AN OBSERVATION ON UNBOUNDED LINEAR OPERATORS WITH AN ARBITRARY SPECTRUM

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Abstract. We furnish a simple way of constructing an unbounded closed linear operator in a complex Banach space, whose spectrum is an arbitrary nonempty closed, in particular compact, subset of the complex plane.

1. Introduction

By Gelfand’s spectral radius theorem (see, e.g., [3]), the spectrum of a bounded linear operator on a complex Banach space is a nonempty compact subset of the complex plane. Conversely, an arbitrary nonempty compact set in \( \mathbb{C} \) is the spectrum of a such an operator (cf. [3, Theorem 5.7]).

While the spectrum of an unbounded closed linear operator in a complex Banach space is known to be a closed subset of the complex plane, including \( \emptyset \) and \( \mathbb{C} \), (see, e.g., [1,3]), the natural question is whether an arbitrary closed set in \( \mathbb{C} \) is the spectrum of a such an operator. Of particular interest are unbounded closed linear operators with nonempty compact spectra.

We give a simple way of constructing an unbounded closed linear operator in a complex Banach space, whose spectrum is any given nonempty closed, in particular compact, subset of the complex plane.

2. Preliminaries

For a closed linear operator \( A \) in a complex Banach space \( X \), the set

\[
\rho(A) := \{ \lambda \in \mathbb{C} \mid \exists R(\lambda, A) := (A - \lambda I)^{-1} \in L(X) \}
\]

(\( I \) is the identity operator on \( X \), \( L(X) \) is the space of bounded linear operators on \( X \) and its complement \( \sigma(A) := \mathbb{C} \setminus \rho(A) \) are called the operator’s resolvent set and spectrum, respectively.

The spectrum is partitioned into three pairwise disjoint subsets, \( \sigma_p(A) \), \( \sigma_c(A) \), and \( \sigma_r(A) \), called the point, continuous, and residual spectrum of \( A \), respectively, as follows:

\[
\sigma_p(A) := \{ \lambda \in \mathbb{C} \mid A - \lambda I \text{ is not injective, i.e., } \lambda \text{ is an eigenvalue of } A \},
\]

\[
\sigma_c(A) := \{ \lambda \in \mathbb{C} \mid A - \lambda I \text{ is injective, not surjective, and } R(A - \lambda I) = X \},
\]

\[
\sigma_r(A) := \{ \lambda \in \mathbb{C} \mid A - \lambda I \text{ is injective and } R(A - \lambda I) \neq X \}
\]
(\(R(\cdot)\) is the range of an operator and \(\tau\) is the closure of a set) (see, e.g., [1,3]).

In the complex Hilbert space \(l_2\) of square-summable sequences, the operator of multiplication by a sequence \((m_n)_{n \in \mathbb{N}} \in \mathbb{C}^\mathbb{N}\)

\[ M(x_n)_{n \in \mathbb{N}} := (m_n x_n)_{n \in \mathbb{N}} \]

with maximal domain

\[ D(M) := \{(x_n)_{n \in \mathbb{N}} \in l_2 \mid (m_n x_n)_{n \in \mathbb{N}} \in l_2\} \]

is a (densely defined) closed linear operator, which is bounded iff the multiplier sequence is bounded, i.e.,

\[ \sup_{n \in \mathbb{N}} |m_n| < \infty. \]

In fact, the operator \(M\) is normal (see, e.g., [2]) and

\[ \sigma(M) = \{m_n\} \]

with

\[ \sigma_p(M) = \{m_n\}, \quad \sigma_c(M) = \{m_n\} \setminus \{m_n\}, \quad \text{and} \quad \sigma_r(M) = \emptyset, \]

where \(\{m_n\}\) is the set of values of \((m_n)_{n \in \mathbb{N}}\) [3].

Thus, by choosing such multiplier sequence \((m_n)_{n \in \mathbb{N}}\) that \(\{m_n\}_{n \in \mathbb{N}}\) is a countable dense subset of an arbitrary nonempty closed set \(\sigma\) in the complex plane, we obtain the multiplication operator \(M\) in \(l_2\) with

\[ \sigma(M) = \sigma, \]

the normal operator \(M\) being bounded whenever the set \(\sigma\) is compact (see, e.g., [2]).

In the complex Banach space \(Y := L_p(0, 1)\) (\(1 \leq p < \infty\)) or \(Y := C[0, 1]\), the latter equipped with the maximum norm

\[ C[0, 1] \ni x \mapsto \|x\|_\infty := \max_{0 \leq t \leq 1} |x(t)|, \]

the differentiation operator

\[ Dx := x' \]

with domain

\[ D(D) := \{x \in L_p(0, 1) \mid x(\cdot) \in AC[a, b], \ x' \in L_p(0, 1), \ x(0) = 0\} \]

or

\[ D(D) := \{x \in C^1[0, 1] \mid x(0) = 0\} \]

respectively, is an unbounded closed linear operator with

\[ \sigma(D) = \emptyset \]

and

\[ [R(\lambda, D)y](t) = \int_0^t e^{\lambda(t-s)} y(s) \, ds, \ \lambda \in \mathbb{C}, y \in Y, \]

(cf. [3]), the operator being densely defined in \(L_p(0, 1)\) (\(1 \leq p < \infty\)) but not in \(C[0, 1]\).
3. **Unbounded Linear Operator with an Arbitrary Spectrum**

Let $\sigma$ be arbitrary nonempty closed set in the complex plane, $M$ be the multiplication operator in the complex space $l^2$ relative to a multiplier sequence $(m_n)_{n \in \mathbb{N}} \in \mathbb{C}^\mathbb{N}$ such that $\{m_n\}$ is a countable dense subset of $\sigma$, and hence, $\sigma(M) = \overline{\{m_n\}} = \sigma$.

and $D$ be the differentiation operator in the complex space $L_p(0,1)$ ($1 \leq p < \infty$) or $C[0,1]$ with $\sigma(D) = \emptyset$ (see Preliminaries).

In the complex Banach space $X \oplus Y$, where $X := l^2$ and $Y := L_p(0,1)$ ($1 \leq p < \infty$) or $Y := C[0,1]$, equipped with the norm

$$X \oplus Y \ni (x,y) \mapsto \|(x,y)\| := \|x\|_X + \|y\|_Y,$$

the operator matrix

$$\begin{bmatrix} M & 0 \\ 0 & D \end{bmatrix}$$

defines an unbounded closed linear operator $A$, whose domain

$$D(A) := D(M) \oplus D(D)$$
is dense in $X \oplus Y$ whenever $D(D)$ is dense in $Y$, with

$$\sigma(A) = \sigma(M) \cup \sigma(D) = \sigma(M) = \overline{\{m_n\}} = \sigma$$

and, as is easily seen,

$$\sigma_p(A) = \sigma_p(M) = \{m_n\}, \quad \sigma_c(A) = \sigma_c(M) = \overline{\{m_n\}} \setminus \{m_n\}, \quad \text{and} \quad \sigma_r(A) = \emptyset$$

(see Preliminaries).

**Remarks 3.1.**

- Thus, the spectrum of an unbounded closed linear operator in a complex Banach space can be an arbitrary closed subset of the complex plane, including $\emptyset$ and $\mathbb{C}$, (cf. [3, Theorem 5.7]).

- If, in the prior example, $Y := L_2(0,1)$, $X \oplus Y$ is a Hilbert space relative to the inner product

$$X \oplus Y \ni (x_1,y_1), (x_2,y_2) \mapsto \langle (x_1,y_1), (x_2,y_2) \rangle := \langle x_1, x_2 \rangle_X + \langle y_1, y_2 \rangle_Y.$$  

**References**

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[2] ———, *Linear Operators. Part II: Spectral Theory. Self Adjoint Operators in Hilbert Space*, Interscience Publishers, New York, 1963.

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