ON CONVEXITY OF SOLUTIONS OF ORDINARY DIFFERENTIAL EQUATIONS.

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Abstract. We prove a result on the convex dependence of solutions of ordinary differential equations on an ordered finite-dimensional real vector space with respect to the initial data.

1. Introduction

Let $E$ be a finite-dimensional real vector space ordered by a closed proper cone $C$. Let $0 < T \leq \infty$, $U \subset E$ be a non-empty open set, and $f: [0, T) \times U \to E$ be a locally Lipschitz continuous map. For any $x \in U$, the differential equation

\begin{equation}
\dot{\psi}(t) = f(t, \psi(t))
\end{equation}

has a unique maximally extended solution $\psi_f(t, x)$ satisfying $\psi_f(0, x) = x$. This solution is defined on a semi-interval $[0, \theta_f(x))$, where $0 < \theta_f(x) \leq T$. For any $t \geq 0$, we set $D_f(t) = \{ x \in U : t < \theta_f(x) \}$.

Let $D \subset E$. A map $g: D \to E$ is called quasi-monotone increasing [11] if the implication

\[ x \leq y, l(x) = l(y) \implies l(g(x)) \leq l(g(y)) \]

holds for all $x, y \in D$ and $l \in C^*$, where $C^* = \{ l \in E^* : l(x) \geq 0 \text{ for any } x \in C \}$ is the dual cone of $C$ ($E^*$ is the dual space of $E$). A map $g: D \to E$ is called convex if $D$ is convex and

\begin{equation}
g(\lambda x + (1 - \lambda)y) \leq \lambda g(x) + (1 - \lambda)g(y)
\end{equation}

for all $x, y \in D$ and $\lambda \in [0, 1]$. A set $D \subset E$ is said to be order regular if the relations $x \in D$ and $y \leq x$ imply that $y \in D$.

Our aim is to prove the next theorem.

Theorem 1. Let $U \subset E$ be a nonempty order-regular convex open set. Let $0 < T \leq \infty$ and $f: [0, T) \times U \to E$ be a continuous map. If $f(t, \cdot)$ is quasi-monotone increasing and convex for all $t \in [0, T)$, then $D_f(t)$ is convex for any $t \in [0, T)$, and $\psi_f(t, \cdot)$ is convex thereon.

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1A set $C$ in a real vector space $E$ is called a cone if $\lambda C \subset C$ for any $\lambda > 0$. A cone $C$ is said to be proper if $C + C \subset C$ and $C \cap (-C) = \{0\}$. A cone $C$ induces a partial order on $E$ if and only if it is proper.
In the formulation of Theorem 1, we do not require the local Lipschitz continuity of \( f \) because the latter is ensured by continuity and convexity (see Lemma 2 below). Note that the quasi-monotonicity of \( f \) is a sufficient but not necessary condition for Theorem 1 to hold. For example, if \( f(t,x) = f(x) \) is a linear map, then \( \psi_f(t,x) \) is linear and hence convex in \( x \), but \( f \) may be not quasi-monotone increasing in this case. On the other hand, at least in the autonomous case \( f(t,x) = f(x) \), the convexity of \( f \) is necessary to maintain the validity of Theorem 1. Indeed, let \( f \) be locally Lipschitz, \( x,y \in U \) and \( z = \lambda x + (1-\lambda)y \) with \( 0 \leq \lambda \leq 1 \). Suppose \( D_f(t) \) is convex for any \( t \in [0,T) \), and \( \psi_f(t,\cdot) \) is convex thereon. Then we have

\[
\frac{\psi_f(t,z) - z}{t} \leq \lambda \frac{\psi_f(t,x) - x}{t} + (1-\lambda) \frac{\psi_f(t,y) - y}{t}
\]

for \( t \) small enough. Passing to the limit \( t \to 0 \) in this inequality, we get \( f(z) \leq \lambda f(x) + (1-\lambda)f(y) \), i.e., \( f \) is convex.

The question of convex dependence of solutions of \( (1) \) on initial data was first addressed in [7], and then pursued in [3, 4]. In the last two papers, \( E \) was assumed to be an ordered Banach space and it was shown (for differentiable \( f \) in [5] and for general locally Lipschitz continuous \( f \) in [4]) that \( \psi_f(t,\cdot) \) is convex on any convex domain contained in \( D_f(t) \) (in Appendix A to this paper, we give a very simple proof of this result). Here, we strengthen this result in the finite-dimensional case by proving the convexity of \( D_f(t) \). Moreover, keeping in mind possible applications (see, e.g., an example in Section 5), we consider arbitrary open convex order-regular domains \( U \) rather than the case \( U = E \) studied in [5, 4].

The paper is organized as follows. In Section 2 we show that the conditions imposed on \( f \) in Theorem 1 ensure its local Lipschitz continuity. In Section 3 we prove Theorem 1 in the case, where \( f \) is differentiable in the second variable. For this, we combine the technique developed in [5] with the well-known “blow-up property” of ODEs in finite dimensions: as \( t \to \theta_f(x) \) for some \( x \in U \), the maximal solution \( \psi_f(t,x) \) of \( (1) \) must approach the boundary of the domain \([0,T) \times U\) on which \( f \) is defined. In Section 4 we get rid of the differentiability assumption and prove Theorem 1 in the general case. Finally, in Section 5 we illustrate Theorem 1 by a concrete example of ODEs naturally arising in the theory of stochastic processes.

2. Convexity and local Lipschitz continuity

Let \( 0 < T \leq \infty \) and \( \| \cdot \| \) be a norm on \( E \). Let \( U \subset E \) be a non-empty open set. Recall that a map \( f : [0,T) \times U \to E \) is called locally Lipschitz if

\[
L_{t,K}(f) = \sup_{0 \leq \tau \leq t, x_1,x_2 \in K, x_1 \neq x_2} \frac{\| f(\tau,x_2) - f(\tau,x_1) \|}{\| x_2 - x_1 \|} < \infty
\]

for any compact set \( K \subset U \) and any \( t \in [0,T) \).

**Lemma 2.** Let \( f : [0,T) \times U \to E \) be a continuous map such that \( f(t,\cdot) \) is convex on \( U \) for all \( t \in [0,T) \). Then \( f \) is locally Lipschitz continuous.

**Proof.** Since \( C \) is closed and \( C \cap (-C) = \{0\} \), the set \( C \setminus \{0\} \) is contained in an open half-space of \( E \). This implies that the dual cone \( C^* \) has a nonempty interior (see, e.g., [10], Section I.4.4, Lemma 1). Let \( l_1,\ldots,l_n \in C^* \) be a basis of \( E^* \). Let the real-valued functions \( f_1,\ldots,f_n \) on \([0,T) \times U\) be defined by the relations \( f_j(t,x) = l_j(f(t,x)) \). Clearly, \( f_j \) are continuous on \([0,T) \times U\) and \( f_j(t,\cdot) \)
are convex on $U$ for any $t \in [0, T)$. Let $e_1, \ldots, e_n \in E$ be the dual basis of $l_1, \ldots, l_n$: $l_j(e_k) = \delta_{jk}$. Then we have

$$f(t, x) = \sum_{j=1}^{n} f_j(t, x)e_j.$$  

Hence, it remains to prove that $f_j$ are locally Lipschitz continuous, i.e., satisfy (4) with $| \cdot |$ in the numerator replaced with $| \cdot |$. Clearly, it suffices to check (3) in the case $K = B_{x,r}$, where $B_{x,r} \subset U$ is a closed ball of radius $r > 0$ centered at $x \in U$. Let $r' > r$ be such that $B_{x,r'} \subset U$. By the continuity of $f_j$, there is $m > 0$ such that $|f_j(\tau, x)| \leq m$ for any $\tau \in [0, t]$ and $x \in B_{x,r'}$. By (12) Corollary 2.2.12], we have

$$|f_j(\tau, x_2) - f_j(\tau, x_1)| \leq \frac{2m r' + r}{r' - r} ||x_2 - x_1||$$  

for any $x_1, x_2 \in B_{x,r}$ and $\tau \in [0, t]$. The lemma is proved. 

\[ \Box \]

### 3. The Differentiable Case

In the rest of the paper, we assume that $T \in (0, \infty]$ is fixed and set $I = [0, T)$.

Our consideration is essentially based on the next comparison result that is a particular case of a more general theorem proved by Volkmann (11) in the setting of normed vector spaces.

**Lemma 3.** Let $U \subset E$ be an open set. Let $f: I \times U \to E$ be a continuous locally Lipschitz map such that $f(t, \cdot)$ is quasi-monotone increasing on $U$ for all $t \in I$. Let $0 < t_0 \leq T$ and $x, y: [0, t_0) \to U$ be differentiable maps such that $x(0) \leq y(0)$ and

$$\dot{x}(t) - f(t, x(t)) \leq \dot{y}(t) - f(t, y(t)), \quad 0 \leq t < t_0.$$  

Then we have $x(t) \leq y(t)$ for all $t \in [0, t_0)$.

In fact, this comparison statement is essentially equivalent to quasi-monotonicity (3), but the above formulation is enough for our purposes. The next lemma is a simple generalization of a well-known result for scalar-valued convex functions.

**Lemma 4.** Let $U \subset E$ be an open convex set. A differentiable function $g: U \to E$ is convex on $U$ if and only if

$$g(y) - g(x) \geq g'(x)(y - x), \quad x, y \in U.$$  

**Proof.** Let $h = y - x$ and $\lambda \in (0, 1)$. If $g$ is convex on $U$, then

$$g(x + \lambda h) = g((1 - \lambda)x + \lambda y) \leq (1 - \lambda)g(x) + \lambda g(y).$$  

This implies that

$$\frac{g(x + \lambda h) - g(x)}{\lambda} \leq g(y) - g(x).$$  

In view of the closedness of $C$, passing to the limit $\lambda \to 0$ yields (4). Conversely, let (4) hold and $z = \lambda x + (1 - \lambda)y$. Then we have

$$g(x) - g(z) \geq -(1 - \lambda)g'(z)h, \quad g(y) - g(z) \geq \lambda g'(z)h.$$  

Multiplying the left and right estimates by $\lambda$ and $1 - \lambda$ respectively and summing the resulting inequalities, we obtain (4). The lemma is proved. 

For differentiable functions, we have the following characterization of quasi-monotonicity (3 Theorem 5).
Lemma 5. Let $U \subset E$ be open and convex. A differentiable function $g: U \to E$ is quasi-monotone increasing on $U$ if and only if the linear map $g'(x): E \to E$ is quasi-monotone increasing for any $x \in U$.

Suppose $f: I \times U \to E$ is a continuous map such that $f(t, \cdot)$ is differentiable on $U$ for all $t \in I$ and the derivative $f'(t, \cdot)$ is continuous on $I \times U$ (here and below, $f'(t, \cdot)$ denotes the derivative of the map $x \to f(x, t)$ with respect to $x$ for fixed $t$). Then $f$ is locally Lipschitz, and we have

$$L_{t,K}(f) = \sup_{0 \leq \tau \leq t, x \in K} \|f'(\tau, x)\|$$

for any $t \in I$, and for any compact convex set $K \subset U$ with a nonempty interior. Given $x \in U$ and $0 \leq t < \theta_f(x)$, we define the linear map $B^x(t): E \to E$ by setting

$$B^x(t) = f'(t, \psi_f(t, x)).$$

For $x \in U$ and $y \in E$, we denote by $w^x_y(t)$ the solution of the initial value problem

$$\dot{w}^x_y(t) = B^x(t)w^x_y(t), \quad 0 \leq t < \theta_f(x), \quad w^x_y(0) = y.$$

Clearly, $w^x_y$ is linear in $y$. For the norm of $w^x_y$, we have the standard bound (see, e.g., [2], Chapter IV, Lemma 4.1)

$$\|w^x_y(t)\| \leq \|y\| \exp \left( \int_0^t \|B^x(\tau)\| \, d\tau \right) , \quad 0 \leq t < \theta_f(x).$$

Lemma 6. Let $U \subset E$ be a convex open set and $f: I \times U \to E$ be a continuous map such that $f(t, \cdot)$ is differentiable on $U$ for all $t \in I$ and the derivative $f'(t, \cdot)$ is continuous on $I \times U$. Suppose $f(t, \cdot)$ is convex and quasi-monotone increasing on $U$ for all $t \in I$. For any $x, y \in U$, we have

$$w^x_y(t) \leq \psi_f(t, y) - \psi_f(t, x) \leq w^y_x(t), \quad 0 \leq t < t_0,$$

where $t_0 = \min(\theta_f(x), \theta_f(y))$.

Proof. It suffices to prove the left inequality in (9) because it implies the right one after interchanging $x$ and $y$. Let $s(t) = \psi_f(t, y) - \psi_f(t, x)$. By Lemma 4, we have

$$\dot{s}(t) = f(t, \psi_f(t, y)) - f(t, \psi_f(t, x)) \geq B^x(t)s(t), \quad 0 \leq t < t_0.$$

By Lemma 5, the map $B^x(t)$ is quasi-monotone increasing for any $t \in [0, t_0)$ and, therefore, the desired inequality follows from (9) and Lemma 3. The lemma is proved. □

Since $E$ is finite-dimensional, the closed ordering cone $C$ is normal. In terms of the partial order induced by $C$, this means that there exists $\mu_C > 0$ such that the implication

$$0 \leq x \leq y \implies \|x\| \leq \mu_C \|y\|$$

holds for all $x, y \in E$.

If $f$ is continuously differentiable in the second variable, Theorem 1 follows from the next lemma.

Lemma 7. Let $U$ and $f$ be as in Lemma 6 and suppose in addition that $U$ is order-regular. Let $x, y \in U$, $\lambda \in [0, 1]$, and $z = \lambda x + (1 - \lambda)y$. Let $t_0 = \min(\theta_f(x), \theta_f(y))$. Then we have $\theta_f(z) \geq t_0$ and

$$\psi_f(t, z) \leq \lambda \psi_f(t, x) + (1 - \lambda)\psi_f(t, y), \quad 0 \leq t < t_0.$$
Let $0 \leq t < t_0$ and $K \subset U$ be a compact convex set with a nonempty interior such that $\psi_f(\tau, x)$ and $\psi_f(\tau, y)$ lie in $K$ for all $\tau \in [0, t]$. Then

\begin{equation}
\|\psi_f(t, z)\| \leq R_K \left[ 1 + \mu C e^{L_1 \kappa(f) t} \right],
\end{equation}

where $R_K = \sup_{\xi \in K} \|\xi\|$.

Proof. Let $\tau_0 = \min(\theta_f(x), \theta_f(y), \theta_f(z))$. Since $z - x = (1 - \lambda)(y - x)$ and $z - y = -\lambda(y - x)$, it follows from Lemma 6 that

\begin{equation}
(1 - \lambda)w_{y-x}^z(t) \leq \psi_f(t, z) - \psi_f(t, x) \leq (1 - \lambda)w_{y-x}^z(t),
\end{equation}

\begin{equation}
-\lambda w_{y-x}^z(t) \leq \psi_f(t, z) - \psi_f(t, y) \leq -\lambda w_{y-x}^z(t),
\end{equation}

for any $0 \leq t < \tau_0$. Multiplying the first and second inequalities by $\lambda$ and $1 - \lambda$ respectively and adding the results, we get

\begin{equation}
-\lambda (1 - \lambda)v(t) \leq \psi_f(t, z) - \psi_f(t, y) \leq 0, \quad 0 \leq t < \tau_0,
\end{equation}

where $u, v : [0, t_0) \rightarrow E$ are given by

\begin{equation}
u(t) = \lambda \psi_f(t, x) + (1 - \lambda)\psi_f(t, y), \quad v(t) = w_{y-x}^z(t) - w_{y-x}^z(t).
\end{equation}

In view of (10), it follows from (13) that

\begin{equation}
\|\psi_f(t, z)\| \leq \|u(t)\| + \|\psi_f(t, z) - u(t)\| \leq \|u(t)\| + \mu C \lambda (1 - \lambda)\|v(t)\|, \quad 0 \leq t < \tau_0.
\end{equation}

Suppose that $\tau_0 < t_0$. Then we obviously have $\tau_0 = \theta_f(z)$. Since both $u$ and $v$ are continuous on $[0, t_0)$, it follows from (15) that $\psi_f(t, z)$ is bounded on $[0, \theta_f(z)]$. This implies that we can choose a sequence $t_k \uparrow \tau_0$ such that $\psi_f(t_k, z)$ converge to some $x_0 \in E$ as $k \rightarrow \infty$. By (13), we have $\psi_f(t_k, z) \leq u(t_k)$ for all $k$. As $C$ is closed, it follows that $x_0 \leq u(\tau_0)$. We hence have $x_0 \in U$ because $U$ is order-regular and $u(\tau_0) \in U$ by the convexity of $U$. On the other hand, we cannot have $x_0 \in U$ because $\psi_f(t, z)$ is a maximal solution and must approach the boundary of $I \times U$ as $t \rightarrow \theta_f(z)$ (see [2], Chapter II, Theorem 3.1). This contradiction shows that

\begin{equation}
\tau_0 = t_0.
\end{equation}

Combining this relation with (13) and (14), we obtain (11). Let $t \in [0, t_0)$ and $K \subset U$ be a convex compact set with a nonempty interior such that both $\psi_f(\tau, x)$ and $\psi_f(\tau, y)$ lie in $K$ for any $\tau \in [0, t]$. It follows from (14), (8), (4), and (5) that

\begin{equation}
\|v(t)\| \leq 2\|y - x\| e^{L_1 \kappa(f) t} \leq 4R_K e^{L_1 \kappa(f) t}.
\end{equation}

In view of (10), inserting this estimate and the obvious inequalities $\|u(t)\| \leq R_K$ and $\lambda (1 - \lambda) \leq 1/4$ in (13) yields (12). The lemma is proved.

4. Proof of Theorem 11

To pass from continuously differentiable to arbitrary continuous functions, we shall need some results concerning the continuous dependence of solutions of (11) on the map $f$. Recall that Eq. (11) possesses a maximal solution satisfying a given initial condition if the function $f : I \times U \rightarrow E$ is continuous. Note however that such a solution may be not unique if $f$ is not locally Lipschitz continuous.

The next lemma easily follows from Theorem 3.2 in Chapter II of [2].
Lemma 8. Let \( U \subset E \) be open. Let \( f, f_1, f_2, \ldots \) be continuous maps from \( I \times U \) to \( E \). Suppose \( f \) is locally Lipschitz and \( f_n \) converge to \( f \) uniformly on all compact subsets of \( I \times U \). Let \( \psi_n \in C^1([0, \theta_n), U) \) be maximal solutions of
\[
\dot{\psi}_n(t) = f_n(t, \psi_n(t))
\]
such that \( \psi_n(0) \) converge to some \( u \in U \) as \( n \to \infty \). Then we have
\[
\theta_f(u) \leq \lim \theta_n.
\]
Let \( 0 < a < \theta_f(u) \) and \( n_0 \) be such that \( \theta_n > a \) for \( n > n_0 \). Then the sequence \( \psi_{n_0+k}(t), k = 1, 2, \ldots, \) converges to \( \psi_f(t, u) \) uniformly on \([0, a] \) as \( k \to \infty \).

Lemma 9. Let \( U \subset E \) be open. Let \( f, f_1, f_2, \ldots \) be continuous maps from \( I \times U \) to \( E \). Suppose \( f \) is locally Lipschitz and \( f_n \) converge to \( f \) uniformly on compact subsets of \( I \times U \). Let \( 0 < a < T \) and \( \psi_n \in C^1([0, a], U) \) be solutions of (17) such that \( \psi_n(0) \) converge to some \( u \in U \) as \( n \to \infty \). If for some compact set \( K \subset U \), \( \psi_n(t) \in K \) for all \( t \in [0, a] \), then \( \theta_f(u) > a \), and we have \( \psi_n(t) \to \psi_f(t, u) \) and \( \psi_n(t) \to \dot{\psi}_f(t, u) \) uniformly on \([0, a] \).

Proof. Since \( f_n \) are uniformly bounded on the compact set \( Q = [0, a] \times K \), Eq. (17) implies that \( \dot{\psi}_n \) are uniformly bounded. Hence, \( \psi_n \) are uniformly equicontinuous. By the Arzela-Ascoli theorem, it follows that the sequence \( \psi_n \) is relatively compact in \( C^0[0, a] \). Let \( \psi_{nk} \) be a subsequence of \( \psi_n \) uniformly converging to a function \( \psi \). Obviously, \( \psi(0) = u \) and \( \psi(t) \in K \) for \( t \in [0, a] \). Fix \( \varepsilon > 0 \). Because \( f \) is uniformly continuous on \( Q \), there exists a \( \delta > 0 \) such that \( \|f(t, x_1) - f(t, x_2)\| < \varepsilon /2 \) for any \( (t, x_i) \in Q \) such that \( \|x_2 - x_1\| < \delta \). Let \( k_0 \) be such that \( \|\psi_{nk}(t) - \psi(t)\| < \delta \) and \( \|f_{nk}(t, x) - f(t, x)\| < \varepsilon /2 \) for all \( (t, x) \in Q \) and \( k \geq k_0 \). Then we have
\[
\|f_{nk}(t, \psi_{nk}(t)) - f(t, \psi(t))\| \leq \|f_{nk}(t, \psi_{nk}(t)) - f(t, \psi(t))\| + \|f(t, \psi_{nk}(t)) - f(t, \psi(t))\| < \varepsilon, \quad t \in [0, a],
\]
for any \( k \geq k_0 \), and in view of (17), the sequence \( \psi_{nk}(t) \) converges to \( f(t, \psi(t)) \) uniformly on \([0, a] \). On the other hand, the uniform convergence of \( \psi_{nk} \) implies that \( \psi \) is continuously differentiable and \( \dot{\psi} \) is the limit of \( \dot{\psi}_{nk} \). This means that \( \psi \) satisfies (1). Since \( f \) is locally Lipschitz, this implies that \( \psi \) is the restriction of \( \psi_f(\cdot, u) \) to \([0, a] \) and, therefore, \( \theta_f(u) > a \). We thus see that all uniformly converging subsequences of \( \psi_n \) have the same limit. As the sequence \( \psi_n \) is relatively compact, we conclude that \( \psi_n(t) \to \dot{\psi}_f(t, u) \) uniformly on \([0, a] \). Replacing \( \psi_{nk} \) with \( \psi_n \) in the above proof, we obtain the uniform convergence of \( \psi_n \). The lemma is proved. □

Proof of Theorem 1

Proof. For \( \kappa > 0 \), we set \( U(\kappa) = \{ \xi \in U : B_{\xi, \kappa} \subset U \} \), where \( B_{\xi, \kappa} \) is the closed ball of radius \( \kappa \) centered at \( \xi \). Clearly, the set \( U(\kappa) \) is open, convex, and order-regular for any \( \kappa > 0 \). Let \( t \in I, x, y \in D_f(t) \) and \( z = \lambda x + (1 - \lambda)y \) for some \( \lambda \in [0, 1] \). We have to show that \( \theta_f(z) > t \) and inequality (11) holds. Let \( S \subset U \) be a convex compact set whose interior contains \( \psi_f(t, \tau) \) and \( \psi_f(t, y) \) for all \( \tau \in [0, t] \). Choose \( \kappa > 0 \) such that \( S \subset U(\kappa) \).

Let \( \rho \) be a nonnegative smooth function on \( E \) such that \( \rho(\xi) = 0 \) for \( \|\xi\| > 1 \) and \( \int_E \rho(\xi) \, d\xi = 1 \). For any positive \( \varepsilon \leq \kappa \), we define the map \( f_\varepsilon : I \times U(\kappa) \to E \) by setting
\[
f_\varepsilon(\tau, \xi) = \int_E f(\tau, \xi - \varepsilon \eta) \rho(\eta) \, d\eta.
\]
Let \( \phi \) denote the restriction of \( f \) to \( I \times U(\kappa) \). Clearly, \( f_x \) are smooth in the second variable and converge to \( \phi \) uniformly on compact subsets of \( I \times U(\kappa) \) as \( \varepsilon \to 0 \). It is straightforward to check that \( f_x \) are convex quasi-monotone increasing maps on \( U(\kappa) \) such that

\[
L_{t,S}(f_x) \leq L_{t,S_x}(f),
\]

where \( S_\kappa \) is the closed \( \kappa \)-neighborhood of \( S \). Our choice of \( \kappa \) ensures that \( t < \min(\theta_\phi(x), \theta_\phi(y)) \). Let \( t_x = \min(\theta_\phi(x), \theta_\phi(y)) \). By Lemma 8 there exists \( 0 < \varepsilon_0 \leq \kappa \) such that \( t_x > t \) for any \( 0 < \varepsilon \leq \varepsilon_0 \) and \( \psi_{f_x}(\cdot, x) \rightarrow \psi_f(\cdot, x) \) and \( \psi_{f_x}(\cdot, y) \rightarrow \psi_f(\cdot, y) \) uniformly on \([0, t] \) as \( \varepsilon \to 0 \). Decreasing \( \varepsilon_0 \) if necessary, we can ensure that \( \psi_{f_x}(\tau, x) \) and \( \psi_{f_x}(\tau, y) \) lie in \( S \) for all \( \tau \in [0, t] \) and \( \varepsilon \in (0, \varepsilon_0] \). It follows from Lemma 7 that \( \theta_{f_x}(z) \geq t_x > t \) and

\[
\psi_{f_x}(\tau, z) \leq \lambda \psi_{f_x}(\tau, x) + (1 - \lambda) \psi_{f_x}(\tau, y),
\]

\[
||\psi_{f_x}(\tau, z)|| \leq R_S \left[ 1 + \mu_C e^{\lambda L_{t,S}(f)} \right]
\]

for any \( 0 \leq \tau \leq t \) and \( 0 < \varepsilon \leq \varepsilon_0 \). Let \( \tau > 0 \) and \( K = (S - C) \cap \{ \xi \in E : ||\xi|| \leq r \} \).

\[ S \text{ is compact and } C \text{ is closed, } S - C \text{ is closed and, therefore, } K \text{ is compact. The order-regularity of } U(\kappa) \text{ implies that } K \subset U(\kappa). \]

By (20) and (21), we have \( \psi_{f_x}(\tau, z) \in K \) for all \( 0 \leq \tau \leq t \) if \( r \) is large enough. It follows from Lemma 9 that \( \theta_\phi(z) \geq t_x > t \) and \( \psi_{f_x}(\tau, z) \rightarrow \psi_\phi(\tau, z) \) uniformly on \([0, t] \). Obviously, \( D_\phi \subset D_f \) and \( \psi_\phi \) is the restriction of \( \psi_f \) to \( D_\phi \). Hence \( \theta_f(z) \geq \theta_\phi(z) > t \) and passing to the limit \( \varepsilon \to 0 \) in inequality (20) for \( \tau = t \) yields (11). The theorem is proved.

5. Example

As an illustration, we give an example of a system of ODEs that arises naturally in the theory of stochastic processes and satisfies all conditions of Theorem 1. We consider a so-called affine process evolving on the state space \( C := \mathbb{R}_0^d \) (see [1]). Such a process \( X = (X_t)_{t \geq 0} \), can be regarded as a multi-type extension of the singe-type continuously branching process of [4], which arises as a continuous-time limit of a classical Galton-Watson branching process. \( X \) is defined as a stochastically continuous, time-homogeneous Markov process starting at \( X_0 \in C \), with the property that the moment generating function is of the form

\[
E \left[ e^{x \cdot X_t} \right] = e^{\psi(t, x) \cdot X_0}
\]

for all \((t, x) \in \mathbb{R}_0^t \times \mathbb{R}^d\), and where \( \psi : \mathbb{R}_{\geq 0} \times \mathbb{R}^d \to \mathbb{R}^d \cup \{ \infty \} \). We assume that the time-derivative of \( \psi(t, x) \) at \( t = 0 \),

\[
f(x) := \left. \frac{\partial}{\partial t} \psi(t, x) \right|_{t=0}
\]

exists and is a continuous function on the set \( U = \{ x \in \mathbb{R}^d : f(x) < \infty \} \). In this case the map \( \psi(t, x) \) satisfies the following differential equation:

\[
\frac{\partial}{\partial t} \psi(t, x) = f(\psi(t, x)), \quad \psi(0, x) = x.
\]

\footnote{We set \( \psi(t, x) = \infty \), whenever the left side of (22) is infinite. Note that for \((t, x) \in \mathbb{R}_{\geq 0} \times (-\infty, 0)^d\) it is always guaranteed that \( \psi(t, x) \) is finite.}
Moreover, the components of the map \( f(x) \) are of so-called Levy-Khintchine type (cf. [8] Theorem 8.1):

\[
f_i(x) = \frac{\alpha_i}{2} x_i^2 + x \cdot \beta_i - c_i + \int_{C \setminus \{0\}} (e^{x \cdot \xi} - 1 - x \cdot \xi) \mu_i(d\xi),
\]

with \( I \), the indicator function, where, for all \( i \in \{1, \ldots, d\} \),

- \( \alpha_i \in \mathbb{R}_{\geq 0} \);
- \( \beta_i \in \mathbb{R}^d \) with \( \beta_i^k - \int_{|\xi| \leq 1} \xi_k \mu_i(d\xi) \geq 0 \) for all \( k \neq i \);
- \( c_i \in \mathbb{R}_{\geq 0} \);
- \( \mu_i(d\xi) \) are Borel measures on \( C \setminus \{0\} \) assigning finite mass to the set \( \{ \xi \in C : |\xi| > 1 \} \) and satisfying the integrability condition

\[
\int_{\xi \in C, 0 < |\xi| \leq 1} \left( \sum_{k \neq i} |\xi_k| + |\xi_i|^2 \right) \mu_i(d\xi) < \infty
\]

on its complement.

The above conditions are both necessary and sufficient for the existence of \( X \) and referred to as admissibility conditions (see [1]).

In the following we consider the ordering on \( \mathbb{R}^d \) induced by the cone \( \mathbb{R}_{\geq 0}^d \).

**Proposition 10.** The domain \( U \) is convex and order-regular and the map \( f(x) \) is convex and quasi-monotone increasing thereon.

**Proof.** We make use of the following representations of \( f_i(x) \):

\[
f_i(x) = \log \int_{\mathbb{R}^d} e^{x \cdot \xi} p_i(d\xi) = f_i^1(x) + \int_{C \setminus \{0\}, |\xi| > 1} e^{x \cdot \xi} - 1 \mu_i(d\xi),
\]

where \( p_i(d\xi) \) is an infinitely divisible, substochastic measure on \( \mathbb{R}^d \), and \( f_i^1(x) \) is a function on \( \mathbb{R}^d \) that can be extended to an entire function on \( \mathbb{C}^d \). The representation as \( \log \int_{\mathbb{R}^d} e^{x \cdot \xi} p_i(d\xi) \) is an immediate consequence of the Levy-Khintchine formula, and its analytic extension to exponential moments [8] Theorem 8.1, Theorem 25.17]. The second representation of \( f_i(x) \) follows directly from [8, Lemma 25.6]. To show that \( f_i(x) \) is convex, apply Hölder’s inequality:

\[
f_i(\lambda x + (1 - \lambda) y) = \log \int_{\mathbb{R}^d} e^{\lambda x \cdot \xi} e^{(1 - \lambda) y \cdot \xi} p_i(d\xi) \leq \lambda \log \int_{\mathbb{R}^d} e^{x \cdot \xi} p_i(d\xi) + (1 - \lambda) \log \int_{\mathbb{R}^d} e^{y \cdot \xi} p_i(d\xi) = \lambda f_i(x) + (1 - \lambda) f_i(y)
\]

for all \( x, y \in \mathbb{R}^d \) and \( \lambda \in (0, 1) \). We show next that the domain \( U \) is order-regular. Assume that \( x \in U \), i.e. \( f_i(x) < \infty \) for all \( i \), and let \( y \leq x \). Using the second representation in [24] it is clear that \( f_i^1(y) < \infty \). But also the integral with respect to \( \mu_i(d\xi) \) is finite, since the integrand is dominated by \( (e^{x \cdot \xi} - 1) \mathbf{1}_{|\xi| \geq 1} \), whose integral is finite by assumption. We conclude that \( f_i(y) = \infty \), and thus that \( y \in U \), i.e., \( U \) is order-regular. Finally we show that \( f(x) \) is also quasi-monotone increasing.
Assume that $y \leq x$ with $y_i = x_i$ for some $i \in \{1, \ldots, d\}$. It follows that

$$f_i(x) - f_i(y) = \sum_{k \neq i} (x_k - y_k) \cdot \left( \beta_k^i - \int_{\xi \in C, 0 < |\xi| \leq 1} \xi_k \mu_\lambda(d\xi) \right) + \int_C \left( e^{x \cdot \xi} - e^{y \cdot \xi} \right) \mu_\lambda(d\xi) \geq 0,$$

where we have made use of the admissibility conditions given above.

\[ \square \]

**APPENDIX A**

In this section we give a very simple proof of the convexity result [4] for ODEs in ordered normed spaces. Let $E$ be a real normed space (not necessarily finite-dimensional) ordered by a proper closed cone $C$. As shown in [11], Lemma 3 holds for $E$ if one of the following conditions is satisfied:

1. $C$ has a non-empty interior,
2. $E$ is complete,
3. $C$ is a distance set (i.e., for every $x \in E$, there is $y \in C$ such that $\|x - y\|$ is equal to the distance from $x$ to $C$).

As above, let $T \in (0, \infty]$ and $I = [0, T)$. Theorem 1 in [4] follows immediately from the next result.

**Theorem 11.** Let $E$ be an ordered normed space such that one of the above conditions is satisfied. Let $U \subseteq E$ be an open convex set and $f: I \times U \to E$ be a continuous locally Lipschitz map such that $f(t, \cdot)$ is quasi-monotone increasing and convex on $U$ for all $t \in I$. Let $0 < t_0 \leq T$ and $x_1, x_2, x_3: [0, t_0) \to U$ be differentiable maps such that

$$\dot{x}_i(t) = f(t, x_i(t)), \quad i = 1, 2, 3,$$

and $x_3(0) = \lambda x_1(0) + (1 - \lambda)x_2(0)$ for some $\lambda \in [0, 1]$. Then $x_3(t) \leq \lambda x_1(t) + (1 - \lambda)x_2(t)$ for all $t < t_0$.

**Proof.** Set $z(t) = \lambda x_1(t) + (1 - \lambda)x_2(t)$ for $t < t_0$. By the convexity of $f$,

$$\dot{z}(t) - f(t, z(t)) = \lambda \dot{x}_1(t) + (1 - \lambda)\dot{x}_2(t) - f(t, \lambda x_1(t) + (1 - \lambda)x_2(t)) \geq$$

$$\geq \lambda (\dot{x}_1(t) - f(t, x_1(t))) + (1 - \lambda)(\dot{x}_2(t) - f(t, x_2(t))) = 0 = \dot{x}_3(t) - f(t, x_3(t))$$

for all $t < t_0$. Since $z(0) = x_3(0)$, the above-mentioned analogue of Lemma 3 for normed spaces implies that $z(t) \geq x_3(t)$. The theorem is proved. \[ \square \]

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