BOUNDS FOR THE SECOND HANKEL DETERMINANT OF CERTAIN BI-UNIVALENT FUNCTIONS

H. ORHAN, N. MAGESH AND J. YAMINI

ABSTRACT. In the present work, we propose to investigate the second Hankel determinant inequalities for certain class of analytic and bi-univalent functions. Some interesting applications of the results presented here are also discussed.

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1. Introduction

Let \(A\) denote the class of functions of the form

\[
 f(z) = z + \sum_{n=2}^{\infty} a_n z^n
\]

which are analytic in the open unit disc \(U = \{ z : z \in \mathbb{C} \text{ and } |z| < 1 \}\). Further, by \(S\) we will show the family of all functions in \(A\) which are univalent in \(U\).

Some of the important and well-investigated subclasses of the univalent function class \(S\) include (for example) the class \(S^*(\beta)\) of starlike functions of order \(\beta\) in \(U\) and the class \(K(\beta)\) of convex functions of order \(\beta\) in \(U\). By definition, we have

\[
 S^*(\beta) := \left\{ f : f \in A \text{ and } \Re \left( \frac{zf'(z)}{f(z)} \right) > \beta; \; z \in U; \; 0 \leq \beta < 1 \right\}
\]

and

\[
 K(\beta) := \left\{ f : f \in A \text{ and } \Re \left( 1 + \frac{zf''(z)}{f'(z)} \right) > \beta; \; z \in U; \; 0 \leq \beta < 1 \right\}.
\]

The arithmetic means of some functions and expressions is very frequently used in mathematics, specially in geometric function theory. Making use of the arithmetic means Mocanu \[22\] introduced the class of \(\alpha\)-convex (\(0 \leq \alpha \leq 1\)) functions (later called as Mocanu-convex functions) as follows:

\[
 \mathcal{M}(\alpha) := \left\{ f : f \in S \text{ and } \Re \left( (1 - \alpha) \frac{zf'(z)}{f(z)} + \alpha \left( 1 + \frac{zf''(z)}{f'(z)} \right) \right) > 0; \; z \in U \right\}.
\]

In \[21\], it was shown that if the above analytical criteria holds for \(z \in U\), then \(f\) is in the class of starlike functions \(S^*(0)\) for \(\alpha\) real and is in the class of convex functions \(K(0)\) for \(\alpha \geq 1\). In general, the class of \(\alpha\)-convex functions determines the arithmetic bridge between starlikeness and convexity.

It is well known that every function \(f \in S\) has an inverse \(f^{-1}\), defined by

\[
 f^{-1}(f(z)) = z \quad (z \in U)
\]

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where
\[ f(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \ldots. \]

A function \( f \in A \) is said to be bi-univalent in \( U \) if both \( f(z) \) and \( f^{-1}(z) \) are univalent in \( U \). Let \( \sigma \) denote the class of bi-univalent functions in \( U \) given by (1.1).

For \( 0 \leq \beta < 1 \), a function \( f \in \sigma \) is in the class \( S^*_\sigma(\beta) \) of bi-starlike function of order \( \beta \), or \( K_{\sigma,\beta} \) of bi-convex function of order \( \beta \) if both \( f \) and \( f^{-1} \) are respectively starlike or convex functions of order \( \beta \). Also, a function \( f \) is in the class \( M^*_\alpha(\beta) \) of bi-Mocanu convex function of order \( \beta \) if both \( f \) and \( f^{-1} \) are respectively Mocanu convex function of order \( \beta \).

For a brief history and interesting examples of functions which are in (or which are not in) the class \( \sigma \), together with various other properties of the bi-univalent function class \( \sigma \) one can refer the work of Srivastava et al. [26] and references therein. Various subclasses of the bi-univalent function class \( \sigma \) were introduced and non-sharp estimates on the first two coefficients \( |a_2| \) and \( |a_3| \) in the Taylor-Maclaurin series expansion (1.1) were found in several recent investigations (see, for example, [2, 7, 10, 16, 19, 23]). However, the problem to find the coefficient bounds on \( |a_n| \) \((n = 3, 4, \ldots)\) for functions \( f \in \sigma \) is still an open problem.

For integers \( n \geq 1 \) and \( q \geq 1 \), the \( q \)-th Hankel determinant, defined as
\[
H_q(n) = \begin{vmatrix}
  a_n & a_{n+1} & \cdots & a_{n+q-1} \\
  a_{n+1} & a_{n+2} & \cdots & a_{n+q-2} \\
  \vdots & \vdots & \ddots & \vdots \\
  a_{n+q-1} & a_{n+q-2} & \cdots & a_{n+2q-2}
\end{vmatrix} \quad (a_1 = 1).
\]

The Hankel determinant plays an important role in the study of singularities (see [8]). This is also an important in the study of power series with integral coefficients [4, 8]. The properties of the Hankel determinants can be found in [27]. The Hankel determinants \( H_2(1) = a_3 - a_2^2 \) and \( H_2(2) = a_2a_4 - a_3^2 \) are well-known as Fekete-Szegő and second Fekete-Szegő functional respectively. Further Fekete and Szegő [9] introduced the generalized functional \( a_3 - \delta a_2^2 \), where \( \delta \) is some real number. In 1969, Keogh and Merkes [14] discussed the Fekete-Szegő problem for the classes \( S^* \) and \( K \). Recently, several authors have investigated upper bounds for the Hankel determinant of functions belonging to various subclasses of univalent functions [1, 6, 13, 15, 17, 18] and the references therein. On the other hand, Zaprawa [28, 29] extended the study on Fekete-Szegő problem for certain subclasses of bi-univalent function class \( \sigma \). Following Zaprawa [28, 29], the Fekete-Szegő problem for functions belonging to various other subclasses of bi-univalent functions were considered in [3, 12, 20]. Very recently, the upper bounds of \( H_2(2) \) for the classes \( S^*_\sigma(\beta) \) and \( K_{\sigma,\beta} \) were discussed by Deniz et al. [7].

Next we state the following lemmas we shall use to establish the desired bounds in our study.

**Lemma 1.1.** [24] If the function \( p \in \mathcal{P} \) is given by the series
\[
p(z) = 1 + c_1z + c_2z^2 + c_3z^3 + \cdots,
\]
then the following sharp estimate holds:

\[ |c_k| \leq 2, \quad k = 1, 2, \ldots. \]

**Lemma 1.2.** [11] If the function \( p \in \mathcal{P} \) is given by the series (1.3), then

\[
\begin{align*}
2c_2 &= c_1^2 + x(4 - c_1^2) \\
4c_3 &= c_3^2 + 2c_1(4 - c_1^2)x - c_1(4 - c_1^2)x^2 + 2(4 - c_1^2)(1 - |x|^2)z
\end{align*}
\]

for some \( x, z \) with \( |x| \leq 1 \) and \( |z| \leq 1 \).

Inspired by the works of [7, 28] we consider the following subclass of the function class \( \sigma \).

For \( 0 \leq \alpha \leq 1 \) and \( 0 \leq \beta < 1 \), a function \( f \in \sigma \) given by (1.1) is said to be in the class \( \mathcal{M}_\sigma^\alpha(\beta) \) if the following conditions are satisfied:

\[
\Re \left( (1 - \alpha) \frac{zf''(z)}{f(z)} + \alpha \left( 1 + \frac{zf''(z)}{f'(z)} \right) \right) \geq \beta \quad (z \in \mathbb{U})
\]

and for \( g = f^{-1} \)

\[
\Re \left( (1 - \alpha) \frac{wg''(w)}{g(w)} + \alpha \left( 1 + \frac{wg''(w)}{g'(w)} \right) \right) \geq \beta \quad (w \in \mathbb{U}).
\]

The class was introduced and studied by Li and Wang [16], further the study was extended by Ali et al. [2]. In this paper we shall obtain the functional \( H_2(2) \) for functions \( f \) belongs to the class \( \mathcal{M}_\sigma^\alpha(\beta) \) and its special classes.

### 2. Bounds for the second Hankel determinant

We begin this section with the following theorem:

**Theorem 2.1.** Let \( f \) of the form (1.1) be in \( \mathcal{M}_\sigma^\alpha(\beta) \). Then

\[
|a_2a_4 - a_3^2| \leq \begin{cases} 
\frac{4(1-\beta)^2}{3(1+\alpha)^3(1+3\alpha)} \left[ 4(1 - \beta)^2 + (1 + \alpha)^2 \right] ; & \\
\beta \in \left[ 0, 1 - \frac{(1+\alpha)[3(1+3\alpha)+\sqrt{9(1+3\alpha)^2-48(1+\alpha)(1+3\alpha)+128(1+2\alpha)^2}]}{16(1+2\alpha)} \right] ; & \\
\frac{(1-\beta)^2}{(1+\alpha)(1+3\alpha)} \left[ (1-\beta)^2(13+7\alpha)-12(1-\beta)(1+\alpha)(1+2\alpha)(1+3\alpha)-4(1+\alpha)^2(9\alpha^2+8\alpha+2) \right] ; & \\
\beta \in \left[ 1 - \frac{(1+\alpha)[3(1+3\alpha)+\sqrt{9(1+3\alpha)^2+128(1+2\alpha)^2}]}{32(1+2\alpha)} \right] ; &
\end{cases}
\]

**Proof.** Let \( f \in \mathcal{M}_\sigma^\alpha(\beta) \). Then

\[
(1 - \alpha) \frac{zf''(z)}{f(z)} + \alpha \left( 1 + \frac{zf''(z)}{f'(z)} \right) = \beta + (1 - \beta)p(z)
\]

and

\[
(1 - \alpha) \frac{wg''(w)}{g(w)} + \alpha \left( 1 + \frac{wg''(w)}{g'(w)} \right) = \beta + (1 - \beta)q(w),
\]

where \( p, q \in \mathcal{P} \) and defined by

\[
p(z) = 1 + c_1z + c_2z^2 + c_3z^3 + \ldots
\]

and

\[
q(z) = 1 + d_1w + d_2w^2 + d_3w^3 + \ldots
\]
It follows from (2.1), (2.2), (2.3) and (2.4) that

\[ (1 + \alpha)a_2 = (1 - \beta)c_1 \]
\[ 2(1 + 2\alpha)a_3 - (1 + 3\alpha)a_2^2 = (1 - \beta)c_2 \]
\[ 3(1 + 3\alpha)a_4 - 3(1 + 5\alpha)a_2a_3 + (1 + 7\alpha)a_2^3 = (1 - \beta)c_3 \]

and

\[ -(1 + \alpha)a_2 = (1 - \beta)d_1 \]
\[ (3 + 5\alpha)a_2^2 - (2 + 4\alpha)a_3 = (1 - \beta)d_2 \]
\[ (12 + 30\alpha)a_2a_3 - (10 + 22\alpha)a_2^3 - (3 + 9\alpha)a_4 = (1 - \beta)d_3. \]

From (2.5) and (2.8), we find that

\[ c_1 = -d_1 \]

and

\[ a_2 = \frac{1 - \beta}{1 + \alpha}c_1. \]

Now, from (2.6), (2.9) and (2.12), we have

\[ a_3 = \frac{(1 - \beta)^2}{(1 + \alpha)^2}c_1^2 + \frac{1 - \beta}{4 + 8\alpha}(c_2 - d_2). \]

Also, from (2.7) and (2.10), we find that

\[ a_4 = \frac{(2 + 8\alpha)(1 - \beta)^3}{(3 + 9\alpha)(1 + \alpha)^3}c_1^3 + \frac{5(1 - \beta)^2}{8(1 + \alpha)(1 + 2\alpha)}c_1(c_2 - d_2) + \frac{1 - \beta}{6(1 + 3\alpha)}(c_3 - d_3). \]

Then, we can establish that

\[ |a_2a_4 - a_3^2| = \frac{-1}{3}(1 + \alpha)^3(1 + 3\alpha)(1 + 3\alpha)\ c_1^4 + \frac{(1 - \beta)^3}{8(1 + \alpha)^2(1 + 2\alpha)}c_1^2(c_2 - d_2) \]
\[ + \frac{(1 - \beta)^2}{6(1 + \alpha)(1 + 3\alpha)}c_1(c_3 - d_3) - \frac{(1 - \beta)^2}{16(1 + 2\alpha)^2}(c_2 - d_2)^2. \]

According to Lemma 1.2 and (2.11), we write

\[ c_2 - d_2 = \frac{(4 - c_1^2)}{2}(x - y) \]

and

\[ c_3 - d_3 = \frac{c_1^2}{2} + \frac{c_1(4 - c_1^2)(x + y)}{2} - \frac{c_1(4 - c_1^2)(x^2 + y^2)}{4} + \frac{(4 - c_1^2)(1 - |x|^2)z - (1 - |y|^2)w}{2} \]

for some \( x, y, z \) and \( w \) with \( |x| \leq 1, |y| \leq 1, |z| \leq 1 \) and \( |w| \leq 1 \). Using (2.16) and (2.17) in (2.15) we have
\[ |a_2a_4 - a_3^2| = \left| \frac{-(1 - \beta)^4c_4^4}{3(1 + \alpha)^3(1 + 3\alpha)} + \frac{(1 - \beta)^3c_3^2(4 - c_1^2)(x - y)}{16(1 + \alpha)^2(1 + 2\alpha)} + \frac{(1 - \beta)^2c_1^2}{6(1 + \alpha)(1 + 3\alpha)} \times \left[ \frac{c_1^2}{2} + c_1(4 - c_1^2)(x + y) - c_1(4 - c_1^2)(x^2 + y^2) \right] \right| \]

\[ \leq \frac{(1 - \beta)^4}{3(1 + \alpha)^3(1 + 3\alpha)} + \frac{(1 - \beta)^2c_1^2}{12(1 + \alpha)(1 + 3\alpha)} + \frac{(1 - \beta)^2c_1(4 - c_1^2)}{6(1 + \alpha)(1 + 3\alpha)} \]

Since \( p \in \mathcal{P} \), so \( |c_1| \leq 2 \). Letting \( c_1 = c \), we may assume without restriction that \( c \in [0, 2] \). Thus, for \( \gamma_1 = |x| \leq 1 \) and \( \gamma_2 = |y| \leq 1 \), we obtain

\[ |a_2a_4 - a_3^2| \leq T_1 + T_2(\gamma_1 + \gamma_2) + T_3(\gamma_2^2 + \gamma_2^2) + T_4(\gamma_1 + \gamma_2)^2 = F(\gamma_1, \gamma_2), \]

\[ T_1 = T_1(c) = \frac{(1 - \beta)^4}{3(1 + \alpha)^3(1 + 3\alpha)}c^4 + \frac{(1 - \beta)^2c_1^2}{12(1 + \alpha)(1 + 3\alpha)} + \frac{(1 - \beta)^2c(4 - c^2)}{6(1 + \alpha)(1 + 3\alpha)} \geq 0 \]

\[ T_2 = T_2(c) = \frac{(1 - \beta)^3c_3^2(4 - c_1^2)}{16(1 + \alpha)^2(1 + 2\alpha)} + \frac{(1 - \beta)^2c_1^2(4 - c_1^2)}{12(1 + \alpha)(1 + 3\alpha)} \geq 0 \]

\[ T_3 = T_3(c) = \frac{(1 - \beta)^2c_1^2(4 - c_1^2)}{24(1 + \alpha)(1 + 3\alpha)} + \frac{(1 - \beta)^2c(4 - c^2)}{12(1 + \alpha)(1 + 3\alpha)} \leq 0 \]

\[ T_4 = T_4(c) = \frac{(1 - \beta)^2(4 - c^2)^2}{64(1 + 2\alpha)^2} \geq 0 \]

Now we need to maximize \( F(\gamma_1, \gamma_2) \) in the closed square \( S := \{(\gamma_1, \gamma_2) : 0 \leq \gamma_1 \leq 1, 0 \leq \gamma_2 \leq 1\} \) for \( c \in [0, 2] \). We must investigate the maximum of \( F(\gamma_1, \gamma_2) \) according to \( c \in (0, 2) \), \( c = 0 \) and \( c = 2 \) taking into account the sign of \( F_{\gamma_1\gamma_1}, F_{\gamma_2\gamma_2} - (F_{\gamma_1\gamma_2})^2 \).

Firstly, let \( c \in (0, 2) \). Since \( T_3 < 0 \) and \( T_3 + 2T_4 > 0 \) for \( c \in (0, 2) \), we conclude that

\[ F_{\gamma_1\gamma_1}, F_{\gamma_2\gamma_2} - (F_{\gamma_1\gamma_2})^2 < 0. \]

Thus, the function \( F \) cannot have a local maximum in the interior of the square \( S \).

Now, we investigate the maximum of \( F \) on the boundary of the square \( S \).

For \( \gamma_1 = 0 \) and \( 0 \leq \gamma_2 \leq 1 \) (similarly \( \gamma_2 = 0 \) and \( 0 \leq \gamma_1 \leq 1 \)) we obtain

\[ F(0, \gamma_2) = G(\gamma_2) = T_1 + T_2\gamma_2 + (T_3 + T_4)\gamma_2^2 \]

(i) The case \( T_3 + T_4 \geq 0 \) : In this case for \( 0 < \gamma_2 < 1 \) and any fixed \( c \) with \( 0 < c < 2 \), it is clear that \( G'(\gamma_2) = 2(T_3 + T_4)\gamma_2 + T_2 > 0 \), that is, \( G(\gamma_2) \) is an increasing function. Hence, for fixed \( c \in (0, 2) \), the maximum of \( G(\gamma_2) \) occurs at \( \gamma_2 = 1 \) and

\[ \max G(\gamma_2) = G(1) = T_1 + T_2 + T_3 + T_4. \]
(ii) The case \( T_3 + T_4 < 0 \): Since \( T_2 + 2(T_3 + T_4) \geq 0 \) for \( 0 < \gamma_2 < 1 \) and any fixed \( c \) with \( 0 < c < 2 \), it is clear that \( T_2 + 2(T_3 + T_4) < 2(T_3 + T_4)\gamma_2 + T_2 < T_2 \) and so \( G'(\gamma_2) > 0 \). Hence for fixed \( c \in (0, 2) \), the maximum of \( G(\gamma_2) \) occurs at \( \gamma_2 = 1 \) and

Also for \( c = 2 \) we obtain

\[
F(\gamma_1, \gamma_2) = \frac{4(1 - \beta)^2}{3(1 + \alpha)^3(1 + 3\alpha)} \left[ 4(1 - \beta)^2 + (1 + \alpha)^2 \right].
\]

Taking into account the value (2.18) and the cases \( i \) and \( ii \), for \( 0 \leq \gamma_2 < 1 \) and any fixed \( c \) with \( 0 \leq c \leq 2 \),

\[
\max G(\gamma_2) = G(1) = T_1 + T_2 + T_3 + T_4.
\]

For \( \gamma_1 = 1 \) and \( 0 \leq \gamma_2 \leq 1 \) (similarly \( \gamma_2 = 1 \) and \( 0 \leq \gamma_1 \leq 1 \)), we obtain

\[
F(1, \gamma_2) = H(\gamma_2) = (T_3 + T_4)\gamma_2^2 + (T_2 + 2T_4)\gamma_2 + T_1 + T_2 + T_3 + T_4.
\]

Similarly, to the above cases of \( T_3 + T_4 \), we get that

\[
\max H(\gamma_2) = H(1) = T_1 + 2T_2 + 2T_3 + 4T_4.
\]

Since \( G(1) \leq H(1) \) for \( c \in (0, 2) \), max \( F(\gamma_1, \gamma_2) = F(1, 1) \) on the boundary of the square \( S \). Thus the maximum of \( F \) occurs at \( \gamma_1 = 1 \) and \( \gamma_2 = 1 \) in the closed square \( S \).

Let \( K : (0, 2) \to \mathbb{R} \)

\[
K(c) = \max F(\gamma_1, \gamma_2) = F(1, 1) = T_1 + 2T_2 + 2T_3 + 4T_4.
\]

Substituting the values of \( T_1, T_2, T_3 \) and \( T_4 \) in the function \( K \) defined by (2.19), yields

\[
K(c) = \frac{-6(1 - \beta)(1 + \alpha)(1 + 2\alpha)(1 + 3\alpha) + (16(1 - \beta)^2(1 + 2\alpha)^2 + 8(1 + \alpha)^2(1 + 2\alpha)^2 + 3(1 + \alpha)^3(1 + 3\alpha))c^4 + 24(1 + \alpha)[(1 - \beta)(1 + 2\alpha)(1 + 3\alpha) + 2(1 + \alpha)(1 + 2\alpha)^2 - (1 + \alpha)^2(1 + 3\alpha)]c^2 + 48(1 + \alpha)^3(1 + 3\alpha)}{48(1 + \alpha)^3(1 + 2\alpha)^2(1 + 3\alpha)}.
\]

Assume that \( K(c) \) has a maximum value in an interior of \( c \in (0, 2) \), by elementary calculation, we find

\[
K'(c) = \frac{-6(1 - \beta)(1 + \alpha)(1 + 2\alpha)(1 + 3\alpha) + (16(1 - \beta)^2(1 + 2\alpha)^2 + 8(1 + \alpha)^2(1 + 2\alpha)^2 + 3(1 + \alpha)^3(1 + 3\alpha))c^3 + 24(1 + \alpha)[(1 - \beta)(1 + 2\alpha)(1 + 3\alpha) + 2(1 + \alpha)(1 + 2\alpha)^2 - (1 + \alpha)^2(1 + 3\alpha)]c}{12(1 + \alpha)^3(1 + 2\alpha)^2(1 + 3\alpha)}.
\]

After some calculations we concluded following cases:

**Case 1.** Let

\[
[16(1-\beta)^2(1+2\alpha)-6(1-\beta)(1+\alpha)(1+3\alpha)](1+2\alpha)+(1+\alpha)^2[3(1+\alpha)(1+3\alpha)-8(1+2\alpha)^2] \geq 0,
\]

that is,

\[
\beta \in \left[0, 1 - \frac{(1 + \alpha)[3(1 + 3\alpha) + \sqrt{9(1 + 3\alpha)^2 - 48(1 + \alpha)(1 + 3\alpha) + 128(1 + 2\alpha)^2}]}{16(1 + 2\alpha)}\right].
\]
Therefore $K'(c) > 0$ for $c \in (0, 2)$. Since $K$ is an increasing function in the interval $(0, 2)$, maximum point of $K$ must be on the boundary of $c \in (0, 2)$, that is, $c = 2$. Thus, we have 

$$\max_{0 < c < 2} K(c) = K(2) = \frac{4(1 - \beta)^2}{3(1 + \alpha)^3(1 + 3\alpha)} \left[ 4(1 - \beta)^2 + (1 + \alpha)^2 \right].$$

**Case 2.** Let 

$$[16(1 - \beta)^2(1 + 2\alpha) - 6(1 - \beta)(1 + \alpha)(1 + 3\alpha)](1 + 2\alpha) + (1 + \alpha)^2[3(1 + \alpha)(1 + 3\alpha) - 8(1 + 2\alpha)^2] < 0,$$

that is, 

$$\beta \in \left[ 1 - \frac{(1 + \alpha)[3(1 + 3\alpha) + \sqrt{9(1 + 3\alpha)^2 - 48(1 + \alpha)(1 + 3\alpha) + 128(1 + 2\alpha)^2}]}{16(1 + 2\alpha)}, 1 \right].$$

Then $K'(c) = 0$ implies the real critical point $c_0_1 = 0$ or 

$$c_0_2 = \sqrt{\frac{-12(1 + \alpha)[(1 - \beta)(1 + 2\alpha)(1 + 3\alpha) + 2(1 + \alpha)[1 + 2\alpha]^2 - (1 + \alpha)^2(1 + 3\alpha)]}{16(1 - \beta)^2(1 + 2\alpha) - 6(1 - \beta)(1 + \alpha)(1 + 3\alpha)(1 + 2\alpha) + (1 + \alpha)^2[3(1 + \alpha)(1 + 3\alpha) - 8(1 + 2\alpha)^2]}}.$$

When 

$$\beta \in \left( 1 - \frac{(1 + \alpha)[3(1 + 3\alpha) + \sqrt{9(1 + 3\alpha)^2 - 48(1 + \alpha)(1 + 3\alpha) + 128(1 + 2\alpha)^2}]}{16(1 + 2\alpha)}, 1 \right)$$

we observe that $c_0_2 \geq 2$, that is, $c_0_2$ is out of the interval $(0, 2)$. Therefore, the maximum value of $K(c)$ occurs at $c_0_1 = 0$ or $c = c_0_2$ which contradicts our assumption of having the maximum value at the interior point of $c \in (0, 2)$.

When 

$$\beta \in \left( 1 - \frac{(1 + \alpha)[3(1 + 3\alpha) + \sqrt{9(1 + 3\alpha)^2 + 128(1 + 2\alpha)^2}]}{32(1 + 2\alpha)}, 1 \right)$$

we observe that $c_0_2 < 2$, that is, $c_0_2$ is an interior of the interval $[0, 2]$. Since $K''(c_0_2) < 0$, the maximum value of $K(c)$ occurs at $c = c_0_2$. Thus, we have 

$$\max_{0 < c < 2} K(c) = K(c_0_2) = \frac{(1 - \beta)^2}{(1 + \alpha)(1 + 3\alpha)} \left[ (1 - \beta)^2(1 + 3\alpha) - 12(1 - \beta)(1 + \alpha)(1 + 2\alpha)(1 + 3\alpha) - 4(1 + \alpha)^2(9\alpha^2 + 8\alpha + 2) \right].$$

This completes the proof. \qed

**Remark 2.2.** For $\alpha = 0$ and $\alpha = 1$, Theorem 2.1 would reduce to a known results in [7, Theorem 2.1, Theorem 2.3].

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