Thermodynamics, Phase Transition and Joule Thomson Expansion of novel 4-D Gauss Bonnet AdS Black Hole

Kartheek Hegde,* Naveena Kumara A.,† Ahmed Rizwan C.L.,‡ and Ajith K.M.§

Department of Physics, National Institute of Technology Karnataka, Surathkal 575 025, India

Md Sabir Ali¶

Department of Physics, Indian Institute of Technology, Ropar, Rupnagar, Punjab 140 001, India

Abstract

We investigate the thermodynamic behaviour of the four dimension Gauss Bonnet black hole, proposed in [Phys. Rev. Lett. 124, 081301 (2020)], in the AdS background. We study the thermodynamics in extended phase space, where the cosmological constant is taken as the thermodynamic pressure. The black hole exhibits a phase transition similar to that of van der Waals system. The phase transition is investigated via isotherms in $P - V$ diagram, Gibbs free energy and specific heat plots. The charged and neutral cases are considered separately to observe the effect of charge on critical behaviour. In both cases the van der Waals like behaviour is exhibited. We also study the throttling process of the black hole analytically using isenthalpic and inversion curves.

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*Electronic address: hegde.kartheek@gmail.com
†Electronic address: naviphysics@gmail.com
‡Electronic address: ahmedrizwancl@gmail.com
§Electronic address: ajithkm@gmail.com
¶Electronic address: alimd.sabir3@gmail.com
I. INTRODUCTION

The notion that black holes are thermodynamic systems is well known since past few decades [1–4]. Especially, the thermodynamics of AdS black holes have gained significant importance due to the AdS/CFT correspondence [5–7]. The AdS black holes exhibit the thermodynamic properties which are quite different from that of the asymptotically flat or de Sitter black holes. The early study on the phase transition is the so called Hawking-Page phase transition [8], between large black holes and radiation, interpreted as the confinement/deconfinement phase transition of gauge field in the context of AdS/CFT correspondence [9]. In the later developments, it was found that, the phase transitions of AdS black holes are analytically similar to that of the van der Waals system [10, 11]. Further, The identification of cosmological constant as the thermodynamic variable pressure [12, 13] has led to the beginning of a new branch called black hole chemistry. A wide variety of applications on the phase transition properties were found using this identification [14–16] (see the review article [16] and references therein for more details).

Recently a novel four dimensional Einstein-Gauss-Bonnet gravity theory was proposed by D. Glavan and C. Lin [17], which triggered great interest among the researchers since it bypasses the Lovelock’s theorem and avoids Ostrogradsky instability. The main feature of this theory is that for a positive GB coupling constant the static spherically symmetric solution is free from the singularity problem. Since the theory is in equal footing with Einstein’s general relativity, it’s natural to probe the theoretical and observational aspects of this classical modified gravity theory. The quasi-normal modes of scalar, electromagnetic and gravitational perturbations were studied and it is reported that, as to the change of the coupling parameter $\alpha$, the damping rate is more sensitive characteristic than the real part[18]. The authors also investigated the stability and shadow of the black hole. Also the inner most circular orbits and photon sphere of the black hole are analysed by studying the geodesics [19]. A rotating analogy of this black hole using Newman-Janis algorithm was constructed recently [20]. The authors investigated the shadows of both rotating and non rotating cases. In the same study, by modelling the $M87^*$ as the rotating black hole of the novel theory, the possible range of GB coupling parameter is constrained. Among the studies extending the four dimensional Gauss Bonnet theory, the most interesting is the recent proposal of charged version in AdS background [21]. In this article we are intended to
study the assorted thermal properties of four dimensional Gauss Bonnet black hole in AdS background for both the charged and neutral case.

The paper is organised as follows. In the next section we present the elementary thermodynamics of the black hole starting from the metric details. In section III we study the phase transitions and critical behaviour of the black hole. In section IV the Joule Thomson expansion of the black hole system is sought. We conclude the paper with discussions on the insights and results in section V.

II. THERMODYNAMICS OF THE BLACK HOLE

We study the thermodynamic properties of four dimensional AdS Gauss Bonnet black hole for both neutral and charged cases. The action for the Einstein-Gauss-Bonnet gravity theory is given by [17],

\[ S = \frac{1}{16\pi} \int d^D x \sqrt{-g} \left( \frac{M_P^2}{2} R - \Lambda + \frac{\alpha}{D-4} G \right) \]  

(1)

where \( M_P \) is the reduced Planck mass which characterises the gravitational coupling strength, \( \Lambda \) is the cosmological constant, \( G \) is the Gauss Bonnet invariant and \( \alpha \) is a dimensionless coupling constant. The charged version of this theory in the AdS background, Einstein-Maxwell Gauss-Bonnet gravity, was obtained by adding the Maxwell term \( F^{\mu\nu} F_{\mu\nu} \) [21]. The spherically symmetric solution to this action reads as,

\[ ds^2 = -f(r)dt^2 + \frac{1}{f(r)} dr^2 + r^2 d\Omega^2_{D-2}. \]  

(2)

Under the limit \( D \to 4 \) the metric function has the form,

\[ f(r) = 1 + \frac{r^2}{2\alpha} \left( 1 - \sqrt{1 + 4\alpha \left( -\frac{1}{l^2} + \frac{2M}{r^3} - \frac{Q^2}{r^4} \right)} \right) \]  

(3)

where \( Q \) is the charge of the black hole. In the above expression \( M \) is the mass of the black hole, the explicit form for which is obtained by using the condition \( f(r_+) = 0 \),

\[ M = \frac{r^3}{2l^2} + \frac{Q^2}{2r} + \frac{\alpha}{2r} + \frac{r}{2} \]  

(4)

The AdS length \( l \) is related to the cosmological constant as \( \Lambda = -\frac{3}{l^2} \). In the extended phase space the cosmological constant is treated as thermodynamic variable pressure using the
relation $P = -\frac{A}{8\pi}$. The Hawking temperature of the black hole is related to the surface gravity $\kappa$, which is calculated as

$$T = \frac{\kappa}{2\pi} = \frac{f'(r_+)}{4\pi} = -\frac{\alpha - 8\pi Pr^4 + Q^2 - r^2}{4\pi r^3 + 8\pi \alpha r}. \quad (5)$$

With these we can write the first law for the black hole,

$$dM = TdS + VdP + \Phi dQ + A d\alpha \quad (6)$$

where $\Phi$ and $A$ are the potentials conjugate to $Q$ and $\alpha$, respectively. From this the black hole entropy can be calculated,

$$S = \int_0^{r_+} \frac{1}{T} dM = \pi r_+^2 + 4\pi \log(r_+) \quad (7)$$

and also the thermodynamic volume,

$$V = \left(\frac{\partial M}{\partial P}\right)_{S,Q,\alpha} = \frac{4}{3}\pi r_+^3. \quad (8)$$

We see that the entropy has a logarithmic correction term whereas the thermodynamic volume remain same as the geometric volume.

### III. PHASE TRANSITION OF THE BLACK HOLE

The equation of state can easily be obtained as,

$$P = \frac{Q^2}{8\pi r^4} + \frac{\alpha}{8\pi r^4} + \frac{\alpha T}{r^3} - \frac{1}{8\pi r^2} + \frac{T}{2r}. \quad (9)$$

This equation of state has a functional behaviour similar to that of van der Waals case. To make it dimensionally correct we need to rescale it as follows,

$$P \rightarrow \frac{hc}{l_P^2}, \quad T \rightarrow \frac{hc}{k_B} T. \quad (10)$$

The comparison with the van der Waals equation of state also renders the relation between the specific volume $v$ and the horizon radius as $v = 2l_P r_+$. At the critical point,

$$\left(\frac{\partial P}{\partial v}\right)_T = \left(\frac{\partial^2 P}{\partial v^2}\right)_T = 0. \quad (11)$$

The critical temperature, critical pressure and critical volume are obtained from these conditions,

$$T_c = \left(\frac{8\alpha + 3Q^2 - \sqrt{48\alpha^2 + 9Q^4 + 48\alpha Q^2}}{48\pi \alpha^2}\right) \sqrt{6\alpha + 3Q^2 + \sqrt{48\alpha^2 + 9Q^4 + 48\alpha Q^2}}; \quad (12)$$
\[ P_c = \frac{9\alpha + 6Q^2 + \sqrt{48\alpha^2 + 9Q^4 + 48\alpha Q^2}}{24\pi \left( 6\alpha + 3Q^2 + \sqrt{48\alpha^2 + 9Q^4 + 48\alpha Q^2} \right)^2}; \]  
\[ V_c = \frac{4}{3\pi} \left( 6\alpha + 3Q^2 + \sqrt{48\alpha^2 + 9Q^4 + 48\alpha Q^2} \right)^{3/2}. \]

In all the expressions we have written so far with \( Q \), the neutral case is obtained by simply putting \( Q = 0 \). The \( P - v \) diagram is shown in figure 1, which shows a critical behaviour. The isotherms below critical temperature \( T_c \), has three distinct regions. The negative slope regions are stable and positive slope regions are unstable phases. A first order phase transition takes place between the stable phases, namely small black hole (SBH) and large black hole (LBH). The unstable region disappears for temperature above the critical value. The critical behaviour is present for both neutral and charged Gauss Bonnet black holes. This result is interesting because for four dimensional RN AdS black holes, charge is the key property to display van der Waals like phase transitions. In our case the Gauss Bonnet parameter \( \alpha \) plays similar role as in five dimensional Gauss Bonnet AdS black holes [22].

![Figure 1: \( P - V \) isotherms for neutral and charged Gauss Bonnet AdS black hole in four dimension.](image)

The phase transition can also be understood by investigating the behaviour of Gibbs free energy. The Gibbs free energy \( G \) for the black hole under consideration is calculated by using the standard expression \( G = M - TS \) and the result is,

\[ G = \frac{1}{6} \left( \frac{3\left( r^2 + 4\alpha \log(r) \right) \left( \alpha - 8\pi Pr^4 + Q^2 - r^2 \right) (\alpha - 8\pi Pr^4 + Q^2 - r^2)}{2 \left( r^3 + 2\alpha r \right)} + 8\pi Pr^3 + \frac{3Q^2}{r} + \frac{3\alpha}{r} + 3r \right). \]  

Here \( r_+ \) is considered as function of \( (P, T) \) from equation of state. We plot Gibbs free energy for changing temperatures at different pressure values (figure 2).
The presence of characteristic swallow tail behaviour for both neutral and charged Gauss Bonnet black hole indicates that there is a first order phase transition in the system. For the temperatures smaller than the critical values there is a small black hole (SBH)-large black hole (LBH) first-order phase transition. In principle it is possible to represent the phase structure with a coexistence curve in the $P - T$ plane. However, it is not feasible to obtain an analytical expression for the four dimensional Gauss Bonnet black hole. This is in contrast to the five dimensional (uncharged) case where analytical solution exists \cite{23}. The coexistence curve separates the SBH and LBH phases below critical point. It terminates at the critical point which is the second order phase transition point.

Another interesting thermodynamic quantity that we need to explore in the context of phase transition and thermal stability is the specific heat. A positive heat capacity corresponds to a stable system whereas a negative heat capacity indicates the instability of the system under small perturbation. For the black hole we consider, the specific heat at constant pressure is

$$ C_P = T \left( \frac{\partial S}{\partial T} \right)_P = \frac{2\pi (2\alpha + r^2)^2 (-\alpha + 8\pi Pr^4 - Q^2 + r^2)}{2\alpha^2 + 8\pi Pr^6 + r^4(48\pi\alpha P - 1) + Q^2 (2\alpha + 3r^2) + 5\alpha r^2}. \quad (16) $$

This is plotted against the horizon radius in fig 3.

Figure 2: $G - T$ diagram for neutral (left) and charged (right) Gauss Bonnet AdS black hole in four dimension.
Figure 3: Behaviour of specific heat of four dimensional Gauss Bonnet AdS black hole. The blue (solid) lines are of charged black hole while the red (dashed) curves are of neutral black hole. Fig 3(a) is for $P < P_c$, fig 3(b) is for $P = P_c$ and fig 3(c) is for $P > P_c$. We have taken $Q = 1$ in charged case and $\alpha = 1$ in both neutral and charged case.

We have considered charged and neutral black hole simultaneously since there is no phenomenal difference in the behaviour of specific heat between both the cases. However quantitative changes are apparent from the plots. For $P < P_c$ there exists two diverging points which separates three regions. The phases with positive specific heat in the lower radius and higher radius regions are stable. The intermediate phase with negative $C_P$ value is unstable phase. Therefore for the pressure lower than critical value there are three phases possible, namely, small black hole (SBH), intermediate black hole (IBH) and large black hole (LBH). The phase transition takes place between the stable phases, SBH and LBH. When $P = P_c$ the two divergent points merge to form a single divergence, removing the unstable
region. For \( P > P_c \) the heat capacity is always positive and there exists no divergences. This implies the black hole is stable and there is no phase transition. These results are similar to the black holes in RN AdS and Gauss Bonnet theory in higher dimensions.

IV. JOULE THOMSON EXPANSION OF THE BLACK HOLE

An interesting feature of a van der Waals system is, heating or cooling during the throttling process. Since the phase structure of the 4D Gauss Bonnet black hole system is analogous to that of van der Waals system, it is worth examining the Joule Thomson expansion of the black hole. It is already known that the AdS black holes exhibit the throttling process [24–26]. The characteristic feature of this adiabatic irreversible expansion is that the enthalpy in the initial and final state remain the same. In the extended phase space the black hole mass is equivalent to enthalpy. Hence the isenthalpic curves, the locus of all points corresponding to the initial and final equilibrium states of the same enthalpy, are constant mass curves. The slope of an isenthalpic curve is the Joule Thomson coefficient, which is given by,

\[
\mu_J = \left( \frac{\partial T}{\partial P} \right)_M = \frac{1}{C_p} \left[ T \left( \frac{\partial V}{\partial T} \right)_P - V \right].
\]

(17)

\( \mu_J = 0 \) defines the inversion temperature,

\[
T_i = V \left( \frac{\partial T}{\partial V} \right)_P.
\]

(18)

After a simple calculation we obtain,

\[
T_i = \frac{2\alpha^2 + 8\pi P (r^6 + 6\alpha r^4) + Q^2 (2\alpha + 3r^2) - r^4 + 5\alpha r^2}{12\pi r (2\alpha + r^2)^2}.
\]

(19)

From equation (5) we have

\[
T_i = -\frac{\alpha - 8\pi P r^4 + Q^2 - r^2}{4\pi r^3 + 8\pi \alpha r}.
\]

(20)

We solve equation (19) and equation (20) \( r_+ \) and chose an appropriate root. Substituting that to equation 19 we obtain a lengthy expression for inversion curve, the result is shown in figure 4. The neutral and charged Gauss Bonnet black holes shows similar behaviour, suggesting that the electric charge \( Q \) does not play any role in this phenomenon. Rather, the Joule Thomson expansion is guided by the GB coupling constant \( \alpha \). The isenthalps also shown in the same set of diagrams. These are known as crossing diagrams. To the left of the
inversion curve the region corresponds to cooling and to the right is the region of heating, during the throttling process.

Figure 4: Inversion curve and isenthalpic curves for neutral and charged Gauss Bonnet AdS black hole in four dimension. The red monotonous line is the inversion curve. The slope changing curves, which have maxima on the inversion curve, are the isenthalps for different $M$ values.

V. DISCUSSIONS

It is widely believed that the higher dimensional Gauss Bonnet theories are nontrivial. However recent proposal of Einstein-Gauss-Bonnet gravity theory is naively formulated in four dimensions by re-scaling of the GB coupling constant prior to the dimensional reduction. We have studied the thermodynamics of asymptotically AdS black hole in the (3+1) dimensional Einstein-Gauss-Bonnet theory. We observed that several thermodynamical properties are similar to higher dimensional Gauss Bonnet, RN AdS and Kerr AdS black holes. It is sometimes argued that the charge of the black hole plays key role in the phase transitions. The relevance of our study lies in the fact that, firstly, there is one more good candidate in four dimension displaying van der Waals like phase transition without necessarily having the charge, like Kerr AdS black hole. Secondly, more importantly, the black hole we have considered is free from the singularity problem. The phase transition is observed from the $P−V$ criticality, Gibbs free energy and specific heat behaviours. Below the critical point a first order phase transition between a small black hole and a large black hole is found. The black hole exhibits throttling process, in which the expansion may lead to heating or cooling.
As an extension of our work the other related thermodynamic studies like thermodynamic geometry and the underlying microstructure can be studied for this black hole.

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