Asymptotic Entropy of Random Walks on Free Products

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Abstract

Suppose we are given the free product $V$ of a finite family of finite or countable sets. We consider a transient random walk on the free product arising naturally from a convex combination of random walks on the free factors. We prove the existence of the asymptotic entropy and present three different, equivalent formulas, which are derived by three different techniques. In particular, we will show that the entropy is the rate of escape with respect to the Greenian metric. Moreover, we link asymptotic entropy with the rate of escape and volume growth resulting in two inequalities.

Key words: Random Walks, Free Products, Asymptotic Entropy.

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1 Introduction

Suppose we are given a finite family of finite or countable sets $V_1, \ldots, V_r$ with distinguished vertices $o_i \in V_i$ for $i \in \{1, \ldots, r\}$. The free product of the sets $V_i$ is given by $V := V_1 \ast \ldots \ast V_r$, the set of all finite words of the form $x_1 \ldots x_n$ such that each letter is an element of $\bigcup_{i=1}^{r} V_i \setminus \{o_i\}$ and two consecutive letters arise not from the same $V_i$. We consider a transient Markov chain $(X_n)_{n \in \mathbb{N}_0}$ on $V$ starting at the empty word $o$, which arises from a convex combination of transition probabilities on the sets $V_i$. Denote by $\pi_n$ the distribution of $X_n$. We are interested in whether the sequence $\mathbb{E}[-\log \pi_n(X_n)]/n$ converges, and if so, to compute this constant. If the limit exists, it is called the asymptotic entropy. In this paper, we study this question for random walks on general free products. In particular, we will derive three different formulas for the entropy by using three different techniques.

Let us outline some results about random walks on free products: for free products of finite groups, Mairesse and Mathéus [21] computed an explicit formula for the rate of escape and asymptotic entropy by solving a finite system of polynomial equations. Their result remains valid in the case of free products of infinite groups, but one needs then to solve an infinite system of polynomial equations. Gilch [11] computed two different formulas for the rate of escape with respect to the word length of random walks on free products of graphs by different techniques, and also a third formula for free products of (not necessarily finite) groups. The techniques of [11] are adapted to the present setting. Asymptotic behaviour of return probabilities of random walks on free products has also been studied in many ways; e.g. Gerl and Woess [10, 28], Sawyer [24], Cartwright and Soardi [5], and Lalley [18], Candellero and Gilch [4].

Our proof of existence of the entropy involves generating functions techniques. The techniques we use for rewriting probability generating functions in terms of functions on the factors of the free product were introduced independently and simultaneously by Cartwright and Soardi [5], Woess [28], Voiculescu [27] and McLaughlin [22]. In particular, we will see that asymptotic entropy is the rate of escape with respect to a distance function in terms of Green functions. While it is well-known by Kingman’s subadditive ergodic theorem (see Kingman [17]) that entropy (introduced by Avez [11]) exists for random walks on groups whenever $\mathbb{E}[-\log \pi_1(X_1)] < \infty$, existence for random walks on other structures is not known a priori. We are not able to apply Kingman’s theorem in our present setting, since we have no (general) subadditivity and we have only a partial composition law for two elements of the free product. For more details about entropy of random walks on groups we refer to Kaimanovich and Vershik [14] and Derriennic [7].

An important link between drifts and harmonic analysis was obtained by Varopoulos [26]. He proved that for symmetric finite range random walks on groups the existence of non-trivial bounded harmonic functions is equivalent to a non-zero rate of escape. Karlsson and Ledrappier [16] generalized this result to symmetric random walks with finite first moment of the step lengths. This leads to a link between the rate of escape and the entropy of random walks, compare e.g. with Kaimanovich and Vershik [14] and Erschler [8]. Erschler and Kaimanovich [9] asked if drift and entropy of random walks on groups vary continuously on the probability measure, which governs the random walk. We prove real-analyticity of the entropy when varying the probability measure of constant support; compare also with the recent work of Ledrappier [19], who simultaneously proved this property for finite-range random walks on free groups.

Apart from the proof of existence of the asymptotic entropy $h = \lim_{n \to \infty} \mathbb{E}[-\log \pi_n(X_n)]/n$ (Theorem 3.7), we will calculate explicit formulas for the entropy (see Theorems 3.7, 3.8, 5.1 and Corollary 4.2) and we will show that the entropy is non-zero. The technique of our proof of exis-
tence of the entropy was motivated by Benjamini and Peres [2], where it is shown that for random walks on groups the entropy equals the rate of escape w.r.t. the Greenian distance; compare also with Blachère, Haïssinsky and Mathieu [3]. We are also able to show that, for random walks on free products of graphs, the asymptotic entropy equals just the rate of escape w.r.t. the Greenian distance (see Corollary 3.3 in view of Theorem 3.7). Moreover, we prove convergence in probability free products of graphs, the asymptotic entropy equals just the rate of escape w.r.t. the Greenian the sequence $-\frac{1}{n} \log \pi_n(X_n)$ to $\ell$ (see Corollary 3.11), and we show also that $\ell$ can be computed along almost every sample path as the limes inferior of the aforementioned sequence (Corollary 3.9). In the case of random walks on discrete groups, Kingman’s subadditive ergodic theorem provides both the almost sure convergence and the convergence in $L_1$ to the asymptotic entropy; in the case of general free products there is neither a global composition law for elements of the free product nor subadditivity. Thus, in the latter case we have to introduce and investigate new processes. The question of almost sure convergence of $-\frac{1}{n} \log \pi_n(X_n)$ to some constant $\ell$, however, remains open. Similar results concerning existence and formulas for the entropy are proved in Gilch and Müller [12] for random walks on directed covers of graphs. The reasoning of our proofs follows the argumentation in [12]: we will show that the entropy equals the rate of escape w.r.t. some special length function, and we deduce the proposed properties analogously. In the present case of free products of graphs, the reasoning is getting more complicated due to the more complex structure of free products in contrast to directed covers, although the main results about existence and convergence types are very similar. We will point out these difficulties and main differences to [12] at the end of Section 3.2. Finally, we will link entropy with the rate of escape and the growth rate of the free product, resulting in two inequalities (Corollary 5.4).

The plan of the paper is as follows: in Section 2 we define the random walk on the free product and the associated generating functions. In Section 3 we prove existence of the asymptotic entropy and give also an explicit formula for it. Another formula is derived in Section 4 with the help of double generating functions and a theorem of Sawyer and Steger [25]. In Section 5 we use another technique to compute a third explicit formula for the entropy of random walks on free products of (not necessarily finite) groups. Section 6 links entropy with the rate of escape and the growth rate of the free product. Sample computations are presented in Section 7.

2 Random Walks on Free Products

2.1 Free Products and Random Walks

Let $\mathcal{I} := \{1, \ldots, r\} \subseteq \mathbb{N}$, where $r \geq 2$. For each $i \in \mathcal{I}$, consider a random walk with transition matrix $P_i$ on a finite or countable state space $V_i$. W.l.o.g. we assume that the sets $V_i$ are pairwise disjoint and we exclude the case $r = 2 = |V_1| = |V_2|$ (see below for further explanation). The corresponding single and $n$-step transition probabilities are denoted by $p_i(x, y)$ and $p_i^{(n)}(x, y)$, where $x, y \in V_i$. For every $i \in \mathcal{I}$, we select an element $o_i$ of $V_i$ as the “root”. To help visualize this, we think of graphs $X_i$ with vertex sets $V_i$ and roots $o_i$ such that there is an oriented edge $x \to y$ if and only if $p_i(x, y) > 0$. Thus, we have a natural graph metric on the set $V_i$. Furthermore, we shall assume that for every $i \in \mathcal{I}$ and every $x \in V_i$ there is some $n_x \in \mathbb{N}$ such that $p_i^{(n_x)}(o_i, x) > 0$. For sake of simplicity we assume $p_i(x, x) = 0$ for every $i \in \mathcal{I}$ and $x \in V_i$. Moreover, we assume that the random walks on $V_i$
are uniformly irreducible, that is, there are \( \epsilon_0(i) > 0 \) and \( K_i \in \mathbb{N} \) such that for all \( x, y \in V_i \)

\[
p_i(x, y) > 0 \quad \Rightarrow \quad p_i^{(k)}(x, y) \geq \epsilon_0(i)
\]

for some \( k \leq K_i \). (2.1)

We set \( K := \max_{i \in \mathcal{I}} K_i \) and \( \epsilon_0 := \min_{i \in \mathcal{I}} \epsilon_0(i) \). For instance, this property is satisfied for nearest neighbour random walks on Cayley graphs of finitely generated groups, which are governed by probability measures on the groups.

Let \( V_i^x := V_i \setminus \{ o_i \} \) for every \( i \in \mathcal{I} \) and let \( V^x := \bigcup_{i \in \mathcal{I}} V_i^x \). The free product is given by

\[
V := V_1 \ast \ldots \ast V_r
\]

\[
= \left\{ x_1 x_2 \ldots x_n \big| n \in \mathbb{N}, x_j \in V_i^x, x_j \in V_k^x \Rightarrow x_{j+1} \notin V_k^x \right\} \cup \{ o \}. \quad (2.2)
\]

The elements of \( V \) are “words” with letters, also called blocks, from the sets \( V_i^x \) such that no two consecutive letters come from the same \( V_i \). The empty word \( o \) describes the root of \( V \). If \( u = u_1 \ldots u_m \in V \) and \( v = v_1 \ldots v_n \in V \) with \( u_m \in V_i \) and \( v_1 \notin V_i \) then \( u v \) stands for their concatenation as words. This is only a partial composition law, which makes defining the asymptotic entropy more complicated than in the case of free products of groups. In particular, we set \( u o_i := u \) for all \( i \in \mathcal{I} \) and \( o u := u \). Note that \( V_i \subseteq V \) and \( o_i \) as a word in \( V \) is identified with \( o \). The block length of a word \( u = u_1 \ldots u_m \) is given by \( \| u \| := m \). Additionally, we set \( \| o \| := 0 \). The type \( \tau(u) \) of \( u \) is defined to be \( i \) if \( u_m \in V_i^x \); we set \( \tau(o) := 0 \). Finally, \( \bar{u} \) denotes the last letter \( u_m \) of \( u \). The set \( V \) can again be interpreted as the vertex set of a graph \( \mathcal{X} \), which is constructed as follows: take copies of \( \mathcal{X}_1, \ldots, \mathcal{X}_r \) and glue them together at their roots to one single common root, which becomes \( o \). Inductively, at each vertex \( v_1 \ldots v_k \) with \( v_k \in V_i \) attach a copy of every \( \mathcal{X}_j \), \( j \neq i \), and so on. Thus, we have also a natural graph metric associated to the elements in \( V \).

The next step is the construction of a new Markov chain on the free product. For this purpose, we lift \( P_i \) to a transition matrix \( \bar{P}_i \) on \( V \): if \( x \in V \) with \( \tau(x) \neq i \) and \( v, w \in V_i \), then \( \bar{P}_i(x v, x w) := p_i(v, w) \). Otherwise we set \( \bar{P}_i(x, y) := 0 \). We choose \( 0 < \alpha_1, \ldots, \alpha_r \in \mathbb{R} \) with \( \sum_{i \in \mathcal{I}} \alpha_i = 1 \). Then we obtain a new transition matrix on \( V \) given by

\[
P = \sum_{i \in \mathcal{I}} \alpha_i \bar{P}_i.
\]

The random walk on \( V \) starting at \( o \), which is governed by \( P \), is described by the sequence of random variables \( (X_n)_{n \in \mathbb{N}_0} \). For \( x, y \in V \), the associated single and \( n \)-step transition probabilities are denoted by \( p(x, y) \) and \( p^{(n)}(x, y) \). Thus, \( P \) governs a nearest neighbour random walk on the graph \( \mathcal{X} \), where \( P \) arises from a convex combination of the nearest neighbour random walks on the graphs \( \mathcal{X}_i \).

Theorem 3.3 in \cite{11} shows existence (including a formula) of a positive number \( \ell_0 \) such that \( \ell_0 = \lim_{n \to \infty} \| X_n \| / n \) almost surely. The number \( \ell_0 \) is called the rate of escape w.r.t. the block length. Denote by \( \pi_n \) the distribution of \( X_n \). If there is a real number \( h \) such that

\[
h = \lim_{n \to \infty} \frac{1}{n} \mathbb{E} \left[ -\log \pi_n(X_n) \right],
\]

then \( h \) is called the asymptotic entropy of the process \( (X_n)_{n \in \mathbb{N}_0} \); we write \( \mathbb{N}_0 := \mathbb{N} \setminus \{0\} \). If the sets \( V_i \) are groups and the random walks \( P_i \) are governed by probability measures \( \mu_i \), existence of the asymptotic entropy rate is well-known, and in this case we even have \( h = \lim_{n \to \infty} -\frac{1}{n} \log \pi_n(X_n) \) almost surely; see Derriennic \cite{7} and Kaimanovich and Vershik \cite{14}. We prove existence of \( h \) in the case of general free products.
2.2 Generating Functions

Our main tool will be the usage of generating functions, which we introduce now. The Green functions related to \( P_i \) and \( P \) are given by

\[
G_i(x_i, y_i | z) := \sum_{n \geq 0} p_i^{(n)}(x_i, y_i) z^n \quad \text{and} \quad G(x, y | z) := \sum_{n \geq 0} p^{(n)}(x, y) z^n,
\]

where \( z \in \mathbb{C}, x_i, y_i \in V_i \) and \( x, y \in V \). At this point we make the basic assumption that the radius of convergence \( R \) of \( G(\cdot, \cdot | z) \) is strictly bigger than 1. This implies transience of our random walk on \( V \). Thus, we may exclude the case \( r = 2 = |V_1| = |V_2| \), because we get recurrence in this case. For instance, if all \( P_i \) govern reversible Markov chains, then \( R > 1 \); see [29] Theorem 10.3. Furthermore, if \( R > 1 \) holds also if there is some \( i \in \mathcal{I} \) such that \( p_i^{(n)}(o_i, o_i) = 0 \) for all \( n \in \mathbb{N} \).

The first visit generating functions related to \( P_i \) and \( P \) are given by

\[
F_i(x_i, y_i | z) := \sum_{n \geq 0} \mathbb{P}[Y_n = y_i, \forall m \leq n - 1 : Y_m^{(i)} \neq y_i | Y_0^{(i)} = x_i] z^n \quad \text{and} \quad F(x, y | z) := \sum_{n \geq 0} \mathbb{P}[X_n = y, \forall m \leq n - 1 : X_m \neq y | X_0 = x] z^n,
\]

where \( (Y_n^{(i)})_{n \in \mathbb{N}_0} \) describes a random walk on \( V_i \) governed by \( P_i \). The stopping time of the first return to \( o \) is defined as \( T_o := \inf\{m \geq 1 | X_m = o\} \). For \( i \in \mathcal{I} \), define

\[
\overline{H}_i(z) := \sum_{n \geq 1} \mathbb{P}[T_o = n, X_1 \notin V_i^X] z^n \quad \text{and} \quad \xi_i(z) := \frac{\alpha_i z}{1 - \overline{H}_i(z)}.
\]

We write also \( \xi_i := \xi_i(1) \), \( \xi_{\min} := \min_{i \in \mathcal{I}} \xi_i \) and \( \xi_{\max} := \max_{i \in \mathcal{I}} \xi_i \). Observe that \( \xi_i < 1 \); see [11] Lemma 2.3. We have \( F(x_i, y_i | z) = F_i(x_i, y_i | \xi_i(z)) \) for all \( x_i, y_i \in V_i \); see Woess [29] Prop. 9.18c.

Thus,

\[
\xi_i(z) := \frac{\alpha_i z}{1 - \sum_{j \in \mathcal{I} \setminus \{i\}} \sum_{o \in V_j} \alpha_j p_j(o_j, s) z F_j(s, o_j | \xi_j(z))}.
\]

For \( x_i \in V_i \) and \( x \in V \), define the stopping times \( T_{x_i}^{(i)} := \inf\{m \geq 1 | Y_m^{(i)} = x_i\} \) and \( T_x := \inf\{m \geq 1 | X_m = x\} \), which take both values in \( \mathbb{N} \cup \{\infty\} \). Then the last visit generating functions related to \( P_i \) and \( P \) are defined as

\[
L_i(x_i, y_i | z) := \sum_{n \geq 0} \mathbb{P}[Y_n^{(i)} = y_i, T_{x_i}^{(i)} > n | Y_0^{(i)} = x_i] z^n,
\]

\[
L(x, y | z) := \sum_{n \geq 0} \mathbb{P}[X_n = y, T_x > n | X_0 = x] z^n.
\]

If \( x = x_1 \ldots x_n, y = x_1 \ldots x_n x_{n+1} \in V \) with \( \tau(x_{n+1}) = i \) then

\[
L(x, y | z) = L_i(o_i, x_{n+1} | \xi_i(z)); \tag{2.3}
\]

this equation is proved completely analogously to [29] Prop. 9.18c. If all paths from \( x \in V \) to \( w \in V \) have to pass through \( y \in V \), then

\[
L(x, w | z) = L(x, y | z) \cdot L(y, w | z);
\]
this can be easily checked by conditioning on the last visit of $y$ when walking from $x$ to $w$. We have the following important equations, which follow by conditioning on the last visits of $x_i$ and $x$, the first visits of $y_i$ and $y$ respectively:

$$
G_i(x_i, y_i | z) = G_i(x_i, x_i | z) \cdot L_i(x_i, y_i | z) = F_i(x_i, y_i | z) \cdot G_i(y_i, y_i | z),
$$

$$
G(x, y | z) = G(x, x | z) \cdot L(x, y | z) = F(x, y | z) \cdot G(y, y | z).
$$

(2.4)

Observe that the generating functions $F(\cdot, \cdot | z)$ and $L(\cdot, \cdot | z)$ have also radii of convergence strictly bigger than 1.

3 The Asymptotic Entropy

3.1 Rate of Escape w.r.t. specific Length Function

In this subsection we prove existence of the rate of escape with respect to a specific length function. From this we will deduce existence and a formula for the asymptotic entropy in the upcoming subsection.

We assign to each element $x_i \in V_i$ the “length”

$$
l_i(x_i) := -\log L(o, x_i | 1) = - \log L_i(o_i, x_i | \xi_i).
$$

We extend it to a length function on $V$ by assigning to $v_1 \ldots v_n \in V$ the length

$$
l(v_1 \ldots v_n) := \sum_{i=1}^n l_{\tau_i(v_i)} = - \sum_{i=1}^n \log L(o, v_i | 1) = - \log L(o, v_1 \ldots v_n | 1).
$$

Observe that the lengths can also be negative. E.g., this can be interpreted as height differences. The aim of this subsection is to show existence of a number $\ell \in \mathbb{R}$ such that the quotient $l(X_n)/n$ tends to $\ell$ almost surely as $n \to \infty$. We call $\ell$ the rate of escape w.r.t. the length function $l(\cdot)$.

We follow now the reasoning of [11] Section 3. Denote by $X_n^{(k)}$ the projection of $X_n$ to the first $k$ letters. We define the $k$-th exit time as

$$
e_k := \min \{ m \in \mathbb{N}_0 \mid \forall n \geq m : X_n^{(k)} \text{ is constant} \}.
$$

Moreover, we define $W_k := X_{e_k}$, $\tau_k := \tau(W_k)$ and $k(n) := \max \{ k \in \mathbb{N}_0 \mid e_k \leq n \}$. We remark that $\| X_n \| \to \infty$ as $n \to \infty$, and consequently $e_k < \infty$ almost surely for every $k \in \mathbb{N}$; see [11] Prop. 2.5. Recall that $\tilde{W}_k$ is just the last letter of the random word $X_{e_k}$. The process $(\tau_k)_{k \in \mathbb{N}}$ is Markovian and has transition probabilities

$$
\hat{q}(i, j) = \frac{\alpha_i \xi_i}{\alpha_j \xi_j} \frac{1 - \xi_j}{1 - \xi_i} \left( \frac{1}{1 - \xi_j G_j(o_j, o_j | \xi_j)} - 1 \right)
$$

for $i \neq j$ and $\hat{q}(i, i) = 0$; see [11] Lemma 3.4. This process is positive recurrent with invariant probability measure

$$
\nu(i) = C^{-1} \cdot \frac{\alpha_i (1 - \xi_i)}{\xi_i} (1 - (1 - \xi_i) G_i(o_i, o_i | \xi_i)),
$$

where $C := \sum_{i \in \mathcal{A}} \frac{\alpha_i (1 - \xi_i)}{\xi_i} (1 - (1 - \xi_i) G_i(o_i, o_i | \xi_i))$;

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see \cite{11} Section 3. Furthermore, the rate of escape w.r.t. the block length exists almost surely and is given by the almost sure constant limit

$$\ell_0 = \lim_{n \to \infty} \frac{\|X_n\|}{n} = \lim_{k \to \infty} \frac{k}{e_k} = \frac{1}{\sum_{i,j \in \mathcal{G}, i \neq j} \nu(i) \alpha_j \frac{1 - \xi_j}{1 - \xi_i} \gamma'_{i,j}(1)}$$

(see \cite{11} Theorem 3.3), where

$$\gamma_{i,j}(z) := \frac{1}{\alpha_i} \frac{\xi_i(z)}{\xi_j(z)} \left( \frac{1}{1 - \xi_j(z)} G_j(o_j, o_j | \xi_j(z)) - 1 \right).$$

**Lemma 3.1.** The process \((\hat{W}_k, \tau_k)_{k \in \mathbb{N}}\) is Markovian and has transition probabilities

$$q((g, i), (h, j)) = \begin{cases} \frac{\alpha_j \xi_j \xi_i}{\alpha_i} \frac{1 - \xi_j}{1 - \xi_i} L_j(o_j, h | \xi_j), & \text{if } i \neq j, \\ 0, & \text{if } i = j. \end{cases}$$

Furthermore, the process is positive recurrent with invariant probability measure

$$\pi(g, i) = \sum_{j \in \mathcal{G}} \nu(j) q((*, j), (g, i)).$$

**Remark:** Observe that the transition probabilities \(q((g, i), (h, j))\) of \((\hat{W}_k, \tau_k)_{k \in \mathbb{N}}\) do not depend on \(g\). Therefore, we will write sometimes an asterisk instead of \(g\).

**Proof.** By \cite{11} Section 3, the process \((\hat{W}_k, e_k - e_{k-1}, \tau_k)_{k \in \mathbb{N}}\) is Markovian and has transition probabilities

$$\bar{q}((g, m, i), (h, n, j)) = \begin{cases} \frac{1 - \xi_i}{1 - \xi_i} \sum_{s \in V_j} k_j^{(n-1)}(s)p(s, h), & \text{if } i \neq j, \\ 0, & \text{if } i = j, \end{cases}$$

where \(k_j^{(n)}(s) := \mathbb{P}[X_n = s, \forall l \leq n : X_l \notin V_i^\times | X_0 = o] \) for \(s \in V_i^\times \setminus V_i\). Thus, \((\hat{W}_k, \tau_k)_{k \in \mathbb{N}}\) is also Markovian and has the following transition probabilities if \(i \neq j:\)

$$q((g, i), (h, j)) = \sum_{n \geq 1} \bar{q}((g, *, i), (h, n, j)) = \frac{1 - \xi_j}{1 - \xi_i} \sum_{s \in V_j} k_j^{(n-1)}(s)p(s, h)$$

$$= \frac{1 - \xi_j}{1 - \xi_i} \sum_{s \in V_j} L_j(o_j, s | \xi_j) p(s, h) = \frac{\alpha_j}{\alpha_i} \frac{\xi_j}{1 - \xi_i} \frac{1 - \xi_j}{1 - \xi_i} L_j(o_j, h | \xi_j).$$

In the third equality we conditioned on the last visit of \(o\) before finally walking from \(o\) to \(s\) and we remark that \(h \in V_j^\times\). A straight-forward computation shows that \(\pi\) is the invariant probability
measure of \((\vec{W}_k, \tau_k)_{k \in \mathbb{N}}\), where we write \(\mathcal{A} := \{(g, i) \mid i \in \mathcal{I}, g \in V_i^x\}:
\[
\sum_{(g, i) \in \mathcal{A}} \pi(g, i) \cdot q((g, i), (h, j)) = \sum_{(g, i) \in \mathcal{A}} \sum_{k \in \mathcal{I}} \nu(k) \cdot q((\ast, k), (g, i)) \cdot q((\ast, i), (h, j))
\]
\[
= \sum_{i \in \mathcal{I}} q((\ast, i), (h, j)) \sum_{k \in \mathcal{I}} \nu(k) \sum_{g \in V_i^x} q((\ast, k), (g, i))
\]
\[
= \sum_{i \in \mathcal{I}} q((\ast, i), (h, j)) \sum_{k \in \mathcal{I}} \nu(k) \cdot \check{q}(k, i)
\]
\[
= \sum_{i \in \mathcal{I}} q((\ast, i), (h, j)) \cdot \nu(i) = \pi(h, j).
\]

Now we are able to prove the following:

**Proposition 3.2.** There is a number \(\ell \in \mathbb{R}\) such that
\[
\ell = \lim_{n \to \infty} \frac{l(X_n)}{n} \quad \text{almost surely.}
\]

**Proof.** Define \(h : \mathcal{A} \to \mathbb{R}\) by \(h(g, j) := l(g)\). Then \(\sum_{\lambda=1}^k h(\vec{W}_\lambda, \tau_\lambda) = \sum_{\lambda=1}^k l(\vec{W}_\lambda) = l(W_k)\). An application of the ergodic theorem for positive recurrent Markov chains yields
\[
\frac{l(W_k)}{k} = \frac{1}{k} \sum_{\lambda=1}^k h(\vec{W}_\lambda, \tau_\lambda) \to C_h := \int h \, d\pi,
\]
if the integral on the right hand side exists. We now show that this property holds. Observe that the values \(G_j(o_j, g | \xi_j)\) are uniformly bounded from above for all \((g, j) \in \mathcal{A}:\)
\[
G_j(o_j, g | \xi_j) = \sum_{n \geq 0} P_j^{(n)}(o_j, g) \xi_j^n \leq \frac{1}{1 - \xi_j} \leq \frac{1}{1 - \xi_{\max}}.
\]
For \(g \in V_*^x\), denote by \(|g|\) the smallest \(n \in \mathbb{N}\) such that \(p_j^{(n)}(o_{\tau(g)}, g) > 0\). Uniform irreducibility of the random walk \(P_i\) on \(V_i\) implies that there are some \(\epsilon_0 > 0\) and \(K \in \mathbb{N}\) such that for all \(j \in \mathcal{I}, \ x_j, y_j \in V_j\) with \(p_j(x_j, y_j) > 0\) we have \(p_j^{(k)}(x_j, y_j) \geq \epsilon_0\) for some \(k \leq K\). Thus, for \((g, j) \in \mathcal{A}\) we have
\[
G_j(o_j, g | \xi_j) \geq \epsilon_0 |g| \xi_j^K \geq (\epsilon_0 \xi_{\max}^{K}) |g|.
\]
Observe that the inequality \(|g| \cdot \log(\epsilon_0 \xi_{\min}) < \log 1/(1 - \xi_{\max})\) holds if and only if \(|g| < \log(1 - \xi_{\max})/\log(\epsilon_0 \xi_{\min})\). Define the sets
\[
M_1 := \{g \in V_*^x \mid |g| \geq \log(1 - \xi_{\max})/\log(\epsilon_0 \xi_{\min})\}, \quad M_2 := \{g \in V_*^x \mid |g| < \log(1 - \xi_{\max})/\log(\epsilon_0 \xi_{\min})\}.
\]
Recall Equation (2.4). We can now prove existence of \( \int h \, d\pi \):

\[
\int |h| \, d\pi = \sum_{(g,j) \in \mathcal{A}} \left| \log L_j(o_j, g|\xi_j) \right| \cdot \pi(g,j)
\]

\[
\leq \sum_{(g,j) \in \mathcal{A}} \left| \log G_j(o_j, g|\xi_j) \right| \cdot \pi(g,j) + \sum_{(g,j) \in \mathcal{A}} \left| \log G_j(o_j, o_j|\xi_j) \right| \cdot \pi(g,j)
\]

\[
\leq \sum_{(g,j) \in \mathcal{A} : g \in M_1} \left| \log G_j(o_j, g|\xi_j) \right| \cdot \pi(g,j)
\]

\[
+ \sum_{(g,j) \in \mathcal{A} : g \in M_2} \left| \log G_j(o_j, g|\xi_j) \right| \cdot \pi(g,j) + \max_{j \in \mathcal{Z}} \log G_j(o_j, o_j|\xi_j)
\]

\[
\leq \sum_{(g,j) \in \mathcal{A} : g \in M_1} \left| \log G_j(o_j, g|\xi_j) \right| \cdot \pi(g,j)
\]

\[
+ \sum_{(g,j) \in \mathcal{A} : g \in M_2} \left| \log G_j(o_j, g|\xi_j) \right| \cdot \pi(g,j) + \max_{j \in \mathcal{Z}} \log G_j(o_j, o_j|\xi_j)
\]

\[
\leq \sum_{(g,j) \in \mathcal{A} : g \in M_1} \left| \log G_j(o_j, g|\xi_j) \right| \cdot \pi(g,j)
\]

\[
+ \left| \log(1 - \xi_{\text{max}}) \right| + \max_{j \in \mathcal{Z}} \log G_j(o_j, o_j|\xi_j) < \infty,
\]

since \( \sum_{(g,j) \in \mathcal{A}} |g| \cdot \pi(g,j) < \infty \); see [11] Proof of Prop. 3.2. From this follows that \( l(W_k)/k \) tends to \( C_n \) almost surely. The next step is to show that

\[
\frac{l(X_n) - l(W_{k(n)})}{n} \xrightarrow{n \to \infty} 0 \quad \text{almost surely.} \tag{3.1}
\]

To prove this, assume now that we have the representations \( W_{k(n)} = g_1 g_2 \cdots g_{k(n)} \) and \( X_n = g_1 g_2 \cdots g_{k(n)} \cdots g_{|X_n|} \). Define \( M := \max\{ |\log(\epsilon_0 \xi_{\text{min}}^K)|, |\log(1 - \xi_{\text{max}})| \} \). Then:

\[
\left| l(X_n) - l(W_{k(n)}) \right| = \left| \sum_{i = k(n) + 1}^{||X_n||} \left| \log L_{\tau(i)}(o_{\tau(i)}, g_i | \xi_{\tau(i)}) \right| \right|
\]

\[
\leq \sum_{i = k(n) + 1}^{||X_n||} \left| \log \frac{G_{\tau(i)}(o_{\tau(i)}, g_i | \xi_{\tau(i)})}{G_{\tau(i)}(o_{\tau(i)}, o_{\tau(i)} | \xi_{\tau(i)})} \right|
\]

\[
\leq \sum_{i = k(n) + 1}^{||X_n||} \left| \log G_{\tau(i)}(o_{\tau(i)}, g_i | \xi_{\tau(i)}) \right|
\]

\[
+ \sum_{i = k(n) + 1}^{||X_n||} \left| \log G_{\tau(i)}(o_{\tau(i)}, g_i | \xi_{\tau(i)}) \right|
\]

\[
+ (||X_n|| - k(n)) \cdot \left| \log(1 - \xi_{\text{max}}) \right|
\]
we can conclude:

\[ \frac{\|X_n\|}{\|X_n\|} \left( \log(e_0 \xi_{\text{min}}^K) \right) \]

\[ + \sum_{i=0}^{\|X_n\|} \left( \log(1 - \xi_{\text{max}}^i) + (\|X_n\| - k(n)) \cdot \log(1 - \xi_{\text{max}}^i) \right) \]

\[ \leq \sum_{i=0}^{\|X_n\|} \left( g_i \cdot M + \sum_{i=0}^{\|X_n\|} M + (\|X_n\| - k(n)) \cdot M \right) \]

\[ \leq 3 \cdot M \cdot (n - e_k(n)). \]

Dividing the last inequality by \( n \) and letting \( n \to \infty \) provides analogously to Nagnibeda and Woess [23, Section 5] that \( \lim_{n \to \infty} \frac{(l(X_n) - l(W_k(n)))}{n} = 0 \) almost surely. Recall also that \( k/e_k \to \ell_0 \) and \( e_k(n)/n \to 1 \) almost surely; compare [23 Proof of Theorem D] and [11] Prop. 3.2, Thm. 3.3. Now we can conclude:

\[ \frac{l(X_n)}{n} = \frac{l(X_n) - l(W_k(n))}{n} + \frac{l(W_k(n))}{n} \xrightarrow{n \to \infty} C_h \cdot \ell_0 \quad \text{almost surely.} \quad (3.2) \]

We now compute the constant \( C_h \) from the last proposition explicitly:

\[ C_h = \sum_{(g,j) \in \mathcal{A}} l(g) \cdot \sum_{i \in \mathcal{V}} v(i) \cdot q((s,i),(g,j)) \]

\[ = \sum_{i \in \mathcal{V}} \sum_{j \in \mathcal{V}} -\log L_j(o_j, g|x_j) \cdot v(i) \frac{\alpha_j}{\alpha_i} \frac{1 - \xi_j}{1 - \xi_i} L_j(o_j, g|x_j). \quad (3.3) \]

We conclude this subsection with the following observation:

**Corollary 3.3.** The rate of escape \( \ell \) is non-negative and it is the rate of escape w.r.t. the Greenian metric, which is given by \( d_{\text{Green}}(x, y) := -\log F(x, y|1) \). That is,

\[ \ell = \lim_{n \to \infty} \frac{1}{n} \log F(e, X_n|1) \geq 0. \]

**Proof.** By (2.4), we get

\[ \ell = \lim_{n \to \infty} \frac{1}{n} \log F(e, X_n|1) - \frac{1}{n} \log G(X_n, X_n|1) + \frac{1}{n} \log G(o, o|1). \]

Since \( F(e, X_n|1) \leq 1 \) it remains to show that \( G(x, x|1) \) is uniformly bounded in \( x \in V \): for \( v, w \in V \), the first visit generating function is defined as

\[ U(v, w|z) = \sum_{n \geq 1} \mathbb{P}[X_n = w, \forall m \in \{1, \ldots, n - 1\} : X_m \neq w | X_0 = v] z^n. \quad (3.4) \]

Therefore,

\[ G(x, x|z) = \sum_{n \geq 0} U(x, x|z)^n = \frac{1}{1 - U(x, x|z)}. \]
Since \( U(x, x|z) < 1 \) for all \( z \in [1, R) \), \( U(x, x|0) = 0 \) and \( U(x, x|z) \) is continuous, strictly increasing and strictly convex, we must have \( U(x, x|1) \leq \frac{1}{R} \), that is, \( 1 \leq G(x, x|1) \leq (1 - \frac{1}{R})^{-1} \). This finishes the proof.

### 3.2 Asymptotic Entropy

In this subsection we will prove that \( \ell \) equals the asymptotic entropy, and we will give explicit formulas for it. The technique of the proof which we will give was motivated by Benjamini and Peres [2], where it is shown that the asymptotic entropy of random walks on discrete groups equals the rate of escape w.r.t. the Greenian distance. The proof follows the same reasoning as in Gilch and Müller [12].

Recall that we made the assumption that the spectral radius of \((X_n)_{n \in \mathbb{N}_0}\) is strictly smaller than 1, that is, the Green function \( G(o, o|z) \) has radius of convergence \( R > 1 \). Moreover, the functions \( \xi_i(z) \), \( i \in \mathcal{I} \), have radius of convergence bigger than 1. Recall that \( \xi_i = \xi_i(1) < 1 \) for every \( i \in \mathcal{I} \). Thus, we can choose \( \varrho \in (1, R) \) such that \( \xi_i(\varrho) < 1 \) for all \( i \in \mathcal{I} \). We now need the following three technical lemmas:

**Lemma 3.4.** For all \( m, n \in \mathbb{N}_0 \),

\[
p^{(m)}(o, X_n) \leq G(o, o|\varrho) \cdot \left( \frac{1}{1 - \max_{i \in \mathcal{I}} \xi_i(\varrho)} \right)^n \cdot \varrho^{-m}.
\]

**Proof.** Denote by \( \mathcal{C}_\varrho \) the circle with radius \( \varrho \) in the complex plane centered at 0. A straightforward computation shows for \( m \in \mathbb{N}_0 \):

\[
\frac{1}{2\pi i} \oint_{\mathcal{C}_\varrho} z^m \frac{dz}{z} = \begin{cases} 1, & \text{if } m = 0, \\ 0, & \text{if } m \neq 0. \end{cases}
\]

Let be \( x = x_1 \ldots x_t \in V \). An application of Fubini’s Theorem yields

\[
\frac{1}{2\pi i} \oint_{\mathcal{C}_\varrho} G(o, x|z) z^{-m} \frac{dz}{z} = \frac{1}{2\pi i} \oint_{\mathcal{C}_\varrho} \sum_{k \geq 0} p^{(k)}(o, x) z^{k - m} \frac{dz}{z}
\]

\[
= \frac{1}{2\pi i} \sum_{k \geq 0} p^{(k)}(o, x) \oint_{\mathcal{C}_\varrho} z^{k - m} \frac{dz}{z} = p^{(m)}(o, x).
\]

Since \( G(o, x|z) \) is analytic on \( \mathcal{C}_\varrho \), we have \( |G(o, x|z)| \leq G(o, x|\varrho) \) for all \( |z| = \rho \). Thus,

\[
p^{(m)}(o, x) \leq \frac{1}{2\pi} \cdot \varrho^{-m} \cdot G(o, x|\varrho) \cdot 2\pi \varrho = G(o, x|\varrho) \cdot \varrho^{-m}.
\]

Iterated applications of equations (2.3) and (2.4) provide

\[
G(o, x|\varrho) = G(o, o|\varrho) \prod_{k=1}^{||x||} L_{\tau(x_k)}(o_{\tau(x_k)}, x_k|\xi_i(\varrho)) \leq G(o, o|\varrho) \left( \frac{1}{1 - \max_{i \in \mathcal{I}} \xi_i(\varrho)} \right)^{||x||}.
\]

Since \( ||X_n|| \leq n \), we obtain

\[
p^{(m)}(e, X_n) \leq G(o, o|\varrho) \cdot \left( \frac{1}{1 - \max_{i \in \mathcal{I}} \xi_i(\varrho)} \right)^n \cdot \varrho^{-m}.
\]

\[\square\]
Lemma 3.5. Let \((A_n)_{n \in \mathbb{N}}, (a_n)_{n \in \mathbb{N}}, (b_n)_{n \in \mathbb{N}}\) be sequences of strictly positive numbers with \(A_n = a_n + b_n\). Assume that \(\lim_{n \to \infty} -\frac{1}{n} \log A_n = c \in [0, \infty)\) and that \(\lim_{n \to \infty} b_n/q^n = 0\) for all \(q \in (0, 1)\). Then \(\lim_{n \to \infty} -\frac{1}{n} \log a_n = c\).

Proof. Under the made assumptions it can not be that \(\liminf_{n \to \infty} a_n/q^n = 0\) for every \(q \in (0, 1)\). Indeed, assume that this would hold. Choose any \(q > 0\). Then there is a subsequence \((a_{n_k})_{k \in \mathbb{N}}\) with \(a_{n_k}/q^{n_k} \to 0\). Moreover, there is \(N_q \in \mathbb{N}\) such that \(a_{n_k}, b_{n_k} < q^{n_k}/2\) for all \(k \geq N_q\). But this implies

\[-\frac{1}{n_k} \log(a_{n_k} + b_{n_k}) \geq -\frac{1}{n_k} \log(q^{n_k}) = - \log q.

The last inequality holds for every \(q > 0\), yielding that \(\limsup_{n \to \infty} -\frac{1}{n} \log A_n = \infty\), a contradiction. Thus, there is some \(N \in \mathbb{N}\) such that \(b_n < a_n\) for all \(n \geq N\). We get for all \(n \geq N\):

\[-\frac{1}{n} \log(a_n + b_n) \leq -\frac{1}{n} \log(a_n) = -\frac{1}{n} \log\left(\frac{1}{2} a_n + \frac{1}{2} a_n\right)
\leq -\frac{1}{n} \log\left(\frac{1}{2} a_n + \frac{1}{2} b_n\right) \leq -\frac{1}{n} \log \frac{1}{2} - \frac{1}{n} \log(a_n + b_n).

Taking limits yields that \(-\frac{1}{n} \log(a_n)\) tends to \(c\), since the leftmost and rightmost side of this inequality chain tend to \(c\).

For the next lemma recall the definition of \(K\) from (2.1).

Lemma 3.6. For \(n \in \mathbb{N}\), consider the function \(f_n : V \to \mathbb{R}\) defined by

\[f_n(x) := \begin{cases} -\frac{1}{n} \log \sum_{m=0}^{Kn^2} p^{(m)}(o, x), & \text{if } p^{(n)}(o, x) > 0, \\ 0, & \text{otherwise}. \end{cases}
\]

Then there are constants \(d\) and \(D\) such that \(d \leq f_n(x) \leq D\) for all \(n \in \mathbb{N}\) and \(x \in V\).

Proof. Assume that \(p^{(n)}(o, x) > 0\). Recall from the proof of Corollary 3.3 that we have \(G(x, x|1) \leq (1 - \frac{1}{R})^{-1}\). Therefore,

\[\sum_{m=0}^{Kn^2} p^{(m)}(o, x) \leq G(o, x|1) \leq F(o, x|1) \cdot G(x, x|1) \leq \frac{1}{1 - \frac{1}{R}},\]

that is

\[f_n(x) \geq -\frac{1}{n} \log \frac{1}{1 - \frac{1}{R}}.
\]

For the upper bound, observe that, by uniform irreducibility, \(x \in V\) with \(p^{(n)}(o, x) > 0\) can be reached from \(o\) in \(N_x \leq K \cdot |x| \leq Kn\) steps with a probability of at least \(\varepsilon_0^{|x|}\), where \(\varepsilon_0 > 0\) from (2.1) is independent from \(x\). Thus, at least one of the summands in \(\sum_{m=0}^{Kn^2} p^{(m)}(o, x)\) has a value greater or equal to \(\varepsilon_0^{|x|} \geq \varepsilon_0^n\). Thus, \(f_n(x) \leq -\log \varepsilon_0\). 

Now we can state and prove our first main result:
Theorem 3.7. Assume $R > 1$. Then the asymptotic entropy exists and is given by

$$h = \ell_0 \cdot \sum_{g \in V_x} l(g) \pi(g, \tau(g)) = \ell.$$ 

Proof. By (2.4) we can rewrite $\ell$ as

$$\ell = \lim_{n \to \infty} -\frac{1}{n} \log L(o, X_n|1) = \lim_{n \to \infty} -\frac{1}{n} \log \frac{G(o, X_n|1)}{G(o, o|1)} = \lim_{n \to \infty} -\frac{1}{n} \log G(o, X_n|1).$$

Since

$$G(o, X_n|1) = \sum_{m \geq 0} p^{(m)}(o, X_n) \geq p^{(n)}(o, X_n) = \pi_n(X_n),$$

we have

$$\liminf_{n \to \infty} -\frac{1}{n} \log \pi_n(X_n) \geq \ell. \quad (3.5)$$

The next aim is to prove $\limsup_{n \to \infty} -\frac{1}{n} \mathbb{E} [\log \pi_n(X_n)] \leq \ell$. We now apply Lemma 3.5 by setting

$$A_n := \sum_{m \geq 0} p^{(m)}(o, X_n), \quad a_n := \sum_{m=0}^{Kn^2} p^{(m)}(o, X_n) \text{ and } b_n := \sum_{m \geq Kn^2+1} p^{(m)}(o, X_n).$$

By Lemma 3.4,

$$b_n \leq \sum_{m \geq Kn^2+1} G(o, o|\varrho) \left( \frac{1}{1 - \max_{i \in \mathcal{I}} \xi_i(\varrho)} \right)^n \cdot \left( \frac{1}{1 - \max_{i \in \mathcal{I}} \xi_i(\varrho)} \right)^n \cdot \varrho^{-Kn^2-1} \cdot \frac{1}{1 - \varrho^{-1}}.$$ 

Therefore, $b_n$ decays faster than any geometric sequence. Applying Lemma 3.5 yields

$$\ell = \lim_{n \to \infty} -\frac{1}{n} \log \sum_{m=0}^{Kn^2} p^{(m)}(o, X_n) \text{ almost surely.}$$

By Lemma 3.6 we may apply the Dominated Convergence Theorem and get:

$$\ell = \int \lim_{n \to \infty} -\frac{1}{n} \log \sum_{m=0}^{Kn^2} p^{(m)}(o, X_n) d\mathbb{P}$$

$$= \lim_{n \to \infty} \int -\frac{1}{n} \log \sum_{m=0}^{Kn^2} p^{(m)}(o, X_n) d\mathbb{P}$$

$$= \lim_{n \to \infty} -\frac{1}{n} \sum_{x \in V} p^{(n)}(o, x) \log \sum_{m=0}^{Kn^2} p^{(m)}(o, x).$$

Recall that Shannon’s Inequality gives

$$-\sum_{x \in V} p^{(n)}(o, x) \log \mu(x) \geq -\sum_{x \in V} p^{(n)}(o, x) \log p^{(n)}(o, x)$$

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for every finitely supported probability measure $\mu$ on $V$. We apply now this inequality by setting
\[
\mu(x) := \frac{1}{K_{n+1}} \sum_{m=0}^{Kn^2} p^{(m)}(o, x)
\]
\[
\ell \geq \limsup_{n \to \infty} -\frac{1}{n} \sum_{x \in V} p^{(n)}(o, x) \log (Kn^2 + 1) - \frac{1}{n} \sum_{x \in V} p^{(n)}(o, x) \log p^{(n)}(o, x)
\]
\[
= \limsup_{n \to \infty} \frac{1}{n} \int \log \pi_n(X_n) d\mathbb{P}.
\]
Now we can conclude with Fatou's Lemma:
\[
\ell \leq \int \liminf_{n \to \infty} \frac{-\log \pi_n(X_n)}{n} d\mathbb{P} \leq \liminf_{n \to \infty} \int \frac{-\log \pi_n(X_n)}{n} d\mathbb{P}
\]
\[
\leq \limsup_{n \to \infty} \int \frac{-\log \pi_n(X_n)}{n} d\mathbb{P} \leq \ell. \quad (3.6)
\]
Thus, $\lim_{n \to \infty} \frac{1}{n} \mathbb{E}[-\log \pi_n(X_n)]$ exists and the limit equals $\ell$. The rest follows from (3.2) and (3.3).

We now give another formula for the asymptotic entropy which shows that it is strictly positive.

**Theorem 3.8.** Assume $R > 1$. Then the asymptotic entropy is given by
\[
h = \ell_0 \cdot \sum_{g, h \in V^*} -\pi(g, \tau(g)) q((g, \tau(g)), (h, \tau(h))) \log q((g, \tau(g)), (h, \tau(h))) > 0.
\]

**Remarks:** Observe that the sum on the right hand side of Theorem 3.8 equals the entropy rate (for positive recurrent Markov chains) of $(\widetilde{W}_k, \tau_k)_{k \in \mathbb{N}}$, which is defined by the almost sure constant limit
\[
h_Q := \lim_{n \to \infty} -\frac{1}{n} \log \mu_n((\widetilde{W}_1, \tau_1), \ldots, (\widetilde{W}_n, \tau_n)),
\]
where $\mu_n((g_1, \tau_1), \ldots, (g_n, \tau_n))$ is the joint distribution of $((\widetilde{W}_1, \tau_1), \ldots, (\widetilde{W}_n, \tau_n))$. That is, $h = \ell \cdot h_Q$. For more details, we refer e.g. to Cover and Thomas [8 Chapter 4].

At this point it is essential that we have defined the length function $l(\cdot)$ with the help of the functions $L(x, y|x)$ and not by the Greenian metric.

**Proof.** For a moment let be $x = x_1 \ldots x_n \in V$. Then:
\[
l(x) = -\log n \prod_{j=1}^{n} L_{\tau(x_j)}(o_{\tau(x_j)}, x_j | \xi_{\tau(x_j)})
\]
\[
= -\log n \prod_{j=2}^{n} \frac{\alpha_{\tau(x_j)} \xi_{\tau(x_{j-1})} 1 - \xi_{\tau(x_j)}}{\alpha_{\tau(x_{j-1})} \xi_{\tau(x_j)} 1 - \xi_{\tau(x_{j-1})}} L_{\tau(x_j)}(o_{\tau(x_j)}, x_j | \xi_{\tau(x_j)})
\]
\[
- \log L_{\tau(x_1)}(o_{\tau(x_1)}, x_1 | \xi_{\tau(x_1)}) + \log \frac{\xi_{\tau(x_1)} \alpha_{\tau(x_1)} (1 - \xi_{\tau(x_1)})}{\alpha_{\tau(x_1)} \xi_{\tau(x_1)} (1 - \xi_{\tau(x_1)})}
\]
\[
= -\log n \prod_{j=2}^{n} q((x_{j-1}, \tau(x_{j-1})), (x_j, \tau(x_j)))
\]
\[
- \log L_{\tau(x_1)}(o_{\tau(x_1)}, x_1 | \xi_{\tau(x_1)}) + \log \frac{\xi_{\tau(x_1)} \alpha_{\tau(x_1)} (1 - \xi_{\tau(x_1)})}{\alpha_{\tau(x_1)} \xi_{\tau(x_1)} (1 - \xi_{\tau(x_1)})}. \quad (3.7)
\]
We now replace \(x\) by \(X_{e_k}\) in the last equation: since \(l(X_n)/n\) tends to \(h\) almost surely, the subsequence \((l(X_{e_k})/e_k)_{k \in \mathbb{N}}\) converges also to \(h\). Since \(\min_{i \in S} \xi_i > 0\) and \(\max_{i \in S} \xi_i < 1\), we get

\[
\frac{1}{e_k} \log \frac{\xi_{\tau(x_k)} a_{\tau(x_k)} (1 - \xi_{\tau(x_k)})}{a_{\tau(x_k)} \xi_{\tau(x_k)} (1 - \xi_{\tau(x_k)})} \xrightarrow{k \to \infty} 0 \quad \text{almost surely,}
\]

where \(x_1 := X_{e_1}\) and \(x_k := \tilde{W}_k = \tilde{X}_{e_k}\). By positive recurrence of \((\tilde{W}_k, \tau_k)_{k \in \mathbb{N}}\), an application of the ergodic theorem yields

\[
-\frac{1}{k} \log \prod_{j=2}^k \pi ([\tilde{W}_{j-1}, \tau_{j-1}), (\tilde{W}_j, \tau_j)) \xrightarrow{n \to \infty} h' := -\sum_{g, h \in V_*^\times; \tau(g) \neq \tau(h)} \pi (g, \tau(g)) q ((g, \tau(g)), (h, \tau(h))) \log q ((g, \tau(g)), (h, \tau(h))) > 0 \text{ a.s.,}
\]

whenever \(h' < \infty\). Obviously, for every \(x_1 \in V_*^\times\)

\[
\lim_{k \to \infty} -\frac{1}{e_k} \log L_{\tau(x_1)} (o_{\tau(x_1)}, x_1 | \xi_{\tau(x_1)}) = 0 \quad \text{almost surely.}
\]

Since \(\lim_{k \to \infty} k/e_k = \ell_0\) we get

\[
h = \lim_{k \to \infty} l(X_{e_k})/e_k = \lim_{k \to \infty} l(X_{e_k})/k \cdot e_k = h' \cdot \ell_0,
\]

whenever \(h' < \infty\). In particular, \(h > 0\) since \(\ell_0 > 0\) by [11] Section 4.

It remains to show that it cannot be that \(h' = \infty\). For this purpose, assume now \(h' = \infty\). Define for \(N \in \mathbb{N}\) the function \(h_N : (V_*^\times)^2 \to \mathbb{R}\) by

\[
h_N (g, h) := N \wedge (-\log q ((g, \tau(g)), (h, \tau(h))).
\]

Then

\[
-\frac{1}{k} \sum_{j=2}^k \log h_N (\tilde{X}_{e_{j-1}}, \tilde{X}_e) \xrightarrow{k \to \infty} h'_N := -\sum_{g, h \in V_*^\times; \tau(g) \neq \tau(h)} \pi (g, \tau(g)) q ((g, \tau(g)), (h, \tau(h))) \log h_N (g, h) \quad \text{almost surely.}
\]

Observe that \(h'_N \to \infty\) as \(N \to \infty\). Since \(h_N (g, h) \leq -\log q ((g, \tau(g)), (h, \tau(h)))\) and \(h' = \infty\) by assumption there is for every \(M \in \mathbb{R}\) and almost every trajectory of \((\tilde{W}_k)_{k \in \mathbb{N}}\) an almost surely finite random time \(T_q \in \mathbb{N}\) such that for all \(k \geq T_q\)

\[
-\frac{1}{k} \sum_{j=2}^k \log q ((\tilde{W}_{j-1}, \tau_{j-1}), (\tilde{W}_j, \tau_j)) > M. \quad (3.8)
\]
On the other hand side there is for every $M > 0$, every small $\epsilon > 0$ and almost every trajectory an almost surely finite random time $T_L$ such that for all $k \geq T_L$

$$-\frac{1}{e_k} \sum_{j=1}^{k} \log L(\tau(X_{e_j}) (o_{\tau(X_{e_j})}, X_{e_j} | \tau(X_{e_j}))) \in (h - \epsilon, h + \epsilon)$$

and

$$-\frac{1}{e_k} \sum_{j=2}^{k} \log q((\tilde{X}_{e_{j-1}}, \tau_{j-1}), (\tilde{X}_{e_j}, \tau_j))$$

$$= -\frac{k}{e_k} \sum_{j=2}^{k} \log q((\tilde{X}_{e_{j-1}}, \tau_{j-1}), (\tilde{X}_{e_j}, \tau_j)) > \ell_0 \cdot M.$$ 

Furthermore, since $\min_{i \in \mathcal{I}} \xi_i > 0$ and $\max_{i \in \mathcal{I}} \xi_i < 1$ there is an almost surely finite random time $T_\epsilon \geq T_L$ such that for all $k \geq T_\epsilon$ and $x_1 = X_{e_1}$ and $x_k = \tilde{X}_{e_k}$

$$-\frac{1}{e_k} \log \frac{\xi_{\tau(x_1)} \alpha_{\tau(x_1)} (1 - \xi(x_1))}{\alpha_{\tau(x_1)} \xi(x_1) (1 - \xi(x_1))} \in (-\epsilon, \epsilon)$$

and

$$\frac{1}{e_k} \log L(\tau(x_1) (o_{\tau(x_1)}, x_1 | \tau(x_1))) \in (-\epsilon, \epsilon).$$

Choose now $M > (h + 3\epsilon)/\ell_0$. Then we get the desired contradiction, when we substitute in inequality (3.7) the vertex $x$ by $X_{e_k}$ with $k \geq T_\epsilon$, divide by $e_k$ on both sides and see that the left side is in $(h - \epsilon, h + \epsilon)$ and the rightmost side is bigger than $h + \epsilon$. This finishes the proof of Theorem 3.8.

**Corollary 3.9.** Assume $R > 1$. Then we have for almost every path of the random walk $(X_n)_{n \in \mathbb{N}_0}$

$$h = \liminf_{n \to \infty} \frac{-\log \pi_n(X_n)}{n}.$$ 

**Proof.** Recall Inequality (3.5). Integrating both sides of this inequality yields together with the inequality chain (3.6) that

$$\int \liminf_{n \to \infty} \frac{-\log \pi_n(X_n)}{n} - h \, d\mathbb{P} = 0,$$

providing that $h = \liminf_{n \to \infty} -\frac{1}{n} \log \pi_n(X_n)$ for almost every realisation of the random walk. 

The following lemma gives some properties concerning general measure theory:

**Lemma 3.10.** Let $(Z_n)_{n \in \mathbb{N}_0}$ be a sequence of non-negative random variables and $0 < c \in \mathbb{R}$. Suppose that $\liminf_{n \to \infty} Z_n \geq c$ almost surely and $\lim_{n \to \infty} \mathbb{E}[Z_n] = c$. Then the following holds:

1. $Z_n \xrightarrow{\mathbb{P}} c$, that is, $Z_n$ converges in probability to $c$.

2. If $Z_n$ is uniformly bounded then $Z_n \xrightarrow{L_1} c$, that is, $\int |Z_n - c| \, d\mathbb{P} \to 0$ as $n \to \infty$. 

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Proof. First, we prove convergence in probability of \((Z_n)_{n \in \mathbb{N}}\). For every \(\delta_1 > 0\), there is some index \(N_{\delta_1}\) such that for all \(n \geq N_{\delta_1}\)
\[
\int Z_n \, d\mathbb{P} \in (c - \delta_1, c + \delta_1).
\]
Furthermore, due to the above made assumptions on \((Z_n)_{n \in \mathbb{N}}\) there is for every \(\delta_2 > 0\) some index \(N_{\delta_2}\) such that for all \(n \geq N_{\delta_2}\)
\[
\mathbb{P}[Z_n > c - \delta_1] > 1 - \delta_2. \tag{3.9}
\]
Since \(c = \lim_{n \to \infty} \int Z_n \, d\mathbb{P}\) it must be that for every arbitrary but fixed \(\epsilon > 0\), every \(\delta_1 < \epsilon\) and for all \(n\) big enough
\[
\mathbb{P}[Z_n > c - \delta_1] \cdot (c - \delta_1) + \mathbb{P}[Z_n > c + \epsilon] \cdot (\epsilon + \delta_1) \leq \int Z_n \, d\mathbb{P} \leq c + \delta_1,
\]
or equivalently,
\[
\mathbb{P}[Z_n > c + \epsilon] \leq \frac{c + \delta_1 - \mathbb{P}[Z_n > c - \delta_1] \cdot (c - \delta_1)}{\epsilon + \delta_1}.
\]
Letting \(\delta_2 \to 0\) we get
\[
\limsup_{n \to \infty} \mathbb{P}[Z_n > c + \epsilon] \leq \frac{2\delta_1}{\epsilon + \delta_1}.
\]
Since we can choose \(\delta_1\) arbitrarily small we get
\[
\mathbb{P}[Z_n > c + \epsilon] \xrightarrow{n \to \infty} 0 \quad \text{for all } \epsilon > 0.
\]
This yields convergence in probability of \(Z_n\) to \(c\).

In order to prove the second part of the lemma we define for any small \(\epsilon > 0\) and \(n \in \mathbb{N}\) the events
\[
A_{n,\epsilon} := [\lvert Z_n - c \rvert \leq \epsilon] \quad \text{and} \quad B_{n,\epsilon} := [\lvert Z_n - c \rvert > \epsilon].
\]
For arbitrary but fixed \(\epsilon > 0\), convergence in probability of \(Z_n\) to \(c\) gives an integer \(N_{\epsilon} \in \mathbb{N}\) such that \(\mathbb{P}[B_{n,\epsilon}] < \epsilon\) for all \(n \geq N_{\epsilon}\). Since \(0 \leq Z_n \leq M\) is assumed to be uniformly bounded, we get for \(n \geq N_{\epsilon}\):
\[
\int \lvert Z_n - c \rvert \, d\mathbb{P} = \int_{A_{n,\epsilon}} \lvert Z_n - c \rvert \, d\mathbb{P} + \int_{B_{n,\epsilon}} \lvert Z_n - c \rvert \, d\mathbb{P} \leq \epsilon + \epsilon (M + c) \xrightarrow{\epsilon \to 0} 0.
\]
Thus, we have proved the second part of the lemma. \(\square\)

We can apply the last lemma immediately to our setting:

**Corollary 3.11.** Assume \(R > 1\). Then we have the following types of convergence:

1. Convergence in probability:
\[
-\frac{1}{n} \log \pi_n(X_n) \xrightarrow{\mathbb{P}} h.
\]

2. Assume that there is \(c_0 > 0\) such that \(p(x, y) \geq c_0\) whenever \(p(x, y) > 0\). Then:
\[
-\frac{1}{n} \log \pi_n(X_n) \xrightarrow{L^1} h.
\]
Proof. Setting \( Z_n = \frac{1}{n} \log \pi_n(X_n) \) and applying Lemma 3.10 yields the claim. Note that the assumption \( p(x, y) \geq c_0 \) yields \( 0 \leq -\frac{\log \pi_n(X_n)}{n} \leq -\log c_0. \)

The assumption of the second part of the last corollary is obviously satisfied if we consider free products of finite graphs.

The reasoning in our proofs for existence of the entropy and its different properties (in particular, the reasoning in Section 3.2) is very similar to the argumentation in [12]. However, the structure of free products of graphs is more complicated than in the case of directed covers as considered in [12]. We outline the main differences to the reasoning in the aforementioned article. First, in [12] a very similar rate of escape (compare [12] Theorem 3.8] with Proposition 3.2) is considered, which arises from a length function induced by last visit generating functions. While the proof of existence of the rate of escape in [12] is easy to check, we have to make more effort in the case of free products, since \( -\log L_i(o_i, x|1) \) is not necessarily bounded for \( x \in V_i \). Additionally, one has to study the various ingredients of the proof more carefully, since non-trivial loops are possible in our setting in contrast to random walks on trees. Secondly, in [12] the invariant measure \( \pi(g, \tau(g)) \) of our proof collapses to \( \nu(\tau(g)) \), that is, in [12] one has to study the sequence \( (\tau(W_k))_{k \in \mathbb{N}} \), while in our setting we have to study the more complex sequence \( (W_k, \tau(W_k))_{k \in \mathbb{N}} \); compare [12] proof of Theorem 3.8] with Lemma 3.1 and Proposition 3.2.

4 A Formula via Double Generating Functions

In this section we derive another formula for the asymptotic entropy. The main tool is the following theorem of Sawyer and Steger [25] Theorem 2.2]:

**Theorem 4.1** (Sawyer and Steger). Let \( (Y_n)_{n \in \mathbb{N}_0} \) be a sequence of real-valued random variables such that, for some \( \delta > 0 \),

\[
\mathbb{E} \left( \sum_{n \geq 0} \exp(-rY_n - sn) \right) = \frac{C(r, s)}{g(r, s)} \quad \text{for } 0 < r, s < \delta,
\]

where \( C(r, s) \) and \( g(r, s) \) are analytic for \( |r|, |s| < \delta \) and \( C(0, 0) \neq 0 \). Denote by \( g'_r \) and \( g'_s \) the partial derivatives of \( g \) with respect to \( r \) and \( s \). Then

\[
\frac{Y_n}{n} \underset{n \to \infty}{\longrightarrow} \frac{g'_r(0, 0)}{g'_s(0, 0)} \text{ almost surely.}
\]

Setting \( z = e^{-s} \) and \( Y_n := -\log L(o, X_n|1) \) the expectation in Theorem 4.1 becomes

\[
\mathcal{E}(r, z) = \sum_{x \in V} \sum_{n \geq 0} p^{(n)}(o, x)L(o, x|1)^rz^n = \sum_{x \in V} G(o, x|z)L(o, x|1)^rz.
\]

We define for \( i \in \mathcal{I}, r, z \in \mathbb{C} \):

\[
\mathcal{L}(r, z) := 1 + \sum_{n \geq 1} \sum_{x_1, \ldots, x_n \in V \setminus \{o\}} \prod_{j=1}^n L_{\tau(x_j)}(o_{\tau(x_j)}, x_j|\xi_{\tau(x_j)}(z)) \cdot L_{\tau(x_j)}(o_{\tau(x_j)}, x_j|\xi_{\tau(x_j)})^rz,
\]

\[
\mathcal{L}_i^+(r, z) := \sum_{x \in V^i} L_i(o_i, x|\xi_i(z))L_i(o_i, x|\xi_i)^rz.
\]
Finally, \( \mathcal{L}_i(r, z) \) is defined by

\[
\mathcal{L}_i^+(r, z) \cdot \left( 1 + \sum_{n \geq 2} \sum_{x_2, \ldots, x_n \in V \setminus \{o\}, j=2}^n L(x_i) (o, o|_i, x_j|_i \xi_{i} (z)) \cdot L(x_i) (o, o|_i, x_j|_i \xi_{i} (z))^{r} \right).
\]

With these definitions we have \( \mathcal{L}(r, z) = 1 + \sum_{i \in \mathcal{I}} \mathcal{L}_i(r, z) \) and \( \mathcal{E}(r, z) = \frac{G(o, o|x) \cdot \mathcal{L}(r, z)}{1 - \mathcal{L}(r, z)} \). Simple computations analogously to \([11] \text{ Lemma 4.2, Corollary 4.3}\) yield

\[
\mathcal{E}(r, z) = \frac{G(o, o|x)}{1 - \mathcal{L}(r, z)}, \quad \text{where} \quad \mathcal{L}^+(r, z) = \sum_{i \in \mathcal{I}} \frac{\mathcal{L}_i^+(r, z)}{1 + \mathcal{L}_i^+(r, z)}.
\]

We now define \( C(r, z) := G(o, o|x) \) and \( g(r, z) := 1 - \mathcal{L}^+(r, z) \) and apply Theorem \([4.1]\) by differentiating \( g(r, z) \) and evaluating the derivatives at \((0, 1)\):

\[
\frac{\partial g(r, z)}{\partial r} \bigg|_{r=0, z=1} = -\sum_{i \in \mathcal{I}} \frac{\sum_{x \in V_i} L_i(o, o_i| \xi_i) \cdot \log L_i(o, o_i| \xi_i)}{(1 + \sum_{x \in V_i} L_i(o, o_i| \xi_i))^2} = -\sum_{i \in \mathcal{I}} G_i(o, o_i| \xi_i) \cdot (1 - \xi_i)^2 \cdot \sum_{x \in V_i} \log L_i(o, o_i| \xi_i) \cdot (1 - \xi_i)^2.
\]

\[
\frac{\partial g(r, z)}{\partial s} \bigg|_{r=0, s=0} = \sum_{i \in \mathcal{I}} \left[ \frac{\partial}{\partial z} \left( 1 - (1 - \xi_i(z)) G_i(o, o_i| \xi_i(z)) \right) \right]_{z=1} = \sum_{i \in \mathcal{I}} \xi_i'(1) \cdot G_i(o, o_i| \xi_i) - (1 - \xi_i) G_i(o, o_i| \xi_i).
\]

**Corollary 4.2.** Assume \( R > 1 \). Then the entropy can be rewritten as

\[
h = \frac{\frac{\partial g(r, z)}{\partial x}(0, 1)}{\frac{\partial g(r, z)}{\partial s}(0, 1)}.
\]

\(\square\)

### 5 Entropy of Random Walks on Free Products of Groups

In this section let each \( V_i \) be a finitely generated group \( \Gamma_i \) with identity \( e_i = o_i \). Without loss of generality we assume that the \( V_i \)'s are pairwise disjoint. The free product is again a group with concatenation (followed by iterated cancellations and contractions) as group operation. We write \( \Gamma_i^x := \Gamma_i \setminus \{e_i\} \). Suppose we are given a probability measure \( \mu_i \) on \( \Gamma_i \setminus \{e_i\} \) for every \( i \in \mathcal{I} \) governing a random walk on \( \Gamma_i \), that is, \( p_i(x, y) = \mu_i(x^{-1} y) \) for all \( x, y \in \Gamma_i \). Let \((\lambda_i)_i \in \mathcal{I} \) be a family of strictly positive real numbers...
if \( \tau \). Denote by \( \text{Recall that} \) and if \( \tau \).

Thus, Analogously, \( \mu^{(n)} \) denotes the \( n \)-th convolution power of \( \mu \). The random walk on \( \Gamma \) starting at the identity \( e \) of \( \Gamma \) is again denoted by the sequence \( (X_n)_{n\in\mathbb{N}_0} \). In particular, the radius of convergence of the associated Green function is strictly bigger than 1; see [29, Theorem 10.10, Corollary 12.5]. In the case of free products of groups it is well-known that the entropy exists and can be written as

\[
h = \lim_{n \to \infty} \frac{-\log \pi_n(X_n)}{n} = \lim_{n \to \infty} \frac{-\log F(e, X_n[1])}{n};
\]

see Derriennic [7], Kaimanovich and Vershik [14] and Blachère, Haïssinsky and Mathieu [3]. For free products of finite groups, Mairesse and Mathéus [21] give an explicit formula for \( h \), which remains also valid for free products of countable groups, but in the latter case one needs the solution of an infinite system of polynomial equations. In the following we will derive another formula for the entropy, which holds also for free products of infinite groups.

We set \( l(g_1 \ldots g_n) := -\log F(e, g_1 \ldots g_n[1]) \). Observe that transitivity yields \( F(g, gh[1]) = F(e, h[1]) \). Thus,

\[
l(g_1 \ldots g_n) = -\log \prod_{j=1}^{n} F(g_1 \ldots g_{j-1}, g_j[1]) = -\sum_{j=1}^{n} \log F(e, g_j[1]).
\]

First, we rewrite the following expectations as

\[
\mathbb{E}l(X_n) = \sum_{i \in I} \sum_{g \in \Gamma_i} \alpha_i \mu_i(g) \sum_{h \in \Gamma} l(h) \mu^{(n)}(h),
\]

\[
\mathbb{E}l(X_{n+1}) = \sum_{i \in I} \sum_{g \in \Gamma_i} \alpha_i \mu_i(g) \sum_{h \in \Gamma} l(gh) \mu^{(n)}(h).
\]

Thus,

\[
\mathbb{E}l(X_{n+1}) - \mathbb{E}l(X_n) = \sum_{i \in I} \sum_{g \in \Gamma_i} \alpha_i \mu_i(g) \int (l(gh) - l(h)) \, d\mu^{(n)}(h)
\]

\[
= \sum_{i \in I} \sum_{g \in \Gamma_i} \alpha_i \mu_i(g) \int -\log \frac{F(e, gX_n[1])}{F(e, X_n[1])} \, d\mu^{(n)}. \quad (5.1)
\]

Recall that \( \|X_n\| \to \infty \) almost surely. That is, \( X_n \) converges almost surely to a random infinite word \( X_\infty \) of the form \( x_1 x_2 \ldots \in \left( \bigcup_{i=1}^{N} \Gamma_i \times \right) \), where two consecutive letters are not from the same \( \Gamma_i \).

Denote by \( X_\infty^{(1)} \) the first letter of \( X_\infty \). Let be \( g \in \Gamma_i \). For \( n \geq e_1 \), the integrand in (5.1) is constant: if \( \tau(X_\infty^{(1)}) \neq i \) then

\[
\log \frac{F(e, gX_n[1])}{F(e, X_n[1])} = \log F(e, g),
\]

and if \( \tau(X_\infty^{(1)}) = i \) then

\[
\log \frac{F(e, gX_n[1])}{F(e, X_n[1])} = \log \frac{F(e, gX^{(1)}_\infty[1])}{F(e, X^{(1)}_\infty[1])}.
\]
By [11] Section 5], for $i \in \mathcal{I}$ and $g \in \Gamma_i^\times$, 

$$\varrho(i) := \mathbb{P}[X_1^{(1)} \in \Gamma_i] = 1 - (1 - \xi_i)G_i(o,i,\xi_i) \quad \text{and} \quad \mathbb{P}[X_\infty^{(1)} = g] = F(o,i,\xi_i)G_i(o,i,\xi_i) = (1 - \xi_i)G_i(o,i,g,\xi_i).$$

Recall that $F(e,g|1) = F_i(o_i,g,\xi_i)$ for each $g \in \Gamma_i$. We get:

**Theorem 5.1.** Whenever $h_i := -\sum_{g \in \Gamma_i} \mu_i(g) \log \mu_i(g) < \infty$ for all $i \in \mathcal{I}$, that is, when all random walks on the factors $\Gamma_i$ have finite single-step entropy, then the asymptotic entropy $h$ of the random walk on $\Gamma$ is given by

$$h = -\sum_{i \in \mathcal{I}} \sum_{g \in \Gamma_i} a_i \mu_i(g) \left[ (1 - \varrho(i)) \log F_i(o_i,g,\xi_i) + (1 - \xi_i)G_i(o_i,o_i,\xi_i) \mathcal{F}(g) \right],$$

where

$$\mathcal{F}(g) := \sum_{g' \in \Gamma_i^\times} F_i(o_i,g'|\xi_i) \log \frac{F_i(o_i,gg'|\xi_i)}{F_i(o_i,g'|\xi_i)} \quad \text{for } g \in \Gamma_i. \quad (5.2)$$

**Proof.** Consider the sequence $\mathbb{E}(X_{n+1}) - \mathbb{E}(X_n)$. If this sequence converges, its limit must equal $h$. By the above made considerations we get

$$\mathbb{E}(X_{n+1}) - \mathbb{E}(X_n) \xrightarrow{n \to \infty} -\sum_{i \in \mathcal{I}} \sum_{g \in \Gamma_i} \mu_i(g) \left[ (1 - \varrho(i)) \log F_i(o_i,g,\xi_i) + \sum_{g' \in \Gamma_i^\times} \mathbb{P}[X_\infty^{(1)} = g'] \log \frac{F_i(o_i,gg'|\xi_i)}{F_i(o_i,g'|\xi_i)} \right],$$

if the sum on the right hand side is finite. We have now established the proposed formula, but it remains to verify finiteness of the sum above. This follows from the following observations:

**Claim A:** $-\sum_{i \in \mathcal{I}} \sum_{g \in \Gamma_i} a_i \mu_i(g) (1 - \varrho(i)) \log F_i(o_i,g,\xi_i)$ is finite.

Observe that $F_i(o_i,g,\xi_i) \geq \mu_i(g) \xi_i$ for $g \in \operatorname{supp}(\mu_i)$. Thus,

$$0 < -\sum_{g \in \Gamma_i} \mu_i(g) \log F_i(o_i,g,\xi_i) \leq -\sum_{g \in \Gamma_i} \mu_i(g) \log \mu_i(g) \xi_i = h_i - \log \xi_i.$$ 

This proves Claim A.

**Claim B:** $\sum_{i \in \mathcal{I}} \sum_{g \in \Gamma_i} a_i \mu_i(g) (1 - \xi_i) \sum_{g' \in \Gamma_i^\times} G_i(o_i,g'|\xi_i) \log \left| \frac{F_i(o_i,gg'|\xi_i)}{F_i(o_i,g'|\xi_i)} \right|$ is finite.

Observe that $F_i(o_i,gg'|\xi_i)/F_i(o_i,g'|\xi_i) = G_i(o_i,gg'|\xi_i)/G_i(o_i,g'|\xi_i)$. Obviously,

$$\mu_i^{(n)}(g') \xi_i^n \leq G_i(o_i,g'|\xi_i) \leq \frac{1}{1 - \xi_i} \quad \text{for every } n \in \mathbb{N} \text{ and } g' \in \Gamma_i.$$
For $g \in \Gamma$ set $N(g) := \{ n \in \mathbb{N}_0 \mid \mu^{(n)}(g) > 0 \}$. Then:

$$0 < \sum_{g' \in \Gamma_i^*} \mathbb{P}[X_{\infty}^{(1)} = g'] \cdot | \log G_i(o_i, g' \mid \xi_i) |$$

$$= \sum_{g' \in \Gamma_i^*} (1 - \xi_i) \cdot G_i(o_i, g' \mid \xi_i) \cdot | \log G_i(o_i, g' \mid \xi_i) |$$

$$= \sum_{g' \in \Gamma_i^*} (1 - \xi_i) \cdot \sum_{n \in N(g')} \mu^{(n)}_i(g') \cdot \xi^n_i \cdot | \log G_i(o_i, g' \mid \xi_i) |$$

$$\leq \sum_{g' \in \Gamma_i^*} (1 - \xi_i) \cdot \sum_{n \in N(g')} \mu^{(n)}_i(g') \cdot \xi^n_i \cdot \max\{-\log(\mu^{(n)}_i(g') \xi^n_i), -\log(1 - \xi_i)\}$$

$$\leq (1 - \xi_i) \sum_{n \in N(g')} n \xi^n_i \cdot \frac{1}{n} \sum_{g' \in \Gamma_i} \mu^{(n)}_i(g') \log \mu^{(n)}_i(g') - (1 - \xi_i) \log \xi_i \sum_{n \geq 1} n \xi^n_i$$

$$(*)$$

$$- (1 - \xi_i) \log(1 - \xi_i) \sum_{n \geq 1} \xi^n_i.$$ 

Recall that $h_i < \infty$ together with Kingman’s subadditive ergodic theorem implies existence of a constant $H_i \geq 0$ with

$$\lim_{n \to \infty} \frac{1}{n} \sum_{g \in \Gamma_i} \mu^{(n)}_i(g) \log \mu^{(n)}_i(g) = H_i. \quad (5.3)$$

Thus, if $n \in \mathbb{N}$ is large enough, the sum $(*)$ is in the interval $(H_i - \varepsilon, H_i + \varepsilon)$ for any arbitrarily small $\varepsilon > 0$. That is, the sum $(*)$ is uniformly bounded for all $n \in \mathbb{N}$. From this follows that the rightmost side of the last inequality chain is finite.

Furthermore, we have for each $g \in \Gamma_i$ with $\mu^{(n)}_i(g) > 0$:

$$0 < \sum_{g' \in \Gamma_i^*} \mathbb{P}[X_{\infty}^{(1)} = g'] \cdot | \log G_i(o_i, g' \mid \xi_i) |$$

$$= \sum_{g' \in \Gamma_i^*} (1 - \xi_i) \cdot G_i(o_i, g' \mid \xi_i) \cdot | \log G_i(o_i, g' \mid \xi_i) |$$

$$= \sum_{g' \in \Gamma_i^*} (1 - \xi_i) \cdot \sum_{n \in N(g')} \mu^{(n)}_i(g') \cdot \xi^n_i \cdot | \log G_i(o_i, g' \mid \xi_i) |$$

$$\leq \sum_{g' \in \Gamma_i^*} (1 - \xi_i) \cdot \sum_{n \in N(g')} \mu^{(n)}_i(g') \cdot \xi^n_i \cdot \max\{-\log(\mu^{(n)}_i(g') \xi^n_i), -\log(1 - \xi_i)\}$$

$$\leq -(1 - \xi_i) \sum_{n \in N(g')} \xi^n_i \cdot \sum_{g' \in \Gamma_i} \mu^{(n)}_i(g') \log \mu^{(n)}_i(g') - (1 - \xi_i) \cdot \log \xi_i \sum_{n \geq 1} (n + 1) \xi^n_i$$

$$- \log \mu_i(g) - \log(1 - \xi_i).$$
If we sum up over all \( g \) with \( \mu(g) > 0 \), we get:

\[
- \sum_{g \in \mathcal{G}} \sum_{n \in \mathbb{N}(g')} \alpha_i \mu_i(g)(1 - \xi_i) \sum_{g' \in \mathcal{G}_i} \xi_i^n \sum_{g' \in \mathcal{G}_i} \mu_i^{(n)}(g') \log \mu_i^{(n)}(g')
\]

\( (I) \)

\[
- \sum_{g \in \mathcal{G}} \sum_{n \in \mathbb{N}(g')} \alpha_i \mu_i(g)(1 - \xi_i) \log \xi_i \sum_{n \geq 1} (n + 1) \xi_i^n
\]

\( (II) \)

\[
- \sum_{g \in \mathcal{G}} \sum_{i \in \mathcal{I}_i} \mu_i(g) \log \mu_i(g) - \sum_{i \in \mathcal{I}_i} \alpha_i \log(1 - \xi_i).
\]

\( (III) \)

Convergence of \((I)\) follows from \([5.3]\), \((II)\) converges since \( \xi_i < 1 \) and \((III)\) is convergent by assumption \( h_i < \infty \). This finishes the proof of Claim B, and thus the proof of the theorem. \( \square \)

Erschler and Kaimanovich \([9]\) asked if drift and entropy of random walks on groups depend continuously on the probability measure, which governs the random walk. Ledrappier \([19]\) proves in his recent, simultaneous paper that drift and entropy of finite-range random walks on free groups vary analytically with the probability measure of constant support. By Theorem 5.1 we are even able to show continuity for free products of finitely generated groups, but restricted to nearest neighbour random walks with fixed set of generators.

**Corollary 5.2.** Let \( \Gamma_i \) be generated as a semigroup by \( S_i \). Denote by \( \mathcal{P}_i \) the set of probability measures \( \mu_i \) on \( S_i \) with \( \mu_i(x_i) > 0 \) for all \( x_i \in S_i \). Furthermore, we write \( \mathcal{A} := \{(\alpha_1, \ldots, \alpha_r) \mid \alpha_i > 0, \sum_{i \in \mathcal{I}_i} \alpha_i = 1\} \). Then the entropy function

\[
h : \mathcal{A} \times \mathcal{P}_1 \times \cdots \times \mathcal{P}_r \to \mathbb{R} : (\alpha_1, \ldots, \alpha_r, \mu_1, \ldots, \mu_r) \mapsto h(\alpha_1, \ldots, \alpha_r, \mu_1, \ldots, \mu_r)
\]

is real-analytic.

**Proof.** The claim follows directly with the formula given in Theorem 5.1 the involved generating functions \( F_i(\alpha_i, g|z) \) and \( G_i(\alpha_i, o_i|z) \) are analytic when varying the probability measure of constant support, and the values \( \xi_i \) can also be rewritten as

\[
\xi_i = \sum_{k_1, \ldots, k_r, l_1, \ldots, l_r, |S_i| \geq 1} x(k_1, \ldots, k_r, l_1, \ldots, l_r, |S_i|) \prod_{i \in \mathcal{I}_i} \alpha_i^{k_i} \prod_{j=1}^{|S_i|} \mu_i(x_{i,j})^{l_j},
\]

where \( S_i = \{x_{i,1}, \ldots, x_{i,|S_i|}\} \). This yields the claim. \( \square \)

**Remarks:**

1. Corollary 5.2 holds also for the case of free products of finite graphs, if one varies the transition probabilities continuously under the assumption that the sets \( \{x_i, y_i \in V_i \times V_i \mid p_i(x_i, y_i) > 0\} \) remain constant: one has to rewrite \( \xi_i \) as power series in terms of (finitely many) \( p_i(x_i, y_i) \) and gets analyticity with the formula given in Theorem 3.7.
2. Analyticity holds also for the drift (w.r.t. the block length and w.r.t. the natural graph metric) of nearest neighbour random walks due to the formulas given in [11] Section 5 and 7.

3. The formula for entropy and drift given in Mairesse and Mathéus [21] for random walks on free products of finite groups depends also analytically on the transition probabilities.

6 Entropy Inequalities

In this section we consider the case of free products of finite sets $V_1, \ldots, V_r$, where $V_i$ has $|V_i|$ vertices. We want to establish a connection between asymptotic entropy, rate of escape and the volume growth rate of the free product $V$. For $n \in \mathbb{N}_0$, let $S_0(n)$ be the set of all words of $V$ of block length $n$. The following lemmas give an answer how fast the free product grows.

Lemma 6.1. The sphere growth rate w.r.t. the block length is given by

$$s_0 := \lim_{n \to \infty} \frac{\log |S_0(n)|}{n} = \log \lambda_0,$$

where $\lambda_0$ is the Perron-Frobenius eigenvalue of the $r \times r$-matrix $D = (d_{i,j})_{1 \leq i,j \leq r}$ with $d_{i,j} = 0$ for $i = j$ and $d_{i,j} = |V_j| - 1$ otherwise.

Proof. Denote by $\hat{D}$ the $r \times r$-diagonal matrix, which has entries $|V_1| - 1, |V_2| - 1, \ldots, |V_r| - 1$ on its diagonal. Let $1$ be the $(r \times 1)$-vector with all entries equal to 1. Thus, we can write

$$|S_0(n)| = 1^T \hat{D} D^{n-1} 1.$$

Let $0 < v_1 \leq 1$ and $v_2 \geq 1$ be eigenvectors of $D$ w.r.t. the Perron-Frobenius eigenvalue $\lambda_0$. Then

$$|S_0(n)| \geq 1^T \hat{D} D^{n-1} v_1 = C_1 \cdot \lambda_0^{n-1},$$

$$|S_0(n)| \leq 1^T \hat{D} D^{n-1} v_2 = C_2 \cdot \lambda_0^{n-1},$$

where $C_1, C_2$ are some constants independent from $n$. Thus,

$$\frac{\log |S_0(n)|}{n} = \log |S_0(n)|^{1/n} \xrightarrow{n \to \infty} \log \lambda_0.$$

Recall from the Perron-Frobenius theorem that $\lambda_0 \geq \sum_{i=1, i \neq j}(|\Gamma_i| - 1)$ for each $j \in \mathcal{S}$; in particular, $\lambda_0 \geq 1$. We also take a look on the natural graph metric and its growth rate. For this purpose, we define

$$S_1(n) := \{x \in V \mid p(n)(o, x) > 0, \forall m < n : p(m)(o, x) = 0\},$$

that is, the set of all vertices in $V$ which are at distance $n$ to the root $o$ w.r.t. the natural graph metric.

We now construct a new graph, whose adjacency matrix allows us to describe the exponential growth of $S_1(n)$ as $n \to \infty$. For this purpose, we visualize the sets $V_1, \ldots, V_r$ as graphs $\mathcal{X}_1, \ldots, \mathcal{X}_r$ with vertex sets $V_1, \ldots, V_r$ equipped with the following edges: for $x, y \in V_i$, there is a directed edge from $x$ to $y$ if and only if $p_i(x, y) > 0$. Consider now directed spanning trees $\mathcal{T}_1, \ldots, \mathcal{T}_r$ of the graphs $\mathcal{X}_1, \ldots, \mathcal{X}_r$. 

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such that the graph distances of vertices in $F_i$ to the root $o_i$ remain the same as in $X_i$. We now investigate the free product $F = F_1 \ast \cdots \ast F_r$, which is again a tree. We make the crucial observation that $F$ can be seen as the directed cover of a finite directed graph $F$, where $F$ is defined in the following way:

1. The vertex set of $F$ is given by $\{o\} \cup \bigcup_{i \in I} V^x_i$ with root $o$.

2. The edges of $F$ are given as follows: first, we add all edges inherited from one of the trees $F_1, \ldots, F_r$, where $o$ plays the role of $o_i$ for each $i \in I$. Secondly, we add for all $i \in I$ and every $x \in V^x_i$ an edge from $x$ to each $y \in V^x_j$, $j \neq i$, whenever there is an edge from $o_i$ to $y$ in $F_j$.

The tree $F$ can be seen as a periodic tree, which is also called a tree with finitely many cone types; for more details we refer to Lyons [20] and Nagnibeda and Woess [23]. Now we are able to state the following lemma:

**Lemma 6.2.** The sphere growth rate w.r.t. the natural graph metric defined by

$$s_1 := \lim_{n \to \infty} \frac{\log |S_1(n)|}{n}$$

exists. Moreover, we have the equation $s_1 = \log \lambda_1$, where $\lambda_1$ is the Perron-Frobenius eigenvalue of the adjacency matrix of the graph $F$.

**Proof.** Since the graph metric remains invariant under the restriction of $V$ to $F$ and since it is well-known that the growth rate exists for periodic trees (see Lyons [20, Chapter 3.3]), we have existence of the limit $s_1$. More precisely, $|S_1(n)|^{1/n}$ tends to the Perron-Frobenius eigenvalue of the adjacency matrix of $F$ as $n \to \infty$. For sake of completeness, we remark that the root of $F$ plays a special role (as a cone type) but this does not affect the application of the results about directed covers to our case. \qed

For $i \in \{0, 1\}$, we write $B_i(n) = \bigcup_{k=0}^n S_i(k)$. Now we can prove:

**Lemma 6.3.** The volume growth w.r.t. the block length, w.r.t. the natural graph metric respectively, is given by

$$g_0 := \lim_{n \to \infty} \frac{\log |B_0(n)|}{n} = \log \lambda_0, \quad g_1 := \lim_{n \to \infty} \frac{\log |B_1(n)|}{n} = \log \lambda_1$$

**Proof.** For ease of better readability, we omit the subindex $i \in \{0, 1\}$ in the following, since the proofs for $g_0$ and $g_1$ are completely analogous. Choose any small $\varepsilon > 0$. Then there is some $K_{\varepsilon}$ such that for all $k \geq K_{\varepsilon}$

$$\lambda^k e^{-k\varepsilon} \leq |S(k)| \leq \lambda^k e^{k\varepsilon}.$$ 

Write $C_{\varepsilon} = \sum_{i=0}^{K_{\varepsilon} - 1} |S(i)|$. Then for $n \geq K_{\varepsilon}$:

$$|B(n)|^{1/n} = \sqrt[1/n]{\sum_{k=0}^n |S(k)|} \leq \sqrt[1/n]{C_{\varepsilon} + \sum_{k=K_{\varepsilon}}^n \lambda^k e^{k\varepsilon}} = \lambda e^{\varepsilon} \sqrt[1/n]{\frac{C_{\varepsilon} + \sum_{k=K_{\varepsilon}}^n \lambda^{n-k} e^{(n-k)\varepsilon}}{\lambda^n e^{n\varepsilon}}} \leq \lambda e^{\varepsilon} \sqrt[1/n]{\frac{C_{\varepsilon}}{\lambda^n e^{n\varepsilon}}} + (n - K_{\varepsilon} + 1) \to \lambda e^{\varepsilon}.$$
In the last inequality we used the fact \( \lambda \geq 1 \). Since we can choose \( \varepsilon > 0 \) arbitrarily small, we get 
\[
\limsup_{n \to \infty} |B(n)|^{1/n} \geq \lambda.
\]
Analogously:
\[
|B(n)|^{1/n} \geq \sqrt[n]{e} \sum_{k=0}^{n} \lambda^k e^{-k \varepsilon} = \lambda\sqrt[n]{e} \sum_{k=0}^{n} \frac{e^{-k \varepsilon}}{\lambda^k} \to \lambda e^{-\varepsilon}.
\]
That is, 
\[
\lim_{n \to \infty} \frac{1}{n} \log |B(n)| = \log \lambda.
\]
\[\square\]

For \( i \in \{0, 1\} \) define \( l_i : V \to \mathbb{N}_0 \) by \( l_i(x) = \|x\| \) and \( \ell_i(x) = \inf\{m \in \mathbb{N}_0 \mid p^{(m)}(o, x) > 0\} \). Then the limits \( \ell_i = \lim_{n \to \infty} l_i(X_n)/n \) exist; see [11], Theorem 3.3, Section 7.II. Now we can establish a connection between entropy, rate of escape and volume growth:

**Corollary 6.4.** \( h \leq g_0 \cdot \ell_0 \) and \( h \leq g_1 \cdot \ell_1 \).

**Proof.** Let be \( i \in \{0, 1\} \) and \( \varepsilon > 0 \). Then there is some \( N_\varepsilon \in \mathbb{N} \) such that for all \( n \geq N_\varepsilon \)
\[
1 - \varepsilon \leq P\left( \{ x \in V \mid -\log \pi_n(x) \geq (h - \varepsilon)n, l_i(x) \leq (\ell_i + \varepsilon)n \} \right) \leq e^{-(h-\varepsilon)n} \cdot |B_i((\ell_i + \varepsilon)n)|.
\]
That is,
\[
(h - \varepsilon) + \frac{\log(1 - \varepsilon)}{n} \leq (\ell_i + \varepsilon) \cdot \frac{\log|B_i((\ell_i + \varepsilon)n)|}{(\ell_i + \varepsilon)n}.
\]
If we let \( n \) tend to infinity and make \( \varepsilon \) arbitrarily small, we get the claim. \[\square\]

Finally, we remark that an analogous inequality for random walks on groups was given by Guivarc’h [13], and more generally for space- and time-homogeneous Markov chains by Kaimanovich and Woess [15], Theorem 5.3.

### 7 Examples

#### 7.1 Free Product of Finite Graphs

Consider the graphs \( \mathcal{X}_1 \) and \( \mathcal{X}_2 \) with the transition probabilities sketched in Figure 7.1. We set \( \alpha_1 = \alpha_2 = 1/2 \). For the computation of \( \ell_0 \) we need the following functions:

\[
\begin{align*}
F_1(g_1, o_1|z) &= \frac{z^2}{1 - z^2} / 2, & F_2(h_1, o_2|z) &= \frac{z^2}{1 - z^2} / 2, \\
\xi_1(z) &= \frac{z/2}{1 - \frac{z(1)^2}{1 - \frac{1}{1 - \xi_2(1)^2}/2}} , & \xi_2(z) &= \frac{z}{1 - \frac{z(1)^2}{1 - \frac{1}{1 - \xi_2(1)^2}/2}}.
\end{align*}
\]

Simple computations with the help of [11], Section 3 and MATHEMATICA allow us to determine the rate of escape of the random walk on \( \mathcal{X}_1 \ast \mathcal{X}_2 \) as \( \ell_0 = 0.41563 \). For the computation of the entropy, we need also the following generating functions:

\[
\begin{align*}
L_1(o_1, g_1|z) &= \frac{z}{1 - z^2}, & L_1(o_1, g_2|z) &= \frac{z^2}{1 - z^2} / 2, & L_2(o_2, h_1|z) &= \frac{z}{1 - z^3 / 2}, \\
L_2(o_2, h_2|z) &= \frac{z^2}{1 - z^3 / 2}, & L_2(o_2, h_3|z) &= \frac{z^3}{1 - z^3 / 2}.
\end{align*}
\]

Thus, we get the asymptotic entropy as \( h = 0.32005 \).
From this system we obtain explicit formulas for \( \hat{\mathcal{A}} \).

Moreover, we need the value \( \xi \). This value can be computed analogously to [11] Section 6.2, that is, \( \xi \) has to be computed numerically from the equation

\[
\frac{\xi}{2 - 2\xi} = \xi G_1(\xi) = \frac{\xi}{1 - \frac{2}{3}\xi F(a|\xi) - \frac{1}{3}\xi F(c|\xi)}.
\]

Solving this equation with Mathematica yields \( \xi = 0.55973 \). To compute the entropy we have to evaluate the functions \( F(g|z) \) at \( z = \xi \) for each \( g \in \mathbb{Z} \). For even \( n \in \mathbb{N} \), we have the following

---

**Figure 1**: Finite graphs \( X_1 \) and \( X_2 \)

### 7.2 \((\mathbb{Z} \times \mathbb{Z}/2) \ast (\mathbb{Z} \times \mathbb{Z}/2)\)

Consider the free product \( \Gamma = \Gamma_1 \ast \Gamma_2 \) of the infinite groups \( \Gamma_i = \mathbb{Z} \times (\mathbb{Z}/2\mathbb{Z}) \) with \( a_i = 1/2 \) and \( \mu_i((\pm 1, 0)) = \mu_i((0, 1)) = 1/3 \) for each \( i \in \{1, 2\} \). We set \( a := (1, 0), b := (1, 1), c := (0, 1) \) and \( \lambda(x, y) := x \) for \( (x, y) \in \Gamma_i \). Define

\[
\hat{F}(a|z) := \sum_{n \geq 1} \mathbb{P}[Y_n = a, \forall m < n : \lambda(Y_m) < 1 \mid Y_0 = (0, 0)] z^n,
\]

\[
\hat{F}(b|z) := \sum_{n \geq 1} \mathbb{P}[Y_n = b, \forall m < n : \lambda(Y_m) < 1 \mid Y_0 = (0, 0)] z^n,
\]

where \((Y_n)_{n \in \mathbb{N}_0}\) is a random walk on \( \mathbb{Z} \times \mathbb{Z}/2 \) governed by \( \mu_1 \). The above functions satisfy the following system of equations:

\[
\hat{F}(a|z) = \frac{z}{3} \left( 1 + \hat{F}(b|z) + \hat{F}(a|z)^2 + \hat{F}(b|z)^2 \right),
\]

\[
\hat{F}(b|z) = \frac{z}{3} \left( \hat{F}(a|z) + \hat{F}(a|z)\hat{F}(b|z) + \hat{F}(b|z)\hat{F}(a|z) \right).
\]

From this system we obtain explicit formulas for \( \hat{F}(a|z) \) and \( \hat{F}(b|z) \). We write \( F(n, j|z) := F_1((0, 0), (n, j)|z) \) for \((n, j) \in \mathbb{Z} \times \mathbb{Z}/2 \). To compute the entropy rate we have to solve the following system of equations:

\[
F(a|z) = \frac{z}{3} \left( 1 + F(b|z) + F(a|z)F(a|z) + F(b|z)F(b|z) \right),
\]

\[
F(b|z) = \frac{z}{3} \left( F(c|z) + F(a|z) + F(a|z)F(b|z) + F(b|z)F(a|z) \right),
\]

\[
F(c|z) = \frac{z}{3} \left( 1 + 2 F(b|z) \right).
\]

Moreover, we need the value \( \xi_1(1) = \xi_2(1) = \xi \). This value can be computed analogously to [11] Section 6.2, that is, \( \xi \) has to be computed numerically from the equation

\[
\frac{\xi}{2 - 2\xi} = \xi G_1(\xi) = \frac{\xi}{1 - \frac{2}{3}\xi F(a|\xi) - \frac{1}{3}\xi F(c|\xi)}.
\]
Moreover, we define \( \hat{F} \) on the first step and solving for odd \( n \in \mathbb{N} \),

\[
F((\pm n, 0) | \xi) = \sum_{k=0}^{n/2} \binom{n}{2k} \hat{F}(b|\xi)^{2k} \hat{F}(a|\xi)^{n-2k} + \\
\sum_{k=0}^{n/2-1} \binom{n}{2k+1} \hat{F}(b|\xi)^{2k+1} \hat{F}(a|\xi)^{n-2k-1} F(c|\xi),
\]

\[
F((\pm n, 1) | \xi) = \sum_{k=0}^{n/2-1} \binom{n}{2k+1} \hat{F}(b|\xi)^{2k+1} \hat{F}(a|\xi)^{n-2k-1} + \\
\sum_{k=0}^{n/2} \binom{n}{2k} \hat{F}(b|\xi)^{2k} \hat{F}(a|\xi)^{n-2k} F(c|\xi).
\]

Moreover, we define \( \check{F} := \mathbb{P}[\exists n \in \mathbb{N} : \lambda(X_n) = 1] \). This probability can be computed by conditioning on the first step and solving

\[
\check{F} = \frac{\xi}{3} (1 + \hat{F} + \check{F}^2),
\]

that is, \( \check{F} = 0.24291 \). Observe that we get the following estimations:

\[
F_1(o, g | \xi) \leq \hat{F}^{\lambda(g)} \] for \( g \in \mathbb{Z} \times \mathbb{Z}_2 \),
\[
F_1(o, g | \xi) \geq \hat{F}^{\lambda(g)-1} \cdot \min \{ F_1(o_1, a|\xi), F_1(o_1, b|\xi) \} \] for \( g \in (\mathbb{Z} \times \mathbb{Z}_2) \setminus \{(0,0), c\} \).

These bounds allow us to cap the sum over all \( g' \in \Gamma_i^x \) in (5.2) and to estimate the tails of these sums. Thus, we can compute the entropy rate numerically as \( \hat{h} = 1.14985 \).

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