On magnetohydrodynamic gauge field theory

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Abstract

Clebsch potential gauge field theory for magnetohydrodynamics is developed based on the theory of Calkin (1963 Can. J. Phys. 41 2241–51). It is shown how the polarization vector \( P \) in Calkin’s approach naturally arises from the Lagrange multiplier constraint equation for Faraday’s equation for the magnetic induction \( B \), or alternatively from the magnetic vector potential form of Faraday’s equation. Gauss’s equation, (divergence of \( B \) is zero) is incorporated in the variational principle by means of a Lagrange multiplier constraint. Noether’s theorem coupled with the gauge symmetries is used to derive the conservation laws for (a) magnetic helicity, (b) cross helicity, (c) fluid helicity for non-magnetized fluids, and (d) a class of conservation laws associated with curl and divergence equations which applies to Faraday’s equation and Gauss’s equation. The magnetic helicity conservation law is due to a gauge symmetry in MHD and not due to a fluid relabelling symmetry. The analysis is carried out for the general case of a non-barotropic gas in which the gas pressure and internal energy density depend on both the entropy \( S \) and the gas density \( \rho \). The cross helicity and fluid helicity conservation laws in the non-barotropic case are nonlocal conservation laws that reduce to local conservation laws for the case of a barotropic gas. The connections between gauge symmetries, Clebsch potentials and Casimirs are developed. It is shown that the gauge symmetry functionals in the work of Henyey (1982 Phys. Rev. A 26 480–3) satisfy the Casimir determining equations.

Keywords: magnetohydrodynamics, gauge symmetries, conservation laws, helicities, Casimirs, Clebsch Potentials
1. Introduction

In this paper we investigate the role of gauge symmetries and gauge transformations in magnetohydrodynamics (MHD) based in part on the work of Calkin (1963). Tanehashi and Yoshida (2015) used the MHD Casimirs of magnetic helicity and cross helicity for MHD and the fluid helicity to determine gauge transformations. They used Clebsch representations for the fluid velocity and the magnetic field and related the gauge transformations to the non-canonical Poisson bracket for MHD of Morrison and Greene (1980, 1982). Our aim is to provide a description of gauge symmetries in MHD using a constrained variational principle, in which the mass, entropy, Lin constraint, and Faraday’s law are included in the variational principle using Lagrange multipliers. We include a Lagrange multiplier term to ensure $\nabla \cdot \mathbf{B} = 0$ where $\mathbf{B}$ is the magnetic field induction.

The MHD model of Calkin (1963) is different from most MHD formulations, since it takes into account the polarization $\mathbf{P}$ and polarization charge density $\rho_c$ of the plasma, as well as quadratic electric field terms in the Lagrangian (i.e. it includes displacement current effects). In particular, Calkin (1963) uses the charge continuity equation:

$$\frac{\partial \rho_c}{\partial t} + \nabla \cdot (\mathbf{J} + \rho_c \mathbf{u}) = 0,$$

where $\mathbf{u}$ is the fluid velocity and $\rho_c$ is the charge density. The charge density $\rho_c$ and non-advection current $\mathbf{J}$ are related to the polarization $\mathbf{P}$ through the equations:

$$\mathbf{J} = \frac{\partial \mathbf{P}}{\partial t} - \nabla \times (\mathbf{u} \times \mathbf{P}) + \mathbf{u} \nabla \cdot \mathbf{P},$$

$$\nabla \cdot \mathbf{P} = -\rho_c.$$

Note that (1.1)–(1.3) are consistent. Taking the divergence of (1.2) and taking into account (1.3) gives the charge continuity equation (1.1). A derivation of (1.2) is given in Panofsky and Phillips (1964). In the non-relativistic MHD model the quasi-neutrality of the plasma (the plasma approximation) is invoked, and $\rho_c$ is set equal to zero. Calkin retains the electric field energy density in his Lagrangian, which is related to the displacement current, which is usually neglected in non-relativistic MHD. The result (1.2) may be written as:

$$\left( \frac{\partial}{\partial t} + \mathcal{L}_\mathbf{u} \right) (\mathbf{P} \cdot d\mathbf{S}) \equiv \frac{d}{dt} (\mathbf{P} \cdot d\mathbf{S}) = \mathbf{J} \cdot d\mathbf{S},$$

where $\mathcal{L}_\mathbf{u} = \mathbf{u} \cdot \nabla$ is the Lie derivative following the flow, and $d/dt = \partial/\partial t + \mathbf{u} \cdot \nabla$ is the Lagrangian time derivative following the flow. The quantity:

$$\mathbf{P} \cdot d\mathbf{S} = P_y dy \wedge dz + P_z dz \wedge dx + P_x dx \wedge dy,$$

is the polarization 2-form. In the usual MHD, non-relativistic limit $\nabla \cdot \mathbf{P} = 0$ (i.e. $\rho_c = 0$), and (1.2) simplifies to:

$$\mathbf{J} = \frac{\partial \mathbf{P}}{\partial t} - \nabla \times (\mathbf{u} \times \mathbf{P}),$$

which resembles Faraday’s equation (but for $\mathbf{P}$ rather than $\mathbf{B}$), in which $\mathbf{J} \neq 0$ is the source term. We show that (1.2) or (1.6) arises as the evolution equation for a vector Lagrange multiplier used in enforcing Faraday’s equation in the variational principle.

Magnetic helicity is a key quantity in describing the topology of magnetic fields (e.g. Moffatt (1969, 1978), Moffatt and Ricca (1992), Berger and Field (1984), Finn and Antonsen (1985) and Arnold and Khesin (1998)). The magnetic helicity $H_M$ is defined as:
\[ H_M = \int_V \omega_1 \wedge d\omega_1 = \int_V d^3 x \mathbf{A} \cdot \mathbf{B}, \]  
(1.7)

where \( \omega_1 = \mathbf{A} \cdot d\mathbf{x} \) is the magnetic vector potential 1-form, \( d\omega_1 = \mathbf{B} \cdot d\mathbf{S} \) is the magnetic field 2-form; \( \mathbf{B} = \nabla \times \mathbf{A} \) is the magnetic induction, \( \mathbf{A} \) is the magnetic vector potential and \( V \) is the isolated volume containing the field. Magnetic helicity is an invariant of MHD (e.g. Elsässer (1956), Woltjer (1958), Moffatt (1969, 1978)). In (1.7) it is assumed that the normal magnetic field component \( B_n \) vanishes on the boundary \( \partial V \). The first form of magnetic helicity in (1.7) is known as the Hopf invariant. It is also a Casimir of MHD. A Casimir \( C \) is a functional that has the property \( \{F, C\} = 0 \), for a general functional \( F \), where \( \{F, G\} \) is the Hamiltonian Poisson bracket for the Hamiltonian system (e.g. Padhye and Morrison (1996a, 1996b), Morrison (1998) and Hameiri (2004)).

Cross helicity in MHD is defined as the integral:

\[ H_C = \int_V d^3 x \mathbf{u} \cdot \mathbf{B}, \]  
(1.8)

In (1.8) it is assumed that \( \mathbf{B} \cdot \mathbf{n} = 0 \) on the boundary \( \partial V \) of the volume \( V \). It is a Casimir for barotropic MHD (e.g. Padhye and Morrison (1996a, 1996b)). It is also referred to as a rugged invariant in MHD turbulence theory (e.g. Mattheus and Goldstein (1982)). For a barotropic equation of state for the gas \( dH_C/dt = 0 \) for a volume \( V = V_m \) moving with the flow. For a non-barotropic equation of state for the gas,

\[ \frac{dH_C}{dt} = \int_V d^3 x \mathbf{T} \mathbf{B} \cdot \nabla \mathbf{S}, \]  
(1.9)

which reduces to \( dH_C/dt = 0 \) for a barotropic gas. One can define a modified form of cross helicity for a non-barotropic gas as:

\[ H_{CNB} = \int_V d^3 x \left( \mathbf{u} + r \nabla \mathbf{S} \right) \cdot \mathbf{B}, \quad \text{where} \quad \frac{dr}{dt} = -T(x, t), \]  
(1.10)

where \( T \) is the temperature of the gas (e.g. Webb et al (2014a, 2014b), Yahalom (2016, 2017a), (2017b)). Note that

\[ \frac{dH_M}{dt} = 0, \quad \frac{dH_{CNB}}{dt} = 0. \]  
(1.11)

Webb et al (2014a, 2014b) derived MHD conservation laws (CL’s) using the Lie dragging approach of Tur and Yanovsky (1993) and also by using Noether’s theorems. In particular, they derived the cross helicity CL for both barotropic and non-barotropic MHD. Yahalom (2016, 2017a), (2017b) and Yahalom (2013) investigated the magnetic helicity and cross helicity conservation laws for both barotropic and non-barotropic equations of state for the gas. He gave interpretations of these conservation laws in terms of generalized Aharonov–Bohm effects. The Aharonov–Bohm effect describes the change in phase of the wave function in quantum mechanics, depending on the path integral of \( \int \mathbf{A} \cdot d\mathbf{x} \) about an isolated magnetic flux in the vicinity of the path (Aharonov and Bohm 1959). Yahalom expresses his results in terms of the magnetic helicity per unit magnetic flux, and the cross helicity per unit magnetic flux and in terms of the magnetic metage (metage is a measure of distance along a field line which is the intersection of two Euler surfaces, see e.g. appendix E).

These conservation laws provides tests for symplectic MHD codes (see Krauss et al (2016) for reduced MHD, and Krauss and Maj (2017) for 2D MHD), which are designed to preserve magnetic helicity, cross helicity and energy, and are sometimes known as geometric
integrators. They also preserve the Hamiltonian Poisson bracket structure of the equations. A brief discussion of Yahalom’s approach is described in appendix E. Yahalom (2013, 2017a, 2017b) provides a semi-analytical example where the magnetic helicity per unit magnetic flux is related to the jump of the metage potential as one moves along a field line formed by the intersection of two Euler potential surfaces. This type of example is in principle very useful for the testing of MHD codes, against theoretical ideas of magnetic helicity and Clebsch potentials. The conservation law for non-barotropic cross helicity, provides a simple challenge for MHD codes, since it can be formulated in the context of the usual MHD equations, by the addition of a single non-local variable \( r \) which satisfies 
\[
\frac{dr}{dt} \equiv r_t + \mathbf{u} \cdot \nabla r = -T(x,t)
\]
where \( T(x,t) \) is the temperature of the gas ((Yahalom 2017a, 2017b) uses \( \sigma = -r \)). There is also a fluid dynamical analog of this conservation law in ideal fluid dynamics in which \( B \) is replaced by the generalized vorticity \( \Omega = \omega + \nabla r \times \nabla S \) where \( \omega = \nabla \times \mathbf{u} \) is the fluid vorticity (e.g. Mobbs (1981)). Testing the magnetic helicity conservation law requires the calculation of the magnetic vector potential \( \mathbf{A} \), which depends on the gauge. To obtain a physically meaningful topological result requires further analysis (e.g. Berger and Field (1984), Moffatt and Ricca (1992), Arnold and Khesin (1998) and Webb et al (2010a)).

Section 2 introduces the Eulerian variational principle, using Lagrange multipliers to enforce the mass, entropy advection, Lin constraints, Gauss’ law, and Faraday’s law. Section 3, gives the determining equations for gauge symmetries. The main idea is that the gauge symmetries do not alter the physical variables. A central equation is the Clebsch potential expansion for the fluid velocity \( \mathbf{u} \), which follows from requiring that the action is stationary for Eulerian variations \( \delta \mathbf{u} \) of the fluid velocity.

Section 3 investigates the conditions for Lie transformations to leave the action and the physical variables invariant up to a divergence transformation by using the approach of Calkin (1963).

Section 4 describes Noether’s theorem, which requires the action remain invariant up to a divergence transformation. This condition gives conservation laws of the MHD system due to the gauge symmetries. We derive the magnetic helicity conservation law, and the cross helicity conservation laws using Noether’s theorem. These two conservation laws were derived by Calkin (1963). Our formulation is slightly more general than Calkin (1963), who considered the case of a barotropic gas for which the pressure \( p = p(\rho) \) whereas we use a non-barotropic gas for which \( p = p(\rho, S) \) where \( \rho \) is the density and \( S \) is the entropy of the gas. For a barotropic gas the cross helicity conservation law is a local conservation, but it is a nonlocal conservation law for a non-barotropic gas.

Section 5 explores the connection between gauge symmetries and Casimirs based on the gauge field theory of Henyey (1982). We show that the condition that the fluid velocity remain invariant coupled with the conditions that the density, entropy and magnetic field and Lin constraint variables \( \mu^k \) do not change under gauge transformations implies that the gauge transformation generating functionals are Casimirs.

Section 6 concludes with a summary and discussion. Some of the important results established in the paper are: (i) A version of Noether’s theorem describing gauge symmetries in MHD; (ii) A gauge symmetry associated with the Lagrange multiplier \( \Gamma \) that enforces Faraday’s equation, and the Lagrange multiplier \( r \) that enforces \( \nabla \cdot \mathbf{B} = 0 \) give rise via Noether’s theorem to the magnetic helicity conservation law (this symmetry is not a fluid relabelling symmetry). Calkin (1963) derived the magnetic helicity conservation law for a more complicated version of MHD involving the polarization vector \( \mathbf{P} \) (for usual MHD \( \nabla \cdot \mathbf{P} = 0 \)); (iii) An analysis of the gauge symmetry determining equations (appendix B) shows the connection of these equations to the Lie symmetry structure of the steady MHD equations established by Bogoyavlenskij (2002) and Schief (2003);
(iv) the gauge symmetry functionals of Henyey (1982) are shown to satisfy the Casimir determining equations (appendix D); (v) A gauge symmetry associated with $\Gamma$ and $\nu$ and an arbitrary potential $\Lambda(x, t)$ is related to fluid conservation laws obtained by Cheviakov (2014). Cheviakov studied conservation laws for fluid models containing an equation subsystem of the form $N_t + \nabla \times M = 0$ and $\nabla \cdot N = 0$. A nonlocal conservation law with conserved density $D = B \cdot \nabla \phi$ ($\phi$ is the potential in Bernoulli’s equation) is derived. This method can be used to obtain the potential vorticity conservation law for MHD of Webb and Mace (2015). (vi) The mass continuity equation, the Eulerian entropy conservation equation, Gauss’s equation and Faraday’s equation may be obtained by using Noether’s theorem and gauge symmetries. The mass continuity equation can be regarded as due to symmetry breaking of a fluid relabelling symmetry (Holm et al. 1998).

Detailed calculations are given in appendices A–E. Appendix A derives the MHD momentum equation from the Clebsch variational equations. Appendices B and C derive solutions of the gauge symmetry equations. In particular, appendix B gives solutions of the equations analogous to the steady state Faraday equation, Gauss law and the mass continuity equation. Faraday’s equation may be described by a 2 dimensional simple Abelian Lie algebra which is integrable by Frobenius theorem (see Bogoyavlenskij (2002), Schief (2003) and Webb et al. (2005)). An alternative solution method was presented in appendix C. Appendix D shows that the variational equations for gauge potential functionals $F$ (e.g. Henyey (1982)) are equivalent to the Casimir determining equations (e.g. Hameiri (2004)). Appendix E describes the work of Yahalom (2016, 2017a), (2017b).

2. MHD variational principles

There are a variety of different variational principles that describe MHD. We use the action:

$$ A = \int \ell \, d^3 x \, dt, \quad (2.1) $$

where

$$ \ell = \left( \frac{1}{2} \rho u^2 - \varepsilon (\rho, S) - \frac{B^2}{2\mu} - \rho \Phi(x) \right) + \phi \left( \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho u) \right) + \beta \left( \frac{\partial S}{\partial t} + u \cdot \nabla S \right) + \sum_k \lambda_k \left( \frac{\partial \mu_k}{\partial t} + u \cdot \nabla \mu_k \right) + \nu \nabla \cdot B + \Gamma \cdot \left( \frac{\partial B}{\partial t} - \nabla \times (u \times B) + \epsilon_1 u \nabla \cdot B \right). \quad (2.2) $$

The Lagrangian (2.2) consists of the kinetic minus the potential energy of the MHD fluid, including the gravitational potential energy term $\rho \Phi(x)$ due to an external gravitational field (e.g. in stellar wind theory, $\Phi(x)$ is the gravitational potential energy due to the star). The other terms in (2.2) are constraint terms where $\phi, \beta, \lambda_k (k < 3), \nu$ and $\Gamma$ are Clebsch potentials that enforce mass, entropy advection, Lin constraint, Gauss’s law ($\nabla \cdot B = 0$) and Faraday’s law in the MHD limit.

Lin constraints Lin (1963), Cendra and Marsden (1987) were originally introduced in fluid dynamics to describe fluid vorticity in Clebsch expansions for the velocity $u$ of the form:

$$ u = \nabla \phi - r \nabla S - \sum_{k=1}^3 \lambda_k \nabla \mu_k, \quad (2.3) $$
where \( r = \beta / \rho \). The necessity of the terms \(- \sum_{k=1}^{3} \lambda^k \nabla \mu^k \) in the expansion becomes obvious when one calculates the vorticity of the fluid, by taking the curl of \( u \), to obtain

\[
\omega = \nabla \times u = -\nabla r \times \nabla S - \sum_{k=1}^{3} \nabla \lambda^k \times \nabla \mu^k.
\]  

(2.4)

From (2.4) the term \(- \nabla r \times \nabla S\) represents the vorticity induced by the non-barotropic effects in which the fluid density and pressure gradients are misaligned (alternatively fluid spin is produced by the misalignment of the temperature an entropy gradients (e.g. Pedlosky (1987)). However, the fluid element could also have non-zero vorticity due to the initial conditions. This latter source of vorticity is represented by the sum over \( k \) from \( k = 1 \) to \( k = 3 \) in (2.4). Further analysis (e.g. Cendra and Marsden (1987) and Holm et al (1998)) links the Clebsch expansion for \( u \) with a fluid relabeling diffeomorphism and a momentum map in which the Lagrange multipliers and the constraint equation variables are canonically conjugate variables (see also Zakharov and Kuznetsov (1997)).

We use the variational principle (2.1) and (2.2) both with \( \epsilon_1 = 0 \) in sections 3 and 4, and with \( \epsilon_1 = 1 \) in section 5. The case \( \epsilon_1 = 0 \) gives the usual form of Faraday’s equation for the case \( \nabla \cdot B = 0 \). The case \( \epsilon_1 = 1 \) and \( \nu = 0 \) is useful in exploring the effect of \( \nabla \cdot B \neq 0 \), which is of interest in numerical MHD codes, in which \( \nabla \cdot B \neq 0 \) may be generated by the approximation scheme (e.g. Powell et al (1999), Janhunen (2000), Dedner et al (2002) and Webb et al (2010b)).

In (2.2) Faraday’s equation is written in the form:

\[
\frac{\partial B}{\partial t} - \nabla \times (u \times B) + \epsilon_1 u \nabla \cdot B = 0,
\]  

(2.5)

For the case \( \epsilon_1 = 1 \), equation (2.5) is equivalent to the equation:

\[
\left( \frac{\partial}{\partial t} + L_u \right) B \cdot dS = \left( \frac{\partial B}{\partial t} - \nabla \times (u \times B) + u \nabla \cdot B \right) \cdot dS = 0,
\]  

(2.6)

where \( B \cdot dS \) is the magnetic flux through the area element \( dS \) which is advected with the flow. Faraday’s equation (2.6) can be described using the Calculus of exterior differential forms (e.g. Tur and Yanovsky (1993) and Webb et al (2014a)). It expresses the fact that the magnetic flux \( B \cdot dS \) is conserved moving with the background flow. Note that \( \nabla \cdot B = 0 \) (Gauss’s law) is enforced by the Lagrange multiplier \( \nu \).

Yoshida (2009) studied the Clebsch expansion \( u = \nabla \phi + \alpha \nabla \beta \) and the completeness of the expansion. He showed that the Clebsch expansion:

\[
u = \nabla \phi + \sum_{j=1}^{N} \alpha_j \nabla \beta^j,
\]  

(2.7)

is in general complete if \( N = n - 1 \) where \( n \) is the number of independent variables (i.e. the number of independent space variables). In some cases, it is necessary to control the boundary values of \( \phi, \alpha_j, \) and \( \beta^j \), then \( N = n \) (see e.g. Tanehashi and Yoshida (2015) who use Clebsch variables for both \( u \) and \( B \) in MHD gauge theory). In the remainder of the paper we use the Einstein summation convention for repeated indices.

The variational equations obtained by varying \( \phi, \beta, \) and \( \lambda^k \) and \( \nu \) gives the equations:
\[
\frac{\delta A}{\delta \phi} = \rho_t + \nabla \cdot (\rho u) = 0, \quad \frac{\delta A}{\delta \beta} = S_t + u \cdot \nabla S = 0,
\]
\[
\frac{\delta A}{\delta \lambda^k} = \frac{d\mu_k}{dt} = 0, \quad \frac{\delta A}{\delta \nu} = \nabla \cdot B = 0,
\]
(2.8)

Varying \( \Gamma \) gives Faraday’s equation (2.3). Equations (2.3) and (2.8) are the constraint equations.

Varying \( u \) in the variational principle (2.2) gives the Clebsch representation for \( u \) as:
\[
u = \nabla \phi - r \nabla S - \tilde{\lambda} \nabla \mu^k + b \times (\nabla \times \Gamma) - \epsilon_1 \Gamma \frac{\nabla \cdot B}{\rho},
\]
(2.9)

where
\[
\begin{align*}
    r &= \frac{\beta}{\rho}, \\
    \tilde{\lambda}^k &= \frac{\lambda^k}{\rho}, \\
    b &= \frac{B}{\rho}.
\end{align*}
\]
(2.10)

By varying \( B \) in the action principle gives:
\[
\frac{\delta A}{\delta B} = - \left( \frac{\partial \Gamma}{\partial t} - u \times (\nabla \times \Gamma) + \nabla (\nu + \epsilon_1 \Gamma \cdot u) + \frac{B}{\mu} \right) = 0,
\]
(2.11)

for the evolution of \( \Gamma \).

Varying \( S \) gives the equation:
\[
\frac{\delta A}{\delta S} = - \left[ \frac{\partial \beta}{\partial t} + \nabla \cdot (\beta u) + \rho T \right] = 0 \quad \text{or} \quad \frac{dr}{dt} = -T,
\]
(2.12)

where \( T \) is the temperature of the gas and \( r = \beta/\rho \). Here \( d/dr = \partial/\partial t + u \cdot \nabla \) is the Lagrangian time derivative following the flow.

Varying \( \mu^k \) gives:
\[
\frac{\delta A}{\delta \mu^k} = - \left\{ \frac{\partial \lambda^k}{\partial t} + \nabla \cdot (\lambda^k u) \right\} = 0 \quad \text{or} \quad \frac{d\tilde{\lambda}^k}{dt} = 0,
\]
(2.13)

where
\[
\begin{align*}
    \tilde{\lambda}^k &= \frac{\lambda^k}{\rho}.
\end{align*}
\]
(2.14)

Varying \( \rho \) results in Bernoulli’s equation:
\[
\frac{\delta A}{\delta \rho} = - \left\{ \frac{d\phi}{dt} - \left[ \frac{1}{2} u^2 - h - \Phi(x) \right] \right\} = 0,
\]
(2.15)

where \( \phi \) is the velocity potential, \( h = \epsilon_\rho = (P + \epsilon)/\rho \) is the enthalpy of the gas and \( P \) is the gas pressure.

By taking the curl of (2.11), we obtain:
\[
\frac{\partial \tilde{\Gamma}}{\partial t} - \nabla \times (u \times \tilde{\Gamma}) = -J \quad \text{where} \quad J = \frac{\nabla \times B}{\mu} \quad \text{and} \quad \tilde{\Gamma} = \nabla \times \Gamma.
\]
(2.16)

Equation (2.16) has the same form as (1.2) giving the current \( J \) in terms of the polarization \( P \) for the case \( \nabla \cdot P = 0 \) in which \( P \to -\tilde{\Gamma} \), and \( \rho_c = 0 \). Thus, we identify \( P = -\tilde{\Gamma} = -\nabla \times \Gamma \).
If one uses the Lagrangian density
\[
\ell_2 = \left( \frac{1}{2} \rho \dot{u}^2 - \epsilon(\rho, S) \right) - \frac{B^2}{2\mu} - \rho \Phi(x) + \phi \left( \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \dot{u}) \right) \\
+ \beta \left( \frac{\partial S}{\partial t} + \dot{u} \cdot \nabla s \right) + \lambda \left( \frac{\partial \mu}{\partial t} + \dot{u} \cdot \nabla \mu \right) \\
+ \gamma \cdot \left( \frac{\partial A}{\partial t} - \dot{u} \times (\nabla \times A) + \nabla (u \cdot A) \right),
\]  
(2.17)
to replace \( \ell \) in the action (2.1), then the equation:
\[
\frac{\partial A}{\partial t} - \dot{u} \times (\nabla \times A) + \nabla (u \cdot A) = 0,
\]  
(2.18)
replaces the Faraday equation constraint. It corresponds to using a gauge in which the electric field potential \( \psi \) is given by \( \psi = u \cdot A \). Here the one-form \( \alpha = A \cdot dx \) is Lie dragged with the background flow. The curl of (2.18) gives Faraday’s equation. The action \( A_2 \) corresponding to \( \ell_2 \) has Clebsch expansion for \( u \) of the form:
\[
u = \nabla \phi - r \nabla S - \tilde{\lambda} \nabla \mu + b \times \gamma + \nabla \cdot \nabla A \rho,
\]  
(2.19)
where \( \phi \) is the velocity potential.
By varying \( A \) in \( A_2 \) we obtain the equation:
\[
\frac{\partial \gamma}{\partial t} - \nabla \times (u \times \gamma) + u \nabla \cdot \gamma = -J \equiv -\nabla \times B \rho.
\]  
(2.20)
In this approach, we identify \( P = -\gamma \).

2.1. The MHD momentum equation

The momentum equation for the fluid arises from the variational equations (2.8)–(2.16). The analysis is carried out for the case \( \epsilon_1 = 1 \) (i.e., for the case where \( \nabla \cdot B \neq 0 \)). The results also apply for the case \( \epsilon_1 = 0 \) and \( \Theta = \nabla \cdot B / \rho = 0 \). The MHD momentum equation is:
\[
\frac{\partial}{\partial t} (\rho \dot{u}) + \nabla \cdot \left( \rho \dot{u} \otimes \dot{u} + \left( p + \frac{B^2}{2\mu_0} \right) I - \frac{B \otimes B}{\mu_0} \rho \right) = -\rho \nabla \Phi,
\]  
(2.21)
where \( \Phi(x) \) is the external gravitational potential. By using the mass continuity equation, (2.21) reduces to the equation:
\[
\frac{d\dot{u}}{dt} = -\frac{1}{\rho} \nabla p + \frac{J \times B}{\rho} + B \frac{\nabla \cdot B}{\mu_0 \rho} - \nabla \Phi.
\]  
(2.22)
By using the identities:
\[
-\frac{1}{\rho} \nabla p = T \nabla S - \nabla h, \quad u \cdot \nabla u = -u \times \omega + \nabla \left( \frac{1}{2} u^2 \right),
\]  
(2.23)
where \( \omega = \nabla \times u \) is the fluid vorticity and \( h = \epsilon_\rho(\rho, S) \) is the gas enthalpy, (2.22) takes the form:
\[
\Delta \equiv \frac{\partial \dot{u}}{\partial t} - u \times \omega + \nabla \left( h + \Phi + \frac{1}{2} u^2 \right) - T \nabla S - J \times B - B \frac{\nabla \cdot B}{\mu_0 \rho} = 0.
\]  
(2.24)
For the case $\epsilon_1 = 1$, $b = B/\rho$ is an invariant vector field that is Lie dragged by the flow, i.e.
\[
\left( \frac{\partial}{\partial t} + L_\mu \right) b = \frac{\partial b}{\partial t} + [u, b] \equiv \frac{\partial b}{\partial t} + u \cdot \nabla b - b \cdot \nabla u = 0.
\]
Equation (2.25) is equivalent to Faraday’s equation, taking into account the mass continuity equation. The result (2.25) holds for $\nabla \cdot B = 0$ and for the case $\nabla \cdot B \neq 0$.

In appendix A, we show how the momentum equation (2.24) arises from the variational equations (2.8)–(2.16). In physical applications $\nabla \cdot B = 0$. In the non-canonical Poisson bracket (e.g. Morrison and Greene (1980, 1982) and Chandre et al (2013)), the effect of $\nabla \cdot B \neq 0$, has been investigated for theoretical reasons (see e.g. Squire et al (2013) for a discussion in equation (13) et sequation). The effect of $\nabla \cdot B \neq 0$ is important in numerical MHD codes, where the effect of numerically generated $\nabla \cdot B \neq 0$ needs to be minimized in order to produce accurate solutions (e.g. Powell et al (1999), Janhunen (2000), Dedner et al (2002), Webb et al (2010b)). Evans and Hawley (1988), Balsara (2004), Balsara and Kim (2004) and Stone and Gardiner (2009) use a staggered grid approach in which the magnetic flux is calculated on the faces of the computational cell in order to minimize $\nabla \cdot B \neq 0$ errors.

3. Gauge symmetries

In this section we study gauge symmetries for MHD for the case $\nabla \cdot B = 0$ (i.e. $\epsilon_1 = 0$ in the Lagrangian variational principle (2.1) and (2.2)). Gauge symmetries in fluid dynamics and MHD are similar to fluid relabeling symmetries. Both these two types of symmetries do not change the ‘physical variables’, e.g. $\rho$, $u$, $B$, $p$, and $S$, but they do allow for the hidden variables behind the scenes to change. Thus, for both gauge symmetries and for fluid relabeling symmetries, the Eulerian variations are zero, i.e.
\[
\delta \rho = \delta u = \delta B = \delta p = \delta S = 0.
\]

For both these symmetries, the change in the action should be zero, modulo a pure divergence term to order $\epsilon$, where $\epsilon$ is the infinitesimal version of the group parameter describing the deviation from the identity transformation. Note that the Euler Lagrange equations are invariant under the addition of a pure divergence term to the Lagrangian. Lie symmetries are important in Noether’s theorems, in which conservation laws are related to symmetries of the action. In the present analysis, the variables $M = \{ \phi, \beta, \lambda, \mu, \nu, \Gamma \}$ are allowed to change. If there are continuous functions defining the transformations, then Noether’s second theorem is used to obtain conservation laws. Lie point symmetries of the MHD equations and combinations of the scaling symmetries give rise to conservation laws (e.g. Webb and Zank (2007)). Potential symmetries of the equations can give rise to nonlocal conservation laws (e.g. Akhatov et al (1991) and Bluman et al (2010)). We restrict our attention to gauge symmetries.

Consider the gauge symmetries for the electric ($E$) and magnetic ($B$) fields in Maxwell’s equations and in MHD. In particular, Gauss’s equation, and Faraday’s equation:
\[
\nabla \cdot B = 0, \quad \nabla \times E + B_t = 0,
\]
are satisfied by choosing
\[
B = \nabla \times A, \quad E = -A_t - \nabla \psi,
\]
where $A$ is the magnetic vector potential and $\psi$ is the electrostatic potential. Equations (3.2) and (3.3) remain invariant under the gauge
\[ A' = A + \nabla \lambda \quad \text{and} \quad \psi' = \psi - \lambda_t. \quad (3.4) \]

The transformations (3.4) leave \( E \) and \( B \) invariant, i.e.
\[ E' = E, \quad \text{and} \quad B' = B. \quad (3.5) \]

The gauge transformations (3.4) in classical electromagnetism are associated with the charge conservation law (e.g. Calkin (1963)).

In the following analysis, we use the notation:
\[ \delta \alpha = \epsilon V^\alpha, \quad (3.6) \]
relating the variable \( \alpha \) to its infinitesimal Lie group (or Lie pseudo group) generator \( V^\alpha \).

For gauge symmetry transformations, the physical variables (i.e. \((\rho, u, S, B, J)\)) do not change. However, the Lagrange multipliers and \( \mu^k \) in general change under gauge transformations. From (2.9)
\[ \delta u = \epsilon V^\mu = \epsilon \left[ \nabla \psi^\phi - \nabla S - \left( \tilde{\lambda}^k \nabla \mu^k + \tilde{\lambda}^k (V^\mu)^k \right) + \mathbf{b} \times (\nabla \times \mathbf{V}^\Gamma) \right] = 0. \quad (3.7) \]

From (2.11), the condition \( \delta \mathbf{B} = 0 \) implies:
\[ \frac{\partial V^\Gamma}{\partial t} - \mathbf{u} \times [\nabla \times (V^\Gamma)] + \nabla V^\nu = 0. \quad (3.8) \]

Equations (2.8), (2.13), (2.12) and (2.15) require:
\[ \frac{d}{d\tau} V^\mu^k = \frac{d}{d\tau} \tilde{\lambda}^k = 0, \quad \frac{d}{d\tau} V^\nu = \frac{d}{d\tau} \psi^\phi = 0. \quad (3.9) \]

Equations (3.9) require \( V^\mu^k, \tilde{\lambda}^k, V^\nu \) and \( \psi^\phi \) to be functions only of the Lagrange labels.

The condition (3.7) for \( \delta \mathbf{u} = 0 \) can be reduced to a simpler form by introducing:
\[ G = \epsilon K = \delta \phi - \tilde{\lambda}^k \delta \mu^k \equiv \epsilon \left[ \psi^\phi - \tilde{\lambda}^k V^\mu^k \right], \quad (3.10) \]
in (3.7) (this definition of \( G \) is like a Legendre transformation). We obtain:
\[ \nabla K = \nabla (\psi^\phi) - (\nabla \tilde{\lambda}^k) V^\mu^k - \tilde{\lambda}^k \nabla V^\mu^k. \quad (3.11) \]

Re-arranging (3.11) gives:
\[ \nabla V^\phi - \tilde{\lambda}^k \nabla V^\mu^k = \nabla K + \nabla \left( \tilde{\lambda}^k \right) V^\mu^k, \quad (3.12) \]

Using (3.12), (3.7) reduces to:
\[ \nabla K + \nabla (\tilde{\lambda}^k) V^\mu^k - \tilde{\lambda}^k \nabla \mu^k - \psi^\phi \nabla S + \mathbf{b} \times (\nabla \times \mathbf{V}^\Gamma) = 0. \quad (3.13) \]

From (3.10)
\[ \psi^\phi = K + \tilde{\lambda}^k V^\mu^k. \quad (3.14) \]

Using (3.14) and (3.9) it follows that:
\[ \frac{dK}{d\tau} = 0, \quad (3.15) \]
(because \( \psi^\phi, \tilde{\lambda}^k \) and \( V^\mu^k \) are advected invariants). In appendix C, we show, that if \( K = K(\tilde{\lambda}, \mu, S) \) and \( \mathbf{b} \times (\nabla \times \mathbf{V}^\Gamma) = 0 \) then \( K \) satisfies the Hamiltonian like equations:
\[
\frac{d\mu}{d\epsilon} = -\frac{\partial K}{\partial \lambda}, \quad \frac{d\lambda}{d\epsilon} = -\frac{\partial K}{\partial \mu}, \quad \frac{dr}{d\epsilon} = -\frac{\partial K}{\partial S}, \quad \frac{dS}{d\epsilon} = -\frac{\partial K}{\partial r} = 0, \quad (3.16)
\]

where \( V^\alpha = d\alpha / d\epsilon \) is the Lie symmetry generator for the generic variable \( \alpha \). This of course does not imply \( K \) is the Hamiltonian for the system of determining equations \((3.7)\)–\((3.15)\).

Once the gauge field Lie determining equations \((3.7)\)–\((3.15)\) are solved for the Lie generators \((V^\mu, V^{\tilde{\lambda}}_k, V^\rho, V^\Gamma, V^\nu)\), conservation laws for the extended MHD system, can be obtained by using Noether’s theorem. A form of Noether’s theorem, suitable for this purpose is given below.

4. Gauge symmetries and conservation laws

In this section we use the gauge symmetries and Noether’s theorem to derive conservation laws associated with the gauge symmetries of section 3. As in section 3, we consider only the case where \( \nabla \cdot \mathbf{B} = 0 \) (i.e. \( \epsilon_1 = 0 \) in the variational principle \((2.1)\) and \((2.2)\)).

Starting from the action \((2.2)\), we look at the change in the action induced by infinitesimal changes in the fields \(\{\phi, \tilde{\lambda}^k, \mu^k, r, \nu, \Gamma\}\). The change in the action is:

\[
A' - A = \int (\ell' - \ell) \, d^3xdt, \quad (4.1)
\]

where

\[
\ell' - \ell = \delta \phi \left( \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) \right) + \rho \left( \delta \tilde{\lambda}^k \frac{d\mu^k}{d\epsilon} + \tilde{\lambda}^k \frac{d}{d\epsilon} \delta \rho \right) + \rho d(\delta \gamma) \cdot (\nabla \times (\mathbf{u} \times \mathbf{B})) + \delta \nu \nabla \cdot \mathbf{B} + O(\epsilon^2). \quad (4.2)
\]

By noting that \( d/d\epsilon(\delta \mu^k) = 0 \), and integrating \((4.2)\) by parts, gives:

\[
\ell' - \ell = \frac{\partial}{\partial t} \left\{ \rho \delta \phi + \rho S \delta r + \rho \mu^k \delta \tilde{\lambda}^k + \delta \gamma \cdot \mathbf{B} \right\} + \nabla \cdot \left\{ \rho \mathbf{u} \left( \delta \phi + S \delta r + \mu^k \delta \tilde{\lambda}^k \right) + \mathbf{B} \delta \nu + \delta \gamma \times (\mathbf{u} \times \mathbf{B}) \right\}
- \rho \frac{d}{dt} \delta \phi - S \frac{d}{dt} \delta r - \rho \mu^k \frac{d}{dt} \delta \tilde{\lambda}^k
- \left( \mu^k \delta \tilde{\lambda}^k + S \delta r \right) \left( \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) \right)
- \mathbf{B} \cdot \left( \frac{\partial \delta \gamma}{\partial t} - \mathbf{u} \times (\nabla \times \delta \gamma) + \nabla (\delta \nu) \right). \quad (4.3)
\]

By using the gauge symmetry determining equations \((3.8)\) and \((3.9)\) and noting that the non-divergence terms in \((4.2)\) vanish because of the gauge symmetry equations \((3.8)\) and \((3.9)\), the expression for \( \ell' - \ell \) reduces to the sum of the pure time and space divergence terms. Using the notation \( \delta \alpha = \epsilon V^\alpha \) (\( \alpha \) is a generic variable, that is varied in \((4.1)\)–\((4.3)\)), we obtain:
\[
\ell' - \ell = \epsilon \left\{ \frac{\partial}{\partial t} \left[ \rho V^\phi + \rho SV^r + \rho \mu^s V^{\bar{\lambda}} + V^\Gamma \cdot B \right] 
+ \nabla \cdot \left[ \rho u \left( V^\phi + SV^r + \mu^s V^{\bar{\lambda}} \right) + B V^\nu + V^\Gamma \times (u \times B) \right] \right\} 
\equiv \epsilon \left( \frac{\partial W^0}{\partial t} + \nabla \cdot W \right), \tag{4.4} \]

where

\[
W^0 = \rho \left( V^\phi + SV^r + \mu^s V^{\bar{\lambda}} \right) + V^\Gamma \cdot B, \\
W = \rho u \left( V^\phi + SV^r + \mu^s V^{\bar{\lambda}} \right) + B V^\nu + V^\Gamma \times (u \times B). \tag{4.5} \]

One can add a pure divergence term to (4.4) since it does not alter the Euler–Lagrange equations. Thus, more generally,

\[
\delta \ell = \epsilon \left( \frac{\partial W^0}{\partial t} + \nabla \cdot W + \frac{\partial \Lambda^0}{\partial t} + \nabla \cdot \Lambda \right), \tag{4.6} \]

gives a more general form for the allowed changes in \( \ell \), where \( \Lambda^0 \) and \( \Lambda \) are to be determined.

For stationary variations, (4.6), and for a finite transformation, not involving arbitrary functions, (4.6) gives the conservation law:

\[
\frac{\partial}{\partial t} (W^0 + \Lambda^0) + \nabla \cdot (W + \Lambda) = 0, \tag{4.7} \]

which is Noether’s first theorem for the case of a divergence symmetry of the action involving the gauge potential \( \Lambda^0 \) and flux \( \Lambda \).

Below, we give examples of the use of (4.4)–(4.7) in which we derive (a) the magnetic helicity conservation law and (b) the cross-helicity conservation law, (c) the unmagnetized fluid helicity conservation equation and other examples. The magnetic helicity and cross helicity conservation laws were determined by Calkin (1963) for an isobaric gas equation of state and for his modified, non-neutral, MHD type equations. The method used to obtain solutions of the Lie determining equations (3.8)–(3.13) are outlined in appendices B and C.

### 4.1. Magnetic helicity

The magnetic helicity conservation law arises from the solution of (3.8)–(3.15), for which:

\[
V^\Gamma = A, \quad V^\nu = \psi, \quad K = 0, \quad V^{\bar{\lambda}} = V^\phi = V^r = 0. \tag{4.8} \]

With this choice of parameters the conserved density \( W^0 \) and flux \( W \) in (4.5) become:

\[
W^0 = V^\Gamma \cdot B = A \cdot B, \\
W = B \psi + A \times (u \times B) = B \psi + (A \cdot B)u - (A \cdot u)B \\
= (A \cdot B)u + (\psi - A \cdot u)B. \tag{4.9} \]

The resultant conservation law, using Noether’s theorem (4.7) gives:

\[
\frac{\partial}{\partial t} (A \cdot B) + \nabla \cdot [(A \cdot B)u + (\psi - A \cdot u)B] = 0. \tag{4.10} \]
which is the Eulerian form of the magnetic helicity conservation law, where \( h_M = A \cdot B \) is the magnetic helicity density. Integration of (4.10) over a volume \( V \) moving with the fluid, in which \( B \cdot n = 0 \) on the boundary \( \partial V \) of \( V \) gives the Lagrangian magnetic helicity conservation law \( dH_M/dt = 0 \), where \( H_M \) is given by (1.7).

4.2. Cross helicity

Using solutions of the Lie determining equations (3.8)–(3.15) (see appendix B: set \( k_2 = k_1 = 1 \), \( \Lambda = \phi \) in (B.17) and (B.18)):

\[
V^\phi = K + \lambda b \cdot \nabla \mu, \quad \text{where} \quad B \cdot \nabla K = 0,
\]
\[
V^\mu = b \cdot \nabla \mu, \quad V^\lambda = b \cdot \nabla \lambda, \quad V^r = 0,
\]
\[
V^\Gamma = \nabla \phi + \gamma \nabla K - \lambda \nabla \mu, \quad V^\nu = -d\phi/dt + u \cdot V^\Gamma,
\]

in Noether’s theorem (4.5) gives:

\[
W^0 = \rho[K + b \cdot \nabla(\lambda \mu)] + h_c, \quad W = uW^0 - \frac{d\phi}{dt}B,
\]

where

\[
h_c = B \cdot (u + r\nabla S),
\]

is the generalized cross helicity density for non-barotropic flows (Webb et al. 2014a, 2014b), Yahalom (2016), and \( \phi \) is the velocity potential in the Clebsch expansion for \( u \) which satisfies Bernoulli’s equation (2.15). We obtain:

\[
\frac{\partial W^0}{\partial t} + \nabla \cdot W = \frac{\partial h_c}{\partial t} + \nabla \cdot \left[ u h_c + B \left( h + \Phi - \frac{1}{2}u^2 \right) \right] = 0,
\]

which is the generalized cross helicity conservation law derived by Webb et al. (2014a, 2014b) (an equivalent form of this conservation law is derived by Yahalom (2016), see also appendix E). In the derivation of (4.14) we have used the conservation law:

\[
\frac{\partial (\rho \Lambda^0)}{\partial t} + \nabla \cdot (\rho u \Lambda^0) = \rho \frac{d\Lambda^0}{dt} = 0 \quad \text{where} \quad \Lambda^0 = K + b \cdot \nabla(\lambda \mu).
\]

Note that \( dK/dt = 0 \) and \( d(b \cdot \nabla(\lambda \mu))/dt = 0 \) in (4.15).

Equation (4.14) may be written in the more explicit form:

\[
\frac{\partial}{\partial t} \left[ B \cdot (u + r\nabla S) \right] + \nabla \cdot \left[ B \cdot (u + r\nabla S)u + B \left( h + \Phi - \frac{1}{2}u^2 \right) \right] = 0,
\]

which is the generalized cross helicity conservation law for non-barotropic flows, in which \( p = p(\rho, S) \). Equation (4.16) is a nonlocal conservation law, because:

\[
\frac{dr}{dt} \equiv \left( \frac{\partial r}{\partial t} + u \cdot \nabla r \right) = -T,
\]

where \( T \) is the temperature of the gas.

Integration of (4.16) over a volume \( V \) moving with the fluid for which \( B \cdot n = 0 \) on the boundary \( \partial V \) of \( V \), gives the generalized non-barotropic cross helicity conservation law:
\[ \frac{dH_{\text{CNB}}}{dt} = 0 \text{ where } H_{\text{CNB}} = \int_V (\mathbf{u} + r \nabla S) \cdot \mathbf{B} \, d^3x, \]  
(4.18) 

(see also appendix E and Yahalom (2017a, 2017b)).

4.3. The gauge symmetry \( \mathbf{V}^\Gamma = \nabla \Lambda \) and \( \mathbf{V}^\nu = -\Lambda_t \)

Set \( k_2 = 1, k_1 = 0, K = 0 \) in (B.17) and (B.18) in appendix B. Using the results in appendix B, we obtain solutions of (3.7)–(3.15) of the form:

\[ \mathbf{V}^\Gamma = \nabla \Lambda, \quad \mathbf{V}^\nu = -\Lambda_t, \quad \mathbf{V}^\mu = \mathbf{V}^\phi = K = 0. \]  
(4.19)

Use of Noether’s theorem (4.7) gives the conservation law:

\[ \frac{\partial}{\partial t} (\mathbf{B} \cdot \nabla \Lambda) + \nabla \cdot [-(\Lambda_t \mathbf{B} + \nabla \Lambda \times (\mathbf{u} \times \mathbf{B})] = 0. \]  
(4.20)

This conservation law holds for all potentials \( \Lambda(x, t) \), where \( \Lambda(x, t) \) is not necessarily related to the MHD equations. One might regard (4.20) as a trivial conservation law. However, if \( \Lambda \) is related to the MHD equations, it does give rise to interesting conservation laws. The conservation law (4.20) can be written in the form:

\[ \frac{\partial}{\partial t} (\mathbf{B} \cdot \nabla \Lambda) + \nabla \cdot \left[ -(\Lambda_t + \mathbf{u} \nabla \Lambda) \mathbf{B} + (\mathbf{B} \cdot \nabla \Lambda) \mathbf{u} \right] = 0. \]  
(4.21)

Some examples of the use of (4.21) are discussed below.

**Example 1.** If \( \Lambda \) is advected with the flow, then \( d\Lambda/dt = 0 \). In this case (4.21) reduces to:

\[ \frac{\partial}{\partial t} (\mathbf{B} \cdot \nabla \Lambda) + \nabla \cdot [(\mathbf{B} \cdot \nabla \Lambda) \mathbf{u}] = 0. \]  
(4.22)

Thus, if \( \Lambda = S \) then

\[ \frac{\partial}{\partial t} (\mathbf{B} \cdot \nabla S) + \nabla \cdot [(\mathbf{B} \cdot \nabla S) \mathbf{u}] = 0. \]  
(4.23)

There are many examples of physically significant scalars advected with the flow. For example

\[ \frac{d}{dt} \left( \frac{\mathbf{A} \cdot \mathbf{B}}{\rho} \right) = 0 \quad \text{if} \quad \psi = \mathbf{A} \cdot \mathbf{u} \quad \text{and} \quad \mathbf{E} = -\mathbf{A}_t - \nabla \psi = -\mathbf{u} \times \mathbf{B}. \]  
(4.24)

Thus, the choice \( \Lambda = \mathbf{A} \cdot \mathbf{B}/\rho \) satisfies \( d\Lambda/dt = 0 \) and gives rise to a physically relevant conservation law of the form (4.22).

**Example 2.** Setting \( \Lambda = \phi \), the conservation law (4.21) becomes:

\[ \frac{\partial}{\partial t} (\mathbf{B} \cdot \nabla \phi) + \nabla \cdot \left[ (\mathbf{B} \cdot \nabla \phi) \mathbf{u} - \mathbf{B} \frac{d\phi}{dt} \right] = 0. \]  
(4.25)

Using Bernoulli’s equation (2.15):

\[ \frac{d\phi}{dt} = \frac{1}{2} \mathbf{u}^2 - h - \Phi, \]  
(4.26)

in (4.25) gives the conservation law:
\[ \frac{\partial}{\partial t}(B \cdot \nabla \phi) + \nabla \cdot \left( (B \cdot \nabla \phi)u + B \left( h + \Phi - \frac{1}{2}u^2 \right) \right) = 0. \] (4.27)

This conservation law is a nonlocal conservation law because

\[ \phi = \int_0^t \left( \frac{1}{2}u^2 - h - \Phi \right) \, dt + \phi_0, \] (4.28)

is given by a Lagrangian time integral in which the memory of the flow plays a crucial role (\( \phi_0 = \phi(x_0,0) \) describes the initial data for the integral (4.28)).

Equation (4.20) is a special case of a class of conservation laws for fluid systems obtained by Cheviakov (2014). He showed that the system:

\[ \frac{\partial N}{\partial t} + \nabla \times M = 0 \quad \text{and} \quad \nabla \cdot N = 0, \] (4.29)

has conservation laws of the form:

\[ \frac{\partial}{\partial t}(N \cdot \nabla F) + \nabla \cdot (M \times \nabla F - F_t N) = 0, \] (4.30)

where \( F(x,t) \) is an arbitrary function of \( x \) and \( t \), not necessarily related to the system (4.29). In the MHD application (4.20) to Faraday’s equation,

\[ N = B, \quad M = -(u \times B), \quad F = \phi. \] (4.31)

Webb and Mace (2015) using a fluid relabelling symmetry, and Noether’s second theorem, derived the conservation law:

\[ \frac{\partial}{\partial t} \left( \omega \cdot \nabla \psi \right) + \nabla \cdot \left[ (\omega \cdot \nabla \psi)u - \left( T \nabla S + \frac{J \times B}{\rho} \right) \times \nabla \psi \right] = 0, \] (4.32)

where \( \psi \) is a scalar advected with the flow, (i.e. \( d\psi/dt = 0 \)), and \( \omega = \nabla \times u \) is the fluid vorticity. Here, \( J = \nabla \times B/\mu_0 \) is the current and \( T \) is the temperature of the gas. The conservation law (4.32) corresponds to the choices:

\[ N = \omega = \nabla \times u, \quad F = \psi(x,t), \]
\[ M = -u \times \omega - \left( T \nabla S + \frac{J \times B}{\rho} \right), \] (4.33)

in Cheviakov’s potential vorticity type equation (4.30). Rosenhaus and Shankar (2016) develop Noether’s second theorem for quasi-Noether systems, and describe the conservation laws obtained by Cheviakov (2014) and Cheviakov and Oberlack (2014).

### 4.4. Fluid helicity

For an ideal, non-barotropic fluid, \((B = 0)\), the Clebsch expansion for \( u \) and related equations of use are:
\[ \mathbf{u} = \nabla \phi - r \nabla S - \tilde{\lambda}^k \nabla \mu^k, \quad \mathbf{w} = \mathbf{u} + r \nabla S, \]
\[ \Omega = \nabla \times \mathbf{w} = -\nabla \tilde{\lambda}^k \nabla \mu^k \equiv \omega + \nabla r \times \nabla S, \]
\[ \frac{d\mathbf{r}}{dt} + T = 0, \quad \omega = \nabla \times \mathbf{u}, \quad (4.34) \]
where \( \omega \) is the fluid vorticity. The vorticity 2-form \( \Omega \cdot dS \) is Lie dragged with the flow, i.e.
\[ \frac{d}{dt} (\Omega \cdot dS) = \left( \frac{\partial}{\partial t} + L_{\mathbf{u}} \right) (\Omega \cdot dS) = \left[ \frac{\partial \Omega}{\partial t} - \nabla \times (\mathbf{u} \times \Omega) + \mathbf{u} \nabla \cdot \Omega \right] \cdot dS = 0, \quad (4.35) \]
(e.g. Webb et al. (2014a)). Note that:
\[ \nabla \cdot \Omega = 0. \quad (4.36) \]
Equations (4.35) and (4.36) show that \( \Omega \) is analogous to \( \mathbf{B} \) in MHD in Faraday's equation, and in Gauss's equation \( \nabla \cdot \mathbf{B} = 0 \).
Thus, using the analogy:
\[ \mathbf{B} \rightarrow \Omega, \quad \mathbf{b} = \frac{\mathbf{B}}{\rho} \rightarrow \frac{\Omega}{\rho}, \quad (4.37) \]
it follows that the fluid helicity equation for non-barotropic fluids has the form:
\[ \frac{\partial}{\partial t} \left[ (\mathbf{u} + r \nabla S) \cdot \Omega \right] + \nabla \cdot \left[ (\mathbf{u} + r \nabla S) \cdot \Omega \mathbf{u} + \Omega \left( h + \Phi - \frac{1}{2} \mathbf{u}^2 \right) \right] = 0, \quad (4.38) \]
(see e.g. Mobbs (1981) and Webb et al. (2014a, 2014b) for vorticity theorems for non-barotropic fluids). Equation (4.38) is analogous to the cross helicity conservation law (4.16). In fact, one can derive the generalized fluid helicity conservation law (4.38) by using the analysis of (4.11) (see Webb et al. (2014a, 2014b) for alternative proofs of (4.38)). For the case of an isobaric equation of state for the gas, (i.e. \( p = p(\rho) \)), (4.38) reduces to the usual kinetic fluid helicity conservation law:
\[ \frac{\partial}{\partial t} \left[ (\mathbf{u} \cdot \omega) \mathbf{u} + \nabla \cdot \left( (\mathbf{u} \cdot \omega) \mathbf{u} + \omega \left( h + \Phi - \frac{1}{2} \mathbf{u}^2 \right) \right) \right] = 0. \quad (4.39) \]
The fluid helicity conservation law (4.38) was derived by Webb et al. (2014b), by using a fluid relabelling symmetry

### 4.5. Basic conservation laws

Noether's theorem (4.5)–(4.7) covers the basic MHD conservation laws. For example, setting \( K = V^\phi = \text{const.} = c_1 \) and all other symmetry generators in (4.5)–(4.7) equal to zero, gives the mass conservation law: \( c_1 [\rho_t + \nabla \cdot (\rho \mathbf{u})] = 0 \). Holm et al. (1998) describe the mass conservation law as being a consequence of symmetry breaking of the fluid relabelling symmetries (this interpretation is consistent with the above derivation, as \( V^\phi = 0 \) gives no conservation law, but for \( V^\phi = c_1 \) gives the mass conservation law). For \( V^\nu = 1 \) and all other generators zero, in (4.5)–(4.7) gives Gauss's law \( \nabla \cdot \mathbf{B} = 0 \). For \( V^\nu = 1 \) and \( K = S \), and all other generators zero, gives the entropy conservation law in the form: \( 2[(\rho S)_t + \nabla \cdot (\rho \mathbf{u} S)] = 0 \). For the choice \( V^F = \mathbf{k} = \text{const.} \), and all other generators zero, gives Faraday's law in the form:
\[ \mathbf{k} \cdot [\mathbf{B}_t - \nabla \times (\mathbf{u} \times \mathbf{B})] = 0. \]
5. Gauge symmetries and Casimirs

Henyey (1982) investigated the role of gauge symmetries in MHD using a Clebsch variable formulation of the equations. Henyey used the fact that the Clebsch variable formulation yields canonical equations for the Hamiltonian, in which the physical variables $\rho$, $B$ and $S$ can be regarded as canonical coordinates and the Lagrange multipliers are the corresponding canonical momenta. He considered gauge transformations in which the canonical coordinates ($\rho$, $S$, $B$) are invariant, but the canonical momenta (the Lagrange constraint variables) are allowed to change. Padhye and Morrison (1996a, 1996b) showed that gauge transformations are related to the MHD Casimirs. Hameiri (2004) gives a thorough discussion of the MHD Casimirs. The Casimirs are functionals $C$ that have zero Poisson bracket with other functionals of the variables.

5.1. Henyey’s approach

In the symmetry group literature (e.g. Bluman and Kumei (1989)), a gauge symmetry is sometimes referred to as a divergence symmetry, in which the action is invariant under a Lie transformation of the variables, and involves a change in the Lagrangian density of the form $L' = L + \epsilon \nabla \cdot A$. If the $\Lambda^\alpha = 0$, the symmetry is known as a variational symmetry. Henyey (1982) does not enforce $\nabla \cdot B = 0$, and omits the Lin constraint terms $\lambda^k \mu^k / dt$ used in our analysis. He used functionals $F$ of the physical variables ($\rho, S, B$) which act as canonical coordinates $(q^\alpha, p^\alpha)$ and the Lagrange multipliers ($\phi, \beta, \Gamma$) act as canonical momenta (see e.g. Zakharov and Kuznetsov (1997)). We include $\mu^k$ as canonical coordinates and the $\lambda^k$ as canonical momenta, and impose $\nabla \cdot B = 0$ by using Lagrange multipliers.

For a finite dimensional Hamiltonian system, the change in the Lagrangian $\ell$, denoted by $\delta \ell = \ell' - \ell$ due to a canonical transformation corresponding to a gauge potential $F$ has the form:

$$\delta \ell = \frac{dF}{dt},$$

(5.1)

where $F$ is a functional of the canonical coordinates $(q^\alpha, p^\alpha)$. In the MHD case we set

$$F = F(\rho, S, B, \mu^k, \phi, \beta, \Gamma, \lambda^k).$$

(5.2)

In classical mechanics (e.g. Goldstein (1980), ch 9), the Lagrangian $\ell$ is related to the Hamiltonian $H(q, p, t)$ by the Legendre transformation:

$$\ell = p_k \dot{q}^k - H(q, p, t),$$

(5.3)

where we use the Einstein summation convention for repeated indices $k$. The Lagrangian in the new coordinates has the form:

$$\ell' = P_k \dot{Q}^k - K(Q^k, P_k, t),$$

(5.4)

where $K(Q^k, P_k, t)$ is the new Hamiltonian, (note $K$ in this section has a different meaning than that used in section 4) and $F$ is the generating function for the canonical transformation. For the transformation (5.1) the Euler–Lagrange equations do not change under a divergence transformation (e.g. Bluman and Kumei (1989), Olver (1993)). If $F = F_1(q^k, Q^k, t)$ (5.3)–(5.4) give the equation:
Collecting the \( \dot{q}_k \), \( \dot{Q}_k \) and remaining terms in (5.5) gives the canonical transformation equations:

\[
p_k = \frac{\partial F_1}{\partial q_k}, \quad P_k = -\frac{\partial F_1}{\partial Q_k}, \quad K = H + \frac{\partial F_1}{\partial t},
\]

(5.6)

The gauge function \( F_1(q_k, Q_k, t) \) defines the new canonical momentum variables \( P_k \) in terms of the other variables (see Goldstein (1980), ch 9 for other possible choices of the gauge function \( F \)).

Henyey (1982) considered gauge transformations in which the canonical coordinates \( q^\alpha = Q^\alpha \) do not change (these are the physical variables \( \rho, S \) and \( B \) and the Lin constraint variables \( \mu_k \)), but the canonical momenta variables \( (\phi, \beta, \Gamma, \lambda^k) \) do change. Because MHD is an infinite dimensional Hamiltonian system (e.g. Morrison and Greene (1980, 1982) and Holm and Kupershmidt (1983a)), it is necessary to use variational derivatives instead of partial derivatives in (5.6). The MHD variational equations (2.8)–(2.16) are invariant under the infinitesimal gauge transformations:

\[
(\delta \phi, \delta \beta, \delta \Gamma, \delta \lambda^k) = \epsilon (F_\rho, F_\Sigma, F_B, F_{\mu_k}), \quad (\delta \rho, \delta S, \delta B, \delta \mu_k) = (0, 0, 0, 0),
\]

(5.7)

where \( F = F_1(\rho, S, B, \mu_k) \) is the gauge function (\( F = -F_1 \) in the analogy (5.6)). Here we use the notation:

\[
\delta P_\alpha = P_\alpha - p_\alpha = \epsilon V_\rho = \epsilon \frac{\delta F}{\delta \rho}, \quad \delta Q_\alpha = Q_\alpha - \dot{Q}_\alpha = \epsilon F_\rho,
\]

(5.8)

(we sometimes use \( p'_\alpha \equiv P_\alpha \) to denote the transformed canonical momenta).

In (2.1), the fluid velocity is given by the Clebsch expansion:

\[
u = \nabla \phi - \frac{\beta}{\rho} \nabla S - \frac{\lambda^k}{\rho} \nabla \mu^k + \frac{B \times (\nabla \times \Gamma)}{\rho} - \frac{\Gamma \nabla \cdot B}{\rho}.
\]

(5.9)

The gauge transformation (5.7) is required to leave \( \nu \) invariant to \( O(\epsilon) \), i.e.

\[
\delta \nu = \epsilon \left\{ \nabla V_\phi - \frac{V_\beta}{\rho} \nabla S - \frac{V_\lambda^k}{\rho} \nabla \mu^k + \frac{B \times (\nabla \times V_\Gamma)}{\rho} - \frac{V_\Gamma \nabla \cdot B}{\rho} \right\} \equiv \epsilon \left\{ \nabla F_\rho - \frac{F_{\Sigma}}{\rho} \nabla S - \frac{F_{\mu_k}}{\rho} \nabla \mu^k + \frac{B \times (\nabla \times F_\Gamma)}{\rho} - \frac{F_\Gamma \nabla \cdot B}{\rho} \right\} = 0.
\]

(5.10)

There are further invariance conditions on the Euler Lagrange equations (2.8)–(2.15) due to the gauge transformations, namely:

\[
\frac{d}{dt} \nu^\mu = 0, \quad \frac{d}{dt} \nu^\phi = 0, \quad \frac{d}{dt} \nu^r = 0,
\]

(5.11)

\[
\frac{\partial}{\partial t} \nabla \Gamma - \nu \times (\nabla \times \nabla \Gamma) + \nabla (\nu^\nu + \nu^r \cdot \nu) = 0.
\]

(5.12)

These equations are the same as (3.8) and (3.9). Note that:
In section 4, we allowed both $\lambda^k$ and $\mu^k$ to vary (i.e. $V^{\lambda^k} \neq 0$ and/or $V^{\mu^k} \neq 0$ as possibilities, since both $\mu^k$ and $\lambda^k$ were not identified as physical variables). Equation (5.11) may be expressed in terms of the variational derivatives of $F$.

Equation (5.10) may be written as:

$$\rho \nabla F_\rho - F_\mu \nabla \mu^k - F_S \nabla S + B \times (\nabla \times F_B) - F_B \nabla \cdot B = 0,$$

or in the form:

$$\rho \nabla F_\rho - F_\mu \nabla \mu^k - F_S \nabla S + B \cdot (\nabla F_B)^T - B \cdot F_B - F_B \nabla \cdot B = 0,$$

which is equivalent to Henyey (1982), equation (24) (the $\mu^k$ terms are not present in Henyey (1982)). By noting that

$$F_A = \nabla \times F_B,$$

Note that $B = \nabla \times A$. Equation (5.14) may be expressed as:

$$\rho \nabla F_\rho - F_\mu \nabla \mu^k - F_S \nabla S + \omega \times F_u + F_A - F_B \nabla \cdot B = 0,$$

which is useful in the case of magnetic helicity functionals.

**Proposition 5.1.** The invariance condition (5.14) in Henyey (1982) gauge transformation can be written in the form:

$$\rho \nabla F_\rho - F_S \nabla S + \omega \times F_u + \bar{F}_A - F_B \nabla \cdot B = 0,$$

where $\omega = \nabla \times u$ is the fluid vorticity. The functional $F(\rho, u, S, B)$ in (5.18) is equivalent to (i.e. has the same value as) the functional $F = F(\rho, S, \mu, B; \phi, \beta, \lambda, \Gamma)$. This result, coupled with the gauge transformations:

$$\delta \rho = F_\phi = -\nabla \cdot F_u = 0, \quad \delta S = F_\beta = -\frac{\bar{F}_u \cdot \nabla S}{\rho} = 0,$$

$$\delta \mu = F_\lambda = -\frac{\bar{F}_u \cdot \nabla \mu}{\rho} = 0,$$

$$\delta B = F_\Gamma = \nabla \times (\bar{F}_M \times B) - \bar{F}_M \nabla \cdot B = 0,$$

where

$$\bar{F}_M = \frac{\bar{F}_u}{\rho},$$

are the Casimir determining equations (see Hameiri (2004), Morrison (1998), Padhye and Morrison (1996a, 1996b), Holm and Kupershmidt (1983a, 1983b)).

The detailed proof of proposition 5.1 is given in appendix D.

Henyey (1982) observed that the gauge symmetry determining equation (5.14) has solutions:

$$F = \int d^3x \left( F_\rho (\nabla \cdot B, x) + \rho G(S, b \cdot \nabla S, b \cdot \nabla (b \cdot \nabla S), \ldots) \right),$$
where \( \mathbf{b} = \mathbf{B}/\rho \) (in Heney’s analysis \( \nabla \cdot \mathbf{B} = 0 \) is not imposed, but it is noted that if \( \nabla \cdot \mathbf{B} = 0 \) at time \( t = 0 \) then \( \nabla \cdot \mathbf{B} = 0 \) for all \( t > 0 \)).

It was shown in section 4, that the local magnetic helicity conservation law (4.10) arises in the Calkin approach by choosing \( \mathbf{V}^\Gamma = \mathbf{A}, \mathbf{V}^\nu = \mathbf{\psi} \) and the other Lie symmetry generators in (4.8) are set equal to zero in Noether’s theorem. If one chooses the gauge of \( \mathbf{A} \) such that \( \mathbf{\psi} = \mathbf{A} \cdot \mathbf{u} \) then the one-form \( \alpha = \mathbf{A} \cdot d\mathbf{x} \) is Lie dragged with the flow, and in that case the conservation law (4.10) can be written in the form:

\[
\frac{d}{dt} \left( \frac{\mathbf{A} \cdot \mathbf{B}}{\rho} \right) = 0.
\] (5.22)

Padhye and Morrison (1996a, 1996b) give a class of MHD Casimir solutions of (5.18)–(5.20) of the form:

\[
C[\rho, S, \mathbf{A}] = \int_V \rho G (\mathbf{A} \cdot \mathbf{b}, \mathbf{b} \cdot \nabla S, \mathbf{b} \cdot \nabla (\mathbf{b} \cdot \nabla S), \mathbf{b} \cdot \nabla (\mathbf{A} \cdot \mathbf{b}), \ldots) \, d^3x,
\] (5.23)

for the gauge case where (5.22) applies (see e.g. Gordin and Petviashvili (1987) and Gordin and Petviashvili (1989)).

6. Concluding remarks

In this paper we have explored the origin of conservation laws in MHD using the Clebsch gauge field theory approach of Calkin (1963). One of the main motivations was to relate the charged fluid extended MHD model developed by Calkin (1963) to the more commonly used quasi-neutral, Clebsch approach to MHD (e.g. Zakharov and Kuznetsov (1997), Holm and Kupershmidt (1983a) and Morrison (1998)). A second motivation was to understand more clearly the gauge symmetry responsible for the magnetic helicity conservation law (4.10) which does not arise as a fluid relabelling symmetry conservation law. In gauge transformations the physical variables \((\rho, \mathbf{u}, \mathbf{B}, S)\) do not change, but the Lagrange multipliers and the Lin constraint variables are allowed to change.

In Calkin (1963) the electric current \( \mathbf{J} \) is expressed in terms of the polarization vector \( \mathbf{P} \) (see (1.1)–(1.3), in which the charge density \( \rho_c \) is given by \( \nabla \cdot \mathbf{P} = -\rho_c \)). Equation (1.6) for \( \mathbf{P} \) has the form of Faraday’s equation for \( \mathbf{P} \) in which the current \( \mathbf{J} \) acts as a source term (see e.g. Panofsky and Phillips (1964)). We show that the curl of the Clebsch variable \( \Gamma \) behaves like \( \mathbf{P} \) (i.e. \( \mathbf{P} = -\nabla \times \Gamma \) in the case \( \nabla \cdot \mathbf{P} = 0 \)). A similar result (equation (2.20)) may be obtained if Faraday’s equation is expressed in terms of the magnetic vector potential \( \mathbf{A} \).

The Lie symmetry determining equations for the Clebsch variables \( \phi, r, \lambda^k, \mu^k, \nu, \) and \( \Gamma \) follow from requiring the physical variables \((\rho, \mathbf{u}, \mathbf{B}, S)\) to have zero variations under the transformations. Requiring the variation of the action to be zero to \( O(\epsilon) \) to within a divergence transformation of the Lagrangian then gives Noether’s theorem, which was used to obtain conservation laws for: (a) magnetic helicity, (b) cross helicity, (c) fluid helicity for a non-magnetized fluid and (d) a class of conservation laws associated with Faraday’s equation.

The latter conservation laws are a special case of conservation laws for curl and divergence systems of equations derived by Cheviakov (2014) (see e.g. Rosenhaus and Shankar (2016) for an account involving Noether’s second theorem for quasi-Noether systems, and Webb and Mace (2015) for a discussion of potential vorticity type conservation laws in MHD).

Section 5 extended the gauge transformation approach to MHD of Heneyey (1982). In this formulation, the physical variables \((\rho, S, \mathbf{B})\) and Lin constraint variables \( \mu^k \) act as canonical coordinates, and the corresponding Lagrange multipliers \((\phi, \beta, \Gamma, \lambda^k)\) correspond to canonical
momenta. The canonical coordinates \((\rho, S, B, \mu^k)\) and the fluid velocity \(u\) do not change, but the canonical momenta (the Lagrange multipliers) do change. We showed that the Henyey approach gives the Casimir determining equations derived by Hameiri (2004) and others.

The present approach can be expanded to take into account the MHD Lie point symmetries. Calkin (1963) used the space-time invariances of the action to derive momentum conservation equation, the energy conservation equation, and the angular momentum conservation equation, associated with space translation invariance, time translation invariance and rotational invariance of the action (see e.g. Morrison (1982) and Webb et al (2005) for the 10 Galilean Lie point symmetries). Webb and Zank (2007, 2009) noted that the scaling Lie point symmetries for special equations of state for the gas can be combined to give another set of conservation laws. Akhatov et al (1991), Bluman et al (2010), and Webb and Zank (2009) have derived nonlocal conservation laws associated with potential symmetries of the gas dynamic equations. The generalized helicity and cross helicity conservation laws for a non-barotropic gas correspond to nonlocal potential symmetries due to the Lagrange multiplier \(r\) used to impose the entropy conservation equation (see also Mobbs (1981)).

Yahalom (2013) discusses magnetic helicity by using an analogy with the Aharonov–Bohm (AB) effect in quantum mechanics. This is related to the magnetic helicity and cross helicity of the flow. The interpretation of cross helicity for non-barotropic flows and magnetic helicity as generalized AB effects is given in Yahalom (2016, 2017a), (2017b), and in appendix E. In Calkin (1963) the polarization \(P\) is used to describe the MHD variational principle, but in our approach the polarization \(P\) is a Lagrange multiplier enforcing Faraday’s equation (see introduction).

The present analysis provides: (a) a direct derivation of the magnetic helicity conservation law using Noether’s theorem and a gauge transformation symmetry (see e.g. Calkin (1963)) and (b) it provides a link between MHD and gauge field theories (e.g. Jackiw (2002), Jackiw et al (2004), Kambe (2007, 2008)), Banerjee and Kumar (2016)). Tanehashi and Yoshida (2015), use the known Casimirs for barotropic MHD, and the non-canonical Poisson bracket of Morrison and Greene (1980, 1982) to uncover gauge symmetries in MHD, by using a Clebsch variable expansion for both \(u\) and \(B\). Araki (2016) provides an alternative viewpoint of fluid relabelling symmetries in MHD involving generalized vorticity and normal mode expansions for ideal incompressible fluids and MHD by using integro-differential operators acting on the generalized velocities.

The multi-symplectic approach to fluid dynamics has been explored by Hydon (2005), Bridges et al (2005, 2010), Cotter et al (2007), Webb (2015) and Webb and Anco (2016). The exact connection of these approaches to the present approach remains to be explored.

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Appendix A

The momentum equation (2.24) is a consequence of the variational equations (2.8)–(2.16). The right handside of the momentum equation (2.24) may be written in the form:
\[ \Delta = \nabla \left\{ \frac{d\phi}{dt} - \left[ \frac{1}{2} \mu^2 - (h + \Phi) \right] \right\} - \left( \frac{dr}{dt} + \Gamma \right) \nabla S \\
\Delta - r \nabla \left( \frac{dS}{dt} \right) - \nabla \mu_k \frac{d\lambda^k}{dt} - \lambda^k \nabla \left( \frac{d\mu^k}{dt} \right) \\
+ b \times \left( \frac{d\tilde{\Gamma}}{dt} - \nabla \times \left( u \times \tilde{\Gamma} \right) \right) + \left\{ \frac{\partial b}{\partial t} + [u, b] \right\} \times \tilde{\Gamma} \\
- \Theta \left\{ \frac{\partial \Gamma}{\partial t} - u \times \Gamma + \nabla(\nu + \Gamma \cdot u) + \frac{B}{\mu_0} \right\} - \Gamma \frac{d\Theta}{dt} + \Theta \nabla \nu \\
+ \nabla \left\{ u \left[ u - \left( \nabla \phi - r \nabla S - \lambda^k \nabla \mu^k + b \times \tilde{\Gamma} - \Gamma \Theta \right) \right] \right\} + Q. \quad (A.1) \]

where

\[ \Theta = \nabla \cdot \frac{B}{\rho}, \quad (A.2) \]

and

\[ Q = b \times \left[ \nabla \times (u \times \tilde{\Gamma}) \right] + u \times \left[ \nabla \times (\Gamma \times b) \right] \\
+ \tilde{\Gamma} \times [u, b] + \nabla \left( u \cdot (b \times \tilde{\Gamma}) \right) = 0. \quad (A.3) \]

One can verify that \( Q = 0 \) by collecting the derivatives of \( u, b \) and \( \tilde{\Gamma} \) separately. An alternative proof that \( Q = 0 \) is an identity, may carried out by using the Calculus of exterior differential forms (see below). In \( (A.3) \) we have dropped the \( \nabla \cdot \tilde{\Gamma} \) terms because \( \tilde{\Gamma} = \nabla \times \Gamma \) is a curl, and hence has zero divergence.

Note that by taking the divergence of the generalized Faraday equation \( (2.3) \) we obtain the conservation law

\[ \frac{\partial}{\partial t} (\nabla \cdot B) + \nabla \cdot (u \nabla \cdot B) = 0 \quad \text{or} \quad \frac{\partial}{\partial t} (\rho \Theta) + \nabla \cdot (\rho u \Theta) \equiv \rho \frac{d\Theta}{dt} = 0. \quad (A.4) \]

A variant of the identity \( (A.3) \) was given by Calkin (1963) in establishing a generalized vorticity equation and a generalized momentum equation.

A more symmetric way to write \( Q \) in \( (A.3) \) is:

\[ Q = \sum \left\{ b \times \left[ \nabla \times (u \times \tilde{\Gamma}) \right] + (u \times b) \nabla \cdot \tilde{\Gamma} + \right\} + \nabla [u \cdot (b \times \tilde{\Gamma})] = 0 \quad (A.5) \]

where the symbol \( \odot \) means cyclically permute \( (b, u, \tilde{\Gamma}) \) and then sum. It is interesting to note that \( \nabla \times Q = 0 \).

Below, we derive the identity \( (A.3) \) by using the algebra of exterior differential forms. The theory and formulae for the algebra of exterior differential forms was developed by Elie Cartan in his study of Lie symmetry transformations and their connection to differential geometry (see e.g. Marsden and Ratiu (1994) and Holm (2008)). We use the results listed in Webb et al (2014a) in our analysis. A basic formula in this theory is Cartan’s magic formula, for the Lie derivative of a differential form \( \omega \) with respect to a vector field \( V \), namely:

\[ L_V (\omega) = V \cdot d\omega + d(V \cdot \omega). \quad (A.6) \]

Here \( L_V = d/dt \) denotes the Lie derivative with respect to a vector field \( V \), which is tangent to a curve with curve parameter \( \epsilon \) (usually \( \epsilon \) corresponds to the infinitesimal parameter of some set of curves associated with a Lie symmetry group). The quantity \( d\omega \) is the exterior derivative
of the differential form $\omega$. If $\omega$ is a $p$-form, then $d\omega$ is a $p+1$-form, and $\mathcal{L}_V\omega$ is a $p$-form. The symbol $\wedge$ denotes the anti-symmetric wedge product and $\mathbf{V}\cdot\omega$ denotes the $(p-1)$-form that results from contracting the vector field $\mathbf{V}$ with the $p$-form $\omega$. A related formula to (A.6) is:

$$\mathcal{L}_V (u_\omega) = \mathcal{L}_V (u_\omega) + u_\omega \mathcal{L}_V (\omega) \equiv [\mathbf{V}, u]_\omega + u_\omega \mathcal{L}_V (\omega) \quad (A.7)$$

where $\mathcal{L}_V = [\mathbf{V}, u]$ is the left Lie bracket of $\mathbf{V}$ and $u$.

To prove (A.3), we start by using (A.7), with $\mathbf{V} \rightarrow \mathbf{b}$, and $\omega \rightarrow \beta$ where

$$\beta = \bar{\Gamma} \cdot d\mathbf{s} = \bar{\Gamma}^y dy \wedge dz + \bar{\Gamma}^x dz \wedge dx + \bar{\Gamma}^z dx \wedge dy, \quad (A.8)$$

$b = B/\rho$, $\bar{\Gamma} = \nabla \times \bar{\Gamma}$ and $u$ is the fluid velocity. We obtain:

$$\mathcal{L}_b (u_\beta) = [b, u]_\beta + u_\beta \mathcal{L}_b (\beta). \quad (A.9)$$

Our strategy is to evaluate the individual terms in (A.9) to obtain (A.3). Using the formulas:

$$u_\beta = u_\beta (\bar{\Gamma} \cdot d\mathbf{s}) = - \left( u \times \bar{\Gamma} \right) \cdot dx,
[b, u]_\beta = - [b, u] \times \bar{\Gamma} \cdot dx,
\mathcal{L}_b (\beta) = \mathcal{L}_b (\bar{\Gamma} \cdot d\mathbf{s}) = [-\nabla \times (b \times \bar{\Gamma}) + b(\nabla \cdot \bar{\Gamma})] \cdot d\mathbf{s}, \quad (A.10)$$

we obtain:

$$u_\beta \mathcal{L}_b (\beta) = -u \times \left\{ -\nabla \times (b \times \bar{\Gamma}) + b(\nabla \cdot \bar{\Gamma}) \right\} \cdot dx. \quad (A.11)$$

Next we evaluate $\mathcal{L}_b (u_\beta)$ in (A.9). Writing $\gamma = u_\beta$ and using Cartan’s magic formula (A.6) we obtain:

$$\mathcal{L}_b (u_\beta) = \mathcal{L}_b (\gamma) = b_\cdot d\gamma + d(b_\cdot \gamma). \quad (A.12)$$

Using the results:

$$\gamma = - (u \times \bar{\Gamma}) \cdot dx, \quad d\gamma = -\nabla \times (u \times \bar{\Gamma}) \cdot d\mathbf{s},
[b, u]_\gamma = (b \cdot \nabla)_\cdot \left( u \times \bar{\Gamma} \cdot dx \right) = - [b, u] \times \bar{\Gamma} \cdot dx
b_\cdot d\gamma = b \times \left( \nabla \times (u \times \bar{\Gamma}) \right) \cdot dx. \quad (A.13)$$

in (A.12) gives the result:

$$\mathcal{L}_b (u_\beta) = b \times \left[ \nabla \times (u \times \bar{\Gamma}) \right] \cdot dx - d \left( b \cdot (u \times \bar{\Gamma}) \right). \quad (A.14)$$

Substituting the results (A.14), (A.12), (A.10) in (A.9) then gives the identity:

$$b \times \left[ \nabla \times (u \times \bar{\Gamma}) \right] \cdot dx - d \left( b \cdot (u \times \bar{\Gamma}) \right)
= -[b, u] \times \bar{\Gamma} \cdot dx + \left\{ u \times \left[ \nabla \times (b \times \bar{\Gamma}) \right] - (u \times b) \nabla \cdot \bar{\Gamma} \right\} \cdot dx. \quad (A.15)$$

The result (A.15), can be written in the form $Q \cdot dx = 0$ where

$$Q = b \times \left[ \nabla \times (u \times \bar{\Gamma}) \right] + u \times \left[ \nabla \times (\bar{\Gamma} \times b) \right] + \bar{\Gamma} \times [u, b] + (u \times b) \nabla \cdot \bar{\Gamma} + \nabla \left( u \cdot (b \times \bar{\Gamma}) \right) = 0. \quad (A.16)$$

The result (A.16) reduces to (A.3) for the case $\nabla \cdot \bar{\Gamma} = 0$. 

23
Appendix B

There are different methods that can be used to obtain solutions of the Lie determining equations (3.8)–(3.15). In this appendix we use a method that has affinities with the steady MHD flows investigated by Bogoyavlenskij (2002), Schief (2003), Golovin (2010, 2011). These ideas were used by Webb et al (2005) Webb and Zank (2007) and Webb and Mace (2015) for fluid relabelling symmetries in MHD. This method allows the function H in (3.13) to have a general form involving the fluid labels, and the advected invariants. Other solutions of the Lie determining equations are given in appendix C.

First consider the solution of (3.13), namely:

\[ \nabla K + V^\mu \nabla \tilde{\lambda} - V^\lambda \nabla \mu - V^r \nabla S + b \times (\nabla \times V^\Gamma) = 0. \]  
(B.1)

Taking the scalar product of (B.1) with \( b \) gives the equation:

\[ b \cdot \nabla K + V^\mu (b \cdot \nabla \tilde{\lambda}) - V^\lambda (b \cdot \nabla \mu) - V^r (b \cdot \nabla S) = 0. \]  
(B.2)

One way in which (B.2) can be satisfied is if:

\[ V^\mu = k_1 b \cdot \nabla \mu, \quad V^\lambda = k_1 b \cdot \nabla \tilde{\lambda}, \quad b \cdot \nabla K = V^r (b \cdot \nabla S). \]  
(B.3)

For simplicity, we consider the case:

\[ V^r = 0 \quad \text{and} \quad b \cdot \nabla K = 0. \]  
(B.4)

Using (B.3)–(B.4) in (B.1), (B.1) reduces to the equation:

\[ \nabla K + b \times \left( \nabla \times \left( V^\Gamma + k_1 \tilde{\lambda} \nabla \mu \right) \right) = 0. \]  
(B.5)

Writing

\[ Q = \nabla \times \left( V^\Gamma + k_1 \tilde{\lambda} \nabla \mu \right), \]  
(B.6)

equation (B.5) takes the form:

\[ Q \times b = \nabla K. \]  
(B.7)

Equation (B.7) is reminiscent of the steady MHD form of Faraday’s equation \( E = -u \times B = -\nabla \psi \) analyzed by Schief (2003) and Bogoyavlenskij (2002). Note from (B.6) that

\[ \nabla \cdot Q = \nabla \cdot (\rho Q) = 0 \quad \text{where} \quad \rho Q = Q, \]  
(B.8)

which resembles the steady mass continuity equation. Similarly,

\[ \nabla \cdot B = \nabla \cdot (\rho b) = 0, \]  
(B.9)

is analogous to the mass continuity equation.

Taking the curl of (B.7) gives:

\[ \nabla \times \left( \hat{Q} \times b \right) = 0, \]  
(B.10)

which is analogous to Faraday’s equation \( \nabla \times E = 0 \) for steady MHD flows (\( E = -u \times B \)). Using (B.8) in (B.10) we obtain:

\[ \nabla \times \left( \hat{Q} \times b \right) = -\rho \left[ \hat{Q}, b \right] = 0. \]  
(B.11)
where
\[
\hat{Q} \cdot \nabla = \frac{\partial}{\partial \alpha}, \quad b \cdot \nabla = \frac{\partial}{\partial \gamma}.
\] (B.13)

are directional derivatives \(\partial/\partial\alpha\) and \(\partial/\partial\gamma\) in the Maxwell surfaces \(K = \text{const.}\) Schief (2003), Bogoyavlenskij (2002) and Webb et al (2005).

From Webb et al (2005), it follows that
\[
Q = \nabla \gamma \times \nabla K, \quad B = \nabla K \times \nabla \alpha.
\] (B.14)

are solutions of the determining equations for \(Q\) and \(B\). From (B.14):
\[
Q \times b = \rho \hat{Q} \times b = (\nabla \gamma \times \nabla K) \times b = (b \cdot \nabla \gamma) \nabla K - (b \cdot \nabla K) \nabla \gamma = \nabla K,
\] (B.15)

which verifies (B.7) (note that \(b \cdot \nabla \gamma = 1\) and \(b \cdot \nabla K = 0\)). Using (B.6) and (B.14), we obtain:
\[
Q = \nabla \times \left( V^{\Gamma} + k_1 \hat{\lambda} \nabla \mu \right) = \nabla \times (\gamma \nabla K).
\] (B.16)

Uncurling (B.16) we obtain the solution
\[
V^{\Gamma} = -k_1 \hat{\lambda} \nabla \mu + \gamma \nabla K + k_2 \nabla \Lambda,
\] (B.17)

as a solution for \(V^{\Gamma}\) where \(k_2\) is an arbitrary constant, and \(\Lambda(x, t)\) is an arbitrary function of \(x\) and \(t\). Substitution of the solution (B.17) for \(V^{\Gamma}\) in (3.8) and assuming \(d\gamma/dt = 0\), we obtain:
\[
V^{\nu} = -k_2 \frac{\partial \Lambda}{\partial t} - k_1 \hat{\lambda} (u \cdot \nabla \mu) + \gamma (u \cdot \nabla K) = -k_2 \frac{d\Lambda}{dt} + u \cdot V^{\Gamma}.
\] (B.18)

The solutions (B.17) and (B.18) for \(V^{\Gamma}\) and \(V^{\nu}\) are used in section 4 to obtain MHD conservation laws via Noether’s theorem.

Appendix C

In this appendix, we present solutions of the Lie determining equations (3.8)–(3.15) which in general, are different than those presented in appendix B.

By assuming that the function \(K\) in (3.14) has the functional form \(K = K(\bar{\lambda}, \mu, S)\) (3.13) may be reduced to the equation:
\[
\nabla \bar{\lambda} \left( K_{\bar{\lambda}} + V^{\nu} \right) + \nabla \mu \left( K_{\mu} - V^{\lambda} \right) + \nabla S \left( K_{S} - V^{\nu} \right) + b \times \left( \nabla \times V^{\Gamma} \right) = 0.
\] (C.1)
Equation (C.1) possesses a simple class of solutions of the form:

\[ V^\mu = -K_\lambda + d_1 (b \cdot \nabla \mu) + d_2 (b \cdot \nabla S), \]
\[ V^\lambda = K_\mu + d_1 (b \cdot \nabla \lambda) - d_3 (b \cdot \nabla S), \]
\[ V^r = K_\lambda + d_2 (b \cdot \nabla \lambda) + d_3 (b \cdot \nabla \mu), \]
\[ V^\Gamma = -\left[d_1 \lambda \nabla \mu + d_2 \lambda \nabla S + d_3 \mu \nabla S\right] + \chi A + \nabla \Lambda, \tag{C.2} \]

where \( d_1, d_2, d_3, \) and \( \chi \) are constants and \( B = \nabla \times A. \)

The solution for \( V^\nu \) satisfying (3.8) has the form:

\[ V^\nu = u \cdot \left[-d_1 \lambda \nabla \mu - d_2 \lambda \nabla S - d_3 \mu \nabla S\right] + \chi \psi - \Lambda_t. \tag{C.3} \]

This class of solutions are different from the solutions in appendix B, where the constraints \( V^r = 0 \) and \( B \cdot \nabla K = 0 \) were imposed. To derive the solutions for \( V^\mu, V^\lambda, \) and \( V^r, \) we first took the scalar product of (C.1) with \( b \) to determine the effect of compatibility conditions parallel to \( b. \)

In the derivation of (C.1)–(C.3), there is a critical equation:

\[ \nabla \times V^\Gamma = -\left[d_1 \lambda \nabla \mu + d_2 \lambda \nabla S + d_3 \mu \nabla S\right] + \chi B. \tag{C.4} \]

In (C.4), \( d_1, d_2, d_3, \) \( \chi \) are not necessarily constants, in which case it is necessary to uncurl (C.4).

The solutions (C.1)–(C.3) may give the magnetic helicity, cross helicity and arbitrary potential \( \Lambda(x, t) \) conservation laws by appropriate choice of the parameters in Noether’s theorem.

**Appendix D**

In this appendix, we prove proposition 5.1. We write:

\[ F(\rho, S, \mu, \beta, \lambda, \Gamma) = \bar{F}(\rho, S, u, B), \tag{D.1} \]

taking into account the constraints. The transformation of variational derivatives may be effected by noting that:

\[ \int \left(F_{\rho} \delta \rho + F_{\phi} \delta \phi + F_{S} \delta S + F_{\beta} \delta \beta + F_{\lambda} \delta \lambda + F_{\mu} \delta \mu + F_{\Gamma} \delta \Gamma \right) d^3x \]
\[ = \int \left(\bar{F}_{\rho} \delta \rho + \bar{F}_{S} \delta S + \bar{F}_{u} \delta u + \bar{F}_{B} \delta B\right) d^3x. \tag{D.2} \]

Using (2.9) or (5.9) to determine \( \delta u \) in (D.2), integrating by parts and dropping surface terms gives the formulae:
\[ F_\rho = F_\rho + \vec{F}_u \cdot \left[ r \nabla S + \lambda \nabla \mu + \vec{\Gamma} \times \vec{b} + \Gamma \frac{\nabla \cdot \vec{B}}{\rho} \right]. \]

\[ F_\phi = - \nabla \cdot \vec{F}_u, \quad F_\phi = \vec{F}_S + \nabla \cdot (r\vec{F}_u), \quad F_\beta = - \frac{\vec{F}_u \cdot \nabla S}{\rho}. \]

\[ F_\mu = \nabla \cdot (\lambda \vec{F}_u), \quad F_\lambda = - \frac{\vec{F}_u \cdot \nabla \mu}{\rho}, \]

\[ F_\Gamma = \nabla \times (\vec{F}_u \times \vec{b}) - \vec{F}_u \nabla \cdot \frac{\vec{B}}{\rho}, \]

\[ F_B = \vec{F}_B + \nabla \left( \frac{\Gamma \cdot \vec{F}_u}{\rho} \right) + \vec{\Gamma} \times \vec{F}_u \rho. \] (D.3)

From (5.7) and (5.8) the variations of \( \delta \rho, \delta S, \delta \mu \) and \( \delta B \) are related to the \( F \) and \( \bar{F} \) variations by the formulae:

\[ \delta \rho = F_\phi = - \nabla \cdot \vec{F}_u = 0, \quad \delta S = F_\beta = - \frac{\vec{F}_u \cdot \nabla S}{\rho} = 0, \]

\[ \delta \mu = F_\lambda = - \frac{\vec{F}_u \cdot \nabla \mu}{\rho} = 0, \]

\[ \delta B = F_\Gamma = \nabla \times (\vec{F}_u \times \vec{b}) - \vec{F}_u \nabla \cdot \frac{\vec{B}}{\rho} = 0. \] (D.4)

In (D.1)–(D.4) we have dropped reference to the index \( k \) which would apply if there are several Clebsch Lin constraint variables \((\mu^k, \lambda^k)\). The basic idea can be illustrated with one Lin constraint Clebsch pair.

Equation (D.4) apply to fluid relabelling symmetries (e.g Padhye and Morrison (1996a, 1996b)). Equation (D.4) may be written as:

\[ \delta \rho = - \nabla \cdot \left( \rho \hat{v}^x \right), \quad \delta S = - \hat{v}^x \cdot \nabla S, \]

\[ \delta B = \nabla \times \left( \hat{v}^x \times \vec{B} \right) - \hat{v}^x \nabla \cdot \vec{B} = 0, \] (D.5)

where

\[ \hat{v}^x = \frac{1}{\rho} \vec{F}_u \equiv \vec{F}_M, \] (D.6)

defines the canonical Lie symmetry operator (vector field):

\[ \hat{v}^x = \hat{v}^x \frac{\partial}{\partial x}, \quad \hat{v}^x = \hat{v}^x - \hat{v}^0 \frac{\partial x}{\partial x_0}. \] (D.7)

associated with the Lagrangian map \( \vec{x} = \vec{x}(x_0, t) \) between the Lagrangian fluid labels \( x_0 \) and the Eulerian position \( \vec{x} \) of the fluid element at time \( t \) (e.g. Newcomb (1962), Webb et al (2005) and Webb and Zank (2007)), where \( dx/dt = u(x, t) \) is formally integrated to give the solution \( \vec{x} = \vec{x}(x_0, t) \) where \( \vec{x} = x_0 \) at time \( t = 0 \). For fluid relabeling symmetries \( \hat{v}^x = 0 \) in (D.7) Webb and Zank (2007). In (D.6) \( \vec{F}_M = \delta \vec{F} / \delta \vec{M} \) is the variational derivative of \( \vec{F} \) with respect to the mass flux \( \vec{M} = \rho \vec{u} \) (i.e. \( \rho \) and \( \vec{M} \) are regarded as independent variables rather than \( \rho \) and \( \vec{u} \)).

Casimirs \( C \) are functionals which have zero non-canonical Poisson brackets with all other functionals, \( F \) defined on the system, i.e.
\{C, F\} = 0, \text{ for all functional } F \tag{D.8}

(e.g. Holm et al (1985) and Hameiri (2004)).

Using (D.3) and (D.4) \( F_\rho, F_\mu \) and \( F_S \) reduce to:

\[
\begin{align*}
F_\rho &= \bar{F}_\rho + \frac{\bar{F}_u}{\rho} \cdot \left( \bar{\Gamma} \times b + \frac{\Gamma \cdot B}{\rho} \right), \\
F_\mu &= \bar{F}_u \cdot \nabla \bar{\lambda}, \\
F_S &= \bar{F}_S + \bar{F}_u \cdot \nabla r.
\end{align*}
\tag{D.9}
\]

The condition (D.5) that \( \delta B = 0 \) may be written in the Lie bracket form:

\[
\rho[b, \bar{F}_M] \equiv \rho(b \cdot \nabla \bar{F}_M - \bar{F}_M \cdot \nabla b) = 0. \tag{D.10}
\]

To obtain the Casimir determining equation (5.18), note that (5.14) and (D.3)–(D.9) together give (5.14) in the form:

\[
\rho \nabla \left\{ \bar{F}_\rho + \frac{\bar{F}_u}{\rho} \cdot \left( \bar{\Gamma} \times b + \frac{\Gamma \cdot B}{\rho} \right) \right\} \\
- \left( \bar{F}_u \cdot \nabla \bar{\lambda} \right) \nabla \mu - (F_S + \bar{F}_u \cdot \nabla r) \nabla S \\
+ B \times \left\{ \nabla \times F_B + \nabla \times \left( \bar{\Gamma} \times \bar{F}_u \rho \right) \right\} \\
- \left\{ \bar{F}_B + \nabla \left( \frac{\Gamma \cdot F_u}{\rho} \right) \right\} \nabla \cdot B = 0. \tag{D.11}
\]

By using the formulae:

\[
\omega = \nabla \times u = -\nabla r \times \nabla S - \nabla \bar{\lambda} \times \nabla \mu - \nabla \times \left( \bar{\Gamma} \times b \right) - \nabla \times \left( \frac{\Gamma \cdot B}{\rho} \right), \tag{D.12}
\]

and

\[
\omega \times \bar{F}_u = - (\bar{F}_u \cdot \nabla r) \nabla S - \left( \bar{F}_u \cdot \nabla \bar{\lambda} \right) \nabla \mu + \bar{F}_u \times \left[ \nabla \times (\bar{\Gamma} \times b) + \frac{\Gamma \cdot B}{\rho} \right]. \tag{D.13}
\]

(D.11) reduces to:

\[
\rho \nabla \bar{F}_\rho - F_S \nabla S + B \times (\nabla \times \bar{F}_B) - \bar{F}_B \nabla \cdot B + \omega \times \bar{F}_u + R = 0, \tag{D.14}
\]

where

\[
R = \rho \left\{ \nabla \left[ \bar{F}_M \cdot \bar{\Gamma} \times b + (\bar{F}_M \cdot \Gamma) \Theta \right] \\
- F_M \times \nabla \times \left[ \bar{\Gamma} \times b + \Gamma \Theta \right] + b \times \left[ \nabla \times (\bar{\Gamma} \times F_M) \right] \\
- \left\{ \nabla (\Gamma \cdot \bar{F}_M) + (\bar{\Gamma} \times F_M) \right\} \Theta \right\} \\
\equiv - \rho \left\{ b \times \left[ \nabla \times (\bar{F}_M \times \bar{\Gamma}) \right] + \bar{F}_M \times [\nabla \times (\bar{\Gamma} \times b)] \\
+ \nabla [\bar{F}_M \cdot (b \times \bar{\Gamma})] + \Gamma \bar{F}_M \cdot \nabla \Theta \right\}, \tag{D.15}
\]

\[
28
\]

G M Webb and S C Anco

J. Phys. A: Math. Theor. 50 (2017) 255501
and

$$\Theta = \frac{\nabla \cdot \mathbf{B}}{\rho}. \tag{D.16}$$

By taking the divergence of $\delta \mathbf{B}$ in (D.5) we obtain:

$$\nabla \cdot \delta \mathbf{B} = -\nabla \cdot [\mathbf{F}_M \nabla \cdot \mathbf{B}] = -\nabla \cdot (\rho \mathbf{F}_M \Theta) = -\rho \mathbf{F}_M \cdot \nabla \Theta = 0. \tag{D.17}$$

Using (D.10), (D.17) and (A.16), with $\mathbf{u} \rightarrow \mathbf{F}_M$, we find $R = -\rho Q = 0$ as $Q = 0$ for the case $\mathbf{u} \rightarrow \mathbf{F}_M$ in (A.16). Equation (D.14) then reduces to the Casimir determining equation (5.18). This completes the proof of proposition 5.1.

Appendix E

In this appendix we discuss the work of Yahalom (2013, 2016), Yahalom (2017a, 2017b) on magnetic helicity $H_M$ and non-barotropic cross helicity $H_{CNB}$. Yahalom developed a five Clebsch variable variational principle for MHD (at first sight there appears to be 8 Clebsch variables involved). Yahalom (2017a, 2017b) uses the action:

$$A = \int \left\{ \frac{1}{2} \rho \mathbf{u}^2 - \rho e(\rho, S) + \frac{B^2}{2 \mu_0} \right\} + \phi \left[ \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) \right] - \rho \alpha \frac{d \chi}{d t} - \rho \beta \frac{d \eta}{d t} - \rho \sigma \frac{d S}{d t}$$

$$- \frac{B}{\mu_0} \cdot \nabla \chi \times \nabla \eta \right\} d^3 x \, dt. \tag{E.1}$$

The stationary point conditions $\delta A / \delta \mathbf{B} = 0$ and $\delta A / \delta \mathbf{u} = 0$ gives the Clebsch expansions:

$$\mathbf{B} = \nabla \chi \times \nabla \eta,$nabla \phi + \alpha \nabla \chi + \beta \nabla \eta + \sigma \nabla S, \tag{E.2}$$

for $\mathbf{B}$ and $\mathbf{u}$ (we use $r = -\sigma$ in our formulation). It is straightforward to write down the other variational equations by varying $\rho, S$ and the Clebsch variables in the variational principle (see e.g. Yahalom (2017a, 2017b)).

The magnetic field Clebsch variable expansion (E.2) is also used by Sakurai (1979). The magnetic vector potential $\mathbf{A}$ and $\mathbf{B}$ have the forms:

$$\mathbf{A} = \chi \nabla \eta + \nabla \zeta, \quad \mathbf{B} = \nabla \chi \times \nabla \eta. \tag{E.3}$$

For a non-trivial magnetic field topology there does not exist a global $\mathbf{A}$ (i.e. $\chi, \eta, \zeta$ are not global single valued functions of $x$). Notice from (E.2) that the magnetic helicity density $h_m = \mathbf{A} \cdot \mathbf{B}$ has the form:

$$h_m = \mathbf{A} \cdot \mathbf{B} = \nabla \zeta \cdot \nabla \chi \times \nabla \eta = \frac{\partial (\zeta, \chi, \eta)}{\partial (x, y, z)}. \tag{E.4}$$

Thus $h_m \neq 0$ only if $\chi, \eta$ and $\zeta$ are independent functions of $x$. Semenov et al (2002) argue that the field topology changes due to jumps in $\zeta$ in magnetic fields with non-trivial topology for generalized versions of the MHD topological soliton (see also Kamchatnov (1982)). A similar jump in $\zeta$ occurs in the non-global $\mathbf{A}$ for the magnetic monopole (Urbanbekte (2003)).

Yahalom (2013, 2017a), (2017b) introduces a further independent magnetic field potential, $\mu$ (called the metage) which represents distance or affine parameter along the magnetic field
line formed by the intersection of the η = const. and χ = const. Euler potential surfaces. Thus, we obtain:

\[ \nabla \zeta = \frac{\partial \zeta}{\partial \chi} \nabla \chi + \frac{\partial \zeta}{\partial \eta} \nabla \eta + \frac{\partial \zeta}{\partial \mu} \nabla \mu. \quad (E.5) \]

Using (E.5) in (E.4) gives:

\[ h_m = A \cdot B = \frac{\partial \zeta}{\partial \mu} \nabla \mu \cdot \nabla \chi \times \nabla \eta = \frac{\partial \zeta}{\partial \mu} \left| \frac{\partial (\chi, \eta, \mu)}{\partial (x, y, z)} \right|. \quad (E.6) \]

The volume integrated magnetic helicity:

\[ H_M = \int_V A \cdot B \, d^3x = \int_V \frac{\partial \zeta}{\partial \mu} \, d\chi \wedge d\eta \wedge d\mu. \quad (E.7) \]

However, the magnetic flux:

\[ d\Phi_B = B \cdot dS = (\nabla \chi \times \nabla \eta) \cdot dS = d\chi d\eta. \quad (E.8) \]

To prove (E.6) note that:

\[ dS = r_\chi \times r_\eta \, d\chi d\eta \quad \text{and} \quad B = \nabla \chi \times \nabla \eta, \]

\[ B \cdot dS = (\nabla \chi \times \nabla \eta) \cdot (r_\chi \times r_\eta) \, d\chi d\eta = d\chi d\eta. \quad (E.9) \]

The last step in (E.9) follows by setting \( (q^1, q^2, q^3) = (\chi, \eta, \mu) \) and noting:

\[ \frac{\partial q^a}{\partial x^b} \frac{\partial x^b}{\partial q^a} = \delta^a_b \quad \text{which implies} \quad e^a \cdot e_b = \delta^a_b. \quad (E.10) \]

where \( e^a = \nabla q^a \) and \( \epsilon^a_b = \partial r^a / \partial x^b \).

Using (E.6) in (E.7) and integrating over \( \mu \) along the field line, we obtain:

\[ H_M = \int [\zeta] \, d\chi d\eta \equiv \int [\zeta] \, d\Phi_B, \quad (E.11) \]

where \([\zeta]\) is the jump in \( \zeta \) between the two ends of the field line (the field lines can be closed or open). Equation (E.11) gives the invariant:

\[ [\zeta] = \frac{dH_M}{d\Phi_B}. \quad (E.12) \]

which is the magnetic helicity per unit magnetic flux. Equation (E.12) shows that for a closed field line, the jump in \([\zeta]\) is non-zero for a non-trivial magnetic helicity. Yahalom (2013, 2016), Yahalom (2017a, 2017b) refers to (E.12) as the MHD ‘magnetic Aharonov–Bohm effect’, in analogy with the Aharonov–Bohm effect in quantum mechanics.

Yahalom (2013, 2016), Yahalom (2017a, 2017b) and Webb et al. (2014a, 2014b) developed conservation laws for cross helicity and a generalized cross helicity for both barotropic and non-barotropic MHD. The cross helicity \( H_C \) is is given by:

\[ H_C = \int_V u \cdot B \, d^3x. \quad (E.13) \]

The differential form of the cross helicity evolution equation from (4.16) is:

\[ \frac{\partial}{\partial t} (u \cdot B) + \nabla \cdot \left[ (u \cdot B)u + B \left( h + \Phi - \frac{1}{2} \mu^2 \right) \right] = T(B \cdot \nabla S). \quad (E.14) \]
Integration of (E.14) over the volume $V$ co-moving with the fluid, and assuming $\mathbf{B} \cdot \mathbf{n} = 0$ on $\partial V$, where $\mathbf{n}$ is the outward normal to $\partial V$, gives the helicity evolution equation:

$$\frac{dH_C}{dt} = \int_V T(\mathbf{B} \cdot \nabla S) \, d^3x.$$  \hspace{1cm} (E.15)

Thus, $dH_C/dt = 0$ for barotropic flows where $\nabla S = 0$. For non-barotropic flows, we define the generalized cross helicity as:

$$H_{CNB} = \int_V (\mathbf{u} - \sigma \nabla S) \cdot \mathbf{B} \, d^3x,$$  \hspace{1cm} (E.16)

(in our notation $\sigma = -r$). Equation (E.14) then gives:

$$\frac{dH_{CNB}}{dt} = 0, \quad \text{where} \quad \frac{d\sigma}{dt} = T(x,t).$$  \hspace{1cm} (E.17)

Using the Clebsch expansions (E.2) for $\mathbf{u}$ and $\mathbf{B}$, we obtain:

$$H_C = \int \mathbf{B} \cdot \nabla \phi \, d^3x + \int \sigma \mathbf{B} \cdot \nabla S \, d^3x,$nabla \phi \, d^3x + \int \sigma \mathbf{B} \cdot \nabla S \, d^3x,$$

$$= \int [\phi] d\Phi_B + \int \sigma \frac{\partial S}{\partial \mu} d\mu d\Phi_B.$$  \hspace{1cm} (E.18)

Also

$$H_{CNB} = H_C - \int \sigma \mathbf{B} \cdot \nabla S \, d^3x = H_C - \int \sigma \frac{\partial S}{\partial \mu} d\mu d\Phi_B.$$  \hspace{1cm} (E.19)

Here $[\phi]$ is the jump in the Clebsch potential across the surface where the multi-valued function $\phi$ jumps (for simplicity we assume that there is one such surface, but there could be many such surfaces). From (E.18) and (E.19)

$$\frac{dH_C}{d\Phi_B} = [\phi] + \int \sigma \frac{\partial S}{\partial \mu} d\mu \equiv [\phi] + \oint \sigma dS,$n\partial S}{\partial \mu} d\mu \equiv [\phi] + \oint \sigma dS,$$

$$\frac{dH_{CNB}}{d\Phi_B} = [\phi].$$  \hspace{1cm} (E.20)

The net upshot of the analysis is that $dH_{CNB}/d\Phi_B$ is an advected topological invariant (note $d[\phi]/dt = 0$ follows from the variational equation $\delta A/\delta \rho = 0$). These results are described in more detail in Yahalom (2017a, 2017b)).

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