Iterative Approach to Gravitational Lensing Theory

Thomas P. Kling, Ezra T. Newman, and Alejandro Perez

Dept. of Physics and Astronomy,
University of Pittsburgh, Pgh, PA, 15260, USA

(March 24, 2022)

Abstract

We develop an iterative approach to gravitational lensing theory based on approximate solutions of the null geodesic equations. The approach can be employed in any space-time which is “close” to a space-time in which the null geodesic equations can be completely integrated, such as Minkowski space-time, Robertson-Walker cosmologies, or Schwarzschild-Kerr geometries. To illustrate the method, we construct the iterative gravitational lens equations and time of arrival equation for a single Schwarzschild lens. This example motivates a discussion of the relationship between the iterative approach, the standard thin lens formulation, and an exact formulation of gravitational lensing.

1. INTRODUCTION

In several recent papers [1,2], an exact approach to gravitational lensing has been developed in which a parametric representation of the past light-cone of an observer is viewed as the fundamental gravitational lens equations. In this approach, one would solve, in principle, the null geodesic equations of general relativity in an arbitrary space-time and construct the past light-cone of an observer. The lens and time of arrival equations then follow from a particular parametric representation of the past light cone. The exact approach
can be implemented in only a few space-times possessing a high degree of symmetry, such as Minkowski space, Robertson-Walker cosmologies, or Schwarzschild/Kerr geometries. We will refer to space-times in which the null geodesic equations can be integrated exactly as integrable geometries.

The exact approach stands in contrast to the generally accepted thin lens approximation in which one does not attempt to find the exact null geodesics. In this approximation, one introduces sharp bending at isolated points near the lens along a geodesic trajectory of a background space-time – usually either Minkowski space or a cosmological model. The thin lens approximation has proved extremely valuable to practicing astrophysicists as a relatively accurate method which is easy to implement [3].

Our goal in this paper is to propose a middle ground in which one systematically finds approximate null geodesics in geometries “close” to integrable geometries and uses these approximate geodesics as the basis for gravitational lensing calculations. These geometries need not correspond to any solution of the Einstein equations; our method should apply to any approximate metric which can be considered as a perturbation of an exact integrable geometry. A key component of this project is a perturbation method known as the “variation of constants,” which can be found in several classical mechanics textbooks [4,5]. This method is based on techniques from Hamiltonian mechanics and is described in Section II. Though it is often used for perturbations of periodic or quasi-periodic orbits, it is suited to finding approximations to arbitrary trajectories, regardless of their type.

In Section III, the method is specialized to the problem of finding approximate null geodesics in space-times close to integrable ones, and turning these geodesics into lens equations. As an example of the method, we apply the iterative approach to a single Schwarzschild lens in Section IV. In a final section, we discuss the relationship between the iterative, thin lens, and exact formulations of lensing theory.
II. FINDING APPROXIMATE TRAJECTORIES

In this section, we describe a method to obtain approximate trajectories for a perturbed Hamiltonian. The discussion largely follows a similar discussion in Goldstein’s Classical Mechanics [4].

Throughout this paper, we refer to an unperturbed Hamiltonian, $H_o$, and a perturbed Hamiltonian,

$$H(x^a, p_a) = H_o(x^a, p_a) + \Delta H(x^a, p_a).$$

(1)

In addition, we will make no distinction between the time coordinate, $t$, and the spatial coordinates, $x^i$. The base space coordinates, $x^a$, are thus $x^a = (t, x^i)$, and the momentum coordinates, $p_a$, include $p_0$, which is canonically conjugate to $t$. A parameter, $\lambda$, will be an affine parameterization of the trajectory, and is a linear function of $t$ if the Hamiltonian does not depend on time. A dot derivative will always refer to a total derivative with respect to $\lambda$, while partial $\lambda$ derivatives will always be explicitly written.

The method begins by assuming that one can find a complete solution, $F_o(x^a, P_a, \lambda)$, to the Hamilton-Jacobi equation for the unperturbed Hamiltonian,

$$H_o(x^a, \frac{\partial F_o}{\partial x^a}) + \frac{\partial F_o}{\partial \lambda} = 0,$$

(2)

with $n$ constants of integration, $P_a$. When

$$\det \frac{\partial^2 F_o}{\partial x^a \partial P_b} \neq 0,$$

(3)

$F_o(x^a, P_a, \lambda)$ is the generator of a (parameter dependent) canonical transformation, $(x^a, p_a) \Rightarrow (X^a, P_a)$. The transformation is defined by

$$X^a = \frac{\partial F_o}{\partial P_a} \equiv X^a(x^a, P_a, \lambda)$$

(4)

$$p_a = \frac{\partial F_o}{\partial x^a} \equiv p_a(x^a, P_a, \lambda),$$

(5)

with the inversion of Eq.(4) being
\[ x^a = \chi^a(X^a, P_a, \lambda). \] (6)

When the inversion is substituted into Eq.(5), one has the canonical coordinate transformation in the form

\[ x^a = \chi^a(X^a, P_a, \lambda) \]
\[ p_a = p_a(x^a(X^a, P_a, \lambda), P_a, \lambda) = \pi_a(X^a, P_a, \lambda). \] (7)

It is important to note that Eq.(7) represents the solution to the equations of motion of the unperturbed Hamiltonian in terms of \(2n\) constants, \((X^a, P_a)\), which represent the initial location and momentum, and the affine parameter, \(\lambda\).

Since the property of being a canonical transformation is independent of the particular form of the Hamiltonian, we may apply the canonical transformation generated by the solution to the unperturbed Hamilton-Jacobi equation to the perturbed Hamiltonian. Under any canonical transformation generated by the complete solution, \(F_o(x^a, P_a, \lambda)\), the Hamiltonian becomes

\[ H'(X^a, P_a, \lambda) = H(x^a(X^a, P_a, \lambda), p_a(X^a, P_a, \lambda)) + \frac{\partial F_o}{\partial \lambda}. \] (8)

Using \(H = H_o + \Delta H\), we have

\[ H'(X^a, P_a, \lambda) = \Delta H(x^a(X^a, P_a, \lambda), p_a(X^a, P_a, \lambda)), \] (9)

because \(F_o(x^a, P_a, \lambda)\) was chosen to satisfy the Hamilton-Jacobi equation of the unperturbed Hamiltonian.

Hamilton’s equations of motion for \(X^a\) and \(P_a\) are

\[ \dot{X}^a(X^a, P_a, \lambda) = \frac{\partial H'}{\partial P_a}(X^a, P_a, \lambda) \]
\[ \dot{P}_a(X^a, P_a, \lambda) = -\frac{\partial H'}{\partial X^a}(X^a, P_a, \lambda). \] (10)

These equations are exact. We note that if the perturbation, \(\Delta H\), was zero, the solution to Eqs.(10) would be that \(X^a\) and \(P_a\) are constants. When \(\Delta H\) is not zero, these “constants”
become functions of initial conditions, \((X^a_o, P^o_a)\), and the parameter, \(\lambda\). For this reason, this method is referred to as the “variation of constants.”

In principle, the solution of the equations of motion, Eqs. (10), would be \(2n\) functions of the form

\[
X^a = X^a(X^a_o, P^o_a, \lambda) \\
P_a = P_a(X^a_o, P^o_a, \lambda),
\]

(11)

where \(X^a_o\) and \(P^o_a\) are \(2n\) initial values. Substituting Eqs. (11) into the canonical transformation, Eqs. (7), yields the solution to Hamilton’s equations for \(x^a\) and \(p_a\) as functions of initial conditions and \(\lambda\).

Note that we have made no assumption about the value of the unperturbed Hamiltonian. As an example, suppose that we wish to solve a Kepler type problem, where \(H_o\) represents a spherically symmetric Hamiltonian and \(\Delta H\) represents a perturbation. Since the value of \(H_o\) has not been determined, this method applies to perturbations of bounded or unbounded orbits.

Since the equations for \(X^a\) and \(P_a\), Eqs. (10), are exact, they tend to be as difficult to solve as Hamilton’s equations in the original phase space variables, \((x^a, p_a)\). Typically, the variation of constants method is not used to find exact solutions, but rather it leads to an approximation procedure.

The natural approximation procedure associated with the variation of constants method is to solve Hamilton’s equations, Eqs. (10), by successive iteration. To iterate, one must choose an appropriate initial trajectory, \((X^a_o(\lambda), P^o_a(\lambda))\), as the “zeroth iterate” and insert these functions of \(\lambda\) into the right hand side of Eq. (10). In this way, the right hand sides of Eqs. (10) become functions of the parameter \(\lambda\). The first iterate approximation is then obtained by simple integration on \(\lambda\). We want to emphasize that any trajectory, suitably close to the exact result, can be used as the zeroth iterate. In practice, general knowledge of the physical situation will guide the choice of the zeroth iterate.

One natural choice is to take the zeroth iterate as given by \(X^a\) and \(P_a\) as constants,
\( X^a = X_o^a \) and \( P_a = P_o^a \), which is the result obtained when \( \Delta H = 0 \). The first iterate then is given by

\[
X_1^a(X_o^a, P_o^a, \lambda) = X_o^a + \int_0^\lambda d\lambda' \left( \frac{\partial H'}{\partial P_a}(X_o^a, P_o^a, \lambda') \right)
\]

\[
P_1^a(X_o^a, P_o^a, \lambda) = P_o^a - \int_0^\lambda d\lambda' \left( \frac{\partial H'}{\partial X_a}(X_o^a, P_o^a, \lambda') \right).
\]

(12)

As we will discuss in Section IV, this choice is not the most appropriate initial trajectory for the application to lensing. Instead, we will take a sequence of constants, \((X_o^a, P_o^a)\), for different ranges in \( \lambda \) as the zeroth iterate. These constants will be selected from values predicted by the thin lens approximation.

The \( n \)th iterate is given in the same way: \( X^a \) and \( P_a \) are replaced by \( X^a_{n-1}(X_o^a, P_o^a, \lambda') \) and \( P_a^{n-1}(X_o^a, P_o^a, \lambda') \) on the right hand side of Eqs.(10) after the derivative is taken. In this way, the right hand sides of Eqs.(10) are always functions of \( \lambda' \) and initial values, \((X_o^a, P_o^a)\), and can be integrated immediately. The \( n \)th iterate approximate solution to Hamilton’s equations in the original phase space variables is obtained by substituting \( X^a_n \) and \( P_a^n \) into the canonical transformation, Eqs.(7).

In the limit that \( n \to \infty \), the iterative solution, \((X_n^a, P_n^a)\), always converges to the exact solution for small \( \lambda \) values if the initial trajectory is chosen as constant values. However, even in these converging cases, at any finite \( n \), the iterative solution may differ greatly from the exact solution at large \( \lambda \) values. For example, Goldstein [4] considers a harmonic oscillator perturbation,

\[
\Delta H = \frac{1}{2} m\omega^2 x^2,
\]

on a free Hamiltonian in one dimension,

\[
H_o = \frac{p^2}{2m}.
\]

Using particular initial conditions, he finds that the 2nd iterate solution in the original variables given by
\[ x_2 = \frac{p_0}{m\omega} \left( \omega t - \frac{\omega^3 t^3}{3!} + \frac{\omega^5 t^5}{5!} \right), \]
\[ p_2 = p_0 \left( 1 - \frac{\omega^2 t^2}{2!} + \frac{\omega^4 t^4}{4!} \right), \]

As \( n \to \infty \), the limit converges to
\[ x \to \frac{p_0}{m\omega} \sin \omega t \quad p \to p_0 \cos \omega t. \]

However, at any finite \( n \), the solution runs away to infinity, quickly separating from the exact result. This is because the finite \( n \) iterative solution is simply the first few terms in a Taylor series expansion of the exact result.

To develop a method which gives reasonable results at finite \( n \), one may attempt to adjust the standard iterative method using general knowledge of the features of the exact solution. Such corrections are required in the application of the variation of constants method to gravitational lensing when accurate results at low \( n \) are desired. These corrections, which will be discussed in Section IV.C, involve the consecutive use of a series of unperturbed orbits.

**III. APPLICATION TO GRAVITATIONAL LENSING**

The iterative approach to gravitational lensing will be an approximation to the exact approach and will remain within its overall perspective and framework [1,2]. In Section III.A, we outline the key elements of the exact approach, while the iterative approach will be developed in Section III.B. We show how to turn null geodesics into lens equations in Section III.C. For clarity, we leave any corrections of the iterative method at low \( n \) to the discussion in the final subsection of Section IV.

**A. Exact Gravitational Lensing**

In the exact approach, one considers a space-time, \((M, g^{ab})\), containing various lenses (travelling on specified world lines) whose properties are hidden in the metric \( g^{ab} \), and an
observer moving on a world line described by

\[ x^a = X^a_0(\tau). \]  

(13)

The past light-cone of this observer is generated by null geodesics originating at \( X^a_0(\tau) \), and may be described parametrically as

\[ x^a = X^a(\tau, \theta, \phi, \lambda) \]

or

\[ t = T(\tau, \theta, \phi, \lambda) \]  

(14)

\[ x^i = \mathcal{X}^i(\tau, \theta, \phi, \lambda) \quad i = 1, 2, 3. \]  

(15)

In these equations, the proper time, \( \tau \), labels the points along the world line of the observer, and \( \lambda \) represents an affine parameter along the null geodesics of the observer’s past light-cone. The angles, \( (\theta, \phi) \), represent the sphere of null directions seen by the observer, or the observer’s celestial sphere.

The time equation, Eq.(14), describes the coordinate emission time of a light ray emitted by a source at \( x^i \) arriving at the observer at time \( \tau \). This equation, often referred to as the “time of arrival equation,” is important in astrophysics, among other reasons, because it can be used, in conjunction with observations and a lens model, to place bounds on the Hubble constant, see for example [6,7].

The three space-like equations, Eqs.(15), can be considered as a generalized, exact form of the gravitational lens equations, because they map observed directions into source positions. By giving a relationship between the affine parameter and some observable distance scale – possibly the redshift distance, \( R \) – one can express the generalized, exact lens equations as

\[ x^i = \mathcal{X}^i(\tau, \theta, \phi, R) \quad i = 1, 2, 3, \]  

(16)

which is similar the traditional form of lens equations, where the \( (\tau, \theta, \phi, R) \) are all observable quantities. In [1,2], Eq.(16) is referred to as the exact lens equation. This step is
necessary for comparison with astrophysical observations because the affine parameter has no measurable meaning, but will not be required in our discussion.

As mentioned, the general perspective taken in exact gravitational lensing is that the lens properties are built into the form of the metric, $g^{ab}$. This means that there can be no reference to any background space-time or any quantity “in the absence of the lens.” To find the explicit form of the lens equations and time of arrival equation, one must solve the null geodesic equations associated with $g^{ab}$, giving initial conditions for past directed rays on the observer’s world line.

Therefore, we begin with the full geodesic equations: four second order coupled differential equations for the four base space coordinates, $x^a$. These equations can alternatively be viewed as a set of eight coupled first order differential equations on the cotangent bundle, $(x^a, p_a)$, which are derived from a Hamiltonian. To make them into the null geodesic equations, initial conditions must be chosen so that the Hamiltonian vanishes.

The Hamiltonian for geodesic orbits is

$$H = g^{ab}(x^a)p_a p_b, \tag{17}$$

with associated Hamilton’s equations:

$$\dot{x}^a = 2g^{ab}p_b$$

$$\dot{p}_a = -g^{bc}_{\ ;a} p_b p_c. \tag{18}$$

Solutions to Hamilton’s equations have the form

$$x^a = \tilde{x}^a(x^a_0, p^a_0, \lambda)$$

$$p_a = \tilde{p}_a(x^a_0, p^a_0, \lambda), \tag{19}$$

where $x^a_0$ is an initial position representing the location of the observer and $p^a_0$ gives the initial direction of the geodesic. The coordinates, $x^a$, represent the space-time position along the geodesic, and $p_a$ is the cotangent vector to this geodesic.
Null geodesics correspond to the geodesics whose value of the Hamiltonian is zero. Since the Hamiltonian does not depend on the parameter $\lambda$, its value will be preserved along the geodesics, and can be set to zero by constraining the initial conditions:

$$H(x^a_o, p^a_o) = g^{ab}(x^a_o)p^a_o p^b_o = 0.$$  \hspace{1cm} (20)

Equation (20) is quadratic in $p^a_o$ and can always be solved for $p^a_o = p^a_o(x^a_o, p^b_i)$. Substituting this into Eqs.(19) gives

$$x^a = \tilde{x}^a(x^a_o, p^a_o(x^a_o, p^b_i), p^b_i, \lambda) = x^a(x^a_o, p^b_i, \lambda),$$  \hspace{1cm} (21)

which are null geodesics.

**B. Iterative Method for Null Geodesics**

In general, it is impossible to solve Hamilton’s equations, Eqs.(18), in closed form. However, if one is able to consider the metric of physical interest, $g^{ab}$, to be close to some metric, $g^{ab}_o$, for which one can solve Hamilton’s equations, then the method of variation of constants gives a way to systematically approximate the null geodesics.

Formally, we suppose that we have two space-times, $(M, g^{ab})$ and $(M, g^{ab}_o)$ where

$$g^{ab} = g^{ab}_o + h^{ab},$$  \hspace{1cm} (22)

and $h^{ab}$ is “small.” For the sake of discussion, we will suppose that $(M, g^{ab}_o)$ represents either Minkowski space-time or a Robertson-Walker geometry, since these are the usual background space-times in gravitational lensing. We form the associated Hamiltonians in both space-times:

$$H_o(x^a, p_a) = g^{ab}_o (x^a)p_a p_b$$  \hspace{1cm} (23)

$$H(x^a, p_a) = g^{ab}(x^a)p_a p_b = g^{ab}_o (x^a)p_a p_b + h^{ab}(x^a)p_a p_b.$$  \hspace{1cm} (24)

By finding a complete solution to the Hamilton-Jacobi equation in $(M, g^{ab}_o)$,
\[ g^{ab}_{o} \frac{\partial F_{o}}{\partial x^{a}} \frac{\partial F_{o}}{\partial x^{b}} + \frac{\partial F_{o}}{\partial \lambda} = 0, \]  

(25)

we obtain a generating function of canonical transformations, \( F_{o}(x^{a}, P_{a}, \lambda) \). Because Minkowski space-time and the Robertson-Walker geometries are integrable geometries, we will always be able to solve Eq. (25) and find an explicit canonical transformation in the form of Eqs. (7):

\[ x^{a} = \tilde{x}^{a}(X^{a}, P_{a}, \lambda) \]

(26)

\[ p_{a} = \tilde{p}_{a}(X^{a}, P_{a}, \lambda). \]

(27)

Using the explicit form of the canonical transformation, Eqs. (26, 27), we transform the Hamiltonian in the perturbed space, Eq. (24), obtaining

\[ H'(X^{a}, P_{a}, \lambda) = H(X^{a}, P_{a}, \lambda) + \frac{\partial F_{o}}{\partial \lambda} \]

\[ = \left( h^{ab}(\tilde{x}^{a}(X^{a}, P_{a}, \lambda)) \right) \tilde{p}_{a}(X^{a}, P_{a}, \lambda) \tilde{p}_{b}(X^{a}, P_{a}, \lambda). \]

(28)

Hamilton’s equations of motion for \( X^{a} \) and \( P_{a} \) are

\[ \dot{X}^{a}(X^{a}, P_{a}, \lambda) = 2 \left( h^{bc}(\tilde{x}^{a}(X^{a}, P_{a}, \lambda)) \right) \tilde{p}_{b}(X^{a}, P_{a}, \lambda) \left( \frac{\partial \tilde{p}_{c}}{\partial P_{a}}(X^{a}, P_{a}, \lambda) \right) \]

\[ + \left( \frac{\partial h^{bc}}{\partial P_{a}}(\tilde{x}^{a}(X^{a}, P_{a}, \lambda)) \right) \tilde{p}_{b}(X^{a}, P_{a}, \lambda) \tilde{p}_{c}(X^{a}, P_{a}, \lambda) \]

\[ \equiv \Xi^{a}(X^{a}, P_{a}, \lambda) \]

(29)

\[ \dot{P}_{a}(X^{a}, P_{a}, \lambda) = - \left( \frac{\partial h^{bc}}{\partial X^{a}}(\tilde{x}^{a}(X^{a}, P_{a}, \lambda)) \right) \tilde{p}_{b}(X^{a}, P_{a}, \lambda) \tilde{p}_{c}(X^{a}, P_{a}, \lambda) \]

\[ (-2h^{bc}(\tilde{x}^{a}(X^{a}, P_{a}, \lambda))) \tilde{p}_{b}(X^{a}, P_{a}, \lambda) \left( \frac{\partial \tilde{p}_{c}}{\partial X^{a}}(X^{a}, P_{a}, \lambda) \right) \]

\[ \equiv \Upsilon_{a}(X^{a}, P_{a}, \lambda), \]

(30)

We now use the iterative procedure of Section II to solve these equations. For the moment, we will choose the zeroth iterate as \( X^{a} \) and \( P_{a} \) taking constant values, \( (X_{o}^{a}, P_{o}^{a}) \), under the integral. The \( n \)th iterate will be given by
\[ X_n^a(X_o^a, P_o^a, \lambda) = X_o^a + \int_0^\lambda d\lambda' \Xi^a(X_{n-1}^a, P_{a}^{n-1}, \lambda') \]  

where \( X_{n-1}^a \) and \( P_{a}^{n-1} \) are functions of initial conditions and \( \lambda' \).

When the \( n \)th iterate is substituted into Eq.(26), the result is an approximate solution to the geodesic equations of \((M, g^{ab})\) with initial conditions \((X_o^a, P_o^a)\):

\[ x^a = \tilde{x}^a(X_n^a(X_o^a, P_o^a, \lambda), P_a^n(X_o^a, P_o^a, \lambda), \lambda). \]  

These approximate geodesics may not be approximately null. In analogy to the exact case, we should require that

\[ H(X_o^a, P_o^a) = 0, \]  

which we can solve for \( P_0^a \) as a function of the other initial conditions,

\[ P_0 = P_0^a(X_o^a, P_i^a). \]  

Substituting \( P_0^a(X_o^a, P_i^a) \) into Eq.(33) gives approximate null geodesics:

\[ x^a = \tilde{x}^a(X_n^a(X_o^a, P_o^a, \lambda), P_a^n(X_o^a, P_o^a, \lambda), \lambda) = x^a(X_o^a, P_i^a, \lambda). \]  

It is important to realize that, while the iteration method introduces a “background space-time,” \((M, g_o^{ab})\), the null geodesics one seeks are the null geodesics of the physical space-time, \((M, g^{ab})\). The spirit of exact gravitational lensing is that no reference to the background should be made. In the iterative approach, a background is introduced only to facilitate finding the approximate geodesics of \((M, g^{ab})\) and plays no further role. All physical quantities are computed in the physical space-time using \( g^{ab} \), not \( g_o^{ab} \); the null condition is fixed by solving Eq.(34) for \( P_0^a \), in which the physical metric appears. Likewise, angles are to be computed from the inner product of two vectors which lie in the tangent space of \((M, g^{ab})\). Hence, the iterative approach remains within the spirit of the exact approach because it does not refer in any physical way to a background space-time.
The \( n \)th iterate, approximate null geodesics can thus be used as the basis for an approximate gravitational lensing theory under the spirit of the exact approach. For these purposes, the \( X_o^a \) in Eq. (36) must be chosen as the space-time position representing the location of an observer, and the \( P_i^a \) are used to parametrize the directions which an observer can see. Sources are then located at \( x^a \), along the past directed null geodesics described by Eq. (36). In the next subsection, we describe the procedure for turning (either an exact or an approximate) solution to the null geodesic equations into the form of a time of arrival equation and lens equations given by Eqs. (14, 15).

C. Exact and Approximate Lens Equations

In this subsection, we describe how either an exact or an approximate solution to the null geodesic equations in an arbitrary space-time gives rise to gravitational lens equations. The null geodesics, exactly or approximately, are described by

\[
\begin{align*}
x^a &= x^a(x_o^a, p_o^i, \lambda) \quad (37) \\
p_a &= p_a(x_o^a, p_o^i, \lambda), \quad (38)
\end{align*}
\]

where the \( x_o^a \) is an initial space-time location, the three \( p_o^i \) describe the spatial direction of the null geodesic at that point, and \( \lambda \) is an affine parameter along the null geodesic, where \( \lambda = 0 \) corresponds to the initial position. The null tangent vectors to these geodesics, \( p^a = g^{ab}p_b = \dot{x}^a \), are taken as past directed. The scaling of \( \lambda \) has not yet been chosen.

To turn Eqs. (37) into a time of arrival equation and lens equations in the form of Eqs. (14, 15), we introduce an observer by taking a time-like curve parametrized by the proper time, \( \tau \),

\[
\begin{align*}
x^a &= X_o^a(\tau), \quad (39)
\end{align*}
\]

as a one parameter family of initial positions in Eqs. (37), or \( x_o^a = X_o^a(\tau) \), for the past directed null geodesics.
The tangent vector to this world line,

\[ v^a = \frac{dX^a_0(\tau)}{d\tau} \]

can be used to scale the \( p^a_i \) (and consequently \( \lambda \)). Because the magnitude of the \( p^a_i \) is irrelevant, we can fix its length by requiring that

\[ v^a p^0_a = -1. \]  (40)

This implies that, at any point along the observer’s world-line, the three \( p^a_0 \) are functions of only two parameters. These two parameters can be taken as two independent “observation angles,” \((\theta, \phi)\). Thus, the null cotangent vector at \( X^a_0(\tau) \) can be expressed as

\[ p^a_0 = p^0_0(\theta, \phi). \]  (41)

By replacing \( x^a_0 \) with \( X^a_0(\tau) \) and \( p^i_0 \) with Eq.(41), in Eq.(37), one obtains a time of arrival equation and lens equations in the form of Eqs.(14, 15):

\[ t = x^0(X^a_0(\tau), p^0_0(\theta, \phi), \lambda) \equiv T(\tau, \theta, \phi, \lambda) \]

\[ x^i = x^i(X^a_0(\tau), p^0_0(\theta, \phi), \lambda) \equiv X^i(\tau, \theta, \phi, \lambda) \quad i = 1, 2, 3. \]  (42)

As an example, we consider an observer moving along a world line given by \( X^a_0(\tau) = (\tau, 0, 0, +z_o) \) in a linearized Schwarzschild space-time. From Eq.(10), we see that \( p^0_0(x^a_0, p^0_i) = -1 \) for a past directed null geodesic. This normalization can be obtained in the previous section by dividing all components of \( P_a \) by the constant value of \( P_0 \) given in Eq.(33).

Since \( p^a_0 \) is null, with \( p^0_0 = -1 \), its spatial part will have the length:

\[ |p^a_i| = \sqrt{-g^{ij} p^0_i p^0_j} = \sqrt{1 + \frac{2m}{|z_o|}}. \]  (43)

Thus, we can write the spatial part of the null covector at the observer as
\[ p_i^o = \sqrt{1 + \frac{2m}{|z_o|} u_i(\theta, \phi)}, \]

where \( u_i(\theta, \phi) \) is a unit vector parametrized by two observation angles, \((\theta, \phi)\).

From the spherical symmetry of Schwarzschild space-time, the lens, source, and observer can be taken as lying in the same spatial plane, and any ray with an initial direction in this plane will remain in the plane. Fixing this plane as the \( \hat{x}-\hat{z} \) plane fixes one of the two observation angles, say \( \phi = 0 \). An observer on the \( +\hat{z} \) axis will observe the lens located at the origin in the direction of the unit spatial vector, \( n^i = -\frac{x^i_o}{\sqrt{-g_{ij} x^i_o x^j_o}} = -\delta^{i3} \). In terms of \( u_i \) and \( n^i \), the remaining observation angle, \( \theta \), is defined by

\[ \cos \theta = u_i n^i. \]  \hspace{1cm} (44)

Since \( n^i \) is proportional to \( \delta^{i3} \), we have

\[ u_z = -\sqrt{1 - \frac{2m}{|z_o|}} \cos \theta. \]

Using \( \sqrt{-g^{ij}} u_i u_j = |u| = 1, u_y = 0 \), and the linearized metric given in Section IV.B, one finds the final component on the momentum,

\[ u_x = \sqrt{1 - \left(1 - \frac{2m}{|z_o|}\right)^2 \cos^2 \theta}. \]

We note that the observation angle is determined relative to the physical metric of the space-time, \( g^{ab} \). This is consistent with the notion that the iterative and exact methods should not make reference to a background space-time.

**IV. THE SCHWARZSCHILD LENS**

To demonstrate the iterative method, we apply this procedure to the case of the Schwarzschild lens, considering the Schwarzschild metric as a perturbation of Minkowski
space-time. Throughout this section we will work in \((t, x, y, z)\) coordinates which, while awkward for the Schwarzschild metric, facilitate the iterative procedure and comparisons with the thin lens approach. In the iterative calculation, we work with the linearized version of the Schwarzschild metric, since there are no significant differences for the large impact parameters usually discussed.

The section will be divided into three subsections. First, we outline the thin lens approximation and its application to the Schwarzschild lens. The second subsection develops the application of the iterative method to the Schwarzschild lens. Finally, we discuss the choice of the zeroth iterate when one wishes to use low \(n\) values. This subsection makes use of results from both of the first two subsections.

**A. The Thin Lens Approximation**

Here, we briefly outline the standard thin lens approximation. Most theoretical details about the thin lens approximation can be found in the book by Falco, Schneider and Ehlers.

The key assumption in the thin lens approximation is that the null geodesics in the space-time of interest travel along the null geodesic path of the background space except at isolated points near the lens where sharp bending occurs. For a Schwarzschild lens in a Minkowski background, the geodesic travels in rectilinear motion from the observer, \(O\), to a point in the lens plane, \(I\), and then from \(I\) to the source location at \(O\), as is shown in Fig. 1.

The effect of the lens is twofold. First, the trajectory is instantaneously bent at \(I\) by a bending angle computed as the angle between the two asymptotic paths of the true null geodesic. For a Schwarzschild lens, the bending angle is simply \(\alpha = 4Gm/(c^2r_o)\), where \(r_o\) is the radius of closest approach, \(m\) is the mass, and the physical factors \(G\), and \(c\) have been restored. Second, the lens affects the time of arrival by introducing a gravitational time delay relative to the time required to traverse the same path in the Minkowski background.

The lens equation (of the thin lens approximation) is the map from the space of observed
light-ray directions to the source positions (on the source plane). Often the map is not invertible, as there could be multiple images for a given point source. In practice, one develops a model for the lens, usually in terms of a potential for the mass distribution, and from this model determines the amount of bending a null geodesic will encounter along its trajectory from past null infinity to future null infinity. This bending angle, $\alpha$, in Fig. 4, is amount of bending of the ray at the lens plane. An angle $\beta$ locates the true source location in the background space-time.

For a single Schwarzschild lens, one determines from Euclidean geometry that the lens equation is

$$\beta = \theta - \frac{4GmD_{ls}}{c^2 D_s D_l} \theta,$$  \hspace{1cm} (45)

where $\theta$ is the image observation angle, and $D_l$, $D_s$, and $D_{ls}$ represent the distances between the lens and observer, the source and observer, and the lens and source, respectively. Generically, there will be two values of $\theta$ for a given value of $\beta$, so that there will be two distinct paths joining the source and observer, with one passing on each side of the lens.

In the thin lens approximation, the time of arrival is determined by the coordinate time which elapses along the path taken in the thin lens approximation. Although the path, up to a sharp bending in the lens plane, is determined by the background space-time, the time of arrival is computed relative to the (usually linearized) physical metric including the lens. The time of arrival for a ray emitted from the source at time $t_s$ and passing close to a Schwarzschild lens is [3]

$$t = t_s + \frac{1}{c} \int \left(1 + \frac{2Gm}{c^2 r(l)}\right) dl,$$ \hspace{1cm} (46)

where $r(l)$ represents the Euclidean distance from the lens to a point along the trajectory. This distance is expressed as a function of the Euclidean distance along the trajectory, $l$, and the integral is taken along the rectilinear path from O to I to S. A time delay is the difference between the two times computed by Eq.(46) along the two paths determined by Eq.(45).
B. Iterated Schwarzschild Null Geodesics

This subsection develops the iterative approach in the case of the single Schwarzschild lens along the lines of Section III.B. In this subsection, the zeroth iterate will be chosen as the solution to Hamilton’s equations in the unperturbed metric, or \((X^a_o, P^a_o) = \text{constant}\). A better choice of the zeroth iterate is proposed in the next subsection. We begin with the linearized Schwarzschild metric, which is given by

\[
g^{ab}(x^a) = \begin{pmatrix}
1 + \frac{2m}{r} & 0 & 0 & 0 \\
0 & -1 + \frac{2mx^2}{r^3} & \frac{2mx}{r^3} & \frac{2mz}{r^3} \\
0 & \frac{2my}{r^3} & -1 + \frac{2my^2}{r^3} & \frac{2myz}{r^3} \\
0 & \frac{2mxz}{r^3} & \frac{2myz}{r^3} & -1 + \frac{2mz^2}{r^3}
\end{pmatrix}
\]  

in \((t, x, y, z)\) coordinates. Here, the signature of the metric is \((+, -, -, -)\), and \(r = \sqrt{x^2 + y^2 + z^2}\). The Hamiltonians of Minkowski space-time and the linearized Schwarzschild space-time are

\[
H_o(x^a, p_a) = \eta^{ab}p_a p_b = (p_0)^2 - \sum_{i=1}^3 (p_i)^2
\]

\[
H(x^a, p_a) = g^{ab}(x^a) p_a p_b = H_o(x^a, p_a) + h^{ab}(x^a) p_a p_b,
\]

and \(h^{ab}\) is

\[
h^{0a} = \frac{2m\delta^{0a}}{r},
\]

\[
h^{ij} = \frac{2mx^i x^j}{r^3}.
\]

The complete solution to Hamilton-Jacobi equation of \(H_o(x^a, p_a)\) is

\[
F_o(x^a, P_a, \lambda) = x^a P_a - \eta^{ab}P_a P_b \lambda,
\]

and the canonical coordinate transformation associated with Eq.(51) is given by

\[
x^a = X^a + 2\eta^{ab}P_b \lambda \equiv \tilde{x}^a(X^a, P_a, \lambda)
\]

\[
p_a = P_a.
\]
Applying this canonical transformation to the Hamiltonian \( H(x^a, p_a) \) yields

\[
H'(X^a, P_a, \lambda) = h^{ab}(\tilde{x}^a(X^a, P_a, \lambda)) P_a P_b,
\]

and Hamilton’s equations of motion for \( X^a \) and \( P_a \) are

\[
\dot{X}^a(X^a, P_a, \lambda) = 2h^{ab}(\tilde{x}^a(X^a, P_a, \lambda)) P_b + \frac{\partial h^{bc}(\tilde{x}^a(X^a, P_a, \lambda))}{\partial P_a} P_b P_c \\
\equiv \Xi^a(X^a, P_a, \lambda)
\]

\[
\dot{P}_a(X^a, P_a, \lambda) = -\frac{\partial h^{bc}(\tilde{x}^a(X^a, P_a, \lambda))}{\partial X^a} P_b P_c \equiv \Upsilon_a(X^a, P_a, \lambda).
\]

We solve Hamilton’s equations by the iterative procedure, as in Section III. The zeroth iterate will be constant values, \( X^a = X^a_0 = x^a_0 \) and \( P_a = P^a_0 = p^a_0 \). The \( n \)th iterate is given by

\[
X^a_n(X^a_0, P^a_0, \lambda) = X^a_0 + \int_0^\lambda d\lambda' \Xi^a(X^a_{n-1}, P^n_{a-1}, \lambda') \\
P^a_n(X^a_0, P^a_0, \lambda) = P^a_0 + \int_0^\lambda d\lambda' \Upsilon_a(X^a_{n-1}, P^n_{a-1}, \lambda').
\]

The approximate geodesics given by Eq.(56) are not necessarily null; we must require that \( H(X^a_0, P^a_0) = 0 \). Using the metric, Eq.(47), and the Hamiltonian, Eq.(49), yields

\[
P^a_0 = \left( \frac{\sum_{i=1}^3 (P^a_i)^2 + \frac{2m}{r_o} (X^i_0 P^a_i)^2}{1 + \frac{2m}{r_o}} \right)^{1/2}
\]

where \( r_o = \sqrt{\delta_{ij} X^i_0 X^j_0} \). When Eqs.(56) and Eq.(57) are substituted into the canonical transformation, Eqs.(52), the \( n \)th iterate, initially null geodesics are obtained.

Finally, we note that when the initial conditions satisfy the null condition, Eq.(57), the resulting trajectories are approximately null in \((M, g^{ab})\). Geodesics in the unperturbed background with these same initial conditions will be space-like. Hence, a cone which is null in \((M, g^{ab})\) will be fully inside the light cone of \((M, g^{ab})\) due to the converging effect of the positive mass in the Schwarzschild lens.
C. Choice of the zeroth iterate for low $n$

As mentioned at the end of Section II, the iterative method can fail to produce accurate results at low $n$ and large $\lambda$ values regardless of whether the method converges in general. There are two ways in which such a failure can be anticipated in the application to lensing.

First, when the zeroth order trajectory is chosen as one set of constants, $X^a = X^a_o$ and $P_a = P^o_a$, the trajectory is a “straight line,” corresponding to a solution to the equations of motion of the unperturbed Hamiltonian. In the Schwarzschild case, the first order trajectory is a “curve” which bends in the vicinity of the lens. These two trajectories become increasingly separated as $\lambda$ grows. For $n = 1$, the integrands in Eqs. (56) are evaluated along the zeroth order trajectory, and the “force” acting on the “photon” becomes increasingly incorrect as $\lambda$ grows. Hence, when the zeroth order trajectory is far from the first iterate trajectory, one may anticipate poor results.

It turns out that for a single Schwarzschild lens, this particular source of error is reasonably insignificant because the value of the “force” is decreasing faster than $1/\lambda$ as $\lambda$ grows. However, in a situation where there are multiple lenses lying in more than one lens plane, the error becomes very significant, and the force vector may point in the wrong direction altogether.

A second way to anticipate troubles is to note that while the true Hamiltonian is conserved under a flow along the actual geodesic, it will not be conserved under an approximate flow. If one finds the exact solution for $(X^a, P^a)$, and sets the initial conditions as in Eq. (36), the value of the perturbed Hamiltonian, Eq. (24), will remain zero along the trajectory. However, at any finite $n$, the value of

$$H(X^a_o, P^o_i, \lambda) = H(x^n(X^a_o, P^o_i, \lambda), p^n(X^a_o, P^o_i, \lambda))$$

will drift away from zero because the $n$th iterate solution is not truly a null solution to the geodesic equations. Hence, at large $\lambda$, the $n$th iterate solution may not closely approximate a null geodesic.
The basic reason for these problems is that, so far, we have chosen zeroth iterate in a non-physical way, as a straight line path corresponding a single solution to the equations of motion for geodesics in the unphysical space-time, \((M, g^{ab}_o)\). This solution separates from the first iterate shortly after passing the lens plane.

The simplest choice of the zeroth iterate is the solution of the equations of motion of the unperturbed Hamiltonian, but on the other hand, the zeroth iterate should remain close to the first iterate trajectory. As we have discussed, the thin lens path is given by two connected solutions of the equations of motion in \((M, g^{ab}_o)\) and remains close to the exact solution.

Therefore, we may avoid these problems by choosing the thin lens path as the zeroth iterate. This is accomplished by choosing different sets of constant values, \((X^a_o, P^a_o)\), in different \(\lambda\) ranges as described below. Making this choice insures the first iterate trajectory will always remain close to the zeroth iterate regardless of the number of lenses.

From a kinematical standpoint, the entire thin lens trajectory is determined by the choice of \(X^i_o\) and \(P^i_o\). These values determine the straight line path from the observer at O to I in Fig. 5. They also determine a new set of six parameters, \((X^i_{lp}, P^i_{lp})\) which represent the location of the intersection of the original straight line path and the lens plane, \((lp)\), and the new momenta determining the straight line path joining I and S.

As Schwarzschild space-time is static, the \(X^0 = T\) and \(P^0\) equations separate from the spatial equations. Due to spherical symmetry, we can assume that the spatial path of the geodesic lies in the \(\hat{x}\)-\(\hat{z}\) plane. In this case, the thin lens trajectory is determined by

\[
\begin{align*}
    x &= X^1_o - 2P^1_o \lambda & 0 < \lambda < \lambda_l \\
    z &= X^3_o - 2P^3_o \lambda & 0 < \lambda < \lambda_l
\end{align*}
\]

and

\[
\begin{align*}
    x &= X^1_{lp} - 2P^1_{lp} \lambda & \lambda_l < \lambda < \lambda_{source} \\
    z &= X^3_{lp} - 2P^3_{lp} \lambda & \lambda_l < \lambda < \lambda_{source}
\end{align*}
\]
If the initial conditions represent an observer on the $\hat{z}$ axis at $X^1_o = 0$, and $X^3_o = z_o$, the lens plane will be $z = 0$, and we have

$$\lambda_l = \frac{z_o}{2 P^o_3},$$
$$X^1_{lp} = -\frac{P^o_1 z_o}{P^o_3},$$
$$x^3_{lp} = 0. \quad (63)$$

The scaling for $\lambda$ and the spatial part of the $p_o$ is determined by

$$v^o p_a = -1, \quad (64)$$

as in Section III.C. Then, by choosing the new $P^{lp}_i$ such at the angle between $P^{lp}_i$ and the original $P^o_i$ at $X^i_{lp}$ is the bending angle, $\alpha = 4 G m/(c^2 r_o)$,

$$\cos \alpha = \frac{g^{ij}_o P^{lp}_i P^o_j}{\sqrt{(g^{ij}_o P^{lp}_i P^o_j)(g^{ij}_o P^o_i P^o_j)}}, \quad (65)$$

both the $P^{lp}_i$ are determined.

In summary, we may always pick the initial conditions as $(X^1_o = 0, X^3_o = z_o, P^o_1 = p^o_x, P^o_3 = p^o_z)$. This fixes the value of $\lambda_l$ reaching the lens plane and $X^1_{lp}$. Eq.(64) and Eq.(65) determine the $P^{lp}_i$ in terms of $(z_o, p^o_x, p^o_z)$. Hence, by freely choosing initial values at the observer, the entire zeroth iterate path is determined.

The first iterate solution will then be represented by the integrals, Eqs.(56), where for $0 < \lambda < \lambda_l$ the constant values are fixed at the initial point, and for $\lambda_l < \lambda < \lambda_{source}$, the values are reset. The initial $P^o_0$ is fixed as in Eq.(57) at the initial point along the first leg, and then reset at $X^i_{lp}$ with the new values. This insures that the value of the Hamiltonian, $H = g^{ab} p_a p_b$ will remain close to zero along the trajectory.

As an example, in Fig. 2, we plot a first iterate trajectory where the initial conditions are taken from the thin lens path as mentioned in this section. This figure also shows a thin lens trajectory with similar initial conditions in dashed lines. The mass is taken to be 1.0 in geometrical units and the range in $\hat{z}$ is 100 units. With these parameter choices, the
differences between the first iterate and thin lens are greatly enhanced compared to what one would see for a more physical choice of distance scale.

We see that the first iterate trajectory is a smooth path, while the thin lens path has a kink at the lens plane. This demonstrates one fundamental difference between the thin lens and iterative approaches, namely, that the iterate approach represents a continuous bending process while the thin lens approach reduces the action of the gravitational force to a single lens plane. We also note that there can be a significant difference in the distance of closest approach between the two methods, which leads to a difference in the bending angle and possible location of a source.

The $n$th iterate time elapsed along the trajectory is formally given by

$$t_n = t_0 + \int d\lambda 2 P_0 \left( 1 + \frac{2m}{r_{n-1}} \right), \quad (66)$$

from Eq.(56), the form of the perturbation, $h^{ab}$, and the canonical transformation. Here, $r_{n-1}$ is defined as the radial coordinate distance from the origin to the point along path of the $n-1$ iterate. It is not difficult to compute the second iterate for the time by first computing the first iterate trajectories above, and using these in the right hand side of Eq.(56). In this case, the integration over $\lambda$ will also contain two segments, corresponding to the two segments above.

V. DISCUSSION

Gravitational lensing is a very active field of observational research and is becoming an increasingly important tool for probing the structure of lenses and the cosmological parameters. Each day, new observations and analysis expand the list of lensing candidates, providing a rich field of study.

While the thin lens approximation has proven remarkably accurate when applied to the current observational data, it relies heavily on the weak field and small angle approximations. With the ever expanding lists of candidates and improving observational techniques, it is
not difficult to imagine that the limits of the thin lens approximation may be approached in the future. Virbhadra and Ellis attempt to improve upon the thin lens approximation by removing the small angle approximations [9], and other methods may try to "fix up" the thin lens approach from within its overall framework. An another approach was developed by Ted Pyne and Mark Birkinshaw [10,11], where the null geodesic equations are solved pertubatively. One may also numerically solve the geodesic equations, as in the "direct integration method" of Tomita, et al [12,13].

The iterative method provides a different approach. Although it works within the framework of exact gravitational lensing, the iterative approach is reasonably easy to apply to a wide range of lensing scenarios. In addition, when the thin lens trajectory is taken as the zeroth order path, the iterative method represents possible improvement upon standard lensing equations, with additional accuracy obtained by increasing the level of iteration.

We are preparing a close study of the iterative and thin lens approaches in a wide variety of space-time settings, including multiple lenses in multiple lens planes. Of particular interest are the time delays in the multiple lens plane configurations. It is entirely conceivable that the thin lens approximation may be accurate beyond any observational limits, and no noticeable differences may be found with the iterative approach. However, we take courage in the fact that the history of gravitational lensing is filled with false prophecies of the unlikely nature of detection.

Acknowledgments

The authors would like to thank Al Janis, Simonetta Frittelli, Jurgen Ehlers, and Jörg Fraudiener for their helpful advice and suggestions. Alejandro Perez would like to thank FUNDACION YPF. This work was supported under grants Phy 97-22049 and Phy 92-05109.
REFERENCES

[1] S. Frittelli and E. T. Newman, Phys. Rev. D 59, 124001, (1999).

[2] J. Ehlers, S. Frittelli, and E. T. Newman, *Gravitational Lensing From a Space-Time Perspective*, to appear in the Festchrift for John Stachel, Ed. J. Renn, Kluwer Academic Publishers, (2000)

[3] P. Schneider, J. Ehlers, and E. E. Falco, *Gravitational Lenses*, (Springer-Verlag, New York, Berlin, Heidelberg, 1992).

[4] H. Goldstein, *Classical Mechanics*, (Addison-Wesley, Reading, Massachusetts, 1980).

[5] H. C. Corben and P. Stehle, *Classical Mechanics*, (Dover, New York, 1977.)

[6] C. D. Impey, et al, ApJ, 509, 551, (1998).

[7] K. H. Chae, ApJ, (1999). (in press, astro-ph/9906179)

[8] S. Frittelli, T. P. Kling, and E. T. Newman, to be published in Phys. Rev. D, (2000).

[9] K. S. Virbhadra and G. F. R. Ellis, astro-ph/9904193

[10] T. Pyne and M. Birkinshaw, ApJ, 415, 459, (1993).

[11] T. Pyne and M. Birkinshaw, ApJ, 458, 46, (1996).

[12] K. Tomita, P. Premadi, and T.T Nakamura, Prog. Theor. Phys. Supplement No. 133, 85, (1999).

[13] K. Tomita, H. Asada, and T. Hamana, Prog. Theor. Phys. Supplement No. 133, 155, (1999).
FIG. 1. The general picture used in the thin lens approximation. A past directed geodesic emitted at the observer O travels rectilinearly to I, is bent by the bending angle, $\alpha$, and then travels to a source at S. Distances $D_l$, $D_s$, and $D_{ls}$ separate the lens and observer, source and observer, and lens and source, while $\beta$ locates the source and $\theta$ is an observation angle.
FIG. 2. The spatial path of the thin lens and first iterate approximate null geodesic when the first iterate takes the thin lens approximation as its zeroth iterate.