Elimination of the spin supplementary condition in the effective field theory approach to the post-Newtonian approximation

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The present paper addresses open questions regarding the handling of the spin supplementary condition within the effective field theory approach to the post-Newtonian approximation. In particular it is shown how the covariant spin supplementary condition can be eliminated at the level of the potential (which is subtle in various respects) and how the dynamics can be cast into a fully reduced Hamiltonian form. Two different methods are used and compared, one based on the well-known Dirac bracket and the other based on an action principle. It is discussed how the latter approach can be used to improve the Feynman rules by formulating them in terms of reduced canonical spin variables.

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I. INTRODUCTION

Recently important progress has been made in the analytic treatment of self-gravitating spinning compact objects in general relativity using different methods. One of these methods is based on an effective field theory (EFT) point of view, which was applied, e.g., within the post-Newtonian (pN) approximation to non-spinning compact objects [1–5] and to spinning objects [6–15]. An advantage of this approach is that some of the very sophisticated and systematic techniques for perturbative calculations used in high energy physics can be applied in a straightforward manner. The present paper addresses certain open questions regarding the handling of the spin supplementary condition (SSC) within this approach, though some aspects may be applicable to other approaches as well. In particular it is shown how the covariant SSC can be eliminated at the level of the potential (which is subtle in various respects) and how the dynamics can be cast into a fully reduced Hamiltonian form. For a review of spin in relativity and the problem of the SSC see, e.g., [16–18].

In classical Newtonian mechanics the spin of an object is described by a 3-dimensional antisymmetric tensor or by its dual vector. In some situations (e.g., when the dynamics depends on the spin but not on the absolute orientation of the objects) it is very convenient to associate a Poisson bracket representation of the so(3) Lie algebra (i.e., the angular momentum algebra) with the spin variables and describe the spin dynamics in terms of a function generating the time evolution via these Poisson brackets. This function may be a Hamiltonian or a Routhian [19, 20]. (The latter is a Hamiltonian for a part of the variables only and a Lagrangian for the remaining variables.) Such a formulation of spin is also desirable in the relativistic case. But in this case the spin is given by a 4-dimensional antisymmetric tensor, so the best one can immediately achieve is to relate it to the so(1,3) Lie algebra of the SO(1,3) Lorentz group. However, it is well-known that the fixation of a representative worldline or center of the spinning objects is equivalent to a supplementary condition on the spin components. With this SSC the independent components of the 4-dimensional antisymmetric spin tensor are given by its 3-dimensional spatial part. But the reduction of the so(1,3) algebra for the spin components to a so(3) algebra is subtle and must be discussed within the framework of constrained Hamiltonian dynamics. Using the Dirac bracket approach this was performed for flat spacetimes [21] and for test-spinning objects in curved spacetime [22]. For self-gravitating spinning objects the reduction succeeded to linear order in spin with the help of an action principle [23, 24] and agrees with a construction via generators of
the global Poincaré algebra valid to next-to-next-to-leading pN order [24–26]. But this derivation is focused on the canonical formalism of Arnowitt, Deser, and Misner (ADM) [27, 28] and is restricted to the Schwinger time gauge family of vierbein gauges [29] (which unfortunately does not include the gauge used in the EFT approach).

A consistent way to deal with the (covariant) SSC within the EFT method is described in [8] (based on developments in [6], see also [13] and [7] for a sophisticated Kaluza-Klein-like reduction of the components of the metric tensor for further construction of the Feynman rules). There a Routhian generates the spin evolution via the so(1, 3) algebra and the center of mass motion is given by Euler-Lagrange equations (see also [30]). The covariant SSC is eliminated at the level of the equations of motion. In the present paper we show how the covariant SSC can consistently be eliminated already at the level of the potential using two different methods. One method is based on the well-known Dirac bracket and the other on an action principle. It is then straightforward to obtain a fully reduced Hamiltonian form of the dynamics. This is very convenient, e.g., for deriving the fully reduced equations of motion (where the length of the 3-dimensional spin vector is constant without further variable transformations) or for an implementation into the effective one body formalism, see [31–33] and references therein. We treat the pN next-to-leading order (NLO) spin-orbit, spin(1)-spin(2), and spin(1)-spin(1) level here. The relevant potentials were derived in [8–14] within the EFT formalism. Recently also the corresponding source multipole moments were calculated [34] (see also [35, 36]). The NLO spin-orbit and spin(1)-spin(2) dynamics was obtained earlier in [25, 37–40] and extended to arbitrary many objects in [41]. The NLO spin(1)-spin(1) Hamiltonian for binary black holes was given with correct center of mass motion in [42, 43]. In [44] the NLO spin(1)-spin(1) dynamics for general compact objects (including neutron stars) was reproduced and put into fully reduced Hamiltonian form. Higher orders in spin were treated in [43, 45]. For radiation-reaction effects on the motion of a binary due to spin see, e.g., [46] and references therein. Very recently even the next-to-next-to-leading order (NNLO) spin-orbit [47] and spin(1)-spin(2) level was tackled, the latter simultaneously by a potential within the EFT approach [15] and by a fully reduced Hamiltonian [48]. An extension of the results in the present article should be useful to relate these two results at NNLO spin(1)-spin(2). Notice that above results were obtained only very recently compared to the first treatments of self-gravitating spinning objects within the post-Minkowskian [49] and post-Newtonian [50, 51] approximations (see also [52–56]).

It should be noted that the basic approach used in the present paper was already described in [44]. Therein also the result for the transformation to canonical variables was used in advance, but the presentation of the derivation was reserved for the present paper (some more details can also be found in [57]). In the meantime the basic approach was also applied in [9, 13], but still the transformation to canonical variables was not derived from general principles.

The paper is organized as follows. In Sect. II an overview of the problem addressed in the present article is given. In Sect. III the known potentials are transformed into non-reduced canonical form, i.e., still with a so(1, 3) Poisson bracket algebra for the spins. In Sect. IV the Dirac bracket for the covariant SSC is calculated and transformed to standard canonical form, i.e., with a so(3) Poisson bracket algebra for the spins. An alternative elimination procedure via an action principle is performed in Sect. V. At the end of Sect. V it is discussed how this alternative approach can be used to improve the Feynman rules of the EFT formalism by formulating them in terms of reduced canonical spin variables. In Sect. VI the non-reduced Hamiltonians obtained earlier are transformed into fully reduced ones and compared with other results. Finally conclusions and outlook are given in Sect. VII.

Our units are such that $c = 1$, where $c$ is the velocity of light and also the implicit inverse pN expansion parameter with the formal counting rule $1/c^n \sim \frac{n}{2} pN$ order. Adapted to EFT convention we work in the spacetime signature $–2$ which is important to remember especially when working on the action level, see Sect. V. Three different frames are utilized in this article, denoted by different indices. Greek indices ($\alpha, \mu, \ldots$) refer to the coordinate frame, lower case Latin indices from the beginning of the alphabet ($a, b, \ldots$) belong to the local Lorentz frame, and upper case Latin indices from the beginning of the alphabet ($A, B, \ldots$) denote the so called body-fixed Lorentz frame. Lower case Latin indices from the middle of the alphabet ($i, j, \ldots$) are used for the spatial part of the mentioned frames and are running through ($i = 1, 2, 3$). In order to distinguish the three frames when splitting them into spatial and time part, we write $a = (0), (i)$ for Lorentz indices (or $a = (0), (1), (2), (3)$ in more detail), $A = [0], [i]$ for the body-fixed frame, and $\mu = 0, i$ for the coordinate frame. Letters $I$ and $J$ are body labels, i.e. $I, J \in \{1, 2\}$ and $z \equiv (z^I)$ denotes a point in the 3-dimensional Euclidean space $\mathbb{R}^3$ endowed with a standard Euclidean metric $(z^I z_J = \delta_{IJ} z_I z_J)$ and a scalar product (denoted by a dot), so $z_I \in \mathbb{R}^3$ denotes the position of the $I$th body. Indices appearing twice in a product are implicitly summed over its range, except for label indices of the objects. Round and square brackets are also used for index symmetrization and antisymmetrization, respectively, e.g., $A^{(\mu \nu)} \equiv \frac{1}{2} \left( A^{\mu \nu} + A^{\nu \mu} \right)$. We also define $r_I := z_I - z_I$, $r_I := [r_I]$, $n_I := [r_I]/r_I$; and for $I \neq J$, $r_J := z_J - z_J$, which is the distance vector of the two bodies. Likewise we define $r_{IJ} := [r_{IJ}]$, $n_{IJ} := [r_{IJ}]/r_{IJ} = |r_{IJ}|$ stands for the length of a vector. The linear momentum of the $I$th body is denoted by $p_I = (p_{I1})$, and $m_I$ denotes its mass parameter. An overdot, as in $\dot{z}$, means the total time derivative. The spin vector of the $I$th body is denoted by $S_I$ and is always supposed to be referred to the local Lorentz frame in all potentials and Hamiltonians that are considered in the present article. The connection between the antisymmetric spin tensor $S_{(i,j)}$ and the spin vector is made by usage of the totally antisymmetric $c$-symbol $\epsilon_{ijk} = \frac{1}{2} (i - j)(j - k)(k - i)$ as $S_{(i,j)} = \epsilon_{ijk} S_{(k)}$. Each body is within our approximation completely characterized by
its three parameters mass, momentum and spin. We associate to them the following relative pN order

\[ m_N \sim O \left( \frac{1}{c^2} \right), \quad p_N \sim O \left( \frac{1}{c^2} \right), \quad S_N \sim O \left( \frac{1}{c^2} \right), \]

yielding the formal pN order for the Newtonian (N), leading-order (LO) and next-to-leading-order (NLO) Hamiltonians in question

\[ H_N \sim O \left( \frac{1}{c^2} \right), \quad H_{LO} \sim O \left( \frac{1}{c^2} \right), \quad H_{NLO} \sim O \left( \frac{1}{c^2} \right), \]

starting with the non-relativistic Newtonian Hamiltonian for two interacting bodies in canonical conjugate variables:

\[ H_N = \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} - \frac{Gm_1m_2}{r_{12}}. \]

Notice that \( H_N \) defines the zeroth pN order for Hamiltonians, so the pN orders of the LO and NLO Hamiltonians must be counted relative to \( H_N \) (which is at \( 1/c^4 \) in the formal counting given above).

II. SETUP OF THE PROBLEM

In [8] an EFT approach to include spin within the pN approximation is given, based on developments in [5] (see also [13]). This approach is able to deliver a pN approximate description of the dynamics of spinning compact objects via a Routhian, namely a function which is a Lagrangian for the objects 3-dimensional position \( z \) and a Hamiltonian for the 4-dimensional antisymmetric spin tensor \( S_{ab} \) (in this section we may drop the object labels). Therefore this Routhian generates the spin evolution via an \( so(1,3) \) Poisson bracket algebra and the center of mass motion is given by Euler-Lagrange equations (such an approach was already used in [30]). In order to allow a treatment of the covariant SSC via Dirac brackets one needs a canonical description of all variables, including \( z \). Therefore we first perform a standard Legendre transformation in order to replace the velocity \( v = \dot{z} \) by its generalized momentum \( p \), which is of course the canonical conjugate of \( z \). (Eventually accelerations or even higher time derivatives must be eliminated using the method in [58] first.)

It is important that \( S_{ab} \) is actually the generalized momentum of the 4-dimensional angular velocity tensor defined by

\[ \Omega^{ab} = \Lambda^a \dot{A}^{ab}. \]  

The latter is build from a Lorentz matrix \( \Lambda^{Aa} \in SO(1,3) \),

\[ \Lambda_{ab} \Lambda^{ab} = \eta_{ab}, \]

which relates the body-fixed and the local Lorentz frames; to be more precise \( \Lambda^{Aa} \) is a representation of elements of \( SO(1,3) \). Here \( \eta_{ab} = \text{diag}(1, -1, -1, -1) \) is the Lorentz metric with signature \(-2\). Notice that this property of the Lorentz matrix makes the angular velocity tensor antisymmetric. The complete Poisson brackets read, see, e.g., [21] or Sect. V below,

\[ \{ z^i, p_j \} = \delta_{ij}, \]

\[ \{ \Lambda^A_{\; b}, S_{cd} \} = \eta_{bc} \Lambda^A_{\; d} - \eta_{bd} \Lambda^A_{\; c}, \]

\[ \{ S_{ab}, S_{cd} \} = S_{ca} \eta_{bd} - S_{da} \eta_{bc} + S_{ab} \eta_{cd} - S_{cd} \eta_{ab}, \]

and all other zero. We will call these brackets canonical here, though they are not canonical in a strict sense (e.g., the spin components are not commuting). But it is possible to relate \( \Lambda^{Aa} \) and \( S_{ab} \) to variables for which the phase space structure is manifest, see Sect. 3.A in [21]. For example, one can parametrize \( \Lambda^{Aa} \) by independent angle-type variables. The time derivatives of these variables are angular velocities and their generalized momenta are canonically conjugate to the angle-type variables. (These angular velocities are also contained in \( \Omega^{ab} \), but with prefactors depending on the angle-type variables.) Notice that [8] uses a different sign convention for the spin part of the Poisson bracket shown above. Further the Hamiltonian does actually not depend on \( \Lambda^{Aa} \) in our case, so one can ignore \( \Lambda^{Aa} \) and its Poisson brackets for obtaining the equations of motion. For the present article, however, \( \Lambda^{Aa} \) is of crucial importance.

The Poisson brackets above are fully canonical, but the degrees of freedom are not fully reduced. An SSC must be imposed, which corresponds to a choice for the representative worldline of the compact object. A similar condition
must be given for $\Lambda^{Aa}$ [21]. According to [21] the covariant supplementary conditions read (consistent with the SSC used in [8])

$$S^{ab}p_b^D = 0, \quad \eta^{[0]}A = \Lambda^{Ab} \frac{p_D^b}{m_D},$$  \hspace{1cm} (II.4)

where $p_D^b$ is Dixon’s momentum of the compact object and $m_D$ the dynamical mass. It holds $m_D^2 = g_{\mu\nu}p^\mu_D p^\nu_D$, where $g_{\mu\nu}$ is the 4-dimensional metric. From (5.13) in [24] we get

$$p_D^\mu = m \frac{u_\mu}{\sqrt{u_\nu u^\nu}} + \mathcal{O}(S^2), \quad m = m_D + \mathcal{O}(S^2),$$  \hspace{1cm} (II.5)

where $m$ is the constant mass parameter of the object. It should be noted that the mass-shell constraint is already implicitly eliminated within the approach in [8], as a gauge-fixing for the worldline parameter was performed. (Indeed, only a 3-dimensional canonical momentum $p_i$ is defined by the Legendre transformation mentioned above, but not its time component $p_0$.) The worldline parameter was chosen to be the coordinate time in [8], $u^0 = 1$. Notice that (II.4) guarantees that in the rest frame the spin tensor contains the 3-dimensional spin $S^{(i)(j)}$ only (i.e., the mass dipole part $S^{(0)(i)}$ vanishes) and that in the rest frame $\Lambda^{Ab}$ describes a pure 3-dimensional rotation (no Lorentz boosts). This obviously reduces the degrees of freedom to the physically relevant ones, which are given by $S^{(0)(i)}$ and $\Lambda^{(i)(j)}$.

The most prominent way to handle the constraints (II.4) on the phase space described by (II.3b, II.3c) is provided by the Dirac bracket, denoted by $\{\cdot,\cdot\}_D$ here. It is straightforward to calculate the Dirac bracket for the current situation in a pN approximate way. Notice that only three components of each condition in (II.4) are independent, so one has six independent constraints for each particle. To the considered approximation the derivation will turn out to be very similar to [21], with the notable exception that in [21] the mass-shell constraint together with the gauge-fixing of the worldline parameter was also treated using the Dirac bracket. The Dirac bracket is essentially the Poisson brackets of the reduced phase space, but the variables used above are not canonical any more with respect to the Dirac bracket. Our next step is thus to transform $z^i$, $p_i$, $S^{(i)(j)}$, and $\Lambda^{(i)(j)}$ to new canonical variables denoted by a hat such that

$$\{\hat{z}^i, \hat{p}_j\}_D = \delta^i_j,$$  \hspace{1cm} (II.6a)

$$\{\hat{\Lambda}^{(i)(j)}, \hat{S}^{(m)(n)}\}_D = -\delta_{im} \hat{\Lambda}^{(i)(n)} + \delta_{jn} \hat{\Lambda}^{(i)(m)},$$  \hspace{1cm} (II.6b)

$$\{\hat{S}^{(i)(j)}, \hat{S}^{(k)(l)}\}_D = \delta_{jl} \hat{S}^{(i)(k)} - \delta_{ik} \hat{S}^{(j)(l)} + \delta_{lk} \hat{S}^{(j)(i)} - \delta_{jk} \hat{S}^{(i)(l)},$$  \hspace{1cm} (II.6c)

and all other zero. The algebra for the spin was reduced from so(1,3) to so(3) and the Lorentz matrix $\Lambda^{Aa} \in SO(1,3)$ was transformed into $\hat{\Lambda}^{(i)(j)} \in SO(3)$. These variables are a suitable generalization of the Newton-Wigner variables defined in flat spacetime [59], but they are not canonically equivalent to the Newton-Wigner variables introduced in [10] (for a discussion see [60]). Notice that the transformation to canonical variables is highly ambiguous, i.e., one may always perform a canonical transformation. It turns out that we are able to choose $\hat{p}_i = \hat{p}_i$ here. Further we will actually not derive the transformation to $\hat{\Lambda}^{(i)(j)}$ via Dirac brackets, as the Hamiltonian does not depend on it anyway. For the same reason the components of the spin tensor in the body-fixed frame $\hat{S}^{(i)(j)}$ and thus the spin length $s$ given by $2s^2 = \hat{S}^{(i)(j)} \hat{S}^{(i)(j)}$ are conserved. It should be noted that in [22] a canonical Newton-Wigner SSC for test-bodies in curved spacetime was handled by Dirac brackets directly (and consistently implemented into the action). However, from a general point of view it is very convenient to start with a covariant SSC, as this manifestly displays the covariance of the effective theory, in particular when higher dimensional operators are included in the worldline action. In the following the SSC is always assumed to be the covariant one, if not otherwise stated.

Spin in relativity can also be treated by an action principle [61–63], see also [6, 21, 23, 24, 64] and appendix A of [65]. It is indeed possible to derive the transformation to reduced canonical variables using an action approach. The Poisson brackets (II.3b, II.3c) are essentially represented by a term in the action of the form $p_iz^i + \frac{1}{2}S_{ab}\hat{\Lambda}^{Aa}\hat{\Lambda}^{Ab}$. After the supplementary conditions (II.4) are inserted, one must find new variables such that this term takes on the form $\hat{p}_iz^i + \frac{1}{2}\hat{S}_{(i)(j)}\hat{\Lambda}^{(i)(k)}\hat{\Lambda}^{(j)(k)}$, which precisely represents the reduced brackets (II.6b, II.6c). Here it is important that $\hat{\Lambda}^{(i)(j)} \in SO(3)$ must be a 3-dimensional rotation matrix, $\hat{\Lambda}^{(i)(j)}\hat{\Lambda}^{(k)(j)} = \delta_{ij}$. This approach is very similar to [23, 24]. Notice that one needs the transformation from $\Lambda^{(i)(j)}$ to $\hat{\Lambda}^{(i)(j)}$ for the action approach, which is not necessary for the Dirac bracket approach.
The effective potential usually depends on velocities and positions when referred to a Lagrangian $L$ defined as the difference between the non-relativistic Newtonian kinetic part $T_N$ and the effective potential $V_{\text{eff}}$:

$$ L_{\text{eff}} = T_N - V_{\text{eff}} = \frac{m_1}{2} \dot{v}_1^2 + \frac{m_2}{2} \dot{v}_2^2 - V_{\text{eff}}. \quad (\text{III.1}) $$

The effective potential is the only part of the Lagrangian which is pN expanded and is further decomposed into different spin contributions. The first step is to arrive at a reduced canonical Hamiltonian when starting with a non-reduced effective potential $V_{\text{eff}} = V_{\text{eff}}(x_1, \mathbf{v}_1, S_{I(0)(j)}, S_{I(0)(k)})$ in pN approximation is to Legendre transform it only with respect to the velocities/momenta (the spin variables are formally kept unchanged by this procedure) to a non-reduced effective Hamiltonian $H_{\text{eff}} = H_{\text{eff}}(x_1, p_1, S_{I(0)(j)}, S_{I(0)(k)})$. In both expressions the SSC is not yet imposed, so that $S_{I(0)(j)}$ will be treated as an independent variable. Furthermore the effective Hamiltonians are not supposed to contain any time derivatives of the variables except for the one of the position variable being defined as the velocity. If the case arose that a spin variable (including the constrained $S_{I(0)(j)}$ ones) would carry a time derivative one could either replace it with its lower order equations of motion or one could shift it onto positions and/or velocities in the same term by neglecting total time derivatives which serve as surface terms in the action. The last procedure ensures to leave us only with time derivatives of variables which are not further subject to a constraint, because the mass-shell constraint is already eliminated, when performing Legendre transformation. Those variables can therefore be treated differently aside from the Dirac bracket formalism but rather with a fully reduced Poisson bracket and a subleading constraint is already eliminated, when performing Legendre transformation. Those variables can therefore be treated differently aside from the Dirac bracket formalism but rather with a fully reduced Poisson bracket and a subleading order equations of motion for eliminating higher order time derivatives as outlined in \cite{58, 66}. The Dirac bracket formalism will be carried out below for the SSCs in order to find a canonical set of variables after using the SSCs. The effective potential $V_{\text{eff}}$ for two interacting bodies is pN expanded up to NLO spin effects:

$$ V_{\text{eff}} = V_{\text{pp}} + V_{\text{SO}}^{\text{LO}} + V_{S_1^2}^{\text{LO}} + V_{S_2^2}^{\text{LO}} + V_{S_1S_2}^{\text{LO}} + V_{S_1S_2}^{\text{NLO}} + V_{S_1S_2}^{\text{NLO}} + V_{S_1S_2}^{\text{NLO}}. \quad (\text{III.2}) $$

$V_{\text{pp}}$ is the point particle interaction potential. This again is decomposed into

$$ V_{\text{pp}} = V_N + V_{\text{EIH}}^{1\text{pN}} + V_{2\text{pN}}, \quad (\text{III.3}) $$

starting with the Newtonian potential

$$ V_N = \frac{G m_1 m_2}{r_{12}}, \quad (\text{III.4}) $$

and continuing with the Einstein-Infeld-Hoffmann (EIH) potential ($V_{\text{EIH}}^{1\text{pN}} = -L_{\text{EIH}}$) \cite{1, 67}

$$ L_{\text{EIH}} = \frac{1}{8} \sum_a m_a \mathbf{v}_a^4 + \frac{G m_1 m_2}{2r_{12}} \left[ 3 \left( \mathbf{v}_1^2 + \mathbf{v}_2^2 \right) - 7 (\mathbf{v}_1 \cdot \mathbf{v}_2) - (\mathbf{v}_1 \cdot \mathbf{n}_{12}) (\mathbf{v}_2 \cdot \mathbf{n}_{12}) \right] - \frac{G^2 m_1 m_2 (m_1 + m_2)}{2r_{12}^2}. \quad (\text{III.5}) $$

The 2pN point potential potential is in fact not needed for the Legendre transformation, the interested reader can find it in \cite{2}. The leading order (LO) spin contributions are decomposed into the LO spin-orbit (SO) contribution in non-reduced form, see \cite{12}:

$$ V_{\text{SO}}^{\text{LO}} = \frac{G m_2}{r_{12}^2} n_{12}^I \left[ S_1^{(j)(0)} + S_2^{(j)(k)} (v_1^k - 2v_2^k) \right] + (1 \leftrightarrow 2), \quad (\text{III.6}) $$

the LO spin(1)-spin(2) contribution, e.g. from \cite{8}

$$ V_{S_1S_2}^{\text{LO}} = \frac{G}{r_{12}^2} \left[ 3 (S_1 \cdot \mathbf{n}_{12}) (S_2 \cdot \mathbf{n}_{12}) - (S_1 \cdot S_2) \right], \quad (\text{III.7}) $$

and the LO spin(I)-spin(I) (finite size) contributions from \cite{10, 36}

$$ V_{S_i^2}^{\text{LO}} = C_{Qi} \frac{G m_1}{2m_1 r_{12}^I} \left( 3 (S_i \cdot \mathbf{n}_{12})^2 - S_i^2 \right), \quad (\text{III.8}) $$
with the spin quadrupole constant $C_{QJ}$, which is chosen such that it is equal to one for black holes and correspondingly bigger for white dwarfs or neutron stars. The Legendre transformation with respect to the velocities/momenta is done by using the formula

$$H_{\text{eff}} = \mathbf{v} \cdot \mathbf{p} + \mathbf{v} \cdot \mathbf{p}_{2} - L_{\text{eff}}$$

$$= \mathbf{v} \cdot \mathbf{p} + \mathbf{v} \cdot \mathbf{p}_{2} - \frac{1}{2}m_{1}\mathbf{v}_{1}^{2} - \frac{1}{2}m_{2}\mathbf{v}_{2}^{2} + V_{\text{eff}}$$

$$= \frac{\mathbf{p}^{2}}{2m_{1}} + \frac{\mathbf{p}_{2}^{2}}{2m_{2}} - \frac{1}{2m_{1}}(\mathbf{p}_{1} - m_{1}\mathbf{v}_{1})^{2} - \frac{1}{2m_{2}}(\mathbf{p}_{2} - m_{2}\mathbf{v}_{2})^{2} + V_{\text{eff}}$$

with

$$\mathbf{p}_{J} = \frac{\partial L_{\text{eff}}}{\partial \mathbf{v}_{J}} = m_{J}\mathbf{v}_{J} - \frac{\partial V_{\text{eff}}}{\partial \mathbf{v}_{J}}.$$  

(III.10)

This leaves us with the expression

$$H_{\text{eff}} = \frac{\mathbf{p}^{2}}{2m_{1}} + \frac{\mathbf{p}_{2}^{2}}{2m_{2}} - \frac{1}{2m_{1}}\left(\frac{\partial V_{\text{eff}}}{\partial \mathbf{v}_{1}}\right)^{2} - \frac{1}{2m_{2}}\left(\frac{\partial V_{\text{eff}}}{\partial \mathbf{v}_{2}}\right)^{2} + V_{\text{eff}}$$

(III.11)

$$= \frac{\mathbf{p}^{2}}{2m_{1}} + \frac{\mathbf{p}_{2}^{2}}{2m_{2}} + H(\partial \mathbf{v})^{2} + V_{\text{eff}},$$

(III.12)

$$H(\partial \mathbf{v})^{2} \equiv \frac{1}{2m_{1}}\left(\frac{\partial V_{\text{eff}}}{\partial \mathbf{v}_{1}}\right)^{2} - \frac{1}{2m_{2}}\left(\frac{\partial V_{\text{eff}}}{\partial \mathbf{v}_{2}}\right)^{2}$$

(III.13)

with $\mathbf{v}_{J} = \mathbf{v}_{J} (\mathbf{p}_{J}, \mathbf{p}_{J})_{1pN}$, so it is sufficient to know the momentum up to 1pN order, because at 2pN level correction terms to the Hamiltonian will be induced only by the square of the velocity derivative of the potential. The momentum is given by

$$\mathbf{p}_{J} = m_{J}\mathbf{v}_{J} - \frac{\partial (V_{\text{eff}}/1pN + V_{\text{LO}}^{S})}{\partial \mathbf{v}_{J}} + \mathcal{O}(c^{-7})$$

(III.14a)

$$= \left(1 + \frac{1}{2}\mathbf{v}_{J}^{2}\right)m_{J}\mathbf{v}_{J} + \frac{Gm_{J}m_{J}}{2r_{IJ}}[6\mathbf{v}_{J} - 7\mathbf{v}_{J}  - (\mathbf{n}_{IJ} \cdot \mathbf{v}_{J})\mathbf{n}_{IJ}]

+ \frac{G}{r_{IJ}^{3}}[m_{J}(\mathbf{n}_{IJ} \times \mathbf{s}_{I}) + 2m_{I}(\mathbf{n}_{IJ} \times \mathbf{s}_{J})] + \mathcal{O}(c^{-7}).$$

(III.14b)

The derivative of the potential is given by

$$\frac{\partial V_{\text{eff}}}{\partial \mathbf{v}_{J}} = -\left[\frac{m_{J}}{2}\mathbf{v}_{J}^{2}\mathbf{v}_{J} + \frac{Gm_{J}m_{J}}{2r_{IJ}}[6\mathbf{v}_{J} - 7\mathbf{v}_{J} - (\mathbf{n}_{IJ} \cdot \mathbf{v}_{J})\mathbf{n}_{IJ}]

+ \frac{G}{r_{IJ}^{3}}[m_{J}(\mathbf{n}_{IJ} \times \mathbf{s}_{I}) + 2m_{I}(\mathbf{n}_{IJ} \times \mathbf{s}_{J})] \right] + \mathcal{O}(c^{-7}).$$

(III.15)

After evaluating its square and replacing the velocity by inverting Eq. (III.14b)

$$\mathbf{v}_{J} = \left(1 - \frac{1}{2}\frac{\mathbf{p}_{J}^{2}}{m_{J}^{2}}\right)\frac{\mathbf{p}_{J}}{m_{J}} - \frac{Gm_{J}}{2r_{IJ}}[6\frac{\mathbf{p}_{J}}{m_{J}} - 7\frac{\mathbf{p}_{J}}{m_{J}} - (\mathbf{n}_{IJ} \cdot \mathbf{p}_{J})\mathbf{n}_{IJ}]

- \frac{G}{r_{IJ}^{3}}[\frac{m_{J}}{m_{J}}(\mathbf{n}_{IJ} \times \mathbf{s}_{I}) + 2(\mathbf{n}_{IJ} \times \mathbf{s}_{J})] + \mathcal{O}(c^{-5}).$$

(III.16)
the Legendre transformation can be performed. The contribution $H_{(\theta V)^2}$ from (III.13) then reads

$$
H_{(\theta V)^2} = \frac{G}{r_{12}^2} \left( \frac{1}{m_1^2} p_1^2 (n_{12} \cdot (p_1 \times S_2)) + \frac{m_2}{2 m_1} p_1^2 (n_{12} \cdot (p_1 \times S_1)) \right)
+ \frac{G^2}{r_{12}^2} \left( \frac{3m_2^2}{m_1} n_{12} \cdot (p_1 \times S_1) - \frac{7m_2}{2} n_{12} \cdot (p_2 \times S_1) 
- 7m_1 n_{12} \cdot (p_2 \times S_2) + 6m_2 n_{12} \cdot (p_1 \times S_2) \right)
+ \frac{G^2}{r_{12}^2} \left( \frac{m_2^2}{2m_1} (S_1^2 + (S_1 \cdot n_{12})^2) + 2m_1 (S_2^2 + (S_2 \cdot n_{12})^2) \right)
+ 2m_2 \left( - (S_1 \cdot S_2) + (S_1 \cdot n_{12}) (S_2 \cdot n_{12}) \right) + (1 \leftrightarrow 2) + O(e^{-10}).
$$

(III.17)

Certainly the replacement of the velocities through (III.16) has also to be done in the LO spin-orbit and the EIH potential to arrive at the fully correct NLO Hamiltonians. The LO spin-orbit potential (III.6) yields the contribution $H_{LOSSO}^{v\rightarrow p}$ excluding the SSC, which remains still untouched:

$$
H_{LOSSO}^{v\rightarrow p} = \frac{G}{r_{12}^2} S_1 \cdot \left[ \frac{m_2}{m_1} (1 - \frac{p_1^2}{2m_1^2}) (n_{12} \times p_1) - \left( 1 - \frac{p_2^2}{m_2^2} \right) (n_{12} \times p_2) \right]
+ \frac{G^2}{r_{12}^2} \left[ - m_2 \left(7 + \frac{3m_2}{m_1}\right) S_1 \cdot (n_{12} \times p_1) 
+ (6m_1 + 7m_2) S_1 \cdot (n_{12} \times p_2) 
- m_2 \left(4 + \frac{m_2}{m_1}\right) ((S_1 \cdot n_{12})^2 - S_1^2) 
- 2(m_1 + m_2) \left( (S_1 \cdot n_{12}) (S_2 \cdot n_{12}) \right) 
- (S_1 \cdot S_2) \right] + (1 \leftrightarrow 2) + O(e^{-10}).
$$

(III.18)

Likewise the EIH potential gives rise to the contribution $H_{EIH}^{v\rightarrow p}$:

$$
H_{EIH}^{v\rightarrow p} = \frac{G}{r_{12}^2} \left( \frac{1}{m_2} p_2^2 ((p_2 \times S_1) \cdot n_{12}) - \frac{m_2}{2 m_1} p_1^2 ((p_1 \times S_1) \cdot n_{12}) \right)
+ \frac{G^2}{r_{12}^2} \left( \left( - \frac{7m_2}{3m_1} \right) ((p_1 \times S_1) \cdot n_{12}) + \left( 6m_1 + \frac{7m_2}{2} \right) ((p_2 \times S_1) \cdot n_{12}) \right) 
+ (1 \leftrightarrow 2) + O(e^{-10}).
$$

(III.19)

Now we are able to evaluate from the effective potentials for the NLO spin-orbit, spin(1)-spin(2) and spin(1)-spin(1) case their effective Hamiltonian counterparts $H_{\text{eff}} = H_{\text{eff}}(x_i, p_i, S_{I(i)(j)}, S_{I(i)}(i))$, which result from Legendre transformation with respect to the velocities/momenta only. So these Hamiltonians still remain non-reduced in phase space as long as the SSC is not imposed.

**A. The non-reduced NLO spin-orbit Hamiltonians**

We consider two effective potentials from literature, one from Levi [13], and the other one from Porto [12]. Both will be subject to a Legendre transformation to arrive at $H_{\text{eff}}(x_i, p_i, S_{I(i)(j)}, S_{I(i)}(i))$. These Hamiltonians still depend on the SSC, so they are far from being canonical and deserve only formally to be called Hamiltonians, in the sense that they are the result of a Legendre transformation of the potentials but only to a subset of variables, likewise the reduction of unphysical degrees of freedom is only achieved on a subspace of phase space. Both Levi and Porto include in their NLO potentials the SSC term arising from the LO SO potential, because the elimination of the SSC is itself subject to a pN expanded expression and will therefore lift LO expressions in the potentials to NLO ones and so one. For this reason it is important to keep track of the SSC in all terms where it is present.
1. The non-reduced NLO spin-orbit Hamiltonian of the potential of Levi

We start with the effective NLO SO potential of Levi as given in [13] Eq. (109) \( V_{\text{SO}(L)}^{\text{NLO}} = -L_{\text{SO}(L)}^{\text{NLO}} \):

\[
L_{\text{SO}(L)}^{\text{NLO}} = \frac{Gm_2}{r_{12}^2} \mathbf{S}_1 \cdot \left[ \frac{1}{2} \mathbf{v}_1 \times \mathbf{n}_{12} \left( \frac{1}{2} \mathbf{v}_1 \cdot \mathbf{v}_2 - \frac{1}{2} \mathbf{v}_2^2 - \frac{3}{2} (\mathbf{v}_1 \cdot \mathbf{n}_{12}) (\mathbf{v}_2 \cdot \mathbf{n}_{12}) \right) \right. \\
+ \mathbf{v}_2 \times \mathbf{n}_{12} \left( \mathbf{v}_1 \cdot \mathbf{v}_2 - \frac{3}{2} (\mathbf{v}_1 \cdot \mathbf{n}_{12}) (\mathbf{v}_2 \cdot \mathbf{n}_{12}) \right) + \left. \frac{1}{2} \mathbf{v}_1 \cdot \mathbf{n}_{12} + \mathbf{v}_2 \cdot \mathbf{n}_{12} \right] \\
+ \frac{G^2 m_2}{r_{12}^2} \mathbf{S}_1 \cdot \left[ \frac{1}{2} \mathbf{v}_1 \times \mathbf{n}_{12} \left( 2m_1 - \frac{1}{2} m_{12} \right) + \mathbf{v}_2 \times \mathbf{n}_{12} \left( 2m_2 \right) \right] \\
+ \frac{Gm_2}{r_{12}^2} \left[ S_1^{(0)(i)} n_i \left( 1 - \frac{3}{2} \mathbf{v}_1 \cdot \mathbf{v}_2 + \frac{3}{2} v_2^2 - \frac{3}{2} (\mathbf{v}_1 \cdot \mathbf{n}_{12}) (\mathbf{v}_2 \cdot \mathbf{n}_{12}) \right) \right] \\
+ S_1^{(0)(i)} v_i^2 \left( -\frac{3}{2} \mathbf{v}_1 \cdot \mathbf{n}_{12} \right) + \frac{3}{2} S_1^{(0)(i)} v_i^2 - \frac{G^2 m_2}{r_{12}^2} S_1^{(0)(i)} n_i \left[ m_1 + 2m_2 \right] .
\]  

This potential owns a term with a time derivative of \( S_1^{(0)(i)} \). According to our agreed rule we shift it onto positions and velocities in the same term yielding

\[
L_{\text{SO}(L)}^{\text{NLO}} = \frac{Gm_2}{r_{12}^2} \mathbf{S}_1 \cdot \left[ \frac{1}{2} \mathbf{v}_1 \times \mathbf{n}_{12} \left( \frac{1}{2} \mathbf{v}_1 \cdot \mathbf{v}_2 - \frac{1}{2} \mathbf{v}_2^2 - \frac{3}{2} (\mathbf{v}_1 \cdot \mathbf{n}_{12}) (\mathbf{v}_2 \cdot \mathbf{n}_{12}) \right) \right. \\
+ \mathbf{v}_2 \times \mathbf{n}_{12} \left( \mathbf{v}_1 \cdot \mathbf{v}_2 - \frac{3}{2} (\mathbf{v}_1 \cdot \mathbf{n}_{12}) (\mathbf{v}_2 \cdot \mathbf{n}_{12}) \right) + \left. \frac{1}{2} \mathbf{v}_1 \cdot \mathbf{n}_{12} + \mathbf{v}_2 \cdot \mathbf{n}_{12} \right] \\
+ \frac{G^2 m_2}{r_{12}^2} \mathbf{S}_1 \cdot \left[ \frac{1}{2} \mathbf{v}_1 \times \mathbf{n}_{12} \left( 2m_1 - \frac{1}{2} m_{12} \right) + \mathbf{v}_2 \times \mathbf{n}_{12} \left( 2m_2 \right) \right] + \frac{Gm_2}{r_{12}^2} \left[ S_1^{(0)(i)} n_i \left( 1 - \frac{3}{2} \mathbf{v}_1 \cdot \mathbf{v}_2 + \frac{3}{2} v_2^2 - \frac{3}{2} (\mathbf{v}_1 \cdot \mathbf{n}_{12}) (\mathbf{v}_2 \cdot \mathbf{n}_{12}) \right) \right] \\
+ S_1^{(0)(i)} v_i^2 \left( -\frac{3}{2} \mathbf{v}_1 \cdot \mathbf{n}_{12} \right) + \frac{3}{2} S_1^{(0)(i)} v_i^2 - \frac{G^2 m_2}{r_{12}^2} S_1^{(0)(i)} n_i \left[ m_1 + 2m_2 \right] .
\]  

Next we insert lower order equations of motion to eliminate the acceleration term. By counting the pN order of this term in the potential it is clear that only the Newtonian equation of motion is needed as replacement which reads

\[
a_2 = \frac{Gm_1 \mathbf{n}_{12}}{r_{12}^2} .
\]  

The resulting potential will then be subject to a Legendre transformation meaning we replace velocities in \((\text{III.20b})\) by momenta indicated by \( (v \rightarrow p) \) in \( V_{\text{SO}(L)}^{\text{NLO}(v ightarrow p)} \) and add the specific NLO spin-orbit contributions from \((\text{III.17}), (\text{III.18}) \) and \((\text{III.19})\), indicated by \( \varsigma \), to arrive at the effective Hamiltonian \( H_{\text{SO}(L)}^{\text{NLO}(\text{eff})}(\mathbf{x}_1, p_1, \mathbf{S}_{1ab}, S_1^{(0)(i)})\):

\[
H_{\text{SO}(L)}^{\text{NLO}(\text{eff})} \simeq H_{(\partial V)^2} + H_{\text{LOS}}^{\text{NLO}(\text{eff})} + H_{\text{EIH}}^{\text{NLO}(\text{v ightarrow p})} + V_{\text{SO}(L)}^{\text{NLO}(\text{v ightarrow p})}
\]  

\((\text{III.22})\)
leading to the result

\[
H_{SO(L)}^{\text{NLO(\text{eff})}} = \frac{G}{r_{12}^2} \left( -\frac{m_2}{2m_1} p_2^2((p_1 \times S_1) \cdot n_{12}) - \frac{3}{2m_1^2}(p_1 \cdot n_{12})(p_2 \cdot n_{12})((p_1 \times S_1) \cdot n_{12}) 
\right.
\]

\[
\quad + \frac{1}{2m_1^2}(p_1 \cdot p_2)((p_1 \times S_1) \cdot n_{12}) - \frac{1}{2m_1m_2} p_2^2((p_1 \times S_1) \cdot n_{12})
\]

\[
\quad + \frac{3}{m_1m_2}(p_1 \cdot n_{12})(p_2 \cdot n_{12})((p_2 \times S_1) \cdot n_{12}) + \frac{1}{m_1m_2}(p_1 \cdot p_2)((p_2 \times S_1) \cdot n_{12})
\]

\[
\quad + \frac{1}{m_1}(p_1 \cdot n_{12})((p_1 \times S_1) \cdot p_2) + \frac{1}{m_1m_2}(p_2 \cdot n_{12})((p_1 \times S_1) \cdot p_2)
\]

\[
\quad + \frac{G^2}{r_{12}^2} \left( -5m_2((p_1 \times S_1) \cdot n_{12}) - \frac{7m_2^2}{2m_1}((p_1 \times S_1) \cdot n_{12}) + 6m_1((p_2 \times S_1) \cdot n_{12})
\right.
\]

\[
\quad + \frac{11m_2}{2}((p_2 \times S_1) \cdot n_{12}) - \frac{G}{m_1r_{12}^2} \left[ S_1^{(0)(i)} n_{12}^i \left( 1 - \frac{3}{2}p_1 \cdot p_2 + \frac{3}{2}p_2^2 
\right.
\right.
\]

\[
\quad \left. - \frac{3}{2}(p_1 \cdot n_{12})(p_2 \cdot n_{12}) + S_1^{(0)(i)}p_2^i \left( -\frac{3}{2}p_1 \cdot n_{12} \right) \right] + \frac{G^2m_2}{r_{12}^2} S_1^{(0)(i)} n_{12}^i \left[ m_1 + 2m_2 \right]
\]

\[
\left. \frac{3}{2} S_1^{(0)(i)} \left[ \frac{Gm_1m_2n_{12}^i}{r_{12}^2} \frac{(p_1 \cdot n_{12})p_2^i}{m_1r_{12}^2} + \frac{(p_2 \cdot n_{12})p_1^i}{m_2r_{12}^2} \right] \right).
\]

2. The non-reduced NLO spin-orbit Hamiltonian of the potential of Porto

The second alternative potential we find in Porto [12] Eq. (53)

\[
V_{SO(P)}^{\text{NLO}} = \frac{Gm_2}{r_{12}^2} \left\{ S_1^{(0)(0)} \left( 1 + 2v_2^2 - 2v_1 \cdot v_2 - \frac{3}{2}(v_2 \cdot n_{12})^2 - \frac{G}{r}(3m_1 + 2m_2) \right)
\right.
\]

\[
\quad + \left( 1 - \frac{3}{2}(v_2 \cdot n_{12})^2 + \frac{G}{2r}(4m_1 - m_2) \right) S_1^{(0)(j)} v_1^j
\]

\[
\quad - \left( 2 - 2v_1 \cdot v_2 - 3(v_2 \cdot n_{12})^2 + 2v_2^2 - \frac{G}{2r}(2m_1 + 5m_2) \right) S_1^{(0)(j)} v_2^j \right\} n_{12}^i
\]

\[
\left. + S_1^{(0)(0)} (v_1 - v_2)^i v_2^j n_{12} + S_1^{(0)(j)} v_1^j v_2^j n_{12} \right) + (1 \leftrightarrow 2).
\]
Notice that accelerations were already eliminated by inserting equations of motion. Legendre transformation yields like in the case of Levi’s potential:

\[ H_{\text{SO}(P)}^{\text{NLO}} \simeq H_{(\partial V)^2} + H_{\text{LOSO}} + H_{\text{EI}P}^{\text{NLO}} + V_{\text{SO}(P)}^{\text{NLO}(\gamma \to p)} \]

\[ H_{\text{SO}(P)}^{\text{NLO}} = \frac{G}{r_{12}^2} \left[ -\frac{m_2}{2m_1} p_2^2 ((p_1 \times S_1) \cdot n_{12}) - \frac{3}{2m_1 m_2} (p_2 \cdot n_{12})^2 ((p_1 \times S_1) \cdot n_{12}) \right. \]

\[ + \frac{3}{m_2} (p_2 \cdot n_{12})^2 ((p_2 \times S_1) \cdot n_{12}) + \frac{2}{m_1 m_2} (p_1 \cdot p_2) ((p_2 \times S_1) \cdot n_{12}) \]

\[ - \frac{1}{m_2} p_2^2 ((p_2 \times S_1) \cdot n_{12}) + \frac{1}{m_2} (p_2 \cdot n_{12})((p_1 \times S_1) \cdot p_2) \]

\[ + \left( -m_2 + \frac{3}{2m_2} (p_2 \cdot n_{12})^2 + \frac{2}{m_1} (p_1 \cdot p_2) - \frac{2}{m_2} p_2^2 \right) S_1^{(0)(i)} n_{12} \]

\[ - \frac{1}{m_1} (p_2 \cdot n_{12}) S_1^{(0)(i)} p_1 \]

\[ + \frac{G^2}{r_{12}^2} \left[ -5m_2 ((p_1 \times S_1) \cdot n_{12}) - \frac{7m_2^2}{2m_1} ((p_1 \times S_1) \cdot n_{12}) + 7m_1 ((p_2 \times S_1) \cdot n_{12}) \right. \]

\[ + 6m_2 ((p_2 \times S_1) \cdot n_{12}) + 3m_1 m_2 S_1^{(0)(i)} n_{12} + 2m_2 S_1^{(0)(i)} n_{12} \right] . \]

**B. The non-reduced NLO spin(1)-spin(2) Hamiltonian**

We take the NLO spin(1)-spin(2) potential \( V_{\text{SO}(P)}^{\text{NLO}} \) of Porto/Rothstein from [8] Eq. (56) (modulo the non-SSC-dependent LO spin-orbit terms while keeping the important SSC dependent LO spin-orbit term)

\[ V_{S_1 S_2 (P)}^{\text{NLO}} = - \frac{G}{r_{12}^2} \left[ (\delta^{ij} - 3n_{12}^i n_{12}^j) \left( S_1^{(i)(0)} S_2^{(j)(0)} + \frac{1}{2} \mathbf{v}_1 \cdot \mathbf{v}_2 S_1^{(i)(k)} S_2^{(j)(k)} \right) \right. \]

\[ + \epsilon_1^m v_2^k S_1^{(i)(k)} S_2^{(j)(m)} - \epsilon_1^k v_2^m S_1^{(i)(k)} S_2^{(j)(m)} + S_1^{(i)(0)} S_2^{(j)(0)} (v_2^k - v_1^k) + S_1^{(i)(k)} S_2^{(j)(0)} (v_1^k - v_2^k) \]

\[ + \frac{1}{2} S_1^{(i)(0)} S_2^{(j)(0)} \left( 3 \mathbf{v}_1 \cdot \mathbf{n}_{12} \mathbf{v}_2 \cdot \mathbf{n}_{12} (\delta^{ij} - 5n_{12}^i n_{12}^j) \right. \]

\[ + 3 \mathbf{v}_1 \cdot \mathbf{n}_{12} (v_2^i n_{12}^j + v_2^j n_{12}^i) + 3 \mathbf{v}_2 \cdot \mathbf{n}_{12} (v_1^i n_{12}^j + v_1^j n_{12}^i) - v_1^i v_2^j - v_2^i v_1^j \]

\[ + \left. 3n_{12}^i v_2 \cdot \mathbf{n}_{12} - v_2^i S_1^{(0)(k)} S_2^{(k)(l)} + (3n_{12}^i \mathbf{v}_1 \cdot \mathbf{n}_{12} - v_1^i S_1^{(0)(k)} S_2^{(k)(l)} \right] \]

\[ \left. + \left( \frac{G}{r_{12}^2} - \frac{3(m_1 + m_2) G^2}{r_{12}^2} \right) \right] S_1^{(j)(k)} S_2^{(i)(l)} (\delta^{ki} - 3n_{12}^i n_{12}^j) + \frac{G m_2}{r_{12}^2} n_{12}^i S_1^{(j)(0)} - \frac{G m_1}{r_{12}^2} n_{12}^i S_2^{(j)(0)} . \]
and apply to it the same Legendre transformation procedure as in the preceding section yielding

\[
H_{S_1, S_2}^{\text{NLO}(\text{eff})} \simeq H_{\partial W} + H_{\text{LOSO}} + H_{\text{EHII}}^{\nu \to p} + V_{S_1, S_2}^{\text{NLO}(\nu \to p)} \\
H_{S_1, S_2}^{\text{NLO}(\text{eff})} = \frac{G}{m_1 m_2 r_{12}^2} \left( -\frac{15}{2} (p_1 \cdot n_{12}) (p_2 \cdot n_{12}) (S_1 \cdot n_{12}) (S_2 \cdot n_{12}) \right. \\
- \frac{3}{2} (p_1 \cdot p_2) (S_1 \cdot n_{12}) (S_2 \cdot n_{12}) + \frac{9}{2} (p_1 \cdot n_{12}) (S_1 \cdot p_1) (S_2 \cdot n_{12}) \\
- \frac{3}{2} (p_2 \cdot n_{12}) (S_1 \cdot n_{12}) (S_2 \cdot p_1) - \frac{3}{2} (p_1 \cdot n_{12}) (S_1 \cdot p_2) (S_2 \cdot n_{12}) + \frac{3}{2} (S_1 \cdot p_2) (S_2 \cdot p_1) \\
+ \frac{9}{2} (p_1 \cdot n_{12}) (S_1 \cdot n_{12}) (S_2 \cdot p_2) - \frac{5}{2} (S_1 \cdot p_1) (S_2 \cdot p_2) \\
- \frac{3}{2} (p_1 \cdot n_{12}) (p_2 \cdot n_{12}) (S_1 \cdot S_2) + \frac{3}{2} (p_1 \cdot p_2) (S_1 \cdot S_2) \\
\left. + \frac{G}{r_{12}^2} \left( \frac{3}{m_1} ((p_1 \times S_2) \cdot n_{12}) - \frac{3}{m_2} ((p_2 \times S_2) \cdot n_{12}) \right) S_1^{(0)(i)} n_{12}^{(i)} \right) (\text{III.27})
\]

\[
- \frac{3}{m_2} (p_2 \cdot n_{12}) \epsilon_{ijk} n_{12}^i S_2^{(j)} S_1^{(0)(k)} - \frac{1}{m_1} \epsilon_{ijk} p_{1i} S_2^{(j)} S_1^{(0)(k)} + \frac{2}{m_2} \epsilon_{ijk} p_{2i} S_2^{(j)} S_1^{(0)(k)} \\
+ \left( \frac{3}{m_2} ((p_2 \times S_1) \cdot n_{12}) - \frac{3}{m_1} ((p_1 \times S_1) \cdot n_{12}) \right) S_2^{(0)(i)} n_{12}^{(i)} \\
+ 3 S_1^{(0)(i)} n_{12}^i S_2^{(0)(j)} n_{12}^j - S_1^{(0)(i)} S_2^{(0)(i)} \\
- \frac{3}{m_1} (p_1 \cdot n_{12}) \epsilon_{ijk} n_{12}^i S_2^{(j)} S_1^{(0)(k)} + \frac{2}{m_1} \epsilon_{ijk} p_{1i} S_2^{(j)} S_1^{(0)(k)} - \frac{\epsilon_{ijk} p_{2j}^i S_1^{(j)} S_2^{(0)(k)}}{m_2} \\
+ \frac{G m_2}{r_{12}^2} n_{12}^{(j)(0)} S_1^{(j)(0)} - \frac{G m_1}{r_{12}^2} n_{12}^{(j)} S_2^{(j)(0)} \\
+ \frac{G^2}{r_{12}^4} \left( 7 (m_1 + m_2) (S_1 \cdot S_2) - 13 (m_1 + m_2) (S_1 \cdot n_{12}) (S_2 \cdot n_{12}) \right).
\]

where \(\simeq\) indicates here focusing only on NLO spin(1)-spin(2) terms with the SSC untouched.

C. The non-reduced NLO spin(1)-spin(1) Hamiltonian

We adopt the NLO spin(1)-spin(1) potential of Porto and Rothstein that was calculated in [10] up to a missing contribution stemming from an acceleration term, which was corrected in [11]. The potential reads according to Eq. (49) of the arXiv with the LO spin-orbit SSC term included:

\[
V_{S_1}^{\text{NLO}} = C_{\text{Q1}} \frac{G m_2}{2 m_1 r^3} \left[ S_1^{(j)(0)} S_1^{(0)(j)} (3 n^j n^j - \delta^{ij}) \right. \\
- 2 S_1^{(k)(0)} \left[ (v_1 \times S_1)^k - 3 (n \cdot v_1) (n \times S_1)^k \right] \right. \\
+ C_{\text{Q1}} \frac{G m_2}{2 m_1 r^3} \left[ S_1^2 \left( 6 (n \cdot v_1)^2 - \frac{15}{2} n \cdot v_1 n \cdot v_2 + \frac{13}{2} v_1 \cdot v_2 - \frac{3}{2} v_2^2 - \frac{7}{2} v_1^2 - 2 a_1 \cdot r \right) \right. \\
+ (S_1 \cdot n)^2 \left[ \frac{9}{2} (v_1^2 + v_2^2) - \frac{21}{2} v_1 \cdot v_2 - \frac{15}{2} n \cdot v_1 n \cdot v_2 \right] + 2 v_1 \cdot S_1 v_1 \cdot S_1 \\
- 3 v_1 \cdot S_1 v_2 \cdot S_1 - 6 n \cdot v_1 n \cdot S_1 v_1 \cdot S_1 + 9 n \cdot v_2 n \cdot S_1 v_1 \cdot S_1 \\
+ 3 n \cdot v_1 n \cdot S_1 v_2 \cdot S_1 \right] + C_{\text{Q1}} \frac{m_2 G^2}{2 r^4} \left( 1 + \frac{4 m_2}{m_1} \right) \left( S_1^2 - 3 (S_1 \cdot n)^2 \right) \\
- \frac{G^2 m_2}{r^4} (S_1 \cdot n)^2 + \left( \tilde{a}_{1(1)}^{\text{SO}} \right)^i S_1^{(i)(i)} + v_1 \times S_1 \cdot \tilde{a}_{1(1)}^{\text{SO}} + \frac{G m_2}{r_{12}^2} n_{12}^{(i)j} S_1^{(j)(0)},
\]
with the \( S_1 \)-dependent part of the acceleration of the local frame
\[
\tilde{a}_i^{so} = \frac{m_2 G}{m_1 r_i^3} \left[ -3 \mathbf{v} \times \mathbf{S}_1 + 6(\mathbf{v} \times \mathbf{S}_1) \cdot \mathbf{n} + 3 \mathbf{n} \cdot (\mathbf{v} \times \mathbf{S}_1) \right].
\] (III.29)

The acceleration term with \( a_1 \) appearing is eliminated by using the Newtonian EOMs for two bodies:
\[
a_1 = -\frac{G m_2 \mathbf{n}_{12}}{r_{12}^2}.
\] (III.30)

This potential will be Legendre transformed like the other potentials above resulting in the effective SSC-dependent Hamiltonian (\( \simeq \) indicates here sole focus on spin(1)-spin(1) terms with the SSC-dependent terms untouched)
\[
H^{\text{NLO (eff)}}_{S_1^2(P)} \simeq H_{\text{LOSO}} + H_{\text{LO LO}} + H_{\text{EHII}} + y^{\text{NLO (v-v)}}_{S_1^2(P)},
\] (III.31)

which results in
\[
H^{\text{NLO (eff)}}_{S_1^2(P)} = \frac{G^2 m_2}{r_{12}^3} \left[ \left( \frac{2 + C_{Q_1}}{2} \right) + \left( \frac{1}{2} + 3C_{Q_1} \right) \frac{m_2}{m_1} \right] \mathbf{S}_1^2
- \left( \frac{3 + 3C_{Q_1}}{2} + \frac{1}{2} + 6C_{Q_1} \right) \frac{m_2}{m_1} \left( \mathbf{S}_1 \cdot \mathbf{n}_{12} \right)^2 + \frac{G}{r_{12}^2} \left[ -\frac{C_{Q_1} m_2}{2 m_1} S_1^{(0)(i)} S_1^{(0)(i)} + \frac{3C_{Q_1} m_2}{2 m_1} \left( S_1^{(0)(i)} n_{12}^1 \right)^2
+ \left( \frac{3(1 - C_{Q_1}) m_2}{m_1^2} \right) (\mathbf{p}_1 \times \mathbf{S}_1) \cdot \mathbf{n}_{12} \right] S_1^{(0)(i)} n_{12}^1 + \frac{3C_{Q_1} m_2}{2 m_1} \left( S_1^{(0)(i)} n_{12}^1 \right)^2
- \left( \frac{3(1 - C_{Q_1}) m_2}{m_1^2} \right) (\mathbf{p}_1 \times \mathbf{S}_1) \cdot \mathbf{n}_{12} \right] S_1^{(0)(i)} n_{12}^1 + \frac{3C_{Q_1} m_2}{2 m_1} \left( S_1^{(0)(i)} n_{12}^1 \right)^2
+ \frac{G}{r_{12}^2} \left( -6 + \frac{9}{4} C_{Q_1} \right) \mathbf{p}_1^2 \left( \mathbf{S}_1 \cdot \mathbf{n}_{12} \right)^2 + (9 - 3C_{Q_1}) (\mathbf{p}_1 \cdot \mathbf{n}_{12}) (\mathbf{S}_1 \cdot \mathbf{n}_{12}) (\mathbf{S}_1 \cdot \mathbf{p}_1)
+ (-3 + C_{Q_1}) (\mathbf{S}_1 \cdot \mathbf{p}_1)^2 + (-3 + 3C_{Q_1}) (\mathbf{p}_1 \cdot \mathbf{n}_{12})^2 \mathbf{S}_1^2 + \left( 3 - \frac{7C_{Q_1}}{4} \right) \mathbf{p}_1^2 \mathbf{S}_1^2
\right] (III.32)
\]

Notice that while the LO potential (III.8) is purely spin quadrupole dependent (via the constant \( C_{Q_1} \)) the corresponding NLO potential (III.28) is not, likewise in the case of the Hamiltonian (III.32).

These non-reduced Hamiltonians are now fit for further phase-space reduction procedures, either by Dirac brackets or by the action principle in Eq. (V.1).

IV. REDUCTION VIA DIRAC BRACKETS

Canonical formalisms in the presence of constraints were analyzed in a very general way by Dirac [68–71], for reviews see also [21, 72–74]. A very important tool developed in this area is nowadays called the Dirac bracket and is further explained in the following. Other important contributions to constrained Hamiltonian dynamics were made by Bergmann and his collaborators, e.g., the notion of primary constraints and the understanding of gauge transformations [75]. Early contributions were already made by Rosenfeld [76], e.g., the discovery of what is nowadays called the Dirac or total Hamiltonian. For a historical review see [77].
A. Construction of the Dirac bracket

The Hamiltonian formulation of a dynamical system needs the construction of a Poisson bracket type structure of the dynamics. In the case of a constraint dynamical systems this is not an easy task, but the needed formalism is available as developed by Bergmann and Dirac. In the following we shall present the formalism to the extent we will need it. At the beginning let us treat the following variables \((z_I, p_{IJ}, S_{K}^{ab}, \Lambda_{L}^{AB})\) as unconstrained. The time derivative of any function \(Q\) of these variables reads

\[
\dot{Q} = \frac{\partial Q}{\partial z_I^i} \dot{z}_I^i + \frac{\partial Q}{\partial p_{JI}} \dot{p}_{JI} + \frac{\partial Q}{\partial S_{Iab}} \dot{S}_{Iab} + \frac{\partial Q}{\partial \Lambda_{L}^{AB}} \dot{\Lambda}_{L}^{AB}.
\]  

(IV.1)

If one wants to restrict the time derivative to the independent degrees of freedom in all variables, one still has to reduce to the six degrees of freedom in the Lorentz matrices, which can be achieved by writing (IV.1) in the following way

\[
\dot{Q} = \frac{\partial Q}{\partial z_I^i} \dot{z}_I^i + \frac{\partial Q}{\partial p_{JI}} \dot{p}_{JI} + \frac{\partial Q}{\partial S_{Iab}} \dot{S}_{Iab} - \frac{\partial Q}{\partial S_{Iab}} \Lambda_{I}^A \partial_{\Lambda_{I}^A} \Omega_{AB}^{ac} \quad \text{with} \quad \Omega_{AB}^{ac} = \Lambda_{L}^{Aa} \Lambda_{I}^{A} c. \tag{IV.2}
\]

Taking into account, see Sect. V,

\[
\dot{z}_I^i = \frac{\partial H_{\text{eff}}}{\partial S_{Iab}}, \quad \dot{p}_{JI} = -\frac{\partial H_{\text{eff}}}{\partial S_{Iab}}, \quad \dot{S}_{Iab} = 4S_{Ic[a} \eta_{bd]} \partial S_{Icd} - 2\eta_{[a} \Lambda_{I}^A \partial S_{Icd} \Lambda_{I}^A c, \tag{IV.3}
\]

\[
\dot{\Lambda}_{A}^{a} = \Lambda_{Ab} \Omega_{B}^{ab}, \quad \Lambda_{Ab} \Lambda_{B} = \delta_{A}^{B}, \quad \Omega_{I}^{ab} = -2 \frac{\partial H_{\text{eff}}}{\partial S_{Iab}} \tag{IV.4}
\]

where, as seen from the previous equations, generally \(H_{\text{eff}} = H_{\text{eff}}(z_I, p_{IJ}, S_{K}^{ab}, \Lambda_{L}^{AB})\) will hold:

\[
\dot{Q} = \{Q, H_{\text{eff}}\}. \tag{IV.5}
\]

The Poisson bracket \(\{\cdot, \cdot\\}\) as defined here has its standard properties which comprise bilinearity, fulfillment of the Leibniz rule and of the Jacobi identity when performed with standard canonical variables having vanishing Poisson brackets among themselves. Hereof one can derive a chain rule analogon for the Poisson bracket when applied to a continuous function \(H\) depending on a canonical variable \(\xi\) which reads

\[
\{Q, H(\xi)\} = \{Q, \xi\} \frac{\partial H}{\partial \xi}. \tag{IV.6}
\]

Applying this formula one can read off all the Poisson brackets, see Sect. V for a thorough derivation of them. The results are (V.10)-(V.12) and read

\[
\{z_I^i, p_{IJ}\} = \delta_{ij} \delta_{IJ}, \tag{IV.7a}
\]

\[
\{S_{Iab}, S_{Icd}\} = S_{Ica[} \eta_{bd]} - S_{Ida[} \eta_{bc]} - S_{Idb[} \eta_{ac]} - S_{Iad[} \eta_{bc]}, \tag{IV.7b}
\]

\[
\{S_{Iab}, \Lambda_{I}^A\} = \eta_{ac} \Lambda_{I}^A - \eta_{cb} \Lambda_{I}^A. \tag{IV.7c}
\]

all other zero. The physical evolution of the variables needs a SSC. We use the covariant one by Tulczyjew written in the local frame, Eq. (II.4), as this SSC is the one implemented in the potentials shown in the last section (i.e., this SSC is conserved by the time evolution given by the potentials). Referring to (II.4) we may replace Dixon’s 4-momentum with the 4-velocity in the SSCs due to their equivalence to our approximation, cf. (II.5), while avoiding unnecessary confusion when dealing with Dixon’s 4-momentum whose 3-components \(p_{I}^{\nu} = \frac{\partial L}{\partial \dot{u}_{I}^{\nu}}\) are essentially different from the canonical ones defined by Legendre transformation \(p_{I}^{\nu} = \frac{\partial L}{\partial u_{I}^{\nu}}\). This means that for the considered approximation the Tulczyjew SSC is equivalent to the Mathisson-Pirani one, \(S_{ab} u_{b} = 0\). With the 4-velocity we get

\[
S_{ab} u_{b} = 0 \iff S_{(i)(j)} u_{b} = 0 \iff S_{(i)(j)} u_{(j)} = S_{(i)(j)} u_{(j)} = 0, \tag{IV.8a}
\]

\[
\Lambda_{[a} u_{a} = 0 \iff \Lambda_{[a} u_{a} + \Lambda_{[a} u_{a} u_{b} u_{b} = 0, \tag{IV.8b}
\]

\[
\Lambda_{[a} u_{a} = \frac{u_{a}}{\sqrt{g_{ab} u_{b}}}. \tag{IV.8c}
\]
Let us call the SSCs, including the Λ-relations, \( \Phi_A = 0 \), \((A = 1, 2, ..., 12)\). Then the Poisson brackets

\[
\{\Phi_A, \Phi_B\} \equiv C_{AB}
\]

(IV.9)

with the inverse matrix \( C^{AB} \), i.e. \( C^{AD}C_{DB} = \delta_B^A \), are most important. Notice, the assumed non-degeneracy of \( C_{AB} \) even for \( \Phi_A = 0 \) makes the SSCs to be second class constraints. The Dirac brackets are defined in the form

\[
\{Q_i, Q_j\}_D = \{Q_i, \Phi_A\}C^{AB}\{\Phi_B, Q_j\}.
\]

(IV.10)

The constrained evolution for the independent variables reads

\[
\dot{z}_I^a = \{z_I^a, H_{\text{eff}}\}_D, \quad \dot{\tilde{p}}_{Ij} = \{p_{Ij}, H_{\text{eff}}\}_D, \quad \dot{\hat{S}}_{I(j)(k)} = \{S_{I(j)(k)}, H_{\text{eff}}\}_D.
\]

(IV.11)

The Dirac bracket therefore satisfies all the laws known from the Poisson bracket, which turns it also into a Lie bracket and it leads to the correct equations of motion together with the Hamiltonian \( H_{\text{eff}} \). The Dirac bracket can thus be used as substitute for the Poisson bracket. However, whereas one may use the constraints only after all Poisson brackets were calculated, the second class constraints \( \Phi_A = 0 \) can be used before an application of the Dirac bracket without changing the result, e.g. one has \( \{Q, \Phi_a\}_D = 0 \) for all \( Q \)'s and \( \Phi_a \) thus preserving the constraints in time when using the Hamiltonian for \( Q \). Notice if one restricts to use the Dirac bracket instead of the Poisson bracket, the second class constraints \( \Phi_a = 0 \) can be used off-shell to solve for certain phase space variables and eliminate them from all quantities, thus reducing the actual degrees of freedom. The transition to new variables \( (\hat{z}_I^a, \hat{p}_{Ij}, \hat{S}_{K}^{I(j)}) \) which fulfill

\[
\{\hat{z}_I^a, \hat{p}_{Ij}\}_D = \delta_{IJ},
\]

(IV.12a)

\[
\{\hat{\Lambda}_I^{I(j)}(\hat{S}_I^{I(j)})^{\mu(n)}, \hat{S}_I^{I(j)}\}_D = -\delta_{IJ}\hat{\Lambda}_I^{I(j)}(\hat{S}_I^{I(j)})^{\mu(n)} + \delta_{Jn}\hat{\Lambda}_I^{I(j)}(\hat{S}_I^{I(j)})^{\mu(n)},
\]

(IV.12b)

\[
\{\hat{S}_I^{I(j)}(\hat{S}_I^{I(j)})^{l(k)}, \hat{S}_I^{I(j)}\}_D = \delta_{IJ}\hat{S}_I^{I(j)}(\hat{S}_I^{I(j)})^{l(k)} - \delta_{Ik}\hat{S}_I^{I(j)}(\hat{S}_I^{I(j)})^{l(k)} + \delta_{Jk}\hat{S}_I^{I(j)}(\hat{S}_I^{I(j)})^{l(k)} - \delta_{Il}\hat{S}_I^{I(j)}(\hat{S}_I^{I(j)})^{l(k)},
\]

(IV.12c)

all other brackets being zero, results in a standard canonical representation. They will be called Newton-Wigner variables, because it can be shown that the ones proposed by those two in [59] represent the only possible standard canonical set of variables at least in Special Relativity. For an extension of definition of those variables to General Relativity see the comment [60]. The ‘unhattet’ variables corresponding to the covariant SSC (IV.8a)-(IV.8c) are from now on shortly dubbed ‘covariant’ variables. More details are given in the application later on. The careful reader should have noticed that the space and momentum variables \( (z_I^a, p_{Ij}) \) were assumed non-constrained throughout. This was done for simplicity reasons and because the pure spinless case is well known. On the other side, the presented form is just the one delivered by researchers using the EFT approach, see later on.

Now the supplementary conditions (IV.8a)-(IV.8c) need to be explicitly known in dependence of positions and canonical momenta in coordinate space when eliminating them in the Hamiltonians. Upon agreement the spin is always defined in the local Lorentz frame, so what we need are the 4-velocities in the coordinate frame mapped into the local frame, which is linked to the coordinate one by a vierbein transformation matrix. By decomposing the curved spacetime metric \( g_{\mu\nu} \) according to

\[
g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu},
\]

(IV.13)

into the Minkowski background field part \( \eta_{\mu\nu} \) and a perturbation part \( h_{\mu\nu} \), whose indices are raised and lowered with \( \eta_{\mu\nu} \), we can derive for \( u_I^a \) the expanded expression

\[
u_I^a = e^a_{\mu}(z_I^a)u_I^\mu = \eta^{\mu\nu}\left(\eta_{\mu\nu} + \frac{1}{2}h_{\mu\nu} - \frac{1}{8}h_{\mu\nu}\eta^{\rho\sigma}h_{\rho\sigma} + \ldots\right)u_I^\nu
\]

(IV.14)

up to a certain pN order. (Notice that a vierbein is not uniquely fixed by the metric, but the one used here is entering the derivation of the Feynman rules.) Expressed in variables defined in an Euclidean flat space coordinate frame the vierbein components are given up to all terms necessary for our declared approximation according to [60] by (see also [78])

\[
e_{(0)0} = 1 + \frac{1}{2}h_{00} = 1 - G_{m_1}^{m_1}r_1 - G_{m_2}^{m_2}r_2 + \mathcal{O}(c^{-4}),
\]

(IV.15a)

\[
e_{(0)i} = \frac{1}{2}h_{0i} = 2Gv_i^{m_1}r_1 - Gs_{I(j)(j)}n_i^{I(j)} + (1 \leftrightarrow 2) + \mathcal{O}(c^{-5}),
\]

(IV.15b)

\[
ed_{(i)j} = -\delta_{ij} + \frac{1}{2}h_{ij} = -\delta_{ij}\left(1 + G_{m_1}^{m_1}r_1 + G_{m_2}^{m_2}r_2\right) + \mathcal{O}(c^{-4}).
\]

(IV.15c)
The velocities depending on the momenta follow from Legendre transformation using the pN expanded Lagrangian for two spinning compact body interaction leading to Eq. (III.16). Notice the vierbein is chosen to be symmetric so that local Lorentz indices and coordinate indices get indistinguishable yielding the 4-velocities in Euclidean flat space coordinates and depending on canonical momenta which replace the coordinate velocities via Eq. (III.16) indicated by \( \simeq \)

\[
u_i = \nu_i^0 = \frac{c_0}{\nu_i^0} u_i^0 = c_{00}^0 u_i^0 + c_{0i}^0 u_i^0 = 1 - \frac{G m_2}{r_{12}} + \mathcal{O}(c^{-4}) \quad \text{(IV.16a)}
\]

\[
u_i^j = c_{ij}^0 u_i^0 = -c_{ij}^0 u_i^0 - c_{ij}^0 u_i^0,
\]

\[
\simeq \frac{p_{i1}}{m_1} \left( 1 - \frac{p_1^2}{2 m_1^2} \right) - \frac{2 G m_2 p_{i1}}{m_1 r_{12}} + \frac{3 G p_{2i}}{2 r_{12}} + \frac{G n_{12}^i (p_2 \cdot n_{12})}{2 r_{12}}
\]

\[
\frac{G n_{12}^i}{r_{12}^2} \left( S_{2}^{i(j)} + \frac{m_2}{m_1} S_{1}^{i(j)} \right) + \mathcal{O}(c^{-5}). \quad \text{(IV.16c)}
\]

The covariant SSC (IV.8a) is then pN expanded to yield

\[
0 = S_1^{(0)(i)} + S_1^{(0)(j)} \frac{u_j^0}{u_i^0} \quad \text{(IV.17a)}
\]

\[
0 = S_1^{(0)(i)} + S_1^{(0)(j)} \left[ v^i_1 + 2 G (v_1^i - v_2^i) \frac{m_2}{r_{12}} + G S^{j(0)}_2 \frac{m}{r_{12}} \right] + \mathcal{O}(c^{-8}) \quad \text{(IV.17b)}
\]

\[
0 = S_1^{(0)(i)} + S_1^{(0)(j)} \left[ \frac{p_{i1}}{m_1} \left( 1 - \frac{p_1^2}{2 m_1^2} \right) + \frac{G}{2 r_{12}} \left( 3 p_{2j} - 2 \frac{m_2}{m_1} p_{1j} + (n_{12} \cdot p_2) n_{12}^j \right) \right.
\]

\[
\left. - \frac{G n_{12}^k}{r_{12}^2} \left( S_{2}^{i(k)} + \frac{m_2}{m_1} S_{1}^{i(k)} \right) \right] + \mathcal{O}(c^{-8}). \quad \text{(IV.17c)}
\]

Likewise the Lorentz matrix constraint (IV.8c) is pN expanded to give

\[
0 = \Lambda_1^{(0)(i)} - \frac{u_i^0}{\sqrt{\left( u_i^0 \right)^2}} \quad \text{(IV.18a)}
\]

\[
0 = \Lambda_1^{(0)(i)} - \left( 1 - v_1^2 \right)^{-\frac{1}{2}} v_1^i - 2 G (v_1^i - v_2^i) \frac{m_2}{r_{12}} - G S^{j(0)}_2 \frac{m_2}{r_{12}} + \mathcal{O}(c^{-5}) \quad \text{(IV.18b)}
\]

\[
0 = \frac{p_{i1}}{m_1} - \frac{G}{2 r_{12}} \left( 3 p_{2j} - 2 \frac{m_2}{m_1} p_{1j} + (n_{12} \cdot p_2) n_{12}^j \right) + \frac{G n_{12}^k}{r_{12}^2} \left( S_{2}^{i(k)} + \frac{m_2}{m_1} S_{1}^{i(k)} \right) + \mathcal{O}(c^{-5}). \quad \text{(IV.18c)}
\]

These 6 constraints per body (12 in total) are comprised into a set of 12 elements \( \Phi_A = 0, A = 1, \ldots, 12 \), which enables us to construct the Dirac bracket out of them according to the standard rule, if we know the Poisson brackets between the various quantities which enter the constraints and the Dirac bracket, which are given by (IV.7a)-(IV.7c) e.g., we have the following Poisson brackets between the constrained variables:

\[
\{ \Lambda_1^{(j)}, S_1^{(m)(n)} \} = -\delta_{mj} \Lambda_1^{(n)} + \delta_{nj} \Lambda_1^{(m)}, \quad \text{(IV.19a)}
\]

\[
\{ \Lambda_1^{(0)}, S_1^{(0)(k)} \} = \Lambda_1^{(k)}, \quad \text{(IV.19b)}
\]

\[
\{ \Lambda_1^{(j)}, S_1^{(0)(k)} \} = \Lambda_1^{(0)} \delta_{jk}, \quad \text{(IV.19c)}
\]

After applying all Poisson brackets, the constraints can be used in the results of the Dirac brackets. The constraints \( \Phi_A = 0 \) are given above by (IV.17c) for \( A = 1, 2, 3 \) for object 1 and \( A = 7, 8, 9 \) for object 2 and by (IV.18c) for \( A = 4, 5, 6 \) for object 1 and \( A = 10, 11, 12 \) for object 2. The matrix \( C_{AB} \) from the definition (IV.9) is decomposed in a Minkowski part \( C_{(0)AB} \) and a curvature correction part linear in \( G, C_{(1)AB} \), approximated by the needed pN order:

\[
C_{AB} = C_{(0)AB} + C_{(1)AB} \quad \text{(IV.20a)}
\]
Referring to [21] the Minkowski parts $C_{(0)AB}$ and $C^{AC}_{(0)}$ can be obtained analytically exact with the formal definition $p^0_i = \sqrt{m_i^2 + p_i^2}$ yielding:

$$C^{AB} = C^{AB}_{(0)} - C^{AC}_{(0)} C^{CD}_{(0)} C^{DB}_{(0)}.$$  \hspace{1cm} (IV.20b)

The notation is adapted to the one used in [21], whence the same calculation was done in the Minkowski case which should be included here as a limiting case. As a matter of fact we will only need the Minkowski part $C_{(0)AB}$ for our calculation, because all terms up to the next-to-leading pN order $c^{-b}$ of the Dirac bracket are produced by the vectors to the left and to the right of the inverse matrix $C$, which can be checked for all possible Dirac brackets, so no curvature terms of $C^{AB}$ will contribute to the Dirac bracket to NLO. Interestingly the object $\mathcal{P}_{ij}$ also appears in surface terms of the stress-energy tensor algebra in Minkowski spacetime, see Eq. (A7) in [25].

### B. Transition to Newton-Wigner variables

The Dirac bracket reads in reduced approximation to linear order in $G$ (indicated by $\simeq$)

$$\{Q_1, Q_2\}_D \simeq \{Q_1, Q_2\} - \{Q_1, \Phi_A\} C^{AB}_{(0)} \{\Phi_B, Q_2\}.$$  \hspace{1cm} (IV.22)

A list of all possible combinations of quantities which may enter the Dirac bracket and their results is given in the appendix A. Assuming that we have calculated the Dirac brackets between all the variables entering the Hamiltonian the transition to Newton-Wigner variables $\hat{z}_1$, $\hat{p}_1$, $\hat{S}_1$, $\hat{z}_2$, $\hat{p}_2$, and $\hat{S}_2$, can be performed. The Dirac brackets of these new variables should be standard canonical fulfilling their natural standard commutation relations

$$\{\hat{z}_i, \hat{p}_{Jj}\}_D = \delta_j^i \delta_{IJ},$$  \hspace{1cm} (IV.23a)

$$\{\hat{S}_i^{(j)(l)}, \hat{S}_j^{(k)(l)}\}_D = \delta_{ij} \delta_{jk} \hat{S}_i^{(l)(l)} + \delta_{jk} \hat{S}_j^{(l)(l)} - \delta_{il} \hat{S}_i^{(j)(k)} - \delta_{kl} \hat{S}_j^{(j)(k)},$$  \hspace{1cm} (IV.23b)

and all other possible brackets should vanish. For the sake of completion we also list the standard Poisson bracket relation for the standard canonical Lorentz matrix $\hat{\Lambda}^{(i)(j)}$, which happens to be a pure 3-dimensional rotation matrix fulfilling the standard Poisson bracket relation

$$\{\hat{\Lambda}_i^{(j)(k)}, \hat{S}_j^{(l)(k)}\}_D = \hat{\Lambda}_j^{(i)(l)} \delta_{ij} - \hat{\Lambda}_j^{(l)(i)} \delta_{kj},$$

while all other possible brackets with $\hat{\Lambda}^{(i)(j)}$ being zero. The Newton-Wigner variables that are assumed to exist up to the considered order are fixed up to canonical transformations which opens up the possibility to choose a representation that leaves the momenta unchanged, $\hat{p}_I = \hat{p}_I$. In order to find the position and spin variable transformation we make a general ansatz with undetermined coefficients, e.g., for the position variable the general ansatz (excluding the
Minkowski term which is known exactly) reads to the pN order of $e^{-5}$

$$z^\\prime_1 = z^\\prime_1 + \frac{p_{1k} S_1(i)(k)}{r_{12}} + \frac{p_{2k} S_1(i)(k)}{r_{12}} + \frac{n^k_{12} (p_1 \cdot \hat{n}_{12}) S_1(i)(k)}{r_{12}} + \frac{n^k_{12} (p_2 \cdot \hat{n}_{12}) S_1(i)(k)}{r_{12}} + \frac{\xi_6 S_1(i)(k)}{r_{12}} + \frac{\xi_9 S_1(i)(k) S_1(i)(l) p_{lk}}{r_{12}^2} + \frac{\xi_0 S_1(i)(l) S_1(i)(l) p_{lk}}{r_{12}^2} + \frac{\xi_4 n^k_{12} (p_1 \cdot \hat{n}_{12}) S_1(i)(k)}{r_{12}^2} + \frac{\xi_5 n^k_{12} S_1(i)(k)}{r_{12}^2} + \frac{\xi_6 S_1(i)(l) S_1(i)(l) p_{lk}}{r_{12}^2} + \frac{\xi_0 S_1(i)(l) S_1(i)(l) p_{lk}}{r_{12}^2} + (1 \leftrightarrow 2) (\xi_6, \ldots, \xi_{16}) + O(e^{-5}). \quad \text{(IV.24)}$$

It is worth mentioning that to first order ‘covariant’ and Newton-Wigner variables agree, so that the hat on the variables of the $G$-terms can also be thought of as erased when trying to cancel similar terms in the Dirac bracket relations. The ansatz (IV.24) contains 30 coefficients to be determined which in general fulfill no symmetries among themselves and are independent of each other, although most of them will be set to zero. Notice the coefficients also depend on mass factors and are therefore not dimensionless. In actuality it is more practical to choose a shorter ansatz which focuses on terms which are present in the Dirac brackets, because only those terms have to be cancelled in order to arrive at canonical Dirac brackets, all other terms in the general ansatz are a priori zero. The ansatz (IV.24) is therefore to be inserted into the Dirac brackets which contain the position variable, e.g., (A.1a). It turns out that the fulfillment of the crucial Poisson/Dirac bracket relation (IV.23a) is already enough to uniquely fix all coefficients in (IV.24), all other Dirac bracket relations that would make $\hat{z}$ canonical are then automatically fulfilled and serve merely as consistency checks. The coefficients for (IV.24) are then given as

$$\xi_1 = \frac{m_2}{m_1^2}, \quad \xi_2 = -\frac{3}{2m_1}, \quad \xi_4 = -\frac{1}{2m_1}, \quad \xi_6 = \frac{m_2}{m_1^2}, \quad \xi_{11} = -\frac{1}{m_1}, \quad \text{(IV.25)}$$

and all other coefficients are zero. The complete transformation formula (including the pN expanded Minkowski term) reads

$$z^\\prime_1 = z^\\prime_1 + \frac{1}{2m_1^2} G \frac{\hat{n}_{12} \cdot \hat{p}_2}{m_1 r_{12}} \left[ -\frac{p_1^2}{4m_1^4} + \left( 1 - \frac{p_1^2}{4m_1^4} \right) + \frac{2 G m_2 p_{1k} S_1(i)(k)}{m_1 r_{12}} + \frac{3}{2} \frac{G p_{2k} S_1(i)(k)}{m_1 r_{12}} \right] + \frac{G \hat{n}_{12} \cdot \hat{p}_2 S_1(i)(k)}{m_1 r_{12}} + \frac{G m_2 \hat{n}_{12} S_1(i)(l) S_1(i)(l)}{m_1 r_{12}} + \frac{G \hat{n}_{12} S_1(i)(l) S_1(i)(l)}{m_1 r_{12}} + O(e^{-5}). \quad \text{(IV.26)}$$

For the spin variable we make a similar ansatz, which we insert into (A.1j) and demand fulfillment of (IV.23b) which again uniquely determines the spin transformation reading

$$S_1(i)(j) = \hat{S}_1(i)(j) - \frac{p_1 [\hat{S}_1(i)(k)] p_{lk}}{m_1^2} \left( 1 - \frac{p_1^2}{4m_1^4} \right) + \frac{2 G m_2 (p_{1k} S_1(i)(k))}{m_1 r_{12}} + \frac{3}{2} \frac{G p_{2k} S_1(i)(k)}{m_1 r_{12}} + \frac{G \hat{n}_{12} \cdot \hat{p}_2 S_1(i)(k)}{m_1 r_{12}} + \frac{G m_2 \hat{n}_{12} S_1(i)(l) S_1(i)(l)}{m_1 r_{12}} + \frac{G \hat{n}_{12} S_1(i)(l) S_1(i)(l)}{m_1 r_{12}} + O(e^{-8}). \quad \text{(IV.27)}$$

The transformation of the Lorentz matrix is derived in the following section.

V. REDUCTION VIA AN ACTION PRINCIPLE

Spin in relativity can also be treated by an action principle [61–63], see also [6, 21, 23, 24, 64] and appendix A of [65]. It is indeed possible to derive the transformation to reduced canonical variables using an action approach. Let
us start with the effective action for two interacting spinning compact objects in curved space

$$S_{\text{eff}} = \int dt L_{\text{eff}} = \int dt \left( p_{1i} \dot{z}_1^i + p_{2i} \dot{z}_2^i - \frac{1}{2} S_{1ab} \Omega_1^{ab} - \frac{1}{2} S_{2ab} \Omega_2^{ab} - H_{\text{eff}} (z_J, p_J, S_{Kab}, \Lambda_{\text{eff}}^A) \right).$$  \hfill (V.1)

The variables in the Hamiltonian $H_{\text{eff}} (z_J, p_J, S_{Kab}, \Lambda_{\text{eff}}^A)$ are all independent from each other, no SSC is imposed yet on this stage. The variables in this action span therefore a too large phase space, because of the redundant $S_{I(0)i}$ and $\Lambda_{(0)A}$ degrees of freedom, which makes the phase space still unphysical, but will give us some insight into the non-reduced Poisson brackets between all the variables of the action. As usual the spin tensor $S_{ab}$ is defined locally through a projection onto a local Lorentz basis by a vierbein $e^a$ which fulfills the condition

$$e_a^\mu e_b^\nu g_{\mu \nu} = \eta_{ab}, \quad \eta_{ab} = \text{diag}(1, -1, -1, -1), \quad e_a^\mu e_b^\nu \eta_{ab} = g_{\mu \nu}$$  \hfill (V.2)

with the Lorentz indices $(a, b, c, d)$ and spacetime coordinate indices $(\mu, \nu, ..)$

$$(a, b, c, d) \in \{(0), (i)\} \quad \text{and} \quad i \in \{1, 2, 3\}, \quad (\mu, \nu, \sigma, ...) \in \{0, 1, 2, 3\}.$$  \hfill (V.3)

We need still another index label appearing in the definition of the angular velocity tensor $\Omega_{ab}$, whose generalized momentum is the spin tensor $S_{ab}$, which involves the Lorentz matrix $\Lambda_{Ab}$ and therefore rotations and boosts with capital letters labeling body-fixed Lorentz indices in the sense that

$$\Lambda_{A\mu} \Lambda^{A\nu} = g_{\mu \nu}, \quad \Lambda_{Aa} \Lambda^{A_b} = \eta_{ab} \quad \text{with} \quad (A, B, ..) \in \{0, [i]\}.$$  \hfill (V.4)

Notice that $\Lambda_{AB}$ is time dependent. Now the definition is (the dot marking total time derivative)

$$\Omega^{ab} \equiv \Lambda^{A}_a \dot{\Lambda}^{A_b} \quad \text{making} \quad \Omega^{ab} = -\Omega^{ba} = \Omega^{[ab]}.$$  \hfill (V.5)

The minus sign in front of the spin kinematic term $\frac{1}{2} S_{Iab} \Omega_I^{ab}$ in the action (V.1) is due to the sign convention for the antisymmetric angular velocity tensor $\Omega^{ab}$ and the used signature. Indeed, variation of the action gives (dropping boundary terms)

$$\delta S_{\text{eff}} = \int dt \left[ \dot{z}_1^i - \frac{\partial H_{\text{eff}}}{\partial p_{1i}} \delta p_{1i} - \dot{p}_{1i} + \frac{\partial H_{\text{eff}}}{\partial \dot{z}_1^i} \right] \delta z_1^i - \left( \frac{1}{2} \Omega_1^{ab} + \frac{\partial H_{\text{eff}}}{\partial S_{1ab}} \right) \delta S_{1ab} + \left( \frac{1}{2} \dot{S}_{1ab} - S_{1ca} \Omega_1^{ab} + \frac{1}{2} \left( \frac{\partial H_{\text{eff}}}{\partial \dot{A}_1^{A_c} \Lambda_1^{A_b}} - \frac{\partial H_{\text{eff}}}{\partial \dot{A}_1^{A_b} \Lambda_1^{A_c}} \right) \right) \delta \theta_1^{ab}$$  \hfill (V.6)

where the variation of $\Lambda^{AB}$ was written in terms of the antisymmetric symbol $\delta \theta^{ab} = \Lambda^{A}_a \delta \Lambda^{AB}$ and we used the relation $\delta \Omega^{ab} = \frac{\partial}{\partial t} \delta \theta^{ab} + 2 \Omega^{[a} \delta \theta^{bc]}$. The variation of the Hamiltonian with respect to the independent degrees of freedom of the Lorentz matrices is also achieved by usage of $\delta \theta^{ab}$ according to

$$\delta H_{\text{eff}} = \frac{\partial H_{\text{eff}}}{\partial \dot{A}_1^{A_b} \Lambda_1^{A_a}} \delta \Lambda_1^{A_b} = \frac{\partial H_{\text{eff}}}{\partial \dot{A}_1^{A_b} \Lambda_1^{A_a}} \Lambda_1^{A_b} \delta \Lambda_1^{A_a} = \frac{\partial H_{\text{eff}}}{\partial \dot{A}_1^{A_b} \Lambda_1^{A_a}} \Lambda_1^{A_c} \delta \Lambda_1^{A_a}$$  \hfill (V.7)

$$= - \frac{1}{2} \left( \frac{\partial H_{\text{eff}}}{\partial \dot{A}_1^{A_b} \Lambda_1^{A_a}} - \frac{\partial H_{\text{eff}}}{\partial \dot{A}_1^{A_a} \Lambda_1^{A_b}} \right) \delta \theta^{ab}.$$  \hfill (V.8)

The equations of motion are therefore given by

$$\dot{z}_1^i = \frac{\partial H_{\text{eff}}}{\partial p_{1i}}, \quad \dot{p}_{1i} = -\frac{\partial H_{\text{eff}}}{\partial \dot{z}_1^i}, \quad \dot{S}_{1ab} = 4 S_{1ca}[\eta_{bd}, \dot{S}_{1cd}] = 2 \eta_{[a} \Lambda_1^{A_b]} \frac{\partial H_{\text{eff}}}{\partial A_1^{A_c}}$$  \hfill (V.9)

and from the definition

$$z_1^i = (z_1^i, H_{\text{eff}}), \quad p_{1i} = (p_{1i}, H_{\text{eff}}), \quad S_{1ab} = \{S_{1ab}, S_{1cd}\} = S_{1ca} \eta_{bd} - S_{1da} \eta_{bc} + S_{1ab} \eta_{ac} - S_{1cb} \eta_{ad}.$$  \hfill (V.10)

we can thus read off the Poisson brackets (see also (IV.7))

$$\{z_1^i, p_{1j}\} = \delta_{ij} \delta t_J, \quad \{S_{1ab}, S_{1cd}\} = S_{1ca} \eta_{bd} - S_{1da} \eta_{bc} + S_{1ab} \eta_{ac} - S_{1cb} \eta_{ad}.$$  \hfill (V.11)
\[ \{S_{a(b}, \Lambda^A_{c)} \} = \eta_{ac} \Lambda^A_{a} - \eta_{bc} \Lambda^A_{b}, \]

(V.12)

and all other brackets are zero. The goal is to transform (V.1) into the canonical form

\[ \dot{S}_{\text{eff}} = \int dt \dot{L}_{\text{eff}} = \int dt \left( \hat{p}_{i} \dot{z}^{i}_{1} + \hat{p}_{2i} \dot{z}^{i}_{2} - \frac{1}{2} \hat{S}^{(0)(j)}_i \dot{\Lambda}^{(i)}_{j} - \frac{1}{2} \hat{S}^{(2)(j)}_i \dot{\Omega}^{(i)}_{j} - H_{\text{can}} \left( \hat{z}_{j}, \hat{p}_{j}, \bar{S}_{K}, \dot{\bar{\Lambda}}^{(j)}_{L} \right) \right), \]

(V.13)

with \( \hat{S}^{(i)(j)}_i = \epsilon_{ijk} \hat{S}_{(k)}, \epsilon_{ijk} = \frac{1}{6} (i - j)(j - k)(k - i), \) and \( \dot{\Omega}^{(i)(j)} = \hat{\Lambda}^{(i)}_{(k)} \dot{\Lambda}^{(j)}_{(l)} \). Notice that with the used conventions the formula for the angular velocity vector \( \dot{\Omega}^{(i)} = -\dot{\Omega}^{(i)} = -\frac{1}{2} \epsilon_{ijk} \dot{\Omega}^{(i)(k)} \) involves a minus sign. The hat labels functions depending on canonical position variables \( \{ \hat{z}^{i}_{j}, \hat{\Lambda}^{(i)}_{j} \} \) with their generalized momenta \( \{ \hat{p}_{i}, \hat{S}_{j}^{(i)(j)} \} \) in reduced phase space, meaning an appropriate SSC is imposed to get rid of \( S_{I(0)(i)} \) and leading to canonical conjugate variables at the same time. Variation of the action is completely analogous to above calculation. Only the 4-indices \( a, b, \ldots \) have to be replaced by 3-indices \( i, j, \ldots \), so the Poisson brackets (V.10)-(V.12) translate into

\[ \{ \hat{z}^{i}_{j}, \hat{p}_{j} \} = \delta^{i}_{j} \delta_{IJ}, \]

(V.14)

\[ \{ \hat{\Lambda}^{(i)}_{j}, \hat{S}_{j}^{(m)(n)} \} = -\delta_{jm} \hat{\Lambda}^{(i)}_{n} + \delta_{jn} \hat{\Lambda}^{(i)}_{m}, \]

(V.15)

\[ \{ \hat{S}^{(i)(j)}_{j}, \hat{S}_{j}^{(k)(l)} \} = \delta_{jl} \hat{S}^{(i)(k)}_{j} - \delta_{jk} \hat{S}^{(i)(l)}_{j} + \delta_{ik} \hat{S}^{(j)(l)}_{j} - \delta_{il} \hat{S}^{(j)(k)}_{j}, \]

all other zero. We used \( \eta_{i(i)(j)} = -\delta_{ij} \) and the antisymmetry of the spin tensor. So let us make the reduction in phase space explicit. We impose the covariant SSC, or to put it more exact the Mathisson-Pirani SSC with the 4-velocity coupled to the 4-dimensional spin tensor. As already mentioned, the reason is that the considered potentials are only valid for this SSC. This reduced action is then ready to be transformed to (V.13) while emerging with the Newton-Wigner SSC. First examine the term \( \frac{1}{2} S_{ab} \Omega^{ab} \) (with suppressed particle label) and make a decomposition into time and space parts by using the supplementary conditions fixing the frame of reference (see also (IV.8))

\[ S_{ab} u^b = 0 \iff S_1^{(i)(j)} u^0 = 0 \iff S_1^{(0)(j)} + S_1^{(j)(0)} \frac{u^0}{u^0} = 0, \]

(V.17a)

\[ \Lambda^{[i]} u_a = 0 \iff \Lambda^{[i]}_0 + \Lambda^{[j]}_j u^j u^0 = 0, \]

(V.17b)

\[ \Lambda^{[i]} u_a = \frac{u_a}{\sqrt{u^0 u^0}}. \]

(V.17c)

Here and in the following \( u^a \) should be understood as given in terms of the canonical momentum \( p \) (not in terms of \( v \)), the pN approximate relations are given by Eqs. (IV.16a) and (IV.16c). Notice that we could have also chosen (II.4) as SSC with \( \rho^{(0)}_D \) already eliminated by the mass-shell constraint \( \rho^{(0)} = \sqrt{m^2 + \rho^{D}_D p^D} \). But again due to the difference between Dixon’s momentum and the canonical one in the Legendre transformation it is easier to work with the 4-velocity in the SSCs. As the full derivation of the variable transformation formulae are quite cumbersome we have put the details in the Appendix B and shall present here only the key steps and results. We start with the insertion of the constraints (V.17a)-(V.17c) leading us the following ‘naive’ reduced expression of the spin coupling term in the action

\[ \frac{1}{2} S_{ab} \Omega^{ab} = -\frac{1}{2} S_1^{(j)(k)} \frac{u^k u^l}{u^0 u^0} \Omega^{(j)(k)}_l - \frac{1}{2} S_{1}^{(j)(k)} \frac{u^k u^l}{u^0 u^0} \Lambda^{[j]}_l \Lambda^{[i]}_0 + \frac{1}{2} S_1^{(j)(k)} \Omega^{(i)(k)}_l \]

(V.18)

with \( \Omega^{(i)(k)}_l \equiv \Lambda^{[k]}_l \Lambda^{[i]}_0 \). Notice the formal difference to the definition of \( \Omega^{(k)(i)}_l \) from (V.5), \( \Omega^{(k)(i)}_l \) is therefore not necessarily antisymmetric, which is actually an unwanted feature. After insertion of \( \Lambda^{[i]}_0 \) by using (V.17b) and further algebraic manipulation we end up with the expression:

\[ \frac{1}{2} S_{ab} \Omega^{ab} = \left( S_{1}^{(j)(k)} + S_{1}^{(j)(k)} \frac{u^k u^l}{u^0 u^0} - S_{1}^{(j)(k)} \frac{u^k u^l}{u^0 u^0} \right) \frac{\Omega^{(j)(k)}}{2} + \frac{1}{2} S_1^{(j)(k)} \frac{u^k u^l}{u^0 u^0} \Omega^{(i)(k)}_l \]

(V.19)

Next thing to do is to redefine variables so that the canonical structure of (V.13) is produced. Obviously one should start by shifting \( \Omega^{(i)(j)} \) to \( \Omega^{(i)(j)} \), which should be antisymmetric in order to be the correct velocity variable belonging
to the spin tensor. This is achieved by a redefinition of the Lorentz matrix according to

$$\Lambda^{[i]}_{(j)} = \hat{\Lambda}^{[i]}_{(j)} \left( \eta^{(i)}_{(k)} - \frac{u_{(k)}u_{(j)}}{u(u + u_{(0)})} \right), \quad \text{with} \quad u_{(i)}u_{(j)} = \mathbf{u}^2 \quad \text{so that} \quad u_{a}u^{a} = \mathbf{u}^2 = u_{(0)}^2 + \mathbf{u}^2. \quad (V.20)$$

After a further redefinition of the spin tensor to the canonical (hatted) one according to

$$S_{(i)(j)} = \hat{S}_{(i)(j)} - \hat{S}_{(i)(k)} \frac{u_{(j)k}}{u(u + u_{(0)})} + \hat{S}_{(j)(k)} \frac{u_{(i)k}}{u(u + u_{(0)})}, \quad (V.21)$$

we arrive at the following reduced expression for the spin coupling term with one term, the $Z$-term, left to be cancelled by a proper position variable shift, because this term includes a local acceleration

$$\frac{1}{2} S_{ab} \Omega^{ab} = \frac{1}{2} \hat{S}_{(i)(j)} \hat{\Omega}^{(i)(j)} - Z \quad \text{with} \quad Z \equiv \hat{S}_{(i)(j)} \frac{u_{(i)j}}{u(u + u_{(0)})}. \quad (V.22)$$

Notice that we could have also used the principle of general covariance to arrive at (V.22), because all the equations up to (V.22) look the same in the special relativistic case for global Minkowski spacetime, where the round brackets around the local indices of the corresponding variables are erased to yield coordinate indices. The principle of general covariance in this case would be to rewrite the round brackets around the coordinate indices to arrive at valid expression in curved spacetime of general relativity. Indeed we recover the result of Hanson and Regge [21] for the transformation to the Newton-Wigner spin variable $\hat{S}$ and Lorentz matrix $\hat{\Lambda}^{[i]}_{(j)}$ in the special relativistic case, when the 4-velocity $u^a$ in our formulae is replaced by Dixon’s momentum $p_h^a$ (II.5). The same will be true for the transformation to the Newton-Wigner position variable in the special relativistic case.

To solve for the momenta in (V.22) one has to insert the vierbein which is perturbatively calculated to the needed pN order, see Eqs. (IV.15a)-(IV.15c). First we make an expansion of $\mathbf{Z}$ in powers of $\mathbf{u}^2$ (in the sense of a post-Newtonian approximation)

$$Z = \hat{S}_{(i)(j)} \frac{u_{(i)j}}{u(u + u_{(0)})} = \hat{S}_{(i)(j)} \frac{u_{(i)j}}{u(u + u_{(0)})} \left( \frac{1}{2} u^2_{(0)} - \frac{3u^2}{8u_{(0)}^2} + O(u^4, c^{-10}) \right). \quad (V.23)$$

The goal is to find the shift of the position variable to its canonical one only approximately to linear order in $G$ and to leading order in spin-orbit, spin(1)-spin(2) and spin(1)-spin(1) interaction. We insert (IV.16a) into (V.23) and pN expand the result up to the order $c^{-8}$ leading to

$$Z_1 \simeq \frac{1}{2} \hat{S}_{(i)(j)} u_{(i)} u_{(j)} \left( 1 + 2G \frac{m_2}{r_{12}} \right) + O(c^{-10}). \quad (V.24)$$

Next we insert (IV.16c) into this equation yielding the approximate expression

$$Z_1 \simeq \frac{1}{2} \hat{S}_{(i)(j)} u_{(i)} u_{(j)} + m_2 \frac{2G}{r_{12}} \left[ \frac{G \hat{P}_{12}}{2r_{12}} + \frac{m_2 n_2}{m_1 r_{12}^2} - \frac{m_2 n_2 S^{(i)(k)}}{m_1 r_{12}^3} \right] \hat{\mathbf{p}}_{ij} + O(c^{-10}). \quad (V.25)$$

We eliminate the time derivative of $u_{(i)}$ by shifting it on-shell (i.e. we neglect total time derivatives symbolized by $\simeq$) onto the momentum leaving us also with a time derivative of the canonical spin, which we will have to deal with later when we reconsider the spin redefinition. So

$$Z_1 \simeq \frac{1}{2} \hat{S}_{(i)(j)} p_{i1} u_{(j)} - \frac{G m_2 n_2 S^{(i)(k)}}{m_1 r_{12}^3} \frac{\hat{p}_{ij}}{m_1} + O(c^{-10})$$

and

$$Z_1 \simeq \frac{1}{2} \hat{S}_{(i)(j)} p_{i1} u_{(j)} - \frac{G m_2 n_2 S^{(i)(k)}}{m_1 r_{12}^3} \frac{\hat{p}_{ij}}{m_1} + O(c^{-10}). \quad (V.26)$$
Now we are ready to return to the action (V.1), wherein we insert Eqs. (V.22) and (V.25) leading to the expression (for particle 1)

\[ S_{\text{eff}} = \int dt \left( p_{1i} \dot{z}_1^i - \frac{1}{2} S_{1ab} \dot{O}_{1}^{ab} - H_{\text{eff}} \right) \]

\[ \approx \int dt \left( - \dot{p}_{1j} \dot{z}_1^j - \frac{1}{2} S_{(ij)(j)} \dot{O}_{1}^{ij} - \frac{1}{2m_1} \dot{S}_{1(i)(j)} p_{1i} u_{1}^{(j)} + \frac{\dot{S}_{1(i)(j)}}{2m_1} \left[ \frac{p_{1i} m_1}{m_1 r_{12}} - 2Gm_2 p_{1i} \right] \right) \]

\[ + \frac{3Gp_{2i}}{r_{12}} + \frac{Gn_{12}(p_2 - n_1)}{r_{12}} - \frac{2Gm_2 n_{12} S_{1(i)(k)}}{m_1 r_{12}^2} - \frac{2Gm_2 n_{12} S_{1(ik)}}{r_{12}^2} \bigg) \dot{p}_{1j} - H_{\text{eff}} \bigg) \].

This enables us to read off the position coordinate shift

\[ z_1^i = z_1^i + \frac{\dot{S}_{1(i)(j)} u_{1}^{(j)}}{2m_1} \left[ \frac{p_{1i}}{m_1 r_{12}} + 3Gp_{2i} + \frac{Gn_{12}(p_2 - n_1)}{r_{12}} - \frac{2Gm_2 n_{12} S_{1(i)(k)}}{m_1 r_{12}^2} \right] \]

\[ - \frac{2Gm_2 n_{12} S_{1(ik)}}{r_{12}^2} \bigg) + O(c^{-6}) \quad \text{(V.28)} \]

This formula coincides with Eq. (IV.26) when spin and position variables on the right hand side of Eq. (V.28) are provided with a hat in the highest pN terms (meaning all the linear in p terms here), which is allowed when working in a perturbative scheme. Again the Minkowski term here is shown for pedagogical reasons only, because the Minkowski case can be treated exactly and in order to arrive at the pN order one has to include one higher Minkowski term in (V.28), which changes the coefficient when transforming it perturbatively to the left of Eq. (V.28). For the Minkowski case we state that we are always able to write

\[ u^{(i)} = \frac{p_i}{\sqrt{m^2 + p^2}} \quad \text{(V.29)} \]

It follows from Legendre transformation of the point particle Lagrangian (\( u^{(i)} \equiv v^i \) in SRT)

\[ S_{\text{pp}} = \int dt L = m \int dt \sqrt{1 - v^2} \quad \text{(V.30)} \]

or from the 3-components of Dixon’s momentum (II.5) which happens to be the same as the canonical one but only in the Minkowski case. Then the addition to the action reads

\[ Z = \frac{\dot{S}_{(k)(j)} u^{(k)}}{u(u + u_{(0)})} \left[ \frac{\dot{p}_{k} p_{j}}{m^2 + p^2} \frac{\dot{p}_{k}}{\sqrt{m^2 + p^2}} \right] \quad \text{(V.31)} \]

This whole contribution can be absorbed by redefining the position variable as

\[ z^i = \dot{z}^i + \frac{1}{\sqrt{m^2 + p^2}} \left( \delta_{ij} - \frac{p_j p_i}{m^2 + p^2} \right) \frac{\dot{S}_{(k)(j)} u^{(k)}}{m u(u + u_{(0)})} \]

\[ = \dot{z}^i + \frac{1}{m^2 + p^2} \left( \delta_{ij} - \frac{p_j p_i}{m^2 + p^2} \right) \frac{\dot{S}_{(k)(j)} p_k}{m u(u + u_{(0)})} \]

\[ = \dot{z}^i + \frac{\dot{S}_{(k)(j)} p_k}{m(m + \sqrt{m^2 + p^2})} \quad \text{with} \quad u \equiv \sqrt{1 - v^2} \quad \text{and} \quad u_{(0)} \equiv 1 \quad \text{(V.32)} \]

This is exactly the formula (B.11) from [21]. The expanded expression yields

\[ z^i = \dot{z}^i - \frac{\dot{S}_{(j)(k)} p_k}{2m^2} \left( 1 - \frac{p^2}{4m^2} \right) + O(c^{-6}) \quad \text{(V.33)} \]

which thus matches the Minkowski terms in Eq. (IV.26).
As already mentioned the spin and the Lorentz matrix need another redefinition in order to cancel the term $-\frac{1}{2m} \hat{S}_{(i)(j)} p_{1i} u_{1j}^{(1)}$ from the action. This is achieved by an infinitesimal rotation $\omega^{(i)(j)} = -\omega^{(j)(i)}$ of the local basis so that the canonical spin and Lorentz matrices are corotated according to

$$-\frac{1}{2} \dot{\hat{S}}_{(i)(j)} \dot{\hat{S}}^{(i)(j)} \to -\frac{1}{2} \left[ \hat{S}_{(i)(j)} + \omega_{(i)}^{(m)} \hat{S}_{(m)(j)} + \omega_{(j)}^{(m)} \hat{S}_{(i)(m)} \right] \dot{\hat{S}}^{(i)(j)}$$  \hspace{1cm} (V.34)

As described in the Appendix you can read off

$$\omega_{(i)}^{(m)} = -\omega_{(j)}^{(m)} = \frac{1}{2} \frac{p_{1i} u_{1j}^{(m)}}{m_1} - \frac{1}{2} \frac{p_{1m} u_{1j}^{(i)}}{m_1}$$  \hspace{1cm} (V.35)

and use Eq. (V.21) to determine the final spin redefinition to our approximation:

$$S_{1(i)(j)} = \hat{S}_{1(i)(j)} - \hat{S}_{1(j)(k)} \frac{u_{1j}^{(k)}}{2u_{1(i)}} + \hat{S}^{(j)(k)} \frac{u_{1j}^{(k)}}{2u_{1(i)}} + \frac{p_{1j} u_{1j}^{(m)}}{m_1} \hat{S}_{(m)(j)} + \frac{p_{1j} u_{1j}^{(m)}}{m_1} \hat{S}_{(i)(m)}$$  \hspace{1cm} (V.36)

Next we insert Eq. (IV.16a) for $u_{1j}^{(1)}$ in the second term and Eq. (IV.16c) for $u_{1j}^{(i)}$ the third term of (V.36) and make a pN expansion up to the order $c^{-7}$:

$$S_{1(i)(j)} = \frac{1}{2} \hat{S}_{1(i)(j)} - \frac{1}{2} \hat{S}_{1(j)(k)} \frac{u_{1j}^{(k)}}{2u_{1(i)}} \left( 1 + 2G \frac{m_2}{r_{12}} \right)$$

$$+ \frac{p_{1j} u_{1j}^{(m)}}{m_1} \left[ \frac{p_{1k}}{m_1} - \frac{2G m_2 p_{1k}}{m_1 r_{12}} + \frac{3G p_{1k}}{2r_{12}} + \frac{G n_{12}^{(k)} (p_2 \cdot n_{12})}{m_1 r_{12}^2} - \frac{G m_2 n_{12}^{(k)(l)}}{m_1 r_{12}^2} \right] - (i \leftrightarrow j) + O(c^{-9})$$  \hspace{1cm} (V.37)

We are left with an insertion of the remaining 4-velocities. Again utilizing Eq. (IV.16c) a further expansion of (V.37) yields

$$S_{1(i)(j)} = \frac{1}{2} \hat{S}_{1(i)(j)} + \hat{S}_{1(j)(k)} \frac{1}{2} \left[ \frac{p_{1k} p_{1j}}{m_1^2} - \frac{G m_2 p_{1k} p_{1j}}{m_1 r_{12}^2} + \frac{3G p_{1j} p_{2k}}{2m_1 r_{12}} + \frac{G p_{1j} n_{12}^k (p_2 \cdot n_{12})}{2m_1 r_{12}^2} \right] - (i \leftrightarrow j) + O(c^{-9})$$  \hspace{1cm} (V.38)

The last step involves providing all spin and position variables with a hat in the highest pN terms of (V.38), so that we end up with the transformation formula

$$S_{1(i)(j)} = \frac{1}{2} \hat{S}_{1(i)(j)} + \hat{S}_{1(j)(k)} \frac{1}{2} \left[ \frac{p_{1k} p_{1j}}{m_1^2} - \frac{G m_2 p_{1k} p_{1j}}{m_1 r_{12}^2} + \frac{3G p_{1j} p_{2k}}{2m_1 r_{12}} + \frac{G p_{1j} n_{12}^k (p_2 \cdot n_{12})}{2m_1 r_{12}^2} \right]$$

$$+ \frac{G m_2 n_{12}^{(k)} p_{1j}}{m_1 r_{12}^2} + \frac{G n_{12}^{(k)(l)} p_{1j}}{m_1 r_{12}^2} - (i \leftrightarrow j) + O(c^{-9})$$  \hspace{1cm} (V.39)

Notice that Eq. (V.39) (or rather Eq. (IV.27)), because one has to extend the Minkowski term to arrive at the order $c^{-7}$) is already high enough in pN order to be used for transforming effective NNLO Hamiltonians to canonical ones, whereas for that purpose Eq. (V.28) needs to be extended to the order $c^{-10}$, which in turn means to calculate the vierbein components (IV.15a)-(IV.15c) to higher pN orders. In order to get the Minkowski case right one starts with of Eq. (V.21) and replaces the 4-velocities by Eq. (V.29). The result can be expanded to match the Minkowski terms in Eq. (IV.27).

The most important relation derived in the present section is equation (V.22). To arrive at this expression the transformations of Lorentz matrix and spin tensor shown in (V.20) and (V.21) were used. Notice that no approximation was used to arrive at (V.22), only the validity of the supplementary conditions (V.17a) and (V.17b) is required. The
calculation following (V.22) is devoted to the absorption of the term \( Z \) by various further variable transformations, most notably by a redefinition of the position. This is necessary due to the presence of the acceleration \( \ddot{u}^{(i)} \) in \( Z \). However, within a perturbative context such terms in the action can be treated in a simpler way by the method developed in [58, 66], which in most cases amounts to an insertion of lower order equations of motion. This actually corresponds to an implicit redefinition of variables, so a comparison with the Dirac bracket approach is more difficult in this case and therefore we have not proceeded in this way here (e.g., it is likely that the canonical momentum is implicitly redefined). However, for an application at even higher pN orders one can take \(-Z\) as an addition to the Hamiltonian, after \( \ddot{u}^{(i)} \) therein was eliminated using the equations of motion. Also a mixed approach may be useful. One can always write

\[
\ddot{u}^{(i)} = \frac{\ddot{p}_i}{m} + \mathcal{V}^i, \tag{V.40}
\]

where \( \mathcal{V}^i \) includes all pN corrections to this relation, see (IV.16c) for the NLO case. \( Z \) then reads

\[
Z = \frac{\dot{S}_{(1)j}^{(i)} u^{(i)}}{u(u + u_{(0)})} \left( \frac{\dot{p}_i}{m} + \mathcal{V}^i \right). \tag{V.41}
\]

The first term can now be absorbed by a transformation of the position of the form

\[
z^i = \hat{z}^i + \frac{\dot{S}_{(1)j}^{(i)} u^{(j)}}{m u(u + u_{(0)})} \tag{V.42}
\]

while the second term is considered as a contribution to the Hamiltonian. This mixed approach thus consists of an explicit redefinition of the position implementing the leading order flat space transformation to the Newton Wigner position, followed by implicit variable redefinitions due to the insertion of equations of motion.

The discussion in the last paragraph can even serve to modify the Feynman rules of the EFT formalism to use reduced canonical variables from the very beginning. First let us explain why such a modification of the Feynman rules provides an improvement. It is most desirable to formulate the Feynman rules in terms of reduced spin variables (either reduced canonical variables from the very beginning. First let us explain why such a modification of the Feynman rules

1 Notice that the Routhian in [8] is therefore not manifestly covariant. By adding \(-\frac{1}{2}S_{ab}\Omega^{ab}\) to it one can get back to the manifestly covariant Lagrangian contained in Eq. (1) of [8], see also Sect. III in [13] and Sect. 5.2.2 in [24].
VI. FINAL COMPARISONS OF POTENTIALS WITH HAMILTONIANS

In this last section we make use of the transformation formulae we have found throughout the previous sections, which enable us to transform all the non-reduced effective Hamiltonians from Sect. III to reduced ones depending on standard canonical variables. We are especially interested in the effective NLO Hamiltonians, which shall be compared with their ADM counterparts that were calculated directly within the ADM approach, see [25, 39, 42–44]. As the variable transformation of the covariant variables to Newton-Wigner ones has to be done in all Hamiltonians up to 2pN order we will again get NLO correction terms stemming from all the subleading order Hamiltonians starting with the Newtonian Hamiltonian from (1.3). Therein we replace the position variable \( z \) by \( \hat{z} \) utilizing Eq. (IV.26) or (V.28). The leading-order spin-orbit correction term emerging from this procedure is labeled as \( H_{\text{LOSO}}^N \). The upper index refers to the Hamiltonian where the variable replacement is made and lower one is reference to the pN order and the specific Hamiltonian which is to be corrected by that. Likewise we label all the NLO correction terms and get

\[
H_{\text{LOSO}}^N = \frac{G}{r_{12}^2} \left[ \frac{m_1}{2m_2} ((p_2 \times \hat{S}_2) \cdot \hat{n}_{12}) - \frac{m_2}{2m_1} ((p_1 \times \hat{S}_1) \cdot \hat{n}_{12}) \right],
\]

(VI.1)

\[
H_{\text{NLOSO}}^N = \frac{G}{r_{12}^2} \left[ \frac{m_2}{8m_1^2} p_1^2 ((p_1 \times \hat{S}_1) \cdot \hat{n}_{12}) - \frac{m_1}{8m_2^2} p_2^2 ((p_2 \times \hat{S}_2) \cdot \hat{n}_{12}) \right]
+ \frac{G^2}{r_{12}^2} \left[ \frac{m_2^2}{m_1} ((p_1 \times \hat{S}_1) \cdot \hat{n}_{12}) - \frac{3m_2}{2} ((p_2 \times \hat{S}_1) \cdot \hat{n}_{12}) \right]
\]

\[
+ \frac{3m_2}{2} ((p_1 \times \hat{S}_2) \cdot \hat{n}_{12}) - \frac{m_2^2}{m_2} ((p_2 \times \hat{S}_2) \cdot \hat{n}_{12}) \right],
\]

(VI.2)

\[
H_{\text{NLOSO}}^{S_1 S_2} = \frac{G}{m_1 m_2 r_{12}^2} \left[ -\frac{3}{4} (p_1 \cdot p_2) (\hat{S}_1 \cdot \hat{n}_{12}) (\hat{S}_2 \cdot \hat{n}_{12}) + \frac{3}{4} (p_1 \cdot \hat{n}_{12}) (\hat{S}_1 \cdot p_2) (\hat{S}_2 \cdot \hat{n}_{12}) 
- \frac{1}{2} (\hat{S}_1 \cdot p_2) (\hat{S}_2 \cdot p_1) + \frac{3}{4} (p_2 \cdot \hat{n}_{12}) (\hat{S}_1 \cdot \hat{n}_{12}) (\hat{S}_2 \cdot p_1) 
- \frac{3}{4} (p_1 \cdot \hat{n}_{12}) (p_2 \cdot \hat{n}_{12}) (\hat{S}_1 \cdot \hat{n}_{12}) + \frac{1}{2} (p_1 \cdot p_2) (\hat{S}_1 \cdot \hat{S}_2) \right]
+ \frac{G^2 (m_1 + m_2)}{r_{12}^2} \left[ (\hat{S}_1 \cdot \hat{n}_{12}) (\hat{S}_2 \cdot \hat{n}_{12}) - (\hat{S}_1 \cdot \hat{S}_2) \right],
\]

(VI.3)

\[
H_{\text{NLOSO}}^{S_1 S_2} = \frac{G m_2}{r_{12}^2} \left[ \frac{3}{8} p_1^2 (\hat{S}_1 \cdot \hat{n}_{12})^2 - \frac{3}{4} (p_1 \cdot \hat{n}_{12}) (\hat{S}_1 \cdot \hat{n}_{12}) (\hat{S}_1 \cdot p_1) 
+ \frac{1}{4} (\hat{S}_1 \cdot p_1)^2 + \frac{3}{8} (p_1 \cdot \hat{n}_{12})^2 \hat{S}_1^2 - \frac{1}{4} \hat{p}_1^2 \hat{S}_1^2 \right]
+ \frac{G^2 m_2^2}{m_1 r_{12}^2} \left[ (\hat{S}_1 \cdot \hat{n}_{12})^2 - \hat{S}_1^2 \right].
\]

(VI.4)

The same replacements of position variables must be performed in the EIH potential giving rise to the following correction terms to the NLO spin-orbit Hamiltonian:

\[
H_{\text{EIH}}^\text{NLOSO} = \frac{G}{r_{12}} \left[ -\frac{3m_2}{4m_1^3} p_1^3 + \frac{3}{4m_1} (p_1 \cdot \hat{n}_{12}) (p_2 \cdot \hat{n}_{12}) \right] ((p_1 \times \hat{S}_1) \cdot \hat{n}_{12})
+ \left( \frac{7}{4m_1^3} (p_1 \cdot p_2) - \frac{3}{4m_1 m_2^2} \right) ((p_1 \times \hat{S}_1) \cdot \hat{n}_{12}) 
\]

\[
- \frac{1}{4m_1^3} (p_1 \cdot \hat{n}_{12}) ((p_1 \times \hat{S}_1) \cdot p_2) \right] + \frac{G^2}{r_{12}^2} \left( \frac{m_2}{2} + \frac{m_2^2}{2m_1} \right) ((p_1 \times \hat{S}_1) \cdot \hat{n}_{12}).
\]

(VI.5)
Next we replace variables in the LO spin-orbit Hamiltonian from Eq. (III.6), where we first have to insert the covariant SSC from Eq. (IV.17c) to arrive at a consistent expression. The spin replacement by Eq. (IV.27) yields the correction term

\[
H_{\text{NLO SO}} = \frac{G}{r_{12}} \left( \frac{n_2}{m_1} \hat{\mathbf{n}}_{12} \cdot (\mathbf{p}_1 \times \hat{\mathbf{s}}_1) - \frac{1}{n_1^2} (\mathbf{p}_1 \cdot \mathbf{p}_2) \hat{\mathbf{n}}_{12} \cdot (\mathbf{p}_1 \times \hat{\mathbf{s}}_1) \right) \\
- \frac{1}{m_1^2} (\hat{\mathbf{n}}_{12} \cdot \mathbf{p}_1) (\mathbf{p}_2 \times \hat{\mathbf{s}}_1) \right) + (1 \leftrightarrow 2). \tag{VI.6}
\]

Equally replacing the position variables by Eq. (IV.26) in this Hamiltonian gives further correction terms for the NLO \( S_1S_2 \) and NLO \( S_1^2 \) Hamiltonians

\[
H_{\text{NLO S1S2}} = \frac{G}{r_{12}} \left[ \frac{1}{m_1^2} \left( -3p_1^2(S_1 \cdot \hat{n}_{12})(S_2 \cdot \hat{n}_{12}) + 3(p_1 \cdot \hat{n}_{12})(S_1 \cdot \mathbf{p}_1)(S_2 \cdot \hat{n}_{12}) + 2p_1^2(S_1 \cdot \hat{S}_2) \\
+ 3(p_1 \cdot \hat{n}_{12})(S_1 \cdot \hat{n}_{12})(S_2 \cdot \mathbf{p}_1) - 2(S_1 \cdot \mathbf{p}_1)(S_2 \cdot \mathbf{p}_1) - 3(p_1 \cdot \hat{n}_{12})^2(S_1 \cdot \hat{S}_2) \\
+ \frac{1}{m_2^2} \left( -3p_2^2(S_2 \cdot \hat{n}_{12})(S_1 \cdot \hat{n}_{12}) + 3(p_2 \cdot \hat{n}_{12})(S_2 \cdot \mathbf{p}_2)(S_1 \cdot \hat{n}_{12}) + 2p_2^2(S_1 \cdot \hat{S}_2) \\
- 2(S_2 \cdot \mathbf{p}_2)(S_1 \cdot \mathbf{p}_2) - 3(p_2 \cdot \hat{n}_{12})^2(S_1 \cdot \hat{S}_2) + 3(p_2 \cdot \hat{n}_{12})(S_2 \cdot \hat{n}_{12})(S_1 \cdot \mathbf{p}_1) \\
+ \frac{1}{m_1 m_2} \left( 6(p_1 \cdot \mathbf{p}_2)(S_1 \cdot \hat{n}_{12})(S_2 \cdot \mathbf{n}_{12}) - 6(p_1 \cdot \hat{n}_{12})(S_1 \cdot \mathbf{p}_2)(S_2 \cdot \hat{n}_{12}) \\
- 6(p_2 \cdot \hat{n}_{12})(S_1 \cdot \hat{n}_{12})(S_2 \cdot \mathbf{p}_1) + 4(S_1 \cdot \mathbf{p}_2)(S_2 \cdot \mathbf{p}_1) \\
+ 6(p_1 \cdot \hat{n}_{12})(p_2 \cdot \hat{n}_{12})(S_1 \cdot \hat{S}_2) - 4(p_1 \cdot \mathbf{p}_2)(S_1 \cdot \hat{S}_2) \right) \right], \tag{VI.7}
\]

\[
H_{\text{NLO S1}} = \frac{G}{r_{12}} \left[ \frac{m_2}{m_1} \left( -3p_1^2(S_1 \cdot \hat{n}_{12})^2 + 6(p_1 \cdot \hat{n}_{12})(S_1 \cdot \hat{n}_{12})(S_1 \cdot \mathbf{p}_1) - 2(S_1 \cdot \mathbf{p}_1)^2 \\
- 3(p_1 \cdot \hat{n}_{12})^2(S_1 \cdot \hat{S}_2) + 2p_1^2S_1^2 + \frac{1}{m_1^2} \left( 3(p_1 \cdot \mathbf{p}_2)(S_1 \cdot \hat{n}_{12})^2 \\
- 3(p_2 \cdot \hat{n}_{12})(S_1 \cdot \hat{n}_{12})(S_1 \cdot \mathbf{p}_1) - 3(p_1 \cdot \hat{n}_{12})(S_1 \cdot \hat{n}_{12})(S_1 \cdot \mathbf{p}_1) \\
+ 2(S_1 \cdot \mathbf{p}_1)(S_1 \cdot \mathbf{p}_2) + 3(p_1 \cdot \hat{n}_{12})(p_2 \cdot \hat{n}_{12})S_1 - 2(p_1 \cdot \mathbf{p}_2)S_1^2 \right) \right] \tag{VI.8}
\]

Spin replacement in the LO \( S_1S_2 \) and LO \( S_1^2 \)-Hamiltonian leads to correction terms

\[
H_{\text{NLO S1S2}} = \frac{G}{r_{12}} \left[ \frac{1}{m_1} \left( \frac{3}{m_1^2}(S_2 \cdot \hat{n}_{12})(S_2 \cdot \mathbf{n}_{12}) - \frac{3}{2}(p_1 \cdot \hat{n}_{12})(S_2 \cdot \mathbf{n}_{12}) \\
+ \frac{1}{2}(S_1 \cdot \mathbf{p}_1)(S_2 \cdot \mathbf{p}_1) - \frac{1}{2}p_1^2(S_1 \cdot \hat{S}_2) \right) + \frac{1}{m_2^2} \left( \frac{3}{2}p_2^2(S_2 \cdot \hat{n}_{12})(S_1 \cdot \hat{n}_{12}) \\
- \frac{3}{2}(p_2 \cdot \hat{n}_{12})(S_2 \cdot \mathbf{p}_2)(S_1 \cdot \hat{n}_{12}) + \frac{1}{2}(S_2 \cdot \mathbf{p}_2)(S_2 \cdot \mathbf{p}_2) - \frac{1}{2}p_2^2(S_1 \cdot \hat{S}_2) \right) \right], \tag{VI.9}
\]

\[
H_{\text{NLO S1}} = \frac{GC_{Q_1} m_2}{2m_1^2r_{12}^2} \left[ 3p_1^2(S_1 \cdot \hat{n}_{12}) - 3(p_1 \cdot \hat{n}_{12})(S_1 \cdot \hat{n}_{12})(S_1 \cdot \mathbf{p}_1) + (S_1 \cdot \mathbf{p}_1)^2 - p_1^2S_1^2 \right], \tag{VI.10}
\]

whilst the replacement of the position variable in these two Hamiltonians leads to no new NLO correction terms.
We adopt Eq. (III.23) and make the transition to a canonical Hamiltonian, which depends solely on the hatted variables \( (\hat{s}_I, \hat{p}_J, \hat{S}_K) \). First thing to do is to eliminate the \( S^{(i)} \) variables by imposing the SSC from Eq. (IV.17c). The result is the effective Hamiltonian still depending on \( (x_I, p_J, S_K) \):

\[
H_{\text{NLO}}^{\text{eff}, \text{red}} = \frac{GM_2}{r_{12}^3} \left[ \frac{p_1}{m_1} \times \hat{n}_{12} \left( \frac{p_1^2}{m_1^2} + \frac{p_1 \cdot p_2}{m_1 m_2} - \frac{p_2^2}{m_2^2} \right) \right] + \frac{p_2 \times \hat{n}_{12}}{m_2} \left( \frac{-p_1 \cdot p_2}{m_1 m_2} - \frac{3(p_1 \cdot n_{12})(p_2 \cdot \hat{n}_{12})}{m_1 m_2} \right) + \frac{G^2 m_2}{r_{12}^3} \left[ \frac{p_1 \times p_2}{m_1 m_2} \left( \frac{p_1 \cdot \hat{n}_{12}}{m_1} - \frac{p_2 \cdot n_{12}}{m_2} \right) \right] \right]
\]

(VI.11)

To make the transition of this Hamiltonian complete we put a hat on its variables and add up to it all the needed correction terms via

\[
H_{\text{NLO}}^{\text{can}} = H_{\text{NLO}}^{\text{eff}} + H_{\text{NLOSO}}^{\text{NLO}} + H_{\text{NLOSO}}^{\text{LLO}} + H_{\text{NLOSO}}^{\text{EIH}}
\]

(VI.12)

which results in

\[
H_{\text{NLOSO}}^{\text{can}} = \frac{GM_2}{r_{12}^3} \left[ \frac{p_1 \times \hat{n}_{12}}{m_1} \left( \frac{5p_1^2}{8m_1^2} + \frac{5p_1 \cdot p_2}{4m_1 m_2} - \frac{p_2^2}{4m_2^2} \right) \right] + \frac{p_2 \times \hat{n}_{12}}{m_2} \left( \frac{-p_1 \cdot p_2}{4m_1 m_2} - \frac{3(p_1 \cdot \hat{n}_{12})(p_2 \cdot \hat{n}_{12})}{4m_1 m_2} \right) + \frac{G^2 m_2}{r_{12}^3} \left[ \frac{p_1 \times p_2}{m_1 m_2} \left( \frac{5p_1 \cdot \hat{n}_{12}}{2m_2} + \frac{5p_2 \cdot \hat{n}_{12}}{2m_1} \right) \right]
\]

(VI.13)

which is to be compared with the NLO spin-orbit ADM canonical Hamiltonian first derived by Damour, Jaranowski and Schäfer [39], which reads

\[
H_{\text{NLOSO}}^{\text{DIS}} = \frac{GM_2}{r_{12}^3} \left[ \frac{p_1 \times \hat{n}_{12}}{m_1} \left( \frac{5p_1^2}{8m_1^2} + \frac{3p_1 \cdot p_2}{4m_1 m_2} - \frac{3p_2^2}{4m_2^2} \right) \right] + \frac{p_2 \times \hat{n}_{12}}{m_2} \left( \frac{-p_1 \cdot p_2}{4m_1 m_2} - \frac{3(p_1 \cdot \hat{n}_{12})(p_2 \cdot \hat{n}_{12})}{4m_1 m_2} \right) + \frac{G^2 m_2}{r_{12}^3} \left[ \frac{p_1 \times p_2}{m_1 m_2} \left( \frac{5p_1 \cdot \hat{n}_{12}}{2m_2} + \frac{5p_2 \cdot \hat{n}_{12}}{2m_1} \right) \right]
\]

(VI.14)

If both Hamiltonians are correct and therefore to generate the same equations of motion, the difference between these two should equal an infinitesimal canonical transformation, which in turn involves a generator function \( g \) that is to be chosen appropriately as outlined in e.g., [13]. So it should hold

\[
\Delta H_{\text{NLOSO}}^{\text{can}} = H_{\text{NLOSO}}^{\text{DIS}} - H_{\text{NLOSO}}^{\text{can}} \approx \{ H_N, g \} = \frac{DG}{df},
\]

(VI.15)

where for the generator function one can make the following general ansatz to canonically transform a NLO spin-orbit Hamiltonian:

\[
g = \frac{GM_2}{r_{12}^3} \left[ \frac{p_1}{m_1} \times \hat{n}_{12} \left( \frac{\gamma_2 p_1 \cdot \hat{n}_{12}}{m_1} + \frac{\gamma_3 p_2 \cdot \hat{n}_{12}}{m_2} \right) \right] + \frac{p_2 \times \hat{n}_{12}}{m_2} \left( \frac{\gamma_4 p_1 \cdot \hat{n}_{12}}{m_1} + \frac{\gamma_5 p_2 \cdot \hat{n}_{12}}{m_2} \right)
\]

(VI.16)

It turns out Eq. (VI.15) can be fulfilled by choosing the coefficients to be

\[
\gamma_1 = \frac{1}{2}, \quad \gamma_2 = 0, \quad \gamma_3 = -\frac{1}{2}, \quad \gamma_4 = 0, \quad \gamma_5 = 0.
\]

(VI.17)

We kindly note that the same agreement of (VI.11) with (VI.14) was already achieved by Levi [13]. However, the transformation to canonical variables was found in [13] by comparing with the Hamiltonian in [39], whereas here
we derived it from general principles. For this reason our Eq. (IV.26) also contains a term which is irrelevant for translation invariant quantities like the Hamiltonian (but would be needed, e.g., for the center of mass), whereas such a term was omitted in Eq. (121) of [13] as it is not needed there. Notice in [39] there was also a variable transformation formula determined to achieve a comparison with a result which was obtained in harmonic coordinates and with a spin whose length is non-conserved. Their transformation formula (6.11), which was uniquely determined and comprises non-canonical and canonical transformations, contains a similar irrelevant term when comparing translation invariant quantities. By our derivation of Eq. (IV.26) from general principles we also achieved direct justification of their transformation formula, that was left as future work by the authors.

B. The NLO spin-orbit Hamiltonian of Porto

We also want to compare the result of Porto with the ADM NLO spin-orbit Hamiltonian from Eq. (VI.14). We start out from the non-reduced effective Hamiltonian from Eq. (III.25) and insert the covariant SSC from Eq. (IV.17c), which results in the reduced effective NLO Hamiltonian in 'covariant' variables

\[
H_{NLO(\text{eff})}^{\text{SO}(P), \text{red}} = \frac{G_{m_2}}{r_{12}^2} S_1 \cdot \left[ \frac{p_1 \times n_{12}}{m_1} \left( \frac{5 m_2^2}{2 m_1^3} + \frac{5 p_1 \cdot p_2}{4 m_1 m_2} - \frac{5 p_2^2}{4 m_2^2} + \frac{3 (p_2 \cdot n_{12})^2}{m_2^2} \right) \right.
\]

\[
+ \frac{p_2 \times n_{12}}{m_2} \left( -3 \frac{(p_2 \cdot n_{12})^2}{m_2^2} + \frac{p_2^2}{m_2^2} - \frac{2 (p_1 \cdot p_2)}{m_1 m_2} \right) \right] \]

\[
+ G_{m_2}^2 \frac{S_1}{r_{12}^2} \cdot \left[ \frac{p_1 \times \hat{n}_{12}}{m_1} \left( \frac{15}{2} n_1 + 5 m_2 \right) + \frac{p_2 \times \hat{n}_{12}}{m_2} \left( -7 m_1 - \frac{15}{2} m_2 \right) \right] + [1 \leftrightarrow 2].
\]

The transition to the canonical is made as in the case of Levi's effective Hamiltonian by adding up the missing pieces and putting a hat on the variables of Porto's Hamiltonian:

\[
H_{NLO(\text{can})}^{\text{SO}(P)} = H_{NLO(\text{eff})}^{\text{SO}(P), \text{red}} + H_{NLOSO}^{\text{NLO}} + H_{NLOSO}^{\text{EIH}} + H_{NLOSO}^{\text{LOS}}
\]

The result is

\[
H_{NLO(\text{can})}^{\text{SO}(P)} = \frac{G_{m_2}}{r_{12}^2} S_1 \cdot \left[ \frac{p_1 \times \hat{n}_{12}}{m_1} \left( \frac{15}{2} n_1 + 5 m_2 \right) + \frac{p_2 \times \hat{n}_{12}}{m_2} \left( -7 m_1 - \frac{15}{2} m_2 \right) \right] + [1 \leftrightarrow 2].
\]

Again for the difference \( \Delta H_{NLOSO}^{\text{NLO}} \) there should exist a generator function Eq. (VI.16) with determined coefficients. The coefficients for this case read

\[
\gamma_1 = \frac{3}{2}, \quad \gamma_2 = 0, \quad \gamma_3 = \frac{1}{2}, \quad \gamma_4 = 0, \quad \gamma_5 = -1,
\]

which means we have achieved 'on-shell' agreement with the ADM Hamiltonian.

C. The NLO spin(1)-spin(2) Hamiltonian of Porto and Rothstein

For reason of completeness we also compare the NLO spin(1)-spin(2) Hamiltonian which results from the corresponding effective potential that Porto and Rothstein have calculated, with the corresponding ADM Hamiltonian. We refer to Eq. (III.27) and insert the SSC from Eq. (IV.17c) to arrive at the reduced effective Hamiltonian, which
reads

\[
H_{NLO,\text{red}}^{\text{eff}} = \frac{G}{r_1^2} \left[ \frac{1}{m_1} \left( 3p_{1z}^2(S_1 \cdot n_{12})(S_2 \cdot n_{12}) - 3(p_1 \cdot n_{12})(S_1 \cdot p_1)(S_2 \cdot n_{12}) \right. \\
- 3(p_1 \cdot n_{12})(S_1 \cdot n_{12})(S_2 \cdot p_1) + 2(S_1 \cdot p_1)(S_2 \cdot p_1) \\
+ 3(p_1 \cdot n_{12})^2(S_1 \cdot S_2) - 2p_{1z}^2(S_1 \cdot S_2) \left) + \frac{1}{m_2} \left( 3p_{2z}^2(S_1 \cdot n_{12})(S_2 \cdot n_{12}) \right. \\
- 3(p_2 \cdot n_{12})(S_1 \cdot p_2)(S_2 \cdot n_{12}) - 3(p_2 \cdot n_{12})(S_1 \cdot n_{12})(S_2 \cdot p_2) \\
+ 2(S_1 \cdot p_2)(S_2 \cdot p_2) + 3(p_2 \cdot n_{12})^2(S_1 \cdot S_2) - 2p_{2z}^2(S_1 \cdot S_2) \left) \\
+ \frac{1}{m_1 m_2} \left( \frac{15}{8} (p_1 \cdot n_{12})(p_2 \cdot n_{12})(S_1 \cdot n_{12})(S_2 \cdot n_{12}) \\
- \frac{21}{2} (p_1 \cdot p_2)(S_1 \cdot n_{12})(S_2 \cdot n_{12}) + \frac{9}{2} (p_2 \cdot n_{12})(S_1 \cdot p_1)(S_2 \cdot n_{12}) \\
- \frac{9}{2} (p_1 \cdot n_{12})(p_1 \cdot n_{12})(S_2 \cdot n_{12}) - \frac{5}{2}(S_1 \cdot p_1)(S_2 \cdot p_2) \\
- \frac{9}{2} (p_2 \cdot n_{12})(S_1 \cdot n_{12})(S_2 \cdot p_1) - \frac{5}{2}(S_1 \cdot p_2)(S_2 \cdot p_1) \\
+ \frac{9}{2} (p_1 \cdot n_{12})(S_1 \cdot n_{12})(S_2 \cdot p_2) - \frac{9}{2}(p_1 \cdot n_{12})(p_2 \cdot n_{12})(S_1 \cdot S_2) \\
+ \frac{11}{2} (p_1 \cdot p_2)(S_1 \cdot S_2) \right) \right] + \frac{G^2 (m_1 + m_2)}{r_1^2} \left[ 5(S_1 \cdot S_2) - 11(S_1 \cdot n_{12})(S_2 \cdot n_{12}) \right].
\]

We transform this ‘covariant’ Hamiltonian to the canonical one by adding up to it the corresponding correction terms which follow from the variable transformation and replace the ‘covariant’ variables with Newton-Wigner ones in the original Hamiltonian

\[
H_{NLO,\text{red}}^{\text{can}} = H_{NLO,\text{red}}^{\text{eff}} + \sum H_{NLO,\text{red}}^{\text{S1,S2}} + H_{NLO,\text{S1,S2}}^{\text{LOS}} + H_{NLO,\text{S1,S2}}^{\text{LOS}},
\]

the result being

\[
H_{NLO,\text{can}}^{\text{S1,S2,red}} = \frac{G}{r_1^2} \left[ \frac{1}{m_1} \left( 3p_{1z}^2(S_1 \cdot n_{12})(S_2 \cdot n_{12}) - 3(p_1 \cdot n_{12})(S_1 \cdot p_1)(S_2 \cdot n_{12}) \right. \\
+ \frac{1}{2} (S_1 \cdot p_1)(S_2 \cdot p_1) - \frac{1}{2} p_{1z}^2(S_1 \cdot S_2) \left) + \frac{1}{m_2} \left( 3p_{2z}^2(S_1 \cdot n_{12})(S_2 \cdot n_{12}) \right. \\
- 3(p_2 \cdot n_{12})(S_1 \cdot p_2)(S_2 \cdot n_{12}) - 3(p_2 \cdot n_{12})(S_1 \cdot n_{12})(S_2 \cdot p_2) \\
+ 2(S_1 \cdot p_2)(S_2 \cdot p_2) + 3(p_2 \cdot n_{12})^2(S_1 \cdot S_2) - 2p_{2z}^2(S_1 \cdot S_2) \left) \\
+ \frac{1}{m_1 m_2} \left( \frac{15}{8} (p_1 \cdot n_{12})(p_2 \cdot n_{12})(S_1 \cdot n_{12})(S_2 \cdot n_{12}) \\
- \frac{21}{4} (p_1 \cdot p_2)(S_1 \cdot n_{12})(S_2 \cdot n_{12}) + \frac{9}{2} (p_2 \cdot n_{12})(S_1 \cdot p_1)(S_2 \cdot n_{12}) \\
- \frac{3}{4} (p_1 \cdot n_{12})(S_1 \cdot p_2)(S_2 \cdot n_{12}) - \frac{5}{2}(S_1 \cdot p_1)(S_2 \cdot p_2) \\
- \frac{3}{4} (p_2 \cdot n_{12})(S_1 \cdot n_{12})(S_2 \cdot p_1) - (S_1 \cdot p_2)(S_2 \cdot p_1) \\
+ \frac{9}{2} (p_1 \cdot n_{12})(S_1 \cdot n_{12})(S_2 \cdot p_2) + \frac{3}{4}(p_1 \cdot n_{12})(p_2 \cdot n_{12})(S_1 \cdot S_2) \\
+ 2(p_1 \cdot p_2)(S_1 \cdot S_2) \right) \right] + \frac{G^2 (m_1 + m_2)}{r_1^2} \left[ 5(S_1 \cdot S_2) - 11(S_1 \cdot n_{12})(S_2 \cdot n_{12}) \right].
\]
This Hamiltonian shall be compared with the corresponding ADM Hamiltonian which was calculated by us in [25, 40]. It reads

\[
H_{\text{ADM(can)}}^{\text{ADM(can)}} = \frac{G}{r_{12}^2} \left[ \frac{1}{m_1^2} \left( 3p_1^2(\hat{S}_1 \cdot \hat{n}_{12})(\hat{S}_2 \cdot \hat{n}_{12}) - 3(p_1 \cdot \hat{n}_{12})(\hat{S}_1 \cdot p_1)(\hat{S}_2 \cdot \hat{n}_{12}) \right) \right.
\]

\[
- 3(p_1 \cdot \hat{n}_{12})(\hat{S}_1 \cdot \hat{n}_{12})(\hat{S}_2 \cdot p_1) + \frac{3}{2}(\hat{S}_1 \cdot p_1)(\hat{S}_2 \cdot p_1) - \frac{3}{2}p_1^2(\hat{S}_1 \cdot \hat{S}_2) \right)
\]

\[
+ \frac{1}{m_2^2} \left( 3p_2^2(\hat{S}_1 \cdot \hat{n}_{12})(\hat{S}_2 \cdot \hat{n}_{12}) - 3(p_2 \cdot \hat{n}_{12})(\hat{S}_1 \cdot \hat{n}_{12})(\hat{S}_2 \cdot \hat{n}_{12}) \right) - 3(p_2 \cdot \hat{n}_{12})(\hat{S}_1 \cdot \hat{n}_{12})(\hat{S}_2 \cdot p_2) + \frac{3}{2}(\hat{S}_1 \cdot p_2)(\hat{S}_2 \cdot p_2) - \frac{3}{2}p_2^2(\hat{S}_1 \cdot \hat{S}_2) \right)
\]

\[
+ \frac{1}{m_1 m_2} \left( - \frac{15}{2}(p_1 \cdot \hat{n}_{12})(p_2 \cdot \hat{n}_{12})(\hat{S}_1 \cdot \hat{n}_{12})(\hat{S}_2 \cdot \hat{n}_{12}) \right)
\]

\[
- \frac{21}{4}(p_1 \cdot p_2)(\hat{S}_1 \cdot \hat{n}_{12})(\hat{S}_2 \cdot \hat{n}_{12}) + \frac{9}{2}(p_1 \cdot \hat{n}_{12})(\hat{S}_1 \cdot p_1)(\hat{S}_2 \cdot \hat{n}_{12})
\]

\[
+ \frac{9}{4}(p_1 \cdot \hat{n}_{12})(\hat{S}_1 \cdot p_2)(\hat{S}_2 \cdot \hat{n}_{12}) - \frac{5}{2}(\hat{S}_1 \cdot p_1)(\hat{S}_2 \cdot p_2)
\]

\[
+ \frac{9}{4}(p_2 \cdot \hat{n}_{12})(\hat{S}_1 \cdot \hat{n}_{12})(\hat{S}_2 \cdot p_1) - (\hat{S}_1 \cdot p_2)(\hat{S}_2 \cdot p_1)
\]

\[
+ \frac{9}{4}(p_1 \cdot \hat{n}_{12})(\hat{S}_1 \cdot \hat{n}_{12})(\hat{S}_2 \cdot p_2) - \frac{21}{4}(p_1 \cdot \hat{n}_{12})(p_2 \cdot \hat{n}_{12})(\hat{S}_2 \cdot \hat{S}_2)
\]

\[
+ 4(p_1 \cdot p_2)(\hat{S}_1 \cdot \hat{S}_2) \right] + \frac{G^2(m_1 + m_2)}{r_{12}^4} \left[ 6(\hat{S}_1 \cdot \hat{S}_2) - 12(\hat{S}_1 \cdot \hat{n}_{12})(\hat{S}_2 \cdot \hat{n}_{12}) \right].
\]

To generate the difference \( \Delta H_{\text{NLO(can)}}^{\text{ADM(can)}} = H_{\text{ADM(can)}}^{\text{ADM(can)}} - H_{\text{NLO(can)}}^{\text{NLO(can)}} \) of the ADM Hamiltonian to the one resulting from an effective potential we make another ansatz for the generator function \( g \):

\[
g_{\text{NLO(can)}}^{\text{can}} = \frac{G}{r_{12}} \left[ \frac{\gamma_1}{m_1}(\hat{S}_1 \cdot p_1)(\hat{S}_2 \cdot \hat{n}_{12}) - \frac{\gamma_2}{m_1}(\hat{S}_1 \cdot \hat{n}_{12})(\hat{S}_2 \cdot p_1) + \frac{\gamma_3}{m_1}(p_1 \cdot \hat{n}_{12})(\hat{S}_1 \cdot \hat{S}_2) + (1 \leftrightarrow 2) \right].
\]

The coefficients can be uniquely determined and yield

\[
\gamma_1 = 0, \quad \gamma_2 = -1, \quad \gamma_3 = -1, \quad \gamma_4 = 0.
\]

An alternative potential in the covariant SSC was derived by Levi [9] and the procedure to obtain a fully reduced Hamiltonian used in the present paper (and briefly presented before in [44]) was already applied therein. Therefore we will not repeat it here. However, the transformation to canonical variables was found in [9] by comparing with the Hamiltonian in [40], whereas here we derived it from general principles.

Notice that in [40] it was found that the result in [79] was incomplete, however, once completed it also fully agrees with the corresponding NLO spin(1)-spin(2) ADM Hamiltonian [8]. Though the result in [79] was derived within the EFT approach, it was not based on the Routhian method but on a direct insertion of a Newton-Wigner SSC.\(^2\) The alternative NLO spin(1)-spin(2) potential derived by Levi even agrees exactly with the corresponding ADM Hamiltonian if a suitable Newton-Wigner SSC is used at the action level [9]. More work is needed to understand why such a direct insertion of a Newton-Wigner SSC leads to correct fully reduced Hamiltonians in some cases.

\(^2\) However, in [79] this procedure is not justified. For a proper way to directly implement a Newton-Wigner SSC in an action see [22], in particular Eqs. (3.7) and (3.12) therein.
D. The NLO spin(1)-spin(1) Hamiltonian of Porto and Rothstein

The last Hamiltonian we want to verify is the effective NLO spin(1)-spin(1) Hamiltonian from Eq. (III.32). From it we eliminate $S_1^{(0)(i)}$ by Eq. (IV.17c) resulting in

\[
H_{S_1,\text{red}}^{\text{NLO}(\text{eff})} = C_Q \frac{G}{r_{12}^2} \left[ \frac{m_2}{m_1^2} \left( \frac{3}{4} p_1^2 (S_1 \cdot n_{12})^2 - 3(p_1 \cdot n_{12})(S_1 \cdot n_{12})(S_1 \cdot p_1) + (S_1 \cdot p_1)^2 \right) + \frac{9}{2} (p_1 \cdot n_{12})^2 S_1^2 - \frac{7}{4} p_1^2 S_1^2 \right] + \frac{1}{m_1^2} \left( - \frac{15}{4} (p_1 \cdot n_{12})(p_2 \cdot n_{12})(S_1 \cdot n_{12})^2 - \frac{21}{4} (p_1 \cdot p_2)(S_1 \cdot n_{12})^2 \right)
\]

\[
+ \frac{g}{2} (p_2 \cdot n_{12})(S_1 \cdot n_{12})(S_1 \cdot p_1) - \frac{3}{2} (S_1 \cdot p_1)(S_1 \cdot p_2) + \frac{3}{2} (p_1 \cdot n_{12})(S_1 \cdot n_{12})(S_1 \cdot p_2) - \frac{15}{4} (p_1 \cdot n_{12})(p_2 \cdot n_{12})S_1^2 \]  

(VI.28)

\[
+ \frac{13}{4} (p_1 \cdot p_2) S_1^2 \right) + \frac{1}{m_1 m_2} \left( \frac{9}{4} p_2^2 (S_1 \cdot n_{12})^2 - \frac{3}{4} p_2^2 S_1^2 \right]
\]

\[
+ \frac{G^2 m_2}{r_{12}^2} \left[ \left( 2 + \frac{1}{2} C_{Q1} + \frac{m_2}{m_1} \left( \frac{1}{2} + 3C_{Q1} \right) \right) S_1^2 + \left( -3 - \frac{3}{2} C_{Q1} + \frac{m_2}{m_1} \left( \frac{1}{2} + 6C_{Q1} \right) \right) (S_1 \cdot n_{12})^2 \right].
\]

(VI.28)

We proceed to the canonical Hamiltonian by collecting all matching correction terms and by replacing covariant variables with Newton-Wigner ones in the above Hamiltonian $H_{S_1,\text{red}}^{\text{NLO}(\text{eff})}$

\[
H_{S_1,\text{red}}^{\text{NLO}(\text{can})} = H_{S_1,\text{red}}^{\text{NLO}(\text{eff})} + H_{NLO}^{\text{NLO}(\text{can})} + H_{\text{LOS}}^{\text{NLO}(\text{can})} + H_{\text{LOS}^2}^{\text{NLO}(\text{can})}. 
\]

(VI.29)

The result is

\[
H_{S_1,\text{red}}^{\text{NLO}(\text{can})} = C_Q \frac{G}{r_{12}^2} \left[ \frac{m_2}{m_1^2} \left( - \frac{21}{8} + \frac{9}{4} C_{Q1} \right) p_1^2 (S_1 \cdot n_{12})^2 \right] + \frac{9}{2} (p_1 \cdot n_{12})^2 S_1^2 + \frac{7}{4} p_1^2 S_1^2 \left( - \frac{15}{4} C_{Q1} \right)
\]

\[
+ \left( - \frac{21}{8} + \frac{9}{2} C_{Q1} \right) (p_1 \cdot n_{12})^2 S_1^2 + \left( 7 \frac{9}{4} C_{Q1} \right) (p_1 \cdot n_{12})^2 S_1^2 \right] + \frac{1}{m_1^2} \left( - \frac{15}{4} C_{Q1} \right) (p_1 \cdot n_{12})^2 (p_2 \cdot n_{12})^2 \left( - \frac{21}{4} \right) (p_1 \cdot p_2)^2 \left( S_1 \cdot n_{12} \right)^2 + \left( 3 + \frac{3}{2} C_{Q1} \right)
\]

\[
+ \left( 2 - \frac{3}{2} C_{Q1} \right) (S_1 \cdot p_1)(S_1 \cdot p_2) + \left( 3 - \frac{15}{4} C_{Q1} \right) (p_1 \cdot n_{12})(S_1 \cdot n_{12}) S_1^2 \right] + \left( - \frac{2}{4} \right) \left( C_{Q1} \right) (p_1 \cdot n_{12})^2 \left( - \frac{3}{4} C_{Q1} \right)
\]

\[
+ \frac{G^2 m_2}{r_{12}^2} \left[ \left( 2 + \frac{1}{2} C_{Q1} + \frac{m_2}{m_1} \left( \frac{1}{2} + 3C_{Q1} \right) \right) S_1^2 + \left( -3 - \frac{3}{2} C_{Q1} + \frac{m_2}{m_1} \left( \frac{1}{2} + 6C_{Q1} \right) \right) (S_1 \cdot n_{12})^2 \right].
\]

(VI.30)

The comparison with the ADM NLO spin(1)-spin(1) Hamiltonian was already briefly addressed [44], and we also refer to [44] for details on the derivation of the ADM Hamiltonian. Notice the two misprints in Eq. (32) of [44] if one compares it with (IV.27). The round bracket term should be corrected to read

\[
\left( 1 - \frac{3}{4m_1^2} \right) \rightarrow \left( 1 - \frac{3p_1^2}{4m_1^2} \right),
\]

(VI.31)

and the third term has a wrong particle label, rather it should read

\[
\frac{3G}{m_1 r_{12}} p_{1[i]} (\hat{S}_{1(j)(k)}) p_{1k} \rightarrow \frac{3G}{m_1 r_{12}} p_{1[i]} (\hat{S}_{1(i)(j)}) p_{2k}.
\]

(VI.32)
It turned out that with the following general ansatz for the generator function

\[ g_{\text{can}}^{\text{NLO}} s_2^2 = \frac{G m_1}{r_{12}^2 m_2^2} \left[ \gamma_1(p_1 \cdot n_{12}) \hat{S}_1^1 + \gamma_2(p_2 \cdot n_{12}) \hat{S}_2^2 + \gamma_3(\hat{S}_1 \cdot n_{12})(\hat{S}_1 \cdot p_1) 
\right. \\
\left. + \gamma_4(\hat{S}_1 \cdot n_{12})(\hat{S}_1 \cdot p_2) + \gamma_5(p_1 \cdot n_{12})(\hat{S}_1 \cdot n_{12})^2 + \gamma_6(p_2 \cdot n_{12})(\hat{S}_1 \cdot n_{12})^2 \right] \quad (VI.33) \]

of an infinitesimal canonical transformation which belongs to this Hamiltonian, the difference to the ADM Hamiltonian

\[ \Delta H_{\text{NLO}} s_2^2 = \{ H_N, g_{\text{can}}^{\text{NLO}} s_2^2 \} \quad (VI.34) \]

can be generated by choosing the coefficients to beat lower orders, this issue is just semantics, but at higher orders it is of practical relevance

\[ \gamma_1 = -\frac{1}{2} + C_{Q_1}, \gamma_2 = 0, \gamma_3 = \frac{1}{2}, \gamma_4 = 0, \gamma_5 = 0, \gamma_6 = 0. \quad (VI.35) \]

We have thus shown rather clearly how one can transform a non-reduced effective potential at NLO to a canonical Hamiltonian in a very replicable and systematically way without losing oneself in subtleties.

**VII. CONCLUSIONS AND OUTLOOK**

In this paper we have presented two different methods to transform non-reduced effective potentials, that can be calculated within the EFT approach, for spinning compact binary systems in general relativity to fully reduced canonical Hamiltonians, which then depend only on the physical degrees of freedom. The main subject we focused on were potentials of constrained systems with redundant spin degrees of freedom that had to be eliminated. The key difficulty was to achieve this scheme on curved spacetime at least perturbatively up to next-to-leading order. The first method involved working with Dirac brackets in Sect. IV to reduce the phase-space variables and the second method was to redefine variables in such a way that the action in Sect. V becomes fully reduced. Both methods yielded the same transformation formulae for spin (IV.27,V.39) and position variable (IV.26,V.28) although the action approach is much more transparent and provides quite general transformation formulae that possess validity in full general relativity given that the vierbein field of curved spacetime is known. Furthermore we could determine how the Lorentz matrices should transform in a gravitational field in order to arrive at a standard canonical representation, see Eq. (V.20). Thus our formalism is also valid for effective potentials/Hamiltonians that depend on the Lorentz matrices, which would describe an asymmetric behavior of the physical system in consideration.

Interestingly the method with the Dirac brackets needs no transformation of the Lorentz matrices, which was expected, because the Hamiltonians are all independent of the Euler angles and the Dirac brackets operate on the level of the equations of motion, which are in turn generated by these Hamiltonians. In contrast the action approach needs the redefinition of the Lorentz matrices via Eq. (V.20) because the action depends on them while coupling to the spin. Due to the knowledge of the vierbein field to next-to-leading pN order we could explicitly calculate the transformation formulae for transforming all next-to-leading order effective potentials known to date for two self-gravitating spinning compact objects to canonical Hamiltonians while eliminating the spin supplementary condition and performing a Legendre transformation. In the near future we plan to go to next-to-next-to-leading pN order in the transformation formulae in order to compare the non-reduced NNLO spin(1)-spin(2) effective potential by Levi [15] with the fully reduced NNLO spin(1)-spin(2) canonical ADM Hamiltonian calculated by Hartung and Steinhoff in [48]. Clearly for this purpose it is much easier to start from the generally valid formulae found by transforming the action directly instead of elaborating more on Dirac bracket calculations. The only piece missing is the vierbein field to next-to-next-to-leading pN order, which can be calculated, e.g., from the metric in harmonic coordinates [37] and corrections [38], and poses no conceptual problem. It is interesting to note that now after 37 years we finally succeeded with the proposal made by Hanson and Regge [21] in their conclusion section to “attempt to include gravitation in our formalism” initiated in [8, 22, 30, 63, 64]. Furthermore at the end of Sect. V it was discussed how Eq. (V.22) can be used to improve the Feynman rules of the EFT formalism by formulating them in terms of reduced canonical spin variables.

One future task would be to extend this formalism for dipole (spin) interaction that we developed especially for weak deformation effects to higher multipoles that would account for stronger rotational deformation effects which will play an important role when the binary system is very close to the merger phase. Right before the merger there is also no more guarantee for the spin length to be conserved which would also be an attractive topic to deal with, because so far there has been no algebraic treatment of a change in the spin length, only analytically starting with Teukolsky’s analysis [80].
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Appendix A: Dirac brackets

Here we list all possible combinations of quantities to enter the Dirac bracket and their results except the ones containing the Λ-matrices. The reason is that all Hamiltonians of interest in this work do not own any Λ-dependency, so there is no need to canonicalize the Λ-matrices explicitly here with the Dirac brackets besides all the other variables ($z_1^i, p_{1i}, S_1^j$, $S_{12}^{ij}$), which really appear in the Hamiltonians. This is different when eliminating the SSC and transforming to canonical variables on the action level, see Sect. V. Then the first thing one must do is to transform the Λ-matrices to canonical ones being the hatted Λ-matrices. The results of the Dirac brackets read

$$\{z_1^i, p_{1j}\}_D \simeq \delta_{ij} + \frac{G}{m_1^2} \left( -\frac{m_2^2}{m_1^2} n_{12}^i p_{1k} S_1^{(i)(k)} + \frac{3}{2m_1^2} n_{12}^i p_{2k} S_1^{(i)(k)} - \frac{1}{2m_1^2} n_{12}^i p_{2j} S_1^{(i)(k)} \right)$$

$$\{- \frac{1}{2m_1} (p_{2j} \cdot n_{12}) S_1^{(i)(j)} + \frac{3}{2m_1^2} n_{12}^j (p_{2j} \cdot n_{12}) n_{12}^i S_1^{(i)(k)} \}$$

$$+ \frac{G}{m_1 r_{12}} \left( 3 n_{12}^i n_{12}^j S_1^{(i)(j)} S_2(k)(l) - S_1^{(i)(k)} S_2(j)(k) \right)$$

$$+ \frac{Gm_2}{m_1^2 r_{12}} \left( 3 n_{12}^i n_{12}^j S_1^{(i)(j)} S_1(k)(l) - S_1^{(i)(k)} S_1(j)(k) \right) + \mathcal{O} (c^{-5}) , \quad (A.1a)$$

$$\{z_1^i, S_1^{(j)}(k)\}_D \simeq \frac{1}{m_1^2} p_{1k} p_{3j} S_1^{(j)(k)} - 2G \frac{m_2 S_1^{(j)(k)}}{m_1^2 r_{12}} + \mathcal{O} (c^{-2}) , \quad (A.1b)$$

$$\{z_1^i, S_1^{(j)(k)}\}_D \simeq \frac{1}{m_1^2} p_{1k} p_{3j} S_1^{(j)(k)} - \frac{4 m_2}{m_1} S_1^{(j)(k)} p_{1k} + (p_{2j} \cdot n_{12}) S_1^{(j)(k)} n_{12j} \right)$$

$$+ \frac{2G S_1^{(j)(k)} S_2(k)(l) n_{12l}}{m_1^2 r_{12}} + \frac{2G m_2 S_1^{(j)(k)} S_1(k)(l) n_{12l}}{m_1^2 r_{12}} + \mathcal{O} (c^{-5}) , \quad (A.1c)$$

$$\{z_1^i, z_2^j\}_D \simeq \frac{G}{2m_1 r_{12}} \left( 3 S_1^{(j)(j)} + S_1^{(k)(k)} n_{12k} n_{12r} \right)$$

$$+ \frac{G}{m_2 r_{12}} \left( 3 S_1^{(j)(j)} - S_2^{(j)(k)} n_{12k} n_{12r} \right) + \mathcal{O} (c^{-2}) , \quad (A.1d)$$

$$\{z_1^i, p_{2j}\}_D \simeq \delta_{ij} - \{x_1^i, p_{1j}\}_D , \quad (A.1e)$$

$$\{z_1^i, S_2^{(j)(k)}\}_D \simeq \frac{3 G S_2^{(j)(k)} p_{2j}}{m_2 r_{12}} + \frac{G S_2^{(j)(k)} p_{2j} n_{12l}^2}{m_2 r_{12}} + \frac{2 G S_2^{(j)(k)} S_1^{(j)(k)} n_{12l}}{m_1 r_{12}^2}$$

$$+ \frac{2 G S_2^{(j)(k)} S_1^{(j)(k)} n_{12l}}{m_1 r_{12}^2} + \mathcal{O} (c^{-5}) , \quad (A.1f)$$

$$\{p_{1i}, p_{1j}\}_D = \{p_{1i}, p_{2j}\}_D = \{p_{2i}, p_{2j}\}_D = 0 , \quad (A.1g)$$

$$\{p_{1i}, S_1^{(j)(k)}\}_D \simeq \frac{G}{m_1 r_{12}^2} \left[ (p_{2j} \cdot n_{12}) S_1^{(j)(k)} p_{1k} \right]$$

$$+ \frac{p_{1l} S_1^{(k)(l)}}{m_1 n_{12}^l} \left( \frac{2 m_2}{m_1} n_{12}^k - 3 p_{2l} n_{12}^k + p_{2l} n_{12}^l - 3 (p_{2j} \cdot n_{12}) n_{12}^l \right)$$

$$+ \frac{2 G p_{1l} S_1^{(j)(l)}}{m_1 n_{12}^l} \left( 3 n_{12}^l n_{12}^k S_2^{(l)(k)} - S_2^{(j)(k)} \right) \quad (A.1h)$$
\[ \begin{align*}
\{p_{14}, S_{(i)j}^{(k)}\}_D &= (\{p_{14}, S_{1(i)j}^{(k)}\}_D) \quad (1 \leftrightarrow 2), \\
\{S_{1(i)}^{(j)}, S_{1(k)}^{(l)}\}_D &\simeq \mathcal{P}_{\mu} S_{1(i)k} - \mathcal{P}_{\mu} S_{1(i)l} + \mathcal{P}_{\nu k} S_{1(j)}^{(l)} - \mathcal{P}_{\nu l} S_{1(j)}^{(k)} \\
&\quad + \frac{8Gm_2 p_{1}[k] S_{(i)j}^{(m)}}{m_1 r_D^{12}} + \frac{6G}{m_1 r_D^2} \left( p_{2j} S_{(i)j}^{(k)} p_{1l} + p_{1j} S_{(i)j}^{(k)} p_{2l} \right) \\
&\quad + \frac{2G p_{1j} S_{1(i)j}^{(k)} S_{2(k)}^{(m)}}{m_1 r_D^{12}} + \frac{2G n_{12 j} S_{1(j)}^{(k)} S_{2(j)}^{(m)}}{m_1 r_D^{12}}
\end{align*} \]
\[ \text{(A.1i)} \]
\[ \begin{align*}
\{S_{1}^{(i)} S_{2(k)}^{(l)}\}_D &\simeq \frac{4G}{m_1 r_D^{12}} \left( p_{1j} S_{1(j)}^{(m)} n_{12 k} S_{1(j)}^{(l)} + n_{12 j} S_{1(j)}^{(m)} p_{1k} S_{1(j)}^{(l)} \right) \\
&\quad + \frac{4G}{m_1 r_D^{12}} \left( p_{2j} S_{2(j)}^{(m)} n_{12 k} S_{2(j)}^{(l)} + p_{1j} S_{2(j)}^{(m)} p_{2k} S_{2(j)}^{(l)} \right) + \mathcal{O} \left( e^{-\phi} \right)
\end{align*} \]
\[ \text{(A.1j)} \]
\[ \begin{align*}
\{S_{1}^{(i)} S_{2}^{(k)}\}_D &\simeq \frac{4G}{m_1 r_D^{12}} \left( p_{2j} S_{1(j)}^{(m)} n_{12 k} S_{2(j)}^{(l)} + n_{12 j} S_{1(j)}^{(m)} p_{2k} S_{2(j)}^{(l)} \right) \\
&\quad + \frac{4G}{m_1 r_D^{12}} \left( p_{1j} S_{2(j)}^{(m)} n_{12 k} S_{1(j)}^{(l)} + p_{1j} S_{2(j)}^{(m)} p_{2k} S_{1(j)}^{(l)} \right) + \mathcal{O} \left( e^{-\phi} \right)
\end{align*} \]
\[ \text{(A.1k)} \]

All remaining Dirac brackets follow via exchange of particle indices 1 with 2 in all terms.

Appendix B: Full derivation of the transition to canonical variables via an action principle

In this section we shall elaborate in great detail on the reduction of the transition formulae to canonical variables presented in Sect. V. The first step involves the insertion of the constraints (V.17a)-(V.17c) into the spin coupling term in the action, which yields

\[ \begin{align*}
\frac{1}{2} S_{ab} \Omega^{ab} &= \frac{1}{2} S_{ab} A^a \tilde{A}^b \tilde{A} \tilde{A}^b \\
&= \frac{1}{2} S_{(0)}^{(k)} A_{(i)}^{(k)} \tilde{A}_{(j)}^{(i)} \tilde{A}_{(j)}^{(i)} + \frac{1}{2} S_{(j)}^{(k)} A_{(i)}^{(j)} \tilde{A}_{(i)}^{(j)} + \frac{1}{2} S_{(k)}^{(j)} A_{(i)}^{(k)} \tilde{A}_{(j)}^{(k)} \\
&\quad \text{(V.17a)} \\
&\rightarrow \frac{1}{2} S_{(k)}^{(j)} \frac{u^{(k)}(u)}{u^{(k)}(u)} A_{(i)}^{(k)} \tilde{A}_{(j)}^{(i)} + \frac{1}{2} S_{(j)}^{(k)} \frac{u^{(k)}(u)}{u^{(k)}(u)} A_{(i)}^{(j)} \tilde{A}_{(i)}^{(j)} + \frac{1}{2} S_{(k)}^{(j)} A_{(i)}^{(k)} \tilde{A}_{(j)}^{(k)} \\
&= \frac{1}{2} S_{(j)}^{(k)} \frac{u^{(k)}(u)}{u^{(k)}(u)} \tilde{A}_{(j)}^{(i)} + \frac{1}{2} S_{(k)}^{(j)} \frac{u^{(k)}(u)}{u^{(k)}(u)} A_{(i)}^{(j)} \tilde{A}_{(i)}^{(j)} + \frac{1}{2} S_{(k)}^{(j)} \tilde{A}_{(j)}^{(i)}
\end{align*} \]

with \( \tilde{A}_{(j)}^{(i)} ) \equiv A_{(i)}^{(k)} \tilde{A}_{(j)}^{(k)} \). Notice the formal difference to the definition of \( \Omega^{(k)(i)} \) from (V.5). \( \tilde{A}_{(j)}^{(i)} \) is therefore not necessarily antisymmetric, which is actually an unwanted feature. To solve for \( \tilde{A}_{(j)}^{(i)} \) we use (V.17b):

\[ \tilde{A}_{(j)}^{(i)} = - \tilde{A}_{(i)}^{(k)} \frac{u^{(k)}(u)}{u^{(k)}(u)} - \tilde{A}_{(i)}^{(k)} \frac{u^{(k)}(u)}{u^{(k)}(u)} \tilde{u}(0). \]

\[ \text{(B.2)} \]

So

\[ \begin{align*}
\frac{1}{2} S_{ab} \Omega^{ab} &= - \frac{1}{2} S_{(j)}^{(k)} \frac{u^{(k)}(u)}{u^{(k)}(u)} \tilde{A}_{(j)}^{(i)} + \frac{1}{2} S_{(k)}^{(j)} \tilde{A}_{(j)}^{(i)} \\
&\quad - \frac{1}{2} S_{(j)}^{(k)} \frac{u^{(k)}(u)}{u^{(k)}(u)} A_{(i)}^{(j)} \left( - \tilde{A}_{(j)}^{(i)} \frac{u^{(k)}(u)}{u^{(k)}(u)} - \tilde{A}_{(i)}^{(k)} \frac{u^{(k)}(u)}{u^{(k)}(u)} \tilde{u}(0) + \tilde{A}_{(i)}^{(k)} \frac{u^{(k)}(u)}{u^{(k)}(u)} \tilde{u}(0) \right)
\end{align*} \]

\[ \text{(B.3)} \]

We make use of (V.4) to find

\[ \begin{align*}
\gamma^{(j)} &= \Lambda_{A}^{(i)} A_{(j)}^{(i)} = \Lambda_{[0]}^{(i)} A_{(j)}^{(i)} + \Lambda_{[k]}^{(i)} A_{(j)}^{(k)} \quad \text{(V.17c)} \\
&= \frac{u^{(j)}(u)}{u^{(i)}(u)} + \Lambda_{[k]}^{(i)} A_{(j)}^{(k)}.
\end{align*} \]

\[ \text{(B.4)} \]
Because of antisymmetry of the spin tensor the first term with the 4-velocities is irrelevant for (B.3). We are left with

\[
\frac{1}{2}S_{ab}\Omega^{ab} = \frac{1}{2}S_{(j)(k)}u^{(k)}u_{(l)}\hat{\Omega}^{(l)(j)} + \frac{1}{2}S_{(k)(l)}\hat{\Omega}^{(k)(l)} + \frac{1}{2}S_{(j)(k)}\hat{\Omega}^{(j)(k)} + \frac{1}{2}S_{(k)(l)}\hat{\Omega}^{(l)(k)} - \frac{1}{2}S_{(j)(k)}\hat{\Omega}^{(j)(k)} + \frac{1}{2}S_{(k)(l)}\hat{\Omega}^{(l)(k)} - \frac{1}{2}S_{(j)(k)}\hat{\Omega}^{(j)(k)}
\]

\[
= \left( S_{(i)(j)} + S_{(j)(k)}\frac{u^{(k)}\hat{u}_{(l)}}{u^{(l)}u_{(0)} - S_{(j)(k)}\frac{u^{(k)}\hat{u}_{(l)}}{u^{(l)}u_{(0)}} \right) + \frac{1}{2}S_{(j)(k)}\frac{u^{(k)}\hat{u}_{(l)}}{u^{(l)}u_{(0)}}.
\]

Next thing to do is to redefine variables so that the canonical structure of (V.13) is produced. Obviously one should start by shifting \(\hat{\Omega}^{(j)(i)}\) to \(\hat{\Omega}^{(i)(j)}\), which should be antisymmetric in order to be the correct velocity variable belonging to the spin tensor. To find the correct redefinition it is useful to make an ansatz for the intrinsic redefinition of Lorentz matrices with an unknown function \(\xi\) according to

\[
\Lambda^{[i](j)} = \hat{\Lambda}^{[i](k)} \left( \eta^{(j)}_{(k)} + \xi u^{(j)}_{(i)} \right)
\]

so that \(\hat{\Lambda}^{[k](i)}\hat{\Lambda}^{[i](j)} = \delta_{ij}\) (B.6)

transforming \(\hat{\Lambda}^{[i](j)}\) into a 3-dimensional rotation matrix yielding

\[
\hat{\Omega}^{(i)(j)} = \hat{\Lambda}^{(i)(k)} \hat{\Lambda}^{(k)(j)} = -\hat{\Omega}^{(j)(i)}.
\]

It follows

\[
\eta^{(j)}_{(i)} = \Lambda_{A(i)}A^{(j)} = \Lambda_{[0](i)}\lambda^{[0](j)} + \Lambda_{[k](i)}\lambda^{[k](j)}
\]

\[
= \frac{u^{(j)}_{(i)}}{u^{(a)}u_{(a)}} + \left( \hat{\Lambda}^{[k](i)} + \xi \hat{\Lambda}^{[k](i)}u^{(l)}u_{(i)} \right) \left( \hat{\Lambda}^{[k](i)} + \xi \hat{\Lambda}^{[k](p)}u^{(p)}u_{(i)} \right)
\]

\[
= \frac{u^{(j)}_{(i)}}{u^{(a)}u_{(a)}} + \eta^{(j)}_{(i)} + \xi \eta^{(j)}_{(i)}u^{(p)}u_{(i)} + \xi \eta^{(j)}_{(i)}u^{(l)}u_{(i)} + \xi u^{(j)}_{(i)}u^{(l)}u_{(i)}u_{(j)} + \xi u^{(j)}_{(i)}u^{(p)}u_{(i)}u_{(i)}u_{(j)}
\]

\[
= \eta^{(j)}_{(i)} + \frac{u^{(j)}_{(i)}}{u^{(a)}u_{(a)}} + 2\xi u^{(j)}_{(i)}u^{(l)}u_{(i)} + \xi^2 u^{(j)}_{(i)}u^{(l)}u_{(i)}u_{(i)}u_{(j)}.
\]

For convenience we define

\[
u^{(i)}_{(i)} = \frac{u^{(i)}}{u^{(2)}} \quad \text{so that} \quad u^{(i)}u^{(a)} = u^{(2)}u^{(a)} + u^{2}.
\]

(B.9)

To find and expression for \(\xi\) we have to solve the equation

\[
\frac{1}{u^{2}} + 2\xi + \xi^2 u^{2} = 0 \iff \xi^2 + \frac{2}{u^{2}}\xi + \frac{1}{u^{2}u^{2}} = 0,
\]

so the solution is twofold

\[
\xi_{1,2} = -\frac{1}{u(u + su_{(0)})} \quad \text{with} \quad s \in \{1, -1\}.
\]

(B.11)

Taking the limit \(u^{2} = 0\) the function \(\xi\) should still be regular because this state corresponds to the rest frame limit which is therefore an adequate physical limit to take. This argument leaves us with the sole solution

\[
\xi = \frac{1}{u(u + u_{(0)})}.
\]

(B.12)

So the Lorentz matrices undergo a redefinition

\[
\Lambda^{[i](j)} = \hat{\Lambda}^{[i](k)} \left( \eta^{(j)}_{(k)} + \frac{u^{(j)}_{(i)}}{u(u + u_{(0)})} \right).
\]

(B.13)
Next we take the time derivative of this expression

\[
\hat{\Lambda}^{[i\langle j]} = \left( \hat{\Lambda}^{[i\langle j]} - \hat{\Lambda}^{[i\langle k]} \frac{u_{(k)}^{(j)}}{u(u + u_{(0)})} \right) \\
= \hat{\Lambda}^{[i\langle j]} - \hat{\Lambda}^{[i\langle k]} \frac{u_{(k)}^{(j)}}{u(u + u_{(0)})} - \hat{\Lambda}^{[i\langle k]} \left( \frac{\dot{u}_{(k)}^{(j)}}{u(u + u_{(0)})} + \frac{u_{(k)} u^{(j)}}{(u + u_{(0)})^2} \right) + u_{(k)}^{(j)} \left( \frac{1}{u(u + u_{(0)})} \right).
\] (B.14)

It follows

\[
\hat{\Omega}^{(i\langle j]} = \Lambda^{[k\langle i} \hat{\Lambda}^{[k\rangle j]} \\
= \left( \Lambda^{[k\langle i} - \Lambda^{[k\langle l]} \frac{u_{(l)}^{(j)}}{u(u + u_{(0)})} \right) \\
\times \left[ \hat{\Lambda}^{[k\rangle j]} - \hat{\Lambda}^{[k\rangle (p)} \frac{u_{(p)}^{(j)}}{u(u + u_{(0)})} - \hat{\Lambda}^{[k\rangle (p)} \left( \frac{\dot{u}_{(p)}^{(j)}}{u(u + u_{(0)})} + \frac{u_{(p)} u^{(j)}}{(u + u_{(0)})^2} \right) + u_{(p)}^{(j)} \left( \frac{1}{u(u + u_{(0)})} \right) \right].
\] (B.15)

The Symbol \([=]\) means we neglect all terms symmetric by interchanging \(i \leftrightarrow j\) and keep those being antisymmetric by this interchange, because only those will contribute when projected on to the spin tensor \(S^{(i\langle j)}\) or equally antisymmetric expressions. We insert above expression into Eq. (B.5), which leads to a rather long expression. We define

\[
\frac{1}{2} S_{ab} \Omega^{ab} - \frac{1}{2} S_{(j)(k)} \frac{u^{(k)} \hat{u}^{(j)}}{u_{(0)} u_{(0)}} \equiv \mathcal{W}
\] (B.16)

and expand it

\[
2\mathcal{W} = \left( S_{(i)(j)} + S_{(i)(k)} \frac{u_{(k)}^{(j)}}{u_{(0)} u_{(0)}} - S_{(j)(k)} \frac{u_{(k)}^{(j)}}{u_{(0)} u_{(0)}} \right) \hat{\Omega}^{(i\langle j]} \\
= \left( S_{(i)(j)} + S_{(i)(k)} \frac{u_{(k)}^{(j)}}{u_{(0)} u_{(0)}} - S_{(j)(k)} \frac{u_{(k)}^{(j)}}{u_{(0)} u_{(0)}} \right) \\
\times \left( \hat{\Omega}^{(i\langle j]} - \hat{\Omega}^{(i\langle l)} \frac{u_{(l)}^{(j)}}{u(u + u_{(0)})} - \hat{\Omega}^{(i\langle l)} \frac{u_{(l)}^{(j)}}{u(u + u_{(0)})} + \frac{u_{(l)}^{(j)} \hat{u}^{(j)}}{u(u + u_{(0)})^2} \right).
\] (B.17)
We demand from which follows the spin redefinition

Again we make an ansatz for the spin redefinition with an unknown function $\chi$ as the supposed canonical spin which yields

So the result is

Again we make an ansatz for the spin redefinition with an unknown function $\chi$ according to the rule

with $\hat{S}(i)(j)$ as the supposed canonical spin which yields

We demand

from which follows the spin redefinition

and

(B.18)

(B.19)

(B.20)

(B.21)

(B.22)

(B.23)

(B.24)
Hence the spin coupling term is reduced to the expression

\[ \frac{1}{2} S_{ab} \Omega^{ab} = \frac{1}{2} \hat{S}^{(i)(j)} \hat{\Omega}^{(i)(j)} + \frac{1}{2} \hat{S}^{(i)(j)} \frac{u^2 u^{(i)\dot{u}^{(j)}}}{u_0^2 (u + u_0)^2} \left( 1 - \frac{u^2}{u(u + u_0)} \right) + \frac{1}{2} \hat{S}^{(i)(j)} \frac{u^{(i)\dot{u}^{(j)}}}{u_0^2} \left( 1 - \frac{u^2}{u(u + u_0)} \right) \]

\[ = \frac{1}{2} \hat{S}^{(i)(j)} \hat{\Omega}^{(i)(j)} + \frac{1}{2} \hat{S}^{(i)(j)} \frac{u^2 u^{(i)\dot{u}^{(j)}}}{u_0^2 (u + u_0)^2} \frac{u_0}{u} - \frac{1}{2} \hat{S}^{(i)(j)} \frac{u^{(i)\dot{u}^{(j)}}}{u_0^2} \frac{u}{u} \]

\[ = \frac{1}{2} \hat{S}^{(i)(j)} \hat{\Omega}^{(i)(j)} - \frac{1}{2} \hat{S}^{(i)(j)} \frac{u^{(i)\dot{u}^{(j)}}}{u(u + u_0)} , \quad \text{(B.25)} \]

giving

\[ \frac{1}{2} S_{ab} \Omega^{ab} = \frac{1}{2} \hat{S}^{(i)(j)} \hat{\Omega}^{(i)(j)} - Z \quad \text{with} \quad Z \equiv \hat{S}^{(i)(j)} \frac{u^{(i)\dot{u}^{(j)}}}{u(u + u_0)} \quad \text{(B.26)} \]

To solve for the momenta in (B.26) one has to insert the vierbein which is perturbatively calculated to the needed pN order, see Eqs. (IV.15a)-(IV.15c). First we make an expansion of \( Z \) in powers of \( u^2 \) (in the sense of a post-Newtonian approximation)

\[ Z = \hat{S}^{(i)(j)} \frac{u^{(i)\dot{u}^{(j)}}}{u(u + u_0)} = \hat{S}^{(i)(j)} u^{(i)\dot{u}^{(j)}} \left( \frac{1}{2u_0^2} - \frac{3u^2}{8u_0^4} + \mathcal{O}(u^4, c^{-10}) \right) . \quad \text{(B.27)} \]

Next we insert (IV.16a) into (B.27) and pN expand the result up to the order \( c^{-8} \) leading to

\[ Z_1 \simeq \frac{1}{2} \hat{S}^{(i)(j)} u^{(i)\dot{u}^{(j)}} \left( 1 + 2Gm_2 \right) + \mathcal{O}(c^{-10}) . \quad \text{(B.28)} \]

After that we insert (IV.16c) into this equation yielding the approximate expression

\[ Z_1 \simeq \frac{1}{2} \hat{S}^{(i)(j)} u^{(i)\dot{u}^{(j)}} + \frac{Gm_2}{m_1 r_{12}} \hat{S}^{(i)(j)} u^{(i)\dot{u}^{(j)}} + \frac{3Gp_2}{2r_{12}} \hat{S}^{(i)(j)} u^{(i)\dot{u}^{(j)}} - \frac{Gm_2 n_{12} S_{12}^{(i)(k)}}{m_1 r_{12}^2} \]

\[ + \frac{Gm_2}{r_{12}^2} \hat{S}^{(i)(k)} u^{(i)\dot{u}^{(j)}} + \frac{Gm_2}{m_1 r_{12}} \hat{S}^{(i)(j)} u^{(i)\dot{u}^{(j)}} + \hat{S}^{(i)(j)} u^{(i)\dot{u}^{(j)}} \]

\[ \simeq \hat{S}^{(i)(j)} p_{1i} \left[ \frac{1}{2m_1} u^{(i)\dot{u}^{(j)}} + \frac{Gm_2}{m_1 r_{12}} \frac{p_{1j} i_{1j}}{m_1} + \hat{S}^{(i)(j)} \right] \]

\[ \simeq \frac{1}{2m_1} \hat{S}^{(i)(j)} p_{1i} u^{(i)\dot{u}^{(j)}} + \frac{Gm_2}{m_1 r_{12}} \frac{p_{1j} i_{1j}}{m_1} + \hat{S}^{(i)(j)} \]

\[ \simeq \frac{1}{2m_1} \hat{S}^{(i)(j)} p_{1i} \dot{u}^{(i)\dot{u}^{(j)}} + \frac{Gm_2}{m_1 r_{12}} \frac{p_{1j} i_{1j}}{m_1} + \hat{S}^{(i)(j)} \frac{p_{1j} i_{1j}}{m_1} + \mathcal{O}(c^{-10}) . \quad \text{(B.29)} \]

We eliminate the time derivative of \( u^{(i)\dot{u}^{(j)}} \) by shifting it on-shell (i.e. we neglect total time derivatives symbolized by \( \approx \)) onto the momentum leaving us also with a time derivative of the canonical spin, which we will have to deal with
later when we reconsider the spin redefinition.

\[
\begin{align*}
Z_1 \approx & \frac{1}{2m_1} \dot{S}_{1(i)(j)} p_{1i} u_1^{(j)} - \frac{1}{2m_1} \dot{S}_{1(i)(j)} p_{1i} u_1^{(j)} + \frac{1}{2} \dot{S}_{1(i)(j)} \left[ \frac{3Gp_{2i}}{r_{12}} + \frac{Gn_{12}^2 (p_2 \cdot n_{12})}{2r_{12}} \right] + O(c^{-10}) \\
\approx & \frac{1}{2m_1} \dot{S}_{1(i)(j)} p_{1i} u_1^{(j)} + \frac{1}{2} \dot{S}_{1(i)(j)} \left[ \frac{3Gp_{2i}}{m_1 r_{12}} + \frac{3Gp_{2i}}{2r_{12}} + \frac{Gn_{12}^2 (p_2 \cdot n_{12})}{2r_{12}} \right] + O(c^{-10}) \\
= & \frac{1}{2m_1} \dot{S}_{1(i)(j)} p_{1i} u_1^{(j)} + \frac{1}{2} \dot{S}_{1(i)(j)} \left[ \frac{3Gp_{2i}}{m_1 r_{12}} + \frac{3Gp_{2i}}{2r_{12}} + \frac{Gn_{12}^2 (p_2 \cdot n_{12})}{r_{12}} \right] + O(c^{-10}).
\end{align*}
\]

(B.30)

Now we are ready to return to the action (V.1), wherein we insert Eqs. (B.26) and (B.29) leading to the expression (for particle 1)

\[
S_{\text{eff}} = \int dt \left( \pi_{1i} z_1^i \frac{1}{2} S_{1ab} \Omega_{1}^{ab} - H_{\text{eff}} \right)
\]

\[
\approx \int dt \left( -\dot{p}_{1j} z_1^j + \frac{1}{2} \dot{S}_{1(j)(j)} \Omega_{1}^{(j)} - \frac{1}{2m_1} \dot{S}_{1(i)(j)} p_{1i} u_1^{(j)} + \frac{1}{2m_1} \dot{S}_{1(i)(j)} \left[ \frac{3Gp_{2i}}{m_1 r_{12}} + \frac{3Gp_{2i}}{2r_{12}} + \frac{Gn_{12}^2 (p_2 \cdot n_{12})}{m_1 r_{12}} \right] \dot{p}_{1j} - H_{\text{eff}} \right). 
\]

(B.31)

This enables us to read off the position coordinate shift (for the Minkowski case only the leading order term is shown)

\[
z_1^i = z_1^i + \frac{\dot{S}_{1(i)(j)} }{2m_1} \left[ \frac{\pi_{1i}}{m_1} - \frac{2Gm_2 p_{1i}}{m_1 r_{12}} + \frac{3Gp_{2i}}{r_{12}} + \frac{Gn_{12}^2 (p_2 \cdot n_{12})}{m_1 r_{12}} \right] \dot{p}_{1j} + O(c^{-6}). 
\]

(B.32)

The spin and the Lorentz matrix need another redefinition in order to cancel the term \(- \frac{1}{2m_1} \dot{S}_{1(i)(j)} p_{1i} u_1^{(j)}\) from the action. This is achieved by an infinitesimal rotation \(\omega^{(i)(j)} = -\omega^{(j)(i)}\) of the local basis so that the canonical spin and Lorentz matrices are correlated according to

\[
-\frac{1}{2} \dot{S}_{(i)(j)} \Omega^{(i)(j)} \rightarrow \frac{1}{2} \left[ \dot{S}_{(i)(j)} + \omega^{(i)(m)} \dot{S}_{(m)(j)} + \omega^{(j)(m)} \dot{S}_{(i)(m)} \right] \Omega^{(i)(j)}
\]

\[
= \frac{1}{2} \left[ \dot{S}_{(i)(j)} + \omega^{(i)(m)} \dot{S}_{(m)(j)} + \omega^{(j)(m)} \dot{S}_{(i)(m)} \right] \Lambda_{[1]}^{(i)} \Lambda_{[2]}^{(j)}
\]

\[
= \frac{1}{2} \left[ \dot{S}_{(i)(j)} + \omega^{(i)(m)} \dot{S}_{(m)(j)} + \omega^{(j)(m)} \dot{S}_{(i)(m)} \right] \left( \Lambda_{[1]}^{(i)} + \omega^{(i)(1)} \Lambda_{[2]}^{(1)} \right)
\]

\[
\approx -\frac{1}{2} \dot{S}_{(i)(j)} \Omega^{(i)(j)} + \frac{1}{2} S_{(i)(j)} \omega^{(j)(i)} + \frac{1}{2} S_{(i)(j)} \omega^{(j)(i)}
\]

\[
\approx -\frac{1}{2} \dot{S}_{(i)(j)} \Omega^{(i)(j)} - \frac{1}{2} \dot{S}_{(i)(j)} \omega^{(j)(i)}
\]

(B.33)
We identify
\[ \omega_i^{(m)} = -\omega_j^{(m)} = \frac{1}{2} p_{1i} u_{1i}^{(m)} - \frac{1}{2} p_{1m} u_{1i}^{(i)} \]  
and use Eq. (B.23) to determine the final spin redefinition to our approximation:
\[ S_1(i) = \frac{1}{2} \hat{S}_1(i) - \frac{1}{2} \hat{S}_1(k) u_1^{(k)} u_1^{(k)} + \frac{1}{2} \hat{S}_1(k) u_1^{(m)} + \frac{1}{2} \hat{S}_1(i) u_1^{(m)} + \frac{1}{2} \hat{S}_1(k) - (i \leftrightarrow j). \]  
Next we insert Eq. (IV.16a) for \( u(i) \) in the second term and Eq. (IV.16c) for \( w(i) \) the third term of (B.35) and make a pN expansion up to the order \( c^{-9} \):
\[ S_1(i) = \frac{1}{2} \hat{S}_1(i) - \frac{1}{2} \hat{S}_1(k) u_1^{(k)} u_1^{(k)} + \frac{1}{2} \hat{S}_1(k) u_1^{(m)} + \frac{1}{2} \hat{S}_1(i) u_1^{(m)} + \frac{1}{2} \hat{S}_1(k) - (i \leftrightarrow j) + O(c^{-9}). \]  
We are left with an insertion of the remaining 4-velocities. Again utilizing Eq. (IV.16c) a further expansion of (B.36) yields
\[ S_1(i) = \frac{1}{2} \hat{S}_1(i) + \frac{1}{2} \hat{S}_1(k) p_{1i} u_1^{(k)} - \frac{1}{2} \hat{S}_1(k) p_{1k} u_1^{(m)} + \frac{1}{2} \hat{S}_1(i) u_1^{(m)} + \frac{1}{2} \hat{S}_1(k) p_{1i} u_1^{(m)} + \frac{1}{2} \hat{S}_1(k) - (i \leftrightarrow j) + O(c^{-9}). \]  
The last step involves providing all spin and position variables with a hat in the highest pN terms of (B.37), so that we end up with the transformation formula
\[ S_1(i) = \frac{1}{2} \hat{S}_1(i) + \frac{1}{2} \hat{S}_1(k) p_{1i} u_1^{(k)} - \frac{1}{2} \hat{S}_1(k) p_{1k} u_1^{(m)} + \frac{1}{2} \hat{S}_1(i) u_1^{(m)} + \frac{1}{2} \hat{S}_1(k) - (i \leftrightarrow j) + O(c^{-9}). \]
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