Estimates in $L_p$ for solutions of SPDEs with coefficients in Morrey classes

N. V. Krylov

Abstract

For solutions of a certain class of SPDEs in divergence form we present some estimates of their $L_p$-norms and the $L_p$-norms of their first-order derivatives. The main novelty is that the low-order coefficients are supposed to belong to certain Morrey classes instead of $L_p$-spaces. Our results are new even if there are no stochastic terms in the equation.

Keywords  Stochastic PDEs · Morrey classes · Singular coefficients

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1 Introduction

In this paper we come back to an old problem of estimating the $L_p$-norm in $x$ of solutions $u_t(x)$ of Itô stochastic partial differential equations of second order. The first such estimates appeared in [9] in 1977 and were achieved by using Itô’s formula for $\|u_t\|_{L_p}^p$ and integrating by parts. This method of obtaining such estimates was generalized, extended, and applied in many papers of which we cite, probably, the most recent [2, 3, 13], and [14] containing vast lists of references. The closest to the present article are some computations in [8] which is about strong solutions of Itô stochastic equations rather than about SPDEs. There, as in the present article, the main emphasis is on the drift coefficients from a Morrey class, which contains functions with critical singularities. In this connection it is also worth mentioning [5] and some references therein where methods different from integration by parts are used to treat strong solutions of Itô stochastic equations with singular drift. We apply integration by parts to SPDEs which may be just nonrandom usual parabolic equations. Even in
this case, as far as the author can tell, the presented results are new although similar results for the elliptic equations can be found in [11].

Let \( d, d_1 \) be integers, \( d \geq 3 \), \( \mathbb{R}^d \) be a \( d \)-dimensional Euclidean space of points \( x = (x^1, \ldots, x^d) \), and

\[
\mathbb{R}^{d+1} = \{ (t, x) : t \in \mathbb{R}, x \in \mathbb{R}^d \}.
\]

Let \( \mathcal{B}(\mathbb{R}^d) \) be the Borel \( \sigma \)-field on \( \mathbb{R}^d \), \( L_p = L_p(\mathbb{R}^d) \).

Let \((\Omega, \mathcal{F}, P)\) be a complete probability space equipped with a filtration \( (\mathcal{F}_t)_{t \geq 0} \) of complete \( \sigma \)-fields \( \mathcal{F}_t \subset \mathcal{F} \). We suppose that on our probability space we are given a \( d_1 \)-dimensional Wiener process \( w_t \), which is a Wiener process relative to the filtration \( (\mathcal{F}_t)_{t \geq 0} \). By \( \mathcal{P} \) we denote the predictable \( \sigma \)-field on \( \Omega \times (0, \infty) \) (we follow the terminology in [4]). Fix a bounded stopping time \( \tau \).

Assume that on \( \Omega \times (0, \infty) \times \mathbb{R}^d \) we are given \( \mathcal{P} \otimes \mathcal{B}(\mathbb{R}^d) \)-measurable \( \mathbb{R}^d \)-valued functions \( \sigma^k_t(x) = (\sigma^i_{tk}(x)) \), \( \beta^i_t(x) = (\beta^j_{it}(x)) \), \( f^i_t(x) = (f^j_{it}(x)) \), \( d \times d \)-matrix valued functions \( \sigma^i_j(x) = (\sigma^i_{jk}(x)) \), \( \mathbb{R}^{d_1} \)-valued functions \( \nu^k_t(x) = (\nu^j_{tk}(x)) \), \( g^k_t(x) = (g^j_{tk}(x)) \), and real-valued functions \( c_t(x) \), \( f^i_t(x) \). Introduce

\[
L_t u_t = D_i (a^i_{jk} D_j u_t + \beta^i_{tk} u_t) + b^i_t D_t u_t + c_t u_t, \quad M^k_t u_t = \sigma^i_{tk} D_i u_t + \nu^j_{tk} u_t,
\]

where and everywhere below the common summation convention is used and \( D_i = \partial/\partial x^i \).

Our goal is to investigate the equation

\[
du_t = (L_t u_t + D_t f^i_t + f^i_t) dt + (M^k_t u_t + g^k_t) dw^k_t, \quad t \leq \tau, \quad (1.1)
\]

with initial condition

\[
u \quad (1.2)
\]

\textbf{Remark 1.1} In the literature the very popular condition on \( b \) (and \( \beta \)) is that \( b \in L_{p_0,q_0}(T) \), that is, for \( T \in (0, \infty) \),

\[
\left( \int_0^T \left( \int_{\mathbb{R}^d} |b_t(x)|^{p_0} dx \right)^{q_0/p_0} dt \right)^{1/q_0} < \infty \quad (1.3)
\]

with \( p_0, q_0 \in [2, \infty] \) satisfying

\[
\frac{d}{p_0} + \frac{2}{q_0} = 1.
\]

Observe that, if \( p_0 > d \) and we take an arbitrary constant \( \hat{N} \) and introduce

\[
\lambda(t) = \hat{N} \left( \int_{\mathbb{R}^d} |b_t(x)|^{p_0} dx \right)^{1/(p_0-d)},
\]
then for (the notation will be explained later)

\[ b^M_t(x) = b_t(x) I_{|b_t(x)| \geq \lambda(t)} \]

we have

\[
\int_{B_\rho} |b^M_t(x)|^d \, dx \leq \lambda^{d-p_0}(t) \int_{B_\rho} |b_t(x)|^{p_0} \, dx \leq N(d)^{d-p_0} \rho^{-d},
\]

where \( B_\rho \) is a ball of radius \( \rho \in (0, \infty) \),

\[
\int_{B_\rho} \ldots = \frac{1}{|B_\rho|} \int_{B_\rho} \ldots, \quad |B_\rho| = \text{Vol} B_\rho.
\]

Here \( N(d)^{d-p_0} \) can be made arbitrarily small if we choose \( \hat{N} \) large enough. In addition, for \( b^B_t = b_t - b^M_t \) \( |b^B_t| \leq \lambda(t) \) and

\[
\|b^B_t\|_{L^{\infty,2}(T)}^2 \leq \int_0^T \lambda^2(t) \, dt = \hat{N}^2 \int_0^T \left( \int_{\mathbb{R}^d} |b_t(x)|^{p_0} \, dx \right)^{q_0/p_0} \, dt < \infty.
\]

This shows that the assumptions we are going to impose on \( b \) (see Definition 1.5 and Assumption 1.6) are weaker than (1.3) if \( p_0 > d \).

If \( p_0 = d \) (and \( q_0 = \infty \)) our assumptions are weaker if (1.3) is combined with the requirement that the family \( |b_t|^d, t \in (0, \infty) \), be uniformly integrable (say, case of \( b \) independent of \( t \)).

Recall that \( W^n_p \) are Sobolev spaces of functions \( u \) on \( \mathbb{R}^d \) such that \( u \) and all its (generalized) derivatives of order \( \leq n \) belong to \( L^p \). The norm in \( W^n_p \) is introduced in a natural way.

**Definition 1.2** A \( W^n_p \)-valued function \( u \), defined on the stochastic interval \( (0, \tau] \), is called a solution of (1.1)-(1.2) if \( u \) is predictable on \( (0, \tau] \),

\[
\int_0^\tau \|u_t\|_{W^n_p}^p \, dt < \infty,
\]

(a.s.) and for each \( \varphi \in C_0^\infty(\mathbb{R}^d) \) for almost all \( \omega \in \Omega \)

\[
(u_t, \varphi) = (v, \varphi) + \int_0^t (\sigma_s^{ik} D_i u_s + v_s^k u_s + g_s^k, \varphi) \, dw_s^k + \int_0^t \left( (b_s^i D_i u_s + c_s u_s + f_s, \varphi) - (\alpha_s^{ij} D_j u_s + \beta_s^i u_s + \xi_s^i, D_i \varphi) \right) \, ds \quad (1.4)
\]

for all \( t \in [0, \tau(\omega)] \), where \( (u, v) \) denotes the integral over \( \mathbb{R}^d \) of \( uv \).

Our standing assumption is that Eq. (1.1) is uniformly nondegenerate. Set \( \alpha_t(x) = (\alpha_t^{ij}(x)) \), where \( \alpha_t^{ij} = \sigma_t^{ik} \sigma_t^{jk} \). Note that we do not suppose that \( \alpha_t \) is symmetric.
Assumption 1.3 There is a constant $\delta > 0$ such that for all values of arguments and $\lambda \in \mathbb{R}^d$

$$|a_t| \leq \delta^{-1}, \quad (2\alpha_{ij} - \alpha_i^{ij})\lambda^i\lambda^j \geq \delta |\lambda|^2.$$  \hspace{1cm} (1.5)

Another standing assumption concerns free terms. We fix $p \in [2, d)$ (recall that $d \geq 3$) and recall that $L_p = L_p(\mathbb{R}^d)$.

Assumption 1.4 We have

$$E \int_0^\tau (\|f_t\|^p_{L_p} + \|f_t\|^p_{L_p} + \|g_t\|^p_{L_p}) \, dt < \infty.$$ \hspace{1cm} (1.6)

The assumptions on the coefficients of $L$ and $M^k$ are more delicate. Fix $r \in (p, d], R_0 \in (0, \infty)$, and let $\mathbb{B}_\rho$ denote the set of balls in $\mathbb{R}^d$ with radius $\rho$.

Definition 1.5 We call a real- or vector- or else tensor-valued function $f_t(x)$ given on $\Omega \times (0, \infty) \times \mathbb{R}^d$ admissible if it is represented as $f_t = f_{M}^t + f_{B}^t$ (“Morrey part” of $f$ plus its “bounded part”) with $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^d)$-measurable $f_{M}^t$, $f_{B}^t$ such that there exists a constant $\tilde{f} < \infty$ for which

$$\left( \int_B |f_{M}^t|^r \, dx \right)^{1/r} \leq \tilde{f} \rho^{-1},$$

whenever $t > 0$, $B \in \mathbb{B}_\rho$ and $\rho \leq R_0$, and there exists a predictable $\tilde{f}_t \geq \sup_{x \in \mathbb{R}^d} |f_{B}^t(x)|$ such that the integral

$$\int_0^\tau |\tilde{f}_t|^2 \, dt$$

is a bounded function on $\Omega$.

Assumption 1.6 $\beta, b, \zeta := c_+^{1/2}, \nu$ are admissible.

Here is our first main result about an a priori estimate in which for convenience of applications we introduce a measurable, $\mathcal{F}_t$-adapted process $\mu_t \geq 0$ such that the integral

$$\int_0^\tau \mu_t \, dt$$

is a bounded function on $\Omega$.

Theorem 1.7 Under the above assumptions suppose that a solution $u_t$ of (1.1)-(1.2) exists. Then with probability one $u_t \wedge \tau$ is continuous as an $L_p$-valued function and there exists a constant $N_0 = N_0(p, \delta, d, r)$ such that if

$$N_0(\hat{\beta} + \hat{b} + \hat{\zeta} + \hat{\nu}) \leq 1,$$ \hspace{1cm} (1.7)
then for any stopping time $\gamma \leq \tau$

$$Ee^{-\Phi_\gamma} \int_{\mathbb{R}^d} |u_t|^p \, dx + (\delta/16)E \int_0^{\gamma} e^{-\Phi_s} \int_{\mathbb{R}^d} |DU_s|^2 \, dx \, ds$$

$$\leq E \int_{\mathbb{R}^d} |u_0|^p \, dx + NE \int_0^{\gamma} e^{-\Phi_s} \int_{\mathbb{R}^d} (|f_s|^p + |s|^p + |g_s|^p)$

$$-\mu_s |u_s|^p) \, dx \, ds,$$

(1.8)

$$E \sup_{t \leq \tau} e^{-\Phi_t} \int_{\mathbb{R}^d} |u_t|^p \, dx \leq N \int_{\mathbb{R}^d} |u_0|^p \, dx$$

$$+ NE \int_0^{\tau} e^{-\Phi_s} \int_{\mathbb{R}^d} (|f_s|^p + |s|^p + |g_s|^p) \, dx \, ds,$$

(1.9)

where $U_t = |u_t|^{p/2}$,

$$\Phi_t = (B + 1) \int_0^t \lambda_s \, ds, \quad \phi_t = \int_0^t (B\lambda_s + \mu_s) \, ds, \quad \lambda_t = 1 + \beta t^2 + \beta t^2 + \gamma t^2 + \nu t^2,$$

and the constants $N, B$ depend only on $p, r, \delta, d, R_0$.

We prove this theorem in Sect. 3.

**Example 1.8** (Example 1.3.1 [10]) Condition (1.7) requires, in particular, $\hat{b}$ to be sufficiently small. It turns out that this smallness assumption is essential. For instance, $u_t(x) = \exp(-|x|^2/(4t))$ satisfies

$$\frac{\partial}{\partial t} u_t(x) = \Delta u_t(x) - \frac{d}{|x|^2} x^i D_i u_t(x),$$

(1.10)

where $|b_t(x)| = d/|x|$ is admissible with any $r < d$. In this situation (1.8) fails. However, if we take $\epsilon b$ in place of $b$ our equation will satisfy (1.7) and the estimate becomes available if $\epsilon$ is sufficiently small.

Similarly, one cannot allow $\hat{b}$ and $\hat{\lambda}$ to be large. Indeed, Eq. (1.10) can be rewritten as

$$\frac{\partial}{\partial t} u_t = \Delta u_t - D_t \left( \frac{d}{|x|^2} x^i u_t \right) + \frac{d(d-2)}{|x|^2} u_t.$$

Note that $|b| = \epsilon/|x|$ is not allowed in [10] and [11], so it seems that Theorem 1.7 is new even if there are no stochastic terms.

Our next result is about estimating $Du_t$. For that we need to impose a stronger assumptions than Assumptions 1.4 and 1.6. By $Df(x)$ we mean the collection of the first-order Sobolev derivatives of (vector- or tensor-valued) $f(x)$ and by $|Df(x)|$ we mean any fixed norm in the space where $Df(x)$ takes its values. Similar meaning will be given later to $D^2 f$ and $|D^2 f|$.
Assumption 1.9 We have $\beta = f \equiv 0$ and

$$E \int_0^T \int_{\mathbb{R}^d} (|f_t|^p + |g_t|^p + |Df_t|^p + |Dg_t|^p) \, dx \, ds < \infty.$$  

Introduce $c_t = |Dc_t|^{1/2}$.

Assumption 1.10 The functions $Da, D\sigma, b, \varsigma, c, \nu, D\nu$ are admissible.

Theorem 1.11 Under the above assumptions suppose that a solution $u_t$ of (1.1)-(1.2) exists and

$$E \int_0^T \|u_t\|_{W^2_p}^p \, dt < \infty. \quad (1.11)$$

Then (a.s.) $Du_{t\wedge \tau}$ is a continuous $L_p$-valued function and there exists a constant $N_0 = N_0(p, \delta, d, r)$ such that, if

$$N_0(\widehat{Da} + \widehat{D\sigma} + \widehat{b} + \widehat{\varsigma} + \widehat{c} + \widehat{\nu} + \widehat{D\nu}) \leq 1, \quad (1.12)$$

then

$$G + H \leq NK, \quad (1.13)$$

where

$$G = E \sup_{t \leq \tau} e^{-\Psi_t} \int_{\mathbb{R}^d} |Du_t|^p \, dx,$$

$$H = E \int_0^T e^{-\Psi_s} \int_{\mathbb{R}^d} (|Du_s|^p - 2|D^2u_s|^2 + \Lambda_s |Du_s|^p) \, dx \, ds,$$

$$K = E \int_{\mathbb{R}^d} (|u_0|^p + |Du_0|^p) \, dx$$

$$+ E \int_0^T e^{-\Psi_s} \int_{\mathbb{R}^d} (|f_s|^p + |g_s|^p + |Df_s|^p + |Dg_s|^p) \, dx \, ds,$$

$$\Psi_t = C \int_0^t \Lambda_s \, ds, \quad \Lambda_t = 1 + \widehat{Da}^2_t + \widehat{D\sigma}^2_t + \widehat{b}^2_t + \widehat{\varsigma}^2_t + \widehat{c}^2_t + \widehat{\nu}^2_t + \widehat{D\nu}_t^2$$

and the constants $N, C$ depend only on $p, r, \delta, d, R_0$.

We prove this theorem in Sect. 4.

Remark 1.12 Somewhat similar result is presented in Theorem 2.1 of [3]. There are, however, several distinctions. Let alone systems, in [3] the equations can degenerate and because of that the assumptions on the smoothness of $a$ and $\sigma$ are stronger. Also the coefficients of lower order terms like $b, c$ are assumed to be bounded. We heavily use the nondegeneracy of our equation.
For nondegenerate equations [6] provides a similar result with a weaker assumption on $a$ but with bounded lower order coefficients and with an assumption on $\sigma$ which, basically, requires its continuity in $x$. Under our assumptions $\sigma$ can have rather wild discontinuities. Incidentally, one of the aims of this article is to develop some tools on the way of proving that under our assumptions on $\sigma$ and $b$ the stochastic equation $\,dx_t = \sigma_t^k(x_t) \,dw^k_t + b_t(x_t) \,dt$ with deterministic $\sigma$ and $b$ and any nonrandom initial condition has a strong solution provided that the matrix $(\sigma_t^{ik} \sigma_t^{jk})$ is uniformly nondegenerate. The way we mean is similar to the one in [8].

2 Auxiliary results

For $r \in (1, d]$ and two real-valued functions $b$ and $\zeta$ on $\mathbb{R}^d$ set

$$\hat{b} := \|b\|_{E_{r,1}}, \quad \hat{\zeta} := \|\zeta\|_{E_{r,1}},$$

where, for any function $f$,

$$\|f\|_{E_{r,1}} := \sup_{\rho > 0} \|f\|_{L^r(B)}^{\rho}, \quad \|f\|_{L^r(B)}^{\rho} = \int_B |f|^\rho \,dx.$$

For functions $u, v$ on $\mathbb{R}^d$ we write $u \prec v$ if the integral of $u$ over $\mathbb{R}^d$ is less than or equal to that of $v$. If the integrals are equal, we write $u \sim v$. Also set $p' = p/(p - 1)$.

**Lemma 2.1** Let $u, v \in C_0^\infty$, then

$$1 < p < r \implies |bu|^p < N\hat{b}^p|Du|^p, \quad (2.1)$$

$$p, p' \in (1, r) \implies \int_{\mathbb{R}^d} |v \zeta^2 u| \,dx \leq N\hat{\zeta}^2\|Dv\|_{L^{p'}}\|Du\|_{L^p}, \quad (2.2)$$

$$1 < p' < r \implies \int_{\mathbb{R}^d} |vbDu| \,dx \leq N\hat{b}\|Dv\|_{L^{p'}}\|Du\|_{L^p}, \quad (2.3)$$

where the constants $N$ depend only on $d$, $r$, $p$.

**Proof** Estimate (2.1) is proved in [1]. To prove (2.2), it suffices to observe that

$$\int_{\mathbb{R}^d} |v \zeta^2 u| \,dx \leq \|\zeta\|_{L^{p'}}\|\zeta u\|_{L^p},$$

and then use (2.1). Similarly (2.3) is proved on the basis of

$$\int_{\mathbb{R}^d} |vbDu| \,dx \leq \|bv\|_{L^{p'}}\|Du\|_{L^p}.$$

The lemma is proved. \qed
Remark 2.2 Estimate (2.1) does not hold, in general, if \( p = r \). Still the restriction on \( r \) can be lifted, however, if \( r > d \), then \( \hat{b} = \infty \) unless \( b = 0 \).

Remark 2.3 Estimate (2.3) shows, in particular, that \( g := b_i D_i u \), with \( \hat{|b|} < \infty \), can be represented as \( D_i f^i + f \), where the \( L_p \)-norms of \( f^i \), \( f \) are controlled by that of \( Du \). Indeed, for \( \Lambda = (1 - \Delta)^{1/2} \) the estimate implies that \( h := \Lambda^{-1} g \in L_p \) and \( g = \Lambda h = (1 - \Delta)^{-1} \Lambda^{-1} h = D_i f^i + f \), where \( f = \Lambda^{-1} h, f^i = -D_i \Lambda^{-1} h \).

Lemma 2.4 If \( f \) is admissible, then for any \( t \) and \( u \in C_0^\infty \) we have

\[
|f_t|^2 |u|^2 < N(d, r) \hat{f}^2 |Du|^2 + (N(d, r) \hat{R}_0^2 \hat{f}^2 + 2 \hat{f}^2)|u|^2.
\]  

Proof We have \( |f_t|^2 \leq 2 |f_t|^2 + 2 |f_t|^2 \) and this shows how \( 2 |f_t|^2 |u|^2 \) appeared in (2.4). By a version of the Chiarenza–Frasca theorem in [1] (see, for instance, Lemma 3.5 in [8])

\[
|f_t|^2 |u_t|^2 < N \hat{f}^2 |Du_t|^2 + N \hat{R}_0^2 \hat{f}^2 |u_t|^2.
\]

This proves the lemma. \( \square \)

Corollary 2.5 If \( f \) is admissible, then for any \( t, \varepsilon > 0 \) and \( u \in C_0^\infty \) we have

\[
|f_t| |Du_t| |u_t| \leq \varepsilon |Du_t|^2 + \varepsilon^{-1} |f_t|^2 |u_t|^2 \\
< (\varepsilon + N \varepsilon^{-1} \hat{f}^2) |Du_t|^2 + N \varepsilon^{-1} (\hat{f}^2 + \hat{R}_0^2 \hat{f}^2) |u_t|^2.
\]

Remark 2.6 The above results are stated for simplicity only for \( u, v \in C_0^\infty \). Observe that, say, (2.1) does not hold for any even smooth function. Constants make its right-hand side vanish. However, (2.1) is certainly also true if \( u \in W_2^p \), because \( \hat{b} \) is a constant, one can approximate \( u \) in the \( W_2^p \)-norm by functions from \( C_0^\infty \), and one can use Fatou’s lemma to deal with the left-hand side.

Other results admit similar easy extensions if we recall that \( \hat{f} \) and \( \hat{f}_t \) are independent of \( x \).

3 Proof of theorem 1.7

In light of Lemma 2.1 and Remark 2.3, Theorem 2.1 of [7] is applicable, which in particular implies the \( L_p \)-continuity of \( u_t \wedge \tau \). By this theorem also, after introducing \( \theta_t = \text{sign} u_t \) we obtain for \( t \leq \tau \) that

\[
(1/p) \int_{\mathbb{R}^d} |u_t|^p dx = (1/p) \int_{\mathbb{R}^d} |u_0|^p dx + \int_0^t \int_{\mathbb{R}^d} J_s dx ds + m_t \\
+ \int_0^t \int_{\mathbb{R}^d} \theta_s |u_s|^{p-1} (L_s u_s + D_i f_s^i + f_s) dx ds,
\]
where (recall that $U_t = |u_t|^{p/2}$)

$$J_t := ((p-1)/2)|u_s|^{p-2}\sum_k (M^k_su_t + g^k_s)^2$$

$$\leq ((p-1)/2)|u_t|^{p-2}\sigma_i^{ik}D_iu_t\sigma_i^{jk}D_ju_t + (\delta/(2p))|DU_t|^2$$

$$+ N|u_t|^p|\nu_t|^2 + N|u_t|^{p-2}|g_t|^2,$$

$$m_t := \int_0^t \int_{\mathbb{R}^d} \theta_s |u_s|^{p-1}(M^k_su_s + g^k_s)) \, dx \, dw^k_s.$$ (3.1)

Here and below constants $N$ denote constants depending only on $d$, $r$, $p$, $\delta$.

From the start we may assume that $\hat{b}$, $\hat{\beta}$, $\hat{\nu}$, $\hat{\nu} \leq 1$ (in order to be able to drop terms like $\hat{f}^2$ in the second term on the right in (2.4)). Note that

$$I_1 := \theta_t |u_t|^{p-1}D_i(a^i_j D_j u_t) + ((p-1)/2)|u_t|^{p-2}\sigma_i^{ik}D_iu_t\sigma_i^{jk}D_ju_t$$

$$+ (\delta/(2p))|DU_t|^2 \sim \frac{2(p-1)}{p^2} (2a - \alpha)_{ij} D_i U_t D_j U_t$$

$$+ (\delta/(2p))|DU_t|^2 \leq \hat{\delta}|DU_t|^2,$$ (3.2)

where $\hat{\delta} = \delta/(2p)$.

Next, by Corollary 2.5

$$I_2 := \theta_t |u_t|^{p-1}D_i(\beta^i_j u_t) + \theta_t |u_t|^{p-1}b^i_j D_i u_t$$

$$\sim -(p-1)\theta_t |u_t|^{p-1}\beta^i_j D_i u_t + \theta_t |u_t|^{p-1}b^i_j D_i u_t$$

$$\leq (NU_t(|\beta_t| + |b_t|))|DU_t|$$

$$\leq (\hat{\delta}/4 + N_1(\hat{\beta}^2 + \hat{b}^2))|DU_t|^2 + N(1 + \hat{\beta}^2 + \hat{b}^2)|U_t|^2.$$ (3.3)

We subject $\hat{\beta}$ and $\hat{b}$ to

$$N_1(\hat{\beta}^2 + \hat{b}^2) \leq \hat{\delta}/4$$ (3.3)

and then get that

$$I_2 \leq (\hat{\delta}/2)|DU_t|^2 + N(1 + \hat{\beta}^2 + \hat{b}^2)|u_t|^p.$$ (3.4)

By Lemma 2.4 (observe that $|DU|^2 \leq N|Du|^p + N|u|^p$)

$$I_3 := \theta_t |u_t|^{p-1}c_t u_t + N|u_t|^p|\nu_t|^2 \leq \hat{\nu}_t^2 U_t^2 + N|\nu_t|^2 U_t^2$$

$$< N_2(\hat{\nu}_t^2 + \hat{\nu}^2)|DU_t|^2 + N(1 + \hat{\nu}_t^2 + \hat{\nu}^2)|U_t|^2.$$ (3.5)

We subject $\hat{\nu}$ and $\hat{\nu}$ to

$$N_2(\hat{\nu}_t^2 + \hat{\nu}^2) \leq \hat{\delta}/4$$ (3.5)
and then get that

\[ I_3 \leq (\delta/4)|DU_t|^2 + N(1 + \xi_t^2 + \nu_t^2)|u_t|^p. \] (3.6)

Next,

\[ I_4 := |u_t|^{p-1}\theta_t D_t f_t^i N |u_t|^{p/2-1} D_t U_t \leq (\delta/8)|DU_t|^2 + N|f_t|^2|u_t|^{p-2} \]
\[ \leq (\delta/8)|DU_t|^2 + N|U_t|^2 + N|f_t|^p. \] (3.7)

In estimating the last remaining terms we use elementary inequalities

\[ |u_t|^{p-1} f_t + N|u_t|^{p-2}g_t^2 \leq N|u_t|^p + N(|f_t|^p + |g_t|^p). \] (3.8)

Upon combining this with (3.2), (3.4), (3.6), and (3.7), we get that, if condition (1.7) is satisfied for an appropriate \( N_0 \), then in the stochastic differential form

\[ \frac{1}{p}d \left( \int_{\mathbb{R}^d} |u_t|^p \, dx \right) \leq \int_{\mathbb{R}^d} \left( - (\delta/8)|DU_t|^2 + N(|f_t|^p + |f_t|^p + |g_t|^p) \right) \, dx \, dt \]
\[ + \left( B/p \right) \int_{\mathbb{R}^d} \lambda_t |u_t|^p \, dx \, dt + dm_t. \] (3.9)

It follows that for \( t \leq \tau \)

\[ e^{-\phi_t} \int_{\mathbb{R}^d} |u_t|^p \, dx + (\delta/8) \int_0^t e^{-\phi_s} \int_{\mathbb{R}^d} |DU_s|^2 \, dx \, ds \]
\[ \leq \int_{\mathbb{R}^d} |u_0|^p \, dx + N \int_0^t e^{-\phi_s} \int_{\mathbb{R}^d} (|f_s|^p + |f_s|^p + |g_s|^p - \mu_t|u_t|^p) \, dx \, ds \]
\[ + p \int_0^t e^{-\phi_s} \, dm_s. \]

This yields (1.8) because, as we know from [7], \( m_t \) is a martingale. To prove (1.9) we follow E. Pardoux ([12]). Estimate (3.9) and the Davis’s inequality imply that

\[ E \sup_{t \leq \tau} e^{-\Phi_t} \int_{\mathbb{R}^d} |u_t|^p \, dx \leq E \int_{\mathbb{R}^d} |u_0|^p \, dx \]
\[ + NE \int_0^\tau e^{-\Phi_s} \int_{\mathbb{R}^d} (|f_s|^p + |f_s|^p + |g_s|^p) \, dx \, ds \]
\[ + NE \left( \int_0^\tau e^{-2\Phi_s} \sum_k \left( \int_{\mathbb{R}^d} |u_s|^{p-1} |M_s^{k} u_s + g_s^{k}| \, dx \right)^2 \, ds \right)^{1/2}. \] (3.10)
Here \(|u_s|^{p-1} = |u_s|^{p/2}|u_s|^{p/2-1}\) and the last term is dominated by

\[
NE\left( \int_0^\tau e^{-2\Phi_s} \|u_s\|^p_{L^p} \int_{\mathbb{R}^d} J_s \, dx \, ds \right)^{1/2} \\
\leq NE\left( \sup_{s \leq \tau} e^{-\Phi_s} \|u_s\|^p_{L^p} \right)^{1/2} \left( \int_0^\tau e^{-\Phi_s} \int_{\mathbb{R}^d} J_s \, dx \, ds \right)^{1/2} \\
\leq NE\left( \sup_{s \leq \tau} e^{-\Phi_s} \|u_s\|^p_{L^p} \right)^{1/2} \left( E \int_0^\tau e^{-\Phi_s} \int_{\mathbb{R}^d} J_s \, dx \, ds \right)^{1/2}
\]

It is seen from (3.1) and the estimates of \(I_3\) and (3.8) that

\[
J_t \leq N |D| u_t|^2 + N |u_t|^p |v_t|^2 + N |u_t|^{p-2} |g_t|^2 \\
< N |D| u_t|^2 + N (1 + \bar{v}_t^2) |u_t|^p + N |g_t|^p.
\]

The last expression is under control due to (1.8) with \(\mu_t = \lambda_t \geq 1 + \bar{v}_t^2\). Hence coming back to (3.10) we immediately obtain (1.9). This proves the theorem.

**Remark 3.1** It follows from the proof that in the definition of \(\phi_t\) one can take any constant \(\geq B\) in place of \(B\).

**4 Proof of theorem 1.11**

We prove the theorem after some discussion.

**Lemma 4.1** For any \(n \geq 1, p_i > 0, i = 1, \ldots, n, \) and \(\kappa \geq d + 1 + p_1 + \ldots + p_n\) there exists a constant \(N = N(d, \kappa, n, p_i)\) such that for any \(A_i \in \mathbb{R}^d, i = 1, \ldots, n,\) we have

\[
|A_1|^{p_1} \cdot \ldots \cdot |A_n|^{p_n} \leq N \int_{\mathbb{R}^d} h(\eta) |(\eta, A_1)|^{p_1} \cdot \ldots \cdot |(\eta, A_n)|^{p_n} \, d\eta, \quad (4.1)
\]

where \(h(\eta) = (1 + |\eta|^\kappa)^{-1}\) and \((\eta, A_i)\) is the scalar product of \(\eta\) and \(A_i\).

**Proof** By dividing both parts by \(|A_1|^{p_1}\) we see that we may assume that \(|A_1| = 1\). After that we may also assume that \(|A_i| = 1\) for all \(i\). Then the result easily follows by contradiction. The lemma is proved.

By applying this lemma to \(A_1 = Du\) and \(A_2 = Dv_{x,t}\) and then summing up with respect to \(i\) we get the following, where

\[
\eta \mapsto u(\eta) := \eta^j D_j u.
\]
Corollary 4.2  For any smooth functions \( u, v \) on \( \mathbb{R}^d \), on \( \mathbb{R}^d \) we have for \( p, q > 0 \), \( \kappa \geq d + 1 + (p + q)d \)

\[
|Du|^p \leq N \int_{\mathbb{R}^d} h(\eta)|u_\eta|^p \, d\eta,
\]

\[
|Du|^p |D^2v|^q \leq N \int_{\mathbb{R}^d} h(\eta)|u_\eta|^p |Dv_\eta|^q \, d\eta.
\]

Proof of Theorem 1.11 Fix \( \kappa \geq d + 1 + p \) and for functions \( u(x, \eta) \) and \( v(x, \eta) \) on \( \mathbb{R}^{2d} \) let us write

\[
\text{if } \hat{R}^{2d} h(\eta) u(x, \eta) \, dx \, d\eta \leq \hat{R}^{2d} h(\eta) v(x, \eta) \, dx \, d\eta.
\]

We also write \( u \sim_{\kappa} v \) if the above integrals coincide. In these terms Corollary 4.2 implies that

\[
|\eta|^p |Du|^p \prec_{\kappa} N|u_\eta|^p, \quad |\eta|^p |Du|^{p-2} |D^2v|^2 \prec_{\kappa} N|u_\eta|^{p-2} |Dv_\eta|^2.
\]  

(4.2)

Introduce

\[
v_t = v_t(x, \eta) = \eta^i D_i u_t(x), \quad \theta_t = \text{sign } v_t, \quad V_t = |v_t|^{p/2},
\]

\[
W_t^2 := |\eta|^p |Du_t|^{p-2} |D^2u_t|^2, \quad \gamma_t^i = D_i a_t^{ij} + b_t^i.
\]

Observe that thanks to (4.2)

\[
|\eta|^p |Du|^p \prec_{\kappa} N|v_t|^p, \quad W_t^2 \preccurlyeq_{\kappa} N|DV_t|^2, \quad |DV_t|^2 \leq NW_t^2.
\]  

(4.3)

Owing to (1.11) we can substitute \( \phi(\eta) \) in place of \( \phi \) into (1.4) which yields

\[
dv_t = (L_t v_t + a_t^{ij} D_{ij} u_t + \gamma_t^{ij} D_i u_t + c_t(\eta) u_t + f_t(\eta)) \, dt
\]

\[
+ (M_t v_t + \sigma_t^{ik} D_{ik} u_t + \nu_t(\eta) u_t + s_t(\eta)) \, dw_t^k.
\]

Again in light of Lemma 2.1, Theorem 2.1 of [7] is applicable to \( v_t \), which in particular implies the \( L_{p,\eta} \)-continuity of \( V_{t,\eta} \) and thus of \( Du_{t,\eta} \). By this theorem also we obtain for \( t \leq \tau \) that

\[
\frac{1}{p} \int_{\mathbb{R}^{2d}} h(\eta)|v_t|^p \, dx \, d\eta = \frac{1}{p} \int_{\mathbb{R}^{2d}} h(\eta)|v_0|^p \, dx \, d\eta
\]

\[
+ \int_0^t \int_{\mathbb{R}^{2d}} h(\eta)\theta_s |v_s|^{p-1} (L_s v_s + a_s^{ij} D_{ij} u_s) \, dx \, d\eta \, ds
\]

\[
+ \int_0^t \int_{\mathbb{R}^{2d}} h(\eta)\theta_s |v_s|^{p-1} (\gamma_s^{ij} D_i u_s + c_s(\eta) u_s + f_s(\eta)) \, dx \, d\eta \, ds
\]

\[
+ \int_0^t \int_{\mathbb{R}^{2d}} h(\eta) J_s \, dx \, d\eta \, ds + m_t,
\]
where

\[
J_t = ((p - 1)/2)|v_t|^2 \sum_k (M_t^k v_t + \sigma^{ik}_{t(\eta)} D_i u_t + v^k_{t(\eta)} u_t + g^k_{t(\eta)})^2 \\
\leq ((p - 1)/2)|v_t|^2 \sum_k (\sigma^{ik}_t D_i v_t)^2 + N|v_t|^2 |v_t|^2 |D\sigma_t|^2 |Du_t|^2 + |DV_t|^2 + |Dg_t|^2, \\
m_t = \int_0^t \int_{\mathbb{R}^{2d}} h(\eta) J^{k}_{s} d\eta d\nu^k, \\
J^k_t = \theta_t |v_t|^{p-1} (M^k_t v_t + \sigma^{ik}_{t(\eta)} D_i u_t + v^k_{t(\eta)} u_t + g^k_{t(\eta)}).
\]

Next, we may assume that \( \tilde{D}a, \tilde{D}\sigma, \tilde{b}, \tilde{c}, \tilde{\nu}, \tilde{D}v \leq 1 \) and first deal with the terms containing \( a \) and \( \sigma \) but not their derivatives. Their sum is (see (4.4)) less than

\[
I_1 := \theta_t |v_t|^{p-1} D_i (a_{ij} D_j v_t) + ((p - 1)/2)|v_t|^2 \sum_k (\sigma^{ik}_t D_i v_t)^2 \\
+ (\delta/(2p))|DV_t|^2 \sim \kappa - ((p - 1)/p^2)(4a_{ij} - 2\sigma^{ij}_t) D_i V_t D_j V_t \\
+ (\delta/(2p))|DV_t|^2 \\
\leq - (\delta/(2p))|DV_t|^2 \sim \kappa - \delta(|DV_t|^2 + W^2_t),
\]

where \( \delta = \delta(p, d, \delta) > 0 \) and in the last inequality we used (4.3).

By Lemma 2.4

\[
I_2 := \theta_t v_t^{p-1} a_{ij} D_i u_t \leq (V_t|D\sigma_t|)|\eta|^{p/2}|Du_t|^{p/2-1}|2u_t| \\
\leq (\delta/2)W^2 + NV^2_t|D\sigma_t|^2 \sim \kappa (\delta/2)W^2 + N\tilde{D}a^2|DV_t|^2 + N(1 + \tilde{D}a^2)V^2_t.
\]

We subject \( \tilde{D}a \) to

\[
N\tilde{D}a^2 \leq \delta/2
\]

and get that

\[
I_2 \sim \kappa (\delta/2)(|DV_t|^2 + W^2_t) + N(1 + \tilde{D}a_t^2)V^2_t.
\]

Next, come the terms containing the products of \( \gamma, b \) and their derivatives:

\[
I_3 := \theta_t |v_t|^{p-1} b^i_j D_i v_t + \theta_t |v_t|^{p-1} \gamma^i_{t(\eta)} D_i u_t \sim \kappa (2/p)(V_t^i b^i_j) D_i V_t \\
- \eta^j \gamma^i (p - 1)|v_t|^{p-2} D_j v_t D_i u_t + |v_t|^{p-1} D_{ij} u_t.
\]

Here by Corollary 2.5

\[
(2/p)(V_t^i b^i_j) D_i V_t \sim \kappa (\delta/8)|DV_t|^2 + N\tilde{b}^2|DV_t|^2 + N\tilde{b}^2V^2_t.
\]
By Corollary 4.2 the remaining term in $I_3$ is dominated by

$$N|\gamma_t||\eta|^p|Du|^{p-1}|D^2u|$$

$$\prec \kappa N(|Da_t| + |b_t|)V_t(|\eta|^{p/2}|Du_t|^{p/2-1}|D^2u_t|)$$

$$\leq N(|Da_t|^2 + |b_t|^2)V_t^2 + (\hat{\delta}/4)W_t^2$$

$$\prec \kappa N_2(\hat{D}a_t^2 + \hat{b}_t^2)|DV_t|^2 + (\hat{\delta}/4)W_t^2 + N(1 + \bar{D}a_t^2 + \bar{b}_t^2)V_t^2.$$

We subject $\hat{D}a$ and $\hat{b}$ to

$$N_2(\hat{D}a^2 + \hat{b}^2) \leq \hat{\delta}/8,$$  \hspace{1cm} (4.8)

and get that

$$I_3 \prec \kappa (\hat{\delta}/4)(|DV_t|^2 + W_t^2) + N(1 + \bar{D}a_t^2 + \bar{b}_t^2)V_t^2.$$  \hspace{1cm} (4.9)

The term with $|Du_t|^2$ is

$$I_4 := N|\eta|^2|v_t|^{p-2}|D\sigma_t|^2|Du_t|^2 \leq N|\eta|^p|D\sigma_t|^2(|Du_t|^{p/2})^2$$

$$\prec \kappa N_3\hat{D}\sigma^2 W_t^2 + N(1 + \bar{D}\sigma_t^2)V_t^2.$$

We subject $\hat{D}\sigma$ to

$$N_3\hat{D}\sigma^2 \leq \hat{\delta}/8,$$  \hspace{1cm} (4.10)

and get that

$$I_4 \prec \kappa (\hat{\delta}/8)(|DV_t|^2 + W_t^2) + N(1 + \bar{D}\sigma_t^2)V_t^2.$$  \hspace{1cm} (4.11)

Now come the terms which do not contain $Du_t$. Their sum is

$$I_5 := I_6 + I_7 + I_8,$$

where

$$I_6 := \theta_t|v_t|^{p-1}(c_t v_t + c_t(\eta)u_t), \quad I_7 := N|v_t|^2|v_t|^p + N|\eta|^2|v_t|^{p-2}|DV_t|^2|u_t|^2,$$

$$I_8 := \theta_t|v_t|^{p-1}f_t(\eta) + N|\eta|^2|v_t|^{p-2}|Dg_t|^2.$$

We have

$$I_6 \leq \bar{\theta}_t^2 V_t^2 + \bar{\theta}_t^2(V_t^2 + |\eta|^p|u_t|^p) \prec \kappa N_4(\bar{\theta}_t^2 + \bar{\theta}_t^2)|DV_t|^2$$

$$+ NV_t^2|\eta|^p|DU_t|^2 + N((1 + \bar{\theta}_t^2)|\eta|^p|u_t|^p + (1 + \bar{\theta}_t^2 + \bar{\theta}_t^2)V_t^2)$$
We subject $\hat{c}^2$ and $\hat{\varsigma}^2$ to

$$N_4(\hat{\varsigma}^2 + \hat{c}^2) \leq \delta/16 \quad (4.12)$$

and get that

$$I_6 \prec \kappa \left( \delta/16 \right)(|DV_t|^2 + W_t^2) + N(1 + \hat{\varsigma}^2_t + \hat{c}^2_t)V_t^2 + N|\eta|^p |DU_t|^2 + N(1 + \hat{\varsigma}^2_t)|\eta|^p U_t^2. \quad (4.13)$$

Similarly we deal with $I_7$ and see that, if for some well defined $N_5$

$$N_5(\hat{\varsigma}^2 + \hat{D}v^2) \leq \delta/32, \quad (4.14)$$

then

$$I_7 \prec \kappa \left( \delta/32 \right)(|DV_t|^2 + W_t^2) + N(1 + \hat{\varsigma}^2_t + \hat{D}v^2)V_t^2 + N|\eta|^p |DU_t|^2 + N(1 + \hat{\varsigma}^2_t)|\eta|^p U_t^2. \quad (4.15)$$

In what concerns $I_8$, obviously,

$$I_8 \leq V_t^2 + N|\eta|^p (|Df_t|^p + |Dg_t|^p).$$

Thus, given that (4.6), (4.8), (4.10), (4.12), and (4.14) hold, we have

$$d|v_t|^p \prec \kappa - \hat{\varsigma}(|DV_t|^2 + W_t^2) dt + (C - 1) \Lambda_t |v_t|^p dt + N|\eta|^p (|Df_t|^p + |Dg_t|^p + |DU_t|^2 + (1 + \hat{\varsigma}^2_t + \hat{D}v^2_t)U_t^2) dt + J_t^k d\omega_t^k, \quad (4.16)$$

where $\delta = \delta/32$.

It follows that

$$d \left( e^{-\Psi_t} |v_t|^p \right) \prec \kappa - \delta e^{-\Psi_t} (|DV_t|^2 + W_t^2) dt - \Lambda_t e^{-\Psi_t} |v_t|^p dt + N e^{-\Psi_t} |\eta|^p (|Df_t|^p + |Dg_t|^p + |DU_t|^2 + (1 + \hat{\varsigma}^2_t + \hat{D}v^2_t)U_t^2) dt + e^{-\Psi_t} J_t^k d\omega_t^k. \quad (4.17)$$

We convert this into the integral form with integrals with respect to $x, \eta,$ and $t$, then take expectations of the resulting inequality. We also use Corollary 4.2. Then we get

$$H = E \int_0^T e^{-\Psi_t} \int_{\mathbb{R}^d} \left( |Du_t|^{p-2} |D^2u_t|^2 + \Lambda_t |Du_t|^p \right) dxdt \leq NE |Du_0|^p \quad (4.18)$$

$$+ NE \int_0^T e^{-\Psi_t} \int_{\mathbb{R}^d} \left( |Df_t|^p + |Dg_t|^p + |DU_t|^2 \right) dt + (1 + \hat{\varsigma}^2_t + \hat{D}v^2_t)U_t^2 dt.$$
We may assume that 
\[ C/\Lambda_1 t \geq B\lambda t + 1 + \bar{c}_t^2 + \bar{D}\nu_t^2 \]
and then, owing to (1.8), the term in (4.18) containing \(|DU_t|^2 + (1 + \bar{c}_t^2 + \bar{D}\nu_t^2)U_t^2\) is estimated by
\[
NE|u_0|^p + NE \int_0^\tau e^{-\Psi t} \int_{\mathbb{R}^d} (|f_t|^p + |g_t|^p) \, dx \, dt \leq NK,
\]
so that \( H \leq NK \).

Next, we come back to (4.17) and, after converting it, take sup with respect to \( t \leq \tau \) and take expectations in the resulting inequality. We also use Corollary 4.2 and the result of the above treatment of (4.18). Then we obtain
\[
G \leq NK + I, \tag{4.19}
\]
where
\[
I = E \sup_{t \leq \tau} \left| \int_0^t e^{-\Psi s} \int_{\mathbb{R}^{2d}} h(\eta) J_s^k \, dx \, d\eta \, dw_s^k \right|.
\]

By Davis’s inequality
\[
I \leq NE \left( \int_0^\tau e^{-2\Psi t} \sum_k \left( \int_{\mathbb{R}^{2d}} h(\eta) J_t^k \, dx \, d\eta \right)^2 \, dt \right)^{1/2}
\]
\[
= NE \left( \int_0^\tau \sum_k e^{-2\Psi t} \left( \int_{\mathbb{R}^{2d}} h(\eta) \partial_t |v_t|^p - 1 \left( M_t^k v_t \right. \right. \right. \\
\left. \left. \left. + a_{t(\eta)}^k D_i u_t + v_{t(\eta)}^k u_t + g_{t(\eta)}^k \right) \, dx \, d\eta \right)^2 \, dt \right)^{1/2}
\]
\[
\leq NE \left( \sup_{t \leq \tau} e^{-\Psi t} \int_{\mathbb{R}^d} |Du_t|^p \, dx \right)^{1/2} \left( \int_0^\tau e^{-\Psi t} \int_{\mathbb{R}^{2d}} h(\eta) J_t \, dx \, d\eta \, dt \right)^{1/2}
\]
\[
\leq NG^{1/2} \left( E \int_0^\tau e^{-\Psi t} \int_{\mathbb{R}^{2d}} h(\eta) J_t \, dx \, d\eta \, dt \right)^{1/2}.
\]

Above we actually estimated \( J_t \) splitting it into parts. It is seen from these estimates that
\[
J_t \prec_k N|DV_t|^2 + I_4 + I_7 + N|\eta|^2 |u_t|^p - 2 |Dg_t|^2
\]
\[
\prec_k N|\eta|^p \left( |Du_t|^p - 2 |D^2 u_t|^2 + \Lambda t |Du_t|^p \right) + N|\eta|^p |DU_t|^2
\]
\[
+ N(1 + \bar{D}\nu_t^2) |\eta|^p U_t^2 + N|\eta|^p |Dg_t|^p.
\]

Recalling our argument about (4.18) we see that \( I \leq NG^{1/2} K^{1/2} \), which after being substituted into (4.19), yields that \( G \leq NK \) as well. This proves (1.13) and the theorem. \( \square \)
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Conflict of interest  The manuscript contains no data and there is no conflict of interest.

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