DUAL QUADRANGLES IN THE PLANE

IRINA BUSJATSKAJA AND YURY KOCHETKOV

Abstract. We consider quadrangles of perimeter 2 in the plane with marked directed edge. To such quadrangle $Q$ a two-dimensional plane $\Pi \in \mathbb{R}^4$ with orthonormal base is corresponded. Orthogonal plane $\Pi^\perp$ defines a plane quadrangle $Q^\circ$ of perimeter 2 and with marked directed edge. This quadrangle is defined uniquely (up to rotation and symmetry). Quadrangles $Q$ and $Q^\circ$ will be called dual to each other. The following properties of duality are proved: a) duality preserves convexity, non convexity and self-intersection; b) duality preserves the length of diagonals; c) the sum of lengths of corresponding edges in $Q$ and $Q^\circ$ is 1.

1. Introduction

We follow the work [1] (see also the bibliography there). Let $Q$ be a quadrangle with perimeter 2 and with marked directed edge in plane $\mathbb{R}^2$. It means that we indicate the first vertex and the direction of going around of $Q$.

Remark 1.1. If perimeter of a quadrangle is not 2, then we made a dilation with some positive $\alpha$.

Let $Q = ABCD$ and $A$ be the first vertex. Vectors $\overline{AB}$, $\overline{BC}$, $\overline{CD}$ and $\overline{DA}$ we will consider as complex numbers $z_1$, $z_2$, $z_3$ and $z_4$, respectively. Then

$$z_1 + z_2 + z_3 + z_4 = 0, \quad \text{and} \quad |z_1| + |z_2| + |z_3| + |z_4| = 2.$$

Remark 1.2. The above complex description of $Q$ is invariant with respect to a translation.

In what follows we will consider only non degenerate quadrangles (with one exception in Section 4), i.e. quadrangles with non-collinear successive edges.

Let’s define complex numbers $u_1, u_2, u_3, u_4$ in the following way: a) $u_k^2 = z_i$, $k=1,2,3,4$; b) $u_1$ we choose arbitrarily; c) the rotation from $u_k$ to $u_{k+1}$, $k = 1, 2, 3$ is in the same direction as the rotation from $z_k$ to $z_{k+1}$. Let $u_k = a_k + i b_k$, $k = 1, 2, 3, 4$, then

$$\sum_k (a_k^2 + b_k^2) = 2 \quad \text{and} \quad \sum_k [a_k^2 - b_k^2] + 2i a_kb_k = 0,$$

i.e. $\tilde{a} = (a_1, a_2, a_3, a_4)$ and $\tilde{b} = (b_1, b_2, b_3, b_4)$ are a pair of orthonormal vectors in $\mathbb{R}^4$. Let $\Pi = (\tilde{a}, \tilde{b})$ be the linear hull. The two-dimensional plane $\Pi$ uniquely defines its orthogonal complement — the two-dimensional plane $\Pi^\perp$. An orthonormal base $(\tilde{c}, \tilde{d})$ of $\Pi^\perp$ in its turn defines a quadrangle $Q^\circ$ of perimeter 2, which will be called the quadrangle dual to the quadrangle $Q$.

We will prove the following properties of the quadrangle duality.
• The dual quadrangle $Q^\circ$ is defined uniquely up to rotation and reflection (Theorem 2.1).
• The change of the first vertex and the direction of the going around of $Q$ does not change the dual quadrangle $Q^\circ$ (Theorem 2.2).
• The duality preserves: a) convexity (Corollary 5.1); b) non-convexity (Theorem 5.1); c) self-intersection (Theorem 4.1).
• The sum of lengths of corresponding edges (in the sense of Section 3) of $Q$ and $Q^\circ$ is 1 (Theorem 6.1).
• The lengths of the corresponding diagonals of $Q$ and $Q^\circ$ are equal (Theorem 7.1).
• Parallelograms are self-dual (Theorem 8.1).

2. General remarks

Our definition of the dual quadrangle $Q^\circ$ is not strictly correct, because the base $(\bar{c}, \bar{d})$ of $\Pi^\perp$ is not unique: it is defined up to a rotation and up to the order of base vectors.

**Theorem 2.1.** The plane $\Pi^\perp$ uniquely defines the dual quadrangle up to a rotation and up to a reflection.

**Proof.** Let a base $(\bar{e}, \bar{f})$ of $\Pi^\perp$ be obtained by the rotation of $(\bar{c}, \bar{d})$ on an angle $\alpha$. Thus,

$$e_k = c_k \cos(\alpha) - d_k \sin(\alpha), \quad f_k = c_k \sin(\alpha) + d_k \cos(\alpha), \quad k = 1, 2, 3, 4,$$

i.e.

$$e_k + i f_k = (c_k + i d_k)e^{i \alpha} \Rightarrow (e_k + i f_k)^2 = (c_k + i d_k)^2 e^{2i\alpha}, \quad k = 1, 2, 3, 4.$$

Hence, the rotation of base of $\Pi^\perp$ on angle $\alpha$ implies the rotation of $Q^\circ$ on angle $2\alpha$.

Let us now consider the base $(\bar{d}, \bar{e})$, instead of the base $(\bar{c}, \bar{d})$, then

$$\text{Re} \left( (d_k + i c_k)^2 \right) = -\text{Re} \left( (c_k + i d_k)^2 \right), \quad k = 1, 2, 3, 4,$$

$$\text{Im} \left( (d_k + i c_k)^2 \right) = \text{Im} \left( (c_k + i d_k)^2 \right), \quad k = 1, 2, 3, 4,$$

i.e. this change of base implies the reflection of $Q^\circ$ with respect to the axis $OY$. \(\square\)

Let $ABCD$ be the quadrangle $Q$, where $A$ is the first vertex and the order $ABCD$ defines the direction of going around. Let $(\bar{c}, \bar{d})$ be the base of $\Pi^\perp$ and $KLMN$ be vertices of $Q^\circ$ ($K$ is the first vertex and the order $KLMN$ defines the direction of going around).

**Theorem 2.2.** The dual of quadrangle $Q$ does not depend on the choice of the first vertex and on the direction of the going around.

**Proof.** Let us consider the going around of $Q = ABCD$ in the same direction, but the first vertex be $B$, i.e. $Q = BCDA$. Complex numbers $z_1, z_2, z_3, z_4$ are the same, but in order $z_2, z_3, z_4, z_1$. Complex numbers $u_2, u_3, u_4$ are the same, but the last one may be $u_1$ or $-u_1$. If the last number is $u_1$, then $\Pi = ((a_2, a_3, a_4, a_1), (b_2, b_3, b_4, b_1))$ and $\Pi^\perp = ((c_2, c_3, c_4, c_1), (d_2, d_3, d_4, d_1))$. Thus, if $KLMN$ is the original dual quadrangle, then $LMNK$ is the new one, but the same. If the last number is $-u_1$, then $\Pi = ((a_2, a_3, a_4, -a_1), (b_2, b_3, b_4, -b_1))$ and $\Pi^\perp = ((c_2, c_3, c_4, -c_1), (d_2, d_3, d_4, -d_1))$, i.e. the result is the same because $(-c_1 - i d_1)^2 = (c_1 + i d_1)^2$. 


Now let us consider the going around in the opposite direction, hence, \( Q = ADCB \).
In this case we must consider complex numbers \(-z_4, -z_3, -z_2, -z_1\) and their square roots \( \pm i u_4, \pm i u_3, \pm i u_2, \pm i u_1 \). Thus
\[
\Pi = \langle \mp(a_4, a_3, a_2, a_1), \pm(b_4, b_3, b_2, b_1) \rangle
\]
and
\[
\Pi^\perp = \langle (c_4, c_3, c_2, c_1), (d_4, d_3, d_2, d_1) \rangle,
\]
i.e. \( Q^\circ = KNML \).

**Remark 2.1.** The rotation of \( Q \) does not change the plane \( \Pi \).

**Corollary 2.1.** Let \( Q^\circ \) is dual to \( Q \) and \((Q^\circ)^\circ \) is dual to \( Q^\circ \), then \( Q = (Q^\circ)^\circ \) up to a rotation and up to a reflection.

3. **The main construction**

In this section, using the knowledge of lengths of edges and angles of the quadrangle \( Q \) we will construct the base of the plane \( \Pi^\perp \).

Let \( Q = ABCD \) be positioned in the following way: \( A \) is at the origin, \( B \) is in the positive real axis, \( C \) and \( D \) are in the upper half-plane. Let \( |AB| = s_1 \), \( |BC| = s_2 \), \( |CD| = s_3 \) and \( |DA| = s_4 \). 4-dimensional vectors \( \bar{a} = (a_1, a_2, a_3, a_4) \) and \( \bar{b} = (0, b_2, b_3, b_4) \) will be considered as quaternions \( a \) and \( b \). Let \( v = (0, a_2, a_3, a_4) \),

\[
|v|^2 = a_2^2 + a_3^2 + a_4^2 = 1 - a_1^2 = 1 - s_1.
\]

We consider quaternion products \( g = a \cdot v = (-a_2^2 - a_3^2 - a_4^2, a_1, a_2, a_3, a_4) = (s_1 - 1, a_1, a_2, a_3, a_4) \) and \( h = b \cdot v = (0, b_3a_4 - b_4a_3, b_4a_2 - b_2a_4, b_2a_3 - b_3a_4) \). The corresponding vectors \( \bar{g} \) and \( \bar{h} \) constitute an orthogonal base of \( \Pi^\perp \) (not orthonormal, because \( |g| = |h| = \sqrt{1 - s_1} \)).

Let \( Q \) be a convex quadrangle

![Figure 1](image-url)

Then
\[
z_1 = s_1, \ z_2 = s_2 \exp(\pi - \beta_2), \ z_3 = s_3 \exp(2\pi - \beta_2 - \beta_3), \ z_4 = s_4 \exp(\pi + \beta_1),
\]
where \( \angle DAB = \beta_1, \ \angle ABC = \beta_2, \ \angle BCD = \beta_3 \) and \( \angle CDA = \beta_4 \). Thus,
\[
\bar{a} = [\sqrt{s_1}, \sqrt{s_2} \sin(\gamma_2), -\sqrt{s_3} \cos(\gamma_2 + \gamma_3), -\sqrt{s_4} \sin(\gamma_1)],
\]
\[
\bar{b} = [0, \sqrt{s_2} \cos(\gamma_2), \sqrt{s_3} \sin(\gamma_2 + \gamma_3), \sqrt{s_4} \cos(\gamma_1)],
\]
where \( \gamma_k = \beta_k/2, \ k = 1, 2, 3, 4, \) and
\[
\bar{g} = [s_1 - 1, \sqrt{s_1s_2} \sin(\gamma_2), -\sqrt{s_1s_3} \cos(\gamma_2 + \gamma_3), -\sqrt{s_1s_4} \sin(\gamma_1)],
\]
\[
\bar{h} = [0, -\sqrt{s_3s_4} \cos(\gamma_4), \sqrt{s_2s_4} \sin(\gamma_1 + \gamma_2), -\sqrt{s_2s_3} \cos(\gamma_3)].
\]
If our quadrangle $Q$ is non convex

\[ z_1 = s_1, \quad z_2 = s_2 \exp(\pi - \beta_2), \quad z_3 = s_3 \exp(\beta_3 - \beta_2), \quad z_4 = s_4 \exp(180 + \beta_1), \]

where $\angle DAB = \beta_1$, $\angle ABC = \beta_2$, $\angle BCD = \beta_3$ and $\angle CDA = \beta_4$. And

\[ \tilde{a} = [\sqrt{s_1}, \sqrt{s_2} \sin(\gamma_2), \sqrt{s_3} \cos(\gamma_3 - \gamma_2), -\sqrt{s_4} \sin(\gamma_1)], \]
\[ \tilde{b} = [0, \sqrt{s_2} \cos(\gamma_2), \sqrt{s_3} \sin(\gamma_3 - \gamma_2), \sqrt{s_4} \cos(\gamma_1)]. \]

Thus,

\[ \tilde{g} = [s_1 - 1, \sqrt{s_1 s_2} \sin(\gamma_2), \sqrt{s_1 s_3} \cos(\gamma_3 - \gamma_2), -\sqrt{s_1 s_4} \sin(\gamma_1)], \]
\[ \tilde{h} = [0, -\sqrt{s_3 s_4} \cos(\gamma_4), \sqrt{s_2 s_4} \sin(\gamma_1 + \gamma_2), \sqrt{s_2 s_3} \cos(\gamma_3)]. \]

If at last our quadrangle $Q$ is self-intersecting

\[ z_1 = s_1, \quad z_2 = s_2 \exp(\pi - \beta_2), \quad z_3 = s_3 \exp(\beta_3 - \beta_2), \quad z_4 = s_4 \exp(\beta_1 - \pi), \]

where $\angle ABC = \beta_2$, $\angle BCD = \beta_3$, $\angle CDA = \beta_4$, $\angle DAB = \beta_1$. And

\[ \tilde{a} = [\sqrt{s_1}, \sqrt{s_2} \sin(\gamma_2), \sqrt{s_3} \cos(\gamma_3 - \gamma_2), \sqrt{s_4} \sin(\gamma_1)], \]
\[ \tilde{b} = [0, \sqrt{s_2} \cos(\gamma_2), \sqrt{s_3} \sin(\gamma_3 - \gamma_2), \sqrt{s_4} \cos(\gamma_1)]. \]

Thus,

\[ \tilde{g} = [s_1 - 1, \sqrt{s_1 s_2} \sin(\gamma_2), \sqrt{s_1 s_3} \cos(\gamma_3 - \gamma_2), \sqrt{s_1 s_4} \sin(\gamma_1)], \]
\[ \tilde{h} = [0, \sqrt{s_3 s_4} \cos(\gamma_4), \sqrt{s_2 s_4} \sin(\gamma_1 + \gamma_2), \sqrt{s_2 s_3} \cos(\gamma_3)]. \]
4. Self-intersecting quadrangles

**Theorem 4.1.** The quadrangle, dual to a self-intersecting quadrangle, is also self-intersecting.

**Proof.** Let \( Q \) be a self-intersecting quadrangle \( ABCD \) (see Figure 3) and \( Q^o = KLMN \) — its dual. As \( z_2 \) belongs to the upper half-plane, then \( a_2 > 0 \) and \( g_2 > 0 \). As \( a_3b_4 - a_4b_3 < 0 \), because of the clockwise turn from \( z_3 \) to \( z_4 \), then \( h_2 > 0 \) and \((g_2 + ih_2)^2 \) belongs to the upper half-plane. Thus, \( M \) also belongs to the upper half-plane (\( K \) is at origin and \( L = (1 - s_1, 0) \)).

As the turn from \( z_2 \) to \( z_3 \) is clockwise, then \( a_2b_3 - a_3b_2 < 0 \) and \( h_4 > 0 \). As \( g_4 > 0 \), then \((g_4 + ih_4)^2 \) belongs to the upper half-plane. Thus, \( N \) belongs to the lower half-plane (the direction of the vector \( NK \) is up), i.e. \( Q^o \) cannot be convex — \( M \) and \( N \) belong to different half-planes with respect to \( KL \).

Now we will demonstrate that \( Q^o \) is self-intersecting. Let us consider the following highly symmetric quadrangle \( Q_0 = ABCD \)

![Figure 4](image)

where all angles \( \beta_k, k = 1, 2, 3, 4 \), are \( \pi/3 \) (see Figure 3). Its dual \( Q^o_0 = KLMN \)

![Figure 5](image)

is the same quadrangle, rotated clockwise on \( \pi/3 \). Let us assume that there exists a self-intersecting quadrangle \( Q_1 \) with non-convex dual (see below)

![Figure 6](image)
Let $Q_t$, $0 \leq t \leq 1$, be a continuous family of non-degenerate self-intersecting quadrangles, that connects $Q_0$ with $Q_1$. We construct this family by moving vertices $B$, $C$ and $D$. Then the continuous family $Q_t^\circ$ connects $Q_0^\circ$ with $Q_1^\circ$. Hence, for some $\alpha$, $0 < \alpha < 1$, the dual quadrangle $Q_\alpha^\circ$ must be degenerate:

![Figure 7](image)

Now we will consider quadrangles $(Q_t^\circ)^\circ$ and take the limit for $t \to \alpha$. Let us consider the quadrangle $Q_t^\circ$ in the left part of Figure 7 and let (with some abusing of the notation) $|KL| = s_1$, $|LM| = s_2$, $|MN| = s_3$, $|NK| = s_4$, $\angle NKL = \beta_1$, $\angle KLM = \beta_2$, $\angle LMN = \beta_3$, $\angle MNK = \beta_4$. Then

$$a_t = [\sqrt{s_1}, \sqrt{s_2} \sin(\gamma_2), -\sqrt{s_3} \cos(\gamma_2 + \gamma_3), \sqrt{s_4} \sin(\gamma_2 + \gamma_3 - \gamma_4)],$$

$$b_t = [0, \sqrt{s_2} \cos(\gamma_2), \sqrt{s_3} \sin(\gamma_2 + \gamma_3), \sqrt{s_4} \cos(\gamma_2 + \gamma_3 - \gamma_4)].$$

When $t \to \alpha$, then $Q_t^\circ$ on the left (Figure 7) is transformed into $Q_\alpha^\circ$ on the right, with angles $\angle MKL = \beta_1$, $\angle KLM = \beta_2$ and $\angle LMK = \beta_3$. When $t \to \alpha$, then $\beta_4 \to 0$, $\beta_2 \to \beta_2$, $\beta_3 \to \beta_3$ and $\beta_1 \to \pi - \beta_1$. Hence,

$$\bar{a}_\alpha = [\sqrt{s_1}, \sqrt{s_2} \sin(\bar{\gamma}_2), -\sqrt{s_3} \sin(\bar{\gamma}_1), \sqrt{s_4} \cos(\bar{\gamma}_1)],$$

$$\bar{b}_\alpha = [0, \sqrt{s_2} \cos(\bar{\gamma}_2), \sqrt{s_3} \cos(\bar{\gamma}_1), \sqrt{s_4} \sin(\bar{\gamma}_1)].$$

Thus,

$$\bar{g}_\alpha = [\sqrt{1 - s_1}, \sqrt{s_1 s_2} \sin(\bar{\gamma}_2), -\sqrt{s_1 s_3} \sin(\bar{\gamma}_1), \sqrt{s_1 s_4} \cos(\bar{\gamma}_1)],$$

$$\bar{h}_\alpha = [0, \sqrt{s_3 s_4}, -\sqrt{s_2 s_4} \cos(\bar{\gamma}_1 + \bar{\gamma}_2), -\sqrt{s_2 s_3} \sin(\bar{\gamma}_1 + \bar{\gamma}_2)].$$

The quadrangle, constructed with the use of vectors $\bar{g}_\alpha$ and $\bar{h}_\alpha$, belongs to our family $Q_t$ (Corollary 2.1). Now let us consider complex numbers $(g_\alpha)_3 + i(h_\alpha)_3$, $(g_\alpha)_4 + i(h_\alpha)_4$ and compute the product

$$\frac{(g_\alpha)_3}{(g_\alpha)_3} \cdot \frac{(h_\alpha)_4}{(g_\alpha)_4} = \frac{\sqrt{s_2 s_4} \cos(\bar{\gamma}_1 + \bar{\gamma}_2)}{\sqrt{s_1 s_3} \sin(\bar{\gamma}_1)} - \frac{\sqrt{s_2 s_4} \sin(\bar{\gamma}_1 + \bar{\gamma}_2)}{\sqrt{s_1 s_4} \cos(\bar{\gamma}_1)} = -\frac{s_2 \sin(\bar{\beta}_3)}{s_1 \sin(\bar{\beta}_1)} = -1$$

(because in triangle the ratio of an edge to the sine of the opposite angle is constant and equal to the diameter of the circumscribed circle). As this product is $-1$ then the corresponding vectors are orthogonal. Thus the squaring of this complex numbers produces collinear vectors. Hence, the quadrangle $(Q_\alpha^\circ)^\circ$ is degenerate. But it cannot be so, because it belongs to our non-degenerate family. $\square$
5. Non-convex quadrangles

Theorem 5.1. If $Q$ is non-convex quadrangle, then its dual is also non-convex.

Proof. Let $Q$ be a quadrangle in Figure 2. As $z_2$ belongs to the upper half-plane, then $a_2 > 0$, thus $g_2 > 0$. As $a_3b_4 - a_4b_3 > 0$, because of the counter clockwise turn from $z_3$ to $z_4$, then $h_2 < 0$, i.e. $(g_2 + i h_2)^2$ belongs to the lower half-plane. Thus, the vertex $M$ lies in the lower half-plane.

As $z_4$ belongs to the lower half-plane, then $a_4 < 0$, $b_4 > 0$, thus $g_4 < 0$. As $a_2b_3 - a_3b_2 < 0$, because of the clockwise turn from $z_2$ to $z_3$, then $h_4 > 0$, i.e. $(g_4 + i h_4)^2$ belongs to the lower half-plane. Thus, the vertex $N$ lies in the upper half-plane, i.e. vertices $M$ and $N$ lie in different half-planes with respect to edge $KL$. Hence, the quadrangle $KLMN$ cannot be convex. But by Theorem 4.1, it cannot be self-intersecting, so it is non-convex. □

Corollary 5.1. The dual to a convex quadrangle is also convex.

6. Edges

Theorem 6.1. Let $Q = ABCD$ be a convex quadrangle and $Q^\circ = KLMN$ be its dual. Then $|AB| + |KL| = 1$, $|BC| + |LM| = 1$, $|CD| + |MN| = 1$ and $|DA| + |NK| = 1$.

Proof. As $|g| = |h| = \sqrt{1 - s_1}$, then have to prove that $|(g_2 + i h_2)^2| = g_2^2 + h_2^2 = (1 - s_1)(1 - s_2)$. Let $|AC| = l$, then

$$g_2^2 + h_2^2 = s_1s_2\sin^2(\gamma_2) + s_3s_4\cos^2(\gamma_4) = \frac{s_1s_2 - s_1s_4\cos(\beta_2) + s_3s_4 + s_3s_4\cos(\beta_4)}{2} = \frac{s_1s_2 + (l^2 - s_1^2 - s_2^2)/2 + s_3s_4 - (l^2 - s_3^2 - s_4^2)/2}{2} = \frac{(s_3 + s_4)^2 - (s_3 - s_4)^2}{4} = (s_3 + s_4 - s_3 - s_4) = (1 - s_2)(1 - s_1).$$

The same reasoning proves that $|NK| = 1 - s_4$. As perimeters of $Q$ and $Q^\circ$ are 2, then $|MN| = 1 - s_3$. □

Remark 6.1. The same reasoning proves the theorem for non-convex and self-intersecting quadrangles.

7. Diagonals

Theorem 7.1. Let $Q = ABCD$ be a convex quadrangle and $Q^\circ = KLMN$ be its dual, then $|AC| = |KM|$ and $|BD| = |LN|$, i.e. the duality preserves lengths of diagonals.

Proof. Let $l = |AC| = |z_1 + z_2| = |z_3 + z_4|$. We will prove, that $|g_1^2 + (g_2 + i h_2)^2| = (1 - s_1)l$. At first we will find the real part of the complex number $(g_2 + i h_2)^2$:

$$\text{Re}(g_2 + i h_2)^2 = g_2^2 - h_2^2 = s_1s_2\sin^2(\gamma_2) - s_3s_4\cos^2(\gamma_4) = s_1s_2(1 - \cos(\beta_2) - s_3s_4(1 + \cos(\beta_4))/2 = 2s_1s_2 - s_1^2 - s_2^2 - 2s_3s_4 + l^2 - s_3^2 - s_4^2)/4 = 2l^2 - (s_1 - s_2)^2 - (s_3 + s_4)^2]/4.$$
Now the real part of \( g_1^2 + (g_2 + i h_2)^2 \) is
\[
(1 - s_1)^2 + [2t^2 - (s_1 - s_2)^2 - (s_3 + s_4)^2]/4 =
\]
\[
= [4(1 - s_1)^2 + 2t^2 - (s_1 - s_2)^2 - (s_3 + s_4)^2]/4 =
\]
\[
= [2(1 - s_1)^2 + 2l^2 + (1 - 2s_1 + s_2)(1 - s_2) +
\]
\[
+ (1 - s_1 + s_3 + s_4)(1 - s_1 - s_3 - s_4)]/4 =
\]
\[
= [2(1 - s_1)^2 + 2l^2 + (1 - 2s_1 + s_2)(1 - s_2) + (s_2 - 1)(3 - 2s_1 - s_2)]/4 =
\]
\[
= [2(1 - s_1)^2 + 2l^2 - 2(1 - s_2)^2]/4 = [l^2 - (s_1 - s_2)(s_3 + s_4)]/2.
\]

Now we will find the square of the imaginary part of \( g_1^2 + (g_2 + i h_2)^2 \):
\[
4s_1s_2s_3s_4 \sin^2(\gamma_2) \cos^2(\gamma_4) =
\]
\[
= s_1s_2(1 - \cos(\beta_2)s_3s_4(1 + \cos(\beta_4)) =
\]
\[
= (2s_1s_2 + t^2 - s_1^2 - s_2^2)(2s_3s_4 + s_3^2 + s_4^2 - l^2)/4 =
\]
\[
= (l^2 - (s_1 - s_2)^2)((s_3 + s_4)^2 - l^2)/4.
\]

At last we can find \(|g_1^2 + (g_2 + i h_2)^2|^2\):
\[
[(l^2 - (s_1 - s_2)(s_3 + s_4)) + (l^2 - (s_1 - s_2)^2)((s_3 + s_4)^2 - l^2)]/4 =
\]
\[
= [l^2(2(s_1 - s_2)(s_3 + s_4) + (s_3 + s_4)^2 + (s_1 - s_2)^2)]/4 =
\]
\[
= l^2[(s_3 + s_4 - s_1 + s_2)^2]/4 = l^2(1 - s_1)^2.
\]

Analogously, we can prove that \(|g_1^2 + (g_4 + i h_4)^2|^2 = (1 - s_1) \cdot |BD| \). \(\square\)

**Remark 7.1.** The statement of this theorem is also valid for non-convex and self-intersecting quadrangles. The reasoning is the same.

## 8. Special cases

**Theorem 8.1.** The dual to a trapezoid is a trapezoid.

**Proof.** Let \( Q = ABCD \) be a trapezoid, where \( AB \parallel CD \):

![Figure 8](image)

Here \( z_3 \) is a negative real number, hence, \( u_3 = \alpha i, \alpha > 0 \), hence, \( g_3 = 0 \), hence \( (g_3 + i h_3)^2 \) is a negative real number. \(\square\)

**Theorem 8.2.** The dual to a parallelogram is the same parallelogram.

**Proof.** Let \( Q = ABCD \) be a parallelogram. As \(|AB| + |BC| = 1\), then \(|KL| = |BC|\) and \(|LM| = |AB|\). It remains to note that \(|AC| = |KM|\). \(\square\)
9. The geometric construction

Given a convex quadrangle $Q$ it is easy to construct the dual $Q^o$, using ruler and compass.

Let $Q = ABCD$ be a convex quadrangle

![Figure 9](image.png)

with diagonal $AC$. Let $|AB| = s_1$, $|BC| = s_2$, $|CD| = s_3$ and $|DA| = s_4$. Using compass we construct the point $B_1$: a) it is in the same half-plane (with respect to $AC$) as point $B$; b) $|B_1A| = (s_2 + s_3 + s_4 - s_1)/2$; c) $|B_1C| = (s_1 + s_3 + s_4 - s_2)/2$. In the same way we construct the point $D_1$: a) it is in the same half-plane (with respect to $AC$) as point $D$; b) $|D_1A| = (s_1 + s_2 + s_3 - s_4)/2$; c) $|D_1C| = (s_1 + s_2 + s_4 - s_3)/2$. Then $AB_1CD_1$ will be the required dual.

**References**

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E-mail address: ibusjatskaja@hse.ru, yukochetkov@hse.ru