On the non-existence for quantum LDPC codes of type IEEE802.16e with rates 1/2 and 2/3B

Manabu HAGIWARA
Research Center for Information Security (RCIS),
National Institute of
Advanced Industrial Science and Technology (AIST),
Akihabara-Daibiru 11F,
1-18-13 Sotokanda, Chiyoda-ku, Tokyo, Japan.
Email: hagiwara.hagiwara@aist.go.jp

Hideki IMAI
Advanced Industrial Science and Technology (AIST)
and Chuo University,
1-13-27 Kasuga, Bunkyo-ku,
Tokyo, Japan.
Email: h-imai@aist.go.jp

I. INTRODUCTION

In this paper, we discuss a construction of CSS codes derived from pairs of practical irregular LDPC codes. Intersection studies between quantum error-correcting codes and LDPC codes are current hot research topics [1], [3], [4], [5], [6], [7], [8], [9]. One of aims of quantum error-correcting code theory is to construct quantum error-correcting codes with small length and high error-correcting performance. The reason why LDPC codes have almost achieved a theoretical error-correction limit, called a Shannon limit, with various error-correcting performances as classical error-correcting codes. In fact, LDPC codes are current hot research topics [1], [3], [4], [5], [6].

The previous researches on quantum LDPC codes have tried to make a CSS code with a pair of classical codes of type IEEE802.16e. To our regret, we proved that it was impossible to construct a CSS code if one of classical codes was of type IEEE802.16e with rate 1/2 and 2/3B. We would like to report the discussion on its impossibility in this paper. This is the first paper to analyze the possibility of a CSS code construction by using two irregular LDPC codes which are practically useful.

II. PRELIMINARIES

A. CSS codes

CSS (Calderbank-Shor-Steane) codes are quantum codes constructed by a pair of classical linear codes $C$ and $D$ which satisfy the following condition (T), called twisted relation:

(T) $D^\perp \subseteq C$,

where $D^\perp$ is the dual code of $D$. The dual code $D^\perp$ of $D$ is defined by the following:

$$D^\perp := \{d' | d \times d'^T = 0, \forall d \in D\},$$

where $d'^T$ is the transposed vector of $d'$. A function $\langle \cdot, \cdot \rangle$, defined by $\langle x, y \rangle := x \times y^T$, gives an inner product over the binary field $\mathbb{F}_2$. The dual code $D^\perp$ is regarded as an orthogonal space of $D$ via $\langle \cdot, \cdot \rangle$. Note that $D^\perp$ is equal to a linear code generated by a parity-check matrix $H_D$ of $D$. In other words,

$$D^\perp = \{xH_D | x \in \mathbb{F}_2^m\},$$

where $m$ is the column size of $H_D$.

Details of the construction of a “quantum” CSS code by such a pair of “classical” codes $C$ and $D$ is written in [13].

Remark 2.1: In general, a pair of linear codes forming a CSS code is denoted by $C_1$ and $C_2^\perp$. For simplicity, we use $C$ (resp. $D$) instead of $C_1$ (resp. $C_2^\perp$).
B. Quasi-Cyclic LDPC Codes

A quasi-cyclic LDPC code with circulant matrices (QC-LDPC code) is defined by a binary matrix $H$ of size $m$-by-$n$, where $n$ is the length of the code and $m$ is the number of parity check bits in the code [2]. The number of systematic bits is $k = n - m$. The matrix $H$ is defined as:

$$H = \begin{bmatrix}
P_{0,0} & P_{0,1} & \ldots & P_{0,L-1} \\
P_{1,0} & P_{1,1} & \ldots & P_{1,L-1} \\
\vdots & \vdots & \ddots & \vdots \\
P_{J-1,0} & P_{J-1,1} & \ldots & P_{J-1,L-1}
\end{bmatrix}$$

where $P_{i,j}$ is one of a set of $z$-by-$z$ circulant permutation matrices or a $z$-by-$z$ zero matrix and $z$ is a positive integer. The matrix $H$ is expanded from a binary base matrix $H_b$ of size $J$-by-$L$, where $m = zJ$ and $n = zL$. The base matrix is expanded by replacing each 1 in the base matrix with a $z$-by-$z$ circulant permutation matrix, and each 0 with a $z$-by-$z$ zero matrix. The circulant permutations used are circular right shifts, and the set of permutation matrices contains the $z \times z$ identity matrix and circular right shifted versions of the identity matrix. Because each permutation matrix is specified by a single circular right shift, the binary base matrix information and permutation replacement information can be combined into a single compact model matrix $H$. The model matrix $H$ is the same size as the binary base matrix $H_b$, with each binary entry $(i, j)$ of the base matrix $H_b$ replaced to create the model matrix $H$. Each 0 in $H_b$ is replaced by a symbol $\infty$ to denote a $z \times z$ all-zero matrix, and each 1 in $H_b$ is replaced by a circular shift size $p(i, j) \geq 0$. The model matrix $H$ can then be directly expanded to $H$. Denote a circular permutation matrix of single shift by $I(1)$, i.e.

$$I(1) = \begin{bmatrix} 1 \\ 1 \\ \ddots \\ 1 \\ 1 \end{bmatrix},$$

and $I(b) = I(1)^b$. A circulant permutation $P_{i,j}$ in $H$ is $I(p(i, j))$.

**Example 2.2:** Let $H$ be a parity-check matrix of size 6-by-10 with the following form:

$$H = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}.$$  

Then $H$ defines a quasi-cyclic LDPC code with circulant matrices of size 2. The base matrix $H_b$ associated to $H$ is a matrix of size 3-by-5:

$$H_b = \begin{bmatrix}
0 & 1 & 1 & 1 & 0 \\
1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 0 & 1
\end{bmatrix}.$$  

The model matrix $H$ associated to $H$ is a matrix of size 3-by-5:

$$H = \begin{bmatrix}
\infty & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
1 & 0 & \infty & 0 & 0
\end{bmatrix}.$$  

Denote a set $\{0, 1, \ldots, z - 1\}$ by $[z]_\infty$. A model matrix is a matrix over $[z]_\infty$. We define a operator $\ominus$ on $[z]_\infty$. For $a, b \in \{0, 1, \ldots, z - 1\}$, we define $a \ominus b$ as an usual integer operation modulo $z$. For $a \in \{0, 1, \ldots, z - 1\}$, we define $a \ominus \infty = \infty - a = \infty - \infty = \infty$. For vectors $v = (v_0, v_1, \ldots, v_{L-1})$ and $u = (u_0, u_1, \ldots, u_{L-1})$, we define $v - u = (v_0 - u_0, v_1 - u_1, \ldots, v_{L-1} - u_{L-1})$. For example, we have $(\infty, 0, 1, 0, \infty) - (1, 0, 1, \infty, 0) = (\infty, 0, 0, \infty, \infty)$.

C. Quasi-Cyclic LDPC codes of type IEEE802.16e

In the standardization IEEE802.16e, quasi-cyclic LDPC codes are chosen as one of optimal error-correcting codes and six model matrices are written to define quasi-cyclic LDPC codes. For each model matrix, 19 kinds of the size of circulental matrices are chosen. Totally, 114 quasi-cyclic LDPC codes are chosen in IEEE802.16e standards. These six model matrices are designed to satisfy the following conditions. In this paper, we call a quasi-cyclic LDPC code (a LDPC matrix, a base matrix, and a model matrix) which satisfies the following condition of type IEEE802.16e.

**Remark 2.3:** The conditions below is not a characterization but a generalization of the 6 model matrices in IEEE802.16e. In fact, the set of model matrices which satisfy the condition below contains the 6 model matrices in IEEE.

The base matrix $H_b$ of a quasi-cyclic LDPC matrix is partitioned into two sections, where $H_{b1}$ corresponds to the systematic bits and $H_{b2}$ corresponds to the parity-check bits, such that $H_b = (H_{b1})_{J \times (L - J)} (H_{b2})_{J \times J}$.

Section $H_{b2}$ is further partitioned into two sections, where $h_b$ is a vector whose weight is three, and $H_{b2}'$ has a diagonal structure with matrix elements at row $i$, column $j$ equal to 1 for $i = j$, 1 for $i = j + 1$, and 0 elsewhere:

$$H_{b2} = [h_b | H_{b2}'] = \begin{bmatrix}
h_b(0) & 1 & \vdots & \vdots & \vdots & 1 \\
h_b(1) & 1 & 1 & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
h_b(J - 1) & 1 & 1 & \vdots & \vdots & \vdots & 1
\end{bmatrix}.$$  

The base matrix has $h_b(0) = 1, h_b(J - 1) = 1$, and a third value $h_b(j) = 1$ with some $0 < j < J - 1$. In particular, the non-zero sub-matrices are circularly right shifted by a particular circular shift value. Each 1 in $H_{b2}'$ is assigned a shift size of 0, and is replaced by a $z \times z$ identity matrix when expanding to $H$. The two located at the top and the bottom of $h_b$ are assigned equal shift sizes, and the third 1 in the middle of $h_b$ is given an unpaired shift size.

**Example 2.4:** We pick up two base matrices $H_b(1/2)$ and $H_b(2/3B)$ written in the standardization IEEE802.16e. The
matrix \(H_{b(1/2)}\) defines quasi-cyclic LDPC codes of rate 1/2 and has the following form:

\[
H_{b(1/2)} = \begin{bmatrix}
01100000110011000000000000
01001110010011000000000000
00111010001101100000000000
10100000110000110000000000
00100110001101000000000000
00110000110001100000000000
10100011000010001000000000
00110010001100000000000000
10001100001100000000000000
01001000110000000000000000
\end{bmatrix}.
\]

The matrix \(H_{b(2/3B)}\) defines quasi-cyclic LDPC codes of rate 2/3 and has the following form:

\[
H_{b(2/3B)} = \begin{bmatrix}
101010101010101011000000
010101010101010110000000
011010101010101010001100
101010101010101010001100
010101010101010101001110
010101010101010101001100
001101010101010101001110
010101010101010101001100
\end{bmatrix}.
\]

D. Preliminaries of the twisted condition for Quasi-Cyclic LDPC codes

Recall that a CSS code is defined by two classical linear codes. Denote the two classical codes by \(C\) and \(D\). We denote a quasi-cyclic LDPC matrix, the model matrix, and the base matrix associated with \(C\) by \(H_C\), \(H_{b(C)}\), and \(H_C\), respectively. Similarly, we use the notation \(H_\mathcal{D}\), \(H_{b(\mathcal{D})}\), and \(H_\mathcal{D}\) to denote a quasi-cyclic LDPC matrix, the model matrix, and the base matrix associated with \(D\), respectively.

In [7], a necessary and sufficient condition for QC-LDPC codes \(C\) and \(D\) to satisfy (T) in terms of the model matrices has been obtained. We quote the necessary and sufficient condition from [7]. Denote \(j\) th rows of the model matrices \(C\) and \(D\) by \(c_j\) and \(d_j\), respectively. We call a vector \(v = (v_0, v_1, \ldots, v_{L-1})\) over \(\{0, 1, \ldots, z - 1\} \cup \{\infty\}\) multiplicity-even if each symbol except for \(\infty\) appears even times in \(\{v_0, v_1, \ldots, v_{L-1}\}\). For example, a vector \((0, 0, 1, 1, 1, \infty, 1)\) is multiplicity-even.

*Theorem 2.5 (Prop. 3.1. and Sec. IV(D)): Let \(C\) and \(D\) be quasi-cyclic LDPC codes. The codes \(C\) and \(D\) satisfy (T) if and only if \(c_j - d_k\) is multiplicity even for any row \(c_j\) of the model matrix \(H_C\) and any row \(d_k\) of the model matrix \(H_D\).*

III. A NECESSARY CONDITION FOR THE TWISTED CONDITION IN TERMS OF THE BASE MATRICES

*Proposition 3.1: Let \(C\) and \(D\) be quasi-cyclic LDPC codes with circulant matrices of size \(z\). Let \(y\) be a positive integer such that \(y\) divides \(z\). Let \(C'\) (resp. \(D'\)) be a quasi-cyclic LDPC code with the model matrix same as \(C\) (resp. \(D\)) and with circulant matrices of size \(y\). If \(C\) and \(D\) satisfy the twisted condition then \(C'\) and \(D'\) satisfy the twisted condition.*

Proof: For any positive integers \(a, b\), if \(a = b \pmod{z}\) then \(a \mod{b} \pmod{y}\). It implies that if a vector \(v\) over \([z]_{\infty}\) is multiplicity even then the vector \(v\) is multiplicity even as a vector over \([y]_{\infty}\).

Let \(c_j\) (resp. \(d_k\)) be the \(i\) th row of the model matrix \(H_C\) (resp. \(H_D\)) of \(C\) (resp. \(D\)). By Theorem 2.5 \(c_j - d_k\) is multiplicity even as a vector over \([z]_{\infty}\). Thus \(c_j - d_k\) is multiplicity even as a vector over \([y]_{\infty}\).

*Corollary 3.2: Let \(C\) and \(D\) be quasi-cyclic LDPC codes with circulant matrices. Let \(H_{b(C)}\) (resp. \(H_{b(D)}\)) be the base matrix of \(C\) (resp. \(D\)). Let \(C'\) (resp. \(D'\)) be a linear code with a parity-check matrix \(H_{b(C)}\) (resp. \(H_{b(D)}\)). If \(C\) and \(D\) satisfy the twisted condition, then \(C'\) and \(D'\) satisfy the twisted condition, i.e. we have

\[
H_{b(C)} \times H_{b(D)}^T = 0.
\]

Proof: The base matrix \(H_{b(C)}\) is a quasi-cyclic LDPC matrix which has the model matrix same as that of \(C\) with circulant matrices of size 1. By Proposition 3.1 \(C'\) and \(D'\) satisfy the twisted condition.

Remember that two linear codes satisfy the twisted condition if and only if these parity-check matrices \(H_1\) and \(H_2\) are orthogonal to each other i.e.

\[
H_1 \times H_2^T = 0.
\]

Thus we have \(H_{b(C)} \times H_{b(D)}^T = 0\).

IV. NON-EXISTENCE RESULTS FOR A PAIR OF QC-LDPC CODES OF TYPE IEEE802.16E WITH RATE 1/2 AND 2/3

The column (resp. row) weight distribution is defined as the distribution of Hamming weights of the columns (resp. the rows) for a given parity-check matrix. The weight distribution is one of the important parameters for the design of LDPC codes. In fact, the performance of LDPC codes with well-optimized weight distribution is very close to the asymptotic theoretical bounds [10]. Note that the weight distribution of the parity-check matrix of quasi-cyclic LDPC code with circulant matrices is the same as that of its base matrix. There are \(t_w\) columns (resp. rows) of Hamming weight \(w\) in the base matrix if and only if there are \(t_wz\) columns (resp. rows) of Hamming weight \(w\) in associated parity-check matrix, where \(z\) is the size of the circulant matrices.

In this section, we discuss our problem, which is a construction of a pair of irregular LDPC codes to derive a CSS code, under the following conditions:

- Classical code \(C\) and \(D\) are quasi-cyclic LDPC codes with circulant matrices of size \(z\).
- The associated low-density parity-check matrices of \(C\) and \(D\) have the same “row-weight” distributions.
- The weight of columns is more than or equal to 2. This condition arises from a standard decoding method, which is called a sum-product decoding. If one of a column weight is less than 2, then the sum-product decoding does not work well [12].
In this paper, denote the condition above by (I). The condition (I) does not impose the limitation on the base matrices.

In subsection IV-A we discuss the impossibility on construction of $D$ under $H_{b(C)} := H_{b(2/3B)}$. Similarly, in subsection IV-B the impossibility on construction under $H_{b(C)} := H_{b(1/2)}$ is discussed.

A. Construction Impossibility on $D$ under $H_{b(C)} := H_{b(2/3B)}$

Let $C$ be a quasi-cyclic LDPC code with the base matrix $H_{b(C)} := H_{b(2/3B)}$.

It is required that the base matrix $H_{b(D)}$ of $D$ satisfies:

$$H_{b(C)} \times H_{b(D)}^T = 0.$$ 

It implies that any row $x \in \mathbb{F}_2^{24}$ of $H_{b(D)}$ satisfies:

$$H_{b(C)}x = 0.$$ 

Define a matrix $M$ as follows:

$$M := \begin{bmatrix} 11111111 \\ 01111111 \\ 00111111 \\ 00011111 \\ 00001111 \\ 00000111 \\ 00000011 \\ 11111110 \end{bmatrix}.$$ 

Then, for a vector $x \in \mathbb{F}_2^{24}$, we have

$$H_{b(C)}x = 0 \iff MH_{b(C)}x = 0.$$ 

By an easy calculation, we obtain

$$MH_{b(C)} = \begin{bmatrix} 000000000000000100000000 \\ 101010101010101100000000 \\ 111111111111110011000000 \\ 010101010110100001000000 \\ 000000000000000001000000 \\ 101010101010101000000000 \\ 111111111111110100000000 \\ 010101010110101000000000 \end{bmatrix}.$$ 

By considering the first and the fifth rows, we obtain a necessary condition $x_{17} = x_{21} = 0$ for $H_{b(C)}x = 0$, where $x = (x_1, x_2, \ldots, x_{24})$.

Thus the weight of 17th column and 21st column of $H_{b(D)}$ must be 0, in particular is less than 2. Therefore it is impossible to construct $D$ which satisfies the condition (I) under $H_{b(C)} = H_{b(2/3B)}$. By the result, we immediately have the

**Corollary 4.1:** Let $C$ be one of 19 quasi-cyclic LDPC codes which has the “base matrix” written as rate 2/3B codes in IEEE802.16e. Then there is no quasi-cyclic LDPC code $D$ which satisfies (I).

**Proof:** $C$ is a particular example which satisfies our condition.

B. Construction Impossibility on $D$ under $H_{b(C)} := H_{b(1/2)}$

Let $C$ be a quasi-cyclic LDPC code with its base matrix $H_{b(C)} := H_{b(1/2)}$.

By the twisted condition, for the base matrix $H_{b(D)}$ of a quasi-cyclic LDPC code $D$, the following equation is required:

$$H_{b(C)} \times H_{b(D)}^T = 0.$$ 

It implies that any row $x \in \mathbb{F}_2^{24}$ of $H_{b(D)}$ satisfies:

$$H_{b(C)}x = 0.$$ 

The weight of each row of $H_{b(D)}$ is 6 or 7. By the definition of a quasi-cyclic LDPC code of type IEEE802.16e, two nonzero entries of any row of $H_{b(D)}$, except for the below, appear in adjacent position to each other. The weight of the right-side 11 bits of a row of $H_{b(D)}$ is one or two and we denote the weight of the right-side by $\text{wt}_{13}$. 

Define three sets $X_1, X_2$ and $X_3$:

$$X_1 := \{x \in \mathbb{F}_2^{24} | \text{wt}_{13}(x) = 6 \text{ or } 7\},$$

$$X_2 := \{x \in \mathbb{F}_2^{24} | x_i = x_{i+1} = 1 \text{ for some } 14 \leq i \leq 23\},$$

$$X_3 := \{x \in \mathbb{F}_2^{24} | \text{wt}_{13}(x) = 1 \text{ or } 2\}.$$ 

For simplicity, we denote $X := X_1 \cap X_2 \cap X_3$. Then each row of $H_{b(D)}$ belongs to the set $X$.

We can determine the set $X$ by using a personal computer. In fact, we verify $X$ consists of six elements. The right-side 12-bits of each element of $X$ have the form $(0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)$. On the other hand, eleven kinds of the right-side 12-bits are required to satisfy (I). It shows that it is impossible to construct $H_{b(D)}$ which satisfies (I) under the assumption $H_{b(C)} := H_{b(1/2)}$.

By the result, we immediately have the

**Corollary 4.2:** Let $C$ be one of 19 quasi-cyclic LDPC codes which has the “base matrix” written as rate 2/3B codes in IEEE802.16e. Then there is no quasi-cyclic LDPC code $D$ which satisfies (I).

**Proof:** $C$ is a particular example which satisfies our condition.

V. Conclusion

In this paper, we discuss the conditions to construct a pair of quasi-cyclic LDPC codes, in particular of type IEEE802.16e, to derive a CSS code. The key of our discussion is to analyze the base matrices of quasi-cyclic LDPC codes. By our research, we find the impossibility to construct a pair of quasi-cyclic LDPC codes to derive a CSS code under conditions that (I) holds and one of the base matrices is the same as an IEEE802.16e LDPC base matrix with rate 1/2 or 2/3B. Note that our results are more general. The 32 quasi-cyclic LDPC codes, which have the model matrix written as rate 1/2 and rate 2/3B in IEEE802.16e, are typical examples of our results.

We should remember that six base matrices, totally 96 quasi-cyclic LDPC codes, are descried in the standardization form of IEEE802.16e. Hope has been left for the other four LDPC codes, totally 76 quasi-cyclic LDPC codes. We leave these possibilities as open problems.
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