LOW MACH NUMBER LIMIT OF THE FULL COMPRESSIBLE HALL-MHD SYSTEM

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Abstract. In this paper we study the low Mach number limit of the full compressible Hall-magnetohydrodynamic (Hall-MHD) system in $T^3$. We prove that, as the Mach number tends to zero, the strong solution of the full compressible Hall-MHD system converges to that of the incompressible Hall-MHD system.

1. Introduction. In this paper we consider the low Mach number limit of the following full compressible Hall-magnetohydrodynamic (Hall-MHD) system ([24]):

$$
\partial_t \rho + \text{div} (\rho u) = 0, 
$$

$$
\partial_t (\rho u) + \text{div} (\rho u \otimes u) + \frac{1}{\epsilon^2} \nabla \rho - \mu \Delta u - (\lambda + \mu) \nabla \text{div} u = \text{rot} b \times b, 
$$

$$
\partial_t (\rho e) + \text{div} (\rho e u) - \text{div} (\kappa \nabla T) + p \text{div} u = \epsilon^2 \left( \frac{\mu}{2} |\nabla u + \nabla u^t|^2 + \lambda (\text{div} u)^2 + |\text{rot} b|^2 \right), 
$$

$$
\partial_t b + \text{rot} (b \times u) + \text{rot} \left( \frac{\text{rot} b \times b}{\rho} \right) = \Delta b, 
$$

$$
\text{div} b = 0. 
$$

Here $\rho, u, p, e, T$ and $b$ denote the density, velocity, pressure, internal energy, temperature, and magnetic field, respectively. The physical constants $\mu$ and $\lambda$ are the shear viscosity and bulk viscosity of the flow and satisfy $\mu > 0$ and $\lambda + \frac{2}{3} \mu \geq 0$. $\kappa > 0$ is the heat conductivity. $\epsilon > 0$ is the (scaled) Mach number. $\nabla u^t$ is the

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transpose of the $\nabla u$. For simplicity, we shall consider the case that the fluid is a polytropic ideal gas, that is

$$e := C_V T, \quad p := R \rho T$$

with $C_V > 0$ and $R > 0$ being the specific heat at constant volume and the generic gas constant, respectively.

The applications of the Hall-MHD system cover a very wide range of physical objects, for example, magnetic reconnection in space plasmas, star formation, neutron stars, and geo-dynamo, see [22, 23, 25]. Due to its physical importance and mathematical interest, there are a lot of results on the Hall-MHD system. For the incompressible Hall-MHD system, for example, the regularity criteria of the solutions were obtained in [6, 11, 12, 15, 26], and the global existence of small solution and global weak solutions were given in [3–5]. For the isentropic compressible Hall-MHD system, the local existence of strong solutions, global existence of small solutions were first obtained in [9] and the low Mach number limit problem was discussed in [21, 27]. Very recently, the local well-posedness and a blow-up criterion of strong solutions to the 3D compressible full Hall-MHD system (1)-(5) with positive density was obtained in [10].

When the Hall effect term $\text{rot} \left( \frac{\text{rot} b \times b}{\rho} \right)$ is neglected, the system (1)-(5) reduces to the well-known full compressible MHD system, which has received many studies [2, 8, 13, 14, 16, 17, 19]. The local strong solution was obtained by Fan-Yu [13]. The global weak solutions was obtained by Fan-Yu [14], Ducomet-Feireisl [8] and Hu-Wang [16] respectively. The low Mach number limit problem was studied by Jiang-Ju-Li [18] in $\mathbb{T}^3$ for well-prepared initial data, Jiang-Ju-Li-Xin [19] in $\mathbb{R}^3$ for ill-prepared initial data, and Cui-Ou-Ren [2] in a bounded domain for well-prepared initial data.

In this paper we study the low Mach number limit to the full Hall-MHD system (1)-(5) with well-prepared initial data in $\mathbb{T}^3$. In the following, we introduce the new unknowns $\sigma$ and $\theta$ with $\rho := 1 + \epsilon \sigma, \quad T := 1 + \epsilon \theta$.

Then the system (1)-(5) can be rewritten as

$$\begin{align*}
\partial_t \sigma + \text{div} (\sigma u) + \frac{1}{\epsilon} \text{div} u &= 0, \\
\rho \partial_t u + \rho u \cdot \nabla u + \frac{R}{\epsilon} (\nabla \sigma + \nabla \theta) + R \nabla \sigma (\sigma \theta)
- \lambda \Delta u - (\lambda + \mu) \nabla \text{div} u &= \text{rot} b \times b, \\
C_V \rho (\partial_t \theta + u \cdot \nabla \theta) + R (\rho \theta + \sigma) \text{div} u + \frac{R}{\epsilon} \text{div} u - \kappa \Delta \theta
&= \epsilon \left( \frac{\mu}{2} |\nabla u + \nabla u^t|^2 + \lambda (\text{div} u)^2 + |\text{rot} b|^2 \right), \\
\partial_t b + \text{rot} (b \times u) + \text{rot} \left( \frac{\text{rot} b \times b}{\rho} \right) &= \Delta b, \quad \text{div} b = 0.
\end{align*}$$

We impose the initial conditions to the system (8)-(11) as

$$(\sigma, u, \theta, b)(\cdot, 0) = (\sigma_0, u_0, \theta_0, b_0) \quad \text{in} \quad \mathbb{T}^3. \quad (12)$$

A local existence result for (8)-(12) in the following sense can be shown in a similar way to that in [28]. Thus we omit the details of the proof.
Proposition 1.1 (Local existence). Let $\epsilon \in (0, 1)$. Suppose that the initial data $(\sigma_0, u_0, \theta_0, b_0)$ satisfy that $1 + \epsilon \sigma_0^m > m > 0$ for some positive constant $m$, and

$$
\tilde{c}_t^k \sigma^e(0), \tilde{c}_t^k u^e(0), \tilde{c}_t^k \theta^e(0), \tilde{c}_t^k b^e(0) \in H^{2-k} (\mathbb{T}^3), \quad k = 0, 1, 2.
$$

Then there exists a positive constant $T^e > 0$ such that the problem (8)-(12) has a unique solution $(\sigma^e, u^e, \theta^e, b^e)$ satisfying that $1 + \epsilon \sigma^e > 0$ in $\mathbb{T}^3 \times (0, T^e)$, and for $k = 0, 1, 2$,

$$
\tilde{c}_t^k \sigma^e \in C([0, T^e]; H^{2-k}), \tilde{c}_t^k u^e, \tilde{c}_t^k \theta^e, \tilde{c}_t^k b^e \in C([0, T^e]; H^{2-k}) \cap L^2(0, T^e; H^{3-k}).
$$

Remark 1.1. To simplify the statement, we have used $\tilde{\partial}_i u(0)$ to signify the quantity $\tilde{\partial}_i u(0)$ obtained through equation (9), and $\tilde{\partial}_t^2 u(0)$ is given recursively by $\tilde{\partial}_i^2 (9)$ in the same manner. Similarly, we can define $\tilde{\partial}_i \sigma(0), \tilde{\partial}_i b(0), \tilde{\partial}_i \theta(0), \tilde{\partial}_i^2 \sigma(0), \tilde{\partial}_i^2 b(0)$ and $\tilde{\partial}_i^2 \theta(0)$.

Denote

$$
\|u\|_{k,j} := \sum_{i=0}^j \|\tilde{\partial}_i^k u(0)\|_{H^{k-i}(\mathbb{T}^3)}, \quad \|u\|_{k,j}(0) := \sum_{i=0}^j \|\tilde{\partial}_i^k u(0)\|_{H^{k-i}(\mathbb{T}^3)}.
$$

The main result of this paper is stated as follows, which shows the uniform estimates of strong solutions to (8)-(12), and the corresponding low Mach number limit.

Theorem 1.2. Let $(\sigma^e, u^e, \theta^e, b^e)$ be the unique solution obtained in Proposition 1.1. Assume further that the initial data $(\sigma_0^e, u_0^e, \theta_0^e, b_0^e)$ satisfy

$$
\|(\sigma^e, u^e, \theta^e, b^e)\|_{2,2}(0) + \|(1 + \epsilon \sigma_0^e)^{-1}\|_{L^\infty} \leq D_0.
$$

Then there exist positive constants $T_0$ and $D$ such that $(\sigma^e, u^e, \theta^e, b^e)$ satisfy the uniform estimates:

$$
\sup_{0 \leq t \leq T_0} \left( \|(\sigma^e, u^e, \theta^e, b^e)\|_{2,2} + \|(1 + \epsilon \sigma^e)^{-1}\|_{L^\infty} \right) (t) + \left( \int_0^{T_0} \|(u^e, \theta^e, b^e)\|_{3,2}^2 \, dt \right)^{1/2} \leq D,
$$

with $D_0, T_0$ and $D$ independent of $\epsilon > 0$. (7) and (14) imply $\rho^e \to 1$ and $T^e \to 1$ in certain Sobolev space as $\epsilon \to 0$. Furthermore, $(\sigma^e, u^e, \theta^e, b^e)$ converge to $(\sigma, u, \theta, b)$ in certain Sobolev space as $\epsilon \to 0$, and there exists a function $\pi(x, t)$ such that $(u, b, \pi)$ in $C([0, T_0; H^2])$ solves the following problem of the incompressible Hall-MHD equations:

$$
\begin{align*}
\tilde{\partial}_t u + u \cdot \nabla u + \nabla \pi - \mu \Delta u &= \text{rot } b \times b, \quad \text{div } u = 0, \\
\tilde{\partial}_t b + \text{rot } (b \times u) + \text{rot } (\text{rot } b \times b) &= \Delta b, \quad \text{div } b = 0,
\end{align*}
$$

(15) $(u, b)(\cdot, 0) = (u_0, b_0)$ in $\mathbb{T}^3$,

where $u_0, b_0$ are the weak limits of $u^e_0$ and $b^e_0$, respectively, in $H^2$ with $\text{div } u_0 = \text{div } b_0 = 0$ in $\mathbb{T}^3$.

We will denote

$$
M^e(t) := \sup_{0 \leq s \leq t} \left( \|(\sigma^e, u^e, \theta^e, b^e)(s)\|_{2,2} + \|(1 + \epsilon \sigma^e)^{-1}(s)\|_{L^\infty} \right)
$$

$$
+ \left( \int_0^t \|(u^e, \theta^e, b^e)\|_{3,2}^2 \, ds \right)^{1/2},
$$

$$
M^e_0 := M^e(t = 0).
$$
Similarly to those in [1, 7, 20], it suffices to show the following theorem to get the uniform estimates in (14). We will give the details on the proof of Theorem 1.3 in section 3 based on Theorem 1.2.

**Theorem 1.3.** Let $T^*$ be the maximal time of existence for the problem (8)-(12) given in Proposition 1.1. Then for any $t \in [0, T^*)$, we have
\[
M^*(t) \leq C_0(M_0^*) \exp[t^\frac{1}{2}C(M^*(t))],
\]
for some given nondecreasing continuous functions $C_0(\cdot)$ and $C(\cdot)$.

The novelty of our paper lies in that the system has the Hall term with strong nonlinearity and therefore the difference between our paper and the references [2, 7, 19, 20] is that we will bound some new terms $I_i$ ($i = 1, \cdots, 6$) coming from the Hall term.

However, the only different term between our system and that in [2] is the Hall term. On the other hand, we will only use the formulation of $M^*(t)$, which is same as that in [2] in our estimates. For example, we mainly use $\sup_{0 \leq s \leq t} \|b'(s)\|_{L^2} \leq M^*(t)$ and \(\left(\int_0^t \|b'(s)\|_{L^2}^2 ds\right)^{\frac{1}{2}} \leq M^*(t)\). Thus we can use the very similar method as that in [2] to show some estimates (17), (18), (19), (22), (25), (26), (28), (30), (34), (35), (36) and (37) and we omit the details below.

2. **Proof of Theorem 1.3.** This section is devoted to the proof of Theorem 1.3, we only need to show the inequality (16). We shall use some ideas developed in [2, 7, 19, 20], say, we will use $\omega := \text{rot } u$ and $J := \text{rot } b$ to show a priori estimates.

Below we shall drop the super script “c” of $\rho^c, \sigma^c, \upsilon^c, \theta^c$, etc. for the sake of simplicity; moreover, we write $M^*(t)$ and $M^*(0)$ as $M$ and $M_0$, respectively. Since the physical constants $\kappa, \nu, \nu'$, and $R$ do not bring any essential difficulties in our arguments, we shall take $\kappa = C_V = R = 1$ for presentation simplicity.

First, by taking the same calculations as that in [2], we get
\[
\left(\|\rho\|_{L^2}^2 + \|\rho^{-}\|_{L^\infty}\right)(t) \leq C_0(M_0) \exp(\sqrt{t}C(M)),
\]
\[
\|\sigma, u, \theta, b(t)\|_{L^2}^2 + \|u, \theta, b\|_{L^2(0, t; H^1)}^2 \leq C_0(M_0) \exp(\sqrt{t}C(M)),
\]
\[
\|\nabla \sigma, u, \nabla \theta(t)\|_{L^2}^2 + \|\nabla u, \Delta \theta\|_{L^2(0, t; L^2)}^2 \leq C_0(M_0) \exp(\sqrt{t}C(M)).
\]

Next, we estimate $\text{rot } (u, \text{rot } b)$. We have

**Lemma 2.1.** For any $0 \leq t \leq \min\{T^*, 1\}$, we have
\[
\|\text{rot } u, \text{rot } b(t)\|_{L^2}^2 + \|\text{rot } u, \text{rot } b\|_{L^2(0, t; L^2)}^2 \leq C_0(M_0) \exp(\sqrt{t}C(M)).
\]

**Proof.** Let $\omega := \text{rot } u$ and $J := \text{rot } b$. Applying rot to (9), we see that
\[
\rho(\partial_t \omega + u \cdot \nabla \omega) - \mu \Delta \omega = K + b \cdot \nabla J - J \cdot \nabla b,
\]
where $K := -(\partial_j \rho \partial_i u_i - \partial_i \rho \partial_j u_j) - [\partial_j (\rho u_k) \partial_k u_i - \partial_i (\rho u_k) \partial_k u_j]$.

Testing (21) by $\omega$ and carrying same computations as that in [2], we find that
\[
\|\sqrt{\omega} \omega(t)\|_{L^2}^2 + \|\text{rot } \omega\|_{L^2(0, t; L^2)}^2 \leq C_0(M_0) \exp(\sqrt{t}C(M)).
\]

Applying rot to (11), we infer that
\[
\partial_t J - \Delta J + \text{rot } (b \times u) + \text{rot }^2 \left(\frac{\text{rot } b \times b}{\rho}\right) = 0.
\]
Lemma 2.2. For any $J$, we deduce that
\[
\frac{1}{2}\|J(t)\|_{L^2}^2 + \|\nabla J\|_{L^2(0,t;L^2)}^2 \\
\leq C_0(M_0) + \left|\int_0^t \left[ \text{rot} (b \times u) + \text{rot} \left( \frac{\text{rot} b \times b}{\rho} \right) \right] \text{rot} J \, dx \, dt \right| \\
\leq C_0(M_0) + C \int_0^t \left( \|b\|_{L^2} \|u\|_{H^1} + \|u\|_{L^2} \|b\|_{H^1} + \|b\|_{L^2} \|b\|_{H^2} \\
+ \|\nabla b\|_{L^4}^2 + \|\nabla \rho\|_{L^6} \|\text{rot} b\|_{L^6} \|b\|_{L^6} \right) \|\text{rot} J\|_{L^2} \, dt \\
\leq C_0(M_0) + tC(M). 
\]
(24)

From (22) and (24), we get (20). \hfill \Box

By taking the very similar calculations to that that in [2], we obtain
\[
\|(\partial_t \sigma, \partial_t u, \partial_t b, \partial_t \theta)(t)\|_{L^2}^2 + \|(\text{rot} \partial_t u, \text{div} \partial_t u, \text{rot} \partial_t b, \nabla \partial_t \theta)\|_{L^2(0,t;L^2)}^2 \\
\leq C_0(M_0) \exp(\sqrt{t}C(M)), 
\]
(25)

\[
\|(\nabla \text{div} u, \nabla^2 \theta)(t)\|_{L^2}^2 + \|(\nabla \partial_t \sigma, \nabla \partial_t \theta)\|_{L^2(0,t;L^2)}^2 \\
\leq C_0(M_0) \exp(\sqrt{t}C(M)). 
\]
(26)

Lemma 2.2. For any $0 \leq t \leq \min\{T^*, 1\}$, we have
\[
\|(\text{rot} \omega, \text{rot} J)(t)\|_{L^2}^2 + \|(\Delta \omega, \Delta J)\|_{L^2(0,t;L^2)}^2 \\
\leq C_0(M_0) \exp(t^{\frac{1}{4}}C(M)). 
\]
(27)

Proof. Testing (21) by $\Delta \omega$, then by calculating as that in [2], one has
\[
\|\sqrt{\rho} \omega\|_{L^2}^2(t) + \|\Delta \omega\|_{L^2(0,t;L^2)}^2 \leq C_0(M_0) + \sqrt{t}C(M). 
\]
(28)

Testing (23) by $\Delta J$, we have
\[
\frac{1}{2}\|\text{rot} J(t)\|_{L^2}^2 + \|\Delta J\|_{L^2(0,t;L^2)}^2 \leq C_0(M_0) + \left|\int_0^t \left[ \text{rot}^2 (b \times u) \Delta J \, dx \, dt \right] \right| \\
\leq C_0(M_0) + I_1 + I_2. 
\]
(29)

We bound $I_1$ and $I_2$ as follows.

\[
I_1 \leq C \int_0^t \|b\|_{H^2} \|u\|_{H^2} \|\Delta J\|_{L^2} \, dx \, dt \leq \sqrt{t}C(M), 
\]

\[
I_2 = \left|\int_0^t \int \frac{\text{rot} b \times b}{\rho} \Delta^2 \text{rot} b \, dx \, dt \right| \\
= \left|\int_0^t \int \Delta \left( \frac{\text{rot} b \times b}{\rho} \right) \Delta \text{rot} b \, dx \, dt \right| \\
\leq C \int_0^t \left( \|\nabla \text{rot} b\|_{L^2} \left\| \frac{b}{\rho} \right\|_{L^6} + \|\text{rot} b\|_{L^6} \left\| \frac{b}{\rho} \right\|_{L^2} \right) \|\nabla^3 b\|_{L^2} \, dx \, dt \\
\leq C \int_0^t \left( \|\nabla^2 b\|_{L^3} \left\| \frac{b}{\rho} \right\|_{L^6} + \|\text{rot} b\|_{L^6} \left\| \frac{b}{\rho} \right\|_{L^2} \right) \|\nabla^3 b\|_{L^2} \, dx \, dt \\
\leq C(M) \int_0^t \left( \|\nabla^2 b\|_{L^3} + \|\text{rot} b\|_{L^6} \right) \|\nabla^3 b\|_{L^2} \, dx \, dt 
\]
We bound

By the same calculations as that in [2], we have

\[ \int_0^t \| \text{rot } \hat{\partial}_t \omega \|_{H^1} ds \leq C_0(M_0) \exp(\sqrt{t} C(M)). \tag{30} \]

**Lemma 2.3.** For any \( 0 \leq t \leq \min\{T^*, 1\} \), we have

\[ \| (\hat{\partial}_t \omega, \hat{\partial}_t J)(t) \|_{L^2}^2 + \| (\text{rot } \hat{\partial}_t \omega, \text{rot } \hat{\partial}_t J) \|_{L^2(0,t;L^2)}^2 \leq C_0(M_0) \exp(\sqrt{t} C(M)). \tag{31} \]

**Proof.** Applying \( \hat{\partial}_t \) to (21), testing by \( \hat{\partial}_t \omega \), doing as that in [2], one has

\[ \frac{1}{2} \left\| \hat{\partial}_t J(t) \right\|_{L^2}^2 + \left\| \nabla \hat{\partial}_t J \right\|_{L^2(0,t;L^2)}^2 \leq C_0(M_0) + \left| \int_0^t \int \hat{\partial}_t \text{rot} (b \cdot \nabla u - u \cdot \nabla b - \nabla \text{div} u) \cdot \hat{\partial}_t J dx ds \right| + \left| \int_0^t \int \text{rot} ^2 \hat{\partial}_t \left( \frac{\text{rot } b \times b}{\rho} \right) \text{rot } \hat{\partial}_t b dx ds \right| \]

\[ = C_0(M_0) + I_3 + I_4. \tag{33} \]

We bound \( I_3 \) and \( I_4 \) as follows.

\[ I_3 = \left| \int_0^t \int \hat{\partial}_t \text{rot} (b \cdot \nabla u - u \cdot \nabla b - \nabla \text{div} u) \cdot \hat{\partial}_t J dx ds \right| \leq C \sqrt{t} \| \hat{\partial}_t b \|_{L^\infty(0,t;H^1)} \left( \| \hat{\partial}_t b \|_{L^2(0,t;H^2)} \| u \|_{L^\infty(0,t;H^2)} + \| b \|_{L^\infty(0,t;H^2)} \| \hat{\partial}_t u \|_{L^2(0,t;H^2)} \right) \leq \sqrt{t} C(M), \]

\[ I_4 = \left| \int_0^t \int \text{rot} ^2 \hat{\partial}_t \left( \frac{\text{rot } b \times b}{\rho} \right) \text{rot } \hat{\partial}_t b dx ds \right| = \left| \int_0^t \int \hat{\partial}_t \left( \frac{\text{rot } b \times b}{\rho} \right) \delta \hat{\partial}_t b dx ds \right| \leq \sum_i \int \int \hat{\partial}_t \text{rot} b \times b \cdot \hat{\partial}_t \hat{\partial}_t b dx ds \]

\[ \leq C \int_0^t \left( \| \nabla \hat{\partial}_t b \|_{L^2} + \| \nabla^2 b \|_{L^2} + \| \nabla \rho (|b| |\nabla \hat{\partial}_t b| + |\nabla b|) \|_{L^2} \right) \]

\[ \leq C \int_0^t \left( \| \nabla \hat{\partial}_t b \|_{L^2} + \| \nabla^2 b \|_{L^2} + \| \nabla \rho (|b| |\nabla \hat{\partial}_t b| + |\nabla b|) \|_{L^2} \right) \]

\[ \leq C \left( \| \nabla b \|_{L^\infty} + \| \nabla \hat{\partial}_t b \|_{L^2} + \| \nabla \hat{\partial}_t b \|_{L^2} + \| \nabla \hat{\partial}_t b \|_{L^2} + \| \nabla \hat{\partial}_t b \|_{L^2} + \| \nabla \hat{\partial}_t b \|_{L^2} + \| \nabla \hat{\partial}_t b \|_{L^2} + \| \nabla \hat{\partial}_t b \|_{L^2} \right) \]

\[ \leq C(M) \int_0^t (\| \nabla b \|_{L^\infty} + \| b \|_{L^2} + \| \nabla \hat{\partial}_t b \|_{L^2} + 1) \| \nabla \hat{\partial}_t b \|_{L^2} ds \]
\[ \leq C(M) \int_0^t \left( \| \Delta b \|_{L^2}^2 + \| \nabla \partial_i b \|_{L^2}^2 + \| \nabla \partial_i J \|_{L^2}^2 + 1 \right) \left \| \nabla \partial_i J \right \|_{L^2} ds \]

\[ \leq C(M) \int_0^t \left( \| b \|_{H^3}^2 + \| \nabla \partial_i J \|_{L^2}^2 + 1 \right) \left \| \nabla \partial_i J \right \|_{L^2} ds \]

\[ \leq \frac{1}{2} \left \| \nabla \partial_i J \right \|_{L^2(0,t;L^2)}^2 + \sqrt{t} C(M). \]

We point out the cancellation of the triple product like

\[ \int \left( \partial_i \partial_i (\mathbf{rot} b) \times \frac{b}{\rho} \right) - \partial_i \partial_i (\mathbf{rot} b) \leq 0 \]

has been used in \( I_4 \) and a similar idea will also be used in \( I_6 \) below. Inserting the above estimates into \((33)\) and using \((32)\) lead to \((31)\).

By taking the same calculations as that in [2], we arrive at

\[ \| (\nabla \partial_i \sigma, \mathbf{div} \partial_i u, \nabla \partial_i \theta)(t) \|_{L^2}^2 + \| (\nabla \mathbf{div} \partial_i u, \partial_i \theta) \|_{L^2(0,t;L^2)}^2 \leq C_0(M_0) \exp(t \frac{1}{2} C(M)), \]

\[ \| \nabla^2 \theta(t) \|_{L^2}^2 + \| \nabla^2 \mathbf{div} u \|_{L^2(0,t;L^2)}^2 \leq C_0(M_0) \exp(\sqrt{t} C(M)), \]

\[ \| \Delta \theta \|_{L^2(0,t;H^1)} \leq C_0(M_0) \exp(t \frac{1}{2} C(M)), \]

\[ \| (\partial_i^2 \sigma, \partial_i^2 u, \partial_i^2 \theta)(t) \|_{L^2}^2 + \| (\partial_i^2 u, \partial_i^2 \theta) \|_{L^2(0,t;H^1)}^2 \leq C_0(M_0) \exp(t \frac{1}{2} C(M)). \]

Finally, we estimate \( \partial_i^2 b \) in order to close the energy estimate.

**Lemma 2.4.** For any \( 0 \leq t \leq \min\{ T^*, 1 \} \), we have

\[ \| \partial_i^2 b(t) \|_{L^2}^2 + \| \partial_i^2 u \|_{L^2(0,t;H^1)}^2 \leq C_0(M_0) \exp(\sqrt{t} C(M)). \]

**Proof.** Applying \( \partial_i^2 b \) to \((11)\), testing by \( \partial_i^2 b \), we reach

\[ \frac{1}{2} \left \| \partial_i^2 b(t) \right \|_{L^2}^2 + \left \| \partial_i^2 b \right \|_{L^2(0,t;H^1)}^2 \leq C_0(M_0) + \left | \int_0^t \int \partial_i^2 \mathbf{rot} \partial_i^2 (b \times u) dx ds \right | + \left | \int_0^t \int \partial_i^2 b \mathbf{rot} \partial_i^2 \left ( \frac{\mathbf{rot} b \times b}{\rho} \right ) dx ds \right | \]

\[ = C_0(M_0) + I_5 + I_6. \]

We bound \( I_5 \) and \( I_6 \) as follows.

\[ I_5 = \left | \int_0^t \int \partial_i^2 \mathbf{rot} \partial_i^2 (b \times u) dx ds \right | \]

\[ = \left | \int_0^t \int \partial_i^2 \mathbf{rot} b \partial_i^2 (b \times u) dx ds \right | \]

\[ \leq \int_0^t \| \partial_i^2 \mathbf{rot} b \|_{L^2} \left ( \| b \|_{L^\infty} \| \partial_i^2 b \|_{L^2} + \| \partial_i b \|_{L^\infty} \| \partial_i u \|_{L^\infty} + \| b \|_{L^\infty} \| \partial_i^2 u \|_{L^2} \right ) dx ds \]

\[ \leq C(M) \int_0^t \| \partial_i^2 b \|_{L^2} ds \]

\[ \leq \sqrt{t} C(M), \]

\[ I_6 = \left | \int_0^t \int \partial_i^2 \mathbf{rot} \partial_i^2 \left ( \frac{\mathbf{rot} b \times b}{\rho} \right ) dx ds \right | \]
for 0 \leq t \leq \min\{T^*, T_1\}, we have (16), where $M_0^{\epsilon} \leq D_0$ for $0 < \epsilon \leq 1$. In the sequence, we choose $D > C_0(D_0)$ and next $T_1 \leq 1$ such that

$$C_0(D_0) \exp(T_1^{\frac{1}{2}}C(D)) < D. \tag{40}$$

Let $t < \min\{T^*, T_1\}$. By combining the inequalities (16) and (40) with the assumption $M^*(0) = M_0^{\epsilon}$, we have that $M^*(t) \neq D$. Besides, we can assume without restriction that $D_0 \leq D$, so that $M^*(0) \leq D$. Since the function $M^*(t)$ is continuous, we obtain

$$M^*(t) \leq D \quad \text{for} \quad t < \min\{T^*, T_1\} \quad \text{and} \quad 0 < \epsilon \leq 1. \tag{41}$$

Then $T^* > T_1$ for $0 < \epsilon \leq 1$. Otherwise, by using the uniform estimates in (41) and applying Proposition 1.1 repeatedly, one can extend the time interval of existence to $[0, T_1]$, which contradicts to the maximality of $T^*$. Therefore, $M^*(t) \leq D$ for any $t \in [0, T_1]$ where $T_1$ is independent of $0 < \epsilon \leq 1$. Clearly, the conclusion is also true for $T^* = \infty$ by applying the same argument. This completes the proof.

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