Special Isothermic Surfaces and Solitons

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Abstract. We establish a correspondence between Darboux’s special isothermic surfaces of type \((A, 0, C, D)\) and the solutions of the second order p.d.e. \(\Phi \Delta \Phi - |\nabla \Phi|^2 + \Phi^4 = s, s \in \mathbb{R}\). We then use the classical Darboux transformation for isothermic surfaces to construct a Bäcklund transformation for this equation and prove a superposition formula for its solutions. As an application we discuss 1 and 2-soliton solutions and the corresponding surfaces.

1. Introduction

The theory of isothermic surfaces in conformal geometry has been the focus for intense research over the past years due to its relation with the theory of integrable systems; see for instance \([6, 9, 10, 14, 15, 19, 20, 21]\) and the literature therein. Among isothermic surfaces Darboux \([17]\) distinguished the class of special isothermic surfaces. These were introduced in connection with the problem of isometric deformation of quadrics and have been investigated by Bianchi \([3, 4]\) and Calapso \([12]\). An isothermic immersion \(f : U \subset \mathbb{R}^2 \to \mathbb{R}^3\) admits conformal curvature line coordinates \(x, y\) for which its first and second fundamental forms read

\[
I = \Theta^2 (dx^2 + dy^2), \quad II = \Theta^2 (h_1 dx^2 + h_2 dy^2),
\]

where \(\Theta\) is a nowhere vanishing smooth function and \(h_1, h_2\) denote the principal curvatures. The immersion \(f\) is called special of type \((A, B, C, D)\) if its mean curvature \(H\) satisfies the equation

\[
4\Theta^2 |\nabla H|^2 + M^2 + 2AM + 2BH + 2CL + D = 0,
\]

where \(A, B, C, D\) are real constants and \(M = -HL\) being \(L = \Theta^2(h_1 - h_2)\). Examples include surfaces of constant mean curvature (cmc).

This article has its origins in the two seminal papers of Bianchi \([3, 4]\) on isothermic surfaces and deals with special isothermic surfaces of type \((A, 0, C, D)\), henceforth

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simply referred to as special isothermic. These surfaces are invariant under the group of conformal transformations. Since the work of Bianchi, it has been known that umbilic free special isothermic surfaces are conformally equivalent to cmc surfaces in 3-dimensional space forms. Examples of special isothermic surfaces with umbilic lines can be constructed by revolving elastic curves in the hyperbolic half-plane about the boundary at infinity (cf. Section 3).

We prove that special isothermic immersions are in correspondence with the solutions $\Phi(x, y)$ of the partial differential equation

$$\Phi \Delta \Phi - |\nabla \Phi|^2 + \Phi^4 = s, \quad s \in \mathbb{R}. \tag{1.2}$$

Further, we show how the geometric properties of the Darboux transformation of special isothermic surfaces apply to obtain analytic results for this equation. As for the study of Darboux transforms of special isothermic surfaces, our work is related to that of Hertrich-Jeromin–Pedit [20] which discusses the case of cmc immersions in Euclidean space and that of Hertrich-Jeromin–Musso–Nicolodi [21] concerning immersions with cmc $H, |H| = 1$, in hyperbolic 3-space.

The special isothermic surface equation (1.2) can be reformulated as a zero-curvature equation

$$P(m)_y - Q(m)_x - [P(m), Q(m)] = 0 \tag{1.3}$$

involving an auxiliary parameter $m$. This equation expresses the compatibility condition for a linear differential system

$$V_x = -P(m)V, \quad V_y = -Q(m)V \tag{1.4}$$

in Minkowski 5-space, where the Lax pair matrices $P(m), Q(m)$ take values in $\mathfrak{so}(4, 1)$ (cf. Section 3). In this representation, for every solution $\Phi$ of (1.2), the spectral parameter $m$ describes a family of special isothermic surfaces which are second order conformal deformations of each other and are obtained by integrating the loop of $\mathfrak{so}(4, 1)$-valued 1-forms $P(m)dx + Q(m)dy$.

The linear system (1.4) have the conserved quantities

$$C_1(V) = -2v^0v^4 + \sum_{a=1}^{3}(v^a)^2, \quad C_2(V) = \Phi^2v^0 - mv^3 + \frac{s}{2}\Phi^{-2}v^4. \tag{1.5}$$

The classical Darboux transforms of a special isothermic immersion with potential $\Phi$ are constructed from the solutions $V$ of the linear system (1.4) satisfying $C_1(V) = 0$ (cf. [3, 4] and the recent papers [4, 15, 10, 21, 20]). Moreover, if $V$ satisfies the additional constraint $C_2(V) = 0$, the Darboux transforms of special isothermic surfaces are still special isothermic. Analytically, this is equivalent to the statement that if $\Phi$ is a solution of (1.2) and if $V$ is a solution of the linear system satisfying $C_1(V) = C_2(V) = 0$, then $(v^3/v^4)\Phi$ is again a solution which can be regarded as a T-transform of $\Phi$.

1 Observe that cmc immersions in 3-dimensional space forms are obtained as T-transforms (spectral deformations) of minimal immersions in space forms (Willmore isothermic surfaces). For a recent account of these facts see [21, 13, and the monograph of Burstall [9].

2 For a special isothermic immersion, the corresponding solution is the Calapso potential $\Phi$ defined by $\Phi^2(dx^2 + dy^2) = (1/4)(h_1 - h_2)^2I$ which only depends on the conformal class of the induced metric.

3 They amount to a special isothermic immersion together with its T-transforms [23] (see also Theorem 4.6 below).
as a Bäcklund transformation of (1.2). This description furnishes a procedure for generating new solutions of (1.2) by solving (1.4).

In Section 7 we prove a superposition formula for the solutions of (1.2). Namely, if \( \Phi \) is a known solution and \( \Phi_1, \Phi_2 \) are solutions respectively generated from \( \Phi \) by solutions \( V \) and \( W \) of (1.4) corresponding to different values of the spectral parameter, then

\[
\Phi_3 = \Phi - \frac{(h - k)v^3w^3}{v^1w^1 + v^2w^2 + v^3w^3 - v^0w^0 - v^4w^4} \left[ \frac{\Phi_2 - \Phi_1}{\Phi_1\Phi_2} \right]
\]

represents a new solution of (1.2). This formula, which geometrically amounts to the Bianchi permutability theorem for special isothermic immersions, shows that, after the first step, the procedure for obtaining new solutions can be carried out without the quadratures associated with the Bäcklund transformation; it also provides a method for obtaining the multisoliton solutions of equation (1.2) by algebraic means only. As an application, we compute the 1 and 2-soliton solutions arising from the trivial solution \( \Phi = 1 \) of (1.2) with \( s = 1 \).

The Bäcklund transformation for the fourth-order equation defining isothermic immersions — the so-called Calapso equation — as well as its 1-solitons have been previously considered in [27] and have been used by Bernstein [2] to obtain explicit examples of non-special isothermic tori with spherical curvature lines (cf. examples in Section 6).

2. The conformal compactification of Euclidean space

Let us begin by recalling some basic facts. The one-point conformal compactification \( \mathcal{M} = \mathbb{R}^3 \cup \{ \infty \} \) of Euclidean space is classically realized as the projectivization of the light cone \( \mathcal{L} \) of Minkowski 5-space \( \mathbb{R}^5_1 \) with Lorentz scalar product \( \langle , \rangle \):

\[
\mathcal{M} \cong \mathbb{P}(\mathcal{L}) = \{ [X] \in \mathbb{P}^4 : \langle X, X \rangle = 0 \}.
\]

We consider linear coordinates \( x^0, \ldots, x^4 \) such that

\[
\langle X, Y \rangle = - (x^0y^4 + x^4y^0) + x^1y^1 + x^2y^2 + x^3y^3 = g_{ij}x^iy^j,
\]

and identify \( \mathcal{M} \) with \( \mathbb{P}[\mathcal{L}] \) by means of the conformal map

\[
j : (p^1, p^2, p^3) \in \mathbb{R}^3 \mapsto \left[ \left( 1, p^1, p^2, p^3, \frac{1}{2}||p||^2 \right) \right] \in \mathbb{P}[\mathcal{L}], \quad j(\infty) = \langle 0, 0, 0, 0, 1 \rangle.
\]

The linear action of the pseudo-orthogonal group \( G \cong SO(4,1) \) of (2.1) descends to a transitive action on \( \mathcal{M} \) by conformal (Möbius) diffeomorphisms. In this model for \( \mathcal{M} \), the de Sitter space \( S^4_1 = \{ Y \in \mathbb{R}^5_1 : \langle Y, Y \rangle = 1 \} \) parametrizes the 2-spheres in \( \mathcal{M} \) by

\[
Y \mapsto \mathbb{P}((Y)^\perp \cap \mathcal{L})
\]

and \( G \) acts transitively on the set of 2-spheres.

A Möbius frame is a basis \( B = (B_0, \ldots, B_3) \) of \( \mathbb{R}^3_1 \) such that the vectors form the columns of a matrix of \( G \). Geometrically, the unit space-like vectors \( B_1, B_2, B_3 \)

\footnotetext{4}{For more information on the Calapso equation we refer the reader to [11, 10, 14, 9].}
represent 2-spheres which intersect orthogonally, and \([B_0], [B_4] \in \mathcal{M}\) their intersection points. Regarding \(B_0, \ldots, B_4\) as \(\mathbb{R}^5\)-valued functions defined on \(G\), there are unique 1-forms \(\{\omega^I_J\}_{0 \leq I, J \leq 4}\) such that
\[
dB_I = \omega^I_J B_J, \quad \omega^K_I g_{IJ} + \omega^J_K g_{KI} = 0
\]
and satisfying the structure equations
\[
d\omega^I_J = -\omega^I_K \wedge \omega^K_J.
\]
\((\omega^I_J) = B^{-1} dB\) is the Maurer-Cartan form of \(G\) with values in the Lie algebra \(G\) of \(G\).

**Remark 2.1.** The set \(\hat{L}\) of all \(X \in L\) such that \(x^4 \neq 0\) can be given a Lie group structure. For, let \(X \in \hat{L}\) and define
\[
g^+(X) = 
\begin{pmatrix}
\frac{1}{x^4} & x^1/x^4 & x^2/x^4 & x^3/x^4 & x^0 \\
0 & 1 & 0 & 0 & x^1 \\
0 & 0 & 1 & 0 & x^2 \\
0 & 0 & 0 & 1 & x^3 \\
0 & 0 & 0 & 0 & x^4 \\
\end{pmatrix}.
\]

The mapping
\[
g^+ : X \in \hat{L} \mapsto g^+(X) \in G
\]
is a smooth embedding which induces a Lie group structure on \(\hat{L}\). The group operation is given by
\[
X \star Y = \left( x^0 y^4 + \frac{1}{x^4} (y_0 + \sum_{j=1}^{3} x_j y_j), x^1 + x^4 y^1, x^2 + x^4 y^2, x^3 + x^4 y^3, x^4 y^4 \right)^t.
\]
In particular, \(X^{-1} = (x^0, \frac{-x^1}{x^4}, \frac{-x^2}{x^4}, \frac{-x^3}{x^4}, \frac{1}{x^4})^t\) and \(1 = (0, 0, 0, 0, 1)^t\).

### 3. The Lax pair

**Definition 3.1.** Throughout the paper a solution \(\Phi\) of equation (1.2) will be referred to as a **wave potential** with **character** \(s\).

Let \(U \subset \mathbb{R}^2\) be a simply connected domain with coordinates \((x, y)\), and let \(\Phi : U \to \mathbb{R}\) be a nowhere vanishing smooth function. For \(m \in \mathbb{R}\), we define \(P(m), Q(m) : U \to \mathfrak{g}\) by
\[
P(m) = \begin{pmatrix}
2\Phi^{-1} \Phi_x & m\Phi^{-1} - \frac{1}{2} \Phi^{-3} & 0 & 0 & 0 \\
\Phi^{-1} \Phi_y & 0 & -\Phi^{-1} \Phi_y & -\Phi & m\Phi^{-1} - \frac{1}{2} \Phi^{-3} \\
0 & \Phi^{-1} \Phi_y & 0 & 0 & 0 \\
0 & \Phi & 0 & 0 & 0 \\
0 & 0 & \Phi^{-1} \Phi_x & 0 & 0 \\
\end{pmatrix}
\]
\[
Q(m) = \begin{pmatrix}
2\Phi^{-1} \Phi_y & 0 & -m\Phi^{-1} - \frac{1}{2} \Phi^{-3} & 0 & 0 \\
0 & \Phi^{-1} \Phi_x & 0 & 0 & 0 \\
\Phi & -\Phi^{-1} \Phi_x & 0 & \Phi & -m\Phi^{-1} - \frac{1}{2} \Phi^{-3} \\
0 & 0 & \Phi^{-1} \Phi_y & 0 & 0 \\
0 & 0 & \Phi & 0 & -2\Phi^{-1} \Phi_y \\
\end{pmatrix}
\]

A straightforward calculation gives:
Lemma 3.2. Equation (1.4) is equivalent to the matrix Lax equation
\begin{equation}
(P(m)y - Q(m)x - [P(m), Q(m)]) = 0,
\end{equation}
where \( P(m), Q(m) \) are as defined above.

This is the compatibility condition for the linear system
\begin{equation}
V_x = -P(m)V, \quad V_y = -Q(m)V
\end{equation}
in Minkowski 5-space. The linear system (3.2) is referred to as the \( D_m \)-system associated to \( \Phi \). The \( G \)-valued one-form \( \beta(m) := P(m)dx + Q(m)dy \) satisfies the Maurer-Cartan equation \( d\beta(m) + \beta(m) \wedge \beta(m) = 0 \) and can consequently be integrated to a frame \( B(m) : U \to G \), uniquely determined up to left multiplication by a constant element of \( G \), such that
\begin{equation}
dB(m) = \beta(m)B.
\end{equation}
We call \( B(m) \) a normal frame field for the wave potential \( \Phi \) with spectral parameter \( m \).

Because of (3.3), the components \( v^0, \ldots, v^4 \) of any constant vector \( X \in \mathbb{R}_5^1 \) with respect to a normal frame \( B(m) \) provide a solution \( V \) of the \( D_m \)-system. From (3.3) it also follows that
\begin{equation}
b(m) = \frac{s}{2} \Phi^{-2}B(m)_0 + mB(m)_3 + \Phi^2B(m)_4
\end{equation}
is a constant vector; we call \( b(m) \) the pointing vector of the normal frame \( B(m) \). The kinetic energy \( C_1(V) \) and the linear momentum \( C_2(V) \) of a solution \( V \) of the linear system (3.2) are defined by
\begin{align}
C_1(V) &= -2v^0v^4 + (v^1)^2 + (v^2)^2 + (v^3)^2, \\
C_2(V) &= \Phi^2v^0 - mv^3 + \frac{s}{2} \Phi^{-2}v^4.
\end{align}
If \( X \in \mathbb{R}_5^1 \) is the initial condition of \( V \) with respect to the normal frame \( B(m) \), then \( C_1(V) = \langle X, X \rangle \) and \( C_2(V) = -\langle b(m), X \rangle \). Thus, \( C_1 \) and \( C_2 \) are two first integrals of the \( D_m \)-system.

4. Potentials and special isothermic surfaces

Let \( f : U \to \mathbb{R}^3 \) be an isothermic immersion with conformal principal coordinates \( x, y \) and let
\begin{equation}
I = \Theta^2(dx^2 + dy^2), \quad II = \Theta^2(h_1dx^2 + h_2dy^2),
\end{equation}
be its fundamental forms, where \( \Theta \) is a nowhere vanishing smooth function and \( h_1 \) and \( h_2 \) are the principal curvatures.

Remark 4.1. Recall that the notion of an isothermic immersion is conformally invariant, that is, if \( f \) is an isothermic immersion and \( A \in G \) is a conformal diffeomorphism, then \( A \circ f \) is also isothermic \( ^3 \). In the following we will not make any distinction between isothermic immersions in \( \mathbb{R}^3 \) or in \( \mathcal{M} \).

Following Bianchi \( ^3 \) we set
\begin{align}
2H = h_1 + h_2, \quad L = \Theta^2(h_1 - h_2), \quad M = -HL.
\end{align}
and give the following

\footnotetext{The linear system (3.2) is gauge equivalent to a special case of the system considered by Darboux and Bianchi for the construction the Darboux transformation (4, 10) (see also Section 1).}
**Definition 4.2.** \( f \) is called *special isothermic* of type \((A, B, C, D)\) if there exist real constants \( A, B, C, D \) such that
\begin{equation}
4\Theta^2|\nabla \Theta|^2 + M^2 + 2AM + 2BH + 2CL + D = 0.
\end{equation}

We shall assume that \( f : U \to \mathbb{R}^3 \) is *generic*, i.e., either one of the following conditions hold: \( L_xL_y \neq 0 \), or \( L_x \neq 0 \) and \( L_y = 0 \), or \( L_y \neq 0 \) and \( L_x = 0 \), or else \( L_x = L_y = 0 \).

Next, let \( \Phi = \frac{1}{2}(h_1 - h_2)\Theta \) be the *Calapso potential*\(^6\) of the isothermic immersion \( f \).

We are now in a position to state:

**Proposition 4.3.** Let \( f : U \to \mathbb{R}^3 \) be a special isothermic immersion of type \((A, 0, C, D)\). Then the Calapso potential \( \Phi \) is a wave potential with character \( D/4 \).

**Proof.** The Gauss and Codazzi equations
\begin{equation}
\Theta \Delta \Theta - |\nabla \Theta|^2 + h_1h_2\Theta^4 = 0, \quad \Theta \neq 0,
\end{equation}

\begin{equation}
(h_1)_y = -\frac{\Theta}{\Theta'}(h_1 - h_2), \quad (h_2)_x = \frac{\Theta}{\Theta'}(h_1 - h_2),
\end{equation}

imply
\begin{equation}
L_x = 2\Theta^2H_x, \quad L_y = -2\Theta^2H_y, \quad L_{xy} = \Theta^{-1}(\Theta_yL_x + \Theta_xL_y),
\end{equation}

and
\begin{equation}
M_x = -h_1L_x, \quad M_y = -h_2L_y.
\end{equation}

Then, \((4.2)\) becomes

\begin{equation}
\Theta^{-2}|\nabla L|^2 + M^2 + 2AM + 2CL + D = 0.
\end{equation}

Since
\begin{equation}
2\Phi = (h_1 - h_2)\Theta, \quad L = 2\Phi \Theta, \quad M = -2\Phi H\Theta,
\end{equation}
equation \((1.3)\) also reads
\begin{equation}
\Theta^{-2}|\nabla \Phi|^2 = -\Phi^{-2}|\nabla \Phi|^2 - 2\Phi^{-1}\Theta^{-1}\nabla \Phi \cdot \nabla \Theta + A\Phi^{-1}H\Theta
\end{equation}
\begin{equation}
- H^2\Theta^2 - C\Phi^{-1}\Theta - \frac{D}{4}\Phi^{-2}.
\end{equation}

The derivative of \((1.3)\) with respect to \(x\) and \(y\) yields
\begin{equation}
2\Theta^{-1}L_x \left( (\Theta^{-1}L_x)_x + \Theta^{-2}L_y\Theta_y - \Theta(M + A)h_1 + C\Theta \right) = 0,
\end{equation}
\begin{equation}
2\Theta^{-1}L_y \left( (\Theta^{-1}L_y)_y + \Theta^{-2}L_x\Theta_x - \Theta(M + A)h_2 + C\Theta \right) = 0.
\end{equation}

When \( L_xL_y \neq 0 \) the last two equations imply
\begin{equation}
(\Theta^{-1}L_x)_x + (\Theta^{-1}L_y)_y + \Theta^{-2}(L_x\Theta_x + L_y\Theta_y) - 2\Theta H(M + A) + 2C\Theta = 0.
\end{equation}

This equation combined with \((4.6)\) and \((4.7)\) gives the result. A similar argument applies in the other cases. \( \Box \)

\(^6\)For more information on the Calapso potential see \([11, 2, 3, 10, 24]\).

\(^7\)Note that the functions \( \Theta^{-1}, -h_1\Theta^2, h_2\Theta^2 \) are related to the Christoffel transformation of \( f \). Thus, from \((4.3)\) it follows that the Christoffel transform of an isothermic immersion of type \((A, 0, C, D)\) is special of type \((A, C, 0, D)\), see also \([4]\).
Remark 4.4. In the setting of the previous proposition, notice that when \( L_x = L_y = 0 \) the immersion \( f \) has constant mean curvature. When instead \( L_x = 0 \) and \( L_y \neq 0 \) the Calapso potential \( \Phi = \Phi(y) \) is a function of the variable \( y \) alone and \( f \) is an isometric canal surface. By a classical result of Darboux \([16]\), \( f \) is then conformally equivalent to either a cone, a cylinder, or a surface of revolution. If \( f \) is a surface of revolution, one can prove that the profile curve lies in the hyperbolic half-plane, that its arclength is proportional to \( y \), and that its curvature is parametrized by the Calapso potential. Further, equation (1.2) reduces to the Euler-Lagrange equation of the total square curvature functional \( \alpha \mapsto \frac{1}{2} \int s_\alpha^2 \) on smooth curves \( \alpha \) with fixed length. Therefore, the surface defined by \( f \) is obtained by revolving an elastic curve (possibly a free elastic one) in the hyperbolic half-plane about the boundary at infinity. This description indicates, in particular, how to construct special isothermic surfaces with umbilic lines (see also Babich–Bobenko \([1]\)). Recall that rotational Willmore isothermic surfaces arise from free elastic curves in the hyperbolic 2-plane \([8]\). See also Langer–Singer \([22]\) and Pinkall \([28]\). For the case of Willmore canal surfaces we refer to \([25]\).

As for the converse, consider first the following:

Definition 4.5. For any \( X \in \mathcal{L} \), let \( \pi_X : \mathcal{L} \to \mathcal{M} \) be the conformal map defined by \( \pi_X = j^{-1} \circ \pi \circ g^+(X)^{-1} \), where \( \pi : \mathcal{L} \to \mathbb{P}[\mathcal{L}] \) denote the canonical projection.

Theorem 4.6. Let \( B(m) : U \to G \) be a normal frame field with wave potential \( \Phi \) and character \( s \), and let \( X \in \mathcal{L} \) be a constant vector. Then, \( f_m = \pi_X \circ B(m)_0 : U \to \mathcal{M} \) is a special isothermic immersion of type \((-2m, 0, 2(b(m), X), 4s)\) with Calapso potential \( \Phi \).

Proof. Let \( V \) be the solution of the \( D_m\)-system with initial condition \( X \), that is \( X = B(m)V \). Then, the fundamental forms of \( f_m = \pi_X \circ B(m)_0 \) can be read off the Maurer–Cartan form of the Euclidean frame \( B(m)g^+(V) \). These are computed to be

\[
I = \frac{\Phi^2}{(v^4)^2} ((dx)^2 + (dy)^2), \quad II = \frac{\Phi^2}{(v^4)^2} ((v^4 - v^3)(dx)^2 - (v^4 + v^3)(dy)^2).
\]

This implies that \( f_m \) is an isometric immersion and that

\[
H = -v^3, \quad L = \frac{2}{v^4}\Phi^2, \quad M = 2\frac{v^3}{v^4}\Phi^2.
\]

Next, by using the constraint \( \langle V, V \rangle = 0 \) and the conservation of the linear momentum \( C_2(V) = -\langle b(m), X \rangle \), a direct computation shows that

\[
4\Theta^2|\nabla H|^2 + M^2 - 4mM + 4\langle b(m), X \rangle L + 4s = 0.
\]

Thus \( f \) is special isothermic of type \((-2m, 0, 2(b(m), X), 4s)\). \( \square \)

Remark 4.7. The solution \( V \) in the proof of the theorem can be expressed in terms of the Euclidean invariants of \( f \) by the following formulae:

\[
v^0 = \frac{1}{h_1 - h_2} \left( \frac{4(H_x)^2}{(h_1 - h_2)^2\Theta^2} + \frac{4(H_y)^2}{(h_1 - h_2)^2\Theta^2} + H^2 \right),
\]

\[
v^1 = \frac{2H_x}{(h_1 - h_2)\Theta}, \quad v^2 = \frac{2H_y}{(h_1 - h_2)\Theta}, \quad v^3 = -H, \quad v^4 = \frac{1}{2}(h_1 - h_2).\]
Remark 4.8. Given an umbilic free immersion $f : U \to \mathbb{R}^3$, there exist a canonical frame field $B_f : U \to G$ along $f$: the central frame field of the immersion. The construction of such a frame is due to Bryant [7]. If $f$ is special isothermic of type $(A, 0, C, D)$ with Calapso potential $\Phi$, then $B_f$ is exactly the normal frame field with wave potential $\Phi$ and spectral parameter $m = -A/2$.

5. The Bäcklund transformation and 1-soliton solutions

Definition 5.1. Let $\Phi : U \to \mathbb{R}$ be a wave potential with character $s$. A solution $V$ of the linear system (3.2) satisfying $C_1(V) = C_2(V) = 0$ is said to be an $m$-system of transforming functions for the potential $\Phi$. The corresponding Bäcklund transform is defined by

$$E(\Phi, V) = \frac{v^3}{v^4} \Phi$$

Figure 1. A Bäcklund transform $\Psi(x, y)$ of the wave potential $\sqrt{2} \text{Sech}(\sqrt{2}y)$.

Theorem 5.2. The Bäcklund transform $E(\Phi, V)$ is a wave potential with character $s$.

Proof. Let $\hat{\Phi} = E(\Phi, V)$. The first derivatives of $\hat{\Phi}$ are given by

$$\hat{\Phi}_x = \frac{v^3}{v^4} \left[ -\Phi_x + v^1 \left( \frac{1}{v^4} - \frac{1}{v^3} \right) \Phi^2 \right], \quad \hat{\Phi}_y = \frac{v^3}{v^4} \left[ -\Phi_y + v^2 \left( \frac{1}{v^4} + \frac{1}{v^3} \right) \Phi^2 \right],$$

Note that $m$-transforming functions do exist only if $m^2 - s \geq 0$. If $m^2 - s > 0$, then the set of $m$-transforming functions with potential $\Phi$ is a 3-dimensional cone. If $m^2 - s = 0$, the transforming functions are of the form

$$V = r \left( \frac{s}{2} \Phi^{-2}, 0, 0, m, \Phi^2 \right)^t$$

for a constant $r \neq 0$. Observe that the last component $v^4$ of a system of transforming functions never vanishes. If, in addition, the character $s$ is positive, then also the third component $v^3$ never vanishes.
From this equation and the constraint\[\langle V, V \rangle = 0\] we now use the constraint of (1.2) with\[s = 1\]. In this case the 1-form\[B\] is defined by\[\tilde{\Phi} = m \sinh(\eta)\cos(\eta y) - \cosh(\eta)\sin(\eta y)\].

Now set\[\tilde{\Phi}^{-1}\Delta \tilde{\Phi} - \tilde{\Phi}^{-2} |\nabla \tilde{\Phi}|^2 = -\tilde{\Phi}^2 + \frac{v^4}{(v^3)^2} (2mv^3 - 2\Phi^2 v^0)\].

From this equation and the constraint\[\Phi^2 v^0 = mv^3 + \frac{s}{2}\Phi^{-2} v^4 = 0\]
the result follows.

\[\Box\]

**Remark 5.3.** Let \(\Phi\) be a potential with character \(s \neq 0\), then the complementary potential \(\Phi^*\) is defined by \(\sqrt{|s|} \Phi^{-1}\). This is a new potential with the same character of \(\Phi\). Note that \(\Phi^*\) can be obtained as the Bäcklund transform of \(\Phi\) with respect to the system of transforming functions

\[V = \left(\frac{s}{2} \Phi^{-2}, 0, \sqrt{|s|}, \Phi^2 \right)^t.\]

**Example 5.4 (One-soliton solutions).** Consider the trivial solution \(\Phi = 1\) of (1.2) with \(s = 1\). In this case the 1-form \(\beta(m)\) corresponding to the Lax pair is given by

\[\beta(m) = \begin{pmatrix}
0 & (m - \frac{1}{2}) dx & -(m + \frac{1}{2}) dy & 0 & 0 \\
0 & 0 & -dx & (m - \frac{1}{2}) dx & 0 \\
0 & 0 & dy & -(m + \frac{1}{2}) dy & 0 \\
0 & dx & -dy & 0 & 0 \\
0 & dx & dy & 0 & 0
\end{pmatrix}.
\]

Now set\[\zeta = \sqrt{2|m-1|}, \quad \eta = \sqrt{2|m+1|}.
\]
By solving a system of first order linear differential equations with constant coefficients, the normal framing \(B(m)\) is computed to be

\[B(m) = \begin{pmatrix}
\frac{\eta \cosh(\zeta) + 2}{\sqrt{2} \zeta} & \frac{\sinh(\zeta)}{\sqrt{2}} & 0 & -\frac{\eta \cosh(\zeta) + 2m}{\sqrt{2} \eta \zeta} & \frac{(2m-1) \eta \cosh(\zeta) + 2}{\sqrt{2} \eta \zeta} \\
\frac{\sqrt{2} \zeta \eta}{\cosh(\zeta)} & 0 & -\frac{\eta \cosh(\zeta) + 2m}{\sqrt{2} \zeta} & \frac{\sqrt{2} \eta \zeta}{\sinh(\zeta)} & \frac{(2m-1) \eta \cosh(\zeta) + 2}{\sqrt{2} \eta \zeta} \\
-\frac{\eta \cosh(\zeta) - 2}{\sqrt{2} \zeta} & \frac{\sinh(\zeta)}{\sqrt{2}} & 0 & \frac{\eta \cosh(\zeta) - 2m}{\sqrt{2} \eta \zeta} & \frac{(2m-1) \eta \cosh(\zeta) - 2}{\sqrt{2} \eta \zeta} \\
\frac{\eta}{\cos(\eta y)} & 0 & -\frac{\cos(\eta y)}{\sqrt{2}} & \frac{\eta}{\sin(\eta y)} & \frac{2 \eta}{\sqrt{2}} \\
\frac{\eta}{\sin(\eta y)} & 0 & \frac{\cos(\eta y)}{\sqrt{2}} & \frac{\eta}{\sin(\eta y)} & \frac{2 \eta}{\sqrt{2}}
\end{pmatrix},
\]
In order to find the solutions of (3.2) satisfying the constraint \(C_1 = C_2 = 0\), we may assume that \(m > 1\). Note that the pointing vector of the normal framing \(B(m)\) is the space-like vector \((1, 0, 0, 0, -1)^t\). Thus, the transforming functions of the \(D_m\)-system are given by

\[ V(m, a^1, a^2, a^3) = G \cdot B(m)^t \cdot GX(a^1, a^2, a^3), \]

where \(X(a^1, a^2, a^3) \in \mathcal{L}\) is defined by

\[ X(a^1, a^2, a^3) = \left( \sqrt{a^1 + a^2 + a^3}, a^1, a^2, a^3, \sqrt{a^1 + a^2 + a^3} \right)^t \]

and where \(G = (g_{ij})\) is as in (2.1). It follows that

\[
\begin{align*}
v^0 &= \frac{2m+1}{2\eta} (a^2 \cos(\eta y) + a^3 \sin(\eta y)) + \frac{2m-1}{\sqrt{2}\eta \zeta} (\|a\| \cosh(\zeta x) + \frac{a^1}{\sqrt{2}} \sinh(\zeta x)) \\
v^1 &= a^1 \cosh(\zeta x) - \|a\| \sinh(\zeta x) \\
v^2 &= a^1 \cos(\eta y) - a^2 \sin(\eta y) \\
v^3 &= \frac{a^2}{\eta} \cos(\eta y) + \frac{a^3}{\eta} \sin(\eta y) + \frac{\|a\|}{\zeta} \cosh(\zeta x) - \frac{a^1}{\zeta} \sinh(\zeta x) \\
v^4 &= \frac{\|a\|}{\zeta} \cosh(\zeta x) - \frac{a^1}{\zeta} \sinh(\zeta x) - \frac{a^2}{\eta} \cos(\eta y) - \frac{a^3}{\eta} \sin(\eta y),
\end{align*}
\]

where \(\|a\| = \sqrt{(a^1)^2 + (a^2)^2 + (a^3)^2}\).

From this we obtain the following formula for the one-soliton solutions

\[
\Phi(m, a^1, a^2, a^3) = \frac{\zeta (a^2 \cos(\eta y) + a^3 \eta \sin(\eta y)) + \eta (\|a\| \cosh(\zeta x) - a^1 \sinh(\zeta x))}{\zeta (a^2 \cos(\eta y) + a^3 \eta \sin(\eta y)) + \eta (\|a\| \cosh(\zeta x) - a^1 \sinh(\zeta x))}.
\]
6. The geometry of the Bäcklund transformation

In this section we briefly describe the geometric transformation of special isothermic surfaces corresponding to the Bäcklund transformation.

DEFINITION 6.1. A curved flat framing is a smooth map \( A : U \to G \) such that \( A^{-1}dA \) takes the form

\[
\begin{pmatrix}
0 & -r\Theta^{-1}dx & r\Theta^{-1}dy & 0 & 0 \\
\Theta dx & 0 & \Theta^{-1}(\Theta y dy - \Theta z dz) & -h_1 \Theta dx & -r\Theta^{-1}dx \\
\Theta dy & \Theta^{-1}(\Theta x dx - \Theta y dy) & 0 & -h_2 \Theta dy & r\Theta^{-1}dy \\
0 & h_1 \Theta dx & h_2 \Theta dy & 0 & 0 \\
0 & \Theta dx & \Theta dy & 0 & 0 \\
\end{pmatrix},
\]

where \( \Theta, h_1, h_2 \) are smooth functions with \( \Theta(p) \neq 0 \), for each \( p \in U \), and \( r \) is a constant referred to as the spectral parameter of the curved flat. For short, the connection form \( A^{-1}dA \) will be denoted by \( \alpha_r(\Theta, h_1, h_2) \).

The functions \( \Theta, h_1 \) and \( h_2 \) satisfy the isothermic Gauss-Codazzi system

\[
\begin{align*}
\Theta \Delta(\Theta) &= |\nabla \Theta|^2 - h_1 h_2 \Theta^4, \\
\Theta((h_1)_y) &= \Theta_y(h_2 - h_1), \\
\Theta((h_2)_x) &= \Theta_x(h_1 - h_2).
\end{align*}
\]

(6.1)

Figure 3. Special isothermic surface with wave potential \( \Psi(x, y) \) (see Figure 1) and spectral parameter \( m = 1 \).

REMARK 6.2. Note that if \( A : U \to G \) is a curved flat framing, then \( f = [A_0] : U \to \mathcal{M} \) and \( \bar{f} = [A_4] : U \to \mathcal{M} \) are isothermic immersions with Calapso potentials

\[
\Phi = \frac{1}{2}(h_1 - h_2)\Theta, \quad \bar{\Phi} = -\frac{1}{2}(h_1 + h_2)\Theta.
\]

For the general notion of a curved flat we refer to Ferus–Pedit [18].
respectively. We say that \( A \) is a curved flat framing along \( f \) and call \( \Phi \) the Calapso potential of \( A \). The dual frame field is defined to be \( \bar{A} = AJ \), where \( J \in G \) is given by
\[
\begin{pmatrix}
0 & 0 & 0 & 0 & 1 \\
0 & -1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 0 \\
\end{pmatrix}.
\]
\[
(6.2)
\]
In particular, \( \bar{A} \) is a curved flat framing along \( \bar{f} \). The mapping \( \sigma := A_3 : U \to S^4_1 \subset \mathbb{R}^4_1 \) defines a sphere congruence whose envelopes are \( f \) and \( \bar{f} \). Moreover, the correspondence induced by \( \sigma \) preserves the curvature lines and is conformal. This amounts to saying that \( \sigma \) is a Darboux congruence. Accordingly, \( \bar{f} \) is called a Darboux transform of \( f \). Any Darboux transform of \( f \) arises in this way. \( \Box \)

**Remark 6.3.** The first envelope \( f : U \to \mathcal{M} \) of a curved flat framing \( A \) is a special isothermic immersion if and only if, in addition to (6.1), the triple \( (\Theta, h_1, h_2) \) satisfies an equation of the form
\[
4\Theta^2|\nabla H|^2 + M^2 + 2AM + 2CL + D = 0,
\]
where \( A, C, D \) are real constants and \( M = -HL \) being \( L = \Theta^2(h_1 - h_2) \) and \( 2H = h_1 + h_2 \). We then say that \( A \) is a special curved flat framing of type \( (A, 0, C, D) \). Notice that both the envelopes of the curved flat are special isothermic immersions if and only if \( C = 0 \). In this case we shall say that \( \bar{f} \) is a special Darboux transform of \( f \).

We now have:

**Proposition 6.4.** Let \( f : U \to \mathcal{M} \) be a special isothermic immersion of type \( (A, 0, C, D) \) with deformation parameter \( m \) and Calapso potential \( \Phi \), and let \( B(m) : U \to G \) be the Bryant’s central frame field along \( f \). Next, let \( V : U \to \mathcal{L} \) be any solution of the \( D_h \)-linear system
\[
dV = -\beta(h)V, \quad h \neq m,
\]
satisfying the constraint \( C_2(V) = 0 \). Then
\[
\bar{f} := [v^0B(m)_0 + v^1B(m)_1 + v^2B(m)_2 + v^3B(m)_3 + v^4B(m)_4]
\]
defines a special isothermic immersion which is a special Darboux transform of \( f \) such that
\[
\bar{\Phi} = E(\Phi, V).
\]

**Proof.** The connection form \( \alpha \) of the framing \( A = B(m)g^+(V) \)
\[
\alpha = g^+(V)^{-1}d[g^+(V)] + g^+(V)^{-1}\beta(m)g^+(V)
\]
takes the form \( \alpha = \alpha_r(\Theta, h_1, h_2) \), with
\[
(6.4) \quad \Theta = \frac{\Phi}{v^4}, \quad h_1 = v^4 - v^3, \quad h_2 = -(v^4 + v^3), \quad r = h - m.
\]
\cite{10}The result that an isothermic surface together with a Darboux transform form a curved flat in the pseudo-Riemannian symmetric space of pairs of distinct points in \( \mathcal{M} \) has been proved in \cite{10}. For the interpretation of the Darboux transforms as dressing transformations of loop groups we refer to Burstall \cite{10}.
It then follows that $A$ is a special curved flat framing of type $(-2h, 0, -2C_2(V), 4s)$, where $s$ denotes the character of $\Phi$. The first envelope $[A_0]$ represents the original immersion $f$ and the second envelope $[A_4]$ represents the Darboux transform $\tilde{f}$. If $C_2(V) = 0$, then $\tilde{f}$ is a special isothermal immersion and $\Phi = E(\Phi, V)$.

**Remark 6.5.** $\tilde{f}$ has deformation parameter $m$ and the normal frame field $\bar{B}(m) : U \to G$ along $\tilde{f}$ is given by

$$\bar{B}(m) = B(m)g^+(V)Jg^+(T),$$

where $T : U \to \mathcal{L}$ is the smooth map defined by

$$T = \left(\frac{\Phi^2v^0v^4}{(h-m)(v^3)^2}, \frac{v^1v^4}{(v^3)^2}, \frac{v^2v^4}{v^3}, \frac{v^4}{v^3}, \frac{(h-m)\Phi^{-2}(v^4)^2}{v^3}\right).$$

**Example 6.6 (Special Darboux transforms of Dupin cyclides).** Special isothermic maps with Calapso potential $\Phi = 1$ are given (up to the action of the conformal group) by the following formulae (where $m$ is the deformation parameter of the family)

- if $m > 1$:

$$f_m(x, y) = \left(\frac{\sqrt{2}\eta\sinh(\zeta x)}{2 + \eta\cosh(\zeta x)}, \frac{\sqrt{2}\zeta\cos(\eta y)}{2 + \eta\cosh(\zeta x)}, \frac{\sqrt{2}\zeta\sin(\eta y)}{2 + \eta\cosh(\zeta x)}\right),$$

- if $m = 1$:

$$f_1(x, y) = \left(\frac{8x}{4x^2 + 1}, \frac{4\cos(2y)}{4x^2 + 1}, \frac{4\sin(2y)}{4x^2 + 1}\right),$$

- if $0 \leq m < 1$:

$$f_m(x, y) = \left(\frac{\sqrt{2}\eta\cos(\zeta x)}{2 + \eta\sin(\zeta x)}, \frac{\sqrt{2}\zeta\cos(\eta y)}{2 + \eta\sin(\zeta x)}, \frac{\sqrt{2}\zeta\sin(\eta y)}{2 + \eta\sin(\zeta x)}\right).$$

The surfaces $S_m \subset \mathbb{R}^3$ parametrized by the maps $f_m : \mathbb{R}^2 \to \mathbb{R}^3$ are the Dupin cyclides. It is a classical result that Dupin cyclides are conformally equivalent to either a circular cone, a circular cylinder, or a torus of revolution. The normal frame field along $f_m$ is computed to be:

- if $m > 1$:

$$B(m) = \begin{pmatrix}
\eta\cosh(\zeta x) + 2 & \sinh(\zeta x) & 0 & \eta\cosh(\zeta x) + 2m \\
\sqrt{2}\eta \sinh(\zeta x) & \sqrt{2} & \eta\sinh(\zeta x) & \sqrt{2} \\
\sqrt{\zeta} \cos(\eta y) & 0 & -\sin(\eta y) & \sqrt{\zeta} \cos(\eta y) \\
\eta\cosh(\zeta x) - 2 & \sinh(\zeta x) & 0 & \eta\cosh(\zeta x) - 2m
\end{pmatrix},$$

where $\sqrt{\zeta} = \sqrt{h - m}(v^3)^2 - 2v^4$.
• if $m = 1$:

$$B(1) = \begin{pmatrix}
\frac{1}{8}(4x^2 + 1) & x & 0 & -\frac{1}{8}(4x^2 - 3) & \frac{1}{16}(4x^2 + 9) \\
x & 1 & 0 & -x & \frac{x}{2} \\
\frac{1}{2}\cos(2y) & 0 & -\sin(2y) & \frac{1}{2}\cos(2y) & -\frac{1}{4}\cos(2y) \\
\frac{1}{2}\sin(2y) & 0 & \cos(2y) & -\frac{1}{2}\sin(2y) & -\frac{1}{4}\sin(2y) \\
1 & 0 & 0 & -1 & \frac{1}{2} \\
\end{pmatrix};$$

• if $0 \leq m < 1$:

$$B(m) = \begin{pmatrix}
\frac{2+\eta\sin(\zeta x)}{\sqrt{2\zeta}} & \cos(\zeta x) & 0 & -\frac{2m+\eta\sin(\zeta x)}{\sqrt{2\zeta}} & \frac{2+(2m-1)\eta\sin(\zeta x)}{2\sqrt{2\zeta}} \\
\frac{\zeta}{\eta} & -\sin(\zeta x) & 0 & -\frac{\zeta}{\eta} & -\frac{\eta}{\zeta} \\
\frac{\zeta}{\sin(\eta x)} & 0 & -\sin(\eta y) & \frac{\zeta}{\sin(\eta y)} & -\frac{\eta}{\zeta} \\
\frac{2-\eta\sin(\zeta x)}{\sqrt{2\zeta\eta}} & \cos(\zeta x) & 0 & -\frac{2m+\eta\sin(\zeta x)}{\sqrt{2\zeta\eta}} & \frac{2-(2m-1)\eta\sin(\zeta x)}{2\sqrt{2\zeta\eta}} \\
\frac{1}{\sqrt{2\zeta\eta}} & 0 & \cos(\eta y) & \frac{1}{\sqrt{2\zeta\eta}} & \frac{\eta}{\zeta} \\
\end{pmatrix}.$$  

**Figure 4.** Darboux transforms of Dupin cyclides; special isothermic surface with wave potential $\Phi(4/3, 0, 1/100, 1/300)$ and spectral parameter $m = 1$.

According to the above discussion, the special Darboux transforms of $f_m$ are given by

$$(6.9) \quad D(h, a^1, a^2, a^3)(f_m) = [B(m)V(h, a^1, a^2, a^3)],$$

where $V(h, a^1, a^2, a^3) : \mathbb{R}^2 \to L$ is the solution of the $D_h$-linear system for $\Phi = 1$ with initial condition $X(a^1, a^2, a^3)$ given by (6.4). Thus, $D(h, a^1, a^2, a^3)(f_m)$ is a special isothermic immersion with Calapso potential $\Phi(h, a^1, a^2, a^3)$. 
7. The superposition formula and two-soliton solutions

**Theorem 7.1.** Let $\Phi : U \to \mathbb{R}$ be a wave potential with character $s$, and let $V = (v^0, ..., v^4)^t$ and $W = (w^0, ..., w^4)^t$ be two systems of $h$ and $k$-transforming functions, respectively, $h \neq k$. Let $\Phi_1 = E(\Phi, V)$ and $\Phi_2 = E(\Phi, W)$ be the corresponding Bäcklund transforms. Then,

$$\Phi_3 = \Phi - (h - k) \frac{v^3 w^3}{\langle V, W \rangle} \left[ \frac{\Phi_2 - \Phi_1}{\Phi_1 \Phi_2} \right].$$

is a wave potential with character $s$.

**Proof.** Let $B(m) : U \to G$ be the normal frame with potential $\Phi$ and spectral parameter $m$, $m \neq h$. Consider the curved flat framing $A := B(m)g^+(V)$ and its dual framing $\tilde{A} := AJ$; then $\tilde{A}$ is a curved flat framing and the corresponding normal frame $\tilde{B}(m)$ is computed to be

$$\tilde{B}(m) = \tilde{A}g^+(V) = B(m)g^+(V)Jg^+(T),$$

where $T : U \to \mathcal{L}$ is the smooth map defined as in (6.5). Next, consider the map $Y : U \to \mathcal{L}$ given by

$$Y = T - 1 * Y = \left( -\langle T, Y \rangle, \frac{y^1 t^4 - y^4 t^1}{t^4}, \frac{y^2 t^4 - y^4 t^2}{t^4}, \frac{y^3 t^4 - y^4 t^3}{t^4}, \frac{y^4}{t^4} \right)^t.$$

Then $\tilde{A} := \tilde{A}g^+(Y)$ is a curved flat framing with spectral parameter $k - m$ and Calapso potential $\Phi_1$. Further, let $L : U \to \mathcal{L}$ be defined by

$$L = T^{-1} * Y = \left( -\langle T, Y \rangle, \frac{y^1 t^4 - y^4 t^1}{t^4}, \frac{y^2 t^4 - y^4 t^2}{t^4}, \frac{y^3 t^4 - y^4 t^3}{t^4}, \frac{y^4}{t^4} \right)^t,$$

so that

$$\tilde{L} = [\tilde{A}g^+(T)][g^+(T^{-1})g^+(Y)] = \tilde{B}(m)g^+(L).$$
Since $B(m)g^+(L)$ is a curved flat framing with Calapso potential $\Phi_1$ and spectral parameter $k-m$, then $L$ is a solution of the $D_k$-system with potential $\Phi_1$. Combining (6.5), (7.2), (7.3) and using the constraints $C_2(V) = C_2(W) = 0$, it is a computational matter to check that also $L$ satisfies the constraint

$$C_2(L) = \Phi_1^2 L^0 - kL^3 + \frac{s}{2} \Phi_1^{-2} L^4 = 0.$$ 

This implies that $L$ is a system of $D_k$-transforming functions for the potential $\Phi_1$. In particular, the Bäcklund transform $E(\Phi_1, L)$ is a new solution of the differential equation (1.2). On the other hand, $E(\Phi_1, L)$ is computed to be

$$(7.4)$$

$$E(\Phi_1, L) = \frac{(v^1 w^1 + v^2 w^2 + v^3 w^3 - v^0 w^4 - v^4 w^0) \Phi^2 - (h-k)(w^3 v^4 - w^4 v^3)}{(v^1 w^1 + v^2 w^2 + v^3 w^3 - v^0 w^4 - v^4 w^0) \Phi},$$

from which follows that

$$E(\Phi_1, L) = \Phi - v^3 w^3 \frac{(h-k)}{\langle V, W \rangle} \left[ \frac{\Phi_2 - \Phi_1}{\Phi_1 \Phi_2} \right].$$

\[\Box\]

Remark 7.2. The wave potential $\Phi_3$ can be realized as the Calapso potential of the special isothermic immersion

$$(7.5)$$

$$\left[ (B(m)g^+(V)Jg^+(Y))_1 \right],$$

which can be explicitly computed from the normal frame $B(m)$ and the two sets $V$ and $W$ of transforming functions.

Example 7.3 (Two-solitons and the corresponding isothermic surfaces). Two-soliton solutions can be computed by means of (5.4) and (7.1). We then obtain a six-parameter family of wave potentials given by

$$\Phi = 1 - \frac{(h-k)(w^3 v^4 - w^4 v^3)}{\langle V, W \rangle},$$

where $V$ and $W$ are the $h$ and $k$-transforming functions corresponding to the initial conditions $X(a^1, a^2, a^3)$ and $X(b^1, b^2, b^3)$. Special isothermic immersions with spectral parameter $m$ and Calapso potentials $\Phi$ can be constructed by using (7.5), (5.4) and the explicit formulae for the central frame fields $B(m)$ of Dupin cyclides.

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