Abstract

In scattering experiments, physicists observe so-called resonances as peaks at certain energy values in the measured scattering cross sections per solid angle. These peaks are usually associated with certain scattering processes, e.g., emission, absorption, or excitation of certain particles and systems. On the other hand, mathematicians define resonances as poles of an analytic continuation of the resolvent operator through complex dilations. A major challenge is to relate these scattering and resonance theoretical notions, e.g., to prove that the poles of the resolvent operator induce the above-mentioned peaks in the scattering matrix. In the case of quantum mechanics, this problem was addressed in numerous works that culminated in Simon’s seminal paper [33] in which a general solution was presented for a large class of pair potentials. However, in quantum field theory the analogous problem has been open for several decades despite the fact that scattering and resonance theories have been well-developed for many models. In certain regimes these models describe very fundamental phenomena, such as emission and absorption of photons by atoms, from which quantum mechanics originated. In this work we present a first non-perturbative formula that relates the scattering matrix to the resolvent operator in the massless Spin-Boson model. This result can be seen as a major progress compared to our previous works [14] and [12] in which we only managed to derive a perturbative formula.

Keywords: Scattering Theory; Resonance Theory; Spin-Boson Model; Multiscale Analysis

1 Introduction

In this work we analyze the massless Spin-Boson model which describes a two-level atom interacting with a second-quantized massless scalar field. We derive a non-perturbative expression of the scattering matrix in terms of the resolvent operator for one-boson
processes, and thus, prove an analogous result that was obtained by Simon in [33] for the N-body Schrödinger operator in this particular model of quantum field theory. More precisely, we show that the pole of a meromorphic continuation of the integral kernel of the scattering matrix is located precisely at the resonance energy. The objective in this result is to contribute to the understanding of the relation between resonance and scattering theory. In our previous works [14] and [12], we were already able to derive perturbative results of this kind in case of the massless and massive Spin-Boson models, respectively. However, both results are only given in leading order with respect to the coupling constant. The present work can be seen as a major improvement of these perturbative results because it provides a closed and non-perturbative formula that connects the integral kernel of the scattering matrix elements for one-boson processes in terms of the dilated resolvent.

Our results are based on the well-established fields of scattering and resonance theories and the numerous works in the classical literature of which we want to give a short overview here. Resonance theory, in the realm of quantum field theory, has been developed in a variety of models; see, e.g., [6, 8, 7, 4, 9, 11, 26, 27, 32, 21, 15, 25, 29, 23, 13]. In these works, several techniques have been invented for massless models of quantum field theory in order to cope with the absence of a spectral gap. Scattering theory has also been developed in various models of quantum field theory (see, e.g., [23, 22, 16, 25, 24]) and in particular in the massless Spin-Boson model (see, e.g., [17, 18, 19, 20, 10, 14, 12]). In [5], a rigorous mathematical justification of Bohr’s frequency condition was derived using an expansion of the scattering amplitudes with respect to powers of the fine structure constant for the Pauli-Fierz model. In [10], the photoelectric effect has been studied for a model of an atom with a single bound state, coupled to the quantized electromagnetic field. A related problem is studying the time-evolution in models of quantum field theory. In [11], this question has been addressed for the Spin-Boson model. A good overview has been given in [34].

This work heavily relies on the multiscale analysis carried out in [13] as well as on the results in [14]. We do not repeat any of those proofs here but rather focus on the core argument to derive the above mentioned non-perturbative formula. However, throughout this work, we give precise references to any of the utilized theorems and lemmas which also contain all technical details.

1.1 The Spin-Boson model

In this section we introduce the considered model and state preliminary definitions, well-known tools and facts from which we start our analysis. If the reader is already familiar with the introductory Sections 1.1 until 1.3 of [13], these subsections can be skipped.

The non-interacting Spin-Boson Hamiltonian is defined as

$$H_0 := K + H_f, \quad K := \begin{pmatrix} e_1 & 0 \\ 0 & e_0 \end{pmatrix}, \quad H_f := \int d^3 k \omega(k) a(k)^* a(k).$$

(1.1)

We regard $K$ as an idealized free Hamiltonian of a two-level atom. As already stated in the introduction, its two energy levels are denoted by the real numbers $0 = e_0 < e_1$. 

2
Furthermore, $H_f$ denotes the free Hamiltonian of a massless scalar field having dispersion relation $\omega(k) = |k|$, and $a, a^{*}$ are the annihilation and creation operators on the standard Fock space. For a precise definition we refer to [14, Section 1.1]. Below, we sometimes call $K$ the atomic part, and $H_f$ the free field part of the Hamiltonian. The sum of the free two-level atom Hamiltonian $K$ and the free field Hamiltonian $H_f$ is named “free Hamiltonian” $H_0$. The interaction term reads

$$V := \sigma_1 \otimes (a(f) + a(f)^*), \quad \sigma_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

where the boson form factor is given by

$$f : \mathbb{R}^3 \setminus \{0\} \to \mathbb{R}, \quad k \mapsto e^{-\frac{k^2}{2\Lambda^2}|k|^{-\frac{1}{2}} + \mu}. \quad (1.3)$$

In our case, the gaussian factor in (1.3) acts as an ultraviolet cut-off for $\Lambda > 0$ being the ultraviolet cut-off parameter and in addition the fixed number

$$\mu \in (0, 1/2) \quad (1.4)$$

yields a regularization of the infrared singularity at $k = 0$ which is a technical assumption chosen such that we can apply the results obtained in [13]. Note that the form factor $f$ only depends on the radial part of $k$. To emphasize this, we often write $f(k) \equiv f(|k|)$.

The full Spin-Boson Hamiltonian is then defined as

$$H := H_0 + gV \quad (1.5)$$

for some coupling constant $g > 0$ on the Hilbert space

$$\mathcal{H} := K \otimes \mathcal{F}[h], \quad K := \mathbb{C}^2, \quad \mathcal{F} := \mathbb{C}$$

where

$$\mathcal{F}[h] := \bigoplus_{n=0}^{\infty} \mathcal{F}_n[h], \quad \mathcal{F}_n[h] := h^\otimes n, \quad h := L^2(\mathbb{R}^3, \mathbb{C}) \quad (1.6)$$

denotes the standard bosonic Fock space, and superscript $\otimes n$ denotes the $n$-th symmetric tensor product, where by convention $h^\otimes 0 \equiv \mathbb{C}$. Note that we identify $K \equiv K \otimes 1_{\mathcal{F}[h]}$ and $H_f \equiv 1_K \otimes H_f$ in our notation (see Notation 1.1 below).

An element $\Psi \in \mathcal{F}[h]$ can be represented as a sequence $(\psi^{(n)})_{n\in \mathbb{N}_0}$ of wave functions $\psi^{(n)} \in h^\otimes n$. The state $\Psi$ with $\psi^{(0)} = 1$ and $\psi^{(n)} = 0$ for all $n \geq 1$ is called the vacuum and is denoted by

$$\Omega := (1, 0, 0, \ldots) \in \mathcal{F}[h]. \quad (1.8)$$

Note that $a$ and $a^{*}$ fulfill the canonical commutation relations:

$$\forall h, l \in h, \quad [a(h), a^{*}(l)] = \langle h, l \rangle_2, \quad [a(h), a(l)] = 0, \quad [a^{*}(h), a^{*}(l)] = 0. \quad (1.9)$$
Let us recall some well-known facts about the introduced model. It is well-known that $K, H_f, H_0, H$ are self-adjoint and bounded below on the domains $K, D(H_f), D(H_0), D(H)$, respectively (see, e.g., [14, Proposition 1.1]). The spectrum of $K$ consists of two eigenvalues $e_0$ and $e_1$ and the corresponding eigenvectors are

$$\varphi_0 = (0, 1)^T \quad \text{and} \quad \varphi_1 = (1, 0)^T \quad \text{with} \quad K\varphi_i = e_i\varphi_i, \quad i = 0, 1. \quad (1.10)$$

The spectrum of $H_f$ is $\sigma(H_f) = [0, \infty)$ and it is absolutely continuous (see [31]). Consequently, the spectrum of $H_0$ is given by $\sigma(H_0) = [e_0, \infty)$, and $e_0, e_1$ are eigenvalues embedded in the absolutely continuous part of the spectrum of $H_0$ (see [30]).

**Notation 1.1.** In this work we omit spelling out identity operators whenever unambiguous. For every vector spaces $V_1, V_2$ and operators $A_1$ and $A_2$ defined on $V_1$ and $V_2$, respectively, we identify

$$A_1 \equiv A_1 \otimes 1_{V_2}, \quad A_2 \equiv 1_{V_1} \otimes A_2. \quad (1.11)$$

In order to simplify our notation further, and whenever unambiguous, we do not utilize specific notations for every inner product or norm that we employ.

### 1.2 Complex dilation

In this section we shortly introduce the method of complex dilation which is a key tool for proving our main result. For a more detailed presentation we refer to [14, Section 1.2]. We start by defining a family of unitary operators on $\mathcal{H}$ indexed by $\theta \in \mathbb{R}$.

**Definition 1.2.** For $\theta \in \mathbb{R}$, we define the unitary transformation

$$u_\theta : \mathfrak{h} \rightarrow \mathfrak{h}, \quad \psi(k) \mapsto e^{-\frac{3\theta}{2}} \psi(e^{-\theta}k). \quad (1.12)$$

Similarly, we define its canonical lift $U_\theta : \mathcal{F}[\mathfrak{h}] \rightarrow \mathcal{F}[\mathfrak{h}]$ by the lift condition $U_\theta a(h)^* U_\theta^{-1} = a(u_\theta h)^*, \ h \in \mathfrak{h}, \text{ and } U_\theta \Omega = \Omega$. This defines $U_\theta$ uniquely. With slight abuse of notation, we also denote $1_K \otimes U_\theta$ on $\mathcal{H}$ by the same symbol $U_\theta$.

We say that $\Psi \in \mathcal{F}[\mathfrak{h}]$ is an analytic vector if the map $\theta \mapsto \Psi^\theta := U_\theta \Psi$ has an analytic continuation from an open connected set in the real line to a (connected) domain in the complex plane.

We define the family of transformed Hamiltonians, for $\theta \in \mathbb{R}$,

$$H^\theta := U_\theta H U_\theta^{-1} = K + H_f^\theta + gV^\theta, \quad (1.13)$$

where

$$H_f^\theta := \int d^3k \omega^\theta(k)a^*(k)a(k), \quad V^\theta := \sigma_1 \otimes \left( a(f^\theta) + a(f^\theta)^* \right) \quad (1.14)$$

and

$$\omega^\theta(k) := e^{-\theta}|k|, \quad f^\theta : \mathbb{R}^3 \setminus \{0\} \rightarrow \mathbb{R}, \quad k \mapsto e^{-\theta(1+\mu)}e^{-e^{2\theta} \frac{2\mu}{\Lambda^2} |k|^{-\frac{3}{2}+\mu}}. \quad (1.15)$$
Eqs. (1.15), (1.14) and the right-hand side of (1.13) can be defined for complex \( \theta \) (see, e.g., [14, Lemma 1.4]). For sufficiently small coupling constants and \( \theta \in S \), where \( S \) is a suitable subset of the complex plane defined in (A.3) below, it has been shown that \( H^{\theta} \) has two non-degenerate eigenvalues \( \lambda^{\theta}_0 \) and \( \lambda^{\theta}_1 \) with corresponding rank one projectors denoted by \( P^{\theta}_0 \) and \( P^{\theta}_1 \), respectively; see, e.g., [13, Proposition 2.1]. Note that there the \( \theta \)-dependence was omitted in the notation. For convenience of the reader, we make it explicit in this paper. The corresponding dilated eigenstates can, therefore, be written as

\[
\Psi^{\theta}_i := P^{\theta}_i \varphi_i \otimes \Omega, \quad i = 0, 1.
\] (1.16)

where the eigenstates \( \varphi_i \) of the free atomic system are given in (1.10), and \( \Omega \) is the bosonic vacuum defined in (1.8). In our notation \( \Psi^{\theta}_i \) is not necessarily normalized. We know from [13, Theorem 2.3] that the eigenvalues \( \lambda^{\theta}_i \) are independent of \( \theta \) as long as \( \theta \) belongs to \( S \) and, therefore, we suppress it in our notation writing \( \lambda^{\theta}_i \equiv \lambda_i \). In the case that \( i = 1 \), it is necessary that 0 does not belong to \( S \). This is not required if \( i = 0 \), and in this situation we extend the set \( S \), with the same notation, to an open connected set that contains 0 (see [13, Definition 1.4 and Remark 2.4]). From this, it is easy to see that \( \Psi^{\theta=0} = \Psi_\lambda_0 \) - as introduced above.

### 1.3 Scattering theory

Finally, we give a short review of scattering theory which is necessary to state the main result in Section 2. For a more detailed introduction we refer to [14, Section 1.3].

**Definition 1.3** (Basic components of scattering theory). We denote by \( \mathfrak{h}_0 \) the set of smooth complex-valued functions on \( \mathbb{R}^3 \) with compact support contained in \( \mathbb{R}^3 \setminus \{0\} \).

We define the following objects:

(i) For \( h \in \mathfrak{h}_0 \) and \( \Psi \in \mathcal{K} \otimes \mathcal{D}(H^{1/2}_f) \), the asymptotic annihilation operators

\[
a_{\pm}(h)\Psi := \lim_{t \to \mp \infty} a_t(h)\Psi, \quad a_t(h) := e^{itH}a(h_t)e^{-itH}, \quad h_t(k) := h(k)e^{-it\omega(k)}.
\] (1.17)

Moreover, we define the asymptotic creation operators \( a^\ast_{\pm}(h) \) as the respective adjoints.

(ii) The asymptotic Hilbert spaces

\[
\mathcal{H}^\pm := \mathcal{K}^\pm \otimes \mathcal{F}[\mathfrak{h}] \quad \text{where} \quad \mathcal{K}^\pm := \{ \Psi \in \mathcal{H} : a_{\pm}(h)\Psi = 0 \ \forall h \in \mathfrak{h}_0 \}.
\] (1.18)

(iii) The wave operators

\[
\Omega_{\pm} : \mathcal{H}^\pm \to \mathcal{H}
\]

\[
\Omega_{\pm} \Psi \otimes a^\ast(h_1)...a^\ast(h_n)\Omega := a^\ast_{\pm}(h_1)...a^\ast_{\pm}(h_n)\Psi, \quad h_1, ..., h_n \in \mathfrak{h}_0, \quad \Psi \in \mathcal{K}^\pm.
\] (1.19)
(iv) The scattering operator $S := \Omega^+ \Omega^-$. The limit operators $a_\pm$ and $a^*_\pm$ are called asymptotic outgoing/ingoing annihilation and creation operators. The existence of the limits in (1.17) and their properties (for example that $\Psi_{\lambda_0} \in K^\pm$) are well-known (see e.g. [23, 22, 16, 25, 24, 17, 18, 19, 20, 10]). For a detailed proof we refer to [14, Lemma 4.1]. We can thus define the following scattering matrix coefficients for one-boson processes:

$$S(h, l) = \|\Psi_{\lambda_0}\|^{-2} \langle a^+_h(h)\Psi_{\lambda_0}, a^+_l(l)\Psi_{\lambda_0}\rangle, \quad \forall h, l \in h_0,$$

where the factor $\|\Psi_{\lambda_0}\|^{-2}$ appears due to the fact that, as already mentioned above, in our notation, the ground state $\Psi_{\lambda_0}$ is not necessarily normalized. In addition, it will be convenient to work with the corresponding transition matrix coefficients for one-boson processes given by

$$T(h, l) = S(h, l) - \langle h, l \rangle_2 \quad \forall h, l \in h_0. \quad (1.21)$$

Physically, these matrix coefficients may be interpreted as transition amplitudes of the scattering process in which an incoming boson with wave function $l$ is scattered at the two-level atom into an outgoing boson with wave function $h$. Notice that the transition matrix coefficients of multi-boson processes can be defined likewise but in this work we focus on one-boson processes only.

## 2 Main results

We are now able to state our main result which provides the precise relation between the one-boson transition matrix elements and the resolvent of the complex dilated Hamiltonian. The corresponding proofs will be provided in Section 3.

**Theorem 2.1 (Scattering Formula).** For sufficiently small $g$, $\theta$ in a suitable subset $S \subset \mathbb{C}$ (see (A.3)), and for all $h, l \in h_0$, the transition matrix coefficients for one-boson processes are given by

$$T(h, l) = \int d^3kd^3k' \overline{h(k)}l(k')\delta(\omega(k) - \omega(k'))T(k, k'),$$

where

$$T(k, k') = -2\pi ig^2 f(k)f(k') \|\Psi_{\lambda_0}\|^{-2} \left( \left\langle \sigma_1 \Psi_{\lambda_0}^\theta, \left(H^\theta - \lambda_0 - |k'|\right)^{-1} \sigma_1 \Psi_{\lambda_0}^\theta \right\rangle \right) + \left\langle \sigma_1 \Psi_{\lambda_0}^\theta, \left(H^\theta - \lambda_0 + |k'|\right)^{-1} \sigma_1 \Psi_{\lambda_0}^\theta \right\rangle. \quad (2.2)$$

The integral with respect to the Dirac’s delta distribution distribution $\delta$ in (2.1) is to be understood as

$$T(h, l) = \int_0^\infty d|k| \int d\Sigma d\Sigma' \overline{h(|k|, \Sigma)}l(|k|, \Sigma')T(|k|, \Sigma, |k|, \Sigma'),$$

(2.3)
where we have introduced spherical coordinates $k = (|k|, \Sigma)$ with $\Sigma$ being the solid angle and $T(k, k') \equiv T(|k|, \Sigma, |k|, \Sigma')$ is given by (2.2). Notice that (2.2) is not defined for $k = 0$ or $k' = 0$. However, since we take $h, l \in h_0$, the expression (2.1) is well-defined. Representing such matrix elements in terms of a distribution kernel is convenient (in our case, e.g., it makes the energy conservation apparent) and also frequently used in the literature. In particular, similar distribution kernels in a closely related model have been studied in [10, 14].

Remark 2.2. In a similar vein as in [14], we can apply perturbation theory together with the spectral properties obtained in [13] in order to deduce a result as [14, Theorem 2.2] from Theorem 2.1 above. Then, one can again see the Lorentzian shape of the integral kernel which was explained in detail in [17].

In the remainder of this work we denote by $C$ any generic (indeterminate), positive constant that might change from line to line but does not depend on the coupling constant.

3 Proof of the main result

In the remainder of this work we provide the proof of Theorem 2.1. This section has three parts: In Section 3.1, we recall a preliminary formula for the scattering matrix coefficients; c.f. Theorem 3.1 below, which was proven in [13, Theorem 4.3]. This formula together with several technical ingredients provided in Section 3.2 and 3.3 pave the way for the proof of our main result given in Section 3.4.

3.1 Preliminary scattering formula

The following theorem has been proven in [13, Theorem 4.3].

Theorem 3.1 (Preliminary Scattering Formula). For $h, l \in h_0$, the transition matrix coefficient for one-boson processes $T(h, l)$ defined in (1.21) fulfills

$$T(h, l) = \lim_{t \to -\infty} \int d^3kd^3k' \overline{h}(k)l(k')\delta(\omega(k) - \omega(k'))T_t(k, k')$$

(3.1)

for the integral kernel

$$T_t(k, k') = -2\pi ig f(k) \|\Psi_{\lambda_0}\|^2 \langle \sigma_1 \Psi_{\lambda_0}, a_t(k')^\dagger \Psi_{\lambda_0} \rangle.$$  

(3.2)

The integral in (3.1) is to be understood as

$$T(h, l) = -2\pi ig \|\Psi_{\lambda_0}\|^2 \left\langle \sigma_1 \Psi_{\lambda_0}, a_- (W)^\dagger \Psi_{\lambda_0} \right\rangle$$

(3.3)

for $W \in h_0$ given by

$$\mathbb{R}^3 \ni k \mapsto W(k) := |k|^2l(k) \int d\Sigma \overline{h}(|k|, \Sigma) f(|k|, \Sigma)$$

(3.4)

using spherical coordinates $k = (|k|, \Sigma)$ with $\Sigma$ being the solid angle.
3.2 General ingredients for the proof of the main theorem

Here, we state some general results which are applied in the proof of our main result, see Section 3.4. Most of the statements in this section are formulated without motivation. However, their importance becomes clear later in Section 3.4. At first, we recall a representation formula of the time-evolution operator similar to the Laplace transform representation (see, e.g., [2]). This formula is an important ingredient for the proof of the perturbative scattering formula in [14] and it plays a relevant role in the present work. For a detailed proof we refer to [14, Lemma 4.5].

Lemma 3.2. For \( \epsilon > 0, \nu = \text{Im} \theta > 0 \) and sufficiently large \( R > 0 \), we consider the concatenated contour \( \Gamma(\epsilon, R) := \Gamma_-(\epsilon, R) \cup \Gamma_c(\epsilon) \cup \Gamma_d(R) \) (see Figure 1), where

\[
\begin{align*}
\Gamma_-(\epsilon, R) &:= [-R, \lambda_0 - \epsilon] \cup [\lambda_0 + \epsilon, R], \\
\Gamma_d(R) &:= \left\{ -R - ue^{\frac{i\nu}{4}} : u \geq 0 \right\} \cup \left\{ R + ue^{-\frac{i\nu}{4}} : u \geq 0 \right\}, \\
\Gamma_c(\epsilon) &:= \left\{ \lambda_0 - ee^{-it} : t \in [0, \pi] \right\}.
\end{align*}
\]

The orientations of the contours in (3.5) are given by the arrows depicted in Figure 1.

Then, for all analytic vectors \( \phi, \psi \in \mathcal{H} \) (analytic in a – connected – domain containing 0) and \( t > 0 \), the following identity holds true:

\[
\langle \phi, e^{-itH} \psi \rangle = \frac{1}{2\pi i} \int_{\Gamma(\epsilon, R)} dz e^{-itz} \left\langle \psi, (H^{\theta} - z)^{-1} \phi \right\rangle.
\]

Figure 1: An illustration of the contour \( \Gamma(\epsilon, R) := \Gamma_-(\epsilon, R) \cup \Gamma_c(\epsilon) \cup \Gamma_d(R) \).

In this paper we use a non-standard definition of the Fourier transform and its inverse:

\[
\mathfrak{F}[u](x) := \int_{\mathbb{R}} ds \, u(s) e^{-isx}, \quad \mathfrak{F}^{-1}[u](x) := (2\pi)^{-1} \int_{\mathbb{R}} ds \, u(s)e^{isx},
\]

where \( u \in S(\mathbb{R}, \mathbb{C}) \) (the Schwartz space). We utilize the same symbols (and names) for their dual transformation on \( S'(\mathbb{R}, \mathbb{C}) \) (the space of tempered distributions). We identify, as usual, functions \( f \in L^p(\mathbb{R}, \mathbb{C}) \) (for some \( p \in [1, \infty] \)) with their induced tempered distributions in \( S'(\mathbb{R}, \mathbb{C}) \) \((f(u) = fuf)\) and, similarly, we identify functions \( f \in L^1_{\text{loc}}(\mathbb{R}, \mathbb{C}) \) with their induced distributions in \( \left(C^\infty_0(\mathbb{R}, \mathbb{C})\right)'\). We denote by \( \Theta \) the
Heaviside function (or distribution, or tempered distribution) and by $\delta$ the Dirac $\delta$ distribution (or tempered distribution):

$$\Theta(x) := \begin{cases} 1 & \text{for } x \geq 0, \\ 0 & \text{for } x < 0, \end{cases} \quad \Theta(u) = \int_0^\infty u(x)dx, \quad \delta(u) = u(0), \tag{3.8}$$

for $u \in S^\prime(\mathbb{R}, \mathbb{C})$.

**Lemma 3.3.** We denote by $(PV(1/\bullet)) \in S^\prime(\mathbb{R}, \mathbb{C})$ the principal value:

$$(PV(1/\bullet))(\varphi) \equiv PV\int_\mathbb{R} ds s^{-1} \varphi(s) := \lim_{\eta \to 0^+} \int_{\mathbb{R}\setminus[-\eta,\eta]} ds s^{-1} \varphi(s) \quad \forall \varphi \in S(\mathbb{R}, \mathbb{C}). \tag{3.9}$$

It follows that

$$\hat{\mathcal{F}}[\Theta] = \pi \delta - i PV(1/\bullet). \tag{3.10}$$

The above result can be shown using methods from standard distribution theory. However, for the sake of completeness, we present a proof in Appendix B.

### 3.3 Key estimates

In this section we establish two key estimates for the proof of the main theorem. We point out to the reader that they strongly rely on the results obtained in [13]. However, for simplicity and due to the fact that the important features have already been studied in [14, Section 4.3], we omit the details related to the multiscale analysis carried out in [13], and give precise references instead.

**Definition 3.4.** (c.f. [14, Definition 4.6]) For every fixed numbers $\rho_0 \in (0, 1)$ and $\rho \in (0, \min(1, e_1/4))$ satisfying (A.13), we define the sequences

$$\rho_n := \rho_0 \rho^n, \quad \epsilon_n := 20 \rho_1^{1+\mu/4}, \quad \forall n \in \mathbb{N}. \tag{3.11}$$

**Lemma 3.5.** Set $G \in C^\infty_0(\mathbb{R}\setminus\{0\}, \mathbb{C})$, $n \in \mathbb{N}$ large enough and $\eta > 0$ small enough such that $G(x) = 0$, for $|x| \leq 2(\epsilon_n + \eta)$. We define

$$T_{n,R}(\eta) := \int_{\Gamma_{-(\epsilon_n, R)}} dz u(z) \int_\mathbb{R} dr \frac{G(r)}{z - \lambda_0 - r} \left(1 - \mathbb{I}_{I_\eta}(z)(r)\right), \tag{3.12}$$

where $\mathbb{I}_{I_\eta}(z)$ is the characteristic function of the set $I_\eta(z) := [z - \lambda_0 - \eta, z - \lambda_0 + \eta]$, $\Gamma_{-(\epsilon_n, R)}$ is defined in (3.5) and

$$u : \mathbb{C}^+ \setminus \{\lambda_0\} \to \mathbb{C}, \quad z \mapsto u(z) := \left\langle \sigma_1 \Psi^{\theta}_{\lambda_0}, \left(H^\theta - z\right)^{-1} \sigma_1 \Psi^{\theta}_{\lambda_0} \right\rangle. \tag{3.13}$$

Then, for sufficiently large $R$ (independent of $n$ and $\theta \in \mathcal{S}$), there is a constant $C$ (that does not depend on $n$, but it does depend on $G$, $\theta$, $e_1$ and $m$ – see above (A.9) below) such that

$$\left|T_{n,R}(\eta) - \pi i \int_\mathbb{R} dr G(r)u(r + \lambda_0)\right| \leq C \left(\rho_n^{\mu/8} + \frac{1}{R} + \eta\right). \tag{3.14}$$
Proof. The integrand in (3.12) is absolutely integrable with respect to the variables \( z \) and \( r \) because the singularity is cut off by the characteristic function. We apply Fubini’s theorem to get

\[
T_{n,R}(\eta) = \int_{\mathbb{R}} \int_{\Gamma_{-}(\epsilon_n,R)} \frac{1}{z - \lambda_0 - r} \left( 1 - \mathbb{1}_{I}(z)(r) \right) \, dz \, dG(r).
\]

(3.15)

Next, we analyze the inner integral above for \( r \) in the support of \( G \). Set \( \Gamma_{(r)} \) the half circle in the upper half complex plane with center \( r + \lambda_0 \) and radius \( \eta \). Moreover, set \( \Gamma^{(R)} \) the half-circle in the upper half complex plane with center \( 0 \) and radius \( R \). As depicted in Figure 2, the two half circles \( \Gamma_{c}(\epsilon_n) \) and \( \Gamma_{(r)} \) do not intersect each other for all \( r \) in the support of \( G \). This is a consequence of the assumption that the support of \( G \) does not intersect with the interval \( (-2(\epsilon_n + \eta), 2(\epsilon_n + \eta)) \). Moreover, we find that both half circles \( \Gamma_{c}(\epsilon_n) \) and \( \Gamma_{(r)} \) are contained in \( \Gamma^{(R)} \) for large enough \( R \) (the value of \( R \) can be chosen uniformly with respect to \( n \) and \( \theta \in S \), but it depends on the support of \( G \) independent of \( n \) and \( \theta \in S \), but dependent on the support of \( G \)).

Note that there is a constant \( C \) (that depends on the support of \( G \), but not on \( n \), \( \theta \in S \), \( \rho \) and \( \rho_0 \)) such that (see (A.12))

\[
\left| \frac{1}{z - \lambda_0 - r} \right| \leq \frac{C}{R^2}, \quad \forall z \in \Gamma^{(R)}.
\]

(3.16)

Moreover, there is a constant \( C \) (that depends on the support of \( G \), but not on \( n \), \( \rho \) and \( \rho_0 \)) such that (see (A.15))

\[
\left| \frac{1}{z - \lambda_0 - r} \right| \leq \frac{CC^{n+1}}{\rho_n}, \quad \forall z \in \Gamma_{c}(\epsilon_n),
\]

(3.17)

where \( \rho_n = \rho_0 \rho^n \) and \( \rho_0 > 0 \), \( 0 < \rho < 1 \) and \( C > 0 \) are specific numbers defined in [13] Definition 4.1 and 4.2] and fulfilling (A.13). We know from (A.10) and (A.11) that the only spectral point of \( H^{\theta} \) in \( \mathbb{C}^+ \) is \( \lambda_0 \). Hence, there is a constant \( C \) (that depends on the support of \( G \), but not on \( n \)) such that

\[
|u(z) - u(\lambda_0 + r)| \leq C\eta, \quad \forall z \in \Gamma_{(r)}.
\]

(3.18)
A direct calculation shows that
\[
\int_{\Gamma_r} dz \frac{u(\lambda_0 + r)}{z - \lambda_0 - r} = -u(\lambda_0 + r)i\pi. \tag{3.19}
\]

We choose the contour which follows the following set of points \( \Gamma^R \cup \Gamma^{(r)} \cup \Gamma_c(\epsilon_n) \) along the mathematical positive orientation. This is a closed contour where the function \( z \mapsto \frac{u(z)}{z - \lambda_0 - r} \) is continuous, and an it is analytic on its interior. Then, it follows from Cauchy’s integral formula that (notice that, for \( z \) in the real numbers, \( \Re \) \( \Gamma^R \cap \Gamma^{(r)} \cup \Gamma_c(\epsilon_n) \) along the mathematical positive orientation. This is a closed contour where the function \( (z - \lambda_0 - r) = -u(\lambda_0 + r)i\pi. \tag{3.20} \)

which together with (3.15)-(3.19) imply the desired result, we additionally use Definition 3.4 and (A.13) to estimate the integral over \( \Gamma_c(\epsilon_n) \).

**Lemma 3.6.** Let \( n \geq 2 \) and \( R > 0 \) be large enough. For \( 0 < q < 1 < Q < \infty \) and \( \zeta \in S(\mathbb{R}, \mathbb{C}) \), we define
\[
A(Q,n,R) := \int_Q^Q ds \zeta(s) \int_{\Gamma^R(n,R)} dz e^{-is(z-\lambda_0)} \left\langle \sigma_1 \Psi^q_{\lambda_0}, (H^\theta - z)^{-1} \sigma_1 \Psi^q_{\lambda_0} \right\rangle. \tag{3.21}
\]

Then, the limits \( A(Q,\infty,\infty) := \lim_{n,R \to \infty} A(Q,n,R) \) and \( A(\infty,n,R) := \lim_{Q \to \infty} A(Q,n,R) \) exist and they are uniform with respect to \( Q \) and \( (n, R) \), respectively. Moreover, there is a constant \( C \) (independent of \( n, q, Q \) and \( R \)) such that
\[
|A(Q,n,R) - A(\infty,n,R)| \leq C/Q. \tag{3.22}
\]

Additionally, the limits
\[
\lim_{Q \to \infty} \lim_{n,R \to \infty} A(Q,n,R), \quad \lim_{n,R \to \infty} A(\infty,n,R) \tag{3.23}
\]
exist and they are equal.

**Proof.** For \( 0 < q < Q < \infty \), \( n \in \mathbb{N} \) and \( R \in \mathbb{R}^+ \) sufficiently large, we write
\[
A(Q,n,R) = A^{(1)}(Q,n,R) + A^{(2)}(Q,n,R), \tag{3.24}
\]
where
\[ A^{(1)}(Q, n) := \int_{q}^{Q} ds \, \zeta(s) \int_{I_n} dz \, e^{-is(z-\lambda_0)} \left< \sigma_1 \Psi_{\lambda_0}^\theta, \left( H^\theta - z \right)^{-1} \sigma_1 \Psi_{\lambda_0}^\theta \right>, \quad (3.25) \]
\[ A^{(2)}(Q, R) := \int_{q}^{Q} ds \, \zeta(s) \int_{I_1} dz \, e^{-is(z-\lambda_0)} \left< \sigma_1 \Psi_{\lambda_0}^\theta, \left( H^\theta - z \right)^{-1} \sigma_1 \Psi_{\lambda_0}^\theta \right>. \quad (3.26) \]

Here, we split the domain of integration \( \Gamma_- (\epsilon_n, R) = I_1 \cup I_n \), where \( I_1 := [-R, R] \setminus (\lambda_0 - \epsilon_1, \lambda_0 + \epsilon_1) \) and \( I_n := [\lambda_0 - \epsilon_1, \lambda_0 + \epsilon_1] \setminus (\lambda_0 - \epsilon_n, \lambda_0 + \epsilon_n) \). We analyze first (3.26). We obtain from the integration by parts formula (in the variable \( s \)) together with \( e^{-is(z-\lambda_0)} = i(z - \lambda_0)^{-1} \partial_s e^{-is(z-\lambda_0)} \) that there is a constant \( C \) such that, for \( \bar{Q} > Q \),
\[ A^{(2)}(\bar{Q}, R) - A^{(2)}(Q, R) \]
\[ = i \int_{q}^{Q} ds \, \zeta(s) \int_{I_1} dz \, (z - \lambda_0)^{-1} \partial_s e^{-is(z-\lambda_0)} \left< \sigma_1 \Psi_{\lambda_0}^\theta, \left( H^\theta - z \right)^{-1} \sigma_1 \Psi_{\lambda_0}^\theta \right>. \]
\[ = i \int_{I_1} dz \, \left( \zeta(\bar{Q})e^{-i\bar{Q}(z-\lambda_0)} - \zeta(Q)e^{-iQ(z-\lambda_0)} \right) (z - \lambda_0)^{-1} \left< \sigma_1 \Psi_{\lambda_0}^\theta, \left( H^\theta - z \right)^{-1} \sigma_1 \Psi_{\lambda_0}^\theta \right>
- i \int_{Q}^{\bar{Q}} ds \, (\partial_s \zeta(s)) \int_{I_1} dz \, (z - \lambda_0)^{-1} e^{-is(z-\lambda_0)} \left< \sigma_1 \Psi_{\lambda_0}^\theta, \left( H^\theta - z \right)^{-1} \sigma_1 \Psi_{\lambda_0}^\theta \right>. \quad (3.27) \]

Since \( \zeta \in \mathcal{S}(\mathbb{R}, \mathbb{C}) \), there is a constant \( C \) such that, for all \( s \in \mathbb{R} \), \( |\zeta(s)|, |\partial_s \zeta(s)| \leq C/(1 + s^2) \), and hence, there is a constant \( C \) such that
\[ \left| A^{(2)}(\bar{Q}, R) - A^{(2)}(Q, R) \right| \leq CQ^{-1} \int_{I_1} dz \, |z - \lambda_0|^{-1} \left< \sigma_1 \Psi_{\lambda_0}^\theta, \left( H^\theta - z \right)^{-1} \sigma_1 \Psi_{\lambda_0}^\theta \right>. \quad (3.28) \]

It follows from (A.12) and (A.15) that there is a constant \( C \) (independent of \( n, R, q \) and \( Q \)) such that
\[ \left| A^{(2)}(\bar{Q}, R) - A^{(2)}(Q, R) \right| \leq C/Q. \quad (3.29) \]

Similarly, using that \( \zeta \in \mathcal{S}(\mathbb{R}, \mathbb{C}) \), we find a constant \( C \) (independent of \( n, R, q \) and \( Q \)) such that
\[ \left| A^{(1)}(\bar{Q}, n) - A^{(1)}(Q, n) \right| \leq CQ^{-1} \int_{I_n} dz \left| \left< \sigma_1 \Psi_{\lambda_0}^\theta, \left( H^\theta - z \right)^{-1} \sigma_1 \Psi_{\lambda_0}^\theta \right> \right| \]
\[ \leq CQ^{-1} \sum_{j=1}^{n-1} \int_{I_{j,j+1}} dz \left| \left< \sigma_1 \Psi_{\lambda_0}^\theta, \left( H^\theta - z \right)^{-1} \sigma_1 \Psi_{\lambda_0}^\theta \right> \right|, \quad (3.30) \]

where \( I_{j,j+1} := [\lambda_0 - \epsilon_j, \lambda_0 + \epsilon_j] \setminus (\lambda_0 - \epsilon_{j+1}, \lambda_0 + \epsilon_{j+1}) \). We observe from (A.15) together with Definition 3.4 that there is a constant \( C \) (independent of \( n, R, q \) and \( Q \)) such that
\[ \left| A^{(1)}(\bar{Q}, n) - A^{(1)}(Q, n) \right| \leq CQ^{-1} \sum_{j=1}^{\infty} \int_{I_{j,j+1}} dz \, \frac{C^{j+2}}{\rho_{j+1}} \leq CQ^{-1} \sum_{j=1}^{\infty} \frac{C^{j+2} \epsilon_j}{\rho_{j+1}}. \quad (3.31) \]
From Definition 3.4 and (A.13), we obtain that
\[
\left| A^{(1)}(\tilde{Q}, n) - A^{(1)}(Q, n) \right| \leq C/Q. \tag{3.32}
\]
This together with (3.29) implies that there is a constant \( C \) such that
\[
\left| A(\tilde{Q}, n, R) - A(Q, n, R) \right| \leq C/Q. \tag{3.33}
\]
Consequently, the limit \( \lim_{Q \to \infty} A(\tilde{Q}, n, R) \) exists and it converges uniformly with respect to \( n \) and \( R \). We denote the limit by \( A(\infty, n, R) = \lim_{Q \to \infty} A(\tilde{Q}, n, R) \). It follows that (3.22) holds true.

For fixed \( Q \) and \( \tilde{n} > n \) and \( \tilde{R} > R \), we have
\[
\left| A(Q, \tilde{n}, \tilde{R}) - A(Q, n, R) \right| \leq \left| A(Q, \tilde{n}, \tilde{R}) - A(Q, \tilde{n}, R) \right| + \left| A(Q, \tilde{n}, R) - A(Q, n, R) \right|. \tag{3.34}
\]
For \( \tilde{n} \) and \( \tilde{R} \) large enough, employing a similar calculation as in (3.28), we get from (3.24), (3.25), (3.26) that there is a constant \( C \) (that does not depend on \( Q \)) such that
\[
\left| A(Q, \tilde{n}, \tilde{R}) - A(Q, n, R) \right| = \left| A^{(2)}(Q, \tilde{n}, \tilde{R}) - A^{(2)}(Q, R) \right| \leq C/R, \tag{3.35}
\]
and furthermore, similarly as in (3.31), we obtain that there is a constant \( C \) such that
\[
\left| A(Q, \tilde{n}, R) - A(Q, n, R) \right| = \left| A^{(1)}(Q, \tilde{n}) - A^{(1)}(Q, n) \right| \leq C \sum_{j=n}^{\tilde{n}-1} \frac{C_{j+2} \epsilon_j}{\rho_{j+1}}, \tag{3.36}
\]
and consequently, it follows from Definition 3.4 together with (A.13) that there is a constant \( C \) (that does not depend on \( Q \)) such that
\[
\left| A(Q, \tilde{n}, R) - A(Q, n, R) \right| \leq C/n. \tag{3.37}
\]
This together with (3.34) and (3.35) yields that there is a constant \( C \) (that does not depend on \( Q \)) such that
\[
\left| A(Q, \tilde{n}, \tilde{R}) - A(Q, n, R) \right| \leq C(R^{-1} + n^{-1}). \tag{3.38}
\]
We conclude that the limit \( A(Q, \infty, \infty) := \lim_{n, R \to \infty} A(Q, n, R) \) exists (uniformly with respect to \( Q \)). This completes the first part of the lemma.

Now we prove the second part of the lemma. At first, we show the existence of the limit \( \lim_{n, R \to \infty} A(\infty, n, R) \). For \( \tilde{n} > n \) and \( \tilde{R} > R \), we estimate
\[
\left| A(\infty, \tilde{n}, \tilde{R}) - A(\infty, n, R) \right| \leq \left| A(\infty, \tilde{n}, \tilde{R}) - A(\infty, \tilde{n}, R) \right| + \left| A(\infty, \tilde{n}, R) - A(Q, n, R) \right| + \left| A(Q, n, R) - A(\infty, n, R) \right|. \tag{3.39}
\]
For $\epsilon > 0$, we take $Q_0 > 0$ such that for all $Q \geq Q_0$
\[ |A(\infty, \tilde{n}, \tilde{R}) - A(Q, \tilde{n}, \tilde{R})| \leq \epsilon/3 \quad \text{and} \quad |A(\infty, n, R) - A(Q, n, R)| \leq \epsilon/3. \quad (3.40)\]
We obtain from (3.38) that, for $\epsilon > 0$, there are constants $n_0, R_0 > 0$ such that, for all $n, \tilde{n} \geq n_0$ and $R, \tilde{R} \geq R_0$,
\[ |A(Q, \tilde{n}, \tilde{R}) - A(Q, n, R)| \leq \epsilon/3. \quad (3.41)\]
This together with (3.40) and (3.39) yields that, for $\epsilon > 0$, there are $n_0 > 0$ and $R_0 > 0$
such that, for $n \geq n_0$ and $R \geq R_0$, we have
\[ |A(\infty, \tilde{n}, \tilde{R}) - A(\infty, n, R)| \leq \epsilon. \quad (3.42)\]
This implies the existence of the limit $\lim_{n, R \to \infty} A(\infty, n, R) = A(\infty, \infty, \infty)$. We fix $\epsilon > 0$.
According to (3.42) we obtain that for large enough $n, R$, $|A(\infty, \infty, \infty) - A(\infty, n, R)| < \epsilon/3$. Since $\lim_{Q \to \infty} A(Q, n, R) = A(\infty, n, R)$ uniformly with respect to $n, R$, then for large enough $Q$ (independently of $n, R$) $|A(\infty, n, R) - A(Q, n, R)| < \epsilon/3$. Moreover, because $A(\infty, \infty, \infty) = \lim_{n, R \to \infty} A(Q, n, R)$ (uniformly with respect to $Q$), for large enough $n, R$ (independently of $Q$) we have that $|A(Q, n, R) - A(\infty, \infty, \infty)| < \epsilon/3$. We conclude that there are $n \in \mathbb{N}$, $R > 0$ and $Q > 0$ such that, for $n \geq n$, $Q \geq Q$ and $R \geq R$, we have
\[ |A(\infty, \infty) - A(Q, n, R)| \leq |A(\infty, \infty, \infty) - A(\infty, n, R)| + |A(\infty, n, R) - A(Q, n, R)| + |A(Q, n, R) - A(\infty, \infty, \infty)| < \epsilon. \quad (3.43)\]
This proves that $\lim_{Q \to \infty} A(Q, \infty, \infty) = A(\infty, \infty, \infty)$ and completes the proof of the second part of the lemma. \hfill \Box

**Remark 3.7.** The absolute value of the integrand in the definition of $A(Q, n, R)$ in Lemma 3.6 is
\[ |\zeta(s)| \left| \left\langle \sigma_1 \Psi_{\lambda_0}^\theta, \left( H^\theta - z \right)^{-1} \sigma_1 \Psi_{\lambda_0}^\theta \right\rangle \right|. \quad (3.44)\]
and since the norm of the resolvent operator behaves as $|1/z|$ for large $|z|$, it is expected that the integral of (3.44) over $\Gamma_{-}(e_\infty, R)$ diverges as $R$ tends to infinity. A uniform bound of the from (3.22) is possible because the oscillatory factor $e^{-is(z-\lambda_0)}$ is being integrated: we treat $A(Q, n, R)$ as an oscillatory integral, and use the usual tools from this area (we use a clever division of the integration domain, apply integration by parts in different forms and interchange orders of integration). This is only possible if the variable $s$ is integrated (otherwise we loose the power of the oscillatory factor and we cannot perform integration by parts in the way we do). This is the reason why do not differentiate with respect to $Q$ and utilize the fundamental theorem of calculus (which is called Cook method in the context of scattering theory), since the derivative of $A(Q, n, R)$ with respect to $Q$ does not contain an integration with respect to $s$.
3.4 Proof of Theorem 2.1

Proof of Theorem 2.1. Let $h, l \in \mathfrak{h}_0$; see Definition 1.3. Recall the definition of $W$ given in (3.4) and the form factor $f$ in (1.3). Thanks to the fact that $f \in C^\infty(\mathbb{R}^3 \setminus \{0\}, \mathbb{C})$, we find that

$$hf, lf, W \in \mathfrak{h}_0.$$  \hfill (3.45)

Theorem 3.1, i.e., Equation (3.3) together with (A.2) yields

$$T(h, l) = -2\pi i g \|\Psi_{\lambda_0}\|^{-2} \langle a_-(W)\sigma_1 \Psi_{\lambda_0}, \Psi_{\lambda_0} \rangle = -2\pi i g \|\Psi_{\lambda_0}\|^{-2} \langle [a_-(W), \sigma_1] \Psi_{\lambda_0}, \Psi_{\lambda_0} \rangle,$$  \hfill (3.46)

and furthermore, recalling that $\omega(k) = |k|$, and (A.1), we obtain that

$$T(h, l) = -2\pi i g \|\Psi_{\lambda_0}\|^{-2} \int_0^\infty ds \int_{\mathbb{R}^3} e^{isz} \sigma_1 \Psi_{\lambda_0}, \Psi_{\lambda_0} \rangle \quad (3.47)$$

where we use the abbreviations

$$T^{(j)} := \lim_{q \to 0^+} \lim_{Q \to \infty} T^{(j), q, Q} \quad (3.48)$$

for $j = 1, 2$ with

$$T^{(1), q, Q} := -2\pi i \int_q^Q \int d^3k W(k) f(k) e^{isz(\|k\|+\lambda_0)} \langle \sigma_1 \Psi_{\lambda_0}, e^{-isz} \sigma_1 \Psi_{\lambda_0} \rangle \quad (3.49)$$

and

$$T^{(2), q, Q} := -2\pi i \int_q^Q \int dr G(r) e^{isz(\|k\|+\lambda_0)} \langle \sigma_1 \Psi_{\lambda_0}, e^{-isz} \sigma_1 \Psi_{\lambda_0} \rangle \quad (3.50)$$

Here, we use the notation

$$G : \mathbb{R} \to \mathbb{C}, \quad r \mapsto G(r) := \begin{cases} \int d\Sigma d\Sigma' r^4 \tilde{h}(r, \Sigma) l(r, \Sigma') f(r)^2 & \text{for } r \geq 0 \\ 0 & \text{for } r < 0, \end{cases} \quad (3.51)$$

where we write spherical coordinates $k = (r, \Sigma)$ and $k' = (r', \Sigma')$ in (3.1) and (3.4) recalling the definition of $W$ and that $f(k) \equiv f(|k|)$ only depends on the radial coordinate $r = |k|$. Thanks to (3.45), we observe

$$G \in C^\infty_c(\mathbb{R} \setminus \{0\}, \mathbb{C}) \subset \mathcal{S}(\mathbb{R}, \mathbb{C}). \quad (3.52)$$
Term $T^{(1),q,Q}$: Theorem 2.3 guarantees that $\Psi_{\lambda_0}$, and therefore, also $\sigma_1 \Psi_{\lambda_0}$ is an analytic vector (see Definition 1.2). As pointed out earlier, for the ground state, we can take the set $\mathcal{S}$ to be a neighborhood of 0 which allows us to apply Lemma 3.2 and find

$$T^{(1),q,Q} = - \int_q^Q ds \int dr G(r)e^{is(r+\lambda_0)} \int_{\Gamma_{(r, R)}} dz \, e^{-isz} \left< \sigma_1 \Psi_{\lambda_0}^\theta, (H^\theta - z)^{-1} \sigma_1 \Psi_{\lambda_0}^\theta \right>.$$  

(3.53)

Here, $\Gamma(\epsilon_n, R) = \Gamma_- (\epsilon_n, R) \cup \Gamma_\epsilon (\epsilon_n) \cup \Gamma_d (R)$ is the contour defined in Lemma 3.2 i.e., (3.5), for sufficiently large $R > 0$ and $n > 2$. We split the term

$$T^{(1),q,Q} = T^{(1),q,Q}_{\epsilon_n, R} + T^{(1),q,Q}_{\epsilon_n} + T^{(1),q,Q}_{R}$$  

(3.54)

according to the different contours parts, see (3.5), in the $dz$-integrals:

$$T^{(1),q,Q}_{\epsilon_n, R} := - \int_q^Q ds \int J(s) \int_{\Gamma_- (\epsilon_n, R)} dz \, e^{-isz} \left< \sigma_1 \Psi_{\lambda_0}^\theta, (H^\theta - z)^{-1} \sigma_1 \Psi_{\lambda_0}^\theta \right>,$$  

(3.55)

$$T^{(1),q,Q}_{\epsilon_n} := - \int_q^Q ds \int_{\Gamma_{\epsilon_n}} dz \, e^{-isz} \left< \sigma_1 \Psi_{\lambda_0}^\theta, (H^\theta - z)^{-1} \sigma_1 \Psi_{\lambda_0}^\theta \right>,$$  

(3.56)

$$T^{(1),q,Q}_{R} := - \int_q^Q ds \int J(s) \int_{\Gamma_d (R)} dz \, e^{-isz} \left< \sigma_1 \Psi_{\lambda_0}^\theta, (H^\theta - z)^{-1} \sigma_1 \Psi_{\lambda_0}^\theta \right>,$$  

(3.57)

and we use the definition

$$J : \mathbb{R} \to \mathbb{C}, \quad s \mapsto J(s) = \int dr G(r)e^{is(r+\lambda_0)}.$$  

(3.58)

We observe that, thanks to (3.52), we have $J \in \mathcal{S}(\mathbb{R}, \mathbb{C})$ which implies

$$|J(s)| \leq C(1 + |s|^2)^{-1}$$  

(3.59)

for some constant $C$. Moreover, we have (see (A.12))

$$\left| e^{-isz} \left< \sigma_1 \Psi_{\lambda_0}^\theta, (H^\theta - z)^{-1} \sigma_1 \Psi_{\lambda_0}^\theta \right> \right| \leq C \|\Psi_{\lambda_0}\| \frac{2^{s \text{Im} z}}{|z - \epsilon_1|}, \quad \forall z \in \Gamma_d (R).$$  

(3.60)

**Contribution $T^{(1),q,Q}_{\epsilon_n}$ in (3.56):** Using (3.59), we may start with the bound

$$|T^{(1),q,Q}_{\epsilon_n}| \leq C \sup_{s \in [q, Q]} \left| \int_{\Gamma_{\epsilon_n}} dz \, e^{-isz} \left< \sigma_1 \Psi_{\lambda_0}^\theta, (H^\theta - z)^{-1} \sigma_1 \Psi_{\lambda_0}^\theta \right> \right|.$$  

(3.61)

It follows from (A.15) together with Definition 3.4 that there is a constant $C$ such that, for $s \in [q, Q]$, we have

$$\left| \int_{\Gamma_{\epsilon_n}} dz \, e^{-isz} \left< \sigma_1 \Psi_{\lambda_0}^\theta, (H^\theta - z)^{-1} \sigma_1 \Psi_{\lambda_0}^\theta \right> \right| \leq C e^{\epsilon_n Q} \frac{\epsilon_n}{\rho_n} C^{n+1} \leq C e^{\epsilon_n Q} \rho_n^{\mu / 8},$$  

(3.62)

where we use (A.13). In conclusion, we have that, for all $0 < q < Q < \infty$,

$$\lim_{n \to 0} T^{(1),q,Q}_{\epsilon_n} = 0.$$  

(3.63)
**Contribution** $T_{n}^{(1), q, Q}$ in (3.57): Using (3.59) again, we find

$$|T_{n}^{(1), q, Q}| \leq C \int_{q}^{Q} \frac{1}{1 + |s|^{2}} \left| \int_{\Gamma_{d}(R)} dz \ e^{-isz} \left\langle \sigma_{1} \Psi_{\lambda_{0}}, \left( H^{\theta} - z \right)^{-1} \sigma_{1} \Psi_{\lambda_{0}} \right\rangle \right|. \quad (3.64)$$

For $s \in [q, Q]$, we observe that there is a constant $C$ such that (see (A.12))

$$\left| \int_{\Gamma_{d}(R)} dz \ e^{-isz} \left\langle \sigma_{1} \Psi_{\lambda_{0}}, \left( H^{\theta} - z \right)^{-1} \sigma_{1} \Psi_{\lambda_{0}} \right\rangle \right| \leq C \int_{0}^{\infty} du \ e^{-su \sin(\nu/4)}. \quad (3.65)$$

Thereby, as in (3.65), we obtain the estimate

$$\lim_{R \to \infty} \int_{q}^{Q} \frac{1}{1 + |s|^{2}} \left| \int_{\Gamma_{d}(R)} dz \ e^{-isz} \left\langle \sigma_{1} \Psi_{\lambda_{0}}, \left( H^{\theta} - z \right)^{-1} \sigma_{1} \Psi_{\lambda_{0}} \right\rangle \right| \leq \lim_{R \to \infty} \frac{C}{R} \int_{q}^{Q} \frac{1}{1 + |s|^{2}} \frac{1}{|s|} = 0. \quad (3.66)$$

Then, we conclude for all $0 < q < Q < \infty$

$$\lim_{R \to \infty} T_{n}^{(1), q, Q} = 0. \quad (3.67)$$

This together with (3.63) and (3.54) yields that for all $0 < q < Q < \infty$

$$T^{(1), q} = \lim_{n, R \to \infty} T_{n, R}^{(1), q, Q}. \quad (3.68)$$

Note that $J \in S(\mathbb{R}, \mathbb{C})$. Therefore, we are in the position to apply Lemma 3.6 and find

$$T^{(1), q, \infty} := \lim_{Q \to \infty} T^{(1), q, Q} = \lim_{Q \to \infty} \lim_{n, R \to \infty} T^{(1), q, Q} = \lim_{n, R \to \infty} T^{(1), q, \infty}, \quad (3.69)$$

where

$$T^{(1), q, \infty}_{n, R} := \lim_{Q \to \infty} T^{(1), q, Q}_{n, R} = -\int_{q}^{\infty} ds \ J(s) \int_{\Gamma_{d}(R)} dz \ e^{-isz} \left\langle \sigma_{1} \Psi_{\lambda_{0}}, \left( H^{\theta} - z \right)^{-1} \sigma_{1} \Psi_{\lambda_{0}} \right\rangle. \quad (3.70)$$

For fixed $n$ and $R$, the function $z \mapsto e^{-isz} \left\langle \sigma_{1} \Psi_{\lambda_{0}}, \left( H^{\theta} - z \right)^{-1} \sigma_{1} \Psi_{\lambda_{0}} \right\rangle$ is bounded in $\Gamma_{-}(\epsilon_{n}, R)$. Then, thanks to (3.59), we may apply Fubini’s theorem and find:

$$T^{(1), q, \infty}_{\epsilon_{n}, R} = -\int_{\Gamma_{d}(\epsilon_{n}, R)} dz \left\langle \sigma_{1} \Psi_{\lambda_{0}}, \left( H^{\theta} - z \right)^{-1} \sigma_{1} \Psi_{\lambda_{0}} \right\rangle \int_{q}^{\infty} ds \int dr \ G(r)e^{is(r + \lambda_{0} - z)}$$

$$= -\int_{\Gamma_{d}(\epsilon_{n}, R)} dz \left\langle \sigma_{1} \Psi_{\lambda_{0}}, \left( H^{\theta} - z \right)^{-1} \sigma_{1} \Psi_{\lambda_{0}} \right\rangle \int ds \Theta(s - q) \int dr \ G^{(z)}(r)e^{-isr}. \quad (3.71)$$

In the last step, we use the coordinate transformation $r \to z - \lambda_{0} - r$ and the notation

$$G^{(z)} : \mathbb{R} \to \mathbb{C}, \quad r \mapsto G^{(z)}(r) := G(z - \lambda_{0} - r) \quad z \in \mathbb{R}. \quad (3.72)$$
Then, it follows from (3.52) together with (3.7) that
\[
\int ds \Theta(s - q) \int dr G^{(z)}(r)e^{-iqr} = \int ds \Theta(s) \int dr G^{(z)}(r)e^{-iqr} - \Theta(\delta[G^{(z)}q]) = \Theta[\delta(\Theta[G^{(z)}q])], \tag{3.73}
\]
where, for \( q > 0 \), we define
\[
G^{(z)}(r) := G^{(z)}(r)e^{-iqr}. \tag{3.74}
\]
Thanks to (3.52), we have for \( z \in \mathbb{R} \) and \( q \geq 0 \)
\[
G^{(z)}q \in C^\infty_c(\mathbb{R} \setminus \{ z - \lambda_0 \}, \mathbb{C}) \subset \mathcal{S}(\mathbb{R}, \mathbb{C}). \tag{3.75}
\]
It follows from Lemma 3.3 that for \( z \in \mathbb{R} \)
\[
\int ds \Theta(s - q) \int dr G^{(z)}(r)e^{-iqr} = \pi \delta(G^{(z)}q) - i \left( \text{PV} \left( 1/\bullet \right) \right)(G^{(z)}q). \tag{3.76}
\]
This together with (3.71) yields that
\[
T^{(1),q,\infty}_{\epsilon_n,R} = T^{(1,1),q,\infty}_{\epsilon_n,R} + T^{(1,2),q,\infty}_{\epsilon_n,R}, \tag{3.77}
\]
where
\[
T^{(1,1),q,\infty}_{\epsilon_n,R} := -\pi \int_{\Gamma_{-}(\epsilon_n,R)} dz \left\langle \sigma_1 \Psi^\theta_{\lambda_0}, \left( H^\theta - z \right)^{-1} \sigma_1 \Psi^\theta_{\lambda_0} \right\rangle G(z - \lambda_0) \tag{3.78}
\]
\[
T^{(1,2),q,\infty}_{\epsilon_n,R} := i \int_{\Gamma_{-}(\epsilon_n,R)} dz \left\langle \sigma_1 \Psi^\theta_{\lambda_0}, \left( H^\theta - z \right)^{-1} \sigma_1 \Psi^\theta_{\lambda_0} \right\rangle \lim_{\eta \to 0^+} \int_{\mathbb{R}[\eta]} dr \frac{G(z - \lambda_0 - r)e^{-iqr}}{r} \tag{3.79}
\]
In the following, we shall compute both contributions explicitly.

**Contribution** \( T^{(1,1)}_{\epsilon_n,R}(h,l) \): It follows from (3.52) that there are numbers \( M > \kappa > 0 \) such that \( \text{supp} \ G \subset [\kappa, M] \). Recall that everything so far holds for any choice of \( n, R > 0 \) large enough. For the rest of this proof we will restrict this choice to \( R > M \) and \( n > 0 \) large enough such that \( \epsilon_n < \kappa/4 \). In this setting, we may turn the \( dz \)-integral in an indefinite one, exploiting, the compact support of \( G \) and the definition of the contour \( \Gamma_{-}(\epsilon_n,R) \). We thus obtain
\[
T^{(1,1),q,\infty}_{\epsilon_n,R} = -\pi \int_{\Gamma_{-}(\epsilon_n,R)} dz \left\langle \sigma_1 \Psi^\theta_{\lambda_0}, \left( H^\theta - z \right)^{-1} \sigma_1 \Psi^\theta_{\lambda_0} \right\rangle G(z - \lambda_0) \tag{3.80}
\]
\[
= -\pi \int_{\Gamma_{-}(\epsilon_n,R)-\lambda_0} dz \left\langle \sigma_1 \Psi^\theta_{\lambda_0}, \left( H^\theta - \lambda_0 - z \right)^{-1} \sigma_1 \Psi^\theta_{\lambda_0} \right\rangle G(z) \tag{3.81}
\]
\[
= -\pi \int_{0}^{\infty} dz \left\langle \sigma_1 \Psi^\theta_{\lambda_0}, \left( H^\theta - z \right)^{-1} \sigma_1 \Psi^\theta_{\lambda_0} \right\rangle G(z) \tag{3.82}
\]
Lemma 3.5. We recall Definition 3.4 and notice that $T^{(1,2)}_{n,R}(h,l)$ in this case one has to consider the Hamiltonian replaced by $T$. In this section we collect the relevant results of [14] and [13] which are used in the proofs.

Contribution $T^{(1,2)}_{n,R}(h,l)$: In order to calculate $T^{(1,2)}_{n,R}(h,l)$ we can now fall back to Lemma 3.5. We recall Definition 3.4 and notice that $0 < \epsilon_n < \kappa/4$ for sufficiently large $n$. Then, as a direct consequence of Lemma 3.5, we find (for sufficiently large $R$)

$$
\lim_{n,R \to \infty} T^{(1,2)}_{n,R} = i \lim_{n,R \to \infty, \eta \to 0} T_{n,R}(\eta)
= -\pi \int_{\mathbb{R}}^{} \text{d}r G(r) e^{-ir\eta} \left\langle \sigma_1 \Psi^{\overline{\theta}}_{\lambda_0}, \left( H^\theta - \lambda_0 - r \right)^{-1} \sigma_1 \Psi^\theta_{\lambda_0} \right\rangle 
= -\pi \int_0^\infty \text{d}z \left\langle \sigma_1 \Psi^{\overline{\theta}}_{\lambda_0}, \left( H^\theta - \lambda_0 - z \right)^{-1} \sigma_1 \Psi^\theta_{\lambda_0} \right\rangle G(z) e^{-iqz},
$$

(3.81)

where $T_{n,R}(\eta)$ is defined in (3.12).

Collecting the contributions of (3.77), i.e., (3.80) and (3.81), we establish the identity $T^{(1)} = \lim_{q \to 0^+} \lim_{n,R \to \infty} T^{(1),q,\infty}_{n,R}$

$$
= -\pi \lim_{q \to 0^+} \int_0^\infty \text{d}z \left\langle \sigma_1 \Psi^{\overline{\theta}}_{\lambda_0}, \left( H^\theta - \lambda_0 - z \right)^{-1} \sigma_1 \Psi^\theta_{\lambda_0} \right\rangle G(z)(1 + e^{-iqz}) 
= -2\pi \int_0^\infty \text{d}z \left\langle \sigma_1 \Psi^{\overline{\theta}}_{\lambda_0}, \left( H^\theta - \lambda_0 - z \right)^{-1} \sigma_1 \Psi^\theta_{\lambda_0} \right\rangle G(z) 
= -2\pi \int d^3k d^3k' h(k) f(k) f(k') \delta(|k| - |k'|) \left\langle \sigma_1 \Psi^{\overline{\theta}}_{\lambda_0}, \left( H^\theta - \lambda_0 - |k'| \right)^{-1} \sigma_1 \Psi^\theta_{\lambda_0} \right\rangle.
$$

(3.82)

In the third line we applied the dominated convergence theorem which is justified by (3.52). Moreover, we have inserted the definition of $G$ using the symbolic notation of the Dirac-delta distribution in the last step.

Term $T^{(2)}$: The second term $T^{(2)}$ can be inferred by repeating the calculation with $\theta$ replaced by $\overline{\theta}$ and reflecting the path of integration $\Gamma(\epsilon_n, R)$ on the real axis when applying Lemma 3.2. In this case one has to consider the Hamiltonian $H^{\overline{\theta}}$ whose spectrum is given by mirroring the spectrum of $H^\theta$ at the real axis. Due to the similarity of the calculation, we omit a proof but only state the result.

$$
T^{(2)} = 2\pi \int d^3k d^3k' h(k) h(k') f(k) f(k') \delta(|k| - |k'|) \left\langle \sigma_1 \Psi^\theta_{\lambda_0}, \left( H^{\overline{\theta}} - \lambda_0 + |k'| \right)^{-1} \sigma_1 \Psi^{\overline{\theta}}_{\lambda_0} \right\rangle.
$$

(3.83)

The relative sign in comparison with (3.82) is due to the opposite mathematical orientation of the contour. Inserting (3.82) and (3.83) in (3.47) completes the proof.

A Collection of previous results used in this work

In this section we collect the relevant results of [13] and [18] which are used in the proofs contained in this work.
A.1 Scattering Theory

Let $\Psi \in \mathcal{K} \otimes D(H_f^{1/2})$ and $h, l \in \mathfrak{h}_0$. Then, we recall from [14, Lemma 4.1] that

$$a_-(h)\Psi = a(h)\Psi + ig\int_{-\infty}^{0} ds e^{isH} \langle h_s, f \rangle_2 \sigma_1 e^{-isH}\Psi.$$  \hfill (A.1)

It can be shown by integration by parts that there is a constant $C$ such that $|\langle h_s, f \rangle_2| \leq C/(1 + s^2)$ for $s \in \mathbb{R}$ (see [14, Eq. (C.7)]). Hence, the integral above is convergent. Moreover, it is proven in [14, Lemma 4.1 (iv)] that

$$a_\pm(h)\psi_{\lambda_0} = 0.$$ \hfill (A.2)

A.2 Spectral Properties

We define

$$S := \left\{ \theta \in \mathbb{C} : -10^{-3} < \text{Re} \theta < 10^{-3} \text{ and } \nu < \text{Im} \theta < \pi/16 \right\},$$ \hfill (A.3)

where $\nu \in (0, \pi/16)$ is a fixed number (see [13, Definition 1.4]).

In order to specify some of the spectral properties of $H^\theta$, we define certain regions in the complex plane:

**Definition A.1.** (c.f. [14, Definition 3.2]) For fixed $\theta \in S$, we set $\delta = e_1 - e_0 = e_1$ and define the regions

$$A := A_1 \cup A_2 \cup A_3,$$ \hfill (A.4)

where

$$A_1 := \{ z \in \mathbb{C} : \text{Re} z < e_0 - \delta/2 \},$$ \hfill (A.5)

$$A_2 := \left\{ z \in \mathbb{C} : \text{Im} z > \frac{1}{8} \delta \sin(\nu) \right\},$$ \hfill (A.6)

$$A_3 := \{ z \in \mathbb{C} : \text{Re} z > e_1 + \delta/2, \text{Im} z \geq -\sin(\nu/2) (\text{Re}(z) - (e_1 + \delta/2)) \},$$ \hfill (A.7)

and for $i = 0, 1$, we define

$$B_i^{(1)} := \left\{ z \in \mathbb{C} : |\text{Re} z - e_i| \leq \frac{1}{2} \delta, \frac{1}{2} \rho_1 \sin(\nu) \leq \text{Im} z \leq \frac{1}{8} \delta \sin(\nu) \right\}.$$ \hfill (A.8)

These regions are depicted in Figure 3

For a fixed $m \in \mathbb{N}$, $m \geq 4$, we define the cone

$$\mathcal{C}_m(z) := \left\{ z + xe^{-i\alpha} : x \geq 0, |\alpha - \nu| \leq \nu/m \right\}.$$ \hfill (A.9)
It follows from the induction scheme in [13, Section 4] that \( \lambda_i \in B_1^{(1)} \), and moreover, [13, Theorem 2.7] together with [13, Lemma 3.13] yields
\[
\sigma(H^{\theta}) \subset C \setminus \left[ A \cup (B_{0}^{(1)} \setminus C_{m}(\lambda_{0})) \cup (B_{1}^{(1)} \setminus C_{m}(\lambda_{1})) \right].
\] (A.10)

For \( g \) small enough, we recall from [14, Eq. (3.13)] that there is constant \( c > 0 \) such that
\[
\text{Im} \lambda_1 < -g^2 c < 0.
\] (A.11)

In the following we collect some important resolvent estimates. The region \( A \) is far away from the spectrum, and therefore, resolvent estimates in this region are easy. In [13, Lemma 3.2], we prove that there is a constant \( C \) (that does not depend on \( n, g, \rho_0 \) and \( \rho \)) such that
\[
\left\| \frac{1}{H^{\theta} - z} \right\| \leq C \frac{1}{|z - e_1|}, \quad \forall z \in A.
\] (A.12)

As in [14, Eq. (3.31)], we select the auxiliary numbers \( \rho \)
\[
C^8 \rho_0^{\mu} \leq 1, \quad C^8 \rho^{\mu} \leq 1/4, \quad \text{(and hence} \quad C \rho^{\frac{1}{2}(1+\mu/4)} \leq 1),
\] (A.13)

where
\[
\nu = \frac{\mu/4}{(1 + \mu/4)} \in (0, 1).
\] (A.14)

In [14, Lemma 4.7] we show that for all \( n \in \mathbb{N} \), a fixed (arbitrary) \( m \geq 4 \) and \( \theta \in S \), there is a constant \( C \) (that depends on \( m \)) such that
\[
\left\| \frac{1}{H^\theta - z} \sigma_{1} \Psi_{\lambda_{0}} \right\| \leq C C^{n+1} \frac{1}{\rho_n},
\] (A.15)
for every $z \in B_0^{(1)} \setminus C_m(\lambda_0 - 2\rho_n^{1+\mu/4}e^{-iv})$, where the cone $C_m$ is defined in (A.9). It can be seen from [14, Lemma 4.7] that $C$ does not depend on $n$, $\rho_0$ and $\rho$. Here, we recall from [14, Eq. (4.51)] that

$$C_m(\lambda_0 - 2\rho_n^{1+\mu/4}e^{-iv}) \cap (\overline{C_m} + \lambda_0 - i2\sin(\nu)\rho_n^{1+\mu/4}) \subset D(\lambda_0, \epsilon_n) \subset D(\lambda_0, 2\epsilon_n) \subset B_0^{(1)}. \tag{A.16}$$

**B Proof of Lemma 3.3**

*Proof of Lemma 3.3.* For $\alpha > 0$, we define $g_\alpha \in S'(\mathbb{R}, \mathbb{C})$ by

$$g_\alpha : S(\mathbb{R}, \mathbb{C}) \to \mathbb{C}, \quad \varphi \mapsto g_\alpha(\varphi) = \int_0^\infty dx e^{-\alpha x} \varphi(x). \tag{B.1}$$

It follows from (3.7) that for $\varphi \in S(\mathbb{R}, \mathbb{C})$

$$\mathcal{F}[g_\alpha](\varphi) = g_\alpha(\mathcal{F}[\varphi]) = \int_0^\infty dx e^{-\alpha x} \mathcal{F}[\varphi](x) = \int_0^\infty dx e^{-\alpha x} \int_{\mathbb{R}} ds \varphi(s)e^{-i\alpha x}. \tag{B.2}$$

The integrand on the right-hand side of (B.2) is absolutely integrable because of $\varphi \in S(\mathbb{R}, \mathbb{C})$, and hence, the Fubini-Tonelli theorem yields that

$$\mathcal{F}[g_\alpha](\varphi) = \int_{\mathbb{R}} ds \varphi(s) \int_0^\infty dx e^{-x(\alpha + is)}. \tag{B.3}$$

This together with

$$\int_0^\infty dx e^{-x(\alpha + is)} = \frac{1}{\alpha + is} = \frac{\alpha}{(\alpha^2 + s^2)} - i\frac{s}{(\alpha^2 + s^2)} \tag{B.4}$$

implies that

$$\mathcal{F}[g_\alpha](\varphi) = G_\alpha^{(1)}(\varphi) - iG_\alpha^{(2)}(\varphi), \tag{B.5}$$

where

$$G_\alpha^{(1)}(\varphi) = \int_{\mathbb{R}} ds \frac{\alpha}{(\alpha^2 + s^2)} \varphi(s) \tag{B.6}$$

and

$$G_\alpha^{(2)}(\varphi) = \int_{\mathbb{R}} ds \frac{s}{(\alpha^2 + s^2)} \varphi(s). \tag{B.7}$$

Using the coordinate transformation $s \to \alpha s$ we obtain that

$$\lim_{\alpha \to 0^+} G_\alpha^{(1)}(\varphi) = \lim_{\alpha \to 0^+} \int_{\mathbb{R}} ds \frac{\varphi(\alpha s)}{1 + s^2} = \varphi(0) \int_{\mathbb{R}} ds \frac{1}{1 + s^2} = \pi \varphi(0) = \pi \delta(\varphi), \tag{B.8}$$
finally, for some $\alpha$, where the second step follows from the dominated convergence theorem together with the continuity of $\varphi$. Moreover, we have

$$G^{(2)}_\alpha(\varphi) = G^{(2,1)}_\alpha(\varphi) + G^{(2,2)}_\alpha(\varphi), \quad \text{(B.9)}$$

where

$$G^{(2,1)}_\alpha(\varphi) := \int_{\mathbb{R} \setminus [-\alpha^8, \alpha^8]} ds \frac{s}{(\alpha^2 + s^2)} \varphi(s) \quad \text{(B.10)}$$

and

$$G^{(2,2)}_\alpha(\varphi) := \int_{\alpha^8} ds \frac{s}{(\alpha^2 + s^2)} \varphi(s). \quad \text{(B.11)}$$

We treat these two terms separately. At first, we obtain

$$\left| G^{(2,2)}_\alpha(\varphi) \right| \leq \int_{-\alpha^8}^{\alpha^8} ds \left| \frac{s}{(\alpha^2 + s^2)} (\varphi(s) - \varphi(0)) + \frac{|\varphi(0)|}{2} \ln(1 + \alpha^8) - \ln(1 - \alpha^8) \right| \quad \text{(B.12)}$$

where we have used the coordinate transformation $s' = s^2$ for the second term in the last line. Then, we obtain

$$\left| G^{(2,2)}_\alpha(\varphi) \right| \leq 2\alpha^{14} \sup_{s \in [-\alpha^8, \alpha^8]} |\varphi(s) - \varphi(0)| + \frac{\varphi(0)}{2} \ln(1 + \alpha^8) - \ln(1 - \alpha^8) \quad \text{(B.13)}$$

Note that $\ln(\cdot)$ is continuous close to 1 and $\sup_{s \in [-\alpha^8, \alpha^8]} |\varphi(s) - \varphi(0)| < \infty$ since a continuous function has a maximum on a compact set. We conclude

$$\lim_{\alpha \to 0^+} G^{(2,2)}_\alpha(\varphi) = 0. \quad \text{(B.14)}$$

Finally, for some $R > 0$, we obtain

$$G^{(2,1)}_\alpha(\varphi) = \int_{[-R,R] \setminus [-\alpha^8, \alpha^8]} ds \frac{s}{(\alpha^2 + s^2)} (\varphi(s) - \varphi(0)) + \int_{[-R,R] \setminus [-\alpha^8, \alpha^8]} ds \frac{s}{(\alpha^2 + s^2)} \varphi(0) + \int_{\mathbb{R} \setminus [-R,R]} ds \frac{s}{(\alpha^2 + s^2)} \varphi(s). \quad \text{(B.15)}$$

Due to symmetry, the second term vanishes independently of $R$, and moreover, the mean value theorem implies that

$$|\varphi(s) - \varphi(0)| \leq |s| \|\varphi'\|_\infty. \quad \text{(B.16)}$$

Altogether, this yields that

$$\left| \frac{s}{(\alpha^2 + s^2)} (\varphi(s) - \varphi(0)) \chi_{[-R,R] \setminus [-\alpha^8, \alpha^8]}(s) \right| \leq \|\varphi'\|_\infty \chi_{[-R,R]}(s), \quad \text{(B.17)}$$

$$\left| \frac{s}{(\alpha^2 + s^2)} \varphi(s) \chi_{[-R,R]}(s) \right| \leq \frac{\phi(s)}{s} \chi_{[-R,R]}(s), \quad \text{(B.18)}$$

23
where $\chi_A$ is the characteristic (indicator) function of the set $A$. This allows us to apply the dominated convergence theorem in order to find

$$\lim_{\alpha \to 0^+} G^{(2,1)}_\alpha(\varphi) = \text{PV} \int_{\mathbb{R}} ds \frac{1}{s} \varphi(s) = (\text{PV}(1/\bullet))(\varphi).$$

(B.19)

This together with (B.14), (B.9), (B.8) and (B.5) implies that

$$\lim_{\alpha \to 0^+} \mathfrak{F}[g_\alpha](\varphi) = \pi \delta(\varphi) - i (\text{PV}(1/\bullet))(\varphi) \quad \forall \varphi \in \mathcal{S}(\mathbb{R}, \mathbb{C}).$$

(B.20)

We conclude the proof by (3.7) which yields

$$\lim_{\alpha \to 0^+} \mathfrak{F}[g_\alpha](\varphi) = \lim_{\alpha \to 0^+} g_\alpha(\mathfrak{F}[\varphi]) = \Theta(\mathfrak{F}[\varphi]) = \mathfrak{F}[\Theta](\varphi) \quad \forall \varphi \in \mathcal{S}(\mathbb{R}, \mathbb{C}).$$

(B.21)

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References

[1] V. Bach, M. Ballesteros, and J. Fröhlich, *Continuous renormalization group analysis of spectral problems in quantum field theory*, J. Funct. Anal. 268 (2015), no. 5, 749–823.

[2] V. Bach, M. Ballesteros, and A. Pizzo, *Existence and construction of resonances for atoms coupled to the quantized radiation field*, ArXiv perprint: arXiv:1302.2829 (2013).

[3] [4], *Existence and construction of resonances for atoms coupled to the quantized radiation field*, Adv. Math. 314 (2017), 540–572.

[4] V. Bach, T. Chen, J. Fröhlich, and I. M. Sigal, *Smooth Feshbach map and operator-theoretic renormalization group methods*, J. Funct. Anal. 203 (2003), 44–92.

[5] V. Bach, J. Fröhlich, and A. Pizzo, *An infrared-finite algorithm for rayleigh scattering amplitudes, and bohr’s frequency condition*, Comm. Math. Phys. (2007).
[6] V. Bach, J. Fröhlich, and I. M. Sigal, *Mathematical theory of nonrelativistic matter and radiation*, Lett. Math. Phys. **34** (1995), no. 3, 183–201.

[7] ______, *Quantum electrodynamics of confined nonrelativistic particles*, Adv. Math. **137** (1998), no. 2, 299–395.

[8] ______, *Renormalization group analysis of spectral problems in quantum field theory*, Adv. Math. **137** (1998), no. 2, 205–298.

[9] ______, *Spectral analysis for systems of atoms and molecules coupled to the quantized radiation field*, Comm. Math. Phys. **207** (1999), no. 2, 249–290.

[10] V. Bach, F. Klopp, and H. Zenk, *Mathematical analysis of the photoelectric effect*, Adv. Theor. Math. Phys. **5** (2001), 969–999.

[11] V. Bach, J. S. Møller, and M. C. Westrich, *Beyond the van hove timescale*, preprint in preparation (2019).

[12] M. Ballesteros, D.-A. Deckert, J. Faupin, and F. Hänle, *One-boson scattering processes in the massive spin-boson model*, arXiv:1810.09135 (2018).

[13] M. Ballesteros, D.-A. Deckert, and F. Hänle, *Analyticity of resonances and eigenvalues and spectral properties of the massless spin-boson model*, J. Funct. Anal. **276** (2019), no. 8.

[14] ______, *Relation between the resonance and the scattering matrix in the massless spin-boson model*, Comm. Math. Phys. **370** (2019), 249–290.

[15] M. Ballesteros, J. Faupin, J. Fröhlich, and B. Schubnel, *Quantum electrodynamics of atomic resonances*, Comm. Math. Phys. **337** (2015), no. 2, 633–680.

[16] J.-F. Bony, J. Faupin, and I.M. Sigal, *Maximal velocity of photons in non-relativistic QED*, Adv. Math. **231** (2012), no. 5, 3054–3078.

[17] W. De Roeck, M. Griesemer, and A. Kupiainen, *Asymptotic completeness for the massless Spin-Boson model*, Adv. Math. **268** (2015), 62–84.

[18] W. De Roeck and A. Kupiainen, *Approach to ground state and time-independent photon bound for massless Spin-Boson models*, Ann. Henri Poincaré **14** (2013), no. 2, 253–311.

[19] ______, *Minimal velocity estimates and soft mode bounds for the massless spin-boson model*, Ann. Henri Poincaré **16** (2015), no. 2, 365–404.

[20] J. Dereziński and C. Gérard, *Asymptotic completeness in quantum field theory. Massive Pauli-Fierz Hamiltonians*, Rev. in Math. Phys. **11** (1999), no. 4, 383–450.

[21] J. Faupin, *Resonances of the confined hydrogen atom and the lamb-dicke effect in non-relativistic qed*, Ann. Henri Poincaré **9** (2008), 743–773.
[22] J. Faupin and I. M. Sigal, *Minimal photon velocity bounds in non-relativistic quantum electrodynamics*, J. Stat. Phys. **154** (2014), no. 1-2, 58–90.

[23] _____, *On Rayleigh scattering in non-relativistic quantum electrodynamics*, Comm. Math. Phys. **328** (2014), no. 3, 1199–1254.

[24] J. Fröhlich, M. Griesemer, and B. Schlein, *Asymptotic completeness for Rayleigh scattering*, Ann. Henri Poincaré **3** (2002), 107–170.

[25] _____, *Asymptotic completeness for Compton scattering*, Comm. Math. Phys. **252** (2004), no. 1, 415–476.

[26] J. Fröhlich, M. Griesemer, and I. M. Sigal, *Spectral renormalization group*, Rev. in Math. Phys. **21** (2009), 511–548.

[27] M. Griesemer and D. Hasler, *On the smooth Feshbach-Schur map*, J. Funct. Anal. **254** (2008), no. 9, 2329–2335.

[28] A. Pizzo, *One-particle (improper) states in nelson’s massless model*, Ann. Henri Poincaré **4** (2003), 439–86.

[29] _____, *Scattering of an infraparticle: The one particle sector in nelson’s massless model*, Ann. Henri Poincaré **6** (2005), 553–606.

[30] M. Reed and B. Simon, *Methods of modern mathematical physics i: Analysis of operators*, Academic Press, 1978.

[31] _____, *Methods of modern mathematical physics ii: Fourier analysis, self-adjointness*, Academic Press, 1978.

[32] I. M. Sigal, *Ground state and resonances in the standard model of the non-relativistic QED*, J. Stat. Phys. **134** (2009), no. 5-6, 899–939.

[33] B. Simon, *Resonances in n-body quantum systems with dilatation analytic potentials and the foundations of time-dependent perturbation theory*, Ann. of Math. Sec. Series **97** (1973), no. 2, 247–274.

[34] H. Spohn, *Dynamics of Charged Particles and their Radiation Field*, 1 ed., Cambridge University Press, Cambridge, 2008 (English).