The role of diffeomorphisms in the integration over a finite dimensional space of geometries

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Starting from the De Witt supermetric and limiting ourselves to a family of geometries characterized by a finite number of geometric invariants we extract the unique integration measure. Such a measure turns out to be a geometric invariant, i.e. independent of the gauge fixed metric used for describing the geometries. The measure is also invariant in form under an arbitrary change of parameters describing the geometries. The additional functional integration on the conformal factor makes the measure independent of the free parameter intervening in the De Witt supermetric. The differences between the case $D = 2$ and $D > 2$ are evidenced.

1. INTRODUCTION

We deal here with the regularization of the functional integral over geometries, obtained by considering a finite dimensional space of geometries i.e. by limiting the integration to a family of geometries described by a finite number $N$ of geometric invariants which we shall denote by $l_i$.

To specify the measure we must state which is the basic integration variable. Following the lessons of gauge theories we shall assume as basic integration variable the metric $g_{\mu\nu}$ which shall play the role analogous to the field $A_{\mu}$ of gauge theories. As in Wilson formulation gauge invariance is treated exactly here the invariance under diffeomorphisms will be dealt with exactly. Obviously this makes the original degrees of freedom infinite and our purpose will be that the derive a formula in which only the finite geometrical degrees of freedom appear.

Essential is to state a diffeomorphism invariant distance among two nearby field configurations and our choice will be the De Witt supermetric

$$\left(\delta g, \delta g\right) =$$

$$\int \sqrt{g(x)} d^D x \delta g_{\mu\nu}(x) G^{\mu\nu\mu'\nu'}(x) \delta g_{\mu'\nu'}(x)$$

(1)

where

$$G^{\mu\nu\mu'\nu'} = g^{\mu\mu'} g^{\nu\nu'} + g^{\mu\nu'} g^{\nu\mu'} - \frac{2}{D} g^{\mu\nu'} g^{\mu\nu'} + C g^{\mu\nu'} g^{\mu'\nu'}$$

(2)

whose very well known property is to be the unique ultralocal diffeomorphism invariant distance.

We recall that the De Witt distance in invariant only under “rigid” diffeomorphisms i.e. diffeomorphisms which are independent of the parameters whose variations generate the $\delta g_{\mu\nu}$. We shall see however that the final result will be invariant under the general diffeomorphism.

A distance in the space of metrics induces a volume in the space of metrics as it happens in finite dimensional spaces and this will be the origin of the integration measure.

In the developments of sect.2 we shall need a positive definite metric; this restricts our treatment to the Euclidean gravity with the parameter $C$ appearing in the De Witt supermetric subject to the restriction $C > 0$.

Related ideas are found in [4] even if there are some differences with respect to [3] and in [5] where however one is concerned with the continuum problem. Details of the material presented are found in [4].
2. INTEGRATION OVER THE DIFFEOMORPHISMS

Everything that follows here will be a direct consequence of the De Witt supermetric and of the restriction on the geometries to belong to a finite dimensional family. We shall denote with $\bar{g}_{\mu\nu}(x, l)$ a gauge fixed metric, i.e. a metric which describes the geometry characterized by the invariants $l_i$; the final results will be independent of the choice of $\bar{g}_{\mu\nu}(x, l)$. The general metric describing our geometries is given by

$$g_{\mu\nu}(x, l, f) = [f^* \bar{g}_{\mu\nu}(l)](x) \equiv \bar{g}_{\mu'\nu'}(x'(x), l) \frac{\partial x'^\mu}{\partial x^\mu'} \frac{\partial x'^\nu}{\partial x^\nu'}.$$  \hspace{1cm} (3)

The problem is now to factorize in $D[g_{\mu\nu}]$ the infinite gauge volume and a term $J(l) \prod_{i=1}^N \delta l_i$. Straightforward generalization of the finite dimensional procedure gives for the Jacobian $J(l)$

$$1 = J(l) \int \prod_i d\delta l_i D[\xi] e^{-\frac{1}{2} \delta g, \delta g}.$$ \hspace{1cm} (4)

In order to compute such a Jacobian it is useful to decompose the general variation of the metric in two pieces which are orthogonal according to the De Witt supermetric

$$\delta g_{\mu\nu}(x) = F(\xi)_{\mu\nu} + \left( f^* \frac{\partial \bar{g}_{\mu\nu}(l)}{\partial l_i} \delta l_i \right)(x)$$

$$= \left[ (F\xi)_{\mu\nu} + F(F^\dagger F)^{-1} F^\dagger \frac{\partial \bar{g}_{\mu\nu}}{\partial l_i} \delta l_i \right] +$$

$$\left[ 1 - F(F^\dagger F)^{-1} F^\dagger \right] \frac{\partial \bar{g}_{\mu\nu}}{\partial l_i} \delta l_i$$ \hspace{1cm} (5)

where

$$F(\xi)_{\mu\nu} = \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu.$$ \hspace{1cm} (6)

Substituting into (4) gives for the Jacobian $J(l)$

$$J(l) = \text{det}(t^i, t^j) \frac{1}{2} \text{Det}(F^\dagger F)^{\frac{1}{2}}$$ \hspace{1cm} (7)

where

$$t^i_{\mu\nu} = \left[ 1 - F(F^\dagger F)^{-1} F^\dagger \right] \frac{\partial \bar{g}_{\mu\nu}}{\partial l_i}.$$ \hspace{1cm} (8)

Thus the measure \ref{(4)} is the product of two factors. The first factor counts the number of different geometries which belong to the element $\prod_i \delta l_i$ while the second one is a true functional determinant giving the gauge volume of each geometry, i.e. the number of different metrics which describe a fixed geometry characterized by the parameters $l_i$. The second factor is the analogous of the Liouville action in the treatment of two dimensional gravity in the conformal gauge \ref{eq:liouville}.

Both terms are invariant under diffeomorphisms also when we let them depend on the parameters $l_i$. In the proof of such a property \ref{eq:liouville} the projector appearing on the l.h.s. of eq. \ref{eq:liouville} plays an essential role; as a result both terms are geometric invariants, i.e. they are functions only of the $l_i$ and do not depend on the original gauge fixed metric $\bar{g}_{\mu\nu}$.

Obviously due to \ref{eq:liouville} the expression $J(l) \prod dl_i$ is invariant under a change of the set of parameters $l_i$ into any other set of parameters which describe the same geometries. The source of such invariance is that the metric $g_{\mu\nu}$ and not the parameters $l_i$ have been chosen as fundamental integration variables.

As the $l_i$ are geometric invariants no diffeomorphism can connect two points on the gauge fixing surface $\bar{g}_{\mu\nu}(x, l)$ and thus no Gribov problem arises in this scheme.

In the Regge case if we give the $l_i$ the meaning of the link lengths, it is well known \ref{eq:liouville} that for a zero measure set of values of the $l_i$, there are changes $\delta l_i$ which leave the geometry unchanged. Not only such a set of values is of zero measure but in addition on such set the term $\text{det}(t^i, t^j)$ vanishes because in such a case a $\xi$ exists such that $\frac{\partial \bar{g}_{\mu\nu}}{\partial l_i} \delta l_i = (F\xi)_{\mu\nu}$ which gives $t_i \delta l_i = 0$ and thus $\text{det}(t^i, t^j) = 0$.

Great simplifications would occur if a gauge could be found in which

$$\bar{F}^\dagger \frac{\partial \bar{g}_{\mu\nu}}{\partial l_i} = 0.$$ \hspace{1cm} (9)

The analogous problem in gauge theories is to find a surface in the space of field configurations which is orthogonal to all gauge fibers. Such a surface in general, i.e. for a sufficiently rich choice of $A(x, l)$ does not exist for non abelian theories. This is also true in gravity where one can give in $D > 2$ simple non pathological examples of families of geometries for which such a gauge fixing surface in
the space of field configurations does not exist (see ref. 4 Appendix A). A gauge satisfying (3) can be found only for simple minisuperspace models.

As $F^P$ depends on $C$ through the De Witt metric, both terms in (3) are $C$ dependent and one can show that such a dependence does not cancel out (4). As already shown in [5] such a dependence disappears once one integrates over all conformal variations of our family of metrics. Thus in addition to a finite number of parameters $\tau_i$ which describe deformations transverse (i.e. non collinear) to the orbits generated by the Weyl and diffeomorphism groups, we consider the deformations induced by the Weyl group [3]. Inclusion of the conformal factor produces the following changes: the operator $F$ is replaced by its conformal version $P$ defined by

$$P(\xi)_{\mu\nu} = F(\xi)_{\mu\nu} - \frac{g_{\mu\nu}}{D} \gamma^{\lambda\nu} F(\xi)_{\lambda\rho}$$ (10)

and $\partial g_{\mu\nu}$ is replaced by

$$k_{\mu\nu} = \frac{\partial g_{\mu\nu}}{\partial \tau_i} - \frac{g_{\mu\nu}}{D} \gamma^{\alpha\beta} \frac{\partial g_{\alpha\beta}}{\partial \tau_i}$$ (11)

reaching for the integration measure $J(\sigma, \tau)$

$$\frac{1}{D} \left[ \det \left( k^i, (1 - P^P P^P) k^j \right) \right]^{\frac{1}{2}}.$$ (12)

The next job is to extract the dependence on the conformal factor $\sigma$ of the two terms appearing in eq. (12), induced by the dependence of $P$

$$P = e^{2\sigma} \tilde{P} e^{-2\sigma}, \quad P^P = e^{-D\sigma} \tilde{P} e^{(D-2)\sigma}$$ (13)

being $\tilde{P}$ the operator on the background metric. Here the first fundamental difference appears between the case $D = 2$ and $D > 2$. Ker$(P^P)$ is the analogous of the pure Teichmüller deformations in two dimensions and in $D > 2$ the dimensions of Ker$(P^P)$ is infinite. By using eq. (13) the dependence of the second factor in (12) on $\sigma$ can be given in terms on an $N \times N$ matrix which however involves the properties of the whole space Ker$(P^P)$. In $D = 2$ instead the dimension of such a subspace is always finite dimensional and the dependence of such a factor on $\sigma$ can be taken into account explicitly.

With regard to the term $\Det(P^P P)$ it is given by the usual $Z$-function expression

$$\log \Det(P^P P) = -\frac{d}{ds} Z(0) = -\frac{d}{ds} \left[ \frac{1}{(s)} \int_0^\infty dt t^{s-1} \Tr(e^{-tP^P P}) \right].$$ (14)

Being $P^P P$ an elliptic operator for any $D$, expression (14) is mathematically well defined [6]. The procedure which works in $D = 2$ is to compute its variation with respect to $\sigma$ and then to integrate back the result [5]. The variation of eq. (14) is

$$-\delta \log \Det(P^P P) = \gamma e^{2\sigma} Z P^P P(0) + \text{Finite}_{\sigma \to 0} \{(2 + D)\Tr(e^{-tP^P P} \delta \sigma) - D \Tr(e^{-tP^P P} \delta \sigma) \}.$$ (15)

Here the second fundamental difference occurs: it is easily seen that in $D = 2$ not only the operator $P^P P$ is elliptic but also the operator $PP^P$ appearing in the second term is elliptic, while in $D > 2$ $PP^P$ is no longer elliptic which makes the usual heat kernel technology inapplicable. Thus while $\Det(P^P P)$ is a mathematically well defined object its explicit dependence on $\sigma$, for $D > 2$ is not jet known.

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