Relational motivation for conformal operator ordering in quantum cosmology

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Abstract
Operator ordering in quantum cosmology is a major as-yet unsettled ambiguity with not only formal but also physical consequences. We determine the Lagrangian origin of the conformal invariance that underlies the conformal operator-ordering choice in quantum cosmology. This arises particularly naturally and simply from relationalist product-type actions (such as the Jacobi action for mechanics or Baierlein–Sharp–Wheeler-type actions for general relativity), for which all that is required is for the kinetic and potential factors to rescale in compensation to each other. These actions themselves mathematically sharply implement philosophical principles relevant to whole-universe modelling, so that the motivation for conformal operator ordering in quantum cosmology is thereby substantially strengthened. Relationalist product-type actions also give emergent times which amount to recovering Newtonian, proper and cosmic time in various contexts. The conformal scaling of these actions directly tells us how emergent time scales; if one follows suit with the Newtonian time or the lapse in the more commonly used difference-type Euler–Lagrange or Arnowitt–Deser–Misner-type actions, one sees how these too obey a more complicated conformal invariance. Moreover, our discovery of the conformal scaling of the emergent time permits relating how this simplifies equations of motion with how affine parametrization simplifies geodesics.

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1. Introduction
In the ‘Magic without Magic’ tribute volume to John Wheeler [1], Charles Misner both presents general relativity (GR) as involving curves conformal to geodesics (‘parageodesics’) on superspace, and then follows this up by arguing for conformal operator ordering in quantum cosmology.
Operator ordering is an ambiguity that arises in promoting a classical theory to a quantum one. As this affects the form of the quantum mechanical wave equation, it is indeed a physical rather than just formal ambiguity, and in the case of quantum cosmology and quantum gravity, we cannot presently resort to experiments to determine the operator ordering chosen by nature, unlike in the ordinary QM of atoms and molecules. Thus, how one may operator order is a major issue in quantum gravity and quantum cosmology [1–16]. One desirable condition pointed out by DeWitt [2] is for one’s operator ordering to be invariant under coordinate changes. As regards the combination

\[ N_{ab}^{\xi}(Q^c) P_a P_b \]  

in one’s quadratic energy or Hamiltonian constraint\(^1\), this desirable condition is fulfilled e.g. (if one limits the complexity of the derivatives involved to linearity in second-order derivatives) by promotion to a member of the family:

\[ \nabla^2 - \xi \text{Ric}(M) \]

(1)

\[ \nabla^2 = \frac{1}{\sqrt{M}} \nabla \left( \sqrt{M} N_{ab}^{\xi} \nabla Q^a \right) \]  

(2)

([1, 4–6]). The conformal operator ordering is a special \( k \)-dependent member of this family: the \( \xi = \{k - 2\}/4\{k - 1\} \) case. Misner arrives at this from the starting point of introducing a new parageodesic structure:

\[ ds^2, \text{Ric}(M) \longrightarrow f^{-2} ds^2, f^2 \text{Ric}(M). \]

(3)

In the present paper, I show how the philosophically well motivated and mathematically sharply implemented relational approach’s product-type action principles serve as a natural and deeper starting point for this structure from which conformal operator ordering follows. NB that this is in combination with (rather than instead of) DeWitt’s coordinate invariance condition. Though its being revealed to possess philosophical foundations of this kind alongside the technical advantages already found in the above-cited papers, I argue that the case to adopt the conformal ordering is considerably strengthened. Additionally, the classical part of my working allows for more to be said about Misner’s issue of GR-involving curves conformal to geodesics on superspace; in particular, it offers new insights as to the nature of emergent time. (This arises as, in various contexts, ‘Jacobi–Barbour–Bertotti’ emergent time [17, 18], the semiclassical approach to the Problem of Time’s WKB time [13, 14, 19], proper time and cosmic time.)

1.1. Relationalism and product-type actions

I first explain the relational approach. Relationalism concerns alternative foundations for physics to absolutism; which of these the real world may possess has been (and remains) the subject of a long debate [20, 21]. I use relationalism in Leibniz–Mach–Barbour’s sense of the word [17, 18, 21–25].\(^2\) This is based on the following first principles.

A physical theory is temporally relational if there is no meaningful primary notion of time for the whole system thereby described (e.g. the universe) [17, 23]. This is implemented by using actions that are manifestly reparametrization invariant while also being free of extraneous time-related variables (such as external Newtonian time or the geometrodynamical formulation of GR’s lapse coordinate [28]).

\(^1\) The \( Q^a \) are configuration variables with \( a \) a multi-index over particle and/or field species. \( P_a \) are the corresponding conjugate momenta. \( M_{ab} \) is the in general \( Q^a \)-dependent kinetic metric of configuration space with determinant \( M \), inverse \( N^{ab} \), line element \( ds^2 \), dimension \( k \), Laplacian \( \nabla^2 \) and Ricci scalar \( \text{Ric}(M) \). \( \nabla \) denotes partial derivative for finite theories and functional derivative for field theories. \( \xi \) is just a parameter. I use round brackets for functions and square brackets for functionals.

\(^2\) See [26] for Rovelli’s distinct use of the same word and [27] for a brief comparison of the two.
A physical theory is configurationally relational if a certain group $G$ of transformations that act on the theory’s configuration space $Q$ are physically meaningless [17, 23, 29–32]. As subcases corresponding to various mathematical forms for, and physical interpretations of, $G$, this includes spatially relational and internally relational (in the usual sense of gauge theory). Spatial relationalism suffices for the examples covered in this paper. One indirect implementation of this involves using arbitrary-$G$-frame-corrected quantities rather than ‘bare’ $Q$-configurations (see four paragraphs down for how this implementation indeed works out). The examples in this paper have product-type reparametrization-invariant actions $^3$:

\[ S_{\text{product}} = 2 \int d\lambda \int_{\Sigma} d\Sigma \sqrt{TW}. \] (4)

Here, the kinetic term $T$ takes the form

\[ T = \sum_{a,b} M_{ab} \circ g Q^a \circ g Q^b / 2, \] (5)

where $\circ = d/d\lambda$ and $\lambda$ is a label time, and $\circ g = \circ - \rightarrow G \circ g$ for $\rightarrow G \circ g$ the group action of $G$ corresponding to the generators $\circ g$ (this is how configurational relationalism with respect to $G$ is implemented). $W[Q^i]$ is minus the potential (possibly up to an additive constant energy, as explained in the examples below). These actions are clearly free of extraneous time variables and reparametrization invariant (and thus temporally relation) since changing from $\lambda$ to another $\lambda'$ straightforwardly cancels as $T$ is homogeneous quadratic. Indeed, one could write the action in a parametrization irrelevant way (i.e. one which makes no reference whatsoever to any label-time parameter $\lambda$):

\[ S_{\text{product}} = 2 \int \int_{\Sigma} d\Sigma \sqrt{W} \sum_{a,b} M_{ab} d_{\lambda} Q^a d_{\lambda} Q^b / 2 \] (6)

for $d_{\lambda} \equiv d - \rightarrow G d_{\lambda}$.

One issue is how these product-type actions are related to the difference-type actions more usually found in physics:

\[ S_{\text{difference}} = \int dt \int_{\Sigma} d\Sigma \{ T_t - V \}. \] (7)

Here, $T_t$ is the kinetic term that is usually$^4$ homogeneous quadratic in the velocities,

\[ T_t = \frac{1}{2} \sum_{a,b} M_{ab} \frac{d_{\lambda} Q^a}{dt} \frac{d_{\lambda} Q^b}{dt}, \] (8)

and $V[Q^i]$ is the potential term. Equivalence is established as follows. Action (7) leads to action (4) by, firstly (parametrization): adjoining $t$ to the configuration space so that $dt/d\lambda$ now features in the action. Secondly, provided that $V$ is independent of $t$ and $d_{\lambda} Q^a / dt$, which can be held to be the case for fundamental classical physics of the universe as a whole, Routhian reduction [34] subsequently serves to eliminate $dt/d\lambda$ from the variational equation for $t$.

Moreover, there is a good case for using product-type actions rather than difference-type actions as regards consideration of whole-universe fundamental physics [18, 22, 35]—the setting for quantum cosmology.

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$^3$ $\Sigma$ is the notion of space of extent (3-space for field theories and trivial for particle theories, for which $\int_{\Sigma} d\Sigma$ is taken to become $\times 1$).

$^4$ One can include a sufficient set of fields in this context to study classical fundamental physics (there being no difficulty [30, 33] with additionally incorporating terms linear in the velocities into this scheme and this paper’s workings, e.g. permitting fermionic as well as bosonic matter coupled to GR).
How do relational actions work? The momenta are

\[ P_a = \sqrt{\frac{W}{T}} M_{ab} \circ_g Q^b = M_{ab} \ast_g Q^b. \]  \tag{9}

Here, \( \ast_g = \sqrt{\frac{W}{T}} \circ_g = \sqrt{\frac{W}{T} \ast \Gamma_g} \) and \( \ast = \frac{d}{dt^{\text{BB}}} = \sqrt{\frac{W}{T} \circ} = \sqrt{\frac{W}{d}} \frac{d}{\sum_{a,b} M_{ab} \circ_g d \circ_g Q^b/2} \) for the 'Jacobi–Barbour–Bertotti emergent time' \([18, 24, 36, 37]\) that is found to considerably simplify the momenta and Euler–Lagrange equations that follow from the relational principles. In this sense such a \( t^{\text{BB}} \) is a privileged parametrization; it coincides with the conventional difference action’s \( t \), but is now to be considered as emergent and provided by the entirety of the model universe’s contents\(^5\). The temporal-relationalism-implementing reparametrization invariance then implies that there must be at least one primary constraint \([38]\); the product-type form of the action in this paper then gives this to be one ‘Pythagorean’ constraint (per space point in the field-theoretic case)

\[ N^{ab} P_a P_b/2 + W = 0. \]  \tag{10}

Note that this is quadratic and not linear in the momenta (this underlies the frozen formalism aspect of the Problem of Time \([13, 14]\)). On the other hand, variation with respect to the configurational-relationalism-implementing arbitrary \( G \)-frame variables produces secondary linear constraints. These use up two degrees of freedom per generator \( g \), thus indeed sending one to the requisite quotient space \( Q/G \). Examples will clarify significance of this section’s actions, constraints and emergent times.

The equations of motion that follow from (4) are the momentum–velocity relations

\[ P_a = \sqrt{\frac{W}{T}} M_{ab} \circ_g Q^b \]  \tag{11}

and (for \( \nabla^a \equiv \nabla^a Q_a \) and \( \Gamma^{abc} \) the configuration space Christoffel symbols) the Euler–Lagrange equations

\[ \circ \{ \sqrt{\frac{W}{T} \circ_g Q^a} \} + \sqrt{\frac{W}{T}} \Gamma^{abc} \circ_g Q^b \circ_g Q^c = \sqrt{T/W} \nabla^a W + M_{bc} \sqrt{\frac{W}{T} \circ_g Q^b \nabla^a \{G_{cG} Q^c \}}. \]  \tag{12}

1.2. Examples of product-type actions

Next, I give examples of product-type actions.

**Example 1** (Temporal relationalism only). The Jacobi action \([34]\) for the Newtonian mechanics of \( N \) particles with positions \( q_I, I = 1 \) to \( N \), is

\[ S_{\text{Jacobi}} = 2 \int \ast \frac{d\lambda}{\sqrt{T[\Upsilon + E]}}. \]  \tag{13}

Here

\[ T = \sum_{I=1}^N m_I \{ \omega q_I \}^2/2, \]  \tag{14}

\( U(Q^c) \) is minus the potential term, \( V \), and \( E \) is the total energy of the system, and I use \( \ast \) in place of \( t \) for mechanical theories. \([34]\) lucidly covers how to obtain this from the Euler–Lagrange difference-type action \( S = \int dt [T - V] \) by Routhian reduction. For the recovery from (13) of the usual Euler–Lagrange formalism from this but with the emergent \( t^{\text{BB}} \) now taking over

\[^5\] Barbour furthermore considers \([18, 24]\) this to be the time standard such that isolated observers who choose to use it obtain clocks that march in step with each others’. However, as I have never seen this quantitatively demonstrated, it will play no further part in this paper.
the role of Newtonian absolute time, see e.g. [18, 36]. The quadratic constraint is in this case the energy constraint

\[ \sum_{I=1}^{N} \frac{p_I^2}{2m_I} + V = E. \]  

(15)

There being no nontrivial \( G \) in this case, there are no linear constraints.

**Example 2A.** One could consider versions of the Jacobi action (13) for ‘relational particle mechanics’ theories in which the velocities now come with arbitrary Euclidean [17, 18, 22, 27, 35, 39–43] or similarity [27, 31, 32, 40, 42–48] group frame corrections that implement spatial relationalism as well. Thus, these are temporally and spatially relational mechanical theories. The quadratic constraint continues in this case to be an energy constraint of the form in the preceding example. Also, variation with respect to the auxiliary variables \( g \) produces constraints that are linear in the momenta. For Euclidean relational particle mechanics, these are \( \mathcal{P} = \sum_{I=1}^{N} p_I = 0 \) (zero total momentum for the model universe) and \( \mathcal{L} = \sum_{I=1}^{N} q_I \times p_I = 0 \) (zero total momentum for the model universe), while similarity relational particle mechanics has these again alongside \( \mathcal{D} = \sum_{I=1}^{N} q_I \cdot p_I = 0 \). By analogy with example 3 below, relational particle mechanics are additionally useful models [13, 19, 36, 37, 41–45, 49–51] for the Problem of Time in quantum gravity and other issues of interest in quantum cosmology [3, 52, 53].

**Example 2B.** Relational particle mechanics can be cast in reduced form in spatial dimension 1 or 2. Here all the above constraints can be eliminated, producing reduced kinetic terms of the form \( T_{\text{red}} = M_{ab}\circ Q^a\circ Q^b/4 \) for \( M_{ab} \), the usual metric on \( S^{N-2} \) for \( N \) particles in 1D and the Fubini–Study metric on \( \mathbb{C}P^{N-2} \) for \( N \) particles in 2D. The quadratic constraint is still an energy constraint of form (10) built from inverses of the above-mentioned metrics.

**Example 3A.** An action (see e.g. [23, 33]) for a geometrodynamical formulation of GR (in terms of 3-metrics \( h_{\mu\nu}(x^w) \) on a fixed topology \( \Sigma \), for simplicity taken to be compact without boundary; \( x^w \) are spatial coordinates) is

\[ S_{\text{GR}}^{BFO-A} = 2 \int d\lambda \int_{\Sigma} d^3 x \sqrt{h} \sqrt{T_{\text{BFO-A}}} \{ \text{Ric}(h) - 2\Lambda \}. \]  

(16)

Here,

\[ T_{\text{BFO-A}}^{\text{GR}} = M^{\mu\nu\rho\sigma} \circ h_{\mu\nu} \circ h_{\rho\sigma} / 4, \quad \circ h_{\mu\nu} = \circ h_{\mu\nu} - \text{Diff}_{\Sigma} h_{\mu\nu} = \circ h_{\mu\nu} - \mathcal{L}_{\mathcal{F}} h_{\mu\nu}, \]  

(17)

where \( M^{\mu\nu\rho\sigma} = h^{\mu\rho} h^{\nu\sigma} - h^{\mu\sigma} h^{\nu\rho} \) (the GR configuration space metric, alias inverse of the undensitized DeWitt supermetric [3]), \( \text{Diff} \) is the group of 3-diffeomorphisms on \( \Sigma \), \( \mathcal{L}_{\mathcal{F}} \) is the Lie derivative with respect to the ‘velocity of the frame’ \( F_{\mu} \), \( \text{Ric}(h) \) is the Ricci 3-scalar corresponding to \( h_{\mu\nu} \), \( h \) is the determinant of \( h_{\mu\nu} \) and \( \Lambda \) is the cosmological constant. This action would be the better-known Baierlein–Sharp–Wheeler (BSW) [54] one if the kinetic term were, rather, \( T_{\text{BSW}} \) which is the same up to being built out of shift corrections \( \beta^a (x^w) \) in place of ‘velocities of the frame’ \( \circ F^a (x^w) \). \(^6\) However, it would not then be manifestly temporally relational. Moreover, the BSW action is equivalent to the even more familiar ‘Lagrangian ADM’ [56] action:

\[ S_{\text{GR}}^{\text{ADM}} = 2 \int d\lambda \int_{\Sigma} d^3 x \sqrt{h} \alpha \{ T_{\text{ADM}} / \alpha^2 + \text{Ric}(h) - 2\Lambda \} \]  

(18)

\(^6\) There are various other equivalent pairs of principles of dynamics objects in this paper that are related to each other by the one using auxiliary frame velocities where the other uses auxiliary coordinates (and, sometimes additionally, the auxiliary velocity of the instant, \( \alpha \), in place of the auxiliary lapse coordinate, \( \alpha \)). For details of how the equivalence of each of these pairs works out, see [55].
for $T_{\text{ADM}}$ taking the same form as the above $T_{\text{BSW}}$. The former follows from the latter by elimination of the Lagrange multiplier coordinate lapse $\alpha$ from its own variational equation. Parallely [55], one can also obtain the BFO-A action from a now also-unfamiliar action that is the Lagrange–ADM’s equivalent pair in the sense of footnote 6:

$$S_{\text{GR}}^A = \int d\lambda \int_{\Sigma} d^3x \sqrt{h} \omega [T_{\text{GR}}^A / \omega] + \text{Ric}(h) - 2\Lambda$$

(for $T_{\text{GR}}^A$ taking the same form as $T_{\text{BFO-A}}$). The former now follows from the latter by using Routhian reduction to eliminate $\dot{I}$ (an even closer parallel of the equivalence at the end of example 1 than the preceding coordinate elimination). The quadratic constraint is now the GR Hamiltonian constraint

$$\mathcal{H} \equiv N_{\mu\nu\rho\sigma} \pi^\mu \pi^\nu / \sqrt{h} = \sqrt{h} \{\text{Ric}(M) - 2\Lambda\} = 0$$

for $N_{\mu\nu\rho\sigma}$ the inverse of $M_{\mu\nu\rho\sigma}$ (i.e. the undensitized DeWitt supermetric itself), while the linear constraint from variation with respect to $F^\mu$ is the GR momentum constraint,

$$L_\mu = -2D_\nu \pi^\nu_\mu = 0.$$

**Example 3B.** Each of the pairs $(16, \text{BSW})$ and $(18, 19)$, $A$ becomes indistinguishable for minisuperspace, giving the relational product-type action

$$S = 2\text{Vol}(\Sigma) \int d\lambda \sqrt{T} [\text{Ric}(h) - 2\Lambda], \quad T = M_{\mu\nu\rho\sigma} h_{\mu\nu} \circ h_{\rho\sigma} / 4$$

(for $\text{Vol}(\Sigma)$ the spatial volume of the universe). Therein, $M_{\mu\nu\rho\sigma} (h_{\gamma\delta}(x^\omega))$ collapses to an ordinary $6 \times 6$ matrix $M_{ab}$ or further in the diagonal case (to a $3 \times 3$ matrix)—the ‘minisupermetric’. Note that this case also is trivial as regards configurational relationalism (no $\circ F$ dependence and thus no momentum constraint).

NB example 3 makes it clear that GR, in a geometrodynamical guise that is fairly standard, is relational [23] in the sense used in the present paper. Moreover, Euclidean relational particle mechanics is a restriction of Newtonian mechanics to zero total angular momentum, while Newtonian mechanics with total angular momentum not necessarily zero also appears if one considers an isolated ‘island universe’ subsystem within Euclidean relational particle mechanics. These issues are relevant to the absolute versus relative motion debate; they are one way in which this debate is still alive today. Finally, I note that different formulations at the classical level can lead to inequivalences at the quantum level, whereby these alternative formulations might be of relevance in (toy modelling of) quantum gravity.

### 1.3. Banal-conformal invariance of product-type actions and its consequences

Product-type actions contain the product of a $T$ factor and a $V$ factor. Then one can pass factors from one to the other: $TW = T\Omega^0W = T\Omega^{2(1-1)}W = (T\Omega^2)[\Omega^{-2}W] = \tilde{T}\tilde{W}$, which is based on $0 = 1 - 1$ (the factor of 2 being for later convenience). Thus, under the simple and natural banal-conformal transformation

$$T \longrightarrow \tilde{T} = \Omega^2T, \quad W \longrightarrow \tilde{W} = W / \Omega^2,$$

product-type actions are preserved: $\tilde{S}_\text{product} = S_\text{product}$. (28) can be interpreted as $T$ being a banal-conformal vector and $V$ a banal–conformal covector. Moreover the first of these can be viewed as

$$M_{ab} \longrightarrow \tilde{M}_{ab} = \Omega^2M_{ab},$$

7 While everybody knows that $1 - 1 = 0$, it takes a mathematician to fruitfully apply $0 = 1 - 1$. 6
so that it is the kinetic metric that is a banal-conformal vector. (I term this transformation ‘banal’ because it is a simple transformation of a compensatory nature—the scaling of $T$ compensates for that of $W$. ‘Banal’ has been used for simple compensatory transformations before [31], though not for the particular case of this paper (other than by me [27]).) This paper views relational actions (action (13) for relational particle mechanics, or (16) for GR including simplification by which it is equivalent to BSW’s action in the case of minisuperspace) and this simple property thereof as a new, deeper starting point for Misner’s argument for conformal operator ordering. My banal-conformal factor $\Omega_1$ amounts to a recovery of Misner’s $f$ of equation (3), so that, firstly, I identify Misner’s conformal transformation to be a banal one, and, secondly, point out that this is a simple and natural feature of product-type actions which, as explained above, themselves rest on the philosophically interesting relational first principles.

In section 2, I show that it immediately follows that the derivative with respect to the emergent time function can be regarded as a banal covector. Also, if one then considers this scaling property to carry over to the difference-type action formulations’ timefunction, a more complicated manifestation of banal-conformal invariance is discovered for difference-type actions. Clearly, performing such a transformation should not (and does not) affect one’s classical equations of motion. Moreover, working through how the scaling of $T$, $W$, and the timefunction conspire to cancel out at the level of the classical equations of motion reveals interesting connections between the simplifying effects of using the emergent time function on the equations of motion and those of the rather better-known affine parametrization [57, 58]. Section 2 ends by preparing for quantization by discussing how momenta, constraints and Hamiltonian-type objects banal scale.

If one furthermore wishes for this banal-conformal invariance—displayed simply and naturally by relationalism-implementing product actions for whole-universe fundamental physics—to continue to hold at the quantum level (section 3), then this alongside the otherwise theoretically desirable (and fairly standard) requirement that one’s quantum theory should not depend on how $Q$ is coordinatized, then one is led to the operator ordering for (1) that is based on the conformally invariant modification of the Laplacian. This conformal ordering had been previously suggested by e.g. Misner [1] (cf the start of the introduction), and also by, Halliwell [8], Moss [9] and Ryan–Turbiner [15]. The present paper also advocates this, via its naturality from product-type actions which themselves rest on the relationalist first principles. On the other hand, Kuchař [4] and Henneaux–Pilati–Teitelboim [5] have advocated the Laplacian ordering itself ($\xi = 0$). So have Page [10], Louko [11] and Barvinsky [12], however their specific examples are two dimensional, for which the Laplacian and conformal orderings coincide. Wiltshire advocates both [53]. Christodoulakis and Zanelli [7] consider the case with an arbitrary $\xi$, as do Hawking and Page [6], albeit the latter then also pass to a 2D example for which $\xi$ drops out. Finally, all of these orderings coincide to $O(\hbar)$ [12].

2. Classical consequences of banal invariance

Upon inspection (see e.g. [18]), the equations of motion (12) following from the product-type action (4) simplify for particular choices of parameter in two generally different ways (the two coincide if the potential is constant):

\[ (A) \]

\[ \frac{d}{d\lambda} \left\{ \sqrt{W} \frac{dQ^a}{d\lambda} \right\} = \frac{d^2 Q^a}{d\lambda^2} + \frac{1}{2\sqrt{WT}} \frac{dW}{d\lambda} \frac{dQ^a}{d\lambda} - \frac{1}{2} \frac{dT}{T^3} \frac{dW}{d\lambda} \frac{dQ^a}{d\lambda} \] versus \[ \frac{d^2 Q^a}{dm^2} \]

which corresponds to $d/d\mu = \sqrt{W/T} d/d\lambda$, where parameter $\mu$ we denote by $t^{BB}$, emergent Jacobi–Barbour–Bertotti time [18, 23]. In the case of a mechanical theory, this
Emergent time turns out also to imply conservation of energy and amounts to a recovery of Newtonian time; it is also aligned with the mechanics case’s emergent semiclassical (WKB) time \[36\]. In the case of geometrodynamics, this emergent time amounts to a recovery of local proper time, as well as being aligned with the geometrodynamical emergent semiclassical (WKB) time \[37\], and corresponding to cosmic time in the case of homogeneous cosmology.

(B) \( \nabla^a W \) being nonzero versus it being zero, the latter corresponding to ‘the dynamical curve being an affinely parametrized geodesic on configuration space’. In this case I denote the time parameter by \( t_{\text{aff-geo}} \).

Consequences of banal-conformal invariance are as follows.

1) \( d/d(\text{emergent time}) \) is a banal-conformal covector:

\[
\tilde{\gamma} \equiv \sqrt{W/T} = \sqrt{[W/\Omega^2]/\Omega^2 T} = \Omega^{-2} \sqrt{W/T} = \Omega^{-2} \gamma.
\]
Thus, the emergent time \[18, 36, 50\] depends on the choice of the banal-conformal factor. In the case of geometrodynamics, one can also think of \( d/\lambda \) being invariant and \( 1/\tilde{\gamma} = 1/\Omega^2 \gamma \) so that the velocity of the instant scales as a banal-conformal vector \( \gamma I \longrightarrow \Omega^2 \gamma I \) (or, equivalently, the emergent lapse coordinate scales as a banal-conformal vector \( N \longrightarrow \Omega^2 N \)). To not confuse ‘\( t_{\text{JBB}} \) as present in the previous literature’ and the banal covector discovered in this paper, I denote the latter by \( \tilde{\gamma} \), rather. I also use the notation

\[
\ast \equiv d/\tilde{\gamma} = \sqrt{W/T}.
\]

2) Next observe that, provided that its timefunction scales as \( \tilde{\gamma} \) does, the difference-type Euler–Lagrange action is also banal-conformally invariant (albeit in a more complicated way):

\[
S_{\text{difference}} = \int \int \Sigma \left[ \tilde{T} \gamma - \tilde{V} \right] d\tilde{t} = \int \int \Sigma [\tilde{M}_{ab} \tilde{\gamma} Q^a \gamma Q^b / 2 - \tilde{V}] d\tilde{t} \\
= \int \int \Sigma [\Omega^2 M_{ab} \Omega^{-2} \gamma Q^a \gamma Q^b / 2 - \Omega^{-2} V] \Omega^2 d\tilde{t} \\
= \int \int \Sigma [M_{ab} \gamma Q^a \gamma Q^b / 2 - V] d\tilde{t} = S_{\text{difference}}.
\]

3) The Euler–Lagrange equations of motion following from (4), will clearly be invariant under the full banal-conformal transformation \((T, W, \ast) \longrightarrow (\tilde{T}, \tilde{W}, \tilde{\gamma}) = (\Omega^2 T, \Omega^{-2} W, \Omega^{-2} \gamma)\), as the action that they follow from is. (Note that in the case of the relational formulation, the banal transformation of \( T, W \) directly implies the full banal transformation, so that these are not distinct entities here.) This seemingly trivial extra fact does however generate some interesting comments when one looks at the details of the cancellations at the level of the Euler–Lagrange equations themselves.

Let us begin with the largely sufficient finite and trivially spatially relational case (i.e. mechanics with temporal relationalism only, fully reduced 1D or 2D relational particle mechanics and minisuperspace). Then the Euler–Lagrange equations are

\[
D^2 Q^a / D\tilde{t}^2 \equiv **Q^a + \Gamma^a_{bc} Q^b Q^c = \delta^a W,
\]
which is the geodesic equation modulo the right-hand-side term (\( D/D\tilde{t} \) being the absolute derivative with respect to \( \tilde{t} \)). The path of motion is not in general an affinely parametrized geodesic (‘simplification (B)’); however, a banal-conformal transformation to a such exists, in the following sense.
Case (i): if $W$ is prescribed as a constant, then (24) is the geodesic equation, which is the
case in mechanics if $V$ is constant and in (for the moment) minisuperspace GR if $\text{Ric}(h)$
is constant. Indeed, this corresponds to having an action proportional to $\int ds \sqrt{T}$ and so
to $\int ds$ for $\text{d}x^2$ the line element corresponding to the kinetic metric $M_{ab}$.

Case (ii): if not, banal-conformal transform with $\Omega^2 = \kappa W$ for $\kappa$ constant so
$T \rightarrow \tilde{T} = \kappa WT$ and $W \rightarrow \tilde{W} = W/\kappa W = 1/\kappa$. This corresponds to obtaining
an action $S \propto \int d\tilde{x}$ and passing from $\mathcal{M}_{\text{BH}}$ to $t^\text{aff-geo}$, that is from banal-conformal
factor $\Omega^2 = 1$ to banal-conformal factor $\Omega^2 = \kappa W$. Thus, simplifications (A) and (B)
are related by a banal transformation. Moreover, case (ii) has range of validity caveats
[1, 59] for regions containing zeros of $W$ as the conformal transformation’s definition
precludes these; infinities and non-smoothnesses of $W$ can likewise be disruptive. In the
minisuperspace case, this paragraph’s contents were spelled out by Misner [1] following
more partial mention in the earlier work of DeWitt [60]. In mechanics, this is in e.g.
[18, 34].

I then ask the following question. How does performing two transformations—conformal
transformation and non-affine parametrization—each of which complicates the equations of
motion, nevertheless work out to preserve them when applied together?

Understanding this requires looking at the alternative, longer proof of (3) at the level of the
equations of motion themselves. By

$$\Gamma^a_{bc} = \Gamma^a_{bc} + \{2fal_{(b}a_{c)}\Omega - M_{bc}\partial^a\Omega\}/\Omega,$$

symmetry and the definition of $T_j$ in terms of velocities with respect to $\tilde{t}$,

$$\Gamma^a_{bc} \ast Q^b \ast Q^c = \Omega^{-1} \Gamma^a_{bc} \ast Q^b \ast Q^c + 2\Omega^{-1} \{\partial_a \Omega \ast Q^b \ast Q^a - T_i \partial^a \Omega\},$$

which, then, alongside using obvious product rule expressions for $\ast\{\Omega^{-1} Q^a\}$ and $\partial^a \{\Omega^{-1} W\}$
gives

$$0 = \ast Q^a + \Gamma^a_{bc} \ast Q^b \ast Q^c - \partial^a \tilde{W}$$

$$= \Omega^{-1} \{\ast Q^a + \Gamma^a_{bc} \ast Q^b \ast Q^c - \partial^a W\} + 2\Omega^{-1} \{\partial_a \Omega \ast Q^b \ast Q^a + \{W - T_i\} \partial^a \Omega\}. \quad (25)$$

Then the second big bracket cancels by the chain rule and conservation of energy in mechanics:

$$W - T_i = E - V - T_i$$

the Lagrangian form of the Hamiltonian constraint in (for the moment)
minisuperspace GR: $W - \text{TR} = \text{Ric}(h) - 2\Lambda - T_G$.

Next, analyse the above in terms of non-affine parametrization and conformal
transformation subworkings. This reveals the second term in the second large bracket to be
the result of non-affine parametrization. It cancels with the first term, which is one of two
complicating terms from conformal transformation, the other one being $T_i$, which
itself cancels with the banal-conformal transformation’s compensatory conformal scaling
of $W = E - V$ by the conservation of energy or the Hamiltonian constraint. This is
therefore an interesting configuration space generalization of the result by which null geodesics
conformally map to null geodesics [57]. There, the first conformal complication is balanced
by a change of what is the suitable affine parametrization, while the second one vanishes
by the geodesic being null with respect to the indefinite spacetime metric. In our case,
the first of these cancellations continues to occur with the same interpretation, but what
was the null combination (and thus working for indefinite metrics only) becomes, in the
configuration space context, the kinetic term whether for indefinite or definite kinetic metrics,
and the null condition becomes replaced by the energy or Hamiltonian constraint (granted the
banal-conformal transformation’s compensatory scaling of the potential factor $W$). Thus ‘in
indefinite spaces null geodesics conformal map to null geodesics’ becomes ‘in configuration
spaces of whatever signature, paths of motion banal-conformal map to paths of motion’.
Next, let us afford a slight generalization to finite models with non-trivial spatial relationalism. At least Euclidean and similarity relational particle mechanics then have as equations of motion

\[ D^2 g^{Qa} / D\vec{t}^2 \equiv \star g^{\star} g^{Qa} + \Gamma_{bc}^{\alpha} \star g^{Qb} \star g^{Qc} = \partial^\alpha W, \]  

(26)

for \( D/g / D\vec{t} \) the \( G \)-corrected absolute derivative, and then the preceding analysis carries through under \( \ast \rightarrow \ast /\Omega^1 \). To the extent that the previous paths of motion were geodesics, the current paths are 'geodesics provided that we suitably align the constituent snapshots by auxiliary \( G \)-transformations'.

We have determined that solely non-affinely parametrizing or solely rescaling the kinetic metric complicate the equations of motion away from the simple form (24) or (26) that using emergent Jacobi–Barbour–Bertotti time places them in, while performing both of these operations alongside the compensating \( W \) rescaling preserves this simple form, choice of emergent time indeed being nonunique up to this ‘banal’ freedom. Thus, if one’s problem requires rescaling or non-affinely parametrizing, one’s problem may permit one to ‘complete’ the required transformation to a full banal-conformal transformation. Thus, the effect of solely rescaling or solely non-affinely parametrizing kicking one out of the form (24) or (26) is circumvented, and so emergent time’s being a banal covector leads to a robustness result for its property of giving simple equations of motion.

Preservation under full banal transformation means that \( \tilde{t} \) corresponding to any \( \Omega \) carries out simplification (A). One can then pick \( \Omega^2 = \kappa W \) so that \( \tilde{\omega}^\alpha W = \tilde{\omega}^\alpha k = 0 \), and then \( \tilde{\omega} = \Omega^{-2} \kappa \{ \kappa W^{-1} \} = \{ \kappa \sqrt{WT} \}^{-1} \); i.e. so that simplification (B)—taking affine geodesic form rather than additionally containing a \( \tilde{\omega}^\alpha W \) term—also holds. Thus, one has gone from physics with a restricted class of affine parameters under which the equations of motion take the form (24) or (26) to physics with a restricted class of banal-conformal factors under which the equations of motion take geodesic form. Each of these, moreover, is nonunique up to a constant multiplicative time scale (evident in the specification of the geodesic equation forming \( \Omega \)) and an additive constant time origin (evident since what a power of \( \Omega \) scales is \( d/d\vec{t} \) and so \( \vec{t} \) itself has an additive constant of integration more freedom than \( \Omega \) itself [36, 37]). These retain one’s civilization’s freedom of choice for calendar year zero and for unit of time, as should be the case.

Finally, affine transformations send \( t_{\text{old}} \) to \( t_{\text{new}}(t_{\text{old}}) \) subject to (I) nonfreezing and monotonicity, so \( dt_{\text{new}} / dt_{\text{old}} > 0 \) which can be encoded by having it be a square of a quantity \( Y \) with no zeros in the region of use, and (II) this derivative, and hence \( Y \), is a physically reasonable function (to stop the transition damaging the equations of motion). But this can be recast as \( d/dt_{\text{new}} = Y^{-2} d/dt_{\text{old}} \), by which (and other properties matching\(^8\)) we are free to identify this \( Y \) with \( \Omega \), so any affine transformation is of a form that extends to a (full) banal-conformal transformation. If one then chooses to ‘complete’ it to a full banal-conformal transformation, the above calculation can be interpreted as the extra non-affine term being traded for a \( T \) term by having an accompanying conformal transformation of the kinetic metric, and then this being traded for \( \tilde{\omega}^\alpha W \) by energy conservation and the compensating banal-conformal transformation of \( W \). Thus, the freedom to affinely transform the geodesic equation on configuration space can be viewed instead as the freedom to (fully) banal-conformally transform a system’s equation of motion. Thus, the relational approach’s simplicity notion for equations of motion has the same mathematical content as prescribing an affine rather than non-affine parameter for the geodesic equation on configuration space. Thus, the banally related \( \tilde{t} \) corresponds to ‘the set of (generally) nonaffine parameters for the geodesic-like equation of motion on configuration space’, while \( \text{aff} \rightarrow \text{geo} \) indeed remains identified with the much more restricted set (unique up to

\(^8\) It may be interesting to find out whether the one restricts the other’s function space more than usual.
a multiplicative constant time scale and an additive constant time origin) of affine parameters for the geodesic equation on configuration space.

Then part of the argument for emergent time being fixed by the universe’s contents [18] is lost as it is revealed to contain an arbitrary factor. But one can then get back that preciseness by making a choice. \( t^{BB} (\Omega = 1, \text{so that } E \text{ carries no nonconstant factors}) \) and \( t^{aff-geo} \) are then interesting such choices.

(4) In preparation for the passage to QM in section 3, the conjugate momenta are the banal-conformal invariant expressions:

\[
\tilde{P}_a = \sqrt{\tilde{W} / \tilde{T}} \tilde{M}_{ab} \tilde{q}^b = \tilde{M}_{ab} \tilde{q}^b = \tilde{M}_{ab} *_g Q^b = P_a. \tag{27}
\]

Thus, what does banal scale concerns specifically configuration space rather than its phase space generalization.

(5) One obtains as a primary constraint resultant from the reparametrization invariance of the action a quadratic constraint (10). Now, as \( \mathcal{N}^{ab} \) is the inverse of \( M_{ab} \), it scales as a banal-conformal covector

\[
\mathcal{N}^{ab} \rightarrow \tilde{\mathcal{N}}^{ab} = \Omega^{-2} \mathcal{N}^{ab}. \tag{28}
\]

Combining (27) and (28), the quadratic constraint (10) is a banal covector.

(6) In cases with nontrivial configurational relationalism there are also linear constraints from variation with respect to \( G \)-auxiliaries (section 1.2). By (27) the linear momentum constraint of GR, \( L^\mu \), and the relational particle mechanics constraints, \( P^\mu \) and \( D \), are banal-conformally invariant.

(7) The mechanics Hamiltonian \( H \) is the left-hand side of the quadratic energy constraint (10) and as such scales as a banal covector. One then integrates this with respect to \( \vec{t} \).

If one looks to extend this prescription to relational particle mechanics, given that the linear constraints are banal-conformally invariant, one finds that one needs to either: build the total almost-Hamiltonian \(^9\) by using \( *_a \) and \( *_b \) (and \(-*_c \) in the similarity case) as the appending auxiliaries to preserve homogeneity of banal transformation:

\[
A_{\text{total}} = H + *_a \cdot P + *_b \cdot L (-*_c D). \tag{29}
\]

Or, have \( H \) carry a ‘lapse’ prefactor \( d\vec{t} / d\lambda \) and then use \( \circ_a \) and \( \circ_b \) (and \(-\circ_c \) in the similarity case) as the appending auxiliaries, producing an almost-Hamiltonian that is now banal-conformally invariant:

\[
\tilde{A}_{\text{total}} = \circ \vec{H} + \circ_a \cdot P + \circ_b \cdot L (-\circ_c D). \tag{30}
\]

These two views are, moreover, physically equivalent, since the first is to be integrated over \( \vec{t} \) and the second over \( \lambda \).

For GR, \( \mathcal{H} \) scales as a banal covector and the velocity of the instant, \( \circ I \) (or the lapse, \( \alpha \)) as a banal vector. Thus, it is entirely straightforward to append the banal scalar auxiliaries \( \circ F^\mu \) (or \( \beta^\mu \)), to make either the relational total GR almost-Hamiltonian

\[
A_{\text{GR total}} = \int d^3x \{ \circ I \mathcal{H} + \circ F^\mu L_\mu \}. \tag{31}
\]

or the Arnowitt–Deser–Misner total GR Hamiltonian

\[
H_{\text{GR total}} = \int d^3x \{ \alpha \mathcal{H} + \beta^\mu L_\mu \}. \tag{32}
\]

\(^{9}\) Total Hamiltonians have constraints appended by Lagrange multiplier coordinates. In the cyclic velocity auxiliary picture that relationalism instead implies, one has rather what I term an almost-Hamiltonian [55] with cyclic velocities in place of multipliers. I use this terminology since velocities still appear in it, albeit only the velocities of auxiliary quantities.
(Comparing the last two paragraphs, relational particle mechanics likewise admit Hamiltonians if formulated partially non-relationally by use of multiplier coordinates in place of their cyclic velocities. Also note that GR comes already parametrized, so there is no primed–unprimed ambiguity in that case.)

3. Banal invariance in finite QM (relational mechanics, minisuperspace)

At the quantum level, those constraints which remain become wave equations. In this paper’s models, this always includes a quadratic constraint of form (10). This contains a product of $P_a$ and $Q^a$ terms, $\mathcal{N}_{ab}(Q^c)P_aP_b$, which picks up ordering issues in passing to QM. Assume that $Q^a \rightarrow \tilde{Q}^a = Q^a$, $P_a \rightarrow \tilde{P}_a = -i\hbar \partial_a$.

I first consider the more widely well-defined finite case, but leave the working in a general notation that covers both relational particle mechanics and minisuperspace. The Laplacian ordering at the QM level of the classical combination $\mathcal{N}_{ab}(Q^c)P_aP_b$ in the energy/minisuperspace Hamiltonian constraint is

$$D^2 = \frac{1}{\sqrt{M}} \frac{\partial}{\partial Q^a} \left\{ \sqrt{M} \mathcal{N}_{ab} \frac{\partial}{\partial Q^b} \right\}.$$  

(33)

This has the desirable property of (straightforwardly) being independent of coordinate choice on the configuration space $Q$ [2]. This property is not, however, unique to this ordering; one can reorder to include a Ricci scalar curvature term so as to have $D^2 - \xi \text{Ric}(M)$ [1, 2, 6–10, 15]. It is then well known [57] that there is a unique choice of $\xi$ (dependent on the dimension $k \geq 2$ of the mathematical space in question$^{10}$) that leads to the production of a conformally invariant operator and hence of a conformally invariant operator ordering in the quantum application [1, 6–10, 15]. Moreover, in the present paper I identify this conformal invariance associated with operator ordering as being the same as the banal-conformal invariance that is simple and natural in classical relational product-type actions, whereby demanding this operator ordering can be seen as asking to extend this simple and natural invariance to hold also at the quantum level. The operator with the desired properties is, specifically,

$$D_\xi^2 = \frac{1}{\sqrt{M}} \frac{\partial}{\partial Q^a} \left\{ \sqrt{M} \mathcal{N}_{ab} \frac{\partial}{\partial Q^b} \right\} - \frac{k-2}{4(k-1)} \text{Ric}(M).$$  

(34)

Moreover, this operator by itself is still not banal-conformally invariant: it is furthermore required that the wavefunction of the universe $\Psi$ that it acts upon itself transforms in general tensorially under banal-conformal transformations (paralleling [57]):

$$\Psi \rightarrow \tilde{\Psi} = \Omega^{k/2} \Psi.$$  

(35)

Some simple examples are as follows.

(1) For models with 2D configuration spaces such as for the minisuperspace models [6, 10–12], quantum similarity relational particle mechanics of three particles in the plane [46, 48] or of four particles on the line [47], and of three particles on the line with scale [41–43], the conformal value of $\xi = (k-2)/4(k-1)$ collapses to zero, so that Laplacian ordering and conformally invariant wavefunctions suffice. Specific subexample (i) using the relational action of example 2B as starting point, then via preservation of its

$^{10}$ I exclude 1D configuration spaces as physics concerns changes of one configurational variable with respect to another physical variable, in which sense configuration space dimension $k \geq 2$ is required. This is the case at the classical level, without it internal time makes no sense and it is also needed at the quantum level, at least for such as [47] the semiclassical approach and records theory approach to the Problem of Time in quantum gravity.
banal-conformal invariance, the reduced presentation of the similarity relational particle mechanics of three particles in the plane has the conformal-ordered time-independent Schrödinger equation
\[ -\hbar^2 2 D_{\zeta^2}^2 \Psi = -\hbar^2 2 \left\{ \frac{1}{\sin \Theta} \frac{\partial}{\partial \Theta} \left( \sin \Theta \frac{\partial \Psi}{\partial \Theta} \right) + \frac{1}{\sin^2 \Theta} \frac{\partial^2 \Psi}{\partial \phi^2} \right\} = \frac{E}{4} \Psi - \frac{V}{4} \Psi. \] (36)

Here [27], \( \Theta = \arctan(t_1/t_2) \) for \( t_1 \) and \( t_2 \) the magnitudes of the two cluster mass weighted Jacobi interparticle cluster vectors and \( \Phi \) is the relative angle between these vectors. Specific subexample (ii) using the relational action (21) as starting point, then via preservation of its banal-conformal invariance, the isotropic minisuperspace model with scale factor \( a \) and a scalar field \( \phi \) as matter in e.g. [6] has the conformal-ordered time-independent Schrödinger equation
\[ \hbar^2 \left\{ a \frac{\partial}{\partial a} \left( a \frac{\partial \Psi}{\partial a} \right) - \frac{\partial^2 \Psi}{\partial \phi^2} \right\} - \text{const} a^4 \Psi + 2a^6 U(\phi) \Psi = 0. \] (37)

(2) For models with zero Ricci scalar, the conformal ordering coincides with the Laplacian one. A particular case of this [27] is the Euclidean relational particle mechanics of three particles in the plane, which has, using the conformal operator ordering and a banal transformation with \( \Omega^2 = 4I \), the flat time-independent Schrödinger equation
\[ -\hbar^2 2 \left\{ \frac{1}{I^2} \frac{\partial}{\partial I} \left( I^2 \frac{\partial \Psi}{\partial I} \right) + \frac{V}{4I} \right\} \Psi = E I \Psi, \]
where \( I \) is the total moment of inertia of the system. Other examples of flat configuration space models are the scaled RPM of \( N \) particles on a line [43] and the above subexample (ii) again.

(3) If a space has constant Ricci scalar, then the effect of a \( \xi \text{ Ric}(M) \) term, conformal or otherwise, is just something which can be absorbed into redefining the energy in the case of mechanics. In particular, this is the case for relational particle mechanics in 1D as their configuration spaces are \( S^{N-2} \) which are clearly of constant curvature, and for relational particle mechanics in 2D as their configuration spaces are \( CP^{N-2} \) which are Einstein [32] and hence of constant Ricci scalar curvature. The constants to be absorbed into the energies in each of these cases are \( -\hbar^2 (N-2)(N-3)/8 \) and \( -\hbar^2 (N-1)(N-2)(N-3)/(2N-5) \), respectively, for \( N \) the number of particles. Parallely, were Ric(M) proportional to \( \sqrt{h} \) in a GR model, it could likewise be absorbed into redefining the cosmological constant.

However, almost all other minisuperspace models and relational particle mechanics models (e.g., [42, 43]) have \( k \geq 3 \) and nonconstant configuration space Ricci scalar. This is the case e.g. for three particles in 2D if one keeps the geometrically natural presentation, in which case the time-dependent Schrödinger equation is
\[ -\hbar^2 4 \left\{ \frac{1}{I^2} \frac{\partial}{\partial I} \left( I^2 \frac{\partial \Psi}{\partial I} \right) + \frac{3}{16I} \right\} + V \Psi = E \Psi. \] (39)

The present paper is written in support of the choice of ordering made in [42, 43, 46, 47] and more complicated relational particle mechanics models (see e.g. [32, 42]). These are useful models because [47] they possess some midisuperspace-like features that minisuperspace does not possess (nontrivial linear constraint analogue of GR momentum constraint, and a nontrivial notion of localization and therefore of structure formation).

Next, if one sends \( \hat{h} \hat{\Psi} = E \Psi \) to \( \tilde{\hat{h}} \tilde{\hat{\Psi}} = \tilde{E} \tilde{\Psi} = (E/\Omega^2) \tilde{\Psi} \), one’s eigenvalue problem has a weight function \( \Omega^{-2} \) which then appears in the inner product:
\[ \int_{\Sigma} \tilde{\Psi}_1^* \tilde{\Psi}_2 \Omega^{-2} \sqrt{\tilde{M}} d^3x. \] (40)
This inner product additionally succeeds in being banal-conformally invariant, being equal to (cf [1] for the minisuperspace case)

\[ \int_{\Sigma} \psi_{1}^{*} \sqrt{\Omega} \frac{\partial}{\partial \Omega} \psi_{2} \Omega^{-1} \sqrt{\Omega} \delta \frac{\partial}{\partial \Omega} \sqrt{\Omega} d^{4}x = \int_{\Sigma} \psi_{1}^{*} \psi_{2} \sqrt{\Omega} d^{4}x \]  

in the banal representation that is mechanically natural in the sense that \( E \) comes with the trivial weight function, 1.

Generally, \( \tilde{H} = \tilde{\tilde{H}} \) is not in a simple sense self-adjoint with respect to \( \tilde{\langle \cdot \rangle} \), while the mechanically natural \( \tilde{H} \) is, in a simple sense, with respect to \( \langle \cdot \rangle \). More precisely, this is in the sense that

\[ \int \sqrt{\Omega} \frac{\partial}{\partial \Omega} \psi_{1}^{*} \delta \frac{\partial}{\partial \Omega} \psi_{1} \sqrt{\Omega} d^{4}x = \int \sqrt{\Omega} \frac{\partial}{\partial \Omega} \{ \delta \frac{\partial}{\partial \Omega} \psi_{1} \} \psi_{1} + \text{boundary terms}, \]  

which amounts to self-adjointness if the boundary terms can be arranged to be zero, whether by the absence of boundaries in the configuration spaces for one- and two-dimensional relational particle mechanics [32] or by the usual kind of suitable fall-off conditions on \( \Psi \). This is not shared by the \( \Omega \)-inner product as that has an extra factor of \( \Omega^{-1} \), which in general interferes with the corresponding move by the product rule. (\( \sqrt{\Omega} \) does not interfere thus above, since the Laplacian is built out of derivatives that are covariant with respect to the metric \( M_{ab} \).) However, on the premise that solving \( \tilde{H} \psi = \tilde{E} \psi \) is equivalent to solving \( H \psi = E \psi \), the banal-conformal transformation might at this level be viewed as a sometimes-useful computational aid, with the answer then being placed in the mechanically natural banal representation for further physical interpretation.

As regards theories with further, linear constraints, in the case of relational particle mechanics, choosing conformal ordering before and after dealing with the linear constraints does not appear to agree in general (so that arguing for conformal ordering by itself is not a guarantee of unambiguously fixing an ordering). As the structure of the configuration space could impart lucid knowledge whenever the reduction can be done, I would favour performing the reduction and then conformal ordering the reduced configuration space Hamiltonian, as in [42, 46, 47]. In the Dirac quantization approach for relational particle mechanics (‘quantize then constrain’), NB that sending \( P \psi = 0 \) to \( \tilde{P} \tilde{\psi} \) does cause an alteration as \( \tilde{P} \) is a differential operator. The same is the case for the zero total angular momentum constraint \( L \) and the dilational constraint \( D \). On the other hand, the reduced quantization approach (‘constrain and then quantize’) is free of this issue. NB this is an issue which minisuperspace does not model, but relational particle mechanics does model.

Barvinsky [12] investigated that for what ordering these two approaches coincide. On the other hand, e.g. Ashtekar, Horowitz, Romano and Tate [61] have argued for inequivalence of these two approaches to quantization. In any case, to 1 loop (first order in \( \hbar \)) Barvinsky argues that the Laplacian ordering will do the trick. Then, as the \( \xi \text{Ric}(M) \) term contributes only to \( O(\hbar^{2}) \) so that conformal ordering will likewise do to get equivalence between Dirac and reduced quantization equivalence to 1 loop.

4. Comments on quantum geometrodynamics itself

Section 3 contains the use of conformal ordering in minisuperspace, which I would argue is already an important and useful case on which there is substantial literature. For infinite theories like GR, one has not an ordinary but a functional Laplacian:

\[ \mathcal{D}^{2} = \frac{1}{\sqrt{\Omega}} \frac{\delta}{\delta \Omega} \left( \sqrt{\Omega} N_{ab} \frac{\delta}{\delta Q^{b}} \right). \]  

\[ \text{(43)} \]
Using this as one’s ordering for $N_c^{ab}(Q^e)P_a P_b$ continues to have the desirable property of being independent of the coordinate choice on the configuration space. As before, this property is not, however, unique to this ordering: one can include a Ricci scalar curvature term so as to have $D^2 - \xi \text{Ric}(M)$. Proceeding analogously to before, there is then a unique banal-conformally invariant choice among these orderings:

$$D_C^2 = \frac{1}{\sqrt{M}} \frac{\delta}{\delta Q^a} \left( \sqrt{M} N^{ab} \frac{\delta}{\delta Q^b} \right) - \frac{k - 2}{4[k - 1]} \text{Ric}(M),$$

so long as $\Psi$ itself transforms in general tensorially under the conformal transformation

$$\Psi \rightarrow \tilde{\Psi} = \Omega^{\frac{2}{k - 1}} \Psi.$$  \hspace{1cm} (45)

There is now a snag in that $k$ is infinite so (45) becomes ill defined; however, in the operator (44) the coefficient of $\text{Ric}(M)$ merely tends to $1/4$, while the cancellation of $k$ in working (41) also continues to hold in this case, and it is the outcome of this (including its operator expectation counterpart), rather than $\Psi$ itself, that has physical meaning. This gives a Wheeler–DeWitt equation of the form

$$\hbar^2 \left\{ \frac{1}{\sqrt{M}} \frac{\delta}{\delta h_{\mu \nu}} \left( \sqrt{M} N^{\nu \rho \sigma} \frac{\delta}{\delta h_{\rho \sigma}} \right) - \frac{1}{4} \text{Ric}(h) \right\} \Psi + \sqrt{h} \left( \text{Ric}(h) - 2 \Lambda \right) \Psi = 0.$$  \hspace{1cm} (46)

Also in full GR, due to the presence of the linear momentum constraint and the previous section’s insight from relational particle mechanics’ analogous linear constraints, conformal order before and after reduction may differ given the insight from the relational particle mechanics toy models. Moreover one cannot in general perform the reduction here, so the conformal order within the Dirac-type quantization scheme may be questionable.

5. Conclusion

Mechanics and fundamental physics at the classical level can be considered in terms of temporally relational product-type actions, and doing so is useful in considering whole-universe situations—the setting for quantum cosmology. These readily exhibit a banal-conformal invariance under compensating rescalings of the configuration space metric and the potential (alongside the total energy in the case of mechanics). The classical equations of motion resulting from product-type actions simplify for a particular form of emergent time. In mechanics, this amounts to a recovery of Newtonian time from relational premises, while in GR this amounts to a recovery of what is cosmic time or proper time in various contexts. In this paper, we found that this emergent time itself scales when a banal-conformal transformation is performed. Then how a more complicated manifestation of banal-conformal invariance is present in the more commonly used difference-type actions can be deduced, provided that the notions of time in these scale in the same way as the emergent time does. I also clarified that the simplifying effects on the equations of motion through use of the emergent time (e.g. Jacobi–Barbour–Bertotti time) and those through use of affine parametrization of geodesics and dynamical trajectories conspire to cancel via a straightforward (albeit apparently hard to spot) working that hinges on the conservation of energy.

Suppose then that one chooses to retain this banal-conformal invariance—simple and natural from the perspective of relational product actions at the classical level—upon passing to the quantum level. Furthermore, if one adheres to the theoretically desirable and fairly standard tenet that QM be independent of choice of coordinates on configuration space $Q$, then these conditions combine to pick out the operator ordering based on the conformally invariant modification of the Laplacian. While this operator ordering has been suggested
by others previously (as documented in section 1.3), this is the first paper pointing out the
relational underpinning for it. As how one operator order has consequences for the physical
predictions of one’s theory, and there is no established way to prescribe the operator ordering
in the case of (toy models of) quantum gravity, this stronger motivation for conformal ordering
is of wide interest\(^{11}\).

As regards applications to simple toy models, for 2D configuration spaces, conformal
ordering becomes indistinguishable from the also sometimes advocated Laplacian ordering,
while the difference becomes minor for manifolds with constant Ricci scalar. Nor is there any
distinction between these to 1 loop in the semiclassical approach. However, more complicated
modelling situations \([32, 42, 43]\) do have a distinction between Laplacian and conformal
orderings. What is conformal ordering depends in detail (to more than 1 loop) on whether one
Dirac-quantizes or reduced-quantizes. This distinction is already visible in finite but linearly
constrained relational particle models. The present paper mostly contains and supports
the use of relational particle mechanics models, which are useful \([13, 18, 19, 22, 27, 45, 47, 62]\)
though having certain features that are not minisuperspace like but rather midisuperspace like
(nontrivial linear constraint and thus e.g. a Dirac-reduced quantization distinction relevant to
the aforementioned point, as well as nontrivial a notion of localization and thus of structure
formation). These examples are further developed in \([41, 42, 43, 46–48]\).

Inner products, which are the directly physically meaningful constructs in quantum theory,
are found to be suitably banal-conformally invariant. Taking these to be primary, that the
scaling of the wavefunction itself (required for the conformal modification of the Laplacian
operator to actually form a conformally invariant combination) is formally infinite in cases with
infinite configuration space dimensions, appears unproblematic. In particular, this suggests
this paper’s proposed ordering for the Wheeler–DeWitt equation of full geometrodynamics.

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References

\[1\] Misner C W 1972 Magic Without Magic: John Archibald Wheeler ed J R Klauder (San Francisco: Freeman)
\[2\] DeWitt B S 1957 Rev. Mod. Phys. 29 377
\[3\] DeWitt B S 1967 Phys. Rev. 160 1113
\[4\] Kuchař K V 1973 Relativity, Astrophysics and Cosmology ed W Israel (Dordrecht: Reidel)
\[5\] Henneaux M, Pilati M and Teitelboim C 1982 Phys. Lett. B 100 123
\[6\] Hawking S W and Page D N 1986 Nucl. Phys. B 264 185
\[7\] Christodoulakis T and Zanelli J 1986 Nuovo Cim. B 93 1
\[8\] Halliwell J J 1988 Phys. Rev. D 38 2468
\[9\] Moss I 1988 Ann. Inst. H. Poincaré 49 341
\[10\] Page D N 1991 J. Math. Phys. 32 3427
\[11\] Louko J 1988 Ann. Phys. 181 318
\[12\] Barvinsky A O and Krykhn V 1993 Class. Quantum Grav. 10 1957
    Barvinsky A O 1993 Class. Quantum Grav. 10 1985
    Barvinsky A O 1993 Phys. Rep. 10 237

\(^{11}\) There does remain the caveat that QM in general and quantum gravity in particular may have other restrictions on
orderings from such as well-defined existence and self-adjointness of quantum Hamiltonians and of other important
quantum operators. Then possibly another such technical condition could turn out to be incompatible with conformal
ordering at least for some theories/models, but to the best of our knowledge, to date nobody has found any such.
[13] Kuchař K V 1992 Proc. of the 4th Canadian Conf. on General Relativity and Relativistic Astrophysics ed G Kunstatter, D Vincent and J Williams (Singapore: World Scientific)
[14] Isham C J 1993 Integrable Systems, Quantum Groups and Quantum Field Theories ed L A Ibort and M A Rodríguez (Dordrecht: Kluwer) (arXiv:gr-qc/9210011)
[15] Ryan M P and Turbiner A V 2004 Phys. Lett. A 333 30 (arXiv:quant-ph/0406167)
[16] Kuchař K 1978 J. Math. Phys. 19 390
Thiemann T 2007 Modern Canonical Quantum General Relativity (Cambridge: Cambridge University Press)
[17] Barbour J B and Bertotti B 1982 Proc. Roy. Soc. A 382 295
[18] Barbour J B 1994 Class. Quantum Grav. 11 2853
[19] Kiefer C 2007 Quantum Gravity 2nd edn (Oxford: Clarendon) (arXiv:gr-qc/9210011)
[20] Newton I 1999 Philosophiae Naturalis Principia Mathematica (1686 and later editions). For an English translation, see e.g. I B Cohen and A Whitman Berkeley: University of California Press. In particular, see the Scholium on absolute motion therein
[21] Alexander H G (ed) 1956 The Leibniz–Clark Correspondence (Manchester: Manchester University Press)
Berkeley G 1710 The Principles of Human Knowledge
Berkeley G 1721 Concerning Motion (De Motu)
Mach E 1883 Die Mechanik in ihrer Entwickelung, Historisch-kritisch dargestellt (Leipzig: JA Barth)
Mach E 1960 The Science of Mechanics: A Critical and Historical Account of its Development (La Salle, IL: Open Court) (Engl. Transl.)
Barbour J B 1986 Quantum Concepts in Space and Time ed R Penrose and C J Isham (Oxford: Oxford University Press)
Barbour J B 1989 Absolute or Relative Motion? Vol 1: The Discovery of Dynamics (Cambridge: Cambridge University Press)
Barbour J B and Pfister H (eds) 1995 Mach’s Principle: From Newton’s Bucket to Quantum Gravity (Boston: Birkaüser)
Pooley O http://philsci-archive.pitt.edu/archive/00000221/index.html
Pooley O 2003–04 Chronos: Proc. of the Philosophy of Time Society
Pooley O and Brown H R 2002 Br. J. Phil. Sci. 53 183
Barbour J B 1999 The Arguments of Time ed J N Butterfield (New York: Oxford University Press)
Butterfield J N 2002 Br. J. Phil. Sci. 53 289 (arXiv:gr-qc/0103055)
Smolin L 2000 Time and the instant ed R Durie (Manchester: Chinam Press)
Barbour J B forthcoming book
[22] Barbour J B 1999 The End of Time (Oxford: Oxford University Press)
[23] Barbour J B, Foster B Z and Murchadha N ´O 2002 Class. Quantum Grav. 19 3217 (arXiv:gr-qc/0211022)
[24] Barbour J B 2009 arXiv:0903.3489
[25] In fact, I have found evidence that this feature of varying Jacobi-type actions has been known for longer, see e.g. Akhiezer N I 1962 The Calculus of Variations (New York: Blaisdell) pp 188–9 for an English translation of an earlier Russian work
[26] Rovelli C 2004 Quantum Gravity (Cambridge: Cambridge University Press)
[27] Anderson E 2009 Class. Quantum Grav. 26 135020 (arXiv:0809.1168)
[28] See e.g. Misner C W, Thorne K and Wheeler J A 1973 Gravitation (San Francisco: Freedman)
[29] Anderson E and Barbour J B 2002 Class. Quantum Grav. 19 3249 (arXiv:gr-qc/0201092)
Anderson E, Barbour J B, Foster B Z and Murchadha N ´O 2003 Class. Quantum Grav. 20 157 (arXiv:gr-qc/0211022)
Anderson E 2005 General Relativity Research Trends, Horizons in World Physics vol 249, ed A Reimer (New York: Nova) (arXiv:gr-qc/0405022)
Anderson E 2004 Geometrodynamics: Spacetime or Space? PhD Thesis University of London (arXiv:gr-qc/0409123)
Anderson E, Barbour J B, Foster B Z, Kelleher B and Murchadha N ´O 2005 Class. Quantum Grav. 22 1795 (arXiv:gr-qc/0407104)
[30] Anderson E 2003 Phys. Rev. D 68 104001 (arXiv:gr-qc/0302035)
[31] Barbour J B 2003 Class. Quantum Grav. 20 1543 (arXiv:gr-qc/0211021)
[32] Anderson E 2008 Class. Quantum Grav. 25 025003 (arXiv:0706.3934)
[33] Anderson E 2007 Stud. Hist. Phil. Mod. Phys. 38 15 (arXiv:gr-qc/0511070)
[34] Lanczos C 1949 The Variational Principles of Mechanics (Toronto: University of Toronto Press)
[35] Anderson E 2006 Class. Quantum Grav. 23 2469 (arXiv:gr-qc/0511068)
[36] Anderson E 2007 Class. Quantum Grav. 24 2935 (arXiv:gr-qc/0611007)
[37] Anderson E 2007 Class. Quantum Grav. 24 2971 (arXiv:gr-qc/0611008)
[38] Dirac P A M 1964 Lectures on Quantum Mechanics (New York: Yeshiva University)
[39] Gergely L Á 2000 Class. Quantum Grav. 17 1949 (arXiv:gr-qc/0003064)
Gergely L Á and McKain M 2000 Class. Quantum Grav. 17 1963 (arXiv:gr-qc/0003065)
[40] Anderson E 2006 AIP Conf. Proc. 861 285 (arXiv:gr-qc/0509054)
Anderson E 2007 Class. Quantum Grav. 24 5317 (arXiv:gr-qc/0702083)
[41] Anderson E 2010 Proc. 12th Marcel Grossmann Meeting on General Relativity (Paris, 2009) eds T Damour, R Jantzen and R Rulfini (Singapore: World Scientific) at press (arXiv:0908.1983)
[42] Anderson E forthcoming
[43] Anderson E 2010 arXiv:1001.1112
[44] Barbour J B 2003 Decoherence and Entropy in Complex Systems: Proc. of the Conf. (DICE, Piombino, 2002) (Springer Lecture Notes in Physics) ed H-T Elze (arXiv:gr-qc/0309089)
[45] Anderson E 2006 Class. Quantum Grav. 23 2491 (arXiv:gr-qc/0511069)
[46] Anderson E 2009 Class. Quantum Grav. 26 135021 (arXiv:0809.3523)
[47] Anderson E and Franzen A 2009 Class. Quantum Grav. at press (arXiv:0909.2436)
[48] Anderson E 2009 arXiv:0909.2436
[49] Barbour J B and Smolin L unpublished, dating from 1989
[50] Barbour J B 1994 Class. Quantum Grav. 11 2875
[51] Anderson E 2009 Int. J. Mod. Phys. D 18 635 (arXiv:0709.1892)
Anderson E 2008 Proc. of the Second Conf. on Time and Matter ed M O’Loughlin, S Stanič and D Veberič (Nova Gorica, Slovenia: University of Nova Gorica Press) (arXiv:0711.3174)
Smolin L Lecture Course on the Problem of Time, available in video form at http://pirsa.org/C08003
Barbour J B and Foster B Z 2008 arXiv:0808.1223
Gryb S B 2008 arXiv:0804.2900
Gryb S B 2009 Class. Quantum Grav. 26 085015 (arXiv:0810.4152)
[52] Misner C W 1969 Phys. Rev. 186 1319
Misner C W 1970 Relativity: Proc. of the Relativity Conference in the Midwest (Cincinnati, OH, 2–6 June 1969) ed M Carmeli, S I Fickler and L Witten (New York: Plenum)
Hartle J B and Hawking S W 1983 Phys. Rev. D 28 2960
D’Eath P D 1996 Supersymmetric Quantum Cosmology (Cambridge: Cambridge University Press)
[53] Wiltshire D L 1996 Cosmology: The Physics of the Universe ed B Robson, N Visvanathan and W S Woolcock (Singapore: World Scientific) (arXiv:gr-qc/0101003)
[54] Baierlein R, Sharp D and Wheeler J A 1962 Phys. Rev. 126 1864
[55] Anderson E 2008 Class. Quantum Grav. 25 175011 (arXiv:0711.0288)
[56] Arnowitt R, Deser S and Misner C 1962 Gravitation: An Introduction to Current Research ed L Witten (New York: Wiley)
[57] See e.g. Wald R M 1984 General Relativity (Chicago: University of Chicago Press)
[58] See e.g. Stewart J 1991 Advanced General Relativity (Cambridge: Cambridge University Press)
[59] See e.g. Burd A and Tavakol R 1993 Phys. Rev. D 47 5336
[60] DeWitt B S 1970 Relativity: Proc. of the Relativity Conf. in the Midwest (Cincinnati, OH, 2–6 June 1969) ed M Carmeli, S I Fickler and L Witten (New York: Plenum)
[61] Ashtekar A and Horowitz G 1984 J. Math. Phys. 25 1473
Romano J D and Tate R S 1989 Class. Quantum Grav. 6 1487
[62] Smolin L 1991 Conceptual Problems of Quantum Gravity ed A Ashtekar and J Stachel (Boston: Birkhäuser)
Rovelli C 1991 Conceptual Problems of Quantum Gravity ed A Ashtekar and J Stachel (Boston: Birkhäuser) p 292