Entropy computing via integration over fractal measures

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PACS: 05.45, 02.50.Ey
MSC: 28A80, 28D20, 81Q50, 58Fxx

Dedicated to the memory of Marcin Poźniak
We discuss the properties of invariant measures corresponding to iterated function systems (IFSs) with place-dependent probabilities and compute their Rényi entropies, generalized dimensions, and multifractal spectra. It is shown that with certain dynamical systems one can associate the corresponding IFSs in such a way that their generalized entropies are equal. This provides a new method of computing entropy for some classical and quantum dynamical systems. Numerical techniques are based on integration over the fractal measures.
In order to characterize quantitatively properties of a given nonlinear system one often uses the notion of the dynamical entropy. It describes the asymptotic changes of the system entropy in time. Since analytical computing of this quantity is possible only for a limited number of simple models, it is important to develop efficient numerical techniques for this purpose. In this article we propose a method of computing the dynamical entropy by averaging the static Shannon entropy. The integration is performed over a suitably chosen measure, which in the general case displays fractal properties.

I. INTRODUCTION

Chaos in a classical dynamical system can be defined by the positiveness of the Kolmogorov-Sinai (KS) dynamical entropy [1]. This quantity, characterizing dynamical properties of a system, and defined via an asymptotic limit (time tending to infinity) is in general not easy to obtain analytically. On the other hand numerical computing of dynamical entropy from time series requires advanced techniques [2]. Also in the quantum case estimating, so called, coherent states (CS) quantum entropy [3,4] is not a simple task. In the present paper we propose a method of computing dynamical entropy of a system by finding an appropriate iterated function system with the same entropy.

An iterated function system (IFS) consists of a certain number $k$ of functions $F_i, i = 1, \ldots, k$, which act randomly with given probabilities $p_i, i = 1, \ldots, k$. An IFS may therefore be concerned as a combination of deterministic and stochastic dynamics. For sufficiently contracting functions one can prove (under some irreducibility conditions) that IFS generates a unique invariant measure (see Sect. II). Generically this measure is localized on a fractal set. As it was described, e.g., in the elegant book of Barnsley [10] IFSs may be used to produce interesting fractal images, or to encode and transmit graphics via computer. For the majority of commonly analyzed and applied IFSs the probabilities $p_i$ are constant. For example, such IFSs have been used to construct multifractal energy spectra of certain quantum systems [11] and to investigate the one-dimensional random-field Ising model [12] or second order phase transitions [13]. On the other hand, with some classical and quantum dynamical systems one can associate in a natural way IFSs with place-dependent probabilities [14–18,6,19]. In the present paper such IFSs will be called iterated function systems of the second kind, on the analogy of position-dependent gauge transformations [20].

We estimate the Kolmogorov-Sinai and Rényi dynamical entropies of certain IFSs of the second kind using various numerical methods, which can be also applied in the general case. We use similar procedures to analyze the properties of the invariant measures of these IFSs and demonstrate their multifractal character. Eventually, we show that one can calculate the entropy of certain dynamical systems constructing IFSs with the same entropy. We give several examples that illustrate this new method of computing entropy.

This paper is organized as follows. In the next section the definitions of IFSs of the first and second kind are recalled and their basic properties are considered. In Sect. III we discuss briefly several methods of analytical and numerical computing of dynamical entropy of an IFS. In Sect. IV we study generalized dimensions of measures which are invariant under the action of a one-dimensional IFS. Sect. V presents a detailed analysis of a family of IFSs of the second kind and their invariant measures. Certain integrals over these measures are calculated. Moreover we compute the Rényi entropies, generalized dimensions and multifractal spectra of these measures. In the subsequent section we investigate the connection between one-dimensional dynamical systems and IFSs of the second kind. In particular, IFSs associated to asymmetric tent map, logistic map, and "hut map" are analyzed and their entropies are calculated. We also show how one can apply this method to compute CS-quantum entropy. Concluding remarks are contained in the last section.

In this paper we present only the results and numerical calculations. For the proofs we refer the reader to a forthcoming publication.

II. ITERATED FUNCTION SYSTEMS AND THEIR INVARIANT MEASURES

An iterated function system (IFS) is specified by $k$ functions transforming a metric space into itself and $k$ place-dependent probabilities which characterize the likelihood of choosing a particular map at each step of the evolution of the system. Under certain contractivity and irreducibility conditions one can prove the existence of a unique attractive invariant measure for an IFS, as well as ergodic and central limit theorems. Miscellaneous results of this type have been established since late thirties (some of them have been proved independently by several authors) - see for instance [21–23,14,15,24–33] and references therein.

In the present paper we study IFSs $\mathcal{F} = \{F_i, p_i : i = 1, \ldots, k\}$ that fulfil the following (rather strong) assumptions which guarantee veracity of the above mentioned theorems:

General Assumption:
(1) \( X \) is a compact metric space;

(2) \( F_i : X \to X, \ i = 1, \ldots, k \) are Lipschitz functions with the Lipschitz constants \( L_i < 1 \);

(3) \( p_i : X \to [0, 1], \ i = 1, \ldots, k \) are Hölder continuous functions fulfilling
\[ \sum_{i=1}^{k} p_i(x) = 1 \text{ for each } x \in X; \]

(4) \( p_i(x) > 0 \) for every \( x \in X \) and \( i = 1, \ldots, k \).

Such IFSs are often called hyperbolic. Unless otherwise stated we assume that all IFSs under consideration are hyperbolic.

Let us recall briefly several basic facts on IFSs.

The IFS \( \mathcal{F} = \{ F_i, p_i : i = 1, \ldots, k \} \) generates the following Markov operator \( V \) acting on \( M(X) \) (the space of all probability measures on \( X \)):

\[ (V\nu)(B) = \sum_{i=1}^{k} \int_{F_i^{-1}(B)} p_i(\lambda) d\nu(\lambda), \]  

(2.1)

where \( \nu \in M(X) \) and \( B \) is a measurable subset of \( X \). This operator describes the evolution of probability measures under the action of \( \mathcal{F} \). The related Markov process can be defined in the following way. As a probability space we take the code space \( \Omega = \{1, \ldots, k\}^\mathbb{N} \) and we put \( P_x \) for the probability measure on \( \Omega \) given by

\[ P_x(i_1, \ldots, i_n) := P_x(\{\omega \in \Omega : \omega(j) = i_j, j = 1, \ldots, n\}) := p_{i_1}(x)p_{i_2}(F_i(x)) \cdots p_{i_n}(F_{i_{n-1}}(F_{i_{n-2}}(\ldots (F_{i_1}(x))))) \]  

(2.2)

where \( x \in X, i_1 = 1, \ldots, k, j = 1, \ldots, n; n \in \mathbb{N} \). Then the formulae

\[ Z^0_n(\omega) = F_{\omega(n)}(F_{\omega(n-1)}(\ldots (F_{\omega(1)}(x)))) \]  

(2.3)

for \( x \in X, \omega \in \Omega, n \in \mathbb{N} \) define the requested Markov stochastic process on \( (\Omega, \{P_x\}_{x \in X}) \).

One can show that for an IFS which fulfills our assumption there exists a unique invariant probability measure \( \mu \) satisfying the equation \( V\mu = \mu \) (the proof of this claim can be found in [14]). This measure is attractive, i.e., \( V^n\nu \) converges weakly to \( \mu \) for every \( \nu \in M(X) \) if \( n \to \infty \) or, in other words, \( \int_X u dV^n\nu \) tends to \( \int_X u d\mu \) for every continuous \( u : X \to \mathbb{R} \). Thus, in order to obtain the exact value of \( \int_X u d\mu \), it is sufficient to find the limit of the sequence \( \int_X u dV^n\nu \) for an arbitrary initial measure \( \nu \). For instance, taking \( \nu \) equal to a Dirac delta measure \( \delta_x \) for some \( x \in X \) we obtain the integral of \( u \) over the invariant measure \( \mu \) as the limit of the sequence

\[ U_n := \sum_{i_1, \ldots, i_n=1}^{k} P_x(i_1, \ldots, i_n)u(x_{i_1, \ldots, i_n}), \]  

(2.4)

where \( x_{i_1, \ldots, i_n} := F_{i_n}(F_{i_{n-1}}(\ldots (F_{i_1}(x)))) \). After Barnsley [10] (see also [33]) we call this method of computing integrals over the invariant measure deterministic algorithm. To find the integral numerically we can also employ the ergodic theorem for IFSs [21,22,15,26,31,32]. Any initial point \( x \in X \) iterated by the IFS generates a random sequence \( (z_0 = x, z_1, \ldots, z_n, \ldots) \), where \( z_i := Z^x_i(\omega) \). Then

\[ \int_X u(x) d\mu(x) = \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} u(z_i) \]  

(2.5)

with probability one, i.e., except of a set of measure \( P_x \) zero. Moreover, if \( u \) fulfils the Lipschitz condition we can evaluate the rate of convergence in the ergodic theorem utilizing the central limit theorem for IFS [15,30]. This leads to a probabilistic (Monte-Carlo) numerical method of computing integrals over the invariant measure which was called random iterated algorithm by Barnsley [10]. We have successfully applied both techniques to compute numerically various integrals, including those necessary to estimate the dynamical entropy of IFS (see Sect. III).

Finally, let us look at the evolution of densities under the action of IFS. If \( m \) is a finite measure on \( \Omega \) and \( F_i \ (i = 1, \ldots, k) \) are nonsingular (i.e., \( m(A) = 0 \) implies \( m(F_i^{-1}(A)) = 0 \) for each measurable \( A \subseteq X \)), then the IFS \( \mathcal{F} \) generates a Markov operator on the space of densities (with respect to \( m \)) on \( X \), also called the Frobenius-Perron operator [13]. It is so if, for instance, \( X \) is an interval in \( \mathbb{R} \) and \( \{ F_i : i = 1, \ldots, k \} \) are diffeomorphisms. It follows from (2.1) that the Frobenius-Perron operator \( M \) associated with \( \mathcal{F} \) is given in this case by the formula

\[ M[\gamma](x) = \sum_{i} p_i(F_i^{-1}(x)) \gamma(F_i^{-1}(x)) \left| \frac{dF_i^{-1}(x)}{dx} \right|, \]  

(2.6)
where the sum goes over all \(1 \leq i \leq k\) such that \(x \in F_i(X)\), for \(\gamma\) a density and \(x \in X\).

If probabilities \(p_i\) are constant then we will say that an IFS is of the first kind. IFSs with place-dependent probabilities will be called IFSs of the second kind (they also appear in the literature under the name of learning systems).

### III. Entropy of IFS

Let \(\mu\) be the attractive invariant measure for the IFS \(\mathcal{F} = \{F_i, p_i : i = 1, \ldots, k\}\). We define the probability measure \(P_\mu\) on the code space \(\Omega = \{1, \ldots, k\}^\mathbb{N}\) by

\[
P_\mu(i_1, \ldots, i_n) := P_\mu(\{\omega \in \Omega : \omega(j) = i_j, j = 1, \ldots, n\}) := \int_X P_x(i_1, \ldots, i_n) d\mu(x)
\]

(3.1)

for \(i = 1, \ldots, k, j = 1, \ldots, n; n \in \mathbb{N}\).

It is easy to show that this measure is invariant with respect to the shift on \(\Omega\).

Now we can define the partial entropies as

\[
H(n) := - \sum_{i_1, \ldots, i_n = 1}^k P_\mu(i_1, \ldots, i_n) \ln P_\mu(i_1, \ldots, i_n).
\]

(3.2)

and the relative entropies by

\[
G(1) := H(1); \quad G(n) = H(n) - H(n - 1), \quad \text{for } n > 1.
\]

(3.3)

The dynamical entropy of Kolmogorov and Sinai can be extracted from both sequences, i.e., \(K_1 = \lim_{n \to \infty} G(n) = \lim_{n \to \infty} H(n)/n\). The usage of relative entropies is often advantageous, since the convergence of \(H(n)/n\) is slow (usually as \(1/n\)), while in many cases the sequence \(G(n)\) converges to the KS-entropy exponentially fast. Note that the entropy of a stochastic system like an IFS can be defined in several different ways. Here we are interested in the dynamics induced by an IFS in the \(k\)-symbols code space, which leads to the entropy finite and bounded by \(\ln k\).

The concept of dynamical KS-entropy is based on the notion of the Boltzmann-Shannon entropy function which can be multifariously generalized. In this paper we discuss the two versions of Rényi entropy defined in \([3]\) for any real \(q \neq 1\). The first one, often used in the literature, corresponds to the limit of partial entropies:

\[
\hat{K}_q := \lim_{n \to \infty} \frac{1}{n} \frac{1}{1 - q} \ln \left[ \sum_{i_1, \ldots, i_n = 1}^k [P_\mu(i_1, \ldots, i_n)]^q \right].
\]

(3.4)

The other one, based on the notion of Rényi conditional entropy \([39]\), is defined via relative entropies:

\[
K_q := \lim_{n \to \infty} \frac{1}{1 - q} \ln \left[ \sum_{i_1, \ldots, i_n = 1}^k P_\mu(i_1, \ldots, i_n) \left( \frac{P_\mu(i_1, \ldots, i_n)}{P_\mu(i_1, \ldots, i_{n-1})} \right)^{q-1} \right].
\]

(3.5)

For \(q = 0\) both quantities are equal to the topological entropy \(K_0 = \ln k\) and the KS-entropy is obtained for both quantities in the limit \(q \to 1\). On the other hand, in general, the two versions of Rényi entropies are different (see Sects. III and V). The computation of entropy \(K_q\) is more straightforward than \(\hat{K}_q\) and an analytical treatment is possible in some cases, on the other hand, its relation to the thermodynamical formalism seems to be less clear.

Both definitions give us some form of the Rényi dynamical entropy for the dynamics generated by an IFS in the \(k\)-symbols code space and for the specific partition of this space into \(k\) rectangles labeled by the first symbol. Note, however, that if one defines the (partition independent) Rényi dynamical entropy taking simply the supremum over all finite partitions (as for the KS-entropy), this leads to trivial dependence: \(K_q = \infty, q < 1; \quad K_q = K_1, q \geq 1\). Consequently, from now on, we shall discuss only the Rényi entropy for the above mentioned \(k\)-elements partition.

For an IFS of the first kind (with constant probabilities \(p_i\)) both Rényi entropies are equal and can be written down explicitly \([1], [2]\):

\[
K_q = \hat{K}_q = \frac{1}{1 - q} \ln(p_1^q + p_2^q + \cdots + p_k^q).
\]

(3.6)
for \( q \neq 1 \). The KS-entropy is obtained by calculating the limit \( \lim_{q \to 1} K_q \), which gives \( K_1 = \tilde{K}_1 = -\sum_{i=1}^{k} p_i \ln p_i \).

Observe that for an IFS of the first kind the value of the entropy does not depend on the character of functions \( F_i \).

For an IFS of the second kind one cannot directly apply formula (3.6), since the probabilities are place-dependent. A natural generalization for this case is possible \([43]\), viz., one has to average the Rényi entropy performing an integral over the invariant measure \( \mu \)

\[
K_q = \frac{1}{1-q} \ln \int_X \sum_{i=1}^{k} (p_i(x))^q d\mu(x). \tag{3.7}
\]

In the limit \( q \to 1 \), corresponding to KS-entropy, this formula gives

\[
K_1 = -\int_X \sum_{i=1}^{k} p_i(x) \ln[p_i(x)] d\mu(x). \tag{3.8}
\]

Moreover, one can show that the relative entropies converge to the limiting value \( K_q \) exponentially \([43]\). Now, to compute the entropy, it suffices to apply one of the two methods of calculating the integral over the invariant measure of an IFS presented in Sect. II.

To estimate entropy \( \tilde{K}_q \) one may consider the modified IFS \( \mathcal{F}_q = \{F_i, \tilde{p}_i(q) : i = 1, \ldots, k\} \) with the probabilities \( \tilde{p}_i(q) \) proportional to \( p_i^q \), that is, given by the formula

\[
\tilde{p}_i(q)(x) = (p_i(x))^q / \sum_{j=1}^{k} (p_j(x))^q. \tag{3.9}
\]

for \( x \in X, i = 1, \ldots, k \), and \( q \in \mathbb{R} \).

Note that a similar method was used for one-dimensional IFSs of the first kind in \([12]\). It is easy to prove that the IFS \( \mathcal{F}_q \) satisfies our general assumption, and hence, a unique invariant probability measure \( \mu^q \) exists. Then one can derive \([43]\) the following inequality

\[
(1-q) \tilde{K}_q \geq \int_X \ln \sum_{i=1}^{k} (p_i(x))^q d\mu^q(x), \tag{3.10}
\]

which provides a lower bound for entropy \( \tilde{K}_q \) for \( q < 1 \), and an upper bound for \( q > 1 \). In examples we analyze (see Sects. V.C and VI.B) this bound is actually very close to the exact value of the entropy \( \tilde{K}_q \) calculated numerically. Furthermore, the integral on the right-hand side of (3.10) can be relatively easily computed (see Sect. II), whereas the convergence in (3.4) seems to be rather slow, namely as \( n \to -1 \).

IV. DIMENSIONS OF INVARINT MEASURE FOR IFS

In this section we assume that a one-dimensional \((X \subset \mathbb{R})\) IFS \( \mathcal{F} = \{F_i, p_i : i = 1, \ldots, k\} \) is given, where \( F_i \) are diffeomorphisms fulfilling the general assumption from Sect. II and the following separation condition: \( \text{int} F_i(X) \cap \text{int} F_j(X) = \emptyset \) for \( i \neq j, i, j = 1, \ldots, k \), where \( \text{int} F_i(X) \) denotes the interior of the set \( F_i(X) \). Our aim is to calculate the generalized dimensions \( D_q \) of the invariant measure for \( \mathcal{F} \). These quantities were introduced and analyzed by Grassberger, Hentschel, and Procaccia \([44,45]\) (for more information see \([46–48]\)), and \( D_0 \) is just the Hausdorff dimension of the invariant measure. The correlation dimension \( D_2 \) for certain IFSs has been recently studied by Chin, Hunt and Yorke \([43]\).

Let us consider the following pressure function

\[
P(q, \tau)(x) =: \limsup_{n \to \infty} \frac{1}{n} \ln \left[ \sum_{i_1, \ldots, i_n=1}^{k} P_x(i_1, \ldots, i_n)^q \left| (F_{i_n} \circ \ldots \circ F_{i_1})'(x) \right|^{-\tau} \right]. \tag{4.1}
\]

for \( q \geq 0 \) and \( \tau \in \mathbb{R} \).

In the sequel we assume that the limit in (4.1) does not depend on \( x \) and the generalized dimensions \( D_q \) are given by the formula

\[
6
\[ D_q = \frac{\tau(q)}{q - 1}, \]  

where \( \tau(q) = \tau \) is the only solution of the equation \( P(q, \tau) = 0 \). For the IFS of the first kind this assumption was heuristically verified by Halsey et al. [3] (see also [4]). Moreover Bohr and Rand [5, 6] showed that it holds for the IFS generated by expanding maps on the interval ("cookie-cutters").

To estimate the generalized dimension \( D_q \) we can use the technique already introduced in the preceding section. Namely, we consider the modified IFS \( F_{q, \tau} = \{ F_i, \tilde{p}_i(q, \tau) : i = 1, \ldots, k \} \) with the probabilities \( \tilde{p}_i(q, \tau) \) given by

\[ \tilde{p}_i(q, \tau)(x) = (p_i(x))^q |F_i'(x)|^{-\tau} / \sum_{j=1}^{k} (p_j(x))^q |F_j'(x)|^{-\tau}. \]

for \( x \in X, i = 1, \ldots, k, q > 0, \) and \( \tau \in \mathbb{R} \).

Again it is easy to prove that the IFS \( F_{q, \tau} \) fulfils our general assumption, and hence admits a unique invariant probability measure \( \mu^{q, \tau} \). Then one can show [4] that the following inequality holds

\[ (q - 1)D_q \leq (q - 1)\overline{D}_q, \]

where \( (q - 1)\overline{D}_q = \tau \) is the solution of the equation

\[ \int_X \ln \sum_{i=1}^{k} (p_i(x))^q |F_i'(x)|^{-\tau} d\mu^{q, \tau}(x) = 0, \]

for \( q > 0 \).

This provides a lower bound for the generalized dimension \( D_q \) for \( q < 1 \), and an upper bound for \( q > 1 \). In all the cases we study in Sects. V.C and VI.B this bound (which can be relatively easily computed) is actually very close to the value of the dimension \( D_q \) calculated numerically. In order to calculate the generalized dimensions \( D_q \) we use the "box-counting" algorithm, which in this case yields better results than the algorithm of Grassberger and Procaccia [14, 15, 21] applied to the time series extracted from the IFS. Note that if \( |F_i'(x)| \equiv L > 0 \) for all \( i = 1, \ldots, k \), then the generalized entropies and dimensions are related by a simple formula \( K_q = -D_q \ln L \) (a relation between both quantities for IFSs of the first kind was examined in [14]).

Scaling properties of the invariant measure could be described with the aid of its multifractal spectrum \( f(\alpha) = \inf_q \{ \alpha q + (1 - q)D_q \} \) (for more information on multifractal spectrum see [3, 18, 17, 13]).

V. MULTIFRACTALS GENERATED BY IFS OF THE SECOND KIND

A. Cantor measures

Let us consider a family of IFSs \( \{ X = [0, 1], k = 2; F_1(x) = x/3, F_2(x) = (x + 2)/3; p_1(x) = (1 - a) + (2a - 1)x, p_2(x) = a + (1 - 2a)x \} \) for \( x \in X \), where \( a \in [0, 1] \). It is easy to see that these IFSs fulfil our general assumption for \( a \in (0, 1) \), which guarantees the existence of a unique invariant measures \( \mu_a \) and veracity of the other results mentioned in Sects. II, III, and IV. For \( a = 1 \) one can prove the existence of a unique attractive invariant measure as well as the ergodic theorem (but not the central limit theorem) using more refined results which may be found in [15, 24]. On the other hand, for \( a = 0 \), the IFS attracts every measure into a linear combination of two Dirac deltas localized at points 0 and 1. Hence there exists a whole family of invariant measures \( \{ r\delta_0 + (1 - r)\delta_1 : r \in [0, 1] \} \) in this case.

An IFS of the first kind is obtained for \( a = 1/2 \), since the probabilities \( p_1(x) = p_2(x) = 1/2 \) do not depend on \( x \). The invariant measure \( \mu_{1/2} \) is spread uniformly over the Cantor set. The generalized fractal dimension is constant \( D_q = D_0 = \ln 2 / \ln 3 \), which implies a singular multifractal spectrum concentrated at \( \alpha_1 = \ln 2 / \ln 3 \) with \( f(\alpha_1) = \alpha_1 \). The generalized Rényi entropy for this IFS can be directly obtained from \( \{ \tau, \alpha \} \). It gives \( K_q = \ln 2 \) for all \( q \in \mathbb{R} \). The Cantor measure \( \mu_{1/2} \) can thus be called both uniform (constant generalized dimension) and balanced (constant Rényi entropy) [4].

In the case \( a = 1 \) the probabilities are place-dependent \( (p_1(x) = x; p_2(x) = 1 - x) \) and define an IFS of the second kind. In order to understand the nature of the measure \( \mu_1 \), let us consider the iterations \( \gamma_n = M_1(\gamma_{n-1}) \) of the initially uniform density \( \gamma_0 \) with respect to the Frobenius-Perron operator \( M_1 \) given by [24].
We simplify the notation by introducing the ‘box’ functions \( x \rightarrow \Theta_{x,b}(x) := \Theta(x-a)\Theta(b-x) \), with the Heaviside function \( \Theta \) given by \( \Theta(y) = 0 \), for \( y < 0 \), and \( \Theta(y) = 1 \), for \( y \geq 0 \). The uniform density in \( X \) can thus be written as \( \gamma_0 = \Theta_{0,1} \).

Formula (2.6) allows us to obtain for instance the first two iterations of \( \gamma_0 \)

\[
\gamma_1(x) = 9x\Theta_{x,1}(x) + 9(1-x)\Theta_{2-x,1}(x),
\]

and

\[
\gamma_2(x) = 3^2x^2\Theta_{x,1}(x) + 3^2(x-3x^2)\Theta_{2-x,1}(x) + 3^4(3x-2)(1-x)\Theta_{0,2-x,1}(x) + 3^5(1-x)^2\Theta_{1-x,1}^2(x)
\]

for \( x \in [0,1] \).

Similarities and differences between densities approximating the standard Cantor measure \( \mu_{1/2} \) and the measure \( \mu_1 \) are displayed in Fig. 1. Due to constant probabilities, in the first case the measure \( \mu_{1/2} \) covers uniformly the Cantor set (Fig. 1a, 1b and 1c). On the other hand, for the IFS of the second kind, the place-dependent probabilities induce a highly non-uniform distribution of the measure (Fig. 1d, 1e and 1f). For example, in each connected component of the support of the density \( \gamma_n \) it can be expressed as a polynomial in \( x \) of \( n \)-th degree. Note that, if \( \gamma_n \) achieves its maximum at \( x_n \), then \( \gamma_{n+1}(x_n) = 0 \). One may expect, therefore, that the measure \( \mu_1 \) is multifractal.

### B. Integration over fractal measures

Let us now calculate the integrals of certain functions \( u \) over the invariant measures \( \mu_{1/2} \) and \( \mu_1 \). Let us find, for example, the mean (\( u_A(x) = x \)) and the mean square (\( u_B(x) = x^2 \)) for these measures. Iterating the uniform density \( \gamma_0 \) by the Frobenius-Perron operator \( M_{1/2} \) we get the sequences of integrals \( U_{A_n}^{(1/2)} = \int_X u_A(x)\gamma_n(x)dx = 1/2 \) (independently of \( n \)) and \( U_{B_n}^{(1/2)} = \int_X u_B(x)\gamma_n(x)dx = 3(1-3^{-2n-2})/8 \). Consequently, two integrals in question read \( \int_X x\mu_{1/2}(x) = 1/2 \) and \( \int_X x^2d\mu_{1/2}(x) = 3/8 \). Computing integrals over the measure \( \mu_1 \) it is advantageous to start with an initially singular measure. To demonstrate the convergence rate explicitly we take a one parameter family of measures consisting of a combination of two delta peaks localized in both ends of the unit interval: \( \kappa_r = r\delta_0 + (1-r)\delta_1 \), where \( r \in [0,1] \). Iterating this measure with respect to the Markov operator \( V_1 \) given by (2.1) we compute the \( r \)-dependent integrals of both functions \( U_{A_n}^{(1)} = \frac{1}{2}[1+(-\frac{1}{3})^n]-r(-\frac{1}{3})^n \rightarrow \frac{1}{2} \) and \( U_{B_n}^{(1)} = \frac{1}{2}[1+2(-\frac{1}{3})^n]-r(-\frac{1}{3})^n \rightarrow \frac{1}{2} \). Both sequences tend to their limits independently of the parameter \( r \), which contributions into the integral decay with \( n \) as \( 3^{-n} \). Since both measures \( \mu_{1/2} \) and \( \mu_1 \) are symmetric with respect to \( x = 1/2 \), the first moments (\( \mu_1 \)) are equal, however, already the second moments reveal the difference.

In a similar way an integral of a function over the Cantor set may be expressed as a limit of the sum of \( 2^n \) terms (multiplied by the appropriate weights), which probe the function on the ends of the intervals composing the Cantor set. In some cases this result can be put into a form of an infinite product. For example the characteristic function of the uniform Cantor measure \( \mu_{1/2} \) is given by (3.3) (see also (3.5))

\[
\int_0^1 e^{itx}d\mu_{1/2}(x) = e^{it/2}\prod_{n=1}^{\infty} \cos\left(\frac{t}{3^n}\right).
\]

Due to fast convergence this form is particularly useful for numerical evaluation. In general, computing the integrals over multifractal measures generated by IFSs of the second kind one has to rely on numerical methods described in Sect. II. For IFSs with small number of functions the deterministic algorithm based on (2.4) provides more precise results than the random iterated algorithm (2.5). The latter seems to be more efficient for IFSs consisting of many functions.

### C. Computing of entropies and dimensions via integration over the fractal measures

The entropy of the IFSs can be expressed as an integral of the Rényi (or Boltzmann-Shannon) entropy function over the invariant measure \( \mu_q \) (see (3.7) and (3.8)) for \( K_q \), or estimated by the respective integral over the measure \( \mu^q \) (see (3.10)) for \( \tilde{K}_q \). Note that, comparing the latter case with the former, the natural logarithm interchanges with the integral over \( X \).

Numerically computed Rényi entropies \( K_q \) and \( \tilde{K}_q \) of the measure \( \mu_1 \) are displayed on Fig. 2a (however, formula (3.10) is valid only for \( q > 0 \) in this case). As expected, both entropies depend substantially on the Rényi parameter \( q \).
which means that the invariant measure \( \mu_1 \) is not balanced. Note that the inflection point of the curve \( K_q \) is situated not at \( q = 0 \) but at some negative \( q_c \). Making use of the integrals \( U_{A_n}(1) \) and \( U_{B_n}(1) \) we obtain analytical results \( K_2 = (\ln 3)/2 \) and \( K_3 = (\ln 2)/2 \) directly from \( (3.4) \). Rényi entropies allow one to compute the scaling spectra via the Legendre transform: \( g(\alpha) = \ln \inf q(\alpha q + (1 - q)K_q) \) and \( \bar{g}(\alpha) = \inf q(\alpha q + (1 - q)K_q) \) (see Fig. 2b). The common maximum of the scaling spectra gives the topological entropy \( K_0 = \ln 2 \). Observe that the spectrum \( g(\alpha) \) acquires also negative values. This does not contradict the interpretation of the scaling spectrum given by Bohr and Rand [51], which is applicable for \( \bar{g}(\alpha) \).

Let us recall that the Hausdorff dimension \( D_0 \) of \( \mu_1 \) is the same as for the standard Cantor measure \( \mu_{1/2} \) (or any other invariant measure \( \mu_a \) for \( a > 0 \)) and equals \( D_0 = \ln 2/\ln 3 \approx 0.631 \). The generalized dimensions are given by \( D_q = K_q/\ln 3 \) and can be fairly approximated by \( D_q \) (see Sect. IV). In Fig. 2c we compare these quantities with those obtained by the ”box-counting” algorithm and observe that the difference is very small. As expected, the generalized dimension decreases with the Rényi parameter \( q \) (for example \( D_1 \approx 0.47 \) and \( D_2 \approx 0.41 \)), which confirms the multifractal property of the measure \( \mu_1 \) (see also Fig. 2d). For this measure we observed that the numerical algorithm, providing reliable results for \( q \geq 1 \), definitely ceases to work for negative \( q \).

VI. DYNAMICAL SYSTEM AND IFS

Let us consider a dynamical system (quantum or classical) endowed with an invariant measure and a partition of the phase space. We shall look for an IFS with the entropy equal to the entropy of the dynamical system with respect to the given partition. This IFS represents, in a sense, the backward evolution of the system [43]. Having such an IFS we could apply formulae for the entropy of IFS given in Sect. III and so we would obtain a new method of computing the dynamical entropy of the system. We illustrate this procedure on two examples: the Rényi entropy of certain 1D dynamical systems and the coherent states (CS) entropy of certain quantum systems.

A. One-dimensional dynamical systems

Let us consider a piecewise continuously differentiable map \( f : [0, 1] \to [0, 1] \). We assume that there exist subintervals \( A_i \) (\( i = 1, \ldots, k \)) such that \([0, 1] = \bigcup_{i=1}^k A_i\), \( f(A_i) = [0, 1] \), and \( |f'| > 0 \) in the interior of \( A_i \), for each \( i \). Let us suppose that \( f \) admits an absolutely continuous invariant measure \( \mu \) and let us denote its density by \( \rho \). The partition \( \{A_i\}_{i=1}^k \) is generating in this case, i.e., the dynamical entropy with respect to this partition is equal to the dynamical entropy of the system. With the map \( f \) we can associate an IFS \( \mathcal{F} = \{F_i, p_i : i = 1, \ldots, k\} \) given by

\[
F_i(x) = f|_{A_i}^{-1}(x)
\]

and

\[
p_i(x) = \frac{\rho(F_i(x))}{\rho(x)} |F_i'(x)|
\]

for \( x \in [0, 1] \) and \( i = 1, \ldots, k \) (this is a particular case of the general construction from [17, 18]). Note that the functions \( (F_i)_{i=1}^k \) are just continuous branches of the inverse of \( f \) (see Fig. 3).

It is well known that the measure \( \mu \) is also invariant for the IFS \( \mathcal{F} \) \( [3], [13], [18] \). Clearly, the generalized entropies (given by \( (3.4) \) and \( (3.5) \)) are in both cases equal, as the probabilities \( \nu_i(t_1, \ldots, t_n) \) are the same. In general, the most difficult stage in this construction is to show that the IFS \( \mathcal{F} \) satisfies the assumptions which guarantee the truthfulness of formulae \( (3.3) \) and \( (3.8) \).

In the present paper we analyze three examples: asymmetric ”tent” map, ”igloo” map (better known as the logistic map) and ”hut” map given by:

a) Tent map: \( f(y) = y/r \) for \( 0 \leq y < r \) and \( f(y) = (1 - y)/(1 - r) \) for \( r \leq y \leq 1 \) (where \( r \in (0, 1) \) is a parameter) with the constant invariant density \( \rho = 1 \). Then \( F_1(x) = rx \), \( F_2(x) = (r - 1)x + 1 \), \( p_1(x) = r \), and \( p_2(x) = 1 - r \) for \( x \in [0, 1] \) (Fig. 3a,b);

b) Igloo map: \( f(y) = 4y(1 - y) \) for \( y \in [0, 1] \). In this case the invariant density has the form \( \rho(y) = 1/(\pi \sqrt{y(1 - y)}) \) for \( y \in [0, 1] \). Then \( F_1(x) = (1 - \sqrt{1 - x})/2 \), \( F_2(x) = (1 + \sqrt{1 - x})/2 \), and \( p_1(x) = p_2(x) = 1/2 \) for \( x \in [0, 1] \) (Fig. 3c,d);

c) Hut map: \( f(y) = (-1 + \sqrt{9 - 16|y - 1/2|})/2 \) for \( y \in [0, 1] \). The invariant density is given by \( \rho(y) = y + 1/2 \) for \( y \in [0, 1] \). Then \( F_1(x) = (x^2 + x)/4 \), \( F_2(x) = 1 - ((x^2 + x)/4) \), \( p_1(x) = (x^2 + x + 2)/8 \), and \( p_2(x) = (6 - x^2 - x)/8 \) for \( x \in [0, 1] \) (Fig 3e,f).
It is easy to show that all the required assumptions are satisfied here and we can use formulae (3.7) and (3.8) to calculate the entropy:

a) **Tent map**: $K_q = \tilde{K}_q = (\ln(r^q + (1-r)^q))/1 - q$ for $q \neq 1$, and $K_1 = -(r \ln r + (1-r) \ln(1-r))$;

b) **Igloo map**: $K_q = \tilde{K}_q = \ln 2$ for $q \in \mathbb{R}$;

c) **Hut map**: $K_q = \left(\ln(4 - q^{2q+1}/q^{q+1})\right)/(1-q)$ for $q \neq 1$ and $K_1 = 1/2 + 2 \ln 2 - (9/8) \ln 3$.

Note that, for the hut map, one can hardly obtain such an analytical formula for the alternative version of Rényi entropy $\tilde{K}_q$.

Clearly, $K_0 = \ln 2$ for each of the three maps. In the cases a) and c) $D_q = 1$ for each $q$. The dependence of $D_q$ on $q$ in the case b) is presented, e.g., in [2].

A similar technique can be applied to other classes of 1D maps like, for example, ”cookie-cutters” introduced by Bohr and Rand in [51,52]. Let us consider, e.g., a repeller given on the unit interval by $f(x) = 3y$ for $y \in [0,2/3]$ and $f(y) = 3y - 2$ for $y \in [2/3,1]$, for which typical (with respect to the Lebesgue measure) trajectories eventually leave the interval with probability one. Then the measure “uniformly” localized on the Cantor set is the invariant measures for this system. The corresponding IFS given by $\{F_1(x) = x/3, F_2(x) = (x + 2)/3\}$ for $x \in [0,1]$ with constant probabilities $p_1 = p_2 \equiv 1/2$ is just the IFS we discussed in Sect. V.

**B. Quantum systems**

In papers [3,4] we introduced the notion of coherent states (CS) quantum entropy. This quantity may be used to characterize chaos in quantum dynamical systems. Out of entire spectrum of Rényi-like quantum entropies $K_q$, a special meaning may be attached to $K_1$. Namely, the CS–entropy $K_1$ corresponds to the classical KS-entropy. The average of $K_1$ over the set of all structureless quantum systems, represented by unitary matrices distributed uniformly with respect to the Haar measure on $U(N)$, diverges with the matrix size $N$ as $\ln(N)$ [7]. This result provides an argument in favor of the ubiquity of chaos in classical mechanics (which corresponds to the limit $N \to \infty$).

The method of computing the CS–entropy based on the notion of IFS was proposed in [6,19] (but see also [56]). Again we have shown that one can associate with a quantum system and a partition of the phase space an IFS with the same entropy.

Here we present an exemplary IFS obtained for the family of spin coherent states, the identity evolution operator, the quantum number $j = 1/2$, and the partition of the phase (which is the two-dimensional sphere in this case) into two hemispheres (see [3] for details). For this IFS we have: $X = [1,1], F_1(x) = (-3 + 2x)/(6 - 3x), F_2(x) = (3 + 2x)/(6 + 3x), p_1(x) = 1/2 - x/4, p_2(x) = 1/2 + x/4$ for $x \in X$. Large contraction coefficient characteristic for this IFS ensures fast convergence of integrals performed over the measures approximating corresponding invariant measure. It enables us to evaluate numerically the entropy with an enormous precision. For example, the entropy $K_1 \approx 0.66131433271130$, being a quantum counterpart of the classical KS-entropy, is evaluated by the deterministic algorithm (2.3) with 14 significant digits. Such precision could be hardly obtained either with random iterated algorithm (2.4) or with standard techniques of time series analysis (2.5). The quantities characterizing the invariant measure of the IFS: a) Rényi entropies: $K_q, \tilde{K}_q$; b) scaling spectra: $g(\alpha), \tilde{g}(\alpha)$; c) fractal dimensions: $D_q, \tilde{D}_q$; and d) multifractal spectrum $f(\alpha)$ are presented in Fig. 4. The common maximum of the scaling spectra gives the topological entropy $K_0 = \ln 2$, while the same curves intersects the bisectrix at the KS-entropy $K_1$. Fractal dimensions $D_q$ are computed with the aid of the ”box-counting” algorithm and compared with the quantities $\tilde{D}_q$ defined in Sect. IV.

**VII. CONCLUDING REMARKS**

We have analyzed properties of IFSs with place-dependent probabilities and showed that their invariant measures often posses the multifractal property, i.e., the fractal dimension $D_q$ depends substantially on the Rényi parameter $q$.

We have described a method of computing the generalized entropy for such IFSs by integrating the entropy function over their invariant measures. For numerical evaluation of the entropy one can apply the deterministic algorithm (useful for small number of functions) or random iterated algorithm (advantageous for large number of functions in IFS). Numerical calculations performed for generalized Cantor measures have shown superiority of both methods with respect to the standard method of computing entropy from time series generated by IFSs (2.3,2.5). The entropy and the dimension of some IFSs of the second kind studied here display non-trivial scaling properties. The invariant measure for such an IFS may be thus neither uniform nor balanced.
It is possible to attach an IFS of the second kind to certain dynamical systems in such a way that the generalized entropies of their invariant measures are equal. This idea allows us to propose a new method of computing entropy for dynamical systems. In this work we demonstrated its usefulness for some classical (Rényi-type entropy of asymmetric tent map, logistic map, and "hut" map) and quantum (CS-measurement entropy for two hemispheres, $j = 1/2$) systems. The method of computing entropy by integration over fractal measure has been recently applied to other dynamical systems. The tent map with a gap, related to physical problem of communication with chaos, was studied in [59], while the fractal structure of an exemplary repelling system has been analyzed in [60]. Moreover, we used a similar method to compute the dynamical entropy of some systems with stochastic perturbations [61]. This technique may be extended for a wider class of classical and quantum dynamical systems (or even for Markov chains). Such results will be presented in a forthcoming publication [43].

During last few years we enjoyed fruitful collaboration with late Marcin Poźniak. It is also a pleasure to thank Iwo Białynicki-Birula, Łukasz Turski and Daniel Wójcik for inspiring discussions on integration over the fractal measures and for indicating the formula (5.3). One of us (K.Ż.) is thankful to Ed Ott for hospitality during his stay at the University of Maryland and acknowledges the Fulbright Fellowship. Financial support by the Polish Committee of Scientific Research under grant No. P03B 060 013 is gratefully acknowledged.
[1] J.-P. Eckmann and D. Ruelle, "Ergodic theory of chaos and strange attractors," Rev. Mod. Phys. 57, 617-653 (1985).
[2] A. Cohen and I. Procaccia, "Computing the Kolmogorov entropy from time signals of dissipative and conservative dynamical systems," Phys. Rev. A 31, 1872-1882 (1985).
[3] P. Schuster, Deterministic Chaos. An Introduction (VCH Verlagsgesellschaft, Weinheim, 1988).
[4] W. Słomczyński and K. Życzkowski, "Quantum chaos, an entropy approach," J. Math. Phys. 35, 5674-5700 (1994); and erratum, ibid. 36, 5201 (1995).
[5] K. Życzkowski and W. Słomczyński, "Coherent states quantum entropy," in Proc. of the Int. Conf. on Dynamical Systems and Chaos, Tokyo, 23-27 May, 1994, Vol 2, ed. Y. Aizawa et al. (World Scientific, Singapore, 1995), pp. 467-470.
[6] J. Kwapien, W. Słomczyński, and K. Życzkowski, "Coherent states measurement entropy," J. Phys. A 30, 3175-3200 (1997).
[7] W. Słomczyński and K. Życzkowski, "Mean dynamical entropy of quantum maps on the sphere diverges in the semiclassical limit," Phys. Rev. Lett. 80, 1880-1883 (1998).
[8] A. Lasota and J. Myjak, "Generic properties of fractal measures," Bull. Pol. Ac., Math. 42, 283-296 (1994).
[9] T. Szarek, "Generic properties of learning systems," to appear in Ann. Polon. Math.
[10] M. F. Barnsley, Fractals Everywhere (Academic Press, Boston, 1988; second edition 1993).
[11] I. Guarneri and G. Mantica, Multifractal energy spectra and their dynamical implications, Phys. Rev. Lett. 73, 3379-3382 (1994).
[12] J. Bene and P. Szépfalusy, "Multifractal properties in the one-dimensional random-field Ising model," Phys. Rev. A 37, 1703-1707 (1988).
[13] G. Rados, "Emergence of quenched phases and second order transitions for sums of multifractal measures," Phys. Rev. Lett. 75, 2518-2521 (1995).
[14] M. F. Barnsley, S. G. Demko, J. H. Elton, and J. S. Geronimo, "Invariant measures for Markov processes arising from iterated function systems with place-dependent probabilities," Ann. Inst. Henri Poincaré 24, 367-394 (1988); and erratum, ibid. 25, 589 (1989).
[15] M. Iosefescu, G. Grigorescu, Dependence with Complete Connections and Its Applications (Cambridge University Press, Cambridge, 1990).
[16] A. Csordás, G. Györgyi, P. Szépfalusy, and T. Tél, "Statistical properties of chaos demonstrated in a class of one-dimensional maps," Chaos 3, 31-50 (1993).
[17] A. O. Lopes and Wm. D. Withers, "Weight-balanced measures and free energy for one-dimensional dynamics," Forum Math. 5, 161-182 (1993).
[18] P. Góra and A. Boyarsky, "Iterated function systems and dynamical systems," Chaos 5, 634-639 (1995).
[19] W. Słomczyński, "From quantum entropy to iterated function systems," Chaos, Solitons & Fractals 8, 1861-1864 (1997).
[20] E. S. Abers and B. W. Lee, "Gauge theories," Phys. Rep. 9, 1-141 (1973).
[21] T. Kajiser, "On a new contraction condition for random systems with complete connections," Rev. Roum. Math. Pures et Appl. 26, 1075-1117 (1981).
[22] J. H. Elton, "An ergodic theorem for iterated maps," Ergod. Th. & Dynam. Sys. 7, 481-488 (1987).
[23] M. Barnsley and J. E. Elton, "A new class of Markov processes for image encoding," Adv. Appl. Prob. 20, 14-32 (1988).
[24] L. Arnold and H. Crauel, "Iterated function systems and multiplicative ergodic theory," in Diffusion Processes and Related Problems in Analysis, Vol. II, ed. M. Pinsky and V. Wihstutz (Birkhauser, Boston, 1992), pp. 283-305.
[25] S. Grigorescu, "Limit theorems for Markov chains arising from iterated function systems," Rev. Roum. Math. Pures et Appl. 37, 887-899 (1992).
[26] E. Gadde, Stable IFSs with probabilities. An ergodic theorem, Research Rep. 10 (1994), Dept. of Mathematics, Umeå University.
[27] A. Lasota and J. A. Yorke, "Lower bound technique for Markov operators and iterated function systems," Random and Computational Dynamics 2, 41-77 (1994).
[28] W. Jaraczky and A. Lasota, "Invariant measures for fractals and dynamical systems," Bull. Pol. Ac., Math. 43, 347-361 (1995).
[29] A. Lasota, "From fractals to stochastic differential equations," in, Chaos - The Interplay Between Stochastic and Deterministic Behaviour, ed. P. Garbaczewski et al. (Springer, Berlin, 1995), pp. 235-255.
[30] K. Loskot and R. Rudnicki, "Limit theorems for stochastically perturbed dynamical systems," J. Appl. Prob. 32, 459-469 (1995).
[31] B. Forte and F. Mendivil, "A classical ergodic property for IFS: a simple proof," Ergod. Th. & Dynam. Sys. 18, 609-611 (1998).
[32] D. Silvestrov and Ö. Stenflo, Ergodic theorems for iterated function systems controlled by regenerative sequences, J. Theoret. Probab. 11, 589-608 (1998).
[33] A. Öberg, *Approximation of invariant measures for iterated function systems*, U.U.D.M. Doctoral Thesis 5 (1998), Dept. of Mathematics, Umeå University.

[34] A. Edalat, "Power domains and iterated function systems," *Inform. and Comput.* 124, 182-197 (1996).

[35] A. Lasota and M. Mackey, *Chaos, Fractals and Noise* (Springer, Berlin, 1994).

[36] K. Zyczkowski and W. Słomczyński, "Exponential decay of relative entropies to the Kolmogorov-Sinai entropy for the standard map," *Phys. Rev.* E 52, 6879-6880 (1995).

[37] Yu. Kifer, *Ergodic Theory of Random Transformations* (Birkhauser, Boston, 1986).

[38] J. N. Kapur, *Measures of Information and Their Applications* (John Wiley & Sons, New York, 1994).

[39] A. Rényi, "On measures of entropy and information," in *Proc. of the Fourth Berkeley Symp. Math. Statist. Prob. 1960, Vol. I* (University of California Press, Berkeley, 1961), pp. 547-561.

[40] F. Takens and E. Verbitski, "Generalized entropies: Rényi and correlation integral approach," *Nonlinearity* 11, 771-782 (1998).

[41] D. Bessis, G. Paladin, G. Turchetti, and S. Vaienti, "Generalized dimensions, entropies, and Liapunov exponents from the pressure function for strange sets," *J. Stat. Phys.* 51, 109-134 (1988).

[42] J.-M. Ghez, E. Orlandini, M.-C. Tesi, and S. Vaienti, "Dynamical integral transform on fractal sets and the computation of entropy," *Physica* D 63, 282-298 (1993).

[43] W. Słomczyński, Dynamical entropy and iterated function systems, *in preparation*.

[44] H. Hentschel and I. Procaccia, "The infinite number of generalized dimensions of fractals and strange attractors," *Physica* D 8, 435-444 (1983).

[45] P. Grassberger and I. Procaccia, "On the characterization of strange attractors," *Phys. Rev. Lett.* 50, 346-349 (1983).

[46] L. Olsen, *Random Geometrically Graph Directed Self-Similar Multifractals* (Longman, Harlow, 1994).

[47] L. Olsen, "A multifractal formalism," *Adv. Math.* 116, 82-196 (1995).

[48] K. Falconer, *Techniques in Fractal Geometry* (Wiley, Chichester, 1997).

[49] W. Chin, B. Hunt, and J. A. Yorke, "Correlation dimension for iterated function systems," *Trans. Am. Math. Soc.* 349, 1783-1796 (1997).

[50] T. C. Halsey, M. H. Jensen, L. P. Kadanoff, I. Procaccia, and B. I. Schramain, "Fractal measures and their singularities. The characterization of strange sets," *Phys. Rev. A* 33, 1141-1151 (1986).

[51] T. Bohr and D. Rand, "The entropy function for characteristic exponents," *Physica* D 25, 387-398 (1987).

[52] D. Rand, "The singularity spectrum f(α) for cookie-cutters," *Ergod. Th. & Dynam. Sys.* 9, 527-541 (1989).

[53] L. Barreira, Y. Pesin, and J. Schmeling, "On a general concept of multifractality, multifractal spectra for dimensions, entropies, and Lyapunov exponents. Multifractal rigidity," *Chaos* 7, 27-38 (1997).

[54] I. Bálynicki-Birula and D. Wójcik, private communication.

[55] M. Holschneider, "Large-scale renormalisation of Fourier-transforms of self-similar measures and self-similarity of Riesz measures," *J. Math. Anal. Appl.* 200, 307-314 (1996).

[56] M. Fannes, B. Nachtergaele, and L. Slegers, "Functions of Markov processes and algebraic measures," *Rev. Math. Phys.* 4, 39-64 (1992).

[57] P. Grassberger and I. Procaccia, "Estimation of the Kolmogorov entropy from a chaotic signal," *Phys. Rev. A* 28, 2591-2593 (1983).

[58] K. Pawelzik and H. G. Schuster, "Generalized dimensions and entropies from a measured time series," *Phys. Rev. A* 35, 481-484 (1987).

[59] K. Zyczkowski and E. Bolt, "On the entropy devil’s staircase for the family of gap tent maps," *Physica* D, *in press*.

[60] Y.-C. Lai, K. Życzkowski, and C. Grebogi, "Universal behavior in the parametric evolution of chaotic saddles," *Phys. Rev. E* 59, 5261-5265 (1999).

[61] A. Ostruszka, P. Pakoński, W. Słomczyński, and K. Życzkowski, "Dynamical entropy for systems with stochastic perturbations," preprint [chao-dyn 9905041](https://arxiv.org/abs/9905041).
FIG. 1. First three iterations of the uniform density on $X = [0,1]$ by two IFSs $\{F_1(x) = x/3, F_2(x) = (x+2)/3\}$ attracting to the Cantor set. Figures a), b), and c) are obtained for IFS ($a = 1/2$) with constant probabilities $p_1(x) = p_2(x) = 1/2$, while figures d), e), and f) for IFS ($a = 1$) with place-dependent probabilities $p_1(x) = x$, $p_2(x) = 1 - x$.

FIG. 2. Quantities characterizing the invariant measure for "Cantor" IFS of the second kind ($a = 1$): a) Rényi entropies $K_q$ (dashed line), $\tilde{K}_q$ (solid line); b) scaling spectra $g(\alpha)$ (dashed line), $\tilde{g}(\alpha)$ (solid line); c) fractal dimensions $D_q$ (solid line), $D_q$ (stars); d) multifractal spectrum $f(\alpha)$.

FIG. 3. Attaching an IFS to a 1D dynamical system: a) the tent map, and b) functions $F_1$ and $F_2$ of the corresponding IFS; c) and d) analogous pictures for the igloo (logistic) map; e) and f) analogous pictures for the hut map.

FIG. 4. Quantities characterizing the invariant measure of the IFS related to the quantum system: a) Rényi entropies $K_q$ (dashed line), $\tilde{K}_q$ (solid line); b) scaling spectra $g(\alpha)$ (dashed line), $\tilde{g}(\alpha)$ (solid line); c) fractal dimensions $D_q$ (solid line), $D_q$ (stars); d) multifractal spectrum $f(\alpha)$.
\[ \gamma_1 \]

\( a = \frac{1}{2} \)  

\[ \gamma_2 \]

\[ \gamma_3 \]

\( a = 1 \)

\[ \gamma_2 \]

\[ \gamma_3 \]

\[ \gamma_1 \]

\( a = 1 \)

\[ \gamma_2 \]

\[ \gamma_3 \]
Fig. 2

Graph a) shows the function $K_q$ as a function of $q$, with $K_q$ on the y-axis and $q$ on the x-axis. The graph displays a decreasing trend as $q$ increases.

Graph b) illustrates the function $g(\alpha)$, with $\alpha$ on the x-axis and $g(\alpha)$ on the y-axis. The graph features a peak at $\alpha = \kappa_0$.

Graph c) represents $D_q$, with $D_q$ on the y-axis and $q$ on the x-axis. The graph indicates a decreasing trend as $q$ increases.

Graph d) depicts $f(\alpha)$, with $\alpha$ on the x-axis and $f(\alpha)$ on the y-axis. The graph shows an increasing trend reaching a peak at $\alpha = \alpha_0$. 
