Renormalization of the $\sigma - \omega$ model within the framework of U(1) gauge symmetry

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It is shown that the $\sigma - \omega$ model which is widely used in the study of nuclear relativistic many-body problem can exactly be treated as an Abelian massive gauge field theory. The quantization of this theory can perfectly be performed by means of the general methods described in the quantum gauge field theory. Especially, the local U(1) gauge symmetry of the theory leads to a series of Ward-Takahashi identities satisfied by Green’s functions and proper vertices. These identities form an uniquely correct basis for the renormalization of the theory. The renormalization is carried out in the mass-dependent momentum space subtraction scheme and by the renormalization group approach. With the aid of the renormalization boundary conditions, the solutions to the renormalization group equations are given in definite expressions without any ambiguity and renormalized S-matrix elements are exactly formulated in forms as given in a series of tree diagrams provided that the physical parameters are replaced by the running ones. As an illustration of the renormalization procedure, the one-loop renormalization is concretely carried out and the results are given in rigorous forms which are suitable in the whole energy region. The effect of the one-loop renormalization is examined by the two-nucleon elastic scattering.

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I. INTRODUCTION

The quantum hadrodynamics (QHD), as a relativistic quantum field theory for baryons and mesons, has been widely applied to studying various nuclear phenomena including the hadron-hadron interaction, the hadron-nucleus scattering, the nuclei and single-particle properties of nuclei, etc. [1-5]. It is commonly recognized that although the quantum chromodynamics is a fundamental theory for strong interaction, the QHD, as an effective field theory formulated in terms of hadronic degrees of freedom, provides a simple and reliable approach to produce the nuclear observables that are insensitive to the short-range dynamics. There are various QHD models, renormalizable and nonrenormalizable, which were tested in the past to reproduce the empirical nuclear properties and the experimental data. Among these models, the $\sigma - \omega$ model proposed by Walecka [1] has been raising particular interest. This model contains proton, neutron and isoscalar, Lorentz scalar and vector mesons $\sigma$ and $\omega$ and in the tree diagram and nonrelativistic approximations leads to a nucleon-nucleon interaction potential which behaves as short-range repulsion and medium-range attraction. The early development of this model is based on the relativistic mean-field and Hartree approximation and shows that the model is quite successful in the applications to the infinite nuclear matter and atomic nuclei. Since the model is renormalizable, it is necessary to consider higher order perturbative corrections to the results given in the mean field approximation by a certain renormalization procedure. Along this line, a number of efforts were made previously [6-19]. Especially, the efforts were mostly concentrated on the renormalization of the model in the study of the nuclear matter at finite temperature and density. In this renormalization, the loop expansion and spectral function methods were applied to evaluate the loop corrections. However, there are various difficulties to occur in the renormalization [8-16]. For example, in Ref. [12], the authors calculated the nuclear matter energy density up to the two loop level and found enormous contributions arising from the loop terms that alter the description of the nuclear bound state qualitatively. Therefore, it was concluded that ”the loop expansion does not provide a reliable approximation scheme in renormalizable QHD” [19]. To this end, one may ask what is the correct procedure of performing the renormalization for a model of QHD? and how to assess the applicability of a renormalizable model of QHD for which the renormalization is carried out? To answer these questions, it is meaningful to examine the renormalization of a QHD model from different angles and, as suggested in Ref. [19], ”to develop and apply systematic and consistent ”power counting” schemes that lead to more general conserving approximations and to study renormalization group methods that could determine the analytic structure of the ground-state energy functional”.

In this paper, we confine ourself to discussing the renormalization of the $\sigma - \omega$ model by the renormalization group method in the case of zero temperature. The procedure is very similar to that described in our previous work on the QED and QCD renormalizations [20]. The main features of the renormalization given in this paper contain two aspects: (1) the renormalization is based on the U(1) gauge symmetry because the $\sigma - \omega$ model, as argued in the
next section, is exactly of the U(1) local gauge symmetry; (2) The renormalization is carried out by a mass-dependent momentum space subtraction [21-24] which will lead to rigorous renormalized results by the renormalization group method [25-28]. Ordinarily, the massive vector fields such as the ω meson field, the ρ meson field and so on are not viewed as gauge fields because the mass term in the Lagrangian is not gauge-invariant [29-31]. On the contrary, it was pointed out in Refs. [32-34] that a massive vector field must be viewed as a constrained system in the whole space of the vector potential $A_\mu(x)$. This is because a massive vector meson has only three polarization states which need only three spatial components of the vector potential $A_\mu(x)$ to describe them. While, the remaining component of the $A_\mu(x)$ appears to be a redundant degree of freedom which must be eliminated by introducing the Lorentz condition. According to the general principle for constrained systems, the gauge-invariance of a massive Abelian or non-Abelian gauge field should be seen from its action given in the physical space defined by the Lorentz condition. This viewpoint will be explained in more detail in the next section. From this viewpoint, it is easy to see that the $\sigma-\omega$ model is surely of U(1) local gauge symmetry. Therefore, the model may be quantized by the method as used in the gauge field theory. In this paper, we will describe the Lorentz-covariant quantization performed in the both of Hamiltonian and Lagrangian path-integral formalisms by following the procedure proposed in Refs. [32-34]. From this quantization, we obtain an effective action which contains a gauge-fixing term and a ghost term in it and manifests itself to be invariant under a set of BRST transformations [35]. It should be mentioned that the quantum theory of the $\sigma-\omega$ model was set up previously by the method of canonical quantization and in the path-integral formalism [29-31, 36, 37]. Especially, with the time paths being generalized to a manifestly covariant form, a covariant path-integral formulation for the model at finite temperature was achieved in Ref. [37] and led to manifestly covariant Feynman rules for both real and imaginary times. Nevertheless, owing to lack of the gauge-fixing term and ghost term in the effective action, the generating functional given in these quantizations would not exhibit the BRST-symmetry.

As emphasized in Ref. [20], a correct renormalization procedure for a gauge field theory must respects the gauge-symmetry (the Ward-Takahashi identities [38, 39]), the Lorentz-invariance (the energy-momentum conservation) and the mathematical convergence principles. Otherwise, the renormalization would be incorrect. From the gauge-invariance (or say, the BRST-symmetry) of the $\sigma-\omega$ model, we derive a set of Ward-Takahashi (W-T) identities satisfied by the generating functionals, Green’s functions and vertices which provide a firm basis for the renormalization of the model. As mentioned before, in this paper, the renormalization of the $\sigma-\omega$ model will be performed in the mass-dependent momentum space subtraction. The prominent advantage of such a subtraction is that it naturally provides boundary conditions satisfied by the renormalized wave functions, propagators and proper vertices for the quantum $\sigma-\omega$ model. These boundary conditions enable us to uniquely determine the solutions to the renormalization group equations for those renormalized quantities. With the solutions of the renormalization group equations, a S-matrix element can be expressed in the form as given in the tree diagrams provided that the physical parameters in the S-matrix element are replaced by the effective (running) ones. To specify the procedure of the renormalization group method, the one-loop effective physical parameters are concretely calculated and given exact and analytical expressions.

The remainder of this paper is arranged as follows. In Sec. II, we present arguments for the gauge-invariance of the $\sigma-\omega$ model. In Sec. III, the $\sigma-\omega$ model will be respectively quantized in the Hamiltonian and Lagrangian path-integral formalisms. In Sec. IV, we will derive a set of W-T identities obeyed by the generating functionals. In Sec. V, a W-T identity satisfied by the $\omega$ meson propagator will be derived and the renormalization of the propagator will be discussed. In Sec. VI, we will derive a W-T identity satisfied by the vectorial vertex (nucleon–nucleon–$\omega$ meson vertex) and discuss the renormalizations of the vertex and the nucleon propagator. In Sec. VII, the renormalizations of the $\sigma$ meson propagator and the scalar coupling vertex (nucleon–nucleon–$\sigma$ meson vertex) will be derived and discussed. Sec. VIII is used to sketch the renormalization group method and the renormalized S-matrix elements. Sec. IX serves to derive the one-loop effective coupling constants and masses. In the last section, summary and discussions will be made. In Appendix A, the gauge-independence of the S-matrix elements given in the one-loop level will be proved. In Appendix B, we will show the differential cross section of the two-nucleon elastic scattering in the approximation of order $g^2$ and examine the effect of the one-loop renormalization on it.

II. ARGUMENT OF GAUGE-INVARINACE FOR THE $\sigma-\omega$ MODEL

The $\sigma-\omega$ model is described by the following Lagrangian density [1]

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu D_\mu - M)\psi - \frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \frac{1}{2} m_\omega^2 A_\mu A_\mu + \frac{1}{2} \partial^\mu \varphi \partial_\mu \varphi - \frac{1}{2} m_\sigma^2 \varphi^2$$ (2.1)

where
\[ \psi = \begin{pmatrix} \psi_p \\ \psi_n \end{pmatrix} \tag{2.2} \]

is the nucleon isospin doublet in which \( \psi_p \) and \( \psi_n \) are the proton and neutron field functions respectively,

\[ D_\mu = \partial_\mu - ig_\nu A_\mu - \frac{i}{4} g_s \gamma_\mu \phi \tag{2.3} \]

is the covariant derivative in which \( A_\mu \) and \( \phi \) stand for the \( \omega \) and \( \sigma \) meson fields, \( g_\nu \) and \( g_s \) designate the vectorial and scalar coupling constants,

\[ F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{2.4} \]

is the vector field strength and \( M, m_\omega \) and \( m_\sigma \) are the masses of nucleon, \( \omega \) meson and \( \sigma \) meson respectively. In the above Lagrangian, the scalar self-couplings are ignored as was done originally in the Walecka model [1].

In the previous, the \( \sigma - \omega \) model was considered to be gauge-non-invariant with respect to the following local U(1) gauge transformations [29-31]

\[ \psi'(x) = e^{ig_\theta(x)} \psi(x), \]
\[ \overline{\psi}(x) = e^{-ig_\theta(x)} \overline{\psi}(x), \]
\[ A'_\mu(x) = A_\mu(x) + \partial_\mu \theta(x), \]
\[ \phi'(x) = \phi(x) \tag{2.5} \]

where \( \theta(x) \) is the scalar parametric function of U(1) group since the mass term of the \( \omega \) meson in the Lagrangian is not gauge-invariant. But, this does not mean that the dynamics of the \( \omega \) meson system is not gauge-invariant. As mentioned in the Introduction, the \( \omega \) meson field must be viewed as a constrained system in the space spanned by the four-dimensional vector potential \( A_\mu(x) \). As we know, a massive gauge field has three polarization states which need only three spatial components of the four-dimensional vector potential \( A_\mu \) to describe them. In the Lorentz-covariant formulation, a full vector potential \( A^\mu(x) \) can be split into two Lorentz-covariant parts: the transverse vector potential \( A^\mu_T(x) \) and the longitudinal vector potential \( A^\mu_L(x) \),

\[ A^\mu(x) = A^\mu_T(x) + A^\mu_L(x) \tag{2.6} \]

where

\[ A^\mu_T(x) = (g^\mu_\nu - \frac{1}{\Box} \partial^\mu \partial^\nu) A_\nu(x), \tag{2.7} \]
\[ A^\mu_L(x) = \frac{1}{\Box} \partial^\mu \partial^\nu A_\nu(x) \tag{2.8} \]

with \( \Box = \partial^\mu \partial_\mu \) being the D’Alembertian operator. The vector potentials \( A^\mu_T(x) \) and \( A^\mu_L(x) \) satisfy the following transverse and longitudinal field conditions (identities):

\[ \partial_\mu A^\mu_T(x) = 0, \tag{2.9} \]
\[ (g^\mu_\nu - \frac{1}{\Box} \partial_\mu \partial_\nu) A^\mu_L(x) = 0 \tag{2.10} \]

and the orthogonality relation

\[ \int d^4x A^\mu_T(x) A^\nu_L(x) = 0 \tag{2.11} \]

which characterizes the linear independence of the two field variables. Since the Lorentz-covariant transverse vector potential \( A^\mu_T(x) \) contains three-independent spatial components, it is sufficient to represent the polarization states of a massive vector boson. Whereas, the Lorentz-covariant longitudinal vector potential \( A^\mu_L(x) \) appears to be a redundant unphysical variable which must be constrained by introducing the Lorentz condition:

\[ \chi \equiv \partial^\mu A_\mu = 0 \tag{2.12} \]
whose solution is

\[ A^\mu_L = 0. \] (2.13)

With this solution, the \( \sigma - \omega \) model Lagrangian may be expressed in terms of the independent dynamical variables \( A^\mu_T(x) \),

\[
\mathcal{L} = \overline{\psi} i \gamma^\mu (i \partial_\mu + g_v A_T^\mu) + \frac{1}{4} g_\ast \gamma^\mu \phi - M \psi - \frac{1}{4} F^\mu\nu F_{\mu\nu} + \frac{1}{2} m_\sigma^2 A^\mu_T A_T^\mu
\] (2.14)

where \( F^\mu\nu \) is defined as in Eq. (2.4) with replacing the \( A^\mu(x) \) by \( A^\mu_T(x) \). The Lagrangian represented above gives a complete description of the dynamics of the \( \sigma - \omega \) model. If we want to represent the dynamics in the whole space of the full vector potential as described by the Lagrangian in Eq. (2.1), the \( \omega \) field must be treated as a constrained system. In this case, according to the general procedure for constrained systems as formulated in Mechanics, the Lorentz condition in Eq. (2.12), as a constraint, must be introduced from the onset and imposed on the Lagrangian in Eq. (2.1) so as to guarantee the redundant degree of freedom to be eliminated from the Lagrangian. Otherwise, the Lagrangian in Eq. (2.1) itself cannot give a complete description for the \( \omega \) field system. From the Lagrangian in Eq. (2.14), one may derive an equation of motion satisfied by the \( \omega \) meson field as follows:

\[ \partial_\mu F^\mu\nu + m_\omega^2 A_\nu = -j^\nu \] (2.15)

where

\[ j^\nu = g_v \overline{\psi} \gamma^\nu \psi \] (2.16)

is the current generated from the nucleon field. The above equation describes the evolution of the independent variable \( A^\mu_T \) with time. In particular, when we take divergence of the both sides of Eq. (2.15), considering the identities in Eq. (2.9) and \( \partial_\mu \partial_\nu F^\mu\nu \equiv 0 \), we immediately obtain the current conservation

\[ \partial_\mu j^\mu = 0 \] (2.17)

which shows that the current is transverse.

Ordinarily, the Lorentz condition is viewed as a consequence of the following \( \omega \) field equation of motion which is derived from the Lagrangian in Eq. (2.1) [29-30, 36]

\[ \partial_\mu F^\mu\nu + m_\omega^2 A_\nu = -j^\nu \] (2.18)

The argument of this viewpoint is as follows. When we take divergence of the equation (2.18) and notice the current conservation, it is found that

\[ m_\omega^2 \partial_\mu A_\mu = 0 \] (2.19)

Since \( m_\omega \neq 0 \), the above equation leads to the Lorentz condition which implies that one component of the vector potential is not independent. It is pointed out here that the above viewpoint actually is an ill-concept and the procedure leading to the Lorentz condition logically is not consistent with the principle established well in the mechanics for constrained systems. In fact, the aforementioned derivation seems to imply that the Lorentz condition has already been included in the Lagrangian denoted in Eq. (2.1). If so, when the Lagrangian is written in the first order form, we should see a term in the Lagrangian which is given by incorporating the Lorentz condition with the aid of the Lagrange multiplier method. Nevertheless, as will be shown in the next section, there is no such a term to appear in the Lagrangian. Moreover, as we know, equations of motion should describe the evolution of the independent variables with time as the equation given in Eq. (2.15) does and should not lead to a constraint condition which implies some variable in the equation is not independent. Therefore, the viewpoint stated above is not reasonable. In accordance with the general principle for constrained systems, the correct procedure is to treat the Lorentz condition as a primary constraint and to impose this condition on the Lagrangian in Eq. (2.1) from the beginning. The necessity of introducing the Lorentz condition can also be seen from the derivation mentioned in Eqs. (2.18) and (2.19). The equation (2.19) can be understood in such a way that if the Lorentz condition is not introduced, there would appear a contradiction that the right hand side of the equation is zero, but the left hand side is not. Only when the Lorentz condition is introduced, the contradiction disappears. In this case, due to the Lorentz condition, the equation (2.19), as a trivial identity, naturally holds and the equation of motion (2.18) can naturally go over to the equation (2.15), exhibiting the self-consistency of the theory. Particularly, in the latter case, when the divergence of the equation (2.18)
is taken and the Lorentz condition is employed, one immediately obtains the current conservation in Eq. (2.17). In addition, we would like to note that for the quantum theory, in the zero-mass limit: $m_\omega \to 0$, the vector field part of the Lagrangian in Eq. (2.1) naturally goes over to the one for the massless vector meson, but, as shown in Sec. V, the vector meson propagator does not and occurs a worse singularity, revealing a severe inconsistence of the theory. Only when the Lorentz condition is introduced initially and incorporated into the Lagrangian by the Lagrange multiplier method, a consistent quantum theory can be constructed.

Now, let us turn to address the gauge-invariance of the $\sigma - \omega$ model. Usually, the gauge-invariance is required to the Lagrangian. From the dynamical viewpoint, as pointed out in Refs. [32-34], the action is of more essential significance than the Lagrangian. This is why in Mechanics and Field Theory, to investigate the dynamical and symmetric properties of a system, one always starts from the action of the system. Similarly, when we examine the gauge-symmetric property of a field system, in more general, we should also see whether the action for the system is gauge-invariant or not. In particular, for a constrained system such as the massive vector field, we should see whether or not the action represented in terms of the independent dynamical variables is gauge-invariant. This point of view is easy to understand from the mechanics for constrained systems. Suppose a mechanical system is described by a Hamiltonian

$$H(\{p_i, q_i\}; i = 1, 2, \cdots, n)$$

which is given in the 2n-dimensional phase space and constraint conditions

$$\varphi_a(p_i, q_i) = 0 (\alpha = 1, 2, \cdots, 2m < 2n)$$

which define a physical phase space of dimension $2(n - m)$ where the system exists and moves only. If the constrained variables can be solved out from the constraint conditions, we may write a Hamiltonian

$$H^*(p_i^*, q_i^*); j = 1, 2, \cdots, n - m$$

which is expressed via the independent variables and gives a complete formulation of the constrained system. Obviously, to examine some symmetry of the constrained system, it is only necessary to see if the Hamiltonian $H^*(p_i^*, q_i^*)$ other than the Hamiltonian $H(p_i, q_i)$ to have the desired symmetry because in contrast to the $H^*(p_i^*, q_i^*)$, the $H(p_i, q_i)$ is not complete for describing the system.

Certainly, in some special cases, the Lagrangian given in the physical space itself is locally gauge-invariant so that the gauge-invariance of the corresponding action is ensured. This situation happens for the massless gauge fields and the massive Abelian gauge field. The gauge transformation of an Abelian gauge field was shown in the third equality in Eq. (2.5). Since $\partial_\mu \theta(x)$ acts as a longitudinal field, according to the decomposition denoted in Eq. (2.6) and considering the independence of the fields $A_{T}^\mu(x)$ and $A_{L}^\mu(x)$, the gauge transformation of the $\omega$ field can be equivalently divided into two transformations:

$$A_{T}^{\mu}(x) = A_{T}^{\mu}(x)$$

$$A_{L}^{\mu}(x) = A_{L}^{\mu}(x) + \partial^\mu \theta(x)$$

Eqs. (2.23) and (2.24) clearly express the fact that the gauge transformation only changes the unphysical longitudinal part of the vector potential, while, the physical transverse vector potential is a gauge-invariant quantity. Furthermore, it is easy to verify that the longitudinal vector potential $A_{T}^{\mu}(x)$, which may be expressed as $A_{L}^{\mu}(x) = \partial^\mu \varphi(x)$ where $\varphi(x)$ is a scalar function, is cancelled in the field strength tensor so that

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu = \partial^\mu A_{L}^{\mu} - \partial^\nu A_{L}^{\nu} = F_{L}^{\mu\nu}$$

This indicates that the longitudinal part of the vector potential has no kinetic energy term in the Lagrangian and hence has no any dynamical meaning. Such a vector potential can only be viewed as a constrained variable. Since the transverse field variable $A_{T}^{\mu}$ is gauge-invariant, the Lagrangian (2.14) which is written in the physical space is manifestly gauge-invariant. Therefore, the action given by this Lagrangian is gauge-invariant. Alternatively, the gauge-invariance may also be seen from the action given by the Lagrangian in Eq. (2.1) which is now constrained by the Lorentz condition. Under the gauge transformation written in Eq. (2.5) and the Lorentz condition denoted in Eq. (2.12), it is easy to find that

$$\delta S = -m^2 \omega \int d^4x \delta \theta A_\mu = 0$$

This indicates that the $\sigma - \omega$ model can surely be set up on the basis of gauge-invariance principle.
III. PATH-INTEGRAL QUANTIZATION OF THE $\sigma - \omega$ MODEL

A. Quantization in the Hamiltonian path-integral formalism

According to the general procedure of dealing with constrained systems, the Lorentz condition (2.12) may be incorporated into the Lagrangian (2.1) by the Lagrange undetermined multiplier method to give a generalized Lagrangian [32, 40]. In the first order formalism [32, 40, 41], this Lagrangian can be written as

$$\mathcal{L}_\lambda = \bar{\psi}(i\gamma^\mu D_\mu - M)\psi + \frac{1}{4}\partial^\mu \varphi \partial_\mu \varphi - \frac{1}{2}m_\sigma^2 \varphi^2 + \frac{1}{4}F^{\mu \nu}F_{\mu \nu}$$

$$- \frac{1}{2}F^{\mu \nu}(\partial_\mu A_\nu - \partial_\nu A_\mu) + \frac{1}{2}m_\sigma^2 A_\mu + \lambda \partial^\mu A_\mu$$

(3.1)

where $A_\mu$ and $F_{\mu \nu}$ are now treated as the mutually independent variables and $\lambda$ is chosen to represent the Lagrange multiplier. Using the canonically conjugate variables defined by

$$\Pi_\psi = \frac{\partial \mathcal{L}}{\partial \dot{\psi}} = i\psi\gamma^0,$$

$$\Pi_{\bar{\psi}} = \frac{\partial \mathcal{L}}{\partial \dot{\bar{\psi}}} = 0,$$

$$\Pi_\varphi = \frac{\partial \mathcal{L}}{\partial \dot{\varphi}} = \dot{\varphi}$$

(3.2)

(3.3)

(3.4)

and

$$\Pi_\mu(x) = \frac{\partial \mathcal{L}}{\partial \dot{A}_\mu} = F_{\mu 0} + \lambda \delta_{\mu 0} = \begin{cases} F_{k 0} = E_k, & \text{if } \mu = k = 1, 2, 3; \\ \lambda = -E_0, & \text{if } \mu = 0, \end{cases}$$

(3.5)

the Lagrangian in Eq. (3.1) may be rewritten in the canonical form

$$\mathcal{L} = E^\mu \dot{A}_\mu + \Pi_\psi \dot{\psi} + \Pi_\varphi \dot{\varphi} + A_0 C - E_0 \chi - \mathcal{H}$$

(3.6)

where $E_\mu = (E_0, E_k)$ is a Lorentz vector,

$$C \equiv \partial^\mu E_\mu + m^2 A_0 + g_\sigma \bar{\psi}\gamma^0 \psi,$$

(3.7)

$\chi$ was defined in (2.12) and $\mathcal{H}$ is the Hamiltonian density expressed by

$$\mathcal{H} = \frac{1}{4}(E_k)^2 + \frac{1}{4}(F_{ij})^2 + \frac{1}{2}m_\sigma^2 [(A_0)^2 + (A_k)^2] + \frac{1}{2}[\Pi_\varphi^2 + (\nabla \varphi)^2$$

$+ m_\sigma^2 \varphi^2] - i\bar{\psi}\gamma^0 \cdot \nabla \psi + M \bar{\psi} \psi - g_\sigma \bar{\psi}\gamma^k \psi A_k - g_\sigma \bar{\psi}\varphi \varphi$

(3.8)

in which $F_{ij}$ was defined in Eq. (2.4). In the above, the four-dimensional and the spatial indices are respectively denoted by the Greek and Latin letters. Eq. (3.6) clearly shows that the terms $A_0 C$ and $E_0 \chi$ are respectively given by incorporating the constraint condition

$$C = 0$$

(3.9)

and the Lorentz condition into the Lagrangian by the Lagrange multiplier method and the Lagrange multipliers $A_0$ and $E_0$ are just the constrained variables themselves in this case. Since the $A_0$ and $E_0$ are a pair of the canonically conjugate unphysical variables, their constraint conditions in Eqs. (2.12) and (3.9) should simultaneously occur in the Lagrangian (3.6). Otherwise, if the Lorentz condition is not introduced, the term $E_0 \chi$ does not appear in the Lagrangian shown in Eq. (3.6) or in Eq. (2.1). In this case, the Lagrangian could not be complete for describing the constrained system under consideration.

From the stationary condition of the action constructed by the Lagrangian (3.6), one may derive the following first-order canonical equations of motion:

$$\dot{A}_k = \partial_k A_0 - E_k,$$

(3.10)
where $H_J$ constrained variables mentioned before, Eq. (2.12) leads to there are altogether twelve equations. They are sufficient to determine the twelve variables including the dynamical completeness of the equations.

The constraint equations obeyed by the constrained variables Green’s functions in terms of the independent canonical variables which are time-derivatives of the dynamical variables $A$, degrees of freedom of polarization for the massive $\omega$ field, while, Eqs. (2.12) and (3.9) can only be regarded as the constraint equations written in Eqs. (2.12) and (3.9). Eqs (3.10) and (3.11) act as the equations of motion satisfied by the independent canonical variables $A_k$ and $E_k(k = 1, 2, 3)$ which precisely describe the three degrees of freedom of polarization for the massive $\omega$ field, while, Eqs. (2.12) and (3.9).

Now, we turn to formulate the quantization performed in the Hamiltonian path-integral formalism for the $\sigma - \omega$ model. In accordance with the general procedure of the quantization, we should first write a generating functional of Green’s functions in terms of the independent canonical variables which are $\psi$, $\bar{\psi}$, $\Pi_\varphi$, $\varphi$, $A_k$ and $E_k(k = 1, 2, 3)$ and one pair constrained variables $A_0$ and $E_0$, showing the completeness of the equations.

One may find the solution to the equation (3.20) as follows:

$$Q(x) = \int d^4y G(x - y)W(y).$$  

(3.23)
From the expressions given in Eqs. (3.19), (3.21) and (3.23), we see that the $E^\mu_L(x)$ is a complicated functional of the variables $A^\mu_L$ and $E^\mu_L$ so that the Hamiltonian $H^\ast(A^\mu_L, E^\mu_L, \cdots)$ is of much more complicated functional structure which is not convenient for constructing the diagram technique in perturbation theory. Therefore, it is better to express the generating functional in Eq. (3.16) in terms of the variables $A_\mu$ and $E_\mu$. For this purpose, it is necessary to insert the following delta-functional into Eq. (3.16) [32, 40, 41]

$$
\delta[A^\mu_L] \delta[E^\mu_L - E^\mu_L(A^\mu_L, \psi, \bar{\psi})] = det M \delta[C] \delta[\psi]
$$

(3.24)

where $M$ is the matrix whose elements are

$$
M(x, y) = \{C(x), \chi(y)\} \equiv \int d^4z \{\delta\chi(x) \delta \chi(y) \over \delta A_\mu(z) \delta \phi^\ast(z) - \delta \chi(y) \delta \chi(x) \over \delta A_\mu(z) \delta \phi^\ast(z)\}
$$

(3.25)

where $\{C(x), \chi(y)\}$ is the Poisson bracket as defined in the second equality in Eq. (3.25). The relation in Eq. (3.24) is easily derived from Eq. (2.12) and (3.9) by applying the property of delta-functional. Upon inserting Eq. (3.24) into Eq. (3.16) and utilizing the Fourier representation of the delta-functional

$$
\delta[C] = \int D(\rho/2\pi)e^{i \int d^4x \rho(x)C(x)}
$$

(3.26)

we have

$$
Z[J^\mu, J, \eta, \bar{\eta}] = \frac{1}{2N} \int D(A_\mu, E_\mu, \psi, \bar{\psi}, \Pi_\phi, \varphi, \rho/2\pi) det M \delta[\chi] \exp\{i \int d^4x [E^\mu_\mu \dot{A}_\mu

+ \Pi_\psi \psi + \Pi_\varphi \varphi + \rho \bar{C} - H(A_\mu, E_\mu, \psi, \bar{\psi}, \Pi_\phi, \varphi) + J^\mu A_\mu + J \varphi + \bar{\eta} \psi + \psi \bar{\eta}]\}
$$

(3.27)

In the above exponent, there is an $E_0$-related term $E_0(\partial_0 A_0 - \partial_0 \rho)$ which permits us to perform the integration over $E_0$, giving a delta-functional

$$
\delta[\partial_0 A_0 - \partial_0 \rho] = det[\partial_0]^{-1}\delta[A_0 - \rho]
$$

(3.28)

The determinant $det[\partial_0]^{-1}$, as a constant, may be put in the normalization constant $N$ and the delta-functional $\delta[A_0 - \rho]$ will disappear when the integration over $\rho$ is carried out. The integrals over $E_0 \Pi_\chi$ are of Gaussian-type and hence easily calculated. After these computations and noticing the expression in Eq. (3.2), we arrive at

$$
Z[J^\mu, J, \eta, \bar{\eta}] = \frac{1}{2N} \int D(A_\mu, \psi, \bar{\psi}, \varphi, \rho/2\pi) det M \delta[\partial_\mu A_\mu] \exp\{i \int d^4x [L + J^\mu A_\mu + J \varphi + \bar{\eta} \psi + \psi \bar{\eta}]\}
$$

(3.29)

where $L$ was written in Eq. (2.1). When employing the familiar expression [41, 42]

$$
det M = \int D(\bar{C}, C)e^{i \int d^4xd4y \bar{C}(x)M(x, y)C(y)} = \int D(\bar{C}, C)e^{i \int d^4x \bar{C}(x)\Box C(x)}
$$

(3.30)

where $\bar{C}(x)$ and $C(x)$ are the mutually conjugate ghost field variables and the following limit for the Fresnel functional

$$
\delta[\partial_\mu A_\mu] = \lim_{\alpha \to 0} C[\alpha] e^{-\alpha \int d^4x (\partial_\mu A_\mu)^2}
$$

(3.31)

where $C[\alpha] \sim \prod_{\alpha} \left( \frac{1}{2\pi \alpha} \right)^{1/2}$ and supplementing the external source terms for the ghost fields, the generating functional in equation (3.29) is finally given in the form

$$
Z[J^\mu, J, \eta, \bar{\eta}, \bar{\xi}, \bar{\xi}] = \frac{1}{2N} \int D(A_\mu, \psi, \bar{\psi}, \varphi, \bar{C}, C) \exp\{i \int d^4x [L_{eff} + J^\mu A_\mu + J \varphi + \bar{\eta} \psi + \psi \bar{\eta} + \bar{\xi} C + C \bar{\xi}]\}
$$

(3.32)

where

$$
L_{eff} = L - \frac{1}{2\alpha} (\partial_\mu A_\mu)^2 + \bar{C} \Box C
$$

(3.33)

which is the effective Lagrangian for the quantized $\sigma - \omega$ model in which the last two terms are the so-called gauge-fixing term and the ghost term, respectively. In Eq. (3.32), the limit $\alpha \to 0$ is implied. Certainly, the theory may be given in arbitrary gauges ($\alpha \neq 0$). In this case, as will be seen shortly, the ghost particle will acquire a spurious mass $\nu = \sqrt{\alpha m_\omega}$. 

8
B. Quantization in the Lagrangian path-integral formalism

Now let us to quantize the $\sigma - \omega$ model in the (second order) Lagrangian path-integral formalism following the procedure proposed in Ref. [32-34, 40]. For later convenience, the Lagrangian in Eq. (2.1) and the Lorentz constraint condition in Eq. (2.12) are respectively generalized to the following forms:

$$\mathcal{L}_\lambda = \mathcal{L} - \frac{\alpha}{2} \lambda^2$$  \hspace{1cm} (3.34)

and

$$\partial^\mu A_\mu + \alpha \lambda = 0$$  \hspace{1cm} (3.35)

where $\lambda(x)$ is an extra function which will be identified with the Lagrange multiplier and $\alpha$ is an arbitrary constant playing the role of gauge parameter. According to the general procedure for constrained systems, Eq. (3.35) may be incorporated into Eq. (3.34) by the Lagrange multiplier method to give a generalized Lagrangian

$$\mathcal{L}_\lambda = \mathcal{L} + \lambda \partial^\mu A_\mu + \frac{1}{2} \alpha \lambda^2.$$  \hspace{1cm} (3.36)

This Lagrangian is obviously not gauge-invariant. However, for building up a correct gauge field theory, it is necessary to require the dynamics of the gauge field to be gauge-invariant. In other words, the action given by the Lagrangian (3.36) is required to be invariant under the gauge transformations shown in Eq. (2.5). By this requirement and applying the constraint condition (3.35), we have

$$\delta S_\lambda = - \frac{1}{\alpha} \int d^4 x \partial^\nu A_\nu(x)(\Box x + \nu^2)\theta(x) = 0$$  \hspace{1cm} (3.37)

where $\nu^2 = \alpha m^2$. From equation (3.35) we see $\frac{1}{\alpha} \partial^\nu A_\nu = -\lambda \neq 0$. Therefore, to ensure the action to be gauge-invariant, the following constraint condition on the gauge group is necessary to be required

$$(\Box x + \nu^2)\theta(x) = 0.$$  \hspace{1cm} (3.38)

The constraint condition in Eq. (3.38) may also be incorporated into the Lagrangian in Eq. (3.36) by the Lagrange undetermined multiplier method. In doing this, it is convenient, as usually done, to introduce the ghost field variable $C(x)$ in such a fashion

$$\theta(x) = \varsigma C(x)$$  \hspace{1cm} (3.39)

where $\varsigma$ is an infinitesimal Grassmann’s number. Based on the above definition, the constraint condition (3.38) can be rewritten as

$$(\Box x + \nu^2)C = 0$$  \hspace{1cm} (3.40)

where the number $\varsigma$ has been dropped. This constraint condition usually is called ghost equation. When the condition (3.40) is incorporated into the Lagrangian (3.36) by the Lagrange undetermined multiplier method, we obtain a more generalized Lagrangian as follows:

$$\mathcal{L}_\lambda = \mathcal{L} + \lambda \partial^\mu A_\mu + \frac{1}{2} \alpha \lambda^2 + \bar{C}(\Box x + \nu^2)C$$  \hspace{1cm} (3.41)

where $\bar{C}(x)$, acting as a Lagrange undetermined multiplier, is the new scalar variable conjugate to the ghost variable $C(x)$. At present, we are ready to formulate the quantization of the $\sigma - \omega$ model. As we learn from the Lagrange undetermined multiplier method, the dynamical and constrained variables as well as the Lagrange multiplier in the Lagrangian (3.41) can all be treated as free ones, varying arbitrarily. Therefore, we are allowed to use this kind of Lagrangian to construct the generating functional of Green’s functions

$$Z[J^\mu, J, \eta, \bar{\eta}, \xi, \bar{\xi}] = \frac{1}{\hbar} \int D(A_\mu, \psi, \bar{\psi}, \phi, \bar{\phi}, C, \lambda) \exp \{ i \int d^4 x \mathcal{L}_\lambda(x) + J^\mu A_\mu + J \phi + \bar{\psi} \psi + \bar{\xi} \xi + \bar{C} \xi + \xi \bar{C} \}.$$  \hspace{1cm} (3.42)

Looking at the expression of the Lagrangian in Eq. (3.41), we see, the integral over $\lambda(x)$ is of Gaussian-type. Upon completing the calculation of this integral, we finally obtain
\[
Z[J^\mu, J, \eta, \bar{\eta}, \xi, \bar{\xi}] = \frac{1}{N} \int \mathcal{D}(A_\mu, \psi, \bar{\psi}, \varphi, \bar{\varphi}, \bar{C}, C) \exp \{i \int d^4x [\mathcal{L}_{\text{eff}}(x) + J^\mu A_\mu + J \varphi + \bar{\eta} \psi + \bar{\psi} \eta + \bar{\xi} C + \bar{C} \xi] \}
\]

(3.43)

where

\[
\mathcal{L}_{\text{eff}} = \mathcal{L} - \frac{1}{2\alpha} (\partial^\mu A_\mu)^2 + \bar{C} (\Box x + \nu^2) C
\]

(3.44)

is the effective Lagrangian given in the general gauges. In the Landau gauge \((\alpha \to 0)\), the Lagrangian (3.44) just goes over to the one given in Eq. (3.33). As proved in Ref. [32], the quantization described in Eqs. (3.34)-(3.44) is equivalent to the quantization performed by the Faddeev-Popov approach [42]. At last of this section, we would like to emphasize that the ghost term in the \(\mathcal{L}_{\text{eff}}\) does not couple to the other fields. But, we do not integrate it out in the generating functional. Keeping this term in the effective action and in the generating functional is helpful to later derivations of W-T identities.

**IV. WARD-Takahashi identities for generating functionals**

**A. BRST-transformation**

In this section, we show that the action and the generating functional in Eq. (3.43) are invariant with respect to a set of BRST transformations which include the infinitesimal gauge transformations of the nucleon, \(\sigma\) meson and \(\omega\) meson fields as well as the transformations for the ghost fields [29-31, 35]. The BRST transformations can be written in the form

\[
\begin{align*}
\delta \psi &= \varsigma \triangle \psi, \\
\delta \bar{\psi} &= \varsigma \bar{\triangle} \psi, \\
\delta A_\mu &= \varsigma \triangle A_\mu, \\
\delta \bar{C} &= \varsigma \bar{\triangle} C, \\
\delta C &= 0, \\
\delta \varphi &= 0
\end{align*}
\]

(4.1)

where

\[
\begin{align*}
\triangle \psi &= ig_\mu C \psi, \\
\bar{\triangle} \psi &= -ig_\mu \bar{C} \psi, \\
\triangle A_\mu &= \partial_\mu C, \\
\bar{\triangle} C &= \frac{1}{\alpha} \partial^\mu A_\mu.
\end{align*}
\]

(4.2)

The above transformations for the nucleon, \(\sigma\) meson and \(\omega\) meson fields can directly be written out from Eqs. (2.5) and (3.39). The transformations for the ghost fields may be found from the stationary condition of the effective action under the BRST transformations for the nucleon, \(\sigma\) meson and \(\omega\) meson fields,

\[
\delta S_{\text{eff}} = \int d^4x \delta \mathcal{L}_{\text{eff}} = \int d^4x \{ (\delta \bar{C} - \frac{\varsigma}{\alpha} \partial^\nu A_\nu)(\Box x + \nu^2) C + \bar{C} (\Box x + \nu^2) \delta C \} = 0.
\]

(4.3)

This expression suggests that when the ghost fields undergo the transformations shown in Eqs. (4.1) and (4.2), the effective action is invariant. It is easy to prove that the integration measure in Eq. (3.43) is also invariant under the BRST-transformations owing to the Jacobian of the transformations being unity.

**B. W-T identity satisfied by the generating functionals for Green’s functions**

When we make the BRST-transformations shown in Eqs. (4.1) to the generating functional in Eq. (3.43) and consider the invariance of the generating functional, the action and the integration measure under the transformations, we obtain an identity such that [29-31]

\[
\frac{1}{N} \int \mathcal{D}(A_\mu, \psi, \bar{\psi}, \varphi, \bar{\varphi}, \bar{C}, C) \int d^4x \{ J^\mu(x) \delta A_\mu(x) + \bar{\eta}(x) \delta \psi(x) + \bar{\eta}(x) \delta \psi(x) + \delta \bar{\psi}(x) \eta(x) + \delta C(x) \xi(x) \} e^{i S_{\text{eff}} + i E \cdot \Phi} = 0
\]

(4.4)
where $E \cdot \Phi$ with $E = (J, J, \eta, \xi, \xi)$ and $\Phi = (A, \varphi, \psi, \bar{\psi}, C, C)$ stands for the external source terms appearing in Eq. (3.43). The Grassmann number $\xi$ contained in the BRST-transformations in Eq. (4.1) may be eliminated by performing a partial differentiation of Eq. (4.4) with respect to $\xi$. As a result, we get a W-T identity as follows:

$$\frac{1}{N} \int D(A, \psi, \bar{\psi}, \varphi, C, C) \int d^4x \{ J^\mu(x) \Delta A_\mu(x) - \bar{\eta}(x) \Delta \psi(x) + \Delta \overline{\psi}(x) \eta(x) + \Delta C(x) \xi(x) \} e^{iS_{eff} + iE \cdot \Phi} = 0. \quad (4.5)$$

In order to represent the composite field functions $\Delta A_\mu, \Delta \bar{\psi}$ and $\Delta \psi$ in Eq. (4.5) in terms of derivatives of the functional $Z$ with respect to external sources, we may, as usual, construct a generalized generating functional by introducing new external sources (called BRST-sources later on) into the generating functional written in Eq. (3.43) by performing a partial differentiation of Eq. (4.4) with respect to $\bar{\eta}$, $\bar{\xi}$, $\eta$, $\xi$, $u^\mu$, $\bar{\tau}$, $v$.

$$Z[J^\mu, J, \eta, \xi, \bar{\tau}, v^\nu, \bar{v}, v]^\mu = \frac{1}{N} \int D(A, \psi, \bar{\psi}, \varphi, C, C) \exp\{iS_{eff} + iE \cdot \Phi + i \int d^4x [u^\mu \Delta A_\mu + \bar{\tau} \Delta \psi + \Delta \overline{\psi} v]\} \quad (4.6)$$

where $u^\mu, \bar{\tau}$ and $v$ are the sources coupled to the functions $\Delta A_\mu, \Delta \bar{\psi}$ and $\Delta \psi$, respectively. Obviously, $u^\mu$ and $\Delta A_\mu$ are anticommuting quantities, while, $\bar{\tau}, v, \Delta \bar{\psi}$ and $\Delta \psi$ are commuting ones. It is easy to verify that the BRST-source terms are invariant under the BRST-transformation because the functions $\Delta A_\mu, \Delta \bar{\psi}$ and $\Delta \psi$ are nilpotent with respect to the BRST-transformations. Thus, we may start from the above generating functional to re-derive the W-T identity. The result is the same as given in Eq. (4.5) except that the external source terms are now extended to include the BRST-sources

$$\frac{1}{N} \int D(A, \psi, \bar{\psi}, \varphi, C, C) \int d^4x \{ J^\mu(x) \Delta A_\mu(x) - \bar{\eta}(x) \Delta \psi(x) + \Delta \overline{\psi}(x) \eta(x) + \Delta C(x) \xi(x) \} \exp\{iS_{eff} + iE \cdot \Phi + i \int d^4x [u^\mu \Delta A_\mu + \bar{\tau} \Delta \psi + \Delta \overline{\psi} v]\} = 0. \quad (4.7)$$

Clearly, Eq. (4.7) may be represented as [29-31]

$$\int d^4x J^\mu(x) \left[ \delta \overline{\psi}(x) + \delta \psi(x) \right] = 0. \quad (4.8)$$

This is the W-T identity satisfied by the generating functional of full Green’s functions.

Apart from the identity in Eq. (4.8), there is another identity called ghost equation. The ghost equation may easily be derived by first making the translation transformation: $C \rightarrow \bar{C} + \bar{\lambda}$ in Eq. (4.6) where $\bar{\lambda}$ is an arbitrary Grassmann variable, then differentiating Eq. (4.6) with respect to $\bar{\lambda}$ and finally setting $\bar{\lambda} = 0$. The result is

$$\frac{1}{N} \int D(A, \bar{C}, C, \bar{\psi}, \psi) \{ \xi(x) + \left( \square_x + \nu^2 \right) C(x) \} \exp\{iS_{eff} + iE \cdot \Phi + i \int d^4x [u^\mu \Delta A_\mu + \bar{\tau} \Delta \psi + \Delta \overline{\psi} v]\} = 0 \quad (4.9)$$

which may be represented in the form [29-31]

$$[\xi(x) + \left( \square_x + \nu^2 \right) \delta \overline{\xi}(x)] Z[J^\mu, \cdots, v] = 0. \quad (4.10)$$

On substituting into Eqs. (4.8) and (4.10) the relation $Z = e^{W}$ where $W$ denotes the generating functional of connected Green’s functions, one may obtain a W-T identity and a ghost equation satisfied by the functional $W$ such that

$$\int d^4x J^\mu(x) \left[ \delta \overline{\psi}(x) + \delta \psi(x) \right] W[J^\mu, \cdots, v] = 0 \quad (4.11)$$

and

$$\xi(x) + \left( \square_x + \nu^2 \right) \delta \overline{\xi}(x) W[J^\mu, \cdots, v] = 0. \quad (4.12)$$
C. W-T identity obeyed by the generating functional for proper vertex functions

The W-T identity in Eq. (4.11) and the ghost equation in Eq. (4.12) may be represented in terms of the generating functional $\Gamma$ for proper (one-particle-irreducible) vertex functions. The functional $\Gamma$ is usually defined by the following Legendre transformation [29-31]

$$\Gamma[A, C, \varphi, \bar{\psi}, \psi; u, \bar{\varphi}, \bar{\psi}, \bar{\psi}, \psi] = W[J, C, \bar{\xi}, \xi, J, \bar{\eta}, \eta, u, \bar{\varphi}, \bar{\psi}, \psi]$$

$$- \int d^4x [J_A^A + \bar{\xi}C + \bar{\xi} + J_C + \bar{\psi} + \bar{\psi}]$$

(4.13)

where $A, C, \varphi, \bar{\psi}$ and $\psi$ are field variables defined by the following functional derivatives:

$$A_\mu(x) = \frac{\delta W}{\delta J_A^A(x)}, \quad \bar{C}(x) = -\frac{\delta W}{\delta \bar{C}(x)}, \quad C(x) = \frac{\delta W}{\delta \psi(x)}$$

$$\bar{\psi}(x) = -\frac{\delta W}{\delta \bar{\psi}(x)}, \quad \psi(x) = \frac{\delta W}{\delta \bar{\psi}(x)}, \quad \varphi(x) = \frac{\delta W}{\delta J_C(x)}$$

(4.14)

From Eq. (4.13), it is not difficult to get the inverse transformations

$$J_\mu(x) = -\frac{\delta \Gamma}{\delta A_\mu(x)}, \quad \bar{\xi}(x) = \frac{\delta \Gamma}{\delta \bar{C}(x)}, \quad \xi(x) = -\frac{\delta \Gamma}{\delta \psi(x)}$$

$$\bar{\eta}(x) = \frac{\delta \Gamma}{\delta \bar{\psi}(x)}, \quad \eta(x) = \frac{\delta \Gamma}{\delta \bar{\psi}(x)}, \quad J_C(x) = -\frac{\delta \Gamma}{\delta J_C(x)}$$

(4.15)

It is obvious that

$$\frac{\delta W}{\delta u} = \frac{\delta \Gamma}{\delta A_\mu}, \quad \frac{\delta W}{\delta v} = \frac{\delta \Gamma}{\delta \psi}, \quad \frac{\delta W}{\delta \bar{\psi}} = \frac{\delta \Gamma}{\delta \bar{\psi}}$$

(4.16)

Employing Eqs. (4.15) and (4.16), Eqs. (4.11) and (4.12) will be represented as

$$\int d^4x \left\{ \frac{\delta \Gamma}{\delta A_\mu} \frac{\delta A_\mu}{\delta \mu} + \frac{\delta \Gamma}{\delta \bar{\psi}} \frac{\delta \bar{\psi}}{\delta \bar{\psi}} + \frac{\delta \Gamma}{\delta \psi} \frac{\delta \psi}{\delta \psi} + \frac{\delta \Gamma}{\delta J_C} \frac{\delta J_C}{\delta C} + \frac{1}{\alpha} \delta A_\mu \frac{\delta \Gamma}{\delta \bar{C}} \right\} = 0$$

(4.17)

and

$$\frac{\delta \Gamma}{\delta \bar{C}} - (\Box + \nu^2)C = 0.$$  

(4.18)

When we define a new functional $\hat{\Gamma}$ in such a manner

$$\hat{\Gamma} = \Gamma + \frac{1}{2\alpha} \int d^4x (\partial^\mu A_\mu)^2,$$

(4.19)

it follows that

$$\frac{\delta \Gamma}{\delta A_\mu} = \frac{\delta \hat{\Gamma}}{\delta A_\mu} + \frac{1}{\alpha} \partial^\mu \partial_\mu A_\mu$$

(4.20)

and

$$\frac{\delta \Gamma}{\delta \Phi} = \frac{\delta \hat{\Gamma}}{\delta \Phi}$$

(4.21)

where $\Phi = \psi, \bar{\psi}, u^\mu, \nu$ and $\bar{\nu}$. Upon inserting Eqs. (4.18)-(4.21) into Eq. (4.17) and noticing $\frac{\delta \Gamma}{\delta u} = \partial^\mu C$, we arrive at

$$\int d^4x \left\{ \frac{\delta \hat{\Gamma}}{\delta A_\mu} \frac{\delta A_\mu}{\delta \mu} + \frac{\delta \hat{\Gamma}}{\delta \bar{\psi}} \frac{\delta \bar{\psi}}{\delta \bar{\psi}} + \frac{\delta \hat{\Gamma}}{\delta \psi} \frac{\delta \psi}{\delta \psi} + \frac{\delta \hat{\Gamma}}{\delta J_C} \frac{\delta J_C}{\delta C} + m^2 \partial^\mu A_\mu C \right\} = 0.$$  

(4.22)

The ghost equation represented through the functional $\hat{\Gamma}$ may be written as

$$\frac{\delta \hat{\Gamma}}{\delta C} - \partial^\mu \frac{\delta \hat{\Gamma}}{\delta \bar{\psi} \delta \psi} - \nu^2 C = 0.$$  

(4.23)

In the Landau gauge, since $\nu = 0$ and $\partial^\mu A_\mu = 0$, Eqs. (4.22) and (4.23) are respectively reduced to
\[ \int d^4x \left\{ \frac{\delta \hat{\Gamma}}{\delta A_\mu} \frac{\delta \hat{\Gamma}}{\delta \psi} + \frac{\delta \hat{\Gamma}}{\delta \psi^c} \frac{\delta \hat{\Gamma}}{\delta \psi} + \frac{\delta \hat{\Gamma}}{\delta \psi^c} \frac{\delta \hat{\Gamma}}{\delta \psi^c} \right\} = 0 \]  \hspace{1cm} (4.24)

and

\[ \frac{\delta \hat{\Gamma}}{\delta C} - \partial^\mu \frac{\delta \hat{\Gamma}}{\delta u^\mu} = 0. \]  \hspace{1cm} (4.25)

These equations formally are the same as those for the massless Abelian gauge field theory [29-31].

From the W-T identities formulated in this section, we may derive various W-T identities obeyed by Green’s functions and vertices, as will be illustrated soon later.

V. W-T IDENTITY FOR \( \omega \) MESON PROPAGATOR AND RENORMALIZATION OF THE PROPAGATOR

The W-T identity satisfied by the \( \omega \) meson propagator can be derived from the identities shown in Eqs. (4.8) and (4.10). By successive differentiations of the identity in Eq. (4.8) with respect to the sources \( J^\nu(y) \) and \( \xi(x) \) and then setting all the sources to be zero, one may obtain

\[ \partial_\nu \left[ \frac{\delta^2 Z}{\delta J^\nu(x) \delta J^\nu(y)} \right] \bigg|_{J^\nu=\xi=\cdots=v=0} = -\alpha \frac{\delta^2 Z}{\delta \xi(x) \delta \xi(y)} \bigg|_{J^\nu=\xi=\cdots=v=0}. \]  \hspace{1cm} (5.1)

Noticing the definitions of the \( \omega \) meson and ghost particle propagators

\[ i D_{\mu\nu}(x-y) = \frac{\delta^2 Z}{i^2 \delta J^\mu(x) \delta J^\nu(y)} \bigg|_{J^\nu=\xi=\cdots=v=0} = \langle 0^+ | T \{ A_\mu(x) A_\nu(y) \} | 0^- \rangle, \]  \hspace{1cm} (5.2)

\[ i \triangle(x-y) = \frac{\delta^2 Z}{\delta \xi(x) \delta \xi(y)} \bigg|_{J^\nu=\xi=\cdots=v=0} = \langle 0^+ | T \{ C(x) C(y) \} | 0^- \rangle \]  \hspace{1cm} (5.3)

(Here and afterward the bold letters represent the operators) and interchanging the coordinate variables and Lorentz indices, Eq. (5.1) may be written as

\[ \partial_\mu D_{\mu\nu}(x-y) = i\alpha \langle 0^+ | T^* \{ \triangle A_\mu(x) C(y) \} | 0^- \rangle = -\alpha \triangle(x-y). \]  \hspace{1cm} (5.4)

where \( T^* \) symbolizes the covariant time-ordering product and the definition of \( \triangle A_\mu \) given in Eq. (4.2) has been considered. Similarly, when taking the derivative of Eq. (4.10) with respect to the source \( \xi(y) \) and then letting all the sources to vanish, we get

\[ (\Box_x + \nu^2) \triangle(x-y) = \delta^4(x-y). \]  \hspace{1cm} (5.5)

This is the equation obeyed by the ghost particle propagator. Differentiating Eq. (5.4) with respect to \( x \) and utilizing Eq. (5.5), we find

\[ \partial_\mu \partial_\nu D_{\mu\nu}(x-y) = -\alpha \Box_x (\Box_x + \nu^2)^{-1} \delta^4(x-y). \]  \hspace{1cm} (5.6)

This just is the W-T identity satisfied by the full \( \omega \) meson propagator.

By the Fourier transformation

\[ D_{\mu\nu}(x-y) = \int d^4k D_{\mu\nu}(k)e^{-ik(x-y)}, \]  \hspace{1cm} (5.7)

Eq. (5.6) becomes

\[ k^\mu k^\nu D_{\mu\nu}(k) = -\frac{\alpha k^2}{k^2 - \nu^2}. \]  \hspace{1cm} (5.8)

The propagator \( D_{\mu\nu}(k) \) may be decomposed into a transverse part and a longitudinal part:
\[ D_{\mu\nu}(k) = D_T(k^2)(g_{\mu\nu} - \frac{k_{\mu}k_{\nu}}{k^2}) + D_L(k^2)\frac{k_{\mu}k_{\nu}}{k^2}. \] (5.9)

Substitution of Eq. (5.9) into Eq. (5.8) gives rise to
\[ D_L(k^2) = -\frac{\alpha}{k^2 - p^2}. \] (5.10)

In comparison of the above expressions with the free propagator which was given in the indefinite-metric approach previously [29] and may easily be derived from the generating functional in Eq. (3.43) by the perturbation method [32]

\[ D^{(0)}_{\mu\nu}(k) = -\frac{g_{\mu\nu} - \frac{k_{\mu}k_{\nu}}{k^2}}{k^2 - m_\omega^2 + i\epsilon} + \frac{\alpha k_{\mu}k_{\nu}}{k^2 - \nu^2 + i\epsilon}, \] (5.11)

one can see that the longitudinal parts in Eqs. (5.9) and (5.11) are the same, implying that the longitudinal part of the \( \omega \) meson propagator does not undergo renormalization.

To derive the expression of the function \( D_T(k^2) \), it is convenient to start from the Dyson equation satisfied by the full \( \omega \) meson propagator [29-31, 36]

\[ D_{\mu\nu}(k) = D^{(0)}_{\mu\nu}(k) + D^{(0)}_{\mu\lambda}(k)\Pi^{\lambda\tau}(k)D_{\tau\nu}(k) \] (5.12)

where \( \Pi^{\lambda\tau}(k) \) stands for the vacuum polarization operator, or say, the self-energy operator of the \( \omega \) meson. Contraction of Eq. (5.12) with \( k^\mu \) and use of the expressions in Eqs. (5.9)-(5.11) yield [1]

\[ k_\lambda \Pi^{\lambda\tau}(k) = 0. \] (5.13)

This is the W-T identity obeyed by the vacuum polarization operator which is a consequence of the gauge symmetry of the theory. The above identity indicates that the operator \( \Pi^{\mu\nu}(k) \) is transverse and therefore can be written in the form

\[ \Pi^{\mu\nu}(k) = (k^2 g_{\mu\nu} - k_{\mu}k_{\nu})\Pi(k^2) \] (5.14)

where \( \Pi(k^2) \) is a scalar function characterizing the vacuum polarization. With the above representation, it is easy to find from Eq. (5.12) that

\[ D_T(k^2) = -\frac{1}{k^2[1 + \Pi(k^2)] - m_\omega^2 + i\epsilon}. \] (5.15)

Thus, the full propagator in Eq. (5.9) can be written as

\[ D_{\mu\nu}(k) = -\frac{g_{\mu\nu} - \frac{k_{\mu}k_{\nu}}{k^2}}{k^2[1 + \Pi(k^2)] - m_\omega^2 + i\epsilon} + \frac{\alpha k_{\mu}k_{\nu}}{k^2 - \nu^2 + i\epsilon}. \] (5.16)

When the gauge parameter \( \alpha \) is taken to be 0 and 1, we obtain the propagators given in the Landau gauge and in the Feynman gauge respectively. When the \( \alpha \) tends to infinity, we have the propagator given in the so-called unitary gauge. In the lowest order perturbative approximation, the latter propagator is of the form [1-5, 29-31]

\[ D^{(0)}_{\mu\nu}(k) = -\frac{g_{\mu\nu} - \frac{k_{\mu}k_{\nu}}{k^2}}{k^2 - m_\omega^2 + i\epsilon}. \] (5.17)

This propagator was originally derived in the canonical quantization from the vacuum expectation value of the time-ordered product of the transverse field operators, \( iD^{(0)}_{\mu\nu}(x - y) = \langle 0 | T\{ A_{T\mu}(x)A_{T\nu}(y)\} | 0 \rangle \) and by making use of the Fourier representation of the transverse field operator \( A_{T\mu}(x) \) in which the \( \omega \) meson momentum \( k \) is put on the mass-shell, \( k^2 = m_\omega^2 \), so that the propagator in Eq. (5.17) is transverse only for this momentum [29, 36]. However, due to the on-shell property of the momentum in Eq. (5.17), when evaluating the contraction \( k^\mu D^{(0)}_{\mu\nu}(k) \), as we see, there appears an indefinite result since the numerator and the denominator in Eq. (5.17) all come to zero. Especially, in the zero-mass limit, there is a serious contradiction that the vector field part of the Lagrangian in Eq. (2.1) is converted to the massless one, but the propagator in Eq. (5.17) does not and is of an awful singularity. In contrast, for the propagator in Eq. (5.11), the momentum is off-shell, \( k^2 \neq m_\omega^2 \). Therefore, for the transverse part of the propagator (or say, the propagator given in the Landau gauge), we have \( k^\mu D^{T}_{\mu\nu}(k) = 0 \), showing a definite result.
Moreover, in the calculation of a loop diagram involving internal ω meson lines in which the momentum of the ω meson line is off-shell, it is necessary to use the propagator in Eq. (5.11). In particular, the good ultraviolet property of the propagator allows us to perform the renormalization safely (in spite of whether the current conservation holds or not). In the zero-mass limit, the propagator in Eq. (5.11) and the vector field part of the Lagrangian in Eq. (3.44) simultaneously go over to the massless ones, exhibiting the logical consistency of the theory.

Now let us discuss renormalization of the ω meson propagator. According to the conventional procedure of renormalization, the divergence included in the functions Π(k²) may be subtracted at a renormalization point, say, k² = μ² where μ may be real or imaginary, corresponding to the subtraction point being timelike or spacelike,

\[ Π(k²) = Π(μ²) + Π_c(k²) \]  

(5.18)

where Π(μ²) and Π_c(k²) are respectively the divergent part and the finite part of the functions Π(k²). The divergent part can be absorbed in the renormalization constant Z₃ which is defined as

\[ Z₃^{-1} = 1 + Π(μ²). \]  

(5.19)

With this definition, on inserting Eq. (5.18) into Eq. (5.16), the ω meson propagator will be renormalized as

\[ D_{μν}(k) = Z₃D_{Rμν}(k) \]  

(5.20)

where

\[ D_{Rμν}(k) = -\{g_{μν} - k_{μ}k_{ν}/k² \over k² - (m_ω^R)² + iε } + α_Rk_{μ}k_{ν}/k² \over (k² - μ² + iε) \} \]

(5.21)

is the renormalized propagator in which m_ω^R is the renormalized mass, α_R the renormalized gauge parameter and Π_R(μ²) denotes the finite correction coming from the loop diagrams. They are defined as

\[ m_ω^R = \sqrt{Z₃m_ω}, α_R = Z₃^{-1}α, Π_R(μ²) = Z₃k²Π_c(k²). \]  

(5.22)

It is noted that the spurious mass ν is a renormalization-invariant quantity, ν² = α_Rm_ω² = ν²_R. Especially, at the renormalization point, Π_R(μ²) = 0, as seen from Eqs. (5.18) and (5.22). In this case, we have a renormalization boundary condition such that

\[ D_{Rμν}(k)|_{k²=μ²} = -\{g_{μν} - k_{μ}k_{ν}/k² \over k² - (m_ω^R)² + iε } + α_Rk_{μ}k_{ν}/k² \over k² - μ² + iε \} \]

(5.23)

which is of the form of free propagator except that the parameters are replaced by the renormalized ones.

VI. W-T IDENTITY FOR THE VECTORIAL VERTEX FUNCTION AND RENORMALIZATION OF THE VERTEX AND THE NUCLEON PROPAGATOR

A. W-T identity for the vectorial vertex function

The W-T identity for the vectorial vertex (nucleon-nucleon-ω meson vertex) can be derived by differentiating the identity in Eq. (4.8) with respect to the sources ξ(x), 7(y) and η(z) and then turning off all the sources. The result derived, written in the operator form, is

\[ \frac{1}{iα} η^μ \partial^μ \langle 0^+ | T\{ A^μ(x)ψ(y)ψ(z) \} | 0^- \rangle \]

\[ = i \langle 0^+ | T\{ C(x)ψ(y)ψ(z) \} | 0^- \rangle + i \langle 0^+ | T\{ C(x)ψ(y)ψ(z) \} | 0^- \rangle \]

(6.1)

where the definitions written in Eq. (4.2) have been used. Similarly, by differentiating the ghost equation in Eq. (4.10) with respect to the sources ξ(y), 7(y) and η(z) and then letting the sources vanishing, one may derive the following equation:

\[ δ^4(x - y) \langle 0^+ | T\{ ψ(y)ψ(z) \} | 0^- \rangle \]

\[ = -i(□x + ν²) \langle 0^+ | T\{ C(x)ψ(y)ψ(z) \} | 0^- \rangle \].

(6.2)
Here it is noted that since there is no coupling between the ghost field and the fermion field, the two fields can not construct a connected Green's function. Therefore, we can write

\[
\langle 0^+ | T\{ C(x)\overline{C}(y)\psi(y)\overline{\psi}(z)\} | 0^- \rangle = \langle 0^+ | T\{ C(x)\overline{C}(y)\} | 0^- \rangle \langle 0^+ | T\{ \psi(y)\overline{\psi}(z)\} | 0^- \rangle 
\]

\[
= - \triangle (x - y) S_F(y - z) 
\]

where

\[
\langle 0^+ | T\{ \psi(x)\overline{\psi}(y)\} | 0^- \rangle = i S_F(x - y) 
\]

is the nucleon propagator. It is easy to verify that once Eq. (6.3) is substituted into the right hand side of Eq. (6.2) and applying Eq. (5.5), we just obtain the expression on the left hand side of Eq. (6.2). Acting on both sides of Eq. (6.1) with the operator \( \Box_x + \nu^2 \) and employing the decomposition in Eq. (6.3) and the ghost equation in Eq. (5.5), we obtain the following W-T identity:

\[
\frac{1}{\alpha g_\nu} (\Box_x + \nu^2) \partial^\rho G_\mu(x, y, z) = i[\delta^4(x - y) - \delta^4(x - z)] S_F(y - z) 
\]

where

\[
G_\mu(x, y, z) = \langle 0^+ | T\{ A_\mu(x)\psi(y)\overline{\psi}(z)\} | 0^- \rangle 
\]

is the three-point Green's function which is connected. This Green's function has the following one-particle irreducible decomposition [29-31]

\[
G_\mu(x, y, z) = \int d^4x' d^4y' d^4z' iD_{\mu\nu}(x - x')iS_F(y - y')i\Gamma^\nu(x', y', z')iS_F(z' - z) 
\]

in which \( \Gamma^\nu(x', y', z') \) is the vectorial proper vertex. On inserting Eq. (6.7) into Eq. (6.5), through the Fourier transformation, we get in the momentum space that

\[
\frac{1}{\alpha g_\nu} (k^2 - \nu^2) k^\mu D_{\mu\nu}(k) S_F(p) \Gamma^\nu(p, q, k) S_F(q) = (2\pi)^4 \delta^4(k + p - q)[S_F(q) - S_F(p)]. 
\]

Considering that the energy-momentum conservation holds at the vertex, we can write

\[
\Gamma_\mu(p, q, k) = (2\pi)^4 \delta^4(k + p - q)ig_\nu[\gamma_\nu + \Lambda_\mu(p, q)] 
\]

With this representation and noticing

\[
(k^2 - \nu^2) k^\mu D_{\mu\nu}(k) = -\alpha k_\nu = \alpha(p - q)_\nu, 
\]

one may obtain from Eq. (6.8) that

\[
(p - q)_\mu[\gamma_\mu + \Lambda_\mu(p, q)] = S_F^{-1}(p) - S_F^{-1}(q). 
\]

It is well-known that the general expression of the nucleon propagator \( S_F(p) \) can be found from the Dyson equation satisfied by the propagator, as was similarly done in Eqs. (5.12)-(5.16) for the \( \omega \) meson propagator. The inverse of the propagator can be written as

\[
S_F^{-1}(p) = p - M - \Sigma(p) 
\]

where \( p = \gamma^\mu p_\mu \) and \( \Sigma(p) \) is the nucleon self-energy. Noticing this expression, when we differentiate the both sides of Eq. (6.11) with respect to \( p^\mu \) and then set \( q = p \), it is found that [1]

\[
\Lambda_\mu(p, p) = -\frac{\partial \Sigma(p)}{\partial p^\mu}. 
\]

This is the W-T identity which establishes the relation between the vectorial proper vertex and the nucleon self-energy. It is interesting to note that the above identity determines the subtraction fashion of the nucleon self-energy. As one knows, the divergence in the vertex \( \Lambda_\mu(p, q) \) may be subtracted at the renormalization point \( \mu \) in such a way...
\[ \Lambda_\mu(p, p) = L \gamma_\mu + \Lambda^\mu_{\mu}(p) \]  
(6.14)

where

\[ L = \Lambda_\mu(p, p) \mid_{p=p} \]  
(6.15)

is a divergent constant. Substituting Eq. (6.14) into Eq. (6.13) and then integrating Eq. (6.13) over \( p_\mu \) from \( p^0_\mu \) to \( p_\mu \), we get

\[ \Sigma(p) = \Sigma(\mu) - L(p - \mu) - (p - \mu)C(p^2) \]  
(6.16)

where we have chosen the \( p^0_\mu \) to meet \( \gamma^\mu p_\mu = \mu \) and set the integral \( \int_{p^0_\mu}^{p_\mu} dp^\mu \Lambda^\mu_{\mu}(p^2) = (p - \mu)C(p^2) \) with the consideration that when \( p = p^0 \), the integral vanishes and \( \Lambda^\mu_{\mu}(p) \) is finite, satisfying the boundary condition \( \Lambda^\mu_{\mu}(p) \mid_{p=p} = 0 \) so that the \( C(p^2) \) is also finite, having the boundary condition \( C(\mu^2) = 0 \). When the divergent constants \( \Sigma(\mu) \) and \( L \) are set to be

\[ \Sigma(\mu) = A \]  
(6.17)

and

\[ L = -B, \]  
(6.18)

Eq. (6.16) will be written in the form

\[ \Sigma(p) = A + (p - \mu)[B - C(p^2)] \]  
(6.19)

This is the formula that gives the uniquely correct way for the subtraction of the nucleon self-energy.

**B. Renormalization of the nucleon propagator and the vectorial vertex**

Based on the representation of the self-energy in Eq. (6.19), the full nucleon propagator may be written as

\[ S_F(p) = \frac{1}{p(1-B) - (M + A - \mu B) + (p-\mu)C(p^2)}. \]  
(6.20)

With the renormalization constant of the nucleon propagator defined by

\[ Z_2^{-1} = 1 - B, \]  
(6.21)

the nucleon propagator will be renormalized as

\[ S_F(p) = Z_2 S^R_F(p) \]  
(6.22)

where

\[ S^R_F(p) = \frac{1}{p - M_R - \Sigma_R(p) + i\epsilon} \]  
(6.23)

is the renormalized propagator in which \( M_R \) and \( \Sigma_R(p) \) are the renormalized mass and the finite renormalization correction respectively. They are separately represented in the following:

\[ M_R = Z_M^{-1} M \]  
(6.24)

where \( Z_M \) is the nucleon mass renormalization constant defined by

\[ Z_M^{-1} = 1 + Z_2 \left[ \frac{A}{M} + (1 - \frac{\mu}{M})B \right] \]  
(6.25)

and

\[ \Sigma_R(p) = -Z_2(p-\mu)C(p^2) \]  
(6.26)
with the boundary condition $\Sigma_R(p) \mid_{p=\mu} = 0$ which leads to the boundary condition of nucleon propagator like this
\[ S^R_F(p) \mid_{p=\mu} = \frac{1}{p-M_R+i\epsilon}. \] (6.27)

Clearly, this propagator is formally the same as the free propagator.

We would like to mention here the renormalization of the vertex function defined by $\hat{\Gamma}_\mu(p,q) = \gamma_\mu + \Lambda_\mu(p,q)$. In view of the subtraction in Eq. (6.14) and the following definition of vertex renormalization constant $Z_1$:
\[ Z_1^{-1} = 1 + L, \] (6.28)
the vertex function will be renormalized as
\[ \hat{\Gamma}_\mu(p,q) = Z_1^{-1}\hat{\Gamma}^R_\mu(p,q) = Z_1^{-1}[\gamma_\mu + \Lambda^R_\mu(p,q)] \] (6.29)
where
\[ \Lambda^R_\mu(p,q) = Z_1\Lambda^c_\mu(p,q). \] (6.30)
is the finite renormalization correction to the vertex. From the boundary condition $\Lambda^c_\mu(p) \mid_{p=\mu} = 0$ mentioned before, it follows $\Lambda^R_\mu(p,q) \mid_{p=q=\mu} = 0$ by which we have
\[ \hat{\Gamma}^R_\mu(p,q) \mid_{p=q=\mu} = \gamma_\mu. \] (6.31)
This just is the boundary condition for the renormalized vertex function $\hat{\Gamma}^R_\mu(p,q)$ under which the vertex is of the form of the bare vertex. In particular, from Eqs. (6.18), (6.21) and (6.28), it is clear to see that
\[ Z_2 = Z_1, \] (6.32)
This is the Ward identity satisfied by the nucleon propagator renormalization constant and the vertex one.

At last, it is pointed out that the identities shown in Eqs. (6.13) and (6.32) and the subtraction represented in Eq. (6.19) formally are the same as those in QED because they are all the consequence of U(1) gauge symmetry. Originally, the identities mentioned above follow from the current conservation. This result is natural because the current conservation, as generally proved in the Gauge Field Theory [29-31], can be derived from the global gauge symmetry or the local gauge symmetry. It would be emphasized that the aforementioned identities hold not only for the case where the $\omega$ meson is considered only, but also for the general case that the $\omega$ meson and the $\sigma$ meson are taken into account together. We take one-loop diagrams to illustrate this point. The one-loop nucleon self-energy in the $\sigma - \omega$ model is represented in Fig. (1a) and Fig. (1b). The one-loop vectorial vertex is shown in Fig. (2a) and Fig. (2b). In the figures, the solid line designates the free nucleon propagator $iS^F(0)$, the wavy line stands for the free $\omega$ meson propagator $iD^{\nu\mu}(0)$ written in Eq. (5.11), the dashed line denotes the free $\sigma$ meson propagator which is of the form
\[ i\triangle(q) = \frac{i}{q^2 - m_\sigma^2 + i\epsilon}. \] (6.33)
the bare vectorial vertex $\Gamma^{(0)}_\mu$ and the bare scalar vertex $\Gamma^{(0)}$ are
\[ \Gamma^{(0)}_\mu = ig_\omega\gamma_\mu, \quad \Gamma^{(0)} = ig_\sigma. \] (6.34)
The above Feynman rules are easily derived from the generating functional in Eq. (3.43) by the perturbation method. Applying the Feynman rules, for the one-loop nucleon self-energy defined by $-i\Sigma(p)$, we have the following expression
\[ \Sigma(p) = i \int d^4k [g_\omega^2\gamma^\nu S^{(0)}_F(p-k)\gamma^\nu D^{(0)}_{\mu\nu}(k) + g_\sigma^2 S^{(0)}_F(p-k)\triangle^{(0)}(k)] \] (6.35)
where the first term and the second one are respectively given by Fig. (1a) and Fig. (1b). While, for the one-loop vectorial vertex, in accordance with the definition in Eq. (6.9), we have
\[ \Lambda_{\mu}(p, q) = i \int d^4k [g_\sigma^2 \gamma^\mu S_F^{(0)}(q - k) \gamma_\mu S_F^{(0)}(p - k) \gamma^\lambda D_{\mu\lambda}^{(0)}(k)] + g_\sigma^2 S_F^{(0)}(q - k) \gamma_\mu S_F^{(0)}(p - k) \triangle^{(0)}(k). \] (6.36)

where the first and second terms are given by Fig. (2a) and Fig. (2b) respectively. By making use of the derivative
\[ \frac{\partial}{\partial p^\mu} S_F(p - k) = -S_F(p - k) \gamma^\mu S_F(p - k), \] (6.37)
it is easy to find that the identity in Eq. (6.13) holds. Thus, the correctness of the W-T identity in Eq. (6.13) which follows from the U(1) gauge symmetry of the model is verified by the perturbative calculation. The identities in Eqs. (6.13) and (6.32) will be helpful to facilitate calculations of the renormalization of the \( \sigma - \omega \) model.

**VII. RENORMALIZATION OF THE \( \sigma \) MESON PROPAGATOR AND THE SCALAR VERTEX**

For later convenience, it is necessary to give a general description for the renormalization of the \( \sigma \) meson propagator and the scalar vertex. We start from the Dyson equation satisfied by the \( \sigma \) meson full propagator
\[ \triangle(q) = \triangle^{(0)}(q) + \triangle^{(0)}(q) \Omega(q) \triangle(q) \] (7.1)
where \( \triangle^{(0)}(q) \) is the \( \sigma \) meson free propagator shown in Eq. (6.33) and \( -i\Omega(q) \) represents the \( \sigma \) meson self-energy. From Eq. (7.1) it may be solved that
\[ \triangle(q) = \frac{1}{q^2 - m_\sigma^2 - \Omega(q) + i\epsilon}. \] (7.2)
The self-energy can be Lorentz-covariantly decomposed into
\[ \Omega(q) = \Omega_1(q^2)q^2 + \Omega_2(q^2)m_\sigma^2. \] (7.3)
The divergence in the \( \Omega(q) \) can be subtracted at the renormalization point \( \mu \) in such a way
\[ \Omega_1(q^2) = \Omega_1(\mu^2) + \Omega_1^c(q^2), \]
\[ \Omega_2(q^2) = \Omega_2(\mu^2) + \Omega_2^c(q^2). \] (7.4)
On substituting Eq. (7.4) in Eq. (7.2), the propagator \( \triangle(q) \) will be renormalized as
\[ \triangle(q) = Z_3^\prime \triangle_R(q) \] (7.5)
where
\[ Z_3^{\prime -1} = 1 - \Omega_1(\mu^2) \] (7.6)
is the renormalization constant of the \( \sigma \) meson propagator,
\[ \triangle_R(q) = \frac{1}{q^2 - (m^{R\sigma})^2 - \Omega_R(q) + i\epsilon}, \] (7.7)
is the renormalized propagator in which
\[ m^{R\sigma} = Z_m^{\prime -1} m_\sigma \] (7.8)
is the renormalized \( \sigma \) meson mass with
\[ Z_m^\prime = |Z_3^{\prime}(1 + \Omega_2(\mu^2))|^{-\frac{1}{2}}, \] (7.9)
being the renormalization constant of the \( \sigma \) meson mass and
\[ \Omega_R(q) = Z_3[q^2 \Omega_1^c(q^2) + m_\sigma^2 \Omega_2^c(q^2)] \] (7.10)
is the finite correction to the renormalized propagator. Obviously, the \( \Omega_R(q) \) has the boundary condition \( \Omega_R(q) \mid_{q^2 = \mu^2} = 0 \) which leads to the boundary condition of the propagator as follows:
\[ \Delta_R(q) \mid_{q^2=\mu^2} = \frac{1}{q^2 - (m_R^2)^2 + i\varepsilon} \]  

(7.11)

This propagator formally is the same as the free propagator in Eq. (6.33).

Analogous to Eq. (6.9) for the vectorial vertex, the scalar vertex can be written as

\[ \Gamma(p, q, k) = (2\pi)^4\delta^4(k+p-q)ig_s\Gamma(p, q) \]  

(7.12)

where

\[ \Gamma(p, q) = 1 + \Lambda(p, q) \]  

(7.13)

in which \( \Lambda(p, q) \) denotes the contribution of all higher order diagrams. When the divergence in the \( \Lambda(p, q) \) is subtracted at the renormalization point \( \mu \), we have

\[ \Lambda(p, q) = L' + \Lambda_c(p, q) \]  

(7.14)

where

\[ L' = \Lambda(p, p) \mid_{p=\mu} \]  

(7.15)

is the divergent constant and \( \Lambda_c(p, q) \) is the finite correction of the \( \Lambda(p, q) \). With the above subtraction, the vertex in Eq. (7.13) will be renormalized as

\[ \hat{\Gamma}(p, q) = Z_1^{-1}\hat{\Gamma}_R(p, q) = Z_1^{-1}[1 + \Lambda_R(p, q)] \]  

(7.16)

where \( Z_1' \) is the renormalization constant of the scalar vertex defined by

\[ Z_1' = 1 + L'. \]  

(7.17)

and \( \Lambda_R(p, q) = Z'\Lambda_c(p, q) \) is the finite correction of the \( \hat{\Gamma}_R(p, q) \) with the boundary condition \( \Lambda_R(p, q) \mid_{p=q=\mu} = 0 \) which yields the boundary condition for the renormalized vertex \( \hat{\Gamma}_R(p, q) \) as follows:

\[ \hat{\Gamma}_R(p, q) \mid_{p=q=\mu} = 1. \]  

(7.18)

This shows that at the renormalization point, the renormalized vertex is reduced to the form of bare vertex.

It is interesting to note that there is an identity which holds between the nucleon self-energy and the scalar vertex. For example, from the expression of the one-loop scalar vertex \( \Lambda(p, q) \) shown in Fig. (3a) and Fig. (3b)

\[ \Lambda(p, q) = ig_s^2 \sum_F^{(0)}(q-k) \Delta^{(0)}(k) S_F(p-k) + g_s^2\gamma^\mu S_F^{(0)}(q-k) S_F^{(0)}(p-k)\gamma^\nu D^{(0)}_{\mu\nu}(k) \]  

(7.19)

and the following derivative

\[ \frac{\partial}{\partial M} S_F(p-k) = S_F(p-k) S_F(p-k), \]  

(7.20)

we find

\[ \Lambda(p, p) = \frac{\partial \Sigma(p)}{\partial M}. \]  

(7.21)

Based on this identity and the expression in Eq. (6.19), the constant defined in Eq. (7.15) can be computed by

\[ L' = \left. \frac{\partial \Sigma(p)}{\partial M} \right|_{p=\mu} = \frac{\partial A}{\partial M}. \]  

(7.22)

This relation will be used to simplify the calculation of renormalization of the \( \sigma - \omega \) model.
Suppose $F_R$ is a renormalized quantity. In the multiplicative renormalization, it is related to the unrenormalized one $F$ in such a way

$$F = Z_F F_R$$  \hspace{1cm} (8.1)$$
where $Z_F$ is the renormalization constant of $F$. The $Z_F$ and $F_R$ are all functions of the renormalization point $\mu = \mu_0 e^t$ where $\mu_0$ is a fixed renormalization point corresponding to the zero value of the group parameter $t$. Differentiating Eq. (8.1) with respect to $\mu$ and noticing that the $F$ is independent of $\mu$, we immediately obtain a renormalization group equation (RGE) satisfied by the function $F_R$ [27–30]

$$\mu \frac{dF_R}{d\mu} + \gamma_F F_R = 0$$  \hspace{1cm} (8.2)$$
where $\gamma_F$ is the anomalous dimension defined by

$$\gamma_F = \mu \frac{d}{d\mu} \ln Z_F.$$  \hspace{1cm} (8.3)$$

We first note here that because the renormalization constant is dimensionless, the anomalous dimension can only depends on the ratio $\sigma = \frac{m_R}{\mu}$, $\gamma_F = \gamma_F(g_R, \sigma)$, where $m_R$ and $g_R$ are the renormalized mass and coupling constant respectively. Next, we note that Eq. (8.2) is suitable for a physical parameter (mass or coupling constant), a propagator, a vertex, a wave function or some other Green function. If the function $F_R$ stands for a renormalized Green function, vertex or wave function, in general, it depends explicitly not only on the scale $\mu$, but also on the renormalized coupling constant $g_R$, mass $m_R$ and gauge parameter $\alpha_R$ which are all functions of $\mu$, $F_R = F_R(p; g_R(\mu), m_R(\mu), \alpha_R(\mu); \mu)$ where $p$ symbolizes all the momenta. Considering that the function $F_R$ is homogeneous in the momentum and mass, it may be written, under the scaling transformation of momentum $p = \lambda p_R$, as follows:

$$F_R(p; g_R, m_R, \alpha_R; \mu) = \frac{D_F}{D_F} F_R(p_0; g_R; m_R; \alpha_R; \mu)$$  \hspace{1cm} (8.4)$$

where $D_F$ is the canonical dimension of $F$. Since the renormalization point is a momentum taken to subtract the divergence, we may set $\mu = \mu_0 \lambda$ where $\lambda = e^t$ which is taken to be the same as in $p = p_0 \lambda$. Noticing the above transformation, the solution to the RGE in Eq. (8.2) can be expressed as

$$F_R(p; g_R, m_R, \alpha_R, \mu_0) = \lambda^{D_F} e^{\int_1^\lambda \gamma_F \lambda} F_R(p_0; g_R(\lambda), m_R(\lambda)\lambda^{-1}, \alpha_R(\lambda); \mu_0)$$  \hspace{1cm} (8.5)$$

where $g_R(\lambda), m_R(\lambda)$ and $\alpha_R(\lambda)$ are the effective (running) coupling constant, mass and gauge parameter, respectively. The solution written above describes the behavior of the function $F_R$ under the scaling of momenta.

How to determine the function $F_R(p_0; \cdots, \mu_0)$ on the right-hand side of Eq. (8.5) when the $F_R(p_0, \cdots)$ stands for a wave function, a propagator or a vertex? This question can be unambiguously answered in the momentum space subtraction scheme. Noticing that the momentum $p_0$ and the renormalization point $\mu_0$ are fixed, but may be chosen arbitrarily, we can, certainly, set $\mu_0 = p^2_0$. With this choice, by making use of the boundary condition satisfied by the propagator, the vertex or the wave function as denoted in Eqs. (5.23), (6.27), (6.31), (7.11) and (7.18), we may write

$$F_R(p_0; g_R, m_R, \alpha_R, \mu) \mid_{\mu_0} = F_R^{(0)}(p_0; g_R, m_R, \alpha_R)$$  \hspace{1cm} (8.6)$$

where the function $F_R^{(0)}(p; g_R, m_R, \alpha_R)$ is of the form of a free propagator, a bare vertex (if the vertex is fundamental, i.e., follows directly from the interaction Lagrangian) or a free wave function and independent of the renormalization point (see the examples given in the preceding sections). In light of the boundary condition in Eq. (8.6) and considering the homogeneity of the function $F_R$ as mentioned in Eq. (8.4), one can write

$$\lambda^{D_F} F_R(p_0; g_R(\lambda), m_R(\lambda)\lambda^{-1}, \alpha_R(\lambda); \mu_0) \mid_{\mu_0} = F_R^{(0)}(p; g_R(\lambda), m_R(\lambda), \alpha_R(\lambda))$$  \hspace{1cm} (8.7)$$

where the renormalized coupling constant, mass and gauge parameter in the function $F_R^{(0)}(p, \cdots)$ become the running ones. With the expression given in Eq. (8.7), Eq. (8.5) will finally be written in the form [20]

$$F_R(p; g_R, m_R, \alpha_R) = e^{\int_1^\lambda \gamma_F \lambda} F_R^{(0)}(p; g_R(\lambda), m_R(\lambda), \alpha_R(\lambda)).$$  \hspace{1cm} (8.8)$$
For a gauge field theory, the anomalous dimensions shown in Eq. (8.8) are all cancelled out in S-matrix elements. To show this point more specifically, let us take the two-nucleon scattering taking place via \( \omega \) meson exchanges as an example. The exact matrix element for the two-nucleon scattering can be written out from the well-known reduction formula which establishes the relation between an on-mass-shell S-matrix element and the corresponding off-mass-shell connected Green’s function [29, 36]. A connected Green’s function may conveniently be derived from the generating functional \( W \) for connected Green’s functions as mentioned in Sec. IV. For the nucleon-nucleon scattering, the S-matrix element is related to the following four-point connected Green’s function

\[
G_c(x_1, x_2; y_1, y_2) = \langle 0^+ | T \{ \bar{\psi}(x_1) \psi(x_2) \bar{\psi}(y_1) \psi(y_2) \} | 0^- \rangle_C
\]

where the subscript \( C \) marks the connectivity of the Green’s function. According to the familiar procedure of irreducible decomposition [29-31], the connected Green’s function in Eq. (8.9) can be decomposed into three one-particle irreducible ones as represented graphically in Figs. (4a)-(4c). In each of the diagrams, there are four external legs which represent the full off-mass-shell nucleon propagators. These propagators will be converted to the full on-mass-shell nucleon wave functions by the reduction formula. The shaded blobs in the diagrams stand for the exact proper (one-particle irreducible) vertices. Let us first concentrate our attention on the diagrams in Figs. (4a) and (4b). These two diagrams represent the two-nucleon scattering taking place in the t-channel via a \( \omega \) meson exchange. The wavy line with a white blob in the figures denotes the full \( \omega \) meson propagator. Considering the well-known fact that a S-matrix element expressed in terms of unrenormalized quantities is equal to that represented by the corresponding renormalized quantities, the scattering amplitude given by Figs. (4a) and (4b) may be written as [20, 29, 36]

\[
T_f^{(1)} = \bar{u}_R(p_1) \gamma_\mu(q_1) \Gamma_R^{\mu}(q_1, p_1) u_R(p_1) i D_{\mu\nu}^R(k) \bar{u}_R(p_2) \Gamma_R^{\nu}(p_2, q_2) u_R(p_2)
\]

where \( k = q_1 - p_1 = p_2 - q_2, u_R^\alpha(p) \), \( \Gamma_R^\mu(q_1, p_1) \) and \( i D_{\mu\nu}^R(k) \) are the nucleon wave function, the proper vectorial vertex and the \( \omega \) meson propagator respectively which are all renormalized. The renormalization constants of the wave function, the propagator and the vertex are denoted by \( \sqrt{Z_2} \), \( Z_3 \) and \( Z_T \) respectively. The constant \( Z_T \) is defined by

\[
Z_T = Z_2^{-1} Z_3^{-\frac{1}{2}}
\]

because the vertex in Eq. (8.10) is now defined by containing a vectorial coupling constant \( g_v^R \) multiplied with an imaginary number \( i \) in it. The renormalized coupling constant is defined as

\[
g_v^R = \frac{Z_2 \sqrt{Z_3}}{Z_1} g_v
\]

On the basis of the formula given in Eq. (8.8), the renormalized nucleon wave function, meson propagator and vertex can be represented in the forms as shown separately in the following. For the nucleon wave function, we have

\[
u_R^\alpha(p) = e^{\int_{\mu_f}^{\mu_0} \frac{d\mu}{2} \gamma_w(\lambda)} u_R^{(0)}(p)
\]

where

\[
u_R^{(0)}(p) = \left( \frac{E + M_R(\lambda)}{2M_R(\lambda)} \right)^{1/2} \left( \frac{1}{\bar{p} \cdot \bar{p}} \right) \varphi_\alpha(\bar{p})
\]

is the renormalized wave function which formally is the same as the free wave function, but, the \( M_R(\lambda) \) in it is a running mass and

\[
\gamma_w = \frac{1}{2} \frac{d}{d\mu} \ln Z_2
\]

is the anomalous dimension of the nucleon wave function.

For the renormalized \( \omega \) meson propagator, we can write

\[
i D_{\mu\nu}^R(k) = e^{\int_{\mu_f}^{\mu_0} \frac{d\mu}{2} \gamma_\omega(\lambda)} i D_{\mu\nu}^{R(0)}(k)
\]

where
\[ iD_{\mu\nu}^{R(0)}(k) = -\frac{i}{k^2 - m_R^2(\lambda) + i\varepsilon} [g_{\mu\nu} - (1 - \alpha_R(\lambda)) \frac{k_{\mu}k_{\nu}}{k^2 - \nu^2 + i\varepsilon}] \]  
(8.17)
is the free propagator with \(m_R^2(\lambda)\) and \(\alpha_R(\lambda)\) in it being the running \(\omega\) meson mass and gauge parameter and

\[ \gamma_3(\lambda) = \mu \frac{d}{d\mu} \ln Z_3 \]  
(8.18)
is the anomalous dimension of the propagator.

For the renormalized vertex, it reads

\[ \Gamma_R^\mu(q_i, p_j) = e^{\int_{\lambda}^\lambda \frac{d\lambda}{\lambda}} \gamma_\nu(\lambda) \Gamma_R^{(0)\mu}(q_i, p_j) \]  
(8.19)
where \(i, j = 1, 2,\)

\[ \Gamma_R^{(0)\mu}(q_i, p_j) = ig_v^R(\lambda)\gamma^\mu \]  
(8.20)
is the bare vertex in which \(g_v^R(\lambda)\) is the running coupling constant and

\[ \gamma_v(\lambda) = \mu \frac{d}{d\mu} \ln Z_v = -\mu \frac{d}{d\mu} \ln Z_2 - \frac{1}{2} \mu \frac{d}{d\mu} \ln Z_3 \]  
(8.21)
is the anomalous dimension of the vertex here the relation in Eq. (8.11) has been used.

Upon substituting Eqs. (8.13), (8.16) and (8.19) into Eq. (8.10) and noticing Eqs. (8.15), (8.18) and (8.21), we find that the anomalous dimensions in the S-matrix element are all cancelled out with each other. As a result of the cancellation, we arrive at

\[ T_{ji}^{(1)} = \pi_{R\sigma}(q_1)\Gamma_R^{(0)\mu}(q_1, p_1)u_{R\alpha}(p_1)\Gamma_R^{(0)}(k)\pi_{R\rho}(q_2)\Gamma_R^{(0)\nu}(q_2, p_2)u_{R\beta}(p_2) \]

\[ -\pi_{R\rho}(q_2)\Gamma_R^{(0)\nu}(q_2, p_1)u_{R\alpha}(p_1)\Gamma_R^{(0)}(k)\pi_{R\sigma}(q_1)\Gamma_R^{(0)\mu}(q_1, p_2)u_{R\beta}(p_2) \]  
(8.22)

This expression clearly shows that the exact t-channel S-matrix element of the two-nucleon scattering can be represented in the form as given by the tree diagrams shown in Figs. (5a) and (5b) provided that all the physical parameters in the matrix elements are replaced by their effective (running) ones.

Next, let us turn to the diagram in Fig. (4c). In the diagram, the shaded blob with four amputated external legs represents the nucleon four-line proper vertex. The direct term of the scattering amplitude given by Fig. (4c) can be represented in terms of the renormalized quantities as follows:

\[ T_{ji}^{(2)} = \pi_{R\rho}(q_1)\pi_{R\sigma}(q_2)\Gamma_R(p_1, p_2; q_1, q_2)u_{R\alpha}(p_1)u_{R\beta}(p_2) \]  
(8.23)

In accordance with Eq. (8.8), the renormalized vertex \(\Gamma_R(p_1, p_2; q_1, q_2)\) is of the form

\[ \Gamma_R(p_1, p_2; q_1, q_2) = e^{\int_{\lambda}^\lambda \frac{d\lambda}{\lambda}} \gamma_\tau(\lambda) \Gamma_R^{(0)}(p_1, p_2; q_1, q_2) \]  
(8.24)
where \(\gamma_\tau(\lambda)\) is the anomalous dimension of the vertex which is determined by the renormalization constant \(Z_\Gamma = Z_2^{-2}\) (which is the inverse of the renormalization constant of the nucleon four-point Green’s function). According to Eq. (8.3),

\[ \gamma_\tau(\lambda) = \mu \frac{d}{d\mu} \ln Z_\Gamma = -2\mu \frac{d}{d\mu} \ln Z_2 \]  
(8.25)

Substituting Eqs. (8.13) and (8.24) into Eq. (8.23) and noticing Eqs. (8.15) and (8.25), we also find that the anomalous dimensions are all cancelled out. Thus, we have

\[ T_{ji}^{(2)} = \pi_{R\rho}(q_1)\pi_{R\sigma}(q_2)\Gamma_R^{(0)}(p_1, p_2; q_1, q_2)u_{R\alpha}(p_1)u_{R\beta}(p_2) \]  
(8.23)

As mentioned in Eq. (8.8), the vertex \(\Gamma_R^{(0)}(p_1, p_2; q_1, q_2)\) is given at \(p_{\alpha0}^2 = q_{\alpha0}^2 = \mu^2\ (i = 1, 2)\) and the physical parameters in it are all running ones. Since the unrenormalized vertex \(\Gamma(p_1, p_2; q_1, q_2)\) is not fundamental, it has a complicated structure, containing a series of tree and loop diagrams [36]. The expression of the vertex \(\Gamma_R^{(0)}(p_1, p_2; q_1, q_2)\) can be determined by the perturbation method. Unlike the loop-expansion, the perturbation series of the S-matrix usually
is expanded in powers of the coupling constant $g_v$. The lowest order approximation of the vertex $\Gamma(p_1, p_2; q_1, q_2)$ is of the order of $g^4$ and contains two terms which are given by the truncated subdiagrams (the box and crossed box diagrams) obtained from Figs. (6e) and (6f) by amputating the external lines. The scattering amplitude given by the tree diagrams in Figs. (6e) and (6f) and their corresponding exchanged counterparts are convergent. We may dress these diagrams by replacing the free wave functions, the free propagators and the bare vertices with the exact wave functions, the full propagators and the rigorous proper vertices. In this way, we obtain a series of loop diagrams. As mentioned before, the dressed wave functions, propagators and vertices in the S-matrix element can all be replaced by the renormalized ones. Therefore, they can be expressed in the forms as given in Eqs. (8.13), (8.16) and (8.19). Due to the cancellation of the anomalous dimensions, we will obtain an expression of the renormalized scattering amplitude which is formally the same as that written from the tree diagrams in Figs. (6e) and (6f) and their exchanged ones. For this reason, the tree diagrams are called skeletons of the dressed diagrams. There are a series of skeleton diagrams (or called tree diagrams) of the vertex $\Gamma(p_1, p_2; q_1, q_2)$ such as the ladder diagrams and some others. But, in practical calculations, it is only feasible to consider the skeleton diagrams given in lower order perturbative approximations. We would like to stress that the skeleton diagrams can all be dressed. The S-matrix elements given by the dressed diagrams can be written out from the corresponding skeleton (tree) diagrams provided that the physical parameters are replaced by the solutions of their RGEs. For other S-matrix elements representing other processes, the conclusion is completely the same. It is noted that a S-matrix element evaluated by the bare S-matrix model is independent of the gauge parameter as illustrated in the appendix A in the one-loop approximation (This is the so-called gauge-independence of S-matrix which is implied by the unitarity of S-matrix elements). This fact indicates that the task of renormalization for the $\sigma - \omega$ model is reduced to find the effective coupling constants and the effective masses by solving their RGEs. These effective quantities completely describe the effect of higher order loop corrections. As an illustration, the effective coupling constants and the effective masses given in the one-loop approximation will be derived and discussed in detail in the next section.

IX. ONE-LOOP EFFECTIVE COUPLING CONSTANTS AND MASSES

For the renormalization of the $\sigma - \omega$ model, we need to derive the effective vectorial coupling constant, the effective scalar coupling constant, the effective nucleon mass and the effective $\omega$ meson and $\sigma$ meson masses. The one-loop expressions of these effective quantities will be derived and discussed in the following subsections.

A. Effective vectorial coupling constant

The RGE for the renormalized vectorial coupling constant $g_v^R$ which appears in the vectorial vertex may be immediately written out from Eq. (8.2) by setting $F = g_v$,

$$\mu \frac{d}{d\mu} g_v^R(\mu) + \gamma_v^\nu(\mu) g_v^R(\mu) = 0. \quad (9.1)$$

In view of the definition shown in Eq. (8.1) and the relation in Eq. (6.32), the renormalization constant defined in Eq. (8.12) will be represented as

$$Z_v^\nu = \frac{Z_1}{Z_2 Z_3^2} = Z_3^{-\frac{1}{4}} \quad (9.2)$$

According to the definition in Eq. (8.3), the anomalous dimension $\gamma_v^\nu(\mu)$ in Eq. (9.1) can be calculated by

$$\gamma_v^\nu = \lim_{\epsilon \to 0} \mu \frac{d}{d\mu} \ln Z_v^\nu = -\frac{1}{2} \lim_{\epsilon \to 0} \mu \frac{d}{d\mu} \ln Z_3 \quad (9.3)$$

where $\epsilon = 2 - \frac{n}{2}$ with $n$ being the dimension of the space in which the regularization is performed. Based on the definition denoted in Eq. (5.19), the renormalization constant $Z_3$ will be given by the subtraction of the $\omega$ meson vacuum polarization (self-energy) operator $-\Pi_{\mu\nu}(k)$. From the one-loop diagram represented in Fig. (7), one may write [1, 29]

$$\Pi_{\mu\nu}(k) = -2ig_v^2 \int \frac{d^4l}{(2\pi)^4} Tr [\gamma_\nu^l (1 - k - M + i\epsilon)(1 - l - M + i\epsilon)] \quad (9.4)$$

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where the factor 2 comes from the fact that the fermion loop can be formed by both of the proton and neutron loops. By the dimensional regularization [28-30, 43], the divergent integral shown above is easily calculated. Then, from the definitions in Eqs. (5.14) and (5.19), it is not difficult to find that in the n-dimensional space, the renormalization constant \( Z_3 \) is expressed as

\[
Z_3 = 1 - \Pi(\mu^2)
= 1 + \frac{g_{\pi}^2}{2\pi^2} (4\pi m_g^2)^\epsilon (2 - \epsilon) \frac{\Gamma(1+\epsilon)}{\epsilon} f_0^1 \frac{dx (x-1)}{[x^{\epsilon}(x-1)+M_R]}
\]

(9.5)

where \( m_g \) is a mass introduced to make the coupling constant to be dimensionless in the n-dimensional space. It is noted here that the factors \((4\pi m_g^2)^\epsilon\) and \(\Gamma(1+\epsilon)\) may all be set to unity because they do not give an effect on the anomalous dimension when we set \( \epsilon \to 0 \) in the final step of the calculation for the anomalous dimension. Inserting Eq. (9.5) into Eq. (9.3), it can be found that

\[
\gamma_g^\nu = -\frac{g_{\pi}^2}{6\pi^2} \left( 1 + 6\sigma^2 + \frac{12\sigma^4}{\sqrt{1 - 4\sigma^2}} \ln \frac{1 + \sqrt{1 - 4\sigma^2}}{1 - \sqrt{1 - 4\sigma^2}} \right)
\]

(9.6)

where \( \sigma = \frac{M}{\mu} \). In this expression, the coupling constant \( g_\pi \) and the nucleon mass \( M \) are unrenormalized. In the approximation of order \( g_{\pi R}^2 \), they can be replaced by the renormalized ones \( g_{\pi R}^R \) and \( M_R \) because in this approximation, as pointed out in the previous literature [28], the lowest order approximation of the relation between the \( g_{\pi}(M) \) and the \( g_{\pi R}(M_R) \) is only necessary to be taken into account. Furthermore, when we introduce the scaling variable \( \lambda \) defined by \( \mu = \mu_0 \lambda \) for the renormalization point and set \( \mu_0 = M_R \) (This can always be done since the \( \mu_0 \) is fixed, but may be chosen at will. The above choice amounts to take the renormalization scale parameter to be the nucleon mass), we have \( \sigma = \frac{M}{\mu_0} = \frac{1}{x} \). Thus, with the \( \gamma_g^\nu \) expressed in Eq. (9.6) and noticing \( \mu_0 \frac{d\sigma}{d\lambda} = \lambda \frac{d\lambda}{d\sigma} \), Eq. (9.1) may be rewritten in the form

\[
\lambda \frac{dg_\pi^R(\lambda)}{d\lambda} = \frac{[g_{\pi R}(\lambda)]^3}{6\pi^2} F_g^\nu(\lambda)
\]

(9.7)

where

\[
F_g^\nu(\lambda) = 1 + \frac{6}{\lambda^2} + \frac{12\lambda^2}{\lambda^4} f(\lambda)
\]

(9.8)

in which

\[
f(\lambda) = \frac{\lambda}{\sqrt{\lambda^2 - 4}} \ln \frac{\lambda + \sqrt{\lambda^2 - 4}}{\lambda - \sqrt{\lambda^2 - 4}}
= \left\{ \begin{array}{ll}
\sqrt{\frac{2\lambda}{\lambda^2 - 4}} \cot^{-1} \sqrt{\frac{\lambda}{\lambda^2 - 4}} & \text{if } \lambda \leq 2,
\sqrt{\frac{2\lambda}{\lambda^2 - 4}} \coth^{-1} \sqrt{\frac{\lambda}{\lambda^2 - 4}} & \text{if } \lambda \geq 2.
\end{array} \right.
\]

(9.9)

Upon substituting Eqs. (9.8) and (9.9) into Eq. (9.7) and then integrating Eq. (9.7) by applying the familiar integration formulas, the effective (running) coupling constant will be found to be

\[
\alpha_R^\nu(\lambda) = \frac{\alpha_R^R(\lambda)}{1 - \frac{4\sigma^2}{4\pi} G(\lambda)}
\]

(9.10)

where \( \alpha_R^\nu(\lambda) = [g_{\pi R}(\lambda)]^2/4\pi \), \( \alpha_R^R(\lambda) = \alpha_R^R(1) \) which is the coupling constant that should be determined by experiment and

\[
G(\lambda) = \int_{\lambda}^{\infty} \frac{d\lambda}{\lambda} F_g^\nu(\lambda)
= 2 + \sqrt{3\pi} - \frac{2}{3} + (1 + \frac{2}{3}) \varphi(\lambda)
\]

(9.11)

in which

\[
\varphi(\lambda) = \sqrt{\lambda^2 - 4} \ln \frac{1}{\lambda + \sqrt{\lambda^2 - 4}}
= \left\{ \begin{array}{ll}
-\sqrt{4 - \lambda^2} \cos^{-1} \frac{\lambda}{2} & \text{if } \lambda \leq 2,
\sqrt{\lambda^2 - 4} \cosh^{-1} \frac{\lambda}{2} & \text{if } \lambda \geq 2.
\end{array} \right.
\]

(9.12)

As mentioned before, the variable \( \lambda \) is also the scaling parameter of momenta, \( p = \lambda p_0 \), and it is convenient to put \( p_0^2 = \mu_0^2 \) so as to apply the boundary condition. Thus, owing to the choice \( \mu_0 = M_R \), we have \( p_0^2 = M_R^2 \) and
\[ \lambda = (\frac{\alpha}{\lambda})^2. \] In this case, it is apparent that when \( \lambda = 1 \), Eq. (9.10) will be reduced to the result given on the nucleon mass shell with the value \( \alpha_R^v(1) = \alpha_R^v \). The behaviors of the effective coupling constants obtained in the timelike and spacelike momentum space subtractions are separately represented in Figs. (8) and (9). The figures show that the effective coupling constant \( \alpha_R^v(\lambda) \) has a singularity at \( \lambda_0 \) (the Landau pole). The position of the pole strongly depends on the parameter \( \alpha_R^v \). By our numerical test, we find, when the \( \alpha_R^v \) is getting smaller and smaller, the \( \lambda_0 \) is getting larger and larger. If the \( \alpha_R^v \) goes to zero, the \( \lambda_0 \) approaches a value near infinity, similar to the case of QED. While, when the \( \alpha_R^v \) is getting larger and larger, the \( \lambda_0 \) moves toward unity; but it can not arrive at unity because \( \alpha_R^v(1) = \alpha_R^v \). In the region \([0,1]\) of \( \lambda \), the \( \alpha_R^v(\lambda) \) has no singularity to appear. In Fig. (8), there are two lines representing the \( \alpha_R^v(\lambda) \) given in the timelike momentum subtraction: one is given by taking \( \alpha_R^v = 1 \) and has a singularity at \( \lambda \approx 1.1385 \); another is obtained by taking \( \alpha_R^v = 0.5 \) and has a singularity at \( \lambda \approx 1.3885 \). When \( \lambda \) goes from \( \lambda_0 \) to zero, the \( \alpha_R^v(\lambda) \) decreases and tends to zero, exhibiting an asymptotically free behavior as we met in QED. In Fig. (9), the two lines represent the \( \alpha_R^v(\lambda) \) given in the spacelike momentum subtraction: one line is obtained by taking \( \alpha_R^v = 1 \) and has a singularity at \( \lambda_0 \approx 26.4689 \); another is obtained by \( \alpha_R^v = 0.5 \) and has a singularity at \( \lambda_0 \approx 280.431 \). When \( \lambda \) goes to zero, the \( \alpha_R^v(\lambda) \) approaches the constant \( \alpha_R^v \). As one knows, the Landau poles mentioned above give a limitation of applicability of the one loop renormalization. That is to say, beyond the region \([0, \lambda_0]\), the \( \alpha_R^v(\lambda) \) is meaningless even though in the limit: \( \lambda \to \infty \), the \( \alpha_R^v(\lambda) \) tends to zero from an opposite direction. In comparison of Fig. (8) with Fig. (9), it is clear to see that the range of applicability for the \( \alpha_R^v(\lambda) \) given in the spacelike momentum subtraction is much larger than the range for the \( \alpha_R^v(\lambda) \) given in the timelike momentum subtraction.

**B. Effective \( \omega \) meson mass**

In Eq. (8.2), when we set \( F = m_\omega \), we have a RGE for the renormalized mass of \( \omega \) meson such that

\[ \mu \frac{d}{d\mu} m_\omega^R(\mu) + \gamma_{m_\omega}^v(\mu)m_\omega^R(\mu) = 0 \]  \tag{9.13}

From the definitions given in Eq. (8.1) and the first equality in Eq. (5.22), we find

\[ Z_m^\omega = Z_3^\omega = Z_3^v \]  \tag{9.14}

so that

\[ \gamma_{m_\omega}^v(\mu) = \gamma_g^v(\mu) = -\frac{[g_0^v(\lambda)]^2}{6\pi^2} F_g^v(\lambda) \]  \tag{9.15}

where \( F_g^v(\lambda) \) was given in Eq. (9.8). With the above expression, Eq. (9.13) can be written as

\[ \frac{dm_\omega^R}{m_\omega^R} = \frac{2\alpha_R^v(\lambda)}{3\pi} F_g^v(\lambda) d\lambda \]  \tag{9.16}

Integrating the above equation, one gets

\[ m_\omega^R(\lambda) = m_\omega^R e^{\int_\lambda^{\lambda_0} \frac{2\alpha_R^v(\lambda)}{3\pi} F_g^v(\lambda) d\lambda} \]  \tag{9.17}

where \( m_\omega^R = m_\omega^R(1) \) is the observed \( \omega \) meson mass. This is just the one-loop result of the effective \( \omega \) meson mass. If we take the approximation \( \alpha_R^v(\lambda) \approx \alpha_R^v \), the above expression becomes

\[ m_\omega^R(\lambda) \approx m_\omega^R e^{\int_\lambda^{\lambda_0} \frac{2\alpha_R^v}{3\pi} G_v(\lambda) d\lambda} \]  \tag{9.18}

where \( G_v(\lambda) \) was given in Eq. (9.11).

To have an insight into the behavior of the effective masses in Eq. (9.17), we take the \( m_\omega^R(\lambda) \) given by taking \( \alpha_R^v = 1 \) as an example. This \( m_\omega^R(\lambda) \) is shown in Fig. (10). In the figure, the solid line represents the effective mass obtained in the spacelike momentum subtraction and the dashed line describes the one given in the timelike momentum subtraction. Comparing Fig. (10) with Figs. (8) and (9), we see that the both effective masses have the same singularities and the same scopes of applicability as the corresponding effective coupling constants \( \alpha_R^v(\lambda) \). Particularly, the position of the singularity strongly depends on the choice of \( \alpha_R^v \) as the \( \alpha_R^v(\lambda) \) does. When \( \lambda \) tends to zero, the \( m_\omega^R(\lambda) \) for the spacelike momentum approaches a nonvanishing value near the \( m_\omega^R \), while, the \( m_\omega^R(\lambda) \) for the timelike momentum goes to zero.

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C. Effective nucleon mass

The RGE for the renormalized nucleon mass, according to Eq. (8.2), can be written as

$$\mu \frac{d}{d\mu} M_R(\mu) + \gamma_M(\mu) M_R(\mu) = 0$$

(9.19)

where $\gamma_M(\mu)$ is the anomalous dimension of nucleon mass. In view of Eqs. (6.25) and (8.3), we have

$$\gamma_M(\mu) = \mu \frac{d}{d\mu} \ln Z_M = -\mu \frac{d}{d\mu} \ln \{1 + Z_2\left[\frac{A}{M} + (1 - \frac{\mu}{M})B\right]\}$$

(9.20)

To determine the constants $A, B$ and $Z_2$ in the one-loop approximation, it is necessary to compute the nucleon self-energy written in Eq. (6.35). By the dimensional regularization, it is not difficult to derive from Eq. (6.35) the following expression

$$\Sigma(p) = (p - \mu)\Sigma_I(p) + \Sigma_2(p)$$

(9.21)

where

$$\Sigma_1(p) = \frac{g_v^2}{(4\pi)^2} \int_0^1 dx \frac{2[(x-1)\mu + 2M]}{\varepsilon \Omega_v(x)} + (1 - \alpha) \frac{g_\omega^2}{(4\pi)^2} \left\{ \int_0^1 dx \int_0^1 dy \int_0^1 dy \left\{ \frac{(1+3xy)}{\varepsilon \Theta_v(x,y)} + \frac{1}{\Theta_v(x,y)} x^2 y^2 [(1 - xy) + M(p + \mu)] - \frac{1}{2} \right\} \right\}$$

(9.22)

and

$$\Sigma_2(p) = \frac{g_\omega^2}{(4\pi)^2} \int_0^1 dx \frac{2[(x-1)\mu + 2M]}{\varepsilon \Theta_v(x)} + (1 - \alpha) \frac{g_\omega^2}{(4\pi)^2} \left\{ \int_0^1 dx \int_0^1 dy \int_0^1 dy \left\{ \frac{(1+3xy)\mu - 2M}{\varepsilon \Theta_v(x,y)} + \frac{1}{\Theta_v(x,y)} x^2 y^2 [(1 - xy) + M]\right\} + \frac{1}{2} (M - \mu) \right\}$$

(9.23)

where

$$\Theta_v(x) = p_v^2 x(x - 1) + M_v^2 x + m_v^2 (1 - x)$$

$$\Theta_v(x, y) = p_v^2 xy(x - 1) + M_v^2 xy + m_v^2 (1 - x) y + \alpha (1 - y)$$

(9.24)

From the definition $A = \Sigma(\mu)$ written in Eq. (6.17) and the expressions in Eqs. (9.21)-(9.23), we find

$$A = A_1 + A_2 + A_3$$

(9.25)

where

$$A_1 = \frac{g_v^2}{(4\pi)^2} \int_0^1 dx \frac{2[(x-1)\mu + 2M]}{\varepsilon \Omega_v(x)}$$

(9.26)

$$A_2 = (1 - \alpha) \frac{g_v^2}{(4\pi)^2} \left\{ \int_0^1 dx \int_0^1 dy \int_0^1 dy \left\{ \frac{(1+3xy)\mu - 2M}{\varepsilon \Theta_v(x,y)} + \frac{1}{\Theta_v(x,y)} x^2 y^2 [(1 - xy) + M\mu^2]\right\} + \frac{1}{2} (M - \mu) \right\}$$

(9.27)

and

$$A_3 = \frac{g_\omega^2}{(4\pi)^2} \int_0^1 dx \frac{x(1 - \mu - M)}{\varepsilon \Omega_v(x)}$$

(9.28)

in which

$$\Omega_v(x) = \mu_v^2 x(x - 1) + M_v^2 x + m_v^2 (1 - x)$$

$$\Omega_v(x, y) = \mu_v^2 xy(x - 1) + M_v^2 xy + m_v^2 (1 - x) y + \alpha (1 - y)$$

$$\Omega_v(x) = \mu_v^2 x(x - 1) + M_v^2 x + m_v^2 (1 - x)$$

(9.29)
The constant $B$ appearing in Eqs. (6.21) and (6.25), according to Eq. (6.19), ought to be computed by

$$B = (p - \mu)^{-1}[(\Sigma(\mu) - A)|_{p=\mu}.$$  

(9.30)

On inserting Eqs. (9.21)-(9.29) into Eq. (9.30) and employing the formula

$$\frac{1}{a^\varepsilon} - \frac{1}{b^\varepsilon} = \int_0^1 dx \frac{\varepsilon(b - a)}{[ax + b(1 - x)]^{1+\varepsilon}},$$

(9.31)

one may derive

$$B = B_1 + B_2 + B_3$$

(9.32)

where

$$B_1 = \frac{g_0^2}{(4\pi)^2} \int_0^1 dx \left\{ \frac{2(x - 1)}{\varepsilon\Omega(x)^{\varepsilon}} - \frac{4}{\Omega(x)} x(x - 1)[(x - 1)\mu^2 + 2M\mu] \right\},$$

(9.33)

$$B_2 = (1 - \alpha) \frac{g_0^2}{(4\pi)^2} \left\{ \int_0^1 dx \int_0^1 dy \left[ \frac{1}{\Omega(\varepsilon)^{\varepsilon}} + \frac{1}{\Omega(x)^{\varepsilon}} x y \mu^2 (2 + 7x - 9x^2 y^2) - 2M\mu (2 - 3xy) \right] + \frac{2}{\Omega(x)^{\varepsilon}} x^3 y^3 (1 - xy) [(1 - xy)\mu^4 + M\mu^3] \right\} - \frac{1}{2} \right\}$$

(9.34)

and

$$B_3 = \frac{g_0^2}{(4\pi)^2} \int_0^1 dx \left\{ \frac{2(x - 1)}{\varepsilon\Omega(x)^{\varepsilon}} - \frac{2}{\Omega(x)} x(x - 1)[(x - 1)\mu^2 - M\mu] \right\}.$$

(9.35)

The terms expressed in Eqs. (9.27) and (9.34) are dependent on the gauge parameter $\alpha$ and look more complicated. However, as demonstrated in the Appendix A, the S-matrix elements evaluated in the $\sigma - \omega$ model are gauge-independent. Therefore, for simplicity, these terms will not be taken into account later on. This means that we limit ourselves to working in the Feynman gauge.

When Eqs. (9.25) and (9.32) with the expressions given in Eqs. (9.26), (9.28), (9.33) and (9.35) are substituted into Eq. (9.20), noticing that $Z_2 \simeq 1$ should be taken in Eq. (9.20) in the approximation of order $g_0^2$, one may find an explicit expression of the anomalous dimension $\gamma_M(\mu)$ through a tedious calculation

$$\gamma_M(\lambda) = \gamma^{(1)}_M(\lambda) + \gamma^{(2)}_M(\lambda)$$

(9.36)

where $\gamma^{(1)}_M(\lambda)$ is derived from the constants in Eqs. (9.26) and (9.33), while $\gamma^{(2)}_M(\lambda)$ is given by the constants in Eqs. (9.28) and (9.35). The expressions of the $\gamma^{(1)}_M(\lambda)$ and $\gamma^{(2)}_M(\lambda)$ are separately described in the following. For the anomalous dimension $\gamma^{(1)}_M(\lambda)$, we have

$$\gamma^{(1)}_M(\lambda) = \frac{\alpha_0}{\pi} \Sigma(\lambda)$$

(9.37)

in which

$$\Sigma(\lambda) = \xi_0^\varepsilon(\lambda) + \sum_{i=1}^4 \xi_i^\varepsilon(\lambda) J_i(\lambda)$$

(9.38)

where the functions $\xi_i^\varepsilon(\lambda)$ are

$$\xi_0^\varepsilon(\lambda) = \frac{1}{2} (3 + \lambda) - \frac{2}{\lambda} + \frac{1}{\lambda^2} (1 - \lambda),$$

(9.39)

$$\xi_1^\varepsilon(\lambda) = \frac{2}{\lambda^2} [2(1 - \beta^2)^2 - 3\beta^2 \lambda^2] (1 - \frac{1}{\lambda}) + \frac{1}{\lambda^3} [2(1 - \beta^2)^3 - 5\beta^2 (1 - \beta^2) \lambda^3 - \beta^2 \lambda^4] (1 - \lambda),$$

(9.40)

$$\xi_2^\varepsilon(\lambda) = \frac{1}{\lambda^2} [\beta^2 \lambda^2 - 3(1 - \beta^2)^2] (1 - \lambda) - \frac{6(1 - \beta^2)^2}{\lambda} (1 - \frac{1}{\lambda}).$$

(9.41)
\[
\xi_1(\lambda) = \frac{4\beta^2}{\lambda^2}[\beta^2\lambda^2 - (1 - \beta^2)^2](1 - \frac{1}{\lambda}) + \frac{2\beta^2}{\lambda^2}[\beta^2\lambda^4 + 2(1 - \beta^2)\beta^2\lambda^2 - (1 - \beta^2)^3](1 - \lambda) \tag{9.42}
\]

and
\[
\xi_4(\lambda) = \frac{4\beta^2}{\lambda^2}[(1 - \beta^2)\lambda^2 - (1 - \beta^2)^2](1 - \frac{1}{\lambda}) - \frac{2\beta^2}{\lambda^2}[(1 - \beta^2)^2 - (1 - \beta^2)^2(1 + 2\beta^2)\lambda^2 + \beta^4\lambda^4](1 - \lambda) \tag{9.43}
\]

with \(\beta = m_\omega/M\) and the functions \(J_i(\lambda)\) are given by the integrals shown below. With defining \(a = (1 - \beta^2)\lambda^{-2}\) and \(b = \beta^2\lambda^{-2}\), we can write
\[
J_1(\lambda) = \int_0^1 dx \frac{x}{x(x-1)+ax+b} = \frac{\lambda^2}{\sqrt{q(\lambda)}} \ln \frac{\lambda^2 - 1 - \beta^2 - \sqrt{2q(\lambda)}}{\lambda^2 - 1 - \beta^2 + \sqrt{2q(\lambda)}} \tag{9.44}
\]

where
\[
q(\lambda) = \lambda^4 - 2(1 + \beta^2)\lambda^2 + (1 - \beta^2)^2, \tag{9.45}
\]

\[
J_2(\lambda) = \int_0^1 dx \frac{x}{x(x-1)+ax+b} = -\ln \beta + \frac{\lambda^2 - 1 + \beta^2}{2\sqrt{q(\lambda)}} \ln \frac{\lambda^2 - 1 - \beta^2 - \sqrt{2q(\lambda)}}{\lambda^2 - 1 - \beta^2 + \sqrt{2q(\lambda)}} \tag{9.46}
\]

\[
J_3(\lambda) = \int_0^1 dx \frac{x}{x(x-1)+ax+b} = [(1 - \beta^2)^2 - (1 + \beta^2)\lambda^2] \frac{\lambda^4}{\beta^4 q(\lambda)} - \frac{2\beta^2}{q(\lambda)^{3/2}} \ln \frac{\lambda^2 - 1 - \beta^2 - \sqrt{2q(\lambda)}}{\lambda^2 - 1 - \beta^2 + \sqrt{2q(\lambda)}} \tag{9.47}
\]

and
\[
J_4(\lambda) = \int_0^1 dx \frac{x}{x(x-1)+ax+b} = -(\lambda^2 + 1 - \beta^2) \frac{x^4}{q(\lambda)} - (\lambda^2 - 1 + \beta^2) \frac{x^4}{q(\lambda)^{3/2}} \ln \frac{\lambda^2 - 1 - \beta^2 - \sqrt{2q(\lambda)}}{\lambda^2 - 1 - \beta^2 + \sqrt{2q(\lambda)}}. \tag{9.48}
\]

For the anomalous dimension \(\gamma_M^{(2)}(\lambda)\), we can write
\[
\gamma_M^{(2)}(\lambda) = \frac{\alpha_R}{2\pi} \Sigma_s(\lambda) \tag{9.49}
\]

in which \(\alpha_R = (g_s^R)^2/4\pi\) and
\[
\Sigma_s(\lambda) = \xi_0^s(\lambda) + \sum_{i=1}^{4} \xi_i^s(\lambda) J_i^R(\lambda) \tag{9.50}
\]

where the functions \(\xi_i^s(\lambda)\) are
\[
\xi_0^s(\lambda) = -\frac{3}{2} + \frac{\lambda}{2} + \frac{1}{\lambda^2} - \frac{\beta_0^2}{\lambda^2}(1 - \lambda) \tag{9.51}
\]

with \(\beta_0 = m_\sigma/M\),
\[
\xi_1^s(\lambda) = \frac{1}{\lambda^2} [2(1 - \beta_0^2)^3 - 5\beta_0^2(1 - \beta_0^2)\lambda^2 - \beta_0^4\lambda^4](1 - \lambda) - \frac{1}{\lambda^2} [2(1 - \beta_0^2)^2 - 3\beta_0^2\lambda^2](1 - \frac{1}{\lambda}) \tag{9.52}
\]

\[
\xi_2^s(\lambda) = \frac{1}{\lambda^2} [\beta_0^2\lambda^2 - 3(1 - \beta_0^2)^2](1 - \lambda) + \frac{2(1 - \beta_0^2)}{\lambda^2}(1 - \frac{1}{\lambda}) \tag{9.53}
\]

\[
\xi_3^s(\lambda) = \frac{2\beta_0^2}{\lambda^2} [(1 - \beta_0^2)^2 - \beta_0^2\lambda^2](1 - \frac{1}{\lambda}) + \frac{2\beta_0^2}{\lambda^2} [\beta_0^2\lambda^4 + 2(1 - \beta_0^2)\beta_0^2\lambda^2 - (1 - \beta_0^2)^3](1 - \lambda) \tag{9.54}
\]

\[
\xi_4^s(\lambda) = \frac{2\beta_0^2}{\lambda^2} [(1 - \beta_0^2)^2 - \beta_0^2\lambda^2](1 - \frac{1}{\lambda}) + \frac{2\beta_0^2}{\lambda^2} [\beta_0^2\lambda^4 + 2(1 - \beta_0^2)\beta_0^2\lambda^2 - (1 - \beta_0^2)^3](1 - \lambda). \tag{9.54}
\]
the effective nucleon mass is real, while, in the spacelike momentum space subtraction, the \( M(0) \) increases a little when \( \lambda \to \infty \). In the figure, the dashed, dotted and solid lines represent the expression in Eq. (9.57) for which the coupling constants are taken as the constants \( \xi^i(\lambda) \). Substituting the \( \xi^i(\lambda) \) as expressed in Eqs. (9.36)-(9.56) into Eq. (9.19) and solving the equation with noticing \( \mu d/\mu = \lambda d/\lambda \), we obtain

\[
M_R(\lambda) = M_R e^{-\int_1^\lambda d\lambda/\lambda \gamma_M(\lambda)} = M_R e^{-\int_1^\lambda d\lambda/\lambda [\alpha^\n_R(\lambda)/\pi \Sigma_\alpha(\lambda) + \alpha^\n_s(\lambda)/2\pi \Sigma_\alpha(\lambda)]}
\]

where \( M_R = M_R(1) \) is the observed nucleon mass. The coupling constants in the above have been taken to be running ones. The \( \alpha^\n_R(\lambda) \) was given in Eq. (9.10), while, the \( \alpha^\n_s(\lambda) \) will be derived in the next subsection.

It would be emphasized that in the timelike momentum space subtraction, the scaling parameter \( \lambda \) is real so that the effective nucleon mass is real, while, in the spacelike momentum space subtraction, the \( \lambda \) is imaginary so that the effective nucleon mass becomes complex one. In the latter case, the \( \lambda \) in the \( \gamma_M(\lambda) \) should be set to be \( i\lambda \). Observing the expressions in Eqs. (9.39)-(4.48) and (8.51)-(9.55), we see that in both subtractions, the functions \( J_i(\lambda) \) are always real. The real and imaginary parts of the \( \gamma_M(\lambda) \) are distinguished by the real and imaginary parts of the functions \( \xi^i(\lambda) \) and \( \xi^s(\lambda) \). In Figs. (11)-(12), we show the behaviors of the effective nucleon masses given by the expression in Eq. (9.57) for which the coupling constants be taken as the constants \( \alpha^\n_R \) and \( \alpha^\n_s \) for simplicity of computation. The effective mass \( M_R(\lambda) \) obtained in the timelike momentum space subtraction is exhibited in Fig. (11). In the figure, the dashed, dotted and solid lines represent the \( M_R(\lambda) \) given by taking \( (\alpha^\n_R, \alpha^\n_s) = (0, 0.5) \), (0.5, 0) and (0.5, 0.5) respectively. From the figure, we see that in the region [0,1] of \( \lambda \), the \( M_R(\lambda) \) almost keeps a constant \( M_R \). Beyond this region, if only the scalar coupling is considered, the \( M_R(\lambda) \) increases up to a maximum at \( \lambda_0 = 4.21 \) and then decreases to zero rather rapidly when \( \lambda \to \infty \). While, if the vectorial coupling enters, the \( M_R(\lambda) \) increases a little when \( \lambda \) goes to a smaller \( \lambda_0 \) and then decreases much rapidly down to zero when \( \lambda \) varies from \( \lambda_0 \) to \( \infty \). The effective mass \( M_R(\lambda) \) obtained in the spacelike momentum space subtraction is shown in Fig. (12). In the figure, the real and imaginary parts of the \( M_R(\lambda) \) are displayed separately. The dashed, solid and dotted lines in the figure represent the real and imaginary parts of the \( M_R(\lambda) \) which are obtained by taking \( (\alpha^\n_R, \alpha^\n_s) = (0.5, 0) \), (0.5, 0.5) and (1, 1) respectively. The figure indicates that in the region [0,1] of \( \lambda \), the real part of the \( M_R(\lambda) \) keeps almost a constant equal to \( M_R \), while, the imaginary part of the \( M_R(\lambda) \) is almost zero. When \( \lambda \) varies from unity to infinity, the real part of the \( M_R(\lambda) \) at first increases smoothly and the imaginary part of the \( M_R(\lambda) \) decreases, then, both of them drastically oscillate and damp to zero. The figure also shows that the stronger the couplings (especially, the vectorial coupling), the larger is the frequency of the oscillation. The appearance of the oscillation implies that the \( M_R(\lambda) \) is invalid to use in the region that the oscillation appears.

D. Effective scalar coupling constant

When setting \( F = g_s \) in Eq. (8.2), one obtains the RGE for the renormalized scalar coupling constant \( g_s^R(\mu) \) which is included in the scalar (nucleon-nucleon–\( \sigma \) meson) vertex

\[
\mu \frac{d}{d\mu} g_s^R(\mu) + \gamma_s(\mu) g_s^R(\mu) = 0
\]

Analogous to the case of vectorial coupling, the anomalous dimension \( \gamma_s(\mu) \) determined by

\[
\gamma_s = \lim_{\varepsilon \to 0} \frac{d}{d\mu} \ln Z_s^\varepsilon
\]

should be calculated from the renormalization constant \( Z_s^\varepsilon \) which is represented as

\[
Z_s^\varepsilon = \frac{Z_1}{Z_2 Z_3}
\]
where $Z_2$, $Z'_2$ and $Z'_3$ were defined respectively in Eqs. (6.21), (7.6) and (7.17). According to these definitions, in the approximation of order $g^2$ and in the Feynman gauge, the $Z'_g$ will be written as

$$Z'_g = 1 + B_1 + B_3 - L'_1 - L'_3 + \Omega_1$$

(9.61)

where $B_1$ and $B_3$ were represented in Eqs. (9.33) and (9.35) respectively, $L'_1$ and $L'_3$ are the parts of the constant $L'$ which can conveniently be determined by the identity in Eq. (7.22). From the identity and the representations written in Eqs. (9.26) and (9.28), it is easy to get

$$L'_1 = \frac{\partial A_1}{\partial M} = \frac{\eta^2}{4\pi^2} \int_0^1 dx \frac{1}{\xi(x)} \frac{1}{[x(1-M) + 2Mx]M}$$

$$L'_3 = \frac{\partial A_3}{\partial M} = -\frac{\eta^2}{16\pi^2} \int_0^1 dx \frac{1}{\xi(x)} \frac{1}{[x(1-M) - 2Mx]M}$$

(9.62)

The one-loop expression of the divergent constant $\Omega_1(\mu^2)$ in Eq. (9.61) can be derived from the $\sigma$ meson one-loop self-energy depicted in Fig. (13). From Fig. (13), it reads

$$\Omega(q) = -2i\eta^2 \int \frac{d^4q}{(2\pi)^4} \text{Tr} \left[ \frac{1}{(1-q-M+i\epsilon)(q-M+i\epsilon)} \right]$$

(9.63)

where the factor 2 also arises from nucleon doublet. By the dimensional regularization, the above integral is easily calculated and expressed as

$$\Omega(q) = \frac{\eta^2}{(4\pi)^2} \int_0^1 dx \frac{24g^2x(x-1)+M^2}{\varepsilon [\mu^2 x(x-1)+M^2]}$$

(9.64)

which gives rise to

$$\Omega_1(\mu^2) = \frac{\eta^2}{(4\pi)^2} \int_0^1 dx \frac{24x(x-1)}{\varepsilon [\mu^2 x(x-1)+M^2]}$$

(9.65)

and

$$\Omega_2(\mu^2) = \frac{\eta^2}{(4\pi)^2} \int_0^1 dx \frac{24M^2/m^2_s}{\varepsilon [\mu^2 x(x-1)+M^2]}$$

(9.66)

On substituting Eq. (9.61) into Eq. (9.59), we get

$$\gamma^s_\lambda = \gamma^s_{g1}(\lambda) + \gamma^s_{g2}(\lambda) + \gamma^s_{g3}(\lambda)$$

(9.67)

where $\gamma^s_{g1}(\lambda)$, $\gamma^s_{g2}(\lambda)$ and $\gamma^s_{g3}(\lambda)$ are separately defined and described in the following. For the $\gamma^s_{g1}(\lambda)$, we have

$$\gamma^s_{g1}(\lambda) = \mu \frac{d}{d\mu} (B_1 - L'_1) = \frac{\alpha^R}{\pi} \Gamma^s(\lambda)$$

(9.68)

where

$$\Gamma^s(\lambda) = \eta^s_0(\lambda) + \sum_{i=1}^4 \eta^s_i(\lambda) J_i(\lambda)$$

(9.69)

in which

$$\eta^s_0(\lambda) = \frac{3}{4\lambda^2} \left[ 2(1-\beta^2)^3 - 3(1-\beta^2)^2 \lambda + (1-\beta^2)(4 - 3\lambda^2) - 3\beta^2 \lambda^3 - 3\beta^2 \lambda^4 \right]$$

$$\eta^s_1(\lambda) = \frac{3}{4\lambda^2} \left[ (1-\beta^2)^2 - 3(1-\beta^2) \lambda + (2-\beta^2) \lambda^2 \right]$$

$$\eta^s_2(\lambda) = \frac{2\lambda}{1-\beta^2} \left[ 3\beta^2 \lambda^3 - 3\beta^2 \lambda^3 - 2(1-\beta^2) \lambda^2 + 3(1-\beta^2)^3 \lambda - (1-\beta^2)^4 \right]$$

(9.70)
and the functions \(J_i(\lambda)\) were given in Eqs. (9.44)-(9.48).

For the \(\gamma_{g2}^s(\lambda)\), we can write

\[
\gamma_{g2}^s(\lambda) = \mu \frac{d}{d\mu} (B_3 - L'_3) = \frac{(g_R)^2}{8\pi^2} \Gamma_1^s(\lambda)
\]

(9.71)

where

\[
\Gamma_1^s(\lambda) = \eta_0^s(\lambda) + \sum_{i=1}^4 \eta_i^s(\lambda) J_i^0(\lambda)
\]

(9.72)
in which

\[
\begin{align*}
\eta_0^s(\lambda) &= -3 + \frac{1}{\lambda^2} 
\eta_1^s(\lambda) &= \frac{1}{\lambda^2} [2(1 - \beta_3^2)^3 - (1 - \beta_3^2)(2 + 5\beta_2^2)\lambda^2] \\
\eta_2^s(\lambda) &= \frac{3}{\lambda^4} [\lambda^2 - (1 - \beta_3^2)^2] \\
\eta_3^s(\lambda) &= \frac{2\beta_3^2}{\lambda^8} [\beta_3^2 \lambda^4 + (1 - \beta_3^2)(1 + 2\beta_2^2)\lambda^2 - (1 - \beta_3^2)^3] \\
\eta_4^s(\lambda) &= -\frac{1}{\lambda^6} [(1 + \beta_3^2)^4 - 2(1 - \beta_3^2)^2(1 + \beta_2^2)\lambda^2 + (1 - \beta_3^2)^4]
\end{align*}
\]

(9.73)

and the functions \(J_i^0(\lambda)\) were defined in Eq. (9.56).

For the \(\gamma_{g3}^s(\lambda)\), By virtue of the expression given in Eq. (9.65), one can get

\[
\gamma_{g3}^s(\lambda) = \mu \frac{d}{d\mu} \Omega_1 = -\frac{(g_R)^2}{8\pi^2} \Gamma_2^s(\lambda)
\]

(9.74)

where

\[
\Gamma_2^s(\lambda) = 2[1 + \frac{6}{\lambda^2} - \frac{12}{\lambda^3\sqrt{\lambda^2 - 4}} \ln \frac{\lambda - \sqrt{\lambda^2 - 4}}{\lambda + \sqrt{\lambda^2 - 4}}]
\]

(9.75)

Based on the anomalous dimension \(\gamma_{g}^s(\lambda)\) given in Eqs. (9.67), (9.68), (9.71) and (9.74), the RGE in Eq. (9.58) may be represented in the form

\[
\frac{dg^R_s(\lambda)}{d\lambda} + P(\lambda) g^R_s(\lambda) + Q(\lambda)(g^R_s(\lambda))^3 = 0
\]

(9.76)

where

\[
P(\lambda) = \frac{\alpha_R^s(\lambda)}{\pi^2} \Gamma^v(\lambda), \quad Q(\lambda) = \frac{1}{8\pi^2 \lambda} [\Gamma_1^s(\lambda) - \Gamma_2^s(\lambda)]
\]

(9.77)

In the above equation, the \(\alpha_R^s(\lambda)\) is a known quantity as given in Eq. (9.10). So, Eq. (9.76) is the equation used to determine the unknown quantity \(g^R_s(\lambda)\) only. To solve the nonlinear equation, we may set

\[
g^R_s(\lambda) = u(\lambda)^{-1/2}
\]

(9.78)

which leads Eq. (9.76) to a linear equation obeyed by the function \(u(\lambda)\)

\[
\frac{du(\lambda)}{d\lambda} - 2P(\lambda) u(\lambda) - 2Q(\lambda) = 0
\]

(9.79)

When setting \(Q(\lambda) = 0\), we obtain a homogeneous equation whose solution is

\[
u(\lambda) = u(\lambda)^{e^2 \int_1^\lambda d\mu P(\lambda)}
\]

(9.80)

In order to seek the solution of Eq. (9.79), we assume

\[
u(\lambda) = v(\lambda)^{e^{2 \int_1^\lambda d\mu P(\lambda)}}
\]

(9.81)
where $v(\lambda)$ is an unknown function needs to be determined from Eq. (9.79). Inserting Eq. (9.81) into Eq. (9.79), we get

$$\frac{dv(\lambda)}{d\lambda} = 2Q(\lambda)e^{-2\int_1^\lambda d\lambda P(\lambda)}$$  \hspace{1cm} (9.82)

Integrating the above equation, one obtains

$$v(\lambda) = u(1) + 2\int_1^\lambda d\lambda Q(\lambda)e^{-2\int_1^\lambda d\lambda P(\lambda)}$$  \hspace{1cm} (9.83)

Combining the expressions in Eqs. (9.78), (9.81) and (9.83), the solution of Eq. (9.76) is finally given in the form

$$\alpha^*_R(\lambda) = \frac{\alpha^*_R K(\lambda)}{1 + \alpha^*_R/\pi G_s(\lambda)}$$  \hspace{1cm} (9.84)

where $\alpha^*_R(\lambda) = (g_s^R(\lambda))^2/4\pi$, $\alpha^*_R = \alpha^*_R(1)$ which is a parameter needed to be determined by fitting the experimental data,

$$K(\lambda) = e^{-\frac{2}{\lambda}}\int_1^\lambda d\lambda/\lambda \alpha^*_R(\lambda)\Gamma^*(\lambda)$$  \hspace{1cm} (9.85)

and

$$G_s(\lambda) = \int_1^\lambda d\lambda/\lambda \Gamma^*(\lambda)K(\lambda)$$  \hspace{1cm} (9.86)

in which

$$\Gamma^*(\lambda) = \Gamma^*_1(\lambda) - \Gamma^*_2(\lambda)$$  \hspace{1cm} (9.87)

The behaviors of the effective coupling constant $\alpha^*_R(\lambda)$ obtained in the timelike and spacelike momentum subtractions are separately displayed in Figs. (14) and (15). For timelike momenta, the $\alpha^*_R(\lambda)$ is real. In Fig. (14), there are four lines representing this $\alpha^*_R(\lambda)$ which are given by four groups of the parameters ($\alpha^*_R, \alpha^*_R$) = (0, 1), (0.5, 1), (0, 0.2) and (0.5, 0.2) respectively. The figure indicates that the $\alpha^*_R(\lambda)$ has a Landau pole $\lambda_0$. The poles for the four lines are respectively located at $\lambda_0 \approx 1.07523$ (for the first lines), 1.8237 and 2.3967. Clearly, the $\lambda_0$ is larger if the both parameters ($\alpha^*_R, \alpha^*_R$) are smaller. In particular, the pole moves to the point near infinity when the both parameters ($\alpha^*_R, \alpha^*_R$) tend to zero. When $\lambda$ goes from $\lambda_0$ to zero and infinity, each $\alpha^*_R(\lambda)$ in Fig. (14) abruptly falls to zero from the opposite directions. For spacelike momenta, The behavior of the effective coupling constant is described by the lines in Fig. (15). The two lines in Figs (15a) and (15b) are given by considering the scalar coupling only with taking $\alpha^*_R = 1$ and 0.2 respectively. In this case, the $\alpha^*_R(\lambda)$ is real and has a pole $\lambda_0$. The poles for the aforementioned lines are located respectively at $\lambda_0 = 5.725$ and 234.6. However, when the vectorial coupling is included, the $\alpha^*_R(\lambda)$ becomes complex. In this case, the pole disappears; instead, there is a maximum to appear as shown in Figs (15c) and (15d). The lines in Figs (15c) and (15d) are given by taking ($\alpha^*_R, \alpha^*_R$) = (0.5, 1) and (0.5, 0.2) respectively. In the figures, the real part of the $\alpha^*_R(\lambda)$ is represented by the solid line and the imaginary part by the dashed line. From Fig. (15c), we see that the real and imaginary parts of the $\alpha^*_R(\lambda)$ have sharp peaks corresponding to the pole of the upper line. The peaks are manifested more clearly by the right amplified lines. Fig. (15d) exhibits that either the real part or the imaginary part varies rather smoothly due to the decrement of the parameters ($\alpha^*_R, \alpha^*_R$) and the larger effect of the vectorial coupling. It is noted that when $\lambda \to 0$, the $\alpha^*_R(\lambda)$ reaches a constant, while, in the limit of $\lambda \to \infty$, the $\alpha^*_R(\lambda)$ goes to zero. In particular, The behaviors of the $\alpha^*_R(\lambda)$ tell us that the smaller the parameters ($\alpha^*_R, \alpha^*_R$), the larger will be the range of applicability and the $\alpha^*_R(\lambda)$ for the spacelike momenta have a larger range of applicability than that for the timelike momenta.

E. Effective $\sigma$ meson mass

In accordance with Eq. (8.2), the RGE for the renormalized $\sigma$ meson mass is

$$\lambda \frac{d}{d\lambda} m^R_\sigma(\lambda) + \gamma^\sigma_m(\lambda)m^R_\sigma(\lambda) = 0$$  \hspace{1cm} (9.88)
\[
\gamma_m^\sigma(\lambda) = \frac{d}{d\mu} \ln Z_m^\sigma. \tag{9.89}
\]

From the definitions given in Eqs. (7.6) and (7.9), it is found that in the approximation of order \(g_s^2\), the renormalization constant \(Z_m^\sigma\) can be written as

\[
Z_m^\sigma = 1 - \frac{1}{2} [\Omega_1(\mu^2) + \Omega_2(\mu^2)] \tag{9.90}
\]

The one-loop expressions of the divergent constants \(\Omega_1(\mu^2)\) and \(\Omega_2(\mu^2)\) were given in Eqs. (9.65) and (9.66). With these expressions, the renormalization constants \(Z_m^\sigma\) in Eq. (9.90) can explicitly be written out. Use of this renormalization constant in Eq. (9.89) yields the \(\sigma\) meson mass anomalous dimension as follows:

\[
\gamma_m^\sigma = -\frac{g_s^2}{4\pi} G_s(\lambda) \tag{9.91}
\]

where

\[
G_s(\lambda) = \frac{6}{\pi} \frac{1}{\beta_0} - \frac{1}{\lambda^2} - 2 \frac{1}{\lambda^2 \beta_0} - \frac{1}{\beta_0} \eta(\lambda) \] \tag{9.92}

in which \(\beta_0 = \frac{m_s}{M}\) and

\[
\eta(\lambda) = \frac{1}{\sqrt{\lambda^2 - 4}} \ln \frac{\lambda - \sqrt{\lambda^2 - 4}}{\lambda + \sqrt{\lambda^2 - 4}} \tag{9.93}
\]

Substituting the above anomalous dimension into Eq. (9.88) and solving the equation, we obtain an expression of the effective \(\sigma\) meson mass such that

\[
m_s^R(\lambda) = m_s^R e^{-\int_1^\lambda \frac{4S_s(\lambda)}{\lambda^2 \beta_0} G_s(\lambda)} \tag{9.94}
\]

where \(m_s^R(\lambda)\) is a mass parameter which needs to be determined by experiment. If the coupling constant \(\alpha_H^R(\lambda)\) is approximately taken to be a constant \(\alpha_H^R\), the integral over \(\lambda\) can easily be calculated. In this case, we have

\[
m_s^R(\lambda) = m_s^R e^{S_s(\lambda)} \tag{9.95}
\]

where

\[
S_s(\lambda) = \frac{2\lambda^2}{\pi} [1 - \frac{1}{\lambda^4} + \frac{2\sqrt{2}}{7}(1 - \frac{2}{\pi^2})\pi
+(\frac{1}{\lambda^2} + \frac{1}{\beta_0^2}) \sqrt{\lambda^2 - 4} \ln \frac{1}{2}(\lambda + \sqrt{\lambda^2 - 4})]. \tag{9.96}
\]

It is seen from Eq. (9.94) that the behavior of the effective mass \(m_s^R(\lambda)\) is intimately related to property of the effective coupling constant \(\alpha_H^R(\lambda)\). To give a view of the behavior of the \(m_s^R(\lambda)\), we take the \(m_s^R(\lambda)\) evaluated from Eq. (9.94) by taking \((\alpha_H^R, \alpha_H^R) = (1, 1)\) as an example. This \(m_s^R(\lambda)\) is shown in Fig. (16). In the figure, the dashed line represents the \(m_s^R(\lambda)\) given in the timelike momentum subtraction. This \(m_s^R(\lambda)\) is real and has a singularity at \(\lambda_0 \approx 1.09758\) which implies that the range of applicability of the \(m_s^R(\lambda)\) is less than \(\lambda = 1\), the nucleon mass scale. When \(\lambda\) tends to zero, the \(m_s^R(\lambda)\) approaches a value which does not deviate from the constant \(m_s^R\) so much. The \(m_s^R(\lambda)\) given in the spacelike momentum subtraction is complex. The solid line in Fig. (16) represents the real part of the \(m_s^R(\lambda)\) which has a maximum near \(\lambda_0 = 5\) which indicates that the \(m_s^R(\lambda)\) is applicable in a wide region of \([0, 5]\). Similar to the coupling constant \(\alpha_H^R(\lambda)\), when the parameters \((\alpha_H^R, \alpha_H^R)\) are taken to be smaller, either the pole or the maximum will be shifted to the point of a large \(\lambda_0\).

**X. SUMMARY AND DISCUSSIONS**

In this paper, it has been argued that the \(\sigma - \omega\) model, as a constrained system, is really of U(1) local gauge symmetry. This enables us to quantize the \(\sigma - \omega\) model by means of the method used for quantizing the gauge field theory. In particular, the gauge symmetry allows us, in a consistent way, to derive various W-T identities which
provide a faithful basis for performing the renormalization of the model. As shown in Sec. V, the W-T identity in Eq. (5.6) satisfied by the $\omega$ meson propagator and the W-T identity in Eq. (5.13) for the vacuum polarization operator determine not only the structures of the propagator and the vacuum polarization operator, but also the renormalization fashion of the propagator and the $\omega$ meson mass as shown in Eqs. (5.14), (5.16), (5.21) and (5.22). Especially, the W-T identity in Eq. (6.13) obeyed by the vertex gives rise to the correct manner of subtraction of the nucleon self-energy as denoted in Eq. (6.19). As shown in Sec.VI, the subtraction in Eq. (6.19) leads to the correct representations for the renormalization constants of nucleon propagator and nucleon mass as shown in Eqs. (6.21) and (6.25). Moreover, the identity in Eq. (6.13) directly yields the important relation between the renormalization constants $Z_1$ and $Z_2$ as written in Eq. (6.32). This relation together with the relation in Eq. (7.22) which follows from the identity in Eq. (7.21) greatly simplify the calculation of the renormalization. It would be mentioned here that in some previous works [7, 15, 17], the subtraction based on the expression $\Sigma(p) = Ap + BM$ was ever used. This subtraction gives the nucleon propagator renormalization constant as $Z_2 = [1 - A(\mu^2)]^{-1}$ and nucleon mass renormalization constant as $Z_M = [Z_2(1 + B(\mu^2))]^{-1}$ which are different from the renormalization constants written in Eqs. (6.21) and (6.25) and therefore the relation in Eq. (6.32) could not be fulfilled in this case. The renormalization of the model under consideration is performed in the mass-dependent momentum space subtraction scheme by the renormalization group approach. The prominent advantage of the subtraction scheme is that it naturally leads to the boundary conditions for the renormalized propagators, vertices and the wave functions. The boundary conditions allow us to give an unique determination of the solutions to the renormalization group equations for the renormalized propagators, vertices and wave functions without any ambiguity. As claimed in the Introduction, we limit ourself in this paper to examine the renormalization of the model at zero temperature by means of the renormalization group method. Since the perturbative series expanded in the powers of coupling constants is chosen to be the starting point of this renormalization, the results of the renormalization would be different from those obtained in the study of nuclear matter by using the loop expansion and the spectral function methods. Hopefully, the renormalization procedure described in this paper will be helpful for applying the renormalization group approach to study the nuclear matter at finite temperature and finite density.

The procedure of renormalization group method was demonstrated by the one-loop renormalization in this paper. Since the renormalization exactly respects the W-T identities, the results obtained are faithful. Especially, the one-loop effective coupling constants and masses are given in the rigorous forms as they are derived from the mass-dependent momentum space subtraction. The subtraction scheme used is, in principle, suitable not only for high energy, but also for low energy, unlike the minimal subtraction scheme [28-31] which is only appropriate in the large momentum limit. In addition, the expressions of the one-loop effective physical quantities derived in this paper are applicable for the both of timelike momentum subtraction and spacelike momentum subtraction. As seen from Figs. (8)-(12) and (14)-(16), the behaviors of the effective quantities given in the timelike subtraction and the spacelike subtraction are much different from one another. In which case we should use the results given in the timelike momentum subtraction or the spacelike momentum subtraction? The answer to this question depends on what process is discussed. For example, when we study the nucleon-nucleon scattering taking place in the t-channel, as mentioned in the Appendix B, the transfer momentum in the boson propagator is spacelike. In this case, it is suitable to take the effective coupling constants and boson masses given in the spacelike momentum subtraction. If we investigate the nucleon-antinucleon annihilation process which takes place in the s-channel, since the transfer momentum is timelike, the effective coupling constants and boson masses given in the timelike momentum subtraction should be used. The effect of the one-loop renormalization is examined by the nucleon-antinucleon elastic scattering whose differential cross section given in the order of $g^2$ is described in the Appendix B and plotted in Fig. 17. In the figure, We only take the differential cross sections given at the laboratory kinetic energies $T_{lab} = 491.9 MeV$ and $575.5 MeV$ as an example. The figure shows that consideration of the one-loop renormalization requires the coupling constants to be smaller in order to fit the experimental data. This actually is a general feature of considering the renormalization effect.

As exhibited in Sec. IX, the one-loop effective physical parameters given in the timelike momentum subtraction and the ones given in the spacelike momentum subtraction not only behave differently, but also have different ranges of applicability because the singularities of the effective quantities given by the two subtractions appear at the different momenta. Especially, the positions of the singularities are strongly dependent on the coupling constants $\alpha^R_v$ and $\alpha^R_s$. The smaller the coupling constants, the larger are the ranges of applicability. It would be mentioned that since the propagators written in Eqs. (5.16), (6.12) and (7.2) are solved from the Dyson equations, the one-loop renormalization actually contains the contribution given by partially summing up a set of chain loop diagrams. Just due to the partial summation, as mentioned before, the coupling constants must be set to be smaller for fitting the experimental data of the nucleon scattering. To this end, it is natural to ask if and how the behaviors of the one-loop effective physical parameters can be modified by considering higher order loop renormalizations? In other words, if the coupling constants would be more smaller and the ranges of applicability of the renormalization would be enlarged when the contributions arising from more higher order loop diagrams are summed up? Obviously, this is an interesting problem worthy to pursuing further. In addition, we would like to address that the $\sigma - \omega$ model

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should be viewed as a restrictive model in which the $\sigma$ field is introduced as a phenomenological field. Aside from the $\sigma - \omega$ model, there are some other models in QHD which are of a certain gauge symmetry. Especially, the model proposed by Sakurai in the early time [44], in our opinion, is most promising to describe the nuclear force because in this model, exchanges of the light mesons, pion and $\rho$ meson, dominate the strong interaction between nucleons. In view of the argument given in Refs. [32-34, 45], the Sakurai's model is a SU(2) gauge field theory which is not only gauge-invariant, but also renormalizable. Certainly, the renormalization of this model may be investigated along the same line as described in this paper. We will discuss this subject in the future. But, it can not be expected that a perturbative investigation could give an ultimate solution to the strong interaction. Just as said in Ref. [46, 47], to resolve the strong interaction, it is adequate to perform a nonperturbative study of the interaction kernel appearing in the relativistic equation whose closed expression can be derived by the procedure as described in Refs. [46, 47].

XI. ACKNOWLEDGMENT

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XII. APPENDIX A: GAUGE-INDEPENDENCE OF S-MATRIX ELEMENTS

The gauge-independence of S-matrix elements computed by a gauge field theory is a well-known fact. For the $\sigma - \omega$ model, as argued in this paper, it actually is a U(1) gauge field theory. So, the same conclusion should hold for the $\sigma - \omega$ model. To convince oneself of this fact, we take the nucleon-nucleon scattering amplitudes up to the one-loop approximation as examples to show that the matrix elements given by the $\sigma - \omega$ model are surely independent of the gauge parameter $\alpha$. The typical Feynman diagrams representing the scattering amplitudes are depicted in Figs. (5) and (6). In Fig. (6), only the diagrams with the internal $\omega$ meson line are necessary to be considered. For the tree diagram in Fig. (5a), the gauge-independence of its S-matrix element is well-known. In fact, the S-matrix element

$$S_1 = \pi_{\sigma'}(q_2)ig_{\gamma\mu}u_{\gamma}(q_1)iD^{\mu\nu}(k)\pi_{\sigma}(p_2)ig_{\gamma\nu}u_{\gamma}(p_1)$$

(A.1)

where $u_{\gamma}(p)$ is the free nucleon wave function can be divided into two parts according to the decomposition of free $\omega$ meson propagator

$$D^{\mu\nu}(k) = D_F^{\mu\nu}(k) + D_\alpha^{\mu\nu}(k)$$

(A.2)

where

$$D_F^{\mu\nu}(k) = -\frac{g^{\mu\nu}}{k^2 - m_\omega^2 + i\epsilon}$$

(A.3)

which is the propagator given in the Feynman gauge and

$$D_\alpha^{\mu\nu}(k) = (1 - \alpha)D_\sigma(k^2)k^\mu k'^\nu$$

(A.4)

which is the $\alpha-$dependent part of the propagator in which

$$D(k^2) = \frac{1}{(k^2 - m_\omega^2 + i\epsilon)(k^2 - \nu^2 + i\epsilon)}.$$  

(A.5)

For the $\alpha-$dependent part of $S_1$, applying the energy-momentum conservation $k = q_1 - q_2 = p_2 - p_1$ and Dirac equation $(p - M)u_{\gamma}(p) = 0$, it is found

$$S_1^\alpha = -\frac{(1 - \alpha)ig_{\gamma\sigma}q_{\gamma}(q_2)k_{\sigma}(q_1)\pi_{\sigma}(p_2)ku_{\gamma}(p_1)}{1}D^{\sigma}(k^2)$$

$$= -(1 - \alpha)ig_{\gamma\sigma}q_{\gamma}(q_1 - q_2)u_{\gamma}(q_1)\pi_{\sigma}(p_2)(p_2 - p_1)u_{\gamma}(p_1)D_\sigma(k^2)$$

$$= 0.$$  

(A.6)

This shows that the $\omega$ meson propagator given in the Feynman gauge is sufficient to use for evaluating the tree diagram matrix element. In the same way, one may prove that the S-matrix element given by the tree-diagram in Fig. (5b) is independent of the gauge parameter as well.
Let us focus on the one-loop diagrams in Fig. (6) where only the direct diagrams are plotted and necessarily to be examined for our purpose. The gauge-independence of the matrix element of Fig. (6a) which contains a $\omega$ meson self-energy in it can easily be proved by using Eq. (A.6). So, we only need to examine the gauge-independence of Figs. (6b)-(6f). The matrix element of Fig. (6b) with a vertex correction in it can be written as

$$S_2^1 = M_\mu(q_1,q_2)A^\mu(p_1,p_2)$$  \hspace{1cm} (A.7)

where

$$M_\mu(q_1,q_2) = \pi_\mu(q_2)ig_\nu\gamma^\nu u_r(q_1)iD_{\nu\mu}(k)$$  \hspace{1cm} (A.8)

which is gauge-independent as shown in (A.6) and

$$A^\mu(p_1,p_2) = \int \frac{d^4k}{(2\pi)^4}\pi_\mu(p_2)ig_\nu\gamma_\rho iS_F(p_2 - k)i\gamma^\mu iS_F(p_1 - k)ig_\sigma u_r(p_1)iD^{\rho\sigma}(k).$$  \hspace{1cm} (A.9)

Replacing $D^{\rho\sigma}(k)$ by the $D^{\rho\sigma}_{\alpha}(k)$ shown in (A.4), we have the following gauge-dependent part of $A^\mu(p_1,p_2)$

$$A^\mu_{\alpha}(p_1,p_2) = -(1 - \alpha)g_\nu^3\int \frac{d^4k}{(2\pi)^4}\pi_\mu(p_2)kS_F(p_2 - k)\gamma^\mu S_F(p_1 - k)ku_r(p_1)D_{\alpha}(k^2)$$  \hspace{1cm} (A.10)

where $k = \gamma^\mu k_\mu$ can be written in the form

$$k = S_F^{-1}(p_i) - S_F^{-1}(p_i - k)$$  \hspace{1cm} (A.11)

Here $i = 1, 2$. Using this relation and Dirac equation, Eq. (A.10) becomes

$$A^\mu_{\alpha}(p_1,p_2) = -(1 - \alpha)g_\nu^3\pi_\mu(p_2)\gamma^\mu u_r(p_1)J$$  \hspace{1cm} (A.12)

where

$$J = \int \frac{d^4k}{(2\pi)^4}D_{\alpha}(k^2) = \lim_{\varepsilon \to 0} \frac{i}{(4\pi)^2} \int_0^1 dx \frac{d\varepsilon}{\varepsilon\{\alpha + (1 - \alpha)x|\gamma^2_{\alpha}\}^\varepsilon}$$  \hspace{1cm} (A.13)

Here the last equality is given by the dimensional regularization. This integral gives a divergent constant without containing any finite number in it. Therefore, it may completely be cancelled out by a counterterm in a renormalization program and gives no contribution to the renormalized S-matrix element. On the other hand, since the integral is independent of momentum, it would not contribute to the anomalous dimension and hence to any physical quantity.

For the matrix element of Fig. (6c) which contains a nucleon self-energy, it may be written as

$$S_2^3 = M_\mu(q_1,q_2)B^\mu(p_1,p_2)$$  \hspace{1cm} (A.14)

where

$$B^\mu(p_1,p_2) = \int \frac{d^4k}{(2\pi)^4}\pi_\mu(p_2)ig_\nu\gamma_\rho iS_F(p_2 - k)i\gamma^\mu iS_F(p_1 + q)ig_\sigma u_r(p_1)iD^{\rho\sigma}(k)$$  \hspace{1cm} (A.15)

in which the $\alpha-$dependent part is of the form

$$B^\mu_{\alpha}(p_1,p_2) = -(1 - \alpha)g_\nu^3\int \frac{d^4k}{(2\pi)^4}\pi_\mu(p_2)kS_F(p_2 - k)kS_F(p_1 + q)\gamma^\mu u_r(p_1)D_{\alpha}(k^2)$$  \hspace{1cm} (A.16)

By employing the relation in Eq. (A.11) and Dirac equation, one may find

$$B^\mu_{\alpha}(p_1,p_2) = (1 - \alpha)g_\nu^3\pi_\mu(p_2)\gamma_\rho S_F(p_2)\gamma^\mu u_r(p_1)J^\nu_2 = 0$$  \hspace{1cm} (A.17)

This is because the integral in it vanishes

$$J^\nu_2 = \int \frac{d^4k}{(2\pi)^4}k^\nu D_{\alpha}(k^2) = 0$$  \hspace{1cm} (A.18)
due to that the integrand is an odd function. Similarly, the \( \alpha \)-dependent part of Fig. (6d) can also be proved to give no contribution to the S-matrix element.

Let us turn to Figs.(6e) and (6f). The matrix elements of the both figures can be respectively represented as

\[
S_2^\omega = g_v^4 \int \frac{d^4k}{(2\pi)^4} \overline{u}_s'(q_1) \gamma^\mu S_F(q_1 - k) \gamma^\nu u_r(q_1) \\
\times \overline{u}_s(p_2) \gamma^\mu S_F(p_1 + k) \gamma^\nu u_r(p_1) D_{\nu\rho}(q - k) D_{\rho\sigma}(k)
\]  
(A.19)

and

\[
S_4^\omega = g_v^4 \int \frac{d^4k}{(2\pi)^4} \overline{u}_s'(q_1) \gamma^\mu S_F(q_1 - k) \gamma^\nu u_r(q_1) \\
\times \overline{u}_s(p_2) \gamma^\mu S_F(p_2 - k) \gamma^\nu u_r(p_1) D_{\nu\sigma}(q - k) D_{\sigma\rho}(k)
\]  
(A.20)

where \( q = p_2 - p_1 = q_1 - q_2 \). Their \( \alpha \)-dependent parts are denoted by \( S_{2\alpha}^\omega \) and \( S_{4\alpha}^\omega \). By making use of the relation in (A.11) and the relation \( q - k = (q_1 - k) - q_2 = p_2 - (k + p_1) \) as well as Dirac equation, it is easy to find

\[
S_{2\alpha}^\omega = -2(1 - \alpha)g_v^4 \overline{u}_s'(q_1) \gamma^\mu u_r(q_1) \overline{u}_s(p_2) \gamma^\mu u_r(p_1) \int \frac{d^4k}{(2\pi)^4} \frac{D_{\nu\mu}(k)}{(q - k)^2 - m_\omega^2 + i\epsilon} \\
-(1 - \alpha)^2 g_v^4 \overline{u}_s'(q_2) \gamma^\mu u_r(q_1) \overline{u}_s(p_2) \gamma^\mu u_r(p_1) \int \frac{d^4k}{(2\pi)^4} k_\mu k_\nu D_{\nu\sigma}(q - k) |D_{\nu\sigma}(k)|^2
\]  
(A.21)

which gives \( S_{2\alpha}^\omega + S_{4\alpha}^\omega = 0 \) so that the sum of \( S_2^\omega \) and \( S_4^\omega \) is independent of the gauge parameter.

XIII. APPENDIX B: CROSS SECTION OF NUCLEON-NUCLEON SCATTERING

To illustrate the effect of the renormalization described in this paper, we evaluate the cross section of the nucleon-nucleon elastic scattering. Here we limit ourself to first consider the cross section given in the approximation of order \( g^2 \). In this approximation, only the tree diagrams denoted in Fig. (5) are concerned. From these diagrams, in the center of mass frame, the differential cross section is easily calculated and represented as follows:

\[
\frac{d\sigma}{d\Omega(\theta, \varphi)} = \frac{1}{S} \left( \alpha_v^2 T_v + \alpha_s^2 T_s - \alpha_v \alpha_s T_{vs} \right)
\]  
(B.1)

where \( S = 4(p^2 + M^2) \) with \( p \) being the nucleon momentum is the squared total energy of the system, \( T_v \) is contributed from the \( \omega \) meson exchange interaction, \( T_s \) is given by the \( \sigma \) meson exchange interaction and \( T_{vs} \) is the cross term related to both of the \( \omega \) meson and \( \sigma \) meson exchanges. They are separately represented as follows:

\[
T_v = \frac{R_1^\omega}{(\Delta_1^\omega)^2} + \frac{R_2^\omega}{(\Delta_2^\omega)^2} + (-1)^{1+t} \frac{R_3^\omega}{\Delta_1^\omega \Delta_2^\omega}
\]  
(B.2)

where

\[
R_1^\omega = M^4 + 2p^2 M^2 \cos \theta + 2M^2 p^2 + 2p^4 + 2p^4 \cos^4 \theta/2,
\]  
(B.3)

\[
R_2^\omega = M^4 - 2p^2 M^2 \cos \theta + 2M^2 p^2 + 2p^4 + 2p^4 \sin^4 \theta/2,
\]  
(B.4)

\[
R_3^\omega = 16(p^4 - M^4),
\]  
(B.5)

\[
\Delta_1^\omega = 4p^2 \sin^2 \theta/2 + m_\omega^2,
\]  
(B.6)

and

\[
\Delta_2^\omega = 4p^2 \cos^2 \theta/2 + m_\omega^2.
\]  
(B.7)

\[
T_s = \frac{R_1^s}{(\Delta_1^s)^2} + \frac{R_2^s}{(\Delta_2^s)^2} + (-1)^{1+t} \frac{R_3^s}{\Delta_1^s \Delta_2^s},
\]  
(B.8)
where
\[ R_1^a = 4(p^2 \sin^2 \theta/2 + M^2), \]  
\[ R_2^a = 4(p^2 \cos^2 \theta/2 + M^2), \]  
\[ R_3^a = 2[2M^2(p^2 + M^2) + p^4(\sin^4 \theta/2 + \cos^4 \theta/2)], \]  
\[ \Delta_1^a = 4p^2 \sin^2 \theta/2 + m_\sigma^2, \]  
\[ \Delta_2^a = 4p^2 \cos^2 \theta/2 + m_\sigma^2, \]
and
\[ T_{vs} = \frac{R_{1}^{us}}{\Delta_1^a} + \frac{R_{2}^{us}}{\Delta_2^a} + (-1)^{I_I} \frac{R_{3}^{us}}{\Delta_1^a} + (-1)^{I_I} \frac{R_{3}^{us}}{\Delta_2^a}, \]
where
\[ R_{1}^{us} = 4M^2[(M^2 + 2p^2)^2 + (M^2 + 2p^2 \cos^2 \theta/2)^2], \]  
\[ R_{2}^{us} = 4M^2[(M^2 + 2p^2)^2 + (M^2 + 2p^2 \sin^2 \theta/2)^2], \]  
\[ R_{3}^{us} = 4[2(M^2 + p^2 \cos^2 \theta/2)^2 - M^2(M^2 + p^2 + p^2 \sin^2 \theta/2)] \]
and
\[ R_{1}^{us} = 4[2(M^2 + p^2 \sin^2 \theta/2)^2 - M^2(M^2 + p^2 + p^2 \cos^2 \theta/2)]. \]

In the above, \((\theta, \varphi)\) are the scattering angles, \(I\) is the isospin of the two-nucleon system, \(\Delta_1^a\) and \(\Delta_2^a\) \((k = 1, 2)\) are respectively given by the \(\omega\) meson and \(\sigma\) meson propagators and \(R_k^a\) \((\alpha = v, s, i = 1, 2, 3)\) are the functions coming from the nucleon spinor matrix elements. In Eqs. (B.2), (B.8) and (B.14), the isospin-related terms arise from the exchanged diagram, while, the remaining terms represent the contribution of the direct diagram.

To consider the renormalization effect on the two-nucleon scattering, as mentioned in Sec. VIII, we may directly replace the coupling constant \(\alpha_v\) and \(\alpha_s\) in Eq. (B.1) by their effective ones \(\alpha_{vR}(\lambda)\) and \(\alpha_{sR}(\lambda)\) and the particle masses \(M, m_\omega\) and \(m_\sigma\) appearing in the propagators and the functions \(R_k^a\) by their effective counterparts \(M_R(\lambda), m_\omega^R(\lambda)\) and \(m_\sigma^R(\lambda)\). To this end, it should be noted that the nucleon spinors in the S-matrix element under consideration are on the mass-shell, satisfying the free nucleon Dirac equation. The momenta \(p_i\) in the spinors are timelike because they meet the relation \(p^2 = M^2\) where \(M\) is real. For the renormalized spinor wave function shown in Eq. (8.14), as easily verified, it also satisfies the Dirac equation and the momentum in the spinor fulfills the relation \(p(\lambda)^2 = M_R(\lambda)^2\) where \(p(\lambda) = (E(\lambda), \vec{p})\) with \(E(\lambda) = (p^2 + M_R(\lambda)^2)^{1/2}\). Therefore, for the nucleon-nucleon scattering, it is adequate to take the effective nucleon mass given in the timelike momentum space subtraction. While, the momenta in the \(\omega\) meson and \(\sigma\) meson propagators, as one knows, are off-shell and spacelike in the t-channel scattering. Therefore, it is appropriate to take the effective coupling constants \(\alpha_{vR}(\lambda)\) and \(\alpha_{sR}(\lambda)\) and the effective meson masses \(m_\omega^R(\lambda)\) and \(m_\sigma^R(\lambda)\) given in the spacelike momentum space subtraction. In this section, we only examine the effect of the one-loop renormalization on the two-proton scattering by using the effective coupling constants and masses presented in Sec. IX. The differential cross sections given at the laboratory kinetic energies \(T_{lab} = 491.9\,MeV\) and \(575.5\,MeV\) are shown in Fig. (17a) and (17b). In the figures, the solid lines and the dashed lines represent respectively the calculated results with and without considering the renormalization effect and the experimental data are taken from Ref. [48]. As shown in Fig. (17a) and (17b), in the case without considering the renormalization effect, the theoretical parameters are taken to be \(M_R = 938\,MeV, \alpha_{vR} = 1.1, \alpha_{sR} = 1.4, m_\omega^R = 782\,MeV\) and \(m_\sigma^R = 580\,MeV\); while in the case of considering the renormalization effect, the parameters must be taken to be \(M_R = 938\,MeV, \alpha_{vR} = 0.55, \alpha_{sR} = 0.62, m_\omega^R = 782\,MeV\) and \(m_\sigma^R = 670\,MeV\). It is clearly seen from the figures that consideration of the renormalization has an effect that to fit the experimental data, the coupling constants must be set to be smaller.
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XV. FIGURE CAPTIONS

Fig. (1): The one-loop nucleon self-energy in the $\sigma - \omega$ model. The solid, wavy line and dashed lines represent the free nucleon, $\omega$ meson and $\sigma$ meson propagators respectively.

Fig. (2): The one-loop vectorial vertices in the $\sigma - \omega$ model. The lines represent the same as in Fig. (1).

Fig. (3): The one-loop scalar vertices in the $\sigma - \omega$ model. The lines mark the same as in Fig. (1).

Fig. (4): The diagrams represent the nucleon four-point one-particle irreducible Green’s function. The solid line with a white blob represents the full nucleon propagator. The wavy line with a white blob denotes the full $\omega$ meson propagator. The shaded blobs represent the proper vertices.

Fig. (5): The tree diagrams of nucleon-nucleon scattering. The first two diagrams represent the interaction generated by the $\omega$ meson exchange. The remaining two diagrams represent the interaction mediated by the $\sigma$ meson exchange.

Fig. (6): Some two-nucleon one-loop Feynman diagrams which are chosen to demonstrate the gauge-independence of the nucleon scattering matrix elements.

Fig. (7): The one-loop diagram of the effective $\omega$ meson self-energy. The solid line marks the free nucleon propagator and the wavy line denotes the free $\omega$ meson propagator.

Fig. (8): The effective one-loop vectorial coupling constants $\alpha^v_R(\lambda)$ given by the timelike momentum space subtraction. The solid line represents the coupling constant given by taking $\alpha^v_R = 0.5$. The dashed line denotes the coupling constant given by $\alpha^v_R = 1$.

Fig. (9): The effective one-loop vectorial coupling constants $\alpha^v_R(\lambda)$ given by the spacelike momentum space subtraction. The solid line represents the coupling constant given by taking $\alpha^v_R = 0.5$. The dashed line denotes the coupling constant given by $\alpha^v_R = 1$.

Fig. (10): The effective one-loop $\omega$ meson masses $m_R^\omega(\lambda)$ given by taking $\alpha^v_R = 1$. The solid line and the dashed line represent the effective masses obtained in the spacelike momentum subtraction and the timelike momentum respectively.

Fig. (11): The effective one-loop nucleon masses $M_R(\lambda)$ obtained in the timelike momentum subtraction. The solid, dashed and dotted lines represent the effective masses given by $(\alpha^v_R, \alpha^s_R) = (0.5, 0), (0, 0.5)$ and $(0.5, 0.5)$ respectively.

Fig. (12): The effective one-loop nucleon masses $M_R(\lambda)$ obtained in the spacelike momentum subtraction. The dashed, solid and dotted lines represent the effective masses given by taking $(\alpha^v_R, \alpha^s_R) = (0.5, 0), (0, 0.5)$ and $(1, 1)$ respectively. The upper figure describes the real part of the $M_R(\lambda)$. Another figure shows the imaginary part of the $M_R(\lambda)$.

Fig. (13): The $\sigma$ meson one-loop self-energy. The solid line represents the free nucleon propagator and the dashed line denotes the free $\sigma$ meson propagator.

Fig. (14): The effective one-loop scalar coupling constants $\alpha^s_R(\lambda)$ obtained in the timelike momentum subtraction. The dashed and solid lines on the left represents the effective coupling constants given by $(\alpha^v_R, \alpha^s_R) = (0, 1)$ and $(0.5, 1)$. The dashed and solid lines on the right denote the effective coupling constants given by $(\alpha^v_R, \alpha^s_R) = (0, 0.2)$ and $(0.5, 0.2)$.

Fig. (15): The effective one-loop scalar coupling constants $\alpha^s_R(\lambda)$ obtained in the spacelike momentum subtraction. The four lines represent the effective coupling constants given by taking $(\alpha^v_R, \alpha^s_R) = (0.1), (0, 0.2), (0.5, 1)$ and $(0, 0.2)$ respectively. The solid and dashed lines denote the real parts and the imaginary parts of the coupling constants respectively.

Fig. (16): The effective one-loop $\sigma$ meson masses $m_R^\sigma(\lambda)$ obtained by taking $(\alpha^v_R, \alpha^s_R) = (1, 1)$. The solid line represents the effective mass given in the spacelike momentum subtraction. The dashed line shows the real part of the $m_R^\sigma(\lambda)$ given in the timelike momentum subtraction.

Fig. (17): The two proton elastic differential cross sections given at the laboratory kinetic energies $T_{lab} = 491.9 MeV$ and $575.5 MeV$. The black squares show the experimental data. The solid lines represent the theoretical values calculated by considering the one-loop renormalization effect. The dashed lines represent the theoretical values without considering the one-loop renormalization effect.

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\[ k \] (a)

\[ p \]

\[ p - k \]

\[ p \]
\( p - q \Rightarrow q - k \Rightarrow \gamma_{\mu} \Rightarrow k \Rightarrow p \)
(a) \[ q - k \]

\[ p - q \]

\[ p - k \]

\[ q \]
\[(a)\]

\[k - q \quad q \quad k\]

\[q_1 \quad k \quad k \quad p_1 \quad p_2 \quad q_2\]
\[(d)\]

\[p_1 = p_2 - q\]
\[ q = p_2 - p_1 \]
\[ q = p_2 - p_1 \]
The diagram illustrates the behavior of $\frac{v}{R}(\lambda)$ as a function of $\lambda$, with two curves representing different values of $\alpha_R$: $\alpha_R = 1$ (dashed line) and $\alpha_R = 0.5$ (solid line). The x-axis represents $\lambda$, and the y-axis represents $\frac{v}{R}(\lambda)/\alpha_R$. The graph shows a significant increase in the value of $\frac{v}{R}(\lambda)$ for both curves as $\lambda$ increases, with the curve for $\alpha_R = 1$ crossing the y-axis at a higher value than the curve for $\alpha_R = 0.5$. The scale on the x-axis ranges from $10^0$ to $10^3$. The graph also includes a boxed notation stating $\frac{v}{R}(\lambda)/\alpha_R$. 
\[ M_R(\lambda)/M_R = \frac{\lambda^{\alpha_R} S^{\alpha_R}}{\lambda^{\alpha/s}} \]

\[ \alpha, s = 0, 0.5 \]
\[ 0.5, 0 \]
\[ 0.5, 0.5 \]
\[ M_{R}(\lambda)/M_R \]
\[
\begin{align*}
\alpha_{R}^v &= 0 \\
\alpha_{S}^v &= 0.2 \\
\alpha_{R}^v &= 0.5 \\
\alpha_{S}^v &= 0.2
\end{align*}
\]
\[
m_{\sigma}(\lambda)/m_{\sigma} \propto \frac{R}{\sigma} \left(\frac{\lambda}{m}\right)
\]
\[ T_{\text{lab}} = 491.9 \text{MeV} \]
\( T_{\text{lab}} = 575.5 \text{ MeV} \)