GENERALIZATION OF THE EHRLING INEQUALITY
AND UNIVERSAL CHARACTERIZATION
OF COMPLETELY CONTINUOUS OPERATORS

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Abstract. The present work is devoted to an extension of the well-known Ehrling inequalities, which quantitatively characterize compact embeddings of function spaces, to more general operators. Firstly, a modified notion of continuity for linear operators, named Ehrling continuity and inspired by the classical Ehrling inequality, is introduced, and then, a necessary and sufficient condition for Ehrling continuity is provided via arguments based on general topology. Secondly, general completely continuous operators between normed spaces are characterized in terms of (generalized) Ehrling type inequalities. To this end, the well-known local metrization of the weak topology (so to speak, a very weak norm) plays a crucial role. Thanks to these results, a universal relation is observed among complete continuity, the very weak norm and generalized Ehrling type inequality.

1. Introduction

We begin with recalling the definitions of complete continuity and compactness of linear operators, which are strictly distinguished throughout this paper.

Definition 1.1. Let $(X, \| \cdot \|_X)$ and $(Y, \| \cdot \|_Y)$ be normed spaces and let $T : X \to Y$ be a linear operator.

(i) $T$ is said to be completely continuous if $T$ maps every weakly convergent sequence $(x_n)_n$ in $X$ to a strongly convergent sequence $(Tx_n)_n$ in $Y$.

(ii) $T$ is said to be compact if $T$ maps any bounded subset of $X$ to a relatively compact subset of $Y$.

Then we make the following general remarks:

Remark 1.2.

(i) It is well known that every compact operator must be completely continuous while the converse does not hold true generally. Throughout the paper, we shall explicitly distinguish these two notions, although they are often mixed in some literature, since they coincide with each other in, e.g., reflexive Banach spaces.

(ii) If a linear operator $T : X \to Y$ is completely continuous, then $T$ is continuous (or bounded), since every strongly convergent sequence in $X$ is weakly convergent, and thus, it is mapped to a strongly convergent sequence in $Y$, i.e., $T$ is continuous.

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In view of applications of Functional Analysis to, e.g., variational problems and PDEs, the complete continuity of linear operators often plays a crucial role, e.g., for constructing solutions for PDEs or for finding minimizers of functionals. For instance, the compactness of embeddings is often used to derive the strong convergence from the weak convergence or boundedness of a minimizing sequence or a sequence of approximate solutions.

Although qualitative aspects of completely continuous operators, i.e., weak convergence implies strong convergence, are often discussed, there are less studies or attempts to reveal their quantitative aspects.

Precisely speaking, we shall consider a completely continuous linear operator $T : X \to Y$, where $(X, \| \cdot \|_X)$ and $(Y, \| \cdot \|_Y)$ are normed spaces. Then we shall seek for a quantitative estimate for the operator $T$ such as

$$\|Tu - Tv\|_Y \leq \omega(u, v) \quad \text{for all } u, v \in X,$$

for some function $\omega : X \times X \to [0, \infty[$ satisfying that $\omega(u, v)$ is infinitesimal as $u \to v$ weakly in $X$. The function $\omega$ may be called a modulus of continuity for the operator $T$ and quantitatively measure the uniform (complete) continuity of $T$.

One of well-known estimates as above for compact embeddings is the so-called Ehrling inequality, which is also known as the J.-L. Lions lemma ([1, p. 173] [3], [7], [5, Lemma 7.6.], [8, p. 269, Theorem 3.5.]) and described as follows:

**Ehrling inequality.** Let $(X, \| \cdot \|_X), (Y, \| \cdot \|_Y)$ and $(Z, \| \cdot \|_Z)$ be normed spaces with embedding $X \hookrightarrow Y$ being compact and $Y \hookrightarrow Z$ being continuous. Then for any $\varepsilon > 0$, there exists a constant $C_\varepsilon > 0$ such that

$$\|u\|_Y \leq \varepsilon \|u\|_X + C_\varepsilon \|u\|_Z$$

for all $u \in X$.

**Remark 1.3.** The embedding $X \hookrightarrow Y$ above is not just completely continuous, but supposed to be compact.

Owing to the arbitrariness of $\varepsilon > 0$ and complete continuity of $X \hookrightarrow Z$, the right-hand side of (1.1) is infinitesimal as $u \to v$ weakly in $X$ and can be regarded as a modulus of continuity of the compact embedding $X \hookrightarrow Y$.

Let us extend the Ehrling inequality (1.1) to more general linear operators, so to speak, the generalized Ehrling type inequality, and making use of it, let us define a new notion of continuity for linear operators, so to speak, the Ehrling continuity.

**Definition 1.4.** Let $(X, \| \cdot \|_1), (X, \| \cdot \|_2)$ and $(Y, \| \cdot \|_Y)$ be normed spaces and let $T : X \to Y$ be a linear operator.

(i) (generalized Ehrling type inequality) The linear operator $T$ is said to satisfy the generalized Ehrling type inequality with the pair $(\| \cdot \|_1, \| \cdot \|_2)$, if for any $\varepsilon > 0$, there exists a constant $C_\varepsilon > 0$ such that

$$(1.2) \quad \|Tu\|_Y \leq \varepsilon \|u\|_1 + C_\varepsilon \|u\|_2 \quad \text{for all } u \in X.$$

(ii) (Ehrling continuity) The linear operator $T$ is said to be Ehrling continuous with the pair $(\| \cdot \|_1, \| \cdot \|_2)$, if $T$ satisfies the generalized Ehrling type inequality (1.2) with the pair $(\| \cdot \|_1, \| \cdot \|_2)$. 
Remark 1.5. The original Ehrling inequality (1.1) means that the compact embedding \( X \hookrightarrow Y \) is Ehrling continuous with the pair \((\| \cdot \|_X, \| \cdot \|_Z)\), where we assign the norm on \( X \) induced by \( \| \cdot \|_Z \) to the (weaker) norm \( \| \cdot \|_2 \) (see also Corollary 1.8 below).

A significance of the Ehrling type continuity (of an operator \( T \)) is that, for any sequence \((u_n)\) in \( X \), \( Tu_n \to 0 \) in \( Y \) provided that \( \sup_n \| u_n \|_1 < \infty \) and \( \| u_n \|_2 \to 0 \), thanks to the arbitrariness of \( \varepsilon > 0 \). In other words, “\( \| \cdot \|_1 \)-boundedness” and “\( \| \cdot \|_2 \)-convergence” yield “\( \| \cdot \|_Y \)-convergence”. Such a continuity of \( T \) with hybrid use of two different norms (or topologies) in \( X \) is in particular useful when the norm \( \| \cdot \|_1 \) is stronger and \( \| \cdot \|_2 \) weaker.

In this paper, we shall develop a general framework to establish generalized Ehrling type inequalities for completely continuous operators. In what follows, we are firstly concerned with the generalized Ehrling type inequality (1.2) from topological points of view. We shall obtain a new topological formulation which completely characterizes the generalized Ehrling type inequality (1.2). More precisely, a necessary and sufficient condition for the Ehrling continuity of linear operators will be provided (see Theorem 1.6 below). Then as a corollary, the classical Ehrling inequality (1.1) is also re-proved via such a topological formulation (see Corollary 1.8 below).

Secondly, the complete continuity of a linear operator \( T : X \to Y \) with \( X \) having a separable dual \( X^* \) is characterized in general settings in terms of the Ehrling continuity. Owing to the separability of the dual \( X^* \), one can construct a norm topology \(| \cdot |_{\Phi} \) which induces a metric topology on \( \{ u \in X; \| u \|_X \leq 1 \} \) equivalent to the weak topology of \( X \) (see Lemma 1.12 below), where \( \Phi = (\phi_k) \) is a dense subset of \( B_{X^*}(1) \). Then with the help of the additional norm \(| \cdot |_{\Phi} \), we shall obtain a universal characterization of completely continuous operators, i.e., a linear operator \( T : (X, \| \cdot \|_X) \to (Y, \| \cdot \|_Y) \), with \((X, \| \cdot \|_X)\) having a separable dual, is completely continuous if and only if \( T \) is Ehrling continuous with the pair \((\| \cdot \|_X, | \cdot |_{\Phi})\), i.e., for any \( \varepsilon > 0 \), there exists \( C_{\varepsilon} > 0 \) such that

\[
\| Tu \|_Y \leq \varepsilon \| u \|_X + C_{\varepsilon} | u |_{\Phi}
\]

for all \( u \in X \) (see Theorem 1.15 below).

Notaion. Throughout the paper, we often use the following notation.

(i) For a normed space \((X, \| \cdot \|_X)\) and its dual \( X^* \), the duality pairing between \( X \) and \( X^* \) is denoted by \( \langle \cdot , \cdot \rangle \).

(ii) For \( r > 0 \) and a normed space \((X, \| \cdot \|_X)\), we denote by \( B_X(r) \) the (strongly) closed ball in \( X \) of radius \( r \) centered at the origin, i.e., \( B_X(r) = \{ u \in X; \| u \|_X \leq r \} \).

(iii) Let \((X, d_X)\) and \((Y, d_Y)\) be metric spaces. We may denote

\[
T : (X, d_X) \to (Y, d_Y)
\]

continuously,

in order to emphasize that \( T \) is continuous with respect to metrics \( d_X \) and \( d_Y \).

(iv) For a normed space \((X, \| \cdot \|_X)\) and a subset \( B \) in \( X \), we regard \((B, \| \cdot \|_X)\) as a metric space with the metric \( d \) being induced by \( \| \cdot \|_X \), i.e., \( d(u,v) := \| u - v \|_X \) for all \( u,v \in B \).

(v) We denote by \( C \) generic non-negative constants, which do not depend on elements of corresponding spaces or sets and may vary from line to line. The constant \( C \) will have subscripts of parameters if we emphasize the dependence of \( C \) on those parameters.
1.1. Main results I. The following theorem gives a topological formulation which completely characterizes the generalized Ehrling type inequality, i.e., a necessary and sufficient condition for the Ehrling continuity of linear operators.

**Theorem 1.6.** Let \((X, \| \cdot \|_1), (X, \| \cdot \|_2), \text{ and } (Y, \| \cdot \|_Y)\) be normed spaces, let \(B := \{u \in X; \|u\|_1 \leq 1\}\) and let \(T : X \to Y\) be a linear operator. Assume that

\[
T|_B : (B, \| \cdot \|_2) \to (Y, \| \cdot \|_Y)
\]

is continuous. Then \(T\) is Ehrling continuous with the pair \((\| \cdot \|_1, \| \cdot \|_2)\), i.e., for any \(\varepsilon > 0\), there exists a constant \(C_\varepsilon > 0\) such that

\[
\|Tu\|_Y \leq \varepsilon \|u\|_1 + C_\varepsilon \|u\|_2 \quad \text{for all } u \in X.
\]

Conversely, if a linear operator \(T\) is Ehrling continuous with the pair \((\| \cdot \|_1, \| \cdot \|_2)\), then the restricted mapping \((1.3)\) is continuous.

Then we further observe that

**Remark 1.7.** In addition to \((1.3)\), if there exists a constant \(C > 0\) such that \(\|u\|_2 \leq C \|u\|_1\) (respectively, \(\|u\|_1 \leq C \|u\|_2\)) for all \(u \in X\), then there exists a constant \(C' > 0\) such that \(\|Tu\|_Y \leq C' \|u\|_1\) (respectively, \(\|Tu\|_Y \leq C' \|u\|_2\)), which implies that \(T\) is a bounded operator from \((X, \| \cdot \|_1)\) to \((Y, \| \cdot \|_Y)\) (respectively, from \((X, \| \cdot \|_2)\) to \((Y, \| \cdot \|_Y)\)).

Owing to Theorem \(1.6\), one can give an alternative proof of the classical Ehrling inequality \((1.1)\).

**Corollary 1.8.** (Ehrling inequality) Let \((X, \| \cdot \|_X), (Y, \| \cdot \|_Y)\) and \((Z, \| \cdot \|_Z)\) be normed spaces with embeddings \(\theta : X \hookrightarrow Y\) being compact and \(\tau : Y \hookrightarrow Z\) being continuous. Set \(B = \{u \in X; \|u\|_X \leq 1\}\). Let \(X\) be also equipped with the norm \(\| \cdot \|_X\) induced by \(\| \cdot \|_Z\), i.e., \(\|u\|_X := \|\tau \circ \theta\|(u)\|_Z\) for all \(u \in X\). Then the canonical injection induced by \(\theta\)

\[
\theta|_B : (B, \| \cdot \|_X) \to (Y, \| \cdot \|_Y)
\]

is continuous, and therefore, the embedding \(\theta\) is Ehrling continuous with the pair \((\| \cdot \|_X, \| \cdot \|_X)\), i.e., for any \(\varepsilon > 0\), there exists a constant \(C_\varepsilon > 0\) such that

\[
\|\theta(u)\|_Y \leq \varepsilon \|u\|_X + C_\varepsilon \|\tau \circ \theta\|(u)\|_Z \quad \text{for all } u \in X.
\]

The corollary above indicates that Theorem \(1.6\) is an extension of the classical Ehrling lemma to more general operators.

**Corollary 1.9.** Under the same conditions as in Corollary \(1.8\) the metric spaces \((B, \| \theta \cdot \|_Y)\) and \((B, \| \cdot \|_X)\) are homeomorphic.

One can easily show the above corollary, since the canonical mapping \((B, \| \theta \cdot \|_Y) \to (B, \| \cdot \|_X)\) is continuous by assumption, and the inverse mapping \((B, \| \cdot \|_X) \to (B, \| \theta \cdot \|_Y)\) is also (sequentially) continuous thanks to the estimate \((1.5)\) and to the arbitrariness of \(\varepsilon > 0\).

Another application of Theorem \(1.6\) is concerned with an example given by H. Brezis in \([1, \text{ p. } 174, \text{ Exercise } 6.13]\) (hereafter we call it Brezis’ example for short), and we shall give an alternative proof for it.

**Corollary 1.10.** Let \((X, \| \cdot \|_X), (Y, \| \cdot \|_Y)\) be normed spaces and let \((X, \| \cdot \|_X)\) be reflexive. Let \(T : X \to Y\) be a linear and completely continuous operator and set \(B = \{u \in \)

\[
\text{(Continued on next page...)}
\]
Assume that $X$ is also equipped with another norm $| \cdot |_X$ weaker than $\| \cdot \|_X$, i.e., for some $C > 0$, $|u|_X \leq C \|u\|_X$ for all $u \in X$. Then

$$T|_B : (B, | \cdot |_X) \to (Y, \| \cdot \|_Y)$$

is continuous, and therefore, $T$ is Ehrling continuous with the pair $(\| \cdot \|_X, | \cdot |_X)$, i.e., for any $\varepsilon > 0$, there exists a constant $C_\varepsilon > 0$ such that

$$\|Tu\|_Y \leq \varepsilon \|u\|_X + C_\varepsilon |u|_X \quad \text{for all } u \in X.$$

**Remark 1.11.** In Brezis’ example, the reflexivity of $(X, \| \cdot \|_X)$ is crucial and provides a weak compactness of $B$ (see also Lemma 2.4 and the proof of Corollary 1.10 below), and a counter example for the non-reflexive case is given in [1, p. 174, Exercise 6.13].

### 1.2. Main results II.

We now move on to providing a general theorem which can guarantee an Ehrling type inequality for any completely continuous operator (whose domain has a separable dual space) by performing a local metrization of weak topology.

It is well known that the weak topology on a normed space with a separable dual is locally metrizable (see [1, Theorem 3.29.] and Lemma 2.3 below).

We need the following lemma on the local metrization of the weak topology. It provides another norm having useful properties, which will be employed to prove the main theorem below.

**Lemma 1.12.** Let $(X, \| \cdot \|_X)$ be a normed space whose dual $X^*$ is separable and let $\Phi = (\phi_k)_k$ be a dense subset of $B_{X^*}(1)$. Set $B_X(r) = \{u \in X; \|u\|_X \leq r\}$, $r > 0$. Define a norm $| \cdot |_\Phi$ on $X$ by

$$|u|_\Phi := \sum_{k=1}^{\infty} 2^{-k} |\langle \phi_k, u \rangle| \quad \text{for } u \in X. \quad (1.6)$$

Then the following holds:

(i) for all $u \in X$, $|u|_\Phi \leq \|u\|_X$;

(ii) the series in the right-hand side of (1.6) converges uniformly on every bounded set in $(X, \| \cdot \|_X)$, i.e., for any bounded subset $K$ in $(X, \| \cdot \|_X)$ and any $\varepsilon > 0$, there exists a number $M = M(K, \varepsilon) \in \mathbb{N}$ such that

$$\sum_{k=M}^{\infty} 2^{-k} |\langle \phi_k, u \rangle| < \varepsilon \quad \text{for all } u \in K;$$

(iii) for a sequence $(u_n)_{n \geq 1}$ and $u$ in $X$, $u_n \rightharpoonup u$ weakly in $(X, \| \cdot \|_X)$ if and only if $\sup_{n \geq 1} \|u_n\|_X < \infty$ and $|u_n - u|_\Phi \to 0$ as $n \to \infty$;

(iv) in particular, the canonical bijection

$$(X, \| \cdot \|_X) \to (X, | \cdot |_\Phi)$$

is completely continuous;

(v) restricted to a closed ball $B_X(r)$ for $r > 0$, the metric topology of $(B_X(r), | \cdot |_\Phi)$ is equivalent to the (relative) weak topology of $(B_X(r), \sigma(X, X^*))$;

(vi) in addition, if $X$ is reflexive, then $(B_X(r), | \cdot |_\Phi)$ is a compact metric space for $r > 0$. 


Then the following holds:

\[ \| \cdot \|_X \text{-convergence} \Rightarrow \sigma(X, X^*) \text{-convergence} \Rightarrow \| \cdot \|_\Phi \text{-convergence}. \]

It is well known that weak convergent sequences do not always converge strongly in general. A natural question is whether or not the convergence in \( \| \cdot \|_\Phi \) implies the convergence in \( \sigma(X, X^*) \). In general, the answer is negative, and there is a counterexample, that is, a sequence convergent in \( \| \cdot \|_\Phi \) but not convergent in \( \sigma(X, X^*) \). For details, see Appendix. Hence it is reasonable to call the norm topology \( (X, \| \cdot \|_\Phi) \) a very weak topology of \( X \).

**Definition 1.13.** (Very weak topology) Let \( (X, \| \cdot \|_X) \) be a normed space whose dual \( X^* \) is separable, let \( \Phi = (\phi_k)_k \) be a dense subset of \( B_{X^*}(1) \) and let \( \| \cdot \|_\Phi \) be given by \( (1.6) \). The norm topology \( (X, \| \cdot \|_\Phi) \) is called the very weak topology of \( X \) associated with \( \Phi \) and the norm \( \| \cdot \|_\Phi \) is called the very weak norm of \( X \) associated with \( \Phi \).

**Remark 1.14.** As is seen in \( (1.9) \), the norm \( \| \cdot \|_\Phi \) depends on the choice of a dense subset \( \Phi = (\phi_k)_k \) in \( B_{X^*}(1) \). However, according to Lemma \( (1.12) \) every very weak norm is equivalent to the weak topology on every closed ball centered at the origin regardless of the choice of a dense subset: i.e., the very weak topology is independent of the choice when restricted to a closed ball centered at the origin. Moreover, as will be remarked in Remark \( (1.19) \) below, the theorem below will show that the choice makes no influence on our main result, that is, Ehrling type inequalities for completely continuous operators in terms of the very weak norm. Thus it suffices to fix one dense subset of \( B_{X^*}(1) \) to define and apply the very weak topology of \( X \).

Now we are in a position to describe our main results on a universal characterization of the complete continuity of linear operators defined on a normed space with a separable dual.

**Theorem 1.15.** Let \( (X, \| \cdot \|_X) \) be a normed space whose dual \( X^* \) is separable, let \( (Y, \| \cdot \|_Y) \) be a normed space and let \( \Phi = (\phi_k)_k \) be a dense subset of \( B_{X^*}(1) \). Set \( B_X(r) = \{ u \in X; \| u \|_X \leq r \} \), \( r > 0 \). Define by \( (1.6) \) the very weak norm \( \| \cdot \|_\Phi \) on \( X \) associated with \( \Phi \). Let \( T : (X, \| \cdot \|_X) \to Y \) be a completely continuous linear operator. Then the following holds:

(i) The operator \( T \) is Ehrling continuous with the pair \( (\|\cdot\|_X, \|\cdot\|_\Phi) \), i.e., for any \( \varepsilon > 0 \), there exists a constant \( C_\varepsilon > 0 \) such that

\[ \| Tu \|_Y \leq \varepsilon \| u \|_X + C_\varepsilon \| u \|_\Phi \quad \text{for all } u \in X. \]  

(ii) If, in addition, \( X \) is reflexive and \( T \) is injective (in this case, \( X \) is separable and reflexive and \( T \) is a compact embedding), then for \( r > 0 \), \( (T(B_X(r)), \| \cdot \|_Y) \) is homeomorphic to \( (B_X(r), \| \cdot \|_\Phi) \), which is a compact metric space, and for any \( \varepsilon > 0 \), there exists a constant \( C_\varepsilon > 0 \) such that for all \( u \in X \),

\[ \| u \|_\Phi \leq \varepsilon \| u \|_X + C_\varepsilon \| Tu \|_Y. \]

Conversely, assume that a linear operator \( T : X \to Y \) satisfies the relation \( (1.7) \) under the same settings as in Theorem \( 1.15 \). Then thanks to the arbitrariness of \( \varepsilon > 0 \), \( T \) turns out to be completely continuous (weak convergence is mapped to strong convergence). Therefore, the relation \( (1.7) \) gives a full characterization of completely
continuous operators in terms of natural two different norms when \((X, \| \cdot \|_X)\) has a separable dual. Hence we get a quantitative estimates of completely continuous operators which we have sought for.

Moreover, we give an important remark.

**Remark 1.16.** We also see that the choice of a dense subset of \(B_{X^*}(1)\) in (1.6) makes no influence on the Ehrling continuity of linear operators. Let \(| \cdot |_\Psi\) be the very weak norm associated with \(\Psi = (\psi_k)_k\) which is a dense subset of \(B_{X^*}(1)\) different from \(\Phi\). Then the canonical bijection
\[
(X, \| \cdot \|_X) \to (X, | \cdot |_\Psi)
\]
is completely continuous. So from the above theorem, for any \(\varepsilon > 0\), there exists a constant \(C > 0\) such that
\[
|u|_\Psi \leq \varepsilon \|u\|_X + C|u|_\Phi \quad \text{for all } u \in X.
\]
Exchanging the roles of \(| \cdot |_\Phi\) and \(| \cdot |_\Psi\), it holds that for any \(\varepsilon > 0\), there is a constant \(C' > 0\) such that
\[
|u|_\Phi \leq \varepsilon \|u\|_X + C'|u|_\Psi \quad \text{for all } u \in X.
\]
Hence generalized Ehrling type inequalities for completely continuous linear operators is irrelevant to the choice of dense subsets \(\Phi\) and \(\Psi\) of \(B_{X^*}(1)\) (to be precise, the constant \(C_\varepsilon\) in (1.7) may depend on the choice).

According to Lemma 1.12 (i) of Theorem 1.15 and Remark 1.16, the very weak topology can be regarded as a universal norm topology on \(X\) in the sense that every completely continuous operator from \(X\) (to some space) can be characterized with the norms \(\| \cdot \|_X\) and \(| \cdot |_\Phi\) via the generalized Ehrling type inequality, and vice versa. Hence in a separable dual case, we have obtained a universal triad: completely continuous operators, the very weak topology and the generalized Ehrling type inequality.

Finally, regarding compact embeddings of reflexive normed spaces with separable dual spaces into normed spaces, we shall observe that the assertion (ii) of Theorem 1.15 provides more detailed information on topology than the classical Ehrling lemma (see also Corollary 1.9).

**Corollary 1.17.** Under the same assumptions as in the assertion (ii) of Theorem 1.15, the metric spaces \((B_X(1), \|T \cdot \|_Y)\) and \((B_X(1), | \cdot |_\Phi)\) are homeomorphic to the (relative) weak topological space \((B_X(1), \sigma(X, X^*))\), i.e., these three topological spaces are homeomorphic to each other.

Indeed, from Lemma 1.12 the metric space \((B_X(1), | \cdot |_\Phi)\) is homeomorphic to the (relative) weak topological space \((B_X(1), \sigma(X, X^*))\), and from the estimate (1.7), (1.8) and the arbitrariness of \(\varepsilon > 0\), the canonical mapping \((B_X(1), \|T \cdot \|_Y) \to (B_X(1), | \cdot |_\Phi)\) and the inverse \((B_X(1), | \cdot |_\Phi) \to (B_X(1), \|T \cdot \|_Y)\) are both (sequentially) continuous, whence follows the topological equivalence.

In contrast to Corollary 1.9 if \((X, \| \cdot \|_X)\) is a reflexive normed space with a separable dual \(X^*\) and is compactly embedded into some normed space \((Y, \| \cdot \|_Y)\), it is remarkable that the metric space \((B_X(1), \|T \cdot \|_Y)\) is homeomorphic to \((B_X(1), \sigma(X, X^*))\) and \((B_X(1), | \cdot |_\Phi)\), even though the latter two spaces stand irrelevantly to the space \(Y\). This observation may indicate that in this situation, the very weak normed space \((X, | \cdot |_\Phi)\)
is a central (universal) space among normed spaces \((Y, \| \cdot \|_Y)\) into which \((X, \| \cdot \|_X)\) is compactly embedded, since the canonical embedding \((X, \| \cdot \|_X) \to (X, | \cdot |_\Phi)\) is compact and the norm \(| \cdot |_\Phi\) induces “locally” the same topology as that induced by \(\|T \cdot \|_Y\); they are even equivalent to the (relative) weak topology.

The rest of the paper is organized as follows. In Section 2, we shall give preliminary facts in order to prove our main results. In Section 3, proofs for main results are provided. In Appendix, we shall compare a very weak topology with strong and weak topologies: we will give an example of a sequence which converges in a very weak topology but does not in the strong topology.

2. Preliminaries

In this section, we recall some preliminary facts in general topology and functional analysis on normed spaces.

2.1. A useful lemma for homeomorphisms. The following well-known lemma is useful to prove that a mapping is a homeomorphism.

**Lemma 2.1.** Every continuous bijection from a compact space onto a Hausdorff space is always a homeomorphism, i.e., the inverse mapping is also continuous.

The lemma above is verified since such a bijection must be a closed mapping (i.e., closed subsets are mapped to closed subsets).

2.2. Density argument via the Hahn-Banach theorem. The following lemma is very useful in finding an element orthogonal to a strict subspace. For more details, we refer the reader to [1, Corollary 1.8].

**Lemma 2.2.** ([1 Corollary 1.8]) Let \(X\) be a normed space and let \(M\) be a subspace of \(X\) such that \(M \neq X\). Then there exists \(\phi \in X^*, \phi \neq 0\), such that
\[
\langle \phi, x \rangle = 0 \quad \text{for all } x \in M.
\]

2.3. Metrizability of the weak topology. The following lemma is one of fundamental facts to prove the main results of the present paper. For more details, we refer the reader to [1 Theorem 3.29] and [2 Section V.5].

**Lemma 2.3.** Let \(X\) be a normed space whose topological dual \(X^*\) is separable. Then a closed ball \(B_X(r)\) equipped with the (relative) weak topology \(\sigma(X, X^*)\) is metrizable.

2.4. Reflexivity and separability. We recall relations among reflexivity and separability on normed spaces (not necessarily restricted on Banach spaces). For proofs, we refer the reader to [6 p.145], [1 pp.225–273] (see also [1 Sections 3.5 and 3.6] in the Banach space case).

**Lemma 2.4.** Let \(X\) be a normed space with its topological dual \(X^*\).

(i) If \(X\) is reflexive, then \(X^*\) is also reflexive and every bounded subset of \(X\) is weakly compact.

(ii) \(X\) is separable whenever \(X^*\) is separable.

3. Proofs of Theorems

Now we are ready to prove our main results.
3.1. Proofs of Main Results I. We prove Theorem 1.6 and Corollaries 1.8 and 1.10.

Proof of Theorem 1.6. Since the mapping \( T : (B, \| \cdot \|_2) \to (Y, \| \cdot \|_Y) \) is continuous particularly at the origin, for any \( \varepsilon > 0 \), there exists \( \delta_\varepsilon > 0 \) such that
\[
T(\{u \in B; \|u\|_2 < \delta_\varepsilon\}) \subset \{u \in Y; \|u\|_Y < \varepsilon\}.
\]
Let \( u \in X \setminus \{0\} \) be arbitrary. One observes
\[
\frac{\delta_\varepsilon}{\delta_\varepsilon \|u\|_1 + \|u\|_2} u \in \{u \in B; \|u\|_2 < \delta_\varepsilon\}.
\]
Indeed,
\[
\frac{\delta_\varepsilon \|u\|_1}{\delta_\varepsilon \|u\|_1 + \|u\|_2} < \delta_\varepsilon \|u\|_1 = 1, \quad \frac{\delta_\varepsilon \|u\|_2}{\delta_\varepsilon \|u\|_1 + \|u\|_2} < \frac{\delta_\varepsilon \|u\|_2}{\|u\|_2} = \delta_\varepsilon.
\]
Hence due to (3.1), one has
\[
\|T\left(\frac{\delta_\varepsilon}{\delta_\varepsilon \|u\|_1 + \|u\|_2} u\right)\|_Y < \varepsilon,
\]
which leads to
\[
\|Tu\|_Y \leq \varepsilon \|u\|_1 + \frac{\varepsilon}{\delta_\varepsilon} \|u\|_2.
\]
The case that \( u = 0 \) is trivial.

Conversely, suppose that a linear operator \( T \) is Ehrling continuous with the pair \((\| \cdot \|_1, \| \cdot \|_2)\) and consider the restricted mapping
\[
T|_B : (B, \| \cdot \|_2) \to (Y, \| \cdot \|_Y).
\]
Let \((u_n)_n\) and \( u \) be in \( B \) such that \( \|u_n - u\|_2 \to 0 \). From the generalized Ehrling type inequality (1.6), it follows that
\[
\|Tu_n - Tu\|_Y \leq \varepsilon \|u_n - u\|_1 + C_\varepsilon \|u_n - u\|_2
\]
and hence, passing to the limit as \( n \to \infty \), one obtains
\[
\lim_{n \to \infty} \|Tu_n - Tu\|_Y \leq 2\varepsilon.
\]
Thanks to the arbitrariness of \( \varepsilon > 0 \), \( T|_B \) turns out to be continuous. \( \Box \)

We can obtain the classical Ehrling inequality as a corollary of Theorem 1.6.

Proof of Corollary 1.8. Set \( B = \{u \in X; \|u\|_X \leq 1\} \). We equip \( X \) with the norm \( | \cdot |_X \) induced by \( \| \cdot \|_Z \), that is, \( |u|_X := \|\tau \circ \theta\|_Z(u) \) for all \( u \in X \) where \( \theta : X \to Y \) is a compact embedding and \( \tau : Y \to Z \) is a continuous embedding, and then we consider a metric space \((B, \| \cdot \|_X)\), for which one has a trivial homeomorphism
\[
(\tau \circ \theta)|_B : (B, \| \cdot \|_X) \to ((\tau \circ \theta)(B), \| \cdot \|_Z); \ u \mapsto (\tau \circ \theta)(u).
\]
In view of Theorem 1.6 it suffices to show that the operator
\[
\theta|_B : (B, \| \cdot \|_X) \to (Y, \| \cdot \|_Y); \ u \mapsto \theta(u),
\]
is continuous. To this end, we shall make use of the following commutative diagram (3.3). Symbols “\( \mapsto \)“ and “\( \leftarrow \mapsto \)“ denote continuous injections and continuous bijections, respectively. The injections \( i_1, i_2 \), and those with no suffix in (3.3) are canonical and continuous injections. Also, \( \leftarrow \mapsto \) denotes a homeomorphism, i.e., a continuous bijection with a continuous inverse.
Since $\theta : (X, \| \cdot \|_X) \to (Y, \| \cdot \|_Y)$ is compact, $\theta(B)$ is relatively compact in $Y$ which means $(\theta(B), \| \cdot \|_Y)$ is a compact metric space. Moreover, the restriction of $\tau$ onto $\theta(B)$, that is, $\tau|_{\theta(B)} : (\theta(B), \| \cdot \|_Y) \to (\tau(\theta(B)), \| \cdot \|_Z)$, is a continuous bijection. From Lemma 2.1, we observe that $\tau|_{\theta(B)} : (\theta(B), \| \cdot \|_Y)$ is a homeomorphism, and hence, the inverse mapping is also continuous. Therefore, from the commutative diagram (3.3), one can find that

$$\theta|_B = i_2 \circ (\tau|_{\theta(B)}^{-1} \circ i_1 \circ (\tau \circ \theta)|_B : (B, | \cdot |_X) \to (Y, \| \cdot \|_Y); \ u \mapsto \theta(u),$$

is continuous. Now from Theorem 1.6 follows the conclusion.

**Remark 3.1.** An essential point of a proof of the classical Ehrling lemma may be that $(\theta(B), \| \cdot \|_Y)$ is a compact metric space due to the compactness of $\theta$ and that thus the mapping $\tau|_{\theta(B)}$ turns out to be a homeomorphism.

**Proof of Corollary 1.10.** Set $B := \{u \in X; \|u\|_X \leq 1\}$. Here we note that if a linear space is furnished with several different norms, then there also arise different weak topologies corresponding to the norms. So we shall pay careful attention to handle such different weak topologies on $X$ and $B$. We denote by $\sigma_1$ and $\sigma_2$ the weak topologies on $X$ associated with norms $\| \cdot \|_X, | \cdot |_X$, respectively, and we denote by $(B, \sigma_j), \ j = 1, 2$, the topological spaces furnished with the relative topologies in $(X, \sigma_j), \ j = 1, 2$, respectively. We set the canonical bijection

$$\theta : (X, \| \cdot \|_X) \to (X, | \cdot |_X); \ u \mapsto \theta(u) := u.$$

Then $\theta$ is a bounded linear bijection due to the assumptions, and moreover, the induced linear operator

$$\hat{\theta} : (X, \sigma_1) \to (X, \sigma_2); \ u \mapsto \hat{\theta}(u) \equiv \theta(u),$$

is also continuous (see also [2, Theorem 3.10,]). We make use of the following commutative diagram (3.4).
(3.4)

All the mappings, except for

\[ T : (X, \| \cdot \|_X) \to (Y, \| \cdot \|_Y), \]
\[ \hat{T} : (X, \sigma_1) \to (Y, \| \cdot \|_Y); \quad u \mapsto \hat{T}u \equiv Tu, \]

are canonical and continuous. Since \( T \) is completely continuous, the mapping \( \hat{T} \) defined above is sequentially continuous. It suffices to show that

\[ T|_B : (B, \| \cdot \|_X) \to (Y, \| \cdot \|_Y); \quad u \mapsto Tu, \]

is continuous. Since \((X, \| \cdot \|_X)\) is reflexive, \((B, \sigma_1)\) turns out to be compact (i.e., \( B \) is weakly compact). Moreover, it is well known that weak topology is Hausdorff; hence so is \((B, \sigma_2)\). Thus Lemma 2.1 implies \( \hat{\theta}|_B \) is a homeomorphism. Therefore, it follows that

\[ T|_B = \hat{T} \circ i_2 \circ (\hat{\theta}|_B)^{-1} \circ i_1|_B : (B, \| \cdot \|_X) \to (Y, \| \cdot \|_Y) \]

is sequentially continuous between metric spaces, and hence, it is also continuous. Consequently, \( T : X \to Y \) is Ehrling continuous with the pair \((\| \cdot \|_X, \| \cdot \|_X)\).

\[ \Box \]

**Remark 3.2.** In the proof mentioned above, the following facts played crucial roles:

(i) \((B, \sigma_1)\) is a compact space due to the reflexivity of \((X, \| \cdot \|_X)\), and thus, the mapping \( \hat{\theta}|_B \) turns out to be a homeomorphism;

(ii) \( \hat{T} \) induced by \( T \) is sequentially continuous, i.e., \( T \) is weakly-strongly sequentially continuous.

### 3.2. Proofs of Main results II.

We shall prove Lemma 1.12 and Theorem 1.15. Some of the assertions of Lemma 1.12 might be well known, and hence, proofs for them will be omitted.

**Proof of Lemma 1.12.** It is well known that the nonnegative function \( | \cdot |_\Phi \) is a norm on \( X \). Let \( K \) be a bounded subset in \((X, \| \cdot \|_X)\). It follows that

\[ |u|_\Phi \leq \sum_{k=1}^{\infty} 2^{-k} \| \phi_k \|_X \cdot \| u \|_X \leq \| u \|_X \leq \sup_{u \in K} \| u \|_X < \infty. \]

Hence the series in (1.6) is uniformly convergent for \( u \in K \). Thus the assertions (i), (ii) are proved.
We next prove the assertion (iii). Assume that a sequence \((u_n)\) in \(X\) is weakly convergent to \(u\). Then from the Uniform Boundedness Principle,
\[
\rho := \sup_{n \geq 1} \|u_n\|_X + \|u\|_X
\]
is finite, and thus for any \(\varepsilon > 0\), there is \(M \in \mathbb{N}\) which depends only on \(\varepsilon\) and \(\rho\) such that
\[
\sum_{k \geq M+1} 2^{-k} |\langle \phi_k, u_n - u \rangle| \leq \rho \sum_{k \geq M+1} 2^{-k} \leq \varepsilon \quad \text{for all } n \in \mathbb{N}.
\]
It follows that
\[
|u_n - u|_\Phi \leq \sum_{k=1}^M 2^{-k} |\langle \phi_k, u_n - u \rangle| + \varepsilon,
\]
and hence, passing to the limit as \(n \to \infty\), one gets
\[
\lim_{n \to \infty} |u_n - u|_\Phi \leq \sum_{k=1}^M 2^{-k} \lim_{n \to \infty} |\langle \phi_k, u_n - u \rangle| + \varepsilon
\]
\[
= \varepsilon.
\]
Thanks to the arbitrariness of \(\varepsilon > 0\), one obtains
\[
\lim_{n \to \infty} |u_n - u|_\Phi = 0.
\]
Conversely, let \(\varepsilon > 0\) be arbitrary and assume that a sequence \((u_n)\) and \(u\) in \(X\) satisfy \(\sup_n \|u_n\|_X < \infty\) and \(|u_n - u|_\Phi \to 0\) as \(n \to \infty\). Fix an arbitrary \(\phi \in B_{X^*}(1)\) and take \(l \in \mathbb{N}\) such that \(\|\phi - \phi_l\|_{X^*} \leq \varepsilon\). It follows that
\[
|\langle \phi, u_n - u \rangle| \leq |\langle \phi - \phi_l, u_n - u \rangle| + |\langle \phi_l, u_n - u \rangle|
\]
\[
\leq (\sup_n \|u_n\|_X + \|u\|_X)\varepsilon + 2^l |u_n - u|_\Phi.
\]
Passing to the limit as \(n \to \infty\), one gets
\[
\lim_{n \to \infty} |\langle \phi, u_n - u \rangle| \leq (\sup_n \|u_n\|_X + \|u\|_X)\varepsilon,
\]
and thanks to the arbitrariness of \(\varepsilon > 0\), one obtains
\[
\lim_{n \to \infty} |\langle \phi, u_n - u \rangle| = 0.
\]
Thus the assertion (iii) is verified.

The assertion (iv) is an immediate consequence of the definition of complete continuity and (iii). The assertion (v) is proved in [1, Theorem 3.29].

As for (vi), assume that \(X\) is reflexive. Due to Lemma 2.21 \((B_X(r), \sigma(X, X^*))\) is a compact space, and thus \((B_X(r), \|\cdot\|)\), homeomorphic to the above, is also compact. The proof is complete.

Employing Theorem 1.6 and Lemma 1.12 we shall prove Theorem 1.15.

**Proof of Theorem 1.15** Set \(B = \{u \in X; \|u\|_X \leq 1\}\). According to (v) of Lemma 1.12 the metric space \((B, \|\cdot\|_\Phi)\) is homeomorphic to \((B, \sigma(X, X^*))\), and \(T\) is (sequentially) continuous from \((B, \|\cdot\|_\Phi)\) to \((Y, \|\cdot\|_Y)\), since \(T\) is completely continuous. Hence, from Theorem 1.6 \(T\) turns out to be Ehrling continuous with the pair \((\|\cdot\|_X, \|\cdot\|_\Phi)\) and the assertion (i) is verified.
Assume that \( X \) is reflexive and \( T \) is injective. Then (vi) of Lemma 1.12 implies that \( (B_X(r), |\cdot|_\Phi) \) is a compact metric space. Since
\[
T : (B_X(r), |\cdot|_\Phi) \to (Y, \|\cdot\|_Y)
\]
is continuous and every continuous mapping preserves compactness, one can immediately observe that \( (T(B_X(r)), \|\cdot\|_Y) \) is a compact metric space. Since the mapping
\[
T : (B_X(r), |\cdot|_\Phi) \to (Y, \|\cdot\|_Y)
\]
is a continuous bijection, we can employ Lemma 2.1 to observe that \( (T(B_X(r)), \|\cdot\|_Y) \) is a compact metric space. Since the mapping
\[
T : (B_X(r), |\cdot|_\Phi) \to (B_X(r), |\cdot|_\Phi)
\]
is a continuous bijection, we can employ Lemma 2.1 to observe that \( T \) is a homeomorphism, i.e., \( (T(B_X(r)), \|\cdot\|_Y) \) is homeomorphic to \( (B_X(r), |\cdot|_\Phi) \), and the inverse mapping
\[
T^{-1} : (T(B_X(r)), \|\cdot\|_Y) \to (B_X(r), |\cdot|_\Phi)
\]
is also continuous. Now for the sake of simplicity we set \( r = 1 \) and \( B = B_X(1) \), and then, for any \( \varepsilon > 0 \), there exists \( \delta_\varepsilon > 0 \) such that
\[
T^{-1}(\{Tu \in T(B); \|Tu\|_Y < \delta_\varepsilon\}) \subset \{u \in B; |u|_\Phi < \varepsilon\}.
\]

Let \( u \in X \setminus \{0\} \) be arbitrary. One has
\[
\delta_\varepsilon \nu \|u\|_X + \|Tu\|_Y u \in \{u \in B; \|Tu\|_Y < \delta_\varepsilon\}.
\]
Indeed, if \( u \neq 0 \), it follows that
\[
\delta_\varepsilon \nu \|u\|_X + \|Tu\|_Y < 1, \quad \delta_\varepsilon \nu \|u\|_X + \|Tu\|_Y < \delta_\varepsilon.
\]
According to (3.5), one obtains
\[
\left| T^{-1} \left( \delta_\varepsilon \nu \|u\|_X + \|Tu\|_Y \right) \right|_\Phi < \varepsilon,
\]
which yields
\[
|u|_\Phi \leq \varepsilon \|u\|_X + \frac{\varepsilon}{\delta_\varepsilon} \|Tu\|_Y.
\]
The case that \( u = 0 \) is trivial, and thus, the proof is complete.

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Appendix A. Strict weakness of the very weak topology

We shall prove that the very weak convergence (see Definition 1.13) is strictly weaker than the weak convergence by employing Lemma 2.2. According to (iii) of Lemma 1.12, it suffices to show the existence of a sequence \( (u_n) \) in \( X \) such that \( \|u_n\|_X \to \infty \) and \( |u_n|_\Phi \to 0 \). Such a sequence exists even if \( X \) is separable and reflexive.

Now we assume that \( (X, \|\cdot\|_X) \) is an infinite dimensional reflexive and separable Banach space and we identify \( X \) with \( X^{**} \). Let \( (\phi_k) \) be a dense subset of \( B_X^{**}(1) \) such that for every \( k \geq 1 \), the finite dimensional subspace
\[
M_k := \text{span}\{\phi_1, \ldots, \phi_k\}, \quad k \in \mathbb{N},
\]
is not dense in $X^*$, i.e., $M_k \neq X^*$ (e.g., $X = L^2([-1, 1])$ with the Fourier basis $(\phi_n)_{n \in \mathbb{N}}$ satisfies these conditions). Applying (ii) of Lemma 1.12 with $K = B_X(1)$, one observes that for any $n \in \mathbb{N}$, there exists $N_n \in \mathbb{N}$ such that

$$\sum_{k \geq N_n+1} 2^{-k} |\langle \phi_k, u \rangle| < \frac{1}{n^2}$$

for all $u \in K$.

From Lemma 2.2, for each $n \in \mathbb{N}$, there exists $\xi_n \in X \setminus \{0\}$ such that

$$\langle \phi, \xi_n \rangle = 0$$

for all $\phi \in M_n$.

It follows from (A.1) and (A.2) that

$$\sum_{k=1}^{\infty} 2^{-k} \left| \frac{n}{\|\xi_n\|_X} \phi_k, \frac{n}{\|\xi_n\|_X} \xi_n \right| < \frac{1}{n},$$

since $\phi_1, \ldots, \phi_{N_n} \in M_n$. Thus one can conclude that

$$\left\| \frac{n}{\|\xi_n\|_X} \xi_n \right\|_\Phi \to 0,$$

although

$$\left\| n \xi_n \right\|_X \to \infty$$

as $n \to \infty$.

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