Confinement, solitons and the equivalence between the sine-Gordon and massive Thirring models

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Abstract

We consider a two-dimensional integrable and conformally invariant field theory possessing two Dirac spinors and three scalar fields. The interaction couples bilinear terms in the spinors to exponentials of the scalars. Its integrability properties are based on the $sl(2)$ affine Kac-Moody algebra, and it is a simple example of the so-called conformal affine Toda theories coupled to matter fields. We show, using bosonization techniques, that the classical equivalence between a $U(1)$ Noether current and the topological current holds true at the quantum level, and then leads to a bag model like mechanism for the confinement of the spinor fields inside the solitons. By bosonizing the spinors we show that the theory decouples into a sine-Gordon model and free scalars. We construct the two-soliton solutions and show that their interactions lead to the same time delays as those for the sine-Gordon solitons. The model provides a good laboratory to test duality ideas in the context of the equivalence between the sine-Gordon and Thirring theories.
1 Introduction

Solitons are believed to have an important role in many non-perturbative aspects of a wide class of quantum field theories, as well as in condensed matter phenomena. The interest is greater in Lorentz invariant theories presenting topological solitons, since in many cases there are strong evidence that such solitons correspond to particle excitations in the quantum spectrum of the theory. The relevance of the solitons to non-perturbative phenomena comes, in general, from the fact that their interactions are inversely proportional to the coupling constants governing the dynamics of the fundamental fields appearing in the Lagrangean. Therefore, the solitons are weakly coupled in the strong regime of the theory, and that is the basic fact underlying several duality ideas. The reason is that one can describe the theory in the strong coupling regime, by replacing the fundamental Lagrangean by another one where the excitations of its fields correspond to the solitonic states. The electromagnetic duality of Montonen-Olive [1], involving magnetic monopoles and gauge particles, is the best example of such behaviour, and has found in supersymmetric gauge theories the best habitat for its implementation [2, 3, 4]. Similar dualities occur in statistical mechanics models and many lower dimensional field theories. The example which is best understood in such context is the quantum equivalence between the sine-Gordon and Thirring models [5, 6], which provide an excellent laboratory to test ideas about the role of solitons in quantum field theories.

In this paper we consider a two dimensional integrable and conformal field theory involving spinor and scalar fields, and presenting topological solitons solutions. The theory is one of the simplest examples of the so-called Conformal Affine Toda models coupled to matter fields proposed in [7]. It has two Dirac spinors coupled to a scalar field which plays the role of a $U(1)$ “gauge” particle. The system is made conformally invariant by the introduction of two other scalar fields constituting what is in general called in the literature a beta-gamma system, with non-positive definite kinetic term. The integrability of the theory is established using a zero curvature formulation of its equations of motion based on the $\hat{sl}(2)$ affine Kac-Moody algebra. The general solution as well as explicit one-soliton solutions have been obtained in [7]. Besides the conformal and local gauge symmetries, the model presents chiral symmetry and some discrete symmetries. However, one of its main properties is that for a large subset of solutions there is an equivalence between the $U(1)$ Noether gauge current, involving the spinors only, and the topological current associated to the scalar “gauge” particle. That fact was established in [7] at the classical level, and has profound consequences in the properties of the theory. It implies that the density of the $U(1)$ charge has to be concentrated in the regions where the scalar field has non vanishing space derivative, i.e. presents large momenta modes. The one-soliton of the theory is of the sine-Gordon type, and therefore the charge density is concentrated inside the soliton. Since the charge carriers are the spinors, it means that if one looks for excitations of the theory around the solitonic state, one would expect the spinors to be confined.

One of the main objectives of this paper is to study the equivalence of the Noether and topological currents at the quantum level, and its consequences for the confinement mechanism. The first thing to be established is if such equivalence is not spoiled by quantum anomalies the currents may present. Fortunately, that issue can be established exactly by using, instead of perturbative approaches, bosonization techniques following the lines of [8].
By bosonizing the spinor field we show that the sector of the theory made of the spinor and "gauge" particles is equivalent to a theory of a free massless scalar and a sine-Gordon field. In addition, the condition for the equivalence of the Noether and topological currents is simply the condition that the free massless scalar should be constant. Therefore, by performing a quantum reduction where the excitations of the free scalar are eliminated, we obtain a submodel where the equivalence of those currents holds true exactly at the quantum level, proving that there are no anomalies. Consequently, we show that in such reduced theory the confinement of the spinor particles does take place.

An important property of the model under consideration is that it possesses another type of spinor particle. That is obtained by fermionizing the sine-Gordon field using the well known equivalence between the sine-Gordon and massive Thirring models [5, 6]. In that scenario the solitons of the sine-Gordon model are interpreted as the spinor particles of the Thirring theory. We are then lead to an interesting analogy with what one expects to happen in QCD. The original spinor particles of our model that get confined inside the solitons play the role of the quarks, and the second spinor particle (Thirring), which are the solitons, play the role of the hadrons. The $U(1)$ Noether charge is also confined and is analogous to color in QCD. In this sense, our model constitute an excellent laboratory to test ideas about confinement, the role of solitons in quantum field theory, and dualities interchanging the role of solitons and fundamental particles.

We also constructed the one-soliton and two-soliton solutions for our theory using two techniques: the dressing transformations and the Hirota’s method. We use the zero curvature representation of the equations of motion with potentials taking value on the affine Kac-Moody algebra $\widehat{sl}(2)$ (see appendix). Then the basic idea is to look for vacuum configurations where the potentials lie in an abelian (up to central terms) subalgebra of $\widehat{sl}(2)$. That constitutes in fact an algebra of oscillators. The solitons are obtained by performing the dressing transformations from those vacuum configurations with group elements which are exponentiations of the eigenvectors of the oscillators [9, 10, 11]. Such procedure leads quite naturally to the definition of tau-functions [9], and then the Hirota’s method can be easily implemented too. We discuss the conditions for the solutions to be real, and evaluate the topological charges. The interactions of the solitons is studied by calculating their time delays. It is attractive, in fact the time delays are the same as those for the sine-Gordon solitons. An interesting aspect of the solutions is that when the two Dirac spinors of the theory are related by a reality condition, then either the soliton or the anti-soliton disappears from the spectrum. We interpret that as indicative of the existence of a duality involving the solitons and the spinor particles.

The paper is organized as follows: in section 2 we summarize the properties of the model, introduced in [7], at the classical level. In section 3 we consider the corresponding quantum field theory and show, using bosonization techniques, the equivalence of the Noether and topological currents as well as the confinement mechanism. The soliton solutions and their interaction (time delays) are studied in section 4. In the appendix we give the zero curvature representation for the model under consideration.
2 The model

Consider the two-dimensional field theory defined by the Lagrangian \[ \frac{1}{k} \mathcal{L} = -\frac{1}{4} \partial_{\mu} \varphi \partial^{\mu} \varphi + \frac{1}{2} \partial_{\mu} \nu \partial^{\mu} \eta - \frac{1}{8} m^2_{\psi} e^{2\eta} + i \bar{\psi} \gamma^\mu \partial_{\mu} \psi - m_{\psi} \bar{\psi} e^{\eta + 2i \varphi} \gamma^5 \psi \] (2.1)

where \( \varphi, \eta \) and \( \nu \) are scalar fields, and \( \psi \) is a Dirac spinor. Notice that, \( \bar{\psi} \equiv \bar{\psi}^T \gamma_0 \), with \( \bar{\psi} \) being a second Dirac spinor. However, in many applications in this paper we shall take \( \bar{\psi} \) to satisfy the reality condition

\[ \bar{\psi} = c_\psi \psi^* \] (2.2)

where \( c_\psi \) is a real dimensionless constant. As we will show, the sign of \( c_\psi \) will have an important role in determining the spectrum of soliton solutions. The corresponding equations of motion are

\[ \partial^2 \varphi = i4m_{\psi} \bar{\psi} \gamma_5 e^{\eta + 2i \varphi} \gamma^5 \psi, \] (2.3)

\[ \partial^2 \nu = -2m_{\psi} \bar{\psi} e^{\eta + 2i \varphi} \gamma^5 \psi - \frac{1}{2} m^2_{\psi} e^{2\eta}, \] (2.4)

\[ \partial^2 \eta = 0, \] (2.5)

\[ i\gamma^\mu \partial_{\mu} \bar{\psi} = m_{\psi} \bar{\psi} e^{\eta + 2i \varphi} \gamma^5 \psi, \] (2.6)

\[ i\gamma^\mu \partial_{\mu} \bar{\psi} = m_{\psi} \bar{\psi} e^{\eta - 2i \varphi} \gamma^5 \bar{\psi}, \] (2.7)

The theory (2.1) was proposed in [7] as an example of a wide class of integrable theories called Affine Toda systems coupled to matter fields\[^1\]. The zero curvature representation, the construction of the general solution including the solitonic ones and many other properties were discussed in [4]. In the appendix [8] we summarize some of those results. Here, we want to discuss some special features of that theory at the quantum level as well as the two-soliton solutions. We start by reviewing the symmetries of (2.1).

Conformal symmetry. The model (2.1) is invariant under the conformal transformations\[^2\]

\[ x_+ \to \hat{x}_+ = f(x_+), \quad x_- \to \hat{x}_- = g(x_-), \] (2.8)

with \( f \) and \( g \) being analytic functions; and with the fields transforming as \[^3\]

\[ \varphi(x_+, x_-) \to \hat{\varphi}(\hat{x}_+, \hat{x}_-) = \varphi(x_+, x_-), \]

\[ e^{-\nu(x_+, x_-)} \to e^{-\nu(\hat{x}_+, \hat{x}_-)} = (f)^{\delta}(g)\delta e^{-\nu(x_+, x_-)}, \]

\[ e^{-\eta(x_+, x_-)} \to e^{-\eta(\hat{x}_+, \hat{x}_-)} = (f)^{\frac{\delta}{2}}(g)^{\frac{\delta}{2}} e^{-\eta(x_+, x_-)}, \]

\[ \psi(x_+, x_-) \to \hat{\psi}(\hat{x}_+, \hat{x}_-) = e^{\frac{\delta}{2}(1+\gamma_5)} \log(f)^{\frac{\delta}{2}}(g)^{\frac{\delta}{2}} \psi(x_+, x_-), \] (2.9)

where the conformal weight \( \delta \), associated to \( e^{-\nu} \), is arbitrary, and \( \bar{\psi} \) transforms in the same way as \( \psi \).

\[^1\]The Lagrangian (2.1) is obtained from (10.18) of [4] by the replacements \( \varphi \to i \varphi \) and \( \nu \to \nu - \frac{i}{2} \varphi \).

\[^2\]We are using \( x_\pm = t \pm x \), and so, \( \partial_\pm = \frac{1}{2} (\partial_t \pm \partial_x) \), and \( \partial^2 = \partial_t^2 - \partial_x^2 = 4 \partial_t \partial_x \).

\[^3\]We take \( \gamma_0 = -i \left[ \begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array} \right], \gamma_1 = -i \left[ \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right], \gamma_5 = \gamma_0 \gamma_1 = \left[ \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right] \).
Left-right local symmetries. The Lagrangian (2.1) is invariant under the local $U(1)_L \otimes U(1)_R$ transformations
\[ \phi \rightarrow \phi + \xi_+ (x_+) + \xi_- (x_-) ; \quad \nu \rightarrow \nu ; \quad \eta \rightarrow \eta \] (2.10)
and
\[ \psi \rightarrow e^{-i(1+\gamma_5)\xi_+(x_+)+i(1-\gamma_5)\xi_-(x_-)} \psi ; \quad \bar{\psi} \rightarrow e^{i(1+\gamma_5)\xi_+(x_+)-i(1-\gamma_5)\xi_-(x_-)} \bar{\psi} \] (2.11)

$U(1)$ global symmetry. Notice that, by taking $\xi_+ (x_+) = -\xi_- (x_-) = -\frac{1}{2} \theta$, with $\theta = \text{const}$., one gets a global $U(1)$ transformation
\[ \phi \rightarrow \phi ; \quad \nu \rightarrow \nu ; \quad \eta \rightarrow \eta ; \quad \psi \rightarrow e^{i\theta} \psi ; \quad \bar{\psi} \rightarrow e^{-i\theta} \bar{\psi} \] (2.12)
The corresponding Noether current is given by
\[ J^\mu = \bar{\psi} \gamma^\mu \psi, \quad \partial_\mu J^\mu = 0. \] (2.13)

Chiral symmetry. In addition, if one takes $\xi_+ (x_+) = \xi_- (x_-) = -\frac{1}{2} \alpha$, with $\alpha = \text{const}$., one gets the global chiral symmetry
\[ \psi \rightarrow e^{i\gamma_5 \alpha} \psi ; \quad \bar{\psi} \rightarrow e^{-i\gamma_5 \alpha} \bar{\psi} ; \quad \phi \rightarrow \phi - \alpha ; \quad \nu \rightarrow \nu ; \quad \eta \rightarrow \eta \] (2.14)
with the corresponding Noether current
\[ J_5^\mu = \bar{\psi} \gamma_5 \gamma^\mu \psi + \frac{1}{2} \partial^\mu \phi ; \quad \partial_\mu J_5^\mu = 0 \] (2.15)

Topological charge. One can shift the $\phi$ field as $\phi \rightarrow \phi + n\pi$, keeping all the other fields unchanged, that the Lagrangian is left invariant. That means that the theory possesses an infinite number of vacua, and the topological charge
\[ Q_{\text{topol.}} \equiv \int dx j^0, \quad j^\mu = \frac{1}{\pi} \epsilon^\mu_{\nu\rho} \partial_\nu \phi \] (2.16)
can assume non trivial values.

CP-like symmetry. Finally the Lagrangian (2.1) is invariant under the transformation
\[ x_+ \leftrightarrow x_- ; \quad \psi \leftrightarrow i \epsilon \gamma_0 \bar{\psi} ; \quad \bar{\psi} \leftrightarrow -i \epsilon \gamma_0 \psi ; \quad \phi \leftrightarrow \phi ; \quad \eta \leftrightarrow \eta ; \quad \nu \leftrightarrow \nu \] (2.17)
where $\epsilon = \pm 1$. Notice that by imposing the reality condition (2.2) one breaks such CP symmetry, for any real value of the constant $\epsilon_\psi$.

Now comes a very interesting property of this model. The conservation of the $U(1)$ vector current (2.13) and of the chiral current (2.15) can be used to show that there exist two charges given by
\[ J = -\bar{\psi}^T (1 + \gamma_5) \psi + \partial_+ \phi, \quad \bar{J} = \bar{\psi}^T (1 - \gamma_5) \psi + \partial_- \phi \] (2.18)
\[ ^4\text{Notice that the CP symmetry is preserved if one takes } \epsilon_\psi = \pm i. \]
satisfying
\[ \partial_- \mathcal{J} = 0 ; \quad \partial_+ \mathcal{J} = 0 \] (2.19)

Notice, from (2.9), that the currents \( \mathcal{J} \) and \( \mathcal{J} \) have conformal weights \((1, 0)\) and \((0, 1)\) respectively. One can now perform a (Hamiltonian) reduction of the model by imposing the constraints
\[ \mathcal{J} = 0 ; \quad \mathcal{J} = 0 \] (2.20)

The degree of freedom eliminated by such reduction does not correspond to the excitations of any of the fields appearing in the Lagragian (2.1). As we show below, it corresponds to the excitations of a free field which is a non-linear combination of those in (2.1).

One can easily check that the constraints (2.20) are equivalent to
\[ \frac{1}{2\pi} e^{\mu \nu} \partial_\nu \varphi = \frac{1}{\pi} \bar{\psi} \gamma^\mu \psi, \] (2.21)

Therefore, in the reduced model, the Noether current (2.13) is proportional to the topological current (2.16).

That fact has profound consequences in the properties of such theory. For instance, it implies (taking \( \bar{\psi} \) to be the complex conjugate of \( \psi \)) that the charge density \( \psi^\dagger \psi \) is proportional to the space derivative of \( \varphi \). Consequently, the Dirac field is confined to live in regions where the field \( \varphi \) is not constant. The best example of that is the one-soliton solution of (2.1) which was calculated in [7] and it is given by
\[
\varphi = 2 \arctan \left( \exp \left( 2m \psi (x - x_0 - vt) / \sqrt{1 - v^2} \right) \right)
\]
\[
\psi = e^{i\theta} \sqrt{m} \exp \left[ m \psi (x - x_0 - vt) / \sqrt{1 - v^2} \right] \left( \frac{1 - v}{1 + v} \right)^{1/4} \left( \frac{1}{1 + ie^{2m \psi (x - x_0 - vt) / \sqrt{1 - v^2}}} \right) \left( \frac{1}{1 - ie^{2m \psi (x - x_0 - vt) / \sqrt{1 - v^2}}} \right)
\]
\[
\nu = -\frac{1}{2} \log \left( 1 + \exp \left( 4m \psi (x - x_0 - vt) / \sqrt{1 - v^2} \right) \right) - \frac{1}{8} m^2 \psi x^+ x_-
\]
\[
\eta = 0
\] (2.22)

and the solution for \( \bar{\psi} \) is the complex conjugate of \( \psi \). Notice that, from (2.16), one indeed has \( Q_{\text{topol.}} = 1 \) for the solution (2.22).^5

Notice that the solution for \( \varphi \) is exactly the sine-Gordon soliton, and therefore \( \partial_\varphi \) is non-vanishing only in a region of size of the order of \( m_\psi^{-1} \). In addition, the solution for \( \psi \) satisfies the massive Thirring model equations of motion [12]. One can check that (2.22) satisfies (2.21), and so is a solution of the reduced model. Therefore, the Dirac field must be confined inside the soliton. One can verify that (2.22) indeed confirms that fact.

We point out that the condition (2.21) together with the equations of motion for the Dirac spinors (2.6)-(2.7) imply the equation of motion for \( \varphi \), namely (2.3). Therefore in the reduced model, defined by the constraints (2.20), one can replace a second order differential equation, i.e. (2.3), by two first order equations, i.e. (2.21).

^5 Notice that we have defined the topological charge in (2.16) as twice that of the reference [7], in order to make it integer.
One could think of equating the chiral current \( (2.15) \) to a topological current associated to a scalar field. However, the equations of motion imply that such a scalar has to be a free field. Therefore, one could consider a submodel of \( (2.1) \) defined by the constraints

\[
\lambda \epsilon^{\mu \nu} \partial_\nu \eta = \bar{\psi} \gamma_5 \gamma^\mu \psi + \frac{1}{2} \partial^\mu \varphi
\]

where \( \lambda \) is a parameter. One can check that such constraints are compatible with the equations of motion. However, we will not use that observation in what follows.

### 3 The quantum theory

The results discussed above are true at the classical level. The question we address now is if they remain true at the quantum level, and the confinement of the Dirac spinor does take place. The answer is affirmative, and can be obtained quite elegantly using bosonization methods \[4, 5\]. Witten has considered a similar model in \[8\], originally proposed by Kogut and Sinclair \[13\], and we shall use his methods. Their model differs from \( (2.1) \) in three points:

1. it does not contain the pair of fields \((\eta, \nu)\), and therefore is not conformally invariant;
2. the sign of the kinetic term of the \( \varphi \) field in \( (2.1) \) is flipped with respect to their corresponding term;
3. their model possesses just one Dirac spinor.

In the considerations about the quantum theory, to be made in this section, we shall impose the reality condition \( (2.2) \) and consequently the Lagrangian \( (2.1) \) will depend on the real constant \( e_\psi \). Therefore, in this section we have \( \bar{\psi} = \psi^\dagger \gamma_0 \).

We introduce a new boson field \( c \) by bosonizing the Dirac spinor \( \psi \) as \[8, 14\]

\[
i : \bar{\psi} \gamma^\mu \partial_\mu \psi : = \frac{\alpha^2}{2\pi} : (\partial_\mu c)^2 : \]

\[
: \bar{\psi}(1 \pm \gamma_5)\psi : = -\frac{\mu}{2\pi} : e^{(\pm i\alpha c)} : \]

\[
: \bar{\psi}^\gamma \mu \psi : = -\frac{\alpha}{\pi} e^{\mu \nu} \partial_\nu c
\]

where \( \alpha \) is a real parameter, and \( \mu \) is a mass parameter used as an infrared regulator.

Rewriting the last term of \( (2.1) \) as

\[
\bar{\psi} e^{\eta + 2i\varphi} \gamma_5 \psi = e^{\eta} \left( \frac{\psi (1 + \gamma_5)}{2} e^{2i\varphi} + \frac{\psi (1 - \gamma_5)}{2} e^{-2i\varphi} \right)
\]

one then obtains that the Lagrangian \( (2.1) \) becomes

\[
\frac{1}{k} \mathcal{L} = -\frac{1}{4} (\partial_\mu \varphi)^2 + \frac{1}{2} \partial_\mu \nu \partial^\nu \eta - \frac{m_\psi^2}{8} e^{2\eta}
\]

\[
+ e_\psi \left( \frac{\alpha^2}{2\pi} (\partial_\mu c)^2 + \frac{\mu m_\psi}{4\pi} e^{\eta} \left( e^{2i(\alpha c + \varphi)} + e^{-2i(\alpha c + \varphi)} \right) \right)
\]

(3.5)

Introduce the linear combinations

\[
\phi \equiv 2 \frac{\alpha c + \varphi}{\sqrt{4\pi - 8e_\psi}} ; \quad \rho \equiv \sqrt{\frac{2}{\pi}} \frac{(2e_\psi \alpha c + \pi \varphi)}{\sqrt{4\pi - 8e_\psi}}
\]

(3.6)
After rescaling the fields as
\[
\rho \to \frac{\rho}{\sqrt{k}}, \quad \nu \to \frac{\nu}{k}, \quad \phi \to \frac{\phi}{\sqrt{k}}
\] (3.7)
we rewrite the Lagrangian as
\[
\mathcal{L} = -\frac{1}{2} \epsilon (e_\psi) (\partial_\mu \rho)^2 + \frac{1}{2} \partial_\mu \nu \partial^\mu \eta - \frac{k}{8} m_\psi^2 e^2 \eta + e_\psi \left( \frac{1}{2} \epsilon (e_\psi) (\partial_\mu \phi)^2 + \frac{m^2}{\beta^2} e^\eta \cos (\beta \phi) \right)
\] (3.8)
where we have introduced
\[
\beta \equiv \sqrt{\frac{4\pi - 8e_\psi}{k}}; \quad m^2 \equiv \frac{\mu}{2\pi} \left( 4\pi - 8e_\psi \right) m_\psi; \quad \epsilon (e_\psi) \equiv \text{sign} \left( 4\pi - 8e_\psi \right)
\] (3.9)
Therefore, \( \rho \) is a free scalar field and \( \phi \) is a sine-Gordon field coupled to \( \eta \). However, in order for the kinetic and potential terms for \( \phi \) to lead to a unitary sine-Gordon theory we shall impose
\[
\epsilon (e_\psi) = 1; \quad e_\psi < \frac{\pi}{2}
\] (3.10)
Clearly, in the limit \( e_\psi \to \pi/2 \), \( \phi \) becomes a massless free field.

We now discuss the quantum version of the reduction (2.20). Since \( \tilde{\psi}^T (1 \pm \gamma_5) \psi = e_\psi \tilde{\bar{\psi}} (\gamma_0 \pm \gamma_1) \psi \) (see (2.2)), it follows from (2.18), (3.3) and (3.6) that \( (\epsilon^{01} = 1) \)
\[
\mathcal{J} = \sqrt{\frac{|4\pi - 8e_\psi|}{2\pi}} \partial_+ \rho; \quad \mathcal{\tilde{J}} = \sqrt{\frac{|4\pi - 8e_\psi|}{2\pi}} \partial_- \rho
\] (3.11)
Therefore, the constraints (2.20) are equivalent to \( \rho = \text{const.} \), and consequently the degree of freedom eliminated by them corresponds to the \( \rho \) field.

The reduction at the quantum level can be realized as follows. Since the scalar field \( \rho \) is decoupled from all other fields in the theory (3.8), we can denote the space of states as \( \mathcal{H} = \mathcal{H}_\rho \otimes \mathcal{H}_0 \), where \( \mathcal{H}_\rho \) is the Fock space of the free massless scalar \( \rho \), and \( \mathcal{H}_0 \) carries the states of the rest of the theory. We shall denote \( \rho = \rho^+ + \rho^- \), where \( \rho^+ \) \( (\rho^-) \) corresponds to the part containing annihilation (creation) operators in its expansion on plane waves. The reduction corresponds to the restriction of the theory to those states satisfying
\[
\partial_+ \rho^+ | \Psi \rangle = 0
\] (3.12)
By taking the complex conjugate one gets
\[
\langle \Psi | \partial_- \rho^- = 0
\] (3.13)
and consequently the expectation value of \( \partial_\pm \rho \) vanishes on such states. Indeed, if \( | \Psi \rangle \) and \( | \Psi' \rangle \) are two states satisfying (3.12), then
\[
\langle \Psi' | \partial_+ \rho | \Psi \rangle = \langle \Psi' | \partial_+ \rho^+ + \partial_+ \rho^- | \Psi \rangle = 0
\] (3.14)
That provides the correspondence with the classical constraints (2.20). Therefore, the Hilbert space of the reduced theory is \( \mathcal{H}_c = | \Psi \rangle \otimes \mathcal{H}_0 \).
For the theory described by $\mathcal{H}_c$, the equivalence between Noether and topological currents, given by (2.21), holds true at the quantum level, since as we have shown before, (2.21) is equivalent to the vanishing of the currents $\mathcal{J}$ and $\bar{\mathcal{J}}$. Notice that such quantum equivalence is exact, since we have not used perturbative or semiclassical methods.

One of the consequences of that quantum equivalence is that in the states of $\mathcal{H}_c$, like the one-soliton (2.22), where the the space derivative of $\varphi$ is localized, one has that the spinor $\psi$ is confined, since from (2.21), $\langle \partial_x \varphi \rangle \sim \langle \psi^\dagger \psi \rangle$. Therefore, we have shown that the confinement of $\psi$ does take place in the quantum theory.

Notice that the quantum reduction (3.12) does not violate the conformal symmetry of the Lagrangian (3.8), because we are not restricting to a non vanishing constant, the expectation value of any quantity of non vanishing conformal weight.

The theory described by (3.8) is certainly non-unitary because the kinetic terms have in general different signs, and so the energy is unbounded from below. The reduction (3.12) eliminates part of the problem, since the kinetic energy associated to the $\rho$ field vanishes in $\mathcal{H}_c$. There remains the non-unitarity associated with the system $(\eta, \nu)$. We can perform a further reduction by restricting ourselves to those states of $\mathcal{H}_0$ satisfying

$$\partial_{\pm} \eta^{(+)} \mid \Psi_0 \rangle = 0 \quad (3.15)$$

where we have denoted the free field $\eta$ as $\eta = \eta^{(+)} + \eta^{(-)}$, with $\eta^{(+)}$ ($\eta^{(-)}$) corresponding to the part containing annihilation (creation) operators in the expansion in terms of plane waves. Consequently, using similar arguments as before, the expectation value of $\partial_{\pm} \eta$ vanishes on such states,

$$\langle \Psi_0 \mid \partial_{\pm} \eta \mid \Psi_0 \rangle = 0 \quad (3.16)$$

In addition, it follows that

$$\langle \Psi_0 \mid e^{\eta} \mid \Psi_0 \rangle = \text{const.} \quad (3.17)$$

Notice that the operator $e^{\eta}$ has conformal weights $\left(-\frac{1}{2}, -\frac{1}{2}\right)$, and since it is present in the Lagrangian (3.8), it follows that such reduction does break the conformal symmetry. Such breaking of symmetry resembles the Higgs mechanism since it generates mass for the $\phi$ field, and that is proportional to the expectation value of $e^{\eta}$.

After the reductions (3.12) and (3.15), the $\nu$ field decouples and we are left with a unitary theory which is that of the sine-Gordon model of the $\phi$ field. In fact, as Coleman has shown, the sine-Gordon theory is unbounded below if $\beta^2 > 8\pi$. Therefore, from (3.9) we have to have

$$\beta^2 < 8\pi \quad \rightarrow \quad k > \frac{|\pi - 2e_\psi|}{2\pi} \quad (3.18)$$

An important property of the model under consideration is that it presents another spinor field which is not confined. Indeed, using bosonization rules again, we can introduce a spinor $\chi$ as

$$i : \bar{\chi} \gamma^\mu \partial_\mu \chi : = \frac{\beta^2}{8\pi} : (\partial \phi)^2 : \quad (3.19)$$

**Notice that for the states of $\mathcal{H}_c$, where (3.14) holds true, one has from (3.6) that $\langle \partial_{\pm} \phi \rangle \sim \langle \partial_{\pm} \varphi \rangle$.**

**For the choice $e_\psi < 0$, the kinetic terms for $\rho$ and $\phi$ have the same sign.**
\[
\chi : \bar{\chi} \chi = -\frac{\bar{\mu}}{2\pi} \cos (\beta \phi) \quad (3.20)
\]

\[
\tilde{\chi} \gamma^\mu \chi = -\frac{\beta}{2\pi} \epsilon_{\mu\nu} \partial_\nu \phi \quad (3.21)
\]

The Lagrangian (3.8) becomes (assuming (3.10))

\[
\mathcal{L} = -\frac{1}{2} (\partial_\mu \rho)^2 + \frac{1}{2} \partial_\mu \nu \partial^\mu \eta - \frac{k}{8} m^2_\psi e^{2\eta} + e_\psi \left( i \bar{\chi} \gamma^\mu \partial_\mu \chi - m_\chi e^{\eta} \bar{\chi} \chi - \frac{g}{2} (\bar{\chi} \gamma^\mu \chi)^2 \right) \quad (3.22)
\]

with

\[
\frac{4\pi}{\beta^2} = 1 + \frac{g}{\pi} \quad m_\chi = \frac{\mu}{\bar{\mu}} km_\psi \quad (3.23)
\]

Therefore, we get a Thirring model coupled to \( \eta \) plus the free scalar field \( \rho \). After the reductions (3.12) and (3.15) the theory becomes the pure massive Thirring model. Notice that \( \chi \) becomes a free massive spinor [5] for \( \beta^2 = 4\pi \) or \( k = | \pi - 2e_\psi | /\pi \).

The properties of the theory (2.1) at the quantum level are quite remarkable. In the weak coupling regime, i.e. small \( k \), the excitations around the vacuum correspond to the spinor \( \psi \) and the “gauge” particle \( \varphi \). The \( U(1) \) symmetry (2.12) is not broken and the charged states correspond to the \( \psi \) particles. Consider now those states satisfying (3.12) and look for the fluctuations around the state corresponding, for instance, to the one-soliton solution (2.22). The \( \psi \) particles disappear from the spectrum since they are confined inside the soliton. The \( \psi \) particles can live outside the soliton only in bound states with vanishing \( U(1) \) charge.

The theory, however, presents another spinor particle corresponding to the excitations of the Thirring field \( \chi \), which have zero \( U(1) \) charge. However, according to Coleman’s interpretation of the sine-Gordon/Thirring equivalence, such excitations correspond to the solitons themselves. Therefore, we can make an analogy with what is expected to happen in QCD. The \( \psi \) and \( \chi \) particles are like the quarks and hadrons respectively. The \( U(1) \) charge is analogous to color in QCD, since it is also confined.

Another important point, first observed in [8], is that although the theory (2.1) presents the chiral symmetry (2.14), it does present massive spinor states. The reason is quite simple and elegant. Using the bosonization rule (3.3) and the field redefinitions (3.6), one gets that the chiral current (2.13) can be written as

\[
J^\mu_5 = \sqrt{\frac{| \pi - 2e_\psi |}{2\pi}} \partial^\mu \rho \quad (3.24)
\]

Therefore, the spinor \( \chi \) does not contribute to such current and so has zero chirality. Therefore, the fact that it does acquire a mass is not incompatible with the chiral symmetry. The spinor with non zero chirality is \( \psi \) and it disappears from the spectrum in the confining sector of the theory. The possibility of having chiral symmetry and massive fermions is another remarkable property of the theory (2.1).

For these reasons the model (2.1) constitute an excellent laboratory to test ideas about confinement, the role of solitons in quantum field theories, and duality transformations interchanging the role of solitons and fundamental particles. With that motivation we shall now study the two-soliton solutions and their interactions.
4 Soliton solutions

It is well known the relevance of localized classical solutions of non-linear relativistic field equations to the corresponding quantum theories. In particular, solitons can be associated with quantum extended-particle states. Therefore, we will examine the classical soliton type solutions to get insight into the quantum spectrum of the model, in much the same way of what is already known in the remarkable sine-Gordon model. We argue that, at the classical level, the solutions for the \( \varphi \) and \( \psi \) fields share some features of the sine-Gordon and the massive Thirring theories, respectively. We have shown in the previous section, using bosonization methods, that at the quantum level the theory (2.1) is equivalent to (3.8). However, the space of classical solutions of the two theories are not the same. One cannot use (3.1)-(3.3), at the classical level, as change of variables to relate classical solutions of one theory to those of the other. The bosonization rules (3.1)-(3.3) are purely quantum relations.

The one-soliton solutions of (2.1) have been calculated in \([7]\) using the dressing transformation method. Here, we construct the two-soliton solutions using the same methods. We start with the zero curvature representation of the equations of motion of the theory

\[
[\partial_+ + A_+, \partial_- + A_-] = 0
\]  
(4.1)

where the indices \( \pm \) stand for the light cone variables \( x_\pm = t \pm x \). For the theory (2.1) the connections \( A_\pm \) live on the \( \hat{sl}(2) \) affine Kac-Moody algebra \( \hat{G} \), and are given in the appendix \( \ref{A} \) (see \( \ref{A.1} \)). Since (4.1) imply the connections must be flat, one can write them as \( A_\mu = -\partial_\mu T T^{-1} \), with \( T \) being a group element obtained by exponentiating \( \hat{G} \). In addition, using an integral gradation of \( \hat{G} \), i.e. \( \hat{G} = \oplus_n \hat{G}_n \), \( n \in \mathbb{Z} \), one can perform a generalized Gauss decomposition of the element \( T\rho T^{-1} \), with \( \rho \) being a given constant group element. Denote by \( \hat{G}_{<0} \), \( \hat{G}_0 \) and \( \hat{G}_{>0} \) the subalgebras generated by elements of grades negative, zero and positive respectively. Then

\[
T\rho T^{-1} = \left(T\rho T^{-1}\right)_{<0} \left(T\rho T^{-1}\right)_0 \left(T\rho T^{-1}\right)_{>0}
\]  
(4.2)

where \( (T\rho T^{-1})_{<0} \), \( (T\rho T^{-1})_0 \) and \( (T\rho T^{-1})_{>0} \) are elements belonging to subgroups whose algebras are \( \hat{G}_{<0} \), \( \hat{G}_0 \) and \( \hat{G}_{>0} \) respectively. One now introduces

\[
T^\rho \equiv \left(T\rho T^{-1}\right)_{>0} T = \left(T\rho T^{-1}\right)_0^{-1} \left(T\rho T^{-1}\right)_{<0} T\rho
\]  
(4.3)

Such relation defines a transformation on the connections

\[
A_\mu = -\partial_\mu T T^{-1} \quad \rightarrow \quad A^\rho_\mu = -\partial_\mu T^\rho T^\rho T^{-1}
\]  
(4.4)

which preserves their grading structure. Therefore, if one knows a solution \( T \) for the zero curvature one obtains a new solution \( T^\rho \), determined by the constant group element \( \rho \). We point out that the fact that the transformed element \( T^\rho \) can be written in two different ways as in (4.3) plays a crucial role in the dressing method. It guarantees that the transformed connection is in the same gauge as the original one, and therefore allows to translate the dressing transformation (4.4) into transformation of the physical fields defining the theory.
A quite general procedure to construct soliton solutions in integrable theories, using such
dressing transformations, is described in [9]. It constitutes a generalization of the so-called
“solitonic specialization” in the context of the Leznov-Saveliev solution for Toda type models
[10, 11, 16]. One starts by looking for a “vacuum” solution such that the connections
\( A_{\pm} \), when evaluated on it, belong to an algebra of oscillators, i.e. an abelian (up to central
terms) subalgebra of the Kac-Moody algebra. One then looks for the eigenvectors \( V_i \), in \( \hat{G} \),
of such oscillators. The solitons belong to the orbits of solutions obtained by the dressing
transformation performed by elements of the form 
\( \rho = e^{V_1} e^{V_2} \ldots e^{V_n} \).

For the theory (2.1) we perform the dressing transformation starting from the vacuum
solution 
\( \varphi = \psi = \bar{\psi} = \eta = 0, \quad \nu = -\frac{1}{8} m_\psi x_+ x_- \equiv \nu_0 \) (4.5)
The connection evaluated on such solution is given by (see (A.1))
\[
A_+^{\text{vac}} = -E_2 \quad A_-^{\text{vac}} = E_2 + \frac{1}{8} m_\psi^2 x_+ C
\] (4.6)
In addition, one has
\[
A_+^{\text{vac}} = -\partial_\pm T_0 T_-^{\text{vac}}^{-1} \quad \text{with} \quad T_0 = e^{x_+ E_2} e^{-x_- E_2}
\] (4.7)
Since \( E_\pm = \frac{1}{4} m_\psi H_\pm \) (see (A.2)) the relevant oscillators are \( H_\pm \) and their algebra is
given by (A.5). The eigenvectors of the oscillators are given by
\[
V_\pm(z) = \sum_{n \in \mathbb{Z}} z^{-n} E_\pm^n
\] (4.8)
and
\[
[E_2, V_\pm(z)] = \pm \frac{1}{2} m_\psi z V_\pm(z),
\]
\[
[E_2, V_\pm(z)] = \pm \frac{1}{2} m_\psi \frac{1}{z} V_\pm(z)
\] (4.9)
Notice that \( V_+(z) \) and \( V_-(z) \) have the same eigenvalues, and such degeneracy is related
to the global \( U(1) \) symmetry discussed in (2.12) [7].

The solutions in the orbit of the vacuum (4.5), under the dressing transformations, are
given by [7]
\[
e^{-i\varphi} = \frac{\tau_1}{\tau_0}, \quad e^{-(\nu-\nu_0) - \frac{i}{4} \varphi} = \tau_0
\] (4.10)
and
\[
\psi = \sqrt{\frac{m_\psi}{4i}} \left( \begin{array}{c}
\tau_R/\tau_0 \\
-\tau_L/\tau_1
\end{array} \right), \quad \bar{\psi} = -\sqrt{\frac{m_\psi}{4i}} \left( \begin{array}{c}
\bar{\tau}_R/\tau_1 \\
\bar{\tau}_L/\tau_0
\end{array} \right)
\] (4.11)
where we have introduced the tau-functions
\[
\tau_0 \equiv <\hat{\lambda}_0 | G | \lambda_0>, \quad \tau_1 \equiv <\hat{\lambda}_1 | G | \lambda_1>
\]
\[
\tau_R \equiv <\hat{\lambda}_0 | E^- G | \lambda_0>, \quad \bar{\tau}_R \equiv <\hat{\lambda}_1 | E^0 G | \lambda_1>
\]
\[
\tau_L \equiv <\hat{\lambda}_1 | G E^0 | \lambda_1>, \quad \bar{\tau}_L \equiv <\hat{\lambda}_0 | GE^- | \lambda_0>
\] (4.12)
and where
\[ G \equiv \mathcal{T}_{\text{vac}} \rho \mathcal{T}_{\text{vac}}^{-1} = e^{x_+ x_2} e^{-x_+ x_2} \mathcal{R} e^{x_- x_2} e^{-x_- x_2} \]  
(4.13)

We have denoted \(| \hat{\lambda}_0 \rangle >\) and \(| \hat{\lambda}_1 \rangle >\) the highest weight states of the two fundamental representations of the affine Kac-Moody algebra \( \hat{sl}(2) \), respectively the scalar and spinor ones. They satisfy
\[ H^0 | \hat{\lambda}_0 \rangle > = 0, \quad H^0 | \hat{\lambda}_1 \rangle = | \hat{\lambda}_1 \rangle >, \quad C | \hat{\lambda}_j \rangle = | \hat{\lambda}_j \rangle > \]  
(4.14)

with \( j = 0, 1 \), and in addition
\[ E^0_+ | \hat{\lambda}_j \rangle > = H^n | \hat{\lambda}_j \rangle > = E^0_\pm | \hat{\lambda}_j \rangle = 0, \quad \text{for } n > 0 \]  
(4.15)

Notice that if one makes the shift \( \rho \to h \rho h^{-1}, \text{with } h = e^{i\theta H^0/2}, \) one gets that \( G \to hGh^{-1} \), since \( h \) commutes with \( E_{\pm2} \). Therefore, the tau-functions transform as
\[ \tau_0 \to \tau_0, \quad \tau_1 \to \tau_1, \quad \tau_R \to e^{i\theta} \tau_R, \quad \tau_L \to e^{i\theta} \tau_L, \quad \tau_\bar{R} \to e^{-i\theta} \tau_\bar{R}, \quad \tau_\bar{L} \to e^{-i\theta} \tau_\bar{L} \]  
(4.16)

which corresponds to the global \( U(1) \) transformations (2.14).

We will be interested in classical solutions satisfying the relations (2.21). In terms of the tau-functions defined above those relations can be written as
\[ \tau_0 \partial_+ \tau_1 - \tau_1 \partial_+ \tau_0 = \frac{1}{2} m_\psi \tau_R \tilde{\tau}_R \]  
\[ \tau_0 \partial_- \tau_1 - \tau_1 \partial_- \tau_0 = \frac{1}{2} m_\psi \tau_L \tilde{\tau}_L \]  
(4.17)

Using (4.12)-(4.13) one can write (4.17) as the algebraic relations
\[ \langle G \rangle_0 < H^1 G \rangle_1 - < G \rangle_1 < H^1 G \rangle_0 > = 2 < E^1_- G \rangle_0 < E^0_+ G \rangle_1 \]  
(4.18)

\[ < G \rangle_0 < G H^{-1} \rangle_1 - < G \rangle_1 < G H^{-1} \rangle_0 > = 2 < GE^1_- \rangle_0 < GE^0_+ \rangle_1 \]

where we have denoted \( \langle \hat{\lambda}_a | X \rangle | \hat{\lambda}_a \rangle = \langle \langle \langle X >_a \rangle \rangle \). These equations determine the group elements \( \rho \) that lead to solutions satisfying the relations (2.21).

One can use (4.10)-(4.11) to write the equations of motion of the theory (2.1) in terms of the tau-functions. The equations of motion (2.3)-(2.4) for \( \varphi \) and \( \nu \) imply that (for \( \eta = 0 \))
\[ \tau_0 \partial_+ \partial_- \tau_0 - \partial_+ \tau_0 \partial_- \tau_0 = \frac{1}{4} m_\psi^2 \tau_L \tilde{\tau}_R \]  
(4.19)
\[ \tau_1 \partial_+ \partial_- \tau_1 - \partial_+ \tau_1 \partial_- \tau_1 = \frac{1}{4} m_\psi^2 \tau_R \tilde{\tau}_L \]  
(4.20)

The equations of motion (2.6)-(2.7) for the Dirac spinors imply (for \( \eta = 0 \))
\[ \tau_0 \partial_- \tau_R - \tau_R \partial_- \tau_0 = -\frac{1}{2} m_\psi \tau_1 \tau_L \]  
(4.21)
\[ \tau_1 \partial_+ \tau_L - \tau_L \partial_+ \tau_1 = \frac{1}{2} m_\psi \tau_0 \tau_R \]  
(4.22)
\[ \tau_1 \partial_- \tau_R - \tau_R \partial_- \tau_1 = \frac{1}{2} m_\psi \tau_0 \tau_L \]  
(4.23)
\[ \tau_0 \partial_+ \tau_R - \tau_R \partial_+ \tau_0 = -\frac{1}{2} m_\psi \tau_1 \tau_R \]  
(4.24)
The relations (4.19)-(4.24) are the Hirota’s bilinear equations [17] for the model (2.1). Notice that the first order equations (4.17) and (4.21)-(4.24) imply that the difference of (4.19) and (4.20) should be satisfied, but not that each one separately should do.

The N-soliton solutions are constructed by taking the constant group element $\rho$ of the dressing transformation to be a product of exponentials of the eigenvectors of $E_{\pm 2}$ (see (4.9)). Since $V_+(z)$ and $V_-(-z)$ have the same eigenvalue, we take the exponential of a linear combination of them, i.e.

$$\rho_{N_{-\text{sol}}} \equiv \prod_{l=1}^{N} e^{V(a_{\pm}^{(l)}, z_l)} \quad \text{with} \quad V(a_{\pm}^{(l)}, z_l) \equiv \sqrt{i} \left( a_{+}^{(l)} V_+(z_l) + a_{-}^{(l)} V_-(z_l) \right); \quad (4.25)$$

In fact, if for a given $l$ (or more than one) either $a_{+}^{(l)}$ or $a_{-}^{(l)}$ vanishes one does not obtain $N$-soliton solutions. For example, in the case of the one-soliton one gets a vanishing solution for $\varphi_7$. The fact that the $N$-soliton solutions need the two degenerate eigenvectors, i.e. $V_+(z)$ and $V_-(-z)$, is what makes them to carry, besides the topological charge, the “electric” $U(1)$ charge (2.13).

The corresponding group element (1.13) becomes

$$G_{N_{-\text{sol}}} = \prod_{l=1}^{N} \exp \left( e^{\Gamma(z_l)} V(a_{\pm}^{(l)}, z_l) \right) \quad (4.26)$$

with

$$\Gamma(z_l) = \frac{1}{2} m_\psi \left( z_l x_+ - \frac{1}{z_l} x_- \right) \equiv \gamma_l (x - v_l t) \quad (4.27)$$

and so

$$\gamma_l = \frac{1}{2} m_\psi \left( z_l + \frac{1}{z_l} \right) = \left( \text{sign } z_l \right) \frac{m_\psi}{\sqrt{1-v_l^2}} \quad \text{with} \quad v_l = \frac{1-z_l^2}{1+z_l^2} \quad (4.28)$$

Notice that one needs to take $z_l$ real for the soliton velocities to be smaller than light speed, i.e. $|v_l| \leq 1$. In fact, the quantities $z_l$ are related to the rapidities $\theta_l$ of the solitons through

$$z_l \equiv \epsilon_l e^{\theta_l} \quad (4.29)$$

with $\epsilon_l = \pm 1$, and so from (4.27)

$$\Gamma(\theta_l) = \epsilon_l m_\psi \left( x \cosh \theta_l + t \sinh \theta_l \right) \quad (4.30)$$

An important point in the calculations involved in the construction of the soliton solutions is that it is much easier to work with the homogeneous (or Fubini-Veneziano) vertex operator construction for $V_{\pm}(z)$. Indeed, in such construction one has that

$$V_+(z) V_+(\zeta) \to 0, \quad V_-(z) V_-(\zeta) \to 0, \quad \text{as} \quad z \to \zeta \quad (4.31)$$

and therefore the exponentials involving $V_{\pm}(z)$ truncate.

---

8See eq. (14.8.14) on page 309 of ref. [18], or eq. (6.2.6) of ref. [19] for the details.
We are interested in the solutions where the field $\varphi$ is real. Using (4.10) one obtains that
\[
-\im \varphi = \log | \frac{\tau}{\tau_0} | + \im \arg \frac{\tau}{\tau_0}
\]
implies
\[
| \tau_0 | = | \tau_1 |
\]
and consequently
\[
\varphi^* = \varphi
\]
implies
\[
| \tau_0 | = | \tau_1 |
\]
(4.32)

and consequently
\[
\varphi = \arctan \left( \frac{\tau_0^* \tau_1 - \tau_0 \tau_1^*}{\tau_0 \tau_1 + \tau_0 \tau_1^*} \right),
\]
or
\[
\varphi = \zeta_0 - \zeta_1 + n\pi
\]
(4.33)

where $n$ is any integer, and where we have denoted $\tau_a = | \tau_a | e^{i\zeta_a}$, $a = 0, 1$.

Using (4.11) one observes that the reality condition (2.2) implies that the tau-functions have to satisfy
\[
\tilde{\tau}_R \tau_1 = -\im e^\psi \tau_0^* \hspace{1cm} \tilde{\tau}_L \tau_0 = \im e^\psi \tau_1^*
\]
(4.34)

4.1 The one-soliton solutions

The one-soliton solution can easily be calculated by evaluating the matrix elements (4.12) with $\rho = e^{V(a^\pm, z)}$, or alternatively by applying the Hirota’s method to the equations (4.19)-(4.24). The result is
\[
\tau_0 = 1 - \frac{i}{4} a_- a_+ e^{2\Gamma(z)} \hspace{1cm} \tau_1 = 1 + \frac{i}{4} a_- a_+ e^{2\Gamma(z)}
\]
\[
\tau_R = \sqrt{i} a_+ z e^{\Gamma(z)} \hspace{1cm} \tilde{\tau}_R = \sqrt{i} a_- e^{\Gamma(z)}
\]
\[
\tau_L = \sqrt{i} a_+ e^{\Gamma(z)} \hspace{1cm} \tilde{\tau}_L = -\sqrt{i} a_- e^{\Gamma(z)}
\]
(4.35)

One can check that such solutions also satisfy (4.17) without any further restriction. Therefore, the one-soliton solution satisfy the condition (2.21) for the equivalence between Noether and topological currents.

If one requires that $\varphi$ should be real, then one gets from (4.32) and (4.33) that $(a_- a_+)$ must be real. Such condition is sufficient to make $\tau_1 = \tau_0^*$. Therefore, using (2.16) and (4.33), one gets that the topological charge of the soliton is
\[
Q_{\text{topol.}} = -\text{sign} (a_- a_+) \quad \text{for } z > 0
\]
\[
Q_{\text{topol.}} = \text{sign} (a_- a_+) \quad \text{for } z < 0
\]
(4.36)

However, if one imposes in addition, the reality condition (2.2) on the spinors one gets from (4.33) that the parameters must satisfy
\[
a_- = -e^\psi a_+^* z
\]
(4.37)

The topological charge in such case is
\[
Q_{\text{topol.}} = \text{sign} e^\psi \quad \text{for any real } z
\]
(4.38)

Notice that in general, solitons are transformed into anti-solitons by parity transformations. However, the theory (2.1) is not invariant under spatial parity, and the CP-like
symmetry (2.17) is broken when the reality condition (2.2) is imposed. In order to have a degenerate vacua and topological solitons we need the factor \(i\) in the exponential of the potential term in (2.1). That makes the Lagrangian complex and so it is very unlikely that (2.1) has a CPT symmetry. The lack of P, CP and CPT symmetries is what reaffirms the existence of only the soliton or anti-soliton solution for a given choice of \(e_\psi\) in (2.2).

We then reach an interesting conclusion: without the reality condition (2.2) the theory (2.1) has two Dirac spinors and it also has the soliton and anti-soliton solutions, since for a given choice of \(e_\psi\) reality condition (2.2) the theory (2.1) losses one Dirac spinor and also one soliton solution, according to (4.36) both signs of the charge are admissible. However, if we impose the

\[ (2.1) \text{ has two Dirac spinors and it also has the soliton and anti-soliton solutions, since} \]

the particular one-soliton solution (2.22) satisfies (4.37) with \(e_\psi = 1\), \(z > 0\), \(\theta\) being the phase of \(a_+\), i.e. \(a_+ = |a_+| e^{i\theta}\), and \(|a_+| \sqrt{z}/2 = \exp \left(-m_\psi x_0/\sqrt{1 - v^2}\right)\).

The mass of these one-soliton solutions was evaluated in [7] and it is given by

\[ M = 2k m_\psi \] (4.39)

where \(k\) is the coupling constant appearing in the Lagrangian (2.1).

### 4.2 The two-soliton solutions

The two-soliton solutions are calculated through the dressing transformation method taking (see (4.25))

\[ \rho_{2-\text{sol}} \equiv e^{V(a^{(1)}_\pm,z_1)} e^{V(a^{(2)}_\pm,z_2)} \quad \text{and so} \quad G_{2-\text{sol}} = e^{(e^{1/2}V(a^{(1)}_\pm,z_1))} e^{(e^{1/2}V(a^{(2)}_\pm,z_2))} \] (4.40)

The explicit solution is calculated by evaluating the matrix elements (4.12). Alternatively, one can apply Hirota’s expansion method [17] to the Hirota’s equation (4.19)-(4.24). The results one obtains are

\[ \tau_0 = 1 - \frac{i}{4} a^{(1)}_+ a^{(1)}_+ e^{2\Gamma(z_1)} - \frac{i}{4} a^{(2)}_+ a^{(2)}_+ e^{2\Gamma(z_2)} - \frac{z_1 z_2}{(z_1 + z_2)^2} \left(a^{(1)}_+ a^{(2)}_- + a^{(1)}_- a^{(2)}_+\right) e^{\Gamma(z_1) + \Gamma(z_2)} \]

\[ - \frac{1}{16} \frac{(z_1 - z_2)^4}{(z_1 + z_2)^2} a^{(1)}_+ a^{(1)}_- a^{(2)}_+ a^{(2)}_- e^{2(\Gamma(z_1) + \Gamma(z_2))} \] (4.41)

\[ \tau_1 = 1 + \frac{i}{4} a^{(1)}_- a^{(1)}_+ e^{2\Gamma(z_1)} + \frac{i}{4} a^{(2)}_- a^{(2)}_+ e^{2\Gamma(z_2)} + \frac{i}{(z_1 + z_2)^2} \left(a^{(1)}_+ a^{(2)}_- z^2_1 + a^{(1)}_- a^{(2)}_+ z^2_2\right) e^{\Gamma(z_1) + \Gamma(z_2)} \]

\[ - \frac{1}{16} \frac{(z_1 - z_2)^4}{(z_1 + z_2)^2} a^{(1)}_+ a^{(1)}_- a^{(2)}_+ a^{(2)}_- e^{2(\Gamma(z_1) + \Gamma(z_2))} \] (4.42)

\[ \frac{\tau_R}{\sqrt{i}} = a^{(1)}_+ z_1 e^{\Gamma(z_1)} + a^{(2)}_+ z_2 e^{\Gamma(z_2)} - \frac{i}{4} \frac{z_2 (z_1 - z_2)^2}{(z_1 + z_2)^2} a^{(1)}_+ a^{(1)}_- a^{(2)}_+ e^{2\Gamma(z_1) + \Gamma(z_2)} \]
g}

\[ -\frac{i}{4} \frac{z_1 (z_1 - z_2)^2}{(z_1 + z_2)^2} a_+^{(1)} a_-^{(2)} a_+^{(2)} e^{\Gamma(z_1) + 2\Gamma(z_2)} \]  

(4.43)

\[ \frac{\tau_L}{\sqrt{i}} = a_+^{(1)} e^{\Gamma(z_1)} + a_-^{(2)} e^{\Gamma(z_2)} + \frac{i}{4} \frac{(z_1 - z_2)^2}{(z_1 + z_2)^2} a_+^{(1)} a_+^{(2)} e^{\Gamma(z_1) + 2\Gamma(z_2)} \]  

+ \frac{i}{4} \frac{(z_1 - z_2)^2}{(z_1 + z_2)^2} a_+^{(1)} a_-^{(2)} e^{2\Gamma(z_1) + \Gamma(z_2)} \]  

(4.44)

\[ \frac{\tilde{\tau}_R}{\sqrt{i}} = a_-^{(1)} e^{\Gamma(z_1)} + a_+^{(2)} e^{\Gamma(z_2)} + \frac{i}{4} \frac{(z_1 - z_2)^2}{(z_1 + z_2)^2} a_-^{(1)} a_-^{(2)} e^{\Gamma(z_1) + 2\Gamma(z_2)} \]  

+ \frac{i}{4} \frac{(z_1 - z_2)^2}{(z_1 + z_2)^2} a_-^{(1)} a_+^{(2)} e^{2\Gamma(z_1) + \Gamma(z_2)} \]  

(4.45)

\[ \frac{\tilde{\tau}_L}{\sqrt{i}} = -\frac{a_+^{(1)}}{z_1} e^{\Gamma(z_1)} - \frac{a_-^{(2)}}{z_2} e^{\Gamma(z_2)} + \frac{i}{4} \frac{(z_1 - z_2)^2}{(z_1 + z_2)^2} a_-^{(1)} a_-^{(2)} e^{\Gamma(z_1) + 2\Gamma(z_2)} \]  

+ \frac{i}{4} \frac{(z_1 - z_2)^2}{(z_1 + z_2)^2} a_-^{(1)} a_+^{(2)} e^{2\Gamma(z_1) + \Gamma(z_2)} \]  

(4.46)

These tau-functions satisfy the Hirota’s equations (4.19)-(4.24) for any value of the constants \( a_+^{(1)}, a_+^{(2)}, z_1 \) and \( z_2 \), which can be even complex. It is interesting to notice that these tau-functions also satisfy the equivalence between the Noether and topological currents (4.17) (or equivalently (2.21)) without any further restrictions.

As we have said the Lorentz transformations in 2d are \( x_\pm \to \hat{x}_\pm \equiv \lambda^{\pm 1} x_\pm \). The corresponding field transformations can be obtained from (2.28)-(2.3) by taking \( f(x_+) = \lambda x_+ \) and \( g(x_-) = \lambda^{-1} x_- \). Therefore, choosing the conformal weight \( \delta \) of the field \( \nu \) to vanish, one conclude from (4.10)-(4.11) that, under Lorentz transformations, the tau-functions transform as

\[ \tau_a(x) \to \tau_a(\hat{x}) = \tau_a(x) \]  

for \( a = 0, 1 \)  

(4.47)

and

\[ \tau_R(x) \to \tau_R(\hat{x}) = \lambda^{-\frac{1}{2}} \tau_R(x) \]  

\[ \tau_L(x) \to \tau_L(\hat{x}) = \lambda^{\frac{1}{2}} \tau_L(x) \]  

(4.48)

and similarly for \( \tilde{\tau}_{R/L} \). One then observes that such transformations can be implemented in the space of solutions by transforming the parameters of the solutions (4.41)-(4.46) as

\[ z_i \to \lambda z_i \]  

\[ a^{(i)}_\pm \to \lambda^{\frac{1}{2}} a^{(i)}_\pm \]  

\( i = 1, 2 \)  

(4.49)

In addition, the global \( U(1) \) transformations (2.12) (see (4.16)) correspond in the space of parameters to

\[ z_i \to z_i \]  

\[ a^{(i)}_\pm \to e^{\pm \theta} a^{(i)}_\pm \]  

\( i = 1, 2 \)  

(4.50)

4.3 The reality conditions

We want \( z_1 \) and \( z_2 \) real in order for the soliton velocities to be smaller than the speed of light \( (c = 1, \text{ see (4.28))}. \) Then, one can easily check that in order for the solutions (4.41)-(4.42)
to satisfy (4.32) one has to have $a_+^{(i)} a_-^{(i)}$ and $a_+^{(2)} a_-^{(2)}$ real, and so

\[ a_+^{(i)} = |a_+^{(i)}| e^{\xi_i}, \quad a_-^{(i)} = |a_-^{(i)}| e^{-i \xi_i}, \quad i = 1, 2 \]  

with $\xi_i$ real being phases, and $\xi_i = \pm 1$ being the sign of $a_+^{(i)} a_-^{(i)}$. In addition, one needs

\[ \bar{\epsilon}_1 z_1 |a_+^{(1)}| |a_-^{(2)}| = \bar{\epsilon}_2 z_2 |a_-^{(1)}| |a_+^{(2)}| \]  

(4.52)

Using (4.28) one observes that (4.52) implies

\[ \bar{\epsilon}_1 \bar{\epsilon}_2 \epsilon_1 \epsilon_2 = 1 \]  

(4.53)

\[ e^{\theta_1 - \theta_2} |a_+^{(1)}| |a_-^{(2)}| = |a_-^{(1)}| |a_+^{(2)}| \]  

(4.54)

The relations (4.51) and (4.52) are the necessary and sufficient conditions for (4.32) to be satisfied. However, they are sufficient to make $\tau_1$ the complex conjugate of $\tau_0$. Therefore,

\[ \varphi^* = \varphi \quad \rightarrow \quad \tau_1 = \tau_0^* \quad \rightarrow \quad \varphi = 2 \zeta_0 + n\pi \]  

(4.55)

Consequently, the conditions (4.34) become

\[ \bar{\tau}_R = -ie_\psi \tau_0^* R, \quad \bar{\tau}_L = ie_\psi \tau_0^* L \]  

(4.56)

One can easily check that the solutions (4.41)-(4.46) satisfy the conditions $\tau_1 = \tau_0^*$ (and so $\varphi$ real) and (4.56) if the parameters satisfy ($z_1, z_2$ being real)

\[ a_+^{(i)} = -e_\psi a_+^{(i)*} z_i, \quad i = 1, 2 \]  

(4.57)

Then it follows that the signs of $z_i$ and $a_+^{(i)} a_-^{(i)}$ are related by

\[ \bar{\epsilon}_i = -\epsilon_i \text{sign} e_\psi, \quad i = 1, 2 \]  

(4.58)

which indeed satisfy (4.53).

### 4.4 The topological charges

For the solutions satisfying the conditions (4.51) and (4.52) we shall denote

\[ |a_+^{(i)}| |a_-^{(i)}| \equiv e^{-2\gamma_i x_0^{(i)}} \quad i = 1, 2 \]  

(4.59)

with $\gamma_i$ defined in (4.28). We then define

\[ \hat{\Gamma} (z_i) \equiv \Gamma (z_i) - \gamma_i x_0^{(i)} \]  

(4.60)

with $\Gamma (z_i)$ being defined in (4.27).

In order to evaluate the topological charges we shall take the solution in the center of mass reference frame, which corresponds to (see (4.28) and (4.29))

\[ z_1 = \epsilon_1 e^{\theta}; \quad z_2 = \epsilon_2 e^{-\theta}, \quad \rightarrow \quad v_1 = -v_2 \quad \text{and} \quad \gamma_2 = \epsilon_1 \epsilon_2 \gamma_1 \]  

(4.61)
Then, using (4.51), (4.54), (4.61), and (4.49), the tau-function \( \tau_0 \) given in (4.41) becomes

\[
\tau_0 = 1 - \epsilon_1 \epsilon_2 \left( \frac{\bar{\epsilon}_1 e^{\theta - \bar{\epsilon}_2 e^{-\theta}}}{e^{\theta} + \epsilon_1 \epsilon_2 e^{-\theta}} \right)^2 \sin (\xi_1 - \xi_2) e^{\hat{\Gamma}(z_1) + \hat{\Gamma}(z_2)} - \frac{\bar{\epsilon}_1 \epsilon_2 (e^{\theta} - \epsilon_1 \epsilon_2 e^{-\theta})^4}{16 (e^{\theta} + \epsilon_1 \epsilon_2 e^{-\theta})^4} e^{2(\hat{\Gamma}(z_1) + \hat{\Gamma}(z_2))}
\]

\[
- i \left( \frac{\bar{\epsilon}_1}{4} e^{2\hat{\Gamma}(z_1)} + \frac{\bar{\epsilon}_2}{4} e^{2\hat{\Gamma}(z_2)} + \epsilon_1 \epsilon_2 \frac{(\epsilon_1 e^{\theta} + \bar{\epsilon}_2 e^{-\theta})^2}{(e^{\theta} + \epsilon_1 \epsilon_2 e^{-\theta})^2} \cos (\xi_1 - \xi_2) e^{\hat{\Gamma}(z_1) + \hat{\Gamma}(z_2)} \right)
\]

(4.62)

In addition, one has \( \tau_1 = \tau_0^* \), when (4.53) is also enforced. Therefore, according to (2.16) and (4.55), the topological charge is \( Q_{\text{topol.}} = \frac{2}{\pi} (\arg \tau_0 (\infty) - \arg \tau_0 (-\infty)) \). We shall use the prescription that the \( \arg \tau_0 \) lies between \( -\pi \) and \( \pi \). When evaluating the charge one should notice that, on the center of mass reference frame, \( \hat{\Gamma}(z_1) + \hat{\Gamma}(z_2) \) does not depend upon \( x \) if \( \epsilon_1 \) and \( \epsilon_2 \) have opposite signs. In addition, for \( \epsilon_1 \epsilon_2 = \bar{\epsilon}_1 \bar{\epsilon}_2 = 1 \) one should pay attention on the sign of the imaginary part of \( \tau_0 \) as it tends to zero, in order to determine if \( \arg \tau_0 \) tends to \( -\pi \) or \( \pi \). In such case, one can write

\[
\text{Im} \tau_0 \to -\frac{1}{2} \bar{\epsilon}_1 \left( 1 + \frac{\cos (\xi_1 - \xi_2)}{\cosh \theta} \right) e^{2m_\psi \epsilon_1 x \cosh \theta + \ldots} \quad \text{as} \quad x \to \epsilon_1 \infty
\]

(4.63)

for \( \epsilon_1 \epsilon_2 = \bar{\epsilon}_1 \bar{\epsilon}_2 = 1 \). Since \( \frac{\cos (\xi_1 - \xi_2)}{\cosh \theta} \geq -1 \), the limit is independent of \( \xi_i \). In fact, the topological charge, for \( \varphi \) real, does not depend upon the phases \( \xi_i \). The results one obtains are

\[
Q_{\text{topol.}} = 2 \bar{\epsilon}_1 \quad \text{for} \quad (\epsilon_1 = \epsilon_2 = -1, \bar{\epsilon}_1 \bar{\epsilon}_2 = 1) \quad \text{or} \quad (\epsilon_1 = -\epsilon_2 = -1, \bar{\epsilon}_1 \bar{\epsilon}_2 = 1)
\]

\[
Q_{\text{topol.}} = -2 \epsilon_1 \quad \text{for} \quad (\epsilon_1 = \epsilon_2 = 1, \bar{\epsilon}_1 \bar{\epsilon}_2 = 1) \quad \text{or} \quad (\epsilon_1 = -\epsilon_2 = 1, \bar{\epsilon}_1 \bar{\epsilon}_2 = -1)
\]

(4.64)

If one now imposes the reality condition (2.2) on the spinors, or equivalently (4.50)-(4.58), one gets

\[
Q_{\text{topol.}} = 2 \text{sign} \epsilon_\psi
\]

(4.65)

for any \( \epsilon_i \) and \( \bar{\epsilon}_i \) satisfying (4.58).

We then have a situation similar to the one-soliton solutions, since in that case the imposition of the condition (2.2) makes either the soliton or anti-soliton to disappear from the spectrum. Here the same thing happens, since (2.2) chooses the sign of \( \epsilon_\psi \) and then either the charge 2 or -2 solutions disappear. However, a new feature emerges. We saw we could have (without (2.2)) soliton and anti-soliton solutions with \( \varphi \) real. We have not found here a charge zero solution corresponding to the scattering of a soliton and anti-soliton for \( \varphi \) real. Such solution exists however, for \( \varphi \) complex but asymptotically real as we show below.

### 4.4.1 Asymptotically real solutions

One can easily verify that for the solutions (4.41)-(4.46) the field \( \varphi \) is always real asymptotically, i.e. \( \tau_0 \to \tau_1 \) as \( x \to \pm \infty \). In addition, one also has that \( \psi, \bar{\psi} \to 0 \) as \( x \to \pm \infty \). Therefore, the topological charge is always real for those solutions.
Let us now consider a solution of the type \((4.41)-(4.46)\) that satisfies the conditions \((4.51)\) and \((4.54)\) but not \((4.53)\) (and of course not \((2.2)\) too). In the center of mass reference frame (see \((4.61)\)) \(\tau_0\) is given by \((4.62)\) and \(\tau_1\) by

\[
\tau_1 = 1 - \frac{\bar{e}_2}{e_1} \left( \frac{e_1 e^\theta - \bar{e}_2 e^{-\theta}}{e^\theta + \bar{e}_2 e^{-\theta}} \right)^2 \sin (\xi_1 - \xi_2) e^{\hat{\Gamma}(z_1) + \hat{\Gamma}(z_2)} - \frac{\bar{e}_1 \bar{e}_2}{16} \left( \frac{e^\theta - e_1 \bar{e}_2 e^{-\theta}}{e^\theta + \bar{e}_2 e^{-\theta}} \right)^4 e^{2(\hat{\Gamma}(z_1) + \hat{\Gamma}(z_2))} \\
+ \frac{i}{4} \left( \frac{\bar{e}_1}{4} e^{2\hat{\Gamma}(z_1)} + \frac{\bar{e}_2}{4} e^{2\hat{\Gamma}(z_2)} + \frac{\bar{e}_1 \bar{e}_2}{e^\theta + \bar{e}_2 e^{-\theta}} \right) \cos (\xi_1 - \xi_2) e^{\hat{\Gamma}(z_1) + \hat{\Gamma}(z_2)}
\]

\((4.66)\)

Since \(|\tau_0| \equiv |\tau_1|\) holds true asymptotically, the topological charge is then given by \(Q_{\text{topol}} = \frac{1}{\pi} (\arg \tau_0 - \arg \tau_1) (\infty) - (\arg \tau_0 - \arg \tau_1) (-\infty)\). Again, using the prescription that the argument of \(\tau_i\) varies from \(-\pi\) to \(\pi\), one gets that

\[
Q_{\text{topol}} = 0 \quad \text{for } (\epsilon_1 \epsilon_2 = 1, \bar{\epsilon}_1 \bar{\epsilon}_2 = -1) \text{ or } (\epsilon_1 \epsilon_2 = -1, \bar{\epsilon}_1 \bar{\epsilon}_2 = 1)
\]

\((4.67)\)

Therefore, such solution corresponds to the scattering of a soliton and an anti-soliton.

### 4.5 The breather solutions

The space-time dependence of the solutions \((4.41)-(4.46)\) are given through the exponentials \(\exp (\gamma_l (x - vt))\), \(l = 1, 2\). Therefore, in order to have solutions periodic in time one needs \(\gamma_l v_l\) to be pure imaginary. Writing \(z_l = |z_l| e^{i\theta_l}\), one observes from \((4.28)\) that for that to happen one needs \(|z_l| = 1\), and so \(\gamma_l v_l = -i m^\psi \sin \theta_l\). In order for the solution to present just one frequency one needs to have \(\sin \theta_2 = \pm \sin \theta_1\). We do not want \(z_1 = z_2\) because \((4.41)-(4.46)\) becomes a one-soliton solution. In addition, we do not want \(z_1 = -z_2\) because the solution \((4.41)-(4.46)\) will present singularities. Therefore, the only possibilities of having solutions periodic in time is to have

\[
z_1 = e^{i\theta}; \quad z_2 = e^{-i\theta}; \quad \epsilon = \pm 1
\]

\((4.68)\)

Then one gets from \((4.27)\) and \((4.28)\) that

\[
\Gamma(z_1) = \gamma x + i\omega t \quad \Gamma(z_2) = \epsilon (\gamma x - i\omega t)
\]

\((4.69)\)

with

\[
\gamma \equiv m^\psi \cos \theta \quad \omega \equiv m^\psi \sin \theta
\]

\((4.70)\)

#### 4.5.1 The case \(\epsilon = 1\)

We are interested in the cases where the field \(\varphi\) is real, and so from \((4.10)\) we need \(|\tau_0| \equiv |\tau_1|\). Imposing the conditions \((4.68)-(4.70)\) into \((4.41)-(4.42)\) one observes that in order for \(\varphi\) to be real one needs

\[
a_+^{(2)} a_-^{(2)} = (a_+^{(1)} a_-^{(1)})^* \quad \theta + \xi_+^{(1)} + \xi_-^{(2)} = n\pi \quad n \in \mathbb{Z}
\]

\((4.71)\)

and

\[(4.72)\]
where we have denoted
\[ a^{(i)}_\pm \equiv |a^{(i)}_\pm| e^{i\xi^{(i)}_\pm} \quad i = 1, 2 \] (4.73)

It follows that the conditions (4.71) and (4.72) are sufficient to make \( \tau_1 = \tau_0^* \), and from (4.41) one gets
\[
\tau_0 = 1 + (-1)^{n+1} S \left( R - \frac{1}{R} \right) \frac{\sin \theta}{\cos^2 \theta} e^{2x_{m_{\psi}} \cos \theta} - S^2 \tan^4 \theta e^{4x_{m_{\psi}} \cos \theta} + i S \left( \frac{(-1)^{n+1}}{\cos \theta} \left( R + \frac{1}{R} \right) - 2 \cos \left( 2\omega t + \xi^{(1)}_+ + \xi^{(1)}_- \right) \right) e^{2x_{m_{\psi}} \cos \theta} \] (4.74)

where the integer \( n \) is the same as in (4.72), and we have introduced
\[
S \equiv \frac{1}{4} |a^{(1)}_+| |a^{(1)}_-| ; \quad R \equiv \frac{|a^{(1)}_+|}{|a^{(2)}_+|} \] (4.75)

Therefore, using (2.16) and (4.10) one gets that the topological charge of such breather solution is
\[
Q_{\text{topol.}} = 2 (-1)^{n+1} \] (4.76)

with \( n \) given by (4.72).

Now, if besides the reality of \( \varphi \), one imposes the condition (2.2) on the spinors, one gets that the parameters have to satisfy
\[
 a^{(1)}_- = -e_{\psi} a^{(2)*}_+ e^{i\theta} ; \quad a^{(2)}_- = -e_{\psi} a^{(1)*}_+ e^{-i\theta} \] (4.77)

which to be compatible with (4.71) and (4.72), one needs \( \exp \left( i \left( \theta + \xi^{(1)}_+ + \xi^{(2)}_- \right) \right) = -\text{sign} e_{\psi} \). Then, the topological charge becomes
\[
Q_{\text{topol.}} = 2 \text{ sign} e_{\psi} \] (4.78)

Consequently, we have for the breather solutions an effect similar to what happens in the two-soliton solutions (see discussions below (4.38) and (4.65)). For \( \varphi \) real we have breathers with topological charges \( \pm 2 \). However, if one imposes the reality condition (2.2) on the spinors, one gets that only one charge is allowed.

### 4.5.2 The case \( \epsilon = -1 \)

One can check that for the choice \( \epsilon = -1 \) in (4.68) one can not obtain non trivial solutions such that \( \varphi \) is real. However, one does have that \( \varphi \) is asymptotically real. It is quite easy to verify that the topological charge vanishes for any choice of the parameters \( a^{(i)}_\pm \). So,
\[
Q_{\text{topol.}} = 0 ; \quad \text{for} \ \epsilon = -1, \ \text{and any} \ a^{(i)}_\pm, i = 1, 2 \] (4.79)
4.6 The time delays

Solitons are classical solutions that travel with constant speed without dispersion and that keep their form under scattering, the effect of it being only a phase shift or a displacement in its position. We now show that the solitons we have been working with are true solitons and indeed fulfill the above requirements. We shall calculate the so-called time delays of the scattering of two solitons, using the procedures of [20].

We consider two solitons that in the distant past are well apart, then collide near \( t = 0 \) and then separate again in the distant future. Therefore, except for the region where the scattering occurs, the solitons are free and so travel with constant velocity. Let us denote the trajectories of one of the solitons before and after the collision as (since the velocity is the same)

\[
x = vt + x(I) \quad \text{and} \quad x = vt + x(F)
\]

(4.80)

The lateral displacement at fixed time is measured by

\[
\Delta(x) \equiv x(F) - x(I)
\]

(4.81)

The time delay is defined by

\[
\Delta(t) \equiv t(F) - t(I) = -\frac{\Delta(x)}{v}
\]

(4.82)

with the intercepts of the trajectories with the time axis being given by \( t(F) = -x(F)/v \) and \( t(I) = -x(I)/v \). The lateral displacement and the time delay are not Lorentz invariant, and so we consider the invariant

\[
E \Delta(x) = -p \Delta(t)
\]

(4.83)

with \( E \) and \( p = vE \) being the energy and momentum of the soliton respectively. Since \( E \) is positive it follows \( \Delta(x) \) has the same sign in any reference frame, with only its strength being frame dependent. The time delay on the other hand may change the sign under Lorentz transformations. One can show that the lateral displacements and time delays for the two solitons participating in the collision have to satisfy [20]

\[
E_1 \Delta_1(x) + E_2 \Delta_2(x) = 0; \quad p_1 \Delta_1(t) + p_2 \Delta_2(t) = 0
\]

(4.84)

with the sub-indices \( i \) labeling the quantities associated to particle \( i \). Notice therefore that, since the energies are positive, the lateral displacements have opposite signs. Clearly, in the center of momentum frame (comf), where \( p_1 + p_2 = 0 \), the time delays are equal, i.e. \( \Delta_1^{(\text{comf})}(t) = \Delta_2^{(\text{comf})}(t) \). Therefore, from (4.83) we have that \( E_1^{(\text{comf})} \Delta_1^{(\text{comf})}(x)/p_1 = -E_2^{(\text{comf})} \Delta_2^{(\text{comf})}(x)/p_1 = -\Delta_1^{(\text{comf})}(t) = -\Delta_2^{(\text{comf})}(t) \). Consequently, since \( \Delta(x) \) has the same sign in any frame, it follows that, if particle 1 moves to the right faster then particle 2 (such that \( p_1 \) is positive in the cmof), then \( -\Delta_1(x) \), \( \Delta_2(x) \), \( \Delta_1^{(\text{comf})}(t) \) and \( \Delta_2^{(\text{comf})}(t) \) all have the same sign. The physical interpretation of that sign is related to the character (attractive or repulsive) of the interaction forces. Indeed, if the force is attractive then particle 1 will accelerate as it approaches particle 2 and then decelerate. That means that \( \Delta_1(x) \) is positive and so the common sign negative. Therefore, attractive forces lead to a negative time delay.
in the center of momentum reference frame, and clearly repulsive forces a positive time delay. These considerations assume that the two particles pass through each other, and there is no reflection. However, when the masses of the two particles are equal there is the possibility of occurring reflection.

Let us now evaluate the time delays for the two-soliton solutions given in (4.41)-(4.46). We choose the particle 1 to move faster to the right than particle 2, i.e. \( v_1 > v_2 \), with \( v_1 > 0 \), and therefore from (4.28) and (4.29) \( z_1 < |z_2| \) or \( \theta_1 < \theta_2 \). We then track the soliton 1 in time \( \tau_1 \), i.e. we hold \( x = x_1 t \) fixed as time varies. Then one gets \( x - v_1 t = \text{const.} + (v_1 - v_2)t \), with const. \( = x - v_1 t \). We then get that, if \( \epsilon_2 = 1 \), \( e^{\Gamma_2(z_2)} \to 0 \) as \( t \to -\infty \), and \( e^{\Gamma_2(z_2)} \to \infty \) as \( t \to \infty \). For the case \( \epsilon_2 = -1 \) the limits get interchanged.

Therefore, taking \( \epsilon_2 = 1 \), one can check that the solution (4.41)-(4.46) becomes, in the limit \( t \to -\infty \), the one-soliton solution (4.35) with the identifications \( a_{\pm}^{(1)} \equiv a_{\pm} \) and \( z_1 \equiv z \). Now, in the limit \( t \to \infty \), one can also verify that the ratios \( \tau_0/\tau_1 \), \( \tau_R/\tau_0 \), \( \tau_L/\tau_1 \), \( \tau_R/\tau_1 \) and \( \tau_L/\tau_0 \) for the solutions (4.41)-(4.46) tend to the corresponding ratios for the one-soliton solution (4.35) with the same identification of parameters, and with the replacement

\[
e^{\Gamma_1(z_1)} \to \left( \frac{z_1 - z_2}{z_1 + z_2} \right)^2 e^{\Gamma_1(z_1)}
\]  

(4.85)

The only difference being the fact that the ratio \( \tau_0/\tau_1 \) gets a relative minus sign, meaning that the \( \varphi \) field gets shifted by \( \pi \) (see (4.10)) during the scattering process. Then one observes from (4.10)-(4.11) that the relevant effect of the scattering on the solutions for the fields \( \varphi \), \( \psi \) and \( \tilde{\psi} \) is a lateral displacement of the soliton 1 given by

\[
\gamma_1 (x - v_1 t) \to \gamma_1 \left( x - v_1 t + \frac{1}{\gamma_1} \ln \left( \frac{z_1 - z_2}{z_1 + z_2} \right)^2 \right)
\]  

(4.86)

If one takes \( \epsilon_2 = -1 \) instead, one observes that the direction of the arrow in (4.86) reverses. Therefore, using (4.81), (4.28) and (4.29) one sees that the lateral displacement for the soliton 1 is given by

\[
\Delta_1(x) = -\frac{\epsilon_1 \epsilon_2}{\cosh \theta_1} \ln \left( \frac{e^{(\theta_1 - \theta_2)/2} - \epsilon_1 \epsilon_2 e^{-(\theta_1 - \theta_2)/2}}{e^{(\theta_1 - \theta_2)/2} + \epsilon_1 \epsilon_2 e^{-(\theta_1 - \theta_2)/2}} \right)^2
\]  

(4.87)

Since \( \epsilon_1 \epsilon_2 = \pm 1 \), one observes that \( \Delta_1(x) \) is in fact independent of such signs, and so

\[
\Delta_1(x) = -\frac{1}{\cosh \theta_1} \ln \left( \frac{\theta_1 - \theta_2}{2} \right)^2
\]  

(4.88)

Notice that the solutions (4.41)-(4.46) are symmetric under the interchange of the indices 1 and 2 of the parameters \( a_{\pm}^{(i)} \) and \( z_i \). Therefore, if we track the soliton 2, i.e. keep \( x - v_2 t \) fixed as time varies, but under the same kinematical conditions, i.e. \( v_1 > v_2 \), with \( v_1 > 0 \), then we obtain that

\[
\Delta_2(x) = \frac{1}{\cosh \theta_2} \ln \left( \frac{\theta_1 - \theta_2}{2} \right)^2
\]  

(4.89)

If we reverse the kinematical conditions, i.e. take \( v_2 > v_1 \), with \( v_2 > 0 \), then the signs of both \( \Delta_i(x) \), \( i = 1, 2 \), reverse. The mass of the one-soliton solutions is given in (4.33), and
therefore the energies of the solitons are $E_i = 2 k m_\psi \cosh \theta_i$, $i = 1, 2$. Consequently, one sees that the $\Delta_i(x)$'s do indeed satisfy (4.84), i.e.

$$E_1 \Delta_1(x) = -E_2 \Delta_2(x) = -\text{sign} (v_1 - v_2) 2 k \ln \left( \tanh \left( \frac{\theta_1 - \theta_2}{2} \right) \right)^2$$  \hspace{1cm} (4.90)

In addition, using (4.82), (4.88) and (4.89) one gets that the time delays are given by (assuming $v_1 > v_2$, with $v_1 > 0$)

$$\Delta_1(t) = -\frac{1}{m_\psi \sinh \theta_1} \ln \left( \tanh \left( \frac{\theta_1 - \theta_2}{2} \right) \right)^2; \hspace{1cm} \Delta_2(t) = \frac{1}{m_\psi \sinh \theta_2} \ln \left( \tanh \left( \frac{\theta_1 - \theta_2}{2} \right) \right)^2$$  \hspace{1cm} (4.91)

Notice that the hyperbolic tangent varies from $-1$ to $1$ and therefore the logarithm of its square is always negative, and so from (4.88) one sees that $\Delta_1(x)$ for $v_1 > v_2$, with $v_1 > 0$, is positive. Therefore, from the considerations made above we conclude that the forces between the solitons is attractive. In addition, it is independent of their topological charges. In fact, the time delays we have obtained coincide with those of the sine-Gordon theory by identifying $k$ with a positive constant multiplied by the inverse of the square of the sine-Gordon coupling constant.

### Appendix: The zero curvature

The equations of motion of the theory (2.1) can be represented as a zero curvature (4.1) with connections given by \[7\]

$$A_+ = -B \left( E_2 + F_1^+ \right) B^{-1}, \hspace{1cm} A_- = -\partial_- B B^{-1} + E_{-2} + F_1^-.$$  \hspace{1cm} (A.1)

where

$$B = e^{i\varphi H^0} e^{(\nu - \frac{i}{2} \varphi) C} e^{\eta Q} \hspace{1cm} E_{\pm 2} \equiv \frac{1}{4} m_\psi H_{\pm 1}^0$$  \hspace{1cm} (A.2)

and

$$F_1^+ = \sqrt{i m_\psi} \left( \psi_R E_0^0 + \bar{\psi}_R E_1^1 \right), \hspace{1cm} F_1^- = \sqrt{i m_\psi} \left( \psi_L E_{-1}^0 - \bar{\psi}_L E_{-1}^0 \right).$$  \hspace{1cm} (A.3)

We have written the Dirac spinors as

$$\psi = \begin{pmatrix} \psi_R \\ \psi_L \end{pmatrix}; \hspace{1cm} \bar{\psi} = \begin{pmatrix} \bar{\psi}_R \\ \bar{\psi}_L \end{pmatrix}$$  \hspace{1cm} (A.4)

and have denoted by $H^n$, $E_\pm^n$, $D$ and $C$ the Chevalley basis generators of the $sl(2)$ affine Kac-Moody algebra. The commutation relations are

$$[H^m, H^n] = 2 m C \delta_{m+n,0},$$  \hspace{1cm} (A.5)

$$[H^m, E_\pm^n] = \pm 2 E_{\mp 0}^{m+n},$$  \hspace{1cm} (A.6)

$$[E_+, E_-^n] = H_{m+n} + m C \delta_{m+n,0},$$  \hspace{1cm} (A.7)

$$[D, T^m] = m T^m, \hspace{1cm} T^m \equiv H^m, E_\pm^m.$$  \hspace{1cm} (A.8)
The generator $Q$ is the grading operator for the principal gradation and given by $Q \equiv \frac{1}{2}H^0 + 2D$.

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References

[1] C. Montonen and D.I. Olive, *Phys. Lett.* **72B** (1977) 117-120.
[2] C. Vafa and E. Witten, *Nucl. Phys.* **B431** (1994) 3-77, [hep-th/9408074](https://arxiv.org/abs/hep-th/9408074).
[3] N. Seiberg and E. Witten, *Nucl. Phys.* **B431** (1994) 484-550, [hep-th/9408099](https://arxiv.org/abs/hep-th/9408099); *Nucl. Phys.* **B426** (1994) 19; [hep-th/9407087](https://arxiv.org/abs/hep-th/9407087).
[4] A. Sen, *Phys. Lett.* **329B** (1994) 217-221; *Int. J. Mod. Phys.* **A9** (1994) 3707, [hep-th/9402002](https://arxiv.org/abs/hep-th/9402002).
[5] S. Coleman, *Phys. Rev.* **D11** (1975) 2088.
[6] S. Mandelstam, *Phys. Rev.* **D11** (1975) 3026.
[7] L.A. Ferreira, J-L. Gervais, J.Sánchez Guillen and M.V.Saveliev, *Nucl. Phys.* **B470** (1996) 236-288; [hep-th/9512105](https://arxiv.org/abs/hep-th/9512105).
[8] E. Witten, *Nucl. Phys.* **B145** (1978) 110-118.
[9] L.A. Ferreira, J. Luis Miramontes and J. Sánchez Guillen, [hep-th/9606060](https://arxiv.org/abs/hep-th/9606060); *J. Math. Phys.* **38** (1997) 882-901.
[10] D.I. Olive, N. Turok and J.W.R. Underwood; *Nucl. Phys.* **B401** (1993) 663, [hep-th/9305160](https://arxiv.org/abs/hep-th/9305160); *Nucl. Phys.* **B409** (1993) 509.
[11] D.I. Olive, J.W.R. Underwood and M.V. Saveliev, [hep-th/9212123](https://arxiv.org/abs/hep-th/9212123); *Phys. Lett.* **311B** (1993) 117.
[12] S.J. Orfanidis and R. Wang, *Phys. Lett.* **57B** (1975) 281; S.J. Orfanidis, *Phys Rev.* **D14** (1976) 472.
[13] J. Kogut and D. Sinclair, *Phys. Rev.* **D12** (1975) 1742-1753.
[14] E. Abdalla, M.C.B. Abdalla and K.D. Rothe, *Non-perturivative methods in two-dimensional quantum field theory*, World Scientific, Singapore, 1991.
[15] H. Aratyn, C.P. Constantinidis, L.A. Ferreira, J.F. Gomes e A.H. Zimerman; [hep-th/9212086](https://arxiv.org/abs/hep-th/9212086); *Nuclear Physics* **B406**, 727 (1993).
[16] A.N. Leznov and M.V. Saveliev; *Group-Theoretical Methods for Integration of Non-Linear Dynamical Systems*, Progr. in Phys. Ser. Vol. 15 (Birkhäuser, Basel, 1992).

[17] R. Hirota, *Direct methods in soliton theory*, in “Solitons” (R.K. Bullough and P.S. Caudrey, eds.), Topics in Current Physics, p. 157, Springer-Verlag (1980); J. Phys. Soc. Japan **33** (1972) 1459.

[18] V.G. Kac, Infinite dimensional Lie algebras, 3rd ed., Cambridge University Press, Cambridge, 1990.

[19] P. Goddard and D. Olive, *Int. J. Mod. Phys. A* **1** (1986) 303-414.

[20] A. Fring, P.R. Johnson, M.A.C. Kneipp and D. I. Olive: hep-th/9405034. *Nucl. Phys. B* **1994** [FS430] (597-614).