Tilting Preenvelopes and Cotilting Precovers in General Abelian Categories

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Abstract
We consider an arbitrary Abelian category $\mathcal{A}$ and a subcategory $\mathcal{T}$ closed under extensions and direct summands, and characterize those $\mathcal{T}$ that are (semi-)special preenveloping in $\mathcal{A}$; as a byproduct, we generalize to this setting several classical results for categories of modules. For instance, we get that the special preenveloping subcategories $\mathcal{T}$ of $\mathcal{A}$ closed under extensions and direct summands are precisely those for which $(-, \mathcal{T})$ is a right complete cotorsion pair, where $\perp^1 \mathcal{T} := \ker(\text{Ext}^1_{\mathcal{A}}(\cdot, \mathcal{T}))$. Particular cases appear when $\mathcal{T} = V \perp^1 := \ker(\text{Ext}^1_{\mathcal{A}}(V, \cdot))$, for an Ext$^1$-universal object $V$ such that Ext$^1_{\mathcal{A}}(V, \cdot)$ vanishes on all (existing) coproducts of copies of $V$. For many choices of $\mathcal{A}$, we show that these latter examples exhaust all the possibilities. We then show that, when $\mathcal{A}$ has an epigenerator, the (semi-)special preenveloping torsion classes $\mathcal{T}$ given by (quasi-)tilting objects are exactly those for which any object $T \in \mathcal{T}$ is the epimorphic image of some object in $\perp^1 \mathcal{T}$ (and the subcategory $\mathcal{B} := \text{Sub}(\mathcal{T})$ of subobjects of objects in $\mathcal{T}$ is reflective) and they are, in turn, the right constituents of complete cotorsion pairs in $\mathcal{A}$ (resp., $\mathcal{B}$). In a final section, we apply the results when $\mathcal{A} = \text{mod-}R$ is the category of finitely presented modules over a right coherent ring $R$, something that gives new results and raises new questions even at the level of classical tilting theory in categories of modules.

Keywords Tilting · Cotilting · Precovering · Preenveloping · Abelian category · Universal extension

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1 Introduction

Co/tilting modules arose in the context of finitely generated modules over finite dimensional algebras, as an attempt to extend to notion of Morita bimodules (see [13, 14, 32]). In this context, equivalences of full module categories were replaced by suitable counter-equivalences of torsion pairs. These results were successively extended to categories of modules over arbitrary rings, but still considering tilting modules with strong finiteness conditions (see [20, 40]). Few years after their introduction, it was noted that co/tilting modules and complexes could be used to construct equivalences between derived categories (see [18, 30, 31, 36, 46]). The last step in this generalization process was that of introducing large (i.e., not finitely generated) counterparts of co/tilting modules and complexes (see [4, 5, 22, 24, 56, 61]).

In parallel to co/tilting theory, the idea of left and right approximations of modules originated in the realm of finitely generated modules over Artin algebras, with the work of Auslander, Smalø and Reiten [8, 9]. On the other hand, the corresponding general notions (for arbitrary modules over general rings) were independently discovered by Enochs [27] and named (pre)envelopes and (pre)covers, see also [29, 59]. This general theory is very useful to give a uniform treatment of, e.g., injective and pure injective hulls, or projective, injective and flat covers.

Our starting point in this note is the observation that, when we restrict our attention to categories of modules, known characterizations relate (quasi-)tilting torsion pairs with (semi-)special preenveloping torsion classes (see [7]) and these with cotorsion pairs. Dual results do hold for (quasi-)cotilting modules. Our main goal in this paper is to understand this relation in the context of arbitrary Abelian categories. In this framework we study, for an Abelian category \( A \) and a subcategory \( T \) closed under extensions and direct summands, the relationships between the following properties (and their respective duals):

- \( T \) is the torsion class given by a (quasi-)tilting object;
- \( T \) is (semi-)special preenveloping;
- \( T \) is the right constituent of a cotorsion pair.

Our main results show that the special preenveloping subcategories \( T \) closed under extensions and direct summands are the right constituents of right complete cotorsion pairs and, when \( A \) has an epi-generator, the special preenveloping torsion classes are of the form \( T = V_{\perp} := \text{Ker}(\text{Ext}^1_A(V, -)) \) for a suitable object \( V \in \mathcal{T} \) (see Theorem 5.14 and Corollary 5.16). Among them, the tilting torsion classes are precisely the special preenveloping torsion classes \( T \) such that \( \perp T \) generates \( T \) (see Corollary 6.12), while quasi-tilting torsion classes \( T \) such that \( \text{Sub}(T) \) is a reflective subcategory of \( A \) are precisely the semi-special preenveloping torsion classes \( T \) such that \( \perp T \) generates \( T \) (see Proposition 6.11).

Let us note that, unlike in the context of module categories, our proofs do not rely on the existence of a projective generator nor on the presence of an injective cogenerator. Furthermore, we do not assume exactness of (co)products and this forces us to develop new tools to control the exactness of certain coproducts (see Section 5.1) and a general theory of universal extensions (see Section 5.2). With these results we are able to give new proofs for the correspondences mentioned above, that hold in our completely general setting.

The organization of the paper goes as follows:

In Sections 2 and 3, we introduce the needed concepts and tools in general Abelian categories, with special attention to those Abelian categories that have an epi-generator, a class
that includes all (Ab.3) and Hom-finite Abelian categories with a generator. Furthermore, we discuss Hom- and Ext^1-finite Abelian categories.

In Section 4 we study (co)reflective subcategories in an Abelian category, with special emphasis on the subcategories of the form Sub(\mathcal{X}) and Quot(\mathcal{X}), consisting of the objects which are isomorphic, respectively, to subobjects and quotients of objects in \mathcal{X}, for a subcategory \mathcal{X} \subseteq \mathcal{A}.

In Section 5 we study universal extensions and introduce the notion of Ext^1-universal object. These concepts are fundamental for the main result of the paper, Theorem 5.14, and its Corollary 5.16, from which most of the subsequent results are a consequence. Let us remark that, to construct an effective theory of universal extensions in the case of “big” (i.e., cocomplete) Abelian categories, we need to have some control on the exactness of infinite coproducts. For this, we introduce and study a “formal derived functor” of the coproduct in Section 5.1.

In Section 6 we define (quasi-)tilting objects in arbitrary Abelian categories and show that tilting objects are Ext^1-universal (see Proposition 5.7). We show that our general concept of (quasi-)tilting object coincides with the classical ones appearing in the literature in module categories and in Hom- and Ext^1-finite Abelian categories. When the ambient Abelian category \mathcal{A} has an epi-generator, Proposition 6.11 and Corollary 6.12 identify the (semi-)special preenveloping torsion classes which are given by (quasi-)tilting objects, thus extending known results in module categories. In the final Section 6.4 we just state the duals of Theorem 5.14 and Proposition 6.11, that give as a consequence Proposition 6.20, a result that extends [16, Thm. 3.5] and [63, Thm. 3.5] from module categories to Grothendieck categories.

In the final section, we focus on the case when \mathcal{A} = \text{mod-}R is the category of finitely presented (right) modules over a right coherent ring \textit{R}. It turns out that the tilting objects \textit{V} of \text{mod-}R are, up to equivalence, the classical (=finitely presented) (1-)tilting \textit{R}-modules whose associated torsion pair in \text{Mod-}R restricts to \text{mod-}R, equivalently, such that \text{End}(\textit{V}_R) is a right coherent ring (see Corollary 6.7). We do not know if all classical tilting modules over a right coherent ring satisfy this latter property.

2 Basic Definitions and Notation

Throughout the paper, all subcategories are implicitly assumed to be full. Furthermore, we fix the following conventions:

- the symbol \mathcal{A} denotes an Abelian category;
- given two objects \textit{A} and \textit{B} in \mathcal{A}, we let \mathcal{A}(\textit{A}, \textit{B}) := \text{Hom}_\mathcal{A}(\textit{A}, \textit{B});
- \text{Proj-}\mathcal{A} denotes the subcategory of projective objects in \mathcal{A};
- the symbol \textit{R} always denotes a (unitary and associative) ring;
- \text{Mod-}\textit{R} and \text{R-Mod} (resp., \text{mod-}\textit{R} and \text{R-mod}) denote the categories of (finitely presented) right and left \textit{R}-modules, respectively;
- \text{Proj-}\textit{R} (resp., proj-\textit{R}) denotes the category of (finitely presented) projective right \textit{R}-modules;
- given a ring \textit{R} and \textit{M}, \textit{N} \in \text{Mod-}\textit{R}, we let \text{Hom}_\text{Mod-}\textit{R}(\textit{M}, \textit{N}) := \text{Hom}_{\text{Mod-}\textit{R}}(\textit{M}, \textit{N}) and \text{End}_\text{Mod-}\textit{R}(\textit{M}) := \text{End}_{\text{Mod-}\textit{R}}(\textit{M}), and we use similar notations for left modules.

Given a class of objects \mathcal{X} in an Abelian category \mathcal{A}, we let

- \mathcal{X}^\perp := \{ \textit{A} \in \mathcal{A} : \mathcal{A}(\textit{X}, \textit{A}) = 0, \ \forall \textit{X} \in \mathcal{X} \};
- \mathcal{X}^\perp := \{ \textit{A} \in \mathcal{A} : \mathcal{A}(\textit{A}, \textit{X}) = 0, \ \forall \textit{X} \in \mathcal{X} \};
\[ \text{Add}_A(\mathcal{X}) \text{ (resp., add}_A(\mathcal{X}) \text{)} \text{ denotes the class of direct summands of coproducts, taken in } A, \text{ of (finite) families of objects taken from } \mathcal{X}; \]

\[ \text{Gen}_A(\mathcal{X}) \text{ (resp., gen}_A(\mathcal{X}) \text{)} \text{ is the class of quotients of objects in } \text{Add}_A(\mathcal{X}) \text{ (add}_A(\mathcal{X}) \text{));} \]

\[ \text{Pres}_A(\mathcal{X}) \text{ (resp., pres}_A(\mathcal{X}) \text{)} \text{ is the class of cokernels of morphisms in } \text{Add}_A(\mathcal{X}) \text{ (add}_A(\mathcal{X}) \text{)).} \]

When the category \( A \) is clear from the context, we generally omit it as a subscript. Finally, \( \text{Cogen}(\mathcal{X}), \text{Copres}(\mathcal{X}), \text{cogen}(\mathcal{X}) \) and \( \text{copres}(\mathcal{X}) \) are defined dually.

**Remark 2.1** Unless otherwise stated, in this paper Abelian categories are not assumed to have (all small) (co)products. So the (co)products that appear in the definitions of, say, \( \text{Pres}(\mathcal{X}) \) and \( \text{Copres}(\mathcal{X}) \) are just those that exist in \( A \). This same convention applies throughout the paper.

### 2.1 Abelian Exact Subcategories

A subcategory \( B \) of an Abelian category \( A \) is said to be an **Abelian exact subcategory** of \( A \) when it is Abelian and the inclusion functor \( B \to A \) is exact. Equivalently, \( B \) is closed under taking finite co/products, kernels and cokernels in \( A \).

**Definition 2.2** Given a subcategory \( \mathcal{X} \) of an Abelian category \( A \), define:

- \( \text{Sub}(\mathcal{X}) \subseteq A \), the class of objects that embed in some object in \( \mathcal{X} \);
- \( \text{Quot}(\mathcal{X}) \subseteq A \), the class of objects that are epimorphic images of some object in \( \mathcal{X} \).

The class \( \mathcal{X} \) is said to be **generating** (resp., **cogenerating**) if \( A = \text{Quot}(\mathcal{X}) \) (resp., \( A = \text{Sub}(\mathcal{X}) \)).

The classes of the form \( \text{Sub}(\mathcal{X}) \) and \( \text{Quot}(\mathcal{X}) \) can be used to construct natural examples of Abelian exact subcategories of \( A \):

**Example 2.3** Let \( A \) be an Abelian category and \( \mathcal{X} \) a subcategory closed under finite co/products, then \( \text{Quot}(\text{Sub}(\mathcal{X})) = \text{Sub}(\text{Quot}(\mathcal{X})) \) is an Abelian exact subcategory of \( A \). Indeed, given \( M \in \text{Sub}(\text{Quot}(\mathcal{X})) \), there is an embedding \( M \to Q \) and an epimorphism \( X \to Q \), for some \( X \in \mathcal{X} \). Consider then the following pullback diagram:

\[
\begin{array}{ccc}
S & \rightarrow & M \\
\downarrow & & \downarrow \\
X & \rightarrow & Q
\end{array}
\]

that shows that \( M \in \text{Quot}(\text{Sub}(\mathcal{X})) \), hence \( \text{Sub}(\text{Quot}(\mathcal{X})) \subseteq \text{Quot}(\text{Sub}(\mathcal{X})) \). The converse inclusion follows by a dual argument, using a pushout square. Hence, \( \text{Quot}(\text{Sub}(\mathcal{X})) \) is closed under taking quotients, subobjects and it is easily seen to be also closed under taking finite co/products. In particular, we obtain the following statements:

- if \( \mathcal{X} \) is also closed under quotients, \( \text{Sub}(\mathcal{X}) \) is an Abelian exact subcategory of \( A \);
- if \( \mathcal{X} \) is also closed under subobjects, \( \text{Quot}(\mathcal{X}) \) is an Abelian exact subcategory of \( A \).
2.2 Ext-Groups

For each positive integer \( n \), we denote by \( \text{Ext}_A^n(\cdot, \cdot) \) the big group of (equivalence classes of) \( n \)-fold extensions, with the usual Baer sum, in the category \( A \) (see [39]). Given a class of objects \( \mathcal{X} \) in \( A \), we use the following notations:

\[
\mathcal{X}^\perp \! n = \{ A \in A : \text{Ext}_A^n(X, A) = 0, \forall X \in \mathcal{X} \}
\]

and

\[
\perp \! n \mathcal{X} = \{ A \in A : \text{Ext}_A^n(A, X) = 0, \forall X \in \mathcal{X} \}.
\]

When \( \mathcal{X} = \{ X \} \), for simplicity we will write \( X^\perp \! n \) (resp. \( \perp \! n X \)) instead of \( X^\perp \! n \) (resp. \( \perp \! n X \)).

Let us start with the following elementary remark that will be often useful hereafter.

**Remark 2.4** Let \( A \) be an Abelian category, \( X \in A \) and \( \{ Y_i \}_I \) a family of objects whose coproduct exists in \( A \). Then, the following canonical map is injective

\[
\phi : \coprod_I A(X, Y_i) \longrightarrow A(X, \coprod_I Y_i).
\]

In fact, if \( (f_i)_I \in \coprod_I A(X, Y_i) \) and we take \( J := \{ i \in I : f_i \neq 0 \} \), which is a finite subset of \( I \), then \( f := \phi((f_i)_I) \) decomposes as \( \iota_J \circ f_J : X \rightarrow \prod_I Y_J \cong (\coprod_I Y_J) \hookrightarrow \coprod_I Y_i \) where \( f_J \) is given by the universal property of (finite) products and \( \iota_J \) is the canonical section. If \( \phi((f_i)_I) = f = 0 \) then \( f_J = 0 \), which implies that \( J = \emptyset \) and so \( (f_i)_I = 0 \).

A similar “embedding property” also holds for \( \text{Ext}^1 \)-groups, as the following lemma shows:

**Lemma 2.5** Let \( A \) be an Abelian category and \( \{ V_i \}_I \) a non-empty family of objects whose coproduct exists in \( A \). Then, for any \( X \in A \), the following canonical morphism of big Abelian groups is injective:

\[
\Phi : \text{Ext}_A^1(\coprod_I V_i, X) \longrightarrow \prod_I \text{Ext}_A^1(V_i, X).
\]

In particular, the following assertions hold true:

1. given \( V \in A \) and a set \( J \neq \emptyset \) such that \( V^{(J)} \) exists in \( A \), then \( V^\perp \! 1 = (V^{(J)})^\perp \! 1 \);
2. if \( \mathcal{X} \) is a subcategory of \( A \), then \( \perp \! 1 \mathcal{X} \) is closed under taking coproducts in \( A \).

**Proof** Assertion (2) clearly follows from the initial statement of the lemma. As for assertion (1), let us fix \( j \in J \). Then we have a decomposition \( V^{(J)} \cong V \oplus V^{(J \setminus \{j\})} \) which, in turn, gives a decomposition of big Abelian groups

\[
\text{Ext}_A^1(V^{(J)}, X) \cong \text{Ext}_A^1(V, X) \oplus \text{Ext}_A^1(V^{(J \setminus \{j\})}, X).
\]

This gives the inclusion “\( \geq \)” of the desired equality. The inclusion “\( \subseteq \)” follows directly from the initial statement of the lemma.

To prove the initial statement, let \([\epsilon]\) be an element of the kernel of \( \Phi \), represented by the following short exact sequence in \( A \):

\[
\epsilon : \quad 0 \rightarrow X \xrightarrow{u} Y \xrightarrow{p} \coprod_I V_i \rightarrow 0. \tag{1}
\]

For each \( j \in I \), if \( i_j : V_j \rightarrow \coprod_I V_i \) is the \( j \)-th inclusion into the coproduct, the map \( t_j^\ast := \text{Ext}_A^1(i_j, X) \) decomposes as follows:

\[
\text{Ext}_A^1(\coprod_I V_i, X) \xrightarrow{\Phi} \prod_I \text{Ext}_A^1(V_i, X) \xrightarrow{\pi_j} \text{Ext}_A^1(V_j, X),
\]
where \( \pi_j \) is the \( j \)-th projection from the product in \( \text{Ab} \). Therefore, \( \iota^*_j(\epsilon) = 0 \), which implies that \( \iota_j \) has the following factorization:

\[
V_j \xrightarrow{\lambda_j} Y \xrightarrow{p} \coprod_I V_i,
\]

for some morphism \( \lambda_j : V_j \to Y \). We obtain a family of morphisms \( \{\lambda_j : V_j \to Y\}_I \) in \( \mathcal{A} \) which, by the universal property of coproducts, gives a unique morphism \( \lambda : \coprod_I V_i \to Y \) such that \( \lambda \circ \iota_j = \lambda_j \) for all \( j \in I \). Thus \( p \circ \lambda = \text{id}_{\coprod_I V_i} \), showing that the sequence Eq. 1 splits, and so \( [\epsilon] = 0 \).

Recall now that, given \( n \in \mathbb{N} \) and an object \( X \) in an Abelian category \( \mathcal{A} \), we say that the projective dimension of \( X \) is less than or equal to \( n \), in symbols \( \text{pd}_{\mathcal{A}}(X) \leq n \), if \( X \perp_{n+1} \mathcal{A} \).

**Corollary 2.6** Let \( \mathcal{A} \) be an Abelian category, \( V \in \mathcal{A} \), and \( I \) a set for which \( V^{(I)} \) exists in \( \mathcal{A} \). If there is a cogenerating class \( \mathcal{X} \subseteq V^{\perp_1} \) which is closed under taking quotients, then \( \text{pd}_{\mathcal{A}}(V^{(I)}) \leq 1 \).

**Proof** Given \( A \in \mathcal{A} \) and using that \( \mathcal{X} \) is cogenerating one gets, by a suitable pushout construction, that any element \( [\epsilon] \in \text{Ext}^2_\mathcal{A}(V^{(I)}, A) \) is represented by an exact sequence of the form:

\[
\epsilon : 0 \to A \xrightarrow{f} X \xrightarrow{f'} Y \xrightarrow{f''} V^{(I)} \to 0,
\]

where \( X \) is in \( \mathcal{X} \). But \( \text{Im}(f) =: X' \) is also in \( \mathcal{X} \) since this class is closed under taking quotients. It follows that \([\epsilon] \in \text{Ext}^1_{\mathcal{A}}(X', A) \circ \text{Ext}^1_{\mathcal{A}}(V^{(I)}, X')\), where \( \circ \) denotes the Yoneda product. Then, \( \text{Ext}^1_{\mathcal{A}}(V^{(I)}, X') = 0 \) (see Lemma 2.5(1)), and so \( \text{Ext}^2_{\mathcal{A}}(V^{(I)}, A) = 0 \). \( \square \)

### 2.3 Complexes, Homotopy Category and Derived Category

For an Abelian category \( \mathcal{A} \) and \( B \) an additive subcategory of \( \mathcal{A} \), we denote by \( \text{Ch}(\mathcal{A}) \), \( \text{Ch}(B) \), \( K(\mathcal{A}) \) and \( K(B) \), the category of, cohomologically graded, cochain complexes, and the homotopy category over \( \mathcal{A} \) and \( B \), respectively. Moreover, we denote by \( D(\mathcal{A}) \), the derived category of \( \mathcal{A} \). We use the suffixes \( b, +, \) and \( - \) to denote the corresponding subcategories consisting of bounded, bounded below and bounded above complexes. For instance, \( K^+(\mathcal{A}) \) denotes the homotopy category of bounded below complexes. We also use the symbol \( K^{-, b}(B) \) to denote the subcategory of \( K^-(B) \) whose objects have almost all cohomology groups equal to zero.

Recall that the bounded above derived category \( D^{-}(\mathcal{A}) \) is well-defined whenever \( \mathcal{A} \) has enough projectives and, in fact, the canonical composition

\[
K^{-}(\text{Proj-}\mathcal{A}) \hookrightarrow K^{-}(\mathcal{A}) \xrightarrow{q} D^{-}(\mathcal{A})
\]

is a triangulated equivalence (see [60]). Dual considerations hold for the bounded below derived category of an Abelian category with enough injectives. Finally, let us recall that, when \( \mathcal{A} \) has enough projectives, \( I \) is a set such that \( I \)-coproducts exist in \( \mathcal{A} \) and \( \{V_i\}_I \) is a family of objects in \( \mathcal{A} \), then we can compute the coproduct of the stalk complexes \( V_i[0] \) in \( D^{-}(\mathcal{A}) \) as follows. We start by fixing, for each \( j \in I \), a projective resolution
2.4 Torsion Pairs

A torsion pair \( t = (\mathcal{T}, \mathcal{F}) \) in an Abelian category \( \mathcal{A} \) is a pair of subcategories, called the torsion and the torsion-free class, respectively, such that \( \mathcal{T}^\perp = \mathcal{F}, \mathcal{T}^\perp = \mathcal{F} \) and such that, for any \( A \in \mathcal{A} \), there is a (necessarily functorial) short exact sequence

\[
0 \to T_A \to A \to F_A \to 0, \quad \text{with } T_A \in \mathcal{T} \text{ and } F_A \in \mathcal{F}.
\]

As a consequence, the class \( \mathcal{T} \) (resp., \( \mathcal{F} \)) is closed under taking quotients (resp., subobjects), extensions and all small coproducts (resp., small products) that exist in \( \mathcal{A} \). We let

\[
t: \mathcal{A} \to \mathcal{T} \quad \text{and} \quad (1 : t): \mathcal{A} \to \mathcal{F}
\]

be the functors acting on objects by \( A \mapsto T_A \) and \( A \mapsto F_A \), respectively. They are, respectively, right and left adjoints to the corresponding inclusions.

Recall that a torsion pair \( t = (\mathcal{T}, \mathcal{F}) \) in \( \mathcal{A} \) is said to be hereditary (resp., cohereditary, of finite type) provided \( \mathcal{T} \) is closed under taking subobjects (resp., \( \mathcal{F} \) is closed under taking quotients, \( \mathcal{F} \) is closed under taking direct limits). Furthermore, \( \mathcal{T} \) is said to be a TTF class if it is both a torsion and a torsion-free class; in this last case, \((\perp \mathcal{T}, \mathcal{T}, \mathcal{T}^\perp)\) is said to be a TTF triple.

If \( \mathcal{A} \) is a Grothendieck category (see Section 3.2) and \( t = (\mathcal{T}, \mathcal{F}) \) is a hereditary torsion pair, then we denote by \( Q_t: \mathcal{A} \to \mathcal{A}/\mathcal{T} \) the Gabriel quotient of \( \mathcal{A} \) over \( \mathcal{T} \) (which is always an exact functor), and we let \( S_t: \mathcal{A}/\mathcal{T} \to \mathcal{A} \) be the fully faithful right adjoint of \( Q_t \).

If \( \mathcal{A} = \text{Mod-}R \) is the category of right \( R \)-modules over a ring \( R \), then we can consider the ring of quotients \( \psi: R \to R_t \) of \( R \) with respect to the Gabriel topology associated with \( t \) (see [55, Chapter IX]). We get the following diagram:

\[
\begin{array}{ccc}
\text{Mod-}R & \overset{Q_t}{\longrightarrow} & \text{(Mod-}R)/\mathcal{T} \\
\psi^t \downarrow & & \psi^t \circ S_t \downarrow \\
\text{Mod-}R_t & \overset{Q_t \circ \psi_s}{\longrightarrow} & \text{(Mod-}R)/\mathcal{T} \\
\end{array}
\]

where \( \psi_s := - \otimes_{R_t} R_t \) is the restriction of scalars along \( \psi \), and \( \psi^t := \text{Hom}_R(R_t, -) \) its right adjoint. This shows that \( Q_t \circ \psi_s \) is exact (as it is a composition of two exact functors) and that \( \psi^t \circ S_t \) is its right adjoint, which is fully faithful (see [55, pages 199 and 217], where this functor is denoted by \( j \)). Hence, letting \( \mathcal{T}' := \ker(Q_t \circ \psi_s) = (\psi_s)^{-1}(\mathcal{T}) \) and \( t' := (\mathcal{T}', (\mathcal{T}')^\perp) \), we obtain a hereditary torsion pair in \( \text{Mod-}R_t \), an equivalence of categories \((\text{Mod-}R)/\mathcal{T} \cong (\text{Mod-}R_t)/\mathcal{T}' \) and, identifying these two categories, we also get two natural isomorphisms of functors \( Q_{t'} \cong Q_t \circ \psi_s \) and \( S_{t'} \cong \psi^t \circ S_t \). Our original torsion pair \( t \) is said to be perfect if \( Q_{t'} \) is an equivalence of categories or, equivalently, if \( \mathcal{T}' = 0 \) (see [55, Prop. XI.3.4] for other equivalent conditions).

2.5 Cotorsion Pairs

Cotorsion pairs were first introduced by Salce [49] in categories of modules. A pair \((\mathcal{X}, \mathcal{Y})\) of subcategories of an Abelian category \( \mathcal{A} \) is said to be a cotorsion pair when \( \mathcal{Y} = \mathcal{X}^{\perp 1} \) and \( \mathcal{X} = (\perp \mathcal{Y}) \).

A cotorsion pair is called right (resp., left) complete when, for each object
A ∈ ℱ, there is a short exact sequence 0 → A → Y → X → 0 (resp., 0 → Y → X → A → 0), where X ∈ ℱ′ and Y ∈ ℱ. Some authors say instead that (ℱ′, ℱ) has enough injectives (resp., projectives) when (ℱ′, ℱ) is right (resp., left) complete (see [35]).

A cotorsion pair is complete if it is both left and right complete.

### 2.6 Covers and Envelopes

Let ℱ be an Abelian category and recall that a morphism φ: X → Y in ℱ is said to be left minimal if each morphism ψ: Y → Y such that ψ ◦ φ = φ is an isomorphism. The notion of a morphism being right minimal is defined dually.

Let now X be a subcategory of ℱ. A morphism φ: M → XM with XM ∈ X is:

- an X-preenvelope if, for any Y ∈ ℱ, the induced map ℱ(XM, Y) → ℱ(M, Y) is surjective;
- a semi-special X-preenvelope if it is a X-preenvelope and Coker(φ) ∈ ⊥1X;
- a special X-preenvelope if it is a monomorphic semi-special X-preenvelope;
- an X-envelope if it is a left minimal X-preenvelope.

If ℱ is such that every object has an X-preenvelope (resp., semi-special preenvelope, special preenvelope, envelope), then ℱ is said to be preenveloping (resp., semi-special preenveloping, special preenveloping, enveloping). The dual notions are those of X-precover, semi-special X-precover, special X-precover, X-cover and any subcategory ℱ for which those notions exist for all objects in ℱ is called precovering, semi-special precovering, special precovering, and covering.

(Pre)envelopes and (pre)covers usually take the name of the classes over which they are constructed. Thus, the notions of injective (pre)envelopes, pure-injective (pre)envelopes, flat (pre)covers, etc. appear naturally in the categories where the corresponding classes can be defined.

**Example 2.7** Let (ℱ′, ℱ) be a right (resp., left) complete cotorsion pair in an Abelian category ℱ. For each A ∈ ℱ, consider an exact sequence 0 → A → Y → X → 0 (resp., 0 → Y → X → A → 0) as above. Then, u (resp., p) is a special ℱ-preenvelope (resp., special ℱ-precover).

Let us conclude this subsection with a technical but useful lemma:

**Lemma 2.8** Let ℱ be an Abelian category, ℱ ⊆ ℱ a subcategory, and X ∈ ℱ.

1. If X embeds in a product of objects of ℱ, then any ℱ-preenvelope of X is a monomorphism;
2. when ℱ is closed under taking coproducts, any coproduct of ℱ-preenvelopes, when it exists, is a ℱ-preenvelope;
3. suppose that Cogen(ℱ) = ℱ, then the following assertions hold:
   1. if φ: A → B is a morphism in ℱ such that the map
      \[ φ^*: ℱ(B, T) → ℱ(A, T) \]
      is surjective for all T ∈ ℱ, then φ is a monomorphism.
   2. any coproduct of ℱ-preenvelopes, when it exists, is a monomorphism.
Proof (1). Let $\phi : X \to T$ be a $T$-preenvelope and $\psi : X \to \prod_I T_i$ an embedding, for some family $(T_i)_I$ in $T$. Put $\psi_j := \pi_j \circ \psi$, where $\pi_j : \prod_I T_i \to T_j$ is the $j$-th projection, for each $j \in I$. Then, there is a map $\alpha_j : T \to T_j$ such that $\psi_j = \alpha_j \circ \phi$, for all $j \in I$. By the universal property of products, we get a map $\alpha : T \to \prod_I T_i$ such that $\pi_j \circ \alpha = \alpha_j$, for all $j \in I$. Hence, $\alpha \circ \phi = \psi$, since $\pi_j \circ \alpha \circ \phi = \alpha_j \circ \phi = \psi_j = \pi_j \circ \psi$, for all $j \in I$. Therefore, $\phi$ is a monomorphism as so is $\psi$.

(2). Let $(f_i : X_i \to T_i)_I$ be a family of $T$-preenvelopes, such that $\coprod_I X_i$ and $\coprod_I T_i$ exist in $A$. For each $i$ in $I$, let $\epsilon_{X_i} : X_i \to \coprod_I X_i$ and $\epsilon_{T_i} : T_i \to \coprod_I T_i$ be the inclusions in the respective coproducts. Given a morphism $\phi : \coprod_I X_i \to T'$ with $T' \in T$, there exists, for each $i \in I$, a morphism $\psi_i : T_i \to T'$ such that $\phi \circ \epsilon_{X_i} = \psi_i \circ f_i$. Hence, there is a unique map $\psi : \coprod_I T_i \to T'$ such that $\psi \circ \epsilon_{T_i} = \psi_i$ for all $i \in I$, and so $\psi \circ (\coprod_I f_i) = \phi$. Assertion (2) then follows, as $\coprod_I T_i \in T$.

(3). We just need to prove (3.1), for then (3.2) is clearly a consequence of assertions (1) and (3.1). Fix a monomorphism $\mu : A \hookrightarrow \prod_I T_i$ for some family $(T_i)_I$ in $T$. Due to the hypothesis on $\phi$, we then have that the morphism

$$A(\phi, \prod_I T_i) : A(B, \prod_I T_i) \cong \prod_I A(B, T_i) \longrightarrow \prod_I A(A, T_i) \cong A(A, \prod_I T_i)$$

is surjective since products are exact in Ab. We get a morphism $\hat{\mu} : B \to \prod_I T_i$ such that $\hat{\mu} \circ \phi = \mu$. Then, $\phi$ is a monomorphism since so is $\mu$.

3 Background on Abelian Categories

Throughout this paper we try to work in Abelian categories which are as general as possible but, from time to time, a few additional hypotheses will be needed. In this section we introduce some of these hypotheses and describe the main classes of examples we are interested in. The hypotheses we introduce will go in two, essentially opposite, directions: on the one hand we will study hypotheses for “big” categories (e.g., (co)completeness, exactness of infinite (co)products, etc.) and, on the other hand, we will study some finiteness conditions (e.g., Hom- and Ext$^1$-finiteness, generators, etc.). Let us start with the following definition:

Definition 3.1 An Abelian category $\mathcal{A}$ is said to be

- Ext$^1$-small provided Ext$^1_A(A, B)$ is a set (as opposed to a proper class) for each pair of objects $A$ and $B$ in $\mathcal{A}$;
- well-powered if the lattice $\mathcal{L}_{\mathcal{A}}(A) := \{\text{subobjects of } A\}$ is a set for all $A \in \mathcal{A}$.

The notion of well-powered category is classical (see, e.g., [39]). Note that both notions introduced in the above definition are self-dual. For this reason, when working in categories satisfying these hypotheses, some concepts and statements will have equivalent dual versions that, unless necessary for our purposes, will not be explicitly given.

3.1 Categories with a Generator

In this first subsection we study some notions of “generator” for an Abelian category. Usually (say, in categories of modules) these notions coincide but, when we work in full generality, some care is needed.
Definition 3.2 A set of objects $\mathcal{G}$ in a category $\mathcal{A}$ is said to be a set of generators when the functors $\{\mathcal{A}(G, -) : \mathcal{A} \to \text{Ab}\}$ are jointly faithful. Furthermore, $\mathcal{G}$ is a set of (finite) epi-generators when $\text{Gen}(\mathcal{G}) = \mathcal{A}$ (resp., $\text{gen}(\mathcal{G}) = \mathcal{A}$). An object $G$ is a (finite) epi-generator if so is the set $\{G\}$. The concepts of (finite, mono-)cogenerator and of set of (finite, mono-)cogenerators are defined dually.

It is not difficult to verify that an epi-generator is also a generator, but the converse might not be true in general. Anyway, we will see that the two notions coincide in our main cases of interest (see Lemmas 3.7 and 3.9). Note also that, if $G$ is an epi-generator of an Abelian category $\mathcal{A}$, then $\mathcal{A} = \text{Gen}(G) = \text{Pres}(G)$, so the following lemma applies to categories with an epi-generator.

Lemma 3.3 Let $\mathcal{A}$ be an Abelian category, $G \in \mathcal{A}$ and $I$ an infinite set for which the coproduct $G^{(I)}$ exists in $\mathcal{A}$. The following assertions hold:

1. if $X \in \text{Pres}(G)$, then the coproduct $X^{(I)}$ exists in $\mathcal{A}$;
2. if $\text{Pres}_I(G)$ denotes the subcategory of those objects $X$ that fit into an exact sequence of the form $G^{(I)} \to G^{(I)} \to X \to 0$, then any $I$-indexed family $(X_i)_I$ in $\text{Pres}_I(G)$ has a coproduct in $\mathcal{A}$ that belongs to $\text{Pres}_I(G)$;
3. if $G$ is a finite epi-generator of $\mathcal{A}$, then $\mathcal{A}$ has $I$-coproducts.

Proof (1). Consider a presentation

$$G^{(J)} \xrightarrow{p} G^{(K)} \to X \to 0$$

and let $\Lambda$ be a set with cardinality $|\Lambda| = \max\{|I|, |J|, |K|\}$ so, by construction, the coproduct $G^{(\Lambda)}$ exists in $\mathcal{A}$. Assume, without loss of generality, that $J = \Lambda = K$. Furthermore, note that $|I \times \Lambda| = |\Lambda|$, so the coproducts in the following exact sequence exist in $\mathcal{A}$:

$$(G^{(\Lambda)})^{(I)} \xrightarrow{p^{(I)}} (G^{(\Lambda)})^{(I)} \to \text{CoKer}(p^{(I)}) \to 0.$$  

Finally, the same argument used in the proof of [42, Lem. 3.1] shows that $\text{CoKer}(p^{(I)})$ represents a coproduct of $|I|$-many copies of $X$ in $\mathcal{A}$.

(2) Let $(G^{(I)} \xrightarrow{f_i} G^{(I)} \xrightarrow{p_i} X_i \to 0)_I$ be an associated family of presentations. Then we get a morphism $\bigsqcup_I f_i : (G^{(I)})^{(I)} \to (G^{(I)})^{(I)}$. This map is well-defined since $(G^{(I)})^{(I)} \cong G^{(I \times I)} \cong G^{(I)}$. By the mentioned argument of [42, Lem. 3.1], we get that $\text{CoKer}(\bigsqcup_I f_i)$ is a coproduct of the family $(X_i)_I$ in $\mathcal{A}$, and obviously it belongs to $\text{Pres}_I(G)$.

(3) is a consequence of the fact that, when $G$ is a finite epi-generator, $\text{Pres}_I(G) = \mathcal{A}$. □

When an Abelian category has an epi-generator, then there is the following useful characterization of some preenveloping classes:

Lemma 3.4 Let $\mathcal{A}$ be an Abelian category with an epi-generator $G$ and $\mathcal{T}$ a subcategory closed under taking coproducts and quotients (e.g., a torsion class) in $\mathcal{A}$. Take $A \in \mathcal{A}$
and suppose that $\mu : G \to T_G$ is a (semi-special) $T$-preenvelope. Fix also an epimorphism $\pi : G^{(I)} \to A$, for some set $I$, and consider the following pushout diagram (see Lemma 3.3):

\[
\begin{array}{ccccccc}
G^{(I)} & \xrightarrow{\mu^{(I)}} & T_G^{(I)} & \xrightarrow{\text{Coker}(\mu^{(I)})} & 0 \\
\downarrow{\pi} & & \downarrow{\text{p.o.}} & & \\
A & \xrightarrow{\mu_A} & T_A & \xrightarrow{\text{Coker}(\mu_A)} & 0 \\
\end{array}
\]

Then $\mu_A$ is a (semi-special) $T$-preenvelope, so the following assertions are equivalent:

1. $G$ has a (semi-special) $T$-preenvelope;
2. $T$ is (semi-special) preenveloping in $A$.

Moreover, $T$ is special preenveloping if, and only if, it is cogenerating (see Definition 2.2) and $G$ has a special $T$-preenvelope.

**Proof** Note that $T_G^{(I)}$ exists by Lemma 3.3. The fact that $\mu_A$ is a (semi-special) $T$-preenvelope follows from the fact that $\mu^{(I)}$ is a (semi-special) $T$-preenvelope by Lemma 2.8(2) (and Lemma 2.5(2)), and by the universal property of pushouts. Finally, the special preenveloping case now follows by Lemma 2.8(3).

Let us conclude with the following lemma, that shows that having a finite epi-generator which is furthermore a projective object is quite a strong requirement:

**Lemma 3.5** An Abelian category $A$ has a projective finite epi-generator if, and only if, it is equivalent to $\text{mod-}R$ for some right coherent ring $R$.

**Proof** Let $P$ be a projective finite epi-generator of $A$, then the category $\text{proj-}A$ coincides with $\text{add}(P)$. Consider now the restricted Yoneda functor

\[ \mathcal{A} \longrightarrow [\text{add}(P)^{\text{op}}, \text{Ab}] \text{ such that } A \mapsto \mathcal{A}(\cdot, A)|_{\text{add}(P)}. \]

This functor induces an equivalence of categories from $\mathcal{A}$ to the category $\text{mod-}\text{add}(P)$ of finitely presented additive functors $\text{add}(P)^{\text{op}} \to \text{Ab}$, that is, the functors $F$ that admit an exact sequence

\[ \text{add}(P)(\cdot, P_1) \longrightarrow \text{add}(P)(\cdot, P_0) \longrightarrow F \longrightarrow 0, \]

with $P_1, P_0 \in \text{add}(P)$ (see [52, Prop. 2.2]). The fact that $\text{mod-}\text{add}(P)$ is Abelian is equivalent to say that the (skeletally small) additive category $\text{add}(P)$ has weak kernels (see [52, Lem. 2.1]). Bearing in mind that $\mathcal{A}(P, \cdot)$ induces an equivalence of categories $\text{add}(P) \cong \text{add}(R_R) = \text{proj-}R$ for the ring $R := \text{End}_A(P)$, it follows that $\text{proj-}R$ has weak kernels, which is equivalent to say that $R$ is right coherent. Since the categories of projectives in $\mathcal{A}$ and in the Abelian category $\text{mod-}R$ are equivalent it follows, again by [52, Prop. 2.2], that $\mathcal{A}$ and $\text{mod-}R$ are equivalent.

A strategy to construct non-trivial examples of finite epi-generators is the following:

**Example 3.6** Let $R$ be a right coherent ring, $t = (T, F)$ a hereditary torsion pair of finite type in $\text{Mod-}R$ and $Q_t : \text{Mod-}R \to \mathcal{G} := (\text{Mod-}R)/T$ the corresponding Gabriel quotient. Then, the following statements hold true:
the category \( \mathcal{A} := \text{fp}(\mathcal{G}) \) of finitely presented objects in \( \mathcal{G} \) is Abelian (that is, \( \mathcal{G} \) is locally coherent);
(2) \( Q_t(R) \) is a finite epi-generator in \( \mathcal{A} \);
(3) \( Q_t(R) \) is projective in \( \mathcal{A} \) if, and only if, \( t \) is perfect.

Proof Part (1) follows by [37, Prop. A.6], while (2) follows by [52, Prop. 2.10]. For (3), consider the notation of diagram Eq. 2. If \( t \) is perfect, then \( S_t \) is an exact functor, in fact, it has both a left and a right adjoint (see [55, Prop. XI.3.4]). Therefore, its left adjoint \( Q_t \) sends projectives to projectives, showing that \( Q_t(R) \) is projective. On the other hand, suppose that \( Q_t(R) \) is projective and let us verify that \( T' = 0 \). Indeed, consider \( T \in T' \) and choose a free presentation of \( T \) in \( \text{Mod-R}_t \):

\[
\begin{array}{c}
R(t) \xrightarrow{\phi} R(t) \\
\text{onto}
\end{array}
\]

Since \( Q_t \) is exact and \( Q_t(T) = 0 \), we have that \( Q_t(\phi) : Q_t(R(t)) \rightarrow Q_t(R(t)) \) is an epimorphism and, since \( Q_t(R(t)) \cong Q_t(R) \) is projective, \( Q_t(\phi) \) is a split epimorphism. Hence, \( \phi \cong (S_t \circ Q_t)(\phi) \) is a (split) epimorphism, showing that \( T = \text{CoKer}(\phi) = 0 \).

3.2 (Co)Complete Abelian Categories

Recall that an Abelian category \( \mathcal{A} \) is said to be (Ab.3), (Ab.4), (Ab.5) (and dually, (Ab.3*), (Ab.4*), (Ab.5*)) if, respectively, it is cocomplete, it is (Ab.3) and it has exact coproducts, and it is (Ab.3) and it has exact directed colimits (it is complete, it is (Ab.3*) and it has exact products, and it is (Ab.3*) and it has exact inverse limits). Let us remind the reader of the following result in category theory.

**Lemma 3.7** Let \( \mathcal{A} \) be an (Ab.3) Abelian category. Then,

(1) an object \( G \) in \( \mathcal{A} \) is a generator if and only if it is an epi-generator;
(2) if \( \mathcal{A} \) has a generator, then it is well-powered and (Ab.3*).

Proof (1). Given a generator \( G \) and an object \( X \in \mathcal{A} \), the natural homomorphism

\[
G(\mathcal{A}(G,X)) \longrightarrow X
\]

is an epimorphism. Hence, \( G \) is also an epi-generator.

(2). By [55, Prop. IV.6.6], we know that \( \mathcal{A} \) is well-powered. On the other hand, given a set \( I \), viewed as a small category, consider the constant diagram functor \( \kappa_I : \mathcal{A} \rightarrow \mathcal{A}^I \). Applying a version of Freyd’s Special Adjoint Functor Theorem, e.g. the dual of [15, Thm. 3.3.4], one can see that \( \kappa_I \) has a right adjoint, which is the \( I \)-product functor \( \prod_I : \mathcal{A}^I \rightarrow \mathcal{A} \).

An Abelian category is said to be Grothendieck if it is (Ab.5) and it has a generator or, equivalently, a set of generators. Note that, by Lemma 3.7, any generator in a Grothendieck category \( \mathcal{A} \) is also an epi-generator and then \( \mathcal{A} \) is bicomplete and well-powered. Furthermore, it is known that Grothendieck categories have enough injectives.

3.3 Hom- and Ext\(^1\)-Finite Categories

There are several classes of natural examples of Grothendieck categories, e.g. categories of modules (over unitary rings, or even small preadditive categories) or of quasi-coherent sheaves over a scheme. These are, in general, “big” categories. On the other hand, in this
paper we are also interested in other classes of “smaller” Abelian categories like, e.g., those of finitely generated modules over Artin algebras. The finiteness properties introduced in this subsection are designed to mimic the properties of this kind of “smaller” examples.

**Definition 3.8** An Abelian category $\mathcal{A}$ is said to be Hom-finite (resp., Ext$^1$-finite) when there is a commutative ring $R$ such that $\mathcal{A}$ is $R$-linear and $\mathcal{A}(A, B)$ (resp., Ext$^1_\mathcal{A}(A, B)$) is finitely generated as an $R$-module, for all objects $A, B \in \mathcal{A}$.

Of course, all Ext$^1$-finite categories are also Ext$^1$-small. As it happens in cocomplete Abelian categories, in Hom-finite Abelian categories any generator is also an epi-generator:

**Lemma 3.9** Let $\mathcal{A}$ be a Hom-finite Abelian category. Then,

1. an object $G$ in $\mathcal{A}$ is a generator if and only if it is a finite epi-generator;
2. if $\mathcal{A}$ has a projective generator $P$, then $\mathcal{A}$ is Ext$^1$-finite.

**Proof** Fix a ground commutative ring $R$ with respect to which $\mathcal{A}$ is Hom-finite.

(1). Given a generator $G$ and an object $X \in \mathcal{A}$, the $R$-module $\mathcal{A}(G, X)$ is finitely generated, so there is a finite subset $\{f_1, \ldots, f_n\} \subseteq \mathcal{A}(G, X)$ such that every morphism $f \in \mathcal{A}(G, X)$ can be written as $f = a_1 f_1 + \cdots + a_n f_n$, for some $a_i \in R$ and $i = 1, \ldots, n$. Consider the coproduct $G^{(n)}$, and denote by $\iota_i : G \to G^{(n)}$ the $i$-th inclusion ($i = 1, \ldots, n$). Then there is a unique morphism $\phi : G^{(n)} \to X$ such that $\phi \circ \iota_i = f_i$, for all $i = 1, \ldots, n$, giving an exact sequence

$$\begin{CD}
G^{(n)} @> \phi >> X @> p >> \text{CoKer}(\phi) @>>> 0.
\end{CD}$$

To show that $\phi$ is an epimorphism, we have to verify that CoKer$(\phi) = 0$ or, equivalently, that $p = 0$. Since $G$ is a generator, this is the same as showing that $\mathcal{A}(G, p) : \mathcal{A}(G, X) \to \mathcal{A}(G, \text{CoKer}(\phi))$ is the trivial map. Apply $\mathcal{A}(G, -)$ to the sequence Eq. 5 to get the following two maps in Mod-$R$:

$$\begin{CD}
\mathcal{A}(G, G^{(n)}) @> \mathcal{A}(G, \phi) >> \mathcal{A}(G, X) @> \mathcal{A}(G, p) >> \mathcal{A}(G, \text{CoKer}(\phi)).
\end{CD}$$

To conclude, note that $\mathcal{A}(G, p) \circ \mathcal{A}(G, \phi) = \mathcal{A}(G, p \circ \phi) = 0$, while $\mathcal{A}(G, \phi)$ is surjective by construction. Hence, $\mathcal{A}(G, p) = 0$, as desired.

(2). Consider two objects $A$ and $B$ in $\mathcal{A}$, and a short exact sequence

$$\begin{CD}
0 @>>> \text{Ker}(\pi) @>>> p^{(\alpha)} @> \pi >> \mathcal{A}(\pi) @>>> 0.
\end{CD}$$

Applying the functor $\mathcal{A}(\pi, -)$, we get the following long exact sequence in Mod-$R$:

$$\begin{CD}
\cdots @>>> \mathcal{A}(p^{(\alpha)}, B) @>>> \mathcal{A}(\text{Ker}(\pi), B) @>>> \text{Ext}^1_\mathcal{A}(A, B) @>>> \text{Ext}^1_\mathcal{A}(p^{(\alpha)}, B) = 0,
\end{CD}$$

where $\text{Ext}^1_\mathcal{A}(P^{(\alpha)}, B) = 0$ since $P^{(\alpha)}$ is projective. Now, $\mathcal{A}(\text{Ker}(\pi), B)$ is finitely generated over $R$, so also its epimorphic image $\text{Ext}^1_\mathcal{A}(A, B)$ has to be a finitely generated $R$-module.

Our next goal is to show that Hom-finite categories are very special (see Proposition 3.11), but let us first remind the reader of the following result, for which it is essential to assume that the ring is commutative (see [12]):
Lemma 3.10 Let $R$ be a non-zero commutative ring, $M$ a non-zero finitely generated $R$-module and $I$ an infinite set. Then, $M$ is not isomorphic to $M^I$ in Mod-$R$.

Proof $M$ has a maximal submodule, say $N$ (see [2, Coro. 10.5]). Then, there is a maximal ideal $m$ of $R$, together with an isomorphism $M/N \cong R/m$. It follows that $Mm \subseteq N \subseteq M$. This implies that $M^I m \subseteq (Mm)^I \subseteq M^I$ and, therefore, there exists an epimorphism $\pi : M^I / M^I m \to M^I / (Mm)^I \cong (M/Mm)^I$, where the last isomorphism is due to the (Ab,4*) condition of Mod-$R$. Suppose, looking for a contradiction, that $M \cong M^I$ in Mod-$R$. Then $\pi$ induces an epimorphism $\pi' : M/Mm \to (M/Mm)^I$ of $R/m$-vector spaces, which is absurd since $M/Mm$ is finite dimensional over $R/m$. □

Proposition 3.11 Let $A$ be a Hom-finite Abelian category. The following assertions hold true:

(1) $A(A, A)$ is a left and right coherent ring, for every object $A \in A$;
(2) $A(A, B)$ is finitely presented both as a left $A(B, B)$- and right $A(A, A)$-module, for all $A, B \in A$;
(3) given an infinite set $I$, there is no non-zero $X \in A$ such that the coproduct $X^I$ exists in $A$.

Proof Fix a commutative ring $R$ with respect to which $A$ is Hom-finite. For $A, B, C \in A$ and $E := A(A, A)$, the right $E$-module $A(A, B)$ is finitely generated (as any set of generators over $R$ also generates $A(A, B)$ over $E$). Thus, given $f : B \to C$ and denoting $f_* := A(A, f) : A(A, B) \to A(A, C)$, the right $E$-module $\text{Ker}(f_*) \cong A(A, \text{Ker}(f))$ is finitely generated.

(1). We just prove that $E$ is right coherent, the proof of left coherence is symmetric. It is well-known (and easily checked) that the functor $A(A, -) : A \to \text{Mod-}R$ induces an equivalence $\text{sum}(A) \cong \text{free-}E$, where $\text{sum}(A) \subseteq A$ is the subcategory of finite coproducts of copies of $A$ and free-$E$ denotes the category of finitely generated free $E$-modules. Hence, any morphism $\varphi : E^n \to E$ in Mod-$E$ is of the form $\varphi = f_n$, for a unique morphism $f : A^n \to A$ in $A$. By the initial discussion, $\text{Ker}(\varphi)$ is a finitely generated right $E$-module. Thus, each finitely generated right ideal of $E$ is finitely presented, that is, $E$ is right coherent.

(2). We just verify that $A(A, B)$ is a finitely presented right $E$-module, the rest can be proved similarly. Let \{f_1, \ldots, f_t\} be a set of generators of $A(A, B)$ as an $R$-module and consider the induced map $f := (f_1, \ldots, f_t) : A^n \to B$. By the initial discussion, $\text{Ker}(f_*)$ is a finitely generated right $E$-module and, therefore, $A(A, B) \cong \text{Im}(f_*)$ is finitely presented over $E$.

(3). Suppose that $X^I$ exists. Then, replacing $X$ by $X^I$ if necessary, we can assume that $X \cong X^I$. This gives isomorphisms $A(X, X) \cong A(X^I, X) \cong A(X, X)^I$ in $A(X, -)$-Mod, which are also isomorphisms of $R$-modules. Since $R$ is commutative, this contradicts Lemma 3.10. □

3.4 Krull-Schmidt Categories

Recall that an additive category with splitting idempotents $C$ is Krull-Schmidt when each of its objects is a finite coproduct of (indecomposable) objects whose endomorphism rings are local. Equivalently, when $\text{End}_C(X)$ is a semiperfect ring for each object $X \in C$ (see [17, Thm. A.1]). The following is the key result that makes Krull-Schmidt Abelian categories particularly convenient for our study of preenveloping torsion classes.
Lemma 3.12 Let \( \mathcal{A} \) be a Krull-Schmidt Abelian category and \( \mathcal{T} \subseteq \mathcal{A} \) a subcategory closed under direct summands. Then, \( \mathcal{T} \) is preenveloping if and only if it is enveloping.

Proof Under our hypotheses, it is well-known that, for any morphism \( f : A \to B \) in \( \mathcal{A} \), there is a decomposition \( B \cong B_1 \oplus B_2 \) such that \( f \) decomposes matricially as
\[
f = (f_1, 0)^t : A \to B_1 \oplus B_2,
\]
and such that \( f_1 : A \to B_1 \) is left minimal (see, e.g., [38, Prop. 1.2]). Hence, if \( f : A \to B \) is a \( \mathcal{T} \)-preenvelope, then \( f_1 \) is a \( \mathcal{T} \)-envelope.

Recall that a ring \( R \) is **Krull-Schmidt** when its category of finitely presented modules \( \text{mod-} R \) is Krull-Schmidt. This is a two-sided notion by a classical duality, due to Auslander and Gruson-Jensen, between the categories \( \text{fp}(\text{mod-}(R^{\text{op}}), \text{Ab}) \) and \( \text{fp}(\text{mod-} R, \text{Ab}) \) of finitely presented functors to \text{Ab} from finitely presented left and right \( R \)-modules, respectively (see [34, Sec. 5]).

Example 3.13 The following are examples of Krull-Schmidt rings (we address the reader to the cited references for the definitions of each of the following types of rings).

1. Perfect and, more generally, semiperfect rings \( R \) such that the descending sequence
\[
aR \supseteq a^2 R \supseteq \ldots \supseteq a^n R \supseteq \ldots
\]
(or, equivalently, its left version) is stationary, for all \( a \in R \) (see [47, Thm. 8]). This includes all one-sided Artinian rings so, in particular, all Artin algebras.
2. Complete semiperfect (two-sided) Noetherian rings (see [48, Thm. B]).
3. Rings that are finitely generated as modules over complete Noetherian local domains (see [58]).

4 Co/reflective Subcategories of Abelian Categories

In this section we study some criteria for a subcategory of an Abelian category \( \mathcal{A} \) to be reflective, also relating this property with that of being (pre)enveloping. In the last part of the section we apply these results to classes of the form \( \text{Sub}(\text{Gen}(V)) \), for some object \( V \in \mathcal{A} \). As a motivation, let us anticipate that the results of this section will be fundamental for our study of quasi-tilting objects (see, e.g., Proposition 6.11).

4.1 Generalities on Co/reflective Subcategories

A subcategory \( \mathcal{B} \) of a category \( \mathcal{A} \) is said to be **reflective** (resp., **coreflective** when the inclusion \( \mathcal{B} \to \mathcal{A} \) has a left (resp., right) adjoint.

Lemma 4.1 Let \( \mathcal{B} \) be a reflective subcategory of an Abelian category \( \mathcal{A} \), and let
\[
\iota : \mathcal{B} \to \mathcal{A}, \quad \tau : \mathcal{A} \to \mathcal{B} \quad \text{and} \quad \rho : \text{id}_\mathcal{A} \Rightarrow \iota \circ \tau
\]
denote the inclusion, its left adjoint, and the unit of the adjunction, respectively. If \( \mathcal{B} \) is closed under quotients in \( \mathcal{A} \), then \( \rho \) is epimorphic, that is, \( \rho_A : A \to \iota(\tau(A)) \) is an epimorphism for all \( A \in \mathcal{A} \).
Proof Let $A \in \mathcal{A}$ and consider the following exact sequence:

$$A \xrightarrow{\rho_A} i(\tau(A)) \longrightarrow \text{CoKer}(\rho_A) \longrightarrow 0.$$ 

Since $\tau$ is a left adjoint, it is right exact and the fact that $\tau(\rho_A)$ is an isomorphism implies that $\tau(\text{CoKer}(\rho_A)) = 0$. On the other hand, since $\mathcal{B}$ is closed under taking quotients, $\text{CoKer}(\rho_A) \in \mathcal{B}$, implying that $\text{CoKer}(\rho_A) \cong i(\tau(\text{CoKer}(\rho_A))) = 0$. □

Let us recall from [15, Prop. 3.1.3], the following useful criterion: a subcategory $\mathcal{B} \subseteq \mathcal{A}$ is reflective if, and only if, any object $A \in \mathcal{A}$ has a $\mathcal{B}$-reflection $\rho_A : A \to B_A$ in $\mathcal{B}$, that is, $B_A \in \mathcal{B}$ and the following map is an isomorphism for all $B \in \mathcal{B}$:

$$\mathcal{A}(\rho_A, B) : \mathcal{A}(B_A, B) \longrightarrow \mathcal{A}(A, B).$$

A $\mathcal{B}$-coreflection $B_A \to A$ is just a $\mathcal{B}^{\text{op}}$-reflection in $\mathcal{A}^{\text{op}}$.

**Proposition 4.2** Let $\mathcal{A}$ be an (Ab.3), well-powered Abelian category and $\iota : \mathcal{B} \to \mathcal{A}$ the embedding of a subcategory closed under quotients. Then, $\mathcal{B}$ is coreflective if, and only if, it is closed under taking coproducts in $\mathcal{A}$. In this case, denote

$$\sigma : \mathcal{A} \longrightarrow \mathcal{B} \quad \text{and} \quad \eta : \iota \circ \sigma \Rightarrow \text{id}_{\mathcal{A}}$$

the right adjoint to the inclusion and the corresponding unit. Suppose that $\mathcal{B}$ is abelian. The following statements hold true:

1. $\mathcal{B}$ is (Ab.3) and $\eta$ is monomorphic, i.e., $\eta_X$ is a monomorphism for all $X \in \mathcal{A}$;
2. if $\mathcal{B}$ is also closed under taking kernels (or, equivalently, under subobjects), then $\mathcal{B}$ is an (Ab.4) (resp., (Ab.5), Grothendieck) Abelian category whenever $\mathcal{A}$ is so.

**Proof** It is a general fact that coreflective subcategories are closed under coproducts. On the other hand, suppose that $\mathcal{B}$ is closed under taking coproducts and let us verify that any given object $X \in \mathcal{A}$ admits a coreflection $\sigma(X) \to X$. Indeed, define

$$\sigma(X) := \sum \{ Y : Y \in B \cap L_{\mathcal{A}}(X) \}$$

(where $L_{\mathcal{A}}(X)$ is a set since $\mathcal{A}$ is well-powered). Then, $\sigma(X)$ is the image of the canonical morphism

$$\bigsqcup_{Y \in B \cap L_{\mathcal{A}}(X)} Y \longrightarrow X$$

induced by the inclusions $Y \hookrightarrow X$, so $\sigma(X) \in \mathcal{B}$. We now verify that the inclusion $\eta_X : \sigma(X) \to X$ is the desired coreflection: let $B \in \mathcal{B}$ and consider the following map

$$\mathcal{B}(B, \sigma(X)) = \mathcal{A}(B, \sigma(X)) \xrightarrow{\mathcal{A}(B, \eta_X)} \mathcal{A}(B, X) = \mathcal{A}(i(B), X).$$

Then, $\mathcal{A}(B, \eta_X)$ is injective by the left exactness of $\mathcal{A}(B, -)$, while it is surjective since, given a morphism $f : B \to X$, $f(B)$ is clearly a subobject of $\sigma(X)$, so $f$ factors through $\eta_X$.

1. As $\mathcal{B}$ is closed under coproducts in $\mathcal{A}$, and $\mathcal{A}$ is (Ab.3), coproducts do exist in $\mathcal{B}$ and they are computed as in $\mathcal{A}$, so $\mathcal{B}$ is (Ab.3). Moreover, the counit $\eta : \iota \circ \sigma \Rightarrow \text{id}_{\mathcal{A}}$ is defined by the above inclusions $\eta_X : \sigma(X) \hookrightarrow X$ (for all $X \in \mathcal{A}$), so it is clearly monomorphic.

2. Suppose now that $\mathcal{B}$ is also closed under taking kernels. Then each subobject $B' \subseteq B$ of an object $B \in \mathcal{B}$ is the kernel of the projection $p : B \to B/B'$, which is a morphism in $\mathcal{B}$ since this subcategory is closed under taking quotients. Therefore, $\mathcal{B}$ is also closed under taking subobjects. $\mathcal{B}$ is clearly an Abelian exact subcategory of $\mathcal{A}$. Furthermore, since the
inclusion \( B \rightarrow A \) is a left adjoint, it preserves all colimits. Hence, whenever coproducts (resp., directed colimits) are exact in \( A \), these colimits are also exact in \( B \), hence \( B \) is (Ab.4) (resp., (Ab.5)) whenever \( A \) is (Ab.4) (resp., (Ab.5)). It remains to prove that, if \( A \) is Grothendieck, with a generator \( G \), then also \( B \) has a generator. For each \( B \in B \), there is a canonical epimorphism \( \epsilon : G^{(A(G,B))} \rightarrow B \). For each finite subset \( F \subset A(G,B) \), the image \( BF := \text{Im}(\epsilon_F) \) of the composition \( \epsilon_F : G^{(F)} \rightarrow G^{(A(G,B))} \rightarrow B \), is an object of \( B \). The corresponding family \( \{BF\} \) is then directed and, by the (Ab.5) condition, \( B = \sum_F BF \), and so \( B \) is a quotient of \( \bigsqcup F BF \). Let
\[
S := \text{Quot}\{G^n : n \in \mathbb{N}\} \cap B.
\]
Being \( A \) well-powered, \( S \) is skeletally small, and it generates \( B \) by the above argument. \( \square \)

We now apply the above proposition and its dual in a situation that we will encounter frequently through the paper.

**Corollary 4.3** Let \( A \) be a well-powered and \( B \) be a subcategory closed under taking subobjects, quotients and finite coproducts, e.g. \( B = \text{Sub}(X) \), for \( X \) closed under taking quotients and finite coproducts, or \( B = \text{Quot}(Y) \) for \( Y \) closed under subobjects and finite coproducts. Then:

1. if \( A \) is (Ab.3), then \( B \) is coreflective if, and only if, \( B \) is closed under coproducts. In this case, if \( \sigma : A \rightarrow B \) is right adjoint to the inclusion functor, then the counit \( \eta : \iota \circ \sigma \Rightarrow \text{id}_A \) is monomorphic and \( B \) is also (Ab.3). Moreover, \( B \) is (Ab.4), (Ab.5) or Grothendieck whenever \( A \) is so;
2. if \( A \) is (Ab.3*), then \( B \) is reflective if, and only if, \( B \) is closed under products. In this case, if \( \tau : A \rightarrow B \) is left adjoint to the inclusion functor, then the unit \( \rho : \text{id}_A \Rightarrow \iota \circ \tau \) is epimorphic and \( B \) is (Ab.3*). Moreover, \( B \) is (Ab.4*), (Ab.5*) or (Ab.5*) with an injective cogenerator whenever \( A \) is so;
3. if \( A \) is bicomplete, i.e. (Ab.3) and (Ab.3*), and \( B \) is coreflective (resp., reflective), then \( B \) is bicomplete, but products (resp., coproducts) in \( B \) need not be computed as in \( A \).

**Proof** Assertion (1) follows by Proposition 4.2 and assertion (2) is just its dual. As for assertion (3), the “in-brackets” statement is dual to the one “not-in-brackets”. We just prove the latter, for which it is enough to prove that if \( B \) is coreflective, then \( B \) has products. Indeed, if \( (B_\lambda)_{\lambda \in \Lambda} \) is a family of objects in \( B \) and we consider \( [P, (\pi_\lambda : P \rightarrow \iota(B_\lambda))_{\lambda \in \Lambda}] \), which is the product of the family \( (\iota(B_\lambda))_{\lambda \in \Lambda} \) in \( A \), then it is well-known, and easy to prove, that \( [\sigma(P), (\sigma(\pi_\lambda) : \sigma(P) \rightarrow (\sigma \circ \iota)(B_\lambda) \cong B_\lambda)_{\lambda \in \Lambda}] \) is a product in \( B \). \( \square \)

**Corollary 4.4** Let \( A \) be a well-powered (Ab.3*) Abelian category, and \( T \subseteq A \) a subcategory such that \( B := \text{Sub}(T) \) is closed under taking products in \( A \) (e.g., if \( T \) is closed under taking products). Then, \( B \) is reflective in \( A \).

**Proof** It is a direct consequence of the dual of Proposition 4.2. \( \square \)

Let us conclude this subsection with the following example showing that, if instead of completeness and well-powerdness of \( A \), we assume a strong enough finiteness condition on the objects, then we can still reach the same conclusion of Corollary 4.4.
Example 4.5 Let $\mathcal{A}$ be an Abelian category and suppose that all the objects of $\mathcal{A}$ are Artinian. If $\mathcal{T} \subseteq \mathcal{A}$ is a subcategory closed under finite co/products, then $\mathcal{B} := \text{Sub}(\mathcal{T})$ is reflective.

**Proof** Let $A \in \mathcal{A}$ and consider the following class:

$$S_A := \{ K \leq A : K = \text{Ker}(\phi) \text{ for some } \phi : A \to B, \ B \in \mathcal{B} \}.$$  

By the DCC on subobjects of $A$, $S_A$ contains a minimal element. If $S_1$ and $S_2$ are two such minimal elements, and we fix monomorphisms $A/S_1 \hookrightarrow T_1$ and $A/S_2 \hookrightarrow T_2$ with $T_1, T_2 \in \mathcal{T}$, then we get a monomorphism $A/(S_1 \cap S_2) \hookrightarrow T_1 \oplus T_2$, with $T_1 \oplus T_2 \in \mathcal{T}$. It follows that $S_1 \cap S_2 \in S_A$ and, by minimality, $S_1 = S_1 \cap S_2 = S_2$, so there is a unique minimal element $S_A$ in $S_A$ which is then its minimum. Let $\pi : A \to B_A := A/S_A$ be the natural projection, it is routine to check that $\mathcal{A}(\pi, B) : \mathcal{A}(B_A, B) \to \mathcal{A}(A, B)$ is an isomorphism for all $B \in \mathcal{B}$, so this is the desired reflection. \hfill \Box

### 4.2 The Category of Subobjects of a (Pre)Enveloping Class

In this subsection we continue the study of the reflective condition of subcategories of the form $\text{Sub}(\mathcal{T})$, but specializing to the case when $\mathcal{T}$ is (pre)enveloping.

**Proposition 4.6** Let $\mathcal{A}$ be an Abelian category and $\mathcal{T} \subseteq \mathcal{A}$ a preenveloping subcategory. Then, $\mathcal{B} := \text{Sub}(\mathcal{T})$ is reflective in $\mathcal{A}$.

**Proof** For any object $A \in \mathcal{A}$, we have to construct a $\mathcal{B}$-reflection $\rho_A : A \to B_A$. Indeed, take a $\mathcal{T}$-preenvelope $\mu_A : A \to TA$ and consider its epi-mono factorization

$$A \xrightarrow{\rho_A} B_A \xleftarrow{\iota_A} TA.$$  

Clearly, $B_A \in \mathcal{B}$ and, since $\mathcal{A}(-, B)$ is left exact, $\mathcal{A}(\rho_A, B)$ is injective for all $B \in \mathcal{B}$. To prove that it is also surjective, fix an object $B \in \mathcal{B}$, an embedding $\iota : B \to T$ with $T \in \mathcal{T}$ and a morphism $f : A \to B$. Since $\mu_A$ is a $\mathcal{T}$-preenvelope, there exists a morphism $g : T_A \to T$ such that $\iota \circ f = g \circ \mu_A$. Consider then the following diagram:

$$
\begin{array}{ccc}
0 & \xrightarrow{\iota} & B_A \\
\downarrow & & \downarrow \\
A & \xrightarrow{\mu_A} & T_A & \xrightarrow{\pi_A} & \text{Coker}(\mu_A) & \xrightarrow{0} \\
\downarrow f & & \downarrow & & \downarrow h \\
0 & \xrightarrow{\iota} & B & \xrightarrow{\pi} & \text{Coker}(\iota) & \xrightarrow{0}
\end{array}
$$

which is commutative, and where the two rows are exact. Here the morphism $h$ is constructed noting that $0 = \pi \circ \iota \circ f = \pi \circ g \circ \mu_A = \pi \circ g \circ \iota_A \circ \rho_A$, so $\pi \circ g \circ \iota_A = 0$ (since $\rho_A$ is an epimorphism) and then, by the universal property of the cokernel, $\pi \circ g$ factors uniquely through $\pi_A$. Once we have $h$, the universal property of kernels gives the unique morphism $\bar{f}$ such that $\iota \circ \bar{f} = g \circ \iota_A$. We then have that $\iota \circ \bar{f} \circ \rho_A = g \circ \iota_A \circ \rho_A = g \circ \mu_A = \iota \circ f$, from which we get that $f = \bar{f} \circ \rho_A$ since $\iota$ is a monomorphism. Therefore, $\mathcal{A}(\rho_A, B)$ is also a surjective map. \hfill \Box
The following corollary is especially useful in Krull-Schmidt categories, where all the preenveloping classes are enveloping (see Lemma 3.12).

**Corollary 4.7** Let \( \mathcal{A} \) be an Abelian category and \( \mathcal{T} \) a torsion class in \( \mathcal{A} \). If \( \mathcal{T} \) is enveloping (and cogenerating) then it is semi-special (resp., special) preenveloping.

**Proof** Bearing in mind that, when \( \mathcal{T} \) is preenveloping, \( \mathcal{B} := \text{Sub}(\mathcal{T}) \) is reflective (see Proposition 4.6), the proof reduces to the case when \( \mathcal{T} \) is cogenerating. In this latter case the proof goes as for categories of modules (see [28, Lem. 2.1.13]). \( \square \)

### 4.3 The Subcategory of Objects Sub(co)generated by a Given Object

Recall from Section 1 the definition of the subcategories \( \text{Gen}(M) \) and \( \text{Cogen}(M) \), for a given object \( M \) in an Abelian category \( \mathcal{A} \). We enlarge these classes to form new subcategories to which we apply the results of the previous subsections.

**Definition 4.8** Let \( X \) and \( M \) be two objects in an Abelian category \( \mathcal{A} \). We say that \( X \) is subgenerated by \( M \) if it embeds in an object of \( \text{Gen}(M) \). We denote by \( \text{Gen}_\mathcal{A}(M) := \text{Sub}(\text{Gen}_\mathcal{A}(M)) \) the subcategory of objects subgenerated by \( M \).

The objects subcogenerated by \( M \) are defined dually. The subcategory of objects subcogenerated by \( M \) is denoted by \( \text{Cogen}_\mathcal{A}(M) := \text{Quot}(\text{Cogen}_\mathcal{A}(M)) \). When the ambient category \( \mathcal{A} \) is clear from the context, the subscript will be generally avoided.

**Lemma 4.9** Let \( \mathcal{A} \) be an Abelian category with a projective epi-generator \( P \) and \( \mathcal{T} \subseteq \mathcal{A} \) a subcategory closed under quotients, coproducts and extensions. If \( \mathcal{T} = \text{Gen}(V) \), for an object \( V \in \mathcal{A} \) such that \( P \) has an \( \text{Add}(V) \)-preenvelope, then \( \mathcal{B} := \text{Sub}(\mathcal{T}) = \text{Gen}(V) \) is reflective in \( \mathcal{A} \).

**Proof** Let \( f : P \to V^{(I)} \) be an \( \text{Add}(V) \)-preenvelope. It is enough to prove that it is also a \( \mathcal{T} \)-preenvelope. In fact, if that happens, then \( \mathcal{T} \) is a preenveloping class in \( \mathcal{A} \) by Lemma 3.4 and so Proposition 4.6 applies. Let \( u : P \to T \) be any morphism, where \( T \in \mathcal{T} = \text{Gen}(V) \). Fix an epimorphism \( p : V^{(J)} \to T \). By the projectivity of \( P \), we get a map \( v : P \to V^{(J)} \) such that \( p \circ v = u \). Since \( f \) is an \( \text{Add}(V) \)-preenvelope, we then get a map \( w : V^{(I)} \to V^{(J)} \) such that \( w \circ f = v \). It follows that \( u = p \circ v = p \circ w \circ f \), thus showing that \( f \) is a \( \mathcal{T} \)-preenvelope. \( \square \)

Note that, putting \( \mathcal{A}' = \text{Gen}(M) \) (resp., \( \mathcal{A}' = \text{Cogen}(M) \)) in Corollary 4.3, one gets assertions about \( \overline{\text{Gen}}(M) \) (resp., \( \overline{\text{Cogen}}(M) \)) whose statement is left to the reader. Nevertheless, we want to emphasize the following consequence.

**Corollary 4.10** Let \( \mathcal{A} \) be a well-powered Abelian category and \( M \in \mathcal{A} \). Then:

1. if \( \mathcal{A} \) is (Ab.5), then \( \overline{\text{Gen}}(M) \) is a Grothendieck category;
2. if \( \mathcal{A} \) is (Ab.5) and (Ab.3*), then \( \overline{\text{Cogen}}(M) \) is (Ab.5);
3. if \( \mathcal{A} \) is a Grothendieck category, so is \( \overline{\text{Cogen}}(M) \).

**Proof** (1). By Corollary 4.3(1), we know that \( \overline{\text{Gen}}(M) \) is (Ab.5). We just need to exhibit a set of generators. The strategy is completely analogous to the case when \( \mathcal{A} \) is a category of
modules (see [62, Sec. 15]). Indeed, consider the following class:

\[ S := \text{Sub}\{ M^n : n \in \mathbb{N} \} \]

which is skeletally small since \( \mathcal{A} \) is supposed to be well-powered. Given \( T \in \text{Gen}(\mathcal{M}) \), fix a monomorphism \( \lambda : T \to M^{(1)}/K \). Furthermore, let \( \lambda_F : M^{(F)}/(K \cap M^{(F)}) \to M^{(1)}/K \) be the induced monomorphism, for any finite subset \( F \subseteq I \). Then \( M^{(1)}/K \) is the directed union of the objects \( L_F := \text{Im}(\lambda_F) \). Since \( \mathcal{A} \) is supposed to be (Ab.5), we obtain that \( T \) is the directed union of the \( \lambda^{-1}(L_F) \) (see [55, Prop. V.1.1]). But each \( \lambda^{-1}(L_F) \) is isomorphic (via \( \lambda \)) to an object of \( \mathcal{S} \). Therefore \( T \) is a quotient of a coproduct of objects in \( \mathcal{S} \), showing that \( \mathcal{S} \) is a skeletally small class of generators of \( \text{Gen}(\mathcal{M}) \).

(2) and (3). Note that \( \mathcal{B} := \text{Cogen}(\mathcal{M}) \) is closed under taking subobjects and quotients in \( \mathcal{A} \). By Proposition 4.2 it is enough to check that \( \mathcal{B} \) is closed under taking coproducts and, since coproducts are exact in \( \mathcal{A} \), it is clearly sufficient to prove that \( \text{Cogen}(\mathcal{M}) \) is closed under taking coproducts. For that, note first that, given a family \( \{ X_i \}_I \) of objects in \( \mathcal{A} \), the canonical morphism \( \prod_I X_i \to \prod_I X_i \) is a monomorphism (see [44, Coro. 8.10]). Let then \( \{ T_i \}_I \) be a family of objects in \( \text{Cogen}(\mathcal{M}) \) and fix monomorphisms \( \lambda_i : T_i \hookrightarrow M^{J_i} \), for suitable sets \( J_i (i \in I) \). We obtain a monomorphism

\[ \coprod_I T_i \hookrightarrow \coprod_I M^{J_i} \xrightarrow{\text{can}} \prod_I M^{J_i} , \]

showing that \( \coprod_I T_i \in \text{Cogen}(\mathcal{M}) \). \( \square \)

5 Semi-special Preenveloping Classes

In this section we introduce the main tools that will be necessary in Section 6. In particular, in Section 5.2, we develop a theory of universal extensions and \( \text{Ext}^1 \)-universal objects in an Abelian category \( \mathcal{A} \). This can be done with relative ease when we impose sufficiently strong finiteness conditions on \( \mathcal{A} \) (see, e.g., Proposition 5.7) but, to treat the case of “big” Abelian categories (see, e.g., Proposition 5.10), we need a preliminary study of some exactness properties of infinite coproducts, that we carry on in Section 5.1. Finally, we characterize the semi-special preenveloping classes in an arbitrary Abelian category \( \mathcal{A} \) in Section 5.3 (see Theorem 5.14 and Corollary 5.16 for the main results in this direction).

5.1 A Formal Derived Functor of Coproducts

In a cocomplete Abelian category, all coproducts exist but, in general, their derived product may fail to exist. Nevertheless, we introduce the following notation:

**Notation 5.1** Let \( \mathcal{A} \) be an Abelian category and let \( I \) be a set such that \( I \)-coproducts exist in \( \mathcal{A} \). Let \( \{ V_i \}_I \) be a family of objects and \( \mathcal{S} \) a subcategory of \( \mathcal{A} \). We shall write

\[ \coprod_I V_i \in \mathcal{S} \]

when, given any family \( \{ 0 \to X_i \xrightarrow{u_i} Y_i \to V_i \to 0 \}_I \) of short exact sequences in \( \mathcal{A} \), the kernel of the induced morphism \( \coprod_I u_i : \coprod_I X_i \to \coprod_I Y_i \) lies in \( \mathcal{S} \). When \( S = \{ 0 \} \), we write \( \coprod_I V_i = 0 \). This applies to the case when \( V_i = V \), for all \( i \in I \) and \( V \in \mathcal{A} \), in which case we write \( \coprod_I V \in \mathcal{S} \) or \( \coprod_I V = 0 \), when \( \mathcal{S} = \{ 0 \} \).
We now want to show that, when the first derived functor of the coproduct exists in \( \mathcal{A} \), then Notation 5.1 means exactly what one expects (see Proposition 5.2 for details). Before proceeding, let us fix some terminology. Indeed, let \( \mathcal{A} \) be an Abelian category with enough projectives. Given an object \( V \in \mathcal{A} \), a **projective presentation** of \( V \) is a short exact sequence in \( \mathcal{A} \)

\[
0 \to \Omega \to P \to V \to 0 \quad \text{with} \quad P \in \text{Proj}(\mathcal{A}).
\]

**Proposition 5.2** Let \( \mathcal{A} \) be an Abelian category with enough projectives, \( I \)-coproducts exist in \( \mathcal{A} \), and let \( \{V_i\}_I \) and \( S \) be a family of objects and a subcategory of \( \mathcal{A} \), respectively. Consider the following assertions:

1. \( \bigsqcup I V_i \in S \);
2. for any family of projective presentations \( 0 \to \Omega_i \to P_i \to V_i \to 0 \) we have that \( \text{Ker}(\bigsqcup I u_i) \in S \);
3. for some family of projective presentations \( 0 \to \Omega_i \to P_i \to V_i \to 0 \) we have that \( \text{Ker}(\bigsqcup I u_i) \in S \);
4. \( H^{-1}(\bigsqcup I V_i[0]) \in S \), where \( \bigsqcup I V_i[0] \) denotes the coproduct in \( D^-(\mathcal{A}) \) (see Section 2.3).

Then the implications \( "(1)\Rightarrow(2)\Leftrightarrow(3)\Rightarrow(4)" \) hold true. Furthermore, when \( S \) is closed under taking quotients in \( \mathcal{A} \), all assertions are equivalent.

**Proof** Consider a family of projective presentations \( \{0 \to \Omega_i \to P_i \to V_i \to 0\}_I \). For any \( j \in I \), one can choose a projective resolution \( s_j : P^\bullet_j \to V_j[0] \), where

\[
\cdots \to P^{-2}_j \xrightarrow{d^{-2}_j} P^{-1}_j \xrightarrow{d^{-1}_j} P^0_j \cong P_j \xrightarrow{\pi_j} \Omega_j \xrightarrow{u_j} 0
\]

such that \( P^0_j \cong P_j \) and with \( \Omega_j \) (necessarily) isomorphic to the 0-th boundary of \( P^\bullet_j \). As coproducts are right-exact, \( \bigsqcup I \pi_i \) is an epimorphism and, as \( \bigsqcup I u_i \circ \bigsqcup I \pi_i = \bigsqcup I d^{-1}_i \), we can deduce that \( \text{Ker}(\bigsqcup I u_i) \cong \text{Ker}(\bigsqcup I d^{-1}_i) \) and \( \text{Im}(\bigsqcup I u_i) = \text{Im}(\bigsqcup I d^{-1}_i) \).

Therefore, the short exact sequence \( 0 \to \text{Ker}(\bigsqcup I u_i) \to \bigsqcup I \Omega_i \to \text{Im}(\bigsqcup I u_i) \to 0 \), can be rewritten as follows:

\[
0 \to \text{Ker}(\bigsqcup I d^{-1}_i) / \text{Ker}(\bigsqcup I \pi_i) \to \bigsqcup I \Omega_i \to \text{Im}(\bigsqcup I d^{-1}_i) \to 0.
\]

Using again the right exactness of coproducts, we get that \( \text{Im}(\bigsqcup I d^{-2}_i) = \text{Ker}(\bigsqcup I \pi_i) \), so we have the following short exact sequence:

\[
0 \to H^{-1}(\bigsqcup I P^*_i) \to \bigsqcup I \Omega_i \to \text{Im}(\bigsqcup I d^{-1}_i) \to 0,
\]

where \( \bigsqcup I P^*_i \) is computed as in \( \text{Ch}(\mathcal{A}) \). Using the inclusion \( \text{Im}(\bigsqcup I d^{-1}_i) \hookrightarrow \bigsqcup I P^0_i \), we get

\[
0 \to H^{-1}(\bigsqcup I P^*_i) \to \bigsqcup I \Omega_i \xrightarrow{\bigsqcup I u_i} \bigsqcup I P_i.
\]

Since \( H^{-1}(\bigsqcup I P^*_i) \cong H^{-1}(\bigsqcup I V_i[0]) \) does not depend on the choice of the projective resolutions, the equivalence of assertions (2), (3) and (4) is clear. Moreover, they are clearly implied by (1).
We next prove the implication “(2)⇒(1)”, assuming that \( S \) is closed under taking quotients in \( \mathcal{A} \). Let \( \{ 0 \to X_i \xrightarrow{u_i} Y_i \xrightarrow{p_i} V_i \to 0 \} \) be a family of short exact sequences in \( \mathcal{A} \), and fix epimorphisms \( \pi_i : P_i \to Y_i \), where each \( P_i \) is a projective object. For each \( j \in I \), we get the following commutative diagram with exact rows, where the left square is bicartesian:

\[
\begin{array}{cccccc}
0 & \to & \Omega_j & \xrightarrow{\bar{u}_j} & P_j & \xrightarrow{\bar{p}_j} & V_j & \to & 0 \\
\rho_j & \downarrow & & & & & & & \\
0 & \to & X_j & \xrightarrow{u_j} & Y_j & \to & V_j & \to & 0.
\end{array}
\]

Bearing in mind that the coproduct functor preserves pushouts, we also get the following commutative diagram with exact rows, where the central square is cocartesian:

\[
\begin{array}{cccccc}
0 & \to & K & \xrightarrow{\bigvee \Omega_i} & \bigvee P_i & \xrightarrow{\bigvee \pi_i} & \bigvee V_i & \to & 0 \\
\rho_i & \downarrow & & \text{p.o.} & & & & & \\
0 & \to & L & \xrightarrow{\bigvee X_i} & \bigvee Y_i & \to & \bigvee V_i & \to & 0.
\end{array}
\]

By (2) we know that \( K \in S \) so we should prove that the map \( f \) in the above diagram is an epimorphism, since then one concludes that \( L \in S \), being \( S \) closed under quotients. For this, consider the following diagram

\[
\begin{array}{cccccc}
\bigvee \Omega_i & \xrightarrow{\bigvee \bar{u}_i} & \bigvee P_i & \xrightarrow{\bigvee \pi_i} & \bigvee V_i & \to & 0 \\
\rho_i & \downarrow & & & \text{p.o.} & & & \\
\bigvee X_i & \xrightarrow{\bigvee u_i} & \bigvee Y_i & \to & \bigvee V_i & \to & 0.
\end{array}
\]

where the first line is the epi-mono factorization of \( \bigvee \bar{u}_i \), the rest of the diagram is constructed by two successive pushouts, using that the juxtaposition of two cocartesian squares is cocartesian. Note that square on the right-hand side is bicartesian, so the map \( N \to \bigvee Y_i \) is a monomorphism. Hence, the kernel of the map \( \bigvee X_i \to N \) is isomorphic to \( \text{Ker}(\bigvee u_i) \). Therefore, we obtain the the following commutative diagram with exact rows:

\[
\begin{array}{cccccc}
0 & \to & K & \xrightarrow{\bigvee \Omega_i} & M & \to & 0 \\
f & \downarrow & & \text{p.o.} & & & \\
0 & \to & L & \xrightarrow{\bigvee X_i} & N & \to & 0.
\end{array}
\]

We can now conclude by the dual of [44, Lem. 2.5.3].

To further support the intuition that \( \bigvee^{\mathbb{I}}V \) is a good substitute for the derived functor of \( I \)-coproducts, we show in the following example that, if coproducts are “exact enough”, then \( \bigvee^{\mathbb{I}}V \) vanishes.

**Example 5.3** Let \( \mathcal{A} \) be an Abelian category and \( V \in \mathcal{A} \). If either one of the following two conditions hold, then \( \bigvee^{\mathbb{I}}V = 0 \), for any set \( I \) for which all \( I \)-coproducts of exact sequences ending at \( V \) exist:
(1) if $\mathcal{A}$ is (Ab.4);
(2) if $V^\perp_1$ is a cogenerating subcategory of $\mathcal{A}$.

Proof Consider a family $\{0 \rightarrow A_i \rightarrow B_i \rightarrow V \rightarrow 0\}_I$ of short exact sequences in $\mathcal{A}$ whose coproduct exists in $\mathcal{A}$. If $\mathcal{A}$ is (Ab.4), then the coproduct morphism $u: \coprod I A_i \rightarrow \coprod I B_i$ is a monomorphism and we are done. Suppose now that $V^\perp_1$ is cogenerating. Then, for each $X \in V^\perp_1$, and each $j \in I$, the induced map $\mathcal{A}(B_j, X) \rightarrow \mathcal{A}(A_j, X)$ is surjective. Hence, the morphism

$$\mathcal{A}(\coprod I B_i, X) \cong \prod I \mathcal{A}(B_i, X) \rightarrow \prod I \mathcal{A}(A_i, X) \cong \mathcal{A}(\coprod I A_i, X) \quad (6)$$

is surjective because products are exact in Ab. Since $V^\perp_1$ is a cogenerating class, the coproduct morphism $u: \coprod I A_i \rightarrow \coprod I B_i$ is a monomorphism by Lemma 2.8(3.1).

Remark 5.4 It is not at all uncommon to find cocomplete non-(Ab.4) Abelian categories $\mathcal{A}$ with an object $V$ satisfying condition (2) of the above example. For instance, if $\mathcal{G}$ is any non-(Ab.4*) Grothendieck category, then, for any torsion pair $t = (\mathcal{T}, \mathcal{F})$ of finite type in $\mathcal{G}$ such that $\mathcal{F}$ is generating, this latter class is of the form $\mathcal{F} = \text{Cogen}(Q) = ^{-1}Q$, for some (1-)cotilting object (see [25, Thm. 3.10] and the dual of Definition 6.1). We refer the reader to [Op.Cit.] for a good supply of examples when $\mathcal{G} = \text{Qcoh-}X$ is the category of quasi-coherent sheaves over a Noetherian scheme with an ample family of line bundles, that is generally not (Ab.4*). Choosing $\mathcal{A} := \mathcal{G}^{\text{op}}$ and taking $V := Q$ as an object of $\mathcal{A}$, condition (2) of Example 5.3 is clearly satisfied.

Finally we can show that, when we assume that $\coprod I V_i = 0$, then we can say more about the morphism $\Phi$ introduced in Lemma 2.5.

Proposition 5.5 Let $\mathcal{A}$ be an Abelian category, $X \in \mathcal{A}$, $I$ a set such that $I$-coproducts exist in $\mathcal{A}$, and $(V_i)_I \subseteq \mathcal{A}$ a family such that $\coprod I V_i = 0$. Then, the canonical morphism of big Abelian groups

$$\Phi: \operatorname{Ext}^1_{\mathcal{A}}(\coprod I V_i, X) \rightarrow \prod I \operatorname{Ext}^1_{\mathcal{A}}(V_i, X)$$

is an isomorphism. In particular, if $V$ is any object such that $\coprod I V = 0$, then the canonical morphism $\operatorname{Ext}^1_{\mathcal{A}}(V^{(I)}, X) \rightarrow \operatorname{Ext}^1_{\mathcal{A}}(V, X)^I$ is an isomorphism.

Proof By Lemma 2.5, we need to prove that $\Phi$ is surjective. Let $([\epsilon_j])_I \in \prod I \operatorname{Ext}^1_{\mathcal{A}}(V_i, X)$ and let the component $[\epsilon_j]$ be represented by a short exact sequence

$$\epsilon_j: 0 \rightarrow X \xrightarrow{u_j} Y_j \xrightarrow{p_j} V_j \rightarrow 0,$$

for each $j \in I$. The condition $\coprod I V_i = 0$ gives that $\coprod I u_j: \coprod I X \rightarrow \coprod I V_i$ is a monomorphism. We next consider the pushout of this monomorphism and the co-diagonal map $\nabla: X^{(I)} \rightarrow X$. Then, for each index $j \in I$, we get the following commutative diagram
with exact rows, where the lower left square is the mentioned pushout and the three upper vertical arrows are the \( j \)-th inclusions into the respective coproducts:

\[
\begin{array}{ccccccccc}
\varepsilon_j : & 0 & \longrightarrow & X & \overset{u_j}{\longrightarrow} & Y_j & \overset{p_j}{\longrightarrow} & V_j & \longrightarrow & 0 \\
\downarrow{\iota_j} & & \downarrow{\mu_j} & & \downarrow{\lambda_j} & & \downarrow{\nu} & & \downarrow{\eta} \\
0 & \longrightarrow & X(I) & \longrightarrow & \biguplus_i Y_i & \longrightarrow & \biguplus_i V_i & \longrightarrow & 0 \\
\end{array}
\]

Denote by \([\varepsilon]\) the element of \( \operatorname{Ext}^1_\mathcal{A}(\biguplus_i V_i, X) \) represented by the lower row of the diagram. Since \( \nabla \circ \iota_j = \text{id}_X \), the juxtaposition of the two right-most squares is a pullback. This just says that the morphism \( \lambda^*_j : \operatorname{Ext}^1_\mathcal{A}(\lambda_j, X) : \operatorname{Ext}^1_\mathcal{A}(\biguplus_i V_i, X) \rightarrow \operatorname{Ext}^1_\mathcal{A}(V_j, X) \) takes \([\varepsilon]\) \( \mapsto \) \([\varepsilon_j]\). Since \( \Phi \) is induced by the \( \lambda^*_j \), with \( j \) varying in \( I \), we see that \( \Phi([\varepsilon]) = ([\varepsilon_i])_I \). Hence, \( \Phi \) is surjective.

### 5.2 Universal Extensions

We start extending the following definition, which is usually given in categories of modules.

**Definition 5.6** Let \( \mathcal{A} \) be an Abelian category. Given \( A \) and \( B \in \mathcal{A} \), we say that a short exact sequence of the form

\[
0 \rightarrow A \overset{u}{\longrightarrow} X \overset{p}{\longrightarrow} B^{(J)} \rightarrow 0,
\]

for some non-empty set \( J \), is a **universal extension** of \( B \) by \( A \) if the following equivalent (for the equivalence see the comment right below) conditions hold:

- (UE) the map \( \operatorname{Ext}^1_\mathcal{A}(B, u) : \operatorname{Ext}^1_\mathcal{A}(B, A) \rightarrow \operatorname{Ext}^1_\mathcal{A}(B, X) \) is the zero map;
- (UE') the connecting morphism \( \mathcal{A}(B, B^{(J)}) \rightarrow \operatorname{Ext}^1_\mathcal{A}(B, A) \) is surjective;
- (UE'') the map \( \operatorname{Ext}^1_\mathcal{A}(B, p) : \operatorname{Ext}^1_\mathcal{A}(B, X) \rightarrow \operatorname{Ext}^1_\mathcal{A}(B, B^{(J)}) \) is injective.

An object \( B \) of \( \mathcal{A} \) is said to be (left) \( \operatorname{Ext}^1 \)-universal when a universal extension of \( B \) by any other object exists in \( \mathcal{A} \).

Note that, to prove that the conditions (UE), (UE') and (UE'') are all equivalent, it is enough to consider the following long exact sequence:

\[
\cdots \rightarrow \mathcal{A}(B, X) \rightarrow \mathcal{A}(B, B^{(J)}) \rightarrow \operatorname{Ext}^1_\mathcal{A}(B, A) \rightarrow \operatorname{Ext}^1_\mathcal{A}(B, X) \rightarrow \operatorname{Ext}^1_\mathcal{A}(B, B^{(J)}) \rightarrow \cdots
\]

In the following proposition and its corollary we give an easy characterization of \( \operatorname{Ext}^1 \)-universal objects in categories that have suitable finiteness conditions on Hom-sets.

**Proposition 5.7** Let \( \mathcal{A} \) be an Abelian category, \( V \in \mathcal{A} \) and consider the following assertions:

1. \( \operatorname{Ext}^1_\mathcal{A}(V, A) \) is finitely generated as a right \( \operatorname{End}_\mathcal{A}(V) \)-module, for all \( A \in \mathcal{A} \);
2. \( V \) is \( \operatorname{Ext}^1 \)-universal.
Then, the implication “(1)⇒(2)” holds true and, whenever \( \mathcal{A}(V, A) \) is finitely generated as a right \( \text{End}_A(V) \)-module for all \( A \in \mathcal{A} \), also the converse implication is verified.

**Proof** (1)⇒(2) is an easy adaptation of a well-known argument from the Representation Theory of Artin Algebras (see, for example, the proof of [14, Lem. 2.1]).

(2)⇒(1) under the extra hypothesis. Let \( A \in \mathcal{A} \) be any object. By condition (UE') in the definition of universal extension, with \( B = V \), we can deduce that \( \text{Ext}^1_A(V, A) \) is finitely generated as a right \( \text{End}_A(V) \)-module, as it is a quotient of \( \mathcal{A}(V, V^{(J)}) \).

**Corollary 5.8** Let \( \mathcal{A} \) be a Hom-finite Abelian category over a commutative ring \( R \). Then, an object \( V \in \mathcal{A} \) is \( \text{Ext}^1 \)-universal if, and only if, \( \text{Ext}^1_A(V, A) \) is finitely generated as an \( R \)-module for all \( A \in \mathcal{A} \).

**Proof** As \( \mathcal{A} \) is Hom-finite over the commutative ring \( R \), then the \( R \)-algebra \( \text{End}_A(V) = \mathcal{A}(V, V) \) is finite (i.e., finitely generated as an \( R \)-module). Hence, an \( \mathcal{A}(V, V) \)-module is finitely generated if and only if it is finitely generated as an \( R \)-module. Now apply Proposition 5.7.

Our following result shows that Ext\(^1\)-universality is related with smallness of some Ext\(^1\) groups. The study of the formal derived functors of coproducts carried on in the previous subsection turns out to be essential to understand that relation.

**Proposition 5.9** Let \( \mathcal{A} \) be an (Ab.3) Abelian category. If \( V \in \mathcal{A} \) is an Ext\(^1\)-universal object then \( \text{Ext}^1_A(V, A) \) is a set (as opposed to a proper class), for all \( A \in \mathcal{A} \). Moreover, if \( \bigsqcup I V = 0 \), for any set \( I \), the converse implication also holds.

**Proof** Suppose that \( V \) is Ext\(^1\)-universal and let \( A \in \mathcal{A} \). Then, by condition (UE') in the definition of universal extension, the big Abelian group \( \text{Ext}^1_A(V, A) \) is a quotient of \( \mathcal{A}(V, A) \), and it is therefore “small”. On the other hand, assume that \( \text{Ext}^1_A(V, A) \) is “small” for all \( A \in \mathcal{A} \) and that \( \bigsqcup I V = 0 \) for all sets \( I \). Then, for each element \( [e] \in E := \text{Ext}^1_A(V, A) \), we choose a short exact sequence

\[
e : \quad 0 \to A \to B_e \to V \to 0
\]

representing \( [e] \). By the hypotheses on \( V \), the coproduct of these sequences, with \( [e] \) varying in \( E \), gives a short exact sequence

\[
0 \to \bigoplus_{[e] \in E} A^{(E)} \xrightarrow{\mu} \bigoplus_{[e] \in E} B_e \xrightarrow{p} V^{(E)} \to 0.
\]

By taking the pushout of \( \mu \) and the codiagonal \( \nabla : A^{(E)} \to A \), and using standard properties of pushouts, we get an exact sequence

\[
0 \to A \xrightarrow{v} B \xrightarrow{q} V^{(E)} \to 0,
\]
which we claim is a universal extension of $V$ by $A$. Indeed, for each $[e'] \in E$, we have the following commutative diagram with exact rows,

$$
\begin{array}{c}
0 \rightarrow A \rightarrow B \rightarrow V \rightarrow 0 \\
0 \rightarrow A(E) \rightarrow B(E) \rightarrow V(E) \rightarrow 0 \\
0 \rightarrow A \rightarrow B \rightarrow V(E) \rightarrow 0
\end{array}
$$

where $\iota_{e'}$ is the inclusion into the coproduct and where the lower left square is cocartesian. Note also that $\nabla \circ \iota_{e'} = \text{id}_A$. This implies that the following square is cartesian

$$
\begin{array}{c}
B_{e'} \rightarrow V \\
\phi \circ \iota_{e'} \downarrow \rightarrow V(E) \\
B \rightarrow V(E)
\end{array}
$$

Hence, the connecting morphism $w : A(V, V(E)) \rightarrow \text{Ext}_A^1(V, A)$ takes $\iota_{e'} \mapsto [e']$, for each $[e'] \in E$. Then $w$ is surjective and the universal extension of $V$ by $A$ always exists.

The following result shows that, under suitable hypotheses on $V$, the condition $\coprod I V = 0$ has strong consequences.

**Proposition 5.10** Let $A$ be an Abelian category and $V \in A$ such that $\text{Ext}_A^1(V, V(I)) = 0$, for all sets $I$ for which $I$-coproducts exist in $A$. Then,

1. if $A$ is Hom-finite, the following are equivalent:
   1.1 $V$ is $\text{Ext}_A^1$-universal;
   1.2 $V \perp^1$ is a cogenerating subcategory of $A$;
2. if $A$ is (Ab.3), the following conditions are equivalent:
   2.1 $\text{Ext}_A^1(V, A)$ is a set (as opposed to a proper class), for all $A \in A$, and $\coprod I V = 0$;
   2.2 $V \perp^1$ is a cogenerating subcategory of $A$.

If these equivalent conditions hold, then $V$ is $\text{Ext}_A^1$-universal.

**Proof** We prove simultaneously the implications “(1.2)$\Rightarrow$(1.1)” and “(2.2)$\Rightarrow$(2.1)”. Indeed, by Example 5.3, we know that $\coprod I V = 0$ for all sets $I$, when condition (2.2) holds. Hence, from Corollary 5.8 and Proposition 5.9, we only need to prove that $\text{Ext}_A^1(V, A)$ is a finitely generated $R$-module in part (1), and a set (as opposed to a proper class) in part (2), for all $A \in A$. To see this, using that $V \perp^1$ is cogenerating, we can choose an exact sequence $0 \rightarrow A \rightarrow T_A \rightarrow B \rightarrow 0$, where $T_A \in V \perp^1$. This gives an exact sequence

$$
\mathcal{A}(V, T_A) \rightarrow \mathcal{A}(V, B) \rightarrow \text{Ext}_A^1(V, A) \rightarrow \text{Ext}_A^1(V, T_A) = 0,
$$

and the result follows. Let us now verify the implications “(1.1)$\Rightarrow$(1.2)” and “(2.1)$\Rightarrow$(2.2)”. By Proposition 5.9, we know that, in both situations, the universal extension of $V$ by $A$
exists, for all objects $A \in \mathcal{A}$. Let then

$$0 \to A \xrightarrow{u} B \xrightarrow{p} V(J) \to 0$$

be a universal extension, so that $\text{Ext}^1_A(V, p)$ is a monomorphism. Then $B \in V^\perp$ since $\text{Ext}^1_A(V, V(J)) = 0$.

The final statement in part (2) follows by Proposition 5.9.

\[\Box\]

### 5.3 Special Preenvelopes and Cotorsion Pairs

In this subsection we study the relation between cotorsion pairs and semi-special preenveloping torsion classes. Let us start with the following technical lemmas:

**Lemma 5.11** Let $\mathcal{A}$ be an Abelian category and $\mathcal{T}$ a subcategory closed under extensions, such that $\mathcal{B} := \text{Sub}(\mathcal{T})$ is closed under quotients in $\mathcal{A}$ (or, equivalently, it is an Abelian exact subcategory). Then, the following statements hold true:

1. For any object $X \in \text{gen}(\mathcal{T}) = \text{Quot}(\mathcal{T})$, the inclusion functor $\iota : \mathcal{B} \hookrightarrow \mathcal{A}$ induces an isomorphism $\text{Ext}^1_B(X, B) \cong \text{Ext}^1_A(X, B)$, for all $B \in \mathcal{B}$;
2. $\perp \mathcal{T} \cap \mathcal{B} \subseteq \text{Ker}(\text{Ext}^1_B(-, \mathcal{T}))$;
3. $\perp \mathcal{T} \cap \text{gen}(\mathcal{T}) = \text{Ker}(\text{Ext}^1_B(-, \mathcal{T})) \cap \text{gen}(\mathcal{T})$;
4. a morphism $\mu : B \to T$, with $B \in \mathcal{B}$, is a special $\mathcal{T}$-preenvelope in $\mathcal{B}$ if, and only if, it is a semi-special $\mathcal{T}$-preenvelope in $\mathcal{A}$.

**Proof** (1). For all $B, B' \in \mathcal{B}$, the canonical morphism $\text{Ext}^1_B(B', B) \to \text{Ext}^1_A(B', B)$ is a monomorphism. Let us show that it is also surjective when $B' = X \in \text{gen}(\mathcal{T})$. Let $0 \to B \xrightarrow{u} A \xrightarrow{p} X \to 0$ be an exact sequence and consider first the case when $X = T \in \mathcal{T}$. We fix a monomorphism $v : B \hookrightarrow T'$ with $T' \in \mathcal{T}$, taking the pushout of $u$ and $v$ and, using that $\mathcal{T}$ is closed under extensions, we see that $A \in \mathcal{B}$, so that the above sequence also belongs in $\text{Ext}^1_B(T, B)$. For a generic $X \in \text{gen}(\mathcal{T}) = \text{Quot}(\mathcal{T})$, we fix an epimorphism $\pi : \tilde{T} \to X$, with $\tilde{T} \in \mathcal{T}$. By taking the pullback of $p$ and $\pi$, we get an exact sequence $0 \to B \to \tilde{B} \to \tilde{T} \to 0$ together with an epimorphism $\tilde{B} \to A$. By the previous paragraph we get that $\tilde{B} \in \mathcal{B}$, so that $A \in \mathcal{B}$, since $\mathcal{B}$ is closed under quotients.

(2). It follows from the fact that the canonical morphism $\text{Ext}^1_B(B', B) \to \text{Ext}^1_A(B', B)$ is a monomorphism, for all $B, B' \in \mathcal{B}$.

(3). The inclusion “$\subseteq$” is a consequence of (2), while “$\supseteq$” is a consequence of (1).

(4). The morphism $\mu$ is a semi-special $\mathcal{T}$-preenvelope in $\mathcal{A}$ (resp., in $\mathcal{B}$) if, and only if, we have that $\text{Coker}(\mu) \in \perp \mathcal{T} \cap \text{gen}(\mathcal{T})$ (resp., $\in \text{Ker}(\text{Ext}^1_B(-, \mathcal{T})) \cap \text{gen}(\mathcal{T})$). Now, apply part (3) and use that each $\mathcal{T}$-preenvelope in $\mathcal{B} = \text{Sub}(\mathcal{T})$ is monomorphic.

In the following example we show that there are some subtleties when taking $\text{Ext}^1$-orthogonals in a subcategory:

**Example 5.12** In the setting of Lemma 5.11, the inclusion $\perp \mathcal{T} \cap \mathcal{B} \subseteq \text{Ker}(\text{Ext}^1_B(-, \mathcal{T}))$ may be strict.

**Proof** Combining [41, Thm. 6.2] and its proof with Corollary 5.2 in [Op.Cit.], we obtain an example of a finite dimensional algebra $R$ with a finitely generated $R$-module $V$ such that
the stalk complex \( V[0] \) is a projective generator of the heart of the Happel-Reiten-Smalø \( t \)-structure in \( \mathbf{D}(R) \) associated with the torsion pair \((\text{Gen}(V), V^\perp)\). Then, by [42, Prop. 3.8], we know that \( V \) is quasi-tilting (see Section 6). Moreover, it is shown in [Op.Cit.] that the trace \( \text{tr}(R) =: I \) is a 2-nilpotent ideal of \( R \) that coincides with \( \text{ann}_R(V) \). Hence, \( \mathcal{B} := \text{Gen}(V) = \text{Mod-}R/I \), so that \( R/I \in \text{Ker}(\text{Ext}_B^1(\_, T)) \). However, the canonical exact sequence \( 0 \to I \leftrightarrow R \to R/I \to 0 \) does not split in \( \text{Mod-}R \), thus proving that \( R/I \notin \perp \mathcal{T} \cap \mathcal{B} \), since \( I \in \text{Gen}(V) = \mathcal{T} \).

\[ \] for some object \( B \). We then get the following commutative diagram with exact rows:

\[ \begin{array}{ccccccc}
0 & \longrightarrow & Q & \longrightarrow & T_Q & \longrightarrow & V_Q & \longrightarrow & 0 \\
\end{array} \]

such that \( \lambda \) is a special \( \mathcal{T} \)-preenvelope of \( Q \) in \( \mathcal{B} \), then the following statements hold true:

1. \( \mathbb{H}_I \lambda : \mathbb{H}_I Q \longrightarrow \mathbb{H}_I T_Q \) is a special \( \mathcal{T} \)-preenvelope in \( \mathcal{B} \), for each non-empty set \( I \) for which the needed coproducts exist, where \( \mathbb{H}_I \) denotes the coproduct in \( \mathcal{B} \);
2. \( V_Q \) is \( \text{Ext}^1 \)-universal in \( \mathcal{B} \);
3. \( \mathcal{T} = \text{Ker}(\text{Ext}_B^1(V_Q, \_)) = \text{Gen}_B(T_Q) \).

**Proof** (1) follows from Lemmas 2.8 and 3.4. To prove parts (2) and (3) we will use the same construction. Indeed, given \( B \in \mathcal{B} \), there is a non-empty set \( I \) and an epimorphism \( \pi : \mathbb{H}_I Q \twoheadrightarrow B \). We then get the following commutative diagram with exact rows:

\[ \begin{array}{ccccccc}
0 & \longrightarrow & \mathbb{H}_I Q & \longrightarrow & \mathbb{H}_I T_Q & \longrightarrow & \mathbb{H}_I V_Q & \longrightarrow & 0 \\
\pi \downarrow & & \mathbb{H}_I \lambda \downarrow & & \mathbb{H}_I \text{p.o.} \downarrow & & \mathbb{H}_I V_Q \downarrow & & 0 \\
0 & \longrightarrow & B & \longrightarrow & T' & \longrightarrow & \mathbb{H}_I V_Q & \longrightarrow & 0 \\
\end{array} \]

where the left square is a pushout, and the coproducts in the first row exist since \( \mathcal{B} \) is closed under taking subobjects, so \( \mathcal{B} = \text{Gen}(Q) = \text{Pres}(Q) \) and Lemma 3.3 applies. Now, since \( \mathcal{T} \) is closed under coproducts in \( \mathcal{B} \) and quotients and \( T' \in \mathcal{T} \) we get that \( \text{Ext}_B^1(V_Q, T') = 0 \). This implies that the bottom row is a universal extension in \( \mathcal{B} \), thus proving (2). It remains to verify part (3): since \( \lambda \) is a special \( \mathcal{T} \)-preenvelope of \( Q \), we know that \( \mathcal{T} \subseteq \text{Ker}(\text{Ext}_B^1(V_Q, \_)) \). On the other hand, let \( B \in \text{Ker}(\text{Ext}_B^1(V_Q, \_-)) \) and note that, when taking the diagram in Eq. 7, the lower row splits by the choice of \( B \) and Lemma 2.5(1). In particular, \( B \) is an epimorphic image of \( \mathbb{H}_I T_Q \), so that \( B \in \text{Gen}_B(T_Q) \). Consequently, we obtain \( \text{Ker}(\text{Ext}_B^1(V_Q, \_-)) \subseteq \text{Gen}_B(T_Q) \), and so assertion (3) follows from the following inclusions: \( \mathcal{T} \subseteq \text{Ker}(\text{Ext}_B^1(V_Q, \_-)) \subseteq \text{Gen}_B(T_Q) \subseteq \mathcal{T} \).

We are now ready for the main result of this section:

**Theorem 5.14** Let \( \mathcal{A} \) be an Abelian category, \( \mathcal{T} \) a subcategory closed under extensions and direct summands and such that \( \mathcal{B} := \text{Sub}(\mathcal{T}) \) is closed under quotients. Consider the following assertions:

1. \( \mathcal{B} \) is reflective in \( \mathcal{A} \) and \( \mathcal{T} = \mathcal{B} \cap V^\perp \) for some object \( V \) which is \( \text{Ext}^1 \)-universal in \( \mathcal{B} \) and satisfies that \( \mathbb{H}_I V \in V^\perp \), for all sets \( I \) for which \( \mathbb{H}_I V \) exists;
2. \( \mathcal{T} \) is semi-special preenveloping in \( \mathcal{A} \);
3. \( \mathcal{B} \) is reflective in \( \mathcal{A} \) and \( \mathcal{T} \) is special preenveloping in \( \mathcal{B} \);

\[ \square \] Springer
(4) \( \mathcal{B} \) is reflective in \( \mathcal{A} \) and \( (\text{Ker}(\text{Ext}^1_{\mathcal{B}}(-, T)), T) \) is a right complete cotorsion pair in \( \mathcal{B} \).

Then, the implications “\((1) \Rightarrow (2) \Leftrightarrow (3) \Leftrightarrow (4)\)” hold true. When, in addition, \( T \) is closed under quotients, \( \mathcal{B} \) has an epi-generator and \( T \) is closed under coproducts in \( \mathcal{B} \), all assertions are equivalent.

**Proof** (2) \( \Rightarrow \) (3). It follows from Lemma 5.11(4) and Proposition 4.6.

(3) \( \Rightarrow \) (4). To show that \( (\text{Ker}(\text{Ext}^1_{\mathcal{B}}(-, T)), T) \) is a cotorsion pair we have to show that, given \( B \in \mathcal{B} \) such that \( \text{Ext}^1_{\mathcal{B}}(X, B) = 0 \), for all \( X \in \text{Ker}(\text{Ext}^1_{\mathcal{B}}(-, T)) \), then \( B \in T \). For this, take a special \( T \)-preenvelope \( \mu_B : B \to T_B \), so that \( \text{Coker}(\mu_B) \in \text{Ker}(\text{Ext}^1_{\mathcal{B}}(-, T)) \).

Thus, the following sequence is split exact
\[
0 \to B \xrightarrow{\mu_B} T_B \to \text{Coker}(\mu_B) \to 0.
\]

Hence \( B \) is a summand of \( T_B \) and so it belongs to \( T \).

(4) \( \Rightarrow \) (2). Let \( \tau : \mathcal{A} \to \mathcal{B} \) be the left adjoint to the inclusion \( \iota : \mathcal{B} \hookrightarrow \mathcal{A} \), and denote by \( \eta : \iota^! \mathcal{A} \Rightarrow \iota \circ \tau \) the unit of the adjunction. Then, \( \eta \) is an epimorphism by Lemma 4.1. Take now an object \( T \in T \) and an object \( A \in \mathcal{A} \). Consider the natural isomorphism
\[
\eta_A^* := \mathcal{A}(\eta_A, \iota(T)) : \mathcal{A}(\iota((\tau(A))), \iota(T)) \to \mathcal{A}(A, \iota(T)).
\]

It shows that any morphism \( f : A \to T \) in \( \mathcal{A} \) factors through \( \eta_A \). Thus, to prove that \( A \) has a \( T \)-preenvelope, it is enough to check that any object of \( \mathcal{B} \) has a \( T \)-preenvelope. But this is a direct consequence of the right completeness of the cotorsion pair \( (\text{Ker}(\text{Ext}^1_{\mathcal{B}}(-, T)), T) \) (see Example 2.7). Furthermore, for each short exact sequence
\[
0 \to (\iota \circ \tau)(A) \xrightarrow{\lambda_A} T_A \xrightarrow{\pi_A} X_A \to 0,
\]
with \( T_A \in T \) and \( X_A \in \text{Ker}(\text{Ext}^1_{\mathcal{B}}(-, T)) \), the map \( \mu_A := \lambda_A \circ \eta_A : A \to T_A \) is a \( T \)-preenvelope such that \( \text{Coker}(\mu_A) = \text{Coker}(\lambda_A) \in \text{Ker}(\text{Ext}^1_{\mathcal{B}}(-, T)) \cap \text{gen}(T) \subseteq \perp T \) (see Lemma 5.11(3)).

(1) \( \Rightarrow \) (3). Note that \( \mathcal{H}_I J V \in T \), for any set \( I \) for which the coproduct exists, and then, by Lemma 5.11(1), we have a natural isomorphism \( \text{Ext}^1_{\mathcal{B}}(\mathcal{H}_I J V, -) \cong \text{Ext}^1_{\mathcal{A}}(\mathcal{H}_I J V, -) \mid_{\mathcal{B}} \) of functors \( \mathcal{B} \to \text{Ab} \). Consider a universal extension:
\[
0 \to B \xrightarrow{\lambda_B} T_B \xrightarrow{p_B} \mathcal{H}_I J V \to 0 \quad \text{in} \ B.
\]

By Definition 5.6 and the mentioned natural isomorphism, the map
\[
\text{Ext}^1_{\mathcal{A}}(V, p_B) : \text{Ext}^1_{\mathcal{A}}(V, T_B) \hookrightarrow \text{Ext}^1_{\mathcal{A}}(V, \mathcal{H}_I J V) = 0
\]
is injective. Therefore \( T_B \in \mathcal{B} \cap V_{\perp T} = T \). Hence, \( \lambda_B \) is a special \( T \)-preenvelope in \( \mathcal{B} \) since \( \text{Ext}^1_{\mathcal{B}}(\mathcal{H}_I J V, -) \) vanishes on \( T \) (see Lemma 2.5).

The implication “\((2 \& 4) \Rightarrow (1)\)” under the stronger assumptions, is a consequence of Lemma 5.13, using also Lemma 5.11(1) to see that \( \text{Ker}(\text{Ext}^1_{\mathcal{B}}(V_Q, -)) = V_Q\perp \cap \mathcal{B} \), since \( V_Q \in T \).

**Remark 5.15** Let \( \mathcal{A} \) be an Abelian category with a (projective) epi-generator \( G \). If \( \mathcal{B} \) is a reflective subcategory of \( \mathcal{A} \), and \( \tau : \mathcal{A} \to \mathcal{B} \) is the left adjoint to the inclusion \( \iota : \mathcal{B} \hookrightarrow \mathcal{A} \), then \( \tau(G) \) is a (projective) epi-generator of \( \mathcal{B} \). In this case, if \( T \subseteq \mathcal{B} \) is a subcategory which is closed under coproducts in \( \mathcal{A} \), then it is also closed under coproducts in \( \mathcal{B} \).
In particular, if \( \mathcal{A} \) is an Abelian category with an epi-generator and \( \mathcal{T} \) is a subcategory closed under coproducts, quotients and extensions (e.g., a torsion class), then \( \mathcal{T} \) satisfies one of the assertions of the above theorem if, and only if, it satisfies all the others.

If in the above theorem the class \( \mathcal{T} \) is cogenerating in \( \mathcal{A} \), that is, \( \mathcal{A} = \text{Sub}(\mathcal{T}) \) then some of the conditions get simplified and we get the following straightforward corollary:

**Corollary 5.16** Consider the following assertions for an Abelian category \( \mathcal{A} \) and a subcategory \( \mathcal{T} \):

1. \( \mathcal{T} = \mathcal{V}^\perp_1 \) for some object \( \mathcal{V} \) that is \( \text{Ext}^1 \)-universal in \( \mathcal{A} \) and satisfies that \( \mathcal{V}(I) \in \mathcal{V}^\perp_1 \), for all sets \( I \) for which the coproduct exists in \( \mathcal{A} \);
2. \( \mathcal{T} \) is special preenveloping in \( \mathcal{A} \) and it is closed under extensions and direct summands;
3. \( (\mathcal{T}^\perp_1, \mathcal{T}) \) is a right complete cotorsion pair in \( \mathcal{A} \).

The implications \( “(1)\Rightarrow(2)\Leftrightarrow(3)” \) hold true. When, in addition, \( \mathcal{A} \) has an epi-generator and \( \mathcal{T} \) is closed under coproducts and quotients (e.g., when \( \mathcal{T} \) is a torsion class), all assertions are equivalent.

Our final result in this section shows that preenveloping hereditary torsion classes are familiar:

**Corollary 5.17** Let \( \mathcal{A} \) be an Abelian category and \( \mathcal{T} \) a subcategory. The following are equivalent:

1. \( \mathcal{T} \) is a (semi-special) preenveloping hereditary torsion class;
2. \( \mathcal{T} \) is a TTF class in \( \mathcal{A} \);
3. \( \mathcal{T} \) is a (semi-special) precoversing cohereditary torsion-free class.

**Proof** We just prove the equivalence \( “(1)\Leftrightarrow(2)” \) since the equivalence \( “(2)\Leftrightarrow(3)” \) will then follow by duality. Both (1) and (2) imply that \( \mathcal{T} = \text{Sub}(\mathcal{T}) \) and \( \mathcal{T} \) is a hereditary torsion class, something that we assume in the sequel. According to Theorem 5.14 and Proposition 4.6, the subcategory \( \mathcal{T} \) is (semi-special) preenveloping if and only if it is reflective in \( \mathcal{A} \). The task is hence reduced to check that, if \( \mathcal{T} \) is a reflective subcategory, then it is a TTF class. Suppose then that \( \mathcal{T} \) is reflective and let \( \sigma : \mathcal{A} \rightarrow \mathcal{T} \) be the left adjoint to the inclusion \( \iota : \mathcal{T} \hookrightarrow \mathcal{A} \). By Lemma 4.1, the unit \( \eta : \text{id}_\mathcal{A} \rightarrow \iota \circ \sigma \) is an epimorphism and the induced map \( \mathcal{A}(\eta_\mathcal{A}, \mathcal{T}) : \mathcal{A}(\sigma(A), \mathcal{T}) \rightarrow \mathcal{A}(A, \mathcal{T}) \) is an isomorphism for all \( A \in \mathcal{A} \) and \( \mathcal{T} \in \mathcal{T} \). Given an object \( A \in \mathcal{A} \), consider the exact sequence

\[
0 \rightarrow c(A) \xrightarrow{j_A} A \xrightarrow{\eta_A} (\iota \circ \sigma)(A) \rightarrow 0,
\]

where \( c(A) := \text{Ker}(\eta_\mathcal{A}) \). We shall prove that \( c(A) \in \perp \mathcal{T} \), from which we immediately conclude that \( (\perp \mathcal{T}, \mathcal{T}) \) is a torsion pair in \( \mathcal{A} \). Indeed, let \( f : c(A) \rightarrow \mathcal{T} \) be a morphism with \( T \in \mathcal{T} \) and take the pushout of \( j_A \) and \( f \):

\[
\begin{array}{ccccccccc}
0 & \xrightarrow{c(A)} & A & \xrightarrow{\eta_A} & (\iota \circ \sigma)(A) & \xrightarrow{0} \\
\downarrow{f} & & \downarrow{\text{PO}} & & \downarrow{g} & & \downarrow{\text{PO}} & & \downarrow{0} \\
0 & \xrightarrow{T} & \mathcal{T}' & \xrightarrow{\omega} & (\iota \circ \sigma)(A) & \xrightarrow{0}
\end{array}
\]
Note that $T'$ is in $\mathcal{T}$ since $\mathcal{T}$ is closed under extensions. In particular, since $A(\eta_A, T')$ is an isomorphism, there exists a morphism $\tilde{g} : (t \circ \sigma)(A) \to T'$ such that $g = \tilde{g} \circ \eta_A$. Hence, we get that $v \circ f = g \circ j_A = \tilde{g} \circ \eta_A \circ j_A = 0$ and, being $v$ a monomorphism, we conclude that $f = 0$.

Let us conclude the section with the following remark:

**Remark 5.18** In the setting of Theorem 5.14, suppose that $B$ is a reflective subcategory of $A$ such that $B$ has an epi-generator. Then, the cotorsion pair $(\text{Ker}(\text{Ext}^1_B(-, T)), T)$ in part (4) of the theorem is complete if, and only if, such an epi-generator can be chosen in $\text{Ker}(\text{Ext}^1_B(-, T))$.

**Proof** Suppose first that the cotorsion pair is complete and fix an epi-generator $Q$ of $B$. Then we have an exact sequence $0 \to TQ \to WQ \to Q \to 0$ in $B$, with $TQ \in T$ and $WQ \in \text{Ker}(\text{Ext}^1_B(-, T))$. It follows that $W := WQ$ is an epi-generator of $B$ belonging to $\text{Ker}(\text{Ext}^1_B(-, T))$. Conversely, assume that $Q \in \text{Ker}(\text{Ext}^1_B(-, T))$. The same argument in the proof of assertion (5) in [51, Thm. 2.13] also works here since, by Lemma 2.5, we know that the class $\text{Ker}(\text{Ext}^1_B(-, T))$ is closed under taking coproducts in $B$. Then the cotorsion pair $(\text{Ker}(\text{Ext}^1_B(-, T)), T)$ is complete. □

### 6 (Quasi-)Tilting Preenvelopes

In module categories, quasi-tilting torsion pairs are very much related to preenveloping torsion classes (see [7]). A similar relation still holds in our general setting, but with some more subtleties. Let us start by defining (quasi-)tilting objects in Abelian categories:

**Definition 6.1** Let $A$ be an Abelian category. A torsion pair $t = (\mathcal{T}, \mathcal{F})$ is called **quasi-tilting** when there is an object $V$ such that the following condition holds:

\[(\text{QT}) \quad \mathcal{T} = \text{Gen}(V) = \text{Pres}(V) = \overline{\text{Gen}(V)} \cap V^\perp.
\]

In this case $V$ is said to be a **quasi-tilting** object. Furthermore, the torsion pair $t$ is **tilting** when

\[(\text{T1}) \quad \text{there is a quasi-tilting object } V \text{ such that } \mathcal{T} = \text{Gen}(V);
\]

\[(\text{T2}) \quad \mathcal{T} \text{ is cogenerating}.
\]

In this case, $V$ is said to be a **(1-)tilting object**.

Note that, for a tilting object $V$, one easily deduces from the definition that:

$$\text{Gen}(V) = \text{Pres}(V) = V^\perp.$$

and this is a torsion class. Anyway, there are some subtleties in the above definition that we discuss in the following remark:

**Remark 6.2** The usual way to define quasi-tilting objects in categories of modules is the following: given a ring $R$, a right $R$-module $V$ is quasi-tilting provided

$$\text{Gen}(V) = \overline{\text{Gen}(V)} \cap V^\perp.$$  \hspace{1cm} (8)

In that particular setting, it is then a consequence of Eq. 8 that $\text{Gen}(V)$ is a torsion class, and that the equality $\text{Pres}(V) = \text{Gen}(V)$ holds. On the other hand, when working in an arbitrary
Abelian category \( \mathcal{A} \), if we take an object \( V \in \mathcal{A} \) that just satisfies the equality Eq. 8, we cannot guarantee in general that \( \text{Gen}(V) \) is a torsion class (see Example 6.3 below) nor the equality \( \text{Gen}(V) = \text{Pres}(V) \). That is our motivation for explicitly including these facts in the above definition. However, the definition can be given as in modules in many Abelian categories that appear “in nature” (see Corollary 6.9).

**Example 6.3** Let \( \mathcal{G} \) be a locally coherent Grothendieck category, \( V \) a finitely presented tilting object in \( \mathcal{G} \), and \((\text{Gen}(V), V^\perp)\) the associated torsion pair. Letting \( \mathcal{A} := \text{fp}(\mathcal{G}) \), we have that \( V \in \mathcal{A} \) and \( \text{Gen}_\mathcal{A}(V) = \text{Ker}(\text{Ext}_\mathcal{A}^1(V, -)) \) but there are cases in which \( \text{Gen}_\mathcal{A}(V) \) is not a torsion class in \( \mathcal{A} \).

**Proof** A situation of this kind is described in [43, Prop. 8.19], let us briefly recall it here. If \( R \) is a left semihereditary ring that is not right coherent (see [43, Ex. 8.19] for an explicit example) then the class \( \mathcal{F} \) of flat objects in \( \text{Mod-}R \) is a generating torsion-free class and the associated torsion pair \( \tau := (\mathcal{T} := \perp_1 \mathcal{F}, \mathcal{F}) \) in \( \text{Mod-}R \) restricts to \( \text{mod-}R \). Consider now the associated Happel-Reiten-Smalø \( \tau \)-structure

\[
\tau_\mathcal{t} = (D^{\leq -1}(\text{Mod-}R) \ast \mathcal{T}[0], \mathcal{F}[1] \ast D^{>0}(\text{Mod-}R))
\]

in \( D(\text{Mod-}R) \), and let \( \mathcal{G} := \mathcal{H}_\mathcal{t} = \mathcal{F}[1] \ast \mathcal{T}[0] \) be its heart, which is a locally coherent Grothendieck category by [43, Thm. 7.3]. Then \( V := R[1] \) is a finitely presented tilting object in \( \mathcal{G} \) whose associated torsion pair is \( \mathcal{t}' := (\mathcal{F}[1], \mathcal{T}[0]) \). If \( \text{Gen}_\mathcal{A}(V) \) were a torsion class in \( \mathcal{A} \), then \( \mathcal{t}' \) would restrict to \( \mathcal{A} \), which, again by [43, Thm. 7.3], would imply that \( \text{Mod-}R \) is locally coherent. This is false since \( R \) is not right coherent by assumption. \( \square \)

**Proposition 6.4** Let \( \mathcal{A} \) be an Abelian category, \( V \) a quasi-tilting object of \( \mathcal{A} \) and \( \mathcal{T} = \text{Gen}(V) \) the associated torsion class. The following assertions hold:

1. \( \mathcal{B} := \text{Sub}(\mathcal{T}) \) is an Abelian exact subcategory of \( \mathcal{A} \) and \( V \) is a tilting object of \( \mathcal{B} \);
2. \( \text{Add}(V) = \perp_1 \mathcal{T} \cap \mathcal{T} \);
3. if \( V' \) is a second quasi-tilting object in \( \mathcal{A} \), the following are equivalent:
   1. \( \text{Gen}(V) = \text{Gen}(V') \);
   2. \( \text{Add}(V) = \text{Add}(V') \).

**Proof** (1). By Example 2.3, we know that \( \mathcal{B} \) is an Abelian exact subcategory. Furthermore, the pair \((\mathcal{T}, \mathcal{T}^\perp \cap \mathcal{B})\) is a torsion pair in \( \mathcal{B} \) and the coproducts of copies of \( V \) in \( \mathcal{A} \) and in \( \mathcal{B} \) are the same, so that, \( \text{Pres}_\mathcal{B}(V) = \text{Pres}(V) \). On the other hand, by Lemma 5.11(1) and condition (QT), we have that \( \text{Ker}(\text{Ext}_\mathcal{B}^1(V, -)) = \mathcal{B} \cap V^\perp = \mathcal{T} \). Therefore, we have that \( \mathcal{T} = \text{Pres}(V) = \text{Pres}_\mathcal{B}(V) \subseteq \text{Gen}_\mathcal{B}(V) = \text{Gen}(V) = \text{Ker}(\text{Ext}_\mathcal{B}^1(V, -)) = \mathcal{T} \), so that \( V \) is a tilting object of \( \mathcal{B} \).

(2). Since \( V \) is a tilting object of \( \mathcal{B} \), \( \text{Add}(V) \subseteq \text{Ker}(\text{Ext}_\mathcal{B}^1(V, -)) \cap \mathcal{T} = \perp_1 \mathcal{T} \cap \mathcal{T} \) (see Lemma 5.11(3)). On the other hand, if \( X \in \perp_1 \mathcal{T} \cap \mathcal{T} \) then, due to the equality \( \text{Pres}(V) = \mathcal{T} \), we have an exact sequence \( 0 \to T \to V(I) \to X \to 0 \), for some set \( I \) and some \( T \in \text{Gen}(V) = \mathcal{T} \). But this sequence splits since \( X \in \perp_1 \mathcal{T} \), so that \( X \in \text{Add}(V) \).

(3). It follows from assertion (2). \( \square \)

The above proposition motivates us to give the following definition:

**Definition 6.5** Let \( \mathcal{A} \) be an Abelian category. Then, two (quasi-)tilting objects \( V \) and \( V' \) in \( \mathcal{A} \) are said to be **equivalent** provided \( \text{Add}(V) = \text{Add}(V') \).
6.1 Properties of Tilting Objects

In view of Theorem 5.14 and Proposition 6.4(1), it is natural to ask whether tilting objects are Ext\(^1\)-universal. In fact, this is true in full generality, as we show in the following lemma:

**Lemma 6.6** Let \(\mathcal{A}\) be an Abelian category and \(V \in \mathcal{A}\) a tilting object. Then, \(V\) is Ext\(^1\)-universal. Furthermore, given \(A \in \mathcal{A}\) and a universal extension \(0 \to A \to T \to V^{(I)} \to 0\) of \(V\) by \(A\), we have that \(T \in V^\perp 1 = T\).

**Proof** Let \(A\) be an object in \(\mathcal{A}\). From the fact that \(\text{Gen}(V)\) is a cogenerating class, we obtain an exact sequence in \(\mathcal{A}\) of the form:

\[
0 \to A \xrightarrow{v} T \xrightarrow{p} T' \to 0,
\]

for some \(T \in \text{Gen}(V)\). Now, since \(\text{Gen}(V)\) is a torsion class, also \(T' \in \text{Gen}(V) = \text{Pres}(V)\), so that, there is an epimorphism \(q : V^{(I)} \to T'\) in \(\mathcal{A}\) with \(\text{Ker}(q) \in \text{Gen}(V)\), for some \(I\) set.

Hence, we obtain the following commutative diagram in \(\mathcal{A}\):

\[
\begin{array}{ccccccccc}
0 & \rightarrow & A & \xrightarrow{v} & T & \xrightarrow{p} & T' & \rightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & A & \xrightarrow{v} & T & \xrightarrow{p} & T' & \rightarrow & 0 \\
\end{array}
\]

From the fact that \(\text{Gen}(V)\) is a torsion class in \(\mathcal{A}\), we obtain that \(Z \in \text{Gen}(V) = V^\perp 1\), as it is an extension of \(T\) and \(\text{Ker}(q)\) and both objects belong in \(\text{Gen}(V)\). Hence, the second row in the above diagram is a universal extension of \(V\) by \(A\).

For the final statement, let \(A \in \mathcal{A}\) and consider a universal extension \(0 \to A \to T \to V^{(I)} \to 0\) of \(V\) by \(A\). Then, applying \(\mathcal{A}(V, -)\), one gets:

\[
0 \to \mathcal{A}(V, A) \to \mathcal{A}(V, T) \to \mathcal{A}(V, V^{(I)}) \xrightarrow{\text{Ext}^1_A(V, A)} \text{Ext}^1_A(V, T) \leftrightarrow \text{Ext}^1_A(V, V^{(I)}) = 0,
\]

so \(T \in V^\perp 1 = T\). \(\square\)

We can now reformulate the definition of tilting object in an Abelian category. In particular, when our ground category \(\mathcal{A}\) has an epi-generator, we can characterize tilting objects by properties that are very similar to those usually taken as a definition in categories of modules.

**Proposition 6.7** Let \(\mathcal{A}\) be an Abelian category, \(V \in \mathcal{A}\) and consider the following assertions:

1. \(V\) is a tilting object;
2. the following conditions hold:
   1. \(\text{Gen}(V) = V^\perp 1\);
Hence, the short exact sequence in Eq. 9, show that sequence

\[ 0 \to G \to V_0 \to V_1 \to 0 \]

such that \( V_0, V_1 \in \text{Add}(V) \), and all coproducts of this sequence that exist in \( \mathcal{A} \) are exact.

Then, the equivalence “(1)\(\Leftrightarrow\)(2)” holds and, if \( \mathcal{A} \) has an epi-generator, all assertions are equivalent.

Proof (1)\(\Rightarrow\)(2). Let \( A \in \mathcal{A} \) and consider the following short exact sequence

\[ 0 \to T \xrightarrow{\iota} A \xrightarrow{\pi} A/T \to 0 \]

where \( T \in \mathcal{T} := \text{Gen}(V) \) and \( A/T \in \mathcal{F} \). Using that \( \mathcal{T} = \text{Pres}(V) \), choose an exact sequence

\[ 0 \to T' \to V^{(J)} \xrightarrow{q_T} T \to 0, \quad \text{where} \quad T' \in \mathcal{T}. \]

We get an epimorphism \( \mathcal{A}(X, V^{(J)}) \xrightarrow{A(X,q_T)} \mathcal{A}(X, T) \to \text{Ext}^1_{\mathcal{A}}(X, T') = 0 \), for all \( X \in \text{Add}(V) \subseteq \mathcal{T} \), which shows that \( q_T \) is a \( \text{Add}(V) \)-precover of \( T \). Consider the composition \( q_A := \iota \circ q_T : V^{(J)} \to T \hookrightarrow A \). Then, for each \( X \in \text{Add}(V) \), the map \( \mathcal{A}(X, q_A) = \mathcal{A}(X, \iota) \circ \mathcal{A}(X, q_T) \) is surjective as \( \mathcal{A}(X, q_T) \) is surjective and \( \mathcal{A}(X, \iota) \) is an isomorphism. Hence, \( q_A \) is the desired \( \text{Add}(V) \)-precover.

(2)\(\Rightarrow\)(1). We first prove that \( \mathcal{T} = \text{Gen}(V) \) is a torsion class. Let \( A \in \mathcal{A} \) and fix an \( \text{Add}(V) \)-precover \( p : X \to A \). It easily follows that \( \text{Im}(p) = \text{tr}_V(A) \) is the trace of \( V \) in \( A \).

We then get a short exact sequence

\[ 0 \to \text{tr}_V(A) \xrightarrow{\iota} A \xrightarrow{\pi} A/\text{tr}_V(A) \to 0. \]

Applying the functor \( \mathcal{A}(V, -) \), we then get the following exact sequence

\[ 0 \to \mathcal{A}(V, \text{tr}_V(A)) \xrightarrow{\cong} \mathcal{A}(V, A) \xrightarrow{0} \mathcal{A}(V, A/\text{tr}_V(A)) \to \text{Ext}^1_{\mathcal{A}}(V, \text{tr}_V(A)) = 0, \]

where the last term is trivial since \( \text{tr}_V(A) \in \text{Gen}(V) = V^\perp \). Hence, \( A/\text{tr}_V(A) \in V^\perp = \text{Gen}(V)^\perp \), showing that \( (\text{Gen}(V), V^\perp) \) is a torsion pair. It remains to check that \( \text{Gen}(V) = \text{Pres}(V) \). Indeed, let \( T \in \mathcal{T} \), choose an (epimorphic) \( \text{Add}(V) \)-precover \( q : X \to T \), take the following exact sequence

\[ 0 \to \ker(q) \to X \to T \to 0, \quad (9) \]

and apply \( \mathcal{A}(V, -) \) to get the following induced long exact sequence:

\[ 0 \to \mathcal{A}(V, \ker(q)) \to \mathcal{A}(V, X) \to \mathcal{A}(V, T) \xrightarrow{0} \text{Ext}^1_{\mathcal{A}}(V, \ker(q)) \hookrightarrow \text{Ext}^1_{\mathcal{A}}(V, X) \to \cdots, \]

where the map \( \mathcal{A}(V, q) : \mathcal{A}(V, X) \to \mathcal{A}(V, T) \) is surjective by definition of precover, so the connecting map \( \mathcal{A}(V, T) \to \text{Ext}^1_{\mathcal{A}}(V, \ker(q)) \) is necessarily trivial, showing that there is an inclusion \( \text{Ext}^1_{\mathcal{A}}(V, \ker(q)) \hookrightarrow \text{Ext}^1_{\mathcal{A}}(V, X) \). But in fact \( \text{Ext}^1_{\mathcal{A}}(V, X) = 0 \) as \( X \in \text{Add}(V) \subseteq V^{\perp_1} \), so also \( \text{Ext}^1_{\mathcal{A}}(V, \ker(q)) = 0 \), showing that \( \ker(q) \in V^{\perp_1} = \text{Gen}(V) \). Hence, the short exact sequence in Eq. 9, shows that \( T \in \text{Pres}(V) \).

We assume in the sequel that \( G \) is an epi-generator of \( \mathcal{A} \).

\( C.E. \) Parra et al.
(1,2)⇒(3). Conditions (3.1) and (3.3) are clear, while (3.2) follows by Corollary 2.6. As for (3.4), let us start with a universal extension of $V$ by $G$ (which exists by Lemma 6.6)

$$0 \to G \xrightarrow{u} T \xrightarrow{p} V(J) \to 0,$$

with $T \in V_{\perp 1} = \mathcal{T}$. Now, given an epimorphism $q: V(I) \to T$, put $\widehat{G} := \text{Ker}(p \circ q)$ and we consider the exact sequence

$$0 \to \widehat{G} \xrightarrow{\widehat{u}} V(I) \xrightarrow{\mu} V(J) \to 0.$$  \hfill (10)

Then, $\widehat{u}$ is a special $\mathcal{T}$-preenvelope since $\text{Ext}^1_A(V(J), -)$ vanishes on $\mathcal{T}$. By Lemma 2.8(3) we know that $u(K): \widehat{G}(K) \to (V(I))(K) \cong V(I \times K)$ is a monomorphism, for each set $K$ for which those coproducts exist in $\mathcal{A}$. Hence, the sequence in Eq. 10 satisfies condition (3.4) for the generator $\widehat{G}$.

(3)⇒(2). Conditions (3.1) and (3.2) imply that $T := \text{Gen}(V) \subseteq V_{\perp 1}$ since $\text{Ext}^1_A(V, -)$ is then a right exact functor. On the other hand, let $u: G \to V_0$ be the monomorphism given by $u(I)$ with $V_0 = V(I)$, $V_1 = V(J)$. Consider an arbitrary $A \in \mathcal{A}$ and an epimorphism $p: G(I) \to A$, for some set $I$, Lemma 3.4 gives a special $\mathcal{T}$-preenvelope $\mu_A: A \to T_A$, with $\text{Coker}(\mu_A) \cong V(I)$. In particular $\mathcal{T}$ is a cogenerating subcategory. When $A \in V_{\perp 1}$ in this argument, we get that $\mu_A$ is a section since then $\text{Ext}^1_A(V(I), A) = 0$. This implies that $V_{\perp 1} \subseteq \mathcal{T}$. Therefore, $\text{Gen}(V) = V_{\perp 1}$ is cogenerating.

\[ \square \]

Remark 6.8 If, in Proposition 6.7, $\mathcal{A}$ has a projective epi-generator $P$ (e.g., if $\mathcal{A}$ is a category of modules) and in the proof of “(1,2)⇒(3)” we take $G = P$, then the universal extension

$$0 \to P \xrightarrow{u} T \xrightarrow{p} V(J) \to 0$$

has the property that $T \in \mathcal{T} \cap V_{\perp 1} = \text{Add}(V)$ and $u$ is a (special) $\mathcal{T}$-preenvelope. By putting $V_0 = T$ and $V_1 = V(J)$, one then gets the sequence of condition (3.4) with $G = P$.

Our next result shows that two of the classical definitions of tilting objects are equivalent, and agree with ours, for a large class of Abelian categories $\mathcal{A}$ appearing “in nature”.

Corollary 6.9 Let $\mathcal{A}$ be an Abelian category which is either $\text{Ext}^1$-small and (Ab.4) (e.g., it is (Ab.3*) Abelian with an injective cogenerator) or Hom- and $\text{Ext}^1$-finite. Then, the following are equivalent for an object $V \in \mathcal{A}$:

1. $V$ is a tilting object;
2. $\text{Gen}(V) = V_{\perp 1}$;
3. the following conditions hold:
   - (3.1) the projective dimension of $V$ is $\leq 1$;
   - (3.2) $\text{Ext}^1_{\mathcal{A}}(V, V(I)) = 0$, for all sets $I$ (enough $\text{Ext}^1_{\mathcal{A}}(V, V) = 0$ in the Hom-finite case);
   - (3.3) $V_{\perp 1} := \text{Ker}(\mathcal{A}(V, -)) \cap \text{Ker}(\text{Ext}^1_{\mathcal{A}}(V, -)) = 0$.

Proof Under both sets of hypotheses, the subcategory $\text{Add}(V)$ is precovering.

(1)⇒(2) is clear.
We have already noticed that Add($V$) is precovering. Moreover, by Proposition 5.10 and Corollary 5.8, we know that $V^\perp$ is cogenerating. Then, assertion (2) of Proposition 6.7 holds.

(1,2) $\iff$ (3) follows as for Grothendieck categories (see [21]).

The following is an interesting consequence of the previous results:

**Corollary 6.10** Let $\mathcal{A}$ be an Abelian category and $V \in \mathcal{A}$ a tilting object.

1. If $\mathcal{A}$ is (Ab.3) then $\mathcal{A}$ is Ext$^1$-small.
2. Suppose that $R$ is a coherent commutative ring and that $\mathcal{A}(A, B)$ is a finitely presented $R$-module, for all $A, B \in \mathcal{A}$ (e.g., any Hom-finite category $\mathcal{A}$ over a commutative Noetherian ring). Then, Ext$^1_{\mathcal{A}}(A, B)$ is a finitely presented $R$-module, for all $A, B \in \mathcal{A}$. In particular, $\mathcal{A}$ is Ext$^1$-finite.

**Proof** Let us start recalling that each tilting object is Ext$^1$-universal (see Lemma 6.6).

(1). By Proposition 5.9, we know that Ext$^1_{\mathcal{A}}(V, B)$ is a set, for each $B \in \mathcal{A}$. Consider then $A \in \mathcal{A}$ and take a universal extension $0 \to A \to T_A \to V(I) \to 0$ with $T_A \in V^\perp = T$.

We get an exact sequence of (a priori big) Abelian groups

$$\text{Ext}^1_{\mathcal{A}}(V(I), B) \to \text{Ext}^1_{\mathcal{A}}(T_A, B) \to \text{Ext}^1_{\mathcal{A}}(A, B) \to \text{Ext}^2_{\mathcal{A}}(V(I), B) = 0$$

(see Corollary 2.6 for the last equality). Then our task reduces to prove that Ext$^1_{\mathcal{A}}(T, B)$ is a set, for all $T \in \mathcal{T}$ and $B \in \mathcal{A}$. By hypothesis, we have that $\mathcal{T} = \text{Pres}(V)$, so we can fix an exact sequence $0 \to T' \to V(J) \to T \to 0$, with $T' \in \mathcal{T}$, leading to an exact sequence of (a priori big) Abelian groups $\mathcal{A}(T', B) \to \text{Ext}^1_{\mathcal{A}}(T, B) \to \text{Ext}^1_{\mathcal{A}}(V(I), B)$, where the outer terms are sets.

(2). By Corollary 5.8, we know that Ext$^1_{\mathcal{A}}(V, B)$ is a finitely generated $R$-module, for each $B \in \mathcal{A}$. Proceeding as in the proof of part (1) (and using Proposition 3.11 to show that the set $I$ has to be finite), we are reduced to prove that Ext$^1_{\mathcal{A}}(T, B)$ is a finitely presented $R$-module, for all $T \in \mathcal{T}$ and $B \in \mathcal{A}$. Since $V$ is Ext$^1$-universal, we can fix a universal extension $0 \to B \to T_B \xrightarrow{\pi} V(r) \to 0$.

This gives an exact sequence of (a priori big) $R$-modules

$$\mathcal{A}(T, T_B) \to \mathcal{A}(T, V(r)) \to \text{Ext}^1_{\mathcal{A}}(T, B) \to \text{Ext}^1_{\mathcal{A}}(T, T_B) \xrightarrow{\pi_*} \text{Ext}^1_{\mathcal{A}}(T, V(r)),$$

where $\pi_* = \text{Ext}^1_{\mathcal{A}}(T, \pi)$. Bearing in mind that mod-$R$ is closed under extensions in Mod-$R$ and that $R$ is coherent, our task is further reduced to prove that Ext$^1_{\mathcal{A}}(T, T')$ is a finitely presented $R$-module, for all $T, T' \in \mathcal{T}$. Now, use that $\mathcal{T} = \text{Pres}(V)$ again, to construct an exact sequence

$$0 \to \tilde{T} \to V(s) \to T \to 0,$$

with $\tilde{T} \in \mathcal{T}$.

Applying the contravariant functor $\mathcal{A}(-, T') : \mathcal{A} \to \text{Mod}-R$, we then get an exact sequence

$$\mathcal{A}(V(s), T') \to \mathcal{A}(\tilde{T}, T') \to \text{Ext}^1_{\mathcal{A}}(T, T') \to 0,$$

and so Ext$^1_{\mathcal{A}}(T, T')$ is a finitely presented as desired. □

### 6.2 (Quasi-)Tilting Torsion Classes

The goal of this subsection is to identify the (semi-)special preenveloping torsion classes which are given by (quasi-)tilting objects.
Proposition 6.11 Let $\mathcal{A}$ be an Abelian category and $\mathcal{T}$ a torsion class in $\mathcal{A}$ such that $\mathcal{B} := \text{Sub}(\mathcal{T})$ is a reflective subcategory of $\mathcal{A}$ that has an epi-generator (e.g., if $\mathcal{A}$ has an epi-generator, see Remark 5.15). Then, the following assertions are equivalent:

1. $\mathcal{T} = \text{Gen}(V)$, for a quasi-tilting object $V$;
2. $\mathcal{T}$ is semi-special preenveloping in $\mathcal{A}$ and $\mathcal{B}$ contains an epi-generator that is in $\text{Ker}(\text{Ext}_B^1(-, \mathcal{T}))$;
3. $(\text{Ker}(\text{Ext}_B^1(-, \mathcal{T})), \mathcal{T})$ is a complete cotorsion pair in $\mathcal{B}$;
4. $\mathcal{T}$ is semi-special preenveloping in $\mathcal{A}$ and $\perp^1 \mathcal{T}$ generates $\mathcal{T}$, that is, each object of $\mathcal{T}$ is the epimorphic image of one in $\perp^1 \mathcal{T}$.

Proof (1)$\Rightarrow$(4). By the implication "(1)$\Rightarrow$(2)" in Theorem 5.14, to see that $\mathcal{T}$ is semi-special preenveloping in $\mathcal{A}$ it is enough to check that $\mathcal{B}$ is reflective (which we know by assumption), and that $\mathcal{T} = \mathcal{B} \cap V^{-1}$ with $V \in \mathcal{T}$ an object which is $\text{Ext}_B^1$-universal in $\mathcal{B}$ (which is also true by the definition of quasi-tilting, Proposition 6.4(1) and Lemma 6.6). The fact that $\perp^1 \mathcal{T}$ generates $\mathcal{T}$ easily follows since $V \in \perp^1 \mathcal{T}$.

(4)$\Rightarrow$(2). Let us fix an epi-generator $Q$ of $\mathcal{B}$. By Theorem 5.14, $\mathcal{T}$ is special preenveloping in $\mathcal{B}$ so, by Lemma 5.13, we have an exact sequence

$$
0 \rightarrow Q \xrightarrow{\lambda} T_Q \xrightarrow{} V_Q \xrightarrow{} 0
$$

where $\lambda$ is a special $\mathcal{T}$-preenvelope of $Q$ in $\mathcal{B}$, and

$\mathcal{T} = \text{Gen}(T_Q) = \text{Ker}(\text{Ext}_B^1(V_Q, -)) = \mathcal{B} \cap V^{-1} \mathcal{T}$

(see Lemma 5.11(1) for the last equality). Take now an epimorphism $\rho : X_Q \rightarrow T_Q$, with $X_Q \in \perp^1 \mathcal{T}$. Fix also a left adjoint $\tau : \mathcal{A} \rightarrow \mathcal{B}$ to the inclusion $\iota : \mathcal{B} \hookrightarrow \mathcal{A}$, and recall that the unit $u : \text{id}_A \rightarrow \iota \circ \tau$ of this adjunction is an epimorphism (see Lemma 4.1). Consider now the epimorphism

$$
\tau(X_Q) \xrightarrow{\tau(\rho)} T_Q \equiv T_Q \quad \text{in } \mathcal{B}.
$$

We claim that $\tau(X_Q) \in \text{Ker}(\text{Ext}_B^1(-, \mathcal{T}))$. In fact, given $T \in \mathcal{T}$ and a short exact sequence

$$
0 \rightarrow T \rightarrow B \rightarrow \tau(X_Q) \rightarrow 0 \quad \text{in } \mathcal{B}, \tag{11}
$$

we get a commutative diagram with exact rows in $\mathcal{A}$:

$$
\begin{array}{ccccccc}
0 & \rightarrow & T & \rightarrow & B & \rightarrow & \tau(X_Q) & \rightarrow & 0 \\
& \downarrow\scriptscriptstyle{\iota(T)} & \downarrow\text{PO.PB} & \downarrow\tau(X_Q) & & \downarrow\iota \circ \tau & \\
0 & \rightarrow & \iota(T) & \rightarrow & \iota(B) & \rightarrow & (\iota \circ \tau)(X_Q) & \rightarrow & 0 \\
\end{array}
$$

where the square on the right-hand side is then bicartesian. By the choice of $X_Q$, the upper row splits, and it is mapped by $\tau$ onto the sequence Eq. 11. Therefore, Eq. 11 also splits and so $\tau(X_Q) \in \text{Ker}(\text{Ext}_B^1(-, \mathcal{T}))$. Consider now the following commutative diagram with exact rows in $\mathcal{B}$:

$$
\begin{array}{ccccccc}
0 & \rightarrow & \widetilde{Q} & \rightarrow & \tau(X_Q) & \rightarrow & V_Q & \rightarrow & 0 \\
& \downarrow\scriptscriptstyle{q} & \downarrow\text{PO.PB} & \downarrow\tau(\rho) & & \downarrow\tau(\rho) & \\
0 & \rightarrow & Q & \rightarrow & T_Q & \rightarrow & V_Q & \rightarrow & 0 \\
\end{array}
$$
where the map \( q \) is an epimorphism since the left square is bicartesian. Then \( \tilde{Q} \) is an epi-generator of \( B \). On the other hand, by Corollary 2.6, we have that the projective dimension of \( V_Q \) in \( B \) is less or equal than one. We then get an exact sequence of functors \( T \rightarrow \text{Ab} \) as follow

\[
0 = \text{Ext}^1_B(\tau(X_Q), -) \rightarrow \text{Ext}^1_B(\tilde{Q}, -) \rightarrow \text{Ext}^2_B(V_Q, -) = 0.
\]

It then follows that \( \tilde{Q} \in \text{Ker}(\text{Ext}^1_B(-, T)) \).

(2) \( \Leftrightarrow \) (3) is Remark 5.18.

(2–3) \( \Rightarrow \) (1). Let us fix an epi-generator \( Q \) of \( B \) such that \( Q \in \text{Ker}(\text{Ext}^1_B(-, T)) \). A sin the proof of the implication “(4)\( \Rightarrow \) (2)”, we consider a short exact sequence

\[
0 \rightarrow Q \xrightarrow{\lambda} T_Q \xrightarrow{\pi} V_Q \rightarrow 0,
\]

where \( \lambda \) is a special \( T \)-preenvelope in \( B \). Now, \( Q \) and \( V_Q \) belong in \( \perp^1 T \cap B \), so that \( T_Q \in T \cap \perp^1 T \). We then have \( \text{Gen}(T_Q) = V_Q^1 \cap B = T \subseteq T_Q^1 \cap B \). By taking \( V := T_Q \oplus V_Q \), we get that \( \text{Gen}(V) = \text{Gen}(T_Q) = T \) and \( B \cap V^1 = B \cap T_Q^1 \cap V_Q^1 = B \cap V_Q^1 = T \). Furthermore, \( \lambda \) is in particular a \( \text{Gen}(V) \)-preenvelope of \( Q \), whose cokernel is \( V_Q \in \text{Add}(V) \). Thus, by Lemma 3.4, we have that for every object \( B \) in \( B \), there is \( \mu_B : B \rightarrow T_B \) a \( T \)-preenvelope of \( B \) whose cokernel is in \( \text{Add}(V) \). Let \( T \in \text{Gen}(V) \) and we consider an exact sequence in \( A \) of the form:

\[
0 \rightarrow K \xrightarrow{\mu} V^{(I)} \xrightarrow{p} T \rightarrow 0.
\]

for some \( I \) set. Note that \( K \in B \), and hence, we can consider \( \mu_K : K \rightarrow T_K \) a \( T \)-preenvelope of \( K \) such that \( \text{Coker}(\mu_K) \in \text{Add}(V) \). Consider the following commutative diagram with exact rows:

\[
\begin{array}{ccccccccc}
0 & \rightarrow & K & \xrightarrow{\mu} & V^{(I)} & \xrightarrow{p} & T & \rightarrow & 0 \\
& & \downarrow{\mu_K} & & \downarrow{\text{p.o.}} & & \downarrow{\text{q}} & & \\
0 & \rightarrow & T_K & \rightarrow & Z & \rightarrow & T & \rightarrow & 0 \\
& & \downarrow{q} & & \downarrow{\text{Coker}(\mu_K)} & & & & \\
& & & \text{Coker}(\mu_K) & & & & & \\
\end{array}
\]

From Lemma 2.5, we obtain that \( Z \cong V^{(I)} \oplus \text{Coker}(\mu_K) \) and so \( Z \in \text{Add}(V) \). Therefore \( T \in \text{Pres}(V) \) since \( T_K \in \text{Gen}(V) \). It follows that \( \text{Pres}(V) = \text{Gen}(V) = T \), so that \( V \) is a quasi-tilting object.

As an immediate consequence, we get:

\begin{corollary}
Let \( A \) be an Abelian category with an epi-generator and \( t = (T, \mathcal{F}) \) a torsion pair in \( A \). The following assertions are equivalent:

(1) \( t \) is a tilting torsion pair;

(2) \( \perp^1 T, T \) is a right complete cotorsion pair and \( \perp^1 T \) contains an epi-generator of \( A \);

(3) \( t \) is a cotorsion pair such that \( \text{Coker}(\mu_K) \in \text{Add}(V) \).
\end{corollary}
(3) \((-1, \mathcal{T}, \mathcal{T})\) is a complete cotorsion pair in \(\mathcal{A}\);

(4) \(\mathcal{T}\) is special preenveloping in \(\mathcal{A}\) and \((-1, \mathcal{T})\) generates \(\mathcal{T}\).

In particular, the assignment \([V] \mapsto \text{Gen}(V)\) gives a one-to-one correspondence

\[
\{ \text{Equivalence classes of tilting objects in } \mathcal{A} \} \quad \xymatrix{ \ar@<0.5ex>[r] & } \quad \{ \text{Special preenveloping torsion classes } \mathcal{T} \text{ in } \mathcal{A} \text{ such that } (-1, \mathcal{T}) \text{ generates } \mathcal{T} \}
\]

### 6.3 Quasi-Tilting Objects Versus Semi-special Preenveloping Torsion Classes

In this subsection we apply Proposition 6.11 to some cases where we can guarantee that the torsion class \(\mathcal{T}\) has the property that \(\mathcal{B} := \text{Sub}(\mathcal{T})\) is a reflective subcategory of \(\mathcal{A}\).

**Corollary 6.13** Let \(\mathcal{A}\) be an (Ab.3) Abelian category with a generator, (whence it is bicomplete and well-powered by Lemma 3.7). The assignment \([V] \mapsto \text{Gen}(V)\) gives a bijection between the equivalence classes of quasi-tilting objects \(V\) such that \(\text{Gen}(V)\) is closed under taking products and the semi-special preenveloping torsion classes \(\mathcal{T}\) such that \((-1, \mathcal{T})\) generates \(\mathcal{T}\).

When \(\mathcal{A}\) is also (Ab.4*) (e.g., when \(\mathcal{A}\) has a projective generator), the mentioned quasi-tilting objects are precisely the strongly finendo ones, i.e. those for which \(V(I)^{J} \in \text{Gen}(V)\), for all sets \(I, J\).

**Proof** By Corollary 4.4, we know that if \(V\) is a quasi-tilting object as indicated and \(\mathcal{T} = \text{Gen}(V)\), then \(\mathcal{B} := \text{Sub}(\mathcal{T})\) is a reflective subcategory. Moreover, we know that \(\mathcal{B}\) is also (Ab.3) with a generator. Then, Proposition 6.11 gives that \(\mathcal{T}\) is semi-special preenveloping and \((-1, \mathcal{T})\) generates \(\mathcal{T}\). Conversely, if we have a torsion class \(\mathcal{T}\) as the latter, then it is closed under products, which implies, again by Corollary 4.4, that \(\mathcal{B} = \text{Sub}(\mathcal{T})\) is a reflective subcategory. By Proposition 6.11, we conclude that \(\mathcal{T} = \text{Gen}(V)\), for a quasi-tilting object \(V\) such that \(\text{Gen}(V)\) is closed under products.

For the last statement, when \(\mathcal{A}\) is (Ab.4*), we just need to prove that, if \(V\) is strongly finendo, then \(\text{Gen}(V)\) is closed under products. Indeed, let \((T_{\lambda})_{\Lambda}\) be a family in \(\text{Gen}(V)\) and fix epimorphisms \(p_{\lambda} : V(I_{\lambda}) \to T_{\lambda}\) (with \(\lambda \in \Lambda\)). There is no loss of generality in assuming that \(I_{\lambda} = I_{\mu} := I\), for all \(\lambda, \mu \in \Lambda\). The (Ab.4*) condition gives an induced epimorphism \(\prod p_{\lambda} : (V(I))^{\Lambda} \to \prod T_{\lambda}\), which implies that \(\prod T_{\lambda} \in \mathcal{T}\), since \(V\) is strongly finendo.

**Remark 6.14** Let us start with the same setting of Corollary 6.13, that is, \(\mathcal{A}\) is an (Ab.3) Abelian category with a generator, and call an object \(X \in \mathcal{A}\) **finendo** when \(X^{I} \in \text{Gen}(X)\), for all sets \(I\). If \(\mathcal{A}\) is also (Ab.4*), \(V\) is quasi-tilting and the canonical map \(V(I) \to V^{I}\) is a monomorphism for all sets \(I\) (e.g., when \(\mathcal{A}\) is (Ab.5)), then \(V\) is finendo if and only if it is strongly finendo (as defined in Corollary 6.13). Indeed, in such case the canonical map \((V(I))^{I} \to (V^{I})^{I} \simeq V^{I \times I}\) is a monomorphism which implies, by the (Ab.4*) condition, that \(\text{Gen}(V)\) is closed under products. This, in turn, implies that \(\text{Gen}(V) = \overline{\text{Gen}}(V) \cap V^{(-1)}\) is closed under products, and hence that \(V\) is strongly finendo. Therefore, when \(\mathcal{A}\) is an (Ab.4*) Grothendieck category, Corollary 6.13 gives a bijection between equivalence classes of finendo quasi-tilting objects and semi-special preenveloping torsion classes such that \((-1, \mathcal{T})\) generate \(\mathcal{T}\). In fact, when \(\mathcal{A} = \text{Mod}-\mathcal{R}\) is a module category (in which case \(\mathcal{A}\) is Grothendieck and (Ab.4*)), it is well-known that \(X\) is finendo exactly when it is finitely generated as a left module over \(\text{End}_{\mathcal{R}}(X)\) (see [23, Lemma after Prop. 1.5]), so Corol-
lary 6.13 extends the known results for categories of modules (see [5, Coro. 3.8], and also [7, Thm. 2.1] for the tilting case).

**Corollary 6.15** Let $\mathcal{A}$ be a $\text{Hom}$-finite Abelian category with a generator such that all objects have finite length. Then, the assignment $[V] \mapsto \text{Gen}(V)$ gives a bijection between equivalence classes of quasi-tilting objects and enveloping (semi-special) preenveloping torsion classes $\mathcal{T}$ such that $\perp_1 \mathcal{T}$ generates $\mathcal{T}$. Furthermore, this bijection restricts to a one-to-one correspondence between equivalence classes of tilting objects and special enveloping torsion classes $\mathcal{T}$ such that $\perp_1 \mathcal{T}$ generates $\mathcal{T}$.

**Proof**  Bearing in mind that $\mathcal{A}$ is a Krull-Schmidt category, we know by Lemma 3.12 that all preenveloping classes closed under direct summands are enveloping. Furthermore, by Corollary 4.7, we know that the enveloping torsion classes are exactly the semi-special preenveloping and that these are special precisely when they are cogenerating. Now, by Example 4.5, we know that $\text{Sub}(\mathcal{X})$ is a reflective subcategory, for all torsion classes $\mathcal{X}$ in $\mathcal{A}$. The result is then a direct consequence of Proposition 6.11.

**Corollary 6.16** Let $\mathcal{A}$ be an Abelian category with a projective epi-generator $P$. Then, the assignment $[V] \mapsto \text{Gen}(V)$ gives a bijection between equivalence classes of quasi-tilting objects $V$ such that $P$ has an $\text{Add}(V)$-preenvelope and semi-special preenveloping torsion classes in $\mathcal{A}$. This bijection restricts to one between equivalence classes of tilting objects and special preenveloping torsion classes. Moreover, when no non-zero infinite coproduct $X^{(I)}$ exists in $\mathcal{A}$ (e.g., if $\mathcal{A}$ is Hom-finite, see Proposition 3.11), those (quasi-)tilting objects are precisely the $V$ for which $\mathcal{A}(P, V)$ is finitely generated as a left $\text{End}_\mathcal{A}(V)$-module.

**Proof** If $V$ is a quasi-tilting object such that $P$ has an $\text{Add}(V)$-preenvelope and $\mathcal{T} := \text{Gen}(V)$, then we know by Lemma 4.9 that $\mathcal{B} = \text{Sub}(\mathcal{T})$ is a reflective subcategory. On the other hand, if $\mathcal{T}$ is a (semi-)special preenveloping torsion class in $\mathcal{A}$, by Proposition 4.6, we have that $\mathcal{B} = \text{Sub}(\mathcal{T})$ is reflective. Note that, in both cases, if $\tau : \mathcal{A} \to \mathcal{B}$ is the left adjoint to the inclusion, then $\tau(P)$ is a projective epi-generator of $\mathcal{B}$, so that, the bijection is then a direct consequence of the Proposition 6.11 since $P \in \perp_1 \mathcal{X}$, for every class of objects $\mathcal{X}$ in $\mathcal{A}$.

Finally, when no non-zero infinite coproduct $X^{(I)}$ exists in $\mathcal{A}$, it is a standard fact that a morphism $f = (f_1, \ldots, f_n) : P \to V^n$ is an $\text{Add}(V)(= \text{add}(V))$-preenvelope if, and only if, $\{f_1, \ldots, f_n\}$ is a set of generators of $\mathcal{A}(P, V)$ as a left $\text{End}_\mathcal{A}(V)$-module.

6.4 A Look at the Duals: (Quasi-)Cotilting Precovers

We leave to the reader the literal dualization of the bijections of the previous subsections between classes of (quasi-)tilting objects and classes of (semi-)special torsion classes in Abelian categories. Here we only emphasize a few of them. The following one is a particular case of the dual of Theorem 5.14 (see also Proposition 6.11).

**Theorem 6.17** Let $\mathcal{A}$ be an $\text{Ext}^1$-small $(\text{Ab.3}^*)$ Abelian category with a cogenerator (so $\mathcal{A}$ is bicomplete and well-powered), let $\mathcal{F}$ be a torsion-free class in $\mathcal{A}$ and $\mathcal{C} := \text{Quot}(\mathcal{F})$. Then, the following are equivalent:

1. $\mathcal{F}$ is closed under taking coproducts in $\mathcal{A}$ and $\mathcal{F} = \mathcal{C} \cap \perp_1 Q$, for some $Q \in \mathcal{F}$;

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(2) \( \mathcal{F} \) is semi-special precovering in \( \mathcal{A} \);
(3) \( \mathcal{F} \) is closed under coproducts and \( (\mathcal{F}, \text{Ker}(\text{Ext}^1_C(\mathcal{F}, -))) \) is a left complete cotorsion pair in \( \mathcal{C} \).

In this case, \( \mathcal{F} \) is the torsion-free class of a quasi-cotilting torsion pair if and only if the cotorsion pair of assertion (3) is complete in \( \mathcal{C} \), if and only if \( \mathcal{F} \perp^1 \) cogenerates \( \mathcal{F} \).

**Proof** By Proposition 4.2, we know that \( \mathcal{C} \) is a coreflective subcategory if, and only if, it is closed under taking coproducts in \( \mathcal{A} \), a fact that is guaranteed whenever \( \mathcal{F} \) is closed under taking coproducts. So that in all the conditions, we have that \( \mathcal{C} \) is a coreflective subcategory (for the condition (2), we use the dual of the Proposition 4.6). Thus, the result is a direct consequence of the duals of Theorem 5.14 and Proposition 6.11 whenever the object \( Q \) of the condition (1) is \( \text{Ext}^1 \)-universal in \( \mathcal{C}^{\text{op}} \). But, in such condition, we have that \( \text{Ker}(\text{Ext}^1_C(\mathcal{C}, Q)) = \mathcal{C} \cap \perp^1 Q = \mathcal{F} \), that is, \( \text{Ker}(\text{Ext}^1_C(\mathcal{C}, Q)) \) is a generating class of \( \mathcal{C} = \text{Quot}(\mathcal{F}) \). Therefore, combining the duals of Example 5.3 and Proposition 5.9, we obtain that \( Q \) is \( \text{Ext}^1 \)-universal in \( \mathcal{C}^{\text{op}} \). \( \square \)

As an immediate consequence, we get (see Lemma 2.5(2)):

**Corollary 6.18** In the same setting of Theorem 6.17, the following assertions are equivalent:

1. \( \mathcal{F} \) is a generating class such that \( \mathcal{F} = \perp^1 Q \), for some \( Q \in \mathcal{F} \) (resp., \( \mathcal{F} = \perp^1 Q \) for \( Q \) cotilting);
2. \( \mathcal{F} \) is a special precovering torsion-free class (such that \( \mathcal{F} \perp^1 \) cogenerates \( \mathcal{F} \));
3. \( (\mathcal{F}, \mathcal{F} \perp^1) \) is a left complete (resp., complete) cotorsion pair.

When \( \mathcal{A} \) is (Ab.3*) with an injective cogenerator, whence even (Ab.4) (see Lemma 3.7 and use the fact that any (Ab.3) Abelian category with an injective cogenerator is (Ab.4)), we get the following consequence of Corollaries 6.13, 6.18 and Theorem 6.17.

**Corollary 6.19** Let \( \mathcal{A} \) be an (Ab.3*) Abelian category with an injective cogenerator. The assignment \( [Q] \mapsto \text{Cogen}(Q) \) gives a one-to-one correspondence:

\[
\left\{ \text{equivalence classes of strongly cofinendo quasi-cotilting objects in } \mathcal{A} \right\} \xrightarrow{1:1} \left\{ \text{semi-special precovering torsion-free classes in } \mathcal{A} \right\}
\]

(where an object is strongly cofinendo if it is strongly finendo in \( \mathcal{A}^{\text{op}} \)). This bijection restricts to the following one:

\[
\left\{ \text{equivalence classes of cotilting objects in } \mathcal{A} \right\} \xrightarrow{1:1} \left\{ \text{special precovering torsion-free classes in } \mathcal{A} \right\}.
\]

When \( \mathcal{A} = \mathcal{G} \) is a Grothendieck category, one can say even more. We refer the reader to [43] for the definition of cosilting object and cosilting torsion pair in such a category.

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Proposition 6.20 Let \( \mathcal{G} \) be a Grothendieck category. For a torsion pair \( t = (\mathcal{T}, \mathcal{F}) \), the following assertions are equivalent:

1. \( t \) is of finite type, i.e. \( \mathcal{F} \) is closed under taking direct limits;
2. \( t \) is a quasi-tilting torsion pair;
3. \( t \) is a cosilting torsion pair;
4. \( \mathcal{F} \) is (semi-special) precovering;
5. \( \mathcal{F} \) is covering.

In particular, the assignment \([Q] \mapsto \text{Cogen}(Q)\) induces a bijection between equivalence classes of quasi-tilting (resp., cosilting) objects and (pre)covering torsion-free classes in \( \mathcal{G} \).

Proof The equivalences \((1) \iff (2) \iff (3)\) are \([43, \text{Thm. 4.1}]\) and \((2) \iff (4)\) follows from Corollary 6.19, bearing in mind that in a Grothendieck category all objects are strongly cofinendo. On the other hand, \((1,4) \implies (5)\) follows from a theorem of Enochs and Xu (see \([26, \text{Thm. 1.2}]\)). The implication \((5) \implies (4)\) is clear.

Remark 6.21 In the setting of module categories, the above result was obtained independently by Breaz and Žemlička \([16, \text{Thm. 3.5}]\) and Zhang and Wei \([63, \text{Thm. 3.5}]\) (for more on this, see also \([3, \text{Thm. 3.8 and Coro. 3.9}]\)).

7 Finitely Presented Modules over Right Coherent Rings

In this last section we fix a (usually right coherent) ring \( R \) and we take a deeper look at the results of the previous sections when \( \mathcal{A} := \text{mod-}R \) is the category of finitely presented (right) \( R \)-modules (by Lemma 3.5 all Abelian categories with a projective finite epi-generator are of this kind).

7.1 Basic Facts and Notations

In what follows we will mostly work in two categories: the category of all right \( R \)-modules \( \text{Mod-}R \) and that of finitely presented right \( R \)-modules \( \text{mod-}R \). To avoid confusion, let us fix the following notational conventions:

- we use the symbol \( \bigsqcup I \) to denote coproducts in the category \( \text{Mod-}R \), while the symbol \( \coprod I \) denotes coproducts in \( \text{mod-}R \). For any non-empty set \( I \), denote by \( \coprod I V \) the coproduct of \(|I|\)-many copies of \( V \) in \( \text{mod-}R \), when such a coproduct exists;
- all throughout this section, the symbols Sub, Gen, Gen, and Pres, without suffix, will be meant in the category \( \text{Mod-}R \). We will use the suffix “mod-\( R \)” (e.g., \( \text{Sub}_{\text{mod-}R}(-) \)) when we want them taken in mod-\( R \).

Let us start noting that mod-\( R \) may have infinite coproducts.

Proposition 7.1 Let \( R \) be any ring and \( I \) an infinite set. The following assertions are equivalent:

1. \( \text{mod-}R \) has \( I \)-coproducts;
2. the coproduct of \(|I|\)-many copies of \( R \) exists in \( \text{mod-}R \);
3. \( R \) is Morita equivalent to a ring \( S \) such that \( S \) is isomorphic to \( S^I \) in \( S\text{-Mod} \).
Proof (1)⇔(2). Since $R$ is a finite epi-generator of mod-$R$, this is a consequence of Lemma 3.3(3).

(3)⇒(1, 2). Since assertion (1) is Morita invariant, replacing $R$ by $S$ if necessary, we can assume that $R \cong R^l$ in $R$-Mod. We then get a family of orthogonal idempotents $(e_i)_I$ such that each $Re_i$ is isomorphic to $R$ (although not canonically) and the canonical map $\rho: R \to \prod_i Re_i$, such that $r \to (re_i)_I$, is an isomorphism in $R$-Mod. We then have isomorphisms

$$e_i R \cong \text{Hom}_R(Re_i, R_R) \cong \text{Hom}_R(R_R, R_R) \cong R, \quad \text{in mod}-R.$$ 

We claim that $R_R$, together with the inclusions $\iota_i: e_i R \hookrightarrow R$ (with $i \in I$), gives a coproduct of the $e_i R$ in mod-$R$. Take an arbitrary object $X \in \text{mod}-R$ and consider the induced map

$$\varphi_X: X \cong \text{Hom}_R(R, X) \xrightarrow{(\iota_i)_I} \prod_i \text{Hom}_R(e_i R, X) \cong \prod_i X e_i,$$

which is clearly identified with the map $x \to (xe_i)_I$. Let now $U: \text{mod}-R \to \text{Ab}$ be the forgetful functor and $F: \text{mod}-R \to \text{Ab}$ the functor that acts on objects by $F(X) := \prod_i X e_i$. Then, there is a natural transformation $\varphi: U \Rightarrow F$ where, for each $X \in \text{mod}-R$, $\varphi_X: X \to \prod_i X e_i$ is defined as above. The subcategory $\mathcal{C} \subseteq \text{mod}-R$ of all the $X$ for which $\varphi_X$ is an isomorphism contains $R$ since $\varphi_R = \rho$, and is clearly closed under finite coproducts. Moreover, since all the $e_i R$ are projective in mod-$R$, and products are exact in Ab, $\mathcal{C}$ is also closed under taking cokernels. It then follows that $\mathcal{C} = \text{mod}-R$, so that $\varphi$ is a natural isomorphism and our claim is settled.

(2)⇒(3). Let $P := \bigsqcup_i R$ be the coproduct of $|I|$-many copies of $R$ in mod-$R$. Then $P$ is projective in mod-$R$. As $P$ is finitely presented, there is an epimorphism $R^n \to P$ and, as $P$ is projective in mod-$R$, this morphism splits (both in mod-$R$ and in Mod-$R$). Hence, $P$ is projective also in Mod-$R$ and $R$ is a direct summand of $P$. Therefore, $P$ is a progenerator of Mod-$R$ and, moreover, we have that $\bigsqcup_i P \cong P$. Hence,

$$S = \text{Hom}_R(P, P) \cong \text{Hom}_R(\bigsqcup_i P, P) \cong \text{Hom}_R(P, P)^l \cong S^l$$

in S-Mod, where $S = \text{End}_R(P_R)$, and this is a ring that is Morita equivalent to $R$.

Example 7.2 If $K$ is a field and $V = K(I)$, for $I$ an infinite set, then the ring $R = \text{End}_K(V)$ is a von Neumann regular ring (whence coherent on both sides) such that

$$R^l \cong \text{Hom}_K(V^{(l)}, V) \cong \text{Hom}_K(V, V) \cong R$$

as left $R$-modules. More generally (see [12]), if $A$ is any ring and $M$ is an $A$-module such that $M^{(l)} \cong M$, then $R^l \cong R$ in $R$-Mod, when $R := \text{End}_A(M)$.

7.2 Tilting Objects in mod-$R$ Versus Classical Tilting Modules

In this subsection we start the comparison between the classical tilting modules (as defined below) and the tilting objects in the Abelian category $\mathcal{A} := \text{mod}-R$, as in Definition 6.1. Let us start recalling the classical definition of tilting in a category of modules:

Definition 7.3 Let $R$ be any ring. A classical tilting right $R$-module is a right $R$-module $V$ satisfying the following conditions:

1. **(CT.1)** there is a short exact sequence

$$0 \to P_1 \longrightarrow P_0 \longrightarrow V \to 0,$$
with $P_0$ and $P_1$ finitely generated projective right $R$-modules;

(CT.2) $\text{Ext}^1_{\text{Mod-}R}(V, V) = 0$ or, equivalently, $\text{Ext}^1_{\text{Mod-}R}(V, V^{(I)}) = 0$ for all sets $I$;

(CT.3) there is a short exact sequence

$$0 \to R \to V_0 \to V_1 \to 0$$

such that $V_0, V_1 \in \text{add}(V)$.

It is well-known, and easy to see, that for an arbitrary ring $R$, the right $R$-module $V$ is classical tilting if, and only if, it is finitely presented and it is a tilting object of $\text{Mod-}R$ in the sense of Definition 6.1. On the other hand, when $R$ is right coherent (so $\text{mod-}R$ is Abelian), it is not clear when a classical tilting right $R$-module $V$ is also a tilting object of the Abelian category $\text{mod-}R$. In this subsection we study the relation between these two concepts. Before proceeding with the following auxiliary result, let us recall that a right $R$-module $V$ is said to be finendo, provided it is finitely generated as a left module over $\text{End}_R(V)$ (see also Remark 6.14).

**Lemma 7.4** Let $R$ be a right coherent ring, $V$ a finitely presented quasi-tilting right $R$-module (i.e., a quasi-tilting object of $\text{Mod-}R$ that is finitely presented) and $t_V := (\text{Gen}(V), V^\perp)$ the associated torsion pair in $\text{Mod-}R$. Suppose also that the torsion pair $t_V$ restricts to $\text{mod-}R$. Then, the following assertions hold true:

1. $\text{Gen}(V) \cap \text{mod-}R = \text{Gen}_{\text{mod-}R}(V) = \overline{\text{Gen}(V)} \cap \text{mod-}R$;
2. $V$ is a quasi-tilting object of $\text{mod-}R$;
3. if $V$ is finendo, then $\overline{\text{Gen}_{\text{mod-}R}(V)}$ is reflective in $\text{mod-}R$.

**Proof** (1). We obviously have the inclusions $\text{Gen}(V) \cap \text{mod-}R = \text{gen}(V) \cap \text{mod-}R \subseteq \text{Gen}_{\text{mod-}R}(V)$. On the other hand, since $\text{Gen}(V) \cap \text{mod-}R$ is a torsion class in $\text{mod-}R$, also the converse holds:

$$\text{Gen}(V) \cap \text{mod-}R = \text{Gen}_{\text{mod-}R}(V). \quad (12)$$

By [19, Lem. 4.4], we know that then $\text{Gen}(V) = \lim_{\rightarrow} (\text{gen}(V) \cap \text{mod-}R)$, from which one easily deduces that $\overline{\text{Gen}(V)} \cap \text{mod-}R \subseteq \overline{\text{Gen}_{\text{mod-}R}(V)} = \text{Sub}_{\text{mod-}R}(\text{gen}(V) \cap \text{mod-}R)$. But then,

$$\overline{\text{Gen}(V)} \cap \text{mod-}R = \overline{\text{Gen}_{\text{mod-}R}(V)}, \quad (13)$$

since the converse inclusion is obvious.

(2). From the equality $\text{Gen}(V) = \overline{\text{Gen}(V)} \cap V^\perp$ and part (1) we get that

$$\text{Gen}_{\text{mod-}R}(V) = \text{Gen}(V) \cap \text{mod-}R = \overline{\text{Gen}(V)} \cap V^\perp \cap \text{mod-}R = \overline{\text{Gen}_{\text{mod-}R}(V)} \cap \text{Ker}(\text{Ext}^1_{\text{mod-}R}(V, -)).$$

To prove that $V$ is a quasi-tilting object of $\text{mod-}R$, we need to prove that $\text{Gen}(V) \cap \text{mod-}R \subseteq \text{pres}(V)$, for then we have that $\text{Gen}_{\text{mod-}R}(V) = \text{Pres}_{\text{mod-}R}(V)$. Indeed, given $T \in \text{Gen}(V) = \text{Pres}(V)$, there is a presentation

$$V^{(\Lambda)} \overset{f}{\longrightarrow} V^{(I)} \longrightarrow T \to 0.$$

We apply an argument due to Lazard (see [43, Lem. 1.11]), that we just outline here, to show that $T \in \lim_{\rightarrow} \text{pres}(V)$. Given finite subsets $\Lambda' \subseteq \Lambda$ and $I' \subseteq I$ denote, respectively, by

$$\iota_{\Lambda'} : V^{(\Lambda')} \to V^{(\Lambda)} \quad \text{and} \quad \epsilon_{I'} : V^{(I')} \to V^{(I)}$$

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the inclusions into the coproduct, and define the following set

\[ \Upsilon := \{ (\Lambda', I') : \Lambda' \subseteq \Lambda, I' \subseteq I \text{ finite, such that } f \circ \iota_{\Lambda'} \text{ factors through } \epsilon_{I'} \}. \]

Endow \( \Upsilon \) with the product order. It is routine to check that \( \Upsilon \) is directed and that

\[ T \cong \text{Coker}(f) \cong \lim_{\longrightarrow} \text{Coker}(f(\Lambda', I')) \in \lim_{\longrightarrow} \text{pres}(V). \]

Finally, if we suppose that \( T \) is also finitely presented, then there is some \( (\Lambda', I') \in \Upsilon \) such that the canonical map \( u := u_{(\Lambda', I')} : \text{Coker}(f(\Lambda', I')) := T' \rightarrow T \) is a retraction. We then get a commutative diagram in \( \text{mod-}R \) with exact rows:

\[
\begin{array}{cccccc}
0 & \rightarrow & T'' & \rightarrow & V(I') & \rightarrow & T' & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow u & & \downarrow & & \downarrow & \\
0 & \rightarrow & U & \rightarrow & V(I') & \rightarrow & T & \rightarrow & 0
\end{array}
\]

By the Snake Lemma, we have an exact sequence \( 0 \rightarrow T'' \rightarrow U \rightarrow \text{Ker}(u) \rightarrow 0 \), where the outer terms are in \( \mathcal{T} \cap \text{mod-}R \). It follows that \( U \in \mathcal{T} \cap \text{mod-}R \), and so \( T \in \text{pres}(V) \).

(3). Let \( \mathcal{T}_0 := \text{Gen}(V) \cap \text{mod-}R = \text{Gen}_{\text{mod-}R}(V) \). Since \( V \) is finendo, we can choose a finite set \( \{ f_1, \ldots, f_n \} \) of generators of \( \text{End}_R(V) \). It is routine to check that \( f := (f_1, \ldots, f_n) : R \rightarrow V^n \) is an add\((V)\)-preenvelope. Hence, by Lemma 4.9, \( \text{Gen}_{\text{mod-}R}(V) = \text{Sub}_{\text{mod-}R}(\mathcal{T}_0) \) is reflective in \( \text{mod-}R \).

**Proposition 7.5** Let \( R \) be a right coherent ring and \( V \in \text{mod-}R \). The following are equivalent:

1. \( V \) is a quasi-tilting object of \( \text{mod-}R \) and \( \text{Gen}_{\text{mod-}R}(V) \) is reflective in \( \text{mod-}R \);
2. there is a (possibly infinite) set \( J \) such that \( V_J := \bigoplus J V \) is a finitely presented finendo quasi-tilting \( R \)-module and the torsion pair \( \text{tv}_J := (\text{Gen}(V_J), V_J^\perp) \) in Mod-\(R \) restricts to \( \text{mod-}R \).

In this case, the restricted torsion pair \( (\text{Gen}(V_J) \cap \text{mod-}R, V_J^\perp \cap \text{mod-}R) \) is the torsion pair associated with \( V \) in \( \text{mod-}R \), that is, \( \text{Gen}(V_J) \cap \text{mod-}R = \text{Gen}_{\text{mod-}R}(V) \).

**Proof** Note that \( \text{Add}_{\text{mod-}R}(V) = \text{Add}_{\text{mod-}R}(V_J) \), whenever \( V_J \) exists. Therefore \( V \) is a quasi-tilting object of \( \text{mod-}R \) if, and only if, so is \( V_J \). In that case the associated torsion pairs in \( \text{mod-}R \) coincide.

(2)\( \Rightarrow \)(1). It follows from Lemma 7.4(2,3), together with the above comment.

(1)\( \Rightarrow \)(2). Put \( \mathcal{T}_0 := \text{Gen}_{\text{mod-}R}(V) \) and \( B^0 := \text{Sub}_{\text{mod-}R}(\mathcal{T}_0) \), and fix a left adjoint \( L : \text{mod-}R \rightarrow B^0 \) to the inclusion \( \iota : B^0 \rightarrow \text{mod-}R \). Then, \( L(R) \) is a projective generator of \( B^0 \) and, by Proposition 6.4, we know that \( V \) is tilting in \( B^0 \). By Proposition 6.7 and Remark 6.8, we have an exact sequence

\[ 0 \rightarrow L(R) \overset{\lambda}{\rightarrow} V_0 \rightarrow V_1 \rightarrow 0 \quad \text{in } B^0, \]

where \( V_0, V_1 \in \text{Add}_{\text{mod-}R}(V) \), that is kept exact for any coproduct that might exist in \( \text{mod-}R \). Consider now the unit \( \mu : \text{id}_{\text{mod-}R} \Rightarrow \iota \circ L \), that is an epimorphism (see Lemma 4.1). By considering the composition \( u := \iota(\lambda) \circ \mu_{R} : R \rightarrow (\iota \circ L)(R) \rightarrow \iota(V_0) = V_0 \), we obtain an exact sequence

\[ R \overset{u}{\rightarrow} V_0 \rightarrow V_1 \rightarrow 0 \quad \text{in } \text{mod-}R, \]

and one easily sees that \( u \) is a semi-special \( \mathcal{T}_0 \)-preenvelope. We fix now a (possibly infinite) set \( J \) such that \( V_J := \bigoplus J V \) exists in \( \text{mod-}R \) and \( V_0, V_1 \) are both direct summands of \( V_J \).
Then, $u$ is an add($V_J$)-preenvelope, which implies that $V_J$ is finendo. We claim that

$$\mathcal{T}_0 = \text{gen}(V_J) \cap \text{mod-}R.$$ 

By the comment at the beginning of this proof, the inclusion “⊇” is obvious. Conversely, let $T \in \mathcal{T}_0$ and fix an epimorphism $p : R^n \to T$. Then, $p$ factors through $u^n : R^n \to V_0^n$, since the latter is a $\mathcal{T}_0$-preenvelope. This implies that $T$ is an epimorphic image of $V_0^n$, and so it belongs in $\text{gen}(V_J) \cap \text{mod-}R$. Put now

$$\mathcal{T} := \lim_{\longrightarrow} \mathcal{T}_0 = \lim_{\longrightarrow}(\text{gen}(V_J) \cap \text{mod-}R),$$

that is a torsion class in $\text{mod-}R$ by [19, Lem. 4.4]. We then have that $\mathcal{T} \subseteq \text{Gen}(V_J)$ and the reverse inclusion is obvious since $V_J \in \mathcal{T}$ and $\mathcal{T}$ is closed under coproducts and quotients. On the other hand, $V_J^{⊥1}$ is closed under direct limits, since $V_J$ is finitely presented and $R$ right coherent. We then get that $\mathcal{T} = \text{Gen}(V_J) \subseteq \overline{\text{Gen}(V_J)} \cap V_J^{⊥1}$. We need just to verify the converse inclusion and the proof will be finished (see Remark 6.2). Now, by Lemma 5.11(1), we have that $\overline{\text{Gen}(V)} \cap V_J^{⊥1} = \text{Ker(Ext}^1_{\mathcal{B}}(V_J, -))$, where $\mathcal{B} := \text{Sub}(\mathcal{T}) = \overline{\text{Gen}(V_J)}$. Furthermore, we have that $\mathcal{B}' = \mathcal{B} \cap \text{mod-}R$. In fact, the inclusion $\mathcal{B}' \subseteq \mathcal{B} \cap \text{mod-}R$ is clear, while the converse follows from the fact that $\mathcal{T} = \lim_{\longrightarrow} \mathcal{T}_0$. On the other hand, since $u$ is an add($V_J$)-preenvelope, we immediately get that $a := \text{Ker}(u)$ is the annihilator of $V_J$, i.e.

$$a = \{r \in R : V_J \cdot r = 0\}.$$

In particular, $a$ is a two-sided ideal of $R$ such that $R/\alpha \cong \text{Im}(u) \subseteq \mathcal{B}$. This implies that $\text{mod-}R/a \subseteq \mathcal{B}$, with the obvious identification. But the converse is also true, since $a$ annihilates $\mathcal{T} = \text{Gen}(V)$, and hence $B \cdot a = 0$, for all $B \in \mathcal{B}$. Hence, $\mathcal{B} = \text{mod-}R/a$. On the other hand, since $a$ is the kernel of a morphism in $\text{mod-}R$, we have that $R/a \in \text{mod-}R$, and so $\text{mod-}R/a \subseteq \text{mod-}R$ and $\mathcal{B}' = \mathcal{B} \cap \text{mod-}R = \text{mod-}R/a \cap \text{mod-}R = \text{mod-}R/a$. Moreover, we have that $R/a$ is a right coherent ring (see [57, Coro. 3.6]) and, by the the last paragraph, we also have that

$$\overline{\text{Gen}(V_J)} \cap V_J^{⊥1} = \text{Ker(Ext}^1_{\mathcal{B}}(V_J, -)) = \text{Ker(Ext}^1_{\text{mod-}R/a}(V_J, -)).$$

We will be done if we prove that $V_J$ is tilting in $\text{mod-}R/a$. We already know that $V_J$ is a tilting object of $\mathcal{B}' = \text{mod-}R/a$, which implies that conditions (CT.1) and (CT.2) of Definition 7.3 hold for $V_J$ in $\text{mod-}R/a$. But, by the choice of the set $J$ and the definition of $a$, we have a short exact sequence $0 \to R/a \to V_0 \to V_1 \to 0$, so that also condition (CT.3) holds, thus ending the proof.

The results of this subsection naturally raise the question of whether the torsion pair in $\text{mod-}R$ induced by a finendo quasi-tilting module, in particular, by a classical tilting $R$-module, restricts to $\text{mod-}R$. The following results will clarify this question. Let us first recall that an object $V^\bullet$ of the derived category $D(R) := D(\text{mod-}R)$ is a **classical tilting complex** when it satisfies the following three conditions:

1. (CTC.1) the functor $D(R)(V^\bullet, -) : D(R) \to \text{Ab}$ preserves coproducts;
2. (CTC.2) $D(R)(V^\bullet, V^\bullet[n]) = 0$ for all integers $n \neq 0$;
3. (CTC.3) if $X^\bullet \in D(R)$ is a complex such that $D(R)(V^\bullet, X^\bullet[n]) = 0$, for all $n \in \mathbb{Z}$, then $X^\bullet = 0$.

We refer the reader to [11] for the definition of $t$-structure in a triangulated category and what it means for it to restrict to a triangulated subcategory.
Proposition 7.6 Let $R$ be a right coherent ring, $V^* \in \mathbf{D}(R)$ a classical tilting complex and take its endomorphism ring $S := \text{End}_{\mathbf{D}(R)}(V^*)$. The following assertions are equivalent:

1. the $t$-structure $((V^*)^\perp_{\geq 0}, (V^*)^\perp_{<0})$ generated by $V^*$ in $\mathbf{D}(R)$ restricts to $\mathbf{D}^b(\text{mod-}R)$;
2. $S$ is a right coherent ring.

Proof Up to replacing $V^*$ by a quasi-isomorphic complex, we can assume that $V^*$ is a complex of $S$-$R$-bimodules, and then we have an induced triangulated equivalence

$$ F := \mathbb{R}\text{Hom}_R(V^*, -) : \mathbf{D}(R) \longrightarrow \mathbf{D}(S) $$

and, by adapting the proof of [45, Prop. 8.1], we know that it restricts to an equivalence

$$ F|_{\mathcal{K}^{-,b}(\text{proj-}R)} : \mathcal{K}^{-,b}(\text{proj-}R) \longrightarrow \mathcal{K}^{-,b}(\text{proj-}S). $$

Furthermore, $F$ takes the $t$-structure $((V^*)^\perp_{\geq 0}, (V^*)^\perp_{<0})$ to the canonical $t$-structure $(\mathbf{D}^{\geq 0}(S), \mathbf{D}^{\leq 0}(S))$ in $\mathbf{D}(S)$. Hence, it takes the pair

$$ \tau_R := (\mathcal{K}^{-,b}(\text{proj-}R) \cap (V^*)^\perp_{\geq 0}, \mathcal{K}^{-,b}(\text{proj-}R) \cap (V^*)^\perp_{<0}) $$

do subcategories of $\mathcal{K}^{-,b}(\text{proj-}R)$ to the pair

$$ \tau_S := (\mathcal{K}^{-,b}(\text{proj-}S) \cap \mathbf{D}^{\leq 0}(S), \mathcal{K}^{-,b}(\text{proj-}S) \cap \mathbf{D}^{\leq 0}(S)) $$

do subcategories $\mathcal{K}^{-,b}(\text{proj-}S)$. With this in mind, we can now proceed with the proof.

(1)$\Rightarrow$(2). Since $R$ is right coherent, there is a triangulated equivalence

$$ \mathcal{K}^{-,b}(\text{proj-}R) \cong \mathbf{D}^b(\text{mod-}R). $$

Assertion (1) is then equivalent to say that $\tau_R$ is a $t$-structure in $\mathcal{K}^{-,b}(\text{proj-}R)$, which is then equivalent to say that $\tau_S$ is a $t$-structure in $\mathcal{K}^{-,b}(\text{proj-}S)$. Take now a morphism $f : S^n \rightarrow S$ in $\text{Mod-S}$ and consider it as a complex

$$ Y_f^* : \cdots \longrightarrow 0 \longrightarrow S^n \xrightarrow{f} S \longrightarrow 0 \longrightarrow \cdots, $$

concentrated in degrees 0 and 1. The associated $\tau_S$-truncation triangle, which coincides with the truncation triangle with respect to the canonical $t$-structure $(\mathbf{D}^{\geq 0}(S), \mathbf{D}^{\leq 0}(S))$, takes the form

$$ (\text{Ker}(f))[0] \longrightarrow Y_f^* \longrightarrow (\text{CoKer}(f))[-1] \longrightarrow (\text{Ker}(f))[1]. $$

It then follows that $(\text{Ker}(f))[0] \in \mathcal{K}^{-,b}(\text{proj-}S)$, which implies that $\text{Ker}(f)$ admits a projective resolution in $\text{Mod-S}$ whose terms are finitely generated (projective) $S$-modules. In particular, $\text{Ker}(f)$ is finitely presented, and so $S$ is right coherent.

(2)$\Rightarrow$(1). Since $R$ is a right coherent ring, we know that there is a triangulated equivalence

$$ \mathcal{K}^{-,b}(\text{proj-}R) \cong \mathbf{D}^b(\text{mod-}R) \quad \text{and} \quad \mathcal{K}^{-,b}(\text{proj-}S) \cong \mathbf{D}^b(\text{mod-}S). $$

Therefore, $F$ restricts to an equivalence $\mathbf{D}^b(\text{mod-}R) \cong \mathbf{D}^b(\text{mod-}S)$. Since the canonical $t$-structure of $\mathbf{D}(S)$ restricts to $\mathbf{D}^b(\text{mod-}S)$, we conclude that $((V^*)^\perp_{\geq 0}, (V^*)^\perp_{<0})$ restricts to $\mathbf{D}^b(\text{mod-}R)$.

We are now in position to determine when the torsion pair associated with a quasi-tilting finitely presented right $R$-module $V$ restricts to $\text{mod-}R$.

Corollary 7.7 Let $R$ be a right coherent ring and $V$ a finitely presented quasi-tilting right $R$-module. Consider the following assertions:

1. the associated torsion pair $t_V := (\text{Gen}(V), V^\perp)$ in $\text{Mod-R}$ restricts to $\text{mod-}R$;
(2) \( \text{add}(V) \) is a precovering subcategory of \( \text{mod-}R \);
(3) \( \text{Hom}_R(V, X) \) is a finitely presented right \( \text{End}_R(V_R) \)-module, for all \( X \in \text{mod-}R \);
(4) \( \text{End}_R(V_R) \) is a right coherent ring.

The implications "(1) \( \Leftrightarrow \) (2) \( \Leftrightarrow \) (3)" hold true and when, in addition, \( V \) is finendo, the implication "(3) \( \Rightarrow \) (4)" also holds. When \( V \) is a (classical) tilting \( R \)-module, then all assertions are equivalent.

**Proof**  (1) \( \Rightarrow \) (2). Let \( T := \text{Gen}(V) = \text{Pres}(V) \). Then, as \( T_0 := T \cap \text{mod-}R = \text{pres}(V) \) (see Lemma 7.4), we can conclude by the argument in the proof of implication "(1) \( \Rightarrow \) (2)" of Proposition 6.7, with \( J \) a finite set in this case.

(2) \( \Rightarrow \) (1). The torsion radical in \( \text{Mod-}R \) associated with \( t_V \) is the trace of \( V \), i.e., \( t(M) = t_V(M) \) for all \( M \in \text{mod-}R \). If now \( X \in \text{mod-}R \), then any morphism \( V \rightarrow X \) factors through any fixed \( \text{add}(V) \)-precover \( p : V^n \rightarrow X \). It follows that \( t(X) = \text{Im}(p) \), which is a finitely presented module since it is the image of a morphism in \( \text{mod-}R \). Therefore, \( t_V \) restricts to \( \text{mod-}R \).

(2) \( \Leftrightarrow \) (3) is folklore.

(1) \( \Leftrightarrow \) (4) if \( V \) is a (classical) tilting \( R \)-module. This is a consequence of Proposition 7.6. Indeed, each classical tilting \( R \)-module \( V \) is a classical tilting complex, when identified with the stalk complex \( V[0] \) in \( \text{D}(R) \). Moreover, we have an obvious ring isomorphism \( \text{End}(V_R) \cong \text{End}_{\text{D}(R)}(V[0]) \). In that case the \( t \)-structure \( (V[0] \perp_{-0}, V[0] \perp_{<0}) \) in \( \text{D}(R) \) is precisely the so-called Happel-Reiten-Smalø \( t \)-structure associated with the torsion pair \( t_V = (\text{Gen}(V), V \perp) \) of \( \text{Mod-}R \) generated by \( V \) (see the proof of [22, Prop. 5.2]). Then, by [50, Prop. 5.1], we know that \( (V[0] \perp_{<0}, V[0] \perp_{<0}) \) restricts to \( \text{D}^b(\text{mod-}R) \) if, and only if, \( t_V \) restricts to \( \text{mod-}R \).

(1–3) \( \Rightarrow \) (4) assuming that \( V \) is finendo. In this case, what happens is that \( B := \text{Sub}(T) \) is a reflective subcategory of \( \text{Mod-}R \) (see Lemma 4.9 and Remark 6.14). In particular, if

\[
\alpha = \text{ann}(V_R) := \{ r \in R : V \cdot r = 0 \},
\]

we have that \( R/\alpha \in B \) since we have a monomorphism \( R/\alpha \hookrightarrow V^V \) and \( V^V \in \text{Gen}(V) = T \). It immediately follows that \( B = \text{Mod-}R/\alpha \). On the other hand, if \( \{ v_1, \ldots, v_l \} \) is a finite set of generators of \( V \) as a left \( \text{End}(V_R) \)-module, then the morphism \( \mu : R \rightarrow V^l \), defined by \( r \mapsto (v_1r, \ldots, v_lr) \) is an \( \text{add}(V) \)-preenvelope of \( R_R \). This implies that \( \alpha = \text{Ker}(\mu) \), so this is a finitely generated right ideal of \( R \). It follows that \( R/\alpha \) is a right coherent ring (see [57, Coro. 3.6]) and that \( \text{mod-}R/\alpha \subset \text{mod-}R \), with the obvious abuse of notation. Recall that, by Proposition 6.4, \( V \) is a tilting object in \( B \), and hence \( V \) is a classical tilting \( R/\alpha \)-module whose associated torsion pair \( (\mathcal{T}, V \perp \cap \text{mod-}R/\alpha) \) in \( \text{Mod-}R/\alpha \) restricts to \( \text{mod-}R/\alpha \). By the equivalence "(1) \( \Leftrightarrow \) (4)" in the tilting case, we conclude that \( \text{End}_{R/\alpha}(V) = \text{End}_R(V_R) \) is right coherent.

**Remark 7.8** Since any ring which is derived equivalent to \( R \) is isomorphic to the endomorphism ring of a classical tilting complex in \( \text{D}(R) \), Proposition 7.6 naturally leads to the question of whether "right coherencc" is a derived invariant property for rings. That is, if the equivalent conditions of the proposition always hold. Jeremy Rickard, to whom we are grateful, has shown to us a counterexample. However, as far as we know, the following restricted version of this question is still open.

**Question 7.9** Is "right coherence" a property of rings which is invariant under classical (1)-tilting equivalences? Equivalently (see Proposition 7.6), given a right coherent ring \( R \)
and a classical (1)-tilting right $R$-module $V$, does the torsion pair $(\text{Gen}(V), V^\perp)$ in $\text{Mod}-R$ restrict always to $\text{mod}-R$?

**Remark 7.10** The answer to the above question is negative when $\text{Mod}-R$ is replaced by a locally coherent Grothendieck category (see Example 6.3).

### 7.3 Construction of Nontrivial Classical Tilting Modules over Coherent Rings

We want to show that, given any ring $A$, there is a canonical way of giving plenty of examples of non-trivial (i.e., not progenerators) classical tilting modules over the (upper) triangular matrix ring

$$T_n(A) := \begin{bmatrix} A & A & \cdots & A \\ 0 & A & \cdots & A \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & A \end{bmatrix}$$

that, in case $T_n(A)$ (or, equivalently, $A$) is right coherent (see Corollary 7.12), always give rise to a torsion pair that restricts to $T_n(A)$-$\text{mod}$.

**Lemma 7.11** Let $A$ and $B$ be rings and $AM_B$ an $A$-$B$-bimodule that is finitely generated as a right $B$-module. Consider the ring $R := \begin{bmatrix} A & M \\ 0 & B \end{bmatrix} \cong \begin{bmatrix} B & 0 \\ 0 & M \end{bmatrix}$. The following assertions are equivalent:

1. $R$ is right coherent;
2. $A$ and $B$ are right coherent and $M$ is a coherent right $B$-module, i.e., any finitely generated $B$-submodule of $M$ is finitely presented.

**Proof** The two-sided ideal $I := \begin{bmatrix} 0 & M \\ 0 & 0 \end{bmatrix}$ is finitely generated as a right ideal since $M_B$ is finitely generated. Moreover $R/I$ is isomorphic to $A \times B$, so that $R/I$ is right coherent if and only if so are $A$ and $B$. Now use [33, Thm. 2].

**Corollary 7.12** Let $A$ be a ring and $n > 0$ an integer. Then, the triangular matrix ring $T_n(A)$ is right (resp., left) coherent if, and only if, so is $A$.

**Proof** The statement about right coherence follows by Lemma 7.10 and an easy induction. On the other hand, in the situation of that lemma, the opposite algebra of $\begin{bmatrix} A & M \\ 0 & B \end{bmatrix}$ is isomorphic to $\begin{bmatrix} B^{\text{op}} & M \\ 0 & A^{\text{op}} \end{bmatrix}$, where $M$ is viewed as a $B^{\text{op}}$-$A^{\text{op}}$-bimodule in the obvious way. It follows that, if $M$ is finitely generated as a left $A$-module, then $\begin{bmatrix} A & M \\ 0 & B \end{bmatrix}$ is left coherent if, and only if, $B$ and $A$ are left coherent and $M$ is a coherent left $A$-module. Applying this dual version of the lemma, again by an easy induction argument, we conclude that $T_n(A)$ is left coherent if, and only if, so is $A$.

**Example 7.13** Let $A$ by a ring, $n > 0$ an integer and consider the triangular matrix ring $T_n(A)$ and its ideal $I$ of strictly upper triangular matrices. If $e_1$ is the idempotent matrix...
having 1 in its (1, 1)-entry and zero elsewhere, then $V = \bigoplus_{k=1}^{n}(e_1 T_n(A)/e_1 I^k)$ is a classical tilting right $T_n(A)$-module that is not a progenerator. In case $A$ is right coherent, the associated tilting torsion pair restricts to mod-$T_n(A)$.

**Proof** The inclusion $\iota : T_n(A) \hookrightarrow M_n(A)$ is a ring epimorphism, where $M_n(A)$ denotes the full matrix ring of $A$ (see [53, Prop. 2.10]). Denote by $e_i$ ($i = 1, \ldots, n$) the idempotent matrix having 1 in its $(i, i)$-entry and zero elsewhere. We have that $M_n(A) \cong (e_1 T_n(A))^{(n)}$ as right $T_n(A)$-modules and $M_n(A) \cong (T_n(A)e_i)^{(n)}$ as left $T_n(A)$-modules, so that $M_n(A)$ is projective on both sides as a module over $T_n(A)$. It follows from [6, Thm. 3.5] that $V' := M_n(A) \oplus (M_n(A)/T_n(A))$ is classical tilting, both as a right and as left $T_n(A)$-module. Furthermore, we have an isomorphism $M_n(A)/T_n(A) \cong \bigoplus_{k=1}^{n-1}(e_1 T_n(A)/e_1 I^k)$ in Mod-$T_n(A)$. Therefore, since $I^n = 0$, we have that $V := \bigoplus_{k=1}^{n}(e_1 T_n(A)/e_1 I^k)$ is a classical (1-)tilting right $T_n(A)$-module which is equivalent to $V'$. Moreover, when $k < n$ (i.e., $I^k \neq 0$) the quotient $e_1 T_n(A)/e_1 I^k$ cannot be projective in Mod-$T_n(A)$ since this would imply that $e_1 I^k$ is a direct summand of $e_1 T_n(A)$ that, due to the nilpotency of $I$, is contained in the radical rad($T_n(A)$). This is absurd. Then $V$ is not a progenerator of Mod-$T_n(A)$.

When $A$ is right coherent, we know by Corollary 7.11 that $T_n(A)$ is right coherent. Note now that then any morphism of right $T_n(A)$-modules $(e_1 T_n(A)/e_1 I^k) \rightarrow (e_1 T_n(A)/e_1 I^l)$ is induced by left multiplication by a matrix $X = (x_{i,j})_{1 \leq i, j \leq n} \in e_1 T_n(A)e_1$ (that is, $x_{i,j} = 0$ whenever $i + j > 2$). Given that $X \cdot I^k \subseteq I^l$, whenever $X \neq 0$ (i.e., $x_{1,1} \neq 0$) and $k < l$, and that $e_1 T_n(A)e_1 \cong A$, we readily see that End$(V'_{T_n(A)})$ is isomorphic to the lower triangular matrix ring

$$T'_{n}(A) := \begin{bmatrix}
A & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
A & \cdots & A & 0 \\
A & \cdots & A & A
\end{bmatrix}.$$  

It is easy to see that this matrix ring is isomorphic to $T_n(A)$, and hence it is a right coherent ring. Therefore, End$(V'_{T_n(A)})$ is a right coherent ring and Corollary 7.7 applies. $\square$

### 7.4 Quasi-Tilting Preenvelopes in mod-$R$

The goal of this last subsection is to identify the quasi-tilting objects $V$ of mod-$R$ that give rise to semi-special preenveloping torsion classes, equivalently (see Proposition 6.11) such that Gen$_{\text{mod-}R}(V)$ is a reflective subcategory of mod-$R$. We start with the following result, which is a trivial translation of Proposition 7.5, bearing in mind that “classical” and “finitely presented” are synonymous terms when referred to a tilting right $R$-module.

**Corollary 7.14** Let $R$ be a right coherent ring and $V \in \text{mod-}R$. The following are equivalent:

1. $V$ is a tilting object of mod-$R$;
2. there is a (possibly infinite) set $I$ such that $V_I := \bigsqcup_I V$ exists, it is a classical tilting module and the torsion pair $t_V := (\text{Gen}(V_I), V_I^\perp)$ in Mod-$R$ restricts to mod-$R$.

In this case, the restricted torsion pair $(\text{Gen}(V_I) \cap \text{mod-}R, V_I^\perp \cap \text{mod-}R)$ is the torsion pair in mod-$R$ associated with $V$.

We are now ready to parametrize the (semi)special preenveloping torsion classes in mod-$R$.  

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Proposition 7.15 Let $R$ be a right coherent ring. Then, the assignment $[V] \mapsto \text{pres}(V)$ gives a one-to-one correspondence between:

1. equivalence classes of finitely presented finendo quasi-tilting (resp., classical tilting) right $R$-modules $V$ whose associated torsion pair in $\text{Mod-}R$ restricts to $\text{mod-}R$;
2. semi-special (resp., special) preenveloping torsion classes in $\text{mod-}R$.

When $R$ is also a Krull-Schmidt ring, the semi-special (resp., special) preenveloping torsion classes in $\text{mod-}R$ are exactly the (pre)enveloping (resp., cogenerating (pre)enveloping) ones.

Proof By Proposition 6.11 and Theorem 5.14, the assignment $[V] \mapsto T_V = \text{Genmod-}R(V)$ establishes a bijection between the equivalence classes of quasi-tilting objects $V$ of $\text{mod-}R$ such that $\text{Genmod-}R(V)$ is reflective and the semi-special preenveloping torsion classes in $\text{mod-}R$. By Proposition 7.5, any such $[V]$ is represented by a finitely presented finendo quasi-tilting $R$-module $V$ whose associated torsion pair in $\text{Mod-}R$ restricts to $\text{mod-}R$. Moreover, by Lemma 7.4, in such case $T_V := \text{pres}(V)$. It is obvious that, for quasi-tilting $R$-modules $V$ and $V'$ as indicated, the equality $\text{pres}(V) = \text{pres}(V')$ is tantamount to their equivalence. Therefore, the general bijection is clear.

The restricted version now follows from the fact that, by [40, Thm. 1.5], any classical (=finitely presented) tilting $R$-module is finendo. On the other hand, when $R$ is also Krull-Schmidt, we know that “enveloping” (resp., “cogenerating enveloping”) and “(semi-special) preenveloping” (resp., “special preenveloping”) are synonymous terms in $\text{mod-}R$.

Recall that a Noether (resp., Artin) algebra is an algebra $A$ over some commutative Noetherian (resp., Artinian) ring $K$ such that $A$ is finitely generated (=finitely presented) as a $K$-module.

Corollary 7.16 Let $R$ be a ring that is finitely presented over a commutative coherent ring $K$ (e.g., a Noether or an Artin algebra). Then, the assignment $[V] \mapsto \text{pres}(V)$ gives a one-to-one correspondence between the equivalence classes of finitely presented quasi-tilting (resp., tilting) right $R$-modules and the semi-special (resp., special) preenveloping torsion classes in $\text{mod-}R$. When $R$ is in addition Krull-Schmidt (see Example 3.13(2,3)), the latter torsion classes are precisely the enveloping (resp., cogenerating and enveloping) ones.

Proof In this situation the category $\text{mod-}R$ is Hom-finite over $K$, which, by Proposition 3.11 and Corollary 7.7, implies that $R$ is left and right coherent, and that any torsion pair in $\text{Mod-}R$ given by a classical tilting right $R$-module restricts to $\text{mod-}R$. On the other hand, each $M \in \text{mod-}R$ is a finitely presented $K$-module and any finite set of generators of $M$ as a $K$-modules also generates it as a left $\text{End}(M_A)$-modules. In particular, all finitely presented right $R$-modules are finendo. The result then follows from Proposition 7.14 since the Krull-Schmidt part is obvious.

Remark 7.17 Note that, if $R$ is right Noetherian in Proposition 7.14, then all torsion pairs in $\text{Mod-}R$ restrict to $\text{mod-}R$.

As a consequence of Proposition 7.14, we can give a new proof of a nice result, which is already known for finite dimensional algebras over a field (see [5, Prop. 3.15]) and, furthermore, we extend it to Artin algebras. The reader is referred to [1, Def. 0.1] for the definition of support $\tau$-tilting $\Lambda$-module and to [5] or [3] for the definition of silting module.
**Corollary 7.18** Let $\Lambda$ be an Artin algebra and let $V$ be a finitely generated (=finitely presented) $\Lambda$-module. The following conditions are equivalent:

1. $V$ is quasi-tilting;
2. $V$ is silting;
3. $V$ is support $\tau$-tilting.

**Proof** $(2)\Leftrightarrow(3)$ is [3, Ex. 2.4(2)].

$(1)\Leftrightarrow(3)$. Note that any finitely generated $\Lambda$-module is finendo and mod-$\Lambda$ is a Krull-Schmidt category. On the other hand, a torsion class in mod-$\Lambda$ is (pre)enveloping if, and only if, it is functorially finite in the terminology of [1]. Therefore, by Proposition 7.14, the assignment $[W] \mapsto \text{pres}(W) = \text{gen}(W)$ gives a bijection between add-equivalence classes of finitely generated quasi-tilting $\Lambda$-modules and functorially finite torsion classes in mod-$\Lambda$. Its inverse takes $T$ to $[W]$, where, by Proposition 6.4, $W$ may be chosen to be the (finite) direct sum of one isomorphic copy of each indecomposable $\Lambda$-module in $\mathcal{T} \cap \mathcal{T}$. This particular $W$ is denoted by $P(T)$ in [1] and, by [1, Thm. 2.7], the assignment $T \mapsto P(T)$ gives a bijection between functorially finite torsion classes in mod-$\Lambda$ and basic support $\tau$-tilting $\Lambda$-modules. Therefore, $V$ is quasi-tilting if, and only if, it is add-equivalent to a basic support $\tau$-tilting $\Lambda$-module.

**Remark 7.19** The bijection in the tilting case of Corollary 7.16 for Artin algebras is part of the main theorem of [54]. Note that Smalø’s bijection also with covering torsion-free classes in mod-$R$ in that case is valid because of the duality between mod-$R$ and $R$-mod, that is no longer available for a general $R$ as in Corollary 7.16.

Let us conclude with the following example:

**Example 7.20** There exist right coherent rings $R$ and torsion classes $T_0$ in mod-$R$ such that $T := \lim T_0$ is a special preenveloping torsion class in Mod-$R$ while $T_0$ is not so in mod-$R$ (see, e.g., [19, Lem. 4.4] to see that $T$ is a torsion class in Mod-$R$). Therefore $T_0$ is a cogenerating preenveloping torsion class that is not special preenveloping in mod-$R$.

Indeed, in [10] the authors give an example of a pure-projective tilting module $T$ over a Noetherian ring that is not Add-equivalent to a finitely presented tilting module. In such case $T = \text{Gen}(T)$ is special preenveloping in Mod-$R$ and $T_0 = T \cap \text{mod-}R$ has the property that $T = \lim T_0$. However, according to Proposition 7.15, $T_0$ is not special preenveloping in mod-$R$.

**References**

1. Adachi, T., Iyama, O., Reiten, I.: $\tau$-tilting theory. Compositio Mathematica 150(3), 415–452 (2014)
2. Anderson, F.W., Fuller, K.R.: Rings and Categories of Modules, 2nd edn. Springer, Berlin (1992)
3. Hügel, L.A.: On the abundance of silting modules. “Surveys in Representation Theory of Algebra”, Contemp. Math. 716, 1–23 (2018)
4. Hügel, L.A., Coelho, F.U.: Infinitely generated tilting modules of finite projective dimension. Forum Mathematicum 13, 239–250 (2001)
5. Hügel, L.A., Marks, F., Vitória, J.: Silting modules. Int. Math. Res. Not. 2016(4), 1251–1284 (2015)
6. Hügel, L.A., Sánchez, J.: Tilting modules arising from ring epimorphisms. Algebra Represent. Theory 14, 217–246, 2011 (2015)
7. Hügel, L.A., Tonolo, A., Trlifaj, J.: Tilting preenvelopes and cotilting precovers. Algebra Represent. Theory 4(2), 155–170 (2001)
8. Auslander, M., Reiten, I.: Applications of contravariantly finite subcategories. Adv. Math. 86(1), 111–152 (1991)
9. Auslander, M., Smalø, S.O.: Preprojective modules over artin algebras. J. Algebra 66(1), 61–122 (1980)
10. Bazzoni, S., Herzog, I., Přihoda, P., Šaroch, J., Trlifaj, J.: Pure projective tilting modules. Doc. Math. 25, 401–424 (2020)
11. Beilinson, A.A., Bernstein, J., Deligne, P.: Faisceaux pervers. Astérisque 100, 5–171 (1982)
12. Bergman, G.M.: Two statements about infinite products that are not quite true. In 'Groups, Rings and Algebras' (in honor of Donald S. Passman). William Chin, James Osterburg and Declan Quin (eds). Contemp. Maths 420, 35–58 (2006)
13. Brenner, S., Butler, M.C.R.: Generalizations of the bernstein-gelfand-ponomarev reflection functors. In: Dlab, V., Gabriel, P. (eds.) Representation Theory II, pp. 103–169. Springer Berlin Heidelberg, Berlin, Heidelberg (1980)
14. Bongartz, K.: Tilted algebras. In: Representations of Algebras, pp. 26–38. Springer (1981)
15. Borceux, F.: Handbook of Categorical Algebra 1: Basic Category Theory (Encyclopedia of Mathematics and its Applications). Cambridge University Press, Cambridge (1994)
16. Breaz, S., Žemlička, J.: Torsion classes generated by silting modules. Arkiv. Math 56(1), 15–32 (2018)
17. Chen, X.-W., Ye, Y., Zhang, P.: Algebras of derived dimension zero. Commun. Algebra 36(1), 1–10 (2008)
18. Cline, E., Parshall, B., Scott, L.: Derived categories and morita theory. J. Algebra 104(2), 397–409 (1986)
19. Crawley-Bovey, W.: Locally finitely presented additive categories. Comm. Algebra 22(5), 1641–1674 (1994)
20. Colby, R.R., Fuller, K.R.: Tilting, cotilting, and serially tilted rings. Commun. Algebra 18(5), 1585–1615 (1990)
21. Colpi, R.: Tilting in Grothendieck categories. Forum Mathematicum 11(6), 735–759 (1999)
22. Colpi, R., Gregorio, E., Mantese, F.: On the heart of a faithful torsion theory. J. Algebra 307(2), 841–863 (2007)
23. Colpi, R., Menini, C.: On the structure of *-modules. J. Algebra 158(2), 400–419 (1993)
24. Colpi, R., Trlifaj, J.: Tilting modules and tilting torsion theories. J. Algebra 178(2), 614–634 (1995)
25. Čoupek, P., Šťovíček, J.: Cotilting sheaves on Noetherian schemes. Math. Z. 296, 275–312 (2020)
26. El Bashir, R.: Covers directed colimits. Algebra Represent. Theory 9, 423–430 (2006)
27. Enochs, E.E.: Injective and flat covers, envelopes and resolvents. Isr. J. Math 39(3), 189–209 (1981)
28. Gobel, R., Trlifaj, J.: Approximations and endomorphism algebras of modules. De Gruyter (2006)
29. Golan, J.S., Teply, M.L.: Torsion-free covers. Isr. J. Math 15(3), 237–256 (1973)
30. Happel, D.: On the derived category of a finite-dimensional algebra. Commentarri Mathematici Helvetici 62(1), 339–389 (1987)
31. Happel, D.: Triangulated Categories in the Representation of Finite Dimensional Algebras, vol. 119. Cambridge University Press, Cambridge (1988)
32. Happel, D., Ringel, C.M.: Tilted algebras. Trans. Am. Math. Soc. 274(2), 399–443 (1982)
33. Harris, M.E.: Some results on coherent rings II. Glasgow Math. J. 8(2), 123–126 (1967)
34. Herzog, I.: The Ziegler spectrum of a locally coherent Grothendieck category. Proc. Lond. Math. Soc. (3) 74(3), 503–558 (1997)
35. Hovey, M.: Cotorsion pairs, model category structures, and representation theory. Math. Z. 241, 553–592 (2002)
36. Keller, B.: Deriving dg categories. Ann. Sci. École Norm. Sup.(4) 27(1), 63–102 (1994)
37. Krause, H.: The Spectrum of a Module Category. Vol. 707. American Mathematical Soc. (2001)
38. Krause, H., Saorín, M.: On minimal approximations of modules. In ‘Trends in Representation Theory of Finite Dimensional Algebras’, Proc. Seattle Conference, 1997, Edward L. Green and Birge Huisgen-Zimmermann (eds). Contemp. Maths 229, 228–236 (1998)
39. Mac Lane, S.: Categories for the Working Mathematician, Volume 5 of Graduate Texts in Mathematics, 2nd edn. Springer, New York (1998)
40. Miyashita, Y.: Tilting modules of finite projective dimension. Math. Z. 193(1), 113–146 (1986)
41. Parra, C.E., Saorín, M.: On hearts which are module categories. J. Math. Soc. Jpn. 68(4), 1421–1460 (2016)
42. Parra, C.E., Saorín, M.: The HRS tilting process and Grothendieck hearts of τ-structures. Contemp. Maths 769, 209–241 (2021)
43. Parra, C.E., Saorín, M., Virili, S.: Locally finitely presented and coherent hearts. arXiv:1908.00649v2 (2021)
44. Popescu, N.: Abelian Categories with Applications to Rings and Modules, vol. 3. London Math. Soc. Monogr. 3, Academic Press (1973)
45. Rickard, J.: Morita theory for derived categories. J. Lond. Math. Soc. 2(39), 436–456 (1989)
46. Rickard, J.: Derived equivalences as derived functors. J. Lond. Math. Soc. 2(1), 37–48 (1991)
47. Rowen, L.H.: Finitely presented modules over semiperfect rings. Proc. Am. Math. Soc. 97(1), 1–7 (1986)
48. Rowen, L.H.: Finitely presented modules over semiperfect rings satisfying ACC-∞. J. Algebra 107(1), 284–291 (1987)
49. Salce, L.: Cotorsion Theories for Abelian Groups. Symposia Mathematica, Vol. XXIII (Conf. Abelian Groups and their Relationship to the Theory of Modules, INDAM, Rome, 1977), pp. 11–32. Academic Press, London-New York (1979)
50. Saorín, M.: On locally coherent hearts. Pac. J. Math. 287(1), 199–221 (2017)
51. Saorín, M., Štovíček, J.: On exact categories and applications to triangulated adjoints and model structures. Adv. Math. 228(2), 968–1007 (2011)
52. Saorín, M., Štovíček, J.: t-structures with Grothendieck hearts via functor categories. arXiv:2003.01401v2 (2020)
53. Silver, L.: Noncommutative localizations and applications. J. Algebra 7, 44–76 (1967)
54. Smaš, S.O.: Torsion theories and tilting modules. Bull. Lond. Math. Soc. 16(5), 518–522 (1984)
55. Stenström, B.: Rings and modules of quotients. In: Rings of Quotients, pp. 195–212. Springer (1975)
56. Štovíček, J.: Derived equivalences induced by big cotilting modules. Adv. Math. 263, 45–87 (2014)
57. Swan, R.G.: K-theory of coherent rings. Unpublished paper, 1–14
58. Swan, R.G.: Induced representations and projective modules. Ann. Math., 552–578 (1960)
59. Teply, M.L.: Torsion-free covers ii. Isr. J. Math. 23(2), 132–136 (1976)
60. Verdier, J.-L.: Des catégories dérivées des catégories abéliennes. Astérisque (1996)
61. Wei, J.: Semi-tilting complexes. Isr. J. Math. 194(2), 871–893 (2013)
62. Wisbauer, R.: Foundations of module and ring theory. Gordon and Breach Science Publishers (1991)
63. Zhang, P., Wei, J.: Cosilting complexes and AIR-cotilting modules. J. Algebra 491, 1–31 (2017)

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