Estimates of Solutions to the Perturbed Stokes System

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Abstract

In this paper we derive local estimates of solutions of the Perturbed Stokes system. This system arises as a reduction of the Stokes system near a curved part of the boundary of the domain if one applies a diffeomorphism flattening the boundary. The estimates obtained in the paper play the crucial role in the investigation of partial regularity of weak solutions to the Navier-Stokes system near a curved part of the boundary of the domain.

1 Introduction

Let $B_R^+ := \{ x \in \mathbb{R}^n : |x| < R, x_n > 0 \}$ be a half-ball in $\mathbb{R}^n$, $n \geq 2$, and assume $Q_R^+ = B_R^+ \times (-R^2, 0)$. For any $x \in \mathbb{R}^n$, $x = (x_1, \ldots, x_{n-1}, x_n)$ we denote by $x' \in \mathbb{R}^{n-1}$ the vector $x' := (x_1, \ldots, x_{n-1})$. Denote $S_R := \{ x' \in \mathbb{R}^{n-1} : |x'| < R \}$ and assume $\varphi : S_R \rightarrow \mathbb{R}$ is a sufficiently smooth function. In this paper we obtain local estimates for the following system which we call the Perturbed Stokes system:

\[
\begin{cases}
\partial_t v - \hat{\Delta}_\varphi v + \hat{\nabla}_\varphi p = f \\
\hat{\nabla}_\varphi \cdot v = 0
\end{cases}
\text{ in } Q_R^+. \tag{1.1}
\]

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Here \( v, f : Q^+_R \to \mathbb{R}^n \) are vector fields, \( p : Q^+_R \to \mathbb{R} \) is a scalar function, \( \hat{\Delta}_\varphi \) and \( \hat{\nabla}_\varphi \) are the differential operators with variable coefficients defined via a function \( \varphi \) by formulas
\[
\hat{\Delta}_\varphi v := \Delta v - 2v_{,an} \varphi_{,a} + v_{,mn} |\nabla' \varphi|^2 - v_{,n} \Delta' \varphi,
\hat{\nabla}_\varphi \cdot v := \text{div} v - v_{,a,n} \varphi_{,a},
\hat{\nabla}_\varphi p := \nabla p - p_{,n} \left( \begin{array}c \nabla' \varphi \\ 0 \end{array} \right).
\] (1.2)

Here we assume summation from 1 to \( n - 1 \) over repeated Greek indexes and \( \nabla' \) and \( \Delta' \) denote the gradient and Laplacian with respect to \( (x_1, \ldots, x_{n-1}) \) variables. We will also make use of the differential operator
\[
\hat{\nabla}_\varphi v = \nabla v - v_{,n} \otimes \left( \begin{array}c \nabla' \varphi \\ 0 \end{array} \right),
\] (1.3)
where for any \( a, b \in \mathbb{R}^n \) the symbol \( a \otimes b \) denotes the \( n \times n \)-matrix with components \( (a_i b_j) \), \( i, j = 1, \ldots, n \).

In this paper we study the problem (1.1) assuming \( v \) satisfies the slip boundary condition on the plane \( \{ x_n = 0 \} \):
\[
v|_{x_n=0} = 0.
\] (1.4)

The Perturbed Stokes system arises as a reduction of the usual Stokes system in a domain near a point belonging to the curved part of the boundary if the latter is a graph of \( \varphi \). Namely, assume \( (u, q, \tilde{f}) \) satisfy the Stokes system in \( \Omega_R \times (-R^2, 0) \),
\[
\begin{align*}
\partial_t u - \Delta u + \nabla q &= \tilde{f} & \text{in } & \Omega_R \times (-R^2, 0),
\text{div} u &= 0 & \text{in } & \Omega_R \times (-R^2, 0).
\end{align*}
\] (1.5)

We assume \( \Omega_R \) is described in the appropriate Cartesian coordinate system by relations
\[
\Omega_R = \left\{ y \in \mathbb{R}^n : |y'| < R, \varphi(y') < y_n < \varphi(y') + \sqrt{R^2 - |y'|^2} \right\},
\]
and we impose the slip boundary condition on \( u \):
\[
u|_{x_n=\varphi(y')} = 0.
\] (1.6)

In this paper we assume \( \varphi \) is of class \( W^3_\infty \) (i.e. its second derivatives are Lipschitz continuous) and the Cartesian coordinate system is chosen in such a way that the following relations hold
\[
\varphi(0) = 0, \quad \nabla \varphi(0) = 0, \quad \|\varphi\|_{W^3_\infty(S_R)} \leq \mu.
\] (1.7)
Now we apply the diffeomorphism flattering the boundary, or, in other words, we introduce new coordinates \( x = \psi(y) \) by formulas

\[
\psi : \Omega_R \to B_R^+, \quad x = \psi(y) = \begin{pmatrix} y' \\ y_n - \varphi(y') \end{pmatrix}, \quad (1.8)
\]

\( y \in \Omega_R \iff x \in B_R^+ \).

Denote

\[ v := u \circ \psi^{-1}, \quad p := q \circ \psi^{-1}. \]

Then for \( x = \psi(y) \) we have relations

\[
\nabla q(y) = \nabla \varphi p(x), \quad \Delta u(y) = \Delta \varphi v(x), \quad \text{div} u(y) = \langle \nabla \varphi \cdot v \rangle(x).
\]

Hence the Stokes system (1.5), (1.6) in \( \Omega \times (-R^2,0) \) in \( y \)-variables transfers to the Perturbed Stokes system (1.1), (1.4) in \( Q_R^+ \) in \( x \)-variables.

Now we introduce some functional spaces: assume \( 1 \leq s, l < +\infty \).

Assume \( \Omega \subset \mathbb{R}^n \), \( Q_T = \Omega \times (0,T) \) and let \( L_{s,l}(Q_T) \) be the anisotropic Lebesgue space equipped with the norm

\[
\|f\|_{L_{s,l}(Q_T)} := \left( \int_0^T \left( \int_\Omega |f(x,t)|^s \, dx \right)^{l/s} \, dt \right)^{1/l},
\]

and denote

\[
W_{s,l}^{1,0}(Q_T) \equiv L_{l}(0,T;W_s^1(\Omega)) = \{ u \in L_{s,l}(Q_T) : \nabla u \in L_{s,l}(Q_T) \},
\]

\[
W_{s,l}^{2,1}(Q_T) = \{ u \in W_{s,l}^{1,0}(Q_T) : \nabla^2 u, \, \partial_t u \in L_{s,l}(Q_T) \}.
\]

We equip these spaces with the following norms:

\[
\|u\|_{W_{s,l}^{1,0}(Q_T)} = \|u\|_{L_{s,l}(Q_T)} + \|\nabla u\|_{L_{s,l}(Q_T)},
\]

\[
\|u\|_{W_{s,l}^{2,1}(Q_T)} = \|u\|_{W_{s,l}^{1,0}(Q_T)} + \|\nabla^2 u\|_{L_{s,l}(Q_T)} + \|\partial_t u\|_{L_{s,l}(Q_T)}.
\]

We also denote by \( W_{s,l}^{-1}(\Omega) \) the conjugate space to \( \overset{\circ}{W}_{s,l}^1(\Omega) \) equipped with the norm

\[
\|f\|_{W_{s,l}^{-1}(\Omega)} = \sup_{w \in \overset{\circ}{W}_{s,l}^1(\Omega), \|w\|_{W_{s,l}^1(\Omega)} \leq 1} |\langle f, w \rangle|,
\]

and we denote by \( L_{l}(0,T;W_{s,l}^{-1}(\Omega)) \) the space of measurable functions \( f : [0,T] \to W_{s,l}^{-1}(\Omega) \) such that the following norm is finite:

\[
\|f\|_{L_{l}(0,T;W_{s,l}^{-1}(\Omega))} = \left( \int_0^T \|f(\cdot,t)\|_{W_{s,l}^{-1}(\Omega)}^l \, dt \right)^{1/l}.
\]
Definition 1.1. Assume \(1 < s, l < +\infty\) and \(f \in L_{s,l}(Q^+_R)\). We say that the functions \((v, p)\) are the strong solution of the problem (1.1), (1.4), if they belong to the spaces

\[
v \in W^{2,1}_{s,l}(Q^+_R), \quad p \in W^{1,0}_{s,l}(Q^+_R),
\]

satisfy the equations (1.1) a.e. in \(Q^+_R\) and satisfy the boundary conditions (1.4) in the sense of traces.

Definition 1.2. Assume \(1 < s, l < +\infty\) and \(f \in L^l(-R^2, 0; W^{-1,s}(B^+_R))\). We say that the functions \((v, p)\) are the generalized solution of the problem (1.1), (1.4), if they belong to the spaces

\[
v \in W^{1,0}_{s,l}(Q^+_R), \quad p \in L^{s,l}(Q^+_R),
\]

\((v, p)\) satisfy (1.1) in the sense of distributions and \(v\) satisfies the boundary condition (1.4) in the sense of traces.

Note that though \(\hat{\Delta}_\varphi\) and \(\hat{\nabla}_\varphi\) are the operators with variable coefficients, the function \(\varphi\) is independent of \(x_n\) and thus these operators possess the properties

\[
\int_{B^+_R} \hat{\Delta}_\varphi v \cdot w \, dx = - \int_{B^+_R} \hat{\nabla}_\varphi v : \hat{\nabla}_\varphi w \, dx = \int_{B^+_R} v \cdot \hat{\Delta}_\varphi w \, dx,
\]

\[
\int_{B^+_R} \hat{\nabla}_\varphi p \cdot w \, dx = - \int_{B^+_R} p \hat{\nabla}_\varphi \cdot w \, dx
\]

for any \(v \in W^2_s(B^+_R), p \in W^1_s(B^+_R), w \in C^\infty_0(B^+_R)\). Hence for equations (1.1) with variable coefficients there is no problem to define solutions “in the sense of distributions” in the usual way (similar to PDEs with constant coefficients) by putting all differential operators \(\hat{\Delta}_\varphi\) and \(\hat{\nabla}_\varphi\) on a smooth test function.

Remark also that if \((v, p)\) is a generalized solution to (1.1), then the following identity holds in \(\mathcal{D}'(Q^+_R)\) (i.e. in the sense of distributions):

\[
\partial_t v = f + \text{div} \left( \nabla v - p I \right) + \\
+ \frac{\partial}{\partial x_n} \left( -2 v_{\alpha \alpha} \varphi_\alpha + v_{n|\nabla' \varphi|^2} - v \Delta' \varphi + p \left( \begin{array}{c} \nabla' \varphi \\ 0 \end{array} \right) \right).
\]
This identity implies that \( \partial_t v \in L_t(-R^2,0;W_s^{-1}(B_R^+)) \) and the estimate

\[
\|\partial_t v\|_{L_t(-R^2,0;W_s^{-1}(B_R^+))} \leq C \left( \|f\|_{L_t(-R^2,0;W_s^{-1}(B_R^+))} + \|v\|_{W^{1,0}_s(Q_R^-)} + \|p\|_{L_s(Q_R^-)} \right) \tag{1.9}
\]

holds. In particular, from that it is possible to choose the representative of \( v \) so that

\[
\forall \ w \in \tilde{W}_s^{-1}(\Omega) \quad t \mapsto \int_{B_R^+} v(x,t) \cdot w(x) \ dx \text{ is continuous on } [-R^2,0].
\]

Hence we can assume that every generalized solution \((v,p)\) satisfies the integral identity

\[
\int_{B_R^+} v(x,t) \cdot \eta(x,t) \ dx \bigg|_{t=-R^2}^{t=0} + \int_{Q_R^+} \left( -v \cdot \partial_t \eta + \nabla \varphi \cdot v + \nabla \varphi \eta \right) \ dx \ dt = \int_{-R^2}^{0} \langle f(t), \eta(t) \rangle \ dt + \int_{Q_R^+} p \nabla \varphi \cdot \eta \ dx \ dt \tag{1.10}
\]

for any \( \eta \in C^\infty(Q_R^+) \) such that \( \eta|_{\partial B_R^+ \times (-R^2,0)} = 0 \).

In the paper we explore the following notations

- \( \partial \Omega \) is a boundary of a domain \( \Omega \subset \mathbb{R}^n \)
- \( \partial' Q_R^+ = (\partial B_R^+ \times (-R^2,0)) \cup (B_R^+ \times \{ t = -R^2 \}) \)
- We assume summation from 1 to \( n \) over repeated Latin indexes and summation from 1 to \( n-1 \) over repeated Greek indexes.
- The indexes after comma imply the derivatives with respect to the corresponding spatial variables.
- \( a \cdot b = a_i b_i \) is the scalar product of vectors \( a, b \in \mathbb{R}^n \)
- \( A: B = A_{ij} B_{ij} \) is the scalar product of matrices \( A, B \in M^{n \times n} \).
2 Main Results

In this section we formulate four theorems which are main results of the present paper. At the end of this section we give some comments to these results.

**Theorem 2.1.** Assume \( \Omega \subset \mathbb{R}^n \) is a bounded domain with smooth boundary which is diffeomorphic to a ball and denote \( Q_T = \Omega \times (0,T) \). Suppose \( s, l \in (1, \infty) \). There is a positive constant \( \mu_1 \) (depending on \( \Omega, T, s, l, n \)) such that for any function \( \varphi \in W^{3}_{\infty}(\Omega) \) which is independent on \( x_n \)-variable and satisfies the condition

\[
\| \varphi \|_{W^{3}_{\infty}(\Omega)} \leq \mu_1 \quad (2.1)
\]

and for any \( f \) and \( g \) satisfying conditions

\[
f \in L^{s,l}(Q_T), \quad (2.2)
\]
\[
g \in W^{1,0}_{s,l}(Q_T), \quad (2.3)
\]
\[
\partial_t g \in L^{s,l}(Q_T), \quad (2.4)
\]
\[
\int_{\Omega} g(x,t) \, dx = 0, \quad \text{a.e. } t \in (0,T), \quad g(\cdot,0) = 0, \quad (2.5)
\]

the problem

\[
\begin{aligned}
\partial_t u - \hat{\Delta} \varphi u + \hat{\nabla} \varphi q &= f & \text{in } Q_T, \\
\hat{\nabla} \varphi \cdot u &= g & \text{in } Q_T, \\
u|_{\partial \Omega \times (0,T)} &= 0, \quad u|_{t=0} = 0,
\end{aligned}
\]

(2.6)

has the unique solution \( u \in W^{2,1}_{s,l}(Q_T), \quad q \in W^{1,0}_{s,l}(Q_T), \quad \int_{\Omega} q(x,t) \, dx = 0, \quad \text{for a.e. } t \in (0,T) \) and the estimate

\[
\| u \|_{W^{2,1}_{s,l}(Q_T)} + \| \nabla q \|_{L^{s,l}(Q_T)} \leq C_* \left( \| f \|_{L^{s,l}(Q_T)} + \| g \|_{W^{1,0}_{s,l}(Q_T)} + \| \partial_t g \|_{L^{s,l}(0,T;W^{1,0}_{s,l}(\Omega))}^{1/s'} \right) \quad (2.7)
\]

holds with some constant \( C_* > 0 \) depending only on \( \Omega, T, n, s, l \).

**Theorem 2.2.** Suppose \( s, l \in (1, \infty), \) and \( 0 < r < R \) are fixed. There exists a positive constant \( \mu_2 \) (depending only on \( n, s, l, r, R \)) such that if \( \varphi \in W^{3}_{\infty}(S_R) \) satisfies (1.7) with \( \mu \leq \mu_2 \) then for any \( f \in L^{s,l}(Q_R) \),
and any strong solution \( v \in W_{s,l}^{2,1}(Q_R^+) \), \( p \in W_{s,l}^{1,0}(Q_R^+) \) to the system (1.1), (1.4) in \( Q_R^+ \), the following local estimate holds:

\[
\|v\|_{W_{s,l}^{2,1}(Q_R^+)} + \|\nabla p\|_{L_{s,l}(Q_R^+)} \leq C \left( \|f\|_{L_{s,l}(Q_R^+)} + \|\nabla v\|_{L_{s,l}(Q_R^+)} + \inf_{b \in L_4(-R^2,0)} \|p - b\|_{L_{s,l}(Q_R^+)} \right),
\]

(2.8)

where \( b \) is a function of \( t \)-variable and the constant \( C \) depends only on \( n, s, l, r, R \).

**Theorem 2.3.** Suppose \( s, l \in (1, \infty) \), and \( 0 < r < R \) are fixed. There exists a positive constant \( \mu_3 \) (depending only on \( n, s, l, r, R \)) such that if \( \varphi \in W_3^3(S_R) \) satisfies (1.7) with \( \mu \leq \mu_3 \) then for any \( f \in L_{s,l}(Q_R^+) \) and any generalized solution \( v \in W_{s,l}^{1,0}(Q_R^+) \), \( p \in L_{s,l}(Q_R^+) \) to the system (1.1), (1.4) in \( Q_R^+ \) the following inclusions hold: \( v \in W_{s,l}^{2,1}(Q_R^+) \), \( p \in W_{s,l}^{1,0}(Q_R^+) \).

**Theorem 2.4.** Suppose \( s, l, m \in (1, \infty) \), \( m \geq s \) and \( 0 < r < R \) are fixed. There exists a positive constant \( \mu_4 \) (depending only on \( n, s, l, r, m, R \)) such that if \( \varphi \in W_3^3(S_R) \) satisfies (1.7) with \( \mu \leq \mu_4 \) then for any \( f \in L_{m,l}(Q_R^+) \) and any generalized solution \( v \in W_{s,l}^{2,1}(Q_R^+) \), \( p \in W_{s,l}^{1,0}(Q_R^+) \) to the system (1.1), (1.4) in \( Q_R^+ \) we have the inclusions \( v \in W_{m,l}^{2,1}(Q_R^+) \), \( \nabla p \in L_{m,l}(Q_R^+) \) and the following local estimate holds:

\[
\|v\|_{W_{m,l}^{2,1}(Q_R^+)} + \|\nabla p\|_{L_{m,l}(Q_R^+)} \leq C \left( \|f\|_{L_{m,l}(Q_R^+)} + \|\nabla v\|_{L_{s,l}(Q_R^+)} + \inf_{b \in L_4(-R^2,0)} \|p - b\|_{L_{s,l}(Q_R^+)} \right),
\]

(2.9)

with some constant \( C \) depending only on \( n, s, l, m, r, R \).

**Remark.** The constants \( \mu_i \) controlling the smallness of the \( W_3^3 \)-norm of the function \( \varphi \) in Theorems 2.2–2.4 depend on the domain (or on the size of the half-cylinders \( Q_r^+ \) and \( Q_R^+ \)). Nevertheless, for applications to the investigation of the Stokes and the Navier-Stokes systems near the point at the curved part of the boundary this is not a serious obstacle (in contrast with the smoothness assumption that \( \varphi \) is of class \( W_3^3 \)) because of the following scaling property of the Perturbed Stokes system: if \( (v, p, f, \varphi) \) satisfy (1.1) in the cylinder \( Q_R^+ \)
with \( \phi \) satisfying (1.7) then the functions

\[
\begin{align*}
v^R(x, t) &= Rv(Rx, R^2t), & p^R(x, t) &= R^2p(Rx, R^2t), \\
f^R(x, t) &= R^3f(Rx, R^2t), & \phi^R(x') &= \frac{1}{R} \phi(Rx')
\end{align*}
\]  

(2.10)

satisfy the Perturbed Stokes system in \( Q^+ \) and from Taylor decomposition of the function \( \phi^R \) one can obtain for \( R \leq 1 \)

\[
\phi^R(0) = 0, \quad \nabla' \phi^R(0) = 0, \quad \|\phi^R\|_{W^3_\infty(S_1)} \leq \mu R.
\]

Hence, one can take canonical domain (say, \( Q^+_R = Q^+_1 \), \( Q^+_r = Q^+_1/2 \)) and compute the constants \( \mu_i = \mu_i \) for these particular domains. We emphasize that \( \mu_i \) are constants depending only on \( n, s, l, m \). Then we consider the Stokes system (1.5), (1.6) near a point of the \( W^3_\infty \)-smooth boundary without any restrictions on the curvature of the boundary (i.e. the constant \( \mu \) in (1.7) can be arbitrary large). After that we choose \( R \) in (1.5) so small that the following estimates hold:

\[
\mu R \leq \mu_i^*, \quad i = 2, 3, 4.
\]

(2.11)

Making change of variables (1.8) we obtain functions \((v, p, f, \phi)\) which satisfy the Perturbed Stokes system (1.1), (1.4) in \( Q^+_R \). At this step our Perturbed Stokes system is not a small perturbation of the usual Stokes system (i.e. so far smallness conditions of Theorems 2.2–2.4 are not satisfied). Then we make the scaling (2.10) and obtain functions \((v^R, p^R, f^R, \phi^R)\) which satisfy the Perturbed Stokes system in \( Q^+ \) and also satisfy the smallness assumptions (2.11). So, we can apply results of Theorems 2.2–2.4 to the functions \((v^R, p^R, f^R, \phi^R)\). Then we recover information about the original functions \((v, p)\).

Now we give some comments to Theorems 2.1 — Theorem 2.4.

Theorem 2.1 in the case of the Stokes system (i.e. for \( \phi \equiv 0 \)) was proved in [2]. The generalization in the case of a “small perturbation” of the Stokes system is quite obvious. The proof is presented in Section 3.

Theorem 2.2 presents a local estimate for strong solutions to the Perturbed Stokes system. In the case of the usual Stokes system near a plane part of the boundary such estimates were originally proved in [3]. In [7] the same estimates were proved for solutions to the Stokes system near curved part of the boundary. In our approach Theorem 2.2 follows from Theorem 2.1 by arguments presented in [2]. We reproduce these arguments in Section 4 just for completeness.
In Theorem 2.3 we prove that any generalized solution is actually a strong one. In the case of the Stokes system this result originally was proved in [5]. In Section 5 to obtain similar result for the Perturbed Stokes system we use new approach based on the estimates obtained in Theorem 2.1. Probably this section contains main novelty of the present paper.

Finally, in Section 6 we obtain improved local estimate of solution to the Perturbed Stokes system. Estimate in Theorem 2.4 turns out to be the crucial step in investigation of boundary regularity of solutions to the nonlinear Navier-Stokes system, see [4] and [6]. For the Stokes system this estimate was originally obtained in [3] in the case of plane boundary, and after that in [7] in the case of curved boundary. In our approach we obtain the corresponding estimate for solutions to the Perturbed Stokes system (under certain conditions that guarantee smallness of the “perturbation”) as a direct consequence of our Theorems 2.3 and 2.2.
3 Proof of Theorem 2.1

We will derive Theorem 2.1 from the following result

**Theorem 3.1.** Suppose $s, l \in (1, \infty)$. Assume $a_{ijkl}, b_{ijk}, c_{ij}, d_{ij} \in L_\infty(Q_T)$ and consider the problem

$$\begin{cases}
\partial_t w_i - a_{ijkl}w_{j,km} + b_{ijk}w_{j,k} + c_{ij}w_j + d_{ij}q_j = f_i, & \text{in } Q_T, \\
\text{div } w = 0, \\
w|_{t=0} = 0, \quad w|_{\partial \Omega \times (0,T)} = 0.
\end{cases}$$

(3.1)

There is a constant $\mu_0 > 0$ (depending on $\Omega, T, s, l$) such that if the coefficients $a_{ijkl}, b_{ijk}, c_{ij}, d_{ij}$ satisfy the estimate

$$\sup_{z \in Q_T} \left( |a_{ijkm}(z) - \delta_{ij} \delta_{km}| + |d_{ij}(z) - \delta_{ij}| + |b_{ijk}(z)| + |c_{ij}(z)| \right) \leq \mu_0,$$

(3.2)

for all $i, j, k, m = 1, \ldots, n$, then for any $f$ satisfying conditions (2.2) the problem (3.1) has the unique solution $w \in W^{2,1}_{s,l}(Q_T), q \in W^{1,0}_{s,l}(Q_T), \int_\Omega q \, dx = 0$ a.e. $t \in \Omega$, and the estimate

$$\|w\|_{W^{2,1}_{s,l}(Q_T)} + \|q\|_{W^{1,0}_{s,l}(Q_T)} \leq C \|f\|_{L_{s,l}(Q_T)}.$$ 

(3.3)

holds with some constant $C > 0$ depending only on $\Omega, T, n, s, l$.

**Proof of Theorem 3.1:** Denote by

$$\mathcal{H} := \left\{ (w, q) \in W^{2,1}_{s,l}(Q_T) \times W^{1,0}_{s,l}(Q_T) : \text{div } w = 0 \text{ a.e. in } Q_T, \\
w|_{\partial \Omega \times (0,T)} = 0, \quad w|_{t=0} = 0, \quad \int_\Omega q \, dx = 0 \text{ a.e. } t \in (0,T) \right\}$$

the Banach space equipped with the norm

$$\|(w, q)\|_{\mathcal{H}} := \|w\|_{W^{2,1}_{s,l}(Q_T)} + \|q\|_{W^{1,0}_{s,l}(Q_T)}.$$ 

For any $f \in L_{s,l}(Q_T)$ denote by $(w, q) \in \mathcal{H}$ the unique strong solution to the Stokes system:

$$\begin{cases}
\partial_t w - \Delta w + \nabla q = f, \\
\text{div } w = 0, \\
w|_{t=0} = 0, \quad w|_{\partial \Omega \times (0,T)} = 0,
\end{cases}$$

(3.4)
and consider the bijective operator
\[ A_0 : \mathcal{H} \to L_{s,l}(Q_T), \quad A_0(w, q) := f. \]
Then we know (see [8], Theorem 1.1) that there is a positive constant \( C_\ast \) such that
\[ C_\ast \| (w, q) \|_\mathcal{H} \leq \| A_0(w, q) \|_{L_{s,l}(Q_T)} \]
for any \((w, q) \in \mathcal{H}\). Hence the linear operator \( A_0 \) is invertible and its inverse operator is bounded from \( L_{s,l}(Q_T) \) to \( \mathcal{H} \).

Consider now the operator
\[ A_1 : \mathcal{H} \to L_{s,l}(Q_T) \]
determined by the system (3.1). The system (3.1) can be reduced to the system (3.4) with the right-hand side \( \tilde{f} \), where
\[ \tilde{f}_i = f_i + (a_{ijkl} - \delta_{ij} \delta_{kl})w_{j,kl} - b_{ijk}w_{j,k} - c_{ij}w_j - (d_{ij} - \delta_{ij})q_j, \]
and due to conditions (3.2)
\[ \| \tilde{f} - f \|_{L_{s,l}(Q_T)} \leq C\mu_0 \| (w, q) \|_\mathcal{H}, \quad (3.5) \]

Then for every \( f \in L_{s,l}(Q_T) \) we have
\[ \| (A_1 - A_0)A_0^{-1}f \|_{L_{s,l}(Q_T)} = \| \tilde{f} - f \|_{L_{s,l}(Q_T)} \leq \]
\[ \leq C\mu_0 \| (w, q) \|_\mathcal{H} \leq \frac{C\mu_0}{C_\ast} \| f \|_{L_{s,l}(Q_T)}. \]
Choosing now \( \mu_0 < \frac{C_\ast^2}{C^2} \) we obtain \( \| (A_1 - A_0)A_0^{-1} \|_{L_{s,l}(Q_T) \to L_{s,l}(Q_T)} \leq \frac{1}{2} \) and hence there exists \( A_1^{-1} : L_{s,l}(Q_T) \to \mathcal{H} \) which is a bounded operator. Theorem 3.1 is proved. □

**Proof of Theorem 2.1:** Let \( L_\varphi = \nabla \psi \) where \( \psi \) is introduced in (1.8). Note that \( L_\varphi \) is a smooth matrix and it is non-degenerate. Denote \( w := L_\varphi u \). Then
\[ \nabla_\varphi \cdot u = \text{div} \ w \quad \text{a.e. in} \quad Q_T \]
and the system (2.6) can be reduced to the form
\[
\begin{aligned}
\left\{
\begin{array}{l}
\partial_t w - L_\varphi \Delta_\varphi (L_\varphi^{-1} w) + L_\varphi \nabla_\varphi q = L_\varphi f \\
\text{div} \ w = g \\
w|_{t=0} = 0, \quad w|_{\partial \Omega \times (0,T)} = 0.
\end{array}
\right.
\end{aligned}
\quad (3.6)
\]
Note that this is a system of type (3.1) but with non-zero divergence. The coefficients \( a_{ijkm}, b_{ijk}, c_{ij}, d_{ij} \) arising in this system depend on the derivatives of \( \varphi \) of the first, second and third orders, due to the condition (2.1) they are bounded and satisfy the conditions (3.2).

Using the result of paper [2] (see section 4, estimate (4.1) in the cited paper) we can find the function \( W \in W^{2,1}_{s,l}(Q_T) \) such that

\[
\begin{align*}
\text{div} W &= g \quad \text{a.e. in } Q_T, \\
W|_{\partial \Omega \times (0,T)} &= 0, \\
W|_{t=0} &= 0, \\
\|W\|_{W^{2,1}_{s,l}(Q_T)} &\leq C\left(\|g\|_{W^{1,0}_{s,l}(Q_T)} + \|\partial_t g\|_{L^1(0,T;W^{1,1}_{s,l} (\Omega))}^{1/s} \right).
\end{align*}
\]

Then we consider the problem

\[
\begin{cases}
\partial_t \tilde{w} - \mathcal{L}_\varphi \Delta_\varphi (\mathcal{L}_\varphi^{-1} \tilde{w}) + \mathcal{L}_\varphi \nabla_\varphi q = \tilde{f} & \text{in } Q_T, \\
\text{div } \tilde{w} = 0 & \text{in } Q_T, \\
\tilde{w}|_{t=0} = 0, \\
\tilde{w}|_{\partial \Omega \times (0,T)} = 0, \\
\tilde{f} := \mathcal{L}_\varphi f - \left( \partial_t W - \mathcal{L}_\varphi \Delta_\varphi (\mathcal{L}_\varphi^{-1} W) \right) & \in L_{s,l}(Q_T),
\end{cases}
\]

which has the unique solution \((\tilde{w}, q) \in W^{2,1}_{s,l}(Q_T) \times W^{1,0}_{s,l}(Q_T)\) due to Theorem 3.1. Now we take \( w := \tilde{w} + W \) and see that \((w, q) \in W^{2,1}_{s,l}(Q_T) \times W^{1,0}_{s,l}(Q_T)\) satisfy all equations in (3.6). The uniqueness of this solution follows from Theorem 3.1, and the estimate

\[
\|w\|_{W^{2,1}_{s,l}(Q_T)} + \|q\|_{W^{1,0}_{s,l}(Q_T)} \leq C\left(\|f\|_{L_{s,l}(Q_T)} + \|g\|_{W^{1,0}_{s,l}(Q_T)} + \|\partial_t g\|^{1/s}_{L^1(0,T;W^{1,1}_{s,l} (\Omega))} \right)
\]

follows from the corresponding estimates of \( \tilde{w} \) and \( W \). From this estimate taking into account \( u = \mathcal{L}_\varphi^{-1} w \) and \( W^3_{\infty} \)-smoothness of \( \varphi \) we obtain (2.7). Note that only here we need \( W^3_{\infty} \)-smoothness for the function \( \varphi \). Theorem 2.1 is proved. □
4 Proof of Theorem 2.2

The estimate (2.8) follows from the estimate (2.7) by the arguments used in the paper [2]. We reproduce this proof here just for the sake of completeness. Within this section C denotes positive constants which can depend only on \( n, r, R, s, l \) and can be different from line to line.

Take arbitrary \( \rho_1, \rho_2 \) such that
\[
  r \leq \rho_1 < \rho_2 \leq R - \frac{1}{10}(R - r).
\]

Consider a cut-off function \( \zeta \in C_0^\infty(Q_R^+) \) such that
\[
  0 \leq \zeta \leq 1 \quad \text{in} \quad Q_R^+, \quad \zeta \equiv 1 \quad \text{in} \quad Q_{\rho_1}^+, \quad \zeta \equiv 0 \quad \text{in} \quad Q_R^+ \setminus Q_{\rho_2}^+,
\]
\[
  \|\nabla^k \zeta\|_{L^\infty(Q_R^+)} \leq \frac{C}{(\rho_2 - \rho_1)^k}, \quad k = 1, 2,
\]
\[
  \|\partial_t \zeta\|_{L^\infty(Q_R^+)} \leq \frac{C}{\rho_2 - \rho_1}, \quad \|\partial_t \nabla \zeta\|_{L^\infty(Q_R^+)} \leq \frac{C}{(\rho_2 - \rho_1)^2}.
\]

Let \((v, p)\) be a solution to the system (1.1), (1.4). Fix arbitrary functions \( b \in L_t(-R^2, 0) \) of \( t \)-variable and denote \( \tilde{p} := p - b \). Let \( \Omega \) be a smooth domain such that \( B_R \supset \Omega \subset B_R^+ \). Consider functions \( u := \zeta v, q := \zeta \tilde{p} \). Then \((u, q)\) is a solution to the initial-boundary problem of type (2.6), but in domain \( \Omega \times (-R^2, 0) \) instead of \( \Omega \times (0, T) \) and with “right hand sides” \( f, g \) in (2.6) equal to \( \tilde{f}, \tilde{g} \), where
\[
  \tilde{f} = \zeta f + v(\partial_t \zeta - \Delta \varphi \zeta) - 2(\nabla\varphi v)\nabla\varphi \zeta + \tilde{p} \nabla\varphi \zeta, \quad \tilde{g} = v \cdot \nabla\varphi \zeta.
\]

Applying the estimate (2.7) to the functions \((u, q, \tilde{f}, \tilde{g})\) and taking into account that \( \zeta \equiv 1 \) on \( Q_{\rho_1}^+ \), \( \frac{1}{\rho_2 - \rho_1} \geq C \), we obtain
\[
  \|v\|_{W_2^{s, 1}(Q_{\rho_1}^+)}^s \leq C \|f\|_{L_{s, l}(Q_{\rho_1}^+)}^s + \frac{C}{(\rho_2 - \rho_1)^{2s}} \left( \|v\|_{W^{1, 0}_{s, l}(Q_{\rho_1}^+)}^s + \|p\|_{L_{s, l}(Q_R^+)}^s \right) + C \left( \|\nabla(v \cdot \nabla\varphi \zeta)\|_{L_{s, l}(Q_{\rho_1}^+)}^s + \|\partial_t(v \cdot \nabla\varphi \zeta)\|_{L_{s, l}(Q_{\rho_1}^+)}^s \right) \|\partial_t(v \cdot \nabla\varphi \zeta)\|_{L_{s, l}(Q_{\rho_1}^+)}^{s-1} \|\partial_t(v \cdot \nabla\varphi \zeta)\|_{L_{s, l}(Q_{\rho_1}^+)}^{s-1} \|\partial_t(v \cdot \nabla\varphi \zeta)\|_{L_{s, l}(Q_{\rho_1}^+)}^{s-1} \|\partial_t(v \cdot \nabla\varphi \zeta)\|_{L_{s, l}(Q_{\rho_1}^+)}^{s-1}.
\]

Taking into account estimates
\[
  \|\nabla(v \cdot \nabla\varphi \zeta)\|_{L_{s, l}(Q_{\rho_1}^+)}^s \leq \frac{C}{(\rho_2 - \rho_1)^{2s}} \|v\|_{W^{1, 0}_{s, l}(Q_{\rho_1}^+)}^s,
\]
\[
  \|\partial_t(v \cdot \nabla\varphi \zeta)\|_{L_{s, l}(Q_{\rho_1}^+)} \leq \frac{C}{(\rho_2 - \rho_1)^2} \left( \|\partial_t v\|_{L_{s, l}(Q_{\rho_1}^+)} + \|v\|_{L_{s, l}(Q_{\rho_1}^+)} \right),
\]
\[
  \|\partial_t(v \cdot \nabla\varphi \zeta)\|_{L_{s, l}(Q_{\rho_1}^+)}^{s-1} \leq \frac{C}{(\rho_2 - \rho_1)^{2s-2}} \left( \|\partial_t v\|_{L_{s, l}(Q_{\rho_1}^+)}^{s-1} + \|v\|_{L_{s, l}(Q_{\rho_1}^+)}^{s-1} \right),
\]

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The inequality (4.2) implies that

\[ \|v\|^{s,2,1}_{W_{s,l}^{1,0}(Q^{+}_{R})} \leq C \|f\|^{s}_{L_{s,l}(Q^{+}_{R})} \]

\[ + \frac{C}{(\rho_2 - \rho_1)^{2s}} \left( \|v\|^{s,1,0}_{W_{s,l}^{1,0}(Q^{+}_{R})} + \|\bar{p}\|^{s}_{L_{s,l}(Q^{+}_{R})} + \|\partial_{t}v\|^{s}_{L_{1}(-R^{2},0;W_{s,l}^{-1}(B_{R}^{+}))} \right) \]

\[ + \frac{C}{(\rho_2 - \rho_1)^{2s}} \|\partial_{t}v\|^{s-1}_{L_{1}(-R^{2},0;W_{s,l}^{-1}(B_{R}^{+}))} + \|v\|^{s-1}_{L_{s,l}(Q^{+}_{R})} . \]

Estimating the last term in the right-hand side of (4.1) via the Young inequality \( ab \leq \varepsilon a^{\gamma} + C_{\varepsilon} b^{\gamma} \) we obtain the estimate

\[ \frac{C}{(\rho_2 - \rho_1)^{2s}} \|\partial_{t}v\|^{s}_{L_{s,l}(Q^{+}_{R})} \left( \|\partial_{t}v\|^{s-1}_{L_{1}(-R^{2},0;W_{s,l}^{-1}(B_{R}^{+}))} + \|v\|^{s-1}_{L_{s,l}(Q^{+}_{R})} \right) \leq \varepsilon \|\partial_{t}v\|^{s}_{L_{s,l}(Q^{+}_{R})} + \frac{C_{\varepsilon}}{(\rho_2 - \rho_1)^{2s}} \left( \|\partial_{t}v\|^{s}_{L_{1}(-R^{2},0;W_{s,l}^{-1}(B_{R}^{+}))} + \|v\|^{s}_{L_{s,l}(Q^{+}_{R})} \right) ; \]

where the constant \( \varepsilon > 0 \) can be chosen arbitrary small. Therefore,

\[ \|v\|^{s,2,1}_{W_{s,l}^{1,0}(Q^{+}_{R})} \leq C \|f\|^{s}_{L_{s,l}(Q^{+}_{R})} + \varepsilon \|\partial_{t}v\|^{s}_{L_{s,l}(Q^{+}_{R})} + \frac{C_{\varepsilon}}{(\rho_2 - \rho_1)^{2s}} \left( \|\partial_{t}v\|^{s}_{L_{1}(-R^{2},0;W_{s,l}^{-1}(B_{R}^{+}))} + \|v\|^{s}_{L_{s,l}(Q^{+}_{R})} \right) . \]

and by virtue of (1.9)

\[ \|v\|^{s,2,1}_{W_{s,l}^{1,0}(Q^{+}_{R})} \leq \varepsilon \|\partial_{t}v\|^{s}_{L_{s,l}(Q^{+}_{R})} + \frac{C_{\varepsilon}}{(\rho_2 - \rho_1)^{2s}} \left( \|f\|^{s}_{L_{s,l}(Q^{+}_{R})} + \|v\|^{s}_{W_{s,l}^{1,0}(Q^{+}_{R})} + \|\bar{p}\|^{s}_{L_{s,l}(Q^{+}_{R})} \right) . \]

Now let us introduce the monotone function \( \Psi(\rho) := \|v\|^{s}_{W_{s,l}^{1,0}(Q^{+}_{R})} \) and the constant

\[ A := C_{\varepsilon} \left( \|f\|^{s}_{L_{s,l}(Q^{+}_{R})} + \|v\|^{s}_{W_{s,l}^{1,0}(Q^{+}_{R})} + \|\bar{p}\|^{s}_{L_{s,l}(Q^{+}_{R})} \right) . \]

The inequality (4.2) implies that

\[ \Psi(\rho_1) \leq \varepsilon \Psi(\rho_2) + \frac{A}{(\rho_2 - \rho_1)^{\alpha}}, \quad \forall \rho_1, \rho_2 : \quad R_1 \leq \rho_1 \leq \rho_2 \leq R_0, \]

for \( \alpha = 2ss' \) and for \( R_1 = r, R_0 = R - \frac{1}{10}(R - r) \). Now we shall take an advantage of the following lemma (which can be easily proved by iterations if one takes \( \rho_k := R_0 - 2^{-k}(R_0 - R_1) \);
Lemma 4.1. Assume $\Psi$ is a nondecreasing bounded function which satisfies the inequality (4.3) for some $\alpha > 0$, $A > 0$, and $\varepsilon \in (0, 2^{-\alpha})$. Then there exists a constant $B$ depending only on $\varepsilon$ and $\alpha$ such that

$$\Psi(R_1) \leq \frac{BA}{(R_0 - R_1)^\alpha}.$$ 

Fixing $\varepsilon = 2^{-4s's'}$ in (4.2) and applying Lemma 4.1 to our function $\Psi$, we obtain the estimate

$$\|v\|_{W^{2,1}_{s,l}(Q^+_r)} \leq C_s \left( \|f\|_{L_{s,l}(Q^+_R)}^s + \|v\|_{W^{1,p}_{s,l}(Q^+_R)}^s + \|\bar{p}\|_{L_{s,l}(Q^+_R)}^s \right).$$

Then from (1.1) we obtain that $\nabla \varphi p \in L_{s,l}(Q^+_r)$. Taking into account (1.2) and $\|\varphi\|_{W^{3}_{s,l}(S_R)} \leq \mu_2$ we get

$$\nabla p \in L_{s,l}(Q^+_r), \quad \|\nabla p\|_{L_{s,l}(Q^+_r)} \leq c \left( \|v\|_{W^{2,1}_{s,l}(Q^+_r)} + \|f\|_{L_{s,l}(Q^+_r)} \right).$$

Theorem 2.2 is proved. □
5 Proof of Theorem 2.3

For the presentation convenience we fix $R = 1$ and $r = \frac{1}{2}$. The extension of our proof to the case of general $0 < r < R$ is straightforward.

Let $\rho_m \to +0$ be an arbitrary sequence. Extend all functions $v, p, f$ from $Q^+$ to the set $B^+ \times \mathbb{R}$ by zero. For any extended function $v$ denote by $v^m$ the mollification of the function $v$ with respect to $t$ variable:

$$v^m(x,t) := (\omega_{\rho_m} * v)(x,t) \equiv \int_{\mathbb{R}} \omega_{\rho_m}(t-\tau)v(x,\tau)\,d\tau,$$

where $\omega_{\rho}(t) = \frac{1}{\rho}\omega(t/\rho)$, and $\omega \in C_0^\infty(-1,1)$ is a smooth kernel normalized by the identity $\int_0^1 \omega(t)dt = 1$.

As $v \in W_{s,l}^{1,0}(Q^+), \ p \in L_{s,l}(Q^+), \ f \in L_{s,l}(Q^+)$ we have

$$v^m \to v \text{ in } W_{s,l}^{1,0}(Q^+), \ p^m \to p \text{ in } L_{s,l}(Q^+), \ f^m \to f \text{ in } L_{s,l}(Q^+). \quad (5.1)$$

Let us fix arbitrary $\delta \in (0, \frac{1}{12})$. Then for any $\rho_m < \delta$ and any $\eta \in C^\infty(\bar{Q}^+)$

$$\partial_t(\omega_{\rho_m} * \eta)(x,t) = (\omega_{\rho_m} * \partial_t \eta)(x,t), \quad \forall \ x \in B^+, \ t \in (-1+\delta,-\delta).$$

Let us take in (1.10) $\eta = \omega_{\rho_m} * \tilde{\eta}$ where $\tilde{\eta} \in C^\infty(\bar{Q}^+)$ is an arbitrary function vanishing on $\partial B^+ \times (-1,0)$ and on $B^+ \times (-1,-1+\delta)$ and $B^+ \times (-\delta,0)$. Using the property of convolution

$$\int_{Q^+} g \cdot (\omega_{\rho_m} * h) \, dxdt = \int_{Q^+} (\omega_{\rho_m} * g) \cdot h \, dxdt,$$

$$\forall \rho_m < \delta, \ g \in L_1(Q^+), \ h \in C^\infty(\bar{Q}^+), \ supp \ h \subset \bar{B}^+ \times [-1+\delta,-\delta],$$

and taking into account the fact that convolution with respect to $t$ commutes with the differential operators $\Delta_\varphi, \nabla_\varphi$, we obtain the identity

$$- \int_{Q^+} v^m \cdot (\partial_t \tilde{\eta} + \tilde{\Delta}_\varphi \tilde{\eta}) \, dxdt = \int_{Q^+} (f^m \cdot \tilde{\eta} + p^m \nabla_\varphi \cdot \tilde{\eta}) \, dxdt \quad (5.2)$$
which holds for all \( \tilde{\eta} \in C^\infty(\bar{Q}^+) \) such that \( \tilde{\eta}|_{x_n=0} = 0 \) and \( \tilde{\eta} \) vanishes on \( B^+ \times (-1, -1 + \delta) \), \( B^+ \times (-\delta, 0) \), and near the set \( \partial B^+ \times (-1, 0) \), where \( \partial B^+ := \{ x \in \mathbb{R}^n : |x| = R, x_n > 0 \} \).

Let \( \zeta \in C^\infty(\bar{Q}^+) \) be a cut-of function vanishing in \( Q^+ \setminus Q^+_{5/6} \) and such that \( \zeta \equiv 1 \) in \( Q^+_{2/3} \). Denote \( u^m := \zeta v^m \), \( q^m := \zeta p^m \). Then from (5.2) we obtain that \( (u^m, q^m) \) satisfy the integral identity

\[
- \int_{B^+ \times (-1, -\delta)} u^m \cdot (\partial\eta + \Delta \varphi \eta) \, dx dt = \int_{B^+ \times (-1, -\delta)} (f_0^m \cdot \eta + q^m \nabla \varphi \cdot \eta) \, dx dt
\]

for any \( \eta \in C^\infty(\bar{B}^+ \times [-1, -\delta]) \) such that \( \eta|_{\partial B^+ \times (-1, -\delta)} = 0 \) and \( \eta|_{B^+ \times \{ t = -\delta \}} = 0 \). Here by \( f_0^m \) we denote the expression

\[
f_0^m = f^m \zeta - v^m \partial_t \zeta + v^m \Delta \varphi \zeta - 2 \nabla \varphi u^m \partial_t \zeta - p^m \nabla \varphi \zeta. \tag{5.3}
\]

Moreover, \( u^m \) also satisfies the identity

\[
\nabla \varphi \cdot u^m = g^m \text{ a.e. in } Q^+,
\]

where we denote

\[
g^m = v^m \cdot \nabla \varphi \zeta.
\]

Assume \( \Omega \subset \mathbb{R}^3 \) is a smooth domain such that \( B^+_{5/6} \subset \Omega \subset B^+ \) and denote \( \tilde{Q}^+ := \Omega \times (-1, 0) \). As \( v^m \) is smooth with respect to \( t \) variable for each fixed \( m \in \mathbb{N} \) the functions \( f^m, g^m \) possess the properties

\[
f_0^m \in L_{s,l}(\tilde{Q}^+), \quad g^m \in W_{s,l}^{1,0}(\tilde{Q}^+), \quad \partial_t g^m \in L_{s,l}(\tilde{Q}^+), \quad \int_\Omega g^m(x, t) \, dx = 0.
\]

From Theorem 2.1 we obtain that for any \( m \in \mathbb{N} \) there exists a strong solution \( \tilde{u}^m \in W_{s,l}^{2,1}(\tilde{Q}^+), \tilde{q}^m \in W_{s,l}^{1,0}(\tilde{Q}^+) \) to the problem

\[
\begin{cases}
\partial_t \tilde{u}^m - \Delta \varphi \tilde{u}^m + \nabla \varphi \tilde{q}^m = f_0^m & \text{in } Q^+, \\
\nabla \varphi \cdot \tilde{u}^m = g^m & \\
\tilde{u}^m|_{\partial \tilde{Q}^+} = 0.
\end{cases} \tag{5.4}
\]

Note that as \( \zeta \equiv 1 \) in \( Q^+_{2/3} \), we have the identity \( g^m \equiv 0 \) in \( Q^+_{2/3} \). So, functions \( (\tilde{u}^m, \tilde{q}^m) \) satisfy all assumptions of Theorem 2.2 in \( Q^+_{2/3} \) and hence by Theorem 2.2 with \( r = \frac{1}{3}, R = \frac{2}{3} \) we obtain the estimate

\[
\| \tilde{u}^m \|_{W_{s,l}^{2,1}(Q^+_{1/2})} + \| \nabla \tilde{q}^m \|_{L_{s,l}(Q^+_{1/2})} \leq C \left( \| f_0^m \|_{L_{s,l}(Q^+_{1/3})} + \| u^m \|_{W_{s,l}^{1,0}(Q^+_{2/3})} + \| \tilde{q}^m - b \|_{L_{s,l}(Q^+_{2/3})} \right)
\]

where \( C \) is a constant independent of \( m \).
where the constant $C$ does not depend neither on $m$ nor on $\delta$ and $b \in L_1(-\frac{4}{3}, 0)$ is arbitrary.

As every strong solution of the Perturbed Stokes system is a generalized one, from (1.10) we obtain that $(\tilde{u}^m, \tilde{q}^m)$ satisfy the integral identity

$$- \int_{\tilde{Q}^+} \tilde{u}^m \cdot (\partial_t \eta + \hat{\Delta}_\varphi \eta) \, dx \, dt = \int_{\tilde{Q}^+} (f_0^m \cdot \eta + \tilde{q}^m \hat{\nabla}_\varphi \cdot \eta) \, dx \, dt$$

for all $\eta \in C^\infty(\tilde{Q}^+)$ such that $\eta|_{\partial \Omega \times (-1, 0)} = 0$ and $\eta|_{\Omega \times \{t=0\}} = 0$. Hence the differences $w^m := u^m - \tilde{u}^m$, $\pi^m := q^m - \tilde{q}^m$ are a generalized solution to the Perturbed Stokes system (1.1) in $\Omega \times (-1, -\delta)$ satisfying the integral identity

$$- \int_{\Omega \times (-1, -\delta)} w^m \cdot (\partial_t \eta + \hat{\Delta}_\varphi \eta) \, dx \, dt = \int_{\Omega \times (-1, -\delta)} \pi^m \hat{\nabla}_\varphi \cdot \eta \, dx \, dt,$$

$$\hat{\nabla}_\varphi \cdot w^m = 0 \quad \text{a.e. in} \quad \Omega \times (-1, -\delta) \tag{5.6}$$

for any $\eta \in W_{s, t}^{2, 1}(\Omega \times (-1, -\delta))$ such that $\eta|_{\partial \Omega \times (-1, -\delta)} = 0$ and $\eta|_{\Omega \times \{t=-\delta\}} = 0$. Denote $\kappa = \min\{s, t\} > 1$. As $u^m, \tilde{u}^m \in L_{s, 1}(\tilde{Q}^+)$ and $q^m, \tilde{q}^m \in L_{s, 1}(\tilde{Q}^+)$ we have $w^m = u^m - \tilde{u}^m \in L_{s, 1}(\tilde{Q}^+)$ and $\pi^m = q^m - \tilde{q}^m \in L_{s, 1}(\tilde{Q}^+)$. Hence $|w^m|^s - 2 w^m \in L_{s, 1}(\tilde{Q}^+)$, and using Theorem 2.1 we can find functions $\eta \in W_{s, t}^{2, 1}(\Omega \times (-1, -\delta))$ and $\kappa \in W_{s, t}^{1, 0}(\Omega \times (-1, -\delta))$ such that

$$\begin{cases}
\partial_t \eta + \hat{\Delta}_\varphi \eta + \hat{\nabla}_\varphi \kappa = |w^m|^s - 2 w^m, & \text{in} \quad \Omega \times (-1, -\delta),
\hat{\nabla}_\varphi \cdot \eta = 0, & \text{in} \quad \Omega \times (-1, -\delta),
\eta|_{\partial \Omega \times (-1, -\delta)} = 0, \quad \eta|_{t=-\delta} = 0.
\end{cases}$$

Substituting this $\eta$ as a test function into the identity (5.6) we obtain $w^m = 0$ in $\Omega \times (-1, -\delta)$. Hence $u^m = \tilde{u}^m \in W_{s, 1}^{2, 1}(\Omega \times (-1, -\delta))$. Hence from (5.6) we obtain

$$\int_{\Omega \times (-1, -\delta)} \pi^m \hat{\nabla}_\varphi \cdot \eta \, dx \, dt = 0, \quad \forall \eta \in L_1((-1, -\delta); W_{s, t}^{1, 0}(\Omega)). \tag{5.7}$$

Correcting, if necessary, function $\tilde{q}^m$ by a constant, we can assume that $\int_{\Omega} \pi^m \, dx = 0$ for a.e. $t \in (1, -\delta)$. As $\pi^m \in L_{s, 1}(\Omega)$ for a.e.
holds for any $\delta \in (0, \frac{1}{2})$ with $C$ independent on $\delta$. The last inequality provides the required properties of $(v, p)$. Theorem 2.3 is proved. $\square$

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6 Proof of Theorem 2.4

As usually, for the presentation convenience we fix \( R = 1 \) and \( r = \frac{1}{2} \). For any \( k = 0, 1, \ldots \) denote \( s_k = \frac{n_k}{n} k_s \) if \( n > k_s \) and \( \frac{n_k}{n} k_s < m \) and \( s_k = m \) otherwise. Denote also \( N = \min \{ k \in \mathbb{N} : s_k = m \} \) and \( \rho_k = \frac{1}{2} + \frac{1}{2^{k+1}} \).

Using Theorem 2.2 and Theorem 2.3 we see that if \( (v, p) \in W^{1,0}_{s_k,l}(Q^+_{\rho_k}) \times L_{s_k,l}(Q^+_{\rho_k}) \) is a generalized solution of the problem (1.1), (1.4) in \( Q^+_{\rho_k} \), then \( (v, p) \in W^{2,1}_{s_k,l}(Q^+_{\rho_k+1}) \times W^{1,0}_{s_k,l}(Q^+_{\rho_k+1}) \) and the following estimate holds:

\[
\|v\|_{W^{2,1}_{s_k,l}(Q^+_{\rho_k+1})} + \|\nabla p\|_{L_{s_k,l}(Q^+_{\rho_k+1})} \leq C \left( \|f\|_{L_{m,l}(Q^+)} + \|v\|_{W^{1,0}_{s_k,l}(Q^+)} + \|p - b\|_{L_{s_k,l}(Q^+)} \right),
\]

where \( b \in L_{1}(-1, 0) \) is an arbitrary function of \( t \)-variable. Moreover, due to the imbedding \( W^{1}_{s_k}(B^+_{\rho_k+1}) \hookrightarrow L_{s_k+1}(B^+_{\rho_k+1}) \) we obtain the estimate

\[
\|v\|_{W^{1,0}_{s_k+1,l}(Q^+_{\rho_k+1})} + \|p\|_{L_{s_k+1,l}(Q^+_{\rho_k+1})} \leq C \left( \|v\|_{W^{2,1}_{s_k,l}(Q^+_{\rho_k+1})} + \|p\|_{W^{1,0}_{s_k,l}(Q^+_{\rho_k+1})} \right).
\]

Iterating (6.1) and (6.2) from \( k = 0 \) to \( k = N \) we finally obtain the estimate

\[
\|v\|_{W^{2,1}_{s_N,l}(Q^{1/2})} + \|\nabla p\|_{L_{s_N,l}(Q^{1/2})} \leq C^N \left( \|f\|_{L_{m,l}(Q^+)} + \|v\|_{W^{1,0}_{s_0,l}(Q^+)} + \|p - b\|_{L_{s_0,l}(Q^+)} \right).
\]

This estimate is equivalent to (2.9). Theorem 2.4 is proved. \( \square \)
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