Escape Metrics and Its Applications

Zhen-Hu Ning, Fengyan Yang and Xiaopeng Zhao

Abstract  Geodesics escape is widely used to study the scattering of hyperbolic equations. However, there are few progresses except in a simply connected complete Riemannian manifold with nonpositive curvature.

We propose a kind of complete Riemannian metrics in \( \mathbb{R}^n \), which is called as escape metrics. We expose the relationship between escape metrics and geodesics escape in \( \mathbb{R}^n \). Under the escape metric \( g \), we prove that each geodesic of \( (\mathbb{R}^n, g) \) escapes, that is, \( \lim_{t \to +\infty} |\gamma(t)| = +\infty \) for any \( x \in \mathbb{R}^n \) and any unit-speed geodesic \( \gamma(t) \) starting at \( x \). We also obtain the geodesics escape velocity and give the counterexample that if escape metrics are not satisfied, then there exists an unit-speed geodesic \( \gamma(t) \) such that \( \lim_{t \to +\infty} |\gamma(t)| < +\infty \).

In addition, we establish Morawetz multipliers in Riemannian geometry to derive dispersive estimates for the wave equation on an exterior domain of \( \mathbb{R}^n \) with an escape metric. More concretely, for radial solutions, the uniform decay rate of the local energy is independent of the parity of the dimension \( n \). For general solutions, we prove the space-time estimation of the energy and uniform decay rate \( t^{-1} \) of the local energy. It is worth pointing out that different from the assumption of an Euclidean metric at infinity in the existing studies, escape metrics are more general Riemannian metrics.

Keywords  Escape metrics, Geodesics escape, Wave equation, Dispersive estimates

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1 Introduction and Main results

1.1 Notations

Let $O$ be the original point of $\mathbb{R}^n$ ($n \geq 2$) and

$$r(x) = |x|, \quad x \in \mathbb{R}^n$$

(1.1)

be the standard distance function of $\mathbb{R}^n$. Moreover, let $\langle \cdot, \cdot \rangle$, div, $\nabla$, $\Delta$ and $I_n = (\delta_{i,j})_{n \times n}$ be the standard inner product of $\mathbb{R}^n$, the standard divergence operator of $\mathbb{R}^n$, the standard gradient operator of $\mathbb{R}^n$, the standard Laplace operator of $\mathbb{R}^n$ and the unit matrix.

Suppose that $(\mathbb{R}^n, g)$ is a smooth complete Riemannian manifold with

$$g = \sum_{i,j=1}^{n} g_{ij}(x)dx_idx_j, \quad x \in \mathbb{R}^n,$$

(1.2)

such that

$$G(x) \frac{\partial}{\partial r} = \frac{\partial}{\partial r}, \quad |x| \geq r_c,$$

(1.3)

where $r_c$ is a positive constant and

$$G(x) = (g_{ij}(x))_{n \times n}, \quad x \in \mathbb{R}^n.$$

(1.4)

Denote

$$\langle X, Y \rangle_g = \langle G(x)X, Y \rangle, \quad |X|_g^2 = \langle X, X \rangle_g, \quad X, Y \in \mathbb{R}^n_x, x \in \mathbb{R}^n.$$

(1.5)
Let \((r, \theta) = (r, \theta_1, \theta_2, \ldots, \theta_{n-1})\) be the polar coordinates of \(x \in \mathbb{R}^n\) in the Euclidean metric. Note that

\[
\left< \frac{\partial}{\partial r}, \frac{\partial}{\partial r} \right>_g (x) = 1, \quad \left< \frac{\partial}{\partial r}, \frac{\partial}{\partial \theta_i} \right>_g (x) = 0 \quad \text{for} \quad 1 \leq i \leq n-1, \ |x| \geq r_c. \tag{1.6}
\]

Then

\[
g = dr^2 + \sum_{i,j=1}^{n-1} \gamma_{ij}(r, \theta) d\theta_i d\theta_j, \quad |x| \geq r_c, \tag{1.7}
\]

which implies \(|x| - r_c\) is the geodesic distance function of \((\mathbb{R}^n, g)\) from \(x\) with \(|x| \geq r_c\) to \(r_c(x/|x|)\).

Let \(\Upsilon(x) = (\gamma_{ij})(n-1) \times (n-1)(x), \ |x| \geq r_c. \tag{1.8}\)

Let \(S(r)\) be the sphere in \(\mathbb{R}^n\) with a radius \(r\). Then

\[
\left< X, \frac{\partial}{\partial r} \right>_g = \left< X, \frac{\partial}{\partial r} \right> = 0, \quad \text{for} \quad X \in S(r)_x, \ |x| \geq r_c. \tag{1.9}
\]

Let \(D\) be the Levi-Civita connection of the metric \(g\) and \(H\) a vector field, then the covariant differential \(DH\) of the vector field \(H\) is a tensor field of rank 2 as follow:

\[
DH(X, Y)(x) = \left< D_Y H, X \right>_g \quad X, Y \in \mathbb{R}^n_x, \ x \in \mathbb{R}^n. \tag{1.10}
\]

Finally, we set \(\text{div}_g, \nabla_g\) and \(\Delta_g\) as the divergence operator of \((\mathbb{R}^n, g)\), the gradient operator of \((\mathbb{R}^n, g)\) and the Laplace–Beltrami operator of \((\mathbb{R}^n, g)\), respectively.

### 1.2 Escape metrics

In this paper, we introduce escape metrics as follows.

**Definition 1.1** We say \(g\) is an escape metric if

\[
\left< \left( \frac{1}{2} \frac{\partial G(x)}{\partial r} \right) X, X \right>_g \geq \alpha(x)|X|^2_g \quad \text{for} \quad X \in S(r)_x, \ |x| \geq r_c, \tag{1.11}
\]

\[
D^2r^2(X, X) \geq 2\rho_c|X|^2_g \quad \text{for} \quad X \in \mathbb{R}^n_x, \ |x| < r_c, \tag{1.12}
\]

where \(r_c\) is given by (1.3), \(\rho_c \leq 1\) is a positive constant, \(D^2r^2\) is the Hessian of \(r^2\) in the metric \(g\) and \(\alpha(x)\) is a smooth function defined on \(|x| \geq r_c\) satisfying

\[
\alpha(x) > \frac{1}{r}, \quad |x| \geq r_c. \tag{1.13}
\]

**Remark 1.1** If \(r_c = 0\) in (1.3), then \(r(x) = |x|\) is the geodesic distance function of \((\mathbb{R}^n, g)\) from \(x\) to the original point \(O\), (1.12) always holds true for sufficiently small \(r_c\).

**Remark 1.2** Escape metrics can be checked by the following Proposition 2.1.
For an exterior domain of $\mathbb{R}^n$, we introduce exterior escape metrics as follows.

**Definition 1.2** We say $g$ is an **exterior escape metric** if

$$
\left\langle \left( \frac{1}{2} \frac{\partial G(x)}{\partial r} \right) X, X \right\rangle \geq \alpha(x)|X|^2_g \quad \text{for} \quad X \in S(r)_x, \quad |x| \geq r_c,
$$

(1.14)

where $r_c$ is given by (1.3) and $\alpha(x)$ is a smooth function defined on $|x| \geq r_c$ satisfying

$$
\alpha(x) > -\frac{1}{r}, \quad |x| \geq r_c.
$$

(1.15)

**Remark 1.3** Exterior escape metrics can be checked by the following Proposition 2.2.

### 1.3 Geodesics escape

One of the important application of escape metrics is to solve geodesics escape problem. Geodesics escape is a very important research topic and there are already some essential researches on it. Bangert[1] proved the existence of escaping geodesic without self-intersections in a complete Riemannian manifold homeomorphic to the plane; Fernandez and Melian [12] studied the quantity of the escaping geodesics of a 2-dimensional, oriented and noncompact Riemann surface of constant negative curvature; Gony[13] gave the Hausdorff dimension of the terminal points of escaping geodesics in a hyperbolic manifold.

The assumption that each geodesic of $(\mathbb{R}^n, g)$ escapes, known as non-trapping assumption, that is, $\lim_{t \to +\infty} |\gamma(t)| = +\infty$ for any $x \in \mathbb{R}^n$ and any unit-speed geodesic $\gamma(t)$ starting at $x$, is widely used to study the scattering of hyperbolic equation (see for example [2, 4, 5, 6, 7, 11, 14, 15, 16, 17, 28, 30, 33] and the references cited therein). It is well-known that all geodesics escape for a simply connected complete Riemannian manifold with nonpositive curvature. However, for general metrics, since the geodesic is dependent on the nonlinear ordinary differential equation, it is hard to check the non-trapping assumption. Therefore, some effective criterions are further needed to make the non-trapping assumption checkable for general metrics.

The following two theorems show how geodesics escape under escape metrics in $\mathbb{R}^n$.

**Theorem 1.1** Let $g$ be an escape metric such that

$$
r \alpha(x) + 1 \geq \rho_0, \quad |x| \geq r_c,
$$

(1.16)

where $\rho_0 \leq \rho_c$ is a positive constant and $\rho_c$ is given by (1.12). Let

$$
c_0 = \sup_{|x| \leq r_c} r |Dr|_g(x).
$$

(1.17)

Then, for any $x \in \mathbb{R}^n$ and any unit-speed geodesic $\gamma(t)$ starting at $x$, there exists $c(x) > 0$ such that

$$
|\gamma(t)| \geq \rho_0 t - \max \{|x|, c_0\}, \quad \forall t > c(x).
$$

(1.18)
which implies
\[ \lim_{t \to +\infty} |\gamma(t)| = +\infty. \tag{1.19} \]

**Theorem 1.2** Let \( g \) be an escape metric and
\[
f(y) = \inf_{r \leq |x| \leq y} \left\{ \alpha(x) + \frac{1}{|x|} \right\}, \quad y \geq r_c. \tag{1.20}\]
Then, for any \( x \in \mathbb{R}^n \) and any unit-speed geodesic \( \gamma(t) \) starting at \( x \), there exists \( c(x) > 0 \) such that
\[ |\gamma(t)| \geq \int_{c(x)}^t \left( 1 - \frac{2}{1 + e^{2 \int_0^y f(|x_0|+z)dz}} \right) dy + |\gamma(c(x))|, \quad t > c(x). \tag{1.21}\]
which implies
\[ \lim_{t \to +\infty} |\gamma(t)| = +\infty. \tag{1.22} \]
Specially, if
\[ \int_{r_c}^{+\infty} f(z)dz = +\infty. \tag{1.23} \]
Then
\[ \lim_{t \to +\infty} \frac{|\gamma(t)|}{t} = 1. \tag{1.24} \]

For exterior escape metrics, we have similar conclusions with escape metrics.

**Theorem 1.3** Let \( g \) be an exterior escape metric. Then, for any \( |x| \geq r_c \) and any unit-speed geodesic \( \gamma(t) \) starting at \( x \), the following conclusions hold true:

- If \( |x| = r_c \) and \( (\gamma'(t), \frac{\partial}{\partial y})_g \geq 0 \), then
  \[ \lim_{t \to +\infty} |\gamma(t)| = +\infty. \tag{1.25} \]

- If \( |x| > r_c \), then
  \[ \lim_{t \to +\infty} |\gamma(t)| = +\infty, \tag{1.26} \]
  or there exist positive constants \( c(x), t_0 = t_0(x, \gamma'(0)) \) such that
  \[ 0 < t_0 \leq c(x) \quad \text{and} \quad |\gamma(t_0)| = r_c. \tag{1.27} \]

**Remark 1.4** If the geodesic is reflected at \( |x| = r_c \), then for exterior escape metrics, for any \( |x| \geq r_c \) and any unit-speed geodesic \( \gamma(t) \) starting at \( x \),
\[ \lim_{t \to +\infty} |\gamma(t)| = +\infty. \tag{1.28} \]

The following theorem shows the necessity of escape metrics for geodesics escape.
**Theorem 1.4** Assume that
\[
\frac{\partial G(x)}{\partial r} = -\frac{2}{R_0} \left( I_n - \frac{x \otimes x}{|x|^2} \right) G(x), \quad |x| = R_0, \tag{1.29}
\]
where $R_0 > r_c$ is a positive constant. Then, for any $x \in S(R_0)$ and any unit-speed geodesic $\gamma(t)$ starting at $x$ with
\[
\gamma'(0) \in S(R_0)_x, \tag{1.30}
\]
we have
\[
\gamma(t) \in S(R_0), \quad \forall t \geq 0. \tag{1.31}
\]

### 1.4 Wave equation on an exterior domain

In the following, we apply exterior escape metrics to study the dispersive estimates for the wave equation on an exterior domain.

Let $\Omega$ be an exterior domain in $\mathbb{R}^n$ with a compact smooth boundary $\Gamma$. Assume that
\[
|x| \geq r_c \quad \text{for any} \quad x \in \Omega, \tag{1.32}
\]
where $r_c$ is given by (1.3). Denote
\[
r_0 = \inf_{x \in \Gamma} |x|, \quad r_1 = \sup_{x \in \Gamma} |x|. \tag{1.33}
\]
Then $r_1 \geq r_0 \geq r_c$.

Consider the following system.
\[
\begin{aligned}
&u_{tt} - \Delta_g u = 0 \quad (x, t) \in \Omega \times (0, +\infty), \\
u|\Gamma = 0 \quad t \in (0, +\infty), \\
u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x) \quad x \in \Omega.
\end{aligned} \tag{1.34}
\]
Define the energy of the system (1.34) by
\[
E(t) = \frac{1}{2} \int_\Omega \left( u_t^2 + |\nabla_g u|^2 \right) dx_g, \tag{1.35}
\]
where
\[
dx_g = \sqrt{\det(G(x))} dx. \tag{1.36}
\]
For $a > r_1$, the local energy for the system (1.34) is defined by
\[
E(t, a) = \frac{1}{2} \int_{\Omega(a)} \left( u_t^2 + |\nabla_g u|^2 \right) dx_g, \tag{1.37}
\]
where $\Omega(a) = \{x| x \in \Omega, |x| \leq a\}$. In this paper, we are interested in how $E(t, a)$ decays.

If $g \equiv \sum_{i=1}^n dx_i^2$, the system (1.34) is known as constant coefficient. In the case of constant coefficient, this problem has a long history and a wealth of results were obtained, see for example [10, 18, 19, 20, 21, 24, 25, 26, 27, 31, 32] and so on.
For the general metric $g$, there exist few studies of the system (1.34). To the best of our knowledge, Vodev [30] proved uniform local energy decay estimates on unbounded Riemannian manifold with nontrapping metrics assumption. In [8], Christianson obtained a sub-exponential local energy decay estimate on a complete, non-compact, odd-dimensional Riemannian manifold with one trapped hyperbolic geodesic and a Euclidean metric outside a compact set. Later, Christianson [9] derived a polynomial local energy decay rate on a non-compact Riemannian manifold with a hyperbolic thin trapped set. Vasy [29] considered the weighted energy decay for the wave equation at low frequency on scattering manifolds. Besides, we refer the reader to [3, 5, 23, 28, 33] for a series of Strichartz estimates for wave equations. Due to the limit that the metric $g$ is assumed as an Euclidean metric at infinity, we are motivated to study the dispersive estimates for the system (1.34) under a general Riemannian metric.

As is known, the multiplier method is a simple and effective tool for energy estimate on PDEs. In particular, the celebrated Morawetz’s multipliers introduced by [24] have been extensively used for studying the energy decay of the wave equation with constant coefficients, see [10, 19, 21, 25, 32] and many others. Therefore, we establish Morawetz multipliers in Riemannian geometry to derive dispersive estimates for the system (1.34).

Define

$$
\tilde{H}^1_0(\Omega) = \left\{ w(x) : w|_{\Gamma} = 0, \; \int_{\Omega} |\nabla_g w|^2_g dx_g < +\infty \right\}.
$$

(1.38)

It is well-known that the system (1.34) is well-posed with

$$
\begin{align*}
&u_t \in C([0, +\infty), L^2(\Omega)) \quad \text{and} \quad u \in C([0, +\infty), \tilde{H}^1_0(\Omega)).
\end{align*}
$$

(1.39)

Then

$$
E(t) = E(0), \; \forall t > 0.
$$

(1.40)

The finite speed of propagation property of the wave can be stated as follows.

**Theorem 1.5** Let initial datum $(u_0, u_1)$ satisfy

$$
\begin{align*}
u_0(x) &= u_1(x) = 0, \quad |x| \geq R_0.
\end{align*}
$$

(1.41)

where $R_0 > r_1$ is a constant. Then

$$
\begin{align*}
u(x, t) &= 0 \quad \text{for} \; |x| \geq R_0 + t.
\end{align*}
$$

(1.42)

The following Assumption (A) and Assumption (B) are the main assumptions for the wave equation.

**Assumption (A)** Let $g$ be an exterior escape metric such that

$$
\alpha(x) = \frac{m_1 - 1}{r} \quad \text{for} \; |x| \geq r_0,
$$

(1.43)

$$
\det (G(x)) = \eta_0(\theta) r^{2m_2 - 2(n - 1)} \quad \text{for} \; |x| \geq r_0.
$$

(1.44)

where $m_1, m_2$ are positive constants, $\eta_0(\theta)$ is a positive function.
Remark 1.5 From the following formulas (3.17) and (3.24), we have
\[
\frac{m_2}{r} = \frac{n - 1}{r} + \frac{\partial \ln \sqrt{\det(G(x))}}{\partial r} = \Delta_g r = trD^2 r
\]
\[
\geq (n - 1) \left( \alpha(x) + \frac{1}{r} \right) = \frac{(n - 1)m_1}{r}, \quad |x| \geq r_0. \tag{1.45}
\]
Then
\[
m_2 \geq (n - 1)m_1. \tag{1.46}
\]

Example 1.1 Let \( r_0 = r_c \) and \( g \) satisfy
\[
G(x) = \frac{x \otimes x}{|x|^2} + r^{2(m_1 - 1)} \left( I_n - \frac{x \otimes x}{|x|^2} \right), \quad |x| \geq r_0. \tag{1.47}
\]
Then
\[
\partial \partial G(x) = \frac{\partial}{\partial r}, \quad |x| \geq r_0, \tag{1.48}
\]
\[
\left\langle \left( \frac{1}{2} \frac{\partial G(x)}{\partial r} \right) X, X \right\rangle = m_1 - 1 |X|_g^2 \quad \text{for} \quad X \in S(r)_x, \quad |x| \geq r_0, \tag{1.49}
\]
\[
det(G(x)) = r^{2(m_1 - 1)(n - 1)} \quad \text{for} \quad |x| \geq r_0. \tag{1.50}
\]

Assumption (B) Let \( g \) be an exterior escape metric such that
\[
\alpha(x) = m_1 r^{-s_1} - \frac{1}{r} \quad \text{for} \quad |x| \geq r_0, \tag{1.51}
\]
\[
det(G(x)) = \eta_0(\theta) r^{-2(n - 1)} e^{2m_2 \int_{r_c}^{r} y^{-s_2} dy} \quad \text{for} \quad |x| \geq r_0, \tag{1.52}
\]
where \( \eta_0(\theta) \) is a positive function, \( m_1, m_2, s_1, s_2 \) are positive constants such that
\[
s_1 > 1, \quad 0 < s_2 \leq 1 \quad \text{and} \quad (s_2 + 1)r_0^{s_2 - 1} < m_2. \tag{1.53}
\]

Remark 1.6 From the following formulas (3.17) and (3.24), we have
\[
m_2 r^{-s_2} = \frac{n - 1}{r} + \frac{\partial \ln \sqrt{\det(G(x))}}{\partial r} = \Delta_g r = trD^2 r
\]
\[
\geq (n - 1) \left( \alpha(x) + \frac{1}{r} \right) = (n - 1)m_1 r^{-s_1}, \quad |x| \geq r_0. \tag{1.54}
\]
Then
\[
m_2 \geq (n - 1)m_1 r_0^{(s_2 - s_1)}. \tag{1.55}
\]

Example 1.2 Let \( r_0 = r_c \) and \( g \) satisfy
\[
G(x) = \frac{x \otimes x}{|x|^2} + e^{2 \int_{r_c}^{r} m_2 y^{-2} dy - 2(n - 1)} \left( I_n - \frac{x \otimes x}{|x|^2} \right), \quad |x| \geq r_0, \tag{1.56}
\]
where \( s_1, s_2, m_2 \) are given by (1.53). Then
\[
G(x) \frac{\partial}{\partial r} = \frac{\partial}{\partial r}, \quad |x| \geq r_0. \tag{1.57}
\]
\[
\left\langle \left( \frac{1}{2} \frac{\partial G(x)}{\partial r} \right) X, X \right\rangle = \left( \frac{m_2 r^{-s_2}}{n - 1} - 1 \right) |X|_g^2 \\
\geq \left( m_1 r^{-s_1} - \frac{1}{r} \right) |X|_g^2 \quad \text{for} \quad X \in S(r) \quad \text{for} \quad |x| \geq r_0,
\]

where \( m_1 \) is a positive constant such that

\[
m_2 \geq (n - 1) m_1 r_0^{(s_2 - s_1)}.
\]

**Remark 1.7** More examples of Assumption (A) and Assumption (B) can be obtained by solving \( \text{det}(G(x)) \) from the following (2.17) with a given \( \alpha(x) \).

In order to facilitate the discussion, we define

\[\nu(x) \text{ is the unit normal vector outside } \Omega \text{ in } (\mathbb{R}^n, g) \text{ for } x \in \Gamma,\]

and

\[\nu(x) = \frac{\partial}{\partial r} \text{ for } |x| > r_1.\]

**Theorem 1.6** Suppose that the following three conditions hold:

Assumption (A) holds true with \( m_1 > \frac{1}{2} \),

\[
\frac{\partial r}{\partial \nu} \leq 0, \quad x \in \Gamma,
\]

\[u_0(x) = u_1(x) = 0, \quad |x| \geq R_0,
\]

where \( R_0 > r_1 \) is a constant.

Then there exists positive constant \( C(a, R_0) \) such that

\[E(a, t) \leq \frac{C(a, R_0)}{t} E(0), \quad \forall t > 0.\]

**Theorem 1.7** Suppose that the following three conditions hold:

Assumption (B) holds true,

\[
\frac{\partial r}{\partial \nu} \leq 0, \quad x \in \Gamma,
\]

\[u_0(x) = u_1(x) = 0, \quad |x| \geq R_0,
\]

where \( R_0 > r_1 \) is a constant.

Then there exists positive constant \( C(R_0) \) such that

\[\int_0^{+\infty} \int_{\Omega} r^{-s_1} \left( u_1^2 + |\nabla_g u|_g^2 \right) dx_g dt \leq C(R_0) E(0).\]
The organization of the rest of our paper goes as follows. In Section 2, we will show how to check escape metrics and exterior escape metrics. Then some multiplier identities and key lemmas for our problem will be given in Section 3. The technical details of the proofs of the results for geodesics escape will be provided in Section 4. Finally, Section 5 is devoted to the wave system (1.34), in which the propagation property, the uniform local energy decay for radial solutions, the uniform local energy decay and the space-time energy estimation for general solutions are present in sequence.

2 Checking the Metric

Escape metrics can be checked by the following proposition.

**Proposition 2.1** Let

\[
G(x) = W(x) + e^{\int_{r_c}^{r} 2\alpha(y, \theta) dy} (I_n - W(r_c, \theta)) \\
+ e^{\int_{r_c}^{r} 2\alpha(y, \theta) dy} \int_{r_c}^{r} 2e^{-\int_{r_c}^{r_c} 2\alpha(z, \theta) dz} Q(y, \theta)(I_n - W(y, \theta)) dy, \quad x \in \mathbb{R}^n, \tag{2.1}
\]

where

\[
W(x) = \frac{x \otimes x}{|x|^2}, \quad x \in \mathbb{R}^n \setminus O, \tag{2.2}
\]

\[
\alpha(x) \text{ is a smooth function defined on } \mathbb{R}^n \text{ satisfying}
\]

\[
\alpha(x) > -\frac{1}{r}, \quad |x| \geq r_c, \tag{2.3}
\]

\[
\alpha(x) = 0, \quad |x| < r_c, \tag{2.4}
\]

and \(Q(x) = (q_{i,j})_{(n-1) \times (n-1)}(x)\) is a smooth, symmetric, nonnegative definite matrix function defined on \(\mathbb{R}^n\) satisfying

\[
Q(x) = 0, \quad |x| < r_c. \tag{2.5}
\]

Then

\[
G(x) \frac{\partial}{\partial r} = \frac{\partial}{\partial r}, \quad x \in \mathbb{R}^n, \tag{2.6}
\]

\[
D^2 r^2(X, X) = 2|X|_g^2 \quad \text{for} \quad X \in \mathbb{R}^n_x, \quad |x| < r_c, \tag{2.7}
\]

\[
\left\langle \left(\frac{1}{2} \frac{\partial G(x)}{\partial r}\right) X, X \right\rangle \geq \alpha(x)|X|_g^2 \quad \text{for} \quad X \in S(r)_x, \quad |x| \geq r_c. \tag{2.8}
\]

**Proof.** Note that

\[
G(x) = I_n, \quad |x| < r_c. \tag{2.9}
\]

Then

\[
D^2 r^2(X, X) = 2|X|_g^2 \quad \text{for} \quad X \in \mathbb{R}^n_x, |x| < r_c. \tag{2.10}
\]
the equality (2.7) holds.

Note that
\[ \frac{\partial}{\partial r} = \frac{x}{|x|}. \]  
(2.11)

Then
\[ W(x) \frac{\partial}{\partial r} = \frac{\partial}{\partial r}, \quad x \in \mathbb{R}^n \setminus O. \]  
(2.12)

Hence
\[ G(x) \frac{\partial}{\partial r} = \frac{\partial}{\partial r}, \quad x \in \mathbb{R}^n \setminus O. \]  
(2.13)

With (2.9), the equality (2.6) holds.

Note that
\[ W(x) X = 0 \quad \text{for} \quad X \in S(r)_x, \quad x \in \mathbb{R}^n \setminus O, \]  
(2.14)

and
\[ \frac{1}{2} \frac{\partial G(x)}{\partial r} = \left( \alpha(x) G(x) + Q(x) \right) (I_n - W(x)), \quad |x| \geq r_c. \]  
(2.15)

Then
\[ \left\langle \left( \frac{1}{2} \frac{\partial G(x)}{\partial r} \right) X, X \right\rangle = \alpha(x)|X|^2_g + \langle Q(x) X, X \rangle \]  
\[ \geq \alpha(x)|X|^2_g \quad \text{for} \quad X \in S(r)_x, \quad |x| \geq r_c. \]  
(2.16)

The inequality (2.8) holds true. □

By a similar proof with Proposition 2.1, exterior escape metrics can checked by the following proposition.

**Proposition 2.2** Let
\[ G(x) = W(x) + e^{\int_{r_c}^r 2\alpha(y,\theta)dy} P(r_c, \theta)(I_n - W(r_c, \theta)) + e^{\int_{r_c}^r 2\alpha(y,\theta)dy} \int_{r_c}^r 2e^{-\int_{r_c}^y 2\alpha(z,\theta)dz} Q(y, \theta)(I_n - W(y, \theta))dy, \quad |x| \geq r_c, \]  
(2.17)

where
\[ W(x) = \frac{x \otimes x}{|x|^2}, \quad x \in \mathbb{R}^n \setminus O, \]  
(2.18)

\[ \alpha(x) \quad \text{is a smooth function defined on} \quad |x| \geq r_c \quad \text{satisfying} \]  
\[ \alpha(x) > -\frac{1}{r}, \quad |x| \geq r_c. \]  
(2.19)

\[ P(x) = (p_{i,j})_{(n-1) \times (n-1)}(x) \quad \text{is a smooth, symmetric, positive definite matrix function} \]  
\[ \text{defined on} \quad |x| = r_c \quad \text{and} \quad Q(x) = (q_{i,j})_{(n-1) \times (n-1)}(x) \quad \text{is a smooth, symmetric, nonnegative} \]  
\[ \text{definite matrix function defined on} \quad |x| \geq r_c. \]

Then
\[ \left\langle \left( \frac{1}{2} \frac{\partial G(x)}{\partial r} \right) X, X \right\rangle \geq \alpha(x)|X|^2_g \quad \text{for} \quad X \in S(r)_x, \quad |x| \geq r_c. \]  
(2.21)
3 Multiplier Identities and Key Lemmas

Lemma 3.1 Suppose that \( u(x,t) \) solves the system (1.34) and \( \mathcal{H} \) is a \( C^1 \) vector field defined on \( \overline{\Omega} \). Then
\[
\int_0^T \int_{\partial \Omega(a)} \frac{\partial u}{\partial \nu} \mathcal{H}(u) d\Gamma_g dt + \frac{1}{2} \int_0^T \int_{\partial \Omega(a)} \left( u_t^2 - |\nabla_g u|^2 \right) \langle \mathcal{H}, \nu \rangle_g d\Gamma_g dt
= (u_t, \mathcal{H}(u))_0^T + \int_0^T \int_{\Omega(a)} D\mathcal{H}(\nabla_g u, \nabla_g u) dx_g dt
+ \frac{1}{2} \int_0^T \int_{\Omega(a)} \left( u_t^2 - |\nabla_g u|^2 \right) div_g \mathcal{H} dx_g dt,
\]

(3.1)

where
\[
(u_t, \mathcal{H}(u))_0^T = \int_0^T \int_{\Omega(a)} u_t \mathcal{H}(u) dx_g |_{t=0}^T, \quad d\Gamma_g = \sqrt{det(G(x))} d\Gamma.
\]

(3.2)

Moreover, assume that \( P \in C^2(\overline{\Omega}) \) and \( Q \in C^1(\overline{\Omega} \times [0, +\infty)) \). Then
\[
\int_0^T \int_{\Omega(a)} \left( u_t^2 - |\nabla_g u|^2 \right) P dx_g dt = \frac{1}{2} \int_0^T \int_{\partial \Omega(a)} \frac{\partial P}{\partial \nu} d\Gamma_g dt \int_0^T \int_{\Omega(a)} u_t^2 \Delta_g P dx_g dt
- \int_0^T \int_{\partial \Omega(a)} P u \frac{\partial u}{\partial \nu} d\Gamma_g dt + (u_t, uP)_0^T.
\]

(3.3)

and
\[
\int_{\Omega(a)} \left( u_t^2 + |\nabla_g u|^2 \right) Q dx_g |_{0}^T = \int_0^T \int_{\Omega(a)} \left( u_t^2 + |\nabla_g u|^2 \right) Q dx_g dt
-2 \int_0^T \int_{\Omega(a)} u_t \nabla_g Q dx_g dt
+2 \int_0^T \int_{\partial \Omega(a)} Q u_t \frac{\partial u}{\partial \nu} d\Gamma_g dt.
\]

(3.4)

\[
\textbf{Proof.} \text{ Firstly, we multiply the wave equation in (1.34) by } \mathcal{H}(u) \text{ and integrate over } \Omega(a) \times (0,T), \text{ noting that}
\[
\langle \nabla_g f, \nabla_g (\mathcal{H}(f)) \rangle_g = \nabla_g f \langle \nabla_g f, H \rangle_g = D^2 f(H, \nabla_g f) + DH(\nabla_g f, \nabla_g f)
= D^2 f(H, \nabla_g f) + \frac{1}{2} H(\nabla_g f^2) + DH(\nabla_g f, \nabla_g f)
= \frac{1}{2} H(\nabla_g f^2) + DH(\nabla_g f, \nabla_g f)
= DH(\nabla_g f, \nabla_g f) + \frac{1}{2} \text{div}_g(|\nabla_g f|^2 H) - \frac{1}{2} |\nabla_g f|^2 \text{div}_g H.
\]

(3.5)

the equality (3.1) follows from Green’s formula.

Secondly, we multiply the wave equation in (1.34) by \( Pu \) and integrate over \( \Omega(a) \times (0,T) \). The equality (3.3) follows from Green’s formula. Finally, the equality (3.4) follows from Green’s formula. \( \square \)
Lemma 3.2

\[ D^2 r(X, X) = \left\langle \left( \frac{1}{2} \frac{\partial G(x)}{\partial r} \right) X, X \right\rangle + \frac{1}{r} |X|^2_g \quad \text{for} \quad X \in S(r)_x, |x| \geq r_c, \quad (3.6) \]

where \( D^2 r \) is the Hessian of \( r \) in the metric \( g \).

**Proof.** Let \( \Upsilon(x) \) be given by (1.8). Denote

\[ \Re = \left\{ x \mid |x| \geq r_c, \det (\Upsilon(x)) \neq 0 \right\}. \quad (3.7) \]

Let \( x \in \Re \) and denote \( \theta_n = r \), we have

\[ \gamma_{ni}(x) = \gamma_{in}(x) = \left\langle \frac{\partial}{\partial \theta_i}, \frac{\partial}{\partial \theta_n} \right\rangle_g(x) = 0 \quad 1 \leq i \leq n - 1, \quad \gamma_{nn}(x) = 1. \quad (3.8) \]

Let \((\gamma^i_j)_{n \times n}(x) = (\gamma_{ij})^{-1}_{n \times n}(x)\). Then

\[ \gamma^{ni}(x) = \gamma^{in}(x) = 0 \quad 1 \leq i \leq n - 1, \quad \gamma^{nn}(x) = 1. \quad (3.9) \]

We compute Christoffel symbols as

\[ \Gamma^k_{in} = \frac{1}{2} \sum_{l=1}^{n} \gamma^{kl} \left( \frac{\partial (\gamma_{il})}{\partial r} + \frac{\partial (\gamma_{nl})}{\partial \theta_i} - \frac{\partial (\gamma_{ni})}{\partial \theta_l} \right) = \frac{1}{2} \sum_{l=1}^{n-1} \gamma^{kl} \frac{\partial (\gamma_{il})}{\partial r}, \quad (3.10) \]

for \( 1 \leq i, k \leq n - 1 \), which give

\[ D \frac{\partial}{\partial \theta_i} \frac{\partial}{\partial r} = \frac{1}{2} \sum_{k=1}^{n-1} \left( \sum_{l=1}^{n-1} \gamma^{kl} \frac{\partial (\gamma_{il})}{\partial r} \right) \frac{\partial}{\partial \theta_k}. \quad (3.11) \]

Then for \( X = \sum_{i=1}^{n-1} X_i \frac{\partial}{\partial \theta_i} \in S(r)_x \), we deduce that

\[ D^2 r(X, X) = \sum_{i,j=1}^{n-1} \left\langle D \frac{\partial}{\partial \theta_i} \frac{\partial}{\partial r}, D \frac{\partial}{\partial \theta_j} \frac{\partial}{\partial r} \right\rangle_g X_i X_j \]

\[ = \frac{1}{2} \sum_{i,j,k,l=1}^{n-1} \gamma^{kl} \frac{\partial (\gamma_{il})}{\partial r} \gamma_{kj} X_i X_j \]

\[ = \frac{1}{2} \sum_{i,j=1}^{n-1} \frac{\partial (\gamma_{ij})}{\partial r} X_i X_j. \quad (3.12) \]

Note that

\[ \gamma_{ij}(x) = \left\langle G(x) \frac{\partial}{\partial \theta_i}, \frac{\partial}{\partial \theta_j} \right\rangle, \quad \nabla_\theta \frac{\partial}{\partial \theta_i}(x) = \frac{1}{r} \frac{\partial}{\partial \theta_i}(x), 1 \leq i \leq n - 1. \quad (3.13) \]

Then

\[ \frac{\partial (\gamma_{ij})}{\partial r} = \left\langle \frac{\partial G(x)}{\partial r} \frac{\partial}{\partial \theta_i}, \frac{\partial}{\partial \theta_j} \right\rangle + \left\langle G(x) \left( \nabla_\theta \frac{\partial}{\partial \theta_i} \right), \frac{\partial}{\partial \theta_j} \right\rangle + \left\langle G(x) \frac{\partial}{\partial \theta_i}, \nabla_\theta \frac{\partial}{\partial \theta_j} \right\rangle \]

\[ = \left\langle \frac{\partial G(x)}{\partial r} \frac{\partial}{\partial \theta_i}, \frac{\partial}{\partial \theta_j} \right\rangle + \frac{2}{r} \left\langle G(x) \frac{\partial}{\partial \theta_i}, \frac{\partial}{\partial \theta_j} \right\rangle. \quad (3.14) \]
It follows from (3.12) that
\[
D^2 r(X, X) = \frac{1}{2} \sum_{i,j=1}^{n-1} \frac{\partial (\gamma_{ij})}{\partial r} X_i X_j \\
= \left( \left( \frac{1}{2} \frac{\partial G(x)}{\partial r} \right) X, X \right) + \frac{1}{r} |X|^2_g,
\]
(3.15)

Note that
\[
\mathbb{R} \text{ is dense in } |x| \geq r_c.
\]
(3.16)
The equality (3.6) holds true. ☐

**Lemma 3.3** Let \( g \) be an escape metric or an exterior escape metric. Then
\[
D^2 r(X, X) \geq \left( \alpha(x) + \frac{1}{r} \right) |X|^2_g \text{ for } X \in S(r)_x, |x| \geq r_c.
\]
(3.17)

**Proof.** With (3.6), we obtain
\[
D^2 r(X, X) \geq \left( \alpha(x) + \frac{1}{r} \right) |X|^2_g \text{ for } X \in S(r)_x, |x| \geq r_c.
\]
(3.18)
The inequality (3.17) holds true. ☐

**Lemma 3.4** Let \( g \) be escape metric. Then
\[
D^2 r^2(X, X) \geq h(x)|X|^2_g \text{ for } X \in \mathbb{R}_x^n, x \in \mathbb{R}_n,
\]
(3.19)
where
\[
h(x) = \begin{cases} 
2\rho_c, & |x| < r_c, \\
\min\{2, 2r\alpha(x) + 2\}, & |x| \geq r_c.
\end{cases}
\]
(3.20)

**Proof.** With (3.6), we obtain
\[
D^2 r(X, X) \geq \left( \alpha(x) + \frac{1}{r} \right) |X|^2_g \text{ for } X \in S(r)_x, |x| \geq r_c.
\]
(3.21)
Note that
\[
D^2 r^2 = 2Dr \otimes Dr + 2rD^2 r.
\]
(3.22)
Then
\[
D^2 r^2(X, X) \geq \min\{2, 2r\alpha(x) + 2\} |X|^2_g \text{ for } X \in \mathbb{R}_x^n, |x| \geq r_c.
\]
(3.23)
With (1.12), the inequality (3.19) holds true. ☐

**Lemma 3.5** Let \( g \) be an escape metric or an exterior escape metric. Then
\[
\Delta_g r = \frac{n-1}{r} + \frac{\partial \ln \sqrt{\det (G(x))}}{\partial r}, \quad |x| \geq r_c.
\]
(3.24)

**Proof.** With (1.3), for \(|x| \geq r_c\), we have
\[
\Delta_g r = \text{div}_g \nabla_g r = \frac{1}{\sqrt{\det (G(x))}} \text{div} \left( \sqrt{\det (G(x))} \frac{\partial}{\partial r} \right) \\
= \frac{n-1}{r} + \frac{\partial \ln \sqrt{\det (G(x))}}{\partial r}.
\]
(3.25)
The equality (3.24) holds true. ☐
4 Proofs for Geodesics Escape

**Lemma 4.1** Let \( \hat{\Omega} \) be a bounded domain of \( \mathbb{R}^n \). Assume that there exists a \( C^1 \) vector field \( H \) on \( \mathbb{R}^n \) satisfying

\[
DH(X, X) \geq \rho_0 |X|^2_g \quad \text{for all} \quad X \in \mathbb{R}^n, \quad x \in \hat{\Omega},
\]

where \( \rho_0 \) is a positive constant. Then, for any \( x \in \hat{\Omega} \) and any unit-speed geodesic \( \gamma(t) \) starting at \( x \), if

\[
\gamma(t) \in \hat{\Omega}, \quad 0 \leq t \leq t_0,
\]

then

\[
t_0 \leq \frac{\sup\{|H|_g(x)| \ x \in \overline{\Omega}\} + |H|_g(x)}{\rho_0}.
\]

**Proof.** Note that

\[
|\gamma'(t)|_g = 1, \quad D\gamma'(t)\gamma'(t) = 0 \quad \forall t \geq 0.
\]

Then

\[
\langle H, \gamma'(t) \rangle_g \big|_0^{t_0} = \int_0^{t_0} \gamma'(t)\langle H, \gamma'(t) \rangle_g dt = \int_0^{t_0} DH(\gamma'(t), \gamma'(t))dt \geq \rho_0 t_0.
\]

We have

\[
t_0 \leq \frac{\sup\{|H|_g(x)| \ x \in \overline{\Omega}\} + |H|_g(x)}{\rho_0}.
\]

\(\square\)

**Lemma 4.2** Assume that

\[
D^2 r^2(X, X) \geq 2\rho_0 |X|^2_g \quad \text{for all} \quad X \in \mathbb{R}^n, \ x \in \mathbb{R}^n,
\]

where \( \rho_0 \leq 1 \) is a positive constant. Let

\[
c_0 = \sup_{|x| \leq r_c} r|Dr|_g(x).
\]

Then, for any \( x \in \mathbb{R}^n \) and any unit-speed geodesic \( \gamma(t) \) starting at \( x \), there exists \( c(x) > 0 \) such that

\[
|\gamma(t)| \geq \rho_0 t - \max\{|x|, c_0\}, \quad \forall t > c(x).
\]

which implies

\[
\lim_{t \to +\infty} |\gamma(t)| = +\infty.
\]

**Proof.** Note that

\[
r|Dr|_g(x) = r(x), \quad |x| \geq r_c.
\]

Hence

\[
r|Dr|_g(x) \leq \max\{r(x), c_0\}, \quad x \in \mathbb{R}^n.
\]
Let $H = Dr^2$, it is easy to see that
\[
\langle H, \gamma'(z) \rangle_g |^t_0 = \int_0^t \gamma'(z) \langle H, \gamma'(z) \rangle_g dz = \int_0^t DH(\gamma'(z), \gamma'(z)) dz \geq 2\rho_0 t. \tag{4.13}
\]

Then
\[
2\rho_0 t \leq 2 \max \{|x|, c_0\} + 2 \max \{\gamma(t), c_0\}. \tag{4.14}
\]

Let
\[
c(x) = \frac{\max\{|x|, c_0\} + c_0}{\rho_0}, \tag{4.15}
\]

the estimate (4.9) holds. \(\square\)

**Lemma 4.3** Let $\rho_0 \leq 1$ be a positive constant. Assume that
\[
D^2 r^2(X, X) \geq 2\rho_0 |X|^2_g, \quad X \in \mathbb{R}^n, |x| < r_c, \tag{4.16}
\]
\[
D^2 r(X, X) \geq f(r)|X|^2_g \quad \text{for} \quad X \in S(r), |x| \geq r_c, \tag{4.17}
\]

where $f \in C([r_c, +\infty))$ is a decreasing positive function.

Then, for any $x \in \mathbb{R}^n$ and any unit-speed geodesic $\gamma(t)$ starting at $x$, there exists $c(x) > 0$ such that
\[
|\gamma(t)| \geq \int_{c(x)}^t \left(1 - \frac{2}{1 + e^2 \int_0^y f(|x| + y) dy} \right) dy + |\gamma(c(x))|, \quad t > c(x), \tag{4.18}
\]

which implies
\[
\lim_{t \to +\infty} |\gamma(t)| = +\infty. \tag{4.19}
\]

Specially, if
\[
\int_{r_c}^{+\infty} f(z) dz = +\infty. \tag{4.20}
\]

Then
\[
\lim_{t \to +\infty} \frac{|\gamma(t)|}{t} = 1. \tag{4.21}
\]

**Proof.** Let $x$ be a fixed point. From Lemma 3.4 and Lemma 4.1, there exists $c(x) > 0$, for any unit-speed geodesic $\gamma(t)$ starting at $x$, there exist $0 \leq t_1, t_2 \leq c(x)$ such that
\[
|\gamma(t_1)| = \max\{|x|, r_c\} + \frac{1}{2}, \quad |\gamma(t_2)| = \max\{|x|, r_c\} + \frac{3}{2}. \tag{4.22}
\]

Let $t_0$ satisfy
\[
t_0 = \sup \left\{ t \ \middle| \ t_1 \leq t \leq t_2, \ |\gamma(t)| = \max\{|x|, r_c\} + 1 \right\}. \tag{4.23}
\]

Then
\[
|\gamma(t_0)| = \left\langle \frac{\partial}{\partial r}, \gamma'(t_0) \right\rangle_g \geq 0. \tag{4.24}
\]

Let
\[
h(t) = \left\langle \frac{\partial}{\partial r}, \gamma'(t) \right\rangle_g, \tag{4.25}
\]
we have \(-1 \leq h \leq 1, \, h(t_0) \geq 0\). For \(t \geq t_0\), we deduce that

\[
h_t = \gamma'(t) \left( \frac{\partial}{\partial r}, \gamma'(t) \right)_g = D^2 r(\gamma'(t), \gamma'(t)) = D^2 r \left( \gamma'(t) - h \frac{\partial}{\partial r}, \gamma'(t) - h \frac{\partial}{\partial r} \right) \\
\geq f(|\gamma(t)|)(1 - h^2).
\]  

which implies \(h_t \geq 0\), \(h\) is increasing on \([t_0, +\infty)\). Then

\[
h(t) \geq h(t_0) \geq 0, \quad t > t_0.
\]

Note that \(f\) is decreasing and

\[
|\gamma(t)| \leq |\gamma(t_0)| + \int_{t_0}^{t} h(z) dz \geq |\gamma(t_0)|, \quad t > t_0.
\]

It follows from (4.26) that

\[
h_t \geq f(|x_0| + t)(1 - h^2), \quad t \geq t_0.
\]

Then

\[
\ln \frac{1 + h_t}{1 - h_{t_0}} \geq 2 \int_{t_0}^{t} f(|x_0| + t) dt, \quad t \geq t_0.
\]

We have

\[
h(t) \geq 1 - \frac{2}{1 + \frac{1 + h(t)}{1 - h(t_0)} e^{2 \int_{t_0}^{t} f(|x_0| + z) dz}} \\
\geq 1 - \frac{2}{1 + e^{2 \int_{t_0}^{t} f(|x_0| + z) dz}}, \quad t \geq t_0.
\]

Hence, for \(T \geq c(x_0)\),

\[
|\gamma(T)| = \int_{c(x_0)}^{T} h(t) dt + |\gamma(c(x_0))| \\
\geq \int_{c(x_0)}^{T} \left( 1 - \frac{2}{1 + e^{2 \int_{t_0}^{t} f(|x_0| + z) dz}} \right) dt + |\gamma(c(x_0))|.
\]

The estimate (4.18) holds.

If

\[
\int_{r_c}^{+\infty} f(z) dz = +\infty,
\]

then

\[
\lim_{t \to +\infty} \left( 1 - \frac{2}{1 + e^{2 \int_{t_0}^{t} f(|x_0| + z) dz}} \right) = 1.
\]

From (4.33),

\[
\lim_{t \to +\infty} \frac{|\gamma(t)|}{t} = 1.
\]
The estimate (4.21) holds. □

Proof of Theorem 1.1
With Lemma 3.4, we have
\[ D^2 r^2(X, X) \geq 2\rho_0 |X|^2 \quad \text{for} \quad X \in \mathbb{R}^n_x, x \in \mathbb{R}^n. \] (4.37)
The estimate (1.18) follows from Lemma 4.2. □

Proof of Theorem 1.2
With Lemma 3.3, we have
\[ D^2 r^2(X, X) \geq 2\rho_c |X|^2 \quad \text{for} \quad X \in \mathbb{R}^n_x, |x| < r_c, \] (4.38)
\[ D^2 r(X, X) \geq f(r)|X|^2_g \quad \text{for} \quad X \in S(r)_x, |x| \geq r_c. \] (4.39)
The estimates (1.21) and (1.24) follow from Lemma 4.3. □

Proof of Theorem 1.3 By a similar proof with the Theorem 1.2, the conclusions in Theorem 1.3 hold true.

Proof of Theorem 1.4 With (3.6), we obtain
\[ D^2 r(X, X) = 0 \quad \text{for} \quad X \in S(R_0)_x, |x| = R_0. \] (4.40)
Let \( \tilde{g} \) be a Riemannian metric induced by \( g \) in \( S(R_0) \) and \( \tilde{D} \) be the associated Levi-Civita connection.
Let \( \tilde{\gamma}(t) \) be a unit-speed geodesic in \( (S(R_0), \tilde{g}) \) starting at \( x \in S(R_0) \), then
\[ \left\langle \tilde{\gamma}'(t), \frac{\partial}{\partial r} \right\rangle_{\tilde{g}} = 0, \quad \tilde{D}_{\tilde{\gamma}(t)} \tilde{\gamma}'(t) = 0. \] (4.41)
Therefore,
\[ D_{\tilde{\gamma}(t)} \tilde{\gamma}'(t) = \tilde{D}_{\tilde{\gamma}(t)} \tilde{\gamma}'(t) + \left\langle D_{\tilde{\gamma}(t)} \tilde{\gamma}'(t), \frac{\partial}{\partial r} \right\rangle_{g} \frac{\partial}{\partial r} = \tilde{D}_{\tilde{\gamma}(t)} \tilde{\gamma}'(t) - D^2 r(\tilde{\gamma}'(t), \tilde{\gamma}'(t)) \frac{\partial}{\partial r} = 0, \] (4.42)
which implies \( \tilde{\gamma}(t) \) is also the geodesic of \( (\mathbb{R}^n, g) \). Then
\[ \gamma(t) = \tilde{\gamma}(t) \in S(R_0), \quad \forall t \geq 0, \] (4.43)
for unit-speed geodesic \( \gamma(t) \) of \( (\mathbb{R}^n, g) \) satisfying
\[ \gamma(0) = \tilde{\gamma}(0), \quad \gamma'(0) = \tilde{\gamma}'(0). \] (4.44)
□
5 Proofs for the Wave Equation

5.1 The propagation property

Proof of Theorem 1.4

Let

\[ \tilde{E}(t) = \frac{1}{2} \int_{\Omega \setminus \Omega(R_0 + t)} (u_t^2 + |\nabla g u|^2_g) \, dx_g. \]  

Then

\[ \begin{aligned}
\tilde{E}'(t) &= \frac{1}{2} \frac{d}{dt} \int_{R_0 + t}^{+\infty} dz \int_{x \in \Omega, |x| = z} (u_t^2 + |\nabla g u|^2_g) \, dx_g \\
&= -\frac{1}{2} \int_{|x| = R_0 + t} \left( u_t^2 + |\nabla g u|^2_g \right) \, d\Gamma_g - \frac{1}{2} \int_{|x| = R_0 + t} u_t u_r \, d\Gamma_g \\
&\leq -\frac{1}{2} \int_{|x| = R_0 + t} \left( u_t^2 + |\nabla g u|^2_g \right) \, d\Gamma_g + \frac{1}{2} \int_{|x| = R_0 + t} \left( u_r^2 + u_r^2 \right) \, d\Gamma_g \\
&\leq 0.
\end{aligned} \]  

(5.2)

Noting that \( \tilde{E}(0) = 0 \), we have \( u(x, t) = 0, \quad |x| \geq R_0 + t. \)

5.2 Uniform decay of local energy for radial solutions

In this chapter, we shall study the differences of the decay rate between the constant coefficient wave equation and the wave equation on Riemannian manifold for radial solutions.

Let \( \Gamma = \{ x | x \in \mathbb{R}^n, |x| = r_0 \} \) and let \( u_0(x), u_1(x) \) be of compact support.

It is well-known that the local energy for the constant coefficient wave equation

\[ \begin{cases}
  u_{tt} - \Delta u = 0 & (x, t) \in \Omega \times (0, +\infty), \\
  u|_{\Gamma} = 0 & t \in (0, +\infty), \\
  u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x) & x \in \Omega
\end{cases} \]  

(5.3)

has a uniform decay rate as follows: For even dimensional space, the uniform decay rate of the local energy is polynomial and for odd dimensional space, the uniform decay rate of the local energy is exponential. See [24],[25].

Let \( u_0(x) = u_0(r), u_1(x) = u_1(r) \) and let \( u(r, t) \) solve the following system

\[ \begin{cases}
  u_{tt} - u_{rr} - (\Delta g r) u_r = 0 & (r, t) \in (r_0, +\infty) \times (0, +\infty), \\
  u(r_0) = 0, \\
  u(0) = u_0(r), \quad u_t(0) = u_1(r) & x \in \Omega
\end{cases} \]  

(5.4)

Note that

\[ \Delta_g u = \text{div}_g \left( u_r \frac{\partial}{\partial r} + \nabla g u \right) = u_{rr} + \Delta_g r u_r \quad \text{for} \quad x \in \Omega. \]  

(5.5)
Then $u(x,t) = u(r(x),t)$ solves the system (1.34). The energy and the local energy for the system (1.34) can be rewritten as

$$E(t,a) = \frac{1}{2} \int_{\Omega(a)} \left( u_t^2 + u_r^2 \right) dx_g, \quad (5.6)$$

$$E(t) = \frac{1}{2} \int_{\Omega} \left( u_t^2 + u_r^2 \right) dx_g. \quad (5.7)$$

Using the conclusion of scattering theory of the constant coefficient wave equation ([20]), for any positive integer $k$, we have:

- If $\Delta_g r = \frac{2k}{r}$ in (5.4), the decay rate of the local energy for the system (1.34) is exponential, whether the dimension $n$ is even or odd.

- If $\Delta_g r = \frac{2k-1}{r}$ in (5.4), the decay rate of the local energy for the system (1.34) is polynomial, whether the dimension $n$ is even or odd.

**Remark 5.1** Let $g$ satisfy

$$\det(G(x)) = C_1(\theta) r^{4k-2(n-1)}, \quad |x| \geq r_0, \quad (5.8)$$

where $C_1(\theta)$ is a positive function. With (3.24), we have

$$\Delta_g r = \frac{n-1}{r} + \frac{\partial \ln \sqrt{\det(G(x))}}{\partial r} = \frac{2k}{r}, \quad |x| \geq r_c. \quad (5.9)$$

**Remark 5.2** Let $g$ satisfy

$$\det(G(x)) = C_2(\theta) r^{4k-2n}, \quad |x| \geq r_0, \quad (5.10)$$

where $C_2(\theta)$ is a positive function. With (3.24), we have

$$\Delta_g r = \frac{n-1}{r} + \frac{\partial \ln \sqrt{\det(G(x))}}{\partial r} = \frac{2k-1}{r}, \quad |x| \geq r_c. \quad (5.11)$$

### 5.3 Uniform decay of local energy for general solutions

**Lemma 5.1** Let $u \in \widehat{H}^1_0(\Omega)$ be of compact support. Then

$$\int_{\Omega} uu_r dx_g = -\int_{\Omega} \frac{m_2}{2r} u^2 dx_g. \quad (5.12)$$

**Proof.** With (3.24), we have

$$\Delta_g r = \frac{n-1}{r} + \frac{\partial \ln \sqrt{\det(G(x))}}{\partial r} = \frac{m_2}{r}, \quad |x| \geq r_0. \quad (5.13)$$

Then for $|x| \geq r_0$,

$$\div_g u^2 \frac{\partial}{\partial r} = 2uu_r + u^2 \Delta_g r = 2uu_r + \frac{m_2}{r} u^2. \quad (5.14)$$

Integrating over $\Omega$, we have

$$\int_{\Omega} uu_r dx_g = -\int_{\Omega} \frac{m_2}{2r} u^2 dx_g. \quad (5.15)$$

The estimate (5.12) holds. □
Lemma 5.2 Let $u_0, u_1$ be of compact support. Let $u$ solve the system (1.34). Then

$$
\int_{\Omega} r \left( u_t^2 + |\nabla_y u|_g^2 \right) dx_g \leq \int_0^T \int_{\Omega} (u_t^2 + u_g^2) dx_g dt + \int_{\Omega} r \left( u_1^2 + |\nabla_y u_0|_g^2 \right) dx, \quad \forall t \geq 0. \tag{5.16}
$$

Proof. Let $Q = (r-t)$, letting $a \to +\infty$, it follows from (3.4) that

$$
\int_{\Omega} (r-t) \left( u_t^2 + |\nabla_y u|_g^2 \right) dx_g \bigg|_0^T = - \int_0^T \int_{\Omega} (u_t^2 + |\nabla_y u|_g^2) dx_g dt - 2 \int_0^T \int_{\Omega} u_t u_t dx_g dt
$$

$$
\leq - \int_0^T \int_{\Omega} |\nabla_{\Gamma_g} u|_g^2 dx_g dt. \tag{5.17}
$$

Then

$$
\int_{\Omega} (r-t) \left( u_t^2 + |\nabla_y u|_g^2 \right) dx_g \leq - \int_0^T \int_{\Omega} |\nabla_{\Gamma_g} u|_g^2 dx_g dt + \int_{\Omega} r \left( u_1^2 + |\nabla_y u_0|_g^2 \right) dx. \tag{5.18}
$$

Simple calculation shows that

$$
\int_{\Omega} r \left( u_t^2 + |\nabla_y u|_g^2 \right) dx_g \leq T E(0) - \int_0^T \int_{\Omega} |\nabla_{\Gamma_g} u|_g^2 dx_g dt + \int_{\Omega} r \left( u_1^2 + |\nabla_y u_0|_g^2 \right) dx
$$

$$
= \int_0^T \int_{\Omega} (u_t^2 + u_t^2) dx_g dt + \int_{\Omega} r \left( u_1^2 + |\nabla_y u_0|_g^2 \right) dx. \tag{5.19}
$$

The estimate (5.16) holds. 

Lemma 5.3 Let $u_0, u_1$ be of compact support. Let $u$ solve the system (1.34). Then

$$
\int_{\Omega} e^{r-t} \left( u_t^2 + |\nabla_y u|_g^2 \right) dx_g \leq \int_{\Omega} e^{r} \left( u_1^2 + |\nabla_y u_0|_g^2 \right) dx_g, \quad \forall t \geq 0. \tag{5.20}
$$

Proof. Let $Q = e^{r-t}$, letting $a \to +\infty$, it follows from (3.4) that

$$
\int_{\Omega} e^{r-t} \left( u_t^2 + |\nabla_y u|_g^2 \right) dx_g \bigg|_0^T = - \int_0^T \int_{\Omega} e^{r-t} \left( u_t^2 + |\nabla_y u|_g^2 \right) dx_g dt
$$

$$
- 2 \int_0^T \int_{\Omega} e^{r-t} u_t u_t dx_g dt
$$

$$
\leq - \int_0^T \int_{\Omega} e^{r-t} \left( u_t^2 + |\nabla_y u|_g^2 \right) dx_g dt + \int_{\Omega} \int_{\Omega} e^{r-t} (u_t^2 + u_t^2) dx_g dt
$$

$$
\leq 0. \tag{5.21}
$$

Then

$$
\int_{\Omega} e^{r-t} \left( u_t^2 + |\nabla_y u|_g^2 \right) dx_g \leq \int_{\Omega} e^{r} \left( u_1^2 + |\nabla_y u_0|_g^2 \right) dx_g. \tag{5.22}
$$

The estimate (5.20) holds. 

□
**Lemma 5.4** Let all the assumptions in Theorem 1.6 hold. Let \( u \) solve the system \((1.34)\). Then
\[
\int_0^T \int_\Omega \left( u_t^2 + u_r^2 + (2m-1)\left| \nabla_{\Gamma_g} u \right|^2 \right) \, dx \, dt \leq \int_\Omega r \left( u_t^2 + u_r^2 \right) \, dx + C(R_0)E(0). \tag{5.23}
\]

**Proof.** Let \( H = r \frac{\partial}{\partial r} \) in \((3.1)\). With \((1.63), (3.17)\) and \((3.24)\), we deduce that
\[
DH(\nabla_g u, \nabla_g u) = DH \left( u_r \frac{\partial}{\partial r}, u_r \frac{\partial}{\partial r} \right) + DH(\nabla_{\Gamma_g} u, \nabla_{\Gamma_g} u) = u_r^2 + rD^2_r \left( \nabla_{\Gamma_g} u, \nabla_{\Gamma_g} u \right) = u_r^2 + m_1 |\nabla_{\Gamma_g} u|_g^2, \quad |x| \geq r_0, \tag{5.24}
\]
and
\[
\text{div} \, H = 1 + r \Delta_g r = 1 + r \left( n - \frac{1}{r} + \frac{\partial \ln \sqrt{\det G(x)}}{\partial r} \right) = 1 + m_2, \quad |x| \geq r_0. \tag{5.25}
\]
From \((3.1)\), we obtain
\[
\int_0^T \int_{\partial \Omega(a)} \frac{\partial u}{\partial \nu} H(u) d\Gamma_g dt + \frac{1}{2} \int_0^T \int_{\partial \Omega(a)} \left( u_t^2 - |\nabla_{\Gamma_g} u|_g^2 \right) \langle H, \nu \rangle_g d\Gamma_g dt \nonumber
\]
\[
= (u_t, H(u)) \big|_0^T + \int_0^T \int_{\Omega(a)} \left( u_r^2 + m_1 |\nabla_{\Gamma_g} u|_g^2 \right) dx \, dt 
+ \int_0^T \int_{\Omega(a)} \frac{m_2}{2} \left( u_t^2 - |\nabla_{\Gamma_g} u|_g^2 \right) dx \, dt. \tag{5.26}
\]
Let \( P = \frac{m_2}{2} \), substituting \((3.3)\) into \((5.26)\), letting \( a \to \infty \), we derive that
\[
\int_0^T \int_{\Gamma} \frac{\partial u}{\partial \nu} H(u) d\Gamma_g dt + \frac{1}{2} \int_0^T \int_{\Gamma} \left( u_t^2 - |\nabla_{\Gamma_g} u|_g^2 \right) \langle H, \nu \rangle_g d\Gamma_g dt 
onumber
\]
\[
+ \int_0^T \int_{\Gamma} P u \frac{\partial u}{\partial \nu} d\Gamma_g dt - \frac{1}{2} \int_0^T \int_{\Gamma} u_t^2 \frac{\partial P}{\partial \nu} d\Gamma_g dt 
= (u_t, ru_r + \frac{m_2}{2} u) \big|_0^T + \frac{1}{2} \int_0^T \int_{\Omega(a)} \left( u_t^2 + u_r^2 + (2m_1 - 1) |\nabla_{\Gamma_g} u|_g^2 \right) dx \, dt. \tag{5.27}
\]
Set
\[
\Pi_\Gamma = \int_0^T \int_{\Gamma} \frac{\partial u}{\partial \nu} H(u) d\Gamma_g dt + \frac{1}{2} \int_0^T \int_{\Gamma} \left( u_t^2 - |\nabla_{\Gamma_g} u|_g^2 \right) \langle H, \nu \rangle_g d\Gamma_g dt 
+ \int_0^T \int_{\Gamma} P u \frac{\partial u}{\partial \nu} d\Gamma_g dt - \frac{1}{2} \int_0^T \int_{\Gamma} u_t^2 \frac{\partial P}{\partial \nu} d\Gamma_g dt. \tag{5.28}
\]
Since \( u|_\Gamma = 0 \), we obtain \( \nabla_{\Gamma_g} u|_\Gamma = 0 \), that is,
\[
\nabla_g u = \frac{\partial u}{\partial \nu} \quad \text{for} \quad x \in \Gamma. \tag{5.29}
\]
Similarly, we have
\[ \mathcal{H}(u) = \langle \mathcal{H}, \nabla_g u \rangle_g = \frac{\partial u}{\partial \nu} \langle \mathcal{H}, \nu \rangle_g \quad \text{for} \quad x \in \Gamma. \] (5.30)

Using the formulas (5.29) and (5.30) in the formula (5.28) on the portion \( \Gamma \), we arrive at
\[ \Pi_\Gamma = \frac{1}{2} \int_0^T \int_{\Gamma} \left( \frac{\partial u}{\partial \nu} \right)^2 d\Gamma_g dt = \frac{1}{2} \int_0^T \int_{\Gamma} r \left( \frac{\partial u}{\partial \nu} \right)^2 \frac{\partial r}{\partial \nu} d\Gamma_g dt \leq 0. \] (5.31)

Substituting (5.31) into (5.27), we have
\[ \left( u_t, ru_r + \frac{m_2}{2} u \right) \int_0^T + \frac{1}{2} \int_\Omega \left( u_t^2 + u_r^2 + (2m_1 - 1)|\nabla_{\Gamma_g} u|_g^2 \right) dx_g dt \leq 0. \] (5.32)

With (5.12) and (5.32), we deduce that
\[
\begin{align*}
&\int_0^T \int_\Omega \left( u_t^2 + u_r^2 + (2m_1 - 1)|\nabla_{\Gamma_g} u|_g^2 \right) dx_g dt \\
&\leq 2 \int_\Omega r \left( u_t + \frac{m_2}{2r} u \right) \left| dx_g + C(R_0) E(0) \right.
\leq \int_\Omega r \left( u_t^2 + u_r^2 + \frac{m_2}{r} u u_r + \frac{m_2^2}{4r^2} u^2 \right) dx_g + C(R_0) E(0) \\
&= \int_\Omega r \left( u_t^2 + u_r^2 \right) dx_g + \int_\Omega \left( m_2 u u_r + \frac{m_2^2}{4r} u^2 \right) dx_g + C(R_0) E(0) \\
&= \int_\Omega r \left( u_t^2 + u_r^2 \right) dx_g - \int_\Omega \frac{m_2}{4r} u^2 dx_g + C(R_0) E(0) \\
&\leq \int_\Omega r \left( u_t^2 + u_r^2 \right) dx_g + C(R_0) E(0). \tag{5.33}
\end{align*}
\]

Then, the estimate (5.23) holds. \( \square \)

**Proof of Theorem 1.6**

Substituting (5.16) into (5.23), we obtain
\[ \int_0^T \int_\Omega |\nabla_{\Gamma_g} u|_g^2 dx_g dt \leq C(R_0) E(0). \] (5.34)

Substituting (5.34) into (5.23), we obtain
\[ T E(T) \leq \int_\Omega r(u_t^2 + u_r^2) dx_g + C(R_0) E(0) \leq \int_{\Omega \setminus \Omega(a)} r \left( u_t^2 + |\nabla_g u|_g^2 \right) dx_g + C(a, R_0) E(0). \] (5.35)

With (5.20), we deduce that
\[
T E(T, a) \leq \int_{\Omega \setminus \Omega(a)} (r - T) \left( u_t^2 + |\nabla_g u|_g^2 \right) dx_g + C(a, R_0) E(0) \leq \int_\Omega e^{r-T} \left( u_t^2 + |\nabla_g u|_g^2 \right) dx_g + C(a, R_0) E(0) \leq C(a, R_0) E(0). \tag{5.36}
\]

The estimate (1.66) holds.
5.4 Space-time energy estimation for general solutions

**Lemma 5.5** Let $u \in \tilde{H}^1_0(\Omega)$ be of compact support. Then

$$
\int_{\Omega} r^{-s_2} uu_r dx_g \leq - \int_{\Omega} \frac{m_2 r^{-2s_2}}{4} u^2 dx_g.
$$

(5.37)

**Proof.** With (3.24), we have

$$
\Delta_g r = \frac{n - 1}{r} + \frac{\partial \ln \sqrt{\det (G(x))}}{\partial r} = m_2 r^{-s_2}, \quad |x| \geq r_0.
$$

(5.38)

Then for $|x| \geq r_0$,

$$
div_g u^2 r^{-s_2} \frac{\partial}{\partial r} = 2r^{-s_2} uu_r - s_2 r^{-s_2-1} u^2 + u^2 r^{-s_2} \Delta_g r
$$

$$
= 2r^{-s_2} uu_r + (m_2 r^{-2s_2} - s_2 r^{-s_2-1}) u^2.
$$

(5.39)

Integrating over $\Omega$, we have

$$
2 \int_{\Omega} r^{-s_2} uu_r dx_g = - \int_{\Omega} r^{-2s_2} (m_2 - s_2 r^{-s_2-1}) u^2 dx_g.
$$

(5.40)

Note that

$$
s_2 r^{-s_2-1} \leq s_2 r_0^{-s_2-1} < \frac{(s_2 + 1)}{2} r_0^{-s_2-1} < \frac{m_2}{2}, \quad r \geq r_0.
$$

(5.41)

The estimate (5.37) holds. $\square$

**Lemma 5.6** Let Assumption (B) holds true. Let

$$
P = r^{-s_2} \left( N + m_1 (s_1 - 1)^{-1} (r_0^{-s_1} - r^{-s_1}) \right), \quad |x| \geq r_0.
$$

(5.42)

where $N$ is a positive constant. Then, for sufficiently large $N$,

$$
\Delta_g P \leq 0, \quad |x| \geq r_0.
$$

(5.43)

**Proof.** Let

$$
h(r) = \left( N + m_1 (s_1 - 1)^{-1} (r_0^{-s_1} - r^{-s_1}) \right), \quad r \geq r_0.
$$

(5.44)

Then,

$$
N \leq h(r) \leq N + m_1 (s_1 - 1)^{-1} r_0^{-s_1}, \quad r \geq r_0,
$$

(5.45)

$$
h'(r) = m_1 r^{-s_1} \quad r \geq r_0.
$$

(5.46)

With (3.24), we have

$$
\Delta_g r = \frac{n - 1}{r} + \frac{\partial \ln \sqrt{\det (G(x))}}{\partial r} = m_2 r^{-s_2}, \quad |x| \geq r_0.
$$

(5.47)

We deduce that

$$
\Delta_g P = \Delta_g (r^{-s_2} h(r)) = div_g \left( -s_2 r^{-s_2-1} h(r) + m_1 r^{-s_1-s_2} \right) \frac{\partial}{\partial r}
$$

$$
= \left( s_2 (s_2 + 1) r^{-s_2-2} - m_2 s_2 r^{-2s_2-1} \right) h(r) - m_1 (s_1 + 2s_2) r^{-s_1-s_2-1} + m_1 m_2 r^{-s_1-2s_2}
$$

$$
\leq s_2 r^{-2s_2-1} \left( (s_2 + 1) r^{s_2-1} - m_2 \right) h(r) + m_1 m_2 r^{-s_1-2s_2}, \quad |x| \geq r_0.
$$

(5.48)
Note that
\[ r^{s_2-1} \leq r_0^{s_2-1}, \quad r^{1-s_1} \leq r_0^{1-s_1}, \quad |x| \geq r_0, \quad (5.49) \]
\[ (s_2 + 1)r_0^{s_2-1} - m_2 < 0. \quad (5.50) \]
Then
\[ r^{2s_2+1} \Delta_g P \leq s_2 \left( (s_2 + 1)r_0^{s_2-1} - m_2 \right) h(r) + m_1 m_2 r^{1-s_1} \]
\[ \leq s_2 \left( (s_2 + 1)r_0^{s_2-1} - m_2 \right) h(r) + m_1 m_2 r_0^{1-s_1} \]
\[ \leq N s_2 \left( (s_2 + 1)r_0^{s_2-1} - m_2 \right) + m_1 m_2 r_0^{1-s_1}, \quad |x| \geq r_0. \quad (5.51) \]
For sufficiently large \( N \),
\[ \Delta_g P \leq 0, \quad |x| \geq r_0. \quad (5.52) \]
The estimate (5.43) holds. \( \square \)

**Proof of Theorem 1.7**

Let
\[ h(r) = \left( N + m_1(s_1 - 1)^{-1}(r_0^{1-s_1} - r^{1-s_1}) \right), \quad r \geq r_0. \quad (5.53) \]
where \( N \geq 1 \) is a positive constant. Then,
\[ N \leq h(r) \leq N + m_1(s_1 - 1)^{-1}r_0^{1-s_1}, \quad r \geq r_0, \quad (5.54) \]
\[ h'(r) = m_1 r^{-s_1} \quad r \geq r_0. \quad (5.55) \]
Let
\[ \mathcal{H} = h(r) \frac{\partial}{\partial r} \text{ in } (3.1), \quad (5.56) \]
With (1.67), (3.17) and (3.24), we deduce that
\[ D\mathcal{H}(\nabla_g u, \nabla_g u) = D\mathcal{H}(u_r \frac{\partial}{\partial r}, u_r \frac{\partial}{\partial r}) + D\mathcal{H}(\nabla_{\Gamma_g} u, \nabla_{\Gamma_g} u) \]
\[ = m_1 r^{-s_1} u_r^2 + h(r) D^2 r(\nabla_{\Gamma_g} u, \nabla_{\Gamma_g} u) \]
\[ \geq m_1 r^{-s_1}|\nabla_g u|^2_g, \quad |x| \geq r_0, \quad (5.57) \]
and
\[ \text{div } \mathcal{H} = m_1 r^{-s_1} + h(r) \Delta_g r = m_1 r^{-s_1} + h(r) \left( \frac{n-1}{r} + \frac{\partial \ln \sqrt{\det (G(x))}}{\partial r} \right) \]
\[ = m_1 r^{-s_1} + m_2 r^{-s_2} h(r), \quad |x| \geq r_0. \quad (5.58) \]
From (3.1), we have
\[ \int_0^T \int_{\partial \Omega(t)} \frac{\partial u}{\partial r} \mathcal{H}(u)d\Gamma_g dt + \frac{1}{2} \int_0^T \int_{\partial \Omega(t)} \left( u_t^2 - |\nabla_g u|^2_g \right) \langle \mathcal{H}, \nu \rangle_g d\Gamma_g dt \]
\[ = (u_t, \mathcal{H}(u)) \bigg|_0^T + \frac{1}{2} \int_0^T \int_{\Omega(t)} D\mathcal{H}(\nabla_g u, \nabla_g u) dx_g dt + \int_0^T \int_{\Omega(t)} \frac{\text{div } \mathcal{H}}{2} \left( u_t^2 - |\nabla_g u|^2_g \right) dx_g dt \]
\[ \geq (u_t, \mathcal{H}(u)) \bigg|_0^T + \frac{1}{2} \int_0^T \int_{\Omega(t)} m_1 r^{-s_1} \left( u_t^2 + |\nabla_g u|^2_g \right) dx_g dt \]
\[ + \frac{1}{2} \int_0^T \int_{\Omega(t)} m_2 r^{-s_2} h(r) \left( u_t^2 - |\nabla_g u|^2_g \right) dx_g dt. \quad (5.59) \]
Let $P = \frac{1}{2}m_2r^{-s_2}h(r)$, substituting (3.3) into (5.59), letting $a \to \infty$, we obtain
\[
\int_0^T \int_{\Gamma} \frac{\partial u}{\partial \nu} \mathcal{H}(u) d\Gamma_g dt + \frac{1}{2} \int_0^T \int_{\Gamma} \left( u_t^2 - |\nabla g u|^2 \right) \langle \mathcal{H}, \nu \rangle_g d\Gamma_g dt \\
+ \int_0^T \int_{\Gamma} Pu \frac{\partial u}{\partial \nu} d\Gamma_g dt - \frac{1}{2} \int_0^T \int_{\Gamma} u_t^2 \frac{\partial P}{\partial \nu} d\Gamma_g dt \\
\geq (u_t, \mathcal{H}(u) + Pu) \bigg|_0^T + \int_0^T \int_{\Omega} \frac{m_1r^{-s_1}}{2} \left( u_t^2 + |\nabla g u|^2 \right) dx_g dt \\
- \frac{1}{2} \int_{\Omega} u^2 \Delta_g P dx_g dt.
\] (5.60)

Set
\[
\Pi_\Gamma = \int_0^T \int_{\Gamma} \frac{\partial u}{\partial \nu} \mathcal{H}(u) d\Gamma_g dt + \frac{1}{2} \int_0^T \int_{\Gamma} \left( u_t^2 - |\nabla g u|^2 \right) \langle \mathcal{H}, \nu \rangle_g d\Gamma_g dt \\
+ \int_0^T \int_{\Gamma} Pu \frac{\partial u}{\partial \nu} d\Gamma_g dt - \frac{1}{2} \int_0^T \int_{\Gamma} u_t^2 \frac{\partial P}{\partial \nu} d\Gamma_g dt.
\] (5.61)
Since $u|_{\Gamma} = 0$, we obtain $\nabla \Gamma, u|_{\Gamma} = 0$, that is,
\[
\nabla g u = \frac{\partial u}{\partial \nu} \quad \text{for} \quad x \in \Gamma.
\] (5.62)

Similarly, we have
\[
\mathcal{H}(u) = \langle \mathcal{H}, \nabla g u \rangle_g = \frac{\partial u}{\partial \nu} \langle \mathcal{H}, \nu \rangle_g \quad \text{for} \quad x \in \Gamma.
\] (5.63)

Using the formulas (5.62) and (5.63) in the formula (5.61) on the portion $\Gamma$, we obtain
\[
\Pi_\Gamma = \frac{1}{2} \int_0^T \int_{\Gamma} \left( \frac{\partial u}{\partial \nu} \right)^2 \langle \mathcal{H}, \nu \rangle_g d\Gamma_g dt = \frac{1}{2} \int_0^T \int_{\Gamma} h(r) \left( \frac{\partial u}{\partial \nu} \right)^2 \frac{\partial r}{\partial \nu} d\Gamma_g dt \leq 0.
\] (5.64)

Substituting (5.64) into (5.60), for sufficiently $N$, with (5.43) we deduce that
\[
(u_t, \mathcal{H}(u) + Pu) \bigg|_0^T + \int_0^T \int_{\Omega} \frac{m_1r^{-s_1}}{2} \left( u_t^2 + |\nabla g u|^2 \right) dx_g dt \leq 0.
\] (5.65)

With (5.37), we obtain
\[
\int_0^T \int_{\Omega} m_1r^{-s_1} \left( u_t^2 + |\nabla g u|^2 \right) dx_g dt \\
\leq 2 \int_{\Omega} h(r) \left| u_t \left( u_r + \frac{m_2}{2r^{s_2}} u \right) \right| dx_g + C(R_0)E(0) \\
\leq C \int_{\Omega} \left| u_t (u_r + \frac{m_2r^{-s_2}}{2} u) \right| dx_g + C(R_0)E(0) \\
\leq C \int_{\Omega} \left( u_t^2 + u_r^2 + m_2r^{-s_2} u u_r + \frac{m_2^2r^{-2s_2}}{4} u^2 \right) dx_g + C(R_0)E(0) \\
= C \int_{\Omega} (u_t^2 + u_r^2) dx_g + m_2C \int_{\Omega} \left( r^{-s_2} u u_r + \frac{m_2^2r^{-2s_2}}{4} u^2 \right) dx_g + C(R_0)E(0) \\
\leq C \int_{\Omega} (u_t^2 + u_r^2) dx_g + C(R_0)E(0) \\
\leq C(R_0)E(0).
\] (5.66)

The estimate (1.70) holds. $\square$

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