Renormalization of the Cabibbo-Kobayashi-Maskawa Matrix at One-Loop Level

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We have investigated the present renormalization prescriptions of the Cabibbo-Kobayashi-Maskawa (CKM) quark mixing matrix. Based on one prescription which is formulated by comparing with the fictitious case of no mixing of quark generations, we propose a new prescription which can make the physical amplitude involving quark’s mixing gauge independent and ultraviolet finite. Compared with the previous prescriptions this prescription is very simple and suitable for actual calculations. Through analytical calculations we also give a strong Proof for the important hypothesis that in order to keep the CKM matrix gauge independent the unitarity of the CKM matrix must be preserved.

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I. INTRODUCTION

Because of the difference between quark’s mass eigenstates and its electroweak eigenstates, the Cabibbo-Kobayashi-Maskawa (CKM) quark’s mixing matrix must be introduced in the standard model [1]. Although the effects of the renormalization of CKM matrix is small due to the unitarity of the CKM matrix (GIM mechanism [2]) - if effects of the quark masses are neglected, it is an theoretically important question in standard model. Along with the accurate measurement of the CKM matrix elements develops very quickly [3], the renormalization of CKM matrix becomes more important at present. Until now many people have discussed this problem using different methods [4, 5, 6, 7, 8, 9]. The early prescription constructed the CKM counterterm by quark’s wave-function renormalization constants (WRC) [6]. But recent calculation has shown that this prescription leads to the physical amplitudes involving quark’s mixing gauge-dependent [7, 8]. Another kind of prescription is to renormalize the CKM matrix by comparing with the fictitious case of no quark’s mixing [4, 9]. But Ref.[4] has pointed out that such prescription will break up the unitarity of the CKM matrix. Although a revised prescription to keep the unitarity of the CKM matrix [4] is present, it is very complex and unsuitable for actual calculations. So we propose a new prescription which is very simple and suitable for actual calculations compared with the previous prescriptions. This is discussed in section 2. In section 3 we have discussed the relationship between the unitarity and the gauge independence of the CKM matrix through analytical calculations. Lastly we give our conclusions.

II. ONE-LOOP CKM MATRIX RENORMALIZATION

The gist of Ref.[4, 9] is to renormalize the transition amplitude of W gauge boson decaying into two quarks in proportion to the same amplitude except for eliminating the quark-mixing effects, since such an amplitude is gauge independent and ultraviolet finite without introducing CKM matrix renormalization. At one-loop level the decay amplitude of $W^+ \rightarrow u_i d_j$ is

$$T_1 = A_L[V_{ij}(F_L + \frac{\delta g}{g} + \frac{1}{2} \delta Z_W + \frac{1}{2} \delta Z_{iL} + \frac{1}{2} \delta Z_{jL}) + \sum_{k \neq i} \frac{1}{2} \delta Z_{ik} V_{kj} + \sum_{k \neq j} \frac{1}{2} V_{ik} \delta Z_{kj} + \delta V_{ij}]$$

$$+ V_{ij}[A_R F_R + B_L G_L + B_R G_R]$$

(1)

with $g$ and $\delta g$ the $SU(2)$ coupling constant and its counterterm, $\delta Z_W$ the W boson WRC, $\delta Z_{iL}$ and $\delta Z_{jL}$ the left-handed up-type and down-type quark’s WRC [10], and

$$A_L = \frac{g}{\sqrt{2}} \bar{u}_i(p_1) \gamma_L \nu_j(q - p_1)$$

$$B_L = \frac{g}{\sqrt{2}} \bar{u}_i(p_1) \frac{e \cdot p_1}{M_W} \gamma_L \nu_j(q - p_1)$$

(2)

with $e^\mu$ the W boson polarization vector, $\gamma_L$ and $\gamma_R$ the left-handed and right-handed chiral operators, and $M_W$ the W boson mass. Similarly, replacing $\gamma_L$ by $\gamma_R$ in Eqs.(2) we define $A_R$ and $B_R$, respectively. $F_{L,R}$ and $G_{L,R}$ are four form factors which come from the contributions of the irreducible electroweak one-loop diagrams for $W^+ \rightarrow u_i d_j$ (see Fig.1) Since each diagram in Fig.1 contains only one vertex of W boson coupling to quarks, the form factors $F_{L,R}$ and
\(G_{L,R}\) are free of CKM matrix element. On the other hand, \(F_R\) and \(G_{L,R}\) are gauge independent \([7, 9]\).

The main idea of Ref.\([4, 9]\) is to choose the CKM counterterm to make the amplitude \(T_1\) in proportion to the fictitious amplitude of \(W^+ \rightarrow u_i \bar{d}_j\) in such a model that no quark-mixing effect is introduced. The ratio is the CKM matrix element \(V_{ij}\). As we know if there is no quark-mixing effect, the amplitude \(W^+ \rightarrow u_i \bar{d}_j\) must be gauge independent and ultraviolet finite without introducing any CKM matrix renormalization. So such a renormalized result of \(T_1\) is acceptable. Ref.\([9]\) has suggested that such amplitude can be obtained from the amplitude \(T_1\) by some
Compared with Eq. (1) and (4) the CKM counterterm is the decaying amplitude Feynman amplitudes, and used the mathematica package FeynCalc lines in a Feynman diagram. Since in Eq. (3) we have recognized u leptons, no CKM matrix element is present at the fermion line which connects with the external fermion lines and quark’s mixing and not having quark’s mixing. In the case of “not having quark’s mixing”, just like the case of \(m\) in what degree \(\delta V\) places of the CKM matrix elements except for \(V_{ij}\). So the non-diagonal fermion’s WRC in the terms \(\sum_k \frac{1}{2} \delta Z_{ik}^{uL} V_{kj} + \sum_k \frac{1}{2} V_{ik} \delta Z_{kj}^{dL}\) disappear and the diagonal fermion’s WRC must be added an operator \([l]\) to remove the left CKM matrix elements.

4. Since only the quarks in the same generation of the external-line quarks can contribute to the amplitude, the contribution of quark \(d_i\) in \(\delta Z_{ij}^{uL}[l]m_{d,i} \rightarrow m_{d,j}\) and the contribution of quark \(u_j\) in \(\delta Z_{jj}^{dL}[l]m_{u,j} \rightarrow m_{u,i}\) (see Fig.2) must be changed to the contributions of quarks \(d_j\) and \(u_i\). This is realized by the operations \(m_{d,i} \rightarrow m_{d,j}\) and \(m_{u,j} \rightarrow m_{u,i}\).

Compared with Eq. (1) and (4) the CKM counterterm is

\[
\delta V_{ij} = -\frac{1}{2} \sum_k [\delta Z_{ik}^{uL} V_{kj} + V_{ik} \delta Z_{kj}^{dL}] + \frac{1}{2} V_{ij} [\delta Z_{ii}^{uL}[l]m_{d,i} \rightarrow m_{d,j}\] + \frac{1}{2} \delta Z_{jj}^{dL}[l]m_{u,j} \rightarrow m_{u,i}\]

Our calculations have proved that this CKM counterterm is gauge independent and makes the physical amplitude \(T_1\) in Eq. (4) ultraviolet convergent.

Does this prescription keep the unitarity of the CKM matrix? Here we will do some analytical calculations to show in what degree \(\delta V\) satisfies the unitary condition. We have split the bare CKM matrix element \(V_{ij}^0\) into \(V_{ij}^0 = V_{ij} + \delta V_{ij}\) and keep the unitarity of the renormalized CKM matrix \(V\) in the previous calculations. So the unitarity condition of the bare CKM matrix \(V^0\) requires

\[
\sum_k (\delta V_{ik} V^*_{jk} + V_{ik} \delta V^*_{jk}) = \sum_k (V^*_k \delta V_{kj} + \delta V^*_k V_{kj}) = 0
\]

At one-loop level only four diagrams have contributions to the CKM counterterm in Eq. (5), as shown in Fig.2. We have used the mathematica package FeynArts to draw the Feynman diagrams and generate the corresponding Feynman amplitudes, and used the mathematica package FeynCalc to calculate these Feynman amplitudes. It is
The one- and two-order results are shown below:

\[
\delta V_{ij}^{(1)} = \frac{\alpha(6\Delta - 11)}{128\pi M_W^2 s_W^2} \left[ -2 \sum_{k,l \neq j} m_{d,j} m_{u,k} V_{il} V_{kl}^* V_{kj}^{(1)} + 2 \sum_{k,l} m_{d,j} m_{u,k} V_{il} V_{kl}^* V_{kj}^{(1)} - 2 \sum_{k \neq i} m_{u,i} m_{d,l} V_{il} V_{kl}^* V_{kj}^{(2)} \right] + m_{d,i} m_{u,k} + m_{u,i} - m_{d,i} m_{d,j} m_{u,k} (4m_{d,j}^2 + 6m_{d,j}^4 + 12m_{d,j}^4, i = k) - \sum_{k} m_{u,k} m_{u,k} (6 \ln m_{u,k} m_{u,k}^* V_{kl}^* V_{kl}^* V_{kj}^{(1)} + 3m_{u,k}^2 + 12m_{u,k}^2 m_{u,k}^2) V_{il} V_{kl}^* V_{kj}^{(2)} + 2 m_{d,i} m_{u,k} (4m_{d,j}^2 + 6 \ln m_{u,k}^2 m_{u,k}^* V_{kl}^* V_{kl}^* V_{kj}^{(1)} + 3m_{u,k}^2) V_{il} V_{kl}^* V_{kj}^{(2)} + 2 m_{u,k} m_{u,i} (4m_{u,i}^2 + 6 \ln m_{d,l} m_{d,i} m_{u,i} m_{u,i} m_{u,i}^* V_{kl}^* V_{kl}^* V_{kj}^{(1)} + 3m_{u,i}^2) V_{il} V_{kl}^* V_{kj}^{(2)} + 2 m_{u,i} m_{u,i} m_{d,l} (4m_{u,i}^2 + 6 \ln m_{u,i}^2 m_{u,i}^* V_{kl}^* V_{kl}^* V_{kj}^{(1)} + 3m_{u,i}^2) V_{il} V_{kl}^* V_{kj}^{(2)}\right].
\]

(7)

\[
delta V_{ij}^{(2)} = \frac{\alpha}{64\pi M_W^2 s_W^2} \left[ V_{ij} (12 \ln m_{d,j}^2 m_{d,j} + 6m_{d,j} + 12 \ln m_{u,i}^2 m_{u,i} + 6m_{u,i} + 32m_{d,j}^2 m_{u,i} - \sum_{k} m_{k} m_{k} (6 \ln m_{u,k} m_{u,k}^* V_{kl}^* V_{kl}^* V_{kj}^{(1)} + 3m_{u,k}^2 + 12m_{u,k}^2 m_{u,k}^2) V_{il} V_{kl}^* V_{kj}^{(2)} + 2 m_{d,i} m_{u,k} (4m_{d,j}^2 + 6 \ln m_{u,k}^2 m_{u,k}^* V_{kl}^* V_{kl}^* V_{kj}^{(1)} + 3m_{u,k}^2) V_{il} V_{kl}^* V_{kj}^{(2)} + 2 m_{u,k} m_{u,i} (4m_{u,i}^2 + 6 \ln m_{d,l} m_{d,i} m_{u,i} m_{u,i} m_{u,i}^* V_{kl}^* V_{kl}^* V_{kj}^{(1)} + 3m_{u,i}^2) V_{il} V_{kl}^* V_{kj}^{(2)} + 2 m_{u,i} m_{u,i} m_{d,l} (4m_{u,i}^2 + 6 \ln m_{u,i}^2 m_{u,i}^* V_{kl}^* V_{kl}^* V_{kj}^{(1)} + 3m_{u,i}^2) V_{il} V_{kl}^* V_{kj}^{(2)}\right].
\]

(8)

where the superscript (1) and (2) denote the one and two order results of \(\delta V_{ij}\) about the series \(m_{quark}^2/M_W^2\). The \(R_\xi\)-gauge has been used. Noted that \(\delta V_{ij}^{(1)}\) and \(\delta V_{ij}^{(2)}\) are both gauge independent. Replacing \(\delta V\) with \(\delta V^{(1)} + \delta V^{(2)}\) in Eq.(6), we find it satisfies the unitary condition.

But when we consider the three-order result of \(\delta V\) about the series \(m_{quark}^2/M_W^2\), we find that

\[
\sum_k (\delta V_{ki}^{(3)*} V_{kj} + V_{ki} \delta V_{kj}^{(3)}) = \frac{9\alpha}{128\pi M_W^6 s_W^2} \left[ \sum_k m_{u,k}^2 (m_{d,j}^4 - 2m_{d,j}^2 m_{d,i}^2 + m_{d,j}^4 + 2m_{d,j}^2 m_{u,k}^2 + m_{d,j}^4 m_{u,k}^2) V_{kl}^* V_{kj}^{(3)} - 2 \sum_{k,l,n} m_{u,k}^2 m_{d,i} m_{u,n}^2 (V_{kl}^* V_{kl}^* V_{kj}^{(3)}) \right] \neq 0
\]

(9)

This result shows that \(\delta V\) doesn’t comply with the unitary condition. But from this result we can see that the deviation of \(\delta V\) from the unitary condition is very small, since the quark’s masses are very small compared with \(M_W\) (except for \(m_1\)). Calculating till to five-order result of \(\delta V\) about the series \(m_{quark}^2/M_W^2\), we find the largest deviation of \(\delta V\) from 0 is proportional to \(\alpha V_{3\neq j}^* m_{j} m_{j}^* M_{W}^2 M_{W}^4 \sim 10^{-7}\), which is very small compared with the present measurement precision of the CKM matrix elements. Thus in actual calculations we can use Eq.(5) as the definition of the CKM counterterm. Comparing with the prescription in Ref.[4] one can see that our prescription is very simple and suitable for actual calculations.
III. RELATIONSHIP BETWEEN THE UNITARITY AND THE GAUGE INDEPENDENCE OF THE CKM MATRIX

It has been proved that any physical parameter’s counterterm must be gauge independent \[13\] if they don’t break up the theory structure. So the CKM matrix counterterm is also gauge independent \[11\] if it doesn’t break up the standard model’s structure, i.e. the unitarity of the CKM matrix. There has been proposed a negative proposition that in order to keep the gauge independence of the CKM counterterm the CKM renormalization prescription must keep the unitarity of the CKM matrix \[11, 12\]. Here we want to give a Proof for this hypothesis by analytical calculations.

In general one will encounter a question: if the CKM counterterms in lower-loop levels are gauge independent but don’t satisfy the unitary condition, will they change the gauge dependence of the CKM counterterms in the higher-loop levels? If the answer is ”yes”, the above hypothesis must be true. In order to study this question we consider the most simple case: the effect of the unitarity of the one-loop CKM counterterm to the gauge independence of the two-loop CKM counterterm.

In order to elaborate this problem clearly we express the amplitude of $W^+ \rightarrow u_d d_j$ as

$$T(V^0) = T(V + \delta V) = T(V) + T'(V)\delta V + \frac{1}{2} T''(V)\delta V^2 + \cdots$$

\[\text{(10)}\]

where the superscript $'$ denotes the partial derivative with respect to the CKM matrix elements. Since the CKM counterterm has been written out apparently, the amplitude $T$ on the right-hand side of Eq.(12) doesn’t contain CKM counterterm any more. To two-loop level, this equation becomes

$$T_2(V^0) = T_2(V) + T'_1(V)\delta V_1 + \delta V_2 A_L$$

\[\text{(11)}\]

where the subscripts 1 and 2 denote the 1-loop and 2-loop results. Since $T_2(V^0)$ must be gauge independent, the gauge dependence of $\delta V_2$ is determined by the gauge dependence of $T'_1(V)\delta V_1$ and $T_2(V)$. In the following we will firstly prove that $T_2(V)$ is gauge independent.

Using the facts that the one-loop formfactors $F_R$ and $G_{L,R}$ are gauge independent and don’t contain CKM matrix element, and the terms in the first bracket of Eq.(1) are gauge independent \[11\], one get

$$T'_1(V)\delta V_1|_\xi = \left[-\frac{\delta V_{ij}}{2\delta V_{ij}} \sum_{k \neq i} \delta Z_{ik}^{uL}V_{kj} + \sum_{k \neq j} V_{ik}\delta Z_{kj}^{dL}\right] + \left[\frac{V_{ij}}{2} \sum_{k,j} \left(\frac{d(\delta g)}{dv_{kl}} + \frac{d(\delta Z_W)}{dv_{kl}} + \frac{d(\delta Z_{ik}^{uL})}{dv_{kl}} + \frac{d(\delta Z_{kj}^{dL})}{dv_{kl}}\right)\delta V_{kl}\right.$$

$$+ \frac{1}{2} \left(\sum_{k \neq i} \delta Z_{ik}^{uL} \delta V_{kj} + \sum_{l,m,k \neq i} \frac{d(\delta Z_{lm}^{dL})}{dv_{lm}} \delta V_{km} V_{kj} + \sum_{k \neq j} V_{ik} \delta Z_{kj}^{dL} + \sum_{l,m,k \neq j} V_{ik} \frac{d(\delta Z_{lm}^{dL})}{dv_{lm}} \delta V_{lm}\right) A_L$$

\[\text{(12)}\]

where the subscript 1 of $\delta V_1$ on the right-hand side of Eq.(12) has been omitted and the subscript $\xi$ on the left-hand side of Eq.(12) denotes the gauge-dependent part of the quantity. Omitting the imaginary part of the quark’s self-energy functions, one obtain

$$T'_1(V)\delta V_1|_\xi = \frac{\alpha A_L}{32\pi M_W^2 m_{d,j}^2 m_{u,k}^2} \sum_{i,k,l} \left(\delta V_{il} V_{kj}^* + \delta V_{ik} V_{lj}^*\right) V_{kj} \left[-\xi W M_W^4 m_{u,k}^2 + 2 \xi W m_{d,j}^2 M_W^2 + 2 \xi W m_{u,k}^2 M_W^2 - m_{d,j}^4 + m_{u,k}^4 + 2 m_{d,j}^2 m_{u,k}^2\right]$$

$$\times (\xi W M_W^2 - m_{d,j}^2 + m_{u,k}^2) \arctan \frac{-\xi W M_W^4 + 2 \xi W m_{d,j}^2 M_W^2 + 2 \xi W m_{u,k}^2 M_W^2 - m_{d,j}^4 + m_{u,k}^4 + 2 m_{d,j}^2 m_{u,k}^2}{\xi W M_W^2 + m_{d,j}^2 - m_{u,k}^2}$$

$$+ \frac{\alpha A_L}{32\pi M_W^2 m_{d,j}^2 m_{u,k}^2} \sum_{i,k,l} \left(\delta V_{ik} V_{lj} + \delta V_{il} V_{kj}\right) V_{kj} \left[-\xi W M_W^4 + 2 \xi W m_{d,j}^2 M_W^2 + 2 \xi W m_{u,k}^2 M_W^2 - m_{d,j}^4 + m_{u,k}^4 + 2 m_{d,j}^2 m_{u,k}^2\right]$$

$$\times (\xi W M_W^2 - m_{d,j}^2 + m_{u,k}^2) \arctan \frac{-\xi W M_W^4 + 2 \xi W m_{d,j}^2 M_W^2 + 2 \xi W m_{u,k}^2 M_W^2 - m_{d,j}^4 + m_{u,k}^4 + 2 m_{d,j}^2 m_{u,k}^2}{\xi W M_W^2 + m_{d,j}^2 - m_{u,k}^2}$$

\[\text{(13)}\]

with $\xi W$ the $W$ boson gauge parameter. It is easy to see that when $\delta V_1$ satisfies the unitary condition of Eq.(6), $T'_1(V)\delta V_1$ is gauge independent. By the precondition that the CKM counterterm must be gauge independent if it
doesn’t break up the unitarity of the CKM matrix, we conclude that $T_2(V)$ is gauge independent. Because in this case both $\delta V_2$ and $T'_1(V)\delta V_1$ are gauge independent, the gauge independence of the right-hand side of Eq.(11) yields $T_2(V)$ is also gauge independent.

Since $T_2(V)$ is free of CKM matrix counterterms (see above), it must be gauge independent in spite of whether $\delta V_1$ satisfies the unitarity condition. According to Eq.(13), if $\delta V_1$ doesn’t satisfy the unitary condition, $T'_1(V)\delta V_1$ will be gauge dependent, thus $\delta V_2$ must be gauge dependent in order to make the right-hand side of Eq.(11) gauge independent. So we have proved the conclusion that if the one-loop CKM counterterm doesn’t keep the unitarity of the CKM matrix the two-loop CKM counterterm must be gauge dependent. This conclusion is a strong Proof for the hypothesis that in order to keep the gauge independence of the CKM counterterm the CKM renormalization prescription must keep the unitarity of the CKM matrix.

IV. CONCLUSION

In summary, We have investigated the present CKM matrix renormalization prescriptions and found the prescriptions proposed in Ref.[4, 9] have some defect - either making the physical amplitude involving quark’s mixing ultraviolet divergent in non-diagonal case or too complex. So we propose a new prescription which can make the physical amplitude involving quark’s mixing ultraviolet finite and gauge independent and keeps the unitarity of the CKM matrix at a very high precision. The most important property is our prescription is very simple compared with the previous prescriptions, so is suitable for actual calculations. On the other hand we have studied the relationship between the unitarity and the gauge independence of the CKM matrix. Through analytical calculations we have given a strong Proof for the important hypothesis that in order to keep the gauge independence of the CKM matrix the CKM renormalization prescription must keep the unitarity of the CKM matrix.

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