BOUNDARY COMBINATORICS OF ORTHOGONAL MODULAR 4-FOLDS

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Abstract. We study combinatorial problems related to the singularities and boundary components of toroidal compactifications of orthogonal modular varieties. In particular, those associated with the moduli of algebraic deformation generalised Kummer 4-folds.

1. Introduction

Toroidal compactifications of modular varieties provide a rich source of combinatorics. Two natural problems are to count boundary components, and to describe the singular locus. Both of these problems have been studied in detail for Siegel modular 3-folds [HKW91, HKW93], but there are fewer results for orthogonal modular varieties. In particular, while there are some results for orthogonal modular varieties of large dimension [Sca87], less is known about those of small dimension. The purpose of this article is to study these problems for orthogonal modular 4-folds $F_{2d}$ associated with the moduli of deformation generalised Kummer varieties of dimension 4, with polarisation of split type, and degree $2d$, where $d = p^2$ for an odd prime $p$.

In Theorem 5.7 we produce bounds for the number of boundary curves of $F_{2p^2}$. In Theorem 5.12 we bound the number of components of the singular locus of a toroidal compactification $F_{2p^2}$ tor in the neighbourhood of a boundary curve. To the best of the author’s knowledge, these are the first such results for $F_{2d}$, and the first such results for orthogonal modular 4-folds.

2. Notation

$(\bigoplus_i(\frac{\cdot}{\cdot}), \bigoplus_i C_a_i)$ The finite quadratic form $\bigoplus_i(\frac{\cdot}{\cdot})$ (with values in $\mathbb{Q}/2\mathbb{Z}$) on the abelian group $\bigoplus_i C_a_i$.

$\chi_X$ The characteristic polynomial of the matrix $X$.

$C_r$ The cyclic group of order $r$.

$\text{div}(x)$ The divisor of $x \in M$ for a lattice $M$.

(i.e. the positive generator of the ideal $(x, M)$.)

$D(M)$ The discriminant group of the (even) lattice $M$ [Nik79].

$D_N$ A connected spinor component of the domain $\Omega_N = \{[x] \in \mathbb{P}(N \otimes \mathbb{C}) \mid (x, x) = 0, (x, \bar{x}) > 0\}$ for a lattice $N$ of signature $(2, n)$.

$F_{2d}$ The orthogonal modular variety $D_{h^\perp}/O^+(L, h)$ where $h \in L$ is primitive, of split type, and degree $2d$.

$\hat{\Gamma}$ The intersection $\Gamma \cap \hat{O}(M)$ for $\Gamma \subset O(M)$.

$\Gamma^+$ The intersection $\Gamma \cap O^+(M)$ for $\Gamma \subset O(M)$.

$h^\perp$ The orthogonal complement $h^\perp \subset L$ for $h \in L$.

$L$ The lattice $U^{\oplus 3} \oplus (-2(n + 1))$.

$L_{2x, 2y}$ The lattice $U^{\oplus 2} \oplus (-2x) \oplus (-2y)$.

$M^\dagger$ The dual of the lattice $M$.

$O(L, h)$ The group $\{g \in O(L) \mid gh = h\}$ for $h \in L$.

$O(M)$ The orthogonal group of a lattice $M$.

$O^+(M)$ The kernel of the real spinor norm on $O(M \otimes \mathbb{R})$.

$\hat{O}(M)$ The stable orthogonal group $\hat{O}(M) = \{g \in O(M) \mid g|D(M) = id\}$.

$\phi$ The Euler $\varphi$-function.

$\Phi_r$ The $r$-th cyclotomic polynomial.

2010 Mathematics Subject Classification: Primary 14G35; Secondary 14M27.

Key words and phrases: orthogonal modular variety; generalised Kummer variety; toroidal compactification.
3. Moduli of Deformation Generalised Kummer varieties

Let $A$ be an abelian surface, and let $A^{[n+1]}$ be the Hilbert scheme parametrising $(n+1)$-points on $A$. The Hilbert scheme $A^{[n+1]}$ inherits an addition from $A$, and so there is a projection

$$p : A^{[n+1]} \rightarrow A.$$  

The fibre $p^{-1}(0)$ is an irreducible symplectic (compact hyperkähler) manifold known as a generalised Kummer variety [Bea83]. A deformation $X$ of $p^{-1}(0)$ is an irreducible symplectic manifold known as a deformation generalised Kummer variety.

There is a lattice structure $L$ on $H^2(X,\mathbb{Z})$ defined by the Beauville-Bogomolov-Fujiki [Bea83] form. By the results of Rapagnetta [Rap08], $L$ is equal to

$$L = U^{\mathbb{R}} \oplus \langle -2(n+1) \rangle,$$

where $\langle -2(n+1) \rangle$ is the rank 1 lattice generated by a vector of square length $-2(n+1)$ and $U$ is the hyperbolic plane. A basis $\{e,f\}$ for $U$ is said to be standard or a standard basis for $U$ if its Gram matrix is given by

$$
\begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix}.
$$

A choice of ample line bundle $L \in \text{Pic}(X)$ defines a polarisation for $X$. The first Chern class of $L$ defines a vector $h := c_1(L) \in L$. The degree $2d$ of $L$ is defined as the square length $h^2$, and the polarisation type of $L$ is defined as the $O(L)$-orbit of $h$. We assume all polarisations are primitive; that is, the vector $h$ is primitive in the lattice $L$.

Let $O^+(L,h)$ be the subgroup of $O(L,h)$ consisting of all elements of spinor norm 1, and let $D_{h^\perp}$ be a connected component of the quadric

$$\Omega_{h^\perp} = \{ [x] \in \mathbb{P}(h^\perp \otimes \mathbb{C}) \mid (x,x) = 0, (x,\overline{x}) > 0 \}$$

preserved by the kernel of the real spinor norm on $O(h^\perp \otimes \mathbb{R})$.

There is a GIT quotient $M$ parametrising deformation generalised Kummer varieties of fixed dimension and polarisation type. By Theorem 3.8 of [GHS13], for every connected component $M'$ of $M$ there is a finite-to-one dominant morphism $\psi : M' \rightarrow \mathcal{F}$ to an orthogonal modular variety $\mathcal{F}$ (i.e. a quotient of a Hermitian symmetric domain of type IV by an arithmetic subgroup of $O(2,m)$). Here, the orthogonal modular variety $\mathcal{F}$ is given by

$$\mathcal{F} = O^+(L,h) \backslash D_{h^\perp}.$$  

**Proposition 3.1.** Suppose $h \in L$ is primitive of length $2d > 0$ with $\text{div}(h) = f$. Let $g = (2(n+1)f^{-1},2df^{-1})$, $w = (g,f)$, $g = w(g)$, and $f = w(f)$. Then $2(n+1) = fg_n = w^2 f_1 g_1 n_1$ and $2d = fg_d = w^2 f_1 g_1 d_1$, where $(n_1,d_1) = (f_1,g_1) = 1$.

1. If $g_1$ is even, then there exists and only if $(d_1,f_1) = (f_1,n_1) = 1$ and $d_1/n_1$ is a quadratic residue modulo $f_1$. Moreover, the number of $O(L)$-orbits of $h$ with fixed $f$ is equal to $w_+(f_1) \phi(w_-(f_1)) 2^{\rho(f_1)}$, where $w = w_+(f_1)w_-(f_1), w_+(f_1)$ is the product of all powers of primes dividing $(w,f_1)$, $\rho(n+1)$ is the number of prime factors of $n+1$, and $\phi$ is the Euler function.

2. If $g_1$ is odd and $f_1$ is even, or $f_1$ and $d_1$ are both odd, then such an $h$ exists if and only if $(d_1,f_1) = (t_1,2f_1) = 1$ and $-d_1/n_1$ is a quadratic residue modulo $2f_1$. The number of $O(L)$-orbits is equal to $w_+(f_1) \phi(w_-(f_1)) 2^{\rho(f_1)/2}$ if $f_1$ is even, and to $w_+(f_1) \phi(w_-(f_1)) 2^{\rho(f_1)}$ if $f_1$ and $d_1$ are both odd.

3. If $g_1$ and $f_1$ are both odd and $d_1$ is even, then such an $h$ exists if and only if $(d_1,f_1) = (n_1,2f_1) = 1$, $-d_1/(4f_1)$ is a quadratic residue modulo $f_1$, and $w$ is odd. In such a case, the number of $O(L)$-orbits of $h$ is equal to $w_+(f_1) \phi(w_-(f_1)) 2^{\rho(f_1)}$.

4. If $c \in \mathbb{Z}$ (determined modulo $f$) satisfies $(c,f) = 1$ and $b = (d+c^2(n+1))/f^2$, then

$$h^\perp \cong 2U \oplus B,$$

where

$$B = \begin{pmatrix}
-2b & c^{2(n+1)}/f \\
c^{2(n+1)}/f & -2t
\end{pmatrix}.$$
Proof. Identical to Proposition 3.6 of [GHS10]. (Noting that the Beauville lattice of a deformation generalised Kummer variety and an irreducible symplectic manifold of $K3^{[2(n+1)]}$-type differ by a factor of $2E_8(-1)$.)

Definition 3.2. A polarisation determined by a primitive vector $h \in L$ is said to be split, or $h$ is said to be split, if $\text{div}(h) = 1$.

Corollary 3.3. If $h \in L$ is split then,

(1) the polarisation type of $h$ is uniquely determined by the length $h^2$;

(2) the lattice $h^±$ is isomorphic to $L_{2(n+1),2d} := 2U \oplus (-2(n-1)) \oplus (-2d)$.

Definition 3.4. Let $F_{2d}$ denote the modular variety $F_{2d} = O^+(L,h) \setminus D_{h^±}$ where $h \in L$ is split and $h^2 = -2d$.

4. The group $O(L,h)$

From now on, we assume all polarisations are split and fix $2(n+1) = 6$. Throughout, we shall use $h_x$ to denote split $h \in L$ of degree $x$.

Proposition 4.1. If $h = h_{2d}$ and $d > 2$, then

$$O(L,h) \cong \{ g \in O(L_{6,2d}) \mid g^{v*} = v^* + L_{6,2d} \},$$

where $v$ generates the $(-2d)$ factor of $L_{6,2d}$ and $v^* = (2d)^{-1}v \in L^\vee$. Moreover, if $d = p^2$ for an odd prime $p$, then $O(L,h) \leq O(L_{6,2d})$.

Proof. (c.f. Proposition 3.12 of [GHS10].) By noting that $O(L,h)$ acts on both $\langle h \rangle$ and $\langle h^\perp \rangle$, but trivially on $\langle h \rangle$, we can immediately identify $O(L,h)$ with a subgroup of $O(L_{6,2d})$.

There is an inclusion of abelian groups

$$L/\langle h \rangle \bigoplus \langle h^\perp \rangle \subset \langle h^\vee \rangle/\langle h \rangle \bigoplus (\langle h^\perp \rangle)^\vee/\langle h^\perp \rangle = D(\langle h \rangle) \bigoplus D(\langle h^\perp \rangle)$$

defined by the series of overlattices

$$\langle h \rangle \bigoplus h^\perp \subset L \subset L^\vee \subset \langle h^\vee \rangle \bigoplus (h^\perp)^\vee,$$

and so the isotropic subgroup $H = L/\langle (h) \bigoplus h^\perp \rangle$ can be regarded as a subgroup of $D(\langle h \rangle) \bigoplus D(\langle h^\perp \rangle)$. Let $p_h$ and $p_{h^\perp}$ denote the corresponding projections $p_h : H \to D(\langle h \rangle)$, $p_{h^\perp} : H \to D(\langle h^\perp \rangle)$. By Proposition 3.1 we can assume that $h$ is given by $h = e_3 + df_3 \in U \oplus \langle -6 \rangle$. Let $k'_1 = (2d)^{-1}k_1$, $k'_2 = 6^{-1}k_2$, $k'_3 = (2d)^{-1}h$, and $k_1 = e_3 - df_3$, where $k_2$ is a generator of the $\langle -6 \rangle$ factor of $L$. Take a basis $\{ e_1, f_1, e_2, f_2, k'_1, k'_2 \}$ for $(h^\perp)^\vee$. By direct calculation, $H = \langle k'_3 - k'_1, d(k'_1 + k'_3) \rangle \oplus (h) \bigoplus h^\perp$, $p_{h^\perp}(H) = \langle k'_1 \rangle$, and $D(\langle h^\perp \rangle) = \langle k'_1 \rangle \bigoplus \langle k'_2 \rangle$. By applying Corollary 1.5.2 of [Nik79],

$$O(L,h) \cong \{ g \in O(h^\perp) \mid g|_{p_{h^\perp}(H)} = \text{id} \},$$

and the first part of the claim follows.

For the second part of the claim, embed $L_{6,2p^2} \subset L_{6,2}$ by identifying factors of $2U \oplus \langle -6 \rangle$ and mapping

$$L_{6,2p^2} \ni t + ak_1 \mapsto t + bu + apk \in L_{6,2},$$

where $t \in 2U \oplus \langle -6 \rangle$, $k$ generates $\langle -2 \rangle \subset L_{6,2}$, and $a, b \in \mathbb{Z}$. Define the totally isotropic subspace $M \subset D(L_{6,2p^2})$ by $M = L_{6,2}/L_{6,2p^2} \subset D(L_{6,2p^2})$. By the above, if $g \in O(L,h)$, then $g(k'_1) = k'_1 + L_{6,2p^2}$. As $M \subset \langle k'_1 \rangle + L_{6,2p^2} \subset D(L_{6,2p^2})$ and $g(L_{6,2p^2}) = L_{6,2p^2}$, then $g$ extends to a unique element of $O(L_{6,2})$.

Let $p$ be an odd prime. We use an idea in [Kon93] (who attributes it to O’Grady) to bound the index $|O(L_{6,2}) : O(L_{6,2p^2})|$. This involves considering the finite quadratic space $Q_p$, defined by

$$Q_p = L_{6,2}/pL_{6,2} \subset L_{6,2p^2}/pL_{6,2},$$

and a number of classical results on orthogonal groups of finite type (which can be found in [Die71], but are stated below for the convenience of the reader).

For $i \in \mathbb{N}$, let $H_i$ denote hyperbolic planes over $\mathbb{F}_p$, and let $V_\theta$ denote the quadratic space $(u,v)$ whose bilinear form is given by $(u,u) = 1$, $(u,v) = 0$, and $(v,v) = \theta$ where $-\theta \notin (\mathbb{F}_p^*)^2$.

A non-degenerate quadratic space $V$ over a finite field $\mathbb{F}_q$ of odd order $q$ is uniquely determined by $\dim V$ and the discriminant $\Delta = \det B \in \mathbb{F}_q^2/(\mathbb{F}_q^*)^2$, where $B$ is the bilinear form on $V$. 
If \( \dim V = 2m \) and \( \epsilon = (-1)^m \Delta \in \mathbb{F}_q^*/(\mathbb{F}_q^*)^2 \), then \( V \) is isomorphic to
\[
\begin{cases}
V_{2m} = H_1 \oplus \ldots \oplus H_m & \text{if } \epsilon = 1 \\
V_{2m} = V_0 \oplus H_1 \oplus H_2 \oplus \ldots \oplus H_{m-1} & \text{if } \epsilon = -1.
\end{cases}
\]
If \( \dim V = 2m + 1 \), there is a single isomorphism class for \( V \) given by
\[
V^{2m+1} = H_1 \oplus \ldots \oplus H_m \oplus \{ \theta \}
\]
for \( 0 \neq \theta \in \mathbb{F}_q^* \).

We also need to know the order of \( O^+(V) \). As in [Dic74],
\[
\begin{align*}
|O^+(V^{2m+1})| &= (q^{2m} - 1)q^{2m-1}(q^{2m-2} - 1) \ldots (q^2 - 1)q \\
|O^+(V_{2m})| &= (q^{2m-1} - eq^{m-1})(q^{2m-2} - 1)q^{2m-3} \ldots (q^2 - 1)q.
\end{align*}
\]

**Lemma 4.2.** If \( u, v \in \mathcal{Q}_p \) and \( u^2 = v^2 \in \mathbb{F}_p^*/(\mathbb{F}_p^*)^2 \) for \( p > 3 \), then \( u \) and \( v \) are equivalent under \( O(L_{6,2}) \).

**Proof.** Let \( \{e_i, f_i, e_2, f_1, v_1, v_2\} \) be a basis for \( L_{6,2} \) where \( v_1, v_2 \) are the respective generators of \( \langle -6 \rangle \), \( \langle -2 \rangle \) and \( \{e_i, f_i\} \) are standard bases for the two copies of \( U \).

We define elements of \( O(L_{6,2}) \). For \( e \in L_{6,2} \) isotropic and \( a \in \mathbb{E} \subset L_{6,2} \) there are elements \( t(e, a) \in O(L) \) (known as *Eichler transgressions*) defined by
\[
t(e, a) : v \mapsto v - (a, v)e + (e, v)a - \frac{1}{2}(a(a))v \in \mathbb{E}.
\]

(See [Eic74] or §3 of [GHS09].) The action of \( t(e_2, v_1) \) and \( t(e_2, v_2) \) on \( w = (w_1, w_2, w_3, w_4, w_5, w_6) \in L_{6,2} \) are given by
\[
\begin{align*}
t(e_2, v_1) : w &\mapsto (w_1, w_2, w_3 + 3w_4 + 6w_5, w_4, w_5 + w_4, w_6) \\
t(e_2, v_2) : w &\mapsto (w_1, w_2, w_3 + w_4 + 2w_6, w_4, w_5 + w_6).
\end{align*}
\]

One can also obtain elements of \( O(L_{6,2}) \) by the trivial extension of elements in \( O(2U) \) for an embedding \( 2U \subset L_{6,2} \). In particular, if \( (w, x, y, z) \) is taken on the standard basis \( \{e_i, f_i\} \) of \( U + U \) then the map
\[
(w, x, y, z) \mapsto \begin{pmatrix} w & -y \\ z & x \end{pmatrix},
\]
identifies \( M_2(\mathbb{Z}) \) with \( U \oplus U \) (where the inner product on \( M_2(\mathbb{Z}) \) is defined by \( \det \)). An element \( (A, B) \in SL(2, \mathbb{Z}) \times SL(2, \mathbb{Z}) \) defines an element in \( O(U + U) \) by the mapping
\[
(A, B) : \begin{pmatrix} w & -y \\ z & x \end{pmatrix} \mapsto A \begin{pmatrix} w & -y \\ z & x \end{pmatrix} B^{-1}.
\]

Let \( x = (x_1, x_2, x_3, x_4, x_5, x_6) \in L_{6,2}/pL_{6,2} \) be non-zero. We can assume that \( x_4 \neq 0 \) by (if required) applying \( t(e_2, v_1) \) or \( t(e_2, v_2) \), or permuting \( \{x_1, x_2, x_3, x_4\} \) by elements in \( O(2U) \). By rescaling \( x \) so that \( x_4 = 1 \), and by repeated application of \( t(e_2, v_1) \) and \( t(e_2, v_2) \), the element \( x \) can be transformed to an element of the form \( (x_1', x_2', x_3', x_4', 0, 0) \), and so can be identified with an element of \( 2U \). Because of the existence of a Smith normal form for the associated matrix \( \Pi \), \( x \) can be mapped to an element of the form \( (r, s, 0, 0, 0, 0) \) by using the image of \( SL(2, \mathbb{Z}) \times SL(2, \mathbb{Z}) \) in \( O(2U) \). One can assume \( x \) is of the form \((1, a, 0, 0, 0, 0)\) by rescaling (if necessary).

Now suppose \( u, v \in L_{6,2}/pL_{6,2} \) are given by \( u = (1, a, 0, 0, 0, 0) \) and \( v = (1, b, 0, 0, 0, 0) \). If \( ab^{-1} \in (\mathbb{F}_p^*)^2 \), then there exists \( \mu, \lambda \in \mathbb{F}_p \) such that \( (\mu \lambda)^2 = (\lambda \lambda)^2 \). Define \( \hat{u} \) and \( \hat{v} \) by \( \hat{u} = \mu u \equiv (u_1, u_2, 0, 0, 0, 0), \)
\( \hat{v} = \lambda v \equiv (v_1, v_2, 0, 0, 0, 0) \) and suppose (without loss of generality) that \( \hat{u} - \hat{v} = (r, s, 0, 0, 0, 0) \) is non-zero. Take representatives for \( r, s \) modulo \( p \), and let
\[
d = \begin{cases} r & \text{if } s = 0 \\ s & \text{if } r = 0 \\ \gcd(r, s) & \text{otherwise.} \end{cases}
\]

If \( r_1, r_2, s_1, s_2 \) are solutions to \( r_2u_1 + r_1u_2 = d \) and \( s_2v_1 + y_2v_2 = d \) taken modulo \( p \), define the elements \( u', v', w \in e_1^T \cap f_1^T \subset L_{6,2} \) by \( u' = (r_1, r_2, 0, 0, 0, 0), v' = (s_1, s_2, 0, 0, 0, 0) \), and \( w = (d^{-1}r, d^{-1}s, 0, 0, 0, 0) \) where \( r' \equiv r \mod p \) and \( s' \equiv s \mod p \). Then, over \( \mathbb{F}_p \), \((\hat{u}, u') = d, (\hat{v}, v) = d, \) and \( t(e_2, v')t(f_2, w)t(e_2, u') : \hat{u} \mapsto \hat{v} \). The result follows. \( \square \)
There is a partial compactification where we require, following [GHS13]. Let the rational boundary components

\[ F \cap \text{Ad}(\Gamma) = \mathbb{Q} \cap \text{Ad}(\Gamma) \]

of \( \Gamma \) extends. The compactification \( N \) stabiliser of \( \Gamma \) extends. The compactification \( O(2, n) \) lattice of signature (2, n) metric spaces corresponding to \( \Gamma \)-equivalence classes of to tally isotropic planes and isotropic lines in \( M \). Because of the Langlands decomposition for the parabolic subgroup \( N(F) \), the domain \( D_M(F) \) decomposes as

\[ D_M(F) = F \times V(F) \times U(F) \mathbb{C}, \]

where \( V(F) \) is the complex vector space \( W(F)/U(F) \).

**Theorem 4.3.** Let \( p > 3 \) be prime. Then \( O^+(L, h_{2p^2}) \) is of finite index in \( O^+(L, h_2) \) and \( \text{O}^+(L, h_2) : O^+(L, h_{2p^2}) \leq 16(p^5 + p^2) \).

**Proof.** There is a natural homomorphism \( O(L_{6,2}) \to O(L_{6,2}/pL_{6,2}) \). If \( v, w \in \mathbb{Q}_p \) and \( v^2 = w^2 \mod (F_p)^2 \) then, by Lemma [4.2], \( v \sim w \) under the action of \( O(L_{6,2}) \) and so \( v \sim w \) under the action of \( O(L_{6,2}/pL_{6,2}) \). The group \( O(L, h_{2p^2}) \subset O(L_{6,2}) \) stabilises a hyperplane \( \Pi \subset \mathbb{Q}_p \) and, as \( \text{Stab}_{O(L_{6,2}/pL_{6,2})}(\Pi) = O(L_{6,2}/pL_{6,2}) \), by the orbit-stabiliser theorem,

\[ |O(L_{6,2}) : \text{Stab}_{O(L_{6,2})}(\Pi)| = |O(L_{6,2}/pL_{6,2}) : O(L_{6,2}/pL_{6,2})| \]

and

\[ |O^+(L_{6,2}) : \text{Stab}_{O^+(L_{6,2})}(\Pi)| = |O^+(L_{6,2}/pL_{6,2}) : O^+(L_{6,2}/pL_{6,2})|. \]

By Lemma [4.1], \( O(L, h_{2p^2}) \subset O(L_{6,2}) \) and so,

\[ O^+(L_{6,2}/2p^2) \leq O^+(L, h_{2p^2}) \leq \text{Stab}_{O^+(L_{6,2})}(\Pi) \leq O^+(L_{6,2}/2p^2). \]

As \( O(D(L_{6,2}/2p^2)) \cong C_2^{2\beta} \), then

\[ |\text{Stab}_{O(L_{6,2})}(\Pi) : O(L, h_{2p^2})| = |O(L_{6,2}/2p^2) : O(L_{6,2}/2p^2)| = 8, \]

and so,

\[ |O^+(L_{6,2}) : O^+(L, h_{2p^2})| \leq 8|O^+(L_{6,2}/pL_{6,2}) : O^+(L_{6,2}/pL_{6,2})| \leq 8(p^5 - 1)p^2(p^2 - 1)p^3 \leq 8(p^5 + p^2). \]

\[ \square \]

## 5. Toroidal Compactifications

**5.1. Overview.** Toroidal compactifications are defined fully in [AMRT10]. We state only the results we require, following [GHS13]. Let \( F \) denote the orthogonal modular variety \( D_M/\Gamma \), where \( M \) is a lattice of signature (2, n) and (without loss of generality) \( \Gamma \) is a neat normal subgroup of \( O^+(M) \). There is a partial compactification \( D_M^* \) of \( D_M \) (taken in the compact dual \( D_M^* \)) to which the action of \( \Gamma \) extends. The compactification \( D_M^* \) admits a decomposition

\[ D_M^* = D_M \sqcup \bigcup \Pi F_{\Pi} \sqcup \bigcup _{\ell} F_{\ell}, \]

where the rational boundary components \( F_{\Pi} \) (boundary curves) and \( F_{\ell} \) (boundary points) are symmetric spaces corresponding to \( \Gamma \)-equivalence classes of totally isotropic planes and isotropic lines in \( M \otimes \mathbb{Q} \).

For a rational boundary component \( F \) (given by some \( F_{\Pi} \) or \( F_{\ell} \) in [3]), let \( N(F) \subset O(2, n) \) be the stabiliser of \( F \), \( W(F) \) be the unipotent radical of \( N(F) \), and \( U(F) \) be the centre of \( W(F) \). Denote the intersections of \( N(F), U(F), W(F) \) with \( \Gamma \) by \( N(F)_Z, U(F)_Z, \) and \( W(F)_Z \). Let \( D_M(F) \) denote the domain \( U(F) : D_M \subset D_M^* \). Because of the Langlands decomposition for the parabolic subgroup \( N(F) \), the domain \( D_M(F) \) decomposes as

\[ D_M(F) = F \times V(F) \times U(F)_\mathbb{C}, \]

where \( V(F) \) is the complex vector space \( W(F)/U(F) \).
If $\mathcal{D}_M(F)'$ is the quotient $\mathcal{D}_M(F)' = \mathcal{D}_M(F)/U(F)_C$, then the spaces $\mathcal{D}_M(F)$, $\mathcal{D}_M(F)'$, and $F$ are related by the diagram

$$
\begin{array}{ccc}
\mathcal{D}_M(F) & \xrightarrow{\pi'_F} & \mathcal{D}_M(F)'\\
\pi_F & & \downarrow p_F \\
F & & \\
\end{array}
$$

where $\pi_F$, $p_F$, and $\pi'_F$ are the natural projections onto $F$, $F$, and $\mathcal{D}_M(F)$, respectively.

The space

$$(4) \quad \pi'_F : \mathcal{D}_M(F) \to \mathcal{D}_M(F)'$$

is a principal homogeneous space for $U(F)_C$ and admits an $N(F)_Z$-action. By taking the quotient of (1) by $U(F)_Z \subset N(F)_Z$, one obtains the principal fibre bundle

$$(5) \quad \mathcal{D}_M(F)/U(F)_Z \to \mathcal{D}_M(F)'$$

whose fibre is equal to the algebraic torus $T(F) := U(F)_C/U(F)_Z$.

There is a real cone $C(F) \subset U(F)$. By taking a fan $\Sigma$ in the closure of $C(F)$ and then replacing the torus $T(F)$ in the fibre bundle (5) with the toric variety $X_{\Sigma(F)}$ one obtains a new bundle over $\mathcal{D}_M(F)$ with fibre $X_{\Sigma(F)}$.

One constructs a partial compactification $\mathcal{F}(F)^{tor}$ for $\mathcal{F}$ in a neighbourhood of $F$ by taking the closure of $\mathcal{D}_M(U(F)_Z)$ in the new bundle, and then taking the quotient by $N(F)_Z$. A toroidal compactification for $\mathcal{F}$ is obtained by taking the set of partial compactifications over all rational boundary components $F$ and gluing by identifying the copies of $\mathcal{F}$ contained in each one.

### 5.2. Invariants associated with $F$.

**Definition 5.1.** If $M$ is a lattice and $E \subset M$ is a primitive totally isotropic sublattice, let $H_E := E^{\perp \perp}/E \subset D(M)$ where $E^{\perp \perp} \subset M^\vee$.

**Lemma 5.2.** Let $E \subset L_{6,2p^2}$ be a primitive totally isotropic sublattice of rank 2 corresponding to the boundary component $F$. Then there exists a $\mathbb{Z}$-basis $\{v_1, \ldots, v_6\}$ of $L_{6,2p^2}$ so that $\{v_1, v_2\}$ is a basis for $E$ and $\{v_1, \ldots, v_4\}$ is a basis for $E^{\perp}$. Furthermore, the basis can be chosen so that the Gram matrix

$$(6) \quad Q = ((v_i, v_j)) = \begin{pmatrix} 0 & 0 & A \\ 0 & B & C \\ A & C & D \end{pmatrix}$$

where

$$A = \begin{pmatrix} a_1 & 0 \\ 0 & a_1a_2 \end{pmatrix},$$

$a_1$, $a_2$ are the elementary divisors of the group $D(L_{6,2p^2})/H_E^\perp$, and $B$ is the quadratic form on $E^{\perp}/E$.

Moreover,

$$\{a_1, a_1a_2\} \subset \{(1,1), (1,p), (1,2p)\}.$$

**Proof.** As the lattices $E$ and $E^{\perp}$ are primitive in $L_{6,2p^2}$, the existence of a basis on which $Q$ assumes the form of (6) is immediate. The Smith normal form of the matrix $A$ embeds $\langle v_5, v_6 \rangle$ in the dual $\langle v^*_5, v^*_6 \rangle$. Therefore, the elementary divisors of $A$ correspond to the elementary divisors of the abelian group $\langle v^*_5, v^*_6 \rangle/\langle v_5, v_6 \rangle$ (c.f. [GHS07]).

If $H_E = E^{\perp \perp}/E \subset D(L_{6,2p^2})$, then $H_E + L_{6,2p^2} = \langle v_1, \ldots, v_4 \rangle$ in $D(L_{6,2p^2})$, and so $\langle v^*_5, v^*_6 \rangle/\langle v_5, v_6 \rangle \cong D(L_{6,2p^2})/H_E^\perp$.

As $E$ is totally isotropic in $L_{6,2p^2}$, then $H_E$ is totally isotropic in $D(L_{6,2p^2})$. If $D(L_{6,2p^2})$ is identified with $((-1/6) \oplus (-1/2p^2), C_6 \oplus C_{2p^2})$, then $(x, y) \in D(L_{6,2p^2})$ is isotropic if and only if

$$p^2x^2 + 3y^2 = 0 \mod 12p^2.$$

As $(3, p) = 1$ then $p|y$, $p^2x^2 + 3p^2y_1^2 = 0 \mod 12p^2$, and $x^2 + py_1^2 = 0 \mod 6$. 
By considering squares modulo 6, we conclude that \( x = 0 \) or 3, and that \( x \) and \( y \) must have equal parities. Therefore, the isotropic elements of \( D(L_{6,2p^2}) \) are given by the set
\[
(x, y) \in \{(0, 2kp), (3, (2k + 1)p) \mid k \in \mathbb{Z}\};
\]
the primitive isotropic subspaces of rank 1 in \( D(L_{6,2p^2}) \) are generated by \( x_1 := (0, 2p) \) and \( x_2 := (3, p) \); and the single primitive totally isotropic subspace of rank 2 is generated by \( \langle x_1, x_2 \rangle \).

If \( H_E = \langle x_1 \rangle \) then
\[
H_E^\perp = \{(a, b) \in D(L_{6,2p^2}) \mid 6b \equiv 0 \mod{6p}\},
\]
and so, \( p \mid b \) and \( H_E^\perp = \langle (1, 0), (0, b) \rangle \cong C_6 \oplus C_{2p} \).

If \( H_E = \langle x_2 \rangle \) then
\[
H_E^\perp = \{(a, b) \in D(L_{6,2p^2}) \mid pa + b \equiv 0 \mod{2p}\},
\]
and so, \( p \mid b, 2 \mid (a + b) \), and \( H_E^\perp = \langle (1, p), (2, 0) \rangle \).

We conclude that
- (1) if \( H_E = \langle 0 \rangle \), then \( H_E^\perp = D(L_{6,2p^2}) \) and \( D(L_{6,2p^2})/H_E^\perp \cong \{0\}; \)
- (2) if \( H_E = \langle x_1 \rangle \), then \( H_E^\perp = \langle (1, 0), (0, p) \rangle \cong C_6 \oplus C_{2p} \) and \( D(L_{6,2p^2})/H_E^\perp \cong C_p; \)
- (3) if \( H_E = \langle x_2 \rangle \), then \( H_E^\perp = \langle (1, p) \rangle \cong C_3 \oplus C_{2p} \) and \( D(L_{6,2p^2})/H_E^\perp \cong C_2 \oplus C_p; \)
- (4) if \( H_E = \langle x_1, x_2 \rangle \), then \( H_E^\perp = \langle (1, p) \rangle \cong C_3 \oplus C_{2p} \) and \( D(L_{6,2p^2})/H_E^\perp \cong C_2 \oplus C_p, \)
and the result follows.

**Lemma 5.3.** There exists a basis \( \{v_1, \ldots, v_6\} \) for \( L_{6,2p^2} \otimes \mathbb{Q} \) so that \( \{v_1, v_2\} \) is a \( \mathbb{Z} \)-basis for \( E \), \( \{v_1, \ldots, v_4\} \) is a \( \mathbb{Z} \)-basis for \( E^\perp \), and
\[
Q = ((v_i, v_j)) = \begin{pmatrix} 0 & 0 & A \\ 0 & B & 0 \\ A & 0 & 0 \end{pmatrix}
\]
where \( A \) and \( B \) are as in Lemma 5.2.

**Proof.** (Essentially as in Lemma 2.24 of [GHS07].) Suppose \( C \) and \( D \) are as in Lemma 5.2 If \( R := -B^{-1}C \in \text{det} B^{-1}M_2(\mathbb{Z}) \) and \( R' \in \text{det} B^{-1}M_2(\mathbb{Z}) \) satisfies
\[
D - tCB^{-1}C + tR'A + tAR' = 0,
\]
then the required base change is given by the matrix
\[
M = \begin{pmatrix} I & 0 & R' \\ 0 & I & R \\ 0 & 0 & I \end{pmatrix}.
\]

5.3. **Counting boundary components.** We determine the \( \text{O}(L_{6,2}) \)-orbits of primitive totally isotropic rank 2 sublattices of \( L_{6,2} \) along the lines of [Sca87]. As in [Sca87], the \( \text{O}(L_{6,2}) \)-orbits of primitive isotropic vectors in \( L_{6,2} \) can be determined in a straightforward manner by using the Eichler criterion ([10 Lic74]), and so we omit the calculation here.

**Lemma 5.4.** (Lemma 4.1 of [Bri83]) If \( M \) is a non-degenerate even lattice, and \( E \subset M \) is a primitive totally isotropic sublattice, then the discriminant form of the lattice \( E^\perp / E \) is isomorphic to \( H_E^\perp / H_E \subset D(M) \).

**Lemma 5.5.** If the rank 2 sublattice \( E \subset L_{6,2} \) is primitive and totally isotropic, then \( E^\perp / E \cong \langle -6 \rangle \oplus \langle -2 \rangle \) or \( E^\perp / E \cong A_2(-1) \).

**Proof.** The lattice \( E^\perp / E \) is negative definite and, by Lemma 5.3 \( D(E^\perp / E) \cong H_E^\perp / H_E \). Identify \( D(L_{6,2}) \) with \( C_6 \oplus C_2 \). If \( (a, b) \in D(L_{6,2}) \) is isotropic, then \( a^2/6 + b^2/2 = 0 \mod 2 \mathbb{Z} \), and so, \( (a, b) = (0, 0) \) or \( (a, b) = (3, 1) \). If \( H_E = \{(0, 0)\} \), then \( H_E^\perp / H_E = D(L_{6,2}) \) with discriminant form \( \langle -1/6 \rangle \oplus \langle 1/2 \rangle, C_6 \oplus C_2 \). If \( H_E = \{(3, 1)\} \), then \( H_E^\perp = \langle (1, 1) \rangle \) and \( H_E^\perp / H_E \cong \langle 2, 0 \rangle \) with discriminant form \( \langle 1/3 \rangle, C_3 \). By tables in [CS99], the two negative definite even lattices of determinant 12 are \( \langle -6 \rangle \oplus \langle -2 \rangle \).
and

\[
\begin{pmatrix}
-4 & -2 \\
-2 & -4
\end{pmatrix}.
\]

The discriminant form of \([5] \) is inequivalent to \((1/2)^{12} \oplus (-1/3), C_2^{12} \oplus C_3\). Therefore, if \(H_E = \langle (0, 0) \rangle\) then \(E^\perp / E \cong \langle -6 \rangle \oplus \langle -2 \rangle\); if \(H_E = \langle (3, 1) \rangle\) then, from tables in \([9] \), \(E^\perp / E \cong A_2(-1)\). \(\square\)

**Lemma 5.6.** If \(E \subset L_{6,2}\) is a primitive totally isotropic sublattice of rank 2, then there exists a \(\mathbb{Z}\)-basis \(\{v_1, \ldots, v_6\}\) of \(L_{6,2}\) so that \(\{v_1, v_2\}\) is a basis for \(E\), \(\{v_1, \ldots, v_4\}\) is a basis for \(E^\perp \subset L_{6,2}\), and the Gram matrix

\[
Q = ((v_i, v_j)) = \begin{pmatrix}
0 & 0 & P \\
0 & B & C \\
P & C & D
\end{pmatrix}.
\]

Moreover,

1. if \(H_E = \langle (1, 1) \rangle\), then \(B = \langle -6 \rangle \oplus \langle -2 \rangle\), \(C = D = 0\), and

\[
P = \begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix}.
\]

2. if \(H_E = \langle (3, 1) \rangle\), then \(B = A_2(-1)\),

\[
C = \begin{pmatrix}
0 & 0 \\
c & 0
\end{pmatrix}, \quad D = \begin{pmatrix}
2d & 0 \\
0 & 0
\end{pmatrix}, \quad P = \begin{pmatrix}
0 & 1 \\
3 & 0
\end{pmatrix},
\]

for \(c \in \{0, 1, 2\}\), \(d \in \{0, 1, 2\}\).

**Proof.** As in Lemma 5.2, take a basis \(\{v_1, \ldots, v_6\}\) for \(L_{6,2}\) so that \(\{v_1, v_2\}\) is a basis for \(E\), and \(\{v_1, \ldots, v_4\}\) is a basis for \(E^\perp\). Suppose that

\[
Q = ((v_i, v_j)) = \begin{pmatrix}
0 & 0 & A_0 \\
0 & B_0 & C_0 \\
A_0 & C_0 & D_0
\end{pmatrix}.
\]

By Lemma 5.5 \(H_E = \langle (0, 0) \rangle\) or \(H_E = \langle (3, 1) \rangle\).

If \(H_E = \langle (0, 0) \rangle\) then, because of the existence of a Smith normal form, there exist integral matrices \(U\) and \(Z\) so that

\[
UA_0Z = \begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix}.
\]

By Lemma 5.5 there exists \(X \in \text{GL}(2, \mathbb{Z})\) so that \(tX B_0 X = B = \langle -6 \rangle \oplus \langle -2 \rangle\). Therefore, the matrix \(g_1 := \text{diag}(U, X, Z)\) transforms \(Q\) to \(Q'\) where

\[
Q' = t g_1 Q g_1 = \begin{pmatrix}
0 & 0 & A \\
0 & B & C_1 \\
tA & tC_1 & D_1
\end{pmatrix},
\]

and

\[
A = \begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix}.
\]

The integral matrix \(g_2\) defined by

\[
g_2 = \begin{pmatrix}
I & -tA' C_1 & 0 \\
0 & I & 0 \\
0 & 0 & I
\end{pmatrix}
\]

transforms \(Q'\) to \(Q''\), where

\[
Q'' = \begin{pmatrix}
0 & 0 & A \\
0 & B & 0 \\
tA & 0 & D_2
\end{pmatrix}.
\]

If \(g_3\) is an integral matrix of the form

\[
g_3 = \begin{pmatrix}
I & 0 & W \\
0 & I & 0 \\
0 & 0 & I
\end{pmatrix},
\]
then $g_3$ sends $D_2 \mapsto D_2 + tWA + tAW$. One checks that the set $\{tWA + tAW \mid W \in M_2(\mathbb{Z})\}$ contains all matrices of the form

$$\begin{pmatrix} 2a & b \\ b & 2c \end{pmatrix}$$

for $a, b, c \in \mathbb{Z}$. Therefore, there exists integral $W$ and $d'_{11}, d'_{22} \in \{0, 1\}$ so that $d'_{11} \equiv d_{11} \mod 2$, $d'_{22} \equiv d_{22} \mod 2$, and

$$g_3 : D_2 \mapsto \begin{pmatrix} d'_{11} & 0 \\ 0 & d'_{22} \end{pmatrix}.$$  

As the form $Q$ is even, both $d_{11}$ and $d_{22}$ are even, and so there exists $W$ so that $g_3$ sends $D_2$ to 0. Therefore, the matrix $g_3g_2g_1$ gives the base change required in the statement of the theorem.

If $H_E = \langle(3, 1)\rangle$ then, because of the Smith normal form, there exist $U, Z \in \text{GL}(2, \mathbb{Z})$ such that

$$UA_0Z = \begin{pmatrix} 0 & 1 \\ 3 & 0 \end{pmatrix}.$$  

Moreover, there exists $X \in \text{GL}(2, \mathbb{Z})$ such that $tXB_0X = B = A_2(−1)$. Therefore the matrix $g_4$ given by $g_4 = \text{diag}(tU, X, Z) \in \text{GL}(2, \mathbb{Z})$ transforms $Q$ to $Q'$ where

$$Q' = tQ_1Qg_1 = \begin{pmatrix} 0 & 0 & A \\ 0 & B & C_1 \\ tA & tC_1 & D_1 \end{pmatrix}$$

and

$$A = \begin{pmatrix} 0 & 1 \\ 3 & 0 \end{pmatrix}.$$  

Let $g_5$ be an integral matrix of the form

$$g_5 = \begin{pmatrix} I & S & 0 \\ 0 & I & T \\ 0 & 0 & I \end{pmatrix},$$

for some $S, T \in M_2(\mathbb{Z})$. We claim that $(s_{ij}) := S$ and $(t_{ij}) := T$ can be chosen so that

$$tSA + BT + C_1 = \begin{pmatrix} 0 & 0 \\ a & 0 \end{pmatrix}$$

where $a$ is determined modulo 3. As

$$tSA + TQ + C_1 = \begin{pmatrix} 3s_{21} - 2s_{11} - t_{21} + c_{11} & s_{11} - 2t_{12} - t_{22} + c_{12} \\ 3s_{22} - 2t_{21} + t_{11} + c_{21} & s_{12} - t_{12} - 2t_{22} + c_{22} \end{pmatrix},$$

then the claim about the second column is immediate, as $s_{11}$ and $t_{12}$ are both free.

Let $δ := 2t_{11} + t_{21}$ and $s_{21} = 0$. By taking $t_{11}$ and $t_{21}$ so that $δ = −c_{11}$, the first column of (10) can be mapped to $t\langle(0, 3s_{22} - c_{11} + c_{21}\rangle$. Therefore, with an appropriate choice of $s_{22}$, the matrix $g_5$ transforms $Q'$ to

$$Q'' = \begin{pmatrix} 0 & 0 & A \\ 0 & B & C_0 \\ tA & tC_0 & D_2 \end{pmatrix},$$

where $C_0$ is as in the statement of the theorem.

We next put $D_2$ in the correct form. If $g_6$ is an integral matrix of the form

$$g_6 = \begin{pmatrix} I & 0 & W \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix},$$

then $g_6$ sends

$$D_2 \mapsto D_2 + tWA + tAW.$$  

One checks that the set $\{tWA + tAW \mid W \in M_2(\mathbb{Z})\}$ contains all matrices of the form

$$\begin{pmatrix} 6a & b \\ b & 2c \end{pmatrix}$$

for $a, b, c \in \mathbb{Z}$. Therefore if $(d_{ij}) := D$, there exists $W$ so that

$$g_3 : D_2 \mapsto \begin{pmatrix} d'_{11} & 0 \\ 0 & d'_{22} \end{pmatrix},$$
for \( d_1' \in \{0, \ldots, 5\} \), \( d_2' \in \{0, 1\} \), and where \( d_1'' \equiv d_1 \mod 6 \) and \( d_2'' \equiv d_2 \mod 2 \). As the form \( Q \) is even, then both \( d_1'' \) and \( d_2'' \) are even. Therefore, there exists \( W \) so that \( d_1'' \) is even and \( d_2'' = 0 \). Therefore, the matrix \( g_6g_5g_4 \) gives the base change required in the statement of the theorem. \( \square \)

**Theorem 5.7.** For prime \( p > 3 \), the modular variety \( \mathcal{F}_{2p^2} \) has at most \( 160(p^5 + p^2) \) boundary curves.

**Proof.** If \( E_1, E_2 \subset L_{6,2} \) are totally isotropic primitive sublattices of rank 2 with the same normal form \( \mathcal{F} \) then, by Lemma 5.6, there exists \( g \in O(L_{6,2}) \) so that \( g(E_1) = E_2 \). Therefore, by counting normal forms, there are at most 20 totally isotropic primitive rank 2 sublattices of \( L_{6,2} \) up to \( O^+(L_{6,2}) \)-equivalence. By Theorem 4.3

\[
| O^+(L_{6,2}) : O^+(L, h_{2p^2}) | \leq 8(p^5 + p^2),
\]

and so, up to \( O^+(L, h_{2p^2}) \)-equivalence, there are at most \( 160(p^5 + p^2) \) boundary curves. \( \square \)

5.4. Counting singularities. In this section we count singularities in the boundary of a toroidal compactification of \( \mathcal{F}_{2p^2} \). We shall only consider the compactification in a neighbourhood of a boundary curve. The structure of the boundary in a neighbourhood of a boundary point is different, and presents additional toric considerations.

Throughout, we assume the boundary component \( F \) corresponds to a rank 2 primitive totally isotropic sublattice \( E \subset L_{6,2p^2} \) and define \( \tilde{N} = a_1a_2 \det B \), where \( a_1, a_2, B \) as are in Lemma 5.2

**Lemma 5.8.** On the basis given in Lemma 5.3, the groups \( N(F), W(F) \) and \( U(F) \) are given by

\[
N(F) = \left\{ \begin{pmatrix} U & V & W \\ 0 & X & Y \\ 0 & 0 & Z \end{pmatrix} \mid ^tU AZ = A, ^tX BX = B, ^tV AZ = 0, ^tY BY + ^tZ AW + ^tW AZ = 0, \det(U) > 0, \right\}
\]

\[
W(F) = \left\{ \begin{pmatrix} I & V & W \\ 0 & I & Y \\ 0 & 0 & I \end{pmatrix} \mid BY + ^tVA = 0, ^tY BY + AW + ^tWA = 0 \right\},
\]

\[
U(F) = \left\{ \begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix} \mid x \in \mathbb{R} \right\}.
\]

Furthermore, if \( g \in N(F) \), then \( g \in N(F) \cap O(L_{6,2p^2}) \) if and only if

\[
M^{-1}gM = \begin{pmatrix} U & V & -VB^{-1}C + W + UR' - R'Z \\ 0 & X & Y - XB^{-1}C + B^{-1}CZ \\ 0 & 0 & Z \end{pmatrix} \in \text{GL}(6, \mathbb{Z})
\]

where \( M \) is as in (7).

**Proof.** The first part follows from direct calculation (as in Section 2.12 of [GHS07]). The second part follows as in Proposition 2.27 of [GHS07]. \( \square \)

As in [Kon93], \( D_{L_{6,2p^2}}(F) \) can be identified with a Siegel domain of the third kind inside \( \mathbb{C} \times \mathbb{C}^2 \times \mathbb{H}^+ \). The identification proceeds by taking homogeneous coordinates \( [t_1 : \ldots : t_6] \) for \( \mathbb{P}(L_{6,2p^2} \otimes \mathbb{C}) \) and mapping \( D_{L_{6,2p^2}}(F) \to \mathbb{P}(L_{6,2p^2} \otimes \mathbb{C}) \) by setting \( t_6 = 1 \) and

\[
\begin{cases}
    t_1 \mapsto z \in \mathbb{C} \\
    t_3 \mapsto w_1 \in \mathbb{C} \\
    t_4 \mapsto w_2 \in \mathbb{C} \\
    t_5 \mapsto \tau \in \mathbb{H}^+ \\
    t_2 \mapsto -2a_1a_2^{-1}(w_1w_2)R(w_1w_2).
\end{cases}
\]

(11)

**Proposition 5.9.** If

\[
g = \begin{pmatrix} U & V & W \\ 0 & X & Y \\ 0 & 0 & Z \end{pmatrix} \in N(F)
\]
is as in Lemma 5.8, where \( Z = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \), then the action of \( g \) on \( \mathcal{D}_L(F) \) is given by

\[
\begin{align*}
    z \mapsto & \quad \frac{z}{\det Z} + (c\tau + d)^{-1} \left( \frac{c}{\det Z} w B w + \frac{1}{2} w \right) + W_{11} \tau + W_{12} \\
    w \mapsto & \quad (c\tau + d)^{-1} (X w + Y (\frac{1}{\tau})) \\
    \tau \mapsto & \quad a\tau + b/c + d.
\end{align*}
\]

Proof. As in [GHS07]. □

Lemma 5.10. Let

\[
Z = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_N,
\]

where \( \Gamma_N \subset \text{SL}(2, \mathbb{Z}) \) is the principal congruence subgroup of level \( N \). Then, on the basis given in Lemma 5.3, the map

\[
Z \mapsto g_Z = \begin{pmatrix} Z' & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & Z \end{pmatrix},
\]

where

\[
Z' = \begin{pmatrix} d & -ca_2 \\ -b/a_2 & a \end{pmatrix},
\]

defines an embedding of \( \Gamma_N \) in \( N(F) \cap O(L_{6,2p^2}) \).

Proof. Suppose \( A, B, C, \) and \( R' \) are as in Lemma 5.2. Let \( P := Z' R' - R' Z \), \( Q := -B^{-1} C + B^{-1} CZ \), and let \( Z, Z' \) be given by (12), (13), respectively. As \( Z' = (AZ^{-1} A^{-1}) \) then, by Lemma 5.8, \( g_Z \in N(F) \cap O(L_{6,2p^2}) \) if and only if both

\[
BY + t VA = 0
\]

and

\[
0 = 0
\]

As \( Z \in \Gamma_N \), then \( Z' \in M_2(\mathbb{Z}) \).

If \( R' = \begin{pmatrix} w & x \\ y & z \end{pmatrix} \),

then

\[
P = \begin{pmatrix} -a_2 cy - a w + dw - cx & -a_2 cz - bw \\ -cz - bw/a_2 & -by + az - dz - bx/a_2 \end{pmatrix}.
\]

As \( Z \in \Gamma_N \), then \( P \in M_2(\mathbb{Z}) \).

As \( Z \equiv I \mod N \), then \( C - CZ \equiv 0 \mod N \). As \( \det B \mid N \) and \( R' \), \( B^{-1} \in \det B^{-1} M_2(\mathbb{Z}) \), then

\[
Q = B^{-1} (CZ - C) \in M_2(\mathbb{Z}),
\]

and the result follows. □

Lemma 5.11. If \( Y \in M_2(N\mathbb{Z}) \) then, on the basis given in Lemma 5.3, there exists \( g_Y \in W(F) \cap O(L_{6,2p^2}) \) of the form

\[
g_Y = \begin{pmatrix} I & * & * \\ 0 & I & Y \\ 0 & 0 & I \end{pmatrix}.
\]

Proof. Fix \( Y \in M_2(N\mathbb{Z}) \). Let \( A, B, \) and \( C \) be as in Lemma 5.2 and \( M \) be as in Lemma 5.3. By Lemma 5.8

\[
g_Y = \begin{pmatrix} I & V & W \\ 0 & I & Y \\ 0 & 0 & I \end{pmatrix}
\]

belongs to \( W(F) \) if and only if both

\[
BY + t VA = 0
\]

and

\[
t YBY + AW + t WA = 0
\]
are satisfied. If \( g_Y \in W(F) \) then, by Lemma 5.8, \( g_Y \in W(F) \cap O(L_{\delta,2p^2}) \) if and only if \( V, Y, \) and \( P \) are integral matrices, where

\[
P := W - VB^{-1}C.
\]

Equation (14) has a solution in \( V \) if \( Y \in M_2(a_1a_2\mathbb{Z}) \), and the matrix \( P \) is integral if \( V \in M_2((\det B)\mathbb{Z}) \). Therefore, by (14), both conditions are satisfied if \( Y \in M_2(N\mathbb{Z}) \).

If

\[
W = \begin{pmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{pmatrix},
\]

then (15) is equivalent to

\[
-\imath YB = AW + \imath WA = \begin{pmatrix} 2a_1w_{11} & a_1w_{12} + a_2w_{21} \\ a_1w_{12} + a_2w_{21} & 2a_1w_{22} \end{pmatrix},
\]

and so has a solution in \( W \) if \( Y \in M_2(N\mathbb{Z}) \).

Suppose \((z, w, \tau)\) are coordinates for \( \mathcal{D}_{L_{\delta,2p^2}}(F) \), as in (11). Local coordinates for a cover \( \mathcal{D}_{2p^2}(F)_{\text{tor}} \) of a toroidal compactification \( \mathcal{F}_{2p^2} \) in a neighbourhood of \( F \) are obtained from \((z, w, \tau)\) by replacing \( z \) with \( u = \exp_{a_1}(z) = e^{2\pi i/a_1} \), and allowing \( u = 0 \). The group acting on the cover is given by \( G(F) := (N(F) \cap O^+(L,h))/ (U(F) \cap O^+(L,h)) \).

Theorem 5.12. If \( g' \in G(F) \) fixes a point \( x = (0, w, \tau) \in \mathcal{D}_{2p^2}(F)_{\text{tor}} \) then, in local coordinates around \( x \), \( g' \) acts by

\[
\text{diag}(\omega_0, \xi\omega_1, \xi\omega_2, \xi^2),
\]

where \( \omega_1, \omega_2, \) and \( \xi \) are 4-th or 6-th roots of unity, and \( \omega_0 \) is a 12-th root of unity. For given \((\omega_1, \omega_2, \xi)\), bounds for the number of connected components of \( \mathcal{D}_{2p^2}(F)_{\text{tor}} \) fixed by some \( g \in G(F) \) acting as in (10) are given in Table 1 and Table 2, where \( J_{N,B} = N\det B \) and \( \omega = e^{2\pi i/3} \). Assuming \( g' \) acts non-trivially, the image of \( x \) in \( \mathcal{F}_{2p^2} \) is singular. For invariants \((a_1, a_2)\) corresponding to \( F \), the values of \( N, \det B, \) and \( K_N \) are given in Table 3.
Proof. Throughout, we assume $g \in N(F)$ represents a finite order element of $G(F)$, and that $g$ fixes the point $x = (0, w, \tau) \in D(F)_{tor}$. By Corollary 2.29 of [GHS07], no element of $G(F)$ acts as a quasi-reflection. Therefore, if $g$ acts non-trivially, by a theorem of Chevalley [Che55], each point in the fixed locus of $G(F)$ on $D_{2p^2}(F)_{tor}$ is singular.

As in Lemma 5.5 let

$$g = \begin{pmatrix} U & V & W \\ 0 & X & Y \\ 0 & 0 & Z \end{pmatrix}$$

where

$$Z = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

and let $\xi = (c\tau + d)^{-1}$, $T = (I - \xi X)$. As $g \in G(F)$ is of finite order, then $U$, $X$, $Z$ are of finite order. By considering rational representations, $o(U)$, $o(X)$, $o(Z) \in \{1, 2, 3, 4, 6\}$ and so the basis of Lemma 5.5 can be chosen so that each of $U$, $X$, $Z$ can be represented by one of

$$\pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}.$$

Indeed, as $Z \in M_2(Z)$ (Lemma 5.8) and $Z$ acts on $\mathbb{H}^+$, then $Z \in SL(2, Z)$ (as noted in [GHS07]).

Suppose $o(Z) \in \{3, 4, 6\}$. By standard results on the elliptic elements of $SL(2, Z)$ [DS05], $\tau$ is $SL(2, Z)$-equivalent to $i$, $\omega$ or $\omega^2$. If $G_k$ is the Eisenstein series of weight $k$, then $G_4(i) \neq 0$, and $G_6(\omega) \neq 0$ (p. 10 [DS05]), and so $\xi$ is a 4-th root of unity if $Z$ is of order 4, and a 6-th root of unity if $Z$ is of order 3 or 6. If $c\tau + d = \pm 1$ then, as both $\{1, \omega\}$ and $\{1, i\}$ are linearly independent over $\mathbb{Q}$,

$$Z = \begin{pmatrix} \pm 1 & b \\ 0 & \pm 1 \end{pmatrix}$$

which implies the contradiction $o(Z) = 1$ or 2. Therefore,

$$\langle 1, \tau \rangle = \begin{cases} 
\langle 1, i \rangle & \text{if } o(Z) = 4 \\
\langle 1, \omega \rangle & \text{if } o(Z) = 3 \text{ or } 6.
\end{cases}$$

The values of $\det T$ against $\chi_X$ and $\xi$ are given in Table 4. By (11),

$$T w = Y \left( \frac{1}{\det B} \right) \subset \frac{\langle 1, \tau \rangle}{\det B} \times \frac{\langle 1, \tau \rangle}{\det B},$$

and so if $\det T \neq 0$, by Table 4

$$w \in \frac{\langle 1, \tau \rangle}{K \det B} \times \frac{\langle 1, \tau \rangle}{K \det B},$$

for appropriate $K \in \{1, 2, 3\}$. Therefore, by using elements of the form $g v$ defined in Lemma 5.11 $w$ can be reduced to one of $(KN \det B)^4$ points modulo $N(F) \cap O(L_{6,2p^2})$.

| $(a_1, a_2)$ | $N$ | $\det B$ | $K_N$ |
|---------------|-----|----------|-------|
| (1, 1)        | 1   | $12p^2$  | $12p^2$ |
| (1, p)        | p   | 12       | $12p$  |
| (1, 2p)       | 2p  | 3        | $6p$   |

Table 3.
If $o(Z) = 1$ or 2, then $\tau \in \mathbb{H}^+$ is free and $\xi = \pm 1$. By Lemma 5.8, $V = 0$, $Y = 0$, and so
\[ T \vec{w} = 0. \]
If $\det T \neq 0$ then $\vec{w} = 0$. We consider each of the cases $\det T = 0$ separately. If $(\chi_X, \xi) = (\Phi_1^2, 1)$ or $(\chi_X, \xi) = (\Phi_2^2, -1)$, then $g$ acts as the identity (as it cannot act as a quasi-reflection); if $(\chi_X, \xi) = (\Phi_1 \Phi_2, 1)$, then $o(Z) = 1$, $\tau$ is free, $w_1 \in (2 \det B)^{-1} \mathbb{Z}$, and $\vec{w}$ can be reduced to one of $2N \det B$ lines by using elements of the form $g \gamma \in N(F) \cap O(L_{6,2^2p^2})$. The case $(\Phi_1 \Phi_2, -1)$ proceeds as for $(\Phi_1 \Phi_2, 1)$. The remaining cases proceed as for $(\chi_X, \xi) = (\Phi_4, -i)$. If $(\chi_X, \xi) = (\Phi_4, -i)$, then $w_1 - iw_2 \in (\det B)^{-1}(1, i)$ and $\vec{w}$ can be reduced to one of $(N \det B)^2$ lines using elements of the form $g \gamma \in N(F) \cap O(L_{6,2^2p^2})$.

By standard results on congruence subgroups (p.13 [DS05]), the index $K_N := |\SL(2, \mathbb{Z}) : \Gamma_N|$ is given by
\[ K_N = N^3 \prod_{p|N} \left( 1 - \frac{1}{p^2} \right). \]
Therefore, if $o(Z) = 3, 4, 6$, then $\tau$ can be reduced to one of $K_N$ cases modulo $N(F) \cap O(L_{6,2^2p^2})$ by using elements of the form $g \gamma \in \Gamma_N \subset N(F) \cap O(L_{6,2^2p^2})$ defined in Lemma 5.10.

By Proposition 5.11, $\hat{O}^+ (L_{6,2^2p^2}) \subset O^+(L, h)$ and one obtains a final bound from (2) by noting that $|O^+(L, h) : O(L_{6,2^2p^2})| \leq 2^4$.

The statement about the action of $g$ in local coordinates follows as in [Kon93], and the values in Table 3 can be calculated from Lemma 5.2. The statement about $\omega_0$ follows by noting that $o(\omega_0)$ divides $\text{lcm}(o(U), o(X), o(Z))$. \hfill $\Box$

Remark 5.13. If $g \in G(F)$ acts as diag$(\omega_0, \xi_1, \xi_2)$ in a neighbourhood of $(0, \vec{w}, \tau)$ and $\omega_0 \neq 1$, then $(0, \vec{w}, \tau)$ is not contained in the closure of the singular locus of $\mathcal{F}_{2p^2}$. The singular locus of $\mathcal{F}_{2p^2}$ can be studied by different methods, as in [Daw17].

6. Acknowledgements

This paper originates from my PhD thesis. I thank Professor G. K. Sankaran for his supervision, and the University of Bath for financial support in the form of a research studentship. I also thank the Riemann Center for Geometry and Physics for providing excellent working conditions as I edited the paper, and for financial support in the form of a Riemann fellowship.

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