The $q$-Harmonic Oscillator and the Al-Salam and Carlitz polynomials

Dedicated to the Memory of Professor Ya. A. Smorodinskiĭ

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Abstract. One more model of a $q$-harmonic oscillator based on the $q$-orthogonal polynomials of Al-Salam and Carlitz is discussed. The explicit form of $q$-creation and $q$-annihilation operators, $q$-coherent states and an analog of the Fourier transformation are established. A connection of the kernel of this transform with a family of self-dual biorthogonal rational functions is observed.

Introduction

Recent development in quantum groups has led to the so-called $q$-harmonic oscillators (see, for example, Refs. [1–7]). Presently known models of $q$-oscillators are closely related with $q$-orthogonal polynomials. The $q$-analogs of boson operators have been introduced explicitly in Refs. [3], [5] and [7], where the corresponding wave functions were constructed in terms of the continuous $q$-Hermite polynomials of Rogers [8,9], in terms of the Stieltjes–Wigert polynomials [10,11] and in terms of $q$-Charlier polynomials of Al-Salam and Carlitz [12], respectively. The model related to the Rogers–Szegő polynomials [13] was investigated in [1,6]. Here we introduce the explicit realization of $q$-creation and $q$-annihilation operators with the aid of another family of the Al-Salam and Carlitz polynomials [12] when eigenvalues of the corresponding $q$-Hamiltonian are unbounded. An attempt to unify $q$-boson operators is also made.

With a great deal of regret we dedicate this paper to the memory of Yacob A. Smorodinskiĭ, who suggested ten years ago that the special case $q = 1$ of this work is interesting and admits a generalization.

1. The Al-Salam and Carlitz Polynomials

The aim of this Letter is to show that the $q$-orthogonal polynomials $U_n^{(a)}(x; q)$ studied by Al-Salam and Carlitz are closely connected with the $q$-harmonic oscillator. To emphasize these relations we use the notation $u_n^{(a)}(x; q) = \mu^{-n} q^{-n(n-1)/2} U_n^{(-\mu)}(x; q)$ for the Al-Salam and Carlitz polynomials. In our notation they can be defined by the three-term recurrence relation of the form

\[ \mu q^n u_{n+1}^{(a)}(x; q) + (1 - q^n) u_{n-1}^{(a)}(x; q) = (x - (1 - \mu) q^n) u_n^{(a)}(x; q), \]

\[ u_0^{(a)}(x; q) = 1, \quad u_1^{(a)}(x; q) = \mu^{-1}(x - 1 + \mu). \]

These polynomials are orthogonal

\[ \int_{-\mu}^{1} u_m^{(a)}(x; q) u_n^{(a)}(x; q) d\alpha(x) = (1 + \mu) q^{-n(n-1)/2} \frac{\langle q; q \rangle_n}{\mu^n} \delta_{mn} \]

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with respect to a positive measure $d\alpha(x)$, where $\alpha(x)$ is a step function with jumps

$$\frac{q^k}{(-q\mu; q)_\infty(q, -q/\mu; q)_k}$$

at the points $x = q^k, k = 0, 1, \ldots$, and jumps

$$\frac{\mu q^k}{(-q/\mu; q)_\infty(q, -q\mu; q)_k}$$

at the points $x = -\mu q^k, k = 0, 1, \ldots$ (see, for example, [12,14,15]). Here the usual notations are

$$\begin{align*}
(a; q)_n &= \prod_{k=0}^{n-1} (1 - aq^k), \\
(a, b; q)_n &= (a; q)_n(b; q)_n, \\
(a; q)_\infty &= \lim_{n \to \infty} (a; q)_n.
\end{align*}$$

The orthogonality relation (2) can also be written in terms of the $q$-integral of Jackson,

$$\int_{-\mu}^{1} u_n^m(x; q) u_n^m(x; q) \tilde{\rho}(x) d_q x = (1 - q)d_n^2 \delta_{mn},$$

where

$$\tilde{\rho}(x) = \frac{(qx, -\mu^{-1}qx; q)_\infty}{(q, -\mu, -q/\mu; q)_\infty}; \quad \mu > 0, \quad 0 < q < 1$$

and

$$d_n^2 = q^{-n(n-1)/2} \frac{(q; q)_n}{\mu^n}.$$  \hspace{1cm} (5)

For the definition of the $q$-integral, see [15]. The “weight function” $\rho(s) = \tilde{\rho}(x)$ in (5) is a solution of the Pearson-type equation $\Delta(\sigma \rho) = \rho \nabla x_1$ with $x(s) = q^s$, $\sigma(s) = (1 - q^s)(\mu + q^s)$ and $\sigma(s) + \tau(s)\nabla x_1(s) = \mu$. The polynomials $y_n(s) = u_n^m(x; q)$ satisfy the hypergeometric-type difference equation in self-adjoint form,

$$\frac{\Delta}{\nabla x_1(s)} \left[ \sigma(s) \rho(s) \frac{\nabla y_n(s)}{\nabla x(s)} \right] + \lambda_n \rho(s) y_n(s) = 0,$$

where

$$\lambda_n = q^{3/2} \frac{q^{-n} - 1}{(1 - q)^2}.$$  \hspace{1cm} (6)

Here $\Delta f(s) = f(s + 1) - f(s) = \nabla f(s + 1)$ and $x_1(s) = x(s + 1/2)$. (For details, see [16–19].) The orthogonality property (2) or (4) can be proved by using standard Sturm–Liouville-type arguments (cf. [16–19]).
The explicit form of the polynomials $u_n^\mu(x; q)$ is

$$
\begin{align*}
  u_n^\mu(x; q) &= 2\varphi_1(q^{-n}, x^{-1}; 0, q, -\frac{q^\mu}{\mu}x) \\
  &= (-\mu^{-1})^n 2\varphi_1(q^{-n}, -\mu x^{-1}; 0, q, qx), \quad x = q^s.
\end{align*}
$$

(7)

It means $u_n^\mu(x; q) = (-\mu^{-1})^n u_n^{1/\mu}(-\mu^{-1}x; q)$. In the limit $q \to 1$ it is easy to see from (1) or (7) that

$$
\lim_{q \to 1} u_n^{(1-q)^\mu}(q^s; q) = 2 {}_0 F_0(-n, -s; -; -1/\mu) = c_n^\mu(s),
$$

(8)

where $c_n^\mu(s)$ are the Charlier polynomials.

2. Model of $q$-Harmonic Oscillator

The Al-Salam and Carlitz polynomials $u_n^\mu(x; q)$ allow us to consider an interesting model of a $q$-oscillator (cf. [7]). We can introduce a $q$-version of the wave functions of the harmonic oscillator as

$$
\psi_n(s) = \tilde{\psi}_n(x) = d_n^{-1}(\tilde{\rho}(x)|x|)^{1/2} u_n^\mu(x; q), \quad x = q^s,
$$

(9)

where $\tilde{\rho}(x)$ and $d_n^2$ are defined in (5) and (6), respectively. These $q$-wave functions satisfy the orthogonality relation

$$
(1 - q)^{-1} \int_{-\mu}^1 \tilde{\psi}_n(x) \tilde{\psi}_m(x) |x|^{-1} d_q x = \delta_{nm},
$$

(10)

which is equivalent to (2) and (4).

The $q$-annihilation $b$ and $q$-creation $b^+$ operators have the following explicit form

$$
\begin{align*}
  b &= (1 - q)^{-\frac{1}{2}} \left[ \mu^s q^{-s} - \sqrt{(1 - q^{s+1})(\mu q^{-1} + q^s)} q^{-s} e^{\partial_s} \right], \\
  b^+ &= (1 - q)^{-\frac{1}{2}} \left[ \mu^s q^{-s} - e^{-\partial_s} \sqrt{(1 - q^{s+1})(\mu q^{-1} + q^s)} q^{-s} \right],
\end{align*}
$$

(11)

where $\partial_s \equiv \frac{d}{ds}$, $e^{\alpha \partial_s} f(s) = f(s + \alpha)$. These operators are adjoint, $(b^+ \psi, \chi) = (\psi, b\chi)$, with respect to the scalar product (10). They satisfy the $q$-commutation rule

$$
b b^+ - q^{-1} b^+ b = 1
$$

(12)

and act on the $q$-wave functions defined in (9) by

$$
b \psi_n = e_n^{1/2} \psi_{n-1}, \quad b^+ \psi_n = e_n^{1/2} \psi_{n+1},
$$

(13)
where

\[ \tilde{e}_n = \frac{1 - q^{-n}}{1 - q^{-1}}. \]

The \( q \)-Hamiltonian \( H = b^+ b \) acts on the wave functions (9) as

\[ H \psi_n = \tilde{e}_n \psi_n \]  

and has the following explicit form

\[ H = (1 - q)^{-1} \left[ \mu q^{-2s} + (1 - q^s)(\mu + q^s) q^{1-2s} - \right. \]

\[ \left. \mu^s q^{-2s} \sqrt{(1 - q^{s+1})(\mu q^{-1} + q^s)} e^{\partial_s} - \mu^s q^{2-2s} \sqrt{(1 - q^s)(\mu q^{-1} + q^{s-1})} e^{-\partial_s} \right]. \]

By factorizing the Hamiltonian (or the difference equation for the Al-Salam and Carlitz polynomials) we arrive at the explicit form (11) for the \( q \)-boson operators. The equations (13) are equivalent to the following difference-differentiation formulas

\[ \mu q^{-s-1} \Delta u_n(x; q) = (1 - q^{-n}) u_n(x; q), \]

\[ q^{-s} \nabla [\rho(s) u_n(x; q)] = \rho(s) u_{n+1}(x; q), \]

respectively. Therefore, the main properties of the Al-Salam and Carlitz polynomials admit a simple group-theoretical interpretation in terms of the \( q \)-Heisenberg–Weyl algebra (12). The symmetric case \( \mu = 1 \) in the above formulas corresponds to the \textit{discrete q-Hermite polynomials} \( H_n(x; q) \) [12,15].

3. The \( q \)-Coherent States

For the model of the \( q \)-oscillator under discussion, by analogy with [7] we can construct explicitly the \textit{q-coherent states} \( | \alpha \rangle \) defined by

\[ b | \alpha \rangle = \alpha | \alpha \rangle, \]

\[ | \alpha \rangle = f_\alpha \sum_{n=0}^{\infty} \frac{\alpha^n \psi_n(s)}{(\tilde{e}_n!)^{1/2}}, \quad \langle \alpha | \alpha \rangle = 1, \]

where

\[ \tilde{e}_n! = \tilde{e}_1 \tilde{e}_2 \ldots \tilde{e}_n = q^{-n(n-1)/2} \frac{(q; q)_n}{(1 - q)^n}, \]

\[ f_\alpha = \left( - (1 - q) | \alpha \rangle | \alpha \rangle \right)^{-1/2}. \]

By using (9) one can obtain

\[ | \alpha \rangle = f_\alpha (\rho|q^s|)^{\frac{1}{2}} \sum_{n=0}^{\infty} u_n(x; q) q^{n(n-1)/2} t^n / (q; q)_n, \quad t = \alpha \mu^{\frac{1}{2}}(1 - q)^{-\frac{1}{2}}. \]
we arrive at the following explicit form for the \( q \)-coherent states

\[
| \alpha \rangle = f_\alpha (\rho | q^s \rangle)^{\frac{1}{2}} \left( \frac{-\alpha (1 - \rho^{-1} q^{1/2} ; q) \alpha (1 - q^{1/2} \rho^{-1} q^{-1/2} ; q)}{\alpha (1 - q^{1/2} \rho^{-1} q^{-1/2} q^s ; q)} \right)_\infty,
\]

where \( \rho (s) = (q^{1+s}, -\rho^{-1} q^{1+s} ; q) / (q, -\rho q / \rho ; q) \). These coherent states are not orthogonal

\[
\langle \alpha | \beta \rangle = \frac{(-\alpha | s \rangle \alpha^* | q \rangle \infty}{(-\alpha | s \rangle, -\alpha | q \rangle \infty} \bigg| \begin{array}{c} \alpha \infty \\ \beta \infty \end{array} \bigg| ; q \infty^{1/2},
\]

where \( * \) denotes the complex conjugate.

4. Analog of the Fourier Transformation

To define an analog of the Fourier transform we begin, in the spirit of Wiener’s approach to the classical Fourier transform [20] (see also [7,21,22]), by deriving the kernel of the form

\[
K_t(x, y) = \sum_{n=0}^{\infty} t^n \tilde{\psi}_n(x) \tilde{\psi}_n(y)
\]

where

\[
\tilde{\psi}_n(x) = \frac{t^n}{(q ; q)_n} u_n^\mu(x ; q) u_n^\nu(y ; q) q^{n(n-1)/2} \frac{u_n^\mu t u_n^\nu}{(q ; q)_n}.
\]

The series can be summed with the aid of the bilinear generating function of Al-Salam and Carlitz [12]

\[
\sum_{n=0}^{\infty} u_n^\mu(x ; q) u_n^\nu(y ; q) q^{n(n-1)/2} \frac{t^n}{(q ; q)_n} \frac{t^n}{(q ; q)_n} u_n^\mu t u_n^\nu = \frac{(-t, t, t ; q)}{(x t / \mu, t / \nu ; q)_\infty} \cdot 3 \varphi_2 \left( \begin{array}{c} x - 1, y - 1, -q t^{-1} \\ q \mu_1(t x)^{-1}, q \mu_2(t y)^{-1} ; q, q \end{array} \right)
\]

( the \( 3 \varphi_2 \)-series is terminating and \( \max (|t / \mu_1|, |t / \mu_2|) < 1 \) ). The answer is

\[
K_t(x, y) = (\hat{\rho}(x) \hat{\rho}(y) | xy \rangle)^{\frac{1}{2}} \frac{(t, t, -\rho t ; q)}{(t x, t y ; q)_\infty} \cdot 3 \varphi_2 \left( \begin{array}{c} x - 1, y - 1, -q(\mu t)^{-1} \\ q(t x)^{-1}, q(t y)^{-1} ; q, q \end{array} \right);
\]

at \( x = q^s \) and at \( x = -\mu q^s \) for \( s = 0, 1, \ldots \), respectively.
In view of (10) and (21),
\[ t^m \tilde{\psi}_m(x) = (1 - q)^{-1} \int_{-\mu}^{1} K_t(x, y) \, \tilde{\psi}_m(y) \, |y|^{-1} \, dq \, y. \tag{24} \]

Letting \( t = i \), we find that the \( q \)-wave functions (9) are eigenfunctions of the following “\( q \)-Fourier transform”,
\[ t^m \tilde{\psi}_m(x) = (1 - q)^{-1} \int_{-\mu}^{1} K_i(x, y) \, \tilde{\psi}_m(y) \, |y|^{-1} \, dq \, y. \tag{25} \]

An easy corollary of (21) or (24) is
\[ (1 - q)^{-1} \int_{-\mu}^{1} K_t(x, y) \, K_{t'}(x', y) \, |y|^{-1} \, dq \, y = K_{tt'}(x, x'). \tag{26} \]

Putting \( t = -t' = i \), we obtain the orthogonality relation of the kernel,
\[ (1 - q)^{-1} \int_{-\mu}^{1} K_i(x, y) \, K^*_i(x', y) \, |y|^{-1} \, dq \, y = \delta_{xx'}, \tag{27} \]
which implies the orthogonality of the rational functions (23) and leads to an inversion formula for the \( q \)-transformation (25). In view of (8), in the limit \( q \to 1^- \) we get one of the “discrete Fourier transforms” considered in [21].

5. Some Biorthogonal Rational Functions

The rational functions (23) have appeared as the kernel of the discrete \( q \)-Fourier transform (25). They admit the following extension. With the aid of the bilinear generating function (22) and the orthogonality property of a special case of the \( q \)-Meixner polynomials, which are dual to the polynomials (7), we obtain the biorthogonality relation,
\[ \int_{-\mu_2}^{1} u(x, y) \, v(x', y) \, \tilde{\rho}(y) \, dq \, y = (1 - q) \, d_x^2 \, \delta_{xx'}, \tag{28} \]
for the \( 3 \phi_2 \)-rational functions of the form
\[ u(x, y) = 3 \phi_2 \left( \frac{x^{-1}, y^{-1}, -qt_1^{-1}}{q \mu_1(t_1 x)^{-1}, q \mu_2(t_1 y)^{-1}; q, q} \right); \tag{29} \]
\[ = 3 \phi_2 \left( \frac{-\mu_1 x^{-1}, -\mu_2 y^{-1}, -qt_2}{-qt_2 x^{-1}, -qt_2 y^{-1}; q, q} \right) \]
at \( x = q^s \) and at \( x = -\mu_1 q^s \) for \( s = 0, 1, \ldots \), respectively, and
\[ v(x, y) = u(x, y)|_{t_1 \leftrightarrow t_2}; \quad t_1 t_2 = \mu_1 \mu_2. \tag{30} \]
Since $\alpha$ with two arbitrary functions $q$ orthogonality property for the big functions [23–27]. It is interesting to compare the biorthogonality relation (28) with the

at $x, x' = \{q^s; s = 0, 1, \ldots\}$ and at $x, x' = \{-\mu_1 q^s; s = 0, 1, \ldots\}$, respectively;

at $x, x' = \{q^s, -\mu_1 q^s\}$ and vice versa, respectively. The squared norm is

$$d_x^2 = \frac{(q, q, -\mu_1, -\mu_2, -q\mu_1^{-1}, -q_2^{-1}; q_\infty)\cdot (t_1^{-1} x, t_2^{-1} x; q_\infty)}{(q, q, -\mu_1, -\mu_2, -q\mu_1^{-1}, -q_2^{-1}; q_\infty)\cdot (t_1^{-1} x, t_2^{-1} x; q_\infty)} |x|^{-1};$$

for $x = q^s$ and for $x = -\mu_1 q^s$, respectively.

The functions (29)–(30) are self-dual and belong to classical biorthogonal rational functions [23–27]. It is interesting to compare the biorthogonality relation (28) with the orthogonality property for the big $q$-Jacobi polynomials [28], which live at the same terminating $3\varphi_2$-level.

### 6. Concluding Remarks

In view of (11), it is natural to introduce operators of the form

$$a = \alpha(s) - \beta(s) e^\delta, \quad a^+ = \alpha(s) - e^{-\delta} \beta(s)$$

with two arbitrary functions $\alpha(s)$ and $\beta(s)$ and to satisfy the commutation rule $aa^+ - qa^+ a = 1$. The result is

$$\alpha(s + 1) = q\alpha(s),$$

$$(1 - q)\alpha^2(s) + \beta^2(s) - q\beta^2(s - 1) = 1$$

and we can choose $\alpha(s) = \varepsilon q^s$ and $\beta(s) = \varepsilon^2 (q^{s+1} - \gamma)(q^s - \delta)$ with $(1 - q)\gamma\delta\varepsilon^2 = 1$. Since

$$(a^+ \psi, \chi) - (\psi, a\chi) = \sum_s \Delta[\beta(s - 1) \psi^*(s - 1) \chi(s)],$$
the corresponding operators are adjoint for the two different cases considered in [7] and in this Letter with 0 < q < 1 and q > 1, respectively.

For $\beta = \text{constant}$ we can try

$$a = e^{\partial} \left( e^{\partial} - \alpha(s) \right), \quad a^+ = \left( e^{-\partial} - \alpha(s) \right) e^{-\partial}$$

and obtain $aa^+ - qa^+a = 1 - q$, when

$$\alpha^2(s + 1) = q\alpha^2(s), \quad \alpha(s + 2) = q\alpha(s),$$

which is satisfied for $\alpha = \varepsilon q^{s/2}$. This case has been considered in [5].

Finally, the operators

$$a = \varepsilon \frac{e^{\gamma\partial} \alpha(s) e^{-\gamma\partial} \beta(s)}{\alpha(s) - \beta(s)}, \quad a^+ = \varepsilon \frac{\alpha(s) e^{-\gamma\partial} + \beta(s) e^{\gamma\partial}}{\alpha(s) - \beta(s)}$$

obey the $q$-commutation rule provided that $\alpha(s)\beta(s) = \pm 1$ and $\alpha(s + 2\gamma) = q^{-1}\alpha(s)$. Therefore, $\alpha = q^{-s}$ for $\gamma = 1/2$ and $\varepsilon^2 = q^{1/2}(1 - q)^{-1}$ (cf. [3]).

We can also introduce the operators

$$a = \alpha^{-1}(s) - \varepsilon \beta^{-1}(s) e^{\partial}, \quad a^+ = \alpha(s) - \varepsilon e^{\partial} \beta(s)$$

and obtain $aa^+ - qa^+a = 1 - q$ if

$$\alpha(s + 1) = q\alpha(s), \quad \beta(s + 2) = q\beta(s),$$

so $\alpha = q^s$ and $\beta = q^{s/2}$. This leads to the Rogers–Szegö polynomials [13] orthogonal on the unit circle (see [1,6]).

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