MULTIPLICATION OF MATRICES OVER LATTICES

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Abstract. We study the multiplication operation of square matrices over lattices. If the underlying lattice is distributive, then matrices form a semigroup; we investigate idempotent and nilpotent elements and the maximal subgroups of this matrix semigroup. We prove that matrix multiplication over nondistributive lattices is antiassociative, and we determine the invertible matrices in the case when the least or the greatest element of the lattice is irreducible.

1. Introduction

Matrix multiplication of matrices over a lattice $L$ can be defined in the same way as for matrices over rings, letting the join operation play the role of addition and the meet operation play the role of multiplication. For notational convenience, we will actually write the lattice operations as addition and multiplication. Thus, throughout the paper, $L = (L; +, ·)$ denotes an arbitrary lattice, and $\mathbb{M}_n(L)$ stands for the set of all $n \times n$ matrices over $L$. To exclude trivial cases, we will always assume without further mention that $L$ has at least two elements and $n \geq 2$. If $L$ has a least and a greatest element (these will be denoted by 0 and 1), then we can define the identity matrix $I \in \mathbb{M}_n(L)$ with ones on the diagonal and zeros everywhere off the diagonal, and it is easy to see that $I$ is indeed the identity element of $\mathbb{M}_n(L)$.

Matrix multiplication is not always associative, and if it is not, then we may ask how far it is from being associative. There are several ways to measure associativity; one of them is the associative spectrum introduced in [1]. The number of possibilities of inserting parentheses (or brackets) into a product $x_1 \cdot \ldots \cdot x_n$ is given by the $(n - 1)$-st Catalan number $C_{n-1} = \frac{1}{n+1} \binom{2n-2}{n-1}$. If multiplication is associative, then all these different bracketings give the same result, but if the multiplication is not associative, then some of the bracketings may induce different $n$-variable term functions. The associative spectrum of a binary operation is the sequence $\{s_n\}_{n=1}^{\infty}$ that counts the number of different term functions induced by bracketings of the product $x_1 \cdot \ldots \cdot x_n$. Clearly $s_1 = s_2 = 1$, and $1 \leq s_n \leq C_{n-1}$ holds for all natural numbers $n$, and we can say that the faster the spectrum grows, the less associative the multiplication is. In particular, if the associative spectrum is the sequence of Catalan numbers, then the multiplication is said to be antiassociative. Of course, there are plenty of operations that fall between the two extreme cases of being associative or antiassociative; examples of associative spectra of various growth rates can be found in [1, 5].

We shall see in Section 2 that there is a dichotomy for matrix multiplication over lattices: if $L$ is distributive, then $\mathbb{M}_n(L)$ is a semigroup, while if $L$ is not distributive, then the multiplication of $\mathbb{M}_n(L)$ is antiassociative. Nonassociativity has some unfortunate consequences: powers of matrices, nilpotent matrices and inverse matrices are not always well-defined. On the other hand, we prove that if $L$ is bounded and at least one of 0 and 1 is irreducible, then inverses are unique (even if $L$ is not distributive), and we describe explicitly the invertible matrices in this case, showing that they form a group isomorphic to the symmetric group $S_n$.

In Section 3 we focus on the semigroup $\mathbb{M}_n(2)$ of $n \times n$ matrices over the two-element lattice $2 = \{0, 1\}$. We can regard a matrix $A \in \mathbb{M}_n(2)$ as the characteristic function of a set $\alpha \subseteq X^n$ where $X := \{1, \ldots, n\}$, thus matrices over 2 correspond to binary relations, and $\mathbb{M}_n(2)$ is isomorphic to the semigroup of binary relations on $X^n$.

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the set $X$. We present various results about this semigroup in Section 3, namely, we describe its idempotent elements, Green’s relations $D$ and $H$ for idempotents, and the maximal subgroups "around" the idempotents. Some of these results appear already in the literature, but we provide elementary proofs for the readers’ convenience, offering a gentle introduction to the structure of the semigroup $M_n(2)$ for non-semigroup theorists.

Section 4 deals with matrices over bounded distributive lattices. Boundedness is not a serious restriction, since most of the time we shall work in a finitely generated sublattice (for instance, in the sublattice generated by the $n^2$ entries of an $n \times n$ matrix), and finitely generated distributive lattices are finite. A matrix $A \in M_n(L)$ can be viewed as a multiple-valued analogue of a binary relation $\alpha \subseteq X^2$. Generalizing results of Section 3 to this multiple-valued setting, we describe idempotents and maximal subgroups around some special idempotents in $M_n(L)$; the full description of maximal subgroups constitutes a topic for further research. We also determine nilpotent matrices over distributive lattices with a meet-irreducible bottom element (this includes chains, which are the most important cases from the viewpoint of applications), and then apply it to solve a problem arising from applications of fuzzy relations.

Some personal remarks from the second author about Ivo Rosenberg: As a graduate student working in clone theory under the supervision of Béla Csákány, I certainly learned the name of Ivo Rosenberg early in my studies. His theorems on maximal and minimal clones are cornerstones of the theory clones, and I always imagined the discoverer of these theorems as an unapproachable “giant”. It is no wonder that I was thrilled to meet him at the AAA58 conference in Vienna in 1999. Unfortunately, it was our first and last personal encounter. We spoke only a few words, and he apologized very kindly for not being able to attend my talk. I was a bit disappointed, but much more astonished, for receiving such friendly apologies from this giant of clone theory as a first-year doctoral student. My talk was about measuring associativity, and our joint paper with Béla Csákány about associative spectra appeared in this journal 20 years ago, in the special issue dedicated to the 65th birthday of Ivo Rosenberg. Now this is a special issue for a much more sad occasion, and I can only hope that this modest contribution is worthy to commemorate Ivo Rosenberg.

2. Matrices over arbitrary lattices

2.1. Antiassociativity of matrix multiplication. First we characterize lattices with associative matrix multiplication. Here, and in the rest of the paper, we always assume that all matrices are square matrices of size at least $2 \times 2$.

**Proposition 2.1.** Multiplication of matrices over a lattice $L$ is associative if and only if $L$ is a distributive lattice.

**Proof.** If $L$ is distributive, then one can prove associativity of matrix multiplication in the same way as it is proved for matrices over commutative rings. In fact, all the usual properties of matrix operations hold in this case (e.g., multiplication is distributive over addition).

If $L$ is not distributive, then $M_3$ or $N_5$ embeds into $L$ (see Figure 1), so it suffices to prove nonassociativity of matrix multiplication over these two lattices. Let us consider the following three matrices from $M_2(M_3)$ or from $M_2(N_5)$:

$$A = \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} c & 0 \\ 0 & 0 \end{pmatrix}. $$

Then it is easy to verify that $(AB)C \neq A(BC)$. For any $n \geq 2$, we can construct matrices attesting the nonassociativity of multiplication in $M_n(L)$ by inserting $A$, $B$ and $C$ into the top left $2 \times 2$ corner of an $n \times n$ matrix and filling all the remaining entries with 0. 

\[\square\]
We can strengthen Proposition 2.1 if $L$ is not distributive, then matrix multiplication over $L$ is not merely nonassociative: it is antiassociative! We derive this as a corollary of the following general proposition.

**Proposition 2.2.** If a binary operation has an identity element, then it is either associative (i.e., the associative spectrum is constant 1) or it is antiassociative (i.e., the associative spectrum consists of the Catalan numbers).

**Proof.** Let $(G; ·)$ be a groupoid with an identity element 1, and assume that $(ab)c ≠ a(bc)$ for some $a, b, c ∈ G$. We prove by induction on $n$ that any two bracketings $p ≠ q$ of size $n$ induce different term operations on $G$. For $n = 1, 2$ this claim is void, and for $n = 3$ it holds by the nonassociativity of the multiplication of $G$. Assume now that different bracketings of size less than $n$ induce different term functions, and let $p, q$ be two distinct bracketings of size $n$.

First we consider the case when the “outermost” multiplication of $p$ and $q$ is at the same place: $p = p_1(x_1, . . . , x_k) · p_2(x_{k+1}, . . . , x_n)$ and $q = q_1(x_1, . . . , x_k) · q_2(x_{k+1}, . . . , x_n)$. Since $p$ and $q$ are not the same term, we have $p_1 ≠ q_1$ or $p_2 ≠ q_2$ (perhaps both). If $p_1 ≠ q_1$, then, by the induction hypothesis, there exist elements $a_1, . . . , a_k ∈ G$ such that $p_1(a_1, . . . , a_k) ≠ q_1(a_1, . . . , a_k)$. This implies

$$p(a_1, . . . , a_k, 1, . . . , 1) = p_1(a_1, . . . , a_k) · p_2(1, . . . , 1)$$

$$= p_1(a_1, . . . , a_k) · 1 = p_1(a_1, . . . , a_k)$$

$$≠ q_1(a_1, . . . , a_k) = q_1(a_1, . . . , a_k) · 1$$

$$= q_1(a_1, . . . , a_k) · q_2(1, . . . , 1)$$

$$= q(a_1, . . . , a_k, 1, . . . , 1),$$

thus the term functions corresponding to $p$ and $q$ are indeed different. If $p_2 ≠ q_2$, then a similar argument can be used, assigning the value 1 to the variables $x_1, . . . , x_k$.

Now assume that the outermost multiplications in $p$ and $q$ are not at the same place: $p = p_1(x_1, . . . , x_k) · p_2(x_{k+1}, . . . , x_n)$ and $q = q_1(x_1, . . . , x_ℓ) · q_2(x_{ℓ+1}, . . . , x_n)$, where $k ≠ ℓ$. We may suppose without loss of generality that $k < ℓ$. Let us put $x_1 = a$, $x_{k+1} = b$, $x_{ℓ+1} = c$, and assign the value 1 to all the remaining variables. Then $p$ evaluates to

$$p_1(a, 1, . . . , 1) · p_2(b, 1, . . . , 1, c, 1, . . . , 1) = a(bc),$$

while $q$ gives the value

$$q_1(a, 1, . . . , 1, b, 1, . . . , 1) · q_2(c, 1, . . . , 1) = (ab)c,$$

proving that $p$ and $q$ induce different term functions, as claimed. □

**Corollary 2.3.** If the lattice $L$ is not distributive, then the multiplication of matrices over $L$ is antiassociative.

**Proof.** Since $L$ is not distributive, it has a sublattice $K$ that is isomorphic to $M_3$ or to $N_5$. The lattice $K$ is bounded, hence $M_n(K)$ has an identity element, thus its multiplication is antiassociative by propositions 2.1 and 2.2. This implies antiassociativity of the multiplication of $M_n(L)$, as it contains $M_n(K)$ as a subgroupoid. □
The following example shows that for nondistributive lattices even the definition of a power of a matrix and the notion of nilpotence can be problematic.

**Example 2.4.** Let $A$ be the following $5 \times 5$ matrix over $M_5$:

$$A = \begin{pmatrix}
0 & a & 0 & 0 & 0 \\
0 & 0 & b & c & 0 \\
0 & 0 & 0 & 0 & b \\
0 & 0 & 0 & 0 & c \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}.$$  

Then we have $(AA)A = 0 \neq A(AA)$. Thus $A$ has two different “cubes”; one of them is zero, the other one is not.

2.2. **Invertible matrices.** As another illustration of the unpleasant consequences of nonassociativity, we present an example of a matrix having several inverses.

**Example 2.5.** Consider the following two matrices over $N_5$:

$$A = \begin{pmatrix}
c & b \\
b & c
\end{pmatrix}, \quad B = \begin{pmatrix}
a & b \\
b & c
\end{pmatrix}.$$  

Then we have $AA = AB = BA = I$, thus $A$ and $B$ are both inverses of $A$.

Let us conclude this section with some positive results: we will prove that if $L$ is a bounded lattice such that at least one of 0 and 1 is irreducible, then inverses in $M_n(L)$ are unique; moreover, the only invertible matrices are the permutation matrices. For any permutation $\pi \in S_n$, we define the *permutation matrix* corresponding to $\pi$ as the matrix $P_{\pi} = (p_{ij})_{i,j=1}^n \in M_n(L)$ given by

$$p_{ij} = \begin{cases} 
1, & \text{if } j = \pi(i);
0, & \text{otherwise.}
\end{cases}$$

**Remark 2.6.** Just as over commutative rings, the matrix $P_{\pi}A$ is obtained from $A$ by permuting its rows according to the permutation $\pi$; similarly, $AP_{\pi}$ is obtained from $A$ by permuting its columns according to the permutation $\pi^{-1}$. In particular, we have $P_{\pi}P_{\sigma} = P_{\pi\sigma}$ for all $\pi, \sigma \in S_n$, and the (unique) inverse of $P_{\pi}$ is $P_{\pi^{-1}}$.

**Theorem 2.7.** Let $L$ be a bounded lattice in which 0 (the bottom element) is meet-irreducible or 1 (the top element) is join-irreducible. Then for all matrices $A,B \in M_n(L)$, we have $AB = I$ if and only if $A = P_{\pi}$ and $B = P_{\pi^{-1}}$ for some permutation $\pi \in S_n$.

**Proof.** The “if” part is clear (see Remark 2.6); so we only prove the “only if” part. Moreover, it suffices to prove that $A = P_{\pi}$; then $B = P_{\pi^{-1}}$ follows by Remark 2.6.

First we make some general observations, assuming only boundedness about the lattice $L$.

Let $A,B \in M_n(L)$ with $AB = I$. Considering the diagonal entries of $AB = I$, we have $\sum_{j=1}^n a_{ij}b_{ji} = 1$ for all $i = 1, \ldots, n$. This implies that for each $i$ there is at least one $j$ such that $a_{ij}b_{ji} \neq 0$. Denoting such an index $j$ by $\pi(i)$, we get a map $\pi: \{1, \ldots, n\} \rightarrow \{1, \ldots, n\}$ such that

1. $a_{\pi(i)} \neq 0$ and $b_{\pi(i)i} \neq 0$ for all $i \in \{1, \ldots, n\}$.

The off-diagonal entries of $AB = I$ yield $\sum_{j=1}^n a_{ij}b_{jk} = 0$ whenever $i \neq k$, hence

2. $a_{ij}b_{jk} = 0$ for all $i,j,k \in \{1, \ldots, n\}$ with $i \neq k$.

Assume first that $1$ is join-irreducible. Then at least one of the summands in $\sum_{j=1}^n a_{ij}b_{ji} = 1$ must be 1, hence we can replace (1) by the following stronger condition:

3. $a_{\pi(i)} = b_{\pi(i)i} = 1$ for all $i \in \{1, \ldots, n\}$.

Now we can see that $\pi$ is injective: if we had $\pi(i) = \pi(k) =: j$ for some $i \neq k$, then (3) would imply that $a_{ij} = b_{jk} = 1$, contradicting (2). In order to prove that $A$
is a permutation matrix, let us consider an entry \( a_{ij} \) in \( A \) with \( j \neq \pi(i) \). Letting \( k = \pi^{-1}(j) \), we have \( b_{jk} = 1 \) by (1); on the other hand, (2) implies \( a_{ij} b_{jk} = 0 \), as \( i \neq k \). Thus \( a_{ij} = 0 \) whenever \( j \neq \pi(i) \), and this together with (1) proves that \( A = P_\pi \).

Suppose next that 0 is meet-irreducible. Then (2) takes the following form:

\[
\begin{aligned}
a_{ij} &= 0 \quad \text{or} \quad b_{jk} = 0
\end{aligned}
\]

Again, \( \pi \) is injective: if we had \( \pi(i) = \pi(k) := j \) for some \( i \neq k \), then (1) would imply that \( a_{ij} \neq 0 \) and \( b_{jk} \neq 0 \), contradicting (2). Just as in the previous case, we can prove that \( a_{ij} = 0 \) whenever \( j \neq \pi(i) \). Indeed, for \( k = \pi^{-1}(j) \) we have \( b_{jk} \neq 0 \) by (1), and then (2) implies \( a_{ij} = 0 \). To show that \( A = P_\pi \), it only remains to prove that \( a_{i\pi(i)} = 1 \) for every \( i \). This follows from the following inequality:

\[
1 = \sum_{j=1}^{n} a_{ij} b_{ji} = a_{i\pi(i)} b_{\pi(i)i} \leq a_{i\pi(i)}.
\]

\[\square\]

**Remark 2.8.** As a consequence of Theorem 2.7, we have that \( AB = I \) implies \( BA = I \) for all matrices \( A, B \in \text{Mat}_n(L) \) if \( L \) satisfies the irreducibility condition of the theorem. For semigroups (and also for rings), the property \( AB = I \implies BA = I \) is called *Dedekind-finiteness.*

**Example 2.9.** Theorem 2.7 is not necessarily valid if neither 0 nor 1 is irreducible.

As an example, let \( A = \begin{pmatrix} a & b \\ b & a \end{pmatrix} \) over the lattice \( \mathbf{2} \times \mathbf{2} \) shown in Figure 2. This lattice is distributive, hence \( \text{Mat}_n(L) \) is a semigroup and inverses are unique. It is easy to verify that \( A \) has an inverse (in fact, we have \( A^{-1} = A \)), even though \( A \) is not a permutation matrix.

![Figure 2. The lattice \( \mathbf{2} \times \mathbf{2} \).](image)

**3. Matrices over the two-element chain**

To each \( n \times n \) matrix \( A \) over the two-element lattice \( \mathbf{2} = \{0, 1\} \), we can associate a binary relation \( \alpha \) defined on the set \( X := \{1, \ldots, n\} \), by letting \( (i, j) \in \alpha \iff a_{ij} = 1 \). Matrix multiplication translates to relational product in this interpretation: if the relations corresponding to \( A, B \in \text{Mat}_n(\mathbf{2}) \) are \( \alpha \) and \( \beta \), then \( AB \) describes the relation \( \alpha \circ \beta \). Thus \( \text{Mat}_n(\mathbf{2}) \) is isomorphic to the semigroup of binary relations on the \( n \)-element set. This semigroup plays a prominent role in semigroup theory; we recall a few of the plethora of results about this semigroup in this section, and we also recast some of the proofs in a simple form.

**Remark 3.1.** We can also regard the relation \( \alpha \subseteq X^2 \) corresponding to \( A \in \text{Mat}_n(\mathbf{2}) \) as the edge set of a directed graph with vertex set \( X \), having \( A \) as its adjacency matrix. We can think of this graph as a transportation network: the vertices are sites, and the edges are (possibly one-way) roads, on which trucks can transport goods between the sites. If \( a_{ii} = 0 \) (i.e., \( (i, i) \notin \alpha \)), then trucks are not allowed to stop at site \( i \), while if \( a_{ii} = 1 \) (i.e., there is a loop \( (i, i) \in \alpha \)), then there is a parking lot at site \( i \), where
trucks can wait as long as they wish. Powers of \( A \) account for routes\(^1\) in our graph: if \( A^\ell = (a_{ij})^\ell \) where \( a_{ij} = \begin{cases} 1 & \text{if and only if there is a directed route of length } \ell \text{ from } i \text{ to } j, \\ 0 & \text{otherwise} \end{cases} \)

3.1. Idempotent matrices. The characterization of idempotent elements of \( M_n(2) \) was given by B. Schein [8] in terms of so-called pseudo-orders. A reflexive transitive relation is called a quasi-order. The symmetric part \( \alpha \cap \alpha^{-1} \) of a quasi-order \( \alpha \) is an equivalence relation, and \( \alpha \) induces a natural partial order on the blocks of this equivalence relation. We usually use the symbol \( \leq \) for a quasi-order on the set \( X \), and we denote the corresponding equivalence relation by \( \sim \). Thus the partially ordered set (poset, for short) corresponding to the quasi-order \( \leq \) is \( (X/\sim, \leq) \). We say that an element \( y \in X \) covers \( x \in X \) (notation: \( x \prec y \)), if \( x/\sim \) is strictly less than \( y/\sim \), and there is no third \( \sim \)-block between them:

\[ x \prec y \iff x \leq y, x \sim y \text{ and } \forall z \in X: x \leq z \leq y \implies x \sim z \text{ or } z \sim y. \]

A quasi-order relation is obtained from a quasi-order by removing some of the loops (i.e., edges of the form \((x, x)\)) in such a way, that loops can be removed only from singleton \( \sim \)-blocks, and it is not allowed to remove loops from both members of a covering pair.

Definition 3.2. Let \( \alpha \subseteq X^2 \) be a binary relation, and let \( Q_\alpha \) denote the set of vertices with a loop: \( Q_\alpha = \{ x \in X : (x, x) \in \alpha \} \). We say that \( \alpha \) is a quasi-order if the reflexive closure \( \alpha \cup \{ (x, x) : x \in X \setminus Q_\alpha \} \) is a quasi-order (we denote this quasi-order by \( \leq \) and we use the symbols \( \sim \) and \( \prec \) for the corresponding equivalence relation and cover relation), and \( Q_\alpha \) satisfies the following two conditions:

(a) \( \forall x \in X \setminus Q_\alpha: x/\sim = \{ x \} \),

(b) \( \forall x, y \in X: x \prec y \implies x \in Q_\alpha \text{ or } y \in Q_\alpha. \)

Remark 3.3. Let us give an interpretation of pseudo-orders in terms of the transportation network outlined in Remark 3.1. A relation \( \alpha \subseteq X^2 \) is a pseudo-order if and only if whenever you drive from site \( x \) to site \( y \),

(a’) you can choose a direct route (formally: if there is a route from \( x \) to \( y \), then \( (x, y) \) is an edge), and

(b’) it is also possible to plan your route so that you will have a chance to take a rest in a parking lot on the way (formally: if there is a route from \( x \) to \( y \), then there is a route that includes a vertex with a loop).

Indeed, condition (a) in Definition 3.2 ensures that the removal of loops from the underlying quasi-order \( \leq \) does not ruin transitivity, thus (a’) holds for every pseudo-order. (Observe that (a’) is actually equivalent to transitivity.) To verify (b’), choose a longest possible route that does not pass through any \( \sim \)-block more than once; then each edge in this route is a covering pair, and at least one member of a covering pair has a loop (if any of them belongs to a non-singleton \( \sim \)-block, then condition (a), otherwise condition (b) provides a loop).

Conversely, let us assume that (a’) and (b’) hold for \( \alpha \), and let us denote the reflexive closure of \( \alpha \) by \( \leq \). Condition (a’) implies that \( \alpha \) is transitive, hence \( \leq \) is also transitive, thus it is a quasi-order. Transitivity of \( \alpha \) also implies that (a) holds. To verify (b), consider a covering pair \( x \prec y \). If the \( \sim \)-block of \( x \) or \( y \) is not a singleton, then condition (a) shows that there is a loop at \( x \) or \( y \). Otherwise, by the definition of covering, no route from \( x \) to \( y \) passes through any vertex other than \( x \) and \( y \). Therefore, the parking lot guaranteed by (b’) must be at \( x \) or at \( y \), and this proves (b).

Now we are ready to state and prove the characterization of idempotent binary relations given by Schein [8]. We will use the description of pseudo-orders given in Remark 3.3.

\(^1\)We use the term route for a sequence of connecting edges (with possible repetitions). The usual terminology would be walk, but we would like to avoid the uncanny image of a walking truck...
**Theorem 3.4.** A matrix over 2 is idempotent if and only if the corresponding binary relation is a pseudo-order.

**Proof.** Let $\alpha$ be the binary relation on $X$ corresponding to the matrix $A \in M_n(2)$. As a preliminary observation, let us note that $\alpha$ is transitive if and only if $\alpha \circ \alpha \subseteq \alpha$, which in turn is equivalent to $A^2 \subseteq A$.

Assume first that $A$ is idempotent. Then $A^2 \subseteq A$, so $\alpha$ is transitive, hence condition (a') of Remark 3.3 holds. Idempotence of $A$ implies $A = A^2 = A^3 = \ldots$, thus whenever there is a route from $x$ to $y$, there are arbitrarily long routes from $x$ to $y$. A long enough route must include a directed cycle, and every vertex of such a cycle has a loop, by transitivity. This proves (b'), therefore $\alpha$ is a pseudo-order.

Now let us suppose that $\alpha$ is a pseudo-order. Then $\alpha$ is transitive by condition (a'), hence $A^2 \subseteq A$. Multiplying this inequality by $A^{m-1}$, we get $A^{m+1} \subseteq A^m$ for every positive integer $m$, thus the powers of $A$ form a decreasing sequence: $A \geq A^2 \geq A^3 \geq \ldots$. Since $M_n(2)$ is a finite set, this sequence cannot be strictly decreasing, i.e., there is a positive integer $\ell$ such that

$$A \geq A^2 \geq A^3 \geq \cdots \geq A^\ell = A^\ell+1 = A^\ell+2 = \cdots = \lim_{m \to \infty} A^m.$$  

Here the limit is understood in the discrete topology on $M_n(2)$, but this is not very important, as an ultimately constant sequence converges in every topology. For every edge $(x, y) \in \alpha$, condition (b') provides a route from $x$ to $y$ with a parking lot on the way. We can park there as long as we wish, before continuing our trip to $y$, thus there are arbitrarily long routes from $x$ to $y$. This means that $A \leq \lim_{m \to \infty} A^m$, which together with the inequalities of (3) implies that $A = A^2 = A^3 = \ldots$, hence $A$ is idempotent. $\square$

**3.2. Green’s relations.** If $e$ is an idempotent element in a semigroup, then there is a maximal subgroup $H_e$ “around” $e$, having $e$ as its identity element. Having determined the idempotents of $M_n(2)$, our next goal is to describe the maximal subgroups corresponding to these idempotents. We will need Green’s equivalence relations $L$, $R$, $H$ and $D$, which can be defined in any semigroup, but we write out the definition for the semigroup $M_n(L)$, where $L$ is a distributive lattice. Two elements $A, B \in M_n(L)$ are in $L$ relation if they generate the same principal left ideal, that is, if and only if there exist $C, D \in M_n(L)$ such that $CA = B$ and $DB = A$. Similarly, the relation $R$ can be defined by $(A, B) \in R$ if and only if there exist $C, D \in M_n(L)$ such that $AC = B$ and $BD = A$. The relation $L \cap R$ is denoted by $H$, and the join $L \lor R$ is denoted by $D$. It is known that $L$ and $R$ commute in every semigroup, thus we have $L \lor R = L \lor R$.

According to Green’s theorem, the maximal subgroups of $M_n(L)$ are precisely the $H$-classes $H_E$ of idempotent matrices $E \in M_n(L)$. Moreover, if two idempotents $E, F$ belong to the same $D$-class, then the groups $H_E$ and $H_F$ are isomorphic. For further background on semigroup theory, and in particular on Green’s relations, see [2].

Green’s relations in $M_n(2)$ can be described in terms of in- and out-neighborhoods in the relations corresponding to matrices over 2. We introduce the following notation for any relation $\alpha \subseteq X^2$:

- $\alpha^+(x) = \{ z \mid (x, z) \in \alpha \} \subseteq X$ is the out-neighborhood of $x \in X$,
- $\alpha^+(Y) = \{ z \mid (y, z) \in \alpha \text{ for some } y \in Y \} = \bigcup_{y \in Y} \alpha^+(y)$ is the out-neighborhood of a set $Y \subseteq X$, and
- $\alpha^+ = \{ \alpha^+(Y) \mid Y \subseteq X \}$ is the set of all outneighborhoods.

The in-neighborhoods $\alpha^-(x)$ and $\alpha^-(Y)$ of vertices and of sets of vertices and the set $\alpha^-$ of all in-neighborhoods are defined dually.

**Remark 3.5.** If $\alpha$ is transitive, then every element of $\alpha^+$ is a “forward-closed” set: $U \in \alpha^+, u \in U, (u, x) \in \alpha$ implies $x \in U$. In fact, it is not hard to see that this condition is equivalent to transitivity. If $\alpha$ is a partial order, then it may be more convenient to denote it by $\leq$, and then forward-closed sets are called **upsets**.
Note that \( \alpha^+ \) and \( \alpha^- \) form lattices under inclusion. The bottom element of both lattices is \( \alpha^+(\emptyset) = \alpha^-(\emptyset) = \emptyset \), but the top elements of the two lattices might be different. The join operation in \( \alpha^+ \) and in \( \alpha^- \) is just the union, but the meet operation need not be the intersection.

The following description of Green’s relations on \( M_n(2) \) is a combination of results obtained by Zaretskii in [9] (see also [7]).

**Proposition 3.6.** Let \( A, B \in M_n(2) \) and let \( \alpha, \beta \subseteq X^2 \) be the corresponding binary relations. Then the following hold:

1. \( (A, B) \in \mathcal{L} \) if and only if \( \alpha^+ = \beta^+ \);
2. \( (A, B) \in \mathcal{R} \) if and only if \( \alpha^- = \beta^- \);
3. \( (A, B) \in \mathcal{H} \) if and only if \( \alpha^+ = \beta^+ \) and \( \alpha^- = \beta^- \);
4. \( (A, B) \in \mathcal{D} \) if and only if \( \alpha^+ \) and \( \beta^+ \) are lattice isomorphic.

Every element of \( \alpha^+ \) is a union of some sets of the form \( \alpha^+(x) \ (x \in X) \), and likewise for \( \alpha^- \), therefore we can reformulate the first two items of the above proposition as follows.

**Proposition 3.7.** Let \( A, B \in M_n(2) \) and let \( \alpha, \beta \subseteq X^2 \) be the corresponding binary relations. Then the following hold:

1. \( (A, B) \in \mathcal{L} \) if and only if for every \( x \in X \) there exist \( Y, Z \subseteq X \) such that \( \alpha^+(x) = \beta^+(Y) \) and \( \beta^+(x) = \alpha^+(Z) \);
2. \( (A, B) \in \mathcal{R} \) if and only if for every \( x \in X \) there exist \( Y, Z \subseteq X \) such that \( \alpha^-(x) = \beta^-(Y) \) and \( \beta^-(x) = \alpha^-(Z) \).

Since maximal subgroups contained in the same \( \mathcal{D} \)-class are isomorphic, we would like to find a “simplest” idempotent in any given \( \mathcal{D} \)-class, and describe the maximal subgroup around this idempotent. For this, we need a way to tell whether two idempotents are \( \mathcal{D} \)-related or not. The following lemma serves this purpose.

**Lemma 3.8.** Let \( A \in M_n(2) \) be an idempotent matrix and let \( \alpha \subseteq X^2 \) be the corresponding pseudo-order relation. Let \( T \) be a complete system of representatives of the blocks of the equivalence relation \( \alpha \cap \alpha^{-1} \subseteq Q_\alpha^2 \). Let \( D = (d_{ij}) \in M_n(2) \) be the diagonal matrix defined by

\[
d_{ij} = \begin{cases} a_{ij}, & \text{if } i = j \in T; \\ 0, & \text{otherwise.} \end{cases}
\]

Then we have \( ADA = A \), and consequently \( DAD \) is an idempotent matrix in the \( \mathcal{D} \)-class of \( A \). The pseudo-order relation corresponding to the matrix \( A_1 := DAD \) is \( \alpha_1 := \alpha \cap T^2 \), and \( \alpha_1 \) is a partial order on \( T \).

**Proof.** The entries of the matrix \( ADA = (c_{ij}) \) can be computed as follows (taking into account that \( d_{kk} = 0 \) whenever \( k \neq \ell \)); for comparison we also write out the entry \( a_{ij} \) from the product \( A = AA \):

\[
c_{ij} = \sum_{k,\ell=1}^n a_{ik}d_{k\ell}a_{\ell j} = \sum_{k=1}^n a_{ik}d_{kk}a_{kj} = \sum_{k=1}^n a_{ik}a_{kk}a_{kj};
\]

\[
a_{ij} = \sum_{k=1}^n a_{ik}a_{kj}.
\]

It is clear that \( c_{ij} \leq a_{ij} \), as \( a_{ik}a_{kk}a_{kj} \leq a_{ik}a_{kj} \) for all \( i, j, k \in X \). The inequality \( c_{ij} \geq a_{ij} \) is equivalent to the implication \( a_{ij} = 1 \implies c_{ij} = 1 \), and we will prove this using of the transportation network interpretation of matrices (see Remark 3.1). If \( a_{ij} = 1 \) (i.e., \((i, j) \in \alpha \)) then there is a (one-step) route from site \( i \) to site \( j \). Therefore, by condition (b’) of Remark 3.3, there is a route from \( i \) to \( j \) that passes through some site \( p \in Q_\alpha \) with a parking lot. Since \( T \) is a complete system of representatives of the blocks of the equivalence relation \( \alpha \cap \alpha^{-1} \), there is a (unique) \( k \in T \) with
(p, k), (k, p) ∈ α. By the transitivity of α, this implies that (i, k), (k, j) ∈ α, hence αik = αkj = αkj = 1. This proves that cij = 1, thus ADA = A, as claimed.

Idempotence of DAD now follows easily (note that D2 = D):

\[(DAD)^2 = DA(DD)AD = DADAD = D(ADA)D = DAD.\]

To prove the relation (A, DAD) ∈ D, we verify that (A, DA) ∈ L and (DA, DAD) ∈ R:

\[A(DA) = A \iff (A, DA) ∈ L;\]
\[(DAD)A = DA \iff (DA, DAD) ∈ R.\]

If A1 = DAD = (bij), then bii = dii djj aij, thus bii = aij if i, j ∈ T, and bii = 0 otherwise. This means that the relation α1 is obtained from α by deleting all edges going into or out from vertices outside T. The choice of the set T ensures that α1 is a partial order on T (and each element of X \ T is an isolated vertex in α1).

The pseudo-order α1 constructed in Lemma 3.8 is a partial order on the set T ⊆ X. Relations of this form (i.e., partial orders on subsets of X) are called reduced idempotents. It was already proved in [6] that if a D-class contains an idempotent, then it also contains a reduced idempotent. We complement this result with a simple criterion to decide whether two pseudo-orders are D-related (see Theorem 3.9 below).

The structure of the poset (T; α1) is independent of the choice of T; let us denote (the isomorphism type of) this poset by T(α). If we use the usual symbol ≤ for this partial order instead of α1, then the out- and in-neighborhood of x ∈ T can be written as:

- \[\alpha^+(x) = \{y ∈ T : y ≥ x\} =: ↑x;\]
- \[\alpha^−(x) = \{y ∈ T : y ≤ x\} =: ↓x.\]

The elements of \[\alpha^+(x)\] (i.e., unions of sets of the form \(↑x\)) are called upsets (cf. Remark 3.8). Thus U ⊆ T is an upset if and only if x ∈ U and y ≥ x implies y ∈ U for all x, y ∈ T. Dually, the members of \(\alpha_1^-\) are called downsets.

**Theorem 3.9.** Let \(A, B ∈ M_n(2)\) be idempotent matrices and let \(α, β ⊆ X^2\) be the corresponding pseudo-order relations. We have (A, B) ∈ D if and only if the posets T(α) and T(β) are isomorphic.

**Proof.** Let α be a pseudo-order on X, and let the set T be defined as in Lemma 3.8. It follows from Lemma 3.8 and Proposition 3.5 that the lattices \(α^+\) and \(α^-\) are isomorphic, and it is clear that the latter is isomorphic to the lattice of upsets of T(α). Thus, by Proposition 3.6 we only need to prove that the posets T(α) and T(β) are isomorphic if and only if their upset lattices are isomorphic. The “only if” part is trivial, and the “if” part follows from the observation that for any poset P, the join- irreducible elements of the upset lattice are exactly the sets of the form \(↑x\) (x ∈ P), and these sets form a poset that is isomorphic to P.

**3.3. Maximal subgroups.** We conclude this section with the promised description of the maximal subgroups of \(M_n(2)\). First we need a simple auxiliary observation.

**Lemma 3.10.** If \((T; ≤)\) is a finite poset and \(f\) is a permutation of \(T\) such that \(f(x) ≥ x\) for all \(x ∈ T\), then \(f = 1_T\).

**Proof.** If \(x\) is a maximal element, then \(f(x) = x\) follows immediately from the assumption \(f(x) ≥ x\). From here, we can proceed downwards, proving by induction on the size of \(↑x\) that \(f(x) = x\) for all \(x ∈ T\).

**Theorem 3.11.** Let \(A ∈ M_n(2)\) be an idempotent matrix with the corresponding pseudo-order \(α ⊆ X^2\). Assume that \(A\) is a reduced idempotent, i.e., \(α\) is a partial order on the set \(T := Q_α ⊆ X\). Then a matrix \(B ∈ M_n(2)\) belongs to the \(H\)-class of \(A\) if and only if it can be written as \(B = PfA\), where \(f\) is a permutation on \(X\) such that \(f(T) = T\) and the restriction of \(f\) to \(T\) is an automorphism of the poset \((T; α)\).
Proof. Let us first interpret the requirements imposed on \( f \) in the theorem. Since all elements of \( X \setminus T \) are isolated in \( \alpha \), a permutation \( f \) of \( X \) is an automorphism of the pseudo-ordered set \((X; \alpha)\) if and only if \( f(T) = T \) and \( f \) is an automorphism of the poset \((T; \alpha)\). Regarding \( f \) as a binary relation, it is clear that \( f \) is an automorphism of \((X; \alpha)\) if and only if \( f \circ \alpha \circ f^{-1} = \alpha \), which is in turn equivalent to \( f \circ \alpha = \alpha \circ f \).

The latter condition can be formulated in terms of matrices as \( P_f A = AP_f \). If this holds, then it is easy to verify that \( B = P_f A \) is \( \mathcal{H} \)-related to \( A \):

\[
B = P_f A \quad \text{and} \quad P_f^{-1} B = A \implies (A, B) \in \mathcal{L}.
\]

\[
B = AP_f \quad \text{and} \quad B P_f^{-1} = A \implies (A, B) \in \mathcal{R}.
\]

Assume now that \( B \in \mathbb{M}_n(2) \) belongs to the \( \mathcal{H} \)-class of \( A \), and let \( \beta \subseteq X^2 \) denote the relation corresponding to \( B \). Then we have \( \alpha^+ = \beta^+ \) and \( \alpha^- = \beta^- \) by Proposition 3.6. If \( x \in X \setminus T \), then \( x \) does not appear in any member of \( \alpha^+ \) or \( \alpha^- \), thus \( x \) is an isolated point in \( \beta \), too. Therefore, it suffices to focus on elements of \( T \).

If \( x \in T \), then \( x \in \alpha^+(x) \), and this implies that if we write \( \alpha^+(x) = \beta^+(y) = \bigcup_{y \in Y} \beta^+(y) \) for a suitable \( Y \subseteq X \) as in Proposition 3.7, then \( x \in \beta^+(y) \) for some \( y \in Y \). Using Proposition 3.7 again, we get a set \( Z \subseteq X \) such that \( \beta^+(y) = \alpha^+(Z) = \bigcup_{z \in Z} \alpha^+(z) \). Therefore, there exists \( z \in Z \) with \( x \in \alpha^+(z) \), and then \( \alpha^+(x) \subseteq \alpha^+(z) \), since \( \alpha \) is transitive. Now we can conclude that \( \alpha^+(x) = \beta^+(y) \):

\[
\alpha^+(x) \subseteq \alpha^+(z) \subseteq \beta^+(y) \subseteq \alpha^+(x).
\]

Note that \( y \in T \), as \( \alpha^+(x) = \beta^+(y) \) is not empty. Letting \( h(x) = y \), we can define a map \( h: T \to T \) such that \( \alpha^+(x) = \beta^+(h(x)) \) for all \( x \in T \). If \( h(x_1) = h(x_2) \), then \( \alpha^+(x_1) = \beta^+(h(x_1)) = \beta^+(h(x_2)) = \alpha^+(x_2) \), and this can hold only if \( x_1 = x_2 \), as \( \alpha \) is antisymmetric. This proves that \( h \) is injective, thus it is also bijective by the finiteness of \( T \). We can extend the inverse of \( h \) to a permutation \( f \) on \( X \) by keeping the elements of \( X \setminus T \) fixed. The defining property \( \alpha^+(x) = \beta^+(h(x)) \) of \( h \) can be expressed in terms of \( f \) as follows:

\[
\forall x, y \in X : (f(x), y) \in \alpha \iff (x, y) \in \beta.
\]

(Note that the above equivalence holds trivially if \( x \) or \( y \) lies outside of \( T \), since each element of \( X \setminus T \) is an isolated vertex in \( \alpha \) as well as in \( \beta \).

Repeating the previous argument for the in-neighborhoods, we obtain a permutation \( g \) on \( X \) such that

\[
\forall x, y \in X : (x, g(y)) \in \alpha \iff (x, y) \in \beta.
\]

Comparing (4) and (5), we see that

\[
\forall x, y \in X : (f(x), y) \in \alpha \iff (x, g(y)) \in \alpha.
\]

In particular, since for every \( x \in T \) we have \( (f(x), f(x)) \in \alpha \), it follows from (6) that \( (x, g(f(x))) \in \alpha \) holds for all \( x \in T \). Applying Lemma 3.3 to the restriction of the map \( fg \) to \( T \), we can draw the conclusion \( g = f^{-1} \) (recall that both \( f \) and \( g \) act identically on \( X \setminus T \)). Now it is easy to deduce from (6) that \( f \) is an automorphism of \( \alpha \):

\[
(x, y) \in \alpha \iff (x, g(f(y))) \in \alpha \iff (f(x), f(y)) \in \alpha.
\]

Finally, let us observe that (1) means that the matrix \( B \) is obtained from \( A \) by permuting its rows by the permutation \( f \), hence \( B = P_f A \) (see Remark 2.6).

\[ \square \]

**Corollary 3.12.** Let \( A \in \mathbb{M}_n(2) \) be an idempotent matrix with the corresponding pseudo-order \( \alpha \subseteq X^2 \). Then the \( \mathcal{H} \)-class containing \( A \) is isomorphic to the automorphism group of the poset \( T(\alpha) \)

**Proof.** By Green’s theorem, the \( \mathcal{H} \)-classes within the same \( D \)-class form isomorphic groups, thus Theorem 3.9 (or Lemma 3.5) allows us to assume without loss of generality that \( A \) is a reduced idempotent. According to Theorem 3.11 the elements of the \( \mathcal{H} \)-class \( H_A \) of \( A \) are of the form \( P_f A \), where \( f \) is a permutation on \( X \) such that \( f(T) = T \) and the restriction of \( f \) to \( T \) is an automorphism of the poset \((T; \alpha)\). We
have seen in the proof of Theorem [3.1] that \( P_f A = AP_f \) holds for such permutations \( f \). Note that the rows of \( A \) indexed by elements of \( X \setminus T \) are all constant zero, and the rows indexed by elements of \( T \) are pairwise distinct, since \( \alpha \) is antisymmetric. Thus \( P_f A \) depends only on the action of \( f \) on \( T \) (see Remark [2.6]). Therefore, the map \( \varphi : \text{Aut}(T(\alpha)) \to HA, f \mapsto P_f A \) is a well-defined bijection, and it is a group homomorphism:

\[
P_{f_1} A \cdot P_{f_2} A = P_{f_1} P_{f_2} AA = P_{f_1 f_2} A.
\]

\[\square\]

4. Matrices over distributive lattices

If \( L = (L; +, \cdot) \) is a bounded distributive lattice with least element 0 and greatest element 1, then, by Birkhoff’s representation theorem, \( L \) can be embedded into the lattice \( \mathcal{P}(\Omega) \) of subsets of a set \( \Omega \) in such a way that 0 is mapped to \( \emptyset \) and 1 is mapped to \( \Omega \). Identifying \( L \) with its embedded image, we can actually assume that \( L \) is a sublattice of \( \mathcal{P}(\Omega) \) with \( 0 = \emptyset \) and \( 1 = \Omega \). This allows us to define a homomorphism \( \Gamma_\omega \) from \( L \) to \( 2 = \{0, 1\} \) for each \( \omega \in \Omega \) by

\[
\Gamma_\omega(a) = \begin{cases} 
1, & \text{if } \omega \in a; \\
0, & \text{if } \omega \notin a.
\end{cases}
\]

We call \( \Gamma_\omega(a) \) the cat of the element \( a \) (at \( \omega \)) \([10]\) (also called section or zero pattern \([3]\)). Since \( a \subseteq \Omega \) is exactly the set of those elements \( \omega \in \Omega \) for which \( \Gamma_\omega(a) = 1 \), every element of \( L \) is uniquely determined by its cuts. Extending \( \Gamma_\omega \) to matrices entrywise, we get cut homomorphisms \( \Gamma_\omega : M_n(L) \to M_n(2) \) for all \( \omega \in \Omega \), and matrices are also uniquely determined by their cuts:

\[
\forall A, B \in M_n(L): A = B \iff [\forall \omega \in \Omega: \Gamma_\omega(A) = \Gamma_\omega(B)].
\]

**Remark 4.1.** Let us give an interpretation of matrices over \( L \) in the spirit of Remark [2.6]. As before, we regard the elements of \( X = \{1, \ldots, n\} \) as sites (cities, stores, etc.), numbered from 1 to \( n \), and we think of the elements of \( \Omega \) as different types of vehicles that can travel between these sites. The entry \( a_{ij} \subseteq \Omega \) of the matrix \( A \in M_n(L) \) determines which vehicles can (or are allowed) to pass through the road from \( i \) to \( j \) (the diagonal entry \( a_{ii} \) is the set of vehicles that can park at site \( i \)). In other words, we have a complete directed graph on \( n \) vertices, and each edge \((i, j)\) has a “capacity” \( a_{ij} \subseteq \Omega \). (In reality, the graph is rarely complete; we can take non-existing connections into account by assigning capacity 0.)

Given a route \( i = v_0 \to v_1 \to \cdots \to v_\ell = j \) of length \( \ell \), the set of vehicles that can travel all the way along this route from \( i \) to \( j \) is the intersection (product) of the capacities of the edges involved in the route, i.e., \( a_{v_0 v_1} \cdots a_{v_{\ell-1} j} \). We will call this element of \( L \) the capacity of the route. The set of vehicles that can go from \( i \) to \( j \) on some route of length \( \ell \) can be computed as the join (sum) of the capacities of the routes of length \( \ell \) from \( i \) to \( j \), which is nothing else but the \((i, j)\)-entry of \( A^\ell \).

4.1. **Idempotent matrices.** From (7) and from the fact that each \( \Gamma_\omega \) is a homomorphism, it follows that a matrix is idempotent if and only if all of its cuts are idempotent:

\[
A = AA \iff \forall \omega \in \Omega: \Gamma_\omega(A) = \Gamma_\omega(AA)
\]

\[
\iff \forall \omega \in \Omega: \Gamma_\omega(A) = \Gamma_\omega(A)\Gamma_\omega(A).
\]

Combining this observation with Theorem [3.3], we get the following description of idempotent matrices over distributive lattices.

**Proposition 4.2.** A matrix \( A \in M_n(L) \) over a distributive lattice \( L \leq \mathcal{P}(\Omega) \) is idempotent if and only the binary relation \( a_\omega \subseteq X^2 \) corresponding to the cut matrix \( \Gamma_\omega(A) \) is a pseudo-order for each \( \omega \in \Omega \).
Although Proposition 4.2 certainly characterizes idempotent matrices, this characterization does not give a complete picture about the idempotent elements of the semigroup $M_n(L)$, since it does not tell us which systems of pseudo-orders $\alpha_\omega (\omega \in \Omega)$ can arise as cuts of idempotent matrices. In full generality perhaps one cannot expect a feasible solution for this problem, but for chains we can give a simple criterion. We represent the $m$-element chain in the power set of $\Omega = \{1, \ldots, m-1\}$ as

$\emptyset \subset \{1\} \subset \{1, 2\} \subset \cdots \subset \{1, 2, \ldots, m-1\}$,

so that the cut homomorphisms are $\Gamma_1, \ldots, \Gamma_{m-1}$.

**Theorem 4.3.** If $L$ is the $m$-element chain, then a matrix $A \in M_n(L)$ is idempotent if and only the binary relations corresponding to the cut matrices $\Gamma_k(A)$ ($k = 1, \ldots, m-1$) form a system of nested pseudo-orders $\alpha_1 \supseteq \cdots \supseteq \alpha_{m-1}$.

**Proof.** Since we represent $L$ by the chain of sets (8), we have the implication $k \in a \implies k-1 \in a$ for all $a \in L$ and $k \in \{2, \ldots, m-1\}$. This implies the inequalities $\Gamma_1(A) \supseteq \cdots \supseteq \Gamma_{m-1}(A)$ for every matrix $A \in M_n(L)$ (idempotent or not), and these inequalities translate to the containments $\alpha_1 \supseteq \cdots \supseteq \alpha_{m-1}$ of the corresponding relations. This together with Proposition 4.2 proves the necessity of the condition formulated in the proposition.

For sufficiency, assume that we have a nested sequence of pseudo-orders $\alpha_1 \supseteq \cdots \supseteq \alpha_{m-1}$ on $X$. Define the matrix $A = (a_{ij})_{i,j=1}^n \in M_n(L)$ by

$a_{ij} = \{k \in \{1, \ldots, m-1\} : (i, j) \in \alpha_k\}$.

Observe that the assumed containments of the relations $\alpha_k$ guarantee that $a_{ij}$ is an element of $L$. Thus $A$ is indeed a matrix over $L$, and the binary relations corresponding to the cuts of $A$ are exactly the relations $\alpha_1, \ldots, \alpha_{m-1}$. Since these are all pseudo-orders, each cut of $A$ is idempotent by Theorem 3.3 and then idempotence of $A$ follows from Proposition 4.2. \qed

### 4.2. Green’s relations and maximal subgroups

We have proved in the previous subsection that a matrix is idempotent if and only if all of its cuts are idempotent. The reason behind this observation is that the definition of idempotence is simply an equality; it does not ask for the existence of certain elements. For “existentially quantified” notions the situation is more complicated. As an example, let us consider the definition of the $R$-relation:

$$(A, B) \in R \iff \exists C, D \in M_n(L) : AC = B \text{ and } BD = A.$$ 

Using the fact that the cut maps are homomorphisms, it follows that $\Gamma_\omega(A)\Gamma_\omega(C) = \Gamma_\omega(B)$ and $\Gamma_\omega(B)\Gamma_\omega(D) = \Gamma_\omega(A)$, hence $\Gamma_\omega(A)$ and $\Gamma_\omega(B)$ are $R$-related in the semigroup $M_n(2)$ for all $\omega \in \Omega$. However, the converse is not necessarily true. Given matrices $C_\omega, D_\omega \in M_n(2)$ such that $\Gamma_\omega(A)C_\omega = \Gamma_\omega(B)$ and $\Gamma_\omega(B)D_\omega = \Gamma_\omega(A)$ for all $\omega \in \Omega$, it is not guaranteed that there exist matrices $C, D \in M_n(L)$ whose cuts are $C_\omega$ and $D_\omega$, respectively. In fact, an example of $R$-inequivalent matrices $A, B \in M_n(2)$ over the three-element chain were presented in [10] such that both of their cuts are $R$-related.

Nevertheless, as illustrated by the following theorem, in some special cases we can recover information about matrices over $L$ from their cuts.

**Theorem 4.4.** Let $L$ be the $m$-element chain, and let $A \in M_n(L)$ be an idempotent matrix such that the binary relations $\alpha_k \subseteq X^2$ corresponding to the cut matrices $\Gamma_k(A)$ ($k = 1, \ldots, m-1$) are partial orders. Then a matrix $B \in M_n(L)$ belongs to the $H$-class of $A$ if and only if it can be written as $B = P_fA$, where $f$ is a permutation on $X$ that is a common automorphism of the posets $(X; \alpha_k)$ ($k = 1, \ldots, m-1$).

**Proof.** Just as in the proof of Theorem 4.11 we can see that $f$ is an automorphism of the poset $(X; \alpha_k)$ if and only if $\Gamma_k(P_f)$ commutes with $\Gamma_k(A)$. According to (7), this holds for every $k$ if and only if $P_fA = AP_f$, and the latter implies that the matrix $B = P_fA$ belongs to the $H$-class of $A$. 

Theorem 4.8. Let $A$ be a matrix $A$ in $\mathbf{M}_n(\mathbb{L})$, where $\mathbb{L}$ is a bounded distributive lattice with a meet-irreducible element $m$. Then $A$ is isomorphic to the group of common automorphisms of the posets $(X; \alpha_k)$ such that $\Gamma_k(A) = \beta_k(\mathbf{M}_n(2))$.

Proof. We are going to use the interpretation of matrices outlined in Remark 4.1. Since each $\Gamma_k$ is a homomorphism, $(\Gamma_k(A), \Gamma_k(B)) \in \mathcal{H}$ holds in the semigroup $\mathbf{M}_n(2)$ for all $k \in \{1, \ldots, m-1\}$.

By Theorem 3.11 for each $k$ there exists an automorphism $f_k$ of the poset $(X; \alpha_k)$ such that $\Gamma_k(A) = P_k \Gamma_k(A)$. We are going to prove that $f_1 = \cdots = f_{m-1}$.

Let us write out (11) for each cut (here $\beta_k$ denotes the binary relation corresponding to the matrix $\Gamma_k(B) \in \mathbf{M}_n(2)$):

(9) $\forall x, y \in X: (f_k(x), y) \in \alpha_k \iff (x, y) \in \beta_k$ $(k = 1, \ldots, m-1)$.

Since $L$ is a chain, the relations $\beta_k$ form a nested sequence (cf. the beginning of the proof of Theorem 4.8):

(10) $\beta_1 \supseteq \cdots \supseteq \beta_{m-1}$.

For every $k \in \{1, \ldots, m-1\}$ and $x \in X$, we have $(f_k(x), f_k(x)) \in \alpha_k$, as $\alpha_k$ was assumed to be a partial order. Using (9), this implies that $(x, f_k(x)) \in \beta_k$, and then $(x, f_k(x)) \in \beta_1$, by (11). Applying (9) with $k = 1$, we can conclude that $(f_1(x), f_k(x)) \in \alpha_1$. We can rewrite this as $(y, f_k(f_1^{-1}(y))) \in \alpha_1$ with the notation $y = f_1(x)$. This holds for every $y \in X$, therefore $f_1 = f_k$ follows from Lemma 3.10.

We have proved that $f := f_1 = \cdots = f_{m-1}$ is a common automorphism of the posets $(X; \alpha_k)$ $(k = 1, \ldots, m-1)$. It remains to prove that $B = P_f A$. By (11), it suffices to show that the cuts of $B$ and $P_f A$ coincide:

\[ \Gamma_k(B) = P_f \Gamma_k(A) = P_f \Gamma_k(A) = \Gamma_k(P_f) \Gamma_k(A) = \Gamma_k(P_f A). \]

(We used the fact that cuts preserve 0 and 1, hence each cut of the permutation matrix $P_f$ is itself.)

\[ \square \]

Corollary 4.5. Let $L$ be the $m$-element chain, and let $A \in \mathbf{M}_n(L)$ be an idempotent matrix such that the binary relations $\alpha_k \subseteq X^2$ corresponding to the cut matrices $\Gamma_k(A) (k = 1, \ldots, m-1)$ are all partial orders. Then the $\mathcal{H}$-class containing $A$ is isomorphic to the group of common automorphisms of the posets $(X; \alpha_k)$ $(k = 1, \ldots, m-1)$.

Remark 4.6. For chains, Theorem 2.7 is a special case of Theorem 4.4. Indeed, if $A = I$, then each $\alpha_k$ is the equality relation on $X$, hence the group of automorphisms is $S_n$.

4.3. Nilpotent matrices. First we give a simple criterion for the nilpotency of a matrix in terms of the underlying directed graph, and then we use it to explicitly describe nilpotent matrices over bounded distributive lattices with a meet-irreducible bottom element.

Lemma 4.7. A matrix $A \in \mathbf{M}_n(L)$ over a bounded distributive lattice $L$ is nilpotent if and only if every cycle in the directed graph corresponding to $A$ has capacity 0.

Proof. We are going to use the interpretation of matrices outlined in Remark 4.1. Suppose first that every cycle has capacity 0. Every route of length $n$ contains a cycle (possibly of length 1), therefore it has capacity 0. Thus we have $A^n = 0$.

Now suppose that there is a cycle $C$ of length $\ell$ with capacity $c \neq 0$. If $c < c$, then trucks of type $\omega$ can drive along $C$. Driving along $C$ several times, we see that the $(i, i)$-entry of $A^{k\ell}$ contains $\omega$ for every vertex $i$ in $C$ and for every natural number $k$. This shows that none of the matrices $A, A^2, A^3, \ldots$ are zero, hence $A$ cannot be nilpotent.

By a strictly upper triangular matrix we mean a matrix $A \in \mathbf{M}_n(L)$ that has zeros below its main diagonal as well as on the main diagonal, i.e., $a_{ij} \neq 0 \iff i < j$.

Theorem 4.8. Let $L$ be a bounded distributive lattice in which 0 (the bottom element) is meet-irreducible. Then a matrix $A \in \mathbf{M}_n(L)$ is nilpotent if and only if it is conjugate to a strictly upper triangular matrix, i.e., there exists a strictly upper triangular matrix $U$ and an invertible matrix $C$ such that $A = C^{-1} U C$. 

Proof. If $U$ is a strictly upper triangular matrix, then we have $U \leq V$, where $V$ is the matrix having ones above the diagonal and zeros on and below the diagonal:

$$
V = \begin{pmatrix}
0 & 1 & \ldots & 1 & 1 \\
0 & 0 & \ldots & 1 & 1 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \ldots & 0 & 1 \\
0 & 0 & \ldots & 0 & 0
\end{pmatrix}.
$$

(11)

In the directed graph corresponding to $V$, we have an edge from $i$ to $j$ if and only if $i < j$. This means that it is impossible to make a route of length $n$, hence $V^n = 0$. Since $U \leq V$, it follows that $U^n = 0$, which implies that $(C^{-1}UC)^n = 0$ for every invertible matrix $C$.

Conversely, let us assume that $A \in M_n(L)$ is a nilpotent matrix. Consider the relation $\alpha \subseteq X^2$ defined by $\alpha := \{(i, j) : a_{ij} \neq 0\}$. (If $L$ is finite, then meet-irreducibility of 0 implies that 0 has a unique upper cover $\alpha$, as 0 is meet-irreducible. This means that $\alpha$ is the reflexive transitive closure of $\alpha$. Since $\alpha$ is an extension of $\alpha$, we have $a_{ij} \neq 0 \implies i < j$ for all $i, j \in X$.

Let $\pi$ be the permutation of $X$ given by $\pi(1) \sqsubseteq \cdots \sqsubseteq \pi(n)$, and let $C = P\pi$. We claim that the matrix $U := CAC^{-1}$ is strictly upper triangular. By the definition of the matrix $C$, we have $u_{ij} = a_{\pi(i)\pi(j)}$, hence

$$
U = \begin{pmatrix}
0 & 1 & \ldots & 1 & 1 \\
0 & 0 & \ldots & 1 & 1 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \ldots & 0 & 1 \\
0 & 0 & \ldots & 0 & 0
\end{pmatrix}.
$$

(The last implication is justified by the definition of $\pi$.) Thus $U$ is indeed strictly upper triangular, and this completes the proof, as $A = C^{-1}UC$.

Remark 4.9. We have seen in the proof of Lemma 4.7 that $A \in M_n(L)$ is nilpotent if and only if $A^n = 0$. This cannot be sharpened: the matrix $V$ given in (11) is nilpotent, but $V^{n-1} \neq 0$.

Example 4.10. Theorem 4.8 does not necessarily remain true without the assumption on the irreducibility of 0. Consider the matrix $A = \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix}$ over the lattice $2 \times 2 \oplus 1$ shown in Figure 3. It is easy to verify that $A^2 = 0$, but $A$ is not a conjugate of a strictly upper triangular matrix. Indeed, by Theorem 4.7, the only invertible matrices in $M_2(L)$ are the permutation matrices, hence the only conjugates of $A$ are itself and the matrix

$$
P_{(12)}^{-1} \cdot A \cdot P_{(12)} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & b \\ a & 0 \end{pmatrix},
$$

and neither of them is upper triangular.

4.4. Fixed point iteration. Our results on nilpotent matrices have some implications on a problem about fuzzy relations raised in 3. The interpretation of a matrix $A \in M_n(L)$ as a directed graph with a capacity assigned to each edge (see Remark 4.11), is almost the same as a fuzzy relation; we only need to regard the entries $a_{ij}$ as membership values instead of capacities.

Each element of a fuzzy set has a membership value, describing to what extent the given element belongs to the fuzzy set. Classically, the membership values are real numbers between 0 and 1, but elements of an arbitrary lattice $L$ can also serve as membership values. (In the case of $L = 2 = \{0, 1\}$, we get back the usual notion of a set, called a crisp set in this framework.)

Thus, a tuple $x \in L^n$ describes a fuzzy subset $F$ of $X = \{1, \ldots, n\}$, and a matrix $A \in M_n(L)$ can be regarded as the membership function of a fuzzy subset $\alpha$ of $X^2$, 

$$
\alpha(i, j) := a_{ij}.
$$
which is called a fuzzy relation. From the definition of matrix multiplication we see that $xA \leq x$ is equivalent to the system of inequalities $x_i a_{ij} \leq x_j (i, j \in X)$, which can be interpreted as follows: “if $i$ belongs to $F$ and $(i, j)$ belongs to $\alpha$ to some extent, then $x_j$ also belongs to $F$ at least to that extent”. If this holds, then we say that the fuzzy set $F$ is closed under the fuzzy relation $\alpha$. (In the crisp case, being closed under $\alpha$ means that if $i \in F$ and $(i, j) \in \alpha$, then $j \in F$.)

The inequality $xA \leq x$ and the equation $xA = x$ were studied in [3] from the viewpoint of fuzzy control. We refer the reader to that paper for more details about fuzzy relations and their applications, and here we focus only on the proposed fixed-point iteration method to find solutions of the equation $xA = x$.

The solutions of $xA = x$ are exactly the fixed points of the “linear transformation” $x \mapsto xA$, hence we can hope that the standard fixed-point iteration method can be used to find solutions. Thus we start with an arbitrary $x \in L^n$, and we form the sequence

$$x, xA, xA^2, xA^3, \ldots, xA^k, \ldots$$

The first problem that arises is whether this sequence converges in some topology. Even if $L$ is an infinite lattice, each entry of each tuple in our sequence belongs to the sublattice generated by the $n + n^2$ elements $x_i, a_{ij} (i = 1, \ldots, n)$, which is finite if $L$ is distributive. Therefore, the only meaningful choice is the discrete topology, and a sequence converges in this topology if and only if it becomes eventually periodic. It follows from the finiteness explained above that $xA^k$ becomes eventually periodic. However, the period can be longer than 1 (consider a permutation matrix, for example), hence the sequence might fail to converge. If $\lim_{k \to \infty}(xA^k)$ exists, then it is easy to see that this limit will be a solution of $xA = x$. Here we face a second problem: it may happen that our sequence converges to the trivial solution $0 = (0, \ldots, 0)$. To avoid this problem, it is natural start with the largest possible initial value $1 = (1, \ldots, 1)$. In the following proposition we show that in this case the limit exists, and it is the greatest solution of $xA = x$.

**Proposition 4.11.** Let $L$ be a bounded distributive lattice, and let $1 = (1, \ldots, 1) \in L^n$. For any matrix $A \in M_n(L)$, the sequence $\{1A^k\}_{k=1}^{\infty}$ is eventually constant, and $\lim_{k \to \infty}(1A^k)$ is the greatest solution of the fixed-point equation $xA = x$.

**Proof.** It is clear that $1 \geq 1A$, and multiplying this inequality by $A^{k-1}$, we get $1A^{k-1} \geq 1A^k$ for every natural number $k$. Therefore, we have a decreasing sequence $1 \geq 1A \geq 1A^2 \geq \ldots$. Since each entry of $1A^k$ belongs to the finite sublattice generated by 1 and the entries of $A$, the sequence $\{1A^k\}_{k=1}^{\infty}$ is eventually periodic, hence eventually constant (as it is decreasing). Thus $1A^\ell = 1A^{\ell+1} = \ldots$ for some natural number $\ell$, and then $\lim_{k \to \infty}(1A^k) = A^\ell$. This immediately implies that $1A^\ell$ is a solution of $xA = x$. Now if $x$ is any other solution, then $x = xA^\ell \leq 1A^\ell$, and this means that $1A^\ell$ is indeed the greatest solution.

**Remark 4.12.** The interpretation of the greatest solution of $xA = x$ in the transportation network setting of Remark [4] is more natural if we work with column
vectors instead of row vectors (or we transpose $A$). If $\lim_{k \to \infty} (A^k 1) = (z_1, \ldots, z_n)$, then $z_i \in L$ is the set of vehicles that can start arbitrarily long trips at $i \in X$. In other words, $z_i$ is the set of vehicles that can reach a directed cycle from $i$.

Proposition 4.11 allows us to completely characterize matrices $A \in \mathbf{M}_n(L)$ having a nonzero fixed point in $L^n$.

Corollary 4.13. For every bounded distributive lattice $L$ and $A \in \mathbf{M}_n(L)$, the fixed-point equation $xA = x$ has a nonzero solution if and only if the matrix $A$ is not nilpotent.

Proof. This follows immediately from Proposition 4.11, since $1 A^\ell = 0$ if and only if $A^\ell = 0$.

If the bottom element of $L$ is irreducible, then combining the above corollary with our earlier results we obtain the following corollary.

Corollary 4.14. Let $L$ be a bounded distributive lattice in which $0$ (the bottom element) is meet-irreducible. Then the following are equivalent for any matrix $A \in \mathbf{M}_n(L)$:

(i) the only solution of the fixed-point equation $xA = x$ is $0$;
(ii) $\lim_{k \to \infty} (1 A^k) = 0$;
(iii) $A$ is nilpotent;
(iv) $A^n = 0$;
(v) $A$ is conjugate to a strictly upper triangular matrix, i.e., there exists a strictly upper triangular matrix $U$ and an invertible matrix $C$ such that $A = C^{-1} UC$;
(vi) one can rearrange the rows and columns of $A$ so that it becomes a strictly upper triangular matrix, i.e., there exists a permutation $\pi \in S_n$ such that $P_\pi^{-1} A P_\pi$ is strictly upper triangular.

Corollary 4.14 applies in particular to chains (which is the most important case for applications) and it shows that the fixed-point equation $xA = x$ has a nontrivial solution except for a few matrices of a very restricted form.

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