Functional Renormalization Prediction of Rupture

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Abstract

We develop theoretical formulas for the prediction of the rupture of systems which are known to exhibit a critical behavior, based solely on the knowledge of the early time evolution of an observable, such as the acoustic emission rate as a function of time or of stress. From the parameterization of such early time evolution in terms of a low-order polynomial, we use the functional renormalization approach introduced by Yukalov and Gluzman to transform this polynomial into a function which is asymptotically a power law. The value of the critical time $t_c$, conditioned on the assumption that $t_c$ exists, is thus determined from the knowledge of the coefficients of the polynomials. We test with success this prediction scheme with respect to the order of the polynomials and as a function of noise.

1 Introduction

To what extent can the material failure of a mechanical system under stress be forecasted? This question has enormous technological interest for its economic and human consequences, especially in the automobile, naval, aeronautics and space industries [1], as well as in the sensitive chemical and nuclear industries to cite a few among many examples.

If rupture occurs brutally without precursors, prediction is impossible. In contrast, Mogi noticed that, for experiments on a variety of materials, the larger the heterogeneity of the material, the stronger and more useful are the precursors to rupture [2]. For a long time, the Japanese research effort for earthquake prediction and risk assessment was based on this very idea [3].

These empirical facts have been put on a firm theoretical basis by using various simplified mechanical models of failure of heterogeneous materials which showed that, increasing the disorder changes rupture from first-order (abrupt) to critical (continuous with power-law properties) [4]. By the term “disorder”, we refer to heterogeneity in material properties (elastic coefficients and rupture thresholds) as well as inhomogeneous pre-stresses. In the presence of long-range elasticity, disorder is found to be always relevant leading to a critical rupture. However, the disorder controls the width of the critical region [5]. The smaller it is, the smaller will be the critical region, which may become too small to play any role in practice.

The potential for predicting rupture thus seems associated with its critical nature. Let us first review past works which support this concept of critical rupture. Theoretically, the concept that rupture is critical was first formulated by Vere-Jones [6] using critical branching models and Allègre et al. [7] using the percolation model and real-space renormalization group (see ref. [8] for a general presentation). The Russian school has also extensively developed this concept [9, 10, 11]. One of us and co-workers have introduced a statistical two-dimensional model of dynamically evolving damage [12], which exhibits the critical time-to-failure dependence of the energy released up to the rupture. Based on a precise numerical description of the many growing interacting micro-cracks with a spatio-temporal organization which is a function of the stress-dependent damage law, one finds that, under a step-function stress loading, the total rate of damage increases on
average as a power-law of the time-to-failure. In this model, rupture is a “critical point” in the statistical physics sense and occurs as the culmination of the progressive nucleation, growth and fusion between microcracks, leading to a fractal network of cracks. This simple model has since then been found to describe quantitatively the experiments on the electric breakdown of insulator-conducting composites and the damage by electromigration of polycrystalline metal films. This led to the proposal and the empirical test on real engineering composite structures that failure in fiber composites is a genuine “critical” point. This critical behavior may correspond to an acceleration of the rate of energy release or to a deceleration, depending on the nature and range of the stress transfer mechanism and on the loading procedure. A generalization is to extend the power-law behavior of the time-to-failure analysis by including corrections in the form of log-periodic modulations. Log-periodicity is the hallmark of a hierarchy of discrete characteristic scales in the rupture process. Mathematically, it corresponds to adding an imaginary part to the exponent $z$ (defined below). Intuitively, the log-periodic oscillations are oscillations that are periodic in the logarithm of the time-to-failure and thus corresponds to an accelerating frequency modulation as the critical time is approached. This acceleration of alternating ups and downs accounts for the succession of damage and quiescent phases self-organizing and culminating in the rupture.

Following this work, this method has been used extensively by the French Aerospace company Aérospatiale on pressure tanks made of kevlar-matrix and carbon-matrix composites used on the European Ariane 4 and 5 rockets. In this application, the method consists in recording acoustic emissions under constant stress rate. The acoustic emission energy as a function of stress is fitted by the critical theory mentioned above. One of the parameters is the time of failure and the fit thus provides a “prediction” when the sample is not brought to failure in the first test. The results indicate that a precision of a few percent in the determination of the stress at rupture is typically obtained using acoustic emission recorded about 20% below the stress for rupture. We now have a better understanding of the conditions, the mathematical properties and physical mechanisms at the basis of log-periodic structures.

The numerical simulations of Sahimi and Arbabi have confirmed that, near the global failure point, the cumulative elastic energy released during fracturing of heterogeneous solids with long-range elastic interactions exhibit a critical behavior with observable log-periodic corrections. Molecular dynamics simulations of the geometry of fracture patterns in a dilute elastic network give similar results: under a uniform strain which drives the fracture to develop by the growth and coalescence of the vacancy clusters in the network, there exists a characteristic time at which a dynamical transition occurs with a power law divergence of the average cluster size. The cluster growth near the critical time also exhibits spatial scaling in addition to the temporal scaling, namely as fracture develops with time, the connectivity length of the clusters increases and diverges at $t_c$. Recent experiments on the rupture of fiber-glass composites have also confirmed the critical scenario. Johansen and Sornette have recently re-analyzed the acoustic emissions recorded during the pressurization of spherical tanks of kevlar or carbon fibers impregnated in a resin matrix wrapped up around a thin metallic liner (steel or titanium) fabricated and instrumented by Aérospatiale-Matra Inc. These experiments were performed as part of a routine industrial procedure, which tests the quality of the tanks prior to shipment. It was found that the seven acoustic emission recordings of seven pressure tanks which was brought to rupture exhibit clear acceleration in agreement with a power-law “divergence” expected from the critical point theory.

At the same time, it became tempting to apply similar considerations to earthquakes. Indeed, over the years there has been a growing evidence that a significant proportion of large and great earthquakes are preceded by a period of accelerating seismic activity of moderate-sized earthquakes. These moderate earthquakes occur during the years to decades prior to the occurrence of the large or great events and over a region much larger than its rupture zone. Sornette and Sammis identified a specific measurable signature of this criticality in terms of a power-law acceleration of the Benioff strain previously interpreted as an exponential mechanical-damage rate. The combined observational and simulation evidence now seems to confirm that the period of increased moment release in moderate earthquakes signals the establishment of long-wavelength correlations in the regional stress field. Large or great earthquakes appear to dissipate a sufficient proportion of the accumulated regional strain to destroy these long wavelength stress correlations. They can thus be considered as different from smaller earthquakes. According to this model, large earthquakes are not just scaled-up version of small earthquakes but play a special role as “critical points”. Recent extensions to the intermediate scale of rockbursts in deep mines confirm the picture.

To summarize these works, there is a rather strong evidence that rupture in heterogeneous media is critical in the sense...
of statistical physics [8]. To what extent can the critical rupture concept be used to predict rupture? Voight noticed in an exciting precursory work that many systems fail by exhibiting a typical relationship relating the second time derivative $d^2\Omega/dt^2$ of some observable $\Omega$ to a positive power of $\Omega$ itself [28, 29]. He then used this relationship to attempt predictions of failures in various materials and of volcanic eruptions. Basically, the relationship he postulates is nothing but a power law

\[ \Omega(t) = (t_c - t)^z, \]

where $t_c$ is the critical time of rupture, $z$ is a critical exponent and $A$ a numerical amplitude. By differentiating twice and eliminating time, we indeed get $d^2\Omega/dt^2 = B \Omega^{z+2}$, where $B = z(z+1) A^{-2/z}$.

As we said above, the critical rupture concept establishes theoretically this law (1) as the result of the cooperative organization of precursory damage preparing the global rupture. If an observable such as the rate of acoustic emissions radiated during loading exhibit an acceleration close to rupture of the form (1) as documented in several experiments [15, 19, 20], it is clear that one can try to fit the data by (1) and get a prediction from the determination of $t_c$.

In practice, the problem is that the fit of a simple power law (1) to a noisy data is rather unstable, so much so that often no fits can be found [30]. The problem comes from the fact that only close to $t_c$ (in some relevant time units) can the power law be clearly distinguished from other parametric accelerating functional forms, such as an exponential. As an exponential has no critical time, this leads to an ill-defined rupture time. The fundamental limitation in using this prediction scheme is thus that $t_c$ is determined only when data is used up to very close to $t_c$. This is thus far from the prediction goal to infer $t_c$ from a distance! One possibility to improve the reliability and range of power law fits far from $t_c$ has been proposed, based on log-periodic corrections to power laws [15, 14, 21, 22, 27].

In the present paper, we propose a different approach to the prediction of $t_c$. In the next section, we give the gist of the method and formulate the problem in precise terms. We also provide a brief summary of the functional renormalization method. In section 3, we test the method in the situation where both the polynomial expansion at early times and the value of $t_c$ is known, to see how the time-dependence is reconstructed by the functional renormalization. In section 4, we present the genuine prediction scheme which determines $t_c$ solely from the knowledge of the first few coefficients of the polynomial fit at early times and the assumption that the late time dynamics is of the power law form (1). In particular, we test how the precision of the prediction improves by adding more terms in the polynomial expansion. Section 5 presents numerical tests of our method compared to direct fits by a power law for noisy data and section 6 concludes.

## 2 Formulation of the problem and outline of the method

The gist of the method is as follows. We assume that we are able to parameterize the early time evolution of an observable, such as the acoustic emission rate as a function of time or of stress by fitting the data to a low-order polynomial $\Omega(t) = a_0 + a_1 t + a_2 t^2 + ...$. We then use the sophisticated functional renormalization approach introduced by Yukalov and Gluzman [31]-[42]. We use the version developed in [38] that transforms a polynomial into an analytic effective sum [39, 40].

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\[ a(t) = \frac{a_0}{\cos(\pi \sigma_0 t / 3D^*)}, \]
and is indeed of the form \( f \) close to \( t_c = 3D^* / 2\sigma_0 \) with \( z = -1 \).

For the convenience of notations, we work in the sequel with dimensionless variables \( f = a/a_0 \) and \( \pi\sigma_0 / 3D^* \), \( t \to t \), so that the solution \( \phi \) becomes

\[
 f_{\text{exact}}(t) = \frac{1}{\cos t}.
\]  \( \tag{3} \)

Let us assume that we have access only to the small time dynamics, captured mathematically by the first terms in the expansion of the solution \( \phi \):

\[
 f_a(t) = 1 + a_1 t^2 + a_2 t^4 + a_3 t^6 + \ldots \quad \text{with} \quad a_1 = 1/2, \quad a_2 = 5/24 \quad \text{and} \quad a_3 = 61 / 720.
\]  \( \tag{4} \)

Since the cosine function is even in \( t \), only even power of \( t \) are present. Knowledge of only the first few terms in \( f_a \) is the relevant situation for instance in an experiment in which the early acoustic emissions are recorded and one would like to infer the subsequent evolution.

To make this paper self-consistent, we first outline the method we use, which is a direct adaptation of ref. [38]. The complete mathematical foundation can be found in earlier publications [31]-[42]. Assume that we are interested in a function \( \phi(x) \) of a real variable \( x \). Let perturbation theory give for this function the perturbative approximations \( p_k(x) \) with \( k = 0, 1, 2, \ldots \), enumerating the approximation order. Define the algebraic transform \( F_k(x, s) = x^k p_k(x) \). This transform changes the powers of the series \( p_k(x) \), thus changing its convergence properties. As a result, the approximation order effectively increases from \( k \) to \( k + s \). The inverse transform is \( p_k(x) = x^{-s} F_k(x, s) \). Define the expansion function \( x = (f(s), x) \) by the equation \( F_0(x, s) = f \), where \( F_0 \) is the first available approximation and \( f \) is a new variable. Substituting \( x(f, s) \) back to \( F_k \), we get \( y_k(f, s) = F_k(x(f, s), s) \). The transformation inverse to the latter reads \( F_k(x, s) = y_k(F_0(x, s), s) \).

Consider the family \( \{ y_k \} \) as a dynamical system in discrete time \( k \), the order of the approximations. The trajectory \( \{ y_k(f(s), s) \} \) of this dynamical system is, by construction, bijective to the approximation sequence \( \{ p_k(f(s, s)) \} \). This system can thus be called the approximation cascade. The next step is to embed the discrete sequence \( \{ y_k(f(s), s) \} \) into a continuous sequence \( \{ y(\tau, f, s) \} \) with \( \tau \in [0, +\infty] \). Thus, the family \( \{ y(\tau, f, s) : \tau \in [0, +\infty] \} \) composes a dynamical system with continuous time, whose trajectory passes through all points of the approximation cascade trajectory. Such a system can thus be called the approximation flow. The evolution equation for a flow can be presented in the functional form \( y(\tau + \tau', f, s) = y(\tau, y(\tau', f, s), s) \). We call this equation the self-similarity relation, which is the central concept of our approach. In this framework, the motion occurs in the space of approximations, where self-similarity is a necessary condition for convergence as a function of “time” defined as the approximation number. The evolution equation for the approximation flow can be rewritten in the differential form and then integrated over time between \( k \) and some \( k^* \). The point \( k^* \) is to be chosen to provide the best approximation \( F_{k^*+1}(x, s) = y(k^*, F_0(x, s), s) \) for the minimal time \( k^* - k \). The cascade velocity \( v_k(y, s) \) in the vicinity of the time \( k \) may be presented by the Euler discretization of the flow velocity giving \( v_k(f, s) = V_k(x(f, s), s) \), with \( V_k(x(f, s), s) = F_{k+1}(x, s) - F_k(x, s) \). The integral form of the evolution equation is

\[
  \int_{F_k}^{F_{k^*+1}} \frac{df}{v_k(f, s)} = k^* - k,
\]  \( \tag{5} \)

where \( F_k = F_k(f, s) \) and \( F_{k+1} = F_{k^*+1}(x, s) \). The approximation \( F_{k^*+1} \) must be reached during the minimal time. When no additional constraints are imposed, then the minimal time corresponds, evidently, to one step: \( k^* = k + 1 \). Finding \( F_k(x, s) \) from \( \{ y_k \} \) and using the inverse transform leads to the self-similar approximation \( p_k^*(x, s) = x^{-s} F_k^*(x, s) \).

Now, by means of the substitution \( s \to s_k \), we have to introduce control functions \( s_k \) which can govern the convergence of the sequence \( \{ p_k^*(x, s_k) \} \). Following the standard procedure [33]-[42] and similar to the steps described above, we may construct an approximation cascade with its corresponding cascade velocity. Convergence of an approximation sequence is, in the language of dynamical theory, the same as the existence of an attracting fixed point for the corresponding approximation cascade. If the cascade trajectory tends to a fixed point, this means that the flow velocity goes to zero as “time” \( k \) goes to infinity. In practice, we cannot, of course, reach the limit \( k \to \infty \), and have to stop at a finite (approximation order) \( k \). Then, the condition to be as close to a fixed point as possible is the minimum of the velocity.
After the control functions are found from the minimal-velocity condition, we substitute them into \( p^*_k \) and obtain the final expression \( f^*_k = p^*_k(x, s_k(x)) \) for the self-similar approximation of the sought function. The practical way of using the minimal-velocity condition is to express it as a minimal difference condition. To check whether the obtained sequence \( \{ f^*_k \} \) converges, we have to analyze whether the corresponding mapping is contracting. The mapping related to the sequence \( \{ f^*_k \} \) is constructed in the standard way \([31]-[42]\) and the contraction, or stability, is analyzed by calculating the mapping multipliers. Ref. \([38]\) has shown how this method of algebraic self-similar renormalization works to obtain accurate estimations of the critical behavior of a large variety of physical systems, starting from virial-type or perturbation expansions containing only second-order terms and derived for a region far from the critical point.

In the sequel, we translate this formalism to the case where \( x \) is now a real time, \( k \) is the order of the polynomial fit to the early time of the signal which plays also the role of “time” for the dynamical flow in the functional space. The goal is to describe as accurately as possible the finite-time singularity, which is equivalent to a critical point in the time domain. Before investigating the predictive power of our approach, we first investigate the possibility of reconstructing as faithfully as possible the time evolution based on the knowledge of the singularity.

3 Reconstruction of the crack dynamics from the knowledge of the small time dynamics and the position of a finite-time singularity

Here, we assume in addition that we know the position of the singularity at \( t = \frac{\pi}{2} \) and its exponent, i.e., that

\[
f(t) = \frac{1}{\pi/2 - t} \quad \text{for} \quad t \to \pi/2.
\]

Note that the amplitude \( A \) of the pole is exactly equal to 1. We apply the Yukalov-Gluzman technique in its version related to crossover phenomena \([39]-[42]\), to obtain the best approximation of \( f(t) \) for arbitrary times based only on the knowledge at small times and \((6)\) at times close to rupture.

3.1 Using only \( f_a(t) = 1 + a_1t^2 \) and \((6)\)

From the expansion \( f_a(t) = 1 + a_1t^2 \), the Yukalov-Gluzman method allows us to build the approximant

\[
f^*_1(t) = \left[ \exp\left( a_1t^2 \right) \right]^{-\beta} + bt^4,
\]

with two unknown parameters \( b \) and \( \beta \), to be determined solely by demanding the existence of a crossover. The exponent \( \beta \) is determined from the condition that \( f_a(t) = 1 + a_1t^2 \) must cross-over to \((6)\). This gives \( \beta = 1 \) for the simple pole \((6)\). The coefficient \( b \) is obtained from the condition of a pole at the known critical point \( t_c = \pi/2 \), which reads \( 1/f_1(t_c) = 0 \). Solving for \( b \), we finally get the first-order approximant

\[
f^*_1(t) = \left[ \exp\left( -a_1t^2 \right) - \frac{t^4}{t_c^4} \exp\left( -a_1t_c^2 \right) \right]^{-1}.
\]

This approximant gives an amplitude \( A_1 = 0.834 \) for the simple pole, only 17% off the exact value \( A = 1 \). Figure \([6]\) plots in logarithmic scale the relative errors between the approximant \( f_1 \) and the exact expression \((3)\) as a function of time \( t \).

3.2 Using \( f_a(t) = 1 + a_1t^2 + a_2t^4 \) and \((6)\)

A similar procedure as in the previous case gives the second-order approximant

\[
f^*_2(t) = \left[ \exp\left( -a_1t^2 \exp\left( \frac{a_2t^2}{a_1} \right) \right) - \frac{t^6}{t_c^6} \exp\left( -a_1t_c^2 \exp\left( \frac{a_2t_c^2}{a_1} \right) \right) \right]^{-1}.
\]
This second-order approximant can be improved by introducing an additional control parameter \( \tau \) (optimal effective time \( k^* - k \)) such that we can enforce the condition that the coefficient \( a_2 \) is preserved in the renormalization procedure. In other words, the expansion of \( f_2(t) \) is now imposed to have the same coefficient of its power \( t^4 \). The expression \( \text{(9)} \) is thus modified into

\[
f_2^\ast(t) = \frac{\exp \left( -a_1 t^2 \exp \left( \frac{a_2}{a_1} \tau t^2 \right) \right) - \frac{t^6}{t_c^6} \exp \left( -a_1 t^2 \exp \left( \frac{a_2}{a_1} \tau t^2 \right) \right) \}}{1}.
\] (10)

The amplitude of the simple pole predicted by this approximant is found equal to 0.898, only 10% off from the exact value 1. The condition that the coefficient of the power \( t^4 \) in the expansion of \( \text{(10)} \) is equal to \( a_2 = 5/24 \) given by \( \text{(4)} \) leads to the improved second approximant of the form \( \text{(10)} \) with

\[
\tau = 1 - \frac{a_1^2}{2a_2}.
\] (11)

Figure 1 plots in logarithmic scale the relative errors between the approximant \( f_2^\ast \) and the exact expression \( \text{(3)} \) as a function of time \( t \).

3.3 Using \( f_a(t) = 1 + a_1 t^2 + a_2 t^4 + a_3 t^6 \) and \( \text{(5)} \)

An extension one step further in the Yukalov-Gluzman procedure gives

\[
f_3^\ast(t) = \left[ \left( \exp \left( a_1 t^2 \exp \left( \frac{a_2}{a_1} t^2 \tau_1 \exp \left( \frac{a_3}{a_2} t^2 \tau_2 \right) \right) \right) \right) \right]^{-\frac{1}{\beta}} + a_8 t^8,
\] (12)

where the two control parameters \( \tau_1 \) and \( \tau_2 \) are to be determined from the condition that \( a_2 \) and \( a_3 \) are conserved by the renormalization. We find

\[
\tau_1 = 1 - \frac{a_1^2}{2a_2}
\] (13)

and

\[
\tau_2 = - \frac{1}{a_3 \tau_1} \left( \frac{a_2 \tau_1^2}{2a_1} + a_1 a_2 \tau_1 + \frac{a_1^3}{6} - a_3 \right).
\] (14)

The coefficient \( a_8 \) is found from the condition on the critical point:

\[
a_8 = - \frac{1}{t_c^8} \exp \left( - \frac{a_1}{\beta} t_c^2 \exp \left( \frac{a_2}{a_1} t_c^2 \tau_1 \exp \left( \frac{a_3}{a_2} t_c^2 \tau_2 \right) \right) \right)
\] (15)

The solution then reads

\[
f_3^\ast(t) = \left[ \exp \left( - \frac{1}{2} t^2 \exp \left( \frac{1}{6} t^2 \exp \left( \frac{11}{60} t^2 \right) \right) \right) - 256 \left( \frac{t}{t_c} \right)^8 \exp \left( - \frac{\pi^2}{8} \exp \left( \frac{\pi^2}{24} \exp \left( \frac{11\pi^2}{240} \right) \right) \right) \right]^{-1}.
\] (16)

This approximant gives an amplitude \( A_3 = 0.9661 \) for the simple pole, only 3.4% off the exact value \( A = 1 \). The quality of the reconstruction of the full function \( \text{(3)} \) can also be checked by comparing the exact value of the next term \( (277/8064) t^8 \) = 0.03435 \( t^8 \) in the expansion \( \text{(3)} \) of the function \( \text{(3)} \). We find 0.034497 \( t^8 \), corresponding to an error of 0.4%.

Figure 1 plots in logarithmic scale the relative errors between the approximants \( f_1^\ast, f_2^\ast \) and \( f_3^\ast \) and the exact expression \( \text{(3)} \).
4 Prediction of the critical time from the knowledge of the small time dynamics

We now assume the knowledge of the expansion (15) up to some order, representing for instance the experimental recording of an acoustic emission signal up to some stress level. In addition, we assume only the existence of a singularity of the form \(1/(t_c - t)^K\) at some value \(t_c\), using the insight from the theory of critical rupture, but do not know a priori neither the position of \(t_c\), nor the value of the exponent \(K\). In other words, we assume that we know that the system is bound to break but we do not know when and how. Our goal is to attempt to determine the critical time and the functional form of the signal on the approach to the critical rupture time from the recording of the early signals.

4.1 Standard approach from Yukalov and Gluzman [38]

Consider an expansion of some function \(\phi\) in powers of some variable \(u\) given by

\[
p_k(u) = \sum_{k=0}^{\infty} b_k u^k, \quad \text{with} \quad b_0 = 1.
\]

The method of algebraic self-similar renormalization [36]-[38] gives the following general recurrence formula for the approximant of order \(k\) as a function of the expansion \(p_{k-1}(u)\) up to order \(k - 1\):

\[
\phi_k^*(u) = p_{k-1}(u) \left[ 1 - \frac{k}{s} b_k u^{k/s} p_{k-1}(u) \right]^{-s/k} \equiv \left[ \frac{-k/s}{p_{k-1}(u)} - \frac{k b_k}{s} u^k \right]^{-s/k},
\]

where, generally speaking, \(s = s_k(u)\), depends on the approximation number and the variable \(u\).

First, let us estimate the position of the critical point. Using only \(f_1(t) = 1 + a_1 t^2\), we can write it as the inverse of a function that is requested to vanish in order to obtain the singularity. Expanding in powers of \(t\) up to first order in \(t\), we get the estimation \(t_{c1} = \sqrt{2} = 1.414\), which should be compared with the exact value \(\pi/2 = 1.571\). Including the next order from the expansion of \(\cos(t)^{-1}\) leads to the improved estimate \(t_{c2} = 1.59\).

Let us now obtain the expansion which will be used as a raw material for renormalization. Including the next order in \(f_2(t) = f_1(t) + a_2 t^4\), inverting it, and expanding in powers of \(t\) up to \(t^4\)-terms, we obtain an expansion \(p_2(t)\) for \(1/f_2\):

\[
p_2(t) = \sum_{k=0}^{2} b_k t^{2k}
\]

\[
b_0 = 1, \quad b_1 = -1/2, \quad b_2 = 1/24,
\]

Note that, in the initial series, all coefficients in the expansion of \(\cos^{-1}\) are positive, giving the worst possible case for resummation. In contrast, the coefficients of the inverted series have alternating signs, which may be better for resummation [38]. Hereafter, we apply the resummation procedure to the function \(F \equiv 1/f\) inverse of \(f\). Correspondingly, \(f^* \equiv (1/F)^*\).

In order to determine the exponent \(K\), we follow Yukalov and Gluzman [38] and construct the two approximants available from the knowledge of the two coefficients \(a_1\) and \(a_2\). They can be readily obtained from the general formula (18), with \(u = t^2\). The first order approximant is

\[
F_1^*(t) = \left( 1 - \frac{b_1}{s_1} t^2 \right)^{-s_1}.
\]

Representing \(p_2(t)\) as \(p_2(t) = 1 + b_1 t^2 (1 + b_2/b_1) t^2\), and applying the general formula to the expression in brackets, we obtain the second order approximant

\[
F_2^*(t) = 1 + b_1 t^2 \left( 1 - \frac{b_2}{b_1 s_2} t^2 \right)^{-s_2}.
\]

Assume further that

\[
s_1 = s_2 = s,
\]
where $s$ is a single control parameter, i.e. limit of the total control function in the critical point, which will play the role of the critical index $K$. As it was explained in \[38\], such an assumption is well justified in the vicinity of a stable fixed point.

We impose the condition of the existence of a critical point, which delivers two equations for $t_c$ and $s$:

$$F_1^*(t_c, s) = 0 \quad \text{and} \quad F_2^*(t_c, s) = 0. \tag{23}$$

The condition of maximum stability of the renormalization amounts to imposing that the difference $F_2^* - F_1^*$ be a minimum with respect to the set of parameters. The minimization of the difference is automatically satisfied when (23) holds, since the difference reaches its smallest possible value, namely zero.

The vanishing of $F_1^*$ given by (24) gives $t_c^2 = s/a_1$. The second condition $F_2^* = 0$ with (21) yields the estimation $s = 1.258$ for the critical index, only 26% off the true value equal to 1. The critical time is given numerically by $t_c = 1.586$, very close to the exact value $\pi/2 = 1.5708$. Note that, as is often found in critical phenomena, an error of less than 1% in the location of the critical point is associated with a much larger error of about 26% on the exponent. In the scheme presented above, it was possible to find $s$ and $t_c$ from to separate equations.

Such convenience does not hold for another approximation scheme, presented below, which has however other advantages such as simplicity. In order to separate the variables, we need an initial guess either for $s$ or for $t_c$. Such an initial guess is provided naturally by the analog of a mean-field approximation.

4.2 Alternative approach: expansion around a “mean-field” approximation

4.2.1 Expansion to the same order as above

An alternative and more transparent approach is first to minimize the distance between approximants and then to verify that (23) holds. This is the reverse order to the previous scheme that solves (23) which then automatically ensures that the distance between the two successive approximant is minimized.

In practice, this is implemented as follows. From (20), we see that the critical point is located at $t_c^2 = s/b_1$, i.e., $s = b_1 t_c^2$. Using only

$$p_1(t) = 1 + b_1 t^2, \tag{24}$$

we estimate $t_c^2 = -1/b_1$, which then yields $s = -1$. Note that this value $-1$ for the exponent always holds for any value of $b_1$. This exponent thus plays a role analog to a mean-field approximation in statistical physics. The fact that the exponent is in the present case equal to the exact value is a mere coincidence.

In the next order,

$$p_2(t) = 1 + b_1 t^2 + b_2 t^4. \tag{25}$$

The two approximants can be derived directly from the general formulas:

$$F_1^*(t) = \left(1 - \frac{b_1}{s_1} t^2\right)^{-s_1},$$

$$F_2^*(t) = \left[(1 + b_1 t^2)^{-2/s_2} - \frac{2b_2}{s_2} t^4\right]^{-s_2/2}. \tag{27}$$

We assume in addition that

$$s_1 = \frac{s_2}{2} = s, \tag{28}$$

thus eliminating a trivial dependence of the control parameter $s$ on the approximation number $k$. Note that (28) is different from (22) because we use a different sequence of approximations $f_1, f_2, ...$. In particular, the condition (28) ensures that $F_1^*(t)$ and $F_2^*(t)$ have the same exponent/control parameter.

In contrast with the previous method of section 4.1, we first minimize the difference $F_2^*(t, s) - F_1^*(t, s)$, and then verify that (23) holds. The difference calculated at the “mean-field” threshold $t_0^2 = -1/b_1$ gives

$$D_1(s) = \frac{1 + s}{s} - \left(\frac{-b_2}{b_1^2 s}\right)^{-s}, \tag{29}$$
which is exactly zero at
\[ s = -1 - \frac{b_2}{b_1^2} = -1.167 . \] (30)

The position of the critical point \( t_c \) can be re-calculated from the condition \( F_3^* (t_c, s) = 0 \), which has a non-trivial solution at
\[ t_c = \sqrt{\frac{s}{b_1}} = \sqrt{-\frac{1}{b_1}} \sqrt{1 + \frac{b_2}{b_1^2}} = 1.528 . \] (31)

Expanding in powers of \( \frac{b_2}{b_1^2} \), we estimate \( t_c \approx \sqrt{-\frac{1}{b_1}} \left( 1 + \frac{b_2}{2b_1^2} \right) = 1.532 \).

Thus, the renormalization scheme used to calculate \( t_c \) and \( K \) corresponds to an expansion around the mean field value \( t_c^2 = 1/b_1 \) and \( K = 1 \) in inverse powers of the dimensionless “Froude” number \( \frac{b_2}{s} \) (see [46] for a definition and use of the Froude number in this context of functional renormalization). In summary, we get the predictions \( t_c = 1.528 \) closer to the exact value \( \pi/2 = 1.5708 \), and the critical exponent \( K = 1.167 \).

**Higher-order expansion** Including the next order in \( f_3(t) = f_2(t) + a_3 t^6 \), inverting it, and expanding in powers of \( t \) up to \( t^6 \)-terms, we obtain an expansion \( p_3(t) \) for \( 1/f_3 \):
\[ p_3(t) = \sum_{k=0}^{3} b_k t^{2k} , \] (32)

with
\[ b_1 = -1/2, \quad b_2 = 1/24, \quad b_3 = -1/720 . \] (33)

The two higher-order approximants can be written as follows:
\[ F_2^* (t, s_2) = \left[ (1 + b_1 t^2)^{-2s_2} - \frac{2 b_2}{s_2} t^4 \right]^{-s_2/2} , \] (34)
\[ F_3^* (t, s_3) = \left[ (1 + b_1 t^2 + b_2 t^4)^{-3s_3} - \frac{3 b_3}{s_3} t^6 \right]^{-s_3/3} . \] (35)

Assume that
\[ \frac{s_2}{2} = \frac{s_3}{3} = s . \] (36)

The difference \( F_3^* (t, s) - F_2^* (t, s) \) calculated at the “mean-field” critical point \( t_0^2 = -1/b_1 \) gives
\[ D_2(s) = \left[ \left( \frac{b_2}{b_1^2} \right)^{-1/s} - \frac{b_3}{s} \left( -\frac{1}{b_1} \right)^{3s} - \left( -\frac{b_2}{b_1^2} \right)^{-s} \right] , \] (37)

which has a zero at \( s = -1.024 \). The position of the critical point \( t_c \) can be re-calculated from the condition \( 1/F_3^* (t_c, s) = 0 \), which has a non-trivial solution at \( t_c = 1.5722 \) determined from the equation
\[ \left( 1 + b_1 t_c^2 + b_2 t_c^4 \right)^{1/s} - \frac{b_3}{s} t_c^6 = 0 \quad (s = -1.024) , \] (38)

which wins over the “trivial solution” of \( 1 + b_1 t^2 + b_2 t^4 = 0 \) at 1.592. Note that at order 2, from the condition \( F_2^* (t_c, s) = 0 \), we could only find the trivial solution \( t_c^2 = 1/b_1 \). It is only at higher order, starting with the order 3 discussed here, that we get corrections to the “mean-field” approximation.
To test the validity of the expansion and the strength of the corrections to the mean-field approximation, let us represent $s$ as $s = -1 + X$, substitute it into equation $D_2(s) = 0$, and expand in powers of $X$, thus assuming that $X$ is small compared to $-1$. Keeping only terms linear in $X$, $X$ can be expressed as follows:

$$X = -\frac{b_3}{b_1 b_2} \frac{1}{1 - \ln \left( \frac{b_3}{b_2^2} \right)} ,$$

(39)

Such an expansion is justified only when $\frac{b_3}{b_1 b_2} \ll 1$ and $1 - \ln \left( \frac{b_3}{b_2^2} \right)$ is significantly different from zero. In our case, $\frac{b_3}{b_1 b_2} = 0.067, 1 - \ln \left( \frac{b_3}{b_2^2} \right) = 2.792$ and thus $X = -0.024$ is small as expected.

Similarly, let us represent $t_c^2$ as $t_c^2 = -1/b_1 + C$, substitute it into the equation $F_3^*(t_c, s) = 0$, and expand in powers of $C$, thus assuming that $C$ is small compared to $-1/b_1$. Keeping only the terms linear in $C$, we get $C = 0.458$, which leads to the value of the critical point $t_c = 1.568$ (close to 1.5722). The expression for $C$ has a simpler structure at the mean-field point $s = -1$:

$$C = -\frac{1}{b_1} \left( \frac{b_3}{b_1 b_2} - 1 \right) \frac{1}{2 - 3 \frac{b_1}{b_2} - \frac{b_3}{b_2}} ,$$

(40)

with an estimate for the critical time given by

$$t_c \approx \sqrt{-\frac{1}{b_1}} \sqrt{1 + \left( \frac{b_3}{b_1 b_2} - 1 \right) \frac{1}{2 - 3 \frac{b_1}{b_2} - \frac{b_3}{b_2}}} = 1.563 .$$

(41)

Thus, the renormalization scheme used to calculate $t_c$ and $K$ corresponds to an expansion around the mean field value $t_c = 1/b_1$ and $K = 1$, in powers of two dimensionless parameters $\frac{b_3}{b_1 b_2}$ and $\frac{b_3}{b_2}$.

In summary, we get the predictions $t_c = 1.5722$ very close to the exact value $\pi/2 = 1.5708$ and $K = 1.024$ only 2% off the exact value 1.

We conclude that the applicability of the scheme presented in this section crucially depends on existence of typical small parameters, while the original scheme of [38] does not have such a dependence and can be applied even if critical indices deviate strongly from the mean-field values.

5 Synthetic tests in the presence of noise

Figure 4 shows one realization of a synthetic noisy data obtained from (3) with a multiplicative noise of variance $10^{-2}$. This noisy data simulates an experiment recording a signal as a function of time or of increasing strain or stress. Our goal is to use this noisy time series up to a maximum value away from $t_c$ to guess using the functional renormalization method what is the critical value $t_c$ of divergence (theoretically equal to $\pi/2 = 1.57...$).

Figure 5 shows the inverse of the function in figure 4, as well as four other realizations, which are used in the functional renormalization scheme developed in the previous sections to predict $t_c$. The five different symbols shown in figure 5 correspond each to a single time series for a specific noise realization. The spread around the theoretical $\cos t$ formula gives a sense of the amplitude of the multiplicative noise.

Figure 6 compares the prediction skill of our procedure described in section 4.2 to that from a direct fit with a power law. Specifically, we plot the predicted value for the critical time $t_c$ as a function of the distance to $t_c$, obtained with the two schemes. In this goal, we generated 1,000 synthetic data sets using expression (3) and modifying it with a multiplicative noise of variance $10^{-3}$ corresponding to a standard deviation of 3.3%, i.e., with the equation $f_i(t) = \frac{1}{\cos(1 + \eta)}$, where $\eta$ is a Gaussian white noise with variance $10^{-3}$. To generate curve a) in figure 6, each data set was fitted by the power law equation $A(t_c - t)^{-\beta}$, with $\beta = 1$ fixed and $A$ and $t_c$ determined from the fit in the time interval $[0.5; T_{\text{last point}}]$. The thick line is the average $t_c$ taken over the 1,000 realizations and the two thin lines are the $\pm$ one standard deviations. The curve b) in figure 6 is the predicted $t_c$ obtained from our resummation technique given in section 4.2 which assumes that the exponent
β is close to 1, using the coefficients \( b_1 \) and \( b_2 \) of the fit with the expansion \( 1 + b_1 t^2 + b_2 t^4 \) to the inverse of each of the synthetic data set \( f_i(t) \). The thick line is the average predicted \( t_c \) and the two thin lines are the ± one standard deviations. The horizontal line at \( t_c = \pi/2 \) is the exact theoretical value for the critical time. Up to very close to the critical time, our resummation method is clearly superior to the power law fit, even when knowing a priori the value of the exponent.

Figure 5 compares the prediction skill of the Yukalov-Gluzman method used in section 4.1 to that of a direct power law fit. Again, 1,000 synthetic time series were generated with multiplicative noise of variance \( 10^{-3} \). The curve a) of figure 5 is obtained by using the general resummation method of section 4.1 which does not assume any specific value of the exponent \( \beta \). We first fit the inverse of each synthetic data set with a parabolic expression \( 1 + b_1 t^2 + b_2 t^4 \) and use these coefficients \( b_1 \) and \( b_2 \) to obtain our prediction \( t_c \). The two thin lines are the ± one standard deviations. The curve b) in figure 5 is obtained by fitting each data set by the power law equation \( A(t_c - t)^{-\beta} \), where \( A, \beta \) and \( t_c \) are all three free parameters determined from the fit in the time interval \([0.5; T_{\text{lastpoint}}]\). The thick line is the average \( t_c \) taken over the 1,000 realizations and the two thin lines are the ± one standard deviations. The horizontal line at \( t_c = \pi/2 \) is the exact theoretical value for the critical time. Note the striking superiority of our resummation method over a direct power law fit.

Figure 6 is the same as figure 4 but with a larger multiplicative noise of variance \( 10^{-2} \). Figure 7 is the same as figure 5 but with a larger multiplicative noise of variance \( 10^{-2} \).

6 Concluding remarks

We have tested two methods for the prediction of the critical time of a singular power law behavior and tested their prediction skills against the direct determination by a power law fit. Our analysis and the numerical tests convincingly demonstrate the value of our approach which provides significant improved forecasting skills. Our tests have however been restricted to an important from a physical viewpoint but still special case of a function which admits an expansion with only even powers of \( t \). The most general situation contains also odd powers of \( t \) which complicates the situation. We intend to report progress in this general case in a forthcoming publication.

Finally, we wish to comment upon a conceptual understanding of the role of the exponent \( s \) used as control functions in the functional renormalization schemes used here. Yukalov and Gluzman [38] first noticed that their functional renormalization frameworks allowed them to propose a novel physical understanding of critical exponents as being directly related to limits of control functions at the critical point. In other words, they appear as physical analogs of the rather abstract mathematical objects given by the control functions. A scale invariant formulation using logarithmic variables allows us to understand this rather surprising observation: critical exponents can be seen to be determined by the initial conditions of an operator of the group of the symmetry of scale invariance [43]. Indeed, a power law function \( \phi(r) \) of the distance \( r \equiv t_c - t \) to the critical point, has the property of scale invariance which reads

\[
\phi(\lambda r) = \lambda^\alpha \phi(r) ,
\]

where \( \lambda \) is an arbitrary magnification factor and \( \alpha \) is a critical exponent. This equation means that the field \( \phi(r) \) is invariant under the homothetical transformation \( r \rightarrow \lambda r \); \( \phi \rightarrow \lambda^{-\alpha} \phi \). Expression \((42)\) can be transformed by picking up an arbitrary reference field \( \phi_0 \) and an arbitrary reference parameter \( r_0 \) and introducing the log-variables \( U = \ln(\phi/\phi_0) \), \( s = r/r_0 \), \( U(0) = \ln (\phi(r_0)/\phi_0) \), \( U(s) - U(0) = \alpha s \). In these variables, equation \((42)\) can also be written:

\[
U(s) = U(s + \mu) + g(\mu) , \quad g(\mu) = U(0) - U(\mu) , \quad \text{for any } \mu .
\]

Reciprocally, any regular function obeying \((43)\) is necessarily of the shape \( U(s) - U(0) = \alpha s \), where

\[
\alpha = U(1) - U(0)
\]

is an arbitrary parameter selected by the initial conditions!
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Figure 1: Relative error $[f_{\text{exact}}(t) - f_i^*(t)]/f_{\text{exact}}(t)$, for the three approximations formulas (8) for $f_1^*(t)$, (10) for $f_2^*(t)$, and (13) for $f_3^*(t)$, where $f_{\text{exact}}(t)$ is given by (3).
Figure 2: Noisy data obtained from (3) with a multiplicative noise of variance $10^{-2}$. The goal is to use this noisy time series up to a maximum value away from $t_c$ to guess what is the critical value $t_c$ of divergence (theoretically equal to $\pi/2 = 1.57...$).
Figure 3: $1/f$ over the function shown in figure 2 as well as four other realizations, which are used in the functional renormalization scheme developed here to predict $t_c$. The five different symbols shown in figure 3 correspond each to a single time series for a specific noise realization. The spread around the theoretical $\cos t$ formula gives a sense of the amplitude of the multiplicative noise.
Figure 3: \( T_c - T_{\text{lastpoint}} \)

Figure 4: This figure shows the predicted value for the critical time \( t_c \) as a function of the distance to \( t_c \), obtained with different schemes, in order to evaluate the value of the resummation procedures proposed here. In this goal, 1,000 synthetic data sets were generated using expression (3) and modifying it with a multiplicative noise of variance \( 10^{-3} \) corresponding to a standard deviation of 3.3%, i.e., with the equation \( f_i(t) = \frac{1}{\cos t} [1 + \eta] \), where \( \eta \) is a Gaussian white noise with variance \( 10^{-3} \). To generate curve a), each data set was fitted by the power law equation \( A(t_c - t)^{-\beta} \), with \( \beta = 1 \) fixed and \( A \) and \( t_c \) determined from the fit in the time interval \([0.5; T_{\text{lastpoint}}]\). The thick line is the average \( t_c \) taken over the 1,000 realizations and the two thin lines are the \( \pm \) one standard deviations. The curve b) is the predicted \( t_c \) obtained from our resummation technique given in section 4.2 which assumes that the exponent \( \beta \) is close to 1, using the coefficients \( b_1 \) and \( b_2 \) of the fit with the expansion \( 1 + b_1 t^2 + b_2 t^4 \) to the inverse of each of the synthetic data set \( f_i(t) \). The thick line is the average predicted \( t_c \) and the two thin lines are the \( \pm \) one standard deviations. The horizontal line at \( t_c = \pi/2 \) is the exact theoretical value for the critical time. Up to very close to the critical time, our resummation method is clearly superior to the power law fit, even when knowing a priori the value of the exponent.
Figure 4: $T_c - T_{\text{lastpoint}}$

Figure 5: Same as figure 4 with 1,000 synthetic time series with multiplicative noise of variance $10^{-3}$, and with different prediction schemes. The curve a) is obtained by using the general resummation method of section 4.1 which does not assume any specific value of the exponent $\beta$. We first fit the inverse of each synthetic data set with a parabolic expression $1 + b_1 t^2 + b_2 t^4$ and use these coefficients $b_1$ and $b_2$ to obtain our prediction $t_c$. The two thin lines are the ± one standard deviations. The curve b) is obtained by fitting each data set by the power law equation $A (t_c - t)^{-\beta}$, where $A$, $\beta$ and $t_c$ are all three free parameters determined from the fit in the time interval $[0.5; T_{\text{lastpoint}}]$. The thick line is the average $t_c$ taken over the 1,000 realizations and the two thin lines are the ± one standard deviations. The horizontal line at $t_c = \pi/2$ is the exact theoretical value for the critical time. Note the striking superiority of our resummation method over a direct power law fit.
Figure 5: $T_c - T_{\text{lastpoint}}$

Figure 6: Same as figure 4 but with a multiplicative noise of variance $10^{-2}$. 
Figure 6: $T_c - T_{\text{lastpoint}}$

Figure 7: Same as figure 5 but with a multiplicative noise of variance $10^{-2}$. 