MIURA MAPS AND INVERSE SCATTERING FOR THE NOVIKOV-VESELOV EQUATION

PETER A. PERRY

Abstract. We use the inverse scattering method to solve the zero-energy Novikov-Veselov (NV) equation for initial data of conductivity type, solving a problem posed by Lassas, Mueller, Siltanen, and Stahel. We exploit Bogdanov’s Miura-type map which transforms solutions of the modified Novikov-Veselov (mNV) equation into solutions of the NV equation. We show that the Cauchy data of conductivity type considered by Lassas, Mueller, Siltanen, and Stahel lie in the range of Bogdanov’s Miura-type map, so that it suffices to study the mNV equation. We solve the mNV equation using the scattering transform associated to the defocusing Davey-Stewartson II equation.

CONTENTS
1. Introduction 1
2. Preliminaries 7
3. Scattering Maps and an Oscillatory $\bar{\partial}$-Problem 9
4. Restrictions of Scattering Maps 11
5. Solving the mNV Equation 15
6. Solving the NV Equation 18
7. Conductivity-Type Potentials 19
Appendix A. Schwarz Class Inverse Scattering for the mNV Equation 22
A.1. Scattering Solutions and Tangent Maps 22
A.2. Expansion Coefficients for $\nu$ 24
A.3. Expansion Coefficients for $\nu^#$ 25
A.4. Inverse Scattering Method for mNV 26
References 28

1. INTRODUCTION
In this paper we will use inverse scattering methods to solve the Novikov-Veselov (NV) equation, a completely integrable, dispersive nonlinear equation in two space and one time $(2+1)$ dimensions, for the class of conductivity type initial data that we define below. Our results solve a problem posed by Lassas, Mueller, Siltanen, and Stahel [40] in their analytical study of the inverse scattering method for the NV equation.

Version of December 21, 2013.
Supported in part by NSF grants DMS-0710477 and DMS-1208778.
Denoting \( z = x_1 + ix_2, \varphi = (1/2)(\partial x_1 + i\partial x_2), \) \( \partial = (1/2)(\partial x_1 - i\partial x_2), \) the Cauchy problem for the NV equation is

\[
q_t + \partial^3 q + \varphi^3 q + \frac{3}{4} \partial \left( q\varphi^{-1} \partial q \right) + \frac{3}{4} \varphi (q\partial^{-1} \varphi q) \tag{1.1}
\]

\[
q|_{t=0} = q_0
\]

where \( q_0 \) is a real-valued function that vanishes at infinity. Up to trivial scalings, our equation is the zero-energy \((E = 0)\) case of the equation

\[
q_t = 4 \text{Re} \left( 4\partial^3 q + \partial(qw) - E\partial q \right)
\]

\[
\partial w = \partial q \tag{1.2}
\]

studied by Novikov and Veselov in [47, 48]. If \( q \) does not depend on \( y \), the zero-energy NV equation (1.1) reduces (after rescaling) to the Korteweg-de Vries (KdV) equation

\[
q_t = \frac{1}{4} q_{xxx} + 6qq_x = 0.
\]

The Novikov-Veselov equation is one of a hierarchy of dispersive nonlinear equations in 2 + 1 dimensions discovered by Novikov and Veselov [47, 48]. In these papers, Novikov and Veselov constructed explicit solutions from the spectral data associated to a two-dimensional Schrödinger problem at a single energy. Novikov conjectured that the inverse problem for the two-dimensional Schrödinger operator at a fixed energy should be completely solvable (see the remarks in [28]), and that inverse scattering for the Schrödinger equation at a fixed energy \( E \) could be used to solve the NV equation at the same energy \( E \) by inverse scattering. In subsequent studies, Grinevich, Grinevich-Manakov, and Grinevich-Novikov [27, 30, 32, 33, 34] further developed the inverse scattering method and constructed multisoliton solutions. Independently, Boiti, Leon, Manna, and Pempinelli [16] proposed the inverse scattering method to solve the NV equation at zero energy with data vanishing at infinity.

It has long been understood that the inverse Schrödinger scattering problem at zero energy poses special challenges (see, for example, the discussion in Part I of supplement 1 in [33], and the comments in §7.3 of [28]). In particular, the scattering transform for the Schrödinger operator at zero energy is known to be well-behaved only for a special class of potentials, the potentials of “conductivity type,” defined as follows.

**Definition 1.1.** A real-valued function \( u \in C^\infty_0(\mathbb{R}^2) \) is called a potential of conductivity type if the equation \((-\Delta + q)\psi = 0\) admits a unique, strictly positive solution normalized so that \( \psi(z) = 1 \) in a neighborhood of infinity.

The class of conductivity type potentials can also be defined for less regular \( q \), but this definition will suffice for the present purpose. This terminology comes from the connection of the Schrödinger inverse problem at zero energy with Calderon’s inverse conductivity problem [19] (see Astala-Päivärinta [5] for the solution to Calderon’s inverse problem for \( \gamma \in L^\infty \), and for references to the literature). The problem is to reconstruct the conductivity \( \gamma \) of a conducting body \( \Omega \subset \mathbb{R}^2 \) from the Dirichlet to Neumann map, defined as follows. Let \( f \in H^{1/2}(\Omega) \) and let \( u \in H^1(\Omega) \)
solve the problem
\[ \nabla \cdot (\gamma \nabla u) = 0, \]
\[ u |_{\partial \Omega} = f. \]

This problem has a unique solution for conductivities \( \gamma \in L^\infty(\Omega) \) with \( \gamma(z) \geq c > 0 \) for a.e. \( z \). The Dirichlet to Neumann map is the mapping
\[ \Lambda_\sigma : H^{1/2}(\partial \Omega) \to H^{-1/2}(\partial \Omega) \]
\[ f \mapsto \gamma \frac{\partial u}{\partial \nu} |_{\partial \Omega}. \]

Nachman \[45\] exploited the fact that \( \psi = \gamma^{1/2}u \) solves the Schrödinger equation at zero energy where \( q = \gamma^{-1/2} \Delta (\gamma^{1/2}) \). The Schrödinger problem also has a Dirichlet to Neumann map defined by the unique solution of
\[ (-\Delta + q) \psi = 0 \]
\[ \psi |_{\partial \Omega} = f \]
which determines the scattering data for \( q \) at zero energy. Note that \( q \) is of conductivity type if we take \( \psi = \gamma^{1/2} \) and extend \( \psi \) to \( \mathbb{R}^2 \setminus \Omega \) setting \( \psi(z) = 1 \). Nachman showed that the scattering transform at zero energy is well-defined only when \( q \) is of conductivity type (we give a precise statement below) and used the inverse scattering transform to reconstruct \( q \) from its scattering data.

An important fact is that, under suitable decay and regularity hypotheses, \( q \) is a potential of conductivity type if and only if \( q \) is a critical potential, i.e., a measurable function \( q \) so that the quadratic form \( -\Delta + q \) is well-defined and nonnegative, but the associated Schrödinger operator does not have a positive Green’s function. Most importantly for our purpose, critical potentials have the following property: if \( q \) is a critical potential, then for any nonzero, nonnegative function \( W \in C_0^\infty(\mathbb{R}^2) \) and any \( \varepsilon > 0 \), \( q - \varepsilon W \) is not critical (for a precise statement and references to the Schrödinger operators literature, see the paper of Gesztesy and Zhao \[25\]). Thus, the set of conductivity-type potentials is nowhere dense in any reasonable function space! For this reason one expects the direct and inverse scattering maps for the Schrödinger operator at zero energy not to have good continuity properties as a function of the potential \( q \).

Let us describe the direct scattering transform \( T \) and inverse scattering transform \( Q \) for the Schrödinger operator at zero energy in more detail (see Nachman \[45\] and Lassas, Mueller, Siltanen, and Stahel \[40\] for details and references). To define the direct scattering map \( T \) on potentials \( q \in C_0^\infty(\mathbb{R}^2) \), we seek complex geometric optics (CGO) solutions \( \psi = \psi(z,k) \) of
\[ (-\Delta + q) \psi = 0. \]
which satisfy the asymptotic condition
\[ \lim_{|z| \to \infty} e^{-ikz} \psi(z,k) = 1. \]
for a fixed \( k \in \mathbb{C} \). Let \( m(z,k) = e^{-izk} \psi(z,k) \). Assuming that the problem \((1.3)-(1.4)\) has a unique solution for all \( k \), we define the scattering transform \( t = Tq \) via the formula
\[ t(k) = \int \frac{e^{i(kz+k\bar{z})}}{A(z,k)} dA(z) \]
where $dA(z)$ is Lebesgue measure on $\mathbb{R}^2$. The surprising fact is that, if $t$ is well-behaved, the solutions $\psi(z, k)$ may be recovered from $t(k)$. This fact leads to an inverse scattering transform $q = Q t$ given by

$$q(z) = \frac{i}{\pi^2} T \left( \int_{C} \frac{t(k)}{k} e^{-i(kz + \overline{\kappa})} m(z, k) \ dA(k) \right).$$

Boiti, Leon, Manna, and Pempinelli [16], proposed an inverse scattering solution to the Novikov-Veselov (mNV) equation using these maps:

$$q(t) = Q \left( e^{it(\gamma^3 + (\overline{\gamma})^3)} (T \rho_0) (\omega) \right)$$

and gave formal arguments to justify it. The maps were further studied by Tsai in [55]. Lassas, Mueller, Siltanen, and Stahel [40], building on results of [39], showed that the scattering transforms are well-defined for certain potentials of conductivity type. For conductivity-type potentials, Lassas et. al. proved that $T$ and $Q$ are inverses, and that (1.7) defines a continuous $L^p(\mathbb{R}^2)$-valued function of $t$ for $p \in (1, 2)$. They conjectured that $q(t)$ is in fact a classical solution of (1.1) if $q_0$ is a smooth, decreasing, real-valued potential of conductivity type but were unable to prove that this was the case.

The fact already mentioned, that conductivity-type potentials are a nowhere dense set in the space of potentials, suggests that studying the NV equation using the maps $T$ and $Q$ is likely to be technically challenging. The following result of Nachman ([45], Theorem 3) makes the difficulty clearer. For given $q_0$, let $E_q$ be the set of all $k$ for which the problem (1.3)-(1.4) does not have a unique solution. Let $L^p_\rho(\mathbb{R}^2)$ denote the Banach space of real-valued measurable functions $q$ with $\|q\|_{L^p_\rho} = \left[ \int (1 + |z|)^p |q(z)|^p \ dA(z) \right]^{1/p}$.

**Theorem 1.2.** [45] Suppose that $q \in L^p_\rho(\mathbb{R}^2)$ for some $p \in (1, 2)$, and $\rho > 1$: The following are equivalent:

(i) The set $E_q$ is empty and $|t(k)| \leq C |k|^\varepsilon$ for some fixed $\varepsilon > 0$ and all sufficiently small $k$.

(ii) There is a real-valued function $\gamma \in L^\infty(\mathbb{R}^2)$ with $\gamma(z) \geq c > 0$ for a.e. $z$ and a fixed constant $c$ so that $q = \gamma^{-1/2} \Delta (\gamma^{1/2})$.

One should think of $\gamma$ as $\psi^2$ where $\psi$ is the unique normalized positive solution of $(-\Delta + q) \psi = 0$ for a potential of conductivity type. Nachman’s result suggests that non-conductivity type potentials will have singular scattering transforms: in [44], Music, Perry, and Siltanen construct an explicit one-parameter deformation $\lambda \mapsto q_\lambda$ of a conductivity type potentials ($q_0$ is of conductivity type, but $q_\lambda$ is not for $\lambda \neq 0$) for which the corresponding family $\lambda \mapsto t_\lambda$ of scattering transforms has an essential singularity at $\lambda = 0$.

We will show that, nonetheless, the formula (1.7) does yield classical solutions of the NV equation for a much larger class of initial data than considered in [40]. We achieve this result by circumventing the scattering maps studied in [40]. Instead, we exploit Bogdanov’s [14] observation that the Miura-type map

$$\mathcal{M}(v) = 2 \partial v + |v|^2$$

takes solutions $u$ of the modified Novikov-Veselov (mNV) equation to solutions $q$ of the NV equation with initial data of conductivity type. Here, the domain of the Miura map is understood to be smooth functions $v$ with $\partial v = \overline{\partial v}$. We will show
that the range of $\mathcal{M}$ contains the conductivity-type potentials studied by Lassas, Mueller, Siltanen and Stahel.

Thus, to solve the NV equation for initial data of conductivity type, it suffices to solve the mNV equation and use the map $\mathcal{M}$ to obtain a solution of NV. The mNV equation is a member of the Davey-Stewartson II (DS II) hierarchy, so the well-known scattering maps for the DS II hierarchy (see [49] and reference therein) can be used to solve the Cauchy problem for mNV. We denote by $\mathcal{R}$ and $\mathcal{I}$ respectively the scattering transform and inverse scattering transform associated to the defocusing Davey-Stewartson II equation (see §3 for the definitions). We show in Appendix A that the function

$$u(t) = \mathcal{I} \big( \exp \left( (\mathcal{S}^3 - \mathcal{S}^3) t \right) (\mathcal{R}u_0)(\circ) \big)$$

is a classical solution of the mNV equation for initial data $u_0 \in \mathcal{S}(\mathbb{R}^2)$.

In order to obtain good mapping properties for the solution map $u_0 \mapsto u(t)$ defined by (1.9), we need local Lipschitz continuity of the maps $\mathcal{I}$ and $\mathcal{R}$ on spaces that are preserved under the flow (compare the treatment of the cubic NLS in one dimension by Deift-Zhou [21], and the Sobolev mapping properties for the scattering maps for NLS proven in [58]). In [49] it was shown that $\mathcal{R}$ and $\mathcal{I}$ are mutually inverse mappings of $H^{1,1}(\mathbb{R}^2)$ into itself where

$$H^{m,n}(\mathbb{R}^2) = \left\{ u \in L^2(\mathbb{R}^2) : (1 - \Delta)^{m/2} u, (1 + |\cdot|)^n u(\cdot) \in L^2(\mathbb{R}^2) \right\}.$$

In order to use (1.9), we need the following refined mapping property of $\mathcal{I}$ and $\mathcal{R}$.

**Theorem 1.3.** The scattering maps $\mathcal{R}$ and $\mathcal{I}$ restrict to locally Lipschitz continuous maps

$$\mathcal{R} : H^{2,1}(\mathbb{R}^2) \to H^{1,2}(\mathbb{R}^2),$$

$$\mathcal{I} : H^{1,2}(\mathbb{R}^2) \to H^{2,1}(\mathbb{R}^2).$$

Theorem 1.3 immediately implies that the solution formula (1.9) defines a continuous map

$$H^{2,1}(\mathbb{R}^2) \to C ([0, T] ; H^{2,1}(\mathbb{R}^2)),
\quad t \mapsto u(t).$$

for any $T > 0$. We say that $u$ is a weak solution of the mNV equation (see (1.10) on $[0, T]$ if

$$-\left( \varphi_t + \partial^3 \varphi + \overline{\varphi} : u \right) + (\varphi, N\mathcal{L}(u)) = 0$$

for all $\varphi \in C_0^\infty (\mathbb{R}^2 \times [0, T])$, where $(\cdot, \cdot)$ denotes the inner product on $L^2(\mathbb{R}^2 \times [0, T])$. We will show that (1.9) defines a weak solution in this sense and that, also, the flow (1.9) leaves the domain of $\mathcal{M}$ invariant. We will prove:

**Theorem 1.4.** For $u_0 \in \mathcal{S}(\mathbb{R}^2)$, the solution formula (1.9) gives a classical solution of mNV. Moreover, if $u_0 \in H^{2,1}(\mathbb{R}^2) \cap L^1(\mathbb{R}^2)$, $\partial u_0 = \partial^3 u_0$, and $\int u_0(z) \ dA(z) = 0$, then $u(t)$ is a weak solution of mNV and the relations $(\partial u) (\cdot, t) = (\partial u)(\cdot, t)$ and $\int u(z, t) \ dA(z) = 0$ hold for all $t$. 

Remark 1.5. Although it is likely well within the reach of current technology (see e.g. [36] for relevant dispersive estimates), there appear to be no uniqueness or local well-posedness result for mNV in the literature. Given such a result, one could conclude from the proof of Theorem 1.4 that the Cauchy problem for mNV is globally well–posed in \( H^{2,1}(\mathbb{R}^2) \).

Now we can solve the NV equation using the solution map for mNV and the Miura map (1.8). We say that \( q \) is a weak solution the NV equation with initial data \( q_0 = 2\partial u_0 + |u_0|^2 \) for all \( \varphi \in C^\infty_0(\mathbb{R}^2) \) if
\[
(\varphi_t + \partial^3 \varphi + |\varphi|^2 \varphi, q) + \frac{3}{4} (\partial \varphi, q\partial^{-1} \partial q) + \frac{3}{4} (|\partial \varphi|, q\partial^{-1} \partial q) = 0
\]
for all \( \varphi \in C^\infty_0(\mathbb{R}^2 \times [0, T]) \). Using Theorem 1.6 we will prove:

**Theorem 1.6.** Suppose that \( q_0 = 2\partial u_0 + |u_0|^2 \) where \( u_0 \in H^{2,1}(\mathbb{R}^2) \cap L^1(\mathbb{R}^2) \), \( \partial u_0 = \overline{\partial u_0} \) and \( \int u_0(z) \, dA(z) = 0 \). Then
\[
q(t) = \mathcal{M} \left( \mathcal{I} \left( e^{2i\mu((\tau)^2 + (\overline{\tau})^2)} (\mathcal{R} u_0)(\tau) \right) \right)
\]
is a weak solution the NV equation with initial data \( q_0 \). If \( u_0 \in \mathcal{S}(\mathbb{R}^2) \), then \( q(t) \) is a classical solution of the NV equation.

The class of initial data covered by Theorem 1.6 includes the conductivity-type potentials considered by Lassas, Mueller, Siltanen, and Stahel. The connection between their work and ours is given in the following theorem.

**Theorem 1.7.** Suppose that \( u_0 \in C^\infty_0(\mathbb{R}^2) \) with \( \int u_0(z) \, dA(z) = 0 \) and \( \partial u_0 = \partial u_0 \), and let \( q_0 = 2\partial u_0 + |u_0|^2 \). Then, for any \( t \),
\[
\mathcal{Q} \left( e^{2i((\tau)^2 + (\overline{\tau})^2)} T q_0(\tau) \right) = \mathcal{M} \left( e^{2i((\tau)^3 - (\overline{\tau})^3)} (\mathcal{R} u_0)(\tau) \right)
\]
and their common value is a classical solution to the Novikov-Veselov equation.

It should be noted that the solution formula (1.12) provides a solution which exists globally in time. On the other hand, Taimanov and Tsarev (see [51, 52, 53, 54]) have used Moutard transformations to construct explicit, nonsingular Cauchy data \( q_0 \) with rapid decay at infinity and having the following properties: (i) the Schrödinger operator \( -\Delta + q_0 \) has nonzero eigenvalues at zero energy and (ii) the solution of (1.1) with Cauchy data \( q_0 \) blows up in finite time.

To close this introduction, we comment on the seemingly restrictive hypothesis in Theorems 1.6 and 1.7. In both theorems, we assume that \( \int u_0 = 0 \). To understand what this assumption means, we recall that if \( \phi_0 = \overline{\partial}^{-1} u_0 \), then the unique, positive, normalized zero-energy solution of the Schrödinger equation (1.3) is given by \( \psi_0 = \exp(\phi_0) \). For \( u_0 \in \mathcal{S}(\mathbb{R}^2) \) say, we have from the integral expression for \( \overline{\partial}^{-1} \) that
\[
\phi_0(z) = -\frac{1}{\pi} \int \frac{u_0(\zeta)}{z} \, d\zeta + \mathcal{O} \left( |z|^{-2} \right)
\]
so that, to leading order
\[
\psi_0 - 1 = -\frac{1}{\pi} \int \frac{u_0(\zeta)}{z} \, d\zeta + \mathcal{O} \left( |z|^{-2} \right)
\]
Recalling that \( \gamma^{1/2}(z) = \psi_0(z) \) we see that the vanishing of \( \int u_0(z) \, dA(z) \) implies that \( \gamma(z) - 1 = \mathcal{O} \left( |z|^{-2} \right) \) as \( |z| \to \infty \). In particular, for conductivities with \( \gamma = 1 \) outside a compact set, \( \int u_0(z) \, dA(z) = 0 \).
Indeed, suppose that 
\[ q = \gamma^{-1/2} \Delta (\gamma^{1/2}) \] in distribution sense, where \( \gamma \in L^\infty(\mathbb{R}^2) \), 
\( \gamma(z) \geq c > 0 \), and suppose further that \( \Delta (\nabla \gamma) \) and \( \gamma - 1 \) belong to \( L^2(\mathbb{R}^2) \). It 
follows that \( \varphi = \log \gamma \in H^{3,1}(\mathbb{R}^2) \) and the function 
\[ u = 2\partial \varphi \] 
belongs to \( H^{2,1} \). We then compute that 
\[ q = 2\partial u + |u|^2. \] If we have stronger decay 
of \( \gamma(z) \) as \( |z| \to \infty \), this will imply additional decay of \( \varphi(z) \) that can be used to 
check \( \int u(z) \, dA(z) = 0 \) by Green’s formula 
\[ \int_\Omega \nabla \varphi \, dA(z) = \frac{1}{2} \int_{\partial \Omega} \varphi (\nu_{x_1} + i \nu_{x_2}) \, d\sigma. \] 
The structure of this paper is as follows. In \( \S 2 \) we recall how the scattering maps 
\( \mathcal{R} \) and \( \mathcal{I} \) for the Davey-Stewartson system are defined, while in \( \S 3 \) we prove that 
\( \mathcal{R} : H^{2,1}(\mathbb{R}^2) \to H^{1,2}(\mathbb{R}^2) \) and 
\( \mathcal{I} : H^{1,2}(\mathbb{R}^2) \to H^{2,1}(\mathbb{R}^2) \) are locally Lipschitz continuous. In \( \S 4 \) we solve the 
mNV equation using the inverse scattering method and prove that, for initial data 
\( u_0 \in H^{2,1}(\mathbb{R}^2) \) with \( \partial u_0 = \overline{\partial u_0} \) and 
\( \int_{\mathbb{R}^2} u_0(z) \, dA(z) = 0 \), the condition \( \partial u = \overline{\partial u} \) holds for all \( t > 0 \). In \( \S 4 \) we show that our class of potentials extends the class 
of conductivity type potentials considered by Lassas, Mueller, Siltanen and Stahel 
\[ 40 \], and that our solution coincides with theirs where the two constructions overlap. 
Appendix \( \mathbf{A} \) sketches the solution of the mNV equation by scattering theory for 
initial data in the Schwarz class.

**Acknowledgements.** The author gratefully acknowledges the support of the 
College of Arts and Sciences at the University of Kentucky for a CRAA travel grant 
and the Isaac Newton Institute for hospitality during part of the time this work 
was done. The author thanks Fritz Gesztesy and Russell Brown for helpful 
conversations and correspondence.

## 2. Preliminaries

**Notation.** In what follows, \( \| \cdot \|_p \) denotes the usual \( L^p \)-norm and \( p' = p/(p-1) \) 
denotes the conjugate exponent. If \( f \) is a function of \( (z, k) \), \( f(z, \cdot) \) (resp. \( f(\cdot, k) \)) 
denotes \( f \) with a generic argument in the \( z \) or \( k \) variable. We will write \( L^p_z \) or \( L^p_k \) 
for \( L^p \)-spaces with respect to the \( z \) or \( k \) variable, and \( L^p_z (L^q_k) \) for the mixed spaces 
with norm
\[ \|f\|_{L^p_z (L^q_k)} = \left( \int \|f(z, \cdot)\|_q^{p'} \, dA(z) \right)^{1/p}. \] 
If \( f \) is a function of \( z \) and \( k \), \( \|f\|_\infty \) denotes \( \|f\|_{L^\infty(\mathbb{R}_z^2 \times \mathbb{R}_k^2)} \).

In what follows, \( \langle \cdot, \cdot \rangle \) denotes the pairing
\[ \langle f, g \rangle = -\frac{1}{\pi} \int f(z)g(z) \, dA(z) \]

We will call a mapping \( f \) from a Banach space \( X \) to a Banach space \( Y \) a **LLCM** 
(locally Lipschitz continuous map) if for any bounded subset \( B \) of \( X \), there is a constant 
\( C = C(B) \) so that, for all \( x_1, x_2 \in B \),
\[ \|f(x_1) - f(x_2)\|_Y \leq C(B) \|x_1 - x_2\|_X. \]
For example, if \( M : X^m \to Y \) is a continuous multilinear map, then 
\[ f \mapsto M(f, f, \ldots, f) \]
is a LLCM from \( X \) to \( Y \).
Cauchy Transforms. The integral operators

\[ P\psi = \frac{1}{\pi} \int \frac{1}{\zeta - z} f(\zeta) \, dm(\zeta), \]
\[ \overline{P}\psi = \frac{1}{\pi} \int \frac{1}{\zeta - z} f(\zeta) \, dm(\zeta) \]

are formal inverses respectively of \( \partial \) and \( \partial \). We denote by \( P_k \) and \( \overline{P}_k \) the corresponding formal inverses of \( \partial_k \) and \( \partial_k \). The following estimates are standard (see, for example, Astala-Iwaniec-Martin [4], §4.3, or Vekua [56]).

**Lemma 2.1.** (i) For any \( p \in (2, \infty) \) and \( f \in L^p \), \( \| Pf \|_p \leq C_p \| f \|_{2p/(p+2)} \). (ii) For any \( p, q \) with \( 1 < q < 2 < p < \infty \) and any \( f \in L^p \cap L^q \), \( \| Pf \|_{\infty} \leq C_{p,q} \| f \|_{L^p \cap L^q} \) and \( Pf \) is Hölder continuous of order \( (p-2)/p \) with

\[ |(Pf)(z) - (Pf)(w)| \leq C_p |z - w|^{(p-2)/p} \| f \|_p. \]

(iii) For \( 2 < p < q \) and \( u \in L^s \) for \( q^{-1} + 1/2 = p^{-1} + s^{-1} \),

\[ \| P(u\psi) \|_q \leq C_{p,q} \| u \|_s \| \psi \|_p. \]

**Remark 2.2.** If \( p > 2 \) and \( u \in L^s \) for \( s \in (1, \infty) \), then estimate (iii) holds true for any \( q > 2 \).

Beurling Transform. The operator

\[ (Sf)(z) = -\frac{1}{\pi} \int \frac{1}{(z - w)^2} f(w) \, dw \]

defined as a Calderon-Zygmund type singular integral, has the property that for \( f \in C_0^\infty(\mathbb{R}^2) \) we have \( S(\partial f) = \partial f \). The operator \( S \) is a bounded operator on \( L^p \) for \( p \in (1, \infty) \) (see for example [4], §4.5.2). This fact allows us to obtain \( L^p \)-estimates on \( \partial \)-derivatives of functions of interest from \( L^p \)-estimates on \( \overline{\partial} \)-derivatives.

**Brascamp-Lieb Type Estimates.** A fundamental role is played by the following multilinear estimate due to Russell Brown [17], who initiated their use in the analysis of the DS II scattering maps. See Appendix A of [49], written by Michael Christ, for a proof of these estimates using the methods of Bennett, Carbery, Christ, and Tao [12, 13]. Define

\[ A_n(\rho, u_0, u_1, \ldots, u_{2n}) = \int_{\mathbb{C}^{2n+1}} \frac{|\rho(\zeta)||u_0(z_0)||\ldots||u(z_{2n})|}{\prod_{j=1}^{2n} |z_{j-1} - z_j|} dA(z), \]

where \( dA(z) \) is product measure on \( \mathbb{C}^{2n+1} \), and set

\[ \zeta = \sum_{j=0}^{2n} (-1)^j z_j. \]

**Proposition 2.3.** [17] The estimate

\[ |A_n(\rho, u_0, u_1, \ldots, u_{2n})| \leq C_n \| \rho \|_2 \prod_{j=0}^{2n} \| u_j \|_2 \]

holds.
Remark 2.4. For $u_1, \ldots, u_{2n} \in S(\mathbb{R}^2)$, define operators $W_j$ by $W_j \psi = P e_k u_j \overline{\psi}$. Proposition 2.3 implies that

$$F(k) = \langle e_k u_0, W_1 W_2 \ldots W_{2n} 1 \rangle$$

is a multilinear $L^2(\mathbb{R}^2)$-valued function of $(u_0, \ldots, u_{2n})$ with

$$\|F\|_2 \leq C 2^n \prod_{j=0}^{2n} \|u_j\|_2.$$

3. Scattering Maps and an Oscillatory $\overline{\partial}$-Problem

First, we recall from [49] that the Davey-Stewartson scattering maps $\mathcal{R}$ and $\mathcal{I}$ are both defined by $\overline{\partial}$-problems: see [49] for full discussion. The inverse scattering method for the Davey-Stewartson II equation was developed Abloiwitz-Fokas [1, 2] and Beals-Coifman [7, 8, 9, 10]. Sung [50] and Brown [17] carried out detailed analytical studies of the map.

For a complex parameter $k$ and for $z = x_1 + ix_2$, let

$$e_k = e^{\pi x_2 k z}.$$

Given $u \in H^{1,1}(\mathbb{R}^2)$ and $k \in \mathbb{C}$, there exists a unique bounded continuous solution of

$$\begin{align*}
\overline{\partial} \mu_1 &= \frac{1}{2} e_k u \mu_2, \\
\overline{\partial} \mu_2 &= \frac{1}{2} e_k u \mu_1,
\end{align*}$$

$$\lim_{|z| \to \infty} (\mu_1(z, k), \mu_2(z, k)) = (1, 0).$$

We then define $r = \mathcal{R}u$ by

$$r(k) = -\frac{1}{\pi} \int e_k(z) u(z) \mu_1(z, k) \, dA(z).$$

On the other hand, it can be shown that

$$\nu_1, \nu_2 = (\mu_1, e_k \mu_2)$$

solve a $\overline{\partial}$-problem in the $k$ variable:

$$\begin{align*}
\overline{\partial}_k \nu_1 &= \frac{1}{2} e_k \overline{\nu_2}, \\
\overline{\partial}_k \nu_2 &= \frac{1}{2} e_k \overline{\nu_1},
\end{align*}$$

$$\lim_{|k| \to \infty} (\nu_1(z, k), \nu_2(z, k)) = (1, 0),$$

and that this solution is unique within the bounded continuous functions. Given $r \in H^{1,1}(\mathbb{R}^2)$, we solve the $\overline{\partial}$-system (3.4) and define $u = \mathcal{I}r$ by

$$u(z) = -\frac{1}{\pi} \int e_{-k}(z) r(k) \nu_1(z, k) \, dA(k).$$

In [49], we proved:

**Theorem 3.1.** The maps $\mathcal{R}$ and $\mathcal{I}$, initially defined on $S(\mathbb{R}^2)$, extend to LLCM’s from $H^{1,1}(\mathbb{R}^2)$ to itself. Moreover $\mathcal{R} \circ \mathcal{I} = \mathcal{I} \circ \mathcal{R} = I$, where $I$ denotes the identity map on $H^{1,1}(\mathbb{R}^2)$.
We now describe three basic tools used in [49] to analyze the generic system

\[
\begin{align*}
\partial w_1 &= \frac{1}{2} e_k \bar{u} w_2, \\
\partial w_2 &= \frac{1}{2} e_k \bar{u} w_1, \\
\lim_{|z| \to \infty} (w_1(z,k), w_2(z,k)) &= (1, 0)
\end{align*}
\]

for unknown functions \(w_1(z,k)\) and \(w_2(z,k)\), where \(k\) is a complex parameter, \(u \in H^{1,1}(\mathbb{R}^2)\). We refer the reader to [49] for the proofs. We don’t state the obvious analogues of the facts below when the roles of \(k\) and \(z\) are reversed, but use them freely in what follows.

1. **Finite \(L^p\)-Expansions.** In [49] it is shown that the system (3.6) has a unique solution in \(L^\infty_z\). This result, and further analysis of the solution, follows from the following facts that we recall from §3 of [49]. Let \(T\) be the antilinear operator

\[
T\psi = \frac{1}{2} Pe_k \bar{u} \psi
\]

which is a bounded operator from \(L^p\) to itself for \(p \in (2, \infty]\) if \(u \in H^{1,1}\) by Lemma 2.1(i). The system (3.6) is equivalent to the integral equation

\[
w_1 = 1 + T^2 w_1
\]

and the auxiliary formula \(w_2 = Tw_1\). The operator \(I - T^2\) has trivial kernel as a map from \(L^p(\mathbb{R}^2)\) to itself for any \(p \in (2, \infty]\), and the estimate

\[
\|T^2\|_{L^p \to L^p} \leq C_p \|u\|_{H^{1,1}}^2 (1 + |k|)^{-1}
\]

holds for any \(p \in (2, \infty]\). For any \(p \in (2, \infty]\), the resolvent \((I - T^2)^{-1}\) is bounded uniformly in \(k \in \mathbb{C}\) and \(u\) in bounded subsets of \(H^{1,1}\) as an operator from \(L^p\) to itself. Note that if \(u \in H^{1,1}\), the expression \(T1 = \frac{1}{2} Pe_k u\) is a well-defined element of \(L^p\) for all \(p \in (2, \infty]\). The unique solution of (3.6) is given by

\[
\begin{align*}
w_1 - 1 &= (I - T^2)^{-1} T^2 1, \\
w_2 &= Tw_1.
\end{align*}
\]

From these facts, one has (see §3 of [49]):

**Lemma 3.2.** (Finite \(L^p\)-expansions) For any positive integer \(N\), the expansions

\[
\begin{align*}
w_1 - 1 &= \sum_{j=1}^N T^{2j} 1 + R_{1,N} \\
w_2 &= \sum_{j=1}^N T^{2j-1} 1 + R_{2,N}
\end{align*}
\]

hold, where the maps

\[
\begin{align*}
u \mapsto (1 + |\cdot|)^N R_{1,N}(\cdot, \cdot), \\
u \mapsto (1 + |\cdot|)^N R_{2,N}(\cdot, \cdot)
\end{align*}
\]

are LLCM’s from \(H^{1,1}(\mathbb{R}^2)\) into \(L^\infty_k(L^p_z)\).
2. Multilinear Estimates. Substituting the expansions into the representation formulas (3.5) and (3.2) leads to expressions of the form
\[ \langle e_w, F_j \rangle \]
where \( e_w \) denotes \( e_k \) or \( e_{-k} \), \( w \) is a monomial in \( u \) and its derivatives, and \( F_j \) denotes \( T^{2j} \) or \( T^{2j+1} \) for \( j \geq 1 \). We assume that \( w \) is bounded in \( L^2 \) norm by a power of \( \| u \|_{H^{2,1}} \). The following fact is an immediate consequence of Remark 2.4.

**Lemma 3.3.** The map \( u \mapsto \langle e_w, F_j \rangle \) is a LLCM from \( H^{2,1}(\mathbb{R}^2) \) to \( L^2_k(\mathbb{R}^2) \).

3. Large-Parameter Expansions. Finally, the following large-\( z \) finite expansions for \( w_1 \) and \( w_2 \) will be useful. We omit the straightforward computational proof.

**Lemma 3.4.** For \( u \in H^{1,1}(\mathbb{R}^2) \),
\[
\begin{align*}
    w_1(z, k) &= 1 - \frac{1}{2\pi z} \int e_k(z') u(z') w_2(z', k) \, dm(z') \\
    &\quad - \frac{1}{4\pi} \int e_k(z') z' u(z') w_2(z', k) \, dm(z') \\
    w_2(z, k) &= 1 - \frac{1}{2\pi z} \int e_k(z') u(z') w_1(z', k) \, dm(z') \\
    &\quad - \frac{1}{4\pi} \int e_k(z') z' u(z') w_1(z', k) \, dm(z')
\end{align*}
\]
and similarly
\[
\begin{align*}
    w_1(z, k) &= \frac{1}{2\pi z} \int e_k(z') u(z') w_2(z', k) \, dm(z') \\
    &\quad - \frac{1}{4\pi} \int e_k(z') z' u(z') w_2(z', k) \, dm(z') \\
    w_2(z, k) &= \frac{1}{2\pi z} \int e_k(z') u(z') w_1(z', k) \, dm(z') \\
    &\quad - \frac{1}{4\pi} \int e_k(z') z' u(z') w_1(z', k) \, dm(z')
\end{align*}
\]
Analogous expansions hold for the \( \overline{\partial} \)-problem in the \( k \) variables.

4. Restrictions of Scattering Maps

In this section we prove Theorem 1.3. In virtue of Theorem 3.1, it suffices to show that the maps \( H^{2,1} \ni u \mapsto |\phi|^2 r(\phi) \) and \( H^{1,2} \ni r \mapsto \Delta u \in L^2 \) are LLCM’s. First, we prove:

**Lemma 4.1.** The map \( u \mapsto |\phi|^2 r(\phi) \) is a LLCM from \( H^{2,1}(\mathbb{R}^2) \) to \( L^2(\mathbb{R}^2) \).

**Proof.** We carry out all computations on \( u \in C_0^\infty(\mathbb{R}^2) \) and extend by density to \( H^{2,1}(\mathbb{R}^2) \). Note that \( \| u \|_p \leq C_p \| u \|_{H^{2,1}} \) for all \( p \in (1, \infty) \) and \( \| \partial u \|_p \leq \| u \|_{H^{2,1}} \) for \( p \in [2, \infty) \). An integration by parts using (3.2) and the identity \( \partial e_k = -ke_k \) shows that (up to trivial factors)
\[
|k|^2 r(k) = -\frac{1}{2} \int e_k(\partial u) - \frac{k}{2} \int e_k(\partial u) \langle m_1 - 1 \rangle - \frac{1}{2} \int |u|^2 \mu_2
\]
where in the last term we used
\[
\overline{\partial} \mu_1 = \frac{1}{2} e_k u \mu_2.
\]
\[
I_1: \text{ This term is the Fourier transform of } \partial \overline{\partial} u \text{ and hence defines a linear map from } H^{2,1} \text{ to } L^2_k.
\]
I_2: An integration by parts using (3.2), the identity \( \partial (e_k) = -ke_k \), and (1.1) again shows that

\[
I_2 = \frac{2}{k} \left[ \int e_k (\partial^2 u) (\frac{1}{p} - 1) + \frac{1}{2} \int \overline{\mu} u \mu_2 \right] = I_{21} + I_{22}.
\]

In \( I_{21} \) we insert \( 1 = \chi + (1 - \chi) \) where \( \chi \in C_0^\infty (\mathbb{R}^2) \) satisfies \( 0 \leq \chi (z) \leq 1, \chi (z) = 1 \) for \( |z| \leq 1 \), and \( \chi (z) = 0 \) for \( |z| \geq 2 \). Drop the unimodular factor \( \frac{2}{k} \) and write \( I_{21} = I_{21}^{\text{in}} + I_{21}^{\text{out}} \) corresponding to this decomposition. Since \( \chi \partial^2 u \in L^p \) for any \( p > 2 \), we may expand

\[
I_{21}^{\text{in}} = \sum_{j=1}^{N} \int e_k (\partial^2 u) \chi (\frac{T^{j+1}}{\mu}) + \int e_k (\partial^2 u) \chi (I - T)^{-1} T^{j+1}.
\]

By Lemma 3.2, Lemma 3.3 and the fact that \( \chi \partial^2 u \in L^p \), each right-hand term defines a LLCM from \( H^{2,1} \) to \( L^1_k \), hence \( u \mapsto I_{21}^{\text{out}} \) is a LLCM. In \( I_{21}^{\text{out}} \), we use Lemma 3.4 to write

\[
(4.2) \quad \int e_k (1 - \chi) \partial^2 u (\frac{1}{p} - 1) = -\frac{1}{2\pi} \left( \int e_k (1 - \chi) (\partial^2 u) z^{-1} \right) \left( \int e_{-k} \overline{\mu} \mu_2 \right) + \frac{1}{2} \left< e_{-k} (1 - \chi) (\partial^2 u) z^{-1}, P_{e_{-k}u} (T \mu_1) \right>.
\]

The first right-hand term in (4.2) is the product of the Fourier transform of the \( L^2 \)-function \( (1 - \chi(z)) (\partial^2 u(z)) z^{-1} \) and the function \( \int e_{-k} \overline{\mu} \mu_2 \). Since \( u \in L^p \) for all \( p > 2 \) while \( u \mapsto \mu_2 \) is a LLCM from \( H^{1,1} \) to \( L^\infty_k (L^p_k) \), the map \( u \mapsto \int e_{-k} \overline{\mu} \mu_2 \) is a LLCM from \( H^{2,1} \) to \( L^\infty_k \), so the first right-hand term in (4.2) defines a LLCM from \( H^{2,1} \) to \( L^1_k \). The second right-hand term in (4.2) may be controlled using Lemmas 3.2 and 3.3. This shows that \( u \mapsto I_{21}^{\text{out}} \), and hence \( u \mapsto I_{21} \), defines a LLCM from \( H^{2,1} \) to \( L^2_k \). Finally, to control \( I_{22} \), we note that \( \overline{\mu} \partial u \in L^p \) for \( p > 2 \). Hence, using Lemma 3.2 we obtain

\[
I_{22} = \sum_{j=0}^{N} \int \overline{\mu} u \ T^{j+1} + \int (\overline{\mu} u) \ (I - T^2) T^{j+1}.
\]

To control terms in the finite sum in (4.3), we write

\[
\int \overline{\mu} u \ T^{j+1} = \left< u \overline{\mu}, P \left[ e_k u \left( \frac{T^{j+1}}{\mu} \right) \right] \right> = -\left< e_{-k} \overline{\mu} \partial u \left( \frac{u \overline{\mu}}{\mu} \right), T^{j+1} \right>.
\]

and apply Lemma 2 since \( \| u \overline{\mu} \partial u \|_{H^{2,1}} \leq C \| u \|_{H^{2,1}} \). The second right-hand term in (4.3) defines a LLCM from \( H^{2,1} \) to \( L^2_k \) by Lemma 3.2. Hence, \( u \mapsto I_2 \) is a LLCM from \( H^{2,1} \) to \( L^2_k \).

I_3: Note that \( |u|^2 \in L^p \) for all \( p > 2 \) and use the expansion of \( \mu_2 \) to write \( I_3 \) as

\[
\sum_{j=1}^{N} \frac{k}{2} \int |u|^2 T_{j+1} - \frac{k}{2} \int |u|^2 (I - T^2)^{-1} T^{2N+3}.
\]
We will show that for any $\chi$ (except that, here, $\exp$ expansion of $\nu$)
we then have that in the second line of (4.4) is
\begin{equation}
\kappa \left( |u|^2, P \left( e_k u \left( T^{2j} 1 \right) \right) \right) = \kappa \left( \left. e_{-k} \bar{u} P \left( |u|^2 \right) \right| , T^{2j} 1 \right)
\end{equation}
where we integrated by parts to remove the factor of $\bar{k}$. The first right-hand term in the second line of (4.4) is
\begin{equation}
\left\langle e_{-k} \bar{u} P \left( |u|^2 \right) , \partial \left( T^{2j} 1 \right) \right\rangle = \left\langle e_{-k} \bar{u} P \left( |u|^2 \right) , e_{-k} \bar{u} P \left( e_k u T^{2j-2} 1 \right) \right\rangle = \left\langle e_{-k} \bar{u} P \left( |u|^2 \right) , T^{2j-2} 1 \right\rangle
\end{equation}
which defines a LLCM from $H^{2,1}$ to $L^2_k$ by Lemma 3.3 since $\bar{u} \in L^2_k$ from $H^{2,1}$ to $L^2_k$. The second right-hand term is treated similarly. Hence $u \mapsto I_3$ is a LLCM from $H^{2,1}$ to $L^2_k$.

Collecting these results, we conclude that $u \mapsto |\phi|^2 r (\phi)$ is a LLCM from $H^{2,1}$ to $L^2_k$.

**Lemma 4.2.** The map $r \mapsto \Delta u$ is a LLCM from $H^{2,1}(\mathbb{R}^2)$ to $L^2(\mathbb{R}^2)$.

**Proof.** Since $r \in H^{1,2}$ we have $kr(k) \in L^p$ for all $p \in (1, 2], r \in L^p$ for all $p \in [1, \infty)$ and $\partial r \in L^p$ for all $p \in [2, \infty)$. A straightforward computation shows that
\begin{align*}
\partial \bar{r} u &= \int |k|^2 e_{-k} r + \int |k|^2 e_{-k} r (\nu_1 - 1) - \int \bar{k} e_{-k} r \partial \nu_1 \\
&\quad + \int k e_{-k} r \partial \nu_1 + \int e_{-k} r \partial \nu_1 \\
&= I_1 + I_2 + I_3 + I_4 + I_5
\end{align*}

where all derivatives are taken with respect to $z$. We now show that each of $I_1$–$I_5$ defines a locally Lipschitz continuous map from $H^{2,1} \ni r$ into $L^2_k$.

$I_1$: This term is the Fourier transform of $\partial \bar{r}$ and hence $L^2_k$.

$I_2$: Inserting $1 = \chi + (1 - \chi)$ in $I_2$, where $\chi$ is as in the proof of Lemma 4.1 (except that, here, $\chi$ is a function of $k$, not $z$), we have $I_2 = I_{21} + I_{22}$ where
\begin{align*}
I_{21} &= \int e_{-k} |k|^2 \chi r (\nu_1 - 1), \quad I_{22} = \int e_{-k} |k|^2 r (1 - \chi) (\nu_1 - 1).
\end{align*}

We will show that $I_{21}$ and $I_{22}$ are both LLCM’s from $H^{1,2}$ to $L^2_k$. Since $|k|^2 \chi r \in L^{p'}$ for any $p > 2$, we can use Lemma 3.2 for $\nu_1 - 1$ together with Lemma 3.3 to conclude that $r \mapsto I_{21}$ is a LLCM from $H^{1,2}$ to $L^2_k$. For $I_{22}$ we use the one-step large-$k$ expansion of $\nu_1 - 1$ (Lemma 3.3):
\begin{align*}
\nu_1(z, k) - 1 &= -\frac{1}{2 \pi k} \int e_k(z) r(k') \nu_2(z, k') \, dm(k') \\
&\quad - \frac{1}{2 \pi k} \int e_k(z) \frac{r(k')}{k - k'} \nu_2(z, k') \, dm(k').
\end{align*}

We then have
\begin{align*}
I_{22} &= \int e_{-k} \bar{r} (1 - \chi) (F_1 + F_2)
\end{align*}
where
\[
F_1(z) = -\frac{1}{2\pi} \int e^{kr(k')} \nu_2(z, k') \, dm(k'),
\]
\[
F_2(z, k) = -\frac{1}{2\pi} \int \frac{e^{kr(z)}}{k-k'} \nu_2(z, k') \, dm(k').
\]

It is easy to see that \( \|F_1\|_{L^\infty} \leq \|r\|_1 \|\nu_2\|_{\infty} \), so that \( r \mapsto F_1 \) is a LLCM from \( H^{1,2} \) to \( L^\infty \). Moreover, \( \int e_{-kr}(1-\chi) \) is the inverse Fourier transform of the \( L^2 \) function \( \Theta(-\Theta(1-\chi(\cdot))) \). Hence, \( r \mapsto \int e_{-kr}(1-\chi) F_1 \) is a LLCM from \( H^{1,2} \) to \( L^2 \).

Next, we may use Lemma 3.2 in \( F_2 \) to conclude that
\[
(4.5) \quad F_2 = -\frac{1}{2} \sum_{j=1}^{N} P_k \left( e_k k^{2j+1} \right) - \frac{1}{2} P_k \left( e_k k^{2} \right) \left( I - T^{2j+1} \right),
\]

The corresponding contributions to \( I_{22} \) from terms in the finite sum from \( 4.5 \) define LLCM’s from \( H^{1,2} \) to \( L^2 \) by Lemma 3.3 while by the remainder estimate in Lemma 3.2, the mapping \( r \mapsto P e_k k^{2} \left( I - T^{-} \right)^{-1} T^{2N+3} \) is a LLCM from \( H^{1,2} \) to \( L^2 \) for \( p > 2 \). Using these estimates we may conclude that \( r \mapsto \int e_{-kr}(1-\chi) F_2 \) is a LLCM from \( H^{1,2} \) to \( L^2 \).

\( I_3 \): Since \( \mu_1 = \nu_1 \), we conclude from (4.1) and \( 3.3 \) that
\[
(4.6) \quad \frac{1}{2} e_k u \mu_2 = \frac{1}{2} u \nu_2
\]

so that
\[
I_3 = -\int e_{-kr} \left( \partial \nu_1 \right) = -\int \left( \partial \nu_2 \right).
\]

Porceeding as in the analysis of \( I_{22} \) in Lemma 4.1, we use the one-step large-\( k \) expansion (Lemma 3.4) to obtain
\[
\nu_2(z, k) = -\frac{1}{2\pi k} \int e^{kr(z)} r_{k^j} \nu_2(z, k') \, dm(k') - \frac{1}{2\pi k} \int \frac{e^{kr(z)}}{k-k'} r_{k^j} \nu_2(z, k') \, dm(k') = F_1 + F_2.
\]

Hence, up to trivial factors,
\[
I_3 = \int e_{-kr} \left( \partial \nu_1 \right) \left( u \left( F_1 + F_2 \right) \right).
\]

By Minkowski’s inequality,
\[
\|I_3\|_{L^2} \leq \frac{1}{2} \int \|r\| \left\| \partial \nu_1 \left( u \left( F_1 + F_2 \right) \right) \right\|_{L^2}
\]

Observe that \( \left\| \partial \nu_1 \left( u \left( F_1 \right) \right) \right\|_{L^2} \leq C \|F_1\|_{L^2} \) (Lemma 3.7) while
\[
\left\| \partial \nu_1 \left( u \left( F_2 \right) \right) \right\|_{L^2} \leq C_p \|u\|_2 \|F_2\|_{L^2} \leq C_p \|u\|_2 \left\| r(\Theta(1-\chi)) \right\|_{2p/(p+2)} \|r\|_\infty
\]
where \( \|u_2\|_\infty \) means \( \|u_2\|_{L^\infty(\mathbb{R}^2 \times \mathbb{R}^2)} \), so that altogether
\[
\|I_3\|_{L^2} \leq C \|u\|_2 \|r\|_{H^{1,2}} (1 + \|v_2\|_\infty).
\]
Thus \( I_3 \in L^2 \). Local Lipschitz continuity of \( I_3 \) follows from the local Lipschitz continuity of \( r \mapsto u \) and \( r \mapsto v_2 \).

\( I_4 \): Using (4.6) again we compute
\[
\int k e_{-k} r \partial \nu_1 = \frac{u}{2} \int e_{-k} kr \nu_2
\]
so it suffices to show that \( r \mapsto \int e_{-k} kr \nu_2 \) is a LLCM from \( H^{1,2} \) to \( L^\infty \). Since \( kr \in L^{p^*} \) for \( p > 2 \), and \( r \mapsto \nu_2 \) is a LLCM from \( H^{1,1} \) to \( L^\infty \), the result follows.

\( I_5 \): Compute
\[
I_5 = \int e_{-k} kr \partial (u \nu_2) = \partial u \int e_{-k} kr \nu_2 + u \int e_{-k} r \partial \nu_2.
\]
The first right-hand term in (4.7) defines a LLCM from \( H^{1,2} \) to \( L^2 \) since \( r \mapsto \partial u \) has this property. Thus, to control the right hand term, it suffices to show that \( r \mapsto \int e_{-k} kr \nu_2 \) defines a LLCM from \( H^{1,2} \) to \( L^\infty \). To see this, note that \( r \in L^{p^*} \) for \( p > 2 \), and \( r \mapsto \nu_2 \) is a LLCM from \( H^{1,1} \) to \( L^\infty \). To control the second right-hand term in (4.7), recall that \( \nu_2 = e_{k} \nu_2 \) so that the second term is written
\[
- u \int k e_{k} r \nu_2 + \frac{|u|^2}{2} \int e_{-k} r \nu_1.
\]
Since \( u \) and \( |u|^2 \) belong to \( L^2 \) it is enough to show that the two integrals in (4.8) define LLCM’s from \( r \in H^{2,1} \) to \( L^\infty \). Since \( kr \in L^{p^*} \) for \( p > 2 \) and \( \nu_2 \) is a LLCM from \( H^{1,2} \) to \( L^\infty \), the first term in (4.8) clearly has this property. Since \( r \in L^1 \) and \( \nu_1 \) is a LLCM from \( r \in H^{2,1} \) to \( L^\infty \), we conclude that the second term also has this property.

\[ \square \]

5. Solving the mNV Equation

In this section we prove Theorem 1.4. Recall that the modified Novikov-Veselov (mNV) equation (1.3) is:
\[
u_t + \left( \partial^3 + \overline{\partial}^3 \right) u + NL(u) = 0
\]
where
\[
NL(u) = \frac{3}{4} (\partial u) \cdot \left( \overline{\partial} \partial^{-1} \left( |u|^2 \right) \right) + \frac{3}{4} \left( \overline{\partial} \partial^{-1} \left( |u|^2 \right) \right) = 3 \left( \partial^3 \overline{\partial} \partial^{-1} \left( \partial u \right) + \frac{3}{4} \partial^{-1} \left( \overline{\partial} \overline{\partial} \partial^{-1} \left( \partial u \right) \right) \right).
\]
By Theorem A.1 for \( u_0 \in \mathcal{S}^1(\mathbb{R}^2) \), the formula
\[
u(z, t) = I \left( \exp \left( \left( \partial^3 - \overline{\partial}^3 \right) t \right) \mathcal{R} u_0(\phi) \right) (z)
\]
gives a classical solution of the mNV equation. By Lipschitz continuity of \( u_0 \to u_0(t) \), this formula extends to \( u_0 \in H^{2,1} \), and exhibits the solution as a continuous curve in \( H^{2,1} \) that depends continuously on the initial data. Since any \( u \) given by (5.2) and \( u_0 \in \mathcal{S}^1(\mathbb{R}^2) \) is a classical solution, such a \( u \) trivially satisfies (1.10). The same fact for \( u(t) \) with \( u_0 \in H^{2,1} \) follows from the density of \( \mathcal{S}(\mathbb{R}^2) \) in \( H^{2,1} \), the continuity of the map (5.2) in \( u_0 \), and an easy approximation argument.
It remains to show that if \( u_0 \in H^{2,1}(\mathbb{R}^2) \cap L^1(\mathbb{R}^2) \) and if, also,
\[
(5.3) \quad \int u_0 dA(z) = 0, \quad \partial u_0 = \overline{\partial u_0},
\]
then \( \partial u = \overline{\partial u} \) for all \( t \). We will show that this holds for initial data \( u_0 \in S(\mathbb{R}^2) \) with the stated properties, and use Lipschitz continuity of the map \( u_0 \to \overline{u}(t) \) defined by \( (5.2) \) to extend to all \( u_0 \in H^{2,1}(\mathbb{R}^2) \cap L^1(\mathbb{R}^2) \) so that the conditions \((5.3)\) hold.

It will be useful to consider the function
\[
\varphi = \Theta^{-1} u
\]
which solves the Cauchy problem
\[
(5.4) \quad \varphi_t = -\partial^3 \varphi - \overline{\partial}^3 \varphi
\]
\[
- \frac{1}{4} (\partial \varphi)^3 - \frac{1}{4} (\overline{\partial} \varphi)^3
\]
\[
+ \frac{3}{4} \partial \varphi \cdot \Theta^{-1} \partial (|\partial \varphi|^2) + \frac{3}{4} \overline{\partial} \varphi \cdot \Theta^{-1} \partial (|\overline{\partial} \varphi|^2)
\]
\[
\varphi|_{t=0} = \varphi_0
\]
Note that the condition \( \partial u_0 = \overline{\partial u_0} \) implies that \( \varphi_0 \) is real. On the other hand, to show that \( \partial u = \overline{\partial u} \), it suffices to show that \( \varphi \) is real for \( t > 0 \). To this end, we consider the function
\[
w = \varphi - \overline{\varphi}
\]
and derive a linear Cauchy problem satisfied by \( w \). We will need to know that \( w \) is \( L^2 \) in the space variables.

**Lemma 5.1.** Suppose that \( u_0 \in S(\mathbb{R}^2) \), that \( u(t) \) solves the mNV equation, and \( \varphi(z,t) = (\Theta^{-1} u)(t) \). Then for each \( t \),
\[
\varphi(z,t) = c_0 \frac{z}{\overline{z}} + O \left( |z|^2 \right)
\]
where \( c_0 = \int u_0(z) \, dA(z) \). If \( c_0 = 0 \), then \( \varphi \in L^2(\mathbb{R}^2) \) for \( t > 0 \).

**Proof.** To see that \( \varphi \) has the stated form if \( u_0 \in S(\mathbb{R}^2) \), we note that \( u(t) \in S(\mathbb{R}^2) \) by the mapping properties of the scattering transform so that
\[
\varphi(z,t) = -\frac{1}{\pi z} \int u(z,t) \, dt + O_t \left( |z|^2 \right)
\]
differentiably in \( z,t \). Let \( c_0(t) = \int u(z,t) \, dA(z) \). Substituting in \((5.4)\) we easily conclude that \( c_0(t) = 0 \). It now follows that \( \varphi(\cdot,t) \in L^2(\mathbb{R}^2) \) as claimed. \( \square \)

Next, we derive a linear Cauchy problem obeyed by \( w \) and derive weighted estimates on \( w \) to show that, if \( w|_{t=0} = 0 \), then \( w(t) = 0 \) identically. It follows that \( \varphi \) is real, and hence \( \partial u = \overline{\partial u} \) for all \( t > 0 \). Using \((5.4)\) and its complex conjugate, we easily see that
\[
(5.5) \quad w_t = -\partial^3 w - \overline{\partial}^3 w + A \partial w + \overline{A} \overline{\partial} w
\]
where
\[
A = \frac{1}{4} \left[ (\partial \varphi)^2 + (\partial \varphi) \cdot (\overline{\partial} \varphi) + (\overline{\partial} \varphi)^2 \right] + \frac{3}{4} \overline{\partial} \varphi \cdot \Theta^{-1} \partial (|\partial \varphi|^2)
\]
Note that \( \partial \varphi, \overline{\partial} \varphi \) belong to \( L^p \) for all \( p \in (1, \infty) \), uniformly locally in \( t \), and that \( A \) is smooth provided that \( u_0 \in S(\mathbb{R}^2) \). We will prove:
Lemma 5.2. Suppose that $A(z, t)$ is a bounded smooth function on $\mathbb{R}^2 \times (0, T)$ and that $\eta(z, t)$ is a bounded smooth nonnegative function with $|A(z, t)| \leq \eta(z, t)$ for $z \in \mathbb{C}$ and $t \in [0, T]$. Let $w$ be a smooth solution of (5.5) with $w(0, t) \in L^2(\mathbb{R}^2)$ for each $t > 0$. Then, there is a constant $C$ so that

$$\sup_{t \in [0, T]} \|w(t)\| \leq e^{CT} \|w(0)\|.$$

Proof. We apply the multiplier method of Chihara [20] (applied to third-order dispersive nonlinear equations; see Doi [22] for a similar pseudodifferential multiplier method applied to Schrödinger-type equations) to (5.5). Let $\eta$ be a function with

$$2|A(z, t)| \leq \eta(z, t),$$

and set

$$p_0(\xi) = \frac{1}{8} (\xi_1^2 - 6\xi_1\xi_2^2),$$

the symbol of the operator $-\partial^3 - \overline{\partial}^3$. With $z = x_1 + ix_2$ and $\lambda > 0$ to be chosen, let

$$\gamma(t, x, \xi) = \left( \int_{-\infty}^{x_1} \eta(y, x_2, t) \ dy + \int_{-\infty}^{x_2} \eta(x_1, y, t) \ dy \right) \times \frac{\partial p_0(\xi)}{\partial \xi_1} \frac{|\xi|}{|\nabla p_0(\xi)|} \chi \left( \frac{|\xi|}{\lambda} \right)$$

where $\chi \in C_0^\infty([0, \infty))$ is a nonnegative function with $\chi(t) = 0$ for $0 \leq t < 1/2$ and $\chi(t) = 1$ for $t \geq 1$. The function $\gamma$ is constructed so that the principal symbol of the commutator $[\gamma, p_0(D)]$ obeys

$$\sigma ([\gamma, p_0(D)]) = \nabla_x \gamma(x, \xi, t) \cdot \nabla_\xi p_0(\xi)$$

$$= \eta(x_1, x_2, t) \cdot |\xi| \chi \left( \frac{|\xi|}{\lambda} \right).$$

By the usual quantization, the pseudodifferential operator $\gamma(t, x, D)$ belongs to the class $OPS^0(\mathbb{R}^n)$. It is easy to see that, also, the symbols

$$k(t, x, \xi) = e^{\gamma(t, x, \xi)},$$

$$\overline{k}(t, x, \xi) = e^{-\gamma(t, x, \xi)}$$

define pseudodifferential operators $K(t) := k(t, x, D)$ and $\overline{K}(t) := \overline{k}(t, x, D)$ in $OPS^0(\mathbb{R}^n)$ with

$$K(t)\overline{K}(t) - I \in OPS^{-1}(\mathbb{R}^n)$$

and

$$\lim_{\lambda \to \infty} \sup_{t \in [0, T]} \|K(t)\overline{K}(t) - I\| = 0.$$

Thus, there is a $\lambda_0 > 0$ so that $K(t)$ is invertible for all $|\lambda| \geq \lambda_0$. We take $|\lambda| \geq \lambda_0$ from now on.

We claim that if $w(t)$ is a solution of the evolution equation (5.5) belonging to $L^2(\mathbb{R}^2)$, the inequality

$$\|K(t)w(t)\| \leq \|K(0)w(0)\| e^{CT}$$

holds for $t \in [0, T]$ and a constant $C$. Since $K(0)$ is invertible for $\lambda$ sufficiently large, this implies that $w(0) = 0$. 


To prove the inequality (5.8), we compute
\[ \frac{d}{dt} \|K(t)w(t)\|^2 = 2 \text{Re} \left( K(t)w(t), \left[ K'(t)K^{-1}(t) \right] K(t)w(t) \right) + 2 \text{Re} \left( K(t)w(t), K(t)L(t)w(t) \right) \]
where
\[ L(t) = -\delta^3 - \overline{\partial^3} + A\partial + \overline{A\partial}. \]
We will show that
\[ K(t)L(t)K(t)^{-1} = -Q_1(t) + Q_2(t) \]
where \( Q_1(t) \in OPS^{1,0}(\mathbb{R}^2) \) which \( q_1(x,\xi) := \sigma(Q_1(t)) \) nonnegative for \( |\xi| \geq 2\lambda \), and \( Q_2(t) \in OPS^0(\mathbb{R}^2) \). If so then by the sharp Gårding inequality [41],
\[ (5.9) \quad \text{Re}(v, Q_1(t)v) \geq -C_1 \|v\|^2 \]
and hence
\[ \frac{d}{dt} \|K(t)w(t)\|^2 \leq C_3 \|K(t)w(t)\|^2 \]
where \( C_3 \) majorizes \( \|Q_2(t)\| + \|K'(t)K^{-1}(t)\| \). The desired result follows from Gronwall’s inequality.

Thus, to finish the proof of (5.8), we need only prove that (5.9) holds. But
\[ K(t)L(t)K(t)^{-1} = K(t)(-p_0(D) + A\partial + \overline{A\partial})K(t)^{-1} \]
The right-hand side has leading symbol \(-q_1(x_1, x_2, \xi, t)\) where
\[ q_1(x_1, x_2, \xi, t) = \nabla_x p_0(\xi) \cdot \nabla_y \gamma(t, x_1, x_2, \xi) + \text{Re} [A(x_1, x_2, t)(\xi_1 - i\xi_2)] \]
which is strictly positive for \( |\xi| \geq 2\lambda \) by (5.7). This completes the proof. \( \square \)

Now suppose that \( u_0 \in S(\mathbb{R}^2) \), \( \partial u_0 = \overline{\partial u_0} \), and \( \int u_0(z) \, dA(z) = 0 \). The function \( \varphi_0 = \overline{\partial^{-1}} u_0 \) is real-valued and if \( u(t) \) solves the mNV equation with Cauchy data \( u_0 \), the function \( \varphi(t) = (\overline{\partial^{-1}} u)(t) \) belongs to \( L^2(\mathbb{R}^2) \) for all \( t \). The same is true of \( w(t) = \varphi(t) - \varphi(t) \), and \( w(0) = 0 \). It now follows from Lemma 5.2 that \( w(t) = 0 \) and \( \varphi(t) \) is real-valued for all \( t \). This implies that \( \partial u = \overline{\partial u} \) for all \( t \).

**Proof of Theorem 1.4.** An immediate consequence of Lemma 5.1, Lemma 5.2, and the above remarks. \( \square \)

### 6. Solving the NV Equation

In this section we prove Theorem 1.6. The key observation is due to Bogdanov [14] and can be checked by straightforward computation. Recall the Miura map \( \mathcal{M} \), defined in (1.8).

**Lemma 6.1.** Suppose that \( u(z,t) \) is a smooth classical solution of (5.1) with
\[ \partial_z u(z, t) = \overline{\partial_z u(z, t)}, \]
and \( \int u(z, t) \, dA(z) = 0 \) for all \( t \). Then, the function
\[ q(z, t) = \mathcal{M}(u(\cdot,t))(z) \]
is a smooth classical solution of (1.1).

**Remark 6.2.** In Bogdanov [14], the mNV and NV are shown to be gauge-equivalent, and the Miura map is computed from the gauge equivalence.
Proof of Theorem 1.6. Pick $u_0 \in H^{2,1}(\mathbb{R}^2) \cap L^1(\mathbb{R}^2)$ so that the conditions $\partial u_0 = 0$ and $\int u_0(z) \, dA(z) = 0$ hold. Let $\{u_{0,n}\}$ be a sequence from $\mathcal{S}(\mathbb{R}^2)$ with $u_{n,0} \to u_0$ in $H^{2,1}(\mathbb{R}^2) \cap L^1(\mathbb{R}^2)$. By local Lipschitz continuity of the scattering maps, for any $T > 0$, the sequence $\{u_n\}$ from $C([0,T]; H^{2,1}(\mathbb{R}^2))$ given by

$$ u_n(z, t) = \mathcal{I} \left( e^{t(\partial q - iq^2)} (\mathcal{R} u_{0,n})(\phi) \right)(z) $$

converges in $C([0,T]; H^{2,1}(\mathbb{R}^2))$ to

$$ u(z, t) = \mathcal{I} \left( e^{t(\partial q - iq^2)} (\mathcal{R} u_0)(\phi) \right)(z). $$

This convergence implies that $q_n(z, t) := \mathcal{M}(u_n(\phi, t))(z)$ converges in $L^2(\mathbb{R}^2)$.

Recall (1.11). Since $q_n \to q$ in $C([0,T]; L^2(\mathbb{R}^2))$ it follows from the $L^2$-boundedness of $\mathcal{S} = \partial \mathcal{S}$ that $q_n \mathcal{S}^{-1} \partial q_n \to q \mathcal{S}^{-1} \partial q$ and $q_n \mathcal{S}^{-1} \mathcal{R} q_n \to q \mathcal{S}^{-1} \mathcal{R} q$ in $C([0,T], L^1(\mathbb{R}^2))$. We conclude that $q$ is a weak solution of the NV equation. \qed

7. Conductivity-Type Potentials

In this section we show that our solution of NV coincides with that of Lassas, Mueller, Siltanen, and Stahel \cite{lassas2007inverse} in the cases they consider, proving Theorem 1.7.

We briefly recall some of the notation and results of Lassas, Mueller, and Siltanen \cite{lassas2007inverse}. Assume first that $q \in \mathcal{C}_0^\infty(\mathbb{R}^2)$ and is of conductivity type. We denote by $\psi(x, \zeta)$ the unique solution of the problem

$$ (-\Delta + q) \psi = 0, $$

$$ \lim_{|z| \to \infty} \left( e^{-i(x \cdot \zeta)} \psi(x, \zeta) - 1 \right) = 0. $$

where $x = (x_1, x_2)$ and $\zeta \in \mathbb{C}^2$ satisfies $\zeta \cdot \zeta = 0$. Here $a \cdot b$ denotes the Euclidean inner product without complex conjugation. Henceforth, we set $\zeta = (k, ik)$ for $k \in \mathbb{C}$, which amounts to choosing a branch of the variety $\mathcal{V} = \{ \zeta \in \mathbb{C}^2 : \zeta \cdot \zeta = 0 \}$. Since $q$ is of conductivity type, it follows from Theorem 3 in \cite{lassas2007inverse} that the problem (7.1) admits a unique solution for each $k \in \mathbb{C}$. We set $z = x_1 + ix_2$ and define

$$ m(z, k) = e^{-ikz} \psi(x, \zeta) $$

for $\zeta = (k, ik)$.

The direct scattering map

$$ \mathcal{T} : q \to t $$

is defined by

$$ t(k) = \int e^{i(kx + k^2)} q(z)m(z, k) \, dA(z) $$

On the other hand, the inverse map

$$ \mathcal{Q} : t \to q $$

is defined by

$$ q(z) = \frac{i}{\pi^2} \mathcal{S}_z \left( \int \frac{t(k)}{k} e^{-i(\zeta z + k\zeta^2)} m(z, k) \, dA(k) \right) $$

where $m(z, k)$ is reconstructed from $t$ via the $\mathcal{T}$-problem

$$ \mathcal{T}_k m(x, k) = \frac{t(k)}{4\pi k} e^{-i(\zeta x + k\zeta^2)}(z) \overline{m(x, k)}. $$
Let

\[ m_n^u(k) = \exp\left(-i^n \left(k^n + \mathbb{K}^n\right) t\right) \]

for an odd positive integer \( n \). In [40], Lassas, Mueller and Siltanen prove:

**Theorem 7.1.** [40] For \( q_0 \in \mathcal{C}_0^\infty(\mathbb{R}^2) \) radial and of conductivity type, \( QT(q_0) = q_0 \). Moreover, if

\[ q(t) := Q(m_n^u T q_0) \]

then \( q(t) \) is a continuous, real-valued potential with \( q(t) \in L^p(\mathbb{R}^2) \) for \( p \in (1, 2) \).

They conjecture that for \( n = 3 \), \( q(t) \) given by (7.8) solves the NV equation, provided that \( q_0 \) obeys the hypotheses of Theorem [40]. We will prove that this is the case (for a larger class of \( q_0 \)) by proving Theorem 1.7.

We will prove Theorem 1.7 in two steps. First, we show that for \( u \in S(\mathbb{R}^2) \) with \( \partial u = \partial u \) and \( \int u_0(z) \ dA(z) = 0 \), the scattering data \( r = \mathcal{R} u \) is related to the scattering transform \( t = \mathcal{T} q \) for \( q = 2\partial u + |u|^2 \) by the identity

\[ t(k) = -2\pi i k \mathcal{R}(ik). \]

Next, we show that for \( t \) of the above form with \( r = \mathcal{R} u \), the identity

\[ (\mathcal{Q} t)(z) = 2(\partial u)(z) + |u(z)|^2. \]

Theorem 1.7 is an easy consequence of these two identities.

The key to both computations is the following construction of complex geometric optics solutions for the potential \( q = 2\partial u + |u|^2 \) from the solutions \( \mu = (\mu_1, \mu_2)^T \) of (3.1). First, suppose that \( \Phi = (\Phi_1, \Phi_2)^T \) is a vector-valued solution of the linear system

\[ \left( \begin{array}{cc} \partial & 0 \\ 0 & \partial \end{array} \right) \Phi = \frac{1}{2} \left( \begin{array}{cc} 0 & u \\ u & 0 \end{array} \right) \Phi. \]

A straightforward calculation shows that the function

\[ \psi = \Phi_1 + \Phi_2 \]

solves the zero-energy Schrödinger equation

\[ (-\Delta + q) \psi = 0 \]

for \( q = 2\partial u + |u|^2 \).

Recall that matrix-valued solutions of (7.9) are related to the solutions \( \mu \) of (3.1) by

\[ \left( \begin{array}{c} \mu_1 \\ \mu_2 \end{array} \right) = \left( \begin{array}{c} \Phi_1 \\ \Phi_2 \end{array} \right) e^{-kz} \]

so that

\[ \Phi_{11} + \Phi_{21} = e^{kz} \mu_1(z, k) + e^{-kz} \mu_2(z, k) \]

solves (7.10). To compute its asymptotic behavior, using \( (\mu_1, \mu_2) \to (1, 0) \) as \( |z| \to \infty \) we conclude that \( e^{-kz} \psi(z, k) \to 1 \) as \( |z| \to \infty \). Hence, denoting by \( \psi \) the solution of the problem (7.10) with \( \zeta = (k, ik) \) for \( k \in \mathbb{C} \), we have

\[ \psi(z, k) = \psi(z, ik) = e^{ikz} \mu_1(z, ik) + e^{-ikz} \mu_2(z, ik) \]
so
\[ m(z, k) = \mu_1(z, k) + e^{-i(kz + ikz)}\mu_2(z, ik). \]

Now, we can prove:

**Lemma 7.2.** Let \( u \in C_0^\infty(\mathbb{R}^2) \) with \( \partial u = \overline{\partial u} \), suppose \( \int u(z) \, dA(z) = 0 \), and let \( q = 2\partial u + |u|^2 \). Then
\[ (Tq) (k) = -2\pi ik (\overline{Ru})(ik). \]

**Proof.** We compute
\[
(Tq) (k) = \int q(z)e^{ikz}\psi(z, k) \, dA(z)
= \int 2(\partial u)(z)e^{i(kz + ikz)}\mu_1(z, ik) \, dA(z)
+ \int 2(\partial u)(z)\mu_2(z, ik) \, dA(z)
+ \int |u(z)|^2 \left( e^{i(kz + ikz)}\mu_1(z, ik) + \mu_2(z, ik) \right) \, dA(z)
= I_1 + I_2 + I_3
\]
where in the first right-hand term we used \( \partial u = \overline{\partial u} \). We can integrate by parts in each of the first two right-hand terms and use (3.1) to obtain
\[
I_1 = -2\pi \int \overline{u(\zeta)} \, dA(\zeta)
= 2(\partial u)(z)
= 2(\partial u)(z)
\]
and
\[
I_2 = - \int |u(z)|^2 e^{i(kz + ikz)}\mu_1(z, ik) \, dA(z)
= T_1 + T_2
\]
Changing variables to \( \zeta = ik \) in \( T_1 \) we recover
\[
T_1 = \frac{2\pi}{\pi} \int \frac{1}{r(\zeta)} e^{-i(kz + ikz)}\mu_1(z, \zeta) \, dA(\zeta)
= 2(\overline{\partial u})(z)
= 2(\partial u)(z)
\]

Next, we analyze the inverse scattering transform \( Q \) defined by (1.6). We will prove:

**Lemma 7.3.** Let \( u \in S(\mathbb{R}^2) \) with \( \partial u = \overline{\partial u} \), and suppose that \( \int u(z) \, dA(z) = 0 \). Let \( r = Ru \) and suppose that \( t \) is given by (7.13). Then
\[ (Qt)(z) = 2(\partial u)(z) + |u(z)|^2. \]

**Proof.** We compute from (1.6), (7.13), and (7.12) that
\[
(Qt)(z) = \frac{2\pi}{\pi} \int \frac{1}{r(ik)} e^{-i(kz + ikz)}\mu_1(z, ik) \, dA(k)
+ \frac{2\pi}{\pi} \int \frac{1}{r(ik)} \mu_2(z, ik) \, dA(k)
= T_1 + T_2
\]
Changing variables to \( \zeta = ik \) in \( T_1 \) we recover
\[
T_1 = \frac{2\pi}{\pi} \int \frac{1}{r(\zeta)} e^{i(kz + ikz)}\mu_1(z, \zeta) \, dA(\zeta)
= 2(\overline{\partial u})(z)
= 2(\partial u)(z)
\]
where we have used (3.3). Using (3.1) in \( T_2 \) we have
\[
T_2 = \frac{1}{\pi} \int r(ik) \, u(z) \, e^{-i(kz+\kappa z)} \mu_1(z,ik) \, dA(k)
\]
\[
= \frac{1}{\pi} |u(z)| \int \kappa \, e^{\kappa z-\zeta z} \mu_1(z,\zeta) \, dA(\zeta)
\]
\[
= |u(z)|^2.
\]
Combining these computations gives the desired result.

**Proof of Theorem 1.7.** For \( u_0 \) satisfying the hypotheses and \( q = 2 \partial u_0 + |u_0|^2 \), we have by Lemma 7.2 that
\[
(T q_0)(k) = -2\pi i \kappa r(ik)
\]
where \( r = R(u_0) \), and hence
\[
e^{-it(k^3+\kappa^3)} (T q_0)(k) = -2\pi i \kappa \left( e^{it((\nu)^3-(\nu_0)^3)} r(\nu) \right)(ik).
\]
We can now apply Lemma 7.3 to conclude that
\[
Q \left( e^{-it((\nu)^3+\nu_0^3)} (T q_0)(\nu) \right) = MI \left( e^{it((\nu)^3-(\nu_0)^3)} r(\nu) \right)
\]
as claimed.

**Appendix A. Schwarz Class Inverse Scattering for the mNV Equation**

In this appendix we develop the Schwarz class inverse theory for the mNV equation, using freely the results and notation of [49] with one exception: we denote the potential by \( u \) rather than \( q \). Our main result is:

**Theorem A.1.** Suppose that \( u_0 \in S(\mathbb{R}^2) \), and let \( R \) and \( I \) be the scattering maps defined respectively by (3.2) and (3.5). Finally, define
\[
u(t) = I \left( e^{it((\nu)^3-\nu_0^3)} (Ru_0)(\nu) \right).
\]
Then \( \nu(t) \) is a classical solution of the modified Novikov-Veselov equation (5.1).

The proof follows the method of Beals-Coifman [8, 9, 10] and Sung [50] but necessitates some long computations.

**A.1. Scattering Solutions and Tangent Maps.** First we recall the solutions \( \nu \) and \( \tilde{\nu} \) of the \( \overline{\partial} \) problem with \( \overline{\partial} \)-data determined by the time-dependent coefficient \( r \) and the formulas from [49] for the tangent maps.

We recall that \( \nu = (\nu_1, \nu_2)^T \) is the unique solution of the \( \overline{\partial} \) problem
\[
(\overline{\partial} \nu_1) = \frac{1}{2} e_k \nu_2,
(\overline{\partial} \nu_2) = \frac{1}{2} e_k \nu_1,
\]
\[
\lim_{|k| \to \infty} \nu(z,k) = (1,0),
\]
where \( r = R(u) \). Here
\[
e_k(z) = e^{\kappa z-kz}.
\]
The function \( \nu^\# = (\nu^\#_1, \nu^\#_2) \) solves the same problem but for \( u^\#(\cdot) = -\overline{u}(\cdot) \) and \( r^\# = R(u^\#) = -\overline{r} \) (see Lemma B.1 in [49]). Thus
\[
\begin{align*}
\overline{\partial}_k \nu^\#_1 &= -\frac{1}{2} \overline{e^k r \nu^\#_1}, \\
\overline{\partial}_k \nu^\#_2 &= -\frac{1}{2} \overline{e^k r \nu^\#_2}, \\
\lim_{|k| \to \infty} \nu^\#(z,k) &= (1,0).
\end{align*}
\]

The tangent map formula gives an expression for \( u \) if \( u = R(r) \) where \( r \) is a \( C^1 \)-curve in \( \mathcal{S}(\mathbb{R}^2) \). Assuming the law of evolution \( \dot{r} = \overrightarrow{k^3 - k^3} r \) and following the calculations in Appendix B of [49], we find that
\[
\begin{align*}
\dot{u} &= 2i(I_1 + \overline{I_2}) \\
I_1 &= \frac{1}{\pi} \int k^3 \overline{\partial}_k \left[ \nu^\#_2(-z,k) \nu_1(z,k) \right] dA(k), \\
I_2 &= -\frac{1}{\pi} \int k^3 \overline{\partial}_k \left[ \nu^\#_1(-z,k) \nu_2(z,k) \right] dA(k).
\end{align*}
\]

As in Appendix B of [49], we evaluate these integrals using the following fact: if \( g \) is a \( C^\infty \) function with asymptotic expansion
\[
\begin{align*}
g(k,k) &\sim 1 + \sum_{\ell \geq 0} \frac{g_{\ell}}{k^{\ell+1}} \\
\lim_{R \to \infty} \left( \frac{1}{|k|^n} \int_{|k| \leq R} k^n \overline{\partial}_k g(k,k) dA(k) \right) &= g_n.
\end{align*}
\]
Using (A.7) we get (noting the \(-\) sign in (A.5))
\[
\begin{align*}
I_1 &= 2 \left[ \nu_1(z,\diamond) \nu^\#_2(-z,\diamond) \right]_3 \\
\overline{I_2} &= 2 \left[ \nu_2(z,\diamond) \nu^\#_1(-z,\diamond) \right]_3
\end{align*}
\]
so that
\[
\dot{u} = 2 \left\{ \left[ \nu_1(z,\diamond) \nu^\#_2(-z,\diamond) \right]_3 + \left[ \nu_2(z,\diamond) \nu^\#_1(-z,\diamond) \right]_3 \right\}
\]
Here \([\diamond]_n\) denotes the coefficient of \( k^{-n-1} \) in an asymptotic expansion of the form (A.6). The formulas
\[
\begin{align*}
\left[ \nu_1(z,\diamond) \nu^\#_2(-z,\diamond) \right]_n &= (\nu^\#_{12})_n + \sum_{j=0}^{n-1} (\nu^\#_{n-j-1})_j (\nu_j)_1 \\
\left[ \nu_2(z,\diamond) \nu^\#_1(-z,\diamond) \right]_n &= (\nu_{12})_n + \sum_{j=0}^{n-1} (\nu_{n-j-1})_j (\nu^\#_j)_2
\end{align*}
\]
will be used in concert with the residue formulae below to obtain the equation of motion.

A.2. Expansion Coefficients for $\nu$. Following the method of Appendix C in [49], we can compute the additional coefficients in the asymptotic expansion

(A.9) \[ \nu \sim (1, 0) + \sum_{\ell \geq 0} k^{-\ell+1} \nu(\ell) \]

needed to compute $\dot{u}$ from the formula (A.8). Let us set $\nu(\ell) = (\nu_{1,\ell}, \nu_{2,\ell}) \top$. We recall from [49] the ‘initial data’

(A.10) \[ \nu_{1,0} = \frac{1}{4} \partial^{-1} \left( |u|^2 \right), \quad \nu_{2,0} = \frac{1}{2} \partial \]

and the recurrence relations

\[ \nu_{2,\ell} = \frac{1}{2} \nu_{1,\ell-1} - \partial \nu_{2,\ell-1} , \]
\[ \nu_{1,\ell} = \frac{1}{2} P (u \nu_{2,\ell}) . \]

The following formulas are a straightforward consequence.

$\ell = 0$:

(A.11) \[ \nu_{1,0} = \frac{1}{4} \partial^{-1} \left( |u|^2 \right) \]

(A.12) \[ \nu_{2,0} = \frac{1}{2} \partial \]

$\ell = 1$:

(A.13) \[ \nu_{1,1} = \frac{1}{16} \partial^{-1} \left( |u|^2 \partial^{-1} \left( |u|^2 \right) \right) - \frac{1}{4} \partial^{-1} (u \partial) \partial \]

(A.14) \[ \nu_{2,1} = \frac{1}{8} \partial^{-1} \left( |u|^2 \right) - \frac{1}{2} \partial \]

$\ell = 2$:

(A.15) \[ \nu_{1,2} = \frac{1}{64} \partial^{-1} \left( |u|^2 \partial^{-1} \left( |u|^2 \partial^{-1} \left( |u|^2 \right) \right) \right) - \frac{1}{16} \left\{ \partial^{-1} \left( \partial \left( \partial^{-1} \left( |u|^2 \right) \right) \right) + \partial^{-1} \left( |u|^2 \partial^{-1} (u \partial) \right) \right\} + \frac{1}{4} \partial^{-1} (u \partial^2 \partial) \]

(A.16) \[ \nu_{2,2} = \frac{1}{32} \partial^{-1} \left( |u|^2 \partial^{-1} \left( |u|^2 \right) \right) - \frac{1}{8} \left\{ \partial \left( \partial^{-1} \left( |u|^2 \right) \right) + \partial^{-1} (u \partial) \right\} + \frac{1}{2} \partial^2 \partial \]

$\ell = 3$:
$$\nu_{2,3} = \frac{1}{128} \overline{u}\overline{v}^{-1} \left( |u|^2 \overline{v}^{-1} \left( |u|^2 \overline{v}^{-1} \left( |u|^2 \right) \right) \right)$$

$$- \frac{1}{32} \left( \overline{u}\overline{v}^{-1} \left( u\overline{v} \left( \overline{v}\overline{v}^{-1} \left( |u|^2 \right) \right) \right) + \overline{u}\overline{v}^{-1} \left( |u|^2 \overline{v}^{-1} (u\overline{v}) \right) \right)$$

$$+ \overline{\partial} \left( \overline{v}\overline{v}^{-1} \left( |u|^2 \overline{v}^{-1} \left( |u|^2 \right) \right) \right) \right)$$

$$+ \frac{1}{8} \left( \overline{u}\overline{v}^{-1} (u\partial \overline{u}) + \partial^2 \left( \overline{u}\overline{v}^{-1} \left( |u|^2 \right) \right) \right) + \partial \left( \overline{u}\overline{v}^{-1} (u\overline{v}) \right) \right)$$

$$- \frac{1}{2} \partial^3 \overline{u}$$

(A.17)

A.3. Expansion Coefficients for $\nu^\#$. The solution $\nu^\#$ corresponds to the potential $-\overline{u}(-z)$. To compute the corresponding residues for $\nu^\# (-z,k)$ we therefore make the following substitutions in the formulas above:

$$\overline{v}^{-1} \rightarrow -\overline{v}^{-1}$$

$$\partial \rightarrow -\partial$$

$$u \rightarrow -\lambda \overline{u}$$

$$\overline{u} \rightarrow -\lambda \overline{u}$$

Thus the overall sign change is $(-1)^{n_u + n_\partial}$ where $n_u$ is the number of factors of $u$ and $\overline{u}$, while $n_\partial$ is the number of factors of $\partial$ and $\overline{v}^{-1}$. There is also an overall factor of $(\lambda)^{n_u}$ i.e. $\lambda$ if $n_u$ is odd, or 1 if $n_u$ is even. Applying these rules we obtain:

$\ell = 0$:

(A.18) $$\nu_{1,0}^\# = -\frac{1}{4} \overline{v}^{-1} \left( |u|^2 \right)$$

(A.19) $$\nu_{2,0}^\# = -\frac{1}{2} u$$

$\ell = 1$:

(A.20) $$\nu_{1,1}^\# = \frac{1}{16} \overline{v}^{-1} \left( |u|^2 \overline{v}^{-1} \left( |u|^2 \right) \right) - \frac{1}{4} \overline{v}^{-1} (u\overline{v}u)$$

(A.21) $$\nu_{2,1}^\# = \frac{1}{8} \overline{u}\overline{v}^{-1} \left( |u|^2 \right) - \frac{1}{2} \partial u$$

$\ell = 2$:

(A.22) $$\nu_{1,2}^\# = \frac{1}{64} \overline{v}^{-1} \left( |u|^2 \overline{v}^{-1} \left( |u|^2 \overline{v}^{-1} \left( |u|^2 \right) \right) \right)$$

$$+ \frac{1}{16} \left\{ \overline{v}^{-1} (u\overline{v} \left( \overline{v}\overline{v}^{-1} \left( |u|^2 \right) \right) \right) + \overline{v}^{-1} \left( |u|^2 \overline{v}^{-1} (u\overline{v}) \right) \right\}$$

$$- \frac{1}{4} \overline{v}^{-1} (u\partial^2 u)$$

(A.23) $$\nu_{2,2}^\# = \frac{1}{32} \overline{u}\overline{v}^{-1} \left( |u|^2 \overline{v}^{-1} \left( |u|^2 \right) \right)$$

$$+ \frac{1}{8} \left\{ \partial \left( u\overline{v}^{-1} \left( |u|^2 \right) \right) + u\overline{v}^{-1} (u\overline{v}) \right\}$$

$$- \frac{1}{2} \partial^2 u$$
\[ \ell = 3: \]

(A.24) \[ \nu_{2,3}^\# = \frac{1}{128} u\overline{\sigma}^{-1}\left(|u|^2 \overline{\sigma}^{-1}\left(|u|^2\right)\right) \]
\[ - \frac{1}{32} \left\{ u\overline{\sigma}^{-1}\left(\overline{\pi}\partial\left(u\overline{\sigma}^{-1}\left(|u|^2\right)\right)\right) + u\overline{\sigma}^{-1}\left(|u|^2 \overline{\sigma}^{-1}\left(\overline{\pi}\partial u\right)\right) \right\} \]
\[ + \partial \left(u\overline{\sigma}^{-1}\left(|u|^2 \overline{\sigma}^{-1}\left(|u|^2\right)\right)\right) \]
\[ + \frac{1}{8} \left\{ u\overline{\sigma}^{-1}\left(\overline{\pi}\partial^2 u\right) + \partial^2 \left(u\overline{\sigma}^{-1}\left(|u|^2\right)\right) + \partial \left(u\overline{\sigma}^{-1}\left(\overline{\pi}\partial u\right)\right) \right\} \]
\[ - \frac{1}{2} \partial^2 u \]

A.4. Inverse Scattering Method for mNV. We now compute the motion of the putative solution
\[ u = \mathcal{I} r \]
if the reflection coefficient evolves according to the law
\[ \dot{r} = -\left(\kappa^2 - \mathcal{K}^2\right) r \]
\[ r|_{t=0} = \mathcal{R} u_0 \]

From (A.8) it is clear that we must compute \( [\nu_1(z, \phi)\nu_{2,3}^\#(-z, \phi)]_3 \) and \( [\nu_2(z, \phi)\nu_{1,1}^\#(-z, \phi)]_3 \).

First, we have

(A.25) \[ [\nu_1(z, \phi)\nu_{2,3}^\#(-z, \phi)]_3 = \nu_{2,3}^\# + \nu_{2,2} \nu_{1,0} + \nu_{2,1} \nu_{1,1} + \nu_{2,0} \nu_{1,2}. \]

From the formulas above we have

(A.26) \[ \nu_{2,2} \nu_{1,0} = -\frac{1}{128} u\overline{\sigma}^{-1}\left(|u|^2 \overline{\sigma}^{-1}\left(|u|^2\right)\right) \cdot \left(\overline{\sigma}^{-1}\left(\overline{\pi}\partial u\right)\right) \]
\[ + \frac{1}{32} \left\{ \partial \left(u\overline{\sigma}^{-1}\left(|u|^2\right)\right) \cdot \left(\overline{\sigma}^{-1}\left(|u|^2\right)\right) + u\overline{\sigma}^{-1}\left(\overline{\pi}\partial u\right) \cdot \left(\overline{\sigma}^{-1}\left(|u|^2\right)\right) \right\} \]
\[ - \frac{1}{8} \partial^2 u \cdot \overline{\sigma}^{-1}\left(|u|^2\right), \]

(A.27) \[ \nu_{2,1} \nu_{1,1} = \frac{1}{128} u\overline{\sigma}^{-1}\left(|u|^2\right) \cdot \overline{\sigma}^{-1}\left(|u|^2 \overline{\sigma}^{-1}\left(|u|^2\right)\right) \]
\[ - \frac{1}{32} \left\{ u\overline{\sigma}^{-1}\left(|u|^2\right) \cdot \overline{\sigma}^{-1}\left(\overline{\pi}\partial u\right) + \partial u \cdot \left(\overline{\sigma}^{-1}\left(|u|^2 \overline{\sigma}^{-1}\left(|u|^2\right)\right)\right) \right\} \]
\[ + \frac{1}{8} \partial u \cdot \overline{\sigma}^{-1}\left(\overline{\pi}\partial u\right), \]

and

(A.28) \[ \nu_{2,0} \nu_{1,2} = -\frac{1}{128} u\overline{\sigma}^{-1}\left(|u|^2 \overline{\sigma}^{-1}\left(|u|^2\right)\right) \]
\[ + \frac{1}{32} \left\{ u\overline{\sigma}^{-1}\left(\partial \left(u\overline{\sigma}^{-1}\left(|u|^2\right)\right)\right) + u\overline{\sigma}^{-1}\left(|u|^2 \overline{\sigma}^{-1}\left(\overline{\pi}\partial u\right)\right) \right\} \]
\[ - \frac{1}{8} u\overline{\sigma}^{-1}\left(\overline{\pi}\partial^2 u\right). \]
Using (A.24) and (A.26)–(A.28) in (A.25) we see that seventh-order terms cancel, while fifth-order terms sum to zero, as may be shown using the identity

$$\mathcal{D}^{-1} f \cdot \mathcal{D}^{-1} g = \mathcal{D}^{-1} \left( f \mathcal{D}^{-1} g + g \mathcal{D}^{-1} f \right),$$

while third-order terms may be simplified using the same identity with \( f = g \). The result is

$$\nu_1(z, \phi) \bar{\nu}_2(\bar{z}, \phi) = \frac{3}{8} \left[ (\partial u) \cdot \left( \mathcal{D}^{-1} \left( \partial \left( |u|^2 \right) \right) \right) \right] + \frac{3}{8} \left[ u \mathcal{D}^{-1} (\mathcal{D} u) \right] - \frac{1}{2} \partial^3 u$$

Next, we compute

$$\nu_1(z, \phi) \nu_2(\bar{z}, \phi) = \nu_2, + \nu_2,_{1,0} \nu_2^* + \nu_2,_{1,1} + \nu_2,_{1,2}. \nu_2,_{1,0}$$

From the formulas above we have

$$\nu_2,_{1,0} = -\frac{\lambda}{128} \mathcal{D}^{-1} \left( |u|^2 \mathcal{D}^{-1} \left( |u|^2 \right) \right) \cdot \mathcal{D}^{-1} \left( |u|^2 \right)$$

$$+ \frac{1}{32} \left\{ \partial \left( \mathcal{D}^{-1} \left( |u|^2 \right) \right) \cdot \mathcal{D}^{-1} \left( |u|^2 \right) + \mathcal{D}^{-1} \left( \mathcal{D} u \right) \cdot \left( \mathcal{D}^{-1} \left( |u|^2 \right) \right) \right\}$$

$$- \frac{\lambda}{8} \partial^2 \mathcal{D}^{-1} \left( |u|^2 \right),$$

$$\nu_2,_{1,1} = \frac{1}{128} \mathcal{D}^{-1} \left( |u|^2 \right) \cdot \mathcal{D}^{-1} \left( |u|^2 \right)$$

$$- \frac{1}{32} \left\{ \mathcal{D}^{-1} \left( |u|^2 \right) \cdot \mathcal{D}^{-1} \left( \mathcal{D} u \right) + \partial \mathcal{D}^{-1} \left( \mathcal{D} u \right) \right\}$$

$$+ \frac{1}{8} \partial \mathcal{D}^{-1} \left( \mathcal{D} u \right),$$

and

$$\nu_2,_{1,2} = -\frac{1}{128} \mathcal{D}^{-1} \left( |u|^2 \right) \cdot \mathcal{D}^{-1} \left( |u|^2 \right)$$

$$+ \frac{1}{32} \left\{ \mathcal{D}^{-1} \left( \mathcal{D} u \right) \cdot \mathcal{D}^{-1} \left( \mathcal{D} u \right) + \mathcal{D}^{-1} \left( \mathcal{D} u \right) \right\}$$

$$- \frac{1}{8} \mathcal{D}^{-1} \left( \mathcal{D} u \right).$$

Using (A.17) and (A.32)–(A.33) in (A.31), noting the cancellation of fifth-order terms, we obtain

$$\nu_2(z, \phi) \nu_2^*(-z, \phi) = \frac{3}{8} \left[ \mathcal{D}^{-1} \left( \partial \left( \mathcal{D} u \right) \right) \right] + \frac{3}{8} \left( \partial \mathcal{D}^{-1} \left( |u|^2 \right) \right)$$

$$- \frac{1}{2} \partial^3 \mathcal{D}^{-1} \left( \mathcal{D} u \right),$$

or upon complex conjugation

$$\nu_2(z, \phi) \nu_2^*(-z, \phi) = \frac{3}{8} \left[ \mathcal{D}^{-1} \left( \partial \left( \mathcal{D} u \right) \right) \right] + \frac{3}{8} \left( \partial \mathcal{D}^{-1} \left( |u|^2 \right) \right)$$

$$- \frac{1}{2} \partial^3 \mathcal{D}^{-1} \left( \mathcal{D} u \right).$$
Using these equations in \((A.37)\), we obtain the mNV equation:

\[
\frac{\partial u}{\partial t} = -\partial^3 u - \overline{\partial^3 u} + \frac{3}{4} \left( \partial u \right) \cdot \left( \partial^{-1} \left( |u|^2 \right) \right) + \frac{3}{4} \left( \overline{\partial u} \right) \cdot \left( \partial^{-1} \left( |u|^2 \right) \right) + \frac{3}{4} \overline{u} \partial^{-1} \left( \overline{\partial u} \right) + \frac{3}{4} u \partial^{-1} \left( \overline{\partial \partial u} \right).
\]

References

[1] Ablowitz, M. J.; Fokas, A. S. Method of solution for a class of multidimensional nonlinear evolution equations. Phys. Rev. Lett. 51 (1983), no. 1, 7–10.
[2] Ablowitz, M. J.; Fokas, A. S. On the inverse scattering transform of multidimensional nonlinear equations related to first-order systems in the plane. J. Math. Phys. 25 (1984), no. 8, 2494–2505.
[3] Ablowitz, Mark J.; Nachman, Adrian I. Multidimensional nonlinear evolution equations and inverse scattering. Solitons and coherent structures (Santa Barbara, Calif., 1985). Phys. D 18 (1986), no. 1-3, 223–241.
[4] Astala, Kari; Iwaniec, Tadeusz; Martin, Gaven. Elliptic partial differential equations and quasiconformal mappings in the plane. Princeton Mathematical Series, 48. Princeton University Press, Princeton, NJ, 2009.
[5] Astala, Kari; Päivärinta, Lassi. Calderón’s inverse conductivity problem in the plane. Ann. of Math. (2) 163 (2006), no. 1, 265–299.
[6] Barceló, Juan Antonio; Barceló, Tomon; Ruiz, Alberto. Stability of the inverse conductivity problem in the plane for less regular conductivities. J. Differential Equations 173 (2001), no. 2, 231–270.
[7] Beals, R.; Coifman, R. R. Scattering and inverse scattering for first order systems. Comm. Pure Appl. Math. 37 (1984), no. 1, 39–90.
[8] Beals, R.; Coifman, R. R. Multidimensional inverse scatterings and nonlinear partial differential equations. Pseudodifferential operators and applications (Notre Dame, Ind., 1984), 45–70, Proc. Sympos. Pure Math., 43, Amer. Math. Soc., Providence, RI, 1985.
[9] Beals, R.; Coifman, R. R. Linear spectral problems, nonlinear equations and the \(\overline{\partial}\)-method. Inverse Problems 5 (1989), no. 2, 87–130.
[10] Beals, R.; Coifman, R. R. The spectral problem for the Davey-Stewartson and Ishimori hierarchies. In: Degasperis, A, Fordy, Allan P., and Lakshmanan, M., eds., Nonlinear Evolution Equations: Integrability and Spectral Methods. Manchester University Press, 1990.
[11] Ben-Artzi, Matania; Koch, Herbert; Saut, Jean-Claude. Dispersion estimates for third order equations in two dimensions. Comm. Partial Differential Equations 28 (2003), no. 11-12, 1943–1974.
[12] Bennett, Jonathan; Carbery, Anthony; Christ, Michael; Tao, Terence. The Brascamp-Lieb inequalities: finiteness, structure and extremals. Geom. Funct. Anal. 17 (2008), no. 5, 1343–1415.
[13] Bennett, Jonathan; Carbery, Anthony; Christ, Michael; Tao, Terence. Finite bounds for Hölder-Brascamp-Lieb multilinear inequalities. Math. Res. Lett. 17 (2010), no. 4, 647–666.
[14] Bogdanov, L. V. The Veselov-Novikov equation as a natural generalization of the Korteweg-de Vries equation. Theoret. Math. Fiz. 70 (1987), no. 2, 309–314. English translation: Theoret. and Math. Phys. 70 (1987), no. 2, 219–223.
[15] Bogdanov, L. V. On the two-dimensional Zakharov-Shabat problem. Theoret. Math. Fiz. 72 (1987), no. 1, 155–159. English translation: Theoret. and Math. Phys. 72 (1987), no. 1, 790–793.
[16] Boiti, M., Leon, J. P., Manna, M., Pempinelli, F. On a spectral transform of a KdV-like equation related to the Schrödinger operator in the plane. Inverse Problems 3 (1987),25–36.
[17] Brown, R. M. Estimates for the scattering map associated with a two-dimensional first-order system. J. Nonlinear Sci. 11 (2001), no. 6, 459–471.
[18] Brown, Russell M.; Nie, Zhongyi. Estimates for a family of multi-linear forms. Journal of Mathematical Analysis and Applications, 377 (2011), 79-87.
[19] Calderón, A. P. On an inverse boundary value problem. Seminar on Numerical Analysis and Its Applications to Continuum Physics. Soc. Brasileira de Matemática (1980), pp. 65–73.

[20] Chihara, Hiroyuki. Third-order semilinear dispersive equations related to deep water waves. Preprint, arXiv:math/0404005 [math.AP].

[21] Deift, Percy; Zhou, Xin. Long-time asymptotics for solutions of the NLS equation with initial data in a weighted Sobolev space. Dedicated to the memory of Jürgen K. Moser. Comm. Pure Appl. Math. 56 (2003), no. 8, 1029–1077.

[22] Doi, Shin-ichi. On the Cauchy problem for Schrödinger type equations and the regularity of solutions. J. Math. Kyoto Univ. 34 (1994), no. 2, 319–328.

[23] Dubrovsky, V. G.; Gramolin, A. V. Gauge-invariant description of some (2+1)-dimensional integrable nonlinear evolution equations. J. Phys. A 41 (2008), no. 27, 275208.

[24] Dubrovsky, D. G., Gramolin, A. V. Gauge-invariant description of several (2+1)-dimensional integrable nonlinear evolution equations. Theor. Mat. Fiz. 160 (2009), 35–48. English translation in Theor. Math. Phys. 160(1) (2009), 905-916.

[25] Gesztesy, F.; Zhao, Z. On positive solutions of critical Schrödinger operators in two dimensions. J. Funct. Anal. 127 (1995), no. 1, 235–256.

[26] J.-M. Ghidaglia and J.-C. Saut, Sur le problème de Cauchy pour les équations de Davey-Stewartson. C. R. Acad. Sci. Paris Sér. I Math. 308 (1989), no. 4, 115–120.

[27] Grinevich, P. G. Rational solitons of the Veselov-Novikov equations are reflectionless two-dimensional potentials at fixed energy. Teoret. Mat. Fiz. 69 (1986), no. 2, 307–310.

[28] Grinevich, P. G. The scattering transform for the two-dimensional Schrödinger operator with a potential that decreases at infinity at fixed nonzero energy. Uspekhi Mat. Nauk 55 (2000), no. 6(336), 3–70; translation in Russian Math. Surveys 55 (2000), no. 6, 1015–1083.

[29] Grinevich, P. G.; Novikov, R. G. Analogues of multisoliton potentials for the two-dimensional Schrödinger operator. Funktsional. Anal. i Prilozhen. 19 (1985), no. 4, 32–42, 95. English translation: Functional Anal. Appl. 19 (1985), no. 4, 276–285.

[30] Grinevich, P. G.; Novikov, R. G. Analogues of multisoliton potentials for the two-dimensional Schrödinger operator, and a nonlocal Riemann problem. Dokl. Akad. Nauk SSSR 286 (1986), no. 1, 19–22. English translation: Soviet Math. Dokl. 33 (1986), no. 1, 9–12.

[31] Grinevich, P. G.; Novikov, S. P. Inverse scattering problem for the two-dimensional Schrödinger operator at a fixed negative energy and generalized analytic functions. Plasma theory and nonlinear and turbulent processes in physics, Vol. 1, 2 (Kiev, 1987), 58–85, World Sci. Publishing, Singapore, 1988.

[32] Grinevich, P. G.; Manakov, S. V. Inverse problem of scattering theory for the two-dimensional Schrödinger operator, the θ-method and nonlinear equations. Funktsional. Anal. i Prilozhen. 20 (1986), no. 2, 14–24, 96. English translation: Functional Anal. Appl. 20 (1986), no. 2, 94–103.

[33] Grinevich, P. G.; Novikov, S. P. A two-dimensional “inverse scattering problem” for negative energies, and generalized-analytic functions. I. Energies lower than the ground state. Funktsional. Anal. i Prilozhen. 22 (1988), no. 1, 23–33, 96; translation in Funct. Anal. Appl. 22 (1988), no. 1, 19–27.

[34] Grinevich, Piotr G.; Novikov, Roman G. Transparent potentials at fixed energy in dimension two. Fixed-energy dispersion relations for the fast decaying potentials. Comm. Math. Phys. 174 (1995), no. 2, 409–446.

[35] Kappeler, Thomas; Perry, Peter; Shubin, Mikhail; Topalov, Peter. The Miura map on the line. Int. Math. Res. Not. 2005, no. 50, 3091–3133.

[36] Kenig, Carlos E.; Ponce, Gustavo; Vega, Luis. Oscillatory integrals and regularity of dispersive equations. Indiana Univ. Math. J. 40 (1991), no. 1, 33–69.

[37] Knudsen, Kim; Tamasan, Alexandru. Reconstruction of less regular conductivities in the plane. Comm. Partial Differential Equations 29 (2004), no. 3–4, 361–381.

[38] Koch, Herbert; Saut, Jean-Claude. Local smoothing and local solvability for third order dispersive equations. SIAM J. Math. Anal. 38 (2006/07), no. 5, 1528–1541 (electronic).

[39] Lassas, M.; Mueller, J. L.; Siltanen, S. Mapping properties of the nonlinear Fourier transform in dimension two. Comm. Partial Differential Equations 32 (2007), no. 4–6, 591–610.

[40] Lassas, Matti; Mueller, Jennifer L.; Siltanen, Samuli; Stahel, Andreas. The Novikov-Veselov Equation and the Inverse Scattering Method, Part I: Analysis. Preprint, arXiv:1105.3903v1 [math.AP].
[41] Lax, P. On stability of difference schemes: A sharp form of Gårding’s inequality. *Comm. Pure Appl. Math.* **19** (1966), 473–492.

[42] Miura, Robert M. Korteweg-de Vries equation and generalizations. I. A remarkable explicit nonlinear transformation. *J. Mathematical Phys.* **9** (1968) 1202–1204.

[43] Miura, Robert M.; Gardner, Clifford S.; Kruskal, Martin D. Korteweg-de Vries equation and generalizations. II. Existence of conservation laws and constants of motion. *J. Mathematical Phys.* **9** (1968), 1204–1209.

[44] Music, Michael E.; Perry, Peter; Siltanen, Samuli. Singularities of the Scattering Transform and Exceptional Sets for the Two-Dimensional Schrödinger Equation with Radial Potentials. *In preparation.*

[45] Nachman, Adrian I. Global uniqueness for a two-dimensional inverse boundary value problem. *Ann. of Math.* **2** (1996), no. 1, 71–96.

[46] Nižnik, L. P. Integration of multidimensional nonlinear equations by the inverse problem method. *Dokl. Akad. Nauk SSSR* **254** (1980), no. 2, 332–335. English translation: *Soviet Phys. Dokl.* **25** (1980), no. 9, 706–708 (1981).

[47] Veselov, A. P.; Novikov, S. P. Finite-gap two-dimensional potential Schrödinger operators. Explicit formulas and evolution equations. *Dokl. Akad. Nauk SSSR* **279** (1984), no. 1, 20–24. English translation: *Soviet Math. Dokl.* **30** (1984), no. 3, 588–591

[48] Novikov, S. P.; Veselov, A. P. Two-dimensional Schrödinger operator: inverse scattering transform and evolutional equations. Solitons and coherent structures (Santa Barbara, Calif., 1985). *Phys. D* **18** (1986), no. 1-3, 267–273.

[49] Perry, Peter. Global well-posedness and large-time asymptotics for the defocussing Davey-Stewartson II equation in $H^1_1(R^2)$. Preprint, arXiv.math/1110.5589 [math.AP], 2011. Submitted to *Journal of Spectral Theory.*

[50] Sung, Li-Yeng. An inverse scattering transform for the Davey-Stewartson II equations. I, II,III. *J. Math. Anal. Appl.* **183** (1994), no. 1, 121–154; no. 2, 289–325; no. 3, 477–494.

[51] Taı̈manov, I. A.; Tsarëv, S. P. Two-dimensional Schrödinger operators with rapidly decaying rational potential and multidimensional $L^2$-kernel. (Russian) *Uspekhi Mat. Nauk* 62 (2007), no. 3(375), 217–218; translation in *Russian Math. Surveys* 62 (2007), no. 3, 631–633.

[52] Taı̈manov, I. A.; Tsarev, S. P. Blowing up solutions of the Veselov-Novikov equation. (Russian) *Dokl. Akad. Nauk* **420** (2008), no. 6, 744–745; translation in *Dokl. Math.* 77 (2008), no. 3, 467–468.

[53] Taı̈manov, I. A.; Tsarev, S. P. Two-dimensional rational solitons constructed by means of the Moutard transformations, and their decay. (Russian) *Teoret. Mat. Fiz.* 157 (2008), no. 2, 188–207; translation in *Theoret. and Math. Phys.* 157 (2008), no. 2, 1525–1541.

[54] Taı̈manov, I. A.; Tsarev, S. P. On the Moutard transformation and its applications to spectral theory and soliton equations. *Sovrem. Mat. Fundam. Napravl.* 35 (2010), 101–117. English translation: *J. Math. Sci. (N. Y.)* **170** (2010), no. 3, 371–387.

[55] Tsai, T.-Y. The Schrödinger operator in the plane. *Inverse Problems* **9** (1993), no. 6, 763–787.

[56] Vekua, I. N. *Generalized analytic functions.* Pergamon Press, London-Paris-Frankfurt; Addison-Wesley Publishing Co., Inc., Reading, Mass. 1962

[57] Zakharov, V. E.; Shabat, A. B. Exact theory of two-dimensional self-focusing and onedimensional self-modulation of waves in nonlinear media. *Soviet Physics JETP* **34** (1972), no. 1, 62–69; translated from *Zh. Ékspер. Teoret. Fiz.* **61** (1971), no. 1, 118–134.

[58] Zhou, X. $L^2$-Sobolev space bijectivity of the scattering and inverse scattering transforms. *Comm. Pure Appl. Math.* **51** (1998), no. 7, 697–731.