On the inverse problem of Möbius geometry on the circle

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Abstract

Any (boundary continuous) hyperbolic space induces on the boundary at infinity a Möbius structure which reflects most essential asymptotic properties of the space. In this paper, we initiate the study of the inverse problem: describe Möbius structures which are induced by hyperbolic spaces at least in the simplest case of the circle. For a large class of Möbius structures on the circle, we define a canonical “filling” each of them, which serves as a natural candidate for a solution of the inverse problem. This is a 3-dimensional (pseudo)metric space Harm, which consists of harmonic 4-tuples of the respective Möbius structure with a distance determined by zig-zag paths. Our main result is the proof that every line in Harm is a geodesic, i.e., shortest in the zig-zag distance on each segment. This gives a good starting point to show that Harm is Gromov hyperbolic with the prescribed Möbius structure at infinity.

Keywords: Möbius structures, cross-ratio, harmonic 4-tuples

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1 Introduction

A Möbius structure on a set $X$ is a class of (semi)metrics whose cross-ratios take one and the same value on every given 4-tuple of points in $X$. Möbius structures naturally arise as geometric structures on the boundary at infinity of hyperbolic spaces. The classical example is the extended Euclidean space $\hat{\mathbb{R}}^n = \mathbb{R}^n \cup \{\infty\}$, which gives rise to the canonical Möbius structure $M_0$ over the sphere $S^n = \hat{\mathbb{R}}^n$, whose group of Möbius transformations is isomorphic to the isometry group of the hyperbolic space $H^{n+1}$.

The inverse problem of Möbius geometry asks to describe Möbius structures which are induced by hyperbolic spaces. The papers [BST14], [BST15] can be regarded as solutions of this problem in the case of rank 1 symmetric spaces. In a general case, it seems very little is known, cp. [BeS17], [BFI18]. Thus we consider a simplest nontrivial case when $X = S^1$ is the circle.

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The class of all Möbius structures on the circle is very large: any extended (semi)metric on \( \hat{\mathbb{R}} \) generates some Möbius structure on \( S^1 \). Note that various hyperbolic cone constructions (see [BoS], [BS07]) give a hyperbolic metric space with prescribed metric at infinity. However, no one of them is equivariant with respect to Möbius transformations of the metric. Thus one can consider the inverse problem as the existence problem of an equivariant hyperbolic cone over a given metric.

Asking more, one should pay an additional price for that: we introduce a set of axioms, which allow to define a reasonable candidate for a solution of the inverse problem. This is the set Harm of harmonic 4-tuples with respect to a given Möbius structure \( M \). It has a natural structure of a 3-dimensional manifold, which in the case of the canonical structure \( M_0 \) is homeomorphic to the projectivized tangent bundle of \( H^2 \). Note that Harm is automatically invariant under Möbius transformations of \( M \).

It follows from our axioms that any pair \((x, y)\) of different points in \( X \) uniquely determines a line \( h = h(x, u) \) in Harm, which consists of all pairs of different points \((z, u)\) such that 4-tuple \( q = ((x, y), (z, u)) \) is harmonic. It turns out that \( h \) is homeomorphic to \( \mathbb{R} \) and, moreover, \( h \) is isometric to \( \mathbb{R} \) with respect to the naturally defined distance

\[
|qq'| = \left| \ln \frac{d(x, z')d(y, z)}{d(x, z)d(y, z')} \right|
\]

\( q' = ((x, y), (z', u')) \), where \( d \) is any metric from \( M \) (\(|qq'| \) is independent of the choice of \( d \)).

The pairs \((x, y), (z, u)\) are called axes of \( q \in \text{Harm} \). Since every harmonic \( q \) has two axes, moving along a line in Harm, there is a possibility to change the axis at any moment. This leads to a notion of special curves in Harm, which are called \( \text{zz-paths} \). Every (finite) \( \text{zz-path} \) \( \sigma \subset \text{Harm} \) consists of a finite number of consecutive sides, every side is a segment of a line, and adjacent sides meet each other at a common harmonic 4-tuple \( q \) as the different axes of \( q \). The point of this construction is that while in general two different \( q, q' \in \text{Harm} \) cannot be connected by a segment of a line, they are always connected by a finite \( \text{zz-path} \).

The length of a \( \text{zz-path} \) \( \sigma \) is the sum of the lengths of its sides \(|\sigma|\). The \( \delta \)-distance on Harm is defined by

\[
\delta(q, q') = \inf_{\sigma} |\sigma|
\]

where the infimum is taken over all \( \text{zz-paths} \) between \( q \) and \( q' \). The \( \delta \)-distance is symmetric, nonnegative and satisfies the triangle inequality. However, it is not clear that \( \delta \) is positive, i.e., \( \delta \) is a pseudometric. Nevertheless, our main result says that lines are geodesics with respect to the \( \delta \)-distance.

**Theorem 1.1.** Every line \( h \subset \text{Harm} \) is a geodesic with respect to the \( \delta \)-distance, i.e. \( \delta(q, q') = |qq'| \) for any \( q, q' \in h \).
This is not at all obvious or trivial. The precise statement of Theorem 1.1 requires to list axioms for Möbius structures under which the theorem is true, see sect. 6. The key property we require from a Möbius structure to satisfy Theorem 1.1 is the Increment axiom, see sect. 2.4. To prove Theorem 1.1 for every line $h \subseteq \text{Harm}$ we define so called midpoint projection of Harm to $h$. The increment axiom allows to show that the midpoint projection decreases distances along $zz$-paths, which leads to Theorem 1.1.

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2 Möbius structures

2.1 Basic notions

Let $X$ be a set. A 4-tuple $q = (x, y, z, u) \in X^4$ is said to be admissible if no entry occurs three or four times in $q$. A 4-tuple $q$ is nondegenerate, if all its entries are pairwise distinct. Let $P_4 = P_4(X)$ be the set of all ordered admissible 4-tuples of $X$, $\text{reg} P_4 \subset P_4$ the set of nondegenerate 4-tuples.

A function $d : X^2 \to \hat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ is called a semi-metric, if it is symmetric, $d(x, y) = d(y, x)$ for each $x, y \in X$, positive outside of the diagonal, vanishes on the diagonal and there is at most one infinitely remote point $\omega \in X$ for $d$, i.e. such that $d(x, \omega) = \infty$ for some $x \in X \setminus \{\omega\}$. Moreover, we require that if $\omega \in X$ is such a point, then $d(x, \omega) = \infty$ for all $x \in X$, $x \neq \omega$. A metric is a semi-metric that satisfies the triangle inequality.

A Möbius structure $M$ on $X$ is a class of Möbius equivalent semi-metrics on $X$, where two semi-metrics are equivalent if and only if they have the same cross-ratios on every $q \in \text{reg} P_4$.

Given $\omega \in X$, there is a semi-metric $d_\omega \in M$ with infinitely remote point $\omega$. It can be obtained from any semi-metric $d \in M$ for which $\omega$ is not infinitely remote by a metric inversion,

$$d_\omega(x, y) = \frac{d(x, y)}{d(x, \omega)d(y, \omega)}.$$

Such a semi-metric is unique up to a homothety, see [FS], and we use notation $|xy|_\omega = d_\omega(x, y)$ for the distance between $x, y \in X$ in that semi-metric. We also use notation $X_\omega = X \setminus \{\omega\}$.

There is a distinguished class of Möbius structures called ptolemaic. The property to be ptolemaic is characterized by the inequality

$$d(x, y)d(z, u) \leq d(x, z)d(y, u) + d(x, u)d(y, z) \quad (1)$$
for every semi-metric $d$ of the Möbius structure and every 4-tuple $q = (x, y, z, u) \in X^4$. The property to be ptolemaic is invariant under any metric inversion, and this invariance can serve as an equivalent definition of ptolemaic Möbius structures. It follows from (I) that any semi-metric of a ptolemaic Möbius structure with infinitely remote point $\omega \in X$ is a metric on $X_\omega$, i.e., it satisfies the triangle inequality.

Every Möbius structure $M$ on $X$ determines the $M$-topology whose sub-base is given by all open balls centered at finite points of all semi-metrics from $M$ having infinitely remote points.

**Example 2.1.** Our basic example is the canonical Möbius structure $M_0$ on the circle $X = S^1$. We think of $S^1$ as the unit circle in the plane, $S^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$. For $\omega = (0, 1) \in X$ the stereographic projection $X_\omega \to \mathbb{R}$ identifies $X_\omega$ with real numbers $\mathbb{R}$. We let $d_\omega$ be the standard metric on $\mathbb{R}$, that is, $d_\omega(x, y) = |x - y|$ for any $x, y \in \mathbb{R}$. This generates a Möbius structure on $X$ which is called canonical. The basic feature of the canonical Möbius structure on $X = S^1$ is that for any 4-tuple $(\sigma, x, y, z) \subset X$ with the cyclic order $\sigma xyz$ we have $d_\sigma(x, y) + d_\sigma(y, z) = d_\sigma(x, z)$. In particular, the canonical Möbius structure is ptolemaic.

### 2.2 An alternative description

The following is an alternative description of a Möbius structure which is convenient in many cases. For any semi-metric $d$ on $X$ we have three cross-ratios

$$q \mapsto cr_1(q) = \frac{|x_1x_3|x_2x_4|}{|x_1x_4|x_2x_3|}; cr_2(q) = \frac{|x_1x_4|x_2x_3|}{|x_1x_2|x_3x_4|}; cr_3(q) = \frac{|x_1x_2|x_3x_4|}{|x_2x_4|x_1x_3|}$$

for $q = (x_1, x_2, x_3, x_4) \in \text{reg } P_4$, whose product equals 1, where $|x_1x_j| = d(x_i, x_j)$. We associate with $d$ a map $M_d : \text{reg } P_4 \to L_4$ defined by

$$M_d(q) = (\ln cr_1(q), \ln cr_2(q), \ln cr_3(q)),$$

(2)

where $L_4 \subset \mathbb{R}^3$ is the 2-plane given by the equation $a + b + c = 0$. Two semi-metrics $d, d'$ on $X$ are Möbius equivalent if and only $M_d = M_{d'}$. Thus a Möbius structure on $X$ is completely determined by a map $M = M_d$ for any semi-metric $d$ of the Möbius structure, and we often identify a Möbius structure with the respective map $M$.

Let $S_n$ be the symmetry group of $n$ elements. The group $S_4$ acts on $\text{reg } P_4$ by entries permutations of any $q \in \text{reg } P_4$. The group $S_3$ acts on $L_4$ by signed permutations of coordinates, where a permutation $\sigma : L_4 \to L_4$ has the sign "$-1$" if and only if $\sigma$ is odd.

The cross-ratio homomorphism $\varphi : S_4 \to S_3$ can be described as follows: a permutation of a tetrahedron ordered vertices $(1, 2, 3, 4)$ gives rise to a permutation of pairs of opposite edges $((12)(34), (13)(24), (14)(23))$. We
denote by \( \text{sign} : S_4 \to \{ \pm 1 \} \) the homomorphism that associates to every odd permutation the sign “\(-1\)”. 

One easily check that any Möbius structure \( M : \text{reg} \mathcal{P}_4 \to L_4 \) is equivariant with respect to the signed cross-ratio homomorphism,

\[
M(\pi(q)) = \text{sign}(\pi) \varphi(\pi) M(q)
\]

for every \( q \in \text{reg} \mathcal{P}_4 \), \( \pi \in S_4 \), where \( \varphi : S_4 \to S_3 \) is the cross-ratio homomorphism.

### 2.3 Monotone Möbius structures

In what follows we assume that Möbius structures we consider are ptolemaic. We say that a Möbius structure \( M \) on \( X = S^1 \) is monotone, if it satisfies the following axioms

- *(T)* Topology: \( M \)-topology on \( X \) is that of \( S^1 \);

- *(M)* Monotonicity: given a 4-tuple \( q = (x, y, z, u) \in X^4 \) such that the pairs \((x, y), (z, u)\) separate each other, we have

\[
|xy| \cdot |zu| > \max\{|xz| \cdot |yu|, |xz| \cdot |yz|\}
\]

for some and hence any semi-metric from \( M \).

A choice of \( \omega \in X \) uniquely determines the interval \( xy \subset X_\omega \) for any distinct \( x, y \in X \) different from \( \omega \) as the arc in \( X \) with the end points \( x, y \) that does not contain \( \omega \). As an useful reformulation of Axiom *(M)* we have

**Corollary 2.2.** Assume for a nondegenerate 4-tuple \( q = (x, y, z, u) \in \text{reg} \mathcal{P}_4 \) the interval \(xz \subset X_u\) is contained in \(xy, xz \subset xy \subset X_u \). Then \( |xz|_u < |xy|_u \).

**Proof.** By the assumption, the pairs \((x, y), (z, u)\) separate each other. Hence, by Axiom *(M)* we have \(|xz||yu| < |xy||zu|\) for any semi-metric from \( M \). In particular, \(|xz|_u < |xy|_u\).

**Lemma 2.3.** Assume a Möbius structure \( M \) on \( X = S^1 \) is monotone. Then \( M(q) \neq (0, 0, 0) \) for every \( q \in \text{reg} \mathcal{P}_4 \).

**Proof.** Assume \( M(q) = (0, 0, 0) \) for \( q = (x, y, z, u) \in \text{reg} \mathcal{P}_4 \). Then in a metric from \( M \) with infinitely remote point \( u \) we have \(|xy|_u = |xz|_u = |yz|_u\). Whatever is the order of \( x, y, z \) on \( X_u = X \setminus \{ u \} \), these equalities contradict the mononicity Axiom *(M)*.  

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2.4 Increment axiom

Increment axiom for monotone Möbius structures has been introduced in [Bu17], where it plays an important role since it implies the time inequality. In this paper, it also plays a key role in solving the inverse problem for Möbius structures on the circle. We briefly recall this axiom and some properties of monotone Möbius structures satisfying it.

We use notation \( \text{reg} \mathcal{P}_n \) for the set of ordered nondegenerate \( n \)-tuples of points in \( X = S^1 \), \( n \in \mathbb{N} \). For \( q \in \text{reg} \mathcal{P}_n \) and a proper subset \( I \subset \{1, \ldots, n\} \) we denote by \( q_I \in \text{reg} \mathcal{P}_k \), \( k = n - |I| \), the \( k \)-tuple obtained from \( q \) (with the induced order) by crossing out all entries which correspond to elements of \( I \).

(I) Increment Axiom: for any \( q \in \text{reg} \mathcal{P}_7 \) with cyclic order \( \text{co}(q) = 1234567 \) such that \( q_{247} \) and \( q_{157} \) are harmonic, we have

\[
\text{cr}_1(q_{345}) > \text{cr}_1(q_{123}).
\]

For definition of harmonic 4-tuples see sect. 3.1. It is proved in [Bu17, Proposition 7.10] that the canonical Möbius structure \( M_0 \) on the circle \( X = S^1 \) satisfies Increment Axiom. Moreover, the class \( \mathcal{I} \) of monotone Möbius structures on the circle which satisfy Axiom (I) contains an open in a fine topology neighborhood of \( M_0 \), see [Bu17, Proposition 7.14].

3 Filling

Here we define a space of harmonic pairs which will serve as a filling of a monotone Möbius structure on the circle.

3.1 Harmonic 4-tuples

Let \( M \) be a monotone Möbius structure on the circle \( X = S^1 \). A 4-tuple \( q \in \text{reg} \mathcal{P}_4 \) is said to be harmonic if \( M(q) \in L_4 \) has a zero coordinate. It follows from Lemma 2.3 for \( q \) harmonic, \( M(q) \) has a unique zero coordinate. Therefore, we have three types of harmonic 4-tuples \( q = (x, y, z, u) \in \text{reg} \mathcal{P}_4 \), determined by conditions

\[
\begin{align*}
(1) \quad |xz| \cdot |yu| &= |xu| \cdot |yz|, \\
(2) \quad |xu| \cdot |yz| &= |xy| \cdot |zu|, \\
(3) \quad |xy| \cdot |zu| &= |xz| \cdot |yu|,
\end{align*}
\]

for some and hence every semi-metric from \( M \), which correspond to the first, the second and the third coordinate of \( M(q) \) respectively.

**Lemma 3.1.** For \( i = 1, 2, 3 \) there is an embedding \( e_i : \text{reg} \mathcal{P}_3 \rightarrow \text{reg} \mathcal{P}_4 \) of the set \( \text{reg} \mathcal{P}_3 \subset X^3 \) of nondegenerate 3-tuples, whose image \( e_i(\text{reg} \mathcal{P}_3) \) is the set of harmonic 4-tuples of type \( (i) \).
Proof. Given \( t = (x_1, x_2, x_3) \in \text{reg} \mathcal{P}_3 \), we take a semi-metric \(| \cdot |_i\) from \( M \) with infinitely remote point \( x_i \). The distance function \( x \mapsto |x_{i+1}x_i|_i \) is continuous on \( X_{x_i} \) (see [Bu17, Lemma 4.1]), thus there is \( y_i \in X_{x_i} \) with \(|x_{i+1}y_i|_i = |y_ix_{i+2}|_i\) (indices are taken modulo 3). By Corollary 2.2, \( y_i \) is uniquely determined and moreover the pairs \((x_i, y_i)\) and \((x_{i+1}, x_{i+2})\) separate each other. Now, we put \( e_i(t) = (y_i, x_1, x_2, x_3) \). By construction, \( e_i(t) \) satisfies
\[
|x_{i+1}y_i| \cdot |x_i x_{i+2}| = |y_ix_{i+2}| \cdot |x_i x_{i+1}|
\]
for any semi-metric from \( M \), and thus \( e_i(t) \) is harmonic of type \((i)\).

Conversely, given a harmonic 4-tuple \( q = (x, y, z, u) \), we take either of \( y, z, u \) as an infinitely remote point and see that \( x \) is the midpoint between remaining two ones for harmonicity type \((1), (2), (3)\) respectively. Therefore, every harmonic 4-tuple of type \((i)\) is \( e_i(t) \) for an appropriate \( t \in \text{reg} \mathcal{P}_3 \). \( \square \)

The set \( \text{reg} \mathcal{P}_3 \subset X^3 \) in the induced topology consists of two connected components each of which is homeomorphic to the unit tangent bundle \( U \mathbb{H}^2 \) of the hyperbolic plane \( \mathbb{H}^2 \), that is, it is the trivial \( S^1 \)-bundle over \( \mathbb{R}^2 \).

By Lemma 3.1, \( e_i(\text{reg} \mathcal{P}_3) \) is the set of harmonic 4-tuples of type \((i)\). Therefore, the set of harmonic 4-tuples consists of six connected components each of which is homeomorphic to \( \mathbb{R}^2 \times S^1 \). The group \( S_4 \) acting on \( \text{reg} \mathcal{P}_4 \) permutes these components with the stabilizer of each one isomorphic to the cyclic group \( \mathbb{Z}_4 \). These facts are not used in what follows, they only describe the general structure of the space of harmonic 4-tuples.

### 3.2 Harmonic pairs

As a topological space, the required filling is defined as the set \( \text{Harm} \) of harmonic pairs. It is convenient to use unordered pairs \((x, y) \sim (y, x)\) of distinct points on \( X = S^1 \), and we denote the set of them by \( aY = S^1 \times S^1 \setminus \Delta/ \sim \), where \( \Delta = \{(x, x) : x \in S^1\} \) is the diagonal. A pair \((a, b) \in aY \times aY\) is harmonic if
\[
|xz| \cdot |yu| = |xu| \cdot |yz|
\]
for some and hence any semi-metric of the Möbius structure, where \( a = (x, y), b = (z, u) \). That is, we use the first type of harmonic 4-tuples to define harmonic pairs. The choice of the type is irrelevant to our construction because different types of harmonicity are permuted with each other by the group \( S_4 \).

Note that the pairs of points \( a, b \) separate each other for every harmonic pairs \((a, b)\). This follows from mononicity of \( M \), see the proof of Lemma 3.1.

The set \( \text{Harm} \) of the harmonic pairs is a 3-dimensional subspace in \( aY \times aY \) given by Equation (4). There is an involution \( \pi(x, y, z, u) = (y, x, u, z) \) acting on the set of harmonic 4-tuples of the first type which factors that set to \( \text{Harm} \). Therefore, \( \text{Harm} \) is homeomorphic the projectivized tangent bundle of \( \mathbb{H}^2 \).
Given $q = (a, b) \in \text{Harm}$, the pair $a \in aY$ is called the left axis and the pair $b \in aY$ the right axis of $q$.

There is a canonical involution $j : \text{Harm} \to \text{Harm}$ without fixed points given by $j(a, b) = (b, a)$. The quotient space we denote by $\text{Harm} := \text{Harm} / j$. In other words, $\text{Harm}$ is the set of unordered harmonic pairs of unordered pairs of points in $X$. Note that $j(q) = (b, a)$ is harmonic with the left axis $b$ and the right axis $a$ for every harmonic pair $q = (a, b) \in \text{Harm}$.

The space $\text{Harm}$ has two canonical structures of a locally trivial bundle $pr_i : \text{Harm} \to aY$ with respect to the factor projections $pr_i : aY \times aY \to aY$, $i = 1, 2$, $pr_1(a, b) = a$, $pr_2(a, b) = b$. It follows from Lemma 3.1 that the fibers of $pr_i$ are homeomorphic to an open arc in $S^1$, i.e. to $\mathbb{R}$. We obviously have $pr_i \circ j = pr_{i+1}$ for $i = 1, 2$, where the indices are taken modulo 2. Both $\mathbb{R}$-bundles $pr_1$, $pr_2$ are nontrivial, i.e. $\text{Harm}$ is not homeomorphic to the product $aY \times \mathbb{R}$.

### 3.3 Lines and zig-zag paths in $\text{Harm}$

A left line $lh_a$, $a \in aY$, in $\text{Harm}$ is the subset $lh_a = pr_1^{-1}(a) \subset \text{Harm}$. The pair $a \in aY$ is called the axis of $lh_a$. Similarly, a right line $rh_b$, $b \in aY$, is the subset $rh_b = pr_2^{-1}(b) \subset \text{Harm}$. The pair $b \in aY$ is called the axis of $rh_b$. Note that $j(lh_a) = rh_a$ and $j(rh_b) = lh_b$.

Every fiber of the fibration $pr_1 : \text{Harm} \to aY$ is a left line, while every fiber of the fibration $pr_2 : \text{Harm} \to aY$ is a right line. Thus every left (right) line is homeomorphic to $\mathbb{R}$. The axis $a$ of $lh_a$ is the common left axis for all $q \in lh_a$. The axis $b$ of $rh_b$ is the common right axis for all $q \in rh_b$.

A line in $\text{Harm}$ is the image of a left line or a right line under the canonical projection $\text{Harm} \to \text{Harm}$. Thus in $\text{Harm}$ we do not distinguish left and right lines. The notion of the axis of a line is preserved by $j$, and we denote by $h_a \subset \text{Harm}$ a line with the axis $a \in aY$.

We say that $b, b' \in aY$ are in the strong causal relation if either of them lies on an open arc in $X$ determined by the other one (more for this terminology see in [Bu17]).

**Lemma 3.2.** For different $q = (a, b)$, $q' = (a, b')$ lying of on a left line $lh_a$, the pairs $b, b' \in aY$ are in the strong causal relation. Conversely, given $b, b' \in aY$ in the strong causal relation, there is a left line $lh_a$ such that $q = (a, b), q' = (a, b') \in lh_a$. Similar properties hold true also for right lines and lines in $\text{Harm}$.

**Proof.** The arguments can be found in [Bu17 Proposition 5.8, Proposition 3.2(b)]]. For convenience of the reader we briefly recall them.

Let $a = (x, y), b = (z, u), b' = (z', u') \in aY$, where $q = (a, b), q' = (a, b')$ lie on a left line $lh_a$. Taking a semi-metric from $M$ with infinitely remote point $x$, we observe that $y$ is the midpoint of the segments $zu, z'u' \subset X_x$. Since $b \neq b'$, we can assume that $z'y \subset zy$. By Axiom (M), $|z'y|_x < |zy|_x$. 

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and thus $|u'y|_x < |uy|_x$. Then again by Axiom (M), $u'y \subset uy$. It follows that $b'$ lies on an open arc in $X$ determined by $b$, i.e., $b, b'$ are in the strong causal relation.

Conversely, Lemma 3.3 implies that for every $b = (z, u) \in aY$ there is a well defined involutive homeomorphism $\rho_b : X \to X$, called the reflection with respect to $b$, that fixes $z, u$, such that the pair $(a, b)$ is harmonic for every $x \in X \setminus b$, where $a = (x, \rho_b(x))$. For $b, b' \in aY$ in the strong causal relation, we take the composition $\rho = \rho_b \circ \rho_{b'}$ of the respective reflection and note that $\rho(b^+) \subset \text{int}(b^+)$, where $b^+ \subset X$ is the closed arc determined by $b$ that does not include $b'$. Thus there is a fixed point $x \in \text{int} b^+$ of $\rho$. Then $a = (x, y) \in aY$, where $y = \rho_{b'}(x)$, is preserved by $\rho_b, \rho_{b'}$, and $q = (a, b), q = (a, b') \in lh_a$.

The pair $a \in aY$ above is called a common perpendicular to $b, b'$. We postpone the proof of uniqueness to sect. 4.1, see Lemma 4.2.

We say that $d \in aY$ separates $b$ and $c \in aY$ if $b$ and $c$ lie on different open arcs in $X$ defined by $d$. Note that in this case $b, c, d$ are in the strong causal relation with each other.

Given a left line $lh_a \subset \text{Harm}$ and distinct $q = (a, b), q' = (a, b') \in lh_a$, we define the left segment $qq' \subset lh_a$ as the union of $q, q'$ and all of $q'' = (a, b'') \in lh_a$ such that $b''$ separates $b, b'$. The points $q, q'$ are the ends of $qq'$. Similarly, we define right segments on a right line. More generally, a segment $qq'$ in $\text{Harm}$ (Hm) is a segment of line in $\text{Harm}$ (Hm). In this case the harmonic pairs $q, q'$ have a common axis.

By the first part of Lemma 3.2 $b, b'$ are in the strong causal relation. Denote by $b^- \subset X$ the open arc determined by $b$ that contains $b'$, and by $(b^-)^-$ the open arc determined by $b'$ that contains $b$. Then $a$ does not meet $b^- \cap (b^-)^-$ because $a, b$ separate each other as well as $a, b'$. By Lemma 3.1 for every $z'' \in b^- \cap (b^-)^-$ there is $u'' \in X$ such that $(a, b'')$ is harmonic, i.e., $(a, b'') \in lh_a$, where $b'' = (z'', u'')$.

Thus $b''$ is in the strong causal relation with $b$ as well as with $b'$. Hence, $u'' \in b^- \cap (b^-)^-$. In other words, the intersection $b^- \cap (b^-)^-$ is invariant under the reflection $\rho_a : X \to X$, see proof of Lemma 3.2. We conclude that the segment $qq' \subset lh_a$ is homeomorphic to the standard segment $[0, 1]$.

A zig-zag path, or zz-path, $S \subset \text{Harm}$ is defined as an alternating finite (maybe empty) sequence of left and right segments $\sigma_i$ in $\text{Harm}$, where consecutive segments $\sigma_i, \sigma_{i+1}$ have a common end. Segments $\sigma_i$ are also called sides of $S$.

**Lemma 3.3.** Given $q, q' \in \text{Harm}$, there is a zz-path $S$ in $\text{Harm}$ with at most five sides that connects $q$ and $q'$.

**Proof.** Let $q = (a, b), q' = (a', b')$. The pairs $a, a' \in aY$ separate $X$ into (at most four) open arcs. Taking $a'' \in aY$ on such an arc, we see that $a''$ is in the strong causal relation with $a$ as well as with $a'$. By Lemma 3.2 there is
a common perpendicular \( \tilde{b} \) to \( a, a'' \), and there is a common perpendicular \( \tilde{b}' \) to \( a', a'' \). Then the pairs \( \tilde{q} = (a, \tilde{b}), \tilde{q}' = (a', \tilde{b}'), \tilde{q}'' = (a'', \tilde{b}'') \) are harmonic, and the alternating sequence

\[
S = q\tilde{q}, \tilde{q}q'', \tilde{q}'q', \tilde{q}''q''
\]
of left and right segments connects \( q, q' \) having at most 5 sides.

A \( zz \)-path in \( Hm \) is the image of a \( zz \)-path in \( Harm \) under the canonical projection \( Harm \to Hm \). This is also an alternating (in obvious sense) finite sequence of segments in \( Hm \), where consecutive segments have a common end. Lemma 3.3 holds true also in \( Hm \).

## 4 Pseudometric on \( Harm \)

### 4.1 Distance between harmonic pairs with common axis

Given two harmonic pairs in \( q, q' \in Harm \) with a common axis, say \( q = (a, b) \) and \( q' = (a', b') \), we define the distance \(|qq'|\) between them as

\[
|qq'| = |j(q)j(q')| = \ln \frac{|xz'| \cdot |yz|}{|xz| \cdot |yz'|}
\]

(5)

for some and hence any semi-metric on \( X \) from \( M \), where \( a = (x, y), b = (z, u), b' = (z', u') \in aY, \) and \( j : Harm \to Harm \) is the canonical involution. Note that

\[
|qq'| = \ln \frac{|zu'| \cdot |yu|}{|zu| \cdot |yu'|} = \ln \frac{|xz| \cdot |yz|}{|xz| \cdot |yz'|} = \ln \frac{|xz| \cdot |yu|}{|xz'| \cdot |yz'|}
\]

(6)

by harmonicity of \( q, q' \). In this way, (5) defines the distance along the left hyperbolic line \( lh_a \subset Harm \) as well as along the right hyperbolic line \( rh_a \subset Harm \).

**Lemma 4.1.** Given \( a \in aY \) and \( q = (a, b), q' = (a', b'), q'' = (a'', b'') \in lh_a \) such that \( b' \) separates \( b \) and \( b'' \), then \(|qq''| = |qq'| + |q'q''|\). A similar property holds true also for right lines.

**Proof.** Let \( a = (x, y), b = (z, u), b' = (z', u'), b'' = (z'', u'') \). In the semimetric from \( M \) with infinitely remote point \( x, y \) is the midpoint of the segments \( zu, z'u', z''u'' \subset X_x \). Using that \( b' \) separates \( b \) and \( b'' \), we can assume without loss of generality that \( z''u'' \subset z'u' \subset zu \). Then \(|yz''|_x < |yz'|_x < |yz|_x \) and thus

\[
|qq'| = \ln \frac{|yz|_x}{|yz'|_x}, \quad |q'q''| = \ln \frac{|yz'|_x}{|yz''|_x}, \quad |qq''| = \ln \frac{|yz|_x}{|yz''|_x}.
\]

Therefore \(|qq''| = |qq'| + |q'q''|\). \(\Box\)
Now, we can prove uniqueness of the common perpendicular.

**Lemma 4.2.** Given \( b, b' \in aY \) in the strong causal relation, there is at most one common perpendicular \( a \in aY \) to \( b, b' \).

**Proof.** Assume there are common perpendiculars \( a = (z, u), a' = (z', u') \in aY \) to \( b, b' \). By the first part of Lemma 3.2, \( a \) and \( a' \) are in the strong causal relation. Let \( b = (x, y), b' = (x', y') \). Using that the pairs \( a, a' \) and \( b, b' \) are in the strong causal relation, we assume without loss of generality that on \( X_x \) we have the following order of points \( zz'yy'uux' \). We denote by \( q_1 = (b, a), q_2 = (b', a), q'_1 = (b, a'), q'_2 = (b', a') \) respective harmonic pairs. Then \( q_1, q_2 \in rh_a, q'_1, q'_2 \in rh_{a'} \), and we have well defined distances \( l = |q_1q_2|, l' = |q'_1q'_2| \). Computing them in a semi-metric of the Möbius structure with infinitely remote point \( x \), we obtain

\[
\epsilon_l = \frac{|zz'|}{|x'u|}, \quad \epsilon_{l'} = \frac{|zz'|}{|x'u'|}.
\]

Using the order of points \( zz'yy'uux' \) on \( X_x \), we have, in particular, that the interval \( z'x' \) is contained in the interval \( zz' \). By Corollary 2.2, \( |zz'| \geq |z'x'| \).

Similarly, \( x'u \subset x'u' \) and hence \( |x'u| \leq |x'u'| \). Thus \( l \geq l' \) and if \( a' \neq a \), the inequality is strong. Applying this argument with infinitely remote point \( y \), we obtain \( l \leq l' \). Therefore \( l = l' \) and \( a = a' \). \( \square \)

### 4.2 Defining a pseudometric metric \( \delta \) on \( Hm \) and \( \text{Harm} \)

It follows from Lemma 4.1 that for harmonic pairs \( q, q' \) on one and the same left (right) line, the length of the segment \( \sigma = qq' \) is equal to the distance \( |qq'| \), that is, it can be computed by any of Equalities \( 3, 4 \).

Let \( S = \{\sigma_i\} \) be a \( zz \)-path in \( \text{Harm} \). We define length of \( S \) as the sum \( |S| = \sum_i |\sigma_i| \) of the length of its sides. Now, we define a distance \( \delta \) on \( \text{Harm} \) by

\[
\delta(q, q') = \inf_S |S|,
\]

where the infimum is taken over all \( zz \)-paths \( S \subset \text{Harm} \) from \( q \) to \( q' \).

**Proposition 4.3.** The distance \( \delta \) on \( \text{Harm} \) is symmetric, \( \delta(q, q') = \delta(q', q) \), \( \delta(q, q) = 0 \), satisfies the triangle inequality,

\[
\delta(q, q'') \leq \delta(q, q') + \delta(q', q''),
\]

and finite \( \delta(q, q') < \infty \), for all \( q, q', q'' \in \text{Harm} \).

**Proof.** The property of \( \delta \) to be symmetric and the triangle inequality immediately follows from the definition. Taking an empty \( zz \)-path, we see that \( \delta(q, q) = 0 \) for any \( q \in \text{Harm} \).

The fact that the distance \( \delta(q, q') \) is finite for every \( q, q' \in \text{Harm} \), follows from Lemma 3.3. \( \square \)
We similarly define the distance on $H_m$, for which we use the same notation $\delta$. The canonical projection $H_{\text{rm}} \to H_m$ is a 2-sheeted covering of 3-manifolds with deck transformation group isomorphic to $\mathbb{Z}_2$ acting by $\delta$-isometries. Then the distance on $H_{\text{rm}}$ is obtained by lifting the distance on $H_m$.

A basic problem is to prove that $\delta$ is nondegenerate, i.e., $\delta(q, q') > 0$ for any distinct $q, q' \in H_{\text{rm}}$. It is not at all clear that this holds even in the case $q, q'$ lie on a line, and moreover that $\delta(q, q') = |qq'|$ in this case.

5 Projections to a line

5.1 $s$-projection and midpoint projection

It follows from Lemma 3.1 that given $a \in aY$ and $x \in X$, $x \not\in a$, there is a uniquely determined $y \in X$ such that the pair $(a, b)$ is harmonic, $(a, b) \in H_m$, where $b = (x, y)$. In this case, we use notation $x_a := b$ and say that $x_a \in h_a$ is the projection of $x$ to the line $h_a$.

We say that a one-parametric family of segments $v_t w_t \subset \mathbb{R}$ is monotone, if its ends $v_t, w_t$ are monotone in the same sense, i.e., $v_t < v_t'$ if and only if $w_t < w_t'$ for $t \neq t'$.

Lemma 5.1. Given two lines $h_a, h_c \subset H_m$ with $a = (z, u) \in aY$, the family of segments $v_a w_a = v_a w_a(p) \subset h_a$ is monotone in $p = (c, d) \in h_c$, where $d = (v, w) \in aY$, as $p$ runs over the segment $z_c u_c \subset h_c$.

Proof. If $c = a$, then there is nothing to prove because $z_c u_c = h_c$ in this case and $v_a = p = w_a$ for any $p \in h_c$. Thus we assume that $c \neq a$.

Another trivial case occurs when the pair $(a, c)$ is harmonic. In that case, $z_c = u_c$, i.e. the segment $z_c u_c$ is degenerate, and for $p = z_c = u_c$, the family $v_a w_a(p) = h_a$ is constant. Thus we assume that the pair $(a, c)$ is not harmonic.

Let $z' = p_c(z), u' = p_c(u)$, where $p_c : X \to X$ is the reflection with respect to $c$ (see the proof of Lemma 3.2 and [Bu17]). Then by definition $z_c = (z, z'), u_c = (u, u')$. Note that $z_c u_c \subset h_c$ is a ray when $a$ and $c$ have a common end.

Let $zu \subset X$ be an open arc determined by $z, u$ that does not contain at least one of the ends of $c$, and let $z'u' \subset X$ be the $p_c$ image of $zu$. Then the ends $v, w \in X$ of $d$ miss the intersection $zu \cap z'u'$ (which is nonempty if and only if the pairs $a, c$ separate each other). We assume without loss of generality that $v \in zu \setminus z'u'$ ($zu \setminus z'u' \neq \emptyset$ by the assumption that $(a, c)$ is not harmonic). Then $w \in z'u' \setminus zu$ because $w = p_c(v)$.

Under our assumption, an order on $z_c u_c$ induces well defined orders on the arcs $zu \setminus z'u'$, $z'u' \setminus zu$ such that $p < p'$ if and only if $v < v'$ and $w < w'$ for $p = (c, d), p' = (c, d'), d = (v, w), d' = (v', w')$. Taking projections on $a$, we see that the family $v_a w_a(p) \subset h_a$ is monotone in $p$. \hfill \Box
We say that \( b \in \mathbb{R} \) is the \( s \)-point of an (oriented) segment \( vw \subset \mathbb{R} \), \( s > 0 \), if \( b \in vw \) and \( |vb|/|bw| = s \). For example, 1-point is the midpoint of a segment \( vw \). In that case, the order of \( v, w \) on \( \mathbb{R} \) is not important.

**Lemma 5.2.** Given \( s > 0 \), \( q = (a, b) \in \text{Harm} \), \( a = (z, u) \in aY \), and a line \( h_c \subset \text{Hm} \), there is a unique \( p = (c,d) \in z_cu_c \subset h_c \), \( d = (v,w) \in aY \), such that \( b \in h_a \) is the \( s \)-point of the (maybe degenerate) segment \( vw \). \( a \subset h_a \).

**Proof.** If \( c = a \), then \( z_cu_c = h_c \), and for every \( p = (c,d) \in h_c \), the segment \( vw \) is degenerate, \( v = w = p \). In this case, we take \( p = (c,b) \). We do not exclude the case when the segment \( z_cu_c \) is degenerate, i.e., \( z_c = u_c \). In this case, the pair \((a,c)\) is harmonic, \( d = a \) and \( vaw_a = h_a \). Therefore, any \( b \in h_a \) is understood as the \( s \)-point of the \( h_a \) is not important.

As in the proof of Lemma 5.1, we always assume that \( v \in zu \setminus z'\), \( d = (v,w) \), where \( z' = \rho_c(z) \), \( w' = \rho_c(u) \). Then \( w \in z'u \setminus zu \). We consider \( vw \subset h_a \) as an oriented segment. This is well defined because by Lemma 5.1 the family \( vaw_a = vaw_a(p) \) in monotone in \( p \in z_cu_c \).

First, we show that any \( b \in h_a \) separates the \( s \)-points \( m''_a \) and \( m''_a \) of segments \( v''aw_a \), \( v''aw_a \) respectively for appropriate \( d' = (v',w') \), \( d'' = (v'',w'') \in z_cu_c \subset h_c \). To this end, note that the \( s \)-point \( m_a \) of the segment \( vw \subset h_a \) approaches \( h_a \) ends at infinity \( z \) or \( u \) as \( d = (v,w) \in z_cu_c \) approaches \( z_c \) or \( u_c \) respectively. Indeed, one of \( v_a, w_a \) stays bounded on \( h_a \) while the other one goes along \( h_a \) to infinity as \( d \to z_c \) or \( u_c \). Thus \( m_a \) goes to \( z \) or \( u \) respectively. (It may happen that one of \( z_c, u_c \in h_c \) is at infinity but not both of them when \( a, c \in aY \) have a common end. In that case, the admissible segment \( z_cu_c \) is a ray on \( h_c \), and both \( v_a, w_a \) together with their \( s \)-point \( m_a \) go to respective end at infinity of \( h_a \) when \( d = (v,w) \in h_c \) goes to the infinite end of the ray).

Second, we conclude that any \( b \in h_a \) is the \( s \)-point of a respective segment \( vaw_a \), \( b = m_a \). By the first part, \( b \) lies between \( m''_a \), \( m''_a \). Now, we move from \( d' \) to \( d'' \) along \( h_c \), i.e. consider \( d_t = (1-t)d'+td'' \in h_c \), \( 0 \leq t \leq 1 \), \( d_t = (v_t,w_t) \). The \( s \)-point \( (m_t)_a \) of \( (v_t)_a(w_t)_a \) varies continuously from \( m''_a \) to \( m''_a \) as \( t \) goes from \( 0 \) to \( 1 \). Therefore, there is \( 0 < \tau \leq 1 \) such that \( b = (m_t)_a \).

Finally, we show that the required \( p = (c,d) \in h_c \) is unique. Indeed, by Lemma 5.1, segments \( vaw_a \), \( waw_a \) goes to respective end at infinity of \( h_a \) when \( d = (v,w) \in h_c \) goes to the infinite end of the ray).

For every \( s > 0 \) and a line \( h_c \subset \text{Hm} \) Lemma 5.2 determines a map \( \text{pr}_c^s : \text{Harm} \to h_c \), which is called the \( s \)-projection to the line \( h_c \). In the case \( s = 1 \) we abbreviate \( \text{pr}_c := \text{pr}_c^1 \), and the map \( \text{pr}_c : \text{Harm} \to h_c \) is called the midpoint projection to \( h_c \). The map \( \text{pr}_c^s \circ \text{j} : \text{Harm} \to h_c \) in general differs from \( \text{pr}_c^s \), thus in \( \text{Hm} \) we have two maybe different projections to the line \( h_c \) depending on the choice of one of the entries of \( q = (a,b) \in \text{Hm} \). However, \( \text{pr}_c^s \) is well defined along any line \( h_a \subset \text{Hm} \), and hence along any \( zz \)-path.
5.2 Equal ratio projection

**Lemma 5.3.** Given $q = (a, b) \in \text{Harm}$, $a = (x, y)$, $b = (z, u) \in aY$, and a line $h_c \subset \text{Hm}$, there is a unique $p = (c, d) \in x_c y_c \cap z_c u_c \subset h_c$, $d = (v, w)$, such that $a \in v_b w_b \subset h_b$, $b \in v_a w_a \subset h_a$ and

$$\frac{|v_b a|}{|a w_b|} = \frac{|v_a b|}{|b w_a|}.$$ 

**Proof.** We fix an orientation of the line $h_c$ and the respective order. Note that one of the segments $x_c y_c$, $z_c u_c$ is degenerate if and only if one of the pairs $(a, c)$ or $(c, b)$ is harmonic. Then there is nothing to prove because $p = (c, a)$, $d = (v, w) = (x, y)$, $a = v_b = w_b \in h_b$, $b \in v_a w_a = h_a$ in the first case, and $p = (c, b)$, $d = (v, w) = (z, u)$, $a \in v_b w_b = h_b$, $b = v_a = w_a \in h_a$, in the second case (and the required equality is understood as $0/0 = \infty/\infty$).

Thus we assume that none of the segments $x_c y_c$, $z_c u_c$ is degenerate. Moreover, their intersection $x_c y_c \cap z_c u_c$ is not empty and also a nondegenerate segment because the pairs $(x, y)$ and $(z, u)$ being harmonic separate each other. Without loss of generality, we assume that $x_c < y_c$, $u_c < z_c$. Then $u_c < y_c$ because the pairs $(x, y)$ and $(z, u)$ separate each other on $X$. Hence every $d \in x_c y_c \cap z_c u_c$ separates $u_c$, $y_c$ and $x_c$, $z_c$.

Furthermore, $c$ cannot separate $(x, y)$, $(z, u)$. Thus we can assume without loss of generality that $c$ does not separate $x$, $z$. We also assume that $v$ lies on the same arc in $X$ determined by $c$ as $x$ and $z$. Then the assumption $d = (v, w) \in x_c y_c \cap z_c u_c$ implies that $v$ lies between $x$ and $z$ on that arc.

It follows that when we are moving along $h_a$ from $x$ to $y$, we meet $v$ earlier than $z$, and $u$ earlier than $w$. Hence, $b \in v_a w_a$. Similarly, when we are moving along $h_b$ from $z$ to $u$, we meet $v$ earlier than $x$, and $y$ earlier than $w$. Hence $a \in v_b w_b$.

To be definite we assume that $x_c y_c \cap z_c u_c = x_c z_c$ (other cases are considered similarly). Thus if $p = (c, d) \in x_c z_c$ goes to $x_c$, then $v_a \to \infty$, while $w_a$ stays bounded on $h_a$, and $v_b \to a$, while $w_b \not\to a$. Setting $s = |v_b a|/|a w_b|$, $t = |v_a b|/|b w_a|$, we see that $(s, t) \to (0, \infty)$ as $p \to x_c$.

Similarly, if $p \to z_c$, then $v_a \to b$, while $w_a \not\to b$, and $v_b \to \infty$, while $w_b$ stays bounded. Therefore, $(s, t) \to (\infty, 0)$ in this case. By continuity, there is $p \in x_c z_c$ with $s = t$. This gives a required $p \in x_c y_c \cap z_c u_c \subset h_c$.

By Lemma 5.1 segments $v_a w_a$, $v_b w_b$ are monotone in $p \in x_c y_c \cap z_c u_c \subset h_c$. Since $b \in v_a w_a$, $a \in v_b w_b$, this implies that the rations $s$, $t$ are monotone. Thus a required $p$ is unique.

For a line $h_c \subset \text{Hm}$, Lemma 5.3 determines a map $\text{pr}_c : \text{Harm} \to h_c$, which is called the equal ratio projection to the line $h_c$. Note that for $q = (a, b) \in \text{Harm}$ we have $\text{pr}_c(q) = \text{pr}_c^q(q)$ for some well determined $s > 0$, where $s$ depends on $q$. 

\[14\]
5.3 Strictly contracting property of the midpoint projection

Increment axiom (I) is only used in the proof of the following proposition which plays a key role in the paper.

**Proposition 5.4.** Given lines $h_a, h_c \subset H_m$, $a \neq c$, and points $d = (v, w)$, $d' = (v', w') \in h_c$ such that the pairs $(v, w')$, $(v', w)$ separate each other, we have

$$\frac{1}{2}(|v_a v'_a| + |w_a w'_a|) > |dd'|.$$

For its proof see [Bu17, Proposition 7.11].

**Lemma 5.5.** The midpoint projection $pr_c : h_a \rightarrow h_c$ to any line $h_c \subset H_m$ is strictly contracting along any line $h_a \subset H_m$, $a \neq c$.

**Proof.** Given $q = (a, b)$, $q' = (a, b') \in h_a$, we let $p = pr_c(q)$, $p' = pr_c(q')$ be the midpoint projections to $h_c$. Then $p = (c, d)$, $p' = (c, d')$ with $d = (v, w)$, $d' = (v', w') \in aY$, so that $c$ and $(v, w)$ separate each other as well as $c$ and $(v', w')$. We assume without loss of generality that $v, v'$ lie on an arc in $X$ determined by $c$, while $w, w'$ lie on the other arc determined by $c$. Then the pairs $(v, w'), (v', w)$ separate each other.

By Proposition 5.4

$$\frac{1}{2}(|v_a v'_a| + |w_a w'_a|) > |dd'|.$$

By definition of the midpoint projection, $d, d' \in z_{c, u_c}$, where $a = (z, u)$. Thus $a$ separates $(v, v')$ and $(w, w')$ (in terms of [Bu17], it means that the event $(z, u) \in aY$ is strictly between events $(v, v')$, $(w, w') \in aY$). By Lemma 3.2, the pairs $d = (v, w)$ and $d' = (v', w') \in h_c$ are in the strong causal relation. Since $(z, u)$ separates $(v, v')$ and $(w, w')$, it follows that moving along $a$, we meet $v, v'$ and $w, w'$ in the same order. Identifying the line $h_a$ with real line $\mathbb{R}$, it means that the signs of $v_a - v'_a$ and $w_a - w'_a$ coincide.

Since $b$ is the midpoint of $v_a w_a$ and $b'$ is the midpoint of $v'_a w'_a$, we obtain

$$|bb'| = \left|\frac{1}{2}(v_a + w_a) - \frac{1}{2}(v'_a + w'_a)\right|$$

$$= \frac{1}{2}|v_a - v'_a + w_a - w'_a|$$

$$= \frac{1}{2}(|v_a v'_a| + |w_a w'_a|).$$

Hence, $|bb'| > |dd'|$ and therefore $|qq'| > |pp'|$. \qed
6 Distance $\delta$ along segments

We assume that a monotone Möbius structure $M$ satisfies Increment axiom (I), and under this assumption we prove Theorem [14.1]

**Proposition 6.1.** For any side $\sigma$ of a closed zz-path $S \subset \text{Harm}$ we have

$$|\sigma| < \sum_{\sigma'} |\sigma'|,$$

where the sum is taken over all sides $\sigma'$ of $S$, $\sigma' \neq \sigma$.

**Proof.** Let $S'$ be a zz-path which is the union of all segments of $S$ excluding $\sigma$, that is, $S = \sigma \cup S'$. The idea is to use the midpoint projection of $S'$ to the line $h_c$ determined by $\sigma$, $\sigma \subset h_c$, and apply Lemma [5.3]. The problem is that the midpoint projections on adjacent segments of $S'$ may not coincide on the common vertex (in the case of the canonical Möbius structure on $S^1$ they coincide). This could create gaps on $\sigma$ which do not covered by the projection and thus prevent the required estimate.

Let $V = V(S')$ be the vertex set $S'$. We fix $\varepsilon > 0$ and for every vertex $v \in V$ we take $\varepsilon_v > 0$ such that $\sum_{v \in V} \varepsilon_v < \varepsilon$. We use the midpoint projection on $S'$ outside of the $\varepsilon_v$-neighborhoods $U_v(\varepsilon_v)$, $v \in V$, of vertices, the equal ratio projection on the vertices, and interpolate between these types of projections inside of $U_v(\varepsilon_v)$ to obtain a continuous projection $\text{pr}_\sigma : S' \to h_c$ with controlled metric properties.

Let $\sigma' \subset S'$ be a side of $S$ different from $\sigma$, $h_a \subset \text{Harm}$ the line containing $\sigma'$, $\sigma' = pq \subset h_a$. If $\sigma'$ is adjacent to $\sigma$, then we assume to be definite that $p \in \text{Harm}$ is the common vertex of $\sigma$, $\sigma'$, in particular, $p = (a, c)$. Then $q = (a, c')$ for some $c' \in aY$. In this case, by Lemmas [5.2], [5.3], the whole $\sigma'$ is projected to $p$, $\text{pr}_\sigma(\sigma') = p$.

Thus we assume that $\sigma'$ is not adjacent to $\sigma$. Let $pq' \subset \sigma'$ be the minimal subsegment containing the $\varepsilon_p$-neighborhood of $p$ in $\sigma'$, $|pq'| = \varepsilon_p$. We define $\text{pr}_\sigma$ on $pq'$ by taking $\text{pr}_\sigma(p) = \text{pr}_{\sigma'}(p)$, $\text{pr}_\sigma(q') = \text{pr}_{\sigma'}(q')$ and $\text{pr}_\sigma(p_r) = \text{pr}_{\sigma'}(p_r)$, where $p_r = (1 - \tau)p + \tau q'$, $s = s(\tau) = (1 - \tau)s_p + \tau$, and $s_p > 0$ is determined by $\text{pr}_{\sigma'}(p)$, $\text{pr}_{\sigma'}(p) = \text{pr}_{\sigma'}^p(p)$ (we take $s_p = 1$ if the adjacent to $\sigma'$ at $p$ segment $\sigma'' \in S'$ is adjacent to $\sigma$).

We have constructed a projection $\text{pr}_\sigma : S' \to h_c$ of the zz-path $S'$ to the line $h_c$. Continuity of $\text{pr}_\sigma$ along sides of $S'$ follows from the uniqueness property of $\text{pr}_\sigma^c$, continuity at common vertices of adjacent sides follows from definition of $\text{pr}_{\sigma'}^c$. Since $\text{pr}_\sigma$ is constant on the sides $\sigma_1$, $\sigma_2$ adjacent to $\sigma$, which are mapped to the vertices of $\sigma$, the continuity of $\text{pr}_\sigma$ implies that the image $\text{pr}_\sigma(S') \subset h_a$ covers $\sigma$. Thus $|\sigma| \leq \sum_{\sigma' \neq \sigma} |\text{pr}_\sigma(\sigma')|$

We decompose the right hand side of this inequality as $\sum_{\sigma' \neq \sigma} |\text{pr}_\sigma(\sigma')| = A + B$, where $A$ is the length of $\text{pr}_\sigma(S' \setminus \cup_{v \in V} U_v(\varepsilon_v))$ and $B$ the length of $\text{pr}_\sigma(\cup_{v \in V} U_v(\varepsilon_v))$. Since $\text{pr}_\sigma$ coincides with the midpoint projection $\text{pr}_{\sigma'}$ on $S'$.
outside of the union $\bigcup_{v \in V} U_v(\varepsilon_v)$, Lemma 5.5 gives $A < |S''| - \sum v \varepsilon_v < |S'|$, where $S'' = S' \setminus (\sigma_1 \cup \sigma_2)$.

Since $\text{pr}_\sigma$ is continuous on $S'$, we can make $B$ arbitrarily small taking $\varepsilon$ sufficiently small, say $B < \delta(\varepsilon) < |\sigma_1| + |\sigma_2|$. Thus $|\sigma| \leq |S''| + \delta(\varepsilon) < |S''| + |\sigma_1| + |\sigma_2| = |S'|$. \hfill $\square$

**Corollary 6.2.** For any $q, q' \in \text{Harm}$ on a line we have $\delta(q, q') = |qq'|$.

**Proof.** Let $S$ be a $zz$-path in $\text{Harm}$ between $q, q'$ different from the segment $qq'$. By definition, the first and last sides of $S$ lie of the line determined by the segment $qq'$. We denote by $\tilde{q}, \tilde{q}'$ the ends of the first and the last sides respectively, and assume that we have the order $q < \tilde{q} < \tilde{q}' < q'$ along the segment $qq'$. Any other order only makes arguments easier.

Let $S'$ be the $zz$-subpath of $S$ between $\tilde{q}, \tilde{q}'$. Then $S'$ together with the segment $\tilde{q}q'$ gives a closed $zz$-path in $\text{Harm}$. By Proposition 6.1 we obtain $|\tilde{q}q'| < |S'|$. On the other hand, $|qq'| = |\tilde{q}q'| + a$ and $|S| = |S'| + a$, where $a = |qq| + |q'q'|$. Hence $|S| > |qq'|$ and thus $\delta(q, q') = |qq'|$. \hfill $\square$

This completes the proof of Theorem 1.1.

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