Repulsive gravity near naked singularities and point massive particles

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Abstract

We investigate the existence of repulsive gravitational acceleration near naked singularities. The investigation is carried out by means of the acceleration tensor, which is a coordinate invariant object. We find that the gravitational acceleration is repulsive in the vicinity of the origin in the Reissner-Nordstrøm and in the Kerr space-times, and attractive at large distances in the expected Newtonian way. We further address the space-time of a point massive particle, which also exhibits repulsive effects near the origin.

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1 Introduction

Under some general physical assumptions, the occurrence of singularities is an inevitable feature of the exact solutions of Einstein’s general relativity. The existence of a singularity may be determined by the emergence of non-spacelike geodesic incompleteness in the space-time as, for example, the sudden disappearance of a timelike geodesic (i.e., an observer) after a finite amount of proper time (for a recent review, see Ref. [1]). In principle, singularities arise in the solutions of the field equations in the form of black holes, or naked singularities. In the vicinity of the singularities the energy densities, the space-time curvature and torsion tensors, as well as any other geometrical field quantities, diverge in powers of $1/r$. This is an indication that the usual description of the space-time, as well as the ordinary laws of physics, break down [1].

The existence of singularities in space-time is connected to one of the most important problems in general relativity, which is the final fate of the complete gravitational collapse of a massive body, such as a star. Investigations on this issue have led to the conclusion that the gravitational collapse of several distinct matter field configurations may end up either as black holes or naked singularities (Ref. [2] and references therein). Therefore, the latter are not ruled out as the final outcome of the physical process. If naked singularities exist in nature as real astrophysical objects, it is important to establish conditions for distinguishing them from black holes. Attempts in this direction have been put forward in several investigations. The purpose is to find out observational differences in the space-time around black holes and naked singularities by analysing: the circular motion of charged test particles in the Reissner-Nordstrøm space-time [3]; and the emergence of accretion disks [2] and of gravitational lensing [4] in the strong field limit, around both types of singularities. The idea behind these analyses is to provide conditions for distinguishing naked singularities from black holes.

Repulsive gravity effects were conjectured to exist around naked singularities in the Reissner-Nordstrøm space-time [3]. The repulsive effects would explain discontinuities in the accretion disks around the singularities. Although these repulsive effects are very speculative, they could exist in the vicinity of naked singularities if the latter are actual physical manifestations of general relativity. Repulsive gravitational effects have not been widely investigated in the literature. A proposal for an invariant definition of repulsive gravity, based on the properties of the Riemann tensor, was presented.
In this article we obtain the gravitational acceleration in the vicinity of naked singularities and point massive particles by means of the acceleration tensor. This tensor arises in the analysis of reference frames in the space-time described by the metric tensor $g_{\mu\nu}$ and by a set of tetrad fields $e^a_\mu$. The tetrad fields yield the frame $e^a_\mu$ that is adapted to a field of observers in space-time, defined by an arbitrary congruence of timelike world lines. The acceleration tensor gives the values of the inertial (i.e., non-gravitational) accelerations that are necessary to maintain the frame in a given inertial state (static, stationary or otherwise). If the frame is maintained static in space-time, then the inertial acceleration is exactly minus the gravitational acceleration imparted to the frame.

Although the procedure to be considered ahead is very simple, it has not been presented in the literature so far. The procedure allows the determination of the emergence of repulsive gravitational acceleration on frames in arbitrary space-times. Reference frames have been discussed in Refs. [6, 7], in connection to the teleparallel equivalent of general relativity [8]. The acceleration tensor $\phi_{ab} = -\phi_{ba}$ is a coordinate independent quantity, but depends on the frame ($a,b$ are $SO(3,1)$ indices). Linear accelerations and rotations are not absolute concepts. They depend on a frame. The acceleration tensor may be used to characterize the frame in space-time. The underlying geometrical structure of this analysis is suitable to the teleparallel equivalent of general relativity, where the gravitational field strength is the torsion tensor $T_{a\mu\nu}$.

In Section 2 we recall the definition and properties of the acceleration tensor. We will consider static or stationary observers in the point massive particle, in the Reissner-Nordstrøm and in the Kerr space-times in Section 3, where we will describe the emergence of a repulsive acceleration near the origin in these space-times. The Reissner-Nordstrøm and Kerr space-times will display naked singularities. Our notation is the following: space-time indices $\mu, \nu, ...$ and $SO(3,1)$ (Lorentz) indices $a, b, ...$ run from 0 to 3. Time and space indices are indicated according to $\mu = 0, i$, $a = (0), (i)$. The tetrad fields are represented by $e^a_\mu$, and the torsion tensor by $T_{a\mu\nu} = \partial_\mu e_{a\nu} - \partial_\nu e_{a\mu}$. The flat, tangent space Minkowski space-time metric tensor raises and lowers tetrad indices and is fixed by $\eta_{ab} = e_{a\mu} e_{b\nu} g^{\mu\nu} = (-1, +1, +1, +1)$. The frame components are given by $e_a^\mu$. 

in Ref. [5].
2 The acceleration tensor

The acceleration tensor is a generalization of the inertial acceleration 4-vector of an observer defined by an arbitrary timelike world line $C$ in space-time. Let $C$ be denoted by $x^\mu(\tau)$, where the parameter $\tau$ is the proper time of the observer. The velocity and acceleration of the observer along $C$ are denoted by $u^\mu = dx^\mu/d\tau$ and $a^\mu = Du^\mu/d\tau$, respectively. The absolute derivative $D/d\tau$ is constructed out of the Christoffel symbols $^0\Gamma_{\alpha\beta}^\mu$.

A frame adapted to the observer is constructed by identifying the timelike component of the frame $e_{(0)}^\mu$ with the velocity $u^\mu$ according to $e_{(0)}^\mu = u^\mu(\tau)/c$ \[9\]. Throughout this article we will adopt $c = 1$. Thus we have

$$a^\mu = \frac{Du^\mu}{d\tau} = \frac{De_{(0)}^\mu}{d\tau} = u^\alpha \nabla_\alpha e_{(0)}^\mu.$$  \hspace{1cm} (1)

Out of the frame $e_a^\mu$ we may obtain the velocity and acceleration of the observer on the timelike world line $C$. Therefore, a given set of tetrad fields, for which $e_{(0)}^\mu$ describes a congruence of timelike curves, is adapted to observers characterized by the velocity field $u^\mu = e_{(0)}^\mu$ and acceleration $a^\mu$.

The acceleration of the whole frame is determined by the absolute derivative of $e_a^\mu$ along $C$. Assuming that the observer carries an orthonormal tetrad frame $e_a^\mu$, the acceleration of the frame along the path is given by \[10\] \[11\]

$$\frac{De_a^\mu}{d\tau} = \phi_{ab} e_b^\mu,$$  \hspace{1cm} (2)

where $\phi_{ab}$ is the antisymmetric acceleration tensor. In analogy with the Faraday tensor, we identify $\phi_{ab} \rightarrow (a, \Omega)$, where $a$ is the translational acceleration ($\phi_{(0)(i)} = a_{(i)}$) and $\Omega$ is the frequency of rotation of the local spatial frame with respect to a non-rotating, Fermi-Walker transported frame \[10\] \[11\]. It follows from Eq. (2) that

$$\phi_{ab} = e_b^\mu \frac{De_a^\mu}{d\tau} = e_b^\mu u^\lambda \nabla_\lambda e_a^\mu.$$  \hspace{1cm} (3)

The acceleration vector $a^\mu$ defined by Eq. (1) may be projected on a frame in order to yield $a^b = e^b_\mu a^\mu = e^b_\mu u^\alpha \nabla_\alpha e_{(0)}^\mu = \phi_{(0)}^b$. Moreover, we find that

$$a^\mu = u^\alpha \nabla_\alpha e_{(0)}^\mu = \frac{d^2x^\mu}{ds^2} + 0\Gamma_{\alpha\beta}^\mu \frac{dx^\alpha}{ds} \frac{dx^\beta}{ds}.$$  \hspace{1cm} (4)
We see that if \( u^\mu = e_{(0)}^\mu \) represents a geodesic trajectory, then the frame is in free fall and \( a^\mu = 0 = \phi_{(0)(i)} \). Therefore we conclude that non-vanishing values of the latter quantities represent inertial accelerations of the frame.

After a number of simple manipulations, it is possible to rewrite the acceleration tensor in the form [6, 7, 8]

\[
\phi_{ab} = \frac{1}{2} [T_{(0)ab} + T_{a(0)b} - T_{b(0)a}],
\]

(5)

where \( T_{abc} = e_b^\mu e_c^\nu T_{a\mu\nu} \), and \( T_{a\mu\nu} = \partial_\mu e_{a\nu} - \partial_\nu e_{a\mu} \) is the torsion tensor of the Weitzenböck space-time. The expression of \( \phi_{ab} \) is invariant under coordinate transformations.

The values of the 6 components of the acceleration tensor may be used to characterize the frame, since \( \phi_{ab} \) is not invariant under local SO(3,1) (Lorentz) transformations [7]. Alternatively, the frame may be characterized (i) by the identification \( u^\mu = e_{(0)}^\mu \) (this equation fixes 3 components, because \( e_{(0)}^0 \) is fixed by normalization), and (ii) by the 3 orientations in the three-dimensional space of the components \( e_{(1)}^\mu, e_{(2)}^\mu, e_{(3)}^\mu \) [6, 8].

For a frame that undergoes the usual translational and/or rotational accelerations in flat space-time, \( \phi_{ab} \) yields the expected, ordinary values [9]. An interesting application of the acceleration tensor is the following. Let us consider a static observer in the Schwarzschild space-time, described by the standard Schwarzschild coordinates \((t, r, \theta, \phi)\). The frame of the observer must satisfy \( e_{(0)}^i = 0 = u^i \). The non-vanishing components of the acceleration tensor form a vector \( a = \phi_{(0)(1)} \hat{x} + \phi_{(0)(2)} \hat{y} + \phi_{(0)(3)} \hat{z} \), which eventually reads [12]

\[
a = a(r) \hat{r} = \frac{m}{r^2} \left( 1 - \frac{2m}{r} \right)^{-1/2} \hat{r},
\]

(6)

where

\[
\hat{r} = \sin \theta \cos \phi \hat{x} + \sin \theta \sin \phi \hat{y} + \cos \theta \hat{z},
\]

and \( \hat{x}, \hat{y}, \hat{z} \) are the usual unit vectors in the asymptotically flat space-time. The expression above represents the inertial acceleration necessary to maintain the frame static in space-time. Therefore it exactly cancels the gravitational acceleration exerted on the frame. The inertial acceleration is oriented along the positive direction of the radial vector \( \hat{r} \). It is well known that \(-a(r)\) is the geodesic acceleration of a free body towards the black hole.
3 Naked singularities and repulsive acceleration

In this section we will consider two types of singularities that arise in the exact solutions of Einstein’s equations. The Reissner-Nordstrøm and Kerr space-times exhibit naked singularities when the physical parameters of the solutions acquire certain values. We will also consider the space-time of the point massive particle recently investigated by Katanaev [13]. It was shown in the latter reference that the Schwarzschild metric in isotropic coordinates is a solution of Einstein’s equations with a δ-type source at the origin. The metric is defined everywhere in $\mathbb{R}^4$, except on the world line at $r = 0$. The latter is not a naked singularity because the space-time is geodesically complete at the origin [13]. The space-time of the point massive particle displays a very interesting repulsive gravitational effect, already noted by Katanaev.

3.1 The Reissner-Nordstrøm naked singularity

The Reissner-Nordstrøm space-time, parametrized by the total mass $m$ and by the charge $Q$, is described by the line element

$$ds^2 = -\alpha^2 dt^2 + \frac{1}{\alpha^2} dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2,$$

where

$$\alpha^2 = 1 - \frac{2m}{r} + \frac{Q^2}{r^2}.$$  

A naked singularity exists if $\alpha^2 > 0$ everywhere in space. This condition takes place if $Q^2 > m^2$, which is assumed in this analysis. In order to obtain the non-vanishing components of the acceleration tensor, we have first to define the frame. We will establish a frame adapted to static observers in space-time, characterized by the velocity field $u^\mu = (u^0, 0, 0, 0)$. This frame is constructed by requiring the condition $e_{(0)}^i = u^i = 0$, which implies $e_{(0)}^0 = 0$. In $(t, r, \theta, \phi)$ coordinates, the tetrad fields $e_{a\mu}$ that satisfy these properties is

$$e_{a\mu} = \begin{pmatrix} -\alpha & 0 & 0 & 0 \\ 0 & \alpha^{-1} \sin \theta \cos \phi & r \cos \theta \cos \phi & -r \sin \theta \sin \phi \\ 0 & \alpha^{-1} \sin \theta \sin \phi & r \cos \theta \sin \phi & r \sin \theta \cos \phi \\ 0 & \alpha^{-1} \cos \theta & -r \sin \theta & 0 \end{pmatrix},$$

(8)
The non-vanishing components of the acceleration tensor are \( \phi(0)(1) \), \( \phi(0)(2) \) and \( \phi(0)(3) \). Definition (5) yields

\[
\phi(0)(i) = T_{(0)(0)(i)} = \varepsilon_{(0)} \varepsilon_{(i)}^{\nu} T_{(0)\nu} ,
\]

where \( i = 1, 2, 3 \). Since the space-time is spherically symmetric, the orientation of the spatial axes of the frame is arbitrary. After simple calculations we obtain \( \phi(0)(1) = (\partial_1 \alpha) \sin \theta \cos \phi \), \( \phi(0)(2) = (\partial_1 \alpha) \sin \theta \sin \phi \), and \( \phi(0)(3) = (\partial_1 \alpha) \cos \theta \). In terms of the vector \( \hat{r} = \sin \theta \cos \phi \hat{x} + \sin \theta \sin \phi \hat{y} + \cos \theta \hat{z} \), we find

\[
a = \phi(0)(1) \hat{x} + \phi(0)(2) \hat{y} + \phi(0)(3) \hat{z} = \frac{1}{\alpha r^2} \left( m - \frac{Q^2}{r} \right) \hat{r} .
\]

We see that for values of the radial coordinate \( r \) such that \( m - Q^2/r < 0 \), or \( r < Q^2/m \), the acceleration \( a \) is directed along the negative direction of the vector \( \hat{r} \). It means that for values of the radial coordinate \( r \) such that \( r < Q^2/m \), the inertial acceleration on the frame has to be directed towards the origin of the coordinate system to maintain the frame static. Therefore the gravitational acceleration is repulsive when \( r < Q^2/m \), and diverges in the limit \( r \to \infty \). For very large values of \( r \) such that \( r >> m \) and \( r >> Q \), we find \( a \approx m/r^2 \hat{r} \), as expected, since the geodesic acceleration is \( -m/r^2 \hat{r} \) in this limit.

### 3.2 The Kerr naked singularity

The Kerr space-time is characterized by the total mass \( m \) and by angular momentum per unit mass \( a \). In terms of the Boyer-Lindquist coordinates, the Kerr space-time is described by the line element

\[
ds^2 = -\frac{\psi^2}{\rho^2} dt^2 - \frac{2 \lambda \sin^2 \theta}{\rho^2} d\phi dt + \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\theta^2 + \frac{\Sigma^2 \sin^2 \theta}{\rho^2} d\phi^2 ,
\]

with the following definitions:

\[
\Delta = r^2 + a^2 - 2mr ,
\]
\[
\rho^2 = r^2 + a^2 \cos^2 \theta ,
\]
\[
\psi^2 = \Delta - a^2 \sin^2 \theta ,
\]
\[ \Sigma^2 = (r^2 + a^2)^2 - \Delta a^2 \sin^2 \theta, \]
\[ \chi = 2amr. \]  
(11)

We assume that \( a > m \). In this case, Eq. (10) represents a naked singularity. The tetrad fields yields the metric tensor according to \( e^a \mu e^b \nu \eta_{ab} = g_{\mu\nu} \). Since \( e^a \mu \) is a kind of square root of \( g_{\mu\nu} \), it may not be defined in every region of the three-dimensional space. For instance, in the case of a black hole, if we choose the observer to be static in space-time, then the tetrad field is defined only in the region \( r > r^*_+ \), where \( r^*_+ = m + \sqrt{m^2 - a^2 \cos^2 \theta} \) represents the external boundary of the ergosphere of the Kerr black hole, because inside the ergosphere it is not possible to maintain any observer in static regime. Inside the ergosphere, all observers are necessarily dragged in circular motion by the gravitational field.

In the context of the naked singularity here considered, we find that the set of tetrad fields that are everywhere defined in the three-dimensional space, except at the origin \( r = 0 \), is the same one considered in Ref. [14]. These tetrad fields satisfy Schwinger’s time gauge condition \( e^{(i)}_\mu = 0 \), and are adapted to a field of observers that rotate in space-time under the action of the gravitational field. The four-velocity of observers that are dragged by the gravitational field in circular motion reads

\[ u^\mu(t, r, \theta, \phi) = \rho \frac{\Sigma}{(\Sigma^2 \Delta + \chi^2 \sin^2 \theta)^{1/2}} (1, 0, 0, \chi), \]  
(12)

where all functions are defined in Eq. (11). The quantity \( \omega(r) = -g_{03}/g_{33} = \chi/\Sigma^2 \) is the dragging velocity of inertial frames.

The tetrad fields that are adapted to observers whose four-velocities are given by Eq. (12), i.e., for which \( e^{(0)}_\mu = u^\mu \), and whose \( e^{(1)}_\mu \), \( e^{(2)}_\mu \) and \( e^{(3)}_\mu \) components in cartesian coordinates are oriented along the unit vectors \( \hat{x}, \hat{y}, \hat{z} \), respectively, are given by

\[
    e_{a\mu} = \begin{pmatrix}
        -A & 0 & 0 & 0 \\
        B \sin \theta \sin \phi & C \sin \theta \cos \phi & D \cos \theta \cos \phi & -E \sin \theta \sin \phi \\
        -B \sin \theta \cos \phi & C \sin \theta \sin \phi & D \cos \theta \sin \phi & E \sin \theta \cos \phi \\
        0 & C \cos \theta & -D \sin \theta & 0
    \end{pmatrix},
\]  
(13)

where
\[ A = \frac{(g_{03}g_{03} - g_{00}g_{33})^{1/2}}{(g_{33})^{1/2}} = \frac{1}{(-g_{00})^{1/2}} = \frac{(\psi^2 \Sigma^2 + \chi^2 \sin^2 \theta)^{1/2}}{\rho \Sigma}, \]

\[ B = -\frac{g_{03}}{(g_{33})^{1/2} \sin \theta} = \frac{\chi}{\rho \Sigma}, \]

\[ C = (g_{11})^{1/2} = \frac{\rho}{\sqrt{\Delta}}, \]

\[ D = (g_{22})^{1/2} = \rho, \]

\[ E = \frac{(g_{33})^{1/2}}{\sin \theta} = \frac{\Sigma}{\rho}. \]  

(14)

The tetrad fields above are asymptotically flat. In Cartesian coordinates they may be written as \( e_{a\mu} \approx \eta_{a\mu} + (1/2)h_{a\mu} \) in the asymptotic limit \( r \to \infty \), where \( h_{a\mu} = h_{a\mu}(1/r) \). Equation (13) constitutes the unique configuration that satisfies the 6 conditions previously discussed, namely, the fixation of \( e_{(0)i} = u_i \) and of the asymptotic orientation of the spatial components \( e_{(i)\mu} \).

We are interested in calculating only the translational acceleration \( a = \phi_{(0)(1)} \hat{x} + \phi_{(0)(2)} \hat{y} + \phi_{(0)(3)} \hat{z} \). In this case we have \( \phi_{(0)(i)} = g^{00}g^{11}e_{(0)0}e_{(i)1}T_{(0)01} + g^{00}g^{22}e_{(0)0}e_{(i)2}T_{(0)02} \), where \( T_{(0)01} = \partial_1 A, T_{(0)02} = \partial_2 A \), and \( \partial_1 = \partial_r, \partial_2 = \partial_\theta \).

The evaluation of \( a \) is long but straightforward. We find that the translational inertial acceleration on the frame given by Eq. (12) is given by

\[ a = \frac{\sqrt{\Delta}}{2\rho A^2} \partial_1(A^2) \hat{r} + \frac{1}{2\rho A^2} \partial_2(A^2) \hat{\theta}, \]  

(15)

where \( \hat{\theta} = \cos \theta \cos \phi \hat{x} + \cos \theta \sin \phi \hat{y} - \sin \theta \hat{z} \), and

\[ \partial_1(A^2) = \frac{2(r - m)}{\rho^2} - \frac{2r}{\rho^2} \left( \frac{\psi^2 \Sigma^2 + \chi^2 \sin^2 \theta}{\rho^2 \Sigma^2} \right) + \frac{4\chi am}{\rho^2 \Sigma^2} \sin^2 \theta \]

\[ -\frac{\chi^2}{\rho^2 \Sigma^4} \sin^2 \theta [4r(r^2 + a^2) - 2a^2(r - m) \sin^2 \theta] , \]  

(16)

\[ \partial_2(A^2) = \frac{2 \sin \theta \cos \theta}{\rho^2} \left[ \frac{\chi^2}{\Sigma^2} (1 + a^2 \sin^2 \theta) - \right. \]

\[ -a^2 \left( 1 - \frac{\psi^2}{\rho^2} \right) + \frac{\chi^2}{\Sigma^4} \Delta a^2 \sin^2 \theta \] .  

(17)
Expression (15) provides the translational inertial acceleration necessary to maintain the frame stationary in space-time. The frame is adapted to an observer in circular motion around the $z$ axis, at a fixed radial coordinate $r$, and rotates according to Eq. (12). The dependence of $a$ on the coordinates $(r, \theta)$ is intricate, but it certainly displays repulsive character near $r = 0$. By inspecting Eq. (15), it is not difficult to conclude that in the limit $r \to 0$ we have

$$a \to -\frac{m}{a^2 (\cos^2 \theta)^{3/2}} \hat{r}.$$  \hspace{1cm} (18)

The limit above comes from the first term on the right hand side of Eq. (16). Since $a$ is directed along the negative direction of $\hat{r}$, the gravitational acceleration on the frame in the vicinity of the origin is repulsive, and is more intense as the observer approaches the equatorial plane determined by $\theta = \pi/2$. At large distances ($r >> m, r >> a$), the gravitational field is attractive in the expected Newtonian way, since $a \approx m/r^2 \hat{r}$ is oriented in the positive direction of $\hat{r}$, and cancels the geodesic acceleration $-a$.

### 3.3 The point massive particle

The space-time of a point massive particle has been addressed by Katanaev [13] as an exact solution of Einstein’s equations, by writing the energy-momentum tensor in terms of a $\delta$ function of a point particle with support at the origin of the coordinate system. Katanaev concluded that the Schwarzschild solution in isotropic coordinates represents the space-time of the point massive particle, which is described by the line element

$$ds^2 = -\alpha^2 dt^2 + \beta^2 [dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2)],$$  \hspace{1cm} (19)

where

$$\alpha^2 = \left(1 - \frac{m}{2r}\right)^2, \quad \beta^2 = \left(1 + \frac{m}{2r}\right)^4.$$  \hspace{1cm} (20)

The parameter $m$ represents the mass of the point particle and arises in the energy-momentum tensor. Katanaev observed that in the range $0 < r < r_*$, where $r_* = m/2$ is the Schwarzschild radius in isotropic coordinates, the gravitational field is repulsive, in contrast to being attractive for $r > r_*$. We will arrive at the same conclusion in the following.
The tetrad fields adapted to static observers in the point massive space-time, in spherical coordinates, are given by,

\[
\epsilon_{\alpha\mu} = \begin{pmatrix}
-\alpha & 0 & 0 & 0 \\
0 & \beta \sin \theta \cos \phi & \beta r \cos \theta \cos \phi & -\beta r \sin \theta \sin \phi \\
0 & \beta \sin \theta \sin \phi & \beta r \cos \theta \sin \phi & \beta r \sin \theta \cos \phi \\
0 & \beta \cos \theta & -\beta r \sin \theta & 0
\end{pmatrix}.
\] (21)

The components of the translational acceleration \(a = \phi_{(0)(1)} \hat{x} + \phi_{(0)(2)} \hat{y} + \phi_{(0)(3)} \hat{z}\) are \(\phi_{(0)(i)} = T_{(0)(0)(i)}\). After simple calculations we obtain

\[
\begin{align*}
T_{(0)(0)(1)} &= \frac{1}{\alpha \beta} \sin \theta \cos \phi (\partial_1 \alpha), \\
T_{(0)(0)(2)} &= \frac{1}{\alpha \beta} \sin \theta \sin \phi (\partial_1 \alpha), \\
T_{(0)(0)(3)} &= \frac{1}{\alpha \beta} \cos \theta (\partial_1 \alpha).
\end{align*}
\] (22)

Altogether, these terms yield

\[
a = \frac{m}{r^2} \left(1 - \frac{m}{2r}\right)^{-1/2} \hat{r} \equiv a(r) \hat{r}. \] (23)

It is easy to verify that

\[
\begin{align*}
r > r_* & : \quad a(r) > 0, \\
r < r_* & : \quad a(r) < 0, \\
r \to r_* + \epsilon & : \quad a(r) \to \infty \quad \text{if} \quad \epsilon \to 0, \\
r \to r_* - \epsilon & : \quad a(r) \to -\infty \quad \text{if} \quad \epsilon \to 0, \\
r \to 0 & : \quad a(r) \to 0.
\end{align*}
\] (24)

The frame must undergo an inertial acceleration \(a(r) < 0\) for \(r < r_*\), in order to be static in space-time. Therefore in the range \(0 < r < r_*\) the gravitational acceleration on the frame is repulsive, and it is possible to show that \(a(r)\) vanishes for \(r = 0\). These results are in agreement with those obtained by Katanaev [13], and indicate a modification of the Newtonian attraction at short distances.
4 Concluding remarks

In this article we have shown that in the vicinity of the Kerr and Reissner-Nordstrom naked singularities, and of the massive point particle, the gravitational field is repulsive, in contrast to the current understanding that gravity is everywhere attractive. This feature was already noted in the study of accretion disks around naked singularities [3]. The analysis was carried out by means of the acceleration tensor presented in Section 2, which plays an important role in the teleparallel equivalent of general relativity [8]. The acceleration tensor is a frame dependent quantity, since acceleration depends on the frame. However, it is invariant under coordinate transformations. In each of the three situations addressed in Section 3, we have obtained the exact expression of the translational inertial acceleration on the frames, in either static or stationary regime. The acceleration tensor provides the values of the inertial accelerations that are necessary to impart to the frame to maintain it in a certain inertial state in space-time. Therefore, for static frames in space-time, the inertial acceleration is precisely minus the gravitational acceleration.

The condition for the emergence of repulsive gravitational acceleration is very simple. One establishes the frame for static or stationary observers, constructs the translational inertial acceleration $\mathbf{a}$,

$$\mathbf{a} = T_{(0)(0)(1)} \hat{x} + T_{(0)(0)(2)} \hat{y} + T_{(0)(0)(3)} \hat{z},$$

and check the sign of each quantity $T_{(0)(0)(i)}$. The gravitational acceleration points in the opposite direction of $T_{(0)(0)(1)} \hat{x}$, $T_{(0)(0)(2)} \hat{y}$ and $T_{(0)(0)(3)} \hat{z}$. The positive or negative sign of $T_{(0)(0)(i)}$ means that the gravitational acceleration is attractive or repulsive, respectively.

It is interesting to investigate the consequences of the results obtained here in the analysis of the formation of accretion disks. In particular, the emergence of $\cos^2 \theta$ in the denominator of Eq. (18) is a curious feature. As the radius of the circular orbit of the frames approaches the Kerr naked singularity on the equatorial plane, the repulsive gravitational acceleration increases, and diverges only in the limit $r \to 0$. If naked singularities are not just hypothetical astrophysical objects, but manifestations of the physical reality, this feature may play a role in the formation of accretion disks around the singularities. It has been conjectured the existence of powerful repulsive forces on the equatorial plane of Kerr-like geometries that display naked
singularities, in contrast to the geodesic behaviour around black holes. It is expected that astrophysical observations will distinguish naked singularities from black holes.

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