Optimal control of the bidomain system (IV):
Corrected proofs of the stability and regularity theorems

Karl Kunisch† and Marcus Wagner‡

Abstract. In a series of papers on optimal control problems for the monodomain as well as for the bidomain equations of cardiac electrophysiology, the authors studied existence of minimizers and derived first-order necessary optimality conditions. The analysis of these control problems was based on a regularity discussion for weak solutions, resulting in a stability estimate and a uniqueness theorem for the monodomain and bidomain system, respectively. Unfortunately, the authors recognized a serious error within the proof of these theorems. However, the present investigation shows that the assertions from [Kunisch/Wagner 12] and [Kunisch/Wagner 13a] can be maintained with minor changes only but the proofs must be subjected to considerable alterations. In [Kunisch/Wagner 13b], the formulation of the related theorems has been corrected without delivering proofs. Therefore, in the present paper we provide a refined regularity discussion of the bidomain system together with corrected proofs.

Key words. PDE constrained optimization, bidomain equations, two-variable ionic models, weak solution, existence theorem, uniqueness theorem.

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1. Introduction.

a) Aim of the paper and main results.

We consider the full bidomain system, which represents a well-accepted description of the electrical activity
of the heart, as given through

\[
\begin{align*}
\frac{\partial \Phi_{tr}}{\partial t} + I_{ion}(\Phi_{tr}, W) - \text{div} \left( M_i \nabla \Phi_i \right) &= I_i \quad \text{for a. a. } (x,t) \in \Omega \times [0,T]; \\
\frac{\partial \Phi_{tr}}{\partial t} + I_{ion}(\Phi_{tr}, W) + \text{div} \left( M_e \nabla \Phi_e \right) &= -I_e \quad \text{for a. a. } (x,t) \in \Omega \times [0,T]; \\
\frac{\partial W}{\partial t} + G(\Phi_{tr}, W) &= 0 \quad \text{for a. a. } (x,t) \in \Omega \times [0,T]; \\
\mathbf{n}^T M_i \nabla \Phi_i &= 0 \quad \text{for all } (x,t) \in \partial \Omega \times [0,T]; \\
\mathbf{n}^T M_e \nabla \Phi_e &= 0 \quad \text{for all } (x,t) \in \partial \Omega \times [0,T]; \\
\Phi_{tr}(x,0) &= \Phi_i(x,0) - \Phi_e(x,0) = \Phi_0(x) \quad \text{and } W(x,0) = W_0(x) \quad \text{for a. a. } x \in \Omega
\end{align*}
\]

on a bounded domain \( \Omega \subset \mathbb{R}^3 \) with the fixed time horizon \( T > 0 \). Further, we consider the monodomain
system

\[
\begin{align*}
\frac{\partial \Phi_{tr}}{\partial t} + I_{ion}(\Phi_{tr}, W) - \frac{\lambda}{1+\lambda} \text{div} \left( M_i \nabla \Phi_{tr} \right) &= \frac{1}{1+\lambda} \left( \lambda I_i - I_e \right) \quad \text{for a. a. } (x,t) \in \Omega \times [0,T]; \\
\frac{\partial W}{\partial t} + G(\Phi_{tr}, W) &= 0 \quad \text{for a. a. } (x,t) \in \Omega \times [0,T]; \\
\mathbf{n}^T M_i \nabla \Phi_{tr} &= 0 \quad \text{for all } (x,t) \in \partial \Omega \times [0,T]; \\
\Phi_{tr}(x,0) &= \Phi_0(x) \quad \text{and } W(x,0) = W_0(x) \quad \text{for a. a. } x \in \Omega
\end{align*}
\]

arising as a special case of (1.1) – (1.6) if the conductivity tensors satisfy \( M_e = \lambda M_i \) with a constant
parameter \( \lambda > 0 \), thus allowing to eliminate \( \Phi_e \) as an independent variable. In a series of papers,\(^{02}\) the
authors investigated optimal control problems related to the dynamics (1.1) – (1.6) and (1.7) – (1.10) together with standard two-variable ionic models, namely the Rogers-McCulloch, FitzHugh-Nagumo and
the linearized Aliev-Panfilov model (see Subsection 2.a) below). Using \( I_e \) as control variable while \( I_i = 0 \),\(^{03}\)
and relying on a weak solution concept for the monodomain as well as for the bidomain system, the authors
studied existence of minimizers and derived first-order necessary optimality conditions.

\(^{01}\) The bidomain model has been considered first in [TUNG 78]. A detailed introduction may be found e. g. in [SUNDNES/LINES/CAI/NIelsen/MARDAL/TVEITO 06], pp. 21 – 56.

\(^{02}\) [KUNISCH/WAGNER 12], [KUNISCH/WAGNER 13A] and [KUNISCH/WAGNER 13B].

\(^{03}\) This setting is due to physiological reasons.
The analysis of the control problems is based on a regularity discussion for the weak solutions, which leads to a stability estimate and a uniqueness theorem for the monodomain and bidomain system, respectively. The regularity of the primal solutions influences the existence proof for the adjoint system as well. Unfortunately, the authors recognized a serious error within the proof of these theorems. The present investigation shows that the assertions from [KUNISCH/WAGNER 12] and [KUNISCH/WAGNER 13A] can be maintained with minor changes only but the proofs must be subjected to considerable alterations. In [KUNISCH/WAGNER 13B], the formulation of the related theorems has been corrected without delivering proofs. Therefore, in the present paper we provide a refined regularity discussion and corrected proofs.

Our main results read as follows:

**Theorem 1.1. (Stability estimate for weak solutions of the monodomain system)** \(^{04}\) We consider the monodomain system in its weak formulation \((2.9) - (2.11)\), assuming that \(\Omega \subset \mathbb{R}^3\) is a bounded Lipschitz domain, and \(M_1:\ cl(\Omega) \rightarrow \mathbb{R}^{3 \times 3}\) is a symmetric, positive definite matrix function with \(L^\infty(\Omega)\)-coefficients, which obeys a uniform ellipticity condition with \(\mu_1, \mu_2 > 0\):

\[
0 \leq \mu_1 \| \xi \|^2 \leq \xi^T M_1(x) \xi \leq \mu_2 \| \xi \|^2 \quad \forall \xi \in \mathbb{R}^3 \quad \forall x \in \Omega.
\]

\(^{04}\) Correction of [KUNISCH/WAGNER 12], p. 1533, Theorem 3.8.

1) Let us specify the Rogers-McCulloch or the FitzHugh-Nagumo model. If two weak solutions \((\Phi_{tr}', W')\), \((\Phi_{tr}'', W'')\) in \((C^0([0, T] \times \Omega), L^2(\Omega) \cap L^2([0, T), W^{1,2}(\Omega)) \times C^0([0, T], L^2(\Omega))\) of the system correspond to initial values \(\Phi_0' = \Phi_0'' = \Phi_0 \in L^2(\Omega), W_0' = W_0'' = W_0 \in L^4(\Omega)\) and inhomogeneities \(I_i', I_i'', I_i''\) and \(I_i''\) in \(L^\infty([0, T], (W^{1,2}(\Omega))^*)\), whose norms are bounded by \(R > 0\), then the following estimates hold:

\[
\|
\Phi_{tr}' - \Phi_{tr}''\|^2
\|^2_{C^0([0, T], L^2(\Omega))} + \|
\Phi_{tr}' - \Phi_{tr}''\|^2_{W^{1,4/3}([0, T], (W^{1,2}(\Omega))^*)}
\]

\[+ \|
\Phi_{tr}' - \Phi_{tr}''\|^2_{L^2([0, T), W^{1,2}(\Omega))} + \|
W' - W''\|^2_{W^{1,2}([0, T], L^2(\Omega))}
\]

\[\leq C \left(\|
I_i' - I_i''\|^2_{L^\infty([0, T], (W^{1,2}(\Omega))^*)} + \|
I_e' - I_e''\|^2_{L^\infty([0, T], (W^{1,2}(\Omega))^*)}\right).
\]

The constant \(C > 0\) does not depend on \(I_i', I_i'', I_i''\) but possibly on \(\Omega, R, \Phi_0, W_0\) and \(p = 4\).

2) If the linearized Aliiv-Panfilov model is specified, then Part 1) remains true provided that \(W'_0 = W''_0 = W_0\) belong to \(W^{3/2,2}([0, T], L^\infty(\Omega))\) instead of \(L^2(\Omega)\).

**Theorem 1.2. (Stability estimate for weak solutions of the bidomain system)** \(^{05}\) We consider the bidomain system in its weak formulation \((2.53) - (2.56)\), assuming that \(\Omega \subset \mathbb{R}^3\) is a bounded Lipschitz domain, \(M_1, M_2:\ cl(\Omega) \rightarrow \mathbb{R}^{3 \times 3}\) are symmetric, positive definite matrix functions with \(L^\infty(\Omega)\)-coefficients, obeying uniform ellipticity conditions with \(\mu_1, \mu_2 > 0\):

\[
0 \leq \mu_1 \| \xi \|^2 \leq \xi^T M_1(x) \xi \leq \mu_2 \| \xi \|^2 \quad \forall \xi \in \mathbb{R}^3 \quad \forall x \in \Omega.
\]

\(^{05}\) [KUNISCH/WAGNER 13B], p. 1082, Theorem 2.4., as a correction of [KUNISCH/WAGNER 13A], p. 959, Theorem 2.7.

\[
\int_\Omega \left(I_i'(x,t) + I_e'(x,t)\right) dx = \int_\Omega \left(I_i''(x,t) + I_e''(x,t)\right) dx = 0 \quad \text{for a. a. } t \in (0, T).
\]
and whose norms are bounded by \( R \) > 0, then the following estimates hold:

\[
\begin{align*}
\| \Phi_{i'} - \Phi_{i''} \|_{L^2([0, T], W^{1,2}(\Omega))}^2 + \| \Phi_{e'} - \Phi_{e''} \|_{C^0([0, T], L^2(\Omega))}^2 \\
+ \| \Phi_{i'} - \Phi_{i''} \|_{W^{1,4/3}([0, T], (W^{1,2}(\Omega))^*)}^2 + \| \Phi_{e'} - \Phi_{e''} \|_{L^2([0, T], W^{1,2}(\Omega))}^2 \\
+ \| W' - W'' \|_{L^2([0, T], L^2(\Omega))}^2 + \| W' - W'' \|_{C^0([0, T], L^2(\Omega))}^2 + \| W' - W'' \|_{W^{1,2}([0, T], L^2(\Omega))}^2 \\
\leq C \left( \| I_i' - I_i'' \|_{L^\infty([0, T], (W^{1,2}(\Omega))^*)}^2 + \| I_e' - I_e'' \|_{L^\infty([0, T], (W^{1,2}(\Omega))^*)}^2 \right).
\end{align*}
\]

\[\text{(1.15)}\]

The constant \( C > 0 \) does not depend on \( I_i', I_i'', I_e', I_e'' \) but possibly on \( \Omega, R, \Phi_0, W_0 \) and \( p = 4 \).

2) If the linearized Aliev-Panfilov model is specified then Part 1) remains true provided that \( W_0' = W_0'' = 0 \) belong to \( W^{1,3/2}(\Omega) \) instead of \( L^4(\Omega) \).

**Theorem 1.3. (Uniqueness of weak solutions of the monodomain system)** \(\text{[60]}\) Consider the monodomain system in its weak formulation (2.9) – (2.11) under the assumptions of Theorem 1.1. Specifying the Rogers-McCulloch or the FitzHugh-Nagumo model, the system admits a unique weak solution

\[
(\Phi_{i'}, W) \in \left( C^0([0, T], L^2(\Omega)) \cap L^2([0, T], W^{1,2}(\Omega)) \cap L^4(\Omega_T) \right) \times C^0([0, T], L^2(\Omega))
\]

in correspondence to initial values \( \Phi_0 \in L^2(\Omega), W_0 \in L^4(\Omega) \) and inhomogeneities \( I_i, I_e \in L^\infty([0, T], (W^{1,2}(\Omega))^*) \). If the linearized Aliev-Panfilov model is specified, this assertion remains true as far as \( W_0 \in W^{1,3/2}(\Omega) \).

**Theorem 1.4. (Uniqueness of weak solutions of the bidomain system)** \(\text{[67]}\) Consider the bidomain system in its weak formulation (2.53) – (2.56) under the assumptions of Theorem 1.2. Specifying the Rogers-McCulloch or the FitzHugh-Nagumo model, the system admits a unique weak solution

\[
(\Phi_{i'}, \Phi_e, W) \in \left( C^0([0, T], L^2(\Omega)) \cap L^2([0, T], W^{1,2}(\Omega)) \cap L^4(\Omega_T) \right)
\]

\[
\times L^2([0, T], W^{1,2}(\Omega)) \times C^0([0, T], L^2(\Omega))
\]

in correspondence to initial values \( \Phi_0 \in L^2(\Omega), W_0 \in L^4(\Omega) \) and inhomogeneities \( I_i, I_e \in L^\infty([0, T], (W^{1,2}(\Omega))^*) \). If the linearized Aliev-Panfilov model is specified, this assertion remains true as far as \( W_0 \in W^{1,3/2}(\Omega) \).

The error to be corrected was the claim that, under the assumptions of Theorems 1.1 – 1.4, the transmembrane potential \( \Phi_0 \) within a weak solution of (1.1) – (1.6) or (1.7) – (1.10) can admit \( L^4([0, T], L^5(\Omega)) \)- or even \( L^\infty([0, T], L^4(\Omega)) \)-regularity. \(\text{[68]}\) In Section 3 below, we will see that the theorems can be proven without relying on this claim.

The paper is structured as follows. We continue with a short collection of notations (Subsect. 1.b) and repeat, for the reader’s sake, the imbedding theorems for Bochner spaces used below (Subsect. 1.c). In Section 2, we start with the description of the ionic models, which will be subsequently used (Subsect. 2.a)). Then we restate the monodomain system in its weak formulation and study the existence and regularity of the weak solutions for the different models (Subsect. 2.b) – d). Subsequently, the weak formulation of

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\(\text{[60]}\) Correction of [Kunisch/Wagner 12], p. 1529, Theorem 2.5.

\(\text{[67]}\) [Kunisch/Wagner 13b], p. 1082, Theorem 2.3, as a correction of [Kunisch/Wagner 13a], p. 959, Theorem 2.8.

\(\text{[68]}\) This error can be traced back to [Kunisch/Wagner 12], p. 1534, (3.39), and p. 1544, (B.14), as well as to [Kunisch/Wagner 13a], p. 964, (2.69) and (2.70), respectively.
the full bidomain system together with existence and regularity results for its weak solutions is provided (Subsect. 2.e – g). In Section 3, we deliver the corrected proof of the stability estimates for the monodomain and bidomain system, respectively (Theorems 1.1. and 1.2.), which imply the uniqueness theorems (Theorems 1.3. and 1.4.). Finally, in Section 4, we list in full detail the corrections to be made in the authors’ previous papers.

b) Notations.

We abbreviate $\Omega \times [0, T]$ by $\Omega_T$. By $L^p(\Omega)$, we denote the space of functions, which are in the $p$th power integrable ($1 \leq p < \infty$), or are measurable and essentially bounded ($p = \infty$), and by $W^{1,p}(\Omega)$ the Sobolev space of functions $\psi : \Omega \to \mathbb{R}$ which, together with their first-order weak partial derivatives, belong to the space $L^p(\Omega, \mathbb{R})$ ($1 \leq p < \infty$). Concerning spaces of Bochner integrable mappings, e.g. $L^2\left([0, T], W^{1,2}(\Omega)\right)$, we refer to [KUNISCH/WAGNER 12], p. 1542. The gradient $\nabla$ is always taken only with respect to the spatial variables $x$. Finally, we use the nonstandard abbreviation “$(\forall) t \in A$”, which has to be read as “for almost all $t \in A$” or “for all $t \in A$ except for a Lebesgue null set”. The symbol $\sigma$ denotes, depending on the context, the zero element or the zero function of the underlying space.

c) Compact imbeddings of Bochner spaces.

**Theorem 1.5.** (Aubin-Dubinskij lemma)\(^{(9)}\) Consider three normed spaces $X_0 \subseteq X \subseteq X_1$ where the imbedding $X_0 \hookrightarrow X$ is compact and the imbedding $X \hookrightarrow X_1$ is continuous. If $p, p' \in (1, \infty)$ then the space

$$Y = \{ f \in L^p\left([0, T], X_0\right) \mid \frac{df}{dt} \in L^{p'}\left([0, T], X_1\right) \}$$

is compactly imbedded into $L^q\left([0, T], X\right)$ for arbitrary $q \in (1, \infty)$.

**Theorem 1.6.** (A generalization of the Aubin-Dubinskij lemma)\(^{(10)}\) Consider three Banach spaces $X_0 \subseteq X \subseteq X_1$ where the imbeddings $X_0 \hookrightarrow X$ and $X \hookrightarrow X_1$ are continuous while $X_0 \hookrightarrow X_1$ is compact. Assume further that there exists a number $0 < \vartheta < 1$ such that

$$\| \psi \|_X \leq C \| \psi \|_{X_0}^{1-\vartheta} \cdot \| \psi \|_{X_1}^{\vartheta} \quad \forall \psi \in X_0 \cap X_1.$$  \hspace{1cm} (1.19)

Then the space

$$Y = \{ f \in L^p\left([0, T], X_0\right) \mid \frac{df}{dt} \in L^{p'}\left([0, T], X_1\right) \}$$

is compactly imbedded into $C^0\left([0, T], X\right)$ provided that $1 \leq p < p' \leq \infty$ and $(1 - \vartheta)/p < \vartheta (1 - 1/p')$.

\(^{(9)}\)[DUBINSKIJ 65], p. 612, Teorema 1, and p. 615, Teorema 2. The formulation is not affected by the corrections, which have recently been presented in [BARRY/SÜLI 12].

\(^{(10)}\)[SIMON 87], p. 90, Corollary 8.
2. Regularity of weak solutions for the monodomain and bidomain system.

a) The ionic models.

The following models for the ionic current $I_{ion}$ and the function $G$ within the gating equation will be considered:

1) The FitzHugh-Nagumo model.\[^{11}\]

\[
I_{ion}(\varphi, w) = \varphi (\varphi - a) (\varphi - 1) + w = \varphi^3 - (a + 1) \varphi^2 + a \varphi + w; \tag{2.1}
\]

\[
G(\varphi, w) = \varepsilon w - \varepsilon \kappa \varphi \tag{2.2}
\]

with $0 < a < 1$, $\kappa > 0$ and $\varepsilon > 0$. Thus the gating variable obeys the linear ODE

\[
\partial W/\partial t + \varepsilon W = \varepsilon \kappa \Phi_{tr}. \tag{2.3}
\]

2) The Rogers-McCulloch model.\[^{12}\]

\[
I_{ion}(\varphi, w) = b \cdot (\varphi - a) (\varphi - 1) + \varphi \cdot w = b \varphi^3 - (a + 1) b \varphi^2 + a b \varphi + \varphi w; \tag{2.4}
\]

\[
G(\varphi, w) = \varepsilon w - \varepsilon \kappa \varphi \tag{2.5}
\]

with $0 < a < 1$, $b > 0$, $\kappa > 0$ and $\varepsilon > 0$. Consequently, the ODE for the gating variable is the same as before.

3) The linearized Aliev-Panfilov model.\[^{13}\]

\[
I_{ion}(\varphi, w) = b \cdot (\varphi - a) (\varphi - 1) + \varphi \cdot w = b \varphi^3 - (a + 1) b \varphi^2 + a b \varphi + \varphi w; \tag{2.6}
\]

\[
G(\varphi, w) = \varepsilon w - \varepsilon \kappa ((a + 1) \varphi - \varphi^2) \tag{2.7}
\]

with $0 < a < 1$, $b > 0$, $\kappa > 0$ and $\varepsilon > 0$. The linear ODE for the gating variable is

\[
\partial W/\partial t + \varepsilon W = \varepsilon \kappa ((a + 1) \Phi_{tr} - \Phi_{tr}^2). \tag{2.8}
\]

b) Weak formulation of the monodomain system and known regularity of weak solutions.

The weak formulation of the monodomain system (1.7) – (1.10) reads as follows:

\[
\int_{\Omega} \left( \frac{\partial \Phi_{tr}}{\partial t} + I_{ion}(\Phi_{tr}, W) \right) \psi dx + \int_{\Omega} \frac{\lambda}{1 + \lambda} \nabla \psi^T M_i \nabla \Phi_{tr} dx = \int_{\Omega} \frac{1}{1 + \lambda} (\lambda I - I_c) \psi dx \tag{2.9}
\]

\[
\forall \psi \in W^{1,2}(\Omega) \ (\forall) t \in [0, T];
\]

\[
\int_{\Omega} \left( \frac{\partial W}{\partial t} + G(\Phi_{tr}, W) \right) \psi dx = 0 \ \forall \psi \in L^2(\Omega) \ (\forall) t \in [0, T] \tag{2.10}
\]

\[
\Phi_{tr}(x, 0) = \Phi_0(x) \ (\forall) x \in \Omega; \ W(x, 0) = W_0(x) \ (\forall) x \in \Omega \tag{2.11}
\]

where $\lambda > 0$. Under the assumptions of Theorem 1.1., the system (2.9) – (2.11) with either the FitzHugh-Nagumo, the Rogers-McCulloch or the linearized Aliev-Panfilov model admits for arbitrary initial values $\Phi_0$, $W_0 \in L^2(\Omega)$ and inhomogeneities $I_i$, $I_c \in L^2([0, T], (W^{1,2}(\Omega))^*)$ at least one weak solution\[^{14}\]

\[
(\Phi_{tr}, W) \in \left(C^0([0, T], L^2(\Omega)) \cap L^2([0, T], W^{1,2}(\Omega)) \cap L^4(\Omega, T)\right) \times \left(C^0([0, T], L^2(\Omega)) \right). \tag{2.12}
\]

\[^{11}\] [FITZHugh 61], together with [NAGumo/ARIMOTO/YOSHIZAWA 62].

\[^{12}\] [ROGERS/MCCULLOCH 94].

\[^{13}\] This model is taken from [BOURGAULT/COUDIERE/PIERRE 09], p. 480. Instead, the original model from [ALIEV/PANFILOV 96] contains a Riccati equation for the gating variable.

\[^{14}\] [KUNiSH/WAGNER 12], p. 1528 f., Theorem 2.2.
Any weak solution satisfies the a-priori estimate\(^{15}\)
\[
\|\Phi_{tr}\|_{C^0([0,T],L^2(\Omega))}^2 + \|\Phi_{tr}\|_{L^2([0,T],W^{1,2}(\Omega))}^2 + \|\Phi_{tr}\|_{L^4(\Omega_T)}^4 + \|\partial \Phi_{tr}/\partial t\|_{L^{4/3}(\{0,T\},(W^{1,2}(\Omega))^*)}^{4/3}
\]
\[
\leq C \cdot \left(1 + \|\Phi_0\|_{L^2(\Omega)}^2 + \|W_0\|_{L^2(\Omega)}^2 + \|I_1\|_{L^2([0,T],(W^{1,2}(\Omega))^*)}^2 + \|I_\varepsilon\|_{L^2([0,T],(W^{1,2}(\Omega))^*)}^2\right) \quad (2.13)
\]
with a constant \(C > 0\), which does not depend on \(\Phi_0\), \(W_0\), \(I_1\) and \(I_\varepsilon\). We will investigate now how to improve the regularity of a given weak solution, depending on the model and the regularity of \(W_0\).

c) Rogers-McCulloch model: improvement of regularity for the weak solutions.

**Proposition 2.1. (Gain of regularity for the transmembrane potential)** Consider the monodomain system with the Rogers-McCulloch model under the assumptions of Theorem 1.1., and let a weak solution \((\Phi_{tr},W)\) of it belong to the spaces in (2.12). Then \(\Phi_{tr}\) belongs to \(L^2\left([0,T)\right) \cap L^0\left([0,T)\right)\) for all \(1 < q < \infty\) and all \(4 \leq r < 6\). In particular, \(\Phi_{tr} \in L^5(\Omega_T)\).

**Proof.** Concerning the first inclusion, note that on \(\Omega \subset \mathbb{R}^3\) the imbedding \(W^{1,2}(\Omega) \hookrightarrow L^6(\Omega)\) is still continuous, cf. [ADAMS/FOURNIER 07], p. 85, Theorem 4.12., Case C. To get the second one, in view of (2.13), we may apply the Aubin-Dubinskij lemma (Theorem 1.5.) to \(\Phi_{tr}\), choosing \(X_0 = W^{1,2}(\Omega)\), \(X = L^\infty(\Omega)\) with \(4 \leq r < 6\), \(X_1 = (W^{1,2}(\Omega))^*, p = 2\) and \(p' = 4/3\).

**Proposition 2.2. (Gain of regularity for the gating variable)** Consider the monodomain system with the Rogers-McCulloch model under the assumptions of Theorem 1.1., and let a weak solution \((\Phi_{tr},W)\) of it belong to the spaces in (2.12). Then \(W\) belongs to \(C^1\left([0,1),L^2(\Omega)\right]\). Moreover, if \(W_0 \in L^4(\Omega)\) then \(W\) belongs even to \(C^0\left([0,T],L^4(\Omega)\right]\), and it holds that
\[
\|W\|^4_{C^0([0,T],L^4(\Omega))} \leq C \left(1 + \|\Phi_0\|_{L^2(\Omega)}^2 + \|W_0\|_{L^2(\Omega)}^4 + \|I_1\|_{L^2([0,T],(W^{1,2}(\Omega))^*)}^2 + \|I_\varepsilon\|_{L^2([0,T],(W^{1,2}(\Omega))^*)}^2\right) .
\]

**Proof.** Observe first that \(W\) admits the representation
\[
W(x,t) = W_0(x) e^{-\varepsilon t} + \varepsilon \kappa e^{-\varepsilon t}\int_0^t \Phi_{tr}(x,\tau) e^{\varepsilon \tau} d\tau ,
\]
from which the \(C^1\left([0,1),L^2(\Omega)\right]\)-regularity of \(W\) follows by differentiation. From the representation (2.15), we derive together with (2.13) the further estimate
\[
\int_\Omega W(t)^4 dx \leq C \|W_0\|_{L^4(\Omega)}^4 + C \|\Phi_{tr}\|_{L^4(\Omega_T)}^4 \quad (2.16)
\]
\[
\leq C \left(1 + \|\Phi_0\|_{L^2(\Omega)}^2 + \|W_0\|_{L^2(\Omega)}^4 + \|I_1'\|_{L^2([0,T],(W^{1,2}(\Omega))^*)}^2 + \|I_\varepsilon'\|_{L^2([0,T],(W^{1,2}(\Omega))^*)}^2\right) ,
\]
which confirms that \(W\) belongs to \(L^\infty\left([0,T),L^4(\Omega)\right]\). In order to confirm that \(\|W(\cdot,t)\|^4_{L^4(\Omega)}\) depends continuously on \(t\), consider for \(s, t \in [0,T]\)
\[
\|W(s)\|^4_{L^4(\Omega)} - \|W(t)\|^4_{L^4(\Omega)} = \left| \int_\Omega \left( W(s)^4 - W(t)^4 \right) dx \right| \quad (2.17)
\]
\[
= \left| \int_\Omega \left( W(s) - W(t) \right) \left( W(s)^3 + W(s)^2 W(t) + W(s) W(t)^2 + W(t)^3 \right) dx \right| \quad (2.18)
\]
\[
\leq \left( \int_\Omega |W(s) - W(t)|^4 dx \right)^{1/4} \left( \int_\Omega |W(s)^3 + W(s)^2 W(t) + W(s) W(t)^2 + W(t)^3|^3 dx \right)^{3/4} . \quad (2.19)
\]
\(^{15}\) [KUNISCH/WAGNER 12], p. 1529, Theorem 2.4.
implies

Estimation of the second term yields:

\[
\int_{\Omega} \left| W(s) - W(t) \right|^4 \, dx = \int_{\Omega} \left| W_0 \left( e^{-\epsilon s} - e^{-\epsilon t} \right) + \epsilon \kappa e^{-\epsilon s} \left( \int_0^s \Phi_{tr} e^{\epsilon \tau} \, d\tau - \int_0^t \Phi_{tr} e^{\epsilon \tau} \, d\tau \right) + \epsilon \kappa \left( e^{-\epsilon s} - e^{-\epsilon t} \right) \right|^4 \, dx
\]

\[
\leq C \left| e^{-\epsilon s} - e^{-\epsilon t} \right|^4 \cdot \left\| W_0 \right\|_{L^4(\Omega)}^4 + C \epsilon^4 \kappa^4 \int_{\Omega} \left| \int_0^s \Phi_{tr} e^{\epsilon \tau} \, d\tau \right|^4 \, dx
\]

\[
\leq C \left| e^{-\epsilon s} - e^{-\epsilon t} \right|^4 \cdot \left\| W_0 \right\|_{L^4(\Omega)}^4 + C \epsilon^4 \kappa^4 \left| e^{-\epsilon s} - e^{-\epsilon t} \right|^4 \cdot \int_{\Omega} \left| t \right|^4 \, dx
\]

\[
\leq C \left| e^{-\epsilon s} - e^{-\epsilon t} \right|^4 \cdot \left\| W_0 \right\|_{L^4(\Omega)}^4 + C \epsilon^4 \kappa^4 \left| e^{-\epsilon s} - e^{-\epsilon t} \right|^4 \cdot \int_{\Omega} \left| t \right|^3 \, dx
\]

\[
= C \left| e^{-\epsilon s} - e^{-\epsilon t} \right|^4 \cdot \left\| W_0 \right\|_{L^4(\Omega)}^4 + C \epsilon^4 \kappa^4 \left| e^{-\epsilon s} - e^{-\epsilon t} \right|^4 \cdot \left\| \Phi_{tr} \right\|_{L^4(\Omega)}^4
\]

\[
\leq C \int_{\Omega} \left( \left| W(s) \right|^4 + \left| W(t) \right|^4 + \left| W(t) \right|^{8/3} + \left| W(s) \right|^{8/3} + \left| W(t) \right|^{8/3} + \left| W(t) \right|^4 \right) \, dx.
\]

From (2.16) we already know that \( \int_{\Omega} \left| W(s) \right|^4 \, dx \) and

\[
\int_{\Omega} \left| W(s) \right|^{8/3} \left| W(t) \right|^{1/3} \, dx \leq \int_{\Omega} \left( \max \left( \left| W(s) \right|, \left| W(t) \right| \right) \right)^4 \, dx
\]

\[
\leq \int_{\Omega} \max \left( \left| W(s) \right|^4, \left| W(t) \right|^4 \right) \, dx \leq \int_{\Omega} \left( \left| W(s) \right|^4 + \left| W(t) \right|^4 \right) \, dx
\]

remain uniformly bounded. Consequently, for every \( \varepsilon > 0 \) we may determine \( \delta(\varepsilon) > 0 \) such that \( |s - t| \leq \delta(\varepsilon) \) implies \( \left\| W(s) \right\|_{L^4(\Omega)}^4 \leq \left\| W(t) \right\|_{L^4(\Omega)}^4 \), and \( W \) belongs to \( C^0[0, T], L^4(\Omega) \).

If the Rogers-McCulloch model is replaced by the FitzHugh-Nagumo model then Propositions 2.1. and 2.2. hold accordingly.

d) Linearized Aliev-Panfilov model: improvement of regularity for the weak solutions.

**Proposition 2.3. (Gain of regularity for the transmembrane potential)** Consider the monodomain system with the linearized Aliev-Panfilov model under the assumptions of Theorem 1.1., and let a weak
solution $\Phi_{tr}, W$ of it belong to the spaces in (2.12). Then $\Phi_{tr} \in L^2([0, T), L^6(\Omega)] \cap L^q([0, T), L^r(\Omega)]$ for all $1 < q < \infty$ and all $4 \leq r < 6$. In particular, $\Phi_{tr} \in L^5(\Omega_T)$.

**Proof.** The proof of Proposition 2.1. may be carried over without alterations. ■

**Proposition 2.4.** *(Gain of regularity for the gating variable)* Consider the monodomain system with the linearized Aliev-Panfilov model under the assumptions of Theorem 1.1., and let a weak solution $(\Phi_{tr}, W)$ of it belong to the spaces in (2.12).

1) Then $W$ belongs to $C^1\left([0, 1], L^1(\Omega)\right)$.

2) If $2 \leq r < 3$ and $W_0 \in L^r(\Omega)$ then $\partial W/\partial t$ belongs to $L^r(\Omega_T)$.

3) If $26/9 < r < 3$ and $W_0 \in W^{1,r/2}(\Omega)$ then $W$ belongs to $L^1\left([0, T), W^{1,r/2}(\Omega)\right) \cap C^0\left([0, T), L^{4/3}(\Omega)\right)$, and $\|W\|_{C^0([0,T], L^{4/3}(\Omega))}$ is bounded by a constant depending on $r$ and the norms $\|\Phi_0\|_{L^2(\Omega)}^2$, $\|W_0\|_{W^{1,r/2}(\Omega)}$, $\|L'_r\|_{L^2([0,T), W^{1,r/2}(\Omega))^*}$ and $\|L'_r\|_{L^2([0,T), (W^{1,r}(\Omega))^*)}$ only.

**Proof.** Note first that, under the assumptions of the proposition, $W$ is represented as

$$W(x, t) = W_0(x) e^{-\varepsilon t} + \varepsilon \kappa e^{-\varepsilon t} \int_0^t \left( (a + 1) \Phi_{tr}(x, \tau) - \Phi_{tr}^2(x, \tau) \right) e^{\varepsilon \tau} d\tau,$$

and (2.13) holds still true.

**Part 1)** By differentiation of (2.29), we get

$$\frac{\partial W}{\partial t}(x, t) = -W_0(x) \varepsilon e^{-\varepsilon t} - \varepsilon \kappa e^{-\varepsilon t} \int_0^t \left( (a + 1) \Phi_{tr}(x, \tau) - \Phi_{tr}^2(x, \tau) \right) e^{\varepsilon \tau} d\tau + \varepsilon \kappa \left( (a + 1) \Phi_{tr}(x, t) - \Phi_{tr}(x, t)^2 \right) \Rightarrow$$

$$\int \left| \frac{\partial W}{\partial t}(x, t) \right| dx \leq \varepsilon e^{-\varepsilon t} \|W_0\|_{L^1(\Omega)} + \varepsilon \kappa e^{-\varepsilon t} \int_0^t \int \left( (a + 1) |\Phi_{tr}(x, \tau)| + |\Phi_{tr}(x, \tau)|^2 \right) e^{\varepsilon \tau} dx d\tau$$

$$+ \varepsilon \kappa \left( (a + 1) \|\Phi_{tr}(t)\|_{L^1(\Omega)} + \|\Phi_{tr}(t)\|_{L^2(\Omega)}^2 \right)$$

$$\leq \varepsilon e^{-\varepsilon t} \|W_0\|_{L^1(\Omega)} + C \varepsilon \kappa e^{-\varepsilon t} \int_0^t \left( \|\Phi_{tr}(\tau)\|_{L^1(\Omega)} + \|\Phi_{tr}(\tau)\|_{L^2(\Omega)}^2 \right) e^{\varepsilon \tau} dx d\tau$$

$$+ C \varepsilon \kappa \left( \|\Phi_{tr}(t)\|_{L^2(\Omega)} + \|\Phi_{tr}(t)\|_{L^2(\Omega)}^2 \right),$$

and $\Phi_{tr} \in C^0\left([0, T], L^2(\Omega)\right)$ implies $\partial W/\partial t \in C^0\left([0, T), L^1(\Omega)\right)$. Thus the claimed $C^1\left([0, 1], L^1(\Omega)\right)$-regularity of $W$ is proved.

**Part 2)** In order to confirm 2), we return to (2.30) and estimate for $2 \leq r < 3$

$$\int \left| \frac{\partial W}{\partial t}(x, t) \right|^r dx \leq C e^{-r \varepsilon t} \|W_0\|_{L^r(\Omega)} + C e^{-r \varepsilon t} \int_0^t \left( (a + 1) \Phi_{tr}(x, \tau) \right)$$

$$- \Phi_{tr}^2(x, \tau) e^{r \varepsilon \tau} d\tau + C \int_0^t \left( (a + 1) |\Phi_{tr}(x, t)|^r + |\Phi_{tr}(x, t)|^{2r} \right) dx$$

$$\leq C e^{-r \varepsilon t} \|W_0\|_{L^r(\Omega)} + C e^{-r \varepsilon t} \int_0^t \left( (a + 1) |\Phi_{tr}(x, \tau)| + |\Phi_{tr}(x, \tau)|^2 \right) e^{r \varepsilon \tau} d\tau$$

$$+ C \left( \|\Phi_{tr}(t)\|_{L^r(\Omega)} + \|\Phi_{tr}(t)\|_{L^2(\Omega)}^2 \right)$$

$$\leq C \|W_0\|_{L^r(\Omega)} + C \int_0^T \left( |\Phi_{tr}(x, \tau)| + |\Phi_{tr}(x, \tau)|^2 \right) e^{r \varepsilon T} d\tau$$

$$+ C \left( \|\Phi_{tr}(t)\|_{L^r(\Omega)} + \|\Phi_{tr}(t)\|_{L^2(\Omega)}^2 \right)$$
Consequently, we find that
\[ L(t) = C \| W(t) \|_{L^r(\Omega)} + C \int_0^T \left( \left| \Phi_{tr}(x, \tau) \right|^r + \left| \Phi_{tr}(x, \tau) \right|^{2r} \right) d\tau \]  
\[ + C \left( \left\| \Phi_{tr}(t) \right\|_{L^r(\Omega)}^2 + \left\| \Phi_{tr}(t) \right\|_{L^{2r}(\Omega)}^2 \right) \]  
\[ \int_0^T \left( \left| \Phi_{tr}(x, \tau) \right|^{2r} \right) d\tau \]  
\[ = C \left( \| W(t) \|_{L^r(\Omega)}^r + \left\| \Phi_{tr} \right\|_{L^r((0, T), L^r(\Omega))}^r + \left\| \Phi_{tr} \right\|_{L^{2r}((0, T), L^{2r}(\Omega))}^r \right. \]  
\[ \left. + \left\| \Phi_{tr}(t) \right\|_{L^r(\Omega)}^2 + \left\| \Phi_{tr}(t) \right\|_{L^{2r}(\Omega)}^2 \right) \]  
\[ \| W \|_{L^r(\Omega_t)} \leq C \left( \| W \|_{L^r(\Omega)} + \left\| \Phi_{tr} \right\|_{L^r(\Omega_t)} + \left\| \Phi_{tr} \right\|_{L^{2r}(\Omega_t)} \right), \]  
which is finite by Proposition 1.6. for all \( 2 \leq r < 3 \).

**Part 3** Let us fix an exponent \( 26/9 < r < 3 \). Note first that the partial generalized derivatives of \( \Phi_{tr} \in L^2((0, T), W^{1,2}(\Omega)) \) satisfy the identity
\[ \frac{\partial}{\partial x_i} \int_0^t \Phi_{tr}(x, \tau) d\tau = \int_0^t \frac{\partial}{\partial x_i} \Phi_{tr}(x, \tau) d\tau \quad (\forall) x \in \Omega \quad (\forall) t \in (0, T), \quad 1 \leq i \leq 3. \]  
Consequently, we find that
\[ \frac{\partial}{\partial x_i} \| W(t) \|_{r/2} \leq C \left| \frac{\partial W_0}{\partial x_i}(x) \right|^{r/2} + C \int_0^t \left( \left| \frac{\partial \Phi_{tr}}{\partial x_i}(x, \tau) \right|^{r/2} + \left| \frac{\partial \Phi_{tr}}{\partial x_i}(x, \tau) \Phi_{tr}(x, \tau) \right|^{r/2} \right) d\tau \]  
\[ \leq C \left| \frac{\partial W_0}{\partial x_i}(x) \right|^{r/2} + C \int_0^t \left( \left| \frac{\partial \Phi_{tr}}{\partial x_i}(x, \tau) \right|^r + \left| \frac{\partial \Phi_{tr}}{\partial x_i}(x, \tau) \right|^2 \right) d\tau \]  
\[ \leq C \left| \frac{\partial W_0}{\partial x_i}(x) \right|^{r/2} + C \int_0^t \left( \left| \frac{\partial \Phi_{tr}}{\partial x_i}(x, \tau) \right|^{r/2} + \left| \frac{\partial \Phi_{tr}}{\partial x_i}(x, \tau) \Phi_{tr}(x, \tau) \right|^{r/2} \right) d\tau \]  
\[ \int_0^t \left| \frac{\partial W}{\partial x_i}(x, \tau) \right|^{r/2} \]  
\[ \leq C \left\| \frac{\partial W_0}{\partial x_i} \right\|_{L^{r/2}(\Omega)}^{r/2} + C \int_0^t \left( \left| \frac{\partial \Phi_{tr}}{\partial x_i}(x, \tau) \right|^{r/2} + \left| \frac{\partial \Phi_{tr}}{\partial x_i}(x, \tau) \Phi_{tr}(x, \tau) \right|^{r/2} \right) d\tau \]  
\[ = C \left\| \frac{\partial W_0}{\partial x_i} \right\|_{L^{r/2}(\Omega)}^{r/2} + C \int_0^t \left( \left| \frac{\partial \Phi_{tr}}{\partial x_i}(x, \tau) \right|^{r/2} + \left| \frac{\partial \Phi_{tr}}{\partial x_i}(x, \tau) \Phi_{tr}(x, \tau) \right|^{r/2} \right) d\tau \]  
\[ \leq C \left\| \frac{\partial W_0}{\partial x_i} \right\|_{L^{r/2}(\Omega)}^{r/2} + C \left\| \frac{\partial \Phi_{tr}}{\partial x_i} \right\|_{L^{r/2}(\Omega_t)}^{r/2} + \left( \int_0^T \left( \left| \frac{\partial \Phi_{tr}}{\partial x_i}(x, \tau) \right|^2 dx \right)^{r/4} \right) \]  
\[ \left( \int_0^T \left( \left| \Phi_{tr}(x, \tau) \right|^{2r/(4-r)} dx \right)^{(4-r)/4} \right) \]  
\[ = C \left\| \frac{\partial W_0}{\partial x_i} \right\|_{L^{r/2}(\Omega)}^{r/2} + C \left\| \frac{\partial \Phi_{tr}}{\partial x_i} \right\|_{L^{r/2}(\Omega_t)}^{r/2} + \left( \left\| \frac{\partial \Phi_{tr}}{\partial x_i} \right\|_{L^{2r/(4-r)}(\Omega_t)}^{2r/(4-r)} \right)^{(4-r)/4} \]  
\[ \left( \left\| \Phi_{tr} \right\|_{L^{2r/(4-r)}(\Omega_t)}^{2r/(4-r)} \right)^{(4-r)/4} \]  
Since \( 2 \leq r < 3 \) implies \( 2 \leq 2r/(4-r) < 6 \), Proposition 2.3. ensures that the last expression is finite. Consequently, \( \| \partial W/\partial x_i \|_{L^r((0, T), L^{r/2}(\Omega))} \) is finite, and \( W \) belongs to \( L^1((0, T), W^{1,r/2}(\Omega)) \). On the
other hand, since the imbedding $W^{1,r/2}(\Omega) \hookrightarrow L^r(\Omega)$ is compact for all $2 \leq r < 3$, $W_0$ belongs to $L^r(\Omega)$ as well, and we know from Part 2) that $\partial W/\partial t$ belongs to $L^2\left[ (0, T), L^r(\Omega) \right]$. Now we are in position to apply the generalized Aubin-Dubinskij lemma (Theorem 1.6.) to $X_0 = W^{1,r/2}(\Omega)$ with $26/9 < r < 3$, $X = L^{8/3}(\Omega)$, $X_1 = L^{47/18}(\Omega)$, $p = 1$, $p' = 2$ and $\vartheta = 47/60$. Then the imbedding $W^{1,r/2}(\Omega) \hookrightarrow L^{47/18}(\Omega)$ is compact. Note that
\[
\frac{3}{8} = (1 - \vartheta) \frac{9}{26} + \vartheta \frac{18}{26} = \frac{13}{60} \cdot \frac{9}{26} + \frac{47}{60} \cdot \frac{18}{26}.
\]
For $\psi \in L^{8/3}(\Omega)$, we find
\[
\| \psi \|_{L^{8/3}(\Omega)} = \left( \int_{\Omega} |\psi|^{8/3} \, dx \right)^{3/8} = \left( \int_{\Omega} |\psi|^{(8/3)\cdot(13/60) - (47/60)} \, dx \right)^{3/8}
\]
\[
\leq \left( \int_{\Omega} |\psi|^{26/9} \, dx \right)^{3/10} \left( \int_{\Omega} |\psi|^{47/18} \, dx \right)^{3/10}
\]
\[
= \| \psi \|_{L^{26/9}(\Omega)} \cdot \| \psi \|_{L^{47/18}(\Omega)} \leq C \| \psi \|_{W^{1,r/2}(\Omega)} \cdot \| \psi \|_{L^{47/60}(\Omega)}
\]
due to the continuous (even compact) imbedding $W^{1,r/2}(\Omega) \hookrightarrow L^{26/9}(\Omega)$ and the continuous imbedding $L^r(\Omega) \hookrightarrow L^{47/18}(\Omega)$. Since $(1 - \vartheta)/p = 13/60 < 47/120 = \vartheta(1 - 1/p')$, we may conclude that $W \in C^0\left[ [0, T], L^{8/3}(\Omega) \right]$. Finally, Theorem 1.6. yields the norm estimate
\[
\sup_{0 \leq t \leq T} \| W(t) \|_{L^{8/3}(\Omega)} \leq C \left( \| W \|_{L^1([0, T], W^{1,r/2}(\Omega))} + \| \partial W/\partial t \|_{L^2([0, T], L^{47/18}(\Omega))} \right),
\]
and this expression is bounded by a constant $C$ depending on $r$, $\| \Phi_0 \|_{L^2(\Omega)}$, $\| W_0 \|_{W^{1,r/2}(\Omega)}$ and the norms of the inhomogeneities $I_i$ and $I_e$.

e) Weak formulation of the bidomain system and known regularity of weak solutions.

The full bidomain system (1.1) – (1.6) can be equivalently stated in a parabolic-elliptic form.\footnote{[KUNISCH/WAGNER 13A], p. 954, (2.1) – (2.6).} To this, the following weak formulation corresponds.
\[
\int_{\Omega} \left( \frac{\partial \Phi_T}{\partial t} \cdot \psi + \nabla \psi^T M_i (\nabla \Phi_T + \nabla \Phi_e) + I_{ion}(\Phi_T, W) \psi \right) \, dx = \int_{\Omega} I_i \psi \, dx \quad \forall \psi \in W^{1,2}(\Omega), \text{ for a. a. } t \in (0, T); \quad (2.53)
\]
\[
\int_{\Omega} \left( \nabla \psi^T M_i \nabla \Phi_T + \nabla \psi^T (M_i + M_e) \nabla \Phi_e \right) \, dx = \int_{\Omega} \left( I_i + I_e \right) \psi \, dx \quad \forall \psi \in W^{1,2}(\Omega) \text{ with } \int_{\Omega} \psi(x) \, dx = 0, \text{ for a. a. } t \in (0, T); \quad (2.54)
\]
\[
\int_{\Omega} \left( \frac{\partial W}{\partial t} + G(\Phi_T, W) \right) \psi \, dx = 0 \quad \forall \psi \in L^2(\Omega), \text{ for a. a. } t \in (0, T); \quad (2.55)
\]
\[
\Phi_T(x, 0) = \Phi_0(x) \quad \text{and} \quad W(x, 0) = W_0(x) \quad \text{for almost all } x \in \Omega. \quad (2.56)
\]
Under the assumptions of Theorem 1.2., the system (2.53) – (2.56) with either the FitzHugh-Nagumo, the Rogers-McCulloch or the linearized Aliev-Panfilov model admits for arbitrary initial values $\Phi_0, W_0 \in L^2(\Omega)$ and inhomogeneities $I_i, I_e \in L^2\left[ (0, T), \left( W^{1,2}(\Omega) \right)^* \right]$, which satisfy the compatibility condition
\[
\int_{\Omega} \left( I_i(x, t) + I_e(x, t) \right) \, dx = 0 \quad \text{for a. a. } t \in (0, T), \quad (2.57)
\]
at least one weak solution\textsuperscript{17}
\[(\Phi_{tr}, \Phi_e, W) \in \left( C^0([0, T], L^2(\Omega)) \cap L^2([0, T], W^{1,2}(\Omega)) \cap L^4(\Omega_T) \right) \times L^2([0, T], W^{2,2}(\Omega)) \times C^0([0, T], L^2(\Omega)) \] (2.58)
with \( \int_{\Omega} \Phi_e(t,x) dx = 0 \) for almost all \( t \in (0, T) \). Any weak solution obeys the a-priori estimate\textsuperscript{18}
\[
\| \Phi_{tr} \|_{C^0([0, T], L^2(\Omega))}^2 + \| \Phi_{tr} \|_{L^2([0, T], W^{1,2}(\Omega))}^2 + \| \Phi_{tr} \|_{L^4(\Omega_T)}^4 + \| \partial \Phi_{tr} / \partial t \|_{L^{4/3}(\Omega_T)}^{4/3} + \| W \|_{C^0([0, T], L^2(\Omega))}^2 + \| \partial W / \partial t \|_{L^2([0, T], (W^{1,2}(\Omega))^*)}^2 \]
\[ \leq C \left( 1 + \| \Phi_0 \|_{L^2(\Omega)}^2 + \| W_0 \|_{L^2(\Omega)}^2 + \| I_i \|_{L^2((0, T), (W^{1,2}(\Omega))^*)}^2 + \| I_e \|_{L^2((0, T), (W^{1,2}(\Omega))^*)}^2 \right) \] (2.59)
with a constant \( C > 0 \) not depending on \( \Phi_0, W_0, I_i \) and \( I_e \). It turns out that the regularity of \( \Phi_{tr} \) and \( W \) within a weak solution \( (\Phi_{tr}, \Phi_e, W) \) of (2.53) – (2.56) can be improved in complete analogy to Subsections 2.c and d).

f) Rogers-McCulloch model: improvement of regularity for the weak solutions.

**Proposition 2.5. (Gain of regularity for the transmembrane potential)** Consider the bidomain system (2.53) – (2.57) with the Rogers-McCulloch model under the assumptions of Theorem 1.2., and let a weak solution \( (\Phi_{tr}, \Phi_e, W) \) of it belong to the spaces in (2.58). Then \( \Phi_{tr} \) belongs to \( L^2([0, T], L^6(\Omega)) \cap L^q([0, T], L^r(\Omega)) \) for all \( 1 < q < \infty \) and all \( 4 \leq r < 6 \). In particular, \( \Phi_{tr} \in L^5(\Omega_T) \).

**Proof.** Under the assumptions of Theorem 1.2., a triple \((\Phi_{tr}, \Phi_e, W)\) forms a weak solution of the bidomain system (2.53) – (2.56) iff \((\Phi_{tr}, W)\) solves the reduced bidomain system\textsuperscript{19}
\[
\int_{\Omega} \left( \frac{\partial \Phi_{tr}}{\partial t} + I_{ion}(\Phi_{tr}(t), W(t)) \right) \psi dx + A(\Phi_{tr}(t), \psi) = \int_{\Omega} S(t) \psi dx \quad \forall \psi \in W^{1,2}(\Omega) \text{ for a. a. } t \in (0, T) ;
\]
\[
\int_{\Omega} \left( \frac{\partial W}{\partial t} + G(\Phi_{tr}, W) \right) \psi dx = 0 \quad \forall \psi \in L^2(\Omega) \text{ for a. a. } t \in (0, T) \]
\[
\Phi_{tr}(x, 0) = \Phi_0(x) \quad \text{and} \quad W(x, 0) = W_0(x) \quad \forall x \in \Omega .
\] (2.60)
(2.61)
(2.62)
where \( A : W^{1,2}(\Omega) \times W^{1,2}(\Omega) \) is the bidomain bilinear form, and \( S(t) \) is defined with the aid of \( I_i \) and \( I_e \), cf. \( \text{[KUNISCH/WAGNER 13A], p. 956 f., (2.22) – (2.25).} \) The structure of the reduced bidomain system and the monodomain system is in complete analogy, and the respective solutions obey the same type of a-priori estimates. Consequently, the proof of Proposition 2.1. may be carried over. \( \blacksquare \)

**Proposition 2.6. (Gain of regularity for the gating variable)** Consider the bidomain system (2.53) – (2.57) with the Rogers-McCulloch model under the assumptions of Theorem 1.2., and let a weak solution \((\Phi_{tr}, \Phi_e, W)\) of it belong to the spaces in (2.58). Then \( W \) belongs to \( C^1([0, 1], L^2(\Omega)) \). Moreover, if \( W_0 \in L^4(\Omega) \) then \( W \) belongs even to \( C^0([0, T], L^4(\Omega)) \), and it holds that
\[
\| W \|_{C^0([0, T], L^4(\Omega))} \leq C \left( 1 + \| \Phi_0 \|_{L^2(\Omega)}^2 + \| W_0 \|_{L^4(\Omega)}^2 + \| I_i \|_{L^2((0, T), (W^{1,2}(\Omega))^*)}^2 + \| I_e \|_{L^2((0, T), (W^{1,2}(\Omega))^*)}^2 \right) .
\] (2.63)
Proof. Since the proof of Proposition 2.2. relies exclusively on the structure of the weak gating equation, which is the same in the monodomain and the reduced bidomain system, as well as on the a-priori estimate for $W$, we may carry over the argumentation without alterations.

g) Linearized Aliev-Panfilov model: improvement of regularity for the weak solutions.

Proposition 2.7. (Gain of regularity for the transmembrane potential) Consider the bidomain system (2.53) – (2.57) with the linearized Aliev-Panfilov model under the assumptions of Theorem 1.1., and let a weak solution $(\Phi_{tr}, W)$ of it belong to the spaces in (2.58). Then $\Phi_{tr} \in L^2([0, T), L^6(\Omega)] \cap L^q([0, T), L^r(\Omega))$ for all $1 < q < \infty$ and all $4 \leq r < 6$. In particular, $\Phi_{tr} \in L^5(\Omega_T)$.

Proof. Even here, the proof of Proposition 2.1. may be carried over.

Proposition 2.8. (Gain of regularity for the gating variable) Consider the bidomain system (2.53) – (2.57) with the linearized Aliev-Panfilov model under the assumptions of Theorem 1.1., and let a weak solution $(\Phi_{tr}, W)$ of it belong to the spaces in (2.58).

1) Then $W$ belongs to $C^1([0, 1), L^1(\Omega)]$.
2) If $2 \leq r < 3$ and $W_0 \in L^r(\Omega)$ then $\partial W/\partial t$ belongs to $L^r(\Omega_T)$.
3) If $26/9 < r < 3$ and $W_0 \in W^{1, r/2}(\Omega)$ then $W$ belongs to $L^1([0, T), W^{1, r/2}(\Omega)] \cap C^0([0, T), L^{8/3}(\Omega)]$, and $\|W\|_{C^0([0, T], L^{8/3}(\Omega))}$ is bounded by a constant depending on $r$ and the norms $\|\Phi_0\|_{L^2(\Omega)}$, $\|W_0\|_{W^{1, r/2}(\Omega)}$, $\|I^r\|_{L^2((0, T), (W^{1, r/2}(\Omega))^*)}$ and $\|I^e\|_{L^2((0, T), (W^{1, r/2}(\Omega))^*)}$ only.

Proof. The proof of Proposition 2.4. may be repeated without changes.

3. Correction of the proof of the stability estimate.

a) Proof of Theorem 1.1.

Part 1) Let us specify within (2.9) – (2.11) the Rogers-McCulloch model. The proof, however, has been organized in such a way that the estimates work in the case of the linearized Aliev-Panfilov model as well. Throughout the following, $C$ denotes a generic positive constant, which may appropriately change from line to line. $C$ will never depend on the data $\Phi_0$, $W_0$, $I_i$ and $I_e$ but, possibly, on $\Omega$ and $p = 4$.

• Step 1. The difference of the parabolic equations. From the parabolic equations, satisfied by the pairs $(\Phi_{tr}', W')$ and $(\Phi_{tr}'', W'')$ for almost all $t \in [0, T]$, we obtain the difference

$$
\langle \frac{d}{dt} (\Phi_{tr}'(t) - \Phi_{tr}''(t) ), \psi \rangle + M (\Phi_{tr}'(t) - \Phi_{tr}''(t), \psi ) + \int_{\Omega} (I_{ion}(\Phi_{tr}'(t), W'(t)) - I_{ion}(\Phi_{tr}''(t), W''(t))) \psi dx
= \langle \frac{1}{1 + \lambda} (\lambda (I_i'(t) - I_i''(t)) - (I_e'(t) - I_e''(t))), \psi \rangle \quad \forall \psi \in W^{1, 2}(\Omega).
$$

(3.1)

Inserting into (3.1) the feasible test function $\psi = \Phi_{tr}'(t) - \Phi_{tr}''(t) \in W^{1, 2}(\Omega)$ and applying the lower estimate for the monodomain bilinear form, we arrive at

$$
\frac{1}{2} \frac{d}{dt} \|\Phi_{tr}' - \Phi_{tr}''\|_{L^2(\Omega)}^2 + \beta \|\Phi_{tr}' - \Phi_{tr}''\|_{W^{1/2}(\Omega)}^2 + \int_{\Omega} (I_{ion}(\Phi_{tr}'(t), W'(t)) - I_{ion}(\Phi_{tr}''(t), W''(t))) (\Phi_{tr}' - \Phi_{tr}'') dx
\leq \frac{1}{1 + \lambda} \langle \lambda (I_i' - I_i''), (\Phi_{tr}' - \Phi_{tr}''), \psi \rangle + \beta \|\Phi_{tr}' - \Phi_{tr}''\|_{L^2(\Omega)}^2.
$$

(3.2)
The first term on the right-hand side will be estimated with the help of the generalized Cauchy inequality as follows:

\[
\left| \left( \frac{1}{1 + \lambda} \left( \lambda \left( I_{t}^{(t)}(t) - I_{t}^{(t)}(t) \right) - \left( I_{t}^{(t)}(t) - I_{t}^{(t)}(t) \right) \right), \Phi_{t}^{(t)} - \Phi_{t}^{(t)} \right) \right| \\
\leq \left\| \frac{1}{1 + \lambda} \left( \lambda \left( I_{t}^{(t)}(t) - I_{t}^{(t)}(t) \right) - \left( I_{t}^{(t)}(t) - I_{t}^{(t)}(t) \right) \right) \right\|_{L^{1,2}(\Omega)} \cdot \left\| \Phi_{t}^{(t)} - \Phi_{t}^{(t)} \right\|_{L^{1,2}(\Omega)}
\]

\[
\leq \varepsilon_{t}^{i} \left( \| \Phi_{t}^{(t)} - \Phi_{t}^{(t)} \|_{L^{1,2}(\Omega)}^{2} + C \left( \| I_{t}^{(t)} - I_{t}^{(t)} \|_{L^{1,2}(\Omega)}^{2} + \| I_{t}^{(t)} - I_{t}^{(t)} \|_{L^{1,2}(\Omega)}^{2} \right) \right)
\]

(3.3)

with arbitrary \( \varepsilon_{t}^{i} > 0 \). The second term will be estimated with the help of the following lemma.

**Lemma 3.1.** For all \( \varphi_{1}, \varphi_{2} \in \mathbb{R} \), the following identity holds:

\[
(\varphi_{1}^{2} - (a + 1) \varphi_{1}^{2} + a \varphi_{1}) - (\varphi_{2}^{2} - (a + 1) \varphi_{2}^{2} + a \varphi_{2}) \\
= (\varphi_{1} - \varphi_{2}) \cdot (\varphi_{1}^{2} + \varphi_{1} \varphi_{2} + \varphi_{2}^{2} - (a + 1) (\varphi_{1} + \varphi_{2}) + a).
\]

(3.5)

Consequently, we find

\[
\int_{\Omega} \left( I_{t}(\Phi_{t}^{(t)}, W_{t}^{(t)}) - I_{t}(\Phi_{t}^{(t)}, W_{t}^{(t)}) \right) \left( \Phi_{t}^{(t)} - \Phi_{t}^{(t)} \right) dx
\]

(3.6)

\[
= \int_{\Omega} \left( \Phi_{t}^{(t)} - \Phi_{t}^{(t)} \right) b \left( \left( \Phi_{t}^{(t)} \right)^{2} + \Phi_{t}^{(t)} - \Phi_{t}^{(t)} \left( \Phi_{t}^{(t)} \right)^{2} + a \right) \left( \Phi_{t}^{(t)} - \Phi_{t}^{(t)} \right) dx
\]

\[
- (a + 1) b \int_{\Omega} \left( \Phi_{t}^{(t)} - \Phi_{t}^{(t)} \right) \left( \Phi_{t}^{(t)} - \Phi_{t}^{(t)} \right) \left( \Phi_{t}^{(t)} - \Phi_{t}^{(t)} \right) dx + \int_{\Omega} \left( \Phi_{t}^{(t)} W_{t}^{(t)} - \Phi_{t}^{(t)} W_{t}^{(t)} \right) \left( \Phi_{t}^{(t)} - \Phi_{t}^{(t)} \right) dx.
\]

Since \( \Phi_{t}^{(t)}(x,t)^{2} + \Phi_{t}^{(t)}(x,t)^{2} + \Phi_{t}^{(t)}(x,t)^{2} \geq 0 \) for almost all \((x,t) \in \Omega_{T} \) and \( a, b > 0 \), the inequalities (3.2), (3.4) and (3.6) imply

\[
\frac{d}{dt} \| \Phi_{t}^{(t)} - \Phi_{t}^{(t)} \|_{L^{2}(\Omega)}^{2} + 2 \beta \| \Phi_{t}^{(t)} - \Phi_{t}^{(t)} \|_{L^{1,2}(\Omega)}^{2}
\]

(3.7)

\[
\leq 2 C \int_{\Omega} \left| \Phi_{t}^{(t)} - \Phi_{t}^{(t)} \right| \cdot \left| \Phi_{t}^{(t)} - \Phi_{t}^{(t)} \right| \cdot \left| \Phi_{t}^{(t)} - \Phi_{t}^{(t)} \right| dx
\]

\[
+ 2 \int_{\Omega} \left( \Phi_{t}^{(t)} - \Phi_{t}^{(t)} \right) \left( \Phi_{t}^{(t)} - \Phi_{t}^{(t)} \right) dx + 2 \int_{\Omega} \left( W_{t}^{(t)} - W_{t}^{(t)} \right) \left( \Phi_{t}^{(t)} - \Phi_{t}^{(t)} \right) dx
\]

\[
+ \varepsilon_{t}^{i} \| \Phi_{t}^{(t)} - \Phi_{t}^{(t)} \|_{L^{1,2}(\Omega)}^{2} + 2 \beta \| \Phi_{t}^{(t)} - \Phi_{t}^{(t)} \|_{L^{2}(\Omega)}^{2}
\]

\[
+ C \left( \| I_{t}^{(t)} - I_{t}^{(t)} \|_{L^{1,2}(\Omega)}^{2} + \| I_{t}^{(t)} - I_{t}^{(t)} \|_{L^{1,2}(\Omega)}^{2} \right).
\]

Now it must be emphasized that (3.7) holds parametrically in \( t \) for almost all fixed \( t \in (0, T) \). In the subsequent applications of the generalized Cauchy’s inequality this will become important since the parameters \( \varepsilon_{i} \) introduced must be chosen in a time-dependent way.

We apply first the generalized Cauchy’s inequality with \( \varepsilon_{2}(t) \) and subsequently Hölder’s inequality to the first term on the right-hand side of (3.7), thus getting

\[
2 C \int_{\Omega} \left| \Phi_{t}^{(t)}(t) - \Phi_{t}^{(t)}(t) \right| \cdot \left| \Phi_{t}^{(t)}(t) - \Phi_{t}^{(t)}(t) \right| \cdot \left| \Phi_{t}^{(t)}(t) - \Phi_{t}^{(t)}(t) \right| dx
\]

\[
\leq C \varepsilon_{2}(t) \left( \int_{\Omega} \left| \Phi_{t}^{(t)} + \Phi_{t}^{(t)} \right| dx \right)^{1/2} \left( \| \Phi_{t}^{(t)} - \Phi_{t}^{(t)} \|_{L^{2}(\Omega)}^{2} \right)^{1/2} + C \varepsilon_{2}(t) \| \Phi_{t}^{(t)} - \Phi_{t}^{(t)} \|_{L^{2}(\Omega)}^{2}
\]

\[
\leq C \varepsilon_{2}(t) \left( \| \Phi_{t}^{(t)} \|_{L^{2}(\Omega)}^{2} + \| \Phi_{t}^{(t)} \|_{L^{2}(\Omega)}^{2} \right) \| \Phi_{t}^{(t)} - \Phi_{t}^{(t)} \|_{L^{2}(\Omega)}^{2} + C \varepsilon_{2}(t) \| \Phi_{t}^{(t)} - \Phi_{t}^{(t)} \|_{L^{2}(\Omega)}^{2}.
\]

(3.8)

(3.9)
Inserting now $\varepsilon_2(t) = \varepsilon_2'(1 + \|\Phi_{t'}(t)\|_{L^3(\Omega)}^2 + \|\Phi_{t''}(t)\|_{L^4(\Omega)}^2)$ with arbitrary $\varepsilon_2' > 0$, we get

$$
\ldots \leq C \varepsilon'_2 \|\Phi_{t'} - \Phi_{t''}\|_{W^{1,2}(\Omega)}^2 + C \varepsilon'_2 \left(1 + \|\Phi_{t'}\|_{L^3(\Omega)}^2 + \|\Phi_{t''}\|_{L^4(\Omega)}^2\right)\|\Phi_{t'} - \Phi_{t''}\|_{L^2(\Omega)}^2.
$$

(3.10)

In order to estimate the second term on the right-hand side of (3.7), let us write

$$
2 \int_\Omega |W'| \cdot |\Phi_{t'} - \Phi_{t''}|^2 \, dx = 2 \int_\Omega \left(|W'|^{2/3} \cdot |\Phi_{t'} - \Phi_{t''}| \right) \left(|W'|^{1/3} \cdot |\Phi_{t'} - \Phi_{t''}| \right) \, dx
$$

(3.11)

$$
\leq C \varepsilon_3(t) \int_\Omega |W'|^{4/3} \cdot |\Phi_{t'} - \Phi_{t''}|^2 \, dx + \frac{C}{\varepsilon_3(t)} \int_\Omega |W'|^{2/3} \cdot |\Phi_{t'} - \Phi_{t''}|^2 \, dx
$$

(3.12)

$$
\leq C \varepsilon_3(t) \left(\int_\Omega |W'|^{8/3} \, dx\right)^{1/2} \left(\int_\Omega |\Phi_{t'} - \Phi_{t''}|^4 \, dx\right)^{1/2}
$$

(3.13)

$$
+ \frac{C}{\varepsilon_3(t)} \left(\int_\Omega |W'|^{8/3} \, dx\right)^{1/2} \left(\int_\Omega |\Phi_{t'} - \Phi_{t''}|^4 \, dx\right)^{1/2}
$$

(3.14)

$$
\leq C \varepsilon_3(t) \|W'\|_{L^{8/3}(\Omega)}^{4/3} \|\Phi_{t'} - \Phi_{t''}\|_{W^{1,2}(\Omega)}^2
$$

(3.15)

+ \frac{C}{\varepsilon_3(t)} \|W'\|_{L^{8/3}(\Omega)}^{4/3} \|\Phi_{t'} - \Phi_{t''}\|_{W^{1,2}(\Omega)}^2 + \frac{C}{\varepsilon_3(t)} \|\Phi_{t'} - \Phi_{t''}\|_{L^2(\Omega)}^2

Our assumption about $W_0$ guarantees the applicability of Proposition 2.2. Consequently, we may ensure that $W' \in C^0[[0,T],L^4(\Omega)] \hookrightarrow C^0[[0,T],L^{8/3}(\Omega)]$. Then with $\varepsilon_3(t) = \varepsilon'_3$, $\tilde{\varepsilon}_3(t) = \varepsilon'_3$, we get

$$
\int_\Omega |W'| \cdot |\Phi_{t'} - \Phi_{t''}|^2 \, dx \leq C \left(\varepsilon'_3 + \frac{\varepsilon'_3^2}{\varepsilon'_3}\right) \|W'\|_{C^0[[0,T],L^{8/3}(\Omega)]} \cdot \|\Phi_{t'} - \Phi_{t''}\|_{W^{1,2}(\Omega)}
$$

(3.16)

+ \frac{C}{\varepsilon'_3} \|\Phi_{t'} - \Phi_{t''}\|_{L^2(\Omega)}^2.

For the third term from the right-hand side of (3.7), we find

$$
2 \int_\Omega (W' - W'') \Phi_{t''} (\Phi_{t'} - \Phi_{t''}) \, dx
$$

(3.17)

$$
\leq C \varepsilon_4(t) \left(\int_\Omega |\Phi_{t''}|^4 \, dx\right)^{1/2} \left(\int_\Omega |\Phi_{t'} - \Phi_{t''}|^4 \, dx\right)^{1/2} + \frac{C}{\varepsilon_4(t)} \|W' - W''\|_{L^2(\Omega)}^2
$$

(3.18)

$$
\leq C \varepsilon_4(t) \|\Phi_{t''}\|_{L^4(\Omega)}^2 \|\Phi_{t'} - \Phi_{t''}\|_{W^{1,2}(\Omega)}^2 + \frac{C}{\varepsilon_4(t)} \|W' - W''\|_{L^2(\Omega)}^2
$$

(3.19)

using the (noncompact but) continuous imbedding $W^{1,2}(\Omega) \hookrightarrow L^6(\Omega)$ and applying again the generalized Cauchy’s inequality with $\varepsilon_4(t) > 0$. Specifying now $\varepsilon_4(t) = \varepsilon'_4/(1 + \|\Phi_{t''}(t)\|_{L^4(\Omega)}^2)$ with arbitrary $\varepsilon'_4 > 0$, we may continue

$$
\ldots \leq C \varepsilon'_4 \|\Phi_{t'} - \Phi_{t''}\|_{W^{1,2}(\Omega)}^2 + \frac{C}{\varepsilon'_4} \left(1 + \|\Phi_{t''}(t)\|_{L^4(\Omega)}^2\right) \|W' - W''\|_{L^2(\Omega)}^2.
$$

(3.20)
Assembling now (3.7), (3.10), (3.16) and (3.20), we arrive at the following inequality:

\[
\frac{d}{dt} \| \Phi_{tr}' - \Phi_{tr}'' \|^2_{L^2(\Omega)} + 2 \beta \| \Phi_{tr}' - \Phi_{tr}'' \|^2_{W^{1,2}(\Omega)} \leq \frac{C}{\varepsilon_1} \left( \| \tilde{I}'_1(t) - \tilde{I}''_1(t) \|^2_{W^{1,2}(\Omega)} + \| \tilde{I}'_2(t) - \tilde{I}''_2(t) \|^2_{W^{1,2}(\Omega)} \right)
\]

\[
+ \frac{C}{\varepsilon_3} \| \Phi_{tr}' - \Phi_{tr}'' \|^2_{W^{1,2}(\Omega)} + 2 \beta \| \Phi_{tr}' - \Phi_{tr}'' \|^2_{L^2(\Omega)}
\]

\[
+ C \frac{\varepsilon_2}{\varepsilon_3} \| \Phi_{tr}' - \Phi_{tr}'' \|^2_{W^{2,1}(\Omega)} + \frac{C}{\varepsilon_3^2} \left( 1 + \| \Phi_{tr}' \|^2_{L^4(\Omega)} + \| \Phi_{tr}'' \|^2_{L^4(\Omega)} \right) \| \Phi_{tr}' - \Phi_{tr}'' \|^2_{L^2(\Omega)}
\]

\[
+ C \left( \frac{\varepsilon'_3 + \varepsilon''_3}{\varepsilon_3} \right) \| \Phi_{tr}' - \Phi_{tr}'' \|^2_{L^4(\Omega)} + \frac{C}{\varepsilon_3^2} \| \Phi_{tr}' - \Phi_{tr}'' \|^2_{L^2(\Omega)}
\]

\[
+ C \frac{\varepsilon_4^2}{\varepsilon_3^2} \| \Phi_{tr}' - \Phi_{tr}'' \|^2_{W^{2,1}(\Omega)} + \frac{C}{\varepsilon_4} \left( 1 + \| \Phi_{tr}'' \|^2_{L^4(\Omega)} \right) \| W' - W'' \|^2_{L^2(\Omega)}.
\]

Now we may fix the numbers \( \varepsilon'_1, \varepsilon'_2, \varepsilon'_3, \varepsilon'_4, \varepsilon''_4 > 0 \) in such a way that the terms with \( \| \Phi_{tr}' - \Phi_{tr}'' \|^2_{W^{1,2}(\Omega)} \) on both sides of (3.21) will be annihilated. We arrive at

\[
\frac{d}{dt} \| \Phi_{tr}' - \Phi_{tr}'' \|^2_{L^2(\Omega)} \leq C \left( 2 \beta + \frac{1}{\varepsilon_2} (1 + \| \Phi_{tr}' \|^2_{L^4(\Omega)} + \| \Phi_{tr}'' \|^2_{L^4(\Omega)}) + \frac{1}{\varepsilon_3^2} \right) \| \Phi_{tr}' - \Phi_{tr}'' \|^2_{L^2(\Omega)}
\]

\[
+ C \frac{1}{\varepsilon_4} \left( 1 + \| \Phi_{tr}'' \|^2_{L^4(\Omega)} \right) \| W' - W'' \|^2_{L^2(\Omega)}
\]

\[
+ C \frac{\varepsilon_2}{\varepsilon_3} \left( \| I'_1(t) - I''_1(t) \|^2_{W^{1,2}(\Omega)} + \| I'_2(t) - I''_2(t) \|^2_{W^{1,2}(\Omega)} \right).
\]

**Step 2. The difference of the gating equations.** Inserting into the difference of the gating equations for \( (\Phi_{tr}', W') \) and \( (\Phi_{tr}'', W'') \),

\[
\left\langle \frac{d}{dt} (W'(t) - W''(t)), \psi \right\rangle = -\varepsilon \int_\Omega (W'(t) - W''(t)) \psi dx + \varepsilon \kappa \int_\Omega (\Phi_{tr}'(t) - \Phi_{tr}''(t)) \psi dx \quad (3.23)
\]

\[
\forall \psi \in L^2(\Omega),
\]

the feasible test function \( \psi = W'(t) - W''(t) \) and applying Cauchy’s inequality to the second term, we get the estimate20

\[
\frac{d}{dt} \| W' - W'' \|^2_{L^2(\Omega)} \leq 2 \left( 2 \varepsilon + \varepsilon \kappa \right) \| W' - W'' \|^2_{L^2(\Omega)} + 2 \varepsilon \kappa \| \Phi_{tr}' - \Phi_{tr}'' \|^2_{L^2(\Omega)}.
\]

**Step 3. The estimates for the differences** \( \| \Phi_{tr}' - \Phi_{tr}'' \|^2_{L^2[(0,T), L^2(\Omega)]} \), \( \| W' - W'' \|^2_{L^2[(0,T), L^2(\Omega)]} \) \( \) and \( \| W' - W'' \|^2_{L^2[(0,T), L^2(\Omega)]} \). After enlarging and equalizing the factors on the right-hand sides, the inequalities (3.22) and (3.24) yield together

\[
\frac{d}{dt} \left( \| \Phi_{tr}' - \Phi_{tr}'' \|^2_{L^2(\Omega)} + \| W' - W'' \|^2_{L^2(\Omega)} \right)
\]

\[
\leq C \left( 2 \beta + \frac{1}{\varepsilon_2} (1 + \| \Phi_{tr}'(t) \|^2_{L^4(\Omega)} + \| \Phi_{tr}''(t) \|^2_{L^4(\Omega)} + \frac{1}{\varepsilon_3^2} + 2 \varepsilon \kappa \right) \| \Phi_{tr}' - \Phi_{tr}'' \|^2_{L^2(\Omega)}
\]

\[
+ C \frac{1}{\varepsilon_4} \left( 1 + \| \Phi_{tr}'(t) \|^2_{L^4(\Omega)} \right) \| W' - W'' \|^2_{L^2(\Omega)}
\]

\[
+ C \frac{\varepsilon_2}{\varepsilon_3} \left( \| I'_1(t) - I''_1(t) \|^2_{W^{1,2}(\Omega)} \right) + \| I'_2(t) - I''_2(t) \|^2_{W^{1,2}(\Omega)} \right).
\]

\[
\frac{d}{dt} \left( \| \Phi_{tr}' - \Phi_{tr}'' \|^2_{L^2(\Omega)} + \| W' - W'' \|^2_{L^2(\Omega)} \right) \leq A(t) \cdot \left( \| \Phi_{tr}' - \Phi_{tr}'' \|^2_{L^2(\Omega)} + \| W' - W'' \|^2_{L^2(\Omega)} \right)
\]

\[
+ C \frac{\varepsilon_2}{\varepsilon_3} \left( \| I'_1(t) - I''_1(t) \|^2_{W^{1,2}(\Omega)} \right) + \| I'_2(t) - I''_2(t) \|^2_{W^{1,2}(\Omega)} \right).
\]

**Note** that here \( \varepsilon > 0 \) is the given one from (2.5).

---

20 Note that here \( \varepsilon > 0 \) is the given one from (2.5).
where
\[ A(t) = C \left( 1 + \frac{1}{\varepsilon_2} \left( 1 + \| \Phi_{tr'}(t) \|_{L^4(\Omega)}^2 + \| \Phi_{tr''}(t) \|_{L^4(\Omega)}^2 \right) + \frac{1}{\varepsilon_3 \varepsilon_3} + \frac{1}{\varepsilon_4} \left( 1 + \| \Phi_{tr''}(t) \|_{L^4(\Omega)}^2 \right) \right) \]  
(3.27)

Gronwall's inequality implies now that
\[
\| \Phi_{tr'}(t) - \Phi_{tr''}(t) \|_{L^2(\Omega)}^2 + \| W'(t) - W''(t) \|_{L^2(\Omega)}^2 \leq e^{\int_0^t A(s) \, ds} \left( \| \Phi_{tr'}(0) - \Phi_{tr''}(0) \|_{L^2(\Omega)}^2 \\
+ \| W'(0) - W''(0) \|_{L^2(\Omega)}^2 \frac{C}{\varepsilon_1} \int_0^t \left( \| I_t' \|_{L^4(\Omega)} \right)^2 + \| I_t'' \|_{L^2(\Omega)}^2 \right) d\tau \\
\leq e^{\tilde{A} T} \frac{C}{\varepsilon_1} \left( \| I_t' - I_t'' \|_{L^2(0,T), (\Omega)}^2 + \| I_t'' \|_{L^2(0,T), (\Omega)}^2 \right) \]  
(3.28)

with
\[
\tilde{A} = \int_0^T A(s) \, ds = CT + \frac{C}{\varepsilon_2} \left( T + \int_0^T \| \Phi_{tr'}(s) \|_{L^4(\Omega)}^2 + \| \Phi_{tr''}(s) \|_{L^4(\Omega)}^2 \right) ds \\
+ \frac{C T}{\varepsilon_3 \varepsilon_4} + \frac{C}{\varepsilon_4} \left( T + \int_0^T \| \Phi_{tr''}(s) \|_{L^4(\Omega)}^2 \right) ds \\
\leq C \left( 1 + \| \Phi_{tr'} \|_{L^4(\Omega)}^4 + \| \Phi_{tr''} \|_{L^4(\Omega)}^4 \right) \\
\leq C \left( 1 + \| \Phi_0 \|_{L^2(\Omega)}^2 + \| W_0 \|_{L^4(\Omega)}^2 + \| I_t' \|_{L^4(\Omega)}^2 \left[ (0,T), (\Omega) \right] - \| I_t'' \|_{L^4(\Omega)}^2 \left[ (0,T), (\Omega) \right] \right) \\
\leq C \left( 1 + \| \Phi_0 \|_{L^2(\Omega)}^2 + \| W_0 \|_{L^4(\Omega)}^2 + 4 R^2 \right) \\
\]  
(3.30)

by (2.13), Proposition 2.2, and the assumption about the uniform bound \(R > 0\) for the norms of the inhomogeneities. Summing up, we obtain from inequality (3.29) the following estimates:

\[
\| \Phi_{tr'} - \Phi_{tr''} \|_{L^\infty \left[ (0,T), L^2(\Omega) \right]} \leq e^{\tilde{A} T} \frac{C}{\varepsilon_1} \left( \| I_t' - I_t'' \|_{L^2(0,T), (\Omega)}^2 \right) \\
+ \| I_t' - I_t'' \|_{L^2(0,T), (\Omega)}^2 \\
\leq C \left( 1 + \| \Phi_{tr'} \|_{L^4(\Omega)}^4 + \| \Phi_{tr''} \|_{L^4(\Omega)}^4 \right) \\
\]  
(3.34)

\[
\| W' - W'' \|_{L^\infty \left[ (0,T), L^2(\Omega) \right]} \leq e^{\tilde{A} T} \frac{C}{\varepsilon_1} \left( \| I_t' - I_t'' \|_{L^2(0,T), (\Omega)}^2 \right) \\
+ \| I_t' - I_t'' \|_{L^2(0,T), (\Omega)}^2 \\
\leq C \left( 1 + \| \Phi_{tr'} \|_{L^4(\Omega)}^4 + \| \Phi_{tr''} \|_{L^4(\Omega)}^4 \right) \\
\]  
(3.35)

\[
\| W' - W'' \|_{L^2 \left[ (0,T), L^2(\Omega) \right]} \leq T e^{\tilde{A} T} \frac{C}{\varepsilon_1} \left( \| I_t' - I_t'' \|_{L^2(0,T), (\Omega)}^2 \right) \\
+ \| I_t' - I_t'' \|_{L^2(0,T), (\Omega)}^2 \\
\leq C \left( 1 + \| \Phi_{tr'} \|_{L^4(\Omega)}^4 + \| \Phi_{tr''} \|_{L^4(\Omega)}^4 \right) \\
\]  
(3.36)

**Step 4. The estimate for the difference \( \| \Phi_{tr'} - \Phi_{tr''} \|_{L^2 \left[ (0,T), W^{1,2}(\Omega) \right]} \):** In (3.21), the numbers \(\varepsilon'_1, ..., \varepsilon'_4 > 0\) may be alternatively chosen in such a way that

\[
\frac{d}{dt} \| \Phi_{tr'} - \Phi_{tr''} \|_{L^2(\Omega)}^2 + \beta \| \Phi_{tr'} - \Phi_{tr''} \|_{W^{1,2}(\Omega)}^2 \leq C \left( 1 + \frac{1}{\varepsilon_2} \left( 1 + \| \Phi_{tr'} \|_{L^4(\Omega)}^2 + \| \Phi_{tr''} \|_{L^4(\Omega)}^2 \right) \right) \\
+ \frac{1}{\varepsilon_3 \varepsilon_3} \| \Phi_{tr'} - \Phi_{tr''} \|_{L^2(\Omega)}^2 + \frac{C}{\varepsilon_4} \left( 1 + \| \Phi_{tr''} \|_{L^4(\Omega)}^2 \right) \| W' - W'' \|_{L^2(\Omega)}^2 \\
+ \frac{C}{\varepsilon_1} \left( \| I_t' \|_{L^2(\Omega)}^2 \right) \left( W^{1,2}(\Omega) \right) + \| I_t'(t) - I_t''(t) \|_{\left( W^{1,2}(\Omega) \right)^*} \\
\]  
(3.37)
This implies the following modification of (3.26):
\[
\frac{d}{dt} \left( \| \Phi_{t'} - \Phi_{t''} \|_{L^2(\Omega)}^2 + \| W' - W'' \|_{L^2(\Omega)}^2 \right) + \| \Phi_{t'} - \Phi_{t''} \|_{W^{1,2}(\Omega)}^2 \\
\leq A(t) \cdot \left( \| \Phi_{t'} - \Phi_{t''} \|_{L^2(\Omega)}^2 + \| W' - W'' \|_{L^2(\Omega)}^2 \right) \\
+ \frac{C}{\varepsilon_1} \left( \| I'_e(t) - I''_e(t) \|_{W^{1,2}(\Omega)}^2 + \| I'_e(t) - I''_e(t) \|_{W^{1,2}(\Omega)}^2 \right).
\] (3.38)

Together with (3.34) and (3.35), we obtain
\[
\frac{d}{dt} \left( \| \Phi_{t'} - \Phi_{t''} \|_{L^2(\Omega)}^2 + \| W' - W'' \|_{L^2(\Omega)}^2 \right) + \| \Phi_{t'} - \Phi_{t''} \|_{W^{1,2}(\Omega)}^2 \\
\leq 2 A(t) e^\Delta \frac{C}{\varepsilon_1} \left( \| I'_e(t) - I''_e(t) \|_{L^2(\Omega)}^2 + \| I'_e(t) - I''_e(t) \|_{L^2[0, T], (w^{1,2}(\Omega))^*} \right) \\
+ \frac{C}{\varepsilon_1} \left( \| I'_e(t) - I''_e(t) \|_{L^2[0, T], (w^{1,2}(\Omega))^*} + \| I'_e(t) - I''_e(t) \|_{L^2[0, T], (w^{1,2}(\Omega))^*} \right).
\] (3.39)

We integrate (3.39) over \([0, T]\) and find, inserting the identical initial values
\[
\| \Phi_{t'}(T) - \Phi_{t''}(T) \|_{L^2(\Omega)}^2 + \| W'(T) - W''(T) \|_{L^2(\Omega)}^2 + \| \Phi_{t'} - \Phi_{t''} \|_{L^2[0, T], (w^{1,2}(\Omega))^*}^2 \\
\leq C \int_0^T A(t) dt \cdot \left( \| I'_e(t) - I''_e(t) \|_{L^2[0, T], (w^{1,2}(\Omega))^*} + \| I'_e(t) - I''_e(t) \|_{L^2[0, T], (w^{1,2}(\Omega))^*} \right) \\
+ \frac{C}{\varepsilon_1} \left( \| I'_e(t) - I''_e(t) \|_{L^2[0, T], (w^{1,2}(\Omega))^*} + \| I'_e(t) - I''_e(t) \|_{L^2[0, T], (w^{1,2}(\Omega))^*} \right)
\leq C \left( \bar{A} + \frac{1}{\varepsilon_1} \right) \left( \| I'_e(t) - I''_e(t) \|_{L^2[0, T], (w^{1,2}(\Omega))^*} + \| I'_e(t) - I''_e(t) \|_{L^2[0, T], (w^{1,2}(\Omega))^*} \right).
\] (3.40)

This implies the desired estimate
\[
\| \Phi_{t'} - \Phi_{t''} \|_{L^2[0, T], (w^{1,2}(\Omega))^*}^2 \\
\leq \frac{C}{\beta} \left( \bar{A} + \frac{1}{\varepsilon_1} \right) \left( \| I'_e(t) - I''_e(t) \|_{L^2[0, T], (w^{1,2}(\Omega))^*} + \| I'_e(t) - I''_e(t) \|_{L^2[0, T], (w^{1,2}(\Omega))^*} \right).
\] (3.42)

Step 5. The estimate for the difference \( \| W' - W'' \|_{L^2(\Omega)}^2 \) \( \rightarrow \) \( \Phi_{t'} - \Phi_{t''} \|_{L^2(\Omega)}^2 \). Into equation (2.21), we insert the test function \( \psi = (\partial W'(t)/\partial t) - (\partial W''(t)/\partial t) \) which, by Proposition 2.2., belongs to \( L^2(\Omega_T) \) and is therefore admissible. Then we get with the generalized Cauchy’s inequality
\[
\langle \frac{\partial W'}{\partial t} - \frac{\partial W''}{\partial t}, \frac{\partial W'}{\partial t} - \frac{\partial W''}{\partial t} \rangle = \| \frac{\partial W'}{\partial t} - \frac{\partial W''}{\partial t} \|_{L^2(\Omega)}^2 \\
\leq 2 \varepsilon \| \frac{\partial W'}{\partial t} - \frac{\partial W''}{\partial t} \|_{L^2(\Omega)}^2 + \frac{\varepsilon}{2 \varepsilon_5} \| W' - W'' \|_{L^2(\Omega)}^2 \\
+ 2 \varepsilon \| \frac{\partial W'}{\partial t} - \frac{\partial W''}{\partial t} \|_{L^2(\Omega)}^2 + \frac{\varepsilon}{2 \varepsilon_6} \| \Phi_{t'} - \Phi_{t''} \|_{L^2(\Omega)}^2
\] (3.43)

for arbitrary \( \varepsilon_5, \varepsilon_6 > 0 \). Fixing the numbers \( \varepsilon_5 \) and \( \varepsilon_6 \) in such a way that \( 2 \varepsilon \varepsilon_5 + 2 \varepsilon \varepsilon_6 = 1/2 \), we find together with (3.34) and (3.35):
\[
\| \frac{\partial W'}{\partial t} - \frac{\partial W''}{\partial t} \|_{L^2(\Omega)}^2 \leq \frac{\varepsilon}{2 \varepsilon_5} \| W' - W'' \|_{L^2(\Omega)}^2 + \frac{\varepsilon \kappa}{2 \varepsilon_6} \| \Phi_{t'} - \Phi_{t''} \|_{L^2(\Omega)}^2 \\
\leq C \left( \| I'_e(t) - I''_e(t) \|_{L^2[0, T], (w^{1,2}(\Omega))^*} \right) + \| I'_e(t) - I''_e(t) \|_{L^2[0, T], (w^{1,2}(\Omega))^*} \right) \\
\Rightarrow \| \frac{\partial W'}{\partial t} - \frac{\partial W''}{\partial t} \|_{L^2[0, T], L^2(\Omega)}^2 \leq C T \left( \| I'_e(t) - I''_e(t) \|_{L^2[0, T], (w^{1,2}(\Omega))^*} \right) \\
+ \| I'_e(t) - I''_e(t) \|_{L^2[0, T], (w^{1,2}(\Omega))^*} \right).
\] (3.44)
Combining (3.45) with (3.36), we get finally:
\[
\| W' - W'' \|_{W^{1,2}[(0, T), L^2(\Omega)]}^2 \leq C \left( \| I'_t - I''_t \|_{L^2[(0, T), (W^{1,2}(\Omega))^\perp]}^2 + \| I'_e - I''_e \|_{L^2[(0, T), (W^{1,2}(\Omega))^\perp]}^2 \right).
\]  
(3.46)

- **Step 6. The estimate for the difference \( \| \Phi_{tr'} - \Phi_{tr''} \|_{W^{1,4/3}[(0, T), (W^{1,2}(\Omega))^\perp]} \). Exploiting the definition of the dual norm, we see that
\[
\int_0^T \| \frac{\partial \Phi_{tr'}(t)}{\partial t} - \frac{\partial \Phi_{tr''}(t)}{\partial t} \|_{W^{1,2}(\Omega)}^{4/3} dt 
\leq C \cdot \left( \int_0^T \sup_{\| \psi \|_{W^{1,2}(\Omega)} = 1} \| \frac{\partial \Phi_{tr'}(t)}{\partial t} - \frac{\partial \Phi_{tr''}(t)}{\partial t} \|_{W^{1,2}(\Omega)}^{4/3} dt \right)^{3/4}
\]

(3.47)

estimating the first and second term by using the continuous imbedding \( L^2[0, T] \hookrightarrow L^{4/3}[0, T] \). Now we estimate the three terms on the right-hand side of (3.49) separately. For the first term, we get
\[
\sup_{\| \psi \|_{W^{1,2}(\Omega)} = 1} \left\| \frac{1}{1 + \lambda} \left( \lambda \left( I'_t(t) - I''_t(t) \right) - \left(I'_e(t) - I''_e(t) \right) \right), \psi \right\|^2
\leq \| \frac{1}{1 + \lambda} \left( \lambda \left( I'_t(t) - I''_t(t) \right) - \left(I'_e(t) - I''_e(t) \right) \right) \|_{W^{1,2}(\Omega)}^2 \leq C \left( \| I'_t(t) - I''_t(t) \|_{W^{1,2}(\Omega)}^2 + \| I'_e(t) - I''_e(t) \|_{W^{1,2}(\Omega)}^2 \right) \rightarrow
\]

(3.50)

\[
\left( \int_0^T \sup_{\| \psi \|_{W^{1,2}(\Omega)} = 1} \left\| \frac{1}{1 + \lambda} \left( \lambda \left( I'_t(t) - I''_t(t) \right) - \left(I'_e(t) - I''_e(t) \right) \right), \psi \right\|^2 dt \right)^{1/2}
\leq C \left( \| I'_t - I''_t \|_{L^2[(0, T), (W^{1,2}(\Omega))^\perp]}^2 + \| I'_e - I''_e \|_{L^2[(0, T), (W^{1,2}(\Omega))^\perp]}^2 \right) \rightarrow
\]

(3.51)

For the second term, we obtain from the continuity of the monodomain bilinear form and (3.42):
\[
\left| M(\Phi_{tr'} - \Phi_{tr''}, \psi) \right|^2 \leq \gamma^2 \| \Phi_{tr'} - \Phi_{tr''} \|_{W^{1,2}(\Omega)}^2 \| \psi \|_{W^{1,2}(\Omega)}^2 \rightarrow
\]

(3.53)

\[
\left( \int_0^T \sup_{\| \psi \|_{W^{1,2}(\Omega)} = 1} \left| M(\Phi_{tr'} - \Phi_{tr''}, \psi) \right|^2 dt \right)^{1/2} \leq C \left( \| \Phi_{tr'} - \Phi_{tr''} \|_{L^2[(0, T), W^{1,2}(\Omega)]}^2 \right)^{1/2}
\]

(3.54)

\[
\leq C \left( \| I'_t - I''_t \|_{L^2[(0, T), (W^{1,2}(\Omega))^\perp]} + \| I'_e - I''_e \|_{L^2[(0, T), (W^{1,2}(\Omega))^\perp]} \right).
\]

(3.55)

\footnote{Note that, in [KUNISCH/WAGNER 12], p. 1547 f., (B.44) and (B.47), the forming of the square root has been overlooked.}
In order to estimate the third term, we write, relying on Lemma 3.1,

$$
\| I_{\text{ion}}(\Phi_{tr}', W') - I_{\text{ion}}(\Phi_{tr}'', W'') \|_{L^{4/3}(\Omega)}
\leq \| b \left( (\Phi_{tr}')^2 + \Phi_{tr}' (\Phi_{tr}'' + (\Phi_{tr}'')^2 - (a + 1) (\Phi_{tr}' + \Phi_{tr}'')) + a \right) (\Phi_{tr}' - \Phi_{tr}'') \|_{L^{4/3}(\Omega)}
+ \| (\Phi_{tr}' - \Phi_{tr}'') W' \|_{L^{4/3}(\Omega)} + \| (W' - W'') \Phi_{tr}'' \|_{L^{4/3}(\Omega)} = J_1 + J_2 + J_3.
$$

(3.56)

For $J_1$, we obtain

$$
J_1 = b \left( \int_{\Omega} ( (\Phi_{tr}')^2 + \Phi_{tr}' (\Phi_{tr}'' + (\Phi_{tr}'')^2 - (a + 1) (\Phi_{tr}' + \Phi_{tr}'')) + a \right) \left( \frac{4}{3} (\Phi_{tr}' - \Phi_{tr}'') \right) \frac{1}{3} dx
\leq b \left( \int_{\Omega} ( (\Phi_{tr}')^2 + \Phi_{tr}' (\Phi_{tr}'' + (\Phi_{tr}'')^2 - (a + 1) (\Phi_{tr}' + \Phi_{tr}'')) + a \right) \frac{4}{3} dx
\cdot \left( \int_{\Omega} (\Phi_{tr}' - \Phi_{tr}'') \frac{4}{3} \right) \frac{3}{4} \frac{1}{3} dx
\leq C \left( 1 + \| \Phi_{tr}'(t) \|_{L^4(\Omega)} + \| \Phi_{tr}''(t) \|_{L^4(\Omega)} \right) \frac{4}{3} \cdot \| \Phi_{tr}' - \Phi_{tr}'' \|_{W^{1,2}(\Omega)}.
$$

(3.58)

(3.59)

(3.60)

due to the continuous imbedding $W^{1,2}(\Omega) \hookrightarrow L^4(\Omega)$. Further, from Proposition 2.2. we get

$$
J_2 = \left( \int_{\Omega} \frac{4}{3} (\Phi_{tr}' - \Phi_{tr}'') \frac{1}{3} dx \right) \frac{3}{4} \leq \left( \int_{\Omega} (W' + W'') \frac{8}{3} \frac{1}{3} dx \right) \frac{3}{8} \left( \int_{\Omega} (\Phi_{tr}' + \Phi_{tr}'') \frac{8}{3} \frac{1}{3} dx \right) \frac{3}{8}
\leq C \| W' \|_{C^0[0,T]} \| \Phi_{tr}' - \Phi_{tr}'' \|_{W^{1,2}(\Omega)}.
$$

(3.61)

(3.62)

due to the continuous imbedding $W^{1,2}(\Omega) \hookrightarrow L^{8/3}(\Omega)$. Since $\| W' \|_{C^0[0,T]} \| L^{8/3}(\Omega) \|$ is uniformly bounded by the norms of the initial data and the bound $R$ of the norms of the inhomogeneities, we arrive at

$$
J_2 \leq C \| \Phi_{tr}' - \Phi_{tr}'' \|_{W^{1,2}(\Omega)}.
$$

(3.63)

Finally, $J_3$ will be estimated through

$$
J_3 = \left( \int_{\Omega} (W' - W'') \frac{4}{3} (\Phi_{tr}'' \frac{4}{3} dx \right) \frac{3}{4} \leq \left( \int_{\Omega} (W' - W'')^2 dx \right) \frac{1}{2} \left( \int_{\Omega} (\Phi_{tr}'')^4 dx \right) \frac{1}{4}
\leq \| W' - W'' \|_{L^2(\Omega)} \cdot \| \Phi_{tr}'' \|_{L^4(\Omega)}.
$$

(3.64)

(3.65)

Summing up, the estimates (3.60), (3.63) and (3.65) imply for the third term in (3.49)

$$
\left( \int_0^T \left( \sup \| I_{\text{ion}}(\Phi_{tr}', W') - I_{\text{ion}}(\Phi_{tr}'', W'') \|_{L^{8/3}(\Omega)} \cdot \| \psi \|_{L^4(\Omega)} \right) \frac{4}{3} dx \right) \frac{3}{4}
\leq C \cdot \left( \int_0^T \left( 1 + \| \Phi_{tr}'(t) \|_{L^4(\Omega)} + \| \Phi_{tr}''(t) \|_{L^4(\Omega)} \right) \frac{4}{3} \cdot \| \Phi_{tr}' - \Phi_{tr}'' \|_{W^{1,2}(\Omega)}
\leq C \| \Phi_{tr}' - \Phi_{tr}'' \|_{W^{1,2}(\Omega)} + \| \Phi_{tr}'' \|_{L^4(\Omega)} \| W' - W'' \|_{L^2(\Omega)} \right) \frac{4}{3} dx \right) \frac{3}{4}
$$

(3.66)
\[
\leq C \cdot \left( \int_0^T \left( 1 + \| \Phi_{tr}'(t) \|^4_{L^1(\Omega)} + \| \Phi_{tr}''(t) \|^4_{L^1(\Omega)} \right)^{8/3} \cdot \| \Phi_{tr}' - \Phi_{tr}'' \|^4_{W^{1,2}(\Omega)} dt \right) \tag{3.67}
\]
\[
+ \left( \int_0^T \| \Phi_{tr}' - \Phi_{tr}'' \|^4_{W^{1,2}(\Omega)} dt \right) \left( \int_0^T \| \Phi_{tr}'' \|^4_{L^2(\Omega)} \| W' - W'' \|^4_{L^2(\Omega)} dt \right)^{3/4}
\]
\[
\leq C \left( \int_0^T \left( 1 + \| \Phi_{tr}'(t) \|^4_{L^1(\Omega)} + \| \Phi_{tr}''(t) \|^4_{L^1(\Omega)} \right)^{8/3} \cdot \| \Phi_{tr}' - \Phi_{tr}'' \|^4_{W^{1,2}(\Omega)} dt \right)^{3/4}
\]
\[
+ C \left( \int_0^T \| \Phi_{tr}' - \Phi_{tr}'' \|^4_{W^{1,2}(\Omega)} dt \right) \left( \int_0^T \| \Phi_{tr}'' \|^4_{L^2(\Omega)} \| W' - W'' \|^4_{L^2(\Omega)} dt \right)^{3/4}
\]
\[
\leq C \left( \int_0^T \left( 1 + \| \Phi_{tr}'(t) \|^4_{L^1(\Omega)} + \| \Phi_{tr}''(t) \|^4_{L^1(\Omega)} \right)^{8/3} \cdot \| \Phi_{tr}' - \Phi_{tr}'' \|^4_{W^{1,2}(\Omega)} dt \right)^{3/4}
\]
\[
+ C \| \Phi_{tr}' - \Phi_{tr}'' \|^4_{L^2[(0,T), W^{1,2}(\Omega)]} + C \left( \int_0^T \| \Phi_{tr}'' \|^4_{L^2(\Omega)} dt \right)^{3/4} \left( \int_0^T \| W' - W'' \|^4_{L^2(\Omega)} dt \right)^{1/2}
\]
\[
\leq C \left( \int_0^T \left( 1 + \| \Phi_{tr}'(t) \|^4_{L^1(\Omega)} + \| \Phi_{tr}''(t) \|^4_{L^1(\Omega)} \right)^{8/3} \cdot \| \Phi_{tr}' - \Phi_{tr}'' \|^4_{W^{1,2}(\Omega)} dt \right)^{3/4}
\]
\[
+ C \| \Phi_{tr}' - \Phi_{tr}'' \|^4_{L^2[(0,T), W^{1,2}(\Omega)]} + C \| \Phi_{tr}'' \|^4_{L^2(\Omega)} \| W' - W'' \|^4_{L^2(\Omega)} \right).\]

By Proposition 2.1., \( \Phi_{tr}' \) and \( \Phi_{tr}'' \) belong to \( L^3\left[ (0, T), L^4(\Omega) \right] \) indeed, and the Aubin-Dubinskij lemma (Theorem 1.5.) yields the norm estimates
\[
\| \Phi_{tr}' \|^4_{L^3\left[ (0, T), L^4(\Omega) \right]} \leq C \left( \| \Phi_{tr}' \|^4_{L^2\left[ (0, T), W^{1,2}(\Omega) \right]} + \| \partial \Phi_{tr}' / \partial t \|^4_{L^2\left[ (0, T), (W^{1,2}(\Omega))^* \right]} \right) \tag{3.71}
\]
\[
\| \Phi_{tr}' \|^4_{L^3\left[ (0, T), L^4(\Omega) \right]} \leq C \left( 1 + \| \Phi_0 \|^2_{L^2(\Omega)} + \| W_0 \|^4_{L^1(\Omega)} + 2 R^2 \right); \tag{3.72}
\]
\[
\| \Phi_{tr}' \|^4_{L^3\left[ (0, T), L^4(\Omega) \right]} \leq C \left( 1 + \| \Phi_0 \|^2_{L^2(\Omega)} + \| W_0 \|^4_{L^1(\Omega)} + 2 R^2 \right). \tag{3.73}
\]

Consequently, (3.70), (3.36) and (3.42) imply
\[
\left( \int_0^T \left( \sup_{I_{on}(\Phi_{tr}', W')} - I_{on}(\Phi_{tr}'', W'') \right) \| \psi \|^4_{L^1(\Omega)} \right)^{4/3} \| \psi \|^4_{L^1(\Omega)} dt \right)^{3/4} \tag{3.74}
\]
\[
\leq C \left( \| I' - I'' \|^4_{L^2\left[ (0, T), (W^{1,2}(\Omega))^* \right]} + \| I' - I'' \|^4_{L^2\left[ (0, T), (W^{1,2}(\Omega))^* \right]} \right). \tag{3.75}
\]

Assembling (3.49), (3.52), (3.55) and (3.74), we obtain
\[
\| \partial \Phi_{tr}' / \partial t - \partial \Phi_{tr}' / \partial t \|^4_{L^2\left[ (0, T), (W^{1,2}(\Omega))^* \right]} \tag{3.75}
\]
\[
\leq C \left( \| I' - I'' \|^4_{L^2\left[ (0, T), (W^{1,2}(\Omega))^* \right]} + \| I' - I'' \|^4_{L^2\left[ (0, T), (W^{1,2}(\Omega))^* \right]} \right). \tag{3.76}
\]

With (3.34), we get finally
\[
\| \Phi_{tr}' - \Phi_{tr}'' \|^4_{L^3\left[ (0, T), (W^{1,2}(\Omega))^* \right]} \tag{3.76}
\]
\[
= \| \Phi_{tr}' - \Phi_{tr}'' \|^4_{L^3\left[ (0, T), (W^{1,2}(\Omega))^* \right]} + \| \partial \Phi_{tr}' / \partial t - \partial \Phi_{tr}' / \partial t \|^4_{L^2\left[ (0, T), (W^{1,2}(\Omega))^* \right]} \tag{3.76}
\]
\[
\leq C \left( \left( \| I' - I'' \|^4_{L^2\left[ (0, T), (W^{1,2}(\Omega))^* \right]} + \| I' - I'' \|^4_{L^2\left[ (0, T), (W^{1,2}(\Omega))^* \right]} \right) \right). \tag{3.77}
\]

as claimed.

**Step 7.** The arguments from [BOURGAULT/Coudière/Pierre 09], p. 478, Subsection 5.3., may be repeated in order to confirm that the left-hand side in (3.34) can be replaced by \( \| \Phi_{tr}' - \Phi_{tr}'' \|^4_{C^0\left[ [0, T], L^2(\Omega) \right]} \).
Note that (3.46) implies a bound for \( \| W' - W'' \|_{C^0[0, T], L^2(\Omega)} \) as well, and the proof is complete. If the Rogers-McCulloch model is replaced by the FitzHugh-Nagumo model then the proof can be repeated with some obvious modifications.

**Part 2)** Let us specify now the linearized Aliev-Panfilov model instead of the Rogers-McCulloch model and consider the additional assumption about the regularity of \( W_0 \). As a consequence, Proposition 2.4., 3) ensures that \( W', W'' \) belong still to \( C^0[[0, T], L^{8/3}(\Omega)] \).

- **Step 1.** Thus from the proof of Theorem 2.1., Step 1 can be taken over without alterations.

- **Step 2.** The estimates (3.23) ff. must be replaced as follows:

\[
\langle \frac{d}{dt} (W'(t) - W''(t)), \psi \rangle = -\varepsilon \int_{\Omega} (W'(t) - W''(t)) \psi \, dx + \varepsilon \kappa (a + 1) \int_{\Omega} (\Phi_{tr}'(t) - \Phi_{tr}''(t)) \psi \, dx \tag{3.78}
\]

Inserting the feasible test function \( \psi = W'(t) - W''(t) \), we obtain

\[
\frac{d}{dt} \left( \| W' - W'' \|_{L^2(\Omega)}^2 \right) \leq 2 \varepsilon \| W' - W'' \|_{L^2(\Omega)}^2 + 2 \varepsilon \kappa (a + 1) \left( \| W' - W'' \|_{L^2(\Omega)}^2 + \| \Phi_{tr}' - \Phi_{tr}'' \|_{L^2(\Omega)}^2 \right) + 2 \varepsilon \int_{\Omega} (\Phi_{tr}' + \Phi_{tr}'')^4 \, dx \tag{3.79}
\]

which, after an appropriate choice of \( \varepsilon \), allows to continue the estimations as above.

- **Step 3.** In the case of the linearized Aliev-Panfilov model, we get from (3.29)

\[
\therefore \leq C \left( 1 + \| \Phi_0 \|_{L^2(\Omega)}^2 + \| W_0 \|_{W^{1,3/2}(\Omega)}^{3/2} + 4 R^2 \right) \tag{3.81}
\]

instead of (3.33).

- **Step 4.** This step can be taken over without changes.

- **Step 5.** Instead of (3.43), we find by inserting the feasible test function \( \psi = (\partial W'(t)/\partial t) - (\partial W''(t)/\partial t) \) into (3.78)

\[
\| \frac{\partial W'}{\partial t} - \frac{\partial W''}{\partial t} \|_{L^2(\Omega)}^2 \leq C \varepsilon \varepsilon_9 \| \frac{\partial W'}{\partial t} - \frac{\partial W''}{\partial t} \|_{L^2(\Omega)}^2 + C \varepsilon \varepsilon_9 \| W' - W'' \|_{L^2(\Omega)}^2 \tag{3.82}
\]

\[
+ C \varepsilon \kappa (a + 1) \left( \varepsilon_9 \| \frac{\partial W'}{\partial t} - \frac{\partial W''}{\partial t} \|_{L^2(\Omega)}^2 + \frac{1}{\varepsilon_9} (a + 1) \| \Phi_{tr}' - \Phi_{tr}'' \|_{L^2(\Omega)}^2 \right) + C \varepsilon_9 \| \frac{\partial W'}{\partial t} - \frac{\partial W''}{\partial t} \|_{L^2(\Omega)}^2 + C \varepsilon_9 \left( \int_{\Omega} (\Phi_{tr}' + \Phi_{tr}'')^4 \, dx \right)^{1/2} \left( \int_{\Omega} (\Phi_{tr}'(t) - \Phi_{tr}''(t))^2 \, dx \right)^{1/2} \tag{3.83}
\]

\[
\leq C \left( \varepsilon_9 + \varepsilon_9 + \varepsilon_9 \right) \| \frac{\partial W'}{\partial t} - \frac{\partial W''}{\partial t} \|_{L^2(\Omega)}^2 + C \varepsilon_9 \| W' - W'' \|_{L^2(\Omega)}^2 + C \varepsilon_9 \| \Phi_{tr}' - \Phi_{tr}'' \|_{L^2(\Omega)}^2 \tag{3.84}
\]

\[
+ C \varepsilon_9 \left( \| \Phi_{tr}' \|_{L^2(\Omega)}^2 + \| \Phi_{tr}''' \|_{L^2(\Omega)}^2 \right) \cdot \| \Phi_{tr}' - \Phi_{tr}'' \|_{L^2(\Omega)}^2 .
\]
Choosing \( \varepsilon'_0, \varepsilon'_{10} \) and \( \varepsilon'_{11} \) > 0 sufficiently small, we obtain

\[
\frac{1}{2} \left\| \frac{\partial W'}{\partial t} - \frac{\partial W''}{\partial t} \right\|_{L^2(\Omega)}^2 \leq \frac{C}{\varepsilon'_0} \left\| W' - W'' \right\|_{L^2(\Omega)}^2 + \frac{C}{\varepsilon'_{10}} \left\| \Phi_{tr'} - \Phi_{tr''} \right\|_{L^2(\Omega)}^2 \\
+ \frac{C}{\varepsilon'_{11}} \left( \left\| \Phi_{tr'} \right\|_{L^4(\Omega)}^2 + \left\| \Phi_{tr''} \right\|_{L^4(\Omega)}^2 \right) \cdot \left\| \Phi_{tr'} - \Phi_{tr''} \right\|_{L^2(\Omega)}^2 
\]  

(3.85)

and the estimates may be continued as before.

- **Step 6 and Step 7.** In Step 6 of the proof of Part 1), no more regularity of \( W', W'' \) as provided by Proposition 2.4., 3) has been exploited. Consequently, the proof can be finished as above. ■

**b) Proof of Theorem 1.2.**

The proof of Theorem 1.2. relies completely on the equivalence of the weak bidomain system (2.53) – (2.57) with the reduced bidomain system (2.60) – (2.62). The monodomain form \( M \) and the bidomain form \( A \) satisfy the same norm estimates, the weak solutions of both systems obey the same type of a-priori estimates, and [Kunisch/Wagner 13A], p. 959 f., Lemma 2.9., yields for arbitrary \( \varepsilon_0 > 0 \) the estimate

\[
\left\| \langle S'(t) - S''(t), \psi \rangle \right\| \leq \frac{C}{2\varepsilon_0} \left( \left\| I_t'(t) - I_t''(t) \right\|_{L^2(W^{1,2}(\Omega))}^2 + \left\| I_e'(t) - I_e''(t) \right\|_{L^2(W^{1,2}(\Omega))}^2 \right) + \frac{3\varepsilon_0}{4} \left\| \psi \right\|_{W^{1,2}(\Omega)}^2 
\]  

(3.87)

for the difference of the right-hand sides where the constant \( C > 0 \) does not depend on \( \varepsilon_0, I_t \) and \( I_e \) and even not on \( \Phi_0 \) and \( W_0 \). Consequently, we may carry over Steps 1 – 7 from the proof of Theorem 1.1. The proof of the estimate for \( \left\| \Phi_{tr'} - \Phi_{tr''} \right\|_{L^2(0,T), W^{1,2}(\Omega)}^2 \) from the difference of the elliptic equations was not influenced by the error to be corrected. Thus we may take over the respective step from [Kunisch/Wagner 13A], pp. 971 ff., (2.117) – (2.125). ■

**c) Proof of Theorems 1.3. and 1.4.**

The stability estimates from Theorems 1.1. and 1.2. yield immediately uniqueness of weak solutions corresponding to right-hand sides and initial values of the assumed regularity. ■

**4. Detailed corrections within the previous papers.**

In the following, we report in detail the corrections to be made in [Kunisch/Wagner 12] and [Kunisch/Wagner 13A]. In [Kunisch/Wagner 13B], already the corrected versions of the theorems appear.

**a) Corrections within [Kunisch/Wagner 12].**

1) [Kunisch/Wagner 12, p. 1529, Theorem 2.5.] In view of Theorem 1.1., 1), the assertion remains true for the monodomain system with the Rogers-McCulloch or the FitzHugh-Nagumo model. If the linearized Aliev-Panfilov model is considered then, by Theorem 1.1., 2), we must assume that \( W_0 \) belongs to \( W^{1,3/2}(\Omega) \) rather than to \( L^4(\Omega) \), cf. Theorem 1.3.
2) [KUNISCH/WAGNER 12, p. 1529 f.] The analytical framework for the optimal control problem (P) remains unchanged as far as the Rogers-McCulloch or the FitzHugh-Nagumo model is considered. If (P) is studied with the linearized Aliev-Panfilov model then we must assume again that \( W_0 \) belongs to \( W^{1,3/2}(\Omega) \).

3) [KUNISCH/WAGNER 12, p. 1532, Theorem 3.5., Corollary 3.6. and 3.7.] Here the same situation arises. In the case of the Rogers-McCulloch or the FitzHugh-Nagumo model, all assertions remain valid without changes; for the linearized Aliev-Panfilov model, the assumption \( W_0 \in W^{1,3/2}(\Omega) \) must be added.

4) [KUNISCH/WAGNER 12, p. 1533, Theorem 3.8.] The stability estimate must be replaced by Theorem 1.1. above.

5) [KUNISCH/WAGNER 12, p. 1533, Theorem 3.10.] Within the formulation of the parabolic existence theorem, there is a transcription error to be corrected. Namely, assumption (c) has to be replaced by

\[(c)’ \quad \text{a}_0 \in L^{r_2}([0, T], L^{q_2}(\Omega)) \text{ for some } 1 \leq r_2 < \infty, 2 < q_2 \leq \infty \text{ satisfying } \frac{1}{r_2} + \frac{n}{2q_2} = 1.\]

6) [KUNISCH/WAGNER 12, pp. 1534 – 1537, Proof of Theorem 3.9.] The proof with the necessary corrections is repeated here.

- **Step 1.** Improved regularity of \( \hat{\Phi}_{tr} \) and \( \hat{W} \). Under the assumptions of the theorem, the adjoint equations read as follows:

\[
\begin{align*}
- \frac{\partial P_1}{\partial t} & - \nabla \cdot \left( \frac{\lambda}{1 + \lambda} M_t \nabla P_1 \right) + \left( 3 b (\hat{\Phi}_{tr})^2 - 2 (a + 1) b \hat{\Phi}_{tr} + a b + \hat{W} \right) P_1 = \varepsilon \kappa P_2 - \frac{\partial r}{\partial \varphi}(\hat{\Phi}_{tr}, \hat{W}); \\
- \frac{\partial P_2}{\partial t} & + \varepsilon P_2 = -\hat{\Phi}_{tr} P_1 - \frac{\partial r}{\partial w}(\hat{\Phi}_{tr}, \hat{W}).
\end{align*}
\]

From Propositions 2.1. and 2.2., we see that \( \hat{\Phi}_{tr} \in L^5(\Omega_T) \) while

\[ \hat{W}(x,t) = W_0(x)e^{-\varepsilon t} + \varepsilon \kappa \int_0^t \hat{\Phi}_{tr}(x,\tau) e^{\varepsilon (\tau-t)} d\tau \] (4.3)

belongs to \( C^0([0, T], L^4(\Omega)) \). \(23)\)

- **Step 2.** For any \( \tilde{P}_1 \in L^5(\Omega_T) \), the terminal problem for the adjoint ODE admits a unique (weak or strong) solution \( P_2 \in C^1((0, T), L^2(\Omega)) \cap C^0([0, T], L^2(\Omega)) \). It is obvious that the problem

\[
- \frac{\partial P_2}{\partial t} + \varepsilon P_2 = -\hat{\Phi}_{tr} \tilde{P}_1 - \frac{\partial r}{\partial w}(\hat{\Phi}_{tr}, \hat{W}) \quad (\forall) \,(x,t) \in \Omega_T, \quad P_2(x,T) \equiv 0
\]

(4.4)

admits the unique solution

\[ P_2(x,t) = - \int_t^T \left( \hat{\Phi}_{tr} \tilde{P}_1 + \frac{\partial r}{\partial w}(\hat{\Phi}_{tr}, \hat{W}) \right) e^{\varepsilon (t-\tau)} d\tau, \] (4.5)

which is continuous in time on \([0, T]\) and even differentiable in time on \((0, T)\). In order to confirm the integrability with respect to \( x \), we estimate

\[
\int_\Omega \left( \hat{\Phi}_{tr}(t) \tilde{P}_1(t) \right)^2 dx \leq \left( \int_\Omega |\hat{\Phi}_{tr}(t)|^4 dx \right)^{1/2} \left( \int_\Omega |\tilde{P}_1(t)|^4 dx \right)^{1/2} \] (4.6)

\(22)\) Cf. again [LADYŽENSKAJA/SOLONNIKOV/URAL’CEVA 88], p. 180, Remark 6.3. In fact, the requirement \( a_0 \in L^2((0, T), L^n(\Omega)) \), which cannot be satisfied, does not appear in the second set of conditions.

\(23)\) The derivation [KUNISCH/WAGNER 12, p. 1534, (3.39) – (3.41)], holds wrong.
where the right-hand side is finite due to the continuous imbedding \( \hat{\Phi}_{tr}(t) \in W^{1.2}(\Omega) \hookrightarrow L^4(\Omega) \). Consequently, \( P_2 \) belongs to the space \( C^1 \left[ (0, T), (0, T), L^2(\Omega) \right] \cap C^0 \left[ [0, T], L^2(\Omega) \right] \).

- **Step 3.** For any \( \tilde{P}_2 \in L^2(\Omega_T) \), the terminal-boundary value problem for the parabolic adjoint equation admits a unique weak solution \( P_1 \in L^2 \left[ (0, T), \mathbb{R}^2(\Omega) \right] \cap L^1 \left[ [0, T], L^2(\Omega) \right] \). In order to confirm this claim, we must check whether the assumptions of \[ \text{Kunisch/Wagner 12, } p. 1533, \text{ Theorem 3.10.} \] are satisfied. Concerning (a), (b), (d), (e) and (f), the arguments from \[ \text{Kunisch/Wagner 12, } p. 1534 \text{ ff.} \] can be maintained. In view of the regularity discussion in Section 2 above, the term

\[
a_0(x, t) = 3 b \left( \hat{\Phi}_{tr} \right)^2 - 2 (a + 1) b \hat{\Phi}_{tr} + a b + \tilde{W}
\]

satisfies condition (c) with \( n = 3 \) and \( r_2 = q_2 = 5/2 \) since \( \| \hat{\Phi}_{tr}^2 \|_{L^{5/2}(\Omega_T)} \) is finite by Proposition 2.1, and \( \| \tilde{W} \|_{L^{5/2}(\Omega_T)} \leq C \| \tilde{W} \|_{L^4(\Omega_T)} \) is finite by Proposition 2.2. Consequently, the application of \[ \text{Kunisch/Wagner 12, } p. 1533, \text{ Theorem 3.10.} \] within the proof of \[ \text{Kunisch/Wagner 12, } p. 1533, \text{ Theorem 3.9.} \] is still justified.

- **Step 4.** For two functions \( P_1', P_1'' \in L^2(\Omega_T) \), the corresponding solutions of the terminal problem for the adjoint ODE satisfy

\[
\| P_2'(t) - P_2''(t) \|_{L^2(\Omega)}^2 \leq C \cdot \int_t^T \| P_1'(\tau) - P_1''(\tau) \|_{L^1(\Omega)}^2 d\tau.
\]

Applying Jensen’s integral inequality and Hölder’s inequality, we may argue that

\[
\int_\Omega | P_2' - P_2'' |^2 \, dx \leq C \int_\Omega \left( \int_t^T | \hat{\Phi}_{tr} | \cdot | P_1' - P_1'' | \, d\tau \right)^2 \, dx
\]

\[
\leq C \int_\Omega \left( \int_0^T | \hat{\Phi}_{tr} |^2 \, d\tau \right) \cdot \left( \int_t^T | P_1' - P_1'' |^2 \, d\tau \right) \leq C \| \hat{\Phi}_{tr} \|_{L^4(\Omega_T)}^2 \cdot \| P_1' - P_1'' \|_{L^1 \left[ [t, T], L^4(\Omega) \right]}^2.
\]

- **Step 5.** For two functions \( P_1', P_1'' \in L^2(\Omega_T) \), the corresponding solutions of the terminal-boundary value problem for the parabolic adjoint equation satisfy

\[
\| P_1'(t) - P_1''(t) \|_{L^1(\Omega)}^2 \leq C \cdot \int_t^T \| P_2'(\tau) - P_2''(\tau) \|_{L^2(\Omega)}^2 d\tau.
\]

Applying \[ \text{Kunisch/Wagner 12, } p. 1533, \text{ Theorem 3.10.} \] to the difference of the linear parabolic equations determining \( P_1' \) and \( P_1'' \), we get the a-priori estimate

\[
C \left( \int_t^T \| P_2' - P_2'' \|_{L^2(\Omega)}^2 \, d\tau \right)^{1/2} \geq \| P_1' - P_1'' \|_{C^0 \left[ [t, T], W^{1,2}(\Omega) \right]} \geq \| P_1' - P_1'' \|_{L^4 \left[ [t, T], L^4(\Omega) \right]}.
\]

- **Step 6.** *Application of Banach’s fixed point theorem.* We consider the operator

\[
\mathcal{I} : \left( L^2 \left[ (0, T), \mathbb{R}^2(\Omega) \right] \times L^2(\Omega_T) \right) \to \left( L^2 \left[ (0, T), L^4(\Omega) \right] \times L^2(\Omega_T) \right),
\]

\[\text{Kunisch/Wagner 12, } p. 1535, \text{ Step 3, (3.45)} \] as well as to \[ \text{Kunisch/Wagner 12, } p. 1537, \text{ after (3.73)} \].

\[\text{Kunisch/Wagner 12, } p. 1555, \text{ (3.47) ff.} \] : Again in here and in the following, we claimed in error that \( \| \hat{\Phi}_{tr} \|_{L^4(\Omega)} \) is essentially bounded.

\[\text{Evans 98, } p. 287, \text{ Theorem 3.} \]
which assigns to a given pair \((P_1, P_2)\) the new pair \((\mathcal{I}P_1, \mathcal{I}P_2)\) arising from the solution \(\mathcal{I}P_2\) of the adjoint ODE after insertion of \(P_1\) and the solution of the adjoint parabolic problem after insertion of \(\mathcal{I}P_2\). Let us prove now the contractivity of this operator. We start with two pairs \((P_1', P_2'), (P_1'', P_2'')\) \(\in L^2\left([0, T), L^4(\Omega)\right) \times L^2(\Omega_T)\). From (4.8) and (4.10), we get

\[
\|\mathcal{I}P_1' - \mathcal{I}P_1''\|^2_{L^4\left([t, T), L^4(\Omega)\right)} \leq C \int_t^T \|\mathcal{I}P_2'(\tau) - \mathcal{I}P_2''(\tau)\|^2_{L^2(\Omega)} d\tau
\]

\[
\leq C \int_t^T \|\hat{\Phi}_{tr}\|^2_{L^4(\Omega_T)} \cdot \|P_1' - P_1''\|^2_{L^4\left([t, T), L^4(\Omega)\right)} d\tau. \tag{4.13}
\]

Defining the functions

\[
f(t) = \|\mathcal{I}P_1' - \mathcal{I}P_1''\|^2_{L^4\left([t, T), L^4(\Omega)\right)} \quad \text{and} \quad \hat{f}(t) = \|P_1' - P_1''\|^2_{L^4\left([t, T), L^4(\Omega)\right)}, \tag{4.14}
\]

this inequality reads as

\[
0 \leq f(t) \leq C \int_t^T \hat{f}(\vartheta) d\vartheta \implies \int_0^T e^{\lambda_1 t} f(t) dt \leq C \int_0^T e^{\lambda_1 t} \left(\int_t^T \hat{f}(\vartheta) d\vartheta\right) dt
\]

\[
= C \left[\frac{1}{\lambda_1} e^{\lambda_1 t}, \int_t^T \hat{f}(\vartheta) d\vartheta\right]_0^T + C \int_0^T \frac{1}{\lambda_1} e^{\lambda_1 t} \hat{f}(t) dt
\]

\[
= C \left(\int_0^T e^{\lambda_1 t} \hat{f}(t) dt - \int_0^T \hat{f}(\vartheta) d\vartheta\right) \leq C \int_0^T e^{\lambda_1 t} \hat{f}(t) dt
\]

since the second member within the brackets is positive. If a sufficiently large \(\lambda_1 > C\) is fixed, we get the relation

\[
\lim_{N \to \infty} \int_0^T e^{\lambda_1 t} \cdot \|\mathcal{I}^N P_1' - \mathcal{I}^N P_1''\|^2_{L^4\left([t, T), L^4(\Omega)\right)} dt = 0, \tag{4.19}
\]

implying

\[
\|\mathcal{I}^N P_1' - \mathcal{I}^N P_1''\|^2_{L^4\left([t, T), L^4(\Omega)\right)} \to 0 \tag{4.20}
\]

for almost all \(0 \leq t \leq T\). Consequently, the sequence \(\{\mathcal{I}^N P_1' - \mathcal{I}^N P_1''\}\) converges in \(L^2\left([0, T), L^4(\Omega)\right)\)-norm to the zero function, and the operator \(\mathcal{I}\) is contractive with respect to its first component on this space.

For the contractivity with respect to the second component, the arguments from Kunisch/Wagner 12, p. 1536 f., (3.65) ff. may be repeated. The proof is complete.

7) [Kunisch/Wagner 12, p. 1537, Remark (2)] In the case of the linearized Aliev-Panfilov model, Kunisch/Wagner 12, p. 1533, Theorem 3.10. can be applied as well: From Proposition 2.4., 3), we get

\[
W \in C^0\left([0, T], L^{8/3}(\Omega)\right), \quad \text{and} \ a_0 \ \text{belongs to} \ L^{5/2}(\Omega_T) \ \text{as required in assumption (c)'}.
\]

Moreover, assumption (d) can be satisfied: It holds that

\[
\int_0^T \int_\Omega |\hat{\Phi}_{tr} P_2|^2 dx dt \leq \int_0^T \|\hat{\Phi}_{tr}(t)\|^2_{L^6(\Omega)} \cdot \|P_2(t)\|^2_{L^3(\Omega)} dt \leq C \|\hat{\Phi}_{tr}\|^2_{L^2\left([0, T), L^6(\Omega)\right)}
\]

(4.21)

since \(P_2 \in C^0\left([0, T], L^3(\Omega)\right)\) by choice of \(\hat{P}_1\) and \(\hat{\Phi}_{tr} \in L^2\left([0, T), L^6(\Omega)\right)\) by Proposition 2.3.

8) [Kunisch/Wagner 12, p. 1542 ff., Appendix B] The proof of the stability estimate must be replaced by the proof of Theorem 1.1. above. The basic error was the inequality [Kunisch/Wagner 12, p. 1544, (B.14)], which holds not true.
b) Corrections within [KUNISCH/WAGNER 13A].

1) [KUNISCH/WAGNER 13A, p. 959, Theorem 2.7.] The stability estimate must be replaced by Theorem 1.2. above.

2) [KUNISCH/WAGNER 13A, p. 959, Theorem 2.8.] In view of Theorem 1.2., 1), the assertion remains true as far as the Rogers-McCulloch or the FitzHugh-Nagumo model is considered. In the case of the linearized Aliev-Panfilov model, we must assume that $W_0$ belongs to $W^{1,3/2}(\Omega)$ rather than to $L^4(\Omega)$, cf. Theorem 1.4.

3) [KUNISCH/WAGNER 13A, pp. 961 ff., Proof of Theorem 2.7.] As mentioned in the proof of Theorem 1.2., [KUNISCH/WAGNER 13A, p. 959 f., Lemma 2.9., and p. 961, Lemma 2.10.] hold true. Steps 1 − 6 of the proof must be changed along the lines of the proof of Theorem 1.2. The basic error entered with the inequalities [KUNISCH/WAGNER 13A, p. 964, (2.69) and (2.70)] which do not hold true.

4) [KUNISCH/WAGNER 13A, p. 973 f., Remark 2), (2.127) − (2.132)] This derivation has to be changed along the lines of (3.82) − (3.86) above.

5) [KUNISCH/WAGNER 13A, p. 976 ff., Theorem 3.3., Corollary 3.4. and 3.5.] In the beginning of [KUNISCH/WAGNER 13A, p. 979, Proof of Theorem 3.3.], it has been claimed that the transmembrane potential $\Phi_{tr}$ within a given weak solution $(\Phi_{tr}, \Phi_e, W)$ of the weak bidomain system can be shown to belong to $L^4([0, T), L^6(\Omega)]$, thus guaranteeing $I_{ion}(\Phi_{tr}, W) \in L^{4/3}[0, T), L^2(\Omega)]$. This derivation holds wrong. The same applies to [KUNISCH/WAGNER 13A, p. 983, Remarks 1) and 2), (3.61) ff.]. Consequently, the assertions cannot be maintained. On the other hand, the theorems remain true if the given weak solution satisfies $\Phi_{tr} \in L^4((0, T), L^6(\Omega))$ or, equivalently, $I_{ion}(\Phi_{tr}, W) \in L^{4/3}((0, T), L^2(\Omega))$ from the outset.

c) Corrections within [KUNISCH/NAGAIH/WAGNER 11].

In view of Theorems 1.1. and 1.3. above, [KUNISCH/NAGAIH/WAGNER 11, p. 260, Theorem 2.2.] and [KUNISCH/NAGAIH/WAGNER 11, p. 261 f., Theorem 3.3. and Corollary 3.4.] remain true without changes.

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