Abstract. The aim of this article is to describe asymptotic profiles for the Kirchhoff equation, and to establish time decay properties and dispersive estimates for Kirchhoff equations. For this purpose, the method of asymptotic integration is developed for the corresponding linear equations and representation formulae for their solutions are obtained. These formulae are analysed further to obtain the time decay rate of $L^p-L^q$ norms of propagators for the corresponding Cauchy problems.

1. Introduction

This article is devoted to several aspects of Kirchhoff equations or Kirchhoff systems, which were discussed in [12, 13, 15]. In particular, we will discuss the asymptotic profiles and dispersion properties, or time decay of $L^p-L^q$ norms of propagators for some relevant classes of hyperbolic equations. These properties are well-known for the wave equations, but several aspects of Kirchhoff equations still remain far from being understood. The global well-posedness of Kirchhoff equations or Kirchhoff systems is known if the data is sufficiently small in some suitable Sobolev spaces of $L^2$ type (see [3, 4, 5, 6, 8, 10, 11, 26, 27, 28]). Up to now, if one takes any large data from these Sobolev spaces, the problem of the global well-posedness is still open.

In this article the asymptotics and the global well-posedness are discussed for small data. The first topic was developed in [15] by relating the problem to the asymptotic behaviour of the Bessel potentials (Theorem from [12] is the anouncement of [15]). More precisely, the first author proved that there exists a solution which is never asymptotically free. Here we say that $u = u(t,x)$ is asymptotically free if it is asymptotically convergent to some solution of the free wave equation as the time goes to $\pm \infty$. From the point of view of the scattering theory all solutions with data satisfying some fast decay conditions in space variables are asymptotically free (see [7, 8, 26]), while the result of [15] states that if the data satisfy the opposite condition to [7, 8, 26], then the scattering theory is not possible. This is stated more precisely in Theorem 2.2. For deriving these asymptotics, we need a delicate analysis of an oscillatory integral associated with Kirchhoff equation, which was introduced by Greenberg and Hu [8] in the one dimensional case (see also [4, 5, 26]), and we will develop an asymptotic expansion of this oscillatory integral.

For further investigations, for example, such as the nonlinear scattering theory, the second topic is very important. This means that there exists a scattering state for Kirchhoff equations or systems with nonlinear perturbations, which can be discussed.
in the standard way but is quite lengthy, hence we do not touch it (see e.g., [17]).

Quite recently, the first author obtained the dispersive estimates for the Kirchhoff equation (see [13]), which will be introduced as Theorem 2.1. The essential point of the proof relies on the stationary phase method together with Littman’s lemma.

Now let us give the precise formulation of Kirchhoff equations considered problems. In 1883 G. Kirchhoff proposed the equation

\[
\begin{align*}
\frac{\partial^2 u}{\partial t^2} - \left(1 + \int_0^L u_x^2 \, dx\right) u_{xx} &= 0,
\end{align*}
\]

for \( u = u(t, x) \) on \( \mathbb{R}_t \times (0, L) \) (see [9]), which describes the nonlinear vibrations of one dimensional elastic strings having the natural length \( L \). For simplicity, all the physical constants are normalised. Generalising the equation (1.1) to a multi-dimensional version, we can consider the Cauchy problem for \( u = u(t, x) \) on \( \mathbb{R}_t \times \mathbb{R}^n \):

\[
\begin{align*}
\frac{\partial^2 u}{\partial t^2} - \left(1 + \int_{\mathbb{R}^n} |\nabla u|^2 \, dx\right) \Delta u &= 0, \\
u(0, x) &= f_0(x), \quad \partial_t u(0, x) = f_1(x),
\end{align*}
\]

where \( \partial_t = \frac{\partial}{\partial t} \), \( \nabla = \left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\right) \) and \( \Delta \) is the standard Laplacian in \( \mathbb{R}^n \) defined by

\[
\Delta = \sum_{j=1}^n \frac{\partial^2}{\partial x_j^2}.
\]

Higher order nonlinear equations of Kirchhoff type are also of great interest, and they can be viewed as dispersion relations for Kirchhoff systems. In particular, since higher order equations are influenced by the geometric properties of characteristics (cf. [22, 23, 24]), in such problem it is important to know how this phenomenon is affected by nonlinearities (this is contrary to the \( L^p-L^p \) estimates, see [19] for a survey of such results).

Thus, let us consider the following nonlinear equation

\[
\begin{align*}
\tilde{L}(t, D_t, D_x, \|\nabla u\|_{L^2}^2) = D_t^m u + \sum_{|\nu|+j=m, j \leq m-1} b_{\nu,j} \left(\|\nabla u(t, \cdot)\|_{L^2}^2\right) D_x^{\nu} D_t^j u = 0,
\end{align*}
\]

for \( t \neq 0 \), with the initial condition

\[
\begin{align*}
D_t^k u(0, x) = f_k(x), \quad k = 0, 1, \ldots, m-1, \quad x \in \mathbb{R}^n,
\end{align*}
\]

where \( D_t = \frac{1}{\sqrt{-1}} \frac{\partial}{\partial t} \) and \( D_x^{\nu} = \left(\frac{\partial}{\partial x_1}\right)^{\nu_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\nu_n} \), \( i = \sqrt{-1} \), for \( \nu = (\nu_1, \ldots, \nu_n) \). We will assume that the symbol of the differential operator \( \tilde{L}(t, D_t, D_x, \|\nabla u\|_{L^2}^2) \) has real and distinct roots \( \tilde{\varphi}_1(t, s, \xi), \ldots, \tilde{\varphi}_m(t, s, \xi) \) for \( \xi \neq 0 \) and \( 0 \leq s \leq \delta \) with \( \delta > 0 \), i.e.

\[
\inf_{j \neq k} \inf_{|\xi|=1, t \in \mathbb{R}, s \in [0, \delta]} |\tilde{\varphi}_j(t, s, \xi) - \tilde{\varphi}_k(t, s, \xi)| > 0.
\]

The detail analysis of the Cauchy problem (1.4)–(1.5) will be done in [16], and we will consider only the Cauchy problem (1.2)–(1.3) of the second order in this article.
We conclude the introduction by fixing the notation used in this article. For \( s \in \mathbb{R} \) and \( 1 \leq p \leq \infty \), let \( L^p_s = L^p_s(\mathbb{R}^n) \) and \( L^p_s = L^p_s(\mathbb{R}^n) \) be the Riesz and Bessel potential spaces with semi-norm or norm

\[
\|u\|_{L^p_s} = \|F^{-1}[\|\xi|^s \hat{u}(\xi)]\|_{L^p(\mathbb{R}^n)} \equiv \|D^s u\|_{L^p(\mathbb{R}^n)},
\]

\[
\|u\|_{L^p_s} = \|F^{-1}[\|\xi|^s \hat{u}(\xi)]\|_{L^p(\mathbb{R}^n)} \equiv \|D^s u\|_{L^p(\mathbb{R}^n)},
\]

respectively. Here \( \hat{\ } \) denotes the Fourier transform, \( F^{-1} \) is its inverse, and \( \langle \xi \rangle = \sqrt{1 + |\xi|^2} \). Throughout this article, we fix the notation as follows:

\[
\dot{H}^s = \dot{L}^2_s, \quad H^s = L^2_s.
\]

We also put, for \( s \geq 1 \),

\[
\dot{X}^s(\mathbb{R}) = C(\mathbb{R}; \dot{H}^s) \cap C^1(\mathbb{R}; \dot{H}^{s-1}) \cap C^2(\mathbb{R}; \dot{H}^{s-2}).
\]

Finally we shall denote by \( S = S(\mathbb{R}^n) \) the Schwartz space on \( \mathbb{R}^n \).

2. Results

In this section we survey the results of [13] and [15] on the Cauchy problem (1.2)–(1.3). In order to state these asymptotics for the solutions to Kirchhoff equation, we refer to a general theorem of Yamazaki (see [26]). For this purpose, let us introduce the set

\[
Y_k := \left\{ \{\phi, \psi\} \in \dot{H}^{3/2} \times H^{1/2} : \{\phi, \psi\}|_{y_k < \infty} \right\}, \quad k > 1,
\]

where

\[
\left| \{\phi, \psi\}|_{y_k} := \sup_{\tau \in \mathbb{R}} (1 + |\tau|)^k \int_{\mathbb{R}^n} e^{i\tau |\xi|} |\xi|^3 |\hat{\phi}(\xi)|^2 d\xi \right|
\]

\[
+ \sup_{\tau \in \mathbb{R}} (1 + |\tau|)^k \int_{\mathbb{R}^n} e^{i\tau |\xi|} |\xi| |\hat{\psi}(\xi)|^2 d\xi
\]

\[
+ \sup_{\tau \in \mathbb{R}} (1 + |\tau|)^k \int_{\mathbb{R}^n} e^{i\tau |\xi|} |\xi|^2 \text{Re} \left( \hat{\phi}(\xi) \overline{\hat{\psi}(\xi)} \right) d\xi.
\]

Then we have the following:

**Theorem A (26).** Let \( n \geq 1 \) and \( s_0 \geq \frac{3}{2} \). If the data \( u_0, u_1 \) satisfy \( u_0 \in \dot{H}^{s_0} \cap H^1 \), \( u_1 \in H^{s_0-1} \), and

\[
(2.1) \quad \delta_1 := \|\nabla u_0\|^2_{L^2} + \|u_1\|^2_{L^2} + \|u_0, u_1\|_{Y_k} \ll 1 \quad \text{for some } k > 1,
\]

then the problem (1.2)–(1.3) has a unique solution \( u(t, x) \in \dot{X}^{s_0}(\mathbb{R}) \) having the following property: there exists a constant \( c_{\pm \infty} \equiv c_{\pm \infty}(u_0, u_1) > 0 \) such that

\[
1 + \|\nabla u(t, \cdot)\|^2_{L^2} = c_{\pm \infty}^2 + O(|t|^{-k+1}) \quad \text{as } t \to \pm \infty.
\]

Furthermore, if \( (2.1) \) holds with \( k > 2 \), then \( c_{+ \infty} = c_{- \infty} := c_{\infty} \) and each solution \( u(t, x) \in \dot{X}^{s_0}(\mathbb{R}) \) is asymptotically free in \( \dot{H}^\sigma \times \dot{H}^{\sigma-1} \) for all \( \sigma \in [1, s_0] \) as \( t \to \pm \infty \), i.e., there exists a solution \( v_\pm = v_\pm(t, x) \in X^{\sigma}(\mathbb{R}) \) of the equation

\[
(\partial_t^2 - c_{\infty}^2 \Delta) v_\pm = 0 \quad \text{on } \mathbb{R} \times \mathbb{R}^n
\]
such that
\[ ||u(t, \cdot) - v_\pm(t, \cdot)||_{\dot{H}^\sigma} + ||\partial_t u(t, \cdot) - \partial_t v_\pm(t, \cdot)||_{\dot{H}^\sigma-1} \rightarrow 0 \quad (t \rightarrow \pm \infty). \]

The inclusions among the classes \( Y_k \) are as follows:
\[ Y_k \subset Y_\ell \quad \text{if} \quad k > \ell > 1, \quad \text{and} \quad S \subset Y_k \quad \text{for all} \quad k \in (1, n + 1). \]

The latter inclusion can be shown by using the asymptotic expansion of oscillatory integral \( I(\tilde{\sigma}(t), 0) \) which was proved in [13]. The definition of \( Y_k \) is somewhat complicated. There are some examples of spaces contained in \( Y_k \). For more details see [15].

Keeping in mind Theorem A, we have \( L^p - L^q \) estimates:

**Theorem 2.1 ([13]).** Let \( n \geq 2 \) and let \( 1 < p \leq 2 \leq q < +\infty \) and \( \frac{1}{p} + \frac{1}{q} = 1 \). Then each solution \( u(t, x) \) in Theorem A with \( k = n + 1 \) has the following properties for all \( \delta > 0 \):
\[ ||\partial_t^\delta \partial_x^\alpha u(t, \cdot)||_{L^q} \leq C(1 + |t|)^{-\left(\frac{\alpha + 1}{p} - \delta\right)} \sum_{i=0,1} ||u_i||_{H^{N_p+j+|\alpha|-i}} \]
where \( N_p = \frac{3n+1}{2} \left( \frac{1}{p} - \frac{1}{q} \right) \), \( j = 0, 1, 2 \), and \( \alpha \) is any multi-index.

Based on Theorem 2.1, we can develop the nonlinear scattering problems for the Kirchhoff equation. But here, we want to exhibit the opposite phenomenon; for this, we will find the asymptotic profiles for the solutions to (1.2)–(1.3). Let us present the definitions of free and non-free waves.

**Definition.**
(i) We say that \( v_\pm = v_\pm(t, x) = \{v_+(t, x), v_-(t, x)\} \) is a \textbf{free wave} if it satisfies the equation
\[ \left( \partial_t^2 - c_{\pm \infty}^2 \Delta \right) v_\pm = 0 \quad \text{on} \quad \mathbb{R} \times \mathbb{R}^n. \]

(ii) Let \( \sigma \geq 1 \). We say that \( v = v(t, x) \) is \textbf{asymptotically free} in \( \dot{H}^\sigma \times \dot{H}^{\sigma-1} \) if it is asymptotically convergent to some free wave \( v_\pm \) in \( \dot{H}^\sigma \times \dot{H}^{\sigma-1} \), i.e.,
\[ ||v(t, \cdot) - v_\pm(t, \cdot)||_{\dot{H}^\sigma} + ||\partial_t v(t, \cdot) - \partial_t v_\pm(t, \cdot)||_{\dot{H}^{\sigma-1}} \rightarrow 0 \quad (t \rightarrow \pm \infty). \]

(iii) Let \( \sigma \geq 1 \). We say that \( w = w(t, x) \) is a \textbf{non-free wave} in \( \dot{H}^\sigma \times \dot{H}^{\sigma-1} \) if it is not asymptotically free.

Theorem A states that each solution \( u \) of (1.2)–(1.3) with initial data satisfying (2.1) with \( k > 2 \), is asymptotically free. On the other hand, the next theorem states that the bound \( k > 2 \) is sharp. More precisely, we have the following:

**Theorem 2.2 ([15]).** Assume that
\[ \text{either} \quad n \geq 2 \quad \text{and} \quad 1 < k \leq 2, \quad \text{or} \quad n = 1 \quad \text{and} \quad 1 < k < 2. \]
Then there exists a solution \( u(t, x) \in \cap_{\sigma \geq 1} \dot{X}^\sigma(\mathbb{R}) \) of (1.2)–(1.3) with data satisfying (2.1), which is a non-free wave in \( \dot{H}^\sigma \times \dot{H}^{\sigma-1} \) for all \( \sigma \geq 1 \).

The proof of Theorems 2.1, 2.2 relies on the representation formulae for the corresponding linear equation. In §3 we will introduce the representation formulae for more general strictly hyperbolic equations. Moreover, the argument of Theorem 2.2 is relating with the asymptotic behaviour of Bessel functions (see [1]).
3. Representation of solutions to linear Cauchy problems

In this section we introduce the representation formulae for more general equations than previously considered by using the asymptotic integration method along the argument of [16]. Let us consider the Cauchy problem for an $m$th order strictly hyperbolic equation with time-dependent coefficients, for function $u = u(t, x)$:

\[
L(t, D_t, D_x)u \equiv D_t^m u + \sum_{|\nu| + j = m} a_{\nu,j}(t) D_t^{\nu} D_x^j u = 0, \quad t \neq 0,
\]

with the initial condition

\[
D_t^k u(0, x) = f_k(x) \in C_0^\infty(\mathbb{R}^n), \quad k = 0, 1, \ldots, m - 1, \quad x \in \mathbb{R}^n.
\]

Denoting by $B^{m-1}(\mathbb{R})$ the space of all functions whose derivatives up to $(m - 1)$th order are all bounded and continuous on $\mathbb{R}$, we assume that each $a_{\nu,j}(t)$ belongs to $B^{m-1}(\mathbb{R})$ and satisfies

\[
\partial_t a_{\nu,j}(t) \in L^1(\mathbb{R}) \quad \text{for all } \nu, j \text{ with } |\nu| + j = m, \text{ and } k = 1, \ldots, m - 1.
\]

Moreover, following the standard definition of equations of the regularly hyperbolic type (e.g. Mizohata [18]), we will assume that the symbol of the differential operator $L(t, D_t, D_x)$ has real and distinct roots $\varphi_1(t; \xi), \ldots, \varphi_m(t; \xi)$ for $\xi \neq 0$, and

\[
L(t, \tau, \xi) = (\tau - \varphi_1(t; \xi)) \cdots (\tau - \varphi_m(t; \xi)),
\]

\[
\inf_{|\xi|=1, j \in \mathbb{R}^n} |\varphi_j(t; \xi) - \varphi_k(t; \xi)| > 0.
\]

By applying the Fourier transform on $\mathbb{R}^n$ to (3.1), we get

\[
D_t^m v + \sum_{j=1}^m h_j(t; \xi) D_t^{m-j} v = 0,
\]

where

\[
h_j(t; \xi) = \sum_{|\nu|=j} a_{\nu,m-j}(t) \xi^\nu, \quad \xi \in \mathbb{R}^n.
\]

This is the ordinary differential equation, homogeneous of $m$th order, with the parameter $\xi = (\xi_1, \ldots, \xi_n)$. As usual, the strict hyperbolicity means that the characteristic roots of (3.6) are real and can be written as $\varphi_1(t; \xi), \ldots, \varphi_m(t; \xi)$ satisfying (3.2)–(3.5). Notice that each $\varphi_\ell(t; \xi)$ has a homogeneous degree one with respect to $\xi$. In this section we will establish the representation formulae for solutions of the Cauchy problem (3.1) in the form of the oscillatory integrals. Let $\widehat{u}(t; \xi)$ be the solution of (3.1) with the initial data $\widehat{f}_k(\xi)$ ($k = 0, \ldots, m - 1$). Let $v_k(t; \xi)$ be the solution of (3.6) with $(D_t^j v_k)(0; \xi) = \delta_{jk}$ for $j, k = 0, 1, \ldots, m - 1$. We set

\[
W(t; \xi) = \begin{pmatrix}
v_0(t; \xi) & v_1(t; \xi) & \cdots & v_{m-1}(t; \xi) \\
v_t v_0(t; \xi) & v_t v_1(t; \xi) & \cdots & v_t v_{m-1}(t; \xi) \\vdots & \vdots & \ddots & \vdots \\
D_t^{m-1} v_0(t; \xi) & D_t^{m-1} v_1(t; \xi) & \cdots & D_t^{m-1} v_{m-1}(t; \xi)
\end{pmatrix}.
\]
Hence, $W(t; \xi)$ is the fundamental matrix of (3.6). Defining
\[
\vartheta_j(t; \xi) = \int_0^t \varphi_j(s; \xi) \, ds, \quad j = 1, \ldots, m,
\]
we introduce the matrix
\[
Y(t; \xi) = \begin{pmatrix}
e^{i\vartheta_1(t; \xi)} & \cdots & e^{i\vartheta_m(t; \xi)} \\
D_t e^{i\vartheta_1(t; \xi)} & \cdots & D_t e^{i\vartheta_m(t; \xi)} \\
\vdots & \ddots & \vdots \\
D_t^{m-1} e^{i\vartheta_1(t; \xi)} & \cdots & D_t^{m-1} e^{i\vartheta_m(t; \xi)}
\end{pmatrix}.
\]
Matrix $Y(t; \xi)$ is the fundamental matrix of a perturbed ordinary differential equation of (3.6):
\[
(D_t - \varphi_1(t; \xi)) \cdots (D_t - \varphi_m(t; \xi)) w = 0.
\]
Then we can write this equation as
\[
D_t^m w + \sum_{j=1}^m h_j(t; \xi) D_t^{m-j} w + \sum_{j=2}^m \widetilde{h}_j(t; \xi) D_t^{m-j} w = 0,
\]
where $\widetilde{h}_j(t; \xi)$ satisfies
\[
\widetilde{h}_j(t; \xi) = \sum_{1 \leq \nu_2 \ldots \nu_j \leq 1} c_{\nu_1 \ldots \nu_j} \varphi_{\nu_1}^{\nu_2} (D_t \varphi_{\nu_2})^{\nu_3} \cdots (D_t^{m-j+1} \varphi_{\nu_j})^{\nu_j}, \quad j = 2, \ldots, m,
\]
with some constants $c_{\nu_1 \ldots \nu_j} \neq 0$. This means that each $e^{i\vartheta_\nu_j(t; \xi)}$ satisfies (3.7), and $e^{i\vartheta_1(t; \xi)}, \ldots, e^{i\vartheta_m(t; \xi)}$ are linearly independent for $\xi \neq 0$ and $t \in \mathbb{R}$. It can be checked that the coefficient $\widetilde{h}_1(t; \xi)$ of $D_t^{m-1} w$ always vanishes for every $m$, by an induction argument on $m$. Then it follows from Proposition 2.4 of [16] (cf. [2, 25]) that there exists the limit
\[
\lim_{t \to \pm \infty} Y(t; \xi) = L_{\pm}(\xi).
\]
Set
\[
L_{\pm}(\xi) = \begin{pmatrix}
\alpha_{0,\pm}^1(\xi) & \alpha_{1,\pm}^1(\xi) & \cdots & \alpha_{m-1,\pm}^1(\xi) \\
\alpha_{0,\pm}^2(\xi) & \alpha_{1,\pm}^2(\xi) & \cdots & \alpha_{m-1,\pm}^2(\xi) \\
\vdots & \vdots & \ddots & \vdots \\
\alpha_{0,\pm}^m(\xi) & \alpha_{1,\pm}^m(\xi) & \cdots & \alpha_{m-1,\pm}^m(\xi)
\end{pmatrix}.
\]
Furthermore, writing
\[
R_{\pm}(t; \xi) = Y(t; \xi)^{-1} W(t; \xi) - L_{\pm}(\xi)
\]
\[
= \begin{pmatrix}
\varepsilon_{0,\pm}^1(t; \xi) & \varepsilon_{1,\pm}^1(t; \xi) & \cdots & \varepsilon_{m-1,\pm}^1(t; \xi) \\
\varepsilon_{0,\pm}^2(t; \xi) & \varepsilon_{1,\pm}^2(t; \xi) & \cdots & \varepsilon_{m-1,\pm}^2(t; \xi) \\
\vdots & \vdots & \ddots & \vdots \\
\varepsilon_{0,\pm}^m(t; \xi) & \varepsilon_{1,\pm}^m(t; \xi) & \cdots & \varepsilon_{m-1,\pm}^m(t; \xi)
\end{pmatrix},
\]
we have
\[
W(t; \xi) = Y(t; \xi) (L_{\pm}(\xi) + R_{\pm}(t; \xi)).
\]
Thus we arrive at

\[ D_t^j v_k(t; \xi) = \sum_{j=1}^{m} \left( \alpha^j_{k,\pm}(\xi) + \varepsilon^j_{k,\pm}(t; \xi) \right) D_t e^{i\theta_j(t; \xi)} \]

\[ = \sum_{j=1}^{m} \left( \alpha^j_{k,\pm}(\xi) + \varepsilon^j_{k,\pm}(t; \xi) \right) p_\ell(\varphi_j(t; \xi)) e^{i\theta_j(t; \xi)} \]

for \( k, \ell = 0, \ldots, m - 1 \), where each \( p_\ell(\varphi_j(t; \xi)) \) is determined by the equation

\[ D_t^\ell e^{i\theta_j(t; \xi)} = p_\ell(\varphi_j(t; \xi)) e^{i\theta_j(t; \xi)}. \]

We note that for the second order equations we have \( m = 2 \) and the next theorem covers the case of the wave equation as a special case, also improving the corresponding result in [13–14]. The result is as follows:

**Theorem 3.1.** Assume that the characteristic roots \( \varphi_1(t; \xi), \ldots, \varphi_m(t; \xi) \) of (3.6) are real and distinct for all \( t \in \mathbb{R} \) and for all \( \xi \in \mathbb{R}^n \setminus \{0\} \), and that they satisfy (3.5). Then there exists \( \alpha^j_{k,\pm}(\xi) \) and \( \varepsilon^j_{k,\pm}(t; \xi) \) determined by (3.5) and (3.6), respectively, such that the solution \( u(t, x) \) of our problem (3.11–3.12) is represented by

\[ D_t^\ell u(t, x) = \sum_{k=0}^{m-1} \sum_{j=1}^{m} \mathcal{F}^{-1} \left[ \left( \alpha^j_{k,\pm}(\xi) + \varepsilon^j_{k,\pm}(t; \xi) \right) p_\ell(\varphi_j(t; \xi)) e^{i\theta_j(t; \xi)} f_k(\xi) \right](x), \quad t \geq 0, \]

for \( \ell = 0, \ldots, m - 1 \), where

\[ |\alpha^j_{k,\pm}(\xi)| \leq c|\xi|^{-k}, \quad |\varepsilon^j_{k,\pm}(t; \xi)| \leq c|\xi|^{-k} \int_{|t|}^{+\infty} \Psi(s) \, ds, \]

and \( \Psi(t) \) is given by

\[ \Psi(t) = \sum_{|\nu|+j=m}^{m-1} \left| \partial_t^\nu a_{\nu,j}(t) \right| \cdots \left| \partial_t^{m-j-1} a_{\nu,j}(t) \right|. \]

For the higher order derivatives of amplitude functions, we have, for \( |\mu| \geq 1 \),

\[ |D_\xi^\mu \alpha^j_{k,\pm}(\xi)| \leq c|\xi|^{-k}, \quad |D_\xi^\mu \varepsilon^j_{k,\pm}(t; \xi)| \leq c \int_0^{|t|} (1+|s|)^{|\mu|} \Psi(s) \, ds |\xi|^{-k}, \quad |\xi| \geq 1, \]

\[ |D_\xi^\mu \alpha^j_{k,\pm}(\xi)| \leq c|\xi|^{-k-|\mu|}, \quad |D_\xi^\mu \varepsilon^j_{k,\pm}(t; \xi)| \leq c \int_0^{|t|} (1+|s|)^{|\mu|} \Psi(s) \, ds |\xi|^{-k-|\mu|}, \quad 0 < |\xi| < 1. \]

If we further assume that

\[ (1 + |t|)^{|\mu|} \partial_t^\mu a_{\nu,j}(t; \xi) \in L^1(\mathbb{R}) \]

for some \( \mu \) with \( |\mu| \geq 1 \), and for all \( \nu, j \), and \( k = 1, \ldots, m - 1 \), then the bound of each \( D_\xi^\mu \varepsilon^j_{k,\pm}(t; \xi) \) is uniform in \( t \).
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