Some classical and quantum aspects of cosmological braneworlds with a time-dependent extra dimension

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Abstract. We discuss certain classical and quantum features of warped braneworld models with a dynamic extra dimension (analytically, wherever possible, otherwise numerically). We are able to demonstrate, that it is indeed possible to construct such generalised braneworld models with different bulk matter sources satisfying the weak energy conditions. The effect of the metric functions - the cosmological scale factor, scale of the extra dimension and the warp factor on geodesic motion are studied in detail. Distinctive features of the geodesics in different braneworld scenarios are thus highlighted. Also the issue of geodesic confinement is addressed. Particle creation, which is essentially a quantum consequence of the dynamic nature of spacetime, is investigated and how the nature of particle production on the brane depends on the nature of warping, type of cosmological evolution as well as the temporal evolution of the extra dimension is illustrated. We, in fact, have found a class of models for which the back reaction due to particle production can be controlled.

1. Introduction
Beginning with Kaluza-Klein [1], a large variety of models with extra dimensions have been proposed over the years. In the recently proposed warped braneworld models [2, 5] our world is viewed as a four dimensional sub-manifold (3-brane) embedded in a curved five dimensional spacetime where the line element on the 3-brane is scaled by a warp factor. The braneworld models seem to provide a viable resolution of the long-standing hierarchy problem in high energy physics and also suggest a dynamic way of compactification [6]. In this article, we briefly discuss few classical and quantum aspects of a so called generalisation of such braneworld scenarios. In Sec. 1 we will investigate whether those generalised braneworld models are compatible with the framework of general relativity or not [7]. In Sec. 2, we analyse the geodesic motion in such universes and address the issue of confinement of test particles [8]. In Sec. 3, we briefly discuss particle creation phenomenon and effects of warped extra dimension on the same [9].

2. Bulk spacetime with a cosmological brane
We have essentially considered the following line element as a generalisation to originally proposed Randall-Sundrum (RS) Braneworld model [3, 4]

\[ ds^2 = e^{2f(\sigma)} \left[ -dt^2 + a^2(t) \left( dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \right) \right] + b^2(t) d\sigma^2 \]  

(1)

where, we have chosen a cosmological line element on the \( \sigma = \text{constant} \) hypersurface. \( a(t) \) is the cosmological scale factor, \( e^{2f(\sigma)} \) is the warp factor and we now have a time-dependent function.
b(t) associated with the extra dimension. In general, such a function, when it is dependent on all the four coordinates \((t, r, \theta, \phi)\), is known as the radion field. It has been demonstrated that for typical choices for the various metric functions, such as a growing \(a(t) = \sqrt{t}\) (radiative) or \(e^t\) (de-Sitter), a stabilisable \(b(t) = c_1 + c_2 e^{-t}\) and \(f(\sigma) = \pm \log(\cosh \sigma)\) which represents a thick brane located at \(\sigma = 0\), the above geometry does satisfy the Weak Energy Condition for specific domains of the parameters involved. This suggests the viability of these choices for physical models.

Next we look for exact solutions of Einstein equations, which gives rise to the type of geometry given by Eq.1, with matter sources of various kinds. To begin with, we look at possible matter sources that may arise if we assume an exponential (in \(\sigma\)) warp factor and some other typical constraints on matter stress energy. The solution space for \([a(t), b(t)]\) is analysed using a dynamical systems approach. Subsequently, we look at solutions with various types of scalar fields as sources eg. bulk normal scalar, the Brans-Dicke scalar and the dilaton scalar. In our approach to obtaining solutions we exploit the known fact that with a conformally related metric and a redefined scalar one can convert BD theory into canonical Einstein Gravity coupled to a scalar. The following table shows some of the typical analytic solutions found.

**Table 1.** The above table shows few of the typical solutions found and the status of the energy conditions through the energy inequality functions \(F_1\), \(F_2\) and \(F_3\) (see [7] for details).

| Bulk Field          | \(f(\sigma) = \text{const.} + \) | \(a(t) \sim\) | \(b(t) \sim\) | \(F_1 \geq 0\) | \(F_2 \geq 0\) | \(F_3 \geq 0\) |
|---------------------|-----------------------------------|----------------|----------------|----------------|----------------|----------------|
| Ordinary Scalar     | 0.25 log(\(\sigma - \sigma_0\))  | \((t - t_0)^{0.48}\) | \((t - t_0)^{-0.43}\) | \(\checkmark\) | \(\times\)   | \(\checkmark\) |
| Brans-Dicke         | 0.37 log(\(\sigma - \sigma_0\))  | \((t - t_0)^{0.43}\) | \((t - t_0)^{-0.50}\) | \(\checkmark\) | \(\times\)   | \(\checkmark\) |
| (for \(\omega = 1\)) | 0.075 log(\(\sigma - \sigma_0\)) | \((t - t_0)^{0.415}\) | \((t - t_0)^{-0.32}\) | \(\checkmark\) | \(\checkmark\) | \(\checkmark\) |
| Dilaton Scalar (\(\omega = -1\)) | 0.5 log(\(\sigma - \sigma_0\)) | \((t - t_0)^{0.53}\) | \((t - t_0)^{-0.82}\) | \(\times\) | \(\times\) | \(\checkmark\) |
|                     | \(-0.5 log(\(\sigma - \sigma_0\)) | \((t - t_0)^{0.51}\) | \((t - t_0)^{-0.18}\) | \(\checkmark\) | \(\checkmark\) | \(\checkmark\) |

In the case with a Brans-Dicke scalar and matter in the bulk, it is observed that, for traceless matter energy momentum, exact and viable analytical solutions can be found with decaying or growing warp factors as well as decelerating/accelerating \(a(t)\). We note that in the case of the dilaton we can have decaying as well as growing warp factor solutions too. We are also able to find desirable solutions with growing \(a(t)\) and decaying \(b(t)\) in several of our examples. We check the energy conditions, the nature of the energy inequality functions \(F_1\), \(F_2\) and \(F_3\) and figure out whether an obtained solution is desirable or not. The details are tabulated in one of the columns in the table commenting on the desirability of the solution. We do find several classes of desirable solutions, which satisfy our requirements and the energy conditions. The bulk singularities that appear in our solutions can be resolved by placing thin branes, where the singularities occur, using proper junction conditions. One can say that those bulk singularities are in fact providing us with places for branes, one of which can be chosen as the Standard Model brane we live in.

With three different functions appearing in the line element, it is always very difficult to find exact solutions. That we have found some is indeed encouraging. We hope to use these solutions in actual brane cosmological scenarios and arrive at relevant conclusions by making use of existing observational data on supernova and CMB anisotropies. We would also like to extend our results to other types of stress energy expressions. A disturbing aspect of our solutions is that they are singular in the bulk, though we do provide ways of resolving them. It remains to be seen whether we can find newer solutions (a) which are not necessarily of a power law type, (b) which have, in appropriate limits, both deceleration and acceleration, (c) which have
a decaying extra dimension stabilisable to a finite value and (d) where the bulk warp factor gives rise to a non-singular bulk metric, similar to RS. In essence, an appropriate combination of different bulk fields, with some dominating over the others in specific time intervals, will then be able to generate the expansion history (with proper decelerating and accelerating phase), of our universe on the brane.

3. Geodesics and confinement of test particles

To investigate the effects of warp factor and the dynamic nature of the bulk on the motion of test particles, we study the geodesics in the geometry given by metric (1). We have assumed the metric functions as earlier. The geodesic equations in such a background can be rewritten as a first order autonomous dynamical system. Analytical insights as well as some specific solutions can therefore be obtained. In particular, the analysis of these equations as a dynamical system shows the role of the warp factor (growing/decaying) in controlling the nature of the solutions (oscillatory/exponential). Apart from a few specific cases, the geodesic equations, in general, can only be solved numerically since they are highly coupled differential equations. However, we are able to demonstrate some qualitative features of the dynamical system through numerical investigations. It is found that, a decaying warp factor results into runaway trajectories (for both timelike and null geodesics) but in presence of a growing warp factor massive particles are constrained to be confined near the brane, while massless particles like photons or gravitons can indeed access the bulk. This observation is important in the context of localisation of fields on the brane. The behaviour of localisation of test particles in a geometry with a growing warp factor is reminiscent of fermion localisation with a growing warp factor. Nature of the effective geodesic potential also clearly indicates existence of bound or confined trajectories. Even in the presence of a decaying warp factor, particles can also be localised, though very special initial conditions are required. The effect of a dynamical (time dependent) extra dimension on the trajectories is found to be largely quantitative. The issue of confinement also puts important constraints on possible dynamic behaviour of the extra dimension.

We mention that it would be useful to investigate geodesic deviation and the kinematics of geodesic flows in a bulk geometry with a thick brane. In the latter case, we need to solve the geodesic and Raychaudhuri equations simultaneously, in order to obtain the evolution and initial condition dependencies of the kinematical quantities: expansion, rotation and shear (or ESR). We believe that such work may improve our understanding about the nature of warped extra dimensions. We hope to report on related work on the above-mentioned aspects, at a later stage.

4. Particle creation in presence of a dynamic extra dimension

Particle creation can be viewed as quantum consequence of existence of a dynamic extra dimension. We have investigated the possible effects of the presence of a warped extra dimension (static or dynamic) on particle production in a 4d universe. Following the standard convention we used the conformal time \( \eta = \int \frac{dt}{a(t)} \) for this study. From the numerical analysis, it is easy to point out the main characteristic features. The warp factor plays its part through the values of the allowed modes arising because of the warped extra dimension. These modes will have discrete values in the case of a two brane model, otherwise it will be a continuum. The location and heights of the peaks in the plots for the particle number density and energy density (as a function of 4D modes) decreases in the presence of an extra dimension. This is a feature of the breaking of conformal invariance. Here we took \( a^2(\eta) = b_1^2 + b_2 \eta^2 \) (which asymptotically reaches a radiative universe) and for the scenario with a time-varying extra dimension, we took \( b^2(\eta) = \frac{b_1^2 + b_2^2 \eta^2}{b_3 + b_2 \eta^2 / b_1^2} \). We note an increase in the heights of the peaks for \( b(\eta, b_3 < 1) \) and a decrease for \( b(\eta, b_3 > 1) \). Though these features look to be case specific and model dependent, it is worth noting that we did not choose \( b(\eta) \) arbitrarily. It is chosen to be equal to \( b_1 \) (or
asymptotically reaching the same value in the dynamic cases) which is the minimum value of \(a(\eta)\) during its evolution. In effect we are assuming some correlation between \(a(\eta)\) and \(b(\eta)\), i.e. there may be a single mechanism which drives both the scale factors and therefore their ought to be a connection between them. We have also found a mechanism (for \(b(\eta, b_3 > 1)\)) to have control over contributions of higher KK modes to the energy density of the universe. We can suppress the production of those higher modes at our will by choosing higher and higher value of \(b_3\). The back reaction thus remains negligible and the classical cosmological model which is our background for all these studies does not break down. To get an idea about the thermal/nonthermal nature of the spectrum of created particles we have tried to fit our numerical data with a thermal profile with a new set of parameters. Our analysis reveals that the spectrum could be assumed to be nearly thermal though marked deviations seem to appear for larger values of 4D modes. As a next step, one should study particle creation in the context of more realistic cosmological models and also further investigate the possibility of creation of other types of particles (like fermions, or spin one particles etc.) on the brane.

We admit, in conclusion, that our results are based on a toy scenario and can only be thought of as a build-up towards the study of more relevant and realistic situations in future. However, we do believe that some of the features observed (eg. possible avoidance of significant backreaction effects and non-thermal nature of the spectral profile) here will indeed be carried over in generically similar situations and can provide pointers towards a better understanding of quantum fields in the presence of a warped extra dimension.

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