PACKABLE SURFACES WITH SYMMETRIES

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Abstract. We discuss several ways of packing a hyperbolic surface with circles (of either varying radii or all being congruent) or horocycles, and note down some observations related to their symmetries (or the absence thereof).

Key words: hyperbolic surface, circle packing, packing density.

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1. Introduction

Let $S_g$ be a hyperbolic genus $g \geq 2$ surface, i.e. a complete compact orientable genus $g$ topological surface endowed with a metric of constant sectional curvature $-1$. Equivalently, if $\Gamma$ is a discrete torsion-free subgroup of $PSL(2, \mathbb{R})$ acting by isometries on the upper half-plane model of the hyperbolic space $\mathbb{H}^2$, then $S_g = \mathbb{H}^2/\Gamma$, provided the quotient is compact.

Let $C$ be a set of geodesic circles embedded in $S_g$ with non-intersecting interiors. Let $T_C$ be the packing graph whose vertices are the centres of the circles in $C$, which are connected by an edge whenever the corresponding circles are tangent.

The surface $S_g$ is called packable by circles in $C$ if its packing graph $T_C$ viewed as embedded in $S_g$ with geodesic edges provides a combinatorial triangulation of $S_g$. Here and below by a (combinatorial) triangulation of $S_g$ we mean $S_g$ as a topological surface with an embedded locally finite graph $T$ such that $S_g \setminus T$ is a union of topological triangles.

The uniformization theorem of Beardon and Stephenson [1, Theorem 4], which is a generalisation of the classical Köbe–Andreev–Thurston theorem, asserts that for every combinatorial triangulation of a topological genus $g \geq 0$ surface with graph $T$ there exists a constant sectional curvature metric on it (with curvature normalized to $-1$, $0$ or $+1$) and a set of geodesic circles $C$ in this metric such that $T_C$ is isomorphic to $T$. For more information about packable surfaces, we refer the reader to the monograph [17].

The circles in $C$ may have different radii, and the packing that they provide is combinatorially “tight”: the tangency relation for circles is transitive. This notion has to be contrasted with the notion of a circle packing where all circles are supposed to be congruent (i.e. isometric to each other).

A circle packing on $S_g$ is a set $P$ of congruent radius $r > 0$ geodesic circles embedded in $S_g$ with non-intersecting interiors such that no more radius $r$ circle can be added to it. We shall assume that, in general, $r$ is less than $\text{inj rad } S_g$, the injectivity radius of $S_g$.

We shall consider only hyperbolic surfaces $S_g$, which implies $g \geq 2$ in the closed case. Then the density of such a packing $P$ equals the ratio of the area of $S_g$ covered by the circles in $P$ to the total area of $S_g$. By applying the Gauß–Bonnet theorem we obtain $\text{Area}(S_g) = 4\pi(g - 1)$. In the case of cusped hyperbolic surfaces we shall consider their
packing by horocycles instead of ordinary (i.e. compact) circles. The definition of packing density in the cusped case remains the same.

Here we would like to stress the fact that circles packings of the hyperbolic space $\mathbb{H}^2$ behave in a drastically different way, and that their packing density is not necessarily a well-defined quantity [4]. This difficulty can be alleviated by studying local densities and packing that are invariant under the action of a co-finite Fuchsian group (i.e. a subgroup $\Gamma < \text{PSL}(2, \mathbb{R})$ such that $\mathbb{H}^2/\Gamma$ has finite hyperbolic area).

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2. COMPACT PACKABLE SURFACES

The Teichmüller space $\mathcal{T}_g$ contains a dense subset of packable surfaces, for all $g \geq 2$, in the compact case [5]. All cusped hyperbolic surfaces are packable [20]. Moreover by [13], if $S_g$ is packable, then $S_g = \mathbb{H}^2/\Gamma$, where $\Gamma < \text{PSL}(2, \mathbb{R} \cap \mathbb{Q})$, so that $S_g$ is defined over algebraic numbers. Thus, being packable puts strong constraints on the metric of $S_g$.

Now, let $O_g$ be an orbifold genus $g$ surface with $k \geq 0$ orbifold points of orders $n_1, \ldots, n_k \geq 2$ or, in another words, let $O_g$ be an orbifold of signature $(g; n_1, \ldots, n_k)$. Then we say that $O_g$ is packable by a set of circles $C$ if all the orbifold points of $O_g$ are circle centres, and the tangency relation between the circles in $C$ is transitive (like in the manifold case). Thus, if $O'_g$ is an orbifold cover of $O_g$, circles lift always to circles, and $O'_g$ is also packable.

In what follows, $S(p, q, r)$ will be the standard notation for the hyperbolic “turnover” orbifold of genus $g = 0$ with three orbifolds points of orders $p, q, r \geq 2$ such that $1/p + 1/q + 1/r < 1$. Such an orbifold can be obtained by identifying two copies of a hyperbolic triangle with angles $\pi/p$, $\pi/q$ and $\pi/r$ along their respective isometric sides. This orbifold is geometrically “rigid” meaning that the hyperbolic metric on it is completely determined by the orbifold angles.

The following is a simple observation that follows from the facts that $S(p, q, r)$ can be packed by three circles, so that any orbifold covering of $S(p, q, r)$ is packable as noted above.

Lemma 2.1. Let a hyperbolic genus $g \geq 2$ surface $S_g$ be a branched covering of $S(p, q, r)$ with $1/p + 1/q + 1/r < 1$. Then $S_g$ is packable.

Indeed, it is well-known that each $S(p, q, r)$ can be obtained from a hyperbolic triangle $T$ with dihedral angles $\pi/p$, $\pi/q$, $\pi/r$ by making a “turnover” (i.e. “glueing” two copies of $T$ isometrically along their boundaries or, equivalently, “doubling” $T$ along its boundary). Let $a, b, c$ be the respective side lengths of such a triangle, as shown in Figure 1. Then the three circular segments of radii $x$, $y$ and $z$ centred at the corresponding vertices becomes circles in $S(p, q, r)$ centred at the orbifold points. Here, we set $x = \frac{a-b+c}{2}$, $y = \frac{a+b-c}{2}$, $z = \frac{-a+b+c}{2}$, and the side lengths $a, b, c$ can be determined from the hyperbolic rule of cosines [13, Theorem 3.5.4].

Let $\text{Iso}^+(S_g)$ be the group of orientation-preserving self-isometries of $S_g$, which is known to be finite and isomorphic to $N_H(\Gamma)/\Gamma$, once $S_g = \mathbb{H}^2/\Gamma$, where $N_H(\Gamma)$ is the normalizer of $\Gamma$ in $H = \text{Iso}^+(\mathbb{H}^2) \cong \text{PSL}(2, \mathbb{R})$. 
One can say that “most” surfaces in $\mathcal{T}_g$ are asymmetric (i.e. have trivial group of self-isometries), if $g \geq 3$, since such surfaces form an open dense subset of $\mathcal{T}_g$. All genus 2 surfaces are hyperelliptic, and thus admit an order 2 isometry: however, most of them (in the above sense) have only their hyperelliptic involution as a non-trivial isometry.

As mentioned above, most surfaces are not packable, although packable ones form a dense subset in $\mathcal{T}_g$. This motivates the question: “Given a surface $S_g \in \mathcal{T}_g$ with a certain number of symmetries, can we guarantee that $S_g$ is packable?”

We also know, by the Hurwitz automorphism theorem, that $|\text{Iso}^+(S_g)| \leq 84(g - 1)$, for any $S_g \in \mathcal{T}_g$ with $g \geq 2$. The next straightforward combinatorial argument shows that given enough symmetries we can always guarantee that $S_g$ covers a turnover orbifold (an observation mentioned earlier in [16]).

**Theorem 2.2.** Let $S_g$ be a hyperbolic genus $g \geq 2$ surface such that $|\text{Iso}^+(S_g)| > 12(g - 1)$. Then $S_g$ is packable.

**Proof.** We have that $S_g$ is a branched covering of $O = S_g/H$, where $H = \text{Iso}^+(S_g)$. Hence, we can apply the Riemann–Hurwitz formula to it.

Let us suppose that $O$ is a genus $h \geq 0$ surface with $k \geq 0$ orbifold points of orders $m_1, \ldots, m_k$. Thus, the ratio $\tau = \text{Area}(O)/(2\pi)$ satisfies

$$\tau = 2h - 2 + \sum_{i=1}^{k} \left(1 - \frac{1}{m_i}\right) = \frac{2g - 2}{|H|} < \frac{1}{6}. \quad (1)$$

First, if $h \geq 2$, then $\tau \geq 2$, which is impossible by the above inequality. If $h = 1$, then in order for $O$ to be an orientable hyperbolic orbifold we need $k \geq 1$. Then, $\tau \geq \frac{1}{2}$.

Finally, let $h = 0$. In this case, the fact that $O$ is an orientable hyperbolic orbifold implies that either we have $k = 3$ with $\sum_{i=1}^{3} \frac{1}{m_i} < 1$, or we have $k = 4$ and $m_1, m_2, m_3 \geq 2$, while
m_4 \geq 3$, or $k \geq 5$ with $m_i \geq 2$, $1 \leq i \leq k$. The latter two possibilities give us $\tau \geq \frac{1}{6}$ and $\tau \geq \frac{1}{2}$.

Thus, our case-by-case check implies that $O = S(m_1, m_2, m_3)$, with $\sum_{i=1}^{3} \frac{1}{m_i} < 1$, and $S_g$ is packable by Lemma 2.1 \hfill \Box

On the other hand, a packable surface does not need to have any symmetries at all.

**Proposition 2.3.** There exists a family of genus $g \geq 3$ packable hyperbolic surfaces $S_g$ such that $\text{Iso}^+(S_g) \cong \{\text{id}\}$, with arbitrarily large values of $g$.

**Proof.** Let $\Sigma = S(2,6,9)$, and let $\Gamma = \pi_1^{orb}(\Sigma) < H = \text{PSL}(2, \mathbb{R})$. By [18], $\Gamma$ is a non-arithmetic group and, by [8], $\Gamma$ is also maximal in $H$, i.e. it is not contained in any other subgroup of $H$ such that $\Gamma \leq H$ with finite index.

Since $\Gamma$ is non-arithmetic, the commensurator $\text{Comm}_H(\Gamma) = \{g \in H \mid g\Gamma g^{-1} \cap \Gamma \text{ has finite index in both } \Gamma \text{ and in } g\Gamma g^{-1}\}$ of $\Gamma$ in $H$ is discrete by [14]. By maximality, $\text{Comm}_H(\Gamma) = \Gamma$. Then we immediately have $N_H(K) = N_\Gamma(K)$ for every finite-index subgroup $K < \Gamma$, where $N_X(Y)$ means the normalizer in $X$ of a subgroup $Y < X$.

Let $K$ be a torsion-free subgroup of $\Gamma$, and let $S_K = \mathbb{H}^2/K$ be the associated hyperbolic surface. Then $\text{Iso}^+(S_K) \cong N_\Gamma(K)/K = N_\Gamma(K)/K$. By [9] Theorem 3(a)], for any finite group $G$ there exists such a subgroup $K < \Gamma$ that $N_\Gamma(K)/K \cong G$. Moreover, it also follows from the proof of [9] Theorem 3(a)] that $K$ can be chosen torsion-free and that there are infinitely many such $K$’s. Since the number of subgroups of bounded index in $\Gamma$ is bounded, then $K$’s will provide surfaces of arbitrarily large genus. The proposition follows. \hfill \Box

**Remark 2.4.** By using GAP [7], among the 335 different genus 3 surfaces that cover the non-arithmetic orbifold $S = S(2,6,9)$ with signature $(0;2,6,9)$, we find 254 ones without automorphisms. In order to obtain these numbers, we need to use LowIndexSubgroupsFpGroup routine to classify all index 18 subgroups of $\pi_1^{orb}(S) = \langle a, b \mid a^2, b^8, (ab)^9 \rangle$ without torsion, and then choose those that correspond to conjugacy classes of length 18. These are self-normalizing, and by the same argument as in Proposition 2.3 we get the smallest examples of packable surfaces without non-trivial isometries, with all genus 2 surfaces being hyperelliptic.

**Remark 2.5.** Note that in Theorem 2.2, we cannot allow $\text{Iso}^+(S_g)$ be $12(g-1)$ as the following example shows. Let $O$ be an orbifold with signature $(0;2,2,2,3)$. The corresponding Fuchsian group is known to be maximal [3], which means that $O$ does not cover a smaller orbifold. We can map $\pi_1^{orb}(O) = \langle a, b, c, d \mid a^2, b^2, c^2, d^3, abcd^{-1} \rangle$ onto the symmetric group $\mathcal{S}_4$ as $a \mapsto (2,3)$, $b \mapsto (1,2)(3,4)$, $c \mapsto (3,4)$, $d \mapsto (1,2,3)$. Then the kernel of this map $\phi$ is torsion-free and corresponds to a cover $S$ of $O$ of degree 24. By the Riemann-Hurwitz formula, $S$ is a genus 3 surface.

Moreover, we can choose $O$ to be non-arithmetic. Indeed, there exists a one-parameter family of orbifolds with signature $(0;2,2,2,3)$ coming from a family of Lambert’s quadrilaterals $Q$ shown in Figure 2. A “double” of the quadrilateral $Q$ along its boundary is an orbifold $O$ with signature $(0;2,2,2,3)$. One of its sides can be given length $\rho$ varied so that $\cosh(\rho)$ is a transcendental number. Then $O$ is non-arithmetic by [19] Theorem 4] (since the trace of $ab$ equals $2 \cosh(\rho)$) and the argument from [10] Theorem 1] shows that $S$ has exactly $|\mathcal{S}_4| = 24$ symmetries.

By [13], Chapter 9] (cf. also [12] Theorem 3] as a more accessible reference), any packable hyperbolic surface has to be defined as $\mathbb{H}^2/\Gamma$ with $\Gamma < \text{PSL}_2(\mathbb{R} \cap \mathbb{Q})$. In our case, however,
Figure 2. The Lambert quadrilateral $Q$: all of the plane angles are right, except one angle of $\frac{\pi}{3}$. One side length $\rho$ is known to be variable within a certain interval. The double of $Q$ along its boundary is the orbifold $O$ with signature $(0; 2, 2, 3)$ and fundamental group $\pi_1(O) = \langle a, b, c, d | a^2, b^2, c^2, d^2, abcd^{-1} \rangle$. The area in $O$ is known to be variable within a certain interval. The double of $Q$ along its boundary is the orbifold $O$ with signature $(0; 2, 2, 3)$ and fundamental group $\pi_1(O) = \langle a, b, c, d | a^2, b^2, c^2, d^2, abcd^{-1} \rangle$.

In this section we shall consider circle packing of surfaces with congruent circles. More precisely, let $S_g$ be a hyperbolic genus $g \geq 2$ surface with a packing of $K$ congruent radius $r$ circles on it, where $0 < r < \text{inj rad } S_g$. Then the area covered by the circles is $2\pi K (\cosh r - 1)$, while the total surface area equals $4\pi (g - 1)$. Then the packing density is simply $\rho(S_g, r) = \frac{K}{2} \cdot \frac{\cosh r - 1}{g - 1}$. The largest packing density of radius $r > 0$ circles associated with a given genus $g \geq 2$ is defined as $\rho(g, r) = \sup_{S_g \in \mathcal{T}_g} \rho(S_g, r)$, where $\mathcal{T}_g = \{S_g \mid \text{inj rad } S_g > r\}$. By convention, supremum over the empty set equals 0. Then the packing density associated solely with the genus $g \geq 2$ is $\rho(g) = \sup_{r>0} \rho(g, r)$.

**Proposition 3.1.** The following limit identity takes place: $\lim_{g \to \infty} \rho(g) = \frac{3}{\pi}$.

**Proof.** The idea is to pick an orbifold $\Sigma = S(p, p, p)$ such that $p \geq 4$ is an arbitrarily large natural number, and construct its manifold cover $S_p$. Since the orbifold fundamental group $\pi_1^\text{orb}(\Sigma) \cong \langle a, b | a^p, b^p, (ab)^p \rangle$ is a finitely generated matrix group then, by Selberg’s lemma, such a cover $S_p$ always exists, and its degree has to be at least $p$ by a simple observation about the order of torsion elements. Thus the genus $g_p$ of $S_p$ satisfies $g_p \geq \text{const} \cdot p$.

Then we obtain a surface $S_p = \mathbb{H}^2/\Gamma$ of genus $g_p$ such that $g_p \to \infty$, and $S_p$ is packable by a set $C_p$ radius $r_p$ circles, such that $\cosh r_p = \frac{1}{2} \csc \frac{\pi}{2p}$. What remains is to compute the ratio of the area covered by circles on $S_p$ to the area of $S_p$. As it follows easily from the covering argument, this is the ratio of three $\frac{1}{2p}$-th pieces of a radius $r_p$ disc to the area of an equilateral triangle with angles $\frac{\pi}{p}$.

The area in $\mathbb{H}^2$ enclosed by a radius $r > 0$ circle equals $2\pi(\cosh r - 1)$. Thus, we obtain

$$\rho(S_p, C_p) = \frac{\frac{3\pi}{p} (\cosh r_p - 1)}{\pi - 3\pi/|p|} = \frac{3}{p - 3} \left( \frac{1}{2} \csc \frac{\pi}{2p} - 1 \right) \to \frac{3}{\pi},$$

as $p \to \infty$, by using the Laurent $\csc(x) = \frac{1}{x} + \frac{\pi}{6} + O(x^2)$ for $\csc(x)$ at $x = 0$.

Moreover, all circles and their Voronoi domains in the packing of $S_p$ are congruent to each other (since the order 3 central symmetry of $\Sigma$ lifts to $S_p$ as the latter is a normal cover). The same holds for the lift of the packing to $\mathbb{H}^2$. The local density of each circle in its Voronoi
domain is then equal to $\rho(S_p, C_p)$. The former can be estimated above by the simplicial packing density $d(2r)$, associated with a regular triangle of side length $2r$. However, the latter is exactly what we already computed above, i.e. simply $\rho(S_p, C_p) = d(2r) \leq d(\infty) = \frac{3}{\pi}$, as stated in \[3, 11\]. The proposition follows.

**Remark 3.2.** Numerically, in Proposition 3.1 we have $\rho(g) \approx 0.954929658551$ for large enough genus $g$, which means that some of the genus $g$ surfaces may be very densely packed. However, some other ones across $T_g$ may be packed quite poorly.

**Remark 3.3.** It is also clear from Proposition 3.1 that the best packing density in $\mathbb{H}^2$ achieved by invariant circle packings (i.e. circle packings invariant under the action of a co-finite Fuchsian group) coincides with the best local packing density achieved by packing congruent horoballs in the ideal triangle $[3, 11]$.

### 4. Surfaces with cusps

As a generalisation of the above facts to the case of hyperbolic surfaces with cusps, let us consider packing by horocycles instead of compact circles. In this case, the above question about a surface being packable can be restated without much alteration. By [20] each cusped hyperbolic surface is known to be packable by circles, while horocycle packings seem to be less studied in this context.

![Figure 3. A surface with 3 cusps at which the corresponding horoballs (shaded) are mutually tangent. By closing up the cusps (which are topologically punctures) with extra points $c_i, i = 1, 2, 3$, we compactify the surface and obtain the packing graph on it with vertices exactly $c_i, i = 1, 2, 3$. The surface depicted above appears to be non-packable.](image)

In the context of horocycle packings of a cusped surface $S$, we suppose that all horocycles are centred in the cusps of $S$, as shown in Figure 3. The packing graph $T$ is formed by taking the completion $\overline{S}$ of $S$, with cusps “filled” by adding a number of points $c_1, \ldots, c_k$ (where $k \geq 1$ is the number of cusps of $S$), and then letting the vertices of $T$ be exactly $c_i$’s, while two vertices are connected by an edge whenever the corresponding horocycles are tangent. Then, we say that $S$ is packable by horocycles if $T$ triangulates $\overline{S}$, i.e. $\overline{S} \setminus T$ is a collection of topological triangles.
Theorem 4.1. Let $S$ be a hyperbolic surface with cusps. Then $S$ is packable by congruent horocycles with packing density $\frac{3}{\pi}$ if and only if $S = \mathbb{H}^2/\Gamma$ for some $\Gamma < \text{PSL}(2, \mathbb{Z})$, up to an appropriate conjugation in $\text{PSL}(2, \mathbb{R})$.

Proof. Let $S$ be packed by congruent horoballs centred at its cusps, so that the packing graph $T$ triangulates $\overline{S}$, then each triangle $T_m$, $m = 1, \ldots, k$, in $\overline{S} \setminus T$ is an ideal triangle that contains some parts of horocycles in its vertices. Let $\rho_m$ be the local density of the horocycles in $T_m$, i.e. the ratio of the area contained in the horocycles to the total area of $T_m$. It is well-known from hyperbolic geometry that all ideal triangles are isometric and have area $\pi$.

Another important fact is that the best local density of horoballs in the ideal triangle $T_m$, which is $\frac{2}{\pi}$, is achieved by 3 congruent horoballs bounded by the respective horocycles $C^m_k$, $k = 1, 2, 3$, centred at the vertices of $T_m$. Each pair $C^m_i$ and $C^m_j$ is tangent at a point $p^m_{ij}$ on the respective side of $T$. The perpendiculars to the sides at $p^m_{ij}$’s intersect in the common point inside $T_m$, which we shall call its centre $O_m$, as shown in Figure 4. Let us call such a configuration of horoballs in $T_m$ the maximal configuration, which is known to be unique. The density and uniqueness of the maximal configuration are discussed in [2, Theorem 4], [6, II.3.2], and [11, Proposition 2.2] (see also the remarks thereafter).

Let $\rho(C)$ be the density of the horocycle packing $C = \{C_1, C_2, \ldots, C_k\}$ of $S$. Then, by assumption, we have that $\rho(C) = \frac{1}{k} \sum_{m=1}^k \rho_m = \frac{3}{\pi}$, while $\rho_m \leq \frac{2}{\pi}$ by the above bound on horoball density in ideal triangles. This implies that $\rho_m = \frac{3}{\pi}$, and each triangle $T_m$ has the maximal density configuration of horoballs. Thus, we can drop 3 perpendiculars from the centre $O_m$ of $T_m$ onto its sides, and split $T_m$ into 6 congruent hyperbolic triangles $\Delta$ with dihedral angles 0, $\pi/3$ and $\pi/2$. Since the surface $S$ becomes tessellated by copies of $\Delta$ such that each copy can be obtained from one of the neighbours by reflecting in one of the sides, we obtain that $S$ covers the reflection orbifold $O_\Delta = \mathbb{H}^2/W(\Delta)$, where $W(\Delta)$ is the Coxeter group generated by reflections in the lines supporting the sides of $\Delta$. Since $S$ is orientable,
then $S \to O_\Delta$ factors through the orientation cover, and we obtain $S = \mathbb{H}^2 / \Gamma \to O_\Delta^+$, where $O_\Delta^+$ is the orientation cover of $O_\Delta$ and $\pi_{\text{orb}}(O_\Delta^+) = \langle a, b, c | abc, b^2, c^3 \rangle \cong \text{PSL}(2, \mathbb{Z})$. Thus, $\Gamma < \text{PSL}(2, \mathbb{Z})$, up to conjugation in $\text{Iso}^+(\mathbb{H}^2) = \text{PSL}(2, \mathbb{R})$.

If, up to conjugation, $\Gamma < \text{PSL}(2, \mathbb{Z})$, then $S$ covers the reflection orbifold $\mathbb{H}^2 / W(\Delta)$, where $\Delta$ is the triangle with 0, $\pi/3$, $\pi/2$ angles as above, and $W(\Delta)$ is its reflection group. We take the maximal horoball bounded by the horocycle $C$ centred at the ideal vertex of $\Delta$ such that $C$ is tangent to the opposite side (exactly at the right-angled vertex). Then the local density of $C$ in $\Delta$ is $\frac{3}{\pi}$ (by a straightforward computation similar to that of Proposition 3.1). Then $C \cap \Delta$ is lifted to a horocycle packing on $S$ with packing density exactly $\frac{3}{\pi}$.

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