Equitable power domination number of total graph of certain graphs

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Abstract. Let \( G(V, E) \), or simply \( G \), be a graph. A set \( S \subseteq V \) is said to be a power dominating set (PDS) if every vertex \( u \in V - S \) is observed by certain vertices in \( S \) by the following two rules: (a) if a vertex \( v \) in \( G \) is in PDS, then it dominates itself and all the adjacent vertices of \( v \) and (b) if an observed vertex \( v \) in \( G \) has \( k > 1 \) adjacent vertices and if \( k - 1 \) of these vertices are already observed, then the remaining one non-observed vertex is also observed by \( v \) in \( G \). A power dominating set \( S \subseteq V \) in \( G \) is said to be an equitable power dominating set (EPDS), if for every vertex \( v \in V - S \) there exists an adjacent vertex \( u \in S \) such that \(|d(u) - d(v)| \leq 1\), where \( d(u) \) and \( d(v) \) represents the degree of \( u \) and degree of \( v \), respectively. The minimum cardinality of an EPDS of \( G \) is called the equitable power domination number (EPDN) of \( G \), denoted by \( \gamma_{epd}(G) \). The vertices and edges of \( G \) are called elements. Two elements of \( G \) are neighbors if they are either incident or adjacent in \( G \). The total graph \( T(G) \) has vertex set \( V(G) \cup E(G) \) and two vertices of \( T(G) \) are adjacent whenever they are neighbors in \( G \). In this paper, we obtain the EPDN of the total graph of certain graphs.

Keywords
Power domination, Equitable power domination, Equitable power domination number, Total graph of a graph.

1. Introduction
The graphs considered in this paper are “non-trivial, simple, finite, connected, and undirected”. The concept of “domination in graphs” was introduced by Hedetniemi and Laskar in 1990 and for more results one can refer to [2, 5, 6, and 7]. Then the concept of equitability in graphs was studied by Swaminathan et al. [9, 10]. The notion of domination has found many applications in engineering and computer science [8]. In the year 1998, Haynes et al. introduced the notion of “power domination” in graphs [3, 7]. A dominating set [6] of \( G \) is a set \( S \) of vertices such that every vertex \( v \) in \( V - S \) has at least one neighbor in \( S \). The minimum cardinality of a dominating set of \( G \) is called the “domination number of \( G \)”, denoted by \( \gamma_d(G) \). The degree \( d(v) \) of a vertex \( v \) in \( G \) is the number of edges of \( G \) incident with \( v \) and any two adjacent vertices \( u \) and \( v \) in \( G \) are said to hold “equitable property” if \(|d(u) - d(v)| \leq 1\). Certain results on degree equitable domination can be found in [9]. A dominating set \( S \subseteq V \) in \( G \) is said to be an “equitable dominating set” [9] if for every \( v \in V - S \) there exists an adjacent vertex \( u \in S \) such that \(|d(u) - d(v)| \leq 1\). The minimum cardinality of an equitable dominating set of \( G \) is called the “equitable domination number of \( G \)”, denoted by \( \gamma_{ed}(G) \).

A set \( S \subseteq V \) is said to be a “power dominating set” (PDS) [3, 10] of \( G \) if every vertex \( u \in V - S \) is observed by some vertices in \( S \) by the following two rules: (a) If a vertex \( v \) in \( G \) is in PDS then it dominates itself and all the adjacent vertices of \( v \). (b) If an observed vertex \( v \) in \( G \) has \( k > 1 \) adjacent vertices and if \( k - 1 \) of these vertices are already observed, then the remaining non-
observed vertex is also observed by \( v \) in \( G \). The minimum cardinality of an power dominating set of \( G \) is called the “power domination number of \( G \)”, denoted by \( \gamma_{pd}(G) \).

In 2017, Banu Priya et al. introduced the notion of “equitable power domination” in graphs [1]. A power dominating set \( S \subseteq V \) in \( G \) is said to be an “equitable power dominating set”, if for every vertex \( v \in V - S \) there exists an adjacent vertex \( u \in S \) such that \( |d(u) - d(v)| \leq 1 \). The minimum cardinality of an EPDS of \( G \) is called the EPDN of \( G \), denoted by \( \gamma_{epd}(G) \). Note that an EPDS of \( G \) is not unique. The total graph \( T(G) \) has vertex set \( V(G) \cup E(G) \) and two vertices of \( T(G) \) are adjacent whenever they are “neighbors in \( G \)”. In this paper, we obtain the EPDN of the total graph of certain graphs.

2. Equitable Power Domination Number of Total Graph of Cycle, Path, Star, and Wheel

This section is devoted for obtaining the EPDN of total graph of some graphs such as path, cycle, star, and wheel. First, we recall the definitions of these graphs for the sake of completeness.

2.1. Definition [4]
A walk of \( G \) is an non-empty alternating sequence of vertices and edges \( v_0, e_1, v_1, e_2, v_2, \ldots, v_{n-1}, e_n, v_n \).

![Figure 1. \( v_0, e_1, v_1, e_2, v_2, e_3, v_3, e_4, v_4, e_5, v_5, e_6 \) is a walk](image)

2.2. Definition [4]
A cycle is a “closed walk” in which no vertex (except the first and last vertex) appears more than once.

2.3. Definition [4]
A path is an open walk in which “no vertex” appears more than once.

2.4. Theorem
The EPDN of the total graph of a path \( P_n \) is 1 for all \( n \).

Proof
Let \( P_n \) be a path with vertex set \( V = \{v_1, v_2, \ldots, v_n\} \) and edge set \( E = \{e_1, e_2, \ldots, e_{n-1}\} \). Construct the total graph of a path with \((T(P_n)) = \{v_1, v_2, \ldots, v_n\} \cup \{e_1, e_2, \ldots, e_{n-1}\} \) and \( E(T(P_n)) = E_1 \cup E_2 \cup E_3 \cup E_4 \), where \( E_1 = E(P_n), \ E_2 = \{(e_i, v_j) \mid 1 \leq i \leq n-1, i+1 \leq j \leq n\}, \ E_3 = \{v_i, v_j\} \mid 1 \leq i, j \leq n-1 \), and \( E_4 = \{(e_i, e_j) \mid 1 \leq i \leq n-2, i+1 \leq j \leq n\} \). Clearly, \( |V(T(P_n))| = 2n - 1 \). The degree of vertices of \( T(P_n) \) are as follows: \( d(v_1) = d(v_n) = 2 \), \( d(v_i) = d(e_{i-1}) = 3 \), and \( d(v_i) = 4 \) for \( 2 \leq i \leq n-1 \). Without loss of generality, let us choose any vertex except \( v_1 \) and \( v_n \) (to get the minimum cardinality), say \( v_2 \). Then by the equitable power domination property, all the remaining vertices in \( T(P_n) \) are observed by its neighboring vertices since \( |d(v_i) - d(v_j)| \leq 1 \ \forall \ 1 \leq i, j \leq n \) and \( |d(e_i) - d(e_j)| \leq 1 \ \forall \ 1 \leq i, j \leq n \). One such example is given in the figure 2. Thus \( \gamma_{epd}(T(P_n)) = 1 \).
2.5. Lemma
The total graph of a regular graph is regular.

2.6. Theorem
The EPDN of the total graph of a cycle, $\gamma_{epd}(T(C_n)) = 1$.

**Proof**
Let $C_n$ be the cycle with vertex set $V = \{v_1, v_2, ..., v_n\}$ and the edge set $E = \{e_1, e_2, ..., e_n\}$. Obtain the total graph of a cycle, $T(C_n)$, as follows: $V(T(C_n)) = V(C_n) \cup E(C_n)$ and $E(T(C_n)) = E_1 \cup E_2 \cup E_3$, where $E_1 = E(C_n)$, $E_2 = \{(e_i e_j): 1 \leq i \leq n - 1, i + 1 \leq j \leq n\} \cup (e_n e_1)$, and $E_3 = \{(v_i v_j): 1 \leq i \leq n\}$. By Lemma 2.5, the total graph of a cycle is regular, so it is enough to choose only one vertex from $V(T(C_n))$ to be in the EPDS. Thus $S = \{e_2\}$ and $\gamma_{epd}(T(C_n)) = 1$.

2.7. Definition
A star is a complete bipartite graph $K_{1,n}$.

2.8. Theorem
The EPDN of the total graph of a star, $\gamma_{epd}(T(K_{1,n})) = n + 2$ for $n \geq 3$.

**Proof**
Let $K_{1,n}$ be the given star with the central vertex $u_0$, pendant vertices $u_1, u_2, ..., u_n$, and the edge set $E = \{e_1, e_2, ..., e_n\}$. Obtain the total graph of a star $K_{1,n}$, denoted by $T(K_{1,n})$, with $V(T(K_{1,n})) = V(K_{1,n}) \cup E(K_{1,n})$ and $E(T(K_{1,n})) = E_1 \cup E_2 \cup E_3$, where $E_1 = E(K_{1,n})$, $E_2 = \{(e_i e_j): 1 \leq i \leq n, i + 1 \leq j \leq n - 1\}$, and $E_3 = \{(e_i v_0): 1 \leq i \leq n\} \cup \{(e_i v_j): 1 \leq i \leq n\}$. One can notice that there are $2n + 1$ vertices in $T(K_{1,n})$. And also $d(v_0) = 2n$ in $T(K_{1,n})$, which does not equitably power dominate any one of the neighbor vertices, so we choose $v_0$ to be in EPDS. The elements in $E_2$ itself form a complete graph with degree $n - 1$ and therefore it is enough to choose one vertex from $E_2$ to be in EPDS, say $e_1$. Further, $d(v_i) = 2$ for all $1 \leq i \leq n$ and no vertex equitably power dominates any of its neighbors. So one has to choose all $v_i, 1 \leq i \leq n$ to be in $S$. Thus $S = \{v_0\} \cup \{e_1\} \cup \{v_i\}$ for $1 \leq i \leq n$. Hence $|S| = n + 2$.

2.9. Theorem
Let \( g_1331/g2364, g_1331/g2320, \ldots, g_1331/g2321 \) \( g_1323/g2364, g_1323/g2320, \ldots, g_1323/g3041 \) be \( g_1324/g1344/g3041 \) where \( g_1324/g1344/g3041 \) is the central vertex. We label the edges are in the rim as \( \{e_1, e_2, \ldots, e_n\} \) and \( n \) edges are in the spoke as \( \{e'_1, e'_2, e'_3, \ldots, e'_n\} \). Thus \( E(W_{1,n}) = e_1, e_2, \ldots, e_n, e'_1, e'_2, e'_3, \ldots, e'_n \). Construct the total graph of a wheel, denoted by \( T(W_{1,n}) \), with \( V(T(W_{1,n})) = V(W_{1,n}) \cup E(W_{1,n}) \). Clearly, \( |V(T(W_{1,n}))| = 3n + 1 \). We obtain edge set \( E(T(W_{1,n})) \) as per the definition of total graph. One can notice that the degree of central vertex in \( T(W_{1,n}) \) is \( 2n \), i.e., \( d(v_0) = 2n \) and moreover, \( d(v) = 6 \) for each \( v \) in \( T(W_{1,n}) \) except \( v_0 \). No vertices adjacent to \( v_0 \) satisfy equitable property. So to obtain an EPDS, one has to choose \( v_0 \) to be in \( S \). Out of the remaining vertices, it is enough to choose 2 vertices from the rim which equitably power dominates the remaining vertices in \( T(W_{1,n}) \). Thus \( |S| = 3 \).

3. Equitable power domination number of crown graph, complete graph, and ladder graph

This section is devoted to obtain the EPDN of the total graph of crown, complete, and ladder graphs. We also recollect the definitions of these graphs.

3.1. Definition
The crown graph, denoted by \( CR_n \), is obtained from \( C_n \) by attaching a pendant edge at all vertices of \( C_n \).

![Figure 4. A crown graph CR_n](image)

3.2. Theorem
The EPDN of the total graph of a crown graph \( CR_n \), \( \gamma_{epd}(T(CR_n)) = 2n + 1 \) for \( n \geq 3 \).

Proof
Let \( CR_n \) be the given crown graph with \( = \{v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_n\} \), where \( v_1, v_2, \ldots, v_n \) are in the rim and \( u_1, u_2, \ldots, u_n \) are the pendent vertices of \( CR_n \). One can notice that the number of edges in \( CR_n \) is \( 2n \). We name the \( n \) edges in the rim as \( \{e_1, e_2, \ldots, e_n\} \) and the remaining \( n \) edges incident with pendent vertices as \( \{e'_1, e'_2, e'_3, \ldots, e'_n\} \). Thus \( E(CR_n) = \{e_1, e_2, \ldots, e_n, e'_1, e'_2, e'_3, \ldots, e'_n\} \). Construct the total graph of a crown graph, denoted by \( T(CR_n) \), as follows: \( V(T(CR_n)) = V(CR_n) \cup E(CR_n) \) and \( E(T(CR_n)) = E_1 \cup E_2 \cup E_3 \cup E_4 \), where \( E_1 = E(CR_n), \ E_2 = \{(e_i, e_j): 1 \leq i \leq n - 1, i + 1 \leq j \leq n\} \cup \{(e_i, e'_i): 1 \leq i \leq n\}, \ E_3 = \{(v_i, e_i): 1 \leq i \leq n\} \cup \{(v_i, e'_i): 1 \leq i \leq n\} \} \) and \( E_4 = \{(e_i, v_j): 1 \leq i \leq n - 1, i + 1 \leq j \leq n\} \cup \{(e_i, v_j): 1 \leq i \leq n - 1, i + 1 \leq j \leq n\} \). Clearly \( |V(T(CR_n))| = 3n \). Notice that degree of \( u_i, 1 \leq i \leq n \) in \( T(W_{1,n}) \) is 2 which do not equitably power dominate any of its neighbors so one has to choose every \( u_i, 1 \leq i \leq n \). Similarly, vertices incident with \( u_i, 1 \leq i \leq n \) are all of degree 4 which do not satisfy EPD property so one must choose vertices incident with \( u_i, 1 \leq i \leq n \). Then from the remaining \( n \) vertices in the rim are of degree 6, it is enough to choose one vertex, say \( v_1 \), which in turn observes the remaining vertices by the property of EPD. One such example is given in the figure 5. Thus \( |S| = 2n + 1 \).
3.4. Theorem [1]
Let $K_n$ be a complete graph. Then $\gamma_{epd}(K_n) = 1$.

3.5. Theorem
The EPDN of the total graph of a complete graph $K_n$, $\gamma_{epd}(T(K_n)) = 2$, for $n \geq 4$.

Proof
Let $K_n$ be a complete graph with $V = \{u_1, u_2, ..., u_n\}$ and $E = \{e_1, e_2, ..., e_m\}$. Obtain the total graph of a complete graph, $T(K_n)$, with $V(T(K_n)) = V(K_n) \cup E(K_n)$. Note that the degree of each vertex of $T(K_n)$ is $2n - 2$ and also the graph is regular. To obtain EPD set $S$, without loss of generality let us choose one vertex, say $u_1$, of $K_n$ in $T(K_n)$, then $u_1$ observes $2n - 2$ vertices which includes all the vertices of $K_n$. Now one can also notice that each observed vertex has more than one non-observed neighbor vertices and so one has to choose one more non-observed vertex, say $w$, in $T(K_n)$ to be in $S$. As $T(K_n)$ being a regular graph, $w$ observes the remaining vertices in $T(K_n)$. Thus $S = \{w, u_1\}$ and hence $\gamma_{epd}(T(K_n)) = 2$.

3.5. Result
Total graph of a complete graph is regular.

3.6. Definition
The ladder graph $L_n$ is defined by $L_n = P_n \times K_2$, where $P_n$ is a path and $\times$ denotes the Cartesian product and $K_2$ is a complete graph.

3.7. Theorem
The EPDN of the total graph of a ladder $P_2 \times P_n$, $\gamma_{epd}(T(P_2 \times P_n)) = \left\lfloor \frac{n}{2} \right\rfloor$ for $n \geq 3$. 
Figure 7. The total graph of $P_2 \times P_6$

**Proof**

Let $P_2 \times P_n$ be a ladder with $V(P_2 \times P_n) = \{u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_n\}$ and $E(P_2 \times P_n) = \{e_1, e_2, \ldots, e_m\}$. Obtain the total graph of a ladder graph, $T(P_2 \times P_n)$, with $V(T(P_2 \times P_n)) = V_1 \cup V_2$ where $V_1 = V(P_2 \times P_n)$ and $V_2 = \{u_1, u_2, u_3, \ldots, u_{n-1}, v_1, v_2, v_3, \ldots, v_{n-1}, e_1, e_2, \ldots, e_n\}$, the set of newly added vertices. One can note that the degrees of the newly added vertices in $T(P_2 \times P_n)$ are 4, 5 and 6 as shown in the Figure 3.3. Without loss of generality, first we choose $e_4$ to be in $S$ which equitably power dominates $u_4, u_4', v_4$, and $v_4'$. Now as the observed vertices have more than two non-observed vertices, they do not equitably power dominate any other vertices. Then we choose $v_4$ to be in $S$ for the sake of minimum cardinality. Further, $v_2'$ equitably dominates $v_2, v_3, e_2, e_3, v_3$ and $v_3'$. Next for the vertices $v_2$ and $u_1'$ the only non-observed vertices are $u_2$ and $u_2$, respectively. So proceeding thus, we obtain that $\gamma_{epd}(T(P_2 \times P_n)) = \left\lfloor \frac{n}{2} \right\rfloor$ for $n \geq 3$.

4. **Conclusion**

The EPDN of the total graph of certain graphs are obtained. Obtaining EPDN of other classes of graphs is still open and this is for future work. Nodes with equal capacity may function very well in the system of phase measurement unit. As minimizing the phase measurement unit in the electric power monitoring system leads to minimizing the cost, we believe that the concept of equitable power domination in graphs may play a vital role in controlling the entire system.

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