Isospin relations for the tau decay modes

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Abstract

Since the fifties, isospin relations have been used in particle physics to understand the properties of multihadrons final states. In the case of the tau lepton, they allow to relate the partial widths of the decay modes to the cross sections of $e^+e^-$ annihilations. A pedagogical introduction to the construction of isospin states for meson systems and an updated review of the use of isospin relations in the study of the tau lepton are presented.

Dedicated to Roberto Salmeron on the occasion of his 80th birthday.

1 Introduction

In 1953, at the Bagnères de Bigorre conference [1], which Roberto Salmeron attended as member of the Manchester group, Dalitz showed the following inequality [2]:

$$\frac{1}{4} \leq \frac{\tau^+ \rightarrow \pi^+\pi^0\pi^0}{\tau^+ \rightarrow \pi^+\pi^+\pi^-} \leq 1.$$  \hspace{1cm} (1)

The $\tau^+$ in Eq. (1) partner of the $\theta^+$ in the celebrated puzzle, is nowadays known as $K^+$. The hypotheses leading to Eq. (1) were the existence of an isospin triplet ($\tau^+, \tau^0, \tau^-$) and the conservation of isospin in the $\tau$ decay. None of them was founded. Nevertheless, owing to the $|\Delta I| = 1/2$ rule, Eq. (1) survived the introduction of the Gell-Mann Nishijima scheme [3,4] and stimulated the discovery of the $K^+ \rightarrow \pi^+\pi^0\pi^0$ decay mode [5].

More than thirty years later, the same inequality was written in a paper [6] devoted to the “calculation of exclusive decay modes of the tau”, but the $\tau$ studied in the paper was the $\tau$ lepton. This coincidence can serve to illustrate the longevity and the generality of the isospin relations. However, in the following pages, we will discuss only their applications [6,7,8,9,10,11] to the decay modes of the $\tau$ lepton.
2 Isospin relations in hadronic final states

The proof of an equation like Eq. 1 starts from the identification of the different possible isospin states for the hadronic system. The amplitudes for a given charge-configuration \((\pi^+\pi^+\pi^-)\) and \((\pi^+\pi^0\pi^0)\) for three \(\pi\)’s with \(Q = 1\) are linear combinations of the isospin amplitudes; thus the partial widths are linear combinations of the squared amplitudes and their interference terms.

Generally, a large fraction of the interference terms are killed by the integration over the phase-space. The remaining terms are bounded by the Schwartz inequality \([12]\). The resulting constraints can be geometrically represented by an allowed convex domain in the space of the charge-configuration fractions, \(f_{cc} = \Gamma_{cc}/\Gamma\). If all the interferences vanish, the domain is a polyhedron, convex hull of the points that describe each isospin state.

Hence, the first step in the construction of the allowed domain is the setting up of a basis for the isospin states, adapted to the implementation of the Pauli principle.

3 Isospin states of \(n\pi\) systems

Such a basis was constructed by Pais \([13]\) with the object of studying the many pion systems produced in \(\bar{p}p\) and \(\bar{p}n\) annihilations. The construction is based on two simple remarks: \(i)\) the representations of the isospin group SU(2) relevant for \(n\pi\) systems are also representations of SO(3), \(ii)\) the group of \(3 \times 3\) orthogonal unimodular matrices, SO(3), is a subgroup of the group of \(3 \times 3\) unimodular matrices, SL(3). Thus, if \(V\) is the three-dimensional space of the isospin states of one pion, the space of the states of \(n\) pions, \(V^{\otimes n}\), supports representations of both SL(3) and the symmetric group \(S_n\), which acts on \(V^{\otimes n}\) by permuting its factors. Standard properties of the representations of linear groups \([14, 15, 16]\) imply the decomposition

\[
V^{\otimes n} = \bigoplus \lambda E_\lambda \otimes F_\lambda, \tag{2}
\]

where the symbol\(^1\) \(\lambda = (\lambda_1, \lambda_2, \lambda_3)\), with \(\lambda_1 \geq \lambda_2 \geq \lambda_3\) and \(\lambda_1 + \lambda_2 + \lambda_3 = n\), is associated to a three-row Young diagram; \(F_\lambda\) is the irreducible representation of \(S_n\) determined by the diagram, and \(E_\lambda\) an irreducible representation of SL(3). For a given \(n\), the correspondence between the representation of SL(3) and \(\lambda\) is one-one. The representation \(E_\lambda\) is characterized by the two numbers: \(\lambda_1 - \lambda_2\) and \(\lambda_2 - \lambda_3\).

As a representation of SO(3), \(E_\lambda\) is no longer irreducible because of the invariance under SO(3) of the contraction operation \([14, 17]\). Its decomposition into irreducible representations of SO(3) reads

\[
E_\lambda = \bigoplus I N_I(\lambda) D_I, \tag{3}
\]

where \(D_I\) is the \((2I + 1)\)-dimensional irreducible representation (integer isospin \(I\)). The multiplicity \(N_I(\lambda)\) was computed by Racah \([17]\). It can be written:

\[
N_I(\lambda) = \phi(\lambda_1 - \lambda_3 - I + 2) - \phi(\lambda_2 - \lambda_3 - I + 1) - \phi(\lambda_1 - \lambda_2 - I + 1), \tag{4}
\]

\(^1\)Also denoted \([13, 15]\) “symmetry class”.

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where \( \phi(x) \) is the greatest integer contained in \( x/2 \) for \( x > 0 \), and 0 for \( x \leq 0 \). For systems of two pions, we get:

\[
\begin{array}{cccc}
\lambda & N_0 & N_1 & N_2 \\
(1,1,0) & 0 & 1 & 0 \\
(2,0,0) & 1 & 0 & 1 \\
\end{array}
\]

and, for three pions,

\[
\begin{array}{cccc}
\lambda & N_0 & N_1 & N_2 & N_3 \\
(1,1,1) & 1 & 0 & 0 & 0 \\
(2,1,0) & 0 & 1 & 1 & 0 \\
(3,0,0) & 0 & 1 & 0 & 1 \\
\end{array}
\]

For any \( \lambda \), Eq. 4 implies the relation

\[ N_0(\lambda) + N_1(\lambda) = 1, \tag{5} \]

which can also be obtained directly from the symmetry (square) of a contraction. If both \( \lambda_1 - \lambda_2 \) and \( \lambda_2 - \lambda_3 \) are even, \( N_0 = 1 \), otherwise \( N_1 = 1 \).

From the equations (2, 3) and (4), we see that the sole degeneracy of the \( n-\pi \) states \( |I_3, \lambda\rangle \) with \( I \leq 1 \), \( (I_3 \text{ and } \lambda \text{ fixed}) \) is due to the permutation symmetry. These states form an irreducible representation of \( S_n \). Since a permutation preserves the charge-configuration \((n^+, n^0, n^-)\) of a \( n-\pi \) system, the irreducibility implies that the Clebsch-like coefficients used to write the isospin states as combinations of charge-configuration states are determined by \( I_3 \) and \( \lambda \) only\(^2\). In other words \([13]\), the coefficients of the charge-configurations are a class property.

Furthermore, since the permutation symmetry properties of the momentum and isospin amplitudes are the same, because of the Pauli principle, integrating over the phase-space kills all the interference terms for a \( n-\pi \) system with \( I \leq 1 \); the allowed domain in the space of the charge-configuration fractions is a polyhedron.

Let’s take the simple example of the \( Q = 1, I = 1 \) three-\( \pi \) system alluded to in the introduction. For \( \text{SO}(3) \), an isospin one is a vector, and any vector made of three vectors can be written

\[ \alpha (\vec{b} \cdot \vec{c}) \vec{a} + \beta (\vec{c} \cdot \vec{a}) \vec{b} + \gamma (\vec{a} \cdot \vec{b}) \vec{c}. \]

As an isospin function, \((\vec{a} \cdot \vec{b}) \vec{c}\) describes an \( I = 1 \) state with the two first \( \pi \) in an \( I = 0 \) state: \((\pi^+ \pi^- - \pi^0 \pi^0 + \pi^- \pi^+) \pi^+ \). It is then straightforward to write the other terms by cyclic permutations and get the ratio

\[ R = \frac{\pi^+ \pi^0 \pi^0}{\pi^+ \pi^- \pi^+} = \frac{|\alpha|^2 + |\beta|^2 + |\gamma|^2}{|\beta + \gamma|^2 + |\alpha + \gamma|^2 + |\alpha + \beta|^2}. \tag{6} \]

\(^2\)The elements of the group algebra used to build an orthogonal basis of the representation give also orthogonal charge configuration states \([13]\).
The symmetry class \( \lambda = (3, 0, 0) \), is associated to the one-dimensional space of completely symmetric states \( (\alpha = \beta = \gamma) \) for which the ratio is \( R = 1/4 \); the class \( \lambda = (2, 1, 0) \) to the two-dimensional orthogonal space \( (\alpha + \beta + \gamma = 0) \), with \( R = 1 \). An example of state in the \( (2, 1, 0) \) class is given by a \( \rho \pi \) system, which is represented by \( (\vec{a} \wedge \vec{b}) \wedge \vec{c} \), where the vector product \( (\wedge) \) is interpreted as the combination of two isospins one into an isospin one.

Thus the charge-configuration fractions are:

\[
f_{\pi^+\pi^0\pi^0} = 1/2 W_{(210)} + 1/5 W_{(300)}, \quad f_{\pi^+\pi^+\pi^-} = 1/2 W_{(210)} + 4/5 W_{(300)},
\]

with \( W_{(210)} + W_{(300)} = 1 \).

The weights \( W_\lambda \) depend on the dynamics. In the Fermi statistical model \(^{18, 19}\), they are proportional to the dimension of the representation \( F_\lambda \) of the permutation group \( S_n \); here \( W_{(210)}^{\text{stat}} = 2/3 \) and \( W_{(300)}^{\text{stat}} = 1/3 \).

If \( I = 2 \) states are allowed, they have to share the symmetry properties of \( I = 0 \) or \( I = 1 \) states, so that interference terms must be taken into account. It can be checked on Eq. \( \text{[2]} \) that, for \( n < 6 \), \( N_2(\lambda) \leq 1 \) for all \( \lambda \). Thus, for \( n < 6 \) and \( I \leq 2 \), the charge-configuration coefficients are determined by \( I \), \( I_3 \) and \( \lambda \) only.

Tables of the coefficients can be found in the literature \(^{13, 20}\). They are computed by explicitly constructing tensors with the required symmetry, as in the previous example, or by more sophisticated methods \(^{13, 20, 21}\).

### 4 Semileptonic decays of the \( \tau \) lepton

The possible hadronic systems\(^3\) (\( h \)) in the semileptonic decay \( \tau \to \nu h \) are: \( n\pi, \eta n\pi, Kn\pi, K\eta n\pi \), and \( K\bar{K}n\pi \).

The properties of the charged weak current imply that the total isospin is 1 for the strangeness zero final states, and 1/2 for the others. Thus the isospin of the \( n\pi \) system is 1 for \( h = n\pi \) and \( h = \eta n\pi; 0 \) or 1 for \( h = Kn\pi; 0 \), 1 or 2 for \( h = K\bar{K}n\pi \).

For all the cases but \( h = K\bar{K}n\pi \) the isospin amplitudes can be labelled by the symmetry class \( \lambda \) only and the interference terms are killed by the integration over phase-space. Thus the partial width for a given charge-configuration \( (cc) \) can be written:

\[
\Gamma_{cc} = \sum_\lambda C_{cc}^{\lambda} \Gamma^\lambda,
\]

where the coefficients \( C_{cc}^{\lambda} \) can be found in tables \(^{10, 13}\).

For positive G-parity systems \( (h = 2n\pi, h = \eta(2n + 1)\pi) \), the weak current is related by an isospin rotation to the electromagnetic current, hence relations can be established between the \( \tau \) partial widths and the cross sections of \( e^+e^- \) annihilations.

For \( h = K\bar{K}n\pi \), a more detailed analysis taking into account the interferences is needed.

\(^3\)The \( \eta \) channels have to be treated separately because \( \eta \) decay violates isospin.
4.1 $\tau \rightarrow \nu(2n + 1)\pi$

For a $3\pi$ system, there are two possible symmetry classes and the isospin constraint is Eq. 1.

Experimentally, the ratio $\pi^-\pi^-\pi^+ / \pi^-\pi^0\pi^0$ is nearly 1 because of the dominance of the $\rho\pi$ intermediate state. A detailed analysis of the final state [22], taking into account the isospin symmetry breaking caused by the difference of the $\pi^0$ and $\pi^\pm$ masses, predicts a ratio 0.985, in good agreement with the measurements [23].

For a $5\pi$ final state, three charge-configurations and four symmetry classes are present. The production of $\eta$ contributes to the $2\pi^-\pi^+\pi^0$ and $\pi^-4\pi^0$ final states.

The comparison of the measured branching ratios [23] (after $\eta$ subtraction) with the allowed domain is made in Fig. 1. It shows the dominance of $\lambda = (2, 2, 1)$, which is due to the $\omega\pi^-\pi^0$ intermediate state.

4.2 $\tau \rightarrow \nu 2n\pi$

The final states of an even number of pions are produced by the vector current, which is related by an isospin rotation to the electromagnetic current. Since the isospin rotation commutes with the permutations, the relation between $\tau$ partial widths and $e^+e^-$ cross sections [6] can be written for each symmetry class $\lambda$:

$$\frac{1}{\Gamma_{\nu e}} \frac{d\Gamma_{\nu 2n\pi}}{dm^2} = \frac{3 \cos^2 \theta_e}{2\pi\alpha^2 m^5} m^2(m^2 - m^2)^2(m^2 + 2m^2)\sigma^\lambda_{e^+e^-\rightarrow 2n\pi}(m^2). \quad (9)$$

There is only one class for $2\pi$ systems. For $4\pi$ final states the possible classes are $\lambda = (3, 1, 0)$ and $\lambda = (2, 1, 1)$. Since there are two charge-configurations for both $\tau$ decays and $e^+e^-$ annihilations, the correspondence between cross sections and partial widths is very simple:

$$\sigma_{2\pi^-2\pi^-} \leftrightarrow 2\Gamma_{\pi^-3\pi^0}$$
$$\sigma_{\pi^+\pi^-2\pi^0} \leftrightarrow \Gamma_{\pi^-\pi^+\pi^0} - \Gamma_{\pi^-3\pi^0}. \quad (9)$$
Figure 2: The decay fractions for $\tau \to \nu 6\pi$. The grey region corresponds to one standard deviation from the measured $B_{3\pi \pm 3\pi^0}/B_{5\pi \pm \pi^0}$ ($\eta$ subtracted). The hatched regions are estimations (1$\sigma$) from $e^+e^-$ annihilations data: ratio $\sigma_{e^+e^-\to 6\pi \pm}/\sigma_{e^+e^-\to 4\pi \pm 2\pi^0}$ for the diagonal band; comparison of $B_{5\pi \pm \pi^0}$ and the total cross section $\sigma_{e^+e^-\to (6\pi^-)}$ for the vertical band.

Thorough comparisons [24] of $\tau$-decay and $e^+e^-$-annihilation data for the two- and four-pion channels, including the consideration of isospin symmetry breaking, have been made recently in order to improve the theoretical determination of the muon anomalous magnetic moment $a_\mu$. They show some discrepancies between the $e^+e^-$ and $\tau$ data, as well as between different $e^+e^-$ experiments.

Four classes can contribute to the $6\pi$ states production, and only three charge-configurations are possible in $\tau$ decays as well as in $e^+e^-$ annihilations. Therefore, even with complete measurements it would not be possible to predict the partial widths from the cross sections or conversely. Nevertheless, with two measurements in $e^+e^-$ annihilations ($\sigma_{3\pi^+3\pi^-}, \sigma_{2\pi^+2\pi^-2\pi^0}$) and in $\tau$ decays ($B_{3\pi^+2\pi^-+\pi^0}, B_{2\pi^-\pi^+3\pi^0}$), it is possible to determine the contributions of the four classes, or, at least, to check the consistency of the different measurements.

Figure 2 displays the allowed region in the plane of the charge-configuration fractions and the ratio $B_{2\pi^-\pi^+3\pi^0}/B_{3\pi^-2\pi^+\pi^0}$ of the measurements [26], after $\eta$ subtraction (grey area).

The large $\sigma_{2\pi^-2\pi^+2\pi^0}/\sigma_{3\pi^-3\pi^+}$ ratio observed in $e^+e^-$ annihilations implies the dominance of $\lambda = (3,2,1)$ and/or $\lambda = (4,1,1)$, quantitatively shown [11] by the diagonal hatched region in Fig. 2. The location of the intersection of the two regions corresponds to $\lambda = (3,2,1)$, in agreement with the observed [26] importance of the $\omega$ production.

Discrepancies appear when the total cross section for $e^+e^-$ annihilations into $6\pi$ is taken into account. It allows an estimation [25] of the total $B_{6\pi}$ branching ratio, which together with the measured $B_{5\pi^+2\pi^-+\pi^0}$ gives the vertical hatched band in Fig. 2 clearly incompatible with the other estimations.

Two hypotheses can be contemplated: either the $e^+e^-$ cross sections are overestimated by a factor of roughly four, or the $e^+e^-$ annihilations into $6\pi$ receive a large contribution from $I = 0$, $\eta 3\pi$ final states. In the second hypothesis, the cross section for
Figure 3: The decay fractions for $\tau \to \nu K\pi\pi$. The experimental point is shown with the $1\sigma$ (39% probability) contour. The solid line is the isospin constraint (Eq. 10).

$e^+e^- \rightarrow \pi^+\pi^- + \text{neutrals}$ would be three times larger than $\sigma_{2\pi-2\pi+2\pi^0}$.

The contribution of the $6\pi$ channel to the estimation of $a_\mu$ is small [24], however the second hypothesis, if true, could have a not completely negligible impact on the estimation.

4.3 $\tau \to \nu Kn\pi$

For $\tau \to \nu K\pi$, the isospin $1/2$ implies the ratio $K^-\pi^0/\bar{K}^0\pi^- = 1/2$, to be compared with the experimental value [23]

$$B_{K^-\pi^0}/B_{\bar{K}^0\pi^-} = 0.51 \pm 0.04.$$  

For $Kn\pi$ systems, the number of charge-configurations is greater than the number of symmetry classes. The resulting relations between branching ratios are

$$B_{K^-\pi^+\pi^-} = \frac{1}{2}B_{K^0\pi^-\pi^0} + 2B_{K^-\pi^0\pi^0}$$

$$B_{\bar{K}^0\pi^-\pi^0} = B_{K^0\pi^-\pi^0} + 2B_{K^-\pi^0\pi^0}$$

for three and four hadron final states. The data for the $K2\pi$ final states are shown in Fig. 3. The location of the experimental point is consistent with the observed dominance of the intermediate states $K^*\pi$ and $K\rho$.

4.4 $\tau \to \nu K\bar{K}n\pi$

In a $K\bar{K}$ system, the possible values of the $K\bar{K}$ isospin are 0 and 1. For $I_{K\bar{K}} = 0$ the isospin of the $n\pi$ system is $I_{n\pi} = 1$; it can be 0, 1 or 2 for $I_{K\bar{K}} = 1$. Both axial and vector currents contribute to the decay, leading to $G = +1$ (V) and $G = -1$ (A) for the G-parity of the hadronic system, and $G_{K\bar{K}} = (-1)^nG$.

Since the $J^P$ quantum numbers are different for the axial and vector currents, there is no V-A interference term in the partial widths.
The G-parity of a $K\bar{K}$ system is related to its isospin and orbital momentum by $G_{K\bar{K}} = (-1)^{I_{K\bar{K}} + l_{K\bar{K}}}$, thus for a given (V or A) current the values of $I_{K\bar{K}}$ associated to $I_{K\bar{K}} = 0$ and $I_{K\bar{K}} = 1$ are different and there is no interference term between $I_{K\bar{K}} = 0$ and $I_{K\bar{K}} = 1$ amplitudes.

This implies [10], for any charge-configuration (cc) of the $n\pi$ system, the equality.

$$\Gamma_{K^0\bar{K}^0(n\pi,cc)} = \Gamma_{K^+K^-(n\pi,cc)}.$$  \hfill (12)

From CPT invariance, $\Gamma_{K^0\bar{K}^0(n\pi,cc)} = \Gamma_{K^0L^0(n\pi,cc)}$, but the ratio $\Gamma_{K^0\bar{K}^0(n\pi,cc)}/\Gamma_{K^0\bar{K}^0(n\pi,cc)}$ is a free parameter depending on dynamics and the respective contributions of V and A currents [10].

Figure 4 shows the agreement of the data [23, 27] with Eq. 12 in the case of the $K\bar{K}\pi$ final state. The ratio $K^+K^-\pi^+/K^0K^-\pi^0$ is found equal to 1 in agreement with the observed dominance of the intermediate state $K^*\pi$.

The isospin amplitudes for a $K\bar{K}n\pi$ final state can be labelled by $I_{K\bar{K}}$, $I_{n\pi}$ and the symmetry class $\lambda$ of the $n\pi$ system.

If no $I_{n\pi} = 2$ amplitude is associated with $\lambda$, there is no possible interference term and the class is described by a point in the space of the charge-configuration fractions.

If $I_{n\pi} = 2$ is possible, there is one interference term, but only one since $m_\pi < 2m_K + 6m_\pi$ (section 3), thus the class is described by a two-dimensional elliptic domain [11].

The allowed domain in the space of the charge-configuration fractions is the convex hull of the points and ellipses associated with the symmetry classes of the $n\pi$ system.

Fig. 5 shows a projection of this multidimensional domain in the case of $n = 2$. It implies the following inequality [11]:

$$B_{K^0K^-\pi^0\pi^0} \leq \frac{3}{4} (B_{K^0K^+\pi^-\pi^-} + B_{K^0K^-\pi^+\pi^-}),$$  \hfill (13)

which, together with the relation $B_{K^0\pi^-\pi^0\pi^0} \leq B_{K^0\pi^-\pi^-\pi^+}$ (Eq. 11), gives the constraint

$$B_{K^0h^-\pi^0\pi^0} < B_{K^0h^-h^-h^+}.$$  \hfill (14)

on topological branching ratios.
Isospin relations have been used in τ lepton physics for nearly twenty years. Their first applications \cite{6, 7} were the estimation of τ branching ratios from $e^+e^-$ annihilation data, the bounding of the contributions of unobserved channels, and the elucidation of the “one prong problem” \cite{28, 29}.

Today, a large number of decay modes are tabulated \cite{23}, the order of magnitude of the smallest measured branching ratios is $10^{-4}$, and the data from τ decays are used \cite{24} to complement and correct the information given by the $e^+e^-$ annihilations.

The observed discrepancies can only be solved by experiment. However, quoting from Blackett’s closing remark at the Bagnères conference \cite{30}: “if the history of scientific discovery is any guide, the same increase of accuracy which will serve to settle our present controversies will equally, surely, itself bring to birth new controversies by leading of some discoveries.”

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