Interpolation sets for Hardy-Sobolev spaces on the boundary of the unit ball of \( \mathbb{C}^n \).

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Abstract

We study the interpolation sets for the Hardy-Sobolev spaces defined on the unit ball of \( \mathbb{C}^n \). We begin by giving a natural extension to \( \mathbb{C}^n \) of the condition that is known to be necessary and sufficient for interpolation sets lying on the boundary of the unit disk. We show that under this condition the restriction of a function in the Hardy-Sobolev space to the set always exists, and lies in a Besov space. We then show that under the assumption that there is an holomorphic distance function for the set, there is an extension operator from these Besov spaces to the Hardy-Sobolev ones.

1 Introduction

In this work we study the boundary interpolation sets for Hardy-Sobolev spaces defined on the unit ball of \( \mathbb{C}^n \). The study of interpolation sets for different spaces is one of the classical subjects of S.C.V. analysis. But in the previous works there are serious restrictions: one considers either sets contained in varieties or sets that have dimension less than one. In this work we study sets not having such restrictions. Even though there are other kinds of restrictions, we believe that one can find here a (perhaps small) step towards the general case.

The study of interpolation sets was begun by Carleson and Rudin (See [Rud,80], chapter 10, for references). They showed (independently) that, for \( n = 1 \), interpolation sets for the ball algebra were precisely those of zero Lebesgue measure. Later, and also for \( n = 1 \), interpolation sets for \( A^\infty(D) \) were described by Alexander, Taylor and Williams in [ATW,71]. In this case the interpolation sets are those satisfying that for any arc \( I \subset \mathbb{T} \),

\[
\frac{1}{|I|} \int_I \log \frac{1}{d(e^{it}, E)} dt \leq C \log \frac{1}{|I|} + C.
\]

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Interpolation sets for the spaces $A_\alpha(D)$ were characterized by Dynkin in \cite{Dyn,80} and Bruna in \cite{Bru,81}. We will say that a closed set $E \subset X$ satisfies the Uniform Hole Condition (UHC-sets, for short) with respect to $X$ if there exists $0 < C < 1$ so that for any $x \in X$ and any ball $B(x, r)$, we have

$$\sup\{d(y, E), y \in B(x, r)\} \geq Cr. \quad (1)$$

The UHC as related to interpolation properties was introduced by Kotchigov, but other equivalent definitions have been introduced by other authors in different contexts. The definition says that a set has holes of a fixed size when looked at at any scale. Dynkin and Bruna proved that, for the spaces $A_\alpha(D)$, $E$ is an interpolation set iff $E$ is a UHC-set. This characterization was obtained by Dynkin for $\alpha \notin \mathbb{N}$ and by Bruna for all $0 < \alpha < +\infty$. Later Dynkin (\cite{Dyn,84}) proved that a set is an interpolation set for the Hardy-Sobolev spaces iff it is a UHC-set.

For $n > 1$ no characterization of boundary interpolation sets is known, not even for the ball algebra. This does not mean that there is no information about interpolation sets. For the ball algebra, Rudin in \cite{Rud,80} devotes all of chapter 10 to these sets, that in this case are the same as peak sets and zero sets. There some examples are given, and one can find some background on the problem.

Also for the ball algebra, Nagel in \cite{Nag,76} proved that any subset of a complex-tangential manifold is an interpolation set. On the other hand, Davie and Øksendal (see \cite{Rud,80}, section 10.5) proved that any set that has, in a sense, dimension less than 1 is an interpolation set. Both results point out to the fact that an interpolation set can be as large as one wants in the complex-tangential directions, but has to be small in the other ones.

The study of zero sets and interpolation sets for spaces other than the ball algebra has been done by several authors. In the case of $A_\alpha(B^n)$ and Hardy-Sobolev spaces results concerning sets contained in varieties were given by Bruna and Ortega in \cite{B-O,86}, \cite{B-O,91}, and \cite{B-O,93}. These works have provided us with our main inspiration. Chaumat and Chollet obtained several results for the space $A^\infty(B^n)$ in, for example, \cite{C-C,86}, and for the Gevrey classes in \cite{C-C,88}.

Our goal was to study interpolation sets for Hardy-Sobolev spaces. But in this case, the first problem was to know, given an $f$ in a Hardy-sobolev space, to which function space defined on the set would the restriction belong. This question, which is in most cases trivial, in this case is not so. However, the results in \cite{B-O,86} showed clearly that the space of the restrictions should be some Besov space. But even in the real case, no general result on restrictions of functions to Besov spaces defined on arbitrary sets is known (however, Jonsson and Wallin in \cite{J-W,84} and Jonsson in \cite{Jon,94} give some partial results). In this article we give a restriction theorem for a general set $E \subset S$. For the restriction to exist, we impose that the Uniform Hole Condition \cite{1} holds. We show that this condition is equivalent to other conditions that will be useful later, and in
particular, that is equivalent to the fact that the set has, in a sense, dimension less than the dimension of $S$.

Once we have done that, we show that under some restrictions, there is an extension operator, thus proving that the given set is interpolating. The restriction we impose is that we assume that there is a holomorphic function behaving like the distance to the set. We give some examples of such functions.

2 Definitions and statement of results

The upper dimension of a set

Let $(X, \rho)$ be a compact pseudo-metric space, with $\text{diam}(X) < +\infty$ (this means that $\rho$ satisfies the triangle inequality with a constant). For $x \in X$, $R > 0$ and $k \geq 1$, let $N(x, R, k)$ be the maximum number of points lying in $B(x, kR)$ separated by a distance greater or equal than $R$.

**Definition 1** We will say that $(X, d) \in \Upsilon_\gamma$ if there exists $C(\gamma) = C(X, d, \gamma)$ such that, for all $x \in X$ and all $0 < R \leq kR \leq 1$,

$$N(x, R, k) \leq C(\gamma)k^\gamma.$$  

We define the upper dimension $\Upsilon(X)$ as

$$\Upsilon(X) = \inf\{\gamma, (X, d) \in \Upsilon_\gamma\}.$$  

This dimension was first introduced by Larman under the name of uniform metric dimension.

We will say that a probability measure $\mu$ lies in $U_\gamma = U_\gamma(X, \rho)$ if there exists $C(\gamma)$ so that for all $x \in X$ and all $0 < R \leq kR \leq 1$,

$$\mu(B(x, kR)) \leq Ck^\gamma \mu(B(x, R)).$$  

$(U_\gamma)$

Note that, by taking $k = 1/R$, $(U_\gamma)$ implies the weaker condition that, for all $x \in X$ and all $0 < R \leq 1$,

$$\mu(B(x, R)) \geq CR^\gamma.$$  

$(U'_\gamma)$

Notice that if for some $\gamma$, $\mu \in U_\gamma$, then $\text{supp}\mu = X$. Moreover, in this case $\mu$ is a doubling measure, that is, there exists $C > 0$ for which:

$$\mu(B(x, 2r)) \leq C\mu(B(x, r)).$$

Let $\mathcal{U} = \cup_\gamma U_\gamma$. It is easily seen (see [V-K, 88]) that $\mathcal{U}$ is precisely the set of all doubling measures with support on $X$.  

3
The lower dimension of a set

**Definition 2** We will say that \((X, d) \in \Lambda\) if there exists \(C(\gamma) = C(X, d, \gamma)\) such that, for \(x \in X\) and \(0 < R \leq kR \leq 1\),

\[ N(x, R, k) \geq C(\gamma)k^\gamma. \] (\(\Lambda\))

Then we define the lower dimension \(\Lambda(X)\) as:

\[ \Lambda(X) = \sup\{\gamma, (X, d) \in \Lambda\}. \]

This dimension was first defined by Larman under the name of minimal dimension.

We will say that a doubling measure \(\mu\) belongs to \(L\) if there exists \(C(\gamma)\) so that for all \(x \in X\) and all \(0 < R \leq kR \leq 1\),

\[ \mu(B(x, kR)) \geq Ck^\gamma \mu(B(x, R)). \] (\(L\))

As before, by taking \(k = 1/R\), condition \((\Lambda)\) implies

\[ \mu(B(x, R)) \leq CR^\gamma. \] (\(L\))

Note that \(L_0\) poses no restriction on \(\mu \in U\).

The following improvement of Volberg and Konyagin’s theorem 1 in [V-K,88] can be found in [B-G,98]:

**Theorem 3** Let \((X, \rho)\) be a pseudo-metric space. Let \((X, \rho) \in \Upsilon_\nu \cap \Lambda_\lambda\), for some \(0 < \lambda \leq \nu < +\infty\). Then for any \(\nu' > \nu\) and \(\lambda' < \lambda\) (or \(\lambda' = 0\) if \(\Lambda(E) = 0\)) there exists \(\mu \in U_{\nu'} \cap L_{\lambda'}\).

The uniform hole condition

We are now going to restrict ourselves to closed subsets \(E \subset S\). On \(S\) we will use the pseudo-metric given by \(d(x, y) = |1 - x\bar{y}|\). In fact, we will consider \(d\) as a function defined on \(B^n \times B^n\), where \(d\), although it is not a metric, satisfies the triangle inequality \(d(x, z) \leq \sqrt{2}(d(x, y) + d(y, z))\) (for this and the following, see [Rud,80], chapter 5).

If we consider \(\rho(x, y) = d(x, y)^{1/2}\), then \(\rho\) is a metric on \(S\). Then using proposition 5.1.4 in [Rud,80], we get that \(U(S, d) = L(S, d) = n\), and that for any \(E \subset S\), \(U(E, d) \leq n\).

Let \(\sigma\) be the normalized Lebesgue measure on \(S\).

**Definition 4** We will say that a closed set \(E \subset S\) satisfies \(\Sigma_s\) if there exists a \(C(s)\) so that for any \(x \in S\), \(R > 0\) and \(0 < \varepsilon < R\),

\[ \sigma(B(x, R) \cap E_\varepsilon) \leq CR^s\varepsilon^{n-s}, \] (\(\Sigma_s\))

where \(E_\varepsilon = \{z \in S, d(z, E) < \varepsilon\}\).
For \( n = 1 \), Bruna ([Bru, 81]), using also results from Dynkin ([Dyn, 80]) proved that \( E \) satisfied the UHC iff there was a \( s < 1 \) so that \( E \) satisfied \( \Sigma_s \), and that both conditions were equivalent to the boundedness of certain integrals. We are going to extend Dynkin and Bruna’s results to \( \mathbb{C}^n, n \geq 1 \). Namely, we are going to prove the following:

**Theorem 5** Let \( E \subset S \) be a closed set. Then the following statements are equivalent:

(a) \( E \) satisfies the Uniform Hole Condition;

(b) There is a \( s < n \) so that \((E, d)\) satisfies \( \Sigma_s \);

(c) \( \Upsilon(E) < n \);

(d) There are an \( a > 0 \) and \( C > 0 \) so that for any \( x \in S \) and \( R > 0 \),

\[
\int_{B(x, R)} d(z, E)^{-a}d\sigma(z) \leq CR^{n-a};
\]

(e) There is a \( C > 0 \) so that for any \( x \in S \) and \( R > 0 \),

\[
\int_{B(x, R)} \log(1 + d(z, E)^{-1}) d\sigma(z) \leq \sigma(B(x, R)) \log \frac{1}{R} + CR^n;
\]

(f) There are \( a < n \) and \( C > 0 \) so that, for any \( x \in E \) and \( R > 0 \),

\[
\int_0^R N(x, \frac{R}{\delta}, \frac{R}{\delta})^{n-a}d\delta \leq CR^{n-a};
\]

(g) There are \( s < n \) and \( C > 0 \) so that, for any \( x \in E \) and \( R > 0 \),

\[
V(B(x, R) \cap E_\varepsilon) \leq CR^n \varepsilon^{n+1-s};
\]

(h) There is an \( a_0 > 1 \) so that for any \( a < a_0 \) there is a \( C > 0 \) so that for any \( x \in S \) and \( R > 0 \),

\[
\int_{B(x, R)} d(z, E)^{-a}dV(z) \leq CR^{n+1-a}.
\]

**The restriction theorem**

Let \( E \subset S = \partial B^n \) be a closed set, and assume \( \Upsilon(E) < n \). We know by theorem 3 that for some \( 0 \leq d \leq s < n \) there is a measure \( \mu \in U_s \cap L_d \). Note that if \( U_d \cap L_d \neq \emptyset \) for some \( d \), then \( E \) is Alhfors regular.

\[\text{5}\]
We will work with functions $f \in H^p_\beta(B^n)$. Recall that if $f \in \Hol(B^n)$, we can define its radial derivative as $Nf(z) = \sum z_j \frac{\partial}{\partial z_j} f(z)$. If $f = \sum_k f_k$ is the homogeneous expansion of $f$, we consider the fractional derivative

$$R^\beta f = \sum_k (k + 1)^\beta f_k$$

so that $R^1 = I + N$. Then for $p \geq 1$ and $\beta > 0$ the Hardy-Sobolev space $H^p_\beta(B^n)$ consists of those holomorphic functions such that

$$\|f\|^p_{H^p_\beta(B^n)} = \sup_{0 < r < 1} \int_S |R^\beta f(r\zeta)|^p d\sigma(\zeta) < +\infty,$$

where $\sigma$ is the normalized Lebesgue measure on $S$.

Let $X = \sum_j a_j(z) \frac{\partial}{\partial z_j} + a_j(z) \frac{\overline{\partial}}{\partial \overline{z}_j}$, where $a_j \in C^\infty(B^n)$, be a vector field. We define its weight $\omega(X)$ as $1/2$ if $X$ is complex-tangential, i.e. $\sum a_j z_j = 0$, and 1 otherwise; for a differential operator $\mathcal{X} = X_1 \cdots X_p$ define $\omega(\mathcal{X}) = \sum \omega(X_j)$. It is then known (see [A-B,88]) that if $\omega(\mathcal{X}) \leq \beta$, $\mathcal{X} f$ has radial limit $\sigma$-a.e. on $S$, even though $\mathcal{X}$ may have order bigger than $\beta$. For such $f$ and $\mathcal{X}$, it makes sense to define the Hardy-Littlewood maximal function of $\mathcal{X} f$ as:

$$M(\mathcal{X} f)(z) = \sup_{\delta > 0} \frac{1}{V(B(z, \delta) \cap B^n)} \int_{B(z, \delta) \cap B^n} |\mathcal{X} f(\zeta)| dV(\zeta),$$

where $z \in S$.

In this context, we want to know under which conditions there is a reasonable way of defining the restriction $\mathcal{X} f|_E$ and in which space of functions it lies. For doing so, we use the results of [B-O,93], where the following is proved:

**Lemma 6** Let $\mu$ be a measure on $S$ satisfying $L^p_d$, for some $d < n + 1$. Then for any differential operator $\mathcal{X}$ with $\omega(\mathcal{X}) < \beta - \frac{n - d}{p}$,

$$\int_S M(\mathcal{X} f)^p(\zeta) d\mu(\zeta) \leq C \|f\|^p_{H^p_\beta(B^n)}.$$

In particular, there exist $\mu$-almost everywhere the limits:

$$\lim_{r \to 1} \mathcal{X} f(r\zeta), \quad \lim_{\delta \to 0} \frac{1}{V(B(\zeta, \delta))} \int_{B(\zeta, \delta)} \mathcal{X} f(\zeta) dV(\zeta),$$

and they are equal.

For a function $f \in H^p_\beta(B^n)$ we can define, as in [B-O,86] and [B-O,93], the non-isotropic Taylor polynomial at a point $\zeta \in E$. This Taylor polynomial $T^n_\alpha f$ is twice as long in the complex-tangential directions. The non-isotropic Taylor polynomial can be defined in an intrinsic way, using the covariant differentials of $f$, as in [B-O,86], or in an explicit way, using local coordinates, as in [B-O,93].
Let us express $T^\alpha_x f(z)$ in coordinates: for a point $\zeta \in S$, let $w_\mu(z, \zeta) = 1 - z\zeta$ be the normal coordinate, and let $w_1, \ldots, w_{n-1}$ coordinate $T^\alpha_x z$. Then because of lemma 3, if $\gamma = (\gamma_1, \ldots, \gamma_n)$ is a multiindex with weight $\omega(\gamma) = \gamma_n + 1/2(\gamma_1 + \ldots + \gamma_{n-1}) < \alpha$,

$$D^\gamma f(\zeta) := \lim_{r \to 1} \frac{\partial^{\gamma}}{\partial w^{\gamma}} f(r\zeta) = \lim_{\delta \to 0} \frac{1}{V(B(\zeta, \delta))} \int_{B(\zeta, \delta)} \frac{\partial^{\gamma}}{\partial w^{\gamma}} f(w) \, dV(w)$$

exists $\mu$-a.e. on $E$. Clearly $\{D^\gamma f(\zeta), \omega(\gamma) < \alpha\}$ determines $df(\mathbb{X})(\zeta)$ if $\omega(\mathbb{X}) < \alpha$ and we have

$$T^\alpha_x f(z) = \sum_{\omega(\gamma) < \alpha} \frac{1}{\gamma!} D^\gamma f(\zeta) w(z, \zeta)^\gamma,$$

where $z \in B$.

In view of that, we define the holomorphic jets of class $B^\alpha_\omega(\mu)$ as those collections $F = (F_\gamma)_{\omega(\gamma) < \alpha}$ of $L^p(\mu)$ functions such that

$$\sum_{\omega(\gamma) < \alpha} \|F_\gamma\|_{L^p(\mu)}^p + \sum_{\omega(\gamma) < \alpha} \int_{E \times E} \frac{|F_\gamma(x) - D^\gamma(T^\alpha_x F)(x)|^p}{d(x, y)^{(\alpha - \omega(\gamma))p-d}} \, d\mu(x) \, d\mu(y)$$

(2)

is finite, where $\mu(x, y) = \mu(B(x, d(x, y)))$.

Note that if $E$ is Ahlfors regular the corresponding Besov space is given by the norm:

$$\|F\|_{B^\alpha_\omega(\mu)} = \sum_{\omega(\gamma) < \alpha} \|F_\gamma\|_{L^p(\mu)} + \int_{E \times E} \frac{|F_\gamma(x) - D^\gamma(T^\alpha_x F)(x)|^p}{d(x, y)^{(\alpha - \omega(\gamma))p+d}} \, d\mu(x) \, d\mu(y).$$

**Remarks:** These Besov spaces with respect to $\mu$ where first introduced by Dynkin in [Dyn,84], when studying the interpolation problem for $H^1$ in $\mathbb{C}$, and later extended to subsets of $\mathbb{R}^n$ by Jonsson in [Jon,94], using only first differences of functions, when studying the restriction of a Besov space on $\mathbb{R}^n$ to a closed set.

On the other hand, we would like to remark that our spaces can be seen as spaces with variable regularity. For example let $E = \Gamma_1 \cup \Gamma_2$ where $\Gamma_1$ is a closed transverse curve whereas $\Gamma_2$ is a closed complex-tangential one, and they are disjoint. Then if we let $\mu$ be the linear Lebesgue measure on $E$, $\mu \in L_{1/2} \cap U_1$. On the other hand, on $\Gamma_1$ $d(x, y) \approx |x - y|$ whereas on $\Gamma_2$ $d(x, y) \approx |x - y|^{1/2}$. Hence, on $\Gamma_1$

$$d(x, y)^{\alpha p - d} \mu(B(x, d(x, y)))^2 \approx |x - y|^\alpha \frac{1}{|x - y|^{1/2} + 2}$$

whereas on $\Gamma_2$

$$d(x, y)^{\alpha p - d} \mu(B(x, d(x, y)))^2 \approx |x - y|^{\frac{1}{2} \alpha} \frac{1}{|x - y|^{1/2} + 2},$$

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so that these spaces are of the Besov kind, but they have different regularity in $\Gamma_1$ and $\Gamma_2$.

Another useful fact, that we will use later without further comment, is that $\mu[x, y] \approx \mu[y, x]$. This is so because

$$\mu(B(x, d(x,y))) \leq \mu(B(y, 4d(x,y))) \leq C\mu(B(y, d(x,y)))$$

because of $U_s$, and if we exchange $x$ for $y$ we get the reverse inequality. Hence our definition is symmetric with respect to $x$ and $y$.

We have seen that for $f \in H^p_\beta(B^n)$ and $\alpha = \beta - n - d/p$, there is a natural way of defining the restriction $D^\gamma f_{|E}$ whenever $\omega(\gamma) < \alpha$. In this case, we could ask ourselves whether $f \in B^\alpha_\mu$. The answer is yes at least if $E$ is Ahlfors-regular, with $2\alpha \notin \mathbb{N}$, or if $\Upsilon(E)$ and $\Lambda(E)$ are close enough (depending on $p$). More precisely, the result is as follows:

**Theorem 7** Let $E \subset S$ be a closed set with $\Upsilon(E) < n$. Assume that between $\beta - \frac{n-\Lambda(E)}{p}$ and $\beta - \frac{n-\Upsilon(E)}{p}$ lies no integer multiple of $\frac{1}{2}$. Take $n > s \geq \Upsilon(E)$ and $d \leq \Lambda(E)$ close enough so that the same is true for $\beta - \frac{n-d}{p}$ and $\beta - \frac{n-s}{p}$. Take any $\mu \in U_s \cap L_d$. Then for any $f \in H^p_\beta(B^n)$ its restriction to $E$ lies in $B^\alpha_\mu$, where $\alpha = \beta - \frac{n-d}{p}$.

**Remarks:** If $E$ is a transverse curve, then the Lebesgue measure on $E$ lies in $U_1 \cap L_1$, and for these curves we recover the results in [B-O,91]. If $E$ is a complex-tangential submanifold of real dimension $d$, then the Lebesgue measure on this submanifold lies in $U_{d/2} \cap L_{d/2}$, so that we recover the results in [B-O,93].

On the other hand, for a general curve $\Gamma$ we get better results than in [B-O,91], because there in that case one gets the same space as the one for a transverse curve, whereas we get a restriction theorem into a space of variable regularity. Namely, they get that the restriction is in a space defined by $2$ but with the metric $|x-y|$, while our spaces are defined by $d(x,y) \geq C|x-y|$. In particular, our spaces are included into the isotropic ones.

The proof of the theorem is based on the representation of $f$ as an integral of $R^\beta f$, together with the use of the Bergman kernels, and the development of $(1 - \frac{z^\gamma}{\varphi})^{-N}$ in a suitable way, plus the bounding of certain integrals.

The restriction on $\beta - \frac{n-d}{p}$ and $\beta - \frac{n-s}{p}$ is more or less natural. Notice that our spaces are defined using only first differences. In the case of an Ahlfors-regular set, the restriction is that $2\alpha \notin \mathbb{N}$, which is the natural one in this case. Thus our restriction is related to the use of first differences.

**The extension theorem**

Consider $H^p_\beta(B^n)$ and $B^\alpha_\mu$, with $\alpha = \beta - \frac{n-d}{p}$, defined as above. We want to prove that, in some cases, for each jet $(F_\gamma)_{\omega(\gamma)<\alpha} \in B^\alpha_\mu$ there exists
Let \( f \in H^p_\beta(B^n) \) so that, for \( \omega(\gamma) < \alpha \), \((D^\gamma f)|_E = F_\gamma\) in \( L^p(d\mu) \). In this chapter we will introduce a condition under which it holds:

**Definition 8** Let \( E \subset S \) be a closed set, and let \( h \in Hol(B) \cap C^\infty(B^n \setminus E) \).

We will say that \( h \) is a holomorphic distance function for \( E \) if:

1. there is a \( C_1 > 0 \) so that
   \[
   C_1^{-1}d(z, E) \leq |h(z)| \leq C_1d(z, E)
   \]
   for all \( z \in B^n \setminus E \);

2. for any differential operator \( \mathcal{X} \) there is a constant \( C(\mathcal{X}) \) so that
   \[
   |\mathcal{X}h(z)| \leq Cd(z, E)^{1-\omega(\mathcal{X})}
   \]
   for all \( z \in B^n \setminus E \).

We will give some examples of such functions in the following subsection.

To prove the following theorem we will work with the Triebel-Lizorkin norms, instead of the Hardy-Sobolev ones. Let \( 0 < p < +\infty \), \( 0 < q < +\infty \), and \( \beta \geq 0 \).

Then the Triebel-Lizorkin space \( HF^p,q_\beta(B^n) \) is the set of holomorphic functions \( f \) on \( B^n \) so that

\[
\|f\|_{p,q,\beta} = \left( \int_S \left( \int_0^1 (1-r^2)^{[(\beta+1-\beta)q-1]} |R^{\beta+1}f(rz)|^q dr \right)^{\frac{p}{q}} d\sigma(z) \right)^{\frac{1}{p}} < +\infty.
\]

It is well known that \( HF^{p,2}_\beta(B^n) = H^p_\beta(B^n) \), and also that \( HF^{p,q_1}_\beta \subset HF^{p,q_2}_\beta \) if \( q_2 \geq q_1 \).

There are two reasons for working with the Triebel-Lizorkin norms. The first one is that it is technically simpler to work with integer powers of \( R \) than to work with \( R^\beta \), for \( \beta \notin \mathbb{N} \). On the other hand, the results we get are more general.

**Theorem 9** Let \( E \subset S \) be a closed set, with \( \Upsilon(E) < n \). Assume that between \( \beta - \frac{n-\Lambda(E)}{p} \) and \( \beta - \frac{n-\Upsilon(E)}{p} \) lies no integer multiple of \( \frac{1}{2} \).

Let \( d \leq \Lambda(E) \) and \( n > s \geq \Upsilon(E) \) be close enough so that this fact is still true for \( \beta - \frac{n-d}{p} \) and \( \beta - \frac{n-s}{p} \), and take \( \mu \in L_d \cap U_s \). Let \( \alpha = \beta - \frac{n-s}{p} \). Assume that there is a holomorphic distance function \( h \) for \( E \). Then for each jet \((F_\gamma)_{\omega(\gamma) < \alpha} \in B^p_\mu(d\mu) \) there is an \( f \in HF^{p,1}_\beta \) so that, for \( \omega(\gamma) < \alpha \), we have \((D^\gamma f)|_E = F_\gamma\) in \( L^p(d\mu) \).

This theorem gives us directly that \( E \) is an interpolation set for the Hardy-Sobolev spaces.

To prove the theorem, we will first construct a function \( g \) satisfying the required growth and interpolation properties, and then we will correct it using a \( \overline{\partial} \) process to get the holomorphic function we are looking for.
On the other hand, it is easily checked that the condition that between $\alpha = \beta - \frac{n-d}{p}$ and $\beta - \frac{n-s}{p}$ lies no integer multiple of $\frac{1}{2}$ is needed only to see that the function we construct lies in $HF^{p,1}_\beta$, and only to deal with the derivatives. Therefore if $\alpha < 1/2$ it is not needed.

Observe that a theorem similar to theorem \ref{thm:ga} but involving the spaces $A_\alpha(B) = \overline{\text{Hol}(B) \cap C^\infty(B^n)}$, had already been proved. For these spaces, Bruna and Ortega in \cite{B-O.86} gave the following:

**Theorem 10 (Bruna-Ortega)** Let $\Gamma$ be a transverse curve, and let $E \subset \Gamma$ with $\Upsilon(E) < 1$. Then $E$ is an interpolation set for $A_\alpha$, for $\alpha \in \mathbb{R}^+ \setminus \mathbb{N}$.

In reading the proof, it is easy to check that the fact that $E$ is contained in a transverse curve is used at two points of it: when, in theorem \ref{thm:ga3}, it is proved that for such a set there is a holomorphic distance function; and in lemma \ref{lem:ga5}, where it is proved that for such a set, condition (d) in theorem \ref{thm:ga} is satisfied. But in the proof of the theorem what is used is that $\Upsilon(E) < n$. Hence, what is proved there is that under the same hypothesis as in theorem \ref{thm:ga}, $E$ is an interpolation set for $A_\alpha(B)$.

Consider now the Besov spaces $A_{p,q,\alpha}(B^n)$. It is well known that if we have two pairs $(q_1, \alpha_1)$ and $(q_2, \alpha_2)$ then $A_{p,q_1,\alpha_1} = A_{p,q_2,\alpha_2}$ whenever $\alpha_2 - \alpha_1 = (q_2 - q_1)/p$. Another remarkable fact about these spaces is that, in a limit sense, $A_{p,0,\alpha} = H^\alpha_p$.

Let $M = \{ z \in B^{n+1}, z_{n+1} = 0 \} \approx B^n$. As $M = B^n$, we have on $M$ the spaces $A_{p,0,\alpha}(M)$. We consider on $B^{n+1}$ the spaces $H^\alpha_p(B^{n+1})$. Beatrous in \cite{Bea.86} proves that there exists a bounded restriction operator $R : H^\alpha_p(B^{n+1}) \to A_{p,0,\alpha}(B^n)$ for $\beta = \alpha - (q - 1)/p$, and that in this case there is also an extension operator $E : A_{p,0,\alpha}(B^n) \to H^\alpha_p(B^{n+1})$ such that $R \circ E = Id$. Therefore, using theorems \ref{thm:ga} and \ref{thm:ga2} we can obtain similar results for these Besov spaces. Moreover, in the process of passing from $\mathbb{C}^n$ to $\mathbb{C}^{n+1}$ we drop the condition $\Upsilon(E) < n$.

**Examples of holomorphic distance functions**

Here we are going to give some examples of sets $E$ for which there is a holomorphic distance function.

**The Chaumat-Chollet example:** Our first example is the one given by Chaumat and Chollet in \cite{C-C.88}, where they construct a holomorphic distance function on $E$ whenever $\Upsilon(E) < 1$.

They proceed as follows: they begin with any set satisfying condition (f) in theorem \ref{thm:ga} for some $n - 1 < a < n$; then they take for each $k$ a $2^{-k}$ covering of
by balls \( \{ B(\zeta_{j,k}, 2^{-k}), j = 1, \ldots, N_k \} \). Then, by defining
\[
\phi(z) = \sum_{k=1}^{\infty} \sum_{j=1}^{N_k} \frac{2^{-k(n-a)}}{2^{-k} + (1 - z \zeta_{j,k})}
\]
they get a holomorphic function \( \phi \) such that \( |\phi(z)| \approx d(z, E)^{a-n} \) and \( \Re(\phi(z)) > 0 \). Hence \( h(z) = \phi(z)^{1/2} \) is the desired function. But saying that there is an \( a \) with \( n - 1 < a < n \) that satisfies (f) in theorem \( \text{[3]} \) is the same as saying that \( \Upsilon(E) < 1 \). Thus we have the following analogue of Davie-Øksendal theorem:

**Corollary 11**  Any set \( E \) with dimension \( \Upsilon(E) < 1 \) is an interpolation set for \( H^p_\beta(B^n) \).

We would like to remark that there is a simpler way of constructing this distance function. Let \( \Upsilon(E) \leq s < 1 \), and take \( \mu \in U_s(E) \). Let \( s < q_1 < q_2 < 1 \). Define
\[
h_q(z) = \int_E \frac{1}{(1 - z \zeta)^q} d\mu(\zeta),
\]
with \( q = q_1, q_2 \). Then as \( q < 1 \), we have that \( \Re(1 - z \zeta)^q \geq C_q |1 - z \zeta|^q \). Using it, is rather simple to see that \( |h_q(z)| \approx d(z, E)^{-q} \mu(B_z) \), and that if we take \( h(z) = h_{q_2}(z)/h_{q_1}(z) \), then \( |h(z)| \approx d(z, E)^{q_2 - q_1} \). As \( h \) takes values on a sector not containing the line \( \{ \Re z < 0, \Im z = 0 \} \), we can take roots of it. Hence we can consider \( h^{1/(q_2 - q_1)} \); and this is the function we were looking for.

**Nagel’s example:** In [Nag,76], Alexander Nagel proves that any compact set \( K \) of a complex-tangential manifold \( M \) is an interpolation set for the ball algebra. He does it by constructing a holomorphic function with specified boundary behaviour, namely:
\[
h_p(z) = \int_M \frac{dx}{F(z, \varphi(x))^p}
\]
where, for \( \frac{2}{n} < p < \frac{2}{n} + \frac{1}{2}, \Re F > 0 \) and \( |F(z, \varphi(x))| \) behaves as nicely so as to get that \( |h_p(z)| \approx d(z, M)^{\frac{2}{n} - p} \). In particular, \( M \) is an interpolation set for \( H^p_\beta(B^n) \).

If \( K \subset M \) is a compact set, then it is an interpolation set for the ball algebra, and also for \( A_n(B^n) \). For if we have a function \( f \) on \( K \), as \( M \) is totally real, we can extend it by any real method to the whole manifold, and then extend it from the manifold to the ball.

On \( B^p_\alpha(\mu) \), though, there was no known general result on the extension of functions from subsets of \( \mathbb{R}^n \) to \( \mathbb{R}^n \). But in [Gud,98] one can find the necessary results, so that we will be able to extend any function first to \( M \), and then from \( M \) to \( H^p_\beta(B^n) \). Therefore, any compact subset of \( M \) is an interpolation set for \( H^p_\beta(B^n) \), with the usual restrictions on the indices.
An interpolation set of Hausdorff dimension $n - \delta$: For each $0 < \delta < 1$ we can build an interpolation set with Hausdorff dimension $n - \delta$. To do so, we consider the variety

$$\Gamma = \{ z \in S, \Im(z_1) = \ldots = \Im(z_n) = 0 \}.$$  

Then the Hausdorff dimension of $\Gamma$ is $n - 1$, and $\Upsilon(\Gamma) = \Lambda(\Gamma) = \frac{n-1}{2}$, because this variety is complex-tangential.

Take $0 < \delta < 1$ and let $C_\delta \subset [-\frac{1}{2}, \frac{1}{2}]$ be the Cantor set with Hausdorff dimension $\delta$. Then for each $t \in C_\delta$ let $\Gamma_t = e^{it}\Gamma$ be the rotation of $\Gamma$, that is $\{e^{it}z, z \in \Gamma\}$. Then

$$d(\Gamma_s, \Gamma_t) = \inf\{ |1 - e^{i(t-s)}x \cdot y|, x, y \in S^{n-1}(\mathbb{R}) \}$$

and as $-1 \leq x \cdot y \leq 1$, this minimum is of the order of $|\sin(t-s)| \approx |t-s|$.

Let $f(z) = \frac{1}{2}(1 + z_1^2 + \ldots + z_n^2)$. Then clearly $f$ is a peak function for $\Gamma$, so that $f(t) = f(e^{-it}z)$ is a peak function for $\Gamma_t$. Now theorem 6.2 in [B-O,86] says that if $M$ is a complex-tangential variety of dimension $n - 1$ and $f \in A^\infty$ is a peak function on $M$, the function $h(z) = 1 - f(z)$ satisfies $|h(z)| \approx d(z, M)$. Then it is easily checked that $h$ is a holomorphic distance function for $M$. Hence for each $t \in C$ the function $1 - f_t(z)$ satisfies $|1 - f_t(z)| \approx d(z, \Gamma_t)$. To do so, let $\mu$ be the Hausdorff measure on $C_\delta$, and let $\delta < q < 1$. Then we define

$$h_q(z) = \int_{C_\delta} \frac{1}{(1 - f_t(z))^q} d\mu(t).$$

This function satisfies:

$$|h_q(z)| \approx d(z, E)^{-(q-\delta)}$$

and $\Re h_q(z) > 0 \quad \forall z \in B$. (3)

Also, for any differential operator $\overline{X}$,

$$|\overline{X}h_q(z)| \leq C(\overline{X})d(z, E)^{\delta-q-\omega(\overline{X})}.$$

Hence if we write $h(z) = h_q(z)$, we have built a holomorphic distance function for $E$, so $E$ is an interpolation set for $H^p_\beta(B^n)$.

We only have to check $\overline{X}$ as the other inequality is proved in the same way. To begin with, we will check the upper inequality.

If $I$ is an interval centered at some $t \in C_\delta$, then $\mu(I) \approx |I|^{\delta}$. Now fix $z \in B$, let $t_0 \in E$ be so that $d(z, E) = d(z, \Gamma_{t_0})$ and for each $t \in C_\delta$ let $z_t \in \Gamma_t$ be so that $d(z, z_t) = d(z, \Gamma_t)$. Let $B_k$ be the set defined by

$$B_k = \{ t \in C_\delta, d(z, \Gamma_t) \leq 2^{k}d(z, \Gamma_{t_0}) \}$$

for $k \geq 0$, and $B_{-1} = \emptyset$. Then if $s \in B_k$, the distance from $s$ to $t_0$ is comparable to $d(\Gamma_s, \Gamma_{t_0})$. Thus, and because of the triangle inequality,

$$|s - t_0| \leq Cd(z_s, z_{t_0}) \leq C(d(z, \Gamma_s) + d(z, \Gamma_{t_0})) \leq C2^k d(z, E),$$

and
so that \( \mu(B_k) \leq C2^{kh}d(z, E)^\delta \). Now if we decompose the integral over \( E \) into the integrals over the coronae \( B_{k+1} \setminus B_k \), and use the previous inequality, we obtain

\[
|h_q(z)| \leq C \sum_{k=0}^{\infty} 2^{-qk}d(z, E)^{-q}\mu(B_k \setminus B_{k-1}) \leq Cd(z, E)^{-q+\delta}
\]

as we wanted to see.

For the other bound in (3), we use that

\[
d(z, \Gamma_t) \leq d(z, e^{i(t-t_0)}z_0) \leq C(d(z, E) + |t-t_0|),
\]

whence \( \{t, |t-t_0| \leq d(z, E)\} \subset \{t, d(z, \Gamma_t) \leq Cd(z, E)\} \). Now, we use that \( \Re(1 - f_i) > 0 \) and \( q < 1 \), so that we can use the bound \( \Re(1 - f_i)^q \geq C_q|1 - f_i|^q \). Thus we can, modulo a constant, enter the modulus inside the integral. Then we can bound the integral by the integral over a smaller set where we can compare \( d(z, E) \) with \( d(z, \Gamma_t) \), and obtain the result.  

\[\blacksquare\]

### 2.1 proof of theorem 5

We begin by proving that (a) implies (b), which is the hardest. To prove it, we will use the following lemma, due to Sawyer and Wheeden ([S-W,92]):

**Lemma 12** Let \((X, d)\) be a separable quasi-metric space, that is, \(d\) satisfies the triangle inequality with constant \(A_0\). Then for \(\lambda = 8A_0^2\) and for any \(m \in \mathbb{Z}\), there are points \(\{x_j^k, k \geq m, j = 1, \ldots, n_j\}\) and Borel sets \(\{X_j^k, k \geq m, j = 1, \ldots, n_j\}\) (where \(n_j \in \mathbb{N} \cup \{\infty\}\)) such that

i) \(B(x_j^k, \lambda^k) \subset X_j^k \subset B(x_j^k, \lambda^{k+1})\);

ii) for any \(k \geq m\), \(\bigcup_{j=1}^{n_j} X_j^k = X\);

iii) given \(i, j, k, \) and \(l\), with \(m \leq k \leq l\), either \(X_j^k \subset X_j^l\) or \(X_j^k \cap X_j^l = \emptyset\).

We will work in \((S, \rho)\). Here \(\rho\) is a metric, so we can apply the previous lemma with any \(A_0 \geq 1\). Note also that there is a constant \(C_0\), depending only on \(n\), so that for any ball \(Q(x, R)\) and any \(y \in Q(x, R)\), if \(r < R\), although \(Q(y, r)\) might not be contained in \(Q(x, R)\), there is a ball \(Q(w, C_0r) \subset Q(x, R) \cap Q(y, r)\). We will write \(K_0\) for the constant appearing in (4).

We take \(\lambda = \max\{8, 4/(K_0C_0)\}\). Fix \(x \in S, R > 0\) and \(0 < \varepsilon < R\), and choose \(m \in \mathbb{Z}\) so that \(\lambda^m \geq 2\varepsilon/(K_0C_0) > \lambda^{m-1}\), and apply lemma 12 with \(\lambda\) and \(m - 4\) to the pseudo-metric space \((Q(x, R), \rho)\). Let \(\{X_j^k, k \geq m - 4, j = 1, \ldots, n_j\}\) be the dyadic decomposition of \(Q(x, R)\) given by the lemma. Let \(k_0 \in \mathbb{Z}\) be the only integer so that \(\lambda^{k_0-1} < 2R \leq \lambda^{k_0}\). This implies that \(X_1^{k_0} = Q(x, R)\) whereas \(n_{k_0-2} > 1\). Write \(U_{k_0-2} = Q(x, R)\). Then \(U_{k_0-2}\) satisfies
1. \( U_k = \bigcup_{X_j^k \subset U_k} X_j^k \);

2. \( E_\varepsilon \cap Q(x, R) \subset U_k \),

with \( k = k_0 \). Fix \( k \leq k_0 - 2 \) and assume we have built \( U_k \) satisfying (1) and (2). Assume also that \( \lambda^k \geq 2\varepsilon/(K_0 C_0) \). We are going to see that we can build \( U_{k-2} \) also satisfying the previous properties and with mass less than a constant times the mass of \( U_k \).

Take a \( j \) so that \( X_j^k \subset U_k \). Then \( Q(x_j^k, \lambda^k) \subset X_j^k \). As \( x_j^k \in Q(x, R) \) there is a ball \( Q(z, C_0 \lambda^k) \subset Q(x_j^k, \lambda^k) \cap Q(x, R) \). Inside \( Q(z, C_0 \lambda^k) \) there must be a ball \( Q(w, K_0 C_0 \lambda^k) \) not intersecting \( E \). If \( K_0 C_0 \lambda^k > 2\varepsilon \), then \( Q(w, K_0 C_0 \lambda^k/2) \cap E_\varepsilon = \emptyset \).

On the other hand, \( X_j^k \) is the union of the sets \( X_i^{k-2} \) contained in it, so that there must be a \( l_0 \) so that \( w \in X_{l_0}^{k-2} \), and then \( \rho(w, x_{l_0}^{k-2}) \leq \lambda^{k-1} \). Hence, \( X_{l_0}^{k-2} \subset Q(x_{l_0}^{k-2}, \lambda^{k-1}) \subset Q(w, 2\lambda^{k-1}) \). But as \( \lambda \geq 4/(K_0 C_0) \), also \( 2\lambda^{k-1} \leq K_0 C_0 \lambda^{k}/2 \), and then \( Q(w, 2\lambda^{k-1}) \subset Q(w, K_0 C_0 \lambda^{k}/2) \). Thus \( X_{l_0}^{k-2} \cap E_\varepsilon = \emptyset \). If we define

\[
\tilde{X}_j^k = \bigcup_{X_i^{k-2} \cap E_\varepsilon \neq \emptyset} X_i^{k-2}
\]
we are omitting at least \( X_{l_0}^{k-2} \). So by writing

\[
U_{k-2} = \bigcup_{X_j^k \subset U_k} \tilde{X}_j^k
\]
we obtain that \( U_{k-2} \) satisfies the same conditions as \( U_k \).

We are going to see that, when passing from \( U_k \) to \( U_{k-2} \) we are taking away at least a fixed part of the mass of \( U_k \). As \( X_j^k \subset Q(x_j^k, \lambda^{k+1}) \), its mass is at most \( C_n \lambda^{(k+1)n} \), whereas the mass of \( X_i^{k-2} \) is at least \( c_n \lambda^{(k-2)n} \), as it contains \( Q(x_{l_0}^{k-2}, \lambda^{k-2}) \). Then \( \sigma(X_{l_0}^{k-2}) \) is at least \( 1/(C\lambda^{3n}) \) of \( \sigma(X_j^k) \). As this is true for any \( j \),

\[
\sigma(U_{k-2}) \leq \left( 1 - \frac{1}{C\lambda^{3n}} \right) \sigma(U_k).
\]

We begin with \( U_{k_0-2} \) and we can go through the previous process while \( \lambda^{k_0-2j} \geq 2\varepsilon/(C_0 K_0) \). As \( m \) is the last integer satisfying this inequality, we can keep on doing it while \( k_0 - 2j \geq m \), that is, while \( 2j \leq k_0 - m \). Take \( j \) to be the last one fulfilling this inequality. For such a \( j \), we can bound \( \sigma(Q(x, R) \cap E_\varepsilon) \) by \( \sigma(U_j) \). Applying the previous bounds, we get that:

\[
\sigma(U_j) \leq \left( 1 - \frac{1}{C\lambda^{3n}} \right)^{j-1} \sigma(U_{k_0-2}) \leq C \left( 1 - \frac{1}{C\lambda^{3n}} \right)^{j_0} \sigma(U_{k_0-2}) \leq C \left( 1 - \frac{1}{C\lambda^{3n}} \right)^{j_0} R^{2n}.
\]
On the other hand, \( \lambda^{n-1} < 2\varepsilon/(K_0C_0) \) and \( \lambda^{k_0} \geq 2R \), so that \( \lambda^{k_0 - m} \geq 2K_0C_0R/\varepsilon \), that is,

\[
(k_0 - m) \log \lambda \geq \log \frac{K_0C_0}{\lambda} + \log \frac{R}{\varepsilon}.
\]

If we write \( s = -\frac{1}{\lambda}(\log (1 - \frac{1}{C\lambda})) \)/\( \log \lambda \), then \( s > 0 \) and, because of the previous inequality,

\[
\left(1 - \frac{1}{C\lambda^m}\right)^{\frac{1}{s}(k_0 - m)} R^{2n} = \exp(-s(k_0 - m) \log \lambda) R^{2n} \leq C\varepsilon^s R^{2n-s}
\]

so that \((E, \rho) \in \Sigma_{2n-s} \), therefore \((E, d) \in \Sigma_{n-\frac{s}{s}} \).

**Remark:** We want to make clear that both the \( s \) for which \( E \in \Sigma_s \) and the related constant depend only on \( n \) and \( K_0 \). This means that if we have two UHC-sets with the same constant \( K_0 \), not only they are both in the same \( \Upsilon_s \), but they satisfy the inequality with the same constant.

**Proof of (b)⇒(c):** Assume \((E, d) \in \Upsilon_s \), and let \( R > \varepsilon > 0 \). We can easily reduce us to the case \( x \in E \). Let \( x_1, \ldots, x_N \) be a maximal set of points in \( Q(x, 2R) \cap E \) with \( \rho(x_i, x_j) \geq \varepsilon \) whenever \( i \neq j \). Using that \((E, \rho) \in \Upsilon_{2s} \), we have that \( N \leq C2^{2s}(R/\varepsilon)^{2s} \). Furthermore, if \( y \in Q(x, R) \) and \( \rho(y, E) < \varepsilon \) and \( x_y \in E \) is such that \( \rho(y, E) = \rho(y, x_y) \), then \( x_y \in Q(x, 2R) \), so that there exists \( x_i \) for which \( x_y \in Q(x_i, \varepsilon) \), and from here:

\[
Q(x, 2R) \cap \{ y, \rho(y, E) < \varepsilon \} \subseteq \bigcup_{i=1}^{N} Q(x_i, 2\varepsilon).
\]

Therefore,

\[
\sigma(Q(x, R) \cap \{ y, \rho(y, E) < \varepsilon \}) \leq \sum_{i=1}^{N} \sigma(Q(x_i, 2\varepsilon)) \leq CR^{2s} \varepsilon^{-2s} \varepsilon^{-2n}.
\]

Hence, taking square roots, we obtain (c).

**Proof of (c)⇒(b):** If \( E \) satisfies \( \Sigma_s \) and for some \( R > 0 \) and \( k \geq 1 \) we have points \( x_1, \ldots, x_N \) lying in \( Q(x, kR) \) with \( \rho(x_i, x_j) \geq R \), then:

\[
\sigma(Q(x, 2kR) \cap \{ y, \rho(y, E) < R \}) \leq Ck^{2s} R^{2s} R^{2n-2s} = Ck^{2s} R^{2n}.
\]

Then again, the balls \( Q(x_i, R/2) \) are mutually disjoint, so:

\[
\sigma(Q(x, 2kR) \cap \{ y, \rho(y, E) < R \}) \geq \sum_{i=1}^{N} \sigma(Q(x_i, R/2)) \geq CN R^{2n},
\]

whence \( N \leq Ck^{2s} \), hence \((E, \rho) \in \Upsilon_{2s} \) and so \((E, d) \in \Upsilon_s \).

The implication \((c)⇒(g)\) is obtained in essentially the same way.
Proof of (b) ⇒ (d): Fix \( x \) and \( R \). If \( d(x, E) \geq 2\sqrt{2}R \), then for any \( y \in B(x, E) \) we have \( d(y, E) \geq R \), and the bound is trivial. If \( d(x, E) \leq 2\sqrt{2}R \), then for any \( y \in B(x, R) \) we have that \( d(y, E) \leq 6R \). Then if we decompose the integral we have to bound into a sum of integrals on coronae of decreasing radii, and apply the trivial bounds to each of these integrals, we obtain the result.

Proof of (d) ⇒ (a): Let \( S_R(x) = \sup\{d(y, E), y \in B(x, R)\} \). Then for any \( z \in B(x, R) \), \( (z, E)^{-a} \geq S_R(x)^{-a} \). Using it to get an inferior bound of the integral in (d) gives us the result. To see that (e) implies (a) we proceed in the same way.

Proof of (b) ⇒ (e): Assume \( d(x, E) \leq 2\sqrt{2}R \), the other case being trivial. Then the decomposition of the integral into integrals over coronae with radii \( 2^{-j}R \), plus the obvious bounds for each of these integrals, gives us the result.

Proof of (c) ⇒ (f): Assume \((E, d) \in \Upsilon_s\), for some \( s < n \). Then using it in the integral we have to bound gives us directly the result.

Proof of (f) ⇒ (c): Fix \( x \in E \), \( R > 0 \) and \( k \geq 1 \). Then the fact that \( N(x, \delta, kR/\delta) \) is a decreasing function of \( \delta \) gives the result.

Proof of (g) ⇒ (h): Just like in (b) ⇒ (d), except that in this case \( a < n + 1 - s \), so that \( a_0 = n + 1 - s > 1 \).

Proof of (h) ⇒ (c): Fix \( x \in E \), \( R > 0 \) and \( 0 < \varepsilon < R \). Then if \( \{t_j, j = 1, \ldots, N\} \) is a set of points in \( B(x, R) \cap E \) with \( d(t_i, t_j) \geq \varepsilon \),

\[
CR^{n+1-a} \geq \sum_{j=1}^{N} \int_{B(t_j, \frac{\varepsilon}{2})} \frac{1}{d(z, E)^a} dV(z) \geq CN\varepsilon^{n+1-a}
\]

so that \( N \leq C(R/\varepsilon)^{n+1-a} \) and then \((E, d) \in \Upsilon_{n+1-a} \), where \( a > 1 \). With this statement we have finished the proof of the theorem.

\[\blacksquare\]

3 Technical lemmas

The following lemmas are going to be later.

**Lemma 13** Let \( z \in \overline{B^n \setminus E} \), and let \( a > 0 \). Then

\[
\int_E \frac{1}{d(y, z)^a} \frac{d\mu(y)}{\mu(B(z, d(y, z)))} \leq Cd(z, E)^{-a}.
\]

**Proof:** This is immediate if we decompose the integral in a sum of integrals over coronae and apply the trivial bounds plus \( U_s \) to each of these integrals.

**Lemma 14** Let \( z \in \overline{B^n \setminus E} \), and let \( a, b, c > 0 \). Then

\[
\int_{E \times E} \frac{d(x, y)^c}{d(x, z)^a d(y, z)^b} \frac{d\mu(x) d\mu(y)}{\mu(x)^2} \leq Cd(z, E)^{c-a-b},
\]

whenever \( c > s \), \( c - a - b < 0 \), \( c - a < d \) and \( c - b < d \).
Proof: We split $E \times E$ into the sets,

$$(E \times E)_1 = \{(x, y) \in E \times E, d(y, z) \leq d(x, z)\}$$

and its complementary. We will bound only the integral over $(E \times E)_1$, as the other is bounded exactly in the same way, changing the roles of $x$ and $y$. Note that, in $(E \times E)_1$, $d(x, y) \leq 2\sqrt{2}d(x, z)$. We will use it to write $(E \times E)_1$ as $(E \times E)_1 = (E \times E)_1^1 \cup (E \times E)_1^2$, where:

$$(E \times E)_1^1 = \{(x, y) \in E \times E, d(x, y) \leq 2\sqrt{2}d(y, z) \leq 2\sqrt{2}d(x, z)\};$$

$$(E \times E)_1^2 = \{(x, y) \in E \times E, 2\sqrt{2}d(y, z) \leq d(x, y) \leq 2\sqrt{2}d(x, z)\}.$$

In $(E \times E)_1^1$, $d(x, z)^{-b} \leq d(y, z)^{-b}$, and also $(E \times E)_1^1 \subset E(y) \times E$, where

$$E(y) = \{x \in E, d(y, x) \leq 2\sqrt{2}d(y, z)\}.$$

Because of $U_s$,

$$\mu(B(y, d(y, z))) = \mu(B(y, d(x, y)) \frac{d(y, z)}{d(x, y)}) \leq C\mu[y, x] \left(\frac{d(y, z)}{d(x, y)}\right)^s,$$

where $\mu[y, x] = \mu(B(y, d(y, x)))$. Using it, and that $d(x, z)^{-b} \leq d(y, z)^{-b}$, the integral over $(E \times E)_1^1$ can be bounded by:

$$\int_E \frac{1}{d(y, z)^{a+b-c}} \int_{E(y)} \frac{d(x, y)^{c-s}}{\mu(B(y, d(x, y)))} d\mu(x) \frac{d\mu(y)}{\mu(B(y, d(y, z)))}.$$

Define, for $j \geq 1$,

$$E_j(y) = \{x \in E, 2^{-j}\sqrt{2}d(y, z) \leq d(y, x) \leq 2^{-j+1}\sqrt{2}d(y, z)\}.$$

Then decomposing the integral into the integral over $E_j$ and using the obvious bounds on each of these integrals, we can bound the previous integral by

$$\int_E \sum_{j=0}^{\infty} \left(\frac{d(y, z)}{d(z, d(y, z))}\right)^{a+b-c} \mu(B(y, \sqrt{2}d(y, z))) d\mu(y).$$

Now this is bounded by:

$$\int_E \frac{1}{d(y, z)^{a+b-c}} \frac{d\mu(y)}{\mu(B(z, d(y, z)))} \leq C d(z, E)^{-(a+b-c)}$$

whenever $c - s > 0$, and $a + b - c > 0$, as we have applied lemma $13$.

In $(E \times E)_1^2$, $d(x, z) \leq 2\sqrt{2}d(x, y)$, so $d(x, z) \approx d(x, y)$. Also $2\sqrt{2}d(y, z) \leq d(x, y)$. Then, using $L_d$,

$$\mu(B(y, d(y, z))) = \mu(B(y, d(y, z)) \frac{d(y, z)}{d(x, y)}) \geq C\mu(B(y, d(y, z))) \left(\frac{d(x, y)}{d(y, z)}\right)^d.$$

Using it, decomposing $E(y)$ into the sets $E_j(y)$ for $j < 0$, and proceeding as before, we obtain the result.
Lemma 15 Let \(a, b, c \geq 0\). If \(c - a - b + n + 1 < 0\), \(c - a + n + 1 > 0\) and \(c - b + n + 1 > 0\), then, for any \(z, w \in B^n\):

\[
\int_{B^n} \frac{(1 - |\zeta|)^c}{|1 - \zeta z|^a|1 - \zeta w|^b} dV(\zeta) \leq C|1 - zw|^{c-a-b+n+1}.
\]

Proof: Split \(B\) into:

\[
B_1 = \{\zeta \in B, |1 - \zeta w| \leq |1 - \zeta z|\}
\]

and its complementary. Clearly it is enough to bound the integral over \(B_1\), as the other one is bounded likewise.

In \(B_1\), we can assume \(b \geq n + \frac{1}{2}\), by changing \(a\) and \(b\) for some \(a'\) and \(b'\) if necessary. Recall that in \(B_1\), \(|1 - \zeta z| \geq C(|1 - \overline{w}z| + (1 - |\zeta|))\). Then if we write \(r = |\zeta|\), what we have to bound is

\[
\int_0^1 \frac{(1 - r^2)^c}{(|1 - z w| + 1 - r)^a} \int_S \frac{1}{|1 - r\zeta w|^b} d\sigma(\zeta) \, dr.
\]

But proposition 1.4.10 in [Rud.80] says that, for \(b > n\),

\[
\int_S |1 - r\zeta w|^{-b} d\sigma(\zeta) \leq C(1 - r^2|w|^2)^{-b+n}.
\]

If we apply this to the last integral, and then use the change of variables \(1 - r = d(z, w)t\), we get:

\[
C|1 - \overline{w}z|^{c+n+1-a-b} \int_0^\infty \frac{t^{c-b+n}}{(1 + t)^a} dt \leq C|1 - \overline{w}z|^{c+n+1-a-b},
\]

the last integral being finite if \(c - b + n > -1\) and \(c + n - a - b < -1\).

Proposition 16 Let \(a > s - n - 1, b < n\) and \(a - b + n + 1 > 0\). Then the integral

\[
\int_B d(z, E)^a d(\zeta, z)^b dV(z)
\]

is bounded independently of \(\zeta \in E\).

Proof: If \(a \geq 0\) this is trivial. Assume \(a < 0\). Then we decompose the integral over \(B\) into the integrals over \(B(\zeta, 2^{-j}) \setminus B(\zeta, 2^{-j-1})\). In each of these integrals, \(d(z, \zeta) \approx 2^{-j}\) and, because of part (h) in theorem 3, the remaining integral can be bounded by \(2^{(n+1+a)}\) whenever \(a > s - n - 1\). Thus our integral is bounded by \(\sum_{j \geq 0} 2^{-(n+1+a-b)}\), which is finite whenever \(n + 1 + a - b > 0\).

The following lemma can be found in [Gud.98]:

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Lemma 17 Let \( 0 < b < a \). There is a constant \( C \) so that for any \( z \in \mathbb{C} \) with \( |z| < 1 \),
\[
\int_0^1 \left( \log \frac{1}{t} \right)^{b-1} \frac{1}{|1-tz|^a} dt \leq C \frac{1}{|1-z|^{a-b}},
\]
whereas if \( b > a \) this integral is bounded by a constant depending only on \( a \) and \( b \).

The following lemma will allow us to compute the Taylor polynomial of a function written as an integral representation:

Lemma 18 For an \( \alpha \) so that \( 2\alpha / \notin \mathbb{Z} \), \( \ell = \lfloor 2\alpha \rfloor \), \( a \in \mathbb{R} \) and \( x, y \in B^n \), if we write \( T^\alpha_y F = T^{NI,\alpha}_y F \), we have:
\[
T^\alpha_y \left( \frac{1}{(1-\zeta z)^a} \right)(x) = \sum_{k=0}^\ell \binom{a+k-1}{k} \frac{(x-y)\zeta^k}{(1-\zeta y)^{a+k}} - \sum_{k>\ell/2} \binom{a+k-1}{k} \sum_{j=\ell-k+1}^k \binom{k}{j} \frac{(x-y)\zeta^j}{(1-\zeta y)^{a+k}} \left[ (x-y)\zeta - (x-y-1)\zeta y \right]^{k-j}.
\]

Proof: In order to compute the non isotropic Taylor polynomial of weight \( \alpha \) of a given \( F \), we begin by computing the isotropic Taylor polynomial of degree \( \ell \) of \( F \). We write it in terms of \( N = \sum_{i=1}^n z_i \frac{\partial}{\partial z_i} \) and \( Y_j = \frac{\partial}{\partial z_j} - z_j N \) (thus \( \omega(N) = 1 \) and \( \omega(Y_j) = \frac{1}{2} \)). Then we will keep only those terms with weight less than \( \alpha \), and we will be done. Recall that if we make this development at a fixed point \( y \), we must write the polynomial in terms of \( (N)_y \) and \( (Y_j)_y \) (we are using here that the values of a tensor at a point depend only on the values of the coefficients at that point). So we must write it in terms of \( N_y = \sum_{i=1}^n y_i \frac{\partial}{\partial z_i} |_{z=y} \) and \( (Y_j)_y = \frac{\partial}{\partial z_j} |_{z=y} - \gamma_j N_y \).

Let \( x, y \in S \), and \( F \in \mathcal{H}ol(B) \). Let \( A = \sum_{i=1}^n (x_i - y_i)(Y_i)_y \) and \( B = (x - y)\gamma N_y \). Thus \( A \) is complex tangential whereas \( B \) is not. Then a straightforward computation shows that:
\[
(T^\ell_y F)(x) = \sum_{k=0}^\ell \frac{1}{k!} (A + B)^k F.
\]

But \( A \) and \( B \) commute, as the coefficients are frozen at \( y \) and partial derivatives commute. Hence
\[
(T^\ell_y F)(x) = \sum_{k=0}^\ell \frac{1}{k!} \sum_{j=0}^k \binom{k}{j} A^{k-j} B^j F.
\]

We compute now the non isotropic Taylor polynomial of weight \( \alpha \), for \( 2\alpha / \notin \mathbb{N} \). Let \( \ell = \lfloor 2\alpha \rfloor \). We begin with the isotropic Taylor polynomial of degree \( \ell \),
and keep those terms with weight less than $\alpha$. Now the weight of $A_{k-j}B^j$ is $\frac{k-j}{2} + j = \frac{k+j}{2}$. So the terms we want are those with $j \leq k$ and $j < 2\alpha - k$, so that $j \leq \ell - k$. For $k \leq \ell/2$, the smaller of the two is $k$, so we have to keep all the terms, whereas for $k > \ell/2$, we have to take away the terms from $\ell - k + 1$ up to $k$. Therefore,

$$T_y^{N\ell,\alpha}F(x) = (T_y^{I,\ell}F)(x) - \sum_{k>\ell/2}^{\ell} \sum_{j=k+1}^{k} \binom{k}{j} A_{k-j}B^jF. \quad (4)$$

Let $\zeta \in B$ be fixed. We want to apply the previous computations to $F_a(z) = (1 - \zeta z)^{-a}$. Note that:

$$N_y F_a = \sum_{i=1}^{n} y_i \frac{\partial}{\partial z_i}_{|z=y} \frac{1}{(1 - \zeta)^a} = \sum_{i=1}^{n} a \left( \frac{y_i \zeta_i}{1 - \zeta} \right)_{|z=y} = a y \zeta F_{a+1}(y),$$

so $B F_a = a[(xy - 1)y\bar{\zeta}]F_{a+1}(y)$. Moreover,

$$(Y_i)_y F_a = \left( \frac{\partial}{\partial z_i}_{|z=y} - y_i N_y \right) F_a = a \zeta F_{a+1}(y) - a y \zeta F_{a+1}(y),$$

thus

$$AF_a = a \left[ (x - y)\zeta - (xy - 1)y\zeta \right] F_{a+1}(y).$$

That is, when we apply $B$ to $F_a$ we get a polynomial in $x$ and $y$ times $F_{a+1}$. From here,

$$B^j F_a = a(a+1) \cdot \cdot \cdot (a+j-1) \left[ (xy - 1)y\zeta \right]^j F_{a+j}(y).$$

Analogously,

$$A^j F_a = a(a+1) \cdot \cdot \cdot (a+j-1) \left[ (x - y)\zeta - (xy - 1)y\zeta \right]^j F_{a+j}(y).$$

By adding up these two things, we get:

$$A^{k-j}B^j F_a = \frac{(a+k-1)!}{(a-1)!} \left[ (xy - 1)y\zeta \right]^j \left[ (x - y)\zeta - (xy - 1)y\zeta \right]^{k-j} F_{a+k}(y).$$

On the other hand,

$$(A + B) F_a = a[(x - y)\zeta] F_{a+1}(y),$$

so that:

$$(A + B)^j F_a = a(a+1) \cdot \cdot \cdot (a+j-1) \left[ (x - y)\zeta \right]^j F_{a+j}(y).$$

Using all of this in the formula 4 gives us the claim. \[\blacksquare\]
Lemma 19 Let \( f \in H^p_\beta(B^n) \cap C(\overline{B^n}) \). Then for \( \alpha < \beta \), \( x, y \in \overline{B^n} \), and \( r \geq 0 \),
\[
T^n_\alpha f(x) = C \int_0^1 \int_B (\log \frac{1}{t})^{\beta-1} R^\beta f(\zeta)(1 - |\zeta|^2)^r T^n_\beta \left( \frac{1}{(1 - t \zeta)^{n+1+r}} \right)(x) dV(\zeta) dt,
\]
where \( C = C(n, r, \beta) \).

**Proof:** We know (see \[\text{Ahe,88}\]) that, for \( f \in C(\overline{B^n}) \), for any \( z \in \overline{B^n} \),
\[
f(z) = C(\beta) \int_0^1 (\log \frac{1}{t})^{\beta-1} R^\beta f(tz) dt.
\]
But theorem 7.1.4 from \[\text{Rud,80}\] says that for \( g \in H^p(B) \), with \( p \geq 1 \), and \( r \geq 0 \),
\[
g(z) = C(n, r) \int_B g(\zeta) \frac{(1 - |\zeta|^2)^r}{(1 - z\zeta)^{n+1+r}} dV(\zeta).
\]
Then for \( \alpha < \beta \), we can differentiate under the integral and get the result. \( \blacklozenge \)

Lemma 20 Let \( \beta > 0 \), \( \alpha = \beta - \frac{n-d}{p} \), \( \gamma < \alpha \), \( C \geq 0 \), \( D > 0 \), and \( r \geq 0 \). Assume \( C + D > \alpha + (s-d)/p = \beta - \frac{n+d}{p} \) and \( C < \alpha \). Then for \( r \) large enough \((r > (D-\alpha)p - n - 1)\), if we write:
\[
I(x, y) = \int_0^1 (\log \frac{1}{t})^{\beta-1} \int_B \frac{|f(\zeta)||(1 - |\zeta|^2)^r d(x, y)^{C+D-\gamma}}{d(x, t\zeta)^{C+n+r+1}} d\mu(\zeta) dt
\]
we have the bound:
\[
\int_{E \times E} \frac{I(x, y)^p}{d(x, y)^{(\alpha-\gamma)p-\delta}} d\mu(x) d\mu(y) \leq C \|f\|_{H^p(B)}^p.
\]

**Proof:** Take \( \delta > 0 \) small enough \((\delta < \frac{n+r+1}{4p} \) and \( \delta < \beta) \). Let \( \lambda = \frac{n+r+1}{4p} - 2\delta \). Because of Hölder's inequality, \( I(x, y)^p \) is bounded by
\[
\int_0^1 (\log \frac{1}{t})^{(\beta-\delta)p-1} \int_B \frac{|f(\zeta)|^p (1 - |\zeta|^2)^r d(x, y)^{(C+D-\gamma)p}}{d(x, t\zeta)^{(C+2\delta)p+n+r+1}} d\mu(\zeta) dt
\]
times the integral
\[
\left( \int_0^1 (\log \frac{1}{t})^{\beta p-1} \int_B \frac{(1 - |\zeta|^2)^r}{d(x, t\zeta)^{\lambda p} d(y, t\zeta)^{4\delta p}} d\mu(\zeta) dt \right)^{\frac{\beta}{p}}.
\]
We want to bound the last integral using lemma 15. It is easily checked that we can apply it whenever \( 0 < 4\delta < \frac{n+r+1}{p} \). To bound the integral with respect to \( t \) we obtain, we use that \( |1 - t^2 a| \approx |1 - ta| \) and then apply lemma 14. In this way we see that \( \beta \) is bounded by \( d(z, E)^{-\delta p} \).
We have to evaluate

\[ \int \int_{E \times E} I(x, y)^p \left( \frac{d\mu(x)}{d(x, y)^{\alpha-\gamma} p-d} \right) \frac{d\mu(y)}{\mu[x, y]^2}. \]  

(6)

If we use the bounds we have obtained, and apply Fubini’s theorem, what we have to bound is:

\[ \int_0^1 \left( \log \frac{1}{t} \right)^{(\beta-\delta)p-1} \int d\mu(x) \mu(y) \frac{1}{d(t\zeta, E)^{\beta-\delta}p+r+1} \right] dt dV(\zeta), \]

where

\[ J(t\zeta) = \int \int_{E \times E} d(x, y)^{(C+D-A-\delta)p+d} \frac{d\mu(x)}{\mu[x, y]^2} \frac{d\mu(y)}{d(t\zeta, E)^{\beta-\delta}p+r+1}. \]

We want to apply lemma 14 to \( J(t\zeta) \). It is easily checked that we can find \( C, D \), and then \( r \) large enough and \( \delta \) small enough so that we can do it. In this case, (6) is bounded by:

\[ \int_B |f(\zeta)|^p (1 - |\zeta|^2)^r \int_0^1 \left( \log \frac{1}{t} \right)^{(\beta-\delta)p-1} \frac{1}{d(t\zeta, E)^{\beta-\delta}p+r+1} dt dV(\zeta). \]

Now if \( \zeta_0 \in E \) satisfies \( d(\zeta, E) = d(\zeta, \zeta_0) \) and \( 0 < t < 1 \), also \( d(t\zeta, E) \approx d(t\zeta, \zeta_0) \).

From this, lemma 17 and the fact that \( 1 - |\zeta|^2 \leq d(\zeta, E) \), we bound (6) by

\[ \int_B |f(\zeta)|^p \frac{1}{d(\zeta, E)^n} dV(\zeta). \]

But, because of part (h) of theorem 3, \( d(z, E)^{-1} dV(z) \) is a Carleson measure, so that this last integral is bounded by \( ||f||_{H^p(E)^n} \). ♣

**Lemma 21** Let \( \zeta, \xi \in E \). Write

\[ S_1 = \{ z, \in B, d(z, \zeta) \leq d(z, \xi) \}. \]

Let \( A, B, C, D, \) and \( F \) be \( \geq 0 \). Then the integral

\[ \int_{S_1} d(z, E)^{-F} \int_0^1 \left( \log \frac{1}{t} \right)^{D-1} \frac{d(tz, E)^C}{d(tz, \zeta)^A d(tz, \xi)^B} dt d\sigma(z) \]

(7)

is bounded by

\[ K d(\zeta, \xi)^{-(A+B-C-D+F-n)} \]

wherever \( A + B - C - D + F - n > 0 \), \( F < n - \Upsilon(E) \), \( A - C - D > 0 \), and \( A - C - D < n - F \). 

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Proof: We split $S_1$ into $S_{11} = B(\zeta, \frac{1}{\sqrt{2}}d(\zeta, \xi))$ and $S_{12}$ its complementary. For $z \in S_1$, $d(z, \zeta) \leq d(z, \xi)$. Thus, and because of triangle's inequality, $d(\zeta, \xi) \leq 2\sqrt{2}d(tz, \xi)$. Then, and as $d(tz, E) \leq d(tz, \zeta)$, the part of the integral corresponding to $S_{11}$ is bounded by

$$d(\zeta, \xi)^{-B} \int_{S_{11}} d(z, E)^{-F} \int_0^1 \left( \frac{1}{t} \right)^{D-1} \frac{1}{d(tz, \zeta)^{A-C}} dt \, d\sigma(z).$$

Now if $A - C - D > 0$ we can apply lemma 17 to the inner integral, and bound the last integral by

$$d(\zeta, \xi)^{-B} \int_{S_{11}} d(z, E)^{-F} \frac{1}{d(z, \zeta)^{A-C-D}} d\sigma(z).$$

Let $B_j = B(\zeta, 2^{-j} \frac{1}{\sqrt{2}}d(\zeta, \xi))$. We split this integral into the integrals over the coronae $B_j \setminus B_{j+1}$. In each of these integrals we bound $d(z, \zeta) \leq 2^{-j}d(z, \xi)$ and then apply part (d) of theorem 5. Thus we obtain the bound

$$d(\zeta, \xi)^{-(A-C-D+F-n)} \sum_{j=0}^{\infty} 2^{-j(A+B-C-D+F-n)},$$

and this sum is bounded whenever $C + D - A - F + n > 0$, which means $A - C - D < n - F$.

For the integral over $S_1 \setminus S_{11}$, we use that $d(tz, \xi) \geq d(z, \xi) \geq d(z, \zeta)$. Hence, and again using that $d(tz, E) \leq d(tz, \zeta)$, the part of the integral corresponding to $S_{12}$ is bounded by

$$\int_{S_{12}} \frac{d(z, E)^{-F}}{d(z, \zeta)^B} \int_0^1 \left( \frac{1}{t} \right)^{D-1} \frac{1}{d(tz, \zeta)^{A-C}} dt \, d\sigma(z).$$

As before, if $A - C - D > 0$ we can apply lemma 17 to the inner integral, and bound the last integral by

$$\int_{S_{12}} \frac{d(z, E)^{-F}}{d(z, \zeta)^{A+B-C-D}} d\sigma(z).$$

Let $B_j = B(\zeta, 2^{-j} \frac{1}{2\sqrt{2}} d(\zeta, \xi))$. We decompose the integral over $S_{12}$ into the integrals over $B_j \setminus B_{j-1}$. Proceeding as before, we can bound this sum of integrals by

$$d(\zeta, \xi)^{-(A+B-C-D+F-n)} \sum_{j=0}^{\infty} 2^{-j(A+B-C-D+F-n)},$$

and this sum is bounded whenever $A + B - C - D + F - n > 0$. 23
4 Proof of theorem

Let \( f \in H^p_0(B^n) \). We already know that its restriction lies in \( L^p(d\mu) \), so we only have to see that the integrals

\[
\int_E \int_E \frac{|f_z(x) - D^\gamma(T_y^a f)(x)|^p \ d\mu(x)\ d\mu(y)}{d(x,y)^{(\alpha(\gamma))p-d} \mu(x,y)^2}
\]

are finite. But because of lemma 19, we have that, for \( r \geq 0 \), and \( F_a(z, \zeta) = (1 - z\zeta)^{-\alpha} \), the difference \( T_y^a f(x) - f(x) \) can be written as

\[
C \int_0^1 \int_B (\log \frac{1}{t})^{d-1} R^\beta f(\zeta)(1 - |\zeta|^2)^r \ [T_y^a(F_N)(tx) - F_N(tx)] \ dV(\zeta)dt,
\]

with \( N = n + 1 + r \). We want to evaluate, for \( x, y \in E \subset S \), \( \zeta \in B \), and \( F_N(\xi) = (1 - \zeta \xi)^{-N} \), the difference \( T_y^a F_N(x) - F_N(x) \). To do so, we use that the derivatives of \( F_N(\xi) \) with respect to \( \xi \) are the same as those of \( F_N(\xi) - F_N(x) \).

Hence:

\[
T_y^a(F_N(x))(x) - F(x) = T_y^a(F_N - F_N(x))(x).
\]

But for \( N \in \mathbb{N} \),

\[
F_N(\xi) - F_N(x) = \sum_{a=1}^{N} \frac{(\xi - x)\zeta}{(1 - \zeta \xi)^a(1 - \zeta x)^{N-a+1}}.
\]

Thus, if we expand \( (\xi - x)\zeta \) as \( (1 - \zeta x) + (1 - \zeta) \),

\[
T_y^a(F_N - F_N(x))(x) = \sum_{a=1}^{N} F_N - a + 1(x) \ [1 - \zeta x]T_y^a(F_a) - T_y^a(F_a-1)](x). \tag{9}
\]

**Claim:**

\[
(1 - \zeta x)T_y^a(F_a)(x) - T_y^a(F_a-1)(x) = -\sum_{k=m}^{\ell} \binom{a+k-1}{k} \binom{k}{k-\ell} \frac{(x\bar{y} - 1)y\bar{z}^{\ell+k}(x - y)\bar{z}^k - (x\bar{y} - 1)y\bar{z}^{2\ell-\ell+1}}{(1 - y\bar{z})^{a+k}} - \\
-\sum_{k=m}^{\ell-1} \binom{a+k-1}{k} \binom{k}{k-\ell} \frac{(x\bar{y} - 1)y\bar{z}^{\ell+k+1}(x - y)\bar{z}^k - (x\bar{y} - 1)y\bar{z}^{2\ell-\ell}}{(1 - y\bar{z})^{a+k}}. \tag{10}
\]

(Recall that \( \ell = [2a]. \))

Assuming this claim, which we will prove later, we can proceed with the proof of the theorem.

We have to bound \( D_x^\gamma(T_y^a F(x) - F(x)) \). We use that for \( a \neq 0 \):

\[
N_x((1 - \zeta z)^a) = a(x\bar{z} - 1) ((1 - \zeta z)^{a-1})|_{z=x} + a ((1 - \zeta z)^{a-1})|_{z=x},
\]

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and also that:

\[(Y_t)_x((1-\zeta z)^a) = a(\zeta_i - x_i) \left((1-\zeta z)^{a-1}\right)_{z=x} - a\xi(x\zeta - 1) \left((1-\zeta z)^{a-1}\right)_{z=x}.
\]

Iterating this we get that for \(X\) as before,

\[|X_x \left((1-\zeta z)^a\right)| \leq C(X)d(x,\zeta)^{a-\omega(X)}. \quad (11)\]

A similar computation shows that:

\[|X_x((z-y)(\zeta - \eta) + (x\zeta - 1)(1-y\zeta)^a)| \leq Cd(x, y)^{\frac{D}{2}}d(y,\zeta)^{\frac{D}{2}}. \quad (12)\]

Therefore, for \(f \in H^p_\beta(B^n)\), using \(\Box\), \(|X_x(T_0^a f - f(x))|\) can be bounded by terms like

\[I_j(x, y) = \int_0^1 \left(\log \frac{1}{t}\right)^{\beta-1} \int_B |R^\beta f(\zeta)(1-|\zeta|^2)^{\prime}\|X T_j\|dV(\zeta) dt\]

where \(j = 0, 1, \text{ and} \),

\[T_j = \frac{[(x\zeta - 1)t \zeta \zeta^2 - (x\zeta - 1)t (x\zeta - 1)\zeta]}{(1-t\zeta)^{N-a+1}(1-t\eta\zeta)^{a+k}}, \]

with \(1 \leq a \leq N, \ell \in [2a], m = [\ell/2], \text{ and } m \leq k \leq \ell\). We will only evaluate the term corresponding to \(T_0\), as the other one can be dealt with likewise.

If \(\omega = \omega(X), \omega_1 = \omega(X'), \omega_2 = \omega(X''), \text{ and } \omega_3 = \omega(X''')\),

\[X T_0 = \frac{1}{(1-t\zeta)^{a+k}} \sum_{\omega_1 + \omega_2 + \omega_3 = \omega} X' \left((1-t\zeta)^{-(N-a+1)}\right) \times \times \times \left((x\zeta - 1)t \zeta \zeta^2 - (x\zeta - 1)t (x\zeta - 1)\zeta\right)^{2k-\ell+1}\]

and from here, using \(\Box\) and \(\Box\)

\[|X T_0| \leq C \sum_{\omega_1 + \omega_2 = \omega} \frac{d(x, y)^{\frac{\ell+1}{2}}}{d(x, t\zeta)^{N-a+1+\omega_1}d(y, t\zeta)^{a+\frac{\ell+1}{2}}}.\]

Now, using that \(1 \leq a \leq N\), these expressions can be bounded by:

\[C \sum_{\omega_1 + \omega_2 = \omega} \frac{d(x, y)^{\frac{\ell+1}{2}}}{d(x, t\zeta)^{N-a+1+\omega_1}d(y, t\zeta)^{a+\frac{\ell+1}{2}}} + \frac{d(x, y)^{\frac{\ell+1}{2}}}{d(x, t\zeta)^{a+\omega_1}d(y, t\zeta)^{N+\frac{\ell+1}{2}}}.\]

We want to apply lemma \(\Box\) to these expressions, for some \(r\) large enough. In the first one, \(\gamma = \omega, C = \omega_1 \geq 0 \text{ and } D = \frac{\ell+1}{2}\), so that \(C + D - \omega = \frac{\ell+1}{2} + \omega_1 - \omega = \frac{\ell+1}{2} - \omega'\). Hence, \(C + D \geq \frac{\ell+1}{2} > \alpha\), and if between \(\alpha\) and \(\beta - \frac{n-\delta}{p}\) lies no integer multiple of \(1/2\) the matching condition is satisfied, and also \(C < \alpha\) trivially. In
the second term, $\gamma = \omega$, $C = \frac{\ell}{2} < \alpha$ and $D = \frac{1}{2} + \omega_1$, so $C + D = \frac{\ell + 1}{2} + \omega_1$ and the requirements of lemma 20 are also fulfilled, so that:

$$\int \int_{E \times E} I_0(x, y)^p \frac{d\mu(x) d\mu(y)}{\mu(x, y)^2} \leq C \| R^\beta f \|_{H^p(B)}^p.$$ 

Proof of the claim: We know that $T_y = T_{y, \ell} - S_\alpha$, where, if we write $Z = (x \overline{\mu} - 1) \overline{y \zeta}$, $V = (x - y \overline{\zeta} - Z$ and $T = (1 - y \zeta)$, we have:

$$T_{y, \ell} \left( \frac{1}{(1 - \zeta x)^a} \right)(x) = \sum_{k=0}^{\ell} \binom{a+k-1}{k} \frac{(V + Z)^k}{T^{a+k}},$$

and, if $m = [\ell/2]$,

$$S_\alpha = \sum_{k=m+1}^{\ell} \binom{a+k-1}{k} \sum_{j=\ell-k+1}^{k} \binom{k}{j} Z^j V^{k-j} \frac{T^j}{T^{a+k}}.$$

Now if we substract the isotropic parts we get:

$$(T + V + Z) \sum_{k=0}^{\ell} \binom{a+k-1}{k} \frac{(V + Z)^k}{T^{a+k}} - \sum_{k=0}^{\ell} \binom{a+k-2}{k} \frac{(V + Z)^k}{T^{a+k-1}}.$$

Rearranging these terms, and using that $\binom{a}{k} - \binom{a-1}{k} = \binom{a-1}{k}$, we see that this term is equal to $-(\ell+1)(V + Z)^{\ell+1} T^{-(a+\ell)}$.

On the other hand,

$$-(1 - \zeta x) S_\alpha + S_{\alpha-1} = (V + Z - T) S_\alpha + S_{\alpha-1}$$

can, rearranging terms and using the properties of the binomial coefficient, be written as

$$\sum_{k=m+1}^{\ell} \binom{a+k-1}{k} \sum_{j=\ell-k+1}^{k} \binom{k}{j} Z^j V^{k-j+1} + Z^j V^{k-j} \frac{T^j}{T^{a+k}} -$$

$$- \sum_{k=m+1}^{\ell} \binom{a+k-2}{k-1} \sum_{j=\ell-k+1}^{k} \binom{k}{j} Z^j V^{k-j} \frac{T^j}{T^{a+k-1}}.$$

We now split from the first sum the term corresponding to $k = \ell$. Next we change $j$ for $j-1$ in the second term. Thus what we have is

$$\binom{a+\ell-1}{\ell} \sum_{j=1}^{\ell} \binom{\ell}{j} Z^j V^{\ell-j+1} + Z^{\ell+1} V^{\ell-j} \frac{T^j}{T^{a+\ell}} +$$

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Rearranging the terms we obtain directly that:

\[ + \sum_{k=m+1}^{\ell-1} \binom{a+k-1}{k} \sum_{j=\ell-k+1}^{k} \frac{(j)}{T^a+k} Z^j V^k-j+1 + Z^{j+1} V^k-j \]

\[ - \sum_{k=m}^{\ell-1} \binom{a+k-1}{k} \sum_{j=\ell-k}^{k+1} ((\binom{j}{\ell-k}) + (k) \binom{j}{\ell-k-1}) \frac{Z^j V^{k+1-j}}{T^a+k}. \]

Next we split the last term and change \( j \) for \( j-1 \) in the latter of the two terms we obtain. We observe also that the first term is precisely \((V + Z)^{\ell+1}\). Thus the last formula can be written as

\[ \binom{a+\ell-1}{\ell} \frac{(V + Z)^{\ell+1}}{T^a+\ell} - \binom{a+\ell-1}{\ell} \frac{V^{\ell+1} + ZV^\ell}{T^a+\ell} + \]

\[ + \sum_{k=m+1}^{\ell-1} \binom{a+k-1}{k} \sum_{j=\ell-k+1}^{k} \frac{(j)}{T^a+k} Z^j V^k-j+1 + Z^{j+1} V^k-j \]

\[ - \sum_{k=m}^{\ell-1} \binom{a+k-1}{k} \sum_{j=\ell-k}^{k+1} ((\binom{j}{\ell-k}) + (k) \binom{j}{\ell-k-1}) \frac{Z^j V^{k+1-j}}{T^a+k}. \]

Next we split in the sums the terms that are not shared. Then we observe that some terms with are formally in the sum, are not really there, because it would be needed that \( \ell - m + 1 \leq m \), so that \( m \geq (\ell + 1)/2 \). But \( m = [\ell/2] \), so this cannot happen. Thanks to this, some terms cancel each other, so that what we have is

\[ \binom{a+\ell-1}{\ell} \frac{(V + Z)^{\ell+1}}{T^a+\ell} - \binom{a+\ell-1}{\ell} \frac{V^{\ell+1} + ZV^\ell}{T^a+\ell} - \]

\[ - \sum_{k=m}^{\ell-1} \binom{a+k-1}{k} ((\binom{k}{\ell-k}) + (k) \binom{k}{\ell-k-1}) \frac{Z^{\ell-k} V^{2k-\ell+1}}{T^a+k} - \]

\[ - \sum_{k=m}^{\ell-1} \binom{a+k-1}{k} \frac{Z^{\ell-k+1} V^{2k-\ell}}{T^a+k}. \]

Rearranging the terms we obtain directly that:

\[ \binom{a+\ell-1}{\ell} \frac{(V + Z)^{\ell+1}}{T^a+\ell} - \sum_{k=m}^{\ell} \binom{a+k-1}{k} \binom{k}{\ell-k-1} \frac{Z^{\ell-k} V^{2k-\ell+1}}{T^a+k} - \]

\[ - \sum_{k=m}^{\ell} \binom{a+k-1}{k} \frac{Z^{\ell-k+1} V^{2k-\ell}}{T^a+k}. \]

Just adding this formula with the formula corresponding to the isotropic Taylor polynomial gives us the result.

\[ \star \]
5 The interpolating function

Let $q > s$. For $z \in \overline{B}^n \setminus E$, let
\[ h_q(z) = \int_E |1 - \zeta z|^{-q} d\mu(\zeta). \] (13)

Let $n(z, \zeta) = (h_q(z)(1 - \zeta z))^{-1}$. Then our real extension has the form:
\[ E(f)(z) = \int_E n(z, \zeta) T^\alpha f(z) d\mu(\zeta), \] (14)

where $T^\alpha$ is the non isotropic Taylor polynomial.

The following lemma about the behaviour of $h_q(z)$ can be found (for the isotropic metric, but the proof is valid for any pseudometric) in [Gud, 98]:

**Proposition 22** Let $x \notin E$, and let $x_0 \in E$ be such that $d(x, E) = d(x, x_0)$. Write $B_x = B(x_0, 3d(x, E))$. Then

(a) For $a \geq 0$, $t \in E$, and $q > s + \alpha$,
\[ \int_E \frac{\rho(y, t)^a}{\rho(y, x)^q} d\mu(y) \leq C \rho(t, x)^a \rho(x, E)^{-q} \mu(B_x); \]
(b) For any $R > 0$ and any differential operator $\mathbb{X}$ there is a $C = C(\mathbb{X}, R)$ so that if $|x| \leq R$,
\[ |\mathbb{X}h_q(x)| \leq C \rho(x, E)^{-q - \omega(\mathbb{X})} \mu(B_x); \]
(c) for any $R > 0$ and any differential operator $\mathbb{X}$ there is a $C = C(\mathbb{X}, R)$ so that if $|x| \leq R$,
\[ |\mathbb{X} h_q(x)| \leq C \rho(x, E)^q \omega(\mathbb{X}) \mu(B_x)^{-1}. \]

Next we study the behaviour of $g$. Part (a) of the following lemma will give us the boundedness of the $\partial$ correction of $g$. Part (b) says that, in a sense, $g$ has finite Triebel-Lizorkin norm. The fact that $g$ interpolates the jet $(E_{\omega(\gamma)} < \alpha)$ up to order $\alpha$ is checked exactly as in the proof of theorem 8 in [Gud, 98], so we will not repeat it.

**Lemma 23** Let $\alpha$ and $\beta$ be as in theorem 3.

(a) For $k > \beta - \frac{n-\omega}{p}$ and $\omega(\mathbb{X}) \leq k$,
\[ \int_{B^n} d(\zeta, E)^{(k-\beta)p-1} |\mathbb{X} g(\zeta)|^p dV(\zeta) < +\infty, \]
and
\[ \int_{B^n} \frac{d(\zeta, E)^{(k-\beta)p-\frac{1}{2}}}{(1 - |\zeta|)^{\frac{1}{2}}} |\mathbb{X} g(\zeta)|^p dV(\zeta) < +\infty. \]
If $\ell = [\beta] + 1$, then:
\[
\int_S \left( \int_0^1 (1 - t^2)^{\ell - \beta - 1}|R^\ell g(tz)| dt \right)^p d\sigma(z) < +\infty.
\]

Proof: We will only prove (a), as (b) is proved in essentially the same way as in [Gud,98].

We will only prove the boundedness of the first integral. The proof of the second inequality is essentially the same, only just a bit more technical. The idea is that, $(1 - |\zeta|)^a$ being integrable for $a < 1$, we can in this case obtain bounds for the integrals with respect to $dV$ similar to those we obtain in the case $a = 0$.

We will split $X g(z)$ in the following way:
\[
X g(z) = \int_E n(z, \zeta) X(T_\zeta^a f(z)) d\mu(z) + \sum_{\omega(X') + \omega(X') = \omega(X)} \int_E X'(n(z, \zeta)) X''(T_\zeta^a f(z)) d\mu(\zeta). \quad (15)
\]

Let now $z \notin E$ be fixed. The term $|X(T_\zeta^a f(z))|$ can be bounded by sums of terms like $|f(\gamma(\zeta))|$, with $\omega(\gamma) < \alpha$, independently of $z$. Then the first term in (15) can be bounded by sums of terms like
\[
\frac{1}{h_q(z)} \int_E \frac{|f(\gamma(\zeta))|}{d(z, \zeta)^\alpha} d\mu(\zeta).
\]
We apply Hölder’s inequality to it, and then use $U_s'$ and proposition 22 to estimate the integral not containing $|f(\gamma(\zeta))|$. Then the previous integral is bounded by
\[
\left( \int_E \frac{|f(\gamma(\zeta))|}{d(z, \zeta)^\alpha} d\mu(\zeta) \right)^\frac{1}{\beta}.
\]
Then we raise it to the power $p$, multiply by $d(z, \zeta)^{p(\ell - \beta) - 1}$, integrate over $B$ and apply Fubini’s theorem. As $s < n$, and whenever $k > \beta - \frac{n - s}{p}$, the integral over $B$ can be bounded using proposition 16 and we are done.

To estimate the second term in (14) we use that $\int X(n(\zeta, \cdot)) d\mu = 0$ if $\omega(X) > 0$. Thus this term is bounded by sums of terms like
\[
\int_E |X'(n(z, \zeta))| |X''(T_\zeta^a f(z) - T_\zeta^a f(z))| d\mu(\zeta), \quad (16)
\]
for any (fixed) $\xi \in E$. A straightforward calculation with the help of proposition 22 (see [Gud,98] for details) shows that
\[
|X'(n(z, \zeta))| \leq C(X') d(z, E)^{g - \omega(X')} d(z, \zeta)^{-g} \mu(B_\zeta). \quad (17)
\]
Thus, if we integrate against $d(z, \xi)^{-q} \mu(\xi)$, divide by $h_q(\xi)$, and apply proposition \ref{prop22} to $h_q$, we get that \ref{eq16} is bounded by

$$
\frac{d(z, E)^{2q - \omega(X')}}{\mu(B')^2} \int_{E \times E} \frac{|X''(T_{\xi} f(z) - T_{\xi} f(z))|}{d(z, \xi)^q d(z, \xi)^q} d\mu(\xi) d\mu(\xi). \tag{18}
$$

Next we use that $T_{\xi}^\alpha f$ is a polynomial in $z$ with degree $\leq 2\alpha$, so it has a development at $\xi$ as:

$$
T_{\xi}^\alpha f(z) = \sum_{|\gamma| \leq 2\alpha} \frac{1}{\gamma!} D^\gamma(T_{\xi} f)(z) w(z, \xi)^\gamma. \tag{19}
$$

Let $\Delta_\gamma(\zeta, \xi) = f_{\gamma}(\xi) - D^\gamma(T_{\xi} f)(\xi)$. Then $T_{\xi}^\alpha f(z) - T_{\xi}^\alpha f(z)$ can be expressed as

$$
\sum_{\omega(\gamma) < \alpha} \frac{1}{\gamma!} \Delta_\gamma(\zeta, \xi) w(z, \xi)^\gamma - \sum_{\omega(\gamma) > \alpha} \frac{1}{\gamma!} D^\gamma(T_{\xi} f)(\xi) w(z, \xi)^\gamma. \tag{20}
$$

Thus \ref{eq18} can be bounded by sums of terms like:

$$
\frac{d(z, E)^{2q - \omega(X')}}{\mu(B')^2} \int_{E \times E} \frac{|\Delta_\gamma(\zeta, \xi)| |X''(w(z, \xi)^\gamma)|}{d(z, \xi)^q d(z, \xi)^q} d\mu(\xi) d\mu(\xi), \tag{21}
$$

with $\omega(\gamma) < \alpha$, and

$$
\frac{d(z, E)^{2q - \omega(X')}}{\mu(B')^2} \int_{E} \frac{|D^\gamma(T_{\xi} f)(\xi)| |X''(w(z, \xi)^\gamma)|}{d(z, \xi)^q d(z, \xi)^q} d\mu(\xi) d\mu(\xi), \tag{22}
$$

with $\omega(\gamma) > \alpha$.

We begin by bounding \ref{eq21}, which is the simplest. To do so, we simply bound $|X''(w(z, \xi)^\gamma)|$ by $d(z, \xi)^{\omega(\gamma) - \omega(X')}$, and $|D^\gamma(T_{\xi} f)(\xi)|$ by sums of terms of the form $|f_{\gamma}(\xi)|$). Next we integrate with respect to $\zeta$, applying the bounds from proposition \ref{prop22}. From here on, we can proceed as for the first term in \ref{eq17}. We will only need that $\omega(\gamma) - (\beta - \frac{n-\alpha}{p}) > 0$. But we know that $\omega(\gamma) > \alpha$, so that under the hypothesis that between $\beta - \frac{n-\alpha}{p}$ and $\beta - \frac{n-\alpha}{p}$ lies no integer multiple of $1/2$, the previous inequality holds.

To bound \ref{eq21}, we begin by using that $|X''(w(z, \xi)^\gamma)|$ can be bounded by $d(z, \xi)^{\omega(\gamma) - \omega(X')}$. Next we raise the integral to the power $p$, and apply Hölder’s inequality to it, with some $A, B > 0$ to be chosen later. The integral not containing $\Delta_\gamma(\zeta, \xi)$ can be bounded using proposition \ref{prop22}. Then \ref{eq20} can be bounded by:

$$
\frac{d(z, E)^{(2q + \omega(\gamma) - \omega(X') - A - B)p}}{\mu(B')^2} \int_{E \times E} \frac{|\Delta_\gamma(\zeta, \xi)|^p d\mu(\xi) d\mu(\xi)}{d(z, \xi)^{(q - A)p} d(z, \xi)^{(q - B)p}}. \tag{23}
$$

We can split $E \times E$ into $(E \times E)_1(z) = \{d(\zeta, z) \leq d(\xi, z)\}$ and $(E \times E)_2$ its complementary. Clearly, we only need to bound the integral over $(E \times E)_1$, the
other one being bounded likewise. On \((E \times E)_1\), it is easily seen that, because of \(U_s\) and \(L_d\),

\[
\mu(B_z) \geq \frac{d(z, E)^s}{d(z, \xi)^s - d(z, \xi)^d} \mu[z, \xi]
\]

(see [3,ud.98] for details).

We use this bound in 22. Then we multiply 22 by \(d(z, E)^{(k - \beta)p - 1}\), integrate over \(B\) and apply Fubini’s theorem. We also use that \(k - \omega(\mathbb{X}) \geq 0\). To finish the proof it is then enough to see that

\[
\int_{\{z, (\xi, \zeta) \in (E \times E)_1(z)\}} \frac{d(z, E)^{(2q - A - B - \beta + \omega(\gamma))p - 1 - 2s}}{d(z, \zeta)^{(q - A)p - 2s + 2d}d(z, \xi)^{(q - B)p}} dV(z)
\]

is bounded by

\[
d(\zeta, \xi)^{-((\alpha - \omega(\gamma))p + d)}.
\]

To check this last assertion, we split the set over which we are integrating into \(B_1 = B(\zeta, d(\zeta, \xi)/4)\) and \(B_2 = B \setminus B_1\) (having in mind that \(d(\zeta, z) \leq d(\zeta, \xi)\)). On \(B_1\), \(d(\zeta, z) \geq d(\zeta, \xi)/4\), and \(d(z, E) \leq d(\zeta, \xi)/4\). Then, if we choose \(A\) so that \((q - A)p - 2(s - d) < n + 1\), we get the result.

On \(B_2\), we use that \(d(\zeta, z)^{-1} \leq d(\zeta, z)^{-1}\), and \(d(z, E) \leq d(\zeta, z)\). Again as \((\alpha - \omega(\gamma))p + d > 0\), we bound the remaining integral and we have done. ♦

### 6 The \(\overline{\partial}\) correction

In what follows, we will follow loosely the article [B-O.93], where the case when \(E\) is a complex-tangential variety, with \(\beta \in \mathbb{N}\) and in the \(H^p_B(B^n)\) spaces, is considered. Our next goal is to modify \(g\) by adding to it a function which is zero up to the necessary order on \(E\) so that it is possible to get a function behaving like \(g\) on \(E\) but holomorphic on \(B\).

We want to see that from \(g = E(f)\) we can get a function lying in \(HF^p_{\beta}\) and interpolating \((F_\gamma)_\omega(\gamma) < \alpha\). To do so, we will use the kernel: \(C_N(\zeta, z) = \Psi_N(\zeta, z)C(\zeta, z)\), where:

\[
\Psi_N(\zeta, z) = \left(\frac{1 - |\zeta|^2}{1 - z\zeta}\right)^{N-1} \sum_{j=0}^{N-1} c_{j, n, N} \left[\frac{(1 - |\zeta|^2)(1 - |z|^2)}{|1 - z\zeta|^2}\right]^j
\]

and \(C(\zeta, z)\) is the Cauchy kernel for the ball, that is:

\[
c_n \left[\frac{(1 - z\zeta)^{n-1}}{|1 - z\zeta|^2 - (1 - |\zeta|^2)(1 - |z|^2)|}\right] \sum_{j=1}^{n} (-1)^{j-1}(\zeta_j - \bar{z}_j) \wedge_{k \neq j} d\zeta_k \wedge_{k=1}^{n} d\zeta_k.
\]

This kernel was introduced by P. Charpentier in [Cha.81] to solve the \(\overline{\partial}\) problem. Namely, if \(\varphi\) is a function with enough regularity defined on \(B^n\), then the
function

\[ U(z) = T_n \varphi(z) = \int_{B^n} C_N(\zeta, z) \wedge \varphi(\zeta) \]

satisfies \( \overline{\partial} U = \varphi \).

In this section we are going to prove the following lemma, that will allow us to finish the proof of the theorem.

**Lemma 24** With the same hypothesis as in theorem \( \text{[3]} \), let \( g = \mathcal{E}(F) \), \( b > 0 \), and \( \eta = \frac{\partial g}{\partial h} \). Let \( U(z) = T_N \eta(z) \). Then for \( b \) large enough, and \( N \) large enough (depending on \( b \)), the following is satisfied:

(a) If \( \ell = [\beta] + 1 \),

\[
\int_{S} \left( \int_{0}^{1} (1 - t^2)^{\ell - \beta - 1} |R^f(h^b U(t\zeta))| dt \right)^p d\sigma(\zeta) < +\infty.
\]

(b) For \( \mu \)-almost every \( \zeta \in E \), if \( \omega(\overline{X}) < \alpha \),

\[
\lim_{\delta \to 0} \frac{1}{V(B(\zeta, \delta))} \int_{B(\zeta, \delta)} X(h^b U)(z) dV(z) = 0.
\]

With this lemma we can easily prove theorem \( \text{[3]} \), for if we define \( f = g - h^b U \), we have

\[ \overline{\partial} f = \overline{\partial} g - h^b \overline{\partial} U = 0, \]

so \( f \) is a holomorphic function. Moreover, part (a) of the lemma together with (b) in lemma \( \text{[23]} \) imply that \( f \in HF^p_{\beta} \). What is more, part (b) says that \( f \) behaves like \( g \) near \( E \), at least up to order \( \alpha \), and in particular interpolates \( (F^\gamma)_\omega(\gamma) < \alpha \).

### 6.1 Technical lemmas

The following lemmas are going to be used to prove lemma \( \text{[24]} \). When \( E \) is a complex-tangential variety, they are proved in \( \text{[B-O,93]} \). In particular, the first proposition, which does not depend on \( E \), needs no proof.

**Proposition 25** Let, for \( a = 0, 1 \),

\[
D_a(\zeta, z) = \left[ \frac{1 - |\zeta|^2}{1 - z \bar{\zeta}} \right]^N \frac{|1 - z \bar{\zeta}|^{n-1}|\zeta - z|^a}{(d(\zeta, z)^2 + [(1 - |\zeta|^2) + (1 - |z|^2)]|\zeta - z|^2)^{n-\frac{1}{2}+\frac{a}{2}}}.
\]

Then the following is satisfied:

1. If \( \zeta \) and \( z \) are near \( S \),

\[
\int_{B(\zeta, \delta)} D_a(\zeta, z) dV(\zeta) = O(\delta^{\frac{1}{2}+\frac{a}{2}}).
\]
2. If \( N - n + 1 \geq 0 \),

\[
D_a(\zeta, z) = O\left( \frac{(1 - |\zeta|^2)^N}{d(\zeta, z)^{N+n+\frac{a}{2}}} \right).
\]

With this proposition, we can prove the following:

**Lemma 26** Let \( i \geq 0 \), \( 1 + j + i < 0 \), \( j + N + n + 1 - s > 0 \), and \( N \geq n - 1 \). Then for \( a = 0, 1 \):

\[
\int_{B^n} D_a(\zeta, z)d(\zeta, z)^id(\zeta, E)^jdV(\zeta) = O\left( d(z, E)^{i+j+1-\frac{a}{2}} \right).
\]

**Proof:** Both inequalities are proved in the same way, by using proposition 25, so we will only prove the one corresponding to \( D_0 \). To do so, we will split the integral into two parts, over \( B_1 = B(z, d(z, E)/2) \) and its complementary \( B_2 \). Over \( B_1 \), \( d(\zeta, z) \) and \( d(\zeta, E) \) can be bounded by \( d(z, E) \), and the remaining integral can be estimated using part (1) of proposition 25, giving us the bound \( d(z, E)^{i+j+1} \).

On the other hand, from part (2) of proposition 25 we get that the integral over \( B_2 \) can be bounded by:

\[
\int_{d(z, E)}^{\infty} \frac{1}{t^{N+n-i+1}} \int_{B(z,t)} d(\zeta, E)^{j+N}dV(\zeta)dt.
\]

Now if we apply part (h) of theorem 3 to the inner integral, the integral with respect to \( t \) can be bounded by \( d(z, E)^{i+j+1} \) whenever \( j + i + 1 < 0 \).

**Lemma 27** If \( i > 0 \), \( 1 + j + i < N \), \( j + n + 1 - s > 0 \), and \( N \geq n - 1 \), then for \( a = 0, 1 \):

\[
\int_{B^n} D_a(\zeta, z)d(\zeta, z)^id(z, E)^jdV(z) = O(d(\zeta, E)^{j}(1 - |\zeta|)^{i+1-\frac{a}{2}}) = O\left( d(\zeta, E)^{i+j+1-\frac{a}{2}} \right).
\]

**Proof:** This lemma is analogous to lemma 26, just swapping \( \zeta \) and \( z \), and with small modifications of the indices.

The following proposition can be found in [B-O,93].

**Proposition 28** Let

\[
E_1(\zeta, z) = \left[ \frac{1 - |\zeta|^2}{1 - |z\zeta|} \right]^N \frac{|1 - z\zeta|^{n-1}}{(d(\zeta, z)^2 + [(1 - |\zeta|^2) + (1 - |z|^2)]|\zeta - z|^2)^{n-\frac{a}{2}}}.
\]

Then the following is satisfied:
1. If \( \zeta \) and \( z \) are near enough to \( S \),

\[
\int_{B(z, \delta)} E_1(\zeta, rz) \, d\sigma(\zeta) = O(\delta^\frac{s}{2});
\]

2. If \( N - n + 1 \geq 0 \),

\[
E_1(\zeta, z) = O\left(\frac{(1 - |\zeta|^2)^N}{d(\zeta, z)^{N+n-\frac{1}{2}}}\right).
\]

Just as before, using this proposition we can prove the following:

**Lemma 29** Let \( i \geq 0 \), \( j \geq 0 \), \( 0 > \delta > s - n, \frac{1}{2} + j + i + \delta < N \), and \( N \geq n - 1 \).

Then, for \( a = 0, 1 \):

\[
\int_S D_a(\zeta, rz) \, d(\zeta, rz)^i d(z, E)^j \, d(\zeta) \, d\sigma(z) = O\left(\frac{(1 - |\zeta|^2)^i + \delta}{|r - |\zeta||^\frac{s}{2}}\right).
\]

**Proof:** As in lemma 26, we will only prove the first inequality. Given the relation between \( D_0 \) and \( E_1 \), and as \( d(\zeta, rz) \geq |r - |\zeta|| \), it is enough to see that:

\[
\int_S E_1(\zeta, rz) \, d(\zeta, rz)^i d(z, E)^j \, d(\zeta) \, d\sigma(z) = O\left(d(\zeta, E)^{i+j+\delta} (1 - |\zeta|)^{i+\frac{1}{2}}\right).
\]

We will split this integral in three parts. We define \( B_1 = B(\zeta, (1 - |\zeta|)/2) \), \( B_2 = B(\zeta, d(z, E)/2) \setminus B_1 \) and \( B_3 = B^n \setminus B_2 \). The integral over \( B_1 \) is bounded by

\[
C(1 - |\zeta|) d(\zeta, E)^{i+j+\delta} \int_{B_1} E_1(\zeta, rz) \, d\sigma(z) \leq C d(\zeta, E)^{i+j+\delta} (1 - |\zeta|)^{i+\frac{1}{2}}.
\]

Because of (2) in proposition 28, the integral over \( B_2 \) can be bounded by

\[
(1 - |\zeta|)^N \int_{S \setminus B_1} d(\zeta, rz)^{-N-n+\frac{1}{2}+i+j} \, d\sigma(z) \leq C(1 - |\zeta|)^{i+j+\frac{1}{2}} d(\zeta, E)^{i+j+\delta}.
\]

In order to bound the integral over \( B_3 \), we use that as \( d(z, E) \leq 2d(z, \zeta) \) also \( d(rz, E) \leq C d(rz, \zeta) \) This, together with (2) from proposition 28 allows us to bound the integral over \( B_3 \) by:

\[
(1 - |\zeta|)^N \int_{d(\zeta, E)}^{+\infty} \frac{1}{t^{N+n-\frac{1}{2}+i+j}} \int_{B(\zeta, t)} d(z, E)^{i+j} \, d\sigma(z) \, dt.
\]

We bound the inner integral using part (d) of theorem 3, and we have done with it.

To prove part (2) of the lemma, it is enough to use that

\[
D_1(\zeta, z) \leq \frac{1}{(1 - |\zeta|)^{\frac{s}{2}}} D_0(\zeta, z)
\]

and use the previous result. ♣
Lemma 30  Let $\zeta \in B^n$, $0 < d < \frac{1}{2}$. Then

$$\int_{\frac{1}{2}}^{1} \frac{(1-t)^{d-1}}{|t-|\zeta||^\frac{1}{2}} dt \leq C(1-|\zeta|)^{-\frac{1}{2}+d}. $$

Proof: We split the integral in three parts, namely:

$$\int_{\frac{1}{2}}^{1} = \int_{|\zeta|+\frac{1}{2}(1-|\zeta|)}^{1} + \int_{|\zeta|-\frac{1}{2}(1-|\zeta|)}^{1} + \int_{\frac{1}{2}}^{1}.$$

In order to bound the first one, we use that there $t - |\zeta| > \frac{1}{2}(1-|\zeta|)$, and the remaining integral is trivially bounded.

To bound the second integral, we use that there $1 - t \geq \frac{1}{2}(1-|\zeta|)$ and $d - 1 < 0$, thus

$$\int_{|\zeta|-\frac{1}{2}(1-|\zeta|)}^{1} \frac{(1-t)^{d-1}}{|t-|\zeta||^\frac{1}{2}} dt \leq C(1-|\zeta|)^{d-1} \int_{|\zeta|-\frac{1}{2}(1-|\zeta|)}^{1} \frac{dt}{|t-|\zeta||^\frac{1}{2}}.$$

But this last integral can be explicitely calculated, whence the result.

To bound the third integral we observe that in case we have to consider it, $|\zeta| - \frac{1}{2}(1-|\zeta|) \geq \frac{1}{2}$, and this leads to $|\zeta| \geq \frac{3}{2}$. Then

$$\int_{\frac{1}{2}}^{1} \frac{|\zeta|+\frac{1}{2}(1-|\zeta|)}{|t-|\zeta||^\frac{1}{2}} dt \leq \int_{0}^{1} \frac{(1-|\zeta|+|\zeta|-t)^{d-1}}{(|\zeta|-t)^\frac{1}{2}} dt.$$

Again a explicit computation of this last integral gives us the result. $\spadesuit$

6.2 Proof of the lemma

Part (b) of the lemma is proved exactly as in [B-O.93], using the previous lemmas, so we will not repeat it.

To prove (a), let $D_0$ and $D_1$ be defined as in proposition 25. Then as seen in [B-O.93], $|XU(z)|$ is bounded by a finite sum of terms like

$$T_a(z) = \int_{B^n} D_a(\zeta, z) d(\zeta, E)^{-b-k+w(\bar{Y})} |\partial g(\zeta)| dV(\zeta),$$

for $a = 0, 1$, where $\omega = \omega(X)$, $\omega \leq k \leq 2\omega$ and $\omega(Y) \leq k$.

Recall that what we have to bound is:

$$\int_{S} \left( \int_{0}^{1} (1-t^2)^{\ell-\beta-1} |R^\ell h^b U(tz)| dt \right)^p d\sigma(z).$$

But as $h$ is a holomorphic distance function, it is enough to bound terms like

$$\int_{S} \left( \int_{0}^{1} (1-t^2)^{\ell-\beta-1} d(tz, E)^{b-m} |XU(tz)| dt \right)^p d\sigma(z),$$
with $m + \omega(\mathcal{X}) \leq \ell$. But as $|XU|$ is bounded by terms like $T_0$ or $T_1$, we have to substitute $|XU|$ by $T_0$ or $T_1$, and bound these integrals. We will bound the terms of the form $T_0$, the other ones being bounded likewise. On the other hand, the integral between 0 and $\frac{1}{2}$ is trivially bounded, thus we need only to bound the integral between $\frac{1}{2}$ and 1.

Let $0 < \delta < \min\{\ell - \beta, 1/2p\}$. Because of lemmas 20 and 37, for $b$ large enough,

$$
\int_0^1 (1 - t^2)^{p(\ell - \beta - \delta) - 1} d(tz, E)^{b + \omega(\mathcal{Y}) - 1} - 1 - (\ell - \beta)p' \times
$$

$$
\times \int_{B^n} D_0(\zeta, tz) d(\zeta, tz)^{k - \omega} d(\zeta, E)^{-b - k + \omega(\mathcal{Y})} dV(\zeta) dt \leq C \int_0^1 (1 - t)^p(\ell - \beta - \delta - 1) d(tz, E)^{-(\ell - \beta)p'} dt \leq C d(z, E)^{-\delta p'}.
$$

Hence, because of Hölder’s inequality,

$$
\left( \int_0^1 (1 - t^2)^{\ell - \beta - 1} d(tz, E)^{b - m} T_1(tz) dt \right)^p \leq Cd(z, E)^{-\delta p} \int_0^1 (1 - t)^{\delta p - 1} d(tz, E)^{b - \frac{p}{2p}(\omega - \omega(\mathcal{Y}) - 1) + (\ell - \beta - m)p} \times
$$

$$
\times \int_{B^n} D_0(\zeta, tz) d(\zeta, tz)^{k - \omega} d(\zeta, E)^{-b - k + \omega(\mathcal{Y})} |\nabla g(\zeta)|^p dV(\zeta) dt.
$$

If we now integrate with respect to $z \in S$ and apply Fubini’s theorem, we get that

$$
\int_S \left( \int_0^1 (1 - t^2)^{\ell - \beta - 1} d(tz, E)^{b - m} T_1(tz) dt \right)^p d\sigma(z) \leq
$$

$$
\leq C \int_{B^n} d(\zeta, E)^{-b - k + \omega(\mathcal{Y})} |\nabla g(\zeta)|^p \int_0^1 (1 - t)^{\delta p - 1} dt dV(\zeta) \times
$$

$$
\times \int_S d(z, E)^{-\delta p} d(tz, E)^{-\frac{p}{2p}(\omega - \omega(\mathcal{Y}) - 1) + (\ell - \beta - m)p} D_1(\zeta, tz) d(\zeta, tz)^{k - \omega} d\sigma(z).
$$

The inner integral can be bounded using lemma 29, and the integral with respect to $t$ using lemma 36. Thus this expression is bounded by

$$
\int_{B^n} d(\zeta, E)^p(\omega(\mathcal{Y}) + 1 - \beta - 1 + p(\ell - \omega - m)) |\nabla g(\zeta)|^p dV(\zeta) \leq
$$

$$
\leq \int_{B^n} d(\zeta, E)^p(\omega(\mathcal{Y}) + 1 - \beta - 1) |\nabla g(\zeta)|^p dV(\zeta),
$$

as $\ell \geq m + \omega$. Now as $\omega(\mathcal{Y}) = \omega(\mathcal{Y}) + 1$ this integral was bounded in lemma 23, so we are done.
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