Suppose you look at today’s stock prices and bet on the value of the first digit. One could guess that a fair bet should correspond to the frequency of $1/9 = 11.11\%$ for each digit from 1 to 9. This is by no means the case, and one can easily observe a strong prevalence of the small values over the large ones. The first three integers 1, 2 and 3 alone have globally a frequency of 60% while the other six values 4, 5, 6, 7, 8 and 9 appear only in 40% of the cases. This situation is actually much more general than the stock market and it occurs in a variety of number catalogs related to natural phenomena. The first observation of this property traces back to S. Newcomb in 1881 [1] but a more precise account was given by F. Benford in 1938 [2,3]. He investigated 20 tables of numbers ranging from the area of lakes and the length of rivers to the molecular weights of molecular compounds. In all cases he found the same behavior for which he guessed the probability distribution $P(n) = \log[(n + 1)/n]$ where $n$ is the value of the first integer. Since then this observation has remained marginal and occasionally reported as a mathematical curiosity [4] with some computer science applications [5] and even tax-fraud detection [6]. In this note we illustrate these observations with the enlightening specific example of the stock market [7,8]. We also identify the general mechanism for the origin of this uneven distribution in the multiplicative nature of fluctuations in economics and in many natural phenomena. This provides a natural explanation for the ubiquitous presence of the Benford’s law in many different phenomena with the common element that their fluctuations refer to a fraction of their values. This brings us close to the problem of the spontaneous origin of scale invariant properties in
various phenomena which is a debated question at the frontier of different fields. Consider the values of the Athens, Madrid, Vienna and Zurich stock markets of January 23, 1998. The stock prices $N$ are expressed in the local currencies. If the values of $N$ were randomly distributed we would expect a uniform distribution for the value of the first digit $n = 1, 2, \ldots, 9$. This would lead to $P(n) = 1/9 = 11.11\%$. In Fig.1a we can see instead that $P(1) = 30\%$ and then the frequency decreases continuously with $n$ until the minimum value $P(9) = 4.5\%$. This is the asymmetric distribution of first digits pointed out by F. Benford in 1938 [2,3] who investigated mostly tables of natural numbers. Here we can see that such a behavior applies to economic data as well. The prevalence of the number 1 in the first digit may suggest ad hoc explanations like the fact that there may be a tendency, in each country, to assign price values of the order of unity in local currency. This hypothesis can be easily checked by expressing the stock prices in a different currency. To this purpose, we repeat the analysis re-expressing the Madrid stocks in Swiss francs and the Zurich ones in Pesetas. Remarkably we observe that the Benford’s distribution is independent on the units adopted. This fact may appear strange at first sight, but it is one of the most common example of scale invariance [4]. This property should be found in the probability distribution of the original stock prices $P(N)$ and then it should reflects also in the distribution of their first digit $P(n)$. Changing the unit of measure of a price corresponds in multiplying it by a factor $b$, in our case the exchange rate between Swiss Francs and Pesetas. If we change our units by a factor $b$, the values $N$ will become $N' = bN$ and the corresponding distribution should be identical to the original one, apart from a constant rescaling factor $A(b)$, that may depend upon $b$ but not on $N$. In mathematical terms scale invariance corresponds to the following functional relation

$$P(N') = P(bN) = A(b)P(N).$$

(1)

It is well known from statistical physics and critical phenomena that the general solution of this equation is any power law behavior with exponent $\alpha$

$$P(N') = N'^{-\alpha} = b^{-\alpha}N^{-\alpha}.$$

(2)
For this type of distributions we can compute the probability of the first digit by observing that for each decade we have the same relative probability for the various integers, so we can write

\[ P(n) = \int_n^{n+1} N^{-\alpha} \, dN = \frac{1}{1-\alpha} \left[ (n+1)^{(1-\alpha)} - (n)^{(1-\alpha)} \right], \tag{3} \]

for \( \alpha \neq 1 \). For \( \alpha = 1 \), we have instead

\[ P(n) = \int_n^{n+1} N^{-1} \, dN = \int_n^{n+1} d(\log N) = \log\left(\frac{n+1}{n}\right), \tag{4} \]

that is precisely Benford’s law as derived from the data analysis. From Eq. (4), we can see that the case \( \alpha = 1 \) corresponds to a uniform distribution in logarithmic space as from Eq (4). This means that if we consider a set of random numbers \( R \) in normal space and then we look at the distribution \( N = \exp(R) \), we have a uniform (random) distribution just in the logarithmic scale. This distribution is actually satisfying the Benford’s law by construction [10].

These observations pose therefore two questions: i) The first is to understand why some data set naturally show scale invariant properties. ii) The second question concerns why, among the various scale invariant distributions corresponding to different \( \alpha \) values, the \( \alpha = 1 \) is the one actually realized in nature. In the following we rationalize the two above questions in the finding of a general mathematical origin of the distribution \( P(N) = 1/N \) that can apply to a variety data set in several fields ranging from economics to physics and geology.

The most general mathematical property which applies to the statistics of a large variety of phenomena is the central limit theorem that governs the probability distribution corresponding to the sum of random numbers. From a dynamical point of view we can consider the value of a variable \( N \) which changes with time by the addition of a random variable \( \xi \), yielding the Brownian process

\[ N(t+1) = \xi + N(t). \tag{5} \]

If the random variable \( \xi \) is symmetrically distributed with finite variance, the probability distribution \( P(N,t) \) to have a given value \( N \) after \( t \) additional steps will be Gaussian with
variance $\sigma \sim t^{1/2}$. In the infinite time limit the variance is diverging and the probability distribution will approach the uniform one. This is indeed very far from the scale invariance properties we are looking for. On the other hand, it easy to realize that the Brownian dynamics does not realistically apply to many stochastic dynamical phenomena. Brownian dynamics is ruled by a noise term whose intensity is independent of the variable value $N$. Fluctuations are in this way independent and related to some external dynamical parameters. Clearly, many systems do not follow such a dynamical description. For instance, it is intuitive to consider that a stock price has fluctuations which are relative to the price itself. In practice, each stock suffers of percent increments. Hence

$$N(t + 1) = \xi N(t),$$

(6)

where $\xi$ is again a stochastic variable that in this case must be positive definite. The nature of this process is completely different from the usual Brownian motion. We can, however, relate the two processes by a simple transformation. If we take the variable in logarithmic space we get:

$$\log N(t + 1) = \log \xi + \log N(t),$$

(7)

If we consider $\log \xi$ the new stochastic variable, we recover a Brownian dynamics in a logarithmic space; i.e. a random multiplicative process corresponds to a random additive process in logarithmic space. This implies that for $t \to \infty$ the distribution $P(\log N)$ approaches a uniform distribution, and by transforming back to linear space we have

$$\int P(\log N)d(\log N) = C \int \frac{1}{N}dN,$$

(8)

where $C$ represents the normalization factor. This immediately gives $P(N) \sim N^{-1}$ as the distribution of variable values $N$. Accordingly, the distribution of first digits $n$ will follow an ideal Benford law.

In order to illustrate numerically this result, we have considered a uniform distribution of numbers in the interval $(0, 999)$. Clearly, such a case corresponds to a uniform distribution for both the actual numbers and their first digits. We start to apply on the numbers
of the starting uniform distribution a multiplicative dynamics as described previously. At each iteration step, Numbers are multiplied or divided by a factor $\Delta$ at random, following a bimodal distribution characterized by two delta function as $p(\xi) = \frac{1}{2} \delta(\xi - \Delta) + \frac{1}{2} \delta(\xi - \Delta^{-1})$. In this way we have that in the logarithmic space the stochastic variable $\log \xi$ has zero mean and it is symmetrically distributed. Then we repeat the updating process of variables, which in logarithmic space corresponds to the random addition of $\pm \log \Delta$. After enough multiplicative steps we can observe the resulting distribution of number $N$. In Fig.2 it is reported the outcome of 100 iteration steps with $\Delta = 1.5$. Both the first digit and the cumulative numbers distribution evolved in the ideal Benford’s law of Eq. (1). This simple exercise shows that the numbers $N$ characterizing some physical quantities or objects, naturally will follow the Benford’s law if their time evolution is ruled by multiplicative fluctuations. It is worth remarking that many physical phenomena shows scale invariant behavior and are characterized by power law distributions. These features are due to cooperative effects, the onset of critical points and other nonlinear dynamical effects [9,11]. Here, however, scale invariance manifests itself with non-trivial power law exponents. Benford’s law will apply to these phenomena in the generalized form of Eq. (3). In this sense, it is interesting to explore also connections with other well known scale invariant features such as the Zipf’s law [12].

The fact that the Benford’s law is naturally explained in terms of a dynamics governed by multiplicative fluctuations can provide new insights on many scale invariant natural phenomena. The understanding of the origin of scale invariance has been one of the fundamental tasks of modern statistical physics. How system with many interacting degrees of freedom can spontaneously organize into critical or scale invariant states is a subject that is of up-surging interest to many researchers. Here we provide a simple dynamical picture of the generation of scale invariant distributions. A multiplicative dynamics might be sort of obvious in the stock pricing, but it is much less clear in the case of lake sizes or molecular weights of chemical compounds. **Acknowledgments:** We thank G.Mussardo for useful discussions and comments. L.P. thanks the hospitality of ICTP where part of this work has been completed. L.P and A.V. acknowledge partial support from the TMR European
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FIG. 1. (Distribution $P(n)$ of the first digits of the stock prices $N$ of Zurich (expressed in Swiss francs). The distribution is strongly asymmetric and it is fairly reproduced by the Benford’s law shown for comparison with circles. Deviations from the ideal Benford’s law are due to statistical noise. Averaging over three data sets (Madrid, Vienna and Zurich stock exchanges) we have that the distribution $P(n)$ shows smaller deviations.
FIG. 2. Probability distribution of the first digit of a flat number distribution after 100 iterative applications of multiplicative noise (see text). The ideal Benford’s law is strikingly satisfied. Small deviations disappear with more iterative noise applications. In the inset it is shown the cumulative distribution of all the numbers $P_c(N) = \int_N^{K^*} P(N')dN'$, where $K^*$ is the upper cut-off; i.e. the largest number generated. For a Benford-like distribution we have $P_c(N) = C(-\log N + \log K^*)$ where $C$ is the normalization factor. This behavior is nicely followed by the generated distribution as it appears from the linear plot on the semi-logarithmic scale.