Angular dispersion can counterbalance normal group-velocity dispersion (GVD) that increases the wave-vector length in a dispersive medium. By tilting the wave vector, angular dispersion reduces the axial wave number in this case to match the pre-GVD value. By the same token, however, angular dispersion fails to counterbalance anomalous GVD, which in contrast reduces the wave-vector length. Consequently, GVD-cancellation via angular dispersion has not been demonstrated to date in the anomalous dispersion regime. Here, structured femtosecond pulsed beams, known as ‘space-time’ wave packets, are designed to realize dispersion-cancellation symmetrically in either the normal- or anomalous-GVD regimes by virtue of non-differentiable angular dispersion inculcated into the pulses. Furthermore, GVD-inversion is also verified by reversing the GVD sign experienced by the field with respect to that dictated by the chromatic dispersion of the medium itself.

1. Introduction

Chromatic dispersion resulting from the wavelength-dependence of the refractive index is an inescapable feature of optical materials, which leads to pulse broadening and distortion.\(^1,2\) One may combat its impact via dispersion compensation or dispersion cancellation. In the former, dispersive broadening and the associated chirp are compensated after (or pre-compensated before) passage through the medium, which is key to the success of chirped pulsed amplification (CPA),\(^3\) for example. Normal group-velocity dispersion (GVD) can be compensated by a pair of gratings or prisms,\(^4\) anomalous GVD by a Martinez stretcher;\(^5\) and almost arbitrary dispersion by a 4f pulse shaper or other techniques.\(^6-11\) In all these cases, it is the group delay dispersion (GDD) that is being neutralized. More challenging, however, is to neutralize dispersion during passage through a dispersive medium, so that the pulse travels invariantly, which we refer to as dispersion cancellation. Such a capability is crucial, for instance, in enabling efficient nonlinear interactions in long crystals. Angular dispersion,\(^12\) whereby each frequency in the pulse is directed at a prescribed angle, has been successfully utilized for this purpose.\(^13\) The resulting field structure after inculcating angular dispersion is typically known as a tilted pulse front (TPF).\(^14\)

To date, however, angular dispersion has been used for dispersion cancellation in only the normal-GVD regime. There have been no reports of dispersion-cancellation in the anomalous-GVD regime, and well-established theoretical considerations suggest the impossibility of such a goal.\(^15\) Indeed, there is strong priori conceptual support for such a claim. Because normal GVD increases the wave-vector length in the dispersive medium by a frequency-dependent amount, changing the wave-vector angle for each frequency can reduce its axial component to the pre-GVD value. Anomalous GVD, on the other hand, reduces the wave-vector length, and no angular tilt can compensate for such a reduction. Consequently, cancelling anomalous GVD has not been reported to date.

Nevertheless, pulsed beams or wave packets that are propagation invariant (diffraction-free and dispersion-free) in dispersive media by virtue of their spatio-temporal field structure have been known to exist theoretically\(^16-21\) in presence of normal\(^22\) and anomalous\(^23\) GVD. These sought-after dispersion-free fields have not been observed to date in linear dispersive media, although evidence for their presence in nonlinear interactions has been reported.\(^24-27\) Indeed, nonlinear interactions in dispersive media have been utilized to produce such spatio-temporally structured fields, and tuning the GVD (by changing the wave-length) has yielded corresponding changes to their spectral structure.\(^27\) However, their propagation invariance in these dispersive media has yet to be confirmed, and changing their spectral structure at fixed GVD has not been observed to date. Such spatiotemporally structured fields have been recently studied systematically in free space and non-dispersive dielectrics under the rubric of “space-time” (ST) wave packets.\(^28-33\) The defining characteristic of ST wave packets is a one-to-one association between the spatial and temporal frequencies. These propagation-invariant wave packets feature a unique set of characteristics including tunable group velocities in absence of chromatic dispersion,\(^34,35\) anomalous refraction,\(^36\), and novel ST Talbot effects.\(^37\) The central characteristics of ST wave packets in free space are now well-understood,\(^31\) and potential applications are being evaluated in optical communications\(^38\) and device physics.\(^39-43\)
In contrast to TPFs, the angular dispersion undergirding ST wave packets is “non-differentiable:” the derivative of the propagation angle is not defined at one wavelength. Such angular dispersion can be produced by a recently developed pulsed-beam shaper that serves as a universal angular-dispersion synthesizer in one dimension. We have recently shown that non-differentiable angular dispersion is the crucial ingredient for tuning the group velocity of a ST wave packet and introducing an arbitrary dispersion profile in free space. This suggests the prospect for dispersion-cancellation in presence of either normal—or anomalous—GVD.

Here, we verify the propagation invariance of ST wave packets symmetrically in both the normal—and anomalous—GVD regimes, the latter for the first time to the best of our knowledge. We identify the spatio-temporal spectral structures needed for achieving GVD-cancellation. The crucial step is to first change the ST wave-packet group velocity via non-differentiable angular dispersion, which opens up a space for subsequent decrease or increase in the axial wave vector relative to the dispersion-free configuration. We sculpt the spatio-temporal spectrum of ≈200-fs (≈16-nm-bandwidth) pulses to produce ST wave packets that are propagation invariant at a wavelength of ≈1 μm in ZnSe as a representative normal-GVD medium, and chirped Bragg mirrors that produce anomalous GVD. Uniquely, the GVD in the medium is cancelled while maintaining independent control over the group velocity of the ST wave packet (in both the subluminal and superluminal regimes). Furthermore, not only is propagation invariance achieved in dispersive media, but dispersion inversion is also verified: the sign of GVD experienced by the wave packet in the medium is reversed. We thus produce wave packets that undergo normal GVD in an anomalously dispersive medium, and vice versa. We expect these results to be particularly useful in tailoring multi-wavelength nonlinear optical interactions in long crystals.

2. The Challenge of Canceling Anomalous GVD via Angular Dispersion

Angular dispersion, whereby each frequency ω in a pulsed field travels at a different angle φ(ω), produces anomalous GVD in free space, which can help cancel normal GVD. In fact, it is commonly understood that angular dispersion cannot produce normal GVD in free space, and thus does not provide the possibility of cancelling anomalous GVD. The illustration in Figure 1 elucidates the origin of this asymmetry between realizing normal and anomalous GVD via conventional angular dispersion.

Consider a plane-wave pulse traveling in a dispersive medium of refractive index n(ω), and expand the wave number k(ω) = n(ω)ω/c around a frequency ω, k(ω + Ω) ≈ n(ω)kω + Ω2, where Ω = ω − ωo. kω = ωo/c, c is the speed of light in vacuum, n(ω) = n(ωo). ωo = 1/2Δωω = ωo is the group velocity and n(ω) the group index, and 1/2Δωω is the GVD coefficient. Throughout, we use the subscript “m” to denote quantities in the dispersive medium, and the subscript “a” for the corresponding quantities in free space. It is clear that normal GVD χ2m > 0 increases the wave-vector length by a frequency-dependent amount 1/2Δωω (Figure 1a), which can be counterbalanced by tilting the wave vector by an angle φ(ω) to reduce the axial component of the wave vector in the medium to the pre-GVD value k(ω) = k(ω)cos[φ(ω)] ≈ n(ω)kω + Ω2 (Figure 1b). To realize this condition in the small-angle limit starting with a plane-wave pulse in free space (Figure 1c), we tilt the wave vector associated with the ω by an angle φ(ω) = φ(a) + Ωx (Figure 1d). The angular dispersion in (e) produces a field in the form of a tilted pulse front (TPF). In the anomalous-GVD regime, the angular-dispersion approach fails because the wave-vector length is reduced, and a tilt cannot increase k to its pre-GVD value. Introducing normal GVD into a plane-wave pulse in free space via conventional angular dispersion renders the field evanescent.

![Figure 1](image-url)
The challenge of cancelling anomalous GVD $k_{2m} < 0$ is now clear. Anomalous GVD reduces $k(\omega)$ in the medium by $-\frac{1}{2}k_{2m}^2\Omega^2$ (Figure 1g), and no angular tilt can increase $k_\perp$ to the pre-GVD value. As shown in Figure 1h, the dispersion curve for normal GVD in free space lies below the light-line, corresponding to an evanescent field. Although this appears to be an insurmountable obstacle, we show here that it is resolved by ST wave packets endowed with non-differentiable angular dispersion.

### 3. Theory of Propagation-Invariant Space-Time Wave Packets in Dispersive Media

#### 3.1. Symmetrized Anomalous and Normal GVD in Free Space via Angular Dispersion

The key to introducing anomalous or normal GVD symmetrically into a pulsed field is to first change the group velocity along the propagation axis. This is done by introducing AD into the pulse, in which case the axial wave number is $k_\parallel = \frac{\omega}{v(\omega)}$. The group velocity $v_\parallel$ is defined as follows: $v_\parallel = \frac{\partial}{\partial \omega} \left( \frac{\omega}{v(\omega)} \right)$. The group velocity $v_\parallel$ is given by $v_\parallel = \frac{\partial}{\partial \omega} \left( \frac{\omega}{v(\omega)} \right)$. For on-axis propagation $\varphi_\parallel \rightarrow 0$, $v_\parallel \rightarrow c$ for any finite value of $\varphi_\parallel$. However, setting $\varphi_\parallel = \varphi_{ST}(\omega_0 + \Omega) \approx \sqrt{2(1 - \Omega)}\Omega/\omega_0$, which is not differentiable at $\omega = \omega_0$, yields a propagation-invariant ST wave packet with a group velocity $v_\parallel = \frac{\Omega}{c}$ and group index $n_{\parallel}$. For a superluminal ST wave packet $\varphi_\perp > c$, $\omega_0$ is the minimum frequency in the spectrum, $\Omega > 0$, and $n_{\parallel} < 1$ (Figure 2a); and for the subluminal counterpart $\varphi_\perp < c$, $\omega_0$ is the maximum, $\Omega < 0$, and $n_{\parallel} > 1$ (Figure 2d).

### 3.2. Coupling from Free Space to a Dispersive Medium

To analyze quantitatively the scenario illustrated in Figure 2, the representation of the spectral support domain of pulsed fields on the surface of the light-cone is a useful guide. In free space, the light-cone is $k^2 = \frac{\omega^2}{c^2}$, where $k_\perp$ is the transverse wave number or spatial frequency (we hold the field uniform along $y$ for simplicity). A monochromatic plane wave is represented by a point on the light-cone surface. Because any physically realizable pulsed beam is formed of a collection of such monochromatic plane waves, its spatio-temporal spectrum will be represented by some area on the light-cone surface that we denote “the spectral
For a propagation-invariant ST wave packet traveling at a group velocity \( \tilde{v}_g \), the spectral support domain is the intersection of the light-cone with a plane \( k_x = k_o + \Omega \tilde{v}_g \) that is parallel to the \( k_x \)-axis and makes an angle \( \theta_g \) (the spectral tilt angle) with the \( k_x \)-axis. \(^{15,48,49} \) where \( \tilde{v}_g = c \tan \theta_g \) (Figure 3a). The spectral projection onto the \( (k_x, \tilde{n}) \)-plane is a straight line, and onto the \( (k_z, \tilde{n}) \)-plane is a conic section that can be approximated by a parabola in the vicinity of \( k_z = 0 \) in the paraxial regime. \(^{19} \)

In presence of GVD due to chromatic dispersion, the dispersion relationship \( k_g^2 + \tilde{k}^2 = (\Omega \tilde{v}_g)^2 \) corresponds to a modified light-cone (Figure 3b,c). At normal incidence on a planar interface, \( k_g \) and \( \Omega \) are invariant, so the \( (k_x, \tilde{n}) \)-projection is the same in free space and the dispersive medium:

\[
\begin{align*}
\frac{\Delta}{c^2} k_x^2 &= \left( k_o + \Omega \tilde{v}_g \right)^2 - \left( n_m k_o + \frac{\Omega}{v_m} + \frac{k_{2m} \Omega^2}{2} \right)^2 - k_x^2 \\
\text{in free space} &
\end{align*}
\]  

However, the \( (k_x, \tilde{n}) \)-projection changes because the light-cone structure has been modified. The ST wave packet that was propagation-invariant in free space now experiences GVD in the dispersive medium. We expand the axial wave number in the medium as \( k_x = n_m k_o + \frac{\Omega}{v_m} + k_{2m} \Omega^2 \), where \( \tilde{v}_g = \tilde{v} \) is the group velocity of the ST wave packet in the medium (which need not be equal to \( \tilde{v}_m \)), and \( k_{2m} \) is the effective GVD coefficient (which can differ from the GVD coefficient \( k_{2m} \) in the medium).

By equating the first-order \( \Omega^2 \) terms in Equation (1), we obtain

\[
1 - \tilde{n}_m = n_m (\tilde{n}_m - \tilde{n})
\]

which can be recognized as the law of refraction for a ST wave packet in a dispersive medium derived in refs. [50, 51] that governs the change in group velocity from \( \tilde{v}_g = \tilde{v} \) in free space to \( \tilde{v} = \tilde{v}_m \) in the medium. Indeed, the quantity \( n_m (\tilde{n}_m - \tilde{n}) \) is a refractive invariant for ST wave packets at normal incidence on planar interfaces between dispersive media, which we have called the "spectral curvature" because it is related to the curvature of the parabolic \( (k_x, \tilde{n}) \)-projection in the vicinity of \( k_x = 0 \). This relationship indicates that a subluminal ST wave packet in free space \( \tilde{v}_g < c \) remains subluminal in the medium \( \tilde{v} < \tilde{v}_m \), and similarly for superluminal wave packets.

Equating the second-order \( \Omega^2 \) terms in Equation (1) yields a relationship between the GVD coefficient of the medium \( k_{2m} \) and the effective GVD coefficient experienced by the wave packet \( k'_{2m} \)

\[
k'_{2m} = k_{2m} + \frac{1}{n_m} \left( \frac{\Delta}{c^2} \right)
\]  

From this we conclude that the ST wave packet experiences the normal or anomalous GVD intrinsic to the medium itself (Figure 3b,c) except for an offset term

\[
\Delta = (\tilde{n}_m - \tilde{n}) - (1 - \tilde{n})
\]

In most cases where the deviation from the luminal limit is small (\( \tilde{n}_m \rightarrow 1 \) and \( \tilde{n} \rightarrow \tilde{n}_m \)), \( \Delta \) can be ignored, and we have \( k'_{2m} \approx k_{2m} \).

3.3. Achieving Dispersion-Free Propagation in Presence of GVD

Dispersion-free propagation in the dispersive medium is achieved by modifying the structure of the ST wave packet in

![Figure 3.](image-url)
free space to introduce GVD of opposite sign to that of the medium: cancelling normal GVD necessitates endowing the ST wave packet in free space with anomalous GVD (Figure 3d,e), and vice versa (Figure 3f,g). In free space, the \((k_x, z)\)-projection is no longer a straight line, but rather takes the form \(k_x = k_o + \frac{\Omega}{c} + \frac{1}{2} k_{2m} \Omega^2\), where \(k_o\) is the GVD coefficient introduced into the ST wave packet. The spectral support domain on the free-space plane is the intersection with a planar curved surface that is parallel to the \(k_x\)-axis

\[
k_x^2 + \left( k_o + \frac{\Omega}{c} + \frac{1}{2} k_{2m} \Omega^2 \right)^2 = \left( k_o + \frac{\Omega}{c} \right)^2
\]  

(5)

The change in the light-cone structure in the medium in conjunction with the invariance of the \((k_x, z)\)-projection can yield a \((k_x, z)\)-projection in the medium that is a straight line. In other words, the curved \((k_x, z)\)-projection in free space has “straightened out” in the medium such that \(k_x = n_{\omega} k_o + \frac{\Omega}{c}\), thus signifying dispersion-free propagation in the medium

\[
k_x^2 + \left( n_{\omega} k_o + \frac{\Omega}{c} \right)^2 = \left( n_{\omega} k_o + \frac{\Omega}{c} \right)^2
\]  

(6)

Equating the \(\Omega\)-terms in Equations (5) and (6) yields \(1 = n_{\omega} = n_{\omega} (n_{\omega} - n)\) as in Equation (2)), whereas equating the \(\Omega^2\)-terms yields

\[
k_{2m} = -n_{\omega} k_{2m} - \frac{\Delta}{c n_{\omega}}
\]  

(7)

That is, dispersion cancellation requires that the dispersion coefficient introduced in free space \(k_{2m}\) must have the opposite sign to that of the medium \(k_{2m}^o\), and to be weighted by the refractive index \(n_{\omega}\). This result is similar to that for a TPF (Appendix) except that the GVD to be cancelled can be either normal or anomalous (in addition to the minor offset term \(\Delta\)).

4. Experimental Section

4.1. Spatio-Temporal Spectral Synthesis

To synthesize dispersive ST wave packets in free space, a universal angular-dispersion synthesizer described in ref. [33] was used and depicted in Figure 4. The synthesis began with plane-wave femtosecond pulses of width \(\approx 100\) fs and bandwidth \(\approx 25\) nm at a central wavelength of \(\lambda_o \approx 1064\) nm (Spark Lasers; Aker). The pulses were spectrally resolved via a diffraction grating (1200 lines per mm) followed by a collimating cylindrical lens of focal length \(f = 500\) mm. At the focal plane of the lens a reflective, phase-only spatial light modulator (SLM; Meadowlark, E19X12) that imparted a 2D phase distribution to the impinging spectrally resolved wave front was placed. Each wavelength occupied a column on the SLM, along which the phase \(\Phi(x, \lambda) = \pm \frac{1}{2} \sin(\phi(\lambda)) x\) was imposed where \(\phi(\lambda)\) is the deflection angle for \(\lambda\) with respect to the z-axis. An example of the phase pattern \(\Phi(x, \lambda)\) is depicted in Figure 4, inset. The upper half of each SLM column deflected the wavelength \(\lambda\) at an angle \(\phi(\lambda)\), while the lower half deflected it at \(-\phi(\lambda)\). This yielded a symmetric spatio-temporal angular spectrum \(\phi(\lambda)\) (Figure 4, inset), and produced a X-shaped wave-packet profile, whereas that for TPFs (Figure 1f) comprised one branch of the X-shaped profile. The wave front retro-reflected from the SLM returned to the grating where the ST wave packet was formed. An imaging system relayed the formed ST wave packet to the dispersive material, and a spatial filter was used to remove the zeroth-order diffraction component from the SLM.

4.2. Dispersive Samples

The normal-GVD medium was ZnSe (Thorlabs; WG71050) comprised of multiple 1-inch-diameter discs of thickness 5 mm each, stacked to a maximum thickness of 30 mm. Using the Sellmeier equation for ZnSe, \(n^2(\lambda) = 4 + \frac{1.92^2}{\lambda^2 - 1000}\) (\(\lambda\) in units of \(\mu m\)).[52] at \(\lambda_o = 1064\) nm an index \(n_{\omega} \approx 2.47\), group index \(n_{g} \approx 2.57\), and GVD parameter \(k_{2m} \approx +607.45\) fs\(^2\) mm\(^-1\) was obtained. It is useful to exploit the dimensionless GVD parameter \(\frac{\Delta}{c n_{\omega}} k_{2m} \approx 0.32\).

The anomalous-GVD sample comprised a pair of chirped Bragg mirrors (Edmund Optics, 12-335) that generated \(-1000\) fs\(^2\) group delay dispersion (GDD) per reflection. By changing the distance separating the two mirrors, the number of reflections could be increased for a given propagation distance before the wave packet emerged, thus controllably increasing the GDD from \(-2000\) to \(-15\) 000 fs\(^2\). The GVD was then taken to be the GDD divided by the total length propagated at a fixed incident angle of 7°, resulting in an effective GVD coefficient of \(k_{2m} \approx -500\) fs\(^2\) mm\(^-1\) and a medium length extending up to 30 mm. The dimensionless GVD parameter here was \(\frac{\Delta}{c n_{\omega}} k_{2m} \approx -0.25\), and \(n_{\omega} \approx n_{g} \approx 1\) because the ST wave packet travelled predominantly in free space.

5. Spectral Measurements

The spatio-temporal spectrum projected onto the \((k_x, \lambda)\)-plane is obtained after implementing a spatial Fourier transform on the
spectrally resolved wave front reflecting back from the SLM (Figure 4). The intensity distribution is then recorded by a CCD camera. The result is a parabola centered at \( k_x = 0 \) of spatial bandwidth \( \Delta k_x \approx 0.23 \text{ rad mm}^{-1} \) (corresponding to a spatial width of \( \Delta \lambda \approx 20 \text{ nm} \) at the pulse center) and a temporal bandwidth of \( \Delta T \approx 16 \text{ nm} \) (corresponding to a temporal linewidth of \( \Delta \tau \approx 200 \text{ fs} \) at the beam center defined as \( 1/e^2 \) full width). We plot in Figure 5 the measured spectra for two classes of ST wave packets: subluminal in Figure 5a,c,e with \( \theta = 44^\circ \) and \( \tilde{v} \approx 0.96c \), and superluminal in Figure 5b,d,f with \( \theta = 46^\circ \) and \( \tilde{v} \approx 1.04c \). In each category we produce three distinct wave packets in free space: 1) a GVD-free wave packet that is propagation invariant (Figure 5a,b); 2) a dispersive ST wave packet endowed with anomalous GVD \( k_{\lambda} < 0 \) (Figure 5c,d); and 3) a dispersive ST wave packet endowed with normal GVD \( k_{\lambda} > 0 \) in free space (Figure 5e,f). In all cases, each spatial frequency \( k_x \) is associated with a single wavelength \( \lambda \). After introducing anomalous GVD \( k_{\lambda} < 0 \), \( |k_x| \) increases with respect to its GVD-free counterpart, corresponding to the required increase in propagation angle (Figure 2b,e). Alternatively, \( |k_x| \) decreases with respect to the GVD-free wave packet after incorporating normal GVD \( k_{\lambda} > 0 \) (Figure 5e,f) corresponding to the required decrease in propagation angle (Figure 2c,f). The spectral gaps in the measurements exist in the laser spectrum (shown in Figure 4, top left corner). Eliminating wavelengths from the spectrum does not impact the GVD-cancellation, which is not related to the spectral amplitudes but stems instead from the association between the wavelengths and the appropriate spatial frequencies.

From these spectra in the \((k_x, \lambda)\)-plane, we extract the spectral projection onto the \((k_x, k_z)\)-plane as illustrated in Figure 2. Therefore, the measurements confirm that the targeted spatio-temporal spectra are indeed produced in free space. We now proceed to verify that the expected propagation dynamics is produced in free space and dispersive media.

6. Dispersive Space-Time Wave Packets in Free Space

We reconstruct the spatio-temporal envelope of the wave-packet intensity profile \( I(x, z, \tau) \) at a given axial plane \( z \) via linear interferometry making use of the initial laser pulses as a reference.\(^{[15]} \) See Figure 4. When the ST wave packet and the reference pulse overlap in space and time, spatially resolved fringes are recorded by a CCD camera whose visibility is used to reconstruct the wave packet profile as an optical delay \( \tau \) is swept in the path of the reference pulse. Furthermore, the profiles of the ST wave packet at different axial planes \( z \) are reconstructed by displacing the CCD camera to the target plane \( z \) and compensating for the relative group delay between the ST wave packet (travelling at \( \tilde{v}_s = c \tan \theta \)) and the reference pulse (travelling at \( c \)).

We start off with a subluminal \((\theta = 44^\circ)\) propagation-invariant ST wave packet in free space, and plot in Figure 6a the measured profile \( I(x, z, \tau) \) at three axial planes in free space \((z = 0, 15, \text{and } 30 \text{ mm})\) reconstructed in a frame traveling at \( \tilde{v}_s \). The X-shaped ST wave packet travels invariantly without distortion. The
on-axis pulsewidth is constant in free space at $\Delta T \approx 200$ fs. However, once this wave packet is coupled to a dispersive medium, pulse broadening is observed; see Figure 6b for the normal-GVD medium and Figure 6c for its anomalous-GVD counterpart. The pulsewidth increases monotonically from $\Delta T \approx 200$ fs to $\Delta T \approx 600$ fs after 30 mm in either medium (Figure 6d,e).

Crucially, accompanying pulse broadening is an asymmetry in the broadening of the spatio-temporal intensity profile between the wave packets in the normal—and anomalous—GVD media. In a normal-GVD medium, the spatio-temporal profile broadens toward later delays with respect to $r = 0$, whereas it broadens toward advanced delays in an anomalous-GVD medium. The distinct field structures that emerge in these two cases allow us to unambiguously delineate the wave packet at the output of the anomalous- and normal-GVD media. In Figure 6d we plot the on-axis pulse profiles $I(0, z; r)$ at $z = 0, 15,$ and 30 mm for the wave packets in Figure 6a–c to highlight this asymmetry, which provides a clear signature of the type of GVD experienced by the wave packet. Note, however, that the rate of increase in pulsewidth $\Delta T$ with distance (Figure 6e) does not depend on the sign of the GVD.[1]

7. Propagation Invariance in Dispersive Media

7.1. Normal-GVD Cancellation

GVD-free propagation in ZnSe in the normal-GVD regime requires introducing anomalous GVD into the ST wave packet in free space. We set $k_0 \approx -1500 \text{ fs}^{-2} \text{ mm}^{-1}$ ($\omega_0 k_{\text{norm}} \approx 0.8$) for a subliminal ST wave packet and monitor its propagation in free space, whereupon it exhibits dispersive temporal broadening (Figure 7a). Moreover, the temporal asymmetry exhibited by the wave packet confirms that it experiences anomalous GVD in free space; compare Figure 7a to Figure 6c. However, once the wave packet is coupled to ZnSe, this behavior is halted, and the wave packet travels GVD-free with a propagation-invariant spatio-temporal profile independently of the distance up to a 30-mm-thick ZnSe sample (Figure 7b). The on-axis pulse broadening in free space depicted in Figure 7c,d is in quantitative agreement with the expectation based on the GVD coefficient introduced, whereas the pulsewidth is constant after GVD-cancellation.

7.2. Anomalous-GVD Cancellation

Propagation invariance in the anomalous-GVD sample requires introducing normal GVD into the ST wave packet in free space. Setting $k_0 \approx 500 \text{ fs}^{-2} \text{ mm}^{-1}$ ($\omega_0 k_{\text{norm}} \approx 0.25$) and monitoring the propagation of the dispersive ST wave packet in free space reveals dispersive temporal broadening (Figure 7e). The temporal asymmetry in the wave packet spreading is consistent with normal GVD (compare Figure 7e to Figure 6b). However, after traversing the chirped mirrors, the wave packet travels GVD-free with a propagation-invariant spatio-temporal profile independently of the sample thickness (Figure 7f). Once again, the broadening in the on-axis pulsewidth $\Delta T$ is in quantitative agreement with the expectation based on the GVD coefficient introduced, whereas the pulsewidth in the medium is constant after GVD-cancellation (Figure 7g,h).

7.3. Independence of the Group Velocity and GVD-Cancellation

The measurements in Figure 7 were carried out at a fixed group velocity $\tilde{v}_g$ in free space. In many nonlinear optical applications that benefit from GVD-cancellation, it is also useful to also control the wave-packet group velocity. This can help group-velocity matching between pulses at disparate wavelengths while exploiting long crystals. In our scheme, the angular dispersion profile $\varphi(\lambda)$ can be controlled almost arbitrarily (Figure 4); each wavelength $\lambda$ can be assigned a propagation angle $\varphi(\lambda)$ independently of all other wavelengths, and we can thus tune $v_g$ and $k_0$ independently.[10] We demonstrate this capability in Figure 8 where we measure the group delay in normal—and anomalous—GVD samples of fixed length ($L = 30$ mm) while varying the spectral tilt angle $\theta_\tilde{v}$ in free space. This results in tuning of the free-space group velocity $\tilde{v}_g = c \tan \theta_\tilde{v}$ and hence the group velocity $\tilde{v}$.
Figure 7. Cancellation of either normal or anomalous GVD. a) Measured spatio-temporal intensity profiles $I(x,z;\tau)$ at $z=0, 15, \text{and } 30 \text{ mm}$ for a ST wave packet endowed with anomalous GVD in free space, and b) in a normal-GVD medium, whereupon dispersion is cancelled. c) On-axis $x=0$ profiles $I(0,z;\tau)$ for the ST wave packets in (a,b). The panels provide the pulse profiles at $z=0$ (where the two coincide), $z=15 \text{ mm}$, and $z=30 \text{ mm}$ (where the pulse has dispersed in free space but not in the medium). d) On-axis pulsewidth $\Delta T$ measured at 5-mm axial intervals. e) Measured spatio-temporal intensity profiles $I(x,z;\tau)$ at $z=0, 15, \text{and } 30 \text{ mm}$ for a ST wave packet endowed with normal GVD in free space, and f) in an anomalous-GVD medium, whereupon dispersion is canceled. g,h) Same as (c,d) but for the ST wave packets in (e,f).

Figure 8. a) Measured group delay $\Delta \tau$ for a ST wave packet with respect to a conventional pulsed plane-wave upon traversing a fixed-thickness medium ($L=30 \text{ mm}$) while tuning the free-space spectral tilt angle $\theta_a$. This wave packet is endowed with anomalous GVD in free space and is coupled to a normal-GVD medium where it is GVD-free. b) Same as (a) but for a ST wave packet endowed with normal GVD in free space traversing a fixed-thickness medium having anomalous GVD. The dashed curves in (a,b) are the theoretical predictions $\Delta \tau = \frac{L}{\gamma}$, where $\gamma$ is determined from Equation (2). In the medium (Equation (2)). Throughout, we maintain GVD-cancellation; that is, the wave packet is invariant after the sample independently of $\gamma$. As $\gamma$ is increased continuously from the subluminal regime ($\theta_a < 45^\circ, \gamma_a < c$) to the superluminal ($\theta_a > 45^\circ, \gamma_a > c$) regime, the group velocity of the wave packet in the medium $\gamma$ also increases, resulting in a concomitant drop in the group delay over the fixed sample length. This confirms that GVD-cancellation and group-velocity-tunability can be maintained independently of each other.

8. Inverting the Group-Velocity Dispersion

When GVD-cancellation is not achieved, the GVD coefficient in the medium $k'_{2m}$ is given by

$$k'_{2m} = k_{2m} + \frac{k_{2a}}{n_m} + \frac{1}{n_m} \frac{\Delta}{\nu_0}$$

In other words, the effective GVD in the medium $k'_{2m}$ combines the intrinsic material GVD $k_{2m}$ (due to chromatic dispersion) with the free-space GVD introduced into the ST wave packet (via non-differentiable angular dispersion), in addition to the negligible offset $\Delta$. By varying $k_{2a}$ we can realize one of three different scenarios. First, the GVD introduced in free space can reinforce the GVD in the medium ($k_{2a}$ has the same sign as $k_{2m}$), leading to GVD enhancement ($|k'_{2m}| > |k_{2m}|$). We present measurements for such a scenario in Figure 9a where we enhance the dispersion in the normal-GVD medium by introducing normal-GVD in free space (Figure 9a), and similarly enhance the dispersion in the anomalous-GVD regime by introducing anomalous GVD...
Figure 9. Enhancing, eliminating, and inverting GVD in a dispersive medium by varying the GVD introduced into the ST wave packet in free space. a–c) Spatio-temporal intensity profiles at \( z = 30 \text{ mm} \) in ZnSe (normal GVD) while varying the free-space GVD: a) \( \cos \Delta k_{zm} = 0.8 \), b) \( \cos \Delta k_{zm} = 0.8 \), and c) \( \cos \Delta k_{zm} = -0.25 \). d–f) Same as (a–c) but for the anomalous-GVD sample: d) \( \cos \Delta k_{zm} = 0.25 \), e) \( \cos \Delta k_{zm} = 0.25 \), and f) \( \cos \Delta k_{zm} = 0.75 \). In (a,d) the GVD experienced by the ST wave packets is enhanced; in (b,e) the GVD is cancelled; and in (c,f) the GVD is inverted.

in free space (Figure 9d). Comparing Figure 9a,d to Figure 6b,c confirms the enhanced pulse broadening. Second, the GVD experienced by the wave packet in the medium can be cancelled \( \Delta k_{zm} = 0 \) by setting \( \Delta k_{zm} = -n_{m} k_{zm} \) (Figure 9b,e), which is the scenario dealt with above in Figure 7.

Third, the effective GVD experienced by the wave packet in the medium can be inverted as shown in Figure 9c,f. By GVD-inversion we mean that the effective GVD coefficient in the medium \( \Delta k_{zm} \) has the opposite sign as that of the intrinsic chromatic dispersion in the medium \( k_{zm} \) (of course, the magnitudes need not be equal). In Figure 9c, the ST wave packet traveling in ZnSe in the normal-GVD regime instead encounters anomalous GVD. Here the anomalous GVD introduced in free space overcomes the normal GVD in the medium and renders it effectively an anomalous-GVD medium. Similarly, the ST wave packet in Figure 9f traveling in the anomalous-GVD medium encounters normal GVD. This can be useful in exploiting media that have desirable nonlinear coefficients for particular interactions but whose sign of GVD at the wavelength of interest is opposite of what is needed.

9. Discussion and Conclusion

We emphasize again the distinction between “dispersion compensation” and “dispersion cancellation.” In the former, after a conventional pulse traverses a dispersive medium, an optical system compensates for the dispersion (in principle of any order or sign) encountered by removing the accumulated spectral phase. This can be accomplished using a 4f spectral phase modulator\( ^{19} \) or other systems. By “dispersion cancellation” we refer to modifying the structure of the optical field by introducing angular dispersion, such that it propagates invariantly in the dispersive medium. Whereas conventional angular dispersion (in TPFs) can cancel normal GVD but not anomalous GVD, we have demonstrated here that non-differentiable angular dispersion (in ST wave packets) enables GVD-cancellation in both the normal and anomalous regimes.

Although the existence of propagation-invariant ST wave packets in presence of normal or anomalous GVD was known theoretically, the lack of experimental strategies for producing non-differentiable angular dispersion precluded putting these predictions to test. Another obstacle faced previously is that the required ST wave packets were of the “baseband” class; that is, their spatial spectra are centered at \( k_{z} = 0 \). Until recently, all experimentally generated ST wave packets in free space were of the “sideband” variety; that is, there spatial spectra are centered at \( k_{z} \neq 0 \) and the low spatial frequencies in the vicinity of \( k_{z} = 0 \) are excluded on physical grounds.\( ^{49} \) Examples include focus-wave modes\( ^{53–55} \) and X-waves.\( ^{56,57} \) Both of these obstacles are overcome by exploiting the universal angular dispersion synthesizer in ref. [33]. Although such an approach introduces arbitrary angular dispersion in one transverse dimension only, recent progress has extended this strategy to both transverse dimensions.\( ^{58–60} \)

Although a previous experiment demonstrated normal-GVD cancellation in silica using modified X-waves,\( ^{61,62} \) no attempts at cancelling anomalous GVD by exploiting focus-wave modes, X-waves, or other sideband ST wave packets have been reported. Baseband ST wave packets have been synthesized via energy-inefficient spatio-temporal amplitude filtering for cancelling anomalous\( ^{61} \) and normal\( ^{64} \) GVD, but propagation invariance in presence of dispersion was not verified. Finally, theoretical studies have uncovered a host of structural field transitions for dispersion-free ST wave packets in dispersive media that have no analogs in free space, including a transition from X-shaped to O-shaped profiles while tuning the group velocity in presence of anomalous GVD.\( ^{23} \) and even more complex transitions in the normal-GVD regime.\( ^{22} \) All such transitions occur at a fixed wavelength (in contrast to refs. [65, 66] where the transition requires changing the GVD sign). None of these phenomena have been observed to date, and an O-shaped ST wave packet has not yet been reported. We anticipate that the work presented here can provide the platform for studying these structural dynamics in dispersive media.

In conclusion, we have realized dispersion-free propagation in dispersive media symmetrically in the normal—and anomalous—GVD regimes. By incorporating non-differentiable angular dispersion into a pulsed field we produce ST wave packets whose group velocity and GVD coefficient can be tuned in free space independently of each other. We have confirmed dispersion-free propagation of 200-fs pulses at a wavelength \( \lambda_{c} \approx 1 \text{ mm} \) in ZnSe (normal GVD) and chirped Bragg mirrors (anomalous GVD). Moreover, because the GVD in the medium combines additively with the GVD introduced into the ST wave packet in free space, we have succeeded in demonstrating
GVD-inversion: the wave packet experiences normal GVD while propagating in a medium in its anomalous-GVD regime, and vice versa. Moreover, we have demonstrated this unprecedented level of GVD control independently of the wave-packet group velocity, which can be tuned separately. These results are useful in multi-wavelength nonlinear interactions and quantum optics in long crystals.

Appendix: Coupling a Tilted-Pulse Front to a Dispersive Medium

In an on-axis TPF in free space, the propagation angle with respect to the z-axis takes the general form \( \phi(z) = \Omega z + \sqrt{2} k z \), with \( \phi(z) = 0 \), \( \phi(0) = \frac{2\pi}{\omega} \frac{\omega_o}{c} k z \), and we similarly expand \( k_0 \) and \( k_z \): \( k_0 (z) = k_0 + \frac{k_0^2}{2} (z + \Delta z) + \ldots \) and \( k_z (z) = k_z + \frac{k_z^2}{2} (z + \Delta z) + \ldots \), where \( k_0^2 = 0 \), \( c_{\omega_0} = c_{\omega_0} k_0^2 \), \( c_{\omega_0} k_z^2 = 2c_{\omega_o} c_{\omega_0} k_0 + c_{\omega_0} c_{\omega_0} k_z^2 \), \( k_0 = \frac{\omega_o}{c} k_0 \), \( c_{\omega_0} = \frac{\omega_o}{c} c_{\omega_0} \), and \( c_{\omega} c_{\omega_0} k_z^2 = c_{\omega} k_z - c_{\omega_0} (\omega_o k_0)^2 \).

Because \( k_0 \) is invariant across a planar interface at normal incidence, matching the first-order expansion coefficients for \( k_0 \) yields \( \phi(z) = n_{\omega_o} \phi_0 \), which can be recognized as the law of refraction for TPFs at normal incidence. Dispersion-free propagation in the medium \( k_0^2 = 0 \) requires that \( \omega_o = \omega_0 = \tan \delta(z) = n_{\omega_o} \), only normal GVD \( k_{\text{norm}} > 0 \) can be cancelled. The corresponding TPF in free space has a GVD coefficient \( k_0^2 = -n_{\omega_o} k_{\text{norm}} \). Therefore, GVD-cancellation requires exercising control over only first-order angular dispersion. To simplify the synthesis of the TPF in free space, we set \( \phi(z) = 0 \) for \( n \geq 2 \). This assumption does not lead to the elimination of \( \phi_0 \), which is given by \( n_{\omega_o} = \frac{2(1 - \omega_o \omega_0) \phi_0}{2} \). The transverse wave number is \( k_0 (z) = \frac{2}{\omega_o} \tan \delta(z) \), which is differentiable with respect to \( \omega_o \) everywhere. In free space, \( k_0 (z) = \frac{2}{\omega_o} \sin \phi (z) \), so that \( \phi_0 (z) = \frac{\omega_o}{\omega} \tan \delta(z) \).

Although the TPF in the material is GVD-free, other dispersion terms nevertheless exist because of the \( \phi_0 \) term. Of course one may eliminate the \( \phi_0 \) term by including an appropriate \( \phi_0 \) term in free space. However, this would add to the complexity of the system. Indeed, no known optical device—besides the universal angular-dispersion synthesizer—has reported independent control over both \( \phi_0 \) and \( \phi_1 \).

Data Availability Statement

The data that support the findings of this study are available from the corresponding author upon reasonable request.

Keywords

angular dispersion, diffraction-free, dispersion cancellation, group velocity dispersion, space-time wave packets

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Conflict of Interest

The authors declare no conflict of interest.

References

[1] B. E. A. Saleh, M. C. Teich, Principles of Photonics, Wiley, Hoboken, NJ 2007.
[2] A. M. Weiner, Ultrafast Optics, John Wiley & Sons, Inc., Hoboken, NJ 2009.
[3] D. Strickland, G. Mourou, Opt. Commun. 1985, 56, 3219.
[4] R. L. Fork, O. E. Martinez, J. P. Gordon, Phys. Lett. 1984, 9, 150.
[5] O. E. Martinez, IEEE J. Quantum Electron. 1987, 23, 39.
[6] W. E. White, F. G. Patterson, R. L. Combs, D. F. Price, R. L. Shepherd, Opt. Lett. 1993, 18, 1343.
[7] B. E. Lemoff, C. P. J. Barty, Opt. Lett. 1993, 18, 1651.
[8] S. Kane, J. Sjever, J. Opt. Soc. Am. B 1997, 14, 661.
[9] S. Kane, J. Sjever, J. Opt. Soc. Am. B 1997, 14, 1237.
[10] A. M. Weiner, Rev. Sci. Instrum. 2000, 71, 1929.
[11] A. P. J. Runge, D. H. Hudson, K. K. K. Tam, C. M. de Sterke, A. Blanco-Redondo, Nat. Photonics 2020, 14, 492.
[12] J. P. Torres, M. Hendrych, A. Valencia, Adv. Opt. Photonics 2010, 2, 319.
[13] S. Szatmári, P. Simon, M. Feuerhake, Opt. Lett. 1996, 21, 1156.
[14] J. A. Fülöp, J. Hebling, In Recent Optical and Photonic Technologies (Ed: K. Y. Kim), InTech, London 2010.
[15] O. E. Martinez, J. P. Gordon, R. L. Fork, J. Opt. Soc. Am. A 1984, 1, 1003.
[16] M. A. Porras, G. Valiulis, P. Di Trapani, Phys. Rev. E 2003, 68, 016613.
[17] M. A. Porras, S. Trillo, C. Conti, P. Di Trapani, Opt. Lett. 2003, 28, 1090.
[18] S. Longhi, Opt. Lett. 2004, 29, 147.
[19] M. A. Porras, P. Di Trapani, Phys. Rev. E 2004, 69, 066606.
[20] D. N. Christodoulides, N. K. Efremidis, P. Di Trapani, B. A. Malomed, Opt. Lett. 2004, 29, 1446.
[21] M. S. Mills, G. A. Siviloglou, N. Efremidis, T. Graf, E. M. Wright, J. V. Moloney, D. N. Christodoulides, Phys. Rev. A 2012, 86, 063811.
[22] S. Malaguti, S. Trillo, Phys. Rev. A 2009, 79, 063803.
[23] S. Malagutti, C. Bellanca, S. Trillo, Opt. Lett. 2008, 33, 1117.
[24] P. Di Trapani, G. Valiulis, A. Piskarskas, O. Jedrzejewicz, J. Trull, C. Conti, S. Trillo, Phys. Rev. Lett. 2003, 91, 093904.
[25] D. Faccio, M. A. Porras, A. Dubietis, F. Bragheri, A. Couairon, P. Di Trapani, Phys. Rev. Lett. 2006, 96, 193901.
[26] D. Faccio, A. Averchi, A. Couairon, M. Kolesik, J. Moloney, A. Dubietis, G. Tamosauskas, P. Polesana, A. Piskarskas, P. Di Trapani, Opt. Express 2007, 15, 13077.
[27] M. A. Porras, A. Dubietis, A. Matijsiušius, R. Piskarskas, F. Bragheri, A. Averchi, P. Di Trapani, J. Opt. Soc. Am. B 2007, 24, 581.
[28] H. E. Kondakci, A. F. Abouraddy, Opt. Express 2016, 24, 28659.
[29] K. J. Parker, M. A. Alonso, Opt. Express 2016, 24, 28669.
[30] M. A. Porras, Opt. Lett. 2017, 42, 4679.
[31] N. K. Efremidis, Opt. Lett. 2017, 42, 5038.
[32] M. A. Porras, Phys. Rev. A 2018, 97, 063803.
