FINITE DIMENSIONAL GROUPS OF LOCAL DIFFEOMORPHISMS

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Abstract

We are interested in classifying groups of local biholomorphisms (or even formal diffeomorphisms) that can be endowed with a canonical structure of algebraic group up to add extra formal diffeomorphisms. We show that this is the case for virtually polycyclic subgroups and in particular finitely generated virtually nilpotent groups of local biholomorphisms. We provide several methods to identify this property and build examples.

As a consequence we generalize results of Arnold, Seigal-Yakovenko and Binyamini on uniform estimates of local intersection multiplicities to bigger classes of groups, including for example virtually polycyclic groups.

1. Introduction

We study the action of groups of self-maps on intersection multiplicities. More precisely, given varieties $V$ and $W$ of complementary dimension of an ambient space $M$ and a subgroup $G$ of self-maps of $M$, we want to identify conditions guaranteeing that $F \mapsto (F(V), W)$ is bounded as a function of $G$. Let us introduce a classical example of an application of such a property. Consider a continuous map $F : M \to M$ and an isolated fixed point $P$ of $F$. The fixed point index of $F$ at $P$ is equal to the topological intersection index of $\Delta$ and $(F \times \text{Id})(\Delta)$ at $(P, P)$ where $\Delta$ is the diagonal of $M \times M$. By considering the iterates $(F \times \text{Id})^n$ with $n \in \mathbb{Z}$ we obtain fixed point indexes for the fixed points of the iterates of $F^n$, i.e. for periodic points. In the context of $C^1$ maps Shub and Sullivan proved that the intersection index of $\Delta$ and $(F \times \text{Id})^n(\Delta)$ at isolated fixed points is uniformly bounded.
Theorem 1.1 ([25]). Let $U$ be an open subset of $\mathbb{R}^m$. Let $F : U \to \mathbb{R}^m$ be a $C^1$ map such that $0$ is an isolated fixed point of $F^n$ for any $n \geq 1$. Then the fixed point index of $F^n$ at $0$ is bounded as a function of $n$.

As an immediate corollary they show that a $C^1$ map $F : M \to M$ defined in a compact differentiable manifold $M$ has infinitely many periodic points if the sequence of Lefschetz numbers $(L(F^n))_{n \geq 1}$ is unbounded.

We denote by $\text{Diff}(\mathbb{C}^n,0)$ the group of germs of biholomorphism defined in a neighborhood of the origin in $\mathbb{C}^n$. We are interested in uniform intersection results in the local holomorphic setting. More precisely, we want to identify subgroups $G$ of $\text{Diff}(\mathbb{C}^n,0)$ satisfying that the set

$$\{(\phi(V),W) : \phi \in G \text{ and } (\phi(V),W) < \infty\}$$

of intersection multiplicities (cf. Definition 5.1) is bounded for any pair of germs of holomorphic varieties $V,W$ defined in a neighborhood of $0$ in $\mathbb{C}^n$.

The first result in this direction is due to Arnold.

Theorem 1.2 ([1, Theorem 1]). Let $\phi \in \text{Diff}(\mathbb{C}^n,0)$. Consider germs of submanifolds $V,W$ of $\mathbb{C}^n,0$ of complementary dimension. Suppose that the intersection multiplicity $\mu_n := (\phi^n(V),W)$ is finite for any $n \in \mathbb{Z}$. Then the sequence $(\mu_n)_{n \in \mathbb{Z}}$ is bounded.

The proof is a consequence of the Skolem-Mahler-Lech theorem on roots of quasipolynomials [26].

The previous result was generalized to the finitely generated abelian case by Seigal and Yakovenko. We denote by $\hat{\text{Diff}}(\mathbb{C}^n,0)$ the group of formal diffeomorphisms (cf. Definition 2.5).

Theorem 1.3 ([23, Theorem 1]). Let $G$ be an abelian subgroup of $\hat{\text{Diff}}(\mathbb{C}^n,0)$ generated by finitely many cyclic and one parameter groups. Consider formal subvarieties $V$, $W$. Then the set

$$\{(\phi(V),W) : \phi \in G \text{ and } (\phi(V),W) < \infty\}$$

is bounded.

The group $\text{Diff}(\mathbb{C}^n,0)$ is a subgroup of $\hat{\text{Diff}}(\mathbb{C}^n,0)$ and hence Theorem 1.3 holds for subgroups of $\text{Diff}(\mathbb{C}^n,0)$ and germs of subvarieties $V$ and $W$. In contrast with Theorem 1.2 notice that it is not necessary to require that all intersection multiplicities are finite. The proof relies on a noetherianity argument (cf. section 5).
An analogous result was proved by Binyamini for the case in which the subgroup $G$ of $\hat{Diff}(\mathbb{C}^n,0)$ is embedded in a group of formal diffeomorphisms that has a natural Lie group structure.

**Theorem 1.4 ([3, Theorem 5]).** Let $G$ be a Lie subgroup (cf. Definition 3.3) of $\hat{Diff}(\mathbb{C}^n,0)$ with finitely many connected components. Consider formal subvarieties $V, W$. Then the set
\[
\{(\phi(V), W) : \phi \in G \text{ and } (\phi(V), W) < \infty\}
\]
is bounded.

Theorem 1.3 has more natural hypotheses (commutativity and finite generation) but Theorem 1.4 is somehow more general. Indeed Binyamini shows that any finitely generated abelian subgroup of $\hat{Diff}(\mathbb{C}^n,0)$ is a subgroup of a Lie group with finitely many connected components [3]. Thus it is interesting to study how to find an extension of a subgroup of $\hat{Diff}(\mathbb{C}^n,0)$ that is also a Lie group. In this paper we characterize the subgroups of $\hat{Diff}(\mathbb{C}^n,0)$ that can be embedded in a Lie group (with finitely many connected components) in a natural way. Moreover we show that every such a group can be canonically embedded in an algebraic matrix group. We call these groups **finite dimensional**.

Let us be more precise. We define a Zariski-closure $\overline{G}$ of a subgroup $G$ of $\hat{Diff}(\mathbb{C}^n,0)$ (cf. Definition 2.12); it is a projective limit of algebraic matrix groups and hence it has a natural definition of dimension. We will say that $G$ is finite dimensional if $\overline{G}$ is finite dimensional (cf. Definition 3.1). If $G$ is finite dimensional then $\overline{G}$ is isomorphic to one of its subgroups of $k$-jets and hence $\overline{G}$ can be interpreted as an algebraic group (Proposition 3.2). Equivalently the group $G$ is finite dimensional if and only if there exists $k_0 \in \mathbb{N}$ such that the coefficients of degree greater than $k_0$ in the Taylor expansion at the origin of the elements of $G$ are polynomial functions on the coefficients of degree less or equal than $k_0$ (Remark 3.4).

Finite dimensional subgroups satisfy uniform local intersection properties.

**Theorem 1.5.** Let $G$ be a finite dimensional subgroup of $\hat{Diff}(\mathbb{C}^n,0)$. Consider ideals $I$ and $J$ of $\hat{O}_n$. Then the set
\[
\{((\phi^*(I)), J) : \phi \in G \text{ and } (\phi^*(I), J) < \infty\}
\]
is bounded.

We define $\hat{O}_n$ as the ring of formal power series with complex coefficients in $n$ variables.
Since an algebraic group is a complex Lie group with finitely many connected components, Theorem 1.5 can be seen as a consequence of Binyamini’s Theorem 1.4. Anyway, the finite dimensional hypothesis provides a simplification of the proof.

We will exhibit different methods to find finite dimensional subgroups of $\widehat{\text{Diff}}(\mathbb{C}^n,0)$ (cf. Theorems 3.1, 4.1...). In particular we identify several algebraic group properties implying that a subgroup of $\widehat{\text{Diff}}(\mathbb{C}^n,0)$ is finite dimensional, including some notable ones. Our main result is the following theorem:

**Theorem 1.6.** Let $G$ be a subgroup of $\widehat{\text{Diff}}(\mathbb{C}^n,0)$ such that either is

- virtually polycyclic or
- virtually nilpotent and generated by finitely many cyclic and one parameter subgroups of $G$.

Then $G$ is finite dimensional. In particular the set

$$\{(\phi^*(I), J) : \phi \in G \text{ and } (\phi^*(I), J) < \infty\}$$

is bounded for any pair of ideals $I$ and $J$ of $\mathcal{O}_n$.

Check Definitions 2.19, 4.1 and 2.18 out for the definitions of virtual property, polycyclic and nilpotent groups respectively.

Notice that in particular Theorem 1.6 applies to finitely generated virtually nilpotent subgroups of $\widehat{\text{Diff}}(\mathbb{C}^n,0)$, i.e. to subgroups of polynomial growth of formal diffeomorphisms.

We introduce several techniques that allow to build finite dimensional subgroups of $\widehat{\text{Diff}}(\mathbb{C}^n,0)$. Every time that we identify such a group we obtain an analogue of Theorem 1.6. Instead of writing down the most general possible result, we prefer to highlight some remarkable properties that imply finite dimensionality.

Let us compare the hypotheses of Theorems 1.4 and 1.5. A priori it could be possible for a subgroup $G$ of $\widehat{\text{Diff}}(\mathbb{C}^n,0)$ to satisfy $\dim G = \infty$ and being the image by a morphism of a real Lie group with finitely many connected components (cf. Definition 3.5). We will see that it never happens (Theorem 4.1) and thus our canonical approach encloses the results by Seigal-Yakovenko and Binyamini [23, 3]. The definition of finite dimensional subgroup of $\widehat{\text{Diff}}(\mathbb{C}^n,0)$ allows us to apply the techniques of the algebraic group theory in the study of local intersection problems.

Our canonical approach makes simpler to analyze whether or not subgroups of $\widehat{\text{Diff}}(\mathbb{C}^n,0)$ are embeddable in algebraic groups. We will relate finite dimensionality with other group properties, namely
• Finite determination (cf. Definition 3.2) properties. We will show that finite determination implies finite dimension under certain closedness properties (Corollary 3.2).

• Finite decomposition properties. A subgroup $G$ of $\hat{\text{Diff}}(\mathbb{C}^n,0)$ is finite dimensional if and only if every element can be written as a word of uniformly bounded length in an alphabet whose letters belong to the union of finitely many cyclic and one parameter subgroups of $\hat{\text{Diff}}(\mathbb{C}^n,0)$ (Theorem 3.1 and Remark 3.10).

• Virtually solvable subgroups of $\hat{\text{Diff}}(\mathbb{C}^n,0)$ with suitable finite generation hypotheses are finite dimensional (Proposition 4.1, Theorems 4.2, 4.3, Corollaries 4.3, 4.4...)

• Decomposition of the group in a tower of extensions of the trivial group.

Let us expand on the final item of the previous list. Consider a normal subgroup $H$ of a subgroup $G$ of $\hat{\text{Diff}}(\mathbb{C}^n,0)$. We can define the codimension of $H$ or equivalently the dimension of the extension $G/H$ as the codimension of $\overline{H}$ in $G$. Such definition is interesting because a tower of finite dimensional extensions is finite dimensional (Proposition 3.1). In particular it is possible to decide whether or not a subgroup of $\hat{\text{Diff}}(\mathbb{C}^n,0)$ is finite dimensional by considering it as a tower of (easier to handle) extensions of the trivial group. We will exhibit some classes of extensions that are finite dimensional, namely

• Finite extensions.

• Finitely generated abelian extensions.

• $G/H$ is a connected Lie group (cf. Definition 3.5).

• Certain subextensions of virtually solvable extensions (Theorems A.2 and A.3).

For instance a virtually polycyclic group is a tower of cyclic and finite extensions of the trivial group and hence it is always finite dimensional. Hence we can apply Theorem 1.5 to show Theorem 1.6. The extension approach provides a method to build examples of finite dimensional subgroups of $\hat{\text{Diff}}(\mathbb{C}^n,0)$.
2. Pro-algebraic groups

Let us explain some of the basic properties of the pro-algebraic subgroups of \( \hat{\text{Diff}}(\mathbb{C}^n, 0) \). Pro-algebraic groups of formal diffeomorphisms have been used in the study of differential Galois theory by Morales-Ramis-Simó [17]. Most of the results in this section can be found in [14] and [19]. We explain them here for the sake of clarity and completeness.

2.1. Formal vector fields and diffeomorphisms. Let us introduce some notations.

Definition 2.1. We denote by \( O_n \) the ring \( \mathbb{C}\{z_1, \ldots, z_n\} \) of germs of holomorphic functions defined in the neighborhood of 0 in \( \mathbb{C}^n \). We denote by \( \mathfrak{m} \) the maximal ideal of \( O_n \).

Analogously we define \( \hat{O}_n \) as the ring of formal power series with complex coefficients in \( n \) variables whose maximal ideal will be denoted by \( \hat{\mathfrak{m}} \).

Next, we define formal vector fields as a generalization of local vector fields.

Definition 2.2. We denote by \( \mathfrak{X}(\mathbb{C}^n, 0) \) the Lie algebra of germs of holomorphic vector fields defined in the neighborhood of 0 in \( \mathbb{C}^n \) that are singular at 0.

Remark 2.1. An element \( X \) of \( \mathfrak{X}(\mathbb{C}^n, 0) \) is of the form

\[
X = f_1(z_1, \ldots, z_n) \frac{\partial}{\partial z_1} + \ldots + f_n(z_1, \ldots, z_n) \frac{\partial}{\partial z_n}
\]

where \( f_1, \ldots, f_n \) belong to the maximal ideal \( \mathfrak{m} \) of \( O_n \). Analogously \( X \) can be interpreted as a derivation of the \( \mathbb{C} \)-algebra \( \mathfrak{m} \).

Definition 2.3. We define the Lie algebra \( \hat{\mathfrak{X}}(\mathbb{C}^n, 0) \) as the set of derivations of the \( \mathbb{C} \)-algebra \( \hat{\mathfrak{m}} \). Analogously we can identify an element \( X \) of \( \hat{\mathfrak{X}}(\mathbb{C}^n, 0) \) with the expression

\[
X = X(z_1) \frac{\partial}{\partial z_1} + \ldots + X(z_n) \frac{\partial}{\partial z_n}
\]

where the coefficients of the vector field belong to \( \hat{\mathfrak{m}} \).

Let us apply the same program to diffeomorphisms.

Definition 2.4. We denote by \( \text{Diff}(\mathbb{C}^n, 0) \) the group of germs of biholomorphism defined in the neighborhood of 0 in \( \mathbb{C}^n \).

Remark 2.2. An element \( \phi \) of \( \text{Diff}(\mathbb{C}^n, 0) \) is of the form

\[
\phi(z_1, \ldots, z_n) = (f_1(z_1, \ldots, z_n), \ldots, f_n(z_1, \ldots, z_n))
\]
where \( f_1, \ldots, f_n \in \mathfrak{m} \) and its linear part \( D_0\phi \) at the origin is an invertible linear map.

**Definition 2.5.** We say that \( \phi \) belongs to the group \( \widehat{\text{Diff}}(\mathbb{C}^n, 0) \) of formal diffeomorphisms if it is of the form

\[
\phi(z_1, \ldots, z_n) = (f_1(z_1, \ldots, z_n), \ldots, f_n(z_1, \ldots, z_n))
\]

where \( f_1, \ldots, f_n \in \widehat{\mathfrak{m}} \) and \( D_0\phi \) is an invertible linear map.

We will use the Krull topology (the \( \widehat{\mathfrak{m}} \)-adic topology) in our spaces of formal objects.

**Definition 2.6.** The sets of the form \( f + \mathfrak{m}^j \) for any choice of \( f \in \mathcal{O}_n \) and \( j \geq 0 \) are a base of open sets of a topology in \( \mathcal{O}_n \), the so called \( \mathfrak{m} \)-adic (or Krull) topology. Since we can interpret formal vector fields and diffeomorphisms as \( n \)-uples of elements in \( \mathfrak{m} \) we can define the Krull topology in \( \mathcal{X}(\mathbb{C}^n, 0) \) and \( \widehat{\text{Diff}}(\mathbb{C}^n, 0) \).

**Remark 2.3.** A sequence \( (f_k)_{k \geq 1} \) of elements of \( \mathcal{O}_n \) converges to \( f \in \mathcal{O}_n \) in the Krull topology if for any \( j \in \mathbb{N} \) there exists \( k_0 \in \mathbb{N} \) such that \( f - f_k \in \mathfrak{m}^j \) for any \( k \geq k_0 \). Convergence of sequences in \( \mathcal{X}(\mathbb{C}^n, 0) \) and \( \widehat{\text{Diff}}(\mathbb{C}^n, 0) \) is analogous.

**Remark 2.4.** It is clear that \( \mathcal{O}_n, \mathcal{X}(\mathbb{C}^n, 0) \) and \( \widehat{\text{Diff}}(\mathbb{C}^n, 0) \) are the closures in the Krull topology of \( \mathcal{O}_n, \mathcal{X}(\mathbb{C}^n, 0) \) and \( \widehat{\text{Diff}}(\mathbb{C}^n, 0) \) respectively.

It is difficult to work with \( \widehat{\text{Diff}}(\mathbb{C}^n, 0) \) since it is an infinite dimensional space. Anyway we can understand \( \widehat{\text{Diff}}(\mathbb{C}^n, 0) \) as a projective limit \( \lim_{\longleftarrow k \in \mathbb{N}} D_k \) where every \( D_k \) is a finite dimensional matrix group for \( k \in \mathbb{N} \). We should interpret \( D_k \) as the group of \( k \)-jets of elements of \( \widehat{\text{Diff}}(\mathbb{C}^n, 0) \). Next, let us explain how to define rigorously the groups \( D_k \) for \( k \in \mathbb{N} \) and how this allows to apply the theory of linear algebraic groups to the groups of formal diffeomorphisms.

Given \( X \in \mathcal{X}(\mathbb{C}^n, 0) \) and \( \phi \in \widehat{\text{Diff}}(\mathbb{C}^n, 0) \) we can associate \( X_k, \phi_k \in \text{GL}(\mathfrak{m}/\mathfrak{m}^{k+1}) \) respectively for any \( k \in \mathbb{N} \). They are given by

\[
\begin{align*}
\frac{\mathfrak{m}}{\mathfrak{m}^{k+1}} &\xrightarrow{X_k} \frac{\mathfrak{m}}{\mathfrak{m}^{k+1}} & \frac{\mathfrak{m}}{\mathfrak{m}^{k+1}} &\xrightarrow{\phi_k} \frac{\mathfrak{m}}{\mathfrak{m}^{k+1}} \\
f + \mathfrak{m}^{k+1} &\mapsto X(f) + \mathfrak{m}^{k+1}, & f + \mathfrak{m}^{k+1} &\mapsto f \circ \phi + \mathfrak{m}^{k+1}.
\end{align*}
\]

The linear map \( X_k \) (resp. \( \phi_k \)) determines and is determined by the \( k \)-jet of \( X \) (resp. \( \phi \)). Moreover \( L_k := \{X_k : X \in \mathcal{X}(\mathbb{C}^n, 0)\} \) is the Lie algebra of the group \( D_k := \{\phi_k : \phi \in \widehat{\text{Diff}}(\mathbb{C}^n, 0)\} \). It is an algebraic subgroup of \( \text{GL}(\mathfrak{m}/\mathfrak{m}^{k+1}) \) since it satisfies

\[
D_k = \{\alpha \in \text{GL}(\mathfrak{m}/\mathfrak{m}^{k+1}) : \alpha(fg) = \alpha(f)\alpha(g) \ \forall f, g \in \mathfrak{m}/\mathfrak{m}^{k+1}\},
\]
Definition 2.7. Given \( k \geq l \geq 1 \) we define the maps \( \pi_k : \hat{\text{Diff}}(\mathbb{C}^n, 0) \to D_k \) and \( \pi_{k,l} : D_k \to D_l \) given by \( \pi_k(\phi) = \phi_k \) and \( \pi_{k,l}(\phi_k) = \phi_l \).

Since \( D_k \) is the group of truncations of elements of \( \hat{\text{Diff}}(\mathbb{C}^n, 0) \) up to level \( k \), we can interpret \( \hat{\text{Diff}}(\mathbb{C}^n, 0) \) as the projective limit of the projective system \( \left( \lim_{\leftarrow} k \in \mathbb{N} D_k, (\pi_{k,l})_{k \geq l \geq 1} \right) \) of algebraic groups and morphisms of algebraic groups \([19, \text{Lemma 2.2}]\). Analogously \( \hat{\mathcal{X}}(\mathbb{C}^n, 0) \) is the projective limit \( \lim_{\leftarrow} \Lambda_k \).

2.2. Exponential map. Given \( X \in \hat{\mathcal{X}}(\mathbb{C}^n, 0) \) we can define its exponential \( \exp(X) \). Indeed given \( (X_k)_{k \geq 1} \in \lim_{\leftarrow} \Lambda_k = \hat{\mathcal{X}}(\mathbb{C}^n, 0) \) the family \( (\exp(X_k))_{k \geq 1} \) defines an element \( \exp(X) \) of \( \hat{\text{Diff}}(\mathbb{C}^n, 0) \equiv \lim_{\leftarrow} D_k \). Equivalently we consider a sequence \( (X_j)_{j \in \mathbb{N}} \) of convergent vector fields that converges to \( X \) in the Krull topology and then we define \( \exp(X) \) as the limit in the Krull topology of \( (\exp(X_j))_{j \in \mathbb{N}} \) where \( \exp(X_j) \) is the time 1 flow of \( X_j \) for \( j \in \mathbb{N} \).

Definition 2.8. We say that a formal vector field \( X \in \hat{\mathcal{X}}(\mathbb{C}^n, 0) \) is nilpotent if its linear part \( D_0 X \) is nilpotent. We denote by \( \hat{\mathcal{X}}_N(\mathbb{C}^n, 0) \) the subset of \( \hat{\mathcal{X}}(\mathbb{C}^n, 0) \) of formal nilpotent vector fields.

Definition 2.9. We say that a formal diffeomorphism \( \phi \in \hat{\text{Diff}}(\mathbb{C}^n, 0) \) is unipotent if its linear part \( D_0 \phi \) is unipotent. We denote by \( \hat{\text{Diff}}_u(\mathbb{C}^n, 0) \) the subset of \( \hat{\text{Diff}}(\mathbb{C}^n, 0) \) of formal unipotent diffeomorphisms. We say that a subgroup \( G \) of \( \hat{\text{Diff}}(\mathbb{C}^n, 0) \) is unipotent if \( G \subset \hat{\text{Diff}}_u(\mathbb{C}^n, 0) \).

Proposition 2.1 (cf. \([5, 16, 9, \text{Th. 3.17}]\)). The map
\[
\exp : \hat{\mathcal{X}}_N(\mathbb{C}^n, 0) \to \hat{\text{Diff}}_u(\mathbb{C}^n, 0)
\]
is a bijection.

Definition 2.10. Given \( \phi \in \hat{\text{Diff}}_u(\mathbb{C}^n, 0) \) we define its infinitesimal generator \( \log \phi \) as the unique formal nilpotent vector field such that \( \phi = \exp(\log \phi) \). We denote \( \phi^t = \exp(t \log \phi) \) for \( t \in \mathbb{C} \).

2.3. Zariski-closure of a group of formal diffeomorphisms.

Definition 2.11. Let \( G \) be a subgroup of \( \hat{\text{Diff}}(\mathbb{C}^n, 0) \). Given \( k \in \mathbb{N} \) we define \( G_k^* = \{ \phi_k : \phi \in G \} \) and \( G_k \) as the Zariski-closure of \( G_k^* \) in \( \text{GL}(\bar{m}/\bar{m}^{k+1}) \).

Let us remark that since \( D_k \) is algebraic, \( G_k \) is a subgroup of \( D_k \) for any \( k \in \mathbb{N} \).
Remark 2.5. Given $k \geq l \geq 1$ the image of the algebraic closure of $G^*_k$ by $\pi_{k,l}$ is the algebraic closure of the image $G^*_l$ (cf. [11 2.1 (f), p. 57]). Hence we obtain $\pi_{k,l}(G_k) = G_l$ for any $k \geq l \geq 1$. In particular $(\lim_{\leftarrow k \in \mathbb{N}} G_k, (\pi_{k,l})_{k \geq l \geq 1})$ is a projective system.

Definition 2.12. Let $G$ be a subgroup of $\widehat{\text{Diff}}(\mathbb{C}^n, 0)$. We define the Zariski-closure $\overline{G}$ of $G$ as $\lim_{\leftarrow k \in \mathbb{N}} G_k$ or in other words

$$\overline{G} = \{ \phi \in \widehat{\text{Diff}}(\mathbb{C}^n, 0) : \phi_k \in G_k \ \forall k \in \mathbb{N} \}.$$

2.4. Definition of pro-algebraic group.

Definition 2.13. Let $G$ be a subgroup of $\widehat{\text{Diff}}(\mathbb{C}^n, 0)$. We say that $G$ is pro-algebraic if $G = \overline{G}$.

Remark 2.6. Since $\pi_{k,l}(G_k) = G_l$ for any $k \geq l \geq 1$, the natural projection $(\pi_k)_{\overline{G}} : \overline{G} \to G_k$ is surjective for any $k \in \mathbb{N}$ (cf. [19 Lemma 2.5], [21 Corollary 3.25]). Thus the Zariski-closure of $\overline{G}$ coincides with $\overline{G}$ and $\overline{G}$ is pro-algebraic. It is the minimal pro-algebraic group containing $G$.

We can characterize pro-algebraic subgroups of $\widehat{\text{Diff}}(\mathbb{C}^n, 0)$.

Proposition 2.2 ([19 Proposition 2.2], cf. [21 Proposition 3.26]). Let $G$ be a subgroup of $\widehat{\text{Diff}}(\mathbb{C}^n, 0)$. Then $G$ is pro-algebraic if and only if $G^*_k$ is an algebraic subgroup of $\text{GL}(\mathbb{m}/\mathbb{m}^{k+1})$ for any $k \in \mathbb{N}$ and $G$ is closed in the Krull topology.

2.5. Lie algebra of a pro-algebraic group. Pro-algebraic groups have a connected component of $\text{Id}$ whose properties are analogous to the connected component of $\text{Id}$ of an algebraic matrix group. This is a particular instance of a more general situation: many analogues of concepts involving algebraic groups can be transferred to the pro-algebraic setting.

Definition 2.14. Let $G$ be a subgroup of $\widehat{\text{Diff}}(\mathbb{C}^n, 0)$. We define $G_{k,0}$ as the connected component of $\text{Id}$ of $G_k$ for $k \in \mathbb{N}$. We define $G_0 = \lim_{\leftarrow k \in \mathbb{N}} G_{k,0}$ or equivalently

$$G_0 = \{ \phi \in \overline{G} : \phi_k \in G_{k,0} \ \forall k \in \mathbb{N} \}.$$

We say that $G_0$ is the connected component of $\text{Id}$ of $\overline{G}$. If $G$ is pro-algebraic then we denote $G_0 = \overline{G}_0$.

Remark 2.7. The group $G_0$ is a finite index normal pro-algebraic subgroup of $\overline{G}$ ([19 Proposition 2.3 and Remark 2.9], cf. [21 Proposition 3.35 and Remark 3.37]).
Proposition 2.3 ([14, Proposition 2]). Let $G$ be a subgroup of $\hat{\text{Diff}}$ ($\mathbb{C}^n, 0$). We consider

$$\mathfrak{g} = \{ X \in \hat{\mathfrak{X}}(\mathbb{C}^n, 0) : \exp(tX) \in G \quad \forall t \in \mathbb{C} \}.$$ 

Then $\mathfrak{g}$ is a Lie algebra and $G_0$ is generated by the set $\exp(\mathfrak{g})$.

Definition 2.15. We say that $\mathfrak{g}$ is the Lie algebra of $G$.

It is natural to consider $G_0$ as the connected component of $Id$ of $G$ since it is a finite index normal subgroup of $G$ that is generated by the exponential of the Lie algebra of $G$.

The Zariski-closure of a cyclic subgroup of $\hat{\text{Diff}}_u(\mathbb{C}^n, 0)$ is connected and one dimensional.

Remark 2.8 ([19, Remark 2.11], cf. [21, Remark 3.30]). Let $\phi$ be a unipotent element of $\hat{\text{Diff}}(\mathbb{C}^n, 0)$. Then $\langle \phi \rangle$ is equal to $\{ \phi^t : t \in \mathbb{C} \}$. In particular the Lie algebra of $\langle \phi \rangle$ is the one dimensional complex vector space generated by $\log \phi$.

The next property is well-known in the finite dimensional setting.

Lemma 2.1. Let $H$ be a finite index subgroup of a pro-algebraic subgroup $G$ of $\hat{\text{Diff}}(\mathbb{C}^n, 0)$. Then $H$ contains $G_0$.

Proof. Given an element $X$ in the Lie algebra $\mathfrak{g}$ of $G$, its one-parameter group $\{ \exp(tX) : t \in \mathbb{C} \}$ is contained in $G$. Since $H$ is a finite index subgroup of $G$, $\{ \exp(tX) : t \in \mathbb{C} \}$ is contained in $H$. Any element of $G_0$ is of the form $\exp(X_1) \circ \ldots \circ \exp(X_m)$ for some $X_1, \ldots, X_m \in \mathfrak{g}$ by Proposition 2.3. Hence $G_0$ is contained in $H$. □

The next result will be used later on to identify pro-algebraic groups.

Theorem 2.1 (Chevalley, cf. [4, section I.2.2, p. 57]). The group generated by a family of connected algebraic matrix groups is algebraic.

2.6. Normal subgroups of pro-algebraic groups. Next results relate the properties of normal subgroups with those of their algebraic closures.

Lemma 2.2. Let $H$ be a normal subgroup of a subgroup $G$ of $\hat{\text{Diff}}(\mathbb{C}^n, 0)$. Then $H$, $G_k$ is a normal subgroup of $\overline{G}$. Moreover $H_k$ is a normal subgroup of $G_k$ for any $k \in \mathbb{N}$.

Proof. Fix $k \in \mathbb{N}$. We have $AH_k A^{-1} = H_k^*$ for any $A \in G_k^*$. We deduce $AH_k A^{-1} = H_k$ for any $A \in G_k^*$. The normalizer of the algebraic subgroup $H_k$ in the algebraic group $G_k$ is algebraic and contains $G_k^*$. Hence it is equal to $G_k$ and then $H_k$ is a normal subgroup of $G_k$ for any $k \in \mathbb{N}$. As a consequence $\overline{H}$ is a normal subgroup of $\overline{G}$. □
Lemma 2.3. Let $H$ be a finite index subgroup of a subgroup $G$ of $\hat{\text{Diff}} (\mathbb{C}^n, 0)$. Then $H$ is pro-algebraic if and only if $G$ is pro-algebraic.

Proof. Since $G_k^*$ and $H_k^*$ are images of $G$ and $H$ respectively by the morphism of groups $\pi_k : \hat{\text{Diff}} (\mathbb{C}^n, 0) \to D_k$, $H_k^*$ is a finite index subgroup of $G_k^*$ for any $k \in \mathbb{N}$.

Suppose $H$ is pro-algebraic. Then $H_k^*$ is algebraic for any $k \in \mathbb{N}$ by Proposition 2.2. Hence $G_k^*$ is algebraic for any $k \in \mathbb{N}$. In order to prove that $G$ is pro-algebraic, it suffices to show that $G$ is closed in the Krull topology. Since $G$ is the union of finitely many left cosets of $H$ and they are all closed in the Krull topology, we deduce that $G$ is closed in the Krull topology.

Suppose $G$ is pro-algebraic. Then $G_0$ is pro-algebraic by Remark 2.7. Moreover $G_0$ is contained in $H$ by Lemma 2.1. Since $G_0$ is a finite index subgroup of $G$ and then of $H$, $H$ is pro-algebraic by the first part of the proof. □

Lemma 2.4. Let $H$ be a finite index normal subgroup of a subgroup $G$ of $\hat{\text{Diff}} (\mathbb{C}^n, 0)$. Then $\overline{H}$ is a finite index normal subgroup of $G$. Moreover $H_k$ is a finite index normal subgroup of $G_k$ for any $k \in \mathbb{N}$.

Proof. Consider $\phi_1, \ldots, \phi_m \in G$ such that $G/H = \{\phi_1 H, \ldots, \phi_m H\}$. We define the group $J = \langle \overline{\phi_1}, \ldots, \overline{\phi_m} \rangle$, it satisfies $J \subset \overline{G}$. Let us show that $J = \overline{G}$ and that $\overline{H}$ is a finite index normal subgroup of $J$.

Since $\overline{H}$ is a normal subgroup of $J$ by Lemma 2.2, every element $\psi$ of $J$ is of the form

$$\psi = \overline{\phi_1^{i_1} \circ \cdots \circ \phi_m^{i_m}} \circ h$$

where $h \in \overline{H}$. The choice of $\phi_1, \ldots, \phi_m$ implies the existence of $1 \leq j \leq m$ and $h' \in H$ such that $\psi = \phi_j \circ (h' \circ h)$. In particular the natural map $G/H \to J/\overline{H}$ is surjective and hence $\overline{H}$ is a finite index normal subgroup of $J$. The group $J$ is pro-algebraic by Lemma 2.3. Since $G \subset J \subset \overline{G}$, we deduce $J = \overline{G}$ and $\overline{H}$ is a finite index normal subgroup of $\overline{G}$. Since $G_k$ and $H_k$ are images of $\overline{G}$ and $\overline{H}$ respectively by the morphism $\pi_k : \overline{G} \to G_k$, $H_k$ is a finite index normal subgroup of $G_k$ for any $k \in \mathbb{N}$. □

2.7. Algebraic properties of the Zariski-closure. The groups $G$ and $\overline{G}$ share many algebraic properties.

Definition 2.16. Let $G$ be a group. Given $f, g \in G$ we define by $[f, g] = fgf^{-1}g^{-1}$ the commutator of $f$ and $g$.

Given subgroups $H, L$ of $G$ we define $[H, L] = \langle [h, l] : h \in H, l \in L \rangle$ as the subgroup generated by the commutators of elements of $H$ and elements of $L$. 
Definition 2.17. Let $G$ be a group. By induction we define the subgroups

$$G^{(0)} = G, \ G^{(1)} = [G^{(0)}, G^{(0)}], \ldots, \ G^{(\ell+1)} = [G^{(\ell)}, G^{(\ell)}], \ldots$$

of the derived series of $G$. We say that $G^{(\ell)}$ is the $\ell$-th derived group of $G$. We use sometimes the notation $G'$ instead of $G^{(1)}$ for the derived group of $G$.

We say that $G$ is solvable if there exists $\ell \in \mathbb{N} \cup \{0\}$ such that $G^{(\ell)} = \{1\}$. We define the derived length of $G$ as the minimum $\ell \in \mathbb{N} \cup \{0\}$ with such a property.

Definition 2.18. Let $G$ be a group. By induction we define the subgroups

$$C^0G = G, \ C^1G = [C^0G, G], \ldots, \ C^{\ell+1}G = [C^{\ell}, G], \ldots$$

of the descending central series of $G$. We say that $G$ is nilpotent if there exists $\ell \in \mathbb{N} \cup \{0\}$ such that $C^{\ell}G = \{1\}$. We define the nilpotence class of $G$ as the minimum $\ell \in \mathbb{N} \cup \{0\}$ with such a property.

Lemma 2.5. Let $G$ be a subgroup of $\hat{\text{Diff}}(\mathbb{C}^n, 0)$. We have

- $G$ is abelian if and only if $\overline{G}$ is abelian.
- $G$ is solvable if and only if $\overline{G}$ is solvable.
- $G$ is nilpotent if and only if $\overline{G}$ is nilpotent.

The two first properties were proved in [14, Lemma 1]. The proof of the last one is completely analogous. All these properties are a consequence of a simple principle: the derived length (resp. the nilpotence class) does not change when we take the Zariski-closure of a matrix group.

Definition 2.19. Let $G$ be a group and $P$ a group property. We say that $G$ is virtually $P$ if there exists a finite index subgroup $H$ of $G$ that satisfies $P$.

Remark 2.9. If the property $P$ is subgroup-closed (for instance solubility or nilpotence) then we can suppose that the group $H$ is a finite index normal subgroup of $G$ (cf. [22, 1.6.9, p. 36]).

Lemma 2.6. Let $G$ be a subgroup of $\hat{\text{Diff}}(\mathbb{C}^n, 0)$. Then the following properties are equivalent:

1. $G$ is virtually nilpotent (resp. solvable).
2. $\overline{G}$ is virtually nilpotent (resp. solvable).
3. $\overline{G_0}$ is nilpotent (resp. solvable).
Proof. Let us show the result in the virtually nilpotent case. The other case is analogous.

Let us show \((1) \implies (2)\). Let \(H\) be a finite index normal nilpotent subgroup of \(G\). Then \(\overline{H}\) is a finite index normal nilpotent subgroup of \(\overline{G}\) by Lemmas 2.4 and 2.5. Hence \(\overline{G}\) is virtually nilpotent.

Let us prove \((2) \implies (3)\). There exists a finite index normal nilpotent subgroup \(J\) of \(\overline{G}\). The group \(J\) contains \(\overline{G}_0\) by Lemma 2.1 and thus \(\overline{G}_0\) is nilpotent.

Let us see that \((3)\) implies \((1)\). Since \(\overline{G}_0\) is a finite index normal subgroup of \(\overline{G}\) by Remark 2.7, \(\overline{G}\) is virtually nilpotent. The group \(\overline{G}\) is a subgroup of \(\overline{G}\) and hence also virtually nilpotent. \(\square\)

2.8. Jordan decomposition of formal diffeomorphisms. Let us consider the multiplicative Jordan decomposition of formal diffeomorphisms in commuting semisimple (or equivalently diagonalizable) and unipotent parts. It was constructed by Martinet in [15]. The analogous decomposition for algebraic matrix groups is called Jordan-Chevalley decomposition since Chevalley showed

**Theorem 2.2** (Chevalley, cf. [4] section I.4.4, p. 83]. Let \(H\) be an algebraic matrix group. Then the semisimple and unipotent parts of the elements of \(H\) also belong to \(H\).

We will see that Chevalley’s theorem also guarantees the closedness of the Jordan decomposition for pro-algebraic subgroups of \(\widehat{\text{Diff}} (\mathbb{C}^n, 0)\).

**Definition 2.20.** We say that \(\phi \in \widehat{\text{Diff}} (\mathbb{C}^n, 0)\) is semisimple if \(\phi_k\) is semisimple for any \(k \in \mathbb{N}\).

**Remark 2.10.** By definition \(\phi \in \widehat{\text{Diff}} (\mathbb{C}^n, 0)\) is unipotent if and only if \(\phi_1\) is unipotent. It is not difficult to show that \(\phi\) is unipotent if and only if \(\phi_k\) is unipotent for any \(k \in \mathbb{N}\) (cf. [21 Proposition 3.12]).

**Remark 2.11.** It is well-known that \(\phi\) is semisimple if and only if \(\phi\) is formally conjugated to a linear diagonal map (cf. [20 Lemma 2.9] [21 Proposition 3.13]).

Given \(\phi \in \widehat{\text{Diff}} (\mathbb{C}^n, 0)\) we consider the multiplicative Jordan decomposition of \(\phi_k\) for \(k \in \mathbb{N}\). The semisimple and unipotent parts \(\phi_{k,s}\) and \(\phi_{k,u}\) of \(\phi_k\) belong to the algebraic group \(D_k\) by Chevalley’s Theorem 2.2. Moreover since \(\pi_{k,l}(\phi_{k,s})\) is semisimple, \(\pi_{k,l}(\phi_{k,u})\) is unipotent and

\[
\phi_l = \pi_{k,l}(\phi_k) = \pi_{k,l}(\phi_{k,s})\pi_{k,l}(\phi_{k,u}) = \pi_{k,l}(\phi_{k,u})\pi_{k,l}(\phi_{k,s}),
\]

we deduce \(\pi_{k,l}(\phi_{k,s}) = \phi_{l,s}\) and \(\pi_{k,l}(\phi_{k,u}) = \phi_{l,u}\) for any \(k \geq l \geq 1\) by uniqueness of the Jordan-Chevalley decomposition. Hence \((\phi_{k,s})_{k \in \mathbb{N}}\)
and \((\phi_{k,u})_{k \in \mathbb{N}}\) define elements \(\phi_s\) and \(\phi_u\) in \(\hat{\text{Diff}}(\mathbb{C}^n,0) = \lim \leftarrow D_k\) respectively. This leads to the next well-known result.

**Proposition 2.4.** Let \(\phi \in \hat{\text{Diff}}(\mathbb{C}^n,0)\). There exist unique elements \(\phi_s\) and \(\phi_u\) in \(\hat{\text{Diff}}(\mathbb{C}^n,0)\) such that
\[
\phi = \phi_s \circ \phi_u = \phi_u \circ \phi_s,
\]
\(\phi_s\) is semisimple and \(\phi_u\) is unipotent.

**Definition 2.21.** Let \(G\) be a subgroup of \(\hat{\text{Diff}}(\mathbb{C}^n,0)\). We say that \(G\) is splittable if \(\phi_s, \phi_u \in G\) for any \(\phi \in G\).

Chevalley’s Theorem 2.2 implies

**Proposition 2.5.** Let \(G\) be a subgroup of \(\hat{\text{Diff}}(\mathbb{C}^n,0)\). Then \(\overline{G}\) is splittable.

Analogously there exists an additive Jordan decomposition for formal vector fields.

**Proposition 2.6.** Let \(X \in \hat{\mathfrak{X}}(\mathbb{C}^n,0)\). There exist unique elements \(X_s\) and \(X_N\) in \(\hat{\mathfrak{X}}(\mathbb{C}^n,0)\) such that
\[
X = X_s + X_N \quad \text{and} \quad [X_s, X_N] = 0,
\]
\(X_s\) is semisimple (i.e. formally conjugated to a linear diagonal vector field) and \(X_N\) is nilpotent.

**Remark 2.12.** It is clear that if \(X = X_s + X_N\) is the additive Jordan decomposition of \(X \in \hat{\mathfrak{X}}(\mathbb{C}^n,0)\) then \(\exp(X) = \exp(X_s) \circ \exp(X_N)\) is the multiplicative Jordan decomposition of \(\exp(X)\).

**Remark 2.13.** Given a semisimple \(\phi \in \hat{\text{Diff}}(\mathbb{C}^n,0)\) it is easy to calculate \(\langle \phi \rangle\). Indeed \(\phi\) is of the form \(\phi(z_1, \ldots, z_n) = (\lambda_1 z_1, \ldots, \lambda_n z_n)\) in some formal system of coordinates. In such coordinates \(\langle \phi \rangle\) coincides with the Zariski-closure of the group \(\langle \text{diag}(\lambda_1, \ldots, \lambda_n) \rangle\) in \(\text{GL}(n, \mathbb{C})\). It can be described in terms of characters. We have
\[
\overline{\langle \phi \rangle} = \{\text{diag}(\mu_1, \ldots, \mu_n) : (\mu_1, \ldots, \mu_n) \in \bigcap_{a \in \mathbb{Z}^n} (\lambda_{a_1}, \ldots, \lambda_{a_n}) \in \ker(\chi_{\overline{a}}) \ker(\chi_{\overline{a}})\}
\]
where given \(\overline{a} = (a_1, \ldots, a_n) \in \mathbb{Z}^n\) we consider the character \(\chi_{\overline{a}} : (\mathbb{C}^*)^n \to \mathbb{C}^*\) defined by \(\chi_{\overline{a}}(\mu_1, \ldots, \mu_n) = \mu_{a_1}^1 \ldots \mu_{a_n}^n\).

Let us calculate the algebraic closure of a cyclic subgroup of \(\hat{\text{Diff}}(\mathbb{C}^n,0)\).

**Definition 2.22.** Given a complex manifold \(M\) we denote its dimension by \(\text{dim} M\). Given a complex vector space \(V\) we denote its dimension by \(\text{dim} V\).
Lemma 2.7. Let $\phi \in \hat{\text{Diff}}(\mathbb{C}^n, 0)$. Then $\langle \phi \rangle$ is an abelian group that is isomorphic to the product $\langle \phi_s \rangle \times \langle \phi_u \rangle$. Moreover $\langle \phi \rangle_k$ is isomorphic to the product $\langle \phi_s \rangle_k \times \langle \phi_u \rangle_k$ and $\dim \langle \phi \rangle_k = \dim \langle \phi_s \rangle_k + \dim \langle \phi_u \rangle_k$ for any $k \in \mathbb{N}$.

Proof. The group $\langle \phi \rangle$ is abelian by Lemma 2.5. By Proposition 2.5 the formal diffeomorphisms $\phi_s$ and $\phi_u$ belong to $\langle \phi \rangle$ and then $\langle \phi \rangle$ contains the group $H := \langle \langle \phi_s \rangle, \langle \phi_u \rangle \rangle$. We claim $\langle \phi \rangle = H$, it suffices to show that $H$ is pro-algebraic.

Remark 2.13 implies that $\langle \phi_s \rangle$ consists of semisimple elements and is closed in the Krull topology. Hence $\langle \phi_s \rangle_k$ is composed of semisimple elements for any $k \in \mathbb{N}$. Analogously $\langle \phi_u \rangle$ is contained in $\hat{\text{Diff}}_u(\mathbb{C}^n, 0)$, is closed in the Krull topology by Remark 2.8 and $\langle \phi_u \rangle_k$ consists of unipotent elements for any $k \in \mathbb{N}$.

The group $H_k^*$ is the image of the morphism

$$\langle \phi_s \rangle_k \times \langle \phi_u \rangle_k \rightarrow D_k$$

$$(\alpha, \beta) \mapsto \alpha \beta$$

of algebraic groups. Hence $H_k^*$ is an algebraic subgroup of $D_k$ for any $k \in \mathbb{N}$ (cf. [4, 2.1 (f), p. 57]). Since $\langle \langle \phi_s \rangle, \langle \phi_u \rangle \rangle$ is abelian, the uniqueness of the Jordan decomposition implies $i$ is injective. Thus $H_k$ is isomorphic to $\langle \phi_s \rangle_k \times \langle \phi_u \rangle_k$ and satisfies

$$\dim H_k = \dim \langle \phi_s \rangle_k + \dim \langle \phi_u \rangle_k$$

for any $k \in \mathbb{N}$. In order to conclude the proof it suffices to show that $H$ is closed in the Krull topology by Proposition 2.2.

Every element $\eta$ of $H$ is of the form $\psi \circ \rho$ where $\psi \in \langle \phi_s \rangle$ and $\rho \in \langle \phi_u \rangle$. Since $H$ is abelian, $\psi$ is semisimple and $\rho$ is unipotent, $\psi \circ \rho$ is the multiplicative Jordan-Chevalley decomposition of $\eta$. Moreover $H$ is isomorphic to $\langle \phi_s \rangle \times \langle \phi_u \rangle$ by uniqueness of the Jordan-Chevalley decomposition. Thus $\langle \phi_s \rangle$ (resp. $\langle \phi_u \rangle$) is the set of semisimple (resp. unipotent) elements of $H$. Given a sequence $(\eta_k)_{k \in \mathbb{N}}$ in $H$ that converges in the Krull topology, the sequences $(\eta_{k,s})_{k \in \mathbb{N}}$ and $(\eta_{k,u})_{k \in \mathbb{N}}$ are contained in $\langle \phi_s \rangle$ and $\langle \phi_u \rangle$ respectively and both converge in the Krull topology. Since $\langle \phi_s \rangle$ and $\langle \phi_u \rangle$ are closed in the Krull topology so is $H$. \hfill $\Box$

3. Finite dimensional groups of formal diffeomorphisms

Our main goal is characterizing the groups $G$ of local diffeomorphisms that can be embedded in finite dimensional Lie groups. We approach this problem from a canonical point of view. Indeed we provide an invariant $\dim G$ of $G$ such that $\dim G < \infty$ implies that the
Zariski-closure $\overline{G}$ of $G$ is algebraic or more precisely that the map $\pi_k : G \to G_k$ is an isomorphism of groups for some $k \in \mathbb{N}$. In such a case $\pi_k^{-1} : G_k \to \overline{G}$ can be interpreted as an algebraic morphism and $\overline{G}$ as a matrix algebraic group (in particular as a complex Lie group with finitely many connected components). On the other hand we will see that a Lie subgroup of $\widehat{\text{Diff}} \left( \mathbb{C}^n, 0 \right)$ with finitely many connected components (cf. Definition 3.5) is finite dimensional (Proposition 3.7 and Lemma 3.3).

There are other advantages of working with the Zariski-closure of a group of local diffeomorphisms. For instance given a normal subgroup $H$ of a group $G \subset \widehat{\text{Diff}} \left( \mathbb{C}^n, 0 \right)$ we can naturally define whether the extension is finite dimensional. A straightforward consequence of the definition is that $G$ is finite dimensional if and only if it is a tower of finite dimensional extensions of the trivial group. Hence it is natural to identify finite dimensional extensions. The following kind of extensions are finite dimensional:

1. $H$ is a finite index subgroup of $G$.
2. $G/H$ is a finitely generated abelian group.
3. $G/H$ is a connected Lie group.

These items are generalizations of the cases treated in [23, 3] in the context of extensions of groups. Thus a natural strategy to show $\dim G < \infty$ for a subgroup $G$ of $\widehat{\text{Diff}} \left( \mathbb{C}^n, 0 \right)$ is decomposing it as a tower of extensions of the types (1), (2) and (3) of the trivial group. This method allows to generalize Theorem 1.6 to much bigger classes of groups.

3.1. Dimensional setting. The first step of our program is defining the dimension of an extension of subgroups of $\widehat{\text{Diff}} \left( \mathbb{C}^n, 0 \right)$.

**Lemma 3.1.** Let $G$ be a subgroup of $\widehat{\text{Diff}} \left( \mathbb{C}^n, 0 \right)$. Consider a subgroup $H$ of $G$. Then $\dim G_k - \dim H_k \leq \dim G_{k+1} - \dim H_{k+1}$ for any $k \in \mathbb{N}$.

**Proof.** Let $\mathfrak{g}_k$ and $\mathfrak{h}_k$ be the Lie algebra of $G_k$ and $H_k$ respectively for $k \in \mathbb{N}$. Since $\pi_{k+1,1} : G_{k+1} \to G_k$ is surjective and we are working in characteristic 0, we obtain $(d\pi_{k+1,1})_{id} : \mathfrak{g}_{k+1} \to \mathfrak{g}_k$ is surjective for any $k \in \mathbb{N}$ (cf. [4, Chapter II.7, p. 105]). Moreover $(d\pi_{k+1,1})_{id}(\mathfrak{h}_{k+1})$ is equal to $\mathfrak{h}_k$ for $k \in \mathbb{N}$. Therefore the linear map $(d\pi_{k+1,1})_{id} : \mathfrak{g}_{k+1}/\mathfrak{h}_{k+1} \to \mathfrak{g}_k/\mathfrak{h}_k$ is surjective. Since

$$\dim \mathfrak{g}_k - \dim \mathfrak{h}_k \leq \dim \mathfrak{g}_{k+1} - \dim \mathfrak{h}_{k+1}$$

we deduce $\dim G_k - \dim H_k \leq \dim G_{k+1} - \dim H_{k+1}$ for any $k \in \mathbb{N}$.
Since \((\dim G_k - \dim H_k)_{k \geq 1}\) is increasing we can define the codimension of \(H\) in \(G\).

**Definition 3.1.** Consider a subgroup \(H\) of a subgroup \(G\) of \(\hat{\Diff}(\mathbb{C}^n, 0)\). We define \(\dim G/H \in \mathbb{Z}_{\geq 0} \cup \{\infty\}\) as

\[
\dim G/H = \lim_{k \to \infty} \dim G_k - \dim H_k.
\]

We say that \(G/H\) is **finite dimensional** or that \(H\) has finite codimension in \(G\) if \(\dim G/H < \infty\). Notice that we can define \(\dim G\) for any subgroup \(G\) of \(\hat{\Diff}(\mathbb{C}^n, 0)\) by considering \(H = \{Id\}\).

**Remark 3.1.** Notice that the definition does not distinguish between a subgroup of \(\hat{\Diff}(\mathbb{C}^n, 0)\) and its Zariski-closure. More precisely, if \(H\) is a subgroup of a group \(G \subset \hat{\Diff}(\mathbb{C}^n, 0)\), we have \(\dim G = \dim \mathcal{G}\) and \(\dim G/H = \dim \mathcal{G}/\mathcal{H}\).

The following result is an immediate consequence of the definition. It will be useful to know whether or not a subgroup of \(\hat{\Diff}(\mathbb{C}^n, 0)\) is finite dimensional in practical applications since it allows to divide the problem in simpler ones.

**Proposition 3.1.** Consider a sequence \(G^1 \subset G^2 \subset \ldots \subset G^m\) of subgroups of \(\hat{\Diff}(\mathbb{C}^n, 0)\). Then we obtain

\[
\dim G^m/G^1 = \dim G^m/G^{m-1} + \ldots + \dim G^3/G^2 + \dim G^2/G^1.
\]

In particular \(G^m/G^1\) is finite dimensional if and only if \(G^{j+1}/G^j\) is finite dimensional for any \(1 \leq j < m\).

Next proposition provides several characterizations of finite dimensional extensions.

**Proposition 3.2.** Let \(G\) be a subgroup of \(\hat{\Diff}(\mathbb{C}^n, 0)\). Let \(H\) be a subgroup of \(G\). The following properties are equivalent:

1. There exists \(k_0 \in \mathbb{N}\) such that \(\phi \in \mathcal{G}\) and \(\phi_{k_0} \in H_{k_0}\) imply \(\phi \in \mathcal{H}\).
2. There exists \(k_0 \in \mathbb{N}\) such that the map \(\tilde{\pi}_{k_0} : \mathcal{G}/\mathcal{H} \to G_{k_0}/H_{k_0}\), induced by \(\pi_{k_0} : \mathcal{G} \to G_{k_0}\), is injective.
3. There exists \(k_0 \in \mathbb{N}\) such that the map \(\tilde{\pi}_{k+1,k} : G_{k+1}/H_{k+1} \to G_k/H_k\) (induced by \(\pi_{k+1,k}\)) is injective for any \(k \geq k_0\).
4. \(G/H\) is finite dimensional.

**Remark 3.2.** Since we are not supposing that \(H\) is normal we consider left cosets. The proposition can be strengthened if \(H\) is normal. Then \(\mathcal{H}\) is normal in \(\mathcal{G}\) and \(H_k\) is normal in \(G_k\) for any \(k \in \mathbb{N}\) by Lemma ~\ref{lem:2.1}. Notice that \(G_k/H_k\) is an algebraic group for any \(k \in \mathbb{N}\) (cf. ~\cite{4}...
section II.6.8, p. 98). Hence \( \hat{\pi}_{k+1,k} \) is a morphism of algebraic groups. Since \( \hat{\pi}_k \) is always surjective for \( k \in \mathbb{N} \), condition (2) is equivalent to \( \hat{\pi}_{k,0} \) being an isomorphisms of groups from \( \overline{G/H} \) onto the algebraic matrix group \( G_{k_0}/H_{k_0} \). Condition (3) is equivalent to \( \hat{\pi}_{k+1,k} \) being a bijective morphism of algebraic matrix groups and then an isomorphism of algebraic groups (cf. [13, Theorem 6, Chapter 3.1.4]).

**Proof.** The first two properties are clearly equivalent.

Let us show (2) \( \implies \) (3). We have \( \hat{\pi}_{k,0} = \hat{\pi}_{k,k_0} \circ \hat{\pi}_{k+1,k} \circ \hat{\pi}_{k+1,0} \) for \( k \geq k_0 \). Since the maps \( \hat{\pi}_{k,k_0}, \hat{\pi}_{k+1,k}, \hat{\pi}_{k+1,0} \) are surjective and \( \hat{\pi}_k \) is injective, \( \hat{\pi}_{k,k_0}, \hat{\pi}_{k+1,k}, \hat{\pi}_{k+1,0} \) are also injective for any \( k \geq k_0 \).

Let us show (3) \( \implies \) (2). The map \( \hat{\pi}_{k,0} \) is equal to \( \hat{\pi}_{k,k_0} \circ \hat{\pi}_k \) for \( k \geq k_0 \). Since \( \hat{\pi}_{k,k_0} = \hat{\pi}_{k+1,k_0} \circ \ldots \circ \hat{\pi}_{k,k-1} \) is a composition of injective maps by hypothesis, \( \hat{\pi}_{k,k_0} \) is injective for \( k \geq k_0 \). Given left cosets \( \overline{\phi H} \) and \( \overline{\eta H} \) such that \( \hat{\pi}_{k,0}(\overline{\phi H}) = \hat{\pi}_{k,0}(\overline{\eta H}) \) we obtain \( \hat{\pi}_k(\overline{\phi H}) = \hat{\pi}_k(\overline{\eta H}) \) for any \( k \geq k_0 \). We deduce \( (\eta^{-1})\phi \in H_k \) for any \( k \geq k_0 \). In particular we have \( \eta^{-1}\phi \in H \) and hence \( \overline{\phi H} = \overline{\eta H} \). Thus \( \hat{\pi}_{k,0} \) is injective.

Let us show (3) \( \implies \) (4). The map \( \hat{\pi}_{k+1,k} \) is an isomorphism of algebraic manifolds for any \( k \geq k_0 \) by the universal mapping property of quotient morphisms (cf. [3, Chapter II.6]). We deduce

\[
\dim G_{k+1} - \dim H_{k+1} = \dim G_k - \dim H_k
\]

for any \( k \geq k_0 \).

Let us show (4) \( \implies \) (3). Let \( \mathfrak{g}_k \) and \( \mathfrak{h}_k \) be the Lie algebra of \( G_k \) and \( H_k \) respectively for \( k \in \mathbb{N} \). There exists \( k_0 \in \mathbb{N} \) such that

\[
\dim G_k - \dim H_k = \dim G_{k_0} - \dim H_{k_0}
\]

for any \( k \geq k_0 \). The linear map \( (d\pi_{k+1,k})_{Id} : \mathfrak{g}_{k+1} \to \mathfrak{g}_k \) is surjective for any \( k \in \mathbb{N} \) by the proof of Lemma 3.11. Since \( (d\pi_{k+1,k})_{Id}(\mathfrak{h}_{k+1}) = \mathfrak{h}_k \) for any \( k \in \mathbb{N} \) the linear map \( (d\pi_{k+1,k})_{Id} : \mathfrak{g}_{k+1} / \mathfrak{h}_{k+1} \to \mathfrak{g}_k / \mathfrak{h}_k \) is well-defined and surjective for any \( k \in \mathbb{N} \). Since both complex vector spaces \( \mathfrak{g}_{k+1} / \mathfrak{h}_{k+1} \) and \( \mathfrak{g}_k / \mathfrak{h}_k \) have the same dimension, the map \( (d\pi_{k+1,k})_{Id} \) is a linear isomorphism for any \( k \geq k_0 \).

Fix \( k \geq k_0 \). Let us show that \( \hat{\pi}_{k+1,k} \) is injective. Let \( A \in G_{k+1} \) such that \( \hat{\pi}_{k+1,k}(AH_{k+1}) = H_k \). We have \( \pi_{k+1,k}(A) \in H_k \). The restriction \( \pi_{k+1,k} : H_{k+1} \to H_k \) is surjective by Remark 2.5 hence there exists \( B \in H_{k+1} \) such that \( \pi_{k+1,k}(A) = \pi_{k+1,k}(B) \). We obtain \( \pi_{k+1,k}(B^{-1}A) = Id \). There exists \( \phi \in \overline{G} \) such that \( \phi_{k+1} = B^{-1}A \) since \( \pi_{k+1} : \overline{G} \to G_{k+1} \) is surjective by Remark 2.6. Since \( \phi_k \equiv Id \) the linear part \( D_0 \phi \) of \( \phi \) at 0 is equal to \( Id \) and thus \( \log \phi \) belongs to the Lie algebra \( \mathfrak{g} \) of \( \overline{G} \) (Remark 2.8) and satisfies \( (\log \phi)_k \equiv 0 \). The property \( (d\pi_{k+1,k})_{Id}((\log \phi)_{k+1}) = 0 \) and the injective nature of \( (d\pi_{k+1,k})_{Id} \)
imply \((\log \phi)_{k+1} \in \mathfrak{h}_{k+1}\). Since \(B^{-1}A = \exp((\log \phi)_{k+1})\) we obtain \(B^{-1}A \in H_{k+1}\) and then \(A \in H_{k+1}\). Hence \(\pi_{k+1,k}\) is injective for any \(k \geq k_0\). \(\square\)

**Remark 3.3.** Notice that the proof of (3) \(\iff\) (4) in Proposition 3.2 implies that \(\pi_{k+1,k} : G_{k+1}/H_{k+1} \to G_k/H_k\) is bijective if and only if \(\dim G_k - \dim H_k = \dim G_{k+1} - \dim H_{k+1}\). Such a property validates our point of view since the dimension determines an extension of the form \(G_k/H_k\) for \(k \in \mathbb{N}\) modulo isomorphism.

Moreover if (3) holds then \(\dim G/H = \dim G_{k_0} - \dim H_{k_0}\). We have \(\dim G/H = \dim G_{k_0} - \dim H_{k_0}\) if (2) holds by the proof of (2) \(\implies\) (3).

**Remark 3.4.** Let \(G\) be a finite dimensional subgroup of \(\widehat{\text{Diff}}(\mathbb{C}^n, 0)\). There exists \(k_0 \in \mathbb{N}\) such that \(\pi_{k,k_0} : G_k \to G_{k_0}\) is an isomorphism of algebraic matrix groups for any \(k \geq k_0\). Consider the Taylor series expansion

\[
\phi(z_1, \ldots, z_n) = (\sum_{i_1 + \ldots + i_n \geq 1} a^1_{i_1 \ldots i_n} z_1^{i_1} \ldots z_n^{i_n}, \ldots, \sum_{i_1 + \ldots + i_n \geq 1} a^n_{i_1 \ldots i_n} z_1^{i_1} \ldots z_n^{i_n})
\]

of \(\phi \in G\). Given \((i_1, \ldots, i_n; j)\) a multi-index such that \(i_1 + \ldots + i_n > k_0\) and \(1 \leq j \leq n\) the function \(a^j_{i_1 \ldots i_n} : G \to \mathbb{C}\) belongs to the affine coordinate ring \(\mathbb{C}[G_{k_0}]\) of \(G_{k_0}\). In other words every coefficient in the Taylor expansion, of an element of \(G\), of degree greater than \(k_0\) is a regular function \(P^j_{i_1 \ldots i_n}\) on the coefficients of degree less or equal than \(k_0\).

Reciprocally suppose that \(a^j_{i_1 \ldots i_n} : G \to \mathbb{C}\) is a polynomial function of the coefficients of degree less or equal than \(k_0\) for any multi-index \((i_1, \ldots, i_n; j)\) such that \(i_1 + \ldots + i_n > k_0\) and \(1 \leq j \leq n\) (meaning that there exists \(P^j_{i_1 \ldots i_n} \in \mathbb{C}[D_{k_0}]\) such that

\[
a^j_{i_1 \ldots i_n}(\phi) = P^j_{i_1 \ldots i_n}((a^m_{i_1 \ldots i_n}(\phi))_{i_1 + \ldots + i_n \leq k_0, 1 \leq m \leq n})
\]

for any \(\phi \in G\). Since all these equations hold true in the Zariski-closure \(G_{i_1 + \ldots + i_n}\) of \(G^*_{i_1 + \ldots + i_n}\), we have that \(\pi_{k,k_0} : \mathbb{C}[G_{k_0}] \to \mathbb{C}[G_k]\) is an isomorphism of \(\mathbb{C}\)-algebras and in particular \(\pi_{k,k_0} : G_k \to G_{k_0}\) is an isomorphism of algebraic groups for any \(k \geq k_0\). Thus \(G\) is finite dimensional.

Let us relate the finite dimension property with a much simpler one, namely the finite determination property.

**Definition 3.2.** Let \(G\) be a subgroup of \(\widehat{\text{Diff}}(\mathbb{C}^n, 0)\). We say that \(G\) has the *finite determination property* if there exists \(k \in \mathbb{N}\) such that \(\phi \in G\) and \(\phi_k \equiv Id\) imply \(\phi \equiv Id\).
Remark 3.5. Let us compare the finite determination and the finite
dimension properties. On the one hand a subgroup $G$ of $\hat{\text{Diff}}(\mathbb{C}^n, 0)$ has the finite determination property if there exists $k \in \mathbb{N}$ such that the projection $\pi_k : G \rightarrow D_k$ is injective. On the other hand $G$ is finite dimensional if there exists $k \in \mathbb{N}$ such that $\pi_k : \overline{G} \rightarrow D_k$ is injective.

Remark 3.6. Notice that a subgroup $G$ of $\hat{\text{Diff}}(\mathbb{C}^n, 0)$ is finite dimensional if and only if $G$ has the finite determination property.

Remark 3.7. Every finite dimensional group has finite determination but in general the reciprocal does not hold true. We define

$$\phi(j)(x, y) = (x, y + d_j x^2 + x^{j+2}) \in \text{Diff}(\mathbb{C}^2, 0)$$

for $j \in \mathbb{N}$. Suppose that the subset $S := \{d_1, d_2, \ldots \}$ of $\mathbb{C}$ is linearly independent over $\mathbb{Q}$. We have $\log \phi(j) = (d_j x^2 + x^{j+2}) \partial/\partial y$ for $j \in \mathbb{N}$. We denote by $G$ the group generated by $\{\phi(1), \phi(2), \ldots \}$. It is an abelian group. Moreover since $S$ is linearly independent over $\mathbb{Q}$, the property $\phi \neq Id$ implies $\phi_2 \neq Id$ for any $\phi \in G$. In particular $G$ has the finite determination property.

By choice the complex Lie algebra generated by $\{\log \phi(1), \log \phi(2), \ldots \}$ is infinite dimensional as a complex vector space. This implies that $\overline{G}$ contains non-trivial elements whose order of contact with the identity is arbitrarily high or in other words that the map $\pi_k : \overline{G} \rightarrow D_k$ is not injective for any $k \in \mathbb{N}$. Hence $\overline{G}$ does not have the finite determination property and $G$ is not finite dimensional.

An example of finite determination group that is not finite dimensional does not exist in dimension 1 (Proposition 4.1).

Remark 3.8. We define

$$\phi(j)(x, y) = (x, y + d_j x^2 + x^{j+2}, z) \in \text{Diff}(\mathbb{C}^3, 0)$$

for $j \in \mathbb{N}$ where the subset $S := \{d_1, d_2, \ldots \}$ of $\mathbb{C}$ is linearly independent over $\mathbb{Q}$. Let $G$ be the group generated by $\{\phi(1), \phi(2), \ldots \}$. Analogously as in the previous example $G$ is finitely determined but it is not finite dimensional. We have

$$\dim_{\mathbb{C}} O_3 \{x = y = 0, \ y = 0 \} = \dim \frac{O_3}{(x, y + d_j x^2 + x^{j+2}, y)} = j + 2$$

for any $j \in \mathbb{N}$. As a consequence finite determination does not suffice to guarantee the uniform intersection property (cf. Theorem 1.5).

Definition 3.3. We say that a subgroup $G$ of $\hat{\text{Diff}}(\mathbb{C}^n, 0)$ is algebraic if $G$ is pro-algebraic and $\dim G < \infty$. 

Remark 3.9. An algebraic subgroup $G$ of $\hat{\text{Diff}}\left(\mathbb{C}^n, 0\right)$ is the image by an algebraic monomorphism of an algebraic matrix group. Given $k_0 \in \mathbb{N}$ such that $\pi_{k_0} : G \to G_{k_0}$ is injective, the map $\pi_{k_0}^{-1} : G_{k_0} \to G$ is an isomorphism of groups (Remark 3.2). Moreover, it is algebraic in every jet space since $\pi_k \circ \pi_{k_0}^{-1} : G_{k_0} \to G_k$ is the inverse of the algebraic isomorphism $\pi_{k,k_0} : G_k \to G_{k_0}$ for any $k \geq k_0$ (Remark 3.2).

The characterization of pro-algebraic groups given by Proposition 2.2 provides a characterization of algebraic subgroups of $\hat{\text{Diff}}\left(\mathbb{C}^n, 0\right)$.

**Lemma 3.2.** Let $G$ be a subgroup of $\hat{\text{Diff}}\left(\mathbb{C}^n, 0\right)$. Then $G$ is algebraic if and only if $G_k$ is algebraic for any $k \in \mathbb{N}$ and the sequence $(\text{dim} G_k)_{k \geq 1}$ is bounded.

**Proof.** The group $G$ is pro-algebraic if and only if $G_k$ is algebraic for any $k \in \mathbb{N}$ and $G$ is closed in the Krull topology by Proposition 2.2. The sufficient condition is obvious. Let us show the necessary condition. It suffices to show that $G$ is closed in the Krull topology. There exists $k_0 \in \mathbb{N}$ such that $\pi_{k+1,k} : G_{k+1} \to G_k$ is injective for any $k \geq k_0$ by Remark 3.3. We deduce that the map $\pi_{k_0} : G \to G_{k_0}$ is injective. Consider a sequence $(\eta_m)_{m \geq 1}$ of elements of $G$ that converge in the Krull topology. Then there exists $m_0 \in \mathbb{N}$ such that $(\eta_m)_{k_0} \equiv (\eta_{m_0})_{k_0}$ if $m \geq m_0$. Therefore $\eta_m \equiv \eta_{m_0}$ for any $m \geq m_0$ and the sequence converges to $\eta_{m_0} \in G$. We obtain that $G$ is closed in the Krull topology. \hfill \Box

Let us provide the first examples of finite dimensional groups. Indeed we will see that cyclic groups and one-parameter groups are always finite dimensional.

**Proposition 3.3.** Let $\phi \in \hat{\text{Diff}}\left(\mathbb{C}^n, 0\right)$. We have $\dim \langle \phi \rangle \leq n$.

**Proof.** We have

$$\dim \langle \phi \rangle_k = \dim \langle \phi_s \rangle_k + \dim \langle \phi_u \rangle_k$$

for any $k \in \mathbb{N}$ by Lemma 2.7. It suffices to show that $\langle \phi_s \rangle$ and $\langle \phi_u \rangle$ are finite dimensional.

Since $\phi_s$ is semisimple, we can suppose up to a formal change of coordinates that $\langle \phi_s \rangle$ is contained in the group of diagonal matrices (Remark 2.13). We deduce $\dim \langle \phi_s \rangle \leq n$.

Since $\phi_u$ is unipotent, we obtain $\langle \phi_u \rangle = \{ \phi_t^k : t \in \mathbb{C} \}$ and in particular $\langle \phi_u \rangle_k = \{ \exp(t \log \phi_{u,k}) : t \in \mathbb{C} \}$ for any $k \in \mathbb{N}$ by Remark 2.8. We deduce $\dim \langle \phi_u \rangle = 1$ if $\phi_u \neq \text{Id}$ and $\dim \langle \text{Id} \rangle = 0$. We get

$$\dim \langle \phi \rangle = \dim \langle \phi_s \rangle + \dim \langle \phi_u \rangle \leq n + 1.$$
In order to show $\dim(\phi) \leq n$ let us prove that $\dim(\phi_u) = n$ implies $\phi_u \equiv Id$. Indeed in such a case $\langle \phi_u \rangle$ is the linear group of diagonal transformations. Every element of such group commutes with $\phi_u$ by Lemma 2.7 and hence $\phi_u$ is linear and diagonal. Since $\phi_u$ is both semisimple and unipotent, it is equal to the identity map. □

The finite dimension of one-parameter groups can be obtained by reduction to the cyclic case. More precisely, we will use that every one-parameter group $G$ of formal diffeomorphisms has cyclic subgroups whose Zariski-closure coincides with $\overline{G}$.

**Proposition 3.4.** Let $X \in \hat{\mathfrak{X}}(\mathbb{C}^n, 0)$. Then there exists $t_0 \in \mathbb{R}$ such that $\{\exp(tX) : t \in \mathbb{C}\} \subset \overline{\langle \exp(t_0X) \rangle}$. In particular we obtain

$$\dim\{\exp(tX) : t \in \mathbb{C}\} \leq n.$$ 

**Proof.** Consider the Jordan decomposition $X = X_s + X_N$ as a sum of commuting formal vector fields such that $X_s$ is formally diagonalizable and $X_N$ is nilpotent. We can suppose $X_s = \sum_{k=1}^n \mu_k \partial/\partial z_k$ where $\mu_1, \ldots, \mu_n \in \mathbb{C}$ up to a formal change of coordinates. We denote

$$D = \{a \in \mathbb{Z}^n : \sum_{k=1}^n a_k \mu_k \neq 0\}$$

where $a = (a_1, \ldots, a_n)$. Given $a \in D$ we define

$$C_a = \{t \in \mathbb{C} : t \sum_{k=1}^n a_k \mu_k \in 2\pi i \mathbb{Z}\};$$

it is a countable set. Consider an element $t_0$ in the complementary of the countable set $\bigcup_{a \in D} C_a$ in $\mathbb{R}^*$. We define

$$\eta(z_1, \ldots, z_n) = \exp(t_0X_s) = (e^{t_0\mu_1}z_1, \ldots, e^{t_0\mu_n}z_n) \in \hat{\text{Diff}}(\mathbb{C}^n, 0).$$

The group $\mathcal{C}$ of characters $\chi_{\underline{a}}(w_1, \ldots, w_n) = w_1^{a_1} \ldots w_n^{a_n}$ with $\underline{a} \in \mathbb{Z}^n$ defined by

$$\mathcal{C} = \{\chi_{\underline{a}} : (e^{t_0\mu_1}, \ldots, e^{t_0\mu_n}) \in \ker(\chi_{\underline{a}})\}$$

satisfies $\mathcal{C} = \{\chi_{\underline{a}} : \sum_{k=1}^n a_k \mu_k = 0\}$ by our choice of $t_0$. The group $\langle \eta \rangle$ consists of the linear diagonal maps $\text{diag}(\lambda_1, \ldots, \lambda_n)$ such that $\chi_{\underline{a}}(\lambda_1, \ldots, \lambda_n) = 1$ for any $\underline{a} \in \mathcal{C}$. Since $(e^{t\mu_1}, \ldots, e^{t\mu_n}) \in \ker(\chi_{\underline{a}})$ for all $\chi_{\underline{a}} \in \mathcal{C}$ and $t \in \mathbb{C}$, the one parameter group $\{\exp(tX_s) : t \in \mathbb{C}\}$ is contained in $\langle \eta \rangle$. We denote $\rho = \exp(t_0X_N)$, it satisfies

$$\langle \rho \rangle = \{\exp(tX_N) : t \in \mathbb{C}\}$$

by Remark 2.8. We denote $\phi = \exp(t_0X) = \eta \circ \rho$. Since $\langle \phi \rangle$ contains

$$\langle \phi_s \rangle \cup \langle \phi_u \rangle = \langle \eta \rangle \cup \langle \rho \rangle,$$
it also contains \( \{ \exp(tX) : t \in \mathbb{C} \} \). Hence \( \dim \{ \exp(tX) : t \in \mathbb{C} \} \leq n \) is a consequence of Proposition 3.3.

The finite dimensional nature of a subgroup of \( \widehat{\text{Diff}}(\mathbb{C}^n,0) \) is related to properties of finite decomposition of the elements of the group in terms of generators. The following results illustrates how a finite writing property allows to decide whether or not \( \dim G < \infty \) by solving simpler problems.

**Proposition 3.5.** Let \( H_1, \ldots, H_m \) and \( G \) be subgroups of \( \widehat{\text{Diff}}(\mathbb{C}^n,0) \). Suppose \( G \subset H_1 \cdots H_m \). Then we have

\[
\dim G \leq \sum_{1 \leq j \leq m} \dim H_j.
\]

Moreover given \( 1 \leq j \leq m \) such that \( H_j \subset G \) we obtain

\[
\dim G/H_j \leq \sum_{k \neq j} \dim H_k.
\]

We denote \( H_1 \cdots H_m = \{ h_1 \circ \ldots \circ h_m : h_j \in H_j \ \forall 1 \leq j \leq m \} \).

**Proof.** Fix \( k \in \mathbb{N} \). Consider the map

\[
\tau_k : H_{m,k} \times H_{m-1,k} \times \cdots \times H_{1,k} \to D_k
\]

\[
(B_m, B_{m-1}, \ldots, B_1) \mapsto B_mB_{m-1}\cdots B_1.
\]

The map \( \tau_k \) is algebraic even if it is not in general a morphism of groups. As a consequence \( \text{Im}(\tau_k) \) is a constructible set whose dimension is less or equal than \( \sum_{1 \leq j \leq m} \dim H_{j,k} \). The Zariski-closure of \( \text{Im}(\tau_k) \) is an algebraic set containing \( G_k \) and then \( G_k \). We deduce \( \dim G_k \leq \sum_{1 \leq j \leq m} \dim H_{j,k} \) for any \( k \in \mathbb{N} \). The results are a direct consequence of Definition 3.1.

**Theorem 3.1.** Let \( G \) be a subgroup of \( \widehat{\text{Diff}}(\mathbb{C}^n,0) \). Suppose there exist \( \psi_1, \ldots, \psi_l \in \widehat{\text{Diff}}(\mathbb{C}^n,0) \), \( X_1, \ldots, X_m \in \widehat{X}(\mathbb{C}^n,0) \) and \( p \in \mathbb{N} \) such that every \( \phi \in G \) is of the form \( \phi = \phi_1 \circ \cdots \circ \phi_q \) where \( q \leq p \) and \( \phi_r \in \cup_{j=1}^l \{ \psi_j \} \cup \cup_{k=1}^m \{ \exp(tX_j) : t \in \mathbb{C} \} \) for any \( 1 \leq r \leq q \). Then \( G \) is finite dimensional.

**Proof.** The result is a straightforward consequence of Propositions 3.5, 3.3 and 3.4.

**Remark 3.10.** The elements of a connected Lie group admit a decomposition in words of uniform length whose letters are taken from the elements of a finite set of cyclic and one-parameter subgroups. This can be seen as a consequence of the Iwasawa-Malcev decomposition of
a connected Lie group (cf. [10, Theorem 6] [13]). Such a property generalizes immediately to Lie groups with finitely many connected components and then to algebraic matrix groups. Since every algebraic group of formal diffeomorphisms is isomorphic to an algebraic matrix group, finite dimension can be interpreted as a finite decomposition property.

3.2. Extensions of groups of formal diffeomorphisms. In this section we study extensions that are always finite dimensional. First, we deal with finite extensions.

**Lemma 3.3.** Let $G$ be a subgroup of $\hat{\text{Diff}}(\mathbb{C}^n,0)$. Consider a finite index subgroup $H$ of $G$. Then we obtain $\dim G/H = 0$.

**Proof.** There exists a subgroup $J$ of $H$ such that $J$ is a finite index normal subgroup of $G$. Since $J_k$ is a finite index normal subgroup of $G_k$ by Lemma 2.2, we obtain $\dim J_k = \dim G_k$ for any $k \in \mathbb{N}$. We deduce $\dim G/J = 0$ and then $\dim G/H = 0$ since $J \subset H$. □

**Corollary 3.1.** Let $G$ be a subgroup of $\hat{\text{Diff}}(\mathbb{C}^n,0)$. Consider subgroups $H,K$ of $G$ such that $H \subset K$ and $K$ is a a finite index subgroup of $G$. Then $\dim G/H = \dim K/H$.

**Proof.** We have $\dim G/H = \dim G/K + \dim K/H = \dim K/H$ by Lemma 3.3. □

Next, we will consider finitely generated abelian extensions. First let us discuss the finite generation hypothesis. A positive dimensional connected Lie group is not finitely generated since it is not countable. Anyway, it is finitely generated by a finite number of one-parameter groups whose infinitesimal generators are the elements of a basis of the Lie algebra. This idea inspires an alternative definition of finitely generated subgroup of $\hat{\text{Diff}}(\mathbb{C}^n,0)$ in which generators can be elements of the group or one-parameter flows.

**Definition 3.4.** Let $H$ be a subgroup of a subgroup $G$ of $\hat{\text{Diff}}(\mathbb{C}^n,0)$. We say that $G$ is **finitely generated in the extended sense** over $H$ if there exist elements $\phi_1, \ldots, \phi_l \in G$ and formal vector fields $X_1, \ldots, X_m \in \hat{\mathfrak{X}}(\mathbb{C}^n,0)$ such that

$$G = \langle H, \phi_1, \ldots, \phi_l, \bigcup_{j=1}^m \{ \exp(tX_j) : t \in \mathbb{R} \} \rangle.$$

We say that $G/H$ is finitely generated in the extended sense if $H$ is normal in $G$. If $G$ is of the form $\langle H, \phi_1, \ldots, \phi_l \rangle$ we say that $G$ is finitely generated over $H$.

We are interested in calculating the dimension of subgroups of $\hat{\text{Diff}}(\mathbb{C}^n,0)$. In this context the new definition of finitely generated group can be reduced to the usual one via the next lemma.
Lemma 3.4. Let $H$ be a subgroup of a subgroup $G$ of $\hat{\text{Diff}}(\mathbb{C}^n,0)$. Suppose that $G$ is finitely generated over $H$ in the extended sense. Then there exists a subgroup $G_+$ of $\hat{\text{Diff}}(\mathbb{C}^n,0)$ such that $H \subset G_+ \subset G$, $\overline{G_+} = \overline{G}$ and $G_+$ is finitely generated over $H$.

Proof. Suppose $G = \langle H, \phi_1, \ldots, \phi_l, \cup_{j=1}^m \{ \exp(tX_j) : t \in \mathbb{R} \} \rangle$. Given $1 \leq j \leq m$ there exists $t_j \in \mathbb{R}$ such that $\psi_j := \exp(t_jX_j)$ satisfies $\{ \exp(tX_j) : t \in \mathbb{C} \} \subset \langle \psi_j \rangle$ by Proposition 3.4. We define $G_+ = \langle H, \phi_1, \ldots, \phi_l, \psi_1, \ldots, \psi_m \rangle$.

It is clear that $H \subset G_+ \subset G$. The choice of $\psi_j$ for $1 \leq j \leq m$ implies $G \subset G_+$. Since $G_+ \subset G$, we obtain $\overline{G} = \overline{G_+}$. □

Proposition 3.6. Let $H$ be a normal subgroup of a subgroup $G$ of $\hat{\text{Diff}}(\mathbb{C}^n,0)$. Suppose $G/H$ is abelian and $G/H$ is finitely generated in the extended sense. Then $G/H$ is finite dimensional.

Proof. We have $G = \langle H, \phi_1, \ldots, \phi_l, \cup_{j=1}^m \{ \exp(tX_j) : t \in \mathbb{R} \} \rangle$. We denote $H_j = \langle \phi_j \rangle$ for $1 \leq j \leq l$ and $J_k = \{ \exp(tX_k) : t \in \mathbb{R} \}$ for $1 \leq k \leq m$. Since $H$ is normal in $G$ and $G/H$ is abelian, we obtain $G = H_1 \ldots H_l J_1 \ldots J_m H$. This implies

$$\dim G/H \leq \sum_{1 \leq j \leq l} \dim H_j + \sum_{1 \leq k \leq m} \dim J_k \leq (l + m)n$$

by Propositions 3.5, 3.3 and 3.4. □

Remark 3.11. Proposition 3.6 implies in particular that Theorem 1.5 can be applied to a finitely generated (in the extended sense) abelian subgroup of $\hat{\text{Diff}}(\mathbb{C}^n,0)$. So Seigal-Yakovenko’s Theorem 1.3 can be understood as a consequence of the finite dimension of finitely generated abelian subgroups of formal diffeomorphisms.

Next, let us consider the third type of extensions, namely extensions that are real connected Lie groups. Of course the first task is finding a proper definition of Lie group for an extension since it is not clear a priori.

Definition 3.5. Let $H$ be a normal subgroup of a subgroup $G$ of $\hat{\text{Diff}}(\mathbb{C}^n,0)$. We say that $G/H$ is a (connected) Lie group if there exists a (connected) Lie group $L$ and a surjective morphism of groups $\sigma : L \to G/H$ such that the map $\sigma_k : L \to D_k/H_k$ induced by $\sigma$ is a morphism of differentiable manifolds for any $k \in \mathbb{N}$.

Notice that $D_k/H_k$ is a smooth algebraic manifold. The map $\pi_k : G \to D_k$ induces a map $\pi_k' : G/H \to D_k/H_k$. The map $\sigma_k$ is equal to $\pi_k' \circ \sigma$. 

Remark 3.12. The previous definition of Lie group coincides with the usual one for a subgroup $G$ of $\widehat{\text{Diff}}(\mathbb{C}^n,0)$, i.e. in the case $H = \{\text{Id}\}$ (cf. [3]).

Let us see that connected Lie group extensions are finite dimensional (Proposition 3.8). Proposition 3.7 is a corollary of Proposition 3.8.

**Proposition 3.7.** Let $G$ be a subgroup of $\widehat{\text{Diff}}(\mathbb{C}^n,0)$. Suppose that $G$ is a connected Lie group. Then $G$ is finite dimensional.

**Proposition 3.8.** Let $H$ be a normal subgroup of a subgroup $G$ of $\widehat{\text{Diff}}(\mathbb{C}^n,0)$. Suppose that $G/H$ is a connected Lie group. Then $G/H$ is finite dimensional.

**Proof.** Let us explain the idea of the proof. Suppose $H = \{\text{Id}\}$ and $L$ (cf. Definition 3.5) is a connected complex Lie subgroup of $\text{GL}(n,\mathbb{C})$. Then the derived group $L'$ is algebraic (cf. [18, Chapter 3.3.3]). We can think of $L$ as an algebraic-by-finitely generated commutative group since $L/L'$ is abelian and finitely generated in the extended sense. Since all these extensions are finite dimensional in a natural way, hence the image of $L$ is finite dimensional.

Consider the real Lie group $L$ and the surjective morphism of groups $\sigma : L \to G/H$ provided by Definition 3.5. Fix $k \in \mathbb{N}$. Let $\mathfrak{g}_k$ and $\mathfrak{h}_k$ be the Lie algebras of $G_k$ and $H_k$. The set $G_k/H_k$ is an algebraic group for any $k \in \mathbb{N}$ (cf. [41, section II.6.8, p. 98]). Let $\mathfrak{g}$ be the Lie algebra of $L$. Consider the map $(d\sigma_k)_l : \mathfrak{g} \to \mathfrak{g}_k/\mathfrak{h}_k$ induced by $\sigma_k$ for $k \in \mathbb{N}$. We define by $\tilde{\mathfrak{g}} = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$ the complexified of the Lie algebra $\mathfrak{g}$. We denote by $(d\tilde{\sigma}_k)_l : \tilde{\mathfrak{g}} \to \mathfrak{g}_k/\mathfrak{h}_k$ the morphism of complex Lie algebras induced by $(d\sigma_k)_l$. Let $\tilde{L}$ be a connected simply connected complex Lie group whose Lie algebra is equal to $\tilde{\mathfrak{g}}$. Then there exists a unique morphism $\tilde{\sigma}_k : \tilde{L} \to G_k/H_k$ of complex Lie groups such that $(d\tilde{\sigma}_k)_l = (d\sigma_k)_l$ for any $k \in \mathbb{N}$ (cf. [18, Chapter 1.2.8]). Notice that $\tilde{\sigma}_k(\tilde{L})$ contains $\sigma_k(L)$.

We denote by $\rho_k : G_k \to G_k/H_k$ the morphism of algebraic groups given by the projection. Since $\tilde{\sigma}_k(\tilde{L})'$ is the derived group of the connected complex Lie group of matrices $\tilde{\sigma}_k(\tilde{L})$, it is an algebraic subgroup of $G_k/H_k$. Hence $\rho_k^{-1}(\tilde{\sigma}_k(\tilde{L})')$ is an algebraic subgroup of $G_k$ such that

$$\dim \rho_k^{-1}(\tilde{\sigma}_k(\tilde{L})') - \dim H_k \leq \dim \tilde{L}' \leq \dim \tilde{L} \leq \dim_{\mathbb{R}} L$$

where $\dim_{\mathbb{R}} L$ is the dimension of the real Lie group $L$. Since $\langle G', H \rangle_k^*$ is contained in $\rho_k^{-1}(\tilde{\sigma}_k(\tilde{L}''))$ and the latter group is algebraic, we obtain $\langle G', H \rangle_k \subset \rho_k^{-1}(\tilde{\sigma}_k(\tilde{L}''))$ and then

$$\dim \langle G', H \rangle_k - \dim H_k \leq \dim_{\mathbb{R}} L$$
for any $k \in \mathbb{N}$. Therefore the extension $\langle G', H \rangle / H$ is finite dimensional. It suffices to show that $G / \langle G', H \rangle$ is finite dimensional by Proposition 3.1.

Let $X_1, \ldots, X_m$ be a basis of $\mathfrak{g}$ (as a real Lie algebra). Fix $k \in \mathbb{N}$. Analogously as in the proof of Proposition 3.4 there exists a countable subset $A_k$ of $\mathbb{R}$ such that the algebraic closure of $\langle \sigma_k(\exp(tX_1)) \rangle$ in $G_k/H_k$ contains the one-parameter group $\sigma_k\{\exp(sX_1) : s \in \mathbb{C}\}$ for any $t \in \mathbb{R}^* \setminus A_k$. We choose $t \in \mathbb{R}^* \setminus \bigcup_{k \geq 1} A_k$ and a representative $\psi_1 \in G$ of the class in $G/H$ defined by $\sigma(\exp(tX_1))$. Analogously we define $\psi_2, \ldots, \psi_m$. It is clear that the extension $\langle G', H, \psi_1, \ldots, \psi_m \rangle / \langle G', H \rangle$ is abelian and finitely generated. It suffices to show $\langle G', H, \psi_1, \ldots, \psi_m \rangle = \mathcal{C}$ by Proposition 3.6. We will prove $\langle H, \psi_1, \ldots, \psi_m \rangle = \mathcal{C}$.

Fix $k \in \mathbb{N}$. We denote $J = \langle H, \psi_1, \ldots, \psi_m \rangle$. The image of the Zariski-closure $J_k$ of $J_k^*$ by the morphism $\rho_k$ is equal to the closure of $\langle J_k^*, H_k \rangle / H_k$ in $G_k/H_k$. The Zariski-closure of $\langle \rho_k(\psi_j) \rangle$ contains $\sigma_k\{\exp(sX_j) : s \in \mathbb{C}\}$ for any $1 \leq j \leq m$ by the choice of $\psi_j$. Hence the closure of $\langle J_k^*, H_k \rangle / H_k$ in $G_k/H_k$ contains $\langle (G_k^*, H_k) / H_k \rangle$. We deduce that the closure of $\rho_k(J_k^*)$ in $G_k/H_k$ is equal to $G_k/H_k$. Thus $\rho_k(J_k)$ is equal to $G_k/H_k$. Since $J_k$ contains $H_k$, we get $J_k = G_k$ for any $k \in \mathbb{N}$. Hence we obtain $\langle H, \psi_1, \ldots, \psi_m \rangle = \mathcal{C}$. □

Remark 3.13. We can recover Binyamini’s Theorem 1.4 in the context of the theory of finite dimensional groups of formal diffeomorphisms by applying Theorem 1.3, Proposition 3.7 and Lemma 3.3.

Binyamini proves that the matrix coefficients of $G$ belong to a noetherian ring by using the Iwasawa-Malcev decomposition of a connected Lie group (cf. [10 Theorem 6]). Let us remark that it is possible to use such decomposition to show Proposition 3.7 (cf. Remark 3.10). Our choice of proof is intended to stress the efficacy of the approach through towers of extensions. Indeed given a connected Lie group $G \subset \text{Diff}(\mathbb{C}^n, 0)$, we showed that its derived group $G'$ is finite dimensional by using classical results of Lie group theory. This allowed us to reduce the problem to treat the finitely generated (in the extended sense) and abelian extension $G/G'$.

Remark 3.14. In the proof of Proposition 3.8 it suffices to consider a linear independent subset $\{X_1, \ldots, X_m\}$ of $\mathfrak{g}$ whose image in $\mathfrak{g}/\mathfrak{g}'$ is a basis. As a consequence we obtain

$$\dim \sigma(L) \leq \dim \tilde{L}' + (\dim \tilde{L} - \dim \tilde{L}')n = \dim \tilde{L} + (\dim \tilde{L} - \dim \tilde{L}')(n-1).$$

This formula reminds the formula in Theorem 4 of [3] in which the role of a maximal connected compact subgroup of $L$ is replaced with the derived group.
Remark 3.15. Subgroups of Lie groups (with finitely many connected components) of formal diffeomorphisms are always subgroups of algebraic groups of formal diffeomorphisms by Proposition 3.7 and Lemma 3.3 Given a subgroup $G$ of $\hat{\text{Diff}}(\mathbb{C}^n,0)$ the existence of an embedding of $G$ in a Lie group of formal diffeomorphisms implies that $G$ is contained in the image by an algebraic monomorphism of an algebraic matrix group (Remark 3.9).

The following theorem about finite determination properties of Lie subgroups of $\hat{\text{Diff}}(\mathbb{C}^n,0)$ is due to Baouendi et al.

**Theorem 3.2** ([2, Proposition 5.1]). Let $G$ be a subgroup of $\hat{\text{Diff}}(\mathbb{C}^n,0)$. Suppose that $G$ is a Lie group with a finite number of connected components. Then $G$ has the finite determination property.

We include a proof since it is an extremely easy application of our techniques.

**Proof.** The group $G$ is finite dimensional by Proposition 3.7 and Lemma 3.3. Hence it has the finite determination property. □

In general the implication finite determination $\implies$ finite dimension does not hold for subgroups $G$ of $\hat{\text{Diff}}(\mathbb{C}^n,0)$ (cf. Remark 3.7). Anyway, it is interesting to explore in which conditions it is true since the former property is much easier to verify. Next, we see that the implication is satisfied, even for extensions, under a property of algebraic closedness for cyclic subgroups.

**Theorem 3.3.** Let $H$ be a normal subgroup of a subgroup $G$ of $\hat{\text{Diff}}(\mathbb{C}^n,0)$. Suppose that the map $\hat{\pi}_{k_0} : G/H \to G_{k_0}/H_{k_0}$ satisfies $(\hat{\pi}_{k_0})_{|(G,H)/H}$ is injective for some $k_0 \in \mathbb{N}$. Furthermore suppose $\langle \phi \rangle \subset \langle G, H \rangle$ for any $\phi \in G$. Then $G/H$ is finite dimensional.

**Proof.** Let $T$ be the subgroup of $\langle G, H \rangle$ generated by $\bigcup_{\phi \in G} \langle \phi \rangle_{k_0}$. The property $\langle \phi \rangle \subset \langle G, H \rangle$ implies that the group $T_k^* = \langle \bigcup_{\phi \in G} \langle \phi \rangle_{k_0} \rangle$ is contained in $\langle G^*_k, H_k \rangle$. Moreover it is a (connected) algebraic group since it is generated by a family of connected algebraic matrix groups (Theorem 2.1) for any $k \in \mathbb{N}$. Theorem 2.1 also implies that $\langle T_k^*, H_{k,0} \rangle$ is algebraic. Since $H_k$ is normal in $G_k$, we deduce that $\langle T_k^*, H_{k,0} \rangle$ is a finite index subgroup of $\langle T_k^*, H_k \rangle$ and in particular $\langle T_k^*, H_k \rangle$ is algebraic for any $k \in \mathbb{N}$. The group $G_k^*$ normalizes the algebraic group $T_k^*$. Thus $T_k^*$ is a normal subgroup of $G_k^*$. As a consequence $\langle T_k^*, H_k \rangle$ is a subgroup of $\langle G_k^*, H_k \rangle$ that is normal in $G_k$ for any $k \in \mathbb{N}$.

The group $(\phi)_{k_0}$ is a finite index normal subgroup of $\langle \phi \rangle$ and $\langle \phi \rangle_{k,0}$ is a finite index normal subgroup of $\langle \phi \rangle_k$ for all $\phi \in G$ and $k \in \mathbb{N}$.
We obtain that all elements of the subgroup \( \langle G^*_k, H_k \rangle / \langle T^*_k, H_k \rangle \) of the algebraic matrix group \( G_k / \langle T^*_k, H_k \rangle \) have finite order. Suppose that the Zariski-closure of a group of matrices of elements of finite order consists only of semisimple elements; we will prove this later on (Lemma 3.5). Since \( G_k / \langle T^*_k, H_k \rangle \) is the Zariski-closure of \( \langle G^*_k, H_k \rangle / \langle T^*_k, H_k \rangle \), the group \( G_k / \langle T^*_k, H_k \rangle \) consists of semisimple elements.

Let us show that \( \hat{\pi}_{k,k_0} : G_k / H_k \to G_{k_0} / H_{k_0} \) is injective for any \( k \geq k_0 \). Such property implies that \( G / H \) is finite dimensional by Proposition 3.2. The hypothesis implies that \( \langle \hat{\pi}_{k,k_0} \rangle |_{\langle G^*_k, H_k \rangle / H_k} \) is injective. Let \( \phi_k H_k \) be an element of the kernel of \( \hat{\pi}_{k,k_0} \) where \( \phi \in \overline{G} \). Since \( \phi_{k_0} \in H_{k_0} \), there exists \( \eta \in \Pi \) such that \( \phi_{k_0} \equiv \eta_{k_0} \) and in particular \( (\phi \circ \eta^{-1})_{k_0} \equiv \text{Id} \). The formal diffeomorphism \( \phi \circ \eta^{-1} \) is unipotent and so is \( (\phi \circ \eta^{-1})_k \). Thus the class of \( (\phi \circ \eta^{-1})_k \) in \( G_k / \langle T^*_k, H_k \rangle \) is unipotent. Since it is also semisimple by the previous discussion, we obtain \( (\phi \circ \eta^{-1})_k \in \langle T^*_k, H_k \rangle \) and then \( \phi_k \in \langle T^*_k, H_k \rangle \subset \langle G^*_k, H_k \rangle \). Since \( \langle \hat{\pi}_{k,k_0} \rangle |_{\langle G^*_k, H_k \rangle / H_k} \) is injective, we obtain \( \phi_k \in H_k \). In particular \( \hat{\pi}_{k,k_0} : G_k / H_k \to G_{k_0} / H_{k_0} \) is injective for any \( k \geq k_0 \).

\[ \square \]

**Corollary 3.2.** Let \( G \) be a subgroup of \( \widehat{\text{Diff}}(\mathbb{C}^n, 0) \) that has the finite determination property. Suppose that \( \langle \phi \rangle \) is contained in \( G \) for any \( \phi \in G \). Then \( G \) is finite dimensional.

Let us remark, regarding Corollary 3.2 that the condition \( \langle \phi \rangle \subset G \) is easy to verify if we know the Jordan-Chevalley decomposition of the elements of \( G \).

**Lemma 3.5.** Let \( G \) be a subgroup of \( \text{GL}(n, \mathbb{C}) \) such that all its elements have finite order. Then all elements of \( \overline{G} \) are semisimple.

**Proof.** The Tits alternative [27] implies that either \( G \) is virtually solvable or it contains a non-abelian free group. Since clearly the second option is impossible, \( G \) is virtually solvable. Hence \( \overline{G}_0 \) is solvable. Since it is also connected, the group \( \overline{G}_0 \) is upper triangular up to a linear change of coordinates by Lie-Kolchkin’s theorem (cf. [8], section 17.6, p. 113).

We denote \( H = G \cap \overline{G}_0 \); it is a finite index normal subgroup of \( G \). The derived group \( H' \) of \( H \) consists of unipotent upper triangular matrices. They are also semisimple by hypothesis and as a consequence \( H' \) is the trivial group and \( H \) is abelian. Moreover \( H \) consists of semisimple elements; hence \( H \) and then \( \overline{H} \) are diagonalizable. Since \( H \) is a finite index normal subgroup of \( G \), \( \overline{H} \) is a finite index normal subgroup of \( \overline{G} \). An element \( A \) of \( \overline{G} \) satisfies \( A^k \in \overline{H} \) for some \( k \in \mathbb{N} \). Since \( A^k \) is semisimple, \( A \) is semisimple for any \( A \in \overline{G} \). \[ \square \]
4. FAMILIES OF FINITE DIMENSIONAL GROUPS

The finite dimensional subgroups of \( \hat{\text{Diff}}(\mathbb{C}, 0) \) can be characterized, they are the solvable groups.

**Proposition 4.1.** Let \( G \) be a subgroup of \( \hat{\text{Diff}}(\mathbb{C}, 0) \). Then the following conditions are equivalent:

1. \( G \) is solvable.
2. \( G \) has the finite determination property.
3. \( G \) is finite dimensional.

**Proof.** The items (1) and (2) are equivalent (cf. [12, Théorème 1.4.1]).

Let us show (1) \( \implies \) (3). The group \( \overline{G} \) is solvable by Lemma 2.5. Since (1) \( \implies \) (2), \( \overline{G} \) has the finite determination property and then \( G \) is finite dimensional.

Let us prove (3) \( \implies \) (1). Since \( G \) is finite dimensional, \( \overline{G} \) has the finite determination property. The property (2) \( \implies \) (1) implies that \( \overline{G} \) and then \( G \) are solvable. \( \square \)

**Remark 4.1.** Any solvable subgroup \( G \) of \( \hat{\text{Diff}}(\mathbb{C}, 0) \) satisfies

\[
\dim G = \dim \overline{G} \leq 2
\]

by the formal classification of such groups (cf. [21, Theorem 5.3] [9, section 6B3]).

**Remark 4.2.** Notice that solvable subgroups of \( \hat{\text{Diff}}(\mathbb{C}^n, 0) \) are not in general finite dimensional for \( n \geq 2 \) (cf. Remark 3.6).

It is very easy to use the extension theorems in section 3 to find a big class of examples of finite dimensional subgroups of formal diffeomorphisms.

**Theorem 4.1.** Let \( G \) be a subgroup of \( \hat{\text{Diff}}(\mathbb{C}^n, 0) \). Suppose that \( G \) has a subnormal series

\[
\{Id\} = G^0 \triangleleft G^1 \triangleleft \ldots \triangleleft G^m = H
\]

such that \( G^{j+1}/G^j \) is either

- finite or
- abelian and finitely generated in the extended sense or
- a connected Lie group (cf. Definition 3.5)

for any \( 0 \leq j < m \). Then \( G \) is finite dimensional.

**Proof.** The result is a straightforward consequence of Proposition 3.1, Lemma 3.3 and Propositions 3.6 and 3.8. \( \square \)
For instance a finite-by-cyclic-by-cyclic-by-finite-by-cyclic group of formal diffeomorphisms is finite dimensional. Anyway we think that it is interesting to apply the extension approach laid out in section 3 to show that several distinguished classes of groups are always finite dimensional. Our first targets are polycyclic groups.

**Definition 4.1.** Let $G$ be a group. We say that $G$ is **polycyclic** if it has a subnormal series as in Equation (1) such that $G^{j+1}/G^j$ is cyclic for any $0 \leq j < m$.

**Remark 4.3.** A group $G$ is polycyclic if and only if is solvable and every subgroup of $G$ is finitely generated (cf. [22, Theorem 5.4.12]).

**Theorem 4.2.** Let $G$ be a virtually polycyclic subgroup of $\hat{\text{Diff}} (\mathbb{C}^n, 0)$. Then $G$ is finite dimensional.

**Proof.** A virtually polycyclic group is a cyclic-by-...-by-cyclic-by-finite group. Therefore $G$ is finite dimensional by Theorem 4.1. \qed

**Definition 4.2.** We say that a group $G$ is **supersolvable** if it has a normal series in which all the factors are cyclic groups.

**Remark 4.4.** The definition is very similar to Definition 4.1. Anyway supersolubility is stronger since we require the groups $G^j$ in Equation (1) to be normal in $G$ for $0 \leq j \leq m$.

**Corollary 4.1.** Let $G$ be a virtually supersolvable subgroup of $\hat{\text{Diff}} (\mathbb{C}^n, 0)$. Then $G$ is finite dimensional.

**Proof.** A virtually supersolvable group is virtually polycyclic. The result is a consequence of Theorem 4.2. \qed

Let us focus now on nilpotent groups.

**Theorem 4.3.** Let $G$ be a virtually nilpotent subgroup of $\hat{\text{Diff}} (\mathbb{C}^n, 0)$. Suppose that $G$ is finitely generated in the extended sense. Then $G$ is finite dimensional. In particular subgroups of $\hat{\text{Diff}} (\mathbb{C}^n, 0)$ of polynomial growth are finite dimensional.

**Proof.** There exists a subgroup $G_+$ of $G$ such that $G_+$ is finitely generated and $\overline{G_+} = \overline{G}$ by Lemma 3.4. Since $G_+$ is a subgroup of $G$, it is virtually nilpotent. Thus up to replace $G$ with $G_+$ we can suppose that $G$ is finitely generated.

Let $H$ be a finite index normal nilpotent subgroup of $G$. Since $H$ is a finite index subgroup of a finitely generated group, it is finitely generated (cf. [22, Theorem 1.6.11]). Any finitely generated nilpotent
group is polycyclic (cf. [11, Theorem 17.2.2]). Therefore $G$ is virtually polycyclic and then finite dimensional by Theorem 4.2.

By a theorem of Gromov [7], the groups of polynomial growth are exactly the finitely generated virtually nilpotent groups. Hence every subgroup of $\hat{\text{Diff}}(\mathbb{C}^n, 0)$ of polynomial growth is finite dimensional. □

We provide a sort of reciprocal of the previous theorem in the setting of unipotent subgroups of formal diffeomorphisms.

**Lemma 4.1.** Let $G$ be a unipotent subgroup of $\hat{\text{Diff}}(\mathbb{C}^n, 0)$. Suppose that $G$ has the finite determination property. Then $G$ is nilpotent.

**Proof.** Since $G$ has the finite determination property, there exists $k \in \mathbb{N}$ such that $\pi_k : G \to G^*_k$ is an isomorphism of groups. Moreover $G^*_k$ is a unipotent algebraic matrix group. Unipotent groups of matrices are always triangularizable and then nilpotent by Kolchin’s theorem (cf. [24, chapter V, p. 35]). Hence $G$ is nilpotent. □

**Corollary 4.2.** Let $G$ be a unipotent subgroup of $\hat{\text{Diff}}(\mathbb{C}^n, 0)$. Suppose that $G$ is finitely generated in the extended sense and finitely determined. Then $G$ is finite dimensional.

**Proof.** The group $G$ is nilpotent by Lemma 4.1. Thus it is finite dimensional by Theorem 4.3. □

**Remark 4.5.** Theorems 4.2, 4.3 and Corollary 4.1 admit straightforward generalizations to extensions. For instance a finitely generated (in the extended sense) virtually nilpotent extension of groups of formal diffeomorphisms is finite dimensional.

Let us focus on solvable subgroups of $\hat{\text{Diff}}(\mathbb{C}^n, 0)$ that are not necessarily polycyclic. In order to show $\dim G < \infty$ for a solvable subgroup of $\hat{\text{Diff}}(\mathbb{C}^n, 0)$ it suffices to consider finite generation properties on the derived groups of $G$.

**Proposition 4.2.** Let $G$ be a solvable subgroup of $\hat{\text{Diff}}(\mathbb{C}^n, 0)$. Suppose that $G^{(\ell)}/G^{(\ell+1)}$ is finitely generated in the extended sense for any $\ell \in \mathbb{Z}_{\geq 0}$. Then $G$ is finite dimensional.

**Proof.** Every extension of the form $G^{(\ell)}/G^{(\ell+1)}$ is abelian and then finite dimensional by Proposition 3.6. Since there exists $\ell$ such that $G^{(\ell)} = \{1\}$, $G$ is finite dimensional by Theorem 4.1. □

In the following our goal is substantially weakening the finite generation hypotheses in Proposition 4.2.
Definition 4.3. We denote
\[ G_s = \{ \phi \in G : \phi = \phi_s \}, \quad G^s = \{ \phi_s : \phi \in G \}, \]
\[ G_u = \{ \phi \in G : \phi = \phi_u \}, \quad G^u = \{ \phi_u : \phi \in G \}, \]
\[ G_s = (G)_s \text{ and } G_u = (G)_u. \]

Notice that \( G_s \) (resp. \( G_u \)) is the set of semisimple (resp. unipotent) elements of \( G \) whereas \( G^s \) (resp. \( G^u \)) is the set of semisimple (resp. unipotent) parts of elements of \( G \). We can have \( G^s \subsetneq G \) (resp. \( G^u \subsetneq G \)) if the group \( G \) is not splitable.

The next results are intended to show that under certain hypotheses a virtually solvable subgroup \( G \) of \( \hat{\text{Diff}}(\mathbb{C}^n,0) \) is finite dimensional if and only if \( G_u \) is finite dimensional.

Lemma 4.2. Let \( H \) be a finite index normal subgroup of a subgroup \( \hat{G} \) of \( \hat{\text{Diff}}(\mathbb{C}^n,0) \). Then \( G_u = H_u \) and \( \langle G^u \rangle = \langle H^u \rangle \). Moreover if \( G \) is virtually solvable then \( G_u \) and \( G^u \) are solvable groups.

Proof. Since \( \overline{H} \) is a finite index normal solvable subgroup of \( G \) by Lemma 2.4, we obtain \( \overline{H}_0 = G_0 \) by Lemma 2.1. The unipotent elements of \( G \) (resp. \( \overline{H} \)) are contained in \( G_0 \) (resp. \( \overline{H}_0 \)) by Remark 2.8. Hence we obtain \( \overline{G}_u = \overline{H}_u \).

Given \( \alpha \in G^u \), there exists \( k \in \mathbb{N} \) such that \( \alpha^k \in H^u \). Since
\[ \alpha \in \langle \alpha^k \rangle = \{ \alpha^t : t \in \mathbb{C} \}, \]
we obtain \( \alpha \in \langle \alpha^k \rangle \subset \langle H^u \rangle \). We deduce \( \langle G^u \rangle \subset \langle H^u \rangle \). It is clear that \( \langle H^u \rangle \subset \langle G^u \rangle \). Thus \( \langle G^u \rangle = \langle H^u \rangle \) holds.

Let us show that \( \overline{G}_u \) is a group if \( G \) is virtually solvable. By the first part of the proof we can suppose that \( G \) is solvable. The group \( \overline{G}_0 \) is solvable by Lemma 2.6 and \( \overline{G}_u \subset \overline{G}_0 \) by Remark 2.8. Moreover since \( G_{1,u} \subset G_{1,0} \) and the latter group is solvable and connected, we can suppose, up to a linear change of coordinates, that all elements of \( \overline{G}_u \) have linear parts that are unipotent upper triangular matrices by Lie-Kolchin’s theorem. Since the set of unipotent upper triangular matrices is a group we deduce that \( \overline{G}_u \) and then \( G_u \) are groups. \( \square \)

Theorem 4.4. Let \( G \) be a virtually solvable subgroup of \( \hat{\text{Diff}}(\mathbb{C}^n,0) \) such that \( G \) is finitely generated over \( G_u \) in the extended sense. Then \( G/G_u \) is finite dimensional. In particular \( G \) is finite dimensional if and only if \( G_u \) is finite dimensional.

Corollary 4.3. Let \( G \) be a finitely generated virtually solvable subgroup of \( \hat{\text{Diff}}(\mathbb{C}^n,0) \) such that \( G_u \) is finitely generated and nilpotent. Then \( G \) is finite dimensional.
Remark 4.6. Notice that the nilpotence of $G_u$ is necessary in Corollary 4.3 by Lemma 4.1. The main advantage of Corollary 4.3 is that we are replacing a property of finite generation for every derived subgroup of $G$ by the analogous property for just $G$ and $G_u$. Both conditions of finite generation can be replaced by finite generation in the extended sense.

Proof of Theorem 4.4. The set $G_u$ is a subgroup of $G$ by Lemma 4.2. Thus $G_u$ is a normal subgroup of $G$.

There exists a subgroup $J$ of $G$ such that $G_u \subset J$, $\overline{J} = \overline{G}$ and $J$ is finitely generated over $G_u$ by Lemma 3.4. It suffices to show $\dim(J/G_u) < \infty$ since $\dim(G/G_u) = \dim(J/G_u)$.

We denote $\overline{K} = J \cap \overline{G_0}$. The group $\overline{K}$ is a finite index normal subgroup of $J$. Since $\overline{G_0}$ is solvable (by Lemma 2.6) and $G_{1,0}$ is connected, we can suppose that all elements of $\overline{G_0}$ have linear parts that belong to the group of upper triangular matrices. Notice that $G_u$ is contained in $\overline{G_0}$. Since $\overline{K}/G_u$ is a finite index normal subgroup of $J/G_u$, the group $\overline{K}/G_u$ is finitely generated. The elements of the derived group $\overline{K}'$ have linear parts that are unipotent upper triangular matrices. Thus $\overline{K}'$ is contained in $G_u$ and $\overline{K}/G_u$ is abelian. We deduce $\dim(\overline{K}/G_u) < \infty$ by Proposition 3.6 and then $\dim(J/G_u) < \infty$ by Corollary 3.1. □

Proof of Corollary 4.3. Since $G_u$ is a normal subgroup of $G$ by Lemma 4.2, it suffices to show $\dim G_u < \infty$ by Theorem 4.4 and Proposition 3.1. The group $G_u$ is finite dimensional by Theorem 4.3. □

Let us see that the finite generation of $G$ can be dropped in Theorem 4.4 if we consider a splittable group.

Theorem 4.5. Let $G$ be a splittable virtually solvable subgroup of $\widehat{\text{Diff}}(\mathbb{C}^n,0)$. Then $G/G_u$ is finite dimensional and $\overline{G_u} = (\overline{G_u})$. In particular $G$ is finite dimensional if and only if $G_u$ is finite dimensional.

Corollary 4.4. Let $G$ be a splittable virtually solvable subgroup of $\widehat{\text{Diff}}(\mathbb{C}^n,0)$. Suppose that either

- $G_u$ is nilpotent and finitely generated in the extended sense or
- $\{\phi^t : t \in \mathbb{C}\} \subset G$ for any $\phi \in G_u$ and $G_u$ has the finite determination property.

Then $G$ is finite dimensional.

We will use the following theorem by Togo.
**Theorem 4.6** ([28]). Let $L$ be a Zariski-connected solvable subgroup of $\text{GL}(n,k)$. Then $L_u$ is a Zariski-closed normal subgroup of $L$. Moreover if $L$ is splittable then $L_u = (L_u)$.

**Proof of Theorem 4.5.** The group $H := G \cap G_0$ is a finite index normal subgroup of $G$. Thus $H$ is a finite index normal subgroup of $G$ by Lemma 2.4. In particular $H$ contains $G_0$ by Lemma 2.1. Since $H \subset G_0$, we obtain $H = G_0$. Moreover $G_0$ is solvable by Lemma 2.6 and then $H$ is solvable. Since $H$ is splittable and $G_u \subset G \cap G_0$, we obtain $G_u \subset H$. We deduce that $H$ is splittable and $H_u = G_u$. Up to replace $G$ with $H$ if necessary we can suppose $G = G_0$ and $G$ is solvable by Lemma 4.2.

Fix $k \in \mathbb{N}$. Since $G$ is splittable, the group $G^*_k$ is solvable. The group $G_k^*$ is solvable (Lemma 2.5) and then $G_k$ is solvable. The property $G = G_0$ implies $G_k = G_{k,0}$. Since the Zariski-closure of $G^*_k$ is solvable and a (connected) smooth irreducible algebraic set, $G^*_k$ is a splittable Zariski-connected solvable group. We can apply Togo’s Theorem 4.6 to the subgroup $G^*_k$ of $G_k$. We obtain

$$G_{k,u} = (G^*_k)_u = (G^*_k)_{u,k} = G_{u,k}$$

for any $k \in \mathbb{N}$. Since $G_u = \lim_{\downarrow k} G_{k,u}$ by Remark 2.10 and $(G_u) = \lim_{\downarrow k} G_{u,k}$ by definition, the equality $G_u = (G_u)$ holds. We have

$$\dim G/G_u = \dim \overline{G}/(G_u) = \dim \overline{G}/G_u.$$

The map $\hat{\pi}_1 : \overline{G}/G_u \to G_1/G_{1,u}$ is injective and thus an isomorphism of groups by Remark 2.10. Therefore $\overline{G}/G_u$ is finite dimensional by Proposition 3.2 and $\dim \overline{G}/G_u = \dim G_1 - \dim G_{1,u}$. We obtain $\dim G/G_u = \dim G_1 - \dim G_{1,u} < \infty$. \qed

**Proof of Corollary 4.4.** It suffices to show that $G_u$ is finite dimensional by Theorem 4.5. In the former case it is a consequence of Theorem 4.3. In the later case we can apply Corollary 3.2 since $\langle \phi \rangle = \{e^t : t \in \mathbb{C}\}$ for any $\phi \in \text{Diff}_u(\mathbb{C}^n, 0)$ (cf. Remark 2.8). \qed

### 4.1. Examples of infinitely dimensional groups.

So far we exhibited distinguished families of virtually solvable groups whose members are finite dimensional. Now let us consider the problem of finding infinite dimensional families of solvable subgroups of $\text{Diff}(\mathbb{C}^n, 0)$.

**Remark 4.7.** Consider the subgroup $\langle \phi, \eta \rangle$ of $\text{Diff}(\mathbb{C}^2, 0)$ generated by $\phi(x,y) = (x, y(1+x))$ and $\eta(x,y) = (x, y+x^2)$. Since $\langle \phi, \eta \rangle^\prime$ is contained in the group $H_1 := \{ (x, y+b(x)) : b \in \mathbb{C}\{x\} \cap (x^2) \}$, we get $\langle \phi, \eta \rangle^{(2)} = \{I\}$. In particular $\langle \phi, \eta \rangle$ is a finitely generated unipotent solvable subgroup of $\text{Diff}(\mathbb{C}^2, 0)$. Since $[(x, y - x^k), \phi] = (x, y + x^{k+1})$ for any
Next we see that solvable subgroups of $\widehat{\text{Diff}} (\mathbb{C}^n, 0)$ of high derived length are never finite dimensional.

**Proposition 4.3.** Let $G$ be a solvable group contained in $\widehat{\text{Diff}}_u (\mathbb{C}^n, 0)$ whose derived length is greater than $n$. Then $G$ does not have the finite determination property. In particular $G$ is not finite dimensional.

**Remark 4.8.** Such groups always exists if $n \geq 2$. Indeed the maximum of the derived lengths of the solvable unipotent subgroups of $\widehat{\text{Diff}} (\mathbb{C}^n, 0)$ is $2n - 1$.

**Remark 4.9.** Notice that given a solvable group $G$, its derived length is the supremum of the derived lengths of all its finitely generated subgroups. Hence there exists a finitely generated subgroup $H$ of $G$ with the same derived length. In particular we can suppose that the examples provided by Proposition 4.3 for $n \geq 2$ are finitely generated.

**Remark 4.10.** Let us provide an example of a group that satisfies the hypotheses of Proposition 4.3. We denote $\phi(x, y) = (x, y(1 + x))$, $\eta(x, y) = (x, y + x^2)$ and $\psi(x, y) = \left(\frac{x}{1 + x}, y\right)$. Consider the subgroup $G := \langle \phi, \eta, \psi \rangle$ of $\text{Diff} (\mathbb{C}^2, 0)$. We define the subgroup

$H_0 = \{x, y(1 + a(x)) + xb(x) : a, b \in \mathbb{C} \{x\} \cap (x)\}$

of $\text{Diff} (\mathbb{C}^2, 0)$. It is clear that $G'$ is contained in $H_0$ and $G^{(2)}$ is contained in the abelian group $H_1$ defined in Remark 4.7. Hence $G$ is a finitely generated unipotent solvable subgroup of $\text{Diff} (\mathbb{C}^2, 0)$ whose derived length is at most 3. Since

$[\psi, \phi] = \left(x, y \frac{1 + 2x}{(1 + x)^2}\right)$, \qquad [\eta^{-1}, \phi] = \left(x, y + x^3\right)$

and

$[[\psi, \phi], [\eta^{-1}, \phi]] = \left(x, y - \frac{x^5}{(1 + x)^2}\right)$,

the diffeomorphism $[[\psi, \phi], [\eta^{-1}, \phi]]$ belongs to $G^{(2)} \setminus \{Id\}$ and hence the derived length of $G$ is equal to 3.

**Proof of Proposition 4.3.** Suppose that $G$ has the finite determination property. Hence $G$ is nilpotent by Lemma 4.1. Therefore the derived length of $G$ is less or equal than $n$ [14, Theorem 5], obtaining a contradiction. \qed
5. Local intersection theory

Let us explain in this section why Theorem 1.5 holds. A priori we could use directly Binyamini’s theorem [3] since an algebraic group is a Lie group with finitely many connected components. Anyway we think that it is instructive to apply our canonical approach to the ideas introduced by Seigal-Yakovenko in [23] (to show Theorem 1.5 for finitely generated in the extended sense abelian subgroups of formal diffeomorphisms).

Consider two formal subschemes $I$ and $J$ of the scheme $\text{spec} \hat{\mathcal{O}}_n$. We can identify $I$ and $J$ with two ideals of the ring $\hat{\mathcal{O}}_n$ of formal power series.

**Definition 5.1.** We define the intersection multiplicity $(I, J)$ as

$$(I, J) = \dim_{\mathbb{C}} \hat{\mathcal{O}}_n/(I + J).$$

**Remark 5.1.** This definition of intersection multiplicity coincides with the usual one if $I$ and $J$ are complete intersections of complementary dimension. It is finite if and only if the usual intersection multiplicity is finite. Moreover it provides an upper bound for the usual intersection multiplicity (cf. [6, Proposition 8.2]). Therefore by showing Theorem 1.5 with Definition 5.1 it will be automatically satisfied for the usual intersection multiplicity.

Next let us show Theorem 1.5. Since we follow Seigal-Yakovenko’s ideas we refer to their paper [23] for details. We are interested in stressing how their point of view fits in the context of the theory of finite dimensional groups of formal diffeomorphisms.

**Proof of Theorem 1.5.** Let $V$ and $W$ be formal subschemes of $\text{spec} \hat{\mathcal{O}}_n$. Suppose that $V$ is given by the ideal $K$ of $\hat{\mathcal{O}}_n$. Given $\phi \in \hat{\text{Diff}}(\mathbb{C}^n, 0)$ the subscheme $\phi^{-1}(V)$ is given by the ideal $\phi^*K = \{ f \circ \phi : f \in K \}$.

There exists $k \in \mathbb{N}$ such that $\pi_k : G \to G_k$ is an isomorphism of groups by Proposition 3.2. The map $\pi_{m,k} : G_m \to G_k$ is an isomorphism of algebraic groups for any $m \geq k$. In particular the affine coordinate rings $\mathbb{C}[G_k]$ and $\mathbb{C}[G_m]$ are isomorphic as $\mathbb{C}$-algebras for any $m \geq k$.

Given an ideal $J$ of $\hat{\mathcal{O}}_n$ the property $\dim_{\mathbb{C}} \hat{\mathcal{O}}_n/J > m$ is equivalent to a system of algebraic equations on the coefficients of the $m$-th jets of the generators of $J$ [23, Lemma 3]. In particular $(\phi^{-1}(V), W) > m$ holds for $\phi \in \hat{\text{Diff}}(\mathbb{C}^n, 0)$ if and only if the coefficients of the $m$-th jet of $\phi$ satisfy a certain system of algebraic equations. More intrinsically we can say that $S_m := \{ \phi \in G : (\phi^{-1}(V), W) > m \}$ defines an ideal $I_m$ of the affine coordinate ring $\mathbb{C}[G_m]$. It also defines an ideal, that we...
denote also by \( I_m \) in \( \mathbb{C}[G_k] \) for any \( m \geq k \). We can suppose \( I_m \subset I_{m'} \) for all \( m' \geq m \geq k \) by replacing \( I_m \) with \( I_k + \ldots + I_m \) for \( m \geq k \). Since \( \mathbb{C}[G_k] \) is noetherian, there exists \( m_0 \geq k \) such that \( I_m = I_{m_0} \) for any \( m \geq m_0 \). In particular \( (\phi^{-1}(V), W) > m_0 \) implies \( (\phi^{-1}(V), W) = \infty \) for any \( \phi \in G \).

**Remark 5.2.** The key point of the proof is showing that the increasing sequence of ideals \( I_1 \subset I_2 \subset \ldots \) is contained in a noetherian ring. Seigal and Yakovenko show that it is contained in a ring of quasipolynomials in their setting \([23]\) whereas Binyamini includes them in a noetherian subring of continuous functions of \( G \) \([3]\). We use that the coefficients of degree greater than \( k \) of the Taylor expansion of the elements of \( G \) are regular functions on the coefficients of degree less or equal than \( k \) if \( G \) is finite dimensional (Remark 3.4). This allows to write all equations defining the ideals \( I_m \) in terms of the coefficients of \( \phi \in G \) of degree less or equal than \( k \). In particular Theorem 1.5 is an immediate consequence of the noetherianity of polynomial rings in finitely many complex variables. More precisely, in the finite dimensional setting the noetherian ring \( \mathbb{C}[G_k] \) containing all the ideals \( I_m \) for \( m \in \mathbb{N} \) is an affine coordinate ring of an algebraic matrix group canonically associated to \( G \).

**Proof of Theorem 1.6.** The hypothesis implies that \( G \) is finite dimensional by Theorems 4.2 and 4.3. Hence the conclusion is a consequence of Theorem 1.5. \( \Box \)

**Appendix A. Solvable groups of formal diffeomorphisms**

In this section we expand the study of virtually solvable subgroups of formal diffeomorphisms of section 4. Our approach is based on considering extensions of groups and hence it makes sense to generalize the results in section 4 to extensions. But we also would like to understand better the phenomenon described in Theorem 4.5. A splittable virtually solvable subgroup \( G \) of \( \hat{\text{Diff}}(\mathbb{C}^n, 0) \) satisfies \( \dim G < \infty \) if and only if \( \dim G_u < \infty \). A natural question is whether or not this “reduction to unipotent” is possible somehow in general or it is peculiar of the splittable case. We will find analogues for the general case (Corollaries A.1 and A.2) and even for extensions (Theorem A.1).

**A.1. Finite dimensional subextensions.** Solvable groups in dimension 1 are finite dimensional by Proposition 4.1. It is natural to study solvable groups and solvable extensions in higher dimensions with the purpose of understanding how far they are of being finite dimensional.
or also where it is concentrated the possible infinite dimensional nature of such groups. These topics are the subjects of this section.

Let us explain our goal a little bit more precisely. Let $H$ be a normal subgroup of a solvable subgroup $G$ of $\text{Diff} (\mathbb{C}^n, 0)$. Suppose that $G/H$ is virtually solvable. We already know that in general the extension $G/H$ is not finite dimensional but anyway we want to find “unipotent” extensions that are finite dimensional if and only if $G/H$ is. Indeed we will see that $\dim G/H < \infty$ is equivalent to $\dim \langle G^u, H \rangle / H < \infty$ (Theorem A.1). We will also show analogues of Theorems 4.4 and 4.5 for extensions.

Analogously as in Corollary 4.4 we want to use Togo’s theorem for virtually solvable extensions even in the non-splittable case. The next linear results are intended to address this issue.

**Lemma A.1.** Let $G$ be a subgroup of $\text{GL}(n, \mathbb{C})$. Then $\langle G^u \rangle$ is normal in $\langle G^s, G^u \rangle$.

**Proof.** Every element of $G^u$ is of the form $A_u$ for some $A \in G$. Given $B \in G$ the property $BAB^{-1} \in G$ implies $BA_uB^{-1} = (BAB^{-1})_u \in G^u$. It is clear that $B_uA_uB_u^{-1} \in \langle G^u \rangle$ and that

$$B_uA_uB_u^{-1} = B_u^{-1}(BA_uB^{-1})B_u \in \langle G^u \rangle$$

for any $B \in G$. We deduce $CA_uC^{-1} \in \langle G^u \rangle$ for all $A_u \in G^u$ and $C \in \langle G^s, G^u \rangle$. Hence we get $C\langle G^u \rangle C^{-1} \subset \langle G^u \rangle$ for any $C \in \langle G^s, G^u \rangle$. □

**Lemma A.2.** Let $G$ be a subgroup of $\text{GL}(n, \mathbb{C})$. Then any element of $\langle G^s, \langle G^u \rangle \rangle$ is of the form $\alpha \beta'$ for some $\alpha \in G^s$ and $\beta' \in \langle G^u \rangle$.

**Proof.** We denote $J = \langle G^s, \langle G^u \rangle \rangle$. Since $G^s$ normalizes $\langle G^u \rangle$ by Lemma A.1 $G^s$ normalizes $\langle G^u \rangle$. Thus $\langle G^u \rangle$ is a normal subgroup of $J$. We deduce that any element of $J$ is of the form $\alpha_1 \ldots \alpha_m \beta'$ where $\alpha_1, \ldots, \alpha_m \in G^s$ and $\beta' \in \langle G^u \rangle$. The element $\alpha_j$ is equal to $(\gamma_j)_s$ for some $\gamma_j \in G$ and any $1 \leq j \leq m$. Since $\langle G^s \rangle$ is normal in $\langle G^s, G^u \rangle$ by Lemma A.1 we obtain

$$\alpha_1 \ldots \alpha_m = \gamma_1 \ldots \gamma_m \beta''$$

for some $\beta'' \in \langle G^u \rangle$. We define $\alpha = (\gamma_1 \ldots \gamma_m)_s$ and $\beta = (\gamma_1 \ldots \gamma_m)_u \beta'' \beta'$. It is clear that $\alpha$ belongs to $G^s$ and $\beta$ belongs to $\langle G^u \rangle$. □

The Zariski-closure of a virtually solvable matrix group $G$ is splittable by Chevalley’s Theorem 2.2. Anyway in the following lemma we provide another extension of $G$ that is splittable and contained in $\overline{G}$.

The main advantage is that in the new extension we can characterize its unipotent elements in terms of $G^u$. 

Lemma A.3. Let $G$ be a virtually solvable subgroup of $\text{GL}(n, \mathbb{C})$. Then $\langle G^s, (G^u)^u \rangle = (G^u)$. In particular $\langle G^s, (G^u) \rangle$ is splittable.

Proof. The group $\overline{G}$ is virtually solvable by Lemma 2.6. Since $G$ normalizes $(G^u)$, so it normalizes $(G^u)$. The normalizer of an algebraic group is algebraic and thus $(G^u)$ is a normal subgroup of $\overline{G}$.

We denote $J = \langle G^s, (G^u) \rangle$. Fix $\gamma \in J$. Let us show $\gamma_u \in (G^u)$. We can suppose that $\gamma$ is of the form $\alpha \beta$ for some $\alpha \in G^s$ and $\beta \in (G^u)$ by Lemma A.2. There exists $k \in \mathbb{N}$ such that $\alpha^k \in \langle \alpha \rangle_0$ where $\langle \alpha \rangle_0$ is the connected component of $\text{Id}$ of $\langle \alpha \rangle$. Since $\langle \alpha^k \rangle$ is a finite index normal subgroup of $\langle \alpha \rangle$ by Lemma 2.4 it contains $\langle \alpha \rangle_0$ by Lemma 2.1. We obtain $\langle \alpha \rangle = \langle \alpha \rangle_0$. It suffices to show $(\gamma_u)^k \in (G^u)$. Indeed since $\langle \eta \rangle = \{ \exp(t \log \eta) : t \in \mathbb{C} \}$ for any unipotent element $\eta$ of $\text{GL}(n, \mathbb{C})$, we obtain

$$\langle \gamma_u^k \rangle = \langle \gamma_u \rangle = \{ \exp(t \log \gamma_u) : t \in \mathbb{C} \}.$$ 

Thus the property $\langle \gamma_u^k \rangle \subseteq (G^u)$ implies $\gamma_u \in (G^u)$. Hence up to replace $\gamma$ with $\gamma^k$ (and as a consequence $\alpha$ with $\alpha^k$) we can suppose $\langle \alpha \rangle = \langle \alpha \rangle_0$.

The group $(G^u)$ coincides with the (connected algebraic) group generated by $\bigcup_{\eta \in G^u} \{ \exp(t \log \eta) : t \in \mathbb{C} \}$ by Chevalley’s Theorem 2.1. Another application of Theorem 2.1 implies that $L := \langle \langle \alpha \rangle, (G^u) \rangle$ is a connected algebraic group. The group $L$ is contained in $\overline{G}$ and hence it is virtually solvable. Since it is connected, $L$ is solvable. The Lie-Kolchin theorem implies that up to a change of basis, the group $L$ is upper triangular. Thus $G^u$ and then $(G^u)$ are contained in the group of unipotent upper triangular matrices. In particular $(G^u)$ is contained in $J_u$. The group $(G^u)$ is normal in $\overline{G}$ and then in $L$ since $L \subset \overline{G}$. We deduce that every element $\eta$ of $L$ is of the form $\eta_1 \eta_2$ where $\eta_1 \in \langle \alpha \rangle$ and $\eta_2 \in (G^u)$. Since algebraic matrix groups are splittable by Chevalley’s Theorem 2.2 we obtain that $\gamma_u$ is of the previous form $\eta_1 \eta_2$ where $\eta_1 \in \langle \alpha \rangle$ and $\eta_2 \in (G^u)$. Since $L$ is upper triangular and $\gamma_u$ and $\eta_2$ are unipotent, $\eta_1$ is unipotent. It is also semisimple since $\eta_1 \in \langle \alpha \rangle$ and $\alpha$ is semisimple. We deduce $\eta_1 \equiv \text{Id}$ and then $\gamma_u \in (G^u)$. We obtain $J_u \subset (G^u)$ and since $(G^u) \subset J_u$ we get $J_u = J_u = (G^u)$.

Now let us identify the unipotent elements of the Zariski-closure of a virtually solvable linear group. The next proposition is a consequence of the aforementioned Togo’s Theorem 4.6 in the setting of virtually solvable (non-necessarily splittable) subgroups of $\text{GL}(n, \mathbb{C})$.

Proposition A.1. Let $G$ be a virtually solvable subgroup of $\text{GL}(n, \mathbb{C})$. Then $\overline{G}_u = (G^u)$. Moreover $\overline{G}_u$ is solvable.
Proof. There exists a finite index normal subgroup $H$ of $G$ that is solvable and such that \( \overline{H} \) is connected. Up to replace $G$ with $H$ we can suppose that these properties are satisfied by $G$ by Lemma 4.2.

We denote $J = \langle G^*, \overline{G^u}\rangle$. We have $G \subset J \subset G$ and then $\overline{J} = \overline{G}$. Since $\overline{G}$ is solvable, $J$ is solvable. The group $J$ is connected and hence $J$ is Zariski-connected. Since $J$ is splittable by Lemma A.3, we obtain $\langle G^u, H \rangle = \langle J^u, H \rangle = \langle G^u \rangle$ by applying Togo’s Theorem 4.6 to $J$. \( \square \)

Let us generalize Proposition A.1 to the setting of extensions of groups of formal diffeomorphisms.

**Proposition A.2.** Let $H$ be a normal subgroup of a subgroup $G$ of $\hat{\text{Diff}}(\mathbb{C}^n, 0)$. Suppose that $G/H$ is virtually solvable. Then $\langle G^u, H \rangle$ is the closure of $\langle G^u, H \rangle$ in the Krull topology.

**Proof.** Since $G^*/H^*$ is virtually solvable, $\langle G^*_k, H_k \rangle/H_k$ is virtually solvable. The group $G_k/H_k$ is the Zariski-closure of $\langle G^*_k, H_k \rangle/H_k$ and hence $G_k/H_k$ is a virtually solvable algebraic matrix group for any $k \in \mathbb{N}$ by Lemma 2.6.

Consider $\langle G^*_k, H_k \rangle/H_k$ as a subgroup of the algebraic matrix group $G_k/H_k$. Then $\langle \langle G^*_k, H_k \rangle/H_k \rangle^u$ is equal to $\langle G^*_k, H_k \rangle/H_k$. Let us apply Proposition A.1 to $\langle G^*_k, H_k \rangle/H_k$. We deduce

$$\frac{\langle (G^*_k)^u, H_k \rangle}{H_k} = \left( \frac{\langle (G^*_k, H_k) \rangle}{H_k} \right)_u = \left( \frac{G_k}{H_k} \right)_u = \langle G_{k,u}, H_k \rangle$$

for any $k \in \mathbb{N}$. We obtain $\langle (G^*_k)^u, H_k \rangle = \langle G_{k,u}, H_k \rangle$ for any $k \in \mathbb{N}$. It follows that

$$\langle H, G^u \rangle_k = \langle H_k^*, (G^*_k)^u \rangle = \langle H_k, (G^*_k)^u \rangle = \langle G_{k,u}, H_k \rangle$$

for any $k \in \mathbb{N}$. Since $\langle G_u, H \rangle_k = \langle G_{k,u}, H_k \rangle = \langle H, G^u \rangle_k$ for any $k \in \mathbb{N}$, the group $\langle G^u, H \rangle$ is the closure of $\langle G_u, H \rangle$ in the Krull topology. \( \square \)

We can identify the unipotent elements of the Zariski-closure of a virtually solvable group of formal diffeomorphisms.

**Corollary A.1.** Let $G$ be a virtually solvable subgroup of $\hat{\text{Diff}}(\mathbb{C}^n, 0)$. Then $\overline{G_u} = \overline{G^u}$. Moreover $\overline{G_u}$ is a solvable group.

**Proof.** We apply Proposition A.2 with $H = \{Id\}$. We obtain that $\overline{G^u}$ is equal to the closure of $\overline{G_u}$ in the Krull topology. Notice that $\overline{G_u}$ is a solvable group by Lemma 4.2. Since $\overline{G_u} = \varprojlim G_{k,u}$, it is closed in the Krull topology. As a consequence we obtain $\overline{G_u} = \overline{G^u}$. \( \square \)
As a corollary of the previous result we show that in order to determine the virtually solvable subgroups of $\hat{\text{Diff}}(\mathbb{C}^n,0)$ that are finite dimensional, it suffices to consider only groups of unipotent elements.

**Corollary A.2.** Let $G$ be a virtually solvable subgroup of $\hat{\text{Diff}}(\mathbb{C}^n,0)$. Then $G$ is finite dimensional if and only if $\langle G^u \rangle$ is finite dimensional.

**Proof.** The group $G$ is finite dimensional if and only if $G$ has the finite determination property. Since $G^u$ contains all the elements of $G$ with identity linear part, the group $G$ has the finite determination property if and only if $G^u$ has the finite determination property. Since $G^u = \langle G^u \rangle$ by Corollary A.1, we deduce that $G^u$ has the finite determination property if and only if $\langle G^u \rangle$ is finite dimensional. $\square$

Next we generalize Corollary A.2 to extensions.

**Theorem A.1.** Let $H$ be a normal subgroup of a subgroup $G$ of $\hat{\text{Diff}}(\mathbb{C}^n,0)$. Suppose that $G/H$ is virtually solvable. Then $\dim G/H < \infty$ if and only if $\dim \langle H,G^u \rangle/H < \infty$.

**Proof.** The sufficient condition is clear since $\langle H,G^u \rangle$ is contained in $G$. Suppose that $\dim \langle H,G^u \rangle/H < \infty$. There exists $k \in \mathbb{N}$ such that the natural map

$$\hat{\pi}_k : \langle H,G^u \rangle/H \to \langle H,G^u \rangle_k/H_k$$

is injective by Proposition 3.2. Let us show that $\hat{\pi}_k : G/H \to G_k/H_k$ is injective. This implies that $G/H$ is finite dimensional by Proposition 3.2.

Let $\phi \in G$ such that $\hat{\pi}_k(\phi H) = 1$ or equivalently $\phi_k \in H_k$. Since $H_k$ is algebraic, it is splittable by Chevalley’s theorem 2.2. In particular we get $\phi_{k,s} \in H_k$ and $\phi_{k,u} \in H_k$. Let $\alpha$ be an element of $H$ such that $\alpha_k = \phi_{k,s}$. The formal diffeomorphism $\alpha^{-1} \circ \phi$ satisfies $(\alpha^{-1} \circ \phi)_k = \phi_{k,u}$ and in particular is unipotent. Since $\alpha^{-1} \circ \phi$ belongs to $G^u$ and this group is contained in $\langle G^u, H \rangle$ by Proposition A.2, we deduce $\alpha^{-1} \circ \phi \in \langle G^u, H \rangle$. Moreover $\hat{\pi}_k'(\alpha^{-1} \circ \phi) = 1$ implies $\alpha^{-1} \circ \phi \in H$ by the injective nature of $\hat{\pi}_k'$. We obtain $\phi \in H$. Thus $\hat{\pi}_k$ is injective. $\square$

Since $\langle H,G^u \rangle \subset G$, we want to replace $\langle H,G^u \rangle$ with $\langle H,G^u \rangle$ in Theorem A.1 to obtain a result analogous to Theorem 4.5 for extensions. We impose two conditions in order to accomplish such a task, namely a “splitting” property (Theorem A.2) and a finite generation one (Theorem A.3).

**Theorem A.2.** Let $H$ be a normal subgroup of a subgroup $G$ of $\hat{\text{Diff}}(\mathbb{C}^n,0)$. Suppose that $G/H$ is virtually solvable and $G^u \subset \langle G^u, H \rangle$. Then
$G/\langle G_u, H \rangle$ is finite dimensional. In particular $G/H$ is finite dimensional if and only if $\langle G_u, H \rangle/H$ is finite dimensional.

**Proof.** Since $G^u \subset \langle G_u, H \rangle$, we obtain

\[
\dim (\langle G_u, H \rangle) = \dim (\langle G_u, H \rangle) = \dim (\langle G_u, H \rangle) = \dim (G_u, H).
\]

The group $\langle G_u, H \rangle$ is the closure of $\langle G_u, H \rangle$ in the Krull topology by Proposition [A.2]. This implies $\langle G_u, H \rangle_1 = \langle G_{1,u}, H_1 \rangle$.

Let us show that $\tilde{\pi}_1 : \overline{G}/\langle G_u, H \rangle \to G_1/\langle G_u, H \rangle_1$ is injective. Let $\phi \in G$ such that $\tilde{\pi}_1(\phi \langle G_u, H \rangle) = 1$. We have $\phi_1 \in \langle G_u, H \rangle_1$. Since $\langle G_u, H \rangle_1 = \langle G_{1,u}, H_1 \rangle$ and $H_1$ is normal in $G_1$, $\phi_1$ is of the form $\alpha \beta$ where $\alpha \in G_{1,u}$ and $\beta \in H_1$. Let $\psi$ be an element of $H$ such that $\psi_1 = \beta$. The formal diffeomorphism $\psi^{-1} \circ \phi$ is unipotent and hence contained in $\overline{G}_u$. Since $\overline{G_u} \subset (\langle G^u, H \rangle = \langle G_u, H \rangle$ by Proposition [A.2] and Equation [2], we obtain $\psi^{-1} \circ \phi \in \langle G_u, H \rangle$. It is clear that $\phi$ belongs to $\langle G_u, H \rangle$ since $\psi \in H$. Hence $\tilde{\pi}_1$ is injective. In particular $G/\langle G_u, H \rangle$ is finite dimensional by Proposition [B.2]. More precisely we get $\dim G/\langle G_u, H \rangle = \dim G_1 - \dim \langle G_u, H \rangle_1$. \hspace{1cm} \(\square\)

**Theorem A.3.** Let $H$ be a normal subgroup of a subgroup $G$ of $\widehat{\text{Diff}} (\mathbb{C}^n, 0)$. Suppose that

- $G/H$ is virtually solvable,
- $G/\langle H, G_u \rangle$ is finitely generated in the extended sense and
- either $G_1^*$ or $H_1^*$ is algebraic.

Then $G/\langle G_u, H \rangle$ is finite dimensional.

**Proof.** Let us show $\dim G/\langle G_u, H \rangle < \infty$ under the following hypotheses:

- There exists a normal subgroup $J$ of $G$ such that $J \subset H$ and $G/J$ is virtually solvable,
- $G/\langle H, G_u \rangle$ is finitely generated in the extended sense and
- $J_1^*$ is algebraic.

There exists a subgroup $G_+$ of $G$ such that $\langle H, G_u \rangle \subset G_+$, $\overline{G}_+ = \overline{G}$ and $G_+ / \langle H, G_u \rangle$ is finitely generated. Up to replace $G$ with $G_+$ we can suppose that $G_1 / \langle H, G_u \rangle$ is finitely generated.

Since $G_1^*/J_1^*$ and $\langle G_1^*, J_1 \rangle/J_1$ are virtually solvable, $G_1/J_1$ is also virtually solvable. Consider the natural maps $\tilde{\tau}_1 : \overline{G} \to G_1/J_1$ and $\tilde{\tau}_1 : \overline{G}/\overline{J} \to G_1/J_1$. There exists a connected finite index normal subgroup $S/J_1$ of $G_1 / J_1$. We define $T = (\tilde{\tau}_1)^{-1}(S/J_1) \cap G$. It is a finite index normal subgroup of $G$ containing $J$. Since $S/J_1$ is connected and virtually solvable, it is solvable. In particular $S/J_1$ is triangularizable by Lie-Kolchin’s theorem. We deduce that the elements of the derived
group \((S/J_1)\)' are unipotent. Therefore \(\hat{\tau}_1(T')\) is a subgroup of \(G_1/J_1\) of unipotent elements. Given any \(\phi \in T'\) we have that \(\phi_1\) is of the form \(\alpha \beta\) with \(\alpha \in G_1,u\) and \(\beta \in J_1\). Since \(J_1 = J_1^*\), there exists \(\psi \in J\) such that \(\psi_1 = \beta\). The formal diffeomorphism \(\psi^{-1} \circ \phi\) belongs to \(T_u\). We deduce

\[(3) \quad T' \subset \langle T_u, J \rangle \subset \langle G_u, H \rangle.\]

The extension \(G/\langle T, G_u, H \rangle\) is finite. It is finite dimensional by Lemma 3.3. Since \(\langle T, G_u, H \rangle/\langle G_u, H \rangle\) is a finite index subgroup of the finitely generated group \(G/\langle G_u, H \rangle\), it is finitely generated (cf. [22 Theorem 1.6.11]). Moreover \(\langle T, G_u, H \rangle/\langle G_u, H \rangle\) is abelian by Property (3). Proposition 3.6 implies that \(\langle T, G_u, H \rangle/\langle G_u, H \rangle\) is finite dimensional. Thus \(G/\langle G_u, H \rangle\) is finite dimensional by Proposition 3.1.

By defining \(J = H\) the case where \(H_1^*\) is algebraic is proved. Let us suppose that \(G_1^*\) is algebraic. Let \(M\) be a finite index normal subgroup of \(G\) containing \(H\) and such that \(M/H\) is solvable. There exists a derived group \(M^{(\ell)}\) of \(M\) contained in \(H\). We define \(J = M^{(\ell)}\). Since derived groups are characteristic, \(J\) is a normal subgroup of \(G\). It is clear that \(J \subset H\) and \(G/J\) is virtually solvable. The group \(M_1^*\) is a finite index normal subgroup of \(G_1^*\) and since the latter group is algebraic, \(M_1^*\) is algebraic. Moreover \(J_1^*\) is the \(\ell\)-th derived group of \(M_1^*\). Derived groups of algebraic groups are algebraic (cf. [4][2.3, p. 58]), thus \(J_1^*\) is algebraic.

\[\square\]

References

[1] V.I. Arnol’d. Bounds for Milnor numbers of intersections in holomorphic dynamical systems. In Topological methods in modern mathematics. Proceedings of a symposium in honor of John Milnor’s sixtieth birthday, held at the State University of New York at Stony Brook, USA, June 14–June 21, 1991, pages 379–390. Houston, TX: Publish or Perish, Inc., 1993.

[2] M.Salah Baouendi, Linda Preiss Rothschild, Jörg Winkelmann, and Dimitri Zaitsev. Lie group structures on groups of diffeomorphisms and applications to CR manifolds. Ann. Inst. Fourier, 54(5):1279–1303, 2004.

[3] Gal Binyamini. Finiteness properties of formal Lie group actions. Transform. Groups, 20(4):939–952, 2015.

[4] Armand Borel. Linear algebraic groups, volume 126 of Graduate Texts in Mathematics. Springer-Verlag, New York, second edition, 1991.

[5] Jean Écalle. Théorie itérative: introduction à la théorie des invariants holomorphes. J. Math. Pures Appl. (9), 54:183–258, 1975.

[6] William Fulton. Intersection theory. 2nd ed. Berlin: Springer, 2nd ed. edition, 1998.

[7] Mikhael Gromov. Groups of polynomial growth and expanding maps. Appendix by Jacques Tits. Publ. Math., Inst. Hautes Étud. Sci., 53:53–78, 1981.
[8] James E. Humphreys. *Linear algebraic groups*. Springer-Verlag, New York, fourth printing, revised edition, 1995. Graduate Texts in Mathematics, No. 21.

[9] Yulij Ilyashenko and Sergei Yakovenko. *Lectures on analytic differential equations*, volume 86 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2008.

[10] Kenkichi Iwasawa. On some types of topological groups. *Ann. Math. (2)*, 50:507–558, 1949.

[11] M. I. Kargapolov and Ju. I. Merzljakov. *Fundamentals of the theory of groups*, volume 62 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1979. Translated from the second Russian edition by Robert G. Burns.

[12] Frank Loray. Pseudo-groupe d’une singularité de feuilletage holomorphe en dimension deux. https://hal.archives-ouvertes.fr/hal-00016434, 2006.

[13] A.I. Mal’tsev. On the theory of the Lie groups in the large. *Mat. Sb., Nov. Ser.* 16:163–190, 1945.

[14] Mitchael Martelo and Javier Ribón. Derived length of solvable groups of local diffeomorphisms. *Mathematische Annalen*, 358(3):701–728, 2014.

[15] Jean Martinet. *Normalisation des champs de vecteurs holomorphes (d’après A. D. Brjuno)*, volume 901 of *Lecture Notes in Math.* Springer, Berlin-New York, 1981.

[16] Jean Martinet and Jean-Pierre Ramis. Classification analytique des équations différentielles non linéaires résonnantes du premier ordre. *Ann. Sci. École Norm. Sup.*, 4(16):571–621, 1983.

[17] Juan J. Morales-Ruiz, Jean-Pierre Ramis, and Carles Simo. Integrability of Hamiltonian systems and differential Galois groups of higher variational equations. *Ann. Sci. École Norm. Sup. (4)*, 40(6):845–884, 2007.

[18] A.L. Onishchik and E.B. Vinberg. *Lie groups and algebraic groups*. Translated from the Russian by D. A. Leites. Berlin etc.: Springer-Verlag, 1990.

[19] Javier Ribón. The solvable length of groups of local diffeomorphisms. Preprint arXiv:1406.0902. DOI 10.1515/crelle-2016-0066. Accepted for publication in Crelle’s Journal.

[20] Javier Ribón. Embedding smooth and formal diffeomorphisms through the Jordan-Chevalley decomposition. *J. Differential Equations*, 253(12):3211–3231, 2012.

[21] Javier Ribón. Algebraic properties of groups of complex analytic local diffeomorphisms. In *VIII Escuela doctoral intercontinental de matemáticas*, pages 185–230. Pontificia Universidad Católica del Perú, 2015.

[22] Derek J.S. Robinson. *A course in the theory of groups*. 2nd ed. New York, NY: Springer-Verlag, 2nd ed. edition, 1995.

[23] Anna Leah Seigal and Sergei Yakovenko. Local dynamics of intersections: V. I. Arnold’s theorem revisited. *Israel J. Math.*, 201(2):813–833, 2014.

[24] Jean-Pierre Serre. *Lie algebras and Lie groups*, volume 1500 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, second edition, 1992. 1964 lectures given at Harvard University.

[25] M. Shub and D. Sullivan. A remark on the Lefschetz fixed point formula for differentiable maps. *Topology*, 13:189–191, 1974.

[26] Th. Skolem. Ein Verfahren zur Behandlung gewisser exponentieller Gleichungen und diophantischer Gleichungen. 8. Skand. Mat.-Kongr., 163-188 (1935)., 1935.

[27] J. Tits. Free subgroups in linear groups. *J. Algebra*, 20:250–270, 1972.
[28] S. Togo. On Cartan subgroups of linear groups. *J. Sci. Hiroshima Univ., Ser. A-I*, 25:63–93, 1961.

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