O-BTZ: Orientifolded BTZ Black Hole

F. Loran
Department of Physics, Isfahan University of Technology, Isfahan 84156-83111, Iran

M. M. Sheikh-Jabbari
2School of Physics, Institute for research in fundamental sciences (IPM), P.O.Box 19395-5531, Tehran, Iran

&Dated: July 29, 2010

Banados-Teitelboim-Zanelli (BTZ) black holes are constructed by orbifolding AdS$_3$ geometry by boost transformations of its $O(2,2)$ isometry group. Here we construct a new class of solutions to AdS$_3$ Einstein gravity, orientifolded BTZ or O-BTZ for short, which in general, besides the usual BTZ orbifolding, involve orbifolding (orientifolding) by a $Z_2$ part of $O(2,2)$ isometry group. This $Z_2$ is chosen such that it changes the orientation on AdS$_3$ while keeping the orientation on its 2D conformal boundary. O-BTZ solutions exhaust all un-oriented AdS$_3$ black hole solutions, as BTZ black holes constitute all oriented AdS$_3$ black holes. O-BTZ, similarly to BTZ black holes, are stationary, axisymmetric asymptotically AdS$_3$ geometries with two asymptotic charges, mass and angular momentum.

PACS numbers: 11.25.Tq, 04.70.-s, 04.20.Dw, 04.60.Rt

Introduction and Summary of results

Although does not have propagating gravitons, AdS$_3$ Einstein gravity has nontrivial black hole solutions, the BTZ black holes [1,2], and hence provides a simple but at the same time rich arena for addressing questions about black holes and quantum gravity in general. Since in three dimensions the number of independent components of the Ricci tensor is the same as that of Riemann curvature, all of the solutions to the equations of motion for pure AdS$_3$ Einstein gravity are locally AdS$_3$ and are obtained as quotients of global AdS$_3$ by (a subgroup of) its $SO(2,2)$ isometry group [2]. For a generic BTZ black hole solution, however, the quotient leads to closed time-like curves (CTC’s) in some regions of the space, which happen to fall behind the inner horizon of the black hole geometry. In order to remove the usual problems with the CTC’s, those regions in the BTZ geometry are cut out of the global AdS$_3$. As was discussed in [2] and we review below, the excised region includes a part of the causal boundary of global AdS$_3$ and renders the BTZ black holes as geodesically incomplete.

Besides the geodesic incompleteness, the BTZ solutions also suffer from a quantum instability. To see the instability one may introduce a free scalar theory in the classical BTZ background geometry. It has been shown that [3], see also [4], expectation value of components of the energy momentum tensor of quantum fluctuations of this field blows up at the inner horizon, signaling an instability at the inner horizon, which is also a Cauchy horizon in the (non-extremal) rotating BTZ black hole. This problem seems to be a generic property of BTZ black holes and independent of the details of the quantum field theory in question. Noting that this instability is originating from inside the inner horizon region [3], if one can cut this region in a consistent manner, the problem with the instability, as well as the geodesic incompleteness issue, might be resolved. However, so far it is not clear how the latter should be performed.

It has been argued that the AdS$_3$/CFT$_2$ provides us with the tools to probe the region inside the horizon [5]. Computations with the CFT$_2$ again reveals the same instability at the inner horizon of the BTZ black hole [8]. The fact that CFT$_2$ correlators blow up at the inner (Cauchy) horizon or at the (orbifold) singularity of the static BTZ has been interpreted as impossibility of probing the region beyond the Cauchy horizon and hence a manifestation of the cosmic censorship conjecture [8].

On the other hand, in the case of pure Lorentzian AdS$_3$ Einstein gravity, which is the case of our interest in this paper, we may have the advantage of a dual 2D conformal field theory (CFT$_2$) description [5]. This dual CFT$_2$, if it exists, resides on the conformal (causal) boundary of AdS$_3$, which is a $1+1$ flat space. The states of this CFT$_2$ are labeled by representations of a Virasoro algebra with equal left and right central charges $c_L = c_R = c$. In the seminal work [6] Brown and Henneaux showed that the Virasoro algebra corresponding to the (proposed) dual CFT$_2$ is directly related to a subset of 3D diffeomorphisms respecting certain boundary conditions, and

\[ c = \frac{3 \ell}{2G}, \]

where $\ell$ is AdS$_3$ radius and $G$ is the 3D Newton constant.

The Brown-Henneaux boundary conditions [7] allow for diffeomorphisms which change the orientation on the AdS$_3$ while preserving the orientation on the conformal boundary as well as diffeomorphisms which preserve AdS$_3$ orientation but change the orientation on the
boundary. The latter set of diffeomorphisms are fixed by the choice of sign for energy and angular momentum in the dual 2D CFT and hence could be discarded. The former, however, may be associated with a well-defined operator in the dual CFT. In this Letter we will focus on such diffeomorphisms.

Motivated by the puzzles (features) of the usual BTZ solutions discussed above, we construct the new class of stationary and axisymmetric “O-BTZ” solutions the line element of which are the same as BTZ everywhere, with the same mass and angular momentum. O-BTZ black holes are obtained by orbifolding or “orientifolding” the orientation changing $\mathbb{Z}_2$ part of the $O(2,2)$ isometry of AdS$_3$. This $\mathbb{Z}_2$ preserves the orientation on the 2D boundary of the AdS$_3$ and has a fixed locus which is a space-like cylinder located in the region between the two horizons of BTZ, the “Space-like Orientifold Cylinder” or SOC for short. As such our O-BTZ black holes completes the results of [1]: All possible black hole solutions to AdS$_3$ Einstein gravity are either BTZ or O-BTZ.

From the usual BTZ geometry viewpoint which serves as the covering space for the O-BTZ, hence, this $\mathbb{Z}_2$ exchanges the region outside the outer horizon and the region inside the inner horizon. Technically, the “orientifold” $\mathbb{Z}_2$ projection in the covering space may be implemented by cutting a BTZ geometry exactly at the middle of the region between its two horizons (where the fixed locus of the orientifold is located) and gluing another copy of the same geometry to it. The O-BTZ geometry is hence the part of BTZ geometry confined between the two orientifold fixed locus which should be viewed as the surfaces at the end of the geometry. As mentioned the O-BTZ everywhere except at the fixed locus of the orientifold projection has the same metric as an ordinary BTZ. O-BTZ geometry in the covering space may be viewed as a solution to AdS$_3$ Einstein gravity with a $\delta$-function source at the fixed locus of the orientifold projection, the SOC.

In this way we obtain a solution which, despite of having the metric of a rotating BTZ (outside its horizon), does not have an inner horizon region. Therefore, these solutions do not have the CTC issue (unlike the standard BTZ geometry). The analysis of [3] can be repeated for our geometry leading to the result that energy momentum tensor for the fluctuations of any given field on the O-BTZ black holes revisited

AdS$_3$ space is a hyperboloid embedded in $R^{2+2}$:

$$- T_1^2 - T_2^2 + X_1^2 + X_2^2 = -\ell^2 .$$

A suitable coordinate for AdS$_3$ is [3]

$$ds^2 = -(\tilde{r}^2 - \ell^2) dt^2 + \ell^2 \frac{d\tilde{r}^2}{\tilde{r}^2 - \ell^2} + r^2 d\phi^2$$

where $t, \phi \in (-\infty, +\infty)$. For global AdS$_3$ this coordinate system should be extended past $\tilde{r}^2 \geq 0$. This may be achieved by replacing $\tilde{r}$ with $\tilde{\rho}$, $\tilde{\rho}^2 = \ell^2 - \tilde{r}^2$, where $\tilde{r}^2$ becomes negative. In the $\tilde{r}^2 \geq \ell^2$ region (region I) $t$ is time-like and $\phi$ is space-like. In the region II, where $\tilde{r}$ and $\tilde{\rho}$ coordinate systems overlap and $0 < \tilde{r}^2 < \ell^2$, $t$ and $\phi$ are both space-like. In region III, where $\tilde{\rho}^2 > \ell^2$, $t$ coordinate becomes space-like while $\phi$ is time-like. Relaxing $\tilde{r}^2 > 0$ condition, $\tilde{r} \leftrightarrow \tilde{\rho}$, $t \leftrightarrow \phi$ coordinate transformation is a $\mathbb{Z}_2$ diffeomorphism which exchanges regions I and III and maps region II to itself. In the embedding $R^{2+2}$ space this $\mathbb{Z}_2$ may be realized as $X_1 \leftrightarrow X_2$, $T_1 \leftrightarrow T_2$. This $\mathbb{Z}_2$ changes the orientation on the boundary while preserving the AdS$_3$ orientation. Later we will introduce another $\mathbb{Z}_2$, the orientifold $\mathbb{Z}_2$, which changes the orientation on AdS$_3$ while keeping the orientation on the 2D boundary.

BTZ black holes [1][2] constitute all classical solutions to the pure AdS$_3$ Einstein gravity (modulo the self-dual orbifold [10]) with a fixed AdS$_3$ orientation and are obtained by orbifolding the original AdS$_3$ by the boosts of its $SO(2,2)$ isometry:

$$T_1 \pm X_1 \equiv e^{\pm \frac{2\pi}{r_+} r_{\ell}} (T_1 \pm X_1) ,$$
$$T_2 \pm X_2 \equiv e^{\pm \frac{2\pi}{r_-} r_{\ell}} (T_2 \pm X_2) .$$

Without loss of generality we assume $r_+ > r_- > 0$. (For $r_+ = r_-$ [10] does not lead to a black hole [2].) For $r_- = 0$, the static BTZ black hole, the above orbifolding has a fixed line at $T_1 = X_1 = 0$, $T_2^2 - X_2^2 = \ell^2$ while for generic $r_- \neq 0$ case the orbifolding is freely acting on AdS$_3$ and we have a smooth geometry. In the coordinate system [3] the BTZ identification (4) is written as

$$(t, \tilde{r}, \phi) \sim (t - 2\pi r_-/\ell, \tilde{r}, \phi + 2\pi r_+/\ell).$$

In the BTZ coordinates the metric takes the form

$$ds^2 = \rho^2 d\tau^2 + \frac{r^2 dr^2}{16G^2J^2 - r^2 \rho^2} + r^2 d\phi^2 - 8G\ell Jd\tau d\phi ,$$

where $\varphi \in [0, 2\pi]$,

$$\rho^2 = 8GM\ell^2 - r^2 ,$$

$1$ The idea of cutting the region with CTC’s and gluing another part to the geometry for removing the CTC problem has been discussed, e.g. see [3]. Note, however, that our construction, although technically and in the covering space of the orientifolding seems similar, has the crucial difference that the geometry on the two sides of the “fixed locus” are related by orientifolding and have exactly the same line element.
and $M$, $J$ are (ADM) mass and angular momentum

$$M = \frac{r_+^2 + r_-^2}{8\ell^2G}, \quad J = \frac{r_+ r_-}{4G\ell}.$$  (8)

The BTZ coordinate system $[5]$ has the advantage that the identification $[6]$ which constructs the BTZ solution is performed only on the $\varphi$ coordinate, $\varphi \equiv \varphi + 2\pi$.

From identification $[5]$ it is clear that there are no closed time-like curves (CTC’s) in the region where $t$ is parameterizing time. This is, however, not sufficient and with the above identification in the $r^2 < 0 \ (\rho^2 > 8GM\ell^2)$ region, which is inside the inner horizon of the BTZ black hole, geometry develops CTC’s. To remove inconsistencies arising from CTC’s one is forced to cut the geometry at $r = 0$ $[2]$. Penrose diagram of the BTZ geometry is depicted in Fig. 1.

**Orientifooled-AdS (O-BTZ) solutions**

As mentioned, the only possible classical solutions to $\text{AdS}_3$ Einstein gravity should necessarily be locally $\text{AdS}_3$ and they are hence all classified by orbifolding $\text{AdS}_3$ by its isometries; which if we also demand preserving the orientation on $\text{AdS}_3$, that is orbifolding with a subgroup of $\text{SO}(2, 2)$. These are BTZ solutions $[1, 2]$ or the self-dual $\text{AdS}_3$ orbifold $[10]$.

The only remaining possibility which we will study here is then to orbifold (orientifold) $\text{AdS}_3$ by the orientation changing $\mathbb{Z}_2$ which is a part of $O(2, 2)$ but not of $\text{SO}(2, 2)$. Moreover, we would like this orientifolding to also commute with that of BTZ $[4]$. Noting $[9]$, this is only possible if the orientifolding is acting on the $\tilde{r}$ coordinate and not $t$ and $\phi$. Explicitly, that is possible if the orientifold $\mathbb{Z}_2$ is $\tilde{r}^2 \leftrightarrow \rho^2$. This $\mathbb{Z}_2$ does not have a simple (linear) realization on $X_i$ and $T_i$ coordinates. In the BTZ coordinate system $[6]$ this orientation changing $\mathbb{Z}_2$ is hence

$$\left(\tau, r^2, \varphi\right) \leftrightarrow \left(\tau, \rho^2, \varphi\right).$$  (9)

(Note that $\varphi$ is compact while $\tau$ is not.)

We use ordinary BTZ geometry as the basis for studying the $\mathbb{Z}_2$ invariant solution in the covering space. As seen in Fig. 2 this $\mathbb{Z}_2$ invariant geometry is indeed a double cover of the O-BTZ geometry, i.e. two O-BTZ geometries or two halves of standard BTZ geometries glued at $r^2 = \rho^2 = 4G\ell^2 M$. The metric for the double cover of O-BTZ is

$$ds^2 = [\rho^2(\Phi) + r^2\theta(-\Phi)]d\tau^2 - 8G\ell J dr d\varphi$$

$$+ [r^2\theta(\Phi) + \rho^2\theta(-\Phi)]d\rho^2 + \frac{r^2 dr^2}{16G^2 J^2 - \frac{r^2 \rho^2}{\ell^2}},$$  (10)

where $\theta(X)$ is the step function and

$$\Phi = r^2 - 4G\ell^2 M = \frac{r^2 - \rho^2}{2}.$$  (11)

In the coordinate system $[11]$ $\tau$ and $\varphi$ are both dimen-
The volume element of the geometry is
\[ dV = \ell dr d\varphi (\theta(\Phi) \ r dr + \theta(-\Phi) \ r d\rho) = \ell r d\sigma d\varphi dr (\theta(\Phi) - \theta(-\Phi)) , \]
where \( \theta(\Phi) \) is the Israel matching condition at the junction. The orientifold projection used is chosen such that it fixes the O-AdS space at \( u = 0 \) which is a space-like cylinder. Away from the fixed locus the metric is the same as for the ordinary BTZ geometry and in particular its asymptotic behavior is the same as for the ordinary BTZ.

\[ R_{\mu\nu} = 64G^2(\ell^2 M^2 - J^2) \text{ diag}(1,0,-1) \delta(\Phi) , \quad (13) \]
in \((\tau, r, \varphi)\) frame. As we see the jump is caused by a 2D object located at \( \Phi = 0 \) with stress tensor
\[ S_{\mu\nu} = T \sqrt{g_2} \text{ diag}(1,0,-1) , \]
where \( g_2 = 16G^2\ell^2(\ell^2 M^2 - J^2) \) is the determinant of the two dimensional \(\tau\varphi\) part of metric \( (10) \) at \( \Phi = 0 \), which is the metric at the junction, and
\[ T = \frac{1}{4\pi G \ell} . \]

Discussion and outlook

We have introduced a new class of locally AdS\(_3\) solutions to AdS\(_3\) Einstein gravity, the O-BTZ black holes. These are BTZ black holes built upon an AdS\(_3\) orientifold projection \( (\text{O-AdS}_3) \) the fixed locus of which is a space-like cylinder. Away from the fixed locus the metric for the O-BTZ is exactly the same as ordinary BTZ and hence they asymptote to O-AdS\(_3\), and are specified by two parameters, ADM mass \( M \) and angular momentum \( J \). The orientifold projection used is chosen such that it keeps the orientation on (conformal) boundary of AdS\(_3\). Having an orientable boundary is important for having a well-defined 2D CFT with a positive definite energy.

As can be seen from the Penrose diagram depicted in Fig.\( \ref{fig:penrose} \), O-BTZ geometries do not have the inner horizon and the region behind it. From the viewpoint of an observer outside the horizon O-BTZ geometry is indistinguishable from an ordinary BTZ and in particular its horizon area is \( 2\pi r_+ \), where \( r_+ \) can be computed in terms of \( M \) and \( J \) using \( (15) \). Therefore, laws of black hole thermodynamics for our solutions is written exactly in the same way as for the ordinary BTZ.

The Brown-Henneaux analysis \( (6) \) only makes use of the asymptotic behavior of diffeomorphisms, that the gravity part of the action is pure AdS\(_3\) Einstein gravity, and is independent of the details of the geometry away from the boundary. Therefore, analysis of \( (6) \) in a straightforward way generalizes to our case and one finds a Virasoro algebra with central charge \( (11) \) for the O-BTZ solutions \( (15) \). This suggests that there is a CFT\(_2\) at the central charge \( (11) \) and that our O-BTZ solutions are states in this CFT.

Many other aspects of the O-AdS\(_3\) and O-BTZ geometries such as geodesic motion in these backgrounds, dis-

\[ \Delta S = \frac{1}{16\pi G} \int dV (R + \frac{2}{\ell^2}) = \frac{1}{\pi} \int d\sigma d\varphi \sqrt{\ell^2 M^2 - J^2} \]

\[ = \int T \sqrt{g_2} d\sigma d\varphi . \quad (16) \]

\footnote{The fact that at the orientifold fixed locus Ricci tensor has a jump is similar to the \( R^2/\mathbb{Z}_k \) orbifold case of the analysis which has been carried out in \( (13) \).}

\footnote{We comment that AdS\(_3\) in Poincaré coordinates, \( ds^2 = \ell^2 u^2 (-dt^2 + dx^2) + \frac{du^2}{u^2} \), \( u > 0 \), which appears in the near horizon limit of D1-D5 system \( (14) \) can be extended in a similar way to beyond \( u^2 > 0 \) region by replacing \( u^2 \) with \( x^2 \). In order to make this a solution to AdS\(_3\) Einstein gravity we need to add an orientifold cylinder with tension \( (14) \) at \( u^2 = 0 \) Note that \( u = 0 \) is a light-like (rather than space-like) direction.}

\footnote{Note that Fig.\( \ref{fig:penrose} \) shows Penrose diagram of O-BTZ geometry in its covering space and that O-BTZ is the region limited to two successive dashed lines.
cussion about the possible dual CFT\textsubscript{2}, extension of proposal made in \cite{16} for the initial (Hartle-Hawking) entangled state to the O-BTZ backgrounds and their relevance for addressing the pure AdS\textsubscript{3} Einstein gravity problems (cf. \cite{5}) will be discussed in upcoming publications.

We would like to thank Vijay Balasubramanian, Jan de Boer, Chethan Krishnan, Kumar S. Narain, Joan Simón, Erik Verlinde for comments and especially Kostas Skenderis for discussions. F.L. would like to thank IPM for hospitality during the course of this project. M.M.Sh-J would like to thank the Abdus-Salam ICTP where this work completed.

\begin{itemize}
  \item Electronic address: loran@cc.iut.ac.ir
  \item Electronic address: jabbari@theory.ipm.ac.ir
\end{itemize}

[1] M. Banados, C. Teitelboim and J. Zanelli, “The Black hole in three-dimensional space-time,” Phys. Rev. Lett. \textbf{69}, 1849 (1992). \texttt{arXiv:hep-th/9204099}.

[2] M. Banados, M. Henneaux, C. Teitelboim and J. Zanelli, “Geometry of the (2+1) black hole,” Phys. Rev. D \textbf{48}, 1506 (1993). \texttt{arXiv:gr-qc/9302012}.

[3] A. R. Steif, “The Quantum stress tensor in the three-dimensional black hole,” Phys. Rev. D \textbf{49}, 585 (1994). \texttt{arXiv:gr-qc/9308032}.

[4] G. Lifschytz and M. Ortiz, “Scalar field quantization on the (2+1)-dimensional black hole background,” Phys. Rev. D \textbf{49}, 1929 (1994). \texttt{arXiv:gr-qc/9310008}.

[5] E. Witten, “Three-Dimensional Gravity Revisited,” \texttt{arXiv:0706.3359 [hep-th]}; A. Maloney and E. Witten, “Quantum Gravity Partition Functions in Three Dimensions,” \texttt{arXiv:0712.0155 [hep-th]}; A. Strominger, “Five Problems in Quantum Gravity,” Nucl. Phys. Proc. Suppl. \textbf{192-193}, 119 (2009). \texttt{arXiv:0906.1313 [hep-th]}.

[6] J. D. Brown and M. Henneaux, “Central Charges in the Canonical Realization of Asymptotic Symmetries: An Example from Three-Dimensional Gravity,” Commun. Math. Phys. \textbf{104}, 207 (1986).

[7] P. Kraus, H. Ooguri and S. Shenker, “Inside the horizon with AdS/CFT,” Phys. Rev. D \textbf{67}, 124022 (2003). \texttt{arXiv:hep-th/0212277}.

[8] V. Balasubramanian, T. S. Levi, “Beyond the veil: Inner horizon instability and holography,” Phys. Rev. D \textbf{70}, 106005 (2004). \texttt{arXiv:hep-th/0405048}; C. Krishnan, “Tomograms of spinning black holes,” Phys. Rev. D \textbf{80}, 126014 (2009). \texttt{arXiv:0911.0597 [hep-th]}.

[9] N. Dukker, B. Fiol and J. Simon, “Goedel’s universe in a supertube shroud,” Phys. Rev. Lett. \textbf{91}, 231601 (2003). \texttt{arXiv:hep-th/0306057}.

[10] O. Coussaert and M. Henneaux, “Self-dual solutions of 2+1 Einstein gravity with a negative cosmological constant,” \texttt{arXiv:hep-th/9407181}; V. Balasubramanian, A. Naqvi and J. Simon, “A multi-boundary AdS orbifold and DLCQ holography: A universal holographic description of extremal black hole horizons,” JHEP \textbf{0408}, 023 (2004). \texttt{arXiv:hep-th/0311237}.

[11] W. Israel, “Singular hypersurfaces and thin shells in general relativity,” Nuovo Cim. B \textbf{44S10}, 1 (1966) [Erratum-ibid. B \textbf{48}, 463 (1967 NUCIA,B44,1,1966)].

[12] R. Mansouri and M. Khorrami, “Equivalence of Darmois-Israel and Distributional-Methods for Thin Shells in General Relativity,” J. Math. Phys. \textbf{37}, 5672 (1996). \texttt{arXiv:gr-qc/9608029}.

[13] D. V. Fursaev and S. N. Solodukhin, “On The Description Of The Riemannian Geometry In The Presence Of Conical Defects,” Phys. Rev. D \textbf{52}, 2133 (1995) \texttt{arXiv:hep-th/9501127}.

[14] J. M. Maldacena, “The large N limit of superconformal field theories and supergravity,” Adv. Theor. Math. Phys. \textbf{2}, 231 (1998) [Int. J. Theor. Phys. \textbf{38}, 1113 (1999)]. \texttt{arXiv:hep-th/9711200}.

[15] F. Loran and M.M. Sheikh-Jabbari, “Orientifolded Locally AdS\textsubscript{3} Geometries,” To appear.

[16] J. M. Maldacena, “Eternal black holes in Anti-de-Sitter,” JHEP \textbf{0304}, 021 (2003). \texttt{arXiv:hep-th/0106112}.