Crossed complexes, free crossed resolutions and graph products of groups

R. Brown ∗ M. Bullejos † T. Porter ∗

November 3, 2018

Abstract

The category of crossed complexes gives an algebraic model of CW-complexes and cellular maps. Free crossed resolutions of groups contain information on a presentation of the group as well as higher homological information. We relate this to the problem of calculating non-abelian extensions. We show how the strong properties of this category allow for the computation of free crossed resolutions of graph products of groups, and so obtain computations of higher homotopical syzygies in this case.

1

Introduction

One aim of this paper is to advertise the category of crossed complexes, and the notion of free crossed resolution, as a working tool for certain problems in combinatorial group theory. Crossed complexes are an analogue of chain complexes of modules over a group ring, but with a non abelian part, a crossed module, at the bottom dimensional part. This allows for crossed complexes to contain in that part the data for a presentation of a group, and to contain in other parts higher homological data. The non abelian nature, and also the generalisation to groupoids rather than just groups, allows for a closer representation of geometry, and this, combined with very convenient properties of the category of crossed complexes, can allow for more and easier calculations than are available in the standard theory of chain complexes of modules.

The notion of crossed complex of groups was defined by A.L.Blakers in 1946 [4] (under the term ‘group system’) and Whitehead [37], under the term ‘homotopy system’ (except that he restricted to the free case). Blakers used these as a way of systematising known properties of relative homotopy groups \( \pi_n(X_n, X_{n-1}, x) \) of a filtered space

\[
X_* : X_0 \subseteq X_1 \subseteq X_2 \subseteq \cdots \subseteq X_n \subseteq \cdots \subseteq X_\infty.
\]

It is significant that he used the notion to establish relations between homotopy and homology of a space. Whitehead was strongly concerned with realisability, that is with the passage between

\*{ R. Brown, T.Porter), Mathematics Division, School of Informatics, University of Wales, Bangor,Gwynedd LL57 1UT, U.K. email: \{r.brown, t.porter\}@bangor.ac.uk
†Departmento da Algebra, Universidad, Granada, Spain. email: bullejos@ugr.es
KEYWORDS: crossed complex, resolution, higher syzygies, graph product.
MATHSCI 2000 CLASS: 55U99, 20J99, 18G55
algebra and geometry and back again. He explored the relations between crossed complexes and
chain complexes with a group of operators and established remarkable realisability properties, some
of which we explain later.

There was another stream of interest in crossed complexes, but in a broader algebraic framework,
in work of Frohlich [23] and Lue [31]. This gave a general formulation of cohomology groups relative
to a variety in terms of equivalence classes of certain exact sequences. However the relation of these
equivalence classes with the usual cohomology of groups was not made explicit till papers of Holt [27]
and Huebschmann [28]. The situation is described in Lue’s paper [32].

Since our interest is in the relation with homotopy theory, we are interested in the case of groups
rather than other algebraic systems. However there is one key change we have to make, as stated
above, namely that we have to generalise to groupoids rather than groups. This makes for a more
effective modelling of the geometry, since we need to use \( CW \)-complexes which are non reduced, i.e.
have more than one 0-cell, for example universal covering spaces, and simplices. It allows for easier
statement of some theorems because the coproduct for groupoids is simply disjoint union. It also
gives the category of crossed complexes better algebraic properties, principally that it is a monoidal
closed category in the sense of having an internal hom which is adjoint to a tensor product. This is a
generalisation of a standard property of groupoids: if \( \mathcal{Gpd} \) denotes the category of groupoids, then for
any groupoids \( A, B, C \) there is a natural bijection, the exponential law,

\[
\mathcal{Gpd}(A \times B, C) \cong \mathcal{Gpd}(A, \mathcal{GPD}(B, C))
\]

where \( A \times B \) is the usual product of groupoids, and \( \mathcal{GPD}(B, C) \) is the groupoid whose objects are
the morphisms \( B \to C \) and whose arrows are the natural equivalences (or conjugacies) of morphisms.
Another advantage of groupoids is that there is a standard model of the unit interval in topology,
namely the groupoid \( I \) with two objects 0,1 and exactly one arrow \( \iota : 0 \to 1 \). This leads to a
homotopy theory for groupoids in terms of morphisms \( I \times B \to C \) (in the case of groups, homotopies
of morphisms are just conjugacies). It also leads to a useful notion of \emph{fibration} of groupoids (see for
example [6]).

This exponential law for groupoids is modelled in the category \( \mathcal{Crs} \) of crossed complexes by a
natural isomorphism

\[
\mathcal{Crs}(A \otimes B, C) \cong \mathcal{Crs}(A, \mathcal{CRS}(B, C));
\]

that is, \( \mathcal{Crs} \) is a monoidal closed category, as proved by Brown and Higgins in [8]. The groupoid
\( \mathcal{I} \) determines a crossed complex also written \( \mathcal{I} \) and so a homotopy theory for crossed complexes in
terms of a cylinder object \( \mathcal{I} \otimes B \) and homotopies of the form \( \mathcal{I} \otimes B \to C \). For our purposes, the key
result is the tensor product \( A \otimes B \); this has a complicated formal definition, reflecting the algebraic
complexity of the definition of crossed complex. However, for the purposes of calculating with free
crossed complexes, it is sufficient to know the boundaries of elements of the free bases, and also the
value of morphisms on these elements. Thus the great advantage is that the free crossed resolutions
model Eilenberg-Mac Lane spaces and their cellular maps (see Corollary 3.5 and Proposition 3.6), and
give modes of calculating with these which would be very difficult geometrically.

The end product of this paper (section 4) is to show how these methods enable one to compute
higher homotopical syzygies for graph products of groups, generalising to higher dimensions the re-
results of [1]. A further paper [12] will apply related methods to obtain free crossed resolutions for
amalgamated sums and HNN-extensions of groups.
1 Definitions and basic properties

A crossed complex $C$ (of groupoids) is a sequence of morphisms of groupoids over $C_0$

$$
\cdots \rightarrow C_n \xrightarrow{\delta_n} C_{n-1} \rightarrow \cdots \rightarrow C_2 \xrightarrow{\delta_2} C_1 \rightarrow \cdots$$

Here $\{C_n\}_{n \geq 2}$ is a family of groups with base point map $\beta$, so that for $p \in C_0$, we have groups $C_n(p) = \beta^{-1}(p)$, and $\delta^0, \delta^1$ are the source and targets for the groupoid $C_1$. We further require given an operation of the groupoid $C_1$ on each family of groups $C_n$ for $n \geq 2$ such that:

(i) each $\delta_n$ is a morphism over the identity on $C_0$;

(ii) $C_2 \to C_1$ is a crossed module over $C_1$;

(iii) $C_n$ is a $C_1$-module for $n \geq 3$;

(iv) $\delta : C_n \to C_{n-1}$ is an operator morphism for $n \geq 3$;

(v) $\delta \delta : C_n \to C_{n-2}$ is trivial for $n \geq 3$;

(vi) $\delta C_2$ acts trivially on $C_n$ for $n \geq 3$.

Because of axiom (iii) we shall write the composition in $C_n$ additively for $n \geq 3$, but we will use multiplicative notation in dimensions 1 and 2 (except when giving the rules for the tensor product). Note that if $a : p \to q, b : q \to r$ in $C_1$ then the composition is written $ab : p \to r$. If further $x \in C_n(p)$ then $x^a \in C_n(q)$ and the usual laws of an action apply. We write $C_1(p) = C_1(p, p)$, and $C_1$ operates on this family of groups by conjugation. The condition (iv) then implies that $\delta_2(x^a) = a^{-1}\delta_2(x)a$, while condition (ii) gives further that $x^{-1}yx = y^{\delta_2(x)}$ for $x, y \in C_2(p), a \in C_1(p, q)$. Consequently $\delta_2(C_2)$ is normal in $C_1$, and $\text{Ker } \delta_2$ is central in $C_2$ and is operated on trivially by $\delta_2(C_2)$.

Let $C$ be a crossed complex. Its fundamental groupoid $\pi_1 C$ is the quotient of the groupoid $C_1$ by the normal, totally disconnected subgroupoid $\delta C_2$. The rules for a crossed complex give $C_n$, for $n \geq 3$, and also $\text{Ker } \delta_2$, the induced structure of $\pi_1 C$-module.

The crossed complex $C$ is reduced if $C_0$ is a singleton, so that all the groupoids $C_n, n \geq 1$ are groups. This was the case considered in [1] and many other sources.

A morphism $f : C \to D$ of crossed complexes is a family of groupoid morphisms $f_n : C_n \to D_n$ ($n \geq 0$) which preserves all the structure. This defines the category $\text{Crs}$ of crossed complexes. The fundamental groupoid now gives a functor $\pi_1 : \text{Crs} \to \text{Gpd}$. This functor is left adjoint to the functor $i : \text{Gpd} \to \text{Crs}$ where for a groupoid $G$ the crossed complex $iG$ agrees with $G$ in dimensions 0 and 1, and is otherwise trivial.

An $m$-truncated crossed complex $C$ consists of all the structure defined above but only for $n \leq m$. In particular, an $m$-truncated crossed complex is for $m = 0, 1, 2$ simply a set, a groupoid, and a crossed module respectively.

One basic algebraic example of crossed complex comes from the notion of identities among relations (For more details on the following, see [1].) Let $\mathcal{P} = \langle X_1 | w \rangle$ be a presentation of a group $G$ where $w$
is a function from a set $X_2$ to $F(X_1)$, the free group on the set $X_1$ of generators of $G$. This gives an epimorphism $\varphi : F(X_1) \to G$, with kernel $N(R)$, the normal closure in $F(X_1)$ of the set $R = w(X_2)$.

Let $H$ be the free $F(X_1)$-operator group on the set $X_2$, so that $H$ is the free group on the elements $(x, u) \in X_2 \times F(X_1)$. Let $\delta' : H \to F(X_1)$ be determined by $(x, u) \mapsto u^{-1}(wx)u$, so that the image of $\delta'$ is exactly $N(R)$. Note that $F(X_1)$ operates on $H$ by $(x, u)^w = (x, uw)$, and for all $h \in H, u \in F(X_1)$ we have

$$CM1) \delta'(hu) = u^{-1}\delta'(h)u.$$ We say that $\delta' : H \to F(X_1)$ is a precrossed module.

We now define Peiffer commutators for $h, k \in H$

$$\langle h, k \rangle = h^{-1}k^{-1}hk^{\delta'h}.$$ Then $\delta'$ vanishes on Peiffer commutators. Also the subgroup $P = \langle H, H \rangle$ generated by the Peiffer commutators is a normal $F(X_1)$-invariant subgroup of $H$. So we can define $C(w) = H/P$ and obtain the exact sequence

$$C(w) \overset{\delta_2}{\to} F(X_1) \overset{\varphi}{\to} G \to 1.$$ The morphism $\delta_2$ satisfies

$$CM2) c^{-1}dc = d^{\delta_2c}$$ for all $c, d \in C(w)$.

The rules CM1, CM2 are the laws for a crossed module, and $\delta_2 : C(w) \to F(X_1)$ is known as the free crossed $F(X_1)$-module on $w$. The injection $i : X_2 \to C(w)$ has the universal property that if $\mu : M \to F(X_1)$ is a crossed module and $v : X_2 \to M$ is a function such that $\mu v = w$, then there is a unique crossed module morphism $\eta : C(w) \to M$ such that $\eta i = w$. The elements of $C(w)$ are ‘formal consequences’

$$c = \prod_{i=1}^{n}(x_i^{e_i})^{u_i}$$

where $n \geq 0, x_i \in X_2, e_i = \pm 1, u_i \in F(X_1), \delta_2(x_i^{e_i})^{u_i} = u_i^{-1}(wx)^{e_i}u$, subject to the crossed module rule $cd = dc^{\delta_2d}, c, d \in C(w)$.

The kernel $\pi(P)$ of $\delta_2$ is abelian and in fact obtains the structure of $G$-module – it is known as the $G$-module of identities among relations for the presentation.

We can now splice to the free crossed module any resolution of $\pi(P)$ by free $G$-modules, and so obtain what is called a free crossed resolution of the group $G$.

This construction is analogous to the usual construction of higher order syzygies and free resolutions for modules, but taking into account the non abelian nature of the group and its presentation, and in particular the action of $F(X_1)$ on $N(R)$.

There is a notion of homotopy for morphisms of crossed complexes defined using the tensor product and the crossed complex $I$. Assuming this we can state one of the basic homological results, namely the uniqueness up to homotopy equivalence of free crossed resolutions of a group $G$.

There is a standard free crossed resolution $F_{st}^n(G)$ of a group $G$ in which $F_1^{st}(G)$ is the free group on the set $G$ with generators $[a], a \in G; F_2^{st}(G)$ is the free crossed $F_1^{st}(G)$-module on $w : G \times G \to F_1^{st}(G)$ given by

$$w(a, b) = [a][b][ab]^{-1}, a, b \in G;$$ for $n \geq 3$, $F_n^{st}(G)$ is the free $G$-module on $G^n$, with

$$\delta_3[a, b, c] = [a, bc][ab, c]^{-1}[a, b]^{-1}[b, c][a]^{-1}.$$
and for \( n \geq 4 \)
\[
\delta_n[a_1, a_2, \ldots, a_n] = [a_2, \ldots, a_n]^{a_1} + \sum_{i=1}^{n-1} (-1)^i[a_1, a_2, \ldots, a_{i-1}, a_i a_{i+1}, a_{i+2}, \ldots, a_n] + (-1)^n[a_1, a_2, \ldots, a_{n-1}].
\]

There is an exact sequence
\[
F_2^s(G) \xrightarrow{\delta_2} F_1^s(G) \xrightarrow{\varphi} G
\]
where \( \varphi([a]) = a, \ a \in G. \)

We can now see the advantage of this setup in considering the notion of non abelian 2-cocycle on the group \( G \) with values in the group \( K \). According to standard definitions, this is a pair of functions \( k^1 : G \to Aut(K), k^2 : G \times G \to K \) satisfying certain properties. But suppose \( G \) is infinite; then it is difficult to know how to specify these functions and check the required properties.

However the 2-cocycle definition turns out to be equivalent to regarding \( k^1, k^2 \) as specifying a morphism of crossed complexes
\[
\cdots \xrightarrow{k^2} F_3^s(G) \xrightarrow{\delta_3} F_2^s(G) \xrightarrow{\delta_2} F_1^s(G) \xrightarrow{\varphi} G \xrightarrow{\partial} Aut(K)
\]

(so that \( \partial k^2 = k^1 \delta_2, k^2 \delta_3 = 0 \)), where \( K \xrightarrow{\partial} Aut(K) \) is the inner automorphism map and so is a crossed module. Further, equivalent cocycles are just homotopic morphisms.

Equivalent data to the above is thus obtained by replacing the standard free crossed resolution by any homotopy equivalent free crossed resolution.

**Example 1.1** We let \( C \) denote the infinite cyclic group. Then \( C \) has a free crossed resolution \( 1 \to \cdots \xrightarrow{\varphi} C \) with one free generator in dimension 1. It is shown in [17] that the finite cyclic group of order \( p \) with generator \( t \), say, \( C_p \) has a free crossed resolution
\[
F(C_p) : \cdots \to \mathbb{Z}[C_p] \xrightarrow{\delta_4} \mathbb{Z}[C_p] \xrightarrow{\delta_3} \mathbb{Z}[C_p] \xrightarrow{\delta_2} C_\infty \xrightarrow{\varphi} C_p
\]
with a free generator \( x_n \) in dimension \( n \) and
\[
\delta_n(x_n) = \begin{cases} x_1^p & \text{if } n = 2; \\ x_{n-1}(1-t) & \text{if } n \text{ is odd}; \\ x_{n-1}(1 + t + t^2 + \cdots + t^{p-1}) & \text{otherwise}. \end{cases}
\]

and where \( \varphi(x_1) = t \). It is shown in [15] how this enables one to recover directly results of [39] on the enumeration and classification of extensions by \( C_p \). In fact a 2-cocycle on \( C_p \) can be specified (up to homotopy) by elements \( k \in K, a \in Aut(K) \) such that \( \partial(k) = a^n \) and \( k = k^a \). The extension \( E \) determined by \( k, a \) is given by
\[
E = (C \ltimes K)/\{(x^n, k^{-1})\}
\]
where the operation yielding the semidirect product is given by \( t \) operating via \( a \).
Similar methods can be used to determine the 3-dimensional obstruction class $l^3 \in H^3(G, A)$ corresponding to a crossed module $\mu : M \to P$ with $\text{Coker } \mu = G, \text{Ker } \mu = A$, provided we have a small free crossed resolution of the group $G$, and so this focuses on methods of constructing such resolutions. This method is successfully applied to the case with $G$ finite cyclic in [17, 18].

2 Relation with topology

In order to give the basic geometric example of a crossed complex we first define a filtered space $X_*$. By this we mean a topological space $X_\infty$ and an increasing sequence of subspaces $X_0 \subseteq X_1 \subseteq \cdots \subseteq X_n \subseteq \cdots \subseteq X_\infty$.

A map $f : X_* \to Y_*$ of filtered spaces consists of a map $f : X_\infty \to Y_\infty$ of spaces such that for all $i \geq 0, f(X_i) \subseteq Y_i$. This defines the category $\text{FTop}$ of filtered spaces and their maps. This category has a monoidal structure in which $(X_* \otimes Y_*)_n = \bigcup_{p+q=n} X_p \times Y_q$,

where it is best for later purposes to take the product in the convenient category of compactly generated spaces, so that if $X_*, Y_*$ are $CW$-spaces, then so also is $X_* \otimes Y_*$. We now define the fundamental, or homotopy, crossed complex functor $\pi : \text{FTop} \to \text{Crs}$. If $C = \pi(X_*)$, then $C_0 = X_0$, and $C_1$ is the fundamental groupoid $\pi_1(X_1, X_0)$. For $n \geq 2$, $C_n = \pi_n X_*$ is the family of relative homotopy groups $\pi_n(X_n, X_{n-1})$ for all $p \in X_0$. These come equipped with the standard operations of $\pi_1 X_*$ on $\pi_n X_*$ and boundary maps $\delta : \pi_n X_* \to \pi_{n-1} X_*$, namely the boundary of the homotopy exact sequence of the triple $(X_n, X_{n-1}, X_{n-2})$. The axioms for crossed complexes are in fact those universally satisfied by this example, though this cannot be proved at this stage (see [3]).

This construction also explains why we want to consider crossed complexes of groupoids rather than just groups. The reason is exactly analogous to the reason for considering non reduced $CW$-complexes, namely that we wish to consider covering spaces, which automatically have more than one vertex in the non trivial case. Similarly, we wish to consider covering morphisms of crossed complexes as a tool for analysing presentations of groups, analogously to the way covering morphisms of groupoids were used for group theory applications by P.J. Higgins in 1964 in [24]. A key tool in this is the use of paths in a Cayley graph as giving elements of the free groupoid on the Cayley graph, so that one moves to consider presentations of groupoids. Further, as is shown by Brown and Razak in [10], higher dimensional information is obtained by regarding the free generators of the universal cover of a free crossed resolution as giving a higher order Cayley graph, i.e. a Cayley graph with higher order syzygies. This method actually yields computational methods, by using the geometry of the Cayley graph, and the notion of deformation retraction of this universal cover.

Thus crossed complexes give a useful algebraic model of the category of $CW$-complexes and cellular maps. This model does lose a lot of information, but its corresponding advantage is that it allows for algebraic description and computation, for example of morphisms and homotopies. This is the key aspect of the methods of [10]. See also the results in Theorems 3.4 - 3.6 here.
Many geometric and algebraic situations are specified by related, and often more complicated, non abelian information, not readily computable by traditional means. As an example, we mention the non abelian tensor product of groups due to Brown and Loday [14].

Thus we can say that crossed complexes:

(i) give a first step to a full non abelian theory;
(ii) have good categorical properties;
(iii) give a ‘linear’ model of homotopy types;
(iv) this model includes all homotopy 2-types;
(v) are amenable to computation;
(vi) give one form of ‘higher dimensional group’.

A further advantage of using crossed complexes of groupoids is that this allows for the category \( \text{Crs} \) to be monoidal closed: there is a tensor product \( - \otimes - \) and internal hom \( \text{CRS}(-,-) \) such that

\[
\text{Crs}(C \otimes D, E) \cong \text{Crs}(C, \text{CRS}(D, E))
\]

for all crossed complexes \( C, D, E \). Here \( \text{CRS}(D, E)_0 = \text{Crs}(D, E) \), the set of morphisms \( D \to E \), while \( \text{CRS}(D, E)_1 \) is the set of ‘1-fold left homotopies’ \( D \to E \). Note that while the tensor product can be defined directly in terms of generators and relations, such a definition makes it not easy to verify essential properties of the tensor product, such as that the tensor product of free crossed complexes is free. The proof of this fact in [9] uses the above adjointness as a necessary step to prove that \( - \otimes D \) preserves colimits.

An important result is that if \( X_*, Y_* \) are filtered spaces, then there is a natural transformation

\[
\eta : \pi(X_*) \otimes \pi(Y_*) \to \pi(X_* \otimes Y_*)
\]

which is an isomorphism if \( X_*, Y_* \) are \( CW \)-complexes (and in fact more generally [3]). In particular, the basic rules for the tensor product are modelled on the geometry of the product of cells \( E^m \otimes E^n \) where \( E^0 \) is the singleton space, \( E^1 \) is the interval \([-1, 1]\) with two 0-cells and one 1-cell, while for \( m \geq 2 \), \( E^m = e^0 \cup e^{m-1} \cup e^m \). This leads to defining relations for the tensor product. To give these we first define a bimorphism of crossed complexes (using additive notation throughout).

A bimorphism \( \theta : (A, B) \to C \) of crossed complexes is a family of maps \( \theta : A_m \times B_n \to C_{m+n} \) satisfying the following conditions, where \( a \in A_m, b \in B_n \):

(i) \( \beta(\theta((a, b))) = \theta(\beta(a, \beta(b))) \) for all \( a \in A, b \in B \).
(ii) \( \theta(a, b^{h_1}) = \theta(a, b)^{\theta(\beta a, \beta h_1)} \) if \( m \geq 0, n \geq 2 \).
(ii)' \( \theta(a^{a_1}, b) = \theta(a, b)^{\theta(a_1, \beta b)} \) if \( m \geq 2, n \geq 0 \).
The tensor product of crossed complexes $A, B$ is then given by the universal bimorphism $(A, B) \to A \otimes B$, $(a, b) \mapsto a \otimes b$. So the rules for the tensor product are obtained by replacing $\theta(a, b)$ by $a \otimes b$ in the above.

**Example 2.1** Let $\langle X|R\rangle, \langle Y|S\rangle$ be presentations of groups $G, H$ respectively and let $C(R) \to F(X)$, $C(S) \to F(Y)$ be the corresponding free crossed modules, regarded as crossed complexes of length 2. Their tensor product $T$ is of length 4 and is given as follows:

- $T_1$ is the free group on generating set $X \sqcup Y$;
- $T_2$ is the free crossed $T_1$-module on $R \sqcup S \sqcup (X \otimes Y)$ with the boundaries on $R, S$ as given before but
  \[ \delta_2(x \otimes y) = y^{-1}x^{-1}yx; \]
- $T_3$ is the free $(G \times H)$-module on generators $r \otimes y, x \otimes s$ with boundaries
  \[ \delta_3(r \otimes y) = r^{-1}y^{-1}(\delta_2r \otimes y), \quad \delta_3(x \otimes s) = (x \otimes \delta_2s)^{-1}x^{-1}s^{-1}x; \]
• $T_4$ is the free $(G \times H)$-module on generators $r \otimes s$, with boundaries

$$\delta_4(r \otimes s) = (\delta_2 r \otimes s) + (r \otimes \delta_2 s),$$

for $x \in X, y \in Y, r \in R, s \in S$.

The conventions here may seem (even are) awkward. They arise from the derivation of the tensor product via another category of 'cubical $\omega$-groupoids with connections', and the formulae are forced by our conventions for the equivalence of the two categories [5, 8]. The important point is that we can if necessary calculate with the formulae (i)-(v), because elements such as $\delta_2 r \otimes y$ may be expanded using the rules for the tensor product. Alternatively, the form $\delta_2 r \otimes y$ may be left as it is since it naturally represents, for example if $\text{dim } y = 1$, a subdivided cylinder.

A related result is that if $C, D$ are free crossed resolutions of groups $C, D$ then $C \otimes D$ is a free crossed resolution of $G \times H$, as proved by Tonks in [33] and recovered in our section 4. This allows for presentations of modules of identities among relations for a product of groups to be read off from the presentations of the individual modules. There is a lot of work on generators for modules of identities (see for example [26]) but not so much on higher syzygies.

As said above, the results of Brown and Higgins are not proved directly, but use a category equivalent to $\text{Crs}$, namely a category of 'cubical $\omega$-groupoids with connections'. It is in the latter category that the exponential law is easy to formulate and prove, as is the construction of the natural transformation $\eta$. However the proof of all the properties of the equivalence is a long story.

In particular if we set $\mathcal{I} = \pi(E^1)$, then a ‘1-fold left homotopy’ of morphisms $D \to E$ is defined to be a morphism $\mathcal{I} \otimes D \to E$. The existence of this ‘cylinder object’ $\mathcal{I} \otimes D$ allows a lot of abstract homotopy theory [24] to be applied immediately to the category $\text{Crs}$. This is useful in constructing homotopy equivalences of crossed complexes, using for example gluing lemmas.

An important construction is the simplicial nerve $NC$ of a crossed complex $C$. This is the simplicial set defined by

$$(NC)_n = \text{Crs}(\pi \Delta^n, C).$$

It directly generalises the nerve of a group. In particular this can be applied to the internal hom functor $\text{Crs}(D, E)$ to give a simplicial set $N(\text{Crs}(D, E))$ and so turn the category $\text{Crs}$ into a simplicially enriched category. This allows the full force of the methods of homotopy coherence to be used [20].

The classifying space $BC$ of a crossed complex $C$ is simply the geometric realisation $|NC|$ of the nerve of $C$. This construction generalises at the same time: the classifying space of a group; an Eilenberg-Mac Lane space $K(A, n)$, $n \geq 2$; the classifying space for local coefficients. It also includes the notion of classifying space $BM$ of a crossed module $M = (\mu : M \to P)$. Every connected $CW$-space has the homotopy 2-type of such a space, and so crossed modules classify all connected homotopy 2-types. This is one way in which crossed modules are naturally seen as 2-dimensional analogues of groups.

3 A Generalised Van Kampen Theorem

This theorem states roughly that the functor $\pi : \text{FTop} \to \text{Crs}$ preserves certain colimits. This allows the calculation of certain homotopically defined crossed complexes, and in particular to see how free crossed complexes arise from $CW$-complexes.
The following definition with one modification is taken from [7]. The modification which we emphasise is that in [7] the assumption is made throughout that all filtered spaces satisfy the condition, there called \( J_0 \), that each loop in \( X_0 \) is contractible in \( X_1 \). It seems desirable to bring this condition into the following connectivity condition.

**Definition 3.1** A filtered space \( X_* \) is called connected if the following conditions \( \varphi(X,m) \) hold for each \( m \geq 0 \):

- \( J_0 \): each loop in \( X_0 \) is contractible in \( X_1 \);
- \( \varphi(X,0) \): If \( j > 0 \), the map \( \pi_0 X_0 \to \pi_0 X_j \), induced by inclusion, is surjective
- \( \varphi(X,m) \), \( (m \geq 1) \): If \( j > m \) and \( \nu \in X_0 \), then the map
  \[
  \pi_m(X_m, X_{m-1}, \nu) \to \pi_m(X_j, X_{m-1}, \nu)
  \]
  induced by inclusion, is surjective.

The following result gives another useful formulation of this condition. We omit the proof.

**Proposition 3.2** A filtered space \( X \) is connected if and only if it is \( J_0 \) and for all \( n > 0 \) the induced map \( \pi_0 X_0 \to \pi_0 X_n \) is surjective and for all \( r > n > 0 \) and \( \nu \in X_0 \), \( \pi_n(X_r, X_n, \nu) = 0 \).

The filtration of a \( \text{CW} \)-complex by skeleta is a standard example of a connected filtered space.

Suppose for the rest of this section that \( X_* \) is a filtered space. Let \( X = X_\infty \).

We suppose given a cover \( \mathcal{U} = \{ U^\lambda \}_{\lambda \in \Lambda} \) of \( X \) such that the interiors of the sets of \( \mathcal{U} \) cover \( X \).

For each \( \zeta \in \Lambda^n \) we set
\[
U^\zeta = U^{\zeta_1} \cap \cdots \cap U^{\zeta_n},
\]
\[
U^\zeta_i = U^\zeta \cap X_i.
\]

Then \( U^\zeta_0 \subseteq U^\zeta_1 \subseteq \cdots \) is called the induced filtration \( U^\zeta_* \) of \( U^\zeta \). Consider the following \( \pi \)-diagram of the cover:
\[
\bigcup_{\zeta \in \Lambda^n} \pi U^\zeta_* \xrightarrow{a} \bigcup_{\lambda \in \Lambda} \pi U^\lambda_* \xrightarrow{c} \pi X_*
\]

Here \( \bigcup \) denotes disjoint union (which is the same as coproduct in the category of crossed complexes); \( a, b \) are determined by the inclusions \( a_\zeta : U^\lambda \cap U^\mu \to U^\lambda, b_\zeta : U^\lambda \cap U^\mu \to U^\mu \) for each \( \zeta = (\lambda, \mu) \in \Lambda^2 \); and \( c \) is determined by the inclusions \( c_\lambda : U^\lambda \to X \).

The following result constitutes a generalisation of the Van Kampen Theorem for the fundamental group (or groupoid).

**Theorem 3.3** (The coequaliser theorem for crossed complexes: Brown and Higgins [9]) Suppose that for every finite intersection \( U^\zeta \) of elements of \( \mathcal{U} \) the induced filtration \( U^\zeta_* \) is connected. Then

- (C) \( X_* \) is connected, and
- (I) in the above \( \pi \)-diagram of the cover, \( c \) is the coequaliser of \( a, b \) in the category of crossed complexes.
The proof of this theorem is not at all straightforward, and uses another category equivalent to that of crossed complexes, called the category of cubical $\omega$-groupoids with connections \cite{5}. It is this category which is adequate for two key elements of the proof, the notion of ‘algebraic inverse to subdivision’, and the ‘multiple compositions of homotopy addition lemmas’ \cite{7}. The setting up of this machinery takes considerable effort.

In this paper we shall take as a corollary that the coequaliser theorem applies to the case when $X$ is a $CW$-complex with skeletal filtration and the $U^\lambda$ form a family of subcomplexes which cover $X$.

In order to apply this result to free crossed resolutions, we need to replace free crossed resolutions by $CW$-complexes. A fundamental result for this is the following, which goes back to Whitehead \cite{38} and Wall \cite{35}, and which is discussed further by Baues in \cite{2, Chapter VI, §7}:

**Theorem 3.4** Let $X_*$ be a $CW$-filtered space, and let $\varphi : \pi X_* \to C$ be a homotopy equivalence to a free crossed complex with a preferred free basis. Then there is a $CW$-filtered space $Y_*$ and an isomorphism $\pi Y_* \cong C$ of crossed complexes with preferred basis such that $\varphi$ is realised by a homotopy equivalence $X_* \to Y_*$. 

In fact, as pointed out by Baues, Wall states his result in terms of chain complexes, but the crossed complex formulation seems more natural, and avoids questions of realisability in dimension 2, which are unsolved for chain complexes.

**Corollary 3.5** If $C$ is a free crossed resolution of a group $G$, then $C$ is realised as free crossed complex with preferred basis by some $CW$-filtered space $Y_*$. 

**Proof** We only have to note that the group $G$ has a classifying $CW$-space $BG$ whose fundamental crossed complex $\pi BG$ is homotopy equivalent to $C$. \hfill \Box

Baues also points out in \cite{2, p.657} an extension of these results which we can apply to the realisation of morphisms of free crossed resolutions.

**Proposition 3.6** Let $X = K(G,1), Y = K(H,1)$ be $CW$-models of Eilenberg-Mac Lane spaces and let $h : \pi(X_*) \to \pi(Y_*)$ be a morphism of their fundamental crossed complexes with the preferred bases given by skeletal filtrations. Then $h = \pi(g)$ for some cellular $g : X \to Y$. 

**Proof** Certainly $h$ is homotopic to $\pi(f)$ for some $f : X \to Y$ since the set of pointed homotopy classes $X \to Y$ is bijective with the morphisms of groups $G \to H$. The result follows from \cite{2, p.657,**} (‘if $f$ is $\pi$-realisable, then each element in the homotopy class of $f$ is $\pi$-realisable’). \hfill \Box

Note that from the computational point of view we will start with a morphism $G \to H$ of groups and then lift that to a morphism of free crossed resolutions. It is important for our methods that such a morphism is exactly realised by a cellular map of the cellular models of these resolutions. This is useful also because a cellular map may be difficult to describe in geometric terms, whereas it is not so hard to write down algebraically the lift of a morphism $C_p \to C_{rp}$ to a morphism $F(C_p) \to F(C_{rp})$, as is done in \cite{?}.

These results give a strategy of weaving between spaces and crossed complexes. The key problem is to prove that a construction on free crossed resolutions yields an aspherical free crossed complex, and so also a resolution. The previous result allows us to replace the free crossed resolutions by $CW$-complexes. We can also replace morphisms of free crossed resolutions by cellular maps. We have a result of Whitehead \cite{36} which allows us to build up $K(G,1)$s as pushouts of other $K(G,1)$s provided
the induced morphisms of fundamental groups are injective. The Coequaliser Theorem now gives that the resulting fundamental crossed complex is exactly the one we want. More precise details are given in the last section.

Note also an important feature of this method: we use colimits rather than exact sequences. This enables precise results in situations where exact sequences might be inadequate, since they often give information only up to extension.

The relation of crossed complex methods to the more usual chain complexes with operators is studied in [10], developing work of Whitehead [37].

4 Application to graph products of groups

Graph products of groups have been studied for example in [1, 19, 21]. The paper [1] obtains generators for the module of identities among relations, while [19] obtains a projective resolution of the graph product given projective resolutions of the individual groups. Of course our aim is for the information to be given in the non-abelian form, where appropriate.

Let \( \Gamma \) be a finite, undirected graph without loops or multiple edges. Suppose the vertices of \( \Gamma \) are well ordered, and suppose given for each vertex \( p \) of \( \Gamma \) a group \( G_p \). The graph product of the groups

\[
G_\Gamma = \prod_{p} G_p
\]

is obtained from the free product of the groups \( G_p \) (in the given order) by adding the relations \([G_p, G_q] = \{1\}\) whenever \((p, q)\) is an edge of \( \Gamma \).

We now consider the problem of constructing a free crossed resolution of the graph product \( G_\Gamma \) given free crossed resolutions \( C_p \) of each group \( G_p \). To this end we define graph products in two other contexts.

First of all we define the nerve \( N(\Gamma) \) to be the simplicial complex whose (ordered) simplices are the complete subgraphs of \( \Gamma \).

Next suppose we are given for each vertex \( p \) of \( \Gamma \) a pointed crossed complex \( C_p \). For each simplex \( \sigma = (p_0, \ldots, p_n) \) of \( N(\Gamma) \) let \( C_\sigma \) be the subcomplex of the tensor product of all the \( C_p \) (in the given vertex order) such that the \( p \)-th coordinate is the base point of \( C_p \) if \( p \notin \sigma \) and is otherwise arbitrary. Let \( C_\Gamma \) be the subcomplex generated by all the \( C_\sigma \) for all \( \sigma \in N(\Gamma) \). This we call the graph tensor product

\[
\otimes^R C_p
\]

of the crossed complexes \( C_p \).

Next suppose we are given for each vertex \( p \) of \( \Gamma \) a pointed CW-space \( X_p \). For each simplex \( \sigma = (p_0, \ldots, p_n) \) of \( N(\Gamma) \) let \( X_\sigma \) be the subset of the product of all the \( X_p \) (in the given vertex order) such that the \( p \)-th coordinate is the base point of \( X_p \) if \( p \notin \sigma \) and is otherwise arbitrary. Let \( X_\Gamma \) be the union of all the \( X_\sigma \) for all \( \sigma \in N(\Gamma) \). This we call the graph product of the spaces \( X_p \).

The main results on these spaces are:

**Theorem 4.1** If each space \( X_p \) is aspherical, then so also is their graph product.
Proof The proof is modelled on that given by Cohen in [19]. If $\Gamma$ is a complete graph the result is clear, since finite products of aspherical spaces are aspherical. We now work by induction on the number of vertices of $\Gamma$.

Suppose $\Gamma$ is not complete. Then there are vertices $p, q$ of $\Gamma$ which do not form an edge of $\Gamma$. Let $V$ be the vertex set of $\Gamma$, and let $\Gamma_0, \Gamma_1, \Gamma_2$ be the full subgraphs of $\Gamma$ on the complements in $V$ of $\{p, q\}, \{p\}, \{q\}$ respectively. Let $X_0, X_1, X_2$ be the corresponding graph products. By the inductive assumption, these are aspherical. It is clear that

$$X_\Gamma = X_1 \cup_{X_0} X_2.$$  

But the maps on fundamental groups induced by the inclusions $X_0 \to X_1, X_0 \to X_2$ are injective. A theorem of Whitehead [36] now implies that $X_\Gamma$ is aspherical. $\square$

**Theorem 4.2** The fundamental crossed complex of the graph product of $CW$-spaces is the graph tensor product of their fundamental crossed complexes.

**Proof** This is immediate from previous results on the fundamental crossed complex of a product of $CW$-spaces, and the Generalised Van Kampen Theorem. $\square$

**Corollary 4.3** If each $C_p$ is a free crossed resolution of a group $G_p$ then the graph tensor product of the crossed resolutions is a free crossed resolution of the graph product of the groups.

Note that our result is stronger than that of Cohen in [19] in that we obtain non abelian information. On the other hand he obtains information on resolutions over arbitrary rings which cannot currently be obtained by our methods.

**Example 4.4** In the case of a direct product $A \times B \times C$ of groups, generators $x, y, z$ of $A, B, C$ respectively give rise to an element of dimension 3 of the corresponding tensor product of free crossed resolutions, namely $x \otimes y \otimes z$. The boundary of this is an identity among relations, and can be worked out explicitly from the formulae for the tensor product. It in fact corresponds to the cubical Homotopy Addition Lemma (HAL) in [5, Lemma 7.1]. This gives another view of Loday’s ‘favourite example’ in [30], which is the case $A = B = C = \mathbb{C}$, the infinite cyclic group.

**Example 4.5** Suppose we take the graph product of four infinite cyclic groups $A, B, C, D$ with generators $x, y, z, w$ and where the graph $\Gamma$ is a square with vertices assigned to the groups $A, B, C, D$ in that order clockwise, say. Each group has a free crossed resolution of length 1, with the same generators, say. The graph tensor product of these resolutions then has free generators: $x, y, z, w$ in dimension 1; $x \otimes y, y \otimes z, z \otimes w, x \otimes w$ in dimension 2; and no generators in dimension 3.

**References**

[1] BAIK, Y.-G., HOWIE, J., AND PRIDE, S., ‘The identity problem for graph products of groups’, *J. Algebra* 162 (1993) 168-177.

[2] BAUES, H.J. *Algebraic homotopy*, Cambridge Studies in Advanced Mathematics, 15. Cambridge University Press, Cambridge, 1989.
[3] BAUES, H.J. AND BROWN, R., ‘On the relative homotopy groups of the product filtration and a formula of Hopf’, *J. Pure Appl. Algebra* 89 (1993) 49-61.

[4] BLAKERS, A.L., ‘Relations between homology and homotopy groups’, *Annals of Math.* 49 (1948) 428-461.

[5] BROWN, R. AND HIGGINS, P.J., ‘The algebra of cubes’, *J. Pure Appl. Algebra* 21 (1981) 233-260.

[6] BROWN, R., *Topology: a geometric account of general topology, homotopy types, and the fundamental groupoid*, Ellis Horwood, Chichester (1988) 460 pp.

[7] BROWN, R. AND HIGGINS, P.J., ‘Colimit theorems for relative homotopy groups’, *J. Pure Appl. Algebra* 22 (1981) 11-41.

[8] BROWN, R. AND HIGGINS, P.J., ‘Tensor products and homotopies for ω-groupoids and crossed complexes’, *J. Pure Appl. Algebra* 47 (1987) 1-33.

[9] BROWN, R. AND HIGGINS, P.J., ‘The classifying space of a crossed complex’, *Math. Proc. Camb. Phil. Soc.*, 110 (1991), 95-120.

[10] BROWN, R. AND HIGGINS, P.J., ‘Crossed complexes and chain complexes with operators’, *Math. Proc. Camb. Phil. Soc.* 107 (1990) 33-57.

[11] BROWN, R. AND HUEBACHMANN, J., ‘Identities among relations’, in *Low-dimensional topology*, ed. R.Brown and T.L.Thickstun, London Math. Soc. Lect. Notes 46, Cambridge University Press, (1982) 153-202.

[12] BROWN, R., MOORE, EMMA, PORTER, T. AND WENSLEY, C.D., ‘Free crossed resolutions for amalgamated sums and HNN-extensions of groups’ (in preparation).

[13] BROWN, R., MOORE, EMMA AND WENSLEY, C.D., ‘Free crossed resolutions for the fundamental groupoid of a graph of groups’ (in preparation).

[14] BROWN, R. AND LODAY, J.-L., ‘Van Kampen Theorems for diagrams of spaces’, *Topology* 26 (1987) 311-335.

[15] BROWN, R. AND PORTER, T., ‘On the Schreier theory of non-abelian extensions: generalisations and computations’, *Proceedings Royal Irish Academy* 96 (1996) 213-227.

[16] BROWN, R. AND RAZAK SALLEH, A., ‘Free crossed resolutions of groups and presentations of modules of identities among relations’, *LMS J Comp. Math.* 2 (1999) 28-61.

[17] BROWN, R. AND WENSLEY, C.D., ‘Finite induced crossed modules and the homotopy 2-type of mapping cones’, *Theory and Applications of Categories* 1 (1995) 51-74.

[18] BROWN, R. AND WENSLEY, C.D., ‘Computing crossed modules induced by an inclusion of a normal subgroup, with applications to homotopy 2-types’, *Theory and Applications of Categories* 2 (1996) 3-16.
[19] Cohen, D.E., ‘Projective resolutions for graph products of groups’, Proc. Edinburgh Math. Soc. 38 (1995) 185-188.

[20] Cordier, J.-M. and Porter, T., ‘Homotopy coherent category theory’, Trans. Amer. Math. Soc. 349 (1997) 1–54.

[21] Dicks, W., ‘An exact sequence for rings of polynomials in partly commuting indeterminates’, J. Pure Appl. Algebra 22 (1981) 215-228.

[22] Ellis, G.J. and Kholodna, Irina, ‘Three-dimensional presentations for the groups of order at most 30’, LMS J. Comp. and Math. 2 (1999) 93-117.

[23] Frohlich, A., ‘Non abelian homological algebra. I. Derived functors and satellites’, Proc. London Math Soc. (3) 11 (1961) 239-275.

[24] Higgins, P. J., ‘Presentations of groupoids, with applications to groups’. Proc. Cambridge Philos. Soc. 60 (1964) 7-20.

[25] Higgins, P. J., ‘The fundamental groupoid of a graph of groups’, J. London Math. Soc. (2) 13 (1976) 145-149.

[26] Hog-Angeloni, C., Metzler, W. and Sieradski, A.J. (Editors), Two dimensional homotopy and combinatorial group theory, London Math. Soc. Lecture Note Series 197, Cambridge University Press, Cambridge (1993).

[27] Holt, D.F., ‘An interpretation of the cohomology groups $H^n(G, M)$’, J. Algebra 60 (1979) 307-320.

[28] Huebschmann, J., ‘Crossed $n$-fold extensions and cohomology, Comm. Math. Helv. 55 (1980) 302-314.

[29] Kamps, H. and Porter, T., Abstract Homotopy and Simple Homotopy Theory, World Scientific, Singapore, (1996).

[30] Loday, J.-L., ‘Homotopical syzygies’, in Une dégustation topologique [Topological morsels]: homotopy theory in the Swiss Alps (Arolla, 1999), 99-127, Contemp. Math. 265, Amer. Math. Soc., Providence, RI, 2000.

[31] Lue, Abraham S.-T., ‘Cohomology of algebras relative to a variety’, Math. Z. 121 (1971) 220-232.

[32] Lue, Abraham S.-T., ‘Cohomology of groups relative to a variety’, J. Algebra 69 (1981) 155-174.

[33] Tonks, A., Theory and applications of crossed complexes, Ph.D. Thesis, University of Wales, Bangor, 1994. Available from http://www.informatics.bangor.ac.uk/public/mathematics/research/tonks/.

[34] Turing, A., ‘The extensions of a group’, Compositio Mathematica 5 (1938) 357-367.
[35] Wall, C.T.C., ‘Finiteness conditions for CW-complexes II’, Proc. Roy. Soc. Ser. A 295 (1966) 149-166.

[36] Whitehead, J.H.C., ‘On the asphericity of regions in a 3-sphere’, Fund. Math. 32 (1939) 149-166.

[37] Whitehead, J.H.C., ‘Combinatorial homotopy II’, Bull. Amer. Math. Soc. 55 (1949) 453-496.

[38] Whitehead, J.H.C., ‘Simple homotopy types’, Amer. J. Math. 72 (1950) 1-57.

[39] Zassenhaus, H. 1949 The theory of groups (trans. by S. Kraverts). New York, Dover.