KdV shock-like waves as invariant solutions of KdV equation symmetries

Vadim R. Kudashev
Institute of Mathematics,
Chernyshevskii 112,
Ufa, 450000,
Russia
E-mail: vadkud@nkc.bashkiria.su

December 30, 1993

Abstract
We consider the following hypothesis: some of KdV equation shock-like waves are invariant with respect to the combination of the Galilean symmetry and KdV equation higher symmetries. Also we demonstrate our approach on the example of Burgers equation.

1 Introduction

1.1 History of a problem [1-14]
In [1] Gurevich and Pitaevskii (G-P) one of the first have formulated the Korteweg-de Vries (KdV) equation shock-like waves problem: "to find solution of KdV equation

\[ u_t + DRu = u_t + uu_x + u_{xxx} = 0, \quad u = u(x,t), \]
\[ R = D^2 + 2u/3 - D^{-1}u_x/3, \quad D \equiv d/dx, \]

which has asymptotic behavior

\[ u^3 = tu - x, \quad \text{as } x \to \infty. \]  

In [2] Gurevich and Pitaevskii have supposed and investigated hypothesis: the first term of asymptotic solution of (1), (2) is a continuous combination of the "external" solution (2) and "inner" solution \( u = \varphi(\theta) = \varphi(\theta + 1) : 

\[ \varphi = 2(r_3 - r_1)dn^2(2K(m)\theta; m) + r_1 + r_2 - r_3, \quad x^+(t) \leq x \leq x^-(t), \]
where $\theta_x = \kappa, \theta_t = \omega = -\kappa U, \kappa \equiv (r_3 - r_1)^{1/2}/K(m), U = (r_1 + r_2 + r_3)/3,$ 
$m = (r_2 - r_1)/(r_3 - r_1)$, $dn$ - is the Jacobi elliptic function, $K(m)$ - is the complete elliptic integral of the first type, $r_1 \leq 0$, $r_1 \leq r_2 \leq r_3$, $r_3 \geq 0$, and $r_i$ are governed by the Whitham-KdV (averaged) equations [3] (we write these in form [4,5])

\[
\begin{align*}
    r_{it} &= \varphi^i(r) r_{ix}, \\
    \varphi^i &= (D_i \omega)/(D_i \kappa), \\
    D_i &\equiv d/dr_i,
\end{align*}
\]

with appropriate boundary conditions. There are not rigorous theorems about exact solution of (1), (2), but the non contradictoriness and self-co-ordination of G-P hypothesis were proved in set works [2,5-11]. Note also that closely connected questions were considered in [12].

The next step was made in [13,14] where it was shown that G-P solution is simultaneous solution of non autonomous ordinary differential equation (which is stationary part of KdV symmetry).

1.2 The problem in question

The problem in question is to find solutions of KdV equation (1) which have asymptotic behavior

\[
    u^{2n+1} = tu - x, \quad as \ |x| \to \infty, \quad |t| \gg 1, \quad n = 1, 2, ...
\]

(5)

The analogous problem was considered in [4,7-11,15] with help of G-P-like hypothesis for "external" and "inner" behavior of the first term for the asymptotic solution.

Let us consider KdV equation symmetry (i.e. $u_{t\tau} = u_{\tau t}$, see, for example, [16]) which is combination of KdV Galilean symmetry and KdV higher order local symmetry

\[
\begin{align*}
    u_{\tau} &= (1 - tu_x) + \gamma_m DR^m u, \quad m = 2n, \\
    \gamma_m &= \prod_{k=1}^{m} 3(k+1)/(2k+1), \quad n = 1, 2, ...
\end{align*}
\]

(6)

The hypothesis in question is: solutions of (1), (5) are invariant with respect to symmetry (6) i.e. $u_{\tau} = 0$, or

\[
(1 - tu_x) + \gamma_m DR^m u = 0, \quad m = 2n, \quad n = 1, 2, ...
\]

(7)

The main goal of this Letter is to show the non contradictoriness and self-co-ordination of this hypothesis with help of G-P-like form for the first term of "external" and "inner" asymptotic solution of (1), (5), (7). Also we well demonstrate our approach to problem on the example of Burgers equation.

The importance of invariant solutions is well known. The main question is to find such combination of symmetries which would be solve physically interesting problem. It was "linear, dispersible" approach [17] which was used in
[13,14] to obtain and study equation (7) as \( n = 1 \). In contrast with [13,14] our suggestion is following: for studying the shock-like waves problem it is useful to investigate such combination of symmetries which would be solve problem in the "dispersionless, nonlinear" limit.

2 External and single-phase inner solutions

2.1 "External" solution

Let us rewrite, for convenience, equations (1), (7) as

\[
 u_t + uu_x + \epsilon^2 u_{xxx} = 0,
\]

\[
 (1 - tu_x) + \gamma_mD[\epsilon^2 D^2 + 2u/3 - D^{-1}u_x/3]m u = 0, \quad m = 2n,
\]

where \( \epsilon \) is arbitrary constant. Our main observation: let \( |\epsilon| \ll 1 \), ( the "dispersionless, nonlinear" limit) then for any \( m \) equations (8), (9) have formal external expansion

\[
 u = \sum_{k=0}^{\infty} \epsilon^{2k} u_k(x,t), \quad as \quad x \to \infty, \quad t \gg 1,
\]

with first term equals (5).

2.2 "Inner" solution

In oscillation region we use solution in single-phase G-P-like form (3), where solutions of (4) are defined by Tsarev’s generalized hodograph method [6] \((m = 2n, \quad n = 1, 2, \ldots)\)

\[
 \beta_{2m+3}\psi^i_{2m+3}(r) = \varphi^i(r)t + x, \quad i = 1, 2, 3,
\]

where \( \beta_{2m+3} = 3\gamma_m/(3 + 2m) \) and [4,5,10]

\[
 \psi^i_{2m+3} = (D_i\omega_{2m+3})/(D_i\kappa), \quad \omega_{2m+3} = -\kappa U_{2m+3},
\]

with \( U_{2m+1} \) (compare with [4]):

\[
 LU_{2m+1} \equiv [\sum_k (2r_k^2 D_k + r_k)]U_{2m+1} = 3(m + 1)U_{2m+3}, \quad U_1 = 1.
\]

Note useful expressions \((s = 1, 2, \ldots):\)

\[
 TU_{2s+1} \equiv (\sum_k D_k)U_{2s+1} = (2s + 1)U_{2s-1}/3,
\]

\[
 SU_{2s+1} \equiv (\sum_k r_k D_k)U_{2s+1} = sU_{2s+1}.
\]
As it is known Whitham-KdV equations (4) are averaged equations for (1), (3). Substituting (3) into (7) and averaging over \( \theta \) we obtain the averaged equations for (3), (7) (compare with [4])

\[
(1 - tr_{ix}) - \gamma_m \psi_{2m+1}^i(r) r_{ix} = 0.
\]

(15)

Note that Whitham-KdV equations (4) admit symmetry

\[
r_{ix} = (1 - tr_{ix}) - \gamma_m \psi_{2m+1}^i(r) r_{ix},
\]

(16)

which is combination of Whitham-KdV equations Galilean symmetry and Whitham-KdV equations higher order (nonclassical [16]) local symmetry. Thus solutions of equations (4) which are described by (15) are invariant with respect to symmetry (16).

Now we have questions: have equations (4), (15) the compatible solutions?, what are these solutions?

**Proposition.** Equations (4), (15) are compatible, i.e. if we rewrite (4) as

\[
r_{it} = \varphi^i(r)/(t + \gamma_m \psi_{2m+1}^i(r)),
\]

(17)

and (15) as

\[
r_{ix} = 1/(t + \gamma_m \psi_{2m+1}^i(r)),
\]

(18)

then \( r_{itx} = r_{ixt} \).

**Remark.** This proposition is true for any semihamiltonian [6] systems, \( \psi_i \) such that (compare with [6])

\[
D_k \psi^i = (\psi^k - \psi^i)(D_k \varphi^i)/(\varphi^k - \varphi^i), \quad i \neq k,
\]

(19)

and \( \varphi^i \) such that \( T \varphi^i = -1 \).

**Lemma.**

\[
T \psi_{2s+1}^i(r) = (2s + 1) \psi_{2s-1}^i(r)/3, \quad s = 1, 2, \ldots; \quad T \varphi^i = -1.
\]

(20)

**Theorem.** Generalized hodograph solutions (11) satisfy equations (15).

**Proof.** The total derivative of (11) with respect to \( x \) gives

\[
\sum_k r_{kx} D_k[\beta_{2m+3} \psi_{2m+3}^i(r) - \varphi^i(r)t]\bigg|_{(11)} = 1.
\]

(21)

As it is known [6,9] the matrix in left side of (21) is diagonal

\[
D_k[\beta_{2m+3} \psi_{2m+3}^i(r) - \varphi^i(r)t]\big|_{(11)} = \delta_{ik} D_i[\beta_{2m+3} \psi_{2m+3}^i(r) - \varphi^i(r)t]\big|_{(11)},
\]

(22)
where $\delta_{ik} = 1$ if $i = k$ and $\delta_{ik} = 0$ if $i \neq k$. From Lemma we have

$$T[\beta_{2m+3}\psi_{2m+3}(r) - \varphi'(r)t] = [\gamma_{m}\psi_{2m+1}(r) + t]. \tag{23}$$

Combining (22) and (23) we obtain

$$D_i[\beta_{2m+3}\psi_{2m+3}(r) - \varphi'(r)t]|_{(11)} = [\gamma_{m}\psi_{2m+1}(r) + t]|_{(11)}. \tag{24}$$

Substituting (22), (24) in (21) we obtain

$$r_{ix}[\gamma_{m}\psi_{2m+1}(r) + t]|_{(11)} = 1.$$ This is the same as (15). Thus generalized hodograph solutions (11) satisfy (15).

Remark. The total derivative of (11) with respect to $t$ gives (17). Note that the analogous theorem is formally true for any $\psi^i_\mu$ from (19) and such that $S\psi^i_\mu = \mu\psi^i_\mu$. It is followed from expression $T\psi^{\mu}_\mu = \gamma^{\mu}_{\mu - 1}$, which is followed from commutation $(TS - ST) = T$. The complete set of the appropriate $\psi^i_\mu$ for any $\mu \geq -3/2$ (I $\neq -1/2$) was obtained in [18].

2.3 Statement

From the above we obtain the main statement: the G-P-like first term (i.e. continuous combination of “external” solution (5) and “inner” solution (3), (11)) of asymptotic solution of (1), (5) satisfies (in framework of G-P-like hypothesis) (7), (15) and may be considered as invariant solution with respect to KdV equation symmetry (6). As it is followed from [8,11] if $m = 2n$ then the plot of the multivalued function $\{r^-, r^+\}$ is $C^1$ smooth near $x^-(t)$ and $x^+(t)$ (here $r^-$ is equal to $u$ from (5) as $x \leq x^-(t)$, and $r^+$ is equal to $u$ from (5) as $x \geq x^+(t)$).

3 Exact solvable simplest examples

3.1 KdV equation case

The simplest case of (7) is $(m = 0, \gamma_0 = 1)$

$$(1 - tu_x) + u_x = 0. \tag{25}$$

From (25) we have solution of KdV equation

$$u = -x/(1 - t),$$

with "initial profile" $u(x, 0) = -x$, and with singularity as $t = 1$. 
3.2 Burgers equation case

Now let us consider problem to find shock-wave solutions of Burgers equation

\[ v_t + vv_x - v_{xx} = 0, \]  

(26)

by the above symmetry approach. The first simplest case gives the same solution as (25). The combination of Galilean symmetry and simplest higher symmetry for (26) [16] gives us the following analog of (6)

\[ v_r = D(x - tv) + D(4v_{xx} - 6vv_x + v^3). \]  

(27)

Solution of (26) which is invariant with respect to symmetry (27) is defined by equation

\[ D(x - tv) + D(4v_{xx} - 6vv_x + v^3) = 0. \]  

(28)

Equation (28) has solution

\[ (x - tv) + (4v_{xx} - 6vv_x + v^3) = 0. \]  

(29)

The "external" expansion for (29) has first term

\[ v^3 = tv - x. \]  

(30)

Substituting \( v = -2V_x/V \) in (29) we obtain equation

\[ 8V_{xxxx} - 2tV_x - xV = 0. \]  

(31)

From (31) we obtain the well known [19] solution of problem (26), (30) with help of integral

\[ V = \int_{-\infty}^{+\infty} \exp(-\lambda^2/8 + t\lambda^2/4 - x\lambda/2)d\lambda. \]  

(32)

The combination of Galilean symmetry and fifth order symmetry of (26) [16] gives equation

\[ (x - tv) + (16v_{xxxx} - 40vv_{xxx} - 80v_vv_{xx} + 40v^2v_{xx} + 60v^2_v + 20v^3v_x + v^5) = 0. \]  

(33)

Solution of (33) has external expansion with first term

\[ v^5 = tv - x. \]  

(34)

Substituting \( v = -2V_x/V \) in (33) we obtain

\[ 32V_{xxxxx} - 2tV_x - xV = 0. \]  

(35)

The solution of (27), (33), (34), (35) is governed by integral

\[ V = \int_{-\infty}^{+\infty} \exp(-\lambda^6/12 + t\lambda^2/4 - x\lambda/2)d\lambda. \]
Acknowledgment

I am grateful to B.I.Suleimanov for many stimulating and useful discussions, and to V.E.Adler, I.Yu.Cherdantsev, V.Yu.Novokshenov and V.V.Sokolov for interest in this study. This work was supported, in part, by a Soros Foundation Grant, and by RFFI grant 93-011-16088.

References

[1] A.V.Gurevich and L.P.Pitaevskii, Zh. Eksp. Teor. Fiz. 60 (1971) 2155.
[2] A.V.Gurevich and L.P.Pitaevskii, JETP 38 (1974) 291.
[3] G.B.Whitham, Proc. Roy. Soc. A283 (1965) 238.
[4] V.R.Kudashev and S.E.Sharapov, The inheritance of KdV symmetries under Whitham averaging and hydrodynamic symmetries of the Whitham equations, preprint IAE-5221/6, Moscow (1990) [in Russian]; Teor. Mat. Fiz. 87 (1991) 40.
[5] V.R.Kudashev, JETP Lett. 54 (1991) 175.
[6] S.P.Tsarev, Sov. Math. Dokl. 31 (1985) 488; Izv. Akad. Nauk SSSR, Ser. Mat. 54 (1990) 1048.
[7] I.M.Krichever, Func. Anal. and its Appl. 22 (1988) 37.
[8] G.V.Potemin, Usp. Mat. Nauk 43 (1988) 211.
[9] B.A.Dubrovin and S.P.Novikov, Usp. Mat. Nauk 44 (1989) 29.
[10] F.R.Tian, Comm. Pure Appl. Math. 44 (1993) 1094.
[11] V.R.Kudashev, JETP Lett., 56 (1992) 320.
[12] P.D.Lax and C.D.Levermore, Comm. Pure Appl. Math. 36 (1983) 253, 571, 809; S.Venakides, Comm. Pure Appl. Math. 38 (1985) 883; R.F.Bikbaev and V.Yu.Novokshenov, Proc. III International Workshop, Kiev, Naukova Dumka, 1988, V.1, P.345.
[13] B.I.Suleimanov, Pis’ma Zh. Eksp. Teor. Fiz. 58 (1993).
[14] B.I.Suleimanov, Zh. Eksp. Teor. Fiz. (1994).
[15] A.V.Gurevich, A.L.Krylov and G.A.El’, Pis’ma Zh. Eksp. Teor. Fiz. 54 (1991) 104.
[16] N.H.Ibragimov, Transformation Groups Applied to Mathematical Physics (Reidel, Dordrecht, 1985); P.J.Olver, Application of Lie Groups to Differential Equations (Springer, Berlin, 1986); A.C.Newell, Solitons in mathematics and physics (Moscow, Mir, 1989).
[17] B.I.Suleimanov, L.T.Habibullin, Teor. Mat. Fiz. 97 (1993) 213.
[18] V.R.Kudashev, Phys. Lett. 171A (1992) 335.
[19] A.M.Ili’u, Matching of asymptotic expansions of solutions of boundary value problems, ”Nauka”, Moscow, 1989; English transl., Amer. Math. Soc., Providence, RI, 1992.