STABILITY OF EQUILIBRIA POINTS FOR A DUMBELL SATELLITE WHEN THE CENTRAL BODY IS OBLATE SPHEROID

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ABSTRACT. The main aim of the present work is to study the positions of the equilibria points and their stability in the frame work of satellite approximation. The significant implication is that the motion around these points is unstable in the linear sense. The principle of angular momentum conservation is used as a tool to reduce the degree of freedom of the dynamical systems of equations. The positions of the relative equilibria are explicitly found as well as necessary and sufficient conditions for stable motion in the linear sense are stated.

1. Introduction

The dumbbell satellite in its most simple structure is composed of two point masses connected by a massless non extensible link. It moves around an object whose gravity central field holds a mutual gravitational attraction with the masses described by the Newton’s universal law of gravitation. The problem of dumbbell satellite is considered a special case of a tethered satellite problem which formally has the same structure of the dumbbell satellite with an extensible link between the two point masses. The dynamics of a dumbbell or tethered satellites have been extensively treated in the literature. We can highlight some significant contributions related to existence and stability of equilibrium points as well as periodic and bifurcations solutions of these problems in the sequel.

Celletti and Sidorenko (2008) investigated the dumbbell satellite’s attitude dynamics when the center of mass moves on a Keplerian trajectory. They found a stable relative equilibrium position in the case of circular orbits which disappears as far as elliptic trajectories are considered. In circular orbits they replaced the equilibrium position by planar periodic motions, which are proved to be unstable with respect to out-of-plane perturbations. Wong and Misra (2008) examined the planar dynamics
of a variable length multi-tether system at the second Sun-Earth Lagrangian point. They determined a closed form solution of the system under some simple tether length functions. They also obtained numerical results for tether pitch libration under more complex tether length functions. In addition they showed that the linear controller can accurately control the spiral motion via numerical simulations. The in-plane periodic solutions of a dumbbell satellite system in elliptic orbits were obtained via bifurcation with respect to the orbital eccentricity, and their trajectories of the searched periodic solutions were projected on the van der Pol plane by Nakaniishi et al. (2011) Zhang et al (2012) applied coincidence degree theory to establish the criteria on the existence of periodic solutions for a tethered satellite system in an elliptical orbit. They presented the uniqueness of periodic solutions for the tethered satellite in a circular orbit. They also addressed the conditions on the global asymptotic stability of the equilibrium states for the tethered satellite system in accordance with the Lyapunov stability theory and Barbashin-Krasovski theory. A simplified model of an orbital cable system equipped with an elevator when the cabin performs periodic "shuttle" motions is studied by Burov et al. (2012), under the assumption that the elevator mass is small compared with the dumbbell mass. They used Poincare's theory to determine the conditions for the existence of families of system periodic motions analytically depending on the arising small parameter and passing into some stable radial steady-state motion of the unperturbed problem as the small parameter tends to zero. They also proved that, for sufficiently small parameter values, each of the radial relative equilibria generates exactly one family of such periodic motions. Moreover they studied the stability of the obtained periodic solutions in the linear approximation. Vera (2013) gave the sufficient conditions for the existence of periodic solutions of a rigid dumbbell satellite placed in the equilateral equilibrium L4 of the restricted three-body problem via averaging theory. The relevance of eccentric reference orbits on the dynamics of a tethered formation and the stability of the formations when a massive cable model is included in the analysis of a multi-tethered satellite formation is discussed and studied by Avanzini et al. (2014). Abouelmagd et al. (2015a) studied the dynamics of a dumbbell satellite moving in a gravity field generated by an oblate body considering the effect of the zonal harmonic parameter. They proved that the trajectory of the mass center of the system is periodic and different from the classical one when the effect of the zonal harmonic parameter is non zero via the Lindstedt-Poincare's technique. Moreover they also completed the classical theory showing that the equations of motion in the satellite approximation can be reduced to Beletsky's equation when the zonal harmonic parameter is
zero. Inspired in Abouelmagd (2012), Abouelmagd et al. (2014), Abouelmagd et al. (2015b) we continue with the study of the stability of motion around the equilibria points and relative equilibria points in the linear sense.

2. Equation of motion

The equations of motion in satellite approximation can be written in the following form, see for details Abouelmagd et al. (2015)

\begin{align}
(1a) \quad m_s (\ddot{r} - r \dot{\theta}^2) &= -k(m_s \left( \frac{1}{r^2} + \frac{3A}{2r^4} \right) + \frac{3\mu l^2}{2r^4}(3 \cos^2 \Theta - 1)) \\
(1b) \quad (m_s r^2 + \mu l^2) \dot{\theta} + \mu l^2 \dot{\phi} = p_\theta \\
(1c) \quad \mu l^2 (\ddot{\theta} + \ddot{\phi}) &= -\frac{3k\mu l^2}{r^3} \cos \Theta \sin \Theta
\end{align}

The above equations represent an autonomous nonlinear dynamical system for the motion of the dumbbell in satellite approximation. This equations can be rewritten in the form

\begin{align}
(2a) \quad \ddot{r} &= r \dot{\theta}^2 - k(m_s \left( \frac{1}{r^2} + \frac{3A}{2r^4} \right) + \frac{3a^2}{4r^4}(3 \cos \phi + 1)) \\
(2b) \quad \dot{\theta} &= -\frac{2r}{r^2} \dot{\theta} + \frac{3ka^2}{2r^5} \sin \phi \\
(2c) \quad \ddot{\phi} &= \frac{4r}{r^2} \dot{\phi} - \frac{3k(r^2 + a^2)}{r^5} \sin \phi
\end{align}

where \( \phi = 2\Theta, a^2 = n_1 n_2 \) and \( n_i = m_3 - l/m_s, i = 1, 2 \)

3. Stability of motion around equilibria points

In this section we find the equilibrium points and study the stability of motion around these points in the linear sense for the dynamical system given by Eqs. (2). Hence we impose that \( r = r_e, \theta = \theta_e \) and \( \phi = \phi_e \) at equilibrium points. Therefor we will have the below conditions.

\begin{align}
(3a) \quad (1 + \frac{3A}{2r_e^2}) + \frac{3a^2}{4r_e^2}(3 \cos \phi_e + 1) &= 0 \\
(3b) \quad \sin \phi_e &= 0
\end{align}

from Equation (3b) we imply that \( \phi_e = n\pi, n \in \mathbb{Z} \). To linearize equations of motion given by System (2), we assume that the perturbations of the configuration variables from their equilibrium points are denoted by
\((x_1, x_2, x_3)\) where \((\dot{r}, \dot{\theta}, \dot{\phi})\) are equivalent \((x_4, x_5, x_6)\). Consequently the linearized system will be controlled by

\[\dot{X} = MX\]  

where

\[
\dot{X} = \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \dot{x}_5 \\ \dot{x}_6 \end{pmatrix}, \quad X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix}, \quad M = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ b_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & b_2 & 0 & 0 & 0 \\ 0 & 0 & b_3 & 0 & 0 & 0 \end{pmatrix}
\]

\(b_1 = \frac{2k}{r_e^3}[1 + \frac{3}{r_e^2} + \frac{3a^2}{2r_e^2}(3\cos\phi_e + 1)]\)

\(b_2 = \frac{3ka^2}{2r_e^5}\cos\phi_e\)

\(b_3 = -\frac{3k}{2r_e^5}(r_e^2 + a^2)\cos\phi_e\)

Eq. (4) represents an autonomists linear dynamical system with six degree of freedom and its characteristic equation is

\[\lambda^2(\lambda^2 - b_1)(\lambda^2 - b_3) = 0\]

It is clear that from Eq. (7) the linearized system has three conjugate pairs of the eigenvalues the first two are equal to zero and the remaining four may be real or pure imaginary or mixed between real and pure roots according to the sign of the quantities of \(b_1\) and \(b_3\). These roots will be ruled by

\[\lambda_{1,2} = 0\]

\[\lambda_{3,4} = \pm\sqrt{b_1}\]

\[\lambda_{5,6} = \pm\sqrt{b_3}\]

since \(\phi_e = n\pi\) then we obtain

\[\Theta_e = \begin{cases} m\pi \\ (2m + 1)\frac{\pi}{2} \end{cases}, \quad m = 0, 1, 2, 3, .....\]

Hence there are two types of equilibrium points first when \(\Theta = m\pi\), in this case the longitudinal axis of dumbbell is elongated in the radial of mass center and faced the attractor center by one of its ends and \(\cos\phi_e = 1\). While in the second type \(\Theta = (2m + 1)\frac{\pi}{2}\) and the longitudinal axis of
dumbbell is elongated in the tangent to the orbit of the center of mass and 
\( \cos \phi_e = -1 \). In the first type we have \( b_1 > 0 \) and \( b_3 < 0 \), therefore the roots of characteristic equation are two roots each of them equals zero and two real conjugate roots as well as two pure imaginary conjugate roots. While in the second type \( b_1 \) may have negative or positive value and \( b_3 > 0 \) and the characteristic equation has also two equal roots with zero value and two conjugate real roots but the other two roots may be real conjugate or pure imaginary conjugate roots. therefor we obtain unstable motion for all cases.

4. Stability of motion around relative equilibria points

The Lagrangian of the dynamical system given by Eqs. (1) does not depend explicitly on the angle \( \theta \) see Abouelmagd et al. (2015) for details, therefore the second equation in Eqs. (1) expressing the angular momentum conservation. This symmetry gives us a permission to reduce the degree of freedom of the dynamical system which associated with the cyclic symmetry of variable \( \theta \). The reduced equations are obtained by setting the angular momentum \( p_\theta \) constant and performing a partial Legendre transformation in the variable \( \theta \), see Marsden and Ratiu (1999) and Bloch (2003). The equilibria points obtained from the stability analysis of reduced equation is called relative equilibria points. From Eqs. (1) the reduced equations of motion can be written in the form

\[
\ddot{r} = \frac{1}{4} \left( \frac{2p - a^2 \dot{\phi}}{r^2 + a^2} \right)^2 - k \left( \frac{1}{r^2} + \frac{3A}{2r^4} \right) + \frac{3a^2}{4r^4} (3 \cos \phi + 1) \tag{10a}
\]

\[
\ddot{\phi} = 2 \dot{r} \left( \frac{2p - a^2 \dot{\phi}}{r^2 + a^2} \right) - \frac{3k}{r^5} (r^2 + a^2) \sin \phi \tag{10b}
\]

where \( p = p_\theta / m_s \)

If we use the same natation in the pervious section, the conditions of equilibrium points in this case are controlled by

\[
\frac{p^2 r_e}{(r^2 + a^2)^2} = k \left( \frac{1}{r^2} + \frac{3A}{2r^4} \right) + \frac{3a^2}{4r^4} (3 \cos \phi_e + 1) \tag{11a}
\]

\[
\sin \phi_e = 0 \tag{11b}
\]

again Eq.(11b) is satisfied when \( \phi_e = n\pi, n \in \mathbb{Z} \).

4.1. Linearized equation of motion. Now our attention is directed to linearize Eqs.(10) to obtain simplified expression which can be handled more easily. For this purpose we impose that the perturbations of the configuration variables from their equilibrium points are denoted by
\((y_1, y_2)\) where \((\dot{r}, \dot{\phi}) \equiv (y_3, y_4)\). Therefore the linearized system will be
governed by

\[(12) \quad \dot{Y} = JY\]

where

\[(13) \quad \dot{Y} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix}, \quad Y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix}, \quad J = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ c_1 & 0 & 0 & c_4 \\ 0 & c_2 & c_3 & 0 \end{pmatrix}\]

\[(14a) \quad c_1 = \frac{p^2}{(r^2 + a^2)^2} - \frac{2p^2 r_e^2}{(r^2 + a^2)^3} + \frac{2k}{r_e^3} \left[1 + \frac{3A}{r_e^2} + \frac{3a^2}{2r_e^2} (3 \cos \phi_e + 1)\right]\]

\[(14b) \quad c_2 = -\frac{3k(r_e^2 + a^2)}{r_e^5} \cos \phi_e\]

\[(14c) \quad c_3 = \frac{4p}{r_e(r_e^2 + a^2)}\]

\[(14d) \quad c_4 = -\frac{p r_e a^2}{(r_e^2 + a^2)^2}\]

It is obvious that Eq.(12) represents an autonomous linear dynamical
system with four degree of freedom and its characteristic equation is

\[(15) \quad \Lambda^4 - (c_1 + c_2 + c_3 + c_4)\Lambda^2 + c_1 c_2 = 0\]

Eq.(15) represents four roots and these roots will be given by

\[(16) \quad \Lambda^2 = \frac{1}{2} [B \pm \sqrt{D}]\]

where

\[(17a) \quad B = (c_1 + c_2 + c_3 + c_4)\]

\[(17b) \quad D = (c_1 - c_2)^2 + 2c_3 c_4 (c_1 + c_2) + (c_3 c_4)^2\]

As aforementioned in the previous section we have two types of relative
equilibrium points. first occurs when \(\Theta = m\pi\) while second appear when
\(\Theta = (2m + 1)\frac{\pi}{2}\) where \((m = 0, 1, 2, 3, ....)\). The system stability depends
on the roots of Eq.(16). If all roots are pure imaginary we will have
periodic stable solution in the proximity of relative equilibrium points.
But the solution will be unstable if any of the roots are real or complex
number.
4.2. Stability conditions. It is obvious that from Eq. (16) the conditions of obtaining stable motion are

\begin{align}
(18a) & \quad B < 0 \\
(18b) & \quad D > 0 \\
(18c) & \quad |B| > \sqrt{D}
\end{align}

In general the collection of these conditions is the guarantee to obtain stable motion if whole of them are satisfied together. While if one of the previous stated conditions is not achieved then we have at least one root, that is not pure imaginary root. In this case the motion will be unbounded and this in turn leads to unstable motion. Therefore we determine the necessary and sufficient conditions for stable motion around the relative equilibrium points through the following two theorems for each type of motion.

**Theorem 1.** In the frame work of the first type of dumbbell satellite motion the necessary and sufficient condition for stable motion in the vicinity of relative equilibrium points is \( B + c_1 < 0 \)

*Proof Theorem 1*

In first type motion \([\cos \phi_e = 1, \ c_2 < 0, \ c_3c_4 < 0]\) and we have three cases:

1. If \( c_1 > 0 \)

   Then \( |c_1 + c_2| < |c_1 - c_2| \), from Eq. (17b) \(|B| < \sqrt{D}\). If \( D > 0 \) we have two conjugate real roots and the other two are conjugate pure imaginary roots whatever \( B \) is negative or positive. But if \( D < 0 \) we have four complex roots every two of them are conjugate whatever \( B \) is also negative or not.

2. If \( c_1 = 0 \)

   There are two equal roots with zero value and the remaining two roots are conjugate pure imaginary or conjugate real according to \( B \) is negative or positive respectively.

3. If \( c_1 < 0 \)

   Then \( |c_1 + c_2| > |c_1 - c_2| \), from Eq. (17b) \(|B| > \sqrt{D}\) and \( D > 0 \). Hence we have four pure imaginary roots if \( B \) is negative. But if \( B \) is positive we have four real roots every two of them are conjugate. Then the conditions for stable motion in this case are \( c_1 < 0 \) and \( B < 0 \). Consequently the necessary and sufficient conditions is \( B + c_1 < 0 \)

**Theorem 2.** In the frame work of the second type of dumbbell satellite motion the necessary and sufficient condition for stable motion in the vicinity of relative equilibrium points is \( B - c_1 < D \)
Proof Theorem 2

In second type motion \[ \cos \phi_e = -1, c_2 > 0, c_3c_4 < 0 \] and we also have three cases:

1. If \( c_1 < 0 \)
   
   Then \( |c_1 + c_2| < |c_1 - c_2| \) from Eq.(17b) \( |B| < \sqrt{D} \). If \( D > 0 \) we have two conjugate real roots and the other two are conjugate pure imaginary roots whatever \( B \) is negative or positive. But if \( D < 0 \) we have four complex roots every two of them are conjugate whatever also \( B \) is negative or not.

2. If \( c_1 = 0 \)
   
   There are also two equal roots with zero value and the remaining two roots are conjugate pure imaginary or real conjugate according to \( B \) is negative or positive respectively.

3. If \( c_1 > 0 \)
   
   Then \( |c_1 + c_2| > |c_1 - c_2| \) from Eq.(17b) \( |B| > \sqrt{D} \). If \( D > 0 \) we have four pure imaginary roots every two of them are conjugate when \( B \) is negative. But if \( D < 0 \) we have four complex roots every two of them are conjugate whatever \( B \) is negative or not. Hence conditions for stable motion in this case are \( c_1 > 0, D > 0 \) and \( B < 0 \) there for the necessary and sufficient condition is \( B - c_1 < D \)

5. Conclusions

In this paper the equilibria points for the dumbbell satellite motion in satellite approximation are found. We found that there are two types of motion around these points. In first motion the symmetric axis of the dumbbell satellite is elongated in the radial of the common center of mass for the dumbbell and faced the attractor center by one of its ends. In second type the symmetric axis of the dumbbell is perpendicular to the radial of mass center. In general we found that the motion is not stable for each type of motion in the linear sense. After that we use the Routh reduction to reduce the degree of freedom from six to four. For the reduced system we also find the equilibria points which in this case called relative equilibria points. The stability motion in the linear sense also around these points are studied. Finally the necessary and sufficient conditions for stable motion around an equilibria points are constructed for every type of motion.

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