Riemann-Roch Theorem and Index Theorem in Non-commutative Geometry *

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1 The classical Riemann-Roch Theorem

The classical Riemann-Roch Theorem is well-known in the function theory as [M.F. Atiyah and Hirzebruch, Riemann-Roch Theorem for differential manifolds, Bull. Amer. Math. Soc. 1959, Vol. 65, 276-281]

Theorem 1 (The Riemann-Roch Theorem)

\[ r(-D) - i(D) = d(D) - g + 1, \]

where \( D \) is a fixed divisor of degree \( d(D) \) on a Riemann surface \( X \) of genus \( g \), \( r(-D) \) is the dimension of the space of meromorphic functions of divisor \( \geq -D \) on \( X \), \( i(D) \) the dimension of the space of meromorphic 1-forms of divisor \( \geq D \) on \( X \).

This theorem can be considered as computing the Euler characteristics of the sheaf of germs of holomorphic sections of the holomorphic bundle, defined by the divisor \( D \), over \( X \). It plays also an important role in classical algebraic geometry.

Theorem 2 Let \( X \) be a nonsingular complex projective algebraic variety, \( c \) its first Chern class, \( \xi \) a holomorphic bundle over \( X \).

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Then the value on $[X]$ of the cohomological class
\[ e^{c/2}.ch\xi.A^{-1}(p_1(X), p_2(X), \ldots) \]
equal to the Euler characteristic of the sheaf of holomorphic sections of the bundle $\xi$.

2 The Riemann-Roch Theorem in Algebraic Topology

In algebraic topology the Riemann-Roch theorem appeared as some measure of noncommutativity of some diagrams relating two generalized homology theories. Let us remind the most general setting of the Riemann-Roch Theorem.

Consider two generalized (co)homology theories $k(X)$ and $h(X)$ and $\tau : h \to k$ be a multiplicative map, sending $1 \in h^0(pt)$ to $1 \in k^0(pt)$. Consider a vector bundle $\xi$, oriented with respect to the both theories, over the base $X$. Denote $T(\xi)$ the Thom space of $\xi$, i.e. the quotient of the corresponding disk bundle $D(\xi)$ modulo its boundary $Sph(\xi)$. Let us consider an $h$-oriented vector bundle $\xi = (E, X, V, p)$. Denote $E'$ the complement to the zero section of $E$. It is easy to see that $h^r(T(\xi)) = h^r(E, E')$.

Theorem 3 (Thom Isomorphism) The Thom homomorphism

\[ t : h^g(X) = h^g(E) \to h^{g+n}(E, E') = \tilde{h}^{g+n}(T(\xi)) \]
is an isomorphism.

Following the Thom isomorphism theorem, there are Thom isomorphisms $t_h^\xi : \tilde{h}^*(T(\xi)) \to h^*(X)$ and $t_k^\xi : \tilde{k}^*(T(\xi)) \to \tilde{k}^*(X)$. The Todd class is defined as $T_\tau(\xi) := (t_k^\xi)^{-1} \circ \tau \circ t_h^\xi(1)$. The most general Riemann-Roch Theorem states:

Theorem 4 For every $\alpha \in h^*(X)$, one has

\[(t_k^\xi)^{-1} \circ \tau \circ t_h^\xi(\alpha) = \tau(\alpha)T_\tau(\xi)\]
The Todd class is therefore some noncommutativity measure of the diagram

\[
\begin{array}{ccc}
\tilde{h}^*(T(\xi)) & \xrightarrow{\tau} & \tilde{k}^*(T(\xi)) \\
\uparrow_{h}^{\xi} & & \uparrow_{k}^{\xi} \\
h^*(X) & \xrightarrow{\tau} & k^*(X)
\end{array}
\]

Example: \( h = k = H^*([.,\mathbb{Z}_2]), \tau = Sq = 1 + Sq^1 + Sq^2 + \ldots, \) then \( T_\tau^\xi = w(\xi) = 1 + w_1(\xi) + w_2(\xi) + \ldots \) - the full Stiefel-Whitney class.

3 The Riemann-Roch Theorem and the Index Theorem of pseudo-differential operators

One of the consequences of the Riemann-Roch Theorem is the fact that the index of the Dirac operator on \( X \) is exactly the Euler characteristics of \( X \). Let us review the classical results from algebraic topology and topology of pseudo-differential operators. With the Riemann-Roch Theorem it is convenient to define the direct image map \( f! \) as the special case of the composition map

\[
\begin{array}{cccc}
h^*(X) & \xrightarrow{t_h} & \tilde{h}^*(T(\nu(X)))) & \xrightarrow{g^*} & \tilde{h}^*(T(\nu(Y)))) & \xrightarrow{t_h^{-1}} & h^*(Y), \\
\end{array}
\]

where \( h \) is the ordinary co-homology, \( X \) and \( Y \) are the oriented manifolds and \( g = f : X \rightarrow Y \).

Theorem 5 Let \( X \) and \( Y \) be two closed manifold, oriented with respect to the both generalized (co)homology theories \( h \) and \( k \) and \( f : X \rightarrow Y \) a continuous map. Then the Todd classes \( T(X) \) and \( T(Y) \) measure noncommutativity of the diagram

\[
\begin{array}{ccc}
h^*(X) & \xrightarrow{\tau} & k^*(X) \\
\downarrow f! & & \downarrow f! \\
h^*(Y) & \xrightarrow{\tau} & h^*(Y)
\end{array}
\]

and more precisely, \( f!(\tau(x)T(X)) = \tau(f!(x)).T(Y) \).

The index of an arbitrary pseudo-differential operator \( D \) is re-
Theorem 6 (Atiyah-Singer-Hirzebruch Index Theorem)

\[ \text{index} D = \langle (\text{ch} \ D \circ \mathcal{T}^{-1}(\mathcal{C}_\tau(X))), [X] \rangle, \]

where \( \mathcal{T}(\mathbb{C}_\tau(X)) \) is the Todd class of complexified tangent bundle, \( \mathcal{T}^{-1}(\mathbb{C}_\tau(X)) = U(p_1(x), p_2(x), \ldots) \), and

\[ U(e_1(x^2), e_2(x^2), \ldots) = -\prod_i \frac{x^2}{(1 - e^{-x^2})(1 - e^{-x^2})}. \]

As a consequence of the previous theorem we have the following result. Let \( X \) denote a \( 2n \)-dimensional oriented closed smooth manifold with spin structure, i.e. a fixed Hermitian structure on fibers, smoothly depending of points on the base \( X \). Denote \( \Omega^k(X) \) the space of alternating differential \( k \)-forms on \( X \), and \( d : \Omega^k(X) \to \Omega^{k+1}(X) \) the exterior differential, \( * \) the Hodge star operator and \( \delta = * \delta \) : \( \Omega^{k-1}(X) \to \Omega^k(X) \) the adjoint to \( d \) operator. The Dirac operator is \( d + \delta \) is a first order elliptic operator and its index is just equal to the Euler characteristic \( \chi(X) \) of the manifold.

\[ \chi(X) := \sum_r (-1)^r \dim H^r(X, \eta). \]

Theorem 7

\[ \text{ind}(d + \delta) = \chi(X) \]

4 Riemann-Roch Theorem in non-commutative geometry

Let us consider an arbitrary algebra \( A \) over the ground field of complex numbers \( \mathbb{C} \) and \( G \) a locally compact group and denote \( dg \) the left-invariant Haar measure on \( G \). The space \( C_c^\infty(G) \) of all continuous functions with compact support on \( G \) with values in \( A \) under ordinary convolution

\[ (f \ast g)(x) := \int_G f(y)g(y^{-1}x)dy, \]

involution
and norm
\[ \| f \| = \sup_{x \in G} |f(x)| \]
form a the so called cross-product \( A \times G \) of \( A \) and \( G \).

**Theorem 8 (Connes-Thom Isomorphism)**
\[ t_K : K_{*+1}(A \times \mathbb{R}) \cong K_*(A), \]
\[ t_{HC} : HC_{*+1}(A \times \mathbb{R}) \cong HC_*(A), \]
\[ t_{HP} : HP_{*+1}(A \times \mathbb{R}) \cong HP_*(A). \]

A consequence of this theorem is the existence of some noncommutative Todd class

**Theorem 9** There exists some Todd class
\[ (T_\tau = (t_K)^{-1} \circ \tau \circ t_{HP}(1) \]
which measures the noncommutativity of the diagram
\[
\begin{array}{ccc}
K_{*+1}(A \times \mathbb{R}) & \xrightarrow{\tau} & HP_{*+1}(A \times \mathbb{R}) \\
\downarrow t_{\mathbb{K}} & & \downarrow t_{HP} \\
K_*(A) & \xrightarrow{\tau} & HP_*(A)
\end{array}
\]

5 Deformation quantization and periodic cyclic homology

Deformation quantization gives us some noncommutative algebras which are deformation of the classical algebras of holomorphic functions on \( X \). For an arbitrary noncommutative algebra \( A \) there are at least two generalized homology theories: the K-theory and periodic cyclic homology. The Connes-Thom isomorphism gives us a possibility to compare the two theories. There appeared some Todd class as the measure of noncommutativity. Let us review some results of P.Bressler, R. Nest and B. Tsygan [alg-geom/9705014v2 3 Jun 1997].

Deformation quantization of a manifold \( M \) is a formal one pa-
algebras $A^h_M$ flat over $\mathbb{C}[[h]]$ together with an isomorphism of algebras $A^h_M \otimes \mathbb{C}[[h]] \mathbb{C} \to \mathcal{O}_M$. The formula
\[
\{f, g\} = \frac{1}{h} \{\hat{f}, \hat{g}\} + h.A^h_M,
\]
where $f$ and $g$ are two local sections of $\mathcal{O}_M$ and $\hat{f}$, $\hat{g}$ are their respective lifts in $A^h_M$, defines a Poisson structure associated to the deformation quantization $A^h_M$.

It is well-known that all symplectic deformation quantization of $M$ of dimension $\dim_{\mathbb{C}} M = 2d$ are locally isomorphic to the standard deformation quantization of $\mathbb{C}^{2d}$, i.e. in a neighborhood $U'$ of the origin in $\mathbb{C}^{2d}$ there is an isomorphism
\[
A^h_{\mathbb{C}^{2d}}(U') = \mathcal{O}_{\mathbb{C}^{2d}}(U')[[h]] \cong A^h_M(U)
\]
of algebras over $\mathbb{C}[[h]]$, continuous in the $h$-adic topology, where the product on $A^h_{\mathbb{C}^{2d}}(U')$ is given in coordinates $x_1, \ldots, x_d, \xi_1, \ldots, \xi_d$ on $\mathbb{C}^{2d}$ by the standard Weyl product $(f \ast g)(x, \xi) =$
\[
\exp\left(\frac{\sqrt{-1}}{2} \sum_{i=1}^{d} \left(\frac{\partial}{\partial \xi_i} \frac{\partial}{\partial y_i} - \frac{\partial}{\partial \eta_i} \frac{\partial}{\partial x_i}\right)\right) f(x, \xi) g(y, \eta)|_{x=y, \xi=\eta}
\]

**Theorem 10 (RRT for periodic cyclic cycles)** The diagram
\[
\begin{array}{ccc}
CC^\text{per}_* (A^h_M) & \xrightarrow{\sigma} & CC^\text{per}_* (\mathcal{O}_M) \\
\downarrow i & & \downarrow \mu_{\mathcal{O}} \cup \hat{A}(TM) \cup e^\theta \\
CC^\text{per}(A^h_M)[h^{-1}] & \xrightarrow{\mu^h_{A^h_M}} & \prod_{p=\infty}^{\infty} \mathbb{C}_M[h^{-1}, h][-2p]
\end{array}
\]
is commutative.

For $\mathcal{D}$- and $\mathcal{E}$-modules of the ring of pseudo-differential operators, take $M = T^*X$ for a complex manifold $X$, and $A^h_{T^*X}$ is the deformation quantization with the characteristic class $\theta = \frac{1}{2} \pi^* c_1(X)$, then $\hat{A}(TM) \cup e^\theta = \pi^* T(TM)$. After the use of Gelfand-Fuch cohomology the computation become available.
6  Noncommutative Chern characters of some quantum algebras

Let us demonstrate the indicated scheme to some concrete cases of quantum half-planes and quantum punctured complex planes. We indicate some results, computed by Do Ngoc Diep and Nguyen Viet Hai [Contributions to Algebra and Geometry, Vol. 42, No 2] and Do Ngoc Diep and Aderemi O. Kuku [arXiv.org/math.QA/0109042].

Canonical coordinates on the upper half-planes. Recall that the Lie algebra \( g = \text{aff}(R) \) of affine transformations of the real straight line is described as follows, see for example [D2]: The Lie group \( \text{Aff}(R) \) of affine transformations of type

\[
x \in R \mapsto ax + b, \text{ for some parameters } a, b \in R, a \neq 0.
\]

It is well-known that this group \( \text{Aff}(R) \) is a two dimensional Lie group which is isomorphic to the group of matrices

\[
\text{Aff}(R) \cong \{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} | a, b \in R, a \neq 0 \}.
\]

We consider its connected component

\[
G = \text{Aff}_0(R) = \{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} | a, b \in R, a > 0 \}
\]

of identity element. Its Lie algebra is

\[
g = \text{aff}(R) \cong \{ \begin{pmatrix} \alpha & \beta \\ 0 & 0 \end{pmatrix} | \alpha, \beta \in R \}
\]

admits a basis of two generators \( X, Y \) with the only nonzero Lie bracket \([X, Y] = Y\), i.e.

\[
g = \text{aff}(R) \cong \{ \alpha X + \beta Y | [X, Y] = Y, \alpha, \beta \in R \}.
\]

The co-adjoint action of \( G \) on \( g^* \) is given (see e.g. [AC2], [Ki1]) by

\[
\langle K(g)F, Z \rangle = \langle F, \text{Ad}(g^{-1})Z \rangle, \forall F \in g^*, g \in G \text{ and } Z \in g.
\]
Denote the co-adjoint orbit of $G$ in $\mathfrak{g}$, passing through $F$ by
\[\Omega_F = K(G)F := \{K(g)F | F \in G\} \].

Because the group $G = \text{Aff}_0(\mathbb{R})$ is exponential (see [D2]), for $F \in \mathfrak{g}^* = \text{aff}(\mathbb{R})^*$, we have
\[\Omega_F = \{K(\exp(U))F | U \in \text{aff}(\mathbb{R})\} \].

It is easy to see that
\[\langle K(\exp U)F, Z \rangle = \langle F, \exp(-\text{ad}_U)Z \rangle \].

It is easy therefore to see that
\[K(\exp U)F = \langle F, \exp(-\text{ad}_U)X \rangle X^* + \langle F, \exp(-\text{ad}_U)Y \rangle Y^* \].

For a general element $U = \alpha X + \beta Y \in \mathfrak{g}$, we have
\[\exp(-\text{ad}_U) = \sum_{n=0}^{\infty} \frac{1}{n!} \begin{pmatrix} 0 & 0 \\ \beta & -\alpha \end{pmatrix}^n = \begin{pmatrix} 1 & 0 \\ L & e^{-\alpha} \end{pmatrix}, \]
where $L = \alpha + \beta + \frac{\alpha}{\beta}(1 - e^\beta)$. This means that
\[K(\exp U)F = (\lambda + \mu L)X^* + (\mu e^{\alpha})Y^* \].

From this formula one deduces [D2] the following description of all co-adjoint orbits of $G$ in $\mathfrak{g}^*$:

- If $\mu = 0$, each point $(x = \lambda, y = 0)$ on the abscissa ordinate corresponds to a 0-dimensional co-adjoint orbit
  \[\Omega_\lambda = \{\lambda X^*\}, \quad \lambda \in \mathbb{R}\].

- For $\mu \neq 0$, there are two 2-dimensional co-adjoint orbits: the upper half-plane \{(\lambda, \mu) \mid \lambda, \mu \in \mathbb{R}, \mu > 0\} corresponds to the co-adjoint orbit
  \[\Omega_+ := \{F = (\lambda + \mu L)X^* + (\mu e^{-\alpha})Y^* \mid \mu > 0\}, \quad (1)\]
  and the lower half-plane \{(\lambda, \mu) \mid \lambda, \mu \in \mathbb{R}, \mu < 0\} corresponds to the co-adjoint orbit
  \[\Omega_- \subseteq \{F = (\lambda + \mu L)Y^* + (\mu e^{-\alpha})X^* \mid \mu < 0\} \quad (2)\].
Denote by $\psi$ the indicated symplectomorphism from $\mathbb{R}^2$ onto $\Omega_+$

$$(p, q) \in \mathbb{R}^2 \mapsto \psi(p, q) := (p, e^q) \in \Omega_+$$

Proposition 11

1. Hamiltonian function $f_Z = \tilde{Z}$ in canonical coordinates $(p, q)$ of the orbit $\Omega_+$ is of the form

$$\tilde{Z} \circ \psi(p, q) = \alpha p + \beta e^q, \text{ if } Z = \begin{pmatrix} \alpha & \beta \\ 0 & 0 \end{pmatrix}.$$

2. In the canonical coordinates $(p, q)$ of the orbit $\Omega_+$, the Kirillov form $\omega_Y$ is just the standard form $\omega = dp \wedge dq$.

Computation of generators $\hat{\ell}_Z$ Let us denote by $\Lambda$ the 2-tensor associated with the Kirillov standard form $\omega = dp \wedge dq$ in canonical Darboux coordinates. We use also the multi-index notation. Let us consider the well-known Moyal $\star$-product of two smooth functions $u, v \in C^\infty(\mathbb{R}^2)$, defined by

$$u \star v = u.v + \sum_{r \geq 1} \frac{1}{r!} \left( \frac{1}{2r} \right)^r P^r(u, v),$$

where

$$P^r(u, v) := \Lambda^{i_1 j_1} \Lambda^{i_2 j_2} \ldots \Lambda^{i_r j_r} \partial_{i_1 i_2 \ldots i_r} u \partial_{j_1 j_2 \ldots j_r} v,$$

with

$$\partial_{i_1 i_2 \ldots i_r} := \frac{\partial^r}{\partial x^{i_1} \ldots \partial x^{i_r}}, \quad x := (p, q) = (p_1, \ldots, p_n, q^1, \ldots, q^n)$$

as multi-index notation. It is well-known that this series converges in the Schwartz distribution spaces $S(\mathbb{R}^n)$. We apply this to the special case $n = 1$. In our case we have only $x = (x^1, x^2) = (p, q)$.

Proposition 12 In the above mentioned canonical Darboux coordinates $(p, q)$ on the orbit $\Omega_+$, the Moyal $\star$-product satisfies the relation

$$i \tilde{Z} \circ i \tilde{T} - i \tilde{T} \circ i \tilde{Z} = i [Z, T], \quad \forall Z, T \in \text{aff}(\mathbb{R}).$$
Consequently, to each adapted chart $\psi$ in the sense of [AC2], we associate a $G$-covariant $\star$-product.

Proposition 13 (see [G]) Let $\star$ be a formal differentiable $\star$-product on $C^\infty(M, \mathbb{R})$, which is covariant under $G$. Then there exists a representation $\tau$ of $G$ in $\text{Aut} \ N[[\nu]]$ such that

$$\tau(g)(u \star v) = \tau(g)u \star \tau(g)v.$$ 

Let us denote by $F_p u$ the partial Fourier transform of the function $u$ from the variable $p$ to the variable $x$, i.e.

$$F_p(u)(x, q) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ipx} u(p, q) dp.$$ 

Let us denote by $F_p^{-1}(u)(x, q)$ the inverse Fourier transform.

Lemma 14

1. $\partial_p F_p^{-1}(p.u) = iF_p^{-1}(x.u)$,
2. $F_p(v) = i\partial_x F_p(v)$,
3. $P^k(\tilde{Z}, F_p^{-1}(u)) = (-1)^k \beta e^q \frac{\partial^k F_p^{-1}(u)}{\partial p^k}$, with $k \geq 2$.

For each $Z \in \text{aff}(\mathbb{R})$, the corresponding Hamiltonian function is $\tilde{Z} = \alpha p + \beta e^q$ and we can consider the operator $\ell_Z$ acting on dense subspace $L^2(\mathbb{R}^2, \frac{d^2 p dq}{2\pi})^\infty$ of smooth functions by left $\star$-multiplication by $i\tilde{Z}$, i.e. $\ell_Z(u) = i\tilde{Z} \star u$. It is then continued to the whole space $L^2(\mathbb{R}^2, \frac{d^2 p dq}{2\pi})$. It is easy to see that, because of the relation in Proposition (12), the correspondence $Z \in \text{aff}(\mathbb{R}) \mapsto \ell_Z = i\tilde{Z} \star$ is a representation of the Lie algebra $\text{aff}(\mathbb{R})$ on the space $N[[\nu]]$ of formal power series in the parameter $\nu = \frac{i}{2}$ with coefficients in $N = C^\infty(M, \mathbb{R})$, see e.g. [G] for more detail.

We study now the convergence of the formal power series. In order to do this, we look at the $\star$-product of $i\tilde{Z}$ as the $\star$-product of symbols and define the differential operators corresponding to $i\tilde{Z}$. It is easy to see that the resulting correspondence is a representation of $g$ by pseudo-differential operators.
Proposition 15 For each $Z \in \text{aff}(\mathbb{R})$ and for each compactly supported $C^\infty$ function $u \in C_0^\infty(\mathbb{R}^2)$, we have

$$\hat{\ell}_Z(u) := \mathcal{F}_p \circ \ell_Z \circ \mathcal{F}_p^{-1}(u) = \alpha \left( \frac{1}{2} \partial_q - \partial_x \right) u + i \beta e^{q - \frac{x}{2}} u.$$

The associate irreducible unitary representations

Our aim in this section is to exponentiate the obtained representation $\hat{\ell}_Z$ of the Lie algebra $\text{aff}(\mathbb{R})$ to the corresponding representation of the Lie group $\text{Aff}_0(\mathbb{R})$. We shall prove that the result is exactly the irreducible unitary representation $T_{\Omega^+}$ obtained from the orbit method or Mackey small subgroup method applied to this group $\text{Aff}(\mathbb{R})$. Let us recall first the well-known list of all the irreducible unitary representations of the group of affine transformation of the real straight line.

Theorem 16 ([GN]) Every irreducible unitary representation of the group $\text{Aff}(\mathbb{R})$ of all the affine transformations of the real straight line, up to unitary equivalence, is equivalent to one of the pairwise non-equivalent representations:

- the infinite dimensional representation $S$, realized in the space $L^2(\mathbb{R}^*, \frac{dy}{|y|})$, where $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$ and is defined by the formula

$$ (S(g)f)(y) := e^{iby} f(ay), \quad \text{where } g = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}, $$

- the representation $U^\varepsilon_\lambda$, where $\varepsilon = 0, 1$, $\lambda \in \mathbb{R}$, realized in the 1-dimensional Hilbert space $C^1$ and is given by the formula

$$ U^\varepsilon_\lambda(g) = |a|^{i\lambda} (\text{sgn } a)^\varepsilon. $$

Let us consider now the connected component $G = \text{Aff}_0(\mathbb{R})$. The irreducible unitary representations can be obtained easily from the orbit method machinery.

Theorem 17 The representation $\exp(\hat{\ell}_Z)$ of the group $G = \text{Aff}_0(\mathbb{R})$ is exactly the irreducible unitary representation $T_{\Omega^+}$ of $G = \text{Aff}_0(\mathbb{R})$. 
associated following the orbit method construction, to the orbit \( \Omega_+ \), which is the upper half-plane \( \mathbb{H} \cong \mathbb{R} \times \mathbb{R}^* \), i. e. +

\[
(\exp(\hat{\ell}_Z)f)(y) = (T_{\Omega_+}(g)f)(y) = e^{iby}f(ay), \forall f \in L^2(\mathbb{R}^*, \frac{dy}{|y|}),
\]

where \( g = \exp Z = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \).

By analogy, we have also

**Theorem 18** The representation \( \exp(\hat{\ell}_Z) \) of the group \( G = \text{Aff}_0(\mathbb{R}) \) is exactly the irreducible unitary representation \( T_{\Omega_-} \) of \( G = \text{Aff}_0(\mathbb{R}) \) associated following the orbit method construction, to the orbit \( \Omega_- \), which is the lower half-plane \( \mathbb{H} \cong \mathbb{R} \times \mathbb{R}^* \), i. e.

\[
(\exp(\hat{\ell}_Z)f)(y) = (T_{\Omega_-}(g)f)(y) = e^{iby}f(ay), \forall f \in L^2(\mathbb{R}^*, \frac{dy}{|y|}),
\]

where \( g = \exp Z = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \).

### 6.1 The group of affine transformations of the complex straight line

Recall that the Lie algebra \( g = \text{aff}(\mathbb{C}) \) of affine transformations of the complex straight line is described as follows, see [D].

It is well-known that the group \( \text{Aff}(\mathbb{C}) \) is a four (real) dimensional Lie group which is isomorphism to the group of matrices:

\[
\text{Aff}(\mathbb{C}) \cong \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \mid a, b \in \mathbb{C}, a \neq 0 \right\}
\]

The most easy method is to consider \( X, Y \) as complex generators, \( X = X_1 + iX_2 \) and \( Y = Y_1 + iY_2 \). Then from the relation \([X, Y] = Y, \text{ we get } [X_1, Y_1] - [X_2, Y_2] + i([X_1 Y_2] + [X_2, Y_1]) = Y_1 + iY_2\). This mean that the Lie algebra \( \text{aff}(\mathbb{C}) \) is a real 4-dimensional Lie algebra, having 4 generators with the only nonzero Lie brackets: \([X_1, Y_1] - [X_2, Y_2] = Y_1; [X_2, Y_1] + [X_1, Y_2] = Y_2\) and we can choose
another basic noted again by the same letters to have more clear Lie brackets of this Lie algebra:

\[[X_1, Y_1] = Y_1; [X_1, Y_2] = Y_2; [X_2, Y_1] = Y_2; [X_2, Y_2] = -Y_1\]

Remark 19 Let us denote:

\[H_k = \{ w = q_1 + iq_2 \in C | -\infty < q_1 < +\infty; 2k\pi < q_2 < 2k\pi + 2\pi \}; k = 0, \pm 1, \ldots \]

\[L = \{ \rho e^{i\varphi} \in C | 0 < \rho < +\infty; \varphi = 0 \} \text{ and } C_k = C \setminus L \]

is a univalent sheet of the Riemann surface of the complex variable multi-valued analytic function \(\text{Ln}(w), (k = 0, \pm 1, \ldots)\) Then there is a natural diffeomorphism \(w \in H_k \Longleftrightarrow e^w \in C_k\) with each \(k = 0, \pm 1, \ldots\) Now consider the map:

\[C \times C \longrightarrow \Omega_F = C \times C^* \]

\[(z, w) \longmapsto (z, e^w),\]

with a fixed \(k \in \mathbb{Z}\). We have a local diffeomorphism

\[\varphi_k : C \times H_k \longrightarrow C \times C_k\]

\[(z, w) \longmapsto (z, e^w)\]

This diffeomorphism \(\varphi_k\) will be needed in the all sequel.

On \(C \times H_k\) we have the natural symplectic form

\[\omega = \frac{1}{2}[dz \wedge dw + d\bar{z} \wedge d\bar{w}], \quad (3)\]

induced from \(C^2\). Put \(z = p_1 + ip_2, w = q_1 + iq_2\) and \((x^1, x^2, x^3, x^4) = (p_1, q_1, p_2, q_2) \in \mathbb{R}^4\), then

\[\omega = dp_1 \wedge dq_1 - dp_2 \wedge dq_2.\]

The corresponding symplectic matrix of \(\omega\) is

\[\wedge = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \quad \text{and} \quad \wedge^{-1} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}\]
We have therefore the Poisson brackets of functions as follows. With each \( f, g \in C^\infty(\Omega) \)

\[
\{f, g\} = \theta_{ij} \frac{\partial f}{\partial x^i} \frac{\partial g}{\partial x^j} = \theta_{12} \frac{\partial f}{\partial p_1} \frac{\partial g}{\partial q_1} + \theta_{21} \frac{\partial f}{\partial q_1} \frac{\partial g}{\partial p_1} + \theta_{34} \frac{\partial f}{\partial p_2} \frac{\partial g}{\partial q_2} + \theta_{43} \frac{\partial f}{\partial q_2} \frac{\partial g}{\partial p_2} = \\
= \frac{\partial f}{\partial p_1} \frac{\partial g}{\partial q_1} - \frac{\partial f}{\partial q_1} \frac{\partial g}{\partial p_1} - \frac{\partial f}{\partial p_2} \frac{\partial g}{\partial q_2} + \frac{\partial f}{\partial q_2} \frac{\partial g}{\partial p_2} = \\
= 2\left[ \frac{\partial f}{\partial z} \frac{\partial g}{\partial w} - \frac{\partial f}{\partial w} \frac{\partial g}{\partial z} \right]
\]

Proposition 20
Fixing the local diffeomorphism \( \varphi_k (k \in \mathbb{Z}) \), we have:

1. For any element \( A \in \text{aff}(\mathbb{C}) \), the corresponding Hamiltonian function \( \tilde{A} \) in local coordinates \((z, w)\) of the orbit \( \Omega_F \) is of the form

\[
\tilde{A} \circ \varphi_k (z, w) = \frac{1}{2} [\alpha z + \beta e^w + \overline{\alpha z} + \overline{\beta e^w}]
\]

2. In local coordinates \((z, w)\) of the orbit \( \Omega_F \), the symplectic Kirillov form \( \omega_F \) is just the standard form \((1)\).

**Computation of Operators** \( \ell_A^{(k)} \).

Proposition 21
With \( A, B \in \text{aff}(\mathbb{C}) \), the Moyal \(*\)-product satisfies the relation:

\[
i \tilde{A} * i \tilde{B} - i \tilde{B} * i \tilde{A} = i[\tilde{A}, \tilde{B}]
\]

(4)

For each \( A \in \text{aff}(\mathbb{C}) \), the corresponding Hamiltonian function is

\[
\tilde{A} = \frac{1}{2} [\alpha z + \beta e^w + \overline{\alpha z} + \overline{\beta e^w}]
\]

and we can consider the operator \( \ell_A^{(k)} \) acting on dense subspace \( L^2(\mathbb{R}^2 \times (\mathbb{R}^2)^*, \frac{dp_1 dq_1 dp_2 dq_2}{(2\pi)^2}) \) of smooth functions by left \(*\)-multiplication by \( i \tilde{A} \), i.e.

\[
\ell_A^{(k)} (f) = i \tilde{A} * f
\]

Because of the relation in Proposition 3.1, we have

Corollary 22

\[
\ell_A^{(k)} = \ell_A^{(k)} + \ell_A^{(k)}\]

(5)
From this it is easy to see that, the correspondence $A \in \text{aff}(C) \mapsto \ell_A^{(k)} = i\tilde{A}$. is a representation of the Lie algebra $\text{aff}(C)$ on the space $N[[\frac{1}{2}]]$ of formal power series, see [G] for more detail.

Proposition 23 For each $A = \begin{pmatrix} \alpha & \beta \\ 0 & 0 \end{pmatrix} \in \text{aff}(C)$ and for each compactly supported $C^\infty$-function $f \in C^\infty_0(C \times H_k)$, we have:

$$\ell_A^{(k)} f := \mathcal{F}_z \circ \ell_A^{(k)} \circ \mathcal{F}_z^{-1}(f) = [\alpha(\frac{1}{2} \partial_w - \partial_\xi) f + \bar{\alpha}(\frac{1}{2} \partial_w - \partial_\xi) f + \frac{i}{2}(\beta e^w - \frac{1}{2} \xi + \bar{\beta} e^w - \frac{1}{2} \bar{\xi}) f]$$

(6)

Remark 24 Setting new variables $u = w - \frac{1}{2} \bar{\xi}; v = w + \frac{1}{2} \bar{\xi}$ we have

$$\hat{\ell}_A^{(k)}(f) = \alpha \frac{\partial f}{\partial u} + \bar{\alpha} \frac{\partial f}{\partial u} + \frac{i}{2}(\beta e^u + \bar{\beta} e^\bar{u}) f|_{(u,v)}$$

(7)

i.e $\hat{\ell}_A^{(k)} = \alpha \frac{\partial}{\partial u} + \bar{\alpha} \frac{\partial}{\partial u} + \frac{i}{2}(\beta e^u + \bar{\beta} e^\bar{u})$, which provides a (local) representation of the Lie algebra $\text{aff}(C)$.

The Irreducible Representations of $\tilde{\text{Aff}}(C)$. Since $\hat{\ell}_A^{(k)}$ is a representation of the Lie algebra $\tilde{\text{Aff}}(C)$, we have:

$$\exp(\hat{\ell}_A^{(k)}) = \exp(\alpha \frac{\partial}{\partial u} + \bar{\alpha} \frac{\partial}{\partial u} + \frac{i}{2}(\beta e^u + \bar{\beta} e^\bar{u}))$$

is just the corresponding representation of the corresponding connected and simply connected Lie group $\tilde{\text{Aff}}(C)$.

Let us first recall the well-known list of all the irreducible unitary representations of the group of affine transformation of the complex straight line, see [D] for more details.

Theorem 25 Up to unitary equivalence, every irreducible unitary representation of $\tilde{\text{Aff}}(C)$ is unitarily equivalent to one the following one-to-another non-equivalent irreducible unitary representations:

1. The unitary characters of the group, i.e the one dimensional unitary representation $U_\lambda, \lambda \in \mathbb{C}$, acting in $\mathbb{C}$ following the formula

$$U_\lambda(z,w) = e^{i \lambda(z \bar{w} - \bar{z} w)}, \forall (z,w) \in \tilde{\text{Aff}}(C), \lambda \in \mathbb{C}.$$
2. The infinite dimensional irreducible representations \( T_\theta, \theta \in S^1 \), acting on the Hilbert space \( L^2(\mathbb{R} \times S^1) \) following the formula:

\[
[T_\theta(z, w)f](x) = \exp(i(\Re(wx) + 2\pi\theta[\frac{\Im(x+z)}{2\pi}]))f(x \oplus z), \quad (8)
\]

Where \((z, w) \in \tilde{\text{Aff}}(C); x \in \mathbb{R} \times S^1 = C \setminus \{0\}; f \in L^2(\mathbb{R} \times S^1); x \oplus z = \Re(x + z) + 2\pi i\{\frac{\Im(x+z)}{2\pi}\}\]

In this section we will prove the following important Theorem which is very interesting for us both in theory and practice.

**Theorem 26** The representation \( \exp(\hat{\ell}^{(k)}_A) \) of the group \( \tilde{\text{Aff}}(C) \) is the irreducible unitary representation \( T_\theta \) of \( \tilde{\text{Aff}}(C) \) associated, following the orbit method construction, to the orbit \( \Omega \), i.e:

\[
\exp(\hat{\ell}^{(k)}_A)f(x) = [T_\theta(\exp A)f](x),
\]

where \( f \in L^2(\mathbb{R} \times S^1); A = \begin{pmatrix} \alpha & \beta \\ 0 & 0 \end{pmatrix} \in \text{aff}(C); \theta \in S^1; k = 0, \pm 1, \ldots \)

**Remark 27** We say that a real Lie algebra \( g \) is in the class \( MD \) if every K-orbit is of dimension, equal 0 or \( \dim g \). Further more, one proved that ([D, Theorem 4.4]) Up to isomorphism, every Lie algebra of class \( MD \) is one of the following:

1. Commutative Lie algebras.
2. Lie algebra \( \text{aff}(\mathbb{R}) \) of affine transformations of the real straight line
3. Lie algebra \( \text{aff}(\mathbb{C}) \) of affine transformations of the complex straight line.

Thus, by calculation for the group of affine transformations of the real straight line \( \text{Aff}(\mathbb{R}) \) in [DH] and here for the group affine transformations of the complex straight line \( \text{Aff}(\mathbb{C}) \) we obtained a description of the co-adjoint \( MD \) orbits.
6.2 $MD_4$-groups

We refer the reader to the results of Nguyen Viet Hai [H3]-[H4] for the class of $MD_4$-groups (i.e. 4-dimensional solvable Lie groups, all the co-adjoint of which are of dimension 0 or maximal). It is interesting that here he obtained the same exact computation for $\ast$-products and all representations.

6.3 $SO(3)$

As an typical example of compact Lie group, the author proposed Job A. Nable to consider the case of $SO(3)$. We refer the reader to the results of Job Nable [Na1]-[Na3]. In these examples, it is interesting that the $\ast$-products, in some how as explained in these papers, involved the Maslov indices and Monodromy Theorem.

6.4 Exponential groups

Arnal-Cortet constructed star-products for this case [AC1]-[AC2].

6.5 Compact groups

We refer readers to the works of C. Moreno [Mo].

7 Algebraic Noncommutative Chern Characters

Let $G$ be a compact group, $\text{HP}_\ast(C\ast(G))$ the periodic cyclic homology introduced in §2. Since $C\ast(G) = \lim_{N} \prod_{i=1}^{N} \text{Mat}_{n_i}(\mathbb{C})$, $\text{HP}_\ast(C\ast(G))$ coincides with the $\text{HP}_\ast(C\ast(G))$ defined by J. Cuntz-D. Quillen [CQ].

Lemma 28 Let $\{I_N\}_{N \in \mathbb{N}}$ be the above defined collection of ideals in $C\ast(G)$. Then

$$K_\ast(C\ast(G)) = \lim_{N} K_\ast(I_N) = K_\ast(\mathbb{C}(\mathcal{T})).$$
where $T$ is the fixed maximal torus in $G$.

First note that the algebraic K-theory of $C^*$-algebras has the stability property
\[ K_\ast(A \otimes M_n(C)) \cong K_\ast(C(T)). \]

Hence,
\[ \lim_{\to} K_\ast(I_{n_i}) \cong K_\ast(\prod_{w = \text{highest weight}} C_w) \cong K_\ast(C(T)), \]
by Pontryagin duality.

J. Cuntz and D. Quillen [CQ] defined the so-called $X$-complexes of $C^*$-algebras and then used some ideas of Fedosov product to define algebraic Chern characters. We now briefly recall their definitions. For a (non-commutative) associate $C^*$-algebra $A$, consider the space of even non-commutative differential forms $\Omega^+(A) \cong RA$, equipped with the Fedosov product
\[ \omega_1 \circ \omega_2 := \omega_1 \omega_2 - (-1)^{\omega_1} d\omega_1 d\omega_2, \]
see [CQ]. Consider also the ideal $IA := \oplus_{k \geq 1} \Omega^{2k}(A)$. It is easy to see that $RA/IA \cong A$ and that $RA$ admits the universal property that any based linear map $\rho : A \to M$ can be uniquely extended to a derivation $D : RA \to M$. The derivations $D : RA \to M$ bijectively correspond to lifting homomorphisms from $RA$ to the semi-direct product $RA \oplus M$, which also bijectively correspond to linear map $\bar{\rho} : \bar{A} = A/C \to M$ given by
\[ a \in \bar{A} \mapsto D(\rho a). \]

From the universal property of $\Omega^1(RA)$, we obtain a bimodule isomorphism
\[ RA \otimes \bar{A} \otimes RA \cong \Omega^1(RA). \]

As in [CQ], let $\Omega^- A = \oplus_{k \geq 0} \Omega^{2k+1} A$. Then we have
\[ \Omega^- A \cong RA \otimes \bar{A} \cong \Omega^1(RA)_\# := \Omega^1(RA)/[(\Omega^1(RA), RA)]. \]
Theorem 29 ([CQ], Theorem 1): There exists an isomorphism of \( \mathbb{Z}/(2) \)-graded complexes
\[
\Phi : \Omega A = \Omega^+ A \oplus \Omega^- A \cong RA \oplus \Omega^1(RA),
\]
such that
\[
\Phi : \Omega^+ A \cong RA,
\]
is defined by
\[
\Phi(a_0 da_1 \ldots da_{2n}) = \rho(a_1) \omega(a_1, a_2) \ldots \omega(a_{2n-1}, a_{2n}),
\]
and
\[
\Phi : \Omega^- A \cong \Omega^1(RA),
\]
\[
\Phi(a_0 da_1 \ldots da_{2n+1}) = \rho(a_1) \omega(a_1, a_2) \ldots \omega(a_{2n-1}, a_{2n}) \delta(a_{2n+1}).
\]
With respect to this identification, the product in \( RA \) is just the Fedosov product on even differential forms and the differentials on the \( X \)-complex
\[
X(RA) : \quad RA \cong \Omega^+ A \to \Omega^1(RA) \cong \Omega^- A \to RA
\]
become the operators
\[
\beta = b - (1 + \kappa)d : \Omega^- A \to \Omega^+ A,
\]
\[
\delta = -N \kappa^2 b + B : \Omega^+ A \to \Omega^- A,
\]
where
\[
N \kappa^2 = \sum_{j=0}^{n-1} \kappa^{2j}, \quad \kappa(da_1 \ldots da_n) := da_n \ldots da_1.
\]
Let us denote by \( IA \triangleleft RA \) the ideal of even non-commutative differential forms of order \( \geq 2 \). By the universal property of \( \Omega^1 \)
\[
\Omega^1(RA/IA) = \Omega^1 RA/((IA)\Omega^1 RA + \Omega^1 RA.(IA) + dIA).
\]
Since \( \Omega^1 RA = (RA)dRA = dRA.(RA) \), then
\[
\Omega^1 RA(IA) \cong IA\Omega^1 RA \mod [RA, \Omega^1 R].
\]
\[
\Omega^1(RA/IA) \# = \Omega^1 RA/([RA, \Omega^1 RA] + IA.dRA + dIA).
\]
For \( IA \)-adic tower \( RA/(IA)^{n+1} \), we have the complex \( \mathcal{X}(RA/(IA)^{n+1}) \):
Define $\mathcal{X}^{2n+1}(RA, IA)$:

$$RA/\left(IA\right)^{n+1} \rightarrow \Omega^1 RA/\left([RA, \Omega^1 RA] + (IA)^{n+1}dRA + d(IA)^{n+1}\right) \rightarrow RA/\left(IA\right)^{n+1},$$

and $\mathcal{X}^{2n}(RA, IA)$:

$$RA/\left((IA)^{n+1} + [RA, IA^n]\right) \rightarrow \Omega^1 RA/\left([RA, \Omega^1 RA] + d(IA)^ndRA\right) \rightarrow RA/\left((IA)^{n+1} + [RA, IA^n]\right).$$

One has

$$b((IA)^ndIA) = [(IA)^n, IA] \subset (IA)^{n+1},$$

$$d(IA)^{n+1} \subset \sum_{j=0}^{n}(IA)^j d(IA)(IA)^{n-j} \subset (IA)^ndIA + [RA, \Omega^1 RA].$$

and hence

$$\mathcal{X}^1(RA, IA) = X(RA, IA),$$

$$\mathcal{X}^0(RA, IA) = (RA/IA)\#.$$}

There is an sequence of maps between complexes

$$\ldots \rightarrow X(RA/IA) \rightarrow \mathcal{X}^{2n+1}(RA, IA) \rightarrow \mathcal{X}^{2n}(RA, IA) \rightarrow X(RA/IA) \rightarrow \ldots$$

We have the inverse limits

$$\hat{X}(RA, IA) := \lim_{\leftarrow} X(RA/\left(IA\right)^{n+1}) = \lim_{\leftarrow} \mathcal{X}^n(RA, IA).$$

Remark that

$$\mathcal{X}^q = \Omega A/F^q\Omega A,$$

$$\hat{X}(RA/IA) = \hat{\Omega} A.$$}

We quote the second main result of J. Cuntz and D. Quillen ([CQ], Thm2), namely:

$$H_i \hat{X}(RA, IA) = HP_i(A).$$

We now apply this machinery to our case. First we have the following.
Lemma 30

\[ \lim_{N \to \infty} HP^*(I_N) \cong HP^*(C(T)). \]

By similar arguments as in the previous lemma 28. More precisely, we have

\[ HP(I_{n_i}) = HP(\prod_{w=\text{highest weight}} C_w) \cong HP(C(T)) \]

by Pontryagin duality.

Now, for each idempotent \( e \in M_n(A) \) there is an unique element \( x \in M_n(\widehat{RA}) \). Then the element

\[ \tilde{e} := x + (x - \frac{1}{2}) \sum_{n \geq 1} \frac{2^n(2n - 1)!!}{n!} (x - x^{2n})^{2n} \in M_n(\widehat{RA}) \]

is a lifting of \( e \) to an idempotent matrix in \( M_n(\widehat{RA}) \). Then the map \([e] \mapsto \text{tr}(\tilde{e})\) defines the map \( K_0 \to H_0(X(\widehat{RA})) = HP_0(A) \). To an element \( g \in \text{GL}_n(A) \) one associates an element \( p \in \text{GL}(\widehat{RA}) \) and to the element \( g^{-1} \) an element \( q \in \text{GL}_n(\widehat{RA}) \) then put

\[ x = 1 - qp, \ \text{and} \ y = 1 - pq. \]

And finally, to each class \([g] \in \text{GL}_n(A)\) one associates

\[ \text{tr}(g^{-1}dg) = \text{tr}(1 - x)^{-1}d(1 - x) = d(\text{tr}(\log(1 - x))) = \]

\[ = -\text{tr} \sum_{n=0}^{\infty} x^n dx \in \Omega^1(A) \# . \]

Then \([g] \mapsto \text{tr}(g^{-1}dg)\) defines the map \( K_1(A) \to HH_1(A) = H_1(X(\widehat{RA})) = HP_1(A) \).

Let \( HP(I_{n_i}) \) be the periodic cyclic cohomology defined by Cuntz-Quillen. Then the pairing

\[ K^\text{alg}_*(C^*(G)) \times \bigcup_N HP^*(I_N) \to \mathbb{C} \]

defines an algebraic non-commutative Chern character.
which gives us a variant of non-commutative Chern characters with values in $HP$-groups.

Theorem 31 Let $G$ be a compact group and $T$ a fixed maximal compact torus of $G$. Then, the Chern character

$$\text{ch}_{\text{alg}} : K_*(C^*(G)) \to HP_*(C^*(G))$$

is an isomorphism, which can be identified with the classical Chern character

$$ch : K_*(C(T)) \to HP_*(C(T))$$

which is also an isomorphism.

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