Improved description of the $2\nu\beta\beta$-decay and a possibility to determine the effective axial-vector coupling constant.

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An improved formalism of the two-neutrino double-beta decay ($2\nu\beta\beta$-decay) rate is presented, which takes into account the dependence of energy denominators on lepton energies via the Taylor expansion. Till now, only the leading term in this expansion has been considered. The revised $2\nu\beta\beta$-decay rate and differential characteristics depend on additional phase-space factors weighted by the ratios of $2\nu\beta\beta$-decay nuclear matrix elements with different powers of the energy denominator. For nuclei of experimental interest all phase-space factors are calculated by using exact Dirac wave functions with finite nuclear size and electron screening. For isotopes with measured $2\nu\beta\beta$-decay half-life the involved nuclear matrix elements are determined within the quasiparticle random phase approximation with partial isospin restoration. The importance of correction terms to the $2\nu\beta\beta$-decay rate due to Taylor expansion is established and the modification of shape of single and summed electron energy distributions is discussed. It is found that the improved calculation of the $2\nu\beta\beta$-decay predicts slightly suppressed $2\nu\beta\beta$-decay background to the neutrinoless double-beta decay signal. Further, a novel approach to determine the value of effective weak-coupling constant in nuclear medium $g_A^\text{e}$ is proposed.

I. INTRODUCTION

The two-neutrino double beta decay ($2\nu\beta\beta$ decay) [1–3],

$$ (A, Z) \rightarrow (A, Z + 2) + 2e^- + 2\overline{v}_e, $$

(1)
a process fully consistent with the standard model of electroweak interaction, is the rarest process measured so far in the nature. It has been observed in twelve even-even isotopes [4].

The $2\nu\beta\beta$-decay is a source of background in experiments looking for a signal of the neutrinoless double beta decay ($0\nu\beta\beta$-decay) [11–13],

$$ (A, Z) \rightarrow (A, Z + 2) + 2e^-, $$

(2)

which observation would prove that neutrinos are Majorana particles, i.e., their own antiparticles.

The inverse half-life of the $2\nu\beta\beta$ decay is commonly presented by the product of a phase-space factor $G^{2\nu}$, fourth power of the effective axial-vector coupling constant $g_A^{\text{eff}}$ and $2\nu\beta\beta$-decay nuclear matrix element (NME) $M_{2\nu}^{\text{GT}}$ as follows:

$$ \left( T_{1/2}^{2\nu} \right)^{-1} = (g_A^{\text{eff}})^4 \left| M_{2\nu}^{\text{GT}} \right|^2 G^{2\nu}. $$

(3)
The matrix element $M_{2\nu}^{\text{GT}}$, which value can be determined from the measured $2\nu\beta\beta$-half-life by making assumption about the value of $g_A^{\text{eff}}$, plays an important role in understanding of the nuclear structure of double beta decay isotopes [5]. Its value is used to adjust the residual part of the nuclear Hamiltonian in calculation of the $0\nu\beta\beta$-decay NME within the proton-neutron Quasiparticle Random Phase Approximation (pn-QRPA) [6,7]. Due to this procedure obtained the results are only weakly sensitive on the size of the model space and chosen type of NN interaction. So far, $2\nu\beta\beta$-decay NMEs have been calculated without the closure approximation only within the Interacting Shell Model (ISM) [5] and the pn-QRPA [9].

The measured single and summed electron differential decay rates of the $2\nu\beta\beta$-decay allow to get valuable information concerning many interesting physical issues. In particular, from the shape of the summed electron distribution we get constraints on the Majoron mode of the $0\nu\beta\beta$-decay [10], the bosonic neutrino component [11], violation of the Lorentz invariance [12]. In addition, a reconstruction of individual electron energies and angular correlations in the NEMO3 experiment allowed to obtain information about the Single State Dominance (SSD) and Higher State Dominance (HSD) hypotheses discussing the importance of various contributions to the $2\nu\beta\beta$-decay NME from transitions through intermediate nuclear states [13,14].

Recently, a significant progress has been achieved in double beta decay experiments. The $2\nu\beta\beta$-decay mode has been measured with high statistics in the GERDA ($^{76}\text{Ge}$) [15], NEMO3 ($^{100}\text{Mo}$) [16], CUORE ($^{130}\text{Te}$) [17], EXO ($^{136}\text{Xe}$) [18] and KamLAND-ZEN ($^{136}\text{Xe}$) [19] experiments. As a consequence there is a request for more accurate description of the $2\nu\beta\beta$-decay process and corresponding differential characteristics. In this contribution we improve the theoretical description of the $2\nu\beta\beta$-decay process by including higher-order contributions to the phase space factors.
process by taking into account the dependence on lepton energies from the energy denominators of nuclear matrix elements, which has been neglected till now. In addition, a novel possibility to determine the effective axial-vector coupling constant $g_A^{\text{eff}}$ will be proposed.

II. THE IMPROVED FORMALISM FOR DESCRIPTION OF DOUBLE-BETA DECAY

In what follows we present improved formulae for the $2\nu\beta\beta$- and $0\nu\beta\beta$-decay half-lives in which the effect of the lepton energies in the energy denominator of NMEs is taken into account.

A. The $2\nu\beta\beta$-decay rate

The inverse half-life of the $2\nu\beta\beta$-decay transition to the $0^+$ ground state of the final nucleus takes the form:

$$T_1^{2\nu} = \frac{m_e}{8\pi^2} \ln \left( \frac{G_\beta m_e^2}{2} \right)^4 \left( g_\text{eff} \right)^4 I^{2\nu}, \quad (4)$$

where $G_\beta = G_F \cos \theta_C$ ($G_F$ is Fermi constant and $\theta_C$ is the Cabbibo angle), $m_e$ is the mass of electron and

$$I^{2\nu} = \frac{1}{m_e^2} \int_{m_e}^{E_i - E_f - m_\nu} F_0(Z_f, E_{e_i}) p_{e_i} E_{e_i} dE_{e_i} \times \int_{m_e}^{E_i - E_f - m_\nu} F_0(Z_f, E_{e_f}) p_{e_f} E_{e_f} dE_{e_f} \times \int_{E_i - E_f - m_\nu}^{E_i - E_f - E_{\nu_2}} E_{\nu_2}^2 E_{\nu_2}^2 A^{2\nu} dE_{\nu_2}. \quad (5)$$

Here, $E_{\nu_2} = E_i - E_f - E_{e_f} - E_{\nu_1}$ due to energy conservation. $E_i, E_f, E_{e_i}$ ($E_{e_i} = \sqrt{E_{e_i}^2 + 4m_e^2}$) and $E_{\nu_1}$ ($i = 1, 2$) are the energies of initial and final nuclei, electrons and antineutrinos, respectively. $F(Z_f, E_{e_i})$ denotes relativistic Fermi function and $Z_f = Z + 2$. $A^{2\nu}$ consists of products of the Gamow-Teller nuclear matrix elements (we neglect the contribution from the double Fermi transitions to the $2\nu\beta\beta$-decay rate), which depends on lepton energies [3]:

$$A^{2\nu} = \left[ \frac{1}{4} |M_{GT}^{K}|^2 + |M_{GT}^{L}|^2 \right],$$

where

$$M_{GT}^{K,L} = m_e \sum_n M_n \frac{E_n - (E_i + E_f)/2}{|E_n - (E_i + E_f)/2|^2 - \varepsilon_{K,L}^2} \quad (6)$$

with

$$M_n = \langle 0_f^+ | \tau^-_m \sigma_m || 1_n^+ \rangle \langle 1_n^+ | \sum_m \tau^-_m \sigma_m || 0_i^+ \rangle, \quad (7)$$

Here, $|0_f^+\rangle$, $|0_i^+\rangle$ are the $0^+$ ground states of the initial and final even-even nuclei, respectively, and $|1_n^+\rangle$ are all possible states of the intermediate nucleus with angular momentum and parity $J^P = 1^+$ and energy $E_n(1^+)$. The lepton energies enter in the factors

$$\varepsilon_K = \frac{(E_{e_2} + E_{\nu_2} - E_{e_1} - E_{\nu_1})}{2}, \quad \varepsilon_L = \frac{(E_{e_1} + E_{\nu_2} - E_{e_2} - E_{\nu_1})}{2}. \quad (8)$$

The maximal value of $|\varepsilon_K|$ and $|\varepsilon_L|$ is the half of $Q$ value of the process ($\varepsilon_{K,L} \in (-Q/2, Q/2)$). For $2\nu\beta\beta$ decay with energetically forbidden transition to intermediate nucleus ($E_n - E_i > -m_e$) the quantity $E_n - (E_i + E_f)/2 = Q/2 + m_e + (E_n - E_i)$ is always larger than half of the Q value.

The calculation of the $2\nu\beta\beta$-decay probability is usually simplified by an approximation

$$M_{GT}^{K,L} \simeq M^{2\nu}_{GT} = m_e \sum_n \frac{M_n}{E_n - (E_i + E_f)/2}, \quad (9)$$

which allows a separate calculation of the phase space factor and nuclear matrix element.

The calculation of $M_{GT}^{2\nu}$ requires to evaluate explicitly the matrix elements to and from the individual $|1_n^+\rangle$ states in the intermediate odd-odd nucleus. In the IBM calculation of this matrix element [20] the sum over virtual intermediate nuclear states is completed by closure after replacing $E_n - (E_i + E_f)/2$ by some average value $E_{av}$:

$$M_{GT}^{2\nu} \simeq \frac{m_e}{E_{av}} M_{GT-cl}^{2\nu} \quad (10)$$

with

$$M_{GT-cl}^{2\nu} = \langle 0_f^+ | \sum_{m,n} \tau^-_m \sigma_m \cdot \bar{\sigma}_n | 0_i^+ \rangle. \quad (11)$$

The validity of the closure approximation is as good as the guess about the average energy to be used. This approximation might be justified, e.g., in the case there is a dominance of transition through a single state of the intermediate nucleus [21].

We get a more accurate expression for the $2\nu\beta\beta$-decay rate by performing the Taylor expansion in matrix elements $M_{GT}^{K,L}$ over the ratio $\varepsilon_{K,L}/(E_n - (E_i + E_f)/2)$. By limiting our consideration to the fourth power in $\varepsilon$ we obtain

$$\left[ T_1^{2\nu} \right]^{-1} = \frac{\Gamma^{2\nu}_0}{\ln (2)} \simeq \frac{\Gamma^{2\nu}_0 + \Gamma^{2\nu}_2 + \Gamma^{2\nu}_4}{\ln (2)}, \quad (12)$$

where partial contributions to the full $2\nu\beta\beta$-decay width $\Gamma^{2\nu}$ associated with the leading $\Gamma^{2\nu}_0$, next to leading $\Gamma^{2\nu}_2$ and next-to-next to leading $\Gamma^{2\nu}_4$ orders is Taylor expansion are given by

$$\frac{\Gamma^{2\nu}_0}{\ln (2)} = \left( g_A^{\text{eff}} \right)^4 \mathcal{M}_0 G_0^{2\nu}, \quad \frac{\Gamma^{2\nu}_2}{\ln (2)} = \left( g_A^{\text{eff}} \right)^4 \mathcal{M}_2 G_2^{2\nu},$$

$$\frac{\Gamma^{2\nu}_4}{\ln (2)} = \left( g_A^{\text{eff}} \right)^4 \left( \mathcal{M}_4 G_4^{2\nu} + \mathcal{M}_2 G_2^{2\nu} \right). \quad (13)$$
The phase-space factors are defined as
\[ G_{N}^{2\nu} = \frac{c_{2\nu}}{m^{2}} \int E_{i}-E_{j}-m_{e} F_{0}(Z_{f}, E_{i}e_{i})e_{c_{i}}dE_{c_{i}} \]
\[ \times \int_{m_{e}}^{E_{i}-E_{j}-E_{c_{2}}} F_{0}(Z_{f}, E_{c_{2}})e_{c_{2}}dE_{c_{2}} \]
\[ \times \int_{0}^{E_{i}-E_{j}-E_{c_{2}}} E_{c_{1}}^{2}e_{c_{1}}^{2}A_{N}^{2}dE_{c_{1}} \]  \( (\text{N}=0, 2, 4, 22) \) (14)
with \( c_{2\nu} = m_{e}(G_{\beta}m_{e}^{2})/(8\pi^{2}\ln 2) \) and
\[ A_{0}^{2\nu} = 1, \quad A_{2\nu}^{2} = \frac{\xi_{K}^{2} + \xi_{L}^{2}}{(2m_{e}^{2})^{2}}, \]
\[ A_{12}^{2\nu} = \frac{\xi_{K}^{2}\xi_{L}^{2}}{(2m_{e}^{2})^{4}}, \quad A_{14}^{2\nu} = \frac{\xi_{K}^{4} + \xi_{L}^{4}}{(2m_{e}^{2})^{4}} \]  \( \xi \equiv \dfrac{G_{\beta}m_{e}^{2}}{4} \quad \text{and} \quad \xi_{K}^{2} = \dfrac{G_{\beta}^{2}m_{e}^{2}}{4} \quad \text{and} \quad \xi_{L}^{2} = \dfrac{G_{\beta}^{2}m_{e}^{2}}{4} \) (15)

The products of nuclear matrix elements are given by
\[ M_{0} = (M_{GT-1}^{2\nu})^{2}, \]
\[ M_{2} = M_{GT-1}^{2\nu}M_{GT-3}^{2\nu}, \]
\[ M_{22} = \frac{1}{3}(M_{GT-3}^{2\nu})^{2}, \]
\[ M_{4} = \frac{1}{3}(M_{GT-3}^{2\nu})^{2} + M_{GT-1}^{2\nu}M_{GT-5}^{2\nu} \]  \( \text{where nuclear matrix elements take the forms} \)
\[ M_{GT-1}^{2\nu} = M_{GT}^{2\nu} \]
\[ M_{GT-3}^{2\nu} = \sum_{n} M_{n} \frac{4m_{e}^{3}}{(E_{n} - (E_{i} + E_{j})/2)^{3}}, \]
\[ M_{GT-5}^{2\nu} = \sum_{n} M_{n} \frac{16m_{e}^{5}}{(E_{n} - (E_{i} + E_{j})/2)^{5}} \]  \( \text{By introducing two ratios of nuclear matrix elements,} \)
\[ \xi_{31}^{2\nu} = \frac{M_{GT-3}^{2\nu}}{M_{GT-1}^{2\nu}}, \quad \xi_{21}^{2\nu} = \frac{M_{GT-5}^{2\nu}}{M_{GT-1}^{2\nu}} \]  \( \text{the} \ 2\nu\beta\beta \text{half-life,} \)
\[ \left( T_{1/2}^{2\nu} \right)^{-1} = (g_{A}^{2\nu})^{2} \left[ \frac{1}{3}(\xi_{31}^{2\nu})^{2} + \frac{1}{2}(\xi_{21}^{2\nu})^{2} + (\xi_{21}^{2\nu})^{2} \right] \]
\[ \times \left( \frac{1}{3}(\xi_{31}^{2\nu})^{2} + \frac{1}{2}(\xi_{21}^{2\nu})^{2} \right) \]  \( \text{is expressed with single NME} \ (M_{GT-1}^{2\nu}) \text{and two ratios of nuclear matrix elements} \ (\xi_{31}^{2\nu} \text{and} \ \xi_{21}^{2\nu}), \text{which have to be calculated by means of the nuclear structure theory, four phase-space factors} \ (G_{A}^{2\nu}, G_{B}^{2\nu}, G_{C}^{2\nu} \text{and} \ G_{D}^{2\nu}), \text{which can be computed with a good accuracy, and the unknown parameter} \ g_{A}^{2\nu}. \)

The inverse lifetime of the 0νββ decay is commonly presented as a product of the total lepton number violating Majorana neutrino mass \( m_{\beta\beta} \), the phase-space factor \( G_{\nu\nu} \), nuclear matrix element \( M_{0\nu\nu}^{\nu\nu} (g_{A}^{\nu\nu}) \) and unquenched axial-vector coupling constant \( g_{A} \) \( (g_{A} = 1.269) \) in fourth power as follows 23:
\[ \left( T_{1/2}^{0\nu} \right)^{-1} = \left( \frac{g_{A}^{2\nu}}{m_{e}} \right)^{2} \left( M_{0\nu\nu}^{\nu\nu} (g_{A}^{\nu\nu}) \right)^{2} G_{\nu\nu}^{\nu\nu} \]  \( (20) \)
where
\[ G_{\nu\nu}^{\nu\nu} = \frac{G_{A}^{4}m_{e}^{2}}{32\pi^{4}R^{2}\ln(2)} \frac{1}{m_{e}^{6}} \times \]
\[ \int_{m_{e}}^{E_{i}-E_{j}-m_{e}} F_{0}(Z_{f}, E_{c_{2}})e_{c_{2}}dE_{c_{2}}dE_{c_{1}} \]  \( \text{with} \ E_{c_{2}} = E_{i} - E_{j} - E_{c_{1}}, \quad p_{c_{1}} = \sqrt{E_{c_{1}} - m_{e}^{2}} \quad (i=1,2). \)

We note that the axial-vector \( g_{A}^{\nu\nu} \) (\( p^{2} \)) and induced pseudoscalar \( g_{P}^{\nu\nu} (p^{3}) \) form factors of nuclear hadron currents \( J_{\nu\nu}^{\nu\nu} \) are "renormalized in nuclear medium". The magnitude and origin of this renormalization is the subject of the analysis of many works, since it tends to increase the 0νββ-decay half-life in comparison with the case in which this effect is absent 22,23.

In derivation of the 0νββ-decay rate in Eq. 20 the standard approximations were adopted: i) a factorization of phase-space factor and nuclear matrix element was achieved by approximation, in which electron wave functions were replaced by their values at the nuclear radius \( R \). ii) the dependence on lepton energies in energy denominators of the 0νββ-decay NME was neglected.

Here, we go beyond the approximation ii). The 0νββ nuclear matrix element contains a sum of two energy denominators:
\[ \frac{1}{p_{0} + E_{n} - E_{i} + E_{c_{1}}} + \frac{1}{p_{0} + E_{n} - E_{i} + E_{c_{2}}} \]  \( (23) \)
where \( p = (p_{0}, p) \) is the four-momentum transferred by the Majorana neutrino (common for all neutrino mass eigenstates, since the neutrino masses \( m_{i} \) can be safely neglected in \( p_{0} = \sqrt{p^{2} + m_{e}^{2}} \approx \mid p \mid \sim 100 \text{MeV} \). By taking advantage of the energy conservation \( E_{i} = E_{f} + E_{c_{1}} + E_{c_{2}} \) (the effect of nuclear recoil is disregarded) the approximation was adopted as follows:
\[ \frac{2}{p_{0} + E_{n} - E_{f} + E_{c_{1}}} \frac{1}{E_{n} - E_{i} + E_{c_{2}}} \approx \frac{2}{p_{0} + E_{n} - E_{i} + E_{c_{2}}} \]  \( (24) \)
with \( \varepsilon = (E_{c_{1}} - E_{c_{2}})/2 \). More accurate expression for the 0νββ-decay half-life is achieved by taking into account
TABLE I. Phase-space factors $G_{0,2,22,4}^{2\nu}$ entering the $2\nu\beta\beta$-decay (0$\nu\beta\beta$-decay) rate in Eq. (12) (Eq. 25). The radial wave-functions $g_{-1}$ and $f_{+1}$ of an electron, which constitute the Fermi function in Eq. (31), were calculated in two approximation schemes: (A) The standard approximation of Doi et al. [2]. (B) The exact solution for a Dirac equation for a uniform charge distribution in nucleus and electron screening is taken into account.

| nucl. el. w.f. | $G_0^{2\nu}$ [yr$^{-1}$] | $G_2^{2\nu}$ [yr$^{-1}$] | $G_{0,2}^{2\nu}$ [yr$^{-1}$] | $G_{2,22}^{2\nu}$ [yr$^{-1}$] | $G_0^{0\nu}$ [yr$^{-1}$] | $G_2^{0\nu}$ [yr$^{-1}$] |
|----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| $^{48}$Ca A | 1.608 $\times 10^{-17}$ | 1.372 $\times 10^{-17}$ | 1.484 $\times 10^{-17}$ | 3.297 $\times 10^{-18}$ | 2.641 $\times 10^{-14}$ | 2.284 $\times 10^{-14}$ |
| $^{76}$Ge A | 3.578 $\times 10^{-20}$ | 1.113 $\times 10^{-20}$ | 2.924 $\times 10^{-21}$ | 6.898 $\times 10^{-22}$ | 2.613 $\times 10^{-15}$ | 6.269 $\times 10^{-16}$ |
| $^{82}$Se A | 1.763 $\times 10^{-18}$ | 7.805 $\times 10^{-19}$ | 4.333 $\times 10^{-19}$ | 9.912 $\times 10^{-20}$ | 1.147 $\times 10^{-14}$ | 5.449 $\times 10^{-15}$ |
| $^{96}$Zr A | 7.777 $\times 10^{-18}$ | 4.292 $\times 10^{-18}$ | 2.974 $\times 10^{-18}$ | 6.774 $\times 10^{-19}$ | 2.423 $\times 10^{-14}$ | 1.422 $\times 10^{-14}$ |
| $^{100}$Mo A | 3.818 $\times 10^{-18}$ | 1.747 $\times 10^{-18}$ | 1.001 $\times 10^{-18}$ | 2.301 $\times 10^{-19}$ | 1.890 $\times 10^{-14}$ | 9.357 $\times 10^{-15}$ |
| $^{110}$Pd A | 1.629 $\times 10^{-19}$ | 3.405 $\times 10^{-20}$ | 8.322 $\times 10^{-21}$ | 2.115 $\times 10^{-21}$ | 5.783 $\times 10^{-15}$ | 1.408 $\times 10^{-15}$ |
| $^{116}$Cd A | 3.314 $\times 10^{-18}$ | 1.318 $\times 10^{-18}$ | 6.546 $\times 10^{-19}$ | 1.522 $\times 10^{-19}$ | 2.064 $\times 10^{-14}$ | 9.061 $\times 10^{-15}$ |
| $^{124}$Sn A | 6.717 $\times 10^{-19}$ | 1.794 $\times 10^{-19}$ | 5.954 $\times 10^{-20}$ | 1.414 $\times 10^{-20}$ | 1.124 $\times 10^{-14}$ | 3.442 $\times 10^{-15}$ |
| $^{128}$Te A | 3.314 $\times 10^{-22}$ | 1.318 $\times 10^{-23}$ | 6.409 $\times 10^{-25}$ | 1.688 $\times 10^{-25}$ | 7.263 $\times 10^{-16}$ | 3.875 $\times 10^{-17}$ |
| $^{130}$Te A | 1.885 $\times 10^{-18}$ | 6.112 $\times 10^{-19}$ | 2.467 $\times 10^{-19}$ | 5.812 $\times 10^{-20}$ | 1.807 $\times 10^{-14}$ | 6.619 $\times 10^{-15}$ |
| $^{134}$Xe A | 2.924 $\times 10^{-22}$ | 1.066 $\times 10^{-23}$ | 4.773 $\times 10^{-25}$ | 1.264 $\times 10^{-25}$ | 7.613 $\times 10^{-16}$ | 3.761 $\times 10^{-17}$ |
| $^{136}$Xe A | 2.347 $\times 10^{-22}$ | 8.553 $\times 10^{-24}$ | 1.915 $\times 10^{-25}$ | 1.014 $\times 10^{-25}$ | 6.100 $\times 10^{-16}$ | 3.008 $\times 10^{-17}$ |
| $^{150}$Nd A | 1.793 $\times 10^{-18}$ | 5.516 $\times 10^{-19}$ | 2.110 $\times 10^{-19}$ | 4.994 $\times 10^{-20}$ | 1.881 $\times 10^{-14}$ | 6.590 $\times 10^{-15}$ |

The additional term in the $0\nu\beta\beta$-decay rate in Eq. (25) is weighted by ratio $\xi_{31}^{0\nu}$, which is given by:

$$\xi_{31}^{0\nu} = \frac{M_{0\nu}^{0\nu}(g_A^{\text{eff}})}{M_{0\nu}^{0\nu}(g_A^{\text{eff}})}.$$  

This ratio is calculated using the two expressions:

$$M_{0\nu}^{0\nu}(g_A^{\text{eff}}) = \frac{R}{2\pi^2 g_A^{\text{eff}}} (2 m_e^2)^2 \sum_n \int e^{ip \cdot (x-y)} \langle \Omega^+_n | J^2 L \rangle (|x\rangle \langle n| J^2 L \rangle |\Omega^+_n \rangle) d^3p \, d^3x \, d^3y.$$  

The next term in Taylor expansion over the quantity $\varepsilon^2/[p_0 + E_n - (E_i + E_f)/2]^2$ in Eq. (24). We end up with:

$$\left( T_{1/2}^{0\nu} \right)^{-1} = \left( \frac{m_{\beta\beta}}{m_e} \right)^2 g_A^{\text{eff}} \left( G_0^{0\nu} + 2 \xi_{31}^{0\nu} G_2^{0\nu} \right),$$  

where

$$G_0^{0\nu} = \sum E_{i-E_f-m_e} \int_{m_e}^{E_i} A_0^{0\nu} F_0(Z_f, E_{c_1}) p_{c_1} E_{c_1} F_0(Z_f, E_{c_2}) p_{c_2} E_{c_2} d E_{c_1} \quad \text{with} \quad A_0^{0\nu} = 1, \quad A_2^{0\nu} = \varepsilon^2/(2m_e)^2.$$  

This expression is used to calculate the $0\nu\beta\beta$-decay rate in the context of the $2\nu\beta\beta$-decay rate.
TABLE II. The $2\nu\beta\beta$- and $0\nu\beta\beta$-decay nuclear matrix elements and ratios of nuclear matrix elements (see Eq. [18] and Eq. [28]) calculated within the p-n-QRPA with partial isospin restoration [20]. $F_0^{2\nu}$, $F_2^{2\nu}$, and $P_4^{2\nu}$ are the leading first, second and third order term contributions to the $2\nu\beta\beta$-decay rate in the Taylor expansion. $T_{1/2}^{2\nu-\text{exp}}$ is the averaged value of the $2\nu\beta\beta$-decay half-life [1] considered in the calculation of the $2\nu\beta\beta$-decay NMEs. $g_A^{\text{eff}}$ is the effective axial-vector coupling constant.

| nucl. | $g_A^{\text{eff}}$ | $M_{0\nu}^{2\nu-\text{GT}}$ | $M_{0\nu}^{2\nu-\text{GT}}$ | $M_{0\nu}^{2\nu-\text{GT}-5}$ | $\xi_{31}^{2\nu}$ | $\xi_{51}^{2\nu}$ | $P_0^{2\nu}$ | $P_2^{2\nu}$ | $P_4^{2\nu}$ | $T_{1/2}^{2\nu-\text{exp}}$ [yr] |
|-------|-----------------|-------------------------------|-------------------------------|-------------------------------|---------------|---------------|-------------|-------------|-------------|-------------------------------|
| $^{48}\text{Ca}$ | 0.800 | 0.0553 | 0.0105 | 0.00163 | 0.1891 | 0.0295 | 0.8456 | 0.1362 | 0.0182 | 4.4 x 10$^{-19}$ |
| $^{100}\text{Mo}$ | 0.800 | 0.292 | 0.123 | 0.0453 | 0.4230 | 0.1553 | 0.8163 | 0.1578 | 0.0259 | 7.1 x 10$^{18}$ |
| $^{136}\text{Xe}$ | 0.800 | 0.0268 | 0.00706 | 0.00232 | 0.2637 | 0.0866 | 0.9190 | 0.0745 | 0.0065 | 2.19 x 10$^{21}$ |

III. CALCULATIONS AND RESULTS

A. Phase-space factors and the QRPA NMEs

The $2\nu\beta\beta$- and $0\nu\beta\beta$-phase-space factors presented in the previous section are associated with the $s_{1/2}$ electron wave function distorted by the Coulomb field:

$$\psi^{(s_{1/2})}(E_e,r) = \left( \frac{g_{-1}(E_e,r)}{f_{+1}(E_e,r)} (\sigma \cdot \mathbf{p}_e) \chi_s \right),$$

where $E_e$ and $\mathbf{p}_e$ are the electron energy and momentum, respectively. $\mathbf{p}_e = \mathbf{p}_0 / |\mathbf{p}_e|$ and $r = |\mathbf{r}|$ is the radial coordinate of the position of the electron. The values of the radial functions $g_{-1}(E_e,r)$ and $f_{+1}(E_e,r)$ at nuclear radius $r = R$ constitute the Fermi function as follows:

$$F_0(Z,E_e) = g_{-1}^2(E_e,R) + f_{+1}^2(E_e,R).$$

Two different approximation schemes for the calculation of radial wave functions $g_{-1}(E_e,R)$ and $f_{+1}(E_e,R)$ are considered.

The approximation scheme A): The relativistic electron wave function in a uniform charge distribution in nucleus is considered. The lowest terms in the power expansion in $r/R$ are taken into account. The Fermi function takes the form

$$F_0 = \left( \frac{\Gamma(3)}{\Gamma(1)\Gamma(1+2\gamma_0)} \right)^2 (2\rho_e R)^{2(\gamma_0-1)} e^{\gamma_0 y} \frac{\Gamma(\gamma_0 + iy)}{\Gamma(\gamma_0 + iy)}^2,$$

where $\gamma_0 = \sqrt{1 - (\alpha)^2}$ and $y = \alpha Z e^2 / \rho_e$.

The approximation scheme B): The exact Dirac wave functions with finite nuclear size and electron screening are used [21]. The effect of screening of atomic electrons is taken into account by the Thomas-Fermi approximation. The numerical calculation is accomplished by the subroutine package RADIAL [22].

In Table II the $2\nu\beta\beta$- and $0\nu\beta\beta$-decay phase-space
factors calculated within approximations A and B are presented for 13 isotopes of experimental interest. We see that all phase-space factors calculated with exact relativistic electron wave functions (the approximation scheme B) are smaller in comparison with those obtained in approximation scheme A. We note that in both approximation schemes the factorization of phase-space factors and nuclear matrix elements is achieved by considering radial electron wave functions at nuclear radius and the difference between them is due to a different treatment of the Coulomb interaction.

In what follows entries B from Table I will be used in calculation of the $2\nu\beta\beta$ differential characteristics and decay rates.

B. Nuclear matrix elements

The $2\nu\beta\beta$- and $0\nu\beta\beta$-decay nuclear matrix elements (see Eqs. (17) and (29)) are calculated within the proton-neutron quasiparticle random phase approximation (QRPA) with isospin restoration [26]. They were obtained by considering the same model spaces and mean fields as in [26]. The G-matrix elements of a realistic Argonne V18 nucleon-nucleon potential are considered. By using the improved theoretical description of the $2\nu\beta\beta$-decay rate in Eqs. (12) and (17) the isoscalar neutron-proton interaction of the nuclear Hamiltonian is adjusted to reproduce correctly the average $2\nu\beta\beta$-decay half-life $T_{1/2}$ for each nucleus and each $g_A^{\text{eff}}$.

In Table II calculated $2\nu\beta\beta$-decay NMEs are presented for $g_A^{\text{eff}} = 0.8, 1.0$ and 1.269 (unquenched value). We see that for all isotopes the inequality $M_{GT-5}^{2\nu} > M_{GT-3}^{2\nu} > M_{GT-1}^{2\nu}$ is valid. The ratios of nuclear matrix elements $e^{2\nu}_B$, $e^{2\nu}_C$, and $e^{0\nu}$ depend only weakly on $g_A^{\text{eff}}$. The largest values $e^{2\nu}_A = 0.56$ and $e^{2\nu}_B = 0.21$ are in the case of $^{100}\text{Mo}$.

The ratio $e^{2\nu}_A$ of nuclear matrix element $M_{GT-3}^{2\nu}$ and $M_{GT-1}^{2\nu}$ (see Eq. (18)) is an important quantity due to a different structure of both nuclear matrix elements. This fact is displayed in Fig. 1 (Fig. 2), where running sum of matrix elements $M_{GT-1}^{2\nu}$ and $M_{GT-3}^{2\nu}$ is plotted as a function of the excitation energy $E_{ex}$ counted from the ground state of the intermediate nucleus. Calculations were performed within the proton-neutron QRPA with isospin restoration [26]. Results are obtained with Argonne V18 potential and for unquenched axial vector coupling constant $g_A = 1.269$.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1.png}
\caption{(Color online) Running sum of the $2\nu\beta\beta$-decay NMEs $M_{GT-1}^{2\nu}$ and $M_{GT-3}^{2\nu}$ (see Eq. (17)) for $^{48}\text{Ca}, ^{116}\text{Cd}, ^{130}\text{Te}$ and $^{136}\text{Xe}$ (normalized to unity) as a function of the excitation energy $E_{ex}$ counted from the ground state of intermediate nucleus. Calculations were performed within the proton-neutron QRPA with isospin restoration [26].}
\end{figure}

In Table II we notice a good convergence of contributions to the $2\nu\beta\beta$-decay rate due to the Taylor expansion. The size of these corrections depends on a given isotope. The largest value of about 25% is found by $^{100}\text{Mo}$.

In the Table II the calculated $0\nu\beta\beta$-decay nuclear matrix elements are presented as well. They were obtained under common assumption that the same $g_A^{\text{eff}}$ governs both modes of double beta decay [3, 26]. The modification of the $2\nu\beta\beta$-decay rate due to the Taylor expansion has only negligible effect on calculation of the $0\nu\beta\beta$-decay NMEs $M_{GT-3}^{0\nu}$ in the context of adjusting the particle-particle interaction strength.

By glancing the Table II we see that the value of $e^{2\nu}_A$ is very small, namely significantly smaller as $e^{2\nu}_B$, as the average momentum of neutrino entering the energy denominator in Eq. (24) is about two orders in magnitude larger when compared to the maximal value $\varepsilon$, which is Q/2. Clearly, in the case of $0\nu\beta\beta$-decay the convergence of the Taylor expansion of the decay rate is fast and the
standard approach given by the leading term in the Taylor expansion is well justified.

C. Energy distributions of emitted electrons

The NEMO3 experiment, which ran for seven years before it stopped taking data in 2010, measured the 2νββ-decay of 100Mo with very high statistics of about 1 million events [16]. Due to high statistics of about tens of thousands of events in the currently running EXO [18], KamLANDZEN [19] (136Xe) and GERDA (76Ge) [15] experiments allow precise determination of the 2νββ-decay energy distributions as well. A similar statistics is expected to be achieved also by the CUORE (130Te) experiment, which has started taking data recently. New perspectives for analysis of 2νββ-decay differential characteristics will be opened by next generation of the double-beta decay experiments like SuperNEMO, nEXO, Legend, which will contain significantly larger amount of double beta decay radioactive source [3, 27].

By considering the leading first and second order terms in the Taylor expansion for the single and summed electron differential decay rate normalized to the full decay rate we get

\[
\frac{1}{\Gamma_{2\nu}} \frac{d\Gamma_{2\nu}}{dE_e} \simeq \frac{1}{\Gamma_{2\nu}} \left( \frac{d\Gamma_0}{dE_e} + \frac{d\Gamma_2}{dE_e} \right),
\]

\[
\frac{1}{\Gamma_{2\nu}} \frac{d\Gamma_{2\nu}}{dE_{ee}} \simeq \frac{1}{\Gamma_{2\nu}} \left( \frac{d\Gamma_0}{dE_{ee}} + \frac{d\Gamma_2}{dE_{ee}} \right),
\]

where

\[
\frac{dG_{2\nu}}{dT_e} = \frac{c_{2\nu}}{m_{e1}^2} F_0(Z_{f,1}, E_{e1}) p_{e1} E_{e1} \int_0^{Q-T_{e1}} \int_0^{Q} F_0(Z_{f,2}, E_{e2}) p_{e2} E_{e2} I_N(T_{e1}, T_{e2}) dT_{e2} dT_{e1},
\]

\[
\frac{dG_{2\nu}}{dT_{ee}} = \frac{c_{2\nu}}{m_{e1}^2} \int_0^{Q} \int_0^{Q} F_0(Z_{f,1}, E_{e1}) p_{e1} E_{e1} F_0(Z_{f,2}, E_{e2}) p_{e2} E_{e2} I_N(T_{e1}, T_{e2}) dV,
\]

(N=0, 2) with

\[
I_N(T_{e1}, T_{e2}) = \int_0^{Q-T_{e1}-T_{e2}} E_{e1} E_{e2}^2 A_{2\nu} dE_{e1},
\]

and

\[
T_{ee} = T_{e1} + T_{e2}, \quad V = Q \frac{T_{e2}}{T_{e1} + T_{e2}}.
\]

Here, \(E_{v2} = E_{v1} - E_f - E_{e1} - E_{e1} - E_{e1} - E_{e1} - E_{e1}\) is determined by the energy conservation. \(T_{ee}\) is a sum of kinetic energies of both electrons (\(T_{e1}\) and \(T_{e2}\)) and \(T_{e}\) represents kinetic energy of any of two emitted electrons.

The single and summed electron differential decay rates normalized to the full width in Eqs. (34) and (35) contain one unknown parameter, namely the ratio \(\xi_{31}\).

We note that partial contributions to the full differential decay rate in Eq. (34) (Eq. (35)) exhibit different behavior as function of \(T_e\) \((T_{ee})\). This fact is displayed in...
and partial differential decay rates \((1/\Gamma) d\Gamma/dT_e\) and \((1/\Gamma) d\Gamma_0/dT_e\) normalized to the full decay rate vs. the kinetic energy of a single electron \(T_e\) (in units of Q-value) for the \(2\nu\beta\beta\)-decay of \(^{48}\text{Ca}\), \(^{116}\text{Cd}\), \(^{130}\text{Te}\) and \(^{136}\text{Xe}\).

Due to this phenomenon there is a possibility to deduce ratio \(\xi_{31}^{2\nu}\) from the measured energy distributions.

For pn-QRPA value of the parameter \(\xi_{31}^{2\nu}\) (see Table II) the full differential decay rate \((1/\Gamma_{31}^{2\nu}) d\Gamma_{31}^{2\nu}/dT_e\) and partial differential decay rates \((1/\Gamma_{31}^{2\nu}) d\Gamma_{0}^{2\nu}/dT_e\), \((1/\Gamma_{31}^{2\nu}) d\Gamma_{2}^{2\nu}/dT_e\) normalized to the full decay rate are presented as function of the kinetic energy of a single electron \(T_e\) (sum of kinetic energy of both electrons \(T_{ee}\) for the eight \(2\nu\beta\beta\)-decay isotopes in Figs. 4 and 5 (6 and 7).

We see that the largest contribution from the additional term due to Taylor expansion to the full differential decay rate is found by the \(2\nu\beta\beta\)-decay of \(^{100}\text{Mo}\), \(^{96}\text{Zr}\), \(^{48}\text{Ca}\), \(^{116}\text{Cd}\), \(^{130}\text{Te}\) and \(^{136}\text{Xe}\). These isotopes are good candidates to measure \(\xi_{31}^{2\nu}\) in double beta decay experiments.

By assuming \(\xi_{31}^{2\nu} = 0.0, 0.4\) and 0.8, the single electron energy distribution and summed electron energy spectrum normalized to the full decay rate for \(2\nu\beta\beta\)-decay of \(^{82}\text{Se}\) and \(^{100}\text{Mo}\) are presented in Fig. 6. We see that
responding curves are close to each other and that high statistics of the $2\nu\beta\beta$-decay experiment is needed to deduce information about the ratio of nuclear matrix elements $\xi_{13}$ from the data. The study performed within the NEMO3 experiment \cite{32} in respect the SSD versus HSD hypothesis \cite{13,14} has shown that it is feasible. It might be that the high statistics achieved by the GERDA \cite{15}, CUORE \cite{17}, EXO \cite{136} and KamlandZEN \cite{136} experiments is sufficient to conclude about the value of $\xi_{13}^2$ for the measured $2\nu\beta\beta$-decay transition.

For some of future double-beta decay experiments the $2\nu\beta\beta$-decay is considered as important background for the signal of the $0\nu\beta\beta$-decay, e.g., in the case of the SuperNEMO experiment. In Fig. 9 the endpoint of the spectrum of the differential decay rate normalized to the full decay rate (1/Γ) dΓ/dT vs. kinetic energy of a single electron $T = T_e$ and the sum of kinetic energies of emitted electrons $T = T_{ee}$ (in units of Q-value), respectively. Results are presented for the $2\nu\beta\beta$-

$^{82}$Se (left panels) and $^{100}$Mo (right panels) by assuming $\xi_{13} = 0.0, 0.40$ and 0.8.

D. Evaluation of the effective axial-vector coupling constant

The calculation of $M_{GT-3}^{2\nu}$ can be more reliable as that of $M_{GT-1}^{2\nu}$, because $M_{GT-3}^{2\nu}$ is saturated by contributions through the lightest states of the intermediate nucleus. Thus, we rewrite the $2\nu\beta\beta$-decay rate as follows:

\begin{equation}
\frac{1}{T_{1/2}}^{-1} \approx (g_A^\text{eff})^4 |M_{GT-3}^{2\nu}|^2 \frac{1}{|\xi_{31}^\text{eff}|^2} \left( G_0^{2\nu} + \xi_{31}^{2\nu} G_2^{2\nu} \right),
\end{equation}

i.e., without explicit dependence on matrix element $M_{GT-1}^{2\nu}$. For sake of simplicity it is assumed that values of involved nuclear matrix elements are real. From Eq. (39) it follows that if $\xi_{31}^{2\nu}$ is deduced from the measured $2\nu\beta\beta$-decay energy distribution and $M_{GT-3}^{2\nu}$ is reliably calculated by nuclear structure theory, the value of the effective axial-vector coupling constant $g_A^\text{eff}$ can be determined from the measured $2\nu\beta\beta$-decay half-life.

Let discuss the value of $\xi_{31}^{2\nu}$ within different approaches before it will be measured by the double-beta decay experiment. Within the SSD hypothesis \cite{13,14,21} it is supposed that the $2\nu\beta\beta$-decay NME is governed by the two virtual transitions: the first one going from the initial $0^+$ ground state to the $1^+$ ground state of the intermediate nucleus and second one from this $1^+$ state to the final $0^+$ ground state. Within this assumption we obtain

\begin{equation}
(g_A^\text{eff})^2 M_{GT-k}^{2\nu} \approx m_e^k \frac{(g_A^\text{eff})^2 M_k}{(E_1 - (E_1 - E_1))^k}
\end{equation}
intermediate nucleus from initial and final ground states entering the double beta decay transition are measured with help of charge-exchange reactions (ChER) \[28\,31\], i.e., via strong interaction due to spin-isospin Majorana force. For \(^{48}\text{Ca},\, ^{76}\text{Ge}\) and \(^{116}\text{Cd}\) the calculated matrix elements \(M^\nu_{\text{GT,1}}\), \(M^\nu_{\text{GT,3}}\) and \(\xi^\nu_{33}\) under the assumption of equal phases for its each individual contribution are presented in Table III. The ChER allow to measure with a reasonable resolution of about tens of keV the Gamow-Teller strengths only up to about 5 MeV, i.e., below the region of the Gamow-Teller resonance, what might be considered as drawback. We note that some questions arise also about the normalization of the Gamow-Teller strengths by the experiment.

The pnQRPA, SSD and CheR predictions for parameter \(\xi^\nu_{33}\) for various isotopes are displayed in Fig. 10. We see that a best agreement among different results occurs by \(^{116}\text{Cd}\). In the case of \(^{48}\text{Ca}\) and \(^{76}\text{Ge}\) there is a significant difference between the pn QRPA and CheR results. We note that within the HSD hypothesis \[13,\, 14\] the value of \(\xi^\nu_{33}\) is equal to zero.

By considering the SSD values for \(\xi^\nu_{33}\) (see Table III) we obtain
\[
g^\text{eff}_{A} (^{100}\text{Mo}) \approx 0.251 \frac{m_k^k}{\sqrt{M^\nu_{\text{GT,3}}}}, \quad g^\text{eff}_{A} (^{116}\text{Cd}) \approx 0.214 \frac{m_k^k}{\sqrt{M^\nu_{\text{GT,3}}}}.
\]
(41)

The corresponding curves are plotted in Fig. 11. It is apparent that if the value of \(M^\nu_{\text{GT,3}}\) would be calculated reliably, e.g. within the interacting shell model, which is known to describe very well the lowest excited states of parent and daughter nucleus participating in double-beta decay process, one could conclude about the value of the effective axial-vector coupling constant \(g^\text{eff}_{A}\) for a given nuclear system. However, we note that the correct value of \(g^\text{eff}_{A}\) can be determined only if \(\xi^\nu_{33}\) deduced from the measured \(2\nu\beta\beta\)-decay energy distribution is considered. In that case the constant on the r.h.s of Eq. (41) might be different.

TABLE III. The nuclear matrix elements \(M^\nu_{\text{GT,k}}\) and \(\xi^\nu_{33}\) calculated from measured \(GT^\pm\) strengths in charge exchange reaction (ChER) under the assumption of a equal phases for its each individual contribution \(28\,31\) and their product with squared effective axial-vector coupling constant \(g^\text{eff}_{A}\), which is determined within the Single State Dominance Hypothesis (SSD hypothesis) \[13\,14\].

| Nucl. | \((g^\text{eff}_{A})^2 M^\nu_{\text{GT,1}}\) | \((g^\text{eff}_{A})^2 M^\nu_{\text{GT,3}}\) | \((g^\text{eff}_{A})^2 M^\nu_{\text{GT,5}}\) | \(\xi^\nu_{33}\) | \(\xi^\nu_{51}\) |
|-------|----------------|----------------|----------------|----------------|----------------|
| \(^{48}\text{Ca}\) | - | - | - | - | - |
| \(^{76}\text{Ge}\) | - | - | - | - | - |
| \(^{100}\text{Mo}\) | \(1.71 \times 10^{-1}\) | \(6.29 \times 10^{-2}\) | \(2.31 \times 10^{-2}\) | \(0.368\) | \(0.135\) |
| \(^{116}\text{Cd}\) | \(1.53 \times 10^{-1}\) | \(4.57 \times 10^{-2}\) | \(1.36 \times 10^{-2}\) | \(0.298\) | \(0.089\) |
| \(^{128}\text{Te}\) | \(1.60 \times 10^{-2}\) | \(5.87 \times 10^{-3}\) | \(2.16 \times 10^{-3}\) | \(0.367\) | \(0.135\) |
| | \(|M^\nu_{\text{GT,1}}| \) | \(|M^\nu_{\text{GT,3}}| \) | \(|M^\nu_{\text{GT,5}}| \) | \(\xi^\nu_{33}\) | \(\xi^\nu_{51}\) |
| \(^{48}\text{Ca}\) | \(4.25 \times 10^{-2}\) | \(2.31 \times 10^{-3}\) | \(1.26 \times 10^{-4}\) | \(0.054\) | \(0.003\) |
| \(^{76}\text{Ge}\) | \(8.61 \times 10^{-2}\) | \(2.20 \times 10^{-2}\) | \(5.61 \times 10^{-3}\) | \(0.255\) | \(0.065\) |
| \(^{100}\text{Mo}\) | \(1.71 \times 10^{-1}\) | \(6.29 \times 10^{-2}\) | \(2.31 \times 10^{-2}\) | \(0.368\) | \(0.135\) |
| \(^{116}\text{Cd}\) | \(1.53 \times 10^{-1}\) | \(4.57 \times 10^{-2}\) | \(1.36 \times 10^{-2}\) | \(0.298\) | \(0.089\) |
| \(^{128}\text{Te}\) | \(1.60 \times 10^{-2}\) | \(5.87 \times 10^{-3}\) | \(2.16 \times 10^{-3}\) | \(0.367\) | \(0.135\) |
In summary, improved formulae for the $2\nu\beta\beta$- and $0\nu\beta\beta$-decay half-lives are presented by taking advantage of the Taylor expansion over the parameters containing the lepton energies of energy denominators. The additional terms due to Taylor expansion in the decay rate have been found significant in the case of the $0\nu\beta\beta$-decay and practically of no importance in the case of the $2\nu\beta\beta$-decay.

Up to first order in the Taylor expansion the $2\nu\beta\beta$-decay rate includes two nuclear matrix elements $M_{2\nu}^{\text{GT-1}}$ and $M_{2\nu}^{\text{GT-3}}$ with energy denominator in the first and third power, respectively. It was shown that the ratio of these matrix elements $\xi_{31}^{2\nu} = M_{2\nu}^{\text{GT-3}}/M_{2\nu}^{\text{GT-1}}$ might be determined experimentally from the shape of the single and sum electron energy distributions, if the statistics of a considered double beta decay experiment allows it. A study of the SSD and HSD hypotheses in the case of the $2\nu\beta\beta$-decay of $^{100}\text{Mo}$ by the NEMO3 experiment has manifested that it is feasible.

A measured value of $\xi_{31}^{2\nu}$ is expected to be an important information about virtual transitions through the states of intermediate nucleus. The calculation of running sum of $M_{2\nu}^{\text{GT-1}}$ and $M_{2\nu}^{\text{GT-3}}$ performed within the pn-QRPA with partial restoration of isospin symmetry showed that $M_{2\nu}^{\text{GT-3}}$ is determined by contributions through the low-lying states of the intermediate nucleus unlike $M_{2\nu}^{\text{GT-1}}$, which is affected significantly also by contributions through transitions over intermediate nucleus from the region of the Gamow-Teller resonance.

Further, the $2\nu\beta\beta$-decay rate was expressed with $M_{2\nu}^{\text{GT-3}}$ and $\xi_{31}^{2\nu}$, i.e. without the explicit dependence on the commonly studied nuclear matrix element $M_{2\nu}^{\text{GT-1}}$. It was suggested that one can get information about the axial-vector coupling constant in nuclear medium $g_{A}^{\text{eff}}$ once $\xi_{31}^{2\nu}$ is deduced from the measured electron energy distribution and $M_{2\nu}^{\text{GT-3}}$ is calculated reliably, e.g. within the ISM.

It goes without saying that improved formula for the $2\nu\beta\beta$-decay half-life will play an important role in accurate analysis of the Majoron mode of the $0\nu\beta\beta$-decay and study of Lorentz invariance violation, bosonic admixture of neutrinos and other effects.

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