Sphaleron of a 4 dimensional $SO(4)$ Higgs model

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Abstract

We construct the finite energy path between topologically distinct vacua of a 4 dimensional $SO(4)$ Higgs model which is known to support an instanton, and show that there is a sphaleron with Chern–Simons number $N_{CS} = \frac{1}{2}$ at the top of the energy barrier. This is carried out using the original geometric loop construction of Manton.
1 Introduction

The basic $SO(4)$ Higgs model $[1]$ in 4 Minkowskian dimensions supports instanton solutions in Euclidean spacetime, that are evaluated numerically $[2]$. This model is arrived at by dimensional descent from the 8 dimensional Yang–Mills action $[3]$ stabilised by the 4-th Chern class. It is also expected that in the static limit this system supports sphaleron solutions, as described in Ref. $[4]$ where such solutions were however not evaluated numerically.

Subsequently, in a different context, monopole solutions to a generalised $SO(3)$ Higgs model (generalising the Georgi–Glashow model in 3 dimensions) were numerically $[5]$ evaluated, which happen to describe the sphaleron solution of the 4 dimensional $SO(4)$ Higgs model concerning us in the present work. We will explain this connection here and will construct the finite energy path between two topologically distinct vacua describing the geometric non contractible loop (NCL) $[6]$, with the sphaleron of Chern–Simons number $N_{CS} = \frac{1}{2}$ at its top. These are the results reported in the present note. Before proceeding to these technical tasks, we note some properties of this model and its extensions, which may be of some physical relevance.

There are two distinct physical justifications for studying this model and some extensions. The first is that some extended versions of it, which certainly $[2]$ support instantons, are capable of describing a Coulomb gas of instantons. This allows the attempt at extending the work of Polyakov $[7]$ exploiting a Coulomb gas of instantons in three dimensions, to four dimensions. (Work in this direction is under active consideration.) The second physical relevance of this model is in the fact that it supports both instantons and sphalerons, in contrast for example to the standard electroweak model which supports only the latter, and whose instantons have shrinking size unless if constrained instantons $[8]$ are employed. It is known that theories which support both finite size instantons and sphalerons on the one hand, and those which support only sphalerons on the other, have quite distinct properties concerning the contribution of periodic instantons $[9]$ relative to that of the sphaleron and the instanton. This situation has been clearly demonstrated in Ref. $[10]$, in the case of various (1+1) dimensional $O(3)$ sigma models with the above described relative properties.

In Section 2, we present the model and explain how the generalised monopole $[5]$ gives the sphaleron solution. This is done using a “geometrical” NCL Ansatz $[6]$. In Section 3, we construct the finite energy barrier as a NCL with the sphaleron at the top. Section 4 is devoted to a discussion of our results in the context of some extensions of the model used here, which help to highlight the physical relevance of these results and point out the theoretical obstacles that must be tackled.

2 The model and its properties

We consider a model in $d = 4$ spacetime dimensions, consisting of a $SO(4)$ gauge field and a Higgs quartet field $\phi^a$. Using Euclidean gamma matrices to represent both the gauge and the Higgs field, the gauge field takes its values in the $so(4)$ algebra with generators $\gamma_{\mu\nu} = -\frac{1}{4} [\gamma_\mu, \gamma_\nu]$ and the Higgs field is written in antihermitean isovector matrix representation, $\Phi = \phi^a \gamma_5 \gamma_\alpha$.

As we are interested in the instanton and (particularly) sphaleron physics of the model, we give its Lagrangian $[1]$ in Euclidean signature,

$$L = \text{tr} \left( S^{\mu\nu\rho\sigma}_{\mu\nu\rho\sigma} + 4\lambda_4 S^{\mu}_{\mu\nu} + 18\lambda_3 S^{\mu}_{\mu\nu} + 54\lambda_2 S^{\mu}_{\mu} + 54\lambda_1 S^4 \right) \quad (1)$$

with

$$S_{\mu\nu\rho\sigma} = \{ F_{\mu\nu} , F_{\rho\sigma} \}$$
The curvature is given by $F_{\mu\nu} = \partial_{[\mu}A_{\nu]} + [A_{\mu}, A_{\nu}]$, and the covariant derivative is defined by $D_\mu \Phi = \partial_\mu \Phi + [A_\mu, \Phi]$. The coupling constants are assumed to be positive, $\lambda_a > 0$. The instanton solutions to this system were evaluated numerically in Ref. [2]. Their salient properties are that they are exponentially localised and, the connection is not asymptotically pure gauge, both in contrast to the BPST [11] instanton. The first of these properties results in the finite size of the instanton, fixed by the absolute scale $\eta$. The second property results because the asymptotic gauge field decays as one half times a pure gauge, rather like a ’t Hooft–Polyakov monopole [12]. A very important consequence is that the curvature field of this instanton features an inverse–square decay, leading to the possibility of constructing a Coulomb gas of instantons [2].

The existence of finite size instantons ensures that the basic model (1) has topologically distinct vacua separated by energy barriers of finite height. To construct a path connecting two vacua “over the barrier” in the space of static configurations with finite energy $E = \int \mathcal{H} \, d^3x$ [4], where $\mathcal{H}$ is the static Hamiltonian

$$\mathcal{H} = \text{tr} \left( 4\lambda_4S^2_{ijkl} + 18\lambda_3S^2_{ij} + 54\lambda_2S^2 + 54\lambda_1S^4 \right),$$

we follow a standard geometrical technique [8] exploiting the topological properties of the model, resulting from the requirement of finite energy.

The Higgs field of any static finite energy configuration has to satisfy $|\Phi|^2 = \eta^2$ at spatial infinity ($\mathbb{R}^3)^\infty = S^2_{\text{space}}$, hence any finite energy configuration defines a mapping $\Phi^\infty : S^2_{\text{space}} \rightarrow S^3_{\text{Higgs}}$. This property allows to consider a one parameter set of static field configurations $\Phi$ parameterised by a loop parameter $\tau \in [0, \pi] := I_{\text{Loop}}$ such that these fields at spatial infinity define a topologically non-trivial mapping

$$\Phi(r \rightarrow \infty) := \Phi^\infty : S^2_{\text{space}} \times I_{\text{Loop}} \sim S^3 \rightarrow S^3_{\text{Higgs}}$$

which maps $S^2_{\text{space}}$ to $S^3_{\text{Higgs}}$ for any fixed value of $\tau \in (0, \pi)$. A simple, geometrically motivated choice of such a mapping is given by

$$\Phi^\infty(\tau, \theta, \phi) = \eta \gamma_5 \gamma_i \cdot \vec{p}(\tau, \theta, \phi), \quad \vec{p}(\tau, \theta, \phi) := \begin{pmatrix} \sin \tau \sin \theta \cos \phi \\ \sin \tau \sin \theta \sin \phi \\ \sin^2 \tau \cos \theta + \cos^2 \tau \\ \sin \tau \cos \tau (\cos \theta - 1) \end{pmatrix}. \quad (5)$$

Finite energy also requires the covariant derivative to vanish at spatial infinity. This fixes the gauge fields $\vec{A}_i, \vec{A}_4 \equiv 0$ (temporal gauge) along the loop at infinity to

$$\vec{A}_i(r \rightarrow \infty) := \vec{A}_i^\infty = -\frac{1}{4\eta^2} [\Phi^\infty, \partial_i \Phi^\infty] = -\rho^\mu \rho^\nu \gamma_{\mu\nu} \quad (6)$$

with $\vec{\rho}_i = \partial_i \vec{\rho}$. The loop itself has to start ($\tau = 0$) and end ($\tau = \pi$) in the vacuum which we choose to be $\Phi^{(V)} = \eta \gamma_5 \gamma_3$, $A_4^{(V)} = 0$. This allows the gauge field along the loop to be chosen proportional to the gauge field at infinity, introducing a radial profile function $f(r)$, whereas the Higgs field has to be deformed to reach the vacua:

$$\Phi = h(r)\Phi^\infty + (1 - h(r))\Psi, \quad \vec{A}_i = (1 + f(r))\vec{A}_i^\infty \quad (7)$$
with $\Psi = \eta \gamma_5 [\gamma_3 \cos^2 \tau - \gamma_4 \sin \tau \cos \tau]$. The profile functions $h(r)$ and $f(r)$ are subject to the boundary conditions

$$h(0) = 0, \quad f(0) = -1; \quad h(r \to \infty) = 1, \quad f(r \to \infty) = 0,$$

resulting from the requirements of regularity at the origin and finite energy. The loop resulting from this construction is noncontractible as $\tilde{\Phi}^\infty$ was chosen to be a topologically nontrivial mapping.

Inserting the ansatz into the static Hamiltonian \([3]\) and multiplying by the radial integration measure, we obtain the radial subsystem

$$H(r)[h, f] = 96 \pi \eta^6 \sin^4 \tau \left[ 4 \lambda_4 \left\{ \frac{1}{\rho^2} \left[ (1 - f^2)h' + 2f'W \right]^2 \sin^2 \tau + [2f'W]^2 \cos^2 \tau \right\} + 6 \lambda_3 \left\{ 2 \left[ (1 - h^2)f' \sin^2 \tau + 2h'W \right]^2 + 2 \left[ (1 - h^2)f' \right]^2 \sin^2 \tau \cos^2 \tau + \frac{1}{\rho^2} \left[ (1 - h^2)(1 - f^2) \sin^2 \tau + 2W^2 \right]^2 \right\} + 36 \lambda_2 \left\{ (1 - h^2)^2 \left[ (\rho h')^2 + 2W^2 \right] \sin^2 \tau \right\} + 9 \lambda_1 \left\{ \rho^2 (h^2 - 1)^4 \sin^4 \tau \right\} \right]$$

with

$$W := h - (1 + f)(\cos^2 \tau + h \sin^2 \tau).$$

It should be emphasised that extrema of the radial subsystem \([9]\) are not necessarily extrema of the static Hamiltonian \([3]\) as the ansatz \((7)\) is in general not spherically symmetric. Nevertheless, besides the (spherically symmetric) vacua for $\tau = 0$ and $\tau = \pi$, the loop ansatz reduces to a spherically symmetric ansatz for $\tau = \frac{\pi}{2}$,

$$\tilde{\Phi}|_{\tau = \frac{\pi}{2}} = \eta h(r) \gamma_5 \gamma_i \hat{x}_i, \quad \tilde{A}|_{\tau = \frac{\pi}{2}} = \frac{1 + f(r)}{r} \hat{\gamma}_{ij} \hat{x}_j.$$ (11)

For this value $\tau = \frac{\pi}{2}$, the radial subsystem loop Hamiltonian \([9]\) reduces to the radial subsystem Hamiltonian of the generalised $SO(3)$ Higgs monopole system \([13]\) (the coupling constants have to be adjusted to $\lambda_a \mapsto \frac{1}{a} \lambda_a$, $a = 1, 2, 3, 4$) for which solutions $(h^{(M)}, f^{(M)})$ are known numerically.

Due to spherical symmetry, inserting these monopole profile functions into the ansatz \((11)\) yields an extremum of the static energy functional \((3)\). Since by construction this extremum appears along a path connecting two vacua, it is expected to be a saddle point. This becomes manifest if one inserts the monopole profile functions $(h^{(M)}, f^{(M)})$ into the loop Hamiltonian \([9]\) and considers the energy $E(\tau) = \int H(r)[h^{(M)}, f^{(M)}]dr$ along the loop which can be calculated numerically using the data known from the monopole analysis \([13]\), e.g. for coupling constants $\lambda_a = \frac{1}{a}$, $a = 1, 2, 3, 4$. Fig. \([4]\) shows that for this loop, $E(\tau)$ really reaches its maximum for $\tau = \frac{\pi}{2}$ which proves that the extremum we found is indeed an instable saddle point, hence the sphaleron of the basic model \([1]\).

It is a special feature of the geometrical NCL construction that the spherically symmetric loop configuration \((11)\) involves only a Higgs triplet $(i = 1, 2, 3)$ and an $so(3)$ gauge field. This “symmetry breakdown” along the loop, together with the fact the the static $SO(4)$ Higgs Hamiltonian \([3]\) is formally equal to the generalised $SO(3)$ Higgs Hamiltonian of Ref.\([13]\) eq. (3) under the coupling constant mapping

$$\lambda_a \mapsto \frac{1}{a} \lambda_a, \quad a = 1, 2, 3, 4,$$ (12)
justifies the interpretation that the generalised \( SO(3) \) Higgs monopole \([13]\) is really “embedded” into the \( SO(4) \) Higgs model under consideration, gaining its instability from the additional gauge degrees of freedom which are excited along the loop, an effect which also occurs in lower dimensional \( SO(d) \) Higgs models with instanton and sphaleron \([14]\).

### 3 NCL, sphaleron and Chern–Simons number

The construction of the geometrical loop in the previous section was guided by the topological properties of static finite energy configurations. The sphaleron itself however, is not a “topological object”. It can be visualised as top of the static energy barrier separating topologically distinct vacua, whereas the instanton interpolates between these vacua in Euclidean spacetime. This instanton is topologically stable due to the existence of a lower bound to the Euclidean action \((\text{II})\),

\[
\int \mathcal{L} \, d^4x \geq \min\{1, \lambda_a\} \lim_{R \to \infty} \int_{S^3(R)} \Omega \quad (13)
\]

\((R^2 = |x_\mu|^2)\) where the Chern–Simons form \(\Omega\), which results from the dimensional reduction of the *fourth* Chern class, is given by \(\Omega = \Omega^{(2)} + \Omega^{(1)} + \Omega^{(0)}\),

\[
\begin{align*}
\Omega^{(2)} &= -\frac{1}{2} \eta^4 \text{tr} \left( \gamma_5 A_\nu \left( F_{\rho\sigma} - \frac{2}{3} A_\rho A_\sigma \right) \right) \, dx^\nu \wedge dx^\rho \wedge dx^\sigma \\
\Omega^{(1)} &= \frac{1}{12} \eta^2 \text{tr} \left( \gamma_5 \Phi S_{\nu\rho\sigma} \right) \, dx^\nu \wedge dx^\rho \wedge dx^\sigma \\
\Omega^{(0)} &= \frac{i}{36} \text{tr} \left( \gamma_5 \Phi S_{[\nu\rho} D_{\sigma]} \Phi \right) \, dx^\nu \wedge dx^\rho \wedge dx^\sigma.
\end{align*}
\quad (14)
\]
Figure 2: Energy along the geometrical NCL in terms of increasing Chern–Simons number $N_{CS}$, $\lambda_a = \frac{1}{a}$

This topological bound allows the classification of the instantons of the model in terms of integer Chern–Pontryagin charge,

$$q = -\frac{1}{8\pi^2\eta^4} \lim_{r \to \infty} \int_{S^3(r)} \Omega = -\frac{1}{8\pi^2\eta^4} \int_{R^4} d\Omega. \tag{15}$$

The connection between instantons and sphalerons as objects relating topologically distinct vacua becomes obvious if one interprets the NCL connecting the vacua through the sphaleron as an object in Euclidean spacetime, treating the loop parameter as time $t = x_0$ depend $\tau = \tau(t)$ such that $\tau(t \to \infty) = 0$, $\tau(t \to -\infty) = \pi$. Splitting the spacetime integral (15) into two parts, consisting of a spatial surface– and a spatial volume–integral yields what is called the Chern–Simons number at time $t_0$,

$$N_{CS}(t_0) = -\frac{1}{8\pi^2\eta^4} \left[ \int_{R^3} \Omega_0 \mid_{t=t_0}^{t=-\infty} + \int_{-\infty}^{t_0} dt \lim_{r \to \infty} \int_{S^2(r)} \hat{\Omega} \right] \tag{16}$$

$(r^2 = |x_i|^2)$ where the three–form $\Omega_0 = \Omega_0^{(2)} + \Omega_0^{(1)} + \Omega_0^{(0)}$ and the two–form $\hat{\Omega} = \hat{\Omega}^{(2)} + \hat{\Omega}^{(1)} + \hat{\Omega}^{(0)}$ in spherical coordinates are found to be

$$\Omega_0^{(2)} = \eta^4 \text{tr} \left( \gamma_5 \left( F_{[r\theta} A_{\phi]} + \frac{2}{3} A_{[r} A_{\theta} A_{\phi]} \right) \right) dr \wedge d\theta \wedge d\phi$$
$$\Omega_0^{(1)} = \frac{1}{2} \eta^2 \text{tr} \left( \gamma_5 \Phi S_{[r\theta} \Phi \right) dr \wedge d\theta \wedge d\phi$$
$$\Omega_0^{(0)} = \frac{i}{6} \text{tr} \left( \gamma_5 \Phi S_{[r} D_{\theta]} \Phi \right) dr \wedge d\theta \wedge d\phi$$
$$\hat{\Omega}^{(2)} = -\eta^4 \text{tr} \left( \gamma_5 \left( F_{[0\theta} A_{\phi]} + \frac{2}{3} A_{[0} A_{\theta} A_{\phi]} \right) \right) d\theta \wedge d\phi$$
\[ \hat{\Omega}^{(1)} = -\frac{1}{2} \eta^2 \text{tr} (\gamma_5 \Phi S_{\theta \phi}) \, d\theta \wedge d\phi \]
\[ \hat{\Omega}^{(0)} = -\frac{i}{6} \text{tr} (\gamma_5 \Phi S_{[\theta \phi]} D_{\phi}) \, d\theta \wedge d\phi. \]  

Integrating over infinite time, the Chern–Simons number equals the Chern–Pontryagin charge, \( N_{CS}(t_0 = \infty) = q \).

Assuming \( \Omega_0(t \to -\infty, x) = 0 \) which fixes \( N_{CS}(t \to -\infty) = 0 \) for the initial vacuum, the volume and surface contributions to the Chern–Simons number along the loop, respectively, are then found to be

\[ \int_{\mathbb{R}^3} \Omega_0 = \frac{8\pi \eta^4}{3} \sin^3 \tau \cos \tau \int_0^\infty \left\{ (h^2 - 3 - \sin^2 \tau) \left[ (1 - f^2 h' - 2(1 - h)f'W) + 6h'W^2 \right] \right\} dr, \]
\[ \int_{t_0}^{t_0} dt \lim_{r \to \infty} \int_{S^2(r)} \hat{\Omega} = -8\pi \eta^4 \left[ \tau(t_0) - \frac{1}{2} \sin 2\tau(t_0) \right]. \]  

In particular, the volume integral contribution vanishes if the loop reaches the sphaleron at time \( t_0 = t_S \), \( \tau(t_S) = \frac{\pi}{2} \), and the “Chern–Simons number of the sphaleron” is found to be \( N_{CS}(t_S) = \frac{1}{2} \). At infinite time, \( t_0 \to \infty \), the volume contribution vanishes, and one immediately finds \( N_{CS}(t_0 \to \infty) = 1 \), relating the NCL interpreted as Euclidean spacetime configuration to the instanton.

The volume integral can be computed numerically, using the monopole profile functions \((h^{(M)}, f^{(M)})\) which were also used to evaluate the energy along the loop in the previous section. This allows to plot the energy barrier between the two topologically distinct vacua in terms of increasing Chern–Simons number as shown in Fig. 4.

### 4 Summary and discussion

We have presented the complete construction of the sphaleron solution to the model (1), together with a finite energy barrier NCL. This completes the verification that this model supports both finite size instantons and sphalerons. Apart from the relevance of this to the study of the periodic instantons [4, 10], some extensions of this model are of some physical relevance, which we now discuss briefly. The results obtained here remain qualitatively unchanged under the extensions discussed below.

The Lagrangian (1) can be called the “basic model”, and is derived from the eight–dimensional generalised Yang–Mills system by dimensional reduction [3]. For physical reasons, it is convenient to consider an “extended model”, adding the term

\[ \mathcal{L}_{\text{ext}} = \text{tr} \left( -\mu_1 (D_\mu \Phi)^2 + \mu_2 S^2 \right) \]

(19)
to the Euclidean Lagrangian (1). The main effect of adding (19) to the “basic model” is to force a time-independent vacuum field, e.g. \( \Phi_{\text{vac}} = \eta \gamma_5 \gamma_4 \). In Ref. [2] another extended version incorporating the term \( \text{tr} F_{\mu \nu} F_{\nu \lambda} F_{\lambda \mu} \) was considered, to enable the construction of a Coulomb gas of instantons. It is interesting to note that in the absence of (19), the instanton now would be power localised, the exponential localisation being restored in the presence of (19).

Another interesting effect of adding (19) to (1) is the resulting spectrum when the Higgs mechanism is applied. Using the notation \( \tilde{\gamma} = (\gamma_\alpha, \gamma_4) \), \( \alpha = 1, 2, 3 \), and expanding around the Higgs vacuum \( \eta \gamma_5 \gamma_4 \)

\[ \Phi = \gamma_5 [\gamma_4 (\eta + v) + \zeta_\alpha \gamma_\alpha], \]

(20)
there appears a mass-like term for an isovector vector field in the (Minkowskian) Lagrangian. It is the $so(3)$ part of the gauge field corresponding to the broken $SO(3)$ subgroup of the gauge group. This part of the $so(4)$ gauge field fluctuation, $W_\mu^\alpha = 2A_\mu^{\alpha 4}$ in

$$A_\mu = A_\mu^{\alpha \beta} \gamma_{\alpha \beta} + W_\mu^{\alpha} \gamma_{\alpha 4},$$

consists of the components that do not commute with $\Phi_{\text{vac}}$. The Higgs field then has one component that stays massive, described by the scalar field $v$ in (20) and the corresponding Goldstone Bosons are swallowed via the gauge $SO(3)$ transformation

$$g = \exp \frac{1}{\eta} \gamma_{\alpha 4} \zeta_\alpha,$$

leaving the mass–like term $\mu_1 \eta^2 W_\mu W^\mu$ in the Lagrangian. Strictly speaking this term is not a mass term in the sense that the accompanying quadratic kinetic term $(\partial_\mu W_\nu - \partial_\nu W_\mu)(\partial^\mu W^\nu - \partial^\nu W^\mu)$ is absent. This is the result of the absence of the usual Yang-Mills term $\text{tr} F_{\mu \nu}^2$ in the Lagrangian (1).

This brings us to the final item of discussion, namely the question of the absence of the YM term $\text{tr} F_{\mu \nu}^2$, in (1). Independently of the dynamics of the $SO(4)$ system with the Higgs field in the 4-vector representation, there exists a Dirac gauge in which the Higgs field can be gauged to a constant at infinity. It follows that the asymptotic $so(4)$ gauge field must decay with the inverse power of $r$, with an inverse–square decay of the curvature. In 4 dimensions, the contribution of the YM term $\text{tr} F_{\mu \nu}^2$ in (1) to the action will then be logarithmically divergent. Thus, the most important property of this model (and its extended versions), namely its suitability for describing a Coulomb gas of instantons, prevents the presence of the usual YM term with the consequence that the gauge field and its massive component are not endowed with a propagator.

To complete this argument one has to eliminate the following possibility: Namely that instead of exploiting the descendent of the fourth Chern–Pontryagin charge (14), one exploits the second Chern–Pontryagin charge $\text{tr} F_{\mu \nu} * F_{\mu \nu}$. In that case, the topological lower bound

$$\text{tr} (F_{\mu \nu}^2 + \text{Higgs terms}) \geq \text{tr} (F_{\mu \nu} * F_{\mu \nu})$$

holds and as a result the curvature strength decays as the inverse power of $r^4$, corresponding to an asymptotically pure gauge connection like the BPST instanton. The problem here is that the finite action condition for the Higgs terms in (22) results in an asymptotic connection field that decays as one half times a pure gauge, which is inconsistent with the finite action requirement on the YM terms, namely that the connection be asymptotically pure gauge.

References

[1] G.M. O’Brien and D.H. Tchrakian, Mod. Phys. Lett. A4 (1989) 1389.

[2] K. Arthur, G.M. O’Brien and D.H. Tchrakian, J. Math. Phys. 38 (1997) 4403.

[3] see for example, D.H. Tchrakian, Yang-Mills Hierarchy, in Proceedings of XXI International Conference on Differential Geometric Methods in Theoretical Physics, Int. J. Mod. Phys. A (Proc. Suppl.) 3A (1993) 584.

[4] G.M. O’Brien and D.H. Tchrakian, Phys. Lett. B 282 (1992) 111.

[5] B. Kleihaus, D. O’Keeffe and D.H. Tchrakian, Phys. Lett. B 427 (1998) 327.
[6] N.S. Manton, Phys. Rev. D 28 (1983) 2019; F.R. Klinkhamer and N.S. Manton, Phys. Rev. D 30 (1984) 2212.

[7] A.M. Polyakov, Nucl. Phys. B 120 (1977) 429.

[8] I. Affleck, Nucl. Phys. B 191 (1981) 429; I. Affleck, M. Dine and N. Seiberg, ibid. B 241 (1984) 493; ibid. B 256 (1985) 557.

[9] S.Yu. Khlebnikov, V.A. Rubakov and P.G. Tinyakov, Nucl. Phys. 367 (1991) 334.

[10] A.N. Kuznetsov and P.G. Tinyakov, Phys. Lett. B 406 (1997) 76.

[11] A.A. Belavin, A.M. Polyakov, A.S. Schwarz and Yu.S. Tyupkin, Phys. Lett. B 59 (1985) 85.

[12] G.’tHooft, Nucl. Phys. B 79 (1974) 276; A.M. Polyakov, JETP Lett. 20 (1974) 194.

[13] B. Kleihaus, D. O’Keeffe and D.H. Tchrakian, Phys. Lett. B 427 (1998) 327.

[14] B. Kleihaus, D.H. Tchrakian and F. Zimmerschied, “$d$–dimensional $SO(d)$–Higgs models with instantons and sphaleron: $d = 2, 3$”, hep-th/9904048.