A Local Stochastic Algorithm for Separation in Heterogeneous Self-Organizing Particle Systems

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Abstract

We investigate stochastic, distributed algorithms that can accomplish separation and integration behaviors in self-organizing particle systems, an abstraction of programmable matter. These particle systems are composed of individual computational units known as particles that each have limited memory, strictly local communication abilities, and modest computational power, and which collectively solve system-wide problems of movement and coordination. In this work, we extend the usual notion of a particle system to treat heterogeneous systems by considering particles of different colors. We present a fully distributed, asynchronous, stochastic algorithm for separation, where the particle system self-organizes into clustered color classes using only local information about each particle’s preference for being near others of the same color. We rigorously analyze the correctness and convergence of our distributed, stochastic algorithm by leveraging techniques from Markov chain analysis, proving that under certain mild conditions separation occurs with high probability. These theoretical results are complemented by simulations.

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1 Introduction

Examples of heterogeneous entities separating and integrating exist at many scales, from molecules exhibiting attractive and repulsive forces, to mixed solutions of varying viscosities, to inherent human biases that influence how we form and maintain social groups. Another example is species such as ants co-mingling peacefully when resources are plentiful but prioritizing the survival of their own colony when resources are scarce. This fundamental behavior of heterogeneous entities separating or integrating in response to environmental stimuli spans remarkably diverse areas of study.

We investigate this phenomenon as it applies to programmable matter, a physical material that can intelligently respond to user input or stimuli from its environment by changing its physical properties to achieve a goal. In our work on self-organizing particle systems, we abstractly envision programmable matter as an ensemble of simple, active computational particles that individually execute distributed, local, asynchronous algorithms to cooperatively achieve macro-scale movement and coordination tasks. In the geometric amoebot model [10,11], these particles move on the infinite triangular lattice, communicate only with immediate neighbors, and have extremely limited computational power. Here we consider heterogeneous particle systems — where particles have different immutable colors — and seek local, distributed algorithms which, when executed independently and concurrently by all particles, result in observable separation or integration of color classes regardless of the starting state. These behaviors can also be viewed as self-organizing sorting or mixing.

To develop distributed algorithms for separation in heterogeneous particle systems, we use concepts from stochastic processes. Of particular relevance is the Schelling model [26,27], which explores how micro-motives of individuals can induce macro-phenomena such as racial segregation in residential neighborhoods. The Ising model of ferromagnetism from statistical physics [28] exhibits a similar dependence of global behavior on a single parameter inducing local preferences. Our work harnesses this interplay between local preferences and global behavior to develop a stochastic, distributed, local, asynchronous algorithm that provably accomplishes separation. The same algorithm is also observed to cause integration with alternate values of certain input parameters.

1.1 Our Results

We present a fully distributed, local, asynchronous algorithm $A$ for heterogeneous particle systems that provably achieves separation. Informally, we say a particle configuration with two colors is separated if we can identify a set $R$ of particles such that $R$ mostly contains particles of color $c_1$, its complement $\overline{R}$ mostly contains particles of color $c_2$, and the boundary between $R$ and $\overline{R}$ is small. If this is the case, we say $R$ and $\overline{R}$ are clusters. To further quantify this, we say a configuration is $(\beta,\delta)$-clustered, for $\beta > 0$ and $\delta < 1/2$, if there are at most $\delta |R|$ particles of color $c_2$ in $R$, at most $\delta |\overline{R}|$ particles of color $c_1$ in $\overline{R}$, and the boundary between $R$ and $\overline{R}$ is of size at most $\beta \sqrt{n}$, where $n$ is the total number of particles.

In the separation problem, we consider an instance $(\sigma_0, \beta, \delta)$, where $\sigma_0$ is an initial configuration of colored particles and $\beta > 0$ and $0 < \delta < 1/2$ are constants. We say a distributed
algorithm solves an instance of the separation problem if, beginning from configuration $\sigma_0$, with all but exponentially small probability it reaches and remains in a set of configurations that are $(\beta, \delta)$-clustered. When the boundary of particle configurations is fixed and small our distributed algorithm $A$ provably solves the separation problem for any $\sigma_0$ with two colors, any $\beta > 4$, and any $\delta < 1/2$. Algorithm $A$ also achieves integration in simulation when using different values of the input parameters, but we do not give rigorous guarantees.

1.2 Related Work

In active programmable matter, individual computational units can control their decisions and movements (as opposed to passive systems whose units have little to no control; see, e.g., [2]). Examples include swarm robotic systems and modular self-reconfigurable robotic systems [16,32] (in particular, metamorphic robots [7,29,30]). Rather than focus on specific instantiations of active programmable matter, the amoebot model [10,11] is an abstraction that enables rigorous study of capabilities and limitations. Efficient (nearly) deterministic algorithms have been developed under this model for tasks such as shape formation [12], leader election [9], object coating [8,13], and distributed computation [23]. These algorithms rely critically on local communication and storing states in memory. For our work on separation, however, we use the stochastic approach to self-organizing particle systems (Section 2.3) which produces algorithms that are nearly oblivious, require almost no communication between particles, and are trivially and significantly more robust to failures (see, e.g., the algorithms for compression [6] and shortcut bridging [1]). Di Luna et al. [14] recently also studied fault tolerance in the amoebot model by considering systems with crash failures. Among other abstractions of programmable matter, the nubot model [31] is most closely related but includes additional capabilities that prohibit direct translation of its results to our setting.

We use a Markov chain to develop our distributed algorithm for separation, and there are several relevant related works in that area. Of particular interest is the classical Schelling housing model [26,27] and related models from statistical physics, such as the Ising model of ferromagnetism [24,28]. In these models, agents (or “sites”) are assigned one of two colors (or “spins”), and each agent prefers to have the same color as its neighbors. Depending on the strength of this preference, an agent may change its color or move to a new location in order to agree with more of its neighbors. These models undergo a phase transition with respect to the preference parameter: at high values the two colors are well integrated, while at low values large monochromatic regions appear. Several variants of the Schelling model have recently been shown to exhibit similarly interesting behavior [4,18] and have received attention from the distributed computing community [22]. Our work also considers local neighborhood information and utilizes threshold-like mechanics, but with three key differences: (i) our particles cannot change their color, so the number of particles of each color is fixed; (ii) we do not assume that every position is occupied as the Schelling model does; and (iii) particles can only move to neighboring locations. Taken together, these constraints define a different and potentially more interesting set of dynamics.

1.3 Our Approach and Techniques

Our distributed algorithm $A$ for the separation problem is derived from a Markov chain using the stochastic approach for self-organizing particle systems, initiated in [6]. This approach, described in detail in Section 2.3, uses concepts from statistical physics to design a Markov chain $\mathcal{M}$ that has some desired convergence behavior and uses knowledge of distributed
computing to transform \( M \) into \( A \), which is local and asynchronous but still exhibits the same behavior as \( M \).

To ensure our algorithm’s long-run behavior is as desired, we use foundational concepts from Markov chains such as Metropolis filters and detailed balance. To prove that \( A \) solves the separation problem for configurations with small boundary, we use a Markov chain analysis technique known as bridging that was developed to analyze molecular mixtures known as colloids [21]. Adapting this approach requires several new innovations to overcome obstacles such as the irregular shapes of particle system configurations, the non-self-duality of the triangular lattice, and the multiple types of adjacencies between differently colored particles. Our analysis is based on a Peierls argument (as does [21]), which has been used, for example, in statistical physics to study uniqueness of Gibbs measures and in computer science to give lower bounds on the mixing times of Markov chains (see, e.g., [5]). While Peierls arguments have also been used in previous work on stochastic algorithms for self-organizing particles systems [1,6], our approach here is of a different flavor, involving three distinct stages.

2 Background

2.1 The Amoebot Model

In the amoebot model, originally proposed in [11] and described in full in [10], programmable matter consists of individual, homogeneous computational elements called particles. In its geometric variant, the underlying geometry is the infinite triangular lattice \( G_\Delta = (V, E) \) (see Fig. 1). Each particle occupies either a single node in \( V \) (i.e., it is contracted) or a pair of adjacent nodes in \( V \) (i.e., it is expanded), as in Fig. 1. Particles move via a series of expansions and contractions: a contracted particle can expand into an unoccupied adjacent node to become expanded, and completes its movement by contracting to once again occupy a single node. An expanded particle’s head is the node it last expanded into and the other node it occupies is its tail; a contracted particle’s head and tail are the same node.

Two particles occupying adjacent nodes are said to be neighbors. Each particle is anonymous, lacking a unique identifier, and has a constant-size local memory which it and its neighbors can directly read from and write to for communication. Particles do not have any global information such as a global compass or estimate of the size of the system.

We assume the standard asynchronous model of computation from distributed computing (see, e.g., [20]), where a system progresses through atomic actions. A classical result

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1 Here, we omit details which are not necessary for our our application of separation.

2 Our past works refer to \( G_\Delta \) as the equilateral triangular grid graph \( G_{eqt} \) or the triangular lattice \( \Gamma \).
under this model states that for any concurrent asynchronous execution of atomic actions, there exists a sequential ordering of actions producing the same end result, provided conflicts that arise in the concurrent execution are resolved. In the amoebot model, an atomic action corresponds to a single particle activation in which a particle can perform an arbitrary, bounded amount of computation involving its local memory and the memories of its neighbors and at most one expansion or contraction. We assume conflicts involving concurrent memory writes or simultaneous particle expansions into the same unoccupied node are resolved arbitrarily such that at most one particle is writing into a given memory location or expanding into a given node at a time. Thus, while in reality many particles may be active concurrently, it suffices when analyzing our algorithm to consider a sequence of activations where only one particle is active at a time.

2.2 Systems of Heterogeneous Particles

We consider a particle system composed of \( n \) heterogeneous particles, generalizing previous work in which all particles were identical and indistinguishable [10]. We model heterogeneity by assuming each particle \( P \) keeps a color \( c(P) \in \{c_1, \ldots, c_k\} \) in its memory that is visible to itself and its neighbors, where \( k < n \) is some small constant. For simplicity, we assume \( k = 2 \) in this paper, though we expect our ideas, algorithms, and proof techniques to generalize to larger \( k \) with little additional effort. These colors can represent anything from differences in equipment between robots in multi-robot systems to demographic diversity in human communities. If particles \( P \) and \( Q \) are neighbors we say they are joined by an edge, and this edge is homogeneous if \( c(P) = c(Q) \) and heterogeneous otherwise.

We define a swap move under the amoebot model that enables adjacent particles of different colors to switch places. For two neighboring contracted particles \( P \) and \( Q \), either \( P \) or \( Q \) can initiate a swap to exchange colors, which can be implemented as follows: \( P \) reads \( x \leftarrow c(Q) \) from the memory of \( Q \), overwrites \( c(Q) \leftarrow c(P) \) in the memory of \( Q \), and finally updates its own color \( c(P) \leftarrow x \). Implementing this swap as an exchange of in-memory attributes is purely for modeling convenience. In systems where individuals have immutable “color” (e.g., ants from different colonies or robots with different hardware), swaps could be realized by some coordinated movement. Adding this natural swap move enables faster convergence of our algorithms in practice, but is not necessary for any results we present.

2.3 The Stochastic Approach to Self-Organizing Particle Systems

The stochastic (Markov chain) approach to self-organizing particle systems was introduced in [6] and further validated its applicability in [1]. Background material on Markov chains can be found in standard textbooks (see, e.g., [19]), while a more focused overview of Markov chain terminology relevant to our work on particle systems can be found in [10]. This approach is motivated by studies from statistical physics that investigate the local micro-behavior causes of global macroscopic phenomena [3,5,25]. Like a spring relaxing, physical systems favor configurations \( \sigma \) that minimize energy, determined by a Hamiltonian \( H(\sigma) \). Each configuration is assigned a weight from the Gibbs distribution: \( w(\sigma) = e^{-B \cdot H(\sigma)} / Z \), where \( B \) is inverse temperature and \( Z = \sum_{\sigma} e^{-B \cdot H(\sigma)} \) is the normalizing constant known as the partition function. Markov chains have been well-studied as a tool for sampling from the Gibbs distribution.

To achieve separation, we define a Hamiltonian \( H(\sigma) \) over particle configurations \( \sigma \) that assigns low values to configurations with many edges and large homogeneous clusters. Using this Hamiltonian, we get Gibbs distribution \( w(\sigma) = e^{-B \cdot H(\sigma)} / Z = \lambda^{c(\sigma)} \cdot \gamma^{a(\sigma)} / Z \), where
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\( e(\sigma) \) is the number of edges and \( a(\sigma) \) is the number of homogeneous edges of \( \sigma \). When \( \lambda \) is large this increasingly favors compressed configurations, while for small \( \lambda \) the opposite is true, just as in [6]. The additional \( \gamma \) factor controls separation, favoring clustered/separated configurations when \( \gamma \) is large and well-mixed configurations when \( \gamma \) is small.

Our main contribution is a Markov chain \( \mathcal{M} \) whose stationary distribution \( \pi(\sigma) = w(\sigma) \) is exactly this Gibbs distribution that, for large \( \lambda \) and large \( \gamma \), favors compressed, separated configurations. We run \( \mathcal{M} \) indefinitely; once we reach its stationary distribution \( \pi \), we continue moving among different configurations but remain at this desirable distribution. We design \( \mathcal{M} \) using a Metropolis filter [17], which gives transition probabilities for a Markov chain so that it converges to some desired stationary distribution. Starting at state \( x \in \Omega \), \( \mathcal{M} \) picks a neighboring state \( y \) uniformly with probability \( 1/(2\Delta) \), where \( \Delta \) is the maximum number of transitions leaving any state, and moves to \( y \) with probability \( \min\{1, \pi(y)/\pi(x)\} \); with the remaining probability, \( \mathcal{M} \) stays at \( x \) and repeats. One can verify \( \pi \) must be the stationary distribution of \( \mathcal{M} \) using detailed balance. However, showing that poorly separated configurations are exponentially unlikely in \( \pi \) is far less immediate and requires significantly more machinery (Section 4).

While calculating \( \pi(y)/\pi(x) \) seems to require global knowledge, this ratio can often be calculated using only local information when many terms cancel out. In our case, the Metropolis probabilities can be written as \( \lambda^{e(y)−e(x)} \cdot \gamma^{a(y)−a(x)} \). If \( x \) and \( y \) only differ in the position of one particle \( P \) or by a swap between neighbors \( P \) and \( Q \) (as is the case for all moves of our Markov chain \( \mathcal{M} \)) then \( \lambda^{e(y)−e(x)} \cdot \gamma^{a(y)−a(x)} \) can be calculated with local information from the neighborhoods of \( P \) and \( Q \). This crucial fact enables us to translate \( \mathcal{M} \) into a distributed algorithm (see Section 3.1).

3 A Stochastic Algorithm for Separation

To achieve separation, our algorithm must create large, mostly monochromatic clusters. It does so using two bias parameters: \( \lambda \), which controls how strongly particles prefer to have more neighbors, and \( \gamma \), which controls how strongly particles prefer to have more neighbors of the same color.

We begin with some terminology. Recall each particle \( P \) has a color \( c(P) \in \{c_1, \ldots, c_k\} \); we assume \( k = 2 \). An arrangement of a particle system is a function \( f : V(G_\Delta) \to \{0, \ldots, k\} \) which maps nodes of the triangular lattice to an index \( i \in \{1, \ldots, k\} \) if they are occupied by the tail of a particle \( P \) with \( c(P) = c_i \), and to 0 otherwise. Two arrangements are equivalent if they are translations of each other; we define a configuration to be an equivalence class of particle system arrangements. Recall a configuration is connected if the subgraph of \( G_\Delta \) induced by its occupied nodes is connected. A hole in a configuration is a maximal finite, connected component of unoccupied nodes. The perimeter \( p(\sigma) \) of a connected hole-free configuration is the length of the walk around its (single external) boundary.

Starting at any connected, hole-free configuration, our stochastic algorithm for separation (Algorithm 1) ensures the particle system remains connected and hole-free throughout its execution, an invariant we prove in Section 3.2. This invariant is achieved using the two local properties for particle moves first introduced in [6]. Taken together, these properties ensure that a particle’s local connectivity with respect to its neighbors does not change as a result of its move. Moreover, they ensure that the resulting Markov chain is reversible, which is necessary for applying established tools from Markov chain analysis.

We use the following notation. For a location \( \ell \), let \( N_i(\ell) \) denote the set of particles of color \( c_i \) adjacent to location \( \ell \). For neighboring locations \( \ell \) and \( \ell' \), let \( N_i(\ell \cup \ell') \) denote
the set $N_i(\ell) \cup N_i(\ell')$, excluding particles occupying $\ell$ and $\ell'$. When ignoring color, let $N(\ell) = \bigcup_i N_i(\ell)$; define $N(\ell \cup \ell')$ analogously. Let $S = N(\ell) \cap N(\ell')$ denote the set of particles adjacent to both locations. The following locally-checkable properties ensure the particle system configuration remains connected and hole-free.

- **Property 1.** $|S| \in \{1, 2\}$ and every particle in $N(\ell \cup \ell')$ is connected to exactly one particle in $S$ by a path through $N(\ell \cup \ell')$.
- **Property 2.** $|S| = 0$, and both $N(\ell) \setminus \{\ell\}$ and $N(\ell') \setminus \{\ell\}$ are nonempty and connected.

These properties need not be verified for swap moves, which do not change which nodes of a configuration are occupied so cannot disconnect the system or create a hole. We can now introduce the Markov chain $\mathcal{M}$ for an instance $(\sigma_0, \sigma, \delta)$ of the separation problem. For input parameters $\lambda, \gamma$ and an initial configuration $\sigma_0$ which we assume is connected and hole-free, repeat Algorithm [1] If $\sigma_0$ has holes, $\mathcal{M}$ will eliminate them and they will not reform; we focus only on what happens after this occurs. We let $\Omega$ be the state space of $\mathcal{M}$, containing all connected hole-free configurations with the same number of particles of each color as $\sigma_0$.

**Algorithm 1 Markov Chain $\mathcal{M}$ for Separation and Integration**

1: Choose particle $P$ uniformly at random from all $n$ particles; let $c_i$ be its color and $\ell$ its location.
2: Choose a neighboring location $\ell'$ and $q \in (0, 1)$ uniformly at random.
3: if $\ell'$ is unoccupied then
4: \hspace{1em} $P$ expands to occupy both $\ell$ and $\ell'$.
5: \hspace{1em} if (i) $\ell$ and $\ell'$ satisfy Property 1 or 2 and (ii) $q < \lambda^{|N(\ell)'| - |N(\ell)'|, \gamma^{|N_i(\ell)'| - |N_i(\ell)'|}$ then
6: \hspace{2em} $P$ contracts to $\ell'$.
7: \hspace{1em} else $P$ contracts back to $\ell$.
8: else if $\ell'$ is occupied by particle $Q$ of color $c_j$ then
9: \hspace{1em} $P$ calculates $|N_j(\ell)|$ and $|N_j(\ell) \setminus \{Q\}|$ and sends these values to $Q$.
10: \hspace{1em} $Q$ calculates $|N_j(\ell') \setminus \{P\}|$ and $|N_j(\ell')|$.
11: \hspace{1em} if $q < \gamma^{|N_j(\ell') \setminus \{P\}| - |N_j(\ell)|, |N_j(\ell') \setminus \{Q\}| - |N_j(\ell')|}$ then $Q$ swaps with $P$.

### 3.1 From Markov Chain to Local, Asynchronous Algorithm

To translate Markov chain $\mathcal{M}$ into a fully local, distributed, asynchronous algorithm $\mathcal{A}$ that is run by each particle concurrently, we decompose the steps of $\mathcal{M}$ into individual particle activations, in which a single particle performs some computation and at most one movement [10]. In particular, this decouples a particle’s expansion and contraction (Steps 4, 6) and $P$ and $Q$’s coordinated swap (Steps 9, 11) each into two particle activations. We utilize flag-locking mechanisms to ensure consistent snapshots of particle neighborhoods.

We first show how to decouple a particle’s expansion and contraction; we used this construction in previous work [16] and recall the important details here. When a particle $P$ expands to occupy both $\ell$ and $\ell'$ (Step 4 of $\mathcal{M}$), it sets its moving flag $f_m$ to True if it is the only expanded particle in its neighborhood and none of its neighbors have their swapping flags set to True (see below), and False otherwise. This completes its first activation. When $P$ is activated again some time later, it checks its flag: if $f_m$ is False, it simply contracts back to $\ell$. Otherwise, it performs the condition checks (ignoring any heads of expanded neighbors) and decides whether to contract to $\ell'$ or back to $\ell$ accordingly, as in Steps 5, 7 of $\mathcal{M}$. It then resets $f_m$ to False and completes its second activation.
To decouple a swap initiated by a particle $P$ with a neighbor $Q$, we first consider the activation of $P$. Each particle has a swapping flag $f_s$ to indicate that it is initiating a swap, and a request flag $f_r$ to indicate that a swap is being initiated with it; both flags are initially False. If $P$ examines its neighbors finds one $f_s$ or $f_m$ set to True, it completes its activation and does nothing; otherwise, it knows there are no moves or other swaps occurring in its neighborhood and sets both $P.f_s$ and $Q.f_r$ to True. It concludes its activation by calculating the values in Step 9 of $M$ and writing them in the memory of $Q$. When $Q$ is activated some time later and observes that its request flag $f_r$ is True, it examines its neighbors. If none have $f_m$ set to True and only $P$ has $f_s = True$, then $Q$ knows no moves or other swaps are occurring in its joint neighborhood with $P$. So, using the values $P$ sent, $Q$ can perform Steps 10 of $M$ locally, deciding whether or not to swap based on the probability calculation. Regardless of whether or not the swap occurs, $Q$ concludes its activation by resetting $P.f_s$ and $Q.f_r$ to False.

Taken together, these flag-locking mechanisms ensure that at most one movement or swap is happening per neighborhood at a time, mimicking the sequential nature of $M$. However, under the assumptions of the asynchronous model of distributed computing, one cannot assume the next particle to be activated is equally likely to be any particle, as we do in Step 1 of $M$. To justify this random activation assumption, we note that uniformly random sequences of particle activations can be accomplished using Poisson clocks with mean 1: a particle activates after a delay $t$ drawn with probability $e^{-t}$ and, after activating, redraws a new delay to its next activation. The exponential distribution is unique in that, if particle $P$ has just activated, it is equally likely that any particle will be the next particle to activate, including particle $P$ (see, e.g., [15]). Asynchronous activation sequences might be better approximated by each particle having its own constant mean for its Poisson clock, meaning some particles are always more likely to be activated next than others. Because this changes nothing in our analysis and the stationary distribution of $M$ is the same in both settings, we assume clocks with mean 1 for simplicity.

### 3.2 Invariants of Markov Chain $M$

We now examine the behavior of $M$, which determines the behavior of distributed $A$.

► **Lemma 3.1.** If $\sigma_0$ is a connected, hole-free configuration, then the current configuration at every iteration of $M$ is also connected and hole-free.

**Proof.** Moves of a particle into an unoccupied location are limited by Properties 1 and 2 exactly as in [6], and their results show such moves cannot disconnect the system or create a hole. Swap moves also cannot disconnect the system or form a hole since they leave the set of occupied vertices unchanged. These are the only two types of moves allowed by $M$. ◄

► **Lemma 3.2.** Markov chain $M$ is ergodic.

**Proof.** First, $M$ is aperiodic: there is always a positive probability of a swap between similarly colored particles, which does not change the state of $M$ because particles of the same color are indistinguishable. To see that the moves of $M$ suffice to move between any states of $\Omega$, we note the set of non-swap moves of $M$ is the same as in [6]. Their results show that for homogeneous particle systems these moves suffice to move between any two connected hole-free configurations. If two configurations are the same when colors are disregarded, swap moves can then be used to transform one into the other without changing the overall shape. As a result, allowed moves can transform any configuration $\sigma$ into any other configuration $\tau$ in $\Omega$, so $M$ is irreducible and thus ergodic. ◄
As $M$ is finite and ergodic, it converges to a unique stationary distribution $\pi$ which we can find using detailed balance.

**Lemma 3.3.** The stationary distribution of $M$ is given by $\pi(\sigma) = \frac{\lambda^e(\sigma) \cdot \gamma^a(\sigma)}{Z}$, where $Z = \sum_{\sigma} \lambda^e(\sigma) \cdot \gamma^a(\sigma)$.

**Proof.** Because $M$ is ergodic with a finite state space, it converges to a unique stationary distribution. We verify this stationary distribution is $\pi$ as claimed via detailed balance, that is, by showing for all $\sigma, \tau \in \Omega$ that $\pi(\sigma) P(\sigma, \tau) = \pi(\tau) P(\tau, \sigma)$, where $P$ is the transition matrix of $M$. It suffices to consider pairs $\sigma, \tau$ where $P(\sigma, \tau) > 0$; because we have carefully defined $M$, especially Properties 1 and 2, this happens if and only if $P(\tau, \sigma) > 0$. There are two cases to consider, one for each type of move allowed by $M$.

If $\sigma$ and $\tau$ differ by a move of particle $P$ of color $c_i$ from location $\ell$ in $\sigma$ to location $\ell'$ in $\tau$, then Steps 1–6 of Algorithm 1 imply

$$P(\sigma, \tau) = \frac{1}{6n} \min\{1, \lambda^{|N(\ell')|-|N(\ell)|} \cdot \gamma^{N_i(\ell')-N_i(\ell)}\},$$

$$P(\tau, \sigma) = \frac{1}{6n} \min\{1, \lambda^{|N(\ell)|-|N(\ell')|} \cdot \gamma^{N_i(\ell)-N_i(\ell')}\}.$$  

Without loss of generality, we suppose $\lambda^{N(\ell')-N(\ell)} \cdot \gamma^{N_i(\ell')-N_i(\ell)} < 1$, so the minimum in $P(\sigma, \tau)$ is this value and the minimum in $P(\tau, \sigma)$ is 1. Noting that

$$e(\sigma) - |N(\ell)| + |N(\ell')| = e(\tau) \quad \text{and} \quad a(\sigma) - |N_i(\ell)| + |N_i(\ell')| = a(\tau),$$

it follows that

$$\pi(\sigma) P(\sigma, \tau) = \frac{\lambda^e(\sigma) \cdot \gamma^a(\sigma)}{Z} \frac{1}{6n} \lambda^{|N(\ell')|-|N(\ell)|} \cdot \gamma^{N_i(\ell')-N_i(\ell)} = \frac{\lambda^e(\tau) \cdot \gamma^a(\tau)}{Z} \frac{1}{6n} = \pi(\tau) P(\tau, \sigma).$$

Now, suppose $\sigma$ and $\tau$ differ by a swap move where $\sigma$ has $P$ of color $c_i$ at location $\ell$ and $Q$ of color $c_j$ at neighboring location $\ell'$ and $\tau$ has $P$ and $Q$ in opposite positions. If $c_i = c_j$, then $\sigma = \tau$ and we are done. When $c_i \neq c_j$, then Steps 1, 2, and 8–11 of Algorithm 1 imply that, because this swap move could be initiated by $P$ or by $Q$,

$$P(\sigma, \tau) = \frac{1}{6n} \min\{1, \gamma^{N_i(\ell') \setminus \{P\}} - |N_i(\ell) + |N_i(\ell') \setminus \{Q\}| - |N_j(\ell)|\}$$

$$+ \frac{1}{6n} \min\{1, \gamma^{N_j(\ell) \setminus \{Q\}} - |N_j(\ell') + |N_j(\ell') \setminus \{P\}| - |N_i(\ell')|\}$$

$$= \frac{1}{3n} \min\{1, \gamma^{N_i(\ell') \setminus \{P\}} - |N_i(\ell) + |N_i(\ell') \setminus \{Q\}| - |N_j(\ell)|\}$$

For $\tau$, which has $P$ in position $\ell'$ and $Q$ in position $\ell$, we can similarly define

$$P(\tau, \sigma) = \frac{1}{3n} \min\{1, \gamma^{N_i(\ell) \setminus \{P\}} - |N_i(\ell') + |N_i(\ell') \setminus \{Q\}| - |N_j(\ell')|\}$$

Due to the differing positions of $P$ and $Q$ in $\sigma$ and $\tau$, the exponential expressions in each of the minimums above are inverses of each other. We note that $e(\sigma) = e(\tau)$, because the total number of edges in the particle configuration remains unchanged by a swap move. Furthermore,

$$a(\sigma) - |N_i(\ell)| - |N_j(\ell')| + |N_i(\ell') \setminus \{P\}| + |N_j(\ell) \setminus \{Q\}| = a(\tau).$$
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Without loss of generality, we assume $\gamma^{(N_i(t')(Q) - |N_i(t')| - |N_i(t)|)} < 1$. This implies the minimum in $P(\tau, \sigma)$ is 1. We conclude

$$\pi(\sigma)P(\sigma, \tau) = \frac{\lambda^{e(\sigma)}\gamma^{a(\sigma)}}{Z} \frac{1}{6n} \gamma^{(N_i(t')(Q) - |N_i(t')| - |N_i(t)|)}$$

$$= \frac{\lambda^{e(\tau)}\gamma^{a(\tau)}}{Z} \frac{1}{6n} = \pi(\tau)P(\tau, \sigma).$$

Detailed balance has been verified for all $\sigma, \tau \in \Omega$, so we conclude $\pi$ is the stationary distribution of $\mathcal{M}$.

We can rewrite this stationary distribution to be more amenable to our proofs. Let $h(\sigma)$ be the number of heterogeneous edges in configuration $\sigma$, and note $e(\sigma) = a(\sigma) + h(\sigma)$ as every edge of $\sigma$ is either homogeneous or heterogeneous.

**Lemma 3.4.** The stationary distribution of $\mathcal{M}$ is also given by $\pi(\sigma) = (\lambda \gamma)^{-p(\sigma)} \cdot \gamma^{-h(\sigma)}/Z$, where $Z = \sum_{\sigma}(\lambda \gamma)^{-p(\sigma)} \cdot \gamma^{-h(\sigma)}$.

**Proof.** We know (6) that $e(\sigma) = 3n - p(\sigma)$ and thus $a(\sigma) = e(\sigma) - h(\sigma) = 3n - p(\sigma) - h(\sigma)$. We substitute both of these in our expression for the stationary distribution found in Lemma 3.3 and simplify to obtain the claimed result:

$$\pi(\sigma) = \frac{\lambda^{e(\sigma)} \cdot \gamma^{a(\sigma)}}{\sum_{\tau \in \Omega^2} \lambda^{e(\tau)} \cdot \gamma^{a(\tau)}} = \frac{\lambda^{3n - p(\sigma)} \cdot \gamma^{3n - p(\sigma) - h(\sigma)}}{\sum_{\tau} \lambda^{3n - p(\tau)} \cdot \gamma^{3n - p(\tau) - h(\tau)}} = \frac{(\lambda \gamma)^{3n} \cdot (\lambda \gamma)^{-p(\sigma)} \cdot \gamma^{-h(\sigma)}}{(\lambda \gamma)^{3n} \sum_{\tau} (\lambda \gamma)^{-p(\tau)} \cdot \gamma^{-h(\tau)}}$$

Simplifying this expression, we reach:

$$\pi(\sigma) = (\lambda \gamma)^{-p(\sigma)} \cdot \gamma^{-h(\sigma)}/Z,$$

where $Z = \sum_{\sigma}(\lambda \gamma)^{-p(\sigma)} \cdot \gamma^{-h(\sigma)}$.

**3.3 Simulations**

We complement our rigorous analysis with simulations. We find that $\mathcal{M}$ exhibits the expected separation behavior for large $\lambda$ and $\gamma$, as well as integration behaviors for other parameter values. We simulated $\mathcal{M}$ on a system of 100 particles with two colors (50 of each). Fig. 2 shows the progression of $\mathcal{M}$ over time with biases $\lambda = 4$, $\gamma = 4$, the regime in which we expect the entire system to compress and individual color classes to segregate. Although the simulation ran for nearly seventy million iterations, much of the progress towards a compressed and segregated system is achieved in the first million iterations. Without swap moves, the time it takes to see segregation appears to be much greater though separation eventually occurs. Fig. 3 compares the resulting particle system configurations after running $\mathcal{M}$ from the same initial configuration for the same number of iterations, varying only the values of $\lambda$ and $\gamma$. We observe four distinct phases: expanded-integrated, expanded-separated, compressed-integrated, and compressed-separated (Figure 3). We will rigorously verify the compressed-separated behavior for large $\lambda$ and $\gamma$.

**4 Proof that Markov Chain $\mathcal{M}$ Achieves Separation**

We prove that, among configurations with small perimeter, those that are separated are exponentially more likely than those that are not in the stationary distribution of $\mathcal{M}$, which is also the stationary distribution of $\mathcal{A}$. We begin with preliminaries about lattice duality and lattice paths that will enable cleaner formulations of our claims.
Figure 2 A 2-color heterogeneous particle system starting in an arbitrary state after — from left to right — 0; 50,000; 1,050,000; 17,050,000; and 68,250,000 iterations of $M$ with $\lambda = 4$ and $\gamma = 4$.

| $\gamma = 0.58$ (Integration) | $\gamma = 5.20$ (Separation) |
|--------------------------------|--------------------------------|
| $\lambda = 0.58$ (Expansion)  |                                |
|                                |                                |
| $\lambda = 5.20$ (Compression) |                                |

Figure 3 A 2-color heterogeneous particle system starting in the leftmost configuration of Figure 2 after 50 million iterations of $M$ for various values of the parameters $\lambda$ and $\gamma$.

4.1 Lattice Duality and Contours

The dual to the triangular lattice $G_\Delta$, obtained by creating a new vertex in every face of $G_\Delta$ and connecting two of these vertices if their corresponding triangular faces have a common edge, is the hexagonal lattice $G_{hex}$; see Fig. 4a. Throughout, by a contour we will mean a walk in $G_{hex}$ that never visits the same vertex twice, except possibly to start and end at the same place; these are also known as self-avoiding walks or, when starting and ending at the same vertex, self-avoiding polygons. Each edge $e \in G_{hex}$ crosses a unique edge $f \in G_\Delta$, and we say an $e$ separates the two locations connected by $f$; because there is this natural bijection between edges of $G_{hex}$ and edges of $G_\Delta$ we will sometimes use them interchangeably. For a configuration $\sigma$, we say an $e \in G_{hex}$ is a boundary edge if it separates a particle of $\sigma$ from an unoccupied location, and $e$ is a heterochromatic edge if it separates two particles of different colors. A contour is a boundary contour if all of its edges are boundary edges and is a heterochromatic contour if all of its edges are heterochromatic. See Fig. 4b for an example of a configuration $\sigma$ with particles of two different colors and its boundary contour (black) and heterochromatic contours (dashed).

For a particle configuration $\sigma$ without holes, its perimeter can be completely described by taking the union of all boundary edges in $G_{hex}$, which yields a boundary contour. A result
Figure 4 (a) The duality between the triangular lattice $G_\Delta$ (gray) and the hexagonal lattice $G_{hex}$ (black). (b) A particle configuration $\sigma$ with 11 black particles and 11 white particles. The boundary contour in dual lattice $G_{hex}$ is thick and black, while all four heterochromatic contours of $\sigma$ are shown by dashed lines.

of [6] implies that if $p(\sigma) = k$, then the length of this boundary contour is $|P(\sigma)| = 2k + 6$.

For a particle configuration with no holes and perimeter determined by boundary contour $P$, we can completely describe the colors of its particles (up to swapping the colors) by giving all heterochromatic edges. Because there are only two colors, these heterochromatic edges form non-intersecting contours. Each (maximal) heterochromatic contour either starts and ends at different places on the boundary contour or is a closed loop; we call the former a crossing contour and the latter an isolated contour. The configuration in Fig. 4(b) has three crossing contours and one isolated contour.

4.2 Statement of Main Result

Let $\Omega_P \subseteq \Omega$ contain all configurations with no holes and boundary contour $P$. Let $\pi_P$ be the stationary distribution conditioned on being in $\Omega_P$. Because all configurations in $\Omega_P$ have the same perimeter, using the definition of $\pi$ given in Lemma 3.4 we see that all terms of the form $(\lambda \gamma)^{-p(\sigma)}$ cancel, yielding $\pi_P(\sigma) = \gamma^{-h(\sigma)}/Z_P$, where $Z_P = \sum_{\sigma \in \Omega_P} \gamma^{-h(\sigma)}$.

We say a configuration $\sigma$ is $\alpha$-compressed if its perimeter is at most $\alpha \cdot p_{\text{min}}$, where $p_{\text{min}}$ is the minimum possible perimeter for the particles in $\sigma$. In this section we prove that, for all $P$ that determine $\alpha$-compressed configurations, non-separated configurations have exponentially small weight according to $\pi_P$. We formally define separation in terms of the existence of clusters as follows.

Definition 4.1. For $\beta > 0$ and $\delta \in (0, 1/2)$, a configuration $\sigma \in \Omega_P$ is $(\beta, \delta)$-clustered if there is a subset $R$ of particles such that:

1. The contours of $bd_{\text{int}}(R)$ have total length at most $\beta \sqrt{n}$;
2. The density of particles of color $c_1$ in $R$ is at least $1 - \delta$; and
3. The density of particles of color $c_1$ not in $R$ is at most $\delta$.

In [6] the authors showed the length of a self-avoiding walk including all but one boundary edge of $\sigma$ in $G_{hex}$ had length $2p(\sigma) + 5$; here we consider (closed) boundary contours with all boundary edges, of total length $2p(\sigma) + 6$. 

3
Here \( \delta \) is a tolerance of having particles of the wrong color within the cluster \( R \), and \( \beta \) is a measure of how small the boundary between \( R \) and \( \overline{R} \), the particles not in \( R \), must be. We note that Condition (1) is equivalent to having at most \( \beta \sqrt{n} \) edges with one endpoint in \( R \) and one endpoint in \( \overline{R} \), which is how we previously defined clustering without referring to contours. We note that this definition is symmetric with respect to the role played by \( \delta \) and some \( \beta < \delta \) in \( R \) measure of how small the boundary between contours. We note that this definition is symmetric with respect to the role played by \( \delta \) and \( \beta \).

We let \( \Psi_{\beta,\delta} \subseteq \Omega_P \) be the configurations in \( \Omega_P \) that are \((\beta,\delta)\)-clustered for some \( \beta > 0 \) and some \( \delta < 1/2 \). We prove (Theorem 4.10) that for \( \gamma \) sufficiently large, as long as \( \mathcal{P} \) is \( \alpha \)-compressed, \( \beta > 4\alpha \), and \( \delta < 1/2 \), with all but exponentially small probability a sample drawn from \( \pi_P \) is in \( \Psi_{\beta,\delta} \):

\[
\pi_P(\Omega_P \setminus \Psi_{\beta,\delta}) \leq \xi^\sqrt{n},
\]

where \( \xi \) is a constant less than one. In the remainder of this section we prove this result.

A simple corollary states that if \( \Omega_{\alpha} \subseteq \Omega \) is the set of configurations that are \( \alpha \)-compressed, then the stationary distribution \( \pi \) restricted to \( \Omega_{\alpha} \) also exhibits this same clustering behavior with high probability; this is easy to prove by decomposing \( \Omega_{\alpha} \) into sets \( \Omega_P \) for each \( \alpha \)-compressed perimeter \( \mathcal{P} \). Thus, when the perimeter is restricted to be small, we provably see separation for large enough values of \( \gamma \). Even if we begin at a non-compressed configuration, we can accomplish separation, with high probability, in two phases. First, run \( \mathcal{M} \) (or \( \mathcal{A} \)) with \( \lambda > 2 + \sqrt{2} \) and \( \gamma = 1 \): this is exactly the compression algorithm of [6], and \( \alpha \)-compression is accomplished with high probability. Then, fix the perimeter, and run \( \mathcal{M} \) with sufficiently large \( \gamma \) to achieve separation. Of course, in distributed asynchronous systems, building a consensus about when to switch between these two stages is difficult. However, if the cue to switch phases is some global state change – for example, a threat to an ant colony is witnessed by all ants and each changes their behavior in response – such problems can be avoided.

### 4.3 Bridging Systems

The crossing contours of a configuration \( \sigma \) separate the particles into simply connected components whose boundary particles all have the same color. We say a face of a particle configuration \( \sigma \) is a maximal simply connected subset \( F \) where all particles in \( F \) next to \( bd(F) \) have the same color, which we call the color of \( F \). For any face \( F \), its maximality implies \( bd_{in}(F) \) is contained in \( \sigma \)'s heterochromatic contours and \( bd_{out} \) \( F \) is contained in \( \sigma \)'s boundary contour.

Let \( (B, I) \) be a collection of contours in \( G_{hex} \) within a face \( F \), where \( B \) contains bridge contours connecting each isolated contour in set \( I \) (a subset of the isolated contours within \( F \)) to the boundary of \( F \). For a given \( (B, I) \), we say particle \( P \) is bridged in face \( F \) if there exists a path through particles of the same color as \( P \) to \( bd(F) \) or to a bridged isolated contour in \( I \). A particle is unbridged if such a path does not exist. We say that \( (B, I) \) is a \( \delta \)-bridge system for face \( F \) if:

1. \( |B| \leq |I|(1 - \delta)/2\delta \), where \( |B| \) is the total number of edges in all the bridge contours in \( B \) and \( |I| \) is the total number of edges in all the bridged isolated contours in \( I \).
2. The number of unbridged particles in \( F \) is \( \leq \delta|F| \), where \( |F| \) is the number of particles in \( F \).

We now show how to find a \( \delta \)-bridge system for any face \( F \).

\[\text{Lemma 4.2.} \] For any face \( F \), there exists a \( \delta \)-bridge system for \( F \).

**Proof.** Fig. 5 gives one example of a face \( F \) and a \( \delta \)-bridge constructed for \( F \).
Without loss of generality, suppose \( F \) is of color \( c_1 \). If \( F \) has only one particle, then \((\emptyset, \emptyset)\) is a \( \delta \)-bridge system for \( F \). We now suppose \( F \) has more than one particle and there exists a \( \delta \)-bridge system for all regions with a smaller number of particles than \( F \). We will iteratively construct a \( \delta \)-bridge system \((B, I)\) for \( F \). To start, let \((B, I) = (\emptyset, \emptyset)\), which satisfies \(|B| \leq |I|(1 - \delta)/2\delta\). Let \( u(F) \) be the unbridged particles for \((B, I)\) in \( F \). If \(|u(F)| \leq \delta|F|\), where \( |F| \) is the number of particles in face \( F \), then \((B, I)\) is a valid \( \delta \)-bridge system for \( F \). If not, we give a procedure for adding to \((B, I)\) that reduces the number of unbridged particles in \( F \) and maintains two invariants: (1) \(|B| \leq |I|(1 - \delta)/2\delta\) and (2) for any \( I \in I \) not surrounded by another contour in \( I \), the face \( F_I \) consisting of all particles inside \( I \) contains at most \( \delta|F_I| \) unbridged particles. Both invariants are true for initial configuration \((\emptyset, \emptyset)\). Repeating this process until \( u(F) \leq \delta|F| \) gives a valid \( \delta \)-bridge for \( F \).

Suppose we are given a bridge system \((B, I)\) for \( F \) that satisfies both invariants but leaves \( u > \delta|F| \) unbridged particles. Let \( F_{\text{ext}} \) be the particles in \( F \) that are not inside any bridged isolated contours in \((B, I)\). We will consider contours \( V \) in \( G_{\text{hex}} \) that stretch vertically across \( F \), from one part of its boundary to another, consisting of alternating down-left and down-right edges. We call such contours vertical contours. We include in set \( V_F \) all (infinite) vertical contours that contain at least one edge inside \( F_{\text{ext}} \); we will only be interested in their intersection with \( F_{\text{ext}} \), which need not be contiguous. For any \( V \in V_F \), let \( V\cap F_{\text{ext}} \) be all particles in \( F_{\text{ext}} \) directly right of \( V \) and let \( V \cap u(F_{\text{ext}}) \) be the unbridged ones. Because \( u(F) > \delta|F| \), applying Invariant (2) we conclude that \( u(F_{\text{ext}}) > \delta|F_{\text{ext}}| \). It follows that there exists \( V \in V_F \) such that \(|V \cap u(F_{\text{ext}})| > \delta|V \cap F_{\text{ext}}|\).

Any \( P \in V \cap u(F_{\text{ext}}) \) must be surrounded by an unbridged isolated contour, as otherwise it would have a monochromatic path to the boundary of \( F \); if there are multiple isolated contours surrounding \( P \), one must be the outermost, encircling all the others. Enumerate all outermost isolated contours surrounding particles in \( u(F_{\text{ext}}) \cap V \) as \( I_j \) for \( j = 1, \ldots, k \). Let \( F_j \) be the face surrounded by \( I_j \), which is of color \( c_2 \). By our induction hypothesis, because \(|F| > |F_j|\) there exists a \( \delta \)-bridge system \((B_j, I_j)\) for \( F_j \). We add to bridge system \((B, I)\) for \( F \) the set of bridges \( \cup_j B_j \) and the set of bridged isolated contours \( \cup_j I_j \). Furthermore, we add to \( B \) all the segments of \( V \) that are left of bridged particles in \( V \cap F_{\text{out}} \), a set we call \( B_0 \), and we add to \( I \) all \( I_j \). Because the number of particles that are newly-bridged by this construction is at least \(|u(F) \cap V|\), we have reduced the number of unbridged particles in \( F \). It only remains to show that this new bridge system satisfies the necessary invariants.

To see that \((B, I)\) satisfies Invariant 2, note that the only new contours \( I_j \in I \) not
surrounded by other contours in $I$ are the $I_j$. All particles that were bridged in any $F_{x_j} = F_j$ are now bridged in $F$, since both the boundary of $F_j$ and the bridged contours in $I_j$ are now bridged contours in $I$. Because $(B_j, I_j)$ is a valid $\delta$-bridge system for $F_{x_j} = F_j$, $F_j$ contains at most $\delta|F_j|$ unbridged particles, as desired.

We now check that $(B, I)$ satisfies Invariant 1. Because $(B_j, I_j)$ is a $\delta$-bridge for $F_j$, $|B_j| \leq |I_j|(1-\delta)/2\delta$ for all $j$. Next, we see that $\sum_j |I_j| \geq 4\sum_j |F_{\text{ext}}\cap I_j|$, as the $I_j$ collectively contain at least two contour edges above and two contour edges below each particle in $u(F_{\text{ext}}) \cap \mathcal{V}$. Because $\mathcal{V}$ satisfies $|\mathcal{V} \cap u(F_{\text{ext}})| \geq \delta|\mathcal{V} \cap F_{\text{ext}}|$, then $\sum_j |I_j| \geq 4\delta|\mathcal{V} \cap F_{\text{ext}}|$. Bridge $B_0$ added to $B$ contains two contour edges for each bridged particle in $F_{\text{ext}} \cap \mathcal{V}$ and at most a $1-\delta$ fraction of the particles in $F_{\text{ext}} \cap \mathcal{V}$ are bridged, so $|B_0|/2 \leq (1-\delta)|\mathcal{V} \cap F_{\text{ext}}|$. Combining the previous two equations,

$$\sum_j |I_j| \geq 4\delta|\mathcal{V} \cap F_{\text{ext}}| \geq 4\delta \left( \frac{1}{2(1-\delta)} |B_0| \right) = \frac{2\delta}{1-\delta} |B_0|.$$  

We conclude that the additions $B_0$ and $B_j$ to $B$ and the additions $I_j$ and $I_j$ to $I$ satisfy

$$|B_0| + \sum_{j=1}^k |B_j| \leq \frac{1-\delta}{2\delta} \sum_{j=1}^k |I_j| + \frac{1-\delta}{2\delta} \sum_{j=1}^k |I_j| = \frac{1-\delta}{2\delta} \sum_{j=1}^k (|I_j| + |I_j|).$$

Thus Invariant 1 is satisfied.

We have added to $(B, I)$ while maintaining both invariants and reducing the number of unbridged particles in $F$. We can continue this process until there are at most $\delta|F|$ unbridged particles in $F$; then, Invariant 1 implies $(B, I)$ is a $\delta$-bridge system for $F$. \hfill $\blacktriangle$

> **Lemma 4.3.** For each $\sigma \in \Omega_P$ with $n$ particles, there exists a $\delta$-bridge system $(B, I)$ for $\sigma$, where $B$ contains bridge contours connecting each isolated contour in set $I$ (a subset of $\sigma$’s isolated contours) to $\sigma$’s boundary contour or to a crossing contour, such that:

- $|B| \leq |I|(1-\delta)/2\delta$, and
- The number of unbridged particles in $\sigma$ is at most $\delta n$.

**Proof.** The crossing contours of $\sigma$ partition $\sigma$ into faces. Construct a $\delta$-bridge system for each of these faces and take their union. \hfill $\blacktriangle$

We now connect the notions of $\delta$-bridges and configurations that are $(\beta, \delta)$-clustered.

> **Lemma 4.4.** Let $\sigma \in \Omega_P \setminus \Psi_{\beta, \delta}$ and let $(B, I)$ be the $\delta$-bridge system for $\sigma$ constructed in Lemma 4.3. Let $x$ be the total length of crossing contours in $\sigma$ and let $y$ be the total length of bridged isolated contours in $I$. Then $x + y > \beta \sqrt{n}$.

**Proof.** Let $\mathcal{F}$ be the set of outermost faces of $\sigma$, that is, those faces of $\sigma$ that contain a particle on $\sigma$’s perimeter. For each $F \in \mathcal{F}$ of color $c_1$, if $P \in F$ is surrounded by $b$ bridged isolated contours then put $P$ in set $R$ if and only if $i + b \equiv 1 \pmod{2}$. Because of how we have carefully defined $R$, inspection shows $bd_{in}(R) = x + y$. Using the properties of $\delta$-bridge system $(B, I)$, one can show the density of particles of color $c_1$ is at least $1 - \delta$ in $R$ and at most $\delta$ outside of $R$. If it were true that $x + y \leq \beta \sqrt{n}$, then $\sigma$ would be $(\beta, \delta)$-clustered, a contradiction as $\sigma \notin \Psi_{\beta, \delta}$. Thus, it must hold that $x + y > \beta \sqrt{n}$. \hfill $\blacktriangle$
4.4 Information Theoretic Argument for Separation

To show the set \( \Omega_P \setminus \Psi_{\beta,\delta} \) of configurations with boundary contour \( P \) that are not \((\beta,\delta)\)-clustered has exponentially small weight under distribution \( \pi_P \), we will define a map \( f = f_2 \circ f_1 \) from this set into \( \Omega_P \) and examine how this map changes weights of configurations. If the number of particles of one color is less than or equal to \( \delta n \), then all configurations in \( \Omega_P \) are \((\beta,\delta)\)-clustered with \( R = \emptyset \) or \( \overline{R} = \emptyset \), so we assume each color class has more than \( \delta n \) particles.

For \( \sigma \in \Omega_P \setminus \Psi_{\beta,\delta} \), let \((B,I)\) be the \( \delta \)-bridge system constructed for \( \sigma \) according to Lemma 4.3. Let \( f_1(\sigma) \) be the (unique) particle configuration that has the same boundary contour \( P \) as \( \sigma \) and particle \( P \) that has color \( c_i \) in \( \sigma \) and is surrounded by \( b \) bridged isolated contours in \( I \) given color \( c_{(i+b)(\text{mod} 2)} \) in \( f_1(\sigma) \). We let \( \text{Im}(f_1(\Omega_P \setminus \Psi_{\beta,\delta})) \) be the set of configurations that \( f_1 \) maps to.

We define \( f_2 \) with domain \( \text{Im}(f_1(\Omega_P \setminus \Psi_{\beta,\delta})) \) to complement all faces of color \( c_2 \) that touch the boundary of the configuration (i.e. that include particles adjacent to \( P \)). The next lemmas explore the composition of these maps \( f_1 \) and \( f_2 \) as applied to configurations \( \sigma \in \Omega_P \setminus \Psi_{\beta,\delta} \).

\textbf{Lemma 4.5.} For any \( \sigma \in \Omega_P \setminus \Psi_{\beta,\delta} \), \( f_2(f_1(\sigma)) \) has boundary contour \( P \), all particles adjacent to \( P \) have color \( c_1 \), and there are at most \( \delta n \) particles of color \( c_2 \).

\textbf{Proof.} The first two claims follow easily from the definitions of \( f_1 \) and \( f_2 \). To see that the last claim holds, we note that any particles of color \( c_2 \) in \( f_2(f_1(\sigma)) \) must have been unbridged by the bridge system \((B,I)\) for \( \sigma \), and there are at most \( \delta n \) such unbridged particles by the definition of a \( \delta \)-bridge system.

\textbf{Lemma 4.6.} Let \( \tau \in \text{Im}(f_2(f_1(\Omega_P \setminus \Psi_{\beta,\delta}))) \). The number of \( \sigma \in \Omega_P \setminus \Psi_{\beta,\delta} \) with crossing contours of total length \( x \) and bridged isolated contours (bridged by a bridging system \((B,I)\)) of total length \( y \) that have \( f_2(f_1(\sigma)) = \tau \) is, for \( p = (|P| - 6)/2 \) the perimeter of any configuration in \( \Omega_P \), at most \( 3p4^{(x+p)(\frac{1+\epsilon}{4})} \).

\textbf{Proof.} Any configuration \( \sigma \in \Omega_P \setminus \Psi_{\beta,\delta} \) has boundary contour \( P \) and perimeter \( p \). One can verify from first principles that \( P \) makes \( p = (|P| - 6)/2 \) left turns and \( p + 6 \) right turns when traversed clockwise. Any bridges or crossing contours that meet \( P \) do so at distinct left turns of \( P \). We can mark each left turn of \( P \) as the start of a bridge, the start of a crossing contour, or neither; the number of ways to do so is \( 3^p \).

Next, we can trace out all crossing contours of \( \sigma \), beginning at the starting points marked along \( P \). In tracing these contours, which do not intersect, at each vertex in \( G_{\text{hex}} \) we make either a left turn or a right turn. Additionally, each vertex along these contours can either be the beginning of a bridge in \( B \), branching in the opposite direction from the contour, or not. Because \( x \) is the total length of \( \sigma \)'s crossing contours, the number of valid ways to do this is at most \( 2^x \times 2^x = 4^x \).

Finally, we trace out the bridges and isolated contours of each face of \( \sigma \) in a depth-first way, beginning at the starting points marked along \( P \) and the crossing contours. Bridges as constructed in Lemma 4.3 always move in the vertical direction, so the direction of the next edge of a bridge, if it exists, is known; at each step we only need to know if the bridge continues or if a bridged isolated contour begins. When tracing out isolated contours, just like with heterochromatic crossing contours, there are four choices for the next step: the direction in which the contour continues (two choices) and whether or not a bridge branches off (two choices). Isolated contours end when they reach an already-constructed bridge, and bridges end when they reach a crossing contour, an already-constructed isolated contour,
or $P$. The number of possibilities for this depth-first traversal of the bridges and isolated contours of $P$ is at most $2^{\lceil |I| \rceil} \leq 2^{|P|} y^4$.

Altogether, any configuration $\sigma \in \Omega_P \setminus \Psi_{\beta, \delta}$ with crossing contours of total length $x$ and bridged isolated contours of total length $y$ that have $f_2(f_1(\sigma)) = \tau$ can be uniquely identified by marking $P$, tracing crossing contours, and tracing bridges and bridged isolated contours. The number of valid ways to do this is at most

$$\frac{3^4 4}{\sqrt{n} + \sqrt{2} n} \leq 3p_4(x + y + \frac{1}{4} n) \leq 3p_4(x + y)(\frac{1 + 3y}{4}) .$$

This is an upper bound on the number of preimages of $\tau$ under $f_2 \circ f_1$ with correct $x$ and $y$. ▶

Any $\tau \in \text{Im}((f_2 \circ f_1)(\Omega_P \setminus \Psi_{\beta, \delta}))$ will not be in $\Omega_P$ because it has too few particles of color $c_2$. We will define $f_3$ such that $f_3(\tau)$ is similar to $\tau$ and has the correct number of particles of each color, but we first need the following lemma.

\textbf{Lemma 4.7.} For a configuration $\tau \in \text{Im}((f_2 \circ f_1)(\Omega_P \setminus \Psi_{\beta, \delta}))$, it is possible to flip the colors of some particles such to yield a configuration with the correct number of particles of each color and at most $4\alpha \sqrt{n}$ more heterogeneous edges than $\tau$.

\textbf{Proof.} Let $p = (|P| - 6)/2$ be the perimeter of $\tau$. Because, by assumption, $P$ is the boundary contour of an $\alpha$-compressed perimeter, we know $p \leq \alpha p_{\text{min}}$. It is easy to see $p_{\text{min}} \leq 4\sqrt{n}$ □

Label the particles of $\tau$ in order from left to right and, within each column, from top to bottom. Flip the colors of particles in this order until there are the correct number of particles of each color: by Lemma 4.5, we begin with at most $b \delta n$ particles of color $c_2$; each flip changes the number of particles of color $c_2$ by one; and flipping all particles yields at least $(1 - \delta)n$ particles of color $c_2$. Because the number of particles of color $c_2$ in any $\tau \in \Omega_P$ is strictly between $\delta n$ and $(1 - \delta)n$, it is possible to achieve this number of particles of color $c_2$ at some intermediate step. Because we flip all particles in one column before flipping any particles in the next column, all heterogeneous edges introduced by this process are in two adjacent columns. If $h$ is the total height of $\tau$ - the vertical difference between its lowest and highest particles - then the number of adjacencies between particles whose color was flipped and particles whose color was not flipped is at most $2h$. This is a lower bound on the number of heterogeneous edges introduced by the flips. The height of a particle configuration is less than half its perimeter, so we conclude the number of new heterogeneous edges is at most $2h \leq p \leq \alpha p_{\text{min}} \leq 4\alpha \sqrt{n}$, as claimed. ▶

Define map $f_3$ on configurations $\tau \in \text{Im}((f_2 \circ f_1)(\Omega_P \setminus \Psi_{\beta, \delta}))$ to complement according to the process of Lemma 4.7 so that $f_3(\tau) = \nu$ at most $n + 1$.

\textbf{Lemma 4.8.} For each $\nu \in \Omega_P$, the number of $\tau \in \text{Im}((f_2 \circ f_1)(\Omega_P \setminus \Psi_{\beta, \delta}))$ such that $f_3(\tau) = \nu$ is at most $n + 1$.

\textbf{Proof.} Given $\nu \in \Omega_P$ and a number $k \in \{0, 1, \ldots, n\}$, complementing the colors of the first $k$ elements (according to the canonical ordering from Lemma 4.7) of $\nu$ yields a configuration that maps to $\nu$ under $f_3$. These $n + 1$ configurations, which may or may not be in $\text{Im}((f_2 \circ f_1)(\Omega_P \setminus \Psi_{\beta, \delta}))$, are the only ones that could map to $\nu$ under $f_3$. ▶

\footnote{One can show that asymptotically $p_{\text{min}} \sim 2\sqrt{3}\sqrt{n}$, but to avoid carrying lower order terms through all subsequent calculations we use the simpler bound of $4\sqrt{n}$.}
Let \( f = f_3 \circ f_2 \circ f_1 \) be a map from \( \Omega_P \setminus \Psi_{\beta,\delta} \) to \( \Omega_P \). For \( \sigma \in \Omega_P \setminus \Psi_{\beta,\delta} \), let \( x \) be the total length of crossing heterochromatic contours in \( \sigma \) and \( y \) be the total length of all isolated contours in \( \sigma \) that are bridged when constructing a \( \delta \)-bridge system according to the process of Lemma 4.2.

**Lemma 4.9.** For \( \sigma \in \Omega_P \setminus \Psi_{\beta,\delta} \), \( h(\sigma) - h(f(\sigma)) \geq (x + y) \left( 1 - \frac{4\alpha}{\beta} \right) \).

**Proof.** Configuration \( f_1(\sigma) \) has \( y \) fewer heterochromatic edges than \( \sigma \), and configuration \( f_2(f_1(\sigma)) \) has \( x \) fewer heterochromatic edges than \( f_1(\sigma) \). When going from \( f_2(f_1(\sigma)) \) to \( f(\sigma) = f_3(f_2(f_1(\sigma))) \), at most \( 4\alpha\sqrt{n} \) heterogeneous edges are added (Lemma 4.7). Using Lemma 4.4, we conclude that

\[
h(\sigma) - h(f(\sigma)) \geq x + y - 4\alpha\sqrt{n} \geq x + y - 4\alpha \left( \frac{x + y}{\beta} \right) \geq (x + y) \left( 1 - \frac{4\alpha}{\beta} \right).\]

\(

\)

We are now ready to prove our main result.

**Theorem 4.10.** For any \( \alpha > 1 \), \( \beta > 4\alpha \), and \( \delta < 1/2 \), there exists \( \gamma^* \) and \( n_0 \) (which depend on \( \alpha \), \( \beta \), and \( \delta \)) such that for all \( \gamma > \gamma^* \) and \( n > n_0 \), for any \( \alpha \)-compressed boundary contour \( \mathcal{P} \), the probability that a particle configuration drawn at random from \( \pi_{\mathcal{P}} \) is not \((\beta,\delta)\)-clustered is at most \( \xi^{\sqrt{n}} \) for some constant \( \xi < 1 \) (\( \xi \) depends on \( \alpha \), \( \beta \), and \( \delta \)).

**Proof.** For any \( \nu \in \Omega_P \), we count the number of configurations in \( \Omega_P \setminus \Psi_{\beta,\delta} \) such that \( f(\sigma) = \nu \). By Lemmas 4.6 and 4.8, the number of such preimages with crossing contours of total length \( x \) and bridged isolated contours of total length \( y \) is at most \( (n + 1)3^p(4^{x+y})\left( \frac{1+3\delta}{\beta} \right) \), where \( p = (|\mathcal{P}| - 6)/2 \) is the perimeter of \( \nu \) (and the perimeter of all configurations in \( \Omega_P \)). As \( p < \alpha p_{\min} < 4\alpha\sqrt{n} \), by Lemma 4.4, \( p < 4\alpha(x + y)/\beta \). We can rewrite the number of preimages in \( f^{-1}(\nu) \) with given values of \( x \) and \( y \) as

\[
(n + 1)3^p(4^{x+y})(\frac{1+3\delta}{\beta}) < (n + 1)3^{4\alpha(x+y)}(\frac{1+3\delta}{\beta}) = (n + 1)4^{(x+y)(\frac{4\alpha\log_4 3 + 1+3\delta}{\beta})}.\]

We now sum over all possible values of \( x + y \). For each possible value of \( x + y \), there are at most \( x + y + 1 \) ways in which each of \( x \) and \( y \) could have contributed to this sum. By Lemma 4.4, \( x + y > \beta\sqrt{n} \), and because the edges counted in \( x + y \) are a subset of all edges in the configuration, \( x + y < 3n \). We conclude, for \( z = x + y \),

\[
|f^{-1}(\nu)| \leq (n + 1) \sum_{z = \lceil \beta\sqrt{n} \rceil}^{3n} (z + 1)4^{(4\alpha\log_4 3 + 1+3\delta)}.
\]

Finally, we see that for any \( \nu \in \Omega_P \),

\[
\sum_{\sigma \in f^{-1}(\nu)} \frac{\pi_{\mathcal{P}}(\sigma)}{\pi(\nu)} = \sum_{\sigma \in f^{-1}(\nu)} \left( \frac{1}{\gamma} \right)^{h(\sigma) - h(f(\sigma))} \leq (n + 1) \sum_{z = \lceil \beta\sqrt{n} \rceil}^{3n} (z + 1)4^{(4\alpha\log_4 3 + 1+3\delta)} \left( \frac{1}{\gamma} \right)^{z(1 - \frac{4\alpha}{\beta})} \leq (n + 1) \sum_{z: \beta\sqrt{n} \leq z \leq 6n} (z + 1)4^{(4\alpha\log_4 3 + 1+3\delta)\gamma^{-1 - \frac{4\alpha}{\beta}}}.
\]

This is exponentially small whenever \( n \) is sufficiently large and \( 4^{\frac{4\alpha\log_4 3 + 1+3\delta}{\beta\gamma - 4\alpha}} < 1 \). This is true if \( \beta > 4\alpha, \delta < 1/2, \) and \( \gamma > \gamma^* = 4^{\frac{4\alpha\log_4 3 + 1+3\delta}{\beta\gamma - 4\alpha}} \).
We expect this lower bound $\gamma^*$ on the values of $\gamma$ needed to guarantee separation can be substantially improved with further efforts.

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