The Radion in the Karch-Randall Braneworld

Ioannis Giannakis\textsuperscript{a}, James T. Liu\textsuperscript{b} and Hai-cang Ren\textsuperscript{a} \textsuperscript{†}

\textsuperscript{(a)} Physics Department, The Rockefeller University  
1230 York Avenue, New York, NY 10021–6399

\textsuperscript{(b)} Michigan Center for Theoretical Physics  
Randall Laboratory, Department of Physics, University of Michigan  
Ann Arbor, MI 48109–1120

Abstract

In a braneworld context, the radion is a massless mode coupling to the trace of the matter stress tensor. Since the radion also governs the separation between branes, it is expected to decouple from the physical spectrum in single brane scenarios, such as the one-brane Randall-Sundrum model. However, contrary to expectations, we demonstrate that the Karch-Randall radion always remains as a physical excitation, even in the single brane case. Here, the radion measures the distance not between branes, but rather between the brane and the anti-de Sitter boundary on the other side of the bulk.

\textsuperscript{†} e-mail: giannak@summit.rockefeller.edu, jimliu@umich.edu, ren@summit.rockefeller.edu
1. Introduction.

The idea that gravity may be trapped on a braneworld with an infinite extra dimension has led to much recent excitement among both particle physicists and cosmologists. From a Kaluza-Klein perspective, it has long been thought that a non-compact extra dimension would lead to a continuous spectrum without any localized gravity. However, as demonstrated by Randall and Sundrum [1,2], the traditional Kaluza-Klein result may be evaded by using a warped compactification of the form

$$ds^2 = e^{2A(y)} \bar{g}_{\mu\nu}(x) dx^\mu dx^\nu + dy^2. \tag{1.1}$$

For the Randall-Sundrum braneworld, the warp factor is given by

$$e^{2A} = e^{-2\kappa|y|}, \tag{1.2}$$

while the brane metric is flat, $\bar{g}_{\mu\nu} = \eta_{\mu\nu}$. This choice ensures that the bulk metric is simply that of AdS$_5$, written in horospherical coordinates, and with cosmological constant $\bar{R}_{MN} = -4\kappa^2 G_{MN}$. The original Randall-Sundrum model [1], denoted RS1, consisted of a compact extra dimension sandwiched between two flat branes, located at $y = 0$ (which we denote the physical brane) and at $y = y_1$ (which we denote the regulator brane). In other words, the range of $y$ in (1.2) is restricted to $y \in [0,y_1]$. However, in a subsequent modification, the second brane was removed ($y_1 \rightarrow \infty$), yielding a one brane model [2], denoted RS2. In both models, the graviton is trapped at the kink in the warp factor at $y = 0$ (i.e. on the physical brane). However it is in the latter that the novel feature of graviton localization with an infinite extra dimension shows up.

Subsequently, it was realized that the fine tuning of brane tensions inherent in the Randall-Sundrum model may be relaxed without destroying the graviton trapping feature. Gravity in the resulting bent braneworld model was studied by Karch and Randall [3], which showed that a massive graviton may be trapped on an AdS$_4$ braneworld, while a massless one would be trapped on a dS$_4$ braneworld. The former massive graviton has been the object of much investigation, particularly in regards to avoidance of the van Dam-Veltman-Zakharov discontinuity [4,5]. In particular, it was demonstrated in [6,7] that the spin-2 discontinuity is absent in an AdS background. Similarly, a possible spin-3/2 discontinuity is also absent [8,9], thus avoiding a possible obstruction to the supersymmetrization of the model. Some novel features remain, however, and indicated by the presence of a quantum discontinuity [10,11], as well as a natural AdS-Higgs mechanism for generation of graviton mass [12,13,14].

For the AdS$_4$/AdS$_5$ braneworld, with four-dimensional cosmological constant $\bar{R}_{\mu\nu} = -3k^2 \bar{g}_{\mu\nu}$, the warp factor $e^{2A}$ of (1.1) takes the form

$$e^{2A} = \frac{k^2}{\kappa^2} \cosh^2 \kappa(|y| - \bar{y}), \tag{1.3}$$
where the constant $\bar{y}$ is determined by continuity across the brane and the normalization requirement that the warp factor on the physical brane at $y = 0$ is one, i.e.

$$\text{sech}^2 \kappa \bar{y} = \frac{k^2}{\kappa^2}.$$ (1.4)

In analogy with the Randall-Sundrum model, we shall denote by KR1 the Karch-Randall model [3] which consists of two AdS$_4$ branes embedded in AdS$_5$ (with the physical brane at $y = 0$ and the regulator brane at $y = y_1$). Similarly, we will use KR2 to denote the model in which the second brane is sent to infinity.

While much attention has been paid to the trapping of a massless graviton on the Randall-Sundrum brane (and a massive one on the Karch-Randall brane), perhaps of no less importance for the braneworld is the possible existence of a radion (spin-0) degree of freedom. Of course, the radion has a natural origin from a Kaluza-Klein point of view. After all, the five-dimensional graviton naturally decomposes into a set of spin-0, spin-1 and spin-2 modes in four dimensions. By imposing $Z_2$ boundary conditions, the massless spin-1 degree of freedom may be projected out. Alternatively, the warped geometry no longer has an isometry generated by $\partial/\partial y$; hence the corresponding graviphoton ought to be absent from the spectrum. However, the radion may in principle survive.

A linearized analysis of both RS1 [15], and KR1 [16], indicates that the radion is in fact present in their spectra. The natural physical picture is that the radion parametrizes the distance between the two branes (physical and regulator). Hence the radion mode corresponds to fluctuations of the relative position between the branes, and is a scalar which couples to the trace of the stress energy tensors on the branes. Along with this picture, it is clear that as we push the second brane in the RS1 model to infinity, the radion mode blows up and in the limit becomes non-normalizable. Hence it is no longer present in the RS2 model, which has only the physical brane at $y = 0$. Another way to understand this is that without a second brane, there is no longer a physically relevant distance that may be measured by the radion. Any shift in $y$ in the RS2 model may be compensated for by a Weyl rescaling of the four-dimensional brane metric.

On the other hand, the physics of removing the second brane in the KR1 model is different from that of the Randall-Sundrum model. In the Randall-Sundrum case, the limit $y \to \infty$ corresponds to a Cauchy horizon of AdS$_5$, as is evident from (1.2). Thus pushing the second brane to infinity in RS1 corresponds to pushing it off to a horizon. On the other hand, as may be seen from (1.3), $y \to \infty$ in the Karch-Randall model corresponds to a blowing up warp factor, and hence to the (partial) boundary of AdS$_5$. In pushing the second KR1 brane to infinity, one is no longer making it disappear behind a horizon, but instead is placing it at the AdS$_5$ boundary, where it remains visible to massless particles. As a result, the radion remains in the physical spectrum in KR2, and now measures the distance between the physical brane and the AdS$_5$ boundary.
To demonstrate the physical difference between the RS1 and KR1 limits, we may compute the time it takes for light to travel from the physical brane \((y = 0)\) to the regulator brane \((y = y_1)\), and back. For the KR1 model, we find that

\[
t = 2 \int_0^{y_1} dy \, e^{-A} = \frac{4}{k} [\tan^{-1} e^{\kappa(y_1 - \bar{y})} - \tan^{-1} e^{-\kappa \bar{y}}]
\]  

(1.5)

As we take the second brane to infinity, \(y_1 \to \infty\), the time \(t\) remains finite, \(t = \frac{4}{k} \tan^{-1} e^{\kappa \bar{y}}\). Thus we observe that in the KR1 model the second brane can never be removed, even when it is pushed to infinity. Note that the Randall-Sundrum braneworld can be recovered from the Karch-Randall one by taking the limit \(k \to 0\) while keeping \(\frac{k}{\kappa} \cosh \kappa \bar{y} = 1\) fixed. In this case, we find that the light travel time in RS1 is \(t = \frac{2}{\kappa} (e^{\kappa y_1} - 1)\). In contrast to the KR1 model, this expression diverges as we take the limit \(y_1 \to \infty\), confirming the physical distinction between the boundary and Cauchy horizon of AdS_5 in the braneworld context.

What we have argued above on physical grounds is that the Karch-Randall radion is present in both the KR1 and KR2 models. In this paper we shall demonstrate that this is true by careful examination of the limit that the \(y\)-coordinate of the regulator brane is taken to infinity, namely \(y_1 \to \infty\). We will examine both the KR1 and RS1 models in the absence of matter on the regulator brane. In the case of KR1, we will demonstrate that the radion mode is retained in the limit, independent of whether Newmann or Dirichlet boundary conditions are imposed on the regulator brane. Furthermore, we will show that the linearized gravity analysis goes smoothly to that of KR2 with a natural boundary condition at \(y = \infty\). On the other hand, we will see that the radion mode in RS1 decouples in the \(y_1 \to \infty\) limit, and that the linearized gravity goes smoothly to that of RS1.

This paper is organized as follows. In section 2, we derive the linearized Einstein equations of the metric fluctuations about the braneworld, investigate the gauge symmetries (diffeomorphisms) that respect the standard (Gauss-normal) form of the metric, and discuss the possible boundary conditions on the branes. In section 3, we investigate the transverse-traceless modes, paying particular attention to boundary conditions for the two brane scenario. We also demonstrate explicitly how to recover the one brane scenario by pushing the regulator brane to infinity. We then explore the details of the radion mode in section 4, especially the KR2 limit of the KR1 brane model. The RS1 limit of KR1, the RS2 limit of RS1 and the RS2 limit of KR2 will be discussed in section 5.

2. Linearized gravity analysis.

A common feature to both the Randall-Sundrum and Karch-Randall models is the presence of an AdS_5 bulk geometry. The main difference between the models then arises in how the 3-branes are embedded in the bulk. In particular, the flat braneworld model arises only with a fine tuning of the brane tensions, while the AdS_4 braneworld corresponds to a
reduced brane tension, in effect providing insufficient brane vacuum energy to compensate for the negative bulk cosmological constant.

Since we are interested in the metric fluctuations in the braneworld, we take as our starting point the general five-dimensional metric

\[ ds^2 = G_{MN}(X) dX^M dX^N. \]  

(2.1)

The action is simply that of gravity with a bulk cosmological constant coupled to the physical and regulator branes

\[ S = K \int d^5 X \sqrt{-G} (\mathcal{R} + 12\kappa^2) + S^{(0)} + S^{(1)}, \]  

(2.2)

where \( \mathcal{R} \) is the scalar curvature computed from the five dimensional metric \( G_{MN} \) and \( K = 1/16\pi G_5 \). The terms \( S^{(0)} \) and \( S^{(1)} \) represent the matter actions on the physical and regulator branes, respectively. The Einstein equation arising from (2.2) is

\[ G_{MN} - 6\kappa^2 G_{MN} = 8\pi G_5 (T_{MN}^{(0)} + T_{MN}^{(1)}), \]  

(2.3)

where \( G_{MN} = \mathcal{R}_{MN} - \frac{1}{2} G_{MN} \mathcal{R} \) is the five-dimensional Einstein tensor and \( T_{MN}^{(s)} \) is the stress tensor of either the physical brane \( (s = 0) \) or the regulator brane \( (s = 1) \).

In order to highlight the physics of the branes and their embedding in the bulk, it is useful to make a gauge choice for the metric \( G_{MN} \) that splits it into components along the brane and perpendicular to it. In particular, we often consider a Gauss-normal coordinate system

\[ ds^2 = g_{\mu\nu}(x,y) dx^\mu dx^\nu + dy^2, \]  

(2.4)

where \( y \) is the direction perpendicular to the brane. For a one brane scenario, the brane (say at \( y = 0 \)) remains straight with respect to this gauge choice. Note, however, that in a two brane model, it is not always possible to keep both branes straight in a single coordinate patch. Furthermore, the rôle that brane bending plays will also be important for the matter coupling of the braneworld. Of course, for the two brane case, it is always possible to cover the physical space between the branes using multiple coordinate patches. For a coordinate system which is straight with respect to the physical brane at \( y = 0 \), the brane stress tensor has the form

\[ T_{\mu\nu}^{(0)} = (\lambda_0 g_{\mu\nu} + T_{\mu\nu}) \delta(y), \]  

(2.5)

where \( \lambda_0 = \frac{3}{4\pi G_5} \tanh\kappa y \) is the brane tension and \( T_{\mu\nu} \) is the stress tensor for additional matter on the brane. On the other hand, the regulator brane stress tensor is given by

\[ T_{\mu\nu}^{(1)} = \lambda_1 g_{\mu\nu} \delta(y - y_1), \]  

(2.6)
where \( \lambda_1 = \frac{3}{\pi G_5} e^{2A(y_1)} \tanh \kappa(y_1 - \bar{y}) \) and the coordinate system is taken to be straight with respect to the regulator brane at \( y = y_1 \). Note that we do not include matter on the regulator brane. In both cases, the stress energy is restricted to lie on the brane directions, so that \( T_{\mu\lambda}^{(s)} = T_{\nu\lambda}^{(s)} = 0 \). Two coordinate patches (one for the vicinity of each brane), and a single set of transition functions ought to be sufficient for the two brane scenario.

We now turn to a linearized gravity analysis of the braneworld. Writing \( G_{MN} = G_{MN}^{(0)} + \delta G_{MN} \), and expanding about a solution to the Einstein equations, (2.3), we find that the action quadratic in metric fluctuations is given by

\[
\Delta S = -\frac{K}{2} \int d^5 X \sqrt{-G^{(0)}} \left[ \delta G^{MN}(\delta G_{MN} - 6\kappa^2 \delta G_{MN}) - 8\pi G_5 \delta g^{\mu\nu} T_{\mu\nu} \delta(y) \right].
\]

(2.7)

To address the braneworld scenario, we consider metric fluctuations of the Gauss-normal form

\[
ds^2 = e^{2A(y)} (\bar{g}_{\mu\nu}(x) + h_{\mu\nu}(x, y)) dx^\mu dx^\nu + dy^2.
\]

(2.8)

A computation of the Ricci tensor and the Ricci scalar to first order in the fluctuation \( h_{\mu\nu} \) yields

\[
\begin{align*}
\mathcal{R}_{\mu\nu} &= -4\kappa^2 e^{2A}(\bar{g}_{\mu\nu} + h_{\mu\nu}) + \frac{1}{4} \bar{g}^\rho\lambda (\nabla_\mu \nabla_\rho h_{\lambda\nu} + \nabla_\nu \nabla_\rho h_{\lambda\mu}) - \frac{1}{2} (\nabla^2 h_{\mu\nu} + \nabla_\mu \nabla_\nu h) \\
&\quad - \frac{1}{2} e^{2A}(\ddot{h}_{\mu\nu} + 4\dot{A} h_{\mu\nu} + \dot{A} \bar{g}_{\mu\nu} \dot{h}) - k^2 (h_{\mu\nu} - \bar{g}_{\mu\nu} h), \\
\mathcal{R}_{4\rho} &= \frac{1}{2} \bar{g}^\nu\lambda (\nabla_\lambda \dot{h}_{\nu\rho} - \nabla_\rho \dot{h}_{\nu\lambda}), \\
\mathcal{R}_{44} &= -4 \kappa^2 - \frac{1}{2} \ddot{h} - \dot{A} \dot{h}, \\
\mathcal{R} &= -12 \kappa^2 e^{-2A} - 8 \ddot{A} - 20 \dot{A}^2 - \ddot{h} - 5 \dot{A} \dot{h} + 3 k^2 \dot{h} - \nabla^2 h + \nabla^\mu \nabla^\nu h_{\mu\nu}.
\end{align*}
\]

(2.9)

Note that the four-dimensional indices \( \mu, \nu, \ldots \) are raised and lowered using the brane metric \( \bar{g}_{\mu\nu} \). Likewise, the covariant derivatives are with respect to this metric, and \( h = \bar{g}_{\mu\nu} h_{\mu\nu} \). Here the dots denote derivatives with respect to the transverse coordinate \( y \).

The Einstein equations, written in Ricci form, are given by \( \mathcal{R}_{MN} + 4\kappa^2 g_{MN} = 8\pi G_5 (T_{MN} - \frac{1}{3} T g_{MN}) \delta(y) \), and become

\[
\begin{align*}
\nabla^2 h_{\mu\nu} - \bar{g}^\rho\lambda (\nabla_\nu \nabla_\lambda h_{\rho\mu} + \nabla_\mu \nabla_\lambda h_{\rho\nu} + \nabla_\mu \nabla_\nu h + 2k^2 (h_{\mu\nu} - \bar{g}_{\mu\nu} h) \\
&\quad + e^{2A}(\ddot{h}_{\mu\nu} + 4\dot{A} h_{\mu\nu} + \dot{A} \bar{g}_{\mu\nu} \dot{h}) = -4\pi G_5 (T_{\mu\nu} - \frac{1}{3} \bar{g}_{\mu\nu} T) \delta(y),
\end{align*}
\]

\[
\begin{align*}
\bar{g}^\nu\lambda (\nabla_\rho \dot{h}_{\nu\lambda} - \nabla_\lambda \dot{h}_{\nu\rho}) &= 0, \\
\ddot{h} + 2\dot{A} \dot{h} &= \frac{4}{3} \pi G_5 T \delta(y).
\end{align*}
\]

(2.10)

The same set of equations was also given in [17]. Here we have included the matter stress energy tensor, \( T_{\mu\nu} \delta(y) \), as a source. This is in addition to the Karch-Randall brane tension itself, which is accounted for by the kink in the warp factor. By integrating the equations (2.10) across the brane, one would obtain junction conditions relating the discontinuity
of the normal derivative of the metric $h_{\mu\nu}$ to the matter sources on the brane. Such conditions are of course equivalent to the Israel matching conditions arising from the matter on the brane. These linearized gravity equations, \( (2.10) \), will be the starting point of our investigation.

Let us keep in mind the fact that the Gauss-normal form of the metric, \( (2.8) \), admits residual gauge transformations

\[
\begin{align*}
  x^\mu &\to x^\mu + \phi^\mu(x) - \frac{1}{k^2} \dot{A}^\mu \chi(x), \\
y &\to y + \chi(x),
\end{align*}
\]

\( (2.11) \)
generated by the $x$-dependent functions $\phi^\mu(x)$ and $\chi(x)$. The transformation of the metric is given by

\[
\begin{align*}
h_{\mu\nu} &\to h_{\mu\nu} + \nabla_\mu \phi_\nu + \nabla_\nu \phi_\mu + \frac{2}{k^2} \dot{A}(-\nabla_\mu \nabla_\nu \chi + k^2 g_{\mu\nu} \chi).
\end{align*}
\]

\( (2.12) \)

While these transformations leave the Gauss-normal form of the metric invariant, the shift $y \to y + \chi(x)$ in \( (2.11) \) is a brane-bending transformation [18,19], since it moves the location of the brane, $y = 0$, to $y = \chi$. The gauge transformation (and brane bending) function $\chi(x)$ is closely related to the radion.

We now consider the requirement of conservation of the brane stress tensor. In particular, we are interested in the bulk divergence $T^{(s):N}_{M\,N}$. This may be computed to first order in $h_{\mu\nu}$. The result is

\[
\begin{align*}
  T^{(s):N}_{\mu\,N} &\equiv 0, \\
  T^{(s):N}_{4\,N} &\propto 2(4 + h) \dot{A}(y) \delta(y - y_s) + \dot{h} \delta(y - y_s).
\end{align*}
\]

\( (2.13) \)

Here, $y_s$ is the location of the brane, \textit{i.e.} $y_s = 0$ for $s = 0$ (the physical brane) and $y_s = y_1$ for $s = 1$ (the regulator brane). Since $\dot{A}(y) \propto \text{sgn}(y - y_s)$ and $\text{sgn}(z) \delta(z)$ may always be regulated to zero, the first term in \( (2.13) \) is unimportant, and we are left with the second term. The vanishing of the divergence then demands $\dot{h}(y_s) = 0$. Thus conservation of energy-momentum demands Newmann boundary conditions in general. However, Dirichlet boundary conditions are also permissible for a traceless mode, $h = 0$.

3. The transverse-traceless modes.

Now that we have set up the linearized gravity equations, we will reexamine the braneworld eigenmode equations paying particular attention to boundary conditions. Since the radion mode is of particular interest, and since its treatment demands care, it will be deferred to the following section. Here we focus on the quasi-zero mode graviton and the corresponding Kaluza-Klein tower.
Based on energy momentum conservation, we always impose Newmann boundary conditions on the physical brane. On the other hand, we consider both Newmann and Dirichlet boundary conditions on the regulator brane. We begin with the two brane scenario and then show how the one brane case may be recovered (for either boundary condition on the regulator brane) by pushing the second brane to infinity.

Since the $y$-dependence of the trace and of the longitudinal components of the metric can be readily determined by the second and third equations of (2.10), we shall focus only on the transverse-traceless components. By writing $h_{\mu\nu}(x,y) = \chi_{\mu\nu}(x)\psi(y)$ where $\chi = 0$ and $\nabla^\mu \chi_{\mu\nu} = 0$, we find that $\chi_{\mu\nu}$ satisfies the transverse-traceless spin-2 equation

$$\nabla^2 \chi_{\mu\nu} + (2k^2 - \mu^2) \chi_{\mu\nu} = 0,$$

while $\psi$ obeys the eigenvalue equation

$$\ddot{\psi} + 4\dot{A}\dot{\psi} = \mu^2 e^{-2A}\psi.$$  

The eigenvalue $\mu^2$ is identified as the four-dimensional mass of the mode. From (2.7), the off-shell action of this mode reads

$$S = \frac{K}{2} ||\psi||^2 \int d^4x \sqrt{-\bar{g}} \bar{h}^{\mu\nu}(\nabla^2 - \mu^2 + 2k^2)h_{\mu\nu} + \frac{1}{2} \int_{y=0}^{y_1} d^4x \sqrt{-\bar{g}} h^{\mu\nu}T_{\mu\nu},$$

where the norm of $\psi(y)$ given by

$$||\psi||^2 = \int_0^{y_1} dy e^{2A}\psi^2.$$  

Note that this is also the norm appropriate to the Sturm-Liouville problem defined by (3.2). In what follows, we shall set the value of $\psi$ to be finite on the physical brane, so that a diverging norm in the limit $y_1 \to \infty$ would imply zero susceptibility to matter on the physical brane (so that the corresponding mode decouples).

We now have a choice of boundary conditions for the regulator brane. Let us first consider imposing Newmann boundary conditions, $\dot{\psi}(y_1) = 0$. In this case, there exists a trivial zero mode, $\mu^2 = 0$, with eigenfunction $\psi = 1$ and corresponding norm

$$||\psi||^2 = \frac{k^2}{4\kappa^3}[\kappa y_1 - \frac{1}{2}\sinh 2\kappa(y_1 - \bar{y}) - \frac{1}{2}\sinh \kappa \bar{y}].$$

For $\mu^2 \neq 0$, we introduce the variable $\zeta = \tanh \kappa(y - \bar{y})$ and the function $\phi = \psi/(1 - \zeta^2)$. The mode equation, (3.2), is then converted into an associated Legendre equation for $\phi$

$$(1 - \zeta^2) \frac{d^2 \phi}{d\zeta^2} - 2\zeta \frac{d\phi}{d\zeta} + \left[\ell(\ell + 1) - \frac{m^2}{1 - \zeta^2}\right] \phi = 0,$$
with \( l = E_0 - 2 \) and \( m = 2 \). Here, we have introduced \( E_0 \), which is the lowest energy eigenvalue for the massive spin-2 representation \( D(E_0, s = 2) \) of the AdS\(_4\) isometry group SO(2,3). The value of \( E_0 \) is related to the four-dimensional mass according to

\[
\mu^2 = E_0(E_0 - 3)k^2 = (l + 2)(l - 1)k^2.
\]

(3.7)

The stability condition of the AdS\(_4\) brane, \( \mu^2 > -\frac{2}{3}k^2 \), rules out complex values for \( l \) and the symmetry of (3.6) with respect to \( l \rightarrow -(l + 1) \) narrows our scope down to \( l > \frac{1}{2} \).

The case of \( l = 1 \), which gives rise to \( \mu^2 = 0 \) has just been discussed and the case with \( l = 0 \) will be postponed to the next section.

For a non-compact \( y \) interval, the range of the new variable \( \zeta \) would lie between the regular singular points \(-1 \) and \( 1 \). Furthermore, the normalizability condition dictates \( l = 2, 3, 4, \ldots \). However, the presence of both the physical and regulator brane cuts off this range, so that \( \zeta \) is constrained to lie in the interval \(-1 + 2\xi < \zeta < 1 - 2\eta \), where

\[
\xi = \frac{1}{2}(1 - \tanh \kappa y), \quad \eta = \frac{1}{2}(1 - \tanh \kappa(y_1 - y)).
\]

(3.8)

For \( \xi \ll 1 \) and \( \eta \ll 1 \) the corresponding value of \( l \) will be shifted slightly away from the above integer sequence and a quasi zero mode is expected to emerge with \( l \approx 1 \). For non-integer \( l \), the most general solution to the associated Legendre equation, (3.6), is a linear combination of \( P^m_l(\zeta) \) and \( P^m_l(-\zeta) \) with \( m = 2 \), \( i.e. \)

\[
\psi(\zeta) = (1 - \zeta^2)[c_1 P^2_l(\zeta) + c_2 P^2_l(-\zeta)].
\]

(3.9)

The quantization condition on \( l \) (and hence the Kaluza-Klein mass spectrum) arises from imposing Newmann boundary conditions, which are simply \( d\psi/d\zeta|_{\zeta=-1+2\xi} = 0 \) for the physical brane and \( d\psi/d\zeta|_{\zeta=1-2\eta} = 0 \) for the regulator brane. Using the representation of the associated Legendre functions in terms of hypergeometric functions

\[
P^2_l(\zeta) = \frac{\Gamma(l + 3)}{8\Gamma(l - 1)}(1 - \zeta^2)^2 F_1(2 - l, l + 3; 3; \frac{1}{2}(1 - \zeta)),
\]

(3.10)

we find that the wavefunction and its derivative on the physical brane are given by

\[
\psi(-1 + 2\xi) \simeq (-)^{l_0}\{ -4\delta l + 2(l - 1)(l + 2)[ -2\delta l\xi + l(l + 1)(1 + (-)^{l_0}c_2)\xi^2] \},
\]

\[
\frac{d}{d\zeta}\psi(-1 + 2\xi) \simeq 2(-)^{l_0}(l - 1)(l + 2)[-\delta l + l(l + 1)(1 + (-)^{l_0}c_2)\xi],
\]

(3.11)

where we have set \( c_1 = 1 \) and \( l = l_0 + \delta l \), with \( l_0 \) an integer and \( \delta l \ll 1 \). The analogous expressions on the regulator brane may be obtained from (3.11) by symmetry

\[
\psi(1 - 2\eta) \simeq \{ -4(-)^{l_0}c_2\delta l + 2(l - 1)(l + 2)[ -2(-)^{l_0}c_2\delta l\eta + l(l + 1)(1 + (-)^{l_0}c_2)\eta^2] \},
\]

\[
\frac{d}{d\zeta}\psi(1 - 2\eta) \simeq -2(l - 1)(l + 2)[(-)^{l_0+1}c_2\delta l + l(l + 1)(1 + (-)^{l_0}c_2)\eta].
\]

(3.12)
As a result, the Newmann–Newmann (NN) boundary conditions yield the relations

\[-\delta l + l(l + 1)(1 + (-)^{l_0}c_2)\xi = 0\]
\[-(-)^{l_0}c_2\delta l + l(l + 1)(1 + (-)^{l_0}c_2)\eta = 0\]

which are satisfied for

\[\delta l = l_0(l_0 + 1)(\xi + \eta), \quad c_2 = (-)^{l_0}\eta/\xi.\]  

Thus the Kaluza-Klein graviton spectrum in the KR1 model with NN boundary conditions is given by

\[E_0 = \begin{cases} 
3, & \text{zero mode graviton,} \\
l_0 + 2 + l_0(l_0 + 1)(\xi + \eta) + \cdots, & l_0 = 1, 2, \ldots
\end{cases}\]  

where \(\xi\) and \(\eta\), as given in (3.8), parametrize the locations of the branes. In this scenario, both the true zero mode graviton \((\psi(y) = 1, l = 1)\) and the \(l_0 = 1\) quasi-zero mode are present in the spectrum.

If on the other hand we demand that the solution satisfies Dirichlet conditions on the regulator brane (ND boundary conditions) we would obtain

\[-\delta l + l(l + 1)(1 + (-)^{l_0}c_2)\xi = 0, \quad c_2 = 0\]

which is solved by

\[\delta l = l_0(l_0 + 1)\xi, \quad c_2 = 0.\]

In this ND case, the trivial zero mode, \(\psi(y) = 1\), is no longer permissible. As a result, the spectrum is simply

\[E_0 = l_0 + 2 + l_0(l_0 + 1)\xi + \cdots, \quad l_0 = 1, 2, \ldots\]

While the true zero mode graviton is only present in the NN case, the massive Kaluza-Klein spectra, (3.15) and (3.18) are similar. Taking the KR2 (single brane) limit, \(y_1 \to \infty\), we see that the trivial NN zero mode graviton becomes non-normalizable and decouples. Furthermore, the Kaluza-Klein spectra, (3.15) and (3.18), converge and yield a single smooth limit which reproduces the KR2 results of [3,20]. In particular, both Kaluza-Klein towers limit to that of the ND case, (3.18), and the resulting quasi-zero mode graviton \((l_0 = 1)\) mass

\[\mu^2 \simeq \frac{3}{2}\frac{k^4}{\kappa^2},\]  

is recovered in the limit, starting from the KR1 model with either NN or ND boundary conditions.
4. The radion mode.

We have seen in the last section that the requirement of imposing homogeneous boundary conditions on both branes (whether NN or ND) gives rise to an eigenvalue problem and hence a discrete Kaluza-Klein mass spectrum. For other values of the mass, $\mu^2$, it is impossible to satisfy both homogeneous boundary conditions simultaneously. This argument that compact extra dimensions give rise to a discrete mass spectrum with a gap is of course just the standard one.

There is a slight subtlety to keep in mind, however, and that has to deal with the fact that the boundary conditions are imposed at separate locations. In particular, since a single Gauss-normal coordinate patch cannot cover both branes simultaneously, we must impose the boundary conditions on separate coordinate patches. Since the physical bulk is unique and unaffected by choice of coordinates, this means in practice that the two simultaneous boundary conditions may be relaxed up to a gauge transformation with respect to each other. The result will not contribute to the physics of the two branes.

This issue of brane bending does not have to be directly addressed in the case of the transverse-traceless modes in vacua, as considered above. However, for other modes, it is often unavoidable. In fact, this is precisely the case for the Karch-Randall radion, which has $\mu^2 = -2k^2$, or equivalently $l = 0$. For $\mu^2 = -2k^2$, it is easy to see that the mode equation (3.2) admits two linearly independent solutions,

$$\psi_1 = \dot{A}, \quad \psi_2 = (\kappa - \dot{A})^2. \quad (4.1)$$

Furthermore, it is important to note that $\psi_1$ matches the $y$ profile of the brane bending gauge transformation (2.11) parametrized by $\chi(x)$. In this sense, $\psi_1$ is at least locally unphysical, although care may have to be taken in extending the transformation into multiple coordinate patches. Note, however, that brane bending in isolation is unphysical, as it only corresponds to choosing different coordinates with which to describe the brane. On the other hand, the second solution, $\psi_2$, cannot be gauged away, and is always physical provided the solution can be made to satisfy appropriate boundary conditions.

For the KR1 radion mode, the solutions $\psi_1$ and $\psi_2$ must be combined to satisfy the appropriate boundary conditions. As we have seen, we may consider either NN or ND boundary conditions. However, in order to describe the solution in the neighborhood of both branes, we will need to introduce two coordinate patches. The first one will be straight with respect to the physical brane (so that the location of the physical brane is at $y = 0$) while the other one will be straight with respect to the regulator brane. The combined solution for the radion in these two coordinate systems will be denoted by the pair $\psi_{\text{phy}}$ and $\psi_{\text{reg}}$, where

$$\psi_{\text{phy}}(\zeta) = c_1\zeta + c_2(1 - \zeta^2),$$

$$\psi_{\text{reg}}(\zeta) = d_1\zeta + d_2(1 - \zeta^2). \quad (4.2)$$
Although these functions are defined in separate coordinate patches, we nevertheless use the same coordinate notation, \( \zeta \), for convenience. One should, however, keep in mind that these are not directly comparable without first performing the appropriate gauge transformation.

While \( \psi_{\text{phy}} \) and \( \psi_{\text{reg}} \) provide the appropriate \( y \) profiles of the radion, the wavefunction on the brane may be denoted \( \chi(x) \). In fact, a residual coordinate transformation of the form (2.11) must necessarily relate the two solutions \( \psi_{\text{phy}} \) and \( \psi_{\text{reg}} \). As a result the metric fluctuations corresponding to the radion mode can be written as

\[
h_{\mu\nu} = (-\nabla_{\mu} \nabla_{\nu} + k^2 \bar{g}_{\mu\nu}) \chi. \tag{4.3}
\]

In addition, the transverse-traceless condition yields the simple equation of motion

\[
(-\nabla^2 + 4k^2) \chi = 0, \tag{4.4}
\]

appropriate to an \( E_0 = 4 \) scalar.

Let us now impose appropriate boundary conditions on \( \psi_{\text{phy}} \) and \( \psi_{\text{reg}} \). Focusing on the NN case, for branes located at \( \zeta = -1+2\xi \) and \( \zeta = 1-2\eta \) (in their respective coordinate patches) we demand

\[
\frac{d}{d\zeta} \psi_{\text{phy}}(-1 + 2\xi) = 0, \quad \frac{d}{d\zeta} \psi_{\text{reg}}(1 - 2\eta) = 0. \tag{4.5}
\]

This is easily solved to obtain \( c_1 = 4(1-\xi)c_2 \) and \( d_1 = 4d_2\eta \). Furthermore, since the \( \psi_2 \) component of the wavefunction is unaffected by the brane bending transformation, we may simultaneously choose \( c_2 = d_2 = 1 \). Then the radion profile takes the form

\[
\psi_{\text{phy}}(\zeta) = 4(1-\xi)\zeta + (1-\zeta)^2, \tag{4.6}
\]
on the physical brane, and

\[
\psi_{\text{reg}}^N(\zeta) = 4\eta\zeta + (1-\zeta)^2, \tag{4.7}
\]
on the regulator brane. Since \( \psi_{\text{phy}} \neq \psi_{\text{reg}} \) (except for the unreasonable case \( \xi + \eta = 1 \) since both parameters are assumed small), this demonstrates that it is impossible to satisfy both boundary conditions simultaneously on a single coordinate patch.

So far, we have been working in Gauss-normal coordinates. However, to examine the off-shell radion action, it is advantageous to transform the metric with a radion fluctuation, (4.3), to the generic form

\[
ds^2 = (e^{2A} + f)\bar{g}_{\mu\nu} dx^\mu dx^\nu + (1 - 2e^{-2A} f) dy^2, \tag{4.8}
\]
which is now straight with respect to both branes. In order to reach this form of the metric from the ‘physical’ Gauss-normal patch, we apply the coordinate transformation

\[ x^\mu \to x^\mu + \epsilon^\mu_{\text{phy}}(x, y) \]
\[ y \to y + u_{\text{phy}}(x, y) \]

(4.9)

to (4.6), where

\[ u_{\text{phy}}(x, y) = \frac{k^2}{\kappa^2} [\dot{A}(0) - \dot{A}(y)] \chi(x), \quad \dot{\epsilon}_{\text{phy}}^\mu(x, y) = -e^{-2A} \nabla^\mu u_{\text{phy}}. \]

(4.10)

Similarly, transforming (4.7) from the ‘regulator’ Gauss-normal patch may be accomplished by taking

\[ x^\mu \to x^\mu + \epsilon^\mu_{\text{reg}}(x, y) \]
\[ y \to y + u_{\text{reg}}(x, y) \]

(4.11)

where

\[ u_{\text{reg}}(x, y) = \frac{k^2}{\kappa^2} [\dot{A}(y_1) - \dot{A}(y)] \chi(x), \quad \dot{\epsilon}_{\text{reg}}^\mu(x, y) = -e^{-2A} \nabla^\mu u_{\text{reg}}. \]

(4.12)

Note that the transformation functions vanish on the appropriate branes, \( u_{\text{phy}}(x, 0) = u_{\text{reg}}(x, y_1) = 0 \), and furthermore the behavior of the metric (4.8) is governed by \( f = \frac{k^4}{\kappa^4} \chi \) for both cases. The radion action is then given by

\[ S_{\text{radion}} = \frac{3}{32 \pi G_5} \left| \psi_{\text{radion}} \right|^2 \int d^4 x \sqrt{-g} f (\nabla^2 - 4k^2) f + \frac{1}{2} \int_{y=0}^{y_1} d^4 x \sqrt{-\bar{g}} \bar{g}^{\mu\nu} T_{\mu\nu}, \]

(4.13)

in agreement with that in [16] and [21]. Here, the radion norm is

\[ \left| \left| \psi_{\text{radion}} \right| \right|^2 = \int_0^{y_1} dy e^{-2A}, \]

(4.14)

and remains finite in the KR2 limit, \( y_1 \to \infty \). As a result, we see that the Karch-Randall radion survives with finite action, even in the limit when the second brane is pushed off to the boundary of AdS_5.

We now turn to the case of ND boundary conditions. For Dirichlet boundary conditions on the second brane, the second condition of (4.5) is replaced by \( \psi_{\text{reg}}(1 - 2\eta) = 0 \). As a result, the corresponding radion wavefunction takes the form

\[ \psi_{\text{reg}}^D(\zeta) = -\frac{4\eta^2}{1 - 2\eta} \zeta + (1 - \zeta)^2. \]

(4.15)

Just as in the NN case, this is inequivalent to \( \psi_{\text{phy}} \) of (4.6). So again a brane bending transformation is unavoidable. However, we observe that in the KR2 limit, \( \eta \to 0 \), the wavefunction \( \psi_{\text{reg}} \to (1 - \zeta)^2 \) is the same regardless of NN or ND boundary conditions.
While the coordinate transformation that brings the radion fluctuation $\psi_{\text{phy}}$ on the physical patch into the form (4.8) remains the same as (4.9), the one on the regulator patch for $\psi_D^\text{reg}$ instead takes the form

$$u_D^\text{reg}(x, y) = -\frac{k^2}{\kappa^2} \dot{A}(y) \chi(x) + \frac{k^4}{2\kappa^2 A(y_1)} \left[ \frac{2\kappa^2}{k^2} - e^{-2A(y_1)} \right] \chi(x).$$

(4.16)

Note that this transformation does not vanish on the regulator brane

$$u_D^\text{reg}(x, y_1) = \frac{k^4}{2\kappa^2} \frac{e^{-2A(y_1)}}{A(y_1)} \chi(x).$$

(4.17)

As a result, even the generic form of the metric, (4.8), is incapable of describing two straight branes, except in the limit $y_1 \to \infty$.

Note that the radion corresponds to $l = 0$ exactly. However, one could ask whether any modes exist for $l$ close to zero which satisfies both boundary conditions simultaneously (thus obviating the need of a brane bending gauge transformation). We demonstrate in the appendix that such hypothetical modes do not exist.

5. Discussion.

For a two brane scenario, whether RS1 or KR1, the radion mode is clearly normalizable (as any mode would be on a compact space), and has a physical interpretation as a modulus for the distance between the branes. For this reason, it is not surprising that it is a scalar mode that is closely related to the brane bending gauge transformation and which couples universally to the trace of the stress tensor on the brane.

Since the radion is connected to the separation of two branes, in the limit when the regulator brane is pushed off to infinity, the fate of the radion is closely tied to that of the brane. As shown in the previous section, the KR1 radion survives this limit (for either NN or ND boundary conditions). At the same time, the second brane has not disappeared from view, but remains attached to the boundary of $\text{AdS}_5$. For the RS1 model, on the other hand, it may be argued that since the second brane is pushed into a Cauchy horizon (so that its presence can no longer be felt), the corresponding radion must disappear from the spectrum in this limit.

This may be seen explicitly in the above analysis by considering the Randall-Sundrum limit of the Karch-Randall model, $k \to 0$. To bring the radion wavefunction $h_{\mu\nu} = (-\nabla_\mu \nabla_\nu + k^2 \bar{g}_{\mu\nu}) \chi \psi$, with $\psi = \psi_{\text{phy}}$, to the familiar form [15] in this limit, let us first perform a gauge transformation

$$h'_{\mu\nu} = h_{\mu\nu} + \frac{k^2}{\kappa^2} \left( 1 + \frac{k^2}{4\kappa^2} \right) \nabla_\mu \nabla_\nu \chi.$$ 

(5.1)
Then we find that

$$\lim_{k \to 0} \frac{4\kappa^4}{k^4} h'_{\mu\nu} = (2e^{2\kappa y} - e^{4\kappa y}) \partial_\mu \partial_\nu \chi + 4\kappa^2 \eta_{\mu\nu} \chi. \quad (5.2)$$

Similarly we find for $\psi = \psi_{\text{reg}}^N$ that

$$\lim_{k \to 0} \frac{4\kappa^4}{k^4} \left( h_{\mu\nu} + \frac{k^2}{\kappa^2} e^{2\kappa y_1} \left[ 1 + \frac{k^2}{4\kappa^2} (2 - e^{2\kappa y_1}) \right] \nabla_\mu \nabla_\nu \chi \right) = (2e^{2\kappa y+2\kappa y_1} - e^{4\kappa y}) \partial_\mu \partial_\nu \chi + 4\kappa^2 e^{2\kappa y_1} \eta_{\mu\nu} \chi. \quad (5.3)$$

where $\chi$ satisfies the massless flat-space scalar equation $\nabla^2 \chi = 0$. The right hand side of (5.2) is in fact the RS1 radion mode. Upon transforming this to the generic coordinate system with two straight branes, (4.8), and noting that $e^{2A} = e^{-2\kappa y}$, we identify the same form of the off-shell radion action as (4.13), however with a norm that diverges like $e^{2\kappa y_1}$ as $y_1 \to \infty$.

In particular, in the Randall-Sundrum limit, the normalization integral pertaining to the Sturm-Liouville problem, (3.4), becomes (up to an overall constant)

$$||\psi||^2 = \int_0^{y_1} dy e^{-2\kappa y} \psi^2, \quad (5.4)$$

and hence the RS1 radion (5.1) is not normalizable in the RS2 limit, $y_1 \to \infty$. Note that the dominant non-normalizable term, $e^{4\kappa y}$, cannot be gauged away through a brane bending transformation. Therefore the radion will decouple in the RS2 limit, unlike for the Karch-Randall model with a single brane.

One may yet worry that the limit is not smooth, so that the KR2 model is physically distinct from the KR1 model where the second brane is pushed to infinity. However, in order to make this claim, one would need a well defined KR2 model to start with, in order to compare with the limiting procedure of KR1. In this case, starting with say the KR2 model, one would have to impose boundary conditions at infinity so that the modes remain normalizable. Although such boundary conditions are not straightforwardly classified as Dirichlet or Newmann, we have seen that the appropriate KR1 modes have well defined and unique asymptotic limits, regardless of NN or ND conditions. As a result, the limiting procedure, $y_1 \to \infty$, applied to the KR1 model will yield normalizable wavefunctions identical to those obtained by solving the KR2 system directly. Hence this limit is well defined, at least for all Karch-Randall modes except the massless KR1 graviton, which becomes non-normalizable in the limit.

The absence of a van Dam-Veltman-Zakharov discontinuity [4,5] for massive gravity in an AdS background [6,12] points to a smooth critical tension (i.e. RS2) limit for the KR2 braneworld. However, the results of [6,12] do not take the radion into account. Thus
it is also necessary to see that the KR2 radion smoothly decouples from the matter source on the brane to ensure a smooth critical tension limit. Of course, the recovery of the full RS2 solution from the KR2 one for a point source on the brane [20], the smoothness of the KR1 → KR2 limit, and the KR1 → RS1 → RS2 limit all indicate that this ought to be the case.

To prove that the radion indeed smoothly decouples, we first note that the KR2 radion action is given by (4.13) in the $y_1 \to \infty$ limit. In this case, the coupling strength of the KR2 radion to matter is inversely proportional to the norm $||\psi_{\text{radion}}||$, given by

$$||\psi_{\text{radion}}||^2 = \int_0^{\infty} dy e^{-2A} = \frac{\kappa}{k^2} (1 + \tanh \kappa y) = \frac{\kappa}{k^2} \left(1 + \sqrt{1 - \frac{k^2}{\kappa^2}}\right).$$

In the RS2 limit, $k \to 0$, the norm diverges, and hence the radion coupling vanishes. Since both the massive graviton and the radion give rise to smooth limits, we conclude that the van Dam-Veltman-Zakharov discontinuity is absent from the KR2 model, in accord with expectations.

A physical understanding of the origin of the KR2 radion may be seen without resorting to the limiting case of the KR1 model. We first recall that for the RS2 scenario, the wrap factor, given by (1.2), is monotonically and uniformly decreasing as one moves into the bulk. So in a sense, there is no preferred location off the brane. On the other hand, for the KR2 model, the warp factor of (1.3) first decreases, but then turns around and increases without bound as one moves into the bulk. The warp factor reaches a minimum at $y = \bar{y}$, which may be viewed as a preferred location in the bulk (based upon the embedding of AdS$_4$ in AdS$_5$). This is especially true in applying the holographic interpretation to the KR2 model, where $\bar{y}$ is the locus separating multiple holographic domains [22,23]. The radion is then connected to the transverse distance between the brane (located at $y = 0$) and the boundary of its holographic domain (located at $y = \bar{y}$). In fact, a four-dimensional CFT analysis of the holographic dual of the KR2 model has previously shown hints of a radion in the spectrum [13]. It would be worthwhile to revisit this analysis and to extract the appropriate radion behavior from the two-point function of the stress tensor of the CFT dual.

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Appendix.

In Section 4, we have argued that the KR braneworld does not admit any additional modes for \( l \) near zero, and hence that we have identified the complete spectrum of the system. Here, we demonstrate explicitly the absence of such a mode with \( l \approx 0 \) that satisfies Neumann boundary conditions on both branes.

Starting with (3.6), we see that the mode equation for \( l \approx 0 \) can be written as
\[
\frac{d^2 \psi}{d\zeta^2} + \frac{2\zeta}{1-\zeta^2} \frac{d\psi}{d\zeta} - \frac{2}{1-\zeta^2} \psi = -\frac{l(l+1)}{1-\zeta^2} \psi \approx -\frac{l}{1-\zeta^2} \psi^{(0)}, \tag{A.1}
\]
where \( \psi^{(0)} \) is the solution to this same equation, however with \( l = 0 \). The first order solution can be written
\[
\psi^{(1)} = \psi^{(0)} + d_1 \zeta + d_2 (1-\zeta)^2
\]
with the coefficients \( d_1 \) and \( d_2 \) given by
\[
d_1 = l \int \zeta \psi^{(0)}(1-\xi)^2 \frac{d\xi}{(1-\xi^2)W} = -l \int \zeta \psi^{(0)}(1+\xi^2), \tag{A.2}
d_2 = -l \int \zeta \psi^{(0)} \frac{\xi}{(1-\xi^2)W} = l \int \zeta \psi^{(0)}(1-\xi^2)^2,
\]
and \( W(\zeta, (1-\zeta)^2) = -1 + \zeta^2 \) is the Wronskian. We now examine the two linearly independent solutions of (A.1). Starting with the zeroth order solution \( \psi^{(0)} = \zeta \), we find
\[
d_1 = -l \left[ \frac{1}{1+\zeta} + \ln(1+\zeta) \right], \quad d_2 = \frac{l}{2} \left[ \frac{\zeta}{1-\zeta^2} - \frac{1}{2} \ln \frac{1+\zeta}{1-\zeta} \right], \tag{A.3}
\]
so that
\[
\psi^{(1)} = \zeta - \frac{l}{2} \left[ \zeta + \frac{1}{2} (1+\zeta)^2 \ln(1+\zeta) - \frac{1}{2} (1-\zeta)^2 \ln(1-\zeta) \right]. \tag{A.4}
\]
Similarly, for \( \psi^{(0)} = (1-\zeta)^2 \), we find
\[
d_1 = -l \left[ \zeta - \frac{4}{1+\zeta} - 4\ln(1+\zeta) \right], \quad d_2 = l \left[ \frac{1}{1+\zeta} + \ln(1+\zeta) \right], \tag{A.5}
\]
and
\[
\psi^{(1)} = (1-\zeta)^2 - l[\zeta^2 - 1 - \zeta - (1+\zeta)^2 \ln(1+\zeta)]. \tag{A.6}
\]

We now write the general solution as a linear combination of even \( \psi_+ \) and odd \( \psi_- \) combinations, \( \psi^{(1)} = c_+ \psi_+ + c_- \psi_- \), where
\[
\psi_+ = 1 + \zeta^2 + l[1 - \zeta^2 + \frac{1}{2} (1+\zeta)^2 \ln(1+\zeta) + \frac{1}{2} (1-\zeta)^2 \ln(1-\zeta)], \tag{A.7}
\]
\[
\psi_- = 2\zeta + l[\zeta + \frac{1}{2} (1+\zeta)^2 \ln(1+\zeta) - \frac{1}{2} (1-\zeta)^2 \ln(1-\zeta)].
\]
Subsequently we demand that the radion wavefunction satisfies Newmann boundary conditions both on the physical brane located at $\zeta = -1 + 2\xi$ and on the regulator brane at $\zeta = 1 - 2\eta$. This requirement leads to the following two equations for $c_+$ and $c_-:$

\begin{align}
[1 - l(1 + \ln 2)]c_- + [-2 + l(1 - 2\ln 2)]c_+ &= 0, \\
[1 - l(1 + \ln 2)]c_- + [2 - l(1 - 2\ln 2)]c_+ &= 0.
\end{align}

(A.8)

Since the determinant of the coefficient matrix for this set of equations is nonvanishing for $l \ll 1$, no non-trivial solutions are possible. This demonstrates the absence of any additional modes near $l \simeq 0$ compatible with NN boundary conditions.

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