A NEW LOOK AT KRZYZ’S CONJECTURE

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Abstract. Recently the author has presented a new approach to solving extremal problems of geometric function theory. It involves the Bers isomorphism theorem for Teichmüller spaces of punctured Riemann surfaces.

We show here that this approach, combined with quasiconformal theory, can be also applied to nonvanishing holomorphic functions from $H^\infty$. In particular this gives a proof of an old open Krzyz conjecture for such functions and of its generalizations.

The unit ball $H^\infty_1$ of $H^\infty$ is naturally embedded into the universal Teichmüller space, and the functions $f \in H^\infty_1$ are regarded as the Schwarzian derivatives of univalent functions in the unit disk.

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1. INTRODUCTORY REMARKS AND RESULTS

1.1. Nonvanishing holomorphic functions and Krzyz’s conjecture. Consider the circular rings

$$A_\rho = \{ \rho < |z| < 1 \}, \quad \rho \geq 0,$$

and denote by $H(\mathbb{D}, A_\rho)$ the collection of holomorphic functions $f$ from the unit disk $\mathbb{D}$ into $A_\rho$. Regarding the points of $A_\rho$ as the constant functions on $\mathbb{D}$, one obtains an embedding of this ring as a subset in the space $H^\infty$ with sup-norm. By $H^\infty_1$ we denote the unit ball of this space.

The collections $H(\mathbb{D}, A_\rho)$ broaden monotonically as $\rho \searrow 0$ giving in the limit the class

$$H^\infty_0 := H(\mathbb{D}, A_0) = \bigcup_\rho H(\mathbb{D}, A_\rho)$$

of all nonvanishing $H^\infty$-functions. This class has been actively investigated in geometric function theory from the 1940s, in view of interesting deep features of nonvanishing (see, e.g., [9], [19]).

In 1968, Krzyz [14] conjectured that for all functions from $H^\infty_0$ the following bound

$$|c_n| \leq 2/e \quad (1)$$

is valid for any $n > 1$, with equality only for the function $\kappa_0(z^n)$ and its pre and post rotations about the origin, where

$$\kappa_0(z) = \exp \left( \frac{z-1}{z+1} \right) = \frac{1}{e} + \frac{2}{e} z - \frac{2}{3e} z^3 + \ldots \quad (2)$$

This conjecture has been investigated by many mathematicians, however it still remains open. The estimate (1) was established only for some initial coefficients $c_n$ including all $n \leq 5$ (see [11], [12], [13]).
For developments related to this problem, see, e.g., [2], [10], [11], [12], [13], [16], [19], [21], [22].

1.2. Main theorem. Put
\[ \alpha_{A_\rho} := \max \{|f'(0)| : f \in H(D, A_\rho)\} \]  
and take a universal holomorphic covering map \( \kappa_\rho : \mathbb{D} \rightarrow A_\rho \), on which this maximal value of \( |f'(0)| \) is attained, i.e., \( |\kappa'_\rho(0)| = \alpha_{A_\rho} \).

Every function \( f \in H(D, A_\rho) \) admits the factorization
\[ f(z) = \kappa_\rho \circ \hat{f}(z), \]  
where \( \hat{f} \) is a holomorphic map of the disk \( \mathbb{D} \) into itself (hence, it also belongs to \( H_1^\infty \)).

In geometric function theory, such a relation is regarded as a subordination of functions \( f \) to \( \kappa_\rho \); it has been investigated mostly for univalent covers \( \kappa_\rho \). Let
\[ k_\rho(z) = c_0^0z + c_1^1z + \cdots + c_n^nz^n + \cdots, \quad |z| < 1. \]

The existence of extremal functions maximizing the coefficient \( c_n(f) \) on \( H(D, A_\rho) \) follows from compactness of these classes in the weak topology determined by the locally uniform convergence on \( \mathbb{D} \).

The main result of this paper is the following theorem, which implies the proof of Krzyz’s conjecture.

**Theorem 1.** For all \( f \in H_0^\infty \) and any \( n > 1 \),
\[ \max |c_n| = k'_0(0) = 2/e, \]  
with equality only for the function \( \kappa_0(z^n) \) and its compositions with pre and post rotations about the origin.

This theorem is extended to the spaces \( H(D, A_\rho) \) with sufficiently small \( \rho \).

**Theorem 2.** There is a number \( r_0, \ 0 < r_0 < 1 \), such that for any \( \rho < r_0 \) every extremal function \( f_0 \) maximizing \( |c_n| \) on the corresponding class \( H(D, A_\rho) \) is of the form
\[ f_0(z) = c_2k_\rho(c_1z^n) \]  
with \( |c_1| = |c_2| = 1 \).

In the case \( \rho > 0 \), we do not have an assertion on uniqueness of the covering map \( \kappa_\rho \) on which the maximal value \( \alpha \) is attained.

1.3. To prove Theorems 1 and 2, we apply a new approach in geometric function theory recently presented in [13]. It involves the Bers isomorphism theorem for Teichmüller spaces of punctured Riemann surfaces.

The unit ball \( H_1^\infty \) of \( H^\infty \) is naturally embedded into the universal Teichmüller space \( T \), and the functions \( f \in H_1^\infty \) are regarded as the Schwarzian derivatives of univalent functions in the unit disk.

In fact, this approach allows one to consider also some more general homogeneous polynomial coefficient functionals than \( c_n(f) \).

2. DIGRESSION TO TEICHMÜLLER SPACES

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[1] Weak compactness of \( H_0^\infty \) is obtained after adding to this class the function \( f(z) \equiv 0 \).
We briefly recall some needed results from Teichmüller space theory on spaces in order to prove Theorem 1; the details can be found, for example, in [3], [5].

2.1. The universal Teichmüller space $T = \text{Teich}(\mathbb{D})$ is the space of quasisymmetric homeomorphisms of the unit circle $S^1$ factorized by Möbius maps; all Teichmüller spaces have their isometric copies in $T$.

The canonical complex Banach structure on $T$ is defined by factorization of the ball of the Beltrami coefficients (or complex dilatations)

$$\text{Belt}(\mathbb{D})_1 = \{ \mu \in L_\infty(\mathbb{C}) : \mu|\mathbb{D}^* = 0, \| \mu \|_1 < 1 \},$$

letting $\mu_1, \mu_2 \in \text{Belt}(\mathbb{D})_1$ be equivalent if the corresponding quasiconformal maps $w^{\mu_1}, w^{\mu_2}$ (solutions to the Beltrami equation $\partial_\bar{z}w = \mu \partial_z w$ with $\mu = \mu_1, \mu_2$) coincide on the unit circle $S^1 = \partial \mathbb{D}^*$ (hence, on $\mathbb{D}^*$). Such $\mu$ and the corresponding maps $w^{\mu}$ are called $T$-equivalent. The equivalence classes $[w^{\mu}]_T$ are in one-to-one correspondence with the Schwarzian derivatives

$$S_w(z) = \left( \frac{w''(z)}{w'(z)} \right)' - \frac{1}{2} \left( \frac{w''(z)}{w'(z)} \right)^2 (w = w^{\mu}(z), \ z \in \mathbb{D}).$$

Note that for each locally univalent function $w(z)$ on a simply connected hyperbolic domain $D \subset \hat{\mathbb{C}}$, its Schwarzian derivative belongs to the complex Banach space $B(D)$ of hyperbolically bounded holomorphic functions on $D$ with the norm

$$\| \varphi \|_B = \sup_D \lambda_D^{-2}(z)|\varphi(z)|,$$

where $\lambda_D(z)|dz|$ is the hyperbolic metric on $D$ of Gaussian curvature $-4$; hence $\varphi(z) = O(z^{-4})$ as $z \to \infty$ if $\infty \in D$. In particular, for the unit disk, $\lambda_D(z) = 1/(1 - |z|^2)$.

The space $B(D)$ is dual to the Bergman space $A_1(D)$, a subspace of $L_1(D)$ formed by integrable holomorphic functions (quadratic differentials $\varphi(z)dz^2$ on $D$), since every linear functional $l(\varphi)$ on $A_1(D)$ is represented in the form

$$l(\varphi) = \langle \psi, \varphi \rangle_D = \int_D \lambda_D^{-2}(z)\overline{\psi(z)}\varphi(z)dxdy$$

(7)

with a uniquely determined $\psi \in B(D)$.

The Schwarzians $S_{w^{\mu}}(z)$ with $\mu \in \text{Belt}(\mathbb{D})_1$ range over a bounded domain in the space $B = B(\mathbb{D}^*)$. This domain models the space $T$. It lies in the ball $\{ \| \varphi \|_B < 2 \}$ and contains the ball $\{ \| \varphi \|_B < 2 \}$. In this model, the Teichmüller spaces of all hyperbolic Riemann surfaces are contained in $T$ as its complex submanifolds.

The factorizing projection

$$\phi_T(\mu) = S_{w^{\mu}} : \text{Belt}(\mathbb{D})_1 \to T$$

is a holomorphic map from $L_\infty(\mathbb{D})$ to $B$. This map is a split submersion, which means that $\phi_T$ has local holomorphic sections (see, e.g., [GL]).

Note that both equations $S_w = \varphi$ and $\partial_\bar{z}w = \mu \partial_z w$ (on $\mathbb{D}^*$ and $\mathbb{D}$, respectively) determine their solutions in $\Sigma_\delta$ uniquely, so the values $w^{\mu}(z_0)$ for any fixed $z_0 \in \mathbb{C}$ and the Taylor coefficients $b_1, b_2, \ldots$ of $w^{\mu} \in \Sigma_\delta$ depend holomorphically on $\mu \in \text{Belt}(\mathbb{D})_1$ and on $S_{w^{\mu}} \in T$.

2.2. The points of Teichmüller space $T_1 = \text{Teich}(\mathbb{D}_*)$ of the punctured disk $\mathbb{D}_* = \mathbb{D} \setminus \{0\}$ are the classes $[\mu]_{T_1}$ of $T_1$-equivalent Beltrami coefficients $\mu \in \text{Belt}(\mathbb{D})_1$ so that the corresponding quasiconformal automorphisms $w^{\mu}$ of the unit disk coincide on both boundary components (unit circle $S^1 = \{ |z| = 1 \}$ and the puncture $z = 0$) and are homotopic on $\mathbb{D} \setminus \{0\}$. This space can be endowed with a canonical complex structure of a complex Banach manifold and embedded into $T$ using uniformization.
Namely, the disk $\mathbb{D}_s$ is conformally equivalent to the factor $\mathbb{D}/\Gamma$, where $\Gamma$ is a cyclic parabolic Fuchsian group acting discontinuously on $\mathbb{D}$ and $\mathbb{D}^*$. The functions $\mu \in L_\infty(\mathbb{D})$ are lifted to $\mathbb{D}$ as the Beltrami $(-1,1)$-measurable forms $\tilde{\mu}d\overline{z}/dz$ in $\mathbb{D}$ with respect to $\Gamma$, i.e., via $(\tilde{\mu} \circ \gamma)\overline{\gamma}'/\gamma' = \tilde{\mu}$, $\gamma \in \Gamma$, forming the Banach space $L_\infty(\mathbb{D}, \Gamma)$.

We extend these $\tilde{\mu}$ by zero to $\mathbb{D}^*$ and consider the unit ball $\text{Belt}(\mathbb{D}, \Gamma)_1$ of $L_\infty(\mathbb{D}, \Gamma)$. Then the corresponding Schwarzians $S_{\tilde{\mu} \mid \mathbb{D}^*}$ belong to $\mathcal{T}$. Moreover, $\mathcal{T}_1$ is canonically isomorphic to the subspace $\mathcal{T}(\Gamma) = \mathcal{T} \cap \mathcal{B}(\Gamma)$, where $\mathcal{B}(\Gamma)$ consists of elements $\varphi \in \mathcal{B}$ satisfying $(\varphi \circ \gamma)(\gamma')^2 = \varphi$ in $\mathbb{D}^*$ for all $\gamma \in \Gamma$.

Due to the Bers isomorphism theorem, the space $\mathcal{T}_1$ is biholomorphically isomorphic to the Bers fiber space

$$\mathcal{F}(\mathcal{T}) = \{(\phi_{\mathcal{T}}(\mu), z) \in \mathcal{T} \times \mathbb{C} : \mu \in \text{Belt}(\mathbb{D})_1, z \in w^\mu(\mathbb{D})\}$$

over the universal space $\mathcal{T}$ with holomorphic projection $\pi(\psi, z) = \psi$ (see [3]).

This fiber space is a bounded hyperbolic domain in $\mathbb{B} \times \mathbb{C}$ and represents the collection of domains $D_\mu = w^\mu(\mathbb{D})$ as a holomorphic family over the space $\mathcal{T}$. For every $z \in \mathbb{D}$, its orbit $w^\mu(z)$ in $\mathcal{T}_1$ is a holomorphic curve over $\mathcal{T}$.

The indicated isomorphism between $\mathcal{T}_1$ and $\mathcal{F}(\mathcal{T})$ is induced by the inclusion map

$$j : \mathbb{D}_s \hookrightarrow \mathbb{D}$$

forgetting the puncture at the origin via

$$\mu \mapsto (S_{w^\mu}, w^\mu(0)) \quad \text{with} \quad \mu_1 = j_\circ \mu := (\mu \circ j_0)_{\overline{j_0}} / \overline{j_0},$$

where $j_0$ is the lift of $j$ to $\mathbb{D}$.

In the line with our goals, we slightly modified the Bers construction, applying quasiconformal maps $F^\mu$ of $\mathbb{D}_s$ admitting conformal extension to $\mathbb{D}^*$ (and accordingly using the Beltrami coefficients $\mu$ supported in the disk) (cf. [23]). These changes are not essential and do not affect the underlying features of the Bers isomorphism (giving the same space up to a biholomorphic isomorphism).

The Bers theorem is valid for Teichmüller spaces $\mathcal{T}(X_0 \setminus \{x_0\})$ of all punctured hyperbolic Riemann surfaces $X_0 \setminus \{x_0\}$ and implies that $\mathcal{T}(X_0 \setminus \{x_0\})$ is biholomorphically isomorphic to the Bers fiber space $\mathcal{F}(\mathcal{T}(X_0))$ over $\mathcal{T}(X_0)$.

Note that $\mathcal{B}(\Gamma_0)$ has the same elements as the space $A_1(\mathbb{D}^*, \Gamma_0)$ of integrable holomorphic forms of degree $-4$ with norm $\|\varphi\|_{A_1(\mathbb{D}^*, \Gamma_0)} = \iint_{\mathbb{D}^*/\Gamma_0} |\varphi(z)| dx dy$; and similar to (10), every linear functional $l(\varphi)$ on $A_1(\mathbb{D}^*, \Gamma_0)$ is represented in the form

$$l(\varphi) = \langle \psi, \varphi \rangle_{\mathbb{D}/\Gamma_0} := \iint_{\mathbb{D}^*/\Gamma_0} (1 - |z|^2)^2 \overline{\psi(z)} \varphi(z) dx dy$$

with uniquely determined $\psi \in \mathcal{B}(\Gamma_0)$.

Any Teichmüller space is a complete metric space with intrinsic Teichmüller metric defined by quasiconformal maps. By the Royden-Gardiner theorem, this metric equals the hyperbolic Kobayashi metric determined by the complex structure (see, e.g., [7], [8]).

We do not use here the finite dimensional Teichmüller spaces corresponding to finitely generated Fuchsian groups.

3. PROOF OF THEOREM 1

We carry out the proof in several stages as a consequence of lemmas.

1. Let $G$ be a ring subdomain of the disk $\mathbb{D}$ bounded by the unit circle $S^1 = \partial \mathbb{D}$ and a Jordan curve $\gamma_G$ separating the origin and $S^1$, and let $H(\mathbb{D}, G)$ denote the subspace of functions $f \in H^\infty_\gamma$ mapping $\mathbb{D}$ into $G$.

We first establish some results characterizing the structure of such sets $H(\mathbb{D}, G)$. 
Lemma 1. Any set $H(\mathbb{D}, G)$ contains an open path-wise connective subdomain $H^0(\mathbb{D}, G)$ which is dense in the weak topology of locally uniform convergence on $\mathbb{D}$, and the universal holomorphic covering map $\kappa_G : \mathbb{D} \to G$ extends to a holomorphic map from $H^\infty_1$ onto $H(\mathbb{D}, G)$ (that is, in $H^\infty_1$-norm).

Proof. We precede the proof of this main lemma by two auxiliary lemmas giving other analytic and geometric features of sets $H^\infty(\mathbb{D}, G)$.

Lemma 2. (a) Every function $f \in H(\mathbb{D}, G)$ admits factorization

$$f(z) = \kappa_G \circ \hat{f}(z),$$

(9)

where $\hat{f}$ is a holomorphic map of the disk $\mathbb{D}$ into itself (hence, from $H^\infty_1$).

(b) Moreover, the relation generates an $H^\infty$-holomorphic map $k_G : \hat{f} \mapsto f$ from $H^\infty_1$ onto $H(\mathbb{D}, G)$.

Proof. (a) Due to a general topological theorem, any map $f : M \to N$, where $M, N$ are manifolds, can be lifted to a covering manifold $\hat{N}$ of $N$, under an appropriate relation between the fundamental group $\pi_1(M)$ and a normal subgroup of $\pi_1(N)$ defining the covering $\hat{N}$ (see, e.g, [17]). This construction produces a map $\hat{f} : M \to \hat{N}$ satisfying

$$f = p \circ \hat{f},$$

(10)

where $p$ is a projection $\hat{N} \to N$. The map $\hat{f}$ is determined up to composition with the covering transformations of $\hat{N}$ over $N$ or equivalently, up to choosing a preimage of a fixed point $x_0 \in \hat{N}$ in its fiber $p^{-1}(x_0)$. For holomorphic maps and manifolds, the lifted map is also holomorphic.

In our special case, $\kappa_G$ is a holomorphic universal covering map $\mathbb{D} \to G$, and the representation (10) provides the equality (9) with the corresponding $\hat{f}$ determined up to covering transformations of the unit disk compatible with the covering map $\kappa_G$.

For a fixed $z \in \mathbb{D}$, each coefficient $c_n$ of $f$ (and hence $f$ itself) is a holomorphic function (polynomial) of the initial coefficients $\hat{c}_0, \hat{c}_1, \ldots, \hat{c}_n$ of cover $\hat{f}$. Holomorphy in the $H^\infty$ norm stated by the assertion (b) is a consequence of a well-known property of bounded holomorphic functions in Banach spaces with sup norm given by the following lemma of Earle [6].

Lemma 3. Let $E, T$ be open subsets of complex Banach spaces $X, Y$ and $B(E)$ be a Banach space of holomorphic functions on $E$ with sup norm. If $\varphi(x, t)$ is a bounded map $E \times T \to B(E)$ such that $t \mapsto \varphi(x, t)$ is holomorphic for each $x \in E$, then the map $\varphi$ is holomorphic.

Holomorphy of $\varphi(x, t)$ in $t$ for fixed $x$ implies the existence of complex directional derivatives

$$\varphi(x, t) = \lim_{\zeta \to 0} \frac{\varphi(x, t + \zeta v) - \varphi(x, t)}{\zeta} = \frac{1}{2\pi i} \int_{|\xi|=1} \frac{\varphi(x, t + \xi v)}{\xi^2} d\xi,$$

while the boundedness of $\varphi$ in sup norm provides the uniform estimate

$$||\varphi(x, t + cv) - \varphi(x, t) - \varphi'(x, t)cv||_{B(E)} \leq M|c|^2,$$

for sufficiently small $|c|$ and $\|v\|_Y$.

The map $k_p : \hat{f} \mapsto f$ is bounded on the ball $H^\infty_1$. Applying Hartog’s theorem on separate holomorphy to the sums $g(z, t) = \hat{f}(z) + \hat{h}(z)$ of $\hat{f} \in H^\infty_1$, $\hat{h} \in H_1$ and $t$ from a region $B \subset \hat{C}$ so that $g(z, t) \in H^\infty_1$, one obtains that $g(z, t)$ are jointly holomorphic in both variables $(z, t) \in \mathbb{D} \times B$. Thus the restriction of the map $k_G$ onto intersection of the ball $H^\infty_1$ with any complex line $L = \{\hat{f} + \hat{h}\}$ is $H^\infty$-holomorphic, and hence this map is holomorphic as the map $H^\infty_1 \to H(\mathbb{D}, G)$, which completes the proof of Lemma 3.
One can also show that the restriction of the extended map $k_G$ to any holomorphic disk
\[ D(\hat{f}) = \{ t\hat{f}/\|\hat{f}\|_\infty : |t| < 1 \}, \quad \hat{f} \in H_1^\infty, \]
is a complex geodesic (cf. [25]), hence a local hyperbolic isometry (preserving such property of the original map $k_G$). We will not use this fact and therefore do not present here its proof.

Now consider the domains $G \subseteq D$. Fix a point $w_0 \in G$ and take for a decreasing sequence $r_n \to 0$ the connected components $G_n^0$ of widening open sets
\[ G_n = \{ w \in G : \text{dist}(w, \partial G) > r_n \}, \quad G_n \subseteq G_{n+1}, \quad n = 1, 2, \ldots, \]
exhausting this domain and containing $w_0$. Let
\[ H^0(D, G) = \bigcup_n H(D, G_n). \]
The set $H^0(D, G)$ is open and contains, in particular, all functions $f \in H(D, G)$ holomorphic on the closed disk $\hat{D}$.

**Lemma 4.** (a) For any fixed $n$, every function $f \in H(D, G_n)$ continuous in the the closed disk $\hat{D}$ has a neighborhood $U(f, \epsilon_n)$ in $H_1^\infty$, which contains only the functions belonging to $H^0(D, G)$.

(b) Each of the sets $H(D, G_n)$ and $H^0(D, G)$ is path-wise connective in $H_1^\infty$; therefore, the union $H^0(D, G)$ is a domain in $H_1^\infty$.

**Proof.** To prove the assertion (a), assume to the contrary that, for some $n = n_0$, such a number $m(n_0)$ does not exist. Then there is a function $f_0 \in H(D, G_{n_0})$, a sequence of functions $f_m \in H(D, G_{n_0})$ convergent to $f_0$ so that
\[ \lim_{m \to \infty} \|f_m - f_0\|_{H^\infty} = 0, \quad (11) \]
and a sequence of points $z_m \in D$ convergent to $z_0 \in D$, for which we will have $f_m(z_m) \in G_m$ for all $m \geq n_0$, and
\[ \lim_{m \to \infty} f_m(z_m) = a \in D \setminus \overline{G}. \quad (12) \]

We approximate $f_m(z)$ by functions $f_{m,r}(z) = f_m(rmz)$ (holomorphic in $\hat{D}$), taking $r_m$ so close to 1 that the equality (11) is preserved for $f_{m,r}$. Then the uniform convergence of $f_m$ and $f_{m,r}$ to $f_0$ on compact subsets of $D$ immediately implies that the limit $a$ in (15) must be equal to $f_0(z_0)$, and therefore it belongs to $G_{n_0}$. This proves part (a).

To show that each $H(D, G_n)$ is path-wise connective, take its arbitrary distinct points $f_1$, $f_2$. By (9),
\[ \tilde{f}_j = \kappa_{G_n} \circ \tilde{f}_j \quad (\tilde{f}_j \in H_1^\infty), \quad j = 1, 2. \]
Connecting the covers $\tilde{f}_1$ and $\tilde{f}_2$ in $H_1^\infty$ by the line interval $l_{1,2}(t) = t\tilde{f}_1 + (1 - t)\tilde{f}_2$ $0 \leq t \leq 1$, one obtains a path $\kappa_{G_n} \circ l_{1,2} : [0, 1] \to H(D, G_n)$ connecting $f_1$ with $f_2$, completing the proof of Lemmas 4 and 1.

Observe that Lemma 4 does not contradict to existence for $f_0 \in H(D, G)$ of sequences $\{f_n\} \in H_1^\infty$ convergent to $f_0$ only locally uniformly in $D$ and taking some values in $D \setminus G$.

Using the homotopy $\tilde{f}_t(z) = t\tilde{f}(tz)$ of the cover functions and representation (9), one concludes that the domain $H^0(D, G)$ is dense in the set $H(D, G)$ in the weak topology. Hence,
\[ \sup_{H^0(D, G)} |J(f)| = \max_{H(D, G)} |J(f)| \]
for any holomorphic functional $J(f)$. This follows also from the fact that all $f \in H(D, G)$ holomorphic on the closure $\overline{G}$ of domain $G$ belong to $H^0(D, G)$. 

\[ \text{Path-wise connective}. \]
Note also that for \( G = A_0 \), the distinguished domain \( H^0(\mathbb{D}, A_0) \) preserves circular symmetry, i.e., it contains the nonvanishing functions \( f \in H^1_+ \) together with their compositions with pre and post rotations about the origin.

2°. In the case of the punctured disk \( \mathbb{D}_* = A_0 \) Lemma 4 admits some strengthening.

**Lemma 5.** Each point \( f \in H_0^0 \) has a neighborhood (ball) \( U(f, \varepsilon) \) in \( H^\infty \), which entirely belongs to \( B \), i.e., contains only nonvanishing functions on the disk \( \mathbb{D} \). Take the maximal balls \( U(f, \varepsilon) \) with such property. Then their union

\[
U_0 = \bigcup_{f \in H^\infty} U(f, \varepsilon)
\]

is a domain in the space \( H^\infty \).

**Proof.** *Openness:* It suffices to show that for each \( r > 1 \) and \( r' < r \), every function \( f \in H_0^\infty \) has a neighborhood \( U(f, \varepsilon(r)) \) in \( H^\infty(\mathbb{D}_{r'}) \), which contains only nonvanishing functions on \( \mathbb{D}_{r'} = \{|z| < r'\} \). For \( r' = 1 \), this gives the first assertion of the lemma.

Assume the contrary. Then (for some \( r > 1 \) and \( r' < r \)) there exist a function \( f_0 \in B_r \) and the sequences of functions \( f_n \in H^\infty(\mathbb{D}_{r'}) \) convergent to \( f_0 \),

\[
\lim_{n \to \infty} \|f_n - f_0\|_{H^\infty(\mathbb{D}_{r'})} = 0
\]  

(13)

and of points \( z_n \in \mathbb{D} \) convergent to \( z_0 \), \(|z_0| \leq r'\) such that \( f_n(z_n) = 0 \) \((n = 1, 2, \ldots)\).

In the case \(|z_0| < r'\), we immediately reach a contradiction, because then the uniform convergence of \( f_n \) on compact sets in \( \mathbb{D}_{r'} \) implies \( f_0(z_0) = 0 \), which is impossible.

The case \(|z_0| = r'\) requires other arguments. Since \( f_0 \) is holomorphic and does not vanish on the closed disk \( \overline{\mathbb{D}}_r \),

\[
\min_{|z| \leq r'} |f_0(z)| = a > 0.
\]

Hence, for each \( z_n \),

\[
|f_n(z_n) - f_0(z_n)| = |f_0(z_n)| \geq a,
\]

and by continuity, there exists a neighborhood \( \mathbb{D}(z_n, \delta_n) = \{|z - z_n < \delta_n\} \) of \( z_n \) in \( \mathbb{D}_{r'} \), in which \(|f_n(z) - f_0(z)| \geq a/2\) for all \( z \). This implies

\[
\|f_n - f_0\|_{H^\infty(\mathbb{D}_{r'})} \geq \max_{\mathbb{D}(z_n, \delta_n)} |f_n(z) - f_0(z)| \geq \frac{a}{2}.
\]

This inequality must hold for all \( n \), contradicting (13).

The **connectedness** of the union \( \mathcal{H} \) is established similar to Lemma 4.

3°. The next step in the proof of Theorem 1 is to construct a holomorphic embedding of the unit ball \( H_1^\infty \) into some Teichmüller spaces.

Any function \( g \) from this ball \( H_1^\infty \) belongs to the space \( \mathcal{B} = \mathcal{B}(\mathbb{D}) \) of hyperbolically bounded holomorphic functions \( f(z) \) regarding as holomorphic quadratic differentials \( f(z)dz^2 \) on the unit disk, with norm

\[
\|f\|_\mathcal{B} = \sup_{\mathbb{D}} (1 - |z|^2)^2 |f(z)| < 1.
\]

Hence, such \( f \) is the Schwarzian derivative \( S_w \) of a univalent function \( w(z) \) in the unit disk \( \mathbb{D} \) solving the differential equation \( S_w = f \). This \( w \) is determined up to a Moebius map of the sphere \( \hat{\mathbb{C}} \).

This implies a holomorphic embedding \( \iota \) of the ball \( H_1^\infty \) into the universal Teichmüller space.

To determine \( w \) uniquely (and ensure the holomorphic dependence of \( w \) from \( S_w \)), we shall use the following normalization.

Consider similar to [13] the family \( \hat{S}(1) \) of univalent functions on \( \mathbb{D} \) which is the completion in the topology of locally uniform convergence on \( \mathbb{D} \) of the set of univalent functions \( w(z) = a_1 z + a_2 z^2 + \ldots \)
with \(|a_1| = 1\), having quasiconformal extensions across the unit circle \(S^1 = \partial \mathbb{D}\) to the whole sphere \(\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}\) which satisfy \(w(1) = 1\).

Equivalently, this family is a disjunct union

\[
\hat{S}(1) = \bigcup_{-\pi \leq \theta < \pi} S_\theta(1),
\]

where \(S_\theta(1)\) consists of univalent functions \(w(z) = e^{i\theta}z + a_2z^2 + \ldots\) with quasiconformal extensions to \(\hat{\mathbb{C}}\) satisfying \(w(1) = 1\) (also completed in the indicated weak topology).

This family is closely related to the canonical class \(S\) of univalent functions \(w(z)\) on \(\mathbb{D}\) normalized by \(w(0) = 0, w'(0) = 1\). Every \(w \in S\) has its representative \(\hat{w}\) in \(\hat{S}(1)\) (not necessarily unique) obtained by pre and post compositions of \(w\) with rotations \(z \mapsto e^{i\alpha}z\) about the origin, related by

\[
w_{\tau,\theta}(z) = e^{-i\theta}w(e^{i\tau}z) \quad \text{with} \quad \tau = \arg z_0, \tag{14}
\]

where \(z_0\) is a point for which \(w(z_0) = e^{i\theta}\) is a common point of the unit circle and the the boundary of domain \((\mathbb{D})\). In the general case, the equality \(w(1) = 1\) in terms of the Carathéodory prime ends.

The relation (14) implies, in particular, that the functions conformal in the closed disk \(\overline{\mathbb{D}}\) are dense in each class \(S_\theta(1)\). Such a dense subset is formed, for example, by the images of the homotopy functions \(\{f_\tau(z) = \frac{1}{2}f(rz)\}\) with real \(r \in (0, 1)\).

The inverted functions \(F(z) = 1/f(1/z)\) for \(f \in \hat{S}(1)\) form the corresponding classes \(\Sigma_\theta(1)\) of nonvanishing univalent functions on the complementary disk

\[
\mathbb{D}^* = \{z \in \hat{\mathbb{C}}: |z| > 1\}
\]

with expansions

\[
F(z) = e^{-i\theta}z + b_0 + b_1z^{-1} + b_2z^{-2} + \ldots, \quad F(1) = 1,
\]

and \(\hat{\Sigma}(1) = \bigcup_\theta \Sigma_\theta(1)\).

Simple computations yield that the coefficients \(a_n\) of \(f \in S_\theta(1)\) and the corresponding coefficients \(b_j\) of \(F(1) = 1/f(1/z) \in \Sigma_\theta(1)\) are related by

\[
b_0 + e^{2i\theta}a_2 = 0, \quad b_n + \sum_{j=1}^n \epsilon_{n,j}b_{n-j}a_{j+1} + \epsilon_{n+2,0}a_{n+2} = 0, \quad n = 1, 2, \ldots,
\]

where \(\epsilon_{n,j}\) are the entire powers of \(e^{i\theta}\). This successively implies the representations of \(a_n\) by \(b_j\) via

\[
a_n = (-1)^{n-1}e_{n-1,0}b_0^{n-1} - (-1)^{n-1}(n-2)e_{1,n-3}b_1b_0^{n-3} + \text{lower terms with respect to } b_0. \tag{15}
\]

The coefficients \(\alpha_n\) of Schwarzians

\[
S_w(z) = \sum_{0}^{\infty} \alpha_n z^n
\]

are represented as polynomials of \(n + 2\) initial coefficients of \(w \in S_\theta(1)\) and, in view of (15), as polynomials of \(n + 1\) initial coefficients of the corresponding \(W \in \Sigma_\theta(1)\); denote these polynomials by \(J_n(w)\) and \(\tilde{J}_n(W)\), respectively. These polynomial functionals are naturally extended to the whole classes \(\hat{S}(1)\) and \(\hat{\Sigma}(1)\).

4°. As was mentioned above, any \(f \in H^\infty\) belongs to the space \(\mathbf{B}\) and

\[
\|f\|_\mathbf{B} = \sup_{\mathbb{D}}(1 - |z|^2)|f(z)| < \|f\|_\infty.
\]

By the well-known Ahlfors-Weill theorem, any \(g \in \mathbf{B}\) with norm \(\|g\|_\mathbf{B} = k < 2\) is the Schwarzian derivative \(S_w = g\) of a function \(w\) which is univalent on the disk \(\mathbb{D}\) and admits \(k\)-quasiconformal
extension across the unit circle \(|z| = 1\) to \(\hat{\mathbb{C}}\) with Beltrami coefficients
\[
\nu_{S_w}(\zeta) = \frac{\partial \overline{w}/\partial \zeta w}{\overline{w}} = -\frac{1}{2}(|\zeta|^2 - 1)\frac{\zeta^2}{\zeta}S_w\left(\frac{1}{\zeta}\right).
\]

We shall denote this holomorphic embedding of the ball \(H_1^\infty\) into the space \(T\) modeled by Schwarzians in \(\mathbb{D}^*\) again by \(\iota\). The image of \(H_1^\infty\) under this embedding is a noncomplete linear subspace in \(B\) so that \(\iota H_1^\infty\) is a complex subset of the unit ball in \(B\), and the image of the distinguished domain \(H^0(\mathbb{D}, A_\rho)\) is a complex submanifold in \(T\).

Another important property of the set \(\iota H^0(\mathbb{D}, A_\rho)\) is given by the following two lemmas.

**Lemma 6.** Let \(f(z) = \sum_{n=0}^\infty c_n z^n \in H^\infty\) and \(s_m(z) = \sum_{n=0}^{m-1} c_n z^n\). Then
\[
\lim_{m \to \infty} \|s_m - f\|_B = 0. \tag{16}
\]

**Proof.** It suffices to consider the functions \(f\) from the ball \(\{\|f\|_B < 1/2\}\). Their coefficients \(c_j\) are estimated by \(|c_n| < 1/2\) for all \(n \geq 0\). Hence,
\[
|s_m(z)| < \frac{1}{2} \sum_{n=0}^{m-1} |z|^n < \frac{1}{2(1 - |z|)},
\]
and
\[
\|s_m\|_B < \frac{1}{2} \sup_{\mathbb{D}}(1 - |z|)(1 + |z|)^2 < 2,
\]
which means that any partial sum \(s_m\) for such \(f\) lies in the ball \(\{g \in B : \|g\|_B < 2\}\) and therefore it also belongs to the space \(T\). Further,
\[
|s_m(z) - f(z)| = |c_m z^m + c_{m+1} z^{m+1} + \ldots| < \frac{1}{2} (|z|^m + |z|^{m+1} + \ldots) = \frac{1}{2} \frac{|z|^m}{1 - |z|} < \frac{1}{2} \frac{1}{1 - |z|},
\]
which implies
\[
\|s_m(z) - f(z)\|_B < \frac{1}{2} \sup_{\mathbb{D}}(1 - |z|)(1 + |z|)^2 < \frac{1}{2^{m-1}},
\]
and (19) follows, proving the lemma.

Lemmas 1 and 5 imply

**Lemma 7.** Any point \(f(z) = \sum_{n=0}^\infty c_n z^n\) from the set \(\iota H^0(\mathbb{D}, A_\rho)\) in \(T\) is approximated in the \(B\)-norm by polynomials \(s_m(z) = c_0 + c_1 z + \ldots + c_m z^m\) with \(m \geq m_0(f)\), which also belong to \(\iota H^0(\mathbb{D}, A_\rho)\) and hence do not have zeros in the disk \(\mathbb{D}\).

\(5^0\). Our next step is to lift both polynomial functionals \(J_n(w)\) and \(\tilde{J}_n(W)\) (equivalently \(c_n\)) onto the Teichmüller space \(T_1\). Letting
\[
\tilde{J}_n(\mu) = \tilde{J}_n(W^\mu), \tag{17}
\]
we lift these functionals from the sets \(S_0(1)\) and \(\Sigma_0(1)\) onto the ball \(\text{Belt}(\mathbb{D})_1\). Then, under the indicated \(T_1\)-equivalence, i.e., by the quotient map
\[
\phi_{T_1} : \text{Belt}(\mathbb{D})_1 \to T_1, \quad \mu \to [\mu]_{T_1},
\]
the functional \(\tilde{J}_n(W^\mu)\) is pushed down to a bounded holomorphic functional \(\tilde{J}_n\) on the space \(T_1\) with the same range domain.
Equivalently, one can apply the quotient map \( \text{Belt}(\mathbb{D})_1 \to T \) (i.e., \( T \)-equivalence) and compose the descended functional on \( T \) with the natural holomorphic map \( \iota_1 : T_1 \to T \) generated by the inclusion \( \mathbb{D}_* \to \mathbb{D} \) forgetting the puncture. Note that since the coefficients \( b_0, b_1, \ldots \) of \( W^\mu \in \Sigma_\theta \) are uniquely determined by its Schwarzian \( S_W^\mu \), the values of \( J_n \) in the points \( X_1, X_2 \in T_1 \) with \( \iota_1(X_1) = \iota_1(X_2) \) are equal.

Now, using the Bers isomorphism theorem, we regard the points of the space \( T_1 \) as the pairs \( X^{W^\mu} = (S^{W^\mu}, W^\mu(0)) \), where \( \mu \in \text{Belt}(\mathbb{D})_1 \) obey \( T_1 \)-equivalence (hence, also \( T \)-equivalence). Denote (for simplicity of notations) the composition of \( J_n \) with biholomorphism \( T_1 \cong F(T) \) again by \( J_n \). In view of (8) and (17), it is presented on the fiber space \( F(T) \) by

\[
J_n(X^{W^\mu}) = J_n(S_W^\mu, t), \quad t = W^\mu(0).
\]  

(18)

This yields a logarithmically plurisubharmonic functional \( \{J_n(S_{W^\mu}, t)\} \) on \( F(T) \).

Note that since the coefficients \( b_0, b_1, \ldots \) of \( W^\mu \in \Sigma_\theta \) are uniquely determined by its Schwarzian \( S_W^\mu \), the values of \( J_n \) in the points \( X_1, X_2 \in T_1 \) with \( \iota_1(X_1) = \iota_1(X_2) \) are equal.

We have to estimate a smaller plurisubharmonic functional arising after restriction of \( J_n(S_{W^\mu}, t) \) onto the images in these spaces of the distinguished convex set \( \iota H^0(\mathbb{D}, A_\rho) \), i.e., the functional (17) on the set of \( S_{W^\mu} \in \iota H^0(\mathbb{D}, A_\rho) \) and corresponding values of \( t = W^\mu(0) \) which runs over some subdomain \( D_{\rho, \theta} \) in the disk \( \{ |t| < 4 \} \).

We denote this restricted functional by \( J_{n,0}(S_{W^\mu}, W^\mu(0)) \) and define in domain \( D_{\rho, \theta} \) the function

\[
u_\theta(t) = \sup_{S_{W^\mu}} |J_{n,0}(S_{W^\mu}, t)|,
\]  

(19)

where the supremum is taken over all \( S_{F^{\rho}} \in \iota H^0(\mathbb{D}, A_\rho) \) admissible for a given \( t = W^\mu(0) \in D_{\rho, \theta} \), that means over the pairs \( (S_{W^\mu}, t) \in F(T) \) with \( S_{F^{\rho}} \in \iota H^0(\mathbb{D}, A_\rho) \) and a fixed \( t \).

Our goal is to establish that this function inherits the subharmonicity of \( J \). This is given by the following basic lemma.

**Lemma 8.** The function \( \nu_\theta(t) \) is subharmonic in the domain \( D_{\rho, \theta} \).

**Proof.** Consider in the set \( \iota H^0(\mathbb{D}, A_\rho) \) its \( m \)-dimensional analytic subsets \( V_m \) corresponding to the partial sums \( s_m \) of functions \( f \in H^0(\mathbb{D}, A_\rho) \) (with \( m \geq m_0(f) \)). Given such \( f \), we define

\[
F(z) = f(1/z)/z^4
\]

and take a univalent solution \( W \in \Sigma_\theta \) of the Schwarzian equation \( S_W(z) = F(z) \) on \( D^* \). Let \( W^\mu \) be one of its quasiconformal extensions onto \( \mathbb{D} \).

Let \( W_m \) and \( W_m^{\mu m} \) be the corresponding functions defined similarly by the partial sums \( s_m \) of \( f \), \( m \geq m_0(f) \). Then the domains \( W_m(\mathbb{D}^*) \) and \( W_m^{\mu m}(\mathbb{D}) \) approximate \( W(\mathbb{D}^*) \) and \( W^\mu(\mathbb{D}) \) uniformly (in the spherical metric on \( \mathbb{C} \)), and the points \( W_m^{\mu m}(0) \) are close to \( W^\mu(0) \).

One can replace the extensions \( W_m^{\mu m} \) by \( \omega_m \circ W_m^{\mu m} \), where \( \omega_m \) is the extremal quasiconformal automorphism of domain \( W_m^{\mu m}(\mathbb{D}) \) moving the point \( W_m^{\mu m}(0) \) into \( W^\mu(0) \) and identical on the boundary of \( W_m^{\mu m}(\mathbb{D}) \) (cf. [21]). This provides for a prescribed \( t = W^\mu(0) \) the points \( W_m^{\mu m}(0) \in F(T) \) corresponding to given \( s_m \in V_m \).

Now, maximizing the function \( \log |J_{n,0}(S_{W_m^{\mu m}}, t)| \) over the manifold \( V_m \), i.e., over \( S_{W_m^{\mu m}} \) (with appropriate \( m \)), one obtains a logarithmically plurisubharmonic function

\[
u_m(t) = \sup_{V_m} |J_{n,0}(S_{W_m^{\mu m}}, t)|, \quad t = W^\mu(0),
\]

in the domain \( D_{\rho, \theta} \) indicated above. We take its upper semicontinuous regularization

\[
u_m(t) = \lim_{t' \to t} \nu_m(t'),
\]
determines a holomorphic surjection of the space $T$.

It is defined and subharmonic in domain $D_{\rho,\theta}$. Lemmas 6 and 7 imply that this function coincides with function (19).

Let $\beta$ be a holomorphic map from $\mathbb{D}_{\rho} = \mathbb{D} \setminus \{0\}$.

We have to establish the value domain of $W(0)$ for $W$ running over $\iota H^0(\mathbb{D}, A_0)$.

First, we apply the following generalization of the above construction. Taking a dense countable subset

$$\Theta = \{\theta_1, \theta_2, \ldots, \theta_m, \ldots\} \subset [-\pi, \pi],$$

consider the increasing unions of the quotient spaces

$$T_m = \bigcup_{j=1}^{m} \Sigma_{\beta_j}/ \sim = \bigcup_{j=1}^{m} \{(S_{W_{\beta_j}}, W_{\beta}^{\mu}(0))\} \simeq T_1 \cup \cdots \cup T_1,$$

(20)

where the equivalence relation $\sim$ means $T_1$-equivalence on a dense subset $\Sigma(1)$ in the union $\Sigma(1)$ formed by all univalent functions $W_{\beta}(z) = e^{-i\theta_j}z + b_0 + b_1z^{-2} + \ldots$ on $\mathbb{D}^*$ (preserving $z = 1$) with quasiconformal extension to $\hat{\mathbb{C}}$, and

$$W_{\theta}^{\mu}(0) := (W_{\theta_1}^{\mu}(0), \ldots, W_{\theta_m}^{\mu}(0)).$$

The Beltrami coefficients $\mu_j \in \text{Belt}(\mathbb{D})_1$ are chosen here independently. The corresponding collection $\beta = (\beta_1, \ldots, \beta_m)$ of the Bers isomorphisms

$$\beta_j : \{(S_{W_{\beta_j}}, W_{\beta_j}^{\mu}(0))\} \to F(T),$$

determines a holomorphic surjection of the space $T_m$ onto $F(T)$.

Taking also in each union (20) the corresponding collection $\iota_m H^\infty_0$ covering $H^0(\mathbb{D}, A_0)$, one obtains in a similar manner to the above the maximal function

$$u(t) = \sup_{\Theta} u_{\theta_m}(t) = \sup\{|J_{n,0}(S_{W_{\theta}^{\mu}}, t)| : \theta \in \bigcup_m \iota_m H^\infty_0\}.\quad (21)$$

It is defined and subharmonic in domain

$$D_\rho = \bigcup_{\Theta} D_{\rho,\theta_m}.$$

Noting that the union of spaces $T_m$ possesses the circular symmetry inherited from the class $\Sigma(1)$, which is preserved under rotations (14), one concludes that this broad domain $D_0$ must be a disk $\mathbb{D}_{r_0} = \{|t| < r_0\}$.

Now we show that in the case of nonvanishing $H^\infty$ functions this radius $r_0$ is naturally connected with the function (2). This requires a covering estimate of Koebe’s type.

Let $G$ be a domain in a complex Banach space $X = \{x\}$ and $\chi$ be a holomorphic map from $G$ into the universal Teichmüller space $T$ modeled as a bounded subdomain of $B$. Consider in the unit disk the corresponding Schwarzian differential equations

$$S_w(z) = \chi(x)\quad (22)$$

and pick their univalent solutions $w(z)$ satisfying $w(0) = w'(0) - 1 = 0$ (hence $w(z) = z + \sum_2^{\infty} a_n z^n$).

Put

$$|a_2^0| = \sup\{|a_2| : S_w \in \chi(G)\},\quad (23)$$
and let \( w_0(z) = z + a_2^0 z^2 + \ldots \) be one of the maximizing functions.

**Lemma 10.** (a) For every indicated solution \( w(z) = z + a_2 + \ldots \) of (22), the image domain \( w(D) \) covers entirely the disk \( \{ |w| < 1/(2|a_2^0|) \} \).

The radius value \( 1/(2|a_2^0|) \) is sharp for this collection of functions, and the circle \( \{ |w| = 1/(2|a_2^0|) \} \) contains points not belonging to \( w(D) \) if and only if \( |a_2| = |a_2^0| \) (i.e., when \( w \) is one of the maximizing functions).

(b) The inverted functions

\[
W(\zeta) = 1/w(1/\zeta) = \zeta - a_2^0 + b_1\zeta^{-1} + b_2\zeta^{-2} + \ldots
\]

map the disk \( D^* \) onto a domain whose boundary is entirely contained in the disk \( \{ |W + a_2^0| \leq |a_2^0| \} \).

**The proof** follows the classical lines of Koebe’s 1/4 theorem (cf. [9]).

(a) Suppose that the point \( w = c \) does not belong to the image of \( D \) under the map \( w(z) \) defined above. Then \( c \neq 0 \), and the function

\[
w_1(z) = cw(z)/(c - w(z)) = z + (a_2 + 1/c)z^2 + \ldots
\]

also belongs to this class, and hence by (23), \( |a_2 + 1/c| \leq |a_2^0| \), which implies

\[|c| \geq 1/(2|a_2^0|)\].

The equality holds only when

\[|a_2 + 1/c| = |1/c| - |a_2| = |a_2^0| \quad \text{and} \quad |a_2| = |a_2^0|\].

(b) If a point \( \zeta = c \) does not belong to the image \( W(D^*) \), then the function

\[W_1(z) = 1/[W(1/z) - c] = z + (c + a_2)z^2 + \ldots\]

is holomorphic and univalent in the disk \( D \), and therefore, \( |c + a_2| \leq |a_2^0| \). The lemma follows.

This lemma implies that the boundary of the range domain of \( W^\mu(0) \) is contained in the disk

\[D_{2|a_2^0|} = \{ W : |W| \leq 2|a_2^0| \}, \quad (24)\]

and, consequently, \( r_0 = 2|a_2^0| \) and touches from inside the circle \( \{ |W| = 2|a_2^0| \} \) at the points corresponding to extremal functions \( W_0 \) maximizing \( |a_2| \) on the closure of the domain \( iH_0^* \).

Generically, the extremal value \( 2|a_2^0| \) of the radius of covered disk can be attained on several functions \( W_0 \).

**7.** We now establish that

\[S_{W_0}(z) = \kappa_0(z). \quad (25)\]

In view of Lemma 1, it is enough to show that

\[S'_{W_0}(0)(z) = c_1^0 \neq 0 \quad (26)\]

(in other words, that the zero set of the functional \( J_1(f) = c_1 \) is separated from the set of rotations (14) of the function \( W_0 \)). This yields that the corresponding function (21), constructed by maximization of functional \( J_1(f) = |c_1| \), is defined and subharmonic on the whole disk \( D_{2|a_2^0|} \), and its maximaum is attained on the boundary circle.

Assume, to the contrary, that \( S'_{W_0}(0)(z) = 0 \). Then, by Lemma 2,

\[S_{W_0}(z) = \kappa_0 \circ \hat{f}_0(z) = c_0 + c_2z^2 + c_3z^3 + \ldots\]

where \( \hat{f}_0 \) is a holomorphic self-map of \( D \) of the form

\[\hat{f}_0(z) = \hat{c}_0 + \hat{c}_2z^2 + \hat{c}_3z^3 + \ldots\]
Since the function
\[ \kappa_2(z) = \kappa_0(z^2) = 1/e + (2/e)z^2 + \ldots \]
also belongs to \( iH_0^1 \), it must be
\[ |c_2^0| > 2/e. \]  
(27)

Now consider the function
\[ \hat{f}_1(z) = \sigma^{-1} \circ \left\{ \hat{f}_0 \circ \sigma(z) \right\} = \hat{c}_0 + \hat{c}_2 z + \hat{c}_3 z^2 + \ldots, \]
where
\[ \sigma(z) = (z - \hat{c}_0)/(1 - \hat{c}_0 z). \]
This function also is a holomorphic self-map of the disk \( D \). Its composition with \( \kappa_0 \) via (10), denoted by \( f_1 \), is a nonvanishing holomorphic self-map of \( D \), and a simple calculation, using (27), yields
\[ f_1'(0) = (\kappa_0 \circ \hat{f}_1)'(0) = |c_2^0| > 2/e, \]
which contradicts to Lemma 1. This proves the relations (25) and (26).

80. Now we can finish the proof of the theorem.

Take \( n = 2 \) and, letting \( f_2(z) = f(z^2) \), consider on \( H_0^\infty \) the plurisubharmonic functional
\[ I_2(f) = \max (|J_2(f)|, |J_2(f_2)|). \]  
(28)

Similar to above, the lift of this functional onto \( T_1 \) generates via (19) a nonconstant radial subharmonic function of on the disk (24). It is logarithmically convex, hence monotone increasing, and attains its maximal value at \( |t| = 2|a_0^0| \).

By Parseval’s equality for the boundary functions \( f(e^{i\theta}) = \lim_{r \to 1} f(re^{i\theta}) \) of \( f \in H_1^\infty \), we have
\[ 1 \geq \frac{1}{2\pi} \int_0^\pi |f(e^{i\theta})|^2 d\theta = \sum_{n=1}^\infty |c_n|^2. \]

Applying it to the function
\[ f(z) = \kappa_0(z) = \sum_{n=0}^\infty c_n^0 z^n \]
and noting that by (2), \( |c_1^0|^2 = 4e^{-2} = 0.541... \), one obtains that for this function,
\[ \sum_{n=2}^\infty |c_n|^2 < 0.5 < |c_1|^2. \]

This implies (in view of the indicated connection of \( a_3^0 \) with \( \kappa_0 \)) that the maximal value of the functional (28) on \( H_0^\infty \) is attained on the functions \( \kappa_0(z), \kappa_2(z) = \kappa_0(z^2) \) and equals
\[ \max (|c_1|^2, |c_2|^2) = 2/e, \]
giving the desired estimate (1) for \( n = 2 \). The extremal extremal function is unique, up to rotations.

Now take \( n = 3 \) and, letting \( f_3(z) = f(z^3) \), consider similar to (28) the functional
\[ I_3(f) = \max (|J_3(f)|, |J_3(f_3)|) \]
Arguing similar to the above case, one obtains
\[ \max_{H_0^\infty} I_3(f) = \max (|c_1^0|^2, |c_3^0|^2) = 2/e, \]
giving the estimate (1) for \( n = 3 \).

Taking subsequently \( n = 4, 5, \ldots \), one obtains by the same arguments that the estimate (1) is valid for all \( n \), completing the proof of Theorem 1.
4. PROOF OF THEOREM 2

In view of the uniform convergence of $H^\infty$ functions on compact subsets of the unit disk, we have for the covering maps $\kappa_\rho : \mathbb{D} \to A_\rho$, that their derivatives $\kappa'_\rho(0)$ are convergent to $\kappa'_0(0) = 2/e < 1$ as $\rho \to 0$.

Taking $\rho < \rho_0$ so small that $\kappa'_\rho(0) < 1$, one can repeat the above arguments applied in the proof of Theorem 1 to the corresponding sets $H^p(\mathbb{D}, A_\rho)$ with such $\rho$.

But now we do not have the assertion on uniqueness of the covering map $\kappa_\rho$ on which the maximal value (3) is attained.

5. REMARK ON THE HUMMEL-SCHEINBERG-ZALCMAN CONJECTURE

The Krzyz conjecture was extended in 1977 by Hummel, Scheinberg and Zalcman to arbitrary Hardy spaces $H^p$, $p > 1$ on the unit disk, for which there is conjectured that the coefficients of nonvanishing functions $f(z) \in H^p$, $p > 1$, with $\|f\|_p \leq 1$ satisfy

$$|c_n| \leq \left(\frac{2}{e}\right)^{1-1/p},$$

with equality for the function

$$f_n(z) = \left[\frac{(1 + z^n)^2}{2}\right]^{1/p} \left[\exp\frac{z^n - 1}{z^n + 1}\right]^{1-1/p}$$

and its rotations (see [II]). As $p \to \infty$, this yields Krzyz’s conjecture for $H^\infty$ (without uniqueness of extremal functions).

This problem also has been investigated by many authors, but it still remains open. The only known results here are that the conjecture is true for $n = 1$ proved by Brown [4] as well as some results for special subclasses of $H^p$, see [4], [5], [21].

Some important intrinsic features of $H^\infty$ functions, essentially involved in the proof of Krzyz’s conjecture, are lost in $H^p$. However, the above arguments can be appropriately modified and completed to include also the Hummel-Scheinberg-Zalcman conjecture. This will be presented in a separate work.

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