GLOBAL WELL-POSEDNESS AND SCATTERING FOR
NONLINEAR SCHRODINGER EQUATIONS WITH COMBINED
NONLINEARITIES IN THE RADIAL CASE

XING CHENG, CHANGXING MIAO, AND LIFENG ZHAO

ABSTRACT. We consider the Cauchy problem for the nonlinear Schrödinger equation with combined nonlinearities, one of which is defocusing mass-critical and the other is focusing energy-critical or energy-subcritical. The threshold is given by means of variational argument. We establish the profile decomposition in $H^1(\mathbb{R}^d)$ and then utilize the concentration-compactness method to show the global well-posedness and scattering versus blowup in $H^1(\mathbb{R}^d)$ below the threshold for radial data when $d \leq 4$.

CONTENTS

1. Introduction 1
2. Variational estimates 6
3. Wellposedness and perturbation theory 14
4. Linear Profile decomposition 20
5. Extraction of a critical element 27
6. Extinction of the critical element 36
7. Blow-up 37
References 39

1. INTRODUCTION

We will consider the Cauchy problem for the nonlinear Schrödinger equation

\[
\begin{aligned}
    i\partial_t u + \Delta u &= |u|^4 u - |u|^{p-1} u, \\
    u(0) &= u_0 \in H^1(\mathbb{R}^d),
\end{aligned}
\] (1.1)

where $u : \mathbb{R} \times \mathbb{R}^d \to \mathbb{C}$ is a complex-valued function, $1 + \frac{4}{d} < p \leq 1 + \frac{4}{d-2}$, $d = 3, 4$ and $1 + \frac{4}{d} < p < \infty$, $d = 1, 2$.

The equation (1.1) has the following mass and energy:

\[
\mathcal{M}(u) = \int_{\mathbb{R}^d} |u|^2 \, dx,
\] (1.2)

\[
\mathcal{E}(u) = \int_{\mathbb{R}^d} \left(\frac{1}{2} |\nabla u|^2 - \frac{1}{p+1} |u|^{p+1} + \frac{d}{2(d+2)} |u|^{2(d+2)} \right) \, dx.
\] (1.3)
The equation (1.1) is a special case of the general nonlinear Schrödinger equation with combined nonlinearities

\[
\begin{aligned}
    i\partial_t u + \Delta u &= \mu_1 |u|^{p_1-1}u + \mu_2 |u|^{p_2-1}u, \\
    u(0) &= u_0 \in H^1(\mathbb{R}^d),
\end{aligned}
\]  

where \(1 < p_1, p_2 \leq 1 + \frac{4}{d-2}\), for \(d \geq 3\), \(1 < p_1, p_2 < \infty\), for \(d = 1, 2\), \(\mu_1, \mu_2 \in \{\pm 1\}\). This equation arises in the study of the Hartree approximation and quasi-classically within the frame of the secondary quantization in the boson gas with many body \(\delta\)-function interaction. It can also be used to describe the effect of saturation of nonlinear refractive index. At the same time, the equation of nuclear hydrodynamics with effective Skyrme’s forces reduces quasi-classically to (1.4). For more physical background, we refer the reader to [5, 26] and the references therein.

The local theory for (1.4) in the energy space follows from the standard method of T. Cazenave, F. B. Weissler [10], see also [9]. Proceeded by [30], T. Tao, M. Visan and X. Zhang considered various cases in [28]. They proved global wellposedness and scattering of the solution to the equation (1.4) for finite energy data when \(\mu_1 = \mu_2 = +1\) and \(1 + \frac{4}{d} \leq p_1 < p_2 \leq 1 + \frac{4}{d-2}\), \(d \geq 3\). The case when \(p_1 = 1 + \frac{4}{d}\) and \(p_2 = 1 + \frac{4}{d-2}\) is the most difficult. In [28], the low frequencies of the solution are well approximated by the \(L^2\) critical problem and the high frequencies are well approximated by the energy-critical problem. The medium frequencies are controlled by the Morawetz estimates.

In [22], C. Miao, G. Xu and L. Zhao considered the case where \(\mu_1 = +1, \mu_2 = -1\) and \(1 + \frac{4}{d} < p_1 < p_2 = 1 + \frac{4}{d-2}\), \(d = 3\). The threshold was given by variational method due to the energy trapping property as in [13]. They established the linear profile decomposition in \(H^1(\mathbb{R}^d)\) in the spirit of [13]. By using this new profile decomposition, they reduced the scattering problem to the extinction of the critical element. The critical element can then be excluded by using the Virial identity. They showed the dichotomy of global wellposedness and scattering versus blow-up phenomenon below the threshold for radial solutions. The radial assumption was removed in dimensions five and higher in [23]. While, for the case of lower dimensions \(d \in \{3, 4\}\), how to remove the radial assumption is still open in this field.

If \(\mu_1 = \mu_2 = -1\) and \(1 + \frac{4}{d} < p_1 < p_2 = 1 + \frac{4}{d-2}\), \(d \geq 5\), the Cauchy problem was considered in [11, 2]. After giving existence of the ground state based on the idea in [7] and [13], they showed a sufficient and necessary condition for the scattering in the spirit of [15] by using the profile decomposition in \(\dot{H}^1(\mathbb{R}^d)\) and the global wellposedness and scattering result in [18].

In this paper, we aim to look for the suitable threshold to study the global wellposedness and scattering versus blowup of (1.1). Before stating the main theorem, we introduce some notations. For \(\varphi \in H^1(\mathbb{R}^d)\), we denote the scaling \((T_\lambda \varphi)(x) = \lambda^{\frac{d}{2}} \varphi(\lambda x)\). For any \(\omega > 0\), we have the Lyapunov functional

\[
S_\omega(\varphi) = \mathcal{E}(\varphi) + \frac{1}{2} \omega \mathcal{M}(\varphi).
\]  

(1.5)
We also denote the scaling derivative of $S_\omega(\varphi)$ by $K(\varphi)$,

$$
K(\varphi) = \mathcal{L} S_\omega(\varphi) = \frac{d}{dx} |_{\lambda = 1} S_\omega(T_\lambda \varphi) = \frac{d}{dx} |_{\lambda = 1} \mathcal{E}(T_\lambda \varphi)
$$

$$
= \int_{\mathbb{R}^d} |\nabla \varphi|^2 - \frac{d(p-1)}{2(p+1)} |\varphi|^{p+1} + \frac{d}{d+2} |\varphi|^{\frac{2(d+2)}{d}} \, dx. \tag{1.6}
$$

Let

$$
m_\omega = \inf \{ S_\omega(\varphi) : \varphi \in H^1(\mathbb{R}^d) \setminus \{0\}, K(\varphi) = 0 \}, \tag{1.7}
$$

then we have

**Proposition 1.1.** For $1 + \frac{4}{d} < p < 1 + \frac{4}{d-2}$, $d \geq 3$, and $1 + \frac{4}{d} < p < \infty$, $d = 1, 2, \omega > 0$, we have $m_\omega > 0$. Moreover, $m_\omega = S_\omega(Q)$, where $Q \in H^1(\mathbb{R}^d)$ is the ground state of

$$
- \omega Q + \Delta Q + |Q|^{p-1}Q - |Q|^\frac{4}{d}Q = 0. \tag{1.8}
$$

**Proposition 1.2.** For $p = 1 + \frac{4}{d-2}$, $d \geq 3$, we have for all $\omega > 0$,

$$
m_\omega = \mathcal{E}^0(W), \tag{1.9}
$$

where

$$
\mathcal{E}^0(W) = \int_{\mathbb{R}^d} \left( \frac{1}{2} |\nabla W|^2 - \frac{d}{2d} |W|^{\frac{2d}{d-2}} \right) \, dx,
$$

and $W \in \dot{H}^1(\mathbb{R}^d)$ is unique positive radial solution of

$$
- \Delta W = |W|^{\frac{4}{d-2}} W. \tag{1.10}
$$

**Remark 1.**

1. When $p = 1 + \frac{4}{d-2}$, we note $m_\omega = \mathcal{E}^0(W)$, which indicates that $m_\omega$ is independent of $\omega$, so we can define $m := m_\omega = \mathcal{E}^0(W)$ in this case. Let

   $$
   A_{\omega,+} = \{ \varphi \in H^1(\mathbb{R}^d) : S_\omega(\varphi) < m_\omega, K(\varphi) \geq 0 \},
   $$

   $$
   A_{\omega,-} = \{ \varphi \in H^1(\mathbb{R}^d) : S_\omega(\varphi) < m_\omega, K(\varphi) < 0 \},
   $$

   $$
   A_+ = \{ \varphi \in H^1(\mathbb{R}^d) : \mathcal{E}(\varphi) < m, K(\varphi) \geq 0 \},
   $$

   $$
   A_- = \{ \varphi \in H^1(\mathbb{R}^d) : \mathcal{E}(\varphi) < m, K(\varphi) < 0 \}.
   $$

2. $A_{\omega,\pm}$ and $A_\pm$ are non-empty. In fact, we note $\varphi = 0$ belongs to both $A_{\omega,+}$ and $A_+$, so $A_{\omega,+}$ and $A_+$ are nonempty. On the other hand, we can easily verify that $\varphi(x) = \epsilon^{-\frac{d}{2}} Q(\epsilon^{-1} x)$ belongs to $A_{\omega,-}$, when $\epsilon$ is sufficiently small. Similarly, by using some truncation of $\varphi = \epsilon^{-\frac{d}{2}} W(\epsilon^{-1} x)$ (still denoted by $\varphi$) to make sure $\varphi \in H^1(\mathbb{R}^d)$, we can show $A_-$ is nonempty.

3. We will give the proof of the following theorem in a unified form regardless of $1 + \frac{4}{d} < p < 1 + \frac{4}{d-2}$, $d \in \{3,4\}$ or $1 + \frac{4}{d} < p < \infty$, $d \in \{1,2\}$ or $p = 1 + \frac{4}{d-2}$, $d = 3,4$. In fact, We note for any $u_0 \in A_+$, we have $\mathcal{E}(u_0) < m$, we can take $\omega > 0$ small enough such that $\mathcal{E}(u_0) + \frac{1}{2}\omega M(u_0) < m$, thus $u_0 \in A_{\omega,+}$. Similarly, for any $u_0 \in A_-$, we can still take $\omega > 0$ small enough such that $u_0 \in A_{\omega,-}$. Thus, our result below in the situation $p = 1 + \frac{4}{d-2}$, $d = 3,4$ can be proved as in the case $1 + \frac{4}{d} < p < 1 + \frac{4}{d-2}$, $d \in \{3,4\}$.

Now, we state our result.
Theorem 1.3. (1) For $1 + \frac{4}{d} < p < 1 + \frac{4}{d-2}$, $d \in \{3, 4\}$, $1 + \frac{4}{d} < p < \infty$, $d \in \{1, 2\}$, and $u_0$ is radial, we have for any $\omega > 0$,
(i) if $u_0 \in A_{\omega,+}$, the solution $u$ to \( (1.1) \) exists globally and scatters in $H^1(\mathbb{R}^d)$;
(ii) if $u_0 \in A_{\omega,-}$, for $d \geq 2$, $p \leq \min(5, 1 + \frac{4}{d-2})$, the solution $u$ blows up in finite time.

(2) For $p = 1 + \frac{4}{d-2}$, $d \in \{3, 4\}$, and $u_0$ is radial, we have
(i) if $u_0 \in A_+$, the solution $u$ to \( (1.1) \) exists globally and scatters in $H^1(\mathbb{R}^d)$;
(ii) if $u_0 \in A_-$, the solution $u$ blows up in finite time.

For the nonlinear Schrödinger equation with combined nonlinearities, we focus on the different roles played by the two nonlinearities. Generally speaking, the main barrier of the local theory is the higher order term while the lower order term is dominant in the global behavior. We prove our main theorem by the compactness-contradiction method initiated by C. E. Kenig and F. Merle [15]. In the argument, the linear profile decomposition plays an important role. In previous works such as [22, 23], the authors considered the equation

$$
\begin{cases}
  i\partial_t u + \Delta u = |u|^{p-1}u - |u|^{\frac{4}{d-2}}u, \\
  u(0) = u_0 \in H^1(\mathbb{R}^d),
\end{cases}
$$

(1.11)

where $1 + \frac{4}{d} < p < 1 + \frac{4}{d-2}$, $d \geq 3$. Since the lower order term is $\dot{H}^s(\mathbb{R}^d)(0 < s = \frac{d}{2} - \frac{2}{p-1} < 1)$ critical, the solution of (1.11) is expected to behave like that of the defocusing equation $i\partial_t u + \Delta u = |u|^{p-1}u$. The critical space of this defocusing equation is $\dot{H}^s(0 < s < 1)$, so it is reasonable to apply the $\dot{H}^s$-profile decomposition to $\langle \nabla \rangle^s u$ for $u \in H^1(\mathbb{R}^d)$. Since the symmetry group in $\dot{H}^s$ is of the same type as in $\dot{H}^1$ case, we may equivalently apply the $\dot{H}^1$-profile decomposition to $\langle \nabla \rangle u$ for $u \in H^1(\mathbb{R}^d)$. However, the lower term in (1.11) is $L^2(\mathbb{R}^d)$-critical. The symmetry group in $L^2(\mathbb{R}^d)$ is different from that in $\dot{H}^1(\mathbb{R}^d)$ due to the Galilean symmetry. Therefore, it is natural that the $L^2$-profile decomposition is applied to $\langle \nabla \rangle u$ for $u \in H^1(\mathbb{R}^d)$ in this paper.

Although the Galilean transform and scaling are encoded in the profile decomposition in $L^2$, they turn out to be excluded in the profile decomposition in $H^1$. In fact, for a typical profile, if the scaling parameter goes to zero then the $\dot{H}^1$ boundedness is violated. If the scaling parameter goes to infinity, then we can show such a profile vanishes. Similar case occurs for the Galilean transform. As a consequence of the linear profile decomposition, we can reduce to almost periodic solution independent of wellposedness and scattering results of energy-critical or mass-critical equations. This is a striking difference from [21, 22], where scaling to zero was excluded by the non-existence of the almost periodic solution to the focusing energy-critical Schrödinger equations. Hence it relies heavily on the results in [15, 18].

We explore the profile decomposition in use to get better estimate (1.13) for the remainder. This estimate provides spacetime control for both the remainder $w_n^k$ and its derivative $|\nabla| w_n^k$. This makes remarkable difference with the profile decomposition for $H^1(\mathbb{R}^d)$ data obtained both in [8] and [22]. In fact, R. Carles and S. Keraani [8] only provided the control over the remainder, while the authors in [22] only provided the control over the derivative of the remainder.
Although we make more delicate analysis due to the stronger control in the
perturbation theorem, we get stronger compactness for the critical element in $H^1(\mathbb{R}^d)$,
which is stronger than the compactness in $\dot{H}^s(0 < s \leq 1)$ obtained in [22, 23].

We expect our result will be extended to higher dimensions ($d \geq 5$) since all
the arguments make sense except the long-time perturbation. However, the exotic
Strichartz estimates in [11, 29] seem useless to establish the long-time perturbation
in our case because the mass-critical term in our equation cannot be controlled
properly in the Sobolev spaces. The radial assumption is expected to be removed
in a forthcoming paper.

The rest of the paper is organized as follows. After introducing some notations and
preliminaries, we give the threshold in Section 2. Moreover, we show the energy-
trapping properties for the set $A_{\omega, \pm}$ in this section. The local wellposedness and
perturbation theory are stated in Section 3. In Section 4, we derive the linear
profile decomposition for data in $H^1(\mathbb{R}^d)$. Then we argue by contradiction. We
reduce to the existence of a critical element in Section 5 and show the extinction
of such a critical element in Section 6. To make the results complete, we show the
existence of blowup solutions in Section 7.

Notation and Preliminaries

We will use the notation $X \lesssim Y$ whenever there exists some positive constant $C$
so that $X \leq CY$. Similarly, we will use $X \sim Y$ if $X \lesssim Y \lesssim X$.

We define the Fourier transform on $\mathbb{R}^d$ to be
$$\hat{f}(\xi) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-ix\xi} f(x) \, dx,$$
and for $s \in \mathbb{R}$, the fractional differential operators $|\nabla|^s$ is defined by $|\nabla|^s f(\xi) = |\xi|^s \hat{f}(\xi)$. We also define $\langle \nabla \rangle^s$ by $\langle \nabla \rangle^s f(\xi) = (1 + |\xi|^2)^{s/2} \hat{f}(\xi)$.

We define the homogeneous Sobolev norms
$$\|f\|_{\dot{H}^s(\mathbb{R}^d)} = \| |\nabla|^s f \|_{L^2(\mathbb{R}^d)},$$
and inhomogeneous Sobolev norms
$$\|f\|_{H^s(\mathbb{R}^d)} = \| \langle \nabla \rangle^s f \|_{L^2(\mathbb{R}^d)}.$$

We use the notation $o_n(1)$ to denote a quantity which tends to 0, as $n \to \infty$.

For $I \subset \mathbb{R}$, we use $L^q_x L^r_t(I \times \mathbb{R}^d)$ to denote the spacetime norm
$$\|u\|_{L^q_x L^r_t(I \times \mathbb{R}^d)} = \left( \int_I \left( \int_{\mathbb{R}^d} |u(t,x)|^r \, dx \right)^{\frac{q}{r}} \, dt \right)^{\frac{1}{q}}.$$
When $q = r$, we abbreviate $L^q_x L^r_t$ as $L^q_{t,x}$.

We also recall Duhamel’s formula
$$u(t) = e^{i(t-t_0)\Delta} u(t_0) - i \int_{t_0}^t e^{i(t-s)\Delta} (iu_t + \Delta u)(s) \, ds.$$
(1.12)

We say that a pair of exponents $(q,r)$ is $L^2$-admissible if $\frac{2}{q} + \frac{d}{r} = \frac{d}{2}$ and $2 \leq q,r \leq \infty$, $(q,r,d) \neq (2, \infty, 2)$, $d \geq 1$. 


Lemma 1.4 (Strichartz estimate, [14]). Let $I$ be a compact time interval and let $u : I \times \mathbb{R}^d \to \mathbb{C}$ be a solution to the forced Schrödinger equation $i\partial_t u + \Delta u = G$ for some function $G$, then we have
\[
\|u\|_{L^q_t L^r_x(I \times \mathbb{R}^d)} \lesssim \|u(t_0)\|_{L^2_x(\mathbb{R}^d)} + \|G\|_{L^q_t L^{r'}_x(I \times \mathbb{R}^d)}
\]
for any $t_0 \in I$ and any $L^2$-admissible exponents $(q, r)$, $(\tilde{q}, \tilde{r})$.

If $I \times \mathbb{R}^d$ is a spacetime slab, we define the Strichartz norm $S^0(I)$ by
\[
\|u\|_{S^0(I)} = \sup \|u\|_{L^q_t L^r_x(I \times \mathbb{R}^d)},
\]
where the sup is taken over all $L^2$-admissible pairs $(q, r)$. When $d = 2$, we need to modify the norm a little, where the sup is taken over all $L^2$-admissible pairs with $q \geq 2 + \epsilon$, for $\epsilon > 0$ arbitrary small. We also define $S^1(I)$ norm by
\[
\|u\|_{S^1(I)} = \|\langle \nabla \rangle u\|_{S^0(I)}.
\]

2. Variational estimates

In this section, we prove Proposition 1.1 and 1.2. We show the existence of the ground state together with the energy-trapping property for $A_{\omega, \pm}$, which will be used to show the scattering and blow-up.

Let $\mathcal{K}(\varphi) = \mathcal{K}^Q(\varphi) + \mathcal{K}^N(\varphi)$, where
\[
\mathcal{K}^Q(\varphi) = \int_{\mathbb{R}^d} |\nabla \varphi|^2 \, dx,
\]
\[
\mathcal{K}^N(\varphi) = -\frac{d(p-1)}{2(p+1)} \int |\varphi|^{p+1} \, dx + \frac{d}{d+2} \int |\varphi|^{\frac{2(d+2)}{d}} \, dx.
\]
We have the following basic fact about $\mathcal{K}(T_\lambda \varphi)$:

Lemma 2.1 (Dynamic behavior of $\mathcal{K}$ under the scaling). For any $\varphi \in H^1(\mathbb{R}^d) \setminus \{0\}$, there exists a unique $\lambda_0(\varphi) > 0$ that
\[
\mathcal{K}(T_\lambda \varphi) \begin{cases} 
> 0, & 0 < \lambda < \lambda_0(\varphi), \\
= 0, & \lambda = \lambda_0(\varphi), \\
< 0, & \lambda > \lambda_0(\varphi).
\end{cases}
\]

Proof. An easy computation gives
\[
\frac{1}{\lambda^2} \mathcal{K}(T_\lambda \varphi) = \int |\nabla \varphi|^2 + \frac{d}{d+2} |\varphi|^{\frac{2(d+2)}{d}} - \frac{d(p-1)}{2(p+1)} \lambda^\frac{d}{2(p-1)-2} |\varphi|^{p+1} \, dx,
\]
which implies $\mathcal{K}(T_\lambda \varphi) > 0$ for $\lambda > 0$ sufficiently small. By
\[
\frac{d}{d\lambda} \left( \frac{1}{\lambda^2} \mathcal{K}(T_\lambda \varphi) \right) = -\frac{d(p-1)}{2(p+1)} \left( \frac{d}{2} (p-1) - 2 \right) \lambda^\frac{d}{2(p-1)-3} \int |\varphi|^{p+1} \, dx < 0,
\]
we see $\frac{1}{\lambda^2} \mathcal{K}(T_\lambda \varphi)$ is monotone decreasing with respect to $\lambda > 0$.

Since
\[
\mathcal{K}(T_\lambda \varphi) = \lambda^2 \left( \|\nabla \varphi\|_{L^2}^2 + \frac{d}{d+2} \|\varphi\|_{L^{\frac{2(d+2)}{d}}}^2 \right) - \lambda^\frac{d}{2} (p-1) \frac{d(p-1)}{2(p+1)} \|\varphi\|_{L^{p+1}}^{p+1},
\]
\[
\to -\infty, \quad \text{as} \quad \lambda \to \infty,
\]
there exists a unique $\lambda_0 > 0$ such that $\mathcal{K}(T_{\lambda_0} \varphi) = 0$ and (2.1) follows. \qed
Now, we show the positivity of $\mathcal{K}$ near 0 in the energy space.

**Lemma 2.2.** For any bounded sequence $\varphi_n \in H^1(\mathbb{R}^d) \setminus \{0\}$ with $\mathcal{K}^Q(\varphi_n) \to 0$, as $n \to \infty$, then for $n$ large enough, we have $\mathcal{K}(\varphi_n) > 0$.

**Proof.** By assumption, $\|
abla \varphi_n\|_{L^2} \to 0$, as $n \to \infty$. Due to the interpolation and Sobolev inequalities, we have

$$
\|\varphi_n\|_{L^{p+1}}^{p+1} \lesssim \|\varphi_n\|_{L^2}^{p+1} \frac{d(p-1)}{d+2} \|
abla \varphi_n\|_{L^2}^{\frac{d(p-1)}{d+2}}.
$$

Since $\frac{d}{2}(p-1) > 2$, for $n$ large enough, we see

$$
\mathcal{K}(\varphi_n) = \int_{\mathbb{R}^d} |\nabla \varphi_n|^2 - \frac{d(p-1)}{2(p+1)} |\varphi_n|^{p+1} + \frac{d}{d+2} |\varphi_n|^{\frac{2(d+2)}{d}}
\geq \int_{\mathbb{R}^d} |\nabla \varphi_n|^2 dx - o\left( \int_{\mathbb{R}^d} |\nabla \varphi_n|^2 dx \right)
\sim \int_{\mathbb{R}^d} |\nabla \varphi_n|^2 dx > 0.
$$

\[\square\]

**Lemma 2.3.** For any $\varphi \in H^1(\mathbb{R}^d)$, we have

\begin{align*}
(2 - \mathcal{L})\mathcal{S}_\omega(\varphi) &= \omega \|\varphi\|_{L^2}^2 + \frac{d(p-1)-4}{2(p+1)} \|\varphi\|_{L^{p+1}}^{p+1}, \\
\mathcal{L}(2 - \mathcal{L})\mathcal{S}_\omega(\varphi) &= \frac{d(p-1)(d(p-1)-4)}{4(p+1)} \|\varphi\|_{L^{p+1}}^{p+1}.
\end{align*}

**Proof.** Direct computation shows that

$$
\mathcal{L}\|\nabla \varphi\|_{L^2}^2 = 2\|\nabla \varphi\|_{L^2}^2,
$$

$$
\mathcal{L}\|\varphi\|_{L^{p+1}}^{p+1} = \frac{d}{2}(p-1)\|\varphi\|_{L^{p+1}}^{p+1},
$$

$$
\mathcal{L}\|\varphi\|_{L^2}^{\frac{2(d+2)}{d}} = 2\|\varphi\|_{L^2}^{\frac{2(d+2)}{d}},
$$

hence we have

\begin{align*}
(2 - \mathcal{L})\mathcal{S}_\omega(\varphi) &= 2\mathcal{S}_\omega(\varphi) - \mathcal{K}(\varphi)
= \omega \|\varphi\|_{L^2}^2 + \frac{d(p-1)-4}{2(p+1)} \int_{\mathbb{R}^d} |\varphi|^{p+1} dx,
\end{align*}

$$
\mathcal{L}(2 - \mathcal{L})\mathcal{S}_\omega(\varphi) = \omega \mathcal{L}\|\varphi\|_{L^2}^2 + \frac{d(p-1)-4}{2(p+1)} \mathcal{L}\|\varphi\|_{L^{p+1}}^{p+1}
= \frac{d(p-1)-4}{2(p+1)} \mathcal{L}\|\varphi\|_{L^{p+1}}^{p+1}
= \frac{(d(p-1)-4)(d(p-1))}{4(p+1)} \|\varphi\|_{L^{p+1}}^{p+1}.
$$

\[\square\]

Due to the lack of positivity of $\mathcal{S}_\omega(\varphi)$, we introduce a non-negative functional

$$
\mathcal{H}_\omega(\varphi) = \left(1 - \frac{d}{2}\right)\mathcal{S}_\omega(\varphi)
= \mathcal{S}_\omega(\varphi) - \frac{1}{2} \mathcal{K}(\varphi)
= \frac{\omega}{2} \|\varphi\|_{L^2}^2 + \frac{d(p-1)-4}{4(p+1)} \|\varphi\|_{L^{p+1}}^{p+1},
$$

where $\omega > 0$ is a constant.
then for any $\varphi \in H^1(\mathbb{R}^d) \setminus \{0\}$, we have $\mathcal{H}_\omega(\varphi) \geq 0$, $\mathcal{L}\mathcal{H}_\omega(\varphi) \geq 0$.

**Proposition 2.4.**

$$m_\omega = \inf\{\mathcal{H}_\omega(\varphi) : \varphi \in H^1 \setminus \{0\}, \mathcal{K}(\varphi) \leq 0\}$$

$$= \inf\{\mathcal{H}_\omega(\varphi) : \varphi \in H^1 \setminus \{0\}, \mathcal{K}(\varphi) < 0\}. \quad (2.5)$$

**Proof.** Since $S_\omega(\varphi) = \mathcal{H}_\omega(\varphi)$ when $\mathcal{K}(\varphi) = 0$,

$$m_\omega \geq \inf\{\mathcal{H}_\omega(\varphi) : \varphi \in H^1 \setminus \{0\}, \mathcal{K}(\varphi) \leq 0\}.$$ 

Claim:

$$m_\omega \leq \inf\{\mathcal{H}_\omega(\varphi) : \varphi \in H^1 \setminus \{0\}, \mathcal{K}(\varphi) < 0\}.$$ 

In fact, $\forall \varphi \in H^1 \setminus \{0\}$ with $\mathcal{K}(\varphi) < 0$, by Lemma 2.1 there exists $0 < \lambda_0 < 1$ that $\mathcal{K}(T_{\lambda_0} \varphi) = 0$, then the fact $\mathcal{L}\mathcal{H}_\omega \geq 0$ implies $S_\omega(T_{\lambda_0} \varphi) = \mathcal{H}_\omega(T_{\lambda_0} \varphi) \leq \mathcal{H}_\omega(\varphi)$.

It suffices to show

$$\inf\{\mathcal{H}_\omega(\varphi) : \varphi \in H^1 \setminus \{0\}, \mathcal{K}(\varphi) \leq 0\} \geq \inf\{\mathcal{H}_\omega(\varphi) : \varphi \in H^1 \setminus \{0\}, \mathcal{K}(\varphi) < 0\}. \quad (2.6)$$

In fact, for any $\varphi \in H^1 \setminus \{0\}$ with $\mathcal{K}(\varphi) \leq 0$, by (2.3), we know that

$$\mathcal{L}\mathcal{K}(\varphi) = 2\mathcal{K}(\varphi) - \frac{d(p-1)(d(p-1)-4)}{4(p+1)}\|\varphi\|_{L^{p+1}}^{p+1} < 0, \quad (2.7)$$

then for any $\lambda > 1$, we have $\mathcal{K}(T_\lambda \varphi) < 0$, and as $\lambda \to 1$,

$$\mathcal{H}_\omega(T_\lambda \varphi) \geq \|T_\lambda \varphi\|_2^2 + \frac{d(p-1)-4}{4(p+1)}\|T_\lambda \varphi\|_{L^{p+1}}^{p+1}$$

$$= \|\varphi\|_2^2 + \frac{d(p-1)-4}{4(p+1)}\lambda\mathcal{K}(\varphi)\|\varphi\|_{L^{p+1}}^{p+1}$$

$$\to \|\varphi\|_2^2 + \frac{d(p-1)-4}{4(p+1)}\|\varphi\|_{L^{p+1}}^{p+1} = \mathcal{H}_\omega(\varphi).$$

This shows (2.6) and completes the proof. \qed

We now give the value of $m_\omega$ for $1 + \frac{4}{d} < p < 1 + \frac{4}{d-2}$, $d \geq 3$, and $1 + \frac{4}{d} < p < \infty$, $d = 1, 2$, namely prove Proposition 1.1.

**Proof of Proposition 1.1.** Let $\varphi_n \in H^1(\mathbb{R}^d)$ be a minimizing sequence for (2.5), namely

$$\mathcal{K}(\varphi_n) \leq 0, \varphi_n \neq 0, \mathcal{H}_\omega(\varphi_n) \searrow m_\omega, \text{ as } n \to \infty.$$ 

Let $\varphi_n^*$ be the Schwartz symmetrization of $\varphi_n$, i.e. the radial decreasing rearrangement. Since the symmetrization preserves the nonlinear parts and does not increase the $H^1$ part, we have

$$\varphi_n^* \neq 0, \mathcal{K}(\varphi_n^*) \leq \mathcal{K}(\varphi_n) \leq 0 \text{ and } \mathcal{H}_\omega(\varphi_n^*) = \mathcal{H}_\omega(\varphi_n) \to m_\omega, \text{ as } n \to \infty.$$ 

Then by Lemma 2.1 and (2.7), there exists $0 < \lambda_n \leq 1$ such that $\psi_n = T_{\lambda_n} \varphi_n^*$ satisfies

$$\psi_n \neq 0, \mathcal{K}(\psi_n) = 0, \mathcal{S}_\omega(\psi_n) = \mathcal{H}_\omega(\psi_n) \to m_\omega, \text{ as } n \to \infty.$$ 

Moreover, direct computation gives

$$\frac{d}{2}\|\psi_n\|_2^2 + \frac{1}{d}\|\nabla \psi_n\|_2^2 \leq \mathcal{S}_\omega(\psi_n) + \frac{4}{d(p-1)-4}\mathcal{H}_\omega(\psi_n),$$

which implies the boundedness of $\psi_n$ in $H^1(\mathbb{R}^d)$.

Then, $\psi_n$ converges weakly to some $\psi$ in $H^1(\mathbb{R}^d)$, up to a subsequence. Since $\psi_n$ is radial, it also converges strongly in $L^q(\mathbb{R}^d)$ for $2 < q < \frac{2d}{d-2}$, $d \geq 3$ and $2 <$
Lemma 2.5. We will show $\lambda$ as $K_0$, as $n \to \infty$, and by Lemma 2.2 we have $K(\psi_n) > 0$ for $n$ large, a contradiction. Since $K(\psi) \leq 0$ and $\psi \neq 0$, we have $H_\omega(\psi) \geq m_\omega$, so we have $H_\omega(\psi) = m_\omega$ and $K(\psi) \leq 0$.

By scaling, we may replace $\psi$ by its rescaling, so that $K(\psi) = 0$, $S_\omega(\psi) = H_\omega(\psi) \leq m_\omega$ and $\psi \neq 0$.

Then $\psi$ is a minimizer and $m_\omega = H_\omega(\psi) > 0$.

By variational theory, there is a Lagrange multiplier $\eta \in \mathbb{R}$ such that $S_\omega'(\psi) = \eta K'(\psi)$. We note

$$0 = K(\psi) = L S_\omega(\psi) = S_\omega'(\psi) L \psi = \eta K'(\psi) L \psi = \eta \cdot L^2 S_\omega(\psi).$$

By (2.3) and $L S_\omega(\psi)$, we have

$$L^2 S_\omega(\psi) = 2 L S_\omega(\psi) - \frac{d(p-1)(d(p-1)-4)}{4(p+1)} \| \psi \|_{L^{p+1}}^{p+1}$$

$$= -\frac{d(p-1)(d(p-1)-4)}{4(p+1)} \| \psi \|_{L^{p+1}}^{p+1} < 0,$$

therefore $\eta = 0$ and $\psi$ is a solution to $-\omega Q + \Delta Q + |Q|^{p-1}Q - |Q|^q |Q| = 0$.

The minimality of $S_\omega(Q)$ among the solutions is clear from (1.7), since every solution $Q$ in $H^1(\mathbb{R}^d)$ of (1.8) satisfies $K(Q) = S_\omega'(Q) L Q = 0$. $\Box$

We now turn to find the ground state in the case $p = 1 + \frac{4}{d-2}$, $d \geq 3$, that is to prove Proposition 1.2.

Let

$$K^0(\varphi) = \int_{\mathbb{R}^d} \left( |\nabla \varphi|^2 - |\varphi|^\frac{2d}{d+2} \right) dx,$$

$$H^0(\varphi) = \frac{1}{d} \| \varphi \|_{L^{\frac{2d}{d+2}}}^{\frac{2d}{d+2}}.$$

We will show

**Lemma 2.5.** For $p = 1 + \frac{4}{d-2}$, $d \geq 3$, $m_\omega = \inf \{ H^0(\varphi) : \varphi \in H^1 \setminus \{0\}, K^0(\varphi) < 0 \}$

$$= \inf \{ H^0(\varphi) : \varphi \in H^1 \setminus \{0\}, K^0(\varphi) \leq 0 \}.$$

**Proof.** Since $K^0(\varphi) \leq K(\varphi)$, $H^0(\varphi) \leq H_\omega(\varphi)$, it follows that

$$m_\omega = \inf \{ H_\omega(\varphi) : \varphi \in H^1 \setminus \{0\}, K(\varphi) < 0 \} \geq \inf \{ H^0(\varphi) : \varphi \in H^1 \setminus \{0\}, K^0(\varphi) < 0 \}.$$

Hence, in order to show the first equality, it suffices to show

$$\inf \{ H_\omega(\varphi) : \varphi \in H^1 \setminus \{0\}, K(\varphi) < 0 \} \leq \inf \{ H^0(\varphi) : \varphi \in H^1 \setminus \{0\}, K^0(\varphi) < 0 \}. \quad (2.11)$$

For any $\varphi \in H^1 \setminus \{0\}$ with $K^0(\varphi) < 0$, taking $T_\lambda \varphi(x) = \lambda^{\frac{d-2}{2}} \varphi(\lambda x)$, we have as $\lambda \to \infty$,

$$K(T_\lambda \varphi) = \int_{\mathbb{R}^d} \left( |\nabla \varphi|^2 - |\varphi|^\frac{2d}{d+2} + \frac{d}{d+2} \lambda^{-\frac{4}{d}} |\varphi|^\frac{2(d+1)}{d} \right) dx \to K^0(\varphi),$$

$$H_\omega(T_\lambda \varphi) = \frac{\lambda}{d} \lambda^{-\frac{4}{d} \frac{2d}{d+2}} \to H^0(\varphi).$$
This gives (2.11) and completes the proof of the first equality. For the second equality, it suffices to show
\[ \inf \{ \mathcal{H}^0(\varphi) : \varphi \in H^1 \setminus \{0\}, \mathcal{K}^0(\varphi) < 0 \} \leq \inf \{ \mathcal{H}^0(\varphi) : \varphi \in H^1 \setminus \{0\}, \mathcal{K}^0(\varphi) \leq 0 \}. \]
(2.12)
For any \( \varphi \in H^1 \setminus \{0\} \) with \( \mathcal{K}^0(\varphi) \leq 0 \) and
\[
\mathcal{L} \mathcal{K}^0(\varphi) = \int \left( 2|\nabla \varphi|^2 - \frac{2d}{d-2} |\varphi|^{\frac{2d}{d-2}} \right) \, dx = 2\mathcal{K}^0(\varphi) - \frac{4}{d-2} \|\varphi\|^{\frac{2d}{d-2}} < 0,
\]
which implies \( \mathcal{K}^0(T_\lambda \varphi) < 0 \), for \( \lambda > 1 \). We also have
\[
\mathcal{H}^0(T_\lambda \varphi) = \frac{1}{\lambda} \lambda^{\frac{2d}{d-2}} \|\varphi\|^{\frac{2d}{d-2}} \to \mathcal{H}^0(\varphi), \text{ as } \lambda \to 1,
\]
so we obtain (2.12) and complete the proof.

**Proof of Proposition 1.2.**

By Lemma 2.5, we have
\[
m_\omega = \inf \left\{ \frac{1}{d} \|\varphi\|^{\frac{2d}{d-2}} : \varphi \in H^1 \setminus \{0\}, \|\nabla \varphi\|_{L^2}^2 \leq \|\varphi\|^{\frac{2d}{d-2}} \right\}
\]
\[
\geq \inf \left\{ \frac{1}{d} \|\nabla \varphi\|_{L^2}^2 : \varphi \in H^1 \setminus \{0\}, \|\nabla \varphi\|_{L^2}^2 \leq \|\varphi\|^{\frac{2d}{d-2}} \right\}
\]
\[
\geq \inf \left\{ \frac{1}{d} \left( \frac{\|\nabla \varphi\|_{L^2}^2}{\|\varphi\|_{L^{\frac{2d}{d-2}}}^{\frac{d}{d-2}}} \right)^{\frac{d-2}{d}} : \varphi \in H^1 \setminus \{0\} \right\}
\]
\[
\geq \inf \left\{ \frac{1}{d} \left( \frac{\|\nabla \varphi\|_{L^2}^2}{\|\varphi\|_{L^{\frac{2d}{d-2}}}^{\frac{d}{d-2}}} \right)^{\frac{d-2}{d}} : \varphi \in H^1 \setminus \{0\} \right\} = \frac{1}{d}(C_0^*)^{-d},
\]
where \( C_0^* \) is the sharp Sobolev constant in \( \mathbb{R}^d \), that is
\[
\|\varphi\|_{L^{\frac{2d}{d-2}}(\mathbb{R}^d)} \leq C_0^* \|\nabla \varphi\|_{L^2(\mathbb{R}^d)}, \forall \varphi \in \dot{H}^1(\mathbb{R}^d), \tag{2.13}
\]
with the equality is attained by \( W \) (see [3], [27]) and \( \frac{1}{d}(C_0^*)^{-d} = \mathcal{E}^0(W) \).

On the other hand, by the density \( H^1(\mathbb{R}^d) \) in \( \dot{H}^1(\mathbb{R}^d) \), we can find \( \varphi_n \in H^1 \setminus \{0\} \), such that \( \varphi_n \to W \) in \( \dot{H}^1(\mathbb{R}^d) \), as \( n \to \infty \). Then we have \( \mathcal{H}^0(\varphi_n) \to \mathcal{H}^0(W) = \mathcal{E}^0(W) \), as \( n \to \infty \), by Lemma 2.5, \( \mathcal{H}^0(\varphi_n) \geq m_\omega \) for \( n \) large enough. So we have \( m_\omega \leq \mathcal{E}^0(W) \). Thus, we obtain (1.3).

Next we show the energy-trapping properties of \( A_{\omega, \pm} \).

**Lemma 2.6.** For \( 1 + \frac{4}{d} < p \leq 1 + \frac{4}{d-2}, \) \( d \geq 3, \) and \( 1 + \frac{4}{d} < p < \infty, d = 1, 2, \) we have for any \( \varphi \in H^1(\mathbb{R}^d) \) with \( \mathcal{K}(\varphi) \geq 0, \)
\[
\frac{d(p-1)-4}{d(p-1)} \int \frac{1}{2}|\nabla \varphi|^2 + \frac{d}{2(d+2)} |\varphi|^{\frac{2(d+2)}{d}} \, dx \leq \mathcal{E}(\varphi) \leq \int \frac{1}{2}|\nabla \varphi|^2 + \frac{d}{2(d+2)} |\varphi|^{\frac{2(d+2)}{d}} \, dx. \tag{2.14}
\]
Proof. On the one hand,
\[ \mathcal{E}(\varphi) = \int \frac{1}{2} |\nabla \varphi|^2 - \frac{1}{p+1} |\varphi|^{p+1} + \frac{d}{2p+2} |\varphi|^{\frac{2(p+2)}{d-2}} \, dx \]
\[ \leq \int \frac{1}{2} |\nabla \varphi|^2 + \frac{d}{2p+2} |\varphi|^{\frac{2(p+2)}{d-2}} \, dx. \]

On the other hand, since
\[ \mathcal{K}(\varphi) = \int |\nabla \varphi|^2 - \frac{d(p-1)}{2(p+1)} |\varphi|^{p+1} + \frac{d}{d+2} |\varphi|^{\frac{2(d+2)}{d-2}} \, dx \geq 0, \]
we have
\[ \frac{1}{p+1} \int |\varphi|^{p+1} \, dx \leq \frac{2}{d(p-1)} \int |\nabla \varphi|^2 + \frac{d}{d+2} |\varphi|^{\frac{2(d+2)}{d-2}} \, dx. \]

So
\[ \mathcal{E}(\varphi) = \int \frac{1}{2} |\nabla \varphi|^2 - \frac{1}{p+1} |\varphi|^{p+1} + \frac{d}{2p+2} |\varphi|^{\frac{2(p+2)}{d-2}} \, dx \]
\[ \geq \int \frac{1}{2} |\nabla \varphi|^2 + \frac{d}{2p+2} |\varphi|^{\frac{2(d+2)}{d-2}} \, dx - \frac{2}{d(p-1)} \int |\nabla \varphi|^2 + \frac{d}{d+2} |\varphi|^{\frac{2(d+2)}{d-2}} \, dx \]
\[ = \int \left( \frac{1}{2} - \frac{2}{d(p-1)} \right) |\nabla \varphi|^2 + \left( \frac{d}{2(p+1)} - \frac{2}{d(p-1)} \right) |\varphi|^{\frac{2(d+2)}{d-2}} \, dx \]
\[ = \frac{d(p-1)-4}{d(p-1)} \int \frac{1}{2} |\nabla \varphi|^2 + \frac{d}{2p+2} |\varphi|^{\frac{2(d+2)}{d-2}} \, dx. \]

\[ \square \]

**Proposition 2.7** (Energy-trapping for \( A_{\omega,-} \)). For \( 1 + \frac{4}{d} < p \leq 1 + \frac{4}{d-2} \), \( d \geq 3 \), \( 1 + \frac{4}{d} < p < \infty \), \( d = 1, 2 \), and \( u_0 \in A_{\omega,-} \). Let \( u \) be the solution of (1.1), and \( I_{\max} \) be the lifespan of \( u \), then
\[ \mathcal{K}(u(t)) < -(m_{\omega} - \mathcal{S}_\omega(u(t))), \quad \forall t \in I_{\max}. \quad (2.15) \]

**Proof.** We first claim that \( \mathcal{K}(u(t)) < 0 \), for \( t \in I_{\max} \). Indeed, since \( u_0 \in A_{\omega,-} \), we have by the mass and energy conservation that \( \mathcal{S}_\omega(u(t)) < m_{\omega} \), for \( t \in I_{\max} \). If \( \mathcal{K}(u(t_0)) \geq 0 \) for some \( t_0 \in I_{\max} \), then there is \( t_1 \in I_{\max} \), such that \( \mathcal{K}(u(t_1)) = 0 \). So we have \( \mathcal{S}_\omega(u(t_1)) \geq m_{\omega} \), which contradicts \( \mathcal{S}_\omega(u(t)) < m_{\omega} \), for all \( t \in I_{\max} \), so we have \( \mathcal{K}(u(t)) < 0 \) for \( t \in I_{\max} \).

Next, we turn to (2.15). By the above claim, for any \( t \in I_{\max} \), there exists \( 0 < \lambda(t) < 1 \) such that \( \mathcal{K}(T_{\lambda(t)}u(t)) = 0 \), which together with the definition of \( m_{\omega} \) shows that \( \mathcal{S}_\omega(T_{\lambda(t)}u(t)) \geq m_{\omega} \). By Lemma 2.3
\[ \mathcal{L}^2 \mathcal{S}_\omega(u(t)) = 2 \mathcal{L} \mathcal{S}_\omega(u(t)) - \frac{d(p-1)(d(p-1)-4)}{4(p+1)} \| u(t) \|_{L^{p+1}}^{p+1} \]
\[ = 2 \mathcal{K}(u(t)) - \frac{d(p-1)(d(p-1)-4)}{4(p+1)} \| u(t) \|_{L^{p+1}}^{p+1} < 0, \]
we have
\[ \mathcal{S}_\omega(u(t)) > \mathcal{S}_\omega(T_{\lambda(t)}u(t)) + (1 - \lambda(t)) \frac{d}{4} \int_{\lambda=1} \mathcal{S}_\omega(T_{\lambda}u(t)) \]
\[ = \mathcal{S}_\omega(T_{\lambda(t)}u(t)) + (1 - \lambda(t)) \mathcal{K}(u(t)) \]
\[ > m_{\omega} + \mathcal{K}(u(t)), \]
so \( \mathcal{K}(u(t)) < -(m_{\omega} - \mathcal{S}_\omega(u(t))) \).

\[ \square \]
Before discussing the energy-trapping for \( A_{\omega,+} \), we first show \( \forall \omega > 0, A_{\omega,+} \) is bounded in \( H^1(\mathbb{R}^d) \).

**Lemma 2.8.** Let \( \omega > 0 \) and \( u \in A_{\omega,+} \), then we have
\[
\|u\|_{L^2}^2 \leq \frac{2m_\omega}{\omega}, \quad \|\nabla u\|_{L^2}^2 \lesssim m_\omega,
\]
and hence
\[
\|u\|_{H^1}^2 \lesssim m_\omega + \frac{m_\omega}{\omega}.
\]

**Proof.** By \( u \in A_{\omega,+} \), we have
\[
\mathcal{H}_\omega(u) \leq \mathcal{S}_\omega(u) \leq m_\omega,
\]
which implies
\[
\|u\|_{L^2}^2 \leq \frac{2m_\omega}{\omega}.
\]
The boundedness of \( \|\nabla u\|_{L^2} \) follows from
\[
m_\omega \geq \mathcal{S}_\omega(u) \geq \mathcal{E}(u) = \int \frac{1}{2} |\nabla u|^2 - \frac{1}{p+1} |u|^{p+1} + \frac{d}{2(d+2)} |u|^{\frac{2(d+2)}{d}} \, dx
\]
\[
\geq \int \frac{1}{2} |\nabla u|^2 + \frac{d}{2(d+2)} |u|^{\frac{2(d+2)}{d}} \, dx - \frac{2}{d(p-1)} \left( \|\nabla u\|_{L^2}^2 + \frac{d}{d+2} \|u\|_{L^2}^{\frac{2(d+2)}{d}} \right)
\]
\[
\geq \left( \frac{1}{2} - \frac{2}{d(p-1)} \right) \|\nabla u\|_{L^2}^2,
\]
where the first inequality is given by \( \mathcal{K}(u) \geq 0 \). \( \square \)

**Proposition 2.9** (Energy-trapping for \( A_{\omega,+} \)). For \( u_0 \in A_{\omega,+} \), let \( u \) be a solution of \( (1.1) \), with \( I_{\text{max}} \) the lifespan, we have \( \delta > 0 \) depending on \( d, p \) and \( \omega \) such that for \( t \in I_{\text{max}} \),
\[
\mathcal{K}(u(t)) \geq \frac{d}{2} \left( \|\nabla u(t)\|_{L^2}^2 + \frac{d}{d+2} \|u(t)\|_{L^2}^{\frac{2(d+2)}{d}} \right), \delta \left( m_\omega - \mathcal{S}_\omega(u(t)) \right)
\]
\[
(2.17)
\]

**Proof.** Direct computation shows that
\[
\mathcal{S}_\omega(T_\lambda u(t)) = \frac{\omega}{2} \|T_\lambda u(t)\|_{L^2}^2 + \mathcal{E}(T_\lambda u(t))
\]
\[
= \frac{\omega}{2} \|u(t)\|_{L^2}^2 + \int \frac{1}{2} \lambda^2 |\nabla u(t)|^2 + \frac{d}{2(d+2)} \lambda^2 |u(t)|^{\frac{2(d+2)}{d}} - \frac{1}{p+1} \lambda^2 |u(t)|^{p+1} \, dx
\]
and
\[
\frac{d^2}{d\lambda^2} \mathcal{S}_\omega(T_\lambda u(t)) = \int |\nabla u(t)|^2 + \frac{d}{d+2} |u(t)|^{\frac{2(d+2)}{d}} - \frac{d(p-1)(d(p-1)-2)}{4(p+1)} \lambda^{\frac{4}{d}(p-1)-2} |u(t)|^{p+1} \, dx.
\]
Since
\[
\mathcal{K}(T_\lambda u(t)) = \int \lambda^2 |\nabla u(t)|^2 - \frac{d(p-1)}{2(p+1)} \lambda^2 |u(t)|^{p+1} + \frac{d}{d+2} \lambda^2 |u(t)|^{\frac{2(d+2)}{d}} \, dx,
\]
we have
\[
\frac{d^2}{d\lambda^2} \mathcal{S}_\omega(T_\lambda u(t)) = -\frac{1}{\lambda^2} \mathcal{K}(T_\lambda u(t)) + \frac{2}{\lambda^2} \left( \mathcal{K}(T_\lambda u(t)) + \frac{(4-d(p-1))(p-1)}{8(p+1)} \int |T_\lambda u(t)|^{p+1} \, dx \right).
\]
\[
(2.18)
\]
Case I.
\[ \mathcal{K}(u(t)) = \frac{(d(p-1)-4)d(p-1)}{8(p+1)} \int |u(t)|^{p+1} \, dx \geq 0, \text{ for } t \in I_{\text{max}}. \] (2.19)
In this case,
\[ \mathcal{K}(u(t)) = \int |\nabla u|^2 - \frac{d(p-1)}{2(p+1)} |u|^{p+1} + \frac{d}{d+2} |u|^\frac{2(d+2)}{d} \, dx \]
\[ \geq \int |\nabla u|^2 + \frac{d}{d+2} |u|^\frac{2(d+2)}{d} \, dx - \frac{4}{d(p-1)-4} \mathcal{K}(u(t)), \]
thus
\[ \mathcal{K}(u(t)) \geq \frac{d(p-1)-4}{d(p-1)} \int |\nabla u(t)|^2 + \frac{d}{d+2} |u(t)|^\frac{2(d+2)}{d} \, dx, \forall t \in I_{\text{max}}. \]

Case II.
\[ \mathcal{K}(u(t)) = \frac{(d(p-1)-4)d(p-1)}{8(p+1)} \int |u(t)|^{p+1} \, dx < 0, \text{ for } t \in I_{\text{max}}. \] (2.20)
In this case,
\[ \|\nabla u(t)\|^2_{L^2} < \frac{d^2(p-1)}{8(p+1)} \int |u(t)|^{p+1} \, dx - \frac{d}{d+2} \int |u(t)|^\frac{2(d+2)}{d} \, dx, \]
which implies \( u(t) \neq 0, \forall t \in I_{\text{max}}, \)
and
\[ \frac{d^2(p-1)}{8(p+1)} \|u(t)\|_{L^{p+1}}^{p+1} \geq \|\nabla u(t)\|^2_{L^2}. \] (2.21)
Since \( u_0 \in A_{\omega^+} \) and \( u(t) \neq 0, \forall t \in I_{\text{max}}, \) together with the definition of \( m_{\omega}, \) we have \( \mathcal{K}(u(t)) > 0 \) by similar argument as in the claim in Proposition 2.7. Therefore, by Lemma 2.20 there is \( \lambda(t) > 1 \) such that
\[ \mathcal{K}(T_{\lambda(t)}u(t)) = 0 \] (2.22)
and
\[ \mathcal{K}(T_{\lambda}u(t)) > 0 \text{ for } 1 \leq \lambda < \lambda(t). \] (2.23)
By (2.22), we have
\[ \|\nabla u(t)\|^2_{L^2} - \frac{d(p-1)}{2(p+1)} \lambda(t) \frac{d}{2(p-1)} \|u(t)\|^{p+1}_{L^{p+1}} + \frac{d}{d+2} \|u(t)\|^\frac{2(d+2)}{d} \frac{L}{L-d} = 0, \]
so by Lemma 2.20 together with the interpolation and Sobolev inequalities, we have
\[ \lambda(t) \frac{d}{2(p-1)} \|u(t)\|^{p+1}_{L^{p+1}} = \frac{2(p+1)}{d(p-1)} \left( \|\nabla u(t)\|^2_{L^2} + \frac{d}{d+2} \|u(t)\|^\frac{2(d+2)}{d} \frac{L}{L-d} \right) \]
\[ \leq \frac{2(p+1)}{d(p-1)} \left( \|\nabla u(t)\|^2_{L^2} + C_d \|u(t)\|^\frac{2}{d} \|\nabla u(t)\|^2_{L^2} \right) \]
\[ \leq \frac{2(p+1)}{d(p-1)} \left( 1 + C_d \left( \frac{2m_{\omega}}{\omega} \right)^\frac{2}{d} \right) \|\nabla u(t)\|^2_{L^2}, \]
where \( C_d \) is some constant depending on \( d \) related to the Sobolev inequality. Thus, we see by (2.21),
\[ \frac{2(p+1)}{d(p-1)} \left( 1 + C_d \left( \frac{2m_{\omega}}{\omega} \right)^\frac{2}{d} \right) \lambda(t)^2 \|\nabla u(t)\|^2_{L^2} \geq \lambda(t)^\frac{2(p-1)}{d} \|u(t)\|^{p+1}_{L^{p+1}} \]
\[ \geq \frac{8(p+1)}{d^2(p-1)^2} \lambda(t)^\frac{2(p-1)}{d} \|\nabla u(t)\|^2_{L^2}, \]
which yields
\[
\lambda(t) \leq \left( \frac{d}{4}(p-1) \left( 1 + C_d \left( \frac{2m_\omega}{\omega} \right)^{\frac{2}{3}} \right) \right)^{\frac{1}{4(p-1)-2}}. \tag{2.24}
\]
Because
\[
d\lambda \left( \frac{1}{X^2} K(T_\lambda u) \right) = -\frac{d(p-1)}{2(p+1)} \left( \frac{d}{2}(p-1) - 2 \right) \lambda^{\frac{d(p-1)-3}{2}} \int |u|^{p+1} \, dx \leq 0,
\]
\[
d\lambda \left( \frac{1}{X^2} \int |T_\lambda u|^{p+1} \, dx \right) = \left( \frac{d}{2}(p-1) - 2 \right) \int |u|^{p+1} \, dx \geq 0,
\]
we have
\[
d\lambda \left( \frac{1}{X^2} \left( K(T_\lambda u) + \frac{(4-d(p-1)d(p-1))}{8(p+1)} \int |T_\lambda u|^{p+1} \, dx \right) \right) \leq 0. \tag{2.25}
\]
Collecting (2.20) and (2.25), we have
\[
\frac{1}{X^2}(K(T_\lambda u(t)) + \frac{(4-d(p-1)d(p-1))}{8(p+1)} \int |T_\lambda u(t)|^{p+1} \, dx) < 0, \text{ for } \lambda \geq 1. \tag{2.26}
\]
Hence, (2.18), (2.23) and (2.26) shows for \(1 \leq \lambda \leq \lambda(t)\),
\[
d^2 \lambda \frac{d}{d\lambda} \mathcal{S}_\omega(T_\lambda u(t))
= -\frac{1}{X^2} K(T_\lambda u(t)) + \frac{2}{X^2} \left( K(T_\lambda u(t)) + \frac{(4-d(p-1)d(p-1))}{8(p+1)} \int |T_\lambda u(t)|^{p+1} \, dx \right)
< -\frac{1}{X^2} K(T_\lambda u(t)) \leq 0. \tag{2.27}
\]
Combining (2.24) and (2.27), we obtain
\[
\left( \left( \frac{d}{4}(p-1) \left( 1 + C_d \left( \frac{2m_\omega}{\omega} \right)^{\frac{2}{3}} \right) \right)^{\frac{1}{4(p-1)-2}} - 1 \right) K(u(t))
\geq (\lambda(t) - 1) \left. \frac{d}{d\lambda} \right|_{\lambda=1} \mathcal{S}_\omega(T_\lambda u(t))
\geq \mathcal{S}_\omega(T_{\lambda(t)} u(t)) - \mathcal{S}_\omega(u(t))
\geq m_\omega - \mathcal{S}_\omega(u(t)).
\]
Thus, there is \(\delta > 0\) depending on \(d, p\) and \(\omega\) that
\[
K(u(t)) \geq \min \left\{ \frac{d(p-1)}{d(p-1)} \left( \| \nabla u(t) \|^2_{L^2} + \frac{d}{d+2} \| u(t) \|^2_{L^{2(d+2)}} \right), \delta \left( m_\omega - \mathcal{S}_\omega(u(t)) \right) \right\}.
\]
\]

\[\square\]

3. Wellposedness and Perturbation Theory

In this section, we present the local wellposedness theory and the perturbation theory for (1.1). We start by recording the wellposedness theory. For the proof we refer to [9, 10, 19].

**Proposition 3.1.**  
(i) (Local existence) Let \(\phi \in H^1(\mathbb{R}^d)\), \(I\) be an interval, \(t_0 \in I\) and \(A > 0\). Assume that
\[
\|\phi\|_{H^1} \leq A.
\]
and there exists $\delta > 0$ depending on $A$ that
\[
\|\langle \nabla \rangle e^{i(t-t_0)\Delta} \phi\|_{L_{t,x}^{\frac{2(d+2)}{d-2}}} (I \times \mathbb{R}^d) \leq \delta,
\]
then there exists a unique solution $u \in C(I, H^1(\mathbb{R}^d))$ to (1.1) such that
\[
u(t_0) = \phi,
\]
\[
\|u\|_{S^1(I)} \lesssim \|\phi\|_{H^1},
\]
\[
\|\langle \nabla \rangle u\|_{L_{t,x}^{\frac{2(d+2)}{d-2}}} (I \times \mathbb{R}^d) \leq 2 \|\langle \nabla \rangle e^{i(t-t_0)\Delta} \phi\|_{L_{t,x}^{\frac{2(d+2)}{d-2}}} (I \times \mathbb{R}^d).
\]
As a consequence, we have the small data global existence: if $\|\phi\|_{H^1}$ is sufficiently small, then $u$ is a global solution with $\|u\|_{S^1(I)} \lesssim \|\phi\|_{H^1}$.

(ii) (Unconditional uniqueness) Suppose $u_1, u_2 \in C(I, H^1(\mathbb{R}^d))$ are two solutions of (1.1) with $u_1(t_0) = u_2(t_0)$ for some $t_0 \in I$, then $u_1 = u_2$.

Let $u \in C((T_{\min}, T_{\max}), H^1(\mathbb{R}^d))$ be the maximal-lifespan solution to (1.1), then we have

(iii) (Conservation laws) For any $t, t_0 \in I_{\max}$,
\[
\mathcal{M}(u(t)) = \mathcal{M}(u(t_0)),
\]
\[
\mathcal{E}(u(t)) = \mathcal{E}(u(t_0)),
\]
\[
\mathcal{S}_\omega(u(t)) = \mathcal{S}_\omega(u(t_0)), \quad \text{for any } \omega > 0,
\]
\[
\mathcal{P}(u(t)) = 3 \int_{\mathbb{R}^d} \nabla u(t,x) \overline{u(t,x)} \, dx = \mathcal{P}(u(t_0)).
\]

(iv) (Blow-up criterion) If $T_{\max} < \infty$, then
\[
\|u\|_{L_{t,x}^{\frac{2(d+2)}{d-2}}(T_{\min}, T_{\max} \times \mathbb{R}^d)} \not\equiv \infty, \quad \forall T_{\min} < T < T_{\max}.
\]

A similar result holds, if $T_{\min} > -\infty$.

(v) (Scattering) If
\[
\|u\|_{L_{t,x}^{\frac{2(d+2)}{d-2}}(T_{\min}, T_{\max} \times \mathbb{R}^d)} < \infty,
\]
then $T_{\max} = \infty$, $T_{\min} = -\infty$, and there exist $u_ \pm \in H^1(\mathbb{R}^d)$ such that
\[
\lim_{t \to \infty} \|u(t) - e^{it\Delta} u_+\|_{H^1} = \lim_{t \to -\infty} \|u(t) - e^{it\Delta} u_-\|_{H^1} = 0.
\]

In the following, we will give the long-time perturbation theory when $d \leq 4$.

**Proposition 3.2** (Long-time perturbation). Let $I$ be a compact time interval and let $w$ be an approximate solution to (1.1) on $I \times \mathbb{R}^d$ in the sense that
\[
i \partial_t w + \Delta w = |w|^{\frac{4}{d}} w - |w|^{p-1} w + e
\]
for some function $e$.

Assume that
\[
\|w\|_{L_t^\infty H_x^1(I \times \mathbb{R}^d)} \leq A_1,
\]
\[
\|w\|_{L_{t,x}^{\frac{2(d+2)}{d-2}}(I \times \mathbb{R}^d)} \leq B
\]
for some $A_1, B > 0$. 


Let $t_0 \in I$ and $u(t_0)$ close to $w(t_0)$ in the sense that
\[ \|u(t_0) - w(t_0)\|_{H^1(I \times \mathbb{R}^d)} \leq A_2 \] (3.5)
for some $A_2 > 0$.

Assume also the smallness conditions
\[ \|\langle \nabla \rangle e^{i(t-t_0)\Delta} (u(t_0) - w(t_0))\|_{L_{t,x}^{2(d+2)}(I \times \mathbb{R}^d)} \leq \delta, \] (3.6)
\[ \|\langle \nabla \rangle e\|_{L_{t,x}^{2(d+2)}(I \times \mathbb{R}^d)} \leq \delta \] (3.7)
for some $0 < \delta \leq \delta_1$, where $\delta_1 = \delta_1(A_1, A_2, B)$ is a small constant. Then there exists a solution $u$ to (1.1) on $I \times \mathbb{R}^d$ with the specified initial data $u(t_0)$ at time $t = t_0$ that satisfies
\[ \|u - w\|_{L_{t,x}^{2(d+2)}(I \times \mathbb{R}^d)} \leq C(A_1, A_2, B)\delta^\alpha, \] (3.8)
\[ \|u - w\|_{S^1(I)} \leq C(A_1, A_2, B)A_2, \] (3.9)
\[ \|u\|_{S^1(I)} \leq C(A_1, A_2, B), \] (3.10)
where $0 < \alpha < \frac{4}{d(p-1)}$.

To show the long-time perturbation theory, we will first give the following short-time perturbation theory.

**Lemma 3.3 (Short-time perturbation).** Let $I$ be a compact time interval and let $w$ be an approximate solution to (1.1) on $I \times \mathbb{R}^d$ in the sense that
\[ i\partial_t w + \Delta w = |w|^\frac{4}{d} w - |w|^{p-1}w + e \]
for some function $e$.

Suppose we also have the energy bound
\[ \|w\|_{L_{t,x}^\infty H^1(I \times \mathbb{R}^d)} \leq A_1 \] (3.11)
for some constant $A_1 > 0$.

Let $t_0 \in I$ and let $u(t_0) \in H^1(\mathbb{R}^d)$ be close to $w(t_0)$ in the sense that
\[ \|u(t_0) - w(t_0)\|_{H^1(I)} \leq A_2 \] (3.12)
for some $A_2 > 0$.

Moreover, assume the smallness conditions
\[ \|\langle \nabla \rangle w\|_{L_{t,x}^{2(d+2)}(I \times \mathbb{R}^d)} + \|w\|_{L_{t,x}^{\frac{(d+2)(p-1)}{2}}(I \times \mathbb{R}^d)} \leq \delta_0, \] (3.13)
\[ \|\langle \nabla \rangle e^{i(t-t_0)\Delta} (u(t_0) - w(t_0))\|_{L_{t,x}^{2(d+2)}(I \times \mathbb{R}^d)} \leq \delta, \] (3.14)
\[ \|\langle \nabla \rangle e\|_{L_{t,x}^{2(d+2)}(I \times \mathbb{R}^d)} \leq \delta \] (3.15)
for some $0 < \delta \leq \delta_0$, where $\delta_0 = \delta_0(A_1, A_2) > 0$ is a small constant.
Then, there exists a solution \( u \in S^1(I) \) to (1.1) on \( I \times \mathbb{R}^d \) with the specified initial data \( u(t_0) \) at time \( t = t_0 \) that satisfies

\[
\| u - w \|_{L_t^{2d+2} L_x^{d+1}(I \times \mathbb{R}^d)} \lesssim \delta^\alpha, \tag{3.16}
\]
\[
\| u - w \|_{S^1(I)} \lesssim A_2 + \delta^\alpha, \tag{3.17}
\]
\[
\| u \|_{S^1(I)} \lesssim A_1 + A_2, \tag{3.18}
\]
\[
\| \langle \nabla \rangle ((i \partial_t + \Delta)(u - w) + e) \|_{L_t^{2d+2} L_x^{d+1}(I \times \mathbb{R}^d)} \lesssim \delta^\alpha, \tag{3.19}
\]

where \( 0 < \alpha < \frac{4}{d(p-1)} \).

**Proof.** By the wellposedness theory, it suffices to prove (3.16)-(3.19) as a priori estimate, that is, we assume that the solution \( u \) already exists and belongs to \( S^1(I) \). By time symmetry, we may assume \( t = t_0 \).

Let \( v = u - w \), then \( v \) satisfies

\[
i \partial_t v + \Delta v = |w + v|^\frac{4}{d} (w + v) - |w + v|^{p-1}(w + v) - |w|^{\frac{4}{d}}w + |w|^{p-1} - e,
\]

and \( v(t_0) = u(t_0) - w(t_0) \).

For \( T \in I \), define

\[
S(T) = \| \langle \nabla \rangle ((i \partial_t + \Delta)v + e) \|_{L_t^{2d+2} L_x^{d+1}([t_0, T] \times \mathbb{R}^d)}.
\]

We will now work entirely on the slab \([t_0, T] \times \mathbb{R}^d\).

\[
\| \langle \nabla \rangle v \|_{L_t^{2d+2} L_x^{d+1}} + \| v \|_{L_t^{2d+2} L_x^{d+1}} \lesssim S(T) + \delta^\alpha, \quad (0 < \alpha < \frac{4}{d(p-1)} < 1)
\]

(3.20)

where we use the fact

\[
\| e^{i(t-t_0)\Delta} v(t_0) \|_{L_t^{2d+2} L_x^{d+1}} \lesssim S(T) + \delta^\alpha
\]

On the other hand, since

\[
|\nabla((i \partial_t + \Delta)v + e)| \lesssim |\nabla w| |v|^{\frac{4}{d}} + |\nabla v| |w + v|^{\frac{4}{d}} + |\nabla w| |v| |w|^{\frac{4}{d} - 1} + |\nabla w| |v|^{p-1} + |\nabla v| |w + v|^{p-1} + |\nabla w| |v| |w|^{p-2}
\]

and

\[
|(i \partial_t + \Delta)v + e| \lesssim |w| |v|^{\frac{4}{d}} + |w + v|^{\frac{4}{d}}|v| + |w| |v|^{p-1} + |w + v|^{p-1}|v|,
\]
we have
\[
S(T) \lesssim \|\langle \nabla \rangle w\|_{L_{t,x}^{2(d+2) \cap L_{t,x}^{d+2(p-1)}}^d}^2 \|w\|_{L_{t,x}^{2(d+2) \cap L_{t,x}^{d+2(p-1)}}^d}^2 + \|\langle \nabla \rangle v\|_{L_{t,x}^{2(d+2) \cap L_{t,x}^{d+2(p-1)}}^d}^2 \|v\|_{L_{t,x}^{2(d+2) \cap L_{t,x}^{d+2(p-1)}}^d}^2
\]
\[
+ \|\langle \nabla \rangle w\|_{L_{t,x}^{2(d+2) \cap L_{t,x}^{d+2(p-1)}}^d}^2 \|w\|_{L_{t,x}^{2(d+2) \cap L_{t,x}^{d+2(p-1)}}^d}^{p-1} + \|\langle \nabla \rangle v\|_{L_{t,x}^{2(d+2) \cap L_{t,x}^{d+2(p-1)}}^d}^2 \|v\|_{L_{t,x}^{2(d+2) \cap L_{t,x}^{d+2(p-1)}}^d}^{p-1}
\]
\[
\lesssim \delta_0 [(S(T) + \delta^\alpha)^{\frac{4}{d}} + \delta_0^4 + (S(T) + \delta^\alpha)^{p-1} + \delta_0^{p-1}]
\]
\[
+ (\delta_0^4 + \delta_0^{p-1})(S(T) + \delta^\alpha) + (S(T) + \delta^\alpha)^{1+\frac{4}{d}} + (S(T) + \delta^\alpha)^p.
\]

By the continuity argument, we can take \(\delta_0 = \delta_0(A_1, A_2)\) sufficiently small, then
\[
S(T) \lesssim \delta^\alpha, \forall T \in I,
\] (3.21)

which implies (3.19). We also have
\[
\|u - w\|_{L_{t,x}^{2(d+2) \cap L_{t,x}^{d+2(p-1)}}^d}^{2(d+2) \cap L_{t,x}^{d+2(p-1)}}^d \lesssim \|e^{i(t-t_0)D}v(t_0)\|_{L_{t,x}^{2(d+2) \cap L_{t,x}^{d+2(p-1)}}^d}^{2(d+2) \cap L_{t,x}^{d+2(p-1)}}^d + \|\langle \nabla \rangle((i\partial_t + \Delta)v + e)\|_{L_{t,x}^{2(d+2) \cap L_{t,x}^{d+2(p-1)}}^d}^{2(d+2) \cap L_{t,x}^{d+2(p-1)}}^d \lesssim \delta^\alpha,
\]
which is (3.16). To obtain (3.17), we see
\[
\|u - w\|_{S^1(I)} \lesssim \|u(t_0) - w(t_0)\|_{H^1} + \|\langle \nabla \rangle((i\partial_t + \Delta)v + e)\|_{L_{t,x}^{2(d+2) \cap L_{t,x}^{d+2(p-1)}}^d}^{2(d+2) \cap L_{t,x}^{d+2(p-1)}}^d \lesssim A_2 + S(t) + \delta \lesssim A_2 + \delta^\alpha.
\]

We now show (3.18). Using Strichartz estimate, (3.11) and (3.13), we get
\[
\|w\|_{S^1(I)} \lesssim \|w(t_0)\|_{H^1} + \|\langle \nabla \rangle(|w|^\frac{4}{d} - |w|^{p-1}w)\|_{L_{t,x}^{2(d+2) \cap L_{t,x}^{d+2(p-1)}}^d}^{2(d+2) \cap L_{t,x}^{d+2(p-1)}}^d + \|\langle \nabla \rangle e\|_{L_{t,x}^{2(d+2) \cap L_{t,x}^{d+2(p-1)}}^d}^{2(d+2) \cap L_{t,x}^{d+2(p-1)}}^d \lesssim A_1 + \|w\|_{L_{t,x}^{2(d+2) \cap L_{t,x}^{d+2(p-1)}}}^{2(d+2) \cap L_{t,x}^{d+2(p-1)}}^d + \delta
\]
\[
\lesssim A_1 + (\delta_0^4 + \delta_0^{p-1}) \|w\|_{S^1(I)} + \delta.
\]

By the continuity argument, we have
\[
\|w\|_{S^1(I)} \lesssim A_1,
\]
provided \(\delta_0\) is sufficiently small depending on \(A_1\). This together with (3.20), (3.21) and \(u = v + w\) yields
\[
\|\langle \nabla \rangle u\|_{L_{t,x}^{2(d+2) \cap L_{t,x}^{d+2(p-1)}}}^{2(d+2) \cap L_{t,x}^{d+2(p-1)}}^d \lesssim A_1.
\]

While, we have by (3.16), (3.20) and (3.21)
\[
\|u\|_{L_{t,x}^{2(d+2) \cap L_{t,x}^{d+2(p-1)}}}^{2(d+2) \cap L_{t,x}^{d+2(p-1)}}^d \lesssim \delta^\alpha.
\]
Combining these with Strichartz estimate, we obtain
\[ \|u\|_{S^1(I)} \lesssim \|u(t_0)\|_{H^1} + \left\| \langle \nabla \rangle (|u|^{\frac{\alpha}{2}} u - |u|^{p-1} u) \right\|_{L^\frac{2(d+2)}{d+2+t}} \]
\[ \lesssim A_1 + A_2 + \|u\|_{L^\infty_t L^\frac{2(d+2)}{d+2}(\mathbb{R}^d)}^\frac{\alpha}{2} \left\| \langle \nabla \rangle u \right\|_{L^\frac{2(d+2)}{d+2}(\mathbb{R}^d)} + \|u\|_{L^\frac{2(d+2)(p-1)}{d+2}(\mathbb{R}^d)} \left\| \langle \nabla \rangle u \right\|_{L^\frac{2(d+2)}{d+2}(\mathbb{R}^d)} \]
\[ \lesssim A_1 + A_2 + \delta_0^\frac{\alpha}{2} A_1 + \delta (p-1) A_1 \lesssim A_1 + A_2, \]
which proves (3.18), provided \( \delta_0 \) is sufficiently small depending on \( A_1 \) and \( A_2 \).

We now show the long-time perturbation theory.

**Proof of Proposition 3.3.** We will derive Proposition 3.2 from Lemma 3.3 by an iterative procedure. First, we will assume without loss of generality that \( t_0 = \inf I \).

Let \( \delta_0 = \delta_0(A_1, 2A_2) \) be as in Lemma 3.3.

The first step is to establish an \( S^1 \) bound on \( w \). In order to do so, we subdivide \( I \) into \( N_0 \sim (1 + M_{\alpha})^{-\frac{d+2}{d}} \) subintervals \( I_k \) such that
\[ \|w\|_{L^\frac{2(d+2)}{d+2}(I_k \times \mathbb{R}^d)} \sim \delta_0. \]

On each subinterval \( I_k \), we have
\[ \|w\|_{S^1(I_k)} \lesssim \|w\|_{L^\infty_t H^1_x} + \left\| \langle \nabla \rangle (|w|^{\frac{\alpha}{2}} w - |w|^{p-1} w) \right\|_{L^\frac{2(d+2)}{d+2+t}(\mathbb{R}^d)} + \left\| \langle \nabla \rangle e \right\|_{L^\frac{2(d+2)}{d+2+t}(I_k \times \mathbb{R}^d)} \]
\[ \lesssim A_1 + \|w\|_{L^\frac{2(d+2)}{d+2}(I_k \times \mathbb{R}^d)} \left\| \langle \nabla \rangle w \right\|_{L^\frac{2(d+2)}{d+2}(I_k \times \mathbb{R}^d)} + \|w\|_{L^\frac{2(d+2)(p-1)}{d+2}(I_k \times \mathbb{R}^d)} \left\| \langle \nabla \rangle w \right\|_{L^\frac{2(d+2)}{d+2}(I_k \times \mathbb{R}^d)} + \delta \]
\[ \lesssim A_1 + (\delta_0^\frac{\alpha}{2} + \delta_0^{p-1}) \|w\|_{S^1(I_k)} + \delta. \]

By the continuity argument, we have \( \|w\|_{S^1(I_k)} \lesssim A_2 \), provided \( \delta_0 \) is sufficiently small depending on \( A_1 \).

Summing these bounds over all the intervals \( I_k \), we obtain \( \|w\|_{S^1(I)} \lesssim C(A_1, B, \delta_0) \), which implies
\[ \left\| \langle \nabla \rangle w \right\|_{L^\frac{2(d+2)}{d+2}(I \times \mathbb{R}^d)} + \|w\|_{L^\frac{2(d+2)(p-1)}{d+2}(I \times \mathbb{R}^d)} \leq C(A_1, B, \delta_0). \]

This allows us to subdivide \( I \) into \( N_1 = C(B, \delta_0) \) subintervals \( J_k = [t_k, t_{k+1}] \) such that
\[ \|w\|_{L^\frac{2(d+2)}{d+2}(J_k \times \mathbb{R}^d)} + \|w\|_{L^\frac{2(d+2)(p-1)}{d+2}(J_k \times \mathbb{R}^d)} \leq \delta_0. \]

Choosing \( \delta_1 = \delta_1(N_1, A_1, A_2) \) sufficiently small, we apply Lemma 3.3 to obtain for each \( k \), and all \( 0 < \delta < \delta_1 \),
\[ \|u - w\|_{L^\frac{2(d+2)}{d+2}(J_k \times \mathbb{R}^d)} \leq C(k) \delta^\alpha, \]
\[ \|u - w\|_{S^1(J_k)} \leq C(k) (A_2 + \delta^\alpha), \]
\[ \|u\|_{S^1(J_k)} \leq C(k) (A_1 + A_2), \]
\[ \left\| \langle \nabla \rangle ((i \partial_t + \Delta)(u - w) + e) \right\|_{L^\frac{2(d+2)}{d+2}(J_k \times \mathbb{R}^d)} \leq C(k) \delta^\alpha, \]
Linear profile decomposition

The linear profile decomposition was first established by H. Bahouri and P. Gérard [4] for the energy critical wave equation in $H^1(\mathbb{R}^d)$. Later, S. Keraani [18] proved the linear profile decomposition for the energy critical Schrödinger equation in $H^1(\mathbb{R}^d)$. At almost the same time, F. Merle and L. Vega [21] gave the linear profile decomposition for the mass critical Schrödinger equation in $L^2(\mathbb{R}^d)$, then R. Carles and S. Keraani [3] established this in $L^2(\mathbb{R})$. In [7], S. Keraani use the $L^2$ profile decomposition to describe the minimal mass blowup solution. Later, P. Bégout and A. Vargas [6] extend these results to $L^2(\mathbb{R}^d)$, $d \geq 3$. In this section, we will give the linear profile decomposition in $H^1(\mathbb{R}^d)$ by using the linear profile decomposition in $L^2(\mathbb{R}^d)$. We first review the linear profile decomposition in $H^1(\mathbb{R}^d)$ in the following.

Lemma 4.1 (Profile decomposition in $L^2(\mathbb{R}^d)$, [6] [3] [21]). Let $\{\phi_n\}_{n \geq 1}$ be a bounded sequence in $L^2(\mathbb{R}^d)$. Then up to passing to a subsequence of $\{\phi_n\}_{n \geq 1}$, there exists a sequence of functions $\phi^j \in L^2(\mathbb{R}^d)$ and $(\theta^j, h^j, t^j, x^j, \xi^j)_{n \geq 1} \subset \mathbb{R}^d \times (0, \infty) \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d$, with

$$
\sum_{j=0}^{k} C(j)\delta^n \lesssim \delta + \sum_{j=0}^{k} C(j)\delta^n.
$$

Here, $C(j)$ depends only on $j, A_1, A_2, \delta_0$.

Choosing $\delta_1$ sufficiently small depending on $N_1, A_1, A_2$, we can continue the inductive argument. This concludes the proof of Proposition 3.2.

4. Linear Profile decomposition

...
where

\[ h_n^j \to h_\infty^j \in \{0, 1, \infty\}, \ h_n^j = 1 \text{ if } h_\infty^j = 1, \tag{4.2} \]

\[ \tau_n^j = -\frac{t_n^j}{(h_n^j)^2} \to \tau_\infty^j \in [-\infty, \infty], \text{ as } n \to \infty, \tag{4.3} \]

\[ \xi_n^j = 0 \text{ if } \limsup_{n \to \infty} |h_n^j \xi_n^j| < \infty. \tag{4.4} \]

such that \( \forall k \geq 1, \) there exists \( r_n^k \in L^2(\mathbb{R}^d), \)

\[ \phi_n(x) = \sum_{j=1}^k T_n^j \phi_j(x) + r_n^k(x), \tag{4.5} \]

here \( T_n^j \) is defined by \( T_n^j \phi(x) = e^{it_n^j} e^{ix \cdot \xi_n^j} e^{-it_n^j \Delta} \left( \frac{1}{(h_n^j)^2} \phi \left( \frac{x - x_n^j}{h_n^j} \right) \right)(x). \) The remainder \( r_n^k \)

satisfies

\[ \limsup_{n \to \infty} \left\| e^{it_n^j} r_n^k \right\|_{L^q_t L^r_x(\mathbb{R} \times \mathbb{R}^d)} \to 0, \text{ as } k \to \infty, \tag{4.6} \]

where \((q, r)\) is \( L^2\)-admissible, and \( 2 < q < \infty \) when \( d \geq 2, \) \( 4 < q < \infty \) when \( d = 1. \) We also have as \( n \to \infty, \)

\[ \left\| e^{it_n^j} T_n^j(\phi_j) e^{it_n^j} T_n^m(\phi^m) \right\|_{L^{\frac{2d}{d-2}}_t L^{\frac{2r}{r-2}}_x(\mathbb{R} \times \mathbb{R}^d)} \to 0, \left\langle T_n^j(\phi_j), T_n^m(\phi^m) \right\rangle_{L^2} \to 0, \text{ for } j \neq m, \tag{4.7} \]

and \( \forall 1 \leq j \leq k, \left\langle T_n^j(\phi_j), r_n^k \right\rangle_{L^2} \to 0, \ (T_n^j)^{-1} r_n^k \to 0 \text{ in } L^2(\mathbb{R}^d). \tag{4.8} \)

As a consequence, we have the mass decoupling property:

\[ \forall k \geq 1, \left\| \phi_n \right\|_{L^2}^2 - \sum_{j=1}^k \left\| \phi_j \right\|_{L^2}^2 - \left\| r_n^k \right\|_{L^2}^2 \to 0. \tag{4.9} \]

**Proof.** We only need to show (4.2), (4.4), (4.6) and (4.8). Other statements in the theorem are stated in the profile decomposition in \( L^2(\mathbb{R}^d) \) proved in [8] [8] [21]. Without loss of generality, we assume that the sequence is up to a subsequence in the following.

To show (4.2), we only need to prove that we may take \( h_n^j \) and \( h_\infty^j \) to be 1 when \( h_\infty^j \in (0, \infty). \) In fact, if \( h_n^j \to h_\infty^j \in (0, \infty), \) as \( n \to \infty, \) we have

\[ e^{it_n^j} e^{ix \cdot \xi_n^j} e^{-it_n^j \Delta} \left( \frac{1}{(h_n^j)^2} \phi_j \left( \frac{x - x_n^j}{h_n^j} \right) \right)(x) \]

\[ = e^{it_n^j} e^{ix \cdot \xi_n^j} e^{-it_n^j \Delta} \left( \frac{1}{(h_\infty^j)^2} \phi_j \left( \frac{x - x_n^j}{h_\infty^j} \right) \right)(x) \]

\[ + e^{it_n^j} e^{ix \cdot \xi_n^j} e^{-it_n^j \Delta} \left( \frac{1}{h_n^j} \phi_j \left( \frac{x - x_n^j}{h_n^j} \right) \right)(x) - e^{it_n^j} e^{ix \cdot \xi_n^j} e^{-it_n^j \Delta} \left( \frac{1}{(h_\infty^j)^2} \phi_j \left( \frac{x - x_n^j}{h_\infty^j} \right) \right)(x). \]

Note that

\[ \left\| e^{it_n^j} e^{ix \cdot \xi_n^j} e^{-it_n^j \Delta} \left( \frac{1}{(h_n^j)^2} \phi_j \left( \frac{x - x_n^j}{h_n^j} \right) \right)(x) - e^{it_n^j} e^{ix \cdot \xi_n^j} e^{-it_n^j \Delta} \left( \frac{1}{(h_\infty^j)^2} \phi_j \left( \frac{x - x_n^j}{h_\infty^j} \right) \right)(x) \right\|_{L^2} \]

\[ = \left\| \phi_j(x) - \left( \frac{h_n^j}{h_\infty^j} \right)^\frac{2}{d} \phi_j \left( \frac{h_n^j}{h_\infty^j} x \right) \right\|_{L^2} \to 0, \text{ as } n \to \infty. \]
So we can put \(e^{it_0} e^{ix \cdot \xi_j} e^{-i t_0 \Delta} \left( \frac{1}{(h_n)^{\frac{d}{2}}} \phi_j \left( \frac{\cdot - \xi_j}{h_n} \right) \right)(x) - e^{it_0} e^{ix \cdot \xi_j} e^{-i t_0 \Delta} \left( \frac{1}{(h_n)^{\frac{d}{2}}} \phi_j \left( \frac{\cdot - \xi_j}{h_n} \right) \right)(x) \) into the remainder term, by the Strichartz estimate. We now shift \(\phi_j(x)\) by \(\frac{1}{(h_n)^{\frac{d}{2}}} \phi_j \left( \frac{x - \xi_j}{h_n} \right)\), and \((\theta_n^j, h_n^j, t_n^j, x_n^j, \xi_n^j)\) by \((\theta_n^j, 1, t_n^j, x_n^j, \xi_n^j)\). It is easy to see that (4.11)–(4.14) are not affected. Thus, we conclude (4.2).

We now show (4.14). If \(h_n^j \xi_n^j \to \xi \in \mathbb{R}^d\), as \(n \to \infty\), for some \(1 \leq j \leq k\). By the Galilean transform

\[ e^{it_0 \Delta} \left( e^{ix \cdot \xi_0} \phi(x) \right) = e^{i(x \cdot \xi_0 - t_0 \cdot |\xi_0|^2)} e^{it_0 \Delta} \phi(x - 2t_0 \xi_0), \]

we have

\[
\begin{align*}
&\frac{1}{(h_n)^{\frac{d}{2}}} e^{i \theta_n^j} e^{i x \cdot \xi_n^j} e^{-i t_n^j \Delta} \left( \frac{1}{(h_n)^{\frac{d}{2}}} \phi_j \left( \frac{\cdot - \xi_j}{h_n} \right) \right) (x) \\
&= \frac{1}{(h_n)^{\frac{d}{2}}} e^{i \theta_n^j} e^{-i t_n^j \Delta} \left( e^{i \xi_n^j \cdot \xi_0} \phi \left( \frac{\cdot - \xi_j}{h_n} \right) \right) (x) - e^{i \theta_n^j} e^{-i t_n^j \Delta} \left( e^{i \xi_n^j \cdot \xi_0} \phi \left( \frac{\cdot - \xi_j}{h_n} \right) \right) (x - 2t_0 \xi_0),
\end{align*}
\]

for the profile decomposition in [6 8 21], the remainder \(r_n^k\) satisfies

\[
\limsup_{n \to \infty} \left\| e^{i \Delta r_n^k} \right\|_{L^2_t L_x^{2(d+2)}} (\mathbb{R} \times \mathbb{R}^d) \to 0, \text{ as } k \to \infty.
\]

By using interpolation, the Strichartz estimate and (4.9), we easily obtain (4.16).

To show (4.8), we see in [6 8 21], we already have

\[
(T_n^j)^{-1} r_n^j \to 0 \text{ in } L^2(\mathbb{R}^d), \text{ as } n \to \infty.
\]

(4.10)

Since \(r_n^k(x) = r_n^j(x) - \sum_{m=j+1}^{k} T_n^m(\phi^m)(x)\), when \(1 \leq j < k\), so by (4.11) and (4.10),

\[
(T_n^j)^{-1} r_n^k = (T_n^j)^{-1} r_n^j - \sum_{m=j+1}^{k} (T_n^j)^{-1} T_n^m \phi^m \to 0 \text{ in } L^2(\mathbb{R}^d), \text{ as } n \to \infty.
\]
We can now show the linear profile decomposition in $H^1(\mathbb{R}^d)$.

**Theorem 4.2.** Let $\{\varphi_n\}$ be a bounded sequence in $H^1(\mathbb{R}^d)$. Then up to passing to a subsequence of $\{\varphi_n\}$, there exists a sequence of functions $\varphi^j \in H^1(\mathbb{R}^d)$ and $(\theta^j_n, t^j_n, x^j_n)_{n \geq 1} \subset \mathbb{R}/2\pi\mathbb{Z} \times \mathbb{R} \times \mathbb{R}^d$, with when $n \to \infty$,

$$
|t^j_n - t^m_n| + |x^j_n - x^m_n| \to \infty, \forall j \neq m, \quad (4.11)
$$

such that for any $k \in \mathbb{N}$, there exists $w^k_n \in H^1(\mathbb{R}^d)$,

$$
e^{it\Delta} \varphi_n = \sum_{j=1}^k e^{it_n^j} e^{i(t-t_n^j)\Delta} \varphi^j(x - x_n^j) + e^{it\Delta} w^k_n. \quad (4.12)
$$

The remainder $w^k_n$ satisfies

$$
\limsup_{n \to \infty} \|\langle \nabla \rangle e^{it\Delta} w^k_n\|_{L^q_t L^r_x(\mathbb{R} \times \mathbb{R}^d)} \to 0, \text{ as } k \to \infty, \quad (4.13)
$$

where $(q, r)$ is $L^2$-admissible, and $2 < q < \infty$ when $d \geq 2$, $4 < q < \infty$ when $d = 1$. Moreover, we have the following decoupling properties: $\forall k \in \mathbb{N}$,

$$
\left\| |\nabla|^s \varphi_n \right\|_{L^2}^2 - \sum_{j=1}^k \left\| |\nabla|^s e^{-it_n^j \Delta} \varphi^j \right\|_{L^2}^2 \to 0, \quad s = 0, 1, \quad (4.14)
$$

$$
\mathcal{E}(\varphi_n) - \sum_{j=1}^k \mathcal{E}(e^{-it_n^j \Delta} \varphi^j) - \mathcal{E}(w^k_n) \to 0, \quad (4.15)
$$

$$
\mathcal{S}_{\omega}(\varphi_n) - \sum_{j=1}^k \mathcal{S}_{\omega}(e^{-it_n^j \Delta} \varphi^j) - \mathcal{S}_{\omega}(w^k_n) \to 0, \quad (4.16)
$$

$$
\mathcal{K}(\varphi_n) - \sum_{j=1}^k \mathcal{K}(e^{-it_n^j \Delta} \varphi^j) - \mathcal{K}(w^k_n) \to 0, \quad (4.17)
$$

$$
\mathcal{H}_{\omega}(\varphi_n) - \sum_{j=1}^k \mathcal{H}_{\omega}(e^{-it_n^j \Delta} \varphi^j) - \mathcal{H}_{\omega}(w^k_n) \to 0, \text{ as } n \to \infty. \quad (4.18)
$$

**Proof.** We divide the proof into four steps.

**Step 1.** Applying Lemma [4.1] to $\{\langle \nabla \rangle \varphi_n\}$, we have

$$
\langle \nabla \rangle \varphi_n(x) = \sum_{j=1}^k e^{it_n^j} e^{i\frac{\xi_n^j}{\epsilon_n^j} \cdot x} e^{-it_n^j \Delta} \left( \frac{1}{(\epsilon_n^j)^{\frac{d}{2}}} \left( \langle \nabla \rangle \varphi^j \left( \frac{-x_n^j}{\epsilon_n^j} \right) \right) \right) (x) + \langle \nabla \rangle w^k_n(x), \quad (4.19)
$$
where
\[ r_n^j = -\frac{h_n^j}{(h_n^j)^2} \to r_\infty^j \in [-\infty, \infty], \]
\[ h_n^j \to h_\infty^j \in \{0, 1, \infty\}, \text{ as } n \to \infty, \text{ and } h_n^j = 1 \text{ if } h_\infty^j = 1, \quad (4.20) \]
\[ \xi_n^j = 0 \text{ if } \limsup_{n \to \infty} |h_n^j \xi_n^j| < \infty, \quad (4.21) \]
\[ \frac{h_n^j}{h_n^m} + \frac{h_n^m}{h_n^j} |\xi_n^j - \xi_n^m| + \frac{|x_n^j - x_n^m|}{(h_n^j)^2} + \frac{2(t_n^j \xi_n^j \xi_n^m)}{h_n^j} \to \infty, \text{ as } n \to \infty, \text{ for } j \neq m, \]
\[ \limsup_{n \to \infty} \|\langle \nabla \rangle e^{it\Delta} w_n^k\|_{L_t^2L_x^q(\mathbb{R} \times \mathbb{R}^d)} \to 0, \text{ as } k \to \infty, \]
where \((q, r)\) is \(L^2\)-admissible and \(2 < q < \infty\) when \(d \geq 2\), \(4 < q < \infty\) when \(d = 1\).

Moreover, we have
\[ \forall \ k \geq 1, \quad \|\langle \nabla \rangle \varphi_n\|_{L^2} - \sum_{j=1}^{k} \|\langle \nabla \rangle \varphi_j\|_{L^2} - \|\langle \nabla \rangle w_n^k\|_{L^2} \to 0, \text{ as } n \to \infty. \]

By (4.19) and the Galilean transform, we have
\[ e^{it\Delta} \varphi_n(x) = \sum_{j=1}^{k} e^{it\Delta} \langle \nabla \rangle^{-1} e^{it\Delta} (\varphi_j^0 + e^{it\Delta} \left( \frac{1}{(h_n^j)^2} \left( \langle \nabla \rangle \varphi_j^0 \left( \frac{-x_n^j}{h_n^j} \right) \right) \right))(x) + e^{it\Delta} w_n^k(x) \]
\[ = \sum_{j=1}^{k} \langle \nabla \rangle^{-1} G_n^j (e^{it\Delta} \langle \nabla \rangle \varphi_j^0)(x) + e^{it\Delta} w_n^k(x), \quad (4.22) \]
where \(G_n^j (e^{it\Delta} \varphi_j)(x) = e^{it\Delta} T_n^j \varphi_j(x)\) and \(T_n^j\) is defined in Lemma 4.1.

By the density of \(C_0^\infty(\mathbb{R}^d)\) in \(H^1(\mathbb{R}^d)\), we can assume that \(\varphi_j^0\) is smooth for all \(j \geq 1\).

**Step 2.** We now show \(h_\infty^j = 1\) and \(\xi_n^j = 0\). By (4.20) and (4.21), we only need to exclude the case \(h_\infty^j = 0, \infty\) and \(\limsup_{n \to \infty} |h_n^j \xi_n^j| = \infty\).

By (4.22), we have
\[ \langle \nabla \rangle^{-1} G_n^j (e^{it\Delta} \langle \nabla \rangle \varphi_j^0) = e^{it\Delta} \left( w_n^{j-1} - w_n^j \right), \]
which implies that
\[ \varphi_j = \langle \nabla \rangle^{-1} e^{-it\Delta} (G_n^j)^{-1} \langle \nabla \rangle e^{it\Delta} \left( w_n^{j-1} - w_n^j \right) \]
\[ = \langle \nabla \rangle^{-1} (T_n^j)^{-1} \langle \nabla \rangle \left( w_n^{j-1} - w_n^j \right). \quad (4.23) \]

We note by (4.8), for \(1 \leq m \leq j\),
\[ (T_n^m)^{-1} \langle \nabla \rangle w_n^j \to 0 \text{ in } L^2(\mathbb{R}^d), \text{ as } n \to \infty, \quad (4.24) \]
so
\[ \langle \nabla \rangle^{-1} (T_n^m)^{-1} \langle \nabla \rangle w_n^j \to 0 \text{ in } H^1(\mathbb{R}^d), \text{ as } n \to \infty. \]

This together with (4.23) yields
\[ \langle \nabla \rangle^{-1} (T_n^j)^{-1} \langle \nabla \rangle w_n^{j-1} \to \varphi_j \text{ in } H^1(\mathbb{R}^d), \text{ as } n \to \infty. \quad (4.25) \]

Meanwhile, since \(\{w_n^{j-1}\}\) is bounded in \(H^1(\mathbb{R}^d)\), there exists a subsequence of \(\{w_n^{j-1}\}\) and \(\psi_j \in H^1(\mathbb{R}^d)\) that
\[ w_n^{j-1} \to \psi_j \text{ in } H^1(\mathbb{R}^d), \text{ as } n \to \infty. \quad (4.26) \]
Combining (4.25), (4.26), we get

\[\langle \nabla \rangle^{-1}(T_n^j)^{-1}\langle \nabla \rangle\psi^j \to \varphi^j \text{ in } H^1(\mathbb{R}^d), \text{ as } n \to \infty.\]

Then by the Rellich-Kondrashov theorem (see [20]), for any ball \(B_K\) centered at the origin with radius \(K\), we have

\[\langle \nabla \rangle^{-1}(T_n^j)^{-1}\langle \nabla \rangle\psi^j \to \varphi^j \text{ in } L^q(B_K), \text{ as } n \to \infty\]

for \(2 \leq q < \infty\) for \(d = 1, 2\) and \(2 \leq q < \frac{2d}{d-2}\) for \(d \geq 3\). So

\[\|\varphi^j\|_{L^q(B_K)} \leq \liminf_{n \to \infty} \|\langle \nabla \rangle^{-1}(T_n^j)^{-1}\langle \nabla \rangle\psi^j\|_{L^q(\mathbb{R}^d)}\]

\[= \liminf_{n \to \infty} (h_n^j)^{\frac{d}{2}} \left\| \left( \frac{1}{(h_n^j)^{\frac{1}{2}}} e^{it_n^j \Delta} e^{-i\langle \cdot,\xi_n^j \rangle} \langle \nabla \rangle\psi^j \right)(h_n^j x) \right\|_{L^q(\mathbb{R}^d)}\]

\[= \liminf_{n \to \infty} (h_n^j)^{\frac{d}{2}} \left( \frac{1}{(h_n^j)^{\frac{1}{2}}} e^{it_n^j \Delta} e^{-i\langle \cdot,\xi_n^j \rangle} \langle \nabla \rangle\psi^j \right)\right\|_{L^q(\mathbb{R}^d)}\]

\[\lesssim \liminf_{n \to \infty} \left\| \frac{|h_n^j|^{\frac{d}{2}}}{(h_n^j)^{\frac{1}{2}}} e^{it_n^j \Delta} e^{-i\langle \cdot,\xi_n^j \rangle} \langle \nabla \rangle\psi^j \right\|_{L^2(\mathbb{R}^d)}\]

\[= \liminf_{n \to \infty} \left\| \frac{|h_n^j|^{\frac{d}{2}}}{(h_n^j)^{\frac{1}{2}}} (\xi + \xi_n^j) \hat{\psi}^j (\xi + \xi_n^j) \right\|_{L^2(\mathbb{R}^d)}. \tag{4.27}

**Case I.** \(|h_n^j\xi_n^j| \to \infty, \text{ as } n \to \infty.\)

When \(h_n^\infty < \infty, \text{ we take } q = 2, \text{ then by the dominated convergence theorem, we have}\)

\[\left\| \frac{1}{(h_n^j)^{\frac{1}{2}}} (\xi + \xi_n^j) \hat{\psi}^j (\xi + \xi_n^j) \right\|_{L^2(\mathbb{R}^d)} = \left\| \frac{1}{(h_n^j)^{\frac{1}{2}}} (\xi - \xi_n^j) \hat{\psi}^j (\xi) \right\|_{L^2(\mathbb{R}^d)} \to 0, \text{ as } n \to \infty.\]

This implies \(\varphi^j = 0 \text{ in } H^1(\mathbb{R}^d).\)

When \(h_n^\infty = \infty, \text{ we take } q = 2, \text{ if } \xi_n^j \to \bar{\xi} \in \mathbb{R}^d, \text{ for any small number } \varepsilon > 0, \text{ we can take } r = r(\varepsilon) > 0 \text{ such that} \)

\[\left\| \frac{1}{(h_n^j)^{\frac{1}{2}}} (\xi - \xi_n^j) \hat{\psi}^j (\xi) \right\|_{L^2(B_r(\bar{\xi}))} \leq \varepsilon \]

by the continuity of the integral. And we use the dominated convergence theorem again to obtain

\[\left\| \frac{1}{(h_n^j)^{\frac{1}{2}}} (\xi - \xi_n^j) \hat{\psi}^j (\xi) \right\|_{L^2(\mathbb{R}^d \setminus B_r(\bar{\xi}))} \to 0.\]

Since \(\varepsilon\) is arbitrary, we have \(\varphi^j = 0 \text{ in } H^1(\mathbb{R}^d).\) If \(|\xi_n^j| \to \infty, \text{ the dominated convergence theorem will guarantee that } \varphi^j = 0 \text{ in } H^1(\mathbb{R}^d).\)

**Case II.** If \(\limsup |h_n^j\xi_n^j| < \infty, \text{ then we obtain } \xi_n^j = 0 \text{ by (4.21).}\)

When \(h_n^\infty = 0, \text{ we can take some } q > 2, \text{ we have} \)

\[\left\| \frac{|h_n^j|^{\frac{d}{2}}}{(h_n^j)^{\frac{1}{2}}} (\xi) \hat{\psi}^j (\xi) \right\|_{L^2(\mathbb{R}^d)} \to 0, \text{ as } n \to \infty,\]

so \(\|\varphi^j\|_{L^q(\mathbb{R}^d)} = 0 \text{ for any } K > 0. \text{ Therefore, } \|\varphi^j\|_{L^q(\mathbb{R}^d)} = 0. \text{ Since } \varphi^j \text{ is smooth, } \varphi^j = 0 \text{ in } H^1(\mathbb{R}^d).\)

When \(h_n^\infty = \infty, \text{ we take } q = 2. \text{ Then by the dominated convergence theorem, we have} \)

\[\left\| \frac{1}{(h_n^j)^{\frac{1}{2}}} (\xi) \hat{\psi}^j (\xi) \right\|_{L^2(\mathbb{R}^d)} \to 0, \text{ as } n \to \infty,\]
which implies $\varphi^j = 0$ in $H^1(\mathbb{R}^d)$.

Combining the above facts, we conclude that $h^j_\infty = 1$ and $\xi^j_n = 0$. This together with (4.22) implies (4.11) and (4.12).

**Step 3.** We now turn to (4.14). By (4.12), we have

$$\varphi_n(x) = \sum_{j=1}^k e^{i\theta_x} e^{-it_n^j \Delta} \varphi^j(x - x^j_n) + w_n^k(x),$$

so we only need to show the orthogonality:

$$\left\langle \mu(\nabla) (e^{i\theta_x} e^{-it_n \Delta} \varphi^j(x - x^j_n)), \mu(\nabla) (e^{i\theta_x} e^{-it_n^m \Delta} \varphi^m(x - x^m_n)) \right\rangle_{L^2} \to 0, \forall j \neq m,$$

$$\left\langle \mu(\nabla) (e^{i\theta_x} e^{-it_n \Delta} \varphi^j(x - x^j_n)), \mu(\nabla) w_n^k(x) \right\rangle_{L^2} \to 0, \text{ for } 1 \leq j \leq k, \text{ as } n \to \infty,$$

for $\mu = \frac{1}{\langle \varphi \rangle_j}$ and $\mu = \frac{\langle \nabla \varphi \rangle_j}{\langle \varphi \rangle_j}$, respectively. This follows from

$$\left\langle \mu(\nabla) (e^{i\theta_x} e^{-it_n \Delta} \varphi^j(x - x^j_n)), \mu(\nabla) (e^{i\theta_x} e^{-it_n^m \Delta} \varphi^m(x - x^m_n)) \right\rangle_{L^2} = 0,$$

where

$$S_{n,j}^m \varphi(x) = e^{i(\theta_x - \theta_x^m)} e^{-it_n \Delta} \varphi(x - (x^j_n - x^m_n)) \to 0, \text{ as } n \to \infty, \forall \varphi \in L^2,$$

by (4.11) and

$$\left\langle \mu(\nabla) (e^{i\theta_x} e^{-it_n \Delta} \varphi^j(x - x^j_n)), \mu(\nabla) w_n^k(x) \right\rangle_{L^2} = \left\langle \mu(\nabla) (e^{i\theta_x} e^{-it_n \Delta} \varphi^j(x - x^j_n)), \mu(\nabla) w_n^k(x) \right\rangle_{L^2} \to 0, \text{ as } n \to \infty, \forall 1 \leq j \leq k, \text{ by } (4.24).$$

**Step 4.** Since we already have (4.14), to obtain (4.15), (4.16), (4.17), (4.18), it suffices to show

$$(4.28) \quad \|\varphi_n\|_{L^{p+1}}^{p+1} - \sum_{j=1}^k e^{-it_n^j \Delta} \varphi^j \|w_n^{p+1} \|_{L^{p+1}} \to 0, \text{ as } n \to \infty, \text{ for } 1 + \frac{4}{d} \leq p \leq 1 + \frac{4}{d - 2}.$$

Suppose that $t^1_1 \in \mathbb{R}$, then the refined Fatou Lemma(see [18] [20]) shows that

$$(4.29) \quad \lim_{n \to \infty} \|\varphi_n\|_{L^{p+1}}^{p+1} - \left\| e^{i\theta_x} e^{-it_n^1 \Delta} \varphi^1 \right\|_{L^{p+1}}^{p+1} = 0.$$

Next, suppose that $t^1_1 = \pm \infty$, then by the dispersive estimate, we obtain

$$\lim_{n \to \infty} \|\varphi_n\|_{L^{p+1}}^{p+1} - \left\| e^{i\theta_x} e^{-it_n^1 \Delta} \varphi^1 \right\|_{L^{p+1}}^{p+1} \leq \lim_{n \to \infty} \|\varphi_n\|_{L^{p+1}}^{p+1} - \|w_n^{p+1}\|_{L^{p+1}}^{p+1} + \lim_{n \to \infty} \left\| e^{-it_n^1 \Delta} \varphi^1 \right\|_{L^{p+1}}^{p+1} \leq \lim_{n \to \infty} \left( \|\varphi_n\|_{L^{p+1}}^{p+1} + \|w_n^{p+1}\|_{L^{p+1}}^{p+1} \right) \left\| e^{-it_n^1 \Delta} \varphi^1 \right\|_{L^{p+1}}^{p+1} \leq \lim_{n \to \infty} \left( \|\varphi_n\|_{H^{1}}^{p+1} + \|w_n^{p+1}\|_{H^{1}}^{p+1} \right) \left\| e^{-it_n^1 \Delta} \varphi^1 \right\|_{L^{p+1}}^{p+1} = 0.$$
Thus, we have proved
\[ \| \varphi_n \|_{L^{p+1}}^{p+1} - \left\| e^{-it_n \Delta} \varphi^1 \right\|_{L^{p+1}}^{p+1} - \| u_n^1 \|_{L^{p+1}}^{p+1} \to 0, \text{ as } n \to \infty. \] (4.30)
Similarly, we can show
\[ \| u_n^1 \|_{L^{p+1}}^{p+1} - \left\| e^{-it_n \Delta} \varphi^2 \right\|_{L^{p+1}}^{p+1} - \| u_n^2 \|_{L^{p+1}}^{p+1} \to 0, \text{ as } n \to \infty, \] (4.31)
which together with (4.30) shows (4.28) when \( k = 2 \). Repeating this procedure, we obtain (1.28) for any \( k \geq 1 \).

5. Extraction of a critical element

In this section, we show the existence of the critical element in the general case by using the profile decomposition and the long-time perturbation theory.

By Proposition 3.1(v), it suffices for Theorem 1.3 to show that any solution \( u \) to (1.1) with \( u_0 \in A_{\omega,+} \) satisfies
\[ \| u \|_{L_t^2 L_x^d} \left\| \int_0^1 \Delta \right\|_{L_t^1 L_x^{\infty}} \leq \infty, \]
where \( I_{\text{max}} \) denotes the maximal interval where \( u \) exists.

To this end, for \( m > 0 \), let
\[ \Lambda_\omega(m) = \sup \left\{ \| u \|_{K(I_{\text{max}})} : u \text{ is a solution to (1.1) with } u_0 \in A_{\omega,+} \text{ and } S_\omega(u) \leq m \right\} \] (5.1)
with \( \| u \|_{K(I)} := \| u \|_{L_t^{2(d+2)} \cap L_x^{(d+2)(p-1)}/(I \times \mathbb{R}^d)} \)
and define
\[ m^*_\omega = \sup \{ m > 0 : \Lambda_\omega(m) < \infty \}. \] (5.2)
If \( u_0 \in A_{\omega,+} \) with \( S_\omega(u_0) \leq m \) sufficiently small, then Lemma 2.8 shows \( \| u_0 \|_{H^1} \ll 1 \). Hence, Proposition 3.1(ii) gives the finiteness of \( \Lambda_\omega(m) \), which implies \( m^*_\omega > 0 \).

Now our aim is to show \( m^*_\omega \geq m_\omega \) defined by (1.7). Suppose by contradiction that \( m^*_\omega < m_\omega \), we will show the existence of the critical element. In fact, by the definition of \( m^*_\omega \), we can take a sequence \( \{ u_n \} \) of solutions (up to time translations) to (1.1) that
\[ u_n(t) \in A_{\omega,+}, \text{ for } t \in I_n, \text{ and } S_\omega(u_n) \to m^*_\omega, \text{ as } n \to \infty, \] (5.3)
\[ \lim_{n \to \infty} \| u_n \|_{K([0,\sup I_n])} = \lim_{n \to \infty} \| u_n \|_{K([\inf I_n,0])} = \infty, \] (5.4)
where \( I_n \) denotes the maximal interval of \( u_n \) including 0.

By Lemma 2.8,
\[ \sup_n \| u_n \|_{H^1(I_n \times \mathbb{R}^d)} \lesssim m_\omega + \frac{m_\omega}{\omega}. \] (5.5)
Applying Theorem 4.2 to \( \{ u_n(0,x) \} \) and obtain some subsequence of \( \{ u_n(0,x) \} \) (still denoted by the same symbol), then there exists \( \varphi^j \in H^1(\mathbb{R}^d) \) and \( (\theta^j_n, t^j_n, x^j_n)_{n \geq 1} \) of sequences in \( \mathbb{R}/2\pi \mathbb{Z} \times \mathbb{R} \times \mathbb{R}^d \), with
\[ t^j_n \to t^j_\infty \in [-\infty, \infty], \]
\[ |t^j_n - t^j_m| + |x^j_n - x^j_m| \to \infty, \quad \forall j \neq m, \text{ as } n \to \infty, \] (5.6)
such that, \( \forall k \in \mathbb{N} \), there exists \( w_n^k \in \mathcal{H}_1^1(\mathbb{R}^d) \),
\[
e^{it\Delta}u_n(0, x) = \sum_{j=1}^{\infty} e^{i\theta_n^j} e^{i(t-t_n^j)\Delta} \varphi_n^j(x - x_n^j) + e^{it\Delta}w_n^k(x). \tag{5.7}
\]
The remainder \( w_n^k \) satisfies
\[
\lim_{n \to \infty} \sup_n \| \langle \nabla \rangle e^{it\Delta}w_n^k \|_{L_t^q L_x^r(\mathbb{R} \times \mathbb{R}^d)} \to 0, \quad \text{as } k \to \infty, \tag{5.8}
\]
where \((q, r)\) is \( L^2 \)-admissible, \(2 < q < \infty\) when \( d \geq 2\) and \( 4 < q < \infty\) when \( d = 1\). Moreover, for any \( k \in \mathbb{N}, s = 0, 1\), we have
\[
\| |\nabla|^s u_n(0) \|^2_{L_2} - \sum_{j=1}^{\infty} \| |\nabla|^s e^{-it\Delta} \varphi_n^j \|^2_{L_2} - \| |\nabla|^s w_n^k \|^2_{L_2} \to 0, \tag{5.9}
\]
\[
\mathcal{E}(u_n(0)) - \sum_{j=1}^{\infty} \mathcal{E}(e^{-it\Delta} \varphi_n^j) - \mathcal{E}(w_n^k) \to 0, \tag{5.10}
\]
\[
\mathcal{S}_\omega(u_n(0)) - \sum_{j=1}^{\infty} \mathcal{S}_\omega(e^{-it\Delta} \varphi_n^j) - \mathcal{S}_\omega(w_n^k) \to 0, \tag{5.11}
\]
\[
\mathcal{K}(u_n(0)) - \sum_{j=1}^{\infty} \mathcal{K}(e^{-it\Delta} \varphi_n^j) - \mathcal{K}(w_n^k) \to 0, \tag{5.12}
\]
\[
\mathcal{H}_\omega(u_n(0)) - \sum_{j=1}^{\infty} \mathcal{H}_\omega(e^{-it\Delta} \varphi_n^j) - \mathcal{H}_\omega(w_n^k) \to 0, \quad \text{as } n \to \infty. \tag{5.13}
\]
Using Strichartz estimate, (5.9) and (5.5), we get
\[
\sup_{k \in \mathbb{N}} \sup_{n \to \infty} \| e^{it\Delta}u_n^k \|_{S^1(\mathbb{R})} \lesssim \sup_{k \in \mathbb{N}} \sup_{n \to \infty} \| w_n^k \|_{\mathcal{H}_1^1} < \infty. \tag{5.14}
\]
Next, we construct the nonlinear profile. We define the nonlinear profile \( w^j \in C((T^j_{\min}, T^j_{\max}), \mathcal{H}_1^1(\mathbb{R}^d)) \) to be the maximal lifespan solution of \( i\partial_t u + \Delta u = |u|^q u - |u|^{p-1} u \) such that
\[
\| w^j(-t_n^j) - e^{-it\Delta} \varphi_n^j \|_{\mathcal{H}_1^1} \to 0, \quad \text{as } n \to \infty. \tag{5.15}
\]
The unique existence of \( w^j \) around \( t = -t_n^j \) is known in all cases, including \( t^j_{\infty} = \pm \infty \) (the latter corresponding to the existence of the wave operators), by using the standard iteration with the Strichartz estimate.

Let
\[
u_n^j = e^{i\theta_n^j} w^j(t - t_n^j, x - x_n^j), \tag{5.16}
\]
then, the lifespan of \( w_n^j \) is \((T_{\min}^j + t_n^j, T_{\max}^j + t_n^j)\).

For the linear profile decomposition (5.7), we can give the corresponding nonlinear profile decomposition
\[
u_n^{\leq k}(t) = \sum_{j=1}^{k} \nu_n^j(t). \tag{5.17}
\]
Lemma 5.2. In the nonlinear profile decomposition when $j$ data global wellposedness and scattering theory together with (5.15), we obtain which shows 

Proof. By (5.5) and (5.9), we have

There exists Lemma 5.1. existence of the critical element. nonlinear profile has finite global Strichartz norm, which is the key to show the

Proof. By Proposition 3.1(5.20), we have

we have

We will show

T

T

large enough, we have the following basic fact about $u^j$:

Lemma 5.1. There exists $j_0 \in \mathbb{N}$ such that $T_{j_{\text{min}}}^j = -\infty$, $T_{j_{\text{max}}}^j = \infty$ for $j > j_0$ and

\[
\sum_{j > j_0} \|u^j\|^2_{S^1(\mathbb{R})} \lesssim \sum_{j > j_0} \|\varphi^j\|^2_{H^1} < \infty. \tag{5.18}
\]

Proof. By (5.5) and (5.9), we have

which shows $\sum_{j = 1}^{\infty} \|\varphi^j\|^2_{H^1} < \infty$, and therefore $\|\varphi^j\|_{H^1} \to 0$, as $j \to \infty$. By the small

data global wellposedness and scattering theory together with (5.15), we obtain when $j$ large enough, $\|u^j\|_{S^1(\mathbb{R})} \lesssim \|\varphi^j\|_{H^1}$ and so we have the desired result. \hfill \Box

Lemma 5.2. In the nonlinear profile decomposition (5.17), if

\[
\|u^j\|_{L_t^\infty L_x^{2(d+2)/4} \cap L_t^2 \cap L_x^{2(d+2)/(d+2)(p-1)}((T_{j_{\text{min}}}^j, T_{j_{\text{max}}}^j) \times \mathbb{R}^d)} < \infty \text{ for } 1 \leq j \leq k, \tag{5.19}
\]

then, we have $T_{j_{\text{min}}}^j = -\infty$, $T_{j_{\text{max}}}^j = \infty$, and

\[
\|u_n^j\|_{S^1(\mathbb{R})} = \|u^j\|_{S^1(\mathbb{R})} \lesssim 1, \tag{5.20}
\]

for $1 \leq j \leq k$, and there exists $B > 0$ such that

\[
\limsup_{n \to \infty} \left( \|u_n^j\|_{L_t^\infty L_x^{2(d+2)/(d+2)(p-1)}((\mathbb{R} \times \mathbb{R}^d))} + \|\nabla\|u_n^j\|_{L_t^{\infty} L_x^2 \cap L_t^2 \cap L_x^{2(d+2)/(d+2)(p-1)}(\mathbb{R} \times \mathbb{R}^d)} \right) \leq B. \tag{5.21}
\]

Proof. By Proposition 3.1(v) and (5.19), we have $T_{j_{\text{min}}}^j = -\infty$, $T_{j_{\text{max}}}^j = \infty$.

Using the Strichartz estimate and (5.19), we get

\[
\|u^j\|_{S^1(\mathbb{R})} \lesssim 1, \text{ for } 1 \leq j \leq k,
\]

which implies (5.20).

We now turn to (5.21). By

\[
\left| \sum_{1 \leq j \leq k} u_n^j \right|^q - \sum_{1 \leq j \leq k} \left| u_n^j \right|^q \leq C_{k,q} \sum_{1 \leq j < m \leq k} \left| u_n^j \right|^{q-1} |u_n^m|, \quad 1 < q < \infty,
\]

we have

\[
\left\| \sum_{1 \leq j \leq k} u_n^j \right\|_{L_t^\infty L_x^{2(d+2)/(d+2)(p-1)}(\mathbb{R} \times \mathbb{R}^d)} \leq \sum_{1 \leq j \leq k} \left\| u_n^j \right\|_{L_t^\infty L_x^{2(d+2)/(d+2)(p-1)}(\mathbb{R} \times \mathbb{R}^d)} \quad \text{and}
\]

\[
+ C_k \sum_{1 \leq j \neq m \leq k} \int_{\mathbb{R} \times \mathbb{R}^d} \left| u_n^j \right|^{2(d+2)/(d+2)(p-1)} \left| u_n^m \right| \, dxdt. \tag{5.22}
\]
We see by (5.20) and Lemma 5.1 that
\[
\sum_{1 \leq j \leq k} \left\| u_j^i \right\|_{L_{t,x}^{(d+2)(p-1)/2}} \lesssim \sum_{1 \leq j \leq j_0} \left\| u_j^i \right\|_{S^1(\mathbb{R})} + \left( \sum_{j > j_0} \left\| \varphi_j^i \right\|_{H^1}^{2(d+2)(p-1)} \right)^{1/(2(d+2)(p-1))} < \infty.
\] (5.23)

Next, we consider the second term on the right side of (5.22). By the Hölder inequality, (5.6) and (5.20), we have
\[
\int \int_{\mathbb{R} \times \mathbb{R}^d} |u_j^i|^{(d+2)(p-1)/2-1} |u_n^m| \, dx \, dt \lesssim \left\| u_j^i u_n^m \right\|_{L_{t,x}^{(d+2)(p-1)}/2} \lesssim \left\| u_j^i u_n^m \right\|_{L_{t,x}^{(d+2)(p-1)-2}} \lesssim \left\| u_j^i u_n^m \right\|_{L_{t,x}^{(d+2)(p-1)-2}} \to 0, \text{ as } n \to \infty.
\] (5.24)

Plugging (5.23) and (5.24) into (5.22), we obtain that there is a sufficiently large, then we can use the long time perturbation theory to give a contradiction.

We argue by contradiction. Assume
\[
\limsup_{n \to \infty} \left\| u_n^k \right\|_{L_{t,x}^{(d+2)(p-1)/2}} \leq B_1.
\]

Similarly, we have
\[
\limsup_{n \to \infty} \left\| (\nabla) u_n^k \right\|_{L_{t,x}^{(d+2)(p-1)/2}} \leq B_1
\]
and
\[
\limsup_{n \to \infty} \left\| (\nabla) u_n^k \right\|_{L_t^\infty L^2_x(\mathbb{R} \times \mathbb{R}^d)} \leq B_1.
\]

Thus, we obtain (5.21).

\[\square\]

**Lemma 5.3** (At least one bad profile). Let \( j_0 \) be as in Lemma 5.1, then there exists \( 1 \leq j \leq j_0 \) such that
\[
\left\| u_j^i \right\|_{L_{t,x}^{2(d+2)}/2 \cap L_{t,x}^{(d+2)(p-1)}/2}^{(T_{\min}^j, T_{\max}^j) \times \mathbb{R}^d} < \infty, \quad \forall \ 1 \leq j \leq j_0.
\]

**Proof.** We argue by contradiction. Assume
\[
\left\| u_j^i \right\|_{L_{t,x}^{2(d+2)}/2 \cap L_{t,x}^{(d+2)(p-1)}/2}^{(T_{\min}^j, T_{\max}^j) \times \mathbb{R}^d} < \infty, \quad \forall \ 1 \leq j \leq j_0.
\]

Combining this with Lemma 5.1, we have
\[
\left\| u_j^i \right\|_{L_{t,x}^{2(d+2)}/2 \cap L_{t,x}^{(d+2)(p-1)}/2}^{(T_{\min}^j, T_{\max}^j) \times \mathbb{R}^d} < \infty, \quad \forall \ 1 \leq j \leq j_0.
\]

This together with Lemma 5.2 implies \( u_j^i \) exists globally in time for \( j \geq 1 \) and hence so does \( u_n^k(t) + e^{it\Delta} u_n^k \).

We now verify \( u_n^k(t) + e^{it\Delta} u_n^k \) is an approximate solution to \( u_n \) when \( n \) and \( k \) large enough, then we can use the long time perturbation theory to give a contradiction.

We see from Lemma 5.2 and (5.14) that there exists \( A_1, B > 0 \) such that
\[
\limsup_{n \to \infty} \left\| (\nabla)(u_n^k(t) + e^{it\Delta} u_n^k) \right\|_{L_t^\infty L^2_x (\mathbb{R} \times \mathbb{R}^d)} \leq A_1,
\] (5.26)
\[
\limsup_{n \to \infty} \left\| u_n^k(t) + e^{it\Delta} u_n^k \right\|_{L_{t,x}^{2(d+2)/2} \cap L_{t,x}^{(d+2)(p-1)/2} \cap L_{t,x}^{2(d+2)/2}} \leq B.
\] (5.27)
Moreover, it follows from (5.7) with $t = 0$ and (5.15) that
\[
\left\| u_n(0) - u_n^{<k}(0) - w_n^k \right\|_{H^1} = \left\| \sum_{j=1}^{k} e^{i\theta_n^j} e^{-i\theta_n^j \Delta} \varphi^j (x - x_n^j) - \sum_{j=1}^{k} e^{i\theta_n^j} \varphi^j (-t_n^j, x - x_n^j) \right\|_{H^1} \\
\leq \sum_{j=1}^{k} \left\| e^{-i\theta_n^j \Delta} \varphi^j - \varphi^j (-t_n^j) \right\|_{H^1} \rightarrow 0, \quad \text{as } n \rightarrow \infty.
\]

Hence,
\[
\left\| u_n(0) - u_n^{<k}(0) - w_n^k \right\|_{H^1} \rightarrow 0, \quad \text{as } n \rightarrow \infty. \tag{5.28}
\]

Next, we claim that as $k \rightarrow \infty$
\[
\lim_{n \rightarrow \infty} \left\| \langle \nabla \rangle \left[ (i\partial_t + \Delta)(u_n^{<k}(t) + e^{it\Delta} w_n^k) - \mathcal{N}(u_n^{<k}(t) + e^{it\Delta} w_n^k) \right] \right\| \frac{2(d+2)}{L_{t,x}} \rightarrow 0, \quad \text{as } k \rightarrow \infty, \tag{5.29}
\]

where $\mathcal{N}(u) = |u|^\frac{4}{d} u - |u|^{p-1} u$.

Before proving this claim, we remark that (5.29) together with the long-time perturbation theory leads to a contradiction. Indeed, by (5.26), (5.27), (5.28) and (5.29), we conclude as a consequence of Proposition 3.2 that
\[
\lim_{n \rightarrow \infty} \left\| u_n \right\| \frac{2(d+2)}{L_{t,x}^{\frac{d+1}{2}}} < \infty
\]
when $n$ large enough, which contradicts (5.24). Hence, Lemma 5.3 holds.

It remains to prove the claim (5.29). Note that
\[
(i\partial_t + \Delta)(u_n^{<k}(t) + e^{it\Delta} w_n^k) - \mathcal{N}(u_n^{<k}(t) + e^{it\Delta} w_n^k) \\
= \sum_{j=1}^{k} (i\partial_t + \Delta) u_n^j - \mathcal{N}(u_n^{<k}(t)) - \mathcal{N}(u_n^{<k}(t) + e^{it\Delta} w_n^k) + \mathcal{N}(u_n^{<k}(t)).
\]

Hence, it suffices to show that
\[
\lim_{n \rightarrow \infty} \left\| \langle \nabla \rangle \left( \sum_{j=1}^{k} (i\partial_t + \Delta) u_n^j - \mathcal{N}(u_n^{<k}(t)) \right) \right\| \frac{2(d+2)}{L_{t,x}^{d+4}} = 0, \quad \forall \ k \in \mathbb{N}, \tag{5.30}
\]

and
\[
\lim_{n \rightarrow \infty} \left\| \langle \nabla \rangle (\mathcal{N}(u_n^{<k}(t) + e^{it\Delta} w_n^k) - \mathcal{N}(u_n^{<k}(t))) \right\| \frac{2(d+2)}{L_{t,x}^{d+4}} \rightarrow 0, \quad \text{as } k \rightarrow \infty. \tag{5.31}
\]

First, we show (5.30). Noting that
\[
(i\partial_t + \Delta) u_n^j = \mathcal{N}(u_n^j),
\]
and using (5.6) and Lemma 5.2 we get as $n \rightarrow \infty$
\[
\left\| \langle \nabla \rangle \left( \sum_{j=1}^{k} (i\partial_t + \Delta) u_n^j - \mathcal{N}(u_n^{<k}) \right) \right\| \frac{2(d+2)}{L_{t,x}^{d+4}} = \left\| \langle \nabla \rangle \left( \sum_{j=1}^{k} \mathcal{N}(u_n^j) - \mathcal{N}(\sum_{j=1}^{k} u_n^j) \right) \right\| \frac{2(d+2)}{L_{t,x}^{d+4}} \rightarrow 0,
\]

and (5.30) follows.
We now turn to \( \text{(5.31)} \). By the Fundamental Theorem of Calculus,

\[
F(u) - F(v) = (u-v) \int_0^1 F_z(v+\theta(u-v))d\theta + (u-v) \int_0^1 F_z(v+\theta(u-v))d\theta,
\]
we obtain

\[
\| \langle \nabla \rangle (N(u_n^{<k} + e^{it\Delta} u_n^k) - N(u_n^{<k})) \|_{L_t W^{2(d+2)/d, 1}}  
\leq \| u_n^{<k} \|_{L_t W^{2(d+2)/d, 1}}^{p-1} \| e^{it\Delta} u_n^k \|_{L_t W^{2(d+2)/d, 1}}^{\frac{d+4}{2(d+2)}} + \| e^{it\Delta} u_n^k \|_{L_t W^{2(d+2)/d, 1}}^{\frac{4}{d+4}} \| e^{it\Delta} u_n^k \|_{L_t L_x}^{\frac{d+4}{2(d+2)}} - \| e^{it\Delta} u_n^k \|_{L_t W^{2(d+2)/d, 1}}^{\frac{d+4}{2(d+2)}} + \| e^{it\Delta} u_n^k \|_{L_t L_x}^{\frac{d+4}{2(d+2)}}. 
\]

(5.33)

For the terms \( \text{(5.34), (5.37)} \), By the Hölder inequality and \( \text{(5.8)} \), we have

\[
\text{(5.34)} \quad \leq \| \langle \nabla \rangle e^{it\Delta} u_n^k \|_{L_t^{d+4}/d, 1}^{p} \| e^{it\Delta} u_n^k \|_{L_t W^{2(d+2)/d, 1}}^{\frac{d+4}{2(d+2)}} + \| e^{it\Delta} u_n^k \|_{L_t L_x}^{\frac{d+4}{2(d+2)}}. 
\]

(5.35)

\[
\text{(5.37)} \quad \leq \left[ \| \nabla \|_{L_t^{d+4}/d, 1} \| e^{it\Delta} u_n^k \|_{L_t W^{2(d+2)/d, 1}}^{p-1} \right. 
\left. + \| e^{it\Delta} u_n^k \|_{L_t L_x}^{\frac{d+4}{2(d+2)}} \right] \cdot \| \nabla e^{it\Delta} u_n^k \|_{L_t L_x}^{\frac{d+4}{2(d+2)}}. 
\]

(5.36)

\[
\text{(5.37)} \quad \leq \| \nabla \|_{L_t^{d+4}/d, 1} \| e^{it\Delta} u_n^k \|_{L_t W^{2(d+2)/d, 1}}^{p-1} + \| e^{it\Delta} u_n^k \|_{L_t L_x}^{\frac{d+4}{2(d+2)}} \right] \cdot \| \nabla e^{it\Delta} u_n^k \|_{L_t L_x}^{\frac{d+4}{2(d+2)}}. 
\]

(5.38)

We now consider the terms of the form

\[
\| u_n^{<k} \|_{L_t W^{2(d+2)/d, 1}}^{p-1} \| e^{it\Delta} u_n^k \|_{L_t L_x}^{\frac{d+4}{2(d+2)}} + \| e^{it\Delta} u_n^k \|_{L_t L_x}^{\frac{d+4}{2(d+2)}}. 
\]

which corresponds to \( \text{(5.33), (5.35)} \).
By the Hölder inequality, (5.8), (5.14), (5.21), we have
\[ \left\| u_n^k \right\|_{L_{t,x}^{2(d+2)}}^{2(d+2)} \lesssim \left\| u_n^k \right\|_{L_{t,x}^{2(d+2)}}^{2(d+2)} \]
\[ \lesssim \left\| u_n^k \right\|_{L_{t,x}^{2(d+2)}}^{2(d+2)} \left\| \nabla^s e^{it\Delta} w_n^k \right\|_{L_{t,x}^{2(d+2)}}^{2(d+2)} \]
\[ \lesssim \left\| u_n^k \right\|_{L_{t,x}^{2(d+2)}}^{2(d+2)} \left\| \nabla^s e^{it\Delta} w_n^k \right\|_{L_{t,x}^{2(d+2)}}^{2(d+2)} \to 0, \text{ as } n \to \infty, \ k \to \infty. \]

Thus, we obtain for \( 1 + \frac{4}{d} \leq q \leq 1 + \frac{4}{d-2}, s = 0, 1, \)
\[ \lim_{n \to \infty} \left\| u_n^k \right\|_{L_{t,x}^{2(d+2)}}^{2(d+2)} \to 0, \text{ as } k \to \infty. \]

We now estimate (5.38). For \( q > 2, \) we have
\[ \left\| u_n^k \right\|_{L_{t,x}^{2(d+2)}}^{2(d+2)} \leq \left\| \nabla^s e^{it\Delta} w_n^k \right\|_{L_{t,x}^{2(d+2)}}^{2(d+2)} \left\| u_n^k \right\|_{L_{t,x}^{2(d+2)}}^{2(d+2)} \left\| \nabla^s e^{it\Delta} w_n^k \right\|_{L_{t,x}^{2(d+2)}}^{2(d+2)} \to 0, \]
as \( n \to \infty, \ k \to \infty. \) Thus (5.31) follows. \( \square \)

We can now show the main result in this section:

**Proposition 5.4** (Existence of the critical element). Suppose \( m^*_\omega < m_\omega, \) then there exists a global solution \( u_c \in C(\mathbb{R}, H^1(\mathbb{R}^d)) \) to (1.1) such that
\[ u_c(t) \in \mathcal{A}_{\omega,+} \text{ and } S_\omega(u_c(t)) = m^*_\omega, \text{ for } t \in \mathbb{R}, \]
\[ \left\| u_c \right\|_{L_{t,x}^{2(d+2)(q-1)}}^{2(d+2)(q-1)} \left\| \nabla^s e^{it\Delta} w_n^k \right\|_{L_{t,x}^{2(d+2)(q-1)}}^{2(d+2)(q-1)} \left\| \nabla^s e^{it\Delta} w_n^k \right\|_{L_{t,x}^{2(d+2)(q-1)}}^{2(d+2)(q-1)} \to \infty. \]

Proof. By Lemma 5.1 Lemma 5.3 and reordering indices, there exists \( J \leq j_0 \) that
\[ \left\{ \begin{array}{l}
\left\| u^j \right\|_{L_{t,x}^{2(d+2)(q-1)}}^{2(d+2)(q-1)} \cap L_{t,x}^{2(d+2)(q-1)}((\infty,0) \times \mathbb{R}^d) = \infty, \text{ for } 1 \leq j \leq J, \\
\left\| u^j \right\|_{L_{t,x}^{2(d+2)(q-1)}}^{2(d+2)(q-1)} \cap L_{t,x}^{2(d+2)(q-1)}((0,\infty) \times \mathbb{R}^d) < \infty, \text{ for } j > J.
\end{array} \right. \]

From (5.39), (5.11) and (5.13), we have
\[ \left\| \langle \nabla \rangle u_n(0) \right\|_{L^2}^2 - \sum_{j=1}^J \left\| \langle \nabla \rangle \varphi^j \right\|_{L^2}^2 - \left\| \langle \nabla \rangle w_n^j \right\|_{L^2}^2 \to 0, \]
\[ \mathcal{S}_\omega(u_n(0)) - \sum_{j=1}^J \mathcal{S}_\omega(e^{-it\Delta} \varphi^j) - \mathcal{S}_\omega(w_n^j) \to 0, \]
\[ \mathcal{H}_\omega(u_n(0)) - \sum_{j=1}^J \mathcal{H}_\omega(e^{-it\Delta} \varphi^j) - \mathcal{H}_\omega(w_n^j) \to 0, \text{ as } n \to \infty. \]
Since $K(u_n(0)) \geq 0$, we have $H_\omega(u_n(0)) \leq S_\omega(u_n(0))$ by (2.4). It follows from (5.43) and (5.3) that
\[
H_\omega(e^{-it^*_j\Delta} \varphi^j), \ H_\omega(w^J_n) \leq S_\omega(u_n(0)) < \frac{m_\omega + m^*_\omega}{2}, \text{ for } 1 \leq j \leq J \text{ and } n \text{ large enough.}
\]
By (5.15), we have
\[
K(e^{-it^*_J\Delta} \varphi^J) > 0, \ K(w^J_n) > 0, \text{ for } 1 \leq j \leq J \text{ and } n \text{ large enough,}
\]
which together with Lemma 2.6 shows
\[
S_\omega(e^{-it^*_j\Delta} \varphi^j) \geq 0, \ S_\omega(w^J_n) \geq 0, \text{ for } 1 \leq j \leq J \text{ and } n \text{ large enough.}
\]
We shall show $J = 1$. Assume for a contradiction that $J \geq 2$. Then, it follows from (5.3) and (5.42) that
\[
\limsup_{n \to \infty} S_\omega(e^{-it^*_j\Delta} \varphi^j) < m^*_\omega, \ \forall 1 \leq j \leq J,
\]
which together with (5.15) shows
\[
S_\omega(w^j(t)) < m^*_\omega, \ \forall 1 \leq j \leq J, \ t \in I^j.
\]
Since $u^j$ is a solution to (1.1), it follows from the definition of $m^*_\omega$ that
\[
\|u^j\|_{L_{t,x}^{\frac{4d+2(p-1)}{2d}}} < \infty, \ \forall 1 \leq j \leq J.
\]
This contradicts (5.41). Thus, we have $J = 1$.
Since $\|u^1\|_{L_{t,x}^{\frac{4d+2(p-1)}{2d}}} (T_{min}^1, T_{max}^1) \times \mathbb{R}^d = \infty$, we have
\[
S_\omega(u^1(t)) \geq m^*_\omega, \text{ for } t \in (T_{min}^1, T_{max}^1).
\]
On the other hand, by (5.15), (5.42), (5.45), we get
\[
S_\omega(u^1(t)) \leq m^*_\omega, \text{ for } t \in (T_{min}^1, T_{max}^1).
\]
Combining this with (5.46), we obtain
\[
S_\omega(u^1(t)) = m^*_\omega, \text{ for } t \in (T_{min}^1, T_{max}^1).
\]
By (5.15), we have
\[
S_\omega(u^1(t)) = \lim_{n \to \infty} S_\omega(e^{-it^*_n\Delta} \varphi^1),
\]
this together with (5.3), (5.42) and (5.47) shows
\[
S_\omega(w^1_n) \to 0, \text{ as } n \to \infty.
\]
Hence, Lemma 2.6 together with (5.44) and (5.48) shows
\[
\|w^1_n\|_{H^1} \to 0, \text{ as } n \to \infty.
\]
We see from (5.47), (5.49) that
\[
\|u^1_n(0,x) - e^{-it^*_n\Delta} \varphi^1(x-x^j_n)\|_{H^1} \to 0, \text{ as } n \to \infty.
\]
Now, we shall show $T_{min}^1 = -\infty, T_{max}^1 = \infty$. Assume for a contradiction that $T_{max}^1 < \infty$. Let $\{t_n\}$ be a sequence in $(T_{min}^1, T_{max}^1)$ such that $t_n \not\to T_{max}^1$ and put
\[
\bar{u}_n(t) = u^1(t + t_n) \text{ and } \bar{I}_n = (T_{min}^1 - t_n, T_{max}^1 - t_n).
\]
We see that \( \{\tilde{u}_n\} \) satisfies
\[
\|\tilde{u}_n\|_{L^2_t L^2_x} = \infty.
\]

Then, we can apply the above argument as deriving (5.51) to this sequence and find that there exists a non-trivial \( \psi \in H^1(\mathbb{R}^d) \), a sequence \( \{\tau_n\} \) with \( \tau_\infty = \lim_{n \to \infty} \tau_n \in [-\infty, \infty] \), \( y_n \in \mathbb{R}^d \) such that
\[
\lim_{n \to \infty} \|u^1(t_n, x) - e^{-i\tau_n \Delta} \psi(x - y_n)\|_{H^1} = \lim_{n \to \infty} \|\tilde{u}_n(0, x) - e^{-i\tau_n \Delta} \psi(x - y_n)\|_{H^1} = 0.
\]

This together with the Strichartz estimate yields
\[
\left\| \left\langle \nabla \right\rangle e^{it\Delta} u^1(t_n, x) - e^{i(t-\tau_n)\Delta} \psi(x - y_n) \right\|_{L^{2+2} t x^{d+2} (\mathbb{R} \times \mathbb{R}^d)} \to 0, \quad \text{as} \quad n \to \infty. \tag{5.51}
\]

**Case 1.** \( \tau_\infty = \pm \infty \). By the dispersive estimate for the free solution, for any compact interval \( I \), we have
\[
\left\| \left\langle \nabla \right\rangle e^{i(t-\tau_n)\Delta} \psi \right\|_{L^{2+2} t x^{d+2} (I \times \mathbb{R}^d)} \to 0, \quad \text{as} \quad n \to \infty,
\]
this together with (5.51) yields
\[
\left\| \left\langle \nabla \right\rangle e^{it\Delta} u^1(t_n) \right\|_{L^{2+2} t x^{d+2} (I \times \mathbb{R}^d)} \to 0, \quad \text{as} \quad n \to \infty. \tag{5.52}
\]

**Case 2.** \( \tau_\infty \in \mathbb{R} \). For any interval \( I \) with \( \tau_\infty \in I \) and \( |I| \ll 1 \), we have by (5.51),
\[
\lim_{n \to \infty} \left\| \left\langle \nabla \right\rangle e^{it\Delta} u^1(t_n) \right\|_{L^{2+2} t x^{d+2} (I \times \mathbb{R}^d)} = \left\| \left\langle \nabla \right\rangle e^{i(t-\tau_\infty)\Delta} \phi \right\|_{L^{2+2} t x^{d+2} (I \times \mathbb{R}^d)} \ll 1. \tag{5.53}
\]

Then, Proposition 3.1 together with (5.52), (5.53) implies that \( u^1 \) exists beyond \( T^1_{\max} \), which is a contradiction. Thus, \( T^1_{\max} = \infty \). Similarly, we have \( T^1_{\min} = -\infty \).

Therefore, \( u^1 \) is a global solution and it is just the desired critical element \( u_c \) satisfying (5.39) and (5.40).

We now show the trajectory of the critical element is precompact in the energy space \( H^1(\mathbb{R}^d) \) modulo spatial translations.

**Proposition 5.5** (Compactness of the critical element). Let \( u_c \) be the critical element in Proposition 5.4, then there exists \( x(t) : \mathbb{R} \to \mathbb{R}^d \) such that \( \{u_c(t, x - x(t)) : t \in \mathbb{R}\} \) is precompact in \( H^1(\mathbb{R}^d) \).

**Proof.** For \( \{t_n\} \subset \mathbb{R} \), if \( t_n \to t^* \in \mathbb{R} \), as \( n \to \infty \), then we see by the continuity of \( u_c(t) \) in \( t \) that
\[
u_c(t_n) \to u_c(t^*) \quad \text{in} \quad H^1(\mathbb{R}^d), \quad \text{as} \quad n \to \infty.
\]
If \( t_n \to \infty \). Applying the above argument as deriving (5.50) to \( u_c(t + t_n) \), there exist \( (t'_n, x'_n) \in \mathbb{R} \times \mathbb{R}^d \) and \( \phi \in H^1(\mathbb{R}^d) \) that
\[u_c(t_n, x) - e^{-it_n \Delta} \phi(x - x'_n) \to 0 \quad \text{in} \quad H^1(\mathbb{R}^d), \quad \text{as} \quad n \to \infty.
\]
(i) If \( t'_n \to -\infty \), then we have
\[
\left\| \left\langle \nabla \right\rangle e^{it\Delta} u_c(t_n) \right\|_{L^{2+2} t x^{d+2} ((0, \infty) \times \mathbb{R}^d)} = \left\| \left\langle \nabla \right\rangle e^{it\Delta} \phi \right\|_{L^{2+2} t x^{d+2} ((-t'_n, \infty) \times \mathbb{R}^d)} + o_n(1) \to 0, \quad \text{as} \quad n \to \infty.
\]
Hence, we can solve (1.1) for \( t > t_n \) globally by iteration with small Strichartz norm when \( n \) large enough, which contradicts
\[
\|u_c\|_{L^\infty_t L^2_x((0,\infty) \times \mathbb{R}^d)} = \infty.
\]

(ii) If \( t'_n \to \infty \), then we have
\[
\|\langle \nabla \rangle e^{it\Delta} u_c(t_n)\|_{L^2_t L^2_x((\infty,0] \times \mathbb{R}^d)} = \|\langle \nabla \rangle e^{it\Delta} \phi\|_{L^2_t L^2_x((\infty,-t'_n] \times \mathbb{R}^d)} + o_n(1) \to 0, \text{ as } n \to \infty.
\]
Hence, \( u_c \) can solve (1.1) for \( t < t_n \) when \( n \) large enough with diminishing Strichartz norm, which contradicts
\[
\|u_c\|_{L^\infty_t L^2_x((\infty,0] \times \mathbb{R}^d)} = \infty.
\]
Thus \( t'_n \) is bounded, which implies that \( t'_n \) is precompact, so is \( u_c(t_n, x + x'_n) \) in \( H^1(\mathbb{R}^d) \).

Similar argument makes sense when \( t_n \to -\infty \), we will omit the proof. \( \square \)

We define for \( R > 0 \), \( x_0 \in \mathbb{R}^d \),
\[
\tilde{E}_{R,x_0}(u(t)) = \int_{|x-x_0| \geq R} |\nabla u(t)|^2 + |u(t)|^2 + |u(t)|^{p+1} \, dx,
\]
then by the compactness of the critical element, we have

**Corollary 5.6.** Let \( u_c \) be the critical element in Proposition 5.4 then for any \( \epsilon > 0 \), there exist \( R_0(\epsilon) > 0 \) and \( x(t) : \mathbb{R} \to \mathbb{R}^d \) such that
\[
\tilde{E}_{R_0,x(t)}(u_c(t)) \leq \epsilon E(u_c), \quad \forall t \in \mathbb{R}.
\]

**Remark 2.** In particular, for the radial data \( u_0 \in H^1 \), by the same argument as in [22], we have \( x(t) \equiv 0 \), i.e.
\[
\tilde{E}_{R_0,0}(u_c(t)) \leq \epsilon E(u_c), \quad \forall t \in \mathbb{R}.
\]

### 6. Extinction of the critical element

In this section, we prove the non-existence of the critical element by deriving a contradiction from Proposition 5.4 and the Virial identity in the radial case.

For a bounded real function \( \phi \in C^\infty(\mathbb{R}^d) \), we can define the virial quantity:
\[
V_R(t) = \int_{\mathbb{R}^d} \phi_R(x)|u(t,x)|^2 \, dx, \quad \text{where } \phi_R(x) = R^2 \phi\left(\frac{|x|}{R}\right), \forall R > 0. \quad (6.1)
\]
Then, for \( u \in C(I; H^1(\mathbb{R}^d)) \), we have
\[
V'_R(t) = 2R \cdot \Im \int_{\mathbb{R}^d} \phi\left(\frac{|x|}{R}\right) \frac{x}{|x|} \cdot \nabla u(t,x) \overline{u(t,x)} \, dx, \quad (6.2)
\]
\[
V''_R(t) = 4R \int \partial_j \partial_k \phi_R(x) \partial_j u(t,x) \overline{\partial_k u(t,x)} \, dx - \int \Delta^2 \phi_R(x)|u_c(t,x)|^2 \, dx - \frac{2(p-1)}{p+1} \int \Delta \phi_R(x)|u(t,x)|^{p+1} \, dx + \frac{4}{d+2} \int \Delta \phi_R(x)|u(t,x)|^{\frac{2(d+2)}{d}} \, dx. \quad (6.3)
\]

**Theorem 6.1.** There does not exist the radial critical element \( u_c \) of (1.1) in Proposition 5.4.
Proof. Let the weight function $\phi$ in (6.1) be a smooth, radial function satisfying $0 \leq \phi \leq 1$, and
\[
\phi(x) = \begin{cases} 
|x|^2, & |x| \leq 1, \\
0, & |x| \geq 2.
\end{cases}
\]

On the one hand, by (6.2), we have
\[
|V_R''(t)| \lesssim R, \quad \forall R > 0, \quad t \in \mathbb{R}.
\tag{6.4}
\]

On the other hand, by (6.3), we have
\[
V_R''(t) = 4 \int \phi''(r)|\nabla u_c(t, x)|^2 \, dx - \int \Delta^2 \phi_R(x)|u_c(t, x)|^2 \, dx
- \frac{2(p-1)}{p+1} \int \Delta \phi_R(x)|u_c(t, x)|^{p+1} \, dx + \frac{4}{d+2} \int \Delta \phi_R(x)|u_c(t, x)|^{\frac{2(d+1)}{d}} \, dx
= 8K(u_c) + \frac{C}{R^2} \int_{|x| \leq 2R} |u_c|^2 \, dx
+ C \int_{|x| \leq 2R} |\nabla u_c(t)|^2 + |u_c(t)|^{p+1} + |u_c(t)|^{\frac{2(d+2)}{d}} \, dx.
\tag{6.5}
\]

By Lemma 2.9 and Lemma 2.6, we have
\[
8K(u_c) \geq \min \left\{ \frac{d(p-1)-4}{d(p-1)} \left( \|\nabla u_c(t)\|_{L^2}^2 + \frac{d}{d+2} \|u_c(t)\|_L^{\frac{2(d+1)}{d}} \right), \delta \left( m_\omega - S_\omega(u_c(t)) \right) \right\}
\gtrsim E(u_c(t)).
\]

Thus, choosing $\epsilon > 0$ small enough and $R = R(\epsilon)$ large enough, then by Corollary 5.6 with $x(t) \equiv 0$, we get
\[
V_R''(t) \gtrsim E(u_c(t)) = E(u_0).
\]

This together with (6.4) implies for $T > 0$,
\[
T \cdot E(u_0) \lesssim \left| \int_0^T V_R''(t) \, dt \right| = |V_R(T) - V_R(0)| \lesssim R.
\]

Taking $T$ large enough, we obtain a contradiction unless $u_c \equiv 0$, which is impossible due to $\|u_c\|_{L^2_{t,x} \cap L^{\frac{2(d+2)}{d}}_{t,x} (\mathbb{R} \times \mathbb{R}^d)} = \infty$.

\[\square\]

7. Blow-up

We will show the blow-up result in Theorem 1.3.

Let the weight function $\phi$ in (6.1) be a smooth, radial function satisfying $\phi(r) = r^2$ for $r \leq 1$, $\phi''(r) \leq 2$ and $\phi(r)$ is constant for $r \geq 3$.

By (6.3), we have
\[
V_R''(t) \leq 4 \int 2|\nabla u|^2 - \frac{d(p-1)}{p+1}|u|^{p+1} + \frac{2d}{d+2}|u|^{\frac{2(d+2)}{d}} \, dx
+ \frac{C}{R^2} \int_{|x| \leq 3R} |u(t, x)|^2 \, dx + C \int_{|x| \leq 3R} |u|^{p+1} + |u|^{\frac{2(d+2)}{d}} \, dx
\tag{7.1}
= 8K(u) + \frac{C}{R^2} \int_{|x| \leq 3R} |u|^2 \, dx + C \int_{|x| \leq 3R} |u(t)|^{p+1} + |u(t)|^{\frac{2(d+2)}{d}} \, dx.
\]
Since $u$ is radial, we have the following radial Sobolev inequalities
\[
\|u\|_{L^{p+1}(|x| \geq R)}^{p+1} \leq \frac{C}{R^{(d-1)p-1}} \|u\|_{L^2(|x| \geq R)} \|\nabla u\|_{L^2(|x| \geq R)}^{p-1},
\]
\[
\|u\|_{L^2(|x| \geq R)}^{2(d+1)p} \leq \frac{C}{R^{2(d-1)p+1}} \|u\|_{L^2(|x| \geq R)} \|\nabla u\|_{L^2(|x| \geq R)}^{2}.
\]
this together with (7.1), the mass conservation and Young’s inequality shows $\forall \epsilon > 0$, there exists $R$ large enough that
\[
V''_R(t) \leq 8\mathcal{K}(u) + \epsilon \|\nabla u(t)\|_{L^2}^2 + \epsilon. \quad (7.2)
\]
By $\mathcal{K}(u) < 0$, energy, mass conservation and Lemma 2.4, we see
\[
\mathcal{K}(u(t)) < - (m_\omega - \mathcal{S}_\omega(u(t))) \quad \forall t \in I_{\text{max}},
\]
thus
\[
\|\nabla u(t)\|_{L^2}^2 < \frac{d(p-1)}{2(p+1)} \|u(t)\|_{L^{p+1}}^{p+1} - (m_\omega - \mathcal{S}_\omega(u)).
\]
So we have by (7.2),
\[
V''_R(t) \leq 8\mathcal{K}(u) + \epsilon \|\nabla u(t)\|_{L^2}^2 + \epsilon
= 16\mathcal{S}_\omega(u) - 8\omega \|u\|_{L^2}^2 + \frac{16-4d(p-1)}{p+1} \|u(t)\|_{L^{p+1}}^{p+1} + \epsilon \|\nabla u(t)\|_{L^2}^2 + \epsilon
< 16\mathcal{S}_\omega(u) - 8\omega \|u\|_{L^2}^2 + \left( \frac{16-4d(p-1)}{p+1} + \frac{d(p-1)}{2(p+1)} \right) \|u(t)\|_{L^{p+1}}^{p+1} - \epsilon (m_\omega - \mathcal{S}_\omega(u)) + \epsilon.
\]
Here we take $\epsilon > 0$ small enough such that $\frac{16-4d(p-1)}{p+1} + \frac{d(p-1)}{2(p+1)} < 0$.

We also note by $\mathcal{K}(u) < 0$ and Proposition 2.4
\[
m_\omega \leq \mathcal{H}_\omega(u(t)) = \frac{\omega}{2} \|u(t)\|_{L^2}^2 + \frac{d(p-1)-4}{4(p+1)} \|u(t)\|_{L^{p+1}}^{p+1},
\]
so
\[
\frac{4(p+1)}{d(p-1)-4} \left( m_\omega - \frac{\omega}{2} \|u(t)\|_{L^2}^2 \right) \leq \|u(t)\|_{L^{p+1}}^{p+1}.
\]
Thus, we have
\[
V''_R(t) \leq 16\mathcal{S}_\omega(u) - 8\omega \|u\|_{L^2}^2 - \epsilon (m_\omega - \mathcal{S}_\omega(u)) + \epsilon
+ \left( \frac{16-4d(p-1)}{p+1} + \frac{d(p-1)-4}{4(p+1)} \right) \frac{4(p+1)}{d(p-1)-4} \left( m_\omega - \frac{\omega}{2} \|u(t)\|_{L^2}^2 \right).
\]
By $\mathcal{S}_\omega(u) < m_\omega$ and the energy, mass conservation, we see there exists $\delta_1 > 0$ small enough such that $\mathcal{S}_\omega(u) \leq (1 - \delta_1)m_\omega$. Then, we get
\[
V''_R(t) \leq 16(1 - \delta_1)m_\omega - 8\omega \|u\|_{L^2}^2 - \epsilon (m_\omega - \mathcal{S}_\omega(u)) + \epsilon
+ \left( \frac{4(p-1)-16}{d(p+1)} + \frac{d(p-1)-4}{2(p+1)} \right) \frac{4(p+1)}{d(p-1)-4} \left( m_\omega - \frac{\omega}{2} \|u\|_{L^2}^2 \right)
= - \left( 16\delta_1 + \epsilon \delta_1 - \frac{4(p+1)}{d(p-1)-4} \frac{d(p-1)-4}{2(p+1)} \right) m_\omega - \frac{d(p-1)-4}{2(p+1)} \frac{4(p+1)}{d(p-1)-4} \frac{\omega}{2} \|u\|_{L^2}^2 + \epsilon
\leq - \left( 16\delta_1 + \epsilon \delta_1 - \frac{2d(p-1)-4}{d(p-1)-4} \right) m_\omega + \epsilon.
\]
We can take $\epsilon > 0$ small enough that $V''_R(t) \leq -4\delta_1 m_\omega$, which implies that $u$ must blow up in finite time.

Remark 3. The blowup is shown for $p \leq 5$, which leads to the restriction of the blowup result to $d \geq 2$. This is a technical restriction. See also [12] [24] [25] for some related discussion.
Acknowledgements The authors were supported by the NSF of China under grant No. 10901148, No.11171033 and 11231006.

References

[1] T. Akahori, S. Ibrahim, H. Kikuchi, and H. Nawa, Existence of a ground state and blow-up problem for a nonlinear Schrödinger equation with critical growth, Differential Integral Equations 25:3-4 (2012), 383-402. MR 2917888
[2] T. Akahori, S. Ibrahim, H. Kikuchi, and H. Nawa, Existence of a ground state and scattering for a nonlinear Schrödinger equation with critical growth, Selecta Math.(N.S.) 19:2 (2013), 545-609. MR 3090237
[3] T. Aubin, Equations différentielles non linéaires et problème de Yamabe concernant la courbure scalaire, J. Math. Pures Appl.(9) 55:3 (1976), 269-296. MR 0431287 (55 #4288)
[4] H. Bahouri and P. Gérard, High frequency approximation of solutions to critical nonlinear wave equations, Amer. J. Math. 121:1 (1999), 131-175. MR 1705001 (2000i:35123)
[5] I. V. Barashenkov, A. D. Gocheva, V. G. Makhankov, and I. V. Puzynin, Stability of the soliton-like bubbles, Phys. D: Nonlinear Phenomena 34:1-2 (1989), 240-254. MR 0982390 (90b:35196)
[6] P. Bégout and A. Vargas, Mass concentration phenomena for the $L^2$-critical nonlinear Schrödinger equation, Trans. Amer. Math. Soc. 359:11 (2007), 5257-5282. MR 2327030 (2008g:35190)
[7] H. Brézis and L. Nirenberg, Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents, Comm. Pure Appl. Math. 36:4 (1983), 437-477. MR 0709644 (84h:35059)
[8] R. Carles and S. Keraani, On the role of quadratic oscillations in nonlinear Schrödinger equations II. The $L^2$-critical case, Trans. Amer. Math. Soc. 359:1 (2007): 33-62. MR 2247881 (2008a:35260)
[9] T. Cazenave, Semilinear Schrödinger equations. Courant Lecture Notes in Mathematics, 10. New York university, Courant Institute of Mathematical Sciences, New York; American Mathematical Society, Providence, RI, 2003. ISBN: 0-8218-3399-5. MR 2002047
[10] T. Cazenave and F. B. Weissler, The Cauchy problem for the critical nonlinear Schrödinger equation in $H^s$, Nonlinear Anal. 14: 10 (1990), 807-836. MR 1055532 (91j:35252)
[11] D. Foschi, Inhomogeneous Strichartz estimates, J. Hyperbolic Differ. Equ. 2: 1 (2005), 1-24. MR 2134950 (2006a:35043)
[12] J. Holmer and S. Roudenko, On blow-up solutions to the 3D cubic nonlinear Schrödinger equation, Appl. Math. Res. Express. AMRX 1(2007): Art. ID abm004. MR2354447 (2008i:35227)
[13] S. Ibrahim, N. Masmoudi, and K. Nakanishi, Scattering threshold for the focusing nonlinear Klein-Gordon equation, Anal. PDE 4: 3 (2011), 405-460. MR 2872122
[14] M. Keel and T. Tao, Endpoint Strichartz estimates, Amer. J. Math. 120: 5 (1998), 955-980. MR 1646048 (2000d:35018)
[15] C. E. Kenig and F. Merle, Global well-posedness, scattering and blow-up for the energy-critical, focusing, non-linear Schrödinger equation in the radial case, Invent. Math. 166: 3 (2006), 645-675. MR 2257393 (2007g:35232)
[16] S. Keraani, On the defect of compactness for the Strichartz estimates of the Schrödinger equations, J. Differential Equations 175: 2 (2001), 353-392. MR 1855973 (2002j:35281)
[17] S. Keraani, On the blow up phenomenon of the critical Schrödinger equation, J. Funct. Anal. 235:1 (2006), 171-192. MR 2216444 (2007e:35260)
[18] R. Killip and M. Visan, The focusing energy-critical nonlinear Schrödinger equation in dimensions five and higher, Amer. J. Math. 132: 2 (2010), 361-424. MR 2654778 (2011e:35357)
[19] R. Killip and M. Visan, Nonlinear Schrödinger equations at critical regularity. Proceedings for the Clay summer school “Evolution Equations”, Eidgenössische technische Hochschule, Zürich, 2008.
[20] E. H. Lieb and M. Loss, Analysis. Graduate Studies in Mathematics, 14. American Mathematical Society, Providence, RI, 2001. ISBN: 0-8218-2783-9. MR 1817225 (2001i:00001)
[21] F. Merle and L. Vega, *Compactness at blow-up time for $L^2$ solutions of the critical nonlinear Schrödinger equation in 2D*, Internat. Math. Res. Notices 8 (1998), 399-425. MR 1628235 (99d:35156)

[22] C. Miao, G. Xu, and L. Zhao, *The dynamics of the 3D radial NLS with the combined terms*, Comm. Math. Phys. 318: 3 (2013), 767-808. MR 3027584

[23] C. Miao, G. Xu, and L. Zhao, *The dynamics of the NLS with the combined terms in five and higher dimensions*, arXiv:1112.4618.

[24] T. Ogawa and Y. Tsutsumi, *Blow-up of $H^1$ solution for the nonlinear Schrödinger equation*, J. Differential Equations 92: 2 (1991), 317-330. MR 1120908 (92k:35262)

[25] T. Ogawa and Y. Tsutsumi, *Blow-up of $H^1$ solutions for the one-dimensional nonlinear Schrödinger equation with critical power nonlinearity*, Proc. Amer. Math. Soc. 111: 2 (1991), 487-496. MR 1045145 (91f:35026)

[26] D. E. Pelinovsky, V. V. Afanasjev, and Y. S. Kivshar, *Nonlinear theory of oscillating, decaying, and collapsing solitons in the generalized nonlinear Schrödinger equation*, Phys. Rev. E 53: 2(1996), 1940-1953.

[27] G. Talenti, Best constant in Sobolev inequality, Ann. Mat. Pura Appl.(4) 110, (1976), 353-372. MR 0463908 (57 #3846)

[28] T. Tao, M. Visan, and X. Zhang, *The nonlinear Schrödinger equation with combined power-type nonlinearities*, Comm. Partial Differential Equations 32: 7-9 (2007), 1281-1343. MR 2354495 (2009f:35324)

[29] M. C. Vilela, *Inhomogeneous Strichartz estimates for the Schrödinger equation*, Trans. Amer. Math. Soc. 359: 5 (2007), 2123-2136. MR 2276614 (2008a:35226)

[30] X. Zhang, *On Cauchy problem of 3-D energy-critical Schrödinger equations with subcritical perturbations*, J. Differential Equations 230: 2 (2006), 422-445. MR 2271498 (2007h:35325)

WU WEN-TSUN Key Laboratory of Mathematics and School of Mathematical Sciences, University of Science and Technology of China, Hefei 230026, Anhui, China

E-mail address: chengx@mail.ustc.edu.cn

Institute of Applied Physics and Computational Mathematics, P. O. Box 8009, Beijing 100088, China

E-mail address: miao_changxing@iapcm.ac.cn

WU WEN-TSUN Key Laboratory of Mathematics and School of Mathematical Sciences, University of Science and Technology of China, Hefei 230026, Anhui, China

E-mail address: zhaolf@ustc.edu.cn