Estimates for periodic Zakharov-Shabat operators

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Abstract

We consider the periodic Zakharov-Shabat operators on the real line. The spectrum of this operator consists of intervals separated by gaps with the lengths $|g_n| \geq 0, n \in \mathbb{Z}$. Let $\mu_\pm^n$ be the corresponding effective masses and let $h_n$ be heights of the corresponding slits in the quasimomentum domain. We obtain a priori estimates of sequences $g_n = (|g_n|), \mu_\pm^n = (\mu_\pm^n), h = (h_n), n \in \mathbb{Z}$ in terms of weighted $\ell^p$-norms at $p \geq 1$. The proof is based on the analysis of the quasimomentum as the conformal mapping.

1 Introduction and main results

Consider the Zakharov-Shabat operator $T$ acting on $L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$ and given by

$$T = J \frac{d}{dt} + V(t), \quad V = \begin{pmatrix} V_1 & V_2 \\ V_2 & -V_1 \end{pmatrix}, \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

where $V$ is a real 1-periodic $2 \times 2$ matrix-valued function of $t \in \mathbb{R}$ and $V \in L^1(0, 1)$. In order to describe our main result we shall introduce some notations and recall some well known facts about the Zakharov-Shabat operator (see [Kr], [LS] for details). The spectrum of $T$ is purely absolutely continuous and is given by the set $\bigcup \sigma_n$, with spectral bands $\sigma_n = [z_n^+, z_n^-], \sigma_{n+1}$ are separated by a gap $g_n = (z_n^-, z_n^+)$ with length $|g_n| \geq 0$. If a gap $g_n$ is degenerate, i.e. $g_n = \emptyset$, then the corresponding spectral bands $\sigma_n, \sigma_{n+1}$ merge. The sequence $\ldots < z_{2n-1}^- \leq z_{2n-1}^+ < z_{2n}^- \leq z_{2n}^+ < \ldots$ is the spectrum of equation $Jf' + Vf = zf$ with the 2-periodic boundary conditions, i.e., $f(t + 2) = f(t), t \in \mathbb{R}$. Here the equality $z_n^- = z_n^+$ means that $z_n^-$ is the double eigenvalue. The eigenfunctions, corresponding to the eigenvalue $z_n^\pm$, are 1-periodic, when $n$ is even and they are antiperiodic, i.e., $f(t + 1) = -f(t), t \in \mathbb{R}$, when $n$ is odd. Introduce the $2 \times 2$-matrix valued fundamental solution $\psi = \psi(t, z)$ of the problem:

$$J\psi' + V\psi = z\psi, \quad \psi(0, z) = I_2, \quad z \in \mathbb{C}, \quad (1.1)$$

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where \( I_2 \) is the identity \( 2 \times 2 \)-matrix. Introduce the Lyapunov function \( \Delta(z) = \frac{1}{2} \text{Tr} \psi(1, z) \). Note that \( \Delta(z_n^\pm) = (-1)^n, n \in \mathbb{Z} \), and the function \( \Delta'(z) \) has exactly one zero \( z_n \in [z_n^-, z_n^+] \) for each \( n \in \mathbb{Z} \). For each \( V \) there exists a unique conformal mapping (the quasi-momentum) \( k : \mathbb{Z} \to K(h) \) such that (see [Mi1])

\[
\cos k(z) = \Delta(z), \quad z \in \mathbb{Z} = \mathbb{C} \setminus \bigcup \gamma_n, \quad K(h) = \mathbb{C} \setminus \bigcup \Gamma_n, \quad \Gamma_n = (\pi n - i h_n, \pi n + i h_n),
\]

\[
k(z) = z + o(1) \quad \text{as} \quad |z| \to \infty,
\]

where \( \Gamma_n \) is the vertical slit and the height \( h_n \geq 0 \) is defined by the equation \( \cosh h_n = (-1)^n \Delta(z_n) \geq 1 \). We emphasize that the introduction of the quasi-momentum \( k(z) \) provides a natural labeling of all gaps \( \gamma_n \) (including the empty ones!) by demanding that \( k(\cdot) \) maps the gap \( g_0 \) on the vertical slit \( \Gamma_0 = (-i h_0, i h_0) \).

For any \( p \geq 1 \) and the weight \( \omega = (\omega_n)_{n \in \mathbb{Z}} \), where \( \omega_n \geq 1 \), we introduce the real spaces

\[
\ell^p_\omega = \{ f = (f_n)_{n \in \mathbb{Z}} : \| f \|_{p,\omega} < \infty \}, \quad \| f \|_{p,\omega} = \sum_{n \in \mathbb{Z}} \omega_n |f_n|^p < \infty.
\]

If the weight \( \omega_n = 1 \) for all \( n \in \mathbb{Z} \), then we will write \( \ell^p_0 = \ell^p \) with the norm \( \| \cdot \|_p \). For each \( V \) we introduce the sequences

\[
h = (h_n)_{n \in \mathbb{Z}}, \quad \gamma = (|\gamma_n|)_{n \in \mathbb{Z}}, \quad J = (J_n)_{n \in \mathbb{Z}}, \quad J_n = |A_n|^\frac{1}{2} \geq 0, \quad A_n = \frac{2}{\pi} \int_{\gamma_n} v(z) dz \geq 0.
\]

For the defocussing cubic non-linear Schrödinger equation (a completely integrable infinite dimensional Hamiltonian system), \( A_n \) is an action variables (see [FM]). Recall the following identities from [KK1]

\[
\frac{2}{\pi} \int_{\mathbb{R}} v(z) dz = \frac{1}{\pi} \int_{\mathbb{C}} \int |z'(k) - 1|^2 dudv = \frac{1}{2} \| V \|^2,
\]

where \( \| V \|^2 = \int_0^1 (V_1^2(t) + V_2^2(t)) dt \). Korotyaev [K1] obtained the two-sided estimates for the case \( \ell^2 \), for example

\[
\| g \|_2 \leq 2 \| h \|_2 \leq \pi \| g \|_2 (2 + \| g \|_2^2),
\]

\[
\frac{1}{\sqrt{2}} \| g \|_2 \leq \| V \| \leq 2 \| g \|_2 (1 + \| g \|_2).
\]

Our main goal is to obtain similar estimates in terms of the \( \ell^p \) and \( \ell^p_\omega \)-norms. Our first results are devoted to estimates in terms of \( \ell^p \)-norms. Let below \( \frac{1}{p} + \frac{1}{q} = 1, p, q \geq 1 \).

**Theorem 1.1.** Let \( V \in L^1(0, 1) \). Then the following estimates hold true:

\[
2^{-p} \| g \|_p \leq \| h \|_p \leq 2 \| g \|_p (1 + \alpha_0^p \| g \|_p^p), \quad p \in [1, 2], \quad \alpha_0^p = \frac{2^{(p+3)p}}{\pi},
\]

\[
\| h \|_p \leq \frac{2}{\pi} C_p^2 \| g \|_q \left( 1 + \left( \frac{2C_p^2}{\pi^2} \right)^\frac{1}{p} \| g \|_q^\frac{2}{p} \right), \quad C_p = \left( \frac{\pi^2}{2} \right)^\frac{1}{p}, \quad p \geq 2, \quad \frac{1}{p} + \frac{1}{q} = 1.
\]
\[
\frac{\|g\|_p}{2} \leq \|J\|_p \leq \frac{2}{\sqrt{\pi}} \|g\|_p (1 + \alpha_0^0 \|g\|_p^p)^{1/2}, \quad p \geq 1,
\]

(1.7)

\[
\frac{\sqrt{\pi}}{2} \|J\|_p \leq \|h\|_p \leq 4 \|J\|_p (1 + \alpha_0^0 2p \|J\|_p^p), \quad p \geq 1.
\]

(1.8)

Estimates (2.1)-(2.4) are new for \( p \in [1, 2) \).

In the case \( z^-_n < z^+_n \) we define the effective masses \( \mu_n^\pm \) by

\[
z(k) - z^\pm_n = \frac{(k - \pi n)^2}{2 \mu_n^\pm} (1 + O(k - \pi n)) \quad \text{as} \quad z \to z^\pm_n.
\]

(1.9)

If \( |g_n| = 0 \), then we set \( \mu_n^\pm = 0 \). Define the sequence \( \mu^\pm = (\mu_n^\pm)_{n \in \mathbb{Z}} \). Our second result is

**Theorem 1.2.** Let \( h \in \ell^0_p, p \in [1, 2] \). Then the following estimates hold true

\[
\|h\|_\infty \leq \min \{2\pi \|\mu\|_\infty, \|J\|_{p,\omega}, 2\|g\|_{p,\omega}(1 + \alpha_0^0 \|g\|_{p,\omega}^p)^{1/2}\},
\]

(1.10)

\[
\|g\|_{p,\omega} \leq 2\|h\|_{p,\omega} \leq c_0 \|g\|_{p,\omega}, \quad c_0 = e^{4\pi \|h\|_\infty}.
\]

(1.11)

\[
\|g\|_{p,\omega} \leq 2\|J\|_{p,\omega} \leq c_0^5 \|g\|_{p,\omega},
\]

(1.12)

\[
\frac{\sqrt{\pi}}{2} \|J\|_{p,\omega} \leq \|h\|_{p,\omega} \leq c_0^5 \sqrt{\frac{\pi}{2}} \|J\|_{p,\omega},
\]

(1.13)

\[
\|g\|_{p,\omega} \leq 2\|\mu\|_{p,\omega} \leq c_0^{18} \|g\|_{p,\omega}.
\]

(1.14)

Estimates (2.5)-(2.9) are new. Introduce the real Hilbert spaces \( \ell^2(m), m \in \mathbb{R} \) of the sequences \( \{f_n\}_{n=1}^\infty \) equipped with the norm \( \|f\|_{(m)}^2 = \sum_{n \geq 1} (2\pi n)^2 f_n^2 \). Korotyaev obtained the two-sided estimates for the \( \ell^2(m) \) norms, \( m \geq 0 \) for the even case \( h_{-n} = h_n, n \in \mathbb{Z} \) ([K2]-[K4]) and for \( \ell^1 \)-norms without symmetry ([K1], ([K6]) (in all these estimates the factor \( c_0 = e^{4\|h\|_\infty/\pi} \) is absent).

**Proposition 1.3.** Let \( V \in L^2(0,1) \). Then

\[
\|h\|_\infty \leq \|V\|,
\]

(1.15)

\[
\|V\|^2 \leq \frac{2}{\pi}\|h\|_p \|g\|_q, \quad p \geq 1,
\]

(1.16)

\[
\|V\|^2 \leq \left(\frac{2}{\pi}\right)^2 \|h\|_p^2 \|g\|_p^2, \quad p \in [1, 2],
\]

(1.17)

\[
\|V\|^2 \leq \frac{2}{\pi}\|h\|_\infty \|g\|_1 \leq \frac{4}{\pi^2} \|g\|_1^2,
\]

(1.18)

\[
\|h\|_\infty \leq \frac{2}{\pi}\|g\|_1, \quad \|g\|_1 \leq 2\|h\|_1.
\]

(1.19)
Note that the comb mappings are used in various fields of mathematics. We enumerate the more important directions:

1) the conformal mapping theory, 2) the Löwner equation and the quadratic differentials, 3) the electrostatic problems on the plane, 4) analytic capacity, 5) the spectral theory of the operators with periodic coefficients, 6) inverse problems for the Hill operator and the Dirac operator, 7) the KDV equation and the NLS equation with periodic initial value problem.

**Example of an electrostatic field.** Consider the system of neutral conductors $\Gamma_n$, $n \in \mathbb{Z}$ on the plane for some $h \in \ell^p_{\omega}$. In other words, we embed the system of neutral conductors $\Gamma_n$ in the external homogeneous electrostatic field $E_0 = (0, -1) \in \mathbb{R}^2$ on the plane. Then on each conductor there exists the induced charge, positive $e_n > 0$ on the lower half of the conductor $\Gamma_n$ and negative $(-e_n) < 0$ on the upper half of the conductor $\Gamma_n$, since their sum equals zero. As a result we have new perturbed electrostatic field $E \in \mathbb{R}^2$. It is well known that $E = iz'(k) = -\nabla y(k)$, $k = u + iv \in K(h)$, $z = x + iy$, where $z(k)$ is the conformal mapping from $K(h)$ onto the domain $Z = \mathbb{C} \cup \gamma g_n$. The function $y(k)$ is called the potential of the electrostatic field in $K(h)$. The density of the charge on the conductor has the form $\rho(k) = |y_u'(k)|/4\pi$, $k \in \Gamma_n$ (see $[LS]$). Thus we obtain the induced charge $e_n$ on the upper half of the conductor $\Gamma_n^+ = \Gamma_n \cap \mathbb{C}_+$ by:

$$e_n = \frac{1}{4\pi} \int_{\Gamma_n^+} x_v'(k) dv = \frac{1}{4\pi} |g_n|.$$  

Introduce the bipolar moment $d_n$ of the conductor $\Gamma_n$ with the charge density $\rho(k)$ by $d_n = \frac{1}{4\pi} \int_{\Gamma_n} vx_v(k) dv \geq 0$. We transform this value into the form

$$d_n = \frac{1}{2\pi} \int_{g_n} v(x) dx = \frac{A_n}{4}.$$  

In the paper $[KK3]$ we study inverse problems for both the charge mapping $h \to e = (e_n)_{n \in \mathbb{Z}}$ and the bipolar moment mapping $h \to J$ acting in $\ell^p_{\omega}, p \in [1, 2]$. In order to solve the inverse problems we need a priori estimates from Theorems $[1]$ and $[2]$.

A priori estimates of potentials in terms of spectral data essentially simplify the proof in the inverse problems. Such simplification was introduced by Garnett and Trubowitz $[GT1]$ and Kargaev and Korotyaev $[KK2]$ and essentially was used in $[K6]-[K9]$.

For the sake of the reader, we briefly recall the results existing in the literature about the a priori estimates. Firstly, we describe a priori estimates for the Hill operators $-\frac{d^2}{dx^2} + P$ in $L^2(\mathbb{R})$ with the real 1-periodic potential $P \in L^2(0, 1)$ and $\int_0^1 P(t) dt = 0$. The spectrum of this operator consists of intervals separated by gaps $\gamma_n, n \geq 1$ with the lengths $|\gamma_n| \geq 0, n \in \mathbb{Z}$. Marchenko and Ostrovski $[MO1-2]$ obtained the estimates: $\|P\| \leq C(1 + \|h\|_{\infty})\|h\|_{(1)}, \|h\|_{(1)} \leq C\|P\|e^{C\|P\|}$ for some absolute constant $C$, where $\|P\|^2 = \int_0^1 P^2(t) dt$ and $\|h\|_{(1)}^2 = \sum (2\pi n)^2 h_n^2$. These estimates are very rough since they used the Bernstein inequality. Using the harmonic measure argument Garnett and Trubowitz $[GT1]$ obtained $\|\gamma\| \leq (4 + \|h\|_{(1)})\|h\|_{(1)}$, where $\gamma = (|\gamma_n|)_{n \in \mathbb{Z}}$ and recall that $h_n$ are heights on the quasimomentum domain. First two-sided estimates (very rough) for $g, h$ were obtained in $[KK2]$. 


Consider a conformal mapping $(z : K_+(h) \to \mathbb{C}_+)$ with asymptotics $z(iv) = iv(1 + o(1))$ as $v \to \infty$, where $k = u + iv \in K(h)$. Here $h = (h_n)_{n \in \mathbb{Z}} \in \ell^\infty$, $h_n \geq 0$ is some sequence and $K_+(h) = \mathbb{C}_+ \cap K(h)$ is the so-called comb domain, where $K(h)$ is given by

$$K(h) = \mathbb{C} \setminus \cup_{n \in \mathbb{Z}} \Gamma_n, \quad \Gamma_n = [u_n - ih_n, u_n + ih_n], \quad u_* = \inf_n (u_{n+1} - u_n) \geq 0,$$

where $u_n, n \in \mathbb{Z}$ is strongly increasing sequence of real numbers such that $u_n \to \pm \infty$ as $n \to \pm \infty$. We fix the sequence $u_n, n \in \mathbb{Z}$ and consider the conformal mapping for various $h \in \ell^\infty$. For fixed $h$ the difference of any two such mappings equals a real constant, but the imaginary part $\Im(z(k))$ is unique. We call such mapping $z(k)$ the comb mapping. Define the inverse mapping $k(z) : \mathbb{C}_+ \to K_+(h)$. It is clear that $k(z)$ is a complex analytic function.

The estimates were obtained:

$$\|\gamma\|_2 \leq 6\|P\|(1 + \|P\|^{1/3}), \quad \|P\| \leq 4\|\gamma\|_2(1 + \|\gamma\|_2^{1/2}),$$

$$2\|h\|_{(1)} \leq \pi\|P\|(1 + \|P\|^{1/3}), \quad \|P\| \leq 3(6 + \|h\|_{\infty})^{1/2}\|h\|_{(1)}.$$

These estimates show the "equivalence" of the values $\|\gamma\|, \|h\|_{(1)}, \|P\|$. Note the author solved the inverse problem and obtained the two-sided estimates for the case $P = y'$, where $y \in L^2(0,1)$. A priori two-sided estimates for the case $P'(m) \in L^2(0,1), m \geq 1$ were obtained by Korotyaev in [K1].

Secondly for the Zakharov-Shabat systems the two-sided estimates were obtained by Korotyaev for $V \in L^2(0,1)$ in [K1] (see (1.3)-(1.4)) and for $V' \in L^2(0,1)$ in [K6]. There are no a priori estimates for the case $P'(m) \in L^2(0,1), m \geq 2$. The proof of a priori estimates in [K1]-[K6] is based on the analysis of the quasimomentum as the conformal mapping.

We shortly describe the proof. In order to prove Theorem 1.1-Proposition 1.3 we use the analysis of a conformal mapping corresponding to quasimomentum of the Zakharov-Shabat operator. That makes it possible to reformulate the problems for the differential operator as the problems of the conformal mapping theory. Then we should study the metric properties of a conformal mapping from $\mathbb{C}_+$ onto a "comb" $K_+(h)$. A similar analysis was done partially in [KK1-3], [K1-3]. In the present paper we use an approach, based on the identities for the Dirichlet integral (1.2) from [KK1] and the estimates from Theorem 2.7 and Lemma 3.1. We emphasize the important role of the Dirichlet integral (this is the energy for the conformal mapping) in this consideration.

We now describe the plan of the paper. In Section 2 we shall obtain some preliminaries results and "local basic estimates" in Theorem 2.7. In Section 3 we shall prove Lemma 3.1 and the main theorems. Moreover, we consider some examples, which describe our estimates.

## 2 Preliminaries

Consider a conformal mapping $z : K_+(h) \to \mathbb{C}_+$ with asymptotics $z(iv) = iv(1 + o(1))$ as $v \to \infty$, where $k = u + iv \in K(h)$. Here $h = (h_n)_{n \in \mathbb{Z}} \in \ell^\infty$, $h_n \geq 0$ is some sequence and $K_+(h) = \mathbb{C}_+ \cap K(h)$ is the so-called comb domain, where $K(h)$ is given by

$$K(h) = \mathbb{C} \setminus \cup_{n \in \mathbb{Z}} \Gamma_n, \quad \Gamma_n = [u_n - ih_n, u_n + ih_n], \quad u_* = \inf_n (u_{n+1} - u_n) \geq 0,$$

where $u_n, n \in \mathbb{Z}$ is strongly increasing sequence of real numbers such that $u_n \to \pm \infty$ as $n \to \pm \infty$. We fix the sequence $u_n, n \in \mathbb{Z}$ and consider the conformal mapping for various $h \in \ell^\infty$. For fixed $h$ the difference of any two such mappings equals a real constant, but the imaginary part $\Im(z(k))$ is unique. We call such mapping $z(k)$ the comb mapping. Define the inverse mapping $k(z) : \mathbb{C}_+ \to K_+(h)$. It is clear that $k(z)$ is a complex analytic function. We define "gaps" $g_n$, "bands" $\sigma_n$ and the "spectrum" $\sigma$ of the comb mapping by:

$$g_n = (z_n^-, z_n^+), \quad (z(u_n - 0), z(u_n + 0)), \quad \sigma_n = [z_{n-1}^+, z_n^-], \quad \sigma = \cup_{n \in \mathbb{Z}} \sigma_n.$$
The function \( u(z) = \text{Re} k(z) \) is strongly increasing on each band \( \sigma_n \) and \( u(z) = u_n \) for all \( z \in [z_n^-, z_n^+] \), \( n \in \mathbb{Z} \); the function \( v(z) = \text{Im} k(z) \) equals zero on each band \( \sigma_n \) and is strongly convex on each gap \( g_n \neq \emptyset \) and has the maximum at some point \( z_n \) given by \( v(z_n) = h_n \). If the gap is empty we set \( z_n = z_n^+ \). The function \( z(\cdot) \) has an analytic extension (by the symmetry) from the domain \( K_+(h) \) onto the domain \( K(h) \) and \( z(\cdot) : K(h) \to z(K(h)) = \mathbb{D} = \mathbb{C} \setminus \bigcup g_n \) is a conformal mapping. These and others properties of the comb mappings it is possible to find in the papers of Levin [Le].

We formulate our second result about the estimates for conformal mappings.

**Theorem 2.1.** Let \( u_* = \inf_n (u_{n+1} - u_n) > 0 \). Then the following estimates hold true

\[
\| h \|_p \leq 2 \| g \|_p (1 + \alpha_p \| g \|_p^p), \quad p \in [1, 2], \quad \alpha_p = \frac{(2 + \pi)^p 2^{p(p+2)}}{\pi u_*^p},
\]

(2.1)

\[
\| h \|_p \leq 2 \pi C_p^2 \| g \|_q \left( 1 + \left[ \frac{2C_p^2}{\pi u_*} \right]^{\frac{2}{p}} \| g \|_p^{\frac{2}{p}} \right), \quad C_p = \left( \frac{\pi}{2} \right)^{1/p}, \quad p \geq 2, \quad \frac{1}{p} + \frac{1}{q} = 1,
\]

(2.2)

\[
\| g \|_p \leq \| J \|_p \leq \frac{2}{\sqrt{\pi}} \| g \|_p (1 + \alpha_p \| g \|_p^{1/p})^2, \quad p \geq 1,
\]

(2.3)

(2.4)

We formulate our second result about the estimates for conformal mappings.

**Theorem 2.2.** Let \( h \in \ell^p, p \in [1, 2] \) and let \( u_* > 0 \). Then the following estimates hold true

\[
\| h \|_{L^\infty} \leq \min \{ 2\pi \| \mu \|_{L^\infty}, \| J \|_{L^{p, \omega}}, 2\pi \| \mu \|_{L^{p, \omega}} \} \frac{1}{\| g \|_{p, \omega}} \left( 1 + \alpha_p \| g \|_{p, \omega}^p \right)^{1/q},
\]

(2.5)

\[
\| g \|_{L^{p, \omega}} \leq 2 \| h \|_{L^{p, \omega}} \leq c^9 \| g \|_{p, \omega}, \quad c = c^{\| h \|_{L^\infty}/u_*},
\]

(2.6)

\[
\| g \|_{L^{p, \omega}} \leq 2 \| J \|_{L^{p, \omega}} \leq c^{18} \| g \|_{p, \omega},
\]

(2.7)

\[
\frac{\sqrt{\pi}}{2} \| J \|_{L^{p, \omega}} \leq \| h \|_{L^{p, \omega}} \leq c^5 \frac{\sqrt{\pi}}{2} \| J \|_{L^{p, \omega}},
\]

(2.8)

\[
\| g \|_{L^{p, \omega}} \leq 2 \| \mu \|_{L^{p, \omega}} \leq c^{18} \| g \|_{p, \omega}.
\]

(2.9)

Define the Dirichlet integral \( I_D \) and the moment \( Q_0 \) by

\[
I_D = \frac{1}{\pi} \int \int_{\mathbb{C}} |z'(k) - 1|^2 \, du \, dv = \frac{1}{\pi} \int \int_{\mathbb{C}} |k'(z) - 1|^2 \, dx \, dy, \quad Q_0 = \frac{1}{\pi} \int_{\mathbb{R}} v(z) \, dz,
\]

where \( k = u + iv, \quad z = x + iy \). The last identity holds since the Dirichlet integral is invariant under the conformal mappings. In order to prove our main theorems we need the following
Proposition 2.3. Let \( h \in \ell^\infty \) and let \( u_* \geq 0 \). Then

\[
\frac{\|h\|_\infty^2}{2} \leq Q_0, 
\]

\[
\pi Q_0 \leq \|h\|_p \|g\|_q, \quad p \geq 1, 
\]

\[
I_D \leq \left(\frac{2}{\pi}\right)^2 \|h\|_p^{2/p} \|g\|_q^{2/p}, \quad p \in [1, 2], 
\]

\[
\pi Q_0 \leq \|h\|_\infty \|g\|_1 \leq \frac{2}{\pi} \|g\|_1^2, 
\]

\[
\|h\|_\infty \leq \frac{2}{\pi} \|g\|_1, \quad \|g\|_1 \leq 2 \|h\|_1. 
\]

**Proof of Theorem 1.1 - Proposition 1.3** follow directly from Theorem 2.1 - Proposition 2.3 and the identity (1.2).

We recall needed results. Below we will use very often the following simple estimate

\[
|g_n| \leq 2h_n, \quad \text{all } n \in \mathbb{Z}, 
\]

see e.g. [MO1], [KK1]. Hence if \( h \in \ell^p_\omega \), then \( \gamma \in \ell^p_\omega \).

For each \( h \in \ell^\infty \) the following estimates and identities hold true

\[
\frac{1}{4} \|g\|^2 \leq 2Q_0 = I_D = \sum A_n = \|J\|^2 \leq \frac{2}{\pi} \sum_{n \in \mathbb{Z}} h_n |g_n|, 
\]

\[
\max\left\{\frac{|g_n|^2}{4}, \frac{|g_n| h_n}{\pi}\right\} \leq A_n = \frac{2}{\pi} \int_{g_n} v(x) dx \leq \frac{2|h_n|}{\pi}, 
\]

see [KK1]. These show that functional \( Q_0 = \frac{1}{\pi} \int_{\mathbb{R}} v(x) dx \) is bounded for \( h \in \ell^2 \).

Below we will sometimes write \( g_n(h), z(k, h), \ldots \) instead of \( g_n, z(k), \ldots \), when several sequences \( h \in \ell^\infty \) are being dealt with. Recall the Lindelöf principle (see [J]), which is formulated in the form, convenient for us (see [KK1]):

Let \( h, \tilde{h} \in \ell^\infty \); and let \( \tilde{h}_n \leq h_n \) for all \( n \in \mathbb{Z} \). Then the following estimates hold:

\[
y(k, \tilde{h}) \geq y(k, h), \quad \text{all } k \in K_+(h), 
\]

\[
Q_0(\tilde{h}) \leq Q_0(h) \quad \text{and if } \quad Q_0(\tilde{h}) = Q_0(h), \quad \text{then } \tilde{h} = h, 
\]

\[
|\sigma_n(\tilde{h})| \geq |\sigma_n(h)|. 
\]

Define the effective masses \( \nu_n \) in the plane \( K(h) \) for the end of the slit \([u_n + ih_n, u_n - ih_n], h_n > 0\) by

\[
k(z) - (u_n + ih_n) = \frac{(z - z_n)^2}{2i\nu_n}(1 + O(z - z_n)) \quad \text{as } \quad z \to z_n. 
\]

Thus we obtain \( \nu_n = 1/|k''(z_n)| \), if \( h_n > 0 \) and we set \( \nu_n = 0 \) if \( |g_n| = 0 \). We show the possibility of the Lindelöf principle in the following Lemma.
Lemma 2.4. For each \(h \in \ell^\infty\) the estimate (2.10) and the following estimate hold true

\[
\nu_n \leq h_n, \quad \text{all } n \in \mathbb{Z}. \tag{2.22}
\]

Proof. Sufficiently to proof for the case \(n = 0\). We apply estimate (2.18) to \(h\) and to the new sequence: \(\tilde{h}_0 = h_0\) and \(\tilde{h}_n = 0\) if \(n \neq 0\). It is clear that \(z(k, \tilde{h}) = \sqrt{(k - u_0)^2 + h_0^2}\) (the principal value). Then (2.18) gives

\[
y(k, h) \leq \text{Im} \left( \sqrt{(k - u_n)^2 + h_0^2} \right), \quad k \in K_+(h).
\]

Then asymptotics (2.21) of the function \(z(k, h)\) as \(k \to u_0 + ih_0\) yields (2.22). In order to prove (2.10) we use (2.19) since \(Q_0(\tilde{h}) = h_0^2/2\). Note that it and (2.19) yield (2.10). \(\blacksquare\)

We recall estimates from [K4].

Theorem 2.5. Let \(h \in \ell^\infty\). Then for any \(r > 0, n \in \mathbb{Z}\) the following estimate holds true

\[
h_n^2 \leq \frac{\pi}{4} \max \left\{ 1, \frac{h_n}{r} \right\} \int \int_{u_n + S_r} |z'(k)| \, |2dudv, \quad S_r = \{ z \in \mathbb{C} : |\text{Re} \, z| < r \}; \tag{2.23}
\]

If in addition, \(\inf_n (u_{n+1} - u_n) = u_* > 0\), then

\[
\frac{\pi}{4} I_D \leq \|h\|_2^2 \leq \frac{\pi^2}{2} \max \left\{ 1, \frac{\|h\|_\infty}{u_*} \right\} I_D \leq \frac{\pi^2}{2} \max \left\{ 1, \frac{I_{1/2}}{u_*} \right\} I_D, \tag{2.24}
\]

\[
\frac{1}{2} \|g\|_2 \leq \|h\|_2 \leq \pi \|g\|_2 \left( 1 + \frac{2}{u_*^2} \|g\|_2^2 \right), \tag{2.25}
\]

\[
\frac{\|g\|_2^2}{2} \leq \|J\|_2 \leq \sqrt{2} \|g\|_2 \left( 1 + \frac{\sqrt{2}}{u_*} \|g\| \right). \tag{2.26}
\]

In order to prove Theorem 2.7 we need the following result about the simple mapping and the domain \(S_r = \{ z \in \mathbb{C} : |\text{Re} \, z| < r \}, \ r > 0\).

Lemma 2.6. The function \(f(k) = \sqrt{k^2 + h^2}, \ k \in \mathbb{C} \setminus [-ih, ih], h > 0\) is the conformal mapping from \(\mathbb{C} \setminus [-ih, ih]\) onto \(\mathbb{C} \setminus [-h, h]\) and \(S_r \setminus [-h, h] \subset f(S_r \setminus [-ih, ih])\) for any \(r > 0\).

Proof. Consider the image of the half-line \(k = r + iv, v > 0\). We have the equations

\[
x^2 + y^2 = \xi \equiv r^2 + h^2 - v^2, \quad xy = rv. \tag{2.27}
\]

The second identity in (2.27) yields \(x > 0\) since \(y > 0\). Then \(x^4 - \xi x^2 - r^2v^2 = 0\), and enough to check the following inequality \(x^2 = \frac{1}{2}(\xi + \sqrt{\xi^2 + 4r^2v^2}) > r^2\). The last estimate follows from the simple relations

\[
(r^2 + h^2 - v^2)^2 + 4r^2v^2 > (r^2 + v^2 - h^2)^2, \quad 4r^2v^2 > 4r^2(v^2 - h^2). \quad \blacksquare
\]

We prove the local estimates for the small slits, which are crucial for us.
Theorem 2.7. Let $h \in \ell^\infty$. Assume that $(u_n - r, u_n + r) \subset (u_{n-1}, u_{n+1})$ and $h_n \leq \frac{r}{2}$, for some $n \in \mathbb{Z}$ and $r > 0$. Then

$$|h_n - |\mu^+_n|| \leq \frac{2 + \pi}{r} |\mu^+_n| \sqrt{I_n}, \quad I_n = \frac{1}{\pi} \int_{u_{n-1} + S_r} |z'(k) - 1|^2 \, du \, dv, \quad (2.28)$$

$$0 \leq h_n - \nu_n \leq \frac{2 + \pi}{r} h_n \sqrt{I_n}, \quad (2.29)$$

$$0 \leq h_n - \frac{|g_n|}{2} \leq \frac{2 + \pi}{r} h_n \sqrt{I_n}. \quad (2.30)$$

**Proof.** Define the functions $f(k) = \sqrt{k^2 + h_n^2}$, $k \in S_r \setminus [-ih_n, ih_n], \phi = f^{-1}$ and $F(w) = z(u_n + \phi(w), h), w = p + iq$, where the variable $w \in G_1 = f((S_r \setminus [-ih_n, ih_n])$. The function $F$ is real for real $w$, then $F$ is analytic in the domain $G = G_1 \cup [-h_n, h_n]$ and Lemma 2.6 yields $S_r \subset G$. Let now $|w_1| = \frac{r}{2}$ and $B_r = \{ z : |z| < r \}$. Then the following estimates hold

$$\sqrt{\pi} \frac{r}{2} F'(w_1) - 1 \leq \left( \int_{B_r} |F'(w) - 1|^2 \, dp \, dq \right)^{1/2} \leq \left( \int_{B_r} |(F(w) - \phi(w))'|^2 \, dp \, dq \right)^{1/2} + \left( \int_{B_r} |\phi'(w) - 1|^2 \, dp \, dq \right)^{1/2}. \quad (2.31)$$

The invariance of the Dirichlet integral with respect to the conformal mapping gives

$$\int_{B_r} |(F(w) - \phi(w))'|^2 \, dp \, dq = \int_{\phi(B_r)} |z'(k) - 1|^2 \, du \, dv \leq \int_{S_r + u_n} |z'(k) - 1|^2 \, du \, dv = \pi I_n. \quad (2.32)$$

Moreover, the identity $2Q_0 = I_D$ implies

$$\frac{1}{\pi} \int_{B_r} |\phi'(w) - 1|^2 \, dp \, dq \leq \frac{1}{\pi} \int_{C} |\phi'(w) - 1|^2 \, dp \, dq = \frac{2}{\pi} \int_{-h_n}^{h_n} \sqrt{h_n^2 - x^2} \, dx = h_n^2. \quad (2.33)$$

Then (2.31)-(2.33) for $|w_1| = \frac{r}{2}$ yields $|F'(w_1) - 1| \leq \frac{2}{r} (\sqrt{I_n} + h_n)$, and (2.28) gives

$$h_n^2 \leq \frac{\pi^2}{4} \cdot \frac{1}{\pi} \int_{S_r + u_n} |z'(k) - 1|^2 \, du \, dv \leq \frac{\pi^2}{4} I_n. \quad (2.34)$$

Then for $|w_1| = \frac{r}{2}$ we have

$$|F'(w_1) - 1| \leq \frac{2}{r} (1 + \frac{\pi}{2}) \sqrt{I_n} = \frac{2 + \pi}{r} \sqrt{I_n}, \quad (2.34)$$

and the maximum principle yields the needed estimates for $|w_1| \leq r/2$.

We prove (2.28) for $\mu^+_n$. The definition of $\mu^+_n$ (see (1.3)) implies

$$F'(h_n) = \lim_{x \to h_n} z'(u_n + g(x)) \cdot g'(x) = \lim_{x \to h_n} \frac{g(x)}{\mu^+_n} \cdot \frac{x}{g(x)} = \frac{h_n}{\mu^+_n}. \quad (3.1)$$
The substitution of the last identity into (2.34) gives (2.28). The proof for $\mu_n$ is similar.

We show (2.29). The definition of $\nu_n$ (see (2.21)) yields

$$(z(k) - z_n)^2 = 2i\nu_n(k - u_n - ih_n)(1 + o(1)) \text{ as } k \to u_n + ih_n,$$

$$\phi(w) - ih_n = -\frac{i}{2h_n}(w - z_n)^2(1 + o(1)) \text{ as } w \to u_n.$$Then we have $F'(0) = \sqrt{\nu_n}$ and the substitution of the last identity into (2.34) shows

$$\left|\sqrt{\nu_n} - 1\right| \leq 2 + \pi r \sqrt{I_n},$$

which implies (2.30).

We prove the estimates in terms of the $\ell^p$-norms.

**Proof of Proposition 2.3.** We have proved (2.10) in Lemma 2.4. Estimate (2.16) and the Hölder inequality yield (2.11). Using (2.10), (2.16) and the Hölder inequality, we obtain

$$(\pi/2)I_D = \pi Q_0 \leq \sum h_n |g_n| \leq \|h\|_{\infty} \left|\sum h_n^{\frac{p}{q}} \|g\|^\frac{p}{p} \|h\|^\frac{q}{p},ight.$$  

$$(\pi/2)I_D \geq \frac{1}{2} \|g\|^\frac{2}{p} \|h\|^\frac{2}{p}, \text{ and } I_D \leq \left(\frac{2}{\pi}\right)^2 \|g\|^\frac{2}{p} \|h\|^\frac{2}{p}.$$Estimate (2.11) at $q = 1$ implies the first one in (2.13). The last result and (2.10) yield the first inequality in (2.14) and then the second one in (2.13). The second estimate in (2.14) follows from $|g_n| \leq 2h_n$, $n \in \mathbb{Z}$ (see (2.15)).

### 3 Proof of the mains theorems

**Proof of Theorem 2.1.** Let $p \in [1, 2]$ and $r = \frac{u_*}{2}$. Estimate (2.23) implies

$$h_n^2 \leq \frac{\pi^2}{4} \max\{1, \frac{h_n}{r}\} I_n, \quad I_n = \frac{1}{\pi} \int_{u_n + S_r} |z'(k) - 1|^2 du dv,$$

(3.1)

Hence

$$h_n \leq \frac{\pi^2}{u_*} I_n, \quad \text{if } h_n > \frac{u_*}{4}, \quad \text{and } h_n \leq \frac{\pi}{2} \sqrt{I_n}, \quad \text{if } h_n \leq \frac{u_*}{4}. \quad (3.2)$$Moreover, (2.30) yields

$$h_n \leq \frac{|g_n|}{2} + 2 \frac{2 + \pi}{u_*} h_n \sqrt{I_n}, \quad \text{if } h_n < \frac{u_*}{4}, \quad (3.3)$$

and then

$$h_n \leq 2\pi \frac{2 + \pi}{u_*} I_n, \quad \text{if } h_n < \frac{u_*}{4}, \quad |g_n| \leq h_n, \quad (3.4)$$
since $h_n \leq \frac{\pi}{2} \sqrt{I_n}$. Hence using (3.4), (3.2), we obtain

$$i f \quad |g_n| \leq h_n \quad \Rightarrow \quad h_n \leq 2\pi \frac{2+\pi}{u_*} I_n = C_1 I_n, \quad C_1 = 2\pi \frac{2+\pi}{u_*}.$$  

The last inequality and (2.10) yield

$$\|h\|_p \leq (\sum_{h_n \leq |g_n|} h_n) \frac{1}{p} + (\sum_{|g_n| \leq h_n} h_n h_n^{p-1}) \frac{1}{p} \leq \|g\|_p + C_1^\frac{1}{2} I_D^{\frac{p+1}{2p}}, \quad h_+ = \|h\|_\infty. \quad (3.5)$$

If we assume that $C_1^\frac{1}{2} I_D^{\frac{p+1}{2p}} \leq \|g\|_p$, then we obtain $\|h\|_p \leq 2 \|g\|_p$.

Conversely, if we assume that $\|g\|_p \leq C_1^\frac{1}{2} I_D^{\frac{p+1}{2p}}$, then (3.3), (2.12) implies

$$\|h\|_p \leq 2C_1^\frac{1}{p} \left[ \frac{2}{\pi} \right]^\frac{1}{p} \|g\|_p^{\frac{2p}{p+1}}.$$  

Hence

$$\|h\|_p^{\frac{1}{p}} \leq 2C_1^\frac{1}{p} \left[ \frac{2}{\pi} \right]^\frac{1}{p} \|g\|_p^{\frac{2p}{p+1}} \quad \Rightarrow \quad \|h\|_p \leq 2^\frac{p}{p+1} C_1^\frac{p}{p} (2/\pi)^{p+1} \|g\|_p^{\frac{1}{p}+p},$$

which yields (2.1).

Let $p \geq 2$. Using inequality (2.24), (2.10) we obtain

$$\|h\|_p \leq (\sum_{h_n \leq |g_n|} h_n^p) \frac{1}{p} \leq C_1^\frac{1}{2} I_D^{\frac{p+1}{2p}}, \quad b = b(h_+), \quad b(t) = \max\{1, t^{-1}\}, \quad h_+ = \|h\|_\infty. \quad (3.6)$$

Consider the case $b \leq 1$. Then (3.6), (2.11) imply

$$\|h\|_p^2 \leq C_2^2 I_D \leq C_2^2 (2/\pi) \|h\|_p \|g\|_q, \quad \text{and} \quad \|h\|_p \leq (2C_2^2)/(\pi) \|g\|_q.$$  

Consider the case $b > 1$. Then the substitution of (2.10), (2.11) into (3.6) yield

$$\|h\|_p \leq C_2^2 I_D \frac{u_*}{p} \|g\|_q \leq u_*^{-\frac{1}{p}} C_2^2 (2/\pi) \|h\|_p \|g\|_q^{\frac{p+1}{2p}},$$

and

$$\|h\|_p^{\frac{p+1}{2p}} \leq C_2^2 u_*^{-\frac{1}{p}} [(2/\pi) \|g\|_q], \quad \|h\|_p \|g\|_q^{\frac{p+1}{2p}} \leq (C_2^2 u_*^{-\frac{1}{p}}) \|g\|_q^{\frac{p+1}{2p}},$$

and combining these two cases we have (2.2).

Estimate $|g_n| \leq 2J_n$ (see (2.17)) yields the first one in (2.3). Relation (2.17) implies

$$\|J\|_p = \sum |J_n|^p \leq \sum (2/\pi)^p h_n^p |g_n|^p \leq (2/\pi)^p \|h\|_p^p \|g\|_p^p,$$

and using (2.1) we obtain the second estimate in (2.3):

$$\|J\|_p \leq \sqrt{2/\pi} \|g\|_p^{1/2} \sqrt{2} \|g\|_p (1 + \alpha_p \|g\|_p^p)^{1/2} = \frac{2}{\sqrt{\pi}} \|g\|_p (1 + \alpha_p \|g\|_p^p)^{1/2},$$

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recall that $\alpha_p = (2^{p+2}(2 + \pi)/u_*)^p/\pi$. Inequality $J_n^2 \leq 4h_n^2/\pi$ (see (2.17)) yields the first estimate in (2.4). Using (2.1) and $\|g\|_p \leq 2\|J\|_p$ (see (2.3)) we deduce that

$$\|h\|_p \leq 2\|g\|_p(1 + \alpha_p\|g\|_p^p) \leq 4\|J\|_p(1 + \alpha_p 2^p\|J\|_p^p).$$

Recall the following identity for $v(z) = \text{Im} k(z), z = x + iy$ from [KK1]:

$$v(x) = v_n(x)(1 + Y_n(x)), \quad Y_n(x) = \frac{1}{\pi} \int_{\mathbb{R}\setminus g_n} \frac{v(t)dt}{|t - x|} v_n(t), \quad v_n(x) = |(x - z_n^+(x - z_n^-)|^\frac{1}{2},$$

for all $x \in g_n = (z_n^-, z_n^+)$. In order to prove Theorem 2.2 we need the following results.

**Lemma 3.1.** Let $h \in \ell^\infty$ and $u_*>0$ and $c = e^{-\|h\|_\infty}/u_*$. Then the following estimates hold:

$$s = \inf |\sigma_n| \leq u_* \leq \frac{\pi s}{2} \max \left\{ e^2, c \frac{s}{\pi} \right\},$$

$$1 + \frac{2\|h\|_\infty}{s\pi} \leq c^9,$$

$$\max_{n \in g_n} Y_n(x) \leq \frac{2\|h\|_\infty}{\pi s}, \quad n \in \mathbb{Z},$$

$$2h_n \leq |g_n|(1 + \max_{n \in g_n} Y_n(x)) \leq |g_n|(1 + \frac{2\|h\|_\infty}{\pi s}) \leq |g_n|c^9, \quad n \in \mathbb{Z}.$$  

**Proof.** Introduce the domain

$$G = \{ z \in \mathbb{C} : h_+ \geq \text{Im} z > 0, \ \text{Re} z \in (-\frac{u_*}{2}, \frac{u_*}{2}) \} \cup \{ \text{Im} z > h_+ \}, \quad h_+ = \|h\|_\infty.$$ 

Let $F$ be the conformal mapping from $G$ onto $\mathbb{C}_+$, such that $F(iy) \sim iy$ as $y \nearrow +\infty$ and let $\alpha, \beta$ be images of the points $\frac{u_*}{2}, \frac{u_*}{2} + ih_+$ respectively. Define the function $f = \text{Im} F$. Fix any $n \in \mathbb{Z}$. Then the maximum principle yields

$$y(k) = \text{Im} z(k, h) \geq f(k - p_n), \quad k \in G + p_n, \quad p_n = \frac{1}{2}(u_{n-1} + u_n).$$

Due to the fact that these positive functions equal zero on the interval $(p_n - \frac{u_*}{2}, p_n + \frac{u_*}{2})$, we obtain

$$\partial_y y(x) = \partial_x x(x) \geq \partial_x f(x - p_n), \quad x \in (p_n - \frac{u_*}{2}, p_n + \frac{u_*}{2}),$$

where $\partial_x = \frac{\partial}{\partial x}$. Then

$$z(u_n) - z(u_{n-1}) \geq \int_{-u_*/2}^{u_*/2} \partial_x f(x) \, dx = 2\alpha > 0,$$
and the estimate \( s \leq u_* \) (see [KK1]) implies \( 2\alpha \leq s \leq u_* \). Let \( w : \mathbb{C}_+ \to G \) be the inverse function for \( F \), which is defined uniquely and the Christoffel-Schwartz formula yields

\[
w(z) = \int_0^z \sqrt{\frac{t^2 - \beta^2}{t^2 - \alpha^2}} dt, \quad 0 < \alpha < \beta.
\]

Then we have

\[
u_* = \int_0^\alpha \sqrt{\frac{\beta^2 - t^2}{\alpha^2 - t^2}} dt, \quad h_+ = \int_\alpha^\beta \sqrt{\frac{\beta^2 - t^2}{t^2 - \alpha^2}} dt. \tag{3.12}
\]

The first integral in (3.12) has the simple double-sided estimates

\[
\alpha = \int_0^\alpha dt \leq \frac{u_*}{2} \leq \int_0^\alpha \frac{\beta dt}{\sqrt{\alpha^2 - t^2}} = \frac{\beta \pi}{2},
\]

that is

\[
2\alpha \leq s \leq u_* \leq \pi \beta. \tag{3.13}
\]

Consider the second integral in (3.12). Let \( \varepsilon = \beta/\alpha \geq 5 \) and using the new variable \( t = \alpha \cosh r, \cosh \delta = \varepsilon \), we obtain

\[
h_+ = \alpha \int_0^\delta \sqrt{\varepsilon^2 - \cosh^2 r} dr \geq \alpha \varepsilon \int_0^{\delta/2} \sqrt{1 - \frac{\cosh^2 r}{\varepsilon^2}} dr \geq \beta \delta \frac{2}{5},
\]

since for \( r \leq \delta/2 \) we have the simple inequality

\[
\frac{\cosh^2 r}{\cosh^2 \delta} \leq e^{-\delta}(1 + e^{-\delta})^2 \leq \varepsilon^{-1}(1 + \varepsilon^{-1})^2.
\]

Due to \( \varepsilon \leq e^{\delta} \) we get \( \varepsilon \leq \exp(5h_+/2\beta) \) and estimate (3.13) implies

\[
\frac{1}{s} \leq \frac{\pi}{2u_*} \exp\left(\frac{5\pi}{2u_*} h_+\right), \quad \text{if} \quad \varepsilon \geq 5. \tag{3.14}
\]

If \( \varepsilon \leq 5 \), then using (3.13) again we obtain

\[
\frac{1}{s} \leq \frac{\varepsilon}{2\beta} \leq \frac{\pi \varepsilon}{2u_*} \leq \frac{\pi}{2u_*} 5, \quad \text{if} \quad \varepsilon \leq 5.
\]

and the last estimate together with (3.14) yield (3.8), (3.9).

Identity (3.7) for \( x \in g_n = (z_n^-, z_n^+) \) implies

\[
\pi Y_n(x) = \int_{-\infty}^{z_n^- - s} \frac{v(t) dt}{|t - x|v_n(t)} + \int_{z_n^+ + s}^{\infty} \frac{v(t) dt}{|t - x|v_n(t)} \leq \int_{-\infty}^{z_n^- - s} \frac{h_+ dt}{|t - z_n^-|^2} + \int_{z_n^+ + s}^{\infty} \frac{h_+ dt}{|t - z_n^+|^2} \leq \frac{2h_+}{s}.
\]

Using (3.7), (3.9) and simple inequality \( v_n(z_n) \leq |g_n|/2 \) we have (3.11). We prove the two-sided estimates of \( h_n, |g_n|, \mu_n^\pm, J_n \) in the weight spaces.
Proof of Theorem 2.2. The first estimate in (2.5) follows from \( h_n \leq 2\pi |\mu_n^+| \) (see [KK1]). The second one in (2.5) follows from \( \|h\|_\infty \leq \sqrt{I_D} = \|J\|_2 \leq \|f\|_p \leq \|f\|_{p,\omega} \) since \( \omega_n \geq 1 \) for any \( n \in \mathbb{Z} \). Moreover, substituting (2.12) into \( \|h\|_\infty \leq \sqrt{I_D} \), using (2.1) and \( \|f\|_p \leq \|f\|_{p,\omega} \) for any \( f \), we obtain the last estimate in (2.5).

Recall that \( c = \exp \|h\|_{u_\omega} \). The first estimate in (2.6) follows from (2.15). Due to (3.11) we get \( 2h_n \leq c^9|g_n| \), which yields the second estimate in (2.6).

The first estimate in (2.7) follows from (2.17). Using (2.17), (3.11) we have \( J_n^2 \leq 2|g_n|h_n/\pi \leq (c^9/\pi)|g_n|^2 \), which gives the second estimate in (2.7).

The first estimate in (2.8) follows from (2.17), (2.15). Using (2.17), (3.11) we obtain \( h_n^2 \leq c^9|g_n|h_n/2 \leq (\pi c^9/2)J_n^2 \), which yields the second inequality in (2.8).

Identity \( 2|\mu_n^\pm| = |g_n|[1 + Y_n(z_n^\pm)]^2 \) (see [KK1]) implies \( 2|\mu_n^\pm| \geq |g_n| \), which yields the first inequality in (2.9). Moreover, using (3.11) we obtain the estimate \( 2|\mu_n^\pm| \leq c^{18}|g_n| \), which gives the second one in (2.9).

Recall that for a compact subset \( \Omega \subset \mathbb{C} \) the analytic capacity is given by

\[
\mathcal{C} = \mathcal{C}(\Omega) = \sup \left[ |f'(\infty)| : f \text{ is analytic in } \mathbb{C} \setminus \Omega, \ |f(k)| \leq 1, \ k \in \mathbb{C} \setminus \Omega \right],
\]

where \( f'(\infty) = \lim_{|k| \to \infty} k(f(k) - f(\infty)) \). We will use the well known Theorem (see [Iv], [Po]).

Theorem (Ivanov-Pommerenke). Let \( E \subset \mathbb{R} \) be compact. Then the analytic capacity \( \mathcal{C}(E) = |E|/4 \), where \( |E| \) is the Lebesgue measure (the length) of the set \( E \). Moreover, the Ahlfors function \( f_E \) (the unique function, which gives sup in the definition of the analytic capacity) has the following form:

\[
f_E(z) = \frac{\exp \left( \frac{1}{2} \phi_E(z) \right) - 1}{\exp \left( \frac{1}{2} \phi_E(z) \right) + 1}, \quad \phi_E(z) = \int_E \frac{dt}{z-t}; \quad z \in \mathbb{C} \setminus E.
\]

We will use the following simple remark: Let \( S_1, S_2, \ldots, S_N \) be disjoint continua in the plane \( \mathbb{C} \); \( D = \mathbb{C} \setminus \bigcup_{n=1}^N S_n \). Introduce the class \( \Sigma'(D) \) of the conformal mapping \( w \) from the domain \( D \) onto \( \mathbb{C} \) with the following asymptotics: \( w(k) = k + [Q(w) + o(1)]/k, \ k \to \infty \). If \( \Omega \subset \mathbb{C} \) is compact; \( D = \mathbb{C} \setminus \Omega, \ g \in \Sigma'(D) \), then \( \mathcal{C}(\Omega) = \mathcal{C}(\mathbb{C} \setminus g(D)) \). It follows immediately from the definition of the analytic capacity.

Let \( \ell^2_{fin} \subset \ell^2 \) be the subset of finite sequences of non negative numbers. Then, using the Ivanov-Pommerenke Theorem and the last remark we obtain

\[
\|g(h)\|_1 = \mathcal{C}(\Gamma(h)), \quad \text{where} \quad h \in \ell^2_{fin}, \ \Gamma(h) = \bigcup [u_n - ih_n, u_n + ih_n];
\]

Now we estimate the Dirichlet integral \( I_D(h) = 2Q_0(h) \) for the case \( u_* \geq 0 \), using a geometric construction.

Theorem 3.2. Let \( h \in \ell^\infty, h_n \to 0 \) as \( |n| \to \infty \); and let \( \tilde{h} = \tilde{h}(h) \) be given by

if \( h = 0 \), then \( \tilde{h} = 0 \),

if \( h \neq 0 \), let an integer \( n_1 \) be such that \( \tilde{h}_{n_1} = h_{n_1} = \max_{n \in \mathbb{Z}} h_n > 0 \); assume that the numbers \( h_{n_1}, h_{n_2}, \ldots, h_{n_k} \) are defined, then \( n_{k+1} \) is given by

\[
\tilde{h}_{n_{k+1}} = h_{n_{k+1}} = \max_{n \in B} h_n > 0, \quad B = \{ n \in \mathbb{Z} : |u_n - u_{n_s}| > h_{n_s}, 1 \leq s \leq k \},
\]

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and let $\tilde{h}_n = 0$, if $n \notin \{n_k, k \in \mathbb{Z}\}$.

Then the following estimates hold:

$$\frac{1}{\pi^2} \|\tilde{h}\|_2^2 \leq Q_0(h) = \frac{I_D(h)}{2} \leq \frac{2\sqrt{2}}{\pi} \|\tilde{h}\|_2^2.$$  (3.18)

**Proof.** The Lindelöf principal yields $Q_0(\tilde{h}) \leq Q_0(h)$. On the other hand open squares $P_k = (u_{n_k} - t_k, u_{n_k} + t_k) \times (-t_k, t_k), t_k \equiv h_{n_k} = \tilde{h}_{n_k}, k \in \mathbb{Z}$, does not overlap. Then applying (2.23) to the function $(z(k, \tilde{h}) - k)$ and $P_k$, we obtain

$$2t_k^2 \leq \pi \int \int_{P_k} |z'(k, \tilde{h}) - 1|^2 \, du \, dv,$$  (3.19)

and

$$Q_0(\tilde{h}) = \frac{1}{2} I_D(\tilde{h}) \geq \frac{1}{\pi^2} \sum_{k \geq 1} t_k^2 = \frac{1}{\pi^2} \|\tilde{h}\|_2^2$$

which yields the first estimate in (3.18).

Let $\Omega_k = \{n \in \mathbb{Z} : u_n \in [u_{n_k} - t_k, u_{n_k} + t_k]\}$. By the Lindelöf principal, the gap length $|g_{n_k}|$ such that $[u_n, u_n + ih_n], n \in \Omega_k$ increases if we take off all another slits. By the Ivanov-Pomerenke Theorem (see above), the sum of new gap lengths equals to $4 \times \text{capacity}$ of the set $E = \cup_{n \in \Omega_k} [u_n - h_n, u_n + h_n]$, which is less than the diameter of the set $E$. Then

$$\sum_{n \in \Omega_k} |g_n(h)| \leq 2\sqrt{2}t_k,$$

and using the last estimate we obtain

$$\pi Q_0(h) \leq \sum_{n \in \mathbb{Z}} h_n |g_n| \leq \sum_{k \geq 1} \sum_{n \in \Omega_k} h_n |g_n| \leq \sum_{k \geq 1} t_k \sum_{n \in \Omega_k} |g_n| \leq 2\sqrt{2} \sum_{k \geq 1} t_k^2 = 2\sqrt{2} \|\tilde{h}\|_2^2,$$  (3.20)

which is less than the diameter of the set $E$, which is less than or equals $2\sqrt{2}t_k$. ■

Note that the proved Theorem shows that estimates (2.25), (2.26) hold true for the weaker conditions on the sequence $u_n, n \in \mathbb{Z}$.

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