Topos Quantum Theory on Quantization-Induced Sheaves

Kunji Nakayama*
Faculty of Law
Ryukoku University
Fushimi-ku, Kyoto 612-8577

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Abstract

In this paper, we construct a sheaf-based topos quantum theory. It is well known that a topos quantum theory can be constructed on the topos of presheaves on the category of commutative von Neumann algebras of bounded operators on a Hilbert space. Also, it is already known that quantization naturally induces a Lawvere-Tierney topology on the presheaf topos. We show that a topos quantum theory akin to the presheaf-based one can be constructed on sheaves defined by the quantization-induced Lawvere-Tierney topology. That is, starting from the spectral sheaf as a state space of a given quantum system, we construct sheaf-based expressions of physical propositions and truth objects, and thereby give a method of truth-value assignment to the propositions. Furthermore, we clarify the relationship to the presheaf-based quantum theory. We give translation rules between the sheaf-based ingredients and the corresponding presheaf-based ones. The translation rules have ‘coarse-graining’ effects on the spaces of the presheaf-based ingredients; a lot of different proposition presheaves, truth presheaves, and presheaf-based truth-values are translated to a proposition sheaf, a truth sheaf, and a sheaf-based truth-value, respectively. We examine the extent of the coarse-graining made by translation.

*e-mail: nakayama@law.ryukoku.ac.jp
1 Introduction

Since Isham \cite{1} applied topos theory to history quantum theory, topos theoretic approach to quantum theory has been studied by many researchers \cite{2,13}. In this approach, quantum theory is reformulated within a framework of intuitionistic (hence, multi-valued) logic. Every physical proposition about a given quantum system is assigned a truth-value without falling foul of the Kochen-Specker no go theorem \cite{19}. Therefore, the topos approach permits some kind of realistic interpretation regarding values of physical quantities that does not require things like the notion of measurement. Because of this, it can provide a promising framework for quantum gravity theory and quantum cosmology.

There are a few different ways of topos approach. Among them, we focus on the formalism made by Döring and Isham \cite{6,7,14,15}. They adopted the topos of presheaves on the category of commutative von Neumann algebras of bounded operators on a Hilbert space. In their theory, the spectral presheaf plays a key role similar to state space of classical physics. As a result of the Kochen-Specker theorem, we cannot assign to every physical quantity of a quantum system a sharply determined value, which means there are no global elements of the spectral presheaf \cite{2,5}. In this respect, the spectral presheaf is largely different from the state space of classical physics, since the latter consists of points, each of which corresponds to a state where every physical quantity has sharply determined value. Nonetheless, the spectral presheaf can work as a state space in that every physical proposition about a given quantum system can be expressed as its subobject, as every physical proposition about a classical system can be identified with its extensional expression, i.e., a subset of state space. By regarding the spectral presheaf as state space and using it in a topos theoretical framework, Döring and Isham succeeded in giving a method that assigns to every physical proposition a truth-value.

Döring and Isham's theory is an abstract, general theory; it does not need to be related to concrete classical systems, like ordinary quantum theories that are axiomatically or algebraically formulated on Hilbert spaces or $C^*$-algebras. This is the case for the other topos quantum theories obtained so far. If quantization of a classical system is taken into consideration, however, some extra structures are induced on the topos on which a quantum theory is formulated. In fact, Nakayama \cite{20} showed that quantization that is given by a function from classical observables to self-adjoint operators on a Hilbert
space naturally induces a Lawvere-Tierney topology on the presheaf topos of Döring and Isham. It is well-known that any Lawvere-Tierney topology defines sheaves, and furthermore, the collection of all such sheaves also forms a topos \([21]\). Thus, from the presheaf topos, we obtain another topos consisting of sheaves via quantization.

One question would arise. Can we construct a quantum theory on the topos of quantization-induced sheaves? One of the purposes of the present paper is to give an affirmative answer to the question. We can construct a topos quantum theory on the quantization-induced sheaves in a way akin to the presheaf-based theory of Döring and Isham. Such a theory could be canonical as a theory of the system quantized from the classical one since quantum observables corresponding to classical ones are identified therein.

Furthermore, the theory on quantization-induced sheaves can be formulated by means of topos-theoretic ingredients smaller than those of the presheaf-based theory. For example, as we will see, the space of truth-values of the quantization-induced topos is smaller than that of the matricial topos of presheaves. This is because, for each sheaf-based truth-value, there are a lot of different presheaf-based ones that can be regarded as its ‘translations’, and conversely, a lot of different presheaf-based truth-values are translated to one and the same sheaf-based one. The same holds for the space of propositions and that of truth objects, because each sheaf-based proposition and each truth object have a lot of different presheaf-based translations. We call these properties coarse-graining made by translation.

Another question would arise. To what degree do the spaces of presheaf-based truth-values, propositions, and truth objects get coarse-grained via translation? In this paper, we answer this question to some extent. We give translation rules between the sheaf-based ingredients and the presheaf-based ones, and for an arbitrarily given sheaf-based one, we explicitly construct corresponding subspaces consisting of its presheaf-based translations that are regarded as the same from the sheaf-based viewpoint.

The present paper is organized as follows. In section 2, we briefly review Nakayama’s result \([20]\) about quantization-induced topologies and sheaves. Further, additional explanation about some related notions that we will need in later sections are given. In section 3, we develop the sheaf-based method of truth-value assignment. This is done along the line of the presheaf-based method given by Döring and Isham \([15]\), which we briefly summarize in appendix A for referential convenience. (We should, however, note that the main purpose of Döring and Isham \([15]\) is not to give the method itself but
to propose a new interpretation for quantum probabilities, which is beyond
the scope of the present paper.) In section 4, we give rules of translation of
the ingredients necessary for truth-value assignment between the sheaf-based
and the presheaf-based cases. In section 5, we deal with the coarse-graining
problem mentioned above. Main results obtained therein are presented by
theorems 5.1, 5.5, and 5.7.

2 Topos of Sheaves Induced by Quantization

In this section, we give a brief review of the results by Nakayama [20] and
some supplementary explanations. Nakayama [20] defines quantization as an
injective map $\upsilon$ from a Lie algebra $O$, a model of classical observables [22], to
self-adjoint operators on a Hilbert space $H$. The quantization map naturally
defines a functor $\phi$ from the category $C(O)$ of sets of commutative classical
observables to the category $V$ of commutative von Neumann algebras of
bounded operators on $H$. The functor $\phi$ assigns to each $C \in C(O)$ the least
commutative von Neumann algebra that includes $e^{i\upsilon(a)}$. We define a functor
$\psi : V \rightarrow C(O)$ by

$$\psi(V) := \{ a \in O \mid e^{i\upsilon(a)} \in V \}. \quad (2.1)$$

The functors $\phi$ and $\psi$ give a Galois connection between $C(O)$ and $V$.

The endofunctor $\flat := \phi \psi : V \rightarrow V$ induces a Grothendieck topology $J$
on $V$, which is defined by for each $V \in V$,

$$J(V) := \{ \omega \in \Omega(V) \mid \flat V \in \omega \}, \quad (2.2)$$

where $\Omega$ is the subobject classifier of the topos $\hat{V} \equiv \text{Set}^{\text{op}}$ of presheaves on $V$.

As is well-known, every Grothendieck topology on $V$ is equivalent to a
Lawvere-Tierney topology on $\hat{V}$. (As for general theory of topoi, see e.g.,
MacLane and Moerdijk’s textbook [21].) The Grothendieck topology (2.2)
gives the Lawvere-Tierney topology $\Omega \overset{j}{\rightarrow} \Omega$ defined by, for each $V \in V$ and $\omega \in \Omega(V)$,

$$j_\upsilon(\omega) := \{ V' \subseteq V \mid \flat V' \in \omega \}, \quad (2.3)$$

where $V' \subseteq V$ means that $V', V \in V$ and $V' \subseteq V$.

Each Lawvere-Tierney topology is equivalent to a closure operator. In
the present case given by (2.3), for each presheaf $Q \in \hat{V}$ and its subobject
$S \in \text{Sub}(Q)$, the closure $\bar{S}$ of $S$ in $Q$ is defined by

$$\bar{S}(V) := \{ q \in Q(V) \mid Q(b(V) \hookrightarrow V)(q) \in S(b(V)) \}. \quad (2.4)$$

Any Lawvere-Tierney topology $j$ on $\hat{V}$ defines sheaves as follows: Let $S \in \text{Sub}(Q)$ be dense in $Q$, that is, $\bar{S} = Q$. Then, a presheaf $R$ is called a sheaf associated with a topology $j$, or simply, $j$-sheaf, if and only if, for any morphism $\lambda \in \text{Hom}(S, R)$, there exists one and only one morphism $\mu \in \text{Hom}(Q, R)$ that makes the diagram

$$\text{dense}$$

commute. All $j$-sheaves and all morphisms between them form a topos, which is denoted by $\text{Sh}_j \hat{V}$.

Sheaves associated with the topology (2.3) are expressed by the functor $b^* : \hat{V} \to \hat{V}$ that is defined by, for each $Q \in \hat{V}$,

$$(b^*Q)(V) := Q(b(V)), \quad (2.6)$$

and for any $V' \subseteq V$,

$$(b^*Q)(V' \hookrightarrow V) := Q(b(V') \hookrightarrow b(V)). \quad (2.7)$$

We can show that a presheaf $Q$ is a $j$-sheaf if and only if $Q$ is isomorphic to $b^*Q$. To make the condition more precise, we define a morphism $Q \xrightarrow{\zeta_Q} b^*Q$ by $(\zeta_Q)_V := Q(b(V) \hookrightarrow V)$. Then, $Q$ is a $j$-sheaf if and only if $\zeta_Q$ is isomorphic. We should note that $\zeta_Q$ is natural with respect to $Q \in \hat{V}$. That is, $\zeta$ is a natural transformation from the identity functor $I : \hat{V} \to \hat{V}$ to the functor $b^* : \hat{V} \to \hat{V}$. Furthermore, we should note that $b^*$ is in fact an associated sheaf functor (a sheafification functor) from $\hat{V}$ to $\text{Sh}_j \hat{V}$.

Returning to the diagram (2.5), we note that the morphism $\mu$ is given by

$$\mu = \zeta_R^{-1} \circ b^*\lambda \circ \zeta_Q, \quad (2.8)$$
since the naturality of $\zeta$ makes the diagram

\[
\begin{array}{c}
S \\
\downarrow \zeta_S \\
b^*S
\end{array}
\begin{array}{c}
\rightarrow
\end{array}
\begin{array}{c}
Q \\
\downarrow \zeta_Q \\
b^*Q
\end{array}
\begin{array}{c}
\rightarrow
\end{array}
\begin{array}{c}
R \\
\downarrow \zeta_R \\
b^*R
\end{array}
\tag{2.9}
\end{array}
\]

commute. Here, this diagram reflects the fact that $b^*S = b^*\bar{S} = b^*Q$.

In our formalism, truth-values of physical propositions are taken on the
subobject classifier $\Omega_j$ of $Sh_j\hat{V}$. That is, they are given as global elements
$1 \rightarrow \Omega_j \in \Gamma\Omega_j$ of $\Omega_j$. As is well-known, $\Omega_j$ is the equalizer of
$\Omega \rightarrow \Omega$ and $\Omega \rightarrow \Omega$. In the present case, $\Omega_j$ is a subobject of $\Omega$ given by
\[
\Omega_j(V) := \{ \omega \in \Omega(V) \mid \forall V' \subseteq V \ (b(V') \in \omega \Rightarrow V' \in \omega) \}.
\tag{2.10}
\]

Since for each $V \in \mathbf{V}$, $\Omega_j(V)$ contains the set $t_V$ of all subalgebras of $V$ as
the top element, the truth arrow $\text{true}_j \in \Gamma\Omega_j$ is given by
\[
(\text{true}_j)_V := t_V.
\tag{2.11}
\]

Later, we will deal with power objects in $Sh_j\hat{V}$. As is well-known, the
power object $\mathbb{P}^j R \equiv \Omega_j^R$ of a $j$-sheaf $R$ can be calculated in $\hat{V}$. That is, for
each $V \in \mathbf{V}$,
\[
(\mathbb{P}^j R)(V) = \text{Hom}(R_{1V}, (\Omega_j)_{1V})
\simeq \text{Hom}(R_{1V}, \Omega_j),
\tag{2.12}
\]
where $R_{1V}$ and $(\Omega_j)_{1V}$ are downward restrictions as presheaves, the definition
of which is given by (2.4) and (2.5). (Since $Sh_j\hat{V}$ is a full subcategory of
$\hat{V}$, $\text{Hom}_{Sh_j\hat{V}}(A, B) = \text{Hom}_{\hat{V}}(A, B)$ for arbitrary sheaves $A$ and $B$. So we
simply write $\text{Hom}(A, B)$ for both of them omitting the subscripts $Sh_j\hat{V}$ and
Also, for $V' \subseteq V$ and $\lambda \in (\mathbb{P}_{j}R)(V)$, $\lambda|_{V'} \equiv (\mathbb{P}_{j}R)(V' \hookrightarrow V)(\lambda)$ is defined as the morphism that makes the diagram

\[
\begin{array}{ccc}
R_{j|_{V'}} & \xrightarrow{\lambda|_{V'}} & \Omega_{j} \\
\downarrow & & \downarrow \\
R_{j|_{V}} & & \\
\end{array}
\]

(2.13)

commute.

In order to give another, more useful expression of the power object $\mathbb{P}_{j}R$, we note that it is a sheaf representing the collection $\text{Sub}_{j}(R)$ of all subsheaves of $R$. Let $Q$ be a presheaf. As we will see below, $\mathbb{P}_{j}(b^*Q)$ can be expressed as

\[
\mathbb{P}_{j}(b^*Q)(V) \simeq \text{Sub}_{j}(b^*(Q_{j|V}))
\]

(2.14)

and

\[
\mathbb{P}_{j}(b^*Q)(V' \hookrightarrow V) : \text{Sub}_{j}(b^*(Q_{j|V})) \rightarrow \text{Sub}_{j}(b^*(Q_{j|V'})); S \mapsto b^*(S_{j|V'}).
\]

(2.15)

In particular, since any $j$-sheaf $R$ satisfies $R \simeq b^*R$, we have

\[
\mathbb{P}_{j}R \simeq \mathbb{P}_{j}(b^*R) \simeq \text{Sub}_{j}(b^*(R_{j|V})).
\]

(2.16)

Expression (2.14) comes from the fact that

\[
\mathbb{P}_{j}(b^*Q)(V) \simeq \text{Hom}(b^*(Q_{j|V}), \Omega_{j}) \\
\simeq \text{Hom}(b^*(Q_{j|V}), \Omega_{j}) \\
\simeq \text{Sub}_{j}(b^*(Q_{j|V})).
\]

(2.17)

Here, the bijectivity between the first and second lines on (2.17) is verified from the commutative diagram

\[
\begin{array}{ccc}
(b^*Q)_{j|V'\hookrightarrow V} & \xrightarrow{\kappa|_{V'}} & (b^*Q)_{j|V} \\
\downarrow & & \downarrow \\
\text{dense} & & \text{dense} \\
\delta^*(Q_{j|V'}) & \xrightarrow{\chi} & \delta^*(Q_{j|V'})
\end{array}
\]

(2.18)
That is, since $\Omega_j$ is a $j$-sheaf, and since, as easily shown, $(\flat Q)_{\downarrow V}$ is dense in $b^* (Q_{\downarrow V})$, $\chi$ is uniquely determined by use of (2.8) for each morphism $\kappa$.

To see consistency between (2.13) and (2.15), let $S^x$ be a subsheaf of $b^* (Q_{\downarrow V})$ corresponding to the characteristic morphism $\chi$. Then, in the diagram

$$
\begin{array}{c}
b^* (S^x_j) \ar[r] & S^x \ar[dr]^{!} & \\
\downarrow^! & & \downarrow^! & \\
\Omega_j \ar[r]_{\chi|_{V'}} & b^* (Q_{\downarrow V'})
\end{array}
$$

(2.19)

the trapezoid at the right hand side is a pullback, and so is the outer square as easily shown. Thus, also the trapezoid at the left hand side is a pullback, which means that $b^* (S^x_j)$ is identified as the subsheaf of $b^* (Q_{\downarrow V'})$ that corresponds to the characteristic morphism $\chi|_{V'} \equiv P_j (b^* Q)(V' \leftrightarrow V)(\chi)$.

3 Truth-Value Assignment on Quantization-Induced Sheaves

In the theory of Döring and Isham [6, 9, 14, 15], the spectral presheaf $\Sigma$, the definition of which is given in appendix $A$, plays a role of state space of a given quantum system. Every physical proposition $P$ is assumed to be representable as a clopen subobject of $\Sigma$, that is, an element of the collection $\text{Sub}_{\text{cl}}(\Sigma)$ of all clopen subobjects of $\Sigma$. For instance, Döring and Isham showed that each projection operator $\hat{P}$, which corresponds to some physical propositions in ordinary quantum theory, naturally defines a clopen subobject $\delta (\hat{P})$ of $\Sigma$ via the ‘daseinization operator’ $\delta$. If we are given a quantum state, we can specify propositions regarded as true. They are represented by a truth object $T$, of which global elements give the truth propositions. If we have $T$, we can assign to every proposition $P$ a truth value via topos-theoretical setting. In appendix $A$ we give a brief explanation of the method of truth-value assignment developed by Döring and Isham [15], the style of
which is helpful for us to construct a sheaf-based theory. (It should be emphasized, however, that the main purpose of [15] is not to give the valuation method summarize in appendix A but to propose a new interpretation of quantum probabilities based on intuitionistic logic, which is beyond the scope of the purpose of the present paper.)

In our formalism, we appropriate the ‘spectral sheaf’ \( b^* \Sigma \) for the role of state space. Namely, every proposition is assumed to be representable as a clopen subsheaf of \( b^* \Sigma \). We, thus, regard \( \operatorname{Sub}_{j cl}(b^* \Sigma) \), the collection of all clopen subsheaves of \( b^* \Sigma \), as a proposition space. It can be internalized to \( \widetilde{\operatorname{Sh}}_j \) as a subsheaf \( P_{j cl}(b^* \Sigma) \) of \( P_j(b^* \Sigma) \) that is defined by

\[
(P_{j cl}(b^* \Sigma))(V) := \operatorname{Sub}_{j cl}(b^*(\Sigma_{\downarrow V})).
\]  

This definition really gives a presheaf because \( (P_{j cl}(b^* \Sigma))(V') \hookrightarrow V) \), i.e., the restriction of \( (P_{j cl}(b^* \Sigma))(V') \hookrightarrow V) \) to \( \operatorname{Sub}_{j cl}(b^*(\Sigma_{\downarrow V})) \), takes values on \( \operatorname{Sub}_{j cl}(b^*(\Sigma_{\downarrow V})). \) Furthermore,

**Proposition 3.1** The presheaf \( P_{j cl}(b^* \Sigma) \) is a \( j \)-sheaf.

Proof. We have

\[
b^*(P_{j cl}(b^* \Sigma))(V) = (P_{j cl}(b^* \Sigma))(b(V)) = \operatorname{Sub}_{j cl}(b^*(\Sigma_{\downarrow b(V)})) = \operatorname{Sub}_{j cl}(b^*(\Sigma_{\downarrow V})) = (P_{j cl}(b^* \Sigma))(V),
\]  

where from the second line to the third, we used (B.3). Furthermore, for each \( S \in \operatorname{Sub}_{j cl}(b^*(\Sigma_{\downarrow V})) \),

\[
(\zeta_{P_{j cl}(b^* \Sigma)})_V(S) = b^*(S_{\downarrow b(V)}) = b^*(S_{\downarrow V}) = S.
\]  

Therefore, \( \zeta_{P_{j cl}(b^* \Sigma)} \) is a natural isomorphism. \( \Box \)

As is well-known, \( \operatorname{Sub}_j(b^* \Sigma) \approx \Gamma(P_j(b^* \Sigma)) := \operatorname{Hom}(1, P_j(b^* \Sigma)) \). That is, every \( S \in \operatorname{Sub}_j(b^* \Sigma) \) has its name \( [S]_j \in \Gamma(P_j(b^* \Sigma)) \) defined by

\[
([S]_j)_V := b^*(S_{\downarrow V}),
\]  

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and every $s \in \Gamma(\mathbb{P}_j(b^*\Sigma))$ has its inverse, i.e., the subsheaf $[s]^{-1}_j$ of $b^*\Sigma$ given by
\[ [s]^{-1}_j(V) := (s_V)(V). \tag{3.5} \]
It is obvious that, for any $S \in \text{Sub}_j(b^*(\Sigma|_V))$, $[S]_j \in \Gamma(\mathbb{P}_j\text{cl}(b^*\Sigma))$ if and only if $S$ is a proposition, i.e., $S \in \text{Sub}_j\text{cl}(b^*(\Sigma|_V))$. Furthermore, for each proposition $P \in \text{Sub}_j\text{cl}(b^*\Sigma)$, the diagram
\[ \begin{array}{ccc}
1 & \rightarrow & \mathbb{P}_j\text{cl}(b^*\Sigma) \\
[P]_j & \downarrow & \downarrow \\
\mathbb{P}_j(b^*\Sigma) & \rightarrow & \mathbb{P}_j(b^*\Sigma)
\end{array} \] \tag{3.6}
commutes. Therefore, $\mathbb{P}_j\text{cl}(b^*\Sigma)$ is a canonical internalization of $\text{Sub}_j\text{cl}(b^*\Sigma)$.

We can express propositions in different ways. To do so, we need to invoke the outer presheaf $O$ of Döring and Isham [7,14,15] and a few related notions. (As for the definition of $O$, see (A.6) and (A.7).) We call the sheafification $♭^*O$ of $O$ the outer sheaf. Furthermore, we call a set
\[ \hat{h}_V = \bigvee_{V \in \mathbf{V}} \{ \hat{h}_V \in (♭^*O)(V) \} \]
for $V \in \mathbf{V}$ a hyper-element of $♭^*O$, if
\[ \hat{h}_V = \hat{h}_{b(V)} \quad \text{and} \quad♭^*O(V' \hookrightarrow V)(\hat{h}_V) \preceq \hat{h}_V'. \tag{3.7} \]
This is a $j$-sheaf counterpart of the notion of hyper-elements (A.12) defined by Döring and Isham [15]. We write $\text{Hyp}_j(b^*O)$ for the collection of all hyper-elements of $b^*O$. Let $\text{Sub}_{j\text{dB}}(b^*O)$ be the collection of all downward closed, Boolean subsheaves of $b^*O$. That is, for all $P \in \text{Sub}_j(b^*O)$, $P \in \text{Sub}_{j\text{dB}}(b^*O)$ if and only if, for any $V \in \mathbf{V}$, $P(V)$ is a downward closed set of $(♭^*O)(V)$ containing a top element. (Obviously, such $P(V)$’s are complete Boolean lattices.) We can regard $\text{Hyp}_j(b^*O)$ and $\text{Sub}_{j\text{dB}}(b^*O)$ as proposition spaces equivalent to $\text{Sub}_j\text{cl}(b^*\Sigma)$. This is because, corresponding to relation (A.10) proved by Döring and Isham [15], the following relation holds:
\[ \text{Sub}_{j\text{dB}}(b^*O) \simeq \text{Hyp}_j(b^*O) \simeq \text{Sub}_j\text{cl}(b^*\Sigma). \tag{3.8} \]
Here, the bijection at the left hand side of (3.8) is realized by a function $c_j : \text{Sub}_{j\text{dB}}(b^*O) \rightarrow \text{Hyp}_j(b^*O)$ that is defined by
\[ (c_j(A))_V := \bigvee A(V). \tag{3.9} \]
To see the right hand side of (3.8), we use the bijections $\alpha_V : O(V) \to \mathcal{C}(\Sigma(V)) (V \in V)$ introduced by Döring and Isham [15]. (For the definition, see (A.15).) These bijections allow us to regard $\{\mathcal{C}(\Sigma(V))\}_{V \in V}$ as a presheaf $\mathcal{C}\Sigma$ isomorphic to the outer presheaf $O$, and $\{\alpha_V\}_{V \in V}$ as a natural isomorphism $\alpha : O \xrightarrow{\sim} \mathcal{C}\Sigma$. Furthermore, $\alpha$ induces a natural isomorphism $\flat^*\alpha : \flat^*O \xrightarrow{\sim} \flat^*(\mathcal{C}\Sigma)$, where $\flat^*(\mathcal{C}\Sigma)(V) = \mathcal{C}\Sigma(\flat(V)) = \mathcal{C}(\Sigma(\flat(V))) = \mathcal{C}(\flat^*(\Sigma(V)))$. Therefore, we obtain a bijection $k_j : \text{Hyp}_j(\flat^*O) \xrightarrow{\sim} \text{Sub}_{j\flat}(\flat^*\Sigma)$ that is given by

$$
(k_j(h))(V) := (\flat^*\alpha)_V(\hat{h}_V)
= \{\sigma \in (\flat^*\Sigma)(V) | \sigma(\hat{h}_V) = 1\}
= \{\sigma \in \Sigma(\flat(V)) | \sigma(\hat{h}_{\flat(V)}) = 1\}, \tag{3.10}
$$

and hence, a bijection $f_j : \text{Sub}_{j\flat}(\flat^*O) \xrightarrow{\sim} \text{Sub}_{j\flat}(\flat^*\Sigma)$ defined by

$$
(f_j(A))(V) := (\flat^*\alpha)_V(\vee A(V)). \tag{3.11}
$$

It is obvious from (3.8) that $k_{j\downarrow V}, c_{j\downarrow V},$ and $f_{j\downarrow V}$, the restrictions of $k_j, c_j,$ and $f_j$, respectively, to subalgebras of $V$, give the relation

$$
\text{Sub}_{j\flat}(\flat^*(O_{\downarrow V})) \simeq \text{Hyp}_j(\flat^*(O_{\downarrow V})) \simeq \text{Sub}_{j\flat}(\flat^*(\Sigma_{\downarrow V})). \tag{3.12}
$$

Therefore, the proposition space $\text{Sub}_{j\flat}(\flat^*\Sigma) \simeq \text{Sub}_{j\flat}(\flat^*O)$ can be internalized also as a subsheaf $\mathbb{P}_{j\flat}(\flat^*O)$ of $\mathbb{P}_j(\flat^*O)$ that is defined by

$$
(\mathbb{P}_{j\flat}(\flat^*O))(V) := \text{Sub}_{j\flat}(\flat^*(O_{\downarrow V})). \tag{3.13}
$$

Every proposition $P \in \text{Sub}_{j\flat}(\flat^*O)$ has its name $[P]_j \in \Gamma(\mathbb{P}_{j\flat}(\flat^*O))$ in $\text{Sh}_{j\flat\hat{V}}$, which is given by $([P]_j)_V := \flat^*(P_{\downarrow V})$.

The daseinization operator $\delta$ introduced by Döring and Isham [6, 7, 14, 15] assigns to each projection operator $\hat{P}$ on $\mathcal{H}$ a global element $\delta(\hat{P})$ of the outer presheaf $O$. (For the definition, see (A.8).) As a counterpart of $\delta$, we introduce a map $\delta_j$, which assigns to each $\hat{P}$ a global element of $\flat^*O$ by

$$
\delta_j(\hat{P})_V := \bigwedge\{\hat{a} \in (\flat^*O)(V) | \hat{P} \preceq \hat{a}\}
= \bigwedge\{\hat{a} \in O(\flat(V)) | \hat{P} \preceq \hat{a}\}
= \delta(\hat{P}_{\flat(V)}). \tag{3.14}
$$
To see that really $\delta_j(\hat{P}) \in \Gamma(b^*O)$, we note that, for $V' \subseteq V$,

$$(\delta_j(\hat{P}))_{V'} = \delta(\hat{P})_{b(V')}$$

$$= \delta(\delta(\hat{P}))_{b(V')}$$

$$= O(b(V') \hookrightarrow b(V))(\delta(\hat{P})_{b(V)})$$

$$= (b^*O)(V' \hookrightarrow V)(\delta_j(\hat{P})_V). \quad (3.15)$$

Because of (3.8) and the fact that $\Gamma(b^*O) \subseteq \text{Hyp}_j(b^*O)$, $\delta_j(\hat{P})$ can be regarded as a proposition sheaf. That is, it defines elements of $\text{Sub}_{j,\text{cl}}(b^*O)$ and $\text{Sub}_{j,\text{cl}}(b^*\Sigma)$ by

$$(\delta_j(\hat{P}))(V) = \{ \hat{a} \in (b^*O)(V) \mid \hat{a} \preceq (\delta_j(\hat{P}))(V) \} \quad (3.16)$$

and

$$(\delta_j(\hat{P}))(V) = \{ \sigma \in (b^*\Sigma)(V) \mid \sigma((\delta_j(\hat{P}))(V)) = 1 \}, \quad (3.17)$$

respectively.

As previously noted, every proposition is represented by a clopen subsheaf of $b^*\Sigma$. We can assign to it a truth-value, a global element of $\Omega_j$, if we are given a collection of truth propositions. It is internalized as a truth sheaf $T_j$, which is a subsheaf of $\mathbb{P}_{j,\text{cl}}(b^*\Sigma)$ that satisfies appropriate properties. We regard a subsheaf $T_j$ of $\mathbb{P}_{j,\text{cl}}(b^*\Sigma)$ as a truth sheaf if and only if $T_j(V)$ is a filter for every $V \in V$. That is, if $T_j(V)$ contains $A \in \text{Sub}_{j,\text{cl}}(b^*(\Sigma_{jV}))$ as an element, then it does also any $B \in \text{Sub}_{j,\text{cl}}(b^*(\Sigma_{jV}))$ such that $A \subseteq B$. Also, if $A, B \in T_j(V)$, then $A \cap B \in T_j(V)$.

Let $\tau_j$ be the characteristic morphism of $T_j$ as a subsheaf of $\mathbb{P}_{j,\text{cl}}(b^*\Sigma)$. That is, the morphism $\mathbb{P}_{j,\text{cl}}(b^*\Sigma) \xrightarrow{\tau_j} \Omega_j$ makes the diagram

$$\begin{array}{ccc}
\mathbb{T}_j & \xrightarrow{1} & 1 \\
\downarrow & & \downarrow \text{true}_j \\
\mathbb{P}_{j,\text{cl}}(b^*\Sigma) & \xrightarrow{\tau_j} & \Omega_j
\end{array} \quad (3.18)
$$

a pullback. The morphism $\tau_j$ is given by, for each $S \in \text{Sub}_{j,\text{cl}}(b^*(\Sigma_{jV}))$,

$$(\tau_j)_V(S) = \{ V' \subseteq V \mid b^*(S_{jV'}) \in T_j(V') \}. \quad (3.19)$$
Given a truth sheaf $\mathbb{T}_j$, we can assign to each proposition $P$ a truth value $\nu(P; \mathbb{T}_j) \in \Omega_j$ as

$$\nu(P; \mathbb{T}_j) = \tau_j \circ [P]_j,$$  \tag{3.20} \quad \text{(3.20)}$$
each V$-element of which is given by

$$\nu(P; \mathbb{T}_j)_V = \{V' \subseteq V \mid b^*(P_{\mathbb{T}_j}) \in \mathbb{T}_j(V')\}. \tag{3.21} \quad \text{(3.21)}$$

Let $\rho$ be a density matrix and $r \in [0, 1]$. Döring and Isham [15] defined generalized truth objects $\mathbb{T}^{\rho \cdot r}$, the definition of which is given by (A.21). Their global elements represent propositions that are only true with probability at least $r$ in the state $\rho$. Following Döring and Isham, we define $\mathbb{T}^{\rho \cdot r}_j \in \text{Sub}_j \text{(b}^* O\text{)}$ by, for each $V \in \mathcal{V}$,

$$\mathbb{T}^{\rho \cdot r}_j(V) := \{A \in \text{Sub}_j \text{(b}^*(O_{\mathbb{T}_j})) \mid \forall V'' \subseteq V \quad (\text{tr}(\rho(\bigvee A(V'))) \geq r)\}. \quad \text{(3.22)}$$

It is easy to see that every $\mathbb{T}^{\rho \cdot r}_j(V)$ is a filter, and as we will see in proposition 3.2 it is really a $j$-sheaf.

When $r = 1$, $\mathbb{T}^{\rho \cdot 1}_j$ gives propositions that are true in the state $\rho$. Further, when $\rho = |\varphi \rangle \langle \varphi |$, $\mathbb{T}^{\rho \cdot |\varphi \rangle \langle \varphi |}_j \equiv \mathbb{T}^{\rho \cdot |\varphi \rangle \langle \varphi |}_j$, the counterpart of (A.21), is given by

$$\mathbb{T}^{\rho \cdot |\varphi \rangle \langle \varphi |}_j(V) := \{A \in \text{Sub}_j \text{(b}^*(O_{\mathbb{T}_j})) \mid \forall V'' \subseteq V \quad (\delta_j(|\varphi \rangle \langle \varphi |)_{V''} \in A(V'))\}$$

$$\simeq \{h \in \text{Hyp}_j \text{(b}^*(O_{\mathbb{T}_j})) \mid \forall V'' \subseteq V \quad (\delta_j(|\varphi \rangle \langle \varphi |)_{V''} \leq \hat{h}_{V''})\}$$

$$\simeq \{h \in \text{Hyp}_j \text{(b}^*(O_{\mathbb{T}_j})) \mid \forall V'' \subseteq V \quad (|\varphi \rangle \langle \varphi | \leq \hat{h}_{V''})\}$$

$$\simeq \{S \in \text{Sub}_j \text{(b}^*(\Sigma_{\mathbb{T}_j})) \mid \forall V'' \subseteq V \quad (|\varphi \rangle \langle \varphi | \leq \text{b}^*(\hat{\alpha})_{\mathbb{T}_j}^{-1}(S(V'))))\}. \quad \text{(3.23)}$$

**Proposition 3.2** For every state $\rho$ and every coefficient $r \in [0, 1]$, $\mathbb{T}^{\rho \cdot r}_j$ is a $j$-sheaf.

Proof. First, we show that $\mathbb{T}^{\rho \cdot r}_j$ is a presheaf. To do so, for each $V \in \mathcal{V}$, let $V'' \subseteq V$ and $V''' \subseteq V''$. Then, since we have $\text{b}(V''') \subseteq V''$, it follows that for any $A \in \text{Sub}_j \text{(b}^*(O_{\mathbb{T}_j}))$,

$$\text{b}^*(A_{\mathbb{T}_j})'(V''') = A_{\mathbb{T}_j}(\text{b}(V''')) = A(\text{b}(V''')) \quad \text{(3.24)}$$

hence,

$$\text{tr}(\rho(\bigvee \text{b}^*(A_{\mathbb{T}_j})(V''')))) = \text{tr}(\rho(\bigvee A(\text{b}(V'''))))). \quad \text{(3.25)}$$

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This means that $A \in \mathbb{T}_j^{\rho,r}(V)$ implies $b^*(A_{1|V'}) \in \mathbb{T}_j^{\rho,r}(V')$, that is, $\mathbb{T}_j^{\rho,r}$ is a presheaf.

Next, let $A \in (b^*\mathbb{T}_j^{\rho,r})(V) = \mathbb{T}_j^{\rho,r}(b(V))$; that is, suppose that for every $V' \subseteq b(V)$, $\text{tr}(\rho(\vee A(V'))) \geq r$. Then, for every $V' \subseteq V$, since $b(V') \subseteq V$, we have
\[ \text{tr}(\rho(\vee A(V'))) = \text{tr}(\rho(\vee A(b(V')))) \geq r, \]
which means $A \in \mathbb{T}_j^{\rho,r}(V)$. Thus, $(b^*\mathbb{T}_j^{\rho,r})(V) \subseteq \mathbb{T}_j^{\rho,r}(V)$ results.

Finally, $\mathbb{T}_j^{\rho,r} \xrightarrow{\zeta_j^{\rho,r}} b^*\mathbb{T}_j^{\rho,r}$ turns out to be a natural isomorphism, because for every $A \in \mathbb{T}_j^{\rho,r}(V)$, $(\zeta_j^{\rho,r})_V(A) = b^*(A_{1|V}) = b^*(A_{1|V}) = A$. \[ \square \]

Let $\mathbb{P}_j,_{db}(b^*O) \xrightarrow{\tau_j^{\rho,r}} \Omega_j$ be the characteristic morphism of $\mathbb{T}_j^{\rho,r}$. From (3.19), we have, for each $A \in \text{Sub}_j,_{db}(b^*(O_{1|V})) = (\mathbb{P}_j,_{db}(b^*O))(V),$

\[ (\tau_j^{\rho,r})_V(A) = \{ V' \subseteq V | \forall V'' \subseteq V' \text{ (tr}(\rho(\vee b^*(A_{1|V'}))(V'')) \geq r) \} \]
\[ = \{ V' \subseteq V | \forall V'' \subseteq V' \text{ (tr}(\rho(\vee A(V'')) \geq r) \}. \]

Therefore, the truth-value of a physical proposition $\delta_j(\hat{P})$ corresponding to a projection operator $\hat{P}$ under the truth sheaf $\mathbb{T}_j^{\rho,r}$ is given by, for each $V \in \mathbb{V},$

\[ \nu_j(\delta_j(\hat{P}); \mathbb{T}_j^{\rho,r}) = \{ V' \subseteq V | \forall V'' \subseteq V' \text{ (tr}(\rho(\delta_j(\hat{P})(V'')) \geq r) \} \]
\[ = \{ V' \subseteq V | \text{tr}(\rho(\delta_j(\hat{P})(V'')) \geq r \}. \] (3.28)

In particular, for $\mathbb{T}_j^{\rho,r} = \mathbb{T}_j^{(\rho)}$, we have

\[ \nu_j(\delta_j(\hat{P}); \mathbb{T}_j^{(\rho)}) = \{ V' \subseteq V | \langle \varphi | (\delta_j(\hat{P})(V'))| \varphi \rangle = 1 \} \]
\[ = \{ V' \subseteq V | | \varphi \rangle \langle \varphi | \leq \delta_j(\hat{P})(V') \} \]
\[ = \{ V' \subseteq V | \delta_j(| \varphi \rangle \langle \varphi |) \in \delta_j(\hat{P})(V') \}. \] (3.29)

### 4 Translation Rules of Propositions, Truth Objects, and Truth-Values

In Section 3 we gave the truth-value function $\nu_j$ that assigns a truth-value to each proposition sheaf $P_j$ under a given truth sheaf $\mathbb{T}_j$. In this and the next sections, we clarify the structural relationship between the present sheaf-based theory and the presheaf-based one. What we show in this section
is that, for each $P_j$ and $T_j$, there are corresponding proposition presheaves $P$ and truth presheaves $T$ that can be regarded as ‘translations’, and that there exists a specific relation between global elements of $\Omega_j$ and $\Omega$, which is satisfied by $\nu_j(P_j; T_j)$ and $\nu(P; T)$ for all such propositions $P_j$ and $P$ and truth objects $T_j$ and $T$. Precisely, we show that they satisfy the following relation:

$$\nu_j(P_j; T_j) = r \circ \nu(P; T),$$

(4.1)

where the morphism $r$ is defined by the epi-mono factorization of $j$,

$$\Omega \xrightarrow{j} \Omega \xrightarrow{r} \Omega_j,$$

(4.2)

that is, $r$ is defined by $r_V(\omega) \equiv j_V(\omega) \in \Omega_j(V)$. In the following, we give concrete translation relationships for proposition objects $P$ and $P_j$ and for truth objects $T$ and $T_j$.

First, we give a definition of translation of propositions. Note that each proposition presheaf $P \in \text{Sub}_{dB}(O)$ is sheafified to a proposition sheaf $\flat^*P \in \text{Sub}_{dB}(O)$. Therefore, it is quite natural to regard $P$ and $P_j$ as each other’s translation if they satisfy

$$\flat^*P = P_j.$$

(4.3)

The following proposition, which is clear from the definition of $\delta_j(\hat{P})$, would suggest (4.3) as a sound definition of translation.

**Proposition 4.1** For every projection operator $\hat{P}$, $\delta(\hat{P})$ and $\delta_j(\hat{P})$ are each other’s translations.

Next, we define translation of truth objects. First, we note that, for each truth sheaf $T_j \in \text{Sub}_{dB}(P_{j,db}(\flat^*O))$, the morphism $\flat^*(P_{db}(\flat^*O)) \xrightarrow{\check{o}_O} P_{j,db}(\flat^*O)$ induces a subsheaf of $\flat^*(P_{db}(\flat^*O))$, which we denote by $\check{o}_O^{-1}(T_j)$, as the pullback of $T_j$ along the morphism $\check{o}_O$; that is,

$$\check{o}_O^{-1}(T_j)(V) := \{ A \in (\flat^*(P_{db}(\flat^*O))(V) | \flat^*A \in T_j(V) \}
= \{ A \in \text{Sub}_{dB}(O_{\flat(V)}) | \flat^*A \in T_j(V) \}.$$

(4.4)
On the other hand, each truth presheaf \( T \in \text{Sub}(\mathbb{P}_{dBO}) \), for which we propose that \( T(V) \) is a filter for every \( V \in V \), has its sheafification \( b^*T \in \text{Sub}_j(b^*(\mathbb{P}_{dBO})) \). We say that \( T \) and \( T_j \) are each other’s translation, if they satisfy
\[
b^*T = g_{O}^{-1}(T_j). \tag{4.5}
\]

To show soundness of the definition \( (4.5) \) of translation, we give the following proposition.

**Proposition 4.2** For every density matrix \( \rho \) and \( r \in [0, 1] \), the corresponding truth presheaf \( T^{\rho, r} \) and the truth sheaf \( T_j^{\rho, r} \) are each other’s translation.

**Proof.** Let \( A \in \text{Sub}_{dB}(O_{\hat{h}(V)}) \) and \( h \in \text{Hyp}(O_{\hat{h}(V)}) \) be its corresponding hyper-element. Then \( A \in (b^*T^{\rho, r})(V) \) if and only if
\[
\forall V' \subseteq V \ b(V) \text{ tr}(\hat{h}_V) \geq r, \tag{4.6}
\]
whereas \( A \in (g_{O}^{-1}(T_j^{\rho, r}))(V) \) if and only if
\[
\forall V' \subseteq V \ V \text{ tr}(\hat{h}_V) \geq r. \tag{4.7}
\]

What we have to prove is that \( (4.6) \) and \( (4.7) \) are equivalent.

Suppose that \( (4.6) \) holds. Then, since for \( V' \subseteq V \), we have \( b(V') \subseteq V \), \( (4.7) \) follows.

Conversely, suppose that \( (4.7) \). Then, in particular,
\[
\text{tr}(\hat{h}_V) \geq r. \tag{4.8}
\]

On the other hand, for every \( V' \subseteq V \),
\[
\hat{h}_V \leq \delta(\hat{h}_V) \leq \hat{h}_V', \tag{4.9}
\]
Thus, we have
\[
 r \leq \text{tr}(\hat{h}_V) \leq \text{tr}(\hat{h}_V'), \tag{4.10}
\]
which implies \( (4.6) \).

Now, let \( P \) and \( T \) be arbitrary translations of \( P_j \) and \( T_j \), respectively. In the following, we prove that they really satisfy \( (4.11) \).
First, note that the names $[P]$ and $[P_j]$ make the diagram commute. Here, the definition of $b^*(\mathbb{P}_O^\Sigma) \xrightarrow{\varphi_O} \mathbb{P}_{j\Sigma}(b^*O)$, the restriction of $b^*(\mathbb{P}_O) \xrightarrow{\varphi_O} \mathbb{P}_j(b^*O)$, is given in appendix B. The commutativity of (4.11) is easily shown as

\[
([P_j]_V) = b^*(b^*P)_V
\]

\[
= b^*(P_j)_V
\]

\[
= b^*((P_\Sigma)_V)_V
\]

\[
= b^*((P_\Sigma)_V^\Sigma)^{\Sigma}_V
\]

\[
= (\varphi_O)_V^{\Sigma}((\zeta_{\Sigma\Sigma})_V^{\Sigma}([P]_V))\),
\]

where we used (B.3) and (B.4).

**Proposition 4.3** Let $\mathbb{P}_O^\Sigma \xrightarrow{\tau} \Omega$ and $\mathbb{P}_{jO}^\Sigma(\Omega) \xrightarrow{\tau_j} \Omega_j$ be the characteristic morphisms of $\mathbb{T}$ and $\mathbb{T}_j$, respectively. Then, the diagram

\[
\begin{array}{ccc}
\mathbb{P}_O^\Sigma & \xrightarrow{b^*} & b^*\Omega \\
\text{\scriptsize \Sigma} & & \text{\scriptsize \Sigma} \\
\mathbb{P}_{jO}^\Sigma(\Omega) & \xrightarrow{\tau_j} & \Omega_j
\end{array}
\]

(4.13)

commutes if and only if equation (4.5) is satisfied.

Proof. First, we note that, for each $A \in b^*(\mathbb{P}_O^\Sigma)(V) = \text{Sub}_{\Omega}(O_{\varphi(V)})$,

\[
(b^*\tau \circ b^*\tau)_V(A) = \{V' \subseteq_V b(V) | b(V') \in \tau_{\Sigma}(A)\}
\]

\[
= \{V' \subseteq_V b(V) | A_{\varphi(V)'} \in \mathbb{T}(b(V'))\}. \quad (4.14)
\]

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and,

$$(\zeta \circ \tau \circ \varrho) V(A) = \{ V' \subseteq V : b(V') \in T_j(V') \}.$$  \hspace{1cm} (4.15)

Suppose that the diagram \((4.13)\) commutes. Then, for each \(V' \subseteq b(V), A_{\varrho V}(V') \in T(\varrho(V'))\) if and only if \(b^*(A_{\varrho V}) \in T_j(V')\). In particular, putting \(V' = b(V)\), we obtain equation \((4.5)\). Conversely, suppose that \((4.5)\) holds. Then, we have, for each \(V \in V\) and \(V' \subseteq b(V), b^*T(V') = \varrho T_j(V')\); that is, for all \(A' \in \text{Sub}(O_{\varrho V}(V)), A' \in T(b(V'))\) if and only if \(b^*A' \in T_j(V')\). In particular, for any \(A \in \text{Sub}_{dB}(O_{\varrho V}(V)), we obtain the condition for the diagram \((4.13)\) to commute, by taking \(A' = A_{\varrho V}(V')\). \(\square\)

To show the relation \((4.1)\), let \(T\) and \(P\) be translations of \(T_j\) and \(P_j\), respectively.

Fitting together \((4.11), (4.13)\), and naturality of \(\zeta\), we have a commutative diagram

\begin{align*}
\begin{array}{c}
P_{dB}O \\
\tau \downarrow \\
\Omega \\
\zeta \downarrow \\
\Omega_j \\
\end{array}
\end{align*}

The outer pentagon of this diagram is just the relation \((4.1)\).

We have proved that for all proposition objects \(P\) and \(P_j\) satisfying \((4.3)\) and truth objects \(T\) and \(T_j\) satisfying \((4.5)\), the truth-values \(\nu(P, T)\) and \(\nu_j(P_j, T_j)\) are related via \((4.11)\). This implies that \(P\) and \(T\) represent virtually the same proposition as \(P_j\) and the same truth object as \(T_j\), respectively, from our sheaf-based viewpoint. In this sense, it is reasonable to call them each other’s translation. Also, we call the same relation between global elements of \(\Omega_j\) and \(\Omega\) as \((4.1)\), that is,

$$\nu_j = r \circ \nu,$$  \hspace{1cm} (4.17)

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the translation rule of global elements, and say that \(\nu_j \in \Gamma\Omega_j\) and \(\nu \in \Gamma\Omega\) are each other’s translation if they satisfy (4.17).

5 Coarse-Graining Properties of Translation

For a proposition \(P_j\) and a truth sheaf and \(T_j\), their translation presheaves \(P\) and \(T\) satisfying (4.3) and (4.5) are not determined uniquely. For such \(P\)'s and \(T\)'s, furthermore, the truth-values \(\nu(P, T)\) take various values. If we consider their sheaf translations, the various truth-values are transformed to one and the same value \(r \circ \nu(P, T)\). In other words, a lot of different propositions, truth objects, and truth-values are not distinguished from the sheaf-based viewpoint. We call this aspect coarse-graining made by translation, the properties of which we observe in the following.

First, let us see coarse-graining of the space \(\Gamma\Omega\) of truth-values. The translation rule (4.17) is equivalent to the condition that for all \(V \in \mathcal{V}\) and \(V' \subseteq \nu V\),

\[
V' \in (\nu_j)_V \iff \nu(V') \in \nu_V. \tag{5.1}
\]

Let \(\gamma(\nu_j)\) be the set of all translations \(\nu \in \Gamma\Omega\) of \(\nu_j\). Note that \(\gamma(\nu_j)\) has an order relation \(\leq\) inherited from \(\Gamma\Omega\). Namely, \(\nu_1 \leq \nu_2\) if and only if \((\nu_1)_V \subseteq (\nu_2)_V\) for all \(V \in \mathcal{V}\). Furthermore, \(\gamma(\nu_j)\) is closed with respect to binary operations on \(\Gamma\Omega\), the join \(\vee\) and the meet \(\wedge\), each of which is defined by \((\nu_1 \vee \nu_2)_V := (\nu_1)_V \cup (\nu_2)_V\) and \((\nu_1 \wedge \nu_2)_V := (\nu_1)_V \cap (\nu_2)_V\), respectively.

Let us define \(\gamma^\vee(\nu_j) \in \Gamma\Omega\) by

\[
\gamma^\vee(\nu_j) := \nu_j \nu_j \Omega_j \twoheadrightarrow \Omega. \tag{5.2}
\]

This is the maximum translation of \(\nu_j\). In fact, it is clear from the definition (2.10) that (5.1) is satisfied if we put \(\nu = \gamma^\vee(\nu_j)\). Furthermore, if \(\nu \in \gamma(\nu_j)\), then, since \(b(V') \in \nu_V\) for every \(V' \in \nu_V\), we have \(V' \in (\nu_j)_V = \gamma^\vee(\nu_j)_V\) from (5.1). Thus, \(\nu \leq \gamma^\vee(\nu_j)\) follows.

Let us define \(\gamma^\wedge(\nu_j)\) by

\[
\gamma^\wedge(\nu_j)(V) := \{V' \subseteq V \mid (\nu_j)_V \cap \mathcal{U}^\wedge(V') \neq \emptyset\}, \tag{5.3}
\]

where, for each \(V \in \mathcal{V}\), \(\mathcal{U}^\wedge(V)\) is defined by

\[
\mathcal{U}^\wedge(V) := \{W \in \mathcal{V} \mid V \subseteq b(W)\}. \tag{5.4}
\]
We can straightforwardly verify that $\gamma^\wedge(\nu) \in \gamma(\nu_j)$. Moreover, $\gamma^\wedge(\nu_j)$ is the least translation of $\nu_j$. To see this, let $\nu \in \gamma(\nu_j)$ and $V' \in \gamma^\wedge(\nu_j)$. Then, we have $V' \subseteq V b(V'')$ for some $V'' \in (\nu_j)$. Furthermore, since for such $V''$, $b(V'') \in \nu_V$ because of (5.1). Thus, we have $V' \subseteq V$, since $V' \subseteq b(V'')$. Conversely, it is easy to show that every $\nu \in \Gamma \Omega$ lying between $\gamma^\wedge(\nu_j)$ and $\gamma^\wedge(\nu_j)'$ satisfies (5.1). On the other hand, every $\nu \in \Gamma \Omega$ is a translation of $r \circ \nu \in \Gamma \Omega_j$. We thus obtain the following result:

**Theorem 5.1** The truth-value space $\Gamma \Omega$ can be expressed as a disjoint union of the lattices $\gamma(\nu_j)$ ($\nu_j \in \Gamma \Omega_j$), each of which is given by

$$\gamma(\nu_j) = \{\nu \in \Gamma \Omega \mid \gamma^\wedge(\nu_j) \leq \nu \leq \gamma^\wedge(\nu_j)\}. \quad (5.5)$$

Next, let us turn to the definition (4.3) of translation of propositions. Let $\nu(P_j)$ be the set of all translation presheaves of $P_j$. It is clear that $\nu(P_j)$ is an ordered set with respect to the inclusion relation defined on $\text{Sub}_O \cong \text{Sub}_O$. That is, $P_1 \subseteq P_2$ if and only if $P_1(V) \subseteq P_2(V)$ for all $V \in \mathbb{V}$. Also, since (4.3) is equivalent to $P(b(V)) = P_j(V)$ for all $V \in \mathbb{V}$, $\nu(P_j)$ is closed for $\vee$ and $\wedge$ defined on $\text{Sub}_O$, where $P_1 \wedge P_2$ and $P_1 \vee P_2$ are defined by $(P_1 \wedge P_2)(V) := P_1(V) \cap P_2(V)$ and $(P_1 \vee P_2)(V) := P_1(V) \cup P_2(V)$, respectively.

Among the translations $P \in \nu(P_j)$, there exists a canonical one $\nu^\wedge(P_j)$. To give the definition, we note the following fact.

**Proposition 5.2** If $h \equiv \{\hat{h}_V \in (b^*O)(V)\}_{V \in \mathbb{V}}$ is a hyper-element of $b^*O$, it is also a hyper-element of $O$.

Proof. Let $h$ be a hyper-element of $b^*O$. Then, since $b(V) \subseteq V$ for every $V \in \mathbb{V}$, we have $\hat{h}_V \in (b^*O)(V) = O(b(V)) \subseteq O(V)$, whereas we have $b(V') \subseteq V'$ for every $V' \subseteq V$. Thus, from (3.7) and (3.15), it follows

$$\delta(\hat{h}_V) \subseteq \delta(\hat{h}_V) \subseteq \delta(\hat{h}_V) \subseteq \delta(\hat{h}_V) \subseteq \delta(\hat{h}_V) \subseteq \hat{h}_V, \quad (5.6)$$

which means that $h$ is a hyper-element of $O$.

Every proposition sheaf $P_j \in \text{Sub}_O(b^*O)$ has its hyper-element $\{\vee P_j(V)\}_{V \in \mathbb{V}}$ of $b^*O$. We define $\nu^\wedge(P_j)$ as the proposition presheaf given by $\{\vee P_j(V)\}_{V \in \mathbb{V}}$ as a hyper-element of $O$:

$$\nu^\wedge(P_j)(V) := \{\hat{a} \in O(V) \mid \hat{a} \leq \vee P_j(V)\}$$

$$= \{\hat{a} \in O(V) \mid \delta(\hat{a}) \in P_j(V)\}$$

$$= ((\zeta_O)V)^{-1}(P_j(V)). \quad (5.7)$$

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Clearly, $\iota'(P_j)$ satisfies $\delta'(\iota'(P_j)) = P$, that is, it is really a translation of $P_j$.

**Proposition 5.3** For every proposition sheaf $P_j \in \text{Sub}_{\text{dB}}(\flat^*O)$, $\iota'(P_j)$ is the largest translation of $P_j$.

Proof. The third line of (5.7) means that $\iota'(P_j)$ is a pullback of $P_j \mapsto \flat^*O$ along the morphism $O \xrightarrow{\zeta} \flat^*O$. That is, it makes the tropezoid in the diagram

\[
\begin{array}{c}
P \downarrow \quad \iota'(P_j) \downarrow \zeta_O \\
\zeta_P \quad \iota'(P_j) \quad \zeta_O \\
\flat^*P \quad \flat^*O \\
\end{array}
\]

a pullback. On the other hand, if $\flat^*P = P_j$, the outer square commutes because of naturality of $\zeta$. We thus obtain an inclusion $P \mapsto \iota'(P_j)$. □

For instance, for every projection operator $\hat{P}$, we have $\hat{\delta}(\hat{P}) \preceq \hat{\delta}(\flat^*P) = \delta_j(\hat{P})$, whereas $\delta_j(\hat{P})$ defines $\iota'(\delta_j(\hat{P}))$ as a hyper element of $O$. Therefore, the proposition presheaf $\delta(\hat{P})$, which is a translation of $\delta_j(\hat{P})$ as previously mentioned, is included by $\iota'(\delta_j(\hat{P}))$.

We define $\iota^*(P_j)$ by, for each $V \in \mathbf{V}$,

\[
\iota^*(P_j)(V) := \begin{cases} \\
\{ \hat{a} \in O(V) \mid \hat{a} \preceq \bigvee \{ \delta(\check{\vee} P_j(W))_V \}_{W \in \mathcal{U}(V)} \} & \text{if } \mathcal{U}(V) \neq \emptyset, \\
\{ \hat{0} \} & \text{if } \mathcal{U}(V) = \emptyset.
\end{cases}
\]

**Proposition 5.4** For every proposition sheaf $P_j$, $\iota^*(P_j)$ is the smallest translation of $P_j$.

Proof. Let $k \in \text{Hyp}_{\text{dB}}(\flat^*O)$ be the hyper-element corresponding to $P_j$. To show that $\iota^*(P_j)$ is a presheaf, we define $h := \{ \hat{h}_V \in O(V) \}_{V \in \mathbf{V}}$ by

\[
\hat{h}_V := \begin{cases} \\
\bigvee \{ \delta(\check{h}_W)_V \}_{W \in \mathcal{U}(V)} & \text{if } \mathcal{U}(V) \neq \emptyset, \\
\hat{0} & \text{if } \mathcal{U}(V) = \emptyset.
\end{cases}
\]
Since for each $V$, $\hat{h}_V$ is the top element of $\iota^\wedge(P_j)(V)$, $\iota^\wedge(P_j)$ is a presheaf if and only if $h$ is a hyper-element of $O$. Let us show this, first.

Suppose that $\mathcal{U}^h(V) \neq \emptyset$. Since $\mathcal{U}^h(V) \subseteq \mathcal{U}^h(V')$ for every $V' \subseteq V$, we have

$$\hat{h}_V = \bigvee \{\delta(\hat{k}_W)_V\}_{W \in \mathcal{U}^h(V)} \leq \bigvee \{\delta(\hat{k}_W)_V\}_{W' \in \mathcal{U}^h(V')}.$$  \hspace{1cm} (5.11)

On the other hand, since $O(V') \subseteq O(V)$, we have, for every $W' \in \mathcal{U}^h(V')$,

$$\delta(\hat{k}_{W'})_V \leq \delta(\hat{k}_{W'})_{V'},$$  \hspace{1cm} (5.12)

hence,

$$\bigvee \{\delta(\hat{k}_{W'})_V\}_{W' \in \mathcal{U}^h(V')} \leq \bigvee \{\delta(\hat{k}_{W'})_{V'}\}_{W' \in \mathcal{U}^h(V')} = \hat{h}_{V'}.$$  \hspace{1cm} (5.13)

From (5.11) and (5.13), we have

$$\delta(h_V)_V \leq \delta(h_{V'})_{V'} = \hat{h}_{V'}.$$  \hspace{1cm} (5.14)

If $\mathcal{U}^h(V) = \emptyset$, then $\delta(h_V)_V = \hat{O} \leq \hat{h}_V$. Thus, $h$ is a hyper-element of $O$, hence really, $\iota^\wedge(P_j)$ a presheaf.

In order to show that $\iota^\wedge(P_j) \in \iota(P_j)$, it suffices to show that $\hat{h}_{b(V)} = \hat{k}_V$ for every $V \in \mathcal{V}$. Since $V \in \mathcal{U}^h(\mathfrak{b}(V))$, we have

$$\hat{h}_{b(V)} = \bigvee \{\delta(\hat{k}_W)_{b(V)}\}_{W \in \mathcal{U}^h(\mathfrak{b}(V))} = (\delta(\hat{k}_V)_{b(V)}) \vee (\bigvee \{\delta(\hat{k}_W)_{b(V)}\}_{W \in \mathcal{U}^h(\mathfrak{b}(V)) \setminus \{V\}}).$$  \hspace{1cm} (5.15)

On the other hand, we have $\delta(\hat{k}_V)_{b(V)} = \delta_j(\hat{k}_b(V))_{b(V)} = \hat{k}_b(V) = \hat{k}_V$, and $\delta(\hat{k}_W)_{b(V)} \leq \hat{k}_b(V) = \hat{k}_V$ for all $W \in \mathcal{U}^h(\mathfrak{b}(V)) \setminus \{V\}$. Thus, $\hat{h}_{b(V)} = \hat{k}_V$ results.

Finally, to show $\iota^\wedge(P_j)$ to be the smallest translation of $P_j$, let $l \in \text{Hyp}(O)$ be the hyper-element corresponding to a translation $P \in \iota(P_j)$. What we have to show is $\hat{h}_V \leq \hat{l}_V$ for all $V \in \mathcal{V}$. It suffices to treat the case where $\mathcal{U}^h(V) \neq \emptyset$. Since we have

$$\delta(\hat{k}_W)_V = \delta(\hat{l}_{b(W)})_V \leq \hat{l}_V$$  \hspace{1cm} (5.16)

for all $W \in \mathcal{U}^h(V)$, it follows that

$$\hat{h}_V = \bigvee \{\delta(\hat{k}_W)_V\}_{W \in \mathcal{U}^h(V)} \leq \hat{l}_V.$$  \hspace{1cm} (5.17)

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Theorem 5.5 The proposition space $\text{Sub}_{\text{db}}(O)$ can be expressed as a disjoint union of the lattices $\mathfrak{v}(P_j)$ ($P_j \in \text{Sub}_{\text{db}}(b^*O)$), each of which is given by

$$\mathfrak{v}(P_j) = \{ P \in \text{Sub}_{\text{cl}}(\Sigma) \mid \mathfrak{v}(P_j) \subseteq P \subseteq \mathfrak{v}^c(P_j) \}. \quad (5.18)$$

Finally, we observe coarse-graining of truth presheaves. Let $\text{Sub}_{\text{filt}}(P_{\text{cl}}\Sigma)$ be the set of all truth presheaves; that is, $\mathbb{T} \in \text{Sub}_{\text{filt}}(P_{\text{cl}}\Sigma)$ means that $\mathbb{T} \in \text{Sub}(P_{\text{cl}}\Sigma)$ and $\mathbb{T}(V)$ is a filter for every $V \in \mathcal{V}$. We first note that we can define $\lor$ and $\land$ on $\text{Sub}_{\text{filt}}(P_{\text{cl}}\Sigma)$. In fact, we define $T_1 \lor T_2 := T_1 \cap T_2$, whereas for $T_1 \land T_2$, we let $(T_1 \land T_2)(V)$ the smallest filter $\mathfrak{F}(T_1(V) \cup T_2(V))$ including $T_1(V) \cup T_2(V)$, that is,

$$(T_1 \lor T_2)(V) := \{ P \cap P' \in \text{Sub}_{\text{db}}(O_{1V}) \mid P, P' \in T_1(V) \cup T_2(V) \}. \quad (5.19)$$

Let $\mathfrak{j}(T_j)$ be the set of all translation presheaves of a truth sheaf $T_j$. Since the translation condition (1.5) is equivalent to $T(b(V)) = (O_{1V})^{-1}(T_j(V))$ for all $V \in \mathcal{V}$, $\mathfrak{j}(T_j)$ is closed for $\lor$ and $\land$ defined above.

Also for every truth sheaf $T_j$, we can define its canonical translation $\mathfrak{j}^c(T_j)$ that is the largest one among the translations satisfying (1.5). It is defined as the pullback of $O_{1V}^{-1}(T_j) \xrightarrow{b^*} (P_{\text{db}}O)$ along the morphism $P_{\text{db}}O \xrightarrow{\xi_{\text{db}}O} b^*(P_{\text{db}}O)$:

$$\mathfrak{j}^c(T_j)(V) = \{ A \in \text{Sub}_{\text{db}}(O_{1V}) \mid (\xi_{\text{db}}O)_V(A) \in O_{1V}^{-1}(T_j(V)) \}$$

$$= \{ A \in \text{Sub}_{\text{db}}(O_{1V}) \mid A_{\mathfrak{i}V(V)} \in O_{1V}^{-1}(T_j(V)) \}$$

$$= \{ A \in \text{Sub}_{\text{db}}(O_{1V}) \mid b^*(A_{\mathfrak{i}V(V)}) \in T_j(b(V)) \}$$

$$= \{ A \in \text{Sub}_{\text{db}}(O_{1V}) \mid b^*(A_{1V}) \in T_j(V) \}. \quad (5.20)$$

Clearly, if $T_j$ is a truth sheaf, every $\mathfrak{j}^c(T_j)(V)$ is a filter, hence, $\mathfrak{j}^c(T_j)$ a truth presheaf.

Next, let us define, for each $V \in \mathcal{V}$ and $W \in \mathcal{U}^b(V)$, $\mathbb{R}_{V;W} \subseteq \text{Sub}_{\text{db}}(O_{1V})$ by

$$\mathbb{R}_{V;W} := \{ A_{1V} \in \text{Sub}_{\text{db}}(O_{1V}) \mid A \in (O_{1V}^{-1}(T_j))(W) \}. \quad (5.21)$$
and $\mathbb{R}_V \subseteq \text{Sub}_{\text{dB}}(O_{iV})$ by

$$\mathbb{R}_V := \bigcup \{ \mathbb{R}_V; W \}_{W \in \mathcal{U}(V)}. \quad (5.22)$$

We define $\mathcal{J}^\wedge(T_j)$ by

$$\mathcal{J}^\wedge(T_j)(V) := \begin{cases} \mathcal{F}^\wedge(V) & \text{if } \mathcal{U}^\wedge(V) \neq \emptyset \\ \emptyset & \text{if } \mathcal{U}^\wedge(V) = \emptyset, \end{cases} \quad (5.23)$$

where $\mathcal{F}^\wedge(V)$ is the smallest filter in $\text{Sub}_{\text{dB}}O_{iV}$ including $\mathbb{R}_V$. 

**Proposition 5.6** For every $T_j \in \text{Sub}_{\text{filt}}(P_{cl}\Sigma)(O_{iV})$, $\mathcal{J}^\wedge(T_j)$ is the smallest translation of $T_j$.

Proof. We prove $\mathcal{J}^\wedge(T_j)$ to be a presheaf. Suppose that $A \in \mathcal{J}^\wedge(T_j)(V)$. This is equivalent to that there exists a finite subset $S_V \subseteq \mathbb{R}_V$ such that $\mathcal{J}^\wedge(T_j)(V) \subseteq A$, since $\mathcal{J}^\wedge(T_j)(V)$ is a filter [23]. Therefore, for every $V' \subseteq V$, we have $\wedge(S_V)_{iV'} \subseteq A_{iV'}$, where $(S_V)_{iV'} \equiv \{ B_{iV'} | B \in S_V \}$ is a finite subset of $\mathbb{R}_V$. Thus, $A_{iV'} \in \mathcal{J}^\wedge(T_j)$.

To show that $\mathcal{J}^\wedge(T_j)$ is a translation of $T_j$, note that $V \in \mathcal{U}^\wedge(T_j)$. We have, for every $W \in \mathcal{U}^\wedge(T_j) \setminus \{ V \},$

$$\mathbb{R}^\vee(W) = \{ A_{iV} | A \in (\mathcal{O})^{-1}(T_j)(W) \} \subseteq (\mathcal{O})^{-1}(T_j)(V), \quad (5.24)$$

whereas,

$$\mathbb{R}^\vee(W) = \{ A_{iV} | A \in (\mathcal{O})^{-1}(T_j)(V) \} = (\mathcal{O})^{-1}(T_j)(V). \quad (5.25)$$

Thus, we obtain

$$\mathbb{R}^\vee(V) = \mathbb{R}^\vee(W) \cup \bigcup \{ \mathbb{R}^\vee(W) | W \in \mathcal{U}^\wedge(T_j) \setminus \{ V \} \} = (\mathcal{O})^{-1}(T_j)(V), \quad (5.26)$$

hence,

$$\mathcal{J}^\wedge(T_j)(\mathcal{O}(V)) = \mathcal{F}^\wedge(V)(\mathbb{R}^\vee(V)) = \mathcal{F}^\wedge(V)((\mathcal{O})^{-1}(T_j)(V)) = (\mathcal{O})^{-1}(T_j)(V), \quad (5.27)$$

where from the second line to the third, we used the fact that $(\mathcal{O})^{-1}(T_j)(V)$ itself is a filter.
Finally, we show that \( j^\wedge(T_j) \) is the smallest translation of \( T_j \). It suffices to show for \( V \in \mathbf{V} \) such that \( \mathcal{U}^p(V) \neq \emptyset \). Let \( T \) be an arbitrary translation of \( T_j \). Suppose that \( A \in j^\wedge(T_j)(V) \). Then, there exists a finite subset \( S_V \) of \( R_V \) such that \( \wedge S_V \subseteq A \). On the other hand, for every \( B \in S_V \), there exists a \( \mathcal{W} \in \mathcal{U}^p(V) \) such that \( B \in R^V; \mathcal{W} \); that is, there exists a \( C \in (\rho_O)^{-1}(T_j)(W) = \mathcal{T}(\mathcal{V}(W)) \) such that \( B = C \). This implies that \( B = \mathcal{T}(V \hookrightarrow \mathcal{V}(W))(C) \in \mathcal{T}(V) \). Thus, \( S_V \subseteq \mathcal{T}(V) \), hence, \( \wedge S_V \in \mathcal{T}(V) \), which implies \( A \in \mathcal{T}(V) \) since \( \mathcal{T}(V) \) is a filter. \( \square \)

**Theorem 5.7** For every truth sheaf \( T_j \), \( j(T_j) \) is a lattice that is given by

\[
j(T_j) = \{ T \in \text{Sub}_{\text{filt}}(\mathcal{P}_d \Sigma) \mid j^\wedge(T_j) \subseteq T \subseteq j^\vee(T_j) \}.
\] (5.28)

Every truth sheaf \( T_j \) determines a lattice of truth presheaves consisting of translations \( T_j \). Not all truth presheaves, however, are not translations of truth sheaves. In fact, if \( T \) is a translation of \( T_j \), \( (\rho_O)_V((b^* T)(V)) = T_j(V) \) needs to be satisfied. However, in general for such \( T \), it only holds that \( (b^* T)(V) \subseteq (\rho_O)^{-1}_V((\rho_O)_V((b^* T)(V))) \). Consequently, the set \( \text{Sub}_{\text{filt}}(\mathcal{P}_d \Sigma) \) of truth presheaves is divided into the pairwise disjoint lattices each of which corresponds to one and the same truth sheaf and the other truth presheaves that fail to be translations.

### A Presheaf-Based Truth-Value Assignment

In this appendix, we give a brief explanation of the truth-value assignment method developed by Döring and Isham [15], for the purpose of convenience for comparison with the present truth-value assignment on \( j \)-sheaves.

The main ingredient is the spectral presheaf \( \Sigma \), which is a presheaf such that, for each \( V \in \mathbf{V} \), \( \Sigma(V) \) is the Gelfand space on \( V \), and for \( V' \subseteq V \) and \( \sigma \in \Sigma(V) \), \( \Sigma(V' \hookrightarrow V)(\sigma) \) is a restriction of \( \sigma \) to \( V' \). The spectral presheaf plays a role of state space; every proposition on a given quantum system is assumed to be representable as a clopen subobject \( S \) of the spectral presheaf \( \Sigma \), where \( S \) is called a clopen subobject of \( \Sigma \) when \( S(V) \) is a closed and open subset of \( \Sigma(V) \). Thus, the collection \( \text{Sub}_{\text{cl}}(\Sigma) \) of all clopen subobjects of \( \Sigma \) can be regarded as a space of propositions. It is internalized to \( \hat{\mathbf{V}} \) by the clopen power object \( \mathcal{P}_d \Sigma \equiv \Omega^{\Sigma} \) of \( \Sigma \), which is expressed as

\[
(\mathcal{P}_d \Sigma)(V) := \text{Sub}_{\text{cl}}(\Sigma_{dV}), \quad (A.1)
\]
and 

\[(\mathbb{P}_\text{cl} \Sigma)(V' \hookrightarrow V) : \text{Sub}_{\text{cl}}(\Sigma_{\downarrow V}) \to \text{Sub}_{\text{cl}}(\Sigma_{\downarrow V'}); S \mapsto S_{\downarrow V}. \quad (A.2)\]

There is a bijection from \(\text{Sub}_{\text{cl}}(\Sigma)\) to \(\Gamma(\mathbb{P}_\text{cl} \Sigma) := \text{Hom}(1, \mathbb{P}_\text{cl} \Sigma)\) which assigns to each proposition \(P\) its name \([P]\) defined by

\[ [P]_V := \hat{P}_V. \quad (A.3) \]

Here, for each presheaf \(Q \in \hat{V}\) and \(V \in V\), we define \(Q_{\downarrow V} \in \hat{V}\) as the downward restriction of \(Q\) to \(V' \subseteq V\):

\[ Q_{\downarrow V}(V') := \begin{cases} Q(V') & \text{if } V' \subseteq V, \\ \emptyset & \text{otherwise}. \end{cases} \quad (A.4) \]

and for each \(V'' \subseteq V',\)

\[ Q_{\downarrow V}(V'' \hookrightarrow V') := \begin{cases} Q(V'' \hookrightarrow V') & \text{if } V'' \subseteq V, \\ \emptyset & \text{otherwise}. \end{cases} \quad (A.5) \]

Döring and Isham gave other ways to express propositions. They are based on the outer presheaf \(O\) that is defined by

\[ O(V) := \mathcal{P}(V) \quad (A.6) \]

and for \(V' \subseteq V\),

\[ O(V' \hookrightarrow V) : O(V) \to O(V'); \hat{P} \mapsto \delta(\hat{P})_{V'}. \quad (A.7) \]

Here, \(\mathcal{P}(V)\) is the set of all projection operators in \(V\) and \(\delta\) the daedainization operator, which assigns to each projection operator \(\hat{P}\) a collection \(\delta(\hat{P}) := \{\delta(\hat{P})_V\}_{V \in \hat{V}}\), each element \(\delta(\hat{P})_V\) of which is defined by

\[ \delta(\hat{P})_V := \bigwedge \{\hat{\alpha} \in \mathcal{P}(V) \mid \hat{P} \preceq \alpha\}. \quad (A.8) \]

Obviously, \(\delta(\hat{P})\) is a global element of the outer presheaf \(O\). Note that for every \(V'' \subseteq V\), it follows

\[ \delta(\delta(\hat{P})_{V'}) = \delta(\hat{P})_{V''}. \quad (A.9) \]

This equality is often used in the text.
Döring & Isham proved that
\[ \text{Sub}_B(O) \simeq \text{Hyp}(O) \simeq \text{Sub}_\text{cl}(\Sigma), \quad (A.10) \]
and hence for every \( V \in \mathcal{V} \),
\[ \text{Sub}_B(O_{\downarrow V}) \simeq \text{Hyp}(O_{\downarrow V}) \simeq \text{Sub}_\text{cl}((\Sigma)_{\downarrow V}). \quad (A.11) \]
Here, \( \text{Sub}_B(O) \) is the collection of subobjects \( B \subseteq O \) such that, for every \( V \in \mathcal{V} \), \( B(V) \subseteq O(V) \) is a downward closed set of \( O(V) \) with a top element. Obviously, it is a complete Boolean lattice. On the other hand, \( \text{Hyp}(O) \) is the collection of all hyper-elements of \( O \), where a hyper-element \( h \) of \( O \) is a collection \( \{ \hat{h}_V \in O(V) \}_{V \in \mathcal{V}} \) that satisfies, for every \( V' \subseteq V \),
\[ O(V' \hookrightarrow V)(\hat{h}_V) = \delta(\hat{h}_V)_{V'} \preceq \hat{h}_{V'}. \quad (A.12) \]
The bijection relation \((A.10)\) is given by the function \( k : \text{Hyp}(O) \xrightarrow{\sim} \text{Sub}_\text{cl}(\Sigma) \) defined by
\[ (k(h))(V) := \alpha_V(\hat{h}_V), \quad (A.13) \]
and \( c : \text{Sub}_B(O) \xrightarrow{\sim} \text{Hyp}(O) \) defined by
\[ c(A)_V := \lor A(V). \quad (A.14) \]
Here, the function \( \alpha_V : \mathcal{P}(V) \to \mathcal{C}l((\Sigma(V)) \), where \( \mathcal{C}l((\Sigma(V)) \) is the collection of all clopen subsets of \( (\Sigma(V)) \), is defined as
\[ \alpha_V(\hat{P}) := \{ \sigma \in (\Sigma(V) \mid \sigma(\hat{P}) = 1 \}. \quad (A.15) \]
Bijections for \((A.11)\) are given as the restrictions of \( k \) and \( c \) to subalgebras of \( V \).
In particular, every projection \( \hat{P} \) defines a proposition presheaf, the global element \( \delta(\hat{P}) \in \Gamma O \subseteq \text{Hyp}(O) \). That is, as an element of \( \text{Sub}_\text{cl}(\Sigma), \delta(\hat{P}) \) is given by
\[ (\delta(\hat{P}))(V) := \alpha_V(\delta(\hat{P})_V) = \{ \sigma \in (\Sigma(V) \mid \sigma(\delta(\hat{P})_V) = 1 \}, \quad (A.16) \]
and as that of \( \text{Sub}_B(O) \),
\[ (\delta(\hat{P}))(V) := c_V^{-1}(\delta(\hat{P})_V) = \{ \hat{a} \in O(V) \mid \hat{a} \preceq \delta(\hat{P})_V \}. \quad (A.17) \]
Each proposition $P \in \text{Sub}_cl(\Sigma)$ is assigned a truth value relative to a truth object $T$, a subobject of $\mathbb{P}_cl\Sigma$ (or, equivalently that of $\mathbb{P}_dB\mathcal{O}$) of which global elements give truth propositions. Let $\tau$ be the characteristic morphism of $T$; that is, the diagram

\[
\begin{array}{ccc}
T & \to & 1 \\
\downarrow \quad & \quad & \downarrow \\
\mathbb{P}_cl\Sigma & \to & \Omega
\end{array}
\]  

(A.18)

is a pullback. Then, for each proposition $P$, its truth-value $\nu(P; T) \in \Gamma\Omega$ is given by

\[
\nu(P; T) = \tau \circ [P],
\]  

(A.19)

or more precisely,

\[
\nu(P; T)_V = \{V' \subseteq V \mid P_{1V} \in T(V')\}.  
\]  

(A.20)

In [15], Döring and Isham defined generalized truth object

\[
T^{\rho, r}(V) := \{A \in \text{Sub}_dB(O_{4\mathcal{V}}) \mid \forall V'' \subseteq V \ (\text{tr}(|\varphi\rangle\langle\varphi|) \leq r)\},
\]  

(A.21)

which gives propositions that are true at least a probability $r \in [0, 1]$ in a mixed state expressed by a density matrix $\rho$. Under the truth presheaf $T^{\rho, r}$, the truth-value of $\delta(\hat{P})$ is evaluated as

\[
\nu_j(\delta(\hat{P}); T^{\rho, r})_V = \{V' \subseteq V \mid \forall V'' \subseteq V \ (\text{tr}(\delta(\hat{P})_{V''}) \geq r)\}
\]  

(A.22)

If we take $\rho = |\varphi\rangle\langle\varphi|$ and $r = 1$, the truth presheaf $T^{\varphi}_V := T^{\varphi}(V)$ and the truth-value of $\delta(\hat{P})$ are given by

\[
T^{\varphi}_V := \{A \in \mathbb{P}_dB\mathcal{O}(V) \mid \forall V'' \subseteq V \ (\delta(|\varphi\rangle\langle\varphi|)_{V''} \in A(V'))\},
\]  

(A.23)

and

\[
\nu(\delta(\hat{P}); T^{\varphi})(V) = \{V' \subseteq V \mid \delta(\hat{P})_{V'} \in T^{\varphi}(V')\} = \{V' \subseteq V \mid \forall V'' \subseteq V' \ (\delta(\langle\varphi|\varphi\rangle)_{V''} \leq \delta(\hat{P})_{V''})\}
\]  

(A.24)

respectively.

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B Mathematical Miscellany

In the text, some properties of the functors $\mathbf{♭} : \mathbf{V} \to \mathbf{V}$ and $\mathbf{♭}^* : \hat{\mathbf{V}} \to \hat{\mathbf{V}}$ are used. In this appendix, we explain them for convenience.

Throughout the text, we use the relation $\mathbf{♭} \mathbf{♭} = \mathbf{♭}$ without notice. Furthermore, the following fact is often used: for any presheaf $Q \in \hat{\mathbf{V}}$ and any subsheaf $A$ of $\mathbf{♭}^* Q$,

$$A = \mathbf{♭}^* A. \quad (B.1)$$

To show this, let $A \in \hat{\mathbf{V}}$ be a subobject of $\mathbf{♭}^* Q$. The closure $\bar{A}$ of $A$ in $\mathbf{♭}^* Q$ is given by

$$\bar{A}(V) = \{ q \in (\mathbf{♭}^* Q)(V) \mid (\mathbf{♭}^* Q)(\mathbf{♭}(V)) \hookrightarrow V(q) \in A(\mathbf{♭}(V)) \}$$

$$= \{ q \in Q(\mathbf{♭}((V))) \mid q \in A(\mathbf{♭}(V)) \}$$

$$= A(\mathbf{♭}(V))$$

$$= \mathbf{♭}^* A(V). \quad (B.2)$$

Thus, $A$ is closed (hence, a sheaf) if and only if $A = \mathbf{♭}^* A$.

When we deal with power objects, the following relations are crucially important: for each $Q \in \hat{\mathbf{V}}$ and $V \in \mathbf{V}$, we have

$$\mathbf{♭}^* (Q_{\downarrow V}) = \mathbf{♭}^* (Q_{\downarrow \mathbf{♭}(V)}), \quad (B.3)$$

$$\mathbf{♭}^* ((\mathbf{♭}^* Q)_{\downarrow V}) = \mathbf{♭}^* (Q_{\downarrow V}), \quad (B.4)$$

and furthermore, for any $V' \subseteq V$,

$$\mathbf{♭}^* ((\mathbf{♭}^* (Q_{\downarrow V}))_{\downarrow V'}) = \mathbf{♭}^* (Q_{\downarrow V'}). \quad (B.5)$$

They can be proved straightforwardly.

In section 4, we treat a relation between $\mathbf{♭}^* \mathbb{P}$ and $\mathbb{P}_j \mathbf{♭}^*$. They are functors from $\hat{\mathbf{V}}$ to $\text{Sh}_j \hat{\mathbf{V}}$, and there exists a canonical natural transformation $\mathbf{♭}^* \mathbb{P} \xrightarrow{\sim} \mathbb{P}_j \mathbf{♭}^*$, which is defined as follows. First, note that, for each presheaf $Q \in \hat{\mathbf{V}}$ and $V \in \mathbf{V}$,

$$\mathbf{♭}^* (\mathbb{P} Q)(V) = \mathbb{P} Q(\mathbf{♭}(V)) \simeq \text{Hom}(Q_{\downarrow \mathbf{♭}(V)}, \Omega), \quad (B.6)$$

and

$$\mathbb{P}_j (\mathbf{♭}^* Q)(V) \simeq \text{Hom}(\mathbf{♭}^* (Q_{\downarrow V}), \Omega_j) = \text{Hom}(\mathbf{♭}^* (Q_{\downarrow \mathbf{♭}(V)}), \Omega_j). \quad (B.7)$$
Let $S$ be a subobject of $Q \downarrow V$ and $Q \downarrow V \xrightarrow{\chi} \Omega$ be the characteristic morphism of $S$ in $\mathcal{V}$. Then, we have the following commutative diagram:

$$
\begin{array}{cccc}
S & \xrightarrow{1} & \Omega \\
\downarrow \iota & & \downarrow \text{true} \\
Q \downarrow V & \xrightarrow{\chi} & \Omega \\
\downarrow \zeta & & \downarrow \zeta \Omega \\
\ast_j Q \downarrow V & \xrightarrow{\ast_j \chi} & \ast_j \Omega & \xrightarrow{\text{true}_j} 1
\end{array}
$$

(B.8)

Here, the top square is a pullback, and hence, so is the bottom one, because of the left-exactness of the associated sheaf functor $\ast_j$. Thus, we define $\zeta Q \downarrow V$ as a function that maps the top square to the bottom one; that is, as a function from $\ast_j(PQ)(V)$ to $(P_j(\ast_j Q))(V)$, it is defined by

$$(\zeta Q \downarrow V)(S) := \ast_j S,$$

and hence, as a function from $\text{Hom}(Q \downarrow V, \Omega)$ to $\text{Hom}(\ast_j(Q \downarrow V), \Omega_j)$,

$$(\zeta Q \downarrow V)(\chi) := \ast_j(r \circ \chi).$$

(B.10)

We can straightforwardly show that $(\zeta Q)(V)$ is natural with respect to $Q \in \mathcal{V}$ and $V \in \mathcal{V}$.

In the text, the case where $Q$ is the outer presheaf $O$ is treated. As easily shown, the restriction of $\zeta O$ to $\ast_j(P_{dB} O)$ takes values on $P_{dB}(\ast_j O)$ and the
Because of this, in section 4 we write $\varrho_O$ for the restriction $\varrho_O|_{\mathcal{O}(\mathcal{P}_dO)}$ described above.

References

[1] C.J. Isham. Topos theory and consistent histories: The internal logic of the set of all consistent sets. *Int. J. Theor. Phys.* **36**, 785 (1997)

[2] C.J. Isham and J. Butterfield. A topos perspective on the Kochen-Specker theorem: I. Internal valuations. *Int. J. Theor. Phys.* **37**, 2669-2733 (1998)

[3] J. Butterfield and C.J. Isham. A topos perspective on the Kochen-Specker theorem: II. Conceptual aspects, and classic analogues. *Int. J. Theor. Phys.* **38**, 837-859 (1999)

[4] J. Hamilton, J. Butterfield and C.J. Isham. A topos perspective on the Kochen-Specker theorem: III. Von Neumann algebras as the base category. *Int. J. Theor. Phys.* **39**, 1413-1436 (2000)

[5] J. Butterfield and C.J. Isham. A topos perspective on the Kochen-Specker theorem: IV. Internal valuations. *Int. J. Theor. Phys.* **41**, 613-639 (2002)

[6] A. D"oring and C.J. Isham. A topos foundation for theoretical physics: I. Formal language for physics. *J. Math. Phys.* **49**, 053515 (2008)

[7] A. D"oring and C.J. Isham. A topos foundation for theoretical physics: II. Daseinization and the liberation of quantum theory. *J. Math. Phys.* **49**, 053516 (2008)
[8] A.Döring and C.J. Isham. A topos foundation for theoretical physics: III. The representation of physical quantities with arrows $\delta^o(A): \Sigma \to \mathbb{R}^\infty$. *J. Math. Phys.* **49**, 053517 (2008)

[9] A.Döring and C.J. Isham. A topos foundation for theoretical physics: IV. Categories of Systems. *J. Math. Phys.* **49**, 053518 (2008)

[10] C.Heunen, N.P.Landsman and B.Spitters. A topos for algebraic quantum theory *Comm. Math. Phys.* **291**, 63 (2009)

[11] C.Flori. A topos formulation of history quantum theory *J. Math. Phys.* **51**, 053527 (2010)

[12] C.Heunen, N.P.Landsman and B.Spitters. in *Deep Beauty* (ed. H. Halvorson), 271 Cambridge University Press, Cambridge (2011)

[13] C.Heunen, N.P.Landsman, B.Spitters and S.Wolters. The Gelfand spectrum of a noncommutative C*-algebra: a topos-theoretic approach *J. Austral. Math. Soc.*, **90**, 39, (2011)

[14] A.Döring and C.J. Isham. “What is a Thing?” Topos Theory in the Foundation of Physics. in *New Structures for Physics* (ed. B.Coecke) 753, Springer, Heidelberg (2011)

[15] A.Döring and C.J. Isham. Classical and quantum probabilities as truth values *J. Math. Phys.* **53**, 032101 (2012)

[16] S.A.M.Wolters. A Comparison of two topos-theoretic approaches to quantum theory *Comm. Math. Phys.* **317**, 3 (2013)

[17] C.Flori. Group action in topos quantum physics *J. Math. Phys.* **54**, 032106 (2013)

[18] C.Flori. *A First Course in Topos Quantum Theory* Springer, Heidelberg (2013)

[19] S.Kochen and E.P.Specker. The problem of hidden variables in quantum mechanics *J. Math. Mech* **17**, 59 (1967)

[20] K.Nakayama. Topologies on quantum topoi induced by quantization *J. Math. Phys.* **54**, 072102 (2013)
[21] S. MacLane and I. Moerdijk. *Sheaves in Geometry and Logic: A First Introduction to Topos Theory*. Springer-Verlag, New York (1992)

[22] C. J. Isham. Topological and global aspect of quantum theory, in *Relativity, Groups and Topology II* (ed. B. S. DeWitt and R. Stora), 1059 North-Holland, Amsterdam (1984)

[23] B. A. Davey and H. A. Priestley. *Introduction to Lattices and Order*. Cambridge University Press, New York (1990)