Abstract The confinement mechanism proposed earlier by the author is applied to problem of arising the so-called scale $\Lambda_{\text{QCD}}$ within the framework of QCD. The natural physical assumption consists of that $1/\Lambda_{\text{QCD}} \sim \langle r \rangle$ where $\langle r \rangle$ is a characteristic size of hadron (radius of confinement). The above confinement mechanism allows us to calculate $\langle r \rangle$ for mesons in terms of quark and gluonic degrees of freedom and this permits to conclude that $\Lambda_{\text{QCD}}$ should slightly change from hadron to hadron.

Keywords Quantum chromodynamics - Confinement

1 Introduction

As is known (see, e.g., [1][2][3]), within the framework of perturbative quantum chromodynamic (QCD) in asymptotical region of large momenta there appears some constant $\Lambda_{\text{QCD}}$ with dimension of mass, $\Lambda_{\text{QCD}}$ being not connected with arbitrary choice of the renormalization point. This is the so-called scale $\Lambda_{\text{QCD}}$ (in what follows we denote it by just $\Lambda$) and it cannot be determined from the theory in asymptotical region of large momenta and should arise in some way in the region of small momenta. The merits of case is contained in the relation between the effective strong coupling constant $\alpha_s$ and the momentum transfer $\sqrt{Q^2}$

$$\alpha_s = \alpha_s(Q^2) = \frac{12\pi}{(33 - 2n_f) \ln(Q^2/\Lambda^2)}$$

(1)

while $n_f$ is number of quark flavours and (1) holds true at the momentum transfer $\sqrt{Q^2} \to \infty$. As is clear from (1), if $Q^2 \to \Lambda^2$ then $\alpha_s \to \infty$ and traditional physical interpretation of it consists in that at distances of order $r \sim 1/\Lambda$ ($r$ is distance between quarks) interaction of quarks becomes very strong and provides the quark confinement so at distances greater than $1/\Lambda$ quarks are unobservable (see,
e.g., [1, 2]). It should be noted, however, that in fact the numerical value of $\Lambda$ depends on the situation where the relation (1) is used [4]. Under the circumstances, appearance of the scale $\Lambda$ should be closely connected with the confinement mechanism and the latter should give an evaluation prescription of the mean radius of confinement $<r>$ which is in essence just a characteristic size of one or another hadron.

It is known, however, at present no generally accepted quark confinement mechanism exists that would be capable to calculate a number of nonperturbative parameters characterizing mesons (masses, radii, decay constants and so on) appealing directly to quark and gluon degrees of freedom related to QCD-Lagrangian. At best there are a few scenarios (directly not connected to QCD-Lagrangian) of confinement that restrict themselves mainly to qualitative considerations with small possibilities of concrete calculation. In view of it in [5, 6, 7] a confinement mechanism has been proposed which was based on the unique family of compatible nonperturbative solutions for the Dirac-Yang-Mills system directly derived from QCD-Lagrangian. The word unique should be understood in the strict mathematical sense. Let us write down arbitrary SU(3)-Yang-Mills field in the form $A = A_\mu dx^\mu = A_\mu^a \lambda_a dx^\mu$ ($\lambda_a$ are the known Gell-Mann matrices, $\mu = t, r, \vartheta, \varphi$, $a = 1, \ldots, 8$ and we use the ordinary set of local spherical coordinates $r, \vartheta, \varphi$ for spatial part of the flat Minkowski spacetime).

In fact in [5, 6, 7] the following theorem was proved:
The unique exact spherically symmetric (nonperturbative) solutions (depending only on $r$ and $r^{-1}$) of SU(3)-Yang-Mills equations in Minkowski spacetime consist of the family of form

\begin{align*}
A_{1t} &\equiv A_1^3 + \frac{1}{\sqrt{3}} A_1^8 = \frac{a_1}{r} + A_1 ,
A_{2t} &\equiv -A_2^3 + \frac{1}{\sqrt{3}} A_2^8 = -\frac{a_2}{r} + A_2 ,
A_{3t} &\equiv -\frac{2}{\sqrt{3}} A_t^8 = \frac{a_1 + a_2}{r} - (A_1 + A_2) ,
A_{1\varphi} &\equiv A_1^3 + \frac{1}{\sqrt{3}} A_1^8 = b_1 r + B_1 ,
A_{2\varphi} &\equiv -A_2^3 + \frac{1}{\sqrt{3}} A_2^8 = b_2 r + B_2 ,
A_{3\varphi} &\equiv -\frac{2}{\sqrt{3}} A_\varphi^8 = -(b_1 + b_2) r - (B_1 + B_2) \quad (2)
\end{align*}

with the real constants $a_j, A_j, b_j, B_j$ parametrizing the family. Besides in [5, 6, 7] it was shown that the above unique confining solutions (2) satisfy the so-called Wilson confinement criterion [8, 9]. Up to now nobody contested this result so if we want to describe interaction between quarks by spherically symmetric SU(3)-fields then they can be only the ones from the above theorem. On the other hand, the desirability of spherically symmetric (colour) interaction between quarks at all distances naturally follows from analysing the $p\bar{p}$-collisions (see, e.g., [10]) where one observes a Coulomb-like potential in events which can be identified with scattering quarks on each other, i.e., actually at small distances one observes the Coulomb-like part of solution (2). Under this situation, a natural assumption will be that the quark interaction remains spherically symmetric at large distances too but then, if trying to extend the Coulomb-like part to large distances in a spherically symmetric way, we shall inevitably come to the solution (2) in virtue of the above theorem.
The aim of the present paper is to some extent to discuss a possible connection between the above confinement mechanism and the appearance of the scale $\Lambda$ in QCD. Section 2 contains a description of the confinement mechanism in question. Section 3 is devoted to how the mean radius of confinement could be computed within the framework under discussion while Section 4 is devoted to discussion and concluding remarks. Finally, Appendix A supplements the present section with a proof of the above uniqueness theorem in the case of SU(3)-Yang-Mills equations.

2 Quark confinement mechanism

The applications of the family (2) to the description of both the heavy quarkonia spectra \[11,12,13,14\] and a number of properties of pions, kaons, $\eta$- and $\eta'$-mesons \[15,16,17,18,19,20\] showed that the confinement mechanism is qualitatively the same for both light mesons and heavy quarkonia. At this moment it can be described in the following way.

The next main physical reasons underlie linear confinement in the mechanism under discussion. The first one is that gluon exchange between quarks is realized with the propagator different from the photon-like one, and existence and form of such a propagator is a direct consequence of the unique confining nonperturbative solutions of the Yang-Mills equations \[5,6,7\]. The second reason is that, owing to the structure of the mentioned propagator, quarks mainly emit and interchange the soft gluons so the gluon condensate (a classical gluon field) between quarks basically consists of soft gluons (for more details see \[5,6,7\]) but, because of the fact that any gluon also emits gluons (still softer), the corresponding gluon concentrations rapidly become huge and form a linear confining magnetic colour field of enormous strengths, which leads to confinement of quarks. This is by virtue of the fact that just the magnetic part of the mentioned propagator is responsible for a larger portion of gluon concentrations at large distances since the magnetic part has stronger infrared singularities than the electric one. In the circumstances physically nonlinearity of the Yang-Mills equations effectively vanishes so the latter possess the unique nonperturbative confining solutions of the Abelian-like form (2) (with the values in Cartan subalgebra of SU(3)-Lie algebra) which describe the gluon condensate under consideration. Moreover, since the overwhelming majority of gluons is soft they cannot leave the hadron (meson) until some gluons obtain additional energy (due to an external reason) to rush out. So we also deal with the confinement of gluons.

As has been repeatedly explained in \[11,12,13,14,15,16,17,18,19,20\], parameters $A_{1,2}$ of solution (2) are inessential for physics in question and we can consider $A_1 = A_2 = 0$. Obviously we have $\sum_{j=1}^{3} A_{jt} = \sum_{j=1}^{3} A_{j\varphi} = 0$ which reflects the fact that for any matrix $T$ from SU(3)-Lie algebra we have $\text{Tr} T = 0$. Also, as has been repeatedly discussed by us earlier (see, e. g., \[11,12,13,14,15,16,17,18,19,20\]), from the above form it is clear that the solution (2) is a configuration describing the electric Coulomb-like colour field (components $A_{3,t}^3$) and the magnetic colour field linear in $r$ (components $A_{3,\varphi}^3$) and we wrote down the solution (2) in the combinations that are just needed further to insert into the Dirac equation (directly derived from QCD-Lagrangian) giving the interaction energy between
two quarks in a meson and which will look as follows [after inserting solution (2)]

\[
\begin{pmatrix}
\partial_t\Psi_1 \\
\partial_t\Psi_2 \\
\partial_t\Psi_3
\end{pmatrix} = iH\Psi \equiv \begin{pmatrix} H_1 & 0 & 0 \\ 0 & H_2 & 0 \\ 0 & 0 & H_3 \end{pmatrix} \begin{pmatrix} \Psi_1 \\ \Psi_2 \\ \Psi_3 \end{pmatrix} = \begin{pmatrix} H_1\Psi_1 \\ H_2\Psi_2 \\ H_3\Psi_3 \end{pmatrix},
\]

where Hamiltonian \( H_j \) is

\[
H_j = \gamma^0 \left( \mu_0 - i\gamma^1 \partial_r - i\gamma^2 \frac{1}{r} \left( \partial_\theta + \frac{1}{2} \gamma^1 \gamma^2 \right) - i\gamma^3 \frac{1}{r \sin \vartheta} \left( \partial_\varphi + \frac{1}{2} \sin \vartheta \gamma^3 \right) \right) - g\gamma^0 \left( \gamma^0 A_{jt} + \gamma^3 \frac{1}{r \sin \vartheta} A_{j\varphi} \right)
\]

with the gauge coupling constant \( g, \mu_0 \) is a mass parameter and one should consider \( \mu_0 = m_{q_1}m_{q_2}/(m_{q_1} + m_{q_2}) \) to be the reduced mass composed of the current masses \( m_{q_1}, m_{q_2} \) of quarks forming a meson (quarkonium) while \( \Psi = (\Psi_1, \Psi_2, \Psi_3) \) with the four-dimensional Dirac spinors \( \Psi_j \) representing the \( j \)th colour component of the meson, so \( \Psi \) may describe relative motion of two quarks in mesons.

Additional considerations show that the unique modulo square integrable (non-perturbative) solutions of the Dirac equation (3) in the field (2) (i.e. relativistic bound states) are (with Pauli matrix \( \sigma_1 \), for more details see \[8\])

\[
\Psi_j = e^{-i\omega_j t} \equiv e^{-i\omega_j t} r^{-1} \begin{pmatrix} F_{j1}(r)\Psi_j(\vartheta, \varphi) \\ F_{j2}(r)\sigma_1\Psi_j(\vartheta, \varphi) \end{pmatrix}, j = 1, 2, 3
\]

with the 2D eigenspinor \( \Phi_j = \begin{pmatrix} \Phi_{j1} \\ \Phi_{j2} \end{pmatrix} \) of the euclidean Dirac operator \( \mathcal{D}_0 \) on the unit sphere \( S^2 \), while the coordinate \( r \) stands for the distance between quarks. The explicit form of \( \Phi_j \) is not needed here and can be found in \[11\] \[12\] \[13\] \[14\] \[15\] \[16\] \[17\] \[18\] \[19\] \[20\]. One can only remark that spinors \( \Phi_j \) form an orthonormal basis in \( L^2(S^2) \) (in what follows we denote \( L^2(F) \) the set of the modulo square integrable complex functions on any manifold \( F \) furnished with an integration measure, then \( L^2(F) \) will be the \( n \)-fold direct product of \( L^2(F) \) endowed with the obvious scalar product).

We can call the quantity \( \omega_j \) relative energy of \( j \)th colour component of meson (while \( \psi_j \) is wave function of a stationary state for \( j \)th colour component). Under this situation, if a meson is composed of quarks \( q_{1,2} \) with different flavours then the energy spectrum of the meson will be given by \( \epsilon = m_{q_1} + m_{q_2} + \omega \) with the current quark masses \( m_{q_1} \) (rest energies) of the corresponding quarks and we should put the interaction energy \( \omega = \omega_j \) for any \( j \) in virtue of Dirac equation (3). On the other hand, \( \omega_j \) is given by (for more details see Refs. \[5\] \[6\] \[7\])

\[
\omega_j = \omega_j(n_j, l_j, \lambda_j) = \frac{A_j g^2 a_j b_j \pm (n_j + \alpha_j) \sqrt{(n_j^2 + 2n_j \alpha_j + A_j^2)\alpha_j^2 + g^2 b_j^2(n_j^2 + 2n_j \alpha_j)}}{n_j^2 + 2n_j \alpha_j + A_j^2}.
\]
where \( a_3 = -(a_1 + a_2), \) \( b_3 = -(b_1 + b_2), \) \( B_3 = -(B_1 + B_2), \) \( A_j = \lambda_j - gB_j, \) \( \alpha_j = \sqrt{A_j^2 - g^2a_j^2}, \) \( n_j = 0, 1, 2, \ldots, \) while \( \lambda_j = \pm (l_j + 1) \) are the eigenvalues of euclidean Dirac operator \( D_0 \) on unit sphere with \( l_j = 0, 1, 2, \ldots, \)

In line with the above we should have the interaction energy \( \omega = \omega_1 = \omega_2 = \omega_3 \) in energy spectrum \( \epsilon = m_{q1} + m_{q2} + \omega \) for any meson (quarkonium) and this at once imposes two conditions on parameters \( a_j, b_j, B_j \) when choosing some experimental value for \( \epsilon \) at the given current quark masses \( m_{q1}, m_{q2}. \)

For example, for reference we shall give the radial parts of (5) at \( n_j = 0 \) (the ground state) that are

\[
F_{j1} = C_j P_j r^{\alpha_j} e^{-\beta_j r} \left(1 - \frac{g b_j}{\beta_j} \right), P_j = g b_j + \beta_j, \\
F_{j2} = iC_j Q_j r^{\alpha_j} e^{-\beta_j r} \left(1 + \frac{g b_j}{\beta_j} \right), Q_j = \mu_0 - \omega_j
\]

with \( \beta_j = \sqrt{\mu_0^2 - \omega_j^2 + g^2 b_j^2} \) while \( C_j \) is determined from the normalization condition \( \int_0^\infty (|F_{j1}|^2 + |F_{j2}|^2) dr = \frac{4}{\beta_j}. \) Consequently, we shall gain that \( \Psi_j \in L^2(\mathbb{R}^3) \) at any \( t \in \mathbb{R} \) and, as a result, the solutions of (6) may describe ground state of mesons or quarkonia. The same holds true for wave functions (5) in general form so that (5) may describe relativistic bound states of mesons (quarkonia) with the energy (mass) spectrum \( \epsilon = m_{q1} + m_{q2} + \omega. \)

### 3 Meson spectroscopy and the scale \( A_{QCD} \)

We would take a sin upon our soul if we did not try using the above unique family of compatible nonperturbative solutions for the Dirac-Yang-Mills system directly derived from QCD-Lagrangian for description of confinement since such an approach to confinement has good mathematical and physical grounds and, as a result, the approach is itself unique, nonperturbative and relativistic from the outset and it appeals immediately to quark and gluon degrees of freedom as should be from QCD-point of view.

Having the off-the-shelf wave functions for mesons of the form (5) we can directly calculate \( < r > \) in accordance with the standard quantum mechanics rules as

\[
<r> = \sqrt{\int r^2 \Psi^\dagger \Psi d^3x} = \sqrt{\sum_{j=1}^{3} \int r^2 \Psi_j^\dagger \Psi_j d^3x}
\]

and the result will, e.g., for the case of ground state (7), be equal to (for more details see \([14,15,16,17,18,19,20]\))

\[
<r> = \sqrt{\frac{3 \cdot 2 \alpha_j^2 + 3 \alpha_j + 1}{6 \beta_j^2}}, \quad (8)
\]

As an illustration of (6) and (8), Tables I and II (taken from \([19]\)) contain the relevant numerical results for charged pions and kaons where it was accepted \( m_u = 2.25 \) MeV, \( m_d = 5 \) MeV, \( m_s = 107.5 \) MeV when computing.
Table 1  Gauge coupling constant, reduced mass $\mu_0$ and parameters of the confining SU(3)-gluonic field for charged pions and kaons

| Particle | $g$  | $\mu_0$ (MeV) | $a_1$ | $a_2$ | $b_1$ (GeV) | $b_2$ (GeV) | $B_1$ | $B_2$ |
|----------|------|----------------|-------|-------|--------------|--------------|-------|-------|
| $\pi^\pm$—$u\bar{d}$, $\bar{u}d$ | 6.09131 | 1.55172 | 0.0473002 | -0.0115497 | 0.179815 | -0.119290 | 0.230 |
| $K^\pm$—$u\bar{s}$, $\bar{u}s$ | 5.30121 | 2.20387 | 0.167182 | -0.0557501 | 0.120150 | 0.131046 | -0.900 |

Table 2  Theoretical and experimental masses and radii for charged pions and kaons

| Particle | Theoret. $\mu$ (MeV) | Experim. $\mu$ (MeV) | Theoret. $<r>$ (fm) | Experim. $<r>$ (fm) |
|----------|-------------------|-------------------|-------------------|-------------------|
| $\pi^\pm$—$u\bar{d}$, $\bar{u}d$ | $\mu = m_u + m_d + \omega_j(0, 0, 1) = 139.570$ | 139.56995 | 0.673837 | 0.672 |
| $K^\pm$—$u\bar{s}$, $\bar{u}s$ | $\mu = m_u + m_s + \omega_j(0, 0, 1) = 493.677$ | 493.677 | 0.544342 | 0.560 |

Then, as was said early in the paper, we should put $1/\Lambda \sim <r>$ with $<r>$ of (8) and it is clear that $\Lambda$ is a function of the parameters of the confining SU(3)-gluonic field (2) and the current quark masses, i.e., $\Lambda$ is expressed through gluonic and quark degrees of freedom, as should be according to the first principles of QCD. Also it is clear that because of $<r>$ slightly changes from hadron to hadron (see Table II) then $\Lambda$ should do it as well. It should be noted that crucial role in generating $<r>$ or $\Lambda$ belongs to the magnetic colour field of (2) (parameters $b_j$). Indeed, as follows from (8) at $|b_j| \to \infty$ we have $<r> \sim \sqrt{\sum_{j=1}^{3} \left(\frac{1}{g|b_j|}\right)^2}$, so in the strong magnetic colour field when $|b_j| \to \infty$, $<r> \to 0$, while the meson wave functions of (5) and (7) behave as $\Psi_j \sim e^{-g|b_j|r}$, i.e., just the magnetic colour field of (2) provides two quarks with confinement. Thus, our confinement mechanism gives an explanation of appearing the scale $\Lambda$ in QCD.

4 Concluding remarks

The results of present paper as well as the ones of [11,12,13,14,15,16,17,18,19,20] allow one to speak about that the confinement mechanism elaborated in [5,6,7] gives new possibilities for considering many old problems of hadronic (meson) physics (such as nonperturbative computation of decay constants, masses and radii of mesons [15,16,17,18,19,20], chiral symmetry breaking [19,20] and so forth) from the first principles of QCD immediately appealing to the quark and gluonic degrees of freedom. This is possible because the given mechanism is based on the unique family of compatible nonperturbative solutions for the Dirac-Yang-Mills system directly derived from QCD-Lagrangian and, as a result, the approach is itself nonperturbative, relativistic from the outset, admits self-consistent non-relativistic limit and may be employed for any meson (quarkonium). Under the circumstances the words quark and gluonic degrees of freedom make exact sense: gluons come forward in the form of bosonic condensate described by parameters $a_j$, $b_j$, $B_j$ from the unique exact solution (2) of the Yang-Mills equations while quarks are represented by their current masses $m_q$. 
Finally, one should say that the unique confining solutions similar to (2) exist for all semisimple and non-semisimple compact Lie groups, in particular, for SU($N$) with $N \geq 2$ and U($N$) with $N \geq 1$ [5]. Explicit form of solutions, e.g., for SU($N$) with $N = 2, 4$ can be found in [4] but it should be emphasized that components linear in $r$ always represent the magnetic (colour) field in all the mentioned solutions. Especially the case U(1)-group is interesting which corresponds to usual electrodynamics. Under this situation, as was pointed out in [6,7] there is an interesting possibility of indirect experimental verification of the confinement mechanism under discussion. Indeed the confining solutions of Maxwell equations for classical electrodynamics point out the confinement phase could be in electrodynamics as well. Though there exist no elementary charged particles generating a constant magnetic field linear in $r$, the distance from particle, after all, if it could generate this electromagnetic field configuration in laboratory then one might study motion of the charged particles in that field. The confining properties of the mentioned field should be displayed at classical level too but the exact behaviour of particles in this field requires certain analysis of the corresponding classical equations of motion. Such a program has been recently realized in [21]. Motion of a charged (classical) particle was studied in the field representing magnetic part of the mentioned solution of Maxwell equations and it was shown that one deals with the full classical confinement of the charged particle in such a field: under any initial conditions the particle motion is accomplished within a finite region of space so that the particle trajectory is near magnetic field lines while the latter are compact manifolds (circles). Those results might be useful in thermonuclear plasma physics (for more details see [21]).

Appendix A

The facts adduced here have been obtained in Refs. [6,7] and we concisely give them only for completeness of discussion.

To specify the question, let us note that in general the Yang-Mills equations on a manifold $M$ can be written as

$$d \ast F = g(\ast F \wedge A - A \wedge \ast F),$$  \hspace{1cm} (A.1)

where a gluonic field $A = A_\mu dx^\mu = A_{a}^\mu \lambda_a dx^\mu$ [$\lambda_a$ are the known Gell-Mann matrices, $\mu = t, r, \vartheta, \varphi$ (in the case of spherical coordinates), $a = 1, ..., 8$], the curvature matrix (field strength) $F = dA + gA \wedge A = F_{a\mu}^\nu \lambda_a dx^\mu \wedge dx^\nu$ with exterior differential $d$ and the Cartan’s (exterior) product $\wedge$, while $\ast$ means the Hodge star operator conforming to a metric on manifold under consideration, $g$ is a gauge coupling constant.

The most important case of $M$ is Minkowski spacetime and we are interested in the confining solutions $A$ of the SU(3)-Yang-Mills equations. The confining solutions were defined in Ref. [5] as the spherically symmetric solutions of the Yang-Mills equations (1) containing only the components of the SU(3)-field which are Coulomb-like or linear in $r$. Additionally we impose the Lorentz condition on the sought solutions. The latter condition is necessary for quantizing the gauge fields consistently within the framework of perturbation theory (see, e. g. Ref. [22]), so we should impose the given condition that can be written in the form $\text{div}(A) = 0$,
where the divergence of the Lie algebra valued 1-form $A = A_\mu dx^\mu = A_\mu^a \lambda_a dx^\mu$ is defined by the relation (see, e. g., Ref. [23])

$$\text{div } A = *(d * A) = \frac{1}{\sqrt{\delta}} \partial_\mu (\sqrt{\delta} g^{\mu\nu} A_\nu).$$  \hspace{1cm} (A.2)

It should be emphasized that, from the physical point of view, the Lorentz condition reflects the fact of transversality for gluons that arise as quanta of SU(3)-Yang-Mills field when quantizing the latter (see, e. g., Ref. [22]).

We shall use the Hodge star operator action on the basis differential 2-forms on Minkowski spacetime with local coordinates $t, r, \vartheta, \varphi$ in the form

\[
\star (dt \wedge dr) = -r^2 \sin \vartheta d\vartheta \wedge d\varphi, \\
\star (dt \wedge d\vartheta) = \sin \vartheta dr \wedge d\varphi, \\
\star (dt \wedge d\varphi) = -\frac{1}{\sin \vartheta} dt \wedge d\vartheta, \\
\star (dr \wedge d\vartheta) = \frac{1}{r^2 \sin \vartheta} dt \wedge dr ,
\]

(A.3) so that on 2-forms $\star^2 = -1$. More details about the Hodge star operator can be found in [23].

The most general ansatz for a spherically symmetric solution is $A = A_t(r) dt + A_r(r) dr + A_\vartheta(r) d\vartheta + A_\varphi(r) d\varphi$. But then the Lorentz condition (A.2) for the given ansatz gives rise to

$$\sin \vartheta \partial_r (r^2 A_r) + \partial_\vartheta (\sin \vartheta A_\vartheta) = 0,$$

which yields $A_r = \frac{C_0}{r^2} - \cot \vartheta \int A_\vartheta(r) dr$ with a constant matrix $C$. But the confining solutions should be spherically symmetric and contain only the components which are Coulomb-like or linear in $r$, so one should put $C = A_\vartheta(r) = 0$. Consequently, the ansatz $A = A_t(r) dt + A_\varphi(r) d\varphi$ is the most general spherically symmetric one.

For the latter ansatz we have $F = dA + gA \wedge A = -\partial_r A_t dt \wedge dr + \partial_r A_\varphi dr \wedge d\varphi + g[A_t, A_\varphi] dt \wedge d\varphi$, where $[\cdot, \cdot]$ signifies matrix commutator.

Then, according to (A.3), we obtain

$$\star F = (r^2 \sin \vartheta) \partial_r A_\vartheta d\vartheta \wedge d\varphi - \frac{1}{\sin \vartheta} \partial_\vartheta A_\varphi dt \wedge d\vartheta - \frac{g}{\sin \vartheta} [A_t, A_\varphi] dr \wedge d\vartheta ,$$  \hspace{1cm} (A.4)

which entails

$$d \star F = \sin \vartheta \partial_r (r^2 \partial_r A_t) dr \wedge d\vartheta \wedge d\varphi + \frac{1}{\sin \vartheta} \partial_\vartheta^2 A_\varphi dt \wedge dr \wedge d\vartheta ,$$  \hspace{1cm} (A.5)

while

$$\star F \wedge A - A \wedge \star F =$$

\[
\left( r^2 \sin \vartheta [\partial_r A_t, A_t] - \frac{1}{\sin \vartheta} [\partial_r A_\varphi, A_\varphi] \right) dt \wedge d\vartheta \wedge d\varphi \\
- \frac{g}{\sin \vartheta} \left( [[A_t, A_\varphi], A_t] dt \wedge dr \wedge d\vartheta + [[A_t, A_\varphi], A_\varphi] dr \wedge d\vartheta \wedge d\varphi \right) .
\]

(A.6)
Under the circumstances the Yang-Mills equations (A.1) are tantamount to the conditions
\[ \partial_r (r^2 \partial_r A_t) = -\frac{g^2}{\sin^2 \vartheta} [[A_t, A_\phi], A_\phi], \quad (A.7) \]
\[ \partial^2_r A_\phi = -g^2 [[A_t, A_\phi], A_t], \quad (A.8) \]
\[ r^2 \sin \vartheta [\partial_r A_t, A_t] - \frac{1}{\sin \vartheta} [\partial_r A_\phi, A_\phi] = 0. \quad (A.9) \]
The key equation is (A.7) because the matrices \( A_t, A_\phi \) depend on merely \( r \) and (A.7) can be satisfied only if the matrices \( A_t = A^a_t \lambda_a \) and \( A_\phi = A^a_\phi \lambda_a \) belong to the so-called Cartan subalgebra of the SU(3)-Lie algebra. Let us remind that, by definition, a Cartan subalgebra is a maximal abelian subalgebra in the corresponding Lie algebra, i.e., the commutator for any two matrices of the Cartan subalgebra is equal to zero (see, e.g., Ref. [24]). For SU(3)-Lie algebra the conforming Cartan subalgebra is generated by the Gell-Mann matrices \( \lambda_3, \lambda_8 \) which are
\[ \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}. \quad (A.10) \]
Under the situation we should have \( A_t = A^a_t \lambda_a + A^a_\phi \lambda_a \) and \( A_\phi = A^a_\phi \lambda_3 + A^a_\phi \lambda_8 \), then \([A_t, A_\phi] = 0 \) and we obtain
\[ \partial_r (r^2 \partial_r A_t) = 0, \partial^2_r A_\phi = 0, \quad (A.11) \]
while (A.9) is identically satisfied and (A.11) gives rise to the solution (2) with real constants \( a_j, A_j, b_j, B_j \), parametrizing the solution which proves the uniqueness theorem of Section 1 for the SU(3) Yang-Mills equations.

The more explicit form of (2) is
\[ A^3_t = \frac{(a_2 - a_1) + A_1 - A_2}{r}, \]
\[ A^8_t = \frac{[A_1 + A_2 - (a_1 + a_2) + B_1 - B_2]}{\sqrt{3}}/2, \]
\[ A^3_\phi = \frac{(b_1 - b_2) + r - B_1 + B_2}{\sqrt{3}}/2, \]
\[ A^8_\phi = \frac{(b_1 + b_2) + r + B_1 + B_2}{\sqrt{3}}/2. \quad (A.12) \]
Clearly, the obtained results may be extended over all SU(N)-groups with \( N \geq 2 \) and even over all semisimple compact Lie groups since for them the corresponding Lie algebras possess just the only Cartan subalgebra. Also we can talk about the compact non-semisimple groups, for example, U(N). In the latter case additionally to Cartan subalgebra we have centrum consisting from the matrices of the form \( \alpha I_N \) (\( I_N \) is the unit matrix \( N \times N \)) with arbitrary constant \( \alpha \).

The most relevant physical cases are of course U(1)- and SU(3)-ones (QED and QCD). In particular, the U(1)-case allows us to build the classical model of confinement (see Ref. [21]).

At last, it should also be noted that the nontrivial confining solutions obtained exist at any gauge coupling constant \( g \), i.e., they are essentially nonperturbative ones.
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