Non-binding of three two-component particles with contact interactions: condition on the heavy-light mass ratio

O. I. Kartavtsev and A. V. Malykh

Joint Institute for Nuclear Research, Dubna, 141980, Russia

(Dated: May 4, 2022)

Abstract

Binding of two heavy fermions interacting with a light particle via the contact interaction is possible only for sufficiently large mass ratio. In this work, the two-variable inequality is derived determining a specific value $\mu^*$, which provides that the three-body bound states do not exist for any mass ratio less than $\mu^*$. Its value is obtained by analyzing the inequality for a total angular momentum and parity $L^P = 1^-$. Other $L^P$ sectors of this problem are considered in a similar way and corresponding specific values of mass ratio are determined. For generality, the method is extended to non-binding of the system consisting of two identical bosons and a distinct particle for different $L^P (L > 0)$ sectors.

PACS numbers: 03.65.Ge, 31.15.ac, 67.85.-d
I. INTRODUCTION

Low-energy few-body dynamics of multi-component ultra-cold quantum gases has attracted much attention, in particular, the two-component three-body system has been investigated, e.g., in [1–10]. In the low-energy limit, the particular form of the short-range interaction of particles becomes insignificant and the zero-range model provides the universal description of the three-body problem. A single parameter of the zero-range model, e.g., a two-body scattering length $a$, can be chosen as a scale, thus all the properties essentially depend only on one parameter, the mass ratio of different particles. Generally, introducing the zero-range model in the few-body problem needs special efforts, which are discussed, e.g., in [9–14].

The three-body spectrum of two identical fermions (bosons) interacting with a distinct particle via the contact interaction was investigated in [2, 5, 9, 15–17] and it was found that the bound states arise only if the mass ratio exceeds some critical value. Besides the numerical results, it is of interest to obtain analytically the mass-ratio value $\mu^*$, below which the bound state does not exist. This work is aimed to determine the value $\mu^*$ in different $L^P$ sectors both for fermionic and bosonic symmetry of identical particles. Notice that the value $\mu^* = 2.617$ for the system containing two identical fermions was obtained in [10] by analyzing the momentum-space integral equation.

II. FORMULATION

Consider a particle 1 of mass $m_1$ interacting with two identical particles 2 and 3 of masses $m_2 = m_3 = m$ via contact interaction. In the framework of zero-range model, the identical fermions do not interact to each other and the same assumption is used for identical bosons. The three-body wave function $\Psi$ satisfies the Schrödinger equation for free motion supplemented by the boundary conditions imposed at zero distance $r$ between the interacting particles,

$$\lim_{r \to 0} \frac{\partial \ln(r \Psi)}{\partial r} = -\text{sign}(a),$$

where $a$ is the two-body scattering length.

In the center-of-mass frame, one defines the scaled Jacobi variables as $x = \sqrt{2\mu} (r_2 - r_1)$ and $y = \sqrt{2\mu} \left( r_3 - \frac{m_1 r_1 + m r_2}{m_1 + m} \right)$, where $r_i$ is a position vector of $i$th particle and the
reduced masses are denoted by \( \mu = \frac{mm_1}{m + m_1} \) and \( \tilde{\mu} = \frac{m(m + m_1)}{m_1 + 2m} \). The units \( \hbar = |a| = 2\mu = 1 \) are chosen, which correspond to the unit two-body binding energy \( \varepsilon_2 = 1 \) for \( a > 0 \). The three-body properties depend on a single parameter - the mass ratio \( m/m_1 \) that for convenience can be interchangeably replaced by the kinematic angle \( \omega \) defined by \( \sin \omega = 1/(1 + m_1/m) \).

Total angular momentum \( L \), its projection \( M \) and parity \( P \) are conserved quantum numbers, which should be used to label the solutions. In addition, it is suitable to introduce the operator \( P_s \), which permutes the identical particles 2 and 3, whose eigenvalue \( P_s = \mp 1 \) designates fermionic or bosonic symmetry. As the zero-range interaction acts in the \( s \)-wave, it is necessary to restrict the consideration only to the case \( P = (-)^L \).

To produce a convenient basis for expansion of the total wave function, let us define a hyper-radius \( \rho \) and hyper-angular variables \( \{ \alpha, \hat{x}, \hat{y} \} \) by \( x = \rho \cos \alpha, y = \rho \sin \alpha, \hat{x} = x/x, \) and \( \hat{y} = y/y \). In these variables the Hamiltonian reads as

\[
H = -\frac{1}{\rho^5} \frac{\partial}{\partial \rho} \left( \rho^5 \frac{\partial}{\partial \rho} \right) + \frac{\Delta_\Omega}{\rho^2},
\]

where \( \Delta_\Omega \) is a Laplace operator on a hyper-sphere, whose explicit form can be found, e.g., in \[2, 9, 18\]. One defines an auxiliary eigenvalue problem on hyper-sphere (for fixed hyper-radius)

\[
(\Delta_\Omega + \gamma^2 - 4) \Phi(\alpha, \hat{x}, \hat{y}; \rho) = 0,
\]

\[
\lim_{\alpha \to \pi/2} \frac{\partial}{\partial \alpha} \left[ \frac{1}{\rho^2} \frac{\partial}{\partial \rho} \left( \rho^2 \frac{\partial}{\partial \rho} \right) \right] \Phi(\alpha, \hat{x}, \hat{y}; \rho) = \rho \text{ sgn}(a),
\]

which solutions inherit symmetry of the total wave function and can be written in the form \[2, 9, 15\]

\[
\Phi(\alpha, \hat{x}, \hat{y}; \rho) = (P_s + P_s^2) \frac{\varphi_L^L(\alpha)}{\sin 2\alpha} Y_{LM}(\hat{y}).
\]

Here \( Y_{LM}(\hat{y}) \) is a spherical function and the explicit dependence on \( L, M, \) and \( P_s \) is suppressed for brevity.

As a result, the eigenvalue problem on the hyper-sphere is reduced to

\[
\left[ \frac{d^2}{d\alpha^2} - \frac{L(L + 1)}{\sin^2 \alpha} + \gamma^2 \right] \varphi_L^L(\alpha) = 0,
\]

\[
\varphi_L^L(0) = 0,
\]

\[
\lim_{\alpha \to \pi/2} \left( \frac{d}{d\alpha} - \rho \text{ sgn}(a) \right) \varphi_L^L(\alpha) = \frac{2(-)^LP_s}{\sin 2\omega} \varphi_L^L(\omega),
\]

\[
\lim_{\alpha \to \pi/2} \left( \frac{d}{d\alpha} - \rho \text{ sgn}(a) \right) \varphi_L^L(\alpha) = \frac{2(-)^LP_s}{\sin 2\omega} \varphi_L^L(\omega),
\]

\[
\lim_{\alpha \to \pi/2} \left( \frac{d}{d\alpha} - \rho \text{ sgn}(a) \right) \varphi_L^L(\alpha) = \frac{2(-)^LP_s}{\sin 2\omega} \varphi_L^L(\omega),
\]

\[
\lim_{\alpha \to \pi/2} \left( \frac{d}{d\alpha} - \rho \text{ sgn}(a) \right) \varphi_L^L(\alpha) = \frac{2(-)^LP_s}{\sin 2\omega} \varphi_L^L(\omega),
\]

\[
\lim_{\alpha \to \pi/2} \left( \frac{d}{d\alpha} - \rho \text{ sgn}(a) \right) \varphi_L^L(\alpha) = \frac{2(-)^LP_s}{\sin 2\omega} \varphi_L^L(\omega),
\]
where the boundary condition (2.6c) follows from (2.4). The solution to Eq. (2.6) is given by the transcendental equation for eigenvalue \( \gamma^2 \) \[9, 18\]

\[
\rho \text{sgn}(a) \Gamma \left( \frac{L + \gamma + 1}{2} \right) \Gamma \left( \frac{L - \gamma + 1}{2} \right) =
2\Gamma \left( \frac{L + \gamma}{2} + 1 \right) \Gamma \left( \frac{L - \gamma}{2} + 1 \right) + P_{s} \frac{(-2)^{1-L} \pi \omega^{L}}{\sin \gamma \pi \cos \omega} \left( \frac{1}{\sin \omega} \frac{d}{d\omega} \right)^{L} \frac{\sin \gamma \omega}{\sin \omega}, \quad (2.7)
\]

where \( \Gamma(x) \) is a Gamma function.

A set of eigenfunctions of an auxiliary problem on a hyper-sphere will be used in expansion of the total wave function,

\[
\Psi(x, y) = \rho^{-5/2} \sum_{n=1}^{\infty} f_{n}(\rho) \Phi_{n}(\alpha, \hat{x}, \hat{y}; \rho), \quad (2.8)
\]

which leads to a system of hyper-radial equations \[2, 9, 18\] for the channel functions \( f_{n}(\rho) \). In the following analysis, the one-channel approximation, \[2.9\]

\[
\left[ \frac{d^2}{d\rho^2} - \frac{\gamma^2(\rho) - 1/4}{\rho^2} + E \right] f(\rho) = 0, \quad (2.9)
\]

where \( \gamma^2(\rho) \) is the lowest eigenvalue of Eq. (2.7), \( f(\rho) \) is the corresponding channel function, and the diagonal term \( \left\langle \frac{\partial \Phi}{\partial \rho} \frac{\partial \Phi}{\partial \rho} \right\rangle \) is skipped.

As shown in \[9, 18\], at some mass ratios, Eq. (2.9) must to be supplemented by the boundary condition (with an additional real-valued three-body parameter) to fix the behavior of the total wave function at the triple collision point (\( \rho \rightarrow 0 \)) in order to make the three-body Hamiltonian self-consistent. In the present work, as will be shown below, the desired specific value \( \mu^{*} \) is smaller the critical mass ratio \( \mu_{r} \) defined as a solution of Eq. (2.7) for \( \gamma^2(0) = 1 \). Therefore, as pointed out in \[9, 18\], the Hamiltonian is completely defined by the condition of square integrability or equivalently \( f(\rho) \rightarrow 0 \) at \( \rho \rightarrow 0 \).

As follows from (2.7), \( \gamma^2(\rho) > 0 \) and the bound states do not exist for any mass ratio \( m/m_{1} \) for two cases: first, for \( a > 0 \) and \( P_{s} = (-)^{L+1} \) and, second, for \( a < 0 \). One should determine the value \( \mu^{*} \) only for \( P_{s} = (-)^{L} \) (odd \( L \) if the identical particles are fermions and even \( L \) if bosons) and the positive scattering length (\( a > 0 \)). Besides, the case \( L = 0 \) for identical bosons (\( P_{s} = 1 \)) is not considered as an infinite number of bound states exist at any mass ratio.
III. THE MASS-RATIO BOUND FOR ABSENCE OF THE THREE-BODY BOUND STATES

A. Basic method

To determine the specific mass-ratio $\mu^*$, it is sufficient to prove that the lower bound of three-body energy exceeds the threshold for any mass ratio smaller than $\mu^*$. This program will be realized in three steps (a) - (c).

(a) Let’s demonstrate that Eq. (2.9) gives a lower bound $E_{LB}$ of the exact three-body bound state energy $E$.

In fact, the lower bound of energy can be obtained for any Hamiltonian, which can be separated into two parts

$$H = T_1 + V_1 (\{x_1\}) + T_2 + V_2 (\{x_1\}, \{x_2\}) .$$  \hspace{1cm} (3.1)

Here the first part $T_1 + V_1$ depends only on the first kind of ("slow") variables $\{x_1\}$, whereas the second part contains kinetic energy $T_2$ depending only on the second kind of ("fast") variables $\{x_2\}$ and the remaining potential $V_2 (\{x_1\}, \{x_2\})$. If $E_{LB}$ is the eigenenergy of $T_1 + V_1 + \varepsilon (\{x_1\})$, where $\varepsilon (\{x_1\})$ is the lowest eigenvalue of $T_2 + V_2 (\{x_1\}, \{x_2\})$, then $E_{LB}$ is the lower bound of the eigenenergy $E$ of $H$, i.e., $E_{LB} \leq E$.

A sketch of a simple proof is based on the operator inequality

$$T_2 + V_2 (\{x_1\}, \{x_2\}) \geq \varepsilon(x_1) .$$  \hspace{1cm} (3.2)

It is defined that $A \geq B$ if $\langle \phi | A | \phi \rangle \geq \langle \phi | B | \phi \rangle$ for any $\phi$. From (3.1) and (3.2) it follows that the sum $T_1 + V_1 + \varepsilon \leq H$, therefore, $E_{LB} \leq E$.

The given proof for energy lower bound is applicable to the problem under consideration, if one uses $\rho$ as "slow" variable $\{x_1\}$, while the angles $\{\Omega\}$ as "fast" variables $\{x_2\}$. Then operator $-\frac{1}{\rho^2} \frac{\partial}{\partial \rho} \left( \rho^5 \frac{\partial}{\partial \rho} \right)$ is used as kinetic-energy operator $T_1$, the potential $V_1 = 0$ and the equation for "slow" variable can be written as Eq. (2.9) by using first term in (2.8). Now Eq. (2.3) with zero-range potential expressed in the form (2.4) is the equation for "fast" variables, where $-\frac{1}{\rho^2} \Delta_0$ is used instead of $T_2$ and $\gamma^2 - 4 / \rho^2$ instead of $\varepsilon$.

The energy lower bound and the proofs were given in the papers dedicated to adiabatic description of the molecule [21, 22], description of atomic hydrogen in a uniform magnetic
field \[19\]. In addition, the basic inequalities in the hyperspherical formalism for the \(N\)-body problem can be found in \[20\].

(b) Consider the properties of reference Hamiltonian \( h = -\frac{d^2}{dx^2} - \frac{1}{4x^2} \), provided the wave function asymptotic is fixed by the condition \( \phi \xrightarrow{x \rightarrow 0} \sqrt{x} + O(x^{3/2}) \). It is well-known that \( h \) is non-negative, i.e., \( \langle \phi | h | \phi \rangle \geq 0 \) for any \( \phi \), that is obvious since \( h \) is a radial part of the two-body two-dimensional kinetic energy operator. Moreover, Hamiltonian \( h \) is a critical one with virtual state at zero energy, which means that the bound state arises if arbitrarily small \( V(x) \) (under the condition \( \int V(x)xdx \leq 0 \)) is added \[23\].

(c) From comparison \((2.9)\) with reference Hamiltonian in (b) one can impose restrictions on the lowest eigenvalue \( \gamma^2 \) that leads to the absence of bound states. Let us mention that \( \gamma^2/\rho^2 \) tends to the two-body threshold energy \( \varepsilon_2 = -1 \) at large \( \rho \) and the three-body bound-state energy must be below \( \varepsilon_2 \). Combining this with statement of (a) and (b), one finds that the bound states are proven to be absent if \( \gamma^2/\rho^2 \) in Eq. \((2.9)\) exceeds the two-body threshold energy \( \varepsilon_2 \), i.e., \( \gamma^2/\rho^2 \geq -1 \). It is convenient to denote \( \gamma = i\kappa \) and finally to obtain the condition

\[
\rho(i\kappa) - \kappa \geq 0, \quad (3.3)
\]

where \( \rho(\gamma) \) is given by \((2.7)\).

Given \( P = P_s = (-1)^L \) and \( a > 0 \) the condition \((3.3)\) by using \( \rho(\gamma) \) \((2.7)\) is written in the form

\[
B_L(\kappa, \omega) = 2\Gamma \left( \frac{L + i\kappa}{2} + 1 \right) \Gamma \left( \frac{L - i\kappa}{2} + 1 \right) - \frac{2^{1-L}\pi \sin(\omega)^L}{\sinh \pi \cos \omega} \left( \frac{1}{\sin \omega d\omega} \right)^L \sinh \kappa \omega - \kappa \Gamma \left( \frac{L + i\kappa + 1}{2} \right) \Gamma \left( \frac{L - i\kappa + 1}{2} \right) \geq 0. \quad (3.4)
\]

As the mass ratio \( m/m_1 \) monotonically increases with increasing \( \omega \) the desired mass-ratio value \( \mu^* \) corresponds to \( \omega^* (\sin \omega^* = \mu^*/(1 + \mu^*)) \), for which the condition \((3.4)\) is satisfied for any \( 0 < \omega \leq \omega^* < \pi/2 \) and \( \kappa > 0 \).

**B. Fermionic system for \( L = 1 \)**

For two identical fermions and third particle in the lowest possible angular momentum \( L = 1 \) and parity \( P = -1 \) the inequality \((3.4)\) takes the form

\[
B_1(\kappa, \omega) = \frac{\pi}{\sinh \pi \kappa} [F(\kappa) - G(\kappa, \omega)] \geq 0, \quad (3.5)
\]
where \( F(x) = (x^2 + 1) \sinh \frac{x \pi}{2} - x^2 \cosh \frac{x \pi}{2} \) and \( G(x, \omega) = 2x \cosh x \omega - \sinh x \omega \). The function \( B_1(x, \omega) \) monotonically decreases with increasing \( \omega \) in the interval \((0, \pi/2)\), i.e., \( \frac{\partial B_1}{\partial \omega} \leq 0 \).

Proof: The derivative of \( B_1(x, \omega) \) is explicitly given by

\[
\frac{\partial B_1(x, \omega)}{\partial \omega} = C(\omega) \left[ x (2 - \tan^2 \omega) \cosh x \omega - (x^2 \tan \omega + 2 \cot \omega) \sinh x \omega \right],
\]

where \( C(\omega) \) is an insignificant positive function. The inequality \( \frac{\partial B_1}{\partial \omega} \leq 0 \) can be written in the form

\[
\sinh^2 x \omega \left[ 4 + x^2 (x^2 + 4) z^2 - x^2 z^3 \right] \geq x^2 z(z - 2)^2,
\]

where the notation \( z = \tan^2 \omega \) is introduced for brevity. By using the inequality \((1 + z) \sinh^2 x \omega \geq x^2 z \) (as follows from \( \sinh x \omega \geq x \sin \omega \)), the inequality (3.7) reduces to

\[
z^3 x^2 (x^2 + 1)(x^2 + 3 - z) \geq 0,
\]

which is valid for any \( x \) if \( z = \tan^2 \omega \leq 3 \). To complete the proof, it is sufficient to take into account from (3.6) that \( \frac{\partial B_1}{\partial \omega} \leq 0 \) also if \( \tan^2 \omega \geq 2 \).

The proved monotonic behavior of \( B_1(x, \omega) \) in \( \omega \) means that the implicit condition \( B_1(x, \omega) = 0 \) determines a single-valued function \( \omega_0(x) \). Since \( B_1(x, \omega) = 0 \) is a lower bound of inequality (3.5), the desired value \( \omega^* \) can be taken as

\[
\omega^* = \min \omega_0(x), \quad 0 \leq x \leq \infty,
\]

which provides the absence of bound states for any \( \omega \leq \omega^* \) or equivalently \( m/m_1 \leq \mu^* \).

Numerical calculation of \( \omega_0(x) \) is presented in Figure 1. This simple function has one minimum \( \omega^* = \omega_0(x^*) \) at the point \( x^* \), which numerical values are presented in Table 1 with corresponding \( \mu^* \). One should emphasize that the condition \( \mu^* < \mu_r \) is fulfilled as in this sector the critical mass ratio \( \mu_r \approx 8.6185769247 \).

C. Higher angular momentum \( L \geq 2 \)

Consider three-body system containing two identical fermions, \( P_s = -1 \), (or bosons, \( P_s = 1 \)) and a distinct particle, for odd (even) \( L = 1, 3, 5 \) \((L = 2, 4)\) and odd (even) parity \( P = (-1)^L \). One supposes, that \( B_L(x, \omega) \) (3.4) is a monotonically decreasing function of
FIG. 1. Dependence $\omega_0(\kappa)$ calculated for different sectors of total angular momentum $L$ and parity $P = (-1)^L$. Odd (even) $L$ corresponds to the three-body systems containing two identical fermions (bosons) and a distinct particle.

$\omega$ also for $L \geq 2$ and, therefore, the implicit condition $B_L(\kappa, \omega) = 0$ also determines a single-valued function $\omega_0(\kappa)$ with a bound $\omega^*$ that provides the absence of bound states for any $\omega \leq \omega^*$ and $m/m_1 \leq \mu^*$.

The dependences $\omega_0(\kappa)$ are calculated numerically and presented in Figure 1. As shown in Figure 1, for any $L$ function $\omega_0(\kappa)$ has one minimum, whose position $\kappa^*$ and value $\omega^* = \omega_0(\kappa^*)$ with corresponding $\mu^*$ are calculated and presented in Table I. Calculation shows that $\kappa^*$, $\omega^*$ and $\mu^*$ increase with increasing $L$ that is not surprising since the mass-

| $L^P$ | $P_s$ | $\kappa^*$ | $\omega^*$ | $\mu^*$ | $\mu_B$ |
|-------|-------|-------------|------------|----------|--------|
| $1^-$ | -1    | 2.17701     | 0.997755   | 5.26002  | 8.17259 |
| $2^+$ | 1     | 3.30822     | 1.243618   | 17.85119 | 22.6369 |
| $3^-$ | -1    | 4.51245     | 1.340135   | 36.75782 | 43.3951 |
| $4^+$ | 1     | 5.74050     | 1.392347   | 61.97274 | 70.457  |
| $5^-$ | -1    | 6.97890     | 1.425184   | 93.49356 | 103.823 |
ratios (numerically calculated in \[2, 18\]) presented in the last column of Table I, at which the first bound state appears, also increase with increasing \(L\). The comparison of two last columns of Table I shows that relative deviation \(\frac{\mu^* - \mu_B}{\mu_B}\) decreases from 0.36 to 0.1 with increasing \(L\) from \(L = 1\) to \(L = 5\). One should emphasize that condition \(\mu^* < \mu_r\) is satisfied for any total angular momentum as can be checked up to \(L = 5\) from the comparison with numbers \(\mu_r \approx 32.947611782, 70.070774958, 119.73121698, 181.86643779\) for \(L = 2, 3, 4, 5\), respectively. It confirms that one does not need to make efforts in order to render the described three-body Hamiltonian self consistent and one can simply use the zero boundary condition at the triple collision point.

Note that, for three-body systems containing identical bosons with \(L = 2\), the Eq. (3.4) can be written in the more simple form

\[
B_2(\kappa, \omega) = \frac{(1 + \kappa^2)^2}{2 \sinh \kappa \pi} \left[ \frac{\kappa(\kappa^2 + 4)}{\kappa^2 + 1} \cosh \frac{\kappa \pi}{2} - \kappa \sinh \frac{\kappa \pi}{2} + 3 \frac{\kappa \cosh \kappa \omega - \sinh \kappa \omega \cot \omega}{(\kappa^2 + 1) \sin^2 \omega} - \frac{2 \sinh \kappa \omega}{\sin 2\omega} \right] \geq 0
\]

(3.10)

and is valid for \(m/m_1 \leq \mu^* \approx 17.851188\) \((\omega \leq \omega^* \approx 1.243618)\).

IV. CONCLUSION

Using the standard "extreme" adiabatic approximation, it was proved that the condition for absence of the three-body bound states is reduced to analysing the inequality for two-variable’s function. Such an analysis was done in detail for positive two-body scattering length \(a > 0\) for a fermionic system with a total angular momentum \(L = 1\) and parity \(P = -1\). As a result, the lower bound of the mass ratio \(\mu^* = 5.2600\) is proved. One should mention that it is closer to numerically calculated precise value \(\mu_B \approx 8.17259\) for appearance of bound state \([9]\) than the proven lower bound \(\mu^* = 2.617\) \([10]\) obtained from the momentum-space integral equations.

Note that the difficulties with formulation of three-body problem, described in \([9, 18]\), do not appear in the present investigation, because the lower bound \(\mu^*\) is below the critical value of the mass ratio \(\mu_r\), starting from which these difficulties appear.

Despite the fact that \(\mu^*\) for \(L^P = 1^-\) gives the lower bound for absence of all bound states in a fermionic system, the analogous lower bounds were numerically calculated for each total angular momenta \(L > 0\) up to \(L = 5\) separately for bosonic as well as for fermionic
systems ($L$ is even for a fermionic system and $L$ is odd for a bosonic system). Note that the described procedure of extracting a specific value of the mass ratio $\mu^\ast$, below which there are no bound states, gives improving with $L$ relative deviation $\frac{\mu^\ast - \mu_B}{\mu_B}$ in respect to value $\mu_B$ (appearance of the first bound state). The method used here can be applicable to many related problems for estimation of lower bounds.

[1] D. S. Petrov, Phys. Rev. A 67, 010703(R) (2003).
[2] O. I. Kartavtsev and A. V. Malykh, J. Phys. B 40, 1429 (2007).
[3] J. Levinsen, T. G. Tieke, J. T. M. Walraven, and D. S. Petrov, Phys. Rev. Lett. 103, 153202 (2009).
[4] K. Helfrich, H.-W. Hammer, and D. S. Petrov, Phys. Rev. A 81, 042715 (2010).
[5] S. Endo, P. Naidon, and M. Ueda, Few-Body Syst. 51, 207 (2011).
[6] Y. Castin and E. Tignone, Phys. Rev. A 84, 062704 (2011).
[7] A. Safavi-Naini, S. T. Rittenhouse, D. Blume, and H. R. Sadeghpour, Phys. Rev. A 87, 032713 (2013).
[8] M. Jag, M. Zaccanti, M. Cetina, R. S. Lous, F. Schreck, R. Grimm, D. S. Petrov, and J. Levinsen, Phys. Rev. Lett. 112, 075302 (2014).
[9] O. I. Kartavtsev and A. V. Malykh, EPL 115, 36005 (2016).
[10] S. Becker, A. Michelangeli, and A. Ottolini, Math. Phys. Anal. Geom. 21, 35 (2018).
[11] R. A. Minlos, ISRN Math. Phys. 2012, 230245 (2012).
[12] R. A. Minlos, Usp. Mat. Nauk 69(3), 145 (2014), [Russ. Math. Surv. 69(3), 539 (2014)].
[13] R. A. Minlos, Mosc. Math. J. 14, 617 (2014).
[14] M. Correggi, G. Dell’antonio, D. Finco, A. Michelangeli, and A. Teta, Math. Phys. Anal. Geom. 18, 32 (2015).
[15] O. I. Kartavtsev and A. V. Malykh, Pis’ma ZhETF 86, 713 (2007), [JETP Lett. 86, 625 (2007)].
[16] K. Helfrich and H.-W. Hammer, J. Phys. B 44, 215301 (2011).
[17] A. Michelangeli and C. Schmidbauer, Phys. Rev. A 87, 053601 (2013).
[18] O. I. Kartavtsev and A. V. Malykh, “Three two-component fermions with contact interactions: correct formulation and energy spectrum,” (2019). [arXiv:1904.04943 [cond-mat.quant-gas]]

10
[19] A. F. Starace and G. L. Webster, Phys. Rev. A \textbf{19}, 1629 (1979).
[20] H. T. Coelho and J. E. Hornos, Phys. Rev. A \textbf{43}, 6379 (1991).
[21] V. Brattsev, Dokl. Akad. Nauk SSSR \textbf{160}, 570 (1965), [Soviet Phys.- Doklady \textbf{10}, 44 (1965)].
[22] S. T. Epstein, J. Chem. Phys. \textbf{44}, 836 (1966).
[23] B. Simon, Ann. Phys. \textbf{97}, 279 (1976).