Analyzing Distributed Join-Idle-Queue: 
A Fluid Limit Approach

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Abstract

In the context of load balancing, Lu et al. [2] introduced the distributed Join-Idle-Queue algorithm, where a group of dispatchers distribute jobs to a cluster of parallel servers. Each dispatcher maintains a queue of idle servers; when a job arrives to a dispatcher, it sends it to a server on its queue, or to a random server if the queue is empty. In turn, when a server has no jobs, it requests to be placed on the idle queue of a randomly chosen dispatcher.

Although this algorithm was shown to be quite effective, the original asymptotic analysis in [2] makes simplifying assumptions that become increasingly inaccurate as the system load increases. Further, the analysis does not naturally generalize to interesting variations, such as having a server request to be placed on the idle queue of a dispatcher before it has completed all jobs, which can be beneficial under high loads.

We provide a new asymptotic analysis of Join-Idle-Queue systems based on mean field fluid limit methods, deriving families of differential equations that describe these systems. Our analysis avoids previous simplifying assumptions, is empirically more accurate, and generalizes naturally to the variation described above, as well as other simple variations. Our theoretical and empirical analyses shed further light on the performance of Join-Idle-Queue, including potential performance pitfalls under high load.

I. INTRODUCTION

The Join-Idle-Queue (JIQ) algorithm is an approach designed to effectively approximate Join-the-Shortest-Queue (JSQ) algorithm in large-scale distributed systems. In this setting jobs coming into a system are sent to one of a group of dispatchers; the dispatchers are in turn responsible for sending the job to a server within a cluster of parallel servers for processing. For the Join-the-Shortest-Queue algorithm, each of the dispatchers would have to keep up-to-date load information on all of the servers, which could be undesirable because of the required communication. For the Join-Idle-Queue algorithm, a server informs a dispatcher when it is idle, having completed all of its jobs. The dispatchers each keep a queue of idle servers, dubbed I-queues to distinguish them from queues for jobs. When a job arrives, it sends the job to an idle server from its I-queue, or to a server chosen uniformly at random if its I-queue is empty. Further details on the model are given in Section II.

The goal of JIQ is to provide excellent load balancing – most jobs are immediately sent to an idle queue – while providing scalability for systems of thousands of servers or larger. In particular, the communication requirements of JIQ are low, and importantly the task of determining the load of various servers is not on the critical path when assigning a job. As noted in [2], alternative schemes requiring a job to communicate to determine the load of servers on the arrival of the job can increase the overall response time, and hence may not be suitable for many practical systems. On the theoretical side, the two-layer approach of load balancing idle servers among dispatchers to allow jobs to be load balanced among servers provides interesting challenges for analysis.

JIQ systems were introduced in [2], and analyzed in the large system limit where the number of servers grows to infinity and the ratio between the number of servers and the number of dispatchers is kept fixed. The analysis presented there, however, assumes that there are only idle processors in the dispatchers’ I-queues. This assumption is inaccurate; indeed, as noted in [2], it “is violated when an idle server receives a random arrival.” They argue that idle servers receive such arrivals relatively rarely, and show empirically that their approximation appears accurate (under sufficiently low arrival rates).

Here we provide a more direct mean-field fluid limit approach that provides an accurate analysis of the large system limit without the assumptions of the previous analysis by deriving a family of differential equations that corresponds to the behavior large system limit. Besides yielding a proper analysis of the basic JIQ system

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analyzed in [2], this approach allows straightforward analysis of many generalizations. For example, under high loads, it may be better to have a server be placed on an I-queue when it has one job left (or, more generally, \( k \) jobs left), rather than waiting for the server to become idle, since it may take some time for the I-queue to present the server with a new job. In [2] it is left as an open question to analyze such variations. We generalize our equations to this setting here, as well as other variations, such as using last-come first-served instead of first-come first-served at the I-queues in order to determine which server to send a job. (In contrast to the analysis of [2], we find once simplifying assumptions are removed, such choices can make a small but noticeable difference.)

Finally, in both our theory and our empirical work, we examine some potential pitfalls of JIQ systems. In particular, under high load, there can be insufficiently many idle queues available, causing degradations in performance as the dispatchers distribute jobs randomly. Even when the expected waiting time remains smaller than alternatives, JIQ systems can exhibit significantly higher variance, which may be unsuitable for some workloads. As we show later in the paper, having a server place itself on an I-queue when it has one job left greatly mitigates this problem, and we would recommend this variation when there is any possibility that the system may enter a highly loaded state.

The main limitation of our analysis is that it applies to the setting of exponentially distributed service times and a Poisson arrival process. While generalizations to other service times and arrival processes are theoretically possible by representing them as mixtures of exponentials (see e.g. [4], [3]), or by considering more general partial differential equations (see e.g. [1], [7], [9]) a more general analysis of JIQ for more general service and arrival processes remains an interesting problem.

Also, we emphasize that here our goal is to derive the appropriate equations, examine implications for the use of JIQ systems and their variations, and demonstrate empirically the accuracy of this approach. One can also formally prove convergence mathematically, but this is not our main goal. There appears to be no impediment to utilizing standard techniques for mean field fluid limit analysis of queueing systems (as in [3], [6], [7], [8]) for JIQ systems, so we leave this aspect to the inclined reader.

II. Preliminaries: Model and Notation

Here we follow much of the model and notation of [2]. We have a system with \( n \) servers and \( m \) dispatchers (or equivalently I-queues), where we fix the ratio \( r = n/m \) and let \( n, m \to \infty \). Jobs arrive as a Poisson process of total rate \( n\lambda \), where \( \lambda < 1 \), and are sent to a dispatcher chosen uniformly at random, so at each dispatcher jobs arrive as a Poisson process of rate \( n\lambda/m \). The service times are exponentially distributed with mean 1. Each dispatcher has an I-queue, which is an ordered list of servers. If the I-queue is non-empty, the dispatcher sends the arriving job to a server on the I-queue; we start by considering the I-queues being governed by the first-come first-served (FCFS) discipline. If the I-queue is empty, the dispatcher sends the job to a server chosen uniformly at random. A server is placed on an I-queue when it finishes all of its jobs (so its job queue is empty) and it is not already on an I-queue, so that at any time it is at most on only a single I-queue. We are assuming here that the dispatcher notes when sending a job whether it was randomly assigned from an empty I-queue, so that a server can track when it is removed from an I-queue, but such an implementation is straightforward.

There are multiple ways in which a server can choose which I-queue to join. In [2], two natural variations are studied. First, on each completion a server can join an I-queue chosen uniformly at random; this is referred to as JIQ-Random. Alternatively, a server can probe the length of \( d \) I-queues and choose the one with the smallest number of jobs. This approach of using a constant number of choices for placement has been studied in many settings, in particular in the context where jobs choose one of a collection of parallel servers for service on arrival, and is referred to as SQ\((d)\). (The approach is sometimes referred to as the power of two choices, or balanced allocations [3], [4], [5]. The variation where jobs arrive to parallel servers and choose one according to SQ\((d)\) is often referred to as the “supermarket model” [3].) In the context of JIQ, this variation is referred to as JIQ-SQ\((d)\). Using SQ\((d)\) to distribute idle servers among I-queues helps ensure that I-queues do not empty, avoiding the need to send incoming jobs to random servers instead of idle ones.
III. The JIQ-Random Model

A. Derivation of Limiting Equations for the JIQ-Random Model

In the equations that follow, we consider the evolution of I-queues and servers. Recall $m$ is the numbers of I-queues, $n$ the number of servers, and $r = n/m$. As done in [2], we consider the regime where $r$ is held constant and $n$ and $m$ are growing to infinity. We use $Q_i(t)$ to represent the number of I-queues with $i$ queued servers, and $q_i(t) = Q_i(t)/m$ be the corresponding fraction of I-queues with $i$ queued servers. For $i \geq 0$ and $j \geq 1$, we use $S_{i,j}(t)$ to represent the number of servers with $i$ jobs that are in position $j$ in one of the I-queues, and similarly we use $s_{i,j}(t)$ to represent the corresponding fraction of servers. Finally, we use $S_{i,\varnothing}(t)$ to represent the number of servers with $i$ jobs that are not currently in an I-queue, and similarly we use $s_{i,\varnothing}(t)$ to represent the corresponding fraction of servers.

To begin, consider a short time interval $dt$, and consider the change in $Q_i$ over that time interval. We use $\Delta Q_i(t)$ to denote $E[Q_i(t + dt) - Q_i(t)]$. Then for suitably short time intervals, we take the probability of an arrival at the I-queues to be $\lambda ndt$ and the total probability of a service event at servers to be $ndt$. Then for $i \geq 1$,

$$\Delta Q_i(t) = \frac{Q_{i+1}(t) - Q_i(t)}{m} \lambda ndt - \frac{Q_i(t) - Q_{i-1}(t)}{m} S_{i,\varnothing}(t) dt.$$  

This equation represents that $Q_i$ increases when an arrival occurs at a I-queue with $i + 1$ servers, and correspondingly decreases when an arrival occurs at a I-queue with $i$ servers. Similarly, $Q_i$ increases when a server not already queued completes its last job and is placed on a I-queue with $i - 1$ servers, and correspondingly decreases when a server not already queued completes its last job and is placed on a I-queue with $i - 1$ servers. Dividing through by $dt$ and using $r = n/m$ yields

$$\frac{\Delta Q_i(t)}{dt} = \lambda r (Q_{i+1}(t) - Q_i(t)) - r (Q_i(t) - Q_{i-1}(t)) s_{1,\varnothing}(t).$$

Dividing through by $m$, and considering the fluid limit, where we replace the expected change over time with the corresponding differential equation, we find

$$\frac{dq_i}{dt} = \lambda r (q_{i+1}(t) - q_i(t)) - r (q_i(t) - q_{i-1}(t)) s_{1,\varnothing}(t).$$  

(1)

For $i = 0$ we have the corresponding equation

$$\frac{dq_0}{dt} = \lambda r q_1(t) - r q_0 s_{1,\varnothing}(t).$$  

(2)

Note that there is no $-\lambda r q_0(t)$ term, since an arrival at an empty I-queue is immediately dispatched to a random server, leaving the I-queue empty.

We now consider the $s_{i,j}$. We use $\Delta S_{i,j}(t)$ to denote $E[S_{i,j}(t + dt) - S_{i,j}(t)]$, where $j$ can include the value $\varnothing$. For $i \geq 1$ and $j \geq 1$, we find

$$\Delta S_{i,j}(t) = (S_{i+1,j}(t) - S_{i,j}(t)) dt + (\lambda ndt) q_0(t) (s_{i-1,j}(t) - s_{i,j}(t)) + (\lambda ndt) S_{i,j+1}(t) - S_{i,j}(t).$$

The first term, $(S_{i+1,j}(t) - S_{i,j}(t)) dt$, represents the change in $S_{i,j}(t)$ due to the completion of jobs at the servers, changing the number of jobs at the server. The second term is the probability that a new job arrives to an empty I-queue, and then is sent randomly to a server with either $i - 1$ or $i$ jobs that is $j$th on its I-queue. The last term represents the change in $S_{i,j}(t)$ due to an arrival that reduces the I-queue length at one of the I-queues holding a server that contributes to $S_{i,j}(t)$ or $S_{i,j+1}(t)$. This equation simplifies in the fluid limit to

$$\frac{ds_{i,j}(t)}{dt} = s_{i+1,j}(t) - s_{i,j}(t) + \lambda q_0(t) (s_{i-1,j}(t) - s_{i,j}(t)) + \lambda r (s_{i,j+1}(t) - s_{i,j}(t)).$$  

(3)

The case where the number of jobs at the server is 0 is similar:

$$\frac{ds_{0,j}(t)}{dt} = s_{0,j}(t) - \lambda q_0(t) s_{0,j}(t) + s_{1,\varnothing}(t) q_{j-1}(t) + \lambda r (s_{0,j+1}(t) - s_{0,j}(t)).$$  

(4)
Here the term $s_{1,0}q_{j-1}$ corresponds to servers with one job and not on an I-queue finishing their job and then joining an I-queue with $j - 1$ other servers.

Finally, we also need to handle servers that are enqueued. For $i \geq 2$,  
\[
\frac{ds_{i,0}(t)}{dt} = s_{i+1,0}(t) - s_{i,0}(t) + \lambda q_0(t)(s_{i-1,0}(t) - s_{i,0}(t)) + \lambda rs_{i-1,1}(t). \tag{5}
\]

Here the first terms $s_{i+1,0}(t) - s_{i,0}(t)$ correspond to job completions at servers with $i + 1$ or $i$ jobs that are not on an I-queue; the next terms $\lambda q_0(t)(s_{i-1,0}(t) - s_{i,0}(t))$ correspond to an arrival at an empty I-queue that lands at a server not on an I-queue; and the term $\lambda rs_{i-1,1}(t)$ corresponds to an arrival at an I-queue where the job at the front goes from having $i - 1$ jobs to $i$ jobs.

Similarly, for $i = 1$,  
\[
\frac{ds_{1,0}(t)}{dt} = s_{2,0}(t) - s_{1,0}(t) - \lambda q_0(t)s_{1,0}(t) + \lambda rs_{0,1}(t). \tag{6}
\]

\[\text{B. Equilibrium in the Limiting JIQ-Random Model}\]

An equilibrium distribution for the above system of equations satisfies that all the derivatives equal 0. We show how to find the equilibrium distribution; let us use $\bar{s}_{i,j}$ and $\bar{q}_i$ for the corresponding values in equilibrium. Our process will be to show that all values $\bar{s}_{i,j}$ can be written in terms of $\bar{s}_{1,\varnothing}$, and then that there is a unique value of $\bar{s}_{1,\varnothing}$ that can be found numerically that satisfies equilibrium conditions.

Various equilibrium conditions arise from first principles and from the equations above. For example, in an equilibrium, $\sum_{j \geq 1} \bar{s}_{0,j} = 1 - \lambda$, since the fraction of idle queues must be $1 - \lambda$.

Equations (1) and (2) yield  
\[
\bar{q}_{i+1} = \bar{q}_i \frac{\bar{s}_{1,\varnothing}}{\lambda};
\]

in an equilibrium $\bar{s}_{1,\varnothing} < \lambda$, since $\lambda$ is the total fraction of servers that are not idle. It follows that $\bar{s}_{1,\varnothing} = \lambda(1 - \bar{q}_0)$.

The final key equation for determining the equilibrium distribution is the following: for $i \geq 0$ and $j \geq 1$,  
\[
\bar{s}_{i,j+1} = \bar{s}_{i,j} \frac{\bar{s}_{1,\varnothing}}{\lambda}.
\]

To see this, consider a server with $i$ jobs and $j$th in queue at some time $t$. The last time it chose an I-queue, suppose instead it had chosen an I-queue with length one larger than the one it actually chose, and then consider coupling all subsequent events except for interchanging the events (arrivals, servers joining) at these two I-queues. From this coupling, and from the fact that the ratio between the equilibrium probability of consecutive I-queue lengths is constant, we see that the relative probability that this server is $j$th in its I-queue instead of $(j + 1)$st is in fact the ratio between the equilibrium probability of consecutive queue lengths, $\bar{s}_{1,\varnothing}/\lambda$.

We then have  
\[
1 - \lambda = \sum_{j \geq 1} \bar{s}_{0,j} = \sum_{k \geq 0} \bar{s}_{0,1} \left( \frac{\bar{s}_{1,\varnothing}}{\lambda} \right)^k = \bar{s}_{0,1} \frac{\lambda}{\lambda - \bar{s}_{1,\varnothing}}.
\]

Hence  
\[
\bar{s}_{0,1} = \frac{(\lambda - \bar{s}_{1,\varnothing})(1 - \lambda)}{\lambda},
\]

and  
\[
\bar{s}_{0,j} = \frac{(\lambda - \bar{s}_{1,\varnothing})(1 - \lambda)s_{1,\varnothing}^{j-1}}{\lambda^j}.
\]
From Equations (6) and (5), we can now solve for the values $\bar{s}_{i,0}$. Setting the derivative to 0 in Equation (6) yields
\[
\bar{s}_{2,0} = \bar{s}_{1,0} + \lambda q_0 \bar{s}_{1,0} - \lambda r \bar{s}_{0,1} \\
= \bar{s}_{1,0} (1 + \lambda q_0) - r (\lambda - \bar{s}_{1,0}) (1 - \lambda) \\
= \bar{s}_{1,0} (1 + r (1 - \lambda) + \lambda - \bar{s}_{1,0}) - r \lambda (1 - \lambda).
\]
Similarly, setting the derivative to 0 in Equation (5) yields
\[
\bar{s}_{i+1,0} = \bar{s}_{i,0} - \lambda q_0 (\bar{s}_{i-1,0} - \bar{s}_{i,0}) - \lambda r \bar{s}_{i-1,1},
\]
and inductively we can solve for all $\bar{s}_{i+1,0}$ in terms of $\bar{s}_{1,0}$.

Equation (4) allows us to find $\bar{s}_{1,1}$ in terms of $\bar{s}_{1,0}$ by looking at when $j = 1$ and setting the derivative to 0:
\[
\bar{s}_{1,1} = \lambda q_0 (t) \bar{s}_{0,1} - \bar{s}_{1,0} q_0 - \lambda r (\bar{s}_{0,2} - \bar{s}_{0,1}).
\]
Equation (4) then again allows us to find the remaining values of $\bar{s}_{i,j}$ inductively, in terms of $\bar{s}_{1,0}$.

We have tested these equations by doing a binary search to find the appropriate value of $\bar{s}_{1,0}$ that leads to a solution where the sum of the $\bar{s}_{i,j}$ values equals one, and have found it matches the results found in our empirical evaluation below in Section III-C. We have not, to this point, been able to prove formally that there is unique fixed point equilibrium to these equations, although our testing of these equations thus far suggests that this is the case. We have also not yet found a way to express the equilibrium in a convenient form in terms of $\lambda$.

C. Empirical Verification of the Equations for the JIQ-Random Model

To test the accuracy of our equations, we ran a simulation of the actual queueing system. In our results below (and throughout the paper), we started at an empty state and ran for 10000 units of time, with 10000 servers and 1000 dispatchers, so that $r = 10$. (We chose $r = 10$ as this was the primary setting used in simulations in [2].) We focus on the average time in system for a job for convenience, although our simulations show more generally that our equations also track the load distribution of servers and I-queues accurately as well. We report the average time for simulations for jobs that complete after time 5000 and before time 10000; this gives the system time to reach approximate equilibrium, for suitable comparison. In reporting the average time in system, we take the average of 1000 trials; we note the standard deviation over experiments is small. The results from our family of differential equations are derived from the average server load and Little’s Law. We started the differential equations with all I-queues empty and all servers with one job initially for convenience. We used the Euler method with a step size of 0.01; experiments with other step sizes led to slightly different results for the average time in system but we found the differences were 0.01% or less; we therefore felt the step size was appropriate for presenting results.

Finally, we also compare with the results from [2]. Specifically, for the case of exponentially distributed service times and Poisson arrivals, based on their assumptions they derive an analytical expression for the average time in system, given by
\[
1 + \frac{\lambda}{(1 - \lambda)(1 + r)}.
\]  

Our results of Table I demonstrate that our fluid limit approach is significantly more accurate, particularly under higher loads, as one might expect given the assumptions use in [2]. In these experiments, the magnitude of our relative error is, in the worst example, about 0.1%, while for the previous approximation the magnitude of the relative error ranges from roughly 1 to 6%.

We have also compared the results from the differential equations with the results we obtain by calculating the equilibrium distribution, and find that they are in agreement to several decimal places; e.g., $s_{1,0}(t)$ and $\bar{s}_{1,0}$ agree for $t = 10000$. The equilibrium distribution calculation does suffer from numerical precision challenges for small values of $\bar{s}_{i,j}$, but we leave this issue for further work.
The differential equation method provides accurate estimates of the average time in system for JIQ-Random at 10000 queues, with relative error of about 0.1% at \( \lambda = 0.99 \) and smaller relative error for lower rates. The results are much more accurate than the still quite good approximation derived in [2].

### IV. Variations of the JIQ Model

In this section, we consider several variations on JIQ that can be modeled using differential equations in a manner similar to our model for JIQ-Random presented in Section III-A. Specifically, we consider having servers place themselves on I-queues before becoming idle, using JIQ-SQ\((d)\), and utilizing other disciplines besides FCFS for the I-queues. One can also naturally combine these variations; here we consider them separately to simplify the exposition. Our focus here is in showing how to generate the relevant equations; we leave additional questions (such as the nature of the equilibrium distribution) for future work.

#### A. Early Assignment of Servers to I-Queues

We may modify the JIQ-Random model to handle servers placing themselves on I-queues before they are idle, such as when they have just one job remaining. Such an approach appears potentially beneficial under very high loads when there are (on average) fewer idle servers than I-queues, since empty I-queues default to the poor strategy of assigning a job to a processor chosen uniformly at random. To make the intuition more concrete, note that when \( r = 10 \) and \( \lambda > 0.9 \), on average less than 1/10 of the servers will be idle, leading to empty I-queues.

Below we modify the equations for the setting where any time a server is not on an I-queue, and its load falls to a threshold \( z \) or below, it places itself on an I-queue. We refer to this as the early threshold variation. While one could also consider a richer framework for I-queues, which either sorted incoming servers by their current load or tracked the load of servers placed on them, here we provide the equations for the simplest generalization.

Deriving a method to analyze this type of strategy was specifically left as an open question in [2]. They examined one such strategy empirically, where their strategy allowed a server to be placed on multiple I-queues. Analyzing strategies where servers can be on multiple I-queues would necessarily complicate our analysis, as then we need to use variables that track the position of the servers in multiple I-queues (e.g., “three-dimensional” variables instead of the “two-dimensional” \( s_{i,j} \)). Although such analysis is possible, we consider here only the natural generalization where a server can only place itself on one I-queue at a time.

The equations for the I-queues remain nearly the same, but now the queue length increases whenever a server not on an I-queue completes servicing a job and has \( z \) or fewer jobs remaining.

\[
\frac{d q_i}{dt} = \lambda r (q_{i+1}(t) - q_i(t)) - r (q_i(t) - q_{i-1}(t)) \sum_{k=1}^{z+1} s_{k,0}(t).
\]

\[
\frac{d q_0}{dt} = \lambda r q_1(t) - r q_0 \sum_{k=1}^{z+1} s_{k,0}(t)
\]
The equations relating to $s_{i,j}$ do not change for $i > z$; but for $i \leq z$, we must consider that a server with $i$ jobs not on an I-queue will be placed on an I-queue on a job completion. Hence for $i > z, j \geq 1$, we have

$$\frac{ds_{i,j}(t)}{dt} = s_{i+1,j}(t) - s_{i,j}(t) + \lambda q_0(t)(s_{i-1,j}(t) - s_{i,j}(t)) + \lambda r(s_{i,j+1}(t) - s_{i,j}(t)).$$

And for $1 \leq i \leq z, j \geq 1$, we have

$$\frac{ds_{i,j}(t)}{dt} = s_{i+1,j}(t) - s_{i,j}(t) + \lambda q_0(t)(s_{i-1,j}(t) - s_{i,j}(t)) + s_{i+1,\emptyset}(t)q_{j-1}(t) + \lambda r(s_{i,j+1}(t) - s_{i,j}(t)).$$

Finally, for $i = 0, j \geq 1$ we have

$$\frac{ds_{0,j}(t)}{dt} = s_{1,j}(t) - \lambda q_0(t)s_{0,j}(t) + s_{1,\emptyset}(t)q_{j-1}(t) + \lambda r(s_{0,j+1}(t) - s_{0,j}(t)).$$

For servers not on an I-queue, the equations remain the same for $i > z$. For $i \leq z$, however, we must now account for the fact that a server with $i$ jobs not on an I-queue will be placed on an I-queue on a job completion. Hence for $i > z, j = \emptyset$ we have

$$\frac{ds_{i,\emptyset}(t)}{dt} = s_{i+1,\emptyset}(t) - s_{i,\emptyset}(t) + \lambda q_0(t)(s_{i-1,\emptyset}(t) - s_{i,\emptyset}(t)) + \lambda rs_{i-1,1}.$$

For $1 \leq i \leq z, j = \emptyset$ we have

$$\frac{ds_{i,\emptyset}(t)}{dt} = -s_{i,\emptyset}(t) + \lambda q_0(t)(s_{i-1,\emptyset}(t) - s_{i,\emptyset}(t)) + \lambda rs_{i-1,1}.$$

In Table II we examine this variation for the setting $z = 1$, again comparing simulation results to the differential equations, as well as to the simulation results for JIQ-Random. The relative error is slightly higher, but the results are again extremely accurate. The key insight, beyond again the accuracy of the fluid limit approach, is that under high load performance improves dramatically by allowing non-idle servers to queue at I-queues. Of course this advantage would be less important for a larger $r$. We have also tried $z = 2$, and obtained slightly better performance than $z = 1$ at $\lambda = 0.99$, but worse performance at lower values of $\lambda$.

### B. Varying the I-Queue Discipline

Intuitively, we might think that first-come first-served might not be the best queuing discipline for the I-Queues, at least under higher loads. This is because servers that have been waiting in I-Queues may have received randomly assigned jobs and may no longer be idle; queues that have just arrived to an I-queue may be more likely to be idle.

| $\lambda$ | Sims | Diff Equations | JIQ-Random |
|----------|------|---------------|------------|
| 0.50     | 1.19356 | 1.19369 | 1.12886  |
| 0.60     | 1.28239 | 1.28254 | 1.17987  |
| 0.70     | 1.40813 | 1.40828 | 1.25888  |
| 0.80     | 1.59780 | 1.59796 | 1.40787  |
| 0.90     | 1.94387 | 1.94377 | 1.83712  |
| 0.95     | 2.33613 | 2.33539 | 2.68138  |
| 0.96     | 2.48165 | 2.48041 | 3.10314  |
| 0.97     | 2.69113 | 2.68903 | 3.80509  |
| 0.98     | 3.04536 | 3.04146 | 5.20555  |
| 0.99     | 3.91664 | 3.90627 | 9.40217  |

**TABLE II**

Comparing the average time in system for JIQ-Random with an early threshold of 1 with results from the differential equations and JIQ-Random. The differential equations yield accurate results, and under high rates performance improves significantly.
Our fluid limit approach allows us to write equations for at least some other I-queue disciplines. Here we analyze last-come first-served (LCFS), with a slight change to our equations and interpretations of the variables. (We can similarly analyze the process where I-queues choose a server in their queue uniformly at random for an arriving job, for example.) Here for $i \geq 0$ and $j \geq 1$, we use $S_{i,j}(t)$ to represent the number of servers $j$th from the front of an I-queue and with $i$ jobs; with LCFS when a server joins an I-queue it is placed at the front of the I-queue and all other jobs move one step further from the front. We use $s_{i,j}(t)$ to represent the corresponding fraction of servers. We use $\hat{S}_{i,\varnothing}(t)$ to represent the number of servers with $i$ jobs that are not currently in an I-queue, and use $s_{i,\varnothing}(t)$ to represent the corresponding fraction. The analysis is similar to that in Section III-A, so we simply provide the final equations.

\[
\frac{dq_i}{dt} = \lambda r(q_{i+1}(t) - q_i(t)) - r(q_i(t) - q_{i-1}(t))s_{1,\varnothing}(t).
\]

\[
\frac{dq_0}{dt} = \lambda r q_1(t) - r q_0 s_{1,\varnothing}(t).
\]

For $i \geq 1$ and $j \geq 2$,

\[
\frac{ds_{i,j}(t)}{dt} = s_{i+1,j}(t) - s_{i,j}(t) + \lambda q_0(t)(s_{i-1,j}(t) - s_{i,j}(t)) + \lambda r(s_{i,j+1}(t) - s_{i,j}(t)) + rs_{1,\varnothing}(t)(s_{i,j-1}(t) - s_{i,j}(t)).
\]

For $i \geq 1$ and $j = 1$,

\[
\frac{ds_{i,1}(t)}{dt} = s_{i+1,1}(t) - s_{i,1}(t) + \lambda q_0(t)(s_{i-1,1}(t) - s_{i,1}(t)) + \lambda r(s_{i,2}(t) - s_{i,1}(t)) + rs_{1,\varnothing}(t)s_{i,1}(t).
\]

For $i = 0$ and $j \geq 2$,

\[
\frac{ds_{0,j}(t)}{dt} = s_{1,j}(t) - \lambda q_0(t)s_{0,j}(t) + rs_{1,\varnothing}(t)s_{0,j-1} - rs_{1,\varnothing}(t)s_{0,j} + \lambda r(s_{0,j+1}(t) - s_{0,j}(t)).
\]

For $i = 0$ and $j = 1$,

\[
\frac{ds_{0,1}(t)}{dt} = s_{1,1}(t) - \lambda q_0(t)s_{0,1}(t) + \lambda r(s_{0,2}(t) - s_{0,1}(t)) - rs_{1,\varnothing}(t)s_{0,1} + s_{1,\varnothing}.
\]

For $i \geq 2$,

\[
\frac{ds_{i,\varnothing}(t)}{dt} = s_{i+1,\varnothing}(t) - s_{i,\varnothing}(t) + \lambda q_0(t)(s_{i-1,\varnothing}(t) - s_{i,\varnothing}(t)) + \lambda r s_{i-1,1}.
\]

For $i = 1$,

\[
\frac{ds_{1,\varnothing}(t)}{dt} = s_{2,\varnothing}(t) - s_{1,\varnothing}(t) - \lambda q_0(t)s_{1,\varnothing}(t) + \lambda r s_{0,1}.
\]

Table III demonstrates that there is a noticeable if small performance difference between LCFS and FCFS when using JIQ-Random, with the intuition that LCFS would perform better being correct. Moreover, the differential equations are able to accurately capture this small performance gap.

C. JIQ-SQ($d$)

Recall that for JIQ-SQ($d$), a server chooses which I-queue to join by choosing $d$ I-queues at random and joining the least loaded (breaking ties randomly). The changes required to model the use of SQ($d$) for I-queues are minimal. However, it becomes easier utilizing a standard change of notation, to keep track of the tails of the distribution [3], [8]. That is, let $\hat{q}_i(t)$ be the fraction of I-queues with at least $i$ jobs, instead of exactly $i$ jobs, at time $t$. Note $\hat{q}_0(t) = 1$.

To begin, to see the change of notation, for the JIQ-Random model, we have for $i \geq 1$:

\[
\frac{d\hat{q}_i}{dt} = -\lambda r(\hat{q}_i(t) - \hat{q}_{i+1}(t)) + r(\hat{q}_{i-1}(t) - \hat{q}_i(t))s_{1,\varnothing}(t).
\]
TABLE III

Comparing average time in system for JIQ-Random with LCFS against FCFS I-queues. The differential equations capture the performance difference between the two approaches.

| λ   | Sims (LCFS) | Diff Equations (LCFS) | JIQ-Random (FCFS) |
|-----|-------------|-----------------------|-------------------|
| 0.50 | 1.10976     | 1.10980               | 1.12886           |
| 0.60 | 1.15732     | 1.15379               | 1.17987           |
| 0.70 | 1.23305     | 1.23310               | 1.25888           |
| 0.80 | 1.37805     | 1.37796               | 1.40787           |
| 0.90 | 1.79941     | 1.79893               | 1.83712           |
| 0.95 | 2.63751     | 2.63559               | 2.68138           |
| 0.96 | 3.05663     | 3.05429               | 3.10314           |
| 0.97 | 3.75659     | 3.75259               | 3.80509           |
| 0.98 | 5.15449     | 5.15006               | 5.20555           |
| 0.99 | 9.35407     | 9.34465               | 9.40217           |

TABLE IV

Comparing average time in system for JIQ-SQ(2) and JIQ-Random. The differential equations yield accurate results, and there is an improvement over JIQ-Random, notably under lower loads.

| λ   | Sims JIQ-SQ(2) | Diff Equations | JIQ-Random |
|-----|----------------|---------------|------------|
| 0.50 | 1.01029        | 1.01033       | 1.12886    |
| 0.60 | 1.02377        | 1.02379       | 1.17987    |
| 0.70 | 1.05558        | 1.05557       | 1.25888    |
| 0.80 | 1.14044        | 1.14027       | 1.40787    |
| 0.90 | 1.46106        | 1.46035       | 1.83712    |
| 0.95 | 2.19241        | 2.19045       | 2.68138    |
| 0.96 | 2.57696        | 2.57420       | 3.10314    |
| 0.97 | 3.23186        | 3.22894       | 3.80509    |
| 0.98 | 4.57810        | 4.57186       | 5.20555    |
| 0.99 | 8.71553        | 8.70009       | 9.40217    |

For the JIQ model using SQ(d) for I-queues, all that changes from the I-queue standpoint is when a server finishes, the probability a server lands on an I-queue with \( i - 1 \) other jobs becomes \((\hat{q}_{i-1}(t))^d - (\hat{q}_i(t))^d)\). Hence for the generalization we have

\[
\frac{d\hat{q}_i}{dt} = -\lambda r(\hat{q}_i(t) - \hat{q}_{i+1}(t)) + r((\hat{q}_{i-1}(t))^d - (\hat{q}_i(t))^d)s_{1,\varnothing}(t).
\]

Correspondingly, from the server side, the only change is to the equation for \( \frac{ds_{0,j}(t)}{dt} \)

\[
\frac{ds_{0,j}(t)}{dt} = s_{1,j}(t) - \lambda q_0(t)s_{0,j}(t) + s_{1,\varnothing}(t)((\hat{q}_{j-1}(t))^d - (\hat{q}_j(t))^d) + \lambda r(s_{0,j+1}(t) - s_{0,j}(t)).
\]

Notice that for convenience we have used the expression \( q_0(t) \) in the equations, although one could write all equations in terms of the \( \hat{q}_i(t) \) as needed.

The results of Table IV will be perhaps at this point unsurprising. The differential equations prove quite accurate, and the performance JIQ-SQ(d) is better than JIQ-Random. However, under high loads the improvements are arguably small because the key issue that there are not enough idle servers to fill the I-queues remains. Distributing the small number of idle servers better does not address this issue as well as the early threshold approach, which increases the number of servers being placed on I-queues.
In [2], the authors compare JIQ systems with supermarket model, or SQ\((d)\), systems, where there are no I-queues; instead, each incoming job probes the load at \(d\) randomly chosen servers, and is assigned to the least loaded server from its \(d\) choices. They note that JIQ systems often perform better, in terms of average time in system. This is true when the system load is sufficiently low, as most jobs are assigned to idle servers. However, under very high load, when I-servers are often empty, we see that the default to random assignment can significantly increase average time in system, so that it becomes worse than SQ\((d)\). (This phenomenon motivated our examination of the early threshold variation.)

Even when JIQ systems perform better in terms of average time in system, however, the variance of JIQ systems can be higher than comparable SQ\((d)\) systems. In particular there are likely to be some number of “unlucky” servers that receive a large number of randomly assigned jobs, and thereby find themselves with long queues. For example, in all of the JIQ systems, an arrival is assigned randomly with probability \(q_0(t)\), and hence if \(\bar{q}_0\) represents an equilibrium probability of an empty I-queue, servers obtain randomly assigned arrivals at a rate of \(\lambda \bar{q}_0\). As such, in JIQ systems the tails of queues fall at most geometrically, while in SQ\((d)\) systems they are known to fall doubly exponentially [3], [8]; for example, even with a supermarket model with just two choices, the fraction of queues with load \(j\) or greater falls like \(\lambda^{2^{j-1}}\), in the asymptotic limit as the system grows large. The maximum loaded server is therefore likely to have significantly more jobs in JIQ systems under high loads.

These results are apparent in both the results of the fluid limit models and in simulations. Here we present simulation results for convenience in Table V, although the results are entirely similar using the results from the differential equations to obtain the variance for an arriving job at time 10000 (when the system is essentially at equilibrium). We compare SQ\((2)\) with JIQ-Random, and with JIQ-Random with a threshold of one. The high variance of JIQ-Random is apparent, as well as its poor relative performance under very high loads. The early threshold greatly mitigates these flaws with JIQ, although under sufficiently high loads even with the early threshold the variance increases beyond SQ\((2)\), as expected.

Table V is not the entire story. When we look at the maximum load over our simulation, we also see the performance difference clearly, even with an early threshold. For SQ\((2)\), the maximum load at the end time 10000 over all of our simulations when \(\lambda = 0.99\) is 10, and when \(\lambda = 0.8\) it is 6. For JIQ-Random with an early threshold \(z = 1\), the maximum load at the end time of our simulation when \(\lambda = 0.99\) is always over 20, and is sometimes over 30; when \(\lambda = 0.8\), it is 11.

We emphasize that under larger values of \(r\), which may be suitable for real-world systems, these concerns are lessened, as higher arrival rates are needed for there to be large numbers of empty I-queues.

| \(\lambda\) | SQ\((2)\) Avg | JIQ\(z = 1\) Avg | JIQ-Random Avg | SQ\((2)\) Var | JIQ\(z = 1\) Var | JIQ-Random Var |
|---|---|---|---|---|---|---|
| 0.50 | 1.26572 | 1.19356 | 1.12886 | 1.49234 | 1.39094 | 1.26742 |
| 0.60 | 1.40747 | 1.28239 | 1.17987 | 1.74541 | 1.56715 | 1.39094 |
| 0.70 | 1.61446 | 1.40813 | 1.25888 | 2.11398 | 1.81205 | 1.56210 |
| 0.80 | 1.94744 | 1.59780 | 1.40787 | 2.70517 | 2.17632 | 1.93743 |
| 0.90 | 2.61442 | 1.94387 | 1.83712 | 3.86808 | 2.86939 | 3.26202 |
| 0.95 | 3.38375 | 2.33613 | 2.68138 | 5.14687 | 3.81179 | 6.94056 |
| 0.96 | 3.64960 | 2.48165 | 3.10314 | 5.56989 | 4.22673 | 9.31200 |
| 0.97 | 4.00222 | 2.69113 | 3.80509 | 6.11533 | 4.89627 | 14.0488 |
| 0.98 | 4.51796 | 3.04536 | 5.20555 | 6.88176 | 6.23157 | 26.4497 |
| 0.99 | 5.43592 | 3.91664 | 9.40217 | 8.16057 | 10.6235 | 87.1031 |

**TABLE V**

Comparing JIQ-Random and JIQ-Random with an early threshold of 1 with SQ\((d)\), for both the average and the variance of the time in system. The early threshold reduces both the average and variance under high load, although under high enough loads even the early threshold version has higher variance than SQ\((d)\).
VI. Conclusion

We have derived fluid limit equations that describe the distributed Join-Idle-Queue process under Poisson arrivals and exponentially distributed service times, showing that these systems are amenable to asymptotic analysis. Our methodology allows us to obtain very accurate estimates for reasonably sized finite systems. By examining variations, we have shown the generality of this method, and in particular we have analyzed the compelling approach of serving I-queues while still processing one or more jobs. Based on its reasonable performance under low loads and its significantly better performance in terms of both expectation and variance under higher loads, we recommend strongly that builders of systems that want to use the distributed JIQ approach examine the use of an early threshold.

On the theoretical side, for many of the JIQ variations we have examined, it is worthwhile to consider whether the asymptotic equilibrium distributions have a useful or pleasant form that could aid in rapidly calculating system performance or making comparisons among system. We leave this for future work.

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