Moving in a crowd: human perception as a multiscale process

Annachiara Colombi, Marco Scianna
Department of Mathematical Sciences
Politecnico di Torino
Corso Duca degli Abruzzi 24, 10129 Torino, Italy

Andrea Tosin
Istituto per le Applicazioni del Calcolo “M. Picone”
Consiglio Nazionale delle Ricerche
Via dei Taurini 19, 00185 Rome, Italy

Abstract

The strategic behavior of pedestrians is largely determined by how they perceive and consequently react to neighboring people. Such interpersonal interactions may be dictated by the emotional state of the individuals, the purpose of their trip, the local crowding, the quality of the environment to mention but a few common examples. This issue is addressed in this paper by a mathematical model which combines, in an evolutionary time- and space-dependent way, discrete and continuous effects of pedestrian interactions. Numerical simulations and qualitative analysis suggest that human perception, and its impact on crowd dynamics, can be effectively modeled as a multiscale process based on such a dual representation of groups of agents.

Keywords: human perception, individuality vs. collectivity, multiscale model, measure theory

Mathematics Subject Classification: 35L65, 35Q70, 90B20

1 Introduction

The study of pedestrian behavior, both in normal and in panic conditions, is fostering the development of various multidisciplinary approaches. For decades, human crowds have been studied mainly by means of empirical methods: in particular, data collection has been typically based on direct observations, photographs, and time-lapse films, see for example [2, 3, 12, 16, 18]. Experimental approaches have provided information on a wide range of walking determinants, such as mean speed and preferential direction, reactions to the presence of obstacles and/or attraction points, individual behavior in normal or exceptional

*This author has been funded by a post-doctoral research scholarship awarded by the National Institute for Advanced Mathematics “F. Severi” (INdAM, Italy).
conditions (such as e.g., scarce visibility due to smoke or fog, evacuation, panic), thereby laying the foundations of an important descriptive knowledge.

In relatively recent years, traditional methods for the investigation of pedestrian dynamics began to increasingly interface and integrate with modeling and simulation approaches stemming from Physics, Engineering, and Applied Mathematics, whose aim is to predict pedestrian behaviors. In this respect, theoretical crowd models are required, on the one hand, to capture the strategic decision-based behavior of walking pedestrians and, on the other hand, to be sufficiently versatile to deal with complex real-world scenarios.

The current literature offers a wide range of mathematical approaches to the modeling of crowd dynamics (for recent critical surveys see e.g., [8, Chapter 4] and [11]), most of which feature significant similarities in the basic hypotheses used to describe pedestrian movement. Excluding more complicated strategies, and referring in particular to normal conditions, pedestrians are typically assumed to walk in a desired direction at a preferred speed, both of which are systematically modified by interactions with other individuals as well as with obstacles and walls. The desired velocity is an “optimal” one suitable to reach a certain destination (henceforth also called target). Usually it is understood as the velocity adopted by a normally behaving individual, who assesses his/her path toward the target considering the arrangement of the environment (obstacles, walls) in the absence of other pedestrians, see e.g., [15, 21]. However, more sophisticated models can also be proposed, which take into account the ability of walkers to evaluate the instantaneous distribution of other surrounding people and possibly to anticipate their movements, see e.g., [9]. Interpersonal interactions reflect instead the intention of a pedestrian to avoid collisions with neighboring individuals, see e.g., [6, 7, 14]. In some cases they also account for the tendency to closely follow some of his/her group mates, i.e., the so-called social behavior described in [13, 17]. Interactions are typically regarded as an additive correction to the prescribed desired velocity. Different possibilities are proposed for instance in [1, 10], where the choice of the local direction of movement is modeled as a unique decision process considering simultaneously both the trend toward the target and the one toward less congested areas. In addition, interactions are usually assumed to be nonlocal and anisotropic in space: pedestrians are able to react to even far individuals (although their visual field is definitely limited) but they normally pay attention only to the crowd distribution “ahead”, i.e., in the direction of the destination to be reached.

The models mentioned so far treat crowds at various observation and representation scales, from the microscopic (discrete, particle-based) to the macroscopic (continuous, density-based) or the mesoscopic (kinetic, statistical) one. More recently, multiscale models have been introduced [7, 8], which take advantage of a dual discrete/continuous representation of the crowd to reproduce realistic dynamical effects stemming from a weighted interplay between small and large scales. These models, which have also been successfully employed to study systems of differentiated interacting particles, such as e.g., cell aggregates [4, 5], suggest that the scale of representation of the crowd may play a role in characterizing the interaction dynamics among the individuals also when their total number is not infinite. In fact, if a continuous model is not derived as a limit of a discrete one for an infinite number of pedestrians but is postulated per se as an alternative description of a given finite number of people, then one may expect that the same interaction rules will produce different observable
effects depending on the spatial structure of the distribution of individuals to whom those rules apply. According to this perspective, the scale of the system complements the interaction rules in modeling individual behaviors.

In this paper we pursue such an intuition to introduce in crowd models the concept of human perception, i.e., how a pedestrian experiences the presence of other individuals. This amounts to admitting that a walker reacts differently if s/he interacts with other individuals singularly or with subgroups perceived collectively as a distributed mass. This fact may depend on several factors such as, among others, his/her emotional state (unhurried, hurried, panicky), the purpose of the trip (leisure, commuting), the local crowding, the quality of the environment (visibility conditions). To address this issue, we propose a substantial development of the measure-theoretic modeling approach first introduced in [22] and then improved and expanded in the already cited works [5, 4, 7, 8].

The basic idea is to describe the space distribution of a crowd by means of an abstract measure, which combines a singular/discrete and a regular/continuous component with respect to the ordinary Lebesgue measure. Such components are meant to represent the same group of pedestrians at complementary scales. Therefore they carry the same total mass, and are weighted by coefficients summing to 1 with values in the range \([0, 1]\), which say how much the crowd is perceived as a discrete/pointwise or continuous/smeared entity. In particular, we assume that they model the level of perception of an individual who crosses a static gathering of people and we study how the resulting migratory path depends on the kind of perception s/he activates, which may change in time and space during the movement due to the distribution of the crowd itself or to external environmental conditions.

In more detail, in Section 2 we present the mathematical model and discuss the formalization of the concept of perception in terms of a dual/multiscale representation of the crowd. In Section 3 we perform numerical simulations for different choices of the perception function, which show the ability of the model to reproduce different conceivable migratory paths of the moving pedestrian. In Section 4 we extend the model to the case of social groups, such as e.g., family members or friends, who want to keep the contact while crossing the crowd, and investigate numerically the impact of perception on their self-organized patterns. Finally, in Section 5 we study analytically, in the representative case of a single moving pedestrian, the relationship between the trajectory followed by the walker and the functional law describing his/her perception in terms of a priori stability/continuous-dependence estimates.

2 Mathematical model

The basic model that we propose in this paper considers a single pedestrian in the two-dimensional space \(\mathbb{R}^2\), who has to walk through a dense crowd in order to reach a specific point of interest. In order to better focus on the perception-dependent dynamics of this walker, we assume that the other individuals within the domain do not move. For this reason, we will henceforth refer to them collectively as the static crowd.

The moving pedestrian is described by his/her position \(X_t \in \mathbb{R}^2\) at each time instant \(t \geq 0\). In particular, the mapping \(t \mapsto X_t\) is his/her trajectory issuing from a given point \(X_0\) at the initial time. According to the modeling
framework discussed in the Introduction, we express the dynamics of the walker as the superposition of two contributions:

\[ \dot{X}_t = v_d(X_t) + v_{\text{int}}[\nu](X_t). \]

The desired velocity \( v_d : \mathbb{R}^2 \to \mathbb{R}^2 \) is a vector field which describes his/her preferred strategy to reach his/her destination from the current position. In the simplest case, like the one considered here, it is determined only by the geometry of the domain.

The interaction velocity \( v_{\text{int}} : \mathbb{R}^2 \to \mathbb{R}^2 \) is the correction to \( v_d \) due to interactions with static individuals. Their mass distribution, as perceived by the moving pedestrian, is described by a Radon positive measure \( \nu \), which carries a total mass \( \nu(\mathbb{R}^2) = N \in \mathbb{N} \) corresponding to the total number of individuals composing the static crowd. In particular, we adopt the following expression for \( v_{\text{int}} \):

\[ v_{\text{int}}[\nu](X_t) = \frac{1}{a + \nu(S_R(X_t))} \int_{S_R(X_t)} K(y - X_t) \, d\nu(y). \]  (1)

The interaction kernel \( K : \mathbb{R}^2 \to \mathbb{R}^2 \) describes interpersonal interactions occurring between the moving pedestrian in \( X_t \) and the individuals of the static crowd perceived in his/her sensory region \( S_R(X_t) \). The latter is typically modeled as a circular sector centered in \( X_t \) of radius \( R > 0 \), whose local orientation follows the vector \( v_d(X_t) \). The angular width of the sensory region defines the visual cone of the moving pedestrian. The total contribution of the interactions is then averaged, with a saturating Michaelis-Menten function (being \( a > 0 \)), by the quantity of static crowd within the sensory region of the moving pedestrian. As we will see in a moment, depending on the structure of \( \nu \) this guarantees a balanced coexistence of localized and distributed interactions with the static crowd corresponding to various levels of perception.

On the whole, the resulting equation for the dynamics of the moving pedestrian is:

\[ \dot{X}_t = v_d(X_t) + \frac{1}{a + \nu(S_R(X_t))} \int_{S_R(X_t)} K(y - X_t) \, d\nu(y). \]  (2)

2.1 Modeling perception: the multiscale structure of \( \nu \)

In principle, the measure \( \nu \) does not describe the actual physical distribution of the static individuals. It describes instead their distribution filtered by the per-
ception process of the moving pedestrian, as illustrated in the cartoon of Fig. 1. Remarkably, such a process corresponds to an interaction strategy adopted by the latter, which can be possibly dictated by environmental and/or psychological factors (for instance visibility, emotional state, travel purpose).

If the moving pedestrian perceives and interacts with each static individual singularly then his/her perception is discrete/localized and can be modeled by assuming

\[ \nu = \epsilon := \sum_{k=1}^{N} \delta_{\xi_k}, \]  

(3)

where \( \xi_k \in \mathbb{R}^2 \) is the position of the \( k \)th static individual. If, conversely, the moving pedestrian perceives the static crowd as an indistinguishable ensemble and interacts with subgroups of static individuals then his/her perception is continuous/distributed and can be modeled by assuming

\[ \nu = \rho, \]

where \( \rho \) is an absolutely continuous measure with respect to the Lebesgue measure in \( \mathbb{R}^2 \). With a slight abuse of notation, we denote its density still by \( \rho : \mathbb{R}^2 \to [0, +\infty) \) and we require \( \rho(\mathbb{R}^2) = \int_{\mathbb{R}^2} \rho(x) \, dx = N \).

The moving pedestrian can however modulate, in time and/or space, his/her level of perception of the surrounding individuals according to personal preferences or external stimuli. Hence, the measure \( \nu \) can be generally defined as a combination of the previously discussed discrete/localized and continuous/distributed perceptions by means of a level of perception \( \theta \in [0, 1] \) such that

\[ \nu = C_\theta \left( \theta \epsilon + (1 - \theta) \rho \right). \]  

(4)

The coefficient \( C_\theta > 0 \) is used to enforce the condition \( \nu(\mathbb{R}^2) = N \). Notice that for constant \( \theta \equiv 0, \theta \equiv 1 \) we recover the two basic perceptions introduced above with \( C_0 = 1 \) in both cases. More in general, in order to deal with a variable level of perception \( \theta \) we distinguish two cases.

**Lagrangian perception** The level of perception can be dictated by personal preferences or behavioral strategies actively expressed by the moving pedestrian. In this respect, the state of the walker is not only characterized by his/her position \( X_t \) but also by his/her level of perception \( \theta_t \) itself, which has to be regarded as a Lagrangian variable possibly evolving in time according to some equation to be specified. A simple, yet meaningful, choice consists in assuming that \( \theta_t \) is linked to \( X_t \) by an equation of state of the form

\[ \theta_t = \theta(X_t), \]

where \( \theta : \mathbb{R}^2 \to [0, 1] \) is a given function. For instance, \( \theta \) may account for the local crowding around the moving pedestrian, in such a way that the denser the static crowd the more biased the pedestrian to collectively interact with the surrounding people.

In this case, the perceived crowd measure (4) specializes as

\[ \nu_t = \theta_t \epsilon + (1 - \theta_t) \rho = \theta(X_t) \epsilon + (1 - \theta(X_t)) \rho. \]  

(5)
In particular, $\nu$ turns out to be parametrized by time because the level of perception evolves along the trajectory $t \mapsto X_t$ of the moving pedestrian. Notice that $C_\theta = 1$ since it results

$$\nu_t(\mathbb{R}^2) = \theta(X_t)\epsilon(\mathbb{R}^2) + (1 - \theta(X_t))\rho(\mathbb{R}^2) = N, \quad \forall \ t \geq 0.$$  

Finally, the interaction velocity (1) takes the form

$\nu_{\text{int}}(\nu_t)(X_t) = \frac{1}{a + \nu_t(S_R(X_t))} \left( \theta(X_t) \sum_{\xi_k \in S_n(X_t)} K(\xi_k - X_t) \right.
$$

$$+ \left. (1 - \theta(X_t)) \int_{S_n(X_t)} K(y - X_t)\rho(y) \, dy \right), \tag{6}$$

where $\nu_t(S_R(X_t)) = \theta(X_t) \sum_{k=1}^N \delta_{\xi_k}(S_R(X_t)) + (1 - \theta(X_t)) \int_{S_n(X_t)} \rho(y) \, dy$.

**Eulerian perception**  The level of perception can also be dictated by external stimuli, such as e.g., the presence of smoke or fog, which may affect the way in which the moving pedestrian interacts with other individuals. In this case, the perceived crowd measure (4) becomes

$$d\nu(x) = C_\theta(\theta(x) d\epsilon(x) + (1 - \theta(x))d\rho(x)), \tag{7}$$

where the level of perception $\theta : \mathbb{R}^2 \to [0, 1]$ is a prescribed Eulerian field which, for the sake of simplicity, we assume to be time-independent. For instance, it can be related to a (static) concentration of smoke or fog, which possibly hinders the visibility. With this interpretation, values of $\theta$ close to 0 would stand for locally high concentration, which prevents binary interactions between single individuals in favor of distributed interactions with subgroups of people, whereas values of $\theta$ close to 1 would model the opposite effect.

Since

$$\nu(\mathbb{R}^2) = C_\theta \left( \int_{\mathbb{R}^2} \theta(x) \, dx + \int_{\mathbb{R}^2} (1 - \theta(x)) \, d\rho(x) \right)$$

$$= C_\theta \left( N + \sum_{k=1}^N \theta(\xi_k) - \int_{\mathbb{R}^2} \theta(x)\rho(x) \, dx \right),$$

we see that the condition $\nu(\mathbb{R}^2) = N$ forces

$$C_\theta = \frac{N}{N + \sum_{k=1}^N \theta(\xi_k) - \int_{\mathbb{R}^2} \theta(x)\rho(x) \, dx}. \tag{8}$$

The interaction velocity (1) takes then the form

$$\nu_{\text{int}}[\nu](X_t) = \frac{C_\theta}{a + \nu(S_R(X_t))} \left( \sum_{\xi_k \in S_n(X_t)} \theta(\xi_k)K(\xi_k - X_t) \right.$$

$$\left. + \int_{S_n(X_t)} (1 - \theta(y))K(y - X_t)\rho(y) \, dy \right), \tag{9}$$

where $\nu(S_R(X_t)) = C_\theta \left( \sum_{\xi_k \in S_n(X_t)} \theta(\xi_k) + \int_{S_n(X_t)} (1 - \theta(x))\rho(x) \, dx \right)$.
Remark. A remarkable difference between Lagrangian and Eulerian perceptions is that in the former case the level of perception depends on the conditions in the place occupied by the moving pedestrian, therefore it is constant in the whole sensory region $S_R(X_t)$, cf. (6); while in the latter case it depends on the conditions in the places occupied by the individuals who the moving pedestrian interacts with, hence in principle it changes from point to point in $S_R(X_t)$, cf. (9).

3 Numerical simulations

In this section we perform numerical simulations of model (2) in a rectangular domain $\Omega \subset \mathbb{R}^2$ of size $30 \text{ m} \times 40 \text{ m}$, which is set to reproduce the planimetry of a square or of a large hall. The ordinary differential equation for $X_t$ is integrated using the forward Euler method with fixed time step. The integrals in the expression of $v_{\text{int}}$ are computed as Riemann sums by introducing in $\Omega$ a mesh made of square cells, over which the values of the density $\rho$ are approximated with cell averages. The simulation setup is illustrated in Fig. 2 (left panel). The moving pedestrian, initially located near the bottom edge of the domain – precisely in the point $x_0 = (15, 5)$ m – wants to reach a 2.5 m-wide target (e.g., a door) represented by the segment $[13.75, 16.25]$ m located on the top edge of the domain. In doing so, s/he has to pass through a static crowd of $N = 14$ individuals, who are unhomogeneously distributed in space: they are closer to each other in the lower part of the domain and quite farther in the upper part. Starting from their pointwise locations $\{\xi_k\}_{k=1}^N$ illustrated in Fig. 2, which form the discrete representation $\epsilon$ of the crowd, we construct a coherent\(^1\) continuous representation by defining the density $\rho$ as the superposition of $N$ unit-mass density bumps centered in the $\xi_k$’s:

$$\rho(x) = \sum_{k=1}^N \rho_k(y),$$

---

\(^1\)See Section 5.3 for a rigorous characterization of the coherence between the proposed discrete and continuous representations of the static crowd.
where

$$\rho_k(x) = \begin{cases} 
\frac{3}{\pi \sigma^2} \left( 1 - \frac{|x - \xi_k|}{\sigma} \right) & \text{if } |x - \xi_k| \leq \sigma \\
0 & \text{otherwise.} 
\end{cases}$$  \hspace{1cm} (11)$$

The parameter $\sigma > 0$ is the radius of the support of $\rho_k$, which we fix to $\sigma = 5$ m.

The behavior of the moving pedestrian is determined by (2). In particular, we take the desired velocity to be

$$v_d(X_t) = v_0 \frac{x_d - X_t}{|x_d - X_t|},$$

where $x_d = (15, 40)$ m is the center of the target to be reached and $v_0 = 1.3$ m/s is a comfort speed, which is a realistic value for walkers in a no-hurry state.

Moreover, we define the sensory region $S_R(X_t)$ as the following circular sector:

$$S_R(X_t) = \left\{ x \in \Omega : |x - X_t| \leq R, \ v_d(X_t) \cdot \frac{x - X_t}{|x - X_t|} \geq v_0 \cos \alpha \right\},$$

with sensory radius $R = 3$ m and half visual angle $\alpha = 35^\circ$. The condition

$$v_d(X_t) \cdot \frac{x - X_t}{|x - X_t|} \geq v_0 \cos \alpha$$

ensures that the pedestrian gaze direction is aligned with the direction of the desired velocity, i.e., toward the target.

Finally, we model the interaction kernel as a distance-decaying repulsive one (collision avoidance), see Fig. 3:

$$K(z) = \begin{cases} 
-k_0 \left( \frac{1}{R_b} - \frac{1}{R} \right) \frac{z}{|z|} & \text{if } |z| \leq R_b \\
-k_0 \left( \frac{1}{|z|} - \frac{1}{R} \right) \frac{z}{|z|} & \text{if } R_b < |z| \leq R \\
0 & \text{if } |z| > R
\end{cases} \hspace{1cm} (12)$$

Figure 3: The modulus of the interaction kernel (12) plotted vs. the distance between the interacting individuals. $\bar{K} := k_0(R_b^{-1} - R^{-1})$ is the maximum repulsion strength.
Figure 4: Paths followed by the moving pedestrian to reach the target on the upper edge of the domain and corresponding levels of perception $\theta$. Top-left: fully discrete perception, $\theta \equiv 1$. Top-right: fully continuous perception, $\theta \equiv 0$. Bottom-left: Lagrangian perception, $\theta_t = \theta(X_t)$. Bottom-right: Eulerian perception, $\theta = \theta(x)$. The value $T_{\text{evac}}$ reported in each panel is the time need by the moving pedestrian to reach the target.

where $R$ is the radius of the sensory region, $R_0 = 25$ cm is a typical body radius of adult medium-sized individuals, and $k_0 = 8 \text{ m}^2/\text{s}$ is a proportionality coefficient (basic repulsion strength).

The aim of the next numerical simulations is to study how different types of perception of the static crowd result in different dynamical strategies and migratory paths followed by the moving pedestrian.

A completely discrete/localized perception, given by the constant function $\theta = 1$, makes the moving pedestrian aware of the exact position of each static individual. Consequently s/he can pass among them, thereby reaching the target following an almost straight path, see Fig. 4 (top-left panel). The rationale is that the moving pedestrian feels a well localized microscopic repulsion coming from each static individual within his/her sensory region, hence s/he only needs to maintain a sufficient distance from each of them in order to walk undisturbed.

On the opposite, a completely continuous/distributed perception, given by the constant function $\theta = 0$, forces the pedestrian to circumnavigate the entire crowd, see Fig. 4 (top-right panel). This is due to the fact that s/he considers the ensemble of static individuals as a compact distributed mass, therefore s/he feels a macroscopic repulsion which prevents him/her to even enter the crowd. Notice that in these two cases the Lagrangian and Eulerian points of view (6), (9) are equivalent.
We now address variable levels of perception. We begin by the Lagrangian case, assuming that the moving pedestrian decides whether to react to single individuals or to subgroups depending on the local crowding around him/her. We choose an equation of state of the form
\[
\theta_t = \theta(X_t) = 1 - \left( \frac{\rho(X_t)}{\rho_{\text{max}}} \right)^\gamma, \quad \rho_{\text{max}} := \max_{x \in \Omega} \rho(x), \quad \gamma > 0, \quad (13)
\]
which suggests that the denser the crowd in \( X_t \) the more biased the perception toward the continuum. In particular, we fix \( \gamma = 6 \). As shown in Fig. 4 (bottom-left panel), the moving pedestrian partially enters the static crowd, however s/he remains definitely outside the densest regions. In particular, s/he always stays quite far from the pointwise locations of the static individuals.

Conversely, in the Eulerian case we consider a level of perception which gradually decreases from the bottom to the top of the domain. This is meant to simulate the presence of e.g., a static smoke whose concentration increases toward the target, thereby increasingly inhibiting the visibility of the moving pedestrian. Denoting the space variable by \( x = (x_1, x_2) \), with \( x_1, x_2 \in \mathbb{R} \), we consider the following function \( \theta \) which varies only with the vertical coordinate \( x_2 \):
\[
\theta(x_1, x_2) = \begin{cases} 
1 & \text{if } x_2 \in [0, 15) \text{ m} \\
\frac{x_2}{15 \text{ m}} + 2 & \text{if } x_2 \in [15, 30) \text{ m} \\
0 & \text{if } x_2 \in [30, 40] \text{ m}.
\end{cases} \quad (14)
\]
As shown in Fig. 4 (bottom-right panel), unlike the Lagrangian case the moving pedestrian is now able to pass between the individuals closer to his/her initial position. However, as soon as a continuous contribution to the perception activates (at \( x_2 = 15 \) m), s/he is quickly repelled by the static crowd and forced to circumnavigate the remaining group of individuals to reach the target. Notice that the resulting path is a coherent combination of the trajectories observed in the constant-perception cases.

By analyzing more closely the paths followed by the moving pedestrian in the last two cases, it is possible to notice that the activation of an even small continuous contribution to the perception of the walker changes dramatically his/her behavior, as s/he is forced to suddenly change direction of motion to reach the target. This is due to the “fragility” of the purely discrete interaction dynamics, which, by definition, strongly depend on the precise locations of the static individuals.

Finally, by looking at the times needed to reach the target (see the values of \( T_{\text{evac}} \) reported in each panel of Fig. 4), it is immediate to observe that the quickest “evacuation” corresponds to the purely discrete perception. The rationale is that perceiving the exact positions of the static individuals allows the moving pedestrian to follow the straightest path to his/her destination, a fact which is well reproduced by modeling perception as a multiscale process.

4 Social groups

We now turn to model the perception-dependent movement of a social group of pedestrians crossing an anonymous static crowd. A social group is here
considered in its sociological sense, i.e., as a group of individuals who have social ties and want to walk together. In this respect, their spatial proximity is not occasional but pursued, see [17].

Let $X_i^t \in \mathbb{R}^2$ be the position at time $t \geq 0$ of the $i$th member of a social group comprising $M > 1$ individuals. It makes sense to assume that all group mates have a common desired velocity driving them to the same destination and that they experience the same repulsive interactions with the static crowd. In addition, their dynamics are integrated by an attractive term, which encodes their desire to remain close to each other, and by a short-range mutual repulsion, which allows them to maintain a minimal reciprocal distance necessary for avoiding collisions in case of sudden velocity changes. We write then:

$$\dot{X}_i^t = v_d(X_i^t) + v_{int}[v](X_i^t) + v_{soc}[\vec{X}_t](X_i^t), \quad i = 1, \ldots, M,$$

where the first two terms at the right-hand side are like in (2), while the last term is a social component of the velocity which expresses the interactions among the members of the social group. We denoted $\vec{X}_t := (X_1^t, \ldots, X_M^t)$ for brevity, to recall that social interactions experienced by the $i$th member depend on all other group mates. Specifically, according to the previous discussion we
Figure 6: Paths followed by the social group, and corresponding spatial organization of its members at selected times, in case of variable levels of perception of the static crowd. Top panels: Lagrangian, $\theta_t = \theta(X_t)$. Bottom panels: Eulerian, $\theta = \theta(x)$.

The set:

$$v_{soc}[\vec{X}_t](X^i_t) = \sum_{X^j_t \in B_{Ra}(X^i_t)} k_a (X^j_t - X^i_t) - \sum_{X^j_t \in B_{Rr}(X^i_t) \setminus \{X^i_t\}} k_r \frac{X^j_t - X^i_t}{|X^j_t - X^i_t|^2},$$  \hspace{1cm} (15)

where $k_a = 0.1 \text{ s}^{-1}$, $k_r = 0.5 \text{ m}^2/\text{s}$ are proportionality coefficients (basic attraction and repulsion strengths, respectively), while $B_{Ra}(X^i_t)$ and $B_{Rr}(X^i_t)$ are the attraction and repulsion sensory regions of the $i$th member of the social group, namely balls centered in $X^i_t$ with radii $R_a = 10 \text{ m}$ and $R_r = 50 \text{ cm}$, respectively. In particular, we choose $R_r = 2R_b$ (cf. (12)) as it properly reproduces the minimal distance for short-range repulsion which prevents interpersonal collisions. It is worth stressing that, unlike interactions with the static crowd, social interactions are isotropic: the background idea is that members of the social group look intentionally for each other in every direction.

Focusing on a social group of $M = 3$ members, who start from the initial configuration shown in Fig. 2 (right panel), we test the effect of the four different types of perception introduced in the previous sections.

In case of a fully discrete/localized perception the social group is able to enter
Figure 7: Paths followed by the social group for all four types of perception in case of a lowered social attraction.

the static crowd and to pass in between, as demonstrated by the pedestrian paths reproduced in Fig. 5 (top-left panel). Interestingly, the group mates initially walk side-by-side, then a leader emerges and the remaining two members line up spontaneously behind him/her, see Fig. 5 (top-right panel). Such a group configuration is completely self-organized and can be explained by arguing that it is more convenient for the walkers to follow the path opened by the leader of the group. On the opposite, in case of a fully continuous/distributed perception the social group is forced to collectively circumnavigate the static crowd, see Fig. 5 (bottom-left panel). Several changes of leader are observed, hence an ordered line of pedestrians forms only occasionally, see Fig. 5 (bottom-right panel).

A Lagrangian density-dependent perception of type (13) allows the group mates to enter the more external ring of the static crowd distribution. In this case, a follow-the-leader pattern characterizes most of their migration, see Fig. 6 (top panels). Conversely, as already observed in the previous section, an Eulerian perception of type (14) results in the combination of the above two constant-perception cases: the members of the social group line up and dribble the first static individuals they encounter but then, as soon as a continuous contribution to the perception activates (again at $x_2 = 15$ m), they are repelled and circumnavigate more disorderly the remaining part of the static crowd, see Fig. 6 (bottom panels).

We then investigate how a decrement in the attraction coefficient $k_a$ among the group members, in particular from $0.1$ s$^{-1}$ to $0.05$ s$^{-1}$, affects the results of the model. As captured in the snapshots of Fig. 7, in case of a discrete perception (both extended in the entire domain with $\theta \equiv 1$ and restricted to its bottom part with the Eulerian space variation $\theta = \theta(x)$ given in (14)), the rightmost pedestrian initially separates from his/her group mates to avoid a static individual. Subsequently, walkers quickly regroup and reach the target together. This detailed effect is due to the fact that, with a lower social attraction, a localized repulsion from one static individual can induce a single pedestrian to shortly separate from his/her group mates. Conversely, as soon as a (possibly small) continuous contribution to the perception activates, the dynamics of the social group become again similar to those observed under a higher attraction coefficient (cf. Figs. 5, 6, 7 in the cases $\theta \equiv 0$, $\theta_t = \theta(X_t)$ in (13), and $\theta = \theta(x)$ in (14) for $x_2 > 15$ m).
As it is natural to infer, the social group dynamics described in this section are mainly determined by the attraction term in the social velocity (15). In fact, it is the tendency to stay together which ultimately characterizes a few pedestrians as a group. In order to confirm such an intuition, we set $k_a = 0$ in (15) and test the model in the case of a constant discrete level of perception. As shown in Fig. 8, the time evolutions of the three walkers become highly uncorrelated. Each of them behaves independently of the others, following his/her own preferred path to pass through the static crowd. Their reciprocal distances increase significantly with respect to the previous cases and they finally reach the target in dribs and drabs.

5 The path-perception relationship

The core of the previous sections has been the comparison of different walking paths generated by different choices of the perception strategy. In this section, focusing for simplicity on the case of a single moving pedestrian, we aim at supporting the numerical findings by showing that the relationship between $\theta$ and $X_t$ modeled by (2) is a “stable” one. That is, roughly speaking, that small variations in the level of perception produce small variations in the resulting trajectories.

In order to achieve analytical results we consider, in particular, a smoother version of the interaction velocity (1), which, for a generic $x \in \mathbb{R}^2$, reads (cf. also [23]):

$$v_{\text{int}}[\nu](x) = \frac{1}{a + \int_{\mathbb{R}^2} \eta_{S_R(x)}(y) \, d\nu(y)} \int_{\mathbb{R}^2} K(y-x) \eta_{S_R(x)}(y) \, d\nu(y),$$

(16)

where $\eta_{S_R(x)} : \mathbb{R}^2 \to [0, 1]$ is a smooth cut-off function compactly supported in the sensory region $S_R(x)$. Since $\eta_{S_R(x)}$ can be chosen, via mollification, as close as desired to the characteristic function of $S_R(x)$, this new form of $v_{\text{int}}$ does not significantly change the qualitative behavior of model (2). In fact, we have
heuristically:
\[
\int_{\mathbb{R}^2} \eta_{S_R(x)}(y) \, d\nu(y) \approx \nu(S_R(x)),
\]
\[
\int_{\mathbb{R}^2} K(y-x) \eta_{S_R(x)}(y) \, d\nu(y) \approx \int_{S_R(x)} K(y-x) \, d\nu(y).
\]

Let \( \mathcal{M}_+(\mathbb{R}^2) \) be the cone of positive measures on \( \mathbb{R}^2 \) endowed with the flat metric\(^2\):
\[
d(\nu', \nu'') := \sup_{\varphi \in C_{b,1}(\mathbb{R}^2) \cap \text{Lip}_1(\mathbb{R}^2)} \int_{\mathbb{R}^2} \varphi \, d(\nu'' - \nu'), \quad \nu', \nu'' \in \mathcal{M}_+(\mathbb{R}^2),
\]
where
\[
C_{b,1}(\mathbb{R}^2) = \left\{ \varphi \in C(\mathbb{R}^2) : \|\varphi\|_\infty := \sup_{x \in \mathbb{R}^2} |\varphi(x)| \leq 1 \right\}
\]
is the space of 1-uniformly bounded and continuous functions on \( \mathbb{R}^2 \) and
\[
\text{Lip}_1(\mathbb{R}^2) = \left\{ \varphi \in C(\mathbb{R}^2) : \text{Lip}(\varphi) := \sup_{x_1, x_2 \in \mathbb{R}^2, x_1 \neq x_2} \frac{|\varphi(x_2) - \varphi(x_1)|}{|x_2 - x_1|} \leq 1 \right\}
\]
is the space of Lipschitz continuous functions on \( \mathbb{R}^2 \) with at most unit Lipschitz constant.

Given two perceived crowd measures of the form (cf. (4)):
\[
\nu^i = C_\theta^i (\theta^i \epsilon + (1 - \theta^i) \rho) \in \mathcal{M}_+(\mathbb{R}^2) \quad (i = 1, 2),
\]
which differ in the choice of the perception function \( \theta^i \), we will compare the corresponding trajectories \( t \mapsto X^i_t \) of the moving pedestrian resulting from model (2) with interaction velocity (16) in two steps:

- first, following [23] we will obtain a Lipschitz continuity estimate for the total velocity of the form (cf. Proposition 5.2):
  \[
  |v\nu^2|(x_2) - v\nu^1|(x_1)| \leq C \left( |x_2 - x_1| + d(\nu^1, \nu^2) \right),
  \]
  \( C > 0 \) being a constant, which, by time integration, will lead to an estimate for \( |X^2_t - X^1_t| \) (cf. (19));

- second, we will analyze the term \( d(\nu^3, \nu^2) \) to extract information on \( |\theta^2 - \theta^1| \), thereby obtaining stability estimates for the path-perception relationship (cf. Theorems 5.4, 5.5). The precise form of such estimates will depend on whether the perception is modeled as a Lagrangian variable defined along the trajectory of the moving pedestrian (cf. Section 5.2.1) or as an Eulerian field defined on the whole space \( \mathbb{R}^2 \) (cf. Section 5.2.2).

\(^2\)In [19] it is shown that such a metric coincides with a generalized Wasserstein distance introduced in [20].
5.1 Estimates on the trajectories

To begin with, we establish estimate (17). We recall that, unlike Sections 2, 3, here we consider the following modified form of the total velocity:

\[ v(\nu)(x) = \nu_d(x) + \frac{1}{a + \int_{\mathbb{R}^2} \eta_{S_R(\nu)}(y) d\nu(y)} \int_{\mathbb{R}^2} K(y - x) \eta_{S_R(\nu)}(y) d\nu(y) \]  

(18)

for a generic \( x \in \mathbb{R}^2 \).

**Assumption 5.1** (Properties of the total velocity (18)).

- \( \nu_d : \mathbb{R}^2 \to \mathbb{R}^2 \) is a Lipschitz continuous field, i.e., there exists \( \text{Lip}(\nu_d) \geq 0 \) such that
  \[ |\nu_d(x_2) - \nu_d(x_1)| \leq \text{Lip}(\nu_d) |x_2 - x_1|, \quad \forall x_1, x_2 \in \mathbb{R}^2. \]

- For all \( x \in \mathbb{R}^2 \), \( S_R(x) \subset \mathbb{R}^2 \) is contained in the ball \( B_R(x) \) centered in \( x \) with radius \( R > 0 \) and is isometric to a reference set \( S_R(0) \subset \mathbb{R}^2 \), i.e.,
  \[ \xi_x(z) = R_x z + x, \quad R_x \in \mathbb{R}^{2 \times 2} \]
  which maps \( S_R(x) \) onto \( S_R(0) \): \( \xi_x^{-1}(S_R(x)) = S_R(0) \). The rotation angle characterizing \( R_x \) is the anticlockwise angle formed by the vector \( \nu_d(x) \) with the horizontal direction, cf. also [23].

- \( \eta_{S_R(x)} : \mathbb{R}^2 \to \mathbb{R} \) is such that
  \[ \eta_{S_R(x)} = \eta_{S_R(0)} \circ \xi_x^{-1} \]
  with \( 0 \leq \eta_{S_R(0)} \leq 1 \), \( \eta_{S_R(0)} \) compactly supported in \( S_R(0) \) and Lipschitz continuous in \( \mathbb{R}^2 \), i.e.,
  \[ \eta_{S_R(0)}(z) = 0 \quad \text{if } z \notin S_R(0) \]
  \[ |\eta_{S_R(0)}(z_2) - \eta_{S_R(0)}(z_1)| \leq \text{Lip}(\eta_{S_R(0)}) |z_2 - z_1| \quad \forall z_1, z_2 \in \mathbb{R}^2. \]

- \( K : \mathbb{R}^2 \to \mathbb{R}^2 \) is Lipschitz continuous and bounded in the ball \( B_R(0) \), i.e., there exist constants \( \text{Lip}(K) \), \( \bar{K} > 0 \) such that
  \[ |K(z_2) - K(z_1)| \leq \text{Lip}(K) |z_2 - z_1|, \quad |K(z)| \leq \bar{K}, \quad \forall z_1, z_2, z \in B_R(0). \]

**Remark.** In order to comply with Assumption 5.1, the desired velocity and the interaction kernel specified in Section 3 need to be slightly redefined, however in a way that does not affect significantly their qualitative trends.

We are now in a position to prove:

**Proposition 5.2** (Lipschitz continuity of the total velocity). _Under Assumption 5.1 there exists \( \bar{C} > 0 \) such that (17) holds for all \( x_1, x_2 \in \mathbb{R}^2 \) and all \( \nu^1, \nu^2 \in M_+(\mathbb{R}^2) \)._
Proof. 1. First we establish the Lipschitz continuity of $x \mapsto v[\nu](x)$ for a fixed $\nu \in \mathcal{M}_+(\mathbb{R}^2)$. From (18) we obtain:

$$
|v[\nu](x_2) - v[\nu](x_1)| \leq \text{Lip}(v_\delta) |x_2 - x_1|
$$

$$
+ \frac{1}{a} \left| \int_{\mathbb{R}^2} (K(y - x_2)\eta_{S_n(x_2)}(y) - K(y - x_1)\eta_{S_n(x_1)}(y)) \, d\nu(y) \right|
$$

$$
+ KN \left| \frac{1}{a + \int_{\mathbb{R}^2} \eta_{S_n(x_2)}(y) \, d\nu(y)} - \frac{1}{a + \int_{\mathbb{R}^2} \eta_{S_n(x_1)}(y) \, d\nu(y)} \right|
$$

For the second term at the right-hand side we proceed like in [23, Proposition 3] and find $C > 0$ such that

$$
\left| \int_{\mathbb{R}^2} (K(y - x_2)\eta_{S_n(x_2)}(y) - K(y - x_1)\eta_{S_n(x_1)}(y)) \, d\nu(y) \right| \leq C |x_2 - x_1|.
$$

Concerning instead the third term at the right-hand side, we observe that

$$
\left| \frac{1}{a + \int_{\mathbb{R}^2} \eta_{S_n(x_2)}(y) \, d\nu(y)} - \frac{1}{a + \int_{\mathbb{R}^2} \eta_{S_n(x_1)}(y) \, d\nu(y)} \right|
$$

$$
\leq \frac{1}{a^2} \int_{\mathbb{R}^2} \left| \eta_{S_n(x_2)}(y) - \eta_{S_n(x_1)}(y) \right| \, d\nu(y)
$$

$$
= \frac{1}{a^2} \int_{S_n(x_1) \cap S_n(x_2)} \left| \eta_{S_n(x_2)}(y) - \eta_{S_n(x_1)}(y) \right| \, d\nu(y)
$$

and we rely again on the same technique invoked above [23, Proposition 3] to find another $C > 0$ such that

$$
\leq C |x_2 - x_1|.
$$

Putting all together we get $|v[\nu](x_2) - v[\nu](x_1)| \leq C |x_2 - x_1|$.

2. Next we establish the Lipschitz continuity of $\nu \mapsto v[\nu](x)$ for a fixed $x \in \mathbb{R}^2$. Again from (18) we discover:

$$
|v[\nu^2](x) - v[\nu^1](x)| = \left| \int_{\mathbb{R}^2} K(y - x)\eta_{S_n(x)}(y) \, d(\nu^2 - \nu^1)(y) \right|
$$

$$
= \left| \int_{\mathbb{R}^2} K(R_x z)\eta_{S_n(0)}(z) \, d(\xi_z^{-1} # \nu^2 - \xi_z^{-1} # \nu^1)(z) \right|
$$

where the symbol $#$ denotes the push forward operator. Then we observe that $|K(R_x z)\eta_{S_n(0)}(z)| \leq \bar{K}$ in $\mathbb{R}^2$ and that this function is also Lipschitz continuous in $\mathbb{R}^2$ with Lipschitz constant bounded from above by $\bar{K} \text{Lip}(\eta_{S_n(0)}) + \text{Lip}(K)$ (cf. [8, Lemma 6.2]). Therefore, letting $C := \max\{\bar{K}, \bar{K} \text{Lip}(\eta_{S_n(0)}) + \text{Lip}(K)\}$ we have

$$
\frac{1}{C} K(R_{x^*}) \eta_{S_n(0)} \in C_{b,1}(\mathbb{R}^2) \cap \text{Lip}_1(\mathbb{R}^2)
$$

and, multiplying and dividing by $C$, we conclude the previous calculation as

$$
\leq Cd(\xi_z^{-1} # \nu^1, \xi_z^{-1} # \nu^2) = Cd(\nu^1, \nu^2),
$$

where the last equality follows from the fact that $\xi_z^{-1}$ is an isometry. \qed
The result of Proposition 5.2 can be transferred to the trajectory of the moving pedestrian using the equation \( \dot{X}_t = v[\nu](X_t) \). In the forthcoming calculation, for the sake of generality, we explicitly admit that the perceived crowd measure possibly depends on time, which will be indeed the case for the Lagrangian perception.

Let then \( \nu_1^i, \nu_2^i \in \mathcal{M}_+ (\mathbb{R}^2) \) be two perceived crowd measures. By time integration, the corresponding trajectories followed by the moving pedestrian satisfy \( X_i^t = X_i^0 + \int_0^t v[\nu_i^s](X_i^s) \, ds \), the \( X_i^0 \)'s being the initial positions. Restricting our attention to the case \( X_1^0 = X_2^0 \) and subtracting term by term we discover

\[
|X_2^t - X_1^t| \leq \int_0^t |v[\nu_2^s](X_2^s) - v[\nu_1^s](X_1^s)| \, ds \\
\leq C \left( \int_0^t |X_2^s - X_1^s| \, ds + \int_0^t d(\nu_1^s, \nu_2^s) \, ds \right) \quad \text{(because of Prop. 5.2)}
\]

whence finally, by Gronwall’s inequality,

\[
|X_2^t - X_1^t| \leq C e^{Ct} \int_0^t d(\nu_1^s, \nu_2^s) \, ds. \quad (19)
\]

### 5.2 Estimates on the path-perception relationship

We now pass to study in more detail the distance \( d(\nu_1^t, \nu_2^t) \) between the two perceived crowd measures in terms of the corresponding perception functions \( \theta^1, \theta^2 \). The following general assumptions are in order:

**Assumption 5.3** (Properties of the perception function). We assume that \( \theta : \mathbb{R}^2 \to [0, 1] \) is a Lipschitz continuous function, i.e., there exists a constant \( \text{Lip}(\theta) \geq 0 \)

\[
|\theta(x_2) - \theta(x_1)| \leq \text{Lip}(\theta) |x_2 - x_1|, \quad \forall x_1, x_2 \in \mathbb{R}^2.
\]

In addition, in the Eulerian perception case, we further assume that there exists \( c > 0 \) such that either of the following conditions holds:

- \( c \leq \theta(x) \leq 1 \);
- \( 0 \leq \theta(x) \leq 1 - c \)

for all \( x \in \mathbb{R}^2 \).

#### 5.2.1 Lagrangian perception: the case \( \theta_t = \theta(X_t) \)

In the Lagrangian case the perceived crowd measure is given by (5) and depends explicitly on time. We have:

**Theorem 5.4** (Stability for Lagrangian perception). Let \( \theta_1^t, \theta_2^t \) be two perception functions defined as \( \theta_i^t := \theta^i(X_i^t) \), \( i = 1, 2 \), where the \( \theta^i \)'s satisfy Assumption 5.3. Fix any final time \( 0 < T < +\infty \). Then there exists \( C_T > 0 \) such that

\[
|X_2^t - X_1^t| \leq C_T e^{C_T \ell(\theta^1, \theta^2) d(\rho, \epsilon) t} d(\rho, \epsilon) \| \theta^2 - \theta^1 \|_{\infty}
\]

for all \( t \in [0, T] \), where \( \ell(\theta^1, \theta^2) := \min\{\text{Lip}(\theta^1), \text{Lip}(\theta^2)\} \).

18
Proof. We pick a function \( \varphi \in C_b^1(R^2) \cap \text{Lip}_1(R^2) \) and, using the representation (5) for either perceived crowd measure \( \nu^i_t \), we compute

\[
\int_{R^2} \varphi \, d(\nu^2_t - \nu^1_t) = (\theta^2(X^2_t) - \theta^1(X^1_t)) \int_{R^2} \varphi \, d(\epsilon - \rho),
\]

which, upon taking the supremum over \( \varphi \) of both sides, implies

\[
d(\nu^1_t, \nu^2_t) = |\theta^2(X^2_t) - \theta^1(X^1_t)| \, d(\rho, \epsilon).
\]

Moreover, adding and subtracting \( \theta^2(X^1_t) \) we get:

\[
|\theta^2(X^2_t) - \theta^1(X^1_t)| \leq |\theta^2(X^2_t) - \theta^2(X^1_t)| + |\theta^2(X^1_t) - \theta^1(X^1_t)|
\]

\[
\leq \text{Lip}(\theta^2)|X^2_t - X^1_t| + \|\theta^2 - \theta^1\|_\infty.
\]

Since a similar result holds when adding and subtracting \( \theta^1(X^2_t) \), but with \( \text{Lip}(\theta^2) \) replaced by \( \text{Lip}(\theta^1) \), we conclude

\[
|\theta^2(X^2_t) - \theta^1(X^1_t)| \leq \ell(\theta^1, \theta^2) \, |X^2_t - X^1_t| + \|\theta^2 - \theta^1\|_\infty.
\]

Plugging this into (19) and using that \( t \leq T \) yields

\[
|X^2_t - X^1_t| \leq C e^{CT} d(\rho, \epsilon) \left( t \|\theta^2 - \theta^1\|_\infty + \ell(\theta^1, \theta^2) \int_0^t |X^2_s - X^1_s| \, ds \right),
\]

whence the thesis follows by Gronwall’s inequality after setting \( C_T := C e^{CT} \).

5.2.2 Eulerian perception: the case \( \theta = \theta(x) \)

In the Eulerian case the perceived crowd measure is given by (7) and, at least in the cases considered in the previous sections, does not depend explicitly on time. The constant \( C_\theta \) is defined in (8) but for the next calculations it is convenient to rewrite it as

\[
C_\theta = \frac{N}{N + \int_{R^2} \theta \, d(\epsilon - \rho)}.
\]  

(20)

In order to tackle the analysis of the term \( d(\nu^1_t, \nu^2_t) \) appearing in (19) we preliminarily recall that the following quantity:

\[
\| \cdot \| := \| \cdot \|_\infty + \text{Lip}(\cdot)
\]

is a norm in the space of Lipschitz continuous functions. Then we have:

**Theorem 5.5** (Stability for Eulerian perception). Let \( \theta^1, \theta^2 \) be two perception functions satisfying Assumption 5.3. Then

\[
|X^2_t - X^1_t| \leq \frac{C t e^{C T}}{e} \left( 1 + \frac{1}{cN} (\ell(\theta_1, \theta_2)d(\rho, \epsilon) + N) \right) d(\rho, \epsilon) \|\theta^2 - \theta^1\|
\]

for all \( t \geq 0 \), where now \( \ell(\theta^1, \theta^2) := \min\{\|\theta^1\|, \|\theta^2\|\} \).
Proof. For \( \varphi \in C_{b,1}(\mathbb{R}^2) \cap \text{Lip}_1(\mathbb{R}^2) \) we use the representation (7) to compute:

\[
\int_{\mathbb{R}^2} \varphi(d(\nu^2 - \nu^1)) = \int_{\mathbb{R}^2} \varphi(C_{\theta^2} \theta^2 - C_{\theta^1} \theta^1) d(\epsilon - \rho) + (C_{\theta^2} - C_{\theta^1}) \int_{\mathbb{R}^2} \varphi d\rho
\]

whence, adding and subtracting \( C_{\theta^2} \theta^1 \),

\[
= C_{\theta^2} \int_{\mathbb{R}^2} \varphi(\theta^2 - \theta^1) d(\epsilon - \rho) \\
+ (C_{\theta^2} - C_{\theta^1}) \left( \int_{\mathbb{R}^2} \varphi \theta^1 d(\epsilon - \rho) + \int_{\mathbb{R}^2} \varphi d\rho \right).
\]

Next we observe that, owing to the properties of \( \varphi, \theta^1, \theta^2 \), the functions \( \varphi(\theta^2 - \theta^1) \) and \( \varphi \theta^1 \) are bounded and Lipschitz continuous with Lipschitz constants such that

\[
\text{Lip}(\varphi(\theta^2 - \theta^1)) \leq \|\theta^2 - \theta^1\|, \quad \text{Lip}(\varphi \theta^1) \leq \|\theta^1\|,
\]

which implies

\[
\frac{\varphi(\theta^2 - \theta^1)}{\|\theta^2 - \theta^1\|}, \quad \frac{\varphi \theta^1}{\|\theta^1\|} \in C_{b,1}(\mathbb{R}^2) \cap \text{Lip}_1(\mathbb{R}^2).
\]

Continuing the previous calculation we discover then:

\[
(21) = C_{\theta^2} \|\theta^2 - \theta^1\| \int_{\mathbb{R}^2} \varphi(\theta^2 - \theta^1) d(\epsilon - \rho) \\
+ (C_{\theta^2} - C_{\theta^1}) \left( \|\theta^1\| \int_{\mathbb{R}^2} \varphi \theta^1 d(\epsilon - \rho) + \int_{\mathbb{R}^2} \varphi d\rho \right) \\
\leq C_{\theta^2} \|\theta^2 - \theta^1\| d(\rho, \epsilon) + (C_{\theta^2} - C_{\theta^1}) (\|\theta^1\| d(\rho, \epsilon) + N).
\]

Since a similar result holds when adding and subtracting \( C_{\theta^1} \theta^2 \), but with \( C_{\theta^2} \) in the first term replaced by \( C_{\theta^1} \) and \( \|\theta^1\| \) in the second term replaced by \( \|\theta^2\| \), we conclude

\[
\int_{\mathbb{R}^2} \varphi(d(\nu^2 - \nu^1)) \leq \min \{C_{\theta^1}, C_{\theta^2}\} \|\theta^2 - \theta^1\| d(\rho, \epsilon) \\
+ |C_{\theta^2} - C_{\theta^1}| (\|\theta^1\| d(\rho, \epsilon) + N).
\]

Moreover, the denominators of the constants \( C_{\theta^i} \) in (20) can be estimated as:

\[
N + \int_{\mathbb{R}^2} \theta^i d(\epsilon - \rho) = \int_{\mathbb{R}^2} \theta^i d\epsilon + \int_{\mathbb{R}^2} (1 - \theta^i) d\rho \\
\geq \begin{cases} 
\int_{\mathbb{R}^2} \theta^i d\epsilon \geq cN & \text{if } \theta^i \geq c \\
\int_{\mathbb{R}^2} (1 - \theta^i) d\rho \geq cN & \text{if } \theta^i \leq 1 - c,
\end{cases}
\]

therefore we have, on the one hand, \( C_{\theta^i} \leq \frac{1}{c} \) for both \( i = 1, 2 \) and, on the other
hand,
\[
|C_\theta^2 - C_\theta^1| = \left| \frac{N}{N + \int_{\mathbb{R}^2} \theta^2 d(\epsilon - \rho)} - \frac{N}{N + \int_{\mathbb{R}^2} \theta^1 d(\epsilon - \rho)} \right|
\]
\[
= N \left| \frac{\int_{\mathbb{R}^2} (\theta^2 - \theta^1) d(\epsilon - \rho)}{(N + \int_{\mathbb{R}^2} \theta^2 d(\epsilon - \rho)) (N + \int_{\mathbb{R}^2} \theta^1 d(\epsilon - \rho))} \right|
\]
\[
\leq \frac{1}{c^2 N} \left\| \theta^2 - \theta^1 \right\| \int_{\mathbb{R}^2} \frac{\theta^2 - \theta^1}{\left\| \theta^2 - \theta^1 \right\|} d(\epsilon - \rho)
\]
\[
\leq \frac{1}{c^2 N} \left\| \theta^2 - \theta^1 \right\| d(\rho, \epsilon),
\]
hence from (22), taking the supremum over \( \varphi \),
\[
d(\nu^1, \nu^2) \leq \frac{1}{c} \left( 1 + \frac{1}{c N} \left( \epsilon(\theta^1, \theta^2) d(\rho, \epsilon) + N \right) \right) \left\| \theta^2 - \theta^1 \right\|, \]
which, plugged into (19), yields the thesis.

5.3 Estimate on the discrete/continuous descriptions of the crowd

The estimates provided by Theorems 5.4, 5.5 do not characterize the path-perception relationship only in terms of the function \( \theta \). They also account for the distance between the continuous and discrete descriptions of the static crowd, which, as explained in Section 2.1, are indeed an integral part of the modeling of perception as a multiscale process. The underlying idea is that the measures \( \epsilon \) and \( \rho \) represent two extreme views that the moving pedestrian has of the static crowd, which are then mixed and weighted by the function \( \theta \). Therefore, in order to fully understand the path-perception relationship it is useful to study also the term \( d(\rho, \epsilon) \).

Given the discrete pedestrian distribution (3), in Section 3 it is explained that the corresponding continuous distribution is built as the superposition of \( N \) density bumps \( \rho_k \), cf. (10), such that \( \int_{\mathbb{R}^2} \rho_k(x) dx = 1 \) for all \( k = 1, \ldots, N \).

In particular, by carefully inspecting the expression (11) we see that the \( \rho_k \)'s are of the form
\[
\rho_k(x) = \frac{1}{\sigma^2} \tilde{\rho} \left( \frac{x - \xi_k}{\sigma} \right), \quad k = 1, \ldots, N, \tag{23}
\]
where \( \tilde{\rho} : \mathbb{R}^2 \to \mathbb{R} \) is the function
\[
\tilde{\rho}(x) = \frac{3}{\pi} \left( 1 - |x| \right) \chi_{B_1(0)}(x),
\]
\( \chi_{B_1(0)} \) being the characteristic function of the unit ball centered in the origin.

More in general it is not difficult to see that, given a nonnegative integrable function \( \tilde{\rho} \) with unit integral, the \( \rho_k \)'s obtained from (23) are nonnegative with unit integral as well for any \( \sigma > 0 \). Moreover, if \( \tilde{\rho} \) is compactly supported around the origin then \( \rho_k \) is compactly supported around the point \( \xi_k \), its support shrinking to \( \xi_k \) when \( \sigma \to 0^+ \). Consequently the density \( \rho \) in (10) is, by construction, a “natural” continuous counterpart of the discrete crowd distribution (3). The sense of this statement can be made more precise, with reference to the distance \( d(\rho, \epsilon) \), by means of the following result.
Proposition 5.6 (Discrete vs. continuous descriptions). Let $\tilde{\rho} \in L^1(\mathbb{R}^2)$ be nonnegative and such that
\[
\int_{\mathbb{R}^2} \tilde{\rho}(x) \, dx = 1, \quad \tilde{\mu}_1 := \int_{\mathbb{R}^2} |x| \tilde{\rho}(x) \, dx < +\infty
\]
(that is, $\tilde{\rho}$ is nominally a probability density with finite first order moment $\tilde{\mu}_1$). Define the $\rho_k$'s by (23) for $\sigma > 0$ and the measures $\epsilon, \rho$ by (3), (10), respectively. Then
\[
d(\rho, \epsilon) \leq \tilde{\mu}_1 N \sigma
\]
and, in particular, $d(\rho, \epsilon) \to 0$ when $\sigma \to 0^+$. 

Proof. For $\varphi \in C_{b,1}(\mathbb{R}^2) \cap \text{Lip}_1(\mathbb{R}^2)$ we compute:
\[
\int_{\mathbb{R}^2} \varphi(x) \, d(\epsilon - \rho)(x) = \sum_{k=1}^N \left( \varphi(\xi_k) - \int_{\mathbb{R}^2} \varphi(x) \rho_k(x) \, dx \right)
\]
\[
= \sum_{k=1}^N \left( \varphi(\xi_k) - \frac{1}{\sigma^2} \int_{\mathbb{R}^2} \varphi(x) \tilde{\rho} \left( \frac{x - \xi_k}{\sigma} \right) \, dx \right),
\]
then we make the substitution $z = \frac{x - \xi_k}{\sigma}$ in the integral to find
\[
= \sum_{k=1}^N \left( \varphi(\xi_k) - \int_{\mathbb{R}^2} \varphi(\sigma z + \xi_k) \tilde{\rho}(z) \, dz \right)
\]
and we further use the fact that $\tilde{\rho}$ has unit integral to discover
\[
= \sum_{k=1}^N \int_{\mathbb{R}^2} (\varphi(\xi_k) - \varphi(\sigma z + \xi_k)) \tilde{\rho}(z) \, dz.
\]
Finally, since $\varphi$ is Lipschitz continuous we obtain
\[
\leq N \sigma \int_{\mathbb{R}^2} |z| \tilde{\rho}(z) \, dz,
\]
whence the thesis follows from the finiteness of $\tilde{\mu}_1$ and the arbitrariness of $\varphi$. 

Remark. The integrability condition $\tilde{\mu}_1 < +\infty$ is automatically verified if $\tilde{\rho}$ has compact support.

Proposition 5.6 says that the construction indicated in Section 3, as a particular case of (23), is an appropriate one for obtaining a continuous distribution of the static crowd coherent with the discrete distribution (3). At the same time, in view of Theorems 5.4, 5.5, we deduce that for decreasing $\sigma$ the perception function $\theta$ becomes less and less influential in shaping the trajectory of the moving pedestrian. In the limit case $\sigma \to 0^+$ this can be heuristically argued from (4) (take formally $\epsilon = \rho$); more in general, we have here a rigorous statement of the “stability” of model (2) also with respect to the proposed dual description of the static crowd.
References

[1] J. P. Agnelli, F. Colasuonno, and D. Knopoff. A kinetic theory approach to the dynamics of crowd evacuation from bounded domains. *Math. Models Methods Appl. Sci.*, 25(1):109–129, 2015.

[2] M. Batty. Predicting where we walk. *Nature*, 388:19–20, 1997.

[3] R. L. Carstens and S. L. Ring. Pedestrian capacities of shelter entrances. *Traffic Eng.*, 41(3):38–43, 1970.

[4] A. Colombi, M. Scianna, and L. Preziosi. A measure-theoretic model for cell migration and aggregation. *Math. Model. Nat. Phenom.*, 1(10):4–35, 2015.

[5] A. Colombi, M. Scianna, and A. Tosin. Differentiated cell behavior: a multiscale approach using measure theory. *J. Math. Biol.*, 2014. doi:10.1007/s00285-014-0846-z.

[6] R. M. Colombo, M. Garavello, and M. Lécureux-Mercier. A class of non-local models for pedestrian traffic. *Math. Models Methods Appl. Sci.*, 22(4):1150023 (34 pages), 2012.

[7] E. Cristiani, B. Piccoli, and A. Tosin. Multiscale modeling of granular flows with application to crowd dynamics. *Multiscale Model. Simul.*, 9(1):155–182, 2011.

[8] E. Cristiani, B. Piccoli, and A. Tosin. *Multiscale Modeling of Pedestrian Dynamics*, volume 12 of *MS&A: Modeling, Simulation and Applications*. Springer International Publishing, 2014.

[9] E. Cristiani, F. S. Priuli, and A. Tosin. Modeling rationality to control self-organization of crowds: an environmental approach. *SIAM J. Appl. Math.*, 2015. Accepted.

[10] P. Degond, C. Appert-Rolland, J. Pettré, and G. Theraulaz. Vision-based macroscopic pedestrian models. *Netw. Heterog. Media*, 6(4):809–839, 2013.

[11] D. C. Duives, W. Daamen, and S. P. Hoogendoorn. State-of-the-art crowd motion simulation models. *Transportation Res. C*, 37:193–209, 2013.

[12] B. D. Hankin and R. A. Wright. Passenger flowing in subways. *Oper. Res. Quart.*, 9(2):81–88, 1958.

[13] D. Helbing, I. J. Farkas, P. Molnár, and T. Vicsek. Simulation of pedestrian crowds in normal and evacuation situations. In M. Schreckenberg and S. D. Sharma, editors, *Pedestrian and Evacuation Dynamics*, pages 21–58. Springer, Berlin, 2002.

[14] D. Helbing and A. Johansson. Pedestrian, crowd, and evacuation dynamics. In R. A. Meyers, editor, *Encyclopedia of Complexity and Systems Science*, volume 16, pages 6476–6495. Springer New York, 2009.

[15] B. Maury and J. Venel. Un modèle de mouvements de foule. *ESAIM: Proc.*, 18:143–152, 2007.
[16] M. Moussaïd, D. Helbing, S. Garnier, A. Johansson, M. Combe, and G. Theraulaz. Experimental study of the behavioural mechanisms underlying self-organization in human crowds. Proc. R. Soc. B, 276(1668):2755–2762, 2009.

[17] M. Moussaïd, N. Perozo, S. Garnier, D. Helbing, and G. Theraulaz. The walking behaviour of pedestrian social groups and its impact on crowd dynamics. PLoS One, 5(4):e10047, 2010.

[18] P. D. Navin and R. J. Wheeler. Pedestrian flow characteristics. Traffic Eng., 19(7):30–33, 1969.

[19] B. Piccoli and F. Rossi. On properties of the generalized Wasserstein distance. Preprint (arXiv:1304.7014).

[20] B. Piccoli and F. Rossi. Generalized Wasserstein distance and its application to transport equations with source. Arch. Ration. Mech. Anal., 211(1):335–358, 2014.

[21] B. Piccoli and A. Tosin. Pedestrian flows in bounded domains with obstacles. Contin. Mech. Thermodyn., 21(2):85–107, 2009.

[22] B. Piccoli and A. Tosin. Time-evolving measures and macroscopic modeling of pedestrian flow. Arch. Ration. Mech. Anal., 199(3):707–738, 2011.

[23] A. Tosin and P. Frasca. Existence and approximation of probability measure solutions to models of collective behaviors. Netw. Heterog. Media, 6(3):561–596, 2011.