Boundary value problems for parabolic equations of high order with a changing time direction

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Abstract. We analyze the solvability of boundary value problems for differential equations of high order with a changing direction of time. It is shown that a generalized solution exists in Sobolev spaces. Correctness of boundary value problems with a complete matrix of conjugation conditions is investigated. For such problems, it is shown that the Hölder classes of their solutions essentially depend on both the type of gluing conditions and on the non-integer index Hölder space.

1. Introduction
The paper concerns boundary-value problems for differential equations of high order of the form

$$\frac{\partial^{2m+1} u}{\partial t^{2m+1}} + \sum_{k=0}^{2m} \frac{\partial^k u}{\partial t^k} a_k(t) + Lu = f, \quad (x,t) \in Q = (a,b) \times (0,T),$$

(1)

where the operator \( L \) has the form \( Lu = \frac{1}{g(x)} (L_0 u + \lambda_0 u + Mu) \); the differential operator \( L_0 \) with respect to a variable \( x \) is of the form \( L_0 u = \sum_{i=0}^{2s} \alpha_i(x) \frac{\partial^{i} u}{\partial x^i} \) and \( M \) is its perturbation of the form \( Mu = \sum_{i=0}^{2s-1} b_i(x,t) \frac{\partial^{i} u}{\partial x^i} \).

The equation (1) is considered with the boundary conditions

$$U_j(u) \equiv \sum_{k=0}^{2s-1} \alpha_{jk} \frac{\partial^{k} u(a,t)}{\partial x^k} u(a,t) + \sum_{k=0}^{2s-1} \beta_{jk} \frac{\partial^{k} u(b,t)}{\partial x^k} u(b,t) = 0, \quad j = 1, 2, \ldots, 2s,$$

(2)

$$u^{(i)}(x,0) = u^{(i)}(x,T), \quad i = 0, 1, \ldots, 2m - 1,$$

(3)

where \( \alpha_{jk}, \beta_{jk} \) are some complex constant.

The function \( g(x) \) can be equal to zero and change the sign on the interval \((a,b)\). In particular, for \( m = 0 \), equations with varying direction of time are sufficiently well analyzed.

In this paper we consider the solvability of the problem (1)–(3). Note that similar equations arise in many areas of physics, mechanics and some other applications. Higher-order equations were considered in works [1–3].
In §1 we formulate some notation, auxiliary statements and main results. In §2 we study parabolic equations with a changing time direction by using the theory of singular integral equations [4–7] as well as the systems of these equations [8].

Note that a large number of papers are devoted to the study of linear second-order equations. The general theory of boundary value problems for equations of mixed type with arbitrary coefficients and variety of type changes have been the subject of many studies [9,10].

### 2. Notation and main result

The notation for the functional spaces that we shall use are standard: $W^s_p(Q)$ is the Sobolev space and $H^s(Q)$, $H^{p,p/2s}(Q)$ are Hölder spaces. The symbol $(X, Y)_{θ, p}$ denotes the space constructed by the method of real interpolation for Banach spaces $X$ and $Y$ [11].

We assume that the domain $D(L_0)$ of the operator $L_0$ consists of the functions $u(x) ∈ W^2_2(a, b)$ satisfying (2) and inequality

$$Re(L_0u, u) ≥ δ_0 ||u||^2_{W^2_2(a,b)}, \quad u ∈ D(L_0), \quad δ_0 > 0,$$

where $(·, ·)$ is the scalar product in $L^2_2(a,b)$. We denote by $H_1$ the closure of $D(L_0)$ in the norm $||u||_{H_1} = ||u||_{W^2_2(a,b)}$. Assume that there exists a constant $c > 0$:

$$|(L_0u, v)| ≤ c ||u||_{H_1} ||v||_{H_1}, \quad u, v ∈ D(L_0).$$

This implies the estimate

$$||L_0u||_{H_1'} ≤ c ||u||_{H_1},$$

where $H_1'$ is a negative space constructed from a pair $H_1, L_2(a,b)$.

We introduce the operator $\tilde{L}_0u = \frac{1}{g(x)}(L_0 + λ_0)u$. Assume that $g(x) ∈ L^1_2(a,b)$. We set $F_1 = L^2_2(0,T; H_1)$. By a definition, the space $L^2_2, g(a,b)$ consists of measurable functions such that $||u||_{2_2, g(a,b)}^2 = \int_a^b |g(x)| ||u||^2 dx < ∞$. We define auxiliary space $F_0 = L^2_2(0,T; L^2_2, g(a,b))$. The space $F_0$ becomes the Kerin space if we define an indefinite metric scalar product by the formula

$$[u, v]_0 = \int_Q g(x)u(x,t)v(x,t)dxdt,$$

$$(u, v)_0 = \int_Q |g(x)|u(x,t)v(x,t)dxdt.$$

We have

$$Re[\tilde{L}_0u, u]_0 = Re \int_Q (L_0 + λ_0)u \bar{u}dQ ≥ δ_0 ||u||^2_{F_1} + Reλ_0 ||u||^2_{L^2_2(Q)}, \quad u ∈ D(L_0).$$

By (6) and inequality (4), the operator $L_0 + λ_0$, $Reλ_0 ≥ 0$ can be extended to isomorphism of $F_1$ and $F_1' = L^2_2(0,T; H_1')$. This assertion is a consequence of the Lax-Milgram theorem and inequalities (5) and (7).

We construct the space $H_{-1}$ as the completion of $L^2_2, g(a,b)$ with the norm

$$||u||_{H_{-1}} = \sup_{v ≠ 0} \frac{||[u,v]||_{H_1'}}{||v||_{H_1'}} = ||g(x)u||_{H_1'},
where \([u, v] = \int_a^b g(x)u(x)v(x)dx\) We set \(F_{-1} = L_2(0, T; H_{-1})\). Note that \(F_1\) is densely embedded in \(F_0\) by the condition on the function \(g(x)\). Consider the operator acting from \(F_{-1}\) to \(F_1\) with the domain of definition \(L_2(0, T; H_1)\). Since \(L_0 + \lambda_0\) for \(Re\lambda_0 \geq 0\) is an isomorphism of \(L_2(0, T; H_1)\) and \(L_2(0, T; H'_1)\), then \(L_0\) is an isomorphism of \(F_1\) and \(F_{-1}\).

The following assertion is valid [2, lemma 4.1, chapter 1]

**Lemma 1.** Let \(i\mathbb{R} \subset \rho(\widetilde{L}_0)\). The following estimate of the resolvent \(\|(\widetilde{L}_0 + i\lambda)^{-1}f\|_{F_{-1}} \leq \frac{C}{1+|\lambda|}\|f\|_{F_1}\) holds.

Consider the operator \(A = i\frac{\partial^{2m+1}}{\partial x^{2m+1}}\). Let \(H\) be a Hilbert space with the scalar product \((\cdot, \cdot)_H\). Put \(D(A) = \{u \in W_2^{2m+1}(0, T, H) : u^{(i)}(0) = u^{(i)}(T), \quad i = 0, 1, ..., 2m\}\).

**Lemma 2.** The operator \(A : L_2(0, T; H) \rightarrow L_2(0, T; H)\) is selfadjoint.

We formulate a theorem on solvability of the problem (1)–(3). One has \(L_2,g(a, b) \subset H_{-1}\). There is a self-adjoint operator \(A_0\) in \(H_{-1}\) such that \(D(A_0) = L_2,g(a, b), \|A_0u\|_{H_{-1}} = \|u\|_{L_2,g(a, b)}\). Moreover \(A_0\) is an isomorphism of \(L_2,g(a, b)\) and \(H_{-1}\). Then \(\|v\|_{H_{-1}} = \|A_0^{-1}v\|_{L_2,g(a, b)}\). Equation (1) can be rewritten in the form

\[
u^{(2m+1)} + \widetilde{L}_0u + M_0u = f, \tag{8}
\]

where \(M_0u = \sum_{i=0}^{2m} u^{(i)}a_i(t) + \frac{1}{g(x)} \sum_{j=0}^{2m-1} b_j(x, t) \frac{\partial u}{\partial x^j}\). In the following theorem, we consider the case when \(M_0(u) \equiv 0\), that is the equation (8) has the form

\[
u^{(2m+1)} + \widetilde{L}_0u = f. \tag{9}
\]

**Theorem 1.** Let \(f \in L_2(0, T; F_{-1}), Re\lambda_0 \geq 0\) and \(M_0(u) \equiv 0\). Then there exists a unique solution of the problem (9), (2), (3) from the class \(u \in F_1, u^{(i)}(0) \in (F_1, F_{-1})_{1-\theta_i, 2} (\theta_i = i/(2m+1), i = 1, 2, \ldots, 2m)\) and \(u^{(2m+1)} \in F_{-1}\). The solution satisfies the estimate

\[
\|u\|_{F_1} + \sum_{i=1}^{2m} \|u^{(i)}\|_{(F_1, F_{-1})_{1-\theta_i, 2}} + \|u^{(2m+1)}\|_{F_{-1}} \leq c\|f\|_{F_{-1}},
\]

where \(c\) is some constant that does not depend on \(f\).

We consider the problem of the solvability to (1)–(3). Assume that \(\forall \varepsilon > 0 \exists c(\varepsilon)\), for which an inequality of the form

1) \(\| \sum_{j=0}^{2m} b_j(x, t) \frac{\partial u}{\partial x^j} \|_{F_1} \leq \varepsilon\|u\|_{F_1} + c(\varepsilon)\|u\|_{L_2(Q)}, \quad u \in D(L_0)\) takes place;

2) \(a_j(t) \in L_{p_0}(0, T), \quad j = 0, 1, ..., m, \quad p_0 > 2, \quad g(x) \in L_2(a, b)\).

**Theorem 2.** Let the conditions of theorem 1 be satisfied and conditions 1), 2) hold. We fix \(\varepsilon \in (0, \pi/2)\). Then there is \(\lambda_1 \geq 0\) such that for all \(Re\lambda_0 \geq \lambda_1\), \(arg\lambda_0 \leq \pi/2 - \varepsilon\), the problem (1)–(3) has a unique solution with properties mentioned in theorem 1.

**Remark 1.** The simplest conditions guaranteeing the fulfillment of condition 1) are as follows \(b_j \equiv 0\) at \(j \geq s\), \(b_j(x, t) \in L_\infty((0, T) \times (a, b))\) at \(j \leq m - 1\).
3. Parabolic equation

Consider the case \( m = 0, s = 2 \), corresponding to the parabolic equation fourth order with a changing direction of time associated with the application of the theory of singular integral equations [5–7] as well as systems of these equations [8]. Here, the results are refined and amplified, obtained in [12].

In the domain \( Q^+ = \mathbb{R}^+ \times (0, T) \) we consider the system of equations

\[
\begin{align*}
    u_1^t &= L u_1, \\
    -u_2^t &= L u_2 \\
    L &= -\frac{\partial^4}{\partial x^4},
\end{align*}
\]

(10)

The solution of the system (10) is sought from the space Hölder \( H^p, \frac{p}{4}, x, t (Q^+) \), \( p = 4l + \gamma \), \( 0 < \gamma < 1 \), satisfying the initial conditions

\[
\begin{align*}
    u_1(x, 0) &= \varphi_1(x), \\
    u_2(x, T) &= \varphi_2(x), \quad x > 0,
\end{align*}
\]

(11)

and bonding conditions

\[
T_1 \vec{u}_1(0, t) = T_2 \vec{u}_2(0, t),
\]

(12)

where \( \vec{u}_k = (u_k, u_k^x, u_k^{xx}, u_k^{xxx}) \), \( T_1, T_2 \) are nondegenerate matrices with constant real coefficients.

We assume that \( \varphi_i(x) \in H^p(\mathbb{R}) \) \((i = 1, 2)\). Then the functions

\[
\omega_1(x, t) = \frac{1}{\pi} \int_{\mathbb{R}} U_0(x, t; \xi, 0) \varphi_1(\xi) d\xi, \quad \omega_2(x, t) = \frac{1}{\pi} \int_{\mathbb{R}} U_0(\xi, T; x, t) \varphi_2(\xi) d\xi
\]

are solutions of the equations (10) satisfying conditions (11). We will use the integral representation of the solution for the system (10):

\[
\begin{align*}
    u_1(x, t) &= \int_0^t U_0(x, t; 0, \tau) \alpha_0(\tau) d\tau + \int_0^t U_1(x, t; 0, \tau) \alpha_1(\tau) d\tau + \omega_1(x, t), \\
    u_2(x, t) &= \int_0^T U_0(0, \tau; x, t) \beta_0(\tau) d\tau + \int_0^T U_2(0, \tau; x, t) \beta_1(\tau) d\tau + \omega_2(x, t),
\end{align*}
\]

where \( U_i (i = 0, 1, 2) \) are the fundamental and elementary solutions Bruno Pini.

The gluing conditions (12) can be rewritten in the form

\[
\vec{u}_1(0, t) = (T_1^{-1} \cdot T_2) \vec{u}_2(0, t) = A \vec{u}_2(0, t),
\]

(13)

where \( A \) is a nondegenerate matrix equal to \( T_1^{-1}T_2 \). Let the elements of the matrix \( C \) consist of real numbers \( a_{ij} \). Consider the case of the matrix

\[
A = \begin{pmatrix}
    a_{11} & 0 & 0 & a_{14} \\
    0 & a_{22} & 0 & a_{24} \\
    0 & 0 & a_{33} & a_{34} \\
    0 & 0 & 0 & a_{44}
\end{pmatrix}.
\]

(14)

We also note that in the case of symmetry of the matrix \( A \), the conditions of work [12] are fulfilled. We assume that the coefficients \( a_{ij} \) of the matrix \( A \) satisfy the uniqueness condition for the boundary value problem (10), (11), (14).
Theorem 3. Let the elements of the matrix $A$ of the form (14) are different from zero, and $a_{33} \neq \sqrt{2}a_{22}$, $\varphi_1, \varphi_2 \in H^p$ $(p = 4l + \gamma)$. Then, if $4l$ conditions hold

$$L_s(\varphi_1, \varphi_2) = 0, \quad s = 1, \ldots, 4l,$$

there exists a unique solution of the equation (10) from space

1) $H^{p/4}_{x,l}$, if $0 < \gamma < 1 - 4\theta$;

2) $H^{q/4}_{x,l}$, $q = 4l + 1 - 4\theta$, if $1 - 4\theta < \gamma < 1$;

3) $H^{q-\varepsilon/4}_{x,l}$, if $\gamma = 1 - 4\theta$, where $\varepsilon$ is an arbitrarily small positive constant, satisfying the conditions (11), (13), $\theta = \frac{1}{2} \arctan \left( \frac{\alpha}{\beta} \right) \in (0, \frac{1}{2})$, $a = a_{33}$, $b = a_{33} - \sqrt{2}a_{22}$.

Remark 2. The proof of theorem 1 remains valid if in the general matrix of gluing $A = \{a_{ij}\}$ at least one of three elements $a_{13}, a_{14}, a_{24}$ will be different from zero.

If instead of the matrix $A$ consider a matrix of the form

$$A = \begin{pmatrix} a_{11} & a_{12} & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix},$$

then the theorem on unconditional solvability holds and there exists a unique solution of equation from space $H^{p/4}_{x,l}(Q^+)$ (10) satisfying the conditions (11), (13), (15).

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