Similar transformation of one class of correct restrictions

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Abstract. The description of all correct restrictions of the maximal operator are considered in a Hilbert space. A class of correct restrictions are obtained for which a similar transformation has the domain of the fixed correct restriction. The resulting theorem is applied to the study of n-order differentiation operator with singular coefficients.

1 Introduction

Let a closed linear operator $L$ be given in a Hilbert space $H$. The linear equation

$$Lu = f$$

(1.1)

is said to be correctly solvable on $R(L)$ if $\|u\| \leq C\|Lu\|$ for all $u \in D(L)$ (where $C > 0$ does not depend on $u$) and everywhere solvable if $R(L) = H$. If (1.1) is simultaneously correct and solvable everywhere, then we say that $L$ is a correct operator. A correctly solvable operator $L_0$ is said to be minimal if $R(L_0) \neq H$. A closed operator $\hat{L}$ is called a maximal operator if $R(\hat{L}) = H$ and $\text{Ker} \hat{L} \neq \{0\}$. An operator $A$ is called a restriction of an operator $B$ and $B$ is said to be an extension of $A$ if $D(A) \subset D(B)$ and $Au = Bu$ for all $u \in D(A)$.

Note that if one of the correct restriction $\hat{L}$ of a maximal operator $\hat{L}$ is known, then the inverses of all correct restrictions of $\hat{L}$ have in the form

$$L^{-1}_K f = L^{-1} f + K f,$$

(1.2)

where $K$ is an arbitrary bounded linear operator in $H$ such that $R(K) \subset \text{Ker} \hat{L}$.

Let $L_0$ be some minimal operator, and let $M_0$ be another minimal operator related to $L_0$ by the equation $(L_0 u, v) = (u, M_0 v)$ for all $u \in D(L_0)$ and $v \in D(M_0)$. Then $\hat{L} = M_0^*$ and $\hat{M} = L_0^*$ are maximal operators such that $L_0 \subset \hat{L}$ and $M_0 \subset \hat{M}$. A correct
restriction $L$ of a maximal operator $\hat{L}$ such that $L$ is simultaneously a correct extension of the minimal operator $L_0$ is called a boundary correct extension. The existence of at least one boundary correct extension $L$ was proved by Vishik in [2], that is, $L_0 \subset L \subset \hat{L}$.

The inverse operators to all possible correct restrictions $L_K$ of the maximal operator $\hat{L}$ have the form (1.2), then

$$D(L_K) = \{ u \in D(\hat{L}) : (I - K\hat{L})u \in D(L) \}$$

is dense in $H$ if and only if $\text{Ker}(I + K^*L^*) = \{0\}$. All possible correct extensions $M_K$ of $M_0$ have inverses of the form

$$M_K^{-1}f = (L_K^*)^{-1}f = (L^*)^{-1}f + K^*f,$$

where $K$ is an arbitrary bounded linear operator in $H$ with $R(K) \subset \text{Ker}\hat{L}$ such that $\text{Ker}(I + K^*L^*) = \{0\}$.

The main result of this work is the following

**Theorem 1.1.** Let $L$ be boundary correct extension of $L_0$, that is, $L_0 \subset L \subset \hat{L}$. If $L_K$ is densely defined in $H$ and

$$R(K^*) \subset D(L^*) \cap D(L_K^*),$$

where $K$ and $L$ are the operators in representation (1.2), then $\overline{KL}_K$ is bounded in $H$ and a correct restriction $L_K$ of the maximal operator $\hat{L}$ is similar to the correct operator $A_K = L - \overline{KL}_K L$ on $D(A_K) = D(L)$.

## 2 Preliminaries

In this section, we present some results for correct restrictions and extensions [3] which are used in Section 3.

Let $A$ and $B$ be bounded operators in a Hilbert space $H$. An operators $A$ and $B$ are said to be similar if there exist an invertible operator $P$ such that $P^{-1}AP = B$. Similar operators have the same spectrum. If at least one of two operators $A$ and $B$ is invertible, then the operators $AB$ and $BA$ are similar.

**Lemma 2.1.** Let $L$ be a densely defined correct restriction of the maximal operator $\hat{L}$ in a Hilbert space $H$. Then the operator $KL$ is bounded on $D(L)$ (that is, $\overline{KL}$ is bounded in $H$) if and only if

$$R(K^*) \subset D(L^*).$$

**Proof.** Let $R(K^*) \subset D(L^*)$. Then, by virtue of $(KL)^* = L^*K^*$, we have that $\overline{KL}$ is bounded in $H$, where $\overline{KL}$ is the closure of the operator $KL$ in $H$. Here we have used the boundedness of the operator $L^*K^*$. Then the operator $KL$ is bounded on $D(L)$. Conversely, let $KL$ be bounded on $D(L)$. Then $\overline{KL}$ is bounded on $H$, by virtue of
(KL)^* = (K\overline{L})^* and that (KL)^* is defined on the whole space \( H \). Then the operator \( K^* \) transfers any element \( f \) in \( H \) to \( D(L^*) \). Indeed, for any element \( g \) of \( D(L) \) we have
\[
(Lg, K^*f) = (KgL, f) = (g, (KL)^*f).
\]
Therefore, \( K^*f \) belongs to the domain \( D(L^*) \).

**Lemma 2.2.** Let \( L_K \) be a densely defined correct restriction of the maximal operator \( \hat{L} \) in a Hilbert space \( H \). Then \( D(L^*) = D(L_{K}^*) \) if and only if \( R(K^*) \subset D(L^*) \cap D(L_{K}^*) \), where \( L \) and \( K \) are the operators from representation (1.2).

**Proof.** If \( D(L^*) = D(L_{K}^*) \), then from representation (1.2) we easily get
\[
R(K^*) \subset D(L^*) \cap D(L_{K}^*) = D(L^*) = D(L_{K}^*).
\]
Let us prove the converse. If
\[
R(K^*) \subset D(L^*) \cap D(L_{K}^*),
\]
then we obtain
\[
(L_{K}^*)^{-1}f = (L^*)^{-1}f + K^*f = (L^*)^{-1}(I + L^*K^*)f, \tag{2.1}
\]
\[
(L^*)^{-1}f = (L_{K}^*)^{-1}f - K^*f = (L_{K}^*)^{-1}(I - L_{K}^*K^*)f, \tag{2.2}
\]
for all \( f \) in \( H \). It follows from (2.1) that \( D(L_{K}^*) \subset D(L^*) \), and taking into account (2.2) this implies that \( D(L^*) \subset D(L_{K}^*) \). Thus \( D(L^*) = D(L_{K}^*) \).

**Corollary 2.1.** Let \( L_K \) be any densely defined correct restriction of the maximal operator \( \hat{L} \) in a Hilbert space \( H \). If \( R(K^*) \subset D(L^*) \) and \( \overline{KL} \) is a compact operator in \( H \), then
\[
D(L^*) = D(L_{K}^*). \tag{3}
\]

**Proof.** Compactness of \( \overline{KL} \) implies compactness of \( L^*K^* \). Then \( R(I + L^*K^*) \) is a closed subspace in \( H \). It follows from the dense definiteness of \( L_K \) that \( R(I + L^*K^*) \) is a dense set in \( H \). Hence \( R(I + L^*K^*) = H \). Then from equality (2.1) we get \( D(L^*) = D(L_{K}^*) \).

**Lemma 2.3.** If \( R(K^*) \subset D(L^*) \cap D(L_{K}^*) \), then bounded operators \( I + L^*K^* \) and \( I - L_{K}^*K^* \) from (2.1) and (2.2), respectively, have a bounded inverse defined on \( H \).

**Proof.** By virtue of the density of the domains of the operators \( L_{K}^* \) and \( L^* \) it follows that the operators \( I + L^*K^* \) and \( I - L_{K}^*K^* \) are invertible. Since from (2.1) and (2.2) we have \( \text{Ker} \ (I + L^*K^*) = \{0\} \) and \( \text{Ker} \ (I - L_{K}^*K^*) = \{0\} \), respectively. From representations (2.1) and (2.2) we also note that \( R(I + L^*K^*) = H \) and \( R(I - L_{K}^*K^*) = H \), since \( D(L^*) = D(L_{K}^*) \). The inverse operators \( (I + L^*K^*)^{-1} \) and \( (I - L_{K}^*K^*)^{-1} \) of the closed operators \( I - L_{K}^*K^* \) and \( I + L^*K^* \), respectively, are closed. Then the closed operators \( (I + L^*K^*)^{-1} \) and \( (I - L_{K}^*K^*)^{-1} \), defined on the whole of \( H \), are bounded.
Under the assumptions of Lemma 2.3 the operators $KL$ and $KL_K$ will be (see [3]) restrictions of the bounded operators $\overline{KL}$ and $\overline{KL_K}$, respectively, where the bar denotes the closure of operators in $H$. Thus $(I - L^*_K K^*)^{-1} = I + L^* K^*$ and $(I - \overline{KL_K})^{-1} = I + KL$.

In what follows, we need the following theorem

**Theorem 2.1** (Theorem 1.1 [5, p. 307]). The sequence $\{\psi_j\}_{j=1}^{\infty}$ biorthogonal to a basis $\{\phi_j\}_{j=1}^{\infty}$ of a Hilbert space $H$ is also a basis of $H$.

## 3 Proof of Theorem 1.1

In this section we prove our main result Theorem 1.1.

**Proof.** We transform (1.2) to the form

$$L_K^{-1} = L^{-1} + K = (I + KL)L^{-1}. \quad (3.1)$$

By Lemma 2.1 and Lemma 2.3 the operators $KL$ and $KL_K$ are bounded and $I + KL$ is invertible with

$$(I + KL)^{-1} = I - K L_K.$$ 

Then we have

$$A_K^{-1} = (I + KL)^{-1} L_K^{-1} (I + KL)$$

$$= (I + KL)^{-1} (I + KL) L^{-1} (I + KL) = L^{-1} (I + KL).$$

Hence, by Corollary 1 [6, p. 259] we have $D(A_K) = D(L)$ and

$$A_K = (I - K L_K)L = L - K L_K L.$$

\[ \square \]

**Corollary 3.1.** Suppose the hypothesis of Theorem 1.1 is satisfied. Then a correct extension $L_K^*$ of a minimal operator $M_0$ is similar to the correct operator

$$A_K^* = L^*(I - L_K^* K^*),$$

on

$$D(A_K^*) = \{v \in H : (I - L_K^* K^*)v \in D(L^*)\}.$$ 

## 4 An application of Theorem 1.1 to n-order differentiation operator

In this section, we give some applications of the main result to differential operators.

As a maximal operator $\hat{L}$ in $L^2(0, 1)$, we consider

$$\hat{L}y = y^n,$$
with domain $D(\hat{L}) = W_2^n(0, 1), \ n \in \mathbb{N}$. Then the minimal operator $L_0$ is the restriction of $\hat{L}$ on $D(L_0) = \tilde{W}_2^n(0, 1)$. As a fixed boundary correct extension $L$ of the minimal operator $L_0$, we take the restriction of $\hat{L}$ on

$D(L) = \{y \in \tilde{W}_2^n(0, 1) : y^{(i)}(0) + y^{(i)}(1) = 0, \ i = 1, 2, \ldots, n - 1\}.$

We find the inverse to all correct restrictions of $L_K \subset \hat{L}$

$L_K^{-1} = L^{-1} + K,$

where

$$Kf = \sum_{i=1}^{n} w_i(x) \int_{0}^{1} f(t)\sigma_i(t) \, dt, \ \sigma_i \in L^2(0, 1),$$

and $w_i \in \text{Ker} \hat{L}, \ i = 1, 2, \ldots, n$ are linearly independent functions with the properties

$$w_i^{(k-1)} = \begin{cases} 1, & i = k, \\ 0, & i \neq k, \end{cases} \ i, k = 1, 2, \ldots, n.$$

Then the operator $L_K$ is the restriction of $\hat{L}$ on

$D(L_K) = \{u \in \tilde{W}_2^n(0, 1) : u^{(k-1)}(0) + u^{(k-1)}(1) = \int_{0}^{1} u^{(n)}(t)\sigma_k(t) \, dt, \ k = 1, 2, \ldots, n\}.$

We will consider restrictions of $L_K$ with dense domains in $L^2(0, 1)$, that is,

$$\overline{D(L_K)} = L^2(0, 1).$$

If $R(K^*) \subset D(L^*)$, then by Corollary 3.1 the operators $\overline{KL}$ and $\overline{KL}_K$ will be bounded in $L^2(0, 1)$ (where bar denotes closure). Since $\overline{KL}$ is a compact operator, then by Lemma 2.3 the operator $I + KL$ is invertible and $(I + KL)^{-1} = I - KL_K$. The operator $\overline{KL}$ is bounded if and only if

$$\sigma_i \in D(L^*) = \{\sigma_i \in \tilde{W}_2^n(0, 1) : \sigma_i^{(k-1)}(0) + \sigma_i^{(k-1)}(1) = 0, \ i, k = 1, 2, \ldots, n\}.$$ 

Hence, we have

$$KLy = \sum_{i=1}^{n} w_i(x) \int_{0}^{1} y^{(n)}(t)\sigma_i(t) \, dt = (-1)^n \sum_{i=1}^{n} w_i(x) \int_{0}^{1} y(t)\sigma_i^{(n)}(t) \, dt.$$

We find the operator $KL_K$. For this, we invert the operator

$$(I + KL)y = y + (-1)^n \sum_{i=1}^{n} w_i(x) \int_{0}^{1} y(t)\sigma_i^{(n)}(t) \, dt = u,$$

where $y \in D(L), \ u \in D(L_K)$. Then we can write

$$y = (I - KL_K)u = u - (-1)^n \sum_{i=1}^{n} w_i(x) \sum_{j=1}^{n} \beta_{ij} \int_{0}^{1} u(t)\sigma_j^{(n)}(t) \, dt,$$
where $\beta_{ij}, \ i, j = 1, 2, \ldots, n$ is an elements of the inverse matrix $U^{-1}$ of a matrix $U$.

\[
U = \begin{pmatrix}
1 + (-1)^{n-1} \sigma_1^{(n-1)}(0) & (-1)^{n-1} \sigma_2^{(n-2)}(0) & \cdots & (-1)^{n-3} \sigma_n^{(2)}(0) & -\sigma_1^{(1)}(0) & \sigma_1(0) \\
(-1)^{n-1} \sigma_2^{(n-1)}(0) & 1 + (-1)^{n-2} \sigma_2^{(n-2)}(0) & \cdots & (-1)^{n-2} \sigma_n^{(2)}(0) & -\sigma_2^{(1)}(0) & \sigma_2(0) \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
(-1)^{n-1} \sigma_n^{(n-1)}(0) & (-1)^{n-2} \sigma_n^{(n-2)}(0) & \cdots & \sigma_n^{(2)}(0) & -\sigma_n^{(1)}(0) & 1 + \sigma_n(0)
\end{pmatrix}
\]

Note that the conditions $R(K^*) \subset D(L^*)$ and $\overline{D(L_K)} = L^2(0,1)$ imply that $\det U \neq 0$. Thereby, the operator

\[
KL_K u = (-1)^n \sum_{i=1}^n w_i(x) \sum_{j=1}^n \beta_{ij} \int_0^1 u(t) \sigma_j^{(n)}(t) dt,
\]

is a bounded operator in $L^2(0,1)$. Then the operator $A_K$ has the form

\[
A_K v = Lv - KL_K Lv = v^{(n)} - (-1)^n \sum_{i=1}^n w_i(x) \sum_{j=1}^n \beta_{ij} \int_0^1 v^{(n)}(t) \sigma_j^{(n)}(t) dt,
\]

on

\[
D(A_K) = D(L) = \{ v \in W_2^n(0,1) : v^{(k-1)}(0) + v^{(k-1)}(1) = 0, \ k = 1, 2, \ldots, n \}.
\]

The operator $A_K$ can be written as

\[
A_K v = v^{(n)} - (-1)^n \sum_{i=1}^n w_i(x) \sum_{j=1}^n \beta_{ij} F_j(v),
\]

where

\[
F_j(v) = < F_j, v > = \int_0^1 v^{(n)}(t) \sigma_j^{(n)}(t) dt, \ j = 1, 2, \ldots, n.
\]

It can be seen that $F_j \in W_2^{-n}(0,1)$ in the sense of Lions-Margins (see [7]). We transform the boundary conditions of $L_K$ to the form

\[
U \begin{pmatrix}
u(0) + u(1) \\
u^{(1)}(0) + u^{(1)}(1) \\
\vdots \\
u^{(n-1)}(0) + u^{(n-1)}(1)
\end{pmatrix} = \begin{pmatrix}
\int_0^1 u(t) \sigma_1^{(n)}(t) dt \\
\int_0^1 u(t) \sigma_2^{(n)}(t) dt \\
\vdots \\
\int_0^1 u(t) \sigma_n^{(n)}(t) dt
\end{pmatrix}
\]

Then we get

\[
u^{(i-1)}(0) + u^{(i-1)}(1) = \sum_{j=1}^n \beta_{ij} \int_0^1 u(t) \sigma_j^{(n)}(t) dt, \ i = 1, 2, \ldots, n, \quad (4.1)
\]

where $u \in D(L_K)$, $\sigma_j^{(n)} \in L^2(0,1)$, $j = 1, 2, \ldots, n$. The boundary condition (4.1) is regular in Shkalikov sense (see [3]). Then, by virtue of [3], the operator $L_K$ has a
system of root vectors forming a Riesz basis with brackets in $L^2(0, 1)$. Thereby the operator $A_K$, being similar to the operator $L_K$, also has a basis with brackets property. The eigenvalues of these operators coincide. If $\{u_k\}_1^\infty$ are eigenfunctions of the operator $L_K$, then the eigenfunctions $v_k$ of the operator $A_K$ are related to them by the relations

$$u_k = (I + KL)v_k = v_k + (-1)^n \sum_{i=1}^n w_i(x) \int_0^1 v_k(t)\sigma_i^{(n)}(t) dt, \quad k = 1, 2, \ldots, n.$$ 

If, in particular, we take

$$\sigma_i^{(n)}(x) = \text{sign}(x-x_i), \quad 0 < x_i < 1, \quad i = 1, 2, \ldots, n,$$

then we get

$$F_i(v) = -2v^{(n-1)}(x_i), \quad i = 1, 2, \ldots, n.$$ 

By Corollary 3.1, Theorem 2.1, and [9, p. 928], we can assert that the system of root vectors of the adjoint operator $A_K^*$ forms a Riesz basis with brackets in $L^2(0, 1)$.

5 Example in case $n = 2$

If the maximal operator $\hat{L}$ acts as

$$\hat{L}y = -y''$$

on the domain $D(\hat{L}) = W_2^2(0, 1)$, then the minimal operator $L_0$ is the restriction of $\hat{L}$ on $D(L_0) = \hat{W}_2^2(0, 1)$. As a fixed operator $L$ we take the restriction of $\hat{L}$ on

$$D(L) = \{y \in W_2^2(0, 1) : y(0) = y(1) = 0\}.$$ 

Then

$$L^{-1}_K f = L^{-1} f + K f = - \int_0^x (x-t)f(t) dt + x \int_0^1 (1-t)f(t) dt 
+ (1-x) \int_0^1 f(t)\overline{\sigma}_1(t) dt + x \int_0^1 f(t)\overline{\sigma}_2(t) dt,$$

$$K f = (1-x) \int_0^1 f(t)\overline{\sigma}_1(t) dt + x \int_0^1 f(t)\overline{\sigma}_2(t) dt.$$
KL is bounded in $L^2(0, 1)$, if $R(K^*) \subset D(L^*) = D(L)$, that is,

$$\sigma_1, \sigma_2 \in D(L) = \{ \sigma_1, \sigma_2 \in W_2^2(0, 1) : \sigma_1(0) = \sigma_1(1) = \sigma_2(0) = \sigma_2(1) = 0 \},$$

and has the form

$$KLy = -(1 - x) \int_0^1 y(t)\sigma_1''(t) \, dt - x \int_0^1 y(t)\sigma_2''(t) \, dt.$$ 

The operator $KL_K$ is also bounded in $L^2(0, 1)$ and

$$KL_Ku = -\frac{1 - x}{\Delta} \left[ (1 - \sigma_2'(1)) \int_0^1 u(t)\sigma_1'(t) \, dt + \sigma_1'(1) \int_0^1 u(t)\sigma_2'(t) \, dt \right]$$

$$+ \frac{x}{\Delta} \left[ (1 + \sigma_1'(0)) \int_0^1 u(t)\sigma_2'(t) \, dt - \sigma_2'(0) \int_0^1 u(t)\sigma_1'(t) \, dt \right],$$

where

$$\Delta = (1 + \sigma_1'(0))(1 - \sigma_2'(1)) + \sigma_2'(0)\sigma_1'(1).$$

Then the operator $A_K$ has the form

$$A_Kv = -v'' - \frac{1}{\Delta} \left[ \left( (1 - x)(1 - \sigma_2'(1)) - x\sigma_2'(0) \right) \int_0^1 v''(t)\sigma_1'(t) \, dt 
+ \left( (1 - x)\sigma_1'(1) + x(1 + \sigma_1'(0)) \right) \int_0^1 v''(t)\sigma_2'(t) \, dt \right],$$

on

$$D(A_K) = D(L) = \{ v \in W_2^2(0, 1) : v(0) = v(1) = 0 \},$$

where $\sigma_1'', \sigma_2'' \in L^2(0, 1)$.

We rewrite the operator $A_K$ in the form

$$A_Kv = -v'' + a(x)F_1(v) + b(x)F_2(v), \quad (5.1)$$

where

$$a(x) = -\frac{1}{\Delta} \left( (1 - x)(1 - \sigma_2'(1)) - x\sigma_2'(0) \right), \quad F_1(v) = \int_0^1 v''(t)\sigma_1'(t) \, dt,$$

$$b(x) = -\frac{1}{\Delta} \left( (1 - x)\sigma_1'(1) + x(1 + \sigma_1'(0)) \right), \quad F_2(v) = \int_0^1 v''(t)\sigma_2'(t) \, dt.$$

Note that $F_1, F_2 \in W_2^{-2}(0, 1)$ in the sense of Lions-Margins (see [7]).

Further, we see that the operator $L_K$ acts as $\hat{L}$ on the domain

$$D(L_K) = \left\{ u \in W_2^2(0, 1) : \right\}$$

such that

$$L_Ky = -(1 - x) \int_0^1 y(t)\sigma_1''(t) \, dt - x \int_0^1 y(t)\sigma_2''(t) \, dt.$$
If in the particular case we take

\[
\begin{pmatrix}
1 + \sigma_1'(0) & 0 & -\sigma_2'(1) & 0 \\
\sigma_2'(0) & 0 & 1 - \sigma_2'(1) & 0
\end{pmatrix}
\begin{pmatrix}
u(0) \\
u'(0) \\
u(1) \\
u'(1)
\end{pmatrix}
= \begin{pmatrix}
-\int_0^1 u(t)\sigma_2''(t) dt \\
-\int_0^1 u(t)\sigma_2''(t) dt
\end{pmatrix},
\]

and

\[J_{13} = (1 + \sigma_1'(0))(1 - \sigma_2'(1)) + \sigma_2'(0)\sigma_1'(1) = \Delta \neq 0,\]

since \(R(K^*) \subset D(L^*)\) and \(D(L_K) = L^2(0,1)\). Then the left hand of this boundary condition is non-degenerate according to Marchenko [10], hence regularly according to Birkhoff (see [8]). By virtue of Theorem (see [8, p. 15]), the system of root vectors of the operator \(L_K\) form a Riesz basis with brackets in \(L^2(0,1)\). Thus, by virtue of Theorem (1.1) the system of root vectors of \(A_K\) also form a Riesz basis with brackets and the eigenvalues of \(L_K\) and \(A_K\) coincide, and the eigenfunctions are related to each other as follows

\[u_k = v_k - (1 - x) \int_0^1 v_k(t)\sigma_1''(t) dt - x \int_0^1 v_k(t)\sigma_2''(t) dt, \quad k \in \mathbb{N}.
\]

If in the particular case we take

\[
\begin{align*}
\sigma_1''(x) &= \text{sign}(x - x_1) - \text{sign}(x - x_2), \\
\sigma_2''(x) &= x[\text{sign}(x - x_1) - \text{sign}(x - x_2)],
\end{align*}
\]

(5.2)

where \(0 < x_1 < x_2 < 1\), then we get

\[
\begin{align*}
F_1(v) &= 2v'(x_2) - 2v'(x_1), \\
F_2(v) &= 2x_2v'(x_2) - 2x_1v'(x_1) - 2v(x_2) + 2v(x_1),
\end{align*}
\]

in (5.1).

In the case \(n = 2\), by Corollary 3.1 Theorem 2.1 and [9, p. 928], we can assert that the system of root vectors of the operator

\[A^*_K v = (-1)^2 \frac{d^2}{dx^2} \left[v(x) - c(x) \int_0^1 (1 - t)v(t) dt - d(x) \int_0^1 tv(t) dt \right],
\]

on

\[D(A^*_K) = \left\{ v \in L^2(0,1) : v(x) - c(x) \int_0^1 (1 - t)v(t) dt - d(x) \int_0^1 tv(t) dt \in D(L) \right\},
\]

form a Riesz basis with brackets in \(L^2(0,1)\), where

\[
\begin{align*}
c(x) &= -\frac{1}{\Delta} \left[ (1 - \sigma_2'(1))\sigma_1''(x) + \sigma_1'(1)\sigma_2''(x) \right], \\
d(x) &= \frac{1}{\Delta} \left[ (1 + \sigma_1'(0))\sigma_2''(x) - \sigma_1'(0)\sigma_2''(x) \right].
\end{align*}
\]
Note that
\[ \sigma''_1, \sigma''_2 \in L^2(0, 1), \quad D(L) = \{ y \in W^2_0(0, 1) : y(0) = y(1) = 0 \}. \]
For clarity, we consider the special case (5.2), then we have
\[
c(x) = \frac{\text{sign}(x - x_1) - \text{sign}(x - x_2)}{\Delta} \left[ 1 + x_2^3 - x_1^3 \right] - \frac{x_2^2 - x_1^2}{2} x,
\]
\[
d(x) = \frac{\text{sign}(x - x_1) - \text{sign}(x - x_2)}{\Delta} \left[ (1 + x_2 - x_1 - \frac{x_2^2 - x_1^2}{2}) x - \frac{x_2^2 - x_1^2}{2} + \frac{x_3^2 - x_3^3}{3} \right].
\]
The domain of \( A^*_K \) will have the form
\[
D(A^*_K) = \{ v \in L^2(0, 1) \cap W^2_0(0, x_1) \cap W^2_0(x_1, x_2) \cap W^2_0(x_2, 1) : v(0) = v(1) = 0, \]
\[
v(x_1 - 0) - v(x_1 + 0) = -c(x_1 + 0) \int_0^1 (1 - t)v(t) \, dt - d(x_1 + 0) \int_0^1 tv(t) \, dt,
\]
\[
v(x_2 + 0) - v(x_2 - 0) = -c(x_2 - 0) \int_0^1 (1 - t)v(t) \, dt - d(x_2 - 0) \int_0^1 tv(t) \, dt,
\]
\[
v'(x_1 - 0) - v'(x_1 + 0) = -c'(x_1 + 0) \int_0^1 (1 - t)v(t) \, dt - d'(x_1 + 0) \int_0^1 tv(t) \, dt,
\]
\[
v'(x_2 + 0) - v'(x_2 - 0) = -c'(x_2 - 0) \int_0^1 (1 - t)v(t) \, dt - d'(x_2 - 0) \int_0^1 tv(t) \, dt \}
\]
where
\[
c(x_1 + 0) = \frac{2}{\Delta} \left( 1 + x_2^3 - x_1^3 \right) - \frac{x_2^2 - x_1^2}{2} x_1,
\]
\[
d(x_1 + 0) = -\frac{2}{\Delta} \left( (1 + x_2 - x_1 - \frac{x_2^2 - x_1^2}{2}) x_1 - \frac{x_2^2 - x_1^2}{2} + \frac{x_3^2 - x_3^3}{3} \right),
\]
\[
c(x_2 - 0) = \frac{2}{\Delta} \left( 1 + x_2^3 - x_1^3 \right) - \frac{x_2^2 - x_1^2}{2} x_2,
\]
\[
d(x_2 - 0) = -\frac{2}{\Delta} \left( (1 + x_2 - x_1 - \frac{x_2^2 - x_1^2}{2}) x_2 - \frac{x_2^2 - x_1^2}{2} + \frac{x_3^2 - x_3^3}{3} \right),
\]
\[
c'(x_1 + 0) = -\frac{1}{\Delta} (x_2^2 - x_1^2),
\]
\[
d'(x_1 + 0) = \frac{2}{\Delta} \left( 1 + x_1 - x_2 + \frac{x_2^2 - x_1^2}{2} \right),
\]
\[
c'(x_2 - 0) = c'(x_1 + 0), \quad d'(x_2 - 0) = d'(x_1 + 0),
\]
\[
\Delta = 1 + x_2 - x_1 - \frac{x_2^2 - x_1^2}{2} + \frac{x_2^3 - x_1^3}{3} + \frac{x_2 - x_1}{12} ((x_2 - x_1)^3 + 6x_1x_2) \neq 0,
\]
since \( x_1, x_2 \in (0, 1) \).
And the operator \( A^*_K \) acts as follows
\[
A^*_K v = -v''(x) + c''(x) \int_0^1 (1 - t)v(t) \, dt + d''(x) \int_0^1 tv(t) \, dt,
\]
where
\[
c''(x) = \frac{2}{\Delta}\left[1 + \frac{x_2^3 - x_1^3}{3} + \frac{x_2^2 - x_1^2}{2}(x_2 - x_1)\right](\delta'(x - x_1) - \delta'(x - x_2))
- \frac{1}{\Delta}(x_2^2 - x_1^2)(\delta(x - x_1) - \delta(x - x_2)),
\]
\[
d''(x) = \frac{2}{\Delta}\left[(1 + x_2 - x_1 - \frac{x_2^2 - x_1^2}{2})(x_1 - x_2) - \frac{x_2^2 - x_1^2}{2} + \frac{x_2^3 - x_1^3}{3}\right]
\times (\delta'(x - x_1) - \delta'(x - x_2))
- \frac{1}{\Delta}\left(1 + x_2 - x_1 + \frac{x_2^2 - x_1^2}{2}\right)(\delta(x - x_1) - \delta(x - x_2)),
\]
here \(\delta\) is the Dirac delta function.

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