DIRAC’S THEOREM ON SIMPLICIAL MATROIDS

RAUL CORDOVIL, MANOEL LEMOS AND CLÁUDIA LINHARES SALES

Abstract. We introduce the notion of $k$-hyperclique complexes, i.e., the largest simplicial complexes on the set $[n]$ with a fixed $k$-skeleton. These simplicial complexes are a higher-dimensional analogue of clique (or flag) complexes (case $k = 2$) and they are a rich new class of simplicial complexes.

We show that Dirac’s theorem on chordal graphs has a higher-dimensional analogue in which graphs and clique complexes get replaced, respectively, by simplicial matroids and $k$-hyperclique complexes. We prove also a higher-dimensional analogue of Stanley’s reformulation of Dirac’s theorem on chordal graphs.

1. Introduction and notations

Set $[n] := \{1,2,\ldots,n\}$. The simplicial matroids $S^k_n(E)$ on the ground set $E \subseteq \binom{[n]}{k}$ have been introduced by Crapo and Rota \[6, 7\] as one of the six most important classes of matroids. These matroids generalize graphic matroids: more precisely $S^2_n(E)$ is the cycle matroid (or graphic matroid) of the graph $([n], E)$. With the aid of Alexander’s duality theorem for manifolds applied to simplices, they prove the following beautiful isomorphism

\[(1.1) \quad \left[ S^n_k \left( \binom{[n]}{k} \right) \right]^* \simeq S^{n-k}_n \left( \binom{[n]}{n-k} \right), \ X \mapsto [n] \setminus X, \]

where $\left[ S^n_k \left( \binom{[n]}{k} \right) \right]^*$ denotes the dual (or orthogonal) matroid of $S^n_k \left( \binom{[n]}{k} \right)$, see \[7\] Theorem 11.4 or, for an elementary proof (depending only on matrix algebra), \[4\] Theorem 6.2.1. In this paper, we use an equivalent definition of simplicial matroids, see Definition 1.4 below.

We introduce the notion of $k$-hyperclique complexes. These simplicial complexes are a natural higher-dimensional analogue of clique (or flag) complexes (case $k = 2$), see Definition 1.2 below. The $k$-hyperclique complexes
are a rich new class of simplicial complexes of intrinsic interest. To get a better understanding of the structure of $S^n_k(E)$ we attach to it the $k$-hyperclique complex on $[n]$ canonically determined by the family $E$.

In this paper, we introduce the notion of a strong triangulable simplicial matroid, a higher-dimensional generalization of the notion of a chordal graph. We prove an analogue of Dirac's theorem on chordal graphs (see Theorem 3.2) using a natural generalization of a perfect sequence of vertices of a chordal graph (see Theorem 5.2). We prove also a higher-dimensional generalization of Stanley’s reformulation of Dirac’s theorem on chordal graphs (see Theorem 4.3).

Let us set some preliminary notation.

**Definition 1.1.** An (abstract) oriented simplicial complex on the set $[n]$ is a family $\Delta$ of linear ordered subsets of $[n]$ (called the faces of $\Delta$) satisfying the following two conditions. (We identify the linear ordered set $\{v_1, v_2, \ldots, v_m\}$, $v_1 < v_2 < \cdots < v_m$, with the symbol $v_1v_2\cdots v_m$.)

1. Every $v \in [n]$ is a face of $\Delta$;
2. If $F$ is a face of $\Delta$ and $F' \subset F$, then $F'$ is also a face of $\Delta$.

Given two faces $F'$ and $F = i_1i_2\cdots i_m$ the “incidence number” $[F' : F]$ is

$$[F' : F] = \begin{cases} (-1)^j & \text{if } F' = i_1i_2\cdots i_{j-1}i_{j+1}\cdots i_m; \\ 0 & \text{otherwise}. \end{cases}$$

Let $S_d(\Delta)$ denote the $d$-skeleton of $\Delta$, i.e., the family of faces of size $d$ (or $d$-faces) of $\Delta$. A facet is a face of $\Delta$, maximal to inclusion. If nothing in contrary is indicated, we suppose that $S_1(\Delta) = [n]$.

**Definition 1.2.** Let $S_k := \{F_1, F_2, \ldots, F_m\}$ be a family of $k$-subsets of $[n]$. Let $\langle S_k \rangle$ be the simplicial complex such that $F \subseteq [n]$ is a face of $\langle S_k \rangle$ provided that:

1. $|F| < k$; or
2. $|F| \geq k$, every $k$-subset of $F$ belongs to $\{F_1, F_2, \ldots, F_m\}$.

We say that $\langle S_k \rangle$ is the $k$-hyperclique complex generated by the set $S_k = \{F_1, F_2, \ldots, F_m\}$.

We see $\langle S_k \rangle$ as an oriented simplicial complex, with the natural orientation induced by $[n]$.

Note that $\langle S_k \rangle$ is the largest simplicial complex $\Delta$ on the set $[n]$ with the fixed $k$-skeleton $S_k$. (In the hypergraph literature, a family of sets satisfying Property (1.2.2) is said to have the Helly dual $k$-property.)

Throughout this work $S_k$ denotes a family of $k$-subsets of $[n]$ and let $\langle S_k \rangle$ denote the corresponding $k$-hyperclique complex. (So, we have $S_k = S_k(\langle S_k \rangle)$.) The paradigm examples of the $k$-hyperclique complexes are the clique (or flag) complexes (the 2-hyperclique complexes).
Example 1.3. Set \( S_k = \{F_1, F_2, \ldots, F_m\} \). If \( \bigcap_{i=1}^m F_i \neq \emptyset \) then \( F_1, \ldots, F_m \) are facets of \( \langle S_k \rangle \). The other facets of \( \langle S_k \rangle \) are the \((k - 1)\)-subsets of \([n]\) which are in no \( F_i \). For every \( k, n \geq k \geq 1 \), \( 2^{[n]} \) is a (full) \( k \)-hyperclique complex.

Let \( S_\ell \) be a subset of \( \binom{[n]}{\ell} \), where \( n \geq \ell \geq 2 \). Let \( \langle S_\ell \rangle \) be the oriented \( \ell \)-hyperclique complex determined by \( S_\ell \). Let \( F \) be a field. Consider the two vector spaces \( \mathbb{F}^{S_\ell} \) and \( \mathbb{F}^{([n]_{\ell-1})} \) over \( F \). Let us define the (boundary) map

\[
\partial \ell : \mathbb{F}^{S_\ell} \to \mathbb{F}^{([n]_{\ell-1})}
\]

as the vector space map determined by linearity specifying its values in the basis elements:

\[
\partial \ell F = \sum_{F' \in \binom{[n]}{\ell-1}} [F' : F]F',
\]

for every \( F \in S_\ell \). By duality let us define the (coboundary) map

\[
\delta^{\ell-1} : \mathbb{F}^{([n]_{\ell-1})} \to \mathbb{F}^{S_\ell}
\]

as the vector space map determined by linearity specifying its values in the basis elements:

\[
\delta^{\ell-1} F' = \sum_{F \in E} [F' : F]F
\]

for every \( F' \in \binom{[n]}{\ell-1} \). (The symbol \([F' : F]\) denotes the incidence number of the faces \( F' \) and \( F \) in the oriented simplicial complex \( \langle S_\ell \rangle \).)

Definition 1.4. [4] The simplicial matroid \( S_n^k(S_k) \), on the ground set \( S_k \) and over the field \( F \), is the matroid such that

\[
\{X_1, X_2, \ldots, X_m\} \subseteq S_k
\]

is an independent set iff the vectors,

\[
\partial_k X_1, \partial_k X_2, \ldots, \partial_k X_m,
\]

are linearly independent in the vector space \( \mathbb{F}^{([n]_{k-1})} \).

Remark 1.5. [3, 4] Let \((s_{p,q})\) be the matrix whose rows and columns are labeled by the sets of \( \binom{[n]}{k-1} \) and \( S_k \) respectively, with \( s_{p,q} = 0 \) if \( p \not\subseteq q \) and \( s_{p,q} = (-1)^j \) if \( q - p = i_j, q = \{i_1, \ldots, i_j, \ldots i_k\} \). The simplicial matroid \( S_n^k(S_k) \) (over the field \( F \)) is the independent matroid of the columns of the \( \{-1, 1, 0\} \) matrix \((s_{p,q})\), over the field \( F \). If the matrix \((s_{p,q})\) is not totally unimodular, the simplicial matroid depends of the field \( F \). Since the time of Henri Poincaré, it is known that if \( k = 2 \), the matrix \((s_{p,q})\) is totally unimodular. The 2-hyperclique complex \( \langle S_2 \rangle \) is the clique complex of the simple graph \( ([n], S_2) \) and \( S_2^0(S_2) \) it is its corresponding cycle matroid. So \( S_n^0(S_2) \) is a regular (or unimodular) matroid, i.e., it is irrespective of the field \( F \).
If nothing in contrary is said, the simplicial matroids here considered are over the field \( \mathbb{F} \).

For background, motivation, and matroid terminology left undefined here, see any of the standard references [7, 11, 13, 14] or the encyclopedic survey [15, 16, 17]. For a description of the developments on simplicial matroids before 1986, see [4]. See also [2] for an interesting application. For a topological approach to combinatorics, see [1]. Dirac characterization of chordal graphs (see [8]) is treated extensively in Chapter 4 of [9]. For an algebraic proof of Dirac’s theorem, see [10].

2. Simplicial matroids

The following two propositions are folklore and they are included for completeness. For every vector \( v \) of \( \mathbb{F}^E \), where

\[
v = a_1 e_1 + a_2 e_2 + \cdots + a_m e_m \quad (e_i \in E, a_i \in \mathbb{F}^*),
\]

let \( v := \{e_1, e_2, \ldots, e_m\} \) denote the support of \( v \).

**Proposition 2.1.** Let \( S_k \) be a subset of \( \binom{[n]}{k} \) and \( S^n_k(S_k) \) be the corresponding simplicial matroid (over the field \( \mathbb{F} \)). Consider the linear map \( \partial_k : \mathbb{F} S_k \to \mathbb{F}^{\binom{[n]}{k-1}} \). Then

\[
\begin{align*}
(2.1.1) & \quad \text{Each circuit of } S^n_k(S_k) \text{ has at least } k + 1 \text{ elements.} \\
(2.1.2) & \quad \text{For every } (k+1)\text{-face } X \text{ of } \langle S_k \rangle, \partial_{k+1} X \text{ is a circuit of } S^n_k(S_k). \text{ Each circuit with exactly } k + 1 \text{ elements is of this type.} \quad \Box
\end{align*}
\]

For each \( X \in S_{k+1}(\langle S_k \rangle) \) we say that \( \partial_{k+1} X \) is a small circuit of \( S^n_k(S_k) \).

**Proposition 2.2.** Let \( S_k \) be a subset of \( \binom{[n]}{k} \), \( k \geq 2 \), and \( S^n_k(S_k) \) be the corresponding simplicial matroid (over the field \( \mathbb{F} \)). Consider the linear map \( \delta^{k-1} : \mathbb{F}^{\binom{[n]}{k-1}} \to \mathbb{F} S_k \). Then

\[
\begin{align*}
(2.2.1) & \quad \text{The cocircuit space of } S^n_k(S_k) \text{ is generated by the set of vectors } \{\delta^{k-1} V \neq 0 : V \in \binom{[n]}{k-1}\}. \\
(2.2.2) & \quad \text{If non empty, the set } \delta^{k-1} V, V \in \binom{[n]}{k-1}, \text{ is a union of cocircuits of } S^n_k(S_k). \nonumber
\end{align*}
\]

**Proof.** The oriented simplicial complex \( (\binom{[n]}{k}) = 2^{[n]} \) is the oriented full \( k \)-hyperclique complex. The matroid \( S^n_k(\binom{[n]}{k}) \) is the full simplicial matroid on the ground set \( \binom{[n]}{k} \). Consider the linear map

\[
\delta^{k-1} : \mathbb{F}^{\binom{[n]}{k-1}} \to \mathbb{F}^{\binom{[n]}{k}}.
\]

From Isomorphism (1.1), we know that

\[
C^* := \{\delta^{k-1} V : V \in \binom{[n]}{k-1}\}
\]
is a generating set of the cocircuit space of $S^n_k(S'_k)$. The linear map

$$\delta^{k-1} : \mathcal{F}(^{[n]}_k) \to \mathcal{F}S_k$$

is the composition of the map (2.1) and the natural projection

$$\iota : \mathcal{F}(^{[n]}_k) \to \mathcal{F}S_k.$$

So, Assertion (2.2.1) holds. We know that $C^*$ is a cocircuit of $S^n_k(^{[n]}_k)$ iff $C^*$ is the support of a non null vector of $\mathcal{C}$, minimal for inclusion. Note that $S^n_k(S'_k)^*$ is obtained from $S^n_k(^{[n]}_k)^*$ by contracting the set $(^{[n]}_k) \setminus S_k$. So, Assertion (2.2.2) holds.

Throughout this work $V, V', V_1, V_2, \ldots$ denote $(k-1)$-subsets of $[n]$. So, they are $(k-1)$-face of $\langle S_k \rangle$. Let $\langle S_k \rangle \setminus V$ denote the $k$-hyperclique complex $\langle S_k \setminus \delta^{k-1}V \rangle$, i.e., the $k$-hyperclique complex determined by the set $S_k \setminus \delta^{k-1}V$. Note that, for every pair of $(k-1)$-faces $V$ and $V'$, we have:

$$\langle S_k \rangle \setminus V \setminus V' = \langle S_k \rangle \setminus (\delta^{k-1}V \cup \delta^{k-1}V').$$

**Definition 2.3.** Let $\Delta_0 = \langle S_k \rangle$ be a $k$-hyperclique complex such that $S^n_k(S'_k)$ has rank $r$. A sequence $V_1, V_2, \ldots, V_r$ of $(k-1)$-faces of $\Delta_0$ is said to be basic linear sequence when

$$C^*_j := \delta^{k-1}V_j \setminus \bigcup_{i=1}^{j-1} \delta^{k-1}V_i$$

is a cocircuit of $S^n_k(S_k(\Delta_{j-1}))$, for $j \in \{1, 2, \ldots, r\}$, where

$$\Delta_{j-1} := \Delta_{j-2} \setminus V_{j-1}, \ j \in \{2, \ldots, r\}.$$

The following result is a corollary of Proposition 2.2.

**Corollary 2.4.** Let $\langle S_k \rangle$ be a $k$-hyperclique complex such that $S^n_k(S'_k)$ has rank $r$. If $\mathcal{V} = (V_1, V_2, \ldots, V_r)$ is a basic linear sequence of $(k-1)$-faces of $\langle S_k \rangle$, then

$$\beta = \{\delta^{k-1}V_1, \delta^{k-1}V_2, \ldots, \delta^{k-1}V_r\}$$

is a basis of the cocircuit space of $S^n_k(S_k)$.

**Proof.** Suppose that $\beta$ is a dependent set. Choose a dependent subset of $\beta$

$$\{\delta^{k-1}V_{i_1}, \delta^{k-1}V_{i_2}, \ldots, \delta^{k-1}V_{i_s}\},$$

such that $i_1 < i_2 < \cdots < i_s$ and $s$ is minimum. Therefore

$$\delta^{k-1}V_{i_s} \subseteq \bigcup_{j=1}^{s-1} \delta^{k-1}V_{i_j}$$

and $V_{i_s} \not\in \mathcal{V}$, a contradiction. As the cocircuit space of $S^n_k(S'_k)$ has dimension $r$ the result follows. \qed
3. D-perfect $k$-hyperclique complexes

In this section we extend to $k$-hyperclique complexes the notions of “simplicial vertex” and “perfect sequence of vertices”, introduced in the Dirac characterization the clique complexes of chordal graphs, see [8, 9].

**Definition 3.1.** Let $\Delta_0 = \langle S_k \rangle$ be a $k$-hyperclique complex and suppose that the simplicial matroid $S_k^n(\langle S_k \rangle)$ has rank $r$. We say that a $(k-1)$-face $V$ is *simplicial* in $\Delta_0$, if there is exactly one facet $X$ of $\Delta_0$ such that $V \subset X$. We say that $\Delta_0$ is D-perfect if there is a basic linear sequence of $(k-1)$-faces, $V = \{V_1, V_2, \ldots, V_r\}$, such that every $V_i \in V$ is simplicial in the $k$-hyperclique complex $\Delta_{i-1}$ where

$$\Delta_{i-1} := \Delta_{i-2} \cup V_{i-1}, \; i \in \{2, \ldots, r\}.$$  

We will call $V$ a D-perfect sequence of $\Delta_0$.

Chordal graphs are an important class of graphs. The following theorem is one of their fundamental characterizations, reformulated in our language.

**Theorem 3.2.** (Dirac’s theorem on chordal graphs [8, 9]) Let $G = ([n], S_2)$, $S_2 \subseteq \binom{[n]}{2}$ be a graph and $\langle S_2 \rangle$ be its clique complex. Then $G$ is chordal if and only if $\langle S_2 \rangle$ is D-perfect. □

**Proposition 3.3.** Let $V$ be a $(k-1)$-subset of $[n]$. If $V$ is simplicial in the $k$-hyperclique complex $\langle S_k \rangle$ then $\delta^{k-1}V$ is a cocircuit of $S_k^n(\langle S_k \rangle)$.

**Proof.** From Proposition 2.2 we know that $\delta^{k-1}V$ is a union of cocircuits of $S_k^n(\langle S_k \rangle)$. Suppose for a contradiction that there are two different cocircuits $C_1^*$ and $C_2^*$ contained in $\delta^{k-1}V$. Choose elements $F_1 \in C_1^* \setminus C_2^*$ and $F_2 \in C_2^* \setminus C_1^*$. As $V$ is simplicial it follows that $C = \langle F_1 \cup F_2 \rangle$ is a circuit of $S_k^n(\langle S_k \rangle)$ and $C \cap C_1^* = \{F_1\}$, a contradiction to orthogonality. □

The reader can easily see that the converse of Proposition 3.3 is not true.

**Example 3.4.** Set

$S_3 = \{123, 124, 125, 145, 245, 136, 137, 167, 367, 238, 239, 289, 389\}$.

Consider the 3-hyperclique complex $\langle S_3 \rangle$ on the set [9]. From Property (1.2.2) we know that $S_1(\langle S_3 \rangle) = \{1245, 1367, 2389\}$ and $S_0(\langle S_3 \rangle) = \emptyset$. From Property (1.2.1) we can see that the sets of 2-faces and 1-faces of $\langle S_3 \rangle$ are respectively $S_2(\langle S_3 \rangle) = \binom{[9]}{2}$ and $S_1(\langle S_3 \rangle) = \binom{[9]}{1}$. We can see that the set of facets of $\langle S_3 \rangle$ is

$$\{18, 19, 26, 27, 34, 35, 46, 47, 48, 49, 56, 57,$
$$58, 59, 68, 69, 78, 79, 123, 1245, 1367, 2389\}.$$  

Note that $S_3^n(\langle S_3 \rangle)$ has rank 10 and $\langle S_3 \rangle$ is D-perfect with the D-perfect sequence: 45, 67, 89, 15, 14, 16, 17, 28, 29, 12.
Proposition 3.5. Let \(V\) be a \((k - 1)\)-subset of \([n]\). Suppose that \(V\) is not a facet of the \(k\)-hyperclique complex \(\langle S_k \rangle = \langle F_1, F_2, \ldots, F_m \rangle\). Then the following two assertions are equivalent:

1. \(V\) is simplicial in \(\langle S_k \rangle\);
2. The set \(X = \bigcup_{F_i \in \delta^{k-1}V} F_i\) is the unique facet of \(\langle S_k \rangle\) containing \(V\).

Proof. The implication (3.5.2) \(\Rightarrow\) (3.5.1) is clear.

(3.5.1) \(\Rightarrow\) (3.5.2). Let \(X'\) be the unique facet of \(\langle S_k \rangle\) containing \(V\). Then it is clear that \(F_i \subseteq X'\) for each \(F_i\) containing \(V\). We conclude that \(X \subseteq X'\) and so \(X\) is a face of \(\langle S_k \rangle\). Suppose, for a contradiction, that \(X\) is not a face of \(\langle S_k \rangle\). Then there is an \(F \in S_k\) such that \(F \not\subseteq X\) but \(F \subseteq X'\). For every \(x \in F \setminus X\), we know that \(V \cup x \in S_k\) and so \(V \cup x \in \delta^{k-1}V\). We have the contradiction \(F \subseteq X\). Therefore \(X = X'\).

4. Superdense simplicial matroids

A matroid \(M\) on the ground set \([n]\) and of rank \(r\) is called superposable if it admits a maximal chain of modular flats

\[
\text{cl}(\emptyset) = X_0 \subseteq X_1 \subset \cdots \subseteq X_{r-1} \subseteq X_r = [n].
\]

The notion of “superposable lattices” was introduced and studied by Stanley in [12]. For a recent study of supersolvability for chordal binary matroids see [5].

Proposition 4.1. Let \(S^n_k(S_k)\), \(k > 2\), be a simplicial matroid. The matroid \(S^n_k(S_k)\) is superposable iff it does not have circuits.

Proof. All the circuits of \(S^n_k(S_k)\) have at least \(k + 1\) elements. So a hyperplane \(H\) is modular iff \(|S_k \setminus H| = 1\). Indeed if \(F, F' \in S_k \setminus H\), the line \(\text{cl}(\{F, F'\})\) cannot intersect the hyperplane \(H\). From (4.1) we conclude that if \(S^n_k(S_k)\) is superposable then it cannot have circuits. The converse is clear.

So, the notion of supersolvability is not interesting for the class of non-graphic simplicial matroids. The following definition gives the “right” extension of the notion of superposable.

Definition 4.2. Suppose that \(S^n_k(S_k)\) has rank \(r\). A hyperplane \(H\) of \(S^n_k(S_k)\) is said to be dense if there is a simplicial \((k - 1)\)-face, \(V\), of \(\langle S_k \rangle\) such that:

\[
H = S_k \setminus \delta^{k-1}V.
\]

We say that the simplicial matroid \(S^n_k(S_k)\) is superdense if it admits a maximal chain of “relatively dense” flats

\[
\emptyset = X_0 \subset X_1 \subset \cdots \subset X_{r-1} \subset X_r = S_k,
\]

i.e., such that \(X_i\) is a dense hyperplane of \(S^n_k(X_{i+1})\), \(i \in \{0, 1, \ldots, r - 1\}\).

A hyperplane \(H\) of \(S^n_2(S_2)\) is dense if and only if \(H\) is modular. Then \(S^n_2(S_2)\) is superdense if and only if it is superposable. So, Theorem 4.3 below can be seen as higher-dimensional generalization of Stanley’s reformulation of Dirac’s theorem on chordal graphs, see [12].
Theorem 4.3. Let \( \Delta_0 = \langle S_k \rangle \) be a \( k \)-hyperclique complex. Then the following two assertions are equivalent:

(4.3.1) \( \Delta_0 \) is D-perfect;

(4.3.2) \( S^n_k(S_k) \) is superdense.

Proof. (4.3.1) \( \Rightarrow \) (4.3.2). Let \( V = (V_1, V_2, \ldots, V_r) \) be a D-perfect sequence of \( \Delta_0 \). From Proposition 3.3 we know that the sets

\[
C^*_j := \delta^{k-1}V_j \setminus \bigcup_{i=1}^{j-1} \delta^{k-1}V_i, \quad j \in \{1, 2, \ldots, r\},
\]

are cocircuits of \( S^n_k(S_k(\Delta_{j-1})) \) where \( \Delta_{j-1} := \Delta_j \setminus V_j, \quad j \in \{2, 3, \ldots, r\} \).

So, \( V \) determines a maximal chain of flats of \( S^n_k(S_k) \):

\[
\emptyset = X_0 \subseteq X_1 \subseteq \cdots \subseteq X_{r-1} \subseteq X_r = S_k,
\]

where

\[
X_{r-j} = S_k(\Delta_{j-1}) \setminus C^*_j, \quad j = 1, \ldots, r.
\]

As \( V_j \) is simplicial in \( \Delta_{j-1} \), we know that \( X_{r-j} \) is dense in \( S^n_k(S_k(\Delta_{j-1})) \).

So, \( S^n_k(S_k) \) is superdense. The proof of the converse part is similar. \( \square \)

5. Triangulable simplicial matroids

Now we introduce a generalization of the notion of “triangulable” for the classes of simplicial matroids. Given a union of circuits \( D \) of \( S^n_k(S_k) \), let \( \overrightarrow{D} \) denote a vector of \( \mathbb{F}S^* \) whose support is \( D \). Set \( \overrightarrow{D} = D \).

Definition 5.1. Let \( \langle S_k \rangle = \langle F_1, F_2, \ldots, F_m \rangle \) be a \( k \)-hyperclique complex. We say that \( S^n_k(S_k) \) (over the field \( F \)) is triangulable provided that the vector family

\[
\{ \partial_{k+1}X : X \in S_{k+1}(\langle S_k \rangle) \}
\]

spans the circuit space.

Moreover, when generators \( \partial_{k+1}X_1, \partial_{k+1}X_2, \ldots, \partial_{k+1}X_m \) can be chosen such that, for every circuit \( C \), there are non-null scalars \( a_j \in \mathbb{F}^* \) such that

\[
\overrightarrow{C} = \sum_{j=1}^{s} a_j \partial_{k+1}X_{i_j} \quad \text{and} \quad \bigcup_{F_i \in C} F_i = \bigcup_{i=1}^{s} X_{i_j} \quad \text{where} \quad X_{i_j} \in \{X_1, \ldots, X_m'\}
\]

we say that \( S^n_k(S_k) \) is strongly triangulable.

Note that we can replace in Definition 5.1 the circuit \( C \) by a union of circuits \( D \). It is clear that a simple graph \( ([n], S_2) \) is chordal iff \( S^n_2(S_2) \) is strongly triangulable. Theorem 5.2 is the possible generalization of Dirac’s theorem on chordal graphs (see Theorem 3.2 above). Indeed, if \( k > 2 \), the converse of Theorem 5.2 is not true, see the remarks following the theorem.

Theorem 5.2. Let \( \Delta_0 = \langle S_k \rangle = \langle F_1, F_2, \ldots, F_m \rangle \) be a \( k \)-hyperclique complex. If \( \Delta_0 \) is D-perfect, then \( S^n_k(S_k) \) is strongly triangulable.
Proof. The proof is algorithmic. Let \( V = (V_1, \ldots, V_r) \) be a D-perfect sequence. Let \( D \) be a union of circuits of \( S_k^n(S_k) \). Let \( V_i \) the first \((k - 1)\)-face of \( V \) contained in an element of \( D \). From the definitions we know that \( V_i \) is a simplicial \((k - 1)\)-face of \( \Delta_{i-1} \) and \( D \) is a union of circuits of \( S_k^n(S_k(\Delta_{i-1})) \), where
\[
\Delta_{i-1} = \Delta_i \setminus V_{i-1}, \ i \in \{2, \ldots, r\}.
\]
From Proposition 3.3 we know that
\[
C^*_j := \delta^{k-1}V_j \setminus \bigcup_{i=1}^{j-1} \delta^{k-1}V_i
\]
is a cocircuit of \( S_k^n(S_k(\Delta_{i-1})) \). Set \( D \cap C^*_j = \{F_1, F_2, \ldots, F_h\} \) and consider the family of vectors of \( F^S_k \)
\[
\{ \overrightarrow{C}_s = \partial_{k+1}(F_{i_1} \cup F_{i_s}), \ s = 2, \ldots, h \}.
\]
Express a vector \( \overrightarrow{D} \) of support \( D \) in the canonical basis, say
\[
\overrightarrow{D} = a_{i_1}F_{i_1} + a_{i_2}F_{i_2} + \cdots + a_{i_h}F_{i_h} + a_{i_{h+1}}F_j + \cdots + a_{i_m}F_{i_m},
\]
where \( a_{i_\ell} \in \mathbb{F}^*, \ell = 1, \ldots, h, \ a_{i_\ell} \in \mathbb{F}, \ell = h + 1, \ldots, m \) and \( \{F_{i_1}, \ldots, F_{i_m}\} = S_k \). For every \( s \in \{2, 3, \ldots, h\} \), it is possible to choose \( b_s \in \mathbb{F}^* \) such that \( F_{i_s} \) does not belong to the support of \( b_s \overrightarrow{C}_s + \overrightarrow{D} \). As \((C_k \cap D) \cap C^*_j = \{F_{i_1}, F_{i_s}\} \) it follows that \( F_{i_2}, F_{i_3}, \ldots, F_{i_h} \) does not belong to the support of
\[
\overrightarrow{D}' := \overrightarrow{D} + b_2\overrightarrow{C}_2 + b_3\overrightarrow{C}_3 + \cdots + b_h\overrightarrow{C}_h.
\]
The dependent set \( D' \) is a union of circuits and \( D' \cap C^*_j \subseteq \{F_{i_1}\} \). So, by orthogonality we have \( D' \cap C^*_j = \emptyset \). Note that
\[\begin{align*}
(i) & \quad \text{For every } V_j \in V, 1 \leq j \leq i, \text{ no element of } D' \text{ contain } V_j; \\
(ii) & \quad \bigcup_{F_i \in D} F_i = \bigcup_{F_i' \in \bigcup_{s=2}^h C_s \cup D'} F_{i'}.
\end{align*}\]
Replace \( D \) by the set \( D' \) and apply the same arguments. From (i) we know that the algorithm finish. It finishes only if \( D' \) is a small circuit. So the theorem follows. \( \square \)

If \( k > 2 \), the converse of Theorem 5.2 is not true. Indeed consider the triangulation of a projective plane
\[
F_1 = 124, F_2 = 126, F_3 = 134, F_4 = 135, F_5 = 165, F_6 = 235, F_7 = 236, F_8 = 245, F_9 = 346, F_{10} = 456.
\]
Consider the 3-hyperclique complex \( \langle S_k \rangle = \langle F_1, F_2, \ldots, F_{10} \rangle \) on the set \{6\}. The simplicial matroid \( S_k^n(S_k) \) over a field \( \mathbb{F} \) of characteristic different of 2, does not have circuits and then it is (trivially) strongly triangulable. Every 2-face of a \( F_i \) is contained in another \( F_j, j \in \{1, \ldots, 10\}, j \neq i \). The facets of
\( \langle S_k \rangle \) are the sets \( F_1, F_2, \ldots, F_{10} \) and all the 2-faces of \( \langle S_k \rangle \) not contained in an \( F_i \). Then \( \langle S_k \rangle \) does not contain simplicial 2-faces and it is not D-perfect.

We remark that the cycle matroid of a non chordal graph can be triangulable. More generally we have:

**Proposition 5.3.** For any \( n, k, n - 3 \geq k \geq 2 \), there is a \( k \)-hyperclique complex \( \langle S_k \rangle \) such that:

(i) The simplicial matroid \( S^n_k(S_k) \) is triangulable but not strongly triangulable;

(ii) \( \langle S_k \rangle \) does not contain a simplicial \((k - 1)\)-face.

**Proof.** Let \( \langle S_k \rangle \) be the \( k \)-hyperclique complex where

\[
\begin{align*}
S_k &= \left( \binom{12 \cdots (k + 1)}{k} \right) \cup \left( \binom{23 \cdots (k + 2)}{k} \right) \setminus \left( \binom{23 \cdots (k + 2)}{k} \right) \setminus \left( \binom{12 \cdots (k + 1)}{k} \right) \\
&\quad \cup \bigcup_{i=1}^{k+1} \left( \binom{12 \cdots \hat{i} \cdots (k + 1)}{k} \right) \\
&\quad \cup \bigcup_{j=2}^{k+2} \left( \binom{23 \cdots \hat{j} \cdots (k + 2)}{k} \right).
\end{align*}
\]

The simplicial matroid \( S^n_k(S_k) \) has \( 2k \) small circuits,

\[
\begin{align*}
C_i &= \left( \binom{12 \cdots \hat{i} \cdots (k + 1)}{k} \right), \quad i \in \{1, 2, \ldots, k + 1\}, \\
C_j &= \left( \binom{23 \cdots \hat{j} \cdots (k + 2)}{k} \right), \quad j \in \{2, 3, \ldots, k + 2\}.
\end{align*}
\]

The set

\[
C := \left( \binom{12 \cdots (k + 1)}{k} \right) \cup \left( \binom{23 \cdots (k + 2)}{k} \right) \setminus \left( \binom{23 \cdots (k + 2)}{k} \right) \setminus \left( \binom{12 \cdots (k + 1)}{k} \right)
\]

is a circuit, symmetric difference of all the \( 2k \) small circuits. So, the simplicial matroid \( S^n_k(S_k) \) over a field of characteristic 2 is triangulable but not strongly triangulable. The reader can check that do not exist simplicial \((k - 1)\)-faces in \( \langle S_k \rangle \).

\[\square\]

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RAUL CORDOVIL
DEPARTAMENTO DE MATEMÁTICA
INSTITUTO SUPERIOR TÉCNICO
AV. ROVISCO PAIS
1049-001 LISBOA, PORTUGAL
E-mail address: cordovil@math.ist.utl.pt

MANOEL LEVOS
DEPARTAMENTO DE MATEMÁTICA
UNIVERSIDADE FEDERAL DE PERNAMBUCO
RECIFE, PERNAMBUCO CEP 50740-540 - BRASIL
E-mail address: manoel@mat.ufpe.br

CLÁUDIA LINHARES SALES
MDCC, DEPARTAMENTO DE COMPUTAÇÃO
UNIVERSIDADE FEDERAL DO CEARÁ - UFC
CAMPO DE PICI, BLOCO 910
FORTALEZA, CE, BRASIL
E-mail address: linhares@lia.ufc.br