Baxter’s T-Q Relation and Bethe Ansatz of Discrete Quantum Pendulum and Sine-Gordon Model

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Abstract

Using the Baxter’s T-Q relation derived from the transfer matrix technique, we consider the diagonalization problem of discrete quantum pendulum and discrete quantum sine-Gordon Hamiltonian from the algebraic geometry aspect. For a finite chain system of the size $L$, when the spectral curve degenerates into rational curves, we have reduced Baxter’s T-Q relation into a polynomial equation; the connection of T-Q polynomial equation with the algebraic Bethe Ansatz is clearly established. In particular, for $L = 4$ it is the case of rational spectral curves for the discrete quantum pendulum and discrete sine-Gordon model. To these Baxter’s T-Q polynomial equations, we have obtained the complete and explicit solutions with a detailed understanding of their quantitative and qualitative structure. In general the model possesses a spectral curve with a generic parameter. We have conducted certain qualitative study on the algebraic geometry of this high-genus Riemann surface incorporating Baxter’s T-Q relation.
1 Introduction

In the early seventies, R. Baxter proposed the method of $Q$-operator and the $T$-$Q$ relation in his renowned solution of the eight-vertex model and the spin $\frac{1}{2}$ XYZ chain in soluble statistical mechanics [2, 3]. Since then, the method has played a powerful mechanism up to nowadays in the two-dimensional exactly solvable lattice models, and the corresponding quantum spin-chain Hamiltonians. The quantum inverse scattering /algebraic Bethe Ansatz method developed by the Leningrad school in the early eighties [7, 11] systematized earlier results on two-dimensional integrable lattice models, and paved the way for the far-reaching effects in both mathematical and physical development in the past two decades. In the study of massless lattice sine-Gordon model, Izergin-Korepin and Tarasov [12] found, within the framework of quantum inverse scattering method, the $L$-operator ( with $C^N$-operators entries) which satisfies the Yang-Baxter equation for the six-vertex model’s $R$-matrix with the parameter $q$ being the $N$th root of unity: $q^N = 1$, (see the formula (3) of this paper). A slightly modified version of this $L$-operator also appeared in the study of chiral Potts $N$-state model [4]. On the other hand, there are Hamiltonians of physical interest, exhibiting an intimate relationship with certain systems derived from the transfer matrix for a fixed finite size $L$ while $N$ varying. For $L=3$, the Hamiltonian, proposed first in [8] and then investigated in a rigorously mathematical manner in [13], can be interpreted as a generalization of the Hofstadter model [1, 9], a renowned Bloch system with a constant external magnetic field. Through the quantum inverse scattering method, one can calculate the spectrum of the Hamiltonian by solving the (algebraic) Bethe ansatz equation. In [13] we formulate the problem from the algebraic geometry aspect, and consider it as the Baxter’s $T$-$Q$ relation on the spectral curve via the Baxter’s vacuum state [3, 5]. In addition, a general scheme of diagonalizing the transfer matrix of a finite size $L$ by means of the Baxter’s $T$-$Q$ relation ( or the Bethe equation) on the spectral curve was also discovered. At present, it is rather difficult to extract the explicit quantitative data for the spectrum problem by this approach, due to the complicated functional theory of the high-genus spectral curve. Nevertheless, when the spectral curve is totally degenerated into rational curves, we demonstrate in this paper that the polynomial equation derived from the Baxter’s $T$-$Q$ relation gives rise to the usual Bethe ansatz equation in physical literature. And for $L=4$, the transfer matrix produces the Hamiltonions of the discrete quantum pendulum and the discrete quantum sine-Gordon (SG) model, appeared in [8, 10].

In this article we make a thorough study of the discrete quantum pendulum and SG model in the context of Baxter’s $T$-$Q$ relation by using the transfer matrix technique, ( for the Hamiltonians, see [17] of Sect. 2). Though the formulation via this approach can be made on Hamiltonian chains of an arbitrary finite size $L$, the mathematical treatment given in this paper for these two specific models takes advantage of certain special features presented only in the size $L=4$. The spectral curves, upon which the Baxter’s $T$-$Q$ relation is formulated, have a very high genus in general; indeed it is of the order $N^5$. However, by examining their algebraic geometry properties, these spectral curves are found to form a family of algebraic curves covering the elliptic curves, so one could hope that the elliptic function theory would eventually play a role in the solutions of Baxter’s $T$-$Q$ relation. In the rational-spectral-curve case for a finite size $L$, the geometry of spectral curves becomes trivial. However, it still inevitably requires some subtle analysis of the Baxter vacuum state to extract the essential data on solutions of the polynomial equation, then to carry out the necessary algebraic study of a certain ”over-determined” system of $q$-difference equations for a $N$th root of unity $q$, a difficult problem for a general $L$. In the cases of discrete quantum pendulum and SG model, where the size $L$ is equal to 4, we are able to obtain the explicit solutions of Baxter’s $T$-$Q$ polynomial equation by taking the special symmetric structure of polynomial functions into account. The results are complete from both the quantitative and
qualitative aspects, and the method provides a sound mathematical treatment on problems raised in [6, 10]. The main advantage of our approach to these problems is to make use of the explicit Baxter’s vacuum state as the precise form of ”Bethe ansatz ground state”, which was only postulated in the previous articles in literature, e.g. [10]. This approach to the eigenvalue problem of Hamiltonians has also been shown in [13] for \( L=3 \) on the Hofstadter-type models. It seems to us that the use of Baxter’s \( T-Q \) relation would be more fundamental, and mathematically tractable, than the usual Bethe Ansatz technique. Furthermore, in this approach a certain mathematical theory of \( q \)-Sturm-Liouville type problem would possibly arise to entangle with the Baxter’s \( T-Q \) polynomial equation originated from certain physical problems.

This paper is organized as follows: In Sect. 2, we review the basic construction of Baxter’s \( T-Q \) relation using the transfer matrix techniques, with the spectral curve of a high genus depending on the size \( L \) and parameters involved in the Hamiltonians. Some facts in [13] will be recalled here for the sake of completeness. Then we describe the constraints of parameters appeared in the spectral curves for the discrete quantum pendulum and SG model. In Sect. 3, we discuss a canonical procedure of reducing the Baxter’s \( T-Q \) relation to a polynomial equation when the spectral curve is degenerated to rational curves for a finite size \( L \). By converting the parameters to one special case studied in our earlier paper [13], we obtain the Baxter’s \( T-Q \) polynomial equation. In Sect. 4, we apply the conclusion of the previous section to the case \( L=4 \), and with the further parameter constraints of the discrete quantum pendulum and SG model. The symmetric Baxter’s \( T-Q \) polynomial relation is introduced, and a general discussion on the qualitative nature of its solutions is given. In Sect. 5, we construct explicitly the complete solutions of symmetric Baxter’s \( T-Q \) polynomial equations, among which the discrete quantum pendulum and SG model for the rational spectral curve are included. Both the quantitative and qualitative nature of these solutions are revealed in the procedure of the derivation; in particular, the connection of the solutions with the usual Bethe ansatz equation in literature is clarified. In Sect. 6, we consider the discrete quantum pendulum and SG model for the general spectral curves, which are high-genus Riemann surfaces. By examining the geometric properties, we discover a canonical relationship of these curves with elliptic curves. A primitive qualitative analysis is made on the geometry of these spectral curves in connection with the eigenvalue problem of physical models through Baxter’s \( T-Q \) relation. In Sect. 7, we present the conclusion remark with a discussion of our future programs. We end with the appendix of presenting the identification of the sine-Gordon integral in [10] with the Hamiltonion given in this paper.

**Notations.** To present our work, we prepare some notations. In this paper, \( \mathbb{Z}, \mathbb{R}, \mathbb{C} \) will denote the ring of integers, real, complex numbers respectively, \( \mathbb{N} = \mathbb{Z}_{>0}, \mathbb{Z}_N = \mathbb{Z}/N\mathbb{Z} \), and \( i = \sqrt{-1} \). For a positive integers \( n \), we denote \( \mathbb{C}^N \) the tensor product of \( n \)-copies of the vector space \( \mathbb{C}^N \). We use the notation of \( q \)-shifted factorials, 

\[
(a;q)_0 = 1, \quad (a;q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1}), \quad n \in \mathbb{N}.
\]

We shall minimize the repetition of materials in our previous article [13], and so opt to use the same notation conventions as much as possible.

### 2 Transfer Matrix, Baxter Vacuum State and T-Q Equation

In this section we first recall some formulae in quantum inverse scattering method for later use. Most of the materials can be found in [13], including the original references. Then, we specify our discussion on the cases of the discrete quantum pendulum and SG Hamiltonian, the models we shall mainly consider in this work.
In this paper, $N$ will always denote an odd positive integer with $M = \left( \frac{N}{2} \right)$: $N = 2M + 1$ ($M \geq 1$), and $\omega$ is a primitive $N$th root of unity, $q := \omega^2$ with $q^N = 1$, i.e. $q = \omega^{M+1}$. A vector $v$ in $\mathbb{C}^N$ is represented by a sequence of coordinates, $(v_k \mid k \in \mathbb{Z})$, with the $N$-periodic condition: $v_k = v_{k+N}$, i.e. $v = (v_k)_{k \in \mathbb{Z}_N}$. The standard basis of $\mathbb{C}^N$ will be denoted by $\langle k \rangle (k \in \mathbb{Z}_N)$, with the dual basis $\langle k \rangle (k \in \mathbb{Z}_N)$ of $\mathbb{C}^{N^*}$.

Let $Z, X, Y$ be a pair of generators of the Weyl algebra with the $N$th power identity: $ZX = \omega XZ$, $Z^N = X^N = I$, and denote by $Y := ZX$ the element in the Weyl algebra. One has the relations, $XY = \omega^{-1} YX$, $YZ = \omega^{-1} YZ$, and $Y^N = 1$. The canonical representation of the Weyl algebra is the (unique) $N$-dimensional irreducible representation, with the following matrix realization:

$$Z(v)_k = \omega^k v_k \quad X(v)_k = v_{k-1} \quad Y(v)_k = \omega^k v_{k-1}$$

for $v = (v_k) \in \mathbb{C}^N$.

By using the operators $X, Y, Z$ and the identity $I$, one can form a solution of the Yang-Baxter (YB) equation for a slightly modified $R$-matrix of the six-vertex model, appeared first in \([8]\), and then studied in great details in \([13]\). The $L$-operator, depending on a parameter $h = [a : b : c : d]$ in the projective 3-space $\mathbb{P}^3$, is given by the following $2 \times 2$ matrix with operator-valued entries acting on the quantum space $\mathbb{C}^N$:

$$L_h(x) = \begin{pmatrix} aY & xbX \\ xcZ & d \end{pmatrix}, \quad x \in \mathbb{C},$$

and it satisfies the following YB relation:

$$R(x/x')(L_h(x) \otimes_\text{aux} 1)(1 \otimes L_h(x')) = (1 \otimes L_h(x'))(L_h(x) \otimes_\text{aux} 1)R(x/x'),$$

where the script letter "aux" indicates an operation taking on the auxiliary space $\mathbb{C}^2$, and $R(x)$ is the matrix of 2-tensor of the auxiliary space with the following numerical expression,

$$R(x) = \begin{pmatrix} x - x^{-1} & 0 & 0 & 0 \\ 0 & \omega(x - x^{-1}) & \omega - 1 & 0 \\ 0 & \omega - 1 & x - x^{-1} & 0 \\ 0 & 0 & 0 & x - x^{-1} \end{pmatrix}.$$ 

The operator \([8]\) is related to the the following $L$-operator appeared in \([12]\) for the study of sine-Gordon lattice model by using the six-vertex model $R$-matrix and the Weyl operators $U, V$ ($UV = q^{-1} VU$, $U^N = V^N = 1$):

$$L^*_{h}(x) = \begin{pmatrix} aqU & xbV^{-1} \\ xcV & dU^{-1} \end{pmatrix}, \quad R(x) = \begin{pmatrix} x - x^{-1}q & 0 & 0 & 0 \\ 0 & x - x^{-1} & q - q^{-1} & 0 \\ 0 & q - q^{-1} & x - x^{-1} & 0 \\ 0 & 0 & 0 & xq - x^{-1}q^{-1} \end{pmatrix}$$

with the YB relation: $\text{R}(x/x')(L^*_{h}(x) \otimes_\text{aux} 1) \otimes (1 \otimes L^*_{h}(x')) = ((1 \otimes L^*_{h}(x') \otimes (L^*_{h}(x) \otimes_\text{aux} 1))\text{R}(x/x').$

Indeed, by the identification: $Z = VU$, $X = V^{-1}U$, (equivalently, $U = q^{\frac{1}{2}} Y^{-1} Y^2, V = q^{\frac{1}{2}} Z Y^{-1}$), $L_h(x)$ in \([8]\) and $L^*_{h}(x)$ in \([12]\) are related by

$$L^*_{h}(x) = L_h(x) Y^{-\frac{1}{2}} q^\frac{1}{2}.$$ 

By the matrix-product on auxiliary spaces and tensor-product of quantum spaces, the following $L$-operator of a finite size $L$, depending on the parameter $\vec{h} = (h_0, \ldots, h_{L-1}) \in (\mathbb{P}^3)^L$,

$$L_{\vec{h}}(x) = \bigotimes_{j=0}^{L-1} L_{h_j}(x) \quad := L_{h_0}(x) \otimes L_{h_1}(x) \otimes \ldots \otimes L_{h_{L-1}}(x),$$
again satisfies the YB relation \(^2\), hence its trace gives rise to a commuting family of transfer matrices:

\[
T_\tilde{h}(x) = \text{tr}_{aux}(L_\tilde{h}(x)) , \quad x \in \mathbb{C} .
\]

The same conclusion also holds for \(L^*_h(x)\), and \(T^*_h(x)\), defined by

\[
T^*_h(x) = \text{tr}_{aux}(L^*_h(x)) .
\]

By \(^4\), one has the following relation of the above two types of transfer matrices:

\[
T^*_h(x) = T_h(x)D^{-3} , \quad \text{where} \quad D := q^{-L} \otimes Y . \tag{5}
\]

For later use, we now summarize some basic facts on the Baxter’s \(T-Q\) relation and the Baxter vacuum state\(^\ddagger\) over the spectral curve in the diagonalization problem of transfer matrix \(T_h(x)\) (for the details, see \(^\ddagger\)). Applying the following gauge transform on \(L_{h_j}(x)\):

\[
\tilde{L}_{h_j}(x,\xi_j,\xi_{j+1}) = A_jL_{h_j}(x)A_j^{-1} , \quad \text{where} \quad A_j = \begin{pmatrix} 1 & \xi_j - 1 \\ 1 & \xi_j \end{pmatrix} , \quad (0 \leq j \leq L - 1) , \quad \text{and} \quad A_L := A_0 ,
\]

one has the expression:

\[
\tilde{L}_{h_j}(x,\xi_j,\xi_{j+1}) = \begin{pmatrix} F_{h_j}(x,\xi_j - 1,\xi_{j+1}) & -F_{h_j}(x,\xi_j - 1,\xi_{j+1} - 1) \\ F_{h_j}(x,\xi_j,\xi_{j+1}) & -F_{h_j}(x,\xi_j,\xi_{j+1} - 1) \end{pmatrix} .
\]

Here the operator \(F_{h_j}(x,\xi,\xi')\) is defined by \(F_{h_j}(x,\xi,\xi') := \xi'a - xbX + \xi'xcZ - \xi d\). Accordingly, for \(\tilde{h} \in (\mathbb{P}^3)^L\) and \(\tilde{\xi} = (\xi_0,\ldots,\xi_{L-1}) \in (\mathbb{C}^N)^L\), the modified \(L\)-operator becomes

\[
L_{\tilde{h}}(x,\tilde{\xi}) := \bigotimes_{j=0}^{L-1} L_{h_j}(x,\xi_j,\xi_{j+1}) = \begin{pmatrix} \tilde{L}_{\tilde{h}_{11}}(x,\tilde{\xi}) & \tilde{L}_{\tilde{h}_{12}}(x,\tilde{\xi}) \\ \tilde{L}_{\tilde{h}_{21}}(x,\tilde{\xi}) & \tilde{L}_{\tilde{h}_{22}}(x,\tilde{\xi}) \end{pmatrix} , \quad \text{and} \quad \xi_L := \xi_0 .
\]

As the gauge-transform procedure keeps the trace unchanged, we have the relation: \(T_{\tilde{h}}(x) = \text{tr}_{aux}(\tilde{L}_{\tilde{h}}(x,\tilde{\xi}))\). For a given \(\tilde{h}\), we will consider the variable \((x,\tilde{\xi})\) in the curve \(C_{\tilde{h}}\) defined by

\[
C_{\tilde{h}} : \quad \xi_j^N = (-1)^N m_j \xi_{j+1}^{N-1} \alpha_j - x_N b_j N \frac{\xi_{j+1}^{N-1} \alpha_j^N - b_j^N}{\xi_{j+1}^{N-1} \alpha_j^N - d_j^N} , \quad j = 0,\ldots, L - 1 . \tag{6}
\]

We shall call \(C_{\tilde{h}}\) the spectral curve in this paper. Over \(C_{\tilde{h}}\), we have the Baxter vacuum state defined by the following family of vectors in \(\bigotimes_{j=0}^L \mathbb{C}^N\):

\[
|p\rangle := |p_0\rangle \otimes \cdots \otimes |p_{L-1}\rangle \in \bigotimes_{j=0}^L \mathbb{C}^N \quad (p \in C_{\tilde{h}}) ,
\]

where \(|p_j\rangle\) is the vector in \(\mathbb{C}^N\) governed by the relations:

\[
\langle 0 | p_j \rangle = 1 , \quad \frac{\langle m | p_j \rangle}{\langle m - 1 | p_j \rangle} = \frac{\xi_j^{m+1} \alpha_j \omega^m - xb_j}{-\xi_j \xi_{j+1} \alpha_j \omega^m - d_j} . \tag{7}
\]

\(^\ddagger\)The Baxter’s \(T-Q\) relation and the Baxter vacuum state here were called by the Bethe equation, the Baxter vector respectively in \(^\ddagger\).
The constraint of the variables \((x, \vec{\xi})\) in \(C_{\vec{h}}\) ensures that the following properties hold for the Baxter vacuum state,

\[
\bar{L}_{\vec{h};11}(x, \vec{\xi})|p\rangle = |\tau_- p\rangle \Delta_-(p), \quad \bar{L}_{\vec{h};22}(x, \vec{\xi})|p\rangle = |\tau_+ p\rangle \Delta_+(p), \quad \bar{L}_{\vec{h};21}(x, \vec{\xi})|p\rangle = 0,
\]
where \(\Delta_\pm, \tau_\pm\) are (rational) functions and automorphisms of \(C_{\vec{h}}\), defined by

\[
\Delta_-(x, \xi_0, \ldots, \xi_{L-1}) = \prod_{j=0}^{L-1} (d_j - x \xi_{j+1} c_j),
\]

\[
\Delta_+(x, \xi_0, \ldots, \xi_{L-1}) = \prod_{j=0}^{L-1} \frac{\xi_j(a_j d_j - x^2 b_j c_j)}{\xi_{j+1} a_j - x b_j}, \quad \tau_\pm : (x, \xi_0, \ldots, \xi_{L-1}) \mapsto (q^{\pm 1} x, q^{-1} \xi_0, \ldots, q^{-1} \xi_{L-1}).
\]

This implies that, under the action of the transfer matrix, the Baxter vacuum state is the sum of its \(\tau_\pm\)-translations:

\[
T_{\vec{h}}(x)|p\rangle = |\tau_- p\rangle \Delta_-(p) + |\tau_+ p\rangle \Delta_+(p), \quad \text{for } p \in C_{\vec{h}}.
\]

For a common eigenvector \(|\varphi\rangle \in \bigotimes_{j=0}^L C^{N_j}\) of \(T_{\vec{h}}(x)\) \((x \in C)\), its eigenvalue \(\Lambda(x)\) is a polynomial of \(x\), i.e., \(\Lambda(x) \in C[x]\); while its values on the Baxter’s vacuum state, \(Q(p) := \langle \varphi|p\rangle\) \((p \in C_{\vec{h}})\), is a rational function on \(C_{\vec{h}}\). Then the following relation hold for \(\Lambda(x)\) and \(Q(p)\):

\[
\Lambda(x)Q(p) = \Delta_-(p)Q(\tau_- p) + \Delta_+(p)Q(\tau_+ p), \quad \text{for } p \in C_{\vec{h}},
\]

which will be called the Baxter’s \(T\)-\(Q\) relation for \(T_{\vec{h}}(x)\). Note that the \(|\varphi\rangle\) is also a common eigenvector of \(T_{\vec{h}}^n(x)\) with the eigenvalue \(\Lambda^*(x) = q^n \Lambda(x)\), where \(q^n\) is the \(D^{\pm 1}\)-eigenvalue of \(|\varphi\rangle\).

The Baxter’s \(T\)-\(Q\) relation for \(T_{\vec{h}}(x)\) becomes

\[
\Lambda^*(x)Q(p) = \Delta^*(p)Q(\tau_- p) + \Delta^+_*(p)Q(\tau_+ p), \quad \text{for } p \in C_{\vec{h}},
\]

with \(\Lambda^*(x) = q^n \Lambda(x)\), \(\Delta^*_-(p) = q^n \Delta_-(p)\), and \(\Delta^*_+(p) = q^n \Delta_+(p)\) for \(n \in \mathbb{Z}_N\). For \(L=4\), one has

\[
T_{\vec{h}}(x) = T_0 + x^2 T_2 + x^4 T_4, \quad T_{\vec{h}}^*(x) = T_0^* + x^2 T_2^* + x^4 T_4^*, \quad \text{for } p \in C_{\vec{h}}.
\]

where \(T_{2j}\) are operators of \(\bigotimes_{j=0}^4 C^{N_j}\) with the expressions:

\[
T_0 = a_0 a_1 a_2 a_3 \omega^2 D + d_0 d_1 d_2 d_3, \quad T_2 = a_0 a_1 b_2 c_3 Y \otimes Y \otimes X \otimes Z + b_0 c_1 a_2 a_3 X \otimes Z \otimes Y \otimes Y + a_0 b_1 c_2 a_3 Y \otimes X \otimes Z \otimes Y + a_0 b_1 d_2 c_3 Y \otimes X \otimes 1 \otimes Z + b_0 d_1 c_2 a_3 X \otimes 1 \otimes Z \otimes Y + b_0 d_1 d_2 c_3 X \otimes 1 \otimes 1 \otimes Z + (a_2 Y \leftrightarrow d_j, b_j X \leftrightarrow c_j Z), \quad T_4 = b_0 c_1 b_2 c_3 D C^{-1} + c_0 b_1 c_2 b_3 C, \quad C := Z \otimes X \otimes Z \otimes X.
\]

One can also obtain the expressions of \(T_{2j}^*\) from \(\bar{T}_{2j}\). Note that the operators \(C, D, T_2, T_4\) commute each other.

For the study of discrete quantum pendulum and SG model in this paper, we shall restrict ourselves on the case \(L=4\) with the following constraint in \(\vec{h}\),

\[
a_j d_j = q^{-1}, \quad b_j c_j = -k^{-1}, \quad \text{for } j = 0, 2, \quad a_j d_j = q^{-1}, \quad b_j c_j = -k, \quad \text{for } j = 1, 3.
\]

\(^2\)The convention we use here is in tune with the one in \(\bar{\vec{h}}\).
where \( k \) is a complex parameter. Then the operators \( T_{2j}, T_{2j}^* \) in (12) now take the following forms:

\[
T_0 = \frac{1}{d_0d_1d_3} D + d_0d_1d_2 d_3 , \quad T_4 = \frac{c_{1c_3}}{k^2c_0c_2} DC^{-1} + \frac{k^2c_0c_2}{c_{1c_3}} C , \\
-T_2 = - \frac{k c_{0}d_1d_2 d_3}{d_0d_1d_3} U_1 + \frac{d_0d_1d_2 d_3}{k c_{0}} U_2 + \frac{kd_0d_1d_3}{d_0d_1d_3} U_3 + \frac{d_0d_1d_2 d_3}{k c_{0}} U_4 + \frac{kd_0d_1d_3}{d_0d_1d_3} U_5 - \frac{k d_0d_1d_3}{d_0d_1d_3} U_2^{-1} \\
+ \frac{c_{0}}{kd_0d_1d_3} DU_3^{-1} + \frac{k c_0}{kd_0d_1d_3} DU_4^{-1} + \frac{k d_{0}c_{0}}{k c_{0}d_{0}c_{3}} DV_1^{-1} + \frac{d_0d_1}{kd_0d_1d_3} V_4 + \frac{q d_0d_1}{k d_0d_1d_3} D V_4^{-1} ; \\
\]

\[
T_{2}^* = - \frac{k c_{0}d_1d_2 d_3}{d_0d_1d_3} D_z^1 + d_0d_1d_2 d_3 D_z^1 , \quad T_4^* = \frac{c_{1c_3}}{k^2c_0c_2} D_z^1 C^{-1} + \frac{k^2c_0c_2}{c_{1c_3}} D_z^1 C , \\
-T_2^* = \frac{k c_{0}d_1d_2 d_3}{d_0d_1d_3} D_z^1 U_1 + \frac{d_0d_1d_2 d_3}{k c_{0}} D_z^1 U_2 + \frac{kd_0d_1d_3}{d_0d_1d_3} D_z^1 U_3 + \frac{d_0d_1d_2 d_3}{k c_{0}} D_z^1 U_4 + \frac{kd_0d_1d_3}{d_0d_1d_3} D_z^1 U_5 - \frac{k d_0d_1d_3}{d_0d_1d_3} U_2^{-1} \\
+ \frac{c_{0}}{kd_0d_1d_3} D_z^1 U_3^{-1} + \frac{k c_0}{kd_0d_1d_3} D_z^1 U_4^{-1} + \frac{k d_{0}c_{0}}{k c_{0}d_{0}c_{3}} DV_1^{-1} + \frac{d_0d_1}{kd_0d_1d_3} V_4 + \frac{q d_0d_1}{k d_0d_1d_3} D V_4^{-1} ,
\]

where \( U_j, V_j \) are operators defined by

\[
U_1 = Z \otimes X \otimes 1 \otimes 1 , \quad U_2 = 1 \otimes Z \otimes X \otimes 1 , \quad U_3 = 1 \otimes 1 \otimes Z \otimes X , \quad U_4 = X \otimes 1 \otimes 1 \otimes Z , \\
V_1 = 1 \otimes Z \otimes Y \otimes X , \quad V_2 = X \otimes 1 \otimes Z \otimes Y , \quad V_3 = Y \otimes X \otimes 1 \otimes Z , \quad V_4 = Z \otimes Y \otimes X \otimes 1 .
\]

It is easy to see that the following relations hold among the above operators,

\[
U_{j+1} U_j = \omega U_j U_{j+1} , \quad V_{j+1} V_j = \omega^2 V_j V_{j+1} , \quad (U_5 := U_1, V_5 := V_1), \\
U_1 U_3 = C , \quad U_2 U_4 = C^{-1} D , \quad V_1 V_3 = V_2 V_4 = \omega D .
\]

Under the constraint (13), the functions \( \Delta_{\pm} \) in (8) become

\[
\Delta_(x, \xi_0, \ldots, \xi_3) = \Pi_{j=0}^3 (d_j - x\xi_{j+1} c_j) , \quad \Delta_+(x, \xi_0, \ldots, \xi_3) = \prod_{j=0}^3 ((x\xi_{j} - d_{j}c_{j}) - (1+x^2\xi_{j} - 4q^{-1} + x^2\xi_{j} - 4q^{-1})) .
\]

By (7), the Baxter vacuum state \(| p \rangle = \otimes_{j=0}^3 | p_j \rangle \) have the following expression,

\[
| m \rangle | p_j \rangle = \begin{cases} 
\frac{\epsilon_m \epsilon_j d_m (x\xi_{j} + c_{j}) c_{j}^m (x\xi_{j} - c_{j}) c_{j}^m (q^{-1}) \omega^{-1}}{\epsilon_{j} d_{j} (x\xi_{j} + c_{j}) c_{j}^m (x\xi_{j} - c_{j}) c_{j}^m (q^{-1}) \omega^{-1}} & \text{for even } j, \\
\frac{\epsilon_m \epsilon_j d_m (x\xi_{j} - c_{j}) c_{j}^m (x\xi_{j} + c_{j}) c_{j}^m (q^{-1}) \omega^{-1}}{\epsilon_{j} d_{j} (x\xi_{j} - c_{j}) c_{j}^m (x\xi_{j} + c_{j}) c_{j}^m (q^{-1}) \omega^{-1}} & \text{for odd } j .
\end{cases}
\]

In this paper we shall mainly study the diagonalization problem of \( T_{2j}^* \), or equivalently \( T_{2j} \), under the condition (13), plus the following further constraints on parameters \( d_j, c_j \) and the operator \( C \):

\[
d_0d_1d_2d_3 = 1 , \quad c_3^N c_3^N = k^{2N} c_0N c_2 N , \quad C = \frac{c_{1c_3}}{k^2c_0c_2} .
\]

The above properties on the parameters arise from the connection of \( T_{2j}^* \) with the following physical models.

(1) Discrete quantum pendulum. This is the situation under the constraint (16), and with the further identifications:

\[
D = 1 , \quad d_0c_1d_3 d_2 d_3 U_2 = \frac{k c_{0}}{d_0d_1d_3} U_2^{-1} (=: Q_{n-1}) , \quad \frac{c_1}{k c_{0}d_2d_3} U_2^{-1} = \frac{kd_0d_1d_3}{c_3} U_3 (=: Q_{n}) .
\]

By (15), one has

\[
Q_{n-1}Q_n = \frac{d_0d_1c_3}{d_2d_3} \omega^{-1} V_1 = \frac{d_0d_1}{d_2d_3} V_2^{-1} , \quad Q_{n-1}Q_n^{-1} = \frac{c_0d_3}{c_2d_1} V_4 = \frac{c_0d_3}{c_2d_1} \omega V_2^{-1} .
\]
Then $T_{2j}^*$ in (14) become
\[
T_{0}^* = T_4^* = 2, \\
-T_{2}^* = 2(Q_n + Q_{n-1}^{-1} + Q_{n-1} + Q_{n-1}^{-1}) + k(qQ_{n-1} + q^{-1}Q_{n-1}^{-1}) + k^{-1}(qQ_{n-1} + q^{-1}Q_{n-1}Q_{n-1}^{-1}).
\]

The above $-T_{2}^*$ is the Hamiltonian of discrete quantum pendulum in [1], subject to the following evolution equation:
\[
Q_{n+1}Q_{n-1} = \left(\frac{k + qQ_{n}}{1 + qkQ_{n}}\right)^2, \quad Q_nQ_{n-1} = q^2Q_{n-1}Q_{n}.
\]

(II) Discrete sine-Gordon (SG) Hamiltonian. This is the situation under the constraint (16) with one further identification:
\[
\frac{c_1d_3d_4}{kc_2}D^{\frac{1}{2}}U_2 = \frac{kc_0}{c_3d_1d_2}D^{\frac{1}{2}}U_4^{-1}.
\]

In this case, we have
\[
T_{0}^* = D^{\frac{1}{2}} + D^{\frac{1}{2}}, \quad T_{3}^* = D^{\frac{1}{2}} + D^{\frac{1}{2}},
\]
\[
-T_{2}^* = \frac{kc_0d_4d_5}{c_1}D^{\frac{1}{2}}U_1 + \frac{c_1d_3d_4}{kc_2}D^{\frac{1}{2}}U_2 + \frac{kc_0d_4d_5}{c_1d_3d_2}D^{\frac{1}{2}}U_3 + \frac{c_1d_3d_4}{kc_2}D^{\frac{1}{2}}U_4 + \frac{c_1}{kc_0d_4d_5}D^{\frac{1}{2}}U_1^{-1} + \frac{c_1}{kc_0d_4d_5}D^{\frac{1}{2}}U_4^{-1} + \frac{c_3}{kc_0d_4d_5}D^{\frac{1}{2}}V_1 + \frac{c_3}{kc_0d_4d_5}D^{\frac{1}{2}}V_4^{-1}.
\]

The above $-T_{2}^*$ can be identified with the discrete quantum sine-Gordon integral in [1], for which a detailed description will be given in the appendix of this paper.

3 The Baxter’s T-Q Polynomial Equation for Rational Degenerated Spectral Curve

In this section, we derive the Baxter’s $T$-$Q$ polynomial relation for a size $L$ when the spectral curve $C_{\bar{L}}$ is degenerated into rational curves, i.e., $C_{\bar{L}}$ is a disjoint union of finite copies of the base $x$-curve. We shall reduce the general degenerated situation to one special case, which we have already discussed in our previous article [3].

By the rational degenerated spectral curves, we mean the coordinates $\xi_j^N$ of $C_{\bar{L}}$ to be constants, i.e., independent of the variable $x$ for all $j$. Then the parameters $h_j$ and the variables $\xi_j$ are subject to the relations:
\[
\frac{b_j^N d_j^N}{a_j^N c_j^N} = \frac{a_j^{N+1} b_j^{N+1}}{c_j^{N+1} d_j^{N+1}}, \quad \xi_j^N = \frac{a_j^N b_j^N}{c_j^N d_j^N} \quad \text{for} \quad 0 \leq j \leq L - 1.
\]

In this situation, we define
\[
r_j = \sqrt{\frac{b_{j-1} d_{j-1}}{a_{j-1} c_{j-1}}}, \quad j \in \mathbb{Z}_L.
\]

Then $C_{\bar{L}}$ contains the following $\tau_{\pm}$-invariant curve $C$, over which we shall formulate the Baxter’s $T$-$Q$ equation,
\[
C := \{(x, \xi_0, \ldots, \xi_{L-1}) \mid r_0^{-1} \xi_0 = \ldots = r_{L-1}^{-1} \xi_{L-1} = q^l, \quad l \in \mathbb{Z}_N\}.
\]
We shall make the identification of $C$ with $P^1 \times Z_N$ via the following correspondence:

$$C = P^1 \times Z_N, \quad (x, r_0 q^l, \ldots, r_{L-1} q^l) \longleftrightarrow (x, l).$$

Then the automorphisms $T$ and the action $T(x)$ are expressed by

$$\tau_\pm : (x, l) \mapsto (q\pm 1 x, l - 1),$$

and the action $T(x) := T_h(x)$ on $|x, l \rangle$ in (11) now takes the form:

$$T(x)|x, l \rangle = |q^{-1} x, l - 1 \rangle \Delta_-(x, l) + |q x, l - 1 \rangle \Delta_+(x, l), \quad (20)$$

where $\Delta_{\pm}$ are the following rational functions of $x$:

$$\Delta_-(x, l) = (d_0 \cdots d_{L-1}) \prod_{j=0}^{L-1} (1 - x q^j d_j^{-1} c_j r_{j+1}),$$

$$\Delta_+(x, l) = (d_0 \cdots d_{L-1}) \prod_{j=0}^{L-1} \frac{1 - x^2 a_j^{-1} d_j^{-1} b_j c_j}{1 - x q^{-1} a_j^{-1} b_j r_{j+1}}.$$

With the substitutions,

$$(d_0 \cdots d_{L-1})^{-1} T(x) \mapsto T(x), \quad (d_0 \cdots d_{L-1})^{-1} \Delta_{\pm}(x, l) \mapsto \Delta_{\pm}(x, l),$$

the relation (20) still holds for the modified $\Delta_{\pm}$, now with the expressions:

$$\Delta_-(x, l) = \prod_{j=0}^{L-1} (1 - x c_j^* q^j), \quad \Delta_+(x, l) = \prod_{j=0}^{L-1} \frac{1 - x^2 c_j^{*2}}{1 - x c_j^* q^{-1}},$$

where

$$c_j^* := d_j^{-1} c_j r_{j+1} = (a_j^{-1} b_j r_{j+1}^{-1}). \quad (21)$$

Furthermore, one can convert the expression (2) of the Baxter vacuum state over $C$ to the following component-expression of the Baxter’s vector $|x, l \rangle$:

$$\langle k | x, l \rangle = q^{\frac{|k|^2}{2}} \prod_{j=0}^{L-1} \left( \frac{x c_j^* q^{-l-1}; \omega^{-1}}{x c_j^* q^l; \omega} \right)_{k_j}. \quad (22)$$

Here the bold letter $k$ denotes a multi-index vector $k = (k_0, \ldots, k_{L-1})$ for $k_j \in Z_N$, and the square-length of $k$ is defined by $|k|^2 := \sum_{j=0}^{L-1} k_j^2$. Each ratio-term in the above right hand side is given by a non-negative representative for each element in $Z_N$ appeared in the formula. With the above description of $T(x)$ on the Baxter vacuum state $|x, l \rangle$, the discussions of Sect. 4 and Sect. 5 Proposition 2, 3 in [13] can be applied to our present situation. This enables us to state the following result on the Baxter’s $T$-Q equation and its connection with the transfer matrix $T(x)$:

**Theorem 1** Let $f^e, f^o$ be functions on $C$ defined by

$$f^e(x, 2n) = \prod_{j=0}^{L-1} \frac{(x c_j^*; \omega^{-1})_{n+1}}{(x c_j^*; \omega)_{n+1}}, \quad f^o(x, 2n + 1) = \prod_{j=0}^{L-1} \frac{(x c_j^* q^{-1}; \omega^{-1})_{n+1}}{(x c_j^* q; \omega)_{n+1}}.$$
For $x \in \mathbb{P}^1$ and $l \in \mathbb{Z}_N$, we define the following vectors in $\otimes \mathbb{C}^N$, 
\[ |x\rangle^i_t = \sum_{n=0}^{N-1} |x, 2n\rangle f^i(x, 2n)\omega^{jn}, \quad |x\rangle_0^i = \sum_{n=0}^{N-1} |x, 2n+1\rangle f^o(x, 2n+1)\omega^{jn}, \]
\[ |x\rangle^+_t = |x\rangle^i_t q^{-l}u(qx) + |x\rangle^i_t u(x) \quad \text{where} \quad u(x) := \prod_{j=0}^{L-1} (1 - x^N c_j^*) (xc_j^* q^q) M. \]

Then
(i) $|x\rangle^+_t u(qx) = |x\rangle^i_t q^l u(x)$, or equivalently, $|x\rangle^+_t = 2q^{-l}|x\rangle^i_t u(qx) = 2|x\rangle^i_t u(x)$.

(ii) The $T(x)$-transform on $|x\rangle^+_t$ is given by
\[ q^{-l}T(x)|x\rangle^+_t = |q^{-l}x\rangle^+_t \Delta_-(x, -1) + |qx\rangle^+_t \Delta_+(x, 0), \quad l \in \mathbb{Z}_N. \]

(iii) For a common eigenvector $|\varphi\rangle$ of $T(x)$ with the eigenvalue $\Lambda(x)$, the function $Q^+_t(x) := \langle \varphi | x\rangle^+_t$ and $\Lambda(x)$ are polynomials with the properties:
\[ \deg Q^+_t(x) \leq (3M + 1)L, \quad \deg \Lambda(x) \leq 2\left\lfloor \frac{L}{2} \right\rfloor, \quad \Lambda(x) = \Lambda(-x), \quad \Lambda(0) = q^{2l} + 1, \]
and they satisfy the following Baxter’s $T$-$Q$ equation:
\[ q^{-l}\Lambda(x)Q^+_t(x) = \prod_{j=0}^{L-1} (1 - x^N c_j^N)Q^+_t(xq^{-1}) + \prod_{j=0}^{L-1} (1 + x^N c_j^*)Q^+_t(xq). \tag{22} \]

Furthermore, for $0 \leq m \leq M$, $Q^+_{m}(x)$ and $Q^+_N(x)$ are elements in $x^m \prod_{j=0}^{L-1} (1 - x^N c_j^N) \mathbb{C}[x]$.

$\square$

For the rest of this paper, the letter $m$ will always denote an integer between 0 and $M$,
\[ 0 \leq m \leq M. \]

By (iii) of the Theorem, the equation (22) for the sectors $m, N - m$ can be combined into a single one by introducing the polynomials $\Lambda_m(x), Q(x)$ via the relation,
\[ (\Lambda_m(x), x^m \prod_{j=0}^{L-1} (1 - x^N c_j^N)Q(x)) = (q^{-m}\Lambda(x), Q^+_m(x)), \quad (q^m\Lambda(x), Q^+_N(x)). \]

Then the equations (22) for $l = m, N - m$ are equivalent to the following polynomial equation of $Q(x), \Lambda_m(x)$:
\[ \Lambda_m(x)Q(x) = q^{-m} \prod_{j=0}^{L-1} (1 - x^N c_j^N)Q(xq^{-1}) + q^m \prod_{j=0}^{L-1} (1 + x^N c_j^*)Q(xq), \tag{23} \]
with the following constraints of $Q(x)$ and $\Lambda_m(x)$,
\[ \deg Q(x) \leq ML - m, \quad \deg \Lambda_m(x) \leq 2\left\lfloor \frac{L}{2} \right\rfloor, \quad \Lambda_m(x) = \Lambda_m(-x), \quad \Lambda_m(0) = q^m + q^{-m}. \]

By (ii), the above $\Lambda_m(x)$ is indeed the eigenvalue of $T^*(x)$; while (23) corresponds the Baxter’s $T$-$Q$ equation (11) for $T^*(x)$ on the sectors $m, N - m$. 

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For $L$ even, by the construction of $T^*_h(x)$, one can see that the eigenvalues of $T^*_L$ are non-zero, hence $\text{deg } \Lambda_m(x) = L$. In the situations we will consider later on, the polynomial solutions, $Q(x), \Lambda_m(x)$ of the equation (23) possesses certain reciprocal symmetry property. Here we call a polynomial $P(x)$ to be reciprocal if $P^\dagger(x) = P(x)$, where $P^\dagger(x)$ is the polynomial defined by

$$P^\dagger(x) := x^{\text{deg } P} P(x^{-1}) .$$

**Proposition 1** For $L$ even, assume that the polynomial $\Lambda_m(x)$ in (22) is reciprocal, and the parameters $c_j^s$ and the degree $d$ of $Q(x)$ satisfy the following properties:

(i) $\{c_0^s, \ldots, c_{L-1}^s\} = \{-c_0^{s-1}, \ldots, -c_{L-1}^{s-1}\}$ .

(ii) $\prod_{j=0}^{L-1} c_j^s = q^{\frac{L}{2}}$, $q^{d+2m+\frac{L}{2}} = 1$.

Then $Q^\dagger(x)$ is also a solution of (22) for $\Lambda_m(x)$.

**Proof.** By substituting $x$ by $x^{-1}$ in (22), and then multiplying $x^{d+L}$ to the equation, one obtains the relation,

$$\Lambda_m(x)Q^\dagger(x) = q^{m+d} \prod_{j=0}^{L-1} c_j^s \prod_{j=0}^{L-1} (1 + xc_j^{s-1}) Q^\dagger(xq^{-1}) + q^{-m-d-L} \prod_{j=0}^{L-1} c_j^s \prod_{j=0}^{L-1} (1 - xc_j^{s-1}q) Q^\dagger(xq) ,$$

By (ii), we have

$$q^{m+d} \prod_{j=0}^{L-1} c_j^s = q^{-m} , \quad q^{-m-d-L} \prod_{j=0}^{L-1} c_j^s = q^m .$$

Then, by (i), the above equation of $Q^\dagger(x)$ is the same as (23). $\square$

The following algebraic fact was shown in [13] Lemma 6, which we just state here for later use.

**Lemma 1** Let $n$ be an odd positive integer, $A$ be a $n \times n$-matrix with complex entries $a_{i,j}$ satisfying the relations

$$a_{i,j} = (-1)^{i+j+1} a_{n-j+1,n-i+1} , \quad \text{for } 1 \leq i, j \leq n .$$

Then $A$ is a degenerated matrix.

$\square$

### 4 The Baxter’s T-Q Polynomial Relation for L=4

For $L=4$ rational degenerated case, the parameter $\vec{h}$ we discuss later in this paper will subject to the constraint (13), and be confined only to the following situation:

$$qa_j = d_j = 1 , \quad -b_j = c_j = \begin{cases} k^{\frac{1}{2}} & \text{for even } j , \\ k^{\frac{1}{2}} & \text{for odd } j . \end{cases}$$

(24)

Then, by (13) and (21) we have $r_j = (-q)^{\frac{1}{2}}$ for all $j$ , and

$$c_j^s = \begin{cases} (-q)^{\frac{1}{2}} k^{\frac{1}{2}} & \text{for even } j , \\ (-q)^{\frac{1}{2}} k^{\frac{1}{2}} & \text{for odd } j . \end{cases}$$

(25)

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The operators \( T^+_h(x) \) in \( T^+_h(x) \) become

\[
T^+_0 = D^\frac{3}{2} + D^{-\frac{3}{2}}, \quad T^+_1 = D^\frac{1}{2}C^{-1} + D^{-\frac{1}{2}}C, \\
-T^+_2 = D^\frac{3}{2}U_1 + D^{-\frac{3}{2}}U_2 + D^\frac{1}{2}U_3 + D^{-\frac{1}{2}}U_4 + \frac{kq}{\lambda}D^\frac{1}{2}V_1 + \frac{k^{-1}q^{-1}}{\lambda}D^{-\frac{1}{2}}V_4 \\
+ D^\frac{3}{2}U_1' + D^{-\frac{3}{2}}U_2' + D^\frac{1}{2}U_3' + D^{-\frac{1}{2}}U_4' + \frac{kq}{\lambda}D^\frac{1}{2}V_1' + \frac{k^{-1}q^{-1}}{\lambda}D^{-\frac{1}{2}}V_4'.
\]

By \( C^N = 1 \), the polynomial \( \Lambda_m(x) \) in (23) now takes the form:

\[
\Lambda_m(x) = (q^{m+l} + q^{-m-l})x^4 + \lambda x^2 + q^m + q^{-m}, \quad 0 \leq m \leq M, \quad 0 \leq l \leq 2M,
\]

where \( \lambda \) is an eigenvalue of \( -T^+_2 \) for the sectors \((m,l),(N-m,l)\), which can be regarded as the label of the eigenvalues of \( D^\frac{3}{2} \) and \( C^{-\frac{1}{2}} \). The Baxter’s T-Q polynomial equation (23) now takes the form,

\[
\Lambda_m(x) Q(x) = q^{-m} \Delta(x(-q)^{\frac{1}{2}})Q(xq^{-1}) + q^m \Delta(x(-q)^{-\frac{1}{2}})Q(xq),
\]

with

\[
\Delta(x) = (1 + 2cx + x^2)^2, \quad c := \frac{1}{2}(k^\frac{1}{2} + k^{-\frac{1}{2}}),
\]

and \( \text{deg } Q(x) \leq 4M - m \). The polynomials \( \Lambda_m(x), Q(x) \) will be called the eigenvalue and the eigen-polynomial of (23) respectively, whenever \( Q(x) \) is a non-trivial function. Indeed, the value \( \lambda \) in the expression of \( \Lambda_m(x) \) is an eigenvalue of \( -T^+_2 \). In the following, we are going to study the equation (27) for a generic \( \lambda \) in (26).

For the rest of this paper, the parameter \( c \) will always be a generic complex number unless otherwise stated. The polynomial \( Q(x) \) will be denoted by

\[
Q(x) = \sum_{j=0}^{d} \alpha_j x^j, \quad d := \text{deg } Q(x),
\]

and we define \( \alpha_j := 0 \) for \( j \) not between 0 and \( d \). An equivalent formulation of the equation (27) is the following system of difference equations in \( \lambda \) and \( \alpha_j \)’s:

\[
\nu_j \alpha_j + \nu_j \alpha_{j-1} + (\delta_j - \lambda) \alpha_{j-2} + u_j \alpha_{j-3} + \mu_j \alpha_{j-4} = 0, \quad (j \geq 1),
\]

where the coefficients in the above equations are defined by

\[
\nu_j = q^{m+j} + q^{-m-j} - q^m - q^{-m}, \quad \nu_j = 4ci(q^{m+j-\frac{1}{2}} - q^{-m-j+\frac{1}{2}}), \\
\delta_j = -(4c^2 + 2)(q^{m+j-1} + q^{-m-j+1}), \quad \delta_j = -4ci(q^{m+j-\frac{3}{2}} - q^{-m-j+\frac{3}{2}}), \\
u_j = 4ci(q^{m+j-\frac{1}{2}} - q^{-m-j+\frac{1}{2}}), \quad \mu_j = q^{m+j-2} + q^{-m-j+2} - q^{m+l} - q^{-m-l}.
\]

Indeed, for the system (28), it suffices to consider those relations for the index \( j \) between 1 and \( d+3 \). Note that the relations in (28) for \( 2 \leq j \leq d+2 \) give rise to the following eigenvalue problem:
Hence for a solution of the system (28), the $\lambda$ can be regarded as an algebraic function of $c$, and it has a limit as $c$ tends to some special value $c_0$.

**Lemma 2** For equation (27) with a given $c$ (no generic property required), the degree $d$ of $Q(x)$ satisfies the following conditions,

$$1 \leq d \leq 4M - m, \quad q^{d+2} = q^l \text{ or } q^{-2m-l},$$

and the zero-multiplicity of $Q(x)$ at the origin is equal to one of $0, N, N - 2m, 2N - 2m$.

**Proof.** The upper bound $4M - m$ of $d$ is given by the assumption of (27). If $d = 0$, then a non-zero constant is a solution $Q(x)$ of (27), and we have

$$\Lambda_{m,l}(x) = q^{-m}\Delta(x(-q)^{-1}) + q^{m}\Delta(x(-q)^{1/2}).$$

By the even-function property of $\Lambda_{m,l}(x)$, the above relation implies $q^{2m+1} = q^{2m+3} = 1$, hence $q^2 = 1$, a contradiction to the odd assumption on the integer $N$. Therefore $d \geq 1$. Comparing the coefficients of the highest degree of $x$ in (27), one has

$$q^{m+l} + q^{-m-l} = q^{-m-2-d} + q^{m+2+d}.$$ 

This implies $q^{m+2+d} = q^{m+l}$ or $q^{-m-l}$, i.e., $q^{d+2} = q^l$ or $q^{-2m-l}$. Denote $r$ the zero-multiplicity of $Q(x)$ at $x = 0$. By comparing the coefficients of degree $r$ in (27), we have

$$q^m + q^{-m} = q^{-m-r} + q^{m+r},$$

hence $r \equiv 0, -2m \pmod{N}$. Then the conclusion of $r$ follows from $d \leq 4M - m$. \(\square\)

**Remark.** For the results obtained later in this paper on certain special cases, and also on the similar problem of size $L = 3$ in [13], the solution $Q(x)$ in (23) always possesses the property $Q(0) \neq 0$. In this situation, one can write

$$Q(x) = \prod_{j=1}^{d}(x - \frac{1}{z_j}), \quad z_j \neq 0.$$ 

Substituting $x = \frac{1}{z_j}$ in (27), one obtains the following relations of $z_j$s:

$$q^{2m+2+d}(\frac{z_j^2 + 2icq^{1/2}z_j - q}{qz_j^2 - 2icq^{1/2}z_j - 1})^2 = \prod_{n \neq j, n=1}^{d} \frac{z_n - qz_j}{qz_n - z_j}, \quad j = 1, \ldots, d, \quad (31)$$

which is the Bethe ansatz equation appeared in literature, e.g. [8]. \(\square\)

A special case in the above setting happens when $\Lambda_{m,l}(x)$ in (27) is a reciprocal polynomial. For the convenience, the Baxter’s $T$-$Q$ relation (27) will be called a symmetric $T$-$Q$ polynomial relation if the following condition holds:

$$\Lambda_{m,l}(x) = \Lambda_{m,l}(x), \quad \text{equivalently} \quad q^l = 1, \quad q^{-2m}, \quad \text{i.e.,} \quad l \equiv 0, N - 2m \pmod{N}.$$ 

In this situation, (26) becomes

$$\Lambda_{m,l}(x) = q^m + q^{-m} + \lambda x^2 + (q^m + q^{-m})x^4, \quad (32)$$
and the coefficients (29) in the system (28) have the following form,

\[
\begin{align*}
\nu_j &= q^{m+j} + q^{-m-j} - q^m - q^{-m}, \\
\delta_j &= -(4c^2 + 2)(q^{m+j-1} + q^{-m-j+1}), \\
u_j &= 4ci(q^{m+j-\frac{i}{2}} - q^{-m-j+\frac{i}{2}}), \\
\mu_j &= q^{m+j-2} + q^{-m-j+2} - q^m - q^{-m}.
\end{align*}
\]

(33)

Note that by the equalities, \(u_{j+1} = -v_j\) and \(\mu_{j+2} = \nu_j\), the transport of the square matrix in (30) is unchanged after substituting \(c\) by \(-c\). Hence the eigenvalue \(\lambda\) for the symmetric T-Q polynomial relation necessarily becomes an algebraic function of \(c^2\), equivalently, the following property holds for \(\lambda\):

\[
\lambda = \lambda(c) = \lambda(-c)\.
\]

(34)

Furthermore, the relation (27) is unchanged when substituting \((c, x)\) by \((-c, -x)\); this implies that if \(Q(x; c)\) is a solution of (27), so is \(Q(-x; -c)\).

We now determine the qualitative nature of a solution \(Q(x)\) for the symmetric polynomial T-Q equation.

**Lemma 3** For a symmetric T-Q polynomial relation (27), there is no non-trivial solution \(Q(x)\) of degree \(d = N - 2\) with the zero-multiplicity at \(x = 0\) equal to \(N - 2m\).

**Proof.** Otherwise, one has \(m \geq 1\) and

\[
Q(x) = x^{N-2m}\tilde{Q}(x), \quad \text{where } \tilde{Q}(0) \neq 0, \ deg\tilde{Q} = 2m - 2.
\]

(35)

Write \(\tilde{Q}(x) = \sum_{j=0}^{2m-2} \tilde{\alpha}_j x^j\). Then \(\tilde{Q}(x)\) satisfies the relation

\[
\Lambda_{m,t}(x)\tilde{Q}(x) = q^m \Delta(x(-q)^{-\frac{1}{2}})\tilde{Q}(xq^{-1}) + q^{-m} \Delta(x(-q)^{\frac{1}{2}})\tilde{Q}(xq),
\]

or equivalently, the coefficients \(\tilde{\alpha}_j\)s of \(\tilde{Q}(x)\) satisfy the following system of equations,

\[
\tilde{v}_j\tilde{\alpha}_j + \tilde{v}_j\tilde{\alpha}_{j-1} + (\tilde{\delta}_j - \lambda)\tilde{\alpha}_{j-2} + \tilde{u}_j\tilde{\alpha}_{j-3} + \tilde{\mu}_j\tilde{\alpha}_{j-4} = 0, \quad 1 \leq j \leq 2m + 1,
\]

(37)

where \(\tilde{v}_j, \tilde{v}_j, \tilde{\delta}_j, \tilde{u}_j, \tilde{\mu}_j\) are expressed by the similar forms as in (34) by changing \(m\) to \(-m\) in the corresponding term. By \(\tilde{v}_1 \neq 0\), we have \(m \geq 2\). By the equalities,

\[
\tilde{v}_j = \tilde{\mu}_{2m+2-j}, \quad \tilde{v}_j = \tilde{u}_{2m+2-j}, \quad \tilde{\delta}_j = \tilde{\delta}_{2m+2-j},
\]

\(\tilde{Q}^1(x)\) also satisfies the equation (36). In general, for a polynomial \(\tilde{Q}(x)\) of degree \(\tilde{d}\) satisfies (34), \(\tilde{d} \equiv 2m - 2, N - 2 \pmod{N}\), and the minimal possible \(\tilde{d}\) is \(2m - 2\). Hence the dimension of the solution space of \(\tilde{Q}(x)\) with degree \(\leq 2m - 2\) is equal to one. For \(Q(x)\) in (25), \(\tilde{Q}^1(x)\) is a scale-multiple of \(\tilde{Q}(x)\), which implies \(\tilde{Q}^1(x) = \pm \tilde{Q}(x)\). Therefore, \(Q(x)\) is determined by the coefficients \(\tilde{\alpha}_j\) for \(0 \leq j \leq m - 1\), which involve only those equations in (37) with \(1 \leq j \leq m + 1\), subject to one of the following two conditions: \(\tilde{\alpha}_j = \tilde{\alpha}_{2m-2-j}\) for all \(j\), or \(\tilde{\alpha}_j = -\tilde{\alpha}_{2m-2-j}\) for all \(j\). On the other hand, the relation (36) is the same when we substitute \((c, x)\) by \((-c, -x)\), hence \(Q(-x, -c) = Q(x, c)\). Hence we may assume the coefficients \(\tilde{\alpha}_j = \tilde{\alpha}_j(c)\) satisfy the following properties:

\[
\tilde{\alpha}_0(c) = 1, \quad \tilde{\alpha}_j(-c) = (-1)^j\tilde{\alpha}_j(c), \quad \text{for all } j.
\]

(38)

For a solution \(\{\lambda, \tilde{\alpha}_j\}\) of (37) for a generic \(c\), \(\lambda\) is a solution of the eigenvalues problem arisen from those relations for \(2 \leq j \leq m\). Hence \(\lambda = \lambda(c)\), an algebraic function of \(c\) so that the limit of \(\frac{\lambda(c)}{c^2}\),
denoted by $\lambda_\infty$, exists as $c \to \infty$. For $1 \leq j \leq m - 1$, by $\tilde{\nu}_j \neq 0$ one can conclude $\tilde{\alpha}_j(c) = O(c^\ell)$ as $c \to \infty$. Denote

$$a_j = \lim_{c \to \infty} \frac{\tilde{\alpha}_j(c)}{c^\ell}, \quad 0 \leq j \leq m - 1,$$

and

$$\bar{v}_l' = 4i(q^{-m+k-\frac{1}{2}} - q^{m-k+\frac{1}{2}}), \quad \tilde{u}_k' = -4i(q^{-m+k-\frac{1}{2}} - q^{m-k+\frac{1}{2}}), \quad \bar{\delta}_k' = -4(q^{m-k-1} + q^{m-k+1})$$

for $1 \leq k \leq m+1$. By multiplying $c^{-j}$ on (37), and then taking the $c$-infinity limit for $1 \leq j \leq m+1$, one obtains the following matrix relation on $\lambda_\infty$ and $a_j$s,

$$
\begin{pmatrix}
\delta_{m+1} - \lambda_\infty & 0 & 0 & \cdots & 0 \\
\tilde{v}_m' & \delta_m' - \lambda_\infty & 0 & \cdots & \cdots \\
\tilde{\nu}_m & \tilde{v}_m' & \delta_{m-1} - \lambda_\infty & 0 & \cdots \\
0 & \cdots & \cdots & \cdots & \cdots \\
\vdots & \cdots & \cdots & \cdots & \cdots \\
0 & \cdots & \cdots & \cdots & \cdots \\
\end{pmatrix}
\begin{pmatrix}
a_{m-1} \\
a_{m-2} \\
l_{m-1} \\
\vdots \\
a_0 \\
\end{pmatrix} = 0. \quad (39)
$$

Note that $a_0 = 1$, and $\bar{\delta}_j' \neq \bar{\delta}_k'$ for $2 \leq j \neq k \leq m + 1$. The square matrix by deleting the last row in (39) becomes the eigenvalue problem with $m$ distinct eigenvalues, hence

$$\lambda_\infty = \bar{\delta}_l', \text{ for some } 2 \leq l \leq m + 1.$$ 

This implies $a_k = 0$ for $l - 2 < k \leq m - 1$, and $a_{l-2} \neq 0$. In (39), the row containing $\bar{\delta}_{l-1} - \lambda_\infty$ gives the following relation of $a_l$ and $a_{l-1}$:

$$\tilde{v}_{l-1}'a_{l-2} + (\bar{\delta}_{l-1} - \bar{\delta}_l')a_{l-3} = 0. \quad (40)$$

One would expect the right lower square matrix of size $l - 1$ with $\lambda_\infty = \bar{\delta}_l'$ has non-zero determinant. This statement is valid for a small number $l$ by direct computation, hence it leads to a contradiction. However, it is a difficult task to obtain a mathematical proof of such a statement for a general $l$. For our purpose, we are going to provide another way to justify the conclusion of the lemma using (37). First we consider the case $l = m + 1$, then $a_{m-1} \neq 0$. This implies $\bar{Q}(x) = \bar{Q}(x)$. The relation (34) and the $(m+1)$th equation in (37) become

$$
\begin{align*}
\tilde{v}_m' a_{m-1} + (\bar{\delta}_m' - \bar{\delta}_{m+1}')a_{m-2} &= 0, \\
(\bar{\delta}_{m+1} - \lambda(c))\tilde{\alpha}_{m-1} + 2\tilde{u}_{m+1}\tilde{\alpha}_{m-2} + 2\tilde{\mu}_{m+1}\tilde{\alpha}_{m-3} &= 0.
\end{align*}
$$

By the property (34) on $\lambda$, the $c$-infinity limit of $c^{-(m-1)}$-multiple of the second relation in (41) gives rise to the equality,

$$-4a_{m-1} + 2\tilde{u}_{m+1}a_{m-2} = 0,$$

which is incompatible with the first relation of (41). Hence, we may assume $2 \leq l \leq m$. By (38) and $a_k = 0$ for $l - 2 < k$, one has

$$\tilde{\alpha}_k(c) = a_k' c^{k-2} + \text{lower order term}, \quad \text{as } c \to \infty, \quad l - 1 \leq k \leq m - 1.$$
By $\lambda_\infty = \tilde{\delta}_l$, one can take the $c$-infinity limit of $c^{-(l-2)}$-multiple of the $l$th relation in (37), which yields the following identities:

$$
\tilde{\nu}_l a'_l + \tilde{v}_l a'_{l-1} + \frac{\tilde{\delta}}{2} a_{l-2} + \tilde{u}_l a_{l-3} = 0 , \quad \text{when } l \leq m - 1;
$$

$$
\tilde{\nu}_m a_m + \tilde{v}_m a'_{m-1} + \frac{\tilde{\delta}}{2} a_{m-2} + \tilde{u}_m a_{m-3} = 0 , \quad \text{when } l = m.
$$

Similarly the $c$-infinity limit of $c^{-(l-1)}$-multiple of the $(l+1)$th relation gives rise to the following relations:

$$
\tilde{\nu}_{l+1} a'_{l+1} + \tilde{v}_{l+1} a'_{l} + (\tilde{\delta}'_{l+1} - \tilde{\delta}'_l) a'_{l-1} + \tilde{u}'_{l+1} a_{l-2} = 0 , \quad \text{when } l \leq m - 2;
$$

$$
\tilde{v}'_m a'_{m-1} + (\tilde{\delta}'_m - \tilde{\delta}'_{m-1}) a'_{m-2} + \tilde{u}'_m a_{m-3} = 0 , \quad \text{when } l = m - 1;
$$

$$
(\tilde{\delta}'_{m+1} - \tilde{\delta}'_m) a'_{m-1} + \tilde{u}'_{m+1} (a_{m-2} + a_m) = 0 , \quad \text{when } l = m.
$$

For $l = m$, by using $a_m = \pm a_{m-2}$, (40) (43) and the last relation in (42) will lead to a contradiction. For $l \leq m - 2$, we continue the same procedure on the $c$-infinity limit of $c^{-s}$-multiple of the $(s+2)$th relation for $s \geq l$, then obtain

$$
\tilde{\nu}_{s+2} a'_{s+2} + \tilde{v}_{s+2} a'_{s+1} + (\tilde{\delta}'_{s+2} - \tilde{\delta}'_s) a'_s = 0 .
$$

Hence one has the following relations for $a'_k$'s,

$$
\begin{pmatrix}
\tilde{\delta}'_{m+1} - \tilde{\delta}'_l & 0 & 0 & \cdots & 0 \\
\tilde{v}'_m & \tilde{\delta}'_m - \tilde{\delta}'_l & 0 & \cdots & 0 \\
\tilde{\nu}_{m-1} & \tilde{v}'_{m-1} & \tilde{\delta}'_{m-1} - \tilde{\delta}'_l & 0 & \cdots \\
0 & \cdots & \tilde{v}'_{l+3} & \tilde{\delta}'_{l+3} - \tilde{\delta}'_l & 0 \\
\vdots & \cdots & \tilde{v}'_{l+2} & \tilde{\delta}'_{l+2} - \tilde{\delta}'_l & \vdots \\
& \cdots & \tilde{v}'_{l+2} & \tilde{\delta}'_{l+2} - \tilde{\delta}'_l & \vdots \\
\end{pmatrix}
\begin{pmatrix}
a'_{m-1} \\
a'_{m-2} \\
\vdots \\
a'_s \\
\vdots \\
a'_l
\end{pmatrix} = 0,
$$

which implies $a'_s = 0$ for $s \geq l$. The relations (42) (43) become

$$
\tilde{\nu}'_l a'_{l-1} + \frac{\tilde{\delta}'_l}{2} a_{l-2} + \tilde{u}'_l a_{l-3} = 0 , \quad (\tilde{\delta}'_{l+1} - \tilde{\delta}'_l) a'_{l-1} + \tilde{u}'_{l+1} a_{l-2} = 0 ,
$$

together with (41), this provides a contradiction to $a_{l-2} \neq 0$. \qed

For a symmetric (27) equation, the eigen-polynomial $Q(x)$ has the following property:

**Theorem 2** Assume there exists a non-trivial polynomial solution for a symmetric $T$-$Q$ polynomial relation (27) with a given reciprocal polynomial $\Lambda_{m,l}(x)$. Then the equation has one-dimensional solution space, generated by a monic polynomial $Q(x)$ of degree $2N - 2 - 2m$ with $Q(0) \neq 0$ and $Q'(x) = \pm Q(x)$.

**Proof.** Let $Q(x)$ be a non-trivial polynomial solution with the degree $d$. By Lemma 2, we have $d = N - 2, N - 2m - 2, 2N - 2m - 2$. First we are going to show that $d = 2N - 2m - 2$. Otherwise, $d$ is one of two odd integers, $N - 2m - 2$ or $N - 2$. By Lemmas 3 and 8, we may assume $Q(0) \neq 0$ with $a_0 = 1$. The coefficients $a_j$ of $Q(x)$ satisfy the relation (28). When $d = N - 2m - 2$, we have $v_j \neq 0$ for $1 \leq j \leq d$. This implies the $x$-coefficients $a_j$ of $Q(x)(= Q(x;c))$ are polynomials of $c$
and \( \lambda = \lambda(c) \), hence \( \alpha_j = \alpha_j(c) \). As \( c \) tends to 0, the coefficients \( \alpha_j = \alpha_j(0) \) of \( Q(x;0) \) satisfy the corresponding relation (28):

\[
v_j \alpha_j + (\delta_j - \lambda) \alpha_j - 2 + \mu_j \alpha_j - 4 = 0 \quad , \quad 1 \leq j \leq d + 3 ,
\]

with \( \alpha_1 = 0 \). Hence \( \alpha_j = 0 \) for odd \( j \), and the polynomial \( Q(x;0) \) has an even degree \( \leq N - 2m - 2 \), which impossible by Lemma 2. It remains the case when \( d = N - 2 \) with \( Q(0;c) \neq 0 \). Now the dimension of the \( Q(x) \)-solution space of (27) with \( \deg Q(x) \leq N - 2 \) is equal to one. As \( Q(-x; c) = Q(x; c) \), equivalently , \( \alpha_j(-c) = (-1)^j \alpha_j(c) \) for all \( j \). Therefore \( Q(x;0) \) is again a polynomial in \( x \) with an even degree \( \leq N - 2 \), which contradicts to Lemma 2 since \( 2 \leq N - 2 \) implies \( Q(x) \) and \( \lambda(2) = 0 \). Hence we have shown that any solution \( Q(x) \)

Remark. For a polynomial \( Q(x) \) in the above proposition, the roots of \( Q(x) \) are all non-zero; furthermore if \( x_k \) is a root, so is \( x_k^{-1} \). Hence the collection of all roots \( x_k \) (counting multiplicity) is the same as that of \( x_k^{-1} \)'s. The criterion for \( Q^\dagger(x) = -Q(x) \) holds if and only if \( Q(x) \) has the root \( x = 1 \) with a positive odd multiplicity. \( \square \)

5 Solutions of Discrete Quantum Pendulum and Sine-Gordon Model in the Rational Degenerated case

In this section we are going to derive the complete solution of symmetric \( T-Q \) polynomial relation (27); hence we now only consider the sectors \( (m, l) = (m, 0), (m, N - 2m) \). By Theorem 3, we may assume

\[
d = 2N - 2 - 2m \quad , \quad Q^\dagger(x) = \pm Q(x).
\]

Hence the coefficients in (33) possesses the following symmetric relations:

\[
v_{d+4-j} = \mu_j, \quad v_{d+4-j} = u_j, \quad \delta_{d+4-j} = \delta_j .
\]

The system (28) is equivalent to the eigenvalue problem (34) together with one more constraint:

\[
\nu_1 \alpha_1 + v_1 \alpha_0 = 0 , \quad (44)
\]

and the \( \alpha_j \)'s satisfy either one of the following conditions :

\[
\begin{align*}
\alpha_i &= \alpha_{d-i} \quad \text{for } 0 \leq i \leq d, \quad \text{i.e., } Q^\dagger(x) = Q(x); \\
\alpha_i &= -\alpha_{d-i} \quad \text{for } 0 \leq i \leq d, \quad \text{i.e., } Q^\dagger(x) = -Q(x).
\end{align*}
\]

(45)  (46)

Note that the polynomial \( Q(x) \) is determined only by the first half of its coefficients, i.e., \( \alpha_0, \ldots, \alpha_{d/2} \).

In the case (44), one has \( \alpha_d = 0 \). Furthermore, through the transformations:

\[
\begin{align*}
\nu_j, \nu_j, \delta_j, u_j, \mu_j \mapsto \nu_{j'}, u_{j'}, \delta_{j'}, v_{j'}, \nu_{j'}, \quad \text{where } j' := d + 4 - j ; \\
\alpha_i \mapsto \alpha_{d-i} \quad (0 \leq i \leq d) \quad \text{or} \quad \alpha_i \mapsto -\alpha_{d-i} \quad (0 \leq i \leq d),
\end{align*}
\]

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the equations for \( j \geq \frac{d}{2} + 3 \) in (28) follow from those for \( j \leq \frac{d}{2} + 2 \). So we need only to consider the relations for \( 1 \leq j \leq \frac{d}{2} + 2 \) in (28), which are regarded as equations of \( \lambda \) and \( \alpha_k \) for \( 0 \leq k \leq \frac{d}{2} \). Note that the \((\frac{d}{2}+2)\)th equation in (28) has the form:

\[
(\delta_{\frac{d}{2}+2} - \lambda)\alpha_{\frac{d}{2}+2} + u_{\frac{d}{2}+2}(\alpha_{\frac{d}{2}+1} + \alpha_{\frac{d}{2}+1}) + \mu_{\frac{d}{2}+2}(\alpha_{\frac{d}{2}-2} + \alpha_{\frac{d}{2}+2}) = 0 ,
\]

which is a trivial relation in the case (46).

By (44), the rational degenerated case of discrete quantum pendulum and discrete sine-Gordon corresponds to \( C = 1 \), i.e. the sectors with \( l = 0 \) in the symmetric (27) relation; in particular, by (47) and (24), the discrete quantum pendulum is given by \( D = C = 1 \), i.e. \((m,l) = (0,0)\).

**Theorem 3** For the symmetric T-Q polynomial equation (24), there are \( N \) distinct eigenvalues \( \lambda \), each of which has one-dimensional eigenspace generated by a monic eigen-polynomial \( Q(x) \) of degree \( d = 4M - 2m \) with \( Q(0) \neq 0 \) and \( Q'(x) = \pm Q(x) \). Furthermore, there are \((M+1)\) eigen-polynomials \( Q(x) \) of the type \( Q'(x) = Q(x) \), and the rest \( M \) ones are of the type \( Q'(x) = -Q(x) \). In particular, the Baxter’s T-Q polynomial relation of SG model are those for the sectors \((m,l) = (m,0)\), and the discrete quantum pendulum is the one for \((m,l) = (0,0)\).

**Proof**. The relation (44) is a non-trivial constraint for \( 0 \leq m \leq M - 1 \) by \( \nu_1 \neq 0 \); while for \( m = M \), both \( \nu_1 \) and \( v_1 \) are zeros, hence (44) is a redundant one. In this proof, we shall first consider the case with \( m = 0 \), then \( 1 \leq m \leq M - 1 \), and finally on \( m = M \).

(I) \( m = 0 \), i.e. \((m,l) = (0,0)\), which is the rational degenerated case of discrete quantum pendulum. We have \( d = 4M \). Consider the relations for \( 1 \leq j \leq \frac{d}{2} + 2 \) in the system (28) as equations of \( \lambda, \alpha_0, ..., \alpha_{\frac{d}{2}} \). By \( \nu_{\frac{d}{2}+1} = 0 \), the problem is formulated in the following matrix form:

\[
\begin{pmatrix}
v_{\frac{d}{2}+1} & \delta_{\frac{d}{2}+1} - \lambda & u_{\frac{d}{2}+1} & \mu_{\frac{d}{2}+1} & 0 & \cdots & 0 \\
\nu_{\frac{d}{2}} & v_{\frac{d}{2}} & \delta_{\frac{d}{2}} - \lambda & u_{\frac{d}{2}} & \mu_{\frac{d}{2}} & \cdots & \vdots \\
0 & \nu_{\frac{d}{2}-1} & v_{\frac{d}{2}-1} & \delta_{\frac{d}{2}-1} - \lambda & \cdots & \cdots & \vdots \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
\vdots & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \alpha_0
\end{pmatrix}
= \tilde{0}
\]

(48)

together with the constraint (47), which is now given by

\[
(q + q^{-1} - 2)(\alpha_{\frac{d}{2}+2} + \alpha_{\frac{d}{2}-2}) + 4ci(q^\frac{1}{2} - q^{-\frac{1}{2}})(\alpha_{\frac{d}{2}+1} + \alpha_{\frac{d}{2}+1}) - (8c^2 + 4 + \lambda)\alpha_{\frac{d}{2}} = 0 .
\]

Note that the square matrix of size \( N (= \frac{d}{2}+1) \) in (48) satisfies the condition of Lemma 1. Hence, by \( \nu_j \neq 0 \) for \( 1 \leq j \leq \frac{d}{2} \), the system (48) has the one-dimensional eigenspace for any given \( c \) and \( \lambda \), generated by a basis element \( (\alpha_k)_{0 \leq k \leq \frac{d}{2}} \) with \( \alpha_0 = 1 \). In fact, for \( 1 \leq k \leq \frac{d}{2} \), \( \alpha_k \) can be expressed by a polynomial of \( \lambda \) and \( c \), regarded as a polynomial in \( \lambda \) with coefficients in \( \mathbb{C}[c] \), and denoted by \( \alpha_k = p_k(\lambda) \). The \( \lambda \)-degree of \( \alpha_k \) is given by \( \deg p_k(\lambda) = \lfloor \frac{k}{2} \rfloor \). In the case (47), the relation (49) becomes:

\[
2(q + q^{-1} - 2)p_{\frac{d}{2}-2}(\lambda) + 8ci(q^{\frac{1}{2}} - q^{-\frac{1}{2}})p_{\frac{d}{2}-1}(\lambda) - (8c^2 + 4 + \lambda)p_{\frac{d}{2}}(\lambda) = 0 ,
\]
which defines $\lambda$ as an algebraic function of $c$. As the $\lambda$-degree of the above relation is equal to $M + 1$, there are $(M + 1)$ $\lambda$-values for a generic $c$. In the case (46), (49) is a trivial relation. The relation (48) becomes:

\[
\begin{pmatrix}
\delta_{d+1} - \lambda & u_{d+1} & \mu_{d+1} & 0 & \cdots & 0 \\
v_2 & \delta_2 - \lambda & u_2 & \mu_2 & 0 & \cdots & 0 \\
v_d-1 & v_d-1 & \delta_{d-1} - \lambda & u_{d-1} & \mu_{d-1} & \cdots & 0 \\
0 & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & \nu_2 & v_2 & \delta_2 - \lambda \\
0 & \cdots & 0 & 0 & \nu_1 & v_1 & \vdots \\
\end{pmatrix}
\begin{pmatrix}
\alpha_{d+1} \\
\vdots \\
\vdots \\
\alpha_1 \\
\alpha_0 \\
\end{pmatrix} = \vec{0}.
\]

Solutions of the above equation can be obtained from the system (48) alone, and then imposing the following constraint on $\lambda$,

\[\alpha_{d+1} = p_{d+1}(\lambda) = 0.\]

As the $\lambda$-degree of $p_k$ is equal to $M$, the above equation gives rise to $M$ eigenvalues of $\lambda$, with the corresponding eigenvector having the components $\alpha_k = p_k(\lambda), 1 \leq k \leq \frac{d}{2} - 1$. By Theorem 3, the $Q(x)$-eigenspaces are all one-dimensional, hence the conclusion follows immediately.

(II) $1 \leq m \leq M - 1$. We have $l = 0, N - 2m$, and $d = 2N - 2 - 2m$. By (33), for $1 \leq j \leq \frac{d}{2} + 1$ one has

\[\nu_j = 0 \iff j = n := N - 2m.\]

In the case (45), the relations for $1 \leq j \leq \frac{d}{2} + 2$ in the system (28), considered as equations of $\lambda$ and $\alpha_k (0 \leq k \leq \frac{d}{2})$, can be formulated in the following form,

\[
\begin{pmatrix}
S & T \\
0 & U \\
\end{pmatrix}
\begin{pmatrix}
\vec{\psi} \\
\vec{\psi} \\
\end{pmatrix} = \vec{0}, \quad \psi := \begin{pmatrix}
\alpha_{n-1} \\
\vdots \\
\alpha_1 \\
\alpha_0 \\
\end{pmatrix}, \quad \vec{\psi} := \begin{pmatrix}
\alpha_{d+1} \\
\vdots \\
\alpha_{n+1} \\
\alpha_n \\
\end{pmatrix},
\]

and together with the constraint (17). Here $S, U$ are square matrices of the size $(\frac{d}{2} - n + 1)$, $n$
respectively, \( T \) is the \( (d - n + 1) \times n \) matrix, with the following expressions:

\[
S = \begin{pmatrix}
\nu_{d+1} & \nu_{d+1} + \delta_{d+1} - \lambda & u_{d+1} & \mu_{d+1} & 0 & \cdots & 0 \\
\nu_{\frac{d}{2}} & v_{\frac{d}{2}} & \delta_{\frac{d}{2}} - \lambda & u_{\frac{d}{2}} & \mu_{\frac{d}{2}} & 0 & \vdots \\
0 & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & \cdots & 0 & \nu_{n+3} & \nu_{n+3} & \delta_{n+3} - \lambda & u_{n+3} \\
0 & \cdots & \cdots & 0 & \nu_{n+2} & \nu_{n+2} & \delta_{n+2} - \lambda \\
0 & \cdots & \cdots & 0 & \nu_{n+1} & \nu_{n+1} & \nu_{n+1} \\
\end{pmatrix},
\]

\[
U = \begin{pmatrix}
v_n & \delta_n - \lambda & u_n & \mu_n & 0 & \cdots & 0 \\
\nu_{n-1} & v_{n-1} & \delta_{n-1} - \lambda & u_{n-1} & \mu_{n-1} & 0 & \vdots \\
0 & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & \cdots & 0 & \nu_3 & \nu_3 & \delta_3 - \lambda & u_3 \\
0 & \cdots & \cdots & 0 & \nu_2 & \nu_2 & \delta_2 - \lambda \\
0 & \cdots & \cdots & 0 & \nu_1 & \nu_1 & \nu_1 \\
\end{pmatrix},
\]

\[
T = \begin{pmatrix}
0 & \cdots & \cdots & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
0 & \cdots & \cdots & \cdots & \cdots \\
\mu_{n+3} & 0 & \cdots & \cdots & \cdots \\
u_{n+2} & 0 & \cdots & \cdots & \cdots \\
\delta_{n+1} - \lambda & u_{n+1} & \mu_{n+1} & 0 & \cdots & 0 \\
\end{pmatrix}.
\]

Note that there is the term \( \nu_{\frac{d}{2}} \) in the second entry of the first row of \( S \); while \( \mu_{n+2} = 0 \) in the matrix \( T \). By (33), the matrix \( U \) satisfies the condition of Lemma 1 for any \( \lambda \). Hence, by (50), the system \( U\psi = 0 \) inside the relation (51) has the one-dimensional solution space generated by a vector \( \psi \) with

\[
\alpha_0 = 1, \quad \alpha_k = p_k(\lambda) \in \mathbb{C}[c][\lambda], \quad \text{where} \quad \deg p_k(\lambda) = \left[ \frac{k}{2} \right] \quad \text{for} \quad k < n.
\]

Using the above vector \( \psi \), we consider the system

\[
S\tilde{\psi} = -T\psi.
\]

By (50), one can first solve \( \alpha_k \) (\( k > n \)) in terms of \( \alpha_n, \lambda, c \) with the form:

\[
\alpha_k = r_k(\lambda)\alpha_n + q_k(\lambda), \quad \text{where} \quad r_k(\lambda), q_k(\lambda) \in \mathbb{C}[c][\lambda], \quad \deg r_k(\lambda) + \frac{n-1}{2} = \deg q_k(\lambda) = \left[ \frac{k}{2} \right].
\]

Furthermore, \( \alpha_n \) satisfies the following relation:

\[
0 = v_{\frac{d}{2}+1}\alpha_{\frac{d}{2}} + (\nu_{\frac{d}{2}+1} + \delta_{\frac{d}{2}+1} - \lambda)\alpha_{\frac{d}{2}-1} + u_{\frac{d}{2}+1}\alpha_{\frac{d}{2}-2} + \mu_{\frac{d}{2}+1}\alpha_{\frac{d}{2}-3} = r(\lambda)\alpha_n + q(\lambda).
\]
where \( r(\lambda), q(\lambda) \in \mathbb{C}[c][\lambda] \) with \( \deg r(\lambda) + \frac{n-1}{2} = \deg q(\lambda) = \left[ \frac{d+2}{4} \right] \). Hence

\[
\alpha_k = \frac{P_k(\lambda)}{r(\lambda)}, \quad P_k(\lambda) := r(\lambda)q_k(\lambda) - r_k(\lambda)q(\lambda).
\]

By multiplying the \( \alpha_k \) by \( r(\lambda) \), we obtain a solution of (51) for all \( \lambda \) with the new \( \alpha_k, 0 \leq k \leq \frac{d}{2} \), in the form:

\[
\alpha_k = P_k(\lambda), \quad \deg P_k(\lambda) = \left[ \frac{k}{2} \right] + \left[ \frac{d+2}{4} \right] - \frac{n-1}{2};
\]

in particular, \( \deg P_{\frac{d}{2}}(\lambda) = M \). Now the constraint (47) becomes

\[
(\delta_{\frac{d}{2}+2} - \lambda)P_{\frac{d}{2}}(\lambda) + 2u_{\frac{d}{2}+2}P_{\frac{d}{2}-1}(\lambda) + 2\mu_{\frac{d}{2}+2}P_{\frac{d}{2}-2}(\lambda) = 0,
\]

by which one can show that the above relation gives rise to \((M+1)\) \( \lambda \)-values for a generic \( c \).

In the case (46), (47) becomes a trivial relation. We consider the following eigenvalue problem, similar to the one in (51) by changing \( S \) to \( S^- \):

\[
\begin{pmatrix}
S^- & T \\
0 & U
\end{pmatrix}
\begin{pmatrix}
\bar{\psi} \\
\psi
\end{pmatrix}
= 0,
\quad
\psi :=
\begin{pmatrix}
\alpha_{n-1} \\
\vdots \\
\alpha_1 \\
\alpha_0
\end{pmatrix},
\quad
\bar{\psi} :=
\begin{pmatrix}
\alpha_{\frac{d}{2}} \\
\vdots \\
\alpha_{n+1} \\
\alpha_n
\end{pmatrix},
\]

where the matrix \( S^- \) differs from \( S \) only on the (1, 2)-th entry by changing \( \nu_{\frac{d}{2}+1} \) to \(-\nu_{\frac{d}{2}+1} \), i.e.,

\[
S^- =
\begin{pmatrix}
\nu_{\frac{d}{2}+1} & -\nu_{\frac{d}{2}+1} + \delta_{\frac{d}{2}+1} - \lambda & u_{\frac{d}{2}+1} & \mu_{\frac{d}{2}+1} & 0 & \cdots & 0 \\
\nu_{\frac{d}{2}} & v_{\frac{d}{2}} & \delta_{\frac{d}{2}} - \lambda & u_{\frac{d}{2}} & \mu_{\frac{d}{2}} & 0 & \vdots \\
0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & \cdots & \cdots & 0 & \nu_{n+3} & v_{n+3} & \delta_{n+3} - \lambda & u_{n+3} \\
0 & \cdots & \cdots & \cdots & 0 & \nu_{n+2} & v_{n+2} & \delta_{n+2} - \lambda \\
0 & \cdots & \cdots & \cdots & 0 & \nu_{n+1} & v_{n+1}
\end{pmatrix}.
\]

Then the problem (53) with the condition \( \alpha_{\frac{d}{2}} = 0 \) is equivalent to the problem in the case (46).

As in the discussion of the eigenvalue problem (51), there is a solution of (51), \( \alpha_j \)'s, with the form \( \alpha_k = P_k^- (\lambda) \) satisfying the property (52). Then \( P_k^- \) is a degree \( M \) polynomial of \( \lambda \), and its zeros, \( \alpha_{\frac{d}{2}} = P_{\frac{d}{2}}^- (\lambda) = 0 \), give rise to \( M \) \( \lambda \)-values for the case (46). The conclusion now follows from Theorem 4.

(III) \( m = M \). We have \( l = 0, 1 \), and \( d = N - 1 \). As the values of \( \nu_1, v_1, u_{N+2}, \mu_{N+2} \) in (29) are all zeros in this case, the relations of \( j = 1, d + 3 \) in (28) are redundant. Hence the system (28) is equivalent to the eigenvalue problem (30) with \( d = N - 1 \). In the case (47), the collection of
\( \alpha_k, 0 \leq k \leq \frac{d}{2}, \) among the coefficients of \( Q(x) \) is the solution of the following eigenvalue problem:

\[
\begin{pmatrix}
\delta_{M+1} & 2u_{M+1} & 2\mu_{M+1} & 0 & \cdots & 0 & 0 \\
v_{M+1} & \delta_{M+1} + \nu_{M+1} & u_{M+1} & \mu_{M+1} & \cdots & 0 & 0 \\
\nu_M & v_M & \delta_M & u_M & \mu_M & \cdots & 0 \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\vdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & \cdots & 0 & \nu_4 & v_4 & \delta_4 & u_4 & \mu_4 \\
\vdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & \cdots & 0 & \nu_3 & v_3 & \delta_3 & u_3 \\
\vdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & \cdots & 0 & \nu_2 & v_2 & \delta_2 \\
\end{pmatrix}
- \lambda
\begin{pmatrix}
\alpha_M \\
\alpha_{M-1} \\
\vdots \\
\vdots \\
\alpha_0 \\
\end{pmatrix} = 0.
\]

Note that the coefficients in the first and second rows have some extra terms, comparing to the rest of entries. There are \( (M + 1) \lambda \)-eigenvalues for the above relation, which gives rise to the solutions for the case \( [13] \). The rest \( M \lambda \)-values for \( [30] \) are those for the case \( [16] \). Then the conclusion follows from Theorem 4. □

**Remark** By Theorem 3, an eigen-polynomial \( Q(x) \) of the symmetric \( T-Q \) polynomial equation (27) are in sectors \( (m, l) = (m, 0), (m, N - 2m), \) and it has the form,

\[
Q(x) = \prod_{j=1}^{2N-2-2m} (x - \frac{1}{z_j}), \quad z_j \neq 0.
\]

The Bethe ansatz equation (31) of \( z_k \)'s becomes

\[
\left( \frac{z_j^2 + 2icq^2 z_j - q}{q z_j^2 - 2icq^2 z_j - 1} \right)^2 = \prod_{n \neq j, n=1}^{2N-2-2m} \frac{z_n - q z_j}{q z_n - z_j}, \quad 1 \leq j \leq 2N - 2 - 2m.
\]

By the description of \( Q(x) \) in Theorem 3, one has the reciprocal constraint on \( z_j \)'s, i.e., \( \{z_j\}_{j=1}^{2N-2-2m} = \{z_j^{-1}\}_{j=1}^{2N-2-2m} \) (counting the multiplicity). □

## 6 The General Spectral Curve for Discrete Quantum Pendulum and Discrete Sine-Gordon Model

In this section, we are going to explore the geometrical structure of the spectral curve \( C_R \) for the discrete quantum pendulum and SG model. By (13) and (14), the parameter \( \vec{h} \) for the curve \( C_R \) has the following constraints,

\[
a_j = q^{-1} d_j^{-1}, \quad b_j = -k^{-\epsilon_j} c_j^{-1}, \quad (\epsilon_j := (-1)^j), \quad 0 \leq j \leq 3, \\
d_0 d_1 d_2 d_3 = 1, \quad c_1^N c_2^N c_3^N = k^2 N c_0^N c_2^N.
\]

By the discussion in Sect. 8 of (18), the value \( \xi_0^N \) of the curve \( C_R \) are determined by \( x^N, \xi_0^N \), which we denote by \( y := x^N, \eta := \xi_0^N \). The variables \((y, \eta)\) satisfies the following equation of the curve \( \mathcal{B}_R \),

\[
C_R(y) \eta^2 + (A_R(y) - D_R(y))\eta - B_R(y) = 0,
\]
where $A_h, B_h, C_h, D_h$ are polynomials of $y$ given by the relation,

$$
\begin{pmatrix}
-A_h(y) & B_h(y) \\
C_h(y) & -D_h(y)
\end{pmatrix} \equiv \prod_{j=0}^{3} \begin{pmatrix}
-d_j^{-N} & -y c_j^{-N} k^{-c_j N} \\
y c_j^{-N} & -d_j^{-N}
\end{pmatrix}.
$$

In fact, by computation, one has the following expressions of these polynomials,

\begin{align*}
A_h(y) &= - (\delta + y^2 \gamma + y^2 \gamma + y^2 \gamma) + y^2 N c_0^{-N} d_0^{-N} c_2^{-N} d_2^{-N} (\delta + k^N \gamma)^2; \\
D_h(y) &= - (\delta + y^2 \gamma + y^2 \gamma + y^2 \gamma) + y^2 N c_0^{-N} d_0^{-N} c_2^{-N} d_2^{-N} (\delta + k^N \gamma + \gamma - 1)^2; \\
B_h(y) &= y (\delta + y^2 \gamma - 1) \left(k^{-N} c_0^{-N} d_0^{-N} (\delta + k^N \gamma + c_2^{-N} d_2^{-N} (k^N \delta - 1 + \gamma - 1))\right); \\
C_h(y) &= - y (\delta + y^2 \gamma) \left(k^N c_0^{-N} d_0^{-N} (k - N \delta + 1 + \gamma - 1) + c_2^{-N} d_2^{-N} (\delta + k^N \gamma)\right),
\end{align*}

where $\delta, \gamma$ are defined by

$$
\delta := d_0^{-N} d_1^{-N} d_2^{-N}, \quad \gamma := \frac{c_0^{-N} k}{c_1^{-N}} = \frac{c_3^{-N}}{k^N c_2^{-N}}.
$$

Eliminating the $y$-factor in the equation of $B_h$, we obtain an irreducible curve, which will still be denoted by $B_h$ for the convenience of notations, but now with the equation:

$$
B_h : a (y^2 \gamma - \delta) \eta^2 + b y \eta + c (y^2 \gamma - \delta - 1) = 0,
$$

where $a, b, c$ are the parameters defined by

\begin{align*}
a &= k^N c_0^{-N} d_0^{-N} (k^{-N} \delta - 1 + \gamma - 1) + c_2^{-N} d_2^{-N} (\delta + k^N \gamma), \\
b &= -k^N c_0^{-N} d_0^{-N} c_2^{-N} d_2^{-N} (\delta + k^N \gamma)^2 - k^N c_0^{-N} d_0^{-N} c_2^{-N} d_2^{-N} (k^{-N} \delta - 1 + \gamma - 1)^2, \\
c &= -k^N c_0^{-N} d_0^{-N} (\delta + k^N \gamma) + c_2^{-N} d_2^{-N} (k^{-N} \delta - 1 + \gamma - 1).
\end{align*}

The curves $B_h$ form a family of elliptic curves, depending on the four parameters, $\delta, \gamma, k^N c_0^{-N} d_0^{-N}, c_2^{-N} d_2^{-N}$. And $C_h$ is a $\mathbb{Z}_N^3$-cover (branched) over $B_h$, with the covering transformation group containing $\tau_{\pm}$ in (8). For a generic $h$, $C_h$ is a high-genus curve; indeed the genus is equal to $2N^3(N - 1)(N + 2) + 1$.

Now we are going to make a qualitative analysis on solutions of the Baxter’s $T$-$Q$ relation (11) related to the discrete quantum pendulum (17) and SG model (18). All the linear transformations appeared in the expressions of (14) are operators of the vector space $\otimes^4 \mathbb{C}^N$. As $D, C$ and $U$ in the expression of $T_h(x)$ are commuting operators, by $(k^2 c_{1c_3} C)^N = 1$, the eigenvalue of $T_h(x)$ are still expressed in the form of (26) with $\lambda$ depending on $k$; while for the discrete quantum pendulum and SG model, it becomes (12). By the expressions of $D$ and $C$, it is not hard to see that the common eigen-subspaces of $\otimes^4 \mathbb{C}^N$ for the commuting operators $D_h^1, k^2 c_{1c_3} C$ all have the dimension $N^2$. The eigenspace decomposition of $\otimes^4 \mathbb{C}^N$ is denoted by

$$
\otimes^4 \mathbb{C}^N = \bigoplus_{n,n'} \mathbb{E}_{n,n'}, \quad \mathbb{E}_{n,n'} \simeq \mathbb{C}^{N^2},
$$

where $D_h^1, k^2 c_{1c_3} C$ act on $\mathbb{E}_{n,n'}$ by the multiplication of $q^n, q^{n'}$ respectively. By the relations of $U_j$’s and $C, D$ in (13), each $\mathbb{E}_{n,n'}$ is stable under $U_j$. The operators $U_j$ on $\mathbb{E}_{n,n'}$ are determined only by those of $U_1, U_2$, which form the Weyl algebra: $U_2 U_1 = \omega U_1 U_2, U_1^N = U_2^N = 1$. This implies the operator $T_h^1$ in (14) is determined by the representation $\mathbb{E}_{n,n'}$ of the Weyl algebra on each sector, labelled by the eigenvalues of $T_h^1, T_h^1$ corresponding to $(n, n')$. As the irreducible
representation of Weyl algebra is unique, given by the standard one on $\mathbb{C}^N$, $E_{n,n'}$ is isomorphic to the sum of $N$-copies of the standard representation as Weyl algebra modules. In particular, the eigenvalues of $-T_2^*$ on the vector space $E_{n,n'}$ are induced from the standard representation of the Weyl algebra; each eigenvalue gives rise to $N$ eigenvectors in $E_{n,n'}$. By (7) and (54), the Baxter vacuum state $|p\rangle \in \mathbb{C}^N$ for $p \in \mathbb{C}_0^*$ is now defined by $|p\rangle = |p_0\rangle \otimes |p_1\rangle \otimes |p_2\rangle \otimes |p_3\rangle$, where the vector $|p_j\rangle$ in $\mathbb{C}^N$ is given by the conditions:

$$
\langle 0|p_j\rangle = 1, \quad \frac{\langle m|p_j\rangle}{\langle m-1|p_j\rangle} = \frac{\xi_{j+1}k^j c_j q^{2m-1} + xd_j}{-\xi_j(xc_j q^{2m} - d_j)k^j c_j d_j}.
$$

For a generic $\hbar$, the evaluation of vectors of $\mathbb{C}^{*N}$ on the Baxter vacuum state, $* \mapsto \langle *|p\rangle$, induces an isomorphism between $\otimes \mathbb{C}^{*N}$ and a $N^4$-dimensional subspace of rational functions of $\mathbb{C}_0^*$. Through this isomorphism, $E_{n,n'}$ gives rise a $N^2$-dimensional functional space of $\mathbb{C}_0^*$, denoted by $\mathfrak{e}(E_{n,n'})$, with a Weyl-algebra-module structure induced from $E_{n,n'}$. In the Baxter’s $T$-$Q$ equation (1) on $\mathbb{C}_0^*$ with $\Lambda^*(x) = \Lambda_{m,l}(x)$ in (24), the function $Q(p)$ is the $T^*_0(x)$-eigenfunction in $\mathfrak{e}(E_{n,n'})$ for $(n,n') = (m,l), (N - m, N - l)$, with the multiplicity $N$. For the discrete quantum pendulum and SG model, by (17) and (18), the $\Lambda^*(x)$ in Baxter’s $T$-$Q$ relation (11) is the reciprocal polynomial (22). To determine the eigen-functions $Q(p)$ would require the understanding of its zeros and poles by using the expression of the Baxter vacuum state, which has been a difficult task at this moment. The possible role of elliptic function theory of $\mathbb{B}_0^*$ in the solutions of Baxter’s $T$-$Q$ relation on $\mathbb{C}_0^*$, and some further understanding on the eigenvalue problem of the models (17) (18) along this line, would be the core of our future work in this aspect.

7 Conclusions and Perspectives

We have studied the discrete quantum pendulum and discrete sine-Gordon model in the framework of quantum inverse scattering method. The diagonalization problem is governed by the Baxter’s $T$-$Q$ relation, which arises from the Baxter vacuum state on the spectral curve through a general scheme by using the transfer matrix for a fixed finite size $L$. We have demonstrated the role of algebraic geometry in the qualitative study of the Baxter’s $T$-$Q$ relation for $L=3$ in [13], and $L=4$ in this article. In both cases, they have shown an intimate relationship with integrable Hamiltonian spin-chains of physical interest. In this approach, one relies on the spectral curves, depending on the parameters encoded in the corresponding Hamiltonian expression. For generic parameters, the spectral curve, where the Baxter’s $T$-$Q$ relation is formulated, is a high-genus Riemann surface, as demonstrated in Sect. 6. However for both $L = 3$ and 4, the spectral curves possess a common feature that they form certain branched covers over elliptic curves. One might expect to employ the elliptic function theory to the solutions of Baxter’s $T$-$Q$ relation so that the algebraic geometry study could enrich our understanding of the corresponding Hamiltonian spectrum problem. Although this thinking is merely a speculation at present, a possible program along this line could be a challenging one, on which we hope to make progress in future.

When the spectral curve degenerates into rational curves, where the geometry plays little role, we derive the polynomial formulation of Baxter’s $T$-$Q$ relation for a system of a finite size $L$ in Sect. 3. We apply these results to the case $L=4$ in Sect. 4 for the setting of discrete quantum pendulum and sine-Gordon model. In these cases, an extra symmetry has naturally been imposed on the Baxter’s $T$-$Q$ polynomial equation; indeed it is governed by the reciprocal property of the equation. We present a detailed and rigorous mathematical derivation of the solutions in Theorem 8. Surprisingly the conclusion on these polynomial solutions has been much in tune with the
one for \( L = 3 \) on the study of Hofstadter-type model (see Theorem 3 in \cite{13}). Furthermore, the exact connection of the Baxter’s \( T-Q \) polynomial equation with the Bethe ansatz technique in literature has been clarified in all these cases. The results obtained in this paper signal some further mathematical feature of the Baxter’s \( T-Q \) polynomial equation, namely, the novel connection with certain \( q \)-Sturm-Liouville problem at roots of unity \( q^N = 1 \). The facts discovered in this work could be served to demonstrate that a systematic mathematical theory embodied in the Baxter’s \( T-Q \) polynomial equation (or algebraic Bethe Ansatz) would emerge in the study of \( q \)-difference operators. Accordingly, the relationship along this line is now under our consideration.

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**Appendix: Discrete Quantum Sine-Gordon Hamiltonian**

For the consistency of our notion of discrete quantum sine-Gordon Hamiltonian with the ones used in other literature, we make an identification of the discrete sine-Gordon integral in \cite{6} with the \( T_j \)'s of \cite{13} in this paper. In Sect. 5 of \cite{13}, the discrete quantum sine-Gordon Hamiltonian arises from the following commuting operators\cite{6},

\[
A^{(0)} = U_2^{-1}U_1^{-1}, \\
A^{(1)} = U_2Z_2U_1^{-1} + U_1Z_1U_2^{-1} + V_2h_2h_1^*V_1^{-1}, \\
A^{(2)} = U_2Z_2U_1Z_1,
\]

where the lower index \( j = 1, 2 \) indicates the site of operators in the same algebra generated by \( U, V, Z \), subject to the relation \( UV = q^{-\frac{1}{2}}VU \) with \( Z \) the central element, and \( h \) is defined by \( h = k^{-\frac{1}{2}} + k^\frac{1}{2}q^{-\frac{1}{2}}U^2Z \). The sine-Gordon (SG) integral is defined by the operator

\[
\tilde{H} = A^{(1)} + A^{(1)*}.
\]

One can show that

\[
V_2h_2h_1^*V_1^{-1} = q^\frac{1}{2}\frac{1}{2}V_2U_2^2Z_2V_1^{-1} + q^\frac{1}{2}\frac{1}{2}V_2Z_1^{-1}U_1^{-2}V_1^{-1} + kV_2U_2^2Z_2Z_1^{-1}U_1^{-2}V_1^{-1} + k^{-1}V_2V_1^{-1}.
\]

For the convenience in expressing \( \tilde{H} \), we denote

\[
W_1 := q^\frac{1}{2}\frac{1}{2}V_2Z_1^{-1}U_1^{-2}V_1^{-1}, \quad W_2 := U_1Z_1U_2^{-1}, \\
W_3 := q^\frac{1}{2}\frac{1}{2}V_2U_2^2Z_2V_1^{-1}, \quad W_4 := U_2Z_2U_1^{-1}.
\]

Note that

\[
W_2W_4 = A^{(0)}A^{(2)}.
\]  

(56)

We have

\[
kV_2U_2^2Z_2Z_1^{-1}U_1^{-2}V_1^{-1} = kq^{-\frac{1}{2}}A^{(0)}W_3W_2^{-1}, \quad k^{-1}V_2V_1^{-1} = k^{-1}q^{-\frac{1}{2}}A^{(2)}W_4^{-1}W_1,
\]

\footnote{Here we use the sans serif type style, instead of the italic type style in \cite{6}, for operators appeared in the right hand side of the expressions, for the purpose of less confusion on notations used in this paper.}
hence
\[ A^{(1)} = W_1 + W_2 + W_3 + W_4 + k q \frac{1}{2} A^{(0)} W_3 W_2^{-1} + k^{-1} q \frac{1}{2} A^{(2)} W_4^{-1} W_1 . \]

With \( q^\frac{1}{2} = q \), we identify the operators appeared in the above \( A^{(j)} \)s with those in \( T_{2j}^* \)s under the condition \( d_0 d_1 d_2 d_3 = 1 \),

\[
A^{(0)} \leftrightarrow D \frac{1}{2} ; \quad A^{(2)} \leftrightarrow \frac{c_1 c_3}{k c_0 c_2} D \frac{1}{2} C^{-1} ;
\]

\[
W_1 \leftrightarrow \frac{k c_0 d_3}{c_1} D \frac{1}{2} U_1 ; \quad W_2 \leftrightarrow \frac{k c_2}{c_1 d_0 d_3} D \frac{1}{2} U_2^{-1} ;
\]

\[
W_3 \leftrightarrow \frac{k c_0 d_3}{c_3} D \frac{1}{2} U_3 ; \quad W_4 \leftrightarrow \frac{c_3 d_1 d_2}{k c_0} D \frac{1}{2} U_4 ;
\]

and then impose further constraints,

\[
W_4^{-1} \leftrightarrow \frac{c_1 d_0 d_3}{k c_2} D \frac{1}{2} U_2 , \quad A^{(2)} \leftrightarrow D \frac{1}{2} . \tag{57}
\]

By the equalities \( V_1 = U_3 U_2, V_4 = U_2 U_1 \) in (13), the \( SG \)-integral \( \tilde{H} \) becomes \( -T_2^* \) in (18). Then (57) gives rise to the identification,

\[
W_2 = W_4 , \quad A^{(0)} = A^{(2)} ,
\]

or equivalently,

\[
\frac{k c_2}{c_1 d_0 d_3} D \frac{1}{2} U_2^{-1} = \frac{c_3 d_1 d_2}{k c_0} D \frac{1}{2} U_4 , \quad \frac{c_1 c_3}{k^2 c_0 c_2} = C .
\]

Note that the above relations are consistent with the relation \( U_2 U_4 = C^{-1} D \) in (13). Hence we obtain the relations (16) (18).

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