On 1d Quadratic Klein–Gordon Equations with a Potential and Symmetries

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Abstract

This paper is a continuation of the previous work (Pusateri, in: Forum of mathematics, Cambridge University Press) by the first two authors. We focus on 1-dimensional quadratic Klein–Gordon equations with a potential, under some assumptions that are less general than (Pusateri, in: Forum of mathematics, Cambridge University Press), but that allow us to present some simplifications in the proof of the global existence with decay for small solutions. In particular, we can propagate a stronger control on a basic $L^2$-weighted-type norm while providing some shorter and less technical proofs for some of the arguments.

1. Introduction

1.1. Assumptions and main theorem

We consider the equation

$$(\partial_t^2 - \partial_x^2 + V + 1)u = a(x)u^2 + b(x)u^3$$

(1.1)

under the assumptions that the potential $V$ is even (this assumption can be dropped in the case of generic $V$) and rapidly decaying together with its derivatives\(^1\)

$$|\partial_x^\alpha V(x)| \lesssim_{\alpha,N} \langle x \rangle^{-N}, \quad \text{for all } \alpha,N,$$

\(^1\) Smoothness and decay assumptions are stated like this for convenience, but a finite amount of regularity and algebraic decay is sufficient.
and that the functions $a$ and $b$ have the following limits at $\pm \infty$ which they reach rapidly

$$
|\partial_x^\alpha (a(x) - a_{\pm \infty})| \lesssim_{\alpha, N} \langle x \rangle^{-N}, \quad |\partial_x^\alpha (b(x) - b_{\pm \infty})| \lesssim_{\alpha, N} \langle x \rangle^{-N},
$$

for $\pm x > 0$. \hfill (1.2)

We assume that $b$ is even and will specify additional parity assumptions on $a$ below.\footnote{These parity assumptions can be omitted for generic potentials.}

Furthermore, the Schrödinger operator

$$
H = -\partial_x^2 + V
$$

is assumed to have no eigenvalues, and to satisfy the assumption below.

**Assumption 1.1.** One of the three conditions below is satisfied

- The potential $V$ is generic.
- The potential $V$ is exceptional, with an even zero energy resonance, the data and $a(x)$ are odd.
- The potential $V$ is exceptional, with an odd zero energy resonance, the data and $a(x)$ are even.

Readers can refer to Definition 2.1 and (2.1)–(2.2) for the definition of generic and exceptional (or non-generic) potentials.

Finally, we provide the equation with data at time zero:

$$
u(t = 0) = u_0, \quad \partial_t u(t = 0) = u_1.
$$

Our main result is the following:

**Theorem 1.2.** Under the above assumptions, there exists $\varepsilon_0 > 0$ such that, if $(u_0, u_1)$ satisfy

$$
\left\|(\sqrt{H} + 1)u_0, u_1\right\|_{H^4} + \left\langle x \right\rangle \left\|(\sqrt{H} + 1)u_0, u_1\right\|_{H^1} = \varepsilon < \varepsilon_0,
$$

then there exists a unique global solution of (1.1) which decays pointwise and is globally bounded in $L^2$-type spaces: for any time $t$,

$$
\left\|(\sqrt{H} + 1)u(t), \partial_t u(t)\right\|_{L^\infty} \lesssim \varepsilon(t)^{-1/2},
$$

$$
\|u(t)\|_{H^5} + \|\partial_t u(t)\|_{H^4} \lesssim \varepsilon(t)^{p_0},
$$

where $p_0$ is a small number.\footnote{$p_0$ can be chosen of the form $C\varepsilon^2$ for some absolute constant $C > 0$, this constraint arising from the Sobolev energy estimate.}
In particular, our result gives a simpler proof of stability for families of kinks of the double Sine-Gordon equation under odd perturbations, see the discussion in Sect. 1.4.3 in [5]. The proof also gives a more precise description of the asymptotic behavior of $u$, which undergoes a type of modified scattering. To be more specific, the distorted Fourier transform [defined in Sect. 2] of a suitably renormalized profile $f$ [defined in Sect. 5, see (5.6)] satisfies the following: there exists an asymptotic profile $W^\infty = (W^+_{\infty}, W^-_{\infty}) \in \left( (\xi)^{-3/2} L^\infty_\xi \right)^2$ such that, for $\xi > 0$,

\[(\tilde{f}(t, \xi), \tilde{f}(t, -\xi)) = S^{-1}(\xi) \exp \left( -\frac{5i}{12} \text{diag}(\ell^2_{+\infty}|W^\infty_+(\xi)|^2, \ell^2_{-\infty}|W^\infty_-(\xi)|^2) \log t \right) W^\infty(\xi) + O(\varepsilon^2_0(t)^{-\delta_0}) \]

(1.3)
as $t \to \infty$, for some $\delta_0 > 0$, where $S(\xi)$ is the scattering matrix (see Sect. 2 for its definition) and $\ell_{\pm\infty}$ can be determined from $a_{\pm\infty}$ and $b_{\pm\infty}$ (we have $\ell_{\pm\infty} = a_{\pm\infty}$ when $b_{\pm\infty} = 0$).

1.2. Relation to [5]

Denoting $\tilde{f}$ for the distorted Fourier transform associated to $H$ of a function $f$, the assumptions on $H$ made in the previous subsection ensure that $\tilde{u}(0) = 0$ for $u$ solution of (1.1); this was the key hypothesis in the main theorem of [5] on global solutions and asymptotics for (1.1). However, the assumptions made Theorem 1.2 imply more, namely, the generalized eigenfunctions associated to $H$ also vanish at zero frequency, as can be seen from (2.24) to (2.28). This additional cancellation is responsible for stronger local decay (see Sect. 3), or for the vanishing of the quadratic spectral distribution at zero frequency (see Sect. 4). These effects are essentially equivalent; the former provides a more direct physical intuition, while the latter is the main tool which allows to simplify the proof.

In the more general framework of [5], these additional cancellations at zero frequency are absent, and this causes the formation of a singularity in frequency space: the $L^2$ norm of $\partial_\xi \tilde{f}(t, \xi)$ diverges rapidly as $t \to \infty$ close to the frequencies $\pm \sqrt{3}$, due to the resonant interaction $(0, 0) \to \pm \sqrt{3}$ (i.e., when the frequency zero interacts with itself to give the frequency $\pm \sqrt{3}$).

The relevance of the weaker assumptions made in [5] comes from the fact that singularities in frequency space are a very general phenomenon for many classes of 1d problems. In particular, these singularities are expected to appear when the linearized operator has a resonance at the bottom of the continuous spectrum (e.g., the linearized operators for the kinks of the $\phi^4$ and Sine-Gordon models) or a so-called internal mode (e.g., the linearized operator for the kink of the $\phi^4$ model); see the discussion in Sect. 1 of [5] for more on these aspects.

This type of singularity in frequency space requires a special functional framework, like the one of [5], which makes the nonlinear estimates of $L^2$ weighted-type norms technical and lengthy. One of the goals of the present paper is to show how some of these estimates can be simplified in some more specific cases. At the
same time, we can also obtain better bounds by propagating a stronger, and more standard, $L^2$ weighted-type norm.

1.3. Background

Equation (1.1) is motivated by the desire to understand the stability properties of topological solitons (kinks) in equations of the type

$$\partial^2_t \phi - \partial^2_x \phi = W''(\phi),$$

(1.4)

where $W$ is a double-well potential. A famous example is the kink $\tanh(x/\sqrt{2})$ in the $\phi^4$ model

$$\partial^2_t \phi - \partial^2_x \phi = \phi - \phi^3.$$  

(1.5)

Local asymptotic stability for (1.5) and many interesting models of the form (1.4) were established recently by Kowalczyk, Martel and Muñoz [6], see also [7, 8], but some important questions remain to be understood, such as: What is the global-in-space behavior of solutions? What is the rate of decay of perturbations of solitons on long-time intervals? Can one establish global asymptotic stability?

The work of Delort and Masmoudi [3] gives a description of odd perturbations of the kink of (1.5) globally in space, up to times of the order $\epsilon^{-4}$, where $\epsilon$ is the size of the perturbation. This limitation is due to a singularity at a frequency related to that of the internal mode. Lindblad, Soffer and Lührmann [10], later joined by Schlag [9], showed that, for non-generic $V$, solutions of (1.1) with $b = a_{\pm \infty} = 0$ exhibit a logarithmic loss in the pointwise decay, because of the aforementioned singularity at the frequencies $\pm \sqrt{3}$.

We also mention [1, 4], which were earlier attempts to understand nonlinear resonances in the framework of the distorted Fourier transform, and the recent work of Lührmann and Schlag [11] on the stability of the Sine-Gordon kink under odd perturbations, and references therein. All the articles mentioned above only correspond to very recent developments, and we refer to the introduction of [5] and [11] for a more complete overview.

Notation

Most of the necessary notation will be introduced in due course. In Sects. 6 and 8, we are going to use the following standard notation for cutoffs: we fix a smooth even cutoff function $\phi : \mathbb{R} \to [0, 1]$ supported in $[-8/5, 8/5]$ and equal to 1 on $[-5/4, 5/4]$. For $k \in \mathbb{Z}$, we define $\phi_k(\xi) := \phi(2^{-k} \xi) - \phi(2^{-k+1} \xi)$, so that the family $(\phi_k)_{k \in \mathbb{Z}}$ forms a partition of unity, $\sum_{k \in \mathbb{Z}} \phi_k(\xi) = 1$ for $\xi \neq 0$. Let

$$\varphi_1(\xi) := \sum_{k \in 1 \cap \mathbb{Z}} \varphi_k(\xi), \quad \varphi_{\leq a}(\xi) := \varphi_{(-\infty, a]}(\xi), \quad \varphi_{> a}(\xi) := \varphi_{(a, \infty)}(\xi), \quad (1.6)$$

with similar definitions for $\varphi_{\geq a}, \varphi_{< a}$. We will also denote $\varphi_{\sim k}$ a generic smooth cutoff function that is supported around $|\xi| \sim 2^k$, e.g., $\varphi_{[k-2,k+2]}$ or $\varphi_k'$. 

We will denote by $T$ a positive time, and always work on an interval $[0, T]$ for our bootstrap estimates; see Proposition 5.3. To decompose the time integrals, such as those appearing in (5.26), for any $t \in [0, T]$, we will use a suitable decomposition of the indicator function $1_{[0,t]}$ by fixing functions $\tau_0, \tau_1, \ldots, \tau_{L+1} : \mathbb{R} \to [0, 1]$, for an integer $L$ with $|L - \log_2(t + 2)| < 2$, with the properties that

$$\sum_{n=0}^{L+1} \tau_n(s) = 1_{[0,t]}, \quad \text{supp} (\tau_0) \subset [0, 2], \quad \text{supp} (\tau_{L+1}) \subset [t/4, t],$$

and $\text{supp} (\tau_n) \subset [2^{n-1}, 2^{n+1}]$, $|\tau_n'(t)| \lesssim 2^{-n}$, for $n = 1, \ldots, L$.

(1.7)

1.4. Organization of the Article

The emphasis in the present article is on the parts of the argument which can be simplified compared to [5]; for estimates which are unchanged or that can be immediately adapted in the present setting, we simply refer to the corresponding statements in [5].

A quick overview of the spectral theory of one-dimensional Schrödinger operators, and their associated distorted Fourier transform, is given in Sect. 2, followed by a discussion of estimates for the linear Klein–Gordon equation in Sect. 3. Structure theorems for the quadratic spectral distribution are stated in Sect. 4. For these three sections, the main novelty occurs in the case of even potentials with odd or even resonances, where our formulation is new, and allows to see more clearly the cancellation at frequency zero.

Section 5 recapitulates the normal-form transformation, and the ensuing decomposition of nonlinear terms, referring to [5] for further details. From there, the bootstrap argument is laid out in Sect. 5.4: Proposition 5.3 is the heart of the proof of Theorem 1.2, and its proof occupies the remaining sections of the paper.

In the proof of Proposition 5.3, the main difficulty is to estimate the $L^2$-norm of $\partial_\xi \tilde{f}$, and this is where the main simplifications occur. The norm that we can propagate is stronger than the one in [5], and the proofs are shorter and simpler than those in Sects. 8 and 9 of [5]. The (regular) quadratic term is treated in Sect. 6, and the (singular) cubic term is treated in Sect. 7. In the final Sect. 8, we discuss how to estimate remainder terms and the other norms appearing in the bootstrap.

2. Spectral Theory in Dimension One

This section starts with a quick review of the scattering and spectral theory of one-dimensional Schrödinger operators; the reader is referred to [5] (see also references therein) for a more thorough presentation. Some helpful formulas in the case of even or odd function are then established.
2.1. Linear Scattering Theory

Define \( f_+ (x, \xi) \) and \( f_- (x, \xi) \) by the requirements that

\[
H f_\pm := (-\partial_x^2 + V) f_\pm = \xi^2 f_\pm, \quad \text{for all } x \in \mathbb{R}, \quad \text{and} \quad \begin{cases} f_+ (x, \xi) &\sim e^{ix\xi} \quad \text{as } x \to \infty, \\ f_- (x, \xi) &\sim e^{-ix\xi} \quad \text{as } x \to - \infty. \end{cases} \tag{2.1}
\]

Define further that

\[
m_+ (x, \xi) = e^{-i\xi x} f_+(x, \xi) \quad \text{and} \quad m_- (x, \xi) = e^{i\xi x} f_-(x, \xi), \tag{2.2}
\]

so that \( m_\pm \) is a solution of

\[
\partial_x^2 m_\pm + 2i\xi \partial_x m_\pm = Vm_\pm, \quad m_\pm (x, \xi) \to 1 \quad \text{as} \quad x \to \pm \infty. \tag{2.3}
\]

These satisfy the estimates

\[
\left| \partial^\alpha \xi^\beta (m_\pm (x, \xi) - 1) \right| \lesssim \langle x \rangle^{-N} \langle \xi \rangle^{-1-\beta} \quad \text{if} \quad \pm x > -1,
\]

\[
\left| \partial^\alpha \xi^\beta m_\pm (x, \xi) \right| \lesssim \langle x \rangle^{1+\beta} \langle \xi \rangle^{-1-\beta} \quad \text{if} \quad \pm x < 1. \tag{2.4}
\]

Denote \( T(\xi) \) and \( R_{\pm}(\xi) \), respectively, the transmission and reflection coefficients associated to the potential \( V \). These coefficients are such that

\[
\begin{align*}
f_+(x, \xi) &= \frac{1}{T(\xi)} f_-(x, -\xi) + \frac{R_-(\xi)}{T(\xi)} f_-(x, \xi), \\
f_-(x, \xi) &= \frac{1}{T(\xi)} f_+(x, -\xi) + \frac{R_+(\xi)}{T(\xi)} f_+(x, \xi), \tag{2.5}
\end{align*}
\]

and are explicitly given by the formulas

\[
T(\xi) = \frac{2i\xi}{2i\xi - \int V(x) m_\pm (x, \xi) \, dx},
\]

\[
R_{\pm}(\xi) = \frac{\int e^{\mp 2i\xi x} V(x) m_\pm (x, \xi) \, dx}{2i\xi - \int V(x) m_\pm (x, \xi) \, dx}. \tag{2.6}
\]

These formulas are only valid for \( \xi \neq 0 \) a priori, but \( T \) and \( R_{\pm} \) can be extended to be smooth functions on the whole real line. From (2.6) to (2.4), we can see that

\[
|\partial^\beta \xi (T(\xi) - 1)| \lesssim \langle \xi \rangle^{-1-\beta}, \quad |\partial^\beta \xi R_{\pm}(\xi)| \lesssim \langle \xi \rangle^{-N}. \tag{2.7}
\]

Finally, the scattering matrix

\[
S(\xi) = \begin{pmatrix} T(\xi) & R_+(\xi) \\ R_-(\xi) & T(\xi) \end{pmatrix}
\]

is unitary.
2.2. Exceptional and Generic Potentials

Here are some useful lemmas about the behavior of the transmission and reflection coefficients. Readers can refer to Deift–Zhou [2] for details. Recall the definition of the Wronskian

\[ W(f, g) = f'g - fg'. \]

Definition 2.1. We call the potential \( V \)

- **Generic** if \( \int V(x)m_{\pm}(x, 0)dx \neq 0; \)
- **Exceptional** if \( \int V(x)m_{\pm}(x, 0)dx = 0. \)

Lemma 2.2. The four following assertions are equivalent:

(i) \( V \) is generic.
(ii) \( T(0) = 0, \ R_{\pm}(0) = -1. \)
(iii) The Wronskian of \( f_+ \) and \( f_- \) at \( \xi = 0 \) is non-zero.
(iv) The potential \( V \) does not have a resonance at \( \xi = 0 \), in other words there does not exist a bounded non-trivial solution in the kernel of \( -\partial_x^2 + V. \)

Proposition 2.3. (Low energy scattering) If \( V \) is generic, there exists \( \alpha \in i\mathbb{R} \) such that

\[ T(\xi) = \alpha \xi + O(\xi^2). \]  

If \( V \) is exceptional, let

\[ a := f_+(-\infty, 0) \in \mathbb{R} \setminus \{0\}. \]

Then,

\[ T(0) = \frac{2a}{1 + a^2}, \quad R_+(0) = \frac{1 - a^2}{1 + a^2}, \quad \text{and} \quad R_-(0) = \frac{a^2 - 1}{1 + a^2}. \]  

Lemma 2.4. Assume that \( V \) is even and exceptional, with an even zero energy resonance. Then

\[ \langle \xi \rangle^{1 + \beta} \left| \partial_\xi^\beta \frac{T(\xi) - 1}{\xi} \right| + \langle \xi \rangle^N \left| \partial_\xi^\beta R_{\pm}(\xi) \right| \lesssim 1. \]  

Similarly, if the zero energy resonance is odd we have that

\[ \langle \xi \rangle^{1 + \beta} \left| \partial_\xi^\beta \frac{T(\xi) + 1}{\xi} \right| + \langle \xi \rangle^N \left| \partial_\xi^\beta R_{\pm}(\xi) \right| \lesssim 1. \]

Proof. These properties can be verified starting from the formulas (2.6) and making use of Proposition 2.3. From Proposition 2.3, we know that when the zero energy resonance \( f_+(x, 0) \) is even we have \( T(0) = 1 \) since \( a = 1 \) (due to the fact that \( f_+(+\infty, 0) = 1 \) and \( a = f_+(-\infty, 0) \)), so \( T(\xi) - 1 \) is of \( O(\xi) \), and the estimate for \( T(\xi) \) in (2.10) follows from the smoothness of \( T \) and the decay in (2.4). Similarly, for the case when the zero energy resonance is odd, we have \( T(0) = -1 \) since \( a = -1 \), and hence \( T(\xi) + 1 \) is of \( O(\xi) \) and (2.11) follows. The estimates for \( R_{\pm}(\xi) \) can be verified in a similar way using that \( R_{\pm}(0) = 0 \) when \( a = \pm 1. \) \( \square \)
2.3. Flat and Distorted Fourier Transform

The normalization we adopt for the flat Fourier transform is
\[ \hat{F}_\phi(\xi) = \hat{\phi}(\xi) = \frac{1}{\sqrt{2\pi}} \int e^{-i\xi \cdot x} \phi(x) \, dx; \]
its inverse is given by
\[ \hat{F}^{-1}_\phi = \frac{1}{\sqrt{2\pi}} \int e^{i\xi \cdot x} \hat{\phi}(\xi) \, d\xi. \]

We now define the wave functions associated to \( H \) as
\[ \psi(x, \xi) := \frac{1}{\sqrt{2\pi}} \left\{ T(\xi) f_+(x, \xi) \text{ for } \xi \geq 0 \right. \]
\[ \left. + T(-\xi) f_-(x, -\xi) \text{ for } \xi < 0 \right\}. \tag{2.12} \]

The distorted Fourier transform is then defined by
\[ \tilde{F}_\phi(\xi) = \tilde{\phi}(\xi) = \int_\mathbb{R} \psi(x, \xi) \phi(x) \, dx; \tag{2.13} \]
it is self-adjoint and has the inverse
\[ \tilde{F}^{-1}_\phi(x) = \int_\mathbb{R} \psi(x, \xi) \phi(\xi) \, d\xi. \tag{2.14} \]

Let \( \rho \) be an even, smooth, non-negative function, equal to 0 outside of \( B(0, 2) \) and such that \( \int \rho = 1 \). Define \( \chi_{\pm} \) by
\[ \chi_{+}(x) = H \ast \rho = \int_{-\infty}^{x} \rho(y) \, dy, \quad \text{and} \quad \chi_{+}(x) + \chi_{-}(x) = 1, \tag{2.15} \]
where \( H \) is the Heaviside function, \( H = 1_+ = 1_{[0,\infty)} \). Notice that
\[ \chi_{+}(x) = \chi_{-}(-x). \]

With \( \chi_{\pm} \) as above, and using the definition of \( \psi \) in (2.12) and \( f_{\pm} \) and \( m_{\pm} \) in (2.1)–(2.2), as well as the identity (2.5) we can write
for \( \xi > 0 \)
\[ \sqrt{2\pi} \psi(x, \xi) = \chi_{+}(x) T(\xi) m_+(x, \xi) e^{ix\xi} \]
\[ + \chi_{-}(x) \left[ m_-(x, -\xi) e^{ix\xi} + R_-(\xi) m_-(x, -\xi) e^{-i\xi x} \right], \tag{2.16} \]
and
for \( \xi < 0 \)
\[ \sqrt{2\pi} \psi(x, \xi) = \chi_{-}(x) T(-\xi) m_-(x, -\xi) e^{ix\xi} \]
\[ + \chi_{+}(x) \left[ m_+(x, \xi) e^{ix\xi} + R_+(\xi) m_+(x, -\xi) e^{-i\xi x} \right]. \tag{2.17} \]

Below we give some general properties of the distorted Fourier transform. The proofs of these statements can be found, for example, in Proposition 3.6 of [5], and more details can be found in [12].
Proposition 2.5. (Mapping properties of the distorted Fourier transform) With \( \widetilde{F} \) defined in (2.13),

(i) \( \widetilde{F} \) is a unitary operator from \( L^2 \) onto \( L^2 \). In particular, its inverse is

\[
\widetilde{F}^{-1}\phi(x) = \widetilde{F}^*\phi(x) = \int_\mathbb{R} \psi(x, \xi) \phi(\xi) \, d\xi.
\]

(ii) \( \widetilde{F} \) maps \( L^1(\mathbb{R}) \) to functions in \( L^\infty(\mathbb{R}) \) which are continuous at every point except 0, and converge to 0 at \( \pm\infty \).

(iii) \( \widetilde{F} \) maps the Sobolev space \( H^s(\mathbb{R}) \) onto the weighted space \( L^2(\langle \xi \rangle^{2s} \, d\xi) \).

(iv) If \( \tilde{f} \) is continuous at zero, then, for any integer \( s \geq 0 \)

\[
\| \langle \xi \rangle^s \partial_\xi \tilde{f} \|_{L^2} \lesssim \| f \|_{H^s} + \| \langle x \rangle f \|_{H^s}.
\]

2.4. The Wave Operator

The wave operator associated to \( H \) is defined as

\[
W = s - \lim_{t \to \infty} e^{itH} e^{-itH_0}.
\]

(2.18)

It has the following property, see Sect. 3.2.3 in [5]:

Proposition 2.6. The wave operator is unitary on \( L^2 \) and satisfies

\[
W = \widetilde{F}^{-1}\widetilde{F}, \quad W^{-1} = W^* = \widetilde{F}^{-1}\widetilde{F},
\]

(2.19)

and it intertwines \( H \) and \( H_0 \):

\[
f(H) = W f(H_0) W^*.
\]

(2.20)

Moreover, \( W \) and \( W^* \) extend to bounded operators on \( W^{k,p}(\mathbb{R}) \) for any \( k \) and \( 1 < p < \infty \). Furthermore, in the exceptional case, if \( f_+(-\infty, 0) = 1 \), this remains true if \( p = 1 \) or \( \infty \).

2.5. The Case of Even and Odd Functions

Lemma 2.7. (Parity preservation) Let the potential \( V \) be even. Then, we have

\[
f_+(x, \xi) = f_-(-x, \xi), \quad R_+(\xi) = R_-(\xi) \equiv R(\xi), \quad \psi(x, \xi) = \psi(-x, -\xi),
\]

(2.21)

and the distorted Fourier transform preserves evenness and oddness.

Proof. The first identity in (2.21) is a consequence of the uniqueness of solutions for the ODE (2.1), \( R_+ = R_- \) follows directly from (2.6) and \( m_+(x, \xi) = m_-(-x, \xi) \) [see (2.2)]. The identity for \( \psi \) follows from the previous ones and the definition (2.12). The preservation of parity for the distorted Fourier transform then follows from (2.13).

Lemma 2.8. (Formulas with parity) Let the potential \( V \) be even and exceptional.
When $f$ is odd, and the zero energy resonance is even, we have the formulas

$$\tilde{f}(\xi) = \int_{\mathbb{R}} \psi_o(\tilde{x}, \xi) f(x) \, dx,$$

$$f(x) = \int_{\mathbb{R}} \psi_o(x, \xi) \tilde{f}(\xi) \, d\xi,$$

(2.22)

where

$$\psi_o(x, \xi) := 1_+ (\xi) \chi_+ (x) K_o(x, \xi) + 1_- (\xi) \chi_- (x) K_o(-x, -\xi),$$

$$\sqrt{2\pi} K_o(x, \xi) := T(\xi) m_+(x, \xi) e^{ix\xi} - m_+(x, -\xi) e^{-ix\xi} - R(\xi) m_+(x, \xi) e^{i\xi x},$$

(2.23)

We also have that

$$K_o(x, 0) = 0,$$

(2.24)

and can write that

$$f(x) = \int_{0}^{\infty} \left[ \chi_+ (x) K_o(x, \xi) - \chi_- (x) K_o(-x, -\xi) \right] \tilde{f}(\xi) \, d\xi.$$

(2.25)

When $f$ is even, and the zero energy resonance is odd, we have the formulas

$$\tilde{f}(\xi) = \int_{\mathbb{R}^+} \psi_e(\tilde{x}, \xi) f(x) \, dx,$$

$$f(x) = \int_{\mathbb{R}^+} \psi_e(x, \xi) \tilde{f}(\xi) \, d\xi,$$

(2.26)

where

$$\psi_e(x, \xi) := 1_+ (\xi) \chi_+ (x) K_e(x, \xi) + 1_- (\xi) \chi_- (x) K_e(-x, -\xi),$$

$$\sqrt{2\pi} K_e(x, \xi) := T(\xi) m_+(x, \xi) e^{ix\xi} + m_+(x, -\xi) e^{-ix\xi} + R(\xi) m_+(x, \xi) e^{i\xi x},$$

(2.27)

We also have that

$$K_e(x, 0) = 0,$$

(2.28)

and can write that

$$f(x) = \int_{0}^{\infty} \left[ \chi_+ (x) K_e(x, \xi) + \chi_- (x) K_e(-x, -\xi) \right] \tilde{f}(\xi) \, d\xi.$$

(2.29)

Note that there is a slight asymmetry between the cutoffs in $x$ and $\xi$ since we are using smooth cutoffs $\chi_\pm$ in the $x$ variable.
Proof. The formulas (2.22)–(2.23) and (2.26)–(2.27) can be directly verified starting from (2.13) to using (2.17)–(2.17), as well as \( R_+ = R_- \). We show this in the case when \( f \) is odd. We have, for \( \xi > 0 \), and denoting \( \bar{z} = z^* \), that
\[
\tilde{f}(\xi) = \int_{\mathbb{R}} \left[ \chi_+(x) T(\xi) m_+(x, \xi) e^{ix\xi} + \chi_-(x) (m_-(x, -\xi)) e^{i\xi x} 
+ R(\xi)m_-(x, \xi) e^{-i\xi x} \right]^* f(x) \, dx 
= \int_{\mathbb{R}} \chi_+(x) \left[ T(\xi) m_+(x, \xi) e^{ix\xi} - m_-(x, -\xi) e^{-i\xi x} 
- R(\xi)m_-(x, \xi) e^{i\xi x} \right]^* f(x) \, dx 
= \int_{\mathbb{R}} \chi_+(x) K_0(\bar{x}, \xi) f(x) \, dx.
\]
Using also that \( m_-(x, -\xi) = m_+(x, \xi) \), and a similar identity for \( \xi < 0 \), gives us the first formula in (2.22). The property (2.24) follows from Proposition 2.3 since \( T(0) = 1 \) and \( R(0) = 0 \) under our parity assumptions (\( a = 1 \) when the resonance is even).

(2.26)–(2.28) follow similarly by explicit computations, and using again Proposition 2.3 to see that \( a = -1 \) when the resonance is odd and therefore \( T(0) = -1 \) and \( R(0) = 0 \).

\[\] 3. Linear Decay Estimates

Let
\[
B := \sqrt{-\partial_{xx} + V + 1}.
\]
We have the following standard decay estimate:

**Proposition 3.1.** (Pointwise decay) Assume that either \( V \) is generic, or that \( V \) is exceptional with an even, resp. odd, zero energy resonance and that \( f \) is odd, resp. even. Then we have that
\[
\| e^{itB} P_c f \|_{L^\infty} \lesssim (t)^{-1/2} \| \langle \xi \rangle^{3/2} \tilde{f} \|_{L^\infty} + (t)^{-11/20} \| \langle \xi \rangle \partial_\xi \tilde{f} \|_{L^2} + (t)^{-7/12} \| f \|_{H^4}.
\]  

(3.1)

This statement corresponds to Proposition 3.11 in [5], where the factor of \( (t)^{-11/20} \), which is not optimal, is replacing the factor of \( (t)^{-3/4+\beta\gamma} \) (where \( \beta, \gamma \) were chosen so that \( \beta\gamma = 1/4- \)); this latter was present in the cited work due to the “bad” frequencies \( \pm \sqrt{3} \) that required a special treatment in [5], which is not the case in the present article. We also refer the reader to Lemma 2.2 of [10] where a linear estimate similar to (3.2) is proved (for the case \( V = 0 \)); the slightly different rate of decay in front of the weighted \( L^2 \) norm is due to the different handling of the high frequencies.

Next, we provide an improved local decay estimate. It will not be used in this work, but we include it for the sake of completeness and future reference.
Such improved decay is one of the keys to nonlinear stability, but it will be more convenient for us to take another route to proving nonlinear bounds, working in (distorted) Fourier space, rather than using physical space estimates.

**Proposition 3.2.** (Local decay) Assume that either $V$ is generic, or that $V$ is exceptional with an even, resp. odd, zero energy resonance and that $f$ is odd, resp. even. Then we have that

$$\| \langle x \rangle^{-2} e^{itB} P_{c} f \|_{L^{\infty}} \lesssim \langle t \rangle^{-1} \left( \| \partial_{\xi} \tilde{f} \|_{L^{1}_{\xi}} + \| \tilde{f} \|_{L^{1}_{\xi}} + \| f \|_{H^{1}_{x}} \right).$$

(3.3)

**Remark 3.3.** One should notice that, in the more general case treated in [5], or in the exceptional case without the present symmetry assumptions, decay rates as strong as (3.3) are not available.

**Proof of Proposition 3.2.** We are first going to give the details of the proof in the case of an exceptional potential with even resonance and with $f$ odd, and we will then indicate how the other cases can be treated similarly.

Let us bound the first quantity on the left-hand side of (3.3). In view of Sobolev’s embedding, we may assume $t \geq 1$. We look at different cases depending on the parity.

The case of odd $f$. By (2.22)–(2.25), we can write

$$\left( e^{itB} f \right)(x) = \chi_{+}(x) \int_{\mathbb{R}^{+}} e^{it\langle \xi \rangle} K_{o}(x, \xi) \tilde{f}(\xi) \, d\xi - \chi_{-}(x) \int_{\mathbb{R}^{+}} e^{it\langle \xi \rangle} K_{o}(-x, \xi) \tilde{f}(\xi) \, d\xi.$$

We will only bound the first term above, since the other one can be treated identically. Observe that we can write (2.23) as

$$\sqrt{2\pi} K_{o}(x, \xi) = m_{+}(x, -\xi) 2i \sin(x\xi) + [m_{+}(x, \xi) - m_{+}(x, -\xi)] e^{ix\xi} + (T(\xi) - 1 - R(\xi)) m_{+}(x, \xi) e^{ix\xi}.$$

This expression makes the cancellations even more apparent, leading, with the help of (2.4) and (2.10), to the estimates

$$|K_{o}(x, \xi)| \lesssim \min(|x\xi|, 1) + \min(\xi, 1), \quad |\partial_{\xi} K_{o}(x, \xi)| \lesssim \langle x \rangle, \quad \langle x \rangle \leq 1,$$

valid in the support of $\chi_{+}(x)$. Using that $K_{o}(x, 0) = 0 = \tilde{f}(0)$, we can integrate by parts through the identity $e^{it\langle \xi \rangle} = \frac{\langle \xi \rangle}{it\xi} \partial_{\xi} e^{it\langle \xi \rangle}$ to obtain

$$\langle x \rangle \int_{\mathbb{R}^{+}} e^{it\langle \xi \rangle} K_{o}(x, \xi) \tilde{f}(\xi) \, d\xi \lesssim \frac{1}{t \langle x \rangle} \int |K_{o}(x, \xi)| \rho_{\xi} \tilde{f}(\xi) \, d\xi + \frac{1}{t \langle x \rangle} \int \tilde{f}(\xi) \rho_{\xi} \left[ \frac{\langle \xi \rangle}{\xi} K_{o}(x, \xi) \right] \, d\xi.$$
Using the bounds (3.4), this is
\[\lesssim \frac{1}{t(x)^2} \int \frac{|\xi|}{|\xi|} (\min(|x_\xi|, 1) + \min(\xi, 1))|\partial_\xi \tilde{f}(\xi)| \, d\xi + \frac{1}{t} \int |\tilde{f}(\xi)| \, d\xi \lesssim \frac{1}{t} \left[ \|\partial_\xi \tilde{f}\|_{L^1} + \|\tilde{f}\|_{L^1} \right].\]

The case of even \( f \). When \( f \) is even, we can use (2.26)–(2.29) to write that
\[e^{itB} f = \chi_+ (x) \int_{\mathbb{R}^+} e^{it(\xi)} K_e (x, \xi) \tilde{f}(\xi) \, d\xi + \chi_- (x) \int_{\mathbb{R}^+} e^{it(\xi)} K_e (-x, \xi) \tilde{f}(\xi) \, d\xi.\]

Similarly to the odd case, also here we have cancellation at zero (see (2.28)). It suffices again to just look at the first integral. Writing
\[\sqrt{2\pi} K_e (x, \xi) = -m_+(x, \xi) 2i \sin(x\xi) + [m_+(x, -\xi) - m_+(x, \xi)] e^{-ix\xi} + (T(\xi) + 1 + R(\xi)) m_+(x, \xi) e^{ix\xi},\]
we see that estimates like (3.4) hold for \( K_e \) as well, and we can apply the same argument above.

The case of generic \( V \). When \( V \) is generic, we have \( T(0) = 0 \) and \( R_\pm (0) = -1 \) and arguments similar to those above can be applied to the general formula (2.13) with (2.17)–(2.17).

\[\Box\]

4. The Quadratic Spectral Distribution

In this section, we analyze the quadratic (and cubic) spectral distribution and will mainly focus on the analysis that is relevant to the case of odd solutions with an even resonance. At the end of the section, we will also indicate how to analyze similarly the case of even solutions with odd resonances, and the case of generic \( V \). We denote, for \( f \) a complex-valued function,
\[f_+ = f, \quad f_- = \overline{f}.

The Odd Case

Recall that in the case of odd data (and solution) we assume that the zero energy resonance is even. Our starting point is the formula (2.22)–(2.25).

Proposition 4.1. (The odd case) Let \( f, g \in S \) be odd functions, and \( a = a(x) \) be odd and satisfying (1.2). There exists a distribution \( \mu_{\xi_1 \xi_2}^{o} \), \( \iota_1, \iota_2 \in \{+, -\} \), such that, for \( \xi \geq 0 \),
\[\tilde{F}(a f_1 g_2) (\xi) = \int \int_{(\mathbb{R}^+)^2} (\tilde{f})_{\iota_1} (\eta) (\tilde{g})_{\iota_2} (\sigma) \mu_{\xi_1 \xi_2}^{o} (\xi, \eta, \sigma) \, d\eta \, d\sigma, \quad (4.1)\]
with odd extension to \( \xi < 0 \), and such that \( \mu_{\xi_1 \xi_2}^{o} \) can be split into a singular and a regular part as
\[(2\pi) \mu_{\xi_1 \xi_2}^{o} = \mu_{\xi_1 \xi_2}^{o,S} + \mu_{\xi_1 \xi_2}^{o,R},\]
where
• The singular part $\mu_{i_{1i_{2}}}^{o,S}$ is given by

$$\mu_{i_{1i_{2}}}^{o,S}(\xi, \eta, \sigma) = \sum_{\lambda, \mu, \nu \in \{+,-\}} \bar{a}_{\lambda}(\xi)(a_{\mu}(\eta))_{i_{1}}(a_{\nu}(\sigma))_{i_{2}} \times \ell_{+\infty} \left[ \sqrt{\frac{\pi}{2}} \delta(p) + i \varphi^{*}(p, \eta, \sigma) \text{p.v.} \frac{\hat{\phi}(p)}{p} \right]_{p=\lambda \xi - \mu \eta - \nu \sigma}$$

(4.2)

where the coefficients are given as

$$a_{+}(\xi) = T(\xi) - R(\xi), \quad a_{-}(\xi) = -1,$$

(4.3)

$\phi \in S$ is even with integral one, and

$$\varphi^{*}(p, \eta, \sigma) = \varphi_{\leq -D}(p R(\eta, \sigma)), \quad R(\eta, \sigma) = \frac{\langle \eta \rangle}{\langle \eta \rangle} + \langle \sigma \rangle.$$

(4.4)

• The regular part $\mu_{i_{1i_{2}}}^{o,R}$ satisfies, for $\xi, \eta, \sigma > 0$

$$\mu_{i_{1i_{2}}}^{o,R}(\xi, \eta, \sigma) = \frac{\xi \cdot \eta \cdot \sigma}{\langle \xi \rangle \langle \eta \rangle \langle \sigma \rangle} q_{i_{1i_{2}}}^{o}(\xi, \eta, \sigma)$$

(4.5)

where

$$|\partial_{\xi}^{a} \partial_{\eta}^{b} \partial_{\sigma}^{c} q_{i_{1i_{2}}}^{o}(\xi, \eta, \sigma)| \lesssim \sup_{\mu, \nu} \frac{1}{\langle \xi \rangle + \langle \mu \eta + \nu \sigma \rangle^{N}} |R(\eta, \sigma)|^{1+a+b+c},$$

(4.6)

$$|a| + |b| + |c| \leq N.$$

Remark 4.2. (About the formula (4.2)) It is important to note that the formula (4.2) gives a measure which vanishes when one of the three frequencies $\xi, \eta$ or $\sigma$ is zero. To see this when $\eta = 0$, say, we can fix $\iota_{1} = \iota_{2} = +$ since they are irrelevant, and recall that $a_{+}(0) - 1 = 0$, so that $a_{+}(0) = \mu$ and we have that

$$\mu_{++}^{o,S}(\xi, 0, \sigma) = \sum_{\mu} \sum_{\lambda, \nu} a_{\lambda}(\xi) \cdot a_{\nu}(\sigma) \times \ell_{+\infty} \left[ \sqrt{\frac{\pi}{2}} \delta(p) + i \varphi^{*}(p R(0, \sigma)) \text{p.v.} \frac{\hat{\phi}(p)}{p} \right]_{p=\lambda \xi - \nu \sigma} = 0$$

(4.7)

The same argument works symmetrically when $\sigma = 0$, and similarly when $\xi = 0$.

Proof of Proposition 4.1. In what follows we let $\iota_{1} = \iota_{2} = +$ for simplicity and also omit the superscripts $o$ from the formulas, so that, for example, $\mu$ will stand for $\mu_{i_{1i_{2}}}^{o}$. Since $afg(x)$ is odd, from the formulas (2.22) and (2.25), we formally have (4.1) by defining, for $\xi > 0$, that

$$\mu(\xi, \eta, \sigma) = \int_{\mathbb{R}} a(x) \chi_{+}(x) K_{o}(x, \xi) \left[ \chi_{+}(x) K_{o}(x, \eta) - \chi_{-}(x) K_{o}(-x, \eta) \right] \times \left[ \chi_{+}(x) K_{o}(x, \sigma) - \chi_{-}(x) K_{o}(-x, \sigma) \right] dx,$$

(4.8)
and for $\xi < 0$, that

$$
\mu(\xi, \eta, \sigma) = \int_{\mathbb{R}} a(x)\chi_-(x)K_0(-x, -\xi) \left[ \chi_+(x)K_0(x, \eta) - \chi_-(x)K_0(-x, \eta) \right] \\
\times \left[ \chi_+(x)K_0(x, \sigma) - \chi_-(x)K_0(-x, \sigma) \right] dx
= -\mu(-\xi, \eta, \sigma).
$$

(4.9)

Note that the limit at $\xi = 0$ exists and is zero. The limit as $\eta, \sigma \to 0^+$ is also zero.

Recall the formula (2.23) and decompose $K_0(x, \xi)$ into a singular and regular part:

$$
K_0(x, \xi) = K^S_0(x, \xi) + K^R_0(x, \xi),
\sqrt{2\pi} K^S_0(x, \xi) = T(\xi)e^{ix\xi} - e^{-ix\xi} - R(\xi)e^{ix\xi} = \sum_{\lambda \in \{+, -\}} a_\lambda(\xi)e^{i\lambda x\xi},
\sqrt{2\pi} K^R_0(x, \xi) = (T(\xi) - R(\xi))(m_+(x, \xi) - 1)e^{ix\xi} - (m_+(x, -\xi) - 1)e^{-ix\xi}.
$$

(4.10)

Note that, using (2.24), (2.7), and (2.4), on the support of $\chi_+(x)$ we have (see also (3.4))

$$
|K_0(x, \xi)| + |K^S_0(x, \xi)| \lesssim \min(|\xi|, 1) + \min(|x\xi|, 1),
$$

(4.11)

$$
|K^R_0(x, \xi)| \lesssim (x)^{-N+1} \min(|\xi|, 1).
$$

(4.12)

Using the above decomposition, we can write, for $\xi \geq 0$,

$$
\mu(\xi, \eta, \sigma) = \int_{\mathbb{R}} a(x)\chi_+(x)^3 K^S_0(x, \xi)K^S_0(x, \eta)K^S_0(x, \sigma) dx + \mu_{R, 0}(\xi, \eta, \sigma),
$$

(4.13)

where $\mu_{R, 0}$ is a remainder term that will be absorbed in $\mu_R$. For the leading order, we first write $a\chi_+^3 = (a - \ell_{+\infty})\chi_+^3 + \ell_{+\infty}\chi_+^3$, and note that the first function decays as fast as $a - \ell_{+\infty}$; then, we use the fact that

$$
\hat{(\chi_+^3)(\xi)} = \sqrt{\frac{\pi}{2}}\delta(\xi) + \text{p.v.} \frac{\hat{\phi}(\xi)}{i\xi} + \hat{\psi}(\xi)
$$

(4.14)

for some even, smooth, compactly supported functions $\phi, \psi$ with $\phi$ having integral 1. The formula for $\mu^S$ in (4.2) with coefficients as in (4.3) then comes from the first two terms in (4.14), that is,

$$
\ell_{+\infty} \int_{\mathbb{R}} (\chi_+^3(x) - \psi(x)) K^S_0(x, \xi)K^S_0(x, \eta)K^S_0(x, \sigma) dx,
$$

(4.15)
by using the expression for $K^S_o(x, \xi)$ in (4.10) and (4.14), and inserting in addition the cutoff $\varphi^*$ in front of the p.v. term. We are then left with three contributions from (4.13), besides $\mu_{R,0}$, that is,

$$
\mu_{R,1}(\xi, \eta, \sigma) = \int_{\mathbb{R}} (a(x) - \ell_{+\infty})(\chi_+(x)) \frac{3}{2} K^S_o(x, \xi) K^S_o(x, \eta) K^S_o(x, \sigma) \, dx,
$$

(4.16)

$$
\mu_{R,2}(\xi, \eta, \sigma) = \int_{\mathbb{R}} \ell_{+\infty} \psi(x) K^S_o(x, \xi) K^S_o(x, \eta) K^S_o(x, \sigma) \, dx,
$$

(4.17)

$$
\mu_{R,3}(\xi, \eta, \sigma) = \sum_{\lambda, \mu, \nu} a_\lambda(\xi) a_\mu(\eta) a_\nu(\sigma) \ell_{+\infty} f\left[1 - \varphi^*(p, \eta, \sigma)\right]\frac{\tilde{\Phi}(p)}{p} \mid_{p = \lambda, \mu - \nu + \nu_2}. \quad (4.18)
$$

All these can be absorbed in $\mu_R$, as we will explain below.

The remainder $\mu_{R,0}(\xi, \eta, \sigma)$ in (4.13) can be written as a linear combination of terms of two types: one type of terms is of the form

$$
I_{\epsilon_2 \epsilon_3} := \int_{\mathbb{R}} a(x) \chi_+(x) \chi_{\epsilon_2}(x) \chi_{\epsilon_3}(x) K^S_o(x, \xi) K^S_o(\epsilon_2 x, \eta) K^S_o(\epsilon_3 x, \sigma) \, dx,
$$

(4.19)

where $\epsilon_2, \epsilon_3 \in \{+, -\}$ with $\epsilon_2 \cdot \epsilon_3 = -1$, while the other type of terms have the form

$$
II_{A, B, C} := \int_{\mathbb{R}^+} a(x) (\chi_+(x))^3 K^S_o(x, \xi) K^B_o(x, \eta) K^C_o(x, \sigma) \, dx,
$$

(4.20)

where $A, B, C \in \{S, R\}$ with at least one of them equal to $R$.

For the terms of the type (4.19), we notice that $a \chi_+ \chi_{\epsilon_2} \chi_{\epsilon_3}$ is compactly supported so that the property (4.5)--(4.6) with $a = b = c = 0$ and with $N = 0$ follows from (4.11), while for the terms in (4.20), it follows using also (4.12). In order to obtain (4.6) for non-zero $a, b, c$ and for general $N$, it suffices to focus on one specific instance, such as the term

$$
\int a(x) (\chi_+(x))^3 K^S_o(x, \xi) K^R_o(x, \eta) K^S_o(x, \sigma) \, dx,
$$

(4.21)

since all other cases can be treated similarly. From (4.10), we write that

$$
\sqrt{2\pi} K^S_o(x, \xi) = \frac{\xi}{(\xi)} g(\xi) e^{ix\xi} + 2i \sin(x\xi), \quad g(\xi) := \frac{\langle \xi \rangle}{\xi} (T(\xi) - 1 - R(\xi)),
$$

$$
\sqrt{2\pi} K^R_o(x, \xi) = \frac{\xi}{(\xi)} \left[g(\xi)(m_+(x, \xi) - 1)e^{ix\xi} + \int_{-1}^1 \partial_\rho\left[e^{i\rho \langle \rho \rangle}(m_+(x, \rho) - 1)\right](x, s\xi) ds\right].
$$

(4.22)

Observe that the function $g$ is smooth in view of (2.10). Plugging the formulas (4.22) into (4.21) gives various contributions; we single out

$$
J(\xi, \eta, \sigma) := \frac{\xi \cdot \eta \cdot \sigma}{(\xi \eta \sigma)} g(\xi) g(\eta) g(\sigma) \int_{\mathbb{R}} (\chi_+(x))^3 e^{ix(-\xi + \eta + \sigma)} (m_+(x, \eta) - 1) \, dx,
$$

(4.23)
which is representative of all of them; indeed, all the other terms from (4.21) would either involve the \( \sin(x \xi) \) term (which we bound by \( |\xi| \langle x \rangle / \langle \xi \rangle \)) or the integral in (4.22), but these can be handled similarly using the estimates for \( m_+ \) in (2.4).

To prove (4.5)–(4.6) with \( a, b, c = 0 \) we write
\[
e^{ix(-\xi + \eta + \sigma)} = \left( i(-\xi + \eta + \sigma) \right)^{-N} \partial_x^N e^{ix(-\xi + \eta + \sigma)}
\]
integrate by parts repeatedly in (4.23), and use (2.4) to get
\[
\left| \int \chi_+(x)^3 e^{ix(-\xi + \eta + \sigma)} (m_+(x, \eta) - 1) \, dx \right| \lesssim \langle -\xi + \eta + \sigma \rangle^{-N}.
\]
To prove (4.5)–(4.6) for general \( a, b, c \) it suffices to apply derivatives to the integral in (4.23), use the smoothness of \( g \), and then integrate by parts in \( x \) as above, using once again (2.4).

Finally, observe that the above argument works identically for terms as in (4.19); the same holds true if we replace \( a^+ \chi + \chi \epsilon^2 \chi \epsilon^3 \) with \( \psi \) or \( (a - \ell + \infty) \chi^3 + \), which takes care of the terms (4.16) and (4.17).

Eventually, we look at (4.18). Note that, from the definition of \( \varphi^* \) in (4.4), we have \(|p| \gtrsim 1/R(\eta, \sigma)\) on its support. We observe that
\[
\mu_{R,3}(0, \eta, \sigma) = \sum_{\lambda, \mu, \nu} \bar{a}_\lambda(0) a_\mu(\eta) a_\nu(\sigma) \ell_{+ \infty} i \left[ 1 - \varphi^*(p_0, \eta, \sigma) \right] \frac{\hat{\phi}(p_0)}{p_0} \bigg|_{p_0 = -\mu \eta + \nu \sigma} = 0,
\]
since \( a_\lambda(0) = \lambda \). Similarly, the expression vanishes at \( \eta = 0 \) or \( \sigma = 0 \) and, in particular,
\[
\langle \xi \rangle \langle \eta \rangle \langle \sigma \rangle \langle \xi \cdot \eta \cdot \sigma \rangle \mu_{R,3}(\xi, \eta, \sigma)
\]
is bounded. This shows (4.5)–(4.6) for \( a = b = c = 0 \), using also that \( \phi \in S \).

Finally, we notice that when differentiating (4.18) the worst terms are those where the cutoff \( \varphi^* \) is hit, and since we can bound
\[
\left| \partial_\xi^a \partial_\eta^b \partial_\sigma^c \varphi_{> D}(p R(\eta, \sigma)) \right| \lesssim |R(\eta, \sigma)|^{a+b+c}
\]
the claimed bounds follow.

**The Even Case** A result similar to Proposition 4.1 holds in the case of even functions under the assumption that the operator \( H \) has an odd resonance

**Proposition 4.3.** (The even case) If \( f, g \in S \) are even functions, \( a = a(x) \) is even and satisfies (1.2), the same statement as that of Proposition 4.1 holds true, up to modifying the coefficients \( a_{\pm}(\xi) \) in (4.3) as follows:
\[
a_+(\xi) = T(\xi) + R(\xi), \quad \text{and} \quad a_-(\xi) = 1. \tag{4.24}
\]
Proof. The proof can be obtained very similarly to the proof of Proposition 4.1, starting from the distorted Fourier transform formulas (2.26)–(2.29). From (2.27), we can write formulas analogous to (4.10) and (4.22), namely,

\[ Ke(x, \xi) = K^S_e(x, \xi) + K^R_e(x, \xi), \]

\[ \sqrt{2\pi} K^S_e(x, \xi) = T(\xi) e^{ix\xi} + e^{-ix\xi} + R(\xi) e^{ix\xi} = \sum_{\lambda \in \{+, -\}} a_\lambda(\xi) e^{i\lambda x\xi}, \]

\[ \sqrt{2\pi} K^R_e(x, \xi) = (T(\xi) + R(\xi))(m_+(x, \xi) - 1)e^{ix\xi} + (m_+(x, -\xi) - 1)e^{-ix\xi}, \]

(4.25)

where \( a_\lambda \) are the coefficients given in (4.24). Then using (2.28), (2.7), and (2.4), we have the analogue of (4.11)–(4.12), that is,

\[ |K_e(x, \xi)| + |K^S_e(x, \xi)| \lesssim \min(|\xi|, 1) + \min(|x\xi|, 1), \]

\[ |K^R_e(x, \xi)| \lesssim (x)^{-N+1} \min(|\xi|, 1), \]

(4.26)

on the support of \( \chi_+(x) \), and we can write, similarly to (4.22),

\[ \sqrt{2\pi} K^S_e(x, \xi) = \frac{\xi}{|\xi|} g(\xi) e^{ix\xi} - 2i \sin(x\xi), \quad g(\xi) := \frac{|\xi|}{\xi}(T(\xi) + R(\xi) + 1), \]

\[ \sqrt{2\pi} K^R_e(x, \xi) = \frac{\xi}{|\xi|} \left[ g(\xi)(m_+(x, \xi) - 1)e^{ix\xi} \right. + \left. \int_{-1}^1 \partial_\rho \left[ e^{ix\rho(\rho)(m_+(x, \rho) - 1)}(x, s\xi)\right] ds \right]. \]

(4.27)

Using (4.25)–(4.27) and noticing that \( g \) is smooth in view of (2.11), the proof can then proceed as before. \( \square \)

The Generic Case In this case, we can directly borrow Proposition 4.1 from [5], which gives the statement.\(^4\)

**Proposition 4.4.** Let \( f, g \in \mathcal{S} \) be arbitrary functions, with \( a = a(x) \) satisfying (1.2) (but no parity assumptions) and assume that the potential \( V \) is generic. Then there exists a distribution \( \mu_{112}, \iota_1, \iota_2 \in \{+, -\} \), such that

\[ \mathcal{F}(a f_{11} g_{12})(\xi) = \iint_{\mathbb{R}^2} \tilde{f}\tilde{g}_{11}(\eta)(\eta)\mu_{112}(\xi, \eta, \sigma) \, d\eta \, d\sigma, \]

(4.28)

and such that \( \mu_{112}^0 \) can be split into a singular and a regular part as

\[ (2\pi)\mu_{112} = \mu_{112}^S + \mu_{112}^R, \]

where

\(^4\) Compared to [5], we changed slightly the definitions of the singular and regular parts, in accordance with those in Proposition 4.1 and to simplify the notation in the forthcoming sections.
The singular part \( \mu_{\xi_1 \xi_2}^S \) is given by
\[
\mu_{\xi_1 \xi_2}^S(\xi, \eta, \sigma) = \sum_{\lambda, \mu, v, \epsilon} \bar{a}_{\epsilon}^{\lambda}(\xi)(a_{\epsilon}^{\mu}(\eta))_{\xi_1}(a_{\epsilon}^{v}(\sigma))_{\xi_2} \times \ell_{\epsilon}(p) - i \epsilon \phi^*(p, \eta, \sigma) p.v. \frac{\phi(p)}{p}
\]
where \( p = \lambda \xi - \mu \eta - v \sigma \).

(4.29)

where the coefficients are given as
\[
\begin{aligned}
a_{\pm}^-(\xi) &= 1_{+}(\xi) + 1_{-}(\xi) T(-\xi) \\
a_{\pm}^+(\xi) &= 1_{+}(\xi) R_{-}(\xi) \\
a_{+}^+(\xi) &= 1_{-}(\xi) R_{+}(\xi). \\
\end{aligned}
\]

(4.30)

\( \phi \in S \) is even with integral one, and
\[
\varphi^*(p, \eta, \sigma) = \varphi_{\leq -D}(p R(\eta, \sigma)), \quad R(\eta, \sigma) = \frac{\langle \eta \rangle \langle \sigma \rangle}{\langle \eta \rangle + \langle \sigma \rangle}.
\]

(4.31)

Here \( 1_{\pm} \) is the characteristic function of \( \{ \pm \xi \geq 0 \} \).

The regular part \( \mu_{\xi_1 \xi_2}^R \) can be written as a linear combination of terms of the form
\[
1_{\epsilon_1}(\xi) 1_{\epsilon_2}(\eta) 1_{\epsilon_3}(\sigma) \frac{\xi \cdot \eta \cdot \sigma}{\langle \xi \rangle \langle \eta \rangle \langle \sigma \rangle} q_{\xi_1 \xi_2 \xi_3}(\xi, \eta, \sigma)
\]

(4.32)

where \( \epsilon_1, \epsilon_2, \epsilon_3 \in \{+, -, \} \), and for all \( |a| + |b| + |c| \leq N \)
\[
|\partial_{\xi}^a \partial_{\eta}^b \partial_{\sigma}^c q_{\xi_1 \xi_2 \xi_3}(\xi, \eta, \sigma)| \lesssim \sup_{\mu, v} \frac{1}{\langle \xi + \mu \eta + v \sigma \rangle^N} |R(\eta, \sigma)|^{1+a+b+c}.
\]

(4.33)

Remark 4.5. (Reduction to the case of odd symmetry) From now on, we will work under the odd symmetry assumption for (1.1), that is, we assume the initial data is odd, the coefficient \( a = a(x) \) in (1.1) is odd (so that the solution stays odd for all times), and the potential is even with (possibly) an even zero energy resonance.

It is apparent from Propositions 4.1 and 4.3 that the case of even data/solution and an odd resonance can be dealt with in exactly the same way as the odd case.

As for the generic case, a little more care would be needed to deal with the non-smoothness of the coefficients (4.30) at zero; in particular, after applying the normal-form transformation (5.6), one needs to show that the singularities in the cubic symbols in (5.11)–(5.12) only appear in the arguments of the inputs \( (\eta, \sigma, \theta) \).

This is a technical point that requires some careful algebra, but since it was addressed already in Sect. 6 of [5] we can skip it here. A part from this, Proposition 4.4, shows that the generic case can be handled like the odd case as well.
5. The Main Nonlinear Decomposition and Bootstrap

In Sects. 5.1–5.3, we summarize several manipulations which lead to a renormalized form of the equation [see (5.20)] over which the main estimates are performed. Some details are omitted, for which we refer the reader to Sect. 5 of [5]. In Sect. 5.4, we state our main bootstrap proposition which will imply the global bounds of Theorem 1.2.

5.1. The Equation on the Profile

Consider $u = u(t, x)$ a solution of the quadratic Klein–Gordon equation

$$\partial_t^2 u + (-\partial_x^2 + V + 1)u = a(x)u^2, \quad (u, u_t)(t = 0) = (u_0, u_1). \quad (5.1)$$

Note that we are disregarding the cubic terms from (1.1) since they are lower order; moreover, cubic terms that are more complicated than $u^3$ will appear after normal forms and will be treated in detail in what follows.

In order to make the equation first order in time, we first define

$$v = (\partial_t - iB)u, \quad B = \sqrt{-\partial_x^2 + V + 1}, \quad (5.2)$$

which solves

$$\left(\partial_t + iB\right)v = a(x)u^2 \quad \text{or} \quad \left(\partial_t + i\langle \xi \rangle\right)\tilde{v} = \tilde{F}(a(x)u^2).$$

Next, we filter by the linear evolution to obtain the profile

$$g(t, \cdot) = e^{itB}v(t, \cdot), \quad (5.3)$$

which solves

$$\partial_t \tilde{g}(t, \xi) = e^{it\langle \xi \rangle}\tilde{F}(a(x)u^2).$$

Using the quadratic spectral distribution described in Proposition 4.1, we write this explicitly as

$$\partial_t \tilde{g}(t, \xi) = -\sum_{i_1, i_2} \int \int e^{it\Phi_{i_1i_2}(\xi, \eta, \sigma)}\tilde{g}_{i_1}(t, \eta)\tilde{g}_{i_2}(t, \sigma) \frac{1}{4\langle \eta \rangle\langle \sigma \rangle} \mu^{o, S}_{i_1i_2}(\xi, \eta, \sigma) d\eta d\sigma.$$

$$\Phi_{i_1i_2}(\xi, \eta, \sigma) = \langle \xi \rangle - i_1 \langle \eta \rangle - i_2 \langle \sigma \rangle. \quad (5.4)$$

For convenience, we are omitting the limits of integration $\eta, \sigma > 0$ and, from now on, we will also omit the apex $o$ from $\mu^{o, S}_{i_1i_2}$ and similar expression such as $\mu^{S}_{i_1i_2}$. 
5.2. Normal-form Transformation and Renormalized Profile

Some simple calculations show that $\Phi_{t_{12}}$ does not vanish on the support of the distribution $\mu_{t_{12}}^S$, defined in (4.2) and (4.4). In other words, the corresponding interaction is not resonant, and we can define the natural normal-form transformation $T_{t_{12}}(g,g)$ by

$$\tilde{T}_{t_{12}}(g,g)(t) := \int e^{it\Phi_{t_{12}}(\xi,\eta,\sigma)} \tilde{g}(t,\eta) \tilde{g}(t,\sigma) m_{t_{12}}(\xi,\eta,\sigma) d\eta d\sigma,$$

(5.5)

The full normal-form transformation is given by $\sum_{t_{12} \in \{+,-\}} T_{t_{12}}$, and accordingly we define a re-normalized profile

$$f := g - T(g,g), \quad T = \sum_{t_{12} \in \{+,-\}} T_{t_{12}}.$$  (5.6)

It is not hard to check that, under our assumptions, $f$ is odd and thus $\tilde{f}(t,0) = 0$ as well.

Let us write $\partial_t \tilde{g} = Q_R(g,g) + Q_S(g,g)$ where $Q^*$ denotes the bilinear expression of the same form of (5.4) with $\mu^*$ replaced by $(2\pi)^{-1} \mu^*$, for $* = S$ or $R$. Then, from (5.5)–(5.6) we see that $\partial_t \tilde{f} = Q_R(g,g) + \tilde{T}(\partial_t g,g) + \tilde{T}(g,\partial_t g).$ This last equation can be rewritten as

$$\partial_t \tilde{f} = Q_R(g,g) + C^S(g,g,g) + C^R(g,g,g),$$

(5.7)

where the terms on the right-hand side are given as follows:

- **The regular quadratic term** is

$$Q_R(a,b) = \sum_{t_{12} \in \{+,-\}} Q^R_{t_{12}}(a,b),$$

$$Q^R_{t_{12}}[a,b](t,\xi) = \int e^{it\Phi_{t_{12}}(\xi,\eta,\sigma)} q(\xi,\eta,\sigma) \tilde{a}_{t_{12}}(t,\eta) \tilde{b}_{t_{12}}(t,\sigma) d\eta d\sigma,$$

(5.8)

with $q$ satisfying, see (4.5)–(4.6) and (4.4),

$$q(\xi,\eta,\sigma) = q'(\xi,\eta,\sigma) \cdot \frac{\eta}{\langle \eta \rangle} \cdot \frac{\sigma}{\langle \sigma \rangle} \cdot \frac{1}{\langle \eta \rangle + \langle \sigma \rangle},$$

(5.9)

where $q'$ is smooth with

$$|\partial^b_{\eta} \partial^c_{\sigma} q'(\xi,\eta,\sigma)| \lesssim (\xi - \eta - \sigma)^{-N \min(\langle \eta \rangle, \langle \sigma \rangle)^{a+b+c}}.$$  (5.10)

Note that the actual bound above should have the factor $(\inf_{\mu,\nu} \langle \xi - \mu \eta - \nu \sigma \rangle)^{-N}$ but we disregard the signs $\mu, \nu$ since they will play no relevant role in our estimates.
• The singular cubic term is
\[ C^S(a, b, c) = \sum_{\kappa_1, \kappa_2, \kappa_3 \in \{+, -\}} C^S_{\kappa_1 \kappa_2 \kappa_3}(a, b, c) \]
\[ C^S_{\kappa_1 \kappa_2 \kappa_3}(a, b, c)(t, \xi) = \int \int \int e^{it \Phi_{\kappa_1 \kappa_2 \kappa_3}(\xi, \eta, \sigma, \theta)} c^S_{\kappa_1 \kappa_2 \kappa_3}(\xi, \eta, \sigma, \theta) \tilde{a}_{\kappa_1}(t, \eta) \tilde{b}_{\kappa_2}(t, \sigma) \tilde{c}_{\kappa_3}(t, \theta) \, d\eta \, d\sigma \, d\theta, \]
where the symbol \( c^S_{\kappa_1 \kappa_2 \kappa_3} \) can be written as a sum of terms of the type
\[ s(\xi, \eta, \sigma, \theta) \delta(p) + m(\xi, \eta, \sigma, \theta) \hat{\phi}(p), \]
where \( p := \lambda \xi - \mu \eta - \nu \sigma - \rho \theta \),
\[ m(\xi, \eta, \sigma, \theta) = 0 \text{ when } \eta \cdot \sigma \cdot \theta = 0. \]
where we write \( s \) and \( m \) for symbols which are globally Lipschitz, and smooth as long as \( \xi, \eta, \sigma, \theta \) do not vanish. Furthermore, these satisfy
\[ |m(\xi, \eta, \sigma, \theta)| + |s(\xi, \eta, \sigma, \theta)| \lesssim \frac{1}{\langle \eta \rangle \langle \sigma \rangle \langle \theta \rangle}. \]
\[ m(\xi, \eta, \sigma, \theta) = 0 \text{ when } \eta \cdot \sigma \cdot \theta = 0. \]
Precise bounds on derivatives of \( m \) and \( s \) are slightly more complicated to state. However, all the contributions coming from differentiating these symbols (as in Sect. 7 for example) are lower order; in fact, the worst loss that may occur when differentiating them is essentially a factor of the form \( \max(\langle \xi \rangle, \langle \eta \rangle, \langle \sigma \rangle, \langle \theta \rangle) \) which is easy to handle. We refer to Sect. 5.5 in [5] where exact formulas can be found.

• The regular cubic term is given by
\[ C^R(a, b, c) = \sum_{\kappa_1, \kappa_2, \kappa_3} C^R_{\kappa_1 \kappa_2 \kappa_3}(a, b, c) \]
\[ C^R_{\kappa_1 \kappa_2 \kappa_3}(a, b, c)(t, \xi) = \int \int \int e^{it \Phi_{\kappa_1 \kappa_2 \kappa_3}(\xi, \eta, \sigma, \theta)} c^R_{\kappa_1 \kappa_2 \kappa_3}(\xi, \eta, \sigma, \theta) \tilde{a}_{\kappa_1}(t, \eta) \tilde{b}_{\kappa_2}(t, \sigma) \tilde{c}_{\kappa_3}(t, \theta) \, d\eta \, d\sigma \, d\theta, \]
where the symbol \( c^R_{\kappa_1 \kappa_2 \kappa_3} \) can be written as a sum of symbols \( m \) satisfying
\[ |m(\xi, \eta, \sigma, \theta)| \lesssim \frac{1}{\langle \eta \rangle \langle \sigma \rangle \langle \theta \rangle}. \]
This regular term is easier to treat than the two previous ones. In particular, it satisfies better trilinear bounds than \( C^S \) does. Therefore, we will only briefly mention how to treat it in the rest of the proof.

We now reproduce Lemmas 6.11 and 6.13 from [5] for later reference. The statements of these lemmas involve the wave operator associated to \( H \), which is denoted as
\[ \mathcal{W} = \tilde{F}^{-1} \tilde{F}, \quad \mathcal{W}^* = \tilde{F}^{-1} \tilde{F}. \]
Lemma 5.1. For any $p_1, p_2 \in [2, \infty)$ such that $\frac{1}{p_1} + \frac{1}{p_2} < \frac{1}{2}$, and $i_1, i_2 \in \{+, -, \}$,
\[
\|Q^R_{i_1i_2} (f_1, f_2)(t, \xi)\|_{L^2_{\xi}} \lesssim \min \left( \|e^{-it(\partial_x)}\mathcal{W}^* \xi f_1\|_{L^p}, \|e^{-it(\partial_x)}\mathcal{W}^* f_2\|_{L^p}, \|e^{-it(\partial_x)}\mathcal{W}^* f_1\|_{L^p}, \|e^{-it(\partial_x)}\mathcal{W}^* f_2\|_{L^p} \right).
\] (5.17)

Lemma 5.2. For any $\kappa_1, \kappa_2, \kappa_3 \in \{+, -, \}$, for all $p, p_1, p_2, p_3 \in (1, \infty)$ with
\[
\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = \frac{1}{p},
\]
\[
\|e^{-it(\partial_x)} \mathcal{F}^{-1} cS_{\kappa_1\kappa_2\kappa_3} (a, b, c)\|_{L^p} \lesssim \|e^{-it(\partial_x)}\mathcal{W}^* a\|_{L^p} \|e^{-it(\partial_x)}\mathcal{W}^* b\|_{L^p} \|e^{-it(\partial_x)}\mathcal{W}^* c\|_{L^p}.
\] (5.18)

Furthermore, if $p_1 = p_2 = \infty$, and $a, b$ are functions that satisfy the bounds as in the assumptions (5.24) of Proposition 5.3, then, for all $t \in [0, T]$ and $p \in (1, \infty)$, we have that
\[
\|e^{-it(\partial_x)} \mathcal{F}^{-1} cS_{\kappa_1\kappa_2\kappa_3} (a, b, c)\|_{L^p} \lesssim \frac{\varepsilon^2}{t} \|\mathcal{W}^* a\|_{L^p} \|\mathcal{W}^* b\|_{L^p} \|\mathcal{W}^* c\|_{L^p}.
\] (5.19)

The estimates of Lemma 5.2 hold true also for the $\delta$ and p.v. components of that $cS$ separately, see (5.11)–(5.12). Moreover, one can also add derivatives to the estimate (5.18) with a natural statement consistent with product estimates in Sobolev spaces; in particular, we can replace the $L^p$ norm on the left-hand side of (5.18) by the $W^{0+, p}$ norm.

5.3. The Equation for the Renormalized Profile

Using the identity $g = f + T(g, g)$ (see (5.6)), we can write Eq. (5.7) as follows
\[
\partial_t \tilde{f} = Q^R (f, f) + \mathcal{C}(f, f, f) + \mathcal{R}(f, g), \quad \mathcal{C} := cS + cR,
\] (5.20)
where the remainder term $\mathcal{R}$ is given by
\[
\mathcal{R}(f, g) = \mathcal{R}_Q (f, g) + \mathcal{R}_C (f, g),
\] (5.21)
\[
\mathcal{R}_Q (f, g) = Q^R (f, T(g, g)) + Q^R (T(g, g), g),
\] (5.22)
\[
\mathcal{R}_C (f, g) = \mathcal{C}(T(g, g), g) + \mathcal{C}(f, T(g, g), g) + \mathcal{C}(f, f, T(g, g)).
\] (5.23)

All the terms above are simpler to estimate than the other quadratic or cubic terms in (5.20) and can be treated similarly or with more straightforward arguments; we will explain how to handle them in Sect. 8 and will mostly concentrate on $Q^R (f, f)$ and $cS (f, f, f)$ in what follows.
5.4. Bootstrap and Basic a Priori Bounds

Fix two constants $\alpha$ and $p_0$ such that $0 < p_0 \ll \alpha \ll 1$ and recall the smallness assumption in Theorem 1.2. Our main bootstrap proposition, that will immediately imply the main theorem, is the following:

**Proposition 5.3.** Assuming that

$$
\sup_{t \in [0,T]} \left[ (t)^{-p_0} \| \langle \xi \rangle^4 \tilde{f}(t) \|_{L^2} + (t)^{-\alpha} \| \langle \xi \rangle \partial_\xi \tilde{f}(t) \|_{L^2} + \| \langle \xi \rangle^{3/2} \tilde{f}(t) \|_{L^\infty} \right] < 2 \varepsilon_1,
$$

$$
\sup_{t \in [0,T]} \left[ (t)^{-p_0} \| \langle \xi \rangle^4 \tilde{g}(t) \|_{L^2} + (t)^{1/2} \| e^{-it \partial_x} 1_{\pm}(D) \mathcal{W}^* g(t) \|_{L^\infty} \right] < 8 \varepsilon_1,
$$

(5.24)

then we have

$$
\sup_{t \in [0,T]} \left[ (t)^{-p_0} \| \langle \xi \rangle^4 \tilde{f}(t) \|_{L^2} + (t)^{-\alpha} \| \langle \xi \rangle \partial_\xi \tilde{f}(t) \|_{L^2} + \| \langle \xi \rangle^{3/2} \tilde{f}(t) \|_{L^\infty} \right] < \varepsilon_1,
$$

$$
\sup_{t \in [0,T]} \left[ (t)^{-p_0} \| \langle \xi \rangle^4 \tilde{g}(t) \|_{L^2} + (t)^{1/2} \| e^{-it \partial_x} 1_{\pm}(D) \mathcal{W}^* g(t) \|_{L^\infty} \right] < 4 \varepsilon_1.
$$

(5.25)

At the heart of the proof of Proposition 5.3 are the following weighted-type estimates for the quadratic and cubic terms in (5.20), under the assumptions (5.24)

$$
\left\| \langle \xi \rangle \partial_\xi \int_0^t Q_R(s, \xi) \, ds \right\|_{L^2} + \left\| \langle \xi \rangle \partial_\xi \int_0^t C^S(s, \xi) \, ds \right\|_{L^2} \lesssim \varepsilon_1^2 (t)^{\alpha}.
$$

(5.26)

These estimates are proven in Sects. 6 and 7. The bounds on the other two norms—the Sobolev-type norm and the Fourier $L^\infty$ norm—are discussed in Sect. 8.

Here are some immediate consequences of the bootstrap assumptions:

**Lemma 5.4.** Under the assumptions of Proposition 5.3, we have that

(i) (Global decay of the solution) For $v = (\partial_t - iB)u$, we have that

$$
\| v \|_{L^\infty} \lesssim \varepsilon_1 (t)^{-1/2}.
$$

(ii) (Decay for the derivative of the profile)

$$
\| e^{-itB} \partial_t f \|_{L^\infty} \lesssim \varepsilon_1^2 (t)^{-3/2 + 2\alpha},
$$

(5.27)

and

$$
\| \partial_t f(t) \|_{L^2} \lesssim \varepsilon_1^2 (t)^{-1 + \alpha/2}.
$$

(5.28)

**Proof.** Recall that $v = e^{-itB} g$. The first assertion follows from $g = f + T(g, g)$, the dispersive estimate (3.2) applied to $e^{-itB} f$ with the a priori bounds (5.24), and a product estimate for the quadratic term $e^{-itB} T(g, g)$; we refer the reader to Proposition 7.1 in [5] for the details.
Turning to (ii), recall that
\[
e^{-isB} \partial_s f = \tilde{F}^{-1} e^{-is\langle \xi \rangle} Q^{R}(f, f, f) + \tilde{F}^{-1} e^{-is\langle \xi \rangle} C^{S}(f, f, f) + \tilde{F}^{-1} e^{-is\langle \xi \rangle} C^{R}(f, f, f) + \{\text{better terms}\}.
\]
(5.29)

We start with \(Q^{R}\), which can actually be seen to satisfy a better bound of the form \(\langle s \rangle^{-2+2\alpha}\). By Sobolev’s embedding, it suffices to show that, for \(s \geq 1\), we have that
\[
\|\langle \xi \rangle Q^{R}(f, f)(\xi)\|_{L^2} \lesssim s^{-3/2}.
\]
(5.30)

From (5.8) to (5.9), we have that
\[
I(s, \xi) := \langle \xi \rangle Q^{R}(f, f)(\xi) = \int \int \int e^{is\Phi(\xi, \eta, \sigma)} \xi q(\xi, \eta, \sigma) \tilde{f}(\eta) \tilde{f}(\sigma) d\eta d\sigma
= \int \int \int_{\mathbb{R}^3_+} e^{is\Phi(\xi, \eta, \sigma)} \xi q(\xi, \eta, \sigma) \tilde{f}(\eta) \tilde{f}(\sigma) \eta \langle \eta \rangle + \langle \sigma \rangle \sigma \langle \sigma \rangle \xi \langle \eta \rangle + \langle \sigma \rangle q'(\xi, \eta, \sigma) d\eta d\sigma,
\]
(5.31)
where \(q'\) satisfies (5.10). Note that we can use the factors of \(\eta/\langle \eta \rangle\) and \(\sigma/\langle \sigma \rangle\) to integrate by parts in (5.31). Such an argument is performed very similarly in Sect. 6 to bound the term \(L_m\) appearing in (6.6); the only difference between (6.6) and the term above is that (5.31) does not have a low-frequency cutoff \(\chi^l_m\), and (6.6) has a factor of \(s\) and it is integrated in time.

Integrating by parts in (5.31) will then give the main terms
\[
Q_1(s, \xi) = \frac{1}{s^2} \int \int e^{is\Phi(\xi, \eta, \sigma)} \partial_\eta \tilde{f}(\eta) \partial_\sigma \tilde{f}(\sigma) \langle \xi \rangle \langle \eta \rangle + \langle \sigma \rangle q'(\xi, \eta, \sigma) d\eta d\sigma,
Q_2(s, \xi) = \frac{1}{s^2} \int \int e^{is\Phi(\xi, \eta, \sigma)} \partial_\eta \tilde{f}(\eta) \tilde{f}(\sigma) \langle \xi \rangle \langle \eta \rangle + \langle \sigma \rangle \partial_\sigma q'(\xi, \eta, \sigma) d\eta d\sigma.
\]
(5.32)

Using (5.10), gives that
\[
\|Q_1(s)\|_{L^2} \lesssim s^{-2} \int \int |\partial_\eta \tilde{f}(\eta)| |\partial_\sigma \tilde{f}(\sigma)| \frac{1}{\langle \xi - \eta - \sigma \rangle^{N-1}} d\eta d\sigma \lesssim s^{-2} \|\partial_\eta \tilde{f}\|_{L^2} \cdot \|\partial_\sigma \tilde{f}\|_{L^1} \lesssim s^{-2} \epsilon_1 s^{2\alpha},
\]
which is more than sufficient. A similar estimate holds for \(Q_2\).

The remaining cubic terms in (5.29) can be treated using Lemma 5.2, and the comment after its statement, to remedy the lack of the endpoint estimate with \(p = \infty\) as follows: choosing \(1/p < s < \alpha\) and using successively Sobolev’s embedding, Lemma 5.2, the intertwining property and boundedness of the wave
operator, and, finally, Proposition 3.1 together with interpolation to deduce decay of $e^{\pm itB} f$ in $L^3 p$, we get that

\[
\| \mathcal{F}^{-1} e^{-it|\xi|} C S (f, f, f) \|_{L^\infty} \lesssim \| (\partial_x)^s \mathcal{F}^{-1} e^{-it|\xi|} C S (f, f, f) \|_{L^p} \\
\lesssim \| e^{\pm it(\partial_x)} \mathcal{W}^* f \|_{L^3 p}^3 \lesssim \| \mathcal{W}^* e^{\pm itB} f \|_{L^3 p}^3 \\
\lesssim \| e^{\pm itB} f \|_{L^3 p}^3 \left( \varepsilon_1 t^{-1/2 + 1/(3p)} + 2p_0/(3p) \right)^3 \\
\lesssim \varepsilon_1^3 t^{-3/2 + 2\alpha}.
\]

Finally, the $L^2$ estimate (5.28) is a consequence of Eq. (5.29) combined with the Hölder-type bounds of Lemmas 5.1 and 5.2 and the a priori bounds (5.24). □

Proof of Theorem 1.2 from Proposition 5.3. To deduce the main theorem from the above proposition, the first step is to check that the bootstrap assumptions in (5.24) are satisfied at $t = 0$. As far as $g$ is concerned, this can be done easily, since from (5.2)–(5.3) we have that

\[ g(t = 0) = v(t = 0) = u_1 - iBu_0. \]

Then, the assumptions on the second line of (5.24) at $t = 0$ hold true thanks to the smallness assumption in Theorem (1.2), and the boundedness of wave operators on Sobolev spaces, by choosing $\varepsilon_1 = C\varepsilon$ for $C$ large enough. The verification of the assumptions for $f$ on the first line of (5.24) when $t = 0$ is slightly more involved, since $f(t = 0)$ is a nonlinear function of $g(t = 0)$; for this, we refer to [5], where this task is accomplished in estimates (7.1) to (7.5).

Assuming Proposition 5.3 and using a standard continuation argument together with Lemma 5.4, we obtain the main theorem. □

6. The main regular interaction

In this section, we prove the weighted $L^2$ bound for the regular quadratic terms

\[ \left\| (\xi) \partial_\xi \int_0^t Q^R (s, \xi) \, ds \right\|_{L^2} \lesssim \varepsilon_1^2, \tag{6.1} \]

with is even stronger than (5.26) and what is needed for (5.25). Applying $\partial_\xi$ to $Q_R$ in (5.8), we see that the main term is the one where the derivative hits the phase, and therefore we can reduce matters to estimating the expression

\[ I_m(t, \xi) := \int_0^t \int_{(\mathbb{R}^+)^2} s^{\xi} e^{i\Phi_{112}(\xi, \eta, \sigma)} \tilde{f}_{i_1}(\eta) \tilde{f}_{i_2}(\sigma) q_{i_1i_2}(\xi, \eta, \sigma) \, d\eta \, d\sigma \, \tau_m(s) \, ds, \]

\[ \Phi_{112}(\xi, \eta, \sigma) := \langle \xi \rangle - \iota_1 \langle \eta \rangle - \iota_2 \langle \sigma \rangle, \quad \iota_1, \iota_2 \in \{ +, - \}, \tag{6.2} \]

and proving that, for any fixed $m$, we have that

\[ \| I_m(t) \|_{L^2} \lesssim 2^{-\delta m} \varepsilon_1^2, \tag{6.3} \]
for some \( \delta > 0 \) small enough. For simplicity of notation, we have omitted the dependence on the indexes \( \iota_1, \iota_2 \) in the definition of \( I_m \), and we will often disregard them in what follows since they play no major role; the understanding is that the case \( \iota_1 = \iota_2 = + \) is relatively harder than the rest, and it suffices to concentrate on this.

**Decomposition of (6.2)**

Let us define the multipliers
\[
\chi^l_m(\xi, \eta, \sigma) := 1 - \varphi_J(\xi)\varphi_J(\eta)\varphi_J(\sigma),
\]
\[
\chi^h_m(\xi, \eta, \sigma) := 1 - \chi^l_m(\xi, \eta, \sigma),
\]
where
\[
J := -(10\alpha)m.
\]

On the support of \( \chi^l \), we have \( \min(|\xi|, |\eta|, |\sigma|) \lesssim 2^J \), while on the support of \( \chi^h \) we have that \( \min(|\xi|, |\eta|, |\sigma|) \gtrsim 2^J \). Using these, we split (6.2) as follows:

\[
I_m = L_m + H_m + R_m,
\]

where
\[
L_m := \int_0^t \iint e^{is\Phi(\xi, \eta, \sigma)} s\xi \tilde{f}(\eta) \tilde{f}(\sigma) \chi^l_m(\xi, \eta, \sigma) q(\xi, \eta, \sigma) d\eta d\sigma \tau_m(s) ds
\]
(6.6)

is the low-frequency term,
\[
H_m := \int_0^t \iint e^{is\Phi(\xi, \eta, \sigma)} s\xi \tilde{f}(\eta) \tilde{f}(\sigma) \chi^h_m(\xi, \eta, \sigma) q(\xi, \eta, \sigma) \varphi_{\geq 2^J}(\Phi) d\eta d\sigma \tau_m(s) ds
\]
(6.7)

is an almost resonant term, and the remainder \( R_m \) is
\[
R_m := \int_0^t \iint e^{is\Phi(\xi, \eta, \sigma)} s\xi \tilde{f}(\eta) \tilde{f}(\sigma) \chi^h_m(\xi, \eta, \sigma) q(\xi, \eta, \sigma) \varphi_{\geq 2^J}(\Phi) \tau_m(s) d\eta d\sigma \tau_m(s) ds.
\]
(6.8)

**Estimate of (6.6)** To estimate (6.6), we recall (5.9)–(5.10) and write
\[
q(\xi, \eta, \sigma) = q'(\xi, \eta, \sigma) \cdot \frac{\eta}{\langle \eta \rangle} \cdot \frac{\sigma}{\langle \sigma \rangle},
\]
\[
|\partial^b_\eta \partial^c_\sigma q'(\xi, \eta, \sigma)| \lesssim (\xi - \eta - \sigma)^{-N} \min(\langle \eta \rangle, \langle \sigma \rangle)^{b+c} \frac{1}{\langle \eta \rangle + \langle \sigma \rangle}.
\]
(6.9)

Note that we have disregarded the signs \( \mu, \nu \) in (4.6) since these play no relevant role. We then integrate by parts in \( \sigma \) through the identity \( e^{is\sigma} = \langle \sigma \rangle^b \partial_\sigma e^{is\langle \sigma \rangle} \), and similarly in \( \eta \), noting that the boundary terms vanish because \( \tilde{f}(0) = 0 \). Out
of the terms arising from this operation, we single out the following ones, since the remaining terms are either similar or easier:

\[ L_{m,1} := \int_0^t \int_{\mathbb{R}^+} e^{is\Phi} \frac{1}{s} \xi^l \chi_m^f(\xi, \eta, \sigma) \partial_\eta \tilde{f}(\eta) \partial_\sigma \tilde{f}(\sigma) \eta d\sigma d\tau_m(s) ds, \]

(6.10a)

\[ L_{m,2} := \int_0^t \int_{\mathbb{R}^+} e^{is\Phi} \frac{1}{s} \xi^l \chi_m^f(\xi, \eta, \sigma) \partial_\eta \tilde{f}(\eta) \partial_\sigma q^l(\xi, \eta, \sigma) \eta d\sigma d\tau_m(s) ds, \]

(6.10b)

\[ L_{m,3} := \int_0^t \int_{\mathbb{R}^+} e^{is\Phi} \frac{1}{s} \xi^l \partial_\sigma \chi_m^f(\xi, \eta, \sigma) \eta d\sigma \tilde{f}(\sigma) \sigma d\tau_m(s) ds. \]

(6.10c)

To estimate (6.10a) in \( L^2 \), we first use (6.9) to bound

\[ |L_{m,1}(t, \xi)| \lesssim \sup_{s \approx 2^m} \int_{\mathbb{R}^+} \chi_m^f(\xi, \eta, \sigma) \partial_\eta \tilde{f}(\eta) \partial_\sigma \tilde{f}(\sigma) |K(\xi, \eta, \sigma)| \eta d\sigma, \]

\[ K(\xi, \eta, \sigma) := \frac{|\xi|}{\langle \eta \rangle + \langle \xi \rangle - \langle \xi - \eta - \sigma \rangle^N}. \]

Note that \( K(\xi, \eta, \sigma) \lesssim (\xi - \eta - \sigma)^{-N+1} \), and recall that \( \min\{|\xi|, |\eta|, |\sigma|\} \lesssim C_2^J \). For the part \( |\sigma| \lesssim C_2^J \), with Young’s inequality followed by Cauchy–Schwarz, we obtain

\[ \|L_{m,1}(t, \cdot)\|_{L^2} \lesssim \sup_{s \approx 2^m} \left\| \int_{\mathbb{R}^+} |\partial_\eta \tilde{f}(\eta)| |\varphi_{<J+D}(\sigma)| |\partial_\sigma \tilde{f}(\sigma)| \frac{1}{(\xi - \eta - \sigma)^{N-1}} \eta d\sigma \right\|_{L^2_\xi} \]

\[ \lesssim \sup_{s \approx 2^m} \|\partial_\eta \tilde{f}\|_{L^2_\xi} \cdot \|\varphi_{<J+D}(\sigma)\partial_\sigma \tilde{f}\|_{L^1_\xi} \]

\[ \lesssim \sup_{s \approx 2^m} \|\partial_\eta \tilde{f}\|_{L^2_\xi} \cdot 2^{J/2} \|\partial_\sigma \tilde{f}\|_{L^1_\xi} \lesssim \varepsilon_1^2 2^{-3\alpha}. \]

(6.11)

The case \( |\eta| \lesssim C_2^J \) can be dealt with in the same way due to symmetry. For the case \( |\xi| \lesssim C_2^J \), we have \( K(\xi, \eta, \sigma) \lesssim 2^J (\xi - \eta - \sigma)^{-N} \). Hence, by Young’s inequality and Cauchy–Schwarz

\[ \|L_{m,1}(t, \cdot)\|_{L^2} \lesssim \sup_{s \approx 2^m} \int_{\mathbb{R}^+} |\partial_\eta \tilde{f}(\eta)| |\partial_\sigma \tilde{f}(\sigma)| \frac{2^J}{(\xi - \eta - \sigma)^{N}} \eta d\sigma \]

\[ \lesssim \sup_{s \approx 2^m} \|\partial_\eta \tilde{f}\|_{L^2_\xi} \cdot \|\partial_\sigma \tilde{f}\|_{L^1_\xi} \lesssim \varepsilon_1^2 2^{-3\alpha}. \]

A similar argument applies to (6.10b) since \( |\partial_\sigma q^l| \lesssim (\xi - \eta - \sigma)^{-N} \).

For (6.10c), we notice that \( \partial_\sigma \chi_m^f(\xi, \eta, \sigma) = -\varphi_{>J}(\xi)\varphi_{>J}(\eta)\varphi_{<J}(\sigma)2^{-J} \), and use again (6.9) and Young’s inequality to get that

\[ \|L_{m,3}(t)\|_{L^2} \lesssim \sup_{s \approx 2^m} \int_{\mathbb{R}^+} 2^{-J} |\partial_\eta \tilde{f}(\eta)| |\varphi_{>J}(\sigma)\tilde{f}(\sigma)| K(\xi, \eta, \sigma) \eta d\sigma, \]

\[ \lesssim \sup_{s \approx 2^m} 2^{-J} \|\partial_\eta \tilde{f}\|_{L^2_\xi} \cdot 2^{J/2} \|\varphi_{>J}\tilde{f}\|_{L^1_\xi} \]

\[ \lesssim 2^{J/2} \sup_{s \approx 2^m} \|\partial_\eta \tilde{f}\|_{L^2_\xi} \|\partial_\sigma \tilde{f}\|_{L^2_\xi} \lesssim \varepsilon_1^2 2^{-3\alpha}. \]
having applied Hardy’s inequality \( \| \varphi_J \widetilde{f} \|_{L^2} \lesssim 2^J \| \partial_\xi \widetilde{f} \|_{L^2} \) for \( \widetilde{f}(0) = 0 \).

**Estimate of (6.7)** To estimate these terms, we first integrate by parts as in the previous case, and then use Schur’s lemma to take advantage of the restriction on the size of \( |\Phi| \lesssim 2^{2J} \approx (s)^{-20} \). More precisely, integration by parts in \( \eta \) and \( \sigma \) gives terms as in (6.10a)–(6.10c) with the additional cutoff \( \varphi_{<2J}(\Phi) \) and with \( \chi_m^h \) replaced by \( \chi_m^\eta \); we denote these terms by \( H_{m,1}, H_{m,2}, H_{m,3} \), respectively. There are also terms where the cutoff \( \varphi_{<2J}(\Phi) \) is hit by the derivatives, but these are easier to estimate since \( \partial_\eta \varphi_{<2J}(\Phi) = 2^{-2J} \varphi_{<2J}(\Phi)/\langle \eta \rangle \) (and similarly for \( \sigma \)) and one can use the factor \( \eta/\langle \eta \rangle \) to repeat the integration by parts; so we disregard them.

To estimate the term \( H_{m,1} \), we first assume, without loss of generality because of symmetry, that \( |\eta| \geq |\sigma| \). We then bound

\[
|H_{m,1}(t, \xi)| \lesssim \int_0^t \frac{1}{s} \int_{\mathbb{R}^+} K_1(\xi, \eta) |\partial_\eta \widetilde{f}(\eta)| \, d\eta \, \tau_m(s) \, ds
\]

\[
K_1(\xi, \eta) := \int_{\mathbb{R}^+} \chi_m^h(\xi, \eta, \sigma) \varphi_{<2J}(\Phi(\xi, \eta, \sigma)) |\partial_\sigma \widetilde{f}(\sigma)| \, d\sigma,
\]

having used \( |\xi||q'| \lesssim 1 \). Then we note that for any fixed \( |\eta| \gtrsim 2^J \), the region in \( \xi \) with \( |\xi| \gtrsim 2^J \) and \( |\Phi(\xi, \eta, \sigma)| \lesssim A \), with \( A \lesssim 2^{2J} \), has size less than \( CA2^{-J} \), with a symmetric estimate exchanging the roles of \( \xi \) and \( \eta \). Therefore, we have that

\[
\int |K_1(\xi, \eta)| \, d\xi + \int |K_1(\xi, \eta)| \, d\eta \lesssim 2^{2J} \cdot 2^{-J} \cdot \| \partial_\sigma \widetilde{f} \|_{L^1} \lesssim \varepsilon_1 2^{-9am}.
\]

By applying Schur’s Lemma, we obtain the bound

\[
\| H_{m,1}(t, \xi) \|_{L^2} \lesssim 2^{-9am} \| \partial_\eta \widetilde{f} \|_{L^2} \lesssim \varepsilon_1^2 2^{-8am}.
\]

The term \( H_{m,2} \) can be estimated in a similar way, using \( 2^{-J} \| \varphi_{<J} \widetilde{f} \|_{L^1} \lesssim \varepsilon_1 \).

The term \( H_{m,3} \) can also be estimated similarly, since

\[
|H_{m,3}(t, \xi)| \lesssim \int_0^t \frac{1}{s} \int_{\mathbb{R}^+} K_3(\xi, \eta) |\partial_\eta \widetilde{f}(\eta)| \, d\eta \, \tau_m(s) \, ds,
\]

\[
K_3(\xi, \eta) := \varphi_{>J}(\xi) \varphi_{>J}(\eta) \int_{\mathbb{R}^+} 2^{-J} |\varphi_{>J}(\sigma)| \varphi_{<2J}(\Phi(\xi, \eta, \sigma)) |\widetilde{f}(\sigma)| \, d\sigma,
\]

and we have that

\[
\int |K_3(\xi, \eta)| \, d\xi + \int |K_3(\xi, \eta)| \, d\eta \lesssim 2^{2J} \cdot 2^{-J} \cdot \| \widetilde{f} \|_{L^\infty} \lesssim \varepsilon_1 2^{-10am}.
\]
**Estimate of (6.8)** In this case, we can integrate by parts in $s$ using $e^{i s \Phi} = (i \Phi)^{-1} \partial_s e^{i s \Phi}$, and the lower bound on $|\Phi| \gtrsim 2^J$. This gives several terms, but it suffices to focus on the most relevant one, namely,

$$
\int_0^t \int \frac{e^{i s \Phi(\xi, \eta, \sigma)} \xi S}{\Phi} \Phi \partial_s f(\eta) \tilde{f}(\sigma) \varphi(\xi, \eta, \sigma) d\eta d\sigma \tau_m(s) ds,
$$

(6.12)

disregarding the symmetric one where $\partial_s$ hits the other profile, and the easier ones where it hits $s, \tau_m$ and that can be estimated using integration by parts in both $\eta$ and $\sigma$.

Integrating by parts in $\sigma$ in (6.12) gives the terms

$$
R_{m,1} := \int_0^t \int \frac{e^{i s \Phi(\xi, \eta, \sigma)} \xi S}{\Phi} \partial_s f(\eta) \tilde{f}(\sigma) \varphi(\xi, \eta, \sigma) \chi_m(\xi, \eta, \sigma) q(\xi, \eta, \sigma) d\eta d\sigma \tau_m(s) ds,
$$

(6.13a)

$$
R_{m,2} := \int_0^t \int \frac{e^{i s \Phi(\xi, \eta, \sigma)} \xi S}{\Phi} \partial_s f(\eta) \tilde{f}(\sigma) \varphi(\xi, \eta, \sigma) \chi_m(\xi, \eta, \sigma) q'(\xi, \eta, \sigma) d\eta d\sigma \tau_m(s) ds.
$$

(6.13b)

To treat (6.13b), we first write it as

$$
R_{m,2}(t, \xi) = \int_0^t \int \frac{e^{i s \Phi(\xi, \eta, \sigma)} \xi S}{\Phi} m(\xi, \eta, \sigma) \partial_s \tilde{f}(\eta) \tilde{f}(\sigma) d\eta d\sigma \tau_m(s) ds,
$$

$$
m(\xi, \eta, \sigma) := \frac{\xi S}{\eta} \varphi(\xi, \eta, \sigma) \chi_m(\xi, \eta, \sigma) q'(\xi, \eta, \sigma).
$$

Note that we can estimate, for $a + b + c \leq 6$,

$$
|\partial^{a}_{\xi} \partial^{b}_{\eta} \partial^{c}_{\sigma} m(\xi, \eta, \sigma)| \lesssim 2^{-2J} \cdot 2^{-2J(a+b+c)} \lesssim 2^{140 \alpha m}.
$$

(6.14)

This allows us to use the following classical lemma:

**Lemma 6.1.** Let $m$ be supported on $(\xi, \eta, \sigma) \in [t_1, t_1+r_1] \times [t_2, t_2+r_2] \times [t_3, t_3+r_3]$ and satisfy, for $a, b, c \in \{0, 1, 2\}$,

$$
|\partial^{a}_{\xi} \partial^{b}_{\eta} \partial^{c}_{\sigma} m(\xi, \eta, \sigma)| \leq r_1^{-a} r_2^{-b} r_3^{-c}.
$$

Denote

$$
B_m(f, g)(x) := \widehat{f}_{\xi \rightarrow x}^{-1} \int \int m(\xi, \eta, \sigma) \hat{f}(\eta) \hat{g}(\sigma) d\eta d\sigma.
$$

Then the operator norm of $B_m$ can be bounded by

$$
\|B_m\|_{L^p \times L^q \rightarrow L^r} \lesssim 1
$$

if $(p, q, r) \in [1, \infty]$ satisfy $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$.
Letting $\delta$ be small, and denoting $\infty := \delta^{-1}$ and $2^+ := 2(1 - 2\delta)^{-1}$, we estimate using Lemma (6.1) and (6.14), the boundedness of wave operators and Sobolev’s embedding:

\[
\left\| R_{m, 2} \right\|_{L^2} \lesssim \int_{s \approx 2^m} 2^{140am} \cdot \left\| \widehat{F}^{-1} \widehat{F} e^{-isB} \partial_s f \right\|_{L^\infty} \cdot \left\| \widehat{F}^{-1} \partial_\sigma \widehat{F} \right\|_{L^2} ds \\
\lesssim 2^m 2^{140am} \sup_{s \approx 2^m} \left\| e^{-isB} \partial_s f \right\|_{L^\infty} \cdot \left\| \langle \sigma \rangle \partial_\sigma \widehat{F} \right\|_{L^2} \\
\lesssim 2^m \cdot 2^{140am} \cdot \epsilon^{2} 2^{-(3/2)m+10am} \cdot \epsilon 2^{am}
\]

where, for the last inequality, we have used interpolation between (5.27) and (5.28), picking $\delta$ small enough. The bound above for $R_{m, 2}$ is more than sufficient compared to the right-hand side of (6.3) if $\alpha$ is small enough.

The remaining term (6.13a) can be handled similarly, since the bilinear symbol is just $\partial_\sigma m$ which satisfies estimates like (6.14) with an additional harmless factor of $2^{-2J} = 2^{20\alpha m}$.

7. The Main Singular Interaction

We consider here the singular cubic terms defined in (5.11). Their kernel contains either a Dirac $\delta(p)$, or a principal value of $\frac{1}{p}$. We focus on the latter case, which is slightly more involved, and thus consider that

\[
C^S(a, b, c)(t, \xi) := \int\int\int e^{it\Phi_{t_1, t_2, t_3}(\xi, \eta, \sigma, \theta)} m(\xi, \eta, \sigma, \theta) \tilde{a}(t, \eta) \tilde{b}(t, \sigma) \tilde{c}(t, \theta) \frac{\hat{\Phi}(p)}{p} d\eta d\sigma d\theta,
\]

\[
\Phi_{t_1, t_2, t_3}(\xi, \eta, \sigma, \theta) = \langle \xi \rangle - t_1 \langle \eta \rangle - t_2 \langle \sigma \rangle - t_3 \langle \theta \rangle, \quad p = \xi - \lambda \eta - \mu \sigma - \nu \theta.
\]

We aim to show

\[
\left\| \langle \xi \rangle \partial_\xi \int_0^t C^S(f, f, f)(s) \right\|_{L^2} \lesssim \epsilon^3 (t)^a.
\]

Observe that

\[
(\langle \xi \rangle \partial_\xi + X_{\eta, \sigma, \theta}) \Phi_{t_1, t_2, t_3}(\xi, \eta, \sigma, \theta) = p, \quad X_{\eta, \sigma, \theta} := \lambda t_1 \langle \eta \rangle \partial_\eta + \mu t_2 \langle \sigma \rangle \partial_\sigma + \nu t_3 \langle \theta \rangle \partial_\theta.
\]
When applying $\langle \xi \rangle \partial_\xi$ to (7.1), we can use this identity to integrate by parts in $\eta$, $\sigma$ and $\theta$. Since the adjoint satisfies $X^*_{\eta,\sigma,\theta} = -X_{\eta,\sigma,\theta} + \{\text{lower order terms}\}$, we see that

$$\langle \xi \rangle \partial_\xi C^3(f, f, f)(t, \xi) = it \int \int \int e^{it\Phi_{t/23}(\xi, \eta, \sigma, \theta)} m(\xi, \eta, \sigma, \theta) \tilde{f}(\eta) \tilde{f}(\sigma) \tilde{f}(\theta) \hat{\phi}(p) d\eta d\sigma d\theta$$

(7.4a)

$$+ \int \int \int e^{it\Phi_{t/23}(\xi, \eta, \sigma, \theta)} m(\xi, \eta, \sigma, \theta) X_{\eta,\sigma,\theta} \left( \tilde{f}(\eta) \tilde{f}(\sigma) \tilde{f}(\theta) \right) \frac{\hat{\phi}(p)}{p} d\eta d\sigma d\theta$$

(7.4b)

$$+ \int \int \int e^{it\Phi_{t/23}(\xi, \eta, \sigma, \theta)} m(\xi, \eta, \sigma, \theta) \tilde{f}(\eta) \tilde{f}(\sigma) \tilde{f}(\theta) \left( \langle \xi \rangle \partial_\xi + X_{\eta,\sigma,\theta} \right) \frac{\hat{\phi}(p)}{p} d\eta d\sigma d\theta$$

(7.4c)

$$+ \{\text{lower order terms}\}. \quad (7.4d)$$

**Estimate of (7.4a).** The first term in (7.4) does not have a singular kernel and can be estimated by integrating in the “uncorrelated” variables $\eta$, $\sigma$ and $\theta$, relying on the vanishing of $\tilde{f}(0)$ of $m(\xi, \eta, \sigma, \theta)$ when $\eta$, $\sigma$, or $\theta$ is zero, see (5.13). (Each integration in one of the three variables is also similar to the argument in the proof of (3.3).) These integrations by parts arguments give the bound $\| (7.4a) \|_{L^2} \lesssim \varepsilon_1^3(t)^{-2+3\alpha}$, consistent with (7.2).

**Estimate of (7.4b).** For this term is suffices to use the Hölder estimate (5.19) in Lemma 5.2, estimating in $L^2$ the profile that is hit by the derivative, and the other two in $L^\infty$:

$$\left\| \int_0^t (7.4b) \right\|_{L^2} \lesssim \int_0^t \| \langle \xi \rangle \partial_\xi \tilde{f} \|_{L^2} \frac{\varepsilon_1^2}{\langle s \rangle} ds \lesssim \varepsilon_1^3 \int_0^t \langle s \rangle^{-\alpha-1} ds \lesssim \varepsilon_1^3(t)^{\alpha}. \quad (7.5)$$

**Estimate of (7.4c).** For this term, we observe, see (7.3) and (7.1), that

$$\langle \xi \rangle \partial_\xi + X_{\eta,\sigma,\theta} p = \Phi_{t_{1/23}}(\xi, \eta, \sigma, \theta), \quad (7.7)$$

hence

$$\langle \xi \rangle \partial_\xi + X_{\eta,\sigma,\theta} \left[ \frac{\hat{\phi}(p)}{p} \right] = \Phi_{t_{1/23}}(\xi, \eta, \sigma, \theta) \partial_p \left[ \frac{\hat{\phi}(p)}{p} \right]. \quad (7.5)$$

in the sense of distributions. We can then use the factor of $\Phi_{t_{1/23}}$ in the right-hand side above to integrate by parts in $s$, and obtain

$$\int_0^t (7.4c) ds = \int \int \int e^{is\Phi_{s/23}} m(\xi, \eta, \sigma, \theta) \tilde{f}(\eta) \tilde{f}(\sigma) \tilde{f}(\theta) \partial_p \frac{\hat{\phi}(p)}{p} d\eta d\sigma d\theta \bigg|_{s=0}^{s=t} \quad (7.6)$$

$$- \int_0^t \int \int \int e^{is\Phi_{s/23}} m(\xi, \eta, \sigma, \theta) \partial_p \left[ \tilde{f}(\eta) \tilde{f}(\sigma) \tilde{f}(\theta) \right] \frac{\hat{\phi}(p)}{p} d\eta d\sigma d\theta ds. \quad (7.7)$$
To estimate (7.6), we convert the $\partial_p$ into $\partial_\eta$ and integrate by parts in $\eta$. This gives that

$$\int \int \int e^{i \Phi_{1/2}^\varepsilon \partial^1 \partial^2 \partial^3 \eta} f(\eta) f(\sigma) f(\theta) \frac{\tilde{\phi}(p)}{p} d\eta d\sigma d\theta \bigg|_{s=0}^{s=t} = - \int \int \int e^{i \Phi_{1/2}^\varepsilon \partial_\eta \partial^1 \partial^2 \partial^3 \eta} f(\eta) f(\sigma) f(\theta) \frac{\tilde{\phi}(p)}{p} d\eta d\sigma d\theta \bigg|_{s=0}^{s=t} - \int \int \int i \varepsilon^2 \Phi \frac{\partial \eta}{\partial^1 \partial^2 \partial^3 \eta} f(\eta) f(\sigma) f(\theta) \frac{\tilde{\phi}(p)}{p} d\eta d\sigma d\theta \bigg|_{s=0}^{s=t}.$$

Out of the three terms on the right-hand side, the worst is the last one when $s = t$, since it picked up a factor $t$ when the derivative hits the complex exponential. The $L^2$ norm of this term can be estimated using Lemma 5.2 (see also the comment below its statement) by

$$C t \varepsilon^2 \langle t \rangle \| f \|_{L^2} \lesssim \varepsilon^3 \langle t \rangle^{-\alpha},$$

as desired. We also refer the reader to [5, Sect. 9.3] for arguments similar to those above.

The term (7.7) is similar to the one just treated. We may assume that $\partial_s$ hits $\tilde{f}(\sigma)$. Again we convert $\partial_p$ into $\partial_\eta$ and integrate by parts in $\eta$. This causes a loss of $s$ when hitting the exponential phase which is offset by an $L^\infty \times L^2 \times L^\infty$ estimate with $\partial_s \tilde{f}$ placed in $L^2$ and giving $\varepsilon^3 \langle s \rangle^{-1+\alpha/2}$ decay by (5.28).

Estimate of (7.4d). This can be treated through Lemma 5.2. The only small difficulty is the loss of derivatives resulting from the differentiation of the symbol (see the description after (5.13)), but this is easily recovered using the $H^4$ a priori bound from Proposition 5.3, and $p_0 < \alpha$.

### 8. The Remainder Terms

We now explain how to deal with the remainder terms appearing after the renormalization (5.20) and the various splittings of the nonlinear terms.

#### 8.1. Weighted Norm and Bootstrap for $g$

The a priori bootstrap estimates for $g = f + T(g, g)$ in Proposition 5.3 can be closed fairly easily using the slightly better a priori bounds on $f$. To explain this, let us first observe that for all practical purposes one can think that the operator $T$, defined in (5.5)–(5.6), essentially has the form

$$T(a, b) = e^{itB} (e^{-itB} a e^{-itB} b),$$

(8.1)
and, in particular, satisfies standard Hölder bounds of the form
\[
\|e^{-itB}T(a, b)\|_{L^r} \lesssim \|e^{-itB}a\|_{L^q} \|e^{-itB}b\|_{L^p},
\]
\[
1/r = 1/p + 1/q, \quad r, p, q \in (1, \infty).
\]
(8.2)
The validity of (8.2) follows from the fact that the symbol in (5.5) is a δ plus a p.v. contribution near its singularity, see (4.2), divided by a smooth non-vanishing expression. We refer the reader to Sect. 6 of [5] for detailed lower bounds on \(\Phi_{\iota,\iota}\) and related symbol-type estimates; in particular, see Sect. 6.3 in [5] for the proof of the bound (8.2) (in fact, a stronger version of it with a gain of almost a full derivative). With the above notation simplification, we may assume that (5.6) reads as
\[
g = f + T(g, g) = f + e^{itB}(e^{-itB}g)^2.
\]
(8.3)
The Sobolev-type estimate for \(g\) is easy to obtain directly from (8.3) and the a priori Sobolev bound for \(f\) and \(g\) in (5.24). First, observe that by fixing an arbitrary small number \(\delta > 0\), using Sobolev’s embedding, and the boundedness of wave operators (see (5.16)) on \(L^p\), we can estimate
\[
\|e^{-itB}g\|_{L^\infty} \lesssim \|e^{-itB}g\|_{W^{\delta+1/\delta}} \lesssim \|e^{-it(\partial_x)}\mathcal{W}^*g\|_{W^{\delta+1/\delta}} \lesssim \varepsilon_1(t)^{-1/2 + p_0};
\]
for the last inequality we have used interpolation between the bounds in the second line of (5.24). Then we can estimate
\[
\|B^4(e^{-itB}g)^2\|_{L^2} \lesssim \|B^4 g\|_{L^2} \|e^{-itB}g\|_{L^\infty} \lesssim \varepsilon_1^2(t)^{2p_0 - 1/2}.
\]
For the remaining \(L^\infty\) estimate in (5.24), we fix again an arbitrary small number \(\delta\) and similarly to above use Sobolev’s embedding, followed by the boundedness of wave operators and the natural version of (8.2) with derivatives, to obtain
\[
\|e^{-it(\partial_x)}1_{\pm}(D)\mathcal{W}^*T(g, g)\|_{L^\infty} \lesssim \|e^{-it(\partial_x)}1_{\pm}(D)\mathcal{W}^*T(g, g)\|_{W^{\delta+1/\delta}} \lesssim \|e^{-itB}T(g, g)\|_{W^{\delta+1/\delta}} \lesssim \|e^{-itB}B^{\delta+}g\|_{L^2}^2 \lesssim \varepsilon_1^2(t)^{-1+2p_0},
\]
where we again used Sobolev–Gagliardo–Nirenberg interpolation for the last bound. This proves the bounds for \(g\) in (5.25).

Let us also mention for later use that from (8.3) one can establish a weak weighted bound for \(g\) of the form
\[
\|\langle \xi \rangle \partial_\xi \tilde{T}g\|_{L^2} \lesssim \varepsilon_1(t)^{1/2 + \alpha/10};
\]
(8.4)
this is because applying \(\partial_\xi\) to \(\tilde{T}(g, g)\) costs a factor of \(t\) and \(\|(e^{-itB}g)^2\|_{L^2} \lesssim \varepsilon_1^2(t)^{-1/2} \). The bound (8.4) is also helpful in estimating weighted norms of remainder terms, see Sect. 8.2. For more details on the bootstrap for \(g\) and (8.4), we refer the reader to Sects. 7.1 and 7.2 in [5].
8.2. Weighted Estimates for Remainder Terms

The remainder terms $\mathcal{R}$ that arise when we substitute $f$ for $g$ into the leading quadratic and cubic terms, see (5.21), are not hard to estimate in our current functional framework, while their estimates are more lengthy in [5]. We provide a few details below. The two terms on the right-hand side of (5.22) can be thought of as cubic versions of the smooth quadratic terms $Q_R$, and essentially are of the form

$$
\mathcal{R}_Q[a, b, c](t, \xi) := \int_0^t \int (\mathbb{R}^3) e^{it\Phi_{12,13}(\xi, \eta, \sigma, \theta)} \tilde{a}_{11}(\eta) \tilde{b}_{12}(\sigma) \tilde{c}_{13}(\theta) q(\xi, \eta, \sigma, \theta) d\eta d\sigma d\theta ds,
$$

for $(a, b, c) = (g, g, g)$ or $(f, g, g),
$$
(8.5)
$$
where $\Phi_{12,13}$ is as in (5.11), and $q$ is some smooth symbol that vanishes whenever $\eta \cdot \sigma \cdot \theta = 0$. To estimate $\mathcal{R}_Q[g, g, g]$, we can substitute one $g$ with $f$ via (8.3), up to terms that are easier to estimate; for example, they are better versions of the terms (8.7) which we will discuss below. We can then reduce matters to considering $\mathcal{R}_Q[f, g, g]$, and then substituting one more $g$ with $f$ we further reduce to $\mathcal{R}_Q[f, f, g]$. For the cubic expression $\mathcal{R}_Q[f, f, g]$, we can apply simple integration by parts arguments in each of the variables in distorted Fourier space, like those performed in Sect. 6 [see for example (6.6) and the expression resulting after integration by parts (6.10a)–(6.10c)]. With this operation, and using (8.4), we obtain the bound

$$
\| \langle \xi \rangle \partial_\xi \mathcal{R}_Q[f, f, g](t) \|_{L^2} \lesssim s (\varepsilon_1 s^{-1+\sigma/2})^2 \cdot \varepsilon_1 s^{-1/2+\alpha/10},
$$

and it follows that

$$
\| \langle \xi \rangle \partial_\xi \int_0^t \mathcal{R}_Q[f, g, g](s, \xi) ds \|_{L^2} \lesssim \varepsilon_1^3.
$$

(8.6)

Moving on to $\mathcal{R}_C$ in (5.23), we first observe that, in view of (5.14)–(5.15), the higher-order remainder terms corresponding to $C^R$ are quartic version of (8.5), and therefore easier to handle compared to the quartic terms we discuss below. The contributions corresponding to $C^S$ are, up to a permutation of the inputs, of the form $C^S(T(g, g, a, b))$ with $a, b = f$ or $g$. In view of (8.3) and the Hölder estimates of Lemma 5.2, we see that all these are essentially standard fourfold products of the form

$$
\mathcal{R}_C[g, g, a, b] \sim e^{itB} (e^{itB} g \cdot e^{itB} g \cdot e^{itB} a \cdot e^{itB} b), \quad \text{for } a, b = f \text{ or } g,
$$

(8.7)

where $t_j, j = 1, \ldots, 4$ are signs. Note that (8.7) is a quartic version of $C^S$. Also note that the hardest term if the one with all $g$ inputs, $\mathcal{R}_C[g, g, g, g]$. For this, we can substitute one $g$ with $f$ via (8.3) up to terms that are quintic and easier to estimate, reducing matters to estimating $\mathcal{R}_C[g, g, g, f]$; then, similarly substituting one more $g$ with $f$ we can reduce to $\mathcal{R}_C[g, g, f, f]$. To estimate $\mathcal{R}_C[g, g, f, f]$, we can use a “commutation identity” similar to the one in (7.3) which we used for
the main singular cubic term. In particular, we can distribute the $\partial_\xi$ derivative on the Fourier side and obtain identities analogous to those in (7.4) (with four input function instead of three). We can then estimate very similarly to Sect. 7 using (8.4), the a priori bounds (5.24), (5.28) and an identical estimate for $\partial s g$, arriving at the bound

$$\|\langle \xi \rangle \partial_\xi \int_0^t R_C [g, g, f, f] ds \|_{L^2} \lesssim \int_0^t \| \partial_\xi \tilde{g} \|_{L^2} \cdot (\epsilon_1(s)^{-1/2})^3 ds + \epsilon_1 \lesssim \epsilon_1^{\alpha/10}. \tag{8.8}$$

8.3. The Sobolev Norm

The bootstrap on the Sobolev-type norm $\|\langle \xi \rangle^4 \tilde{f} \|_{L^2}$ is substantially easier to obtain than the weighted bound. The estimate for the cubic terms follows directly from Lemma 5.2 and the natural extension of (5.18) with derivatives; see [5, Lemma 6.13].

A slightly less immediate argument is required to estimate the quadratic terms $Q_R$. For this, we need to consider a term of the form

$$I(s, \xi) := \iint e^{is(\langle \xi \rangle - \langle \eta \rangle - \langle \sigma \rangle)} \langle \xi \rangle^4 q(\xi, \eta, \sigma) \tilde{f}(\eta) \partial_\sigma \tilde{f}(\sigma) \eta \langle \eta \rangle \langle \xi \rangle^4 q'(\xi, \eta, \sigma) d\eta d\sigma,$$

see (5.8)–(5.9), where $q'$ satisfies (5.10). It suffices to bound (8.9) in $L^2$ by $\epsilon_1^2(s)^{-1+p_0}$. Note that since $p_0 \ll \alpha$ this does not follow immediately from integration by parts in frequency. We then proceed as follows. First, we may assume, by the symmetry in $\eta$ and $\sigma$, and the decay of the kernel away from $\xi - \eta - \sigma = 0$, that $1 \leq |\xi| \lesssim |\sigma| \leq |\eta|$. Second, we can integrate by parts in $\sigma$ and estimate the main contribution by

$$\left| \iint_{\mathbb{R}^2_+} \frac{1}{s} e^{is(\langle \xi \rangle - \langle \eta \rangle - \langle \sigma \rangle)} \tilde{f}(\eta) \partial_\sigma \tilde{f}(\sigma) \frac{\eta}{\langle \eta \rangle} \frac{\langle \xi \rangle^4}{\langle \eta \rangle + \langle \sigma \rangle} q'(\xi, \eta, \sigma) d\eta d\sigma \right|. \tag{8.10}$$

Then, we see that if we restrict the integration region to $|\eta| \gtrsim (s)^{2\alpha}$, an application of Young’s inequality suffices:

$$\left\| \int_{\mathbb{R}^2_+ \cap \{|\eta| \gtrsim (s)^{2\alpha}\}} \frac{1}{s} e^{is(\langle \xi \rangle - \langle \eta \rangle - \langle \sigma \rangle)} \tilde{f}(\eta) \partial_\sigma \tilde{f}(\sigma) \frac{\eta}{\langle \eta \rangle} \frac{\langle \xi \rangle^4}{\langle \eta \rangle + \langle \sigma \rangle} q'(\xi, \eta, \sigma) d\eta d\sigma \right\|_{L^2} \lesssim s^{-1-2\alpha} \| \langle \xi \rangle^4 \tilde{f} \|_{L^2} \| \partial_\xi \tilde{f} \|_{L^1} \lesssim s^{-1-\alpha + p_0} \epsilon_1^2.$$

When instead \(|\eta| \lesssim \langle s \rangle^{2\alpha}\), we can integrate by parts also in \(\eta\) and bound the main contribution in \(L^2\) by

\[
\left\| \int_{\mathbb{R}^2_+ \cap \{|\eta| \lesssim \langle s \rangle^{2\alpha}\}} \frac{1}{s^2} e^{is(\xi-(\eta-(\sigma))} \partial_\eta \tilde{f}(\eta) \partial_\sigma \tilde{f}(\sigma) \frac{\langle \xi \rangle^4}{\langle \eta \rangle + \langle \sigma \rangle} q'(\xi, \eta, \sigma) \, d\eta \, d\sigma \right\|_{L^2} 
\lesssim s^{-2} \left\| \int_{\mathbb{R}^2_+ \cap \{|\eta| \lesssim \langle s \rangle^{2\alpha}\}} \left| \partial_\eta \tilde{f}(\eta) \right| \left| \partial_\sigma \tilde{f}(\sigma) \right| \frac{1}{\langle \xi - \eta - \sigma \rangle^N} \, d\eta \, d\sigma \right\|_{L^2} 
\lesssim s^{-2+6\alpha} \| \partial_\xi \tilde{f} \|_{L^2} \| \partial_\xi \tilde{f} \|_{L^1} \lesssim s^{-2+8\alpha} \varepsilon_1^2,
\]

which is more than sufficient.

The remainder terms (5.21) can be handled similarly, and in fact are even easier to estimate via direct applications of the Hölder bounds of Lemmas 5.1 and 5.2 and arguing as in Sect. 8.1 to estimate the \(T\) operator.

### 8.4. Fourier \(L^\infty\) Estimates and Asymptotics

Finally, we briefly discuss how to close the bootstrap for the norm \(\| \langle \xi \rangle^{3/2} \tilde{f} \|_{L^\infty}\) in (5.25). The simple observation is that since our a priori assumptions (5.24) are stronger than the assumptions made in [5], but the conclusion we want is exactly the same for this norm, all of the arguments used in [5] apply here. In particular, we can replace the a priori bound (10.31) in [5, Sec. 10.3], that is (recall the notation (1.6))

\[
\| \varphi_{[-5,5]} \partial_\xi \tilde{f} \|_{L^2} \lesssim \varepsilon_1(t)^\rho,
\]

in which \(\rho = \alpha + \beta \gamma = 1/4-\), by the stronger

\[
\| \varphi_{[-5,5]} \partial_\xi \tilde{f} \|_{L^2} \lesssim \varepsilon_1(t)^\alpha,
\]

with \(\alpha > \rho_0\) being small as in Proposition 5.3. Then, under the a priori assumption (5.24) for \(\tilde{f}\), we can prove

\[
\| \langle \xi \rangle^{3/2} \tilde{f} \|_{L^\infty} \leq C \varepsilon_0 + C \varepsilon_1^2.
\]

(8.11)

The argument used to obtain (8.11) is based on the derivation of an (Hamiltonian) ODE for \(\tilde{f}\) which relies on precise asymptotics for the cubic terms \(C^S\); this ODE also gives the asymptotic modified scattering behavior in (1.3). We refer the reader to Proposition 10.1 and [5, Sec 10] for detailed statements and proofs.

As it turns out, we believe that the arguments that have been used in [5] to bound the cubic terms, and obtain their asymptotics in time, cannot be simplified much. One can, however, simplify the estimates for the other higher-order remainder terms \(R(f, g)\), see (5.20)–(5.23), as we explain below.

First, we observe that the weighted bound (6.1) for \(Q^R\) and the weighted bound (8.6) for \(R_Q\) imply the Fourier \(L^\infty\) bound for these terms via the interpolation

\[
\| \langle \xi \rangle^{3/2} h \|_{L^\infty} \lesssim \| \langle \xi \rangle \partial_\xi h \|_{L^2}^{1/2} \| \langle \xi \rangle^2 h \|_{L^2}^{1/2},
\]

and the fact that a bound on \(\| \langle \xi \rangle^2 h \|_{L^2}^{1/2}\)
which decays in time is easy to obtain for $h = Q^R$ or $Q^R$. Indeed, one can obtain $\|\langle \xi \rangle^2 \int_0^t Q^R ds\|_{L^2} \lesssim \varepsilon_1^2$ proceeding in a similar way as in the estimate of $\|\langle \xi \rangle Q^R\|_{L^2}$ in Lemma 5.4; see the proof of (5.30). Hence, we can bound

$$\left\| \langle \xi \rangle^{3/2} \int_0^t Q^R ds \right\|_{L^\infty} \lesssim \left\| \langle \xi \rangle \partial_\xi \int_0^t Q^R ds \right\|_{L^2}^{1/2} \left\| \langle \xi \rangle \int_0^t Q^R ds \right\|_{L^2}^{1/2} \lesssim \varepsilon_1^2.$$ 

For $RQ$, we can obtain $\|\langle \xi \rangle^2 \int_0^t RQ ds\|_{L^2} \lesssim \varepsilon_1^2$ via a similar argument as in the estimate (8.6), since $RQ$ is a cubic version of the quadratic terms $Q^R$, and then conclude using interpolation as above.

Finally, for the term $RC$ in (5.23) we can first substitute all the $g$’s by $f$’s, noticing that the error terms generated in this process are higher-order remainders. Then, we can estimate very similarly to Sect. 7 using the a priori bounds (5.24), and (5.28) to obtain the following analogue of (8.8) with $f$ instead of $g$:

$$\left\| \langle \xi \rangle \partial_\xi \int_0^t RC [f, f, f, f] ds \right\|_{L^2} \lesssim \int_0^t \left\| \partial_\xi \tilde{f} \right\|_{L^2} \cdot (\varepsilon_1 \langle s \rangle^{-1/2})^3 ds + \varepsilon_1^4 \lesssim \varepsilon_1^4.$$ 

Then we obtain $\|\langle \xi \rangle^{3/2} \int_0^t RC ds\|_{L^\infty} \lesssim \varepsilon_1^4$ via interpolation with a simpler Sobolev bound as above.

All the terms on the right-hand side of (5.20) are thus accounted for, Proposition 5.3 follows, and the main Theorem 1.2 is proven.

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