COMPOSITE TUNNEL NUMBER ONE GENUS TWO
HANDLEBODY-KNOTS

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This paper is dedicated to the 70th birthday of Professor Fico González Acuña.

Abstract. We characterize composite tunnel number one genus two handlebody-knots.

1. Introduction

It is a fundamental theorem in Knot Theory that any non-splittable link can be uniquely decomposed into prime links. This theorem was proved for knots by Schubert ([17]) and for links by Hashizume ([5]).

It was expected that two generator knots can not be decomposed into prime knots since two generator knots are most “simple” among all knots. Indeed, Norwood showed that two generator knots are prime ([15]). In contrast to Norwood’s algebraic proof, Scharlemann gave a geometric proof showing that tunnel number one knots are prime ([16]). In this direction, Scharlemann showed that tunnel number one knots are 2-string prime ([16]), and Gordon–Reid showed that tunnel number one knots are n-string prime for all n ([4]). Although Li gave a counterexample to the Rank versus Genus Conjecture for closed orientable hyperbolic 3-manifolds ([10]), it remains to be unknown whether there exists a knot in $S^3$ such that the rank of the fundamental group of the knot exterior is less than the Heegaard genus of it (e.g. [10, Question 2]).

On the other hand, there are two generator, or tunnel number one links which are not prime. Jones showed that composite two-generator links have a Hopf link summand ([8]). Furthermore, Morimoto showed that tunnel number one links are composite if and only if they are connected sums of 2-bridge knots and the Hopf link ([12], cf. [2]). In this direction, Gordon–Reid showed that n-string composite tunnel number one links have a Hopf tangle summand ([4], cf. [3]).

Suzuki generalized Schubert–Hashizume’s result to spatial graphs by proving that every connected graph embedded in $S^3$ can be split along spheres meeting the graph in 1 or 2 points to obtain a unique collection of prime embedded graphs together with some trivial graphs ([19]). Motohashi and Matveev–Turaev proved a prime decomposition theorem for $\theta_n$-curves by decomposing spheres which intersect each edge in one point ([13], [11]). Motohashi also proved a prime decomposition
theorem for handcuff graphs by decomposing spheres which intersects the graph in exactly three points ([14]).

In this direction, we consider a genus two handlebody embedded in the 3-sphere whose exterior admits a genus three Heegaard splitting, and which has a decomposing sphere intersecting the handlebody in two meridian disks.

2. Results

We call an embedding (or the image of it) of a handlebody \( V \) into \( S^3 \) a handlebody-knot. We denote the exterior \( S^3 - \text{int} V \) by \( E(V) \).

The following definition on decomposing spheres for handlebody-knots was given by Ishii, Kishimoto and Ozawa in [7].

**Definition 2.1.** A 2-sphere \( S \) embedded in \( S^3 \) is an \( n \)-decomposing sphere for a handlebody-knot \( V \) if the following conditions hold.

1. \( S \cap V \) consists of \( n \) essential disks of \( V \).
2. \( S \cap E(V) \) consists of an incompressible and not boundary-parallel planar surface in \( E(V) \).

We say that a handlebody-knot \( V \) is reducible if \( E(V) \) is boundary-reducible, i.e. \( \partial V = \partial E(V) \) is compressible in \( E(V) \), and \( V \) is irreducible if it is not reducible (note that \( E(V) \) is always irreducible in the sense that any sphere bounds a 3-ball). It follows that if \( V \) has a 1-decomposing sphere, then \( V \) is reducible. Conversely, Tsukui showed that if \( V \) is reducible, then \( V \) has a 1-decomposing sphere in the case that the genus of \( V \) is two ([21], cf. [9]).

The decomposition by 1-decomposing spheres is unique for a genus two handlebody-knot ([20]) and for a trivial handlebody-knot, i.e. one standardly embedded in \( S^3 \). The uniqueness is not known for a genus \( g \geq 3 \) handlebody-knot. Ishii, Kishimoto and the second author showed that a handlebody-knot whose exterior is boundary-irreducible has a unique maximal unnested set of knotted handle decomposing spheres up to isotopies and annulus-moves ([7]). Moreover, Koda and the second author showed the same uniqueness theorem for arbitrary handlebody-knots ([9]).

When the genus \( g \) of \( V \) is greater than one, the spine of a handlebody-knot \( V \) can not be uniquely determined. However, if we specify \( g - 1 \) mutually non-parallel essential disks in \( V \), then the spine without degree one vertices can be uniquely determined. For the case that the genus of \( V \) is equal to two, if \( V \) has a 2-decomposing sphere \( S \), then by the definition (1), \( S \cap V \) gives two mutually parallel essential disks in \( V \). Therefore, the spine of \( V \) is uniquely determined by a theta-curve or handcuff graph depending on whether the disks of \( S \cap V \) are non-separating or separating in \( V \).

Let \( \{\gamma_1, \ldots, \gamma_l\} \) be a set of mutually disjoint arcs properly embedded in \( E(V) \). We call the set \( \{\gamma_1, \ldots, \gamma_l\} \) an unknotting tunnel system for \( V \) if \( E(V) - \text{int} N(\gamma_1 \cup \ldots \cup \gamma_l) \) is a handlebody. The tunnel number of \( V \) is the minimal number of arcs among all unknotting tunnel systems. We say that a handlebody-knot is trivial if the tunnel number is zero. By Waldhausen’s theorem, any two genus \( g \) trivial handlebody-knots are equivalent up to isotopy of \( S^3 \) ([22]). When the tunnel number is one, we abbreviate \( \{\gamma\} \) to \( \gamma \) and call \( \gamma \) an unknotting tunnel. From the point of view of the tunnel number, the first considerable class of handlebody-knots is the class of tunnel number one handlebody-knots. For a spatial graph \( \Gamma \),
the exterior, unknotting tunnel (system), and tunnel number are defined by that for $N(\Gamma)$.

Let $\Gamma$ be a connected spatial graph in $S^3$, and $S$ a 2-sphere which intersects $\Gamma$ transversely, hence it is disjoint from the vertices of $\Gamma$. Then $(S^3, \Gamma)$ is decomposed by $S$ into two tangles $(B_1, T_1)$ and $(B_2, T_2)$. By a tangle we mean a pair $(B, T)$ consisting of a 3-ball $B$ and a properly embedded graph $T$ in $B$, that is, $\partial B \cap T$ consists of the degree one vertices of $T$. We say that a tangle $(B, T)$ is free if $B - \text{int}\, N(T)$ is a handlebody. A tangle $(B, T)$ is essential if $\partial B \cap E(T)$ is incompressible and not boundary-parallel in $E(T)$, where $E(T) = B - \text{int}\, N(T)$ is the exterior of $T$. We say that a tangle decomposition $(S^3, \Gamma) = (B_1, T_1) \cup_S (B_2, T_2)$ is essential if both tangles $(B_1, T_1)$ and $(B_2, T_2)$ are essential. It follows that if $S$ is an essential tangle decomposing sphere for $\Gamma$, then $S$ is also an $n$-decomposing sphere for the handlebody-knot $V = N(\Gamma)$, where $n = |S \cap \Gamma|$.

Let $\Gamma$ be a theta-curve $\theta$ or a handcuff graph $\phi$ embedded in $S^3$, and $e$ be an edge of $\theta$ or the cut edge of $\phi$. Let $P$ be a 2-sphere which intersects $\Gamma$ in four points or three points of $e$ depending on whether $\Gamma$ is $\theta$ or $\phi$. Then $P$ bounds a tangle $(B, T)$, where $T$ consists of a cycle $C$ attached with two or one edges, and an arc $\alpha$. We say that such a tangle $(B, T)$ is a Hopf tangle with two edges or a Hopf tangle with one edge if $T - \alpha$ is contained in a properly embedded disk $D$ in $B$, and $C$ bounds a subdisk $D'$ of $D$ which intersects $\alpha$ in one point, and $\alpha$ is an unknotted arc in $B$.

**Theorem 2.2.** Let $V$ be an irreducible tunnel number one genus two handlebody-knot in $S^3$. If $V$ has a 2-decomposing sphere $S$, then there exists a spine $\Gamma$ of $V$ which satisfies one of the following conditions.

1. $\Gamma$ is a theta-curve graph which is decomposed by $S$ into a tunnel number zero theta-curve graph $\theta$ and a $(1, 1)$-knot $K$.
2. $\Gamma$ is a handcuff graph which is decomposed by $S$ into a tunnel number zero handcuff graph $\phi$ and a $(1, 1)$-knot $K$, where $S$ intersects $\phi$ in the cut edge of $\phi$.
3. $\Gamma$ is a theta-curve graph which is decomposed by $S$ into a theta-curve graph $\theta$ and a 2-bridge knot $K$, where the connected sum $\Gamma = \theta \#_S K$ is done at an edge $e$ of $\theta$. The theta-curve $\theta$ is decomposed by a 2-sphere $P$ into an essential free 2-string tangle and a Hopf tangle with two edges, where $P$ intersects $\theta$ in the edge $e$.
4. $\Gamma$ is a handcuff graph which is decomposed by $S$ into a handcuff graph $\phi$ and a 2-bridge knot $K$, where the connected sum $\Gamma = \phi \#_S K$ is done at the cut edge $e$ of $\phi$. The handcuff graph $\phi$ is decomposed by a 2-sphere $P$ into an essential free tangle and a Hopf tangle with one edge, where $P$ intersects $\phi$ in the edge $e$.

See Figure 1 for a sketch of the four types of a spine $\Gamma$. Note that if a graph $\Gamma$ satisfies one of the conclusions of the theorem, then in fact it has tunnel number one. Note also that a genus two handlebody-knot could have different spines satisfying different conclusions of the theorem.

Explicit examples of genus two handlebody-knots admitting 2-decomposing spheres can be found in the table given by Ishii, Kishimoto, Morinichi and Suzuki \[6\]. They have produced a table of genus two handlebody-knots which have diagrams with at most six crossings. All these handlebody-knots have tunnel number one,
and it follows by inspection that the handlebody knots $5_4$ and $6_{14}$ satisfy condition (1) of the theorem, while $6_{15}$ and $6_{16}$ satisfy condition (2) of the theorem.

**Remark 2.3.** In the proof of Theorem 2.2, we will show that there exists an unknotting tunnel $\gamma$ for $V$ which intersects $S$ in at most one point. If $|\gamma \cap S| = 0$, then we have a conclusion (1) or (2). If $|\gamma \cap S| = 1$, then we have a conclusion (3) or (4).

![Figure 1. Four types of a spine $\Gamma$](image)

### 3. Main proofs

In this section we prove the following:

**Proposition 3.1.** Let $V$ be an irreducible genus two handlebody-knot with an unknotting tunnel $\gamma$. Suppose that $V$ admits 2-decomposing spheres. Then there is a 2-decomposing sphere $S$ for $V$ which is disjoint from $\gamma$ or intersects it in one point.

Let $V$ be an irreducible tunnel number one genus two handlebody-knot, and let $E(V) = S^3 - \text{int}N(V)$ be its exterior. Let $\gamma$ be an unknotting tunnel for $V$. So $W = S^3 - \text{int}N(V \cup \gamma)$ is a genus 3 handlebody. Let $\alpha$ be the cocore of the tunnel $\gamma$, that is, a curve on $\partial N(\gamma)$ that bounds a disk in $N(\gamma)$ which intersects $\gamma$ transversely in one point. So $E(V)$ is obtained by adding a 2-handle to $W$ along $\alpha$. Note that the curve $\alpha$ is non-separating in $\partial W$. The arc $\gamma$ may have been slid over itself; in that case it can be expressed as $\gamma = \gamma_1 \cup \gamma_2$, where $\gamma_1$ is a simple closed curve, and $\gamma_2$ is an arc with endpoints in $\partial V$ and $\gamma_1$, in this case let $v = \gamma_1 \cap \gamma_2$. 
Let $S$ be a 2-decomposing sphere for $V$. So $S \cap V$ consists of 2 essential disks in $V$, and $\tilde{S} = S \cap E(V)$ is a properly embedded surface in $E(V)$. Note that because $V$ is a genus 2 handlebody, $S$ bounds a 3-ball $B$ in $S^3$, so that $B \cap \Gamma$ consists of a knotted spanning arc in $B$, where $\Gamma$ is a graph such that $N(\Gamma) = V$. It can be assumed that $\tilde{S}$ and $\gamma$ meet in general position, that is, $\tilde{S}$ intersects $\gamma$ transversely in a finite number of points, $\tilde{S}$ is disjoint from $\gamma$ and $\gamma \cap \partial V$, and $\tilde{S} \cap N(\gamma)$ consists of a collection of disjoint disks. Then $\tilde{S}$ meets $\gamma_1$ in, say, $n$ points, and $\gamma_2$ in $m$ points. Define the complexity of $\tilde{S}$ to be $c(\tilde{S}) = n + m$.

Label with $\alpha_1, \alpha_2, \ldots, \alpha_n$ the disks of intersection of $\tilde{S}$ and $N(\gamma_1)$, labeled in order as they occur in $\gamma_1$, starting at $v$ with an arbitrary choice of direction, and label with $\beta_1, \ldots, \beta_m$ the disks of intersection between $\tilde{S}$ and $N(\gamma_2)$, labeled as they occur in $\gamma_2$, going from $v$ to $\partial V$. Denote the components of $\partial \tilde{S}$ by $s_1, s_2$.

Let $\tilde{S}$ be the surface $\tilde{S} - \text{int} N(\gamma)$. Assume that $\tilde{S}$ has been isotoped and $\gamma$ has been isotoped and slided, to make $c(\tilde{S})$ minimal. Suppose also that $S$ is a 2-decomposing sphere so that $\tilde{S}$ has minimal complexity among all the 2-decomposing spheres for $V$. Assume that $c(\tilde{S}) \neq 0$, for otherwise we are finished.

**Claim 3.2.** $\tilde{S}$ is incompressible in $W$.

**Proof.** If $\tilde{S}$ is compressible, then there is a disk $D$ in $W$ with $D \cap \tilde{S} = \partial D$, which is an essential curve on $\tilde{S}$. But in $\tilde{S}$ this curve has to be inessential, i.e. it bounds a disk $D' \subset \tilde{S}$ which intersects $\gamma$. As $E(V)$ is irreducible, the sphere $D \cup D'$ bounds a 3-ball in $E(V)$, and by interchanging $D$ and $D'$, we obtain a surface $\tilde{S}'$ isotopic to $\tilde{S}$ but with lower complexity. \hfill $\square$

Let $D$ be a compression disk for $\partial W$, which exists for $W$ is a handlebody. Consider the intersections between $D$ and $\tilde{S}$, which we assume consist of arcs and circles. Simple closed curves of intersection can be removed, for $\tilde{S}$ is incompressible. There
must exist arcs of intersection, for any compression disk for \( \partial W \) meets \( \partial N(\gamma) \), for otherwise \( \partial V \) would be compressible, and then as \( c(\hat{S}) \neq 0 \), and \( D \) and \( \hat{S} \) intersect. Assume that \( D \) has been isotoped to make this intersection minimal. Label the endpoints of the arcs of intersection in \( D \) with the labels of the disks of \( \hat{S} \cap N(\gamma) \) in which the points lie.

Let \( \delta \) be an outermost arc of intersection in \( D \), which cuts off a disk \( D' \subset D \), so that \( \partial D' = \delta \cup \eta \), where \( \eta \) is an arc on \( \partial W \), and the interior of \( D' \) is disjoint from \( \hat{S} \).

Claim 3.3. Both endpoints of \( \delta \) are in \( \alpha_1, \alpha_n \) or \( \beta_m \). Furthermore,

1. If both endpoints are in \( \alpha_1 \) or \( \alpha_n \), then \( m = 0 \), \( \eta \) intersects \( \partial E(V) \), \( \eta \cap \partial E(V) \) is an essential arc in \( \partial E(V) - N(\gamma) \), and \( \delta \) is an essential arc in \( \hat{S} \).
2. If both endpoints are in \( \beta_m \), then \( \eta \) intersects \( \partial E(V) \), and \( \eta \cap \partial E(V) \) is an essential arc in \( \partial E(V) - N(\gamma) \), and \( \delta \) is an essential arc in \( \hat{S} \).

Proof. There are several possible cases for \( D' \).

Case 1: An end of \( \delta \) is in \( s_i \) and the other in \( s_j \), \( i \neq j \).
Case 2: Both ends of \( \delta \) are in \( s_i \).
Case 3: An end of \( \delta \) is in \( s_i \) and the other in \( \alpha_1, \alpha_n \) or \( \beta_m \).
Case 4: An end of \( \delta \) is in \( \alpha_i \) and the other in \( \alpha_{i+1} \) (or \( \beta_i \) and \( \beta_{i+1} \)), and \( \eta \) is disjoint from \( N(v) \) and from \( \partial N(V) \).
Case 5: An end of \( \delta \) is in \( \beta_1 \) and the other in \( \alpha_1 \) or \( \alpha_n \).
Case 6: An end of \( \delta \) is in \( \alpha_1 \) and the other in \( \alpha_n \), and \( \eta \) intersects \( N(v) \) or \( \partial E(V) \).

The proof that all these cases cannot happen is identical to the one given in [1], Prop. 2.3, Cases 1-6. There are three more cases.

Case 7: Both ends of \( \delta \) are in \( \beta_1 \) and \( \eta \) is disjoint from \( \partial E(V) \).

If \( \eta \) is disjoint from \( N(v) \) then the intersection between \( D \) and \( \hat{S} \) would not be minimal, so it must intersect it. The arc \( \delta \) cuts off a disk \( \hat{E} \) from \( \hat{S} \), so that \( \partial \hat{E} = \delta \cup \beta' \), where \( \beta' \) is a subarc of \( \beta_1 \). Note that \( E \) may contain disks of intersection with \( V \). We may assume that \( E \) contains at most one disk of intersection with \( V \). So \( D' \cup \hat{E} \) is a disk with boundary \( \eta \cup \beta' \). If \( \eta \) is disjoint from the curve \( \alpha_i \), then there is a disk \( E' \subset \partial E' = \eta \cup \beta' \), and which intersects \( \gamma_1 \) in one point. If \( E \) is disjoint from \( N(\gamma_1) \), then \( D' \cup E \cup E' \) is a sphere in \( S^3 \) intersecting \( \gamma_1 \) transversely in one point, which is not possible, so \( E \) must intersect \( \gamma_1 \) an odd number of times. It follows that the disk \( E \) must be disjoint from \( V \) for otherwise the sphere \( D' \cup E \cup E' \) intersects \( V \) in one disk, which is not possible. By isotoping \( \hat{S} \) through the 3-ball bounded by \( D' \cup E \cup E' \), we get a sphere isotopic to \( \hat{S} \), intersecting \( \gamma_1 \) in a new point, but where the disk \( \beta_1 \) and the intersections of \( E \) with \( \gamma_1 \) have been eliminated, getting then a sphere with lower complexity.

Then the arc \( \eta \) intersects \( \alpha \), and we must have that \( n = 0 \). If the algebraic intersection number of \( \eta \) with \( \alpha \) is not \( \pm 1 \), then \( D' \cup E \) can be isotoped so that its boundary lies on \( \partial N(\gamma_1) \), and then \( N(D' \cup E) \cup N(\gamma_1) \) is a punctured lens space, which is not possible. So the arc \( \eta \) must intersect \( \alpha \) exactly once. In this case the disk \( D' \) can be used to isotope \( \gamma \) through \( \hat{S} \), eliminating \( \beta_1 \) and then reducing \( c(\hat{S}) \).

Case 8: Both ends of \( \delta \) are in \( \alpha_1 \) or \( \alpha_n \).

If \( \eta \) is disjoint from \( \partial E(V) \), then the intersection between \( D \) and \( \hat{S} \) would not be minimal, or we can find a sphere intersecting \( \gamma_1 \) in one point, or a sphere isotopic
to $\tilde{S}$ but with lower complexity, as in the proof of Case 7. So assume that $\eta$ meets $\partial E(V)$. So we must have $m = 0$, and $\gamma_1$ can be slid so that $\gamma$ is just an arc. The arc $\eta \cap \partial E(V)$ must be essential in $\partial E(V) - N(\gamma)$, for otherwise $\eta$ could be isotoped to lie in $N(\gamma)$. Finally, note that the arc $\delta$ is essential in $\tilde{S}$, for otherwise it cuts off a disk $E$ from $\tilde{S}$, so that $D' \cup E$ is a disk which can be isotoped to be a compression disk for $E(V)$.

**Case 9:** Both ends of $\delta$ are in $\beta_m$.

If $\eta$ is disjoint from $\partial E(V)$, then the intersection between $D$ and $\tilde{S}$ would not be minimal or $m = 1$ and we are in Case 7. So assume that $\eta$ meets $\partial E(V)$. An argument as in Case 8 shows that $\eta \cap \partial E(V)$ is an essential arc in $\partial E(V) - N(\gamma)$, and that $\eta$ is essential in $\tilde{S}$.

This completes the proof of the claim. □

**Claim 3.4.** There is a circle $w$ of $\Gamma$, associated to $\delta$, which is parallel to an essential circle on $\hat{S}$, that is, there is an annulus $A$ in $E(V)$ with interior disjoint from $\tilde{S}$, so that one component of $\partial A$ is a curve on $\hat{S}$ and the other component is a circle $w$ of $\Gamma$. Furthermore the circle $w$ and the annulus $A$ lie outside of the 3-ball $B$ bounded by $S$. In particular if there is an outermost arc with endpoints in $\alpha_1$ and another one with endpoints in $\alpha_n$, then $n$ must be an even number.

**Proof.** By Claim 3.3 we can assume that the endpoints of $\delta$ both lie in $\alpha_1$, $\alpha_n$ or $\beta_m$. If they lie in $\alpha_1$ or $\alpha_n$, then as the arc $\eta$ must intersect $\partial V$, we can assume that $\gamma$ has been slid so that it is just an arc. In all three cases there is a disk $E \subset \partial N(\gamma)$ so that $D' \cup E$ is an annulus with one boundary component $c_1$ in $\tilde{S}$ and the other $c_0$ in $\partial E(V)$. The curves $c_0$ and $c_1$ are essential in $\tilde{S}$ and $\partial E(V)$ respectively, by Claim 3.3. If $c_0$ is a meridian of $V$, then we can cut off $S$ with $A$ to get two new decomposing spheres. At least one of them must be essential, and the intersection with $\alpha_1$, $\alpha_n$ or $\beta_m$ is eliminated, so we get a 2-decomposing sphere with lower complexity, contradicting the hypothesis.

Suppose then that $c_0$ is not a meridian of $V$. The curve $c_0$ then lies in the boundary of a solid torus $V' \subset V$, this because $S$ divides $V$ into solid tori and 3-balls. If $c_0$ wraps twice or more around this solid torus, we have a punctured lens space, which is formed by the union of $V'$, the annulus $A$, plus a disk in $S$ bounded by $c_1$. As this is not possible, $c_0$ wraps just once around $V'$, and then it is isotopic to its core $w$. This implies that $w$ is isotopic to the curve $c_1$. Finally note that the annulus $A$ can not be contained in $B$, for $B$ contains just an arc of $\Gamma$. □
Claim 3.5. There cannot be 3 parallel arcs in $D$, where one of the arcs is outermost in $D$, and the endpoints of the arcs are all in the $\alpha_i'$s or all in the $\beta_i'$s.

**Proof.** Assume that there are 3 parallel arcs in $D$, say $\delta_1$, $\delta_2$, $\delta_3$, where $\delta_1$ is an outermost arc with both endpoints in $\alpha_1$, and $\delta_2$, and $\delta_3$ have both endpoints in $\alpha_2$ and $\alpha_3$, respectively. If the endpoints of $\delta_1$ are in $\alpha_n$ or $\beta_m$, the proof is similar. Let $D_1$ be the disk cut off by $\delta_1$ in $D$, and let $D_2$ ($D_3$) be the disks in $D$ determined by $\delta_1$ and $\delta_2$ ($\delta_2$ and $\delta_3$ respectively). As in Claim 3.3 there is a disk $E_1 \subset \partial N(\gamma)$, so that $A_1 = D_1 \cup E_1$ is an annulus in $E(V)$ with interior disjoint from $\hat{S}$, so that one component of $\partial A$ is a curve $c_1$ on $\hat{S}$ and the other component is a circle of $\Gamma$. There are disks $E_2$, $E_3$ in $\partial N(\gamma)$, so that $A_2 = D_2 \cup E_2$, $A_3 = D_3 \cup E_3$ are two annuli. The annulus $A_2$ is contained in the 3-ball $B$ bounded by $S$, and the annulus $A_3$ is in the complement of this ball, $\partial A_2 = c_1 \cup c_2$ and $\partial A_3 = c_2 \cup c_3$, where $c_2$ and $c_3$ are essential simple closed curves in $\hat{S}$. The curves $c_2$ and $c_3$ bound an annulus $F \subset \hat{S}$. Consider the sphere $\Sigma = (S - F) \cup A_3$. This is a 2-decomposing sphere for $V$; this sphere is not trivial, because the knotted arc of $\Gamma$ lying in $B$ still remain as a knotted arc inside the 3-ball bounded by $\Sigma$. If the annuli $F$ and $A_3$ are parallel, then in fact $\hat{S}$ and $\Sigma$ are isotopic, but $c(\Sigma) < c(\hat{S})$. If these annuli are not parallel, then $\Sigma$ is a new essential 2-decomposing sphere with $c(\Sigma) < c(\hat{S})$. □

Note that in the above proof, the curves $c_1$ and $c_2$ bound an annulus $F_1$ in $\hat{S}$. But in this case the new sphere $(S - F_1) \cup A_2$ may not be essential.

Claim 3.6. We have that $n \leq 4$, or $m \leq 2$.

**Proof.** In this and next claim we do an outermost fork argument, as in [10]. $D \cap \hat{S}$ consists of a collection of arcs in $D$. We construct a tree in $D$ as follows: take a vertex for each region of $D - \hat{S}$, and connect two vertices if their respective regions are adjacent, that is, they have an arc of $D \cap \hat{S}$ in common. The resultant graph $G$ is a tree, because $D$ is a disc. The ends of the tree, that is the vertices of degree one, correspond to the outermost regions of $D$.

A branch of $G$ is a trajectory that begins in a vertex of degree one of $G$ and finishes in a vertex of degree $> 2$, so that the intermediate vertices of the branch are all of degree 2. If all the vertices of $G$ are of degree 1 or 2, that is, $G$ is a trajectory, then all the arcs are parallel. By Claim 3.5 there can not be 3 parallel arcs, so $G$ has at most 3 vertices and 2 edges, which implies that $n \leq 2$ or $m \leq 2$.

If $G$ is not a trajectory, let $G'$ be the graph obtained by eliminating the branches, that is, by clearing the vertices of degree 1 and 2 of the branches together with the corresponding edges. Let $x$ be a vertex of degree 1 of $G'$ (if vertices of degree 1 do not exist, let $x$ the unique vertex of $G'$). Then at least two branches arrive at $x$, and say, let $r_1$ and $r_2$ be two adjacent branches arriving at $x$, and let $\epsilon$ an arc of $\partial D$ that goes from the outermost region of $r_1$ to the one of $r_2$.

This shows that there are two adjacent sets of parallel arcs of intersection in $D$, each containing an outermost arc. If the corresponding outermost arcs both have endpoints labeled $\alpha_1 - \alpha_1$, then the arc $\epsilon$ must cross labels $1, 2, \ldots, n - 1, n, n, n - 1, \ldots, 2, 1$, and perhaps more labels between $n$ and $n$. Any arc of intersection that leaves these labels corresponds to an edge of $r_1$ or $r_2$, by the selection of the branches. This implies that $r_1 \cup r_2$ has at least $2n$ edges, and then at least one of the branches has $n$ or more edges, that is, correspond to $n$ parallel arcs. Then there will be 3 parallel edges, which is not possible by Claim 3.5 so we must have that
$n \leq 2$. Similarly, if the outermost arcs both have endpoints labeled $\alpha_n - \alpha_n$, or $\beta_m - \beta_m$, then there will be 3 parallel edges, unless $n \leq 2$ or $m \leq 2$. Remember that if an outermost arc have endpoints in $\alpha_1$ or $\alpha_n$, then $m = 0$. If the corresponding outermost arcs have endpoints $\alpha_1 - \alpha_1$ and $\alpha_n - \alpha_n$, then there will be 3 parallel edges, unless $n \leq 4$. \hfill \qed

Claim 3.7. If $m \neq 0$ then $n = 0$.

Proof. Suppose $n \neq 0$. Any outermost arc must have both endpoints in $\beta_m$. Let $G$ and $G'$ be the graphs constructed in Claim 3.6. If $G$ is a trajectory then clearly $n = 0$; so suppose it is not a trajectory and then $G'$ is non-empty. Let $x$ be a vertex of degree 1 of $G'$ (if vertices of degree 1 do not exist, let $x$ be the unique vertex of $G'$). Then at least two branches arrive at $x$, say $r_1$ and $r_2$ are two adjacent branches arriving at $x$, and let $\epsilon$ be an arc of $\partial D$ that goes from the outermost region of $r_1$ to the one of $r_2$. $\epsilon$ must cross labels $m, m - 1, \ldots, 2, 1, 1, 2, \ldots, n - 1, n, 1, 2, \ldots, n - 1, n, 1, 2, \ldots, m$, where the sequence $1, 2, \ldots, n - 1, n$ can be repeated several times. With a simple orientation argument, it can be shown that for a label $i \in \{1, 2, \ldots, n - 1, n\}$ there can not be an arc with ends labeled $i - i$. If in one of the branches arriving at the vertex $x$, there are at least $m + 1 + n/2$ parallel arcs, there will be two parallel arcs with ends labeled $n/2$ and $(n + 1)/2$. The disk bounded by these parallel arcs forms what is commonly called a Scharlemann cycle \((10)\). So, there will be a Scharlemann cycle formed by two parallel arcs, unless all the branches arriving at $x$ have exactly $m + n/2$ parallel arcs. In this case the region $F$ of $D$ corresponding to $x$, have arcs with endpoints labeled $n/2, (n + 1)/2, n/2, (n + 1)/2, \ldots, n/2, (n + 1)/2$ in this order, that is, it is a Scharlemann cycle. As usual, this implies the existence of a punctured lens space embedded in our ambient manifold. This is formed by taking a regular neighborhood $N(S \cup H \cup F)$, where $H$ is a 1-handle attached to $S$ which consists of the part of $N(\gamma_1)$ bounded by $\alpha_{n/2}$ and $\alpha_{(n+1)/2}$. As this is impossible, we conclude that $n$ must be 0. \hfill \qed

Claim 3.8. Suppose that $n \neq 0$. Then $n \leq 2$.

Proof. By Claim 3.6 we can assume that $n \leq 4$. Suppose first that $n = 4$. By Claim 3.6 there are two pairs of parallel edges in $D$, so that the ends of the edges of one of the pairs are labeled $\alpha_1 - \alpha_1$ and $\alpha_2 - \alpha_2$, and the ends of the other arcs are labeled $\alpha_4 - \alpha_4$ and $\alpha_3 - \alpha_3$. As in the proof of Claim 3.6 there are two annuli in $B$, determined by the pairs of parallel edges. One such annulus $A_2$ has as boundary the curves $c_1$ and $c_2$, which are essential in $\hat{S}$. The other annulus $A_3$ has as boundary curves $c_3, c_4$, which are also essential in $\hat{S}$. The curves $c_1$ and $c_2$ bound an annulus $F$ in $\hat{S}$. By interchanging $F$ and $A_2$ we get a new 2-decomposing sphere, which will be essential unless $A_2$ is an annulus that follow the knotted arc of $\Gamma$ lying in $B$. Something similar can be said about the annulus $A_3$, where $c_3$ and $c_4$ bound an annulus $F'$ in $\hat{S}$. So we get a new 2-decomposing sphere with lower complexity, unless $A_2$ and $A_3$ are parallel and follow the knotted arc lying in $B$. If this happens, then suppose, say, that $F' \subset F$. Then the torus $F \cup A_3$ can be isotoped to be disjoint from $\gamma$. But it is an essential torus in the complement of $V \cup N(\gamma)$, which is impossible for $W$ is a handlebody.

Suppose now that $n = 3$. In this case either there are 3 parallel edges, contradicting Claim 3.3 or there is an outermost arc with endpoints in $\alpha_1$, and another outermost arc with endpoints in $\alpha_3$. But this contradicts Claim 3.4. \hfill \qed
Claim 3.9. If \( m \neq 0 \) then \( m = 1 \).

Proof. By Claim 3.4 suppose that \( m = 2 \). Then there is a pair of parallel edges in \( D \), one with labels \( \beta_2 - \beta_2 \), and the other with labels \( \beta_1 - \beta_1 \). As in the proof of Claim 3.8 there is an annulus in \( B \), which implies that there is another 2-decomposing sphere, or that there is an incompressible torus in \( W \), which is not possible. \( \square \)

Claim 3.10. The cases \( n = 2, m = 0 \), or \( m = 1, n = 0 \), are not possible.

Proof. Suppose first that \( n = 2 \) and then \( m = 0 \). If there is a pair of parallel edges in \( D \), one with labels \( \alpha_1 - \alpha_1 \), and the other with labels \( \alpha_2 - \alpha_2 \), then proceed as in Claim 3.4 to show that this is not possible. Suppose then that there is an outermost arc with labels \( \alpha_1 - \alpha_1 \) and another one with labels \( \alpha_2 - \alpha_2 \). Then by Claim 3.4 there are two annuli \( A_1, A_2 \) outside the 3-ball \( B \), so that \( \partial A_i = c_i \cup w_i \), where \( c_i \) is an essential curve embedded in \( \hat{S} \), and \( w_i \) is a curve on \( \partial V \), for \( i = 1, 2 \). Note that \( c_1 \neq c_2 \), but that \( w_1 \) and \( w_2 \) could be isotopic curves in \( \partial V \), in fact, they will be isotopic curves if and only if \( \Gamma \) is a theta-curve. Note that the subarc of \( \gamma \) going from \( \partial V \) to \( \alpha_1 (\alpha_2) \) is contained in the annulus \( A_1 (A_2) \). The curves \( c_1, c_2 \) divide \( \hat{S} \) into 3 annuli. Let \( C_1 (C_2) \) be the annulus whose boundary consists of the curve \( c_1 (c_2) \) and a component of \( \partial \hat{S} \) (the other component of \( \partial \hat{S} \)) and let \( C_3 \) be the annulus bounded by \( c_1 \) and \( c_2 \).

Suppose first that \( \Gamma \) is a theta-curve. Let \( C'_1 = C_1 \cup A_1 \) and \( C'_2 = C_2 \cup A_2 \), and push them to be disjoint from the tunnel \( \gamma \) and from \( B \). Note that \( C'_1 \) and \( C'_2 \) are disjoint annuli properly embedded in the handlebody \( W \), so they are \( \partial \)-compressible. Note that there is a \( \partial \)-compression disk for one of the annuli which is disjoint from the other one. Say, there is a disk \( E \) contained in \( W \), so that \( \partial E = \nu \cup \mu \), where \( \nu \) is an arc on \( \partial W \), \( \mu \) is a spanning arc in \( C'_1 \), and \( E \) is disjoint from \( C'_2 \). Note that \( C'_1 \cup C'_2 \) divides \( W \) into two handlebodies \( W_1 \) and \( W_2 \), where, say, \( B \cap W \) is contained in \( W_1 \). Note that \( E \) must be contained in \( W_2 \), for otherwise the arc \( \nu \) would intersect both components of \( \hat{S} \), i.e., it would intersect \( C'_2 \). By using \( E \), we may assume that there is an arc of \( \Gamma \cap W_2 \) that is isotopic to the arc \( \mu \) on \( E \), and by taking a neighborhood of \( C'_1 \) that contains \( E \), it is not difficult to see that there is a disk \( F \) in \( W_2 \), whose boundary is an essential curve on \( V \), i.e., \( V \) would be reducible, which is not possible.

Suppose now that \( \Gamma \) is a handcuff graph. Let \( C'_1 \) and \( C'_2 \) be defined as above. Consider the annulus \( C''_1 = A_1 \cup C_3 \cup A_2 \), and push it to be disjoint from the tunnel \( \gamma \), from \( B \) and from \( C'_1 \cup C'_2 \). Note that \( C'_1 \cup C''_1 \cup C''_2 \) divides \( W \) into two handlebodies \( W_1 \) and \( W_2 \), where, say, \( B \cap W \) is contained in \( W_1 \). There is a compression disk \( E \) for one of the annuli which is disjoint form the other two annuli. If \( E \) is contained in \( W_1 \), it will be a \( \partial \)-compression disk for \( C'_1 \) but it would imply that the tunnel \( \gamma \) is isotopic to an arc on \( C'_2 \), i.e., it would be disjoint from \( \hat{S} \). If \( E \) is contained in \( W_2 \), then it would be a \( \partial \)-compression disk for \( C'_1 \) or \( C'_2 \), and as before, this will imply that \( V \) is reducible.

Suppose now that \( m = 1 \) and \( n = 0 \). There is an outermost arc in \( D \) with labels \( \beta_1 - \beta_1 \). By Claim 3.4 there is one annulus \( A_1 \) outside the 3-ball \( B \), so that \( \partial A_1 = c_1 \cup w_1 \), where \( c_1 \) is an essential curve embedded in \( \hat{S} \), and \( w_1 \) is a curve on \( \partial V \). Note that the subarc of \( \gamma \) going from \( \partial V \) to \( \alpha_1 \) is contained in the annulus \( A_1 \). The curve \( c_1 \) divides \( \hat{S} \) into 2 annuli \( C_1 \) and \( C_2 \). Let \( C'_1 = C_1 \cup A_1 \) and \( C'_2 = C_2 \cup A_1 \), and push them to be disjoint from the tunnel \( \gamma \) and from \( B \). Note that \( C'_1 \) and \( C'_2 \) are disjoint annuli properly embedded in the handlebody \( W \), so
they are $\partial$-compressible. An argument as in the previous cases shows that this is not possible.

So we conclude that if $S$ is not disjoint from $\gamma$, then it intersects it once, and $\gamma$ is just one arc. This completes the proof of Proposition 3.1.

4. Conclusion

In this section, we characterize composite tunnel number one genus two handlebody-knots. Recall that $V$ is a tunnel number one genus two handlebody-knot in $S^3$ whose exterior is boundary-irreducible, $\gamma$ is an unknotting tunnel for $V$, and $S$ is a 2-decomposing sphere for $V$ which intersects $\gamma$ in at most one point by the previous section. $S$ bounds a 3-ball $B$ such that a spine $\Gamma$ of $V$ intersects $B$ in a knotted arc $k$.

There are four cases to consider:

1. $S \cap \gamma = \emptyset$, and $\Gamma - k$ is connected.
2. $S \cap \gamma = \emptyset$, and $\Gamma - k$ is not connected.
3. $|S \cap \gamma| = 1$, and $\Gamma - k$ is connected.
4. $|S \cap \gamma| = 1$, and $\Gamma - k$ is not connected.

Note that in Cases (1) and (2), the tunnel $\gamma$ lies on the 3-ball $B$. $N(\Gamma \cup \gamma)$ is a genus 3 handlebody and let $W = S^3 - \text{int}N(\Gamma \cup \gamma)$ be the complementary genus 3 handlebody.

In Case (1), we take a spine $\Gamma$ of $V$ as a theta-curve graph. $\Gamma$ is decomposed into a theta-curve graph $\theta$ and a knot $K$. $N(\Gamma \cup \gamma)$ is decomposed by two disks of $S \cap N(\Gamma \cup \gamma)$ into two solid tori $V_1$ and $V_2$. The handlebody $W$ is decomposed by the separating annulus $\tilde{S} = S \cap W$ into two genus 2 handlebodies $W_1$ and $W_2$ (Figure 5).

Since $W_1$ is a handlebody, $\theta$ has a tunnel number 0.

Let $E$ be a boundary-compressing disk for the separating annulus $\tilde{S}$ in $W$. Note that $E \subset W_2$. We consider a torus $T$ obtained from $\partial V_2$ by isotoping it into $\text{int}V_2$ slightly. Then $T$ bounds a solid torus $X$ in $V_2$ and $K \cap X$ is an unknotted arc in $X$. If we attach a pair $(B', t)$ of a 3-ball and an unknotted arc to $(B, k)$, then we have a pair of the 3-sphere and a knot $K$. Since $W_2$ is a genus 2 handlebody and $E$ cuts $W_2$ into a solid torus, $T$ bounds a solid torus $Y = S^3 - \text{int}X$. The arc $t$ is isotopic to an spanning arc of the annulus $\tilde{S}$, which in turn, by isotoping it through $E$, isotopic to an arc lying on $T$. Hence $K$ is a (1,1)-knot, for $K \cap X = k \cap X$ is unknotted in $T$, $K \cap Y$ is unknotted in $Y$.

In Case (2), we take a spine $\Gamma$ of $V$ as a handcuff graph. $\Gamma$ is decomposed into a handcuff graph $\phi$ and a knot $k$. $N(\Gamma \cup \gamma)$ is decomposed by two disks of $S \cap N(\Gamma \cup \gamma)$ into three solid tori $V_1$, $V_2$, and $V_3$. The handlebody $W$ is decomposed by the separating annulus $\tilde{S} = S \cap W$ into two genus 2 handlebodies $W_1$ and $W_2$ (Figure 5). Similarly to Case 1, $\phi$ has a tunnel number 0 and $k$ is a (1,1)-knot.

In Case (3), we take a spine $\Gamma$ of $V$ as a theta-curve graph. $\Gamma$ is decomposed into a theta-curve graph $\theta$ and a knot $k$. The 2-decomposing sphere $S$ for $V$ intersects $N(\Gamma \cup \gamma)$ in 3 disks one of which intersects $\gamma$ in a single point, which divide $V$ into a solid torus and a 3-ball. Note that $\tilde{S} = S \cap W$ is a pair of pants.

Note that by Claim 3.3 there is a boundary-compressing disk $E$ for $\tilde{S}$ in $W$, with interior disjoint from $B$ by Claim 3.2, and so that $E \cap \tilde{S}$ is an arc with both endpoints in the disk of intersection of $S$ with $N(\gamma)$. By isotoping $S$ along $E$, $S$
intersects $N(\Gamma \cup \gamma)$ in 2 disks and one annulus $A_1$ which intersects $\gamma$ in a single point, and $S$ intersects $W$ in two annuli $A_2, A_3$. $N(\Gamma \cup \gamma)$ is decomposed by two disks of $S \cap N(\Gamma \cup \gamma)$ and the annulus $A_1$ into two solid tori $V_1$ and $V_2$. It follows from Claim 3.4 that the core of $A_1$ is parallel to a cycle $w$ of $\theta - k$ in $V_1$. The handlebody $W$ is decomposed by two non-separating annuli $A_2, A_3$ into two genus 2 handlebodies $W_1$ and $W_2$ (Figure 7).

Now we show that $k$ is a 2-bridge knot. We note that $k$ is parallel to an arc on $\partial V_2$, $A_1$ is boundary-compressible, and $A_2, A_3$ are also boundary-compressible in $W_2$. This shows that $k$ has a 2-bridge decomposition as Figure 8.

Since the cycle of $\theta - k$ is parallel to the core of $A_1$, the cycle can be put on $S$. Let $S'$ be a 2-sphere which is obtained from $S$ by putting the cycle on it. Then we take another 2-sphere $P$ which is parallel to $S'$ and bounds a 2-string tangle. Since $W_1$ is a genus 2 handlebody, the 2-string tangle is free. Moreover, if the tangle is not
essential, say, it is a rational tangle, then the handlebody-knot is reducible, this can be seen by untwisting the rational tangle around the loop \( w \). The complementary tangle is a Hopf tangle with two edges connected sum with a 2-bridge knot.

In Case (4), we take a spine \( \Gamma \) of \( V \) as a handcuff graph. \( \Gamma \) is decomposed into a theta-curve graph \( \phi \) and a knot \( k \). The 2-decomposing sphere \( S \) for \( V \) intersects the genus 3 handlebody \( N(\Gamma \cup \gamma) \) in 3 disks one of which intersects \( \gamma \) in a single point, and \( \mathcal{S} = S \cap W \) is a pair of pants.

Note that by Claim 3.3 there is a boundary-compressing disk \( E \) for \( \mathcal{S} \) in \( W \), with interior disjoint from \( B \) by Claim 3.4 and so that \( E \cap \mathcal{S} \) is an arc with both endpoints in the disk of intersection of \( S \) with \( N(\gamma) \). By isotoping \( S \) along \( E \), \( S \) intersects \( N(\Gamma \cup \gamma) \) in 2 disks and one annulus \( A_1 \) which intersects \( \gamma \) in a single point, and \( S \) intersects \( W \) in two annuli \( A_2, A_3 \).

The handlebody \( N(\Gamma \cup \gamma) \) is decomposed by two disks of \( S \cap N(\Gamma \cup \gamma) \) and the annulus \( A_1 \) into three solid tori \( V_1, V_2 \) and \( V_3 \). It follows from Claim 3.4 that the core of \( A_1 \) is parallel to a cycle of \( \phi - k \) in \( V_1 \). The handlebody \( W \) is decomposed by two non-separating annuli \( A_2, A_3 \) into two genus 2 handlebodies \( W_1 \) and \( W_2 \) (Figure 8). Similarly to Case 3, \( k \) is a 2-bridge knot, and \( \phi \) is decomposed by a 2-sphere \( P \) into a free tangle and a Hopf tangle with one edge. If the free tangle is not essential, then there is a compression disk for \( P \), which separates the arc and the other circle of the graph, which then implies that the circle bounds a disk, i.e., again the handlebody-knot is reducible.
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