New generalizations of BCI, BCK and Hilbert algebras

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Abstract

We introduce more generalizations of BCI, BCK and of Hilbert algebras, with proper examples, and show the hierarchies existing between all these algebras, old and new ones. Namely, we found thirty one new generalizations of BCI and BCK algebras and twenty generalizations of Hilbert algebras.

Keywords BCI algebra, BCK algebra, Hilbert algebra, BCH algebra, BCC algebra, BZ algebra, BE algebra, pre-BCK algebra, RM algebra, RML algebra

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Hilbert algebras are particular cases of BCK algebras [18], while BCK algebras are particular cases of BCI algebras.

Hilbert algebras were introduced in 1950, in a dual form, by Henkin [10], under the name “implicative model”, as a model of positive implicative propositional calculus - an important fragment of classical propositional calculus introduced by Hilbert [11], [12]. Cf. A. Diego [7], it was Antonio Monteiro who has given the name “Hilbert algebras” to the dual algebras of Henkin’s implicative models.

BCK algebras and BCI algebras were introduced in 1966 by K. Iséki [19], as algebraic models of BCK-logic and of BCI-logic, respectively. The axioms of the propositional calculus of the BCK-logic are the following:

(B) \((\psi \rightarrow \chi) \rightarrow ((\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \chi))\)
(C) \((\varphi \rightarrow (\psi \rightarrow \chi)) \rightarrow ((\psi \rightarrow \varphi) \rightarrow (\varphi \rightarrow \chi))\)
(K) \(\varphi \rightarrow (\psi \rightarrow \varphi)\).

The axioms of the propositional calculus of the BCI-logic are the above (B), (C) and the following (I):

(I) \(\varphi \rightarrow \varphi\).

Hundred of papers were written on BCK and BCI algebras, and the books [24] and [18] on BCK algebras and the book [15] on BCI algebras.

Most of the commutative algebras of logic (such as residuated lattices, Boolean algebras, MV algebras, Wajsberg algebras, BL algebras, Gödel algebras, product algebras, Hilbert algebras, Heyting algebras, NM algebras, MTL algebras, IMTL algebras, R₀ algebras, weak-R₀ algebras etc.) can be expressed as particular cases of BCK algebras (more precisely, of reversed left-BCK algebras) (see [18]).

The BCK algebras and the commutative groups are particular cases of BCI algebras.

Several generalizations of BCI and BCK algebras were introduced in time, namely:

BCH algebras were introduced in 1983 by Q.P. Hu and X. Li [13]; their examples of proper BCH algebras given in [14] are in fact BCI algebras. There are many papers on BCH algebras since then, but the exact connection between BCH algebras and BCI algebras was not found; we shall establish this connection in this paper.

BCC algebras, also called BIK⁺ algebras, were introduced in 1984 by Y. Komori [22], [23] (see [26]).

BZ algebras, also called weak-BCC algebras, were introduced in 1995 by X.H. Zhang and R. Ye [27].

BE algebras were introduced in 2006 by H.S. Kim and Y.H. Kim [21].

Pre-BCK algebras were introduced in 2010 by D. Buşneag and S. Rudeanu [2]. In pre-BCK algebras, the binary relation \(\leq\) is only a pre-order (i.e. reflexive and transitive). A BCK algebra is just a pre-BCK algebra verifying also the antisymmetry. And as it can be noticed from [2], a pre-BCK algebra is a BE...
In this paper, we introduce more generalizations of BCI, BCK and of Hilbert algebras, with proper examples, and show the hierarchies existing between all these algebras, old and new ones. Namely, we found 31 new generalizations of BCI and BCK algebras and 20 generalizations of Hilbert algebras. With these new generalizations, we better understand the BCI, BCK and the Hilbert algebras.

The paper is organized as follows:

In Section 1, we present a list of properties of BCK algebras which is the starting point of the research; we define the seven old algebras BCI, BCK, BCH, BCC, BZ, BE and pre-BCK in an unifying way by the properties of the list and we draw the hierarchy of the seven old algebras.

In Section 2, which is the core of the paper, we present connections between the properties in the list.

In Section 3, we present new equivalent definitions of BCI and BCK algebras coming from the corresponding logics.

In Section 4, first we find the connection between BCH and BCI algebras and then we introduce step by step the first nine new generalizations of BCI and BCK algebras: the pre-BCC, aBE, RM, pre-BZ, aRM, RME, pre-BCI, RML and aRML algebras. We present the connections between the sixteen old and new algebras. All the mentioned algebras are particular cases of RM and RML algebras.

In Section 5, we introduce other twenty two new RM and RML algebras, generalizations of BCI and BCK algebras respectively, and we present their connections with the previous defined algebras.

In section 6, we introduce twenty generalizations of Hilbert algebras; all are particular cases of RML algebras.

In Section 7, we define the proper algebras mentioned in the paper.

In Sections 8, 9, 10 we present examples of the proper old and new algebras defined in Sections 4, 5, 6 respectively.

In Section 11, we present final remarks.

1 The list. The seven old algebras. Hierarchy 0

Let $A = (A, \rightarrow, 1)$ be an algebra of type $(2,0)$ through this paper, where a binary relation $\leq$ can be defined by: for all $x, y$,

$$ x \leq y \iff x \rightarrow y = 1. $$

Equivalently, let $A = (A, \leq, \rightarrow, 1)$ be a structure where $\leq$ is a binary relation on $A$, $\rightarrow$ is a binary operation on $A$ and $1 \in A$, all connected by: $x \leq y \iff x \rightarrow y = 1$.

1.1 The list of properties

Consider the following list of properties that can be satisfied by $A$ (in fact, the properties in the list are the most important properties satisfied by a BCK algebra), where each property is presented in two equivalent forms, determined by the corresponding two equivalent above definitions of $A$:

- (An) (Antisymmetry) $x \rightarrow y = 1 \Rightarrow x = y$,
- (An') (Antisymmetry) $x \leq y, y \leq x \Rightarrow x = y$;
- (B) $(y \rightarrow z) \rightarrow [(x \rightarrow y) \rightarrow (x \rightarrow z)] = 1$,
- (B') $y \rightarrow z \leq (x \rightarrow y) \rightarrow (x \rightarrow z)$,
- (BB) $(y \rightarrow z) \rightarrow [(z \rightarrow x) \rightarrow (y \rightarrow x)] = 1$,
- (BB') $y \rightarrow z \leq (z \rightarrow x) \rightarrow (y \rightarrow x)$;
- (*) $y \rightarrow z = 1 \Rightarrow (x \rightarrow y) \rightarrow (x \rightarrow z) = 1$,
- (*') $y \leq z \Rightarrow x \rightarrow y \leq x \rightarrow z$.
(**) $y \rightarrow z = 1 \implies (z \rightarrow x) \rightarrow (y \rightarrow x) = 1$,
(***$) y \leq z \implies z \rightarrow x \leq y \rightarrow x$;
(C) $[x \rightarrow (y \rightarrow z)] \rightarrow [y \rightarrow (x \rightarrow z)] = 1$,
(C') $x \rightarrow (y \rightarrow z) \leq y \rightarrow (x \rightarrow z)$;
(D) $y \rightarrow [(y \rightarrow x) \rightarrow x] = 1$,
(D') $y \leq (y \rightarrow x) \rightarrow x$;
(Ex) (Exchange) $x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z)$;
(K) $x \rightarrow (y \rightarrow x) = 1$,
(K') $x \leq y \rightarrow x$;
(L) (Last element) $x \rightarrow 1 = 1$,
(L') (Last element) $x \leq 1$;
(M) $1 \rightarrow x = x$;
(N) $1 \rightarrow x = 1 \implies x = 1$,
(N') $1 \leq x \implies x = 1$;
(Re) (Reflexivity) $x \rightarrow x = 1$ (we prefer here notation (Re) instead of original (I)),
(Re') (Reflexivity) $x \leq x$;
(S) $x = y \implies x \rightarrow y = 1$,
(S') $x = y \implies x \leq y$;
(Tr) (Transitivity) $x \rightarrow y = 1 = y \rightarrow z \implies x \rightarrow z = 1$,
(Tr') (Transitivity) $x \leq y, y \leq z \implies x \leq z$;
(U) $((y \rightarrow x) \rightarrow x) \rightarrow x = y \rightarrow x$.

Remark 1.1 Consider the following property:
(MP) (Modus Ponens) $x = 1$ and $x \rightarrow y = 1 \implies y = 1$,
(MP') (Modus Ponens) $x = 1$ and $x \leq y \implies y = 1$.
Note that (MP) is in fact a reformulation of (N), and vice-versa.

1.2 The seven old algebras: BCI, BCK, BCH, BCC, BZ, BE, pre-BCK

Recall now the following definitions:

Definition 1.2 An algebra $(A, \rightarrow, 1)$ is a:
O1. BCI algebra if it verifies the axioms (BB), (D), (Re), (N), (An) or, equivalently, (BB), (D), (Re), (An), or, equivalently, (BB), (M), (An).
O2. BCK algebra if it verifies the axioms (BB), (D), (Re), (L), (An), or, equivalently, (BB), (M), (L), (An).
O3. BCH algebra if it verifies the axioms (Re), (Ex), (An).
O4. BCC algebra if it verifies the axioms (Re), (M), (L), (B), (An).
O5. BZ algebra if it verifies the axioms (Re), (M), (B), (An).
O6. BE algebra if it verifies the axioms (Re), (M), (L), (Ex).
O7. pre-BCK algebra if it verifies the axioms (Re), (M), (L), (Ex), (*).
Note that, obviously, there are equivalent definitions as structures \((A, \leq, \to, 1)\).

It is known that the binary relation \(\leq\) is an order relation in BZ, BCI, BCC and BCK algebras and that it is only a pre-order in pre-BCK algebras.

Denote by BCI, BCK, BCH, BCC, BZ, BE, pre-BCK the classes of BCI, BCK, BCH, BCC, BZ, BE, pre-BCK algebras respectively.

It is known that:

\[
\begin{align*}
\text{BCI} & \subseteq \text{BCH}; \\
\text{BCI} + (L) = \text{BCK}; & \quad \text{BCC} + (\text{Ex}) = \text{BCK}; \\
\text{BZ} + (\text{Ex}) = \text{BCI}; & \quad \text{BZ} + (L) = \text{BCC}; \\
\text{BE} + (*) = \text{pre-BCK}; & \quad \text{pre-BCK} + (\text{An}) = \text{BCK}.
\end{align*}
\]

It is known that:

**Proposition 1.3** In any BCI algebra, the following properties hold: (Ex), (U), (B), (M), (*) and (**).

### 1.3 Hierarchy 0, of the seven old algebras

We introduce now the following definition.

**Definition 1.4** Let \(A = (A, \to, 1)\) be an algebra of type \((2,0)\) and let \(\leq\) be the associated binary relation defined by \((\#)\).

1. We shall say that \(A\) is **reflexive** if \(\leq\) is reflexive (i.e. it satisfies property (Re) or (Re')).
2. We shall say that \(A\) is **antisymmetrical** if \(\leq\) is antisymmetrical (i.e. it satisfies property (An) or (An')).
3. We shall say that \(A\) is **transitive** if \(\leq\) is transitive (i.e. it satisfies property (Tr) or (Tr')).
4. We shall say that \(A\) is **pre-ordered** if \(\leq\) is a pre-order relation (i.e. it is reflexive and transitive).
5. We shall say that \(A\) is **ordered** if \(\leq\) is a partial-order relation (i.e. it is reflexive, antisymmetrical and transitive).
6. We shall say that \(A\) is a **lattice** if \(\leq\) is a lattice-order relation (i.e. it is a partial-order such that there exists \(\text{sup}(x,y)\) and \(\text{inf}(x,y)\) for each \(x, y \in A\); we shall use the notation \(x \lor y\) for \(\sup(x,y)\) and \(x \land y\) for \(\inf(x,y)\), with \(x \leq y \iff x \lor y = y \iff x \land y = x\).

**Remark 1.5** In a hierarchy of classes of algebras, we shall represent:
- a class of **reflexive** algebras by \(\bigcirc\)
- a class of **antisymmetrical** algebras by \(\bigcirc\)
- a class of **transitive** algebras by \(\bullet\)

Consequently, we shall represent:
- a class of algebras where \(\leq\) is **reflexive and antisymmetrical** and
- a class of **pre-ordered** algebras (i.e. \(\leq\) is reflexive and transitive), respectively, by:

\[
\begin{align*}
\bigcirc & \quad \bigcirc
\end{align*}
\]

**2 Connections between the properties in the list**

### 2.1 Connections

We shall establish several connections between the properties in the above list. These connections are the core of the paper. Based on them, we then introduced the new generalizations claimed in the title of the
Figure 1: Hierarchy 0 of the seven old algebras

 Proposition 2.1 Let \((A, \to, 1)\) be an algebra of type \((2,0)\). Then the following are true:

\begin{enumerate}
    
    \item (Re) implies (S);
    \item (M) implies (N);
    \item (L) + (An) imply (N);
    \item (K) + (An) imply (N);
    \item (C) + (An) imply (Ex);
    \item (Ex) + (Re) imply (C);
    \item (Re) + (Ex) imply (D);
    \item (Re) + (Ex) + (An) imply (M);
    \item (Re) + (Ex) + (An) imply (N);
    \item (Re) + (K) imply (L);
    \item (N) + (K) imply (L);
    \item (Re) + (L) + (Ex) imply (K);
    \item (M) + (L) + (B) imply (K);
    \item (M) + (L) + (**) imply (K);
    \item (Ex) implies (B) \iff (BB);
    \item (Ex) + (B) imply (BB);
    \item (Ex) + (BB) imply (B);
    \item (Re) + (Ex) + (*) imply (BB);
\end{enumerate}
((12.) (N) + (B) imply (*); (12') (M) + (B) imply (*);
(13.) (N) + (*) imply (Tr); (13') (M) + (*) imply (Tr);
(14.) (N) + (B) imply (Tr); (14') (M) + (B) imply (Tr);

(15.) (N) + (BB) imply (**); (15') (M) + (BB) imply (**);
(16.) (N) + (** imply (Tr); (16') (M) + (** imply (Tr);
(17.) (N) + (BB) imply (Tr); (17') (M) + (BB) imply (Tr);

(18.) (M) + (BB) imply (Re); (18') (M) + (BB) imply (D);
(19.) (M) + (B) imply (Re);

(20.) (BB) + (D) + (N) imply (C); (20') (M) + (BB) imply (C);
(21.) (BB) + (D) + (N) + (An) imply (Ex);
(21') (BB) + (D) + (L) + (An) imply (Ex); (21'') (M) + (BB) + (An) imply (Ex);

(22.) (B) + (C) + (K) + (An) imply (Re);
(23.) (BB) + (D) + (Re) + (An) imply (N);

(24.) (Re) + (Ex) + (Tr) imply (**).

Proof. (0.): Suppose \( x = y \); then \( x \to y = y \to y = (Re) 1 \); thus, (S) holds.
(00.): Suppose \( 1 \to x = 1 \). Then, by (M), we get \( x = 1 \), i.e. (N) holds.
(1.): Suppose \( 1 \to x = 1 \). By (L), we also have \( x \to 1 = 1 \). Hence, by (An), we get \( x = 1 \), i.e. (N) holds.

(2.): Suppose \( 1 \to x = 1 \); by (K), we have \( x \to (1 \to x) = 1 \), then \( x \to 1 = 1 \). Hence, by (An), \( x = 1 \), i.e. (N) holds.

(3.): By (C), we have: \( [x \to (y \to z)] \to [y \to (x \to z)] \) = 1 and also \( [y \to (x \to z)] \to [x \to (y \to z)] \) = 1; hence, by (An), we obtain that: \( x \to (y \to z) = y \to (x \to z) \), i.e. (Ex) holds.

(3'): By (Ex), we have:

\[
x \to (y \to z) = y \to (x \to z);
\]

(1)

by (0.), (Re) implies (S); hence, by (S), (1) implies \( x \to (y \to z) \to [y \to (x \to z)] = 1 \), i.e. (C) holds.

(4.): \( y \to [(y \to x) \to z] = (y \to x) \to y = (Re) 1 \), i.e. (D) holds.

(5.): \( x \to (1 \to x) = (Ex) 1 \to (x \to x) = 1 \), i.e. (Re) holds.

We shall prove now that \( (1 \to x) \to x = 1 \) also. Indeed, by (5), (D) holds, consequently, \( 1 \to [(1 \to x) \to x] = 1 \). Since by (4), (N) holds too, we obtain that \((1 \to x) \to x = 1 \).

Applying now (An), we obtain that \( 1 \to x = x \), i.e. (M) holds.

(5'): By above (5), (Re) + (Ex) + (An) imply (M); and by above (0), (M) implies (N); hence, (Re) + (Ex) + (An) imply (N).

(6.): In (K) \( x \to (y \to x) = 1 \), take \( y = x \): we get \( 1 = x \to (x \to x) = (Re) x = 1 \), i.e. (L) holds.

(7.): By (K), we have: \( 1 \to (x \to 1) = 1 \); hence, by (N), we obtain that \( x \to 1 = 1 \), i.e. (L) holds.

(7'): By above (00.), (M) implies (N); and by (7), (N) + (K) imply (L); thus, (L) holds.

(8.): In (Ex) \( (x \to (y \to z) = y \to (x \to z)) \), take \( z = x \): we get \( x \to (y \to x) = y \to (x \to x) = (Re) y \to 1 \equiv 1 \), i.e. (K) holds.

(9.): Take \( y = 1 \) in (B) \( (y \to z) \to [(x \to y) \to (x \to z)] = 1 \); we obtain: \( (1 \to z) \to [(x \to 1) \to (x \to z)] = 1 \); then, by (M), we obtain: \( z \to [(x \to 1) \to (x \to z)] = 1 \); then by (L) and (M) again, we obtain \( z \to (x \to z) = 1 \), i.e. (K) holds.

(9'): By (L'), \( y \leq 1 \) is true and hence, by (**), we get: \( 1 \to x \leq y \to x \), which by (M) means that \( x \leq y \to x \), i.e. (K') holds.

(10.): \( (y \to z) \to [(x \to y) \to (x \to z)] = (Ex) (x \to y) \to [(y \to z) \to (x \to z)] \). Hence, (B) \( \leftrightarrow \) (BB).

(10') Obviously, by (10).

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Then, by (Ex), we obtain:

\[(x \to y) \to [(y \to z) \to (x \to z)] = 1,\]

i.e. (BB) holds.

(12.): Suppose \(y \to z = 1\); then, by (B) \(((y \to z) \to (x \to y) \to (x \to z)] = 1\), it follows that 
\(1 \to [(x \to y) \to (x \to z)] = 1\); hence, by (N), we obtain that \(x \to y) \to (x \to z)] = 1\), i.e. (*) holds.

(12'): By (00.), (M) implies (N); then apply above (12).

(13.): Suppose \(x \to y = 1 = y \to z\), or \(y \to z = 1 = x \to y\); then, by (\(\ast\)), we obtain: \(1 \to (x \to z)] = 1\);

hence by (N), we obtain: \(x \to z = 1\), i.e. (Tr) holds.

(13'): By (00.), (M) implies (N); then apply (13).

(14.): Suppose \(y \to z = 1\) and \(x \to y = 1\); then, by (B), we obtain: \(1 \to [1 \to (x \to z)] = 1\); then, by (\(\ast\)), we obtain: 1 \to (x \to z)] = 1\); then, by (N), we obtain: \(1 \to (x \to z)] = 1\); by (N) again, we obtain: \(x \to z = 1\). Thus, (Tr) holds.

(14'): By (00.), (M) implies (N); then apply (14).

(15.): Suppose \(y \to z = 1\); then, by (BB) \(((y \to z) \to ([z \to x] \to (y \to x)] = 1\), it follows that 
\(1 \to [(z \to x) \to (y \to x)] = 1\); hence, by (N), we obtain \((z \to x) \to (y \to x)] = 1\), i.e. (**) holds.

(15'): By (00.), (M) implies (N); then apply (15).

(16.): Suppose \(y \to z = 1 = z \to x\); then, by (**), we obtain: \(1 \to (y \to x)] = 1\); hence, by (N), we obtain: \(y \to x = 1\), i.e. (Tr) holds.

(16'): By (00.), (M) implies (N); then apply (16).

(17.): Suppose \(y \to z = 1\) and \(z \to x = 1\); then, by (BB), we obtain: \(1 \to [1 \to (y \to x)] = 1\); then, applying (N) twice, we obtain \(y \to x = 1\). Thus, (Tr) holds.

(17'): By (00.), (M) implies (N); then apply (17).

(18.): In (BB) \(((y \to z) \to ([z \to x] \to (y \to x)] = 1\), take \(y = z = 1\); we obtain: \((1 \to 1) \to [(1 \to x] \to (1 \to x)] = 1\), hence, by (M), \(1 \to [x \to x] = 1\), hence by (M) again, \(x \to x = 1\), i.e. (Re) holds.

(18'): In (BB) \(((y \to z) \to ([z \to x] \to (y \to x)] = 1\), take \(y = 1\); we obtain: \((1 \to z) \to [(z \to x) \to (1 \to x)] = 1\), i.e. \(z \to [(z \to x] \to x)] = 1\), by (M); thus, (D) holds.

(19.): In (B) \(((y \to z) \to ([x \to y] \to (x \to z)] = 1\), take \(x = y = 1\); we obtain: \((1 \to z) \to [(1 \to 1) \to (1 \to z)] = 1\), hence, by (M), \(z \to [1 \to z] = 1\), by (M) again, \(z \to z = 1\); thus, (Re) holds.

(20.): (see [20], Theorem 1) We shall use \(\leq\) for a better understanding.

By (BB'), we have: \(y \to z \leq (z \to x) \to (y \to x)\). By (15.), (BB) + (N) imply (**). Then, by (**), we obtain:

\![z \to x) \to (y \to x)] \to u \leq (y \to z) \to u. \tag{2}\]

We substitute in (2): \(x \to y \to x, z \to y \to x, u \to y \to (u \to x)\). Then, we obtain:

\[V_{notation} \leq \leq (y \to (z \to x)] \to (u \to z) \to (y \to (u \to x))\] notation \(W.\)

Then, the left side \(V = 1, \text{ by (2) (with } Y = u, U = y \to (u \to x))\). Thus, \(1 \leq W\); then, by (N'), \(W = 1\), i.e.

\[y \to (z \to x) \leq (u \to z) \to (y \to (u \to x)). \tag{3}\]

Take now \(u = z, z = y \to x \in (3). \text{ Then, we obtain:}

\[y \to ((y \to x) \to x) \leq [z \to (y \to x)] \to [y \to (z \to x)]. \tag{4}\]
By (D), from (4) we obtain:

\[ 1 \leq [z \to (y \to x)] \to [y \to (z \to x)]. \tag{5} \]

From (4), we obtain, by (N') again, that: \([z \to (y \to x)] \to [y \to (z \to x)] = 1\), i.e. (C) holds.

(20'): (M) implies (N), by above (00.), and (M) + (BB) imply (D), by above (18'). Hence, (M) + (BB) imply (BB) + (D) + (N), which imply (C), by above (20).

(21): By above (20.), (BB) + (D) + (N) imply (C); and (C) implies that:

\[ [z \to (y \to x)] \to [y \to (z \to x)] = 1. \tag{6} \]

and also that:

\[ [y \to (z \to x)] \to [z \to (y \to x)] = 1. \tag{7} \]

From (6) and (7), by applying (An), we obtain that \(y \to (z \to x) = z \to (y \to x)\), i.e. (Ex) holds.

(21').: By above (1.), (L) + (An) imply (N); then apply (21.).

(21''): By above (20.'), (M) + (BB) imply (C); by above (3.), (C) + (An) imply (Ex); thus, (M) + (BB) + (An) imply (Ex).

(22): Take \(z = 1\) and \(y = x\) in (B) \([(y \to z) \to [(x \to y) \to (x \to z)] = 1\); we obtain:

\[ (x \to 1) \to [(x \to x) \to (x \to 1)] = 1. \tag{8} \]

By above (2.), (K) + (An) imply (N); by above (7.), (N) + (K) imply (L); by above (3.), (C) + (An) imply (Ex). Then, from (8), we obtain, by (L):

\[ 1 \to [(x \to x) \to 1] = 1. \tag{9} \]

From (9), by (N), we obtain:

\[ (x \to x) \to 1 = 1. \tag{10} \]

On the other hand, \(1 \to (x \to x) \overset{(Ex)}{=} x \to (1 \to 1) \overset{(K)}{=} 1\), hence

\[ 1 \to (x \to x) = 1. \tag{11} \]

From (10) and (11), by applying (An), we obtain \(x \to x = 1\), i.e. (Re) holds.

(23): Suppose that:

\[ 1 \to x = 1; \tag{12} \]

we must prove that \(x = 1\). Indeed, in (BB) \([(y \to z) \to [(z \to x) \to (y \to x)] = 1\), take \(y = x\) and \(z = 1\); we obtain: \((x \to 1) \to [(1 \to x) \to (x \to x)] = 1\), hence, by (12) and (Re), \((x \to 1) \to [1 \to 1] = 1\), hence, by (Re) again,

\[ (x \to 1) \to 1 = 1. \tag{13} \]

Since by (D) we have: \(x \to [(x \to 1) \to 1] = 1\), it follows, by (13), that:

\[ x \to 1 = 1. \tag{14} \]

From (12), (14) and (An), we obtain \(x = 1\). Thus, (N) holds.

(24): Suppose (Tr) holds, i.e. \(X \to Y = 1, Y \to Z = 1\) imply \(X \to Z = 1\).

We must prove that (***) holds, i.e. \(y \to z = 1\) implies \((z \to x) \to (y \to x) = 1\).

Suppose that \(y \to z = 1\); we must prove that \(H =_{notation} (z \to x) \to (y \to x) = 1\).

Indeed, \(H \overset{(Ex)}{=} y \to ((z \to x) \to x)\). Take \(X = y\), \(Z = (z \to x) \to x\) and \(Y = z\); then we have \(H = X \to Z\) and:

\(X \to Y = y \to z = 1\), by hypothesis;

\(Y \to Z = z \to [(z \to x) \to x] \overset{(Ex)}{=} (z \to x) \to (z \to x) \overset{(Re)}{=} 1\).

Hence, by (Tr), it follows that \(X \to Z = 1\), i.e. \(H = 1\).

Now we prove two important results.
**Theorem 2.2** (Generalization of ([2], Lemma 1.2 and Proposition 1.3))

If an algebra \( (A, \to, 1) \) verifies properties (Re), (M), (Ex), then:

\[
(B) \Leftrightarrow (BB) \Leftrightarrow (*).
\]

**Proof.**

By Proposition 2.1 (9.), (Ex) implies that \((B) \Leftrightarrow (BB)\).

By Proposition 2.1 (11.), \((M) + (B) \implies (*).\)

By Proposition 2.1 (10.), \((Re) + (Ex) + (*) \implies (BB).\) Hence, we have:

\[
(*) \implies (BB) \Leftrightarrow (B) \implies (*),
\]

thus \((B) \Leftrightarrow (BB) \Leftrightarrow (*).\)

\( \square \)

**Theorem 2.3**

If an algebra \( (A, \to, 1) \) verifies properties (Re), (M), (Ex), then:

\[
(**) \Leftrightarrow (Tr).
\]

**Proof.**

By Proposition 2.1 (16'), \((M) + (** \implies (Tr).\)

By Proposition 2.1 (24), \((Re) + (Ex) + (Tr) \implies (**).\)

\( \square \)

**Theorem 2.4**

If an algebra \( (A, \to, 1) \) verifies properties (Re), (M), (B), (An), then:

\[
(Ex) \Leftrightarrow (BB).
\]

**Proof.** By Proposition 2.1 (10.), \((Ex) \implies (B) \Leftrightarrow (BB);\) hence, \((Ex) \implies (BB).\)

By Proposition 2.1 (21".), \((M) + (BB) + (An) \implies (Ex),\) hence \((BB) \implies (Ex).\)

It follows immediately that:

**Corollary 2.5** In BZ, BCC algebras we have \((Ex) \Leftrightarrow (BB).\)

The next theorem was proved by professor Michael Kinyon, Department of Mathematics, University of Denver, following our open problems announced in the preprint on arXiv; he proved (i) first by using the automated theorem proving tool Prover9.

**Theorem 2.6** (Michael Kinyon) In any algebra \( (A, \to, 1) \) we have:

(i) \((M) + (BB) \implies (B),\)

(ii) \((M) + (B) \implies (**).\)

**Proof.** (i): By Proposition 2.1 (18'), we have \((M) + (BB) \implies (D).\)

Next, if \((BB)\) is

\[
(x \to y) \to [(y \to z) \to (x \to z)] = 1,
\]

in \((BB)\) set \(x = u\) and \(y = (u \to v) \to v,\) to get:

\[
(u \to [(u \to v) \to v]) \to [[[u \to v) \to v] \to z] \to (u \to z)] \overset{(D)}{=} 1 \to [[[u \to v) \to v] \to z] \to (u \to z)];
\]

\[
(((u \to v) \to v) \to z) \to (u \to z) = 1.
\]

After renaming variables, we get:

\[
((x \to y) \to y) \to z \to (x \to z) = 1.
\]

Next, in \((BB)\) set \(x = u \to v\) and \(y = (v \to w) \to (u \to w),\) to get:

\[
((u \to v) \to [(v \to w) \to (u \to w)]) \to [[[v \to w) \to (u \to w)] \to z] \to ((u \to v) \to z)] \overset{(BB)}{=} 1 \to [[[v \to w) \to (u \to w)] \to z] \to ((u \to v) \to z)] \overset{(M)}{=} \]

\[
1 \to [[[v \to w) \to (u \to w)] \to z] \to ((u \to v) \to z) = 1.
\]
If an algebra \( A \) is bounded; it will be then denoted by \((A, \to, 1)\)

\[ (((v \to w) \to (u \to w)) \to z) \to ((u \to v) \to z) = 1. \]

After renaming variables, we get:

\[ ((x \to y) \to (u \to y)) \to (u \to x) \to z) = 1. \]

Taking \( z = u \to y \) in (b), we get:

\[ ((x \to y) \to (u \to y)) \to (u \to x) \to (u \to y) = 1. \]

Now, in (a) set \( x = v \to w, y = t \to w, z = (t \to v) \to (t \to w) \) to get:

\[ (((v \to w) \to (t \to w)) \to (t \to w)) \to ((t \to v) \to (t \to w)) \to ((v \to w) \to ((t \to v) \to (t \to w))) = 1 \]

(iii) If \( (v \to w) \to ((t \to v) \to (t \to w)) = 1 \), i.e. (B) holds.

(ii): Suppose (B) is

\[ (y \to z) \to [(x \to y) \to (x \to z)] = 1. \]

If \( x \to y = 1 \), then we get from (B):

\[ (y \to z) \to [1 \to (x \to z)] = (y \to z) \to (x \to z) = 1, \text{ i.e. } (** \text{ holds).} \]

2.2 Generalities

Let as above \( A = (A, \to, 1) \) verifying (\#) or \( A = (A, \leq, \to, 1) \).

Definition 2.7

(i) If there exists an element \( 0 \in A \) such that \( 0 \leq x \) for all \( x \in A \), then we shall call it zero of \( A \).

(ii) If there exists a zero of \( A \) and it is unique, denote it by \( 0 \), and if property (L) holds, then we shall say that \( A \) is bounded; it will be then denoted by \((A, \to, 0, 1)\).

(iii) If \( A = (A, \to, 0, 1) \) is bounded, then we define a new operation \( \neg \) on \( A \), called negation, by: for all \( x \in A \),

\[ x^- \text{ def. } x \to 0. \]

(iv) If the negation \( \neg \) verifies the property:

\[ \text{(DN) (Double negation) for all } x \in A, \ (x^-)^- = x, \]

we shall say that \( A \) is involutive or with (DN).

Proposition 2.8 If an algebra \( (A, \to, 1) \) verifies (Ex), then:

\[ (G1) \quad x \to y^- = y \to x^- . \]

Proof. \( x \to y^- = x \to (y \to 0) \overset{(Ex)}{=} y \to (x \to 0) = y \to x^- . \)

Proposition 2.9 If a bounded algebra \( (A, \to, 0, 1) \) verifies (Ex) and (DN), then:

\[ (G2) \quad x \to y = y^- \to x^-, \]

\[ (G3) \quad y^- \to x = x^- \to y. \]

Proof. \( x \to y \overset{(DN)}{=} x \to (y^-)^- \overset{(G1)}{=} y \to x^- . \)

\( y^- \to x \overset{(DN)}{=} y^- \to (x^-)^- \overset{(G2)}{=} x^- \to y. \)

Proposition 2.10 If an algebra \( (A, \to, 1) \) verifies (D), then:

\[ (G4) \quad x \leq (x^-)^-. \]

Proof. \( (x^-)^- = (x \to 0) \to 0 \overset{(D)}{=} x . \)
Proposition 2.11 If an algebra \((A, \rightarrow, 1)\) verifies \((BB)\), then:

\[(G5) \quad x \rightarrow y \leq y^{-} \rightarrow x^{-}.
\]

Proof. \(y^{-} \rightarrow x^{-} = (y \rightarrow 0) \rightarrow (x \rightarrow 0) \geq x \rightarrow y. \square\)

Proposition 2.12 If an algebra \((A, \rightarrow, 1)\) verifies \((**)\), then:

\[(G6) \quad x \leq y \implies y^{-} \leq x^{-}.
\]

Proof. \(x \leq y \overset{(**)}{\implies} (y \rightarrow 0) \leq (x \rightarrow 0)\), i.e. \(x \leq y \implies y^{-} \leq x^{-}. \square\)

Proposition 2.13 If a bounded algebra \((A, \rightarrow, 0, 1)\) verifies \((**)\) and \((DN)\), then:

\[(G7) \quad x \leq y \iff y^{-} \leq x^{-}.
\]

Proof. We have by \((G6)\) \(x \leq y \implies y^{-} \leq x^{-}\) and, similarly, \(y^{-} \leq x^{-} \implies (x^{-})^{-} \leq (y^{-})^{-}\), i.e. \(y^{-} \leq x^{-} \implies x \leq y\), by \((DN)\). Hence, \(x \leq y \iff y^{-} \leq x^{-}. \square\)

Proposition 2.14 If an algebra \((A, \rightarrow, 1)\) verifies \((U)\), then:

\[(G8) \quad ((x^{-})^{-})^{-} = x^{-}.
\]

Proof. \(((x^{-})^{-})^{-} = ((x \rightarrow 0) \rightarrow 0) \overset{(U)}{=} x \rightarrow 0 = x^{-}. \square\)

3 New equivalent definitions of BCI and of BCK algebras, coming from logic

3.1 A new equivalent definition of BCI algebras coming from logic

We shall present now a new equivalent definition of BCI algebras, starting from the axioms \((B)\), \((C)\), \((I)\) of the BCI-logic.

Proposition 3.1 Let \(\mathcal{A} = (A, \rightarrow, 1)\) be an algebra of type \((2,0)\) verifying the following axioms: for all \(x, y, z \in A\),

\[(B) \quad (y \rightarrow z) \rightarrow ((x \rightarrow y) \rightarrow (x \rightarrow z)) = 1,
\]

\[(C) \quad (x \rightarrow (y \rightarrow z)) \rightarrow (y \rightarrow (x \rightarrow z)) = 1,
\]

\[(Re) \quad x \rightarrow x = 1,
\]

\[(An) \quad x \rightarrow y = 1 = y \rightarrow x \text{ imply } x = y,
\]

where the name \((Re)\), coming from "Reflexivity", is used instead of the initial name \((I)\) for the corresponding logical axiom of BCI logic.

Then \(\mathcal{A}\) is a BCI-algebra.

Proof. We shall prove that axioms \((BB)\), \((D)\), \((Re)\), \((N)\), \((An)\) from the first definition of BCI algebras hold. Indeed,

- by Proposition 2.1 (3.), \((C)\) + \((An)\) imply \((Ex)\); then, by Proposition 2.1 (10'), \((B)\) + \((Ex)\) imply \((BB)\);
  thus, \((BB)\) holds;
- by Proposition 2.1 (4.), \((Re)\) + \((Ex)\) imply \((D)\); thus, \((D)\) holds;
- by Proposition 2.1 (5'), \((Re)\) + \((Ex)\) + \((An)\) imply \((N)\); thus, \((N)\) holds. \square

Proposition 3.2 A BCI algebra verifies \((B)\), \((C)\), \((Re)\), \((An)\).
Proof. Suppose that axioms (BB), (D), (Re), (N), (An) from the first definition of BCI algebras hold. Then:
- by Proposition 2.1 (20.), (BB) + (D) + (N) imply (C); thus, (C) holds;
- by Proposition 2.1 (3.), (C) + (An) imply (Ex); then, by Proposition 2.1 (10”), (Ex) + (BB) imply (B); thus, (B) holds.

By Propositions 3.1 and 3.2, an algebra \((A, \rightarrow, 1)\) verifying the axioms (B), (C), (Re), (An) is an equivalent definition, coming from BCI logic, of a BCI algebra. Hence, we have the following

**Theorem 3.3** An algebra \((A, \rightarrow, 1)\) of type \((2,0)\) is a BCI algebra if and only if properties (B), (C), (Re), (An) are satisfied.

### 3.2 A new equivalent definition of BCK algebras coming from logic

We shall present now a new equivalent definition of BCK algebras, starting from the axioms (B), (C), (K) of the BCK-logic.

**Proposition 3.4** Let \(\mathcal{A} = (A, \rightarrow, 1)\) be an algebra of type \((2,0)\) verifying the following axioms: for all \(x, y, z\) \(\in A\),

- \((B)\) \((y \rightarrow z) \rightarrow ((x \rightarrow y) \rightarrow (x \rightarrow z)) = 1\),
- \((C)\) \((x \rightarrow (y \rightarrow z)) \rightarrow (y \rightarrow (x \rightarrow z)) = 1\),
- \((K)\) \(x \rightarrow (y \rightarrow x) = 1\),
- \((An)\) \(x \rightarrow y = 1 = y \rightarrow x\) imply \(x = y\).

Then \(\mathcal{A}\) is a BCK-algebra.

**Proof.** We shall prove that axioms (BB), (D), (Re), (L), (An) from the first definition of BCK algebras hold. Indeed,

- by Proposition 2.1 (2.), (K) + (An) imply (N) and (N) + (K) imply (L); thus, (L) holds.
- by Proposition 2.1 (3.), (C) + (An) imply (Ex) and by Proposition 2.1 (10”), (Ex) + (B) imply (BB); thus, (BB) holds.
- by Proposition 2.1 (22.), (B) + (C) + (K) + (An) imply (Re); thus, (Re) holds.
- Proposition 2.1 (4.), (Re) + (Ex) imply (D); thus, (D) holds.
- \((V)’\) is (An).

We obviously have:

**Proposition 3.5** A BCK algebra verifies properties (B), (C), (K), (An).

By Propositions 3.4 and 3.5, an algebra \((A, \rightarrow, 1)\) verifying the axioms (B), (C), (K), (An) is an equivalent definition, coming from BCK logic, of a BCK algebra. Hence, we have the following

**Theorem 3.6** An algebra \((A, \rightarrow, 1)\) of type \((2,0)\) is a BCK algebra if and only if properties (B), (C), (K), (An) are satisfied.

### 4 The first nine new generalizations of BCI and BCK algebras

#### 4.1 Hierarchy 1, including the old algebras BZ, BCC, BCH, BCI, BCK

It is known that:

**Proposition 4.1** In any BCH algebra, the following properties hold [3]: (D), (M), (N) (see Proposition 2.1 (5), (6), (4) respectively).

We obtain immediately the following corollary (which gives an equivalent, longer definition of BCH algebras):
Corollary 4.2 For any algebra \( (A, \to, 1) \), we have the equivalence:

\[
(Re) + (Ex) + (An) \iff (Re) + (M) + (Ex) + (An).
\]

We prove now the following important result:

Theorem 4.3

\[
\text{BCH} + (B) = \text{BCI}.
\]

Proof.

We must prove that (using the first definition of BCI algebras):

\[
((Re) + (Ex) + (An)) + (B) \iff (BB) + (D) + (Re) + (N) + (An).
\]

\( \Longrightarrow \): Suppose \( (Re) + (Ex) + (An) + (B) \) holds. By Proposition 2.1 (10'), \( (Ex) + (B) \Rightarrow (BB) \). By Proposition 2.1 (4.), \( (Re) + (Ex) \Rightarrow (D) \). By Proposition 2.1 (5'), \( (Re) + (Ex) + (An) \Rightarrow (N) \). Thus, \( (BB) + (D) + (Re) + (N) + (An) \) holds.

\( \Longleftarrow \): Suppose now that \( (BB) + (D) + (Re) + (N) + (An) \) holds. By Proposition 2.1 (21.), \( (BB) + (D) + (An) \) imply \( (Ex) \). By Proposition 2.1 (10'',) \( (Ex) + (BB) \Rightarrow (B) \).

\( \square \)

Remark 4.4 We also have, by Proposition 2.1 (10.):

\[
\text{BCH} + (BB) = \text{BCI}.
\]

It follows, based on Hierarchy 0 (from above Figure 1), that we have the hierarchy, called Hierarchy 1, from Figure 2, where three new algebras, denoted for the moment by “name-1”, “name-2”, “name-3”, have to exist. Note that since property (B) implies transitivity (Tr), it follows that the BCH algebras and these three new algebras are not transitive.
4.2 Hierarchy 2, including the old algebras BE, pre-BCK, BCC, BCK

Recall that pre-BCK algebra is a BE algebra verifying property (*). By ([2], Proposition 1.3), an equivalent definition of pre-BCK algebras is given by the axioms: (M), (L), (Ex), (BB). By ([2], Corollary 1.2), the class of pre-BCK algebras is equational. As the name tells us, pre-BCK algebras are reflexive and transitive algebras, i.e. the binary relation \( \preceq \) defined by \( x \preceq y \Longleftrightarrow x \rightarrow y = 1 \) is a pre-order relation ([2], Corollary 1.1).

By ([2], Proposition 1.2'), BCK algebras are the same as pre-BCK algebras that satisfy property (An). By ([2], Proposition 1.4), an algebra \((A, \rightarrow, 1)\) is a BCK algebra if and only if it is both a pre-BCK algebra and a BCC algebra.

Consequently, it is quite obvious that we have a new hierarchy, called Hierarchy 2, including the old algebras BE, pre-BCK, BCC and BCK, the algebras “name-1” and “name-3”, and two new algebras: one is denoted for the moment by “name-4”, the second is denoted naturally by “pre-BCC”. Note that the algebra denoted by “name-3” in Hierarchy 1 (Figure 2) will receive now naturally the name “aBE”. Denote by pre-BCC, aBE the classes of pre-BCC and aBE algebras respectively. Hierarchy 2 is drawn connected with Hierarchy 1 in Figure 3.

Hence, we have the following definitions:

**Definition 4.5**
1. A pre-BCC algebra is an algebra \((A, \rightarrow, 1)\) verifying the axioms: (Re), (M), (L), (B).
2. An aBE algebra is a BE algebra that is antisymmetric, i.e. is an algebra \((A, \rightarrow, 1)\) verifying the axioms: (Re), (M), (L), (Ex), (An).

As the name says, the binary relation \( \preceq \) in a pre-BCC algebra is a pre-order too.

4.3 Hierarchy 3, including the old algebras BCH, BZ, BCI

From above new algebras and above Hierarchies 0, 1, 2, we can obtain obviously new algebras and hierarchies. Thus, we obtain the new hierarchy called Hierarchy 3, including: the old algebras BCH, BZ and BCI, the algebra “name-2”, and four new algebras, called “RM”, “RME”, “pre-BZ” and “pre-BCI”. Note that the algebra denoted by “name-2” in Hierarchy 1 (Figure 2) will receive now naturally the name “aRM”. Denote by RM, RME, pre-BZ, pre-BCI, aRM the classes of RM, RME, pre-BZ, pre-BCI, aRM algebras respectively. Hierarchy 3 is drawn connected with Hierarchies 1 and 2 in Figure 4.

Hence, we have the following obvious definitions:

**Definition 4.6**
3. A RM algebra is an algebra \((A, \rightarrow, 1)\) verifying the axioms: (Re), (M).
4. A pre-BZ algebra is an algebra \((A, \rightarrow, 1)\) verifying the axioms: (Re), (M), (B).
5. An aRM algebra is an algebra \((A, \rightarrow, 1)\) verifying the axioms: (Re), (M), (An).
6. A RME algebra is an algebra \((A, \rightarrow, 1)\) verifying the axioms: (Re), (M), (Ex).
7. A pre-BCI algebra is an algebra \((A, \rightarrow, 1)\) verifying the axioms: (Re), (M), (Ex), (B).

We check and we have, indeed, that:
\[
\begin{align*}
\text{Pre-BZ} + (\text{An}) &= \text{BZ}, \\
\text{Pre-BCI} + (\text{An}) &= \text{BCI}, \\
\text{RME} + (\text{An}) &= \text{BCH}, \\
\text{aRM} + (\text{B}) &= \text{BZ}, \\
\text{aRM} + (\text{Ex}) &= \text{BCH}.
\end{align*}
\]

4.4 Hierarchy 4 and the final, global hierarchy

We can now complete the Hierarchies 0, 1, 2, 3 by adding a new hierarchy, called Hierarchy 4, which connects obviously (by property (L)) the algebras RM, pre-BZ, RME, pre-BCI from Hierarchy 3 and the algebras name-4, pre-BCC, BE, pre-BCK from Hierarchy 2. Note that the algebra denoted by “name-4” in Hierarchy 2 (Figure 3) will receive now naturally the name “RML”, hence the algebra denoted by
Figure 3: Hierarchies 1 and 2, including Hierarchy 0
Figure 4: Hierarchies 1, 2, 3, including Hierarchy 0
“name-1” receives naturally the name “aRML”. Denote by RML, aRML the class of RML and aRML algebras, respectively. Hierarchy 4 can now be easily drawn.

Before drawing the final, global hierarchy, connecting the Hierarchies 0, 1, 2, 3, 4, recall that by Corollary 2.5, in BZ, BCC algebras we have: (Ex) ⇔ (BB) and that by Theorem 2.2, we have: (Re) + (M) + (Ex) imply (B) ⇔ (BB) ⇔ (*).

Consequently, we have:

**Corollary 4.7** In RM and RML algebras satisfying property (Ex), i.e. in RME, BCH and BE, aBE algebras, we have:

(B) ⇔ (BB) ⇔ (*).

Note that this corollary gives us equivalent definitions for the four algebras pre-BCI, BCI and pre-BCK, BCK.

We are now in position to present the final, global hierarchy (which connects the Hierarchies 0, 1, 2, 3, 4) in Figure 5.

Hence, we can introduce obviously the following new definitions:

**Definition 4.8**
8. A RML algebra is an algebra (A, →, 1) verifying the axioms: (Re), (M), (L).
9. An aRML algebra is an algebra (A, →, 1) verifying the axioms: (Re), (M), (L), (An), i.e. is a RML algebra verifying (An) (Antisymmetry).

**Remarks 4.9**
(i) All the RM algebras from Figure 5 are generalizations of BCK algebras. More precisely, the RM algebras from Hierarchy 3 are generalizations of BCI algebras, while the RM algebras from Hierarchy 2 (i.e. the RML algebras) are called proper generalizations of BCK algebras.
(ii) All RM algebras are at least reflexive.
(iii) While the hierarchy of the four old ordered algebras (BZ, BCI, BCC, BCK) is presented in Figure 1, we can remark the similar hierarchy of the four pre-ordered corresponding algebras (pre-BZ, pre-BCI, pre-BCC, pre-BCK) in Figure 5.

**5 Other twenty two new RM and RML algebras, generalizations of BCI and BCK algebras respectively**

Looking for proper examples of the above mentioned RM and RML algebras, we discovered that the classes of RM and RML algebras are much richer. Thus, we have introduced the following eleven new RM algebras and eleven new RML algebras.

**5.1 New RM and RML algebras, without property (Ex)**

- **New RM algebras, without (Ex)**

  Define the following nine new algebras:

  **Definition 5.1**
10. A tRM algebra is a RM algebra verifying (Tr).
11. A *RM algebra is a RM algebra verifying (*).
12. A RM** algebra is a RM algebra verifying (**).
13. A *RM** algebra is a RM algebra verifying (*), (**).
14. A pre-BBBZ algebra is a RM algebra verifying (B), (BB) (hence also (*), (**), (Tr)).

  **Definition 5.2**
15. An oRM algebra is an aRM algebra verifying (Tr).
16. An *aRM algebra is an aRM algebra verifying (*).
17. An aRM** algebra is an aRM algebra verifying (**).
18. An *aRM** algebra is an aRM algebra verifying (*), (**).
Figure 5: The global hierarchy (connecting Hierarchies 0, 1, 2, 3, 4)
New RML algebras, without (Ex)

Define the following corresponding (connected by (L) with the previous ones) nine new algebras:

**Definition 5.3**
19. A *tRML algebra* is a RML algebra verifying (Tr).
20. A *RML algebra* is a RML algebra verifying (*).
21. A *RML** algebra is a RML algebra verifying (**).
22. A *RML** algebra is a RML algebra verifying (*), (**).
23. A *pre-BBBCC algebra* is a RML algebra verifying (B), (BB) (hence also (*), (**), (Tr)).

**Definition 5.4**
24. An *oRML algebra* is an aRML algebra verifying (Tr).
25. An *aRML algebra* is an aRML algebra verifying (*).
26. An *aRML** algebra is an aRML algebra verifying (**).
27. An *aRML** algebra is an aRML algebra verifying (*), (**).

**Corollary 5.5** By Corollary 2.5, we have:
pre-BBBZ + (An) = pre-BZ + (BB) + (An) = BZ + (BB) = BZ + (Ex) = BCI,
pre-BBBCC + (An) = pre-BCC + (BB) + (An) = BCC + (BB) = BCC + (Ex) = BCK.

Hence, we have, for example, the Hierarchies 3.1 and 2.1 (determined by RM and by RML algebras, considering (B), (*) and (Tr) properties), from the next Figure 6, and we also have the Hierarchies 3.2 and 2.2 (determined by RM and by RML algebras, considering (*) and (**) properties) from the next Figure 7. In both Figures, the connection by (L) between the Hierarchies 3.1 and 2.1 and between the Hierarchies 3.2 and 2.2 is not drawn, in order to simplify the figure.

**Remark 5.6** Jānis Cirulis introduced in 2006 the class of weak BCK-algebras. They were studied in [4] and as algebras with subtraction in [5], [6]. We just noticed that weak BCK-algebras are in fact our aRML** algebras with property (D).

**Remark 5.7** Note that:
tRM + (*) = *RM,
*RM + (B) = pre-BZ,
RM** + (B) = pre-BZ,
*RM** + (B) = pre-BZ;
oRM + (*) = *aRM,
*aRM + (B) = BZ,
aRM** + (B) = BZ,
*aRM** + (B) = BZ;
tRML + (*) = *RML,
*RML + (B) = pre-BCC,
RML** + (B) = pre-BCC,
*RML** + (B) = pre-BCC;
oRML + (*) = *aRML,
*aRML + (B) = BCC,
aRML** + (B) = BCC,
*aRML** + (B) = BCC.

**Open problem 5.8** Find other connections between the above RM algebras.
Figure 6: Hierarchies 3.1 and 2.1, determined by RM and by RML algebras, considering (B), (*) and (Tr) properties
Figure 7: Hierarchies 3.2 and 2.2, determined by RM and by RML algebras, considering (*) and (**) properties
5.2 New RM and RML algebras, with property (Ex)

By Theorem 2.2, in RM and RML algebras satisfying condition (Ex), we have:

\[(B) \iff (BB) \iff (*).\]

By Theorem 2.3, in RM and RML algebras satisfying condition (Ex), we also have that:

\[(**) \iff (Tr).\]

Hence, we have obviously the following:

**Corollary 5.9** In RM and RML algebras satisfying property (Ex), i.e. in RME, BCH and BE, aBE algebras, we have:

\[(**) \iff (Tr).\]

Define now the following last four new algebras:

**Definition 5.10**

28. A RME** algebra is a RME algebra verifying (**).

29. A BCH** algebra is a BCH algebra verifying (**).

30. A BE** algebra is a BE algebra verifying (**).

31. An aBE** algebra is an aBE algebra verifying (**).

Note that Corollary 5.9 gives us equivalent definitions for the four algebras RM**, BCH** and BE**, aBE**.

It follows that we have, for example, the Hierarchies 3.3 and 2.3 from Figure 8; in order to simplify the figure, the connection by (L) between Hierarchies 3.3 and 2.3 is not drawn.

**Remark 5.11** Note that:

RME** + (B) = pre-BCI,

BCH** + (B) = BCI,

BE** + (B) = pre-BCK,

aBE** + (B) = BCK.

6 Generalizations of Hilbert algebras

**Definition 6.1** (see [7])

A Hilbert algebra is an algebra \( A = (A, \rightarrow, 1) \) of type \((2, 0)\), satisfying, for all \( x, y, z \in A \):

\[(h1) \quad x \rightarrow (y \rightarrow x) = 1,\]

\[(h2) \quad ((x \rightarrow y) \rightarrow (x \rightarrow z)) = 1,\]

\[(h3) \quad \text{if } x \rightarrow y = y \rightarrow x = 1, \text{ then } x = y.\]

Note that (h1) is (K) and (h3) is (An).

Let \( A, \rightarrow, 1 \) be an algebra of type \((2, 0)\). Consider the properties: for all \( x, y, z \in A \):

\[(pimpl) \quad x \rightarrow (y \rightarrow z) = (x \rightarrow y) \rightarrow (x \rightarrow z),\]

\[(pi) \quad y \rightarrow (y \rightarrow x) = y \rightarrow x,\]

and also

\[(p-1) \quad [x \rightarrow (y \rightarrow z)] \rightarrow [(x \rightarrow y) \rightarrow (x \rightarrow z)] = 1,\]

\[(p-2) \quad [(x \rightarrow y) \rightarrow (x \rightarrow z)] \rightarrow [x \rightarrow (y \rightarrow z)] = 1.\]

Note that (p-1) is (h2).

We know [20] that in BCK algebras we have the equivalence:

\[(pimpl) \iff (pi)\]
Figure 8: Hierarchies 3.3 and 2.3, determined by RME and BE algebras
and that [18] Hilbert algebras are just those BCK algebras, called positive implicative, verifying the equivalent properties (pimpl) and (pi).

We shall see in this section what happens in the old and new generalizations of BCK algebras with these two properties (pimpl) and (pi).

Proposition 6.2 Let \((A, \rightarrow, 1)\) be an algebra of type \((2,0)\). Then,

(i) \((\text{Re}) + (\text{pimpl}) \implies (L)\);

(ii) \((\text{Re}) + (\pi) \implies (L)\);

(iii) \((\text{Re}) + (M) + (\text{pimpl}) \implies (\pi)\).

Proof.

(i): Take \(y = z = x\) in (pimpl); we obtain: \(x \rightarrow (x \rightarrow x) = (x \rightarrow x) \rightarrow (x \rightarrow x)\), hence, by (Re), we obtain: \(x \rightarrow 1 = 1 \rightarrow 1\), hence, by (Re) again, \(x \rightarrow 1 = 1\), i.e. (L) holds.

(ii): Take \(y = x\) in (pi); we obtain: \(x \rightarrow (x \rightarrow x) = x \rightarrow x\), hence, by (Re), \(x \rightarrow 1 = 1\), i.e. (L) holds.

(iii): Take \(x = y\) in (pimpl); we obtain: \(y \rightarrow (y \rightarrow z) = (y \rightarrow y) \rightarrow (y \rightarrow z)\); then, by (Re), we obtain: \(y \rightarrow (y \rightarrow z) = 1 \rightarrow (y \rightarrow z)\); then, by (M), we obtain: \(y \rightarrow (y \rightarrow z) = y \rightarrow z\), i.e. (pi) holds.

We shall see now in which properties (pi) implies (pimpl).

Proposition 6.3 Let \((A, \rightarrow, 1)\) be an algebra of type \((2,0)\). Then,

(a) \((\text{Re}) + (L) + (\text{Ex}) + (**) \implies (p-2)\);

(b) \((\text{Ex}) + (B) + (*) + (\pi) \implies (p-1)\);

(c) \((p-1) + (p-2) + (\text{An}) \implies (\text{pimpl})\);

(d) \((\text{Re}) + (\text{Ex}) + (B) + (**) + (*) + (L) + (\text{An}) + (\pi) \implies (\text{pimpl})\).

Proof.

(a): By Proposition 2.1, (8), (Re) + (Ex) + (L) imply (K). By (K), we have: \([x \rightarrow y] \rightarrow [(x \rightarrow z) \rightarrow (x \rightarrow z)] = 1\);

hence, by (Ex), we obtain: \([x \rightarrow y] \rightarrow [(x \rightarrow z) \rightarrow (x \rightarrow z)] = 1\),

i.e. (p-2) holds.

(b): We must prove that \([x \rightarrow (y \rightarrow z)] \rightarrow [(x \rightarrow y) \rightarrow (x \rightarrow z)] = 1\).

Denote \(X = (x \rightarrow y) \rightarrow (x \rightarrow z)\). We obtain:

\[X \overset{(\text{Ex})}{=} (x \rightarrow y) \rightarrow [x \rightarrow (x \rightarrow z)] \overset{(Ex)}{=} x \rightarrow [(x \rightarrow y) \rightarrow (x \rightarrow z)].\]

By (B), we have: \((y \rightarrow z) \rightarrow [(x \rightarrow y) \rightarrow (x \rightarrow z)] = 1\); then, by (*), we obtain:

\([x \rightarrow (y \rightarrow z)] \rightarrow [x \rightarrow [(x \rightarrow y) \rightarrow (x \rightarrow z)]] = 1 = [x \rightarrow (y \rightarrow z)] \rightarrow X,\]

hence

\([x \rightarrow (y \rightarrow z)] \rightarrow [(x \rightarrow y) \rightarrow (x \rightarrow z)] = 1,\]

i.e. (p-1) holds.

(c): Obviously.

(d): By above (a), (b), (p-2) and (p-1) hold. Then, by (c), (pimpl) holds.

We then immediately obtain:

Corollary 6.4

(1) Properties (pimpl) and (pi) may hold only in the generalizations of BCK algebras verifying (L), i.e. in the RML algebras.

(2) In RML algebras, (pimpl) implies (pi).

(3) \(\text{BE}**, \ a\text{BE}**\) and pre-BCK, BCK algebras verify (p-2).
In pre-BCK and BCK algebras, $(\pi)$ implies $(\text{p-1})$.

In BCK algebras only, $(\pi)$ implies $(\text{pimpl})$, hence $(\text{pimpl}) \iff (\pi)$.

Hilbert algebras are the only RML algebras where $(\text{pimpl})$ and $(\pi)$ hold and we have $(\text{pimpl}) \iff (\pi)$.

We prove now the following result:

**Proposition 6.5**

$$(\text{pimpl}) + (K) \implies (B).$$

**Proof.**

$$(K') \quad y \to z \leq x \to (y \to z) \overset{(\text{pimpl})}{=} (x \to y) \to (x \to z).$$

Hence, $y \to z \leq (x \to y) \to (x \to z)$, i.e. $(B')$ holds.

**Corollary 6.6**

In proper BE, aBE, BE**, aBE** algebras, property $(\text{pimpl})$ cannot hold.

**Proof.** Since by Proposition 2.1 (8.), $(\text{Re}) + (L) + (\text{Ex})$ imply $(K)$, it follows that in BE algebras and in aBE algebras we have $(K)$, hence in BE** and aBE** algebras we have property $(K)$. In proper BE, aBE, BE**, aBE** algebras, property $(B)$ does not hold. If $(\text{pimpl})$ holds in proper BE, aBE, BE**, aBE** algebras, then, by above Proposition 6.5, $(B)$ holds: contradiction.

**Corollary 6.7**

In proper RML**, *RML**, aRML**, *aRML** algebras, property $(\text{pimpl})$ cannot hold.

**Proof.** Since by Proposition 2.1 (9'), $(M) + (L) + (**)$ imply $(K)$, it follows that in RML**, *RML**, aRML**, *aRML** algebras we have $(K)$. In proper RML**, *RML**, aRML**, *aRML** algebras, property $(B)$ does not hold. If $(\text{pimpl})$ holds in proper RML**, *RML**, aRML**, *aRML** algebras, then, by above Proposition 6.5, $(B)$ holds: contradiction.

**Remarks 6.8**

(1) There are plenty of RML algebras verifying $(\pi)$ and not verifying $(\text{pimpl})$.

(2) The examples we found of RML algebras verifying $(\text{pimpl})$ (hence $(\pi)$) are pre-BCK, BCK and pre-BBBCC algebras.

(3) We know that Hilbert algebras cannot be generalized to the non-commutative case. In non-commutative RML algebras $(A, \rightarrow, \sim, 1)$ (as for example the pseudo-BCK algebras, the pseudo-BE algebras), property $(\pi)$ would become a pair of the properties:

$$(\pi \sim) \quad y \to (y \sim x) = y \sim x,

(\pi \rightarrow) \quad y \sim (y \to x) = y \to x,$$

and since in the non-commutative case, $(\text{Ex})$ becomes:

$$(\text{pEx}) \quad x \to (y \sim z) = y \sim (x \to z)$$

it follows that, for all $x, y$:

$$y \to x \overset{(\pi \rightarrow)}{=} y \sim (y \to x) \overset{(\text{pEx})}{=} y \to (y \sim x) \overset{(\pi \sim)}{=} y \sim x,$$

i.e. $\rightarrow \sim \sim$, hence this non-commutative algebra $(A, \rightarrow, \sim, 1)$ is in fact a commutative one. In other words, the RML algebras verifying $(\text{Ex})$ and $(\pi)$ (i.e. the BE, pre-BCK, aBE, BCK and BE**, aBE** algebras verifying $(\pi)$) cannot be generalized to the non-commutative case.

Let us then introduce the following definitions:

**Definition 6.9**

(1) A **pi-RML algebra** is a RML algebra verifying property $(\pi)$.

(2) A **positive implicative RML algebra**, or a **pimpl-RML algebra** for short, is a RML algebra verifying property $(\text{pimpl})$.

(3) A **generalization of Hilbert algebra** is any pi-RML algebra.

(4) A **proper generalization of Hilbert algebra** is a pi-RML algebra that cannot be generalized to the non-commutative case.
Remarks 6.10

(1) Any pimpl-RML algebra is pi-RML algebra; the converse holds only for BCK algebras.

(2) The pi-RML algebras and the pimpl-RML algebras, their hierarchies and their connections with the corresponding RML algebras, are presented in the next Figures 9, 10, 11 and 12. We shall present in other section examples of all of these generalizations of Hilbert algebras.

(3) By above Remarks 6.8 (3), the proper generalizations of Hilbert algebras are the following pi-RML algebras: pi-BE, pi-pre-BCK, pimpl-pre-BCK, pi-aBE, pi-BCK = pimpl-BCK = Hilbert and pi-BE**.

Open problem 6.11 As Hilbert algebra (see Definition 6.1) is equivalent with pimpl-BCK algebra (= pi-BCK algebra), define a pre-Hilbert algebra as an algebra (A, \rightarrow, 1) equivalent to pimpl-pre-BCK algebra.

Figure 9: The hierarchy connecting Hierarchies 2 and 2H
Figure 10: Hierarchies 2.1 and 2.1 H, determined by RML and by pi-RML algebras, considering (B), (*) and (Tr) properties
Figure 11: Hierarchies 2.2 and 2.2 H, determined by RML and by pi-RML algebras, considering (*) and (***) properties
Figure 12: Hierarchies 2.3 and 2.3 H, determined by BE and pi-BE algebras
7 Proper algebras

In this section, we define the proper algebras discussed in the paper; the definitions are needed in the next sections, where we present examples of proper algebras. The definitions are denoted by: PO1-PO7 (following the corresponding notations O1 - O7), and P1 - P31 (following the corresponding notations 1 - 31), for the generalizations of BCI and BCK algebras and are denoted by Pnm-pi or Pnm-pimpl (following the corresponding notation Pnm), for the corresponding generalizations of Hilbert algebras verifying (pi) or (pimpl).

Hence, we have the following obvious definitions.

**Definition 7.1**

PO1. A proper BCI algebra is a BCI algebra (i.e. verifying (BB), (D), (Re), (N), (An) or, equivalently, (BB), (D), (Re), (An), or, equivalently, (BB), (M), (An)) not verifying (L).

PO2. A proper BCK algebra is a BCK algebra (i.e. verifying (BB), (D), (Re), (L), (An), or, equivalently, (BB), (M), (L), (An)) not verifying (pi) (hence (pimpl)).

PO3. A proper BCH algebra is a BCH algebra (i.e. verifying (Re), (Ex), (An)) not verifying (B), (BB), (*), (**), (Tr), (L).

PO4. A proper BCC algebra is a BCC algebra (i.e. verifying (Re), (M), (L), (B), (An)) not verifying (Ex), (BB), (pi).

PO4-pi. A proper pi-BCC algebra is a pi-BCC algebra (i.e. verifying (Re), (M), (L), (B), (An), (pi)) not verifying (Ex), (pimpl).

PO5. A proper BZ algebra is a BZ algebra (i.e. verifying (Re), (M), (B), (An)) not verifying (L), (Ex), (BB).

PO6. A proper BE algebra is a BE algebra (i.e. verifying (Re), (M), (L), (Ex) (hence (D))) not verifying (An), (Tr), (*) (hence (B)), (**) (hence (BB)), (pi).

PO6-pi. A proper pi-BE algebra is a pi-BE algebra (i.e. verifying (Re), (M), (L), (Ex) (hence (D)), (pi)) not verifying (An), (B), (BB), (**), (Tr), (pimpl).

PO7. A proper pre-BCK algebra is a pre-BCK algebra (i.e. verifying (Re), (M), (L), (Ex) (hence (D)), (pi)) not verifying (An), (Tr).

PO7-pi. A proper pre-BCK algebra is a pi-pre-BCK algebra (i.e. verifying (Re), (M), (L), (Ex), (pi)) not verifying (Tr).

PO7-pimpl. A proper pimpl-pre-BCK algebra is a pimpl-pre-BCK algebra (i.e. verifying (Re), (M), (L), (Ex), (**)), (Tr) not verifying (An).

**Definition 7.2**

P1. A proper pre-BCC algebra is a pre-BCC algebra (i.e. verifying (Re), (M), (L), (B) (hence (*), (Tr))) not verifying (An), (Ex), (BB), (pi).

P1-pi. A proper pi-pre-BCC algebra is a pi-pre-BCC algebra (i.e. verifying (Re), (M), (L), (B) (hence (*), (Tr)) not verifying (An), (Ex), (BB), (pimpl).

P2. A proper aBE algebra is an aBE algebra (i.e. verifying (Re), (M), (L), (Ex) (hence (D)), (An)) not verifying (B), (BB), (**), (Tr), (pi).

P2-pi. A proper pi-aBE algebra is a pi-aBE algebra (i.e. verifying (Re), (M), (L), (Ex) (hence (D)), (pi)) not verifying (B), (BB), (**), (Tr), (pimpl).

P3. A proper RM algebra is an RM algebra (i.e. verifying (Re), (M), (L), (BB), (**), (Tr).

P4. A proper pre-BZ algebra is a pre-BZ algebra (i.e. verifying (Re), (M), (B) (hence (pi)), (Tr))) not verifying (Ex), (An), (L), (BB).

P5. A proper aRM algebra is an aRM algebra (i.e. verifying (Re), (M), (An)) not verifying (L), (Ex), (B), (**), (Tr).

P6. A proper RME algebra is a RME algebra (i.e. verifying (Re), (M), (Ex) (hence (D))) not verifying (An), (L), (B), (BB), (**), (Tr).

P7. A proper pre-BCI algebra is a pre-BCI algebra (i.e. verifying (Re), (M), (Ex), (B) (hence (BB), (**), (Tr)) not verifying (An), (L).
P8. A proper RML algebra is a RML algebra (i.e. verifying (Re), (M), (L)) not verifying (Ex), (An), (B), (*), (BB), (**), (Tr), (pi).
P8-pi. A proper pi-RML algebra is a pi-RML algebra (i.e. verifying (Re), (M), (L), (pi)) not verifying (Ex), (An), (B), (*), (BB), (**), (Tr), (pimpl).
P9. A proper aRML algebra is an aRML algebra (i.e. verifying (Re), (M), (L), (An)) not verifying (Ex), (B), (*), (BB), (**), (Tr), (pi).
P9-pi. A proper pi-aRML algebra is a pi-aRML algebra (i.e. verifying (Re), (M), (L), (An), (pi)) not verifying (Ex), (B), (*), (BB), (**), (Tr), (pimpl).

Definition 7.3

P10. A proper tRM algebra is a tRM algebra (i.e. verifying (Re), (M), (Tr)) not verifying (Ex), (An), (L), (*) (hence (B)), (**) (hence (BB)).
P11. A proper *RM algebra is a *RM algebra (i.e. verifying (Re), (M), (*) (hence (Tr))) not verifying (Ex), (An), (L), (B), (**) (hence (BB)).
P12. A proper RM** algebra is a RM** algebra (i.e. verifying (Re), (M), (**) (hence (Tr))) not verifying (Ex), (An), (L), (BB), (*) (hence (B)).
P13. A proper *RM** algebra is a *RM** algebra (i.e. verifying (Re), (M), (*), (**) (hence (Tr))) not verifying (Ex), (An), (L), (B), (BB).
P14. A proper pre-BBBZ algebra is a pre-BBBZ algebra (i.e. verifying (Re), (M), (B), (BB) (hence (*), (**), (Tr), (D))) not verifying (Ex), (An), (L).
P15. A proper oRM algebra is an oRM algebra (i.e. verifying (Re), (M), (An), (Tr)) not verifying (*), (**), (L).
P16. A proper *aRM algebra is an *aRM algebra (i.e. verifying (Re), (M), (An), (*) (hence (Tr))) not verifying (Ex), (L), (B), (**) (hence (BB)).
P17. A proper aRM** algebra is an aRM** algebra (i.e. verifying (Re), (M), (An), (**)) (hence (Tr))) not verifying (Ex), (L), (BB), (*) (hence (B)).
P18. A proper *aRM** algebra is a *aRM** algebra (i.e. verifying (Re), (M), (An), (An), (**)) (hence (Tr))) not verifying (Ex), (L), (B), (BB).
P19. A proper tRML algebra is a tRML algebra (i.e. verifying (Re), (M), (L), (Tr)) not verifying (An), (Ex), (*) (hence (B)), (**) (hence (BB)), (pi).
P19-pi. A proper pi-tRML algebra is a pi-tRML algebra (i.e. verifying (Re), (M), (L), (Tr), (pi)) not verifying (An), (Ex), (pi), (**), (pimpl).
P20. A proper *RML algebra is a *RML algebra (i.e. verifying (Re), (M), (L)) (hence (Tr))) not verifying (Ex), (An), (Ex), (*), (**) (hence (BB)), (pi).
P20-pi. A proper pi-*RML algebra is a pi-*RML algebra (i.e. verifying (Re), (M), (L)) (hence (Tr)), (pi) not verifying (An), (Ex), (B), (**), (BB), (pimpl).
P21. A proper RML** algebra is a RML** algebra (i.e. verifying (Re), (M), (L), (**)) (hence (Tr))) not verifying (An), (Ex), (B), (**), (BB), (pi).
P21-pi. A proper pi-RML** algebra is a pi-RML** algebra (i.e. verifying (Re), (M), (L), (**)) (hence (Tr)), (pi) not verifying (An), (Ex), (BB), (**), (B), (pimpl).
P22. A proper *RML** algebra is a *RML** algebra (i.e. verifying (Re), (M), (L), (An), (**)) (hence (Tr))) not verifying (An), (Ex), (B), (BB), (pi).
P22-pi. A proper pi-*RML** algebra is a pi-*RML** algebra (i.e. verifying (Re), (M), (L), (An), (**) (hence (Tr)), (pi)) not verifying (An), (Ex), (B), (BB), (pimpl).
P23. A proper pre-BBBCC algebra is a pre-BBBCC algebra (i.e. verifying (Re), (M), (L), (B), (BB) (hence (*), (**), (Tr), (D))) not verifying (An), (Ex), (B), (BB), (pimpl).
P23-pi. A proper pi-pre-BBBCC algebra is a pi-pre-BBBCC algebra (i.e. verifying (Re), (M), (L), (B), (BB) (hence (*), (**), (Tr), (D), (pi)) not verifying (An), (Ex), (pimpl).
P23-pimpl. A proper pimpl-pre-BBBCC algebra is a pimpl-pre-BBBCC algebra (i.e. verifying (Re), (M), (L), (B), (BB) (hence (*), (**), (Tr), (D), (pimpl)) not verifying (An), (Ex).
P24. A proper oRML algebra is an oRML algebra (i.e. verifying (Re), (M), (L), (An), (Tr), (pi)) not verifying (Ex), (An), (Tr), (**), (BB), (**), (B), (pi).
P24-pi. A proper pi-oRML algebra is a pi-oRML algebra (i.e. verifying (Re), (M), (L), (An), (pi)) not verifying (Ex), (An), (Tr), (**), (BB), (**), (B), (pi).
P25. A proper aRML** algebra is an aRML** algebra (i.e. verifying (Re), (M), (L), (An), (Tr), (pi)) not verifying (An), (Tr), (**), (BB), (**), (B), (pi).
P25. A proper *aRML algebra is a *aRML algebra (i.e. verifying (Re), (M), (L), (An), (**) (hence (Tr))) not verifying (B), (** (hence (BB)), (pi).

P25-pi. A proper pi-*aRML algebra is a pi-*aRML algebra (i.e. verifying (Re), (M), (L), (An), (**) (hence (Tr)), (pi)) not verifying (B), (** (hence (BB)), (pimpl).

P26. A proper aRML** algebra is an aRML** algebra (i.e. verifying (Re), (M), (L), (An), (** (hence (Tr))) not verifying (Ex), (BB), (**) (hence (B)), (pi).

P26-pi. A proper pi-aRML** algebra is a pi-aRML** algebra (i.e. verifying (Re), (M), (L), (An), (** (hence (Tr)), (pi)) not verifying (Ex), (BB), (** (hence (B)), (pimpl).

P27. A proper *aRML** algebra is a *aRML** algebra (i.e. verifying (Re), (M), (L), (An), (**) (hence (Tr))) not verifying (Ex), (B), (BB), (pi).

P27-pi. A proper pi-*aRML** algebra is a pi-*aRML** algebra (i.e. verifying (Re), (M), (L), (An), (**) (hence (Tr)), (pi)) not verifying (Ex), (B), (BB), (pimpl).

P28. A proper RME** algebra is a RME** algebra (i.e. verifying (Re), (M), (Ex) (hence (D)), (** (hence (Tr)), not verifying (An), (BB), (**) (hence (B)).

P29. A proper BCH** algebra is a BCH** algebra (i.e. verifying (Re), (M), (Ex) (hence (D)), (** (hence (Tr)), (An)), not verifying (BB), (**) (hence (B)).

P30. A proper BE** algebra is a BE** algebra (i.e. verifying (Re), (M), (L), (Ex) (hence (D)), (** (hence (Tr)), not verifying (An), (BB), (**) (hence (B)), (pi).

P30-pi. A proper pi-BE** algebra is a pi-BE** algebra (i.e. verifying (Re), (M), (L), (Ex) (hence (D)), (** (hence (Tr)), (pi)) not verifying (An), (BB), (** (hence (B)), (pimpl).

P31. A proper aBE** algebra is an aBE** algebra (i.e. verifying (Re), (M), (L), (Ex) (hence (D)), (An), (** (hence (Tr)), not verifying (BB), (** (hence (B)), (pi).

P31-pi. A proper pi-aBE** algebra is a pi-aBE** algebra (i.e. verifying (Re), (M), (L), (Ex) (hence (D)), (** (hence (Tr)), (pi)) not verifying (BB), (** (hence (B)), (pimpl).

In the remaining sections we shall give proper examples of all the algebras, old and new, discussed in the paper. The examples will be presented mainly in the same order as the corresponding algebras were introduced:
- first, the sixteen old and new algebras from Hierarchy 3 (the eight RM algebras, generalizations of BCI algebras) and Hierarchy 2 (the eight RML algebras, proper generalizations of BCK algebras),
- then, the twenty two new algebras from Hierarchies 3.1 - 3.3 (other eleven RM algebras) and Hierarchies 2.1 - 2.3 (other eleven RML algebras),
- finally, the (proper) generalizations of Hilbert algebras (pi-RML algebras and pimpl-RML algebras).

8 Examples of the first sixteen old and new RM and RML algebras PO1 - PO7, P1 - P9

For a coherent presentation of the examples, we shall provide also examples of the seven old algebras.

The examples will be presented in the following order: first the RM algebras from Hierarchy 3, then the RML algebras (Hierarchy 2).

Note that even if the algebras were obtained from the bottom to the top, the examples will be presented from the top to the bottom.

8.1 Examples of the RM algebras from Hierarchy 3

Recall that we have proved that all the algebras from Hierarchy 3 (i.e. not satisfying condition (L)) cannot have the properties (pimpl) and (pi).

In the first subsubsection, we present proper examples of RM algebras not satisfying property (Ex) and in the second subsubsection, we present examples of RM algebras satisfying property (Ex).
8.1.1 Examples of RM algebras without condition (Ex):
proper RM, pre-BZ, aRM, BZ algebras

- Proper RM algebras (P3)

**Example 8.1** Let \( (A = \{a,b,1\}, \to, 1) \) with the following matrix (table) of implication:

\[
\begin{array}{c|ccc}
\rightarrow & a & b & 1 \\
\hline
a & 1 & 1 & a \\
b & 1 & 1 & 1 \\
1 & a & b & 1 \\
\end{array}
\]

Properties (Re), (M), (D) are satisfied. (An) is not satisfied for \((x,y) = (a,b)\). (L) is not satisfied for \(x = a\). (Ex) is not satisfied for \((x,y,z) = (a,b,a)\), (BB) is not satisfied for \((x,y,z) = (a,b,1)\), (***) is not satisfied for \((x,y,z) = (a,b,1)\), (**) is not satisfied for \((x,y,z) = (a,b,1)\), (Tr) is not satisfied for \((x,y,z) = (a,b,1)\).

Hence, \( A \) is a proper RM algebra with the minimum number of elements, three, with (D).

**Example 8.2** Let \( (A = \{a,b,1\}, \to, 1) \) with the following matrix of implication:

\[
\begin{array}{c|ccc}
\rightarrow & a & b & 1 \\
\hline
a & 1 & a & b \\
b & 1 & 1 & 1 \\
1 & a & b & 1 \\
\end{array}
\]

Properties (Re), (M) are satisfied. (Ex) is not satisfied for \((a,b,a)\); (An) is not satisfied by \((a,b)\); (BB) is not satisfied for \((a,b,1)\); (***) is not satisfied for \((a,b,1)\); (***) is not satisfied for \((a,b,1)\); (B) is not satisfied for \((a,b,1)\); (**) is not satisfied for \((a,b,1)\); (**) is not satisfied for \((a,b,1)\); (Tr) is not satisfied for \((a,b,1)\); (D) is not satisfied for \((1,a)\).

Hence, \( A \) is a proper RM algebra with three elements too, without (D).

**Example 8.3** Consider the set \( A = \{0,a,b,1\} \) with the following table of \( \rightarrow \):

\[
\begin{array}{c|ccc}
\rightarrow & 0 & a & b & 1 \\
\hline
0 & 1 & 1 & 1 & 1 \\
a & 1 & 1 & 1 & a \\
b & 1 & 1 & 1 & a \\
1 & 0 & a & b & 1 \\
\end{array}
\]

Then the algebra \( A = (A, \rightarrow, 1) \) verifies properties (Re), (M) and (D). It does not verify properties: (Ex) for \( x = a, y = 0, z = b \); (BB) for \( x = 0, y = a, z = 1 \); (B) for \( x = 0, y = a, z = 1 \); (***) for \( x = a, y = 0, z = 1 \); (***) for \( x = a, y = 0, z = 1 \); (Tr) for \( x = a, y = 0, z = 1 \); (An) for \( x = a, y = b \).

Hence, \( A \) is a proper RM algebra, with four elements, with (D).

**Example 8.4** Consider the set \( A = \{0,a,b,1\} \) with the following table of \( \rightarrow \):

\[
\begin{array}{c|ccc}
\rightarrow & 0 & a & b & 1 \\
\hline
0 & 1 & 1 & 1 & b \\
a & 1 & 1 & a & b \\
b & 0 & a & 1 & 1 \\
1 & 0 & a & b & 1 \\
\end{array}
\]

Then the algebra \( A = (A, \rightarrow, 1) \) verifies properties (Re), (M). It does not verify properties (Ex), (BB), (B), (***) , (Tr), (An), (D).

Hence, \( A \) is a proper RM algebra, with four elements, without (D).

- Proper pre-BZ algebras (P4)
Example 8.5 Consider the set $A = \{a, b, c, d, 1\}$ with the following table of $\rightarrow$:

|   | a | b | c | d | 1 |
|---|---|---|---|---|---|
| a | 1 | a | a | a | a |
| b | a | 1 | b | c | 1 |
| c | a | b | 1 | 1 | 1 |
| d | a | 1 | 1 | 1 | 1 |
| 1 | a | b | c | d | 1 |

Properties (Re), (M), (B) (hence (*), (Tr), (**)) are satisfied. Properties (An) and (L) do not hold obviously. Property (Ex) does not hold for $a, b, d$; (BB) does not hold for $d, b, a$; (D) is not satisfied for $d, a$.

Hence, $A$ is a proper pre-BZ algebra, without (D).

• Proper aRM algebras (P5)

Example 8.6 Consider the set $A = \{a, b, c, 1\}$ with the following table of $\rightarrow$:

|   | a | b | c | 1 |
|---|---|---|---|---|
| a | 1 | a | a | a |
| b | a | 1 | a | 1 |
| c | a | 1 | 1 | a |
| 1 | a | b | c | 1 |

$A$ satisfies (Re), (M), (An). (Ex) is not satisfied for $a, b, c$; (L) is not satisfied for $a$; (BB) is not satisfied for $a, b, c$; (***) does not hold for $c, b, 1$; (B) does not hold for $a, b, c$; (*) is not satisfied for $b, c, b$; (Tr) is not satisfied for $c, b, 1$; (D) is not satisfied for $a, c$.

Hence, $A$ is a proper aRM algebra, without (D).

• Proper BZ algebras (PO5)

Example 8.7 Consider the set $A = \{a, b, c, d, 1\}$ with the following table of $\rightarrow$:

|   | a | b | c | d | 1 |
|---|---|---|---|---|---|
| a | 1 | a | a | a | a |
| b | a | 1 | b | c | 1 |
| c | a | b | 1 | 1 | 1 |
| d | a | 1 | 1 | 1 | 1 |
| 1 | a | b | c | d | 1 |

Properties (Re), (M), (An), (B) (hence (*), (Tr), (**)) are satisfied. Property (Ex) is not satisfied for $b, c, d$, (BB) for $d, 1, b, c$; (D) for $d, b$.

Hence, $A$ is a proper BZ algebra, without (D).

8.1.2 Examples of RM algebras with condition (Ex):

RME, pre-BCI, BCH, BCI

These algebras verify all property (D), by Proposition 2.1 (4.).

• Proper RME algebras (P6)

Example 8.8 Consider the set $A = \{a, b, c, d, 1\}$ with the following table of $\rightarrow$:

|   | a | b | c | d | 1 |
|---|---|---|---|---|---|
| a | 1 | a | a | a | a |
| b | a | 1 | c | d | 1 |
| c | a | b | 1 | 1 | 1 |
| d | a | 1 | 1 | 1 | 1 |
| 1 | a | b | c | d | 1 |
\[ A \] satisfies \((R_e), (M), (E_x)\). It does not satisfies: \((L)\) for \(a\), \((A_n)\) for \(c,d\), \((B B)\) for \(b,c,d\), \((**)\) for \(b,c,d\), \((B)\) and \((*)\) for \(c,d,b\), \((T r)\) for \(c,d,b\).

Hence, \( A \) is a proper \(R M E\) algebra.

- **Proper pre-BCI algebras** \((P 7)\)

**Example 8.9** Consider the set \( A = \{a,b,1\} \) with the following table of \(\rightarrow\):

\[
\begin{array}{c|cccc}
\rightarrow & a & b & 1 \\
\hline
a & 1 & 1 & a \\
b & 1 & 1 & a \\
1 & a & b & 1 \\
\end{array}
\]

Properties \((R_e), (M), (E_x), (B)\) (hence \((B B), (**), (T r)\)) are satisfied. \((L)\) and \((A n)\) are not satisfied obviously.

Hence, \( A \) is a proper pre-BCI algebra.

- **Proper BCH algebras** \((P O 3)\)

**Example 8.10** Consider the set \( A = \{a,b,c,d,1\} \) with the following table of \(\rightarrow\):

\[
\begin{array}{c|ccccc}
\rightarrow & a & b & c & d & 1 \\
\hline
a & 1 & a & a & a & a \\
b & a & 1 & b & 1 & 1 \\
c & a & 1 & 1 & d & 1 \\
d & a & b & c & 1 & 1 \\
1 & a & b & c & d & 1 \\
\end{array}
\]

Properties \((R_e), (M), (E_x), (A n)\) are satisfied. \( A \) does not satisfy \((L)\) for \(a\), \((B B)\) and \((**)\) for \(d,c,b\), \((*)\), \((T r)\) for \(c,b,d\).

Hence, \( A \) is a proper BCH algebra.

- **Proper BCI algebras** \((P O 1)\)

**Example 8.11** Consider the set \( A = \{a,b,1\} \) with the following table of \(\rightarrow\):

\[
\begin{array}{c|ccc}
\rightarrow & a & b & 1 \\
\hline
a & 1 & a & a \\
b & a & 1 & 1 \\
1 & a & b & 1 \\
\end{array}
\]

Properties \((R_e), (A n), (B B), (D), (N)\) are satisfied. \((L)\) is not satisfied for \(a\).

Hence, \( A \) is a proper BCI algebra.

### 8.2 Examples of the RML algebras from Hierarchy 2

Recall that we have proved that only these algebras can have the properties \((pimpl)\) and \((p)\).

In the first subsubsection we present proper examples of RML algebras not satisfying property \((E x)\) and in the second subsubsection we present examples of RML algebras satisfying property \((E x)\).

### 8.2.1 Examples of RML algebras without condition \((E x)\):

- \(R M L\), \(p r e-B C C\), \(a R M L\), \(B C C\)

- **Proper RML algebras** \((P 8)\)
Example 8.12 Consider the set $A = \{a, b, c, 1\}$ with the following table of $ \to $:

|   | a | b | c | 1 |
|---|---|---|---|---|
| a | 1 | a | a | 1 |
| b | a | 1 | 1 | 1 |
| c | 1 | 1 | 1 | 1 |
| 1 | a | b | c | 1 |

$A$ satisfies (Re), (M), (L), (D). It does not satisfy (Ex) for $a, b, b$, (pi) for $b, a$, (BB), (**) for $a, b, c$; (B), (*) for $b, c, a$.
Hence, $A$ is a **proper RML algebra**, with (D).

Example 8.13 Consider the set $A = \{0, a, b, c, 1\}$ with the following table of $ \to $:

|   | 0 | a | b | c | 1 |
|---|---|---|---|---|---|
| 0 | 1 | 1 | 1 | 1 | 1 |
| a | c | 1 | 1 | 1 | 1 |
| b | b | a | 1 | 1 | 1 |
| c | a | 1 | a | 1 | 1 |
| 1 | 0 | a | b | c | 1 |

Then the bounded algebra $A = (A, \to, 0, 1)$ verifies properties (Re), (M), (L), (D), and (DN).
It does not verify properties (Ex), (An), (BB), (B), (*), (**), (Tr), (pi).
The relation $\leq$ is only reflexive; it is not antisymmetrique (not (An)): $a \leq c$ and $c \leq a$, but $a \neq c$ and it is not tranzitive (not (Tr)).
Hence, $A$ is a **proper RML algebra**, with (D), (DN).

- **Proper pre-BCC algebras** (P1)

Example 8.14 Consider the set $A = \{a, b, c, d, 1\}$ with the following table of $ \to $:

|   | a | b | c | d | 1 |
|---|---|---|---|---|---|
| a | 1 | a | a | b | 1 |
| b | 1 | 1 | 1 | b | 1 |
| c | 1 | 1 | 1 | b | 1 |
| d | 1 | 1 | 1 | 1 | 1 |
| 1 | a | b | c | d | 1 |

$A$ satisfies properties (Re), (M), (L), (B) (hence (*), (Tr)) and (**). It does not satisfy (Ex) for $a, b, d$, (An) for $b, c$, (BB) for $d, 1, a$, (pi) for $b, a$, (D) for $d, a$.
Hence, $A$ is a **proper pre-BCC algebra**, with (**), without (D).

- **Proper aRML algebras** (P9)

Example 8.15 Consider the set $A = \{a, b, c, 1\}$ with the following table of $ \to $:

|   | a | b | c | 1 |
|---|---|---|---|---|
| a | 1 | a | a | 1 |
| b | a | 1 | 1 | 1 |
| c | 1 | a | 1 | 1 |
| 1 | a | b | c | 1 |

It verifies (Re), (M), (L), (An), (D). It does not verify (Ex) for $a, b, b$, (pi) for $b, a$, (BB), (**) for $a, b, c$, (B), (*), (Tr) for $b, c, a$.
Hence, $A$ is a **proper aRML algebra**, with (D).

- **Proper BCC algebras** (PO4)
Example 8.16 Consider the set \( A = \{a, b, c, d, 1\} \) with the following table of \( \to \):

\[
\begin{array}{cccc|c}
\rightarrow & a & b & c & d \\
a & 1 & a & a & b & 1 \\
b & 1 & 1 & a & 1 \\
c & 1 & a & 1 & a & 1 \\
d & 1 & 1 & a & 1 & 1 \\
1 & a & b & c & d & 1
\end{array}
\]

\( A \) satisfies properties \((\text{Re}), (\text{M}), (\text{L}), (\text{An}), (\text{B})\) (hence \((\ast), (\text{Tr})\)) and \((**).\) It does not satisfy \((\text{Ex})\) for \(a, c, d\), \((\text{pi})\) for \(b, a\), \((\text{BB})\) for \(d, 1, c\), \((\text{D})\) for \(d, c\).

Hence, \( A \) is a proper BCC algebra, with \((**),\) without \((\text{D}).\)

8.2.2 Examples of RML algebras with condition \((\text{Ex})\):

- Proper BE algebras \((\text{PO6})\)

Example 8.17 Consider the set \( A = \{a, b, c, 1\} \) with the following table of \( \to \):

\[
\begin{array}{cccc|c}
\rightarrow & a & b & c & 1 \\
a & 1 & a & 1 & 1 \\
b & 1 & 1 & a & 1 \\
c & 1 & a & 1 & 1 \\
1 & a & b & c & 1
\end{array}
\]

\( A \) satisfies \((\text{Re}), (\text{M}), (\text{L}), (\text{Ex})\) (hence \((\text{D})\)). It does not satisfy: \((\text{pi})\) for \(b, a\), \((\text{BB})\), \((**)\) for \(c, b, a\), \((\ast)\), \((\text{Tr})\) for \(b, a, c\), \((\text{An})\) for \(a, c\).

Hence, \( A \) is a proper BE algebra.

Example 8.18 Consider the set \( A = \{0, a, b, c, 1\} \) with the following table of \( \to \):

\[
\begin{array}{cccc|c}
\rightarrow & 0 & a & b & c & 1 \\
0 & 1 & 1 & 1 & 1 & 1 \\
a & c & 1 & 1 & 1 & 1 \\
b & b & 1 & 1 & 1 & 1 \\
c & a & a & 1 & 1 & 1 \\
1 & 0 & a & b & c & 1
\end{array}
\]

Then the bounded algebra \( A = (A, \to, 0, 1) \) verifies properties \((\text{Re}), (\text{M}), (\text{L}), (\text{Ex})\) (hence \((\text{D})\)) and \((\text{DN}).\) It does not verify \((\text{An}), (\text{BB}), (\text{B}), (\ast), (**), (\text{Tr}), (\pi).\)

Hence, \( A \) is a proper BE algebra, with \((\text{DN}).\)

- Proper pre-BCK algebras \((\text{PO7})\)

Example 8.19 Consider the set \( A = \{a, b, c, 1\} \) with the following table of \( \to \):

\[
\begin{array}{cccc|c}
\rightarrow & a & b & c & 1 \\
a & 1 & a & a & 1 \\
b & 1 & 1 & 1 & 1 \\
c & 1 & 1 & 1 & 1 \\
1 & a & b & c & 1
\end{array}
\]

Properties \((\text{Re}), (\text{M}), (\text{L}), (\text{Ex})\) (hence \((\text{D})\)), \((\ast)\) (hence \((\text{B}), (\text{BB}), (**), (\text{Tr})\)) are satisfied. It does not satisfy \((\text{An})\) for \(b, c\); \((\pi)\) for \(b, a\).

Hence, \( A \) is a proper pre-BCK algebra.
Example 8.20 Consider the set $A = \{0, a, b, c, 1\}$ with the following table of $\to$:

|   | 0 | a | b | c | 1 |
|---|---|---|---|---|---|
| 0 | 1 | 1 | 1 | 1 | 1 |
| a | c | 1 | 1 | 1 | 1 |
| b | b | 1 | 1 | 1 | 1 |
| c | a | 1 | 1 | 1 | 1 |
| 1 | 0 | a | b | c | 1 |

Then the bounded algebra $\mathcal{A} = (A, \to, 0, 1)$ verifies properties (Re), (M), (L), (Ex) (hence (D)), (*) (hence (B), (BB), (**), (Tr)) and (DN). It does not verify properties (An), (pi). Hence, $\mathcal{A}$ is a **proper pre-BCK algebra**, with (DN).

• **Proper $aBE$ algebras** (P2)

Example 8.21 Consider the set $A = \{a, b, c, 1\}$ with the following table of $\to$:

|   | a | b | c | 1 |
|---|---|---|---|---|
| a | 1 | a | a | 1 |
| b | 1 | 1 | c | 1 |
| c | a | b | 1 | 1 |
| 1 | a | b | c | 1 |

Properties (Re), (M), (L), (An) and (Ex) (hence (D)) are satisfied. It does not satisfy: (pi) for $b, a$, (BB) and (**) for $c, b, a$, (B), (*), (Tr) for $b, a, c$.

Hence, $\mathcal{A}$ is a **proper $aBE$ algebra**.

• **Proper BCK algebras** (PO2)

Example 8.22 Consider the set $A = \{a, b, c, 1\}$ with the following table of $\to$:

|   | a | b | c | 1 |
|---|---|---|---|---|
| a | 1 | a | a | 1 |
| b | 1 | 1 | a | 1 |
| c | 1 | a | 1 | 1 |
| 1 | a | b | c | 1 |

Properties (Re), (M), (L), (Ex) (hence (D)), (An), (BB) (hence (B), (*), (**), (Tr)) are satisfied. It does not satisfy (pi) for $b, a$.

Hence, $\mathcal{A}$ is a **proper BCK algebra**.

One can find many examples of proper BCK algebras in [18].

9 **Examples of the other twenty two new RM and RML algebras** (P10) - (P31)

We present examples of proper algebras.

9.1 **Examples of RM and RML algebras without (Ex)**

9.1.1 **RM algebras without (Ex)**:  
$tRM, \ast RM, RM**, \ast RM**, \text{pre-BBBZ}$,  
oRM, $\ast aRM, aRM**, \ast aRM**

• **Proper $tRM$ algebras** (P10)
Example 9.1 Consider the set $A = \{a, b, c, 1\}$ with the following table of $\rightarrow$:

| $\rightarrow$ | a  | b  | c  | 1  |
|---------------|----|----|----|----|
| a             | 1  | b  | 1  | b  |
| b             | 1  | a  | c  |    |
| c             | 1  | c  | 1  | c  |
| 1             | a  | b  | c  | 1  |

Then the algebra $\mathcal{A} = (A, \rightarrow, 1)$ verifies properties (Re), (M), (Tr). It does not verify: (Ex) for $a, b, a$, (An) for $a, c$, (L) for $x = b$, (**), (BB) for $b, a, c$, (*) for $b, a, c$, (D) for $b, a$.

Hence, $\mathcal{A}$ is a proper tRM algebra, without (D).

• Proper *RM algebras (P11)

Example 9.2 Consider the set $A = \{0, a, b, 1\}$ with the following table of $\rightarrow$:

| $\rightarrow$ | 0  | a  | b  | 1  |
|---------------|----|----|----|----|
| 0             | 1  | 1  | 1  | 1  |
| a             | 1  | 1  | 1  | 1  |
| b             | 0  | a  | 1  | a  |
| 1             | 0  | a  | b  | 1  |

Then the algebra $\mathcal{A} = (A, \rightarrow, 1)$ verifies properties (Re), (M), (*) (hence (Tr)). It does not verify properties (Ex), (An), (L), (B), (**), (***) (hence (BB)), (D).

Hence, $\mathcal{A}$ is a proper *RM algebra, without (D).

• Proper RM** algebras (P12)

Example 9.3 Consider the set $A = \{a, b, c, 1\}$ with the following table of $\rightarrow$:

| $\rightarrow$ | a  | b  | c  | 1  |
|---------------|----|----|----|----|
| a             | 1  | a  | b  | a  |
| b             | b  | 1  | 1  | a  |
| c             | c  | 1  | c  | a  |
| 1             | a  | b  | c  | 1  |

Then the algebra $\mathcal{A} = (A, \rightarrow, 1)$ verifies properties (Re), (M), (**), (***) (hence (Tr)). It does not verify: (Ex) for $a, b, b$, (An) for $b, c$, (L) for $x = a$, (BB) for $a, a, c$, (B) for $a, b, c$, (D) for $c, a$.

Hence, $\mathcal{A}$ is a proper RM** algebra, without (D).

• Proper *RM** algebras (P13)

Example 9.4 Consider the set $A = \{a, b, c, 1\}$ with the following table of $\rightarrow$:

| $\rightarrow$ | a  | b  | c  | 1  |
|---------------|----|----|----|----|
| a             | 1  | 1  | c  | c  |
| b             | 1  | 1  | c  | c  |
| c             | a  | b  | 1  | 1  |
| 1             | a  | b  | c  | 1  |

Then the algebra $\mathcal{A} = (A, \rightarrow, 1)$ verifies properties (Re), (M), (*), (**), (***) (hence (Tr)). It does not verify properties (Ex), (An), (L), (BB), (B), (D).

Hence, $\mathcal{A}$ is a proper *RM** algebra, without (D).

• Proper pre-BBBZ algebras (P14)

Example 9.5 Consider the set $A = \{a, b, 1\}$ with the following table of $\rightarrow$:
Properties (Re), (M), (B), (BB) (hence (*), (**), (Tr), (D)) are satisfied. (Ex) is not satisfied for \(a, b, a\), (An) for \(a, b, (L)\) for \(x = a\).

Hence, \(\mathcal{A}\) is a **proper pre-BBBZ algebra**, with minimum number of elements, three.

**Example 9.6** Consider the set \(A = \{a, b, c, 1\}\) with the following table of \(\rightarrow\):

\[
\begin{array}{ccc}
\rightarrow & a & b \\
\hline
a & 1 & a & a \\
b & b & 1 & 1 & a \\
c & c & 1 & 1 & a \\
1 & 1 & a & b \\
\end{array}
\]

Properties (Re), (M), (B), (BB) (hence (*), (**), (Tr), (D)) are satisfied. (Ex) is not satisfied for \(a, c, b\), (An) for \(b, c\), (L) for \(x = b\).

Hence, \(\mathcal{A}\) is a **proper pre-BBBZ algebra**, one of the very many pre-BBBZ algebras with four elements.

• **Proper oRM algebras** (P15)

**Example 9.7** Consider the set \(A = \{a, b, 1\}\) with the following table of \(\rightarrow\):

\[
\begin{array}{ccc}
\rightarrow & a & b & 1 \\
\hline
a & 1 & a & a \\
b & 1 & 1 & b \\
1 & a & b & 1 \\
\end{array}
\]

Properties (Re), (M), (An), (Tr), (D) are satisfied. \(\mathcal{A}\) does not satisfy (L), (Ex), (**), (D) (hence (B)).

Hence, \(\mathcal{A}\) is a **proper oRM algebra**, with (D).

**Example 9.8** Consider the set \(A = \{a, b, c, 1\}\) with the following table of \(\rightarrow\):

\[
\begin{array}{ccc}
\rightarrow & a & b & c & 1 \\
\hline
a & 1 & a & a & a \\
b & a & 1 & a & a \\
c & b & 1 & 1 & 1 \\
1 & a & b & c & 1 \\
\end{array}
\]

Properties (Re), (M), (An), (Tr) are satisfied. \(\mathcal{A}\) does not satisfy (L) for \(x = a\); (Ex) for \(a, b, a\); (BB), (**), for \(b, c, 1\); (B), (*) for \(b, c, b\); (D) for \(a, b\).

Hence, \(\mathcal{A}\) is a **proper oRM algebra**, without (D).

• **Proper *aRM algebras** (P16)

**Example 9.9** Consider the set \(A = \{a, b, c, 1\}\) with the following table of \(\rightarrow\):

\[
\begin{array}{ccc}
\rightarrow & a & b & 1 \\
\hline
a & 1 & a & a \\
b & b & 1 & 1 \\
1 & a & b & 1 \\
\end{array}
\]

Properties (Re), (M), (An), (*) (hence (Tr)), (D) are satisfied. \(\mathcal{A}\) does not satisfy (L), (Ex), (BB), (**), (B).

Hence, \(\mathcal{A}\) is a **proper *aRM algebra**, with (D).
Example 9.10 Consider the set $A = \{a, b, c, 1\}$ with the following table of $\rightarrow$:

|     | a | b | c | 1 |
|-----|---|---|---|---|
| a   | 1 | a | a | a |
| b   | a | 1 | a | a |
| c   | a | a | 1 | 1 |
| 1   | a | b | c | 1 |

Properties (Re), (M), (An), (*) (hence (Tr)) are satisfied. $A$ does not satisfy (L) for $x = a$, (Ex) for $a, b, a$, (BB), (**) for $b, c, 1$, (B) for $a, b, c$, (D) for $a, b$.
Hence, $A$ is a proper *aRM algebra, without (D).

- Proper aRM** algebras (P17)

Example 9.11 Consider the set $A = \{a, b, c, 1\}$ with the following table of $\rightarrow$:

|     | a | b | c | 1 |
|-----|---|---|---|---|
| a   | 1 | a | a | a |
| b   | a | 1 | a | b |
| c   | a | b | 1 | 1 |
| 1   | a | b | c | 1 |

Properties (Re), (M), (An), (**) (hence (Tr)) are satisfied. $A$ does not satisfy (L) for $x = a$, (Ex) for $a, b, a$, (BB) for $a, b, c$, (B), (*) for $b, c, 1$, (D) for $a, b$.
Hence, $A$ is a proper aRM** algebra, without (D).

- Proper *aRM** algebras (P18)

Example 9.12 Consider the set $A = \{a, b, c, 1\}$ with the following table of $\rightarrow$:

|     | a | b | c | 1 |
|-----|---|---|---|---|
| a   | 1 | a | a | a |
| b   | a | 1 | a | a |
| c   | a | a | 1 | a |
| 1   | a | b | c | 1 |

Properties (Re), (M), (An), (*), (**) (hence (Tr)) are satisfied. $A$ does not satisfy (L) for $x = a$, (Ex) for $a, b, a$, (BB), (B) for $a, b, c$, (D) for $a, b$.
Hence, $A$ is a proper *aRM** algebra, without (D).

9.1.2 RML algebras without (Ex):
- tRML, *RML, RML**, *RML**, pre-BBBCC,
- oRML, *aRML, aRML**, *aRML**

- Proper tRML algebras (P19)

Example 9.13 Consider the set $A = \{a, b, c, 1\}$ with the following table of $\rightarrow$:

|     | a | b | c | 1 |
|-----|---|---|---|---|
| a   | 1 | a | b | 1 |
| b   | a | 1 | 1 | 1 |
| c   | a | 1 | 1 | 1 |
| 1   | a | b | c | 1 |

Properties (Re), (M), (L), (Tr), (D) are satisfied. $A$ does not satisfy (An) for $b, c$; (Ex) for $a, b, b$; (pi) for $b, a$; (BB), (**) for $b, a, 1$; (B), (*) for $a, b, c$.
Hence, $A$ is a proper tRML algebra, with (D).
Example 9.14 Consider the set $A = \{0, a, b, c, 1\}$ with the following table of $\to$:

\[
\begin{array}{c|ccccc}
\to & 0 & a & b & c & 1 \\
\hline
0 & 1 & 1 & 1 & 1 & 1 \\
\hline
a & c & 1 & 1 & 1 & 1 \\
b & b & a & 1 & 1 & 1 \\
c & a & a & 1 & 1 & 1 \\
1 & 0 & a & b & c & 1 \\
\end{array}
\]

Then the bounded algebra $\mathcal{A} = (A, \to, 0, 1)$ verifies properties (Re), (M), (L), (Tr) and (D), (DN). It does not verify properties (An), (Ex), (BB), (B), (\text{*}), (**) (pi).

Hence, $\mathcal{A}$ is a proper tRML algebra, with (DN), and with (D).

Example 9.15 Consider the set $A = \{0, a, b, c, 1\}$ with the following table of $\to$:

\[
\begin{array}{c|ccccc}
\to & 0 & a & b & c & 1 \\
\hline
0 & 1 & 1 & 1 & 1 & 1 \\
\hline
a & c & 1 & 1 & 1 & 1 \\
b & b & a & 1 & 1 & 1 \\
c & a & b & 1 & 1 & 1 \\
1 & 0 & a & b & c & 1 \\
\end{array}
\]

Then the bounded algebra $\mathcal{A} = (A, \to, 0, 1)$ verifies properties (Re), (M), (L), (Tr), and (DN). It does not verify (An) for $b, c$; (Ex) for $b, c, 0$; (BB), (**) for $0, c, b$; (B) for $a, c, (\text{pi})$ for $0, a$.

Hence, $\mathcal{A}$ is a proper tRML algebra, with (DN), without (D).

- Proper *RML algebras (P20)

Example 9.16 Consider the set $A = \{a, b, c, 1\}$ with the following table of $\to$:

\[
\begin{array}{c|ccccc}
\to & a & b & c & 1 \\
\hline
a & 1 & a & a & 1 \\
b & a & 1 & 1 & 1 \\
c & b & a & 1 & 1 \\
1 & a & b & c & 1 \\
\end{array}
\]

Properties (Re), (M), (L), (\text{*}) (hence (Tr)), (D) are satisfied. $\mathcal{A}$ does not satisfy (An) for $b, c$; (Ex) for $a, b, b$; (\text{pi}) for $b, a$; (**) for $0, c, b$; (BB) for $b, a, 1$; (B) for $a, c, 1, b$.

Hence, $\mathcal{A}$ is a proper *RML algebra, with (D).

Example 9.17 Consider the set $A = \{0, a, b, c, 1\}$ with the following table of $\to$:

\[
\begin{array}{c|ccccc}
\to & 0 & a & b & c & 1 \\
\hline
0 & 1 & 1 & 1 & 1 & 1 \\
\hline
a & c & 1 & 1 & 1 & 1 \\
b & b & c & 1 & 1 & 1 \\
c & a & c & 1 & 1 & 1 \\
1 & 0 & a & b & c & 1 \\
\end{array}
\]

Then the bounded algebra $\mathcal{A} = (A, \to, 0, 1)$ verifies properties (Re), (M), (L), (\text{*}) (hence (Tr)) and (D), and (DN). It does not verify properties (An), (Ex), (BB), (B), (**) (pi).

Hence, $\mathcal{A}$ is a proper *RML algebra, with (DN), with (D).

Example 9.18 Consider the set $A = \{0, a, b, c, 1\}$ with the following table of $\to$:
Then the bounded algebra $A = (A, \rightarrow, 0, 1)$ verifies properties (Re), (M), (L), (*) (hence (Tr)), and (DN). It does not verify properties (An), (Ex), (BB), (B), (**), (D), (pi). Hence, $A$ is a proper *RML algebra, with (DN), without (D).

• Proper RML** algebras (P21)

Example 9.19 Consider the set $A = \{a, b, c, d, 1\}$ with the following table of $\rightarrow$:

| $\rightarrow$ | a | b | c | d | 1 |
|---------------|---|---|---|---|---|
| a             | 1 | a | b | 1 |   |
| b             | 1 | 1 | a | b |   |
| c             | 1 | 1 | 1 | 1 |   |
| d             | 1 | 1 | 1 | 1 |   |
| 1             | a | b | c | d | 1 |

Properties (Re), (M), (L), (**), (Tr) are satisfied. $A$ does not satisfy (An) for $c, d$, (Ex) for $a, b, d$, (pi) for $b, a$, (BB) for $d, a, c$, (B), (pi) for $a, c, d$, (D) for $d, a$. Hence, $A$ is a proper *RML** algebra, without (D).

• Proper *RML** algebras (P22)

Example 9.20 Consider the set $A = \{a, b, c, d, 1\}$ with the following table of $\rightarrow$:

| $\rightarrow$ | a | b | c | d | 1 |
|---------------|---|---|---|---|---|
| a             | 1 | a | c | c | 1 |
| b             | 1 | 1 | d | c | 1 |
| c             | a | b | 1 | 1 | 1 |
| d             | a | b | 1 | 1 | 1 |
| 1             | a | b | c | d | 1 |

Properties (Re), (M), (L), (**), (Tr) are satisfied. $A$ does not satisfy (An), (Ex), (pi), (BB), (B), (D). Hence, $A$ is a proper *RML** algebra, without (D).

• Proper pre-BBBCC algebras (P23)

Example 9.21 Consider the set $A = \{a, b, c, d, 1\}$ with the following table of $\rightarrow$:

| $\rightarrow$ | a | b | c | d | 1 |
|---------------|---|---|---|---|---|
| a             | 1 | a | c | c | 1 |
| b             | 1 | 1 | d | c | 1 |
| c             | a | b | 1 | 1 | 1 |
| d             | a | b | 1 | 1 | 1 |
| 1             | a | b | c | d | 1 |

Properties (Re), (M), (L), (BB), (B) (hence (**), (Tr), (D)) are satisfied. $A$ does not satisfy (An) for $x = c$, $y = d$, (Ex) for $a, b, c$, (pi) for $b, a$. Hence, $A$ is a proper pre-BBBCC algebra.

• Proper oRML algebras (P24)
Example 9.22 Consider the set \( A = \{a, b, c, 1\} \) with the following table of \( \rightarrow \):

| \( \rightarrow \) | a | b | c | 1 |
|-----------------|---|---|---|---|
| a               | 1 | a | a | 1 |
| b               | a | 1 | b | 1 |
| c               | 1 | a | 1 | 1 |
| 1               | a | b | c | 1 |

Properties (Re), (M), (L), (An), (Tr), (D) are satisfied. \( A \) does not satisfy: (Ex) for \( a, b, b \); (pi) for \( b, a \); (**) for \( b, c, 1 \); (*) for \( b, c, a \).

Hence, \( A \) is a proper oRML algebra, with (D).

Example 9.23 Consider the set \( A = \{0, a, b, c, 1\} \) with the following table of \( \rightarrow \):

| \( \rightarrow \) | 0 | a | b | c | 1 |
|-----------------|---|---|---|---|---|
| 0               | 1 | 1 | 1 | 1 | 1 |
| a               | c | 1 | 1 | 1 | 1 |
| b               | b | a | 1 | 1 | 1 |
| c               | a | a | a | 1 | 1 |
| 1               | 0 | a | b | c | 1 |

Then the bounded algebra \( A = (A, \rightarrow, 0, 1) \) verifies properties (Re), (M), (L), (An), (Tr) and (D), and (DN). It does not verify properties (Ex), (BB), (B), (**), (*), (pi).

Hence, \( A \) is a proper oRML algebra, with (DN) and with (D).

Example 9.24 Consider the set \( A = \{0, a, b, c, 1\} \) with the following table of \( \rightarrow \):

| \( \rightarrow \) | 0 | a | b | c | 1 |
|-----------------|---|---|---|---|---|
| 0               | 1 | 1 | 1 | 1 | 1 |
| a               | c | 1 | 1 | 1 | 1 |
| b               | b | a | 1 | 1 | 1 |
| c               | a | b | a | 1 | 1 |
| 1               | 0 | a | b | c | 1 |

Then the bounded algebra \( A = (A, \rightarrow, 0, 1) \) verifies properties (Re), (M), (L), (An), (Tr), and (D). It does not verify properties (Ex), (BB), (B), (**), (*), (D), (pi).

Hence, \( A \) is a proper oRML algebra, with (DN), without (D).

- Proper *aRML algebras (P25)

Example 9.25 Consider the set \( A = \{a, b, c, 1\} \) with the following table of \( \rightarrow \):

| \( \rightarrow \) | a | b | c | 1 |
|-----------------|---|---|---|---|
| a               | 1 | a | a | 1 |
| b               | a | 1 | a | 1 |
| c               | a | a | 1 | 1 |
| 1               | a | b | c | 1 |

Properties (Re), (M), (L), (*) (hence (Tr)) are satisfied. \( A \) does not satisfy: (Ex) for \( a, b, b \); (pi) for \( b, a \); (BB), (**) for \( b, a, 1 \); (B) for \( a, 1, b \); (D) for \( b, c \).

Hence, \( A \) is a proper *aRML algebra, without (D).

Example 9.26 Consider the set \( A = \{a, b, 1\} \) with the following table of \( \rightarrow \):

| \( \rightarrow \) | a | b | 1 |
|-----------------|---|---|---|
| a               | 1 | a | 1 |
| b               | a | 1 | 1 |
| 1               | a | b | 1 |
Then the algebra $\mathcal{A} = (A, \rightarrow, 1)$ verifies properties (Re), (M), (L), (An), (*) (hence (Tr)), and (D). It does not verify (Ex), (BB), (**) (B), (pi).

Hence, $\mathcal{A}$ is a proper *aRML algebra, with (D).

Example 9.27 Consider the set $A = \{0, a, b, c, 1\}$ with the following table of $\rightarrow$:

| $\rightarrow$ | 0 | a | b | c | 1 |
|---------------|---|---|---|---|---|
| 0             | 1 | 1 | 1 | 1 | 1 |
| a             | c | 1 | 1 | 1 | 1 |
| b             | b | c | 1 | 1 | 1 |
| c             | a | a | a | 1 | 1 |
| 1             | 0 | a | b | c | 1 |

Then the bounded algebra $\mathcal{A} = (A, \rightarrow, 0, 1)$ verifies properties (Re), (M), (L), (An), (**) (hence (Tr)), and (DN). It does not verify properties (Ex), (BB), (B), (pi).

Hence, $\mathcal{A}$ is a proper *aRML algebra, with (DN), without (D).

Example 9.28 Consider the set $A = \{a, b, c, 1\}$ with the following table of $\rightarrow$:

| $\rightarrow$ | a | b | c | 1 |
|---------------|---|---|---|---|
| a             | 1 | a | c | 1 |
| b             | 1 | 1 | 1 | 1 |
| c             | a | a | 1 | 1 |
| 1             | a | b | 1 | 1 |

Properties (Re), (M), (L), (An), (**) (hence (Tr)) are satisfied. $\mathcal{A}$ does not satisfy: (Ex) for $a, c, b$; (pi) for $b, a$; (BB) for $b, 1, c$; (B), (*) for $a, b, c$; (D) for $b, c$.

Hence, $\mathcal{A}$ is a proper aRML** algebra, without (D).

Example 9.29 Consider the set $A = \{0, a, b, c, 1\}$ with the following table of $\rightarrow$:

| $\rightarrow$ | 0 | a | b | c | 1 |
|---------------|---|---|---|---|---|
| 0             | 1 | 1 | 1 | 1 | 1 |
| a             | c | 1 | 1 | 1 | 1 |
| b             | b | a | 1 | 1 | 1 |
| c             | a | a | b | 1 | 1 |
| 1             | 0 | a | b | c | 1 |

Then the bounded algebra $\mathcal{A} = (A, \rightarrow, 0, 1)$ verifies properties (Re), (M), (L), (An), (**) (hence (Tr)), (D) and (DN). It does not verify properties (Ex), (BB), (B), (pi).

Hence, $\mathcal{A}$ is a proper aRML** algebra, with (DN), with (D).

Example 9.30 Consider the set $A = \{0, a, b, c, 1\}$ with the following table of $\rightarrow$:

| $\rightarrow$ | 0 | a | b | c | 1 |
|---------------|---|---|---|---|---|
| 0             | 1 | 1 | 1 | 1 | 1 |
| a             | c | 1 | 1 | 1 | 1 |
| b             | b | b | 1 | 1 | 1 |
| c             | a | a | c | 1 | 1 |
| 1             | 0 | a | b | c | 1 |

Then the bounded algebra $\mathcal{A} = (A, \rightarrow, 0, 1)$ verifies properties (Re), (M), (L), (An), (*), (**) (hence (Tr)), (D) and (DN). It does not verify properties (Ex), (BB), (B), (pi).

Hence, $\mathcal{A}$ is a proper *aRML** algebra, with (DN), with (D).
Example 9.31 Consider the set $A = \{0, a, b, c, 1\}$ with the following table of $\rightarrow$:

| $\rightarrow$ | 0 | a | b | c | 1 |
|---------------|---|---|---|---|---|
| 0             | 1 | 1 | 1 | 1 | 1 |
| a             | c | 1 | 1 | 1 | 1 |
| b             | b | c | 1 | 1 | 1 |
| c             | a | a | b | 1 | 1 |
| 1             | 0 | a | b | c | 1 |

Then the bounded algebra $A = (A, \rightarrow, 0, 1)$ verifies properties (Re), (M), (L), (An), (**), (*) (hence (Tr)), and (DN). It does not verify properties (Ex), (BB), (B), (D), (pi). Hence, $A$ is a proper *aRML** algebra, with (DN), without (D).

9.1.3 Examples of RM and RML algebras with (Ex):
RME**, BCH**, BE**, aBE**

• Proper RME** algebras (P28)

Example 9.32 Consider the set $A = \{a, b, c, d, 1\}$ with the following table of $\rightarrow$:

| $\rightarrow$ | a | b | c | d | 1 |
|---------------|---|---|---|---|---|
| a             | 1 | a | a | a | a |
| b             | a | 1 | b | d | 1 |
| c             | a | 1 | 1 | 1 | 1 |
| d             | a | 1 | 1 | 1 | 1 |
| 1             | a | b | c | d | 1 |

Properties (Re), (M), (Ex) (hence (D)), (**), (hence (Tr)), (D) are satisfied. $A$ does not satisfy: (An) for $c, d$; (BB) for $a, b, c$; (B), (*) for $b, c, d$. Hence, $A$ is a proper RME** algebra.

• Proper BCH** algebras (P29)

Example 9.33 Consider the set $A = \{a, b, c, d, 1\}$ with the following table of $\rightarrow$:

| $\rightarrow$ | a | b | c | d | 1 |
|---------------|---|---|---|---|---|
| a             | 1 | a | a | a | a |
| b             | a | 1 | b | d | 1 |
| c             | a | 1 | 1 | 1 | 1 |
| d             | a | b | c | 1 | 1 |
| 1             | a | b | c | d | 1 |

Properties (Re), (M), (Ex) (hence (D)), (An), (**), (hence (Tr)) are satisfied. $A$ does not satisfy: (BB) for $d, b, c$; (B), (*) for $b, c, d$. Hence, $A$ is a proper BCH** algebra.

• Proper BE** algebras (P30)

Example 9.34 Consider the set $A = \{a, b, c, 1\}$ with the following table of $\rightarrow$:

| $\rightarrow$ | a | b | c | 1 |
|---------------|---|---|---|---|
| a             | 1 | a | b | 1 |
| b             | 1 | 1 | 1 | 1 |
| c             | 1 | 1 | 1 | 1 |
| 1             | a | b | c | 1 |

Properties (Re), (M), (L), (Ex) (hence (D)), (**), (hence (Tr)) are satisfied. $A$ does not satisfy: (An) for
b, c; (pi) for b, a; (BB) for c, a, b; (B), (*) for a, b, c.
Hence, $\mathcal{A}$ is a proper BE** algebra.

- Proper aBE** algebras (P31)

**Example 9.35** Consider the set $A = \{0, a, b, 1\}$ with the following table of $\to$:

|    | 0 | a | b | 1 |
|----|---|---|---|---|
| 0  | 1 | 1 | 1 | 1 |
| a  | 0 | 1 | b | 1 |
| b  | b | a | 1 | 1 |
| 1  | 0 | a | b | 1 |

Then the algebra $\mathcal{A} = (A, \to, 1)$ verifies properties (Re), (M), (L), (Ex) (hence (D)), (An), (**) (hence (Tr)) (see [1], Example 3.9). It does not verify condition (BB) for $x = a, y = b, z = 0$: $b = b \to 0 = y \to z \not\leq (z \to x) \to (y \to x) = (0 \to a) \to (b \to a) = 1 \to a = a$; it does not verify condition (pi) for $x = 0, y = b$: $1 = b \to b = b \to (b \to 0) = y \to (y \to x) \neq y \to x = b \to 0 = b$. The relation $\leq$ is a lattice order. Hence, $\mathcal{A}$ is a proper aBE** lattice.

We shall represent the set $A$ and the binary relation $\leq$ by the following Hasse diagram:

![Figure 13: Proper aBE** lattice](image)

**Remark:** We have proved that condition (BB) implies (transitivity) (Tr); but (Tr) does not imply condition (BB). Indeed, the relation $\leq$ associated with above $\mathcal{A}$ is transitive, but $\mathcal{A}$ does not verify (BB).

**10 Examples of (proper) generalizations of Hilbert algebras**

**10.1 Examples of pi-RML algebras from Hierarchy 2H**

Recall that we have proved that only RML algebras can have the properties (pi) and (pimpl).

**10.1.1 Examples of pi-RML algebras without condition (Ex):**

- pi-RML, pi-pre-BCC, pi-aRML, pi-BCC

- **Proper pi-RML algebras** (P8-pi)

**Example 10.1** Consider the set $A = \{a, b, c, d, 1\}$ with the following table of $\to$:

|    | a | b | c | d | 1 |
|----|---|---|---|---|---|
| a  | 1 | b | b | b | 1 |
| b  | a | 1 | a | a | 1 |
| c  | a | a | 1 | 1 | 1 |
| d  | a | 1 | 1 | 1 | 1 |
| 1  | a | b | c | d | 1 |

$\mathcal{A}$ satisfies (Re), (M), (L), (pi). It does not satisfy: (An) for $c, d$, (Ex) for $a, c, b$, (pimpl) for $b, c, a$, (BB), (BB)}.
Hence, \( A \) is a **proper pi-RML algebra**, without (D).

- **Proper pi-pre-BCC algebras** (P1-pi)

**Example 10.2** Consider the set \( A = \{a, b, c, d, 1\} \) with the following table of \( \rightarrow \):

| \rightarrow | a | b | c | d | 1 |
|------------|---|---|---|---|---|
| a          | 1 | b | b | b | 1 |
| b          | a | 1 | c | c | 1 |
| c          | a | a | 1 | 1 | 1 |
| d          | a | 1 | 1 | 1 | 1 |
| 1          | a | b | c | d | 1 |

Properties (Re), (M), (L), (B) (hence (*), (Tr), (**)) and (pi) are satisfied. \( A \) does not satisfy (An) for \( c, d \), (Ex) for \( a, b, c \), (pimpl) for \( a, b, c \), (BB) for \( c, b, a \), (D) for \( d, a \).

Hence, \( A \) is a **proper pi-pre-BCC algebra**, without (D).

- **Proper pi-aRML algebras** (P9-pi)

**Example 10.3** Consider the set \( A = \{a, b, c, d, 1\} \) with the following table of \( \rightarrow \):

| \rightarrow | a | b | c | d | 1 |
|------------|---|---|---|---|---|
| a          | 1 | b | b | b | 1 |
| b          | a | 1 | a | a | 1 |
| c          | a | a | 1 | 1 | 1 |
| d          | a | 1 | a | 1 | 1 |
| 1          | a | b | c | d | 1 |

Properties (Re), (M), (L), (An), (pi) are satisfied. \( A \) does not satisfy: (Ex) for \( a, c, b \), (B) for \( a, 1, c \), (BB) for \( b, c, d \), (*) for \( c, d, b \), (** for \( c, d, b \), (Tr) for \( c, d, b \), (pimpl) for \( b, c, a \), (D) for \( b, c \).

Hence, \( A \) is a **proper pi-aRML algebra**, without (D).

- **Proper pi-BCC algebras** (PO4-pi)

**Example 10.4** Consider the set \( A = \{a, b, c, 1\} \) with the following table of \( \rightarrow \):

| \rightarrow | a | b | c | 1 |
|------------|---|---|---|---|
| a          | 1 | b | b | 1 |
| b          | a | 1 | c | 1 |
| c          | 1 | 1 | 1 | 1 |
| 1          | a | b | c | 1 |

Then the algebra \( A = (A, \rightarrow, 1) \) verifies properties (Re), (M), (L), (An), (B) (hence (*), (Tr), (**)) and (pi). It does not verify: (Ex) for \( a, b, c \), (BB) for \( c, b, a \), (pimpl) for \( a, b, c \), (D) for \( c, a \).

Hence, \( A \) is a **proper pi-BCC algebra**, without (D).

10.1.2 Examples of pi-RML algebras with condition (Ex):

- pi-BE, pi-pre-BCK and pimpI-pre-BCK, pi-aBE, pi-BCK = pimpI-BCK = Hilbert algebras

- **Proper pi-BE algebras** (PO6-pi)

**Example 10.5** Consider the set \( A = \{a, b, c, 1\} \) with the following table of \( \rightarrow \):
Then the algebra \( \mathcal{A} = (A, \rightarrow, 1) \) verifies properties (Re), (M), (L), (Ex) (hence (D)) and (pi). It does not verify (An) for \( a, b \), (BB), (***) for \( x = c, y = a, z = b \); (B), (*) for \( a, b, c \); (pimpl) for \( x = a, y = b, z = c \).

Hence, \( \mathcal{A} \) is a **proper pi-BE algebra**.

We shall represent the set \( \mathcal{A} \) and the binary relation \( \leq \) by the following Hasse-type diagram:

![Proper pi-BE algebra](Image)

**Figure 14: Proper pi-BE algebra**

- **Proper pi-pre-BCK (PO7-pi) and pimpl-pre-BCK (PO7-pimpl) algebras**

**Example 10.6** Consider the set \( \mathcal{A} = \{a, b, c, d, 1\} \) with the following table of \( \rightarrow \):

| \( \rightarrow \) | a  | b  | c  | d  | 1  |
|-------------------|----|----|----|----|----|
| a                 | 1  | b  | b  | d  | 1  |
| b                 | a  | 1  | d  | 1  |    |
| c                 | a  | 1  | d  | 1  |    |
| d                 | 1  | b  | c  | 1  | 1  |
| 1                 | a  | b  | c  | d  | 1  |

Then the algebra \( \mathcal{A} = (A, \rightarrow, 1) \) verifies properties (Re), (M), (L), (Ex) (hence (D)), (*) (hence (B), (BB), (**), (Tr)) and (pi). It does not verify properties (An) for \( x = b, y = c \) and (pimpl) for \( x = d, y = a, z = c \).

Hence, \( \mathcal{A} \) is a proper pre-BCK algebra, verifying (pi) and not verifying (pimpl), Hence, it is a **proper pi-pre-BCK algebra**.

**Example 10.7** (This example is taken from [2], from the proof of Proposition 1.1)

Consider the set \( \mathcal{A}_2 = \{c, d, 1\} \) with the following table of \( \rightarrow \):

| \( \rightarrow \) | c  | d  | 1  |
|-------------------|----|----|----|
| c                 | 1  | 1  | 1  |
| d                 | 1  | 1  |    |
| 1                 | c  | 1  |    |

Then the algebra \( \mathcal{A} = (A, \rightarrow, 1) \) verifies properties (Re), (M), (L), (Ex) (hence (D)), (*) (hence (B), (BB), (**), (Tr)) and (pimpl) (hence (pi)). It does not verify (An). \( \leq \) is only a pre-order ([2], the proof of Proposition 1.1).

Hence, \( \mathcal{A} \) is a **proper pimpl-pre-BCK algebra**, with the minimum number of elements, three.

We shall represent the set \( \mathcal{A} \) and the binary relation \( \leq \) by the following Hasse-type diagram:

**Example 10.8** Consider the set \( \mathcal{A} = \{a, b, c, 1\} \) with the following table of \( \rightarrow \):

| \( \rightarrow \) | a  | b  | c  | 1  |
|-------------------|----|----|----|----|
| a                 | 1  | 1  | c  | 1  |
| b                 | 1  | 1  | 1  | 1  |
| c                 | 1  | 1  | 1  | 1  |
| 1                 | a  | b  | c  | 1  |
Then the algebra \( A = (A, \rightarrow, 1) \) verifies properties (Re), (M), (L), (Ex) (hence (D)), (*) (hence (B), (BB), (**), (Tr)) and (pimpl) (hence (pi)). It does not verify (An). The relation \( \leq \) is a pre-order relation. Hence, \( A \) is a proper pimpl-pre-BCK algebra, with four elements.

We shall represent the set \( A \) and the binary relation \( \leq \) by the following Hasse-type diagram:

Figure 16: The proper pimpl-pre-BCK algebra with 4 elements

\begin{itemize}
\item Proper pi-aBE algebras (P2-pi)
\end{itemize}

**Example 10.9** Consider the set \( A = \{a, b, c, 1\} \) with the following table of \( \rightarrow \):

\[
\begin{array}{c|cccc}
\rightarrow & a & b & c & 1 \\
\hline
a & 1 & 1 & c & 1 \\
b & a & 1 & 1 & 1 \\
c & a & b & 1 & 1 \\
1 & a & b & c & 1 \\
\end{array}
\]

Then the algebra \( A = (A, \rightarrow, 1) \) verifies properties (Re), (M), (L), (Ex) (hence (D)), (An) and (pi). It does not verify (BB), (**) for \( x = c, y = a, z = b, (B), (\ast), (Tr) \) for \( a, b, c \), (pimpl) for \( x = a, y = b, z = c \). Hence, \( A \) is a proper aBE algebra, verifying (pi) and not verifying (pimpl), which is linearly-ordered; hence it is a proper pi-aBE algebra. We shall represent the set \( A \) and the binary relation \( \leq \) by the following Hasse-type diagram:

Figure 17: Proper linearly-ordered pi-aBE algebra

**Example 10.10** Consider the set \( A = \{0, a, b, c, 1\} \) with the following table of \( \rightarrow \):
Then the algebra $A = (A, \rightarrow, 1)$ verifies the properties (Re), (M), (L), (Ex) (hence (D)), (An) and (pi). It does not verify (BB), (***) for $x = b, y = 0, z = a$, (B), (*), (Tr) for $0, a, b$, (pimpl) for $x = 0, y = a, z = b$. Hence, $A$ is a **proper pi-aBE algebra**.

We shall represent the set $A$ and the binary relation $\leq$ by the following Hasse-type diagram:

![Hasse diagram](image)

**Figure 18: Proper non-linearly-ordered pi-aBE algebra**

**Remark:** We have proved that condition (pimpl) implies condition (pi); the inverse implication is not true. Indeed, the two above aBE algebras verify (pi), but do not verify (pimpl).

• **Hilbert algebras**

**Example 10.11** Consider the set $A = \{a, b, c, 1\}$ with the following table of $\rightarrow$:

| $\rightarrow$ | 0 | a | b | c | 1 |
|---------------|---|---|---|---|---|
| 0             | 1 | b | 1 | 1 |   |
| a             | 0 | 1 | 1 | c | 1 |
| b             | 0 | a | 1 | c | 1 |
| c             | 0 | a | b | 1 | 1 |
| 1             | 0 | a | b | c | 1 |

Properties (Re), (M), (L), (Ex) (hence (D)), (An), (BB) (hence (B), (**), (*), (Tr)) and (pimpl) (hence (pi)) are satisfied. Hence, $A$ is a **pi-BCK = pimpl-BCK = Hilbert algebra**.

### 10.2 Examples of pi-RML algebras from Hierarchy 2.1 H, 2.2 H and 2.3 H

**10.2.1 Examples of pi-RML algebras without (Ex):**
- pi-tRML, pi-*RML, pi-RML**, pi-*RML**, pi-pre-BBBCC, pimpl-pre-BCC,
- pi-oRML, pi-*aRML, pi-aRML**, pi-*aRML**

• **Proper pi-tRML algebras** (P19-pi)

**Example 10.12** Consider the set $A = \{a, b, c, d, 1\}$ with the following table of $\rightarrow$:
Properties (Re), (M), (L), (Tr), (D) and (pi) are satisfied. \( A \) does not satisfy (An) for \( c,d \); (Ex) for \( a,b,d \); (pimpl) for \( a,b,d \); (BB), (***) for \( b,c,1 \); (B) for \( a,1,c \); (*) for \( b,c,d \).

Hence, \( A \) is a proper pi-tRML algebra, with (D).

• Proper pi-*RML algebras (P20-pi)

Example 10.13 Consider the set \( A = \{a,b,c,d,1\} \) with the following table of \( \rightarrow \):

\[
\begin{array}{lllll}
\rightarrow & a & b & c & d & 1 \\
\hline
a & 1 & b & b & b & 1 \\
b & a & 1 & a & d & 1 \\
c & a & a & 1 & 1 & 1 \\
d & a & a & 1 & 1 & 1 \\
1 & a & b & c & d & 1 \\
\end{array}
\]

Properties (Re), (M), (L), (*) (hence (Tr)), (pi) are satisfied. \( A \) does not satisfy: (Ex) for \( a,c,b \); (pimpl) for \( b,c,a \); (BB) for \( d,b,a \); (B), (*) for \( b,c,d \); (D) for \( b,c \).

Hence, \( A \) is a proper pi-*RML algebra, without (D).

• Proper pi-RML** algebras (P21-pi)

Example 10.14 Consider the set \( A = \{a,b,c,d,1\} \) with the following table of \( \rightarrow \):

\[
\begin{array}{lllll}
\rightarrow & a & b & c & d & 1 \\
\hline
a & 1 & b & b & b & 1 \\
b & a & 1 & a & d & 1 \\
c & a & a & 1 & 1 & 1 \\
d & a & a & 1 & 1 & 1 \\
1 & a & b & c & d & 1 \\
\end{array}
\]

Properties (Re), (M), (L), (**) (hence (Tr)), (pi) are satisfied. \( A \) does not satisfy: (An) for \( c,d \); (Ex), (pimpl) for \( a,b,d \); (BB) for \( d,b,a \); (B) for \( a,1,c \); (D) for \( b,c \).

Hence, \( A \) is a proper pi-RML** algebra, without (D).

• Proper pi-*RML** algebras (P22-pi)

There are no proper pi-*RML** algebras with three, four or five elements (we have run PASCAL programs searching for such an algebra). Finding an example of pi-*RML** algebra with minimum six elements, if it exists, was announced as an open problem in our preprint on arXiv. Professor Michael Kinyon, Department of Mathematics, University of Denver, communicated us the following example of proper pi-*RML** algebra with six elements, found by using the finite model finder Mace4:

Example 10.15 (Michael Kinyon)

Consider the set \( A = \{a,b,c,d,e,1\} \) with the following table of \( \rightarrow \):

\[
\begin{array}{llllll}
\rightarrow & a & b & c & d & e & 1 \\
\hline
a & 1 & b & c & e & e & 1 \\
b & a & 1 & c & a & a & 1 \\
c & 1 & 1 & 1 & 1 & 1 & 1 \\
d & 1 & b & b & 1 & 1 & 1 \\
e & 1 & b & b & 1 & 1 & 1 \\
1 & a & b & c & d & e & 1 \\
\end{array}
\]
Properties (Re), (M), (L), (*), (**), and (pi) are satisfied. \(\mathcal{A}\) does not satisfy: (An) for \(d, c\); (Ex), (pimpl) for \(a, b, d\) (note that (Ex) and (pimpl) are not satisfied for the same elements); (BB) for \(c, a, d\); (B) for \(b, d, c\); (D) for \(c, a, d\).

Hence, \(\mathcal{A}\) is a proper pi-*RML** algebra.

• Proper pi-pre-BBBCC (P23-pi) and pimpl-pre-BBBCC (P23-pimpl) algebras

**Example 10.16** Consider the set \(A = \{a, b, c, d, 1\}\) with the following table of \(\rightarrow\):

\[
\begin{array}{c|ccccc}
\rightarrow & a & b & c & d & 1 \\
\hline
a & 1 & b & b & d & 1 \\
b & a & 1 & 1 & d & 1 \\
c & a & 1 & 1 & d & 1 \\
d & a & c & c & 1 & 1 \\
1 & a & b & c & d & 1 \\
\end{array}
\]

Properties (Re), (M), (L), (BB) (hence (D)), (B) (hence (**), (*), (Tr)), (pi) are satisfied. \(\mathcal{A}\) does not satisfy: (An) for \(b, c\); (Ex), (pimpl) for \(a, d, b\).

Hence, \(\mathcal{A}\) is a proper pi-pre-BBBCC algebra.

**Example 10.17** Consider the set \(A = \{a, b, c, d, 1\}\) with the following table of \(\rightarrow\):

\[
\begin{array}{c|ccccc}
\rightarrow & a & b & c & d & 1 \\
\hline
a & 1 & b & b & 1 & 1 \\
b & a & 1 & a & 1 & 1 \\
c & a & 1 & 1 & a & 1 \\
d & 1 & c & c & 1 & 1 \\
1 & a & b & c & d & 1 \\
\end{array}
\]

Properties (Re), (M), (L), (BB) (hence (D)), (B) (hence (**), (*), (Tr)) and (pimpl) (hence (pi)) are satisfied. \(\mathcal{A}\) does not satisfy: (An) for \(b, c\); (Ex) for \(a, d, b\).

Hence, \(\mathcal{A}\) is a proper pimpl-pre-BBBCC algebra.

• Proper pi-oRML algebras (P24-pi)

**Example 10.18** Consider the set \(A = \{a, b, c, d, 1\}\) with the following table of \(\rightarrow\):

\[
\begin{array}{c|ccccc}
\rightarrow & a & b & c & d & 1 \\
\hline
a & 1 & b & b & b & 1 \\
b & a & 1 & a & a & 1 \\
c & a & a & 1 & d & 1 \\
d & a & 1 & a & 1 & 1 \\
1 & a & b & c & d & 1 \\
\end{array}
\]

Properties (Re), (M), (L), (An), (Tr) and (pi) are satisfied. \(\mathcal{A}\) does not satisfy: (Ex) for \(a, c, b\); (pimpl) for \(a, c, d\); (BB) for \(b, c, d\); (***) for \(b, c, 1\); (B) for \(a, 1, c\); (*) for \(c, d, b\); (D) for \(b, c\).

Hence, \(\mathcal{A}\) is a proper pi-oRML algebra, without (D).

• Proper pi-*aRML algebras (P25-pi)

**Example 10.19** Consider the set \(A = \{a, b, c, d, 1\}\) with the following table of \(\rightarrow\):

\[
\begin{array}{c|ccccc}
\rightarrow & a & b & c & d & 1 \\
\hline
a & 1 & b & b & b & 1 \\
b & a & 1 & a & a & 1 \\
c & a & a & 1 & a & 1 \\
d & a & a & a & 1 & 1 \\
1 & a & b & c & d & 1 \\
\end{array}
\]
Then the algebra $\mathcal{A} = (A, \rightarrow, 1)$ verifies properties (Re), (M), (L), (An), (*) (hence (Tr)) and (pi). It does not verify: (Ex) for $a, c, b$, (BB) for $b, c, 1$, (B) for $a, 1, c$, (**) for $b, c, 1$, (D) for $b, c$, (pimpl) for $b, c, a$. Hence, $\mathcal{A}$ is a proper pi-*aRML algebra, without (D).

- **Proper pi-aRML** algebras (P26-pi)

**Example 10.20** Consider the set $A = \{a, b, c, d, 1\}$ with the following table of $\rightarrow$:

| $\rightarrow$ | a | b | c | d | 1 |
|---------------|---|---|---|---|---|
| a             | 1 | b | b | b | 1 |
| b             | a | 1 | a | d | 1 |
| c             | 1 | 1 | 1 | 1 | 1 |
| d             | a | 1 | a | 1 | 1 |
| 1             | a | b | c | d | 1 |

Then the algebra $\mathcal{A} = (A, \rightarrow, 1)$ verifies properties (Re), (M), (L), (An), (**), (Tr), and (pi). It does not verify: (Ex) for $a, b, d$, (BB) for $d, b, a$, (B) for $b, c, d$, (*) for $b, c, d$, (pimpl) for $a, b, d$, (D) for $d, c$. Hence, $\mathcal{A}$ is a proper pi-aRML** algebra, without (D).

- **Proper pi-*aRML** algebras (P27-pi)

**Example 10.21** Consider the set $A = \{a, b, c, d, 1\}$ with the following table of $\rightarrow$:

| $\rightarrow$ | a | b | c | d | 1 |
|---------------|---|---|---|---|---|
| a             | 1 | b | b | d | 1 |
| b             | a | 1 | c | d | 1 |
| c             | a | 1 | a | 1 | 1 |
| d             | 1 | b | b | 1 | 1 |
| 1             | a | b | c | d | 1 |

Then the algebra $\mathcal{A} = (A, \rightarrow, 1)$ verifies properties (Re), (M), (L), (An), (**), (Tr), and (pi). It does not verify: (Ex) for $a, b, c$, (BB) for $c, b, a$, (B) for $a, c, d$, (D) for $c, a$, and it does not verify condition (pimpl) for $a, b, c$. Hence, $\mathcal{A}$ is a proper pi-*aRML** algebra, without (D).

10.2.2 Examples of pi-RML algebras with (Ex): pi-BE**, pi-aBE** algebras

- **Proper pi-BE** algebras (P30-pi)

**Example 10.22** Consider the set $A = \{a, b, c, d, 1\}$ with the following table of $\rightarrow$:

| $\rightarrow$ | a | b | c | d | 1 |
|---------------|---|---|---|---|---|
| a             | 1 | b | b | d | 1 |
| b             | a | 1 | a | d | 1 |
| c             | 1 | 1 | 1 | 1 | 1 |
| d             | 1 | 1 | 1 | 1 | 1 |
| 1             | a | b | c | d | 1 |

Properties (Re), (M), (L), (Ex) (hence (D)), (**), (Tr) and (pi) are satisfied. $\mathcal{A}$ does not satisfy: (An) for $c, d$, (BB) for $d, a, c$, (B) for $a, c, d$, (Tr) for $a, c, d$, (pimpl) for $a, c, d$. Hence, $\mathcal{A} = (A, \rightarrow, 1)$ is a proper pi-BE** algebra.

- **Proper pi-aBE** algebras (P31-pi)
Example 10.23 Consider the set $A = \{a, b, c, d, 1\}$ with the following table of $\rightarrow$:

|   | a | b | c | d | 1 |
|---|---|---|---|---|---|
| a | 1 | b | b | d | 1 |
| b | 1 | a | c | d | 1 |
| c | 1 | 1 | 1 | 1 | 1 |
| d | 1 | a | b | c | 1 |
| 1 | 1 | 1 | 1 | 1 | 1 |

Then the algebra $\mathcal{A} = (A, \rightarrow, 1)$ verifies properties (Re), (M), (L), (Ex), (An), (**), (Tr) and (pi). It does not verify condition (BB) for $x = d$, $y = a$, $z = c$, condition (B) for $a, c, d$, condition (*) for $a, c, d$; it does not verify (pimpl) for $x = a$, $y = c$, $z = d$. The relation $\leq$ is reflexive, antisymmetric and transitive, hence is an order relation, namely is a lattice order.

Hence, $\mathcal{A}$ is a proper pi-aBE** lattice.

11 Final remarks

Remarks 11.1
(i) There are only $2^1 = 2$ RM algebras ($A = \{a, 1\}, \rightarrow, 1$) with two elements:

|   | a | 1 |
|---|---|---|
| a | 1 |
| 1 | 1 |

One is a special case of BCI algebra, namely the $p$-semisimple BCI algebra with two elements:

|   | a | 1 |
|---|---|---|
| a | 1 |
| 1 | 1 |

the other is a special case of BCK algebra (hence of RML algebra), namely the particular case of Hilbert algebra which is the Boolean algebra with two elements:

|   | a | 1 |
|---|---|---|
| a | 1 |
| 1 | 1 |

(ii) There are $3^2 = 81$ RM algebras ($A = \{a, b, 1\}, \rightarrow, 1$) with three elements:

|   | a | b | 1 |
|---|---|---|---|
| a | 1 | 1 |
| b | 1 | 1 |
| 1 | a | b |

and among them there are $3^2 = 9$ RML algebras with three elements:

|   | a | b | 1 |
|---|---|---|---|
| a | 1 | 1 |
| b | 1 | 1 |
| 1 | a | b |

A Pascal programme has determined all the 81 RM algebras. They are:

- 4 proper RM algebras (2 with (D), 2 without (D)), 2 proper pre-BBBZ algebras, 2 proper pre-BCI algebras (with (D));
- 3 proper BCI algebras (1 with (D), 2 without (D)), 8 proper aRM algebras (4 with (D), 4 without (D)), 8 proper *aRM algebras (2 with (D), 6 without (D)), 24 proper oRM algebras (8 with (D), 16 without (D)), 4 proper aRM** algebras (all without (D)), 17 proper *aRM** algebras (1 with (D), 16 without (D)), 17 proper *aRM** algebras (1 with (D), 16 without (D)).
(D));

- (the 9 RML algebras:) 1 proper pimpl-pre-BCK algebra, 3 proper *aRML algebras (all with (D)), 2 proper BCK algebras, 3 Hilbert algebras.

(iii) There are $4^3 = 262.144$ RM algebras ($A = \{a, b, c, 1\}, \rightarrow, 1$) with four elements:

|   | a | b | c | 1 |
|---|---|---|---|---|
| a | 1 | . | . | . |
| b | . | 1 | . | . |
| c | . | . | 1 | . |
| 1 | a | b | c | 1 |

and among them there are $4^6 = 4.096$ RML algebras with four elements:

|   | a | b | c | 1 |
|---|---|---|---|---|
| a | 1 | . | . | 1 |
| b | . | 1 | . | 1 |
| c | . | . | 1 | 1 |
| 1 | a | b | c | 1 |

Different Pascal programmes can provide the algebras we look for. We can say that there are many proper pre-BBBZ algebras, but there are no proper pre-BZ algebras, proper pre-BCC algebras, proper pre-BBBCC algebras with four elements.

(iv) There are $5^{16} = 152.587.890.625$ RM algebras ($A = \{a, b, c, d, 1\}, \rightarrow, 1$) with five elements:

|   | a | b | c | d | 1 |
|---|---|---|---|---|---|
| a | 1 | . | . | . | . |
| b | . | 1 | . | . | . |
| c | . | . | 1 | . | . |
| d | . | . | . | 1 | . |
| 1 | a | b | c | d | 1 |

and among them there are $5^{12} = 244.140.625$ RML algebras with five elements.

A Pascal programme has found, for example, all the proper pimpl-pre-BBBCC algebras with five elements, which are in number of 60.

(v) There are $6^{20} = 3.656.158.440.062.976$ RML algebras ($A = \{a, b, c, d, e, 1\}, \rightarrow, 1$) with six elements:

|   | a | b | c | d | e | 1 |
|---|---|---|---|---|---|---|
| a | 1 | . | . | . | . | 1 |
| b | . | 1 | . | . | . | 1 |
| c | . | . | 1 | . | . | 1 |
| d | . | . | . | 1 | . | 1 |
| e | . | . | . | . | 1 | 1 |
| 1 | a | b | c | d | e | 1 |

Hence, it is difficult to look for an example of RML algebra with six elements.

Remarks 11.2

1). The properties (*) and (**) are independent, i.e. there are algebras verifying (*) and not verifying (**) (for example, the proper *RM algebras) and there algebras verifying (**) and not verifying (*) (for example, the proper RM** algebras).

2). Properties (BB) and (B) are dependent, namely (BB) $\Rightarrow$ (B). We have examples of RM algebras verifying (B) and not verifying (BB) (namely the proper pre-BZ, BZ, pre-BCC, BCC algebras), we have examples of RM algebras verifying both (BB) and (B) (the proper pre-BBBZ algebras, the proper pre-BBBCC algebras).

3). Properties (B) and (**) are dependent, namely (B) $\Rightarrow$ (**). We have examples of RM algebras verifying (**) and not verifying (B) (for example, the proper RM** algebras), we have examples of RM
algebras verifying both (B) and (**) (the proper pre-BZ, BZ, pre-BCC, BCC algebras).

4). We found very many examples of pre-BBBZ algebras and few examples of pre-BBBCC algebras. We have (see Figure 6): \( \text{pre-BBBCC} = \text{pre-BBBZ} + (L); \) \( \text{pre-BBBZ} + (\text{An}) = \text{BCI} \) and \( \text{pre-BBBCC} + (\text{An}) = \text{BCK} \).

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