Small generating sets for the Torelli group

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Proving a conjecture of Dennis Johnson, we show that the Torelli subgroup \( \mathcal{I}_g \) of the genus \( g \) mapping class group has a finite generating set whose size grows cubically with respect to \( g \). Our main tool is a new space called the handle graph on which \( \mathcal{I}_g \) acts cocompactly.

1 Introduction

Let \( \Sigma_{g,n} \) be a compact connected oriented genus \( g \) surface with \( n \) boundary components. The mapping class group of \( \Sigma_{g,n} \), denoted \( \text{Mod}_{g,n} \), is the group of orientation-preserving homeomorphisms of \( \Sigma_{g,n} \) that fix the boundary pointwise modulo isotopies that fix the boundary pointwise. We will often omit the \( n \) if it vanishes. For \( n \leq 1 \), the Torelli group, denoted \( \mathcal{I}_{g,n} \), is the kernel of the action of \( \text{Mod}_{g,n} \) on \( H_1(\Sigma_{g,n}; \mathbb{Z}) \).

The Torelli group has been the object of intensive study ever since the seminal work of Dennis Johnson in the early ’80’s. See [10] for a survey of Johnson’s work.

Finite generation of Torelli One of Johnson’s most celebrated theorems says that \( \mathcal{I}_{g,n} \) is finitely generated for \( g \geq 3 \) and \( n \leq 1 \) (see [9]). This is a surprising result – though \( \text{Mod}_{g,n} \) is finitely presentable, \( \mathcal{I}_{g,n} \) is an infinite-index normal subgroup of \( \text{Mod}_{g,n} \), so there is no reason to hope that \( \mathcal{I}_{g,n} \) has any finiteness properties. Moreover, McCullough and Miller [13] proved that \( \mathcal{I}_{2,n} \) is not finitely generated for \( n \leq 1 \), and later Mess [14] proved that \( \mathcal{I}_2 \) is an infinite rank free group.

Johnson’s generating set Johnson’s generating set for \( \mathcal{I}_{g,n} \) when \( g \geq 3 \) and \( n \leq 1 \) is enormous. Indeed, for \( \mathcal{I}_g \) (resp. \( \mathcal{I}_{g,1} \)), it contains \( 9 \cdot 2^{2g-3} - 4g^2 + 2g - 6 \) (resp. \( 9 \cdot 2^{2g-3} - 4g^2 + 4g - 5 \)) elements. In [11], Johnson proved that the abelianization of \( \mathcal{I}_g \) (resp. \( \mathcal{I}_{g,1} \)) has rank \( \frac{1}{3}(4g^3 + 5g + 3) \) (resp. \( \frac{1}{3}(4g^3 - g) \)). These give large lower bounds on the size of generating sets for \( \mathcal{I}_{g,n} \); however, there is a huge gap between this cubic lower bound and Johnson’s exponentially growing generating set. At the end of [9] and in [10, page 168], Johnson conjectures that there should be a generating set for \( \mathcal{I}_{g,n} \) whose size grows cubically with respect to the genus. Later, in

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Farb asked whether there at least exists a generating set whose size grows polynomially.

**Main theorem** In this paper, we prove Johnson’s conjecture. Our main theorem is as follows.

**Theorem A** For $g \geq 3$, the group $I_g$ has a generating set of size at most $57\binom{g}{3}$ and the group $I_{g,1}$ has a generating set of size at most $57\binom{g}{3} + 2g + 1$.

The generating set we construct was conjectured to generate $I_{g,n}$ by Brendle and Farb [2]. To describe it, we must introduce some notation. As in Figure 1(a), let $R'_1, \ldots, R'_g$ be $g$ subsurfaces of $\Sigma_g$ each homeomorphic to $\Sigma_{1,1}$ such that the following hold. Interpret all indices modulo $g$.

- If $1 \leq i < j \leq g$ satisfy $i \not\in \{j-1, j+1\}$, then $R'_i \cap R'_j = \emptyset$.
- For all $1 \leq i \leq g$, the intersection $R'_i \cap R'_{i+1}$ is homeomorphic to an interval.

For $1 \leq i < j < k \leq g$, define a subsurface $R_{ijk}$ of $\Sigma_g$ by $R_{ijk} = \Sigma_g \setminus \bigcup_{l \neq i,j,k} R'_l$. Thus $R_{ijk}$ is a genus 3 surface with at most 3 boundary components such that $R'_i, R'_j, R'_k \subset R_{i,j,k}$ (see Figure 1(b)).

If $S$ is a subsurface of $\Sigma_g$, define $\text{Mod}(\Sigma_g, S)$ to be the subgroup of $\text{Mod}_g$ consisting of mapping classes that can be realized by homeomorphisms supported on $S$ and $\mathcal{I}(\Sigma_g, S)$ to equal $\mathcal{I}_g \cap \text{Mod}(\Sigma_g, S)$. The key result for the proof of Theorem A is the following theorem.

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**Theorem B** For $g \geq 3$, the group $\mathcal{I}_g$ is generated by the set

$$\bigcup_{1 \leq i < j < k \leq g} \mathcal{I}(\Sigma_g, R_{ijk}).$$

Using Johnson’s work, it is easy to see that $\mathcal{I}(\Sigma_g, R_{ijk})$ is finitely generated by a generating set with at most 57 generators (see Lemma 2.2). Also, standard techniques (see Lemma 2.1) show that if $\mathcal{I}_g$ has a generating set with $k$ elements, then $\mathcal{I}_g,1$ has a generating set with $k + 2g + 1$ elements. Since there are $\binom{g}{3}$ subsurfaces $R_{ijk}$, Theorem A follows from Theorem B.

**Remark** To illustrate the relative sizes of our generating sets, Johnson’s generating set for $\mathcal{I}_{20}$ contains more than one trillion elements while our generating set for $\mathcal{I}_{20}$ has 64980 elements.

**New proof of Johnson’s theorem** Our deduction of Theorem A from Theorem B depends on Johnson’s theorem that $\mathcal{I}_3$ is finitely generated. However, Hain [6] has recently announced a direct conceptual proof that $\mathcal{I}_3$ is finitely generated. Hain’s proof uses special properties of the moduli space of genus 3 Riemann surfaces and cannot be easily generalized to $g > 3$. Combining this with our paper, we obtain a new proof that $\mathcal{I}_{g,n}$ is finitely generated for $g \geq 3$ and $n \leq 1$.

Our new proof is more conceptual than Johnson’s original one. To illustrate this, we will sketch Johnson’s proof. He starts by writing down an enormous finite subset $S \subset \mathcal{I}_{g,n}$ which is known (from work of Powell [15]) to normally generate $\mathcal{I}_{g,n}$ as a subgroup of $\text{Mod}_{g,n}$. Letting $T$ be a standard generating set for $\text{Mod}_{g,n}$, Johnson then proves via a laborious computation that for $t \in T$ and $s \in S$, the element $tst^{-1} \in \mathcal{I}_{g,n}$ can be written as a word in $S$. This implies that the subgroup $\Gamma'$ of $\mathcal{I}_{g,n}$ generated by $S$ is a normal subgroup of $\text{Mod}_{g,n}$, and thus that $\Gamma = \mathcal{I}_{g,n}$.

**Remark** Our proof of Theorem B appeals to a theorem of [17] whose proof depends on Johnson’s theorem. However, Hatcher and Margalit [12] have recently given a new proof of this result that is independent of Johnson’s work.

**Nature of generators** Some basic elements of $\mathcal{I}_{g,n}$ are as follows (see, eg [16]). If $x$ is a simple closed curve on $\Sigma_{g,n}$, then denote by $T_x \in \text{Mod}_{g,n}$ the Dehn twist about $x$. If $x$ is a separating simple closed curve, then $T_x \in \mathcal{I}_{g,n}$; these are called *separating twists*. If $x$ and $y$ are disjoint homologous nonseparating simple closed curves, then $T_x T_y^{-1} \in \mathcal{I}_{g,n}$; these are called *bounding pair maps*. Following work...
of Birman [1], Powell [15] proved that \( I_g, n \) is generated by bounding pair maps and separating twists for \( g \geq 1 \) and \( n \leq 1 \) (see [16] and [12] for alternate proofs). Johnson’s finite generating set for \( I_g, n \) for \( g \geq 3 \) and \( n \leq 1 \) consists entirely of bounding pair maps. It follows easily from our proofs of Lemma 2.1 and 2.2 that our generating set consists of bounding pair maps and separating twists; see the remark after Lemma 2.2.

The handle graph Our proof of Theorem B is topological. To prove that a group \( G \) is finitely generated, it is enough to find a connected simplicial complex upon which \( G \) acts cocompactly with finitely generated stabilizers. We use a variant on the curve complex. If \( \gamma \) is an oriented simple closed curve on \( \Sigma_g \), then denote by \([\gamma]\) its homology class. Also, if \( \gamma_1 \) and \( \gamma_2 \) are isotopy classes of simple closed curves on \( \Sigma_g \), then denote by \( i_g(\gamma_1, \gamma_2) \) their geometric intersection number, ie the minimal possible number of intersections between two curves in the isotopy classes of \( \gamma_1 \) and \( \gamma_2 \). Finally, denote by \( i_a(\cdot, \cdot) \) the algebraic intersection pairing on \( H_1(\Sigma_g; \mathbb{Z}) \).

Definition Let \( a, b \in H_1(\Sigma_g; \mathbb{Z}) \) satisfy \( i_a(a, b) = 1 \). The handle graph associated to \( a \) and \( b \), denoted \( \mathcal{H}_{a,b} \), is the graph whose vertices are isotopy classes of oriented simple closed curves on \( \Sigma_g \) that are homologous to either \( a \) or \( b \) and where two vertices \( \gamma_1 \) and \( \gamma_2 \) are joined by an edge exactly when \( i_g(\gamma_1, \gamma_2) = 1 \).

We will show that \( \mathcal{H}_{a,b}/I_g \) consists of a single edge (see Lemma 5.2) and that \( \mathcal{H}_{a,b} \) is connected for \( g \geq 3 \) (see Lemma 3.1).

A complication It would appear that we have all the ingredients in place to use the space \( \mathcal{H}_{a,b} \) to prove that \( I_g \) is finitely generated. However, there is one remaining complication. Namely, we do not know the answer to the following question.

Question 1.1 For some \( g \geq 4 \), let \( \gamma \) be the isotopy class of a nonseparating simple closed curve on \( \Sigma_g \). Is the stabilizer subgroup \( (I_g)_\gamma \) of \( \gamma \) finitely generated?

In other words, we do not know if the vertex stabilizer subgroups of the action of \( I_g \) on \( \mathcal{H}_{a,b} \) are finitely generated. Nonetheless, in Section 4 we will prove a weaker statement that suffices to prove Theorem B. The proof of Theorem B is in Section 5.

Smaller generating sets A positive answer to Question 1.1 would likely lead to a smaller generating set for \( I_g \), though of course this depends on the nature of the finite generating sets for the stabilizer subgroups. Let us describe one way this could work. For \( g \geq 3 \), let \( \sigma_g \) be the smallest cardinality of a generating set for \( I_g \). Consider

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$g \geq 4$, and fix an edge $\{\alpha, \beta\}$ of $\mathcal{H}_{a,b}$. The proof of Theorem B shows that $\mathcal{I}_g$ is generated by $(\mathcal{I}_g)_\alpha \cup (\mathcal{I}_g)_\beta$. Let $S$ be a subsurface of $\Sigma_g$ such that $S \cong \Sigma_{g-1,1}$ and $\alpha \cup \beta \subset \Sigma_g \setminus S$. We have $\mathcal{I}(\Sigma_g, S) \cong \mathcal{I}_{g-1,1}$ (see Section 2) and $\mathcal{I}(\Sigma_g, S) \subset (\mathcal{I}_g)_\alpha$ and $\mathcal{I}(\Sigma_g, S) \subset (\mathcal{I}_g)_\beta$. Assume that there exists a finite set $V_\alpha$ (resp. $V_\beta$) such that $\mathcal{I}_g/\alpha$ (resp. $\mathcal{I}_g/\beta$) is generated by $I_{g-1,1}$ can be generated by $\sigma_{g-1} + 2g + 1$ elements. Moreover, it seems likely that there exists some relatively small $K$ such that $|V_\alpha|, |V_\beta| \leq Kg^2$. This would imply that

$\sigma_g \leq \sigma_{g-1} + 2g + 1 + 2Kg^2$.

Iterating this, we would get that

$\sigma_g \leq \sigma_3 + \sum_{i=4}^{g} (2i + 1 + 2Ki^2)$

for $g \geq 4$. This bound is cubic in $g$ (as it needs to be), but as long as $K$ is not too large it is much smaller than $57\left(\frac{g}{3}\right)$.

**Finite presentability** Perhaps the most important open question about the combinatorial group theory of $\mathcal{I}_g$ is whether or not it is finitely presentable for $g \geq 3$. One way of proving that a group $G$ is finitely presentable is to construct a simply-connected simplicial complex $X$ upon which $G$ acts cocompactly with finitely presentable stabilizer subgroups (see, eg [3]). For example, Hatcher and Thurston use this technique in [7] to prove that the mapping class group is finitely presentable.

The handle graph $\mathcal{H}_{a,b}$ appears to be the first example of a useful space upon which $\mathcal{I}_g$ acts cocompactly (of course, there are trivial non-useful examples of such spaces; for example, the Cayley graph of $\mathcal{I}_g$ or a 1-point space). Unfortunately, while $\mathcal{H}_{a,b}$ is connected for $g \geq 3$, it is not simply connected. Indeed, it does not even have any 2–cells (and is not a tree). However, one could probably attach 2–cells to $\mathcal{H}_{a,b}$ to obtain a simply connected complex upon which $\mathcal{I}_g$ acts cocompactly. This would not be enough, however – one would also have to prove that the simplex stabilizer subgroups were finitely presentable. In other words, this complex would provide the inductive step in a proof that $\mathcal{I}_g$ was finitely presentable, but one would still need a base case.

**A complex that does not work** We close this introduction by discussing an approach to Theorem B that does not work. One might think of trying to prove Theorem B using the following complex. Let $a \in H_1(\Sigma_g; \mathbb{Z})$ be a primitive vector. Define $C_a$ to be the graph whose vertices are isotopy classes of oriented simple closed curves $\gamma$ on $\Sigma_g$ such that $[\gamma] = a$ and where two vertices $\gamma$ and $\gamma'$ are joined by an edge if

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It is known ([17, Theorem 1.9]; see [12] for an alternate proof) that $C_g$ is connected for $g \geq 3$. Moreover, $\mathcal{I}_g$ acts transitively on the vertices of $C_g$. However, it does not act cocompactly; indeed, there are infinitely many edge orbits. To see this, consider edges $e_1 = \{\gamma_1, \gamma'_1\}$ and $e_2 = \{\gamma_2, \gamma'_2\}$ of $C_g$. Assume that there exists some $f \in \mathcal{I}_g$ such that $f(e_1) = e_2$. Since $\gamma_1$ is homologous to $\gamma'_1$, the multicurve $\gamma_1 \cup \gamma'_1$ divides $\Sigma_g$ into two subsurfaces $S_1$ and $S'_1$. Similarly, $\gamma_2 \cup \gamma'_2$ divides $\Sigma_g$ into two subsurfaces $S_2$ and $S'_2$. Relabeling if necessary, we have $f(S_1)$ isotopic to $S_2$ and $f(S'_1)$ isotopic to $S'_2$. Since $f \in \mathcal{I}_g$, the images of $H_1(S_1; \mathbb{Z})$ and $H_1(S_2; \mathbb{Z})$ in $H_1(\Sigma_g; \mathbb{Z})$ must be the same, and similarly for $H_1(S'_1; \mathbb{Z})$ and $H_1(S'_2; \mathbb{Z})$. It is easy to see that infinitely many such images occur for different edges of $C_g$, so there must be infinitely many edges orbits. We remark that Johnson proved in [8, Corollary to Lemma 9 on page 250] that the images of $H_1(S_1; \mathbb{Z})$ and $H_1(S'_1; \mathbb{Z})$ in $H_1(\Sigma_g; \mathbb{Z})$ are a complete invariant for the edge orbits.

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2 The Torelli group on subsurfaces

We will need to understand how the Torelli group restricts to subsurfaces. For a general discussion of this, see [16]. In this section, we will extract from [16] results on two kinds of subsurfaces. In Section 2.1, we will show how to analyze subsurfaces like the subsurfaces $R_{ijk}$ from Section 1. In Section 2.2, we will show how to analyze stabilizers of nonseparating simple closed curves (which are supported on the subsurface obtained by taking the complement of a regular neighborhood of the curve).

2.1 Analyzing the subsurfaces $R_{ijk}$

We begin by defining groups $\mathcal{I}_{g,n}$ for $n \geq 2$. There is a map $\text{Mod}_{g,n} \rightarrow \text{Mod}_g$ induced by gluing discs to the boundary components of $\Sigma_{g,n}$ and extending homeomorphisms by the identity. Define $\mathcal{I}_{g,n}$ to be the kernel of the resulting action of $\text{Mod}_{g,n}$ on $H_1(\Sigma_g; \mathbb{Z})$. For the case $n = 1$, the map $H_1(\Sigma_{g,1}; \mathbb{Z}) \rightarrow H_1(\Sigma_g; \mathbb{Z})$ is an isomorphism, so this agrees with our previous definition of $\mathcal{I}_{g,1}$.

Remark In [16], the different definitions of the Torelli group on a surface with boundary are parametrized by partitions of the boundary components. The above definition of $\mathcal{I}_{g,n}$ corresponds to the discrete partition $\{\beta_1, \ldots, \beta_n\}$ of the set $\{\beta_1, \ldots, \beta_n\}$ of boundary components of $\Sigma_{g,n}$.
In [16, Theorem 1.2], a version of the Birman exact sequence is proven for the Torelli group. For \( \mathcal{I}_{g,n} \) with \( g \geq 2 \), it takes the form

\[
1 \rightarrow \pi_1(U_{\Sigma_{g,n}}) \rightarrow \mathcal{I}_{g,n+1} \rightarrow \mathcal{I}_{g,n} \rightarrow 1.
\]

Here \( U_{\Sigma_{g,n}} \) is the unit tangent bundle of \( \Sigma_{g,n} \). The subgroup \( \pi_1(U_{\Sigma_{g,n}}) \) of \( \mathcal{I}_{g,n+1} \) is often called the “disc-pushing subgroup” – the mapping class associated to \( \gamma \in \pi_1(U_{\Sigma_{g,n}}) \) “pushes” a fixed boundary component around \( \gamma \) while allowing it to rotate.

The following is an immediate consequence of (1) and the fact that \( \pi_1(U_{\Sigma_{g,n}}) \) can be generated by \( 2g + 1 \) elements.

**Lemma 2.1** \( \mathcal{I}_{g,1} \) can be generated by \( k + 2g + 1 \) elements if \( \mathcal{I}_g \) can be generated by \( k \) elements.

Now assume that \( S \cong \Sigma_{h,n} \) is an embedded subsurface of \( \Sigma_g \) and that all the boundary components of \( S \) are non-nullhomotopic separating curves in \( \Sigma_g \). For example, \( S \) could be one of the surfaces \( R_{ijk} \) from Section 1. Letting \( \text{Mod}(S) \) be the mapping class group of \( S \), the induced map \( \mathcal{I}(S) \rightarrow \text{Mod}_g \) is an injection. This gives a natural identification of \( \text{Mod}(S) \) with \( \text{Mod}(\Sigma_{g,S}) \). The group \( \mathcal{I}(\Sigma_{g,S}) \) is thus naturally a subgroup of \( \text{Mod}(S) \cong \text{Mod}_{h,n} \), and in [16, Theorem 1.1] it is proven that \( \mathcal{I}(\Sigma_{g,S}) = \mathcal{I}_{h,n} \). Johnson [9] proved that \( \mathcal{I}_3 \) can be generated by 35 elements. Applying (1) repeatedly, we see that \( \mathcal{I}_{3,1} \) can be generated by 42 elements, \( \mathcal{I}_{3,2} \) by 49 elements, and \( \mathcal{I}_{3,3} \) by 57 elements. Since \( R_{ijk} \cong \Sigma_{3,k} \) with \( k \leq 3 \), we obtain the following.

**Lemma 2.2** For all \( 1 \leq i < j < k \leq g \), the group \( \mathcal{I}(\Sigma_g, R_{ijk}) \) can be generated by 57 elements.

**Remark** It is well-known (see, eg [16, Section 2.1]) that the mapping classes corresponding to the generators of \( \pi_1(U_{\Sigma_{g,n}}) \) used to prove Lemmas 2.1 and 2.2 can be chosen to be bounding pair maps and separating twists. Additionally, Johnson’s minimal-size generating set for \( \mathcal{I}_3 \) consists entirely of bounding pair maps, so the generating set for \( \mathcal{I}(\Sigma_g, R_{ijk}) \) in Lemma 2.2 can be taken to consist of bounding pair maps and separating twists.

### 2.2 Stabilizers of nonseparating simple closed curves

Let \( \gamma \) be a nonseparating simple closed curve on \( \Sigma_g \). Define \( \Sigma_{g,\gamma} \) to be the result of cutting \( \Sigma_g \) along \( \gamma \), so \( \Sigma_{g,\gamma} \cong \Sigma_{g-1,2} \). Letting \( \text{Mod}_{g,\gamma} \) be the mapping class group of \( \Sigma_{g,\gamma} \), the natural map \( \Sigma_{g,\gamma} \rightarrow \Sigma_g \) induces a map \( i: \text{Mod}_{g,\gamma} \rightarrow \text{Mod}_g \). Define \( \mathcal{I}_{g,\gamma} = i^{-1}(\mathcal{I}_g) \). The map \( i \) restricts to a surjection \( \mathcal{I}_{g,\gamma} \rightarrow (\mathcal{I}_g)_\gamma \), where \( (\mathcal{I}_g)_\gamma \) is the stabilizer subgroup of \( \gamma \).
Remark In the notation of [16], the group $\mathcal{I}_{g,\gamma}$ corresponds to the Torelli group of $\Sigma_{g-1,2}$ with respect to the “indiscrete partition” $\{\beta, \beta'\}$ of the boundary components $\beta$ and $\beta'$ of $\Sigma_{g,\gamma}$. Also, the kernel of the map $\mathcal{I}_{g,\gamma} \to (\mathcal{I}_g)_{\gamma}$ is isomorphic to $\mathbb{Z}$ and is generated by $T_\beta T_{\beta'}^{-1}$, where $T_\beta$ and $T_{\beta'}$ are the Dehn twists about $\beta$ and $\beta'$, respectively.

In [16, Theorem 1.2], it is proven that for $g \geq 2$ there is a short exact sequence

$$1 \longrightarrow K_{g,\gamma} \longrightarrow \mathcal{I}_{g,\gamma} \longrightarrow \mathcal{I}_{g-1,1} \longrightarrow 1.$$  

Here $K_{g,\gamma} \cong [\pi_1(\Sigma_{g-1,1}), \pi_1(\Sigma_{g-1,1})]$. This exact sequence splits via the inclusion $\mathcal{I}_{g-1,1} \hookrightarrow \mathcal{I}_{g,\gamma}$ induced by the inclusion $\Sigma_{g-1,1} \hookrightarrow \Sigma_{g,\gamma}$ indicated in Figure 2(a). In other words, the following holds.

**Lemma 2.3** $\mathcal{I}_{g,\gamma} = K_{g,\gamma} \times \mathcal{I}_{g-1,1}$ for $g \geq 3$ and $\gamma$ a simple closed nonseparating curve on $\Sigma_g$.

The group $\mathcal{I}_{g-1,1}$ acts on $K_{g,\gamma} < \pi_1(\Sigma_{g-1,1})$ as follows. As is clear from [16, Theorem 1.2], the basepoint for $\pi_1(\Sigma_{g-1,1})$ is as indicated in Figure 2(b). As shown in Figure 2(c), the surface $\Sigma_{g-1,1}$ deformation retracts onto the surface $\Sigma_{g-1,1}$ on which $\mathcal{I}_{g-1,1}$ is supported. After this deformation retract, the basepoint ends up on $\partial \Sigma_{g-1,1}$. Summing up, $\mathcal{I}_{g-1,1}$ acts on $K_{g,\gamma} < \pi_1(\Sigma_{g-1,1})$ via the action of $\text{Mod}_{g-1,1}$ on $\pi_1(\Sigma_{g-1,1})$, where the basepoint for $\pi_1(\Sigma_{g-1,1})$ is on $\partial \Sigma_{g-1,1}$.

### 3 The handle graph is connected

In this section, we prove the following.

**Lemma 3.1** Fix $g \geq 3$. Let $a, b \in H_1(\Sigma_g; \mathbb{Z})$ satisfy $i_a(a, b) = 1$. Then $\mathcal{H}_{a,b}$ is connected.
We will need two lemmas. In the first, if $\epsilon$ is an oriented arc in a surface, then $\epsilon^{-1}$ denotes the arc obtained by reversing the orientation of $\epsilon$.

**Lemma 3.2** Let the boundary components of $\Sigma_{g,2}$ be $\delta_0$ and $\delta_1$. Choose points $v_i \in \delta_i$ for $i = 0, 1$ and let $\epsilon$ be an oriented properly embedded arc in $\Sigma_{g,2}$ whose initial point is $v_0$ and whose terminal point is $v_1$. Then for any $h \in H_1(\Sigma_{g,2}; \mathbb{Z})$, there exists an oriented properly embedded arc $\epsilon'$ in $\Sigma_{g,2}$ whose initial point is $v_0$ and whose terminal point is $v_1$ such that the homology class of the loop $\epsilon' \cdot \epsilon^{-1}$ is $h$.

**Proof** Gluing $(\delta_0, v_0)$ to $(\delta_1, v_1)$, we obtain a surface $S \cong \Sigma_{g+1}$. Let $\alpha$ and $*\alpha$ be the images of $\delta_0$ and $v_0$ in $S$, respectively. The image of $\epsilon$ in $S$ is an oriented simple closed curve $\beta$ with $i_g(\alpha, \beta) = 1$. There is a natural isomorphism $H_1(\Sigma_{g,2}; \mathbb{Z}) \cong [\alpha]_{\perp}$, where the orthogonal complement is taken with respect to $i_a(\cdot, \cdot)$. Under this identification, we can apply [16, Lemma A.3] to find an oriented simple closed curve $\beta'$ on $S$ such that $[\beta'] = [\beta] + h$ and such that $\alpha \cap \beta' = \{*\}$. Cutting $S$ open along $\alpha$, the curve $\beta'$ becomes the desired arc $\epsilon'$.

**Lemma 3.3** Let $a, b \in H_1(\Sigma_g; \mathbb{Z})$ satisfy $i_4(a, b) = 1$. Let $\alpha_1$ and $\alpha_2$ be disjoint oriented simple closed curves on $\Sigma_g$ such that $[\alpha_i] = a$ for $i = 1, 2$. There then exists some oriented simple closed curve $\beta$ on $\Sigma_g$ such that $[\beta] = b$ and $i_g(\alpha_i, \beta) = 1$ for $i = 1, 2$.

**Proof** Let $\beta'$ be any simple closed curve on $\Sigma_g$ such that $i(\alpha_i, \beta') = 1$ for $i = 1, 2$. Orient $\beta'$ so that its intersections with $\alpha_1$ and $\alpha_2$ are positive. Let $X_1$ and $X_2$ be the two subsurfaces of $\Sigma_g$ that result from cutting $\Sigma_g$ along $\alpha_1 \cup \alpha_2$. For $i = 1, 2$, the surface $X_i$ has 2 boundary components and the intersection of $\beta'$ with $X_i$ is an oriented properly embedded arc $\epsilon_i$ running between these boundary components. Also, the induced map $H_1(X_i; \mathbb{Z}) \to H_1(\Sigma_g; \mathbb{Z})$ is an injection, and we will identify $H_1(X_i; \mathbb{Z})$ with its image in $H_1(\Sigma_g; \mathbb{Z})$. The orthogonal complement to $a$ with respect to the algebraic intersection pairing is spanned by $H_1(X_1; \mathbb{Z}) \cup H_1(X_2; \mathbb{Z})$. Since $i(a, b) = i(a, [\beta'])$, the homology class $b - [\beta']$ is orthogonal to $a$. There thus exist $h_i \in H_1(X_i; \mathbb{Z})$ for $i = 1, 2$ such that $b = [\beta'] + h_1 + h_2$. Lemma 3.2 says that for $i = 1, 2$ there exists an oriented properly embedded arc $\epsilon_i'$ in $X_i$ with the same endpoints as $\epsilon_i$ such that the homology class of the loop $\epsilon_i' \cdot \epsilon_i^{-1}$ equals $h_i$. Letting $\beta$ be the loop $\epsilon_1' \cdot \epsilon_2'$, it follows that $[\beta] = [\beta'] + h_1 + h_2 = b$, as desired.

**Proof of Lemma 3.1** Let $\delta$ and $\delta'$ be vertices of $\mathcal{H}_{a,b}$. We will construct a path in $\mathcal{H}_{a,b}$ from $\delta$ to $\delta'$. Without loss of generality, $[\delta] = [\delta'] = 0$. By [17, Theorem 1.9] (see [12] for an alternate proof), we can find a sequence

$$\delta = \alpha_1, \alpha_2, \ldots, \alpha_n = \delta'.$$
of isotopy classes of oriented simple closed curves on $\Sigma_g$ such that $[\alpha_i] = a$ for $1 \leq i \leq n$ and $i_g(\alpha_i, \alpha_{i+1}) = 0$ for $1 \leq i < n$ (this is where we use the condition $g \geq 3$). Lemma 3.3 implies that there exist isotopy classes $\beta_1, \ldots, \beta_{n-1}$ of oriented simple closed curves on $\Sigma_g$ such that $[\beta_i] = b$ and $i_g(\alpha_i, \beta_i) = i_g(\alpha_{i+1}, \beta_i) = 1$ for $1 \leq i < n$. Since $\beta_i$ is adjacent to both $\alpha_i$ and $\alpha_{i+1}$ in $\mathcal{H}_{a,b}$, the desired path from $\delta$ to $\delta'$ is thus

$$\delta = \alpha_1, \beta_1, \alpha_2, \beta_2, \ldots, \beta_{n-1}, \alpha_n = \delta'.$$

\[\square\]

4 Generating the stabilizer of a nonseparating simple closed curve

Let the subsurfaces $R'_i$ of $\Sigma_g$ be as in the introduction. Define $S_i = \overline{\Sigma_g \setminus R'_i}$. The goal of this section is to prove the following lemma.

**Lemma 4.1** Assume that $g \geq 4$. Let $\gamma$ be the isotopy class of a simple closed nonseparating curve on $\Sigma_g$ that is contained in $R'_1$. Then the subgroup $(\mathcal{I}_g)_\gamma$ of $\mathcal{I}_g$ stabilizing $\gamma$ is contained in the subgroup of $\mathcal{I}_g$ generated by $\bigcup_{i=1}^{g} \mathcal{I}(\Sigma_g, S_i)$.  

Before proving this, we need a technical lemma. Set $\pi = \pi_1(\Sigma_g, 1, \ast)$, where $\ast \in \partial \Sigma_g, 1$. Let $T'_1, \ldots, T'_g$ be disjoint subsurfaces of $\Sigma_g, 1$ such that $T'_i \cong \Sigma_1, 1$ and $T'_i \cap \partial \Sigma_g, 1 = \emptyset$ for $1 \leq i \leq g$ (see Figure 3(a)). Define $T_i = \overline{\Sigma_g, 1 \setminus T'_i}$. We have $T_i \cong \Sigma_{g-1, 2}$ and $\ast \in T_i$ for $1 \leq i \leq g$. The maps $\pi_1(T_i, \ast) \to \pi_1(\Sigma_g, 1, \ast)$ and $H_1(T'_i; \mathbb{Z}) \to H_1(\Sigma_g, 1; \mathbb{Z})$ are injective; we will identify $\pi_1(T_i, \ast)$ and $H_1(T'_i; \mathbb{Z})$ with their images in $\pi_1(\Sigma_g, 1, \ast)$ and $H_1(\Sigma_g; \mathbb{Z})$, respectively. Define $K_i = [\pi, \pi] \cap \pi_1(T_i, \ast)$. We then have the following.

**Lemma 4.2** For $g \geq 3$, the group $[\pi, \pi]$ is generated by the $\mathcal{I}_{g, 1}$-orbits of the set $\bigcup_{i=1}^{g} K_i$.

The proof of this will have two ingredients. The first is the following theorem of Tomaszewski. As notation, if $G$ is a group and $a, b \in G$, then $[a, b] := a^{-1}b^{-1}ab$ and $a^b := b^{-1}ab$.

**Theorem 4.3** (Tomaszewski, [20]) Let $F_n$ be the free group on $\{x_1, \ldots, x_n\}$. Then the set

$$\{[x_i, x_j]^{k_i} x_{i+1}^{k_{i+1}} \cdots x_{n}^{k_n} \mid 1 \leq i < j \leq n \text{ and } k_m \in \mathbb{Z} \text{ for all } i \leq m \leq n\}$$

is a free basis for $[F_n, F_n]$. 

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The second is the following lemma about the action of $\mathcal{I}_{g,1}$ on $\pi$. Choose a standard basis $\{\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g\}$ for $\pi$ (as in Figure 3(b)) such that $\alpha_i$ and $\beta_i$ are freely homotopic into $T_i'$ for $1 \leq i \leq g$. Our proof of Lemma 4.2 would be much simpler if the image of $\text{Mod}_{g,1}$ in $\text{Aut}(\pi)$ contained the inner automorphisms – since inner automorphisms act trivially on homology, this would imply that the $\mathcal{I}_{g}$–orbits of $\{[x, y] \mid x, y \in \{\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g\}\}$ generate $[\pi, \pi]$. However, the image of $\text{Mod}_{g,1}$ in $\text{Aut}(\pi)$ does not contain the inner automorphisms since $\text{Mod}_{g,1}$ fixes the loop $\delta = [\alpha_1, \beta_1] \cdots [\alpha_g, \beta_g]$ depicted in Figure 3(b). The following lemma is a weak replacement for this.

**Lemma 4.4** Let $i$ be either 1 or $g$. Consider $h \in H_1(T_i'; \mathbb{Z})$. There then exists some $w \in \langle \alpha_i, \beta_i, \delta \rangle$ and $f \in \mathcal{I}_{g,1}$ such that $[w] = h$ and such that $f(a_j) = a_j^w$ and $f(b_j) = b_j^w$ for $1 \leq j \leq g$ with $j \neq i$.

**Proof** Let $X$ be a regular neighborhood of the curves $\alpha_i \cup \beta_i \cup \partial \Sigma_{g,1}$ depicted in Figure 3(b). Thus $X \cong \Sigma_{1,2}$, the surface $T_i'$ is homotopic into $X$, and the image of $\pi_1(X, \ast)$ in $\pi$ is $\langle \alpha_i, \beta_i, \delta \rangle$. Let $Y = \Sigma_{g,1} \setminus \overline{X}$, so $Y \cong \Sigma_{g-1,1}$ and $X \cap Y \cong S^1$. The key property of $X$ is as follows (this is where we use the assumption that $i$ is either 1 or $g$). There exists some $\ast' \in X \cap Y$, a properly embedded arc $\eta$ in $X$ from $\ast$ to $\ast'$, and elements

$$\{\alpha_j', \beta_j' \mid 1 \leq j \leq g, j \neq i\} \subset \pi_1(Y, \ast')$$

such that $\alpha_j = \eta \cdot \alpha_j' \cdot \eta^{-1}$ and $\beta_j = \eta \cdot \beta_j' \cdot \eta^{-1}$ for $1 \leq j \leq g$ with $j \neq i$. See Figure 3(c) for the case $i = 1$ and Figure 3(d) for the case $i = g$. 

Figure 3: (a) The subsurfaces $T_i'$ (b) The standard basis for $\pi$ (c) The surface $X$ when $i = 1$ (d) The surface $X$ when $i = g$
By Lemma 3.2, there exists an oriented properly embedded arc \( \eta' \) in \( X \) whose endpoints are the same as those of \( \eta \) such that the homology class of \( w := \eta \cdot (\eta')^{-1} \in \pi \) in \( H_1(\Sigma_g; \mathbb{Z}) \) is \( h \). Observe that \( w \in \langle \alpha_i, \beta_i, \delta \rangle \). Also, 
\[
\eta' \cdot \alpha_j \cdot (\eta')^{-1} = w^{-1} \cdot \eta \cdot \alpha_j \cdot \eta^{-1} \cdot w = \alpha_j^w
\]
for \( j \neq i \), and similarly for \( \beta_j \). It is thus enough find some \( f \in \mathcal{I}(\Sigma_g, X) \) such that \( f(\eta) = \eta' \).

The “change of coordinates principle” from [5, Section 1.3] implies that there exists some \( f' \in \text{Mod}(\Sigma_g, X) \) such that \( f'(\eta) = \eta' \). Briefly, an Euler characteristic calculation shows that cutting \( X \) open along either \( \eta \) or \( \eta' \) results in a surface homeomorphic to \( \Sigma_{1,1} \). Choosing an orientation-preserving homeomorphism between these two cut-open surfaces and gluing the boundary components back together in an appropriate way, we obtain some \( f' \in \text{Mod}(\Sigma_g, X) \) such that \( f'(\eta) = \eta' \). See [5, Section 1.3] for more details and many other examples of arguments of this form.

The mapping class \( f' \) need not lie in Torelli; however, it satisfies \( f'([\alpha_j]) = [\alpha_j] \) and \( f'([\beta_j]) = [\beta_j] \) for \( j \neq i \) and \( f'(H_1(T_i'; \mathbb{Z})) = H_1(T_i'; \mathbb{Z}) \). Since the image of \( \text{Mod}(T_i') \) in \( \text{Aut}(H_1(T_i'; \mathbb{Z})) = \text{Aut}(\mathbb{Z}^2) \) is \( \text{SL}_2(\mathbb{Z}) \), we can choose some \( f'' \in \text{Mod}(\Sigma_g, T_i') \) such that \( f'([\alpha_i]) = f''([\alpha_i]) \) and \( f'([\beta_i]) = f''([\beta_i]) \). It follows that \( f := f' \cdot (f'')^{-1} \) lies in \( \mathcal{I}(\Sigma_g, X) \) and satisfies \( f(\eta) = \eta' \), as desired.

**Proof of Lemma 4.2** The generating set for \( [F_n, F_n] \) in Theorem 4.3 depends on an ordering of the generators for \( F_n \). It seems hard to prove the lemma using the generating set corresponding to the standard ordering
\[
(x_1, x_2, \ldots, x_{2g}) = (\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g)
\]
of the generators for \( \pi \cong F_{2g} \). However, consider the following nonstandard ordering on the generators for \( \pi \):
\[
(x_1, x_2, \ldots, x_{2g}) = (\alpha_2, \beta_2, \alpha_1, \beta_1, \ldots, \alpha_g, \beta_g).
\]
Let \( S \) be the generating set for \( [\pi, \pi] \) given by Theorem 4.3 using this ordering of the generators. All the elements of \( S \) lie in \( K_2 \) except for
\[
[\alpha_2, \xi]^{n_2}_2 \beta_2^{m_2} \alpha_1^{m_1} \beta_1^{n_1} \alpha_3^{n_3} \beta_3^{m_3} \ldots \beta_g^{m_g} \quad \text{and} \quad [\beta_2, \xi']^{m_2}_2 \alpha_1^{m_1} \beta_1^{n_1} \alpha_3^{n_3} \beta_3^{m_3} \ldots \beta_g^{m_g};
\]
here \( \xi \in \{\beta_2, \alpha_1, \beta_1, \ldots, \beta_g\} \) and \( \xi' \in \{\alpha_1, \beta_1, \alpha_3, \ldots, \beta_g\} \) and \( n_i, m_i \in \mathbb{Z} \). Letting \( T \subset S \) be the elements in (3), we must show that every \( t \in T \) can be expressed as a product of elements in the \( \mathcal{I}_{g,1} \)-orbit of the set \( \bigcup_{i=1}^g K_i \). Consider \( t \in T \), so either \( t = [\alpha_2, \xi]^{n_2}_2 \beta_2^{m_2} \alpha_1^{m_1} \beta_1^{n_1} \alpha_3^{n_3} \beta_3^{m_3} \ldots \beta_g^{m_g} \) or \( t = [\beta_2, \xi']^{m_2}_2 \alpha_1^{m_1} \beta_1^{n_1} \alpha_3^{n_3} \beta_3^{m_3} \ldots \beta_g^{m_g} \). There are two cases.

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Case 1 \( \zeta \notin \{\alpha_1, \beta_1\} \).

We will do the case where \( t = [\alpha_2, \zeta]^{n_2 \beta_2 m_2} \alpha_1^{n_1} \beta_1^{m_1} \alpha_3^{n_3} \beta_3^{m_3} \); the other case is treated in a similar way. Set \( t' = [\alpha_2, \zeta]^{n_2 \beta_2 m_2} \alpha_3^{n_3} \beta_3^{m_3} \), so \( t' \in K_1 \). By Lemma 4.4, there exists some \( w \in \{\alpha_1, \beta_1, \delta\} \) and \( f \in \mathcal{I}_{g,1} \) such that \([w] = [\alpha_1^{n_1} \beta_1^{m_1}]\) and such that \( f(a_j) = a_j^w \) and \( f(b_j) = b_j^w \) for \( j > 1 \). This implies that \( f(t') = [\alpha_2, \zeta]^{n_2 \beta_2 m_2} \alpha_3^{n_3} \beta_3^{m_3} w \). Now, \( \alpha_3^{n_3} \beta_3^{m_3} \) and \( \alpha_1^{n_1} \beta_1^{m_1} \alpha_3^{n_3} \beta_3^{m_3} \) are homologous, so there exists some \( \theta \in [\pi, \pi] \) such that \( \alpha_3^{n_3} \beta_3^{m_3} w \theta = \alpha_1^{n_1} \beta_1^{m_1} \alpha_3^{n_3} \beta_3^{m_3} \). Moreover, since \( w \in \{a_1, b_1, \delta\} \) we have \( \theta \in K_2 \). Observe now that

\[
\theta^{-1} \cdot f(t') \cdot \theta = [\alpha_2, \zeta]^{n_2 \beta_2 m_2} \alpha_3^{n_3} \beta_3^{m_3} \theta = [\alpha_2, \zeta]^{n_2 \beta_2 m_2} \alpha_1^{n_1} \beta_1^{m_1} \alpha_3^{n_3} \beta_3^{m_3} = t.
\]

We have thus found the desired expression for \( t \).

Case 2 \( \zeta' \in \{\alpha_1, \beta_1\} \).

This case is similar to Case 1. The only difference is that the \( \alpha_1^{n_1} \beta_1^{m_1} \) term of \( t \) is deleted to form \( t' \) instead of the \( \alpha_1^{n_1} \beta_1^{m_1} \) term.

**Proof of Lemma 4.1**

Let \( I \) be the subgroup of \( \mathcal{I}_g \) generated by \( \bigcup_{i=1}^g \mathcal{I}(\Sigma_g, S_i) \).

Using the notation of Section 2, there is a surjection \( \rho: \mathcal{I}_{g,\gamma} \to (\mathcal{I}_g)_{\gamma} \) induced by a continuous map \( \phi: \Sigma_{g,\gamma} \to \Sigma_g \). Define \( X = \phi^{-1}(S_1) \), so \( X \cong \Sigma_{g-1,1} \). Letting \( \mathcal{I}(X) \) be the Torelli group of \( X \), Lemma 2.3 gives a decomposition \( \mathcal{I}_{g,\gamma} = K_{g,\gamma} \rtimes \mathcal{I}(X) \). Clearly \( \rho(\mathcal{I}(X)) = \mathcal{I}(\Sigma_g, S_1) \subseteq I \). Also, Lemma 4.2 implies that \( K_{g,\gamma} \) is generated by the \( \mathcal{I}(X) \)–conjugates of a set \( S \subseteq K_{g,\gamma} \) such that \( \rho(S) \subseteq I \). We conclude that \( \rho(\mathcal{I}_{g,\gamma}) \subseteq I \), as desired.

**5 Proof of main theorem**

We finally prove our main theorem. The key is the following standard lemma, whose proof is similar to that given in [19, (1) of Appendix to Section 3] and is thus omitted.

**Lemma 5.1** Consider a group \( G \) acting without inversions on a connected graph \( X \). Assume that \( X/G \) consists of a single edge \( \mathcal{e} \). Let \( e \) be a lift of \( \mathcal{e} \) to \( X \) and let \( v \) and \( v' \) be the endpoints of \( e \). Then \( G \) is generated by \( G_v \cup G_{v'} \).

To apply this, we will need the following lemma.

**Lemma 5.2** Let \( a, b \in H_1(\Sigma_g; \mathbb{Z}) \) satisfy \( i_a(a, b) = 1 \). Then \( \mathcal{H}_{a,b}/\mathcal{I}_g \) is isomorphic to a graph with a single edge.
The proof is similar to the proofs of [16, Lemma 6.2] and [18, Lemma 6.9], and is thus omitted.

**Proof of Theorem B** Let \( R'_1, \ldots, R'_g \) and \( R_{ijk} \) be the subsurfaces of \( \Sigma_g \) from the introduction. Let \( \Gamma \) be the subgroup of \( \mathcal{I}_g \) generated by \( \bigcup_{1 \leq i < j < k \leq g} \mathcal{I}(\Sigma_g, R_{ijk}) \). Our goal is to prove that \( \Gamma = \mathcal{I}_g \).

The proof will be by induction on \( g \). The base case \( g = 3 \) is trivial, so assume that \( g \geq 4 \) and that the theorem is true for all smaller \( g \) such that \( g \geq 3 \). Choose simple closed curves \( \alpha \) and \( \beta \) in \( R'_1 \) such that \( i_g(\alpha, \beta) = 1 \). Observe that \( R'_1 \) is a closed regular neighborhood of \( \alpha \cup \beta \). Set \( a = [\alpha] \) and \( b = [\beta] \). Clearly \( \mathcal{I}_g \) acts on \( \mathcal{H}_{a,b} \) without inversions. Lemmas 3.1 and 5.2 show that the action of \( \mathcal{I}_g \) on \( \mathcal{H}_{a,b} \) satisfies the other conditions of Lemma 5.1. We deduce that \( \mathcal{I}_g \) is generated by the union \( (\mathcal{I}_g)_\alpha \cup (\mathcal{I}_g)_\beta \) of the stabilizer subgroups of \( \alpha \) and \( \beta \).

Recall that \( S_i = \Sigma_g \setminus R'_i \) for \( 1 \leq i \leq g \). By Lemma 4.1, both \( (\mathcal{I}_g)_\alpha \) and \( (\mathcal{I}_g)_\beta \) are contained in the subgroup generated by \( \bigcup_{i=1}^g \mathcal{I}(\Sigma_g, S_i) \). We must prove that \( \mathcal{I}(\Sigma_g, S_i) \subset \Gamma \) for \( 1 \leq i \leq g \). We will do the case \( i = g \); the other cases are similar.

We have a Birman exact sequence

\[
1 \longrightarrow \pi_1(U \Sigma_{g-1}) \longrightarrow \mathcal{I}(\Sigma_g, S_g) \longrightarrow \mathcal{I}_{g-1} \longrightarrow 1.
\]

By induction, the subset \( \bigcup_{1 \leq i < j < k < g-1} \mathcal{I}(\Sigma_g, R_{ijk}) \) of \( \mathcal{I}(\Sigma_g, S_g) \) projects to a generating set for \( \mathcal{I}_{g-1} \). Also, it is clear that the disc-pushing subgroup \( \pi_1(U \Sigma_{g-1}) \) of \( \mathcal{I}(\Sigma_g, S_g) \) is generated by elements that lie in \( \bigcup_{1 \leq i < j < k \leq g} \mathcal{I}(\Sigma_g, R_{ijk}) \). We conclude that \( \mathcal{I}(\Sigma_g, S_g) \subset \Gamma \), as desired.

\[\square\]

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