AFFINE CONNECTIONS ON 3-SASAKIAN HOMOGENEOUS MANIFOLDS

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Abstract. The space of invariant affine connections on every 3-Sasakian homogeneous manifold of dimension at least 7 is described. In particular, the remarkable subspaces of invariant affine metric connections, and the subclass with skew-torsion, are also determined. To this aim, an explicit construction of all 3-Sasakian homogeneous manifolds is exhibited. The unique 3-Sasakian homogeneous manifolds which admit nontrivial Einstein with skew-torsion invariant affine connections are those of dimension 7, that is, $S^7 = \text{Sp}(2)/\text{Sp}(1)$, $\mathbb{R}P^7 = \text{Sp}(2)/\text{Sp}(1) \times \mathbb{Z}_2$ and the Aloff-Wallach space $W_{7,1} = \text{SU}(3)/U(1)$. For $S^7$ and $\mathbb{R}P^7$, the set of such connections is in one to one correspondence with two copies of the conformal linear transformation group of the Euclidean space, while it is strictly bigger for $W_{7,1}$. In addition, the set of invariant connections with totally skew-symmetric torsion whose Ricci tensor is multiple of the metric, with different factors, on the canonical vertical and horizontal distributions, is fully described on every 3-Sasakian homogeneous manifold. An affine connection satisfying these conditions is distinguished, characterized by parallelizing all the characteristic vector fields associated to the 3-Sasakian structure. This connection is Einstein with skew-torsion for the 7-dimensional examples. Several results have also been adapted to the nonnecessarily homogeneous setting. In this case, the above mentioned sets of affine connections are, in general, only proper subsets satisfying the properties.

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1. Introduction

Almost contact metric structures on $(2n + 1)$-dimensional (smooth) manifolds may be regarded as an analogue of Hermitian structures for odd dimensional manifolds. Recall that, for every almost contact metric structure on a $(2n + 1)$-dimensional manifold, there exists a general method, the cone construction, which permits to obtain a $(2n + 2)$-manifold endowed with an Hermitian structure. Of course, the family of differentiable manifolds with Hermitian structures is wider than the set of Kähler manifolds. In this setting, the following natural question raised. When does the cone construction produce a Kähler manifold? This very special kind of almost contact metric structure is now known as a Sasakian manifold in honour to the Japanese geometer Shigeo Sasaki, who introduced it in 1960. An extensive and complete study of Sasakian manifolds and related topics can be found in the excellent monograph [9].

In the late sixties, the new notion of the 3-Sasakian manifold was introduced as a $(4n + 3)$-dimensional manifold with a family of Sasakian structures parametrized by points on the 2-dimensional

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unit sphere $S^2$ and satisfying several compatibility conditions (Section 2). Every 3-Sasakian manifold $M$ carries 3-orthonormal Killing vector fields which span, at every tangent vector space to $M$, a copy of the Lie algebra $\mathfrak{sp}(1)$, and therefore a 3-dimensional foliation $\mathcal{F}_Q$. Under the assumption that $\mathcal{F}_Q$ is regular, the space of leaves $M/\mathcal{F}_Q$ inherits a hyper-Kähler structure with positive scalar curvature, [26]. That is, $M/\mathcal{F}_Q$ is endowed with a suitable family of integrable complex structures which satisfy the quaternionic identities (see details in [9, Chapter 13]). The relationship with hyper-Kähler structures goes in two ways. In fact, in this setting, starting now with a 3-Sasakian manifold, the cone construction produces a hyper-Kähler structure.

From 1970 to 1975, the study of 3-Sasakian geometry was mainly investigated by the Japanese school. In 1971, a remarkable property for our objectives was achieved: every 3-Sasakian manifold is an Einstein space with positive scalar curvature, [29]. Despite of that particularly relevant result and others, in Boyer and Galicki’s words, “1975 seems to be the year when 3-Sasakian manifolds are relegated to an almost complete obscurity which lasted for about 15 years... The authors [Boyer and Galicki] have puzzled over this phenomenon without any sound explanation” [9, Chapter 13]. Anyway, at the beginning of the nineties, a renewed interest in 3-Sasakian geometry arose in several areas. Let us briefly recall some examples which supported this growing interest. The existence of two different Einstein metrics on 3-Sasakian manifolds was obtained in [10], but only one is 3-Sasakian. In the 7-dimensional case, both Einstein metrics have $G_2$ weak holonomy. Moreover, for 7-dimensional Riemannian manifolds, the existence of three Killing spinors is equivalent to the existence of a 3-Sasakian structure, [23]. Bearing in mind that cones over 3-Sasakian manifolds produce Calabi-Yau manifolds, recent developments on 3-Sasakian geometry also include the Yang-Mills equations on cones over 3-Sasakian manifolds ([24] and references therein). Finally, in the study of the control system of a $n$-dimensional Riemannian manifold $M$ rolling on the sphere $S^n$, without twisting or slipping, and under certain assumption, the manifold $M$ can be endowed with a 3-Sasakian structure, [14]. In fact, several generalizations of 3-Sasakian structures have been recently studied, as the 3-(\(\alpha, \delta\))-Sasakian manifolds, [4], which are 3-Sasakian manifolds when $\alpha = \delta = 1$, or the 3-quasi-Sasakian manifolds [11]. In general lines, they deal with geometric structures which are less rigid than the 3-Sasakian ones, which permits to clarify some geometrical properties of the 3-Sasakian manifolds.

We should recall a key difference between Sasakian and 3-Sasakian manifolds. Indeed, every Sasakian manifold admits a unique metric connection with totally skew-symmetric torsion such that all the tensors involved in the Sasakian structure are parallel, [22] (see Remark 2). Needless to say, this is not the case for a 3-Sasakian manifold $M^{4n+3}$. In fact, for every $\tau \in S^2$, the corresponding Sasakian structure on $M$ admits such connection $\nabla^\tau$, but they do not coincide for different values of the parameter $\tau \in S^2$. Therefore, the 3-Sasakian structure is not parallel for any metric connection with skew-torsion. Thus, it is natural to ask whether there is a best affine metric connection on a 3-Sasakian manifold. This natural question was posed by Cartan in 1924, [12], to look for a connection adapted to the geometry of the space under consideration.

In this paper, we will focus on the existence (or not) of remarkable affine connections on 3-Sasakian manifolds. A specially interesting case is to look for affine metric connections with skew-torsion. In fact, these affine connections share geodesics with the Levi-Civita connection. The nice survey [1] includes both Mathematical and Physical motivations, as well as a wide variety of examples of affine metric connections with torsion. From our approach, among all the affine metric connections with skew-torsion, the nicest choice has been proposed in [2] in the following terms. A triple $(M, g, \nabla)$, where $g$ is a Riemannian metric and $\nabla$ is a metric affine connection with skew-torsion, is said to be Einstein with skew-torsion whenever

$$\text{Sym}(\text{Ric}^{\nabla}) = \frac{s^{\nabla}}{\dim M} g,$$

where $\text{Sym}(\text{Ric}^{\nabla})$ denotes the symmetric part of the Ricci tensor and $s^{\nabla}$ is the scalar curvature of $\nabla$ (Section 2). Clearly, this notion is a wide generalization of the usual Einstein spaces. The main
purpose of this paper is to determine when there are metric affine connections $\nabla$ on a 3-Sasakian manifold $M$ such that, with the underlying Riemannian metric $g$, the triple $(M, g, \nabla)$ is Einstein with skew-torsion. To face this problem, we will pay attention to invariant metric affine connections on 3-Sasakian homogeneous manifolds. At this point, it is a remarkable fact that every $(4n+3)$-dimensional compact regular 3-Sasakian manifold with $n < 4$ is homogeneous. Moreover, it has been conjectured that every regular 3-Sasakian manifold must be homogeneous, [9, p. 498].

The classification of 3-Sasakian homogeneous manifolds (see details in [9, Theorems 13.4.6 and 13.4.7]) shows four families and five exceptional cases, as follows

$$
\frac{Sp(n+1)}{Sp(n)}, \frac{Sp(n+1)}{Sp(n) \times Z_2}, \frac{SU(m)}{SU(m-2) \times U(1)}, \frac{SO(k)}{SO(k-4) \times Sp(1)},
$$

for $n \geq 0$, $m \geq 3$ and $k \geq 7$. In particular, there is a one-to-one correspondence between compact simple Lie algebras and simply-connected 3-Sasakian homogeneous manifolds. In some sense, 3-Sasakian geometry seems to be an interesting geometric point of view to approach to compact simple Lie algebras, specially to the exceptional ones.

Taking into account that every 3-Sasakian homogeneous manifold $M = G/H$ admits a reductive decomposition (though not naturally), the space of invariant affine connections on $M$ can be described in algebraical terms. In fact, in the classical paper [34], starting from a fixed reductive decomposition $g = \mathfrak{h} \oplus \mathfrak{m}$ of the Lie algebra $G$, Katsumi Nomizu established a very fruitful one-to-one correspondence between the set of all invariant connections on $M$ and the set of all bilinear functions $\alpha$ on $\mathfrak{m}$ with values in $\mathfrak{m}$ which are invariant by $Ad(H)$. We will extensively use this correspondence for our purposes.

The paper is organized as follows. Section 2 is devoted to introducing the basic definitions and properties of 3-Sasakian manifolds in order to fix the notations. This section is mainly indebted to [9, Chapter 13]. In particular, we have adopted the very geometric definition of 3-Sasakian structure $\mathcal{S} = \{\xi_\tau, \eta_\tau, \varphi_\tau\}_{\tau \in \mathbb{R}^2}$ as in [9, Definition 13.1.8], where $\mathcal{S}$ is a family of Sasakian structures parametrized on the points of the 2-dimensional sphere $S^2$. Thus, the compatibility conditions on the family of Sasakian structures reduce to the following

$$
g(\xi_\tau, \xi_{\tau'}) = \tau \cdot \tau' \quad \text{and} \quad [\xi_\tau, \xi_{\tau'}] = 2\xi_{\tau \times \tau'},$$

where “$\cdot$” and “$\times$” are the standard inner and cross products in $\mathbb{R}^3$. This section also includes the statement of the classification theorem on 3-Sasakian homogeneous manifolds (Theorem 1) and several basic facts on Riemann-Cartan manifolds and Einstein with skew-torsion connections. Riemann-Cartan manifolds can be seen as a generalization of the Riemann manifolds where the Levi-Civita connection is replaced with a metric affine connection with nonvanishing torsion tensor, in general.

Section 3 continues to give the necessary background, since we have tried to keep the paper as self-contained as possible. Nomizu’s Theorem on invariant connections on homogeneous reductive spaces is stated in the way that we will use it (Theorem 4). Several sets of invariant connections are algebraically characterized. We have enclosed a technical but powerful lemma, which allows us to properly handle covariant derivatives with respect to invariant connections; and several ad hoc algebraical lemmata, in order to compute later the dimensions of the different spaces of invariant connections we are interested in.

Section 4 is the longest part of this paper. In Proposition 13, we exhibit an explicit and complete construction of all the 3-Sasakian homogeneous manifolds $M = G/H$ in a unified way. As far as we know, this construction was not available in the literature. In particular, the Riemannian metric $g$ and the multiplication map $\alpha^q$ corresponding to the Levi-Civita connection via Nomizu’s Theorem are given. Up to a factor, the explicit description of the metrics on 3-Sasakian manifolds was given in [8] (see Remark 14). The reductive decompositions $g = \mathfrak{h} \oplus \mathfrak{m}$ are studied in order to determine the number of free parameters involved in the spaces of invariant connections, metric invariant connections.
and metric with skew-torsion invariant connections, because these numbers coincide, respectively, with the dimensions of the following spaces of \( h \)-module homomorphisms
\[
\text{Hom}_h(m \otimes m, m), \quad \text{Hom}_h(m, m \wedge m), \quad \text{Hom}_h(\wedge^3 m, \mathbb{R}),
\]
when \( H \) is connected. Since they are computed by complexification, it is necessary to know in detail the decomposition of the \( \mathfrak{g}^\mathbb{C} \)-module \( m^\mathbb{C} \) as a sum of irreducible submodules, in each case. It is a remarkable fact that these dimensions do not depend on the particular choice of the invariant metric. In particular, these computations can be applied to every canonical variation along the fibers of the metric \( g \) on each 3-Sasakian homogeneous manifold. Next, Section 4 is subdivided in several subsections. Each one describes, case-by-case, a family of 3-Sasakian homogeneous manifolds and computes the dimensions of the related sets of homomorphisms. A surprising fact on these vector spaces happens: except for the family \( SU(m)/S(U(m-2) \times U(1)) \), and for dimension at least 7, these dimensions are always the same,
\[
\dim \text{Hom}_h(m \otimes m, m) = 63, \quad \dim \text{Hom}_h(m, m \wedge m) = 30, \quad \dim \text{Hom}_h(\wedge^3 m, \mathbb{R}) = 10.
\]
That is, these numbers do not depend on the concrete example we are dealing with. This observation was one of the seminal ideas for this paper. In fact, at the beginning, we were only interested in the family of spheres \( S^{4n+3} = \text{Sp}(n+1)/\text{Sp}(n) \) and we realized that for \( S^7 \) and \( G_2/\text{Sp}(1) \) the numbers 63, 30 and 10 were the same, \([7, 20] \). This feature made us think that the 3-Sasakian structure was behind this numerical coincidence. Then, we found that several remarkable algebraical properties of the reductive decompositions \( g = \mathfrak{h} \oplus \mathfrak{m} \) are common for all 3-Sasakian homogeneous manifolds (see Eqs. (15) and (16), with Remark 28). Section 4 finishes analyzing the remaining family, \( G = SU(m) \) with \( m \geq 3 \), and we found that
\[
\dim \text{Hom}_h(m \otimes m, m) = 99, \quad \dim \text{Hom}_h(m, m \wedge m) = 45, \quad \dim \text{Hom}_h(\wedge^3 m, \mathbb{R}) = 13.
\]
Thus, the exceptional cases now are the manifolds in the family \( SU(m)/S(U(m-2) \times U(1)) \). The concrete computations made to determine the pieces of \( m^\mathbb{C} \) (Section 4.4) shed light on the new invariant torsion tensors.

The main aim of Section 5 is to read the above computations from a more geometrical point of view, and obtain further consequences. Specifically, we provide explicit expressions for all the invariant metric connections with skew-torsion on 3-Sasakian homogeneous manifolds in Corollary 31. By means of Proposition 32, where we compute the symmetric part of the Ricci tensor, we obtain our main result (Theorem 33), namely,

Let \( M = G/H \) be a \((4n+3)\)-dimensional 3-Sasakian homogeneous manifold \((n \neq 0)\). Assume that \( M \) admits an invariant metric connection \( \nabla \) such that \((M, g, \nabla)\) is Einstein with nonzero skew-torsion. Then \( n = 1 \), that is, either \( M = S^7 \), or \( M = \mathbb{RP}^7 \), or \( M \) is the Aloff-Wallach space \( \mathfrak{m}_{7,1} = SU(3)/U(1) \) \([6] \). Moreover, for \( S^7 \) (and \( \mathbb{RP}^7 \)), the set of such connections is parametrized by two copies of the conformal linear transformation group of the Euclidean space, while for \( \mathfrak{m}_{7,1} \), by (the bigger set of) two copies of nonzero elements in
\[
\left\{(c, B) \in \mathbb{R}^3 \times \mathcal{M}_3(\mathbb{R}) : BB^t + cc^t \in \mathbb{R}I_3, \ c^tB = 0 \right\}.
\]
As far as we know, there are no precedents in the literature of homogeneous manifolds in which the set of Einstein connections with skew-torsion is so big and structured.

Finally, Section 6 deals with connections with symmetric Ricci tensor. As already pointed out, the Levi-Civita connection \( \nabla^g \) is not adapted to the 3-Sasakian structure in the sense that \( \nabla^g \xi \neq 0 \) for all the characteristic vector fields \( \xi \). Besides, the 3-Sasakian structure is not parallel for any metric connection with skew-torsion. Then we look for a nontrivial invariant affine connection \( \nabla \) (that is, \( \nabla \) has nonvanishing torsion tensor), Einstein with skew-torsion and such that the characteristic vector fields are \( \nabla \)-parallel. We show in Theorem 42 that there exists a well-adapted one.
Let $M$ be a $(4n + 3)$-dimensional 3-Sasakian manifold ($n \neq 0$) with characteristic vector fields $\{\xi_\tau\}_{\tau \in S^2}$. Then, there exists an affine connection $\nabla^S$ on $M$ such that
\[ \nabla^S \xi_\tau = 0 \text{ for any } \tau, \text{ with skew-torsion given by} \]
\[ \omega_{\xi_\tau}^{\nu} = \eta_1 \wedge d\eta_2 + \eta_2 \wedge d\eta_3 + \eta_3 \wedge d\eta_1 + 4\eta_1 \wedge \eta_2 \wedge \eta_3. \]
Moreover, if $n = 1$, then $(M, g, \nabla^S)$ is Einstein with skew-torsion; and when $M$ is homogeneous, such affine connection is unique among the invariant ones.

The affine connection $\nabla^S$ is not Einstein with skew-torsion for $n > 1$, but it has interesting properties. In fact, its Ricci tensor is always symmetric and satisfies
\[ \text{Ric}^\nabla = \alpha g + \beta \sum_{k=1}^{3} \eta_k \otimes \eta_k \]
for $\alpha, \beta \in \mathbb{R}$ and $\eta_k = g(\xi_k, -)$, where $\{\xi_k\}_{k=1,2,3}$ is an orthonormal basis of characteristic vector fields.

Recall the following definition of [9, Definition 11.1.1]: A contact metric structure $\{\xi, \eta, \varphi, g\}$ on $M$ is said $\eta$-Einstein if there are constants $\alpha, \beta$ such that $\text{Ric}^g = \alpha g + \beta \eta \otimes \eta$. Observe that the Ricci tensors of the Robertson-Walker metrics satisfy a similar property [7, 12.10]. In addition, a similar condition was considered for real hypersurfaces in complex space forms in [13], [27], [30] and [33], and for real hypersurfaces in quaternionic space forms in [32] and [35]. Motivated for these properties, we introduce the notion of $S$-Einstein affine connection on a manifold with a 3-Sasakian structure $S$ (Definition 38). In order to find these connections, we first find out when the Ricci tensor is symmetric in Corollary 40, and then, in Theorem 41, we describe with precision the set of $S$-Einstein invariant affine connections on any 3-Sasakian homogeneous manifold. As a consequence of Theorem 41, any 3-Sasakian manifold (without conditions on the dimension $\geq 7$) possesses a great amount of $S$-Einstein connections, being $\nabla^S$ distinguished among them.

2. Set up

All the manifolds, maps, tensor fields, etc, are assumed to be smooth. Let us briefly recall the basic notions on 3-Sasakian geometry in order to fix some notations. This section is indebted to [9, Chapter 13].

Among the alternative definitions of Sasakian structure, we take the following one. Let $(M, g)$ be a Riemannian manifold with Levi-Civita connection $\nabla^g$. The triple $S = \{\xi, \eta, \varphi\}$ is called a Sasakian structure on $(M, g)$ when $\xi \in \mathfrak{X}(M)$ is a unit Killing vector field, $\varphi$ is the endomorphism field given by $\varphi(X) = -\nabla^g_X \xi$ for all $X \in \mathfrak{X}(M)$, $\eta$ is the 1-form on $M$ metrically equivalent to $\xi$, i.e., $\eta(X) = g(X, \xi)$, and the following condition is satisfied
\[ (\nabla^g_X \varphi)(Y) = g(X, Y)\xi - \eta(Y)X \]
for $X, Y \in \mathfrak{X}(M)$. The vector field $\xi$ and the 1-form $\eta$ are called the characteristic vector field and the characteristic 1-form of the Sasakian structure, respectively. A Sasakian manifold $(M, g)$ endowed with a fixed Sasakian structure $S$. The following consequences of the definition allow to handle properly Sasakian manifolds (see details in [9]),
\[ \varphi^2 = -\text{id} + \eta \otimes \xi, \quad g(\varphi X, \varphi Y) = g(X, Y) - 2\eta(X)\eta(Y), \quad \varphi(\xi) = 0, \quad \eta(\varphi) = 0, \]
\[ g(X, \varphi(Y)) + g(\varphi(X), Y) = 0, \quad d\eta(X, Y) = 2g(X, \varphi(Y)), \]
for all $X, Y \in \mathfrak{X}(M)$.\(^1\)

A 3-Sasakian structure on $(M, g)$ is a family of Sasakian structures $S = \{\xi_\tau, \eta_\tau, \varphi_\tau\}_{\tau \in S^2}$ on $(M, g)$ parametrized by points $\tau \in S^2$ on the 2-dimensional unit sphere and such that, for $\tau, \tau' \in S^2$, the following compatibility conditions are satisfied
\[ g(\xi_\tau, \xi_{\tau'}) = \tau \cdot \tau' \quad \text{and} \quad [\xi_\tau, \xi_{\tau'}] = 2\xi_{\tau \times \tau'}, \]
\(^1\)Our convention is $d\eta(X, Y) = X(\eta(Y)) - Y(\eta(X)) - \eta([X, Y])$.\]
where “·” and “×” are the standard inner and cross products in \( \mathbb{R}^3 \), and we extend the characteristic vector fields from \( S^2 \) to \( \mathbb{R}^3 \) by linearity. The compatibility conditions imply, for all \( \tau, \tau' \in S^2 \),

\[
\varphi_\tau \circ \varphi_{\tau'} - \eta_\tau \otimes \xi_\tau = \varphi_{\tau \times \tau'} - (\tau \cdot \tau') \text{id}, \quad \varphi_\tau(\xi_\tau') = \xi_{\tau \times \tau'}, \quad \eta_\tau \circ \varphi_{\tau'} = \eta_{\tau \times \tau'}.
\]

In order to construct a 3-Sasakian structure on a Riemannian manifold \((M, g)\), we need only to fix three Sasakian structures \( S_k = \{\xi_k, \eta_k, \varphi_k\} \), for \( k = 1, 2, 3 \), such that \( g(\xi_i, \xi_j) = \delta_{ij} \) and \( [\xi_i, \xi_j] = 2\epsilon_{ijk} \xi_k \) (here \( \epsilon_{ijk} \) denotes the sign of the permutation). In fact, it is an easy matter to extend the characteristic vector fields to the whole sphere \( S^2 \) by means of the formula \( \xi_v = \sum_{k=1}^3 v_k \xi_k \) for \( v = (v_1, v_2, v_3)^t \in S^2 \). Throughout this text, we will call 3-Sasakian structure both to \( S \) \( \tau \in S^2 \) and to \( \{S_k\}_{k=1, 2, 3} \). The 3-dimensional fundamental foliation \( F_Q \) is generated by the characteristic vector fields \( \{\xi_\tau\}_{\tau \in S^2} \) and the distribution \( Q = F_Q^\perp \) is called the quaternionic distribution of \( M \).

A remarkable result stated by Kashiwada (cf. [29]) shows that every 3-Sasakian manifold of dimension \( 4n + 3 \) is Einstein, with Ricci tensor satisfying \( \text{Ric}^o = 2(2n + 1)g \).

Given \((M, g, S)\) a 3-Sasakian manifold, the automorphism group \( \text{Aut}(M, g, S) \) is given by

\[
\text{Aut}(M, g, S) = \bigcap_{\tau \in S^2} \text{Aut}(M, g, S_\tau),
\]

where \( \text{Aut}(M, g, S_\tau) \subset \text{Iso}(M, g) \) is the subgroup of isometries \( f \) of \( M \) which preserve \( \xi_\tau \) (and then \( \eta_\tau \) and \( \varphi_\tau \)). That is, \( f_\tau(\xi_\tau(p)) = \xi_\tau(f(p)) \) for every \( p \in M \). We will mainly focus on 3-Sasakian homogeneous manifolds, that is, 3-Sasakian manifolds \( M \) where \( \text{Aut}(M, g, S) \) acts transitively. If \((M, g, S)\) is a 3-Sasakian homogeneous manifold, then the orbits space \( M/F_Q \) is a quaternionic Kähler homogeneous manifold [9, Prop. 13.4.5]. Then, the next classification theorem is achieved ([5]).

**Theorem 1.** Any 3-Sasakian homogeneous manifold is one of the following coset manifolds:

\[
\begin{align*}
\text{Sp}(n+1) & / \text{Sp}(n), & \text{Sp}(n+1) & / \text{Sp}(n) \times \mathbb{Z}_2, & \text{SU}(m) & / (\text{U}(m-2) \times \text{U}(1))^3, & \text{SO}(k) & / (\text{SO}(k-4) \times \text{Sp}(1)), \\
\text{Sp}(1) & / \text{Sp}(1), & \text{Sp}(3) & / \text{SU}(6), & \text{Spin}(12) & / \text{E}_7.
\end{align*}
\]

for \( n \geq 0 \), \( m \geq 3 \) and \( k \geq 7 \) (\( \text{Sp}(0) \) denoting the trivial group).

For the explicit description of the metrics of the 3-Sasakian homogeneous manifolds, see Proposition 13 below.

Observe that \( \dim G/H = 4n + 3 \) in all the cases, with \( n = m - 2 \) for the quotients of the unitary groups, \( n = k - 4 \) in the orthogonal case, and \( n = 2, 7, 10, 16 \) and 28, respectively, in the exceptional cases. The only 3-dimensional 3-Sasakian homogeneous manifolds are the sphere \( S^3 \) and the projective space \( \mathbb{R}P^3 \), which behave different from the rest, because their quaternionic distributions reduce to \( \{0\} \).

As a consequence of Theorem 1, all 3-Sasakian homogeneous manifolds are simply-connected except the real projective spaces \( \mathbb{R}P^{4n+3} \cong \text{Sp}(n+1) / \text{Sp}(n) \times \mathbb{Z}_2 \). In particular, there is a one-to-one correspondence between compact simple Lie algebras and simply-connected 3-Sasakian homogeneous manifolds.

Let us recall the notion of Riemann-Cartan manifold, since this is the second main topic of this paper (cf. [1]). A Riemann-Cartan manifold is a triple \((M, g, \nabla)\), where \((M, g)\) is a Riemannian manifold and \( \nabla \) is a metric affine connection, that is to say, \( \nabla g = 0 \). It is also commonly said that \( \nabla \) is compatible with the metric \( g \). The torsion tensor field of \( \nabla \) is defined as usual by \( T^\nabla(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y] \), for \( X, Y \in \mathfrak{X}(M) \), and it does not vanish in general. Clearly, these manifolds can be seen as a generalization of Riemannian manifolds, since the considered metric affine connection may be different from the Levi-Civita connection \( \nabla^g \), which is characterized by the condition \( T^{\nabla^g} = 0 \). For \((M, g, \nabla)\) a Riemann-Cartan manifold, we set

\[
\omega_\nabla(X, Y, Z) := g(T^\nabla(X, Y), Z),
\]
Then, in a 3-Sasakian manifold, each Sasakian structure whose spinorial properties in dimension 7 have been widely studied in [3].

Remark 2. For every Sasakian structure \( \{ \xi, \eta, \varphi \} \) on \((M, g)\), there is a unique metric connection with totally skew-symmetric torsion \( \nabla^v \) such that \( \nabla^v \xi = 0, \nabla^v \eta = 0 \) and \( \nabla^v \varphi = 0 \), [22]. An explicit formula for this connection is given by

\[
g(\nabla^v_X Y, Z) = g(\nabla^g_X Y, Z) + \frac{1}{2} \eta \wedge d\eta(X, Y, Z).
\]

Then, in a 3-Sasakian manifold, each Sasakian structure \( \{ \xi, \eta, \varphi \} \) has a related metric connection as above, whose spinorial properties in dimension 7 have been widely studied in [3]. □

For any metric affine connection \( \nabla \), we define the difference \((1, 2)\)-tensor \( D = \nabla - \nabla^g \) on \( M \). The torsion \( T^\nabla \) satisfies \( T^\nabla(X, Y) = D(X, Y) - D(Y, X) \), for any \( X, Y \in \mathfrak{X}(M) \). The connections \( \nabla \) and \( \nabla^g \) share the same geodesics if, and only if, the difference tensor \( D \) is skew-symmetric. In such case, we have

\[
\nabla = \nabla^g + \frac{1}{2} T^\nabla.
\]

Given \( \nabla \) an affine connection on \( M \), it is always possible to compute its Ricci tensor, but it is not symmetric in general. Thus, a Riemann-Cartan manifold \((M, g, \nabla)\) is said to be Einstein with skew-torsion (cf. [2]) if the metric affine connection \( \nabla \) has totally skew-symmetric torsion and satisfies

\[
\text{Sym}(\text{Ric}^\nabla) = \frac{s^\nabla}{\dim M} g,
\]

where \( s^\nabla \) is the corresponding scalar curvature given by \( s^\nabla = \sum_{i,j=1}^n g(R^\nabla(e_i, e_j)e_i, e_j) \) for an orthonormal basis and, following [1], \( \text{Sym}(\text{Ric}^\nabla) \) denotes the symmetric part of the Ricci (curvature) tensor of \( \nabla \). The metric connections such that \((M, g, \nabla)\) is Einstein with skew-torsion are the critical points of a variational problem which involves the scalar curvature of the Levi-Civita connection \( \nabla^g \) and the torsion of \( \nabla \) (see details in [2]). For brevity, we will also say that \( \nabla \) is an Einstein with skew-torsion affine connection.

We recall from [22] some curvature identities on Riemann-Cartan manifolds with totally skew-symmetric torsion. Let \( S \in T^{0, 2}(M) \) be the tensor given at \( p \in M \) by

\[
S(X, Y)_p := \sum_{j=1}^n g(T^\nabla(e_j, X_p), T^\nabla(e_j, Y_p)),
\]

where \( \{ e_1, \ldots, e_n \} \) is an orthonormal basis of \( T_p M \) and \( X, Y \in \mathfrak{X}(M) \). The Ricci tensor of the Levi-Civita connection, denoted by \( \text{Ric}^g \), and the the Ricci tensor of \( \nabla \) are related by

\[
\text{Ric}^\nabla = \text{Ric}^g - \frac{1}{4} S + \frac{1}{2} \text{div}(T^\nabla),
\]

where \( \text{div}(T^\nabla) \) denotes the divergence of the torsion form given by

\[
\text{div}(T^\nabla)(X, Y) = \sum_{i=1}^n (\nabla^g_{e_i} \omega_\varphi)(X, Y, e_i).
\]

In particular, since \( \text{Ric}^g \) and \( S \) are symmetric tensors but \( \text{div}(T^\nabla) \) is skew-symmetric, we get

\[
\text{Sym}(\text{Ric}^\nabla) = \text{Ric}^g - \frac{1}{4} S, \quad \text{Skew}(\text{Ric}^\nabla) = \frac{1}{2} \text{div}(T^\nabla).
\]

Then, \( \text{Ric}^\nabla \) is symmetric if, and only if, \( \text{div}(T^\nabla) = 0 \). For instance, this holds if the torsion is \( \nabla \)-parallel.
Nomizu’s Theorem can be stated as follows:

\[ g \circ \pi = \pi^* \delta \]

where \( \pi \) is the projection. If \( H \) is connected, this condition gives a bijection correspondence between these connections and certain homomorphisms of modules. This allows to compute the size of the set of invariant connections.

Let \( G \) be a Lie group acting transitively on a manifold \( M \). We write a dot to denote the action of \( G \) on \( M \) and so, for \( \sigma \in G \), the left translation by \( \sigma \) will be given by \( \tau_\sigma(p) = \sigma \cdot p \). For each \( \sigma \in G \) and \( X \in \mathfrak{X}(M) \), the vector field \( \tau_\sigma(X) \in \mathfrak{X}(M) \) is given at each \( p \in M \) by

\[ (\tau_\sigma(X))_p := (\tau_\sigma)_*(X_{\sigma^{-1} \cdot p}). \]

An affine connection \( \nabla \) on \( M \) is said to be \( G \)-invariant if, for each \( \sigma \in G \) and for all \( X,Y \in \mathfrak{X}(M) \),

\[ \tau_\sigma(\nabla XY) = \nabla \tau_\sigma(X) \tau_\sigma(Y). \]

Let \( H \) be the isotropy subgroup at a fixed point \( o \in M \), so that there exists a diffeomorphism between \( M \) and \( G/H \). The homogeneous space \( M = G/H \) is said to be reductive if the Lie algebra \( \mathfrak{g} \) of \( G \) admits a vector space decomposition

\[ \mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}, \]

for \( \mathfrak{h} \) the Lie algebra of \( H \) and \( \mathfrak{m} \) an \( \text{Ad}(H) \)-invariant subspace (i.e., \( \text{Ad}(H)(\mathfrak{m}) \subset \mathfrak{m} \)). In this case, \( \mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m} \) is called a \textit{reductive decomposition} of \( \mathfrak{g} \). The condition \( \text{Ad}(H)(\mathfrak{m}) \subset \mathfrak{m} \) implies that \( [\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m} \), and both are equivalent conditions when \( H \) is connected. The differential map \( \pi_* \) of the projection \( \pi: G \to M = G/H \) gives a linear isomorphism \( (\pi_*)_*: \mathfrak{m} \to T_oM \), where \( o = \pi(e) \).

Nomizu’s Theorem, [34], can be stated as follows:

**Theorem 4.** Let \( G/H \) be a reductive homogeneous space with a fixed reductive decomposition as in (6). Then, there is a one-to-one correspondence between the set of \( G \)-invariant linear connections \( \nabla \) on \( G/H \) and the vector space of bilinear maps \( \alpha: \mathfrak{m} \times \mathfrak{m} \to \mathfrak{m} \) such that \( \text{Ad}(H)(\mathfrak{m}) \subset \text{Aut}(\mathfrak{m}, \alpha) \). In case \( H \) is connected, this condition \( \text{Ad}(H) \subset \text{Aut}(\mathfrak{m}, \alpha) \) is equivalent to

\[ [A, \alpha(X,Y)] = \alpha([A,X], Y) + \alpha(X,[A,Y]) \]

for all \( X,Y \in \mathfrak{m} \) and \( A \in \mathfrak{h} \) (i.e., \( \text{ad}(\mathfrak{h}) \subset \text{Der}(\mathfrak{m}, \alpha) \)).

We should bear in mind that, under the above identification \( (\pi_*)_*: \mathfrak{m} \), the correspondence given by Nomizu’s Theorem works as follows

\[ \nabla \mapsto \alpha_\psi(X,Y) = \nabla_X Y - [X,Y]_\alpha, \quad X,Y \in \mathfrak{m}, \]

for \( \nabla \) a \( G \)-invariant affine connection on \( G/H \) and \( \alpha_\psi \) the associated bilinear map on \( \mathfrak{m} \). (The key point is that \( L(X,Y) = \nabla_X Y - [X,Y] \) defines a tensor.)

The reductive complement \( \mathfrak{m} \) is an \( \mathfrak{h} \)-module and so, in a natural way, the tensor product \( \mathfrak{m} \otimes \mathfrak{m} \) is also an \( \mathfrak{h} \)-module for \( A \cdot X \otimes Y = [A,X] \otimes Y + X \otimes [A,Y] \). Note that any bilinear map \( \alpha \) as in (7) allows to construct a homomorphism of \( \mathfrak{h} \)-modules \( \tilde{\alpha}: \mathfrak{m} \otimes \mathfrak{m} \to \mathfrak{m} \) by means of \( \tilde{\alpha}(X \otimes Y) = \alpha(X,Y) \) for all \( X,Y \in \mathfrak{m} \), and conversely. We will usually identify \( \alpha \) and \( \tilde{\alpha} \). So, Nomizu’s Theorem can be read in the following terms.

**Corollary 5.** Let \( G/H \) be a reductive homogeneous space with a fixed reductive decomposition as in (6) and \( H \) connected. Then, there is a bijective correspondence between the set of \( G \)-invariant affine connections on \( G/H \) and the vector space \( \text{Hom}_h(\mathfrak{m} \otimes \mathfrak{m}, \mathfrak{m}) \).
From now on, we extensively use the identification \((\pi_*)_e|m: m \to T_oM\). Given \(\nabla\) a \(G\)-invariant affine connection on \(G/H\) and \(\alpha_\nu\) the associated bilinear map as in (8), the torsion and curvature tensors of the \(G\)-invariant affine connection \(\nabla\) are computed in [34] as follows:

\[
\begin{align*}
9. & \quad T^\nabla(X,Y) = \alpha_\nu(X,Y) - \alpha_\nu(Y,X) - [X,Y]_m, \\
10. & \quad R^\nabla(X,Y)Z = \alpha_\nu(X,\alpha_\nu(Y,Z)) - \alpha_\nu(Y,\alpha_\nu(X,Z)) - \alpha_\nu([X,Y]_m,Z) - [[X,Y]_b,Z],
\end{align*}
\]

for any \(X,Y,Z \in m\), where \([,]_b\) and \([,]_m\) denote the composition of the bracket \([m,m] \subset g\) with the projections \(\pi_b\) and \(\pi_m\) of \(g = h \oplus m\) on each factor, respectively. These expressions give \(T^\nabla\) and \(R^\nabla\) at the point \(o = \pi(e)\), but the invariance permits to recover the whole tensors.

In order to deal with the covariant derivative of arbitrary tensors at the point \(o\) with respect to an invariant affine connection \(\nabla\), we include the following technical result.

**Lemma 6.** Let \(M = G/H\) be a reductive homogeneous space with a fixed reductive decomposition as in (6) and \(\nabla\) a \(G\)-invariant affine connection. For every tensor field of type \((1,k)\), \(T \in T^{(1,k)}(M)\), the following formula holds

\[
(\nabla_Z T)(X_1, ..., X_k) = \alpha_\nu(Z, T(X_1, ..., X_k)) - \sum_{i=1}^k T(X_1, ..., \alpha_\nu(Z, X_i), ..., X_k),
\]

where \(Z, X_1, ..., X_k \in m \approx T_oM\).

**Proof.** For every \(X \in g\), let us consider the fundamental vector field \(X^+ \in \mathfrak{x}(M)\) defined by

\[
X^+_p := \left. \frac{d}{dt}\right|_0 (\exp(tX) \cdot p).
\]

A direct computation shows \(X^+_o = (\pi_*)_e(X)\) and therefore \(X \in m\) corresponds with \(X^+_o\) via the identification \((\pi_*)_e|m\). In these terms, the bilinear map \(\alpha_\nu\) corresponding to \(\nabla\) via Nomizu’s Theorem is given by

\[
\alpha_\nu(X,Y) = \nabla_X W^Y - [X^+, W^Y]_o,
\]

for \(X \in m\) and \(W^Y \in \mathfrak{x}(M)\) is an arbitrary extension of \(Y \in T_oM\). The tangent vector \(Y \in m \approx T_oM\) can be extended to the vector field \(W^Y\) in a convenient way. Indeed, we choose the extension \(W^Y\) such that \([X^+, W^Y]_o = 0\). For \(X = 0\), the vector field \(X^+\) vanishes identically and any extension \(W^Y\) satisfies the required property. On the contrary, when \(X \neq 0\), we take a local coordinate system \((x_1, ..., x_n)\) centered at \(o \in M\) and such that \(X^+_o = \frac{\partial}{\partial x_1}|_o\). We have the local coordinate expressions \(X^+ = \sum f_i \frac{\partial}{\partial x_i}\) and \(W^Y = \sum h_i \frac{\partial}{\partial x_i}\). Thus, the conditions \([X^+, W^Y]_o = 0\) and \(W^Y_o = Y\) become

\[
\left. \frac{\partial h_i}{\partial x_j}\right|_o = \sum_{j=1}^n y_j \left. \frac{\partial f_i}{\partial x_j}\right|_o, \quad h_i(o) = y_i, \quad (i = 1, ..., n)
\]

where \(Y = \sum_{i=1}^n y_i \frac{\partial}{\partial x_i}|_o\). Now, it is clear how to construct the vector field \(W^Y\) and we obtain the short formula

\[
\alpha_\nu(X,Y) = \nabla_X W^Y,
\]

for our convenient extension \(W^Y\). Finally, the proof of (11) is an easy matter by using (12). Indeed, a straightforward computation shows

\[
(\nabla_Z T)(X_1, ..., X_k) = \nabla_Z (W^T(X_1, ..., X_k)) - \sum_{i=1}^k T(X_1, ..., \nabla_Z W^X_i, ..., X_k)
\]

\[
= \alpha_\nu(Z, T(X_1, ..., X_k)) - \sum_{i=1}^k T(X_1, ..., \alpha_\nu(Z, X_i), ..., X_k),
\]

by using here suitable extensions of the tangent vectors \(X_1, ..., X_k, T(X_1, ..., X_k) \in T_oM\). □
If our reductive homogeneous space \( G/H \) is endowed with a \( G \)-invariant Riemannian metric \( g \), the identification of \( \mathfrak{m} \) with \( T_eM \) via \( \pi_* \), allows to consider the orthogonal Lie algebra \( \mathfrak{so}(m, g) \) corresponding to the \( \text{Ad}(H) \)-invariant nondegenerate symmetric bilinear map induced by \( g \). In particular, we have \( \text{ad}(\mathfrak{h}) \subset \mathfrak{so}(m, g) \), so we can consider the \( \mathfrak{h} \)-module structure on \( \mathfrak{so}(m, g) \) given by \( (A \cdot \psi)(X) = [A, \psi(X)] - \psi([A, X]) \). Recall also the usual \( \mathfrak{h} \)-module structure on \( \Lambda^2 \mathfrak{m} \) given by \( A \cdot (X \wedge Y) = [A, X] \wedge Y + X \wedge [A, Y] \). Then, \( \Lambda^2 \mathfrak{m} \) and \( \mathfrak{so}(m, g) \) are isomorphic \( \mathfrak{h} \)-modules by means of \( X \wedge Y \mapsto g(X, -)Y - g(Y, -)X \in \mathfrak{so}(m, g) \). The outcome is another version of Nomizu’s Theorem (see [18, Theorem 2.7, Remark 2.8]).

**Lemma 7.** Let \( (G/H, g) \) be a reductive homogeneous Riemannian manifold with reductive decomposition \( \mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m} \) and \( H \) connected. Then, a \( G \)-invariant affine connection \( \nabla \) is metric \((\nabla g = 0)\) if, and only if, the related bilinear map \( \alpha_\nabla : \mathfrak{m} \times \mathfrak{m} \to \mathfrak{m} \) satisfies \( \alpha_\nabla (X, -) \in \mathfrak{so}(m, g) \) for all \( X \in \mathfrak{m} \). Therefore, there is a bijective correspondence between the set of connections on \( G/H \) which are \( G \)-invariant and \( g \)-metric and compatible with \( g \), and the vector space \( \text{Hom}_\mathfrak{h}(\mathfrak{m}, \mathfrak{so}(m, g)) \) (or alternatively with \( \text{Hom}_\mathfrak{h}(\mathfrak{m}, \Lambda^2 \mathfrak{m}) \)).

It will be specially useful for our purposes to also set Nomizu’s Theorem for the case of invariant metric connections with skew-torsion on \( G/H \). Let us denote such set by \( \mathcal{C}_S(G/H, g) \).

**Lemma 8.** Let \( (G/H, g) \) be a reductive homogeneous Riemannian manifold with reductive decomposition \( \mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m} \) and \( H \) connected. There is a one-to-one correspondence between \( \mathcal{C}_S(G/H, g) \) and \( \text{Hom}_\mathfrak{h}(\Lambda^3 \mathfrak{m}, \mathbb{R}) \) \((\mathbb{R} \text{ as trivial } \mathfrak{h}-\text{module})\).

**Proof.** Let us consider the map \( \Theta : \mathcal{C}_S(G/H, g) \to \text{Hom}_\mathfrak{h}(\Lambda^3 \mathfrak{m}, \mathbb{R}) \) given by

\[
\Theta(\nabla)(x \wedge y) = g(\alpha_\nabla(x, y) - \alpha^g(x, y), z),
\]

where \( \alpha^g = \alpha_\nabla \) denotes the bilinear map associated to the Levi-Civita connection by Nomizu’s Theorem. It is easy to conclude that \( \Theta(\nabla) \) is a \( \mathfrak{h} \)-module homomorphism, since \( \alpha_\nabla, \alpha^g \) and \( g \) so are. The injectivity of \( \Theta \) is consequence of the nondegeneracy of \( g \). Finally, given \( \omega \in \text{Hom}_\mathfrak{h}(\Lambda^3 \mathfrak{m}, \mathbb{R}) \), there exists a unique bilinear map \( T : \mathfrak{m} \times \mathfrak{m} \to \mathfrak{m} \) such that \( \omega(x \wedge y) = g(T(x, y), z) \) for any \( x, y, z \in \mathfrak{m} \). We also denote by \( \omega \) and \( T \) the natural extensions of \( \omega \) and \( T \) to the whole manifold \( G/H \) by using the \( G \)-invariance. Now, the invariant connection \( \nabla = \nabla^g + (1/2)T \) has torsion \( T \), is compatible with the metric \( g \), and satisfies \( \Theta(\nabla) = 1/2\omega \). \( \Box \)

Our next target will be to compute the dimensions of

\[
\text{Hom}_\mathfrak{h}(\mathfrak{m} \otimes \mathfrak{m}, \mathfrak{m}), \quad \text{Hom}_\mathfrak{h}(\mathfrak{m}, \mathfrak{m} \wedge \mathfrak{m}), \quad \text{Hom}_\mathfrak{h}(\mathfrak{m} \wedge \mathfrak{m}, \mathfrak{m}),
\]

for each reductive decomposition related to any 3-Sasakian homogeneous manifold \( G/H \) in Theorem 1. These vector spaces will be useful to describe, respectively, the set of \( G \)-invariant affine connections on \( G/H \), those ones which are besides compatible with an invariant metric, and those ones that have also skew-torsion. We need precise descriptions of such reductive decompositions as well as of the involved modules, which will be provided case-by-case in the next section. As mentioned in Introduction, up to the case \( \text{SU}(m)/S(\text{U}(m-2) \times \text{U}(1)) \), the dimensions of the above three vector spaces turn out to be the same for all 3-Sasakian homogeneous manifolds (see Proposition 27).

We finish this section with some ad-hoc facts involving representation theory of complex Lie algebras. (A textbook including the algebraic concepts relative to Lie algebras and their representations is [25], for instance.) These results will be used through the paper to compute the dimensions in (13). Although well-known, we include them here for the sake of completeness. As usual, we denote by \( pV, p \in \mathbb{N} \), to the module which is a direct sum of \( p \) submodules all of them isomorphic to \( V \).

**Lemma 9.** Let \( V \) and \( W \) be \((\text{finite-dimensional})\) modules for a complex Lie algebra \( L \).

i) If \( V \) and \( W \) are irreducible, then the vector space \( \text{Hom}_L(V, W) \) is one-dimensional if \( V \) and \( W \) are isomorphic \( L \)-modules and it is 0 otherwise.
ii) If \( V \cong (\oplus, p_iV_i) \oplus (\oplus, n_jU_j) \) and \( W \cong (\oplus, q_kW_k) \oplus (\oplus, m_jU_j) \), with \( V_i, W_k \) and \( U_j \) (finite-dimensional) irreducible \( L \)-modules which are not isomorphic, then
\[
\dim \text{Hom}_L(V, W) = \sum n_jm_j.
\]

Schur’s lemma gives item i), while item ii) is an immediate consequence. If besides \( L \) is semisimple, all finite-dimensional \( L \)-modules \( V \) and \( W \) are completely reducible (sum of irreducible submodules) and then we can always apply item ii) to them. In particular, \( \dim \text{Hom}_L(V, W) = \dim \text{Hom}_L(W, V) \).

So, for complex representations, such dimensions are computed by finding the irreducible submodules with their corresponding multiplicities. We denote by \( S^nW \) and \( \Lambda^nW \) the \( n \)th symmetric and alternating tensor power of the module \( W \), respectively.

**Lemma 10.** Let \( U \) and \( W \) be modules for a complex Lie algebra \( L \).

a) \( U \otimes U \cong S^2U \oplus \Lambda^2U \).

b) \( \Lambda^n(U \oplus W) \cong \oplus_{k=0}^n \Lambda^{n-k}U \otimes \Lambda^kW \) and \( S^n(U \oplus W) \cong \oplus_{k=0}^n S^{n-k}U \otimes S^kW \).

c) In particular, for \( C \) the trivial \( L \)-module and \( W = 2U \oplus 3C \),
\[
\begin{align*}
W \otimes W &\cong 4(U \otimes U) \oplus 12U \oplus 9C, \\
\Lambda^3W &\cong 3\Lambda^2U \oplus S^2U \oplus 6U \oplus 3C, \\
\Lambda^3W &\cong 2\Lambda^3U \oplus 2(\Lambda^2U \otimes U) \oplus 9\Lambda^2U \oplus 3S^2U \oplus 6U \oplus 3C.
\end{align*}
\]

We will check in Section 4 that for any 3-Sasakian homogeneous manifold \( M \), the complexification of the \( \mathfrak{h} \)-module \( \mathfrak{m} \cong T_\mathfrak{m}M \) is always in the situation of item c). Moreover, under certain technical conditions, we will compute the required dimensions in (13) in a unified way, as follows.

**Corollary 11.** Let \( L \) be a complex Lie algebra and \( U \) an irreducible nontrivial \( L \)-module such that \( \dim \text{Hom}_L(\Lambda^2U, C) = 1 \) and \( \dim \text{Hom}_L(S^2U, U) = \dim \text{Hom}_L(\Lambda^2U, U) = \dim \text{Hom}_L(S^2U, C) = 0 \). Then, for the \( L \)-module \( W = 2U \oplus 3C \), we have
\[
\dim \text{Hom}_L(W \otimes W, W) = 63, \quad \dim \text{Hom}_L(W, \Lambda^2W) = 30.
\]
If, besides, neither \( \Lambda^3U \) nor \( \Lambda^2U \otimes U \) contains any trivial submodule (that is, a submodule isomorphic to \( C \)), then
\[
\dim \text{Hom}_L(\Lambda^3W, C) = 10.
\]

**Proof.** Bearing in mind that \( W \otimes W \cong 4S^2U \oplus 4\Lambda^2U \oplus 12U \oplus 9C \), our assumptions imply that \( W \otimes W \) contains \( 4 + 9 = 13 \) copies of \( C \) and \( 12 \) copies of \( U \). Then, Lemma 9 ii) gives \( \dim \text{Hom}_L(W \otimes W, W) = 12 \cdot 2 + 13 \cdot 3 = 63 \). In a similar way, \( \Lambda^3W \cong 3\Lambda^2U \oplus S^2U \oplus 6U \oplus 3C \) contains \( 3 + 3 = 6 \) copies of \( C \) and \( 6 \) copies of \( U \), so that \( \dim \text{Hom}_L(W, \Lambda^2W) = 6 \cdot 2 + 6 \cdot 3 = 30 \). The third dimension is immediately deduced from Lemma 10 c). \( \square \)

4. 3-Sasakian homogeneous manifolds

Now we provide an explicit and detailed construction of all 3-Sasakian homogeneous manifolds in algebraic terms. These precise descriptions will be as self-contained as possible and will be used throughout the text. The common information is summarized in Proposition 13. Later, the following subsections are devoted to making the computations for each case.

The starting point will be that each 3-Sasakian homogeneous manifold is the total space of a \( \text{Sp}(1) \) or a \( \text{SO}(3) \) principal bundle over a symmetric space, [9, Proposition 13.4.5]. More specifically, for a 3-Sasakian homogeneous manifold \( M = G/H \), the leaves space \( M/\mathcal{F}_Q \) is a symmetric space ([5], [9, Chapter 13]), and we have the natural projection
\[
M = G/H \rightarrow M/\mathcal{F}_Q.
\]

The reductive decomposition related to the 3-Sasakian homogeneous manifold \( M \) is introduced from the \( \mathbb{Z}_2 \)-grading related to the corresponding symmetric space \( M/\mathcal{F}_Q \). To be precise, let us consider
the $\mathbb{Z}_2$-grading on $\mathfrak{g}$ or symmetric decomposition $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ (that is, $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$). The even part $\mathfrak{g}_0 = \mathfrak{sp}(1) \oplus \mathfrak{h}$ is is sum of two ideals, which immediately implies that $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ is a reductive decomposition for $\mathfrak{m} = \mathfrak{sp}(1) \oplus \mathfrak{g}_1$ (see Definition 12).

In order to simplify computations, we would like to point out that, up to the projective spaces $\mathbb{P}^{4n+3} = \mathbb{P}^{n+1}/\mathbb{P}(n) \times \mathbb{Z}_2$, for every 3-Sasakian homogeneous manifold $G/H$ the Lie subgroup $H$ is connected. Therefore, from now on, we assume that $H$ is connected and the projective cases will be studied in Remark 21 in a separate way.

Before the next definition, recall that if $W_i$ is an $L_i$-module, $i = 1, 2$, for $L_i$ any Lie algebra, then $W_1 \otimes W_2$ is an $L_1 \oplus L_2$-module for the action $(x_1 + x_2) \cdot (w_1 \otimes w_2) = (x_1 \cdot w_1) \otimes w_2 + w_1 \otimes (x_2 \cdot w_2)$.

**Definition 12.** A 3-Sasakian data is a pair of Lie algebras $(\mathfrak{g}, \mathfrak{h})$ such that

- $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ is a $\mathbb{Z}_2$-graded compact simple Lie algebra (that is, $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$, $i, j \in \{0, 1\}$) whose even part is sum of two ideals,

$$\mathfrak{g}_0 = \mathfrak{sp}(1) \oplus \mathfrak{h},$$

- and there exists an $\mathfrak{h}^C$-module $W$ such that the complexified $\mathfrak{g}_1^C$-module $\mathfrak{g}_1^C$ is isomorphic to the tensor product of the natural $\mathfrak{sp}(1)^C \cong \mathfrak{sl}(2, \mathbb{C})$-module $\mathbb{C}^2$ given by multiplication on the columns and $W$, that is,

$$\mathfrak{g}_1^C \cong \mathbb{C}^2 \otimes W.$$

In particular, $\dim_{\mathbb{R}} \mathfrak{m} = 4n + 3$ holds for $n = \dim_{\mathbb{C}} W$. We will prove in Remark 28 that Eq. (16) can be obtained as a consequence of Eq. (15) under certain conditions, for instance, when the algebra $\mathfrak{h}$ is semisimple and the $\mathfrak{h}$-module $\mathfrak{g}_1$ is irreducible. These conditions will be satisfied in most of our cases.

**Proposition 13.** Let $M^{4n+3} = G/H$ be a homogeneous space with $H$ connected such that the pair $(\mathfrak{g}, \mathfrak{h})$ is a 3-Sasakian data, being $\mathfrak{g}$ and $\mathfrak{h}$ the Lie algebras of $G$ and $H$ respectively. Take $\mathfrak{m} = \mathfrak{sp}(1) \oplus \mathfrak{g}_1$ (for the subalgebra of type $\mathfrak{sp}(1)$ in (15)).

- i) The decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ is a reductive decomposition.
- ii) Let $g$ be the Riemannian metric on $M$ corresponding to the $\text{Ad}(H)$-invariant inner product on $\mathfrak{m}$ given by

$$g|_{\mathfrak{sp}(1)} = -\frac{1}{4(n+2)}\kappa, \quad g|_{\mathfrak{g}_1} = -\frac{1}{8(n+2)}\kappa, \quad g|_{\mathfrak{sp}(1) \times \mathfrak{g}_1} = 0,$$

where $\kappa$ is the Killing form of $\mathfrak{g}$. Then, the multiplication $\alpha^g: \mathfrak{m} \times \mathfrak{m} \to \mathfrak{m}$ corresponding via Nomizu’s Theorem with the Levi-Civita connection $\nabla^g$ satisfies

$$\alpha^g(X, Y) = \begin{cases} 0 & \text{if } X \in \mathfrak{sp}(1) \text{ and } Y \in \mathfrak{g}_1, \\ \frac{1}{\kappa}[X, Y]_m & \text{if either } X, Y \in \mathfrak{sp}(1) \text{ or } X, Y \in \mathfrak{g}_1, \\ [X, Y]_m & \text{if } X \in \mathfrak{g}_1 \text{ and } Y \in \mathfrak{sp}(1). \end{cases}$$

- iii) Let $\{\xi_i\}_{i=1}^3$ be the $G$-invariant vector fields on $M$ corresponding to the following basis of $\mathfrak{sp}(1) = \mathfrak{su}(2),$

$$\xi_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \xi_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \xi_3 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}.$$

Then, the endomorphism field $\varphi_i = -\nabla^g \xi_i$ satisfies

$$\varphi_i|_{\mathfrak{sp}(1)} = \frac{1}{2} \text{ad} \xi_i, \quad \varphi_i|_{\mathfrak{g}_1} = \text{ad} \xi_i,$$

for each $i = 1, 2, 3$, and $S_i = \{\xi_i, \eta_i, \varphi_i\}$ is a Sasakian structure for $\eta_i = g(\xi_i, -)$. In addition, $S_1, S_2$ and $S_3$ satisfy the compatibility conditions (1), hence providing a 3-Sasakian structure on $M$. 

Proof. Item i) is clear. For item ii), note that the inner product on \(\mathfrak{m}\) defined in (17) is \(\text{Ad}(H)\)-invariant, so that it extends to a \(G\)-invariant Riemannian metric on \(M\), also denoted by \(g\). Next, we show that \(g(\alpha^g(X, Y), Z) + g(Y, \alpha^g(X, Z)) = 0\) for \(X, Y, Z \in \mathfrak{m}\). Indeed, the bilinear map \(\alpha^g\) satisfies
\[
\alpha^g(X, Y) = \frac{1}{2}[X^v, Y^v] + \frac{1}{2}[X^h, Y^h]_{\text{sp}(1)} + [X^h, Y^v],
\]
where the superscripts \(v\) and \(h\) denote the projections from \(\mathfrak{m}\) on \(\text{sp}(1)\) and \(\mathfrak{g}_1\), respectively (note that \([X^h, Y^v] \in \mathfrak{g}_1\)). Now, from (21) and the associativity of the Killing form, one deduces that
\[
g(\alpha^g(X, Y), Z) = -\frac{1}{8(n+2)}\kappa \left( [X^v, Y^v] + [X^h, Y^h]_{\text{sp}(1)} + [X^h, Y^v], Z \right) = -g(\alpha^g(X, Z), Y),
\]
since \(\kappa(\mathfrak{sp}(1), \mathfrak{g}_1) = 0\). Thus, the affine connection corresponding to \(\alpha^g\) is compatible with the metric. Also, one easily shows that its torsion tensor vanishes identically.

The formulae in (20) for the endomorphism fields \(\varphi_i = -\nabla^g\xi_i\) are a straightforward computation from (12) and the explicit expression for \(\alpha^g\). In order to deduce that every \(\mathcal{S}_i = \{\xi_i, \eta_i, \varphi_i\}\) for \(i = 1, 2, 3\), is a Sasakian structure, we are going to prove that the vector fields \(\{\xi_i\}_{i=1}^3\) are \(g\)-orthonormal. Note that \([\xi_i, \xi_j] = 2\epsilon_{ijk}\xi_k\). Now, from (15), we can compute
\[
\kappa(\xi_i, \xi_j) = \text{tr} \text{ad}^2\xi_i = \text{tr} \text{ad}^2\xi_i|_{\text{sp}(1)} + \text{tr} \text{ad}^2\xi_i|_{\mathfrak{g}_1} = -8 + \text{tr} \text{ad}^2\xi_i|_{\mathfrak{g}_1}.
\]
Take into account, by (16), that the action of \(\mathfrak{sp}(1)^C\) on \(\mathfrak{g}_1^C \cong \mathbb{C}^2 \otimes W\) is given by the matrix multiplication on column vectors, that is, \([\xi_i]\left[\begin{array}{c} x \\ y \end{array}\right] = \xi_i\left[\begin{array}{c} x \\ y \end{array}\right] \otimes w\). As \(\xi_i^2 = \left[\begin{array}{cc} -1 & 0 \\ 0 & -1 \end{array}\right]\) for any \(i\), we obtain \(\text{ad}^2\xi_i|_{\mathfrak{g}_1^C} = -\text{id}\), and hence \(\text{ad}^2\xi_i|_{\mathfrak{g}_1} = -\text{id}\). In particular, the Killing form takes the value \(\kappa(\xi_i, \xi_j) = -8 - \text{dim}_{\mathbb{R}} \mathfrak{g}_1 = -8 - 4n\), so that each \(\xi_i\) is a \(g\)-unitary vector field. In a similar way, for \(i \neq j\), we get
\[
\kappa(\xi_i, \xi_j) = \text{tr} (\text{ad}_{\xi_i} \circ \text{ad}_{\xi_j})|_{\text{sp}(1)} + \text{tr} (\text{ad}_{\xi_i} \circ \text{ad}_{\xi_j})|_{\mathfrak{g}_1} = 0 + \text{tr} (\text{ad}_{\xi_i} \circ \text{ad}_{\xi_j})|_{\mathfrak{g}_1}.
\]
And again from (16), we have
\[
\left[\xi_i, \left[\xi_j, \left[\begin{array}{c} x \\ y \end{array}\right] \otimes w \right]\right] = 2\epsilon_{ijk}\xi_k\left[\begin{array}{c} x \\ y \end{array}\right] \otimes w.
\]
Therefore, it holds \(\text{tr}_{\mathfrak{g}_1^C} (\text{ad}_{\xi_i} \circ \text{ad}_{\xi_j}) = 0\) and, as the trace does not depend on the field extension, it yields \(\kappa(\xi_i, \xi_j) = 0\).

Next, we want to show that every \(\xi_i\), \(i = 1, 2, 3\), is a Killing vector field for \(g\). The invariance properties simplify the computations, so we just need to check that for \(X, Y \in \mathfrak{m}\),
\[
g(\varphi_i(X), Y) + g(X, \varphi_i(Y)) = 0,
\]
but this formula can be easily deduced from (20). Finally, it remains to prove the following expression for the covariant derivative of \(\varphi_i\),
\[
(\nabla^g_X\varphi_i)Y = g(X, Y)\xi_i - \eta_i(Y)X.
\]
By Lemma 6 and the invariance of the tensors, this is equivalent to prove, for \(X, Y \in \mathfrak{m}\), that
\[
(\nabla^g_X\varphi_i(Y)) - \varphi_i(\alpha^g(X, Y)) = g(X, Y)\xi_i - \eta_i(Y)X.
\]
As explicit expressions for \(\alpha^g\) and \(\varphi_i\) are given in (18) and (20), respectively, we will check this equation case by case. First, if \(X \in \mathfrak{sp}(1)\) and \(Y \in \mathfrak{g}_1\), then both terms of (22) vanish. Second, for a fixed \(i \in \{1, 2, 3\}\), if \(X \in \mathfrak{g}_1\) and \(Y = \xi_j \in \mathfrak{sp}(1)\) with \(j \neq i\), Eq. (22) reduces to check that
\[
\epsilon_{ijk}\alpha^g(X, \xi_k) - \varphi_i([X, \xi_j]) = -\epsilon_{ijk}\varphi_k(X) + \varphi_i \circ \varphi_j(X) = 0.
\]
But this holds due to the equality $\epsilon_{ijk} \varphi_k = \varphi_i \circ \varphi_j$ on horizontal vectors for $j \neq i$. The subcase $j = i$ can be proved similarly. Third, (22) is a straightforward computation when $X$ and $Y$ belongs to $\{\xi_j\}_{j=1}^3$. Finally, for $X, Y \in \mathfrak{g}_1$, (22) becomes
\[\alpha^g(\varphi_i(Y)) - \varphi_i(\alpha^g(X,Y)) = g(X,Y)\xi_i.\]

As the left-side of this expression can be written as follows
\[Z := \frac{1}{2} [X, [\xi_i, Y]]_{\mathfrak{sp}(1)} - \frac{1}{4} \xi_i, [X, Y]] \in \mathfrak{sp}(1),\]
it suffices to prove that $g(Z, \xi_j) = \delta_{ij} g(X,Y)$, or equivalently, $\kappa(2Z, \xi_j) = \delta_{ij} \kappa(X,Y)$. But this is easy to check due to the associativity of the Killing form and the fact $\kappa(\mathfrak{h}, \mathfrak{sp}(1)) = 0$. □

**Remark 14.** Proposition 13 exhibits, in a unified way, the algebraic structures involved in all $3$-Sasakian homogeneous manifolds. In particular, it clarifies our choice of the metric $g$. We should recall the explicit description of the metric tensor of the $3$-Sasakian homogeneous manifolds given by Bielawski in [8, Theorem 4] as follows,
\[(23) \quad g(X + U, X + U) = -\kappa(X, X) - \frac{1}{2} \kappa(U, U), \]
for all $X \in \mathfrak{sp}(1)$ and $U \in \mathfrak{m}' \subseteq \mathfrak{g}_1$. Needless to say, we have determined the metric $g$ with more precision in Proposition 13, since our metric $g$ is homothetic to the metric in (23).

At this point, it is interesting to note that for $n = 0$, we have $\mathfrak{sp}(1) = \mathfrak{m}$, that is, $\mathfrak{g}_1 = \mathfrak{g}$. For this particular case, the multiplication in $\mathfrak{m}$ corresponding to the Levi-Civita connection coincides with $\frac{1}{2} [\ , \ ]$ and so $M = G/H$ is naturally reductive for our reductive decomposition when $n = 0$. However, the homogeneous spaces $M = G/H$ in Proposition 13 are not naturally reductive whenever $n > 0$. □

**Remark 15.** Every $3$-Sasakian manifold $M$ is Einstein with positive scalar curvature, [29]. Remarkably, every $3$-Sasakian manifold $M$, homogeneous or not, does always admit a second Einstein metric with positive scalar curvature [9, 13.3.3]. That is, every $3$-Sasakian manifold $M$ has at least two distinct homothety classes of Einstein metrics. This second metric, say $\tilde{g}$, was constructed by making a canonical variation of the original one along the fibers of the natural projection $M \to M/\mathcal{F}_Q$ on the leaves space. The Sasakian structures for the original metric are not Sasakian structures on $(M, \tilde{g})$. For any $G$-homogeneous $3$-Sasakian manifold, the new metric $\tilde{g}$ is also $G$-invariant. □

**Remark 16.** Equation (16) implies that $\mathfrak{m}^C$, as $\mathfrak{h}^C$-module, is the sum of three trivial irreducible modules with two copies of certain module $W$. This will allow to apply Lemma 10(c) to all the $3$-Sasakian homogeneous manifolds in order to compute the dimension $\dim_\mathbb{C} \text{Hom}_\mathbb{R}(\Lambda^3 \mathfrak{m}, \mathbb{R}) = \dim_\mathbb{C} \text{Hom}_\mathbb{C}(\Lambda^3 \mathfrak{m}^C, \mathbb{C})$. In spite of this unified treatment, we will make use of the concrete module $W$, which will be computed next jointly with its irreducible summands. The SU-case will be the last one to consider. It is the only case in which the module $W$ is not irreducible, what causes the apparition of new tensors and hence, the existence of more invariant affine connections. For dealing with these extra tensors and their covariant derivatives, the explicit expression given in (18) for the multiplication $\alpha^g$ on $\mathfrak{m}$ corresponding to the Levi-Civita connection will be crucial. □

**Remark 17.** An algebraical comment is in order. The complex module $W$ in the previous Proposition 13 is a *symplectic triple system*, as defined in [37]. This means that there exist a triple product $\ [\ , \ , \ ]$ in $W$ and a symplectic form $(\ , \ ) : W \times W \to \mathbb{C}$ satisfying the following list of identities:
\[
[x, y, z] = [y, x, z], \\
[x, y, z] - [x, z, y] = (x, z)y - (x, y)z + 2(y, z)x, \\
[x, y, [u, v, w]] = [[x, y, u], v, w] + [u, [x, y, v], w] + [u, v, [x, y, w]], \\
([x, y, u], v) = -(u, [x, y, v]),
\]
for any \( x, y, z, u, v, w \in W \). Conversely, given any symplectic triple system \((W, [\ , \ , ]), (\ , \ )\), consider the set of inner derivations \( \text{ind} W \), that is, the linear span of the operators \([x, y, -] \in \mathfrak{sp}(W, (\ , \ ))\).

Then, \( \text{ind} W \) is a Lie subalgebra of \( \mathfrak{gl}(W) \) and \( \mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \) is a \( \mathbb{Z}_2 \)-graded Lie algebra for
\[
\mathfrak{g}_0 = \mathfrak{sp}(V) \oplus \text{ind} W, \quad \mathfrak{g}_1 = V \oplus W,
\]
being \( V \) a two dimensional vector space endowed with a nonzero symplectic form. According to [16, Theorem 5.3], \( \mathfrak{g} \) is a simple algebra if, and only if, \( W \) is simple as a symplectic triple system, which is our situation.

These symplectic triple systems appeared as ingredients in the constructions of 5-graded Lie algebras \( \mathfrak{g} = \oplus_{i=-2}^2 \mathfrak{g}_i \) with one dimensional corners: \( \dim \mathfrak{g}_{4,2} = 1 \). They are strongly related to Freudenthal triple systems and to Faulkner ternary algebras (see [16] for more details and references). A natural question on 3-Sasakian homogeneous manifolds is how the curvature tensor could be naturally expressed by means of this ternary product. \( \square \)

In the following subsections, the dimensions of the vector spaces in (13) will be computed for the homogeneous 3-Sasakian manifolds listed in Theorem 1. These dimensions will provide us the amount of invariant affine connections on each case.

4.1. Case \( \text{Sp}(n+1)/\text{Sp}(n) \). Let \( \mathbb{H} \) be the algebra of quaternions with complex units \{\( j_1, j_2, j_3 \}\). This means \( j_i^2 = -1 \) for all \( 1 \leq i \leq 3 \) and \( j_1 j_2 = j_3 = j_2 j_1 \) for all permutation of the indices \( (1,2,3) \) and \( \epsilon_{ijk} \) the sign of the corresponding permutation. For every \( z = x_0 + x_1 j_1 + x_2 j_2 + x_3 j_3 \in \mathbb{H} \), with \( x_i \in \mathbb{R} \), \( i = 0, 1, 2, 3 \), we denote by \( \overline{z} = x_0 - x_1 j_1 - x_2 j_2 - x_3 j_3 \in \mathbb{H} \) its quaternionic conjugate and by \( \text{tr} z = z + \overline{z} = 2x_0 \) its trace. Also, \( \mathbb{H}_0 \) denotes the set of zero trace quaternions, that is, \( \mathbb{H}_0 = \mathbb{R}j_1 \oplus \mathbb{R}j_2 \oplus \mathbb{R}j_3 \). For any \( n \geq 1 \), let us consider the Euclidean metric \( g \) on \( \mathbb{H}^n \) defined by
\[
g(z, w) = \text{Re} \left( \sum_{k=1}^n z_k \overline{w}_k \right),
\]
for \( z = (z_1, \ldots, z_n)^t, w = (w_1, \ldots, w_n)^t \in \mathbb{H}^n \). The compact symplectic group \( \text{Sp}(n) \) is defined as
\[
\text{Sp}(n) = \{ A \in \mathcal{M}_n(\mathbb{H}) : A \overline{A}^t = I_n \}.
\]
This Lie group acts on the left by matrix multiplication on \( \mathbb{H}^n \) (as column vectors) in such a way that the Euclidean metric \( g \) is preserved.

Let us consider the \((4n+3)\)-dimensional unit round sphere \( S^{4n+3} = \{ z \in \mathbb{H}^{n+1} : g(z, z) = 1 \} \), which inherits from \( g \) the usual Riemannian metric of constant sectional curvature \( +1 \). The action of the Lie group \( \text{Sp}(n+1) \) on \( \mathbb{H}^{n+1} \) restricts to a transitive action on \( S^{4n+3} \) and \( \text{Sp}(n+1) \subset \text{Iso}(S^{4n+3}, g) \). The isotropy group at the point \( o = (0, \ldots, 0, 1)^t \in S^{4n+3} \) is
\[
H = \left\{ \left( \begin{array}{cc} B & 0 \\ 0 & 1 \end{array} \right) : B \in \text{Sp}(n) \right\}.
\]
Moreover, we have a reductive decomposition as in (6) given by
\[
\mathfrak{g} = \mathfrak{sp}(n+1) = \{ A \in \mathcal{M}_{n+1}(\mathbb{H}) : A + A^t = 0 \}, \quad \mathfrak{h} = \left\{ \left( \begin{array}{cc} B & 0 \\ 0 & 0 \end{array} \right) : B \in \mathfrak{sp}(n) \right\} \cong \mathfrak{sp}(n), \quad \mathfrak{m} = \left\{ \left( \begin{array}{cc} -aI_n & z \\ z^t & 0 \end{array} \right) : z \in \mathbb{H}^n, a \in \mathbb{H}_0 \right\} \cong \mathbb{H}^n \oplus \mathbb{H}_0 = \mathfrak{g}_1 \oplus \mathfrak{sp}(1).
\]
By means of the identifications suggested by (24), the action of \( \mathfrak{h} \) on \( \mathfrak{m} \) is the action of \( \mathfrak{sp}(n) \) on \( \mathbb{H}^n \oplus \mathbb{H}_0 \) given by \( B \cdot (z, a) = (Bz, 0) \), for \( B \in \mathfrak{sp}(n), z \in \mathbb{H}^n \) and \( a \in \mathbb{H}_0 \). In particular, the action of \( \mathfrak{h} \) on \( \mathbb{H}_0 \) is trivial.

As expected, the pair of Lie algebras \((\mathfrak{g}, \mathfrak{h})\) is a 3-Sasakian data. Indeed, we have the \( \mathbb{Z}_2 \)-grading \( \mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \) with \( \mathfrak{g}_0 = \mathbb{H}_0 \oplus \mathfrak{h} \) and \( \mathfrak{g}_1 = \mathbb{H}^n \). (Recall that \( (\mathbb{H}_0, [\ , \ ] ) \cong \mathfrak{sp}(1) \) by means of the isomorphism \( j_i \mapsto \xi_i \) for \( 1 \leq i \leq 3 \), with \( \xi_i \) as in (19).) Then, Eq. (16) becomes clear from the next proof.
Lemma 18. Assume $\mathfrak{sp}(n + 1) = \mathfrak{h} \oplus \mathfrak{m}$ with $n \geq 1$ is the reductive decomposition in (24). Then $\dim_{\mathbb{R}} \text{Hom}_{\mathbb{R}}(\mathfrak{m} \otimes \mathfrak{m}, \mathfrak{m}) = 63$, $\dim_{\mathbb{R}} \text{Hom}_{\mathbb{R}}(\mathfrak{m}, \Lambda^2 \mathfrak{m}) = 30$, and $\dim_{\mathbb{R}} \text{Hom}_{\mathbb{R}}(\Lambda^3 \mathfrak{m}, \mathbb{R}) = 10$.

Proof. Put $Z_n = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$. Then, the Lie algebra $\mathfrak{h}^C = \mathfrak{h} \oplus i \mathfrak{m} \cong \mathfrak{sp}(2n, \mathbb{C}) = \{ A \in M_{2n}(\mathbb{C}) : AZ_n + Z_n A^t = 0 \}$ is simple of type $C_n$. Moreover, $\mathfrak{m}^C$ is isomorphic to the module $2V(\lambda_1) \oplus 3C$, where $\lambda_1$ denotes the fundamental weight for $C_n$ (notation as in [25]). In order to apply Corollary 11 to the module $U : = V(\lambda_1)$, we compute

$$S^2 V(\lambda_1) \cong V(2\lambda_1) \text{ for all } n, \text{ and } \Lambda^2 V(\lambda_1) \cong \begin{cases} V(\lambda_2) \oplus \mathbb{C} & \text{if } n \geq 2, \\ \mathbb{C} & \text{if } n = 1. \end{cases}$$

Recall that these computations can also be done by using the LiE online service: http://wwwmathlabo.univ-poitiers.fr/~maavl/LiE/form.html

It is well-known that, if $\lambda$ and $\beta$ are dominant weights, i.e., belonging to $\Lambda^+ = \{ \sum_{i=1}^n m_i \lambda_i : m_i \in \mathbb{Z}_{\geq 0} \}$, the irreducible modules $V(\lambda)$ and $V(\beta)$ are isomorphic if, and only if, $\lambda = \beta$. This implies that $\mathfrak{m}^C$ satisfies the technical conditions in Corollary 11, which guarantee that $\dim_{\mathbb{R}} \text{Hom}_{\mathbb{R}}(\mathfrak{m} \otimes \mathfrak{m}, \mathfrak{m}) = 63$ and $\dim_{\mathbb{R}} \text{Hom}_{\mathbb{R}}(\mathfrak{m}, \Lambda^2 \mathfrak{m}) = 30$. For computing the third dimension, we check

$$\Lambda^3 U \cong \begin{cases} V(\lambda_3) \oplus U & \text{if } n \geq 3, \\ U & \text{if } n = 2, \text{ and } \Lambda^2 U \otimes U \cong \begin{cases} V(\lambda_3) \oplus U & \text{if } n \geq 3, \\ V(\lambda_1 + \lambda_2) \oplus 2U & \text{if } n = 2, \\ U & \text{if } n = 1. \end{cases}$$

Taking into account that any of these modules does not contain any copy of the trivial module $C$, we can again apply Corollary 11 to deduce $\dim_{\mathbb{R}} \text{Hom}_{\mathbb{R}}(\Lambda^3 \mathfrak{m}, \mathbb{R}) = 10$. □

Remark 19. Note that this lemma was proved with different arguments in [20], but after a long chain of computations which made use of 63 parameters. Now, our argument in [20] is replaced with a short proof which avoids the explicit algebraical descriptions of the connections provided by Nomizu’s Theorem. Moreover, there is a slip-up in the survey [20] in the expression of the map $\alpha^g$ related to the Levi-Civita connection, which should coincide with that one in (18). □

Remark 20. We have not studied here the case $n = 0$ because the 3-dimensional sphere is well-known. First, Laquer [31] studied invariant affine connections on Lie groups. And second, our work [18, Section 7] studies the metric connections with skew-torsion on $S^3$ and also, those which are Einstein with skew-torsion. □

Remark 21. To end this subsection, we would like to mention the projective case too. This is the only 3-Sasakian homogeneous manifold which is not simply connected. Here again, the group $\text{Sp}(n + 1)$ acts transitively on $\mathbb{R}P^{4n+3} \cong S^{4n+3}/\mathbb{Z}_2 = \{ [z] : z \in \mathbb{C}^{2n+2}, g(z, z) = 1 \}$ by $[A][z] = [Az]$ for $A \in \text{Sp}(n + 1)$ and $[z] \in \mathbb{R}P^{4n+3}$. Now, the isotropy group is not connected. To be precise, the isotropy group at the point $[(0, \ldots, 0, 1)^t]$ is

$$H = \left\{ \begin{pmatrix} B & 0 \\ 0 & \pm 1 \end{pmatrix} : B \in \text{Sp}(n) \right\},$$

and the reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ is the same as the given one by (24). Bearing in mind that the projection $S^{4n+3} \to \mathbb{R}P^{4n+3}$ commutes with the actions of the Lie group $\text{Sp}(n + 1)$ on $S^{4n+3}$ and $\mathbb{R}P^{4n+3}$, respectively, every $\text{Sp}(n + 1)$-invariant affine connection on $S^{4n+3}$ induces, in a natural way, a $\text{Sp}(n + 1)$-invariant affine connection on $\mathbb{R}P^{4n+3}$. Therefore, the set of invariant affine connections on $S^{4n+3}$ is in one-to-one correspondence with the corresponding set on $\mathbb{R}P^{4n+3}$.

Alternatively, this conclusion can be achieved algebraically. Take $\sigma = \begin{pmatrix} I_n & 0 \\ 0 & -1 \end{pmatrix} \in H$, which does not belong to the connected component $H_0$. As $\text{Ad}(\sigma)(z, a) = (z, -a)$, it is easy to check

$$\text{Ad}(\sigma)(\alpha(X, Y)) = \alpha(\text{Ad}(\sigma)(X), \text{Ad}(\sigma)(Y))$$
for all $X,Y \in \mathfrak{m}$ and any $\mathfrak{h}$-invariant bilinear map $\alpha: \mathfrak{m} \times \mathfrak{m} \to \mathfrak{m}$. □

4.2. **Case** $\frac{SO(k)}{SO(k-4) \times Sp(1)}$ for $k \geq 7$. As mentioned, every homogeneous 3-Sasakian homogeneous manifold is the total space of a $Sp(1)$ or a $SO(3)$ principal bundle over a symmetric space [9, Proposition 13.4.5]. In this case, we have the projection

$$p: \frac{SO(k)}{SO(k-4) \times Sp(1)} \longrightarrow \frac{SO(k)}{SO(k-4) \times SO(4)},$$

so that the base manifold is the real Grassmann manifold of oriented linear 4-subspaces of $\mathbb{R}^k$. To be precise, let us recall the following double covering group homomorphism

$$Sp(1) \times Sp(1) \to SO(4), \quad (q, z) \mapsto T_{(q, z)}: \mathbb{R}^4 \to \mathbb{R}^4 \cong \mathbb{H},$$

where $T_{(q, z)}v = qv\bar{z}$ and $Sp(1) = \{q \in \mathbb{H} : q\bar{q} = 1\}$. The map $T_{(q, z)} = L_q \circ R_z$ is the composition of the left and right quaternionic multiplications $L_q(v) = qv$ and $R_z(v) = v\bar{z}$. Therefore, there are two remarkable ways of considering the group $Sp(1)$ as a subgroup of $SO(4)$. Namely, $Sp(1)^- \subset SO(4)$ given by $q \mapsto T_{(q, 1)}$ and $Sp(1)^+ \subset SO(4)$ given by $q \mapsto T_{(q, 1)}$. The above projection $p$ is defined, in a natural way, from $Sp(1)^- \subset SO(4)$.

**Remark 22.** This point of view permits to give a geometrical meaning for the 3-Sasakian manifold $\frac{SO(k)}{SO(k-4) \times Sp(1)}$. Recall that we also have the induced isomorphism $Sp(1)Sp(1) \to SO(4)$, where as customary $Sp(1)Sp(1) := (Sp(1) \times Sp(1))/\mathbb{Z}_2$. Now, let us consider $\Pi = \frac{SO(k)}{SO(k-4) \times SO(4)}$ with $\Pi = \sigma(SO(k-4) \times SO(4))$. Then, the last four column vectors of $\sigma \in SO(k)$ give an oriented orthonormal basis $\mathcal{B}(\sigma)$ of the 4-subspace $\Pi$ which identifies $\mathcal{B}(\sigma): \mathbb{R}^4 \cong \mathbb{H} \to \Pi$. Thus, the fibre over $\Pi$ can be described by $p^{-1}(\Pi) = \{\mathcal{B}(\sigma) \circ T_{[1, q]} : q \in Sp(1)/\mathbb{Z}_2 \cong SO(3)\}$. Hence, $\frac{SO(k)}{SO(k-4) \times Sp(1)}$ can be seen as a distinguished family of oriented orthonormal basis on every oriented linear 4-subspace of $\mathbb{R}^k$. □

Now we introduce the 3-Sasakian data corresponding to this case. The $\mathbb{Z}_2$-grading on $\mathfrak{g} = \mathfrak{so}(k) = \{A \in M_k(\mathbb{R}) : A + A^t = 0\}$ is given by

$$\mathfrak{g}_0 = \left\{ \begin{pmatrix} B & 0 \\ 0 & C \end{pmatrix} : B \in \mathfrak{so}(k-4), C \in \mathfrak{so}(4) \right\},$$

$$\mathfrak{g}_1 = \left\{ \begin{pmatrix} 0 & D \\ -D^t & 0 \end{pmatrix} : D \in M_{(k-4) \times 4}(\mathbb{R}) \right\}.$$

Now, $\mathfrak{so}(4)$ is not a simple Lie algebra but decomposes as a sum of two copies of $\mathfrak{sp}(1)$. Namely, $\mathfrak{so}(4) = I^- \oplus I^+$ for

$$I^- = \left\{ \begin{pmatrix} 0 & \alpha_1 & \alpha_2 & \alpha_3 \\ -\alpha_1 & 0 & -\alpha_3 & \alpha_2 \\ -\alpha_2 & \alpha_3 & 0 & -\alpha_1 \\ -\alpha_3 & -\alpha_2 & \alpha_1 & 0 \end{pmatrix} : \alpha_i \in \mathbb{R} \right\}, \quad I^+ = \left\{ \begin{pmatrix} 0 & -\alpha_1 & -\alpha_2 & -\alpha_3 \\ \alpha_1 & 0 & -\alpha_3 & \alpha_2 \\ \alpha_2 & \alpha_3 & 0 & -\alpha_1 \\ \alpha_3 & -\alpha_2 & \alpha_1 & 0 \end{pmatrix} : \alpha_i \in \mathbb{R} \right\},$$

where these matrices are just the matrices of $R_q = -R_q$ and $L_q$, respectively, if $q = \alpha_1j_1 + \alpha_2j_2 + \alpha_3j_3 \in \mathbb{H} \cong \mathfrak{sp}(1)$. In other words, the above decomposition of $\mathfrak{so}(4)$ has been obtained from the inclusions $Sp(1)^- \subset SO(4)$ for $I^-$ and $Sp(1)^+ \subset SO(4)$ for $I^+$. In particular, we have the reductive decomposition of $\mathfrak{g} = \mathfrak{so}(k)$ given by $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ for

$$\mathfrak{h} = \left\{ \begin{pmatrix} B & 0 \\ 0 & C \end{pmatrix} : B \in \mathfrak{so}(n), C \in I^- \right\},$$

$$\mathfrak{m} = \left\{ \begin{pmatrix} 0 & D \\ -D^t & X \end{pmatrix} : D \in M_{n \times 4}(\mathbb{R}), X \in I^+ \right\} \cong I^+ \oplus \mathfrak{g}_1,$$
with $n = k - 4$, which of course satisfies $g_0 = I^+ \oplus h \cong \mathfrak{sp}(1) \oplus h$. We use the natural identifications $h \cong \mathfrak{so}(n) \oplus I^- \cong \mathfrak{so}(n) \oplus \mathfrak{sp}(1)$ and $\mathcal{M} \cong \mathcal{M}_{n \times 4}(\mathbb{R}) \oplus I^+ \cong \mathcal{M}_{n \times 4}(\mathbb{R}) \oplus \mathfrak{sp}(1)$ given by
\[
\begin{pmatrix}
B & 0 \\
0 & C
\end{pmatrix} \mapsto (B, C), \quad \begin{pmatrix}
0 & D \\
-D^T & X
\end{pmatrix} \mapsto (D, X).
\]

Thus, taking into account that $[I^-, I^+] = 0$ (as $R_q L_z = L_z R_q$), the adjoint action of the semisimple Lie algebra $h$ on $m$ can be expressed as
\[(B, C) \cdot (D, X) = (BD - DC, 0).\]

If there is no ambiguity, we use $B$ for $(B, 0)$ and $C$ for $(0, C)$. In particular, $m$ decomposes as the sum of $I^+$, which is a trivial 3-dimensional $h$-module, and $\mathcal{M}_{n \times 4}(\mathbb{R})$, which is an irreducible $h$-module.

In order to apply Corollary 11, we have previously checked that $m^C \cong 3C \oplus 2U$ for some irreducible and nontrivial $h^C$-submodule $U$. So, we need to know the decomposition of $\mathcal{M}_{n \times 4}(\mathbb{R})^C \cong \mathcal{M}_{n \times 4}(\mathbb{C})$ as a sum of $h^C$-irreducible submodules. More precisely, we are going to prove that such decomposition is
\[
\mathcal{M}_{n \times 4}(\mathbb{C}) = U_1 \oplus U_2,
\]
for $U_1 := \langle \{i(a|a)ib|b : a, b \in \mathbb{C}\} \rangle$ and $U_2 := \langle \{(-i(a|a)ib|b) : a, b \in \mathbb{C}\} \rangle$ (notation by columns). First, $U_1 \oplus U_2 = \mathcal{M}_{n \times 4}(\mathbb{C})$ is clear since $\mathbb{C}(i, 1) \oplus \mathbb{C}(-i, 1) = \mathbb{C} \times \mathbb{C}$. Second, we also have $h^C U_1 \subset U_1$ (all works analogously for $U_2$): If $B \in \mathfrak{so}(n)^C \cong \mathfrak{so}(n, \mathbb{C})$, then $B \cdot (i(a|a)ib|b) = (iBa|Ba)ib|b) \in U_1$; and, if $C \in I^-$, then $C \cdot (i(a|a)ib|b) = (0, C)((i(a|a)ib|b), 0) = (- (i(a|a)ib|b)C, 0) \in U_1$, since
\[
\begin{align*}
&j_1 \cdot (i(a|a)ib|b) = (a) - ia - b|ib, \\
j_2 \cdot (i(a|a)ib|b) = (ib|b - ia - a), \\
j_3 \cdot (i(a|a)ib|b) = (b - ib)\alpha|a - ia).
\end{align*}
\]

Third, in order to check the irreducibility of each $U_\ell$, we observe that both $U_\ell$ are isomorphic to the $h^C$-module $U = \mathbb{C}^n \times \mathbb{C}^n = \{(a, b) : a, b \in \mathbb{C}\}$ endowed with the action given by
\ishow{- for $B \in \mathfrak{so}(n, \mathbb{C})$, $B \cdot (a, b) := (Ba, Bb)$ (so $U$ is sum of two copies of the natural $\mathfrak{so}(n, \mathbb{C})$-irreducible representation $\mathbb{C}^n$, each column is a copy); \ishow{the action of $(I^-)^C \cong \mathfrak{sl}(2, \mathbb{C}) = \langle H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, F = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} \rangle$ is given by right multiplication: 
\[
(a, b)H = (a, -b), \quad (a, b)E = (0, a), \quad (a, b)F = (-b, 0);
\]

so $U$ is sum of $n$ copies of the two-dimensional $\mathfrak{sl}(2, \mathbb{C})$-module $\mathcal{V}(1)$;\footnote{Recall that there is exactly one irreducible $\mathfrak{sl}(2, \mathbb{C})$-module of each dimension $n + 1$, which is denoted by $\mathcal{V}(n)$. It coincides with $\mathcal{V}(n) \cong \mathbb{C}^n(\mathcal{V}(\lambda_1))$. In particular $\mathcal{V}(1) \cong \mathcal{V}(\lambda_1) \cong \mathbb{C}^2$ is given by the columns.} each row of $(a, b)$ viewed as a matrix in $\mathcal{M}_{n \times 2}(\mathbb{C})$ is such a copy).}

It is enough to call $H = i\mathbf{1}_1$, $E = \frac{1}{2}(j_2 + i\mathbf{1}_3)$ and $F = \frac{1}{2}(j_2 + i\mathbf{1}_3)$ to pass from (25) to (26), thus getting that $U_1$ is isomorphic to $U$. In order to clarify the irreducibility of $U$, let us assume that $0 \neq W \neq U$ is an $h^C$-submodule of $U$. Then $W$ must be an $\mathfrak{so}(n, \mathbb{C})$-submodule isomorphic to $\mathbb{C}^n$, that is, either $W = \{(a, a) \alpha : a \in \mathbb{C}\}$ for some fixed $\alpha \in \mathbb{C}$ or $W = \{(0, a) : a \in \mathbb{C}\}$. But none of these are invariant for the action of $\mathfrak{sl}(2, \mathbb{C})$ given by (26). This finishes the proof that $m^C \cong 3C \oplus 2U$ is the decomposition of $m^C$ as a sum of irreducible $h^C$-submodules.

**Lemma 23.** If $g = h \oplus m$ is the reductive decomposition related to $M = \mathfrak{so}(k)/\mathfrak{so}(k-4) \times \mathfrak{sp}(1)$ for $k \geq 7$, then $\dim \mathfrak{hom}(m \otimes \mathfrak{m}, m) = 63$, $\dim \mathfrak{hom}(m, \Lambda^2 \mathfrak{m}) = 30$, and $\dim \mathfrak{hom}(\Lambda^3 \mathfrak{m}, \mathbb{R}) = 10$.

**Proof.** As mentioned, we can also apply Corollary 11 to the complex (semisimple) algebra $h^C = \mathfrak{so}(n, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C})$ for $n = k - 4 \geq 3$ and to the $h^C$-module $m^C$. First, we observe that $\mathfrak{hom}(U \otimes U, U) = 0$, since, as $\mathfrak{sl}(2, \mathbb{C})$-module, $U$ is isomorphic to $n(1)$ and hence
\[
U \otimes U \cong n \mathcal{V}(1) \otimes n \mathcal{V}(1) \cong n^2 \mathcal{V}(2) \oplus n^2 \mathcal{V}(0).
\]
Thus, $\text{Hom}_C(S^2\mathcal{U}, \mathcal{U}) = \text{Hom}_C(\Lambda^2\mathcal{U}, \mathcal{U}) = 0$. Second, let us see that $\text{Hom}_C(S^2\mathcal{U}, C) = 0$ and that the (complex) dimension of $\text{Hom}_C(\Lambda^2\mathcal{U}, C)$ is one. Indeed, $\mathcal{U}$ is isomorphic to $2C^n$ as $\mathfrak{so}(n, C)$-module, so that $\mathcal{U} \otimes \mathcal{U} \cong 4C^n \otimes C^n$ contains four copies of the trivial module (namely, $\Lambda^2C^n$ is the adjoint module and $S^2C^n$ is sum of a trivial one-dimensional module and $V(2\lambda_1)$). In other words, there is a four parametric family of $\mathfrak{so}(n, C)$-invariant maps $\mathcal{U} \times \mathcal{U} \to C$ given by

$$\rho((a, b), (c, d)) = s_1a'c + s_2a'd + s_3b'c + s_4b'd,$$

for some scalars $s_i \in C$. The map $\rho$ is also $\mathfrak{sl}(2, C)$-invariant if, and only if, $s_1 = s_2 + s_3 = s_4 = 0$. For instance, the equalities $\rho((a, b), (c, d)) + \rho((a, b), (c, d))E = \rho((0, 0), (c, d)) = \rho((a, b), (0, c))$ for all $a, b, c, d \in C^n$. Also $s_1 = 0$ is achieved by changing $E$ with $F$ above. Therefore, we obtain the unique (up to scalar) $\mathfrak{h}^C$-invariant map $\mathcal{U} \times \mathcal{U} \to C$ given by

$$\rho((a, b), (c, d)) = a'd - b'c,$$

which is alternating. This gives the first two desired dimensions.

For the third dimension, that one of $\text{Hom}_C(\Lambda^3m^C, C)$, observe that

$$\Lambda^3m^C \cong 2\Lambda^2\mathcal{U} \oplus 2(\Lambda^2\mathcal{U} \otimes \mathcal{U}) \oplus 9\Lambda^2\mathcal{U} \oplus 3S^2\mathcal{U} \oplus 6\mathcal{U} \oplus C$$

only contains trivial submodules on $\Lambda^2\mathcal{U}$ (one copy of $C$ on each $\Lambda^2\mathcal{U}$), because, paying attention to the $\mathfrak{sl}(2, C)$-action, there are copies of $V(0)$ neither in $V(1) \otimes 3$ nor in $V(2) \otimes V(0) \otimes V(1)$ nor in $\mathcal{U} \otimes V(1) \cong V(2)$. Consequently, there are just $9 + 1$ copies of the trivial one-dimensional module inside $\Lambda^3m^C$. □

### 4.3. Exceptional Cases

We provide a model of the reductive decompositions and hence also of the 3-Sasakian structures related to the exceptional Lie algebras. Our approach will be based on the famous Tits’ unified construction. Actually, it is not necessary for our purposes of computing dimensions, but we include it here for completeness and beauty.

Let $\mathcal{C}$ be a real division composition algebra, that is, $\mathcal{C} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}\}$. Thus $\mathcal{C}$ is endowed with a nonsingular quadratic form $n: \mathcal{C} \to \mathbb{R}$, usually called the norm, such that $n(xy) = n(x)n(y)$. Each element $a \in \mathcal{C}$ satisfies a quadratic equation (with real coefficients) $a^2 - t_C(a)a + n(a)1 = 0$, where $t_C(a) = n(a + 1) - n(a) - 1$ is called the trace. Denote by $C_0 = \{a \in C : t_C(a) = 0\}$ the subspace of traceless elements.

The map $-: \mathcal{C} \to \mathcal{C}$ given by $a \mapsto \bar{a} = t_C(a)\overline{1} - a$ is an involution such that $n(a) = a\overline{a}$ and $t_C(a)\overline{1} = a + \overline{a}$ hold. Furthermore, for any $a, b \in \mathcal{C}$, the endomorphism $D_{a,b} := [l_a, l_b] + [l_a, r_b] + [r_a, r_b]$ is a derivation of $\mathcal{C}$, where $l_a(b) = ab$ and $r_a(b) = ba$ denote the left and right multiplication operators. These are quite representative derivations, since $\text{Der}(\mathcal{C}) = \text{Span}\{D_{a,b} : a, b \in C\} \equiv D_{C,C}$. Their main properties are summarized here:

$$D_{a,b} = -D_{b,a}, \quad D_{a,b,c} + D_{b,c,a} + D_{c,a,b} = 0, \quad [d, D_{a,b}] = D_{d(a),b} + D_{a,d(b)},$$

for any $a, b, c \in \mathcal{C}$ and $d \in \text{Der}(\mathcal{C})$.

Recall that a basis of the octonion algebra $\mathbb{O}$ is $\{1, i, j, k, l, m, n, l\}$, where the product is given by $q_1(q_2l) = (q_2q_1)l$, $(q_1l)(q_2l) = -q_2q_1l$ and $(q_2q_1)l = (q_2q_1)l$ for any $q_1, q_2 \in \mathbb{H} = \{1, i, j, k\}$. The norm is determined by $n(\mathbb{H}, l) = 0$ and $n(l) = 1$, being $n|_{\mathbb{H}}$ the usual norm of the quaternion algebra.

A commutative algebra $\mathcal{J}$ satisfying the Jordan identity $(x^2y)x = x^2(yx)$ is called a Jordan algebra. The relevant Jordan algebras for our purposes are $\mathbb{R}$ and $\mathcal{H}_3(\mathbb{C}) = \{x = (x_{ij}) \in M_3(\mathbb{C}) : x^t \equiv (\overline{y}) = x\}$, for $\mathcal{C}$ one of the previous composition algebras, where the product is given by

$$x \cdot y = \frac{1}{2}(xy + yx),$$
denoting here by juxtaposition the usual product of matrices. (This product \( \cdot \) is usually called *symmetrized* product.) We have a decomposition \( J = R1 \oplus J_0 \), for \( J_0 = \{ x \in J : \text{tr}(x) = 0 \} \) the subspace of traceless matrices, since \( x \ast y := x \cdot y - \frac{1}{2} \text{tr}(x \cdot y)1 \in J_0 \). In particular we have a commutative multiplication \( \ast \) defined on \( J_0 \). Denote by \( R_x : J \to J, y \mapsto y \cdot x \) the multiplication operator, and observe that \([R_x, R_y] \in \text{Der}(J)\) for any \( x, y \in J \).

The beautiful unified Tits’ construction of all the exceptional simple Lie algebras, [36], is reviewed here only for compact real exceptional Lie algebras, although this construction is valid in a wider context. For \( C \) and \( C' \) two real division composition algebras and the Jordan algebra given by either \( J = R \) or \( J = H_3(C') \), consider the vector space

\[
\mathcal{T}(C, J) = \text{Der}(C) \oplus (C_0 \otimes J_0) \oplus \text{Der}(J),
\]

which is made into a (compact) Lie algebra by defining the (bilinear and anticommutative) multiplication \( [\, , \,] \) on \( \mathcal{T}(C, J) \) which agrees with the ordinary commutator in \( \text{Der}(C) \) and \( \text{Der}(J) \) and is specified by:

\[
[D, a \otimes x] = a \otimes D(x),
\]

for all \( d \in \text{Der}(C), D \in \text{Der}(J), a \in C_0 \) and \( x, y \in J_0 \). (If \( J = R \), note that \( \mathcal{T}(C, R) = \text{Der}(C) \).)

Then:

| \( \mathcal{T}(C, J) \) | \( \mathbb{R} \) | \( \mathcal{H}_3(\mathbb{R}) \) | \( \mathcal{H}_3(C) \) | \( \mathcal{H}_3(\mathbb{H}) \) | \( \mathcal{H}_3(\mathbb{O}) \) |
|---|---|---|---|---|---|
| \( \mathbb{H} \) | \( \mathfrak{sp}(1) \) | \( \mathfrak{sp}(3) \) | \( \mathfrak{su}(6) \) | \( \mathfrak{so}(12) \) | \( c_7 \) |
| \( \mathcal{O} \) | \( \mathfrak{g}_2 \) | \( \mathfrak{f}_4 \) | \( c_6 \) | \( c_7 \) | \( c_8 \) |

We will take, for each Jordan algebra \( J^1 = R, J^2 = \mathcal{H}_3(\mathbb{R}), J^3 = \mathcal{H}_3(C), J^4 = \mathcal{H}_3(\mathbb{H}), J^5 = \mathcal{H}_3(\mathbb{O}) \), the Lie algebras constructed by the above process: \( \mathfrak{g}^* = \mathcal{T}(\mathbb{O}, J^*) \) and \( \mathfrak{h}^* = \mathcal{T}(\mathbb{H}, J^*) \), so that \( \mathfrak{h}^* \) can be trivially considered as a subalgebra of \( \mathfrak{g}^* \). If \( \kappa : \mathfrak{g}^* \times \mathfrak{g}^* \to \mathbb{R} \) denotes the (negative definite) Killing form, we will take \( \mathfrak{m}^* \) the orthogonal complement to \( \mathfrak{h}^* \). In order to describe \( \mathfrak{m}^* \) explicitly, we focus first on the case \( J^1 = R \) and \( \mathfrak{g}^1 = \text{Der}(\mathbb{O}) \cong \mathfrak{g}_2 \).

The \( \mathbb{Z}_2 \)-grading on \( \mathbb{O} = \mathbb{O}_0 \oplus \mathbb{O}_1 = \mathbb{H} \oplus \mathbb{H} \) induces a \( \mathbb{Z}_2 \)-grading on \( \text{Der}(\mathbb{O}) \) with

\[
(\mathfrak{g}^1)_0 = \{ d \in \text{Der}(\mathbb{O}) : d(\mathbb{H}) \subset \mathbb{H}, d(\mathbb{H}) \subset \mathbb{H} \},
\]

\[
(\mathfrak{g}^1)_1 = \{ d \in \text{Der}(\mathbb{O}) : d(\mathbb{H}) \subset \mathbb{H}, d(\mathbb{H}) \subset \mathbb{H} \}.
\]

As \( \mathbb{O} \) is generated (as an algebra) by \( \mathbb{H} \), any derivation \( d \in (\mathfrak{g}^1)_0 \) is determined by its restriction to \( \mathbb{H} \), so that

\[
(\mathfrak{g}^1)_0 \to \mathfrak{so}(\mathbb{H}, n) \cong \mathfrak{so}(4), \quad d \mapsto d|_{\mathbb{H}}
\]
is an isomorphism of Lie algebras and then \( (\mathfrak{g}^1)_0 \) is isomorphic to two copies of \( \mathfrak{sp}(1) \) as in Section 4.2. To be precise, we introduce the derivations

\[
\begin{align*}
d^-_a & : \mathbb{O} \to \mathbb{O} \quad d^+_a : \mathbb{O} \to \mathbb{O} \\
q & \mapsto [a, q] \\
q & \mapsto 0
\end{align*}
\]

for any \( a \in \mathbb{H}_0 \) (\( q \in \mathbb{H} \)), so that \( I^* = \{ d^*_a : a \in \mathbb{H}_0 \} \) is isomorphic to \( (\mathbb{H}_0, [\, , \,]) \cong \mathfrak{sp}(1) \) for each \( \sigma \in \{ \pm \} \) and \( (\mathfrak{g}^1)_0 = I^- \oplus I^+ \) is sum of two simple ideals. Consider the subalgebra \( \mathfrak{h}^1 = I^- \), isomorphic to \( \text{Der}(\mathbb{H}) = \mathcal{T}(\mathbb{H}, \mathbb{R}) \). We are in the situation of (15), so that we have \( \mathfrak{m}^1 = I^+ \oplus (\mathfrak{g}^1)_1 \). In order to check Eq. (16), we have to dive a little bit in the module structures. Taking into account that \( D_{\mathbb{O}_1, \mathbb{O}_2} \subset \text{Der}(\mathbb{O}) \), then \( (\mathfrak{g}^1)_1 = D_{\mathbb{H}_0, \mathbb{H}^1} \). As \( I^+ = \{ d \in \text{Der}(\mathbb{O}) : d(\mathbb{H}) = 0 \} \) acts on \( (\mathfrak{g}^1)_1 \) by \( [d^+_a, D_{\mathbb{H}_0, \mathbb{H}^1}] = D_{a,\mathbb{H}^1} \), this tells that \( (\mathfrak{g}^1)_1 = D_{1,\mathbb{H}} \oplus D_{2,\mathbb{H}^1} \) is sum of two (irreducible) modules, each one working as the \( \mathbb{H}_0 \)-module \( \mathbb{H}^1 \) under the left multiplication. (For these arguments, we have used (27).) Hence, the complexification \( (\mathfrak{g}^1)^C_{\mathbb{C}} \) breaks as 4 copies of the \( (I^+) \mathbb{C} \cong \mathfrak{sl}(2, \mathbb{C}) \)-module \( V(1) \). Even more,
it is not difficult to prove ([7] for more details) that \((\mathfrak{g}^s)_1\) is an absolutely irreducible \((\mathfrak{g}^s)_0\)-module whose complexification becomes
\[
(\mathfrak{g}^s)_0^C \cong V(1) \otimes V(3),
\]
the tensor product of the \((S^-)_C\)-module of type \(V(1)\) with the \((S^+_C)\)-module of type \(V(3)\). In particular Eq. (16) is satisfied so that Proposition 13 tells that \(G_2/\text{Sp}(1)\) is a 3-Sasakian manifold. Another consequence is that \((m^s)^C \cong 3\mathbb{C} \oplus 2V(3)\) is the decomposition as a sum of \((\mathfrak{h}^s)^C\)-irreducible submodules, and Corollary 11 can be applied.

All the remaining reductive decompositions for the exceptional cases can be obtained from the above case (note \(\mathfrak{h}^r \subset \mathfrak{h}^s\) and \(\mathfrak{g}^r \subset \mathfrak{g}^s\) if \(r < s\), namely,
\[
\mathfrak{h}^s = I^- \oplus \text{Der}(J^s) \oplus H_0 \otimes J_0^s,
\]
\[
\mathfrak{m}^s = I^+ \oplus D_{10,11} \oplus \mathbb{H} \otimes J_0^s.
\]
Indeed, the next (symmetric) decomposition easily provides a \(\mathbb{Z}_2\)-grading on \(\mathfrak{g}^s = T(\mathbb{O}, J^s)\):
\[
(\mathfrak{g}^s)_0 = \text{Der}(\mathbb{O})_0 \oplus (\mathbb{O}_0)_0 \otimes J_0^s \oplus \text{Der}(J^s) = I^+ \oplus \mathfrak{h}^s,
\]
\[
(\mathfrak{g}^s)_1 = \text{Der}(\mathbb{O})_1 \oplus (\mathbb{O}_0)_1 \otimes J_0^s,
\]
in such a way that \([I^+, \mathfrak{h}^s] = 0\), so that (15) holds. It is well-known that \((\mathfrak{g}^s)_0\) is a \((\mathfrak{g}^s)_0\)-irreducible module (see [28, Chapter 8]), but not absolutely irreducible. To be precise, \((\mathfrak{g}^s)_0^C \cong \mathfrak{sl}(2, \mathbb{C}) \oplus (\mathfrak{h}^s)^C\), where the Lie algebra \((\mathfrak{h}^s)^C\) is isomorphic to
\[
\mathfrak{sl}(2, \mathbb{C}) (A_1), \quad \mathfrak{sp}(6, \mathbb{C}) (C_1), \quad \mathfrak{su}(6, \mathbb{C}) (A_5), \quad \mathfrak{so}(12, \mathbb{C}) (D_6), \quad \mathfrak{e}_7^C (E_7),
\]
if \(s = 1, 2, 3, 4, 5\) respectively; and \((\mathfrak{g}^s)_1^C\) is isomorphic to the tensor product of the natural \(\mathfrak{sl}(2, \mathbb{C})\)-module \(\mathbb{C}^2 \cong V(1)\) with certain irreducible \((\mathfrak{h}^s)^C\)-module \(W_s\) which can be identified [21, Eq. 2.23] with the vector space
\[
W_s = \left\{ \begin{pmatrix} \alpha & x \\ y & \beta \end{pmatrix} : \alpha, \beta \in \mathbb{C}, x, y \in (J^s)^C \right\},
\]
of dimension 4, 14, 20, 32 and 56 respectively. In terms of dominant weights,
\[
(\mathfrak{g}^s)_1^C \cong \begin{cases} 
\mathbb{C}^2 \otimes V(3) & \text{if } s = 1, \\
\mathbb{C}^2 \otimes V(\lambda_3) & \text{if } s = 2, \\
\mathbb{C}^2 \otimes V(\lambda_5) & \text{if } s = 3, \\
\mathbb{C}^2 \otimes V(\lambda_5) & \text{if } s = 4, \\
\mathbb{C}^2 \otimes V(\lambda_7) & \text{if } s = 5.
\end{cases}
\]
This gives Eq. (16), so that Proposition 13 gives the 3-Sasakian structure, where now \(\xi_i = d^+_i \in \text{Der}(\mathbb{O}) \cap \mathfrak{m}^s\) if \(i = 1, 2, 3\) (for all \(s = 1, \ldots, 5\)). In particular, \((m^s)^C \cong 3\mathbb{C} \oplus 2W_s\) is a \((\mathfrak{h}^s)^C\)-module isomorphism, which is the condition to apply Corollary 11.

Two comments are in order. First, there is an abuse of notation, because \(\lambda_i\) is used simultaneously for the fundamental weight relative to different Lie algebras. We think that this is clear from the context. More relevant, for \(s = 4\), is the fact that there are two valid decompositions, the other one being \((m^s)^C \cong 3\mathbb{C} \oplus 2V(\lambda_6)\). Note that \(V(\lambda_5)\) and \(V(\lambda_6)\) are not isomorphic \(D_6\)-modules, but dual, while the own adjoint module \(D_6\) is self-dual. This explains why the existence of one reductive decomposition implies the other one.

With all this information, it is quite easy to compute the desired dimensions:

Lemma 24. If \(\mathfrak{g}^s = \mathfrak{h}^s \oplus \mathfrak{m}^s\) is the reductive decomposition related to the homogeneous manifold \(G_2/\text{Sp}(1), F_4/\text{Sp}(3), E_6/\text{SU}(6), E_7/\text{Spin}(12)\) and \(E_8/E_7\) respectively for \(s = 1, \ldots, 5\), then
\[
\dim_{\mathbb{R}} \text{Hom}_{\mathfrak{h}^s}(m^s \otimes m^s, m^s) = 63, \quad \dim_{\mathbb{R}} \text{Hom}_{\mathfrak{h}^s}(m^s, \Lambda^2 m^s) = 30, \quad \dim_{\mathbb{R}} \text{Hom}_{\mathfrak{h}^s}(\Lambda^3 m^s, \mathbb{R}) = 10.
\]

Proof. Once again, we apply Corollary 11 for \(U\) the \((\mathfrak{h}^s)^C\)-irreducible module of type \(V(3), V(\lambda_3), V(\lambda_5)\) and \(V(\lambda_7)\) respectively (if \(s = 1, \ldots, 5\)). Indeed,
\[
s = 1): \quad S^2U \cong V(6) \oplus V(2) \quad \text{and} \quad \Lambda^2U \cong V(4) \oplus \mathbb{C};
\]
\[ s = 2): \, S^2U \cong V(2\lambda_1) \oplus V(2\lambda_3) \quad \text{and} \quad \Lambda^2U \cong V(2\lambda_2) \oplus \mathbb{C}; \]

\[ s = 3): \, S^2U \cong V(\lambda_1 + \lambda_3) \oplus V(2\lambda_3) \quad \text{and} \quad \Lambda^2U \cong V(\lambda_2 + \lambda_4) \oplus \mathbb{C}; \]

\[ s = 4): \, S^2U \cong V(2\lambda_2) \oplus V(\lambda_2) \quad \text{and} \quad \Lambda^2U \cong V(\lambda_4) \oplus \mathbb{C}; \]

\[ s = 5): \, S^2U \cong V(2\lambda_2) \oplus V(\lambda_1) \quad \text{and} \quad \Lambda^2U \cong V(\lambda_6) \oplus \mathbb{C}. \]

Then, \( \text{Hom}_c(S^2U, U) = \text{Hom}_c(\Lambda^2U, U) = \text{Hom}_c(S^2U, \mathbb{C}) = 0, \) and \( \text{Hom}_c(\Lambda^2U, \mathbb{C}) \) has (complex) dimension one. Now, by Corollary 11, we conclude that \( \dim \text{Hom}_c(m^* \otimes m^*, m^*) = 63 \) and \( \dim \text{Hom}_c(m^*, \Lambda^2m^*) = 30. \) The third dimension is equal to 10, as a consequence of the fact that neither \( \Lambda^3U \) nor \( \Lambda^2U \otimes U \) contains any trivial submodule:

\[ s = 1): \, \Lambda^2U \otimes U \cong V(7) \oplus V(5) \oplus 2V(3) \oplus 2V(1) \quad \text{and} \quad \Lambda^3U \cong V(3); \]

\[ s = 2): \, \Lambda^2U \otimes U \cong V(2\lambda_2 + \lambda_3) \oplus V(\lambda_1 + 2\lambda_2) \oplus V(2\lambda_1 + \lambda_3) \oplus V(\lambda_1 + \lambda_2) \oplus 2V(\lambda_3); \Lambda^3U \cong V(\lambda_1 + 2\lambda_2) \oplus V(\lambda_3); \]

and so on. \( \square \)

4.4. **Case** \( M = \frac{SU(m)}{S(U(m-2) \times U(1))} \) **with** \( m \geq 3. \) This family of homogeneous manifolds has attached the following reductive decomposition, for \( n = m - 2 \geq 1: \)

\( \mathfrak{g} = su(m) = \{ A \in M_m(\mathbb{C}) : A + \bar{A}^t = 0, \text{tr} A = 0 \}, \)

the subalgebra (with matrices written by blocks \( 1 + n + 1 \))

\[
\mathfrak{h} = \left\{ \begin{pmatrix} -\frac{1}{2}B & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & -\frac{1}{2}B \end{pmatrix} : B \in \mathfrak{u}(n) \right\},
\]

and the complementary subspace

\[
\mathfrak{m} = \left\{ \begin{pmatrix} \alpha i & z_2^t & w \\ -\bar{z}_2 & 0 & z_1 \\ -\bar{w} & -\bar{z}_1^t & -\alpha i \end{pmatrix} : z_1, z_2 \in \mathbb{C}^n, w \in \mathbb{C}, \alpha \in \mathbb{R} \right\}.
\]

Observe that, in this case, \( \mathfrak{h} \) is not semisimple. In fact, we have the decomposition \( \mathfrak{h} = Z(\mathfrak{h}) \oplus [\mathfrak{h}, \mathfrak{h}] \)

with a one-dimensional center \( Z(\mathfrak{h}) = \mathbb{R}I_n \) and \( [\mathfrak{h}, \mathfrak{h}] \cong su(n), \) which is simple if \( n \neq 1 \) while is 0 if \( n = 1, \) case in which \( \mathfrak{h} \) is one-dimensional and hence abelian. Also, let us note that \( \kappa(\mathfrak{h}, \mathfrak{m}) = 0 \) for \( \kappa \)

the Killing form of \( \mathfrak{g} \) and \( \dim \mathfrak{m} = 4n + 3. \)

In order to understand \( \mathfrak{m} \) as \( \mathfrak{h} \)-module, we use the above suggested identifications \( \mathfrak{h} \cong \mathfrak{u}(n) \) and \( \mathfrak{m} \cong \mathbb{C}^n \oplus \mathbb{C}^n \oplus su(2) \) given by

\[
(30) \quad \begin{pmatrix} \frac{-1}{2}B & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & \frac{-1}{2}B \end{pmatrix} \rightarrow B, \quad \begin{pmatrix} \alpha i & z_2^t & w \\ -\bar{z}_2 & 0 & z_1 \\ -\bar{w} & -\bar{z}_1^t & -\alpha i \end{pmatrix} \rightarrow (z_1, z_2) + \begin{pmatrix} \alpha i & w \\ -\bar{w} & -\alpha i \end{pmatrix}.
\]

Thus, the action of \( \mathfrak{h} \) on \( \mathfrak{m} \) is translated from the bracket \( [\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m} \) and can be expressed in these terms as

\[
(31) \quad B \cdot \left( z_1, z_2 \right) + \begin{pmatrix} \alpha i & w \\ -\bar{w} & -\alpha i \end{pmatrix} = \left( B + \frac{\text{tr} B}{2} I_n \right) z_1, \left( B - \frac{\text{tr} B}{2} I_n \right) z_2,
\]

for \( I_n \) the identity matrix. (In particular, \( iI_n \cdot \left( z_1, z_2 \right) = \left( 1 + \frac{\text{tr} A}{2} \right) i(z_1, -z_2). \) In other words, \( \mathfrak{m} \) can be decomposed as \( \mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2 \oplus \mathfrak{m}_3, \) the sum of the following \( \mathfrak{h} \)-submodules:

- \( \mathfrak{m}_1 \equiv \{ (z_1, 0) : z_1 \in \mathbb{C}^n \}, \) which is, as \( \mathfrak{h}, \mathfrak{m} \cong su(n) \)-module, isomorphic to the natural module \( \mathbb{C}^n \) (hence irreducible), while the center \( Z(\mathfrak{h}) \) acts scalarily by \( (iI_n)z_1 = (1 + \frac{\text{tr} A}{2}) i z_1; \)
- \( \mathfrak{m}_2 \equiv \{ (0, z_2) : z_2 \in \mathbb{C}^n \}, \) which is, as \( su(n) \)-module, irreducible and isomorphic to \( \mathbb{C}^n^* \), the dual of the natural module, while the center \( Z(\mathfrak{h}) \) acts scalarily by \( (iI_n)z_2 = -\left( 1 + \frac{\text{tr} A}{2} \right) i z_2; \)
- \( \mathfrak{m}_3 \equiv \{ X : X \in su(2) \}, \) which is a trivial 3-dimensional \( \mathfrak{h} \)-module.
Note that, if \( n = 1 \), the decomposition \( m = m_1 \oplus m_2 \oplus m_3 \) still works, but now these pieces are not irreducible. To deal with this case, we simply forget the \([\mathfrak{h}, \mathfrak{h}]\)-action, taking only into account that \((iI_1)z_1 = \frac{\sqrt{2}}{2}i z_1\) and \((iI_1)z_2 = -\frac{\sqrt{2}}{2}i z_2\).

We can do a initial comparison with our previous cases of reductive decompositions related to 3-Sasakian homogeneous manifolds. First, as was mentioned, \( \mathfrak{h} \) is not semisimple. Second, we have the \( \mathbb{Z}_2 \)-grading required for a 3-Sasakian data \( g_0 = \mathfrak{h} \oplus m_3 \) and \( g_1 = m_1 \oplus m_2 \), since \( m_3 \cong \mathfrak{su}(2) \cong \mathfrak{sp}(1) \) and \([m_3, \mathfrak{h}] = 0\). But in this case \( g_1 \) is not \( \mathfrak{h} \)-irreducible, since \( m_1 \) and \( m_2 \) are proper \( \mathfrak{h} \)-submodules. This makes more delicate to prove Eq. (16), as well as our computation of dimensions.

Thus, let us begin to study how is \( \mathfrak{m}^C \) as \( \mathfrak{h}^C \)-module for \( n \neq 1 \). Denote by \( \mathcal{V} \) the \( \mathfrak{sl}(n, \mathbb{C}) \)-natural module \( \mathbb{C}^n \) (that is, the action is given by column multiplication). Let \( \mathcal{V}_+ \) and \( \mathcal{V}_- \) be the \( \mathfrak{gl}(n, \mathbb{C}) = (\mathfrak{sl}(n, \mathbb{C}) \oplus \mathcal{I}_n) \)-modules which are \( \mathcal{V} \) and \( \mathcal{V}^* \) as \( \mathfrak{sl}(n, \mathbb{C}) \)-modules, and where the action of \( I_n \) is \((1 + \frac{\sqrt{2}}{2})\text{id} \) and \(- (1 + \frac{\sqrt{2}}{2})\text{id}\), respectively. Our purpose is to prove that

\[
\mathfrak{m}^C \cong 3 \mathbb{C} \oplus 2 \mathcal{V}_+ \oplus 2 \mathcal{V}_-
\]

is the decomposition of \( \mathfrak{m}^C \) as a sum of irreducible \( \mathfrak{h}^C \)-submodules, so that Lemma 10 can be again applied. More precisely, we prove \( \mathfrak{m}^C_1 \cong \mathfrak{m}^C_2 \cong \mathcal{V}_+ \oplus \mathcal{V}_- \), in spite that \( m_1 \neq m_2 \).

Recall that \( \mathfrak{h}^C = \mathfrak{h} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathfrak{gl}(n, \mathbb{C}) = \mathfrak{su}(n) \oplus \mathfrak{u}(n) \), since any \( A \in \mathfrak{gl}(n, \mathbb{C}) \) can be written as

\[
A = A_0 + A_1 i \quad \text{for} \quad A_0 = \frac{1}{2}(A - A^t), \quad A_1 = -\frac{i}{2}(A + A^t) \in \mathfrak{u}(n).
\]

Besides, if \( A \in \mathfrak{sl}(n, \mathbb{C}) \), then \( A_0, A_1 \in \mathfrak{su}(n) \). The action of \( \mathfrak{gl}(n, \mathbb{C}) \) on \( (\mathbb{C}^n)^C \) obtained as a complexification of an action \( \phi \) of \( \mathfrak{su}(n) \) on \( \mathbb{C}^n \) is defined by

\[
A \circ (z \otimes 1 + w \otimes i) = (A_0 \circ z - A_1 \circ w) \otimes 1 + (A_0 \circ w + A_1 \circ z) \otimes i,
\]

for any \( A \in \mathfrak{h}^C \) (viewed as \( A_0 \circ 1 + A_1 \circ i \)), \( z, w \in \mathbb{C}^n \). In particular, \( I_n \circ (z \otimes 1 + w \otimes i) = (I_n \circ w) \otimes 1 - (I_n \circ z) \otimes i \). This gives, jointly with Eq. (31),

\[
(33) \quad A \cdot (z_1 \otimes 1 - iz_1 \otimes i, 0) = (AZ_1 \otimes 1 - i(AZ_1) \otimes i, 0),
\]

\[
A \cdot (z_1 \otimes 1 + iz_1 \otimes i, 0) = (-A^t z_1 \otimes 1 - i(A^t z_1) \otimes i, 0),
\]

\[
A \cdot (0, z_2 \otimes 1 - iz_2 \otimes i) = (0, -A^t z_2 \otimes 1 + i(A^t z_2) \otimes i),
\]

\[
A \cdot (0, z_2 \otimes 1 + iz_2 \otimes i) = (0, A z_2 \otimes 1 + i(A z_2) \otimes i),
\]

for any \( A \in \mathfrak{sl}(n, \mathbb{C}) \), \( z_1, z_2 \in \mathbb{C}^n \), while \( I_n \) acts scalarly with eigenvalue \( \lambda := 1 + \frac{\sqrt{2}}{2} \) on

\[
(34) \quad \{(z_1 \otimes 1 - iz_1 \otimes i, 0) : z_1 \in \mathbb{C}^n\} \oplus \{(0, z_2 \otimes 1 + iz_2 \otimes i) : z_2 \in \mathbb{C}^n\}
\]

and with eigenvalue \(- (1 + \frac{\sqrt{2}}{2}) \) on

\[
(35) \quad \{(z_1 \otimes 1 + iz_1 \otimes i, 0) : z_1 \in \mathbb{C}^n\} \oplus \{(0, z_2 \otimes 1 - iz_2 \otimes i) : z_2 \in \mathbb{C}^n\}.
\]

Let us denote by \( H : \mathbb{C}^n \times \mathbb{C}^n \rightarrow \mathbb{C} \) the usual Hermitian product given by \( H(u, v) = u^t \bar{v} \), and by \( g : \mathbb{C}^n \times \mathbb{C}^n \rightarrow \mathbb{C} \) the usual scalar product given by \( g(u, v) = u^t v \). Now, it is a direct consequence from (33) that the following maps

\[
\begin{align*}
\mathfrak{m}^C_1 & \cong \mathbb{C}^n \oplus (\mathbb{C}^n)^* \\
(z_1 \otimes 1 - iz_1 \otimes i, 0) & \mapsto z_1 \\
(z_1 \otimes 1 + iz_1 \otimes i, 0) & \mapsto H(\cdot, z_1),
\end{align*}
\]

and

\[
\begin{align*}
\mathfrak{m}^C_2 & \cong (\mathbb{C}^n)^* \oplus \mathbb{C}^n \\
(0, z_2 \otimes 1 - iz_2 \otimes i) & \mapsto g(z_2, \cdot) \\
(0, z_2 \otimes 1 + iz_2 \otimes i) & \mapsto z_2,
\end{align*}
\]

are isomorphisms of \([\mathfrak{h}^C, \mathfrak{h}^C]\)-modules. Also, Eqs. (34) and (35) tell that they are isomorphisms of \( \mathfrak{h}^C \)-modules when we consider the scalar action of \( I_n \in Z(\mathfrak{h}^C) \) on \( \mathbb{C}^n \) and \( (\mathbb{C}^n)^* \) with eigenvalue \( \lambda \) and \(- \lambda \) respectively. This finishes the proof of Eq. (32).
The case \( n = 1 \) has to be considered separately. Denote by \( \mathcal{V}_s \), \( s \in \mathbb{Z} \), the one-dimensional \( \mathbb{C} \)-vector space in which \( I_1 \) acts scalarly with eigenvalue \( 3/2s \). Thus \( \mathfrak{m}^C \) is isomorphic to the \( \mathfrak{gl}(n, \mathbb{C}) \)-module \( 2 \mathcal{V}_1 \oplus 3 \mathcal{V}_0 \oplus 2 \mathcal{V}_{-1} \) (complex dimension 7). It is very useful to observe that \( \mathcal{V}_s \odot \mathcal{V}_t \cong \mathcal{V}_{s+t} \) for all \( s, t \in \mathbb{Z} \).

Now, we are in position to prove

**Lemma 25.** If \( \mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m} \) is the reductive decomposition related to \( M = \frac{\text{SU}(m)}{\text{SU}(m-2) \times \text{U}(1)} \) for \( m \geq 3 \), then \( \dim_{\mathbb{R}} \text{Hom}_h(\mathfrak{m} \oplus \mathfrak{m}, \mathfrak{m}) = 99 \), \( \dim_{\mathbb{R}} \text{Hom}_h(\mathfrak{m}, \Lambda^2 \mathfrak{m}) = 45 \), and \( \dim_{\mathbb{R}} \text{Hom}_h(\Lambda^3 \mathfrak{m}, \mathbb{R}) = 13 \).

**Proof.** Consider first the general case \( n = m - 2 \neq 1 \). On one hand, by Lemma 10c) and b),

\[
\Lambda^2 \mathfrak{m}^C \cong 3 \Lambda^2 (\mathcal{V}_+ \oplus \mathcal{V}_-^*) \oplus 4 \mathcal{V}_+^\perp \oplus 3 \mathcal{V}_+^\perp \oplus 6 \mathcal{V}_+^\perp \oplus 6 \mathcal{V}_-^\perp \oplus 3 \mathcal{V}_-^\perp.
\]

Observe that \( I_n \) acts on \( \Lambda^2 \mathcal{V}_+ \) and on \( S^2 \mathcal{V}_+ \) with eigenvalue \( 2\lambda \), on \( \Lambda^2 \mathcal{V}_+^\perp \) and on \( S^2 \mathcal{V}_+^\perp \) with eigenvalue \( -2\lambda \), and on \( \mathcal{V}_+ \odot \mathcal{V}_-^\perp \) with eigenvalue 0, so that the summand \( 6 \mathcal{V}_+^\perp \) does not contain any copy of the trivial module, because the \( \mathfrak{sl}(n, \mathbb{C}) \)-module \( \mathcal{V} \otimes \mathcal{V}^* \) is isomorphic to the adjoint module \( \mathfrak{sl}(\mathcal{V}) \) direct sum with a copy of the trivial one. Hence

\[
\dim_{\mathbb{R}} \text{Hom}_h(\mathfrak{m} \oplus \mathfrak{m}, \mathfrak{m}) = 3 \dim_{\mathbb{C}} \text{Hom}_{\mathfrak{h}^C}(\Lambda^2 \mathfrak{m}^C, \mathbb{C}) + 2 \dim_{\mathbb{C}} \text{Hom}_{\mathfrak{h}^C}(\Lambda^2 \mathfrak{m}^C, \mathcal{V}^+) 2
= 3(4 + 4) + 2 \cdot 6 \cdot 2 = 45.
\]

Similarly, we check that

\[
\mathfrak{m}^C \otimes \mathfrak{m}^C \cong 4\Lambda^2 (\mathcal{V}_+ \oplus \mathcal{V}_-^*) \oplus 4 S^2 (\mathcal{V}_+ \oplus \mathcal{V}_-^*) \oplus 12 \mathcal{V}_+ \oplus 12 \mathcal{V}_-^* \oplus 9 \mathbb{C},
\]

and hence, taking into consideration that \( S^2 (\mathcal{V}_+ \oplus \mathcal{V}_-^*) \cong S^2 (\mathcal{V}_+^\perp) \oplus S^2 (\mathcal{V}_-^\perp) \oplus (\mathcal{V}_+ \odot \mathcal{V}_-^\perp) \) there is just one copy of \( \mathbb{C} \), then

\[
\dim_{\mathbb{R}} \text{Hom}_h(\mathfrak{m} \oplus \mathfrak{m}, \mathfrak{m}) = 3 \dim_{\mathbb{C}} \text{Hom}_{\mathfrak{h}^C}(\mathfrak{m} \otimes \mathfrak{m}, \mathbb{C}) + 2 \dim_{\mathbb{C}} \text{Hom}_{\mathfrak{h}^C}(\mathfrak{m} \otimes \mathfrak{m}, \mathcal{V}_+) 2
= 3(4 + 4 + 9) + 2 \cdot 12 \cdot 2 = 99.
\]

Finally, the \( \mathfrak{gl}(n, \mathbb{C}) \)-module \( \Lambda^3 \mathfrak{m}^C \) decomposes as a sum of:

- \( 2 \Lambda^2 (\mathcal{V}_+ \oplus \mathcal{V}_-^*) \oplus 2 (\Lambda^2 (\mathcal{V}_+ \oplus \mathcal{V}_-^*) \otimes (\mathcal{V}_+ \odot \mathcal{V}_-^*)) \), which does not contain any copy of the trivial module \( \mathbb{C} \), since \( I_n \in \mathfrak{gl}(n, \mathbb{C}) \) acts with eigenvalue \( \pm \lambda \pm \lambda \pm \lambda \neq 0 \) on any of the submodules;
- 9 copies of \( \Lambda^2 (\mathcal{V}_+ \oplus \mathcal{V}_-^*) \), each one with a trivial 1-dimensional submodule;
- 3 copies of \( S^2 (\mathcal{V}_+ \oplus \mathcal{V}_-^*) \), each one with a trivial irreducible submodule;
- \( 6 \mathcal{V}_- \oplus 6 \mathcal{V}_+^* \oplus \mathbb{C} \).

In particular \( \Lambda^3 \mathfrak{m}^C \) just contains \( 9 + 3 + 1 = 13 \) copies of \( \mathbb{C} \) and the result follows.

Consider finally the case \( n = 1 \). We obtain the same dimensions as for \( n \neq 1 \) for the three sets of homomorphisms, after computing

\[
\mathfrak{m}^C \otimes \mathfrak{m}^C \cong 4 \mathcal{V}_2 \oplus 12 \mathcal{V}_1 \oplus 17 \mathcal{V}_0 \oplus 12 \mathcal{V}_{-1} \oplus 4 \mathcal{V}_{-2};
\]

\[
\Lambda^2 \mathfrak{m}^C \cong \mathcal{V}_2 \oplus 6 \mathcal{V}_1 \oplus 7 \mathcal{V}_0 \oplus 6 \mathcal{V}_{-1} \oplus \mathcal{V}_{-2};
\]

\[
\Lambda^3 \mathfrak{m}^C \cong 3 \mathcal{V}_2 \oplus 8 \mathcal{V}_1 \oplus 13 \mathcal{V}_0 \oplus 8 \mathcal{V}_{-1} \oplus 3 \mathcal{V}_{-2};
\]

and applying Lemma 9 ii). \( \square \)

**Remark 26.** It is a remarkable fact that there are more invariant 3-forms on \( \mathfrak{m} \) than in the remaining 3-Sasakian homogeneous manifolds studied in the previous sections. We can provide an explicit description of the related 13 linear independent 3-forms. Consider first \( \{ \xi_i, \eta_i, \varphi_i \}_{i=1} \) and the metric \( g \) as in Proposition 13. Use our previous identification of \( \mathfrak{m} \) with \( \mathbb{C}^n \oplus \mathbb{C}^n \oplus \mathfrak{su}(2) \), writing the elements as a sum of a pair of vectors and a matrix in \( \mathfrak{su}(2) \). As the Killing form of \( \mathfrak{su}(m) \) is \( \kappa(x, y) = 2m \text{ tr}(xy) \), it is easy to check that

\[
g((z, w), (u, v)) = \frac{1}{2} \text{Re}(z^t \bar{u} + w^t \bar{v}),
\]
for any $z, w, u, v \in \mathbb{C}^n$, and

\begin{equation}
\varphi_1(z, w) = (iz, iw), \quad \varphi_2(z, w) = (-\bar{w}, \bar{z}), \quad \varphi_3(z, w) = (-iw, i\bar{z}).
\end{equation}

This allows to check that the two-forms $\Phi_i$'s restricted to $\mathbb{C}^n \oplus \mathbb{C}^n \leq \mathfrak{m}$ are given by

$$
\Phi_i((z, w), (u, v)) = \begin{cases} 
\text{Im}(z^t\bar{u} + w^t\bar{v})/2 & i = 1, \\
\text{Re}(-z^t\bar{v} + w^t\bar{u})/2 & i = 2, \\
\text{Im}(z^t\bar{v} + w^t\bar{u})/2 & i = 3.
\end{cases}
$$

Consider, finally, $h = (1 + \frac{\vartheta}{2})^{-1} \begin{pmatrix} -4 & 0 & 0 \\
0 & i & 0 \\
0 & 0 & -4 \\
\end{pmatrix} \in \mathfrak{h}$ and the endomorphism $\varphi_0 = \text{ad} h|_{\mathfrak{m}} \in \text{ad} \mathfrak{h}|_{\mathfrak{m}} \subset \text{End}_\mathbb{C}(\mathfrak{m})$. By Eq. (31),

$$
\varphi_0(z, w) = (iz, -iw) \quad \text{and} \quad \varphi_0|_{\mathfrak{su}(2)} = 0,
$$

so that the related $\mathfrak{h}$-invariant 2-form $\Phi_0(\cdot, \cdot) = g(-\cdot, \varphi_0(-\cdot))$ satisfies

$$
\Phi_0((z, w), (u, v)) = \frac{1}{2}\text{Im}(z^t\bar{u} - w^t\bar{v}).
$$

As $[\mathfrak{h}, \mathfrak{sp}(1)] = 0$, we obtain $\varphi_0\varphi_i = \varphi_i\varphi_0$, for any $i = 1, 2, 3$. It is also clear that $g(\varphi_0X, Y) + g(X, \varphi_0Y) = 0$. As a consequence, the extra 3-forms are $\eta_i \wedge \Phi_0$, $i = 1, 2, 3$, which are of course linearly independent. \(\square\)

In order to apply Proposition 13 to assert that $\{\xi_i, \eta_i, \varphi_i\}_{i=1}^3$ is in fact a 3-Sasakian structure, we need to prove that the $\mathfrak{g}_0^C$-module $\mathfrak{g}_1^C$ is isomorphic to the $\mathfrak{g}_0^C \cong \mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(n, \mathbb{C})$-module

$$
\mathfrak{g}_1^C \cong \mathbb{C}^2 \otimes (V_+ \oplus V_+^*).
$$

We know that this is true as $\mathfrak{h}^C$-modules, by Eq. (32). We also know ([25]) that any irreducible $\mathfrak{g}_1^C$-submodule $W_i$ of $\mathfrak{g}_1^C$ is the tensor product of an irreducible $\mathfrak{sl}(2, \mathbb{C})$-module $V_i$ with an irreducible $\mathfrak{sl}(n, \mathbb{C})$-module $U_i$, so that $\mathfrak{g}_1^C = \bigoplus V_i \otimes U_i$ is isomorphic as $\mathfrak{sl}(n, \mathbb{C})$-module to $\bigoplus_i (\dim V_i)U_i \cong 2V_+ \oplus 2V^*_+$. We would like to prove that $\dim V_i = 2$ for any $i$. By dimension count, the only other possibility would be $\dim V_i = 1$ for some $i$, but then there would be a trivial $\mathfrak{sl}(2, \mathbb{C})$-submodule of $\mathfrak{g}_1^C$ of dimension $n$. This would give a contradiction, because the $\mathfrak{sl}(2, \mathbb{C})$-action on $\mathfrak{g}_1^C$ is complexified of the action of $\varphi_i$'s described in Eq. (36), obviously never trivial: $\varphi_i^2|_{\mathfrak{g}_1} = -\text{id}$.

4.5. Some general conclusions. We have shown that $\text{Hom}_\mathbb{C}(\mathfrak{m} \otimes \mathfrak{m}, \mathfrak{m})$ is a real vector space of dimension 63 in all the cases with $G \neq SU(m)$. This was observed in the case $G_2/SU(2)$ ([7]) and on the spheres $S^{4n+3}$ under the action of the symplectic group ([20]). We now provide a unified description of the set $\text{Hom}_\mathbb{C}(\mathfrak{m} \otimes \mathfrak{m}, \mathfrak{m})$, which can be obviously extended to obtain concrete expressions of the invariant affine connections.

**Proposition 27.** Assume the previous situation with $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \neq \mathfrak{su}(m)$. Let $\pi_0 : \mathfrak{m} \to \mathfrak{m}$ be the projection onto the second factor relative to the decomposition $\mathfrak{m} = \mathfrak{sp}(1) \oplus \mathfrak{g}_1$, and $X^h = \pi_0(X)$ if $X \in \mathfrak{m}$. Let $\theta : \mathfrak{m} \times \mathfrak{m} \to \mathfrak{m} \times \mathfrak{m}$ be the interchanging map $\theta(X, Y) = (Y, X)$ and let $\kappa$ be the Killing form of $\mathfrak{g}$. Then a basis of the vector space $\text{Hom}_\mathbb{C}(\mathfrak{m} \otimes \mathfrak{m}, \mathfrak{m})$ is provided by the following bilinear maps

\begin{equation}
\{\alpha_{rst}, \beta_{rs}, \beta_{0s} \circ \theta, \beta_{rs} \circ \theta, \gamma_{0s}, \gamma_{rs} : r, s, t = 1, 2, 3\},
\end{equation}

which are defined, for any $X, Y \in \mathfrak{m}$, by

$$
\alpha_{rst}(X, Y) = \eta_r(X)\eta_s(Y)\xi_t,
$$

$$
\beta_{0s}(X, Y) = \eta_s(X)Y^h, \quad \beta_{rs}(X, Y) = \eta_s(X)\varphi_r(Y^h),
$$

$$
\gamma_{0s}(X, Y) = \kappa(X^h, Y^h)\xi_s, \quad \gamma_{rs}(X, Y) = \eta_r([X^h, Y^h]_{\mathfrak{sp}(1)})\xi_s.
$$

**Proof.** This is simply a corollary of all the previous results, since the set in (37) provides 63 independent elements in $\text{Hom}_\mathbb{C}(\mathfrak{m} \otimes \mathfrak{m}, \mathfrak{m})$. \(\square\)
Remark 28. We have proved that we have a 3-Sasakian data on each of our cases, so that the 3-Sasakian structure is described by Proposition 13. We have used a case-by-case test proving \( \mathfrak{g}^C_{\mathfrak{h}} \cong \mathbb{C}^2 \otimes W \). But now we would like to prove now that some conditions are sufficient to assure Eq. (16) starting with Eq. (15), that is, with the \( \mathbb{Z}_2 \)-grading \( \mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \) whose even part is sum of the ideals \( \mathfrak{g}_0 = \mathfrak{sp}(1) \oplus \mathfrak{h} \). Namely,

If \( \mathfrak{h} \) is semisimple and \( \mathfrak{g}_1 \) is an \( \mathfrak{h} \)-irreducible module (i.e., \( G \neq \mathbb{SU}(m) \)), then Eq. (16) holds.

Indeed, on one hand, as \( [\mathfrak{sp}(1), \mathfrak{h}] = 0 \) and also \( [\mathfrak{sp}(1), \mathfrak{g}_1] \subset [\mathfrak{g}_0, \mathfrak{g}_1] \subset \mathfrak{g}_1 \), this implies that \( \text{ad}(\mathfrak{sp}(1)) \subset \text{End}_\mathbb{h}(\mathfrak{g}_1) \). But the (finite-dimensional) centralizer \( \text{End}_\mathbb{h}(\mathfrak{g}_1) \) is a real division associative algebra (consequently of dimension either 1 or 2 or 4) containing a copy of \( \mathbb{H} \), so that such centralizer has to be isomorphic to \( \mathbb{H} \). If \( \mathfrak{h} \) is simple, [17, Lemma 5.7] tells that \( \mathfrak{g}_1^C \) is sum of two isomorphic irreducible \( \mathfrak{h}^C \)-modules (and it has real dimension multiple of 4). On the other hand, every irreducible module for a complex semisimple algebra is isomorphic to the tensor product of irreducible \( \mathfrak{h} \)-modules. Hence, as \( \mathfrak{h}^C \)-module, \( \mathfrak{g}_1^C \cong (\dim V)U \), so that \( \dim V = 2 \), which was the number of irreducible \( \mathfrak{h}^C \)-submodules of \( \mathfrak{g}_1^C \). As there are only one \( \mathfrak{sp}(1)^C = \mathfrak{sl}(2, \mathbb{C}) \)-module of dimension 2, the natural module \( \mathbb{C}^2 \), then Eq. (16) holds. The proof in the semisimple case is the same one, taking into account that [17, Lemma 5.7] can be adapted to such hypothesis: only the complete reducibility was used through the proof. \( \square \)

5. Invariant Affine Connections with Skew-Torsion

In this section we give a basis of the space \( \text{Hom}_\mathfrak{h}(\mathfrak{m} \wedge \mathfrak{m} \wedge \mathfrak{m}, \mathbb{R}) \), which is used to describe explicitly the set of all invariant affine connections with skew-torsion on a homogeneous 3-Sasakian manifold. We also determine which of these invariant affine connections satisfy the Einstein with skew-torsion condition.

Let \( (M = G/H, g) \) be a 3-Sasakian homogeneous manifold with a fixed reductive decomposition as in (6). We will focus on dimension at least 7, since the 3-dimensional sphere \( S^3 \) has been previously studied in [18]. Let us denote by \( \{ \xi_i, \eta_r, \varphi_i \}_{i=1,2,3} \) the three compatible Sasakian structures on \( M \).

Also, recall the 2-forms \( \Phi_\iota(X, Y, Z) = g(X, \varphi_\iota Y) \), for any \( X, Y, Z \in \mathfrak{X}(M) \). We put \( X^v \) the component of \( X \) in \( \mathcal{Q}^1 = \text{Span}\{\xi_1, \xi_2, \xi_3\} \), i.e., \( X^v = \sum_{j=1}^3 \eta_j(X)\xi_j \), and we consider the unique vectorial product in \( \mathcal{Q}^2 \) such that \( \xi_1 \times \xi_2 = \xi_3 \) (that is, \( \times = \frac{1}{3} [ , ] \)). In particular, \( \eta_r \wedge \Phi_\iota(X_1, X_2, X_3) = \sum_{j=1}^3 \eta_j(X_i)\Phi_\iota(X_{i+1}, X_{i+2}) \) (indices modulo 3).

Denote by \( T^0 \) and \( T^{rs} \) the \((1,2)\)-tensors given as follows

\[
\eta_1 \wedge \Phi_0(X, Y, Z) = g(T^0(X, Y, Z), Z), \quad \eta_r \wedge \Phi_\iota(X, Y, Z) = g(T^{rs}(X, Y, Z), Z),
\]

for any \( X, Y, Z \in \mathfrak{m} \) and \( r, s \in \{1, 2, 3\} \). For the SU-case, it will also be convenient to introduce the tensors \( T^{r0} \) given by (see Remark 26)

\[
\eta_r \wedge \Phi_0(X, Y, Z) = g(T^{r0}(X, Y, Z), Z).
\]

Lemma 29. i) For \( G \neq \mathbb{SU}(m) \), the set \( \{ \eta_1 \wedge \eta_2 \wedge \eta_3 \} \cup \{ \eta_r \wedge \Phi_\iota : r, s = 1, 2, 3 \} \) at the point \( o \) is a basis of \( \text{Hom}_\mathfrak{h}(\mathfrak{m} \wedge \mathfrak{m} \wedge \mathfrak{m}, \mathbb{R}) \).

ii) For \( G = \mathbb{SU}(m), m \geq 3 \), the set \( \{ \eta_1 \wedge \eta_2 \wedge \eta_3 \} \cup \{ \eta_r \wedge \Phi_\iota : r = 1, 2, 3, s = 0, 1, 2, 3 \} \) at the point \( o \) is a basis of \( \text{Hom}_\mathfrak{h}(\mathfrak{m}, \mathbb{R}) \).

iii) For any \( X, Y \in \mathfrak{m}, r = 1, 2, 3, s = 0, 1, 2, 3 \), we have

\[
T^0(X, Y) = X^v \times Y^v, \quad T^{rs}(X, Y) = -\eta_r(X)\varphi_s Y + \eta_s(Y)\varphi_r X + \Phi_\iota(X, Y)\xi_r.
\]

---

\(^3\)Our convention for the exterior product of a \( p \)-form \( \omega_1 \) and a \( q \)-form \( \omega_2 \) is the following

\[
\omega_1 \wedge \omega_2(X_1, \ldots, X_{p+q}) = \frac{1}{p!q!} \sum_{\sigma \in S_{p+q}} (-1)^{|\sigma|} \omega_1(X_{\sigma(1)}, \ldots, X_{\sigma(p)})\omega_1(X_{\sigma(p+1)}, \ldots, X_{\sigma(p+q)}).
\]
Proof. For item i), recall that all G-homogeneous 3-Sasakian manifolds with $G \neq SU(m)$ satisfy $\dim \text{Hom}_0(\Lambda^3 \mathfrak{m}, \mathbb{R}) = 10$. Thus, we only need to check the linear independence of the sets given in i). Thus, consider a linear combination

$$b \eta_1 \wedge \eta_2 \wedge \eta_3 + \sum_{r,s=1}^3 b_{rs} \eta_r \wedge \Phi_s = 0.$$ 

Given a unit vector $X \in \mathcal{Q}$, we take $(Y_1, Y_2, Y_3) = (\xi_1, \varphi_j X, X)$. Then, taking into account that $g(\varphi_j X, \varphi_j X) = \delta_{js} g(X, X) = \delta_{js}$, we compute

$$0 = \left(b \eta_1 \wedge \eta_2 \wedge \eta_3 + \sum_{r,s=1}^3 b_{rs} \eta_r \wedge \Phi_s\right)(Y_1, Y_2, Y_3)$$

$$= \sum_{r,s=1}^3 b_{rs} \sum_{k=1}^3 \eta_r(Y_k) \Phi_s(Y_{k+1}, Y_{k+2}) = \sum_{r,s=1}^3 b_{rs} \delta_{rs} g(\varphi_j X, \varphi_j X) = b_{jj}.$$ 

Now $0 = b \eta_1 \wedge \eta_2 \wedge \eta_3(\xi_1, \xi_2, \xi_3) = b$. This shows the linear independence in case i). The proof of item ii) is similar, bearing in mind that $\dim \text{Hom}_0(\Lambda^3 \mathfrak{m}, \mathbb{R}) = 13$.

Finally, we get the metrically equivalent torsions to the 3-forms $\eta_1 \wedge \eta_2 \wedge \eta_3$ and $\eta_r \wedge \Phi_s$. As

$$\eta_1 \wedge \eta_2 \wedge \eta_3(X_1, X_2, X_3) = \sum_{\sigma \in S_3} (-1)^{[\sigma]} \eta_1(X_{\sigma(1)}) \eta_2(X_{\sigma(2)}) \eta_3(X_{\sigma(3)})$$

$$= \det(X^v_1, X^v_2, X^v_3) = g(X^v_1 \times X^v_2, X^v_3) = g(X^v_1 \times X^v_2, X^v_3),$$

then $T^v(X, Y) = X^v \times Y^v$. Also, since $\Phi_s$ is alternating,

$$\eta_r \wedge \Phi_s(X_1, X_2, X_3) = \eta_r(X_1) g(X_2, \varphi_s X_3) + \eta_r(X_2) g(X_3, \varphi_s X_1) + \eta_r(X_3) g(X_1, \varphi_s X_2)$$

$$= g(X_3, -\eta_r(X_1) \varphi_s X_2 + \eta_r(X_2) \varphi_s X_1 + g(X_1, \varphi_s X_2) \xi_r),$$

so we get the desired expression for $T^{rs}$.

\[ \square \]

Remark 30. Note that our hypothesis $\dim M \geq 7$ is essential for item i). In fact, for $\dim M = 3$ we have $\dim \text{Hom}(\Lambda^3 \mathfrak{m}, \mathbb{R}) = 1$. \[ \square \]

The above explicit description of the basis of $\text{Hom}_0(\Lambda^3 \mathfrak{m}, \mathbb{R})$ and Lemma 8 provide the following geometric description for all $G$-invariant affine connections with skew-symmetric torsion on 3-Sasakian homogeneous manifolds $M$ with $\dim M \geq 7$.

Corollary 31. Let $M = G/H$ be a 3-Sasakian homogeneous manifold with $\dim M \geq 7$. Then, any $G$-invariant affine connection on $M$ with skew-symmetric torsion is given by

i) If $G \neq SU(m)$,

$$\nabla = \nabla^g + \frac{1}{2} \left(a T^o + \sum_{r,s=1}^3 b_{rs} T^{rs}\right),$$

for some $a \in \mathbb{R}$ and $B = (b_{rs}) \in \mathcal{M}_3(\mathbb{R})$.

ii) If $M = SU(m)/S(U(m-2) \times U(1))$ with $m \geq 3$,

$$\nabla = \nabla^g + \frac{1}{2} \left(a T^o + \sum_{r,s=1}^3 b_{rs} T^{rs} + \sum_{l=1}^3 c_l T^{l0}\right),$$

for some $a \in \mathbb{R}$, $(c_1, c_2, c_3)^t \in \mathbb{R}^3$ and $B = (b_{rs}) \in \mathcal{M}_3(\mathbb{R})$.

As usual, for $B = (b_{rs}) \in \mathcal{M}_3(\mathbb{R})$ and $c \in \mathbb{R}^3$, we use the notations $\|B\|^2 = \sum_{r,s=1}^3 b_{rs}^2$ and $\|c\|^2 = \sum_{j=1}^3 c_j^2$. 


Let us recall that the tensor $S \in T^{(0,2)}(M)$ defined in (4) plays a key role in the curvature identities for Riemann-Cartan manifolds with totally skew-symmetric torsion. The following result provides the expressions of the tensors $S$ for the affine connections given in (40).

**Proposition 32.** Let $M^{4n+3} = G/H$ be a 3-Sasakian homogeneous manifold as in Theorem 1, with $\dim M \geq 7$. Let $\nabla$ be an affine connection on $M = G/H$ given by (40), where $c = 0$ when $G \neq SU(m)$, $m \geq 3$.

i) The tensor $S$ is given, if $X, Y \in \mathcal{Q}$, by

$$S|_{\mathcal{Q} \times \mathcal{Q}^1} = 0,$$

$$S(\xi, \xi) = 2\delta_{ik}(a - \text{tr}(B))^2 + 4n\sum_{s=1}^3 b_{is}b_{ks} + 4nc_c k,$$

$$S(X, Y) = 2(||B||^2 + ||c||^2)g(X, Y) + 4\sum_{s,j=1}^3 b_{js}c_j g(\varphi_0 X, \varphi_s Y).$$

ii) The symmetric part of the Ricci tensor is given, if $X, Y \in \mathcal{Q}$, by

$$\text{Sym}(\text{Ric}^\nabla)|_{\mathcal{Q} \times \mathcal{Q}^1} = 0,$$

$$\text{Sym}(\text{Ric}^\nabla)(\xi, \xi) = (4n^2 - \frac{2}{3}(a - \text{tr}(B))^2)\delta_{ik} - n\sum_{s=1}^3 b_{is}b_{ks} - nc_c k,$$

$$\text{Sym}(\text{Ric}^\nabla)(X, Y) = (4n^2 - \frac{2}{3}(||B||^2 + ||c||^2))g(X, Y) - \sum_{s,j=1}^3 b_{js}c_j g(\varphi_0 X, \varphi_s Y).$$

iii) The scalar curvature becomes $s^\nabla = (4n + 2)(4n + 3) - \frac{4}{3}(a - \text{tr}(B))^2 - 3n(||B||^2 + ||c||^2).

**Proof.** Denote by $T = aT^o + \sum_{r,s=1}^3 b_{rs}T^{rs} + \sum_{s=1}^3 c_sT^0$ the torsion of the connection $\nabla$, but put $c_i = 0$ and $\varphi_0 = 0$ when $G \neq SU(m)$. Recall that $\mathcal{Q} = \text{Span}\{\xi_1, \xi_2, \xi_3\} \perp$ is preserved by any $\varphi_s$, $s = 0, \ldots, 3$. By Lemma 29, it is easy to check that

$$T^o(X, Y) = 0, \quad T^{rs}(X, Y) = \Phi_s(X, Y)\xi_r,$$

$$T^o(X, \xi) = 0, \quad T^{rs}(X, \xi_j) = \delta_{ij}\varphi_s X,$$

$$T^o(\xi, \xi) = \xi_1 + 2, \quad T^{rs}(\xi, \xi_j) = -\delta_{ij}\xi_2 + 2,$$

for any $X, Y \in \mathcal{Q}$, $r = 1, 2, 3$, $s = 0, 1, 2, 3$ (so $\delta_{r0} = 0$). In particular, $T(X, Y) \in \mathcal{Q}^1$, $T(X, \xi_j) \in \mathcal{Q}$ and $T(\xi, \xi_j) \in \mathcal{Q}^1$.

Then, by taking $\{\xi_1, \xi_2, \xi_3, V_1, \ldots, V_{4n}\}$ an orthonormal basis of $\mathfrak{m}$, it holds

$$S(X, \xi) = \sum_{j=1}^3 g(T(X, \xi_j), T(\xi, \xi_j)) + \sum_{j=1}^4 g(T(X, V_j), T(\xi, V_j)) = 0.$$

For the sake of simplicity, we write $b_{r0} = c_r$. Thus, we have

$$T(X, \xi_j) = \sum_{s=0}^3 b_{js}\varphi_s X, \quad T(X, Y) = \sum_{r=1}^3 \left[ \sum_{s=0}^3 b_{rs}g(X, \varphi_s Y) \right] \xi_r.$$

According to the definition of the tensor $S$ in (4),

$$S(X, Y) = \sum_{r=1}^3 g(T(X, \xi_r), T(Y, \xi_r)) + \sum_{j=1}^4 g(T(X, V_j), T(Y, V_j))$$

$$= \sum_{r=1}^3 \sum_{s, u=0}^3 g(b_{rs}\varphi_s X, b_{ru}\varphi_u Y) + \sum_{j=1}^4 \sum_{i, r=1}^3 \left[ \sum_{s=0}^3 b_{rs}g(X, \varphi_s V_j)\xi_r \sum_{u=0}^3 b_{tu}g(Y, \varphi_u V_j)\xi_t \right].$$

Taking into account that $\varphi_0 \in \text{so}(\mathfrak{m}, g)$ and that $g(X, Y) = \sum_{j=1}^{4n} g(X, V_j)g(Y, V_j)$ holds for horizontal vectors, then both summands of the above formula are equal and then,

$$S(X, Y) = 2 \left[ \sum_{s, r, u=1}^3 b_{rs}b_{ru}g(\varphi_s X, \varphi_u Y) + \sum_{s, r=1}^3 b_{rs}c_r g(\varphi_s X, \varphi_0 Y) + g(\varphi_0 X, \varphi_s Y) \right] + 2||c||^2 g(X, Y).$$
If we also observe that \( \varphi_s \varphi_u + \varphi_u \varphi_s = -2\delta_{su} \text{id}_Q \) for \( s,u = 1, 2, 3 \) and \( \varphi_0 \varphi_s = \varphi_s \varphi_0 \), we get

\[
S(X, Y) = 2\|B\|^2 + \|c\|^2 g(X, Y) + 4 \sum_{s,j=1}^3 b_{js} c_j g(\varphi_0 X, \varphi_s Y).
\]

For the next steps, we recall from Section 4.4 that \( \varphi_0 \varphi_1(z, w) = \varphi_0(iz, iw) = (-z, w), \varphi_0 \varphi_2(z, w) = \varphi_0(\bar{w}, \bar{z}) = (-\bar{w}, -\bar{z}), \varphi_0 \varphi_3(z, w) = \varphi_0(-\bar{w}, i\bar{z}) = (\bar{w}, \bar{z}) \), and also \( \varphi_0 |_{\text{mas}^3} = 0 \). From the point of view of real linear spaces, the eigenvalues of \( \varphi_0 \varphi_1 |_{\text{mas}^3} \) are \( 1, -1 \), with equal number of \( +1 \) and \( -1 \). Then, \( \text{tr}(\varphi_0 \varphi_1) = 0 \). Also, if we take an orthonormal \( \mathbb{R} \)-basis of \( \text{mas}^3 \) of the form \((e_1, 0), \ldots, (e_n, 0), (ie_1, 0), \ldots, (ie_n, 0), (0, e_1), \ldots, (0, e_n) \), \( (0, ie_1), \ldots, (0, ie_n) \)), a simple computation shows that \( \text{tr}(\varphi_0 \varphi_2) = \text{tr}(\varphi_0 \varphi_3) = 0 \).

We will compute \( S(\xi_s, \xi_k) \) in two steps. On one hand,

\[
4n \sum_{j=1}^{4n} g(T(\xi_s, V_j), T(\xi_k, V_j)) = 4n \sum_{j=1}^3 \sum_{s,u=0} b_{js} b_{ku} g(\varphi_s V_j, \varphi_u V_j)
\]

\[
= 4n \sum_{j=1}^3 \sum_{s,u=1} b_{js} b_{ku} c_{su} + c_k \sum_{s=1}^3 b_{js} b_{ku} g(\varphi_s V_j, \varphi_0 V_j) + c_i \sum_{u=1}^3 b_{ku} g(\varphi_u V_j, \varphi_0 V_j) + c_i c_k
\]

\[
= 4n \sum_{s=1}^3 b_{js} b_{ks} - \text{tr} \left( \sum_{s=1}^3 (c_{js} b_{is} + c_i b_{ks}) \varphi_s \varphi_0 \right) + 4n c_i c_k = 4n \sum_{s=1}^3 b_{js} b_{ks} + 4n c_i c_k.
\]

On the other hand, \( T(\xi_s, \xi_{s+1}) = -T(\xi_{s+1}, \xi_s) = (a - \sum_{r=1}^3 b_{sr}) \xi_{s+2} \). Then, it holds

\[
3 \sum_{j=1}^3 g(T(\xi_s, \xi_j), T(\xi_k, \xi_j)) = \delta_{ik} (\|T(\xi_s, \xi_{s+1})\|^2 + \|T(\xi_s, \xi_{s+2})\|^2) = 2\delta_{ik} (a - \text{tr}(B))^2,
\]

which gives the required expression for \( S(\xi_s, \xi_k) \).

Next, taking into account that every 3-Sasakian manifold is Einstein (in the usual sense), the formula (5) can be claimed to obtain the expression for the symmetric part of the Ricci tensor. Finally, the scalar curvature becomes

\[
s^\nabla = \sum_{k=1}^{4n+3} \text{Sym}(\text{Ric}^\nabla)(e_k, e_k)
\]

\[
= 3 \left[ 4n + 2 - \frac{1}{2} (a - \text{tr}(B))^2 \right] - n\|B\|^2 - n\|c\|^2 + \left( 4n + 2 - \frac{1}{2} (\|B\|^2 + \|c\|^2) \right) 4n
\]

\[
+ \sum_{s,j} b_{js} c_j \text{tr}(\varphi_0 \varphi_s) = (4n + 2)(4n + 3) - \frac{3}{2} (a - \text{tr}(B))^2 - 3n (\|B\|^2 + \|c\|^2).
\]

With this information, we can characterize which of our 3-Sasakian homogeneous manifolds admit invariant Einstein with skew-torsion connections, and describe such connections \( \nabla \) in terms of the coefficients \( a \in \mathbb{R}, B = (b_{su}) \in M_3(\mathbb{R}) \) and \( c \in \mathbb{R}^3 \) in (40).

On one hand, as already stated, any 3-Sasakian manifold \((M^{4n+3}, g)\) is Einstein with \( \text{Ric}^g = 2(2n + 1)g \). Therefore, by (5), the symmetrized of \( \text{Ric}^\nabla \) is a multiple of the metric \( g \) if, and only if, the corresponding tensor \( S \) is so. On the other hand, the scalar curvature \( s^\nabla \) is a constant for every affine connection in (40). Hence, for \( \nabla \) an invariant connection Einstein with skew-torsion on the homogeneous 3-Sasakian manifold \( M^{4n+3} = G/H \), there should exist \( \lambda \in \mathbb{R} \) such that \( S = \lambda g \). Now, the identities (3) and (5) imply that the tensor \( S \) corresponding with such connection \( \nabla \) satisfies

\[
S = 4 \left( 2(2n + 1) - \frac{s^\nabla}{4n + 3} \right) g.
\]
By substituting here the expression of $s^\nabla$ provided by Proposition 32 iii), we get (keep in mind that for convenience we are assuming $c = 0$ whenever $G \neq SU(m)$)

$$\lambda = \frac{6 (a - \text{tr}(B))^2 + 12n(||B||^2 + ||c||^2)}{4n + 3}.$$ \hfill (42)

A direct computation with the expression of $S$ provided by Proposition 32 i) gives

$$3\lambda = \sum_{i=1}^{3} S(\xi_i, \xi_i) = 6 (a - \text{tr}(B))^2 + 4n(||B||^2 + ||c||^2).$$ \hfill (43)

From (42) and (43), we deduce that every Einstein with skew-torsion invariant affine connection $\nabla$ on the homogeneous 3-Sasakian manifold $M^{4n+3} = G/H$ satisfies

$$3 (a - \text{tr}(B))^2 + (2n - 3)(||B||^2 + ||c||^2) = 0.$$ \hfill (44)

This forces either $a = 0$, $B = 0$, $c = 0$ (Levi-Civita connection) or $2n - 3 \leq 0$, so that $n = 1$, in other words, dim $M = 7$.

Now let us assume dim $M = 7$ and $3 (a - \text{tr}(B))^2 = ||B||^2 + ||c||^2$. By Proposition 32 iii), the equation $\text{Sym}(\text{Ric}^\nabla) = \frac{2}{7} g$ is equivalent to

$$\text{Sym}(\text{Ric}^\nabla) = \left(6 - \frac{1}{2}(||B||^2 + ||c||^2)\right) g.$$ 

Now we use the expression of $\text{Sym}(\text{Ric}^\nabla)$ in Proposition 32 ii). Apply it to $(\xi_i, \xi_k)$ to get $\sum_s b_{is} b_{ks} + c_i c_k = \frac{1}{3}(||B||^2 + ||c||^2)\delta_{ik}$, so that

$$BB^t + cc^t = \frac{1}{3}(||B||^2 + ||c||^2)I_3.$$ \hfill (45)

And apply it to $X, Y \in \mathbb{Q}$ to get $c^t B = 0$. Conversely these conditions (that is, $n = 1$, (44), (45) and $c^t B = 0$) imply that $\text{Sym}(\text{Ric}^\nabla) = \frac{2}{7} g$, by a direct application of Proposition 32 ii). We summarize these arguments in the following theorem.

**Theorem 33.** Let $M = G/H$ be a 3-Sasakian homogeneous manifold of dimension at least 7.

i) If $M$ admits a nontrivial $G$-invariant Einstein with skew-torsion affine connection, then dim $M = 7$, that is, either $M = \text{Sp}(2)/\text{Sp}(1) \cong \mathbb{S}^7$, or $M = \mathbb{R}P^7$, or $M = \text{SU}(3)/\text{S}(U(1) \times U(1) = \text{SU}(3)/U(1)$ is the Alhoff-Wallach space $M_{1,1}^7$.

ii) There is a bijective correspondence between the Lie group $\mathbb{Z}_2 \times C_0(3)$ and the set of nontrivial $\text{Sp}(2)$-invariant Einstein with skew-torsion affine connections on $M = \mathbb{S}^7$, given by

$$\pm (1, B = (b_{rs})) \mapsto \nabla^g + \frac{1}{2} \left[ (\text{tr}(B) \pm \sqrt{\frac{||B||^2}{3}}) T^0 + \sum_{r,s=1}^{3} b_{rs} T^{rs} \right].$$ \hfill (46)

In such case, $s^\nabla = 42 - \frac{7}{2} ||B||^2$. The same happens for the projective space $M = \mathbb{R}P^7$.

iii) There is a bijective correspondence between the set of nontrivial $\text{SU}(3)$-invariant Einstein with skew-torsion affine connections on $M = M_{1,1}^7$ and the set $\mathbb{Z}_2 \times \{(c, B) \in \mathbb{R}^3 \times M_3(\mathbb{R}) : BB^t + cc^t \in \mathbb{R}I_3, c^t B = 0\}$, given by

$$\pm (1, c = (c_r), B = (b_{rs})) \mapsto \nabla^g + \frac{1}{2} \left[ a_\pm T^0 + \sum_{r,s} b_{rs} T^{rs} + \sum_r c_r T^{r0} \right],$$ \hfill (47)

where $a_\pm = \text{tr}(B) \pm \sqrt{\frac{||B||^2 + ||c||^2}{3}}$. In such case, $s^\nabla = 42 - \frac{7}{2}(||B||^2 + ||c||^2)$.

**Remark 34.** A relevant fact is that, for an arbitrary 3-Sasakian manifold of dimension 7, any affine connection as in (46) is Einstein with skew-torsion. Thus, the homogeneity is only required to assure that the list is complete. □
Among the Einstein with skew-torsion invariant affine connections in Theorem 33, the scalar curvature $s^\nabla$ equals zero if, and only if, $\|B\|^2 + \|c\|^2 = 12$. Also, $\text{Sym}(\text{Ric}^\nabla)$ is positive definite (resp. negative definite) when $\|B\|^2 + \|c\|^2 < 12$ (resp. $>12$). We are including here the cases $S^7$ and $\mathbb{R}P^7$, with $c = 0$.

The remarkable families of connections corresponding to $B \in O(3)$ in $S^7$, $\mathbb{R}P^7$ and $\mathfrak{M}_{1,1}^7$ are those with torsion equal to

$$(\text{tr}(B) \pm 1) T^\alpha + \sum_{r,s=1}^3 b_{rs} T^{rs};$$

so that $s^\nabla = \frac{3}{2} \nabla$ and $\text{Sym}(\text{Ric}^\nabla)$ is positive definite.

Note that the set of Einstein with skew-torsion invariant connections in the Aloff-Wallach space $\mathfrak{M}_{1,1}^7$ contains strictly $\mathbb{Z}_2 \times \text{CO}(3)$. For instance, we can take the nonzero vector $c = (1, 0, 0)^t$ and the noninvertible matrix $B = \begin{pmatrix} \cos \theta & 0 & 0 \\ -\sin \theta & -\cos \theta & 0 \\ \sin \theta & \cos \theta & 0 \end{pmatrix}$. Indeed, $c^t B = 0$ and $BB^t + cc^t = I_3$.

**Remark 35.** The formulae (46) and (47) are given in terms of an election of three Sasakian compatible structures $\{\xi_i, \eta_i, \varphi_i\}_{i=1,2,3}$ in $S = \{\xi_r, \eta_r, \varphi_r\}_{r \in \mathbb{Z}^2}$. If we consider a second choice $\{\xi'_i, \eta'_i, \varphi'_i\}_{i=1,2,3}$ in $S$ such that $g(\xi'_i, \xi'_j) = \delta_{ij}$ and $[\xi'_i, \xi'_j] = 2\epsilon_{ijk} \xi'_k$, the first condition is equivalent to the fact that the change of basis given by $\xi'_i = \sum_j p_{ij} \xi_j$ is an orthogonal transformation, that is, $P = (p_{ij}) \in O(3)$. The second condition is equivalent to $\det(P) = 1$. Thus a 3-Sasakian structure determines implicitly an orientation on the distribution $Q^\perp$, and we have a natural action of the special orthogonal group $SO(3)$ on the set of compatible triplets of Sasakian structures.

If we write a torsion tensor $T = a T^\alpha + \sum_{r,s=1}^3 b_{rs} T^{rs} + \sum_{i=1}^3 c_i T^{i0} = a T^\alpha + \sum_{r,s=1}^3 b_{rs} T^{rs} + \sum_{i=1}^3 c_i T^{i0}$, by a straightforward computation, we see that

$$a' = \det(P) a = a, \quad B' = PBP^{-1}, \quad c' = Pc.$$

This implies that the sets of Einstein with skew-torsion affine connections on $S^7$ ($\mathbb{R}P^7$) and $\mathfrak{M}_{1,1}^7$ in Theorem 33 should be invariant under the actions $P \cdot (s, B) = (s, PBP^{-1}) \in \mathbb{Z}_2 \times \text{CO}(3)$ and $P \cdot (s, c, B) = (s, Pc, PBP^{-1})$, respectively. This happens trivially, but we point it out because it gives a clue to understand the great amount of connections in (46) and (47), which is a little bit surprising initially. Another property which will be invariant under the SO(3)-action (independent of the choice) is the symmetry of the Ricci tensor, considered in Section 6. □

**Remark 36.** Einstein with skew-torsion invariant connections on $S^7$ have been studied in [15] and [18], where the sphere has been regarded as $S^7 \cong \text{Spin}(7)/G_2$ and $S^7 \cong \text{SU}(4)/\text{SU}(3)$, respectively. As $\text{Sp}(2) \subset \text{SU}(4) \subset \text{Spin}(7)$, those connections should appear in our list. According to these works, the torsion of any nontrivial $\text{SU}(4)$-invariant Einstein with skew-torsion connection is

$$\omega_\phi = r \eta_1 \wedge d\eta_1 + \text{Re}(q) (\eta_2 \wedge d\eta_2 - \eta_3 \wedge d\eta_3) - \text{Im}(q) (\eta_2 \wedge d\eta_3 + \eta_3 \wedge d\eta_2),$$

for $0 \neq r \in \mathbb{R}$ and $q \in \mathbb{C}$ such that $|q|^2 = r^2$. Moreover, it is Spin(7)-invariant if, and only if, $q = -r$. That is, there is $\theta \in [0, 2\pi]$ ($q = re^{i\theta}$) such that they correspond to the following pairs $(s, B)$ in $\mathbb{Z}_2 \times \text{CO}(3)$:

$$(48) \quad s = -1, \quad B = 2r \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta \\ 0 & \sin \theta & \cos \theta \end{pmatrix} \quad \text{and} \quad s = -1, \quad B = 2r \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

respectively. Taking into account the previous remark about the special orthogonal group and its action on the set on Einstein with skew-torsion $\text{Sp}(2)$-invariant affine connections, this means that

\footnote{Be aware that the expression of $\omega_\phi$ on the paper on line have a wrong coefficient $\frac{1}{2}$, corrected in the printed version [18].}
any SU(4) = Spin(6)-invariant Einstein connection is Spin(7)-invariant for a convenient election of \(\{\xi, \eta, \varphi_i\}_{i=1}^3\).

Also, the canonical \(G_2\)-structure on \(\mathbb{S}^7\) studied in [3] is given by

\[
(49) \quad \omega_\varphi = \frac{1}{2} (\eta_1 \wedge d\eta_1 + \eta_2 \wedge d\eta_2 + \eta_3 \wedge d\eta_3) + 4\eta_1 \wedge \eta_2 \wedge \eta_3,
\]

so that \(\omega_\varphi\) corresponds to \((s, B) = (+1, I_3)\) in (46). In particular, the affine connection with torsion \(\omega_\varphi\) is Einstein with skew-torsion. On the other hand, there exists a unique affine connection with skew-torsion \(\nabla^c\) preserving the \(G_2\)-structure, whose torsion form is given by

\[
T^c = \eta_1 \wedge d\eta_1 + \eta_2 \wedge d\eta_2 + \eta_3 \wedge d\eta_3.
\]

Thus, Theorem 33 tells that \(\nabla^c\) is not an Einstein with skew-torsion affine connection.

We think that further study of the remaining orbits could be convenient, since their representatives could behave rather differently. If we denote by \(R_\theta\) the rotation \(R_\theta = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}\), the orbits of \(Z_2 \times \text{CO}(3)\) under the SO(3)-action are precisely

\[
\left\{(s, r \begin{pmatrix} 1 \\ 0 \\ R_\theta \end{pmatrix}), (s, r \begin{pmatrix} -1 \\ 0 \\ R_\theta \end{pmatrix}) : s = \pm 1, r \in \mathbb{R}, r > 0, \theta \in [0, 2\pi) \right\},
\]

where \(r\) has been taken positive to avoid duplicity of the orbits. The orbits which have already appeared are \((-1, r \begin{pmatrix} -1 \\ 0 \\ R_\theta \end{pmatrix})\) and \((-1, r \begin{pmatrix} 1 \\ 0 \\ R_\theta \end{pmatrix})\) in (48), and, as mentioned, \((1, \begin{pmatrix} 1 \\ 0 \\ R_\theta \end{pmatrix})\) in (49).

Coming back to the Aloff-Wallach space and its extra Einstein with skew-torsion invariant connections, that is, those ones with \(c \neq 0\), we can assume that \(c = (r, 0, 0)^t, r > 0\), due to the action of the special orthogonal group. Now, the first row of \(B\) is equal to zero, and the two others can be any pair of orthogonal vectors in \(\mathbb{R}^3\) of length \(r\). So \(B\) is determined by an element in the Stiefel manifold \(V_{3,2}\).

**Remark 37.** Recall that there is a canonical way to associate a Lorentzian metric to any Sasakian manifold, [9, 11.8.1]. If \((\xi, \eta, \varphi)\) is a Sasakian structure on a manifold \((M, g)\), define \(g_L = g - 2\eta \otimes \eta\). This metric was considered in the sphere \(\mathbb{S}^{2n+1}\) in [19], where it is proved that there exist SU\((n + 1)\)-invariant connections which are Einstein with skew-torsion only in dimensions 3 and 7. According to [19] (respectively [18]), the set of SU\((4) = \text{Spin}(6)\)-invariant connections which are Einstein with skew-torsion on \((\mathbb{S}^7, g_L)\) (resp. \((\mathbb{S}^7, g)\)) is parametrized by an ellipsoid (resp. a cone). Obviously, the set of Sp\((2)\)-invariant connections must contain the set of SU\((4)\)-invariant connections, so that it arises the natural question of finding the whole set, in a similar way to Theorem 33, where the whole set \(Z_2 \times \text{CO}(3)\) containing the cone is found. It is not difficult to prove that the set of Sp\((2)\)-invariant Einstein with skew-torsion connections on \((\mathbb{S}^7, g_L)\) is parametrized by the group \(Z_2 \times \text{CO}(2,1)\), for

\[
\text{CO}(2,1) = \{ A \in \mathcal{M}_3(\mathbb{R}) : AHA^t \in \mathbb{R}H, \det(A) \neq 0 \}, \quad H = \begin{pmatrix} -1 & 0 \\ 0 & I_2 \end{pmatrix}.
\]

Here we are considering \(g_L = g - 2\eta_\tau \otimes \eta_\tau\) for any fixed \(\tau \in \mathbb{S}^2\).

6. On the symmetry of the Ricci Tensor

We finish this paper by studying the question about the symmetry of the Ricci tensors corresponding to our families of affine connections with skew-symmetric torsion. To this aim, we have to compute the divergence of the corresponding torsion tensors. In fact, from (5), the Ricci tensor of \(\nabla\) is symmetric just when the divergence of the torsion tensor of \(\nabla\) vanishes. These results will be applied to analyse whether there are connections such that their Ricci tensors are multiple of the metric both in the horizontal (\(Q^+\)) and vertical (\(Q\)) distributions, but in general with different parameters (Definition 38). The following notion is clearly indebted to the notion of \(\eta\)-Einstein as in [9, Definition 11.1.1].
Analogously, we obtain the required expression for the Ricci tensor:

\[ \text{Ric} \nabla = \alpha g + \beta \sum_{k=1}^{3} \eta_k \otimes \eta_k. \]

Surprisingly, we will see in Theorem 41 that there is a big amount of such connections in all the 3-Sasakian homogeneous manifolds.

**Proposition 39.** The divergences of the torsion tensors defined in (38) are:

\[ \text{div}(T^o) = 0, \quad \text{div}(T^{rs}) = \begin{cases} 0 & \text{if } s \in \{r, 0\}, \\ 2\Phi_{r+2} + (4n + 2)\eta_r \wedge \eta_{r+1} & \text{if } s = r + 1, \\ -2\Phi_{r+1} + (4n + 2)\eta_r \wedge \eta_{r+2} & \text{if } s = r + 2. \end{cases} \]

In particular, the tensor \( T = aT^o + \sum_{r,s=1}^{3} b_{rs}T^{rs} + \sum_r e_rT^{r0} \) has zero divergence if, and only if, \( B = (b_{rs}) \) is a symmetric matrix.

**Proof.** Given a point \( p \in M \), let us consider an orthonormal local frame \( (e_1, \ldots, e_{4n+3}) \) of \( TM \), such that \( (\nabla_{e_i}e_j)_p = 0 \). Given \( X, Y \in TM \), we have

\[ \text{div}(T)(X, Y) = \sum_{k=1}^{4n+3} g(e_k, (\nabla_{e_k}^g T)(X, Y)). \]

Since this is a tensorial expression, we can consider \( X = e_i, Y = e_j \) for \( i, j \in \{1, \ldots, 4n + 3\} \), and by using that the frame is parallel at \( p \), we reduce the computation to

\[ \text{div}(T)_p(e_i, e_j) = \sum_{k=1}^{4n+3} g_p(e_k, \nabla_{e_k}^g T(e_i, e_j)). \]

We will not write down the point for the sake of simplicity. We start with the computation of \( \text{div}(T^{rs}) \), for \( r, s \in \{1, 2, 3\} \). First,

\[ \sum_k g(e_k, \nabla_{e_k}^g (\eta_r(e_j) \varphi_s e_i)) = \sum_k [g(\nabla_{e_k}^g \xi_r, e_j) g(e_k, \varphi_s e_i) + \eta_r(e_j) g(e_k, (\nabla_{e_k}^g \varphi_s) e_i)] \]

\[ = \sum_k [-g(\varphi_r e_k, e_j) g(e_k, \varphi_s e_i) + \eta_r(e_j) g(e_k, g(e_k, e_i) \xi_s - \eta_s(e_i) e_k)] \]

\[ = g(\varphi_r e_j, \varphi_s e_i) + \eta_r(e_j) (g(e_i, \xi_s) - (4n + 3)\eta_s(e_i)) = g(\varphi_r e_j, \varphi_s e_i) - (4n + 2)\eta_r(e_j)\eta_s(e_i). \]

Next,

\[ \sum_k g(e_k, \nabla_{e_k}^g (g(e_i, \varphi_s e_j) \xi_r)) = \sum_k [g(e_i, (\nabla_{e_k}^g \varphi_s) e_j) g(e_k, \xi_r) + g(e_i, \varphi_s e_j) g(e_k, \nabla_{e_k}^g \xi_r)] \]

\[ = g(e_i, (\nabla_{e_k}^g \varphi_s) e_j) + g(e_i, \varphi_s e_j) \text{div}(\xi_r) = g(e_i, (\nabla_{e_k}^g \varphi_s) e_j) \]

\[ = \eta_s(e_i) \eta_r(e_j) - \eta_s(e_j) \eta_r(e_i) = \eta_s \wedge \eta_r(e_i, e_j). \]

By joining the information, we get

\[ \text{div}(T^{rs})(e_i, e_j) = \sum_k g(e_k, \nabla_{e_k}^g (-\eta_r(e_i) \varphi_s e_j + \eta_r(e_j) \varphi_s e_i + g(e_i, \varphi_s e_j) \xi_r)) \]

\[ = g(e_i, (\varphi_r \varphi_s - \varphi_s \varphi_r) e_j) + (4n + 1)\eta_r \wedge \eta_s(e_i, e_j). \]

From here, it is trivial that \( \text{div}(T^{rr}) = 0 \). Also, as \( [\varphi_r, \varphi_{r+1}] = 2\varphi_{r+2} - \eta_r \otimes \xi_{r+1} + \eta_{r+1} \otimes \xi_r \), then

\[ \text{div}(T^{rr+1})(e_i, e_j) = 2\Phi_{r+2}(e_i, e_j) + (4n + 2)\eta_r \wedge \eta_{r+1}(e_i, e_j). \]

Analogously, we obtain the required expression for \( \text{div}(T^{rr+2}) \).
Next, we compute $\text{div}(T^\alpha)$ in some steps. As $T^\alpha(X, U) = 0$ for any $X, U \in TM$, $X \in Q$, thus $\text{div}(T^\alpha(X, Y)) = 0$ when $X, Y \in Q$. Also, the compatibility with the metric implies $g(\xi, g^g_{e_X} X) = -g(\nabla^g_{e_X} \xi, X) = g(\xi, V_{e_X} X) = -g(\xi, V_{e_X} X)$, so that

$$\text{div}(T^\alpha(X, \xi)) = \sum_k g(e_k, T^\alpha(\nabla^g_{e_X} \xi, e_k)) = \sum_k g(T^\alpha(e_k, \xi), \nabla^g_{e_X} X) = \sum_k g(T^\alpha(e_k, \xi), e_k) g(\xi, \nabla^g_{e_X} X) = \sum_k g(T^\alpha(\xi, \xi), e_k) g(\xi, \nabla^g_{e_X} X) = 0,$$

since $\sum_k g(T^\alpha(\xi, \xi), \xi) \in g(Q^\perp, Q) = 0$. For the last step, consider a local orthonormal frame $\{e_k\}_{k=1}^{4n+3}$ such that $e_k = \xi_k$, $k = 1, 2, 3$. Recall that $\eta_i(e_k) = 0$ for $k \geq 4$. As $\text{div}(\varphi^\alpha(\xi, e_k)) = 0$, we get

$$\text{div}(T^\alpha(\xi, e_k)) = -\sum_{k=1}^{4n+3} [\eta_1 \wedge \eta_2 \wedge \eta_3(\nabla^g_{e_X} \xi, \eta_1 + \eta_2 \wedge \eta_3(\xi, \nabla^g_{e_X} \xi, e_k)) = 0.$$

Finally, in order to prove that $\text{div}(T^\alpha) = 0$, $r = 1, 2, 3$, we apply the algebraical tools in Section 3 to obtain the covariant derivative of the torsion tensor $T^\alpha$. Lemma 6 allows to work algebraically, using the bilinear map $\alpha^g$ related to the Levi-Civita connection obtained in (18). Again, $\{e_k\}_{k=1}^{4n+3}$ denotes an orthonormal basis of $\mathfrak{m} = T_p M$ such that $\xi_k = e_k$ for $k = 1, 2, 3$. Given $X, Y, Z \in Q$, we compute

$$(\nabla^g_X T^\alpha(\xi, e_k)) = \alpha^g(X, 0) - T^\alpha([X, \xi], e_k) - T^\alpha(\xi, [X, e_k]) = -\delta_{rj} \varphi_0(\alpha^g(X, e_k) + \delta_{rj} \varphi_0(\alpha^g(X, e_k)),$$

and $(\nabla^g_{\xi_k} T^\alpha(\xi, e_k)) = 0$, so that

$$\text{div}(T^\alpha(\xi, e_k)) = \sum_k g(\xi, (\nabla^g_{\xi_k} T^\alpha(\xi, e_k)) = \text{tr}(\delta_{rj} \varphi_0(\varphi - \delta_{rj} \varphi_0(\varphi)) = 0.$$

Second, we have

$$(\nabla^g_X T^\alpha(\xi, e_k)) = \frac{1}{2} \delta_{rj} [X, \varphi_0 Y]_m + g(Y, \varphi_0 \varphi)(X) \xi_r,$$

$$(\nabla^g_{\xi_k} T^\alpha(\xi, e_k)) = \frac{1}{2} T^\alpha([X, \xi_k], e_k) = -\varepsilon_{ks} T^\alpha(Y, e_k) \xi_k \in Q.$$

Next, as $\varphi_0 Y \in Q$ and $\text{ad}(Q)$ interchanges $Q$ and $Q^\perp$, then

$$\text{div}(\nabla^g_{\xi_k} T^\alpha(\xi, e_k)) = \sum_{k=1}^{4n+3} \frac{1}{2} g(\xi, (\nabla^g_{\xi_k} T^\alpha(\xi, e_k) = 0.$$

Third, observe that $T^\alpha(X, Y, Z) = \frac{1}{2} \sum_3 g([X, Y]_m, \xi_k) T^\alpha(\xi, Z) = -\frac{1}{2} g([X, Y]_m, \xi_k) \varphi_0(Z) = \Phi_r(X, Y) \varphi_0 Z$, which gives

$$(\nabla^g_{\xi_k} T^\alpha(\xi, e_k)) = -\Phi_r(X, Y) \varphi_0 X - \Phi_r(X, Y) \varphi_0 Z + \Phi_r(X, Z) \varphi_0 Y,$$

and since $\varphi_0$ commutes with $\varphi_r$, and these maps have zero trace, then

$$\text{div}(T^\alpha(\xi, e_k)) = \sum_{k=1}^{4n+3} g(\xi, \varphi_0 Y) \varphi_0 X - \Phi_r(X, Y) \varphi_0 Z + \Phi_r(X, Z) \varphi_0 Y = -\Phi_0(Y, Z) \varphi_0 Z = g(Y, (\varphi_0 \varphi_0 - \varphi_0 \varphi_r) Z) = 0.$$
Taking all the above into account, the divergence of $T = aT^o + \sum_{r,s=1}^{3} b_{rs} T^{rs} + \sum c_r T^{r0}$ is zero if, and only if, $B$ is symmetric, because $\{ \eta_r \wedge \eta_{r+1}, \Phi_r : r = 1, 2, 3 \}$ is a linearly independent set.

These results are coherent with [18, Remark 5.16], where the divergences of the SU(4)-invariant skew-symmetric torsions are studied. But of course, there is much more generality here.

**Corollary 40.** The invariant connection $\nabla = \nabla^g + aT^o + \sum_{r,s=1}^{3} b_{rs} T^{rs} + \sum c_r T^{r0}$ has symmetric Ricci tensor if, and only if, $B = (b_{rs})$ is a symmetric matrix.

**Theorem 41.** Let $(M = G/H, g)$ be a homogeneous $S$-Sasakian manifold with a $S$-Sasakian structure $S$, being $\dim M \geq 7$.

i) If $G \neq \text{SU}(m)$, any nontrivial $S$-Einstein $G$-invariant affine connection on $M$ is given by

$$\nabla^g + \frac{1}{2} \left( aT^o + \sum_{r,s=1}^{3} b_{rs} T^{rs} \right),$$

for $a \in \mathbb{R}$ and $B \in \text{CO}(3)$ such that $B = B^t$.

ii) If $G = \text{SU}(m)$, any $S$-Einstein $G$-invariant affine connection on $M$ is given by

$$\nabla^g + \frac{1}{2} \left( aT^o + \sum_{r,s=1}^{3} b_{rs} T^{rs} + \sum_{r=1}^{3} c_r T^{r0} \right),$$

for $a \in \mathbb{R}$, $B \in M_3(\mathbb{R})$ and $c \in \mathbb{R}^3$ such that

$$B = B^t, \quad BB^t + cc^t \in \mathbb{R}I_3, \quad c^tB = 0.$$

**Proof.** Take $T = aT^o + \sum_{r,s=1}^{3} b_{rs} T^{rs} + \sum c_r T^{r0}$ a skew-symmetric torsion related to a $S$-Einstein invariant affine $S$-Einstein connection on $M$. We denote $c_r = 0$ if $G \neq \text{SU}(m)$. First, the Ricci tensor must be symmetric, so that $B = (b_{rs})$ is a symmetric matrix by Corollary 40. As $M$ is Einstein (in the usual sense), the $S$-Einstein condition can be written as $S = \lambda g + \beta \sum_{k} \eta_k \otimes \eta_k$ for some scalars $\lambda, \beta \in \mathbb{R}$. For vertical vectors, Proposition 32 gives

$$(\lambda + \beta)\delta_{ij} = S(\xi_i, \xi_j) = 2(a - \text{tr}(B))^2 \delta_{ij} + 4n(\sum_{s=1}^{3} b_{s}b_{js} + c_i c_j),$$

so that $cc^t + BB^t = \frac{2(a - \text{tr}(B))^2 - \lambda - \beta}{4n} I_3$. But, if $X, Y \in Q$, the identity

$$\lambda g(X, Y) = S(X, Y) = g \left( X, 2(||B||^2 + ||c||^2)Y - 4 \sum_{s,j=1}^{3} b_{js}c_j \phi_0 \varphi_s Y \right),$$

jointly with the nondegeneracy of $g$ and the linear independence of $\{ \varphi_s \}_{s=0}^{3}$, imply $\lambda = 2(||B||^2 + ||c||^2)$ and $\sum_{s,j=1}^{3} b_{js}c_j = 0$, that is, $c^tB = 0$. The obtained conditions are of course sufficient.

Observe that the set described in Eq. (50) includes $c = 0$, $B \in \text{CO}(3)$, $B = B^t$. Besides, for $c \neq 0$, condition (50) forces $||B||^2 = 2||c||^2$, $BB^t + cc^t = ||c||^2 I_3$ and $\det(B) = 0$.

Again, we emphasize that the homogeneity is only necessary to guarantee that the found set is the whole set of invariant $S$-Einstein affine connections. But the connections on Theorem 41 i) are $S$-Einstein affine connections in any 3-Sasakian manifold of dimension strictly bigger than 3.

The obtained set of invariant $S$-Einstein connections is invariant for the $SO(3)$-action described in Remark 35. We expected that, because Definition 38 does not depend on the choice of the three compatible Sasakian structures.
6.1. A Distinguished Connection. Recall again the importance of the connections with skew-torsion on Sasakian manifolds (Remark 2), due to the fact that the Levi-Civita connection does not parallelize the characteristic vector fields, while there are connections with skew-torsion which do it. We would like to give an answer to the question of finding a connection that parallelizes all the characteristic vector fields.

Theorem 42. Let $(M,g)$ be a 3-Sasakian manifold of dimension at least 7, with a 3-Sasakian structure $\mathcal{S} = \{\xi_\tau, \eta_\tau, \varphi_\tau\}_{\tau \in S^2}$. There exists an affine connection with skew-torsion $\nabla^S$ on $M$ satisfying
\[ \nabla^S \xi_\tau = 0 \]
for any $\tau \in S^2$, whose torsion is given by
\[ T^\nabla^S = 4T^o + \sum_{r,s=1}^{3} 2T^{rr}. \]

The above connection $\nabla^S$ is:
- Einstein with skew-torsion, with symmetric Ricci tensor, if $\dim M = 7$;
- S-Einstein for arbitrary dimension.

Furthermore, if $M = G/H$ is homogeneous, this is the unique $G$-invariant affine connection with skew-torsion satisfying that every $\xi_\tau$ is parallel.

Proof. Let $\nabla$ be the affine connection on $M = G/H$ given by (40), where $c = 0$ when $G \neq SU(m)$, $m \geq 3$. Using Eq. (41), we get, for a horizontal vector $X \in \mathcal{Q}$,
\[ \nabla_X \xi_i = -\varphi_i X + \frac{1}{2} \left( \sum_{s=1}^{3} b_{is} \varphi_s X + c_i \varphi_0 X \right). \]

Thus, if $\nabla_X \xi_i = 0$ for any $i$, the linear independence of $\{\varphi_s\}_{s=0}^{3}$ gives $c = 0$ and $B = 2I_3$. Assuming this,
\[ \nabla_{\xi_{i+1}} \xi_i = -\varphi_i \xi_{i+1} + \frac{1}{2} \left( aT^o(\xi_{i+1},\xi_i) + \sum_{r=1}^{3} 2T^{rr}(\xi_{i+1},\xi_i) \right) = \left( -1 + \frac{1}{2}(-a+6) \right) \xi_{i+2}, \]
which vanishes if, and only if, $a = 4$. This proves the uniqueness of $\nabla^S$ in the homogeneous case. All that remains is clear. \qed

Note that the connection $\nabla^S$ does not parallelize the endomorphism fields $\varphi_\tau$. Also, it does not coincide with the connections on the sphere $S^7$ mentioned in (48) and (49), since now $(s,B) = (-1,2I_3) \in \mathbb{Z}_2 \times \text{CO}(3)$ according to the correspondence in (46).

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