Origin of the repulsive Casimir force in giant polarization-interconversion materials

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Achieving strong repulsive Casimir forces through engineered coatings can pave the way for micro- and nano-electromechanical applications where adhesive forces currently cause reliability issues. Here, we exploit Lifshitz theory to identify the requirements for repulsive Casimir forces in gyrotropic media for two limiting cases (ultra-strong gyroelectric and non-gyroelectric). We show that the origin of repulsive force in media with strong gyrotropy such as Weyl semi-metals arises from the giant interconversion of polarization of vacuum fluctuations.

I. INTRODUCTION

The Casimir force [1–5] exists between charge-neutral bodies separated by submicron gaps because of the quantum fluctuations of electromagnetic fields. It competes with other forces in the micro-to-nano-meter region. For majority of geometric and material configurations, the Casimir force is known to be attractive. Realizing repulsive Casimir force is not only fundamentally important but also technologically relevant for micro- and nano-electromechanical systems (MEMS and NEMS). From an engineering perspective, attractive Casimir force dominates in the sub-micrometer regime where closely spaced system parts tend to attract each other. Our findings of the mixed attractive-repulsive sign of MEMS and NEMS, to avoid attraction-induced friction. Our findings of the repulsive Casimir force could be applied in the design of MEMS and NEMS, to avoid attraction-induced friction. Our findings of the mixed attractive-repulsive Casimir force could be applied to extract energy from vacuum fluctuations.

The most-studied geometry in the context of Casimir force [6–8] is that of two parallel plates of real, dispersive materials separated by a vacuum gap, where the force is accurately described by Lifshitz theory based on the fluctuation–dissipation theorem [2, 4]. One well-known approach to obtain repulsive Casimir force [9] is to use two plates of dielectric constants $\epsilon_1, \epsilon_2$ separated by a liquid medium of permittivity $\epsilon_l$ instead of vacuum such that $\epsilon_1 < \epsilon_l < \epsilon_2$. Recent works have revealed other approaches such as the use of Teflon-coated metallic plates \cite{10}. Another intriguing approach relies on exploiting topological materials to achieve repulsive Casimir forces using topological materials. These include three dimensional topological insulators and Weyl semi-metals \cite{11–14}, two dimensional Chern insulators and the Graphene family \cite{15–17}. However, the underlying mechanism which gives rise to repulsive Casimir forces in these topological materials has remained unexplored.

In this paper, we elucidate the origin of Casimir repulsion between two ultra-strong gyrotrropic plates (Eq. (3)). Weyl semi-metals are strong (not ultra-strong) gyrotrropic medium, the Casimir force has to be determined numerically. For the repulsive Casimir force, we choose the typical parameters of the dielectric tensor of a Weyl semimetal, which could be realized in machine-learning assisted material growth. Strong gyrotrropic media have recently opened up new promising fundamental and technological avenues for thermal radiation-based devices \cite{18–21} as well.

II. FORMALISM

We start from the Lifshitz theory of the Casimir free energy for two parallel plates separated by a vacuum gap, which is

$$
\frac{E_c(d)}{A} = k_B T \times \sum_n \int \frac{d^2 k}{(2\pi)^2} \ln[\det(1 - \mathbf{R}_1 \mathbf{R}_2 e^{-2k_z d})]
$$

where $\mathbf{R}_1$ and $\mathbf{R}_2$ are the reflection matrix for the plate 1 and plate 2, $E_c(d)$ is the free energy, $A$ is the area, $k_1$ is the absolute value of the imaginary part of the wavevector $k_z$, $T$ is the temperature and $k_B$ is the Boltzmann constant. The sum over $n$ is defined as $1/2(n = 0) + \sum_{n>0}$ with $n$ the index of the Matsubara frequencies $\xi_n = \frac{\pi n k_B T}{\hbar}$.
where \( R_{ss} = R_{TE}/A_{TE} \), \( r_{sp} = R_{TM}/A_{TE} \), \( r_{pp} = R_{TM}/A_{TM} \), and \( r_{ps} = R_{TE}/A_{TM} \). Here \( A_{TE}(R_{TE}) \) are the amplitudes of injecting (reflected) transverse magnetic (TM) mode, and \( A_{TM}(R_{TM}) \) are the amplitudes of injecting (reflected) transverse electric (TE) mode. While the TE mode and the TM mode are orthogonal electromagnetic wave-functions traveling in a vacuum, in other media, the electromagnetic eigenstates may be a mixture of TE and TM modes. Thus at the boundary of a vacuum and another medium, the TE mode to TM mode transfer ratio \( (r_{ps} \text{ and } r_{sp}) \) could be non-zero. The Casimir force is the derivative of the Casimir energy, \( F = -\frac{\partial E_{\text{Cas}}(d)}{\partial d} \). If \( F > 0 \), the force is repulsive, if \( F < 0 \), the force is attractive. The Casimir pressure is defined as

\[
P = F/A = k_B T \sum\limits_n \int \frac{d^2 k}{(2\pi)^2} \left[ \frac{\partial \text{Ln}(L)}{\partial d} \right] \quad (2)
\]

where \( L = \det(1 - R_{1} R_{2} e^{-2ik_{d}d}) \). While in general the repulsive Casimir force should be verified numerically, in two limit cases (ultra-strong-gyrotropic and non-gyrotropic), we provide here the requirement for a repulsive Casimir force in its basic form, with details given in the appendix. For a ultra-strong-gyrotropic, the reflection coefficients \( |r_{sp}| \) and \( |r_{ps}| \) are much larger than \( |r_{ss}| \) and \( |r_{pp}| \), the requirement for a repulsive force is

\[
\sum\limits_n \int \frac{d^2 k}{(2\pi)^2} (r_{sp1} r_{ps2} + r_{ps1} r_{sp2}) < 0 \quad (3)
\]

For a non-gyrotropic isotropic medium, \( r_{sp1} = r_{ps1} = 0 \) and \( r_{sp2} = r_{ps2} = 0 \), the requirement for a repulsive force is

\[
\sum\limits_n \int \frac{d^2 k}{(2\pi)^2} (r_{ss1} r_{ss2} + r_{pp1} r_{pp2}) < 0 \quad (4)
\]

This requirement also holds for the weak gyrotropic material where \( |r_{sp}| \) and \( |r_{ps}| \) are much smaller than \( |r_{ss}| \) and \( |r_{pp}| \).

Based on the Dirac-Maxwell correspondence, we define a Maxwell Hamiltonian and obtain the reflection coefficients \( (r_{ss}, r_{pp}, r_{sp}, r_{ps}) \) by connecting the wave-functions (eigenstates) at the interface. The Maxwell equations for a specific medium (omitting the magneto-electric media) is given by

\[
\begin{bmatrix}
\epsilon & 0 & \frac{\partial}{\partial t} & \mathbf{E} \\
0 & \mu & -\frac{\partial}{\partial t} & \mathbf{H}
\end{bmatrix} = \begin{bmatrix}
\nabla \times \mathbf{H} \\
-\nabla \times \mathbf{E}
\end{bmatrix}
\]

where \( \mathbf{E} \) and \( \mathbf{H} \) are the electric and magnetic fields. Note that \( (\mathbf{S} \cdot \nabla)\mathbf{H} = i\nabla \times \mathbf{S} \) where \( \mathbf{S} \) is the spin-1 matrices with \( S_x, S_y \) and \( S_z \) defined as,

\[
S_x = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ i & 0 & 0 \end{bmatrix}, S_y = \begin{bmatrix} 0 & i & 0 \\ -i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, S_z = \begin{bmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}
\]

Maxwell equations transform into a Dirac-like equation \[21][23]. Assuming a plane wave \( e^{i\mathbf{k} \cdot \mathbf{r} - i\omega t} \) of the electromagnetic field, we obtain the Maxwell Hamiltonian (\( \omega = \xi_n \))

\[
H_{Max} = \begin{bmatrix} 0 & \epsilon^{-1} \mathbf{S} \cdot \mathbf{k} \\ -\mu^{-1} \mathbf{S} \cdot \mathbf{k} & 0 \end{bmatrix}
\]

(6)

For the Casimir force, the permittivity matrix at imaginary Matsubara frequencies \( \omega = i \xi_n \) is relevant. With gyrotropy axis along the \( z \)-direction, the permittivity matrix and its inverse are given below:

\[
\epsilon = \begin{bmatrix} \epsilon_1 & g & 0 \\ -g & \epsilon_1 & 0 \\ 0 & 0 & \epsilon_2 \end{bmatrix}, \epsilon^{-1} = \begin{bmatrix} d_1 & g' & 0 \\ -g' & d_1 & 0 \\ 0 & 0 & d_2 \end{bmatrix}
\]

(7)

Both are real-valued at imaginary Matsubara frequencies. Here \( d_1 = \epsilon_1/(\epsilon_1^2 + g^2) \), \( d_2 = 1/\epsilon_2 \) and \( g' = -g/(\epsilon_1^2 + g^2) \). For a plane wave travelling in the \( k_x - k_z \) plane, with the wave vector \( \mathbf{k} = (k_x, 0, q) \), the eigenvalue is given by

\[
\xi_n^2 = \left[ (d_1 + d_2)k_x^2 + 2d_1 q^2 \mp \sqrt{P} \right] / 2
\]

(8)

here \( P = k_x^2(d_1 - d_2)^2 - 4(k_x^2 + q^2)g^2g'^2 \) and we set \( c = 1 \), so \( \xi_n/c \) is written as \( \xi_n \). We obtain an inverse solution of \( q \) from \( \xi_n \) and \( k_x \), which is

\[
q^2 = 1/[2(d_1^2 + g'^2)] \times \left[ -d_1[(d_1 + d_2)k_x^2 + 2\xi_n^2] - k_x^2 g'^2 \mp \sqrt{\Delta} \right]
\]

(9)

where

\[
\Delta = [d_1(d_1 - d_2) + g'^2]k_x^4 - 4g'^2\xi_n^2(\xi_n^2 + d_2k_x^2)
\]

(10)

This gives four solutions (eigenstates) of electromagnetic waves inside a gyroelectric medium. However, along one specific direction, only two solutions are allowed. The two momenta \( q_1 \) and \( q_2 \) are chosen by the following rules: along the direction \( +z \), the plane waves should decay at infinity. This requires the imaginary part \( \text{Im}(q) > 0 \), but the real part is not restricted, \( \text{Re}(q) > 0 \) or \( \text{Re}(q) < 0 \) : along the direction \( -z \), it requires the imaginary part \( \text{Im}(q) < 0 \), with no restriction on the real part. In both cases we have \( q_1 = -q_2 \). The eigenstate is a function of the momenta \( q_1 \) and \( q_2 \), \( \psi_1 = [e_{x1}, e_{y1}, e_{z1}, h_{x1}, h_{y1}, h_{z1}]^T = [E_1, H_1]^T \) and \( \psi_2 = [e_{x2}, e_{y2}, e_{z2}, h_{x2}, h_{y2}, h_{z2}]^T = [E_2, H_2]^T \), given by \( E_1, 2 = E(q_{1,2}) \) and \( H_1, 2 = H(q_{1,2}) \), with

\[
E(q) = [E_x, -g'q_n^2, -d_2k_x(d_1k_x^2 + d_1q^2 + \xi_n^2)]
\]

(11)
where \( E_x = q[(d_1^2 + g'^2)(k_x^2 + q^2) + d_1 \xi_n^2] \) and
\[
H(q) = i \xi_n [-g'q^2, d_1 k_x^2 + d_1 q^2 + \xi_n^2, g'k_x q] \tag{12}
\]
The details of obtaining the reflection coefficients are given in the appendix. For a non-gyroelectric medium, \( g' = 0 \) and \( d_1 = d_2 \), we obtain \( q = iq_1 \) from Eq. (9) where \( q_1^2 = k_x^2 + \xi_n^2/d_1 = \varepsilon(i \xi_n) \xi_n^2/\varepsilon^2 + k_x^2 \).

III. GYROELECTRIC MEDIUM AND THE MAGNETO-PLASMA MODEL

Now we consider a gyroelectric medium, which has non-zero off-diagonal elements in a permittivity matrix, typically given by the following magneto-plasma model:
\[
\varepsilon = \varepsilon_b - \frac{\omega_p^2}{(\omega + i \Gamma)^2 - \omega_c^2}
\begin{pmatrix}
1 + i \frac{\Gamma}{\omega_c} & -i \frac{\omega_c}{\omega} & 0 \\
-i \frac{\omega_c}{\omega} & 1 + i \frac{\Gamma}{\omega_c} & 0 \\
0 & 0 & \frac{\omega + i \Gamma)^2 - \omega_c^2}{\omega_c(\omega + i \Gamma)}
\end{pmatrix}
\tag{13}
\]
Here \( \omega_p \) is the plasma frequency, \( \omega_c \) is the cyclotron frequency. \( \Gamma \) is the inverse lifetime of the charge carriers inside the medium, microscopically determined by the scattering process from impurities, phonons or other sources. The background dielectric constant \( \varepsilon_b = \varepsilon_\infty + \varepsilon_{\text{inter}} + \varepsilon_{\text{lattice}} \), where \( \varepsilon_\infty \) is the high-frequency limit, \( \varepsilon_{\text{inter}} \) and \( \varepsilon_{\text{lattice}} \) are contributions from inter-band transitions and lattice vibrations, respectively [18]. In the Matsubara frequency domain \( \omega = i \xi_n \), the permittivity matrix (Eq. 7) is described by diagonal coefficients.

IV. STRONG GYROELECTRIC MEDIUM AND THE WEYL SEMIMETAL

The constitutive relations for an ideal Weyl semimetal is given by [21, 23]
\[
D = \varepsilon_w E + \frac{i e^2}{4 \pi^2 \hbar \omega}(2b \times E - 2b_0 B) \tag{14}
\]
where \( \varepsilon_w \) (\( \varepsilon \) in Fig. 3 and Fig. 4) is the diagonal part of the dielectric constant of the Weyl semimetal. The first term in the bracket comes from the anomalous Hall effect, and contributes to the off-diagonal part of the dielectric constant. The second term in the bracket comes from the chiral magnetic effect, for simplicity we set \( b_0 = 0 \). Here \( b \) is the momentum-separation of the two Weyl nodes, and we choose \( b = bk_z \) to be along the \( z \)-direction. The typical frequency \( \omega_b = \frac{e^2 b}{2 \pi \hbar \varepsilon_0} \) is defined from the anomalous Hall effect, where \( \varepsilon_0 \) is the vacuum permittivity. For \( b = 0.5 A^{-1} \), we have \( \omega_b = 6949 \) THz. The microscopic theory of obtaining these parameters is provided in [26].
FIG. 4. (Color online) The integrand \( k_x \times \left[ -\frac{\partial \ln(L)}{\partial d} \right] \) in Eq. (2) (shown in a,b) and the associated reflection coefficients (shown in c,d) as a function of \( k_x \). We set the Matsubara frequency at \( n = 1 \) and choose parameters the same as those of the black and grey curves in Fig. 3. We use the red (blue) color to denote the repulsive (attractive) contribution to the Casimir force respectively in (a) and (b).

TABLE I. Off-diagonal permittivity \( g \) in comparison with diagonal permittivity \( \epsilon_1 \) for magneto-plasma model and Weyl semimetal.

|                  | \( g(n = 1) \) | \( g(n = 2) \) |
|------------------|----------------|----------------|
| Magneto-plasma model | 0.076 ≪ \( \epsilon_1 \) | 0.0096 ≪ \( \epsilon_1 \) |
| Weyl semimetal   | 6.0778 ≳ \( \epsilon_1 \) | 3.039 ≳ \( \epsilon_1 \) |

For the longitudinal optical conductivity, one finds that the optical conductivity increases linearly as a function of the frequency \( \omega \) [30]. Similarly it increases sub-linearly in topological insulators [29]. After dividing by \( \omega \), the dielectric constant \( \epsilon_w \) should not change too much as a function of \( \omega \), therefore, in the Matsubara frequency domain, we choose \( \epsilon_1 = \epsilon_2 \approx 1 \) (see discussions below equation (11) in [14]), and the off-diagonal part of the permittivity matrix to be \( g = \frac{\omega_p}{\xi_n} \). For \( n = 0 \), we add a tiny positive number to \( \xi_0 \) to avoid the divergence.

In Table I we show the absolute values of the gyrotropic coupling (off-diagonal permittivity) for the Matsubara frequencies corresponding to \( n = 1, 2 \) with the diagonal permittivity \( \epsilon_1 \approx 1 \) for both models. For the magneto-plasma model we use \( \omega_p = 120 \) THz and \( \omega_c = 24 \) THz and for the Weyl semimetal we use \( \omega_b = 1000 \) THz. The temperature is assumed to be \( T = 200 \) K.

V. NUMERICAL RESULTS

In Fig. 2, we show that gyrotropy i.e non-reciprocity is not a sufficient condition for repulsive Casimir force. We present numerical results of the Casimir force (pressure), between a silicon plate and a gyrotropic (magneto-plasma) plate which is clearly attractive. The frequency-dependent dielectric constant of silicon is given by \( \epsilon_{Si} = \epsilon(Si)_{\infty} + (\epsilon(Si)_{0} - \epsilon(Si)_{\infty}) \frac{\omega^2}{\epsilon_0(\xi_n^2 + \omega_0^2) + \frac{\omega_p^2}{\xi_n(\xi_n + 1)}} \), where \( \omega_0 = \)
of parallel gyrotropy axes, we plot the Casimir force as a function of the distance between the plates. For the case of parallel gyrotropy axes, we plot the Casimir force with three different parameter sets represented by black, red and grey curves respectively. As the distance $d$ is changed, the Casimir force is tuned from repulsive to attractive, or always repulsive, or tuned from attractive to repulsive, respectively. For the case of anti-parallel gyrotropy axes, the Casimir force is always attractive, as represented by the blue curve. For the black curve ($\epsilon = 1$ and $\omega_b = 3000$ THz), the Casimir force is tuned to be zero at $d_0 = 0.41 \mu m$ (stable equilibrium); for the grey curve ($\epsilon = 1.1$ and $\omega_b = 1000$ THz), the zero-Casimir-force point $d_0 = 0.052 \mu m$ is an unstable equilibrium.

In Fig. 4 we investigate the origin of the stable and unstable equilibria, as shown in Fig. 3 by the black and grey curves respectively. As Casimir forces depend on vacuum fluctuations at all momenta and frequencies, their interpretation is significantly more challenging than conventional narrowband (e.g., laser-driven) coherent optical forces. Therefore, we separately identify the contribution of specific plane waves to explain the origin of the repulsive Casimir force. We present the integrand of Eq. (2) in Fig. 4(b), as a function of $k_z$ and fix the Matsubara frequency at $n = 1$. The black curve is negative (contribute to an attractive Casimir force) for small $k_z$ and positive (contribute to a repulsive Casimir force) for large $k_z$. The grey curve shows an opposite trend. In Fig. 4(c) and Fig. 4(d) we plot the amplitude of the reflection coefficients for the grey curve and black curve respectively. For Fig. 4(c), $|r_{sp}|$ is larger than $|r_{pp}|$ and $|r_{ss}|$ in a narrow range of $k_z$, while for Fig. 4(d) $|r_{sp}|$ is larger than $|r_{pp}|$ (or $|r_{ss}|$) in a broad range of $k_z$.

VI. CONCLUSION

In this paper we develop a theory of Casimir force based on the Lifshitz formula and the Dirac-Maxwell correspondence. We choose a magneto-plasma model and a Weyl semimetal to apply the theory. We find repulsive Casimir force in the latter, which is a strong gyrotropic medium. We also find the repulsive Casimir force is tuned from repulsive to zero at specific distance (equilibrium). This could be applied in the design of MEMS(NEMS) to reduce the friction between tiny parts of devices.
Known that $k_1 > 0$, $L > 0$ and $e^{-2k_1d} > 0$, we use an approximation that $2k_1e^{-2k_1d}$ does not change too much in the sum of $n$ and integral over $k$. The qualitative condition for a repulsive force is

$$\sum_n \int \frac{d^2k}{(2\pi)^2} [(D_1 + D_4) - 2D_1 D_4 e^{-2k_1d}] < 0 \quad (A8)$$

If the absolute value of the reflection coefficient satisfy the condition $|r_{ss}| < 1$ and $|r_{pp}| < 1$, known that $e^{-2k_1d} < 1$, we have $|r_{ss}1_{ss2}| > |r_{ss1}r_{ss2}r_{pp1}r_{pp2}e^{-2k_1d}|$, and $|r_{pp}r_{pp2}| > |r_{ss1}r_{ss2}r_{pp1}r_{pp2}e^{-2k_1d}|$, so $|D_1 + D_4| > 2|D_1 D_4|e^{-2k_1d}$, the dominat term is $D_1 + D_4$, the repulsive force condition simplifies as

$$\sum_n \int \frac{d^2k}{(2\pi)^2} (r_{ss1}r_{ss2} + r_{pp1}r_{pp2}) < 0 \quad (A9)$$

For an ultra-strong gyrotronic material with giant polarization interconversion, we need to start from the Eqn. (A4). In this case, the reflection coefficients $|r_{sp}|$ and $|r_{ps}|$ could be much larger than $|r_{ss}|$ and $|r_{pp}|$, $D_1 \approx r_{sp1}r_{ps2}$, $D_4 \approx r_{ps1}r_{sp2}$ and $D_2 \approx 0$, $D_3 \approx 0$, follow the same logic we have

$$\sum_n \int \frac{d^2k}{(2\pi)^2} (r_{sp1}r_{ps2} + r_{ps1}r_{sp2}) < 0 \quad (A10)$$

**Appendix B: Casimir force between a perfectly conducting plate and an infinitely permeable plate**

For a perfectly conducting plate $\epsilon = \infty$ and $\mu = 1$, and for an infinitely permeable plate $\mu = \infty$ and $\epsilon = 1$. The reflection coefficients at the interface of a perfectly conducting plate and a vacuum is $r_{ss} = -1$ and $r_{pp} = 1$. The reflection coefficients at the interface of an infinitely permeable plate and a vacuum is $r_{ss} = 1$ and $r_{pp} = -1$. For both cases $r_{sp} = r_{ps} = 0$, so no polarization interconversion happens. At zero temperature, the sum of Matsubara frequencies becomes an integral $2\pi k_B T/h \times \sum_n = \int d\xi$, assume the system is isotropic, so the integral over the angle $\theta$ gives $2\pi$, the Casimir pressure is

$$P = \hbar/(2\pi) \int d\xi \int \frac{2\pi kdk}{(2\pi)^2} \left[ -\frac{\partial \ln(L)}{\partial d} \right] \quad (B1)$$

We consider two cases below:

i) Casimir force between two perfectly conducting plates,

$$L = 1 - 2e^{-2k_1d} + e^{-4k_1d} \quad (B2)$$

The derivative of $\ln(L)$ becomes,

$$-\frac{\partial \ln(L)}{\partial d} = -\frac{2k_1}{L} [2e^{-2k_1d} - 2e^{-4k_1d}] \quad (B3)$$

The attractive Casimir pressure is

$$P_C = -\frac{\hbar}{\pi^2} \int kdk \frac{e^{-2k_1d}}{1 - e^{-2k_1d}} \quad (B4)$$

Known that $k_1^2 = k^2 + (\xi/c)^2$, we could introduce polar coordinates according to $k = k_1 \sin(\phi)$ and $\xi/c = k_1 \cos(\phi)$, so $\int_0^\infty d(\xi/c) \int_0^\infty dk = \int_0^\infty k_1 dk_1 \frac{\pi}{2} d\phi$, then the Casimir pressure is

$$P_C = -\frac{\hbar c}{\pi^2} \int_0^{\pi/2} k_1^3 dk_1 \int_0^\infty \sin(\phi) d\phi \frac{e^{-2k_1d}}{1 - e^{-2k_1d}} \quad (B5)$$

Note that the integral

$$\int_0^\infty dx \frac{x^3 e^{-2x}}{1 - e^{-2x}} = \frac{\pi^4}{240} \quad (B6)$$

we have

$$P_C = -\frac{\hbar c \pi^2}{240a^4} \quad (B7)$$

ii) Casimir force between a perfectly conducting plate and an infinitely permeable plate,

$$L = 1 + 2e^{-2k_1d} + e^{-4k_1d} \quad (B8)$$

The derivative of $\ln(L)$ becomes,

$$-\frac{\partial \ln(L)}{\partial d} = -\frac{2k_1}{L} [-2e^{-2k_1d} - 2e^{-4k_1d}] \quad (B9)$$

The repulsive Casimir pressure is

$$P_B = \frac{\hbar}{\pi^2} \int d\xi \int kdk \frac{e^{-2k_1d}}{1 + e^{-2k_1d}} \quad (B10)$$

Follow the same logic in case i) and note that the integral

$$\int_0^\infty dx \frac{x^3 e^{-2x}}{1 + e^{-2x}} = \frac{7\pi^4}{1920} \quad (B11)$$

we have

$$P_B = \frac{7\hbar c \pi^2}{1920d^4} = -(7/8)P_C \quad (B12)$$

For a typical distance $d = 0.2\mu m$, $P_C = -0.8$ Pa.

**Appendix C: Reflection matrix between a vacuum and a gyrotronic plate**

For more general cases with polarization interconversion, we solve the reflection coefficients in the way below. Assume the amplitudes of TM mode and TE mode are $A_{TM}$ and $A_{TE}$ respectively, the injecting electromagnetic wave in the vacuum is,
\[ E_0 = (k_z A_{TM}/\omega, A_{TE}, -k_z A_{TM}/\omega) e^{ik_x x + ik_z z - i\omega t} \] for \( z > 0 \)

\[ H_0 = (-k_z A_{TE}/\omega, A_{TM}, k_z A_{TE}/\omega) e^{ik_x x + ik_z z - i\omega t} \] for \( z > 0 \)

The reflected wave is

\[ E_r = (-k_z R_{TM}/\omega, R_{TE}, -k_z R_{TM}/\omega) e^{ik_x x - ik_z z - i\omega t} \] for \( z > 0 \)

\[ H_r = (k_z R_{TE}/\omega, R_{TM}, k_z R_{TE}/\omega) e^{ik_x x - ik_z z - i\omega t} \] for \( z > 0 \)

The transmitted wave inside a specific medium is given by,

\[ E_t = (e_x, e_y, e_z) e^{ik_x x + i\eta z - i\omega t} \] for \( z < 0 \)

\[ H_t = (h_x, h_y, h_z) e^{ik_x x + i\eta z - i\omega t} \] for \( z < 0 \)

To obtain the Fresnel coefficients for a gyrotropic slab, we first consider the injecting TE mode to be zero (\( A_{TE} = 0 \)) and write down the boundary conditions

\[
(k_z/\omega)(A_{TM} - R_{TM}) = A_1 e_{x1} + A_2 e_{x2}
\]

\[ R_{TE} = A_1 e_{y1} + A_2 e_{y2} \]

\[ (k_z/\omega)R_{TE} = A_1 h_{x1} + A_2 h_{x2} \]

\[ (A_{TM} + R_{TM}) = A_1 h_{y1} + A_2 h_{y2} \] (C1)

from which we obtain the ratio of \( A_1 \) and \( A_2 \),

\[ [(k_z/\omega)e_{y1} - h_{x1}] A_1 = A_2 [h_{x2} - (k_z/\omega)e_{y2}] \] (C2)

where \( \psi_1 = [e_{x1}, e_{y1}, e_{z1}, h_{x1}, h_{y1}, h_{z1}]^T = [E_1, H_1]^T \) and \( \psi_2 = [e_{x2}, e_{y2}, e_{z2}, h_{x2}, h_{y2}, h_{z2}]^T = [E_2, H_2]^T \) are the transmitted electromagnetic wave function inside a specific medium (gyrotropic or magneto-electric), usually obtained numerically. We also obtain the following equations,

\[ (k_z/\omega)(A_{TM} + R_{TM}) = A_1 (k_z/\omega)h_{y1} + A_2 (k_z/\omega)h_{y2} \]

\[ 2(k_z/\omega)(A_{TM}) = A_1 [(k_z/\omega)h_{y1} + e_{x1}] + A_2 [(k_z/\omega)h_{y2} + e_{x2}] \]

\[ 2(k_z/\omega)R_{TM} = A_1 [(k_z/\omega)h_{y1} - e_{x1}] + A_2 [(k_z/\omega)h_{y2} - e_{x2}] \] (C3)

the Fresnel coefficients \( r_{pp} \) and \( r_{ps} \) are obtained,

\[ r_{pp} = R_{TM}/A_{TM} = \frac{A_1 [(k_z/\omega)h_{y1} - e_{x1}] + A_2 [(k_z/\omega)h_{y2} - e_{x2}]}{A_1 [(k_z/\omega)h_{y1} + e_{x1}] + A_2 [(k_z/\omega)h_{y2} + e_{x2}]} \] (C4)

\[ r_{ps} = R_{TE}/A_{TM} = \frac{2(A_1 h_{x1} + A_2 h_{x2})}{A_1 [(k_z/\omega)h_{y1} + e_{x1}] + A_2 [(k_z/\omega)h_{y2} + e_{x2}]} \] (C5)

Then we consider the injecting TM mode to be zero (\( A_{TM} = 0 \)), the boundary conditions are given by,

\[ (k_z/\omega)(-R_{TM}) = C_1 e_{x1} + C_2 e_{x2} \]

\[ (A_{TE} + R_{TE}) = C_1 e_{y1} + C_2 e_{y2} \]

\[ (k_z/\omega)(R_{TE} - A_{TE}) = C_1 h_{x1} + C_2 h_{x2} \]

\[ (R_{TM}) = C_1 h_{y1} + C_2 h_{y2} \] (C6)

from which we obtain the ratio of \( C_1 \) and \( C_2 \),

\[ [(k_z/\omega)h_{y1} + e_{x1}] C_1 = -C_2 [(k_z/\omega)h_{y2} + e_{x2}] \] (C7)

and the following equations,

\[ (k_z/\omega)(A_{TE} + R_{TE}) = C_1 (k_z/\omega)e_{y1} + C_2 (k_z/\omega)e_{y2} \]

\[ 2(k_z/\omega)(A_{TE}) = C_1 [(k_z/\omega)e_{y1} - h_{x1}] + C_2 [(k_z/\omega)e_{y2} - h_{x2}] \]

\[ 2(k_z/\omega)(R_{TE}) = C_1 [(k_z/\omega)e_{y1} + h_{x1}] + C_2 [(k_z/\omega)e_{y2} + h_{x2}] \] (C8)
the Fresnel coefficients $r_{ss}$ and $r_{sp}$ are obtained,

$$r_{ss} = \frac{R_{TE}/A_{TE}}{C_1[(k_z/\omega)e_{y1} - h_{x1}] + C_2[(k_z/\omega)e_{y2} - h_{x2}]}$$  \hspace{1cm} (C9)$$

$$r_{sp} = \frac{R_{TM}/A_{TE}}{-2(C_1e_{x1} + C_2e_{x2})}$$  \hspace{1cm} (C10)$$

[1] H. B. G. Casimir and D. Polder, The Influence of Retardation on the London-van der Waals Forces, Phys. Rev. 73, 360 (1948).

[2] G. M. Lifshitz, The Theory of Moleculary Attractive Forces between Solids, Sov. Phys. JETP, Vol. 2, 1956, pp. 73-83.

[3] S. K. Lamoreaux, Casimir forces: Still surprising after 60 years. Physics Today, 60, 40 (2007).

[4] G. L. Klimchitskaya, U. Mohideen, and V. M. Mostepanenko, The Casimir force between real materials: Experiment and theory, Rev. Mod. Phys. 81, 1827 (2009).

[5] A. O. Sushkov, W. J. Kim, D. A. R. Dalvit, and S. K. Lamoreaux, Observation of the thermal Casimir force, Nature Physics 7,230–233(2011).

[6] V.A. Yampol’skii, S. Savel’ev, Z.A. Mayselis, S.S. Apostolov, F. Nori, Anomalous temperature dependence of the Casimir force for thin metal films, Phys. Rev. Lett. 101, 096803 (2008).

[7] E.G. Galkina, B.A. Ivanov, S. Savel’ev, V.A. Yampol’skii, F. Nori, Drastic change of the Casimir force at the metal-insulator transition, Phys. Rev. B 80, 125119 (2009).

[8] V.A. Yampol’skii, S. Savel’ev, Z.A. Mayselis, S.S. Apostolov, F. Nori, Temperature dependence of the Casimir force for bulk lossy media, Phys. Rev. A 82, 032511 (2010).

[9] K. A. Milton, E. K. Abalo, P. Parashar, N. Pourtolami, I. Brevik, S. A. Ellingsen, Repulsive Casimir and Casimir-Polder Forces, J. Phys. A. 45 (37): 4006 (2012).

[10] Rongkuo Zhao, Lin Li, Sui Yang, Wei Bao, Yang Xia, Paul Ashby, Yuan Wang, Xiang Zhang, Stable Casimir equilibrium and quantum trapping, Science 364, Issue 6444, pp. 984-987 (2019).

[11] Adolfo G. Grushin, Alberto Cortijo, Tunable Casimir repulsion with three dimensional topological insulators, Phys. Rev. Lett. 106, 020403 (2011).

[12] Justin H. Wilson, Andrew A. Allocca, and Victor Galitski, Repulsive Casimir force between Weyl semimetals, Phys. Rev. B 91, 235115 (2015).

[13] Jiang, Qing-Dong; Wilczek, Frank, Chiral Casimir forces: Repulsive, enhanced, tunable. Physical Review B. 99 (12): 125403 (2019).

[14] M. Belén Farias, Alexander A. Zyuzin, Thomas L. Schmidt, Casimir force between Weyl semimetals in a chiral medium, Phys. Rev. B 101, 235446 (2020).

[15] W. K. Tse, and A. H. MacDonald, Phys. Rev. Lett. 109, 236806 (2012).

[16] Pablo Rodriguez-Lopez and Adolfo G. Grushin, Repulsive Casimir Effect with Chern Insulators, Phys. Rev. Lett. 112, 056804 (2014).

[17] Pablo Rodriguez-Lopez, Wilton J.M. Kort-Kamp, Diego A.R. Dalvit and Lilia M. Woods, Casimir force phase transitions in the graphene family, Nature Communications 8: 14699 (2017).

[18] Linxiao Zhu and Shanhui Fan, Persistent Directional Current at Equilibrium in Nonreciprocal Many-Body Near Field Electromagnetic Heat Transfer, Phys. Rev. Lett. 117, 134303 (2016).

[19] Bo Zhao, Cheng Guo, Christina A. C. Garcia, Prineha Narang and Shanhui Fan, Axion-Field-Enabled Nonreciprocal Thermal Radiation in Weyl Semimetals, Nano Letters 2020 20 (3), 1923-1927.

[20] Chimmay Khandekar, Farhad Khosravi, Zhou Li and Zubin Jacob, New spin-resolved thermal radiation laws for nonreciprocal bianisotropic media, New Journal of Physics 22 (12), 123005 (2020).

[21] Konstantin Y. Bliokh, Daria Smirnova and Franco Nori, Quantum spin Hall effect of light, Science 348, 1448 (2015).

[22] T. V. Mechelen and Z. Jacob, Universal spin-momentum locking of evanescent waves, Optica 3, 118 (2016).

[23] Farid Kalhor, Thomas Thundat, and Zubin Jacob, Universal spin-momentum locked optical forces, Appl. Phys. Lett. 108, 061102 (2016).

[24] O. V. Kotov and Y. E. Lozovik, Giant tunable nonreciprocity of light in Weyl semimetals, Physical Review B 98, 195446 (2018).

[25] J.-R. Soh, F. de Juan, M. G. Vergniory, N. B. M. Schröter, M. C. Rahn, D. Y. Yan, J. Jiang, M. Bristow, P. Reiss, J. N. Blandy, Y. F. Guo, Y. G. Shi, T. K. Kim, A. McCollam, S. H. Simon, Y. Chen, A. I. Colden, and A. T. Boothroyd, Ideal Weyl semimetal induced by magnetic exchange, Phys. Rev. B 100, 201102(R).

[26] A.A. Burkov, Anomalous Hall Effect in Weyl Metals, Phys. Rev. Lett. 113, 187202 (2014).

[27] A. Zyuzin, Rakesh P. Tiwari, Intrinsic Anomalous Hall Effect in Type-II Weyl Semimetals, JETP Lett. 103, 717 (2016).

[28] Zhou Li and J. P. Carbotte, Longitudinal and spin-valley Hall optical conductivity in single layer MoS$_2$, Phys. Rev. B 86, 205425 (2012).

[29] Zhou Li and J. P. Carbotte, Hexagonal warping on optical conductivity of surface states in topological insulator Bi$_2$Te$_3$, Phys. Rev. B 87, 155416 (2013).

[30] Phillip E. C. Ashby and J. P. Carbotte, Magneto-optical conductivity of Weyl semimetals, Phys. Rev. B 87, 245131 (2013).

[31] Phillip E. C. Ashby and J. P. Carbotte, Chiral anomaly and optical absorption in Weyl semimetals, Phys. Rev. B 89, 245121 (2014).

[32] Ya.I. Rodionov, K.I. Kugel, F. Nori, Effects of anisotropy
and disorder on the conductivity of Weyl semimetals, Phys. Rev. B 92, 195117 (2015).
[33] Ya. I. Rodionov, K. I. Kugel, B. A. Aronzon and F. Nori, Effect of disorder on the transverse magnetoresistance of Weyl semimetals, Phys. Rev. B 102, 205105 (2020).