FORM FACTORS AND ACTION OF $U_{\sqrt{-1}}(\tilde{sl}_2)$ ON $\infty$-CYCLES

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Abstract. Let $p = \{P_{n,l}\}_{n,l \in \mathbb{Z} \geq 0}$ be a sequence of skew-symmetric polynomials in $X_1, \cdots, X_l$ satisfying $\deg X_j P_{n,l} \leq n - 1$, whose coefficients are symmetric Laurent polynomials in $z_1, \cdots, z_n$. We call $p$ an $\infty$-cycle if

$$P_{n+2,l+1} |_{X_{l+1} = z^{-1}, z_{n-1} = z, z_n = -z} = z^{n-1} \prod_{a=1}^l (1 - X_a^2 z^2) \cdot P_{n,l}$$

holds for all $n, l$.

These objects arise in integral representations for form factors of massive integrable field theory, i.e., the $SU(2)$-invariant Thirring model and the sine-Gordon model. The variables $\alpha_a = -\log X_a$ are the integration variables and $\beta_j = \log z_j$ are the rapidity variables. To each $\infty$-cycle there corresponds a form factor of the above models. Conjecturally all form-factors are obtained from the $\infty$-cycles.

In this paper, we define an action of $U_{\sqrt{-1}}(\tilde{sl}_2)$ on the space of $\infty$-cycles. There are two sectors of $\infty$-cycles depending on whether $n$ is even or odd. Using this action, we show that the character of the space of even (resp. odd) $\infty$-cycles which are polynomials in $z_1, \cdots, z_n$ is equal to the level $(-1)$ irreducible character of $\tilde{sl}_2$ with lowest weight $-\Lambda_0$ (resp. $-\Lambda_1$). We also suggest a possible tensor product structure of the full space of $\infty$-cycles.

1. Introduction

First let us recall the form factor bootstrap approach to massive integrable models [10] in field theory. A form factor is a tower of meromorphic functions

$$f = (f_n(\beta_1, \ldots, \beta_n))_{n \geq 0}$$

in the variables $\beta_1, \ldots, \beta_n \in \mathbb{C}$, satisfying a certain set of axioms, to be referred to as Axiom 1–3. There are two sectors in the space of form factors: the even sector where $f_n = 0$ for all odd $n$, and the odd sector where $f_n = 0$ for all even $n$. With each form factor is associated a local field in the theory. Axiom 1 describes the exchange relation of $\beta_j$ and $\beta_{j+1}$ in $f_n$, and Axiom 2 relates the analytic continuation $f_n(\beta_1, \ldots, \beta_{n-1}, \beta_n + 2\pi i)$ to the cyclic shift $f_n(\beta_n, \beta_1, \ldots, \beta_{n-1})$. These two axioms imply that, for each $n$, $f_n(\beta_1, \ldots, \beta_n)$ is a solution to the quantum Knizhnik-Zamolodchikov (qKZ) equation. Axiom 3 stipulates that $f_{n-2}$ is determined by the residue of $f_n$ at the simple pole $\beta_n = \beta_{n-1} + \pi i$. In this paper we consider the $SU(2)$ invariant Thirring model (ITM). For the details of the axioms in this case, see Section [4].

One of the basic issues in the theory of form factors is to describe all form factors satisfying the three axioms. In recent papers [6, 7], Nakayashiki solved this problem for the case of ITM under some assumptions. Subsequently, in [4] the case of the restricted sine-Gordon model was studied by a different method based on representation theory of quantum affine algebras. For the purpose of introducing the subject, and for making a comparison between the two methods, let us briefly review the results of [6, 7, 4].
Nakayashiki’s approach consists of three steps. The first step is to construct solutions of the qKZ equation by exploiting hypergeometric integrals (cf. Section 5 for the explicit formulas). The solutions are \( \mathfrak{sl}_2 \)-singular vectors. They are parameterized by polynomials \( P_{n,l}(X_1, \ldots, X_l| z_1, \ldots, z_n) \) in two sets of variables, \( X_1, \ldots, X_l \) and \( z_1, \ldots, z_n \). The variables \( X_a \) are related to the integration variables \( \alpha_a = X_a = e^{-\alpha_a} \), and the variables \( z_j \) to \( \beta_j = z_j = e^{\beta_j} \). The polynomial \( P_{n,l} \) is skew-symmetric in \( X_1, \ldots, X_l \) of degree less than or equal to \( n-1 \) in each \( X_a \), and is symmetric in \( z_1, \ldots, z_n \). We call \( P_{n,l} \) a deformed cycle. Among them there are null cycles, i.e., those which give rise to vanishing integrals. The problem of finding all null cycles has been solved by Smirnov and Tarasov.

The second step is to characterize deformed cycles which give rise to minimal form factors in the following sense. A form factor \( f \) is valid if each of the basis elements of minimal cycles. If \( f_n \) and \( f_{n+2} \) are given by deformed cycles \( P_{n,l} \) and \( P_{n+2,l+1} \), respectively, then Axiom 3 (with \( n \) replaced by \( n + 2 \)) is valid if

\[
P_{n+2,l+1}(X_1, \ldots, X_l, z^{-1}| z_1, \ldots, z_n, z, -z) = z^{-n-1} \prod_{a=1}^{l} (1 - X_a^2 z^2) \cdot P_{n,l}(X_1, \ldots, X_l| z_1, \ldots, z_n).
\]

In [7], Nakayashiki constructed directly a tower of deformed cycles \( \{P_{n,l}\} \) satisfying the linking condition, starting from each minimal cycle he has constructed in [6].

In [4], a different approach was taken in the construction of minimal cycles. In that paper, the restricted sine-Gordon model (RSG) was considered. The hypergeometric integrals for ITM and RSG have different form, but the deformed cycles are exactly the same. Let us denote the space of minimal deformed cycles with fixed \( N \) by \( W_N = \oplus_{0 \leq l \leq N} \mathcal{W}_{n,l} \). In [4], the space \( W_N \) was constructed as a representation space of a subalgebra of \( U_{\sqrt{-1}}(\mathfrak{sl}_2) \) over the ring \( R_N \) of symmetric polynomials in \( z_1, \ldots, z_N \), and it was shown that the total space \( W_N \) is created from the constant polynomial \( 1_N \in W_{N,0} \) by the \( R_N \)-linear action of this subalgebra.

In the present paper we continue the study of form factors along the same direction. We generalize the result in several respects. First, we consider deformed cycles \( P_{n,l} \) which are symmetric Laurent polynomials in the variables \( z_1, \ldots, z_n \). Namely, we consider both chiralities simultaneously. We denote the space of deformed cycles in this extended sense by \( \mathcal{W}_{n,l} \), and that of minimal deformed cycles by \( \mathcal{W}_{n,l} \). Second, we consider the action of the full quantum algebra \( U_{\sqrt{-1}}(\mathfrak{sl}_2) \) on
\( \hat{W}_n = \oplus_{0 \leq l \leq n} \hat{W}_{n,l} \). By doing so, we no longer need to introduce multiplication by symmetric Laurent polynomials ‘by hand’, since it is incorporated as a part of this action on the subspace \( \hat{W}_{n,0} \). Finally, and most importantly, we consider also the action of \( U_{\sqrt{-1}}(\hat{\mathfrak{sl}}_2) \) on towers of polynomials.

Let us elucidate the last point. We are interested in a tower of deformed cycles \( p = (P_{n,l})_{n-2l=m} \) satisfying the linking condition (1.3), where \( m \) is a fixed integer. As we have seen, such a sequence gives rise to a form factor satisfying Axiom 3 as well. We will refer to \( p \) as an \( \infty \)-cycle of weight \( m \). (For the precise definition, see Section 3.1.) The following is an example of \( \infty \)-cycles of weight \( m \),

\[
1_m = (1_m, X^{m+1}, X^{m+1} \wedge X^{m+3}, X^{m+1} \wedge X^{m+3} \wedge X^{m+5}, \ldots). 
\]

Here we have used the wedge notation for skew-symmetric polynomials (see (2.11) below). The action of the quantum algebra extends to the space \( \hat{C}_{n,l} \otimes \mathbb{C}(z_1, \ldots, z_n) \) of polynomials whose coefficients are rational functions in \( z_1, \ldots, z_n \). Therefore it acts naturally on sequences of polynomials componentwise. Consider the orbit of the particular \( \infty \)-cycles given above with \( m = 0, 1 \),

\[
(1.4) \quad \hat{Z} := U_{\sqrt{-1}}(\hat{\mathfrak{sl}}_2).1_0 + U_{\sqrt{-1}}(\hat{\mathfrak{sl}}_2).1_1.
\]

We show that any element in this space is an \( \infty \)-cycle, that is, a sequence of deformed cycles (in particular, they are symmetric Laurent polynomials in \( z_1, \ldots, z_n \)), satisfying the linking condition (1.3). The space \( (1.4) \) is filtered by submodules \( \hat{Z}_N \) consisting of \( N \)-minimal \( \infty \)-cycles of weight \( m \) \( (P_{n,l}) \) with \( P_{n,l} = 0 \) for \( n < N \). We show that the natural map

\[
\oplus_{N \geq 0} \hat{Z}_N / \hat{Z}_{N+1} \rightarrow \oplus_{N \geq 0} \hat{W}_N
\]

is an isomorphism of \( U_{\sqrt{-1}}(\hat{\mathfrak{sl}}_2) \)-modules. In particular, any minimal deformed cycle can be lifted to an \( \infty \)-cycle, and hence gives rise to a form factor. This gives an alternative proof of Nakayashiki’s result and extends it in the presence of both chiralities. These are the main results of the present paper.

As was shown in [6], the character of the space of even, polynomial \( \infty \)-cycles, i.e., \( \oplus_{N \text{ even}} \hat{W}_N \), coincides with the character of the level 1 basic representation for \( \hat{\mathfrak{sl}}_2 \). In our convention, it is more natural to think of this character as the level \((-1)\)-character. When we introduce negative powers in \( z_j \), the character becomes ill defined. Instead of dealing with the full space \( \hat{Z} \), we fix a non-negative integer \( L \in \mathbb{Z}_{\geq 0} \) and consider the subspace consisting of \( N \)-minimal deformed cycles \( P_{N,l} \) such that \( (z_1 \cdots z_N)^L P_{N,l} \) are polynomials in \( z_1, \ldots, z_N \). Then the sum of the corresponding characters (over \( N \equiv i \mod 2 \) with \( i = 0, 1 \) fixed) has the product form\( \chi_i(q^{-1}, z) \cdot \chi_0(q, z; L) \), where \( \chi_i(q^{-1}, z) \) is the level \((-1)\)-character and \( \chi_0(q, z; L) \) denotes the character of a Demazure subspace of the level 1 irreducible module with highest weight \( \Lambda_0 \). The latter tends to \( \chi_0(q, z) \) as \( L \rightarrow \infty \). This leads us to conjecture that \( \hat{Z} \) is isomorphic as a \( U_{\sqrt{-1}}(\hat{\mathfrak{sl}}_2) \)-module to the tensor product of level \((-1)\)- and level 1-modules. We plan to address this point in our next paper.

Let us give some remarks. First, the space of \( \infty \)-cycles is not the same as that of form factors. On one hand we should take into account the quotients by null cycles, and on the other hand we should incorporate form factors other than the singular vectors with respect to the \( \hat{\mathfrak{sl}}_2 \)-action. At the level of characters, these two effects cancel each other. The problem of determining the symmetries of the space of form factors themselves is beyond the scope of the present paper.
Second, lifting of a minimal form factor $f_N$ into a tower is not unique. Starting from a given lifting, we can add infinitely many $\infty$-cycles that are minimal and of increasing degrees of minimality, without changing the degree $N$ part $f_N$. Note, however, that such an infinite sum of $\infty$-cycles is not an $\infty$-cycle in our definition of $\hat{\mathbb{Z}}$. It is not clear if Nakayashiki’s extension belongs to our space $\hat{\mathbb{Z}}$ (though it is clear that it belongs to the completion of $\hat{\mathbb{Z}}$). Since form factors are determined only up to null cycles, the identification of form factors is not a simple problem. We illustrate this point by some examples. Form factors corresponding to the $\mathfrak{su}(2)$ currents given in [10] belong to the space $\hat{\mathbb{Z}}$. This is true at the level of $\infty$-cycles. On the other hand, the known formula [10] for form factors corresponding to the energy-momentum tensor arise from $\hat{\mathbb{Z}}$, but only modulo null cycles.

The plan of the paper is as follows. In Section 2, we introduce our notation on the quantum loop algebra at roots of unity, and give an action of $U_{\sqrt{-1}}(\tilde{\mathfrak{sl}}_2)$ on the space of deformed cycles. In Section 3 we introduce the space of $\infty$-cycles. The linking condition is preserved by the action of $U_{\sqrt{-1}}(\tilde{\mathfrak{sl}}_2)$. The main result is stated in Theorem 3.4. We also calculate the character of the truncated space in terms of the level one irreducible characters and the Demazure characters. In Section 4, we apply the results on the $\infty$-cycles to the form factors. Some technical matters are given in Appendices. Appendix A is devoted to the derivation of the $U_{\sqrt{-1}}(\tilde{\mathfrak{sl}}_2)$ action on the tensor product of evaluation modules. In Appendix B we give a proof of a lemma regarding certain null cycles.

2. The space of deformed cycles

2.1. Quantum loop algebra. In this subsection, we recall some basic facts about $U_q(\mathfrak{sl}_2)$ at roots of unity. Our basic reference is [2].

Let $\mathbb{C}(q)$ be the field of rational functions in indeterminate $q$. The quantum loop algebra $U_q(\mathfrak{sl}_2)$ is a $\mathbb{C}(q)$-algebra generated by $x_k^\pm$ ($k \in \mathbb{Z}$), $a_n$ ($n \in \mathbb{Z}\setminus\{0\}$) and $t_1^{\pm 1}$, with the defining relations

\begin{align}
[t_1, a_n] &= 0, \quad [a_m, a_n] = 0, \\
t_1^{\pm 1} x_k^\pm t_1^{\mp 1} &= q^{\pm 2} x_k^\pm, \\
[a_n, x_k^\pm] &= \pm \frac{[2n]}{n!} x_k^{\pm n}, \\
x_{k+1}^\pm x_l^\pm - q^{\pm 2} x_l^\pm x_{k+1}^\pm &= q^{\pm 2} x_k^\pm x_{l+1}^\pm - x_{l+1}^\pm x_k^\pm, \\
[x_k^+, x_l^-] &= \frac{\varphi_{k+l}^+ - \varphi_{k+l}^-}{q - q^{-1}}.
\end{align}

Here

\[\sum_{k \in \mathbb{Z}} \varphi_{k}^\pm z^k = t_1^{\pm 1} \exp \left( \pm (q - q^{-1}) \sum_{n=1}^\infty a_{\pm n} z^n \right),\]

and $[j] = (q^j - q^{-j})/(q - q^{-1})$. We use the notation

\[x^{(n)} = \frac{x^n}{[n]!}, \quad [n]! = \prod_{j=1}^n \frac{[j]}{[j]}.\]
Let $U_{q}^{\pm}$ be the $\mathbb{C}[q, q^{-1}]$-subalgebra of $U_{q}(sl_{2})$ generated by $(x_{n}^{\pm})^{(r)}$ ($n \in \mathbb{Z}$, $r \in \mathbb{Z}_{\geq 0}$). Let $U_{q}^{\text{res}0}$ be the $\mathbb{C}[q, q^{-1}]$-subalgebra generated by $t_{1}^{\pm 1}$, $[t_{1}; n, r]$ ($n \in \mathbb{Z}$, $r \in \mathbb{Z}_{\geq 0}$) and $\tilde{a}_{n}$ ($n \in \mathbb{Z}\setminus\{0\}$), where

$$\tilde{a}_{n} = \frac{n}{|n|}a_{n}, \quad [t_{1}; n, r] = \prod_{s=1}^{r} \frac{t_{1}q^{n-s+1} - t_{1}^{-1}q^{-n+s-1}}{q^{s} - q^{-s}}.$$

Let further $U_{q}^{\text{res}}$ be the $\mathbb{C}[q, q^{-1}]$-subalgebra generated by $U_{q}^{\pm}$ and $U_{q}^{\text{res}0}$. We have the triangular decomposition ([2], Proposition 6.1)

$$U_{q}^{\text{res}} = U_{q}^{\text{res} -} \cdot U_{q}^{\text{res} 0} \cdot U_{q}^{\text{res} +}.$$ 

We will use also $\mathbb{C}[q, q^{-1}]$-subalgebras $B_{q}^{\pm}$ generated by the following elements:

$$B_{q}^{+} : (x_{n}^{+})^{(r)}, (x_{n}^{-})^{(r)}, \tilde{a}_{m}, t_{1}^{\pm 1},$$
$$B_{q}^{-} : (x_{n}^{-})^{(r)}, (x_{n}^{-})^{(r)}, \tilde{a}_{m}, t_{1}^{\pm 1},$$

where $n, r$ run over $\mathbb{Z}_{\geq 0}$ and $m$ over $\mathbb{Z}_{> 0}$, respectively.

Introduce the generating series

$$X_{n+0}(t) := \sum_{n \geq 0} x_{n}^{+}(q^{-1})^{n}, \quad X_{n-0}(t) := \sum_{n < 0} x_{n}^{+}(q^{-1})^{n},$$
$$X_{n+0}(t) := \sum_{n > 0} x_{n}^{-}(q^{-1})^{n}, \quad X_{n-0}(t) := \sum_{n \leq 0} x_{n}^{-}(q^{-1})^{n}.$$

By Corollary 4.6 in [2] and the Remark below it, $U_{q}^{\text{res} +}$ (resp. $U_{q}^{\text{res} -}$) is also generated over $\mathbb{C}[q, q^{-1}]$ by the coefficients of $(X_{n+0}(t))^{(r)}$ and $(X_{n+0}(t))^{(r)}$ (resp. $(X_{n-0}(t))^{(r)}$ and $(X_{n-0}(t))^{(r)}$), where $r$ runs over $\mathbb{Z}_{\geq 0}$.

For any non-zero complex number $\epsilon$, the specialization $U_{\epsilon}$ is defined to be $U_{q}^{\text{res}} \otimes_{\mathbb{C}[q, q^{-1}]} \mathbb{C}$ by the ring homomorphism $\mathbb{C}[q, q^{-1}] \to \mathbb{C}$ sending $q$ to $\epsilon$. The subalgebras $U_{\epsilon}^{\pm} = U_{q}^{\text{res} \pm}$, $U_{\epsilon}^{0} = U_{q}^{\text{res} 0}$ and $B_{\epsilon}^{\pm}$ are defined similarly.

In this paper we restrict to the case $\epsilon = \sqrt{-1}$. Define in $U_{q}^{\text{res}}$

$$a_{n}(t) := \sum_{n \geq 0} \tilde{a}_{n}t^{n},$$

and denote their images in $U_{\sqrt{-1}}$ by the same symbols. The subalgebras $U_{\sqrt{-1}}^{\pm}$, $U_{\sqrt{-1}}^{0}$ and $B_{\sqrt{-1}}^{\pm}$ are generated over $\mathbb{C}$ by (the coefficients of) the following elements:

$$(2.6) \quad U_{\sqrt{-1}}^{+} := X_{n+0}(t), \quad X_{n+0}(t)^{(2)}, \quad X_{n-0}(t), \quad X_{n-0}(t)^{(2)},$$
$$(2.7) \quad U_{\sqrt{-1}}^{0} := t_{1}^{\pm 1}, \quad [t_{1}; n, 2] (n \in \mathbb{Z}), \quad a_{\pm}(t),$$
$$(2.8) \quad U_{\sqrt{-1}}^{-} := X_{n+0}(t), \quad X_{n+0}(t)^{(2)}, \quad X_{n-0}(t), \quad X_{n-0}(t)^{(2)},$$
$$(2.9) \quad B_{\sqrt{-1}}^{+} := X_{n+0}(t), \quad X_{n+0}(t)^{(2)}, \quad X_{n-0}(t), \quad X_{n-0}(t)^{(2)},$$
$$x_{0}, \quad (x_{0})^{(2)}, \quad a_{+}(t), \quad t_{1}^{\pm 1},$$
$$(2.10) \quad B_{\sqrt{-1}}^{-} := X_{n+0}(t), \quad X_{n+0}(t)^{(2)}, \quad X_{n-0}(t), \quad X_{n-0}(t)^{(2)},$$
$$x_{0}, \quad (x_{0})^{(2)}, \quad a_{-}(t), \quad t_{1}^{\pm 1}.$$
We assign the degree and weight to $U_{\sqrt{-1}}$ as follows.
\[
\deg(x_n^\pm)^{(r)} = nr, \quad \deg a_m = m, \quad \deg t_1^{\pm 1} = 0,
\]
\[
\wt(x_n^\pm)^{(r)} = \pm 2r, \quad \wt a_m = 0, \quad \wt t_1^{\pm 1} = 0.
\]

2.2. Representation of $U_{\sqrt{-1}}$ on the space of polynomials. Until the end of this section, we fix a non-negative integer $n$. Let $K_n = \mathbb{C}(z_1, \ldots, z_n)$ be the field of rational functions in $z_1, \ldots, z_n$, and let $\mathcal{R}_n$ be its subring consisting of symmetric Laurent polynomials. Consider the vector space over $K_n$
\[
A_n := \bigoplus_{l=0}^n A_{n,l}, \quad A_{n,l} := \iota\left(\bigoplus_{j=0}^{n-1} K_n X_i^j\right).
\]
By definition we set $A_{n,0} = K_n$ and $A_{n,l} = 0$ if $l > n$. For $0 \leq l \leq n$, we identify an element $P \in A_{n,l}$ with a skew-symmetric polynomial in the variables $X_1, \ldots, X_l$ with coefficients in $K_n$, of degree at most $n-1$ in each $X_j$. We use the wedge product notation for $P_1(X_1, \ldots, X_{l_1}) \in A_{n,l_1}$ and $P_2(X_1, \ldots, X_{l_2}) \in A_{n,l_2}$,
\begin{equation}
(2.11) \quad P_1 \wedge P_2(X_1, \ldots, X_{l_1+l_2}) := \frac{1}{l_1!l_2!} \text{Skew} (P_1(X_1, \ldots, X_{l_1})P_2(X_{l_1+1}, \ldots, X_{l_1+l_2}))
\end{equation}
where Skew stands for the skew-symmetrization
\[
\text{Skew} f(X_1, \ldots, X_m) := \sum_{\sigma \in S_m} (\text{sgn } \sigma) f(X_{\sigma(1)}, \ldots, X_{\sigma(m)}).
\]
We shall introduce a $K_n$-linear action of $U_{\sqrt{-1}}$ on $A_n$. For that purpose, let us prepare some notation. Set
\[
\Theta_n(X) := \prod_{j=1}^n (1 - z_j X),
\]
\[
\Theta_n(X_1, X_2) := \Theta_n(X_1) \Theta_n(X_2) - \Theta_n(-X_1) \Theta_n(-X_2).
\]
Define also
\[
F_n(t|X) := \frac{t}{2(X - t)} \frac{\Theta_n(t, -X)}{\Theta_n(t)},
\]
\[
F_n^{(2)}(t|X_1, X_2) := \frac{t}{X_1 + t X_2 + t X_1 + X_2} \Theta_n(X_1, X_2)
\]
\[
+ \frac{t}{X_1 - t X_2 + t} \Theta_n(-X_1) \Theta_n(t, -X_1) - \frac{t}{X_2 - t X_1 + t} \Theta_n(t, -X_2).
\]
$F_n(t|X)$ is a polynomial in $X$ of degree $n - 1$ because $\Theta_n(t, -X)$ is divisible by $X - t$. At $t = 0$, it has a power series expansion in $t^{\pm 1}$ whose coefficients are symmetric polynomials in $z_1^{\pm 1}, \ldots, z_n^{\pm 1}$. $F_n^{(2)}(t|X_1, X_2)$ has similar properties.

Let us define the action of the generators $g \in U_{\sqrt{-1}}$ on $P \in A_{n,l}$. For $l = 0$ or 1 and $g \in U_{\sqrt{-1}}$, we set
\[
X_{\geq 0}^{(2)}(t).P = 0, \quad X_{\leq 0}^{(2)}(t).P = 0 \quad (l = 0),
\]
\[
X_{\geq 0}^{(2)}(t).P = 0, \quad X_{\leq 0}^{(2)}(t).P = 0 \quad (l = 0, 1).
\]
In the other cases, we define the action as follows.

\[(2.12) \quad t^\pm_1.P := i^{\pm(n-2)}.P,\]
\[(2.13) \quad X_{>0}(t).P := F_n(t) \wedge P,\]
\[(2.14) \quad -4iX_{>0}^{-}(t)(2).P := F^{(2)}_n(t) \wedge P,\]
\[(2.15) \quad a_+(t).P := \sum_{j=1}^{n} \frac{z_j t}{1 - z_j t} P(X_1, \cdots, X_l)\]
\[+ \sum_{p=1}^{t} \left\{ \frac{t}{X_p - t} \left( \frac{\Theta_n(X_p)}{\Theta_n(t)} P(X_1, \cdots, t, \cdots, X_l) - P(X_1, \cdots, X_l) \right) \right\} + (t \to -t),\]
\[(2.16) \quad a_-(t).P := -\sum_{j=1}^{n} \frac{1}{1 - z_j t} P(X_1, \cdots, X_l)\]
\[+ \sum_{p=1}^{t} \left\{ \frac{t}{X_p - t} \left( \frac{\Theta_n(X_p)}{\Theta_n(t)} P(X_1, \cdots, t, \cdots, X_l) - \frac{X_p}{t} P(X_1, \cdots, X_l) \right) \right\} + (t \to -t),\]
\[(2.17) \quad i^{1-n}X_{>0}^+(t).P := \frac{1}{\Theta_n(t)} P(X_1, \cdots, X_{l-1}, t),\]
\[(2.18) \quad i(-1)^{n+1}X_{>0}^+(t)(2).P := \sum_{a=1}^{n} \text{res}_{u = z_a} \left( \frac{P(X_1, \cdots, X_{l-2}, -u, u) du}{\Theta_n(-u) \Theta_n(u) u - t} \right).\]

In (2.13), (2.14), (2.15), (2.17), and (2.18), the right hand sides stand for the power series expansion in \(t\) and in the expansion in \(t^{-1}\). Note that the factor \(X_p \pm t\) in the denominator of (2.15) or (2.17) divides the numerator. Define also \(-X_{<0}^{-}(t).P, -4iX_{<0}^{-}(t)(2).P, -i^{1-n}X_{<0}^+(t).P\) and \(i(-1)^{n+1}X_{<0}^+(t)(2).P\) by the expansion at \(t = \infty\) of the right hand side of (2.13), (2.14), (2.15), (2.17), respectively.

Note, in particular, that

\[(2.19) \quad \tilde{a}_m.P := \left( \sum_{j=1}^{n} z_j^m \right) P \quad (m \in \mathbb{Z} : \text{odd}).\]

**Proposition 2.1.** With the above rule, \(A_n\) is a \(U_{\sqrt{-1}}\)-module.

This action of \(U_{\sqrt{-1}}\) is essentially the one on the \(n\)-fold tensor product of two-dimensional evaluation modules in disguise. We give a proof of Proposition 2.1 in Appendix A.

The following lemma will be used later.

**Lemma 2.2.** On the subspace \(A_{n,0} = K_n\), we have

\[\tilde{a}_m.P = \left( \sum_{j=1}^{n} z_j^m \right) P \quad (m \in \mathbb{Z}\setminus\{0\}, P \in A_{n,0}).\]

We have \(\tilde{R}_n = U_{\sqrt{-1}}^0 1_n\), where \(1_n \in \tilde{R}_n\) denotes the unit.

**Proof.** This is an immediate consequence of (2.13), (2.16) and (2.19). \(\square\)
2.3. Deformed cycles. For the application to form factors, we introduce several submodules of $A_n$

$$A_n \supset \hat{C}_n \supset \hat{D}_n \supset \hat{W}_n$$

to be defined as follows.

Set

$$\hat{C}_n := \bigoplus_{l=0}^{n} \hat{C}_{n,l}, \quad \hat{C}_{n,l} := \bigwedge^l \left( \bigoplus_{j=0}^{n-1} \hat{R}_n X^j \right).$$

An element of $\hat{C}_n$ will be referred to as a deformed cycle. We say that $P \in \hat{C}_{n,l}$ is weakly minimal if

$$P = 0 \text{ if } X_{l-1}^{-1} = -X_{l}^{-1} = z_{n-1} = -z_n,$$

and minimal if

$$P = 0 \text{ if } X_{l}^{-1} = z_{n-1} = -z_n.$$

These conditions arise in the study of form factors (see Lemma 3.2 below). Denote by $\hat{D}_{n,l}$ (resp. $\hat{W}_{n,l}$) the subspace of weakly minimal (resp. minimal) elements of $\hat{C}_{n,l}$. We set

$$\hat{D}_n := \bigoplus_{l=0}^{n} \hat{D}_{n,l}, \quad \hat{W}_n := \bigoplus_{l=0}^{n} \hat{W}_{n,l}.$$

Let further $R_n$ be the subring of $\hat{R}_n$ consisting of symmetric polynomials. Replacing $\hat{R}_n$ by $R_n$ in the above, we define the spaces $C_n = \bigoplus_{l=0}^{n} C_{n,l}$, $D_n = \bigoplus_{l=0}^{n} D_{n,l}$, $W_n = \bigoplus_{l=0}^{n} W_{n,l}$.

**Lemma 2.3.**

(i) The spaces $\hat{D}_n$, $\hat{W}_n$ are $U_{\sqrt{-t}}$-submodules.

(ii) The spaces $D_n$, $W_n$ are $B_{\sqrt{-t}}$-submodules.

**Proof.** From the formulas (2.12)–(2.18), it is clear that $\hat{D}_n$, $\hat{W}_n$ are stable under the action of all generators listed in (2.6), (2.7) and (2.8), except for $X_{t}^0(t)^{(2)}$, $X_{t}^0(t)^{(2)}$. The only subtle point about the latter elements is that, when acted on $P \in \hat{C}_n$, they give rise to symmetric rational functions in $z_1, \ldots, z_n$ which have a pole on $z_a + z_b = 0$ in general. We show that for elements $P \in \hat{D}_n$ these poles do not appear, and that the (weak-) minimality condition is preserved.

As an example, consider the case $Q = i(-1)^n X_{t}^0(t)^{(2)} P, P \in \hat{D}_{n,l}$ with $l \geq 2$. Explicitly we have

$$Q(X_1, \ldots, X_{l-2}|z_1, \ldots, z_n) = \frac{1}{2} \sum_{a=1}^{n} \prod_{b=1}^{n} \frac{1}{1 - z_b^2 / z_a^2} \frac{1}{1 - z_a t} \times P(X_1, \ldots, X_{l-2}, z_a^{-1}, -z_a^{-1}|z_1, \ldots, z_n).$$

Since the right hand side is symmetric in $z_1, \ldots, z_n$, the only possible poles are those at $z_a + z_b = 0$ ($a \neq b$). However, those poles are absent because of (2.20). Let us verify that $Q \in \hat{D}_{n,l-2}[[t]]$. If $l < 4$, there is nothing to show. Suppose $l \geq 4$. 

Setting $z_{n-1} = z$ and $z_n = -z$ we find

$$Q(X_1, \cdots, X_{l-2}|z_1, \cdots, z_{n-2}, z, -z)$$

$$= \frac{1}{2} \sum_{a=1}^{n-2} \frac{P(X_1, \cdots, X_{l-2}, z_a^{-1}, -z_a^{-1}|z_1, \cdots, z_{n-2}, z, -z)}{(1 - tz_a)(1 - z^2/z_a^2)^{l-1}} \prod_{1 \leq i < j \leq n-2}(1 - z_i^2/z_j^2)$$

$$+ \frac{1}{2} \prod_{b=1}^{n-2}(1 - z^2/z_b^2) \left( \frac{z}{1 + tz} \frac{\partial P}{\partial z_{n-1}}(X_1, \cdots, X_{l-2}, z, -z^{-1}|z_1, \cdots, z_{n-2}, z, -z) + \frac{z}{1 - tz} \frac{\partial P}{\partial z_{n-1}}(X_1, \cdots, X_{l-2}, z, -z^{-1}|z_1, \cdots, z_{n-2}, z, -z) \right).$$

Under further specialization $X_{l-3} = -X_{l-2} = z^{-1}$, the first term vanishes by Proposition 2.4, and the rest is 0 by skew symmetry.

In the same way, (2.21) is also preserved. The case $X_{l-2}^+ = (2)$ and the remaining assertions can be shown similarly. □

**Proposition 2.4.** Let $1_n \in \hat{C}_n$ be the unit element. Then we have

$$\hat{W}_n = U_{\sqrt{-1}}1_n, \quad W_n = B^+_{\sqrt{-1}}1_n.$$  

**Proof.** By Lemma 2.3, $U_{\sqrt{-1}}$ acts on $\hat{W}_n$. It is known [4] that $W_n$ is generated from $1_n$ by the action of $X_0^-(t)$, $X_0^+(t)$, $X_{l-2}^+(t)$ and $\rho_{z_l}$, by elements of $R_n$. By Lemma 2.2, $R_n = (B^+_{\sqrt{-1}} \cap U_{\sqrt{-1}}1_n, \rho_{z_l})$, since $X_{l-2}^+(t)$, $X_{l-2}^+(t)$ kill $1_n$, and since the action of $U_{\sqrt{-1}}$ is $R_n$-linear, we obtain $W_n = B^+_{\sqrt{-1}}1_n$. The other assertion follows in a similar manner. □

Consider the tensor product of evaluation representation $\rho_{z_1} \otimes \cdots \otimes \rho_{z_n}$ specialized to $q = \sqrt{-1}$ (see [A]). As explained in Appendix A, the action of $U_{\sqrt{-1}}$ on $\hat{W}_n$ is induced from $\rho_{z_1} \otimes \cdots \otimes \rho_{z_n}$ under the identification of $1_n$ with $\mathfrak{g}_n$. Proposition 2.4 implies that

**Corollary 2.5.** As $U_{\sqrt{-1}}$-module, $\hat{W}_n$ is isomorphic to the subrepresentation of $\rho_{z_1} \otimes \cdots \otimes \rho_{z_n}$ generated by $\mathfrak{g}_n^+.$

Define the degree on $\hat{C}_{n,l}$ by

$$\deg_0 X_n = -1, \quad \deg_0 z_j = 1.$$  

We also assign a weight $m = n - 2l$ to $P \in \hat{C}_{n,l}$. With this definition, $\hat{D}_n$ and $\hat{W}_n$ are bi-graded $U_{\sqrt{-1}}$-modules.

### 3. ACTION OF $U_{\sqrt{-1}}(\mathfrak{g}_2)$ ON THE SPACE OF $\infty$-CYCLES

In this section we introduce certain sequences of cycles which we call $\infty$-cycles, and define an action of $U_{\sqrt{-1}}$ on them.

#### 3.1. Links of cycles

Let $P_{n,l} \in \hat{C}_{n,l}$, $P_{n+2,l+1} \in \hat{C}_{n+2,l+1}$. We say that the pair $(P_{n,l}, P_{n+2,l+1})$ is a link if

$$P_{n+2,l+1}(X_1, \cdots, X_l, z^{-1}|z_1, \cdots, z_n, z, -z)$$

$$= z^{-n-1} \prod_{a=1}^{l+1} (1 - X_a^2 z^{-2}) \cdot P_{n,l}(X_1, \cdots, X_l|z_1, \cdots, z_n).$$

(3.1)
The condition (3.1) appeared in [9] in the following equivalent form.

Lemma 3.1. A pair \((P_{n,l}, P_{n+2,l+1})\) is a link if and only if there exists a

\[ P_{n,l}^*(X_1, \cdots, X_l|X_{l+1}|z_1, \cdots, z_n|z) \]

with the following properties.

(i) \(P_{n,l}^*\) is a skew-symmetric polynomial in \(X_1, \cdots, X_l\) with \(\text{deg}_{X_i} P_{n,l}^* \leq n-1\) \((1 \leq i \leq l)\). It is a polynomial in \(X_{l+1}\) with \(\text{deg}_{X_{l+1}} P_{n,l}^* \leq n + 1\).

(ii) \(P_{n,l}^*\) is a symmetric Laurent polynomial in \(z_1, \cdots, z_n\), and an even Laurent polynomial in \(z\).

(iii) We have

\[
\begin{align*}
(3.2) & \quad P_{n+2,l+1}(X_1, \cdots, X_{l+1}|z_1, \cdots, z_n, z, -z) \\
& = \frac{1}{l!} \text{Skew} \left( \prod_{1 \leq a < b \leq l} (1 - X_a^2 z^2) P_{n,l}^*(X_1, \cdots, X_l|X_{l+1}|z_1, \cdots, z_n|z) \right), \\
(3.3) & \quad P_{n,l}^*(X_1, \cdots, X_l|z^2|z_1, \cdots, z_n|z) = z^{-n-1}P_{n,l}(X_1, \cdots, X_l|z_1, \cdots, z_n). 
\end{align*}
\]

Here \(\text{Skew}\) in \((3.2)\) stands for the skew-symmetrization with respect to \(X_1, \cdots, X_{l+1}\).

Proof. The ‘if’ part is evident. To see the ‘only if’ part, suppose \((3.1)\) holds. Since

\[
\frac{1}{l!} \text{Skew} \left( X_{l+1}^{n+1} \prod_{a=1}^n (1 - X_a^2 z^2) P_{n,l}(X_1, \cdots, X_l|z_1, \cdots, z_n) \right)
\]

has a zero at \(X_{l+1} = \pm z^{-1}\), it can be written as \(\prod_{a=1}^{l+1} (1 - X_a^2 z^2) \cdot Q(X_1, \cdots, X_{l+1}|z_1, \cdots, z_n|z)\). Setting

\[
P_{n,l}^*(X_1, \cdots, X_l|X_{l+1}|z_1, \cdots, z_n|z)
\]

\[
= X_{l+1}^{n+1} P_{n,l}(X_1, \cdots, X_l|z_1, \cdots, z_n) + \frac{1}{l+1} (1 - X_{l+1}^2 z^2) Q(X_1, \cdots, X_{l+1}|z_1, \cdots, z_n|z),
\]

one easily verifies the conditions mentioned above. \(\square\)

The following lemma is obvious from the definition.

Lemma 3.2.

(i) If \((P_{n,l}, P_{n+2,l+1})\) is a link, then \(P_{n,l}\) and \(P_{n+2,l+1}\) are weakly minimal.

(ii) For \(P_{n,l} \in \tilde{C}_n\), \((0, P_{n,l})\) is a link if and only if \(P_{n,l}\) is minimal.

Let us study the action of \(U_{\sqrt{-1}}\) on links.

Proposition 3.3. Let \(P_{n,l} \in \tilde{C}_{n,l}\), \(P_{n+2,l+1} \in \tilde{C}_{n+2,l+1}\) and \(g \in U_{\sqrt{-1}}\). If the pair \((P_{n,l}, P_{n+2,l+1})\) is a link, then \((g P_{n,l}, g P_{n+2,l+1})\) is also a link.

Using the formulas for the action \([2.13] = [2.18]\), it is straightforward to verify the linking condition \((3.1)\). We omit the proof.
3.2. $\infty$-cycles. Let $m$ be an integer. We call a sequence of cycles

$$p = (P_{n,t})_{n,t \geq 0, n-2l = m}, \quad P_{n,t} \in \mathcal{C}_{n,t}$$

an $\infty$-cycle if $(P_{n,t}, P_{n+2,t+1})$ is a link for any $n \geq 0$. The integer $m$ is called the weight of $p$. We denote by $\overline{Z}[m]$ the space of $\infty$-cycles of weight $m$, and set $\overline{Z} = \bigoplus_{m \in \mathbb{Z}} \overline{Z}[m]$. Often we write the component $P_{n,t}$ as $(p)_{n,t}$. For an $\infty$-cycle $p \in \overline{Z}[m]$, define the action of $g \in U(\sqrt{-1})$ by

$$g \cdot p = (g \cdot P_{n,t})_{n,t \geq 0, n-2l = m}.$$ 

Example. For each $m \in \mathbb{Z} \geq 0$, there is a distinguished $\infty$-cycle $1_m \in \overline{Z}[m]$ given by

$$(1_m)_{m,0} = 1_m, \quad 1_m \in \mathbb{Z} \geq 0, \quad 1_m = X^{m+1} \wedge \cdots \wedge X^{m+2l-1} \quad (l \geq 1).$$

From the formula (2.13) we find

$$(i^{-1}mX_{\geq 0}(t)1_m)_{m+2l,t-1} = t^{m+1}X^{m+3} \wedge \cdots \wedge X^{m+2l-1} + O(t^{m+2}),$$

and hence

$$(-1)^{m+1}x_{m+1}^+ 1_m = 1_{m+2}, \quad x_n^+ 1_m = 0 \quad (n \leq m).$$

In particular, for all $m \geq 0$ we have

$$1_m = \begin{cases} (-1)^k x_{2k-1}^+ \cdots x_3^+ x_1^+ 1_0 & (m = 2k), \\ x_{2k-2}^+ \cdots x_3^+ x_1^+ 1_1 & (m = 2k - 1). \end{cases}$$

Next we give a formula for $x_{2k-1}^+ \cdots x_3^+ x_1^+ 1_0 \in \overline{Z}[2k]$ as a nontrivial example. For $k = 1, 2$ the $\infty$-cycles are given by

$$(x_1^+ 1_0)_{2+2l,2+l} = t^{1/2} \left( \sum_{0 \leq a < l+1} e_{2a+1}X^{2a} \right) \wedge X \wedge X^3 \wedge \cdots \wedge X^{2l+1},$$

$$(x_3^+ x_1^+ 1_0)_{4+2l,4+l} = \left( \sum_{0 \leq a_1 < a_2 < l+2} e_{2a_1+3} e_{2a_2+3} e_{2a_2+3} e_{2a_1+1} e_{2a_1+2} \hat{e}_1 \right) \wedge X^{2a_1} \wedge X^{2a_2} \wedge X \wedge X^3 \wedge \cdots \wedge X^{2l+3},$$

where $e_a$ is the $a$-th elementary symmetric polynomial in $z_j$’s. To write down the formula for general $k$, we set

$$w_{2n,r}^{(0)} := (-1)^{\binom{n+1}{2}} \times \sum_{0 \leq a_1 < \cdots < a_r < n} S_{22(r-1),\ldots,202a_r,\ldots,2a_1}(z_1, \ldots, z_2n)X^{2a_1} \wedge \cdots \wedge X^{2ar}.$$ 

Here $S_{22(r-1),\ldots,202a_r,\ldots,2a_1}(z_1, \ldots, z_2n)$ is the Schur polynomial associated with the Young diagram given in the Frobenius notation by $(2(r-1), \ldots, 2,0|2a_r,\ldots,2a_1)$. Then the formula is given by

$$(x_{2k-1}^+ \cdots x_3^+ x_1^+ 1_0)_{2k+2l,2k+l} = t^{k^2/2} w_{2k+2l,k}^{(0)} \wedge X \wedge X^3 \wedge \cdots \wedge X^{2(k+l)-1}.$$ 

In particular we have

$$(x_{2k-1}^+ \cdots x_3^+ x_1^+ 1_0)_{2k,2k} = t^{k^2/2} \prod_{1 \leq j < j' \leq 2k} (z_j + z_{j'}) X^0 \wedge X^1 \wedge \cdots \wedge X^{2k-1}.$$
We have
\[
\text{Proof. It suffices to show the surjectivity. Take any degree of an } \infty \text{ and } i \in \mathbb{Z}, \quad (3.5) 
\]
Similarly we have a filtration by \( B^{-\sqrt{-1}} \) submodules \( \hat{\mathbb{Z}} \), \( \hat{\mathbb{Z}}[m] := \hat{\mathbb{Z}} \cap \hat{\mathbb{Z}}[m] \) and likewise for \( \mathbb{Z} \). We also define the degree of an \( \infty \)-cycle \( p = (P_{n,i}) \) by
\[
(3.6) \quad \deg p = \frac{n^2}{4} + \deg_0 P_{n,i},
\]
where \( \deg_0 \) is defined in (2.22). Since any pair \((P_{n,i}, P_{n+2,i+1})\) in \( p \) is a link, the right hand side above is independent of \( n \).

We say an \( \infty \)-cycle \( p = (P_{n,i}) \) is \( N \)-minimal if \( P_{n,i} = 0 \) for \( n < N \). The space \( \hat{\mathbb{Z}} \) is filtered by \( U_{\sqrt{-1}} \)-submodules \( \hat{\mathbb{Z}}_{N} \) consisting of \( N \)-minimal \( \infty \)-cycles,
\[
\hat{\mathbb{Z}} = \hat{\mathbb{Z}}_{0} \supset \hat{\mathbb{Z}}_{1} \supset \cdots .
\]
Similarly we have a filtration by \( B^{+\sqrt{-1}} \) submodules \( \mathbb{Z}_{N} = \mathbb{Z} \cap \hat{\mathbb{Z}}_{N} \) of \( \mathbb{Z} \). If \( p = (P_{n,i})_{n \geq 0, n-2l = m} \in \hat{\mathbb{Z}}_{N} \), then \( P_{N,l'} \) is \( N \)-minimal where \( l' = \frac{N-m}{2} \). We have therefore natural injective homomorphisms
\[
(3.7) \quad \varphi : \text{gr} \hat{\mathbb{Z}} := \oplus_{N \geq 0} \hat{\mathbb{Z}}_{N}/\hat{\mathbb{Z}}_{N+1} \to \oplus_{N \geq 0} \hat{W}_{N},
\]
which respect the bi-grading \((d,m)\) given by \(-\deg\) and weight. (See Section 3.3 for the reason of the minus sign.) We are now in a position to state the main result of this paper.

**Theorem 3.4.** The map (3.6) (resp. (3.7)) is an isomorphism of bi-graded \( U_{\sqrt{-1}} \)-modules (resp. \( B^{+\sqrt{-1}} \)-modules).

**Proof.** It suffices to show the surjectivity. Take any \( P \in \hat{W}_{N} \). By Proposition 2.20 there exists \( g \in U_{\sqrt{-1}} \) such that \( P = g.1_{N} \). From (3.4), we can choose \( g' \in U_{\sqrt{-1}} \) and \( i \in \{0,1\} \) such that \( 1_{N} = g'.1_{i} \). Then \( p = gg'.1_{i} \) is an \( \infty \)-cycle which is sent to \( P \) under the map (3.6). The case (3.7) is completely parallel. \( \square \)

Theorem 3.4 gives an alternative proof of Nakayashiki’s result [7] and extends it to the case including negative powers of \( z_1, \cdots, z_n \).

**Remark.** Fix \( m \). For any sequence \( p_{j} \in \hat{\mathbb{Z}}[m] \) such that \( (p_{j})_{n, (n-m)/2} = 0 \) for \( n < j \), the infinite sum \( \sum_{j \geq 0} p_{j} \) is well defined and belongs to \( \mathbb{Z}[m] \). Theorem 3.4 implies that conversely any \( p \in \mathbb{Z}[m] \) can be written as such an infinite sum with \( p_{j} \in \hat{\mathbb{Z}}[m] \). \( \square \)

3.3. **Characters.** In this subsection we study the characters of \( \mathbb{Z}, \hat{\mathbb{Z}} \). By a character of a bi-graded vector space \( V = \oplus_{d,m} V_{d,m} \), we mean the generating series
\[
(3.8) \quad \text{ch}_{q,z} V = \sum_{d,m} q^{d}z^{m} \dim V_{d,m}.
\]
In the below we use the bi-grading \((d, m)\) by \(-\deg\) and weight. In order to match our characters to the irreducible characters of \(\hat{\mathfrak{sl}}_2\) we put the minus sign for the degree.

**Theorem 3.5.** For \(i = 0\) or \(1\), we have

\[
\text{ch}_{q, z} Z^{(i)} = \chi_i(q^{-1}, z),
\]

where

\[
\chi_i(q, z) := \frac{1}{(q)_{\infty}} \sum_{m \in 2\mathbb{Z} + i} q^{m^2/4} z^m
\]

is the character of the level 1 integrable module with the highest weight \(\Lambda_i\) of \(\hat{\mathfrak{sl}}_2\).

**Proof.** This is a consequence of the isomorphism (3.7) and Nakayashiki’s result [6] on the character of \(W_n\). \(\square\)

Note that (3.6) implies that the character of \(Z^{(i)}\) is equal to that of the level \(-1\) integrable module with the lowest weight \(-\Lambda_i\).

On the other hand, the character of \(\hat{Z}^{(i)}\) is ill-defined since each weight subspace is infinite dimensional. To get around this inconvenience, we follow the idea of [5] and introduce the truncated character. For each \(L \in \frac{1}{2} \mathbb{Z}_{\geq 0}\), consider the space

\[
\hat{W}_{N,L} := \prod_{j=1}^{N} z_j^{-L} \cdot W_N.
\]

We have \(\hat{W}_{N,0} \subset \hat{W}_{N,1} \subset \cdots\) and \(\hat{W}_N = \cup_{L \in \mathbb{Z}_{\geq 0}} \hat{W}_{N,L}\). We will use below the standard \(q\)-binomial symbol

\[
\left[ \begin{array}{c} m \\ n \end{array} \right] = \begin{cases} \frac{(q)_m}{(q)_n(q)^{m-n}} & (m \geq n \geq 0), \\ 0 & \text{(otherwise)}, \end{cases}
\]

where \((z)_n = \prod_{j=0}^{n-1} (1 - q^j z)\). Recall that each integrable \(\hat{\mathfrak{sl}}_2\)-module has a family of Demazure subspaces parametrized by the elements of the affine Weyl group. Their characters

\[
\chi_i(q, z; L) = \sum_{m \in 2\mathbb{Z} + i} \left[ \begin{array}{c} 2L \\ L + \frac{m}{2} \end{array} \right] q^{m^2/4} z^m \quad (i \equiv 2L \mod 2)
\]

give polynomial finitizations of the full characters in the sense that \(\lim_{L \to \infty} \chi_i(q, z; L) = \chi_i(q, z)\).

**Proposition 3.6.** Let \(L \in \frac{1}{2} \mathbb{Z}_{\geq 0}\), \(i, j = 0, 1\) with \(j \equiv 2L \mod 2\). Then we have

\[
\text{ch}_{q, z} \left( \bigoplus_{N \equiv \frac{1}{2} \mod 2} \hat{W}_{N,L} \right) = \chi_i(q^{-1}, z) \chi_j(q, z; L).
\]

**Proof.** Without loss of generality, we prove (3.10) in the summed form over \(i = 0, 1\). Theorem 3.5 is equivalent to the statement

\[
\text{ch}_{q^{-1}, z} W_N = q^{N^2/4} \sum_{l=0}^{N} \frac{1}{(q)_N} \left[ \begin{array}{c} N \\ l \end{array} \right] z^{N-2l}.
\]
Hence, taking the sum over $i = 0, 1$, the left hand side of (3.10) with $q$ replaced by $q^{-1}$ becomes

$$
\sum_{N \geq l \geq 0} \frac{q^{N^2/4-NL}}{(q)_N} \left[ \begin{array}{c} N \\ l \end{array} \right] z^{N-2l} = \sum_{l,m \geq 0} q^m q^{m^2/4-Lm} \frac{q^{m+l(l-2L)}}{(q)_l(q)_{i+m}}.
$$

The sum over $l$ can be simplified by using the formula (4, Lemma 6.2)

$$
\sum_{l \geq 0} q^l z^{\infty} = \frac{1}{(q)_\infty} \frac{2L}{s} q^{-s-1},
$$

wherein $\frac{2L}{s}$ signifies the $q$-binomial symbol with $q$ replaced by $q^{-1}$. Specializing $z = q^m$ in (3.12), we find that the right hand side of (3.11) becomes

$$
\sum_{m,s \in \mathbb{Z}} q^m q^{m^2/4-Lm} \frac{1}{(q)_\infty} \left[ \begin{array}{c} 2L \\ s \end{array} \right] q^{-s-1} q^{ms}.
$$

Changing $s$ to $s + L - j/2$, and changing $m$ to $m - 2s + j$ afterwards, we obtain

$$
\frac{1}{(q)_\infty} \sum_{m \in \mathbb{Z}} q^m q^{m^2/4} \sum_{s \in \mathbb{Z}} \left[ \begin{array}{c} 2L \\ s+L-j/4 \end{array} \right] q^{-2s-j} z^{-2s-j/2}.
$$

Changing $q$ to $q^{-1}$ we obtain the desired formula.

Letting $L \to \infty$ with $L \in \mathbb{Z}$, we obtain from (3.10) a formal but suggestive expression

$$
\chi_i(q^{-1},z) \chi_0(q,z)
$$

for the would-be ‘character’ of $\widehat{Z}^{(i)}$.

We refer the reader to [1] for similar results on product formulas for shifted characters.

4. **Form factors and $\infty$-cycles**

In this section we discuss the relation between the space of $\infty$-cycles and form factors of the $SU(2)$ invariant Thirring model (ITM).

4.1. **Form factor axioms.** First we recall the setting. Let $V = \mathbb{C}v_+ \oplus \mathbb{C}v_-$ be the vector representation of $\mathfrak{sl}_2 = \mathbb{C}E \oplus \mathbb{C}F \oplus \mathbb{C}H$. The action is given by

$$
Ev_+ = 0, \quad Ev_- = v_+, \quad Fv_+ = v_-, \quad Fv_- = 0, \quad Hv_\pm = \pm v_\pm.
$$

Denote by $P \in \text{End}(V^\otimes 2)$ the permutation operator $P(u \otimes v) = v \otimes u$. The S-matrix of ITM is the linear operator acting on $V^\otimes 2$ defined by

$$
S(\beta) = \frac{\zeta(-\beta) \beta - \pi iP}{\zeta(\beta) \beta - \pi i}.
$$

Here $\zeta(\beta)$ is a certain meromorphic scalar function that accounts for the overall normalization. The precise formula can be found e.g. in [9], eq.(16).

With each local field $\mathcal{O}$ in the theory is associated a tower of functions $f^{\mathcal{O}} = (f_n(\beta_1,\ldots,\beta_n))_{n \geq 0}$ called the form factor of $\mathcal{O}$. The function $f_n(\beta_1,\ldots,\beta_n)$ takes
values in the tensor product $V^\otimes n$. Form factor $(f_n)_{n \geq 0}$ should satisfy the following axioms:

Axiom 1: $f_n(\ldots, \beta_{j+1}, \beta_j, \ldots) = P_{j,j+1}S_{j,j+1}(\beta_j - \beta_{j+1})f_n(\ldots, \beta_j, \beta_{j+1}, \ldots)$.

Axiom 2: $f_n(\beta_1, \ldots, \beta_{n-1}, \beta_n + 2\pi i) = e^{\frac{\pi i}{2}}P_{n,n-1} \cdots P_{2,1}f_n(\beta_n, \beta_1, \ldots, \beta_{n-1})$.

Axiom 3: $\text{res}_{\beta_n = \beta_{n-1} + 2\pi i} f_n(\beta_1, \ldots, \beta_n) = (I + e^{-\frac{\pi i}{2}}S_{n-1,n-2}(\beta_{n-1} - \beta_{n-2}) \cdots S_{1,1}(\beta_1 - \beta_1)) \times f_{n-2}(\beta_1, \ldots, \beta_{n-2}) \otimes (v_+ \otimes v_- - v_- \otimes v_+)$.

Here $P_{j,j+1}$ is the permutation operator acting on the $j$-th and $(j+1)$-st components, and $S_{j,j'}(\beta)$ is the operator acting on the tensor product of the $j$-th and $j'$-th components of $V^\otimes n$ as $S(\beta)$.

The operators $S_{i,j}(\beta)$ and $P_{i,j}$ commute with the action of $\mathfrak{sl}_2$. If $(f_n)_{n \geq 0}$ is a form factor, then $(Ff_n)_{n \geq 0}$ is also a form factor. For that reason, we restrict our considerations to only form factors satisfying the highest weight condition

$$Hf_n(\beta_1, \ldots, \beta_n) = mf_n(\beta_1, \ldots, \beta_n),$$

$$Ef_n(\beta_1, \ldots, \beta_n) = 0$$

for some $m \in \mathbb{Z}_{\geq 0}$ for all $n$. Other form factors can be obtained from these ones by the action of $F \in \mathfrak{sl}_2$.

4.2. The integral formula and $\infty$-cycles. A large class of form factors of the ITM is given in terms of the hypergeometric integral [3]. It has the following structure:

$$\psi(\beta_1, \ldots, \beta_n) := \sum_{\#M = l} v_M \int_{C_\infty} \prod_{p=1}^l d\alpha_p \prod_{p=1}^l \phi(\alpha_p; \beta_1, \ldots, \beta_n)$$

$$\times w_M(\alpha_1, \ldots, \alpha_l|\beta_1, \ldots, \beta_n) P(X_1, \ldots, X_l|z_1, \ldots, z_n),$$

where $0 \leq l \leq n$. Let us explain the notation.

First, $v_M \in V^\otimes n$ denotes the vector

$$v_M := v_{\epsilon_1} \otimes \cdots \otimes v_{\epsilon_n} \in V^\otimes n,$$

where the $\epsilon_j$'s are related to the index set $M = \{m_1, \ldots, m_l\}$ (1 \leq m_1 < \cdots < m_l \leq n) by $M = \{j|\epsilon_j = -\}$. Second, $\phi$ and $w_M$ are fixed functions given by

$$\phi(\alpha; \beta_1, \ldots, \beta_n) := \prod_{j=1}^n \frac{1}{1 - e^{-\alpha - \beta_j}} \frac{\Gamma(\frac{\alpha - \beta_j + \pi i}{2\pi i})}{\Gamma(\frac{\alpha - \beta_j}{2\pi i})},$$

$$w_M := \text{Skew}_{\alpha_1, \ldots, \alpha_l} g_M,$$

$$g_M(\alpha_1, \ldots, \alpha_l|\beta_1, \ldots, \beta_n) := \prod_{p=1}^l \left( \frac{1}{\alpha_p - \beta_{m_p}} \prod_{j=1}^{m_p-1} \frac{\alpha_p - \beta_j + \pi i}{\alpha_p - \beta_j} \prod_{1 \leq p < p' \leq l} (\alpha_p - \alpha_{p'} + \pi i) \right).$$

In the second line, Skew$_{\alpha_1, \ldots, \alpha_l}$ stands for the skew symmetrization with respect to $\alpha_1, \ldots, \alpha_l$. Third, $P \in C_{n,l}$ is a deformed cycle, wherein the variables $X_a, z_j$ are related to the variables $\alpha_a, \beta_j$ by

$$X_a = e^{-\alpha_a}, \quad z_j = e^{\beta_j}.$$
Lastly, the integration contour $C$ goes along the real axis, except that the simple poles of the integrand at

$$\alpha_p = \beta_j - 2\pi i Z_{\geq 0}$$

are located below $C$, and those at

$$\alpha_p = \beta_j - \pi i + 2\pi i Z_{\geq 0}$$

above $C$. These are the only poles of the integrand. The integral converges absolutely if $2l \leq n$.

For a cycle $P_{n,l} \in \hat{C}_{n,l}$ we set

$$f_{P_{n,l}}(\beta_1, \ldots, \beta_n) := c_{n,l} e^{\frac{\pi}{4} \sum_{j=1}^{n} \beta_j} \prod_{1 \leq j < j' \leq n} \zeta(\beta_j - \beta_{j'}) \cdot \psi_{P_{n,l}}(\beta_1, \ldots, \beta_n),$$

where $c_{n,l}$ is a constant defined by

$$c_{n,l} := \begin{cases} \prod_{a=0}^{l} d_{2k+2a, a}, & (m = 2k), \\ \prod_{a=1}^{l} d_{2k+2a-1, a}, & (m = 2k+1), \end{cases}$$

$$d_{n,l} := \frac{2\pi}{\zeta(-\pi i)} (-2\pi i)^{-l-\frac{n}{2}}.$$ 

The precise connection between $\infty$-cycles and form factors is given by the following theorem proved in [9].

**Theorem 4.1.** For an $\infty$-cycle $p = (P_{n,l})$ of weight $m \in \mathbb{Z}_{\geq 0}$, the tower $f_p = (f_{P_{n,l}})$ satisfies the form factor axioms as well as the highest weight conditions (4.1), (4.2).

A local field $\mathcal{O}$ is said to have Lorentz spin $s$ if the homogeneous property

$$f_{\mathcal{O}}(\beta_1 + \Lambda, \ldots, \beta_n + \Lambda) = e^{s \Lambda} f_{\mathcal{O}}(\beta_1, \ldots, \beta_n)$$

holds for its form factor. It is easy to see that the Lorentz spin of $f_p$ is equal to $\deg p$ introduced in (3.5).

Suppose $f_{\mathcal{O}} = (f_n)$ is a form factor of a local field. Then there exists an $N \geq 0$ such that $f_n = 0$ if $n < N$. From Axiom 3 we have

$$\text{Axiom 3’: res}_{\beta_N = \beta_{N-1} + \pi i} f_N(\beta_1, \ldots, \beta_N) = 0.$$

It can be shown [9] that the function $f_{P_{N,l}}$ associated with a minimal cycle $P_{N,l}$ satisfies Axiom 3’. As in [9] [11], we expect the converse to be true, namely that for any $N$-minimal form factor $f = (f_n)_{n \geq 0}$ satisfying (4.1), (4.2), there exists a minimal cycle $P_{N,l} \in \hat{W}_{N,l}$ such that $f_{P_{N,l}} = f_N$. Under this assumption, Theorem 4.4 states that any $f$ (with fixed weight $m$) can be written as $f_p$ for some $\infty$-cycle $p \in \mathbb{Z}$.

4.3. **Realization of the space of local operators.** The space of $\infty$-cycles is not the same as the space of local fields, since there are null cycles. In [13] it is proved that

$$(x_0 \cdot A_{n,l-1} + (x_0^2) \cdot A_{n,l-2}) \cap \tilde{C}_{n,l} = \ker (P_{n,l} \mapsto f_{P_{n,l}}).$$

Now we set

$$\tilde{M}_{n,l} := \tilde{C}_{n,l} / \left((x_0 \cdot A_{n,l-1} + (x_0^2) \cdot A_{n,l-2}) \cap \tilde{C}_{n,l}\right)$$

where
and consider the projection
\[ \hat{C}_{n,l} \rightarrow \hat{M}_{n,l}, \quad P_{n,l} \mapsto \bar{P}_{n,l}. \]

Set
\[ \mathcal{L}[m] := \{ \bar{p} = (\bar{P}_{n,l})| p = (P_{n,l}) \in \hat{Z}[m] \}. \]

**Lemma 4.2.** If \( m < 0 \), then we have \( \mathcal{L}[m] = \{ 0 \} \).

**Proof.** Let \( p = (P_{n,l}) \in \hat{Z}[m] \) be \( N \)-minimal and set \( l' = \frac{N-m}{2} \). Then we have \( P_{N,l'} \in \hat{W}_{N,l'}. \) In [4] it is proved that
\[ \hat{W}_{N,l'}/ \left( x_0^2 \hat{W}_{N,l'-1} + (x_0^2)^2 \hat{W}_{N,l'-2} \right) = 0 \quad \text{if} \quad N - 2l' < 0. \]

Hence there exists an \( \infty \)-cycle
\[ q \in x_0^2 \hat{Z} + (x_0^2)^2 \hat{Z} \]
such that \( p - q \) is \((N+2)\)-minimal. Repeating this argument, we find that for each \( n, l \) we have \( \bar{P}_{n,l} = 0 \). Therefore \( \bar{p} = 0. \) \( \square \)

Now we set \( \mathcal{L} = \oplus_{m \geq 0} \mathcal{L}[m] \). Denote by \( \mathcal{F} \) the space of form factors satisfying (4.1) and (4.2). Then the following map is well defined:
\[ \Phi : \mathcal{L} \rightarrow \mathcal{F}, \quad \bar{p} = (\bar{P}_{n,l}) \mapsto f_p = (f_{P_{n,l}}). \]

We conjecture that the map \( \Phi \) is an isomorphism.

In the end, let us give some examples discussed in [9].

**Identity operator.** The form factor of the identity operator \( I \) is of weight zero and zero minimal. It is given by
\[ f^I = (1, 0, 0, \ldots). \]

Note that \( f_2 = 0 \) and \( f_0 \neq 0 \) does not violate Axiom 3, since the latter becomes trivial for \( n = 2 \),
\[ \text{res}_{\beta_2 = \beta_1 + \pi i} f_2(\beta_1, \beta_2) = (I + (-1)I)(f_0 \otimes (v_+ \otimes v_- - v_- \otimes v_+)) = 0. \]

The form factor \( f^I \) is obtained from the \( \infty \)-cycle
\[ 1_0 = (1_0, X, X \wedge X^3, X \wedge X^3 \wedge X^5, \ldots). \]

This can be seen from (4.4) and the following lemma.

**Lemma 4.3.** Set
\[ \tilde{A}_{2l,l} = \{ P(X_1, \ldots, X_l) \in A_{2l,l}|P(X_1, \ldots, X_{l-1}, 0) = 0 \}. \]

Then
\[ \tilde{A}_{2l,l} \subset x_0^2 \cdot A_{2l,l-1} + (x_0^2)^2 \cdot A_{2l,l-2}. \]

For the proof, see [4].

**su(2) currents.** The \( \infty \)-cycles associated with \( su(2) \) currents \( j^+_{\sigma} \ (\sigma = \pm) \) are of weight two and two minimal. They are given by
\[ (j^+_{\sigma})_{2l+2,l} = (-1)^l \prod_{j=1}^{2l+2} z_j^{-1} X \wedge X^3 \wedge \cdots \wedge X^{2l-1}, \]
\[ (j^+_{\sigma})_{2l+2,l} = X^3 \wedge X^5 \wedge \cdots \wedge X^{2l+1} \ (l \geq 0). \]
It is easy to see that
\[ j_+^+ = -x_1^+ 1, \quad j_-^+ = x_1^+ 1. \]

**Energy-momentum tensor.** Denote by \( T_z \) and \( T_{\bar{z}} \) the holomorphic and anti-holomorphic part of the energy momentum tensor, respectively. They are of weight zero and two minimal. The form factors are obtained from the following sequences of deformed cycles:
\[
(T_z)_{2l,l} = (\sum_{j=1}^{2l} z_j) X^0 \wedge X^1 \wedge X^3 \wedge \cdots \wedge X^{2l-1},
\]
\[
(T_{\bar{z}})_{2l,l} = (-1)^{l-1} (\prod_{j=1}^{2l} z_j^{-1}) (\sum_{j=1}^{2l} z_j^{-1}) X^0 \wedge X^1 \wedge X^3 \wedge \cdots \wedge X^{2l-3} \quad (l \geq 1).
\]

Note that these sequences are not \( \infty \)-cycles. However the following holds modulo null cycles \([4.3]\):
\[
T_z = -ix_1^+ x_1^+ 1, \quad T_{\bar{z}} = -ix_1^+ x_1^- 1.
\]

These equalities can be checked by using Lemma \([4.3]\).

**Appendix A. Polynomial realization of evaluation modules**

In this appendix, we derive an action of the quantum algebra \( U_q(\mathfrak{g}) \) on the space \( A_n = \oplus_{l=0}^{n} A_{n,l} \). The definition of the action is given in subsection \([2.2]\). Here we give the origin of the definition and some details of its derivation.

Recall that the space \( A_{n,l} \) is the space of skew-symmetric polynomials in the variables \( X_1, \ldots, X_l \) of degree less than or equal to \( n-1 \) with coefficients in \( K_n = \mathbb{C}(z_1,\ldots,z_n) \). We use the wedge notation defined in \([2.11]\) for skew-symmetric polynomials. For \( 1 \leq a \leq n \) we set
\[
G_a(X) = \prod_{j=1}^{a-1} (1 + z_j X) \prod_{j=a+1}^{n} (1 - z_j X).
\]

First note the following simple fact (see Remark below eq. \([3.12]\) in \([4]\)).

**Lemma A.1.** The skew-symmetric polynomials \( G_{p_1} \wedge \cdots \wedge G_{p_l} \) \((1 \leq p_1 < \cdots < p_l \leq n)\) constitute a \( K_n \)-basis of \( A_{n,l} \).

Set \( V = \mathbb{C} v_+ \oplus \mathbb{C} v_- \). Define operators \( \sigma^+, \sigma^- \) acting on \( V \):
\[
\sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \sigma^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \sigma^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.
\]

When we consider the tensor product of \( V \), we denote by \( \sigma^*_a \) the operator \( \sigma^* \) acting on the \( a \)-th tensor component. Extending the coefficient ring we set \( \tilde{K}_n = K_n \otimes \mathbb{C}[q,q^{-1}] \), \( \tilde{A}_n = A_n \otimes \mathbb{C}[q,q^{-1}] \). We define the action of \( g \in U_q^{res} \) (see Section \([3.1]\)) on \( V^\otimes n \otimes \tilde{K}_n \) by \( (\rho_{z_1} \otimes \cdots \otimes \rho_{z_n}) \circ \Delta^{(n-1)}(g) \) where \( \rho_z \) is the evaluation representation of \( U_q^{res} \) on \( V \otimes \mathbb{C}[q,q^{-1}] \) such that
\[
(A.1) \quad e_i \mapsto \sigma^+, \quad f_i \mapsto \sigma^-, \quad t_i \mapsto q^{s^+}, \quad e_0 \mapsto z \sigma^-, \quad f_0 \mapsto z^{-1} \sigma^+, \quad t_0 \mapsto q^{-s^+},
\]
and \( \Delta \) is the coproduct defined by
\[
\Delta(e_i) = e_i \otimes 1 + t_i \otimes e_i, \quad \Delta(f_i) = f_i \otimes t_i^{-1} + 1 \otimes f_i, \quad \Delta(t_i) = t_i \otimes t_i.
\]
Here we use the Chevalley generators $e_i, f_i, t_i$, i.e.,
\[
x_0^+ = e_1, x_{-1}^+ = t_1^{-1} f_0, x_0^- = f_1, x_t^- = e_0 t_1, t_0 = t_1^{-1}.
\]
We induce an action of $U_q^\text{res}$ on $\tilde{A}_n$ from the action on $V_q^\otimes n \otimes \tilde{K}_n$ by using an isomorphism between $V_q^\otimes n \otimes \tilde{K}_n$ and $\tilde{A}_n$. The isomorphism is given as follows. Let $\psi_1, \ldots, \psi_n$ be a set of Grassmann variables. We denote by $\Lambda_n$ the Grassmann algebra generated by them. It is an irreducible module over the fermion algebra $\Psi_n$ generated by $\psi_a, \psi^*_a$ such that $[\psi_a, \psi^*_b]_\pm = \delta_{a,b}, [\psi_a, \psi_b]_\pm = [\psi_a, \psi^*_b]_\pm = 0$.

There are isomorphisms of vector spaces over $\tilde{K}_n$:
\[
(A.2) \quad V_q^\otimes n \otimes \tilde{K}_n \simeq \Lambda_n \otimes \tilde{K}_n \simeq \tilde{A}_n.
\]

The first isomorphism is given by the Jordan-Wigner transformation $
\psi_a^* = \sigma_a^+ \prod_{j=a+1}^n (-i\sigma_j^z), \quad \psi_a = \sigma_a^- \prod_{j=a+1}^n (i\sigma_j^z)
$
and the identification of $\psi^*_a \in V_q^\otimes n$ with $1 \in \Lambda_n$. The second isomorphism is given by the identification of the left multiplication of $\psi_a$ on $\Lambda_n \otimes \tilde{K}_n$ with the wedge product $G_a \wedge$ on $\tilde{A}_n$ (see Lemma A.1).

Our goal is to compute the actions of the operators $X^-_{>0}(t), X^-_{<0}(t), X^+_{>0}(t), X^+_{<0}(t), X^-_{>0}(t)^{(2)}, X^-_{<0}(t)^{(2)}, X^+_{>0}(t)^{(2)}, X^+_{<0}(t)^{(2)}, a_+(t)$ (see Section 222) in the limit $\varepsilon \to 0$ by setting $q = ie^\varepsilon$. The computation is elementary but long. We do not give all the details of computation.

By a straightforward calculation we obtain

Lemma A.2. We have the expansion of the end term of the half current $X^-_{>0}(t)$ up to the order $\varepsilon$.

\[
(A.3) \quad q^{-1}x_t^- = \sum a z_a \psi_a + \left( \sum_{a < b} z_a \psi_a \sigma_b^z \right) \varepsilon + O(\varepsilon^2).
\]

Similarly, we have the expansion of the end term of the half current $a_+(t)$ up to the order $\varepsilon^2$.
\[
(A.4) \quad iq^{-1}a_1 = \alpha_0 + \alpha_1 \varepsilon + \alpha_2 \varepsilon^2 + O(\varepsilon^3),
\]
where
\[
\alpha_0 = \sum a z_a, \quad \alpha_1 = -\sum a z_a \sigma_a^z + 4 \sum_{a < b} z_a \psi_a \psi_b^*, \quad \alpha_2 = \alpha_0/2 = 4 \sum_{a < b < c} z_a \psi_a \sigma_b^z \psi_c^*.
\]

The other terms in the half current $X^-_{>0}(t)$ are determined recursively by the equation
\[
(A.5) \quad X^-_{>0}(t) - q^{-1}tx_t^- = \frac{it}{2} [iq^{-1}a_1, X^-_{>0}(t)].
\]

Using (A.3) and (A.4), one can check
Proposition A.3. We have the expansion
\begin{equation}
X^-_{>0}(t) = \gamma_0(t) + \gamma_1(t)\varepsilon + O(\varepsilon^2),
\end{equation}
where
\begin{equation}
\gamma_0(t) = \sum_a A_a(t)\psi_a, \quad \gamma_1(t) = \sum_{a<b} B_{a,b}(t)\psi_a\sigma_b + \sum_{a<b<c} C_{a,b,c}(t)\psi_a\psi_b\psi_c,
\end{equation}
and
\[ A_a(t) = \frac{z_a t}{1 - z_a t} \prod_{j > a} \frac{1 + z_j t}{1 - z_j t}, \quad B_{a,b}(t) = \frac{1 - z_b t}{1 + z_b t} A_a(t), \]
\[ C_{a,b,c}(t) = 8C_{a,b}(t)A_c(t), \quad C_{a,b}(t) = \frac{z_a t}{1 - z_a t} \left( \prod_{a<j<b} \frac{1 + z_j t}{1 - z_j t} \right) \frac{z_b t}{1 - z_b t}. \]

The following equality is useful in this calculation.
\begin{equation}
\sum_{b > a} A_b(t) = \frac{1}{2} \left( \prod_{j > a} \frac{1 + z_j t}{1 - z_j t} - 1 \right).
\end{equation}

Instead of repeating similar calculations for \(X^-_{\leq 0}(t), X^+_{\geq 0}(t), X^+_{<0}(t)\), we can use symmetries. Let \(\alpha\) be an anti-algebra map of the algebra \(\Psi_n \otimes \tilde{K}_n\) given by
\begin{equation}
\alpha(z_a) = z_{n+1-a}, \quad \alpha(\psi_a) = z_{n+1-a}^{-1}\psi_{n+1-a}^*, \quad \alpha(\psi_a^*) = z_{n+1-a}\psi_{n+1-a}.
\end{equation}

We extend it to the space \(\Psi_n \otimes \tilde{K}_n[[t,t^{-1}]]\) of formal series in \(t\) by setting \(\alpha(t) = t\). Similarly, we define an algebra map \(\beta\) by
\begin{equation}
\beta(z_a) = z_{-n+1-a}, \quad \beta(\psi_a) = (-1)^{n+1}\psi_{n+1-a}^*, \quad \beta(\psi_a^*) = (-1)^{n+1}\psi_{n+1-a},
\end{equation}
and extend it to formal series in \(t\) by setting \(\beta(t) = t^{-1}\). Then we have \(\alpha^2 = \text{id}, \beta^2 = (-1)^{(n+1)}\text{id}, \alpha\beta = (-1)^{(n+1)}\beta\alpha\), where \(\varepsilon\) is the parity in the fermion algebra \(\Psi_n\). Namely, \(\varepsilon = 0\) on the even part of \(\Psi_n\), and \(\varepsilon = 1\) on the odd part. We also have \(\alpha(T) = T, \beta(T) = T^{-1}\), where
\[ T = \prod_{a=1}^n (i\sigma_a^*). \]

Proposition A.4. We have
\begin{align}
X^+_{\geq 0}(t) &= -iTt^{-1}\alpha(X^-_{<0}(t)) \mod \varepsilon^2, \\
X^+_{<0}(t) &= iT^{-1}\beta(X^-_{<0}(t)) \mod \varepsilon^2, \\
X^-_{<0}(t) &= t\beta\alpha(X^-_{<0}(t)) \mod \varepsilon^2.
\end{align}

Proof. We use the equalities
\begin{align}
\alpha(q^{-1}a_1) &= q^{-1}a_1 \mod \varepsilon^3, \\
\beta(q^{-1}a_1) &= -qa_{-1} \mod \varepsilon^3.
\end{align}

Let us prove (A.11). It is easy to check
\[-iTt^{-1}\alpha(x_1 q^{-1}t) = x_0^+ \mod \varepsilon^2.\]
By using (A.14), the equation (A.15), which determines \(X^-_{<0}(t)\), is transformed into
\[-iTt^{-1}\alpha(X^-_{<0}(t)) - x_0^+ = -\frac{i}{2}[i q^{-1}a_1, -iTt^{-1}\alpha(X^-_{<0}(t))] \mod \varepsilon^2.\]
From the defining relations of the Drinfeld currents we have
\[ X^+_{\geq 0}(t) - x_0^+ = -\frac{it}{2} [q^{-1}a_1, X^+_{\geq 0}(t)]. \]

Therefore, we obtain (A.11). Other cases are similar. \(\square\)

From Propositions A.5 and A.6 a short calculation leads to

**Proposition A.5.** On the space \(\Lambda_n \otimes \hat{K}_n\), the actions of the half currents in the limit \(q \to \sqrt{-1}\) are given as follow. We write them on the subspace isomorphic to \(\Lambda_{n,t}\) by A.2, and show the equalities as rational functions in \(t\). However, they should be properly understood as equalities of power series in \(t\) for \(X^-_{\geq 0}(t), X^+_{\geq 0}(t)\), and in \(t^{-1}\) for \(X^-_{\leq 0}(t), X^+_{\leq 0}(t)\).

\[(A.16) X^-_{\leq 0}(t) = -X^+_{\leq 0}(t) = \sum_a \frac{z_a t}{1 - z_a} \prod_{j > a} \frac{1 + z_j t}{1 - z_j} \psi_a, \]

\[(A.17) X^-_{\leq 0}(t) = X^+_{\leq 0}(t) = i^n \sum_{a < b} \frac{1}{1 - z_{ab} t} \prod_{a < j < b} \frac{1 + z_j t}{1 - z_j} \psi_a \psi_b, \]

\[(A.18) X^+_{\leq 0}(t) = -X^+_{\leq 0}(t) = -i^n (-1)^{2l-1} \sum_a \frac{1}{1 - z_a t} \prod_{j < a} \frac{1 + z_j t}{1 - z_j} \psi_a, \]

\[(A.19) X^+_{\leq 0}(t) = X^-_{\leq 0}(t) = (-1)^n \sum_{a < b} \frac{1}{1 - z_{ab} t} \prod_{a < j < b} \frac{1 + z_j t}{1 - z_j} \frac{1}{1 - z_{ab} t} \psi_a \psi_b. \]

We set
\[ b_\pm(t) := \sum_{\pm n > 0} a_n (q^{-1} t)^n, \]
\[ b^{(2)}_\pm(t) := \sum_{\pm n > 0} \frac{a_{2n}}{q + q^{-1} (q^{-1} t)^2 n}. \]

**Proposition A.6.** The actions of \( b_\pm(t) \), \( b^{(2)}_\pm(t) \) in the limit \( \varepsilon \to 0 \) are given as follows.

\[(A.20) b_+(t) = -i \sum_{m \geq 1, \text{odd}} \frac{z_a t}{m} \sum_m t^m, \]

\[(A.21) 2b^{(2)}_+(t) = \text{the even part of} \]
\[ \sum_a \frac{z_a t}{1 - z_a} \psi_a - 4 \sum_{a < b} \frac{z_a t}{1 - z_{ab} t} \prod_{a < j < b} \frac{1 - z_j t}{1 + z_j t} \frac{1}{1 - z_{ab} t} \psi_a \psi_b. \]

\[ b_-(t) = -\beta(b_+(t)), \quad b^{(2)}_-(t) = -\beta(b^{(2)}_+(t)). \]

**Proof.** We use the equality
\[ b_+(t) = \frac{1}{q - q^{-1}} \log(1 + (q - q^{-1}) t_1^{-1} x_0^+, X^-_{\leq 0}(t))). \]

A simple calculation shows (A.20) and (A.21). For \( b_-(t) \), we use the properties of the algebra map \( \beta \). (A.17) and (A.18) give
\[ \beta(x_0^+) = -it^{-1} x_0^- \mod \varepsilon^2, \quad \beta(t_1^{-1}) = t_1, \]
and obtain
\[ \beta(b_+(t)) = \frac{1}{q - q^{-1}} \log(1 + (q - q^{-1})t_1[x_0^-, X_0^+(t)]) \mod \varepsilon^2. \]

Since the expression in the right hand side is exactly \(-b_-(t)\), we have (A.22) in the limit \(\varepsilon \to 0.\)

**Proof of Proposition 2.2.** The final step is to rewrite the actions by the second part of the isomorphisms (A.2). It is often useful in the work to note that these actions are symmetric with respect to \(z\).

The actions (2.13) and (2.14) follow from the equalities (A.20). Let us prove (2.15). We rewrite as
\[ \sum_a A_a(t)G_a(x) = F_n(t, X), \]
(A.24)
\[ 4 \sum_{a<b} C_{ab}(t)G_a(X) \wedge G_b(X) = F_n(t|X_1, X_2). \]

Note that there are no poles at \(t = X\) in \(F_n(t, X)\), or \(t = X_1, X_2\) in \(F_n(t|X_1, X_2)\). By induction, we can check the equalities at the pole \(t = z_n^{-1}\) (and therefore at \(t = z_n^{-1}\) for all \(a\)), and at \(t = \infty\).

The action (2.19) follows from (A.20). Let us prove (2.15). We rewrite as \(\sigma_a^* = 1 - 2\psi_a\psi_a^*\) in (A.21). Then, the proof reduces to the case \(l = 1\), which is equivalent to the equality
\[ \frac{-z_a t}{1 + z_a t} G_a(X) - 2 \sum_{b < a} \frac{z_b t}{1 + z_b t} \left( \prod_{b < j < a} \frac{1 - z_j t}{1 + z_j t} \right) \frac{G_b(X)}{1 + z_a t} \]
\[ = -\frac{t}{X + t} \left( \frac{G_a(X) - \Theta_n(X)}{\Theta_n(-t)} G_a(-t) \right). \]

The proof of (2.10) is similar.

Let us prove (2.17). We use (A.11), note that
\[ t^{-1} \alpha(\gamma_0(t)) = \sum_a \frac{1}{1 - z_a t} \prod_{j < a} \frac{1 + z_a t}{1 - z_a t} \psi_a^*. \]

This operator sends \(G_a(X)\) to
\[ \frac{1}{1 - z_a t} \prod_{j < a} \frac{1 + z_a t}{1 - z_a t} = \frac{G_a(t)}{\Theta_n(t)}. \]

Computing the effect of \(T\), we obtain (2.14).

Similarly, the proof of (2.18) reduces to the equality for \(b < a\),
\[ \sum_c \text{res}_{u = z_c} \frac{G_a(-u)G_b(u) - G_a(u)G_b(-u)}{\Theta_n(-u)\Theta_n(u)} \frac{du}{u - t} \]
\[ = \frac{1}{1 - z_b t} \left( \prod_{b < j < a} \frac{1 + z_j t}{1 - z_j t} \right) \frac{1}{1 - z_a t}. \]

The first term with \(G_a(-u)G_b(u)\) vanishes because there are no poles at \(u = z_c^{-1}\) for any \(c\). The second term with \(-G_a(u)G_b(-u)\) can be calculated by the residue at \(u = t\) because there are no pole at \(u = -z_c^{-1} \).
APPENDIX B. PROOF OF LEMMA 4.3

Following the line of [13], we give a proof of Lemma 4.3 by using the fermionic realization of the space $A_{n,l}$ given in Appendix A. Let $\psi_1, \ldots, \psi_n$ be a set of Grassmann variables. Denote by $K_n[\psi_1, \ldots, \psi_n]$ the exterior algebra generated by them over $K_n = \mathbb{C}(z_1, \ldots, z_n)$. Introduce the degree defined by $\text{deg} \psi_a = 1$ and denote by $K_n[\psi_1, \ldots, \psi_n]_l$ the homogeneous component of degree $l$. Then we have the isomorphism

$$\mathcal{C}_{n,l} : K_n[\psi_1, \ldots, \psi_n]_l \rightarrow A_{n,l}.$$ 

See Lemma A.2 for the definition of this isomorphism. The action of $x_0^-$ and $(x_0^-)^{(2)}$ on $A_n = \oplus_{l=0}^{2n} A_{n,l}$ is intertwined with multiplication on $K_n[\psi_1, \ldots, \psi_n]$ with the following elements:

$$x_0^- \leftrightarrow \Sigma_1 = \sum_{a=1}^{n} (-1)^{n-a} \psi_a, \quad i(x_0^-)^{(2)} \leftrightarrow \Sigma_2 = \sum_{1 \leq a < b \leq n} (-1)^{a+b} \psi_a \psi_b.$$ 

Now let us prove the lemma. Set $n = 2l$ in the construction above. Introduce a set of new generators $\{\varphi_a\}_{a=1}^{2l}$ of $K_n[\psi_1, \ldots, \psi_{2l}]$ given by

$$\varphi_a = \psi_a - \psi_{2l} \quad (a = 1, \ldots, 2l - 1), \quad \varphi_{2l} = \psi_{2l}.$$ 

The elements $\Sigma_1$ and $\Sigma_2$ are represented in terms of these generators as follows:

(B.1) \quad $\Sigma_1 = \sum_{a=1}^{2l-1} (-1)^a \varphi_a$,

(B.2) \quad $\Sigma_2 = \tilde{\Sigma}_2 - \Sigma_1 (\varphi_{2l-1} - \varphi_{2l}),$ 

where

$$\tilde{\Sigma}_2 = \sum_{1 \leq a < b \leq 2l-2} (-1)^{a+b} \varphi_a \varphi_b.$$ 

From the definition of the isomorphism $\mathcal{C}_{2l,l}$ we can see that

$$\mathcal{C}_{2l,l}(K_n[\varphi_1, \ldots, \varphi_{2l-1}]_l) = \tilde{A}_{2l,l}.$$ 

From (B.1) we have that

(B.3) \quad $K_n[\varphi_1, \ldots, \varphi_{2l-1}]_l = \Sigma_1 \cdot K_n[\varphi_1, \ldots, \varphi_{2l-2}]_{l-1} + K_n[\varphi_1, \ldots, \varphi_{2l-2}]_{l-2}.$ 

In [13] it is proved that the map

$$K_n[\varphi_1, \ldots, \varphi_{2l-2}]_{l-2} \rightarrow K_n[\varphi_1, \ldots, \varphi_{2l-2}]_l, \quad x \mapsto \tilde{\Sigma}_2 \cdot x$$ 

is an isomorphism. From this fact, (B.2) and (B.3), we find that

$$K_n[\varphi_1, \ldots, \varphi_{2l-1}]_l \subset \Sigma_1 \cdot K_n[\varphi_1, \ldots, \varphi_{2l-2}]_{l-1} + \Sigma_2 \cdot K_n[\varphi_1, \ldots, \varphi_{2l-2}]_{l-2}.$$ 

This completes the proof. \hfill \Box

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