Two-fluid hydrodynamics of cold atomic bosons under the influence of quantum fluctuations at non-zero temperatures

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Abstract
Ultracold Bose atoms is the physical system existing at the small finite temperatures, where the quantum and nonlinear phenomena play crucial role. Bosons are considered to be composed of two different fluids: the Bose–Einstein condensate and the normal fluid (the thermal component). The extended hydrodynamic models are obtained for each fluids, where the pressure evolution equations and the pressure flux third rank tensor evolution equations are obtained along with the continuity and Euler equations. It is found that the pressure evolution equation contains zero contribution of the short-range interaction. The pressure flux evolution equation contains the interaction which simplifies to the quantum fluctuations in the zero temperature limit. The structure of the third rank tensor describing this interaction is obtained in the regime of small temperature and weak interaction. The model is derived via the straightforward calculation of evolution of macroscopic functions using the microscopic many-particle Schrodinger equation in the coordinate representation. Finally, the two-fluid hydrodynamics is constructed in form of four equations for each fluid in order to give model describing the quantum fluctuations in BEC and the thermal effects in the normal fluid.

1. Introduction
Small temperature bosons are studied in terms of two-fluid hydrodynamics consisting of the Bose–Einstein condensate (BEC) and normal fluid [1]. The two-fluid equations are suggested for the liquid 4He. They first obtained phenomenologically by Landau in 1941 (see for instance [2]). Further development is presented by Bogoliubov in 1963 [3] Each fluid is considered in terms of two hydrodynamic equations: the continuity and Euler equations. It is assumed that the BEC can be completely described by the concentration and velocity field, or, in other terms, by the Gross–Pitaevskii equation [1], since BEC is the collection of particles in the single quantum state. However, the normal fluid model requires an truncation of the set of hydrodynamic equations. The normal fluid is the collection of particles thermally distributed over the quantum states with different energies. Therefore, it is described by the thermal or kinetic pressure, and other kinetic functions in addition to the concentration and velocity field. The pressure of the normal fluid appears in the Euler equation as the independent function. Hence, we can use some equation of state for the approximate representation of pressure via the concentration. Overwise, we can derive the pressure evolution equation. Equation for the pressure evolution provides an expression for the pressure perturbations via the perturbations of other functions. Application of the equation of state for pressure makes the model more simple, but equation of state for pressure leads to the less accurate model.

The kinetic pressure for the BECs is equal to zero [4, 5]. Since the kinetic pressure is related to the occupation of the excited states. However, there is the nonzero pressure-like term caused by the quantum fluctuations in the Euler equation [6]. The pressure evolution equation of the weakly interacting bosons contains no interaction, but next equation in the chain of the quantum hydrodynamic equations (the equation for the pressure flux third rank tensor) contains the interaction causing the depletion of the BECs at the zero temperature. Therefore, it is
necessary to consider the pressure flux evolution equation both for the BEC and for the normal fluid at the analysis of the small temperature influence.

The quantum depletion of the BECs is the appearance of the bosons in the excited states while the system is kept at the zero temperature. So, some energy of the collective motion is transferred to the individual motion of bosons. It is caused by the quantum fluctuations related to the interparticle interaction. The quantum fluctuations are considered in literature via the Bogoliubov-de Gennes approach [7–10]. The quantum fluctuations in BECs are studied experimentally as well [11–13]. The theoretical methods are generalized on the dipolar BECs, where the quantum fluctuations plays crucial role at the description of the dipolar BECs of lantanoids. The large scale instability causing the splitting of the cloud of atoms on the number of macroscopic drops is found in such systems. This highly nonlinear phenomena is called the quantum droplet formation [14–33]. The reassemble of bosons to smaller compact groups causes the stabilization of the system. These studies give the motivation for the study of quantum fluctuations. However, here we restrict our analysis with the short-range interaction only and no dipole-dipole interaction is discussed. Moreover, the influence of the finite temperature is considered as the necessary part of complete model.

Non-thermal occupation of excited states requires to compare it with the small thermal effects. While the thermal component or the normal fluid is considered as the second fluid. As the direct manifestation of two-fluid nature of cold bosons there are two distinct sound velocities exist in the finite temperature ultracold Bose gas [1, 34]. The two-fluid model shows that the slower mode (second sound) is associated with the BEC component, while the faster mode (first sound) is associated with the thermal component.

In this paper, the derivation of two-fluid hydrodynamics for the finite temperature bosons in the limit of small temperature, where the large fraction of the bosons is located in the BEC state, is given from the microscopic motion in accordance with the quantum hydrodynamic method [4, 5, 35, 36]. The microscopic dynamics is described by the Schrodinger equation in the coordinate representation. Collection of the macroscopic functions is presented to describe the collective effects in ultracold bosons. It includes the concentration of particles, the velocity field and the pressure tensor. The derivation of basic equations is made for all bosons distributed on the lower energy level and the excited levels as the single fluid. The decomposition on two fluids is made on the microscopic scale. The general structure of equations is obtained for the arbitrary temperature and arbitrary strength of interaction. Next, the approximate calculation of functions presenting the interaction is made for the regime of short-range interaction. Hence, the small parameter related to the small area of action of the interaction potential is used. The further truncation is made at the calculation of the interaction terms for the weak interaction and small temperatures.

Quantum theory has many representations. It includes most widely used coordinate representation, where the wave function can be interpreted that the square of module of the single particle wave function is the probability density to find the particle in the vicinity of the point of space. The occupation of the representation numbers which is very useful for the many-particle systems and applied in the quantum statistics. These representations are focused on the work in the abstract Hilbert space.

The hydrodynamic form of the quantum equations appeared at the early steps of development of quantum theory [37]. One of the main focuses of this representation is the nonstationary evolution of quantum systems in the physical space. Moreover, it includes an attempt to describe the quantum systems in terms of observable only.

Different physical systems are described by the hydrodynamics of different forms including various approximations. However, the majority of hydrodynamic models includes the continuity equation and Euler momentum balance equation. Particularly, the evolution of weakly interacting BECs of neutral atoms at the absence of the thermal cloud and neglecting the quantum fluctuations can be described by continuity and Euler equations only. The extended set of hydrodynamic equations for the cold atomic bosons composed of the BEC and the normal fluid is obtained in this paper for the small temperature regime of weakly interacting bosons. The model is justified on the time scale which is small in compare with the mean free path for inelastic three particle scattering leading to the formation of dimers.

This paper is organized as follows. In section 2 major steps of the derivation of hydrodynamic equations from the Schrodinger equation are demonstrated, where the pressure evolution equation and the third rank tensor evolution equation are obtained along with the continuity and Euler equations. In section 3 the method of calculation of the interaction in the Euler equation, the pressure evolution equation, and the pressure flux evolution equation is demonstrated. Section 4 presents the derived extended two-fluid quantum hydrodynamic model for the ultracold finite temperature bosons. In section 5 the limit regime of derived model is obtained for the BEC under the influence of the quantum fluctuations at the zero temperature. In section 6 a brief summary of obtained results is presented.
2. On the derivation of hydrodynamic equations from microscopic quantum dynamics

2.1. Basic definitions of quantum hydrodynamics and the Euler equation derivation

On the microscopic level we do not have notion of temperature. Hence, we consider systems of interacting bosons governed by the Schrodinger equation $i\hbar \partial_t \Psi = \hat{H}\Psi$ with the following Hamiltonian

$$\hat{H} = \sum_{i=1}^{N} \left( \frac{\hat{p}_i^2}{2m_i} + V_{ext}(\mathbf{r}_i, t) \right) + \frac{1}{2} \sum_{i,j\neq i} U(\mathbf{r}_i - \mathbf{r}_j),$$

(1)

where no specification of the temperature regime is made. In equation (1) we use the following notations: $m_i$ is the mass of $i$-th particle, $\hat{p}_i = -i\hbar \hat{\nabla}_i$ is the momentum of particle. The last term in the Hamiltonian (1) is the boson-boson interaction $U_{ij}$. We do not specify the form of interaction. However, the derivation presented below employs that the interaction has the finite value on the small distances between particles and shows fast decay at the increase of the interparticle distance. No distinguish between bosons in the BEC state and bosons in other states is made at this stage. Separation of all bosons on two subsystems is made in terms of collective variables. Hydrodynamic model usually made for each species of particles. We consider the single species of bosons so all masses are equal to each other $m_i = m$.

Distribution of particles in the traps, waves, solitons, oscillations of form of trapped clouds are described by the concentration of particles. Concentration is the essential macroscopic function both for the classical and the quantum fluids. The module of the macroscopic wave function in the Gross–Pitaevskii equation gives the square root of concentration of bosons in the BEC state. Therefore, we start the derivation of quantum hydrodynamic equations from the definition of concentration. The quantum mechanics is based on the notion of point-like objects in spite the wave nature of quantum objects. So, the eigenfunction of the coordinate operator in the coordinate representation is the delta function $\hat{\delta} = \frac{\delta}{\delta x}$, where normalized wave function is $\psi(x) = \delta(x - x')$. The operation of concentration in the coordinate representation of quantum mechanics is the sum of delta functions $\hat{n} = \sum_{i=1}^{N} \delta(\mathbf{r} - \mathbf{r}_i)$. The operators in quantum mechanics are constructed as the quantization of the corresponding classical dynamic functions. The classical microscopic concentration is the sum of delta functions since each particle is the one unit in the zero volume. It gives us infinite concentration in the point of space, where this particle is located. Therefore, the concentration of $N$-particles is the superpositions of $N$ infinities. While other points of space give the zero contribution in the concentration. This physical picture is mathematically modeled by the sum of $N$ delta-functions.

The transition to description of the collective motion of bosons is made via introduction of the concentration [4, 35]:

$$n = \int d\mathbf{R} \sum_{i=1}^{N} \delta(\mathbf{r} - \mathbf{r}_i)\Psi^\ast(\mathbf{R}, t)\Psi(\mathbf{R}, t),$$

(2)

which is the first collective variable in our model. Other collective variables appear during the derivation. Equation (2) contains the following notations: $d\mathbf{R} = \prod_{i=1}^{N} d\mathbf{r}_i$ is the element of volume in $3N$ dimensional configurational space, with $N$ is the number of bosons. Concentration (2) is the sum of partial concentrations $n = n_\text{B} + n_\text{f}$ describing the distribution of BEC $n_\text{B}$ and normal fluid $n_\text{f}$ in the coordinate space.

The equation for the evolution of concentration (2) can be obtained by acting by the time derivative on function (2). The time derivative acts on the wave functions under the integral while the time derivatives of the wave function are taken from the Schrodinger equation. We obtain the continuity equation for concentration (2) after straightforward calculations:

$$\partial_t n + \nabla \cdot j = 0,$$

(3)

where the new collective function called the current appears as the following integral of the wave function

$$j(\mathbf{r}, t) = \int d\mathbf{R} \sum_{i=1}^{N} \delta(\mathbf{r} - \mathbf{r}_i) \times$$

$$\times \frac{1}{2m_i} (\Psi^\ast(\mathbf{R}, t) \hat{p}_i \Psi(\mathbf{R}, t) + c.c.),$$

(4)

with c. c. is the complex conjugation. The continuity equation (3) is the direct consequence of the nonstationary many-particle Schrodinger equation with Hamiltonian (1). However, this derivation includes some additional assumptions hidden in the Hamiltonian (1). Considered cold bosons are atoms, but their inner structure is not considered here. Hence, the possibility to create the bound states of two atoms (the dimers) at the simultaneous interaction of three atoms is not included. Otherwise, the number of Bose atoms would change. It lead to the nonzero right-hand side of the continuity equation. Moreover, it creates additional effects in other hydrodynamic equations. The present day experiments contain finite number of atoms $N \sim 10^4$ in the trap. The field configurations in the traps are constructed for particular electronic structures of atoms. So, the dimers of...
these atoms are not bound by these traps. The creation of dimers leads to decay of the cloud of cold atoms. This effect leads to characteristic time scale \( \tau \). Equations derived in this paper are correct on the time scale \( \tilde{t} \ll \tau \).

Both introduced collective functions \( n(r, t) \) and \( j(r, t) \) are the quadratic forms of the wave function. Each of them can be split on two parts related to the BEC and the normal fluid. Hence, we have \( n = n_B + n_n \) and \( j = j_B + j_n \). No microscopic definitions are introduced for the partial functions \( n_B, n_n, j_B, \) and \( j_n \). Therefore, the continuity equation (3) splits on two partial continuity equations

\[
\partial_t n_a + \nabla \cdot \vec{j}_a = 0, \tag{5}
\]

where subindex \( a \) stands for \( B \) and \( n \). Some details of derivation of the continuity equation (3) and the Euler equation (which is given below) are presented in the appendix.

We continue the derivation of hydrodynamic equations and consider the time evolution of the particle current (4). We act by time derivative on function \( j \) (4) and use the Schrödinger equation with Hamiltonian (1).

It leads to the general form of the current evolution equation

\[
\partial_t j^\alpha + \partial_{\beta} \Pi^{\alpha \beta} = - \frac{1}{m} n \partial_a V_{\text{ext}} + \frac{1}{m} F_{a,\text{int}}^\alpha, \tag{6}
\]

where

\[
\Pi^{\alpha \beta} = \int dR \sum_{i=1}^N \frac{1}{4m^2} \{ |\Psi_i(R, t)|^2 \hat{j}_i^\alpha \hat{j}_i^\beta \Psi_i(R, t) + \text{c.c.} \}.
\]

is the momentum flux, and

\[
F_{a,\text{int}}^\alpha = - \int (\partial^\alpha U(r - r')) n_2(r, r', t) dr', \tag{8}
\]

with the two-particle concentration

\[
n_2(r, r', t) = \int dR \sum_{i,j=1, j \neq i}^N \delta(r - r_i) \delta(r' - r_j) |\Psi_i(R, t)|^2 |\Psi_j(R, t)|^2. \tag{9}
\]

It is necessary to split equation (6) on two equations for each subsystem of bosons. In current form equation (6) consists of superposition of functions which are the quadratic forms of the wave function. Hence, each term can be split on two parts and we find two similar equations for the partial currents

\[
\partial_t j^\alpha_a + \partial_{\beta} \Pi^{\alpha \beta}_a = - \frac{1}{m} n_a \partial_a V_{\text{ext}} + \frac{1}{m} F_{a,\text{int}}^\alpha. \tag{10}
\]

The first and third terms are proportional to the concentration and the current. Therefore, they require no comments. Some difference between two current evolution equations appears at further analysis of the momentum flux \( \Pi^{\alpha \beta}_a \) and the interaction \( F_{a,\text{int}} \). However, we point out some difference which appears for the momentum flux \( \Pi^{\alpha \beta} \). Its structure is obtained in many papers (see for instance [4] after equation (52), [38] equation (24))

\[
\Pi^{\alpha \beta} = mn^\alpha v^\beta + p^{\alpha \beta} + T^{\alpha \beta}, \tag{11}
\]

where \( p^{\alpha \beta} \) is the kinetic pressure tensor, and \( T^{\alpha \beta} \) is the tensor giving the quantum Bohm potential, its approximate expression can be written in the following form

\[
T^{\alpha \beta}_0 = - \frac{\hbar^2}{4m^2} \left[ \partial_{\alpha} n \partial_{\beta} n - \frac{\partial_{\alpha} n \cdot \partial_{\beta} n}{n} \right]. \tag{12}
\]

Tensor \( T^{\alpha \beta}_0 \) (12) is obtained for noninteracting particles located in the single quantum state.

Basically, the pressure tensor \( p^{\alpha \beta} \) is defined via the wave function \( \Psi(R, t) \). However, it requires some manipulations with the wave function and introduction of the number of intermediate function. Hence, we do not present its explicit form. Nevertheless, the pressure tensor is related to the distribution of bosons on the quantum states with energies above \( E_{\text{min}} \). Therefore, for bosons in the BEC state we have \( p^{\alpha \beta}_B = 0 \) if no quantum fluctuations are considered. If we include the quantum fluctuations we keep \( p^{\alpha \beta}_B = p^{\alpha \beta}_\text{q} = 0 \). Finally, the momentum flux is composed of three terms

\[
\Pi^{\alpha \beta}_B = n_B v^\alpha_B v^\beta_B + T^{\alpha \beta}_B + p^{\alpha \beta}_\text{q}, \tag{13}
\]

where \( T^{\alpha \beta}_B \) is the function of \( n_B \) (12) if there is no interaction. Distribution of particles on different quantum states does not allow to get full expression (12). The first linear term can be strictly derived for any strength of interaction and any distribution on quantum states. However, equation (12) can be used as the equation of state.
The normal fluid bosons have nonzero pressure $p_n^{αβ} \neq 0$. Hence, all terms in presentation (11) exists in this regime.

2.2. The pressure evolution equation

Extending the set of hydrodynamic equations we can derive the equation for the momentum flux evolution. It can be expected that this equation brings extra information for the normal fluid bosons only. However, the quantum fluctuations give some contribution in the evolution of the kinetic pressure of BECs in the limit of zero temperature via the divergence of the third rank tensor. If the temperature is nonzero we have two partial kinetic pressures for the BEC and for the normal fluid. We consider the time evolution of the momentum flux (7) using the Schrödinger equation with Hamiltonian (1).

It gives to the following expression:

$$\partial_t \Pi^{αβ} = \frac{1}{\hbar} \int dR \sum_{i=1}^{N} \delta(r - r_i) \frac{1}{4m^2} [\hat{H}\Psi^*(R, t) \hat{p}_i^α \hat{p}_i^β \Psi(R, t)$$
$$- \Psi^*(R, t) \hat{p}_i^α \hat{p}_i^β \hat{H}\Psi(R, t) + \hat{p}_i^α \hat{H}^* \Psi^*(R, t) \hat{p}_i^β \Psi(R, t)$$
$$- \hat{p}_i^α \Psi^*(R, t) \hat{p}_i^β \hat{H}\Psi(R, t) - \text{c.c.}]$$

(14)

The part of the presented terms contains the Hamiltonian $\hat{H}$ under action of the momentum operators. We permute the Hamiltonian $\hat{H}$ and the operators acting on it. Hence, the result of permutation is presented by terms where no operators act on the Hamiltonian $\hat{H}$. The terms containing the corresponding commutators appear as well. Therefore, all terms are combined in two groups:

$$\partial_t \Pi^{αβ} = \frac{1}{\hbar} \int dR \sum_{i=1}^{N} \delta(r - r_i) \frac{1}{4m^2} [\hat{H}\Psi^*(R, t) \hat{p}_i^α \hat{p}_i^β \Psi$$
$$- \Psi^*(R, t) \hat{p}_i^α \hat{p}_i^β \hat{H}\Psi(R, t) + \hat{p}_i^α \hat{H}^* \Psi^*(R, t) \hat{p}_i^β \Psi$$
$$+ \hat{p}_i^α \Psi^*(R, t) \hat{p}_i^β \hat{H}\Psi(R, t) + \text{c.c.}]$$

(15)

The first group of terms in expression (15) gives the divergence of flux of tensor $\Pi^{αβ}$. The second group of terms contains the commutators. This group leads to the contribution of interaction in the momentum flux evolution.

It gives the momentum flux evolution equation

$$\partial_t \Pi^{αβ} + \partial_j M^{αj} = - \frac{1}{m} j^β V_{ext}$$

(16)

where the momentum flux is the full flux of all bosons $\Pi^{αβ} = \Pi_n^{αβ} + \Pi_{fi}^{αβ}$, the splitting on two subspecies is to be made later,

$$F^{αj} = - \int [\partial^j U(r - r')] \frac{1}{2m} \Psi^*(r', t) \frac{1}{8m_i} \frac{1}{1} [\Psi^*(R, t) \hat{p}_i^α \hat{p}_i^β \Psi(R, t)$$

$$+ \hat{p}_i^α \hat{H}^* \Psi^*(R, t) \hat{p}_i^β \Psi(R, t) + \hat{p}_i^α \Psi^*(R, t) \hat{p}_i^β \hat{H}\Psi(R, t)$$

(17)

is the force tensor field,

$$M^{αj} = \int dR \sum_{i=1}^{N} \delta(r - r_i) \frac{1}{8m_i} [\Psi^*(R, t) \hat{p}_i^α \hat{p}_i^β \Psi(R, t)$$

(18)

is the current (flux) of the momentum flux, and

$$j^α(r', t) = \int dR \sum_{i,j=1}^{N} \delta(r - r_j) \delta(r' - r_i) \times$$

(19)

is the two-particle current-concentration function.

If quantum correlations are dropped function $j^α(r', t)$ splits on product of the current $j^α(r, t)$ and the concentration $n_i(t', t)$. Interaction in the momentum flux evolution equation (16) is presented by symmetrized combinations of tensors $F^{αβ}$, which is the flux or current of force.
Partial momentum flux equations appear as follows
\[ \partial_t \Pi^{\alpha\beta} + \partial_{\alpha} M^{\alpha\beta} = -\frac{1}{m} j^{\alpha} \partial_{\beta} V_{ext} \]
\[ -\frac{1}{m} j^{\alpha} \partial_{\beta} V_{ext} + \frac{1}{m} (F^{\alpha\beta} + F^{\alpha}_{\beta}), \]  
(20)

where \( M^{\alpha\beta} = M^{(\alpha\beta)} + M^{(3\beta)} \), with
\[ M^{(\alpha\beta)} = n_{a} \nu_{a}^{\beta} v_{a}^{\alpha} + v_{a}^{\beta} (p_{a}^{(3)} + T^{(3)}_{a}) + v_{a}^{\beta} (p_{a}^{(3)} + T^{(3)}_{a}) \\
+ v_{a}^{\beta} (p_{a}^{(3)} + T^{(3)}_{a} + Q_{a}^{(3)} + T^{(3)}_{a} + L^{(3)}_{a}). \]  
(21)

The pressure is the average of the square of the thermal velocity, when tensor \( Q_{a}^{(3)} \) is the average of the product of three projections of the thermal velocity. Function \( L^{(3)}_{a} \) presents the quantum-thermal terms If we consider particles being in the BEC state (quantum fluctuations are neglected) we have \( p_{a}^{(3)} = 0, Q_{a}^{(3)} = 0, L^{(3)}_{a} = 0 \). These functions present the contribution of the excited states. They have nonzero values under influence of the quantum fluctuations and thermal effects. For symmetric equilibrium distributions we have \( Q_{a}^{(3)} = 0, L_{a}^{(3)} = 0 \). We generalize this conclusion for nonequilibrium states as the equations of state for these functions. Tensor \( T^{(3)}_{a} \) is
\[ T^{(3)}_{a} = -\frac{\hbar^2}{12m^2} n_{a} (\partial^{\beta} \partial^{\beta} v_{a}^{\alpha} + \partial^{\alpha} \partial^{\alpha} v_{a}^{\beta} + \partial^{\gamma} \partial^{\gamma} v_{a}^{\nu}). \]  
(22)

This definition of tensor \( T^{(3)}_{a} \) differs from equation (27) in [38] by extraction of the quantum Bohm potentials written together with pressure tensors in equation (21). Equation (27) in [38] contains approximate form of the quantum Bohm potential \( T^{(3)}_{a} \). Equation (21) includes the quantum Bohm potential in its general form. Moreover, expression (22) is an exact formula obtained with no assumption about structure of the many-particle wave function like the first term in equation (23) in [38].

Equations (3)–(20) are obtained in the general form. The short-range nature of the inter-particle interaction is not used. Moreover, the traditional hydrodynamic equations are presented via the velocity field and the pressure tensor while equations (3)–(20) are written via the current and the momentum flux.

The method of the introduction of the velocity field in the equations of quantum hydrodynamics of the spinless particles is presented in [4, 38]. The method of calculation of the terms containing the interaction for the short-range interaction limit is also described in [4, 38]. Let us present the results of application of these methods for finite temperature bosons. Moreover, we consider the short-range interaction in the first order by the interaction radius.

2.3. Appearance of the quantum fluctuations in the third rank tensor evolution equation

Derivation of the quantum fluctuations requires the calculation of the time evolution of the current of the momentum flux \( M^{(\alpha\beta)} \) (18). The method of derivation is similar to the equations obtained above. The time derivative of tensor \( M^{(\alpha\beta)} \) acts on the wave function in its definition. The time derivative of the wave function is replaced by the Hamiltonian (1) in accordance with the many-particle microscopic Schrodinger equation \( i\hbar \partial_{t} \Psi = \hat{H} \Psi \). It leads to the following expression:
\[ \partial_{t} M^{(\alpha\beta)} = \frac{1}{\hbar} \int dR \sum_{i=1}^{N} \delta(r - r_{i}) \frac{1}{8m_{i}^{3}} [\hat{H}^{\alpha} \Psi^{*} \cdot \hat{p}_{i}^{\beta} \hat{p}_{i}^{\gamma} \Psi] \\
- \Psi^{*} \hat{p}_{i}^{\alpha} \hat{p}_{i}^{\beta} \hat{p}_{i}^{\gamma} \hat{H} \Psi + \hat{H}^{\alpha \beta} \Psi^{*} \cdot \hat{p}_{i}^{\gamma} \hat{p}_{i}^{\beta} \hat{p}_{i}^{\gamma} \Psi + \hat{p}_{i}^{\alpha} \hat{p}_{i}^{\beta} \hat{H}^{\gamma} \Psi^{*} \cdot \hat{p}_{i}^{\gamma} \Psi \\
- \hat{p}_{i}^{\alpha} \Psi^{*} \cdot \hat{p}_{i}^{\beta} \hat{p}_{i}^{\gamma} \hat{H} \Psi - \hat{H}^{\alpha \beta} \hat{p}_{i}^{\gamma} \Psi^{*} \cdot \hat{p}_{i}^{\gamma} \Psi + \hat{p}_{i}^{\alpha} \hat{p}_{i}^{\beta} \hat{H}^{\gamma} \Psi^{*} \cdot \hat{p}_{i}^{\gamma} \Psi \\
+ \hat{H}^{\alpha \beta} \Psi^{*} \cdot \hat{p}_{i}^{\beta} \hat{p}_{i}^{\gamma} \Psi - \hat{H}^{\alpha \beta} \hat{p}_{i}^{\gamma} \Psi^{*} \cdot \hat{p}_{i}^{\gamma} \Psi - c.c.] \]  
(23)

The part of the presented terms contains the Hamiltonian \( \hat{H} \) under action of the momentum operators. We permute the Hamiltonian \( \hat{H} \) and the operators acting on it. Hence, the result of permutation presented by the terms, where no operators act on the Hamiltonian \( \hat{H} \), and the terms containing the corresponding commutators. Therefore, all terms are combined in two groups:
\[ \partial_{t} M^{(\alpha\beta)} = \frac{1}{\hbar} \int dR \sum_{i=1}^{N} \delta(r - r_{i}) \frac{1}{8m_{i}^{3}} [\hat{H}^{\alpha} \Psi^{*} \cdot \hat{p}_{i}^{\beta} \hat{p}_{i}^{\gamma} \Psi] \\
- \Psi^{*} \hat{H}^{\alpha \beta} \Psi^{*} \cdot \hat{p}_{i}^{\gamma} \hat{p}_{i}^{\beta} \hat{p}_{i}^{\gamma} \Psi + \hat{H}^{\alpha \beta} \Psi^{*} \cdot \hat{p}_{i}^{\beta} \hat{p}_{i}^{\gamma} \Psi^{*} \cdot \hat{p}_{i}^{\gamma} \Psi \\
- \hat{p}_{i}^{\alpha} \Psi^{*} \cdot \hat{H}^{\beta} \Psi - \hat{H}^{\alpha \beta} \Psi^{*} \cdot \hat{p}_{i}^{\gamma} \Psi^{*} \cdot \hat{p}_{i}^{\gamma} \Psi \\
+ \hat{H}^{\alpha \beta} \Psi \Psi^{*} \cdot \hat{p}_{i}^{\beta} \hat{p}_{i}^{\gamma} \Psi - \hat{H}^{\alpha \beta} \Psi \Psi^{*} \cdot \hat{p}_{i}^{\gamma} \Psi - c.c.] \]
The first group of terms leads to the divergence of the flux of tensor $M^{\alpha\beta\gamma}$. The second group of terms containing the commutators presents the interactions.

Final form of tensor $M^{\alpha\beta\gamma}$ evolution equation can be expressed in the following terms:

$$\partial_t M^{\alpha\beta\gamma} + \partial_\alpha R_a^{\alpha\beta\gamma} = \frac{\hbar^2}{4m} n\partial_\alpha \partial_\beta \partial^\alpha V_{ext}$$

$$- \frac{1}{m} \Pi^{\alpha\beta} \partial_\alpha V_{ext} - \frac{1}{m} \Pi^{\alpha\beta} \partial_\beta V_{ext} - \frac{1}{m} \Pi^{\alpha\beta} \partial_\gamma V_{ext}$$

$$+ \frac{1}{m} F_{a\alpha\beta\gamma} + \frac{1}{m} (F^{\alpha\beta\gamma} + F^{\beta\alpha\gamma} + F^{\gamma\alpha\beta}),$$

(25)

where

$$F_{a\alpha\beta\gamma} = \frac{\hbar^2}{4m^2} \int [\partial^\alpha \partial^\beta \partial^\gamma U(r - r')] n_2(r, r', t) dr'$$

(26)

is the quantum force contribution leading to the quantum fluctuations, and

$$F^{\alpha\beta\gamma} = - \int [\partial^\alpha U(r - r')] \Pi^{\beta\gamma}(r, r', t) dr'$$

(27)

is the interaction contribution containing nonzero limit in the classical regime, with

$$\Pi^{\alpha\beta}(r, r', t) = \int dR \sum_{\alpha \beta \gamma} \frac{1}{4m^2} \delta(r - r) \delta(r' - r) \times$$

$$\times [\Psi^* \bar{P}^{\alpha\beta\gamma} \bar{P}^\gamma \Psi + (\bar{P}^{\beta\alpha\gamma} \bar{P}^\gamma \Psi^* + c.c.)].$$

(28)

Tensor $\Pi^{\alpha\beta}(r, r', t)$ can be simplified in the correlationless regime to the following form

$$\Pi^{\alpha\beta}(r, r', t) = \Pi^{\alpha\beta}(r, t) \cdot n(r', t) \cdot n(r, t).$$

However, the correlations caused by the symmetrization of the bosonic many-particle wave function are used below.

Terms $F^{\alpha\beta\gamma}$ and $F_{a\alpha\beta\gamma}$ are the third rank force tensors describing the interparticle interaction. However, equation (25) contains the flux of tensor $M^{\alpha\beta\gamma}$ which is the fourth rank tensor appearing in the following form:

$$R^{\alpha\beta\gamma} = \int dR \sum_{i=1}^{N} \delta(r - r) \frac{1}{16m^4 i} [\Psi^* \bar{P}^{\alpha\beta\gamma} \bar{P}^\gamma \Psi + \bar{P}^{\beta\alpha\gamma} \bar{P}^\gamma \Psi^* + \bar{P}^{\gamma\alpha\beta} \bar{P}^\gamma \Psi + \bar{P}^{\alpha\beta\gamma} \bar{P}^\gamma \Psi^* + c.c.]$$

(29)

equation (25) is obtained for the bosons at the arbitrary temperature. It can be separated on two equations for two subsystems: the BEC and the normal fluid. All terms in equation (25) are additive on the particles. Therefore, they are additive on the subsystems. Hence, the structure of the partial equations is identical to the structure of equation (25):

$$\partial_t M_a^{\alpha\beta\gamma} + \partial_\alpha R^{\alpha\beta\gamma}_a = - \frac{1}{m} n_{\alpha} \partial_\alpha \partial_\beta \partial^\alpha V_{ext}$$

$$- \frac{1}{m} \Pi^{\alpha\beta}_a \partial_\alpha V_{ext} - \frac{1}{m} \Pi^{\alpha\beta}_a \partial_\beta V_{ext} - \frac{1}{m} \Pi^{\alpha\beta}_a \partial_\gamma V_{ext}$$

$$+ \frac{1}{m} F^{\alpha\beta\gamma}_a + \frac{1}{m} (F^{\alpha\beta\gamma}_a + F^{\beta\alpha\gamma}_a + F^{\gamma\alpha\beta}_a),$$

(30)

where subindex $a$ equal $B$ for the BEC and $n$ for the normal fluid.

The fourth rank kinematic tensor $R^{\alpha\beta\gamma}_a$ (29) has the following form after the introduction of the velocity field via the Madelung transformation of the many-particle wave function:

$$R^{\alpha\beta\gamma}_a = n_{\alpha} v_{\alpha\beta\gamma} + T^{\alpha\beta\gamma}_a$$

$$+ v_{\alpha\beta\gamma}(P^{\alpha\beta\gamma}_a + T^{\alpha\beta\gamma}_a) + v_{\beta\alpha\gamma}(P^{\beta\alpha\gamma}_a + T^{\beta\alpha\gamma}_a) + v_{\gamma\alpha\beta}(P^{\gamma\alpha\beta}_a + T^{\gamma\alpha\beta}_a)$$
\[ + v^a_{i} v^b_{j} \left( p^a_{i} + T^a_{i,} \right) + v^a_{k} v^b_{l} \left( T^a_{k} + T^a_{l} \right) + v^a_{i} v^b_{j} \left( R^a_{i} + T^a_{i,} \right) + v^a_{j} \left( q^a_{i} + T^a_{i,} \right) + v^a_{i} \left( q^a_{i} + T^a_{i,} \right) + v^a_{i} q^b_{l} + v^a_{i} q^b_{k} + v^a_{i} q^b_{i} + v^a_{i} q^b_{j} + \]
\[ + Q^a_{i} T^a_{i,} + T^a_{i,} + L^a_{i} . \]

This structure shows some similarity to the representations for the second rank tensor momentum flux (11) and for the third rank tensor (21), where the higher rank tensors are partially transformed via the concentration, velocity field and, if possible, via other tensors of the smaller rank. However, this transformation is partial since there is the tensor of the equal rank, but defined in the comoving frame. Moreover, this final tensor is splitted on few parts. It has two parts for the second rank tensor of the momentum flux, where we have the kinetic pressure (quasi-classical part of thermal nature) and the quantum Bohm potential (the quantum part). There are three parts for the third rank tensor $M^{\alpha \beta \gamma}$. They are the quasi-classical part of thermal nature, the quantum part, and the combined thermal-quantum part. For the fourth rank tensor, we also have three parts: the quasi-classical part of thermal nature $Q^{\alpha \beta \gamma}$, the quantum part $T^{a_{\alpha \beta \gamma}}$, and the combined thermal-quantum part $L^a_{\alpha \beta \gamma}$. 

Developed model shows that the arbitrary quantum system can be modeled via the hydrodynamic equations which are traditionally associated with the fluid dynamics. Quantum systems demonstrate that each particle shows the properties of the wave and this wave-like behavior is incorporated in the quantum hydrodynamic model. This conclusion follows from the fact that the quantum hydrodynamic is derived from the Schrodinger equations which contains these information. These general concept is illustrated for the ultracold bosons, but the quantum hydrodynamic method can be applied to other physical systems. This similarity between quantum behavior and the dynamics of fluids recently found unusual realization. It is experimentally found that classic fluid objects demonstrate the quantum-like behavior [39–41]. It is observed as the millimetric droplet walking on the surface of vibrating fluids, where the motion of droplets is affected by the resonant interaction with their own wave field [40, 41]. Systems of the walking droplets demonstrate various quantum effects [42–44].

3. Contribution of interaction in the quantum hydrodynamic equations

Equations (6), (16), and (25) contain terms describing interaction. Approximate forms of these force fields of different tensor ranks are necessary to get the truncated set of equations. In our case, it is necessary to include the short-range nature of the potential of the interparticle interaction. Moreover, the weak limit interaction is considered. These two assumptions are used to get the simplified forms of $F^{\alpha \beta}, F^{\alpha \beta}, F^{\alpha \beta}$ and $F^{\alpha \beta}$ in this section.

3.1. Interaction terms in the Euler equation

The short-range interaction in the Euler for the single species of quantum particles can be written as the divergence of the symmetric quantum stress tensor $F^{\alpha \beta} = - \partial^\beta \sigma^{\alpha \beta}$. 

The first order by the interaction radius approximation gives the following expression for the quantum stress tensor (see also [4])

\[ \sigma^{\alpha \beta}(r, t) = \frac{1}{2} \int dR \sum_{i,j=2} \delta(r - R^i_j) \frac{\partial U(t)}{\partial r} \Psi^*(R^i_j) \Psi(R^i_j), \] (32)

where $R^i_j = (..., R^i_j, ..., R^i_j, ...)$ with vector $R^i_j$ located at $i$-th and $j$-th places.

Expression (32) can be rewritten in terms of the two-particle concentration

\[ \sigma^{\alpha \beta}(r, t) = -\frac{1}{2} Tr(f_2(r, r', t)) \int dr \frac{\partial U(r)}{\partial r}, \] (33)

where the notion of trace is used

\[ Trf(r, r') = f(r, r). \] (34)

Consideration of the short-range interaction leads to the separation of integral containing the potential of interaction (33). So, the characteristics of interaction do not depend on the motion or position of particles. This integral simplifies in the following way

\[ \int \frac{\partial U}{\partial r} r \, dr = \frac{1}{2} \delta^{\alpha \beta} \int r U' dr = -\delta^{\alpha \beta} \int U dr. \] (35)

The last integral in this expression is denoted as the interaction constant $g = \int U dr$.

The two-particle concentration can be calculated in the weakly interacting limit (see [4])

\[ f_2(r, r', t) = n(r, t)n(r', t) + |\rho(r, r', t)|^2 + \phi(r, r', t). \] (36)
where
\[ n(r, t) = \sum_{f} n_f \varphi_f^*(r, t) \varphi_f(r, t) \] (37)
is the expression of concentration \((2)\) in terms of the single particle wave functions \(\varphi_f(r, t)\),
\[ \rho(r', r, t) = \sum_{f} n_f \varphi_f^*(r, t) \varphi_f(r', t) \] (38)
is the density matrix, and
\[ \varphi(r, r', t) = \sum_{f} n_f (n_f - 1) |\varphi_f(r, t)|^2 |\varphi_f(r', t)|^2. \] (39)
The last term in equation describes interaction of pairs of particles being in the same quantum state. It cannot be seen from the existence of single quantum number \(f\) in all wave are single-particle wave functions (39). The first and second terms are related to particles located in different quantum states. It cannot be seen from equation (36), but it follows from intermediate terms which can be found in [4].

The trace of the two-particle concentration entering the quantum stress tensor has the following form
\[ Tr_{nn}(r, r', t) \approx 2(n^2)' + n^2_B, \] (40)
where the first term on the right-hand side contains symbol \(^\prime\) which means that the product of concentrations is related to the particles being in the different quantum states. Therefore, the first term has no \(n^2_B\) contribution from selfaction of BEC. The dropped terms are described in [45].

Next, we present the explicit contribution of the BEC concentration \(n_B\) and the normal fluid concentration \(n_n\) in the first term on the right-hand side of equation (40):
\[ (n^2)' = ((n_B + n_n)(n_B + n_n))' = (n^2_n + 2n_B n_n). \] (41)
The last term in equation (40) appears for the particles being in BEC from expression (39). Overall, equation (40) gives contribution for the interaction between BEC and normal fluid and for the interaction between bosons belonging to normal fluid.

Full expression of the quantum stress tensor for the bosons at the finite temperature can be written in terms of the concentration of BEC and the concentration of normal fluid:
\[ \sigma^{\alpha\beta} = \frac{1}{2} \delta^{\alpha\beta} (2n_B^2 + 4n_B n_n + n_n^2). \] (42)
If we consider the dynamics of BEC or normal fluid we cannot use the notion of the quantum stress tensor \(\sigma^{\alpha\beta}\) for the interaction of subspecies.

If we consider dynamics of BEC we need to extract force caused by the BEC and normal fluid acting on the BEC. The first (last) term in equation (42) contains the selfaction of the normal fluid (of the BEC). The second term in equation (42) presents the interaction between the BEC and normal fluid. This force is the superposition of a part of the second term in equation (42) and the last term in equation (42):
\[ F^{\alpha}_{nn} = -g n_B \partial^{\alpha} (2n_n + n_B), \] (43)
The second term in equation (42) can be rewritten as follows \(F^{\alpha}_{nn} = -2g (n_B \partial^{\alpha} n_n + n_n \partial^{\alpha} n_B)\). The first part of this expression is used in equation (43).

If we consider the dynamics of normal fluid it means that the source of field in the first term of \(n_n\) can be the normal fluid and the BEC. Hence, the last term gives no contribution in this case in equation (42):
\[ F^{\alpha}_{nn} = -2g n_n \partial^{\alpha} (n_n + n_B). \] (44)
The nonsymmetric decomposition allows to use the notion of the NLSE. It is necessary condition to have the GP equation for the BEC at the finite temperatures. Moreover, the nonsymmetric form is traditionally used in literature [34]. Similar chose is made at the analysis \(j^{\beta}_2\) below.

3.1.1. Nonlinear Schroedinger equations
Dropping the pressure of normal fluid and using the quantum Bohm potential in form (12) for both subspecies we find the closed set of hydrodynamic equations. Introducing the macroscopic wave function for both the BEC and the normal fluid for the potential velocity fields as \(\Phi_B = \sqrt{n_B} e^{i\omega_B t / \hbar}\), where \(\phi_B\) is the potential of the velocity field \(v_B = -\nabla \phi_B\),
\[ i\hbar \partial_t \Phi_B = \left( -\frac{\hbar^2 \nabla^2}{2m} + V_{ext} + g (n_B + 2n_n) \right) \Phi_B, \] (45)
and
\[ \hbar \partial_t \Phi_n = \left( -\frac{\hbar^2 \nabla^2}{2m} + V_{\text{ext}} + 2g(n_b + n_n) \right) \Phi_n. \] (46)

The kinetic energy (the first term on the right-hand side of equations (45) and (46)) corresponds to the application of the noninteracting limit for the quantum Bohm potential for the BEC and for the normal fluid. Let us repeat that the pressure of the normal fluid is formally dropped in equation (46).

Equations (45), (46) partially correspond to equations (17)–(19) given in [1]. Let us discuss this comparison in more details. Equation (45) completely corresponds to equation (17) of [1]. Equation (45) is the GP equation, where the action of the normal fluid on the BEC evolution is presented by the last term \(2gn_n\Phi_n\). The properties of the normal fluid are described in [1] within the concentration of bosons. This concentration is constructed as the superposition of bilinear combinations of partial nonlinear amplitudes. Each amplitude is described by equations (18) and 19 of [1]. The structure of these equations is similar to structure of equation (46). Moreover, in [1], the concentration of normal fluid includes the probability of contribution of partial amplitudes. The probability is given by the Bose–Einstein distribution. Therefore, the concentration in [1] contains detailed connection to the distribution of bosons on the quantum states (this concentration is more like the distribution function, but it has no explicit dependence on the momentum) in contrast with the rather rough connection used in equation (46). As we mentioned above, we formally dropped the pressure of normal fluid in order to get equation (46). Comparison with [1] shows that this is inappropriate step. The pressure keeps some minimal information about distribution of particles over the quantum states. Moreover, the kinetic pressure is the necessary part of accurate hydrodynamic model of the normal fluid.

Therefore, the account of the pressure evolution together with the pressure flux evolution gives the generalization of the model presented with the nonlinear Schrödinger equations (45), (46). Necessity of additional equations is demonstrated in [6, 46, 47] if one wants to include the quantum fluctuations.

3.2. Interaction terms in the pressure evolution equation

General form of the pressure evolution equation contains the interaction via the force second rank tensor field (17). Its main contribution is obtained in the first order by the interaction radius. The result appears in the following form

\[ F^{\alpha\beta}(r, t) = \frac{1}{8m^2} \partial^\gamma \int dR \sum_{i,j=1} \delta(r - R_{ij}) \times \frac{r_i^\alpha r_j^\beta}{|r_i|} \partial U(r_i) \left[ \Psi^\dagger(R', t)(\hat{R}^\alpha_{(1)} + \hat{R}^\alpha_{(2)})\Psi(R', t) + c.c. \right] \]
\[ - \frac{1}{8m^2} \int dR \sum_{i,j=1} \delta(r - R_{ij}) \frac{r_i^\alpha r_j^\beta}{|r_i|} \partial U(r_i) \times \left[ \left( \partial^\gamma_{(1)} - \partial^\gamma_{(2)} \right) \Psi^\dagger(R', t)(\hat{R}^\alpha_{(1)} - \hat{R}^\alpha_{(2)})\Psi(R', t) + \Psi^\dagger(R', t)(\hat{R}^\alpha_{(2)} - \hat{R}^\alpha_{(1)})\Psi(R', t) + c.c. \right]. \] (47)

Expression (47) appears at the expansion of the force tensor field (17) using the short-range nature of the interaction (see [4] for the method described for the force field, or [38] for application of this method to the fermions).

For the force tensor field \( F^{\alpha\beta} \), we can present the intermediate expressions like equations (33) and (36) obtained for the force field \( F^\alpha = -\partial^\beta \sigma^{a\beta} \). However, similar expressions obtained for \( F^{\alpha\beta} \) are rather large. Hence, we start the presentation with the equation similar to equation (40) which is obtained after taking the trace of the intermediate expressions.

Therefore, we obtain the following simplification of equation (47) for the force tensor field \( F^{\alpha\beta} \):

\[ F^{\alpha\beta} = -\frac{g}{4m^2} \partial^\beta [2(n\Lambda^\alpha)' + n_n\Lambda^\alpha_0] \]
\[ + \frac{i}{\hbar} \frac{g}{4m^2} [2(n^\alpha r^{\alpha\beta})' - 2(N^\alpha \Lambda^\beta)' + n_n r^{\alpha\beta}_0 - \Lambda^\alpha_0 \Lambda^\beta_0] + c.c., \] (48)

where we use the intermediate functions \( \Lambda^\alpha \) and \( r^{\alpha\beta} \) with the following definitions:

\[ \Lambda^\alpha = \sum_f n_f \varphi^\dagger_f \hat{p}^\alpha \varphi_f = mj^\alpha - \frac{i}{2} \partial^\alpha n, \] (49)
and

\[
 r^{\alpha\beta} = \sum_f n_f \varphi_f^* \varphi_f^\dagger r^{\alpha\beta} = m^2 \left( m v^\alpha v^\beta + p^{\alpha\beta} - \frac{\hbar^2}{2m^2} \sum_f n_f a_f \partial^\alpha \partial_f a_f \right) \\
- \frac{m\hbar}{2} \left[ \partial^\alpha (mv^\beta) + \partial^\beta (mv^\alpha) \right].
\]

The calculation of functions \(\Lambda^{\alpha}\) and \(r^{\alpha\beta}\) includes the Madelung transformation of the single-particle wave functions \(\varphi(r, t) = \sqrt{\rho} e^{i\phi} \). Next, we use the following definitions of the velocity field and the pressure tensor in terms of the single-particle wave functions \(mv^\alpha = \sum_f n_f a_f^* (\hbar \partial^\alpha S_f / m)\), and \(p^{\alpha\beta} = \sum_f n_f a_f^* u^\alpha f^\dagger u^\beta\), where \(u^\alpha_f = (\hbar \partial^\alpha S_f / m) - v^\alpha\).

Let us represent terms like \((nL^\alpha)^f\) in the explicit form:

\[
 F^{\alpha\beta} = -\frac{g}{2m} \partial^\beta \left[ 2n_a \Lambda^\alpha_n + 2n_b \Lambda^\alpha_n + 2n_B \Lambda^\alpha_n \right] \\
+ \frac{g}{\hbar} \left[ 2n_a r^{\alpha\beta}_n - 2\Lambda^\alpha_a \Lambda^\beta_n + 2n_a r^{\alpha\beta}_n - 2\Lambda^\alpha_a \Lambda^\beta_n \right] \\
\times 2n_B r^{\alpha\beta}_n - 2\Lambda^\alpha_B \Lambda^\beta_n + n_B r^{\alpha\beta}_n - \Lambda^\alpha_B \Lambda^\beta_n + c.c.
\]

Further calculation gives the representation of tensor \(F^{\alpha\beta}\) in terms of hydrodynamic functions:

\[
 F^{\alpha\beta} = -\frac{g}{2m} \partial^\beta \left[ 2n_a j^\alpha_n + 2n_a i^\alpha_n + 2n_B j^\alpha_n + 2n_B i^\alpha_n \right] \\
+ \frac{g}{\hbar} \left[ 2n_a (\partial^\beta j^\alpha_n + \partial^\beta i^\alpha_n) - 2j^\alpha_n \partial^\beta n_a - 2i^\alpha_n \partial^\beta n_a \right] \\
+ 2n_a (\partial^\beta j^\alpha_n + \partial^\beta i^\alpha_n) - 2j^\alpha_n \partial^\beta n_a - 2i^\alpha_n \partial^\beta n_a \\
+ 2n_B (\partial^\beta j^\alpha_n + \partial^\beta i^\alpha_n) - 2j^\alpha_n \partial^\beta n_B - 2i^\alpha_n \partial^\beta n_B \\
+ n_B (\partial^\beta j^\alpha_n + \partial^\beta i^\alpha_n) - 2j^\alpha_n \partial^\beta n_B - 2i^\alpha_n \partial^\beta n_B \right].
\]

The momentum flux evolution equation contains the symmetric combination of the force tensor fields \(F^{\alpha\beta}\):\n
\[
 F^{\alpha\beta} + F^{\beta\alpha} = -\frac{g}{m} \left[ 2\left( j^\alpha_n \partial^\beta n_b + j^\beta_n \partial^\alpha n_b \right) \\
+ 2\left( j^\alpha_n \partial^\beta n_b + j^\beta_n \partial^\alpha n_b \right) + 2\left( j^\alpha_n \partial^\beta n_b + j^\beta_n \partial^\alpha n_b \right) \right].
\]

The zero temperature analysis demonstrates that there is nonzero pressure for the BECs, caused by the quantum fluctuations entering the set of hydrodynamic equations via the evolution of the pressure flux [6, 46, 47]. The pressure also exists for the normal fluid. So, we make the decomposition of the momentum flux evolution equation on two partial equations for \(\Pi^{\alpha\beta}\) and \(\Pi^{\alpha\beta}\). Formally this decomposition is presented with equation (20). To complete this procedure we need to split the force tensor field \(F^{\alpha\beta} + F^{\beta\alpha} = F^{\alpha\beta}_B + F^{\beta\alpha}_B + F^{\alpha\beta}_n + F^{\beta\alpha}_n\), where

\[
 F^{\alpha\beta}_B + F^{\beta\alpha}_B = -\frac{g}{m} \left[ 2\left( j^\alpha_n \partial^\beta n_b + j^\beta_n \partial^\alpha n_b \right) + 2\left( j^\alpha_n \partial^\beta n_b + j^\beta_n \partial^\alpha n_b \right) \right],
\]

and

\[
 F^{\alpha\beta}_n + F^{\beta\alpha}_n = -\frac{g}{m} \left[ 2\left( j^\alpha_n \partial^\beta n_b + j^\beta_n \partial^\alpha n_b \right) + 2\left( j^\alpha_n \partial^\beta n_b + j^\beta_n \partial^\alpha n_b \right) \right].
\]

After extraction of the pressure tensor \(p^{\alpha\beta}\) from the momentum flux evolution \(\Pi^{\alpha\beta}\) we have the extra contribution of the interaction in the pressure evolution equation in compare with equations (20). It contains the following contribution \(F^{\alpha\beta} - v^\alpha F^{\beta\alpha}\).

Using equations (43), (44), (55), (54) find \(F^{\alpha\beta} + F^{\beta\alpha} - v^\alpha F^{\beta\alpha} - v^\beta F^{\alpha\beta} = 0\) for the BECs and for the normal fluid.
The pressure evolution equation is used in [48] for bosons above the critical temperature. Equation (4) of [48] contains the force in the following form $n v \cdot F$ which generally differs from $F^{\alpha \beta} - v^\beta F^\alpha$ obtained above.

### 3.3. The short-range interaction in the third rank tensor evolution equation

The third rank tensor $M^{\alpha \beta \gamma}$ (25) evolution equation contains two kinds of the third rank force tensors $F^{\alpha \beta \gamma}$ (27) and $F_{abg}^{\alpha \beta \gamma}$ (26). Consider them separately.

#### 3.3.1. Quasi-classical third rank force tensor

Tensor $F_{abg}^{\alpha \beta \gamma}$ (26) is proportional to the Planck constant, so it goes to zero in the classical limit. The third rank force tensor $F^{\alpha \beta \gamma}$ (27) is different, it has nonzero limit in the quasiclassical regime. However, we are interested in the value of tensor $F^{\alpha \beta \gamma}$ (27) in one of quantum regimes for the degenerate bosons.

We calculate the third rank force tensor $F^{\alpha \beta \gamma}$ (27) in the first order by the interaction radius. It appears in the following form

$$F^{\alpha \beta \gamma}(r, t) = \frac{1}{8m} \int dR \sum_{i,j} \delta(r - R_{ij}) \frac{r_{ij}^3}{|r_{ij}|} \frac{\partial U(r_{ij})}{\partial |r_{ij}|} \left[ \Psi^\alpha(R', t) \tilde{\Psi}^\beta(R', t) + \tilde{\Psi}^\alpha(R', t) \Psi^\beta(R', t) + c.c. \right]$$

$$- \frac{1}{8m^3} \int dR \sum_{i,j} \delta(r - R_{ij}) \frac{r_{ij}^3}{|r_{ij}|} \frac{\partial U(r_{ij})}{\partial |r_{ij}|} \left[ (\partial_{\alpha} - \partial_{\gamma}) \Psi^\beta(R', t) \tilde{\Psi}^\gamma(R', t) + \tilde{\Psi}^\beta(R', t) \Psi^\gamma(R', t) \right]$$

$$\times \Psi^\alpha(R', t) \tilde{\Psi}^\beta(R', t) + c.c.] + \tilde{\Psi}^\alpha(R', t) \Psi^\beta(R', t) \right]$$

Here, the part of expression for $F^{\alpha \beta \gamma}$ containing the interaction potential appears as the independent multiplier. It has same form as the integral in the Euler equation (35). Hence, tensor $F^{\alpha \beta \gamma}$ is proportional to the Groos-Pitaevskii interaction constant.

Further calculations are made in the weakly interacting limit, following the method described in [4]. It gives the intermediate representation of the third rank force tensor:

$$F^{\alpha \beta \gamma}(r, t) = -\frac{g}{4m} \left[ \Pi^{\alpha \beta} \partial^\gamma n_b + \Pi^{\alpha \beta} \partial^\gamma n + \left( 4 \pi \sum_f n_f \partial^\gamma \tilde{\varphi}^\alpha \tilde{\varphi}^\beta \varphi_f + \frac{2}{h^2} N^{\alpha \beta \gamma} - \frac{1}{h} N^{\alpha \gamma} + c.c. \right) \right]$$

(57)

where

$$\kappa^{\alpha \beta} = \sum_f n_f \tilde{\varphi}_f^\alpha \tilde{\varphi}_f^\beta \varphi_f.$$ 

(58)

Function $\kappa^{\alpha \beta}$ is the nonsymmetric tensor. It has symmetric real part and the antisymmetric imaginary part:

$$\kappa^{\alpha \beta} = m^2 \left( m^2 v^\alpha v^\beta + p^{\alpha \beta} + \frac{h^2}{m^2} \sum_f n_f \partial^\alpha a_f \partial^\beta a_f \right)$$

$$- \frac{1}{2} m h [v^\alpha \partial^\beta n - v^\beta \partial^\alpha n + \sum_f n_f a_f (u^\alpha \partial^\beta a_f - u^\beta \partial^\alpha a_f)].$$

(59)

No specific notation is introduced for the third rank tensor $\sum_f n_f \partial^\gamma \tilde{\varphi}_f^\alpha \tilde{\varphi}_f^\beta \varphi_f$. In our calculations we need its real part multiplied by 2:

$$\sum_f n_f \partial^\gamma \tilde{\varphi}_f^\alpha \tilde{\varphi}_f^\beta \varphi_f + c.c. = 2m^2 \left[ \frac{1}{2} \partial^\gamma n \cdot v^\alpha v^\beta - \partial^\alpha n \cdot v^\beta v^\gamma - \partial^\beta n \cdot v^\alpha v^\gamma - m^2 (\partial^3 v^\alpha + \partial^3 v^\beta) \right]$$

$$+ \sum_f n_f a_f (\partial^\gamma a_f) u^\alpha u^\beta - \frac{1}{2} \partial^\gamma p^{\alpha \gamma} - \frac{1}{2} \partial^\alpha p^{\gamma \gamma} + \frac{1}{2} \sum_f n_f a_f^2 (u^\alpha \partial^\beta a_f + u^\beta \partial^\alpha a_f)$$

$$+ \nu^\alpha \sum_f n_f a_f (\partial^\gamma a_f \cdot u^\beta) + v^\beta \sum_f n_f a_f (\partial^\gamma a_f \cdot u^\alpha) - \frac{h^2}{m^2} \sum_f n_f \partial^\gamma a_f \cdot \partial^\alpha \partial^\beta a_f \right].$$

(60)

Equation (57) includes term $\Pi^{\alpha \beta} \partial^\gamma n_b$ which describes full contribution of the BEC in $F^{\alpha \beta \gamma}$. The second term in equation (57) is the analog of the first term in equation (36). All following terms in equation (57) are the analog of the second term in equation (36). It can be interpreted as the exchange interaction.
Further calculation of the (57) gives the partially truncated expression mainly presented via the macroscopic hydrodynamic functions:

\[ F^{\alpha\beta\gamma}(r, t) = -\frac{g}{4} \left[ 4 \Pi^{\alpha\beta}_{\mu} \partial^\mu n_b + 4 \Pi^{\beta\gamma}_{\mu} \partial^\mu n + \partial^\alpha n \left( \Pi^{\beta\gamma} + \frac{\hbar^2}{4m^2} \partial^\gamma \partial^\beta n \right) \right. \\
+ \partial^\beta n \left( \Pi^{\alpha\gamma} + \frac{\hbar^2}{4m^2} \partial^\gamma \partial^\alpha n \right) + \partial^\gamma n \left( \Pi^{\alpha\beta} - \frac{\hbar^2}{4m^2} \partial^\alpha \partial^\beta n \right) \\
+ n(3 \partial^\alpha n \cdot \psi^\nu \psi^\lambda - 2 \partial^\alpha n \cdot \psi^\nu \psi^\gamma - 2 \partial^\beta n \cdot \psi^\nu \psi^\gamma - n \psi^\nu (\partial^\alpha \psi^\lambda + \partial^\beta \psi^\gamma) - \partial^\beta \rho^\gamma - \partial^\beta \rho^\nu \\
+ \sum_f n_f(\partial^\beta a^\mu_f) u^\mu_f u^\beta_f + \sum_f n_f a^\mu_f a^\rho_f u^\beta_f u^\gamma_f + \frac{3}{2} \psi^\nu \sum_f n_f(\partial^\gamma a^\mu_f \cdot u^\beta_f - \partial^\beta a^\mu_f \cdot u^\gamma_f) \\
\left. + \frac{3}{2} \psi^\nu \sum_f n_f(\partial^\gamma a^\mu_f \cdot u^\beta_f - \partial^\beta a^\mu_f \cdot u^\gamma_f) - \frac{\hbar^2}{m^2} \sum_f n_f \partial^\gamma a^\mu_f \cdot \partial^\alpha \partial^\beta a_f \right], \tag{61} \]

equation for the third rank tensor \( M^{\alpha\beta\gamma} \) evolution contains the symmetric combination of the third rank force tensors (61) which can be presented in the following form:

\[ F^{\alpha\beta\gamma} + F^{\beta\gamma\alpha} + F^{\gamma\alpha\beta} = -\frac{g}{4} \left[ 4(\Pi^{\alpha\beta}_{\mu} \partial^\mu n_b + \Pi^{\beta\gamma}_{\mu} \partial^\mu n + \Pi^{\gamma\alpha}_{\mu} \partial^\mu n) \\
+ \partial^\alpha n \left( 3 \Pi^{\beta\gamma} + \frac{\hbar^2}{4m^2} \partial^\gamma \partial^\beta n \right) + \partial^\beta n \left( 3 \Pi^{\gamma\alpha} + \frac{\hbar^2}{4m^2} \partial^\alpha \partial^\gamma n \right) + \partial^\gamma n \left( 3 \Pi^{\alpha\beta} + \frac{\hbar^2}{4m^2} \partial^\beta \partial^\alpha n \right) \\
- n(\partial^\alpha \Pi^{\beta\gamma} + \partial^\beta \Pi^{\gamma\alpha} + \partial^\gamma \Pi^{\alpha\beta}) - \frac{3 \hbar^2}{4m^2} n \partial^\alpha \partial^\beta \partial^\gamma n \right], \tag{62} \]

The pressure flux evolution equation obtained as the reduction of the third rank tensor \( M^{\alpha\beta\gamma} \) evolution equation contains the following combination of the force fields:

\[ F^{\alpha\beta\gamma} + F^{\beta\gamma\alpha} + F^{\gamma\alpha\beta} = \frac{\hbar^2}{16m^2} \left[ 3n \partial^\alpha \partial^\beta \partial^\gamma n - \partial^\alpha n \cdot \partial^\beta \partial^\gamma n - \partial^\beta n \cdot \partial^\alpha \partial^\gamma n - \partial^\gamma n \cdot \partial^\alpha \partial^\beta n \right], \tag{63} \]

where symbol \([\cdot]\)' specifies that product of functions describing the BEC is excluded similarly equations (40) and (41).

The quantum part can be represented

\[ F^{\alpha\beta\gamma} + F^{\beta\gamma\alpha} + F^{\gamma\alpha\beta} = \frac{\hbar^2}{16m^2} \left[ \partial^\alpha n \partial^\beta \partial^\gamma n - \partial^\beta n \cdot \partial^\gamma \partial^\alpha n + \partial^\gamma n \partial^\alpha \partial^\beta n - \partial^\alpha n \cdot \partial^\beta \partial^\gamma n - \partial^\gamma n \cdot \partial^\alpha \partial^\beta n \right]' \tag{64} \]

It can be considered as \( F^{\alpha\beta\gamma} + F^{\beta\gamma\alpha} + F^{\gamma\alpha\beta} = \tilde{F}^{\alpha\beta\gamma} + \tilde{F}^{\beta\gamma\alpha} + \tilde{F}^{\gamma\alpha\beta} \),

where

\[ \tilde{F}^{\alpha\beta\gamma} = \frac{\hbar^2}{4m^2} \left[ \partial^\alpha (n \Pi^{\beta\gamma}) + \partial^\beta (n \Pi^{\gamma\alpha}) + \partial^\gamma (n \Pi^{\alpha\beta}) \right]' \tag{65} \]

It is the derivative of the second rank tensor. If we consider the zero temperature limit we find \( F^{\alpha\beta\gamma} = -g(4m^2) \Pi^{\alpha\beta}_{\mu} \partial^\mu n_b \).

The transition from the equation of evolution for tensor \( M^{\alpha\beta\gamma} \) to the equation of evolution for tensor \( Q^{\alpha\beta\gamma} \) is made. Tensor \( Q^{\alpha\beta\gamma} \) is the sibling of \( M^{\alpha\beta\gamma} \), but the pressure flux \( Q^{\alpha\beta\gamma} \) is defined in the comoving frame leads to the canceling of such term. Hence, the nonzero contribution comes from the quantum part of the third rank force tensor \( F_{\alpha\beta\gamma}^{\rho} \). The nonzero temperature gives the nonzero contribution of \( F^{\alpha\beta\gamma} \) at the transition to the pressure flux evolution equation.

Equation (65) is obtained for all bosons. We need to separate it on the force acting on the BEC and the force acting on the normal fluid. Finally, we obtain
\[ F_{\alpha \beta}^{\gamma \delta} + F_{\beta}^{\alpha \gamma} + F_{\gamma}^{\alpha \beta} = \frac{g}{4} [n\partial^{\alpha}(n_{\alpha} \Pi_{\beta}^{\gamma}) + \partial^{\gamma}(n_{\alpha} \Pi_{\beta}^{\gamma}) + \partial^{\delta}(n_{\alpha} \Pi_{\beta}^{\gamma})] \\
+ [n\partial^{\alpha}(\Pi_{\beta}^{\gamma}) + \partial^{\gamma}(\Pi_{\beta}^{\gamma}) + \partial^{\delta}(\Pi_{\beta}^{\gamma}) + \Pi_{\beta}^{\alpha} \partial\varphi n_{\alpha} + \Pi_{\beta}^{\gamma} \partial\varphi n_{\delta} + \Pi_{\beta}^{\delta} \partial\varphi n_{\alpha} + \Pi_{\beta}^{\gamma} \partial\varphi n_{\delta}] \\
+ \frac{g\hbar^{3}}{16m^{2}} [3n\partial^{\alpha} \varphi \partial^{\gamma} \varphi n_{\alpha} + 3n\partial^{\alpha} \varphi \partial^{\delta} \varphi n_{\delta} - \partial^{\gamma} \varphi n_{\alpha} \partial^{\delta} \varphi n_{\delta} - \partial^{\gamma} \varphi n_{\delta} \partial^{\delta} \varphi n_{\alpha} - \partial^{\gamma} \varphi n_{\alpha} \partial^{\delta} \varphi n_{\delta} - \partial^{\gamma} \varphi n_{\delta} \partial^{\delta} \varphi n_{\alpha}] \\
- \frac{1}{2} \partial^{\gamma} \varphi n_{\alpha} \cdot \partial^{\delta} \varphi \varphi n_{\delta} - \frac{1}{2} \partial^{\gamma} \varphi n_{\delta} \cdot \partial^{\delta} \varphi \varphi n_{\alpha} - \frac{1}{2} \partial^{\gamma} \varphi n_{\alpha} \partial^{\delta} \varphi n_{\delta} - \frac{1}{2} \partial^{\gamma} \varphi n_{\delta} \partial^{\delta} \varphi n_{\alpha} . \\
\] (66)

and

\[ F_{\alpha \beta}^{\gamma \delta} + F_{\beta}^{\alpha \gamma} + F_{\gamma}^{\alpha \beta} = \frac{g}{4} [n_{\beta} \partial^{\alpha}(\Pi_{\gamma}^{\delta}) + \partial^{\gamma}(\Pi_{\beta}^{\delta}) + \partial^{\delta}(\Pi_{\beta}^{\gamma}) + \Pi_{\beta}^{\alpha} \partial\varphi n_{\gamma} + \Pi_{\beta}^{\gamma} \partial\varphi \varphi \gamma \varphi n_{\delta} + \Pi_{\beta}^{\delta} \partial\varphi \gamma \varphi n_{\alpha} + \Pi_{\beta}^{\gamma} \partial\varphi \gamma \varphi n_{\delta}] \\
+ \frac{g\hbar^{3}}{16m^{2}} [3n_{\beta} \partial^{\alpha} \varphi \partial^{\gamma} \varphi n_{\gamma} + 3n_{\beta} \partial^{\alpha} \varphi \partial^{\delta} \varphi n_{\delta} - \partial^{\gamma} \varphi n_{\alpha} \partial^{\delta} \varphi n_{\delta} - \partial^{\gamma} \varphi n_{\delta} \partial^{\delta} \varphi n_{\alpha} - \partial^{\gamma} \varphi n_{\alpha} \partial^{\delta} \varphi n_{\delta} - \partial^{\gamma} \varphi n_{\delta} \partial^{\delta} \varphi n_{\alpha}] \\
- \frac{1}{2} \partial^{\gamma} \varphi n_{\alpha} \cdot \partial^{\delta} \varphi \varphi n_{\delta} - \frac{1}{2} \partial^{\gamma} \varphi n_{\delta} \cdot \partial^{\delta} \varphi \varphi n_{\alpha} - \frac{1}{2} \partial^{\gamma} \varphi n_{\alpha} \partial^{\delta} \varphi n_{\delta} - \frac{1}{2} \partial^{\gamma} \varphi n_{\delta} \partial^{\delta} \varphi n_{\alpha} . \\
\] (67)

Equations (66) and (67) gives the final expressions for the quasi-classic force fields in the pressure flux evolution equations for the two fluid model.

3.3.2. The third rank force tensor describing the quantum fluctuation

The quantum fluctuations in the zero temperature BECs is caused by tensor \( F_{\alpha \beta}^{\gamma \delta} \) (26). Its major contribution can be found in the first order by the interaction radius approximation. Here, we consider the small nonzero temperature regime of \( F_{\alpha \beta}^{\gamma \delta} \) for the bosons. So, we obtain its generalization for the two-fluid model. The quantum third rank force tensor \( F_{\alpha \beta}^{\gamma \delta} \) (26) is calculated in the first order by the interaction radius

\[
F_{\alpha \beta}^{\gamma \delta} = \frac{\hbar^{2}}{8m^{2}} \int dR \sum_{i,j=1} \delta(r - R_{ij}) \\
\times \phi_{i}^{\alpha} \partial_{i}^{\alpha} \phi_{j}^{\beta} \phi_{j}^{\gamma} U(r_{ij}) \Phi(R', t) \Phi(R', t) . \\
\] (68)

In formula (68) for tensor \( F_{\alpha \beta}^{\gamma \delta} \) we have separation of the integral containing the interaction potential, as we have it at the calculation of other force fields above. However, here we obtain different integral \( \int \phi_{i}^{\alpha} \partial_{i}^{\alpha} \phi_{j}^{\beta} \phi_{j}^{\gamma} U(r_{ij}) \Phi(R', t) \Phi(R', t) \).

Calculation of this integral leads to the second interaction constant given below in the simplified expression for the quantum third rank force tensor

\[
F_{\alpha \beta}^{\gamma \delta} = \frac{\hbar^{2}}{8m^{2}} g_{2} I_{0}^{\alpha \beta \gamma \delta} \partial\varphi n_{2}(r, r', t) , \\
\] (69)

where

\[
g_{2} = \frac{2}{3} \int dr U''(r) , \\
\] (70)

and

\[
I_{0}^{\alpha \beta \gamma \delta} = \delta_{\alpha \beta} \delta_{\gamma \delta} + \delta_{\alpha \gamma} \delta_{\beta \delta} + \delta_{\alpha \delta} \delta_{\beta \gamma} . \\
\] (71)

Calculation of the two-particle concentration leads to the following expression:

\[
F_{\alpha \beta}^{\gamma \delta} = \frac{\hbar^{2}}{8m^{2}} g_{2} I_{0}^{\alpha \beta \gamma \delta} (2n_{2}^{2} + 4n_{\alpha} n_{\beta} + n_{\beta}^{2}) . \\
\] (72)

Here we have \( \partial\varphi (2n_{2}^{2} + 4n_{\alpha} n_{\beta} + n_{\beta}^{2}) \). Next, we open brackets and find \( 4n_{\alpha} \partial\varphi n_{\alpha} + 4n_{\beta} \partial\varphi n_{\beta} + 4n_{\beta} \partial\varphi n_{\alpha} + 2n_{\alpha} \partial\varphi n_{\alpha} \). The source of field is under derivative, while the multiplier in front corresponds to the species under the action of field. Hence the first two terms correspond to the force acting on the normal fluid, while the third and fourth terms correspond to the force acting on the BEC.

Therefore, we can separate expression (72) on two parts corresponding to the BEC and to the normal fluid:

\[
F_{\alpha \beta}^{\gamma \delta} = \frac{\hbar^{2}}{4m^{2}} g_{2} I_{0}^{\alpha \beta \gamma \delta} (2n_{\beta} \partial\varphi n_{\alpha} + n_{\beta} \partial\varphi n_{\beta}) , \\
\] (73)

and

\[
F_{\alpha \beta}^{\gamma \delta} = \frac{\hbar^{2}}{2m^{2}} g_{2} I_{0}^{\alpha \beta \gamma \delta} (n_{\alpha} \partial\varphi n_{\alpha} + n_{\beta} \partial\varphi n_{\alpha}) . \\
\] (74)
4. Hydrodynamic equations for two fluid model of bosons with nonzero temperature

This section provides the final set of equations obtained in this paper. The method of the introduction of the velocity field and corresponding representation of the hydrodynamic equations is not described in this paper. It can be found in number of other papers, majority of details are given in [4, 38].

In this regime, we have two continuity equations:

\[
\partial_t n_B + \nabla \cdot (n_B v_B) = 0, \tag{75}
\]

and

\[
\partial_t n_n + \nabla \cdot (n_n v_n) = 0. \tag{76}
\]

The Euler equation for bosons in the BEC state

\[
m_B (\partial_t + v_B \cdot \nabla) v_B^\alpha + \partial_\alpha T_B^{\alpha\beta} + \partial_\beta p_{\text{df}}^{\alpha\beta} + g n_B \partial_\alpha V_{\text{ext}} = -n_B \partial_\alpha V_n, \tag{77}
\]

where the quantum Bohm potential is given by equation (12) in the noninteracting limit.

The Euler equation for bosons in the excited states corresponding to the nonzero temperature

\[
m_n (\partial_t + v_n \cdot \nabla) v_n^\alpha + \partial_\alpha T_n^{\alpha\beta} + \partial_\beta p_{\text{df}}^{\alpha\beta} + 2 g n_n \partial_\alpha V_{\text{ext}} = -n_n \partial_\alpha V_n, \tag{78}
\]

where the effective pressure tensor \( p_{n,\text{eff}}^{\alpha\beta} = T_{n,\text{eff}}^{\alpha\beta} + p_{\text{df}}^{\alpha\beta} \).

The effective pressure evolution equation for the normal boson fluid is also the part of developed hydrodynamic model

\[
\partial_t p_{n,\text{eff}}^{\alpha\beta} + v_n^\gamma \partial_\gamma p_{n,\text{eff}}^{\alpha\beta} + p_{n,\text{eff}}^{\alpha\gamma} \partial_\gamma v_n^\beta + p_{n,\text{eff}}^{\beta\gamma} \partial_\gamma v_n^\alpha + p_{n,\text{eff}}^{\alpha\beta} \partial_\gamma Q_{n,\text{eff}}^{\alpha\beta\gamma} = 0. \tag{79}
\]

Moreover, we have the pressure (the quantum Bohm potential) \( p_{B,\text{eff}}^{\alpha\beta} = T_{B,\text{eff}}^{\alpha\beta} + p_{\text{df}}^{\alpha\beta} \) evolution equation for the BEC

\[
\partial_t p_{B,\text{eff}}^{\alpha\beta} + v_B^\gamma \partial_\gamma p_{B,\text{eff}}^{\alpha\beta} + p_{B,\text{eff}}^{\alpha\gamma} \partial_\gamma v_B^\beta + p_{B,\text{eff}}^{\beta\gamma} \partial_\gamma v_B^\alpha + p_{B,\text{eff}}^{\alpha\beta} \partial_\gamma Q_{B,\text{eff}}^{\alpha\beta\gamma} = 0. \tag{80}
\]

Let us to point out the following property of the quantum Bohm potential that it satisfies the following equation for the arbitrary species \( a \)

\[
\partial_\alpha T_a^{\alpha\beta} + v_a^\gamma \partial_\gamma T_a^{\alpha\beta} + T_a^{\alpha\gamma} \partial_\gamma v_a^\beta + T_a^{\beta\gamma} \partial_\gamma v_a^\alpha + T_a^{\alpha\beta} \partial_\gamma Q_a^{\alpha\beta\gamma} = 0. \tag{81}
\]

It is expected that the approximate form of the quantum Bohm potential (12) satisfies equation (81) existing at the zero interaction. Hence, we substitute (12) in equation (81) with the zero right-hand-side:

\[
\begin{align*}
\partial^3 \partial^\alpha n_a \cdot (\partial^\alpha v_a^\beta - \partial^\alpha v_a^\beta) & + \partial^3 \partial^\alpha n_a \cdot (\partial^\gamma v_a^\beta - \partial^\gamma v_a^\beta) + \\
&+ \frac{1}{3} \partial_\alpha n_a \cdot (\partial^3 \partial^\alpha v_a^\beta + \partial^3 \partial^\beta v_a^\gamma - \partial^{\alpha\beta} \partial^\gamma v_a^\beta) + \\
&+ \frac{1}{3} n_a [ \Delta (\partial^3 v_a^\alpha + \partial^3 v_a^\beta) - \partial^{\alpha\beta} \partial^3 (\nabla \cdot v_a) ] = 0,
\end{align*} \tag{82}
\]

where the continuity equation is used for the time derivatives of concentration. Next, we use the condition of the potentiality of the velocity field \( v_a = \nabla \phi_a \). We include it in equation (82) and find that this equation is satisfied.

Hence, we obtain the simplified form of the pressure evolution equations, where the traditional quantum Bohm potential is extracted:

\[
\begin{align*}
\partial_t p_{\text{df}}^{\alpha\beta} + v_B^\gamma \partial_\gamma p_{\text{df}}^{\alpha\beta} + p_{\text{df}}^{\alpha\gamma} \partial_\gamma v_B^\beta + p_{\text{df}}^{\beta\gamma} \partial_\gamma v_B^\alpha + p_{\text{df}}^{\alpha\beta} \partial_\gamma Q_{\text{df}}^{\alpha\beta\gamma} = 0, \tag{83}
\end{align*}
\]

and

\[
\begin{align*}
\partial_t p_{n}^{\alpha\beta} + v_n^\gamma \partial_\gamma p_{n}^{\alpha\beta} + p_n^{\alpha\gamma} \partial_\gamma v_n^\beta + p_n^{\beta\gamma} \partial_\gamma v_n^\alpha + p_n^{\alpha\beta} \partial_\gamma Q_n^{\alpha\beta\gamma} = 0. \tag{84}
\end{align*}
\]
equations for the evolution of quantum-thermal part of the third rank tensor are [6, 46]:

\[
\partial_t Q_{\alpha\beta}^{s} + \partial_k (v_k^{s} Q_{\alpha\beta}^{s}) + Q_{\alpha\beta}^{s} \partial_k v_k^{s} + Q_{\alpha\beta}^{s} \partial_k v_k^{s} + Q_{\alpha\beta}^{s} \partial_k v_k^{s} = \frac{\hbar^2}{4m^2} g_2 \rho_0^{s}\delta^3(n_\mathbf{p}_0^s + 2n_\mathbf{p}_0^s + \nabla^2 n_\mathbf{p}_0^s) \\
- \frac{1}{m} n_\mathbf{p}_0^s \partial_{\mathbf{p}_0^s} \partial_{\mathbf{p}_0^s} V_{\text{ext}} + \tilde{F}_{\mathbf{p}_0^s} + \tilde{F}_{\mathbf{p}_0^s} + \tilde{F}_{\mathbf{p}_0^s} + \frac{1}{m} (p_{\mathbf{p}_0^s,\text{eff}} \partial_{\mathbf{p}_0^s,\text{eff}} + \rho_{\mathbf{p}_0^s,\text{eff}} \partial_{\mathbf{p}_0^s,\text{eff}} + \rho_{\mathbf{p}_0^s,\text{eff}} \partial_{\mathbf{p}_0^s,\text{eff}}),
\]

(85)

and

\[
\partial_t Q_{\alpha\beta}^{s} + \partial_k (v_k^{s} Q_{\alpha\beta}^{s}) + Q_{\alpha\beta}^{s} \partial_k v_k^{s} + Q_{\alpha\beta}^{s} \partial_k v_k^{s} + Q_{\alpha\beta}^{s} \partial_k v_k^{s} = \frac{\hbar^2}{2m^2} g_2 \rho_0^{s}\delta^3(n_\mathbf{p}_0^s + 2n_\mathbf{p}_0^s + \nabla^2 n_\mathbf{p}_0^s) \\
- \frac{1}{m} n_\mathbf{p}_0^s \partial_{\mathbf{p}_0^s} \partial_{\mathbf{p}_0^s} V_{\text{ext}} + \tilde{F}_{\mathbf{p}_0^s} + \tilde{F}_{\mathbf{p}_0^s} + \tilde{F}_{\mathbf{p}_0^s} + \frac{1}{m} (p_{\mathbf{p}_0^s,\text{eff}} \partial_{\mathbf{p}_0^s,\text{eff}} + \rho_{\mathbf{p}_0^s,\text{eff}} \partial_{\mathbf{p}_0^s,\text{eff}} + \rho_{\mathbf{p}_0^s,\text{eff}} \partial_{\mathbf{p}_0^s,\text{eff}}),
\]

(86)

where \( \rho_0^{s} \) is not presented explicitly. Tensor \( F_{\mathbf{p}_0^s} \) is given by equations (66) and (67).

Hydrodynamic model for fermions with pressure evolution is derived in [38, 49, 50]. Terms proportional to \( p_{\mathbf{p}_0^s,\text{eff}} \partial_{\mathbf{p}_0^s,\text{eff}} + \rho_{\mathbf{p}_0^s,\text{eff}} \partial_{\mathbf{p}_0^s,\text{eff}} + \rho_{\mathbf{p}_0^s,\text{eff}} \partial_{\mathbf{p}_0^s,\text{eff}} \) appears in the pressure flux evolution equation, but it leads to the contribution beyond the chosen approximation [51, 52].

Term containing the external potential \( -\frac{1}{m} n_\mathbf{p}_0^s \partial_{\mathbf{p}_0^s} \partial_{\mathbf{p}_0^s} V_{\text{ext}} \) goes to zero for the parabolic trap. However, it can give some nontrivial contribution for other forms of potentials.

5. BEC dynamics under the influence of the quantum fluctuations

Developed model shows that there is nontrivial evolution equation for the pressure and the pressure flux of the BEC. Therefore, the well-known model of BEC is extended in spite the fact that the kinetic pressure tensor is expected to be equal to zero due to the zero temperature. However, the quantum fluctuations lead to the nonzero occupation numbers for the excited states.

If we need to consider pure BECs we need to drop the contribution of the normal fluid in the model presented above. Therefore, let us summarize the BEC model in parabolic traps:

\[
\partial_t n_\mathbf{p}_0^s \partial_{\mathbf{p}_0^s} + \nabla \cdot (n_\mathbf{p}_0^s v_\mathbf{p}_0^s) = 0, \\
mm_0 (\partial_t + v_\mathbf{p}_0^s \cdot \nabla) v_\mathbf{p}_0^s + \partial_{\mathbf{p}_0^s} (p_{\mathbf{p}_0^s} + T_{\mathbf{p}_0^s}) \\
+ gn_\mathbf{p}_0^s \partial^\beta n_\mathbf{p}_0^s + n_\mathbf{p}_0^s \partial^\beta V_{\text{ext}} = 0, \\
\partial_t p_{\mathbf{p}_0^s} + v_\mathbf{p}_0^s \partial_{\mathbf{p}_0^s} p_{\mathbf{p}_0^s} + \partial^\beta v_\mathbf{p}_0^s + p_{\mathbf{p}_0^s} \partial_{\mathbf{p}_0^s} v_\mathbf{p}_0^s \\
+ p_{\mathbf{p}_0^s} \partial_{\mathbf{p}_0^s} v_\mathbf{p}_0^s + \partial_{\mathbf{p}_0^s} Q_{\mathbf{p}_0^s,\text{eff}} = 0,
\]

(87)

and

\[
\partial_t Q_{\mathbf{p}_0^s,\text{eff}} + \partial_k (v_k^{s} Q_{\mathbf{p}_0^s,\text{eff}}) + Q_{\mathbf{p}_0^s,\text{eff}} \partial_k v_k^{s} + Q_{\mathbf{p}_0^s,\text{eff}} \partial_k v_k^{s} + Q_{\mathbf{p}_0^s,\text{eff}} \partial_k v_k^{s} = \frac{\hbar^2}{4m^2} g_2 \rho_0^{s}\delta^3(n_\mathbf{p}_0^s + 2n_\mathbf{p}_0^s + \nabla^2 n_\mathbf{p}_0^s).
\]

(88)

This simplified model is reported earlier in [6, 46, 47], where the dipole-dipole interaction is also considered. It has been demonstrated that the quantum fluctuations give mechanisms for the instability of the small amplitude perturbations [6]. The Bogoliubov spectrum of the concentration oscillations is modified by the quantum fluctuations. The second matter wave solution appears in BECs under the influence of the quantum oscillations as the concentration oscillations. In the long-wavelength regime both waves have same real part of the frequency, but there are small imaginary parts. It can be interpreted as the formation of the finite number of particles in the excited states under the action of the quantum fluctuations. So, the equilibrium condition composed of the particles being in the BEC state only is weakly unstable. The single species BECs show the existence of bright and dark solitons, depending on the sign of the interaction. These solitons can be described by the GP equation. However, the dipolar part of the quantum fluctuations creates conditions (in the small area of parameters) for the bright soliton in the repulsive BECs [47]. The developed in previous section model provides proper generalization of earlier model giving the small temperature contribution.
The BECs with no contribution of the thermal effects is considered above in section V, where we deal with BECs under influence of the quantum fluctuations. Further simplification of the model is discussed for the case of neglecting of the quantum fluctuations in equations (87)–(90):

\[ \partial_t n_B + \nabla \cdot (n_B v_B) = 0, \]  

(91)

and

\[ m n_B (\partial_t + v_B \cdot \nabla) v_B^2 + \partial_j T_B^{\beta \beta} + g n_B \partial^\beta n_B + n_B \partial^\alpha V_{\text{ext}} = 0. \]  

(92)

Here we repeat the continuity equation (91) with no modifications. In the Euler equation (92) we dropped the kinetic pressure caused by the quantum fluctuations \( p_{\ell q}^{\alpha \beta} \). Consequently we do not need to consider the pressure evolution equation and the evolution equation for the pressure flux.

6. Conclusion

Revision of the two-fluid model for the finite temperature ultracold bosons has been presented through the derivation of the number of particles, the momentum, the momentum flux, and the third rank tensor balance equations. The derivation has been based on the trace of the microscopic dynamics of quantum particles via the application of the many-particle microscopic Schrodinger equation. Hence, the microscopic Schrodinger equation determines the time evolution for the macroscopic functions describing the collective motion of bosons.

General equations have been derived for the arbitrary strength of interaction and the arbitrary temperatures. The set of equations has been restricted by the third rank kinematic tensor (the flux of pressure). The truncation is made for the low temperatures weakly interacting bosons after the derivation of the general structure of hydrodynamic equations. Therefore, the thermal part of the fourth rank kinematic tensor has been taken equal to zero. Next, the terms containing the interaction potential have been considered for the short-range interaction. The small radius of interaction provides the small parameter for the expansion. The expansion is made in the force field in the Euler equation, the force tensor field in the momentum flux equation, and the third rank force tensor in the pressure flux evolution equation. The first term in expansion on the small interparticle distance has been considered in each expansion, which corresponds to the first order by the interaction radius.

This model allows to gain the quantum fluctuations which is the essential property of the BECs. Moreover, the interaction causing the quantum fluctuations has been consider at the finite temperature.

The functions obtained in the first order by the interaction radius have been expressed via the trace of two-particle functions. The two-particle functions have been calculated for the weakly interacting bosons.

The single species of 0-spin bosons has been considered. Therefore, the single fluid hydrodynamics has been derived. Next, it has been included that the concentration of particles, the current of particles (the momentum density), the momentum flux, and the current of the momentum flux are additive functions. Consequently, they can be easily splitted on two parts: the BEC and the normal fluid of bosons. Hence, the two fluid model of single species of bosons is obtained. This separation on two fluids has been made in general form of equations.

Atomic BECs demonstrates large number of collective phenomena, such as the solitons, vorticities, Landau-Zinner transitions in the double-wells, etc. Particularly, the quantum droplet formations are experimentally observed in dipolar BECs, where it is interpreted as the considerable effect of the quantum fluctuations. It shows the contribution of the excited states on the BEC properties. The small proportion of bosons in the thermally excited states can give noticeable contribution in the collective effects as well.

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Data availability statement

No new data were created or analysed in this study.

Appendix

Equation (1) is written in the Born-Oppenheimer approximation. So, we have localization of electrons near nucleus. The process of the dimer formation leads to the redistribution of electrons to form different structure of
electrons appearing near two nucleus. It would lead to reconstruction of the wave function of atoms $\Psi_{N}(R_{N}, t)$ as well. It transforms to another wave function with the different construction of arguments $\Psi_{N-2,i}(\tilde{R}_{N-1}, t)$, where $\Psi_{i}(R_{i}, t)$ is the wave function of $N$ identical particles, and $\Psi_{N-2,i}(\tilde{R}_{N-1}, t)$ is the wave function of $N-1$ particles, which are $N-2$ identical atoms and one dimer.

In order to obtain the equation for the evolution of concentration (2) we consider the time derivative of the definition of concentration via the many-particle wave function:

$$\partial_{t} n = \frac{i}{\hbar} \int dR_{N} \sum_{i} \delta(r - r_{i}) ((\hat{H}\psi)^{*}\psi - \psi^{*}\hat{H}\psi).$$

(A1)

Here we see that terms in the Hamiltonian (1) give no contribution in $\partial_{t} n$ if they do not depend on the momentum operator. Thus, we see that the first term in Hamiltonian (1) gives the contribution in the concentration evolution. Let us explicitly present few step we make to obtain the continuity equation

$$\partial_{t} n = \frac{i}{\hbar} \int dR_{N} \sum_{i} \delta(r - r_{i}) \frac{1}{2m_{i}} \left[ (p_{i}^{2}\psi)^{*}\psi - \psi^{*}p_{i}^{2}\psi \right]$$

$$= \frac{i}{\hbar} \int dR_{N} \sum_{i} \delta(r - r_{i}) \frac{1}{2m_{i}} \left[ -\frac{\hbar}{i} \nabla(p_{i}\psi)^{*}\psi + (p_{i}\psi)^{*}p_{i}\psi - \psi^{*}p_{i}^{2}\psi \right]$$

$$= \frac{i}{\hbar} \int dR_{N} \sum_{i} \delta(r - r_{i}) \frac{1}{2m_{i}} \left[ -\frac{\hbar}{i} \nabla(p_{i}\psi)^{*}\psi - \psi^{*}p_{i}^{2}\psi \right].$$

(A2)

Here we use that $\nabla \delta(r - r_{i}) = -\nabla \delta(r - r_{i})$, where $\nabla$ is the derivative on space variable $r$. To finish the derivation we make following steps

$$\partial_{t} n = -\int dR_{N} \sum_{i} \delta(r - r_{i}) \frac{1}{2m_{i}} \nabla(\psi^{*}p_{i}\Psi + c.c.)$$

$$= -\int dR_{N} \sum_{i} \nabla \left( \delta(r - r_{i}) \frac{1}{2m_{i}} (\psi^{*}p_{i}\Psi + c.c.) \right)$$

$$+ \int dR_{N} \sum_{i} \nabla \delta(r - r_{i}) \frac{1}{2m_{i}} (\psi^{*}p_{i}\Psi + c.c.)$$

$$= -\nabla \int dR_{N} \sum_{i} \delta(r - r_{i}) \frac{1}{2m_{i}} (\psi^{*}p_{i}\Psi + c.c.).$$

(A3)

We consider integral $\int dR_{N} \nabla f(R) = \int dR_{N-1} dR_{i} \nabla f(R) = 0$. It equals to zero as the integral over whole space of the divergence. Therefore, we can replace operator $\nabla$ outside of the integral since it acts on $\delta(r - r_{i})$ only.

The derivation of the Euler equation shows the general structure which is also used in derivation of over hydrodynamic equations. While the continuity equation derivation presented above is simplified in accordance with simplicity of the operator of concentration depending on coordinates only.

We differentiate the momentum density (4) with respect to time and apply the many-particle Schrodinger equation with Hamiltonian (1) to replace the time derivatives of the wave functions

$$\partial_{t} j = -\int dR_{N} \sum_{i=1}^{N} \delta(r - r_{i}) \frac{1}{2m_{i}} \frac{1}{\hbar} \left( \psi^{*}p_{i}(\hat{H}\psi) - (\hat{H}\psi)^{*}p_{i}\psi + p_{i}^{*}\psi^{*}(\hat{H}\psi) - p_{i}^{*}(\hat{H}\psi)^{*}\psi \right).$$

(A4)

At the next step we have

$$\partial_{t} j = -\int dR_{N} \sum_{i=1}^{N} \delta(r - r_{i}) \frac{1}{2m_{i}} \frac{1}{\hbar} \sum_{k=1}^{N} \psi^{*}p_{i} \frac{p_{k}^{2}}{2m_{k}} \psi$$

$$= \left( \frac{p_{i}^{2}}{2m_{i}} \psi \right)^{*} p_{i} \psi + (p_{i}\psi)^{*} \frac{p_{i}^{2}}{2m_{i}} \psi$$

$$+ \psi^{*} \left[ p_{i}, V_{ext,i} \right] \psi - \left( [p_{i}, V_{ext,i}] \psi \right)^{*} \psi$$
\[ + \frac{1}{\pi} \sum_{p \neq k} \left( \psi^* \left[ P_{pk} \right] \psi - \left[ P_{pk} \right] \psi^* \psi \right). \]  

(A5)

The following commutators have to be calculated to get the interaction in the momentum balance equation

\[ [P_{k}, \varphi_\alpha] \psi = \frac{\hbar}{i} \left( \delta_{k\alpha} \left( \nabla \varphi_\alpha \right) \psi \right), \]

(A6)

and

\[ [P_{k}, G_{lm}] \psi = \frac{\hbar}{i} \left( \delta_{k} \left( \nabla G_{lm} \right) + \delta_{lm} \left( \nabla G_{ki} \right) \right) \psi. \]

(A7)

As the result we have the momentum balance equation

\[ \partial_t \rho_\alpha + \frac{1}{m} \partial^\beta \Pi^{\alpha \beta} = F_\alpha, \]

(A8)

where \( F_\alpha (r, t) \) is the force field and \( \Pi^{\alpha \beta} \) represents the momentum current density tensor or the momentum flux tensor. Equation (A5) contains three group of terms. The first group of terms in the right-hand side of formula (A5) consists of four terms, where three momentum operators \( \vec{p} \) act on the wave functions. This group of terms give the divergency of the momentum flux tensor \( \partial^\beta \Pi^{\alpha \beta} \). The two other groups construct of terms proportional to commutators (A6), (A7). They give the force field \( F_\alpha (r, t) \).

Let us to point out the method of replacing of the external potential from under the integral in equation (A5)

\[ \int dR \sum_{i=1}^{N} \delta (r - \tau_i) V_{ext}(\tau_i, t) \psi^* (R, t) \psi (R, t) \]

\[ = V_{ext}(r, t) \int dR \sum_{i=1}^{N} \delta (r - \tau_i) \psi^* (R, t) \psi (R, t). \]

(A9)

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