On the Weight Spectrum of Pre-Transformed Polar Codes

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Abstract—Polar codes are the first class of channel codes achieving the symmetric capacity of the binary-input discrete memoryless channels with efficient encoding and decoding algorithms. But the weight spectrum of Polar codes is relatively poor compared to RM codes, which degrades their ML performance. Pre-transformation with an upper-triangular matrix (including cyclic redundancy check (CRC), parity-check (PC) and polarization-adjusted convolutional (PAC) codes), improves weight spectrum while retaining polarization. In this paper, we determine the weight spectrum of upper-triangular pre-transformed Polar Codes. In particular, we focus on determining the number of low-weight codewords due to their impact on error-correction performance. Simulation results verify the accuracy of the analysis.

I. INTRODUCTION

Polar codes [1], invented by Ariklan, are a great breakthrough in coding theory. As code length $N = 2^n$ approaches infinity, the synthesized channels become either noiseless or pure-noise, and the fraction of the noiseless channels approaches channel capacity. Thanks to channel polarization, efficient successive cancellation (SC) decoding algorithm can be implemented with a complexity of $O(N\log N)$. However, the performance of polar codes under SC decoding is poor at short to moderate block lengths.

In [2], a successive cancellation list (SCL) decoding algorithm was proposed. As the list size $L$ increases, the performance of SCL decoding approaches that of maximum-likelihood (ML) decoding. But the ML performance of polar codes is still inferior to low minimum distance. Consequently, concatenation of polar codes with CRC [3] and PC [4] were proposed to improve weight spectrum. Recently, Ariklan proposed polarization-adjusted convolutional (PAC) codes [5], which is shown to approach BIAWGN dispersion bound [6] under large list decoding [7].

CRC-Aided (CA) Polar, PC-Polar, and PAC codes can be viewed as pre-transformed Polar codes with upper-triangular transformation matrices [7]. In [8], it is proved that any pre-transformation with an upper-triangular matrix does not reduce the minimum Hamming weight, and a properly designed pre-transformation can reduce the number of minimum-weight codewords. In this paper, we propose an efficient method to calculate the average weight spectrum of pre-transformed polar codes. Moreover, the method holds for arbitrary information sub-channel selection criteria, thus covers Polar codes, RM codes and is not constrained by “partial order” [9]. Our results confirm that the pre-transformation with an upper-triangular matrix can reduce the number of minimum-weight codewords significantly. In the meantime, it enhances error-correcting performance of SCL decoding.

In section II, we review Polar codes and pre-transformed Polar codes. In section III we propose a formula to calculate the average weight spectrum of pre-transformation Polar codes. In section IV the simulation results are presented to verify the accuracy of the formula. Finally we draw some conclusions in section V.

II. BACKGROUND

A. Polar code

Given a B-DMC $W : \{0, 1\} \to \mathcal{Y}$, the channel transition probabilities are defined as $W(y|x)$ where $y \in \mathcal{Y}, x \in \{0, 1\}$. $W$ is said to be symmetric if there is a permutation $\pi$, such that $\forall y \in \mathcal{Y}, W(y|1) = W(\pi(y)|0)$ and $\pi^2 = id$.

Then the symmetric capacity and the Bhattacharyya parameter of $W$ are defined as

$$I(W) \triangleq \sum_{y \in \mathcal{Y}} \sum_{x \in \mathcal{X}} \frac{1}{2} W(y | x) \log \frac{W(y | x)}{\frac{1}{2} W(y | 0) + \frac{1}{2} W(y | 1)}$$

and

$$Z(W) \triangleq \sum_{y \in \mathcal{Y}} \sqrt{W(y | 0)W(y | 1)}$$

Let $F = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$, $N = 2^m$, and $H_N = F^\otimes m$. Starting from $N = 2^m$ independent channels $W$, we obtain $N$ polarized channels $W_N^{(i)}$, after channel combining and splitting operations [11], where

$$W_N \left( y_1^N | u_1^N \right) \triangleq W_N \left( y_1^N | u_1^N H_N \right)$$

and

$$W_N^{(i)} \left( y_1^N, u_1^{i-1} | u_i ight) \triangleq \sum_{u_{i+1}^N \in \mathcal{X}^{N-i}} \frac{1}{2^{N-i}} W_N \left( y_1^N | u_i^N \right)$$

Polar codes can be constructed by selecting the indices of $K$ information sub-channels, denoted by the information set $A = \{I_1, I_2, \ldots, I_K\}$. The optimal sub-channel selection...
criterion for SC decoding is reliability, i.e., selecting the $K$ most reliable sub-channel as information set. Density evolution (DE) algorithm [10], Gaussian approximation (GA) algorithm [11] and the channel-independent PW construction method [12] are efficient methods to find reliable sub-channels. The optimal sub-channel selection criterion for SCL decoding is still an open problem. Some heuristic approaches consider both reliability and row weight, such as RM-Polar codes [13] and PC-Polar codes [4], to improve minimum code distance. Others employ artificial intelligence techniques to find good information set [14] [15].

After determining the information set $\mathcal{A}$, the complement set $\mathcal{A}^c$ is called the frozen set. Let $u_1^N = (u_1, u_2, \ldots, u_N)$ be the bit sequence to be encoded. The information bits are inserted into $u_\mathcal{A}$, and all zeros are filled into $u_{\mathcal{A}^c}$. Then the codeword $x_1^N$ is obtained by $x_1^N = u_1^1 H_N$.

**B. Weight Spectrum of Polar Codes**

There are many prior works to analyze the weight spectrum of Polar codes. In [16], the authors use SCL decoding with a large list size to decode an all-zeros codeword. Codewords within the list are enumerated to estimate the number of low-weight codewords. In [17], the authors improve this approach in terms of memory usage. The above methods only obtain partial weight spectrum. In [18] [19], the probabilistic computation methods are proposed to estimate the weight spectrum of Polar codes.

**C. Weight Spectrum of Polar Cosets**

As in [20], let $u_1^{i-1} \in \{0, 1\}^{i-1}$, $u_i \in \{0, 1\}$, define the polar coset $C_N^{(i)} (u_1^{i-1}, u_i)$ as

$$C_N^{(i)} (u_1^{i-1}, u_i) = \{(u_1', u_i') H_N | u_i' \in \{0, 1\}^{n-i}\}$$

In [21] [22], recursive formulas are proposed to efficiently compute the weight spectrum of $C_N^{(i)} (0_1^{i-1}, 1)$. The weight spectrum of $C_N^{(i)} (0_1^{i-1}, 1)$ is tightly associated with the performance of SC decoding, our analysis of average weight spectrum of pre-transformed Polar codes is based on the polar coset spectrum as well.

**D. Pre-Transformed Polar Codes**

$$T = \begin{bmatrix} 1 & T_{12} & \cdots & T_{1N} \\ 0 & 1 & \cdots & T_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

The above non-degenerate upper-triangular pre-transformation matrix $T$ has all ones on the main diagonal. Let $G_N = TH_N$ and $u_{\mathcal{A}^c} = 0$, the codeword of the pre-transformed Polar codes is given by $x_1^N = u_1^N G_N = u_1^N TH_N$. Let $z_i^N = u_1^N T$, the $i$-th pre-transformed bit is given by $z_i = u_i \oplus \sum_{j=i+1}^{N} u_j T_{ji}$. As seen, $z_i$ is a linear combination of the $i$-th and previous bits, much like a parity-check bit or a dynamic frozen bits [23].

### III. Average Code Spectrum Analysis

In this section, we propose a formula to compute the average weight spectrum of the pre-transformed Polar codes, with focus on the number of low-weight codewords. The average number assumes that $T_{ij}, 1 \leq i < j \leq N$ are i.i.d. Bernoulli($\frac{1}{2}$) r.v., $T_{ij}^i$ are i.i.d. Bernoulli($\frac{1}{2}$) r.v. as well.

**A. Notations and Definitions**

$h_N^{(i)}$ is the $i$-th row vector of $H_N$, and $g_N^{(i)}$ is the $i$-th row vector of $G_N$. The number of codewords with Hamming weight $d$ of the pre-transformed Polar codes is denoted by $N_d(U \times T \times H_N)$. The minimum distance of Polar/RM codes and the pre-transformed codes are denoted by $d_{\min}(U \times H_N)$ and $d_{\min}(U \times T \times H_N)$, respectively. The number of minimum-weight codewords of Polar/RM codes and the pre-transformed codes are denoted by $N_{\min}(U \times H_N)$ and $N_{\min}(U \times T \times H_N)$, respectively.

**B. Code Spectrum Analysis**

The expected number of codewords with Hamming weight $d$ is

$$E[N_d(U \times T \times H_N)] = \sum_{u_1, u_2, \ldots, u_K \in \{0, 1\}^K} P\left(w\left(\sum_{i=1}^K u_I g_N^{(i)} \right) = d\right)$$

$$= \frac{K}{d} \sum_{j=1}^K \sum_{u_1, \ldots, u_{j-1} = 0}^{u_j = 1} \sum_{u_{j+1}, \ldots, u_K \in \{0, 1\}^{K-j}} P\left(w\left(g_N^{(j)} \oplus \sum_{i=j+1}^K u_I g_N^{(i)} \right) = d\right)$$

**Lemma 1.** For $u_{j+1}, \ldots, u_K \in \{0, 1\}^{K-j}$,

$$P\left(w\left(g_N^{(j)} \oplus \sum_{i=j+1}^K u_I g_N^{(i)} \right) = d\right) = P\left(w\left(g_N^{(j)} \right) = d\right)$$

**Proof.** According to the pre-transformation matrix

$$g_N^{(j)} = h_N^{(j)} \oplus \sum_{i=j+1}^K T_{ij} h_N^{(i)}$$

$$g_N^{(j)} \oplus \sum_{i=j+1}^K u_I g_N^{(i)} = h_N^{(j)} \oplus \sum_{i=j+1}^K T_{ij} h_N^{(i)}$$

And

$$T_{ij} = \begin{cases} \sum_{l_i < l, u_{k_i} = 1} T_{lk_i} & i \notin [J_{ij+1}, \ldots, I_K] \\ \sum_{l_i < l, u_{k_i} = 1} T_{lk_i} + u_i & i \in [J_{ij+1}, \ldots, I_K] \end{cases}$$

It is straightforward to see that when $T_{ij}$ are i.i.d. Bernoulli($\frac{1}{2}$) r.v., $T_{ij}^i$ are i.i.d. Bernoulli($\frac{1}{2}$) r.v. as well.
As a result, \( g_N^{(i)} \) and \( g_N \) follow the same distribution too, \( \forall \ u_{I_1}, \ldots, u_{I_K} \in \{0, 1\}^{K-j} \). Therefore \textbf{Lemma 1} holds.

\textbf{Lemma 2.} If \( w(h_N^{(i)}) > d \), \( P\left( w(g_N^{(i)}) = d \right) = 0 \).

\textit{Proof.} Recall that \( g_N^{(i)} = h_N^{(i)} + \sum_{i=I_1}^N T_{I_i} h_N^{(i)} \)

According to \[9\], corollary 1, \( w\left(g_N^{(i)}\right) \geq w\left(h_N^{(i)}\right) > d \)

Therefore \( P\left( w\left(g_N^{(i)}\right) = d \right) = 0 \)

According to \textbf{Lemma 1} and \textbf{Lemma 2}, \(5\) can be further simplified to

\[ E\left[N_d(U \times T \times H_N)\right] = \sum_{1 \leq j \leq K \atop w(h_N^{(i)}) \leq d} 2^{K-j} P\left( w\left(g_N^{(i)}\right) = d \right) \]

(6)

Let \( P(m, i, d) \triangleq P\left( w\left(g_{2m}^{(i)}\right) = d \right) \), \(6\) can be rewritten as

\[ E\left[N_d(U \times T \times H_N)\right] = \sum_{1 \leq j \leq K \atop w(h_N^{(i)}) \leq d} 2^{K-j} P(m, I_j, d) \]

(7)

In particular, let \( P(m, i) \triangleq P\left( w\left(g_{2m}^{(i)}\right) = w\left(h_{2m}^{(i)}\right)\right) \). So if \( d = d_{\min} \), \(6\) can be rewritten as

\[ E\left[N_{\min}(U \times T \times H)\right] = \sum_{1 \leq j \leq K \atop w(h_N^{(i)}) = d_{\min}(U \times H)} 2^{K-j} P(m, I_j) \]

(8)

Let \( A_d \) denote the number of codewords in \( C_N^{(i)} \{0^{i-1}, 1\} \) with Hamming weight \( d \). Clearly, \( 2^{N-1} P(m, i) = A_d \), \( 2^{N-1} P(m, i, d) = A_d \). In \[21\] \[22\], the authors propose recursive formulas to calculate the weight spectrum of Polar cosets.

In \textbf{Theorem 1} and \textbf{Theorem 2}, we investigate the recursive formulas for \( P(m, i) \) and \( P(m, i, d) \), which are similar to the formula in \[22\]. But instead of Polar cosets, we are interested in the pre-transformed Polar codes. For the completeness of the paper, the proofs are in the appendix.

\textbf{Theorem 1.}

\[ P(m, i) = \begin{cases} \frac{2^{w(h_N^{(i)})}}{2^{2m-i}} P(m-1, i) & 1 \leq i \leq 2^{m-1} \\ P(m-1, i-2^{m-1}) & 2^{m-1} < i \leq 2^m \end{cases} \]

(9)

with the boundary conditions \( P(1, 1) = P(1, 2) = 1 \).

With \(5\) and \(9\), we can recursively calculate the average number of minimum-weight codewords. We are also interested in other low-weight codewords on the weight spectrum, since together they determine the ML performance at high SNR. The problem boils down to evaluating the more general formula of \( P(m, i, d) \). As we will see in \textbf{Theorem 2}, the average weight spectrum can be calculated efficiently in the same recursive manner especially for codewords with small Hamming weight.

\textbf{Theorem 2.} If \( 1 \leq i \leq 2^{m-1} \)

\[ P(m, i, d) = \sum_{d' = w(h_{2m}^{(i)}) \atop d' \text{is even}}^{d} P(m-1, i, d') \]

(10)

\[ \text{if } \frac{2^{m-1}}{d} < i \leq 2^m \]

with the boundary conditions \( P(1, 1) = P(1, 2, 1) = 1 \).

\[ P(m, i, d) = \begin{cases} P(m-1, i-2^{m-1}, d/2) & \text{if } d \text{ is even} \\ 0 & \text{if } d \text{ is odd} \end{cases} \]

(10)

\textbf{IV. Simulation}

In this section, we verify the correctness of the recursive formula through simulations. In particular, we employ the "large list decoding" method described in \[10\] to collect low-weight codewords. At first, we randomly generate one thousand pre-transform matrices for RM\(128, 64\), and set \( L = 5 \times 10^3 \) to count the number of minimum-weight codewords for each matrix, and obtain their average \( N_{\min} \). The result is shown in Fig. 1: \( d_{\min} = 16 \), \( N_{\min} = 2768.1 \), \( N_{\text{recursion}} = 2766.9 \).

To show that our recursive formula is applicable for any sub-channel selection criterion we also construct Polar code \(128, 64\) by the PW method \[12\]. The simulation result is shown in Fig. 2: \( d_{\min} = 8 \), \( N_{\min} = 272.64 \), \( N_{\text{recursion}} = 272 \).

As seen, the recursively calculated minimum-weight codeword numbers are very close to ones obtained through simulation.

In Table. I, we display the number of minimum codewords of the original RM/Polar codes, and the average number is recursively calculated. It is shown that pre-transforming significantly reduces the number of minimum-weight code words, especially in RM\(128, 64\). The significant improvement of weight spectrum after pre-transformation explains why the CA-Polar, PC-Polar, and PAC codes outperform the original Polar codes under list decoding with large list size.

The improvement can be observed under different code lengths and rates, as we can see from Fig. 3. In all cases, pre-transformation reduces the number of minimum codewords significantly.
In addition to minimum-weight codewords, we also simulate to verify the accuracy of the formula for other low-weight codewords. The simulation results are shown in Table II for RM(128, 64) and PW(128, 64) respectively, where \( N_{\text{sim}} \) is the simulation results, and \( N_{\text{recur}} \) is the calculation results. In parity-check (PC) codes [4], both reliability and code distance are taken into consideration when selecting the information set. A coefficient \( \alpha \) is used to control the trade-off between reliability and code distance. The larger \( \alpha \) is, the greater code distance is. A parity check pattern can be considered as a realization of the pre-transformation matrix. Take PC-Polar (128, 64) \((\alpha = 1.5)\) as an example, we calculate the average number of low-weight codewords. The result implies that pre-transformation of the pre-transformed code can increase the minimum code distance when the information set is properly chosen, that is, reducing the number of original minimum-codewords to zero. The spectrum of the code ensemble average, the original code and a realization of the pre-transformed code are shown in Table III. In this case, although some rows of \( H_N \) with Hamming weight 8 are selected into the information set, PC-Polar codes can increase the minimum distance from 8 to 12.

**Table I**

| Minimum-weight codewords | \( d_{\text{min}} \) | Original | Pre-transformed |
|--------------------------|----------------------|---------|----------------|
| RM(128,64)               | 16                   | 94488   | 2767           |
| PW(128,64)               | 8                    | 304     | 272            |

**Table II**

|                  | RM(128,64) | PW(128,64) |                  |                  |
|------------------|------------|------------|------------------|------------------|
| \( d \)          | \( N_{\text{sim}} \) | \( N_{\text{recur}} \) | \( d \)          | \( N_{\text{sim}} \) | \( N_{\text{recur}} \) |
| 16               | 2764.5     | 2766.9     | 8                | 272.2            | 272               |
| 18               | 397.1      | 393.5      | 12               | 896.6            | 896               |
| 20               | 80251      | 80182      | 16               | 76812.2          | 77111             |

Note that \( N_{10} = N_{14} = 0 \) for PW(128, 64)

**Table III**

|                  | Average | Original | Pre-transformed |
|------------------|---------|----------|----------------|
| 8                | 0.5     | 32       | 0              |
| 10               | 0.0547  | 0        | 0              |
| 12               | 0.395   | 0        | 48             |
| 14               | 0.27    | 128      | 28             |
| 16               | 0.5250  | 57048    | 5228           |

Fig. 4 and Fig. 5 provide the BLER performances of various constructions under different list sizes, with reference to finite-length performance bounds such as normal approximation (NA), random-coding union (RCU) and meta-converse (MC) bounds [6] [24] [25]. It is observed that reliability is the only contributing factor to decoding performance under SC decoding. Under SCL decoding with list size \( L = 8 \), the PC-Polar codes \((\alpha = 1.5)\) strike a good balance between reliability
and distance, and shows the best decoding performance. When the list size is large enough, both PAC and PC-Polar codes can approach NA bound with their ML performances.

V. CONCLUSION

In this paper, we propose recursively formulas to efficiently calculate the average weight spectrum of pre-transformed Polar codes, which include CA-Polar, PC-Polar and PAC codes as special cases. It is worth mentioning that our formulas work for any sub-channel selection criteria. We found that, with pre-transformation, the average number of minimum codewords decreases significantly, therefore outperforming the original RM/Polar codes under the ML decoding and SCL decoding with large list sizes. Furthermore, as in the instance of PC-Polar codes ($\alpha = 1.5$), the combination of a proper sub-channel selection and pre-transformation has the potential to increase minimum code distance by eliminating minimum-weight codewords.

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A. Proof of Theorem 1

Proof. A trivial examination can prove the correctness of the boundary conditions. Let us focus on deriving the recursive formula.

Case 1: $1 \leq i \leq 2^{m-1}$

$$g_{2^m}^{(i)} = h_{2^m}^{(i)} + \sum_{j=1+1}^{2^{m-1}} T_{ij} h_{2^m}^{(j)} + \sum_{j=2^{m-1}+1}^{2^m} T_{ij} h_{2^m}^{(j)}$$

Let $h_{2^m}^{(i)} \oplus \sum_{j=1+1}^{2^{m-1}} T_{ij} h_{2^m}^{(j)} \oplus [X, 0]$, $\sum_{j=2^{m-1}+1}^{2^m} T_{ij} h_{2^m}^{(j)} \oplus [Y, Y]$, where $0$ is an all-zero row vector of length $2^{m-1}$, $X = (x_1, \ldots, x_{2^{m-1}}), Y = (y_1, \ldots, y_{2^{m-1}})$.

Apparently, $X$ and $Y$ are independent, and $\forall a = (a_1, \ldots, a_{2^{m-1}}) \in \{0, 1\}^{2^{m-1}}$, $P(Y = a) = 2^{2^{m-1}}$. Let $w(X) = d_1$, $w(Y) = d_2$, and $c$ be the number of positions where $X$ and $Y$ are both 1. We have

$$w(g_{2^m}^{(i)}) = w([X \oplus Y, Y]) = w(X \oplus Y) + w(Y) = d_1 + 2d_2 - 2c$$

Because $d_2 \geq c$ and $d_1 \geq w(h_{2^m}^{(i)})$ [9, corollary 1], the equation $w(g_{2^m}^{(i)}) = d_1 + 2d_2 - 2c = w(h_{2^m}^{(i)})$ holds if and only if $d_1 = w(h_{2^m}^{(i)})$, $d_2 = c$. In fact, $d_2 = c$ implies that if $x_i = 0$ then $y_i = 0$, $1 \leq i \leq 2^{m-1}$. Let $\{i_1, \ldots, i_{d_1}\}$ denote the $d_1$ locations where $x_{i_1}, \ldots, x_{i_{d_1}} = 1$, hence the recursive formula is

$$P(m, i) = P(d_1 = w(h_{2^m}^{(i)})) \ast P(d_2 = c | d_1 = w(h_{2^m}^{(i)}))$$

$$= P(m - 1, i) \ast P(y_{i_1}, \ldots, y_{i_{d_1}} \in \{0, 1\}^{d_1}, y_i = 0 \text{ otherwise})$$

$$= P(m - 1, i) \ast \frac{2d_1}{2^{2^{m-1}}}$$

$$= P(m - 1, i) \ast \frac{2w(h_{2^m}^{(i)})}{2^{2^{m-1}}}$$

Case 2: $2^{m-1} < i \leq 2^m$

$$g_{2^m}^{(i)} = \left[ h_{2^m}^{(i)} \oplus \sum_{j=1+1}^{2^{m-1}} T_{ij} h_{2^m}^{(j)} \right]$$

$$= \left[ h_{2^m-1}^{(i-2^{m-1})} \oplus \sum_{j=1+1}^{2^{m-1}} T_{ij} h_{2^m-1}^{(j-2^{m-1})}, h_{2^m-1}^{(i-2^{m-1})} \oplus \sum_{j=1+1}^{2^{m-1}} T_{ij} h_{2^m-1}^{(j-2^{m-1})} \right]$$

$$\oplus \left[ g_{2^m-1}^{(i-2^{m-1})}, g_{2^m-1}^{(i-2^{m-1})} \right]$$

where $X_1 \sim X_2$ means $X_1, X_2$ have the same distribution.

B. Proof of Theorem 2

Proof. (10) is obtained with the observation that $w(h_{N}^{(i)})$ is odd and $\forall i > 1$, $w(h_{N}^{(i)})$ is even.

Case 1: $1 \leq i \leq 2^{m-1}$

Similar to the proof of Theorem 1, let $w(X) = d_1$, $w(Y) = d_2$ and $c$ be the number of positions where $X$ and $Y$ are both 1. Denoted by $V = \{v_1, \ldots, v_c\}$ the set of positions where $X$ and $Y$ are both 1, and $V^c$ its complement. Let $Y_v$ denote the corresponding subvector of $Y$, we have $w(Y_v) = d_2 - c$. Because

$$w(g_{2^m}^{(i)}) = w([X \oplus Y, Y]) = w(X \oplus Y) + w(Y) = d_1 + 2d_2 - 2c = d$$

then $d_2 - c = \frac{d - d_1}{2}$, so $d - d_1$ must be even. No matter what $c$ is, the equation is satisfied if and only if $w(Y_v) = \frac{d - d_1}{2}$.

Based on the above observations, $P(m, i, d)$ can be formulated as

$$P(m, i, d) = \sum_{d' = w(h_{2^m}^{(i)})}^{d} P(m, i, d | w(X) = d') \ast P(w(X) = d')$$

$$= \sum_{d' = w(h_{2^m}^{(i)})}^{d} P(m, i, d | w(X) = d') \ast P(m - 1, i, d')$$

The last equality holds due to $X \sim g_{2^{m-1}}^{(i)}$.

In particular

$$P(m, i, d | w(X) = d') = P \left( w(Y_v) = \frac{d - d_1}{2} \right) = 2^{2^d} \left( \frac{2^{2^{m-1}} - d'}{2^{2^{m-1}}} \right)$$

Consequently, the recursive formula is

$$P(m, i, d) = \sum_{d' = w(h_{2^m}^{(i)})}^{d} P(m - 1, i, d') \ast \frac{2^{2^d} \left( \frac{2^{2^{m-1}} - d'}{2^{2^{m-1}}} \right)}{2^{2^{m-1}}}$$

Case 2: $2^{m-1} < i \leq 2^m$, according to (11)

$$g_{2^m}^{(i)} = \left[ g_{2^m-1}^{(i-2^{m-1})}, g_{2^m-1}^{(i-2^{m-1})} \right]$$

It is straightforward to obtain the recursive formula

$$P(m, i, d) = P(m - 1, i - 2^{m-1}, d/2)$$

