A remark on black holes of Chern-Simons gravities in 2n + 1 dimensions: n = 1, 2, 3

D. H. Tchrakian‡*

‡School of Theoretical Physics, Dublin Institute for Advanced Studies, 10 Burlington Road, Dublin 4, Ireland
*Department of Mathematical Physics, National University of Ireland Maynooth, Maynooth, Ireland

Abstract

It is shown that Chern-Simons gravities in 2n + 1 dimensions admit solutions described by the same lapse function which describes the BTZ black hole in the n = 1 case. This has been carried out explicitly for n = 1, 2, 3. Moreover, it is seen that these solutions are unique.
1 Introduction

Chern-Simons gravities (CSG) derived from non-Abelian Chern-Simons (CS) densities in 2 + 1 dimensions were proposed by Witten in Ref. [1] and they were extended to all odd dimensions by Chamseddine in Refs. [2, 3]. The CSG systems consist of superpositions of gravitational Lagrangians of all possible higher order gravities in the given dimensions, each appearing with a precise real numerical coefficient resulting from the calculus.

The construction of Chern-Simons gravities employs the usual non-Abelian (nA) Chern-Simons (CS) densities, the gauge group being chosen suitably. Non-Abelian CS densities are defined only in odd dimensions since they are derived via a one-step descent [4] from the corresponding Chern-Pontryagin (CP) density in one dimension higher, in even dimensions 1.

Now the CP density for a nA field is the trace of an antisymmetrised product of \(n + 1\) curvature tensors, so it follows that the corresponding CS density resulting from the one-step descent is also the trace of the product of the nA curvatures, but now also of the connections. Thus, unlike the CP density which is gauge invariant, the CS density is gauge invariant [4]. This is the salient contrast between CP and CS densities.

The nA CS densities in question, in \(d = 2n + 1\) dimensions, are derived from CP densities in one dimension higher for nA gauge group \(SO(2n + 2) = SO(d + 1)\). Concretely, we adopt the Dirac gamma matrix representations for the generators

\[
\gamma_{ab} = -\frac{1}{4} \varepsilon_{[a|\gamma_b]} , \quad a = 1, 2, \ldots, d + 1
\]

for the nA \(SO(2n + 2) = SO(d + 1)\) connections. Since this orthogonal group is even, \(2n + 2\) dimensional, in addition to \(\gamma_a\), there exists also a chiral matrix \(\gamma_{2n+3} = \gamma_{d+2}\). Thus in the construction of the CP and CS densities we have the option of inserting the chiral matrix \(\gamma_{2n+3}\) under the trace defining these densities, and anticipating the passage from nA gauge systems to gravitational systems we exercise this option and insert the chiral matrix under the trace defining these densities. The reason for this is that the resulting gravitational systems, described by all Einstein-Hilbert (EH) Lagrangians, must be Euler type and not Pontryagin type densities.

Since most of the concrete work here will be carried out for \(d = 3, 5, 7\), i.e., for \(n = 1, 2, 3\), we list these \(SO(d + 1)\) CS densities \(\Omega^{(n)}_{CS}\)

\[
\Omega^{(2)}_{CS} = \varepsilon^{\lambda\mu\nu}\text{Tr} \gamma_5 A_{\lambda} \left[ F_{\mu\nu} - \frac{2}{3} A_{\mu} A_{\nu} \right] \\
\Omega^{(3)}_{CS} = \varepsilon^{\lambda\mu\rho\sigma}\text{Tr} \gamma_7 A_{\lambda} \left[ F_{\mu\rho} F_{\sigma\sigma} - F_{\mu\rho} A_{\rho} A_{\sigma} + \frac{2}{3} A_{\mu} A_{\nu} A_{\rho} A_{\sigma} \right] \\
\Omega^{(4)}_{CS} = \varepsilon^{\lambda\mu\rho\sigma\tau\kappa}\text{Tr} \gamma_9 A_{\lambda} \left[ F_{\mu\rho} F_{\sigma\tau} F_{\tau\kappa} - \frac{4}{3} F_{\mu\rho} F_{\sigma\tau} A_{\tau} A_{\kappa} - \frac{2}{3} F_{\mu\rho} A_{\rho} A_{\sigma} F_{\tau\kappa} \\
+ \frac{4}{3} F_{\mu\tau} A_{\rho} A_{\sigma} A_{\tau} A_{\kappa} - \frac{8}{35} A_{\mu} A_{\nu} A_{\rho} A_{\sigma} A_{\tau} A_{\kappa} \right] ,
\]

1The space in which the CS density is defined can be a Minkowskian spacetime as in the present case, or, it can be a Euclidean space when the volume integral of the CS density plays the role of the Hopfion charge in a Skyrme-Fadde’ev model.
which we ill employ in the construction of the Chern-Simons gravitational system in these dimensions.

The main aim of the present work is to construct the balck hole solutions of the Chern-Simons gravities (CSG) concretely in \( d = 3, 5, 7 \) dimensions, and to show that these are described uniquely by the same lapse function \( N(r) \) descriing the Banados \textit{et al.} [5] black hole (BH) in \( d = 3 \). It is natural to conjecture thereafter that the same \( N(r) \) describes the BH solutions to CSG systems in all dimensions \( d = 2n + 1 \).

2 Chern-Simons gravities in \( d = 3, 5, 7 \) dimensions

The Chern-Simons gravities (CSG) are described by the most general superposition of higher order gravities, in a given dimension. We refer to the higher order gravities as \( p \)-Einstei-Hilbert (\( p \)-EH) gravities, the integer \( p \) specifying the number of times the Riemann curvature appears in the given Lagrangian \( L_{\text{EH}}^{(p,d)} \). Employing the \textit{Vielbeine} \( e^a_\mu \), with \( \mu = 1, 2, \ldots, d \) labelling the coordinate indices and \( a = 1, 2, \ldots, d \) labelling the frame indices, the \( p \)-EH Lagrangian is defined as

\[
L_{\text{EH}}^{(p,d)} = \varepsilon^{\mu_1\mu_2\ldots\mu_{2p}\nu_1\nu_2\ldots\nu_{d-2p}} e_{a_1 a_2 \ldots a_{2p}b_1 b_2 \ldots b_{d-2p}} e^{b_1 \nu_1} e^{b_2 \nu_2} \ldots e^{b_{d-2p} \nu_{d-2p}} R^{a_1 a_2}_{\mu_1 \mu_2} R^{a_3 a_4}_{\mu_3 \mu_4} \ldots R^{a_{2p-1} a_{2p}}_{\mu_{2p-1} \mu_{2p}}. \tag{4}
\]

Clearly, in the case \( d - 2p = 0 \), the density (4) is a total divergence, and non-trivial Lagrangians are only those for \( d > 2p \). For \( p = 0 \) it is the cosmological term.

Like all purely gravitational systems, the Chern-Simons gravities (CSG) support only black hole (BH) solutions, the best known BH solutions being those found by Banados \textit{et al.} [5] in \( d = 2 + 1 \) dimensions. It is our aim here to extend that result to the next two odd dimensions, \( d = 5 \) and \( d = 7 \), restricted to the radially symmetric case.

We start from the non Abelian (nA) Chern-Simons (CS) densities (1)-(3) in \( d = 3, 5, 7 \) dimensions. To pass from a nA gauge system to gravity we employ the prescription

\[
A_\mu = -\frac{1}{2} \omega^{ab}_\mu \gamma_{ab} + \kappa e^a_\mu \gamma_{a,d+1} \Rightarrow F_{\mu\nu} = -\frac{1}{2} \left( R^{a}_{\mu\nu} - \kappa^2 e^a_{[\mu} e^b_{\nu]} \right) \gamma_{ab}, \tag{5}
\]

restricting our attention to torsionless models. The prescription (5) relates the nA connection and curvature \( (A_\mu, F_{\mu\nu}) \) to the spin-connection and Riemann curvature \( (\omega^{ab}_\mu, R^{ab}_{\mu\nu}) \). The constant \( \kappa \) has dimensions of \( L^{-1} \) and is chosen to be real such that the resulting gravitational system be a dS model. To obtain a AdS model, one just makes the replacement \( \kappa^2 \rightarrow -\kappa^2 \).

Substituting (5) in (1), (2) and (3) and evaluating the traces yields the corresponding Chern-
Simons–Gravitational (CSG) Lagrangians $L_{\text{CSG}}^{(d)}$ in $d = 3, 5, 7$

\begin{align}
L_{\text{CSG}}^{(3)} &= -\kappa \varepsilon^{\mu \nu \lambda} \varepsilon_{abc} \left( R_{\mu \nu}^{ab} - \frac{2}{3} \kappa^2 e_{\mu}^{a} e_{\nu}^{b} \right) e_{\lambda}^{c} \\
&= -\kappa \left( L_{\text{EH}}^{(1,3)} - \frac{2}{3} \kappa^2 L_{\text{EH}}^{(0,3)} \right) \\
L_{\text{CSG}}^{(5)} &= \kappa \varepsilon^{\mu \nu \rho \sigma \lambda} \varepsilon_{abcde} \left( \frac{3}{4} R_{\mu \nu}^{ab} R_{\rho \sigma}^{cd} - \kappa^2 R_{\mu \nu}^{ab} e_{\rho}^{c} e_{\sigma}^{d} + \frac{3}{5} \kappa^4 e_{\mu}^{a} e_{\nu}^{b} e_{\rho}^{c} e_{\sigma}^{d} \right) e_{\lambda}^{e} \\
&= \kappa \left( \frac{3}{4} L_{\text{EH}}^{(2,5)} - \kappa^2 L_{\text{EH}}^{(1,5)} + \frac{3}{5} \kappa^4 L_{\text{EH}}^{(0,5)} \right) \\
L_{\text{CSG}}^{(7)} &= -\kappa \varepsilon^{\mu \nu \rho \sigma \tau \kappa} \varepsilon_{abcdefg} \left( \frac{1}{8} R_{\mu \nu}^{ab} R_{\rho \sigma}^{cd} R_{\tau \kappa}^{ef} - \frac{1}{4} \kappa^2 R_{\mu \nu}^{ab} R_{\rho \sigma}^{cd} e_{\tau}^{e} e_{\kappa}^{f} \\
&\quad + \frac{3}{10} \kappa^4 R_{\mu \nu}^{ab} e_{\rho}^{c} e_{\sigma}^{d} e_{\tau}^{e} e_{\kappa}^{f} - \frac{1}{7} \kappa^6 e_{\mu}^{a} e_{\nu}^{b} e_{\rho}^{c} e_{\sigma}^{d} e_{\tau}^{e} e_{\kappa}^{f} \right) e_{\lambda}^{g} \\
&= -\kappa \left( \frac{1}{8} L_{\text{EH}}^{(3,7)} - \frac{1}{4} \kappa^2 L_{\text{EH}}^{(2,7)} + \frac{3}{10} \kappa^4 L_{\text{EH}}^{(1,7)} - \frac{1}{7} \kappa^6 L_{\text{EH}}^{(0,7)} \right)
\end{align}

The $p$-Einstein-Hilbert Lagrangians $L_{\text{EH}}^{(p,d)}$ in (7), (9) and (11) are those defined by (4). They will be subjected to static radial symmetry and the resulting one dimensional reduced densities will be subjected to variations. Solving the resulting equations of motion will yield the sought after BH solutions.

The equivalent expressions (6), (8) and (10) for $L_{\text{CSG}}^{(d)}$ are also useful, in that they afford a transparent expression for the Euler-Lagrange equations resulting from the variation w.r.t. the Vielbein. Employing the short hand notation

\[
\bar{R}_{\mu \nu}^{ab} = R_{\mu \nu}^{ab} - \kappa^2 e_{[\mu}^{a} e_{\nu]}^{b}
\]

the Einstein equations resulting from CSG Lagrangians (6), (8) and (10) are

\begin{align}
-\kappa \varepsilon^{\mu \nu \lambda} \varepsilon_{abc} \bar{R}_{\mu \nu}^{ab} &= 0 \\
\frac{3}{4} \kappa \varepsilon^{\mu \nu \rho \sigma \lambda} \varepsilon_{abcde} \bar{R}_{\mu \nu}^{ab} \bar{R}_{\rho \sigma}^{cd} &= 0 \\
-\frac{1}{8} \kappa \varepsilon^{\mu \nu \rho \sigma \tau \kappa} \varepsilon_{abcdefg} \bar{R}_{\mu \nu}^{ab} \bar{R}_{\rho \sigma}^{cd} \bar{R}_{\tau \kappa}^{ef} &= 0
\end{align}

which are all gauge covariant equations inspite of the fact that the (gravitational) Chern-Simons Lagrangians they result from are themselves gauge variant. This is expected and it is also clear that the Einstein equation of the CSG in $2n + 1$ dimensions is simply

\[
\bar{R} \wedge \bar{R} \cdots \wedge \bar{R} = 0 , \quad n \text{ times}.
\]
2.1 Spherically symmetric black hole solutions

We now impose static radial symmetry via the generic spherically symmetric line-element in terms of two unknown functions \( N(r) \) and \( \sigma(r) \), with

\[
ds^2 = \frac{dr^2}{N(r)} + r^2 d\Omega_{d-2}^2 - \sigma(r)^2 N(r) dt^2,
\]

(16)

where \( r, t \) are the radial and time coordinates, while \( d\Omega_{d-2}^2 \) denotes the unit metric on \( S^{d-2} \).

The reduced one dimensional Lagrangians \( L_{EH}^{(p,d)} \) pertaining to \( L_{EH}^{(p,d)} \) defined by (4), which are systematically calculated in a slightly different normalisation in Refs. [7, 8, 9], are given here as

\[
L_{EH}^{(p,d)} = (d-2)!(d-2p) \sigma(r) \frac{d}{dr} [r^{d-2p-1}(1 - N(r))^p] .
\]

(17)

The \( p \)-Einstein equations, namely the Euler-Lagrange equations resultin g from the variation of (17) w.r.t. \( N \) and \( \sigma \) yield, respectively

\[
\sigma = 1 , \quad \text{and} \quad (d-2)!(d-2p) \sigma(r) \frac{d}{dr} [r^{d-2p-1}(1 - N(r))^p] = 0 .
\]

(18)

We now replace each term \( L_{EH}^{(p,d)} \) in (7), (9) and (11) by the appropriate one-dimensional \( L_{EH}^{(p,d)} \) given by (17). The result is strikingly simple, yielding the same expression for each \((p,d)\),

\[
L_{CSG}^{(d)} \simeq (-1)^n \kappa \sigma \frac{d}{dr} [(1 - N) - 2\kappa^2 r^2]^n , \quad n = 1, 2, 3 ,
\]

(19)

up to a numerical constant, which can be evaluated only when the CSG Lagrangian is given. In the cases (7), (9) and (11) at hand these coefficients are \((3!)\), \((\frac{9}{2} \cdot 5!)\) and \((15 \cdot 7!)\) for \(d = 3, d = 5\) and \(d = 7\) respectively.

Subjecting (19) to variations w.r.t. \( \sigma \) lead to the quantity in the square bracket in (19) being equal to a constant, i.e., we have the unique solution

\[
N = c - 2\kappa^2 r^2 ,
\]

(20)

where \( c \) is an integration constant, and \( c = 1 \) corresponds to the (A)dS space depending on the sign of \( \kappa^2 \). In view of the unique result (20) for \( d = 3, 5, 7 \), it is reasonable to conjecture that the solution (20) holds for all \( d = 2n + 1 \).

Next, subject (19) to variations w.r.t. \( N \). This yields

\[
[(1 - N) - 2\kappa^2 r^2]^n \sigma' = 0 .
\]

(21)

In dimensions \( d \geq 5 \), (21) does not force \( \sigma' = 0 \) and \( \sigma = 1 \). There are then two options, depending on whether the square bracket in (21) vanishes or not. Now the solution (20) describes a AdS black hole when \( c < 0 \) and \( \kappa^2 < 0 \), in which case the square bracket in (21) does not vanish and hence \( \sigma = 1 \). These are the black hole solutions of CS gravity in \( d = 2n + 1 \) claimed here, which for \( d = 3 \) has been found by Banados et. al. [5].
Alternatively, one may opt for the solution of (21) when instead the square bracket vanishes, in which case \( \sigma(r) \) is not fixed and \( c = 1 \) in (20). These solutions, which appear only in dimensions \( d \geq 5 \), are the 'special degenerate vacuum solutions' given in Ref. [10].

The situation here, where the black holes in all odd dimensions are described by the same lapse function \( N(r) \) given by (20), is reminiscent of the unit charge BPST instantons [11] of the Yang-Mills (YM) system on \( \mathbb{R}^4 \) and the instantons [12] of the \( p \)-YM systems on \( \mathbb{R}^{4p} \), which are described by the same radial structure function \( w(r) \) for all \( p \). This situation holds also for the solitons of the \( O(2p+1) \) Skyrme systems on \( \mathbb{R}^{2p} \), where the Belavin-Polyakov vortices of the \( O(3) \) sigma model on \( \mathbb{R}^2 \) [13] and the unit charge Skyrmions of the \( O(2p+1) \) \( (p \geq 2) \) Skyrme systems on \( \mathbb{R}^{2p} \) [14], are all described by the same chiral function \( f(r) \).

3 Summary

The Chern-Simons gravitational (CSG) Lagrangians \( \mathcal{L}^{(d)}_{\text{CSG}} \) in dimensions \( d = 1, 2, 3 \), listed in (7), (9) and (11), are expressed in terms of the usual \( p \)-Einstein-Hilbert (\( p \)-EH) Lagrangians \( \mathcal{L}^{(p,d)}_{\text{EH}} \) defined by (1) in \( d \) dimensions. Then, each \( \mathcal{L}^{(p,d)}_{\text{EH}} \) is subjected to static radial symmetry and the result substituted in the corresponding CSG Lagrangian (7), (9) or (11). The resulting Euler-Lagrange equations yield the unique black hole solution \( N(r) \) (20), which is the lapse function describing the BTZ [5] black hole. Based on this, it may be reasonable to conjecture this result to hold for all \( d = 2n + 1 \).

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