Solution of boundary problems of structural mechanics with the combined application use of Discrete-Continual Finite Element Method and Finite Element Method

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Abstract. In most structural problems the object is usually to find the distribution of stress in an elastic body produced by an external loading system. The theory of elasticity is a methodology that creates a linear relation between the imposing force (stress) and resulting deformation (strain), for the majority of materials, which behave fully or partially elastically\textsuperscript{[1–11]}. Besides, all governing equations in this theory are linear partial differential ones. The distinctive paper is devoted to

1. Introduction

The object of most structural problems is usually to find the distribution of stress in an elastic body produced by an external loading system. As it is known, the theory of elasticity is a methodology that creates a linear relation between the imposing force (stress) and resulting deformation (strain), for the majority of materials, which behave fully or partially elastically\textsuperscript{[1–11]}. Besides, all governing equations in this theory are linear partial differential ones. The distinctive paper is devoted to
combined semianalytical and numerical static analysis of three-dimensional structures. The stress-strain or constitutive behavior is given for isotropic materials subjected to small deformations. Solution of multipoint (particularly, two-point) boundary problem of three-dimensional theory of elasticity based on combined application of finite element method (FEM) [12–15] and discrete-continual finite element method (DCFEM) [13-16] which are under consideration. The given domain, occupied by considering structure, is embroidered by extended one within so-called method of extended domain [17]. The field of DCFEM application comprises fragments of structure (subdomains) with regular (constant or piecewise constant) physical and geometrical parameters in some dimension ("basic" dimension) [24]. DCFEM presupposes finite element mesh approximation for non-basic dimensions of extended domain while dimension problem remains in the basic one, continual (corresponding correct analytical solution is constructed). FEM is used for approximation of all other subdomains [12]. Discrete (within FEM) and discrete-continual (within DCFEM) approximation models for subdomains and coupled multilevel approximation model for extended domain are constructed. Brief information about software systems and verification samples are presented as well.

2. Formulation of the problem and approximation
Let us consider multipoint (particularly, two-point) boundary problem of elasticity three-dimensional theory, particularly static analysis of three-dimensional structure. Several elements of corresponding notation system for corresponding two-point boundary problem are presented at Figures 1, 2.

![Figure 1. Static analysis of three-dimensional structure (sample).](image-url)
Figure 2. Cross-sections of considering three-dimensional structure: (a) 1–1; (b) 2–2.

Let’s $\Omega$ be domain occupied by structure, $\Omega = \{(x_1, x_2) : (x_1, x_2) \in S_k, \ 0 < x_3 < l_1 \}$,

$$\Omega = \Omega_1 \cup \Omega_2, \quad \Omega_k = \{(x_1, x_2, x_3) : 0 < x_1 < l_1, \ 0 < x_2 < l_2, \ x_{3,k}^b < x_3 < x_{3,k+1}^b\}, \ k = 1, 2 \tag{1}$$

where $x_1, x_2, x_3$ are coordinates ($x_3$ corresponds to basic dimension); $x_{3,k}^b, \ k = 1, 2, ..., n_b$ are coordinates of corresponding boundary points (cross-sections) along basic dimension for multipoint boundary problems

$$n_b > 2 \ (x_{3,1}^b = 0, \ x_{3,2}^b = l_3, \ x_{3,3}^b = l_3 + l_3 = l_3) ; \ \Omega_k, \ k = 1, 2 \tag{2}$$

are subdomains of $\Omega$; $S_k$ is the cross-section of domain $\Omega_k$ perpendicular to basic dimension. In accordance with the method of extended domain, proposed by A.B. Zolotov [24], the given domain isembordered by extended one of arbitrary shape, particularly elementary. Let $\omega_k, \ k = 1, 2$ are extended subdomains, embordering - subdomains $\Omega_k \subset \omega_k, \ k = 1, 2; \ \omega = \omega_1 \cup \omega_2$.

We suppose piecewise constancy of physical and geometrical parameters of one subdomains group from (2) along coordinate $x_3$ (“basic” dimension) without loss of generality. Physical and geometrical parameters of structure can be changed arbitrarily along $x_1, x_2$. It is recommended to use DCFEM for approximation of these subdomains (discrete-continual design model is introduced). Let physical and
geometrical parameters within other group of subdomains from (2) are arbitrary varying. FEM [18, 19] can be used for approximation. Combined application of DCFEM and FEM is advisable.

Operational formulation of elasticity three-dimensional problem is given at [18].

We can introduce the following notations for considering two-point boundary problem (Figures 1, 2): \( x_{1,i,j} = x_{1,i}^d, x_{2,i,j} = x_{2,j}^d, i = 1, 2, ..., N_1^d, j = 1, 2, ..., N_2^d \) are coordinates (along \( x_1 \) and \( x_2 \)) of nodes (nodal lines) of discrete-continual finite elements, which are used for approximation of domain \( \omega_1; (N_1^d - 1) \) and \( (N_2^d - 1) \) are the numbers of discrete-continual finite elements along coordinates \( x_1 \) and \( x_2 \); \( x_{1,i,j,s}^{fe} = x_{1,i}^{fe}, x_{2,i,j,s}^{fe} = x_{2,j}^{fe}, x_{3,i,j,s}^{fe} = x_{3,r}^{fe}, i = 1, 2, ..., N_1^{fe}, j = 1, 2, ..., N_2^{fe}, r = 1, 2, ..., N_3^{fe} \) are coordinates (along \( x_1, x_2 \) and \( x_3 \)) of nodes of finite elements, which are used for approximation of domain \( \omega_2; (N_1^{fe} - 1), (N_2^{fe} - 1) \) and \( (N_3^{fe} - 1) \) is the number of finite elements along coordinates \( x_1, x_2 \) and \( x_3 \). Besides, we have: \( x_{q,i,j,r}^{fe} = x_{q,i,j,s}^{fe}, i = 1, 2, ..., N_1, j = 1, 2, ..., N_2, r = 1, 2, ..., N_3^{fe}, q = 1, 2, 3, x_{1,i} = x_{1,i,j}, x_{2,j} = x_{2,j,i}, i = 1, 2, ..., N_1, j = 1, 2, ..., N_2 \).

### 3. Approximation models for subdomains and domain

Discrete-continual approximation model within DCFEM presupposes mesh approximation for non-basic dimensions of extended domain (along \( x_1, x_2 \)) while (along \( x_3 \)) problem remains continual in the basic dimension. Thus extended subdomain \( \omega_1 \) is divided into discrete-continual finite elements

\[
\omega_1 = \bigcup_{i=1}^{N_1-1} \bigcup_{j=1}^{N_2-1} \omega_{i,j,i,j}; \quad \omega_{i,j,i,j} = \{ (x_1, x_2, x_3) : x_{1,i} < x_1 < x_{1,i+1}, x_{2,j} < x_2 < x_{2,j+1}, x_{3,i}^b < x_3 < x_{3,i}^e \}. \quad (3)
\]

Lame constants and characteristic function for finite element are defined by formulas:

\[
\lambda_{i,j,i} = \theta_{i,j,i} \lambda; \quad \mu_{i,j,i} = \theta_{i,j,i} \mu, \quad \text{where} \quad \theta_{i,j,i} = \begin{cases} 1, & \omega_{i,j,i} \subset \Omega_1, \\ 0, & \omega_{i,j,i} \subset \Omega_2. \end{cases} \quad (4)
\]

Basic nodal unknown functions are displacement components \( u_1^{(i,j)}, u_2^{(i,j)}, u_3^{(i,j)} \) and their derivatives \( v_1^{(i,j)}, v_2^{(i,j)}, v_3^{(i,j)} \) with respect to \( x_3 \) (superscript hereinafter corresponds to the number of considered subdomain i.e. \( \omega_1 \)). Thus for node \((1,i,j)\) we have the following unknown functions: \( u_1^{(1,i,j)}, u_2^{(1,i,j)}, u_3^{(1,i,j)} \) and \( v_1^{(1,i,j)}, v_2^{(1,i,j)}, v_3^{(1,i,j)} \). Bilinear approximation is used for unknown functions within discrete-continual finite element.

DCFEM is reduced at some stage to the solution of systems of \( 6N_1N_2 \) first-order ordinary differential equations (subscript corresponds to the number of subdomain \( \omega_1 \)):

\[
\overline{U}_1(x_3) = A_1 \overline{U}_1(x_3) + \overline{R}_1(x_3), \quad (5)
\]

where \( A_1 \) is global matrix of coefficients of order \( 6N_1N_2 \); \( \overline{R}_1(x_3) \) is the right-side vector of order \( 6N_1N_2 \); \( \overline{U}_1(x_3) \) is global vector of nodal unknown functions,

\[
\overline{U}_1 = \overline{U}_1(x_3) = [ (\overline{u}_1^{(1,1)}(x_3)), (\overline{u}_1^{(1,2)}(x_3)), ..., (\overline{u}_1^{(1,N_1)}(x_3)), (\overline{u}_1^{(2,1)}(x_3)), ..., (\overline{u}_1^{(2,N_2)}(x_3)), (\overline{u}_1^{(3,1)}(x_3)), ..., (\overline{u}_1^{(3,N_3)}(x_3)), (\overline{u}_1^{(4,1)}(x_3)), ..., (\overline{u}_1^{(4,N_4)}(x_3)), ..., (\overline{u}_1^{(6,1)}(x_3)), ..., (\overline{u}_1^{(6,N_6)}(x_3)) ]^T. \quad (6)
\]

\[
\overline{u}_1 = \overline{u}_1(x_3) = [ (\overline{u}_n^{(1,1,1)}), (\overline{u}_n^{(1,1,2)}), ..., (\overline{u}_n^{(1,1,N_1)}), (\overline{u}_n^{(1,2,1)}), ..., (\overline{u}_n^{(1,2,N_2)}), (\overline{u}_n^{(1,3,1)}), ..., (\overline{u}_n^{(1,3,N_3)}), (\overline{u}_n^{(1,4,1)}), ..., (\overline{u}_n^{(1,4,N_4)}), ..., (\overline{u}_n^{(1,6,1)}), ..., (\overline{u}_n^{(1,6,N_6)})) ]^T. \quad (7)
\]
\[ \bar{v}_1 = \bar{v}_1(x_3) = \left[ \begin{array}{l} (\bar{v}_n^{1,1,1})^T \\ \cdots \\ (\bar{v}_n^{1,N_r,1})^T \end{array} \right] \cdots \left[ \begin{array}{l} (\bar{v}_n^{1,1,1})^T \\ \cdots \\ (\bar{v}_n^{1,N_r,1})^T \end{array} \right] \cdots \left[ \begin{array}{l} (\bar{v}_n^{1,1,1})^T \\ \cdots \\ (\bar{v}_n^{1,N_r,1})^T \end{array} \right] \cdots \left[ \begin{array}{l} (\bar{v}_n^{1,1,1})^T \\ \cdots \\ (\bar{v}_n^{1,N_r,1})^T \end{array} \right]^T; \]

\[ \bar{u}^{(i,j)}_n = \bar{u}^{(i,j)}_n(x_1) = \left[ u_1^{(i,j)} \right]^T; \quad \bar{v}^{(i,j)}_n = \bar{v}^{(i,j)}_n(x_3) = \left[ v_1^{(i,j)} \ v_2^{(i,j)} \ v_3^{(i,j)} \right]^T. \]  

Correct analytical solution of (5) is defined by formula

\[ \bar{U}_1(x_3) = E_1(x_3)\bar{C}_1 + \bar{S}_1(x_3), \]

where \( \bar{C}_1 \) is the vector of constants of order \( 6N_1N_2 \),

\[ E_1(x_3) = \varepsilon_1(x_3, x_{3,1}^b) - \varepsilon_1(x_3, x_{3,2}^b), \quad \bar{S}_1(x_3) = \varepsilon_1(x_3) \ast \bar{R}_1(x_3), \]

\( \varepsilon_1(x_3) \) is the fundamental matrix-function of system (5), which is constructed in the special form convenient for problems of structural mechanics [26]; * is convolution notation.

Discrete approximation model within FEM presupposes finite element approximation along \( x_1, x_2 \) and \( x_3 \). Thus extended subdomain \( \omega_2 \) is divided into finite elements

\[ \omega_2 = \bigcup_{j=1}^{N_1-1} \bigcup_{k=1}^{N_2-1} \bigcup_{l=1}^{N_3-1} \omega_{2,i,j,r}; \]

\[ \omega_{2,i,j,r} = \left\{ (x_1, x_2, x_3) : \begin{array}{l} x_{1,1} < x_1 < x_{1,i+1}, \ x_{2,j} < x_2 < x_{2,j+1}, \ x_{3,r} < x_3 < x_{3,r+1} \end{array} \right\}. \]

Lame constants for finite element are defined by formulas:

\[ \lambda_{2,i,j,r} = \theta_{2,i,j,r} \lambda; \quad \mu_{2,i,j,r} = \theta_{2,i,j,r} \mu, \quad \text{where} \quad \theta_{2,i,j,r} = \left\{ \begin{array}{l} 1, \quad \omega_{2,i,j,r} \subset \Omega_2; \\
0, \quad \omega_{2,i,j,r} \subset \Omega_2, \end{array} \right. \]

where \( \theta_{2,i,j,r} \) is the characteristic function of element \( \omega_{2,i,j,r} \).

Basic nodal unknowns are displacement components \( u_1^{(i)}, u_2^{(i)}, u_3^{(i)} \) (superscript hereinafter corresponds to the number of considered subdomain i.e. \( \omega_2 \)). Thus for node \( (2, i, j, r) \) we have the following unknowns: \( u_1^{(2,i,j,r)}, u_2^{(2,i,j,r)}, u_3^{(2,i,j,r)} \). Bilinear approximation of unknowns is used within finite element (conventional three-dimensional parallelepipedic 8-node finite element of three-dimensional problem of elasticity theory [18, 19].

As known, FEM is reduced to the solution of systems of \( 3N_1N_2N_3 \) linear algebraic equations:

\[ K_2 \bar{U}_2 = \bar{R}_2, \]

where \( K_2 \) is global stiffness matrix of order \( 3N_1N_2N_3 \); \( \bar{R}_2 \) is global load vector of order \( 3N_1N_2N_3 \); \( \bar{U}_2 \) is global vector of nodal unknowns,

\[ \{\bar{U}_2\}_{ij} = u^{(i,j,r)}_q; \]

\( i \) is the global index of element of vector \( \bar{U}_2 \); \( k, i, j, r, q \) are corresponding local indexes,

\[ r = \left\lfloor \frac{i_g}{3N_1N_2} \right\rfloor + 1; \quad j = \left\lfloor \frac{i_g - 3(r-1)N_1N_2}{3N_1} \right\rfloor + 1; \quad i = \left\lfloor \frac{i_g - 3(r-1)N_1N_2 - 3(j-1)N_1}{3} \right\rfloor + 1; \]
\[ k = 2; \quad q = i_g - 3(r - 1)N_1N_2 - 3(j - 1)N_1 - 3i_g. \]  

(18)

System (15) can be rewritten for all nodes with indexes \( 1 < r < N_3 \) (i.e. \( x_{3,2} < x < x_{3,3} \)) in the following form (resolving system of \( 2N_1(N_2 - 2) \) linear algebraic equations):

\[ \tilde{K}_2 \tilde{U}_2 = \tilde{R}_2, \]

(19)

where \( \tilde{K}_2 \) is reduced global stiffness matrix of size \([3N_1N_2(N_3 - 2)] \times [3N_1N_2N_3] \); \( \tilde{R}_2 \) is reduced global load vector of order \( 3N_1N_2(N_3 - 2) \).

Boundary conditions at section \( x_3 = x_{3,1}^b \) (hinged edge, Figure 1) has the form \( (3N_1N_2) \) equations:

\[ u_1^{(1,i,j)}(x_{3,1}^b) = 0, \quad u_2^{(1,i,j)}(x_{3,1}^b) = 0, \quad u_3^{(1,i,j)}(x_{3,1}^b) = 0, \quad i = 1, 2, \ldots, N_1; \quad j = 1, 2, \ldots, N_2. \]

(20)

Equations (20) can be rewritten in matrix form:

\[ B_1^i \tilde{U}_1(x_{3,1}^b) = \tilde{g}_1; \quad \{ B_1^i \}_{p,q} = \delta_{p,q}; \quad p = 1, 2, \ldots, 3N_1N_2; \quad q = 1, 2, \ldots, 6N_1N_2; \quad \tilde{g}_1 = 0. \]

(21)

where \( B_1^i \) is matrix (with elements \( \{ B_1^i \}_{p,q} \) ) of boundary conditions of size \( 3N_1N_2 \times 6N_1N_2 \); \( \delta_{p,q} \) is Kronecker delta.

After substitution of (10) into (21) it can be obtained that

\[ Q_1 \tilde{C}_1 = \tilde{G}_1; \quad Q_1 = B_1^i E_1(x_{3,1}^b) + 0; \quad \tilde{G}_1 = \tilde{g}_1 - B_1^i \tilde{S}_1(x_{3,1}^b) + 0, \]

(22)

where \( Q_1 \) is the matrix of size \( 3N_1N_2 \times 6N_1N_2 \); \( \tilde{G}_1 \) is the vector of order \( 3N_1N_2 \).

Boundary conditions at section \( x_3 = x_{3,2}^b \) (perfect contact) has the form \( (6N_1N_2) \) equations:

\[ u_q^{(1,i,j)}(x_{3,2}^b) = u_q^{(2,i,j,r)}, \quad i = 1, 2, \ldots, N_1; \quad j = 1, 2, \ldots, N_2; \quad r = 1; \quad q = 1, 2, 3; \]

(23)

\[ \sigma_{q,3}^{(1,i,j)}(x_{3,2}^b) = \sigma_{q,3}^{(2,i,j,r)}, \quad q = 1, 2, 3; \quad i = 1, 2, \ldots, N_1; \quad j = 1, 2, \ldots, N_2; \quad r = 1; \]

(24)

where \( \sigma_{q,3}^{(1,i,j)}(x_3) \), \( \sigma_{1,3}^{(2,i,j,r)} \) and \( \sigma_{3,3}^{(1,i,j)}(x_3) \) are nodal functions (after corresponding averaging) of stress components \( \sigma_{1,3}(x_3) \), \( \sigma_{2,3}(x_3) \) and \( \sigma_{3,3}(x_3) \) for discrete-continual finite element \( (1, i, j) \); \( \sigma_{1,3}^{(2,i,j,r)} \), \( \sigma_{2,3}^{(2,i,j,r)} \) and \( \sigma_{3,3}^{(2,i,j,r)} \) are nodal stress components \( \sigma_{1,3}, \sigma_{2,3} \) and \( \sigma_{3,3} \) (after corresponding averaging) for finite element \( (2, i, j, r) \); \( r = 1 \).

Equations (24) and (25) can be rewritten in matrix form:

\[ B_2^r \tilde{U}_1(x_{3,2}^b) = 0 = B_2^r \tilde{U}_2, \]

(25)

where \( B_2^r \) is matrix of boundary conditions of size \( 6N_1N_2 \times 6N_1N_2 \), which can be constructed in accordance with so-called method of basis variations [26,27]; \( B_2^r \) is matrix of size boundary conditions \( 6N_1N_2 \times 3N_1N_2N_3 \), which can be constructed in accordance with method of basis variations.

After substitution of (10) into (26) it can be obtained that

\[ Q_2^r \tilde{C}_1 + Q_2^r \tilde{U}_2 = \tilde{G}_2, \]

(26)
where $Q_{2,1}$ and $Q_{2,2}$ are matrices with sizes $6N_1N_2 \times 6N_1N_2$ and $6N_1N_2 \times 3N_1N_2N_3$; $G_2$ is the vector of order $6N_1N_2$,

$$Q_{2,1} = B_2 E_1(x^b_{3,2} - 0); \quad Q_{2,2} = -B_2^*; \quad G_2 = -B_2^* \ddot{S}_1(x^b_{3,2} - 0), \quad (27)$$

Boundary conditions at section $x_3 = x^b_{3,3}$ (hinged edge, Figure 1) has the form ($3N_1N_2$ equations):

$$u_q^{(2,i,j,r)} = 0, \quad q = 1, 2, 3, \quad i = 1, 2, ..., N_1, \quad j = 1, 2, ..., N_2, \quad r = N_3. \quad (28)$$

Equations (25) can be rewritten in matrix form:

$$B_3^* \bar{U}_2 = g^-_3 : \{B_3^*\}_ {p,q} = \delta_{p,q}, \quad p = 1, 2, ..., 3N_1N_2, \quad q = 1, 2, ..., 3N_1N_2N_3; \quad g^-_3 = 0; \quad (29)$$

$B_3^*$ is matrix (with elements $\{B_3^*\}_ {p,q}$) of boundary conditions of size $3N_1N_2 \times 3N_1N_2N_3$.

The total number of equation is equal to $3N_1N_2N_3 + 6N_1N_2$. Corresponding coupled system of $3N_1N_2N_3 + 6N_1N_2$ linear algebraic equations with $3N_1N_2N_3 + 6N_1N_2$ unknowns has the form:

$$\begin{bmatrix}
Q_1 & 0 \\
Q_{2,1} & Q_{2,2} \\
0 & \tilde{K}_2 \\
0 & B_3^*
\end{bmatrix}
\begin{bmatrix}
\ddot{C}_1 \\
\ddot{G}_2 \\
\ddot{R}_2 \\
\ddot{G}_3
\end{bmatrix}
= \begin{bmatrix}
Q_1 & 0 \\
Q_{2,1} & Q_{2,2} \\
0 & \tilde{K}_2 \\
0 & B_3^*
\end{bmatrix}
\begin{bmatrix}
\ddot{C}_1 \\
\ddot{G}_2 \\
\ddot{R}_2 \\
\ddot{G}_3
\end{bmatrix}, \quad (30)$$

where $\tilde{K}_2$ is corresponding reduced global stiffness matrix of size $[3N_1N_2(N_3-1)] \times [3N_1N_2N_3]$; $\tilde{R}_2$ is corresponding reduced global right-side vector of order $3N_1N_2(N_3-1)$ (if boundary conditions (29) are taken into account automatically within construction of global stiffness matrix and global right-side vector corresponding to subdomain $\omega_2$).

Then strain and stress components are computed according to well-known formulas.

4. About multipoint boundary problem

As it is known, two-point boundary problem is a special case of multipoint boundary problem. Generally, the formulation of considering multipoint boundary problem includes three main components: a description of the domain occupied by the structure and the corresponding subdomains; description of the conditions inside the domain and inside subdomains; description of boundary conditions (for boundaries of domain and boundaries between subdomains). We have to adjust notation system by analogy with [23, 24] for multipoint boundary problem. General principles of domain approximation have been already considered. Algorithm of subdomains is numbering presented in [23].

Numbering of finite elements, discrete-continuous finite elements and construction of multilevel approximation model for domain are carried out by analogy with [23]. Corresponding matrices construction of boundary conditions and right-side vectors for various variants of boundary conditions (interface conditions) between subdomains (2) can be done by analogy with [24].

All methods and algorithms, considered in this paper, were realized in software system. The main purpose of Analysis system CSASA3Dm (DCFEM + FEM) is semianalytical structural analysis (static structural analysis of three-dimensional structure within theory of elasticity), based on combined application of FEM and DCFEM. Programming environment is Microsoft Visual Studio 2013 Community and Intel Parallel Studio 2017XE (Fortran programming language) with Intel MKL Library. Software system is designed for Microsoft Windows 8.1/10. Verification samples of structural analysis are presented in [18, 19, 25]. ANSYS Mechanical 15.0 (FEM) was used for
verification purposes. The results of analysis obtained by the ANSYS Mechanical and CSASA2D generally agree well with each other. Besides, it is necessary to note, that DCFEM is more effective in the most critical, vital, potentially dangerous areas of structure in terms of fracture (areas of the so-called edge effects), where some components of solution are rapidly changing functions and their rate of change in many cases can’t be adequately taken into account by the standard FEM.

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