ON THE EXISTENCE OF MINIMAL TORI IN $S^3$
OF ARBITRARY SPECTRAL GENUS.

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Abstract

Harmonic mappings are a generalisation of geodesics, and are defined as the solutions to a natural variational problem. Interest in them began in 1873 with Plateau’s problem of finding surfaces of minimal area bounded by given closed space curves. The field has been studied by mathematicians and physicists ever since, and is now both broad and extremely active. In this dissertation I consider harmonic maps which can be studied using integrable systems, and thus by algebro-geometric means. In particular I focus upon a simple case of both geometric and physical interest, namely harmonic maps \( f \) from a 2-torus (with conformal structure \( \tau \)) to the 3-sphere. In [10] Hitchin showed that (except in the case of a conformal map to a totally geodesic \( S^2 \subseteq S^3 \)) the data \( (f, \tau) \) is in one-to-one correspondence with certain algebro-geometric data. This data consists of a hyperelliptic curve \( X \) (called the spectral curve) together with a projection map \( \pi : X \to \mathbb{CP}^1 \), a pair of holomorphic functions on \( X - \pi^{-1}\{0, \infty\} \), and a line bundle on \( X \), all satisfying certain conditions. He proved (case-by-case) that for \( g \leq 3 \), there are curves of genus \( g \) that support the required data, and hence describe harmonic maps \( f : (T^2, \tau) \to S^3 \). Once such a curve is found, the line bundle may be chosen from a real \((g - 2)\)-dimensional family, and each choice yields a new harmonic map. Of especial interest are conformal harmonic maps as their images are minimal surfaces. I show that for each \( g \geq 0 \), there are countably many conformal harmonic maps \( f : (T^2, \tau) \to S^3 \) whose spectral curves have genus \( g \). All of these harmonic tori have rectangular conformal type, and those with \( g > 2 \) come in real \((g - 2)\)-dimensional families.
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To Mum and Dad
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Chapter 1

Introduction

A map $f : M \to N$ between Riemannian manifolds is harmonic if it is critical for the energy functional $E(f) = \int_M \| df \|^2_{\mathcal{P}} dM$, where $\| \cdot \|_{\mathcal{P}}$ is the norm on $T^*_p M \otimes T_{f(p)} N$ induced from the Riemannian metrics on $M$ and $N$. Examples of harmonic maps include geodesics, harmonic functions and holomorphic and antiholomorphic maps between Kähler manifolds. If $f$ is an isometric immersion, then it is critical for the energy functional if and only if it is critical for the area functional; hence the relevance of harmonic maps to Plateau’s problem. When $M$ is a surface, the harmonic map equations are invariant under conformal changes in the domain metric. Thus a conformal map of $M$ is harmonic precisely when its image is a minimal surface. The conformal invariance also means that one can consider harmonic maps of a Riemann surface.

Harmonic maps from a Riemann surface to a compact Lie group or symmetric space are of particular interest. One reason for this is their relationship with the important Yang-Mills equations of mathematical physics. The harmonic map equations are then locally the self-dual Yang-Mills equations on $\mathbb{R}^4$ with signature $(+, +, -, -)$, invariant under translation in the last two variables. Physicists study harmonic maps of $\mathbb{R}^2$ and $\mathbb{R}^{1,1}$ in order to gain insight into the Yang-Mills equations. Harmonic maps of surfaces also arise in the study of surfaces of geometric interest. For example, the theorem of Ruh and Vilms [13] asserts that a surface has constant mean curvature precisely when its Gauss map is harmonic. Notice that the Gauss map is then a harmonic map to $S^2$, and hence to $S^3$. Similar characterizations exist for both Willmore surfaces and surfaces of constant negative Gauss curvature [2, 11].

The last twenty-five years have seen an explosion of interest in this area. A major theme of this research has been the “classification” of harmonic maps. A series of papers (e.g. [6, 7, 15, 16, 14, 5]) gave descriptions of harmonic maps from $S^2$ to various symmetric spaces in terms of an algebraic curve in an auxiliary complex manifold. Hitchin [10] showed that harmonic maps from a 2-torus (with some complex structure $\tau$) to the 3-sphere (with standard metric) also enjoy an algebro-geometric description. Underlying this is the idea [12, 17, 18, 14] that by the insertion of a parameter into the harmonic map equations, they can be reformulated as a family of equations of a particularly pleasant form, namely as equations of Lax type. These equations linearise on the Jacobian of a hyperelliptic curve, called the spectral curve [9, 3]. Thus to a
harmonic map \( f : (T^2, \tau) \rightarrow S^3 \) there corresponds a hyperelliptic curve. Hitchin also proved that if one begins with a hyperelliptic curve with certain additional data, one can construct a torus \((T^2, \tau)\) and a harmonic map \( f : (T^2, \tau) \rightarrow S^3 \). Hence in order to find harmonic tori in the 3-sphere, one seeks hyperelliptic curves supporting the requisite additional data. Not surprisingly, this is not a trivial problem. Hitchin proved (case-by-case) that for each \( g \leq 3 \), there is a torus \((T^2, \tau)\) and a harmonic map \( f : (T^2, \tau) \rightarrow S^3 \) whose spectral curve has genus \( g \). I show that for each \( g \geq 0 \), there are conformal harmonic maps of tori to \( S^3 \) whose spectral curves have genus \( g \). The importance of conformality is that the conformal harmonic maps are precisely those whose images are minimal surfaces. The precise algebro-geometric statement is given below.

Given a curve \( X: y^2 = x \prod_{i=1}^{g} (x - \alpha_i)(x - \bar{\alpha}_i^{-1}) \), \( |\alpha_i| \neq 0, 1 \), let

- \( \pi \) be the projection to the \( x \)-plane,
- \( \sigma \) be the hyperelliptic involution \((x, y) \mapsto (x, -y)\),
- \( \rho \) be the antiholomorphic involution, \((x, y) \mapsto (x^{-1}, (\prod_{i=1}^{g} \lambda_i \bar{\lambda}_i^{-1})^{\frac{1}{2}} \frac{\bar{y}}{x^{g+1}})\), where we choose the square root so that \( \rho \) fixes the points with \( |x| = 1 \),
- \( \gamma_1 \) be a curve in \( X \) joining the two points in \( \pi^{-1}(1) \), and
- \( \gamma_{-1} \) a curve in \( X \) joining the two points in \( \pi^{-1}(-1) \).

Then for each genus \( g > 0 \), there are countably many curves \( X \) as above possessing meromorphic differentials \( \Theta \) and \( \Psi \) that satisfy the following conditions:

1. \( \Theta \) and \( \Psi \) have double poles at \( \pi^{-1}\{0, \infty\} \), and are holomorphic elsewhere. The principal parts of \( \Theta \) and \( \Psi \) are linearly independent over \( \mathbb{R} \). (The ratio of these gives the conformal type \( \tau \) of the torus.)

2. \( \Theta \) and \( \Psi \) satisfy the symmetry conditions \( \sigma^*\Theta = -\Theta \), \( \sigma^*\Psi = -\Psi \), and the reality conditions \( \rho^*\Theta = \bar{\Theta} \), \( \rho^*\Psi = \bar{\Psi} \).

3. The integrals of \( \Theta \) and \( \Psi \) over \( \gamma_1 \), \( \gamma_{-1} \), and over a basis for the homology of \( X \), are integers.

The last condition is by far the most difficult to satisfy; it demands that transcendental objects (the integrals) be integers. This places a severe restriction on the hyperelliptic curve.

My approach to this problem follows the work of Ercolani, Knörrer and Trubowitz [8], in which they prove that for each even \( g \geq 2 \), there is a constant mean curvature torus in \( \mathbb{R}^3 \) whose spectral curve has genus \( g \).
Chapter 2

Algebro-Geometric Description

In his paper [10], Hitchin showed that specifying a conformal structure \( \tau \) on \( T^2 \), and a harmonic map \( f : (T^2, \tau) \to SU(2) \) (other than a branched conformal map to a totally geodesic \( S^2 \subseteq S^3 \)) is equivalent to specifying certain algebro-geometric data, described in theorems 2.1 and 2.2 below. He in fact gave an algebro-geometric description of harmonic sections of an \( SU(2) \) principal bundle over \( (T^2, \tau) \), and then specified the conditions the data must satisfy in order to correspond to the special case of a harmonic mapping. The purpose of this section is to give an expository account of Hitchin’s work, outlining the ideas, but omitting most proofs.

Given Riemannian manifolds \((M, g)\) and \((N, h)\), the energy \( E(f) \) of \( f : M \to N \) is

\[
E(f) = \frac{1}{2} \int_M \|df\|^2_p dM,
\]

where \( \| \cdot \| \) is the metric on \( T^*_pM \otimes T_{f(p)}N \) induced from \( g \) and \( h \). We say that \( f \) is harmonic if whenever \( \{f_t\} \) is a one-parameter family of smooth maps with \( f_0 = f \),

\[
\frac{d}{dt} \bigg|_{t=0} E(f_t) = 0.
\]

If \( M \) is a surface, then the harmonicity of \( f \) depends only on the conformal class of the metric \( g \), and so we can speak of harmonic maps of Riemann surfaces. If in addition \( f : M \to N \) is conformal, then it is harmonic if and only if \( f(M) \) is a minimal surface.

Let \( M \) be a compact Riemann surface, \( G \) a compact Lie group with bi-invariant metric, and \( f : M \to G \) a harmonic map. We denote the Lie algebra \( T_eG \) of \( G \) by \( \mathfrak{g} \). On \( G \) we define the Maurer-Cartan form \( \omega \) to be the unique \( \mathfrak{g} \)-valued 1-form on \( G \) that is left-invariant and acts as the identity on \( T_eG \). \( \omega \) satisfies the Maurer-Cartan equation

\[
d\omega + \frac{1}{2} [\omega, \omega] = 0,
\]

and for a linear group \( \omega \) is given by \( g^{-1}dg \). Given any smooth map \( f \) from \( M \) to \( G \), we may pull back the Maurer-Cartan form to obtain a \( \mathfrak{g} \)-valued 1-form \( \phi \) on \( M \) satisfying

\[
d\phi + \frac{1}{2} [\phi, \phi] = 0. \tag{2.1}
\]
Conversely, if $U$ is a simply-connected and connected open subset of $M$, then given a $g$-valued 1-form $\phi$ on $U$ satisfying the Maurer-Cartan equation, we may integrate it to obtain a smooth map $f : U \to G$ such that $f^*(\omega) = \phi$, where $f$ is defined only up to left translation. Since the metric on $G$ is left invariant, one expects that there is a condition characterising those $\phi$ that correspond to harmonic maps, and we calculate it below.

For a one-parameter family $\{f_t\}$,

$$\frac{d}{dt} \bigg|_{t=0} E(f_t) = \frac{1}{2} \int_M \frac{d}{dt} \bigg|_{t=0} \|df_t\|^2_p dM$$

$$= \frac{1}{2} \int_M \frac{d}{dt} \bigg|_{t=0} \|df_t f_t^{-1}\|^2_p dM$$

$$= \frac{1}{2} \int_M \frac{d}{dt} \bigg|_{t=0} \langle df_t f_t^{-1}, \ast(df_t f_t^{-1})\rangle_p dM.$$

Writing $f_t = e^{th} f$,

$$\frac{d}{dt} E(f_t) \bigg|_{t=0} = \int_M \langle f dh f^{-1}, \ast f^{-1}(df)\rangle_p$$

$$= \int_M \langle dh, \ast(f^{-1}df)\rangle_p$$

$$= \int_M d\langle h, \ast(f^{-1}df)\rangle_p - \langle h, d \ast(f^{-1}df)\rangle_p,$$

so we see that $f$ is harmonic if and only if

$$d \ast \phi = 0. \quad (2.2)$$

Equations (2.1) and (2.2) may be rewritten in terms of natural connections on $f^*TG$. $TG$ has connections $d_L$ and $d_R$ corresponding to its trivialisations by left and right translation, respectively. These are related by

$$d_R = d_L + ad_\omega, \text{ (where } ad_\omega(a) = [\omega, a])$$

and the Levi-Civita connection $d_A$ is the average of the two:

$$d_A = \frac{1}{2}(d_L + d_R).$$

We will use the same notation for the pull-back of these connections to $f^*TG$, and we may rewrite (2.1) as

$$d_A(\phi) = 0. \quad (2.3)$$

Denoting the $(1,0)$-component of $\frac{1}{2}\phi$ by $\Phi$, we have

$$\frac{1}{2} \phi = \Phi - \Phi^*,$$
and since
\[ *(\Phi - \Phi^*) = \sqrt{-1}(\Phi + \Phi^*), \] (2.4)
we have
\[ [\phi, *\phi] = 0, \]
so that (2.2) is equivalent to
\[ d_A(*\phi) = 0. \] (2.5)
Using (2.4) again, equations (2.3) and (2.5) are
\[ \bar{\partial}_A \Phi - \partial_A \Phi^* = 0 \]
\[ \bar{\partial}_A \Phi + \partial_A \Phi^* = 0 \]
so give
\[ \bar{\partial}_A \Phi = 0. \]
We have also that \( d_L = d_A - \frac{1}{2} \phi \) is flat, and hence
\[ 0 = d_L^2 = (d_A - \frac{1}{2} \phi)^2 = d_A^2 + (\frac{1}{2} \phi)^2, \text{ (from (2.3))} \]
\[ = F_A + (\Phi - \Phi^*)^2, \]
so
\[ F_A = [\Phi, \Phi^*]. \]

These equations may be considered as taking place in a trivial principal \( G \)-bundle over \( M \), as follows. The trivial principal \( G \)-bundle
\[ G \times G \]
\[ \downarrow \]
\[ G \]
with projection \((g_1, g_2) \mapsto g_1\) and action \((g_1, g_2) \cdot h = (g_1, g_2 h)\) has the obvious trivial connection, which we shall denote by \( \nabla_L \), along with the trivial connection \( \nabla_R = g^{-1} \nabla_L g \). More explicitly, writing \( \omega_L \) and \( \omega_R \) for the connection 1-forms of \( \nabla_L \) and \( \nabla_R \), then
\[ \omega_L = g_2^{-1} dg, \]
\[ \omega_R = (g_1 g_2)^{-1} d(g_1 g_2) \]
\[ = \omega_L + Ad_{g_2}^{-1} \omega. \]

We set
\[ \nabla_A = \frac{1}{2}(\nabla_L + \nabla_R). \]
Pulling back under the map \( f \), we obtain a trivial principal bundle \( P \), with connections \( \nabla_L, \nabla_R, \nabla_A \) and \( \phi = Ad_{g_2}(\omega_R - \omega_L) \). Note that \( Ad(P) = f^*(TG) \). Again we may write \( \frac{1}{2} \phi = \Phi - \Phi^* \), and can consider \( \Phi \) to be a section of \( Ad(P) \otimes K \), where \( K \) is the canonical bundle of holomorphic 1-forms on \( M \).
Hitchin seeks solutions to the pair of equations

\[
\begin{align*}
\bar{\partial}_A \Phi &= 0 \\
F_A &= [\Phi, \Phi^*]
\end{align*}
\]  

(2.6)

for a connection \( A \) in a principal \( G \)-bundle \( P \) over \( M \), and section \( \Phi \) of \( \text{Ad}(P) \otimes K \), called a Higgs field. If these data come from a harmonic map \( f : M \to G \), then one additionally has that the connections

\[
\nabla_L = \nabla_A - \Phi + \Phi^* \\
\nabla_R = \nabla_A + \Phi - \Phi^*
\]

are trivial. This is also a sufficient condition for the data \( (A, \Phi) \) to come from a harmonic map as if \( \nabla_L, \nabla_R \) as defined above are trivial then the difference between these trivialisations is a map \( f : M \to G \), and equations (2.6) tell us that \( f \) is harmonic. It is however fruitful to first study general solutions to this pair of equations, and then focus on those corresponding to harmonic maps. The equations (2.6) are locally the self-dual Yang-Mills equations on \( \mathbb{R}^4 \) with signature \((+, +, -, -)\), invariant under translation in the last two variables. The equations (2.6) can be made more conducive to analysis by the introduction of a parameter \( x \in \mathbb{C}^* \), termed the spectral parameter. An easy check reveals that equations (2.6) are equivalent to the statement that for every \( x \in \mathbb{C}^* \), the connections

\[
d_x = d_A + x^{-1}\Phi - x\Phi^*
\]

(2.7)

are flat.

We shall henceforth narrow our focus to the case where \( M \) is a torus with some conformal structure \( \tau \) and \( G = SU(2) \). Then (2.7) defines a family of flat \( SL(2, \mathbb{C}) \) connections. \( SU(2) \cong S^3 \) is a natural and simple case of interest. Ultimately we will assign an algebraic curve to each solution of equations (2.6); the fact that \( SU(2) \) is a rank one symmetric space enables us to describe the solution using a single curve, and the fact that the elements of \( SL(2, \mathbb{C}) \) are \( 2 \times 2 \) matrices gives that this curve is hyperelliptic. We refer the reader to [4] for a study of harmonic maps of complex \( n \)-tori into symmetric spaces, and note in particular their result that any non-conformal harmonic map of a (real) 2-torus into a rank one symmetric space is of finite type. The restriction to tori is however more essential; an integrable systems approach to harmonic maps of higher genus Riemann surfaces has so far proven elusive. In what follows we will make use of the fact that the fundamental group \( \pi_1(T^2) \) is abelian.

Several geometrically interesting properties of a harmonic map \( f : (T^2, \tau) \to S^3 \) can be described in terms of the connection \( A \) and Higgs field \( \Phi \). In particular, \( f \) is branched conformal if and only if \( \det \Phi = 0 \) (with branch points the zeros of \( \Phi \)) and \( f \) maps to a totally geodesic \( S^2 \subseteq S^3 \) if and only if \( A \) is reducible to a \( U(1) \) connection. This latter condition is also equivalent to the existence of a gauge transformation \( g \) leaving \( A \) invariant and satisfying \( g^2 = -1, g^{-1}\Phi g = -\Phi \).

We have a holomorphic family of flat \( SL(2, \mathbb{C}) \) connections on \( T^2 \), and so we study the holonomy of these connections. We shall consider them as connections in a vector
bundle $V$ over $(T^2, \tau)$ with the $SU(2)$ structure exhibited by a quaternionic structure

$$j : V \to V, \ j^2 = -1,$$

and a symplectic form

$$\omega : V \times V \to \mathbb{C}.$$

Then $\Phi$ is a holomorphic section of $\text{End} \ V \otimes K$ with trace zero.

In the case where $\det \Phi = 0$, $\Phi \not\equiv 0$, there are additional holomorphic invariants that we may associate to the solution. We then have $\Phi^2 = 0$, and may define a holomorphic line bundle $L \subseteq V$ by $L \subseteq \ker \Phi$. Thus $\Phi : L^* \cong V/L \to L \otimes K$, so we may consider $\Phi$ as a holomorphic section $u$ of $L^2 \otimes K$. It is natural to ask whether $d_A$ preserves the sub-bundle $L$; the obstruction being the holomorphic section $v$ of $L^{-2} \otimes K$ defined by

$$vs^2 = \omega(\partial_A s, s),$$

where $s$ is any local holomorphic section of $L$. If $v \not\equiv 0$, then the extension of $L$ defining $V$ is non-trivial. The quadratic differential $uv \in H^0(T^2, K^2) \cong \mathbb{C}$ is then a constant multiple of the second fundamental form of $f(T^2) \subseteq S^3$. Assume additionally that $A$ is irreducible. Then $v \in H^0(T^2, L^2 \otimes K)$ is not identically zero, so $\deg L^2 \leq 0$. If $\deg L^2 < 0$, then $u$ vanishes identically, and hence so does $\Phi$. But from $F_A = [\Phi, \Phi^*]$ we have then that $A$ is flat, so is given by a representation of $\pi_1(T^2)$ in $SU(2)$. Since $\pi_1(T^2)$ is abelian, we see that $A$ reduces to a $U(1)$ connection. Thus $\deg L^2 = 0$, and $u$ is nowhere vanishing. Hence a branched conformal harmonic map $f : (T^2, \tau) \to S^3$ whose image does not lie in a totally geodesic $S^2$ is in fact a conformal immersion.

Take $p \in T^2$ and generators $a$ and $b$ for $\pi_1(T^2, p) = \mathbb{Z} \oplus \mathbb{Z}$ that are conjugate to $[0, 1]$ and $[0, \tau]$ respectively. For each $x \in \mathbb{C}^*$, denote by $H(x, p)$ and $K(x, p)$ the holonomy of $d_x$ around $a$ and $b$ respectively, and write $h(x) = \text{tr}H(x, p), \ k(x) = \text{tr}K(x, p)$. Note that the conjugacy classes of $H(x, p)$ and $K(x, p)$ are independent of the choice of base point $p$. Denoting by $\mu(x)$ and $\mu^{-1}(x)$ the eigenvalues of $H(x, p), \mu^2(x) - h(x)\mu(x) + 1 = 0,$ and so

$$\mu(x) = \frac{1}{2} \left( h(x) \pm \sqrt{h(x)^2 - 4} \right).$$

This defines a 2-sheeted branched cover of $\mathbb{C}^*$, which we shall compactify to a cover of $\mathbb{C}P^1$. The resulting curve is algebraic:

**Proposition 2.1** For each $x \in \mathbb{C}^*$, let $H(x)$ be the holonomy of $d_x$ around $[0, 1]$, and $h(x) = \text{tr}H(x)$. The function $h(x)^2 - 4$ has finitely many odd order zeros in $\mathbb{C}^*$.

Unsurprisingly, the proof of this result employs the fact that an elliptic operator on a compact domain has but a finite-dimensional kernel. It is worth remarking that it also utilisesthe full two-dimensional compactness of the torus, rather than the mere one-dimensional compactness of the closed curve $[0, 1]$.

If the holonomy is identically trivial, $H(x) \equiv 1, K(x) \equiv 1$ (possibly after tensoring with a flat $\mathbb{Z}_2$-bundle), then this construction will not yield a two-sheeted
covering, and cannot be described using the methods of [10]. However this occurs if and only if the data \((A, \Phi)\) correspond to a conformal map into a totally geodesic \(S^2 \subseteq S^3\), and such mappings can be described using divisors on \((T^2, \tau)\). We shall assume henceforth that we are not in this trivial case.

Importantly, the branched cover of \(\mathbb{C}^*\) defined by the holonomy \(H(x)\) is the same as that defined by \(K(x)\), this being is a consequence of the fact that the fundamental group of the torus \(T^2\) is abelian. The next step is to study the behaviour of the holonomy of \(d_x\) as \(x \to 0\) and \(\infty\), and thus determine the branching behaviour of the compactified curve.

Now \(A\) is an \(SU(2)\) connection, and thus \(j^{-1}d_Aj = d_A\). Since \(j^{-1}\Phi j = -\Phi^*\), this gives

\[ j^{-1}d_xj = d_{\bar{x}}^{-1}, \]

so

\[ (H(x)^{-1})^* = j^{-1}H(x)j = H(\bar{x}^{-1}), \]

and hence

\[ h(\bar{x}^{-1}) = \overline{h(x)}. \]

Thus the behaviour of the holonomy as \(x \to \infty\) is determined by that as \(x \to 0\). Moreover, one can show that the eigenspaces of the holonomy matrices are determined simply by the holomorphic structure of \(V\), and then use the regularity of this structure to study the holonomy in the limit \(x \to 0\).

**Proposition 2.2** Let \((A, \Phi)\) be a solution of (2.6) with \(\det(\Phi) = -\eta^2dz^2 \neq 0\). For each \(x \in \mathbb{C}^*\), we use \(\mu(x)\) and \(\nu(x)\) to denote the eigenvalues of the holonomy of \(d_x\) around \([0, 1]\) and \([0, \tau]\) respectively. There is a punctured neighbourhood of 0 in \(\mathbb{C}\) such that

\[
\pm \log \mu(x) = \eta x^{-1} + a + xb(x)
\]

\[
\pm \log \nu(x) = \eta \tau x^{-1} + \tilde{a} + \tilde{x}b(x)
\]

where \(b(x)\) and \(\tilde{b}(x)\) are even holomorphic functions.

**Proposition 2.3** Let \((A, \Phi)\) be a solution of (2.6) with \(\det(\Phi) = 0\) and \(A\) irreducible. There is a punctured neighbourhood of 0 in \(\mathbb{C}\) such that

\[
\pm \log \mu(x) = \kappa x^{-\frac{1}{2}} + \sqrt{-1k\pi} + x^{\frac{1}{2}}b(x^{\frac{1}{2}})
\]

\[
\pm \log \nu(x) = \kappa \tau x^{-\frac{1}{2}} + \sqrt{-1k\pi} + x^{\frac{1}{2}}\tilde{b}(x^{\frac{1}{2}})
\]

where \(uv = -\kappa^2dz^2\) is the quadratic invariant defined on page 7.

With these propositions in mind, we define a smooth hyperelliptic curve \(\tilde{X}\) corresponding to each non-trivial solution \((A, \Phi)\) of 2.6. Let \(\alpha_1, \ldots, \alpha_m, \tilde{\alpha}_1, \ldots, \tilde{\alpha}_m\) be the odd order zeros of \(h(x)^2 - 4\).
1. If $\det \Phi \neq 0$, let $\hat{X}$ be the curve
\[ y^2 = \prod_{i=1}^{m} (x - \alpha_i)(x - \bar{\alpha}^{-1}). \]

2. If $\det \Phi = 0$, let $\hat{X}$ be the curve
\[ y^2 = x \prod_{i=1}^{m} (x - \alpha_i)(x - \bar{\alpha}^{-1}). \]

Denote by $\pi : \hat{X} \to \mathbb{C}P^1$ the projection map $(x, y) \mapsto x$.
By definition, $\mu$ and $\nu$ are well-defined functions on $\hat{X}$ and
\[ \Theta := \frac{1}{2\pi \sqrt{-1}} d \log \mu, \quad \Psi := \frac{1}{2\pi \sqrt{-1}} d \log \nu \]
are meromorphic differentials whose only singularities are double poles at $\pi^{-1}\{0, \infty\}$, and which have no residues.

$\hat{X} - \pi^{-1}\{0, \infty\}$ is the Riemann surface of the eigenvalues of the holonomy matrices $H(x, p)$. It supports a natural line bundle for each $p \in T^2$, namely the bundle of eigenspaces of $H(x, p)$:
\[ (E_p)_{(x,y)} \subseteq \ker(H(x,p) - \mu(x,y)). \]

Since the fundamental group $\pi_1(T^2, p)$ is abelian, these eigenspaces are independent of the choice of generator. This line bundle can be naturally extended across $\pi^{-1}\{0, \infty\}$ to give a holomorphic line bundle $E_p$ on $\hat{X}$.

**Proposition 2.4** Let $\hat{X}$ be the smooth hyperelliptic curve associated to a solution $(A, \Phi)$ of (2.6). Then:

1. $\hat{X}$ has a real structure (i.e. an antiholomorphic involution) $\rho : \hat{X} \to \hat{X}$ that commutes with $\pi$ and covers the real structure $x \mapsto x^{-1}$ of $\mathbb{C}P^1$.

2. The hyperelliptic involution $\sigma : \hat{X} \to \hat{X}$ commutes with $\pi$ and has no real fixed points.

3. The differentials $\Theta$ and $\Psi$ satisfy
\[ \sigma^* \Theta = -\Theta, \quad \sigma^* \Psi = -\Psi, \quad \rho^* \Theta = \bar{\Theta}, \quad \rho^* \Psi = \bar{\Psi}. \]

4. $\Theta$ and $\Psi$ are linearly independent over $\mathbb{R}$.

5. For each $p \in T^2$, $j$ gives an isomorphism between $(\sigma \rho)^* E_p$ and $E_p$ whose square is $-1$, so each $E_p$ is quaternionic with respect to the real structure $\sigma \rho$.

6. The periods of $\Theta$ and $\Psi$ are all integers.
The involution \( \rho \) is the lift of \( x \to \bar{x}^{-1} \) that fixes the points in \( \pi^{-1}\{x : |x| = 1\} \). Using the fact that \( d_x \) is unitary for \( x \) on the unit circle,

\[
\rho^*\mu = \bar{\mu}^{-1}
\]

and by definition of \( \hat{X} \),

\[
\sigma^*\mu = \mu^{-1}.
\]

The most important condition in the Proposition above is the last one, as the existence of meromorphic differentials with integral periods places a stringent restriction on \( \hat{X} \).

The algebraic curve that we shall associate to a harmonic map \( f : (T^2, \tau) \to S^3 \) is a possibly singular curve \( X \) (called the spectral curve) of which \( \hat{X} \) is the normalisation. \( X \) will reflect the geometry of the eigenspaces of the holonomy rather than merely that of the eigenvalues. We would like to be able to employ our algebro-geometric description to study families of harmonic maps, and since the limit of a family of smooth curves may be a singular one, it is sensible to allow spectral curves to be singular. The spectral curve will also enable us to compute the degree of the eigenspace bundle. Since \( V \) has rank two, for \( v, w \in V_p \) we have

\[
\omega_p(v, w) = 0 \iff v \text{ and } w \text{ are linearly dependent}.
\]

Thus \( \omega_p \) vanishes precisely at those points \( (x, y) \in \hat{X} \) at which \( (E_p)_{(x,y)} \) and \( (\sigma^*E_p)_{(x,y)} \) coincide as subspaces of \( V_p \). It necessarily vanishes at the branch points of \( \hat{X} \), though for those branch points other than 0 and \( \infty \), it may do so to some odd order \( > 1 \). Suppose that \( \omega \) vanishes to order \( 2k_i + 1 \) at \( \alpha_i \) and \( \bar{\alpha}_i^{-1} \), and to order \( l_j \) at non-branch points \( \beta_j, j = 1, \ldots r \) of \( \hat{X} \).

Then \( X \) is described by the equation

\[
y^2 = \prod_{i=1}^{m} (x - \alpha_i)^{2k_i+1}(x - \bar{\alpha}_i^{-1})^{2k_i+1} \prod_{j=1}^{r} (x - \beta_j)^{2l_j}.
\]

Our definition of \( X \) uses a particular eigenspace bundle \( E_p \), but since for \( x \in C^* \), the eigenspaces of the holonomy matrices \( H(x, p) \) and \( H(x, q) \) are related by parallel translation of \( d_x \), \( X \) is in fact independent of the choice of base point \( p \). It follows from the definition of \( X \) that for each \( p \in T^2 \), \( E_p \) is defined as a bundle over \( X \). Furthermore, from the adjunction formula we may deduce that

\[
deg(E_p^*) = g_a + 1, \text{ where } g_a \text{ is the arithmetic genus of } X.
\]

We may then define a map

\[
l : T^2 \to Pic^{g+1} X
\]

\[
p \mapsto E_p^*
\]

Given the real structure \( \sigma \rho : X \to X \), we say that a holomorphic line bundle \( L \) on \( X \) is real if there is an isomorphism \( i : L \to \sigma \rho^*L \) whose square gives multiplication by a positive scalar on \( L \). \( L \) is quaternionic if there is an isomorphism \( i : L \to \sigma \rho^*L \).
whose square gives multiplication by a negative scalar on $L$. The line bundles $E_p^*$ are all quaternionic, and so if we fix $p \in T^2$, then for each $q \in T^2$, $E_q \otimes E_p^*$ is real. In fact the map
\[
    l \otimes E_p^* : T^2 \rightarrow Pic^0 X \\
    q \mapsto E_q \otimes E_p^*
\]
is linear, and so is a linear map of $(T^2, \tau)$ to a real torus in $Pic^0(X)$. If $g_a(X) \geq 2$, then this map is injective. There is a natural isomorphism
\[
    (V)_p^* \cong H^0(X, E_p^*),
\]
which enables us to reconstruct $V$ from the eigenspace bundles. The space of quaternionic line bundles of degree $g_a(X)$ is connected, and the line bundles $E_p^*$ are non-special. The following is essentially Theorem 8.1 of [10].

**Theorem 2.1** Let $X$ be a hyperelliptic curve $y^2 = P(x)$ of arithmetic genus $g_a$, and let $\pi : X \rightarrow \mathbb{C}P^1$ be the projection $\pi(x, y) = x$. Suppose $X$ satisfies:

1. $P(x)$ is real with respect to the real structure $x \mapsto \bar{x}^{-1}$ on $\mathbb{C}P^1$.
2. $P(x)$ has no real zeros (i.e. no zeros on the unit circle $x = \bar{x}^{-1}$).
3. $P(x)$ has at most simple zeros at $x = 0$ and $x = \infty$.
4. There exist differentials $\Theta$ and $\Psi$ of the second kind on $X$ with periods in $\mathbb{Z}$.
5. $\Theta$ and $\Psi$ have double poles at $\pi^{-1}(0)$ and $\pi^{-1}(\infty)$ and satisfy
   \[
   \sigma^* \Theta = -\Theta, \quad \sigma^* \Psi = -\Psi, \quad \rho^* \Theta = \Theta, \quad \rho^* \Psi = \Psi
   \]
   where $\sigma$ is the hyperelliptic involution $(x, y) \mapsto (x, -y)$ and $\rho$ is the real structure induced from $x \mapsto \bar{x}^{-1}$.
6. The principal parts of $\Theta$ and $\Psi$ are linearly independent over $\mathbb{R}$.

Then, for each point $E_0$ in the Picard variety of line bundles of degree $g_a + 1$ on $X$ which are quaternionic with respect to the real structure $\sigma \rho$, there is a solution of (2.6) for a torus, such that $X$ is the spectral curve of the solution and $\Theta, \Psi$ the corresponding differentials. The solution is, moreover, unique modulo gauge transformations and the operation of tensoring $V$ by a flat $\mathbb{Z}_2$-bundle. (Note that (2) implies that $\rho \sigma$ has no fixed points and so by [1] quaternionic bundles of degree $g_a + 1$ exist.)

Begining with the spectral data $(X, \Theta, \Psi, E_0)$, one may construct a solution $(A, \Phi)$ to (2.6) by carefully reversing the steps outlined above. For details, the reader is referred to [10].

Given such data, we may calculate the eigenvalues of the holonomy using the relations
\[
    \Theta = \frac{1}{2\pi \sqrt{-1}} d \log \mu, \quad \Psi = \frac{1}{2\pi \sqrt{-1}} d \log \nu
\]  \hspace{1cm} (2.8)
on \(X - \pi^{-1}\{0, \infty\}\). The solutions are unique only up to multiplication by a constant, but if we demand that
\[
\mu \sigma^* \mu = 1, \quad \nu \sigma^* \nu = 1, \quad \tag{2.9}
\]
they are defined up to sign. By choosing \(\mu, \nu\) satisfying (2.8) and (2.9), we determine a solution \((A, \Phi)\) of (2.6) up to gauge equivalence, that is, up to a diffeomorphism of the principal bundle \(P\) that covers the identity on \(T^2\) and commutes with the action of \(SU(2)\). As mentioned earlier, this solution corresponds to a harmonic map
\[
f : (T^2, \tau) \to S^3
\]
if and only if the flat connections \(d_1\) and \(d_{-1}\) are trivial. Since they are in any case unitary, this occurs precisely when \(\mu\) and \(\nu\) take the value 1 at all points in \(\pi^{-1}\{1, -1\}\). Given such a solution \((A, \Phi)\), let \(s_1, s_{-1}\) be constant covariant sections of \(P\) with respect to the connections \(d_1\) and \(d_{-1}\) respectively. We define \(f : (T^2, \tau) \to SU(2)\) by
\[
s_1(P) = s_{-1}(P)f(P).
\]
Thus \(f\) is unaffected by gauge transformations, but if we choose different covariant constant sections \(\tilde{s}_1 = s_1h, \tilde{s}_{-1} = s_{-1}k\), then
\[
\tilde{s}_1 = \tilde{s}_{-1}k^{-1}fh,
\]
so the map \(f\) is well-defined modulo right and left actions of \(SU(2)\), or modulo the action of \(SO(4) = SU(2) \times SU(2)/\{\pm 1\}\) on \(S^3\).

The following is Theorem 8.20 of [10].

**Theorem 2.2** Let \((X, \Theta, \Psi, E_0)\) be spectral data satisfying the conditions of Theorem 2.1, where \(X\) is given by \(y^2 = P(x)\). Let \(\mu\) and \(\nu\) be functions on \(X - \pi^{-1}\{0, \infty\}\) satisfying
\[
\Theta = \frac{1}{2\pi \sqrt{-1}}d \log \mu, \quad \Psi = \frac{1}{2\pi \sqrt{-1}}d \log \nu \quad \text{and} \quad \mu \sigma^* \mu = 1, \quad \nu \sigma^* \nu = 1.
\]
Then
1. \((X, \mu, \nu)\) determines a harmonic map from a torus to \(S^3\) if and only if
\[
\mu(x, y) = \nu(x, y) = 1 \quad \text{for all} \quad (x, y) \in \pi^{-1}\{1, -1\}.
\]
2. The map is conformal if and only if \(P(0) = 0\).
3. The torus maps to a totally geodesic 2-sphere if and only if \(g_0\) is odd, \(P(x)\) is an even polynomial, and the point \(E_0 \in \text{Pic}^{g_0+1}(X)\) and the functions \(\mu\) and \(\nu\) on \(X - \pi^{-1}\{0, \infty\}\) are invariant under \(\sigma \gamma\), where \(\gamma\) is the involution of \(X\) defined by \(\gamma(x, y) = (-x, y)\).
4. The harmonic map is uniquely determined by \((X, \mu, \nu, E_0)\) modulo the action of \(SO(4)\) on \(S^3\).
Chapter 3

Conformal Maps

We prove the following theorem:

**Theorem 3.1** For each integer \( g > 0 \) there are countably many conformal harmonic immersions from rectangular tori to \( S^3 \) whose spectral curves have genus \( g \).

We in fact demonstrate the existence of spectral curves possessing an additional symmetry, namely \( x \mapsto \frac{1}{x} \), where \( \pi : X \to \mathbb{C}P^1, (x, y) \mapsto x \). This symmetry induces two holomorphic involutions on \( X \), which we utilise by quotienting out by them. The resulting quotient curves \( C_{\pm} \) are our basic object of study, and we obtain differentials on \( X \) by pulling back differentials from \( C_{\pm} \). We will use proof by induction, in which at each induction step the genera of \( C_{\pm} \) increase by one, and hence the genus \( g \) of \( X \) increases by two. Thus we divide our proof into the even and odd genus cases. Our proofs extend the methods of Ercolani, Knörrer and Trubowitz [8], who showed that for each even genus \( g \geq 2 \) there is a constant mean curvature torus in \( \mathbb{R}^3 \) whose spectral curve has genus \( g \). (There is also a spectral curve construction for constant mean curvature tori.)

### 3.1 Odd Genera

Let \( C_+ = C_+(R, \lambda_1, \bar{\lambda}_1, \ldots, \lambda_n, \bar{\lambda}_n) \) be the curve given by

\[
    w_+^2 = (z - R) \prod_{i=1}^n (z - \lambda_i)(z - \bar{\lambda}_i)
\]

and \( C_- = C_-(R, \lambda_1, \bar{\lambda}_1, \ldots, \lambda_n, \bar{\lambda}_n) \) that given by

\[
    w_-^2 = (z - 2)(z + 2)(z - R) \prod_{i=1}^n (z - \lambda_i)(z - \bar{\lambda}_i)
\]

where we assume that \( R \in (-\infty, -2) \cup (2, \infty) \), \( \lambda_i \neq 2 \) for \( i = 1, \ldots, 2n \) and \( \lambda_i \neq \lambda_j \) for \( i \neq j \). Let \( \pi_{\pm} : C_{\pm} \to \mathbb{C}P^1, (z, w_{\pm}) \mapsto z \) denote the respective projections to the Riemann sphere. (We shall henceforth generally omit the word “respective”.)
Construct $\pi : X \to \mathbb{CP}^1$ as the fibre product of $\pi_+ : C_+ \to \mathbb{CP}^1$ and $\pi_- : C_- \to \mathbb{CP}^1$, that is, let $X := \{(p_+, p_-) \in C_+ \times C_- : \pi_+(p_+) = \pi_-(p_-)\}$, with the induced algebraic structure and the obvious projection $\pi$ to $\mathbb{CP}^1$. Notice then that $X$ is given by the equation

$$y^2 = x(x - r)(x - r^{-1}) \prod_{i=1}^{n}(x - \alpha_i)(x - \alpha_i^{-1})(x - \bar{\alpha}_i)(x - \bar{\alpha}_i^{-1}),$$

where

$$r + r^{-1} = R, \quad \alpha_i + \alpha_i^{-1} = \lambda_i,$$

and $\pi$ is given by

$$\pi : (x, y) \mapsto x.$$

These identifications occur via the maps

$$q_+(x, y) = \left(x + \frac{1}{x}, \frac{y}{x^{n+1}} \right) = (z, w_+)$$

and

$$q_-(x, y) = \left(x + \frac{1}{x}, \frac{(x + 1)(x - 1)y}{x^{n+2}} \right) = (z, w_-).$$

$X$ has genus $2n + 1$ and possesses the holomorphic involutions

$$i_\pm : X \to X \quad \text{with quotient maps} \quad q_\pm : X \to C_\pm.$$

The curves $C_\pm$ are the quotients of $X$ by these involutions, with quotient maps $q_\pm : X \to C_\pm$. $C_\pm$ each possesses a real structure $\rho_\pm$, characterised by the properties that it covers the involution $z \mapsto \bar{z}$ of $\mathbb{CP}^1$ and fixes the points in $\pi_\pm^{-1}[-2, 2]$. These real structures are given by

$$\rho_\pm(z, w_\pm) = (\bar{z}, \mp \bar{w}_\pm), \text{ for } R > 2$$

or

$$\rho_\pm(z, w_\pm) = (\bar{z}, \pm \bar{w}_\pm), \text{ for } R < -2.$$

The cases $R > 2$ and $R < -2$ are similar, but the sign difference carries through to future computations. For simplicity of exposition we assume henceforth that $R > 2$. Then the corresponding real structure on $X$ is given by

$$\rho(x, y) = \left(\frac{1}{x}, \frac{-iy}{x^{2n+1}} \right).$$

Whilst our primary interest lies with curves $C_\pm$ as described above, we consider also curves $C_\pm = C_\pm(R, \lambda_1, \ldots, \lambda_{2n})$ given by

$$w_\pm^2 = (z - R) \prod_{i=1}^{2n}(z - \lambda_i).$$

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and

\[ w_-^2 = (z - 2)(z + 2)(z - R) \prod_{i=1}^{2n} (z - \lambda_i) \]

respectively, where we assume that \( R \in (2, \infty) \), \( \lambda_i \neq \pm 2 \) for \( i = 1, \ldots, 2n \) and that the sets \( \{\lambda_1, \lambda_2\}, \ldots, \{\lambda_{2n-1}, \lambda_{2n}\} \) are mutually disjoint.

Take \((R, \lambda_1, \ldots, \lambda_{2n})\) as described above. Let \( \tilde{a}_0, \ldots, \tilde{a}_n \) be simple closed curves in \( \mathbb{C}P^1 - \{R, 2, -2, \lambda_1, \ldots, \lambda_{2n}\} \), and \( \tilde{c}_1, \tilde{c}_{-1} \) simple closed curves in \( \mathbb{C}P^1 - \{R, \lambda_1, \ldots, \lambda_{2n}\} \), such that

1. \( \tilde{a}_0 \) has winding number one around 2 and \( R \), and winding number zero around the other branch points of \( C_- \),
2. for \( i = 1 \ldots n \), \( \tilde{a}_i \) has winding number one around \( \lambda_{2i-1} \) and \( \lambda_{2i} \), and winding number zero around the other branch points of \( C_\pm \),
3. \( \tilde{c}_1 \) begins and ends at \( z = 2 \), has winding number one around \( R \) and zero around \( \lambda_i \), \( i = 1, \ldots, 2n \),
4. \( \tilde{c}_{-1} \) begins and ends at \( z = -2 \), and has winding number one around \( R \) and each \( \lambda_i \) \( i = 1, \ldots, 2n \).

Choose lifts of the curves \( \tilde{a}_1, \ldots, \tilde{a}_n \) to \( C_+ \) and also of \( \tilde{a}_0, \ldots, \tilde{a}_n \) to \( C_- \). Let \( a_0^\pm, a_1^\pm, \ldots, a_n^\pm \in H_1(C_\pm, \mathbb{Z}) \) denote the homology classes of these lifts. Denote by \( b_0^\pm, b_1^\pm, \ldots, b_n^\pm \) the completions to canonical bases of \( H_1(C_\pm, \mathbb{Z}) \). Choose open curves \( c_1, c_{-1} \) in \( C_+ \) covering the loops \( \tilde{c}_1 \) and \( \tilde{c}_{-1} \).

Denote by \( M_n \) the space of \( 2n + 1 \)-tuples \((R, \lambda_1, \ldots, \lambda_{2n})\) as above together with the choices we have described. Let \( M_{n, \mathbb{R}} \) denote the subset of \( M_n \) such that (see figure 3.1):

1. \( \lambda_{2i} = \bar{\lambda}_{2i-1} \) for \( i = 1, \ldots, n \),
2. for \( i = 1 \ldots n \), \( \tilde{a}_i \) is invariant under conjugation and intersects the real axis exactly twice, both times in the interval \((-2, 2)\),
3. the lifts of \( \tilde{a}_1, \ldots, \tilde{a}_n \) to \( C_+ \) are chosen so that the point where \( \tilde{a}_i \) intersects the \( z \)-axis with positive orientation is lifted to a point where \( \frac{w_+}{\sqrt{-1}} \) is negative,
4. the lifts of \( \tilde{a}_0, \ldots, \tilde{a}_n \) to \( C_- \) are chosen so that the point where \( \tilde{a}_i \) intersects the \( z \)-axis with positive orientation is lifted to a point in \( C_- \) where \( w_- \) is positive,
5. \( c_1, c_{-1} \) begin at points with \( \frac{w_+}{\sqrt{-1}} < 0 \).

For each \( p \in M_{n, \mathbb{R}} \) there is a unique canonical basis \( A_0, \ldots, A_{2n}, B_0, \ldots, B_{2n} \) for the homology of \( X \) such that \( A_0, \ldots, A_{2n} \) cover the homotopy classes of loops \( \tilde{A}_0, \ldots, \tilde{A}_{2n} \) shown in Figure 3.2 and

\[ (q_-)_*(A_0) = 2a_0^-, (q_\pm)_*(A_i) = \mp(q_\pm)_*(A_{n+i}) = a_i^\pm. \]
Figure 3.1: Curves $\tilde{a}_i$, $\tilde{c}_{\pm 1}$ for $p \in M_{n, \mathbb{R}}$.

There are also unique curves $\gamma_1$ and $\gamma_{-1}$ on $X$ such that $(q_\pm)_*(\gamma_{\pm 1}) = c_{\pm 1}$; they project to $\tilde{\gamma}_1$ and $\tilde{\gamma}_{-1}$ of Figure 3.2. Note that $\gamma_1$ connects the two points of $X$ with $x = 1$ whilst $\gamma_{-1}$ connects the two points of $X$ with $x = -1$.

Figure 3.2: The curves $\tilde{A}_i$ and $\tilde{\gamma}_{\pm 1}$.

Denote by $A^{\pm}$ the subgroups of $H_1(C_{\pm}, \mathbb{Z})$ generated by the $a^{\pm}$ classes. Then modulo $A^{\pm}$,

$$(q_-)_*(B_0) \equiv b_0^-, \quad (q_\pm)_*(B_i) \equiv b_i^\pm, \quad (q_{\pm 1})_*(B_n) \equiv \mp b_n^\pm.$$  

For each $p \in M_n$, we will define differentials $\Omega_{\pm}(p)$ on $C_{\pm}(p)$, and show that for each $n$ there exists $p \in M_{n, \mathbb{R}}$ such that appropriate integer multiples of $q_+(\Omega_+(p))$ and
q_−^*(Ω_−(p)) satisfy the conditions of Hitchin’s correspondence. It is easier to argue this way than to perform a similar argument directly on X, for reasons that will become clear.

Let p ∈ M_n and define Ω_± = Ω_±(p) on C_±(p) by:

1. Ω_±(p) are meromorphic differentials of the second kind: their only singularities are double poles at z = ∞, and they have no residues.
2. ∫_{a^-} Ω_−(p) = 0 and ∫_{a^+} Ω_+(p) = 0 for i = 1, ..., n.
3. As z → ∞, Ω_+(p) → z^n w_+ and Ω_−(p) → z^{n+1} w_−.

(We shall generally denote the paths of integration a_±(p), b_±(p), c_±(p) simply by a_±, b_± and c_±, as the point p in question is usually clear from the differential being integrated.) In view of the defining conditions above, we may write

Ω_+ = \prod_{j=1}^n \frac{(z - \zeta_j^+)}{w_+} dz

and

Ω_− = \prod_{j=0}^n \frac{(z - \zeta_j^-)}{w_-} dz.

Define

I_+(p) := \sqrt{-1} \left( \int_{c_1} \Omega_+(p), \int_{c_{-1}} \Omega_+(p), \int_{b_1^+} \Omega_+(p), ..., \int_{b_n^+} \Omega_+(p) \right),

I_−(p) := \left( \int_{b_0^-} \Omega_−(p), \int_{b_1^-} \Omega_−(p), ..., \int_{b_n^-} \Omega_−(p) \right).

Then for p ∈ M_{n,R}, I_+(p) and I_−(p) are real, since then

(ρ_±)_*(b_±^i) = b_±^i mod A^±, (ρ_+)_*(c_±1) = c_±1 mod A^+,

so

\int_{b_1^i} \Omega_+(p) = \int_{(ρ_+)_,(b_1^i)} \Omega_+(p) = \int_{b_1^i} \rho_+^*(Ω_+(p)) = - \int_{b_1^i} \overline{Ω_+(p)}, i = 1 ... n,

and similarly

\int_{c_±1} \Omega_+(p) = - \int_{c_±1} \overline{Ω_+(p)}

and

\int_{b_1^-} \Omega_−(p) = \int_{b_1^-} \overline{Ω_−(p)}, i = 0 ... n.

Given p ∈ M_{n,R}, there are real numbers s_+ and s_- such that \sqrt{-1}s_+q_+^*(Ω_+(p)) and s_-q_−^*(Ω_−(p)) are differentials on X satisfying the conditions of Hitchin’s correspondence if and only if I_+(p) and I_−(p) represent rational elements of \mathbb{R}P^{n+1} and \mathbb{R}P^n respectively.
Theorem 3.2 For each non-negative integer $n$, there exists $p \in M_{n, \mathbb{R}}$ such that

1. $\zeta_j^+(p), j = 1, \ldots, n$ are pairwise distinct, as are $\zeta_j^-(p), j = 0, \ldots, n$.

2. The map 
\[
\phi : M_n \longrightarrow \mathbb{C}P^{n+1} \times \mathbb{C}P^n
\]
\[
p \longmapsto ([I_+(p)], [I_-(p)]).
\]
has invertible differential at $p$.

This gives that the restriction 
\[
\phi |_{M_{n, \mathbb{R}}} : M_{n, \mathbb{R}} \rightarrow \mathbb{R}P^{n+1} \times \mathbb{R}P^n
\]
of $\phi$ to $M_{n, \mathbb{R}}$ also has invertible differential at $p$. Since rationality is a dense condition, the Inverse Function Theorem then implies that for each positive odd integer $g$, there are countably many spectral curves $X$ of genus $g$ each giving rise to a torus $(T^2, \tau)$ and a branched minimal immersion $f : (T^2, \tau) \rightarrow S^3$. The conformal type of the torus is given by 
\[
\tau = \frac{\sqrt{-1}s_+p \cdot p \cdot \infty(q_+^*(\Omega_+(p)))}{s_-p \cdot p \cdot \infty(q_-^*(\Omega_-((p)))}
\]
where $p \cdot p \cdot \infty(q_+^*(\Omega_+))$ denotes the principal part of $q_+^*(\Omega_+)$ at $\infty$. Thus each torus $(T^2, \tau)$ is rectangular.

In fact we shall prove a slightly stronger result, the extra strength residing in a statement that arises naturally from an attempt to prove Theorem 3.2 by induction on $n$, and enables one to complete the induction step. This statement is somewhat lengthy to formulate, and will appear unmotivated at this juncture. Our approach is thus to present an attempt to prove Theorem 3.2 by induction, and derive the necessary modifications. The reader who wishes to view the modified statement at this point is referred to page 31.

**Proof of Theorem 3.2:** We use induction upon $n$. We shall begin with the induction step, in order to formulate the “extra conditions” mentioned above. Suppose then that $\phi$ has invertible differential at $p \in M_{n, \mathbb{R}}$. For $\mu \in (-2, 2), \nu \in \mathbb{R}$, we shall denote by $(p, \mu, \nu)$ the point of $M_{n+1, \mathbb{R}}$ such that 
\[
\lambda_i(p, \mu, \nu) = \lambda_i(p), i = 1, \ldots, n, \quad \lambda_{n+1}(p, \mu, \nu) = \mu + \sqrt{-1}\nu.
\]
Denote by $(p, \mu, \nu)$ the point in $M_{n+1, \mathbb{R}}$ with branch points 
\[
\lambda_i(p, \mu, \nu) = \lambda_i(p), i = 1, \ldots, 2n
\]
and 
\[
\lambda_{2n+1}(p, \mu, \nu) = \mu + \sqrt{-1}\nu, \quad \lambda_{2n+2}(p, \mu, \nu) = \mu - \sqrt{-1}\nu.
\]
We wish to show that for a generic $\mu \in (-2, 2)$ and $\nu$ sufficiently small, $\phi$ has invertible differential at $(p, \mu, \nu)$. (Here “generic” means “outside the zero set of a
Once we have incorporated our “additional condition”, we shall achieve our aim by considering the boundary case $\nu = 0$. Notice that choosing $\pi_\pm : \mathbb{C}^1 \to \mathbb{C}P^1$ to each have an additional branch point of multiplicity two on the interval $(-2, 2)$ corresponds to choosing $\pi : X \to \mathbb{C}P^1$ to have an additional branch point of multiplicity two on the unit circle. For brevity of notation, we shall write $p'_0 = p'_0(p, \mu)$ for $(p, \mu, 0)$.

Let

$$H(\mu) := \begin{pmatrix} I_+(p'_0) & 0 & I_-(p'_0) \\ \frac{\partial}{\partial \mu} I_+(p'_0) & \frac{\partial}{\partial \mu} I_-(p'_0) \\ \vdots & \vdots & \vdots \\ \frac{\partial}{\partial \lambda_i^2} I_+(p'_0) & \frac{\partial}{\partial \lambda_i^2} I_-(p'_0) \end{pmatrix},$$

and

$$h(\mu) := \det H(\mu).$$

$h$ is a real-analytic function of $\mu \in (-2, 2)$ and for each $\epsilon \in (0, \min_{i=1,...,n} |\lambda_i| + 2)$ we may use the above formula to define it as a real-analytic function $h_\epsilon$ of $\mu$ on the curve $L_\epsilon$ shown in figure 3.3.

$$\begin{array}{ccl}
\hline
-2-\epsilon & -2 & -2+\epsilon \\
\hline
\end{array}$$

Figure 3.3: $h_\epsilon$ is a function of $\mu \in L_\epsilon$.

We will show that $h(\mu) \neq 0$ as $\mu \to \infty$ along $L_\epsilon$, by computing asymptotics for each of the vectors in $H(\mu)$. The advantage of passing to the quotient curves $C_\pm$ is the availability of this limiting argument. The interval $(-2, 2)$ possesses a natural extension to a line whereas the unit circle does not. We will also prove that

$$\frac{\partial}{\partial \nu} (I_+(p'_0), I_-(p'_0)) = 0.$$

Then for generic $\mu \in (-2, 2)$, we will have that $h(\mu) \neq 0$, and utilising

$$\frac{\partial}{\partial \nu} (I_+(p, \mu, \nu), I_-(p, \mu, \nu)) = \frac{\partial^2}{\partial \nu^2} (I_+(p'_0), I_-(p'_0)) + O(\nu^2),$$

this gives that for $\nu$ sufficiently small, $d\phi_{(p, \mu, \nu)}$ is invertible.

One simplification provided by choosing $\nu = 0$ is that $C_\pm(p)$ are the respective normalisations of $C_\pm(p'_0)$, with normalisation maps
\[ \Psi_{\pm} : \begin{array}{ccc} C_{\pm}(p) & \longrightarrow & C_{\pm}(p') \\ (z, w_{\pm}(p)) & \longmapsto & (z, (z - \mu)w_{\pm}(p)) \end{array} \]

and that
\[ \Psi_{\pm}^* (\Omega_{\pm}(p')) = \Omega_{\pm}(p), \]
whilst
\[ (\Psi_+)_* (b_i^+(p)) = b_i^+(p'), \]
\[ (\Psi_+)_* (c_{\pm 1}(p)) = c_{\pm 1}(p'), \]
\[ (\Psi_-)_* (b_i^-(p)) = b_i^-(p'), \]

Thus
\[
I_+(p') = \left( I_+(p), \sqrt{-1} \int_{b_{n+1}}^{b_n} \Omega_+(p') \right), \quad (3.3)
\]
\[
I_-(p') = \left( I_-(p), \int_{b_{n+1}}^{b_n} \Omega_-(p') \right). \quad (3.4)
\]

For each point \( q \in M_n \), let \( u_{\pm}(q) \) be local coordinates on \( C_{\pm}(q) \) near \( \pi^{-1}_-(\infty) \) such that
\[ u_{\pm}(q)^2 = z_{\pm}^{-1} \]
and
\[ w_+(q) = u_+(q)^{2n+1} + O(u_+(q)^{2n}) \text{ as } z_+ \to \infty \]
whilst
\[ w_-(q) = u_-(q)^{2n+3} + O(u_-(q)^{2n+2}) \text{ as } z_- \to \infty. \]

Then for \( z \) near \( \infty \),
\[ \Omega_{\pm}(q) = (u_{\pm}(q) + D_{\pm}(q)u_{\pm}(q)^3 + O(u_{\pm}(q)^5))dz, \quad (3.5) \]
where
\[ D_+(q) := \frac{1}{2} R(q) + \sum_{i=1}^{2n} \lambda_i(q) - \sum_{j=1}^{n} \zeta_j^+(q), \quad (3.6) \]
and
\[ D_-(q) := \frac{1}{2} R(q) + \sum_{i=1}^{2n} \lambda_i(q) - \sum_{j=0}^{n} \zeta_j^-(q). \quad (3.7) \]

We assumed that the \( \zeta_j^\pm \) are pairwise distinct, \( j = 1, \ldots, n \). Thus the differentials \( \frac{\Omega_{\pm}(p)}{z - \zeta_j^\pm} \) are a basis for the holomorphic differentials on \( C_+(p) \), and we may define \( c_j^+(p) \) by the equations
\[
\frac{3}{2} \int_{a_j^+} z\Omega_+(p) + \sum_{j=1}^{n} c_j^+(p) \int_{a_j^+} \frac{\Omega_+(p)}{z - \zeta_j^+} = 0, \quad i, j = 1, \ldots, n, \quad (3.8)
\]

\[ \text{and} \]
\[ w_+(q) = u_+(q)^{2n+1} + O(u_+(q)^{2n}) \text{ as } z_+ \to \infty \]
whilst
\[ w_-(q) = u_-(q)^{2n+3} + O(u_-(q)^{2n+2}) \text{ as } z_- \to \infty. \]
Lemma 3.1 As $\mu \to \infty$ along $L_\epsilon$, the following asymptotic expressions hold:

1. $I_+(p_0) = (I_+(p), 4\sqrt{-1}\mu^{1/2} - 4\sqrt{-1}D_+(p)\mu^{-1/2} + O(\mu^{-3/2}))$
   $I_-(p_0) = (I_-(p), 4\mu^{1/2} - 4D_-(p)\mu^{-1/2} + O(\mu^{-3/2}))$

2. $\frac{\partial}{\partial R} I_+(p_0') = \left( \frac{\partial}{\partial R} I_+(p) , \sqrt{-1} \left( -2 + \sum_{j=1}^{n} \frac{\partial \zeta_j^+}{\partial R} \right) \mu^{-1/2} + O(\mu^{-3/2}) \right)$
   $\frac{\partial}{\partial R} I_-(p_0') = \left( \frac{\partial}{\partial R} I_-(p) , \left( -2 + \sum_{j=0}^{n} \frac{\partial \zeta_j^-}{\partial R} \right) \mu^{-1/2} + O(\mu^{-3/2}) \right)$

3. For $i = 1, \ldots, 2n$,
   $\frac{\partial}{\partial \lambda_i} I_+(p_0') = \left( \frac{\partial}{\partial \lambda_i} I_+(p) , \sqrt{-1} \left( -2 + \sum_{j=1}^{n} \frac{\partial \zeta_j^+}{\partial \lambda_i} \right) \mu^{-1/2} + O(\mu^{-3/2}) \right)$, and
   $\frac{\partial}{\partial \lambda_i} I_-(p_0') = \left( \frac{\partial}{\partial \lambda_i} I_-(p) , \left( -2 + \sum_{j=0}^{n} \frac{\partial \zeta_j^-}{\partial \lambda_i} \right) \mu^{-1/2} + O(\mu^{-3/2}) \right)$

4. $\frac{\partial}{\partial \mu} I_+(p_0') = (0, 2\sqrt{-1}\mu^{-1/2} + 2\sqrt{-1}D_+(p)\mu^{-3/2} + O(\mu^{-5/2}))$
   $\frac{\partial}{\partial \mu} I_-(p_0') = (0, 2\mu^{-1/2} + 2D_-(p)\mu^{-3/2} + O(\mu^{-5/2}))$

5. $\frac{\partial}{\partial \nu} I_+(p_0') = 0$

6. $\frac{\partial^2}{\partial \nu^2} I_+(p_0') = \left( \left( -\frac{1}{2} \mu^{-2} + D_+(p)\mu^{-3} \right) I_+(p) - \mu^{-3} \tilde{I}_+(p) + O(\mu^{-4}) \right)$
   $\left( \frac{3\sqrt{-1}}{2} \mu^{-3/2} + \frac{9}{2} \sqrt{-1} D_+(p) \mu^{-5/2} + O(\mu^{-7/2}) \right)$

   $\frac{\partial^2}{\partial \nu^2} I_-(p_0') = \left( \left( -\frac{1}{2} \mu^{-2} + D_-(p)\mu^{-3} \right) I_-(p) - \mu^{-3} \tilde{I}_-(p) + O(\mu^{-4}) \right)$
   $\left( \frac{3}{2} \mu^{-3/2} + \frac{9}{2} D_-(p) \mu^{-5/2} + O(\mu^{-7/2}) \right)$

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Proof of Lemma 3.1: All but the last components of (1)–(4) are applications of equations 3.3 and 3.4. The last components of (1)–(4) involve integrals over the curves \( b_{n+1}^+(p'_0) \). Let \( \Gamma \) denote the circle \(|z| = \mu\), transversed clockwise. For \( \mu \) sufficiently large,

\[
\int_{b_{n+1}^+(p'_0)} \Omega_+(p'_0) = -\int_\Gamma \frac{\prod_{j=1}^n (z - \zeta_j^+)dz}{\sqrt{(z-R) \prod_{i=1}^n (z - \lambda_i)(z - \bar{\lambda}_i)}} \tag{see figure 3.4}
\]

\[
= 4\mu^{1/2} - 4D_\pm(p)\mu^{-1/2} + O(\mu^{-3/2}),
\]

and similarly for \( \int_{b_{n+1}^-(p'_0)} \Omega_-(p'_0) \), which easily gives the remainder of (1)–(4).

![Figure 3.4: We can take a representative of \( b_{n+1}^+(p'_0) \) that projects to a circle.](image)

(5): We employ the notation \( \hat{f} \) to indicate \( \frac{\partial f}{\partial \nu} \big|_{\nu=0} \). We will work only with \( \Omega_+(p'_0) \), but similar arguments apply to \( \Omega_-(p'_0) \). For \( i = 1, \ldots, n+1 \),

\[
\int_{a_i^+} \hat{\Omega}_+(p'_0) = \frac{\partial}{\partial \nu} \bigg|_{\nu=0} \int_{a_i^+} \Omega_+(p'_0) = 0. \tag{3.10}
\]

For \( \nu \) small, we may write

\[
\Omega_+(p, \mu, \nu) = \frac{\prod_{j=1}^{n+1} (z - \zeta_j^+(p, \mu, \nu))dz}{w_+(p, \mu, \nu)},
\]

where \( \zeta_j^+ \) are analytic functions satisfying

\[
\zeta_j^+(p'_0) = \begin{cases} 
\zeta_j(p) & \text{for } j = 1, \ldots, n \\
\mu & \text{for } j = n+1.
\end{cases}
\]
(Due to this accordance, we write simply \( \zeta_j^\pm \) for \( \zeta_j^\pm(p) \) or \( \zeta_j^\pm(p'_0) \).)

Then

\[
\dot{\Omega}_+(p'_0) = \sum_{j=1}^{n+1} \left( \frac{-\zeta_j^+}{z - \zeta_j^+} \right) \prod_{k=1}^{n+1} (z - \zeta_k) \frac{dz}{w_+}.
\]

If \( \mu \) coincides with one of the \( \zeta_j \), \( j = 1, \ldots, n \), then we immediately see that the lift of \( \dot{\Omega}_+ \) to the normalisation \( C_+(p) \) of \( C_+(p'_0) \) is holomorphic. If \( \mu \) does not equal any of the \( \zeta_j^+ \), then we may again conclude that \( \dot{\Omega}_+(p'_0) \) lifts to a holomorphic differential by employing the observation that (3.10) for \( j = n + 1 \) states that \( \dot{\Omega}_+(p'_0) \) has zero residue at \( z = \mu \), and hence \( \dot{\zeta}_{n+1} = 0 \). Equation (3.10) tells us that this lift has zero \( a \)-periods, and so is itself zero. For \( i = 1, \ldots, n \) then,

\[
\frac{\partial}{\partial \nu} \bigg|_{\nu=0} \int_{b_i^+} \Omega_+(p'_0) = \int_{b_i^+} \dot{\Omega}_+(p'_0) = 0.
\]

To compute \( \frac{\partial}{\partial \nu} \bigg|_{\nu=0} \int_{b_i^+} \Omega_+(p'_0) \) we use reciprocity with the holomorphic differential \( \omega(p, \mu, \nu) \) on \( C_+(p, \mu, \nu) \) defined by

\[
\int_{a_i^+} \omega(p, \mu, \nu) = \begin{cases} 0, & \text{for } i = 1, \ldots, n \\ 2\pi \sqrt{-1}, & \text{for } i = n + 1. \end{cases}
\]

To simplify our notation, we write \( p' \) for \( (p, \mu, \nu) \). Let \( \Delta(p') \) denote the polygon formed by cutting \( C_+(p') \) open along representatives of the homology elements \( a_i^+, b_i^+ \), and choose \( q_0 \in \Delta(p') \). Define a holomorphic function \( g_{p'} \) on \( \Delta(p') \) by

\[
g_{p'}(q) := \int_{q_0}^{q} \Omega_+(p').
\]

Reciprocity gives

\[
2\pi \sqrt{-1} \sum_{\text{residues}} g_{p'} \omega(p') = \sum_{i=1}^{n+1} \int_{a_i^+} \Omega_+(p') \int_{b_i^+} \omega(p') - \int_{b_i^+} \Omega_+(p') \int_{a_i^+} \omega(p'),
\]

so

\[
\int_{b_{n+1}^+} \Omega_+(p') = - \sum_{\text{residues}} g_{p'} \omega(p'). \tag{3.11}
\]

Writing

\[
\omega(p') = \kappa(p') \prod_{j=1}^{n} (z - \beta_j(p')) dz \frac{1}{w_+(p')}, \tag{3.12}
\]

(3.11) gives that

\[
\int_{b_{n+1}^+} \Omega_+(p') = 4\kappa(p'). \tag{3.13}
\]

For \( i = 1, \ldots, n + 1 \),

\[
\int_{a_i^+} \dot{\omega} = \left. \frac{\partial}{\partial \nu} \right|_{\nu=0} \int_{a_i^+} \omega(p') = 0,
\]

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and since $\dot{\omega}$ is holomorphic, this implies that $\dot{\omega} = 0$. Now
\[
\dot{\omega} = \left( \dot{\kappa} - \kappa(p'_0) \sum_{j=1}^{n} \frac{\beta_j}{z - \beta_j(p'_0)} \right) \prod_{j=1}^{n}(z - \beta_j(p'_0)) \frac{dz}{w_+(p'_0)},
\]
so this gives $\dot{\kappa} = 0$, (and $\dot{\beta}_j = 0$, $j = 1, \ldots, n$) and hence by (3.13), we have that
\[
\frac{\partial}{\partial \nu} \bigg|_{\nu=0} \int_{b^+_{n+1}} \Omega_+ = 0.
\]

(6): Using the fact that $\dot{\Omega}_+(p'_0) = 0$, a calculation gives that
\[
\ddot{\Omega}_+(p'_0) = -\left( \sum_{j=1}^{n} \frac{\ddot{\zeta}_j^+}{z - \zeta_j^+} - \frac{\ddot{\zeta}_{n+1}^+}{z - \mu} - \frac{1}{(z - \mu)^2} \right) \Omega_+(p'_0).
\]  

Since
\[
\int_{a^+_{n+1}} \ddot{\Omega}_+(p'_0) = \frac{\partial^2}{\partial \nu^2} \bigg|_{\nu=0} \int_{a^+_{n+1}} \Omega_+(p'_0) = 0,
\]
$\ddot{\Omega}_+(p'_0)$ has zero residue at $z = \mu$.

Writing $\Omega_+(p'_0) = k(z)dz$, then from equation (3.14), we have that
\[
\ddot{\zeta}_{n+1}^+ k(\mu) = -\frac{dk}{dz}(\mu)
\]
and hence using equations (3.5) and (3.6),
\[
\ddot{\zeta}_{n+1}^+ = \frac{1}{2\mu} + \frac{D_+(p)}{\mu^2} + O\left( \frac{1}{\mu^3} \right).
\]

We can represent the homology classes $a_{i+}, b_{i+}, i = 1, \ldots, n$ by loops whose projections to $\mathbb{C}P^1$ lie in a fixed compact region $K$ that is independent of $\mu$. Then for $\mu$ sufficiently large and $z \in K$, we have
\[
\frac{1}{z - \mu} = -\frac{1}{\mu} + O\left( \frac{1}{\mu^3} \right)
\]
and
\[
\frac{1}{(z - \mu)^2} = \frac{1}{\mu^2} + \frac{z}{\mu^2} + O\left( \frac{1}{\mu^4} \right)
\]
and substituting these, together with (3.15), into (3.14), we obtain that in $K$,
\[
\ddot{\Omega}_+(p'_0) = \left( - \sum_{j=1}^{n} \frac{\ddot{\zeta}_j^+}{z - \zeta_j^+} - \frac{1}{2\mu^2} + \frac{2D_+(p)}{2\mu^3} - \frac{3z}{2\mu^3} + O\left( \frac{1}{\mu^4} \right) \right) \Omega_+(p'_0).
\]

We will thus obtain an asymptotic expression for $\ddot{\Omega}_+(p'_0)$ by calculating one for $\ddot{\zeta}_j^+(p'_0)$. To do this, we note that both $\Omega_+(p'_0)$ and $\ddot{\Omega}_+(p'_0)$ have trivial $\alpha$-periods, so
\[
\frac{3}{2} \int_{a_i(p'_0)} \Omega_+(p'_0) + \ddot{\zeta}_j^+ \mu^3 \sum_{j=1}^{n} \int_{a_i^+(p'_0)} \frac{\Omega_+(p'_0)}{z - \zeta_j^+} = O\left( \frac{1}{\mu} \right), \text{ for } i = 1, \ldots, n,
\]
or equivalently
\[
\frac{3}{2} \int_{a_i(p)} z \Omega_+(p) + \tilde{\zeta}_j + \mu^3 \sum_{j=1}^n \int_{a_i(p)} \frac{\Omega_+(p)}{z - \zeta_j^+} = O \left( \frac{1}{\mu} \right), \text{ for } i = 1, \ldots, n.
\]

Recalling the definition of \( \tilde{\Omega}_+ \) from equations (3.8) and (3.9), notice that
\[
\ddot{\zeta}_j = \frac{\zeta_j(p)}{\mu^3} + O \left( \frac{1}{\mu^4} \right), \quad (3.19)
\]

From (3.18), (3.8), (3.9) and (3.19),
\[
\Psi_+^* \tilde{\Omega}_+ = -\frac{1}{2\mu^2} \Omega_+ + \frac{1}{\mu^3} \left( D_+(p) \Omega_+ + \tilde{\Omega}_+ \right) + O \left( \frac{1}{\mu^4} \right),
\]
proving all but the last component of (6).

To compute \( \frac{\partial^2}{\partial \nu^2} \bigg|_{\nu=0} \int_{b_{n+1}^+} \Omega_+ \), we again use reciprocity with \( \omega(p'_0) \). Differentiating (3.12), and using \( \dot{\omega} = 0 \) one obtains
\[
\ddot{\omega} = \left( \frac{\ddot{\kappa}}{\kappa(p'_0)} - \frac{1}{(z - \mu)^2} - \sum_{j=1}^n \frac{\ddot{\beta}_j}{z - \beta_j(p'_0)} \right) \omega(p'_0). \quad (3.20)
\]

Taking residues at \( z = \mu \), and utilising
\[
\text{res}_{z=\mu} \omega(p'_0) = 1, \text{ res}_{z=\mu} \ddot{\omega} = 0,
\]
gives
\[
\frac{\ddot{\kappa}}{\kappa(p'_0)} - \sum_{j=1}^n \frac{\ddot{\beta}_j}{z - \beta_j(p'_0)} = \text{res}_{z=\mu} \frac{\omega(p'_0)}{(z - \mu)^2}. \quad (3.21)
\]

Thus in order to obtain an asymptotic expression for \( \ddot{\kappa} \), we first obtain expressions for \( \kappa(p'_0), \beta_j(p'_0) \) and \( \ddot{\beta}_j \). We make the assumption throughout that \( z \in K \). We shall abuse notation and write \( \omega(p'_0) \) for \( \Psi_+^*(\omega(p'_0)) \), where
\[
\Psi_+^* : \quad C_+(p) \quad \longrightarrow \quad C_+(p'_0) \quad (z, w_+(p)) \quad \longmapsto \quad (z, (z - \mu)w_+(p))
\]
is the normalisation map. From (3.12) and (3.16),
\[
\omega(p'_0) = \kappa(p'_0) \left( \frac{-1}{\mu} + O \left( \frac{1}{\mu^2} \right) \right) \frac{\prod_{j=1}^n (z - \beta_j(p'_0)) dz}{w_+(p)}, \quad (3.22)
\]
so
\[
\int_{a_i^+} \frac{\prod_{j=1}^n (z - \beta_j(p'_0)) dz}{w_+(p)} = O \left( \frac{1}{\mu} \right).
\]
\[ \prod_{j=1}^{n}(z - \beta_j(p'_0)) \] is moreover a differential of the second kind on \( C_{+}(p) \) whose only singularity is a double pole at \( z = \infty \), and it approaches \( \frac{z^n}{w_{+}(p)} \) as \( z \to \infty \). Hence

\[
\prod_{j=1}^{n}(z - \beta_j(p'_0)) \, dz + O \left( \frac{1}{\mu} \right) = \Omega_{+}(p),
\]

from which we conclude that

\[ \beta_j(p'_0) = \zeta_j + O \left( \frac{1}{\mu} \right), \quad j = 1, \ldots, n. \] (3.23)

The fact that \( \text{res}_{z=\mu} \omega(p'_0) = 1 \) yields

\[
\kappa(p'_0) \prod_{j=1}^{n}(\mu - \beta_j(p'_0)) = \prod_{j=1}^{n}(\mu - \zeta_j) + O \left( \frac{1}{\mu^{5/2}} \right)
\]

so using (3.6),

\[
\kappa(p'_0) \left( \mu^{-1/2} + D_{+}(p)\mu^{-3/2} + O(\mu^{-5/2}) \right) = 1
\]

and therefore

\[ \kappa(p'_0) = \mu^{1/2} - D_{+}(p)\mu^{-1/2} + O(\mu^{-3/2}). \]

This, along with (3.22) gives that

\[ \omega(p'_0) = \mu^{-1/2}\Omega_{+}(p) + O(\mu^{-3/2}). \] (3.24)

Since \( \int_{a_1^+} \omega(p'_0) = 0 \) and \( \int_{a_i^+} \ddot{\omega} = 0 \), (3.20), (3.24) and (3.17) yield, for \( i = 1, \ldots, n \),

\[
\sum_{j=1}^{n} \beta_j \int_{a_i^+} \frac{\omega(p'_0)}{z - \beta_j(p'_0)} = \int_{a_i^+} \frac{\omega(p'_0)}{(z - \mu)^2}
\]

\[ = \frac{2}{\mu^{7/2}} \int_{a_i^+} z\Omega_{+}(p'_0) + O \left( \frac{1}{\mu^{9/2}} \right). \]

For each \( i \), \( \int_{a_i^+} z\Omega_{+}(p'_0) \) is independent of \( \mu \) and so

\[ \sum_{j=1}^{n} \beta_j \mu^{1/2} \int_{a_i^+} \frac{\omega(p'_0)}{z - \beta_j(p'_0)} = O \left( \frac{1}{\mu^3} \right). \] (3.25)

But by (3.23) and (3.24),

\[ \left( \mu^{1/2} \int_{a_i^+} \frac{\omega(p'_0)}{z - \beta_j(p'_0)} \right)^i_j = \left( \int_{a_i^+} \frac{\Omega_{+}(p)}{z - \zeta_j} \right)^i_j + O \left( \frac{1}{\mu} \right), \]
and the matrix on the right is invertible and has no dependence on \( \mu \). (3.25) hence implies that for \( j = 1, \ldots, n \),

\[
\ddot{\beta}_j(p'_0) = O \left( \frac{1}{\mu^3} \right).
\]

We now have the asymptotics for \( \kappa(p'_0) \), \( \beta_j(p'_0) \) and \( \ddot{\beta}_j \) that we desired earlier, and substituting them into

\[
\frac{\ddot{\kappa}}{\kappa(p'_0)} - \sum_{j=1}^{n} \frac{\ddot{\beta}_j}{z - \beta_j(p'_0)} = \text{res}_{z=\mu} \frac{\omega(p'_0)}{(z - \mu)^2}
\]

from (3.21)

\[
= \frac{1}{2} \frac{d^2}{dz^2} \bigg|_{z=\mu} \left( \frac{\kappa(p'_0) \prod_{j=1}^{n}(z - \beta_j(p'_0))}{w_+(p)} \right)
\]

gives

\[
\ddot{\kappa} = \frac{3}{8\mu^{3/2}} + \frac{9D_+(p)}{8\mu^{3/2}} + O \left( \frac{1}{\mu^{7/2}} \right)
\]

which upon substitution into (3.13) completes the proof of (6).

\( \square \)

We now have asymptotic expressions for each row of the \( 2n + 5 \times 2n + 5 \) matrix \( H(\mu) \) in (3.1), which we wish to show is non-singular in the limit as \( \mu \to \infty \) along \( L_\epsilon \) of figure 3.3. The inductive assumption and Lemma 3.1 tell us that columns 1, \ldots, \( n + 2 \), \( n + 4 \), \ldots, \( 2n + 4 \) of the the first \( 2n + 3 \) rows of \( H(\mu) \) are linearly independent, and that its \( 2n + 4^{th} \) row \( (\frac{\partial}{\partial \mu} I_+(p'_0), \frac{\partial}{\partial \mu} I_-(p'_0)) \) is

\[
(0, \ldots, 0, 2 \sqrt{-1} (\mu^{-1/2} + D_+(p) \mu^{-3/2}) + O(\mu^{-5/2}); \underbrace{2 \sqrt{-1} (\mu^{-1/2} + D_-(p) \mu^{-3/2}) + O(\mu^{-5/2})}_{n + 1 \text{ zeros}}).
\]

Note that the two non-zero entries in this row have leading terms differing only by multiplication by \( \sqrt{-1} \). We find a linear combination of the rows of \( H(\mu) \) that equals

\[
(0, \ldots, 0, \sqrt{-1} \mu^{-5/2} (4\eta^+(p) - 5D_+(p)) + O(\mu^{-7/2}); \underbrace{2 \sqrt{-1} \mu^{-5/2} (4\eta^-(p) - 5D_-(p)) + O(\mu^{-7/2})}_{n + 1 \text{ zeros}}),
\]

where \( \eta^\pm(p) \) are defined in (3.28).

Our matrix is non-singular if

\[
\lim_{\mu \to \infty} 4\eta^+(p) - 5D_+(p) \neq \lim_{\mu \to \infty} 4\eta^-(p) - 5D_-(p)
\]

where the limits are taken along \( L_\epsilon \). This is the “extra condition” referred to earlier, and we will modify the statement we prove by induction to ensure that it is satisfied. First, we find the linear combination yielding this condition.
From Lemma 3.1 we have that
\[
\left( \frac{\partial^2}{\partial \nu^2} I_+(p'_0); \frac{\partial^2}{\partial \nu^2} I_-(p'_0) \right) = (3.26)
\]
\[
\left( \frac{D_+(p)}{\mu^3} - \frac{1}{2\mu^2} \right) I_+(p) - \frac{1}{\mu^3} \left( \hat{I}_+(p) + O\left(\frac{1}{\mu}\right) \right) + \frac{3\sqrt{-1}}{2\mu^{3/2}} + \frac{9\sqrt{-1}D_+}{2\mu^{5/2}} + O\left(\frac{1}{\mu^{7/2}}\right) \\
\left( \frac{D_-(p)}{\mu^3} - \frac{1}{2\mu^2} \right) I_-(p) - \frac{1}{\mu^3} \left( \hat{I}_-(p) + O\left(\frac{1}{\mu}\right) \right) + \frac{3\sqrt{-1}}{2\mu^{3/2}} + \frac{9D_-}{2\mu^{5/2}} + O\left(\frac{1}{\mu^{7/2}}\right)
\]

By the induction hypothesis, there are unique \( \eta^+(p), \chi(p), \xi_i(p), i = 1, \ldots, 2n \) such that
\[
(\hat{I}_+(p); \hat{I}_-(p)) = \eta^+(p)(I_+(p); 0) + \eta^-(p)(0; I_-(p)) + \chi(p) \frac{\partial}{\partial R}(I_+(p); I_-(p)) + \sum_{i=1}^{2n} \xi_i(p) \frac{\partial}{\partial \lambda_i}(I_+(p); I_-(p)).
\]

Thus by (3.26) and Lemma 3.1 there are \( \tilde{\eta}^+(p) = \eta^+(p) + O(\mu^{-1}), \tilde{\chi}(p) = \chi(p) + O(\mu^{-1}), \tilde{\xi}_i(p) = \xi_i(p) + O(\mu^{-1}) \) such that
\[
(\frac{\partial^2}{\partial \nu^2} I_+(p'_0); \frac{\partial^2}{\partial \nu^2} I_-(p'_0)) + (\frac{1}{2\mu^2} + \frac{\tilde{\eta}^+(p) - D_+(p)}{\mu^3})(I_+(p'_0); 0) + \\
(\frac{1}{2\mu^2} + \frac{\tilde{\eta}^+(p) - D_+(p)}{\mu^3})(0; I_-(p'_0)) + \tilde{\chi}(p) \frac{\partial}{\partial R}(I_+(p); I_-(p)) + \sum_{i=1}^{2n} \tilde{\xi}_i(p) \frac{\partial}{\partial \lambda_i}(I_+(p); I_-(p)) = (0, \ldots, 0, g_+(p'_0); 0, \ldots, 0, g_-(p'_0))
\]

where
\[
g_+(\mu) = \frac{3}{2\mu^{3/2}} + \frac{9D_+(p)}{2\mu^{5/2}} + \frac{1}{\mu^2} \frac{\eta^+(p) - D_+(p)}{\mu^3} \left( 4\mu^{1/2} - \frac{4D_+(p)}{\mu^{1/2}} \right) + O\left(\frac{1}{\mu^{7/2}}\right)
\]
\[
= \frac{7}{2\mu^{3/2}} - \frac{3D_+(p)}{2\mu^{5/2}} + \frac{4\eta^+(p)}{\mu^{5/2}} + O\left(\frac{1}{\mu^{7/2}}\right)
\]
and similarly
\[
g_-(\mu) = \frac{7}{2\mu^{3/2}} - \frac{3D_-(p)}{2\mu^{5/2}} + \frac{4\eta^-}{\mu^{5/2}} + O\left(\frac{1}{\mu^{7/2}}\right),
\]
so adding the appropriate multiple of \( \frac{\partial}{\partial \mu}(I_+(p'_0); I_-(p'_0)) \) to (3.29), we have
\[
l(p'_0) := \left( \frac{\partial^2}{\partial \nu^2} I_+(p'_0); \frac{\partial^2}{\partial \nu^2} I_-(p'_0) \right) + (\frac{1}{2\mu^2} + \frac{\tilde{\eta}^+(p) - D_+(p)}{\mu^3})(I_+(p'_0); 0) + \\
+ (\frac{1}{2\mu^2} + \frac{\tilde{\eta}^+(p) - D_+(p)}{\mu^3})(0; I_-(p'_0)) + \frac{\tilde{\xi} \frac{\partial}{\partial R}(I_+(p); I_-(p))}{\mu^3} \right) \]
Proof of Lemma 3.2:

For \( \mu = 1 \),

\[
\frac{1}{\mu^4} \sum_{i=1}^{2n} \xi_i(p) \frac{\partial}{\partial \lambda_i} (I_+(p); I_-(p)) + \frac{7}{4\mu} \frac{\partial}{\partial \mu} (I_+(p_0'), I_-(p_0'))
\]

\[
= \left( 0, \ldots, 0, \frac{\sqrt{-1}(4\eta^+(p) - 5D_+(p))}{\mu^{5/2}} + O \left( \frac{1}{\mu^{1/2}} \right) \right);
\]

\[
\frac{0, \ldots, 0}{n + 2 \text{ zeros}} \frac{(4\eta^-(p) - 5D_-(p))}{\mu^{5/2}} + O \left( \frac{1}{\mu^{1/2}} \right).
\]

(3.30)

We are led therefore, to modify the statement we prove by induction to include the assumption that

\[
\lim_{\mu \to \infty} 4\eta^+(p) - 5D_+(p) \neq \lim_{\mu \to \infty} 4\eta^-(p) - 5D_-(p),
\]

where the limits are taken along \( L_\epsilon \). Of course this modification needs to be such that it is preserved under the induction step. With this in mind, we define \( \eta^\pm(p_0') \) by the condition that

\[
(\hat{I}_+(p_0'); \hat{I}_-(p_0')) - \eta^+(p_0')(I_+(p_0'); 0) + \eta^-(p_0')(0; I_-(p_0')) \in \text{span} \left\{ \frac{\partial}{\partial R} (I_+(p_0'); I_-(p_0')), \frac{\partial}{\partial \lambda_i} (I_+(p_0'); I_-(p_0')), \frac{\partial^2}{\partial \mu^2} (I_+(p_0'); I_-(p_0')) \right\},
\]

and calculate the relationship between \( \eta^+(p_0') - \eta^-(p_0') \) and \( \eta^+(p) - \eta^-(p) \).

Lemma 3.2: As \( \mu \to \infty \) along \( L_\epsilon \),

\[
\hat{I}_+(p_0') = (\hat{I}_+(p), 2\sqrt{-1}\mu^{3/2} + (6\sqrt{-1}D_+(p) - 4\sqrt{-1}\eta^+(p))\mu^{1/2} + O(\mu^{-1/2})),
\]

\[
\hat{I}_-(p_0') = (\hat{I}_-(p_0'), 2\mu^{3/2} + (6D_-(p) - 4\eta_-(p))\mu^{1/2} + O(\mu^{-1/2})).
\]

Proof of Lemma 3.2: For \( \mu \) sufficiently large, we may assume that \( \mu \neq \zeta_j^\pm \), \( j = 1, \ldots, n \). Then

\[
\hat{\Omega}_+(p_0') := \frac{3}{2} z\Omega_+(p_0') + \sum_{j=1}^{n+1} \frac{c_j^+(p_0')\Omega_+(p_0')}{z - \zeta_j^+},
\]

where the \( c_j^+(p_0') \), \( j = 1, \ldots, n + 1 \) are determined by the (non-singular) system of equations

\[
\frac{3}{2} \int_{a_i^+} z\Omega_+(p_0') + \sum_{j=1}^{n+1} c_j^+(p_0') \int_{a_i^+} \frac{\Omega_+(p_0')}{z - \zeta_j^+} = 0, \; i, \; j = 1, \ldots, n + 1.
\]

Taking \( i = n + 1 \) we quickly see that \( c_{n+1}^+(p_0') = 0 \) and \( c_j^+(p_0') = c_j^+(p) \), for \( j = 1, \ldots, n \), so, not surprisingly,

\[
\Psi_+^*(\hat{\Omega}_+(p_0')) = \hat{\Omega}_+(p).
\]

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The argument for $\hat{\Omega}_-(p'_0)$ is similar, and together they prove all but the last components of the lemma above. For these, we again let $\Gamma$ denote the circle $|z| = \mu$, traversed clockwise. For $\mu$ sufficiently large,

$$\int_{b_{n+1}^+} \hat{\Omega}_+(p'_0) = -\int_{\Gamma} \left( \frac{3}{2} + \sum_{j=1}^n \frac{c_j^+(p)}{z - \zeta_j^+} \right) \frac{\prod_{k=1}^n (z - \zeta_k^+)}{\sqrt{(z - R) (z - \lambda_i)}} dz$$

$$= 2\sqrt{-1} \mu^{3/2} + (6\sqrt{-1}D_+(p) - 4\sqrt{-1}\eta^+(p))\mu^{1/2} + O(\mu^{-1/2}),$$

and similarly for $\int_{b_{n+1}^-} \hat{\Omega}_-(p'_0)$.

From Lemma 3.1,

$$(\hat{I}_+(p'_0); \hat{I}_-(p'_0)) - \eta^+(p)(I_+(p'_0); 0) - \eta^-(p)(0; I_-(p'_0)) - \chi(p) \frac{\partial}{\partial R}(I_+(p'_0); I_-(p'_0))$$

$$- \sum_{i=1}^{2n} \xi_i(p) \frac{\partial}{\partial \lambda_i}(I_+(p'_0); I_-(p'_0))$$

$$= (0, 2\sqrt{-1} \mu^{3/2} + (6\sqrt{-1}D_+(p) - 4\sqrt{-1}\eta^+(p))\mu^{1/2} + O(\mu^{-1/2});$$

$$0, 2\mu^{3/2} + (6D_-(p) - 4\eta^-(p))\mu^{1/2} + O(\mu^{-1/2})$$

$$= \Lambda \frac{\partial}{\partial \mu}(I_+(p'_0); I_-(p'_0)) + \Upsilon l(p'_0),$$

(3.31)

where $l(p'_0)$ is defined in (3.30), and by Lemma 3.2, we have that $\Lambda$ and $\Upsilon$ are defined by the equations

$$\begin{pmatrix}
\frac{2\sqrt{-1}}{\mu^{1/2}} (1 + \frac{D_+(p)}{\mu}) + O(\frac{1}{\mu^{3/2}}) & \frac{\sqrt{-1}(4\eta^+(p) - 5D_+(p))}{\mu^{3/2}} + O(\frac{1}{\mu^{5/2}}) \\
\frac{2}{\mu^{3/2}} (1 + \frac{D_-(p)}{\mu}) + O(\frac{1}{\mu^{5/2}}) & \frac{4\eta^-(p) - 5D_-(p)}{\mu^{3/2}} + O(\frac{1}{\mu^{5/2}})
\end{pmatrix}

\begin{pmatrix}
\Lambda \\
\Upsilon
\end{pmatrix}

= \begin{pmatrix}
2\sqrt{-1} \mu^{3/2} + (6\sqrt{-1}D_+(p) - 4\sqrt{-1}\eta^+(p))\mu^{1/2} + O(\frac{1}{\mu^{1/2}}) \\
2\mu^{3/2} + (6D_-(p) - 4\eta^-(p))\mu^{1/2} + O(\frac{1}{\mu^{1/2}})
\end{pmatrix}

so

$$\Lambda = \mu^2 + O(\mu),$$

$$\Upsilon = \frac{4(\eta^+(p) - \eta^-(p)) - 4(D_+(p) - D_-(p))}{5(D_+(p) - D_-(p)) - 4(\eta^+(p) - \eta^-(p))}\mu^3 + O(\mu^2).$$

Using (3.30) and (3.31) then,

$$\eta^+(p'_0) = \eta^+(p) + \Upsilon \left( \frac{1}{2\mu^2} + \frac{\eta^+(p) - D_+(p)}{\mu^3} + O\left(\frac{1}{\mu^4}\right) \right),$$

$$\eta^-(p'_0) = \eta^-(p) + \Upsilon \left( \frac{1}{2\mu^2} + \frac{\eta^-(p) - D_-(p)}{\mu^3} + O\left(\frac{1}{\mu^4}\right) \right).$$
\[
\eta^+(p'_0) - \eta^-(p'_0) = \eta^+(p) - \eta^-(p) + \frac{4(\eta^+(p) - \eta^-(p)) - 4(D_+(p) - D_-(p))}{5(D_+(p) - D_-(p)) - 4(\eta^+(p) - \eta^-(p))} (\eta^+(p) - \eta^-(p) - (D_+(p) - D_-(p))) + O\left(\frac{1}{\mu}\right) \\
= \frac{(D_+(p) - D_-(p)) (3(\eta^+(p) - \eta^-(p)) - 4(D_+(p) - D_-(p)))}{4(\eta^+(p) - \eta^-(p)) - 5(D_+(p) - D_-(p))}.
\]

(3.32)

Defining \( T_p \) to be the linear fractional transformation

\[
T_p : x \mapsto (D_+(p) - D_-(p)) \frac{3x - 4(D_+(p) - D_-(p))}{4x - 5(D_+(p) - D_-(p))},
\]

(3.32) is the statement that

\[
T_p(\eta^+(p) - \eta^-(p)) = \eta^+(p'_0) - \eta^-(p'_0).
\]

Moreover, we know that \( D_\pm(p'_0) = D_\pm(p) \) and hence

\[
T_{p'_0} = T_p.
\]

The statement then, that we prove by induction, is:

**Theorem 3.3** For each positive integer \( m \) and integer \( n \) with \( 0 \leq n \leq m \) there exists \( p \in M_{n,\mathbb{R}} \) such that

1. \( \zeta_j^+(p), j = 1, \ldots, n \) are pairwise distinct, as are \( \zeta_j^-(p), j = 0 \ldots n \),

2. \( \mathbb{R}^{2n+3} \) is spanned by the vectors \((I_+(p), 0), (0, I_-(p)), \frac{\partial}{\partial x_1} (I_+(p), I_-(p)), i = 1, \ldots, 2n\),

3. \( 5(D_+(p) - D_-(p)) + 4T^k_p (\eta_+(p) - \eta_-(p)) \neq 0 \), for \( 0 \leq k \leq m - n \).

For the convenience of the reader, we reiterate here the definitions of many of the objects appearing in Theorem 3.3, so that it may be read immediately after the statement of Theorem 3.2, without recourse to the above arguments. Assume that (1) and (2) of Theorem 3.3 hold for \( p \in M_{n,\mathbb{R}} \). Then the differentials \( \frac{\Omega_\pm(p)}{z - \zeta_j^\pm} \) are a basis for the holomorphic differentials on \( C_\pm(p) \). Thus we may define \( c_j^\pm(p) \) by the equations

\[
\frac{3}{2} \int_{a_i^\pm} z\Omega_{\pm}(p) + \sum_{j=1}^{n} c_j^\pm(p) \int_{a_i^\pm} \frac{\Omega_{\pm}(p)}{z - \zeta_j^\pm} = 0, \ i, j = 1, \ldots, n.
\]

Let

\[
\hat{\Omega}_{\pm}(p) := \frac{3}{2} z\Omega_{\pm}(p) + \sum_{j=1}^{n} c_j^\pm(p) \frac{\Omega_{\pm}(p)}{z - \zeta_j^\pm},
\]

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\[ \hat{I}_+(p) := \sqrt{-1} \left( \int_{c_1} \hat{\Omega}_+(p), \int_{c_{-1}} \hat{\Omega}_+(p), \int_{b^+_1} \hat{\Omega}_+(p), \ldots, \int_{b^+_n} \hat{\Omega}_+(p) \right) \]

and
\[ \hat{I}_-(p) := \left( \int_{b^-_0} \hat{\Omega}_-(p), \int_{b^-_1} \hat{\Omega}_-(p), \ldots, \int_{b^-_n} \hat{\Omega}_-(p) \right). \]

Define \( \eta^\pm(p), \xi(p), \xi_i^\pm(p), i = 1, \ldots, 2n \) by
\[
(\hat{I}_+(p), \hat{I}_-(p)) = \eta^+(p)(I_+(p), 0) + \eta^-(p)(0, I_-(p)) + \xi(p) \frac{\partial}{\partial R} (I_+(p), I_-(p)) + \sum_{i=1}^{2n} \xi_i(p) \frac{\partial}{\partial \lambda_i} (I_+(p), I_-(p)).
\]

Put
\[
D_+(p) := \frac{1}{2} \left( R + 2 \sum_{i=1}^{2n} \lambda_i \right) - \sum_{j=1}^{n} \zeta^+_j,
\]
\[
D_-(p) := \frac{1}{2} \left( R + 2 \sum_{i=1}^{2n} \lambda_i \right) - \sum_{j=0}^{n} \zeta^-_j
\]

and let \( T_p \) be the linear fractional transformation
\[
T_p : x \mapsto (D_+(p) - D_-(p)) \frac{3x - 4(D_+(p) - D_-(p))}{4x - 5(D_+(p) - D_-(p))}.
\]

**Proof of Theorem 3.3:** Fix \( m \), and for \( n < m \) suppose \( p \in M_{n,\mathbb{R}} \) satisfies the conditions of Theorem 3.3. By the above arguments, the set of \( \mu \in (-2, 2) \) such that

(i) for all \( \epsilon \in (0, \min_{i=1,\ldots,n} |\lambda_i + 2|) \), \( h_\epsilon(\mu) \neq 0 \),

(ii) for \( j = 1, \ldots, n \), \( \mu \neq \zeta^+_j \) and

(iii) for \( j = 0, \ldots, n \), \( \mu \neq \zeta^-_j \)

is dense in \((-2, 2)\).

Take such a \( \mu \). Then \( p'_0 = (p, \mu, 0) \) satisfies Theorem 3.3, where in (2) we replace \( \frac{\partial}{\partial x_{2n+2}} (I_+(p'_0); I_-(p'_0)) \) by \( \frac{\partial}{\partial x^2} (I_+(p'_0); I_-(p'_0)) \). Then for \( \nu \) small, utilising (3.2),
\[
\eta^+(p, \mu, \nu) = \eta^+(p'_0) + O(\nu),
\]
\[
D_\pm(p, \mu, \nu) = D_\pm(p'_0) + O(\nu)
\]

and
\[
T_{(p, \mu, \nu)} = T_{p'_0} + O(\nu),
\]
we conclude that \((p, \mu, \nu)\) satisfies Theorem 3.3. It remains to show the existence of \( p \in M_{0,\mathbb{R}} \) verifying (1) and (2) of Theorem 3.3, and such that for no \( k \geq 0 \) do we have
\[
5(D_+(p) - D_-(p)) + 4T_p^k (\eta_+(p) - \eta_-(p)) = 0.
\]
3.1.1 Genus One \((n = 0)\)

We consider pairs \(C_+ = C_+(R)\) and \(C_- = C_-(R)\) given by

\[
w_+^2 = (z - R)
\]

and

\[
w_-^2 = (z + 2)(z - 2)(z - R)
\]

respectively, where \(R > 2\). Writing \(\pi_\pm : (z, w_\pm) \mapsto z\) for the projections to \(\mathbb{CP}^1\), the fibre product of these is the genus one curve \(X = X(R)\), given by

\[
y^2 = x(x - r)(x - \frac{1}{r}),\text{ where } r + \frac{1}{r} = R.
\]

Lemma 3.3 There exists a \(p \in M_{0,3}\) such that

1. \(\mathbb{R}^3\) is spanned by the vectors \((I_+(p), 0), (0, I_-(p))\) and \(\frac{\partial}{\partial R}(I_+(p), I_-(p))\),

2. for all \(k \geq 0\), \(5(D_+(p) - D_-(p)) + 4T^k(p)(\eta^+(p) - \eta^-(p)) \neq 0\).

Proof: The natural limit to consider is \(r \to 1\), i.e. \(R = r + 1/r \to 2\), which suggests setting \(\zeta := z + 2, t := R - 2\). Then \(C_+(t)\) is given by

\[
w_+^2 := \zeta - t,
\]

and \(C_-(t)\) by

\[
w_-^2 := \zeta(\zeta - t)(\zeta + 4).
\]

For each \(t > 0\), choose \(c_1(t), c_{-1}(t)\) and \(a_-(t)\) as shown in figure 3.5. We write

\[
\Omega_-(t) = \frac{(\zeta - s(t))d\zeta}{w_-}, \text{ where } s(t) \text{ is defined by the condition } \int_{a_-(t)} \Omega_-(t) = 0.
\]

Using

\[
\int_{a_-(t)} \Omega_-(t) = 0 \quad \text{and} \quad \frac{\partial}{\partial t} \int_{a_-(t)} \Omega_-(t) = 0,
\]

\[
\Omega_-(t) = \frac{(\zeta - s(t))d\zeta}{w_-}, \text{ where } s(t) \text{ is defined by the condition } \int_{a_-(t)} \Omega_-(t) = 0.
\]

Figure 3.5: The curves \(\tilde{a}^-\) and \(\tilde{c}_{\pm 1}\), for \(n = 0\).
one obtains
\[ s(t) = t + O(t^2). \tag{3.33} \]

Thus
\[
I_-(t) = \int_{b_-(t)} \Omega_- (t) \tag{3.34}
\]
\[
= 2 \int_{-4}^{0} \frac{(\zeta - t) d \zeta}{\sqrt{\zeta (\zeta + 4)(\zeta - t)}} + O(t^2) \tag{3.35}
\]
\[
= 8 + O(t). \tag{3.36}
\]

Now
\[
I_+(t) = \sqrt{-1} (\int_{c_1(t)} \Omega_+ (t), \int_{c_{-1}(t)} \Omega_+(t)), \]
where \( c_1(t) \) is a path in \( C_+(t) \) joining the two points with \( \zeta = 0 \), and \( c_{-1}(t) \) is one joining the two points with \( \zeta = -4 \), both beginning at points with \( \frac{w_+}{\sqrt{-1}} < 0 \).

Then
\[
I_+(t) = 2 \sqrt{-1} \left( \int_{0}^{t} \frac{d \zeta}{\sqrt{\zeta - t}}, \int_{-4}^{t} \frac{d \zeta}{\sqrt{\zeta - t}} \right) \tag{3.37}
\]
\[
= (4t^{1/2}, 4(4 + t)^{1/2}) \tag{3.38}
\]

and
\[
\frac{\partial I_+(t)}{\partial t} = (2t^{-1/2}, 2(4 + t)^{-1/2})
\]
so we see that condition (1) of Lemma 3.3 is satisfied for all \( t > 0 \), in other words for all \( R > 2 \).

We proceed then to calculate \( \lim_{t \to 0} D_\pm (t) \) and \( \lim_{t \to 0} \eta_\pm (t) \). We have
\[
\lim_{t \to 0} D_+(t) = \lim_{t \to 0} \frac{1}{2}(t + 2)
\]
\[
= 1
\]
and
\[
\lim_{t \to 0} D_-(t) = \lim_{t \to 0} \frac{1}{2}(t + 2) - (2 + s(t))
\]
\[
= -1.
\]

Also,
\[
\hat{\Omega}_+ (t) = \frac{3}{2} \hat{\Omega}_+ (t) = \frac{3(\zeta + 2) d \zeta}{2 \sqrt{\zeta - t}},
\]
so
\[
\hat{I}_+(t) = \sqrt{-1} \left( \int_{c_1(t)} \hat{\Omega}_+(t), \int_{c_{-1}(t)} \hat{\Omega}_+(t) \right) \tag{3.39}
\]
\[
= \sqrt{-1} \left( 3 \int_{0}^{t} \frac{(\zeta + 2) d \zeta}{\sqrt{\zeta - t}}, 3 \int_{-4}^{t} \frac{(\zeta + 2) d \zeta}{\sqrt{\zeta - t}} \right) \tag{3.40}
\]
\[
= (12t^{1/2} + 8t^{3/2}, 12(4 + t)^{1/2} + 8(4 + t)^{3/2}). \tag{3.41}
\]
Recall that $\hat{\Omega}_-(t) = \frac{3}{2}(\zeta + 2)\Omega_-(t) + \frac{c_-(t)\Omega_-(t)}{\zeta - s(t)}$, where $c_-(t)$ is defined by
\begin{equation}
\frac{3}{2} \int_{a_-}^{a_-} \frac{(\zeta + 2)(\zeta - s(t))d\zeta}{\sqrt{\zeta(\zeta + 4)(\zeta - t)}} + c_-(t) \int_{a_-}^{a_-} \frac{d\zeta}{\sqrt{\zeta(\zeta + 4)(\zeta - t)}} = 0. \tag{3.42}
\end{equation}

Let $\int_{a_-}^{a_-} \frac{d\zeta}{\sqrt{\zeta(\zeta + 4)(\zeta - t)}} = a_0 + a_1 t + O(t^2)$. Then
\begin{align*}
a_0 &= 2\pi \sqrt{-1} \text{Res}_{\zeta = 0} \frac{1}{\sqrt{\zeta(\zeta + 4)(\zeta - t)}} \\
&= \pi \sqrt{-1},
\end{align*}
and
\begin{align*}
a_1 &= 2\pi \sqrt{-1} \text{Res}_{\zeta = 0} \frac{\partial}{\partial t} \bigg|_{t=0} \frac{1}{\sqrt{\zeta(\zeta + 4)(\zeta - t)}} \\
&= -\frac{\pi \sqrt{-1}}{16},
\end{align*}
so
\begin{equation}
\int_{a_-}^{a_-} \frac{d\zeta}{\sqrt{\zeta(\zeta + 4)(\zeta - t)}} = \pi \sqrt{-1} - \frac{\pi \sqrt{-1}}{16} t + O(t^2),
\end{equation}
and similarly
\begin{align*}
\int_{a_-}^{a_-} \frac{\zeta d\zeta}{\sqrt{\zeta(\zeta + 4)(\zeta - t)}} &= \pi \sqrt{-1} t + O(t^2), \\
\int_{a_-}^{a_-} \frac{\zeta^2 d\zeta}{\sqrt{\zeta(\zeta + 4)(\zeta - t)}} &= O(t^2).
\end{align*}
Substituting these and (3.33) into (3.42), one obtains
\begin{equation}
c_-(t) = O(t^2).
\end{equation}
From this and (3.33),
\begin{align*}
\hat{I}_-(t) &= 3 \int_{-4}^{0} \frac{(\zeta + 2)d\zeta}{\sqrt{\zeta + 4}} + O(t) \\
&= -8 + O(t). \tag{3.44}
\end{align*}
From (3.36), (3.38), (3.44) and (3.41) then
\begin{equation}
\begin{pmatrix}
I_+(t) & 0 \\
0 & I_-(t) \\
\frac{\partial I_+(t)}{\partial t} & \frac{\partial I_-(t)}{\partial t} \\
\frac{\partial \hat{I}_+(t)}{\partial t} & \frac{\partial \hat{I}_-(t)}{\partial t}
\end{pmatrix}
= \begin{pmatrix}
4t^{1/2} & 4(4 + t)^{1/2} & 0 \\
0 & 2t^{-1/2} & 2(4 + t)^{-1/2} & 8 + O(t) \\
2t^{1/2} + 8t^{3/2} & 12(4 + t)^{1/2} + 8(4 + t)^{3/2} & -8 + O(t) & 0 \\
12t^{1/2} + 8t^{3/2} & 12(4 + t)^{1/2} + 8(4 + t)^{3/2} & -8 + O(t) & 0
\end{pmatrix}.
\end{equation}
Upon multiplication of its third row by $t$, its first column by $2t^{-1/2}$ and its second column by $2(4 + t)^{-1/2}$ this matrix becomes
\[
\begin{pmatrix}
2 & 2 & 0 \\
0 & 0 & 8 \\
1 & 0 & 0 \\
6 & 22 & -8
\end{pmatrix} + O(t).
\]
Since $-11(2, 2, 0) + 1(0, 0, 8) + 16(1, 0, 0) + 1(6, 22, -8) = (0, 0, 0)$, then recalling that $\eta^\pm(t)$ are defined by the condition
\[
(\hat{I}_+(t), \hat{I}_-(t)) + \eta^+(t)(I_+(t), 0) + \eta^-(t)(0, I_-(t)) \in \text{span}\left\{ \frac{\partial}{\partial t}(I_+(t), I_-(t)) \right\},
\]
we conclude that
\[
\lim_{t \to 0} \eta^+(t) = -11
\]
and
\[
\lim_{t \to 0} \eta^-(t) = 1.
\]
The linear fractional transformation $T_t$ is defined by
\[
T_t: u \mapsto \frac{-(D_-(t) - D_+(t))(3u + 4(D_-(t) - D_+(t)))}{4u + 5(D_-(t) - D_+(t))},
\]
so letting $T := \lim_{t \to 0} T_t$,
\[
T: u \mapsto \frac{3u - 8}{2u - 5}.
\]
This has a unique fixed point ($u = 2$) and so is conjugate to a translation, in fact denoting the map $u \mapsto \frac{1}{u - 2}$ by $S$, we have
\[
STS^{-1}: u \mapsto u - 2.
\]
Now
\[
4T^k(\lim_{t \to 0}(\eta^-(t) - \eta^+(t))) = 5(\lim_{t \to 0}(D_+(t) - D_-(t)))
\]
\[
\iff
T^k(12) = 5 \cdot 2
\]
\[
\iff
(STS^{-1})^k(\frac{1}{10}) = 2
\]
\[
\iff
\frac{1}{10} - 2k = 2,
\]
which is clearly false for all integers $k \geq 0$. Thus for $t > 0$ sufficiently small, the Lemma holds.

\[\square\]
3.2 Even Genera

Let $C_{\pm} = C_{\pm}(\lambda_1, \ldots \lambda_{2n})$ be the curves given by

$$w_{\pm}^2 = (z \pm 2) \prod_{i=1}^{2n}(z - \lambda_i),$$

where we assume that the sets \(\{\lambda_1, \lambda_2\}, \ldots, \{\lambda_{2n-1}, \lambda_{2n}\}\) are mutually disjoint and that $\lambda_i \neq \pm 2$ for $i = 1, \ldots, 2n$. Denote by $\pi_{\pm} : C_{\pm} \to \mathbb{C}P^1$ the respective projections $(z, w_{\pm}) \mapsto z$ to the Riemann sphere, and define a real structure $\rho_{\pm}$ on $C_{\pm}$ by

$$\rho_{\pm}(z, w_{\pm}) = (\bar{z}, \pm \bar{w}_{\pm}).$$

Choose lifts of the curves $\tilde{a}_1, \ldots, \tilde{a}_n, \tilde{c}_1, \tilde{c}_{-1}$ to $C_{\pm}$ and denote the homology classes of these lifts by $a_{\pm}^1, \ldots, a_{\pm}^n$. Choose also an open curve $c_1$ in $C_+$ covering $\tilde{c}_1$ and an open curve $c_{-1}$ in $C_-$ covering $\tilde{c}_{-1}$. We write $M_n$ for the space of $2n$-tuples $(\lambda_1, \ldots, \lambda_{2n})$ as above together with the choices described, and let $M_n, \mathbb{R}$ denote the subset of $M_n$ such that (see figure 3.6):

1. $\lambda_{2i} = \bar{\lambda}_{2i-1}$ for $i = 1, \ldots, n$,
2. $\tilde{a}_i$ intersects the real axis exactly twice, both times in the interval $(-2, 2)$,
3. the lifts of $\tilde{a}_1, \ldots, \tilde{a}_n$ to $C_+$ are chosen so that the point where $\tilde{a}_i$ intersects the $z$-axis with positive orientation is lifted to a point where $w_+$ is positive,
4. the lifts of $\tilde{a}_1, \ldots, \tilde{a}_n$ to $C_-$ are chosen so that the point where $\tilde{a}_i$ intersects the $z$-axis with positive orientation is lifted to a point where $\frac{w_-}{\bar{w}_{-1}}$ is positive,
5. $c_1$ begins at a point where $w_+$ is positive.
6. $c_{-1}$ begins at a point where $\frac{w_-}{\bar{w}_{-1}}$ is positive.

For $p \in M_n, \mathbb{R}$, we may, as before, construct $\pi : X \to \mathbb{C}P^1$ as the fibre product of $\pi_+ : C_+ \to \mathbb{C}P^1$ and $\pi_- : C_- \to \mathbb{C}P^1$. $X$ is given by the equation

$$y^2 = x \prod_{i=1}^{n}(x - \alpha_i)(x - \alpha_i^{-1})(x - \bar{\alpha}_i)(x - \bar{\alpha}_i^{-1}),$$

where $\alpha_i + \alpha_i^{-1} = \lambda_i$. 37
Figure 3.6: The curves $\tilde{a}_i$ and $\tilde{c}_i$. 

and $\pi$ by

$$\pi : (x, y) \mapsto x.$$ 

These identifications occur via the maps

$$q_\pm(x, y) = \left( x + \frac{1}{x}, \frac{(x \pm 1)y}{x^{n+1}} \right) = (z, w_\pm).$$

$X$ has genus $2n$ and possesses the holomorphic involutions

$$i_\pm : X \longrightarrow X,$$

$$(x, y) \longmapsto \left( \frac{1}{x}, \frac{\pm y}{x^{2n+1}} \right).$$

The curves $C_\pm$ are the quotients of $X$ by these involutions, with quotient maps $q_\pm : X \to C_\pm$, and the real structures $\rho_\pm$ on $C_\pm$ induce upon $X$ the real structure

$$\rho : (x, y) \mapsto (\frac{1}{x}, \frac{\pm y}{x^{2n+1}}).$$

For each $p \in M_{n, \mathbb{R}}$ there is a unique canonical basis $A_1, \ldots, A_{2n}, B_1, \ldots, B_{2n}$ for the homology of $X$ such that $A_1, \ldots, A_{2n}$ cover the homotopy classes of loops $\tilde{A}_1, \ldots, \tilde{A}_{2n}$ shown in Figure 3.7 and

$$(q_\pm)_*(A_i) = \pm (q_\pm)_*(A_{n+i}) = a_i^\pm.$$ 

There are also unique curves $\gamma_1$ and $\gamma_{-1}$ on $X$ such that $(q_\pm)_*(\gamma_{\pm 1}) = c_{\pm 1}$; they project to $\tilde{\gamma}_1$ and $\tilde{\gamma}_{-1}$ of Figure 3.7. For each $p \in M_n$ define differentials $\Omega_\pm = \Omega_\pm(p)$ on $C_\pm(p)$ by: 

Figure 3.7: The curves $\tilde{A}_i$ and $\tilde{\gamma}_\pm$.

1. $\Omega_\pm (p)$ are meromorphic differentials of the second kind: their only singularities are double poles at $z = \infty$, and they have no residues.

2. $\int_{a_i^+} \Omega_\pm (p) = 0$ for $i = 1, \ldots, n$.

3. As $z \to \infty$, $\Omega_\pm (p) \to \frac{\pm dz}{w_\pm (p)}$.

In view of these defining conditions, we may write

$$\Omega_\pm = \frac{\prod_{j=1}^n (z - \zeta_j^\pm)}{w_\pm} dz.$$  

Define

$$I_+ (p) := \left( \int_{c_1} \Omega_+ (p), \int_{b_1^+} \Omega_+ (p), \ldots, \int_{b_n^+} \Omega_+ (p) \right),$$

$$I_- (p) := \sqrt{-1} \left( \int_{c_{-1}} \Omega_- (p), \int_{b_1^-} \Omega_- (p), \ldots, \int_{b_n^-} \Omega_- (p) \right).$$

For $p \in M_{n,R}$, $I_+ (p)$ and $I_- (p)$ are real, since, writing $A^\pm$ for the subgroups of $H_1 (C_\pm, \mathbb{Z})$ generated by the $a_i^\pm$, we then have

$$(\rho_\pm)_* (b_i^\pm) = b_i^\pm \mod A^\pm, \quad (\rho_\pm)_* (c_{\pm 1}) = c_{\pm 1} \mod A^\pm,$$

and

$$\rho^*_\pm (\Omega_\pm) = \pm \overline{\Omega}_\pm.$$
**Theorem 3.4** For each positive integer \( m \) and integer \( n \) with \( 0 \leq n \leq m \) there exists \( p \in M_{n, \mathbb{R}} \) such that

1. \( \zeta_j^+(p), j = 1, \ldots, n \) are pairwise distinct, as are \( \zeta_j^-(p), j = 1 \ldots n \)

2. \( \mathbb{R}^{2n+2} \) is spanned by the vectors \((I_+(p), 0), (0, I_-(p))\) and \( \frac{\partial}{\partial \lambda_i}(I_+(p), I_-(p)) \), \( i = 1, \ldots, 2n \).

3. \( 5(D_+(p) - D_-(p)) + 4T_k^i(\eta^+(p) - \eta^-(p)) \neq 0 \), for \( 0 \leq k \leq m - n \), where \( D_\pm(p), \eta^\pm(p) \) and \( T_p \) are defined as follows:

   Assume that (1) and (2) of this theorem hold for \( p \in M_{n, \mathbb{R}} \). Then the differentials \( \frac{\Omega(p)}{z-\zeta_j} \) are a basis for the holomorphic differentials on \( C_\pm(p) \). Thus we may define \( c_j^+(p) \) by the equations

   \[
   \frac{3}{2} \int_{a_i^+} z \Omega^+(p) + \sum_{j=1}^n c_j^+(p) \int_{a_i^+} \frac{\Omega^+(p)}{z-\zeta_j} = 0, \quad i, j = 1, \ldots, n,
   \]

   and set

   \[
   \hat{\Omega}_\pm(p) := \frac{3}{2} z \Omega^\pm(p) + \sum_{j=1}^n c_j^\pm(p) \frac{\Omega^\pm(p)}{z-\zeta_j},
   \]

   \[
   \hat{I}_+(p) := \left( \int_{c_1} \hat{\Omega}_+(p), \int_{b_1^+} \hat{\Omega}_+(p), \ldots, \int_{b_n^+} \hat{\Omega}_+(p) \right),
   \]

   \[
   \hat{I}_-(p) := \sqrt{-1} \left( \int_{c_1} \hat{\Omega}_-(p), \int_{b_1^-} \hat{\Omega}_-(p), \ldots, \int_{b_n^-} \hat{\Omega}_-(p) \right).
   \]

   Then we define \( \eta^\pm(p) \) by

   \[
   (\hat{I}_+(p), \hat{I}_-(p)) - \eta^+(p)(I_+(p), 0) - \eta^-(p)(0, I_-(-p)) \in \text{span} \left\{ \frac{\partial}{\partial \lambda_i}(I_+(p), I_-(p)), \; i = 1 \ldots n \right\},
   \]

   put

   \[
   D_\pm(p) := \frac{1}{2} \left( 1 + 2 + \sum_{i=1}^{2n} \lambda_i \right) - \sum_{j=1}^n \zeta_j^\pm,
   \]

   and let \( T_p \) be the linear fractional transformation

   \[
   T_p : x \mapsto \left( D_+(p) - D_-(p) \right) \frac{3x - 4(D_+(p) - D_-(p))}{4x - 5(D_+(p) - D_-(p))}.
   \]

**Proof:** For each fixed \( m \), this theorem can be proven by induction. The induction step is both similar to and simpler than that detailed in the odd genus case, and is almost identical to the induction step described in [8]. We thus omit it here, but
demonstrate the existence of \( p \in M_{0,\mathbb{R}} \) satisfying (1) and (2) and such that, for all \( k \geq 0 \), \( 5(D_+(p) - D_-(p)) + 4T^k_p(\eta^+(p) - \eta^-(p)) \neq 0 \).

Consider the curves \( C_\pm \) given by

\[ w^2_\pm = (z \pm 2). \]

Let \( c_1 \) be an open curve on \( C_+ \) from \((2,2)\) to \((2,-2)\), and \( c_-1 \) one on \( C_- \) from \((-2,2\sqrt{-1})\) to \((-2,-2\sqrt{-1})\). We have

\[ \Omega_\pm = \frac{dz}{w_\pm}, \]

which gives

\[ I_\pm = \mp 8. \]

Also,

\[ \widehat{\Omega}_\pm = \frac{3}{2}z\Omega_\pm, \]

which gives rise to

\[ \widehat{I}_\pm = 8. \]

\( \eta^\pm \) are defined by the equation

\[ (\widehat{I}_+, \widehat{I}_-) = \eta^+(I_+, 0) + \eta^-(0, I_-), \]

so

\[ \eta^\pm = \mp 1. \]

We have

\[ D_\pm = \mp 1, \]

and hence the linear transformation \( T \) is given by

\[ T : x \mapsto -\frac{3x + 8}{2x + 5}. \]

The statement

\[ 5(D_+ - D_-) + 4T^k(\eta^+ - \eta^-) = 0 \]

may be written as

\[ T^k(-2) = \frac{5}{4}, \]

but \(-2\) is a fixed point of \( T \), so this is false for all \( k \geq 0 \).
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