Profinite completions of some groups acting on trees

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Abstract

We investigate the profinite completions of a certain family of groups acting on trees. It turns out that for some of the groups considered, the completions coincide with the closures of the groups in the full group of tree automorphisms. However, we introduce an infinite series of groups for which that is not so, and describe the kernels of natural homomorphisms of the profinite completions onto the aforementioned closures of respective groups.

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Introduction

A profinite group is just infinite if its every proper (continuous) quotient is finite; equivalently, if every closed normal subgroup is open. It is hereditarily just infinite if every open subgroup is just infinite. It is known, further, that any finitely generated profinite group which is virtually a pro-p group can be mapped onto a virtually pro-p just infinite group [5].

Another result from [5], based partly on the results of [14], states that any profinite just infinite group either contains an open normal subgroup which is isomorphic to the direct

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product of a finite number of copies of some hereditarily just infinite profinite group, or is a profinite branch group. The latter groups can be defined as profinite groups with a tree-like structure lattice of subnormal subgroups [15].

It is such groups that are the main subject of the present paper. More specifically, we start with certain just infinite self-similar groups which possess a branching structure in the sense mentioned above (a tree-like structure lattice of subnormal subgroups). Such a group $G$ can, in particular, be interpreted as a group acting on a regular rooted tree $T$.

At the start, we do not require $G$ to be profinite. Nevertheless, as a subset of $\text{Aut} T$, the full automorphism group of $T$, $G$ is equipped with the induced topology $\tau_1$. It is proven in [5] that the closure in $\text{Aut} T$ (i.e. the completion with respect to $\tau_1$) of any branch group is a profinite branch group. However, $G$ has its own profinite topology $\tau_2$. Then a very natural question to ask, and it is the first question considered in the paper, appears to be, do the two topologies coincide (for a given $G$)?

This question admits an equivalent form. It is easy to see that the level stabilizers of vertices of $T$ form a base of profinite topology in $\text{Aut} T$ (i.e. a system of neighborhoods of the identity). Define principal congruence subgroups in $G$ to be the intersections of those level stabilizers with $G$. Let us say that $G$ has congruence property if every finite index subgroup of $G$ is a congruence subgroup, i.e. if it contains some principal congruence subgroup. The two topologies coincide if and only if $G$ has congruence property [2], [6]. In the paper we establish that groups from a rather well-known class, $GGS$-groups [7, 8], do possess this property. This is done in Section 2.

The main part of the paper, however, is devoted to studying profinite completions of a certain class of self-similar $p$-groups without the congruence property. The existence of such groups is somewhat unexpected and answers negatively Question 3 of [1]. Once their existence is proven, however, the next interesting question is to investigate the natural homomorphism of the profinite completion of such a group onto its closure in $\text{Aut} T$. Namely, we describe the kernel of this homomorphism, which turns out to be a profinite abelian group of prime exponent. Section 3 is devoted to these questions.

The main results of the paper are stated in Theorems 2.1, 3.1, 3.3, and 3.4. The key steps of the prolonged proof of Theorem 3.4 are Lemma 3.5, Corollaries 3.6 and 3.4, and Proposition 3.2. All necessary definitions are given in Section 1.

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# Main definitions

A self-similar group is, roughly speaking, a group acting in a self-replicating manner on the set of all words in a finite alphabet. More precisely, choose a finite set $A$. Denote by $A^*$ the set of all words of finite length in alphabet $A$.

**Definition 1.1.** A group $G$ acting on $A^*$ is called *self-similar* if for every $a \in A$ and $g \in G$ there exist $b \in A$ and $h \in G$ such that

$$(aw)^g = b(w^h),$$

no matter what $w \in A^*$ we take.

For historical reasons, we prefer to speak about regular rooted trees and their vertices rather than about words. Indeed, the set of all words in a finite alphabet can be naturally identified with a rooted spherically homogeneous tree, where the words correspond to vertices, the root is the empty word, and two vertices are joined by an edge if and only if they have the form $a_1a_2\ldots a_n$ and $a_1a_2\ldots a_n a_{n+1}$ for some $n$ and some $a_i \in A$. The number $n$ is called the *length* of a vertex $u = a_1a_2\ldots a_n$ and is denoted by $|u|$. The set of all vertices of length $n$ is called the $n$th level of $T$.

Suppose that $u = \hat{a_1}\hat{a_2}\ldots \hat{a_n}$ is a vertex. The set of all vertices of the form $\hat{a_1}\hat{a_2}\ldots \hat{a_n} a_{n+1} a_{n+2} \ldots a_{n+m}$, where $m \in N$ and $a_{n+i}$ range over the set $A$, forms a subtree of $T$. We will denote that subtree by $T_u$. It is easy to see that $T_u$ is naturally isomorphic to the same tree $T$ via the map

$$\hat{a_1}\hat{a_2}\ldots \hat{a_n} a_{n+1} a_{n+2} \ldots a_{n+m} \mapsto a_{n+1} a_{n+2} \ldots a_{n+m}.$$ 

This map allows to identify subtrees $T_v$ for all vertices $v$, with one fixed tree $T$.

Consider now an arbitrary subgroup $G$ in $\text{Aut } T$ and a vertex $v$ of $T$. The *stabilizer* of $v$ in $G$ is the subgroup

$$\text{Stab}_G(v) = \{g \in G \mid v^g = v\}.$$ 

Denote also by $\text{Stab}_G(n)$ the subgroup $\cap_{|v|=n} \text{Stab}_G(v)$, which keeps all vertices of level $n$ fixed.

Subgroups $\text{Stab}_G(n)$ are called *principal congruence subgroups* in $G$. A subgroup of $G$ which contains a principal congruence subgroup is called a *congruence subgroup*.

**Definition 1.2.** A group $G \leq \text{Aut } T$ is said to possess the *congruence property* if any its finite index subgroup is a congruence subgroup.

It is easy to see that $\text{Aut } T$ admits a natural map $\phi : \text{Aut } T \to \text{Aut } T \wr \text{Sym}(A)$, where $\text{Sym}(A)$ is the group of all permutations of elements of $A$. Thus, every element $x$ of $\text{Aut } T$ is given by an element $f_x \in \text{Aut } T \times \ldots \times \text{Aut } T$ and a permutation $\pi_x \in \text{Sym}(A)$. The latter permutation is called the *accompanying permutation*, or the *activity*, of $x$ at the root. We write that

$$\phi(x) = f_x \cdot \pi_x.$$
In particular, the restriction of $\phi$ onto $\text{Stab}_{\text{Aut}_T}(1)$ is an embedding (in fact, an isomorphism) of $\text{Stab}_{\text{Aut}_T}(1)$ into the direct product of $|A|$ copies of $\text{Aut}_T$. We will denote this restriction by $\Phi_1$.

It is evident now that $\Phi_1(\text{Stab}_{\text{Aut}_T}(2)) = \underbrace{\text{Stab}_{\text{Aut}_T}(1) \times \ldots \times \text{Stab}_{\text{Aut}_T}(1)}_{|A|}$. Hence we can obtain an isomorphism

$$\Phi_2 = (\Phi_1 \times \ldots \times \Phi_1) \circ \Phi_1 : \text{Stab}_{\text{Aut}_T}(2) \rightarrow \underbrace{\text{Aut}_T \times \ldots \times \text{Aut}_T}_{|A|^2}.$$

Proceeding in this manner, we define for each positive integer $n$ an isomorphism

$$\Phi_n = (\Phi_{n-1} \times \ldots \times \Phi_{n-1}) \circ \Phi_1 : \text{Stab}_{\text{Aut}_T}(n) \rightarrow \underbrace{\text{Aut}_T \times \ldots \times \text{Aut}_T}_{|A|^n}.$$

The above notations allow us to introduce several modifications of the notion of self-similarity for groups acting on trees.

**Definition 1.3.** A group $G \leq \text{Aut}_T$ is called *recursive* if $\phi(G)$ is contained in $G \wr \text{Sym}(A)$ and the map $G \rightarrow G \wr \text{Sym}(A) \rightarrow \text{Sym}(A)$ is onto a transitive subgroup of $\text{Sym}(A)$ (the latter map is the projection).

Geometrically speaking, the latter condition means that $G$ acts transitively on each level of $T$.

**Definition 1.4.** A group $G \leq \text{Aut}_T$ is called *weakly recurrent* if it is recursive and the set-map $G \rightarrow G \wr \text{Sym}(A) \rightarrow G$ is onto for each coordinate.

**Definition 1.5.** A group $G \leq \text{Aut}_T$ is called *recurrent* if it is recursive and $\Phi_1(\text{Stab}_G(1))$ is subdirect product of $|A|$ copies of $G$, i.e. if $\text{Stab}_G(1) \rightarrow \underbrace{G \times \ldots \times G}_{|A|} \rightarrow G$ is onto for each coordinate (the latter map is the projection).

The proposition below readily follows from the above definition:

**Proposition 1.1.** Let $G \leq \text{Aut}_T$ be a recurrent group. Then for every $n \Phi_n(\text{Stab}_G(n))$ is a subdirect product of $|A|^n$ copies of $G$.

It remains to introduce a few more notations before proceeding to the subject matter. Notice that to every $x \in \text{Stab}_{\text{Aut}_T}(v)$ we can assign a unique automorphism $x_v \in \text{Aut}_T$ which is obtained by taking the restriction of $x$ onto the subtree $T_v$ (the notation $x@v$ is also used for $x_v$). Moreover, for each vertex $v$, $x_v$ can be considered as an element of $\text{Aut}_T$. Thus, for each vertex $v$ there is a fixed homomorphism $\varphi_v : \text{Stab}_{\text{Aut}_T}(v) \rightarrow \text{Aut}_T$. We will denote $\varphi_v(\text{Stab}_G(v))$ by $G_v$.

In general, for different vertices $u, v$ of the same length $G_u$ and $G_v$ are conjugate in $\text{Aut}_T$ but may not coincide. For recursive groups they are conjugate in $G$, and for recurrent groups
they coincide with $G$ (this readily follows from Proposition 1.1). The latter case holds for all examples considered in this paper.

The rigid stabilizer of $v$ in $G$ is the subgroup

$$\text{rist}_G(v) = \{ g \in G \mid \text{for any } u \in T \setminus T_v \ u^g = u \}.$$ 

We also denote by $\text{rist}_G(n)$ the subgroup $\prod_{|v|=n} \text{rist}_G(v)$. This is, of course, a normal subgroup in $G$ (unlike the rigid stabilizer of just one vertex). Given an element $g \in \text{rist}_G(v)$, we often denote by $g \ast v$ the element of $\text{rist}_G(v)$ such that $(g \ast v)_v = g$.

**Definition 1.6.** A group $G \leq \text{Aut} T$ is called branch if for all $n$ the index $|G : \text{rist}_G(n)|$ is finite.

Indeed, the tree-like structure lattice of subnormal subgroups mentioned in the Introduction, is given by various $\text{rist}_G(v)$.

Let us now describe our major examples.

### 1.1 Periodic GGS-groups

The term $GGS$-group was introduced by Gilbert Baumslag and refers to Rostislav Grigorchuk, Narain Gupta, and Said Sidki. Those groups act on regular $p$-trees, where $p \geq 3$. We will always assume that $p$ is prime.

$GGS$-groups are a partial case of a wider class of groups introduced in §2. Each $GGS$-group is fully defined by a vector

$$\bar{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_{p-1}) \in \mathbb{GF}(p) \oplus \mathbb{GF}(p) \oplus \ldots \oplus \mathbb{GF}(p)$$

(1.1)

in the following way.

Construct the regular $p$-tree $T_p$ as the tree over the sequence $(A, A, \ldots)$, $A = \{0, 1, 2, \ldots, p-1\}$. Each $GGS$-group $G_{\bar{\alpha}}$ acting on $T_p$ is generated by two automorphisms $a$ and $b$. The decomposition of $a$ is given by

$$\phi(a) = 1 \cdot \pi,$$

where $\pi$ is the cyclic permutation $(01 \ldots p-1)$. The second generator $b$ is in the stabilizer of level 1, and

$$\Phi_1(b) = (a^{\alpha_1}, a^{\alpha_2}, \ldots, a^{\alpha_{p-1}}, b).$$

The vector (1.1) is called the accompanying vector of $G_{\bar{\alpha}}$. The accompanying vector is called symmetric if $\alpha_i = \alpha_{p-i}$ for all $i = 1, 2, \ldots, \frac{p-1}{2}$. (Throughout the paper we will only deal with nonsymmetric accompanying vectors.)

It is immediately seen that each $G_{\bar{\alpha}}$ is recurrent. (Indeed, since it is evidently recursive, it is sufficient to verify conditions of Definition 1.5 for one coordinate only.) In particular, it implies that for each vertex $v$ $(G_{\bar{\alpha}})_v = G_{\bar{\alpha}}$. By §3, all groups $G_{\bar{\alpha}}$ are just infinite; as follows from §8 and §13, they are periodic if and only if $\sum_{i=1}^{p-1} \alpha_i = 0$.

By Proposition 1 from §9, we have
Lemma 1.1. Let $\bar{G}_\alpha$ be a GGS-group. Then
\[ \bar{G}_\alpha/[\bar{G}_\alpha, \bar{G}_\alpha] \cong \langle a \rangle \times \langle b \rangle \cong \mathbb{Z}_p \times \mathbb{Z}_p. \]

Since knowing the structure of rigid stabilizers is often crucial for studying groups acting on trees, we cite the following proposition proved in [10].

Proposition 1.2. If the accompanying vector of a periodic GGS-group $\bar{G}_\alpha$ is nonsymmetric, then for any vertex $u$ in $\Gamma_{\bar{G}_\alpha}(u)$ contains $[\bar{G}_\alpha, \bar{G}_\alpha]$.

We do not reproduce the full proof here, but it is important to notice that the proof is based on the lemma below, which is obtained as a partial case of Theorem 2.7.1 from [12].

Lemma 1.2. Let $\bar{G}_\alpha$ be a periodic GGS-group with nonsymmetric accompanying vector $\bar{\alpha} = (\alpha_1, \ldots, \alpha_{p-1})$. Then we have inclusion
\[ \Phi_1(\gamma_{\bar{G}_\alpha}) \geq [\bar{G}_\alpha, \bar{G}_\alpha] \times \cdots \times [\bar{G}_\alpha, \bar{G}_\alpha]. \quad (1.2) \]

Proposition 1.2 and the property of $\bar{G}_\alpha$ to be recurrent imply that $\bar{G}_\alpha$ is branch.

Later on we will need the following length function on $\bar{G}_\alpha$. Each element $x \in \bar{G}_\alpha$ can be represented (not uniquely) by a word of the form,
\[ a^{\delta_1} b^{\beta_1} \cdots a^{\delta_m} b^{\beta_m} a^{\delta_{m+1}}, \]
where $\delta_2, \ldots, \delta_m, \beta_1, \ldots, \beta_m \in \mathbb{GF}(p) \setminus \{0\}$ and $\delta_1, \delta_{m+1} \in \mathbb{GF}(p)$. The number $m$ will be called the length of such a word. The length $\partial_{\bar{G}_\alpha}(x)$ is defined as the minimal length of words representing $x$.

Also, denote by $\partial_a(x)$ the sum of all $\delta_i$ and by $\partial_b(x)$ the sum of all $\beta_i$ in some representation of $x$ (both sums are considered as elements of $\mathbb{GF}(p)$). By Lemma 1.1, the sums are independent of the choice of a particular representation, so $\partial_a(x)$ and $\partial_b(x)$ are well-defined.

1.2 EGS-groups

The term EGS-group stands for “extended Gupta-Sidki group”, but the word “extended” should not be understood in the usual algebraic sense. While each EGS-group contains some specific GGS-group as a subgroup, it is not an extension of it. However, in a certain sense a periodic GGS-group, or, more precisely, its accompanying vector, does define a unique EGS-group.

Let $\bar{G}_\alpha$ be a periodic GGS-group with generators $a$ and $b$ and accompanying vector $(\alpha_1, \alpha_2, \ldots, \alpha_{p-1})$. The corresponding EGS-group $\Gamma_{\bar{G}_\alpha}$ is generated by automorphisms $a$, $b$, and the automorphism $c$ such that
\[ \Phi_1(c) = (c, a^{\alpha_1}, a^{\alpha_2}, \ldots, a^{\alpha_{p-1}}). \]

It is immediately obvious that each $\Gamma_{\bar{G}_\alpha}$ is recurrent. In particular, it implies that for each vertex $v$ in $\Gamma_{\bar{G}_\alpha}$, $\Gamma_{\bar{G}_\alpha}(v) = \Gamma_{\bar{G}_\alpha}$.

The group $\bar{G}_\alpha$ is called the associated GGS-group of $\Gamma_{\bar{G}_\alpha}$. We also denote by $F_{\bar{G}_\alpha}$ the subgroup generated by $a$ and $c$ and prove the following lemma:
Lemma 1.3. Subgroups $G_{\bar{\alpha}}$ and $F_{\bar{\alpha}}$ are conjugate in $\text{Aut} \, T$.

Proof. It is easy to see that they are conjugated by the automorphism $C = af$, where $f$ is in $\text{Stab}_{\text{Aut} \, T}(1)$ and is defined by the equality,

$$\Phi_1(f) = (C, C, \ldots, C).$$

Indeed, we have that

$$a^C = a,$$

$$b^C = (b^C, (a^{\alpha_1})^C, \ldots, (a^{\alpha_{p-1}})^C) = (b^C, a^{\alpha_1}, \ldots, a^{\alpha_{p-1}}).$$

It follows from the latter equality that

$$b^C = c.$$

Remark 1.1. In many cases, it is possible to prove that these subgroups are not conjugate in $\Gamma_{\bar{\alpha}}$.

Let us establish a few other properties of $EGS$-groups. First of all, by Theorem 2 of [11], all $EGS$-groups are periodic. Also, we have

Lemma 1.4. Let $\Gamma_{\bar{\alpha}}$ be an $EGS$-group. Then

$$\Gamma_{\bar{\alpha}}/[\Gamma_{\bar{\alpha}}, \Gamma_{\bar{\alpha}}] \cong \langle a \rangle \times \langle b \rangle \times \langle c \rangle \cong \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p.$$ 

The proof of this lemma can be found in [10], as well as the proof of the following proposition.

Proposition 1.3. Let $\Gamma_{\bar{\alpha}}$ be an $EGS$-group with nonsymmetric accompanying vector. Then for any vertex $u \text{ rist}_{\Gamma_{\bar{\alpha}}}(u)_a$ contains $[\Gamma_{\bar{\alpha}}, \Gamma_{\bar{\alpha}}]$.

We again stress that the proof comes from the inclusion,

$$\Phi_1([\Gamma_{\bar{\alpha}}, \Gamma_{\bar{\alpha}}]) \supseteq \underbrace{[\Gamma_{\bar{\alpha}}, \Gamma_{\bar{\alpha}}] \times \ldots \times [\Gamma_{\bar{\alpha}}, \Gamma_{\bar{\alpha}}]}_p.$$  \hspace{1cm} (1.3)

Relying on this fact, we also obtain

Proposition 1.4. Let $\Gamma_{\bar{\alpha}}$ be an $EGS$-group with nonsymmetric accompanying vector. Then $\Gamma_{\bar{\alpha}}$ is just infinite.

Proof. Since $\Gamma_{\bar{\alpha}}$ is finitely generated periodic, then so is its derived subgroup. Hence the second derived subgroup has finite index. Then by Theorem 4 of [5] $\Gamma_{\bar{\alpha}}$ is just infinite. ■

By the same reasoning as with $G_{\bar{\alpha}}$, all $\Gamma_{\bar{\alpha}}$ are branch.
2 Profinite completions of GGS-groups

The main goal of this section is to prove that the profinite completions of GGS-groups coincide with their closures in Aut $T$ (the latter group is considered with the profinite topology). We do that by establishing that all periodic GGS-groups possess the congruence property. This is done by proving first that rigid stabilizers are congruence subgroups (a necessary condition always) and then showing that in fact every normal subgroup of groups in question has a “sufficiently large” intersection with some rigid stabilizer.

2.1 Normal subgroups

As we mentioned in Section 2.1 all GGS-groups are recurrent, i.e. for any vertex $u$ $\text{Stab}_{G_{\alpha}}(u)_{u}$ coincides with $G_{\alpha}$. Here we prove essentially that, with the exception of a finite number of vertices, a similar statement holds for any normal subgroup of $G_{\alpha}$.

Lemma 2.1. Let $G_{\alpha}$ be a periodic GGS-group. Then for any $x \in G_{\alpha}$ there is a vertex $u$ such that $\text{Stab}_{X}(u)_{u} \ni b$, where $X$ is the normal closure of $x$ in $G_{\alpha}$.

Proof. Let us prove the statement by induction on length $\partial_{G_{\alpha}}(x)$. The base of induction is $\partial_{G_{\alpha}}(x) \leq 1$.

The elements of length 0 are $a^{i}$, $i \in \text{GF}(p)$. We have

$$\Phi_{1}([a^{i}, b]) = (a^{-\alpha_{p-1}a_{i-1}}, a^{-\alpha_{p-2}a_{i-2}}, \ldots, a^{-\alpha_{p-1}}, a^{-1}, \ldots, a^{-\alpha_{p-1}})(a^{a_{i}}, a^{a_{2}}, \ldots, a^{a_{p-1}}, b) =$$

$$(a^{a_{i}}, a^{a_{2}}, \ldots, a^{-a_{p-1}}, b^{-1}a^{a_{i}}, a^{a_{i+1}}, \ldots, a^{-a_{p-1}}b).$$

If $\alpha_{i} = 0$ or $\alpha_{p-i} = 0$, then for either $u = i - 1$ or $u = p - i - 1$, respectively, we have $\varphi_{u}(X) \ni b$. If $\alpha_{i} \alpha_{p-i} \neq 0$, we proceed to case $\partial_{G_{\alpha}}(x) = 1$.

If $\partial_{G_{\alpha}}(x) = 1$, then $x = a^{i}b^{x}a^{j}$. Conjugating by $a^{i}$ if needed, we can assume that $x = b^{x}a^{j}$.

If $j = 0$, we have the desired fact right away. Suppose $j \neq 0$. But in this case $\varphi_{p-1}(x^{b}) = b^{x}$. Since $p$ is prime, this completes the base of induction.

Suppose now that $x$ is an element of length $m > 1$, and for all elements of smaller length the statement is already proven. If $x$ is in $\text{Stab}_{G_{\alpha}}(1)$, for any vertex $u$ of length 1 $\partial_{G_{\alpha}}(\varphi_{u}(x)) \leq \frac{\partial_{G_{\alpha}}(x)+1}{2} < \partial_{G_{\alpha}}(x)$. Since $G_{\alpha}$ is recurrent, we can apply the inductive assumption to any $\varphi_{u}(x)$.

Suppose that $x$ is not in $\text{Stab}_{G_{\alpha}}(1)$, i.e. $x = ya^{i}$ for some $y \in \text{Stab}_{G_{\alpha}}(1)$ and some $i \in \text{GF}(p)$. Denote $\Phi_{1}(y) = (y_{0}, y_{1}, \ldots, y_{p-1})$. Then we have

$$\Phi_{1}(x^{p}) = (y_{0}y_{\pi^{1}(0)} \ldots y_{\pi^{(p-1)}(0)}, y_{1}y_{\pi^{1}(1)} \ldots y_{\pi^{(p-1)}(1)}, \ldots, y_{p-1}y_{\pi^{1}(p-1)} \ldots y_{\pi^{(p-1)}(p-1)}),$$

where $\pi$ denotes the (nontrivial) accompanying permutation of $a$. As for any element of $\text{Stab}_{G_{\alpha}}(1)$, there is equality

$$\partial_{a}(y_{0}) + \ldots + \partial_{a}(y_{p-1}) = (\alpha_{1} + \ldots + \alpha_{p-1})(\partial_{b}(y_{0}) + \ldots + \partial_{b}(y_{p-1})) = 0.$$
Therefore, since for any \( i \in \mathbb{GF}(p) \) \( a^i \) is transitive on vertices of length 1, \( x^p \) is actually in \( \text{Stab}_{G_\bar{\alpha}}(2) \). On the other hand, for any vertex \( u \) of length 1 \( \partial_{G_\bar{\alpha}}(\varphi_u(x^p)) \leq \partial_{G_\bar{\alpha}}(x) \). Thus, for any vertex \( v \) of length 2 \( \partial_{G_\bar{\alpha}}(\varphi_v(x^p)) \leq \frac{\partial_{G_\bar{\alpha}}(x)+1}{2} < \partial_{G_\bar{\alpha}}(x) \). Again, since \( G_\bar{\alpha} \) is recurrent, we can apply the inductive assumption to any \( \varphi_v(x^p) \). ■

**Proposition 2.1.** Let \( G_\bar{\alpha} \) be a periodic GGS-group. Then for any \( x \in G_\bar{\alpha} \) there is a vertex \( u \) such that \( \text{Stab}_X(u)_a = G_\bar{\alpha} \).

*Proof.* By Lemma 2.1 there is a vertex \( u \) such that \( \text{Stab}_X(u)_a \supseteq b \). Since \( G_\bar{\alpha} \) is recurrent and \( \text{Stab}_X(u) \) is normal in \( G_\bar{\alpha} \), we have \( \text{Stab}_X(u)_a \supseteq b^i \) for any \( i \in \mathbb{GF}(p) \) as well. Thus, if \( v = u(p-1) \) is a vertex adjacent to \( u \), then \( \text{Stab}_X(v)_a \supseteq b, a^{\alpha_1}, a^{\alpha_2}, \ldots, a^{\alpha_{p-1}} \). Since among the numbers \( \alpha_1, \alpha_2, \ldots, \alpha_{p-1} \) there is a nontrivial one, \( \text{Stab}_X(v) = G_\bar{\alpha} \). ■

### 2.2 Congruence subgroups

Consider an arbitrary accompanying vector \( \bar{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_{p-1}) \) of a periodic GGS-group, i.e. one with \( \alpha_1 + \alpha_2 + \ldots + \alpha_{p-1} = 0 \). Denote by \( A_\bar{\alpha} \) the following matrix

\[
A_\bar{\alpha} = \begin{pmatrix}
0 & \alpha_{p-1} & \ldots & \alpha_1 \\
\alpha_1 & 0 & \ldots & \alpha_2 \\
\alpha_2 & \alpha_1 & \ldots & \alpha_3 \\
\vdots & \vdots & \ddots & \vdots \\
\alpha_{p-2} & \alpha_{p-3} & \ldots & \alpha_{p-1} \\
\alpha_{p-1} & \alpha_{p-2} & \ldots & 0
\end{pmatrix}.
\]

This matrix can be viewed as a linear transformation of the vector space \( \mathbb{GF}(p) \oplus \mathbb{GF}(p) \oplus \ldots \oplus \mathbb{GF}(p) \).

**Lemma 2.2.** If a vector \((\beta_0, \ldots, \beta_{p-1})\) is in the kernel of \( A_\bar{\alpha} \) then the sum of its coordinates is zero: \( \sum_{i=0}^{p-1} \beta_i = 0 \).

*Proof.* Notice that the rank of \( A_\bar{\alpha} \) is at least 1, i.e. the dimension of the kernel is no greater than \( p - 1 \). Consider a vector \((\beta_0, \ldots, \beta_{p-1})\) in the kernel. Since each row of \( A \) is obtained by a cyclic shift from any other row, the cyclic space generated by \((\beta_0, \ldots, \beta_{p-1})\) is in the kernel as well. On the other hand, the dimension of this cyclic space is equal to \( p - \deg \text{GCD}(x^p - 1, f(x)) \), where \( f(x) = \beta_0 + \beta_1 x + \ldots + \beta_{p-1} x^{p-1} \).

Since \( p \) is the characteristic of \( \mathbb{GF}(p) \), there is equality \( x^p - 1 = (x - 1)^p \). Suppose that \( \sum_{i=0}^{p-1} \beta_i \neq 0 \). Then 1 is not a root of \( f(x) \), hence \( \text{GCD}(x^p - 1, f(x)) = 1 \). Thus, the dimension of the cyclic space generated by \((\beta_0, \ldots, \beta_{p-1})\) is \( p \). But the dimension of the kernel, which contains this cyclic space, does not exceed \( p - 1 \). This contradiction proves that for any vector from the kernel the sum of its coordinates is equal to zero. ■

We now easily establish the following lemma.
Lemma 2.3. The derived subgroup of any periodic GGS-group \( G_\alpha \) contains principal congruence subgroup \( \text{Stab}_{G_\alpha}(2) \).

Proof. Consider an element \( x \in \text{Stab}_{G_\alpha}(1) \); denote \( \Phi_1(x) = (x_0, x_1, \ldots, x_{p-1}) \). Clearly, \( \partial_b(x) = \sum_{i=0}^{p-1} \partial_b(x_i) \), and vectors \( (\partial_b(x_0), \ldots, \partial_b(x_{p-1})) \) and \( (\partial_\alpha(x_0), \ldots, \partial_\alpha(x_{p-1})) \) are related by the following expression,

\[
\begin{pmatrix}
0 & \alpha_{p-1} & \ldots & \alpha_1 \\
\alpha_1 & 0 & \ldots & \alpha_2 \\
\alpha_2 & \alpha_1 & \ldots & \alpha_3 \\
\vdots & \vdots & \ddots & \vdots \\
\alpha_{p-2} & \alpha_{p-3} & \ldots & \alpha_{p-1} \\
\alpha_{p-1} & \alpha_{p-2} & \ldots & 0
\end{pmatrix}
\begin{pmatrix}
\partial_b(x_0) \\
\partial_b(x_1) \\
\partial_b(x_2) \\
\vdots \\
\partial_b(x_{p-2}) \\
\partial_b(x_{p-1})
\end{pmatrix} =
\begin{pmatrix}
\partial_\alpha(x_0) \\
\partial_\alpha(x_1) \\
\partial_\alpha(x_2) \\
\vdots \\
\partial_\alpha(x_{p-2}) \\
\partial_\alpha(x_{p-1})
\end{pmatrix}.
\]

For \( x \) to be in \( \text{Stab}_{G_\alpha}(2) \), it is necessary and sufficient that \( \partial_\alpha(x_i) = 0 \) for all \( i = 0, 1, \ldots, p-1 \). Thus, \( x \in \text{Stab}_{G_\alpha}(2) \) if and only if vector \( (\partial_b(x_0), \ldots, \partial_b(x_{p-1})) \) over \( \text{GF}(p) \) is in the kernel of \( A_\alpha \). Then by Lemma 2.2 the sum of coordinates of \( (\partial_b(x_0), \ldots, \partial_b(x_{p-1})) \) is equal to zero. Hence, \( \partial_b(x) = 0 \), which, given that \( x \) is in \( \text{Stab}_{G_\alpha}(1) \), implies \( x \in [G_\alpha, G_\alpha] \). □

Theorem 2.1. Any periodic GGS-group with a nonsymmetric accompanying vector has congruence property.

Proof. We need to prove that every finite index subgroup of a given \( G_\alpha \) is a congruence subgroup. This is equivalent to proving that every normal finite index subgroup is a congruence subgroup, and since all \( G_\alpha \) are just infinite, we can simply prove that the normal closure \( X \) of any element \( x \in G_\alpha \) is a congruence subgroup.

By Proposition 2.1 there is a vertex \( u \) such that \( \text{Stab}_X(u) = G_\alpha \). Notice that \( [\text{Stab}_X(u), \text{rist}_{G_\alpha}(u)] \leq \text{rist}_X(u) \). By Corollary 1.2 \( \text{rist}_{G_\alpha}(u) \geq [G_\alpha, G_\alpha] \). Hence, \( \text{rist}_X(u) \geq \varphi_u([\text{Stab}_X(u), \text{rist}_{G_\alpha}(u)]) \geq \gamma_3(G_\alpha) \). By Proposition 1.2 \( \gamma_3(G_\alpha) \geq \Phi_1^{-1}([G_\alpha, G_\alpha] \times \ldots \times [G_\alpha, G_\alpha]) \). Hence, there is inclusion

\[
X \geq \Phi_1^{-1}([G_\alpha, G_\alpha] \times \ldots \times [G_\alpha, G_\alpha]).
\]

However, by Lemma 2.3 \( [G_\alpha, G_\alpha] \) contains \( \text{Stab}_{G_\alpha}(2) \). Therefore, \( X \) contains \( \Phi_1^{-1}([\text{Stab}_{G_\alpha}(2) \times \ldots \times \text{Stab}_{G_\alpha}(2)]) = \text{Stab}_{G_\alpha}(|u| + 3) \). Theorem is proven. □

It follows from the above theorem and Proposition 2 of [6] that the closure \( \bar{G}_\alpha \) of \( G_\alpha \) in \( \text{Aut} T \) coincides with its profinite completion \( \hat{G}_\alpha \). Thus, we have the following description of profinite completions of periodic GGS-groups with nonsymmetric accompanying vectors.

Theorem 2.2. The profinite completion \( \hat{G}_\alpha \) of a periodic GGS-group with nonsymmetric accompanying vector is isomorphic to the projective limit of the following inverse systems of finite \( p \)-groups,

\[
1 \xleftarrow{\pi_1} G_\alpha / \text{Stab}_{G_\alpha}(1) \xleftarrow{\pi_2} G_\alpha / \text{Stab}_{G_\alpha}(2) \xleftarrow{\pi_3} \ldots \xleftarrow{\pi_{n-1}} G_\alpha / \text{Stab}_{G_\alpha}(n) \xleftarrow{\pi_{n+1}} \ldots,
\]
where $\pi_i$’s are the natural projections.

Remark 2.1. A similar theorem can be proven for periodic $GGS$-groups with symmetric accompanying vectors, see [10].

3 Completions and closures of $EGS$-groups

In this section we again treat $EGS$-groups with nonsymmetric accompanying vectors only, often not specifying that.

3.1 Absence of congruence property

Here we establish that $EGS$-groups do not have the congruence property.

Lemma 3.1. Let $\Gamma_\bar{\alpha}$ be an $EGS$-group with nonsymmetric accompanying vector. Then for any natural number $n$ there is an element $t_n$ of coset $b[\Gamma_\bar{\alpha},\Gamma_\bar{\alpha}]$ such that $t_n \equiv c(\mod Stab_{\Gamma_\bar{\alpha}}(n))$.

Proof. Put $t_1 = b$ and $t_2 = b^a$. Suppose $n > 1$ and an element $t_n$ with the required property has already been defined. Notice that $b^{-1}t_n$ is in $[\Gamma_\bar{\alpha},\Gamma_\bar{\alpha}]$. Then by Proposition 1.3 automorphism $x_{n+1} = \Phi_1^{-1}(b^{-1}t_n, 1, \ldots, 1)$ is in $\Gamma_\bar{\alpha}$. Put $t_{n+1} = b^a x_{n+1}$.

We have

$$\Phi_1(t_{n+1}) = (t_n, a^{a_1}, \ldots, a^{a_{p-1}}).$$

Thus,

$$\Phi_1(c^{-1}t_{n+1}) = (c^{-1}t_n, 1, \ldots, 1).$$

Since by assumption $c^{-1}t_n$ is in $Stab_{\Gamma_\bar{\alpha}}(n)$, $c^{-1}t_{n+1}$ is in $Stab_{\Gamma_\bar{\alpha}}(n+1)$.

Finally, it follows from the proof of Proposition 1.3 that $x_{n+1}$ is in $[\Gamma_\bar{\alpha},\Gamma_\bar{\alpha}]$. Therefore $t_{n+1}$ is in $b[\Gamma_\bar{\alpha},\Gamma_\bar{\alpha}]$.

Corollary 3.1. The derived subgroup $[\Gamma_\bar{\alpha},\Gamma_\bar{\alpha}]$ is not a congruence subgroup.

Proof. It follows from Lemma 1.3 that for every $n$ $c^{-1}t_n$ is in $Stab_{\Gamma_\bar{\alpha}}(n)$. However, by Lemma 1.4 it is not in $[\Gamma_\bar{\alpha},\Gamma_\bar{\alpha}]$. Therefore, $[\Gamma_\bar{\alpha},\Gamma_\bar{\alpha}]$ does not contain $Stab_{\Gamma_\bar{\alpha}}(n)$ for any $n$.

Thus, we obtain the following important result.

Theorem 3.1. No periodic $EGS$-group with nonsymmetric accompanying vector has congruence property.

This theorem suggests that for an $EGS$-group $\Gamma_\bar{\alpha}$, its profinite completion and its closure in $\text{Aut} T$ are distinct. However, there is a natural homomorphism of the former onto the latter, which is investigated in Section 3.3.
3.2 Description of the profinite completions

In this section we obtain a description of profinite completions of EGS-groups as projective limits of certain linearly ordered inverse systems of finite groups. Let us introduce a few extra notations first. Denote by \( \mathcal{H}_n, n \geq 1, \) the subgroup of \( \Gamma_\bar{\alpha} \) such that

\[
\Phi_n(\mathcal{H}_n) = [\Gamma_\bar{\alpha}, \Gamma_\bar{\alpha}] \times \ldots \times [\Gamma_\bar{\alpha}, \Gamma_\bar{\alpha}],
\]

by \( \mathcal{T}_n \) the subgroup of \( \Gamma_\bar{\alpha} \leq \Gamma_\bar{\alpha} \) given by

\[
\Phi_n(\mathcal{T}_n) = [G_\bar{\alpha}, G_\bar{\alpha}] \times \ldots \times [G_\bar{\alpha}, G_\bar{\alpha}],
\]

and by \( \mathcal{R}_n \) the subgroup of \( \Gamma_\bar{\alpha} \leq \Gamma_\bar{\alpha} \) given by

\[
\Phi_n(\mathcal{R}_n) = [F_\bar{\alpha}, F_\bar{\alpha}] \times \ldots \times [F_\bar{\alpha}, F_\bar{\alpha}].
\]

It is also natural to denote by \( \mathcal{H}_0, \mathcal{T}_0, \) and \( \mathcal{R}_0 \) subgroups \( [\Gamma_\bar{\alpha}, \Gamma_\bar{\alpha}], [G_\bar{\alpha}, G_\bar{\alpha}], \) and \( [F_\bar{\alpha}, F_\bar{\alpha}], \) respectively.

Subgroups of \( \text{Aut} T \mathcal{H}_n, \mathcal{T}_n, \mathcal{R}_n \) are contained in groups \( \Gamma_\bar{\alpha}, G_\bar{\alpha}, \) and \( F_\bar{\alpha}, \) respectively, by Propositions 1.3, 1.2 and Lemma 1.3. Notice that by formulas (1.3) and (1.2) and Lemma 1.3 these subgroups form descending chains,

\[
\mathcal{H}_0 \geq \mathcal{H}_1 \geq \mathcal{H}_2 \geq \ldots,
\]

\[
\mathcal{T}_0 \geq \mathcal{T}_1 \geq \mathcal{T}_2 \geq \ldots,
\]

\[
\mathcal{R}_0 \geq \mathcal{R}_1 \geq \mathcal{R}_2 \geq \ldots.
\]

The main result of this subsection is based on the following fact.

**Theorem 3.2.** Let \( \Gamma_\bar{\alpha} \) be an EGS-group with nonsymmetric accompanying vector. Then for every normal subgroup \( N \) in \( \Gamma_\bar{\alpha} \) there is \( n \) such that \( N \) contains \( \mathcal{H}_n. \)

The proof of this theorem is decomposed into several easy steps. It generally follows the same scheme of reasoning as the proof of Theorem 2.1.

**Lemma 3.2.** The normal closure in \( \Gamma_\bar{\alpha} \) of the subgroup \( [G_\bar{\alpha}, G_\bar{\alpha}] \) contains \( \mathcal{H}_1. \)

**Proof.** Notice that \( [G_\bar{\alpha}, G_\bar{\alpha}] \) contains all subgroups \( \mathcal{T}_n. \) Choose an \( i \) such that \( \alpha_1 \neq \alpha_{p-i}, \) which is possible to do because the accompanying vector is not symmetric, hence not constant. Now by direct calculation

\[
[[a^{i+1}, b], c] = \Phi_1^{-1}(a_{\alpha_1-\alpha_{p-i}} c, 1, \ldots, 1, b^{-1} a^{\alpha_{i+1}} a^{\alpha_i} 1, \ldots, 1, a^{-\alpha_{p-i-1}} b, a^{\alpha_{p-i-1}}) \equiv \\
\equiv \Phi_1^{-1}(a_{\alpha_1-\alpha_{p-i}} c, 1, \ldots, 1)(\mod \mathcal{T}_1).
\]

The latter equality means that the normal closure of \( [G_\bar{\alpha}, G_\bar{\alpha}] \) in \( \Gamma_\bar{\alpha} \) contains \( \mathcal{R}_1. \)
Now we have

\[
[[a, b], c] = \Phi^{-1}_1([b^{-1}a^{\alpha_1}, c], 1, \ldots, 1, [a^{-\alpha_{p-1}}b, a^{\alpha_{p-1}}]) \equiv \Phi^{-1}_1([b^{-1}a^{\alpha_1}, c], 1, \ldots, 1) \pmod{\mathcal{T}_i} \equiv \\
\equiv \Phi^{-1}_1((b^{-1}, c)^{a^{\alpha_1}}, 1, \ldots, 1) \pmod{\mathcal{R}_i}.
\]

Now, since \(\Gamma_{\bar{a}}\) is generated by \(a, b, c\), \([\Gamma_{\bar{a}}, \Gamma_{\bar{a}}]\) is the normal closure in \(\Gamma_{\bar{a}}\) of elements \([a, b], [a, c], [b, c]\). Thus, since \(\Gamma_{\bar{a}}\) is recurrent, the two equivalences above imply the statement of the lemma. ■

It is easy to obtain a similar statement for \([F_{\bar{a}}, F_{\bar{a}}]\) as well:

**Lemma 3.3.** The normal closure in \(\Gamma_{\bar{a}}\) of the subgroup \([F_{\bar{a}}, F_{\bar{a}}]\) contains \(\mathcal{H}_1\).

**Proof.** Notice that \([F_{\bar{a}}, F_{\bar{a}}]\) contains all subgroups \(\mathcal{R}_n\). Since the accompanying vector is nonsymmetric and therefore non constant, there is \(i\) such that \(\alpha_i \neq \alpha_{i+1}\). We have that

\[
[[a, c], b^{\alpha_i+1}] = \Phi^{-1}_1(1, \ldots, 1, [a^{\alpha_i-\alpha_{i+1}}, b], 1, \ldots, 1).
\]

Hence the normal closure of \([F_{\bar{a}}, F_{\bar{a}}]\) contains \(\mathcal{T}_i\). On the other hand, we have

\[
[[a, c], b^a] = \Phi^{-1}_1([a^{-\alpha_{p-1}}c, b], [c^{-1}a^{\alpha_1}, a^{\alpha_1}], 1, \ldots, 1) \equiv \\
\equiv \Phi^{-1}_1([a^{-\alpha_{p-1}}c, b], 1, \ldots, 1) \pmod{\mathcal{R}_1} \equiv \Phi^{-1}_1([c, b], 1, \ldots, 1) \pmod{\mathcal{T}_i^c}.
\]

Now the desired conclusion follows at once. ■

Combining Lemmas 3.2, 3.3 and definitions of subgroups \(\mathcal{T}_n, \mathcal{R}_n\), we get the following corollary.

**Corollary 3.2.** The normal closures in \(\Gamma_{\bar{a}}\) of subgroups \(\mathcal{T}_n\) and \(\mathcal{R}_n\) contain \(\mathcal{H}_{n+1}\).

Corollary 3.2 will allow us to easily deal with the normal subgroups contained in either \(G_{\bar{a}}\) or \(F_{\bar{a}}\). For remaining ones, we need another preparatory statement.

**Lemma 3.4.** Let \(X\) be the normal closure in \(\Gamma_{\bar{a}}\) of an element \(x \in \Gamma_{\bar{a}} \setminus (G_{\bar{a}} \cup F_{\bar{a}})\). Then there exists a vertex \(u\) such that \(\text{Stab}_X(u)\) contains at least one of \(b, c\).

**Proof.** Let us prove the statement by induction on \(\partial(x)\). Since \(x\) is not in \(G_{\bar{a}}\) or \(F_{\bar{a}}\), its length is at least 2. This is the base of the induction.

If \(\partial(x) = 2\) then, up to conjugation, \(x = a^\alpha b^\beta a^\delta c^\gamma\), where \(\alpha, \delta\) can be equal to 0. If \(\alpha + \delta = 0\) then we have two cases. One is \(\alpha = -1\) and \(\beta = -\gamma\). In this case \(\Phi_1(x) = (b^{-\gamma}c^\gamma, 1, \ldots, 1)\), and we pass to the second case for \(x = b^{-\gamma}c^\gamma\).

The second case is \(\alpha \neq -1\) or \(\beta \neq -\gamma\). In this case there is a vertex of length 1 such that \(x_u\) is a nontrivial element of either \(G_{\bar{a}}\) or \(F_{\bar{a}}\), and the conclusion follows from Lemma 2.1 or from it and Lemma 1.3. If \(\alpha + \delta \neq 0\) then for any \(u\) of length 1 \((x^\mu)_u\) is, up to conjugation, either \(a^{-\mu}b^\beta a^\mu c^\gamma\) or \(a^{-\mu}c^\gamma a^\mu b^\beta\) (\(\mu\) could be zero), and the same reasoning applies.
Suppose now that for all elements of length $\leq n$ the statement is proven, and consider $x$ of length $n+1$. If $x$ is in $\text{Stab}_{\Gamma_{\tilde{a}}}(1)$ then for any vertex $u$ of length 1 $x_u$ has length $\leq \frac{\partial(x)+1}{2} < \partial(x)$, and for $x_u$ the statement is true either by the inductive assumption (if it is not in $G_{\tilde{a}} \cup F_{\tilde{a}}$) or by Lemmas 2.1 and 1.3 (if it is). If $x$ is not in $\text{Stab}_{\Gamma_{\tilde{a}}}(1)$ then, by periodicity, $x^p$ is, and $\partial(x^p) \leq \partial(x)$. Thus, the inductive assumption or Lemmas 2.1, 1.3 can be applied to $(x^p)_u$ for any $u$ of length 2. ■

**Proof of Theorem 3.2** Obviously it is sufficient to prove the theorem for normal closures of all elements $x \in \Gamma_{\tilde{a}}$. Several cases are possible depending on whether $x$ is in $G_{\tilde{a}}$, in $F_{\tilde{a}}$, or is not in either of them.

**Case** $x \in G_{\tilde{a}}$. Then by Theorem 2.1 there is $n$ such that $X \geq T_n$. Now by Corollary 3.2 $X$ contains $\mathcal{H}_{n+1}$.

**Case** $x \in F_{\tilde{a}}$. Then by Theorem 2.1 and Lemma 1.3 there is $n$ such that $X \geq R_n$. By Corollary 3.2 $X$ contains $\mathcal{H}_{n+1}$.

**Case** $x \in \Gamma_{\tilde{a}} \setminus (G_{\tilde{a}} \cup F_{\tilde{a}})$. Then by Lemma 3.3 there is a vertex $u$ such that $\text{Stab}_{X}(u)_u$ contains either $b$ or $c$. Suppose it contains $b$. Then it follows from the proof of Proposition 2.1 that there is a vertex $v$ (in fact, it is one of the vertices adjacent to $u$) such that $\text{Stab}_{X}(v)_v$ contains $G_{\tilde{a}}$. Applying the same reasoning as in the proof of Theorem 2.1 we get that $X$ contains $T_{|v|+1}$. Hence by Corollary 3.2 $X$ contains $\mathcal{H}_{|v|+2}$. In case when $\text{Stab}_{X}(u)_u$ contains $c$ and not $b$, all similar arguments, with respective substitutions, apply. ■

An immediate corollary of the previous theorem is the following one.

**Theorem 3.3.** The profinite completion $\hat{\Gamma}_{\tilde{a}}$ is the projective limit of the following inverse system of finite $p$-groups:

$$1 \leftarrow \varepsilon_1 \Gamma_{\tilde{a}}/\mathcal{H}_1 \leftarrow \varepsilon_2 \Gamma_{\tilde{a}}/\mathcal{H}_2 \leftarrow \varepsilon_3 \Gamma_{\tilde{a}}/\mathcal{H}_3 \leftarrow \cdots \leftarrow \varepsilon_{n} \Gamma_{\tilde{a}}/\mathcal{H}_n \leftarrow \varepsilon_{n+1} \Gamma_{\tilde{a}}/\mathcal{H}_{n+1} \leftarrow \cdots,$$

where $\varepsilon_i$ are the natural projections.

### 3.3 The kernel of the natural homomorphism

**The statement of the problem and some notations** By definition of the projective limit, the group $\hat{\Gamma}_{\tilde{a}}$ is endowed with the canonical homomorphisms $\theta_n : \hat{\Gamma}_{\tilde{a}} \to \Gamma_{\tilde{a}}/\mathcal{H}_n$ such that $\theta_{n-1} = \varepsilon_n \circ \theta_n$. The group $\Gamma_{\tilde{a}}$ is also equipped with canonical projections $\pi_{\mathcal{H}_n}$ onto its factor-group $\Gamma_{\tilde{a}}/\mathcal{H}_n$, and the projections satisfy the property of $\pi_{\mathcal{H}_{n-1}} = \varepsilon_n \circ \pi_{\mathcal{H}_n}$. Hence there is a uniquely defined homomorphism $\hat{\sigma} : \Gamma_{\tilde{a}} \to \hat{\Gamma}_{\tilde{a}}$ with the property that $\pi_{\mathcal{H}_n} = \theta_n \circ \hat{\sigma}$. The kernel of this homomorphism is the intersection $\cap_{i=1}^{\infty} \mathcal{H}_i$. Since this intersection is trivial, $\hat{\sigma}$ is an inclusion.

Since every finite index subgroup of $\text{Aut } T$ contains $\text{Stab}_{\text{Aut } T}(n)$ for some $n$, the closure $\bar{\Gamma}_{\tilde{a}}$ of $\Gamma_{\tilde{a}}$ in $\text{Aut } T$ considered with the profinite topology, is the projective limit of the
Similarly, the group $\bar{\Gamma}$ is endowed with the canonical homomorphisms $\vartheta_n : \bar{\Gamma} \to \Gamma/\text{Stab}_{\alpha}(n)$ such that $\vartheta_{n-1} = \eta_n \circ \vartheta_n$. There is, too, a unique homomorphism $\bar{\sigma} : \bar{\Gamma} \to \bar{\Gamma}$ with the property that $\pi_{\text{Stab}_{\alpha}(n)}(\vartheta_n) = \vartheta_n \circ \bar{\sigma}$ and the trivial kernel $\cap_{i=1}^{\infty} \text{Stab}_{\Gamma}(i)$.

Finally, there are natural projections $\varphi_n : \bar{\Gamma}/\mathcal{H}_n \to \bar{\Gamma}/\text{Stab}_{\alpha}(n)$ such that there is a commutative diagram
\[
\begin{array}{ccc}
\Gamma/\mathcal{H}_{n-1} & \xleftarrow{\varepsilon_n} & \Gamma/\mathcal{H}_n \\
\varphi_{n-1} \downarrow & & \downarrow \varphi_n \\
\Gamma/\text{Stab}_{\alpha}(n-1) & \xleftarrow{\eta_n} & \Gamma/\text{Stab}_{\alpha}(n).
\end{array}
\]

Therefore the set of covering homomorphisms $\varphi_n \circ \vartheta_n$ allows to define a unique homomorphism $\sigma : \hat{\Gamma} \to \bar{\Gamma}$ such that $\varphi_n \circ \vartheta_n = \vartheta_n \circ \sigma$ for all $n$. This homomorphism, or rather, its kernel, is the main object of study in this section.

It is useful to remember that the three described homomorphisms are related by the equality $\sigma \circ \bar{\sigma} = \bar{\sigma}$.

**One presentation of elements of the kernel** The main goal of this section is to obtain a description of $\text{Ker} \sigma$. Since $\hat{\Gamma}$ is a completion of $\bar{\Gamma}$, for every $x \in \hat{\Gamma}$ there exists a sequence $\{g_n\}_{n=1}^{\infty}$, $g_n \in \bar{\Gamma}$, such that the sequence $\hat{\vartheta}(g_n)$ is converging and its limit is equal to $x$. Recall that $\hat{\vartheta}(g_n)$ is converging if and only if for every $N$ there is $n_N$ such that for all $n \geq n_N$ $g_n \mathcal{H}_N = g_{n_N} \mathcal{H}_N$. Likewise, for every $y \in \hat{\Gamma}$ there exists a sequence $\{h_n\}_{n=1}^{\infty}$, $h_n \in \bar{\Gamma}$, such that the sequence $\hat{\vartheta}(h_n)$ is converging and its limit is equal to $y$. Here $\hat{\vartheta}(h_n)$ is converging if and only if for every $N$ there is $n_N$ such that for all $n \geq n_N$ $h_n \text{Stab}_{\alpha}(N) = h_{n_N} \text{Stab}_{\alpha}(N)$.

Thus, $x = \lim_{n \to \infty} \hat{\vartheta}(g_n)$ is in $\text{Ker} \sigma$ if and only if the following two conditions hold:

1. for every $N$ there exists $n_N$ such that for all $n \geq n_N$ $g_n \in g_{n_N} \mathcal{H}_N$;

2. for every $N$ there exists $m_N$ such that for all $m \geq m_N$ $g_m \in \text{Stab}_{\alpha}(N)$.

Replacing $n_N$ with $\max\{n_N, m_N+1\}$, we get the following condition:

$\forall N$ there is $n_N$ such that $g_n \in g_{n_N} \mathcal{H}_N \cap \text{Stab}_{\alpha}(N + 1)$ for all $n$ greater than $n_N$.

Since we can assume that $n_{N-1} > n_N$, this condition implies that $g_n \in (\bigcap_{i=1}^{N} g_i \mathcal{H}_i) \cap \text{Stab}_{\alpha}(N + 1)$. Thus, we can extract a subsequence $\{g_{n_i}\}_{i=1}^{\infty}$ such that for every $N$ $g_{n_{N+1}} \in (\bigcap_{i=1}^{N} g_i \mathcal{H}_i) \cap \text{Stab}_{\alpha}(N + 1)$. Since a profinite group is Hausdorff as a topological space, the limit of a converging sequence coincides with the limit of every its subsequence. Therefore $\text{Ker} \sigma$ consists of limits of all converging sequences $\{\hat{\vartheta}(g_n)\}$, where sequence $\{g_n\}_{n=1}^{\infty}$ is such that

\[
g_n \in \bigcap_{i=1}^{n-1} g_i \mathcal{H}_i \cap \text{Stab}_{\alpha}(n),
\]

and we only need to consider sequences satisfying (3.1). More precisely, we have the following statement.
Lemma 3.5. For every sequence \( \{g_n\} \) such that \( g_n \in \cap_{i=1}^{n-1} g_i H_i \cap \text{Stab}_{\Gamma_\alpha}(n) \) for all \( n \), the limit \( \lim_{n \to \infty} \hat{\sigma}(g_n) \) exists and is in \( \text{Ker} \sigma \). And vice versa, for every \( x \in \text{Ker} \sigma \) there exists a sequence \( \{g_n\} \) satisfying condition \( g_n \in \cap_{i=1}^{n-1} g_i H_i \cap \text{Stab}_{\Gamma_\alpha}(n) \) and such that \( x = \lim_{n \to \infty} \hat{\sigma}(g_n) \).

Proof. The second statement of the lemma has just been proven. Let us prove the first one. Obviously, if a sequence \( \{\hat{\sigma}(g_n)\} \), where \( g_n \) are of the type described in Lemma, converges then its limit is in \( \text{Ker} \sigma \). Now the only argument that we need to ensure the existence of limit \( \lim_{n \to \infty} \hat{\sigma}(g_n), \) is the following evident implication: if \( g_n \in g_{n-1}H_{n-1} \) then \( g_nH_n \subset g_{n-1}H_{n-1} \).

Indeed, then \( g_{n+1} \in g_nH_n \subset g_{n-1}H_{n-1} \), and by induction we get that \( g_{n+k} \in g_{n+k}H_{n+k} \subset g_{n-1}H_{n-1} \) for all \( n \geq 1 \) and for all \( k \geq 0 \). The latter statement is just a reformulation of the convergence condition in \( \hat{\Gamma}_\alpha \). ■

We can now restrict our attention to considering only sequences satisfying (3.1). Which sequences are those? is the most important question now. For instance, an immediate corollary of (3.1) is that for every sequence \( \{g_n\} \) satisfying it the coset \( g_nH_n \) has non-empty intersection with all stabilizers \( \text{Stab}_{\Gamma_\alpha}(m), m \geq n \).

Definition 3.1. We say that a coset \( xH_n \) satisfies the kernel condition if it has non-empty intersection with all stabilizers \( \text{Stab}_{\Gamma_\alpha}(m), m \geq n \).

By Lemma 3.5, every element of \( \text{Ker} \sigma \) can be presented as the limit of some converging sequence \( \{\hat{\sigma}(g_n)\} \), where all \( g_n \) satisfy the kernel condition. Therefore we need to investigate precisely which cosets do satisfy it.

Cosets satisfying the kernel condition We start with the case \( n = 0 \), from which all other cases easily follow.

Lemma 3.6. A coset \( x[\Gamma_\alpha, \Gamma_\alpha] \) satisfies the kernel condition if and only if for some integer \( i \) \( x \equiv (c^{-1}b)^i \pmod{[\Gamma_\alpha, \Gamma_\alpha]} \).

Proof. It follows from Lemma 3.5 that the coset \( c^{-1}b[\Gamma_\alpha, \Gamma_\alpha] \) has non-empty intersection with \( \text{Stab}_{\Gamma_\alpha}(m) \) for all \( m \). Hence all cosets of the form \( (c^{-1}b)^i[\Gamma_\alpha, \Gamma_\alpha] \) have non-empty intersection with \( \text{Stab}_{\Gamma_\alpha}(m) \).

On the other hand, cosets of the form \( a^\alpha b^\beta c^\gamma [\Gamma_\alpha, \Gamma_\alpha] \) with \( \alpha \neq 0 \) obviously have empty intersection with any stabilizer. Consider a coset of the form, \( b^\beta c^\gamma [\Gamma_\alpha, \Gamma_\alpha] \). An arbitrary element of this coset has the form, \( b^\beta c^\gamma g \) for \( g \in [\Gamma_\alpha, \Gamma_\alpha] \). Denoting \( \Phi_1(g) = (g_0, \ldots, g_{p-1}) \) and \( \Phi_1(b^\beta c^\gamma g) = (x_0, \ldots, x_{p-1}) \), we have that

\[
\begin{pmatrix}
\partial_a(x_0) \\
\partial_a(x_1) \\
\vdots \\
\partial_a(x_{p-2}) \\
\partial_a(x_{p-1})
\end{pmatrix}
= \beta
\begin{pmatrix}
\alpha_1 \\
\alpha_2 \\
\vdots \\
\alpha_{p-1} \\
0
\end{pmatrix}
+ \gamma
\begin{pmatrix}
0 \\
\alpha_1 \\
\vdots \\
\alpha_{p-1}
\end{pmatrix}
+ \begin{pmatrix}
\partial_a(g_0) \\
\partial_a(g_1) \\
\vdots \\
\partial_a(g_{p-2}) \\
\partial_a(g_{p-1})
\end{pmatrix}.
\]
Now it is easy to show (see also the proof of Lemma 2.2) that

\[
\begin{pmatrix}
\partial_a(g_0) \\
\partial_a(g_1) \\
\vdots \\
\partial_a(g_{p-2}) \\
\partial_a(g_{p-1})
\end{pmatrix}
= \begin{pmatrix}
0 & \alpha_{p-1} & \ldots & \alpha_1 \\
\alpha_1 & 0 & \ldots & \alpha_2 \\
\alpha_2 & \alpha_1 & \ldots & \alpha_3 \\
\vdots & \vdots & \ddots & \vdots \\
\alpha_{p-2} & \alpha_{p-3} & \ldots & \alpha_{p-1} \\
\alpha_{p-1} & \alpha_{p-2} & \ldots & 0
\end{pmatrix}
\begin{pmatrix}
\partial_b(g_0) + \partial_c(g_0) \\
\partial_b(g_1) + \partial_c(g_1) \\
\vdots \\
\partial_b(g_{p-2}) + \partial_c(g_{p-2}) \\
\partial_b(g_{p-1}) + \partial_c(g_{p-1})
\end{pmatrix}.
\]

The matrix participating in the above equality is precisely \( A_\alpha \) from Section 2.2. Notice that

\[
\beta = \begin{pmatrix}
\alpha_1 \\
\alpha_2 \\
\vdots \\
\alpha_{p-1} \\
0
\end{pmatrix}
= A_\alpha \begin{pmatrix}
0 \\
0 \\
\vdots \\
0
\end{pmatrix},
\gamma = \begin{pmatrix}
0 \\
\alpha_1 \\
\vdots \\
\alpha_{p-2} \\
\alpha_{p-1}
\end{pmatrix}
= A_\alpha \begin{pmatrix}
0 \\
0 \\
\vdots \\
0
\end{pmatrix},
\]

so the equality (4.2) can be rewritten as

\[
\begin{pmatrix}
\partial_a(x_0) \\
\partial_a(x_1) \\
\vdots \\
\partial_a(x_{p-2}) \\
\partial_a(x_{p-1})
\end{pmatrix}
= A_\alpha \begin{pmatrix}
\gamma + \partial_b(g_0) + \partial_c(g_0) \\
\partial_b(g_1) + \partial_c(g_1) \\
\vdots \\
\partial_b(g_{p-2}) + \partial_c(g_{p-2}) \\
\beta + \partial_b(g_{p-1}) + \partial_c(g_{p-1})
\end{pmatrix}.
\]

It follows that if \( b^\beta c^\gamma g \in \text{Stab}_{\Gamma_\alpha}(2) \) then vector \((\gamma + \partial_b(g_0) + \partial_c(g_0), \partial_b(g_1), \ldots, \partial_b(g_{p-2}) + \partial_c(g_{p-2}), \beta + \partial_b(g_{p-1}) + \partial_c(g_{p-1}))\) is in the kernel of the linear transformation \( A_\alpha \). By Lemma 2.2 for every vector from that kernel, the sum of its coordinates should be equal to zero. On the other hand, the sum of coordinates of our vector is

\[
\beta + \gamma + (\partial_b(g_0) + \ldots + \partial_b(g_{p-1})) + (\partial_c(g_0) + \ldots + \partial_c(g_{p-1})) = \beta + \gamma + \partial_b(g) + \partial_c(g) = \beta + \gamma,
\]

because \( g \) is in \([\Gamma_\alpha, \Gamma_\alpha]\). Thus, an element \( b^\beta c^\gamma g \) is in \( \text{Stab}_{\Gamma_\alpha}(2) \) if and only if \( \beta + \gamma = 0 \). In particular, if \( \beta + \gamma \neq 0 \) then the intersection of coset \( b^\beta c^\gamma [\Gamma_\alpha, \Gamma_\alpha] \) with \( \text{Stab}_{\Gamma_\alpha}(2) \) is empty. ■

Notice that for every vertex \( v \) there exists a unique element \( x \in \text{rist}_{\Gamma_\alpha}(v) \) such that \( x_v = c^{-1}b \). Recall that we can denote such \( x \) by \((c^{-1}b) * v\). Let \( CH_n \) denote the subgroup generated by all \((c^{-1}b) * v\) for all \( v \) of length \( n \),

\[
CH_n = \langle (c^{-1}b) * v : |v| = n \rangle.
\]

Notice that all \( CH_n \) are abelian.
Proposition 3.1. A coset \( x\mathcal{H}_n \) satisfies the kernel condition if and only if there exists an element \( y \in \mathcal{CH}_n \) such that \( x \equiv y \pmod{\mathcal{H}_n} \).

Proof. If \( x\mathcal{H}_n \) satisfies the kernel condition then \( x \in \text{Stab}_{\Gamma_\alpha}(n) \). Consider a vertex \( v \) of length \( n \). Obviously, the coset \( x_v[\Gamma_\alpha, \Gamma_\alpha] \) must satisfy the kernel condition. Hence for each \( x_v \) there is \( y_v \in \mathcal{CH}_0 = \langle c^{-1}b \rangle \) such that \( x_v \equiv y_v \pmod{\Gamma_\alpha, \Gamma_\alpha} \). Thus,

\[
x \equiv \prod_{|v|=n} (y_v * v) \in \mathcal{CH}_n \pmod{\mathcal{H}_n}.
\]

Denoting the element on the righthand side by \( y \), we get the desired statement. \( \blacksquare \)

"Canonical" sequences It follows from Proposition 3.1 that we only have to consider sequences \( \{g_n\} \) such that for all \( n \) and for some \( y_n \in \mathcal{CH}_n \), \( g_n \equiv y_n \pmod{\mathcal{H}_n} \). More precisely, we have the following easy fact.

Lemma 3.7. Suppose that \( \{g_n\}, \{y_n\} \) are two sequences such that sequences \( \{\hat{\sigma}(g_n)\}, \{\hat{\sigma}(y_n)\} \) converge, and for all \( n \), \( g_n \equiv y_n \pmod{\mathcal{H}_n} \). Then

\[
\lim_{n \to \infty} \hat{\sigma}(g_n) = \lim_{n \to \infty} \hat{\sigma}(y_n).
\]

Proof. The condition \( g_n \equiv y_n \pmod{\mathcal{H}_n} \) for all \( n \) means that, for every \( N \) and for all \( n \geq N \), \( g_n^{-1}y_n \in \mathcal{H}_n \subset \mathcal{H}_N \). This means that \( \lim_{n \to \infty} \hat{\sigma}(g_n^{-1}y_n) \) is the trivial element of \( \hat{\Gamma_\alpha} \). Since sequences \( \{\hat{\sigma}(g_n)\}, \{\hat{\sigma}(y_n)\} \) converge, this is equivalent to the statement of the lemma. \( \blacksquare \)

Hence we are only interested in sequences \( \{g_n\} \) such that for all \( n \), \( g_n \in \mathcal{CH}_n \). More precisely,

Corollary 3.3. For every element \( x \) of \( \text{Ker} \sigma \) there exists a sequence \( \{g_n\} \) such that \( g_n \in \mathcal{CH}_n \) for all \( n \), sequence \( \{\hat{\sigma}(g_n)\} \) converges, and \( x = \lim_{n \to \infty} \hat{\sigma}(g_n) \). Vice versa, if a sequence \( \{\hat{\sigma}(g_n)\} \) such that \( g_n \in \mathcal{CH}_n \) for all \( n \), converges, then its limit is in \( \text{Ker} \sigma \).

The proof of the corollary follows immediately from Lemma 3.5, Proposition 3.1, and the definition of the kernel condition. Due to the corollary, we now only need to establish when a sequence of the type described in Corollary 3.3 converges, and when two such sequences have the same limit.

Convergence of canonical sequences Although that is not a necessary condition of convergence, due to Lemma 3.5 we can confine ourselves to considering only sequences satisfying (3.1). Since all \( \mathcal{CH}_n \) are in \( \text{Stab}_{\Gamma_\alpha}(n) \) already, we just need to ensure that for all \( n \), \( g_n \) be in \( \bigcap_{i=1}^{n-1} g_i \mathcal{H}_i \). In particular, we want this intersection to be non-empty. Recall also the previously mentioned (Lemma 3.5) fact that

\[
g_n \in \bigcap_{i=1}^{n-1} g_i \mathcal{H}_i \iff g_n \mathcal{H}_n \subset \bigcap_{i=1}^{n-1} g_i \mathcal{H}_i.
\]
Lemma 3.8. Let \( u, v \) be vertices of \( T \) such that \( u \leq v \) and \( n = |u| = |v| - 1 \). Then \((c^{-1}b)^{u}H_{n} \supset ((c^{-1}b)^{v})H_{n+1}\).

Proof. We have, for a suitably chosen \( i \),
\[
((c^{-1}b)^{v})_{u} = (c^{-1}b[b,a])^{a} \equiv c^{-1}b \pmod{[\Gamma_{\alpha}, \Gamma_{\alpha}]}. \]
This means that there is \( x \in [\Gamma_{\alpha}, \Gamma_{\alpha}] \) such that \((c^{-1}b)^{v})_{u} = c^{-1}bx\). Hence
\[
(c^{-1}b) * v = ((c^{-1}b) * u)(x * u),
\]
and since \( x * u \in H_{n} \), we have that
\[
((c^{-1}b) * u)H_{n} = ((c^{-1}b) * v)H_{n} \supset ((c^{-1}b) * v)H_{n+1}.
\]
This proves the lemma. \( \blacksquare \)

Corollary 3.4. Let \( \{g_{n}\} \) be a sequence such that for all \( n \in \mathcal{H}_{n} \), \( g_{n} = \prod_{|v|=n} (c^{-1}b)^{i_{v}} \ast v \). Then the sequence \( \{\sigma(g_{n})\} \) satisfies (3.1) if and only if the following condition holds: for every vertex \( v \), \( i_{v} \equiv i_{v_{1}} + \ldots + i_{v_{p}} \pmod{p}, \) where \( v_{1}, \ldots, v_{p} \) are all vertices of length \( |v| + 1 \) joined to \( v \) by an edge.

Proof. Consider an arbitrary \( m \geq 1 \), and for each vertex \( v \) of length \( m - 1 \) denote by \( v_{1}, \ldots, v_{p} \) the \( p \) vertices of length \( m \) adjacent to it. In particular, each vertex of length \( m \) is written as \( v_{i} \) for some \( v, i \). Then by definition of numbers \( i_{v} \) and Lemma 3.8 we have:
\[
g_{m}H_{m} = \prod_{|v|=m-1} ((c^{-1}b)^{i_{v_{1}}} \ast v_{1} \ldots (c^{-1}b)^{i_{v_{p}}} \ast v_{p})H_{m} = \\
= \prod_{|v|=m-1} ((c^{-1}b)^{i_{v_{1}}} \ast v_{1})H_{m} \ast \ldots \ast ((c^{-1}b)^{i_{v_{p}}} \ast v_{p})H_{m} \subset \\
\subset \prod_{|v|=m-1} ((c^{-1}b)^{i_{v_{1}}} \ast v)H_{m-1} \ast \ldots \ast ((c^{-1}b)^{i_{v_{p}}} \ast v)H_{m-1} = \\
= \prod_{|v|=m-1} ((c^{-1}b)^{i_{v_{1}}} \ast v \ldots (c^{-1}b)^{i_{v_{p}}} \ast v)H_{m-1} = \prod_{|v|=m-1} ((c^{-1}b)^{i_{v_{1}}+\ldots+i_{v_{p}}} \ast v)H_{m-1}.
\]

By Lemma 1.4 cosets \( (c^{-1}b)_{i} [\Gamma_{\alpha}, \Gamma_{\alpha}] \) and \( (c^{-1}b)_{j} [\Gamma_{\alpha}, \Gamma_{\alpha}] \) have non-empty intersection if and only if \( i \equiv j \pmod{p} \). Therefore the coset \( \prod_{|v|=m-1} ((c^{-1}b)^{i_{v_{1}}+\ldots+i_{v_{p}}} \ast v)H_{m-1} \) has non-empty intersection with the coset \( g_{m-1}H_{m-1} \) if and only if \( v_{1} + \ldots + v_{p} \equiv i_{v} \pmod{p} \) for all \( v \). Since two cosets of the same subgroup have non-empty intersection if and only if they coincide, we have inclusion \( g_{m}H_{m} \subset g_{m-1}H_{m-1} \). This guarantees us that, for all \( n \), \( g_{n}H_{n} \) is in \( \bigcap_{i=1}^{n-1} g_{i}H_{i} \). By (3.3), this is equivalent to \( g_{n} \) being in \( \bigcap_{i=1}^{n-1} g_{i}H_{i} \). Finally, since \( g_{n} \in \text{Stab}_{\alpha}(n) \) by definition, \( g_{n} \) satisfies condition (3.1). \( \blacksquare \)

Uniqueness of canonical sequences Any sequence satisfying conditions of Corollary 3.4 is given by the set of indices \( i_{v} \in \{0, 1, \ldots, p-1\} \) placed at each vertex of tree \( T \) and satisfying the “summation condition” stated in that corollary. The next natural question to ask is, is the collection of indices uniquely determined by the limit of the sequence?
Proposition 3.2. Let \( \{g_n\}, \{h_n\} \) be two sequences satisfying the conditions of Corollary 3.4. \( g_n = \prod_{|v|=n}(c^{-1}b)^{i_v} * v, \) \( h_n = \prod_{|v|=n}(c^{-1}b)^{j_v} * v. \) Then \( \lim_{n \to \infty} \hat{\sigma}(g_n) = \lim_{n \to \infty} \hat{\sigma}(h_n) \) if and only if for all \( v i_v \equiv j_v \pmod{p}. \)

Proof. If for all \( v i_v \equiv j_v \pmod{p} \) then \( \hat{\sigma}(g_n) = \hat{\sigma}(h_n), \) and the equality of limits is evident. Suppose that the limits are equal. This means that \( \lim_{n \to \infty} \hat{\sigma}(g_n^{-1}h_n) \) is the trivial element of \( \hat{\Gamma}, \) i.e. that for every \( N \) there is \( n_N \) such that for all \( n \geq n_N \) \( g_n^{-1}h_n \in \hat{H}_N. \) Consider some \( n \geq \max\{n_N, N\}. \) It was shown in the proof of Corollary 3.4 that \( g_n \hat{H}_n \subset g_N \hat{H}_N, \) \( h_n \hat{H}_n \subset h_N \hat{H}_N. \) In particular, \( g_n \in g_N \hat{H}_N \) and \( h_n \in h_N \hat{H}_N. \) Hence \( g_n^{-1}h_n \in (g_N^{-1}h_N) \hat{H}_N, \) and so

\[
g_n^{-1}h_n \in (g_N^{-1}h_N) \hat{H}_N \bigcap \hat{H}_N.
\]

Since the latter two sets are cosets of the same subgroup, they can only have non-empty intersection if they coincide. This means that \( g_N^{-1}h_n \in \hat{H}_N, \) i.e. that for every vertex \( u \) of length \( N i_u \equiv j_u \pmod{p}. \) Since \( N \) is any, this is actually true for any vertex at all. \( \blacksquare \)

At last, description of the kernel It follows from Corollary 3.4 and Proposition 3.2 that Ker \( \sigma \) can be described in the following way. Consider a collection of groups \( \mathbb{Z}_p^{(n)} = \bigoplus_{|v|=n}\mathbb{Z}_p^{(v)}, \) the \( n \)th of which is the direct sum of \( p^n \) exemplars of the cyclic group \( \mathbb{Z}_p. \) The 0th group \( \mathbb{Z}_p^{(0)} \) is trivial. The exemplars are parameterized by vertices of length \( n \) in tree \( T. \) Let \( \pi_u \) be the projection of the corresponding group \( \mathbb{Z}_p^{(n)} \) onto its summand \( \mathbb{Z}_p^{(u)}. \) Consider the map \( \theta_n : \mathbb{Z}_p^{(n)} \to \mathbb{Z}_p^{(n-1)}, n \geq 2, \) defined by the rule

\[
\pi_u(\theta_n(x)) = \sum_{v \geq u, |v|=n} \pi_v(x),
\]

where \( u \) is any vertex of length \( n-1. \) The map \( \theta_1 : \mathbb{Z}_p \to \{0\} \) simply sends everything to zero.

Theorem 3.4. Ker \( \sigma \) is isomorphic to the projective limit \( \mathbb{Z}_p^{(\infty)} \) of the following inverse system of groups,

\[
1 \leftarrow \theta_1 \mathbb{Z}_p^{(1)} \leftarrow \theta_2 \mathbb{Z}_p^{(2)} \leftarrow \theta_3 \mathbb{Z}_p^{(3)} \leftarrow \ldots \leftarrow \theta_n \mathbb{Z}_p^{(n)} \leftarrow \theta_{n+1} \mathbb{Z}_p^{(n+1)} \leftarrow \ldots.
\]

In particular, Ker \( \sigma \) is a profinite abelian group of (prime) exponent \( p. \)

Proof. We establish a map \( \Lambda : \text{Ker} \sigma \to \mathbb{Z}_p^{(\infty)} \) in the following way. Let \( x \) be in Ker \( \sigma. \) Then \( x = \lim_{n \to \infty} \hat{\sigma}(g_n) \) with \( g_n \) as in Corollary 3.4 \( g_n = \prod_{|v|=n}(c^{-1}b)^{i_v} * v. \) Consider map \( \tau_n : \text{Ker} \sigma \to \mathbb{Z}_p^{(n)} \) defined by

\[
\pi_v(\tau_n(x)) = i_v \text{ for each vertex } v \text{ of length } n.
\]

(We assume that all \( i_v \) are taken from the set \( \{0, 1, \ldots, p-1\} \).) Then by definition of the projective limit, the collection of maps \( \tau_n \) defines a unique map \( \Lambda : \text{Ker} \sigma \to \mathbb{Z}_p^{(\infty)}. \) However, we need to make sure that all maps \( \tau_n \) are well-defined (i.e. that \( i_v \) depend on \( x \) only), that they are homomorphic and that \( \Lambda \) is bijective. The first two conditions follow immediately
from Proposition 3.2. To get the bijectivity condition, we construct an inverse map $\Omega$ by putting for each $y \in \tilde{\mathbb{Z}}_p^{(\infty)}$

$$\Omega(y) = \lim_{n \to \infty} \hat{\sigma}(g_n), \text{ where } g_n = \prod_{|v|=n} (c^{-1}b)^{\pi_v(\zeta_n(y))} * v$$

($\zeta_n$ are the canonical maps $\tilde{\mathbb{Z}}_p^{(\infty)} \to \tilde{\mathbb{Z}}_p^{(n)}$). Evidently, this is a well-defined inverse map and a homomorphism. $lacksquare$

It is interesting to note that $\text{Ker } \sigma$ does not depend on a particular vector $\tilde{\alpha}$, only on the number $p$ (as soon as the vector is nonsymmetric and the sum of its coordinates is zero).

Notice also that the group $\tilde{\mathbb{Z}}_p^{(\infty)}$ is rather large. Indeed, for every infinite (strictly descending) path $\gamma$ in tree $T$ it contains an element $b_\gamma$ defined by the rule

$$\pi_v(\zeta_n(b_\gamma)) = \begin{cases} 1, & \text{if } v \in \gamma, \\ 0, & \text{otherwise}. \end{cases}$$

Since for every finite set of paths there is a vertex lying on exactly one of those paths, and $p$ is prime, the subgroup $B^\omega$ generated by all $b_\gamma$ is generated freely by them. Hence this subgroup is isomorphic to the additive group of the space of all finite-support functions on $\partial T$ taking values in $\mathbb{GF}(p)$. Notice also that for any $z \in \tilde{\mathbb{Z}}_p^{(\infty)}$

$$z \equiv \sum_{|v|=n} b_{\gamma_v}^{\pi_v(\zeta_n(z))} \pmod{\text{Ker } \zeta_n},$$

where $\gamma_v$ is the infinite path of the form, $v00\ldots0\ldots$. Hence $B^\omega$ is dense in $\tilde{\mathbb{Z}}_p^{(\infty)}$.

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