TOPOLOGICAL DESCRIPTION OF THE BOREL PROBABILITY SPACE

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Abstract. Given a topological ambient space $X$, we study properties of some popular topology on the Borel probability space $\mathcal{M}(X)$ in this paper. We show that the two types of vague topology are equivalent to each other in case the ambient space $X$ is LCH. The two types of setwise topology induced from two equivalent descriptions of setwise convergence of sequences of probability measures are also equivalent to each other for any ambient space $X$. We give explicit conditions for the two types of vague topology and the two types of setwise topology to be separable or metrizable on $\mathcal{M}(X)$. These conditions are either in terms of the cardinality of the elementary events in the Borel $\sigma$-algebra $\mathcal{B}$ or some direct topological assumptions on the ambient space $X$. We give the necessary and sufficient condition for families of probability measures to be setwisely relatively compact in case $X$ is a compact metric space. There are some extending questions on the topology of the Borel probability space $\mathcal{M}(X)$ at the end of the work.

1. The weak, setwise and TV topology on $\mathcal{M}(X)$

This work grows out of the following question: Let $X$ be a topological space along with its Borel $\sigma$-algebra $\mathcal{B}$, consider the collection of all the probability measures $\mathcal{M}(X)$ on $(X, \mathcal{B})$. How to describe the topological or even geometric structure of $\mathcal{M}(X)$?

The answer of course depends on one’s aims in the application of the topological or geometric description of $\mathcal{M}(X)$ in their situations. Suitable choice of topology of the probability space $\mathcal{M}(X)$ is delicate, and sometimes decisive in solving one’s problems, especially in case when one considers various continuity or other regularity problems on $\mathcal{M}(X)$. For long time it seems the weak topology is the most popular topology on $\mathcal{M}(X)$, while the solutions of one’s problems due to applications of various other topology in due course show the importance of these topology on $\mathcal{M}(X)$. For example, M. Hellwig [He] introduced the topology of convergence in information on measure spaces in order to deal with game or optimization problems, which is studied further in [BG]. While various other topologies on $\mathcal{M}(X)$ are highlighted in their applications, we limit our attention to the vague, weak, setwise and TV topology in this work.

The vague topology on $\mathcal{M}(X)$ is a coarser topology than the weak topology. It is an important kind of topology on $\mathcal{M}(X)$ in dealing with continuity problems in stochastic processes, for example, in judging the moment convergence of sequences of random variables. We focus on two types of vague topology on $\mathcal{M}(X)$. One is in the sense of Kallenberg [Kallen], Klenke [Kle], Daley and Vere-Jones [DV], another is in the sense of Folland [Fol], Lasserre [Las]. A general notion of vague topology is introduced in [BP] by B. Basrak and H. Planinić, whose property is studied there. See also [EP] by R.

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V. Erickson and V. Fabian for a different version of vague topology on \( \mathcal{M}(X) \). We will show the equivalence of our two focused vague topologies in case of the ambient space \( X \) is LCH. We study some topological properties of \( \mathcal{M}(X) \) under the vague topology in this work, such as its separability and metrizability.

Setwise topology as a finer topology than the weak topology on \( \mathcal{M}(X) \), is a relative new concept, which is getting more and more attention recently. Comparing with the vague and weak convergence of sequences of probability measures, the setwise convergence is a more demanding property. Correspondingly, it also guarantees better properties for convergent sequences of probability measures in this sense. It is utilised by M. Gerlac \([\text{Ger}]\) to describe the strong convergence of the powers of the Perron–Frobenius operators. It is an important tool in analysing the continuity properties of the optimal cost in stochastic kernel in stochastic control theory by T. Linder and S. Yüksel \([\text{LY}]\). It also has important applications in the Markov decision processes, according to E. Feinberg, P. Kasyanov and M. Zgurovsky \([\text{FKL}], [\text{FKZ1}], [\text{FKZ3}]\). The notion is further exploited by the author into the dimensional theory in iterated function systems (IFS), which resolves a long-standing problem of R. Mauldin and M. Urbański in the last 90s \([\text{Ma}]\).

We also define two types of setwise topology on \( \mathcal{M}(X) \), which follow naturally from two equivalent descriptions of setwise convergence of sequences of probability measures. These two types of setwise topology are shown to be equivalent to each other for any topological ambient space \( X \). The separability and metrizability of \( \mathcal{M}(X) \) under the setwise topology are decided by the cardinality of elementary events in the Borel \( \sigma \)-algebra \( \mathcal{B} \). We also consider the setwisely relative compactness of families of probabilities in \( \mathcal{M}(X) \).

As the uniform version of the setwise topology, the TV topology on \( \mathcal{M}(X) \) is metrizable by the explicit TV metric. See its application in \([\text{BHP}], [\text{LY}] \) and \([\text{FKZ1}]\).

There are three notes for the readers. The first one is on the ambient space \( X \). We try to set our results on general topological ambient spaces \( X \), while some results do require some assumptions on the ambient space \( X \) (these assumptions are even crucial in the work or results mentioned above). Notable assumptions are these requiring \( X \) to be LCH, compact, metric, separable, normal or complete.

The second note is that although we pay main attention to the vague and setwise topology on \( \mathcal{M}(X) \) (partly because the weak topology is well-studied in various circumstances), we hope it will shed some light on properties of other topology, as well as comparisons of different types of topology induced from equivalent descriptions of convergence of sequences of probability measures on \( \mathcal{M}(X) \).

The last one is that we confine our attention to Borel probability measures \( \mathcal{M}(X) \) in this work, however, some notions and results extend naturally to (locally) finite measures \( \tilde{\mathcal{M}}(X) \) with \( X \) endowed with other \( \sigma \)-algebra \( \mathcal{A} \), simply by normalization, or even to infinite measures in some cases.

The organization of the paper is as following. In Section 2 we introduce the vague, weak, setwise and TV topology on \( \mathcal{M}(X) \). These definitions are induced naturally from the corresponding convergence of sequences of probability measures in \( \mathcal{M}(X) \). The main results of the work are presented in this section—Theorem 2.5, 2.6, 2.7, 2.16, 2.17, 2.18. Section 3 is devoted to the proof of Theorem 2.5 and 2.16, which compare different types
of vague or setwise topology on $\mathcal{M}(X)$. In Section 4 we prove Theorem 2.6, 2.7 and 2.17, which give conditions for the separability and metrizability of $\mathcal{M}(X)$ under the vague or setwise topology in due course. Section 5 is devoted to the proof of Theorem 2.18 on the setwisely relative compactness of families of probabilities in $\mathcal{M}(X)$. In the last section we indicate some further directions of research on the various topology on $\mathcal{M}(X)$.

2. The vague, weak, setwise, TV topology on $\mathcal{M}(X)$ and the main results

We give a summary of some popular topology on $\mathcal{M}(X)$, namely, the vague, weak, setwise and total-variation (TV) topology in this section. These notions presumably differ from each other in fineness. One can expect more desiring properties for convergent sequences of probability measures under finer topology. For example, the uniform Fatou lemma holds for convergent sequences of finite measures under the TV topology, but it does not hold for convergent sequences of finite measures under the setwise topology [FKZ2].

We merely assume the ambient space $X$ is a topological space with the Borel $\sigma$-algebra $\mathcal{B}$, in general. Some assumptions on the ambient space $X$ are required in order to deduce some particular results. The behaviour of a measure is reflected by the integrals of families of functions on $X$ with respect to it. For integrations on a general measure space, see [Li] or [Tay, Chapter 3]. For the Borel $\sigma$-algebra $\mathcal{B}$, choose a family of measurable sets $\mathcal{B}_b \subseteq \mathcal{B}$ called bounded sets. We highlight the following families of functions in this work.

- $C(X) = \{ f : f$ is a continuous function from $X$ to $\mathbb{R} \}$.
- $C_b(X) = \{ f : f$ is a bounded continuous function from $X$ to $\mathbb{R} \}$.
- $C_s(X) = \{ f : f$ is a continuous function from $X$ to $\mathbb{R}$ with support in $\mathcal{B}_b \}$.
- $M_b(X) = \{ f : f$ is a bounded measurable function from $X$ to $\mathbb{R} \}$.
- $M_b(X) = \{ f : f$ is a measurable function from $X$ to $[-1, 1] \}$.

Obviously we can see that,

$$M_1(X) \cup C_b(X) \subseteq M_b(X) \text{ and } C_{sc}(X) \subseteq C_b(X) \subseteq C(X).$$

2.1. Definition.
A sequence of probability measures $\{\nu_n \in \mathcal{M}(X)\}_{n=1}^{\infty}$ is said to converge vaguely to $\nu \in \mathcal{M}(X)$, if

$$\lim_{n \to \infty} \int_X f(x) d\nu_n = \int_X f(x) d\nu$$

for any $f \in C_s(X)$.

Denote the convergence in this sense by

$$\nu_n \overset{v}{\to} \nu$$

as $n \to \infty$. Refer to [BP] for more discussion on the notion. By choosing different families of bounded sets, it integrates some well-known notions of vague convergence of measures. For example, by choosing $\mathcal{B}_b$ to be all the compact sets, that is, let

$$C_{sc}(X) = \{ f : f$ is a continuous function from $X$ to $\mathbb{R}$ with compact support $\}.$$
it means the vague convergence in \([Kle]\). In case of \(X\) being a separable and complete metric space, by choosing \(B\) to be all the metrically bounded sets, that is, let
\[
C_{sb}(X) = \{ f : f \text{ is a continuous function from } X \text{ to } \mathbb{R} \text{ with bounded support}\},
\]
it means the vague convergence in \([Kallen1]\) and \([DV]\).

2.2. Remark.
This notion differs from the one in \([Fol]\) and \([Las1]\). For \(X\) being a locally compact and Hausdorff (LCH) space, a function \(f : X \to \mathbb{R}\) is said to vanish at infinity if \(f^{-1}((\infty, -\epsilon] \cup [\epsilon, \infty))\) is a compact set in \(X\) for any \(\epsilon > 0\). Consider the family of functions
\[
C_0(X) = \{ f : f \text{ is a continuous function from } X \to \mathbb{R} \text{ vanishing at infinity}\}.
\]
According to \([Fol\ p223]\), A sequence of probability measures \(\{\nu_n \in \mathcal{M}(X)\}_{n=1}^{\infty}\) is said to converge vaguely to \(\nu \in \mathcal{M}(X)\), if
\[
\lim_{n \to \infty} \int_X f(x)d\nu_n = \int_X f(x)d\nu
\]
for any \(f \in C_0(X)\). This definition of vague convergence is obviously a stronger notion that the one in \([Kallen1\ Chapter 4]\) and \([Kle\ Definition 13.12]\) since \(C_{sc}(X) \cup C_{sb}(X) \subset C_0(X)\).

We pay attention to two types of vague topology on \(\mathcal{M}(X)\) in this work, following the above mentioned two notions of vague convergence of sequences of probability measures in \(\mathcal{M}(X)\).

2.3. Definition.
The Type-I vague topology \(\mathfrak{W}_{v1}\) on \(\mathcal{M}(X)\) is the topology with basis
\[
W_{v1}(\nu, f, \epsilon) = \{ \theta \in \mathcal{M}(X) : | \int_X f(x)d\theta - \int_X f(x)d\nu | < \epsilon \}
\]
for any \(f \in C_{sc}(X)\) and any real \(\epsilon > 0\).

2.4. Definition.
The Type-II vague topology \(\mathfrak{W}_{v2}\) on \(\mathcal{M}(X)\) is the topology with basis
\[
W_{v2}(\nu, f, \epsilon) = \{ \theta \in \mathcal{M}(X) : | \int_X f(x)d\theta - \int_X f(x)d\nu | < \epsilon \}
\]
for any \(f \in C_0(X)\) and any real \(\epsilon > 0\).

It is obvious that the Type-II vague topology is finer than Type-I vague topology since \(C_{sc}(X) \subset C_0(X)\). Our first result show that the fineness is not strict in some cases.

2.5. Theorem.
Let \(X\) be an LCH space. Then the Type-II vague topology \(\mathfrak{W}_{v2}\) is equivalent to the Type-I vague topology \(\mathfrak{W}_{v1}\) on \(\mathcal{M}(X)\).

Now rename the space \(\mathcal{M}(X)\) as \(\mathcal{M}_{v1}(X)\) and \(\mathcal{M}_{v2}(X)\) under the topology \(\mathfrak{W}_{v1}\) and \(\mathfrak{W}_{v2}\) respectively. Our next result describes the separability and metrizability of the probability space \(\mathcal{M}(X)\) under the vague topology. A set \(A \in \mathcal{B}\) is called an elementary event if it does not contain any non-empty proper subset \(B \in \mathcal{B}\).
2.6. Theorem.
For an LCH space $X$, $\mathcal{M}_{v1}(X)$ $(\mathcal{M}_{v2}(X))$ is separable and metrizable if the Borel $\sigma$-algebra $\mathcal{B}$ admits at most countably many elementary events.

The conclusion is not true for non-LCH space $X$, see Remark 4.5. It is natural to ask the separability and metrizability of $\mathcal{M}_{v1}(X)$ or $\mathcal{M}_{v2}(X)$ in case the Borel $\sigma$-algebra $\mathcal{B}$ has uncountably many elementary events. We are not able to provide a conclusive answer to this question, even if $X$ is LCH. However, we do have some results when $X$ is a compact metric space. The separability of the probability space under vague topology can be found in [Tao, Exercise 1.10.22.] by Tao, see also Proposition 4.8.

2.7. Theorem.
The probability space $\mathcal{M}_{v1}(X)$ $(\mathcal{M}_{v2}(X))$ is separable and metrizable if the ambient space $X$ is a compact metric space.

Remark. The reader is strongly recommended to [Kallen1, Theorem 4.2] for a stronger result on the separability, metrizability and completeness of the space $\mathcal{M}_{v1}(X)$ $(\mathcal{M}_{v2}(X))$, with $X$ being a separable and complete metric space (a compact metric space is separable and complete). The author find Kallenberg’s exciting result after Theorem 2.7, but I will retain it in its form as the proofs differ from each other completely, which will benefit the reader from different point of views.

Now we turn to a presumably stronger notion of convergence of sequences of probability measures than vague convergence on $\mathcal{M}(X)$.

2.8. Definition.
A sequence of probability measures $\{\nu_n \in \mathcal{M}(X)\}_{n=1}^\infty$ is said to converge weakly to $\nu \in \mathcal{M}(X)$, if

$$\lim_{n \to \infty} \int_X f(x)d\nu_n = \int_X f(x)d\nu$$

for any $f \in C_b(X)$.

One is also recommended to [Bil1, Bil2, Kallen2, Kle, Las2, Mat] for more related research topics on the notion. Denote the convergence in this sense by

$$\nu_n \overset{w}{\to} \nu$$

as $n \to \infty$. There is a bouquet of equivalent descriptions on the convergence of measures in this sense, which is called the Portemanteau Theorem, presumably with metric ambient spaces. We list some of them here, with no inclination to be complete, see [Kle, Theorem 13.16], [Bil1, Theorem 2.1] and [HL1, Theorem 1.4.16].

2.9. Theorem (Portemanteau).
For a sequence of measures $\{\nu_n \in \mathcal{M}(X)\}_{n=1}^\infty$ and $\nu \in \mathcal{M}(X)$ on a metric space $X$, the following conditions are equivalent to each other:

(I). $\nu_n \overset{w}{\to} \nu$ as $n \to \infty$.
(II). $\liminf_{n \to \infty} \nu_n(A) \geq \nu(A)$ for any open set $A \in \mathcal{B}$.
(III). $\limsup_{n \to \infty} \nu_n(A) \leq \nu(A)$ for any closed set $A \in \mathcal{B}$.
(IV). $\lim_{n \to \infty} \int_X f d\nu_n = \int_X f d\nu$ for any bounded Lipschitz continuous function $f$.
(V). $\lim_{n \to \infty} \int_X f d\nu_n = \int_X f d\nu$ for any bounded uniformly continuous function $f$. 
(VI). \( \lim_{n \to \infty} \nu_n(A) = \nu(A) \) for any measurable \( A \) such that \( \nu(\partial A) = 0 \), in which \( \partial A \) is the boundary of the set \( A \subset X \).

More equivalent descriptions can be formulated when setting restrictions on the ambient space \( X \) or the \( \sigma \)-algebra \( \mathcal{B} \). For example, if \( X = \mathbb{R} \), then \( \nu_n \xrightarrow{w} \nu \) is equivalent to that, the sequence of distribution functions \( F_n(x) \) of \( \nu_n \) converges to the distribution function \( F(x) \) of \( \nu \) at every continuity point of \( F(x) \) \cite{Bi1, P1}.

2.10. Definition.
The Type-I weak topology \( W_{w1} \) on \( \mathcal{M}(X) \) is the topology with basis
\[
W_{w1}(\nu, f, \epsilon) = \{ \varrho \in \mathcal{M}(X) : |\int_X f(x) d\varrho - \int_X f(x) d\nu| < \epsilon \}
\]
for any \( f \in C_b(X) \) and any real \( \epsilon > 0 \).

This topology is obviously finer than the vague topology \( W_{v1} \) and \( W_{v2} \) on \( \mathcal{M}(X) \). There is a detailed study of some finer form of weak topology and its various applications in \cite{Kal}. The topology \( W_{w1} \) is metrizable if \( X \) is metrizable, for example, by the Prohorov metric (see \cite{Bi1, Section 6}).

2.11. Definition.
A sequence of probability measures \( \{\nu_n \in \mathcal{M}(X)\}_{n=1}^{\infty} \) is said to converge setwisely to \( \nu \in \mathcal{M}(X) \), if
\[
\lim_{n \to \infty} \nu_n(A) = \nu(A)
\]
for any \( A \in \mathcal{B} \).

See for example \cite{Do4, FKZ1, GR, HL1, Las1, LY}. Denote the convergence in this sense by
\[
\nu_n \xrightarrow{s} \nu
\]
as \( n \to \infty \). Obviously the setwise convergence implies weak convergence of sequences of measures due to (2.1).

2.12. Definition.
The induced topology \( W_{s1} \) with subbasis
\[
W_{s1}(\nu, A, \epsilon) = \{ \varrho \in \mathcal{M}(X) : |\varrho(A) - \nu(A)| < \epsilon \}
\]
with \( A \in \mathcal{B} \) and any real \( \epsilon > 0 \) is called the Type-I setwise topology on \( \mathcal{M}(X) \).

An equivalent way of defining the setwise convergence of measures stems from treating the measure as a functional on the space of bounded Borel-measurable functions on \( X \), similar to the way in defining weak convergence.

2.13. Definition.
A sequence of probability measures \( \{\nu_n \in \mathcal{M}(X)\}_{n=1}^{\infty} \) is said to converge setwisely to \( \nu \in \mathcal{M}(X) \), if
\[
\lim_{n \to \infty} \int_X f(x) d\nu_n = \int_X f(x) d\nu
\]
for any \( f \in M_b(X) \).
This is because the simple functions are dense among the bounded Borel-measurable functions on $X$ under the supremum norm. In this way one can also define a topology $\mathcal{W}_{s2}$ on $\mathcal{M}(X)$ as following.

2.14. Definition.
The Type-II setwise topology $\mathcal{W}_{s2}$ on $\mathcal{M}(X)$ is the topology with basis

$$W_{s2}(\nu, f, \epsilon) = \{ \mu \in \mathcal{M}(X) : |\int_X f(x) d\mu - \int_X f(x) d\nu| < \epsilon \}$$

for any $f \in \mathcal{M}_b(X)$ and any real $\epsilon > 0$.

Similar to the Portemanteau Theorem regarding the equivalent descriptions on the weak convergence of measures in $\mathcal{M}(X)$, we have the following setwise Portemanteau Theorem according to Feinberg, Kasyanov and Zgurovsky ([FKZ1, Theorem 2.3]).

2.15. Theorem (Feinberg-Kasyanov-Zgurovsky).
For a sequence of measures $\{\nu_n \in \mathcal{M}(X)\}_{n=1}^{\infty}$ and $\nu \in \mathcal{M}(X)$ with a metric space $X$, the following conditions are equivalent to each other:

(I). $\nu_n \overset{s}{\to} \nu$ as $n \to \infty$.
(II). $\lim_{n \to \infty} \nu_n(A) = \nu(A)$ for any open set $A \in \mathcal{B}$.
(III). $\lim_{n \to \infty} \nu_n(A) = \nu(A)$ for any closed set $A \in \mathcal{B}$.

There are some sufficient conditions to guarantee setwise convergence of measures within certain contexts by J. Lasserre, see [Las1, Lemma 4.1(ii)]. See also the Vitali-Hahn-Saks Theorem [Doo, P31, P155] or [HL2] on setwisely convergent sequences of measures in $\mathcal{M}(X)$. Due to the equivalence of Definition 2.11 and 2.13, it is alluring whether the two types of topology $\mathcal{W}_{s1}$ and $\mathcal{W}_{s2}$ are equivalent to each other on $\mathcal{M}(X)$.

2.16. Theorem.
The Type-I setwise topology $\mathcal{W}_{s1}$ is always equivalent to the Type-II setwise topology $\mathcal{W}_{s2}$ on the probability space $\mathcal{M}(X)$.

Note that two kinds of topology admit the same convergent sequences does not guarantee they are equivalent to each other, since there are examples of spaces with inequivalent topology which admit the same convergent sequences (as one can refer to a classical example by J. Schur [Sch]). Especially one needs to be careful about the case when the two kinds of setwise topology are not metrizable (refer to Theorem 2.17).

In the following we pay attention to the separability and metrizability of the probability space $\mathcal{M}(X)$ under the setwise topology. Rename the space $\mathcal{M}(X)$ as $\mathcal{M}_{s1}(X)$ equipped with the topology $\mathcal{W}_{s1}$, and $\mathcal{M}_{s2}(X)$ equipped with the topology $\mathcal{W}_{s2}$ on $\mathcal{M}(X)$. Part of the following result is due to J. K. Ghosh and R. V. Ramamoorthi [GR, Proposition 2.2.1].

2.17. Theorem.
The topological space $\mathcal{M}_{s1}(X)$ ($\mathcal{M}_{s2}(X)$) is separable or metrizable if and only if the Borel $\sigma$-algebra $\mathcal{B}$ admits at most countably many elementary events.

Now we consider setwisely relative compactness of families of probability measures in $\mathcal{M}(X)$. A family of probability measures $\Xi \subset \mathcal{M}(X)$ is said to be setwisely relatively
compact if there is a setwisely convergent subsequence for every sequence of probability measures in $\Xi$. We give a necessary and sufficient condition for families of probability measures to be setwisely relatively compact when the ambient space $X$ is a compact metric space.

2.18. Theorem.
For a compact metric space $X$, a family of probability measures $\Xi \subset \mathcal{M}(X)$ is setwisely relatively compact if and only if for any sequence of probability measures $\{\nu_n\}_{n=1}^{\infty} \subset \Xi$, there is a subsequence $\{\nu_{n_i}\}_{i=1}^{\infty}$, such that

$$
\limsup_{i \to \infty} \nu_{n_i}(U) = \sup_{K \subset U, K \text{ closed}} \limsup_{i \to \infty} \nu_{n_i}(K)
$$

for any open set $U \subset X$.

The last topology we consider in this work is the total-variation (TV) topology on $\mathcal{M}(X)$. It is induced by the total-variation (TV) metric on $\mathcal{M}(X)$.

2.19. Definition.
The total variation metric is defined by

$$
\|\nu - \varrho\|_{TV} = \sup_{A \in \mathcal{B}} \{|\nu(A) - \varrho(A)|\}
$$

between two measures $\nu, \varrho \in \mathcal{M}(X)$.

One is recommended to [BHP, Doo, FZ1, HL1, PS] for more research topics related with the TV topology on $\mathcal{M}(X)$. Denote by

$$\nu_n \xrightarrow{TV} \nu$$

as $n \to \infty$ for sequences converging in this sense. On comparing the setwise topology with the TV topology on $\mathcal{M}(X)$, the Type-I (Type-II) setwise topology is coarser than the TV topology, and this comparison is strict in some cases. In fact, there are examples of sequences of probability measures converging under the setwise topology but diverging under the TV topology on compact metric spaces.

3. Comparison of the vague topology $\mathcal{W}_{v1}$ with $\mathcal{W}_{v2}$ and the setwise topology $\mathcal{W}_{s1}$ with $\mathcal{W}_{s2}$ on $\mathcal{M}(X)$

This section is devoted to the proof of Theorem 2.15 and Theorem 2.16 on comparison of different types of vague topology and setwise topology. To prove Theorem 2.15, we need some suitable continuous approximation for characteristic functions of compact sets in LCH spaces. The following result is a locally compact version of Urysohn’s Lemma (refer to [Fol, p131]).

3.1. Urysohn’s Lemma.
Let $X$ be an LCH space. For any compact set $K \subset X$, let $U \supseteq K$ be its open neighbourhood, then there exists a continuous map $g_K : X \to [0, 1]$ with compact support $\text{Supp}(g_K) \subset U$, such that

$$g_K(x) = 1$$

for any $x \in K$. 

Now we recall the following separation axioms in general topological spaces.

- A topological space $X$ is called **Hausdorff** or $T_2$ if for any two points $x, y \in X$, there are two disjoint open sets containing $x$ and $y$ respectively.
- A topological space $X$ is called **regular** if for any point $x \in X$ and any closed set $C \subset X$ not containing $x$, there are two disjoint open sets containing $x$ and $C$ respectively.
- A topological space $X$ is called **$T_3$** if it is both Hausdorff and regular.

Note that some people use the terminology 'regular' for 'regular and Hausdorff' ($T_3$) spaces. A topological space $X$ is called **second-countable** if its topology admits a countable basis. The equivalence of the Type-I and Type-II vague topology on $\mathcal{M}(X)$ with an LCH space $X$ attributes essentially to the Urysohn’s Lemma.

**Proof of Theorem 2.5**

> **Proof.** It is enough for us to show that the Type-I vague topology is finer than the Type-II vague topology in case of $X$ being an LCH space. For a measure $\nu \in \mathcal{M}(X)$, consider its neighbourhood $W_{\epsilon,2}(\nu, f, \epsilon)$ for some continuous function $f : X \to \mathbb{R}$ vanishing at infinity and some $\epsilon > 0$ under the Type-II vague topology. Consider the following two sets in $X$,

$$K = f^{-1}\left((\infty, -\frac{\epsilon}{4}] \cup [\frac{\epsilon}{4}, \infty)\right)$$

and

$$U = f^{-1}\left((\infty, -\frac{\epsilon}{8}] \cup (\frac{\epsilon}{8}, \infty)\right).$$

Obviously $K \subset U$ while $K$ is compact and $U$ is open. So according to the Urysohn’s Lemma, there exists a continuous map $g_K : X \to [0,1]$ with compact support $\text{Supp}(g_K) \subset U$, such that

$$g_K(x) = 1$$

for any $x \in K$. Let $g = f \cdot g_K$. It is a continuous map on $X$ with compact support. We claim that the neighbourhood

$$W_{\epsilon,1}(\nu, g, \frac{\epsilon}{8}) \subset W_{\epsilon,2}(\nu, f, \epsilon).$$

To see this, for any probability measure $\varrho \in W_{\epsilon,1}(\nu, g, \frac{\epsilon}{8})$, we have

$$\left| \int_X f(x)d\varrho - \int_X f(x)d\nu \right| = \left| \int_X f(x) - g(x)d\varrho + \int_X g(x)d\varrho - \int_X g(x)d\nu - \int_X f(x) - g(x)d\nu \right|$$

$$\leq \left| \int_X f(x) - g(x)d\varrho \right| + \left| \int_X g(x)d\varrho - \int_X g(x)d\nu \right| + \left| \int_X f(x) - g(x)d\nu \right|$$

$$= \left| \int_{K^c} (1-g_K(x))f(x)d\varrho \right| + \left| \int_X g(x)d\varrho - \int_X g(x)d\nu \right| + \left| \int_K^c (1-g_K(x))f(x)d\nu \right|$$

$$< \frac{3\epsilon}{8},$$
in which $K' = X \setminus K$ is the residual set. (3.1) guarantees the Type-I vague topology is finer than Type-II vague topology with LCH ambient spaces.

It would be an interesting question to ask whether there exists some none-LCH space $X$, such that the Type-II vague topology $W_v^2$ is strictly finer than the Type-I vague topology $W_v^1$ on $M(X)$. We can not give an example of a none-LCH space $X$ for which the strict fineness holds between the two types of vague topology, but we do have an example of a none-LCH space $X$ on which the Urysohn’s Lemma fails.

3.2. Example.

Let $P$ be a collection of polynomials with $n$ variables over an algebraically closed field $F$. Let

$$\mathbb{A}^n = F \times F \times \cdots \times F$$

be the affine space. Suppose the affine variety decided by $P$ in $\mathbb{A}^n$ is not empty. Consider the Zariski topology on the affine space $\mathbb{A}^n$.

$\mathbb{A}^n$ is a compact space under the Zariski topology but it is not Hausdorff in Example 3.2. Moreover, the Urysohn’s Lemma fails on $\mathbb{A}^n$ in this case since any continuous map on $\mathbb{A}^n$ is sure to be a constant map. However, the two types of vague topology are still equivalent to each other on $M(\mathbb{A}^n)$ with the Zariski topology on $\mathbb{A}^n$.

Since Urysohn’s Lemma holds on normal spaces, the following result can be proved in a similar way as Theorem 2.5.

3.3. Corollary.

The Type-II vague topology $W_v^2$ is equivalent to the Type-I vague topology $W_v^1$ on $M(X)$ with $X$ being a normal space.

In the following we show the equivalence of the two types of setwise topology on $M(X)$.

Proof of Theorem 2.16

Proof. Obviously the topology $W_{s2}$ is finer than $W_{s1}$ since any simple function $1_A$ associated with a measurable set $A \in \mathcal{B}$ is a bounded measurable function. In the following we show the inverse is also true.

For a probability measure $\nu \in M(X)$, a function $f \in M_b(X)$ and a small $\epsilon > 0$, consider the neighbourhood $W_{s2}(\nu, f, \epsilon)$ of $\nu$. We will find an open neighbourhood $U_\nu$ of $\nu$ under $W_{s1}$, such that $U_\nu \subset W_{s2}(\nu, f, \epsilon)$, this is enough to justify the Type-I setwise topology $W_{s1}$ is finer than Type-II setwise topology $W_{s2}$.

First, choose an integer $N \in \mathbb{N}$ large enough such that

$$\frac{4\|f\|_\infty}{N} < \epsilon.$$

For $1 \leq i \leq N - 1$, let

$$A_i = f^{-1}\left([-\|f\|_\infty + \frac{2(i-1)\|f\|_\infty}{N}, -\|f\|_\infty + \frac{2i\|f\|_\infty}{N}]\right).$$

Let
\[ A_N = f^{-1}\left([\|f\|_\infty - \frac{2\|f\|_\infty}{N}, \|f\|_\infty]\right)\].

Note that \( A_i \in \mathcal{B} \) for any \( 1 \leq i \leq N \) since \( f \) is measurable, and \( \cup_{1 \leq i \leq N} A_i \) is a disjoint partition of the ambient space \( X \). Now for every \( 1 \leq i \leq N \), consider the open neighbourhood \( W_{s1}(\nu, A_i, \frac{\epsilon}{\|f\|_\infty N}) \) of \( \nu \). Let

\[ U_\nu = \cap_{1 \leq i \leq N} W_{s1}(\nu, A_i, \frac{\epsilon}{\|f\|_\infty N}) \]

be the open neighbourhood of \( \nu \) under \( \mathfrak{M}_{s1} \). We claim that

(3.2) \[ U_\nu \subset W_{s2}(\nu, \epsilon) \]

To see this, for any probability measure \( \varrho \in U_\nu \), compare the integration of \( f \) with respect to \( \nu \) and \( \varrho \) over \( X \), we have

\[
\begin{align*}
\leq & \left| \int_X f d\nu - \int_X f d\varrho \right| \\
\leq & \sum_{1 \leq i \leq N} \left| \int_{A_i} f d\nu - \int_{A_i} f d\varrho \right| \\
\leq & \max \left\{ \sum_{1 \leq i \leq N} \left| \left( -\|f\|_\infty + \frac{2\|f\|_\infty}{N} \right) \nu(A_i) - \left( -\|f\|_\infty + \frac{2\|f\|_\infty}{N} \right) \varrho(A_i) \right|, \\
& \sum_{1 \leq i \leq N} \left| \left( -\|f\|_\infty + \frac{2\|f\|_\infty}{N} \right) \nu(A_i) - \left( -\|f\|_\infty + \frac{2\|f\|_\infty}{N} \right) \varrho(A_i) \right| \right\} \\
\leq & \sum_{1 \leq i \leq N} \left| \left( -\|f\|_\infty + \frac{2\|f\|_\infty}{N} \right) (\nu(A_i) - \varrho(A_i)) \right| + \frac{2\|f\|_\infty}{N} \\
< & \sum_{1 \leq i \leq N} \frac{\epsilon}{\|f\|_\infty N} + \frac{2\|f\|_\infty}{N} \\
= & \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
= & \epsilon.
\end{align*}
\]

3.4. Remark.

Be careful that the type-I setwise topology is defined by a basis in Definition 2.12 while the type-II setwise topology is defined by a subbasis in Definition 2.14 on \( \mathcal{M}(X) \).

Due to Theorem 2.16 we will usually not distinguish the two types of setwise topology in some cases. However, technically, it is more convenient to resort to one of the two types of setwise topology than the other in due course. These also apply to the two types of vague topology on \( \mathcal{M}(X) \) with LCH spaces \( X \) in virtue of Theorem 2.5.

4. Separability and metrizability of the probability space \( \mathcal{M}(X) \) under the vague topology and setwise topology

This section is devoted to separation properties of the probability space \( \mathcal{M}(X) \) under the vague topology and setwise topology, especially the separability and metrizability. We aim to prove Theorem 2.6, 2.7 and Theorem 2.17 in this section. Our strategy is to prove Theorem 2.17 first by establishing some separation and countability properties of the probability space \( \mathcal{M}_{s1}(X) \) (\( \mathcal{M}_{s2}(X) \)). Since the setwise topology is finer than the vague topology, some separation and countability properties of the probability space \( \mathcal{M}_{s1}(X) \) (\( \mathcal{M}_{s2}(X) \)) are inherited naturally by the space \( \mathcal{M}_{s1}(X) \) (\( \mathcal{M}_{s2}(X) \)) in some cases. These properties are applied to the proof of Theorem 2.6 and 2.7.

To prove Theorem 2.17 we need several preceding results on the separation and countability properties of the topological space \( \mathcal{M}_{s1}(X) \) (\( \mathcal{M}_{s2}(X) \)). In virtue of Theorem
all the separation and countability properties are shared by the two spaces $\mathcal{M}_{s1}(X)$ and $\mathcal{M}_{s2}(X)$.

4.1. Lemma.
The topological space $\mathcal{M}_{s1}(X)$ ($\mathcal{M}_{s2}(X)$) is Hausdorff.

Proof. Without loss of generality, suppose that $X$ is not endowed with the trivial topology $\{\emptyset, X\}$. Now for two probability measures $\nu, \varrho \in \mathcal{M}_{s1}(X)$, if $\nu \neq \varrho$, there must exist some $A \in \mathcal{B}$, such that $\nu(A) \neq \varrho(A)$. Without loss of generality suppose $\nu(A) > \varrho(A)$.

Then we have

$$\nu \in W_{s1}(\nu, A, \frac{\nu(A) - \varrho(A)}{4}) \text{ and } \varrho \in W_{s1}(\varrho, A, \frac{\nu(A) - \varrho(A)}{4})$$

while

$$W_{s1}(\nu, A, \frac{\nu(A) - \varrho(A)}{4}) \cap W_{s1}(\varrho, A, \frac{\nu(A) - \varrho(A)}{4}) = \emptyset.$$ 

4.2. Lemma.
The topological space $\mathcal{M}_{s2}(X)$ ($\mathcal{M}_{s1}(X)$) is regular.

Proof. Let $\nu \in \mathcal{M}_{s2}(X)$ and $\Xi_1 \subset \mathcal{M}_{s2}(X)$ be a closed set such that $\nu \notin \Xi_1$. So the residual set $\Xi'_1 = \mathcal{M}_{s2}(X) \setminus \Xi_1$ is an open set such that $\nu \in \Xi'_1$. Then there must exist $f \in M_b(X)$ and $\epsilon > 0$, such that

$$W_{s2}(\nu, f, \epsilon) \subset \Xi'_1.$$

Since

$$W_{s2}(\nu, f, \frac{\epsilon}{4}) \subset W_{s2}(\nu, f, \frac{\epsilon}{2}) \text{ and the closure } \overline{W_{s2}(\nu, f, \epsilon/2)} \subset W_{s2}(\nu, f, \epsilon),$$

we have

$$\nu \in W_{s2}(\nu, f, \frac{\epsilon}{4}) \text{ while } C \subset \overline{W_{s2}(\nu, f, \epsilon/2)}$$

and

$$W_{s2}(\nu, f, \frac{\epsilon}{4}) \cap \overline{W_{s2}(\nu, f, \epsilon/2)} = \emptyset.$$ 

4.3. Lemma.
The topological space $\mathcal{M}_{s1}(X)$ ($\mathcal{M}_{s2}(X)$) is second-countable if the $\sigma$-algebra $\mathcal{B}$ has at most countably many elementary events.

Proof. In case of $\# \mathcal{B}$ being finite, let

$$\{A_1, A_2, \cdots, A_n\} \subset \mathcal{B}$$
be the collection of all the elementary events. Then the set of probability measures
\[
\{ \nu : \nu(A_i) \in \mathbb{Q}, 0 \leq \nu(A_i) \leq 1 \text{ for any } 1 \leq i \leq n \text{ and } \sum_{1 \leq i \leq n} \nu(A_i) = 1 \}
\]
is a dense subset of \( \mathcal{M}_{s1}(X) \). Every measure in the set has a countable neighbourhood basis, which can be used to build a countable basis of \( \mathcal{M}_{s1}(X) \).

Now suppose \( \mathcal{B} \) has countably many elementary events \( \{ A_i \}_{i=1}^\infty \). Consider the collection of finite unions of these elementary events,
\[
\{ B_F = \bigcup_{i \in F} A_i \}_{F \subset \mathbb{N}, \#F < \infty}.
\]
It is a countable set. We claim that the countable set of measures
\[
\prod_1 = \bigcup_{F \subset \mathbb{N}, \#F < \infty, j \in (\mathbb{N} \setminus F)} \{ \nu : \nu(B_F) \in \mathbb{Q} \cap [0, 1], \nu(A_j) = 1 - \nu(B_F) \}
\]
is a countable dense subset in \( \mathcal{M}_{s1}(X) \) (one has the freedom to adjust mass on the elementary events in \( B_F \), but we take only one such measure with respect to individual \( B_F \) in \( \prod_1 \)).

To see this, let \( \varrho \in \mathcal{M}_{s1}(X) \) be a probability measure. For any measurable set \( A \in \mathcal{B} \) and any \( \epsilon > 0 \), consider the neighbourhood \( W_{s1}(\varrho, A, \epsilon) \). Without loss of generality suppose \( A \neq X \) in the following. Now if \( A = B_F \) for some \( F \subset \mathbb{N} \) and \( \#F < \infty \), obviously there exists some \( \nu \in \prod_1 \), such that \( \nu \in W_{s1}(\varrho, A, \epsilon) \). If
\[
A = \bigcup_{i=1}^\infty A_{n_i}
\]
for some \( \{ n_i \}_{i=1}^\infty \subset \mathbb{N} \), then there exists \( k \in \mathbb{N} \) large enough, such that
\[
0 < \varrho(A) - \varrho(\bigcup_{i=1}^k A_{n_i}) < \frac{\epsilon}{4}.
\]
Let \( F^* = \{ n_i \}_{i=1}^k \), so \( B_{F^*} = \bigcup_{i=1}^k A_{n_i} \). Then there exists \( \nu_{F^*} \in \prod_1 \), such that
\[
|\nu_{F^*}(B_{F^*}) - \varrho(B_{F^*})| < \frac{\epsilon}{4}
\]
and
\[
\nu_{F^*}(A \setminus B_{F^*}) = 0.
\]
So
\[
|\nu_{F^*}(A) - \varrho(A)| = |\nu_{F^*}(B_{F^*}) - \varrho(B_{F^*}) + \nu_{F^*}(A \setminus B_{F^*}) - \varrho(A \setminus B_{F^*})|
\]
\[
\leq |\nu_{F^*}(B_{F^*}) - \varrho(B_{F^*})| + |\nu_{F^*}(A \setminus B_{F^*}) - \varrho(A \setminus B_{F^*})|
\]
\[
= |\nu_{F^*}(B_{F^*}) - \varrho(B_{F^*})| + |\varrho(A) - \varrho(\bigcup_{i=1}^k A_{n_i})|
\]
\[
< \epsilon/2.
\]
This implies \( \nu_{F^*} \in W_{s1}(\varrho, A, \epsilon) \), and thus justifies our claim.

To see the topological space \( \mathcal{M}_{s1}(X) \) is second countable, let
\[
\mathcal{B}_s = \{ W_{s1}(\nu, A_i, \frac{\epsilon}{4^j}) : \nu \in \prod_1, 1 \leq i, j < \infty \}
\]
be the countable family of open sets. The proof that it can serve as a subbase of the topological space \( \mathcal{M}_{s1}(X) \) is left to the keen readers.
Now we are in a position to prove Theorem 2.17.

Proof of Theorem 2.17

Proof. If the Borel $\sigma$-algebra $\mathcal{B}$ admits at most countably many elementary events, then there is a dense subset of $\mathcal{M}_{s1}(X)$ according to the proof of Lemma 4.3. If the $\sigma$-algebra $\mathcal{B}$ has uncountably many elementary events, consider the collection of all open neighbourhoods of the Dirac measures $\{\delta_A : A \text{ is an elementary event in } \mathcal{B}\}$. One can see that

$$\{W_{s1}(\delta_A, A, 1/2) : A \text{ is an elementary event in } \mathcal{B}\}$$

is an uncountably disjoint family of open sets. So $\mathcal{M}_{s1}(X)$ is not separable in this case.

As to the metrization, if $\mathcal{B}$ has at most countably many elementary events, considering Lemma 4.1, 4.2 and 4.3, $\mathcal{M}_{s1}(X)$ is metrizable by the Urysohn Metrization Theorem [Mun Theorem 34.1]. If the $\sigma$-algebra $\mathcal{B}$ has uncountably many elementary events, it supports a continuous measure, which does not admit a countable neighbourhood basis according to [GR Proposition 2.2.1(ii)], so $\mathcal{M}_{s1}(X)$ is not metrizable in this case. ■

Now we turn to the proof of Theorem 2.6. Again we first establish some separation properties of the spaces $\mathcal{M}_{v1}(X)$ and $\mathcal{M}_{v2}(X)$.

4.4. Lemma.
The topological space $\mathcal{M}_{v1}(X)$ (or $\mathcal{M}_{v2}(X)$) is Hausdorff if $X$ is LCH.

Proof. For two probability measures $\nu, \varrho \in \mathcal{M}_{v1}(X)$, if $\nu \neq \varrho$, according to the Riesz Representation Theorem (see for example [Kallen3 Theorem 2.22] or [Rud Theorem 2.14]), there exists some $f \in C_{sc}(X)$, such that

$$\int_X f d\nu \neq \int_X f d\varrho.$$

Without loss of generality suppose

$$\int_X f d\nu < \int_X f d\varrho.$$

Now let $\epsilon = \frac{\int_X f d\varrho - \int_X f d\nu}{4}$. One can easily check that

$$W_{v1}(\nu, f, \epsilon) \text{ and } W_{v1}(\varrho, f, \epsilon)$$

are two disjoint open neighbourhoods of $\nu$ and $\varrho$ respectively. ■

4.5. Remark.
Lemma 4.4 needs not to be true without the assumption of $X$ being LCH. For example, consider the affine space $\mathbb{A}^n$ endowed with the Zariski topology on it in Example 3.2. Since any continuous map on $\mathbb{A}^n$ is a constant map, then

$$\int_{\mathbb{A}^n} f d\varrho - \int_{\mathbb{A}^n} f d\nu = 0$$

for any $f \in C(\mathbb{A}^n)$ and any two probability measures $\nu, \varrho \in \mathcal{M}(\mathbb{A}^n)$. In this case a neighbourhood $W_{v1}(\nu, f, \epsilon)$ of $\nu$ is always the whole space $\mathcal{M}(\mathbb{A}^n)$ for any $f \in C(\mathbb{A}^n)$ and $\epsilon > 0$. 
4.6. Lemma.
The topological space $\mathcal{M}_{v1}(X)$ ($\mathcal{M}_{v2}(X)$) is regular.

Proof. The proof of Lemma 4.2 applies in these cases, with the role of the measurable function $f \in M_b(X)$ substituted by a continuous function $f \in C_{sc}(X)$ (or $f \in C_0(X)$). ■

Note that Lemma 4.6 holds for any topological space $X$ instead of only for LCH spaces, comparing with Lemma 4.4.

4.7. Lemma.
The probability spaces $\mathcal{M}_{v1}(X)$ and $\mathcal{M}_{v2}(X)$ are second-countable if the $\sigma$-algebra $\mathcal{B}$ admits at most countably many elementary events.

Proof. First, if the $\sigma$-algebra $\mathcal{B}$ has at most countably many elementary events, one can check that the set

$$\prod_2 = \bigcup_{F \in \mathcal{C}, \#F < \infty, j \in (\mathbb{N} \setminus F)} \left\{ \nu : \nu(B_i) \in \mathbb{Q} \cap [0, 1] \text{ for any } i \in F \cup \{j\} \text{ and } \sum_{i \in F \cup \{j\}} \nu(A_i) = 1 \right\}$$

is still a dense subset in $\mathcal{M}_{v1}(X)$ and $\mathcal{M}_{v2}(X)$, by a similar argument as in the proof of Lemma 4.3. Note that the setwise topology $\mathcal{W}_{v1}$ is finer than the Type-I and Type-II vague topology on $\mathcal{M}(X)$, so $\mathcal{M}_{v1}(X)$ and $\mathcal{M}_{v2}(X)$ are both second-countable in this case. ■

Now we are in a position to prove Theorem 2.6.

Proof of Theorem 2.6.
The separability follows from the fact that $\prod_2$ is a countable dense subset of $\mathcal{M}_{v1}(X)$ and $\mathcal{M}_{v2}(X)$. The metrizability follows from a combination of Lemma 4.4, 4.6 and 4.7 in virtue of the Urysohn Metrization Theorem. ■

Now we go towards our final goal in this section—the proof of Theorem 2.7. Together with Theorem 2.17, these results provide some incisive distinctions between the vague (weak) topology and the setwise topology on the probability space $\mathcal{M}(X)$ for a compact metric space $X$. A locally compact and $\sigma$-compact metric space $X$ admits a countable dense subset, say

$$X_d = \{x_1, x_2, \ldots\}.$$ 

Let $\Xi_2 \subset \mathcal{M}(X)$ be the collection of all the discrete probability measures supported on finite points in $X_d$. One can easily check that $\Xi_2$ is separable under either of the two types of vague topology on it.

4.8. Proposition (Tao).

If $X$ is a locally compact and $\sigma$-compact metric space endowed with a metric $\rho$, then $\Xi_2$ is a dense subset of $\mathcal{M}(X)$ under the Type-I or Type-II vague topology.

Proof. It suffices for us to show that $\Xi_2$ is a dense subset of $\mathcal{M}(X)$ under the Type-II vague topology. Consider a probability measure $\nu \in \mathcal{M}(X)$. First, for any $\epsilon > 0,$
since $X$ is a locally compact and \( \sigma \)-compact metric space, there exists some compact set $X_c \subset X$, such that

\[
\nu(X \setminus X_c) < \epsilon.
\]

Without loss of generality we assume $X_c \neq X$. Since $X_d$ is dense in $X$ and the residual set $X'_c$ of the compact (closed) set $X_c$ is open, we choose some point $x_1 \in X_d \cap X'_c$. Let $I \subset \mathbb{N}$ be the collection of all indexes such that $x_i \in X_c$ for $i \in I$. For any $f \in C_0(X)$ and any $\epsilon > 0$, there exists some $\delta > 0$ independent of $\epsilon$, such that

\[|f(x) - f(y)| < \epsilon\]

for any $\rho(x, y) < \delta$ and $x, y \in X_c$. For any $x \in X$ and $r > 0$, let $B(x, r)$ be the open ball in $X$ centred at $x$ with radius $r$. Let $N_1 \in \mathbb{N}$ be large enough such that

\[\frac{2}{N_1} < \delta.
\]

Since $\bigcup_{i \in I} B(x_i, \frac{1}{N_1})$ covers $X_c$ and $X_c$ is compact, there exists a collection of finite indexes

\[I_{N_2} = \{i_1, i_2, \ldots, i_{N_2}\} \subset I
\]

for some $N_2 \in \mathbb{N}$, such that $\bigcup_{i \in I_{N_2}} B(x_i, \frac{1}{N_1})$ covers $X_c$. Note that for any $i \in I_{N_2}$ we have

\[\max_{x \in B(x_i, \frac{1}{N_1}) \cap X_c} |f(x) - \min_{x \in B(x_i, \frac{1}{N_1}) \cap X_c} f(x)| < \epsilon\]

considering (4.1).

Now define a discrete measure $\nu_\epsilon \in \Xi_2$ supported on $\bigcup_{i \in I_{N_2}} x_i \cup \{x_1\}$ as following.

\begin{itemize}
  \item $\nu_\epsilon(\{x_{i_1}\}) = \nu(B(x_{i_1}, \frac{1}{N_1}) \cap X_c)$.
  \item $\nu_\epsilon(\{x_{i_2}\}) = \nu\left(B(x_{i_2}, \frac{1}{N_1}) \cap X_c \setminus B(x_{i_1}, \frac{1}{N_1})\right)$.
  \item $\nu_\epsilon(\{x_{i_3}\}) = \nu\left(B(x_{i_3}, \frac{1}{N_1}) \cap X_c \setminus \bigcup_{j=1}^{2} B(x_{i_j}, \frac{1}{N_1})\right)$.
  \item \ldots
  \item $\nu_\epsilon(\{x_{i_{N_2}}\}) = \nu\left(B(x_{i_{N_2}}, \frac{1}{N_1}) \cap X_c \setminus \bigcup_{j=1}^{N_2-1} B(x_{i_j}, \frac{1}{N_1})\right)$.
  \item $\nu_\epsilon(\{x_1\}) = \nu(X \setminus X_c)$.
\end{itemize}

Compare the integration of $f$ with respect to $\nu$ and $\nu_\epsilon$ over $X$, we have

\[
\begin{align*}
\int_X f \, d\nu &- \int_X f \, d\nu_\epsilon \\
&= \int_{X_c} f \, d\nu - \int_{X_c} f \, d\nu_\epsilon + \int_{X \setminus X_c} f \, d\nu - \int_{X \setminus X_c} f \, d\nu_\epsilon \\
&= \int_{B(x_{i_1}, \frac{1}{N_1}) \cap X_c} f \, d(\nu - \nu_\epsilon) + \int_{B(x_{i_2}, \frac{1}{N_1}) \cap X_c \setminus B(x_{i_1}, \frac{1}{N_1})} f \, d(\nu - \nu_\epsilon) + \cdots \\
&\quad + \int_{B(x_{i_{N_2}}, \frac{1}{N_1}) \cap X_c \setminus \bigcup_{j=1}^{N_2-1} B(x_{i_j}, \frac{1}{N_1})} f \, d(\nu - \nu_\epsilon) + \int_{X \setminus X_c} f \, d(\nu - \nu_\epsilon) \\
&\leq \epsilon \nu(\{x_{i_1}\}) + \epsilon \nu(\{x_{i_2}\}) + \cdots + \epsilon \nu(\{x_{i_{N_2}}\}) + 2\epsilon \|f\|_\infty \\
&\leq (1 + 2\|f\|_\infty)\epsilon.
\end{align*}
\]

This means that

\[\nu_\epsilon \in W_{\epsilon^2}(\nu, f, (1 + 2\|f\|_\infty)\epsilon),\]

which implies $\Xi_2$ is a dense subset of $\mathcal{M}(X)$ under the Type-II vague topology. \(\blacksquare\)
Equipped with Proposition 4.8, we are ready to prove Theorem 2.7.

Proof of Theorem 2.7

Proof. First note that if $X$ is a compact metric space, then according to Theorem 2.5, it suffices for us to prove the separability and metrizability of either space $M_{v1}(X)$ or $M_{v2}(X)$. If $X$ is a compact metric space, all the probability measures in $M(X)$ are Radon measures. A compact metric space is of course locally compact and $\sigma$-compact, so the conclusion of separability of $M_{v1}(X)$ follows directly from Proposition 4.8.

In the following we show $M_{v1}(X)$ is metrizable in case $X$ is a compact metric space. Considering Lemma 4.4 and Lemma 4.6, in virtue of the Urysohn Metrization Theorem, it suffices for us to show $M_{v1}(X)$ is second-countable. Let $\Pi_3 \subset \Xi_2$ be a countable dense subset of $M_{v1}(X)$. Since

$$C(X) = C_{sc}(X) = C_0(X),$$

on any compact metric space $X$, according to [Tao, Proposition 1.10.20.], $C_{sc}(X)$ is separable. Let $\Xi_3 \subset C_{sc}(X)$ be a countable dense subset (note that [Tao, Proposition 1.10.20.] actually asserts that $C_{sc}(X)$ is separable under the infinite norm on it, this obviously implies the separability under the $L^1$ norm on it). One can easily check that the following collection of open sets is a countable basis under $W_{v1}$,

$$\bigcup_{\nu\in\Pi_3, f\in\Xi_3, n\in\mathbb{N}} W_{v1}(\nu, f, \frac{1}{n}).$$

This justifies that $M_{v1}(X)$ is second-countable.

Theorem 2.7 together with Theorem 2.17 have some interesting applications to the probability spaces on some popular ambient compact metric spaces.

4.9. Corollary.

Let $X = [0, 1]$ or $X = B(0, 1) \subset \mathbb{R}^n$ be endowed with the Euclidean metric on it. Then the probability space $M(X)$ is separable and metrizable under the Type-I or Type-II vague topology, while it is not separable and not metrizable under the Type-I or Type-II setwise topology.

5. Relative compactness of families of probabilities in $M(X)$

In this section we deal with relative compactness of families of probability measures in $M(X)$, especially under the setwise topology. It seems to us that a general condition for families of probability measures to be relatively compact is difficult when we merely assume the ambient space $X$ is a topological space, so we decide to limit our attention to metric ambient spaces in this section.

We first recall some results on relative compactness of families of probability measures in $M(X)$ under the vague and weak topology. In a similar way as we define setwisely relative compactness, we can define vague, weak or TV relative compactness of families of probability measures in $M(X)$. A family $\Xi$ of probability measures in $M(X)$ is said to be tight if for any small $\epsilon > 0$, there exists some compact set $K \subset X$ such that

$$\nu(K) > 1 - \epsilon$$
for any \( \nu \in \Xi \). In case of \( X \) being a metric space, Prohorov gave the following condition for weakly relative compactness of families of probability measures in \( \mathcal{M}(X) \), see [Bill, Theorem 5.1, Theorem 5.2].

5.1. Prohorov’s Theorem.
Let \( X \) be a metric space. If a family of probability measures \( \Xi \subset \mathcal{M}(X) \) is tight, then it is weakly relatively compact. Conversely, if \( X \) is a separable and complete metric space, then \( \Xi \) is tight if it is weakly relatively compact.

For the vaguely relative compactness of families of locally finite measures, see [Kallen1, Theorem 4.2].

As to the setwisely relative compactness of a family \( \Xi \subset \mathcal{M}(X) \), the condition of tightness is obviously inadequate, even if we require \( X \) to be of the best topological space in our consideration.

5.2. Example.
Let \( X = [0, 1] \) endowed with the Euclidean metric on it. Let
\[
\Xi_4 = \{\delta_{\frac{1}{n}}\}_{n=1}^\infty
\]
be the sequence of Dirac measures supported on \( \{\frac{1}{n}\} \) for an individual \( n \in \mathbb{N} \).

In the above example the ambient space \( X \) is a compact, separable, complete metric space, so \( \Xi_4 \) is tight, while one can not find any setwisely convergent subsequence in \( \Xi_4 \).

The reason of obstacle for the appearance of a setwisely convergent subsequence in \( \Xi_4 \) is that there are fractures of mass transportation between some open sets and their closed (compact) subsets when taking limit(sup) along the sequence, that is, for the open set \( U = (0, 1) \subset X \) in Example 5.2, we have
\[
\limsup_{i \to \infty} \delta_{\frac{1}{n_i}}(U) = 1 > \sup_{K \subset U, K \text{ is closed}} \limsup_{i \to \infty} \delta_{\frac{1}{n_i}}(K) = 0
\]
for any subsequence \( \{n_i\}_{i=1}^\infty \subset \mathbb{N} \). This inspires us the condition \( 5.8 \) on setwisely relative compactness of families of probability measures in \( \mathcal{M}(X) \).

5.3. Lemma.
A compact metric space \( X \) admits a countable base \( \mathcal{B}_c \) such that if \( x \in U \) for some open set \( U \subset X \), then there is some \( B \in \mathcal{B}_c \) such that
\[
x \in B \subset \bar{B} \subset U,
\]
in which \( \bar{B} \) is the closure of \( B \).

Proof. First, since a compact metric space is second-countable, we can find a countable base \( \mathcal{B}_{c_1} \). Then according to [Bill, p237, Theorem], there is a countable collection \( \mathcal{B}_{c_2} \) of open sets such that if \( x \in U \) for some open set \( U \subset X \), then there is some \( B \in \mathcal{B}_{c_2} \) such that
\[
x \in B \subset \bar{B} \subset U.
\]
Then the countable base \( \mathcal{B}_c = \mathcal{B}_{c_1} \cup \mathcal{B}_{c_2} \) satisfies the requirement of the lemma. ■
Proof. First, if a family of probability measures $\Xi$ is setwisely relatively compact, then for any sequence $\{\nu_n\}_{n=1}^{\infty} \subset \Xi$, we can find a subsequence $\{\nu_{n_i}\}_{i=1}^{\infty}$, such that
\[ \nu_{n_i} \xrightarrow{\text{s}} \nu \]
as $i \to \infty$ for some probability measure $\nu \in M(X)$. Since
\[ \lim_{i \to \infty} \nu_{n_i}(K) = \nu(K) \quad \text{and} \quad \lim_{i \to \infty} \nu_{n_i}(U) = \nu(U) \]
for any open set $U \subset X$ and closed (any closed set is compact since we are assuming the ambient space $X$ is compact now) set $K \subset U$, we have
\[ \lim_{i \to \infty} \nu_{n_i}(U) = \sup_{K \subset U} \nu_{n_i}(K) \]
as $\nu$ is a regular measure on the metric space $X$. This of course guarantees that the subsequence $\{\nu_{n_i}\}_{i=1}^{\infty}$ satisfies (5.8).

Now we show the inverse is also true. Suppose for any sequence of probability measures $\{\nu_n\}_{n=1}^{\infty} \subset \Xi$, we can find a subsequence $\{\nu_{n_i}\}_{i=1}^{\infty}$ satisfying (5.8) for any open set $U \subset X$. Upon the technique in constructing a weakly convergent sequence of probability measures in proving the Prohorov’s Theorem [Bill, p60], we will find a setwisely convergent subsequence of the sequence $\{\nu_{n_i}\}_{i=1}^{\infty}$. Since $X$ is a compact metric space, it is second-countable. In virtue of Lemma 5.3, let $B_c = \{B_i\}_{i=1}^{\infty}$ be a countable basis such that if $x \in U$ for some open set $U \subset X$, there is some $j \in \mathbb{N}$ such that
\[ x \in B_j \subset \bar{B}_j \subset U, \]
in which $\bar{B}_j$ is the closure of $B_j$. Let
\[ \bar{B}_c = \{B : B = \bigcup_{j=1}^{n} \bar{B}_{i_j} \text{ with } \{i_j\}_{j=1}^{n} \subset \mathbb{N}\}. \]
Now consider the sequence $\{\nu_{n_j}\}_{j=1}^{\infty}$ satisfying (5.8). Since $\bar{B}_c$ is countable, we can find a subsequence $\{\nu_{n_{i_j}}\}_{j=1}^{\infty}$ of $\{\nu_{n_j}\}_{j=1}^{\infty}$, such that $\lim_{j \to \infty} \nu_{n_{i_j}}(B)$ exists for any $B \in \bar{B}_c$. Then we can find a probability measure $\nu \in M(X)$, such that
\[ \nu(U) = \sup_{B \in U, B \in \bar{B}_c} \lim_{j \to \infty} \nu_{n_{i_j}}(B) \leq \liminf_{j \to \infty} \nu_{n_{i_j}}(U) \]
for any open set $U \subset X$. We claim now that the limit of the sequence $\lim_{j \to \infty} \nu_{n_{i_j}}(U)$ exists for any open set $U \subset X$, moreover, we have
\[ \nu(U) = \lim_{j \to \infty} \nu_{n_{i_j}}(U) \]
for any open set $U \subset X$. This is enough to justify
\[ \nu_{n_{i_j}} \xrightarrow{\text{s}} \nu \]
as $j \to \infty$ in virtue of Theorem 2.15 which completes the proof. To show the claim, note that the sequence $\{\nu_{n_{i_j}}\}_{j=1}^{\infty}$ satisfies
\[ \limsup_{j \to \infty} \nu_{n_{i_j}}(U) = \sup_{K \subset U, K \text{ is closed}} \limsup_{j \to \infty} \nu_{n_{i_j}}(K) \]
for any open set $U \subset X$. Since any metric space is normal [Tao], for any closed set $K \subset U$, we can find an open set $\bar{U}_K$, such that $K \subset U_K \subset \bar{U}_K \subset U$. As $U_K$ can be written as an union of sets in $\mathcal{B}_c$ and $K$ is compact, then $\bar{U}_K$ can be written as a finite
union of sets in $B_c$ whose closures are all in $U$. This means that there exists some $B \in \bar{B}_c$ such that $K \subset B$, which guarantees that

$$\sup_{B \subset U, B \in B_c} \lim_{j \to \infty} \nu_{n_j}(B) = \sup_{K \subset U, K \text{ is closed}} \limsup_{j \to \infty} \nu_{n_j}(K)$$

for any open set $U \subset X$. Now combining (5.1), (5.3) and (5.4) together, we have

$$\limsup_{j \to \infty} \nu_{n_j}(U) = \nu(U) \leq \liminf_{j \to \infty} \nu_{n_j}(U)$$

for any open set $U \subset X$, which implies our claim and (5.2).

It seems that checking the condition (5.8) holding for any open $U \subset X$ is a rather tough job in Theorem 2.18, in fact, we only need to check it holds for countably many open $U \subset X$.

5.4. Corollary.
For a compact metric space $X$, there exists a countable collection $B_{cf}$ of open sets in $X$, such that a family of probability measures $\Xi \subset M(X)$ is setwisely relatively compact if and only if for any sequence of probability measures $\{\nu_n\}_{n=1}^{\infty} \subset \Xi$, there is a subsequence $\{\nu_{n_i}\}_{i=1}^{\infty}$, such that (5.8) holds for any $U \in B_{cf}$.

Proof. It suffices for us to show the sufficiency. In case of $X$ being a compact metric space, let $B_{c1}$ be a countable base. Now let $B_{cf} = \{B: B \text{ is a finite union of sets in } B_{c1}\}$.

$B_{cf}$ is a countable set. Now suppose that for any sequence of probability measures $\{\nu_n\}_{n=1}^{\infty} \subset \Xi$, (5.8) holds for some subsequence $\{\nu_{n_i}\}_{i=1}^{\infty}$ on any open set in $B_{cf}$. Since $B_{cf}$ is countable, we can find a subsequence $\{\nu_{n_{i_j}}\}_{j=1}^{\infty}$ of $\{\nu_{n_i}\}_{i=1}^{\infty}$, such that (5.8) exists for any $B \in B_{cf}$. For any open set $U \subset X$, let

$$U = \bigcup_{j=1}^{\infty} A_j$$

with $A_j \in B_{c1}$ for any $j \in \mathbb{N}$. One can show that

$$\limsup_{i \to \infty} \nu_{n_i}(U) = \lim_{m \to \infty} \limsup_{i \to \infty} \nu_{n_i}(\bigcup_{j=1}^{m} A_j).$$

Note that $\bigcup_{j=1}^{m} A_j \in B_{cf}$. According to the assumption, we have

$$\lim_{m \to \infty} \limsup_{i \to \infty} \nu_{n_i}(\bigcup_{j=1}^{m} A_j) = \lim_{m \to \infty} \sup_{K \subset \bigcup_{j=1}^{m} A_j, K \text{ is closed}} \limsup_{i \to \infty} \nu_{n_i}(K) \leq \sup_{K \subset U, K \text{ is closed}} \limsup_{i \to \infty} \nu_{n_i}(K).$$

Now combining (5.5), (5.6) together with the following obvious fact

$$\limsup_{i \to \infty} \nu_{n_i}(U) \geq \sup_{K \subset U, K \text{ is closed}} \limsup_{i \to \infty} \nu_{n_i}(K),$$

we justify (5.8) holds for the subsequence $\{\nu_{n_i}\}_{i=1}^{\infty}$ and any open set $U \subset X$. This finishes the proof in virtue of Theorem 2.18. ■
One is recommended to compare Corollary 5.4 with [FKZ1, Theorem 3.3.] and [FKZ1, Lemma 3.4.].

Theorem 2.18 and Corollary 5.4 of course can be applied to judge setwise convergence of sequences of probability measures in due course, with the aid of the following result.

5.5. Lemma.
For any topological space $X$, a sequence of probability measures $\{\nu_n\}_{n=1}^{\infty} \subset \mathcal{M}(X)$ converges vaguely, setwisely or TV to some $\nu \in \mathcal{M}(X)$ as $n \to \infty$ if and only if every subsequence $\{\nu_{n_i}\}_{i=1}^{\infty}$ of $\{\nu_n\}_{n=1}^{\infty}$ contains a further subsequence $\{\nu_{n_{ij}}\}_{j=1}^{\infty}$, such that $\{\nu_{n_{ij}}\}_{j=1}^{\infty}$ converges vaguely, setwisely or TV to the probability measure $\nu$ as $j \to \infty$ respectively.

Proof. The necessity is obvious, while the sufficiency of the result follows from a simple argument of reduction to absurdity, see [Bill, Theorem 2.6.]. \hfill \blacksquare

Theorem 2.18 together with Lemma 5.5 indicate the following result on judging the setwise convergence of sequences of probability measures in $\mathcal{M}(X)$ with the ambient space being a compact metric space.

5.6. Corollary.
Let $X$ be a compact metric space. Now if for a sequence of probability measures $\{\nu_n\}_{n=1}^{\infty} \subset \mathcal{M}(X)$, there is a subsequence $\{\nu_{n_i}\}_{i=1}^{\infty}$, such that

$$ \lim_{i \to \infty} \sup_{K \subset \partial A, K \text{ is closed}} \nu_{n_i}(K) = \sup_{K \subset \partial A, K \text{ is closed}} \lim_{i \to \infty} \nu_{n_i}(K) $$

for any open set $U \subset X$, and every setwisely convergent subsequence converges to the same probability measures $\nu \in \mathcal{M}(X)$, then

$$ \nu_n \overset{s}{\rightarrow} \nu $$

as $n \to \infty$.

6. SOME FURTHER DISCUSSIONS ON THE TOPOLOGY ON $\mathcal{M}(X)$

In this section we intend to make a discussion on more kinds of topology on $\mathcal{M}(X)$, their relationships and their topological properties. We will formulate some open problems on some further research topics.

One concern is on the comparison between the various topology on $\mathcal{M}(X)$. In the above sections we make a comparison between the the two types of vague topology, as well as the two types of setwise topology on the probability space $\mathcal{M}(X)$. Similar questions can be asked on the other kinds of topology on $\mathcal{M}(X)$. For example, considering the Portemanteau theorem (Theorem 2.9), we define the following three types of weak topology on $\mathcal{M}(X)$ with the ambient space $X$ being a general topological space.

6.1. Definition.
The Type-II weak topology $\mathfrak{W}_{w2}$ on $\mathcal{M}(X)$ is the topology with subbasis

$$ W_{w2}(\nu, A, \epsilon) = \{\varrho \in \mathcal{M}(X) : |\varrho(A) - \nu(A)| < \epsilon \} $$

with $A \in \mathcal{B}$, $\nu(\partial A) = 0$ and any real $\epsilon > 0$.

6.2. Definition.
The Alexandrov topology $\mathfrak{W}_{w3}$ on $\mathcal{M}(X)$ is the topology with subbasis
\[ W_{w3}(\nu, A, \epsilon) = \{ \varrho \in \mathcal{M}(X) : \varrho(A) > \nu(A) - \epsilon \} \]

for any open set \( A \in \mathcal{B} \) and any real \( \epsilon > 0 \).

Refer to [Ale], [Bla] and [Kal] for the Alexandrov topology on \( \mathcal{M}(X) \).

6.3. Definition.
The Type-IV weak topology \( W_{w4} \) on \( \mathcal{M}(X) \) is the topology with basis
\[ W_{w4}(\nu, f, \epsilon) = \{ \varrho \in \mathcal{M}(X) : | \int_X f(x) d\varrho - \int_X f(x) d\nu | < \epsilon \} \]

for any bounded Lipschitz continuous function \( f \) and any real \( \epsilon > 0 \).

6.4. Problem.
How is the comparison of the fineness between the Type-I, Type-II, the Alexandrov and Type-IV weak topology on \( \mathcal{M}(X) \) with \( X \) being a topological space?

Another concern is on the properties of \( \mathcal{M}(X) \) under various topology, for example, its separation properties, metrizability or completeness. Our Theorem 2.6, Theorem 2.7 together with [Kallen1, Theorem 4.2] provide some conditions on the separability and metrizability of the vague topology on \( \mathcal{M}(X) \) with certain assumptions on the ambient space. A remaining problem is the following one.

6.5. Problem.
In case that the ambient space \( X \) is not a separable and complete metric space but its Borel \( \sigma \)-algebra \( \mathcal{B} \) admits uncountably many elementary events, is the probability space \( \mathcal{M}_{v1}(X) \) or \( \mathcal{M}_{v2}(X) \) separable or metrizable?

It is well-known that the Type-II weak topology on the probability space \( \mathcal{M}(X) \) can be induced by the Prohorov metric if the ambient space \( X \) is a separable and complete metric space, see [Bil1, p72]. According to Theorem 2.17 the probability space \( \mathcal{M}(X) \) is metrizable in case the Borel \( \sigma \)-algebra \( \mathcal{B} \) admits at most countably many elementary events, then it is natural to try to give an explicit metric which induces the setwise topology on \( \mathcal{M}(X) \) in this cases.

6.6. Problem.
In the context of Theorem 2.17, can one give an explicit metric which induces the setwise topology in case the probability space \( \mathcal{M}(X) \) is metrizable? How is its relationship with the Prohorov metric and the TV metric on \( \mathcal{M}(X) \)?

We have got a necessary and sufficient condition on setwisely relative compactness of families of probabilities in \( \mathcal{M}(X) \) with the ambient space \( X \) being a compact metric space. The question on non-compact metric space or even non-metrizable space is still open.

6.7. Problem.
Can one give a necessary and sufficient condition for a family of probability measures in \( \mathcal{M}(X) \) to be setwisely relatively compact in case the ambient space \( X \) is a non-compact metric space or even a non-metrizable space?
We suspect that on a non-compact metric space, tightness of the family together with existence of a subsequence satisfying (5.8) on any open $U \subset X$ for any sequence in the family may be sufficient for setwisely relative compactness of the family?

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