The number of countable models of a countable supersimple theory

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Abstract

In this paper, we prove the number of countable models of a countable supersimple theory is either 1 or infinite. This result is an extension of Lachlan’s theorem on a superstable theory.

1 Introduction

The aim of this paper is to extend the following classical result of Lachlan in a supersimple theory context.

Theorem 1.1 Let $T$ be a countable superstable theory. Then the number of (nonisomorphic) countable models of $T$ is either 1 or $\geq \aleph_0$.

In other words, we will prove the following.

Theorem 1.2 Let $T$ be a countable supersimple theory. Then the number of countable models of $T$ is either 1 or $\geq \aleph_0$.

Let us quote a brief history of Lachlan’s Theorem from Baldwin ([1, XIII.2.31]).

‘Theorem 1.1’ has a long history. The first step in this direction is the proof by Baldwin and Lachlan ([2]) that the conclusion holds for countable theories which are $\aleph_1$-categorical. This proof
used many special properties of $\aleph_1$-categorical theories. Then Lachlan ([6]) proved ‘Theorem 1.1’ by a complicated argument using rank. Lascar ([7]) simplified the proof by the use of $U$-rank. Finally, Pillay ([9]) has given an even simpler proof...

Lascar’s proof of Lachlan’s Theorem is essentially using the characteristics of “weight”. Pillay’s proof, according to a personal conversation with him, is actually a translation of Lachlan’s original proof into forking context. Pillay’s proof only uses the basic properties of forking (for example, the notion of weight is not used), together with the Open Map Theorem. However as the Open Map Theorem is no longer true in a simple unstable theory, we are not able to copy the same proof for Theorem 1.2.

**Example 1** Let $M$ be the countable bipartite random graph, consisting of disjoint infinite sets $U, V$ with the relation $R$ between $U, V$. Hence for any finite disjoint subsets $X, Y$ of $U, V$, there is $z \in V$ such that $xRz$ for $x \in X$ and $\neg yRz$ for $y \in Y$, and vice versa. Let $A = \{ a_i | i < \omega \} \subseteq U$. Choose $c \in U \setminus A$ so that $tp(c/A)$ is not isolated. Also select $b \in V$ such that $\neg a_i Rb$ for all $i$, and $cRb$. Then $tp(c/Ab)$ does not fork over $A$, whereas $tp(c/Ab)$ is isolated.

In fact, one can come up with the following version of the Open Map Theorem for a simple theory, using the exactly same proof of the Open Map Theorem for a stable theory with Fact 1.4 (see the proof of 4.27 in [8]). But the following theorem will not be used in this paper.

**Theorem 1.3** Let $T$ be simple, and let $A \subseteq B$. For each formula $\varphi(\bar{x}) \in L(B)$, there is a (partial) type $\Delta(\bar{x})$ over $A$ such that, for each $p \in S(A)$, $\Delta(\bar{x}) \subseteq p$ iff $\varphi(\bar{x})$ is in some nonforking extension of $p$.

**Fact 1.4** ([10]) Let $T$ be simple and let $p(\bar{x}) \in S(A)$. For each $L$-formula $\varphi(\bar{x}, \bar{y})$, there is a corresponding partial type $\Delta(\bar{y})$ over $A$ such that, for any $\bar{c}, \models \Delta(\bar{c})$ iff $\varphi(\bar{x}, \bar{c})$ is in some nonforking extension of $p$.

The main novelty of our argument here is that we find a new proof of Lachlan’s Theorem which uses only the symmetry and transitivity of nonforking. Hence this proof also works for Theorem 1.2.

Now we recall from [4], [5], [11], some basic facts and definitions we need. A type $p$ forks over a set $A$, if there are an $L$-formula $\varphi(\bar{x}, \bar{y})$ and a set of
transitivity

iff

is superstable

and so called Independence Theorem

for any worthwhile to mention. Tuples \( \bar{a}, \bar{b}, \bar{c} \) is simple.

Let \( \bar{a}, \bar{b}, \bar{c} \in A \): if there do not exist \( i, j < \omega \), such that \( p_{i+1} \) is a forking extension of \( p_i \) for each \( i < \omega \). We also recall that \( T \) is unstable if there are a formula \( \psi(x, y) \) and tuples \( \bar{b}_i, \bar{c}_i \) (\( i < \omega \)) such that \( \models \psi(\bar{b}_i, \bar{c}_j) \) iff \( i \leq j \in \omega \). A theory \( T \) is said to be stable if \( T \) is not unstable, and superstable if \( T \) is stable and supersimple. Every stable theory is simple.

In [4], it is shown that, for simple \( T \), nonforking satisfies (i) extension: for any \( p \in S(A) \) and \( A \subseteq B, p \) has a nonforking \( q \) in \( S(B) \), (ii) symmetry: \( \text{tp}(\bar{b}/A\bar{c}) \) does not fork over \( A \) iff \( \text{tp}(\bar{c}/A\bar{b}) \) does not fork over \( A \), and (iii) transitivity: if \( A \subseteq B \subseteq C \) and \( p \in S(C) \), then \( p \) does not fork over \( A \) iff \( p \) does not fork over \( B \) and the restriction of \( p \) to \( B \) does not fork over \( A \). Hence nonforking supplies a nice notion of independence to an arbitrary simple theory. If \( T \) is simple, we say \( \{C_i|i \in I\} \) is independent over \( A \) if for each \( i \in I \) and \( \bar{c} \in C_i \), \( \text{tp}(\bar{c}/A \cup \bigcup\{C_j|j \neq i, j \in I\}) \) does not fork over \( A \).

One of the important properties of nonforking in a simple theory is the so called Independence Theorem, which is not so relevant to this paper, but worth while to mention. Tuples \( \bar{a}, \bar{b} \) are said to have the same Lascar strong type over \( A \) \( (\text{Lstp}(\bar{a}/A) = \text{Lstp}(\bar{b}/A)) \) if there are models \( M_1, ..., M_k \), each of which contains \( A \), and tuples \( \bar{a} = \bar{a}_0, \bar{a}_1, ..., \bar{a}_k = \bar{b} \) such that \( \text{tp}(\bar{a}_{i-1}/M_i) = \text{tp}(\bar{a}_i/M_i) \) for \( 1 \leq i \leq k \). In [5], the following is shown.

**Fact 1.5** (The Independence Theorem for Lascar strong types) Assume that \( T \) is simple. Let \( \{B, C\} \) is independent (over \( \phi \)). If \( \text{Lstp}(\bar{d}) = \text{Lstp}(\bar{e}) \) and \( \text{tp}(\bar{d}/B), \text{tp}(\bar{e}/C) \) both do not fork over \( \phi \), then there is \( \bar{a} \) such that \( \text{tp}(\bar{e}/C) \cup \text{tp}(\bar{d}/B) \subseteq \text{tp}(\bar{a}/BC) \), \( \text{tp}(\bar{a}/BC) \) does not fork over \( \phi \), and \( \text{Lstp}(\bar{a}) = \text{Lstp}(\bar{d}) \).

The notation here is fairly standard. \( T \) is a complete theory with no finite models in a first order language \( L \). Types, denoted by \( p, q \), are \( n \)-types and possibly partial. We fix a huge \( \kappa \)-saturated model \( \bar{M} \), as usual. Tuples \( \bar{a}, \bar{b}, \bar{c}... \in \bar{M} \) are finite. Sets \( A, B, C... \) are subsets of \( \bar{M} \) and models which we mention are elementary submodels of \( \bar{M} \), the cardinalities of all of those are strictly less than \( \kappa \).
Let us recall Pillay’s notion of semi-isolation ([3, §2],[9]). We say $tp(b/a)$ is semi-isolated if there is a formula $\varphi(x, a)$ in $tp(b/a)$ such that $\models \varphi(x, a) \rightarrow tp(b)$. Definition implies the following easy, but important facts.

Fact 2.1 (i) If $tp(b/a)$ is isolated, then $tp(b/a)$ is semi-isolated.

(ii) If $tp(c/b)$ and $tp(b/a)$ are semi-isolated, then $tp(c/a)$ is semi-isolated.

Example 2 (i) The notions semi-isolation and isolation are different. For consider the model $(Z, S)$, where $S$ is the successor function. If $a, b$ are in different chains, then $tp(b/a)$ is not isolated, but semi-isolated.

(ii) Let $L = \{E_i|i < \omega\}$. Let $T$ be a theory saying that $E_0$ is an equivalence relation having two infinite classes, and for each $i < \omega$, equivalence relation $E_{i+1}$ refines every $E_i$-class into exactly two infinite $E_{i+1}$-classes. Then $T$ is superstable. Now if $a, c$ are in the same $E_i$-class for each $i$, and $\neg bE_0a$, then $tp(c/b), tp(b/a)$ are isolated, while $tp(c/a)$ is not isolated (but semi-isolated).

Fact 2.2 Suppose that $tp(b/a)$ is isolated, whereas $tp(a/b)$ is nonisolated. Then $tp(a/b)$ is nonsemi-isolated.

Proof. Suppose that $\varphi(x, a)$ isolates $tp(b/a)$. In order to induce a contradiction, assume that $tp(a/b)$ is semi-isolated witnessed by $\psi(b, y)$. Now as $tp(a/b)$ is nonisolated, there is a formula $\phi(x, y) \in L$ such that $\varphi(b, y) \land \psi(b, y) \land \phi(b, y)$ and $\varphi(b, y) \land \psi(b, y) \land \neg \phi(b, y)$ are both consistent. Moreover both formulas imply $tp(a)$. Hence $\varphi(x, a) \land \phi(x, a)$ and $\varphi(x, a) \land \neg \phi(x, a)$ are both consistent. This contradicts the fact that $\varphi(x, a)$ is a principal formula.

Now we state a key proposition which describes the relationship between isolation and forking in a simple theory.

Proposition 2.3 Assume that $T$ is simple. Let $\bar{a}, \bar{b}$ be two realizations of a complete type over $\phi$. If $tp(b/a)$ is semi-isolated, and $tp(a/b)$ is nonsemi-isolated, then $tp(a/b)$ forks over $\phi$.

Proof. Suppose that $tp(b/a)$ is semi-isolated witnessed by $\varphi(x, \bar{a})$. Let $\bar{c}$ be any tuple such that $tp(c/b) = tp(b/a)$. We claim that $\varphi(c, \bar{x}) \land \varphi(x, \bar{a})$ forks over $\phi$. 

2 Forking and isolation
In this section, we prove Theorem 1.2. Isolated. Thus by Fact 2.1, $Lstp(a/b) = Lstp(b/a)$ and $Lstp(a_i/b_i) = Lstp(b_i/a_i)$. We note that, by Fact 2.1, $Lstp(a_i) = Lstp(b_i)$ for every $i$. Therefore clearly $Lstp(a/b)$, $Lstp(b/a)$ both semi-isolated, and hence again by Fact 2.1, so does $Lstp(a_i/b_i)$. Now as $Lstp(a_i/b_i) = Lstp(b_i/a_i) = Lstp(a_i/b_i)$, and so $Lstp(a/b)$, $Lstp(b/a)$ both do not fork over $\phi$. In fact, whenever $Lstp(ab) = Lstp(bc)$, then $Lstp(b/ab)$, $Lstp(a/bc)$ forks over $\phi$.

**Corollary 2.4** Assume that $T$ is simple. Let $\bar{a}, \bar{b}$ be realizations of a complete type over $\phi$. If $Lstp(\bar{b}/\bar{a})$ is isolated, and $Lstp(\bar{a}/\bar{b})$ is nonisolated, then $Lstp(\bar{a}/\bar{b})$ forks over $\phi$.

**Remark 2.5** (i) The simplicity of $T$ is essential in Proposition 2.3. Let $(M, <, \{c_i\}_{i<\omega})$ be the Ehrenfeucht model having 3 nonisomorphic models. The theory of the model is not simple. Choose $a, b$ such that $c_i < a < b$ for all $i$. Then $Lstp(a) = Lstp(b)$, and $Lstp(ab)$ is isolated whereas $Lstp(a/b)$ is not isolated. But $Lstp(a/b)$, $Lstp(b/a)$ both do not fork over $\phi$. In fact, whenever $Lstp(ab) = Lstp(bc)$, then $Lstp(b/ab)$, $Lstp(a/bc)$ forks over $\phi$.

(ii) In 2.3, the Independence Theorem for Lascar strong types yields a cheap proof, provided there is an additional assumption that $Lstp(a) = Lstp(b)$. Now if $\{a, b\}$ were independent, then there is a common realization $\bar{c}$ of $Lstp(\bar{d}/\bar{a})$ and $Lstp(\bar{e}/\bar{b})$ where $Lstp(\bar{d}/\bar{a}) = Lstp(\bar{e}/\bar{b}) = Lstp(\bar{b}/\bar{e})$ and $Lstp(\bar{d}) = Lstp(\bar{e}) = Lstp(\bar{c})$. We note that $Lstp(\bar{b}/\bar{a}), Lstp(\bar{a}/\bar{d})$, and so $Lstp(\bar{a}/\bar{c})$ are semi-isolated. Thus by Fact 2.1, $Lstp(\bar{b}/\bar{c})$ is semi-isolated, while $Lstp(\bar{b}/\bar{e}) = Lstp(\bar{b}/\bar{c})$, a contradiction.

### 3 Proof of Theorem 1.2

In this section, $T$ will be a countable, non-$\aleph_0$-categorical theory.
Fact 3.1 (folklore) Suppose that $T$ has finitely many nonisomorphic models. Then there is a tuple $\bar{a}$ and a prime model $M$ over $\bar{a}$ such that $tp(\bar{a})$ is nonisolated and every complete $n$-type (for all $n$) over $\phi$ is realized in $M$. Moreover there is a tuple $\bar{b}$ in $M$ such that, $tp(\bar{b}) = tp(\bar{a})$ and $tp(\bar{a}/\bar{b})$ is nonisolated.

Proof. Let $q_0, q_1, q_2, ...$ be an enumeration of all complete types of $T$ over $\phi$. Suppose that $\bar{e}_i | q_i$ and $\bar{d}_i = \bar{e}_0\bar{e}_1...\bar{e}_i$. Now there is a prime model $N_i$ over $\bar{d}_i$ for each $i < \omega$. Thus for some $j < \omega$, $N_j(= M)$ is isomorphic to $N_i$ for infinitely many $i \geq j$. Therefore the prime model $M$ over $\bar{d}_j(= \bar{a})$ realizes every complete types over $\phi$. As $M$ is not prime over $\phi$, $tp(\bar{a})$ is not isolated.

Now since $T(\bar{a})$ is again non $\aleph_0$-categorical, for some tuple $\bar{s}$, $tp(\bar{s}/\bar{a})$ is nonisolated. Let $\bar{s}'\bar{b}(\in M)$ realize $tp(\bar{s}\bar{a})$. Then as $tp(\bar{s}'/\bar{b})$ is nonisolated, $M$ is not prime over $\bar{b}$. Since $M$ is prime over $\bar{a}$, $tp(\bar{a}/\bar{b})$ must not be isolated. \qed

We are ready to prove Theorem 1.2. We will use the same notation in the preceding Fact 3.1. Let $T$ be supersimple, and have finitely many models. We will lead a contradiction.

Claim 3.2 Let $p = tp(\bar{a})$. There are two realizations $\bar{a}_0, \bar{a}_1$ of $p$ such that \{\bar{a}_0, \bar{a}_1\} is independent (over $\phi$), and $tp(\bar{a}_0/\bar{a}_1)$ is nonisolated.

Proof. Let $\bar{c}$ be a realization of $p$ such that $tp(\bar{c}/\bar{a}\bar{b})$ does not fork over $\phi$. Now, by Fact 2.2, $tp(\bar{a}/\bar{b})$ is nonsemi-isolated. Hence, by Fact 2.1, either $tp(\bar{a}/\bar{c})$ or $tp(\bar{c}/\bar{b})$ must not be isolated. Thus $\bar{a}, \bar{c}$ or $\bar{c}, \bar{b}$ are desired two realizations of $p$. \qed

Now in the preceding claim, we may assume $\bar{a}_0, \bar{a}_1$ are in $M$. Moreover, as \{\bar{a}_0, \bar{a}_1\} is independent, $tp(\bar{a}_1/\bar{a}_0)$ is also nonisolated, by Corollary 2.4. Now then $tp(\bar{a}/\bar{a}_0)$, $tp(\bar{a}/\bar{a}_1)$ are both nonisolated; for example if $tp(\bar{a}/\bar{a}_0)$ were isolated, then $M$ is prime over $\bar{a}_0$ and so $tp(\bar{a}_1/\bar{a}_0)$ were isolated, a contradiction. Therefore again by Corollary 2.4, $tp(\bar{a}/\bar{a}_0)$ and $tp(\bar{a}/\bar{a}_1)$ both fork over $\phi$.

Let us here summarize the relationships between three realizations $\{\bar{a}, \bar{a}_0, \bar{a}_1\}$ of $p$.

(1) $\{\bar{a}_0, \bar{a}_1\}$ is independent.
(2) For each \(i = 0, 1\), \(tp(\bar{a}_i/\bar{a})\) is isolated, whereas \(tp(\bar{a}/\bar{a}_i)\) is nonisolated (so nonsemi-isolated). Thus \(\{\bar{a}, \bar{a}_i\}\) is not independent.

Now then we are able to construct a tree \(\{\bar{a}_\sigma|\sigma \in 2^{<\omega}\}\) such that \(\bar{a}_\phi = \bar{a}\) and \(tp(\bar{a}_\sigma\bar{a}_0\bar{a}_1) = tp(\bar{a}\bar{a}_0\bar{a}_1)\) for each \(\sigma \in 2^{<\omega}\) (**). Moreover, the basic properties of nonforking together with (1) enable us to assume that every antichain in the tree is independent, (e.g. \(\{\bar{a}_{01}: |\bar{a}_{01}| = n\text{ for some } n < \omega\}\) is independent). Now by (2) with Fact 2.1, for each \(\sigma \in 2^{<\omega}\) and each \(i = 0, 1\), \(tp(\bar{a}_{\sigma\phi}/\bar{a})\) is semi-isolated. But \(tp(\bar{a}/\bar{a}_{\sigma\phi})\) is nonsemi-isolated, since if it were, then again by Fact 2.1, \(tp(\bar{a}_\sigma/\bar{a}_{\sigma\phi})\) is semi-isolated, contradicting (2) and (**). Hence by Proposition 2.3, \(tp(\bar{a}/\bar{a}_{\sigma\phi})\) forks over \(\phi\).

Conclusively, we have countably many independent realizations of \(p\), each of which is not independent with \(\bar{a}\). Finally, by the symmetry and transitivity of nonforking, there is a sequence of complete types \(\langle p_k|k \in \omega\rangle\) such that \(p_0 = p\) and \(p_{k+1}\) is a forking extension of \(p_k\) for each \(k \in \omega\). This violates supersimplicity of \(T\). Therefore Theorem 1.2 is proved.

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