RELATIVE BENDING ENERGY FOR WEAKLY PRESTRAINED SHELLS

SILVIA JIMÉNEZ BOLAÑOS AND ANNA ZEMLYANOVA

ABSTRACT. In this paper, we derive a dimensionally reduced model for a thin film prestrained with a given incompatible Riemannian metric:

\[ G^h(x', x_3) = I_3 + 2h^\gamma S(x') + 2h^{\gamma/2} x_3 B(x') + h.o.t, \quad \gamma > 2, \]

where \( 0 < h \ll 1 \) is the thickness of the film. The problem is studied rigorously by using a variational approach and establishing the \( \Gamma \)-convergence of the non-Euclidean version of the nonlinear elasticity functional. It is shown that the residual nonlinear elastic energy scales as \( O(h^{\gamma+2}) \) as \( h \to 0 \).

1. INTRODUCTION.

The present study is concerned with the derivation of dimensionally reduced models for limiting behavior of thin films prestrained with a family of incompatible metrics \( G^h \). The motivation behind this study comes from applications for thin objects with internal prestrain such as growing tissues and various manufactured phenomena (for instance, polymer gels, atomically thin graphene layers, and plastically strained sheets). Shape formation driven by internal prestrain is a very active area of research which has been tackled before by various analytic and numerical arguments; see for instance [31, 11, 5, 10, 9, 12, 13, 23, 24].

The dimension reduction problems for thin plates and shells consist of minimizing a nonlinear elastic energy functional representing a mismatch between the deformation of the film and the target Riemannian metric \( G \). In [14, 15], the authors considered the case \( G = I_3 \) and derived nonlinear membrane models, for planar membranes and shells, from the variational formulation of the three-dimensional nonlinear elasticity. They showed that the deformations that minimize, or almost minimize, the total energy weakly converge in a Sobolev space towards deformations that minimize a nonlinear membrane energy, as the thickness of the body goes to zero. The limiting nonlinear membrane energy was obtained by \( \Gamma \)-convergence of the sequence of three-dimensional energies. The notion of \( \Gamma \)-convergence, introduced by Ennio De Giorgi in a series of papers published between 1975 and 1983 [4], has become the standard notion of convergence for variational problems [3].

In the fundamental papers [7, 8], the authors derived the hierarchy of limiting theories of thin plates which arise as \( \Gamma \)-limits of three dimensional nonlinear theory in the classical elasticity. In particular, they were able to show that von Kármán equations can be obtained rigorously from variational formulation of the nonlinear elasticity. The limiting theories differ in their scaling as powers of the thickness \( h \) of the plate or shell depending on the scaling of the applied external forces.

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The work in this area has been further extended to limiting theories for thin shells in [15, 6, 18, 20, 25, 19, 21, 17]. The first results for thin nonlinear elastic shells were obtained in [15] for the scaling $\beta = 0$. The case $\beta = 2$ was investigated in [6]. The nonlinear theory for shells with varying thickness with external loading was studied in [15]. In [20], the authors studied higher order $\beta \geq 4$ energy scaling for thin nonlinear elastic shells. The paper [16] further extends the results of [20], by proving that the equilibria of nonlinear energy functional for a thin shell converge to the equilibria of the von Kármán functional. The intermediate case of $2 < \beta < 4$ was studied in [19]. Finally, the results were summarized and the conjecture about the infinite hierarchy of limiting two-dimensional models was made in [21]. The problem for thin shells has been revisited in [17] to study the Γ-limit for thin shells with certain scaling of the applied forces. In the papers mentioned above, the deformation of the plate or shell has been activated due to an application of external forces and not due to the prestrain.

In the context of the prestrain-driven response, the parallel theories are differentiated by the embeddability properties of the target metrics and, a-posteriori, by the emergence of isometry constraints on deformations with low regularity. In turn, results on thin limit models have ramifications for the three dimensional original model with regard to energy scaling laws, understanding of the role of curvature in identifying the material’s mechanical properties, and in the consequences of the symmetry and the symmetry breaking in the solutions to the resulting Euler-Lagrange equations. The first work to rigorously study non-trivial configurations of thin prestrained flat films was produced in [30]. In this paper, the variational formulation of the problem was introduced by using a nonlinear elasticity functional and the necessary and sufficient conditions for existence of a $W^{2,2}$ isometric immersion were obtained. The targeted metric in [30] does not depend on the thickness variable of the plate. Further results for quantization of the elastic energy for the case of thickness-independent metric have been obtained in [1] and [29]. A survey of available models for the thickness-independent prestrain is given in [28]. The question of immersability for a thickness-dependent Riemann metric satisfying certain general conditions has been decisively answered in [23]. In [24], the dimension reduction for oscillatory metrics was obtained. The prescribed incompatibility metric in this paper exhibits a nonlinear dependence on the transversal variable. Asymptotic theories for shallow shells prestrained with a growth tensor linearly dependent on the transversal variable were studied in [26]. A survey of results available in prestrained elasticity can be found in [22].

In our paper, as in [32, 25, 27], the prestrain metrics $G^h$ (see (2.4)) are perturbations of the flat $I_3$ metric. In particular, in [27] the authors derived a new variational model consisting of minimizing a biharmonic energy of the out-of-plane displacement $v \in W^{2,2}(\Omega, \mathbb{R})$, satisfying the Monge-Ampère constraint $\det \nabla^2 v = f$, where $f = -\text{curl}^T \text{curl} S_{2 \times 2}$ is the linearized Gauss curvature of the Riemannian metrics in (2.4). The work in [27] was done in the parameter range $0 < \gamma < 2$, whereas the case $\gamma = 2$ was treated in [29], leading to the derivation of the Föppl-von Kármán equations accounting for the presence of the prestrain. The main contribution of our paper is to develop the analysis for the parameter range $\gamma > 2$. We identified the asymptotic behavior of the minimizers of $I^h_W(u^h)$ as $h \to 0$ (see (2.2)), through deriving the Γ-limit of the rescaled energies $\frac{1}{h^{\gamma+2}} I^h_W(u^h)$. These new outcomes are
presented in Theorem 3.1 and Theorem 3.2. A generalization of the current results to the growth tensors \( A^h(x', x_3) = I_3 + h^\alpha S(x') + h^{\gamma/2} x_3 B(x') \), with arbitrary powers \( \alpha \) and \( \gamma \), is currently in preparation [2].

This paper is organized as follows. In Section 2 we give the problem formulation. In Section 3, we present the main results of this paper, which are stated in Theorem 3.1 and Theorem 3.2. Finally, in Section 4 and Section 5 we present the proofs of Theorem 3.1 and Theorem 3.2, respectively.

2. Problem formulation: non-Euclidean elasticity model.

Let the smooth invertible growth tensors \( A^h = [A^h_{ij}] : \overline{\Omega}^h \to \mathbb{R}^{3 \times 3}, \det A^h > 0, \)
be defined by:
\[
A^h(x', x_3) = I_3 + h^\gamma S(x') + h^{\gamma/2} x_3 B(x'),
\]
where the scaling exponent \( \gamma > 2 \) and \( I_n \) is the \( n \times n \) identity matrix. Consider a family of three-dimensional thin plates:
\[
\Omega^h = \omega \times (-h/2, h/2) \subset \mathbb{R}^3,
\]
where \( \omega \) is an open bounded set of \( \mathbb{R}^2 \) and \( 0 < h << 1 \), viewed as the reference configurations of thin elastic films. A typical point in \( \Omega^h \) is denoted by \( x = (x_1, x_2, x_3) = (x', x_3), \) where \( x' \in \omega \) and \( |x_3| < h/2. \) The “stretching” and “bending” tensors \( S, B : \mathbb{R}^3 \to \mathbb{R}^{3 \times 3} \) are two given smooth matrix fields.

For a deformation \( u^h : \Omega^h \to \mathbb{R}^3, \) its elastic energy \( I^h_W(u^h), \) defined by:
\[
I^h_W(u^h) = \frac{1}{n} \int_{\Omega^h} W(F)dx
= \frac{1}{h} \int_{\Omega^h} W(\nabla u^h(A^h)^{-1})dx \quad \forall u^h \in W^{1,2}(\Omega^h, \mathbb{R}^3),
\]
is given in terms of the elastic tensor \( F = \nabla u^h(A^h)^{-1} \) (see [31]) accounting for the reorganization of \( \Omega^h \) in response to \( A^h. \)

The elastic energy density \( W : \mathbb{R}^{3 \times 3} \to \mathbb{R}_+ \) is assumed to satisfy the standard conditions of normalization, frame indifference with respect to the special orthogonal group \( SO(3) \) of proper rotations in \( \mathbb{R}^3, \) and second order nondegeneracy given by:
\[
\exists c > 0 \quad \forall F \in \mathbb{R}^{3 \times 3} \quad \forall R \in SO(3) \quad W(R) = 0, \quad W(RF) = W(F), \quad W(F) \geq c \operatorname{dist}^2(F, SO(3)).
\]
We also assume that there exists a monotone nonnegative function \( \nu : [0, +\infty] \to [0, +\infty] \) which converges to zero at 0, and a quadratic form \( Q_3 \) on \( \mathbb{R}^{3 \times 3}, \) with:
\[
\forall F \in \mathbb{R}^{3 \times 3} \quad |W(I_3 + F) - \frac{1}{2} Q_3(F)| \leq \nu(|F|)|F|^2.
\]
If \( W \) is \( C^2 \) regular in a neighborhood of \( SO(3), \) then condition (2.4) is satisfied and, in that case, we have \( Q_3 = D^2W(I_3). \) Note that (2.4) implies that \( Q_3 \) is nonnegative, is positive definite on symmetric matrices, and \( Q_3(F) = Q_3(\operatorname{sym} F) \) for all \( F \in \mathbb{R}^{3 \times 3} \) (see [27] for a proof).

Recall that in (2.2), \( I^h_W(u^h) = 0 \) is equivalent, via (2.3) and the polar decomposition theorem, to:
\[
(\nabla u^h)^T \nabla u^h = (A^h)^T A^h \quad \text{and} \quad \det \nabla u^h > 0 \quad \text{in} \ \Omega^h;
\]
which is equivalent to: \( I^h_w(u^h) = 0 \) if and only if \( u^h \) is an isometric immersion of the Riemannian metric \( G^h = (A^h)^T(A^h) \). Therefore, the quantity:

\[
e_h = \inf \{ I^h_w(u^h) : u^h \in W^{1,2}(\Omega^h, \mathbb{R}^3) \}
\]

measures the residual energy at free equilibria of the configuration \( \Omega^h \) that has been prestrained by \( G^h \). This is consistent with Theorem 2.2 in [30], which observes that \( e_h > 0 \) whenever \( G^h \) has no smooth isometric immersion in \( \mathbb{R}^3 \), i.e. when there is no \( u^h \) with (2.5) or, equivalently, when the Riemann curvature tensor of the metric \( G^h \) does not vanish identically on \( \Omega^h \).

Observe now that \( A^h \) in (2.1) yields:

\[
G^h(x', x_3) = (A^h)^T(A^h)
\]

\[
= I_3 + 2h^\gamma \text{sym} \ S(x') + 2h^{\gamma/2}x_3 \text{sym} \ B(x')
\]

\[
+ 2h^{3\gamma/2}x_3 \text{sym} \ S(x')^T B(x')
\]

\[
+ h^{2\gamma} S(x')^T S(x') + h^\gamma x_3^2 B(x')^T B(x').
\]

(2.6)

2.1. Notation. For a matrix \( F \), \( F_{n \times m} \) denotes its \( n \times m \) principal minor. If \( m = n \), the symmetric part of a square matrix \( F \) is denoted by \( \text{sym}(F) = (F + F^T)/2 \). The superscript \( ^T \) refers to the transpose of a matrix or an operator.

Also, for any \( F \in \mathbb{R}^{3 \times 2} \), we denote by \( F^* \in \mathbb{R}^{3 \times 3} \) the matrix for which \( F^*_{2 \times 2} = F \) and \( F^*_i = F^*_3 = 0 \), for \( i = 1, 2, 3 \). By \( \nabla_{\tan} \) we denote taking derivatives \( \partial_1 \) and \( \partial_2 \) in the in-plane directions \( e_1 = (1,0,0)^T \) and \( e_2 = (0,1,0)^T \). The derivative \( \partial_3 \) is taken in the out-of-plane direction \( e_3 = (0,0,1)^T \).

Finally, we use the Landau symbols \( O(h^\alpha) \) and \( o(h^\alpha) \) to denote quantities which are of the order of, or vanish faster than \( h^\alpha \), as \( h \to 0 \). By \( C \) we denote any universal constant, depending on \( \omega \) and \( W \), but independent of other involved quantities, so that \( C = O(1) \), and it can change values from line to line.

2.2. Formal discussion about scaling. Consider the deformations \( u^h : \Omega^h \to \mathbb{R}^3 \) of the shell \( \Omega^h \) given by:

\[
u^h(x', x_3) = (x', 0)^T + h^{\gamma/2}V(x') + x_3N_h(x') + h.o.t.,
\]

where \( N_h(x') \) is the unit normal to the midplate and \( V : \omega \to \mathbb{R}^3 \). Notice that we have:

\[
\nabla u^h(x', x_3) = (I_3)^* + h^{\gamma/2}\nabla V + N_h \otimes e_3 + x_3 \nabla N_h + h.o.t.,
\]

and, since \( F = \nabla u^h(A^h)^{-1} \), we have that:

\[
\sqrt{F^TF} = I_3 - h^\gamma \text{sym} \ S - x_3h^{\gamma/2}\text{sym} \ B + h^{\gamma/2} \text{sym} \nabla V
\]

\[
+ \frac{h^\gamma}{2} \nabla V^T \nabla V + x_3 \text{sym} \left( \left[ I_3 + h^{\gamma/2} \nabla V^T \right] \nabla N_h \right) + h.o.t.
\]

(2.7)

Now, by (2.7) and using the Taylor expansion for \( W \), we can go back to (2.2) to obtain:

\[
I^h_w(u^h) = \frac{1}{h} \int_{\Omega^h} W(\sqrt{F^TF})dx'dx_3 = \frac{1}{h} \int_{\Omega^h} \left( \frac{1}{2} Q_3(\ast) + h.o.t. \right) dx'dx_3,
\]

(2.8)

where \( Q_3(\ast) = D^2W(Id)(\ast, \ast) \), and \( \ast \) is given by:

\[
\ast = -h^\gamma \text{sym} \ S - x_3h^{\gamma/2}\text{sym} \ B + h^{\gamma/2} \text{sym} \nabla V
\]

\[
+ \frac{h^\gamma}{2} \nabla V^T \nabla V + x_3 \text{sym} \left( \left[ I_3 + h^{\gamma/2} \nabla V^T \right] \nabla N_h \right) + h.o.t.
\]
Thus, we can rewrite (2.8) as:

\[
I_W^h(u^h) = \int_\omega \frac{1}{2} Q_3(-h^\gamma \text{sym}S + h^\gamma/2 \text{sym} \nabla v + \frac{h^\gamma}{2} \nabla V \nabla V) dx' \\
+ \frac{h^2}{24} \int_\omega Q_3(-h^\gamma/2 \text{sym} B - h^\gamma/2 \nabla^2 V + h^\gamma D) dx' + \text{h.o.t.}
\]

\[
= \frac{h^{2\gamma}}{8} \int_\omega Q_3(-2 \text{sym} S + \nabla V^T \nabla V) dx' \\
+ \frac{h^{\gamma+2}}{24} \int_\omega Q_3(-\text{sym} B - \nabla^2 V + h^2 D) dx' + \text{h.o.t.,}
\]

where the matrix \(D\) is given by:

\[
D = \begin{pmatrix}
V_{1,11} & V_{1,2,11} & V_{1,1,12} & V_{1,2,12} \\
V_{1,1,21} & V_{1,2,21} & V_{1,1,22} & V_{1,2,22}
\end{pmatrix}.
\]

Since \(\gamma > 2\), we observe that \(I_W^h(u^h) \approx Ch^{\gamma+2}\) and hence, we expect the \(\Gamma\)-limit of \(1/h^{\gamma+2}I_W^h(u^h)\) to be only the first order change in the linear bending energy:

\[
\frac{1}{24} \int_\omega Q_3(B + \nabla^2 V) dx'.
\]

3. The variational limit in the case \(\gamma > 2\).

As explained before, the main goal of this paper is the identification of the asymptotic behavior of the minimizers of \(I_W^h(u^h)\) as \(h \to 0\), through deriving the \(\Gamma\)-limit of the rescaled energies \(1/h^{\gamma+2}I_W^h(u^h)\).

In [27] the authors derived a new variational model consisting of minimizing a biharmonic energy \(\int_\omega |\nabla^2 v|^2 dx'\) of the out-of-plane displacement \(v \in W^{2,2}(\omega, \mathbb{R})\), satisfying the Monge-Ampère constraint \(\det \nabla^2 v = f\), where \(f = -\text{curl}^T \text{curl} S_{2 \times 2}\) is the linearized Gauss curvature of the Riemannian metrics in (2.6). This work was done in the parameter range \(0 < \gamma < 2\), whereas the case \(\gamma = 2\) was treated in [25], leading to the derivation of the Föppl-von Kármán equations accounting for presence of the prestrain. In what follows, we carry out the analysis for the parameter range \(\gamma > 2\).

We now state the main results of this paper:

**Theorem 3.1.** Let \(A^h\) be given as in (2.7), with an arbitrary exponent \(\gamma > 2\). Assume that a sequence of deformations \(u^h \in W^{1,2}(\Omega^h, \mathbb{R}^3)\) satisfies:

\[
I_W^h(u^h) \leq Ch^{\gamma+2},
\]

where \(W\) fulfills (2.3) and (2.4). Then, there exist rotations \(\hat{R}^h \in SO(3)\) and translations \(\hat{c}^h \in \mathbb{R}^3\) such that, for the normalized deformations:

\[
\tilde{y}^h \in W^{1,2}(\Omega^1, \mathbb{R}^3), \quad \tilde{y}^h(x', x_3) = (\hat{R}^h)^T u^h(x', hx_3) - \hat{c}^h,
\]

the following hold (up to a subsequence that we do not relabel):

(i) \(\tilde{y}^h(x', x_3) \to x'\) in \(W^{1,2}(\Omega^1, \mathbb{R}^3)\)
Theorem 3.2. Assume (2.1), (2.3), and (2.4). Moreover, assume that

\[
\omega \quad \text{and that} \quad \gamma > 2.
\]

Under this condition, for any minimizing sequence \( u^h \in W^{1,2}(\Omega^h, \mathbb{R}^3) \) such that the following hold:

(i) The sequence \( y^h(x', x_3) = u^h(x', hx_3) \) converge in \( W^{1,2}(\Omega^1, \mathbb{R}^3) \) to \( x' \).

(ii) \( V^h(x') = \frac{1}{h^{3/2}} \int_{-h/2}^{h/2} (u^h(x', t) \cdot x') \, dt \) converge in \( W^{1,2}(\Omega, \mathbb{R}^3) \) to \( (0, 0, V_3)^T \).

(iii) Recalling (3.5), one has:

\[
\lim_{h \to 0} \frac{1}{h^{\gamma/2}} I_{W^h}(u^h) = \mathcal{I}_\gamma(V_3).
\]

As a result of Theorem 3.1 and Theorem 3.2 we have:

Theorem 3.3. Assume (2.1), (2.3), and (2.4). Moreover, assume that \( \omega \) is simply connected and that \( \gamma > 2 \). Then there exist a uniform constant \( C \geq 0 \) such that:

\[
e_h = \inf I_{W^h} \leq C h^{\gamma/2}.
\]

Under this condition, for any minimizing sequence \( u^h \in W^{1,2}(\Omega^h, \mathbb{R}^3) \) for \( I_{W^h} \), i.e. when:

\[
\lim_{h \to 0} \frac{1}{h^{\gamma+2}} (I_{W^h}(u^h) - \inf I_{W^h}) = 0,
\]

the convergences (i), (ii) of Theorem 3.1 hold up to a subsequence, and the limit \( V_3 \) is a minimizer of the functional \( \mathcal{I}_\gamma \) defined as in (3.5).

Moreover, for any (global) minimizer \( V_3 \) of \( \mathcal{I}_\gamma \), there exists a minimizing sequence \( u^h \), satisfying (3.7) together with (i), (ii), and (iii) of Theorem 3.2.

4. Proof of Theorem 3.1

Proof. This proof will be separated in different sections. In Section 4.1 we obtain the rotations \( \hat{R}^h \) required by Theorem 3.1. In Sections 4.2, 4.3, and 4.4 we present the proof of statements (i), (ii), and (iii) respectively.
4.1. Construction of the rotations $\tilde{R}^h$. First, we quote the following approximation result (Theorem 4.1), which can be directly obtained from the geometric rigidity estimate found in Theorem 1.6 in [7], in view of the following bounds:

$$\begin{align*}
V ar(A^h) &= \| \nabla_t(A^h)|_{x_3=0} \|_{L^\infty(\omega)} + \| \partial_3 A^h \|_{L^\infty(\Omega)} \leq Ch^2, \\
\| A^h \|_{L^\infty(\Omega)} + \| (A^h)^{-1} \|_{L^\infty(\Omega)} &\leq C.
\end{align*}$$

**Theorem 4.1.** Let $u^h \in W^{1,2}(\Omega^h, \mathbb{R}^3)$ satisfy $\lim_{h \to 0} \frac{1}{h^2} W(u^h) = 0$, which is, in particular, implied by (3.1). Then, there exist matrix fields $R^h \in W^{1,2}(\omega, \mathbb{R}^{3 \times 3})$, such that $R^h(x') \in SO(3)$ for a.e. $x' \in \omega$, and:

$$\frac{1}{h} \int_{\Omega^h} |\nabla u^h(x) - R^h(x') A^h(x)|^2 \, dx \leq Ch^{2+\gamma}, \quad \int_\omega |\nabla R^h|^2 \, dx' \leq C h^\gamma. \quad (4.1)$$

In order to achieve the proof of compactness in Theorem 3.1, we outline similar arguments as in [27], emphasizing the differences and new outcomes of this paper.

Assume (3.1) and let $R^h \in W^{1,2}(\omega, \mathbb{R}^{3 \times 3})$ be the matrix fields given by Theorem 4.1. Define the averaged rotations: $\tilde{R}^h = P_{SO(3)} \int_\omega R^h$. These projections of $\int_\omega R^h$ onto $SO(3)$ are well defined for small $h$ since, by (4.1) and Poincaré’s inequality, we have:

$$\text{dist}^2(\int_\omega R^h, SO(3)) \leq C \int_\omega |\nabla R^h|^2 \, dx' \leq C h^\gamma,$$

which together with Poincaré’s inequality, deliver:

$$\int_\omega |\tilde{R}^h - R^h|^2 \leq C(\int_\omega |\tilde{R}^h - \int_\omega R^h|^2 + \text{dist}^2(\int_\omega R^h, SO(3))) \leq C h^\gamma. \quad (4.2)$$

Now, let:

$$\tilde{R}^h = P_{SO(3)} \int_{\Omega^h} (\tilde{R}^h)^T \nabla u^h, \quad (4.3)$$

which is well defined for small $h$ because:

$$\text{dist}^2(\int_{\Omega^h} (\tilde{R}^h)^T \nabla u^h, SO(3))$$

$$\leq C(\int_{\Omega^h} |\nabla u^h - R^h A^h|^2 + \int_{\Omega^h} |A^h - I_3|^2 + \int_{\Omega^h} |R^h - \tilde{R}^h|^2)$$

$$\leq C h^\gamma, \quad (4.4)$$

where we used (4.1), (2.1), and (4.2). Consequently, using the fact that $I_3 = P_{SO(3)} I_3$, we also obtain:

$$|\tilde{R}^h - I_3|^2 \leq C(\text{dist}^2(\int_{\Omega^h} (\tilde{R}^h)^T \nabla u^h, SO(3)) + |\int_{\Omega^h} (\tilde{R}^h)^T \nabla u^h - I_3|^2)$$

$$\leq C h^\gamma. \quad (4.5)$$

We may now define:

$$\tilde{R}^h = \tilde{R}^h \tilde{R}^h. \quad (4.6)$$

By (4.2), (4.5), and (4.4), it follows that:

$$\int_\omega |\tilde{R}^h - R^h|^2 \leq C\left(\int_\omega |\tilde{R}^h - R^h|^2 + \int_\omega |\tilde{R}^h(I_3 - \tilde{R}^h)|^2\right) \leq C h^\gamma. \quad (4.7)$$
Now, consider:
\[
\| (\bar{R}^h)^T R^h - I_3 \|_{W^{1,2}(\omega)}^2 = \int_\omega |R^h - \bar{R}^h|^2 + \int_\omega |\nabla R^h|^2 \leq C h^\gamma,
\] (4.8)
and observe that from (4.8) we conclude:
\[
\lim_{h \to 0} (\bar{R}^h)^T R^h = I_3 \quad \text{in } W^{1,2}(\omega, \mathbb{R}^{3 \times 3}).
\] (4.9)

**Lemma 4.2.** There exist vectors \( c^h \in \mathbb{R}^3 \) such that, for the re-scaled averaged displacement \( V^h \) defined as in (3.3):
\[
V^h(x') = \frac{1}{h^{7/2}} \int_{-1/2}^{1/2} (\bar{R}^h)^T u^h(x', ht) - c^h - x' dt,
\]
it holds:
\[
\int_\omega V^h dx' = 0, \quad \text{skew} \int_\omega \nabla V^h dx' = 0.
\] (4.10)

**Proof.** To ensure the first statement in (4.10), we can take:
\[
c^h = \frac{1}{|\omega|} \int_\omega \int_{-1/2}^{1/2} [(\bar{R}^h)^T u^h(x', ht) - x'] dt dx'.
\]
For the second statement in (4.10), we observe that:
\[
\nabla V^h = \frac{1}{h^{7/2}} \int_{-1/2}^{1/2} [(\bar{R}^h)^T \nabla_{tan} u^h(x', ht) - (I_3)^*] dt.
\] (4.11)
In view of (4.6), we obtain that:
\[
(\bar{R}^h)^T \int_{\Omega_h} \nabla u^h = (\bar{R}^h)^T \int_{\Omega_h} (\bar{R}^h)^T \nabla u^h;
\]
and, because of (4.3) and (4.4), \((\bar{R}^h)^T \int_{\Omega_h} \nabla u^h\) is symmetric. Hence:
\[
\text{skew} \int_\omega \nabla V^h = h^{-7/2} \text{skew} \int_{\Omega_h} (\bar{R}^h)^T \nabla u^h = 0.
\]
In particular, we see as well that (4.6) is equivalent to:
\[
\bar{R}^h = \mathbb{P}_{SO(3)} \int_{\Omega_h} \nabla u^h.
\]
The above result is true since for a matrix \( F \) sufficiently close to \( SO(3) \), its projection \( R_0 = \mathbb{P}_{SO(3)} F \) coincides with the unique rotation appearing in the polar decomposition of \( F \), that is, \( F = R_0 U \) with \( \text{skew} U = 0 \). \( \square \)

**4.2. Proof of (i) in Theorem 3.1.** We use (3.2), (4.1), (4.4), and (4.5), to obtain:
\[
\left\| (\nabla y^h - I_3)_{3 \times 2} \right\|_{L^2(\Omega^1)}^2 \\
\leq \frac{1}{h} \int_{\Omega^h} |(\bar{R}^h)^T \nabla u^h - I_3|^2 \\
\leq C \left( \frac{1}{h} \int_{\Omega^h} |(\bar{R}^h)^T \nabla u^h - I_3|^2 + \frac{1}{h} \int_{\Omega^h} |\bar{R}^h - I_3|^2 \right) \\
\leq C h^\gamma,
\] (4.12)
Lemma 4.4. The sequence

and also, by (4.1), we get:

\[ \| \partial_3 y^h \|_{L^2(\Omega)}^2 = h \int_{\Omega^h} |\partial_3 y^h|^2 \]

\[ \leq Ch \left( \int_{\Omega} |\nabla y^h - R^h A^h|^2 + \int_{\Omega^h} |A^h|^2 \right) \]

\[ \leq Ch^2. \]  

(4.13)

By the first statement in (4.10), we have:

\[ \int_{\Omega^h} y^h(x) - x' \, dx = h^{\gamma/2} \int_{\omega} V^h \, dx' = 0; \]

hence, using Poincaré’s inequality, (4.12), and (4.13), we have:

\[ \| y^h(x) - x' \|^2_{L^2(\Omega)} \leq C \int_{\Omega} |\nabla y^h - I_3|^2 \]

\[ \leq C \left( \int_{\Omega^h} |\nabla y^h - I_3|^2 + \int_{\Omega^h} |\partial_3 y^h|^2 \right) \]

\[ \leq Ch^2, \]  

(4.14)

and therefore, we obtain the convergence of \( y^h \) to \( x' \) in \( W^{1,2}(\Omega) \).

4.3. Proof of (ii) in Theorem 3.1

Lemma 4.3. \( V^h \) converges (up to a subsequence) weakly in \( W^{1,2}(\omega, \mathbb{R}^3) \).

Proof. Observe that, using the first statement in (4.11) and Poincaré’s inequality, we have:

\[ \| V^h \|^2_{L^2(\omega)} \leq C \| \nabla V^h \|^2_{L^2(\omega)}. \]

Also, by Jensen’s inequality and (4.12), we obtain:

\[ \| \nabla V^h \|^2_{L^2(\omega)} = \left\| \frac{1}{h^{\gamma/2}} \int_{-h/2}^{h/2} \left[ (\tilde{R}^h)^T \nabla \tan u^h(x', t) - (I_3)^* \right] dt \right\|^2_{L^2(\omega)} \]

\[ \leq \frac{C}{h^{\gamma}} \int_{\Omega^h} \left| (\tilde{R}^h)^T \nabla \tan u^h(x', t) - (I_3)^* \right|^2 \]

\[ \leq C. \]

Therefore:

\[ \| V^h \|^2_{W^{1,2}(\omega)} = \| V^h \|^2_{L^2(\omega)} + \| \nabla V^h \|^2_{L^2(\omega)} \leq C. \]

Hence, \( V^h \) is a bounded sequence in the norm of \( W^{1,2}(\omega, \mathbb{R}^3) \), which implies the \( V^h \)

is weakly convergent (up to a subsequence, still called \( V^h \)) to \( V \) in \( W^{1,2}(\omega, \mathbb{R}^3) \). □

Consider the matrix fields \( D^h \in W^{1,2}(\omega, \mathbb{R}^{3 \times 3}) \):

\[ D^h(x') = \frac{1}{h^{\gamma/2}} \int_{-h/2}^{h/2} \left( (\tilde{R}^h)^T R^h(x', t) A^h(x', t) - I_3 \right) dt \]

\[ = h^{\gamma/2} (\tilde{R}^h)^T R^h(x') S(x') + \frac{1}{h^{\gamma/2}} \left( (\tilde{R}^h)^T R^h(x') - I_3 \right). \]  

(4.15)

Lemma 4.4. The sequence \( D^h \) converges (up to a subsequence) weakly in \( W^{1,2}(\omega, \mathbb{R}^{3 \times 3}) \).
Proof. Since \( S \) is smooth and by (4.7), we have that:
\[
\|D^h\|_{L^2(\omega)}^2 = \int_\omega |\frac{h^{\gamma/2}}{2} (\bar{R}^h)^T R^h(x') S(x') + \frac{1}{h^{\gamma/2}} ((\bar{R}^h)^T R^h(x') - I_3)|^2 dx' \\
\leq C \left( h^{\gamma} \int_\omega |S(x')|^2 dx' + \frac{1}{h^{\gamma}} \int_\omega |(R^h(x') - \bar{R}^h)|^2 dx' \right) \\
\leq C.
\]
Also, by (4.1), we have that:
\[
\text{Since } S \text{ is smooth, we have that}
\]
\[
\|\nabla D^h\|_{L^2(\omega)}^2 = \int_\omega |\frac{h^{\gamma/2}}{2} (\bar{R}^h)^T \nabla (R^h(x') S(x')) + \frac{1}{h^{\gamma/2}} (\bar{R}^h)^T \nabla R^h(x')|^2 dx' \\
\leq C \left( h^{\gamma} \int_\omega |\nabla R^h(x')|^2 |S(x')|^2 dx' + h^{\gamma} \int_\omega |\nabla S(x')|^2 dx' + \frac{1}{h^{\gamma}} \int_\Omega |\nabla R^h|^2 \right) \\
\leq C.
\]
Therefore \( \|D^h\|_{W^{1,2}(\omega)}^2 \leq C \) and so, up to a subsequence still called \( D^h \), we obtain:
\[
\lim_{h \to 0} D^h = D \quad \text{weakly in } W^{1,2}(\omega, \mathbb{R}^{3 \times 3})
\] (4.16)
\]
\[
\boxed{\text{The following limit:}}
\]
\[
\lim_{h \to 0} \frac{1}{h^{\gamma/2}} \left( (\bar{R}^h)^T R^h - I_3 \right) = D \quad \text{in } L^q(\omega, \mathbb{R}^{3 \times 3}),
\] (4.17)
for all \( q \geq 1 \), is obtained by noticing that:
\[
\left\| \frac{1}{h^{\gamma/2}} ((\bar{R}^h)^T R^h - I_3) - D \right\|_{L^q(\omega)} \\
\leq \left\| \frac{1}{h^{\gamma/2}} ((\bar{R}^h)^T R^h - I_3) - D^h \right\|_{L^q(\omega)} + \left\| D^h - D \right\|_{L^q(\omega)} \\
\leq h^{\gamma/2} \left\| (\bar{R}^h)^T R^h(x') S(x') \right\|_{L^q(\omega)} + \left\| D^h - D \right\|_{L^q(\omega)}. \] (4.18)
Since \( SO(3) \) is a bounded set, we know that \( \left\| (\bar{R}^h)^T R^h \right\|_{L^\infty(\omega)} \leq C \) for some \( C > 0 \); and since \( S \) is smooth, we have that \( \|S\|_{L^q(\omega)} \) is also bounded. Then, the first term on the right hand side of (4.18) is bounded by \( C h^{\gamma/2} \). For the second term, by Lemma 4.4 the Rellich-Kondrachov theorem, and since \( \omega \) is bounded, we have that \( D^h \) converges strongly in \( L^q(\Omega) \) to \( D \), for \( q \geq 1 \). Therefore we have proven (4.17).

Lemma 4.5. The limiting matrix field \( D \) has skew-symmetric values:
\[
\text{sym } D = \lim_{h \to 0} \text{sym } D^h = 0.
\] (4.19)

Proof. For all \( R \in SO(3) \), we have:
\[
(R - I_3)^T (R - I_3) = 2 I_3 - 2 \text{sym } R = -2 \text{sym}(R - I_3).
\] (4.20)
Now, using (4.20), (4.7), and (4.1), we obtain:

\[
\frac{1}{h^{\gamma/2}} \| \text{sym}((\bar{R}^h)^T R^h - I_3) \|_{L^2(\omega)} = \frac{1}{2h^{\gamma/2}} \left\| (\bar{R}^h)^T R^h - I_3 \right\|^T (\bar{R}^h)^T R^h - I_3 \|_{L^2(\omega)} \leq C \frac{1}{h^{\gamma/2}} \| R^h - \bar{R}^h \|_{W^{1,2}(\omega)}^2 \leq Ch^{\gamma/2}.
\]

Then, it follows that:

\[
\frac{1}{h^{\gamma/2}} \| \text{sym}((\bar{R}^h)^T R^h - I_3) \|_{L^2(\omega)} \to 0
\]
as \( h \to 0 \); and, as a result, \( D \) has skew-symmetric values.

By (4.20), we observe that:

\[
\frac{1}{h^{\gamma/2}} \text{sym} D^h = \text{sym} \left[ (\bar{R}^h)^T R^h (x') S(x') + \frac{1}{h^2} (\bar{R}^h)^T R^h (x') - I_3 \right]
= \text{sym} \left[ (\bar{R}^h)^T R^h (x') S(x') \right] - \frac{1}{2h^{\gamma}} ((\bar{R}^h)^T R^h - I_3)^T ((\bar{R}^h)^T R^h - I_3). \tag{4.21}
\]

First, since \( S \) is smooth, we have:

\[
\left\| (\bar{R}^h)^T R^h (x') S(x') - S(x') \right\|_{L^q(\omega)} \leq \left\| (\bar{R}^h)^T R^h (x') - I_3 \right\|_{L^q(\omega)} S(x') \leq C \left\| (\bar{R}^h)^T R^h (x') - I_3 \right\|_{L^q(\omega)}. \tag{4.22}
\]

Using (4.8) and the Rellich-Kondrachov theorem, it follows that the limit, as \( h \to 0 \), of the right hand side of (4.22) is zero, for \( q \geq 1 \).

Also, observe that:

\[
\left\| \frac{1}{h^2} ((\bar{R}^h)^T R^h - I_3)^T ((\bar{R}^h)^T R^h - I_3) + D^2 \right\|_{L^q(\omega)} \leq \left\| \frac{1}{h^{\gamma/2}} ((\bar{R}^h)^T R^h - I_3)^T \left[ \frac{1}{h^{\gamma/2}} ((\bar{R}^h)^T R^h - I_3)^T - D \right] \right\|_{L^q(\omega)}
+ \left\| \frac{1}{h^{\gamma/2}} ((\bar{R}^h)^T R^h - I_3)^T - D \right\|_{L^q(\omega)}, \tag{4.23}
\]

approaches 0, as \( h \to 0 \), by (4.17).

Therefore, by (4.21), (4.22) and (4.23), we obtain:

\[
\lim_{h \to 0} \frac{1}{h^{\gamma/2}} \text{sym} D^h = \lim_{h \to 0} \text{sym} \left[ (\bar{R}^h)^T R^h (x') S(x') \right] - \frac{1}{2h^{\gamma}} ((\bar{R}^h)^T R^h - I_3)^T ((\bar{R}^h)^T R^h - I_3)
= \text{sym} S + \frac{1}{2} D^2 \quad \text{in } L^q(\omega, \mathbb{R}^{3 \times 3}), \ \forall q \geq 1.
\]
As for the convergence of $V^h$, we have by (4.11) and (4.15) that:

$$\left[\nabla V^h(x') - D^h(x')\right]_{3\times 2} = \frac{1}{h^{\gamma/2}} (\bar{R}^h)^T \int_{-h/2}^{h/2} \left[\nabla_{\tan} u^h(x', t) - R^h(x') A^h(x', t)\right]_{3\times 2} dt, \quad (4.24)$$

which, together with (4.1), imply that:

$$\left\| \left[\nabla V^h(x') - D^h(x')\right]_{3\times 2}\right\|_{L^2(\omega)}^2 \leq C h^{\gamma+1} \int_{\omega} \left|\nabla u^h(x', t) - R^h(x') A^h(x', t)\right|^2 dx \leq C h^2, \quad (4.25)$$

and therefore, by (4.10), the sequence $\nabla V^h$ converges in $L^2(\omega, \mathbb{R}^{3\times 2})$ to $D$. Observe that:

$$\nabla_{\tan} V = D_{3\times 2}, \quad (4.26)$$

and $D \in W^{1,2}(\omega, \mathbb{R}^{3\times 3})$ by (4.16). Then, we obtain $V \in W^{2,2}(\omega, \mathbb{R}^3)$.

Finally, we use (4.19) to conclude that $\text{sym}(\nabla_{\tan} V)_{2 \times 2} = 0$, and so, by Korn’s inequality:

$$\int_{\omega} |(\nabla_{\tan} V)_{2 \times 2}|^2 dx' \leq C \int_{\omega} |\text{sym}(\nabla_{\tan} V)_{2 \times 2}|^2 dx' = 0,$$

which implies $V_{\tan}$ must be constant and, hence, equal to 0 in view of (4.10). Then $V = (0, 0, V_3)$. This ends the proof of (ii) in Theorem 3.1

4.4. **Proof of (iii)** in Theorem 3.1. To prove the lower bound (4.4), we define the rescaled strains $P^h \in L^2(\Omega^1, \mathbb{R}^{3\times 3})$ by:

$$P^h(x', x_3) = \frac{1}{h^{\gamma/2+1}} \left[\left((R^h(x'))^T \nabla u^h(x', h x_3) \left(A^h(x', h x_3)\right)^{-1} - I_3\right)\right].$$

**Lemma 4.6.** The rescaled strains $P^h$ converge (up to a subsequence) weakly in $L^2(\Omega^1, \mathbb{R}^{3\times 3})$.

**Proof.** Observe that, since $A^h$ is smooth, $R^h \in SO(3)$, and using (4.1), we have:

$$\|P^h\|_{L^2(\Omega^1)}^2 \leq C h^{\gamma+3} \int_{\Omega^1} \left|\nabla u^h(x', t) - A^h(x', t)\right|^2 dx' dt \leq C. \quad (4.27)$$

Then, up to a subsequence, we get:

$$\lim_{h \to 0} P^h = P \quad \text{weakly in } L^2(\Omega^1, \mathbb{R}^{3\times 3}). \quad (4.28)$$

We follow similar steps as in [23] to obtain a useful property of the limiting strain $P$. First, by (3.2), we obtain:

$$\frac{(\partial y^h - he_3)}{h^{\gamma/2+1}} = \frac{1}{h^{\gamma/2}} (\bar{R}^h)^T \left[\nabla u^h(x', h x_3) - R^h(x') A^h(x', h x_3)\right] e_3 + \frac{1}{h^{\gamma/2}} (\bar{R}^h)^T R^h(x') \left[A^h(x', h x_3) - I_3\right] e_3 + \frac{1}{h^{\gamma/2}} \left[\left((\bar{R}^h)^T R^h(x') - I_3\right) e_3. \quad (4.29)$$
Let’s study all three terms on the right-hand side of (4.29). For the first term, using (4.1), we get:
\[
\frac{1}{h^3} \int_{\Omega^1} |(\bar{R}^h)^T [\nabla u^h(x', h x_3) - R^h(x') A^h(x', h x_3)] e_3|^2 \, dx' dx_3 \\
\leq \frac{1}{h^{n+3}} \int_{\Omega^2} |\nabla u^h(x', t) - R^h(x') A^h(x', t)|^2 \, dx' dx_3 \\
\leq Ch^2.
\]

For the second term, we have:
\[
\frac{1}{h^7} \int_{\Omega^1} |(\bar{R}^h)^T R^h(x') [A^h(x', h x_3) - I_3] e_3|^2 \, dx' dx_3 \\
\leq \int_{\Omega^1} \left| h^{\gamma/2} S(x') + h x_3 B(x') \right|^2 \, dx' dx_3 \\
\leq Ch^2.
\]

And finally, for the third term, using (4.17), we have that it converges to $De_3$ in $L^2(\omega)$. Therefore:
\[
\lim_{h \to 0} \frac{1}{h^{\gamma/2+1}} (\partial_3 y^h - h e_3) = De_3 \quad \text{in} \quad L^2(\Omega^1, \mathbb{R}^3). \quad (4.30)
\]

For each small $s > 0$, we define the sequence of functions $f^{s,h} \in W^{1,2}(\Omega^1, \mathbb{R}^3)$:
\[
f^{s,h}(x) = \frac{1}{h^{\gamma/2+1}} \frac{1}{s} (y^h(x + s e_3) - y^h(x) - h s e_3). \quad (4.31)
\]

Clearly, (4.31) is equivalent to
\[
f^{s,h}(x) = \frac{1}{h^{\gamma/2+1}} \int_0^s (\partial_3 y^h(x + t e_3) - h e_3) dt,
\]
and, by (4.30), we have that:
\[
\lim_{h \to 0} f^{s,h} = De_3 \quad \text{in} \quad L^2(\Omega^1, \mathbb{R}^3). \quad (4.32)
\]

Also, observe that $\partial_3 f^{s,h}(x) = \frac{1}{s h^{\gamma/2+1}} (\partial_3 y^h(x + s e_3) - \partial_3 y^h(x))$. Then:
\[
\|\partial_3 f^{s,h}(x)\|_{L^2(\Omega^1)} \leq \frac{1}{|s|} \left\| \frac{1}{h^{\gamma/2+1}} (\partial_3 y^h(x + s e_3) - h e_3) - De_3 \right\|_{L^2(\Omega^1)} \\
+ \frac{1}{|s|} \left\| \frac{1}{h^{\gamma/2+1}} (\partial_3 y^h(x) - h e_3) - De_3 \right\|_{L^2(\Omega^1)}
\]
goess to 0 as $h \to 0$, by (4.30). In other words:
\[
\lim_{h \to 0} \partial_3 f^{s,h} = 0 \quad \text{in} \quad L^2(\Omega^1, \mathbb{R}^3). \quad (4.33)
\]

Further, for any $\alpha = 1, 2$, we have:
\[
\partial_\alpha f^{s,h}(x) = \frac{1}{s} (\bar{R}^h)^T R^h(x') [P^h(x', x_3 + s) A^h(x', h x_3 + hs) \\
- P^h(x', x_3) A^h(x', h x_3) \\
+ \frac{1}{h^{\gamma/2+1}} (A^h(x', h x_3 + hs) - A^h(x', h x_3))] e_\alpha;
\]
which, in view of (4.28), (1.9), and since \(A^h \to I_3\) strongly in \(L^2(\omega, \mathbb{R}^{3\times 3})\), yields the weak convergence in \(L^2(\Omega^1, \mathbb{R}^{3\times 2})\) of \(\partial_\alpha f^{s,h}(x)\):

\[
\lim_{h \to 0} \partial_\alpha f^{s,h}(x) = \frac{1}{s} \left( P(x', x_3 + s) - P(x', x_3) \right) e_\alpha \\
+ \frac{1}{s} \lim_{h \to 0} \frac{1}{h^{\gamma/2+1}} \left( I_3 + h^\gamma S(x') + h^{\gamma/2+1}(x_3 + s)B(x') \\
- (I_3 + h^\gamma S(x') + h^{\gamma/2+1}x_3B(x')) \right) e_\alpha \\
= \frac{1}{s} \left( P(x', x_3 + s) - P(x', x_3) \right) e_\alpha + B(x')e_\alpha. \tag{4.34}
\]

Consequently, by (4.32), (4.33), and (4.34), we see that \(f^{s,h}\) converges weakly in \(W^{1,2}(\omega, \mathbb{R}^3)\) to \(De_3\). Hence, the left-hand side in (4.34) equals \(\partial_\alpha(De_3)\):

\[
\partial_\alpha(De_3) = \frac{1}{s} \left( P(x', x_3 + s) - P(x', x_3) \right) e_\alpha + B(x')e_\alpha.
\]

Also, we notice that:

\[
(\partial_3P)e_\alpha = \lim_{s \to 0} \left( \frac{P(x', x_3 + s) - P(x', x_3)}{s} \right) e_\alpha = \partial_\alpha(De_3) - B(x')e_\alpha;
\]

and consequently:

\[
P(x)_{3\times 2} = (\nabla(D(x')e_3) - B(x'))_{3\times 2} x_3 + P_0(x')_{3\times 2}, \tag{4.35}
\]

for some \(P_0 \in L^2(\omega, \mathbb{R}^{3\times 3})\).

We can now finish the proof of Theorem 3.1. Observe that, from (2.4), and using the Taylor expansion of the function \(W(F)\) close to \(F = I_3\), we obtain:

\[
\frac{1}{h^{\gamma+2}} W \left( \nabla u^h(x)(A^h(x))^{-1} \right) \\
= \frac{1}{h^{\gamma+2}} W \left( (R^h(x))^T \nabla u^h(x) (A^h(x))^{-1} \right) \\
= \frac{1}{h^{\gamma+2}} W \left( Id + h^{\gamma/2+1} P^h(x) \right) \\
= \frac{1}{2} Q_3(P^h(x)) + \nu \left( h^{\gamma/2+1} |P^h| \right) \mathcal{O}(|P^h(x)|^2). \tag{4.36}
\]

Consider the sets \(U_\delta = \{ x \in \Omega^1 : h |P^h(x', x_3)| \leq 1 \}\). Note that, using Hölder’s inequality and (4.1), we have:

\[
\int_{\Omega^1} |hP^h| dx \\
= \frac{1}{h^{\gamma/2+1}} \int_{\Omega^1} |(R^h(x'))^T \nabla u^h(x', t) (A^h(x', t))^{-1} - Id| dx \\
\leq \frac{C}{h^{\gamma/2+1}} \left( \int_{\Omega^1} |\nabla u^h(x', h x_3) - R^h(x') A^h(x', h x_3)|^2 dx \right)^{1/2} \\
\leq Ch^{1/2}. \tag{4.37}
\]

Since, by (4.37), \(hP^h\) converges to 0 in \(L^1(\Omega^1)\) as \(h \to 0\), then there exists a subsequence (which we call again \(hP^h\)) such that \(hP^h\) converges to 0 pointwise a.e.
Now, for that subsequence observe that, by (4.37), we obtain:
\[
\int_{\Omega^1} |\chi h - 1|^2 \, dx = \int_{\Omega^1 \setminus \cup h} 1 \, dx \\
\leq \int_{\Omega^1 \setminus \cup h} |hP^h| \, dx \leq \int_{\Omega^1} |hP^h| \, dx \leq C h^{1/2}.
\]
And therefore, \(\chi h\) converges to 1 in \(L^2(\Omega^1)\) as \(h \to 0\). Since \(\lim_{t \to 0} \nu(t) = 0\), using (4.36), and the properties of \(Q_3\), we get:
\[
\liminf_{h \to 0} \frac{1}{h} \int_{\Omega^1} |hP^h(x)| \, dx \\
\geq \liminf_{h \to 0} \frac{1}{h} \int_{\Omega^1} \chi h W \left( \nabla h(x', hx_3) \left( A^h(x', hx_3) \right)^{-1} \right) \, dx \\
= \liminf_{h \to 0} \left( \frac{1}{2} \int_{\Omega^1} Q_3(\chi h P^h(x)) \, dx + o(1) \int_{\Omega^1} |P^h(x)|^2 \, dx \right) \\
\geq \frac{1}{2} \int_{\Omega^1} Q_3(\text{sym} P(x)) \, dx,
\]
where the convergence to 0 of the term \(o(1) \int_{\Omega^1} |P^h(x)|^2 \, dx \) as \(h \to 0\) follows from (4.1); and since, by (4.28), \(\chi h P^h(x)\) converges weakly to \(P\) in \(L^2(\Omega^1, \mathbb{R}^{3 \times 3})\).
Further, by (3.6) and (4.35):
\[
\frac{1}{2} \int_{\Omega^1} Q_3(\text{sym} P(x)) \, dx \\
\geq \frac{1}{2} \int_{\Omega^1} Q_2(\text{sym} P_{2 \times 2}(x)) \, dx \\
= \frac{1}{2} \int_{\Omega^1} Q_2(x_3 \text{ sym} (\nabla D e_3 - B(x'))_{2 \times 2} + \text{sym} P_0(x')_{2 \times 2}) \, dx \\
\geq \frac{1}{24} \int_{\Omega^1} Q_2(\text{sym} (\nabla D e_3)_{2 \times 2} - (\text{sym} B(x'))_{2 \times 2}) \, dx'.
\]

Now, in view of Theorem (3.1) (ii), (4.15), and (4.20) we have:
\[
(D e_3)^T = (D_{1,3}, D_{2,3}, 0)^T = (-D_{3,1}, -D_{3,2}, 0)^T = -(\partial_1 V_3, \partial_2 V_3, 0)^T,
\]
in other words:
\[
(\nabla D e_3)_{2 \times 2} = -\nabla^2 V_3,
\]
which yields the claim in Theorem (3.1) (iii), by (4.38) and (4.39). \(\square\)

5. PROOF OF THEOREM 3.2

Proof. Recalling (3.6) the definition of \(Q_2\), let \(c(F) \in \mathbb{R}^3\) be the unique vector so that:
\[
Q_2(F) = Q_3(F^* + \text{sym}(c \otimes e_3)).
\]
The mapping \(c : \mathbb{R}^{2 \times 2}_{\text{sym}} \to \mathbb{R}^3\) is well-defined and linear, by the properties of \(Q_3\). Also, for all \(F \in \mathbb{R}^{3 \times 3}\), we denote by \(I(F)\) the unique vector in \(\mathbb{R}^3\), linearly depending on \(F\), such that:
\[
\text{sym}(F - (F_{2 \times 2})^*) = \text{sym}(I(F) \otimes e_3).
\]
Let the out-of-plane displacement \( V_3 \) be as in Theorem 3.1. By the Rellich-Kondrachov embedding theorem, since \( V_3 \in W^{2,2}(\omega, \mathbb{R}) \), we have \( V_3 \in W^{1,q}(\omega, \mathbb{R}) \) for all \( 1 \leq q < \infty \). We first prove the result under the additional assumption of \( V_3 \) being smooth up to the boundary in Section 5.1. In Section 5.2, we prove the result for \( V_3 \in W^{2,2}(\omega, \mathbb{R}) \).

### 5.1. Case: \( V_3 \in C^\infty(\bar{\omega}, \mathbb{R}) \)

Define the recovery sequence:

\[
u^h(x', x_3) = \begin{bmatrix} x' \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ h^{\gamma/2} V_3(x') \end{bmatrix} + x_3 \begin{bmatrix} -h^{\gamma/2} (\nabla_{\tan} V_3(x'))^T \\ 1 \end{bmatrix} + \frac{1}{2} h^{\gamma/2} x_3^2 d^1(x'), \tag{5.1}\]

for \((x', x_3) \in \Omega^h\), where the smooth warping field \( d^1 : \bar{\Omega} \to \mathbb{R}^3 \) is given by:

\[
d^1 = l(B) + c \left( -\nabla^2 V_3 - (\text{sym} B)_{2 \times 2} \right). \tag{5.2}\]

Calculating the deformation gradient, we obtain:

\[
\nabla u^h = I_3 + h^{\gamma/2} \nu - h^{\gamma/2} x_3 (\nabla_{\tan}^2 V_3)^* + h^{\gamma/2} \left[ \frac{1}{2} x_3^2 \nabla_{\tan} d^1 \quad x_3 d^1 \right],
\]

where the skew-symmetric matrix field \( \nu \) is given by

\[
\nu = \begin{bmatrix} 0 & -(\nabla_{\tan} V_3)^T \\ \nabla_{\tan} V_3 & 0 \end{bmatrix}.
\]

The convergence statements in (i) and (ii) of Theorem 3.2 are verified by a straightforward calculation:

For (i):

\[
\left\| u^h(x', x_3) - x' \right\|_{W^{1,2}(\Omega, \mathbb{R}^3)} \leq Ch.
\]

For (ii):

\[
\left\| V^h(x') - \begin{bmatrix} 0 & 0 \\ 0 & V_3 \end{bmatrix} \right\|_{W^{1,2}(\Omega, \mathbb{R}^3)} \leq Ch^2.
\]

To prove (iii), we need to estimate the energy of the sequence \( u^h \). In order to do this, we shall use an auxiliary \( SO(3) \)-valued matrix \( R^h = e^{h^{\gamma/2} \nu} \). Clearly, \( R^h = I_3 + h^{\gamma/2} \nu + \frac{h^{2\gamma}}{2} \nu^2 + h.o.t \) and \((R^h)^T = I_3 - h^{\gamma/2} \nu + \frac{h^{2\gamma}}{2} \nu^2 + h.o.t\). Also, recall that \((A^h)^{-1} = Id - h^\gamma S - h^{\gamma/2} x_3 B + h.o.t.\) Hence, we obtain:

\[
(R^h)^T (\nabla u^h)(A^h)^{-1} = I_3 + h^\gamma \left( -\frac{1}{2} \nu^2 - S \right) + h^{\gamma/2} x_3 \left( - (\nabla^2 V_3)^* - B + d^1 \otimes e_3 \right) + h.o.t.
\]

Using the definition of the quadratic form \( Q_3(F) \), Taylor expanding the energy density \( W \) around the identity, and taking into account the uniform boundedness of all the involved functions and their derivatives, we get in (2.2):

\[
I_w^h(u^h) = \frac{1}{h} \int_{\Omega^h} W((R^h)^T (\nabla u^h)(A^h)^{-1}) \, dx
\]

\[
= \frac{h^{2\gamma}}{2} \int_{\Omega} Q_3(\text{sym} \left( -\frac{1}{2} \nu^2 - S \right)) \, dx' + \frac{h^{\gamma+2}}{24} \int_{\Omega} Q_3 \left( \text{sym} \left( - (\nabla^2 V_3)^* - B + d^1 \otimes e_3 \right) \right) \, dx' + h.o.t. \tag{5.3}
\]
In view of (5.3), it follows that:
\[
\frac{1}{h^{\gamma+2}} I_{W}^h(u^h) = \frac{1}{24} \int_{\Omega} Q_3 \left( \text{sym} \left( - (\nabla^2 V_3)^* - B + d^1 \otimes e_3 \right) \right) dx' + O(h^{\gamma-2}).
\]

Observe that, from (5.2), we get that:
\[
\text{sym} \left( - (\nabla^2 V_3)^* - B + d^1 \otimes e_3 \right) = (-\nabla^2 V_3 - (\text{sym} B)_{2 \times 2})^* + \text{sym} \left( (d^1 - l(B)) \otimes e_3 \right),
\]
which implies:
\[
\frac{1}{h^{\gamma+2}} I_{W}^h(u^h) = \mathcal{I}_f(V_3) + O(h^{\gamma-2}), \tag{5.4}
\]
which, in turn, proves (iii) for smooth displacement \( V_3 \).

5.2. Case: \( V_3 \in W^{2,2}(\omega, \mathbb{R}) \). We now consider a sequence \( V_3^n \in C^\infty(\overline{\omega}, \mathbb{R}) \) such that:
\[
||V_3^n - V_3||_{W^{2,2}(\omega, \mathbb{R})} \to 0, \quad \text{as } n \to \infty. \tag{5.5}
\]

Define the sequence \( u^{h,n} \) as in (5.1) using \( V_3^n \) instead of \( V_3 \):
\[
u^{h,n}(x', x_3)
\]
\[
= \begin{bmatrix} x' \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ h^{\gamma/2} V_3^n(x') \end{bmatrix} + x_3 \begin{bmatrix} -h^{\gamma/2}(\nabla V_3^n(x'))^T \\ 1 \end{bmatrix} + \frac{1}{2} h^{\gamma/2} x_3^2 d^{1,n}(x'),
\]

where the smooth warping field \( d^{1,n} : \omega \to \mathbb{R}^3 \) is given by:
\[
d^{1,n} = l(B) + c (-\nabla^2 V_3^n - (\text{sym} B)_{2 \times 2}).
\]

Calculating the deformation gradient, we first obtain:
\[
\nabla u^{h,n} = I_3 + h^{\gamma/2} \mathcal{V}^n - h^{\gamma/2} x_3 (\nabla^2 V_3^n)^* + h^{\gamma/2} \begin{bmatrix} 1/2 x_3^2 \nabla d^{1,n} \\ x_3 d^{1,n} \end{bmatrix},
\]

where the skew-symmetric matrix field \( \mathcal{V}^n \) is given by:
\[
\mathcal{V}^n = \begin{bmatrix} 0 \\ -\nabla V_3^n \end{bmatrix}^T.
\]

Let the sequence \( \{n(h)\} \) be such that \( n(h) \to \infty \) when \( h \to 0 \), and define:
\[
u_h(x', x_3) := u^{h,n(h)}(x', x_3).
\]

Observe that:
\[
\left| \frac{1}{h^{\gamma+2}} I_{W}^h(u_h) - \mathcal{I}_f(V_3) \right| \leq \left| \frac{1}{h^{\gamma+2}} I_{W}^h(u_h) - \mathcal{I}_f(V_3^n) \right| + \left| \mathcal{I}_f(V_3^n) - \mathcal{I}_f(V_3) \right|. \tag{5.6}
\]
We will study the two terms on the right hand side of (5.6). As above, we use an auxiliary $SO(3)$-valued matrix $R^{h,n(h)} = e^{h\gamma/2 V^{n(h)}}$. Hence, we obtain:

$$
\left\| I_3 - (R^{h,n(h)})^T(\nabla u_h)(A^h)^{-1}\right\|_{L^\infty(\Omega,\mathbb{R}^{3x3})} \\
\leq C h^{\gamma/2 + 1} \left( 1 + \left\| \nabla V_3^{n(h)} \right\|_{L^\infty(\omega)} + \left\| \left( \nabla^2 V_3^{n(h)} \right)^* \right\|_{L^\infty(\omega)} \\
+ \left\| \nabla^3 V_3^{n(h)} \right\|_{L^\infty(\omega)} \right).
$$

Let $\epsilon > 0$, we can always find $h_1 > 0$ such that for all $h \leq h_1$:

$$
h^{\gamma/2 + 1} \left( 1 + \left\| \nabla V_3^{n(h)} \right\|_{L^\infty(\omega)} + \left\| \left( \nabla^2 V_3^{n(h)} \right)^* \right\|_{L^\infty(\omega)} \\
+ \left\| \nabla^3 V_3^{n(h)} \right\|_{L^\infty(\omega)} \right) < \frac{\epsilon}{2C},
$$

(5.7)

by reparameterizing the sequence $V_3^{n(h)}$, in order to slow down the rate of convergence to $V_3$.

Then, in a similar way to (5.4), we obtain:

$$
\frac{1}{h^{\gamma+2}} I_W^{h}(u_h) \\
= \mathcal{I}_f(V_3^{n(n)}) + O \left( \min(h^{\gamma-2}, h^2) \left( \nabla V_3^{n(h)}, (\nabla^2 V_3^{n(h)})^*, \nabla^3 V_3^{n(h)} \right) \right),
$$

where the quantity $F \left( \nabla V_3^{n(h)}, (\nabla^2 V_3^{n(h)})^*, \nabla^3 V_3^{n(h)} \right)$ depends only on $L^\infty(\omega)$-norms of $\nabla V_3^{n(h)}$, $\left( (\nabla^2 V_3^{n(h)})^* \right)$ and $\nabla^3 V_3^{n(h)}$.

As above, we can always find $h_2 > 0$ such that for all $h \leq h_2$:

$$
O \left( \min(h^{\gamma-2}, h^2) \left( \nabla V_3^{n(h)}, (\nabla^2 V_3^{n(h)})^*, \nabla^3 V_3^{n(h)} \right) \right) < \frac{\epsilon}{2},
$$

which is possible by slowing down the rate of convergence of the sequence $V_3^{n(n)}$ (by reparametrizing the sequence). Take $h_3 = \min(h_1, h_2)$. Then, for the first term on the right hand side of (5.6), we have:

$$
\left| \frac{1}{h^{\gamma+2}} I_W^{h}(u_h) - \mathcal{I}_f(V_3^{n(h)}) \right| < \frac{\epsilon}{2},
$$

for all $h \leq h_3$.

Also, by (5.5) and (5.6), we obtain for the second term on the right hand side of (5.6):

$$
\left| \mathcal{I}_f(V_3^{n(h)}) - \mathcal{I}_f(V_3) \right| \\
\leq \frac{1}{24} \int_{\omega} \left| Q_2(\nabla^2 V_3^{n(h)} + (\text{sym} B(x'))_{2x2}) \\
- Q_2(\nabla^2 V_3 + (\text{sym} B(x'))_{2x2}) \right| dx' \\
= \frac{1}{24} \int_{\omega} Q_3 \left( ( - \nabla^2 V_3^{n(h)} - (\text{sym} B)_{2x2})^* \\
+ \text{sym} \left( (c ( - \nabla^2 V_3^{n(h)} - (\text{sym} B)_{2x2})) \otimes e_3 \right)\right) \\
- Q_3 \left( ( - \nabla^2 V_3 - (\text{sym} B)_{2x2})^* \\
+ \text{sym} \left( (c ( - \nabla^2 V_3 - (\text{sym} B)_{2x2})) \otimes e_3 \right)\right) \right| dx'.
$$
Define:
\[
T^{n(h)} = \left( - \nabla^2 V_3^n(h) - (\text{sym} B)_{2\times2} \right)^* \\
+ \text{sym}\left( (c \left(- \nabla^2 V_3^n(h) - (\text{sym} B)_{2\times2}\right)) \otimes e_3 \right),
\]
and:
\[
T = \left( - \nabla^2 V_3 - (\text{sym} B)_{2\times2} \right)^* \\
+ \text{sym}\left( (c \left(- \nabla^2 V_3 - (\text{sym} B)_{2\times2}\right)) \otimes e_3 \right),
\]
and notice that:
\[
\left\| T^{n(h)} \right\|_{L^2(\omega)} + \left\| T \right\|_{L^2(\omega)} \leq C.
\]
Then, using Hölder’s inequality and properties of \( Q_3 \), we obtain:
\[
\left| \mathcal{I}_f(V_3^{n(h)}) - \mathcal{I}_f(V_3) \right| \\
\leq \frac{1}{24} \int_\omega \left| Q_3(T^{n(h)}) - Q_3(T) \right| dx' \\
\leq C \left\| T^{n(h)} - T \right\|_{L^2(\omega)} \left( \left\| T^{n(h)} \right\|_{L^2(\omega)} + \left\| T \right\|_{L^2(\omega)} \right) \\
\leq C \left\| T^{n(h)} - T \right\|_{L^2(\omega)},
\]
Now, by (105), there exists \( h_4 > 0 \) such that
\[
\left\| T^{n(h)} - T \right\|_{L^2(\omega)} = \left\| (\nabla^2 V_3^{n(h)} - \nabla^2 V_3)^* \\
+ \text{sym}\left( (c (\nabla^2 V_3^{n(h)} - \nabla^2 V_3)) \otimes e_3 \right) \right\|_{L^2(\omega)} \\
\leq \left\| (\nabla^2 V_3^{n(h)} - \nabla^2 V_3)^* \right\|_{L^2(\omega)} \\
+ \left\| \text{sym}\left( (c (\nabla^2 V_3^{n(h)} - \nabla^2 V_3)) \otimes e_3 \right) \right\|_{L^2(\omega)} \\
< \frac{\epsilon}{2C},
\]
for all \( h \leq h_4 \). Finally, taking \( h^* = \min(h_3, h_4) \), we obtain that for all \( h \leq h^* \) we have:
\[
\left| \frac{1}{2\gamma + 2} J_{W}^{h}(u_h) - \mathcal{I}_f(V_3) \right| < \epsilon.
\]
This concludes the proof of (iii) in Theorem 3.2. \( \square \)

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REFERENCES

[1] Kaushik Bhattacharya, Marta Lewicka, and Mathias Schäffner. Plates with incompatible prestrain. *Archive for Rational Mechanics and Analysis*, 221(1):143–181, Jul 2016.
[2] Silvia Jiménez Bolaños, Marta Lewicka, and Anna Zemlyanova. Limiting theories for shells with weak incompatible growth tensors. In preparation.
[3] Andrea Braides. \( \Gamma \)-convergence for beginners. In *Oxford Lecture Series in Mathematics and its Applications*. University Press, 2002.
[4] Ennio De Giorgi. \( G \)-operators and \( \Gamma \)-convergence. *Proceedings of the International Congress of Mathematicians*, 1,2:1175–1191, 1984.
[5] Efi Efrati, Eran Sharon, and Raz Kupferman. Elastic theory of unconstrained non-Euclidean plates. *Journal of the Mechanics and Physics of Solids*, 57(4):762 – 775, 2009.

[6] G. Friesecke, R. James, Mora. M.G., and S. Müller. Derivation of nonlinear bending theory for shells from three-dimensional nonlinear elasticity by gamma-convergence. *C. R. Math. Acad. Sci. Paris*, 336(8):697–702, 2003.

[7] Gero Friesecke, Richard D. James, and Stefan Müller. A theorem on geometric rigidity and the derivation of nonlinear plate theory from three-dimensional elasticity. *Communications on Pure and Applied Mathematics*, 55(11):1461–1506, 2002.

[8] Gero Friesecke, Richard D. James, and Stefan Müller. A hierarchy of plate models derived from nonlinear elasticity by Gamma-convergence. *Archive for Rational Mechanics and Analysis*, 180(2):183–236, May 2006.

[9] G W Jones and L Mahadevan. Optimal control of plates using incompatible strains. *Nonlinearity*, 28(9):3153–3174, aug 2015.

[10] J. Kim, J. Hanna, M. Byun, C. Santangelo, and R. Hayward. Designing responsive buckled surfaces by halftone gel lithography. *Science*, 335:12011205, 2012.

[11] Yael Klein, Efi Efrati, and Eran Sharon. Shaping of elastic sheets by prescription of non-Euclidean metrics. *Science*, 315(5815):1116–1120, 2007.

[12] Raz Kupferman and Cy Maor. A Riemannian approach to the membrane limit in non-Euclidean elasticity. *Communications in Contemporary Mathematics*, 16, 10 2014.

[13] Raz Kupferman and Jake P. Solomon. A Riemannian approach to reduced plate, shell, and rod theories. *Journal of Functional Analysis*, 266(5):2989 – 3039, 2014.

[14] H Le Dret and Annie Raoult. The nonlinear membrane model as variational limit of nonlinear three-dimensional elasticity. *Journal de Mathématiques Pures et Appliquées. Neuvième Série*, 74, 01 1995.

[15] H Le Dret and Annie Raoult. The membrane shell model in nonlinear elasticity: A variational asymptotic derivation. *Journal of Nonlinear Science*, 6:59–84, 01 1996.

[16] M Lewicka. A note on the convergence of low energy critical points of nonlinear elasticity functionals, for thin shells of arbitrary geometry. *ESAIM: Control, Optimisation and Calculus of Variations*, 17:493–505, 2011.

[17] M. Lewicka, L. Mahadevan, and Mohammad Reza Pakzad. The Monge-Ampère constraint: matching of isometries, density and regularity and elastic theories of shallow shells. *Annales de l’Institut Henri Poincaré (C) Non Linear Analysis*, 34(1):45–67, 2017.

[18] M. Lewicka, M.G. Mora, and M. Pakzad. A nonlinear theory for shells with slowly varying thickness. *C.R. Acad. Sci. Paris, Ser I*, 347:211–216, 2009.

[19] M. Lewicka, M.G. Mora, and M. Pakzad. The matching property of infinitesimal isometries on elliptic surfaces and elliptic theory of thin shells. *Arch. Rational Mech. Anal.*, 200(3):1023–1050, 2011.

[20] M. Lewicka, M.G. Mora, and M. Pakzad. Shell theories arising as low energy gamma-limit of 3d nonlinear elasticity. *Ann. Scuola Norm. Sup. Pisa Cl. Sci.*, 2010:1–43, 9(5).

[21] M. Lewicka and M. Pakzad. The infinite hierarchy of elastic shell models: some recent results and a conjecture. *Fields Institute Communications*, 64:407–420, 2011.

[22] M. Lewicka and R. Pakzad. Prestrained elasticity: from shape formation to Monge-Ampère anomalies. *Notices of the AMS*, January 2016.

[23] Marta Lewicka. Quantitative immersability of Riemann metrics and the infinite hierarchy of prestrained shell models. *eprint arXiv:1812.09850*.

[24] Marta Lewicka and Danka Lucic. Dimension reduction for thin films with transversally varying prestrain: the oscillatory and the non-oscillatory case, 2019.

[25] Marta Lewicka, L. Mahadevan, and Mohammad Reza Pakzad. The Föppl-von Kármán equations for plates with incompatible strains. *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences*, 467(2126):402–426, 2011.

[26] Marta Lewicka, L. Mahadevan, and Mohammad Reza Pakzad. Models for elastic shells with incompatible strains. *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences*, 470(2165):20130604, 2014.

[27] Marta Lewicka, Pablo Ochoa, and Mohammad Reza Pakzad. Variational models for prestrained plates with Monge-Ampère constraint. *Differential Integral Equations*, 28(9/10):861–898, 09 2015.

[28] Marta Lewicka and Annie Raoult. Thin structures with imposed metric. *ESAIM: Proceedings and Surveys*, 2019.
[29] Marta Lewicka, Annie Raoult, and Diego Ricciotti. Plates with incompatible prestrain of high order. *Annales de l’Institut Henri Poincaré C, Analyse non linéaire*, 34(7):1883 – 1912, 2017.

[30] Marta Lewicka and Mohammad Reza Pakzad. Scaling laws for non-Euclidean plates and the $W^{2,2}$ isometric immersions of Riemannian metrics. *ESAIM: Control, Optimisation and Calculus of Variations*, 17(4):1158–1173, 2011.

[31] Emily Rodriguez, Anne Hoger, and Andrew D. McCulloch. Stress-dependent finite growth in soft elastic tissues. *Journal of biomechanics*, 27 4:455–67, 1994.

[32] Bernd Schmidt. Plate theory for stressed heterogeneous multilayers of finite bending energy. *Journal de Mathématiques Pures et Appliquées*, v.88, 107-122 (2007), 88, 07 2007.

DEPARTMENT OF MATHEMATICS, COLGATE UNIVERSITY, 13 OAK DRIVE, HAMILTON, NY 13346 USA

E-mail address: sjimenez@colgate.edu

DEPARTMENT OF MATHEMATICS, KANSAS STATE UNIVERSITY, MANHATTAN, KANSAS, 66506 USA

E-mail address: azem@ksu.edu