Quantum Toda-like regularisation of the Mixmaster anisotropy

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Abstract. Regularisation approach to the study of the quantum dynamics of the Mixmaster universe is presented. This allows to approximate the anisotropy potential with the explicitly integrable periodic 3-particle Toda system. Such approach is based on a covariant Weyl-Heisenberg integral quantization which naturally amplifies the dynamical role of the underlying Toda system by smoothing out the three canyons of the anisotropy potential.

1. Introduction
It is generally recognized [1, 2] that isotropic cosmological solutions become less and less accurate as the universe approaches the initial singularity. The anisotropic solutions, in particular Bianchi models of types VIII and IX appear more suitable for studies of the earliest universe. The spatially closed Bianchi type IX, which includes also some isotropic solutions as a particular case, is an attractive anisotropic model of the initial singularity. The diagonal, surface-orthogonal Bianchi IX model is called the Mixmaster universe and its canonical formulation was given by Misner [3]. The Mixmaster dynamics is given by the Hamiltonian of a particle moving in the 3-dimensional Minkowski spacetime in a time-dependent potential. Elseways, the quantum dynamics of Mixmaster remains poorly understood despite many interesting studies (see e.g. a review [4]). The difficulty lies in the elaborate spatial dependence of the anisotropy potential and the fact that solving the quantum dynamics requires the knowledge of the eigenstates and the eigenvalues for the motion in this potential. The solution to the eigenvalue problem seems to have been so far approximated by means of the harmonic and the steep-wall approximation to the anisotropy potential (see e.g. [5] for their proper derivation).

In this paper we show a new way to approach the Mixmaster dynamics at the quantum level, which is based on a unique approximation to the anisotropy potential. The described approximation is obtained via applying the Weyl-Heisenberg quantisation procedure [6, 7, 8] to the anisotropy Hamiltonian. Such procedure smoothes out and regularises the Bianchi IX potential. Moreover, the resulting anisotropy potential might be split into two separate parts: the first one dominating over the other one several orders of magnitude smaller. Therefore, the latter can be treated as a small perturbation. Furthermore, the dominating part of the potential is shown to be a 2-dimensional representation of a 3-particle periodic Toda lattice. This split opens the door to better understanding of the quantized dynamics, possibly through the perturbation methods.
The outline of the paper is as follows. In Sec. 2 we consider the Bianchi IX Hamiltonian constraint and the anisotropy potential. In Sec. 3 we study the covariant Weyl-Heisenberg quantization of the anisotropy potential and show that the proposed approximation is reaffirmed at the quantum level. In Sec. 4 we describe a link between regularized anisotropy Hamiltonian and a simple periodic 3-particle Toda lattice. We conclude in Sec. 5. The Appendix contains description of quantum Toda lattice.

2. Definition of the model

Let us first recall the definition and the Hamiltonian formulation of the Bianchi type IX model. We assume the line element:

$$\text{d}s^2 = -N^2\text{d}\tau^2 + \sum a_i^2 (\omega^i)^2,$$

(1)

where \(d\omega_i = \frac{1}{2} \epsilon^{ijk} \omega_j \wedge \omega_k\) and \(N, a_i\) are functions of time. The respective Hamiltonian constraint in the Misner variables reads [3]:

$$C = \frac{Ne^{-3\Omega}}{24} \left( \frac{2\kappa}{V_0} \right)^2 \left( -p_\Omega^2 + p^2 + 36 \left( \frac{V_0}{2\kappa} \right)^3 n^2 e^{4\Omega} [V(\beta) - 1] \right), \quad (\Omega, p_\Omega, \beta, p) \in \mathbb{R}^6,$$

(2)

where \(\beta := (\beta_+, \beta_-), p := (p_+, p_-), V_0 = \frac{16\pi^2}{n}\) is the fiducial volume, \(\kappa = 8\pi G\) is the gravitational constant, \(N\) is the nonvanishing and otherwise arbitrary lapse function. In what follows we set \(n = 1\) and \(2\kappa = V_0\). The gravitational Hamiltonian \(C\) resembles the Hamiltonian of a particle in the 3D Minkowski spacetime in a potential. The spacetime variables used in eq. (2) have the cosmological interpretation:

$$\Omega = \frac{1}{3} \ln a_1 a_2 a_3, \quad \beta_+ = \frac{1}{6} \ln \frac{a_1 a_2}{a_3^2}, \quad \beta_- = \frac{1}{2\sqrt{3}} \ln \frac{a_1}{a_2}. \quad (3)$$

Thus, the variable \(\Omega\) describes the isotropic geometry, whereas \(\beta_{\pm}\) describe distortions to the isotropic geometry and are referred to as the anisotropic variables. The potential that drives the motion of the geometry represents the spatial curvature \(\frac{3}{2}R\).

The Hamiltonian constraint (2) is a sum of the isotropic and anisotropic parts, \(C = -C_{\text{iso}} + C_{\text{ani}}\), where (up to a factor)

$$C_{\text{iso}} = p_\Omega^2 + 36e^{4\Omega},$$

(4)

$$C_{\text{ani}} = p^2 + 36e^{4\Omega}V(\beta),$$

(5)

and

$$V(\beta) = \frac{e^{4\beta_+}}{3} \left[ \left( 2 \cosh(2\sqrt{3}\beta_-) - e^{-6\beta_+} \right)^2 - 4 \right] + 1.$$  

(6)

The potential \(V(\beta)\) is plotted in Fig. 1. In the present paper we propose a new approach to the anisotropic Hamiltonian (5) for a fixed value of the isotropic variable \(\Omega\), specifically we study

$$C_{\text{ani}} = p^2 + V(\beta).$$

(7)

The issue of coupling between the anisotropic and isotropic variables (at the quantum level) is addressed elsewhere [9].
3. Quantization of the potential

The so-called canonical quantization works sufficiently well for the basic observables and many typical Hamiltonians. However, it is neither the only possible quantization prescription nor it has to work well for more complex observables, in particular, when the latter admit some sort of singularities. The three channels of the anisotropy potential, which narrow to the zero width, are in a sense singular and one may expect that the quantization would smooth out these singular features. For this reason, we employ a more general quantization framework called the Weyl-Heisenberg integral quantization [6, 7, 8] which, apart from being well-suited for dealing with singular observables, possesses an attractive probabilistic interpretation.

The anisotropy potential (6) can be written as a sum of products of exponentials. The Weyl-Heisenberg integral quantization applied to a single exponential function yields:

$$A_{e^q} = \mathcal{G}_{1/\sigma}[e^q],$$

where $\mathcal{G}_\sigma[F]$ is the Gaussian convolution of the function $F$:

$$\mathcal{G}_\sigma[F](x) := \frac{1}{\sigma \sqrt{2\pi}} \int_{-\infty}^{\infty} dy \ e^{-\frac{(x-y)^2}{2\sigma^2}} F(y),$$

which in the case of the exponential function yields:

$$A_{e^q} = e^{\frac{\sigma^2}{2\sigma^2}} e^{q\sigma}. \tag{10}$$

For the quantization of the two-dimensional anisotropy potential (6) we employ the separable Gaussian distributions

$$\Pi(\beta_+; p_+; \beta_-; p_-) = e^{-\frac{\beta_+^2}{2\sigma^2}} e^{-\frac{\beta_-^2}{2\sigma^2}} e^{-\frac{p_+^2}{2\tau^2}} e^{-\frac{p_-^2}{2\tau^2}}. \tag{11}$$

We obtain

$$V(\beta_+; \beta_-) \mapsto AV(\beta_+; \beta_-) = \frac{1}{3} \left( 2D_+^{12} e^{4\beta_+} \cosh 4\sqrt{3}\beta_- - 4D_+ D_-^3 e^{-2\beta_+} \cosh 2\sqrt{3}\beta_- + D_+^{16} e^{-8\beta_+} - 2D_+^{4} e^{4\beta_+} \right) + 1, \tag{12}$$
where $D_+ = e^{\sigma_+}$ and $D_- = e^{\sigma_-}$. The classical anisotropy potential $V(\beta_+,\beta_-)$ is recovered for $D_+ = D_- = 1$ (or, $\sigma_+,\sigma_- \to \infty$).

Fig. 2 shows the form of the potential (12) for sample values of $D_+$ and $D_-$. The original escape channels became regularized and the whole potential is now fully confining. However it has become anisotropic in the variables $\beta_+$ and $\beta_-$ and its minimum is shifted from the $(0,0)$ position, namely it is at the $(\beta_0,0)$ point, where the value $\beta_0$ is subject to the following equation:

$$-D_+^{16}e^{-8\beta_0} + D_+D_+^3e^{-2\beta_0} - D_+^4e^{4\beta_0} + D_+^4D_-^{12}e^{4\beta_0} = 0$$

(13)

arriving from the condition $\partial A_V(\beta_+\beta_-)/\partial \beta_+ = 0$ for $(\beta_0,0)$. Condition $\partial A_V(\beta_+\beta_-)/\partial \beta_- = 0$ is fulfilled automatically at this point. The value of the potential at the minimum is also shifted from 0 to the value:

$$A_V(\beta_+\beta_-)(\beta_0,0) = \frac{1}{3}e^{-8\beta_0} \left(D_+^{16} - 4D_+D_+^3e^{6\beta_0} + 3e^{8\beta_0} + 2D_+^4(-1 + D_+^{12})e^{12\beta_0}\right).$$

(14)

3.1. Minimum and isotropy preservation

We want our quantization procedure to preserve the basic properties of the classical potential by requiring that (i) the isotropy around its minimum and (ii) the position of the minimum is localized at the point $(0,0)$. Expansion of the potential (12) around the point $(0,0)^1$ reads:

$$A_V(\beta_+\beta_-) \approx \frac{1}{3} \left[D_+^{16} - 2D_+^4 + 2D_+^4D_+^{12} - 4D_+D_+^3 - 8(D_+^4 + D_+^{16} - D_+D_+^3 - D_+^4D_+^{12}) \beta_+ + 8(4D_+^{16} - 2D_+^4 - D_+D_+^3 + 2D_+^4D_+^{12}) \beta_+^2 + 24(2D_+^4D_+^{12} - D_+D_+^3) \beta_+^2 \right] + 1.$$  

(15)

The coefficients in front of $\beta_+^2$ and $\beta_-^2$ are equal if

$$4D_+^{16} - 2D_+^4 - D_+D_+^3 + 2D_+^4D_+^{12} = 3(2D_+^4D_+^{12} - D_+D_+^3).$$

This condition may be rewritten as:

$$(D_+^4 - D_+D_+^3) \left[2(D_+^4 + D_+D_+^3) (D_+^8 + (D_+D_+^3)^2 - 1) \right] = 0.$$  

(16)

1 Expanding around $(\beta_0,0)$ yields similar results but the presentation is far more sophisticated.
For values of $D_+, D_- \geq 1$ (a condition fulfilled by exponents) it has only one unique solution:

$$D_+ = D_- := D.$$  \hfill (17)

The condition (17) also implies the vanishing of the coefficient in front of $\beta_+$ in Eq. (15).

This is consistent with the Eq. (13), which for $\beta_0 = 0$ can be rearranged as:

$$(D_+^4 - D_+ D_-^2) [(D_+^4 + D_+ D_-^2) (D_+^8 + (D_+ D_-^2)^2) + 1] = 0,$$  \hfill (18)

which again for positive values of $D_+$ and $D_-$ has only one unique solution $D_+ = D_-$, the same as the one for symmetry preservation. Thus, there is no shift in the position of the minimum.

The full quantized Bianchi IX potential reads now:

$$A_{V(\beta_+,\beta_-)} = 1 + 3 \left( 2D_+^{16} e^{4\beta_+} \cosh 4\sqrt{3} \beta_- - 4D_+^{4} e^{-2\beta_+} \cosh 2\sqrt{3} \beta_- + D_+^{16} e^{-8\beta_+} - 2D_+^{4} e^{4\beta_+} \right),$$  \hfill (19)

One may verify that it is invariant with respect to the rotations by $2\pi/3$ and $4\pi/3$, and thus, the $C_{3v}$ symmetry is preserved in the full plane.

3.2. Perturbative approach

The quantized potential (19), may be written as a sum of two potentials:

$$A_{V(\beta_+,\beta_-)} = D_+^{16} V_T + D_+^{4} V_p + 1,$$  \hfill (20)

where

$$V_T = \frac{1}{3} \left( 2e^{4\beta_+} \cosh 4\sqrt{3} \beta_- + e^{-8\beta_+} \right),$$  \hfill (21)

$$V_p = -\frac{2}{3} \left( 2e^{-2\beta_+} \cosh 2\sqrt{3} \beta_- + e^{4\beta_+} \right).$$  \hfill (22)

The part $V_T$ significantly dominates over the part $V_p$, as $D = e^{\frac{2}{\sqrt{3}}} > 1$. To be precise, one may verify whether the expression $|V_p/V_T|$ is bounded from above:

$$\left| \frac{V_p}{V_T} \right| = \left| \frac{2XY + 1}{2Y^2 + X^2/2 - 1} \right|,$$  \hfill (23)

where

$$X = e^{-6\beta_+} \geq 0,$$  \hfill (24)

$$Y = \cosh 2\sqrt{3} \beta_- \geq 1.$$  \hfill (25)

Direct computation shows that in the open domain $(X,Y) \in ]0, \infty[ \times ]1, \infty[$ the ratio $\left| \frac{V_p}{V_T} \right|(X,Y)$ does not have any critical points. On the boundary $Y = 1$ it has one maximum at $X = 1$, with $\left| \frac{V_p}{V_T} \right|(1,1) = 2$, which is therefore a unique global maximum. Thus the coefficients $D_+^{16}$ and $D_+^{4}$ additionally strengthen the dynamical role of $V_T$. 
4. Integrable system

After applying the Weyl-Heisenberg integral quantisation scheme to the Hamiltonian (7) we obtain up to a constant (details described in [6]) a new Hamiltonian:

\[ A_{ani} = \frac{1}{2} (p_+^2 + p_-^2) + D^{16} V_T + D^4 V_p. \]  

(26)

In the first order of approximation we deal with Hamiltonian of the following form:

\[ A^0_{ani} = \frac{1}{2} (p_+^2 + p_-^2) + \frac{D^{16}}{3} \left( 2e^{4\beta_+} \cosh 4\sqrt{3}\beta_- + e^{-8\beta_+} \right). \]  

(27)

It has been presented in the papers [10, 11] that the 3-particle periodic Toda system may be reduced to the one described by the above Hamiltonian (27). In what follows we will reproduce the method described in [10] but in reverse direction. The Hamiltonian (27) implies the following set of equation of motion:

\[ \ddot{\beta}_- = -\frac{D^{16}}{3} \left( 8\sqrt{3}e^{4\beta_+} \sinh 4\sqrt{3}\beta_- \right), \]  

(28)

\[ \ddot{\beta}_+ = -\frac{D^{16}}{3} \left( 8e^{4\beta_+} \cosh 4\sqrt{3}\beta_- - 8e^{-8\beta_+} \right). \]  

(29)

Let us introduce new set of position coordinates \((\alpha_1, \alpha_2)\) and also rescale the time coordinate \(\tau\) in the following manner:

\[ \alpha_1 = 4\sqrt{2}\beta_-, \]  

(30)

\[ \alpha_2 = 4\sqrt{2}\beta_+. \]  

(31)

\[ t = \frac{4D^4\sqrt{2}}{3\sqrt{3}}\tau, \]  

(32)

where \(t\) denotes a new time variable.

Equations of motion (28)-(29) expressed in terms of new coordinates (30)-(32) take the following form:

\[ \ddot{\alpha}_1 = -\sqrt{3} \left( \exp \frac{\alpha_2 + \sqrt{3}\alpha_1}{\sqrt{2}} - \exp \frac{\alpha_2 - \sqrt{3}\alpha_1}{\sqrt{2}} \right), \]  

(33)

\[ \ddot{\alpha}_2 = -\frac{\sqrt{2}}{2} e^{-2\alpha_2} \left( \exp \frac{3\alpha_2 + \sqrt{3}\alpha_1}{\sqrt{2}} + \exp \frac{3\alpha_2 - \sqrt{3}\alpha_1}{\sqrt{2}} - 2 \right), \]  

(34)

where the double dot ” denotes derivative with respect to the new time variable \(t\). We may construct a new Hamiltonian, whose corresponding equations of motion are given by (33)-(34) as follows:

\[ H_1 = \frac{1}{2} \left( \gamma_1^2 + \gamma_2^2 \right) + \exp \frac{3\alpha_2 + \sqrt{3}\alpha_1}{\sqrt{2}} + \exp \frac{3\alpha_2 - \sqrt{3}\alpha_1}{\sqrt{2}} + \exp(-\sqrt{2}\alpha_2), \]  

(35)

where we denoted by \((\gamma_1, \gamma_2)\) momenta corresponding to \((\alpha_1, \alpha_2)\). We may extend this two-dimensional system to three dimensions adding additional position variable \(\alpha_3\) with trivial dynamics. Thus the corresponding momentum \(\gamma_3\) should be conserved and the position \(\alpha_3\) should not appear explicitly in the extended Hamiltonian, which reads as

\[ H_2 = \frac{1}{2} \left( \gamma_1^2 + \gamma_2^2 + \gamma_3^2 \right) + \exp \frac{3\alpha_2 + \sqrt{3}\alpha_1}{\sqrt{2}} + \exp \frac{3\alpha_2 - \sqrt{3}\alpha_1}{\sqrt{2}} + \exp(-\sqrt{2}\alpha_2), \]  

(36)
The position variable $\alpha_3$ corresponds to a uniform translation of the two-dimensional original system in the third direction.

We may now perform the final canonical change of variable as follows:

$$q_i = \sum_{i=1}^{3} A_{ji} \alpha_j, \quad p_i = \sum_{i=1}^{3} A_{ij} \gamma_j,$$

where $[A]_{ij} := \begin{bmatrix} 1/\sqrt{6}, -\sqrt{2/3}, 1/\sqrt{6} \\ 1/\sqrt{2}, 0, -1/\sqrt{2} \\ 1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3} \end{bmatrix}^T$.

This transformation preserves the 2-form $\vec{dq} \wedge \vec{dp}$ and leads to a new Hamiltonian given by a formula:

$$H = \frac{1}{2} \left( p_1^2 + p_2^2 + p_3^2 \right) + e^{-(q_1-q_3)} + e^{-(q_2-q_1)} + e^{-(q_3-q_2)}, \quad (37)$$

which is exactly a Hamiltonian of a 3-body periodic Toda lattice [12, 13]. It is the simplest non trivial crystal consisting of three particles (Fig. 3).

In general, a periodic Toda chain or lattice is a system of $N$ equal-mass particles interacting via exponential forces, described by the Hamiltonian:

$$H_T = \frac{1}{2} \sum_{k=1}^{N} p_k^2 + \sum_{k=1}^{N} e^{-(q_k-q_{k+1})} \quad (38)$$

with periodicity condition $q_0 \equiv q_N$ and $q_1 \equiv q_{N+1}$. It is known that the Toda systems are integrable [10, 13, 14] and solutions can be made explicit.

This particular system, with three degrees of freedom, has also three independent conserved quantities: the total momentum $P = p_1 + p_2 + p_3$, the total energy $H$ and an additional third invariant:

$$K = -p_1 p_2 p_3 + p_1 e^{-(q_1-q_2)} + p_2 e^{-(q_1-q_3)} + p_3 e^{-(q_2-q_1)}. \quad (39)$$

which gives rise to the integrability of the system and the removal of chaotic behaviour. The periodic 3-body Toda system has been analysed both on the classical and quantum levels. In the literature, one may find ways to construct classical solutions [14, 15] as well as the corresponding quantum eigenfunctions and eigenvalues [16, 17, 18]. Short summary of the method is provided in the Appendix.

**Figure 3.** A model for the periodic 3-body Toda lattice

5. Discussion

In this work, we described an innovative proposal for approximating the anisotropic part of the quantum Mixmaster Hamiltonian by the integrable 3-particle Toda system. The classical
anisotropic potential of the Bianchi IX space-time model is asymptotically confining except for the three directions in which it tends to zero. These the so called “channels” are problematic at the analytical level. We presented a regularisation procedure that approximated the three exponential walls and smoothed out the escape channels. That led to the potential that can be considered as consisting of two parts, one dominating and the other one being a small perturbation to the former. The main dominating part of the potential lead to the Hamiltonian similar to the 2-dimensional representation of the Hamiltonian for 3-particle periodic Toda lattice (there is also a straightforward transformation procedure from 2D to 3D Hamiltonian described in our other paper [6]).

On the classical level, the similarity between the anisotropy potential and the Toda potential was used to introduce the so-called disturbed Toda lattices [19] that include the anisotropy potential and the Toda one as special cases. Nevertheless, due to the integrability of the Toda lattices this similarity seems too little to be useful in the context of the classical dynamics (though see [20]). On the other hand, it has never been shown justified and exploited on the quantum level. Since the eigenvalue problem for the Toda lattice can be solved, the proposed approximation should be very useful in future investigations of the quantum dynamics.

In the ongoing studies, we calculate explicitly the eigenstates and energy levels of the quantum Mixmaster (see Appendix), starting from the integrable Toda potential. This approximation serves as an exact model, around which we may use standard perturbative methods.

Appendix A. Quantum Toda system

In this Appendix we present an exact method for solving the quantum Toda problem for 3-particle periodic system, following the method of Gutzwiller [17].

The Hamiltonian for a 3-particle periodic Toda lattice reads as follows:

$$H = \frac{1}{2} (p_1^2 + p_2^2 + p_3^2) + e^{-(q_1-q_3)} + e^{-(q_2-q_1)} + e^{-(q_3-q_2)}.$$ (A.1)

The equations of motion for such a system may be written as Lax’s equation [14]:

$$\frac{dL}{dt} = [L, M],$$ (A.2)

where matrices $L$ and $M$ form a so called Lax pair and read as follows:

$$L := \begin{bmatrix} a_1 & a_2 & a_3 \\ a_2 & a_1 & a_3 \\ a_3 & a_2 & a_1 \end{bmatrix}, \quad M := \frac{1}{2} \begin{bmatrix} 0 & a_1 & -a_3 \\ -a_1 & 0 & a_2 \\ a_3 & -a_2 & 0 \end{bmatrix}. $$ (A.3)

Elements of the symmetric matrix $L$ and the skew one $M$ are functions of positions and momenta of a Toda system:

$$a_i := e^{(q_i-q_{i+1})/2} \quad \text{and} \quad b_i := p_i \quad \text{where} \quad i = 1, 2, 3, \quad \text{with} \quad q_0 \equiv q_3, \quad q_4 \equiv q_1.$$ (A.4)

Following the method of Kac and Van Moerbeke [15] one may consider the 2-particle open Toda chain with the first particle removed. The corresponding Lax pair reads as follows:

$$L^* := \begin{bmatrix} a_2 & a_3 \\ a_3 & b_3 \end{bmatrix}, \quad M^* := \frac{1}{2} \begin{bmatrix} 0 & a_2 \\ -a_2 & 0 \end{bmatrix}. $$ (A.5)

The eigenvalues $\mu_1$, $\mu_2$ of the truncated matrix $L^*$ are subject to the evolution equation:

$$\frac{d\mu_{\pm}}{dt} = \pm \frac{\sqrt{\Delta^2(\mu_{\pm}) - 4}}{m'(\mu_{\pm})},$$ (A.6)
where \( m(\mu) = (\mu - \mu_+)(\mu - \mu_-) \), \( m'(\mu) = dm(\mu)/d\mu \). The characteristic polynomial \( \Delta(\lambda) \) of the full matrix \( L \) reads as:

\[
\Delta(\lambda) = \det |\lambda I - L| = \lambda^3 - \lambda^2(b_1 + b_2 + b_3) + \lambda(b_1b_2 + b_2b_3 + b_3b_1 - a_1^2 - a_2^2 - a_3^2)
- b_1b_2b_3 - 2a_1a_2a_3 + b_1a_2^2 + b_2a_3^2 + b_3a_1^2.
\] (A.7)

Coefficients of the polynomial are exactly constants of motion for the 3-particle periodic Toda lattice, e.g. \( \sum_1 b_i = p_1 + p_2 + p_3 = P \) is the conserved total momentum. Without loss of generality one can assume that the center of mass is at rest: \( P = 0 \) and that the center of mass is at the origin: \( q_1 + q_2 + q_3 = 0 \). The equations (A.6) might be obtained from the Hamilton equation [16] generated by

\[
H^*(\nu_+,\nu_-;\mu_+,\mu_-) = \sum \frac{2\cosh \nu_- - \mu_+}{m'(\mu_\pm)}.
\] (A.8)

As the systems generated by (A.1) and (A.8) are equivalent one can first quantize the less dimensional Hamiltonian (A.8). For this purpose one considers first the eigenvalues of the reduced matrix \( L^* \), that are solutions of the following equation:

\[
\det |\mu I - L^*| = \mu^2 - \mu(b_2 + b_3) + b_2b_3 - a_3^2 = 0.
\] (A.9)

Again coefficients of powers of \( \mu \) are constants of motion, thus, substituting definitions of \( a_i \) and \( b_i \) (A.4), we have two constants of motion:

\[
p_2 + p_3 = \text{const.} =: P, \quad p_2p_3 - e^{q_2-q_3} = \frac{1}{4}P^2 - \frac{1}{4}(p_2 - p_3)^2 - e^{q_2-q_3} = \text{const.}
\] (A.10)

Such a system is a well know quantum system whose motion with respect to the centre of mass is described by the Hamiltonian:

\[
\hat{H} := \frac{1}{4}p^2 + e^{2q},
\] (A.11)

where \( p := p_2 - p_3 \) and \( q := \frac{1}{2}(q_2 - q_3) \). Eq. (A.11) leads straightforwardly to the one dimensional Schrödinger’s equation:

\[
-\frac{\hbar}{4}\frac{\partial^2 \psi}{\partial q^2} + e^{2q}\psi = E\psi.
\] (A.12)

The eigenfunctions of the above equation are expressed in terms of modified Bessel functions \( I_n(z) \), where \( z := 2e^q/\hbar \) and \( \pm n := \pm 2i\sqrt{E}/\hbar \). The wave functions that vanish for large values of \( z \) are obtained as differences \( I_n(z) - I_{-n}(z) \). Making use of the power expansion of the Bessel function \( I_n(z) \) and expressing them in terms of the more convenient variables \( \xi = e^{q_2}, \eta = e^{q_3}, \rho = (\sqrt{E} + P/2)/\hbar, \sigma = (-\sqrt{E} + P/2)/\hbar \) lead to the following formula:

\[
\psi_{\rho\sigma}(\xi, \eta) = \xi^{i\rho} \eta^{i\sigma} \sum_{m=0}^{\infty} \frac{(\xi/\hbar^2\eta)^m}{\Gamma(m+1)\Gamma(m+1+i\rho-i\sigma)} - \xi^{i\rho} \eta^{i\sigma} \sum_{m=0}^{\infty} \frac{(\xi/\hbar^2\eta)^m}{\Gamma(m+1)\Gamma(m+1-i\rho-i\sigma)}.
\]

The basic set of wave function for the 3-particle periodic Toda chain is constructed from the wave modes for the open 2-body Toda chain:

\[
\Psi_{\rho\sigma}(\theta, \xi, \eta) = \theta^{-i(\rho+\sigma)} \psi_{\rho\sigma}(\xi, \eta), \tag{A.13}
\]

where the new third variable \( \theta = e^{q_1} \) is introduced, and the 2-particle wave function is given above. One would like to find simultaneous eigenfunctions for two operators, one corresponding
to the energy $A$ and the other one to the conserved quantity $K$, specific to the periodic 3-particle Toda chain:

$$\hat{A} := \frac{1}{2} \left(p_1^2 + p_2^2 + p_3^2\right) + e^{q_1-q_2} + e^{q_2-q_3} + e^{q_3-q_1}, \quad (A.14)$$

$$\hat{K} := p_1p_2p_3 - p_1e^{q_2-q_3} - p_2e^{q_3-q_1} - p_3e^{q_1-q_2}, \quad (A.15)$$

where $p_j = (\hbar/i)\partial/\partial q_j$ and $q_j$ is the multiplication operator. Actions of these operators on wave modes $\Psi_{\kappa,\lambda}$ (where quantum numbers $i\sigma$ and $i\rho$ have been replaced by $\kappa = i\rho + k$ and $\lambda = i\sigma + l$) read as follows:

$$\hat{A}\Psi_{\kappa,\lambda} = -\hbar^2(\kappa^2 + \kappa\lambda + \lambda^2)\Psi_{\kappa,\lambda} + \frac{1}{\hbar(\kappa - \lambda)} \left(-\Psi_{(\kappa+1),\lambda} + \Psi_{(\kappa-1),\lambda} + \Psi_{\kappa,(\lambda+1)} - \Psi_{\kappa,(\lambda-1)}\right),$$

$$i\hat{K}\Psi_{\kappa,\lambda} = \hbar^3\kappa\lambda(\kappa + \lambda)\Psi_{\kappa,\lambda} + \frac{\lambda}{\kappa - \lambda} \left(\Psi_{(\kappa+1),\lambda} - \Psi_{(\kappa-1),\lambda}\right) - \frac{\kappa}{\kappa - \lambda} \left(\Psi_{\kappa,(\lambda+1)} - \Psi_{\kappa,(\lambda-1)}\right). \quad (A.16)$$

Therefore, the full wave function for 3-particle periodic Toda chain may be written as series expansion as:

$$\Psi(\vartheta, \xi, \eta) = \sum_{\kappa=ip+k, k\in\mathbb{Z}}^{\lambda=\sigma+l, l\in\mathbb{Z}} C_{\kappa,\lambda}\Psi_{\kappa,\lambda}(\vartheta, \xi, \eta). \quad (A.17)$$

Here $k$ and $l$ range over all integers. Coefficients $C_{\kappa,\lambda}$ may be written as

$$C_{\kappa,\lambda} := (\kappa - \lambda)r_\kappa s_\lambda, \quad (A.18)$$

where the coefficients $r_\kappa$ and $s_\lambda$ are subject to recursion conditions:

$$r_{\kappa+1} - r_{\kappa-1} = (\hbar^2\kappa^3 + Ah\kappa + iK)r_\kappa, \quad (A.19)$$

$$s_{\lambda+1} - s_{\lambda-1} = (\hbar^3\lambda^3 + Ah\lambda + iK)s_\lambda \quad (A.20)$$

which are direct implication of equations (A.16), that have to be satisfied in order to make the wave function $\Psi$ a simultaneous eigenfunction of the operators $\hat{A}$ and $\hat{K}$.

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