta_i^- , \quad S^+ = \sum_i a_i^+ \theta_i^+ , \quad (4.26) \]
and the bosonic charges
\[ T_\pm = \frac{1}{2} \sum_i (a_i^\pm)^2 \quad (4.27) \]
The bosonic operators $T_\pm$, $T_3$ and $J$ span the algebra $sp(2) \oplus o(2)$, the bosonic subalgebra of $OSp(2; 2)$,
\[ [T_3, T_\pm] = \pm 2T_\pm , \quad [T_-, T_+] = T_3 , \quad (4.28) \]
and $J$ commutes with all bosonic generators.

The nontrivial anticommutators are
\[ \{S^+, Q^+\} = T_+ , \quad \{Q^-, S^-\} = T_- \quad (4.29) \]
\[ \{Q^+, Q^-\} = T_3 + J , \quad \{S^+, S^-\} = T_3 - J , \quad (4.30) \]
(all other anticommutators vanish). The nonvanishing boson-fermion commutators read
\[ [T_+, Q^-] = -S^+ , \quad [T_+, S^-] = -Q^+ \quad (4.31) \]
\[ [T_-, S^+] = Q^- , \quad [T_-, Q^+] = S^- \quad (4.32) \]
\[ [T_3, Q^+] = Q^- , \quad [T_3, Q^-] = -Q^- \quad (4.33) \]
\[ [T_3, S^-] = -S^- , \quad [T_3, S^+] = S^+ \quad (4.34) \]
\[ [J, S^-] = -S^- , \quad [J, S^+] = S^+ \quad (4.35) \]
\[ [J, Q^+] = -Q^+ , \quad [J, Q^-] = Q^- . \quad (4.36) \]

One can rewrite this in a more systematic way by introducing supercharges
\[ Q^{ab} = \sum_i a_i^a \theta_i^b \quad (4.37) \]
and bosonic charges
\[ B^{\alpha\beta} = \frac{1}{2} \sum_i \{a_i^\alpha , a_i^\beta\} \quad (4.38) \]
\[ J^{ab} = \sum_{i \neq j} \left[ \theta_i^a , \theta_j^b \right] A_{ij} \quad (4.39) \]

18
with $\alpha, \beta, a, b = \pm 1$.

It is seen easily that

$$J^{ab} = \delta_{a+b,0} a J$$  \hspace{1cm} (4.40)

The commutation relations (4.28)-(4.36) can be written now as

$$\{Q^{\alpha a}, Q^{\beta b}\} = \delta_{a+b,0} B^{\alpha \beta} + \delta_{\alpha+\beta,0} \beta J_{ab}$$  \hspace{1cm} (4.41)

$$[B^{\alpha \beta}, Q^{\gamma a}] = \delta_{\alpha+\gamma,0} \gamma Q^{\beta a} + \delta_{\beta+\gamma,0} \gamma Q^{\alpha a}$$  \hspace{1cm} (4.42)

$$[B^{\alpha \beta}, B^{\gamma \chi}] = \delta_{\alpha+\gamma,0} B^{\beta \chi} + \delta_{\alpha+\chi,0} \chi B^{\beta \gamma} + \delta_{\beta+\gamma,0} \gamma B^{\alpha \chi} + \delta_{\beta+\chi,0} \chi B^{\alpha \gamma}$$  \hspace{1cm} (4.43)

$$[B^{\alpha \beta}, J^{ab}] = 0$$  \hspace{1cm} (4.44)

$$[J, Q^{\alpha a}] = a Q^{\alpha a}$$  \hspace{1cm} (4.45)

5. Fermionic Covariant Derivatives

In the previous section we supersymmetrized the Calogero model by adding anticommuting harmonic oscillators. We could also ask whether we can extend the extended Heisenberg algebra to an extended super-Heisenberg algebra. To this end, we add operators $D_i$ and $\theta_i$, with the non-trivial (anti-)commutation rules,

$$\{D_i, D_j\} = 2 \delta_{ij} D_i$$  \hspace{1cm} (5.1)

$$\{D_i, \theta_j\} = \delta_{ij}$$  \hspace{1cm} (5.2)

$$[D_i, x_j] = \theta_i A_{ij}$$  \hspace{1cm} (5.3)

A representation of the algebra (5.1)-(5.3) is given by the standard construction,

$$D_i = \partial_i + \theta_i D_i$$  \hspace{1cm} (5.4)

so that

$$D_i = \partial_i + \theta_i \partial_i + \sum_{k \neq i} \frac{\nu \theta_i}{x_i - x_k} (1 - K_{ik})$$  \hspace{1cm} (5.5)

where as previously $K_{ik}$ is the bosonic permutation operator interchanging $x$-coordinates and

$$\partial_i = \theta_i^\dagger \hspace{1cm} \partial_i = \frac{d}{dx_i}$$  \hspace{1cm} (5.6)
From the general viewpoint one can consider the relations (1.5) - (1.11), as defining relations for generating elements of a (new?) class of infinite-dimensional associative algebras, deformed Heisenberg-Weyl algebras. Then the relations (5.1) - (5.3) extend them to deformed super Heisenberg-Weyl algebras. An interesting question then is if there exist physical applications of these algebras in connection with, say, higher-spin theories in two and more dimensions.

Since the bilinears constructed from bosonic and fermionic generating elements were shown in the previous section to form the superalgebra osp(2; 2) with respect to (anti)commutators one can expect that the deformed super Heisenberg-Weyl algebras contain the enveloping algebra of osp(2; 2) at least for sufficiently high \( N \) (for the condition that \( \nu \) is interpreted as an independent Abelian generator). For the simplest case \( N = 2 \) it was shown in [34] that the corresponding associative algebra is isomorphic to the enveloping algebra of osp(1; 2).

Another question that may be interesting to address is whether there exist other representations for the derivatives \( D_i \) and \( D_i \) that obey the crucial relation (5.1) (which in its turn guarantees that the derivatives \( D_i \) are mutually commuting in accordance with (1.7)) thus leading to other deformations of the super-Heisenberg algebra.

A direct analysis shows that there is a one parameter family of such derivatives given by,

\[
\alpha D_i = \partial_i + \theta_i d_i + \sum_{k \neq i} \frac{\nu \theta_i - \alpha \theta_k}{x_i - x_k} (1 - K_{ik})
\]

\[
+ \alpha (\nu - \alpha) \sum_{k : k \neq i, l \neq i, k \neq l} \theta_i \theta_k \theta_l \frac{1}{x_i - x_k} (1 - K_{ik}) \frac{1}{x_i - x_l} (1 - K_{il}) .
\]  

where \( D_i = 0 D_i \). The basic property

\[
\{ \alpha D_i , \alpha D_j \} = 2 \delta_{ij} \alpha D_i
\]

(5.8) can be shown to hold and therefore the \( \alpha D_i \)'s are still commuting.

Their explicit form is

\[
\alpha D_i = d_i + \sum_{l \neq i} \frac{\nu}{x_i - x_l} (1 - K_{il})
\]

\[
+ \frac{\alpha \theta_i \theta_l}{(x_i - x_l)^2} (1 - K_{il}) + \frac{\alpha \theta_i \theta_l}{x_i - x_l} (d_i - d_l) K_{il}
\]

\[
+ \nu \alpha \sum_{k : k \neq i, l \neq i, k \neq l} \theta_i \theta_k + \theta_i \theta_l + \theta_k \theta_l \frac{1}{x_k - x_l} (1 - K_{il}) K_{ik} .
\]  

(5.9)
The bosonic part $H_B = \frac{1}{2} \sum_i ((-\alpha D_i)^2 + x_i^2)$ of the Hamiltonian can be shown to have the following form

\[ H_B = -\frac{1}{2} \sum_i d_i^2 + \frac{1}{2} \sum_i x_i^2 - \frac{1}{2} \sum_{k,l;k\neq l} \left( -\frac{\nu}{(x_k - x_l)^2}(1 - K_{kl}) \right) \]

\[ + \frac{\nu}{x_k - x_l}(d_k - d_l) - 2\frac{\alpha \theta_k \theta_l}{(x_k - x_l)^3}(1 - K_{kl}) + \frac{\alpha \theta_k \theta_l}{(x_k - x_l)^2}(d_k - d_l)(1 - K_{kl}) \]

\[ - \frac{\nu \alpha}{2} \sum_{i,k,l; i\neq k\neq l \neq i} \left( \frac{\theta_k \theta_l + \theta_l \theta_i + \theta_i \theta_l}{(x_k - x_l)(x_l - x_i)} \right) \left( \frac{1}{3} - K_{ik} + \frac{2}{3} K_{ik} K_{il} \right). \quad (5.10) \]

Obviously these new derivatives $\alpha D_i$ lead to some solvable system and the question is if this system is really new or this is another way for describing the original system.

The result is that all these systems are indeed pairwise equivalent. To see this one observes that it is possible to introduce new variables $\alpha x_i$, $\alpha a_i^\pm$, $\alpha \theta_i^\pm$, and $\alpha K_{ij}$

\[ \alpha x_i = x_i + \alpha \sum_k \theta_i \theta_k K_{ik}, \quad (5.11) \]

\[ \alpha a_i^\pm = \frac{1}{\sqrt{2}} (\alpha x_i \pm \alpha D_i), \quad (5.12) \]

\[ \alpha \theta_i^- = \theta_i, \quad (5.13) \]

\[ \alpha \theta_i^+ = \alpha D_i - \theta_i \alpha D_i \quad (5.14) \]

\[ \alpha K_{ij} = \alpha K_{ij}^\theta K_{ij}^{\text{tot}} \]

\[ \alpha K_{ij}^\theta = \frac{1}{2} \left[ \alpha \theta_i^+ - \alpha \theta_j^+ , \alpha \theta_i^- - \alpha \theta_j^- \right] \quad (5.15) \]

such that the fermionic variables obey the standard commutation relations

\[ \{ \alpha \theta_i^\pm , \alpha \theta_j^\pm \} = 0, \quad \{ \alpha \theta_i^+ , \alpha \theta_j^- \} = \delta_{ij} \quad (5.16) \]

and commute with the bosonic variables $\alpha a_i^\pm$, and $\alpha K_{ij}$ while the latter obey the relations analogous to (2.7)

\[ [\alpha a_i^\pm , \alpha a_j^\pm] = 0 \]
\[ [\alpha a_i^-, \alpha a_j^+] = \delta_{ij} (1 + \nu \sum_{l=1}^{N} \alpha K_{il}) - \nu \alpha K_{ij} \] (5.17)

and \( \alpha K_{ij} \) behave as the permutation operators for the bosonic variables \( \alpha a_i^\pm \) obeying the relations analogous (1.8) - (1.11).

To make the relationship between the algebras with different \( \alpha \) explicit let us introduce the operator \( \Theta \)

\[ \Theta = \sum_{i,j : i \neq j} \frac{\theta_i \theta_j}{x_i - x_j} (1 - K_{ij}) \] (5.18)

Then, using the Baker-Hausdorff formula,

\[ e^{-\alpha B} A e^{\alpha B} = \sum_{n=0}^{\infty} \frac{\alpha^n}{n!} A_n \] (5.19)

valid for any operators \( A \) and \( B \) where \( A_0 = A \) and \( A_i = [A_{i-1}, B] \), one can prove that

\[ \alpha A = \exp \left( \frac{\alpha}{2} \Theta \right) A \exp \left( -\frac{\alpha}{2} \Theta \right) \] (5.20)

for any of operators \( \alpha x_i, \alpha D_i, \alpha D_i, \alpha a_i^\pm, \alpha \theta_i^\pm \), and \( \alpha K_{ij} \).

Thus the derivatives \( \alpha D_i \) lead to a solvable system equivalent to (4.19).

One can conjecture that the relations (1.5) - (1.11) and (5.1) - (5.5) exhaust all nontrivial deformations of the (super-) Heisenberg algebra involving the permutation generators.

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