On an evolution system describing self-gravitating particles in microcanonical setting

Robert Stańczy
Instytut Matematyczny, Uniwersytet Wrocławski,
pl. Grunwaldzki 2/4, 50–384 Wrocław, Poland;
stanczr@math.uni.wroc.pl

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Abstract
The global in time existence of solutions of a system describing the interaction of gravitationally attracting particles with a general diffusion term and fixed energy is proved. The presented theory covers the case of the model with diffusion that obeys Fermi–Dirac statistics. Some of the results apply to the dissipative polytropic case as well.

Key words and phrases: Chavanis–Sommeria–Robert model, mean field equations, Fermi–Dirac particles, nonlinear nonlocal parabolic system, local and global solutions.

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1 Introduction

We consider the following initial-boundary value problem

\[ n_t = \nabla \cdot (D (\nabla p(n) + n \nabla \varphi)) \quad \text{in} \quad \Omega \times (0, \infty), \]
\[ \Delta \varphi = n \quad \text{in} \quad \Omega \times (0, \infty), \]
\[ (\nabla p(n) + \nabla \varphi) \cdot \bar{\nu} = \varphi = 0 \quad \text{on} \quad \partial \Omega \times (0, \infty), \]
\[ n(0) = n_0 \geq 0 \quad \text{in} \quad \Omega \subset \mathbb{R}^d. \]
with the pressure in the self-similar form

\[ p(n, \theta) = \theta^{d/2+1} P(n\theta^{-d/2}) \]  

(1.5)

for a given function \( P \) and some diffusion coefficient \( D \), motivated by applications to statistical mechanics and describing self-attracting clouds of particles modelling elliptical galaxies, globular clusters, interstellar medium or cores of neutron stars among others (cf. [12] and references therein). These sorts of problems were considered among others in [5] including numerous pressure formulae coming from statistical mechanics: Maxwell–Boltzmann, Fermi–Dirac, Bose–Einstein and polytropic distributions. The common feature of all these examples is the self-similar profile of the pressure (1.5). In this paper we focus our attention on the Fermi–Dirac model although we formulate the results in a more general setting. The pressure in this model assumes an intermediate form between the linear Maxwell–Boltzmann case at zero and a polytropic, power–like form at infinity. In [3] the authors proved the local and global existence for the specific choice of the diffusion parameter \( D \) corresponding to the Fermi–Dirac statistics in the isothermal case, i.e., with fixed, constant temperature \( \theta \). Moreover, the asymptotic behaviour with the possibility of the evolution towards steady states was addressed therein (cf. also [23]), whereas in [4] some results for nonisothermal case were established. In the aforementioned papers, some \textit{a priori} estimates for the density \( n \) and the pressure \( P \) were also provided. For physical motivations one can see the series of papers of Chavanis and collaborators including [7]–[15]. Related Keller–Segel model in mathematical biology was recently studied in [18], [26] and [27] and the blow-up for large data was proved.

Note that, as a consequence of (1.3), total mass

\[ M = \int_{\Omega} n(x, t) \, dx \]  

(1.6)

is conserved during the evolution of the system.

In the first part of the paper we will extend the results of [3] to allow more general pressure and nonconstant temperature for the dimension \( d \in [2, 4] \) with small mass \( M \). The local and global results for the parabolic perturbation is contained in Proposition 2.1 and the result for the original
parabolic-elliptic problem can be found in Theorem 3.1. As far as steady states are concerned nonexistence results hold for \( d > 2(1 + \sqrt{2}) \) (cf. [21]) we can expect that global existence result holds only for the dimension \( d \leq 4 \) when global minimizer for the entropy functional is attained as proven in [21], [25] or [3]. In fact analogous nonexistence results also hold for a problem related to (1.1)–(1.4) but with constant diffusion parameter \( D \) (cf. [3]). Thus one can conjecture that the gap \( d \in (4, 2(1 + \sqrt{2})) \) is left for the existence of the critical points (possibly unstable) of another type than the extremal ones.

Next, we shall use the aforementioned existence theorems for a given temperature \( \theta(t) \) at time \( t \) to prove the existence theorem 4.8 in the microcanonical (nonisothermal) setting, i.e. with the given energy and the temperature to be determined so that the energy relation (1.12) is satisfied. Steady states for the model were considered, among others, in [22]. Thus, we will show that in low dimensions \( 2 \leq d \leq 4 \) for small mass and domination of the thermal energy a gravo-thermal catastrophe (white dwarfs in a physical interpretation) does not occur for this system, i.e., neither blow-up for the density nor the vanishing of the temperature takes place.

Finally, in Appendix, we gather the properties of some special functions appearing in the Fermi–Dirac model.

First, notice that due to the self-similar structure of the pressure (1.5) for the specific canonical diffusion coefficient \( D = P' \) the system (1.1)–(1.4) can be transformed to the following one (cf. also the Appendix for Fermi–Dirac case and the papers [12], [3], [4] where such \( D \) was used). Thus we arrive at the system

\[
\begin{align*}
  n_t &= \nabla \cdot \left( \theta P'^2 \nabla n + n P' \nabla \varphi \right) \quad \text{in} \quad \Omega \times (0, \infty), \\
  \Delta \varphi &= n \quad \text{in} \quad \Omega \times (0, \infty), \\
  \left( \theta P'^2 \nabla n + n P' \nabla \varphi \right) \cdot \bar{\nu} &= \varphi = 0 \quad \text{on} \quad \partial \Omega \times (0, \infty), \\
  n(0) &= n_0 \geq 0 \quad \text{in} \quad \Omega \subset \mathbb{R}^d,
\end{align*}
\]

where we suppose that \( 2 \leq d \leq 4 \), the temperature is a fixed continuous function \( \theta : [0, \infty) \rightarrow [a, b] \), with some positive numbers \( a \) and \( b \) with values to be determined later. Morover, we look for the solutions of (1.7)–(1.10)
satisfying the energy relation given by
\[ E = \frac{d}{2} \int_{\Omega} \theta^{d/2+1} P(n \theta^{-d/2}) \, dx + \frac{1}{2} \int_{\Omega} n \phi \, dx = \text{const.} \quad (1.11) \]

The steady state problem with the prescribed energy for the linear diffusion was considered, among others, in [2]. The main results of this paper (to be specified in the next sections) can be stated as follows.

**Theorem 1.1** If \( P \) is the Fermi–Dirac pressure and mass \( M > 0 \) is sufficiently small the problem (1.7)-(1.10) admits at least one global weak solution for \( d \leq 3 \) and a local one if \( d = 4 \) for a given continuous function \( \theta(t) \). Moreover, there exists a local weak solution to the problem (1.7)-(1.10) with the energy given by (1.11) for \( d \leq 4 \).

Now we shall sketch the method of proving the above theorem.

First, we regularize the problem to obtain a parabolic system and to apply general Amann theory. Next sign–sensitive a priori bounds together with a bootstrap argument are used to prove global or local existence depending on the dimension \( d \). Then we go with the parameter to infinity and obtain the corresponding existence result in weak sense for the original elliptic–parabolic system.

We introduce a new temperature, call it \( \vartheta \), defined implicitly by the aforementioned energy relation, i.e.
\[ E = \frac{d}{2} \int_{\Omega} \vartheta^{d/2} P(n \vartheta^{-d/2}) \, dx + \frac{1}{2} \int_{\Omega} n \phi \, dx = \text{const.} \quad (1.12) \]

Note the implicit dependence of the ‘new’ temperature \( \vartheta \) on the old one \( \theta \) via \( n, \varphi \) (solving (1.7)-(1.10) for given \( \theta \)) in the above formula. In sections 2 and 3, for given \( \theta \), we solve (1.7)-(1.10) to get \( n, \varphi \). Then, in section 4 for given value of the energy \( E \), we use the implicit formula (1.12) for \( \vartheta \) and ask whether the operator \( \mathcal{T} : \theta \mapsto \vartheta \) defined by (1.12) has a fixed point. The problem of a priori bounds for the temperature \( \theta \), determined by (1.11), was addressed in [3]. In the last section the properties of the special Fermi–Dirac pressure function have been gathered.

**Notation.** By \( C \) we will denote inessential constants, which may vary from one line to another. By \( | \cdot |_p \), for \( p \geq 1 \), we shall denote the standard \( L^p(\Omega) \)
norm. By smoothness we shall always mean $C^2$ regularity and it will apply only to the function $P$ and is explicitly stated at the beginning of the next section. Finally, both $\| \cdot \|_{H^1}$ and $\| \cdot \|_*$ will denote the norm in the Sobolev space $H^1(\Omega)$ with $L^2(\Omega)$ and $L^1(\Omega)$ term correspondingly.

2 The existence result for the perturbation

In this section we follow the lines of the proof of the existence addressed in [3], where the authors considered a specific Fermi–Dirac density $P = P_{FD}$ defined by (5.87), dimension $d = 3$ and a constant temperature $\theta$, whereas here we will just exploit smoothness of the pressure $P \in C^2([-\delta, \infty); [0, \infty))$, the crucial estimates: $a \leq \theta(t) \leq b$ with some $a > 0, b > 0$, and for $z \geq 0$,

$$\max\{ p_0, p_1 z^{2/d} \} \leq P'(z) \leq p_2(1 + z^{2/d}), \quad (2.13)$$
$$zP''(z) \leq p_3(1 + z^{2/d}). \quad (2.14)$$

These assumptions imply that, changing $p_2$ if necessary,

$$\max\{ p_0 z, p_1 z^{1+2/d} \} \leq P(z) \leq p_2(1 + z^{1+2/d}), \quad (2.15)$$
$$zP'(z) \leq CP(z), \quad (2.16)$$
$$zP''(z) \leq CP'(z). \quad (2.17)$$

In order to study the well-posedness of (1.7)–(1.10), for $k \geq 1$ and $P' = P'(n\theta^{-d/2})$, we consider the following regularized initial-boundary value problem

$$n_t = \nabla \cdot (\theta P'^2 \nabla n + nP' \nabla \varphi) \quad \text{in} \quad \Omega \times (0, \infty), \quad (2.18)$$
$$\varphi_t - k \Delta \varphi = -k n \quad \text{in} \quad \Omega \times (0, \infty), \quad (2.19)$$
$$(\theta P'^2 \nabla n + nP' \nabla \varphi) \cdot \vec{n} = 0 \quad \text{on} \quad \partial \Omega \times (0, \infty), \quad (2.20)$$
$$(n(0), \varphi(0)) = (n_0, \varphi_0) \quad \text{in} \quad \Omega \subset \mathbb{R}^d, \quad (2.21)$$

with

$$n_0 \in C^\infty(\bar{\Omega}) \quad \text{such that} \quad n_0 \geq 0, \quad M = |n_0|_1, \quad \text{and} \quad \varphi_0 = 0. \quad (2.22)$$
For this parabolic system we first use the theory developed by Amann [1] to prove the local well-posedness of (2.18)–(2.21) and then the global one. The proposition formulated below and its proof has been adapted from [3] to cover the case of variable temperature \( \theta(t) \), slightly more general pressure \( P(t) \) than the Fermi–Dirac one, and any dimension \( 2 \leq d \leq 4 \).

**Proposition 2.1** If \( d \geq 2 \) and the function \( P \) is smooth and satisfies \( P' \geq p_0 > 0 \) and the temperature is continuous and satisfies \( \theta(t) \geq a \) then the initial-boundary value problem (2.18)–(2.21) has a unique maximal classical solution

\[
(n, \varphi) \in C(\overline{\Omega} \times [0, T_{max}); \mathbb{R}^2) \cap C^2(\overline{\Omega} \times (0, T_{max}); \mathbb{R}^2)
\]

for some \( T_{max} \in (0, \infty] \). In addition, \( n(t) \geq 0 \) for \( t \in [0, T_{max}) \).

Furthermore, \( T_{max} = \infty \) if there are \( \varepsilon > 0 \) and a locally bounded function \( \omega : [0, \infty) \to [0, \infty) \) such that, for every \( T > 0 \), the estimate holds

\[
\|n(t)\|_{C^\varepsilon} + \|\varphi(t)\|_{C^\varepsilon} \leq \omega(T) \quad \text{for} \quad t \in [0, T_{max}) \cap [0, T].
\] (2.23)

It is the case for \( d \leq 3 \) if \( P \) additionally satisfies (2.13)–(2.14) and (2.27) with \( |R'(z)|z^{1/2-1/d} \leq B \) and \( \theta(t) \leq b \) for some constants \( b > 0, B > 0 \). For \( d = 4 \) we claim only the local existence result since the bootstrap argument does not yield (2.23).

**Proof.** We set \( D_0 = (-\delta, \infty) \times \mathbb{R}, u = (n, \varphi) \) with \( u_0 = (n_0, \varphi_0) \), and by the assumptions, define \( a \in C^2(D_0; \mathcal{M}_2(\mathbb{R})) \) and \( f \in C^2(D_0; \mathbb{R}^2) \) by

\[
a(u) = \begin{pmatrix}
\theta \left( P'(n\theta^{-d/2}) \right)^2 & nP'(n\theta^{-d/2}) \\
0 & k
\end{pmatrix}, \quad f(u) = \begin{pmatrix}
0 \\
-k \ n
\end{pmatrix}.
\]

Next, for \( v \in D_0 \), we introduce the operators

\[
\mathcal{A}(v)u = -\sum_{i=1}^d \sum_{j=1}^d \partial_i (a_{ij}(v) \partial_j u),
\]

\[
\mathcal{B}(v)u = b \sum_{i=1}^d \sum_{j=1}^d \nu_i (a_{ij}(v) \partial_j u) + (I_2 - b) \ u,
\]
where $a_{ij}(v) = a(v) \delta_{ij}$, $1 \leq i, j \leq d$, and

$$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$  

Then, an abstract formulation of (2.18)–(2.21) reads

\begin{align*}
  u_t + \mathcal{A}(u)u &= f(u), & (2.24) \\
  \mathcal{B}(u)u &= 0, & (2.25) \\
  u(0) &= u_0. & (2.26)
\end{align*}

Thanks to the strict positivity of $P'(z) \geq p_0$ and the lower bound for the temperature $\theta(t) \geq a$, the eigenvalues of the matrix $a(v)$ are positive for each $v \in D_0$, and the boundary-value operator $(\mathcal{A}, \mathcal{B})$ is of separated divergence form and is normally elliptic in the sense of [1, Section 4] Therefore we may apply [1, Theorem 14.4 and Theorem 14.6] to conclude that, for some $T_{\text{max}} \in (0, \infty]$, (2.24)–(2.26) has a unique maximal classical solution

$$u = (n, \varphi) \in C(\bar{\Omega} \times [0, T_{\text{max}}); D_0) \cap C^2(\bar{\Omega} \times (0, T_{\text{max}}); D_0).$$

Also, since $n_0 \geq 0$ and the first component of $f(u)$ is equal to zero, the comparison principle (see, e.g., [1, Theorem 15.1] or [19, Corollary I.2.1]) implies that $n(t) \geq 0$ for $t \in [0, T_{\text{max}})$. Furthermore, since $f$ does not depend on $\nabla u$ and $n \geq 0$, Theorem 15.3 in [1] ensures that $T_{\text{max}} = \infty$ if there are $\varepsilon > 0$ and a locally bounded function $\omega : [0, \infty) \to [0, \infty)$ such that (2.23) holds true for every $T > 0$. The fact that the assumptions imposed on $P$ guarantee (2.23) requires some preparatory lemmas and is postponed until the end of this section. \hfill \Box

We proceed to present a series of lemmas which will guarantee that (2.23) is satisfied for $2 \leq d \leq 3$ and thus ascertain the global solvability of the perturbed problem. We recall after [5] that the neg-entropy functional $\mathcal{W}$

$$\mathcal{W} = \int_{\Omega} \left( nH - \left( \frac{d}{2} + 1 \right) P\theta^{d/2} \right) dx$$

plays the role of a Lyapunov functional for the original and regularized problem. The function $H(z)$ depending on $z = n\theta^{-d/2}$ is a primitive of $P'(z)/z$.  

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However, in our case this functional is not useful for \textit{a priori} estimate of the density $n$ (contrary to isothermal case \cite{3}) as due to (2.27) it is of too low order in $n$. Indeed, the order is $1 - 2/d$ for the Fermi–Dirac case to be exact (cf. \cite[Lemma 3.6]{4}), and as such does not provide any reasonable \textit{a priori} estimates for the density $n$. On the other hand, it can be used to get \textit{a priori} bounds for the fixed points of the temperature operator $T$ as was done in \cite{5} and is presented in section 4. In the isothermal ($\theta = \text{const}$) case a crucial $L^{1+2/d}$ bound was obtained from the fact that the entropy (other than $\mathcal{W}$) was coercive in this space. As one can see it is not the case for $\mathcal{W}$. For the details of the nontrivial derivation of the entropy $\mathcal{W}$ one can see \cite{5} and for its application to get \textit{a priori} bounds for the temperature - \cite{6} and \cite{24}.

Now, we are going to formulate analogous results to the ones presented in \cite{24} where \textit{a priori} bounds for the limit parabolic-elliptic system, as $k \to \infty$, were obtained.

**Lemma 2.2** Assume that, for $d \geq 2$,

$$P(z) = p_1 z^{1+2/d} + R(z), \quad (2.27)$$

where the lower order term satisfies $|R'(z)| z^{1/2 - 1/d} \leq B$. Then, for any fixed $T > 0$ and any $t \in [0, T] \cap [0, T_{\text{max}})$, the following growth condition holds

$$\frac{d}{dt} \left( \frac{d}{2} \int_\Omega p_1 n^{1+2/d} + \int_\Omega n \varphi + \frac{1}{2} \int_\Omega |\nabla \varphi|^2 \right) + \frac{1}{k} \int_\Omega \varphi_t^2 \leq C \theta^{d/2} \int_\Omega |\nabla \varphi|^2. \quad (2.28)$$

**Proof.** Let $t \in [0, T] \cap [0, T_{\text{max}})$ and recall that both $P$ and $P'$ are the functions of $n \theta^{-d/2}$. Now, we multiply (2.18) by $\frac{d}{2} n^{2/d}$ and integrate over $\Omega$ to obtain

$$\frac{d^2}{2(d+2)} \frac{d}{dt} \int_\Omega n^{1+2/d} dx = -\theta \int_\Omega P'^2 |\nabla n|^2 n^{2/d-1} dx - \int_\Omega P' n^{2/d} \nabla n \cdot \nabla \varphi dx. \quad (2.18)$$

Similarly, multiplying (2.18) by $A \varphi$, we get

$$A \int_\Omega n t \varphi dx = -A \int_\Omega P' n |\nabla \varphi|^2 dx - A \theta \int_\Omega P'^2 \nabla n \cdot \nabla \varphi dx. \quad (2.18)$$



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Summing up the above equalities and using the Hölder inequality

\[
\left| \int_{\Omega} P' \theta^{1/2} n^{1/d-1/2} \nabla n \cdot \nabla \varphi \left( \theta^{-1/2} n^{1/d+1/2} + A P' \theta^{1/2} n^{1/d+1/2} \right) \right| \\
\leq \int_{\Omega} P'^2 n^{2/d-1} |\nabla n|^2 + \frac{1}{4} |\nabla \varphi|^2 \left( \theta^{-1/2} n^{1/d+1/2} + A P' \theta^{1/2} n^{1/d+1/2} \right)^2 ,
\]

taking \( A(d + 2)p_1 = d \) and

\[
\int_{\Omega} n \varphi \ dx = \frac{d}{dt} \left( \int_{\Omega} \frac{n}{2} |\nabla \varphi|^2 \ dx \right) + \frac{1}{k} \int_{\Omega} \varphi^2 \ dx,
\]
we arrive at

\[
\frac{d}{dt} \left( \frac{d}{2} \int_{\Omega} p_1 n^{1+2/d} \ dx + \int_{\Omega} n \varphi \ dx + \frac{1}{2} \int_{\Omega} |\nabla \varphi|^2 \ dx \right) + \frac{1}{k} \int_{\Omega} \varphi^2 \ dx \\
\leq \frac{1}{4A} \int_{\Omega} |\nabla \varphi|^2 \left( A P' \theta^{1/2} n^{-1/d+1/2} - \theta^{-1/2} n^{1/d+1/2} \right)^2 .
\]

This yields the claim, with \( C = \frac{dB^2}{4(d+2)p_1} \), from the assumption on \( R' \) applied to the differentiated pressure \( P'(z) = (p_1(d + 2)/d) z^{2/d} + R'(z) \).

**Remark.** Note that the above theorem holds both in the polytropic case with \( R(z) = 0 \) and, less obviously, in the Fermi–Dirac case as explained below. Indeed, by the properties of Fermi functions (cf. Lemma 5.6 from the Appendix or for more properties see [4, Sec.5]) we get \( |R'(z)| \leq B \) at \( z = \infty \) and \( |R'(z)| \leq B \) at \( z = 0 \), implying the required estimate if \( 2 \leq d \leq 6 \).

Next lemmas will allow us to estimate the right hand side of (2.28).

**Lemma 2.3** For any \( 2 \leq d \leq 4 \) we have

\[
\left| \int_{\Omega} n \varphi \ dx \right| \leq CM^{1/2-1/d} \left( \int_{\Omega} n^{1+2/d} \ dx + \int_{\Omega} |\nabla \varphi|^2 \ dx \right). \tag{2.29}
\]

**Proof.** The proof of (2.29) involves standard Hölder and Sobolev–Gagliardo–Nirenberg inequalities as follows

\[
\left| \int_{\Omega} n^{1/2-1/d} n^{1/2+1/d} \varphi \ dx \right| \leq M^{1/2-1/d} \left( \int_{\Omega} n^{2d/4} \ dx \right)^{\frac{4d}{4d-2}} \\
\leq M^{1/2-1/d} |n|^{\frac{4d}{4d-2}} |\varphi|^{\frac{4d}{4d-2}} \leq CM^{1/2-1/d} \left( \int_{\Omega} n^{1+2/d} \ dx + \int_{\Omega} |\nabla \varphi|^2 \ dx \right).
\]
due to the inequality $\frac{d+2}{d} \leq \frac{d+2}{d}$ and the fact that $H^1_0(\Omega)$ can be imbedded in $L^\frac{2d}{d-2}(\Omega)$. The proof of the case $d = 2$ is straightforward by the Poincaré inequality.

In low dimensions a similar argument leads to another estimates (cf. [24]).

**Lemma 2.4** For $d = 2$ we have

$$\left| \int_\Omega n_1 \varphi \, dx \right| \leq C \int_\Omega n^2 \, dx ,$$

(2.30)

while for $d = 3$ the estimate

$$\left| \int_\Omega n_1 \varphi \, dx \right| \leq CM^{7/3} + \frac{d}{2} \int_\Omega n^{5/3} \, dx ,$$

(2.31)

holds.

Now, we are ready to deduce the following lemma on *a priori* estimates.

**Lemma 2.5** Assume that $2 \leq d \leq 4$, condition (2.27) holds for a smooth function $P$ satisfying $|R'(z)| \leq Bz^{1/d-1/2}$ and the temperature is bounded from above $\theta(t) \leq b$. Then for any $t \in [0, T] \cap [0, T_{\text{max}})$ and sufficiently small data, i.e. mass $M$ if $2 < d \leq 4$ or the Poincaré constant for $d = 2$, we have

$$\frac{d}{2} \int_\Omega p_1 n^{1+2/d} \, dx + \int_\Omega n_1 \varphi \, dx + \frac{1}{2} \int_\Omega |\nabla \varphi|^2 \, dx + \frac{1}{k} \int_0^t \int_\Omega \varphi^2 \, dx \, ds \leq C .$$

(2.32)

Moreover, each of the terms appearing on the left hand side of the above inequality is bounded and the constant $C$ may depend on the initial data. If $d = 3$ the assumption on the smallness of $M$ can be relaxed due to Lemma 2.4.

**Proof.** Starting with the direct consequence of Lemma 2.3 true for sufficiently small mass $M$ and large $C$ (if $d = 2$, instead of making mass $M$ small, we have to assume that the constant from the Poincaré inequality is smaller than 2),

$$\int_\Omega |\nabla \varphi|^2 \, dx \leq C \left( \frac{d}{2} \int_\Omega p_1 n^{1+2/d} \, dx + \int_\Omega n_1 \varphi \, dx + \frac{1}{2} \int_\Omega |\nabla \varphi|^2 \, dx \right)$$

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we plug this into (2.28) and integrate with respect to time to arrive, with possibly a larger $C$ dependent on the initial value of the right hand side of the above inequality, at

$$\frac{d}{2} \int_\Omega p_1 n^{1+2/d} dx + \int_\Omega n \varphi dx + \frac{1}{2} \int_\Omega |\nabla \varphi|^2 dx \leq C \exp \left( \int_0^t C \theta(s)^{d/2} ds \right)$$

which together with an upper bound on $\theta$ ends the proof. Note that (2.29) from Lemma 2.3 shows that the negative term $\int_\Omega n \varphi dx$ is dominated by the positive ones and thus the last claim of the lemma is ascertained. $\square$

The integral version of the estimate (2.28) from Lemma 2.2 follows by an argument similar to the one in the proof of Lemma 2.5 and reads

$$V(t) \leq V(0) \exp \left( C \int_0^t \theta(s)^{d/2} ds \right), \quad (2.33)$$

where $V(t) \overset{df}{=} \frac{d}{2} \int_\Omega p_1 n^{1+2/d} + \int_\Omega n \varphi + \frac{1}{2} \int_\Omega |\nabla \varphi|^2$. Note that for $d = 3$ we can add $CM^{7/3}$ in the definition of $V$ and thus relax the assumption on smallness of mass $M$.

Now, we state similarly as in [3], where only three–dimensional case was treated, a lemma on the improved regularity of $\nabla \varphi$.

**Lemma 2.6** Let $q, \alpha \in (1, \infty)$, $d \geq 2$ and $T > 0$. There is a constant $C$ depending on $q, \alpha, d$ and $T$ such that, for $t \in [0, T_{\text{max}}) \cap [0, T]$,

$$\int_0^t |\nabla \varphi(s)|_q^q ds \leq C |n|_q^{L^q(0,t;L^\frac{d\alpha}{d\alpha+\alpha}(\Omega))}. \quad (2.34)$$

**Proof.** We infer from [17, Corollaire 1.1], as in [3] where for $\alpha = \frac{d(d+2)}{d^2-d-2}$ and $d = 3$ the authors used the bound with $d\alpha/(d+\alpha) = 1 + 2/d$ norm of $n$, that

$$\frac{1}{k} |\varphi_t|_{L^q(0,t;L^{\frac{d\alpha}{d\alpha+\alpha}}(\Omega))} + |\Delta \varphi|_{L^q(0,t;L^{\frac{d\alpha}{d+\alpha}}(\Omega))} \leq C |n|_{L^q(0,t;L^{\frac{d\alpha}{d+\alpha}}(\Omega))}. \quad (2.35)$$

Now, if $t \in [0, T_{\text{max}}) \cap [0, T]$, we get from the above inequality

$$\int_0^t \|\varphi(s)\|_{W^{2, \frac{d\alpha}{d+\alpha}}(\Omega)}^q ds \leq C |n|_{L^q(0,t;L^{\frac{d\alpha}{d+\alpha}}(\Omega))}^q.$$

To conclude we use the imbedding of $W^{2, \frac{d\alpha}{d+\alpha}}(\Omega)$ in $W^{1,\alpha}(\Omega)$. $\square$

Furthermore, an $L^2$-estimate is available for $n$. 11
Lemma 2.7 Let $T > 0$, $2 \leq d \leq 4$, $P$ be smooth and satisfy (2.13) and (2.27). There are constants $c, C > 0$ depending on sufficiently small mass $M$, bounds on $\theta(t) \in [a, b]$ and $P$ and the initial data such that, for $t \in [0, T_{\max}) \cap [0, T],$

$$|n(t)|^2_2 + c \int_0^t |\nabla n(s)|^2_2 \, ds \leq C$$

(2.35)

In fact, a constant $C$ is a function of the integral $\int_0^t |n(s)|_{1+2/d} \, ds$, locally bounded in $t$, which can be estimated by constant due to (2.32).

Proof. Note that the estimate of the first term in (2.35) follows from Lemma 2.5 if $d = 2$. Let $t \in [0, T_{\max}) \cap [0, T]$ and multiply (2.18) by $2n$, and integrate over $\Omega$ to obtain

$$\frac{d}{dt}|n(t)|^2_2 + 2\theta \int_{\Omega} P'' |\nabla n|^2 \, dx = -2 \int_{\Omega} n P' \nabla n \cdot \nabla \varphi \, dx .$$

Next, we have

$$2 \left| \int_{\Omega} n P' \nabla n \cdot \nabla \varphi \, dx \right| \leq \theta \int_{\Omega} P'' |\nabla n|^2 \, dx + \frac{1}{\theta} \int_{\Omega} n^2 |\nabla \varphi|^2 \, dx$$

by the Young inequality, whence

$$\frac{d}{dt}|n(t)|^2_2 + \theta \int_{\Omega} P'' |\nabla n|^2 \, dx \leq \frac{1}{\theta} \int_{\Omega} n^2 |\nabla \varphi|^2 \, dx .$$

(2.36)

For $d = 2$, by Lemma 2.6 and Lemma 2.5 we deduce that

$$|\nabla \varphi|_\infty \leq C$$

(2.37)

and that the integrated with respect to the time variable the right hand side of (2.36) is bounded. Thus the estimate (2.35) is proved in this case. If $d \geq 3$ a longer argument is required. Namely, it follows from the Hölder and Young inequalities that for any $\varepsilon > 0, \alpha > 2$ and some $C = C_{\varepsilon}$

$$\int_{\Omega} n^2 |\nabla \varphi|^2 \, dx \leq |n^2|_{\alpha/(\alpha - 2)} |\nabla \varphi|^2_\alpha \leq \frac{\varepsilon P_0^2}{3} |n^2|_{\frac{\alpha(d+6)}{\alpha+2}} + C |\nabla \varphi|^2_{\frac{2\alpha(d+6)}{6\alpha-2d}} .$$

Then, interpolating with positive $\beta = \frac{4(2\alpha-d-2)}{d(\alpha+2)}$, we get

$$|n^2|_{\frac{\alpha(d+6)}{\alpha+2}} \leq M^\beta |n^{1+2/d}|_{2d/(d-2)}^2$$

(2.38)
and \( P(z) \geq p_1 z^{1+2/d} \), or precisely \( \theta^{d/2+1} P(n \theta^{-d/2}) \geq p_1 n^{1+2/d} \), implies that

\[
\begin{align*}
\int_{\Omega} n^2 |\nabla \varphi|^2 \, dx & \leq \frac{\varepsilon}{3} M^\beta \theta^{d/2+1} P_1^2 |\nabla \varphi|_{\theta^{2a(d+6)/(6a-d)}}^2 + C |\nabla \varphi|_{\alpha}^{2a(d+6)/(6a-d)} \tag{2.39} \\
& \leq \frac{1}{3} M^\beta \|\theta^{d/2+1} P\|_{\alpha}^2 + C |\nabla \varphi|_{\alpha}^{2a(d+6)/(6a-d)}, \tag{2.40}
\end{align*}
\]

where the last inequality follows from the continuous imbedding of \( H^1(\Omega) \) in \( L^{2d/(d-2)}(\Omega) \) with a constant \( \varepsilon^{-1} \) and the norm

\[
\|z\|_*^2 = \int_{\Omega} |\nabla z|^2 \, dx + \left( \int_{\Omega} |z| \, dx \right)^2. \tag{2.41}
\]

Consequently, by (2.36), it follows that, for \( c = 1 - \frac{1}{3} M^\beta \),

\[
dt |n(t)|_2^2 + c \int_{\Omega} \theta P^2 |\nabla n|^2 \, dx \leq \frac{C}{\theta} \left( |\theta^{d/2+1} P|_1^2 + |\nabla \varphi|_{\alpha}^{2a(d+6)/(6a-d)} \right). \tag{2.42}
\]

Now, integration with respect to time, the assumption on growth of \( P \) and Lemma 2.6 with \( q = \frac{2a(d+6)}{6a-d} \), \( \alpha = 1+d/2 \), \( \beta = 0 \), for \( 2 < d < (3 + \sqrt{17})/2 \) (including \( d = 3 \)) yields the estimate by the time integral of \( |n|_{1+2/d} \). Finally, due to Lemma 2.5 providing a bound for \( |n|_{1+2/d} \), the estimate (2.35) is proven for any mass \( M \).

Now, allowing higher dimensions \( 2 < d < 2(1 + \sqrt{2}) \) (including \( d = 4 \)) we use \( q = \frac{2a(d+6)}{6a-d}, \alpha = \frac{d(d+2)}{d^2-d-2} \geq 2, \beta = \frac{d^2+4d+1}{d}, d \alpha/(d + \alpha) = 1 + 2/d, \) to get the estimate by the time integral of \( |n|_{1+2/d} \) but this time for small mass only. To get a bound for \( |n|_{1+2/d} \), assumptions have to be more restrictive, e.g. Lemma 2.5 requires \( 2 \leq d \leq 4 \).

In fact, we have obtained the estimate

\[
|n(t)|_2^2 + c \int_0^t \theta P^2 |\nabla n(s)|_2^2 \, ds \leq C, \tag{2.43}
\]

which due to the estimate \( P' \geq p_0 \) implies (2.35) \( \square \).

**Proof of the global existence part of Proposition 2.1** We are now ready to prove (2.23) and thus obtain the global existence. Let \( T > 0 \) and \( t \in [0, T_{\max}) \cap [0, T] \). We claim that there is \( C > 0 \) depending on \( n_0 \) and \( T \) and bounds on \( \theta \) such that

\[
|nP'|_{L^{2(d+4)/(d+2)}(\Omega \times (0,t))} + |\nabla (nP')|_{L^2(\Omega \times (0,t))} \leq C. \tag{2.44}
\]
Indeed, we infer from assumption (2.13)-(2.17) that $zP'(z) \leq C(1+z^{1+2/d})$ and $zP''(z) \leq C(1+z^{2/d})$ for $z \geq 0$. Consequently, 

$$\sup_{s \in [0,t]} |n(s)P'(s)| \frac{dt}{s^2} + \int_0^t \|n(s)P'(s)\|^2_{H^1} ds \leq C \tag{2.45}$$

holds. Next, we use the continuity of the imbedding of $H^1(\Omega)$ in $L^{2d/(d-2)}(\Omega)$ and an interpolation argument to deduce (2.44).

We now employ a bootstrap argument to show that (2.23) holds true. It follows from (2.15) and the Sobolev imbedding that

$$|n|^{(d+2)/d}_{L^{2+4/d}(0,t;L^{2+4/d}(d+2)/(d-2))} \leq C|\theta^{d/2+1}P|^{(d+2)/d}_{L^{2+4/d}(d+2)/(d-2)}(\Omega) \leq C,$$

which, together with (2.35), leads to

$$\int_0^t |n(s)|^{2+8/d}_{2+8/d} ds \leq \int_0^t |n(s)|^{2+4/d}_{2(d+2)/(d-2)} |n(s)|^{4/d}_{2} ds \leq C.$$

Therefore,

$$|n|^{2+8/d}_{L^{2+8/d}(0,t;\Omega \times (0,t))} \leq C,$$

and we infer from [19, Theorem IV.9.1 and Lemma II.3.3] that

$$|\nabla \varphi|^{(d+2)/(d+4)}_{L^{d+2}(\Omega \times (0,t))} + |\Delta \varphi|^{(d+8)/(d+4)}_{L^{d+2}(\Omega \times (0,t))} \leq C.$$

This estimate and (2.44) ensure that

$$|\nabla (nP') \cdot \nabla \varphi|^{(d+2)/(d+4)}_{L^{d+2}(\Omega \times (0,t))} + |nP' \Delta \varphi|^{(d+4)/(d+2)}_{L^{d+2}(\Omega \times (0,t))} \leq C.$$

Since

$$n_t - \theta \nabla (P^2 \nabla n) = n P' \nabla \varphi + n P' \Delta \varphi; \tag{2.46}$$

we use once more [19, Theorem IV.9.1] to obtain that

$$\|n\|_{W^{2,1}_{d+4/(d+1)}(\Omega \times (0,t))} \leq C,$$

which, in turn, implies that $n \in L^{(d+2)/(d+6-6)}(\Omega \times (0,t))$. With thus improved $n$ we would like to bootstrap once again. The right hand side of (2.46) is in the space $L^q(\Omega \times (0,t))$, with

$$q = d(d+2)(d+4) \min \left\{1/(d^2 + 4d^2 - 12), 1/(2(d+1)(d+3)(d-2)) \right\}.$$
Therefore, for $d = 3$ we finally get the right hand side of (2.46) in $L^q(\Omega \times (0, t))$ with $q = 35/12$ larger than critical $1 + 3/2$ allowing to conclude with

$$\|n\|_{C^\epsilon([0, t])} \leq C,$$

for some $\epsilon > 0$ by [19, Lemma II.3.3]. However, for $d = 4$ one should note that we have obtained from the bootstrap the integrability of the right hand side of (2.46) of the order $q = 48/35$ which is less than we had before, i.e. $q = 8/5$. Thus for $d = 4$ we cannot conclude with the estimate (2.47). □

3 The local existence result for the original elliptic–parabolic problem

In this section we shall subtract a convergent subsequence of solutions to (2.18)–(2.21) obtained in the previous section which will guarantee the following existence result for the limiting problem (1.1)–(1.4) as $k \to \infty$.

**Theorem 3.1** Assume that $M$ is small enough if necessary, and $P$ is smooth and satisfies (2.13)–(2.14) and (2.27). Moreover, let $2 \leq d \leq 4$, $n_0 \in L^2(\Omega)$ and, for given constants $0 < a < b$, $\theta \in C(0; T; [a, b])$. Then there exist a weak local-in-time solution $n \in C(0, T; L^2(\Omega))$, $\varphi \in L^\infty(0, T; H^2(\Omega))$, $\theta^{d/2+1}P \in L^2(0, T; H^1(\Omega))$ of the system (1.7)–(1.10), i.e.

$$\int_\Omega (n - n_0) \chi \, dx + \int_0^t \int_\Omega \nabla \chi \cdot (\theta P^2 \nabla n + n P' \nabla \varphi) \, dx ds = 0,$$

$$\Delta \varphi = n, \quad \varphi = 0 \text{ on } \partial \Omega,$$

for each test function $\chi \in W^{1, 2d/(d-2)}(\Omega)$. Additionally,

$$|n(t)|_{1+2/d} + \|\varphi(t)\|_{H^1} \leq C,$$

$$|n(t)|_2 + \int_t^{t+1} \|\theta^{d/2+1}(s)P(s)\|_{H^1}^2 \, ds \leq C,$$

for any $t \in [0, T)$ where $C$ depends on $n_0$, $\Omega$, $d$, $a$ and $b$. If $d \leq 3$ then the global result can be claimed.
Proof. We follow the lines of the proof from [3], where $d = 3$ and a constant temperature $\theta$ were assumed. We consider $n_0 \in L^2(\Omega)$ such $n_0 \geq 0$ a.e. in $\Omega$ and put $M = |n_0|_1$ (sufficiently small if necessary). Let $(n_{0,k})_{k \geq 1}$ be a sequence of nonnegative functions in $C^\infty(\bar{\Omega})$ approximating $n_0$, i.e.,

$$|n_{0,k}|_1 = M \quad \text{and} \quad \lim_{k \to \infty} |n_{0,k} - n_0|_2 = 0. \quad (3.52)$$

For $k \geq 1$, we denote by $(n_k, \varphi_k)$ the unique classical solution to (2.18)–(2.21) with initial datum $(n_{0,k},0)$ given by Theorem 2.1 and let $P_k = P(n_k\theta^{d/2})$. Owing to (2.5), (2.43) and (3.52) there is $C > 0$ such that

$$|n_k|_2 + \|\varphi_k\|_{H^1} + \frac{1}{k} \int_0^T |(\varphi_k)(s)|_2^2 \, ds + \int_0^T \|\theta^{d/2+1}(s)P_k(s)\|_{H^1}^2 \, ds \leq C. \quad (3.53)$$

Observe that the Hölder inequality, (3.53) and assumptions (2.13), (2.16) imply

$$|\theta P_k^2 \nabla n_k|_{\frac{d}{d+2}} \leq C \left( |n_k^{2/d} \nabla P_k|_{\frac{d}{d+2}} + \theta^{d/2+1} |\nabla P_k|_{\frac{d}{d+2}} \right) \leq C \left( |n_k|_2 |\nabla P_k|_2 + |\nabla P_k|_{\frac{d}{d+2}} \right),$$

$$|n_k P_k \nabla \varphi_k|_{d/2} \leq C |P_k \theta^{d/2} \nabla \varphi_k|_{d/2} \leq C |P_k|_{\frac{d}{d+2}} |\nabla \varphi_k|_{\frac{d}{d+2}},$$

whence, by (3.53) thanks to the imbedding of $H^1(\Omega)$ in $L^{\frac{2d}{d-2}}(\Omega)$,

$$\int_0^T \left( |\theta(s)P_k^2(s)\nabla n_k(s)|_{2d/(d+2)}^2 + |n_k(s)P_k(s) \nabla \varphi_k(s)|_{d/2}^2 \right) \, ds \leq C. \quad (3.54)$$

For $d = 2$ due to (2.37) we get the $L^\infty$ bound for $\nabla \varphi_k$ whence $|n_k P_k^\prime \nabla \varphi_k|_1 \leq c|P_k|_1 \leq C$. We then deduce from the above inequality and equation (2.18) that

$$|(n_k)|_{L^2(0,T;W^{1,2d/(d-2)}(\Omega)'}) \leq C. \quad (3.55)$$

Consequently, owing to (3.55), (2.35) and (3.53) the sequence $(n_k)$ is bounded in $L^2(0,T;H^1(\Omega))$ and in $H^1(0,T;W^{1,2d/(d-2)}(\Omega)^\prime)$. Owing to the compactness of the imbedding of $H^1(\Omega)$ in $L^2(\Omega)$ and to the continuity of the imbedding of $L^2(\Omega)$ in $W^{1,2d/(d-2)}(\Omega)^\prime$, we infer from [20, Corollary 4] that $(n_k)$ is
relatively compact in $L^2(\Omega \times (0, T))$. Therefore, there are $n \in L^2(\Omega \times (0, T))$ and a subsequence of $(n_k)$ such that $n_k \to n$ a.e. and

$$n_k \to n \quad \text{in} \quad L^2(\Omega \times (0, T)) \cap C([0, T]; W^{1,2d/(d-2)}(\Omega)') \quad (3.56)$$

Let $\varphi \in L^\infty(0, T; H^2(\Omega))$ be the solution to

$$\Delta \varphi = n \quad \text{in} \quad \Omega \times (0, T), \quad \varphi = 0 \quad \text{on} \quad \partial \Omega \times (0, T) \quad (3.57)$$

It follows from (2.19) and (3.57) that $\varphi_k - \varphi$ solves the Poisson equation

$$-\Delta(\varphi_k - \varphi) = n - n_k - \frac{1}{k}(\varphi_k)_t$$

with the homogeneous Dirichlet boundary conditions, and the right-hand side of the above equation converges to zero in $L^2(\Omega \times (0, T))$ by (3.53) and (3.56). Therefore,

$$\varphi_k \to \varphi \quad \text{in} \quad L^2(0, T; H^2(\Omega)) \quad (3.58)$$

Combining (3.53) with the convergence results (3.56) and (3.58) finally allow us to conclude that $P_k^2 \nabla n_k$ and $P_k' n_k \nabla \varphi_k$ converge weakly to $P^2 \nabla n$ and $nP \nabla \varphi$ in $L^{2d/(d+2)}(\Omega \times (0, T))$ and $L^{d/2}(\Omega \times (0, T))$, respectively. It is now straightforward to pass to the limit as $k \to \infty$ and conclude that $(n, \varphi)$ is a weak solution to (1.7)–(1.9) as stated in Theorem 3.1.

We may also pass to the limit in (3.50) and use classical lower semicontinuity argument to deduce that (3.50) holds true.

Next, by (2.13) and (2.15) it follows from the conservation of mass, (2.42) and the Poincaré inequality that

$$\frac{d}{dt}|n_k(t)|_2^2 + \gamma (|n_k|_2^2 + \|\theta^{4/2+1}P_k\|^2_{L^1}) \leq C \left(1 + |\nabla \varphi_k|_{\alpha \frac{2\alpha(d+6)}{6d-6}}\right) (3.59)$$

for some positive constant $\gamma$. Integrating with respect to time, we get

$$|n_k(t)|_2^2 \leq |n_{0,k}|_2^2 e^{-\gamma t} + C \int_0^t \left(1 + |\nabla \varphi_k(s)|_{\alpha \frac{2\alpha(d+6)}{6d-6}}\right) e^{\gamma(s-t)} \, ds \quad (3.60)$$
for \( t \geq 0 \). Next, from the Fubini theorem and the double integration of (3.59) we obtain, for \( t \geq 1 \),

\[
\int_t^{t+1} \| \theta^{d/2+1}(s)P_k(s) \|_{H^1}^2 \, ds \leq \int_{t-1}^t \int_{t+2}^{t+2} \| \theta^{d/2+1}(s)P_k(s) \|_{H^1}^2 \, ds \, d\tau
\]

\[
\leq C \left( 1 + \int_{t-1}^t |n_k(\tau)|^2 \, d\tau + \int_{t-1}^{t+2} |\nabla \varphi_k(s)|^2 \frac{2^{(d+6)}}{6^{d-4}} \, ds \right). \tag{3.61}
\]

Now, \(|\nabla \varphi_k|_\alpha\) is bounded in \( L^q(0, t)\) for any \( q \in (1, \infty)\) by Lemma 2.6 and we infer from (3.58) and the continuous imbedding of \( H^2(\Omega)\) in \( W^{1,\alpha}(\Omega)\) that \(|\nabla \varphi_k - \nabla \varphi|_\alpha\) converges to zero in \( L^2(0, t)\). Consequently, by interpolation, \(|\nabla \varphi_k - \nabla \varphi|_\alpha\) converges to zero in \( L^{\frac{2^{(d+6)}}{6^{d-4}}}(0, t)\). Then one can pass to the limit as \( k \to \infty \) in (3.60) and (3.61) with the help of (3.56) and weak convergence arguments for the left-hand sides and conclude that

\[
|n(t)|^2 \leq |n_0|^2 e^{-\gamma t} + C \int_0^t \left( 1 + |\nabla \varphi(s)|^2 \frac{2^{(d+6)}}{6^{d-4}} \right) e^{\gamma(s-t)} \, ds
\]

for \( t \geq 0 \), while for \( t \geq 1 \)

\[
\int_t^{t+1} \| \theta^{d/2+2}(s)P(s) \|_{H^1}^2 ds \leq C \left( 1 + \int_{t-1}^t |n(s)|^2 ds + \int_{t-1}^{t+2} |\nabla \varphi(s)|^2 \frac{2^{(d+6)}}{6^{d-4}} \, ds \right).
\]

Since \( \varphi \) is a solution to equation (1.8), by (3.50) we get taking \( \alpha = \frac{d(d+2)}{d^2-d-2} \)

\[
|\nabla \varphi|_\alpha \leq C \| \varphi \|_{W^{2,1+2/d}} \leq C \, |n|_{1+2/d} \leq C.
\]

Inserting this estimate in the previous two inequalities yields the boundedness of \(|n(t)|^2\) with respect to time, and then (3.51). For \( d = 2 \) we have \( 1+2/d = 2 \) whence the estimate for \(|n|_{1+2/d}\) is sufficient.

\section{Fixed point for the temperature operator \( T \)}

First we recall a lemma on relations between \( n \) and \( \theta \) imposed by (1.12). This should be understood as necessary condition for the density obtained from (1.7)-(1.10) and not as a sufficient condition for admissibility of the
given energy $E$. The lemma on *a priori* bounds is related to the one from [3] in the Fermi–Dirac case and to the ones from [5] and [6] in more general case. Recall from [6, Lemma 3.1] or [24] the following version of these energy estimates.

**Lemma 4.1** Let $\nu = 4/(d(4-d))$ for any $2 \leq d < 4$. Provided that $P(s) \geq p_1 s^{1+2/d}$ for any $\frac{d}{2} p_1 > \varepsilon > 0$ and all $s \geq 0$, the following estimate holds

$$E + CM^{1+\nu} \geq \max \left\{ \varepsilon \int_{\Omega} n^{1+2/d} \, dx, |\varphi|^2 \right\}. \quad (4.62)$$

Moreover, for each $0 < \varepsilon < d/2$, the temperature $\vartheta$ and the density $n$ should satisfy

$$E \geq \varepsilon \int_{\Omega} \vartheta^{d/2+1} P(n \vartheta^{-d/2}) \, dx + \left| \int_{\Omega} n \vartheta \, dx \right| - CM^{1+\nu}. \quad (4.63)$$

Now we shall prove some *a priori* estimate for $L^{1+2/d}$ norm of the solution to BVP (1.7)–(1.10). We derive them directly from these equations since at this moment we cannot directly use the energy *a priori* bounds presented above. Note that these *a priori* estimates for limit functions are better than those for the perturbed parabolic system presented in previous sections (cf. lemmas: 2.2, 2.3, 2.5).

The next lemma can be found in [24] (cf. Lemma 2.1 and 2.2 therein).

**Lemma 4.2** For any $2 \leq d < 2(1 + \sqrt{2})$ and $\varphi$ related to $n$ by (1.8) we have the estimate

$$\left| \int_{\Omega} n \varphi \, dx \right| \leq CM^{1-2/d} \int_{\Omega} n^{1+2/d} \, dx. \quad (4.64)$$

Let $2 \leq d \leq 4$ and assume that

$$P(z) = p_1 z^{1+2/d} + R(z), \quad (4.65)$$

where the lower order term satisfies $|R'(z)|z^{1/2-1/d} \leq B$. Define the ‘asymptotic energy’, i.e. $E^a(t) = \lim_{\vartheta \to 0^+} E(t)$, by

$$E^a(t) = \frac{d}{2} \int_{\Omega} p_1 n^{1+2/d} \, dx - \frac{1}{2} \int_{\Omega} |\nabla \varphi|^2 \, dx. \quad (4.66)$$
Then for any fixed $T > 0$ and any $t \in [0, T]$ the following growth condition for $E^a$ is available

$$\frac{d}{dt} E^a(t) \leq C \theta(t)^{d/2} \int_\Omega |\nabla \varphi|^2 \, dx. \quad (4.67)$$

**Remark.** Note that the above theorem holds both in the polytropic case with $R(z) = 0$ and in the Fermi–Dirac case, since by the properties of Fermi function (cf. Lemma 5.6 the Appendix and [4, Sec.5]) $|R'(z)| \leq B z^{-2/d}$ at $z = \infty$ and $|R'(z)| \leq B$ at $z = 0$.

**Remark.** It should be noted that for the polytropic case the theorem implies the dissipation of the energy, since in this case $E^a = E$.

Applying the estimate (4.64) to $\int_\Omega n \varphi \, dx = - \int_\Omega |\nabla \varphi|^2 \, dx$ and integrating (4.67) from Lemma 4.2 allows us to derive the following corollary (for details and the proof see [24]).

**Corollary 4.3** Under the assumptions of Lemma 4.2 $E^a$ grows like

$$E^a(t) \leq E^a(0) \exp(C(t)) \quad (4.68)$$

where the function $C$ is defined by $C(t) = c M^{1-2/d} \int_0^t (\theta(s))^{d/2} \, ds$. Moreover, $E^a$ is positive if $CM^{1-2/d} < dp_1$ while for $d = 2$ we assume smallness of the Poincaré constant i.e. $C < 2p_1$.

After integrating inequality (4.68) from Corollary 4.3 and using Lemma 4.4 we obtain $L^{1+2/d}$ estimate for the density $n$.

**Corollary 4.4** Under the assumptions of Lemma 4.2 we have for any $2 \leq d < 4$, $\nu = 4/(d(4 - d))$ and any $M > 0$

$$\int_\Omega n^{1+2/d} \, dx \leq CM^{1+\nu} + |E^a(0)| \exp(C(t)),$$

while for small $M > 0$ and any $2 \leq d \leq 4$ the constant $C$ may depend on $M$

$$\int_\Omega n^{1+2/d} \, dx \leq CE^a(0) \exp(C(t))$$

where $CM^{1-2/d} < dp_1$. 20
Corollaries 4.3 and 4.4 allow us to define the new temperature $\vartheta$. This was the subject of the considerations in [24] under physically acceptable property of the pressure $\frac{\partial p}{\partial \vartheta} > 0$ expressed as

$$P(z)z^{-1-2/d} p_2 > 0$$

(4.69)

that guarantees, in particular, the uniqueness of the temperature $\vartheta$ emerging from the energy formula (1.11).

The next theorem claims that that temperature is well defined for some values of the energy (for the proof see Theorem 3.2 in [24]), and the remainder of the section is devoted to proving its compactness.

**Theorem 4.5** Assume that $P$ is smooth and satisfies (2.13), (2.14), and (2.27). Then the temperature operator $T : \theta \mapsto \vartheta$ is formally well defined by (1.11) for small mass $M \ll 1$ and $2 \leq d \leq 4$ all the values of the energy $E$ admissible at $t = 0$. Moreover, for $2 \leq d < 4$, $\nu = 4/(d(4 - d))$ and some positive constants $B, C$, it has to satisfy

$$E < BM^{1+2/d} - CM^{1+\nu}.$$  

(4.70)

Next, we estimate $\vartheta'$ to get the compactness of the operator $T$. By differentiation of the energy relation (1.11) we get

$$\vartheta' = \left( \int_{\Omega} \frac{\partial p}{\partial n} n_t \, dx + \frac{1}{d} \frac{d}{dt} \int_{\Omega} n \varphi \, dx \right) \left( \int_{\Omega} \frac{\partial p}{\partial \vartheta} \, dx \right)^{-1}.$$  

(4.71)

In the following two lemmas we claim the boundedness of both factors in appropriate norms so that $\vartheta'$ be in $L^\gamma$ with some $\gamma > 1$ which guarantees the equicontinuity condition in the classical Arzela-Ascoli theorem.

**Lemma 4.6** Assume that $P$ is a smooth function such that

$$(P(z)z^{-1-2/d})' < 0,$$  

(4.72)

$$(P(z)z^{-1-2/d})'z^{1+2/d} > 0.$$  

(4.73)

Then the first inequality implies (2.16) with a strict inequality and $C = 1 + 2/d$, i.e.,

$$P'(z)z < (1 + 2/d)P(z).$$  

(4.74)
Moreover, the function $p$ is decreasing, convex with respect to $\vartheta$ and satisfies

$$\int_\Omega \frac{\partial p}{\partial \vartheta} \, dx > C \quad (4.75)$$

for some $C > 0$ depending on $M$ and a lower bound for $\vartheta \geq a$ provided that

$$\liminf_{z \to \infty} \left( -P(z)z^{-1-2/d} \right)' z^{A/d+1} \geq C > 0, \quad (4.76)$$
$$\liminf_{z \to 0} \left( -P(z)z^{-1-2/d} \right)' z^{2/d+1} \geq C > 0. \quad (4.77)$$

**Proof.** The formula for the first derivative reads

$$\frac{\partial p}{\partial \vartheta} = -(d/2)\vartheta^{d/2} \left( P'(z)z - (1 + 2/d)P(z) \right), \quad (4.78)$$

or in another form

$$\frac{\partial p}{\partial \vartheta} = -(d/2)\vartheta^{d/2} z^{2+2/d} \left( P(z)z^{-1-2/d} \right)' > 0, \quad (4.79)$$

where $z = n\vartheta^{-d/2}$. Then, the second derivative can be calculated

$$\frac{(2/d)^2 \vartheta^{1-d/2} \partial^2 p}{\partial \vartheta^2} = P''(z)z^2 - (1 + 2/d)P'(z)z + (1 + 2/d)P(z), \quad (4.80)$$

or expressing it in a more concise way

$$\frac{(2/d)^2 \vartheta^{1-d/2} \partial^2 p}{\partial \vartheta^2} = z^2 \left( (P(z)z^{-1-2/d}z^{1+2/d})' \right).$$

Thus the convexity of $p$ with respect to $\vartheta$ follows from the second assumption (which by the way can be deduced from the first assumption or (4.74) under an extra convexity assumption on $P$). Now, by the asymptotics of $P$, i.e. (4.76), (4.77), $\liminf_{n \to 0} \int_\Omega \frac{\partial p}{\partial \vartheta} \, dx \geq CM$ as and $\liminf_{n \to \infty} \int_\Omega \frac{\partial p}{\partial \vartheta} \, dx \geq CM^{-2/d} a$, respectively. Hence, by the convexity of $p$ with respect to $\vartheta$, (4.75) follows. \hfill \Box

To get the bound for $\vartheta'$ we are left to estimate the denominator in (4.71).

**Lemma 4.7** The denominator appearing in (4.71) is bounded in some $L^\gamma$ with $\gamma > 1$, i.e.,

$$\int_0^T \left[ \int_\Omega \frac{\partial p}{\partial n} \, dx + \frac{1}{d} \frac{d}{dt} \int_\Omega n^\varphi \, dx \right]^\gamma \, dt < C.$$
Proof. First, recall that
\[
\int_{\Omega} n_t \varphi \, dx = \frac{d}{dt} \left( \int_{\Omega} n \varphi + \frac{1}{2} |\nabla \varphi|^2 \, dx \right) + \frac{1}{k} \int_{\Omega} \varphi^2 \, dx.
\]
Using (3.55) we get the bound for \(n_t\) in \(L^2(0, T; W^{1,2d/(d-2)}(\Omega))\) so we have to show that both \(\frac{\partial p}{\partial n}\) and \(\varphi\) are bounded in \(L^\xi(0, T; W^{1,2d/(d-2)}(\Omega))\) with some \(\xi > 2\). Using (2.6) and (3.51) with \(\alpha = \frac{2d}{d-2}\) and any \(q > 1\) we get
\[
\int_0^t |\nabla \varphi(s)|^q \, ds \leq C \int_0^t |n(s)|_2^q \, ds \leq C'.
\]
whence \(\varphi \in L^\xi(0, T; W^{1,2d/(d-2)}(\Omega))\) with \(\xi > 2\). Moreover, \(\frac{\partial p}{\partial n} = \theta P' \sim n^{2/d}\) and the function \(\nabla \frac{\partial p}{\partial n} = \theta^{1-d/2} P''\) is bounded in view of the regularity assumption on \(P\), so the claim is guaranteed by the estimates for \(P\) (2.13) and (2.14) and (2.6). Lastly, from (2.32) follows the bound for \(\int_{\Omega} \varphi^2 \, dx\) and from (2.29) and Lemma 2.3 for \(\int_{\Omega} |\nabla \varphi|^2 \, dx\). \(\square\)

Finally, we recall after [6] and [24] \textit{a priori} bounds on the fixed points of the compact operator \(T\) thus guaranteeing the existence result for the problem.

The authors assumed therein, for negative initial values of the entropy
\[
\liminf_{z \to \infty} (H(z) - (d/2 + 1)P(z)/z) > W(0)/M, \quad (4.81)
\]
where \(H'(z)z = P'(z)\) under the following conditions consistent with (2.13)
\[
P(z)/z^{1+2/d} \searrow \varepsilon > 0, \quad (4.82)
\]
\[
\liminf_{z \to 0} P(z)/z > 0. \quad (4.83)
\]
If (2.27) is fulfilled then the highest order terms cancel assuming that the limit exists
\[
\lim_{z \to \infty} \frac{H(z)z - (d/2 + 1)P(z)}{z} = \lim_{z \to \infty} \left( \frac{H(z) - \frac{d}{2} z H'(z)}{z} \right) = \lim_{z \to \infty} \left( G(z) - (d/2) z G'(z) \right) \equiv G_0 \quad (4.84)
\]
\[
H'(z) = h_1(2/d)z^{2/d-1} + G'(z) \quad (4.86)
\]
where
\[
G_0(4.85)
\]
Finally, we recall after [6] and [24] \textit{a priori} bounds on the fixed points of the compact operator \(T\) thus guaranteeing the existence result for the problem.

The authors assumed therein, for negative initial values of the entropy
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\]
\[
H'(z) = h_1(2/d)z^{2/d-1} + G'(z) \quad (4.86)
\]

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with $G'(z) = o(z^{2/d-1})$, we are left with the analysis of the lower order term
$G''(z) = g_1 z^\beta + o(z^\beta)$ with some $\beta < 2/d - 1$. Namely, if $\beta < -1$ then $G_0 = 0$ in (4.84), e.g. for the Fermi–Dirac model in $d = 2$. Otherwise, if $\beta \in [-1, 2/d - 1)$ then the important factor is the sign of $g_1$ which has to be positive, and indeed is, e.g. $g_1 = 1 - 2/d$ and $\beta = -2/d$ in the Fermi–Dirac case for $d \geq 3$, to imply $G_0 = \infty$ and thus to guarantee (4.81).

Thus we have proved the existence of a fixed point for temperature operator $T$ and we can formulate the following existence result in the microcanonical case.

**Theorem 4.8** Assume that $P$ is smooth, satisfies (2.13), (2.14), (2.27), (4.69), (4.82), (4.83) and (4.84) with $G_0 \geq 0$. Then for the negative initial values of the entropy we get the global existence result for (1.7)-(1.10) with the energy constraint (1.11).

5 Appendix on Fermi–Dirac model

First, it should be noted that for the Fermi–Dirac case we have

$$dP_{FD}(z) = \mu f_{d/2} \left( f_{d/2-1}^{-1}(2z/\mu) \right),$$

(5.87)

where $\mu = \eta_0 G\sigma d^{2d/2}$, $G$ is the gravitational constant, $\eta_0$ – a bound for the density in phase space and $f_\alpha$ is the Fermi function of order $\alpha > -1$ defined by

$$f_\alpha(z) = \int_0^{\infty} \frac{x^\alpha}{1 + e^{x-z}} dx.$$

(5.88)

In [4] Lemma 5.1 substitute $z = -\log(\lambda)$ and $f_\alpha(z) = I_\alpha(e^{-z})$ to get

**Lemma 5.1** The following asymptotic relations hold as $z \to \infty$

$$f_\alpha(z) - \frac{z^{\alpha+1}}{\alpha + 1} = O(z^{\alpha-1}),$$

(5.89)

for each $\alpha \geq 0$, while for each $\alpha > -1$

$$z^{-\alpha} \left\{ f_\alpha(z) - \frac{\alpha + 2}{\alpha + 1} f_{\alpha+1}(z) \right\} \to -\frac{\pi^2}{3}.$$

(5.90)
Moreover, we have the recursive relation for the derivatives

\[ f'_\alpha(z) = \alpha f_{\alpha-1}(z). \tag{5.91} \]

Next \cite{3} Lemma 2.2 can be formulated as follows.

**Lemma 5.2** For \( \alpha > \beta \), \( f_\alpha \circ f_\beta^{-1} \) is an increasing convex function.

In conclusion of the above lemma the function \( P_{FD} \) shares the same properties.

**Lemma 5.3** The function \( P_{FD} \), defined by (5.87), is increasing and convex function.

Next, to check that the assumptions of the Lemma \cite{4}, Lemma 5.4 are verified for the Fermi–Dirac case we will need a version of \cite{4}, Lemma 5.3.

**Lemma 5.4** For all \( \beta < \alpha + 1 \) the following inequality holds

\[ f'_{\alpha+\beta} f'_{\alpha-\beta} - (f'_\alpha)^2 > 0. \tag{5.92} \]

Now, we recall after \cite{3}, Lemma 2.1 the following properties of the Fermi–Dirac pressure.

**Lemma 5.5** The function \( P_{FD} \) belongs to \( C^2([0, \infty)) \), is nonnegative, increasing, convex and can be extended to an element (still denoted by \( P_{FD} \)) of \( C^2([-\delta, \infty)) \) for some \( \delta > 0 \).

Next the following asymptotic result holds (cf. \cite{24}) for

\[ P_{FD}(z) = p_1 z^{1+2/d} + R_{FD}(z). \]

**Lemma 5.6** For the Fermi–Dirac pressure \( P_{FD} \) we have at \( z = \infty \)

\[ P'_{FD}(z) = \frac{2}{d} \left( \frac{d}{\mu} z \right)^{2/d} + \mathcal{O}(z^{-2/d}) \tag{5.93} \]

and, in consequence,

\[ P_{FD}(z) = p_1 z^{1+2/d} + \mathcal{O}(z^{1-2/d}). \tag{5.94} \]

where \( p_1 = \frac{2}{d+2} (d/\mu)^{2/d} \). Moreover, at \( z = 0 \), we have

\[ R'_{FD}(z) = \mathcal{O}(1). \tag{5.95} \]
Lemma 5.7 The conditions $\frac{\partial p}{\partial \vartheta} > 0$ and/or $P(z)/z^{1+2/d} \nabla p_2 > 0$ are satisfied for
\[ P_{FD}(z) = (\mu/d) f_{d/2} \circ f_{d/2-1}^{-1} (2z/\mu). \tag{5.96} \]

Proof. Indeed, putting $2z = \mu f_{d/2-1}(x)$, we get
\[ \frac{\partial p}{\partial \vartheta} = \frac{d}{2} \vartheta^{d/2} \left( (\mu/2)(f_{d/2-1}(x))^2 (f_{d/2-1}'(x))^{-1} + (1 + 2/d)(\mu/d) f_{d/2-1}(x) \right). \]

Thus, the condition $\frac{\partial p}{\partial \vartheta} > 0$ is equivalent to
\[ -((\mu/2)f_{d/2-1}(x))^2 + (\mu/d)f_{d/2-1}'(x)(\mu/2)(1 + 2/d)f_{d/2-1}(x) > 0. \]

This, however, follows from Lemma 5.2 (take $\alpha = d/2$ and $\beta = d/2 - 1$) or more explicitly by the property of Fermi functions presented in (5.92), namely
\[ \frac{d}{dx} \left( f_{d/2} \left( z \right) f_{d/2} \left( x \right) f_{d/2-1}(x) \right) < 0. \]

Lastly, we shall trace how (1.7)–(1.10) could be derived, in the Fermi–Dirac case, from (1.1)–(1.4) (used by the authors in [3]) under the assumption (1.5) with a specific diffusion coefficient, used in [3] and [12],
\[ D(\lambda) = \frac{-I_{d/2-1}(\lambda)}{\lambda I_{d/2-1}'(\lambda)}, \tag{5.97} \]

where $I_{\alpha}(e^{-z}) = f_{\alpha}(z)$ and $\lambda = I_{d/2-1}^{-1} \left( \frac{2n}{\mu \sqrt{d}} \right)$. Using the recurrence property (5.91) (cf. also [4, Section 5] and [21, Lemma 1.1]), we get, differentiating formula (5.87), the relation $D = P'$. Furthermore, $\vartheta D = \frac{\partial p}{\partial n}$. Moreover, it should be noted that in [3] and [4] the authors used the following notation $F' = P^2, V = nP'$. Note that $D$ should be defined exactly as in (5.97) but it might differ throughout these papers up to a constant, inessential therein.

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