Formulation of the Classical Mechanics in the Ring of Operators

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Abstract

By making use of the Weyl-Wigner-Groenewold-Moyal association rules, a commutative product and a new quantum bracket are constructed in the ring of operators $\mathcal{F}(\mathcal{H})$. In this way, an isomorphism between Lie algebra of classical observables (with Poisson bracket) and the Lie algebra of quantum observables with this new bracket is established. By these observations, a formulation of the classical mechanics in $\mathcal{F}(\mathcal{H})$ is obtained and is shown to be $\hbar \to 0$ limit of the Heisenberg picture formulation of the quantum mechanics.

I. INTRODUCTION

In this report we are going to answer, in a most general setting, two related questions: (1) What is the analogue of the multiplicative structure of the classical observables (functions defined on a classical phase space) in the quantum formalism?, (2) “What is the image of

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the Poisson bracket (PB) of functions in the ring of operators” (of quantum observables)? These questions, in that way or another, were in the mind of many physicists since the very beginning days of quantum mechanics [1,2] and, to the best of my knowledge, they are not answered yet. Especially the second question was explicitly stated, as is quoted here, in the second of two seminal papers of Ref. [3] and in a figure of which an empty box was used for the image of the PB. Throughout this paper we assume that the phase space is $\mathbb{R}^{2d}, d$ integer.

In the well-known canonical quantization, for the analogue of the multiplication “product $\rightarrow$ anti-commutator” rule works well up to cubic polynomials. For the analogue of PB “PB $\rightarrow (i\hbar)^{-1}$ commutator ([,])” rule works well up to quadratic polynomials of position and momentum variables, and up to observables which are affine functions of the position or of the momentum. According to the Groenewold-Van Howe theorem they lead to inconsistencies for the quartic and cubic polynomials, respectively. Similar obstructions arise for some other phase spaces which have different topology from $\mathbb{R}^{2d}$. For more extensive and technical discussion of this topic we refer to Ref. [2,4] and references therein.

These two questions, which underline the fundamental differences between the classical mechanics and quantum mechanics, will be answered by making use of the Weyl-Wigner-Groenewold-Moyal (WWGM)-quantization scheme (for recent reviews see [5,6]). The WWGM-quantization enables us to carry the quantum theory to a phase space and giving it an autonomous structure [3] with its own “genvalue” equations [7], (quasi)probability distributions [5] and spectral resolution. Quantum information encoded in the noncommutative product of the quantum observables is transferred via the WWGM-association to classical phase space and stored in the noncommutative $\star$-product (see Eq. (16) below) of the classical observables. In this way, to the product of operators corresponds the $\star$-product of functions and to the commutator of operators corresponds the Moyal bracket (MB) of functions. The resulting theory is also referred as the deformation quantization, or as the phase space formulation of the quantum mechanics. On the other hand, this quantization scheme also enables us to carry the classical mechanics on a phase space to a Hilbert space. This paper concentrates on this latter aspect of this quantization, although almost all the literature deals with the former. More concretely, we search for what correspond to the commutative product of functions and to the PB of them in the WWGM-quantization. Our answers to these questions will lead us to a formulation of the classical mechanics in the ring of operators.

For the purposes of this report we mainly consider systems with one degree of freedom ($d = 1$) and the corresponding phase spaces in real coordinates. Generalizing our results to systems with finite or denumerably infinite number of degrees of freedom and to phase spaces with complex coordinates are straightforward. Such a generalization of one of the main results of this paper will be given at the end of section IV. We use the derivative-based approach developed in [8], which is different from the conventional integral-based one but can be considered as a Liouville space formalism [9]. In this formalism operators are represented by (super)kets and superoperators (see Section II below) act on them. The derivative based approach shows that when operators are labeled by some parameters; derivatives with respect to them, multiplication of the operators with them and even integration over them must be considered as superoperators. This is crucial point of the WWGM-quantization for it comes into play by considering the parameters of the group space as the coordinate
functions of a phase space, and these parameters are carried by the used operator basis as labels.

The organization of the paper is as follows. In Section II, we define some important superoperators which are commutative function of their arguments. The importance of these superoperators is made manifest in Section III, which includes a brief review of some fundamental ideas of the WWGM-quantization, and our answer for the first question. As the second main result of this paper, a new quantum bracket is derived in Section IV. In Section V we give some general applications by using this new bracket. We conclude with a brief summary and discussion of results.

II. LIOUVILLIAN SUPEROPERATORS

Let us consider the Heisenberg-Weyl (HW) algebra: \([\hat{q}, \hat{p}] = i\hbar \hat{I}\), where \(\hbar = h/2\pi\), \(\hat{I}\), \(\hat{q}\) and \(\hat{p}\) are the Planck’s constant, the identity operator and the Hermitian position and momentum operators, respectively. Here and henceforth operators and functions of operators acting in \(\mathcal{H}\) are denoted by \(\hat{\text{A}}\) and superoperators by \(\hat{\text{A}}\)

\[
\hat{O}^{(s)}_{nm} = 2^{-(n+m)} \hat{T}^n_{[\hat{q}(s)]} \hat{T}^m_{[\hat{p}(s)]} \tag{1}
\]

where, \(\hat{L}_A\) and \(\hat{R}_A\) being, respectively, multiplication from left and from right by \(\hat{A}\)

\[
\hat{T}_{[\hat{A}(s)]} = (1 + s)\hat{L}_A + (1 - s)\hat{R}_A. \tag{2}
\]

Note that for an arbitrary operator \(\hat{F}\)

\[
[T_{[\hat{q}(s)]}, \hat{T}_{[\hat{p}(s)]}] \hat{F} = 0. \tag{3}
\]

The actions of \(\hat{O}^{(s)}_{nm}\) on the unit operator \(\hat{I}\) and, under the trace sign, on an arbitrary operator \(\hat{F}\) are as follows;

\[
\hat{O}^{(s)}_{nm}(\hat{I}) = \hat{i}^{(s)}_{nm}, \tag{4}
\]

\[
Tr[\hat{O}^{(s)}_{nm}(\hat{F})] = Tr[\hat{i}^{(s)}_{nm}\hat{F}], \tag{5}
\]

where

\[
\hat{i}^{(s)}_{nm} = 2^{-(n+m)} \hat{T}^n_{[\hat{q}(s)]} \hat{T}^m_{[\hat{p}(s)]} \hat{I} = 2^{-(n+m)} \hat{T}^m_{[\hat{p}(s)]} \hat{T}^n_{[\hat{q}(s)]} \hat{I}, \tag{6}
\]

\[
= 2^{-n} \sum_{j=0}^{n} \binom{n}{j} (1 + s)^j (1 - s)^{n-j} \hat{q}^j \hat{p}^m \hat{q}^{n-j} \hat{p}^m, \tag{7}
\]

is the \(s\)-ordered product of a term containing \(n\) factors of \(\hat{q}\) and \(m\) factors of \(\hat{p}\). In obtaining these expressions we note that ordering parameters \(-s\) and \(s\) in the last factors of the first
two lines of the above expressions do not contribute to the results since $\hat{T}^n_{[\hat{A}]_{(\pm s)}} \hat{I} = 2^n \hat{A}^n$. Moreover, in the last two lines we made use of the binomial formula

$$\hat{T}^n_{[\hat{A}]_{(s)}} = \sum_{j=0}^{n} \binom{n}{j} (1+s)^j (1-s)^{n-j} \hat{L}_j \hat{R}^{n-j}_j.$$  \hfill (8)

From Eqs. (6) and (7) we have, for $s = \pm 1$,

$$\hat{i}^{(1)}_{nm} = \hat{L}_q \hat{R}^m_{\hat{p}} \hat{I} = q^n \hat{p}^m, \quad \hat{i}^{(-1)}_{nm} = \hat{L}^m_{\hat{p}} \hat{R}^n_{\hat{q}} \hat{I} = \hat{p}^m \hat{q}^n,$$

and for $s = 0$,

$$\hat{i}^{(0)}_{nm} = 2^{-n} \sum_{j=0}^{n} \binom{n}{j} q^j \hat{p}^m \hat{q}^{n-j} = 2^{-m} \sum_{k=0}^{m} \binom{m}{k} \hat{p}^k \hat{q}^m \hat{p}^{m-k}.$$  \hfill (9)

While relations (9) exhibit the standard ($s = 1$) and antistandard ($s = -1$) rule of ordering, that corresponding to $s = 0$ are two well known expressions of the Weyl, or symmetrically ordered products. In fact, the usual expression known for the Weyl ordered form of $\hat{i}^{(0)}_{nm}$ is a totally symmetrized form containing $n$ factors of $\hat{q}$ and $m$ factors of $\hat{p}$, normalized by dividing by the number of terms in the symmetrized expression. Here we give an example

$$\hat{i}^{(0)}_{12} = \frac{1}{2} (\hat{q}\hat{p}^2 + \hat{p}^2 \hat{q}) = \frac{1}{4} (\hat{q} \hat{p}^2 + 2 \hat{p} \hat{q} \hat{p} + \hat{p}^2 \hat{q}) = \frac{1}{3} (\hat{q} \hat{p}^2 + \hat{p} \hat{q} \hat{p} + \hat{p}^2 \hat{q}).$$

As a simple result of the approach followed here, explicit expressions for many forms of the $s$–ordered products and their equivalents, without using the usual commutation relations, naturally arise by noting only the relation (3). From (7) easily follows that $[\hat{i}^{(s)}_{nm}]^\dagger = \hat{i}^{(-s)}_{nm}$, that is, for general $n$, $m$ integers, $\hat{i}^{(s)}_{nm}$ are Hermitian if and only if $\hat{s} = -s$ ($\hat{s}$ denotes the complex conjugation of $s$). In particular, the Weyl ordered products $\hat{i}^{(0)}_{nm}$ are Hermitian. Since the result is independent of $s$ when both or one of the integers $n$, $m$ is zero, these special monomials are Hermitian for any value of $s$.

Because of Eqs. (4) and (5) we would like to call $\hat{O}^{(s)}_{nm}$ the ordering superoperator. Making use of (1) and (5) we obtain the following relations for their repeated actions

$$\hat{i}^{(s)}_{nm} = \hat{O}^{(s)}_{nm} (\hat{I}) = \hat{O}^{(s)}_{n_1m_1} (\hat{O}^{(s)}_{n_2m_2} (\hat{I})) = \hat{O}^{(s)}_{n_1m_1} (\hat{i}^{(s)}_{n_2m_2}),$$  \hfill (11)

$$Tr[\hat{O}^{(s)}_{nm} (\hat{F})] = Tr[\hat{O}^{(s)}_{n_1m_1} (\hat{O}^{(s)}_{n_2m_2} (\hat{F}))] = Tr[\hat{O}^{(s)}_{n_1m_1} (\hat{O}^{(-s)}_{n_2m_2} (\hat{F}))] = Tr[\hat{F} \hat{O}^{(-s)}_{n_2m_2} (\hat{i}^{(-s)}_{n_1m_1})],$$  \hfill (12)

where $n = n_1 + n_2$, $m = m_1 + m_2$. Here is an example of the relation (11)

$$\hat{i}^{(s)}_{n+1,m+1} = \frac{1}{4} \hat{T}^{(s)}_{[\hat{q}]_{(s)} \hat{T}[\hat{p}]_{(-s)} \hat{T}^{(s)}_{\hat{q} \hat{p}} \hat{T}^{(s)}_{\hat{p} \hat{q}}} \hat{i}^{(s)}_{nm} = \frac{1}{4} \{ (1 - s^2) [\hat{q} \hat{p} \hat{i}^{(s)}_{nm} + \hat{i}^{(s)}_{nm} \hat{p} \hat{q}] + (1 + s^2) \hat{q} \hat{i}^{(s)}_{nm} \hat{p} + (1 - s^2) \hat{p} \hat{i}^{(s)}_{nm} \hat{q} \}.$$  \hfill (13)

Finally in this section, with a phase space function expandable as power series in $p$ and $q$
we associate a Liouvillian superoperator
\[ \hat{\mathcal{O}}_{nm}^{(s)} = \hat{f}(\frac{1}{2} T_{[q]_{(s)}}, \frac{1}{2} T_{[\hat{p}]_{(-s)}}) = \sum_{n,m} c_{nm} \hat{O}_{nm}^{(s)}. \]

Note that like \( f(q,p) \), \( \hat{f}^{(s)} \) is also a commutative function of its arguments, and in this sense the Liouvillian superoperators defined here mimic the fundamental property of the corresponding phase space functions in \( \mathcal{F}(\mathcal{H}) \). \( \hat{O}_{nm}^{(s)} \) is the Liouvillian superoperator corresponding to the monomial \( q^n p^m \). For short, we write \( \hat{f}^{(s)}, \hat{O}_{nm}^{(s)} \) without denoting their arguments.

### III. WWGM QUANTIZATION

Let us denote by \( N = C^\infty(M) \) the vector space of functions defined over a phase space \( M \) and by \( \mathcal{F}(\mathcal{H}) \) the vector space of operators acting in a Hilbert space \( \mathcal{H} \). While with the usual pointwise product \( N \) becomes a commutative and associative algebra, \( \mathcal{F}(\mathcal{H}) \) becomes a noncommutative but associative algebra with respect to usual operator product. A noncommutative but associative algebra structure on \( N \) can be implemented by \(*-\)product; \( \ast_{(-s)} : N \times N \rightarrow N \), explicitly given by [11]

\[ \ast_{(-s)} = \exp \frac{1}{2} \hbar [(1 - s) \partial_p^L \partial_q^R - (1 + s) \partial_q^L \partial_p^R]. \]  

Here we take \( (q,p) \in \mathbb{R}^2 = M \) and use the convention that \( \partial^L \) and \( \partial^R \) are acting on the left (L) and on the right (R), respectively. Thus two different Lie algebras structure can be defined on \( N \); with respect to PB; \( \{,\}_{PB} : N \times N \rightarrow N \) defined by

\[ \{ f, g \}_{PB} = \partial_p f \partial_q g - \partial_q f \partial_p g, \]  

(henceforth the notation \( \partial_x \equiv \partial/\partial x \) will be used) and with respect to \( s\)-MB defined by

\[ \{ f_1(q,p), f_2(q,p) \}_{MB}^{(-s)} = f_1(q,p) \ast_{(-s)} f_2(q,p) - f_2(q,p) \ast_{(-s)} f_1(q,p). \]  

where \( f_1, f_2 \in N \). Let us denote these two Lie algebras by \( N_{PB} \) and \( N_{MB} \), where the subscribes PB and MB refer to the respective brackets. The relations (16) and (18) give the different expressions for the star product and Moyal brackets, that appeared in the literature separately in a unified manner and generalize them for an arbitrary \( s\)-ordering [8].

Despite these two different Lie algebra structure there is only one in \( \mathcal{F}(\mathcal{H}) \) defined with respect to the usual Lie Bracket [\( ,\)], which we denote by \( \mathcal{F}_{LB} \). This is (anti-)homomorphic to Lie algebra \( N_{MB} \) \( \{ q^n p^m, q^k p^l \}_{MB}^{(-s)} \rightarrow [\hat{f}_{nm}^{(s)}, \hat{f}_{kl}^{(s)}] \) (see Eq. (61) of the second paper of Ref. [8]). In the next section we will obtain a new quantum bracket which, quite in parallel with the Lie algebra structures in \( N \), enables us to define a new Lie algebra structure in \( \mathcal{F}(\mathcal{H}) \). Before doing that we have to recall some fundamental relations of the WWGM-quantization.

The above mentioned (anti-)homomorphism between \( N_{MB} \) and \( \mathcal{F}_{LB} \) is established via WWGM quantization rule symbolically defined by linear and invertible map \( \mathcal{M}_s : N \rightarrow \)).
\[ F(\mathcal{H}) \] with inverse \( M^{-1}_s : F(\mathcal{H}) \to N, \) such that \( M_s M^{-1}_s \) and \( M^{-1}_s M_s \) are identity transformations on \( F(\mathcal{H}) \) and \( N, \) respectively. Explicitly we write \( M_s(f) = \hat{F}^{(s)}, \) and \( M^{-1}_s(\hat{F}^{(s)}) = f \) where

\[
\hat{F}^{(s)}(\hat{q}, \hat{p}) = \hbar^{-1} \int \int f(q, p) \hat{\Delta}_{qp}(s) dq dp; \quad f(q, p) = Tr[\hat{F}^{(s)} \hat{\Delta}_{qp}(-s)] \tag{19}
\]

(All the integrals are from \(-\infty\) to \(+\infty\).) The first relation is an expansion of an operator in a complete continuous operator basis

\[
\hat{\Delta}_{qp}(s) = (\hbar/2\pi) \int \int e^{-i(\xi q + \eta p)} \hat{D}(s) d\xi d\eta, \tag{20}
\]

obeying the relations

\[
\int \int \hat{\Delta}_{qp}(s) dq dp = \hbar, \quad Tr[\hat{\Delta}_{qp}(s)] = 1. \tag{21}
\]

Here \( \hat{D}(s) = e^{-i\hbar s \xi /2} \exp(i(\xi \hat{q} + \eta \hat{p})) \) is the \( s \)-parametrized displacement operator. The basis operators \( \hat{\Delta}_{qp} \) are known as the Grossmann-Royer displaced parity operators \([12]\) for \( s = 0 \) and as the Kirkwood bases for \( s = \pm 1 \). Since they form complete operator bases, in the sense that any operator obeying certain conditions can be expanded in terms of them as in the first relation given by (19), they provide a unified approach to different quantization rules \([13,14]\). For special values \( s = 1, 0, -1 \) these are known, respectively, as the standard, the Wigner-Weyl, and the antistandard rules of associations \([3,14]\).

The second relation in Eq. (19) easily follows by multiplying both sides of the first relation by \( \hat{\Delta}_{qp'}(-s) \), and making use of the relation

\[
Tr[\hat{\Delta}_{qp}(s) \hat{\Delta}_{qp'}(-s)] = \hbar \delta(q - q')\delta(p - p'). \tag{22}
\]

Among other nice properties of the \( \hat{\Delta}(s) \) basis we quote the so called differential properties

\[
\partial_q \hat{\Delta}_{qp}(s) = -i \frac{\hbar}{\hbar} \hat{p} \hat{\Delta}_{qp}(s), \quad \partial_p \hat{\Delta}_{qp}(s) = i \frac{\hbar}{\hbar} \hat{q} \hat{\Delta}_{qp}(s) \tag{23}
\]

\[
q \hat{\Delta}_{qp}(s) = \frac{1}{2} T_{[\hat{q}]}(s) \hat{\Delta}_{qp}(s), \quad p \hat{\Delta}_{qp}(s) = \frac{1}{2} T_{[\hat{p}]}(-s) \hat{\Delta}_{qp}(s). \tag{24}
\]

These last relations can be generalized as

\[
q^n p^m \hat{\Delta}_{qp}(s) = \hat{O}_{nm}^{(s)}(\hat{\Delta}_{qp}(s)). \tag{25}
\]

As an illustration, taking the traces of both sides we have \( q^n p^m = Tr[\hat{O}_{nm}^{(s)} \hat{\Delta}_{qp}(-s)] \) which shows that \( M_s(q^n p^m) = \hat{t}_{nm}^{(s)}, \) or \( M^{-1}_s(\hat{t}_{nm}^{(s)}) = q^n p^m. \) More generally, for a function accepting power series expansion as in (14) we see that the corresponding operator in \( s \)-association given by (19) is obtained by simply replacing \( q^n p^m \) by \( \hat{t}_{nm}^{(s)}. \) For these kind of functions a generalization of (25) is

\[
f(q, p) \hat{\Delta}_{qp}(s) = \hat{F}^{(s)}(\hat{\Delta}_{qp}(s)). \tag{26}
\]

Now by multiplying both sides of this relation by another function \( g(q, p) \) we have
\[ g(q, p) f(q, p) \hat{\Delta}_{qp}(s) = \hat{\mathbf{f}}(s)[\hat{\mathbf{g}}(s)(\hat{\Delta}_{qp}(s))] = \hat{\mathbf{g}}(s)[\hat{\mathbf{f}}(s)(\hat{\Delta}_{qp}(s))]. \]

By taking the integral and trace of all sides and making use of the relations (21) we arrive at
\[ \hbar^{-1} \int \int g(q, p) f(q, p) \hat{\Delta}_{qp}(s) dq dp = \hat{\mathbf{f}}(s)[\hat{\mathbf{g}}(s)(\hat{I})] = \hat{\mathbf{g}}(s)[\hat{\mathbf{f}}(s)(\hat{I})] = \hat{\mathbf{f}}(s)(\hat{\mathbf{G}}(s)) = \hat{\mathbf{g}}(s)(\hat{\mathbf{F}}(s)), \]
(27)
\[ g(q, p) f(q, p) = Tr\{[\hat{\mathbf{g}}(s)(\hat{\mathbf{F}}(s))]\hat{\Delta}_{qp}(-s)\} = Tr\{[\hat{\mathbf{f}}(s)(\hat{\mathbf{G}}(s))]\hat{\Delta}_{qp}(-s)\}. \]
(28)

These two relations explicitly answer the first question stated in the introduction. Under the WWGM-association corresponding to an arbitrary s-ordering, to the product of two c-number functions there corresponds an operator which results by action of the Liouvillian superoperator form of one on the other. More formally, in accordance with (19) we obtain
\[ \mathcal{M}_s[g(q, p) f(q, p)] = \hat{\mathbf{g}}(s)(\hat{\mathbf{F}}(s)) = \hat{\mathbf{f}}(s)(\hat{\mathbf{G}}(s)). \]
(29)

As an example, by making use of Eq. (11) we have
\[ \mathcal{M}_s(q^{n_1} p^{m_1} q^{n_2} p^{m_2}) = \hat{\mathcal{O}}^{(s)}_{n_1 m_1} f^{(s)}_{n_2 m_2} = \hat{\mathcal{O}}^{(s)}_{n_2 m_2} f^{(s)}_{n_1 m_1} = \hat{f}^{(s)}_{n_1+n_2, m_1+m_2}. \]

Note that the result is, in general, different from the noncommutative product \( \hat{f}_{n_1 m_1} \hat{f}_{n_2 m_2} \), or from \( \hat{f}_{n_1 m_1} \hat{f}_{n_2 m_2} \).

**IV. DERIVATION OF THE NEW BRACKET**

Taking the derivatives of the second relation in Eq. (19) with respect to \( q \) (and \( p \)) and then making use of Eq. (23) we have
\[ \partial_q f(q, p) = Tr\{[\hat{\mathbf{f}}(s)]\partial_q(\hat{\Delta}_{qp}(-s))\} \]
\[ = -\frac{i}{\hbar} Tr\{[\text{ad}_q \hat{\mathbf{F}}(s)][\hat{\Delta}_{qp}(-s)]\}, \]
(30)
\[ \partial_q g(q, p) = Tr\{[\hat{\mathbf{g}}(s)]\partial_q(\hat{\Delta}_{qp}(-s))\} \]
\[ = \frac{i}{\hbar} Tr\{[\text{ad}_p \hat{\mathbf{G}}(s)][\hat{\Delta}_{qp}(-s)]\}, \]
(31)

where \( \text{ad}_A \) denotes the adjoint action: \( \text{ad}_A \hat{B} = [\hat{A}, \hat{B}] \). These relations show that, if \( \mathcal{M}_s(f) = \hat{\mathbf{F}}(s) \), then \( \mathcal{M}_s(\partial_q f) = -(i/\hbar)\text{ad}_q \hat{\mathbf{F}}(s) \) and \( \mathcal{M}_s(\partial_q f) = (i/\hbar)\text{ad}_p \hat{\mathbf{F}}(s) \). Now by multiplying both sides of Eq. (30) by \( \partial_q g \) and making use of (26) and then of (12) we have
\[ \partial_p f \partial_q g = -\frac{i}{\hbar} Tr\{[\text{ad}_q \hat{\mathbf{F}}(s)][\hat{\Delta}_{qp}(-s)]\}, \]
(32)
\[ = -\frac{i}{\hbar} Tr\{[\hat{\Delta}_{qp}(-s)]\hat{\mathbf{g}}^{(-s)}(\partial_q \hat{\mathbf{F}}(s))\].

By reversing the order of manipulations these relations can be rewritten as
\[ \partial_p f \partial_q g = \frac{i}{\hbar} Tr\{[\hat{\Delta}_{qp}(-s)]\hat{\mathbf{f}}^{(s)}(\partial_p \hat{\mathbf{G}}(s))\}. \]
(33)
where \( \hat{h}_x^{(s)} \) stands for the superoperator associated to the \( \partial_x h \). In a similar way we have

\[
\partial_q f \partial_p g = \frac{i}{\hbar} Tr[\hat{g}_q^{(s)}(\text{ad}_\hat{p})\hat{F}^{(s)}(s)] \hat{\Delta}_{qp}(-s) \tag{34}
\]

\[
= -\frac{i}{\hbar} Tr[\hat{f}_q^{(s)}(\text{ad}_\hat{q})\hat{G}^{(s)}(s)] \hat{\Delta}_{qp}(-s). \tag{35}
\]

Thus by combining Eqs. (32, 33) with (34, 35) we arrive at four differently looking but equivalent expressions for PB of two functions:

\[
\{f, g\}_PB = \frac{i}{\hbar} Tr\left[[\hat{g}_q^{(s)}(\text{ad}_\hat{p})\hat{F}^{(s)}(s)] + \hat{g}_p^{(s)}(\text{ad}_\hat{p})\hat{G}^{(s)}(s)] \hat{\Delta}_{qp}(-s)\right] \tag{36}
\]

\[
= \frac{i}{\hbar} Tr\left[[\hat{g}_q^{(s)}(\text{ad}_\hat{p})\hat{F}^{(s)}(s)] - \hat{f}_q^{(s)}(\text{ad}_\hat{q})\hat{G}^{(s)}(s)] \hat{\Delta}_{qp}(-s)\right] \tag{37}
\]

\[
= \frac{i}{\hbar} Tr\left[[\hat{f}_p^{(s)}(\text{ad}_\hat{p})\hat{G}^{(s)}(s)] - \hat{g}_p^{(s)}(\text{ad}_\hat{p})\hat{F}^{(s)}(s)] \hat{\Delta}_{qp}(-s)\right] \tag{38}
\]

\[
= \frac{i}{\hbar} Tr\left[[\hat{f}_q^{(s)}(\text{ad}_\hat{q})\hat{G}^{(s)}(s)] + \hat{f}_p^{(s)}(\text{ad}_\hat{p})\hat{F}^{(s)}(s)] \hat{\Delta}_{qp}(-s)\right]. \tag{39}
\]

These relations enable us to define a new bracket in \( \mathcal{F}(\mathcal{H}) \) which we denote by \([.,.\]^{(s)}_{PMB} ,\) and call it the Poisson-Moyal bracket (PMB). It is defined as the image of the PB under the WWGM-association:

\[
\mathcal{M}_s(\{f, g\}_PB) = [\hat{F}, \hat{G}]^{(s)}_{PMB}, \tag{40}
\]

and explicitly given by the following four equivalent expressions

\[
[\hat{F}, \hat{G}]^{(s)}_{PMB} = -\frac{i}{\hbar} [\hat{g}_q^{(s)}(\text{ad}_\hat{p})\hat{F}^{(s)}(s)] + \hat{g}_p^{(s)}(\text{ad}_\hat{p})\hat{F}^{(s)}(s)] \tag{41}
\]

\[
= -\frac{i}{\hbar} [\hat{g}_q^{(s)}(\text{ad}_\hat{p})\hat{F}^{(s)}(s)] - \hat{f}_q^{(s)}(\text{ad}_\hat{q})\hat{G}^{(s)}(s)] \tag{42}
\]

\[
= \frac{i}{\hbar} [\hat{f}_p^{(s)}(\text{ad}_\hat{p})\hat{G}^{(s)}(s)] - \hat{g}_p^{(s)}(\text{ad}_\hat{p})\hat{F}^{(s)}(s)] \tag{43}
\]

\[
= \frac{i}{\hbar} [\hat{f}_q^{(s)}(\text{ad}_\hat{q})\hat{G}^{(s)}(s)] + \hat{f}_p^{(s)}(\text{ad}_\hat{p})\hat{F}^{(s)}(s)]. \tag{44}
\]

Obviously, since the WWGM-association is linear, and PB is a Lie bracket i.e., bilinear, antisymmetric and obeying Jacobi identity, so is this new PMB. The four seemingly different but equivalent expressions of this new bracket correspond to trivially equivalent rearrangement of the terms in the right hand side of (17). In contrast, the equivalences of the Eqs. (41-44) are not so trivial.

In the case of many degrees of freedom, in the right hand sides of Eqs. (41-44) \( \hat{q} \) and \( \hat{p} \) are to be labeled with an index and summed over them. For instance, the second and third ones are to be as follows

\[
[\hat{F}, \hat{G}]^{(s)}_{PMB} = -\frac{i}{\hbar} \sum_i [\hat{g}_{q_i}^{(s)}(\text{ad}_{\hat{p}_i})\hat{F}^{(s)}(s)] - \hat{f}_{q_i}^{(s)}(\text{ad}_{\hat{q}_i})\hat{G}^{(s)}(s)] \tag{45}
\]

\[
= \frac{i}{\hbar} \sum_i [\hat{f}_{p_i}^{(s)}(\text{ad}_{\hat{p}_i})\hat{G}^{(s)}(s)] - \hat{g}_{p_i}^{(s)}(\text{ad}_{\hat{q}_i})\hat{F}^{(s)}(s)]. \tag{46}
\]
V. APPLICATIONS

As a general first application we take \( f(q, p) = q^n p^m \) and \( g(q, p) = q^k p^l \); \( n, m, k, l \) integers. These kind of monomials form a basis for the so called \( w_\infty \)-algebra with respect to PB:

\[
\{q^n p^m, q^k p^l\}_P = (mk - nl)q^{n+k-1}p^{m+l-1},
\]

and \( W_\infty \) algebra with respect to \( s - MB \)

\[
\{q^n p^m, q^k p^l\}_MB = \sum_{j=0}^{j_{\text{max}}} \frac{j^2}{j!} [\sum_{r=0}^{r_{\text{max}}} (\frac{j}{r}) f_{srj} a_{nmkl,rj}] q^{n+k-j}p^{m+l-j}.
\]

\( j_{\text{max}} = (n + r_{\text{max}}, l + r_{\text{max}}) \), \( a_{nmkl,rj} = \frac{n!m!k!l!}{(n + r - j)!(m - r)!(k - r)!(l + r - j)!} \).

The restrictions imposed on summations also follows from the expression of \( a_{nmkl,rj} \). In Eq. (44)

\[
f_{srj} = (s^-)^r (-s^+)^{j-r} - (s^-)^{j-r} (-s^+)^r,
\]

is the only factor depending on the chosen rule of ordering, where \( s^\pm = \hbar (1 \pm s)/2 \). The \( w_\infty \)-algebra is the algebra of canonical diffeomorphisms of a phase space that is topologically equivalent to \( \mathbb{R}^2 \), or, since the area element and symplectic form coincide in two dimensions, as the algebra of area preserving diffeomorphisms \( Diff_A \mathbb{R}^2 \). The above given \( W_\infty \)-algebra is quantum (or, \( \bar{\hbar} \)) deformation of this classical \( w_\infty \). More explicitly, one can easily show that

\[
\lim_{\hbar \to 0} (i\hbar)^{-\{ , \}} (-s)^{-s} = \{ , \}_PB.
\]

Now with respect to our new bracket we will obtain an algebra isomorphic to \( w_\infty \)-algebra. This will be denoted by \( F_{PMB} \). From (19) we obtain

\[
\hat{F}^{(s)} = \hat{i}_{nm}, \quad \hat{G}^{(s)} = \hat{i}_{kl},
\]

and by making use of (7)

\[
\text{ad}_q \hat{F}^{(s)} = i\hbar m \hat{i}_{n,m-1}^{(s)}, \quad \text{ad}_p \hat{F}^{(s)} = -i\hbar n \hat{i}_{n-1,m}^{(s)},
\]

\[
\text{ad}_q \hat{G}^{(s)} = i\hbar l \hat{i}_{k,l-1}^{(s)}, \quad \text{ad}_p \hat{G}^{(s)} = -i\hbar k \hat{i}_{k-1,l}^{(s)}.
\]

The corresponding superoperators are as follows

\[
\hat{f}^{(s)}_q = n2^{-(n+m-1)} T^{n-1}_{[\hat{q}]_{(s)}} T^{m}_{[\hat{p}]_{(-s)}} = n \hat{O}_{n-1,m}^{(s)},
\]

\[
\hat{f}^{(s)}_p = n \hat{O}_{n,m-1}^{(s)}, \quad \hat{g}^{(s)}_q = k \hat{O}_{k-1,l}^{(s)}, \quad \hat{g}^{(s)}_p = l \hat{O}_{k,l-1}^{(s)}.
\]
Substituting these relations in anyone of that given in Eq. (41-44) and by using the identities such as (see Eq. (11))

\[ \hat{O}^{(s)}_{k-1,l}(\hat{t}^{(s)}_{n,m-1}) = \hat{O}^{(s)}_{k,l-1}(\hat{t}^{(s)}_{n-1,m}) = \hat{t}^{(s)}_{n+k-1,m+l-1}, \]

we obtain

\[ [\hat{F}, \hat{G}]^{(s)}_{PMB} = (mk - nl)\hat{t}^{(s)}_{n+m-1}. \] (55)

Thus, by comparing with (47) we see that \( F_{PMB} \) is isomorphic to \( w_\infty \)-algebra. Because of (50), or, as can be directly verified, we have

\[ -\lim_{\hbar \to 0} (i\hbar)^{-1}[\_\_\_] = [\_\_\_]_{PMB}. \] (56)

provided that the same ordering convention is used in both sides.

There are some remarkable particular cases of this general application that deserve to be mentioned. \( W_\infty \)-algebra has some abelian and finite or infinite dimensional nonabelian subalgebras for which structure constants are proportional to the first power of \( i\bar{\hbar} \) [8]. These are generated by \( \hat{t}^{(s)}_{nm} \) such that : (i) \( n = 0 \), (ii) \( m = 0 \), (iii) \( n = m \) (Cartan subalgebra) [16], (iv) \( n + m \leq 1 \) (HW-algebra), (v) \( n + m = 2 \) (symplectic algebra \( sp(2) \)), (vi) \( n + m \leq 2 \) (inhomogeneous symplectic algebra \( isp(2) \)), (vii) \( m = 1 \), (viii) \( n = 1 \). The first three are infinite dimensional abelian subalgebras and the last two are isomorphic copy of the well known centerless Virasoro algebra [15]. For all these subalgebras Eq. (56) is of the form

\[ -(i\hbar)^{-1}[\_\_\_] = [\_\_\_]_{PMB}. \]

As a second general application we will carry the classical Hamiltonian equations of motion

\[ \dot{q} = -\{q, H\}_{PB} = \partial_p H, \quad \dot{p} = -\{p, H\}_{PB} = -\partial_q H \] (57)

to \( \mathcal{F}(H) \). Here \( H \equiv H(q,p) \) is the classical Hamiltonian and \( t \) being the time parameter \( \dot{a} \equiv da/dt \). Now applying \( \mathcal{M}_s \) to both sides of Eq. (57) we obtain

\[ \dot{\hat{q}} = \frac{1}{i\hbar}[\hat{q}, \hat{H}^{(s)}], \quad \dot{\hat{p}} = \frac{1}{i\hbar}[\hat{p}, \hat{H}^{(s)}] \] (58)

here \( \hat{H}^{(s)} = \mathcal{M}_s(H) \), and we used the fact that

\[ [\hat{q}, \hat{H}^{(s)}]_{PMB} = (i/\hbar)[\hat{q}, \hat{H}^{(s)}], \quad [\hat{q}, \hat{H}^{(s)}]_{PMB} = (i/\hbar)[\hat{p}, \hat{H}^{(s)}]. \] (59)

Notice that \( \hat{H}^{(s)} \) is Hermitian only for pure imaginary values of \( s \), that is \( (\hat{H}^{(s)})^\dagger = \hat{H}^{(-s)}. \) In particular, when \( H = (p^2/2m) + V(q) \) Eqs. (58) are of the form (Ehrenfest’s Theorem)

\[ \dot{\hat{q}} = \frac{\hat{p}}{m}, \quad \dot{\hat{p}} = -\frac{i}{\hbar}[\hat{p}, \hat{V}(\hat{q})]. \] (60)

Note that these are independent from \( s \).

We would like to call Eqs. (58) the operator form of the Hamilton equations. Assume that the operators belong to Heisenberg picture, these equations are identical to Heisenberg picture equations of motion that can be obtained from
\[
\frac{d\hat{A}_H}{dt} = \frac{\partial \hat{A}_H}{\partial t} + \frac{1}{i\hbar} [\hat{A}_H, \hat{H}],
\]
(61)

by taking \(\hat{A}_H = \hat{q}, \hat{p}\) and \(\hat{H} = \hat{H}^{(s)}\). Here the subscribe \(H\) refers to the Heisenberg picture in which the state vectors are time-independent and the dynamical variables are time-dependent. We should note that the first term in the right side of Eq. (61) is defined as follows [17]

\[
\frac{\partial \hat{A}_H}{\partial t} \equiv (\frac{\partial \hat{A}}{\partial t})_H = \hat{U} \frac{\partial \hat{A}_S}{\partial t} \hat{U}^\dagger,
\]
(62)

where \(\hat{U} = \exp(it\hat{H}/\hbar)\) is the evolution operator and the subscribe \(S\) refers to the Schrödinger picture in which the state vectors are time-dependent and dynamical variables are time-independent (except for a possible explicit time dependence, which is not the case for the position and momentum operators in the Schrödinger picture).

As a result, as far as the dynamics of \(\hat{q}\) and \(\hat{p}\) are concerned, the WWGM-association directly maps the classical Hamilton equations of motion on the Heisenberg picture equations of motion for general Hamiltonian \(\hat{H}^{(s)} = \mathcal{M}_{s}(H)\). Arrival to the Schrödinger equation (see the next section) for the time evolution of states vectors is straightforward by making use of the evolution operator \(\hat{U}\) in the case of \(\bar{s} = -s\). Despite of this application, the fact that the image of the classical mechanics under the WWGM-association is not identical to the Heisenberg picture formulation of the conventional quantum mechanics is made apparent in the next application.

Finally, we consider the equation describing the time evolution of a phase space function \(f \equiv f(q, p; t)\)

\[
\dot{f} = \partial_t f + \{H, f\}_{PB},
\]
(63)

associated with a system described by \(H\). The corresponding equation in \(\mathcal{F}(\mathcal{H})\) is as follows

\[
\dot{\hat{F}}^{(s)} = \partial_t \hat{F}^{(s)} + [\hat{H}, \hat{F}^{(s)}]_{PMB},
\]
(64)

where \(\hat{F}^{(s)} = \mathcal{M}_{s}(f)\). In particular, for \(H = (p^2/2m) + V(q)\), \(f = f(q)\) and \(g = g(p)\) the equations are

\[
\dot{\hat{F}}^{(s)} = \frac{1}{m} \hat{p}^{(s)}(\hat{p}), \quad \dot{\hat{G}}^{(s)} = -\frac{i}{\hbar} \hat{g}^{(s)}(\hat{p}, \hat{V}(\hat{q})].
\]
(65)

Note that the operator form of Hamilton equations are particular case of these last equations. The distinction between the Heisenberg picture formulation of the quantum mechanics and the image of the Hamilton formulation of the classical mechanics is made manifest by (64) by appearance of PMB instead of [ ].

VI. CONCLUSION AND DISCUSSION

The conventional way for finding a quantum system that reduce to a specified classical system in the classical limit is to write the classical Hamilton equations in terms
of the PB and then to replace the PB with commutator brackets in accordance with \( \{f, g\}_\text{PB} \rightarrow (\ii)_{\text{PB}}^{-1}\{\hat{F}, \hat{G}\} \). Although this construction suffers from the obstructions stated in the introduction, the association \( \{f, g\}_\text{PB} \rightarrow [\hat{F}, \hat{G}]_{\text{PMB}}^{(s)} \) is free from them. Note that \( -(\ii)_{\text{PMB}}^{-1}\{\_, \_\}^{(s)} \) reduces to the commutator bracket when one of the entry is \( \hat{q} \), or \( \hat{p} \). A bit more generally, when \( \hat{F} = a\hat{q} + b\hat{p} + c\hat{I} \), a general element of the H-W-algebra, then \( -[\hat{F}, \hat{H}]_{\text{PMB}}^{(s)} = (\ii)_{\text{PMB}}^{-1}\{\hat{F}, \hat{H}^{(s)}\} \). Thus, for time-independent Hamiltonian, as the solutions of Eqs. (58), the time evolution of the basic observables are given by \( \hat{q}(t) = \hat{U}(t, s)\hat{q}(0)\hat{U}(-t, s) \), \( \hat{p}(t) = \hat{U}(t, s)\hat{p}(0)\hat{U}(-t, s) \). Here \( \hat{q}(0) \), and \( \hat{p}(0) \) are time independent position and momentum operators and \( \hat{U}(t, s) = \exp(\ii t\hat{H}^{(s)}/\hbar) \). By noting that \( \hat{U}(t, s) \) is unitary only when \( \bar{s} = -s \), these are the same as that in the Heisenberg picture if \( \hat{q}(0) \) and \( \hat{p}(0) \) are considered to be in the Schrödinger picture and if \( \bar{s} = -s \). In that case, by assuming time-independent state vector \( |\psi(0)\rangle \in \mathcal{H} \) in the Heisenberg picture such that \( \langle \psi(0)|\hat{p}(t)|\psi(0)\rangle = \langle \psi(t)|\hat{p}(0)|\psi(t)\rangle \), we obtain time-dependent state vector \( |\psi(t)\rangle = \hat{U}(-t, s)|\psi(0)\rangle \) obeying the dynamics \( \ii\hbar\partial_{t}|\psi(t)\rangle = \hat{H}(\hat{\psi})(\hat{\psi}) \), i.e., the Schrödinger equation.

On the other hand, while the time evolution of a general observable (not explicitly time-dependent) is given in the Heisenberg picture by \( \hat{F}_H(t) = \hat{U}(t, s)\hat{F}_H(0)\hat{U}(-t, s) \), it is not so in the association scheme \( \text{PB} \rightarrow \text{PMB} \). Instead, if we define \( \text{ad}_{\hat{A}}^{(s)} \) by \( \text{ad}_{\hat{A}}^{(s)} = [[\hat{A}, \hat{B}]_{\text{PMB}}^{(s)} = \{\hat{A}, \hat{B}\}_{\text{PMB}}^{(s)} \), then the time evolution governed by Eq. (64) can be written as \( \hat{F}^{(s)}(t) = \exp(\ii t\text{ad}_{\hat{A}}^{(s)})\hat{F}^{(s)}(0) \) which, because of (50) or (56), is the limiting case of the above given relation.

In the sense described above, the conventional canonical quantization itself can be thought as an \( \hbar \) deformation of the quantization by PMB. This fact is made manifest by the following diagrams:

(i) Hierarchy of Products

\[
f \ast g \neq g \ast f \leq WLGM \rightarrow \text{association} \Rightarrow \hat{F} \ast \hat{G} \neq \hat{G} \ast \hat{F}
\]

(ii) Hierarchy of Brackets

\[
\{f, g\}_\text{PB} \leq \text{association} \Rightarrow \{f, g\}_\text{PMB}
\]

These two diagrams schematically summarize the main points of this report, and exhibit the hierarchies of the products and brackets involved. Here cont. and def. stand for contraction and deformation, respectively and, for the sake of simplicity the commutative operator product derived in Eq. (29) is shown here by \( \tilde{\text{f}}(\tilde{G}) \equiv \hat{F} \ast \hat{G} \). Note that the \( \forall \) product (like the \( \ast \) product and the MB) depends on the ordering parameter \( \bar{s} \), but this is not shown in the diagrams for the same reason. In summary, what we have done here may be considered as the “contraction quantization”, or, as the formulation of the classical mechanics in the ring of operators.
ACKNOWLEDGMENTS

This report has been written during my visit to the Feza Gürsey Institute to which I would like to express my profound gratitude for the hospitality. Also I wish to thank T. Dereli and Ö. F. Dayı for helpful discussions. This work was supported in part by the Scientific and Technical Research Council of Turkey (TÜBITAK).
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[10] In this report we want to map a specified structure defined on $N$ to $\mathcal{F}(\mathcal{H})$. But, for different mapping rules different operators correspond to the same function. To distinguish them we label them by superscript $(s)$. Conversely when the main goal is to map a specified structure defined in $\mathcal{F}(\mathcal{H})$ to $N$, different functions corresponding to the same operator are to be distinguished.

[11] $N[[\nu]]$ being the space of formal power series in the parameter $\nu$ with coefficients in $N$, both the star product and MB can be considered as $N[[\nu]] \times N[[\nu]] \rightarrow N[[\nu]]$. The so called deformation parameter $\nu$ in physical applications corresponds to $\hbar/2$.

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