A MULTIPLICITY RESULT
FOR THE LINEAR
SCHRÖDINGER-MAXWELL EQUATIONS
WITH NEGATIVE POTENTIAL

GIUSEPPE MARIA COCLITE
S. I. S. S. A., via Beirut 2-4, Trieste 34014, Italy
e-mail: coclite@sissa.it

Abstract. In this paper it is proved the existence of a sequence of radial solutions
with negative energy of the linear Schrödinger-Maxwell equations under the action of
a negative potential.

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1. Introduction

In this paper we study the interaction between the electromagnetic field and the wave function related to a quantistic non-relativistic charged particle, that is described by the Schrödinger equation.

In [2, 3, 11] it has been studied the case in which the electromagnetic field is assigned. Here we shall assume that the unknowns of the problem are both the wave function $\psi = \psi(x, t)$ and the gauge potentials $\varphi = \varphi(x, t)$ and $A = A(x, t)$ related to the electromagnetic fields $E, H$ by the equations

$$
E = -\frac{1}{c} \frac{\partial A}{\partial t} - \nabla \varphi, \quad H = \nabla \times A.
$$

Such a situation has been studied by Benci and Fortunato (cfr. [5]) in the case in which the charged particle "lives" in a space region $\Omega$, which is bounded. Here we want to analyze the case in which $\Omega = \mathbb{R}^3$. Moreover we assume that there is an external field deriving from a potential $-V(x)$. We consider the electrostatic case, namely we look for potentials $\varphi$ and $A$ which do not depend on time $t$:

$$
\varphi = \varphi(x), \quad A = A(x), \quad x \in \mathbb{R}^3,
$$

and for standing wave function

$$
\psi(x, t) = u(x)e^{i\omega t}, \quad x \in \mathbb{R}^3, \ t \in \mathbb{R},
$$

where $\omega \in \mathbb{R}$ and $u$ is real valued. In this situation we can assume

$$
A = 0.
$$

It can be shown (cfr. [5]) that $\varphi$, $\omega$ and $u$ are related by the equations

$$
\begin{aligned}
-\frac{1}{2} \Delta u - \varphi u - V(x)u &= \omega u, \quad \text{in } \mathbb{R}^3, \\
\Delta \varphi &= 4\pi u^2, \quad \text{in } \mathbb{R}^3,
\end{aligned}
$$

where $V : \mathbb{R}^3 \to \mathbb{R}$ is a radial positive map, which is the potential of the external action. We shall assume:
\((V_1)\) \(V\) is continuous in \(\mathbb{R}^3 \setminus \{0\}\);

\((V_2)\) \(V \in L^{\frac{4}{3}}(\{|x| \leq 1\})\);

\((V_3)\) \(\lim_{|x| \to +\infty} V(x) = 0\);

\((V_4)\) \(\lim_{|x| \to +\infty} x^2 V(x) = +\infty\).

Observe that the coulumbian potential, that is the most physically interesting one, satisfies \((V_1)\), \((V_2)\), \((V_3)\) and \((V_4)\) (cfr. [13; 14]).

The equations in (1) have a variational structure in fact they are the Euler-Lagrange equations related to the functional:

\[
F_\omega(u, \varphi) = \frac{1}{4} \int_{\mathbb{R}^3} |\nabla u|^2 dx - \frac{1}{2} \int_{\mathbb{R}^3} \varphi u^2 dx - \frac{1}{16\pi} \int_{\mathbb{R}^3} |\nabla \varphi|^2 dx - \frac{1}{2} \int_{\mathbb{R}^3} V(x) u^2 dx - \frac{\omega}{2} \int_{\mathbb{R}^3} |u|^2 dx,
\]

this functional is strongly indefinite; this means that \(F_\omega\) is neither bounded from below nor from above and this indefinitness cannot be removed by a compact perturbation. Moreover \(F_\omega\) does not exhibit symmetry properties. By a suitable variational principle we are reduced to study an even functional which does not exhibit the same indefinitness of \(F_\omega\). The main result of this paper is the following.

**Theorem 1** Let \(V\) satisfy \((V_1)\), \((V_2)\), \((V_3)\) and \((V_4)\) then for all \(\omega < 0\) problem (1) has infinitely many solutions \(\{(u_k, \varphi_k)\}_{k \in \mathbb{N}}\) with \(u_k \in H^1(\mathbb{R}^3)\),

\[
\int_{\mathbb{R}^3} |\nabla \varphi_k|^2 dx < \infty
\]

and such that \(F_\omega(u_k, \varphi_k) < -\frac{\omega}{2}\).

The case in which \(V\) is radially decreasing and belongs to \(L^p(\mathbb{R}^3)\), with \(\frac{3}{2} < p < \infty\), is investigated in [9, Capitolo 6] and the nonlinear case is studied in [10]. Finally we recall that the Maxwell equations coupled with nonlinear Klein-Gordon equation and with Dirac equation have been studied respectively in [6; 12].
2. The Variational Principle

In this section we shall prove a variational principle which permits to reduce (1) to the study of the critical points of an even functional, which is not strongly indefinite. To this end we need some thecnical preliminaries.

We define the space $D^{1,2}(\mathbb{R}^3)$ as the closure of $C_0^\infty(\mathbb{R}^3)$ with respect to the norm
\[
\|u\|_{D^{1,2}} = \left( \int_{\mathbb{R}^3} |\nabla u|^2 dx \right)^{1/2}.
\]
The following lemma holds (cfr. [7, Theorem 2.4]):

**Lemma 2** For all $\rho \in L^1(\mathbb{R}^3) \cap L^r(\mathbb{R}^3)$, with $\frac{6}{5} < r \leq 2$, there exists only one $\varphi \in D^{1,2}(\mathbb{R}^3)$ such that $\Delta \varphi = \rho$. Moreover there results
\[
\|\varphi\|_{D^{1,2}} \leq c(\|\rho\|_{L^1}^2 + \|\rho\|_{L^r}^2)
\]
and the map $\rho \in L^1(\mathbb{R}^3) \cap L^r(\mathbb{R}^3) \mapsto \varphi = \Delta^{-1}(\rho) \in D^{1,2}(\mathbb{R}^3)$ is continuous.

By Lemma 2 and Sobolev inequality, for any given $u \in H^1(\mathbb{R}^3)$ the second equation of (1) has the unique solution
\[
\varphi = 4\pi \Delta^{-1} u^2 (\in D^{1,2}(\mathbb{R}^3)).
\]
For this reason we can reduce (1) to
\[
-\frac{1}{2} \Delta u - 4\pi(\Delta^{-1} u^2) u - V(x) u = \omega u, \quad \text{in } \mathbb{R}^3 \tag{2}
\]
Observe that (2) is the Euler-Lagrange equation of the functional
\[
J_\omega(u) = \frac{1}{4} \int_{\mathbb{R}^3} |\nabla u|^2 dx + \pi \int_{\mathbb{R}^3} |\nabla \Delta^{-1} u^2|^2 dx - \frac{1}{2} \int_{\mathbb{R}^3} V(x) u^2 dx - \frac{\omega}{2} \int_{\mathbb{R}^3} u^2 dx.
\]
Now we set
\[
H^1_r(\mathbb{R}^3) := \{ u \in H^1(\mathbb{R}^3) \mid u(x) = u(|x|), \quad x \in \mathbb{R}^3 \}.
\]
Since
\[
\frac{d}{d\lambda} \left( \int_{\mathbb{R}^3} |\nabla \Delta^{-1} (u + \lambda v)|^2 dx \right) \bigg|_{\lambda=0} = -2 \int_{\mathbb{R}^3} (\Delta^{-1} u|v) dx
\]
easily the following lemma holds.

**Lemma 3** For all \( \omega \in \mathbb{R} \) there results:

i) \( J_\omega \) is even;

ii) \( J_\omega \) is \( C^1 \) on \( H^1(\mathbb{R}^3) \) and its critical points are solutions of (2);

iii) any critical point of \( J_\omega \big|_{H^1_r(\mathbb{R}^3)} \) is also a critical point of \( J_\omega \).

3. Proof of Theorem 1

We begin proving some lemmas.

**Lemma 4** Let \( V \) satisfy \((V_1), (V_2) \) and \((V_3)\) then for all \( \omega < 0 \) the functional \( J_\omega \) is weakly lower semicontinuous in \( H^1_r(\mathbb{R}^3) \). Precisely

\[
u \in H^1_r(\mathbb{R}^3) \mapsto \int_{\mathbb{R}^3} |\nabla u|^2 dx - 2\omega \int_{\mathbb{R}^3} u^2 dx
\]
is weakly lower semicontinuous and

\[
u \in H^1_r(\mathbb{R}^3) \mapsto \int_{\mathbb{R}^3} |\nabla \Delta^{-1} u|^2 dx,
\]

\[
u \in H^1_r(\mathbb{R}^3) \mapsto \int_{\mathbb{R}^3} V(x)u^2 dx
\]
are weakly continuous.

**Proof.** Let \( \omega < 0 \). By a well known argument the functional

\[
u \in H^1_r(\mathbb{R}^3) \mapsto \int_{\mathbb{R}^3} |\nabla u|^2 dx - 2\omega \int_{\mathbb{R}^3} u^2 dx
\]
is weakly lower semicontinuous.
Prove that the functional
\[ u \in H^1_r(\mathbb{R}^3) \mapsto \int_{\mathbb{R}^3} |\nabla \Delta^{-1} u|^2 \, dx \]
is weakly continuous. We just have to observe that the operator
\[ Q : u \in H^1_r(\mathbb{R}^3) \mapsto u^2 \in L^\frac{6}{5}(\mathbb{R}^3) \cap L^2(\mathbb{R}^3) \]
is compact, in fact by the compact embeddings of \( H^1_r(\mathbb{R}^3) \) (cfr. [8, Theorem A.1'; 16]) the operator:
\[ H^1_r(\mathbb{R}^3) \hookrightarrow L^\frac{12}{5}(\mathbb{R}^3) \cap L^4(\mathbb{R}^3) \xrightarrow{Q} L^\frac{6}{5}(\mathbb{R}^3) \cap L^2(\mathbb{R}^3) \]
is compact and by Lemma 2 the following one
\[ \Delta^{-1} : L^\frac{6}{5}(\mathbb{R}^3) \cap L^2(\mathbb{R}^3) \longrightarrow \mathcal{D}^{1,2}(\mathbb{R}^3) \]
is continuous.

Prove that the functional
\[ u \in H^1_r(\mathbb{R}^3) \mapsto \int_{\mathbb{R}^3} V(x)u^2 \, dx \]
is weakly continuous. Let \( \{u_k\} \subset H^1_r(\mathbb{R}^3) \) and \( u \in H^1_r(\mathbb{R}^3) \) such that
\[ u_k \rightharpoonup u \quad \text{weakly in} \quad H^1_r(\mathbb{R}^3). \]
Since
\[ u_k \rightharpoonup u \quad \text{weakly in} \quad L^2(\mathbb{R}^3), \]
there exists \( C > 0 \) such that
\[ \|u_k\|_{L^2} \leq C, \quad \|u\|_{L^2} \leq C. \]
By \( (V_3) \) for all \( \varepsilon > 0 \) there exists \( R > 0 \) such that
\[ |x| \leq R \implies 0 \leq V(x) < \frac{\varepsilon}{C^2} \]
then
\[ \int_{\{|x| \geq R\}} V(x)u_k^2 \, dx < \varepsilon, \quad \int_{\{|x| \geq R\}} V(x)u^2 \, dx < \varepsilon. \quad (3) \]
By the Sobolev inequality clearly
\[ u_k^2 \rightharpoonup u^2 \text{ weakly in } L^3(\mathbb{R}^3), \]
and by \((V_1)\) and \((V_2)\) there results
\[ \int_{\{|x| \leq R\}} V(x)u_k^2\,dx \rightarrow \int_{\{|x| \leq R\}} V(x)u^2\,dx. \]
Then by the previous and (3) we can conclude
\[ \int_{\mathbb{R}^3} V(x)u_k^2\,dx \rightarrow \int_{\mathbb{R}^3} V(x)u^2\,dx. \]
So we are done. ■

**Remark 5** Observe that only for \(3 \leq n < 6\) we are able to prove that the functional
\[ u \in H^1_r(\mathbb{R}^n) \mapsto \int_{\mathbb{R}^n} |\nabla \Delta^{-1} u^2|^2\,dx \]
is weakly continuous by using the compact embedding results for radial solutions (cfr. [8, Theorem A.I'; 16]) and Lemma 2.

**Lemma 6** Let \(V\) satisfy \((V_1)\), \((V_2)\) and \((V_3)\) then for all \(\omega < 0\) the functional \(J_\omega\) is coercive in \(H^1_r(\mathbb{R}^3)\), i.e. for all sequence \(\{u_k\} \subset H^1_r(\mathbb{R}^3)\) such that \(\|u_k\|_{H^1} \rightarrow +\infty\) there results \(\lim_{k} J_\omega(u_k) = +\infty\).

**Proof.** Let \(\omega < 0\). Denote
\[ B' = \{ u \in H^1_r(\mathbb{R}^3) | \|u\|_{H^1} = 1 \}. \]
Let \(\{u_k\} \subset H^1_r(\mathbb{R}^3)\) such that
\[ \|u_k\|_{H^1} \rightarrow +\infty. \]
Denote
\[ u_k = \lambda_k \tilde{u}_k \]
with $\lambda_k \in \mathbb{R}$ and $\tilde{u}_k \in B'$. We have
\[
J_\omega(u_k) = a_k \lambda_k^2 + b_k \lambda_k^4 - c_k \lambda_k^2 + d_k \lambda_k^2
\]
with
\[
a_k = \frac{1}{4} \int_{\mathbb{R}^3} |\nabla \tilde{u}_k|^2 dx \in [0, \frac{1}{4}], \quad b_k = \pi \int_{\mathbb{R}^3} |\nabla \Delta^{-1} \tilde{u}_k|^2 dx \geq 0, \\
c_k = \frac{1}{2} \int_{\mathbb{R}^3} V(x) \tilde{u}_k^2 dx \geq 0, \quad d_k = -\frac{\omega}{2} \int_{\mathbb{R}^3} \tilde{u}_k^2 dx \in [0, -\frac{\omega}{2}].
\]
Observe that by Sobolev inequality $(V_1)$, $(V_2)$ and $(V_3)$ there results
\[
2c_k = \int_{\{|x| \leq 1\}} V(x) \tilde{u}_k^2 dx + \int_{\{|x| > 1\}} V(x) \tilde{u}_k^2 dx \leq \\
\leq \|V\|_{L^2([|x| \leq 1])} \|\tilde{u}_k\|_{L^6}^2 + \sup_{|x| \geq 1} V(x) \|\tilde{u}_k\|_{L^2}^2 \leq \\
\leq (C\|V\|_{L^2([|x| \leq 1])} + \sup_{|x| \geq 1} V(x)) \|\tilde{u}_k\|_{H^1} = (C\|V\|_{L^2([|x| \leq 1])} + \sup_{|x| \geq 1} V(x)),
\]
where $C > 0$ is the Sobolev embedding constant. Since $u \in H^1_{r}(\mathbb{R}^3) \mapsto \int_{\mathbb{R}^3} |\nabla \Delta^{-1} u|^2 dx$ is weakly continuous and $B'$ is bounded in $H^1_{r}(\mathbb{R}^3)$ there exists $\alpha > 0$ such that $b_k \geq \alpha > 0$. Then we can conclude that
\[
\lim_{k} J_\omega(u_k) = +\infty,
\]
and so we are done. \(\blacksquare\)

By a well known argument by the two previous lemma the following holds.

**Lemma 7** Let $V$ satisfy $(V_1)$, $(V_2)$ and $(V_3)$ then for all $\omega < 0$ the functional $J_\omega$ is bounded from below in $H^1_{r}(\mathbb{R}^3)$.

**Lemma 8** Let $V$ satisfy $(V_1)$, $(V_2)$ and $(V_3)$ then for all $\omega < 0$ the functional $J_\omega \big|_{H^1_{r}(\mathbb{R}^3)}$ satisfies the Palais-Smale condition, i.e. any sequence $\{u_k\} \subset H^1_{r}(\mathbb{R}^3)$ such that $\{J_\omega(u_k)\}$ is bounded and $J_\omega(u_k) \big|_{H^1_{r}(\mathbb{R}^3)} \to 0$ contains a covering subsequence.
Proof. Let $\omega < 0$ and $\{u_k\} \subset H^1_r(\mathbb{R}^3)$ such that $\{J_\omega(u_k)\}$ is bounded and $J'_\omega(u_k) \big|_{H^1_r(\mathbb{R}^3)} \to 0$. First of all observe that, by (iii) of Lemma 3, there results

$$J_\omega \big|_{H^1_r(\mathbb{R}^3)}(u) = 0 \iff J'_\omega(u) = 0,$$

then we can suppose

$$J'_\omega(u_k) \to 0.$$

By Lemma 6 $\{u_k\}$ is bounded in $H^1_r(\mathbb{R}^3)$, passing to a subsequence there exists $u \in H^1_r(\mathbb{R}^3)$ such that

$$u_k \rightharpoonup u \quad \text{weakly in } H^1_r(\mathbb{R}^3). \quad (4)$$

Clearly there results

$$J'_\omega(u) = 0. \quad (5)$$

We prove that

$$u_k \to u \quad \text{in } H^1_r(\mathbb{R}^3).$$

By Lemma 4 and (4) there results

$$\int_{\mathbb{R}^3} |\nabla u_k|^2 dx - 2\omega \int_{\mathbb{R}^3} u_k^2 dx =$$

$$= 2\langle J'_\omega(u_k), u_k \rangle - 8\pi \int_{\mathbb{R}^3} |\nabla \Delta^{-1} u_k|^2 dx + 2 \int_{\mathbb{R}^3} V(x) u_k^2 dx \to$$

$$\to -8\pi \int_{\mathbb{R}^3} |\nabla \Delta^{-1} u|^2 dx + 2 \int_{\mathbb{R}^3} V(x) u^2 dx =$$

$$= \int_{\mathbb{R}^3} |\nabla u|^2 dx - 2\omega \int_{\mathbb{R}^3} u^2 dx - 2\langle J'_\omega(u), u \rangle.$$

By (5) and since $\omega < 0$ the thesis is proved.\(\blacksquare\)

Remark 9 Since for all $\omega < 0$ the functional $J_\omega$ is bounded from below and satisfies the Palais-Smale condition there exists at least the critical level $\inf J_\omega$. The assumption $(V_4)$ helps us to obtain the multiplicity of the same ones.
Lemma 10 Let $V$ satisfy $(V_1), (V_2), (V_3)$ and $(V_4)$ then for all $k \in \mathbb{N}\{0\}$, there exist a subspace $V_k \subset H^1_r(\mathbb{R}^3)$ of dimension $k$ and $\nu > 0$ such that

$$\int_{\mathbb{R}^3} \left( \frac{1}{2} |\nabla u|^2 - V(x)u^2 \right) dx \leq -\nu,$$

for all $u \in V_k \cap B$, where

$$B = \left\{ u \in H^1_r(\mathbb{R}^3) \mid \int_{\mathbb{R}^3} |u|^2 dx = 1 \right\}.$$ 

Proof. Let $u$ a smooth map with compact support such that

$$\int_{\mathbb{R}^3} |u|^2 dx = 1, \quad \text{supp}(u) \subset B_2(0) \setminus B_1(0),$$

where

$$B_\rho(x) = \{ y \in \mathbb{R}^3 \mid |x - y| < \rho \}, \quad x \in \mathbb{R}^3, \quad \rho > 0.$$ 

Denote

$$u_\lambda(x) = \lambda^{\frac{3}{2}} u(\lambda x), \quad \lambda > 0, \quad x \in \mathbb{R}^3,$$

and

$$A_\lambda = B_{\frac{1}{\lambda}}(0) \setminus B_1(0), \quad \lambda > 0,$$

there results

$$\int_{\mathbb{R}^3} |u|^2 dx = \int_{\mathbb{R}^3} |u_\lambda|^2 dx = 1, \quad \text{supp}(u_\lambda) \subset A_\lambda.$$

By $(V_1)$ we have

$$\int_{\mathbb{R}^3} \left( \frac{1}{2} |\nabla u_\lambda|^2 - V(x)u_\lambda^2 \right) dx = \int_{\mathbb{R}^3} \left( \lambda^2 \frac{1}{2} |\nabla u|^2 - V(\frac{x}{\lambda})u^2 \right) dx \leq$$

$$\leq \lambda^2 \int_{\mathbb{R}^3} \frac{1}{2} |\nabla u|^2 dx - \inf_{\lambda \in \text{supp} u} V \leq \lambda^2 \int_{\mathbb{R}^3} \frac{1}{2} |\nabla u|^2 dx - \inf_{A_\lambda} V =$$

$$= \lambda^2 \int_{\mathbb{R}^3} \frac{1}{2} |\nabla u|^2 dx - V(x_\lambda),$$
where $x_\lambda$ belongs to the closure of $A_\lambda$ and $V(x_\lambda) = \inf_{A_\lambda}$. By $(V_3)$ and $(V_4)$ there exists $\lambda_0 > 0$ such that

$$\int_{\mathbb{R}^3} \left( \frac{1}{2} |\nabla u_{\lambda_0}|^2 - V(x)u_{\lambda_0}^2 \right) dx < 0.$$

Let $k \in \mathbb{N}\setminus\{0\}$ and $u_1, u_2, \ldots, u_k$ smooth maps with compact supports such that

$$\int_{\mathbb{R}^3} |u_i|^2 dx = 1, \quad \text{supp}(u_i) \subset B_{2i}(0) \setminus B_i(0), \quad i = 1, 2, \ldots, k.$$

Using an analogous argument we are able to find $\lambda_1, \lambda_2, \ldots, \lambda_k > 0$ such that

$$\int_{\mathbb{R}^3} \left( \frac{1}{2} |\nabla u_{\lambda_i}|^2 - V(x)u_{\lambda_i}^2 \right) dx < 0, \quad i = 1, 2, \ldots, k.$$

Let

$$0 < \bar{\lambda} < \min\{\lambda_1, \lambda_2, \ldots, \lambda_k\}$$

and $V_k$ the subspace spanned by $u_{1\lambda}, u_{2\lambda}, \ldots, u_{k\lambda}$. Since the supports of this maps are pairwise disjoint $V_k$ has dimension $k$. Since for all $i = 1, 2, \ldots, k$ and $\lambda \leq \lambda_i$ there results

$$\int_{\mathbb{R}^3} \left( \frac{1}{2} |\nabla u_{i\lambda}|^2 - V(x)u_{i\lambda}^2 \right) dx < 0$$

and $V_k \cap B$ is compact, the thesis is proved.

**Lemma 11** Let $V$ satisfy $(V_1)$, $(V_2)$, $(V_3)$ and $(V_4)$ then for all $\omega < 0$ the functional $J_\omega$ has infinitely many critical points $\{u_k\}_{k \in \mathbb{N}} \subset H^1_r(\mathbb{R}^3)$ such that $J_\omega(u_k) < -\frac{\omega}{2}$.

**Proof.** Let $\omega < 0$ and denote

$$c_\omega^k = \inf\{\sup_{A_\omega}(A) | A \in \mathcal{A}, \gamma(A) \geq k\}, \quad k \in \mathbb{N}\setminus\{0\},$$

with

$$\mathcal{A} = \{A \subset H^1_r(\mathbb{R}^3) | A \text{ closed, symmetric and } 0 \notin A\}$$

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and $\gamma$ is the Genus (cfr. e. g. [1, Definition 1.1]). We have to prove that $c_k^\omega < -\frac{\omega}{2}$ for all $k \in \mathbb{N}$. Let $k \in \mathbb{N}\setminus\{0\}$, by the previous lemma there exist $V_k \subset H^1_r(\mathbb{R}^3)$ subspace of dimension $k$ and $\nu > 0$ such that for all $u \in V_k \cap B$ there results

$$\int_{\mathbb{R}^3} \left( \frac{1}{2} |\nabla u|^2 - V(x) u^2 \right) dx \leq -\nu.$$  

Let $\lambda > 0$ and define

$$h_\lambda : V_k \cap B \longrightarrow H^1_r(\mathbb{R}^3), \quad h_\lambda(u) = \lambda^{\frac{1}{4}} u.$$  

Fixed $u \in V_k \cap B$ and $0 < \lambda < 1$ there results

$$J_\omega(h_\lambda(u)) \leq -\frac{\lambda}{2} \nu + c\lambda^2 - \frac{\omega}{2} \lambda \leq -\frac{\lambda}{2} \nu + c\lambda^2 - \frac{\omega}{2}, \quad (6)$$  

where $c$ is a positive constant. Then there exists $0 < \bar{\lambda} < 1$ such that for all $u \in V_k \cap B$ there results $J_\omega(h_{\bar{\lambda}}(u)) < -\frac{\omega}{2}$. Since $h_{\bar{\lambda}}$ is continuous, odd and $0 \notin V_k \cap B$ we have

$$h_{\bar{\lambda}}(V_k \cap B) \in A. \quad (7)$$  

Since $V_k \cap B$ is compact, (6) and (7) we have

$$\inf J_\omega \leq c_k^\omega \leq \sup J_\omega(h_{\bar{\lambda}}(V_k \cap B)) < -\frac{\omega}{2}.$$  

By Lemma 8 (cfr. [15, Theorem 9.1; 4]) there exists $\{u_k\} \subset B$ sequence of critical points of $J_\omega$ such that $J_\omega(u_k) = c_k^\omega < -\frac{\omega}{2}$. So we are done. 

**Proof of Theorem 1** Since

$$F_\omega(u, 4\pi \Delta^{-1} u^2) = J_\omega(u)$$  

for all $\omega \in \mathbb{R}$ and $u \in H^1(\mathbb{R}^3)$, by Lemma 3 and the previous one the thesis is done. 

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