Virasoro Algebra, Dedekind $\eta$-function and Specialized Macdonald’s Identities

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Abstract

We motivate and prove a series of identities which form a generalization of the Euler’s pentagonal number theorem, and are closely related to specialized Macdonald’s identities for powers of the Dedekind $\eta$–function. More precisely, we show that what we call “denominator formula” for the Virasoro algebra has “higher analogue” for all $c_{s,t}$-minimal models. We obtain one identity per series which is in agreement with features of conformal field theory such as fusion and modular invariance that require all the irreducible modules of the series. In particular, in the case of $c_{2,2k+1}$-minimal models we give a new proof of a family of specialized Macdonald’s identities associated with twisted affine Lie algebras of type $A_{2k}^{(2)}, k \geq 2$ (i.e., $BC_k$-affine root system) which involve $(2k^2 - k)$-th powers of the Dedekind $\eta$-function. Our paper is in many ways a continuation of [math.QA/0309201].

1 Introduction

In 1972 I.G. Macdonald discovered a beautiful series of identities associated to affine root systems [Ma]. From the affine Kac–Moody algebra point of view, Macdonald’s identities are well understood and are instances of what is nowadays called denominator identity [Ka1], [Ka3], often written in the following form:

$$\sum_{w \in W} \epsilon(w)e^{w(\rho) - \rho} = \prod_{\alpha \in \Delta_+} (1 - e^{-\alpha})^{\text{mult}(\alpha)},$$

(1.1)

where $\Delta$ is an affine root system, $\epsilon(w) \in \{\pm 1\}$, $\Delta_+$ is the set of positive roots, $W$ is the affine Weyl group, mult($\alpha$) are the root multiplicities and $\rho$ denotes the half sum of the positive roots [Ka3]. By using various specializations Macdonald obtained several $q$-identities, the most interesting of which involve

$$\eta(q)^{\dim(g)}$$

$$\eta(q) = q^{1/24} \prod_{i \geq 1} (1 - q^i)$$

(i.e., Dyson-Macdonald’s identities [Dy]), appearing on the right hand side of (1.1), where $\dim(g)$ denotes the dimension of a finite dimensional simple Lie algebra $g$, such as for example

$$\dim(g) = 3, 8, 10, 14, 15, 21, 24, ...$$

(1.2)

For instance, in the case of $g = \mathfrak{sl}_2$, a specialized Macdonald’s identity gives a famous Jacobi’s identity

$$\eta(q)^3 = q^{1/8} \sum_{m=0}^{\infty} (-1)^m (2m + 1)q^{m(m+1)/2}.$$
By now there are several different proofs, interpretations, reinterpretations and extensions of Macdonald’s work. Let us mention a few most important contributions. A new hat for these identities was obtained by Kac [Ka1] who placed Macdonald’s identities in the context of affine Lie algebras (cf. [Mo]) as a very special case of the Weyl-Kac character formula for the trivial module (cf. [Ka3]). Garland and Lepowsky generalized an earlier work by Kostant to general Kac-Moody Lie algebras [GL] (cf. [G]), which, in particular, led to Macdonald’s identities. More recently, some methods and results from [GL] were generalized by Borcherds so that they apply to generalized Kac–Moody algebras (for an important application of this theory see [Bo]). There are many different approaches and proofs of Macdonald’s identities and generalizations, such as [As], [Fe1], [Fe2], [Ko], [Lo], [Ka2], [Le1], [Le2], [Fr], etc. In particular, the main result in [Le1] indicates that distinguished powers (1.2) are by no means distinguished from the affine Kac–Moody algebra point of view (see also [Ka3]). (We thank Jim Lepowsky for pointing us to [Le1].)

Unlike the previous approaches, in this paper we obtain some specialized Macdonald’s identities by using the representation theory of the Virasoro algebra. Compared to affine Lie algebras [PS], the Virasoro algebra has a simple geometric interpretation; being the non-trivial central extension of the Lie algebra of polynomial vector fields on the circle, \( \text{Vect}(S^1) \). In spite of this simplicity the highest (or the lowest) weight representation theory turns out to be quite complicated and the most interesting representations of the Virasoro algebra have no (known) geometric realization. Nevertheless, a complete classification of irreducible highest weight modules (including BGG-type resolutions) was given by Feigin and Fuchs [FFu1], [FFu2] and Rocha-Caridi and Wallach who obtained some partial results prior to Feigin and Fuchs’ results [RCW1], [RCW2] (\( c = 1 \) case is due to Kac [KR]). Actually, Rocha-Caridi was the first [RC] who wrote explicitly formulas for graded dimensions (or characters) of irreducible highest weight Virasoro algebra modules (after [FFu1]). Among all the highest weight irreducible modules the most interesting are those parameterized by the central charge (cf. [KR])

\[
c_{s,t} = 1 - \frac{6(s - t)^2}{st}, \quad s, t \in \mathbb{N}, \quad s, t \geq 2, \quad (s, t) = 1,
\]

i.e., minimal models. One of the main reasons why the Virasoro algebra and minimal models have been studied intensively over the last two decades is because of their relevance in conformal field theory [BPZ], [S] (but also in 2-dimensional statistical physics and integrable models). In addition, the Virasoro algebra is related to affine Lie algebras via Sugawara construction (see for instance [Ka3], [KR]), which allows us to obtain all the unitary minimal models (cf. [KR] via coset constructions [GKO], [KW2]) from the integrable highest weight modules for affine Lie algebras. For non-unitary minimal models (considered for instance in this paper) there is a version of GKO-construction which uses cosets associated to representations of affine Lie algebras at admissible levels (certain rational levels above the critical level [KW1]). These admissible level modules are rather mysterious so it is not clear how the work [KW1] helps in understanding non-unitary minimal models. Anyhow, it is known that the minimal models with the central charge \( c_{s,t} \) are sources of rational vertex operator algebras [Wa], [Zh] (even better, these vertex operator algebras are regular [DLM]), genus-zero weakly holomorphic conformal field theories [Hu1], modular invariant theories [Zhu], [Hu3], even modular functors [BFM], so clearly these objects are of crucial importance (cf. [Fea], [BMCS] for different approaches to Virasoro minimal models)

At the abstract level, as in the affine Lie algebra case, the Virasoro algebra has what we call denominator formula; a consequence of the Euler-Poincaré principle applied to a resolution of the trivial Virasoro algebra module (with \( c = 0 \)) in terms of Verma modules [RCW1]. This resolution
1 INTRODUCTION

gives a $q$–series identity equivalent to a classical Euler’s formula (cf. A)

$$\eta(q) = q^{1/24} \sum_{n \in \mathbb{Z}} (-1)^n q^{\frac{3n^2-n}{2}}. \quad (1.4)$$

Notice that $c = 0$ occurs on the list (1.3) for $s = 2, t = 3$, and that this is the only minimal model with $c = 0$ (cf. Section 3).

In general, for every (irreducible) minimal model we can compute its graded trace or the character, but a single irreducible module does not carry a full information unless it “interacts” with other minimal models with the same central charge (e.g., fusion [BPZ], [FHL], etc.). Therefore, as in the central charge zero case, we would like to have some conformal field theoretical formula that takes into account all the minimal models with the same central charge, generalizing the denominator formula (1.4).

In the present paper we address the following question:

What are the conformal field theoretical analogue of (1.4) for other $c, t$–series?

In Theorem 7.2 we give the answer for all $c, t$–minimal models. However, in this paper we will be primarily interested in $c_{2,2k+1}$–minimal models. These models are important because of their combinatorial interpretation (e.g., Andrews-Gordon identities [FFr] and dilogarithm identities [FS]). For these series our “denominator formulas” are equivalent to a series of specialized Macdonald’s identities associated with twisted affine Lie algebras of type $A^{(2)}_{2k}$, $k \geq 2$ (i.e., $BC_k$ affine root system [Ma]). We will prove the following theorem (essentially a formula on p.138, [Ma]):

**Theorem 1.1** ($c_{2,2k+1}$-denominator formula) For every $k \in \mathbb{N}$, $k \geq 2$

$$\eta(q)^{2k-2-k} = C_k (-1)^{\frac{k(k-1)}{2}} \sum_{n \in \mathbb{Z}^k} (-1)^{\sum_{i=1}^k n_i} \chi_D(n) q^{L(n)}, \quad (1.5)$$

where $n = (n_1, ..., n_k) \in \mathbb{Z}^k$,

$$L(n) = \frac{2k^2 - k}{24} + \sum_{i=1}^k \left( \frac{(2k+1)n_i^2}{2} + \frac{(2i-1)n_i}{2} \right), \quad (1.6)$$

$$\chi_D(n) = \prod_{1 \leq i < j \leq k} \left\{ (2i - 1 + n_i(4k + 2))^2 - (2j - 1 + n_j(4k + 2))^2 \right\} \quad (1.7)$$

and

$$C_k = \frac{1}{2^{k(k-1)} \prod_{1 \leq i < j \leq k} (i - j)(i + j - 1)}. \quad (1.8)$$

Notice that $k = 1$ case is not included in our theorem, but our motivation clearly indicates that the Euler’s identity should be added at the beginning of this list of identities.

We also obtain a generalization of (1.1) (see Theorem 7.2). However, at this point we do not fully understand whether some of identities from Theorem 7.2 can be related to other specialized Macdonald’s identities.

Let us elaborate our proof of Theorem 1.1.

- We apply vertex operator algebra methods to study the characters of minimal models [Zh], [M2], [M1], [DMN] and derive differential equations with fundamental system of solutions formed by characters of irreducible modules with $c = c_{2,2k+1}$, $k \geq 2$. 
• By using the Abel’s lemma 2.1 as in [M2], we obtain a list of identities that involve
  \[ \eta(q)^{2k(k-1)}, \ k \geq 2. \]

• We “factor” missing powers of \( \eta(\tau) \) from the denominators of characters of irreducible modules
  which gives a series of identities for
  \[ \eta(q)^{(2k-1)k}, \ k \geq 2. \]

Notice that
  \[ \dim(D_k) = \dim(\mathfrak{so}(2k)) = (2k - 1)k, \ k \geq 1. \]

In the last step we evaluate the Wronskian and express the result in terms of Vandermonde
  determinants. Surprisingly, our identities are not associated to \( D_k^{(1)} \)-series but rather to
  \( A_{2k}^{(2)} \)-series.

In what follows: \( \mathbb{H} \) is the upper half-plane, \( q = e^{2\pi i \tau}, \tau \in \mathbb{H} \), \( \mathbb{N} \) is the set of positive integer
  and \( \mathbb{N}_0 \) is the set of non-negative integers.

2 Determinants

Let

\[
V(x_1, \ldots, x_k) = \begin{vmatrix}
1 & 1 & \cdots & 1 \\
x_1 & x_2 & \cdots & x_k \\
x_1^2 & x_2^2 & \cdots & x_k^2 \\
\vdots & \vdots & \ddots & \vdots \\
x_1^{k-1} & x_2^{k-1} & \cdots & x_k^{k-1}
\end{vmatrix}
\]

denote the Vandermonde determinant associated to \( x_1, \ldots, x_k \). This determinant can be computed
  by using the well-known formula

\[
V(x_1, \ldots, x_k) = \prod_{1 \leq j < i \leq k} (x_i - x_j),
\]

which is equivalent to the Weyl denominator formula for the finite dimensional Lie algebra \( \mathfrak{sl}_k \).

In complex analysis another determinant plays a prominent role. Wronskian (or Wronski)
  determinant associated to a set of analytic functions \( y_1(\tau), \ldots, y_k(\tau) \) is given by

\[
W(y_1, \ldots, y_k) = \begin{vmatrix}
y_1(\tau) & y_2(\tau) & \cdots & y_k(\tau) \\
y_1'(\tau) & y_2'(\tau) & \cdots & y_k'(\tau) \\
y_1''(\tau) & y_2''(\tau) & \cdots & y_k''(\tau) \\
\vdots & \vdots & \ddots & \vdots \\
y_1^{(k-1)}(\tau) & y_2^{(k-1)}(\tau) & \cdots & y_k^{(k-1)}(\tau)
\end{vmatrix}
\]

and it is important in the following fundamental result due to Abel:

**Lemma 2.1** Suppose that \( f_i(\tau) \) are holomorphic functions in \( \mathbb{H} \), and for every \( k \geq 1 \), let

\[ y_1(\tau), \ldots, y_k(\tau) \]
form a fundamental system of solutions for
\[
\left( \frac{d}{d\tau} \right)^k y(\tau) + f_1(\tau) \left( \frac{d}{d\tau} \right)^{k-1} y(\tau) + \cdots + f_k(\tau)y(\tau) = 0,
\]
then
\[
W(y_1, \ldots, y_k) = Ce^{-\int_{\tau_0}^{\tau} f_1(\tau)d\tau},
\]
where \( C = W(y_1(\tau_0), \ldots, y_k(\tau_0)) \) and \( \tau_0 \in \mathbb{H} \) (in fact, \( C \) is some non-zero constant which depends on \( y_1, \ldots, y_k \) and \( \tau_0 \), but not on \( \tau \)).

Of course, as in the case of differential equations, the Wronskian can be used to determine whether the set of functions are linearly independent.

In all our applications \( y_i(\tau) \)'s are analytic in the upper half-plane and meromorphic at infinity (i.e., have \( q \)-expansions). Also, from now on
\[
\ell' = \left( \frac{1}{2\pi i} \frac{d}{d\tau} \right) = \left( q \frac{d}{dq} \right).
\]

The next result shows that Vandermonde and Wronskian determinants are closely related.

**Lemma 2.2** Suppose that \( y_i(\tau), i = 1, \ldots, k \), are holomorphic in \( \mathbb{H} \), with the \( q \)-expansions
\[
y_i(q) = \sum_{n_i \geq \nu_i} a^{(i)}_{n_i} q^{n_i},
\]
where \( a^{(i)}_{n_i} \in \mathbb{C}, \nu_i \in \mathbb{Q}, \) for \( i = 1, \ldots, k \). Then the Wronskian \( W(y_1(\tau), \ldots, y_n(\tau)) \) is holomorphic in \( \mathbb{H} \) and its \( q \)-expansion at infinity is given by
\[
W(y_1, \ldots, y_n) = \sum_{n_1 \geq \nu_1, \ldots, n_k \geq \nu_k} V(n_1, \ldots, n_k) \left( \prod_{i=1}^{k} a_{n_i}^{(i)} \right) q^{n_1+\cdots+n_k}.
\]  

**Proof:** From the definition of Wronskian it is clear that \( W(y_1(\tau), \ldots, y_k(\tau)) \) is holomorphic and meromorphic at infinity (it has a \( q \)-expansion). Thus, the only thing we have to show is (2.1). Let \( A = \{a_{i,j}\} \) be a matrix of order \( k \) and \( r \in \mathbb{N}, 1 \leq r \leq k \). Suppose that \( a_{j,r} = b_{j,r} + c_{j,r} \) for every \( j = 1, \ldots, k \), then
\[
\det(A) = \det(B_r) + \det(C_r),
\]
where \( B_r \) and \( C_r \) are matrices obtained from the matrix \( A \) by replacing the \( r \)-th column by \([b_{1,r}, \ldots, b_{k,r}] \) and \([c_{1,r}, \ldots, c_{k,r}] \), respectively. In \( W(y_1, \ldots, y_n) \) all our entries are sums so we can repeat the previous formula for all the columns and simultaneously factor \( q^{n_i} \) from the \( i \)-th column for every \( i \). The remaining coefficient of \( q^{n_1+\cdots+n_k} \) is the Vandermonde determinant for \( n_1, \ldots, n_k \). \( \square \)

## 3  The Virasoro algebra

In this part we recall a few basic results regarding the Virasoro algebra. For a good introduction to infinite-dimensional Lie algebra theory, the Virasoro algebra and its minimal models see [KR]. Let us recall that \( c_{s,t} \)-minimal models are parameterized by the central charge
\[
c_{s,t} = 1 - \frac{6(s-t)^2}{st},
\]
where \( s, t \in \mathbb{N}, s, t \geq 2, (s, t) = 1, \) and by the weights

\[
h_{s,t}^{m,n} = \frac{(ns - mt)^2 - (s - t)^2}{4st},
\]

where \( 1 \leq m < s, 1 \leq n < t. \) We also let

\[
\tilde{h}_{s,t}^{m,n} = h_{s,t}^{m,n} - \frac{cs,t}{24}.
\]

Once we fix \( cs,t \) there are exactly

\[
\frac{(s-1)(t-1)}{2}
\]

different values of \( h_{s,t}^{m,n} \) for \( 1 \leq m < s, 1 \leq n < t. \) As in [M2], we denote by \( L(c, h) \) the irreducible highest weight module with the central charge \( c \) and the weight \( h, \) and by

\[
\tilde{\text{ch}}_{c,h}(q) = \text{tr}|L(c,h)q^{L(0) - c/24}
\]

the graded dimension or simply the character of \( L(c,h). \) The following result is well known (cf. [RC])

**Theorem 3.1** We have

\[
\tilde{\text{ch}}_{cs,t,h^{m,n}}(q) = q^{(h_{s,t}^{m,n} - cs,t/24)} \sum_{r \in \mathbb{Z}} \frac{q^{rstr^2 + r(ns - mt) - q^{rstr^2 + r(ns + mt) + mn}}}{(q)_\infty}.
\]

(3.1)

It is not hard to see that the expression

\[
\sum_{r \in \mathbb{Z}} \left( q^{rstr^2 + r(ns - mt)} - q^{rstr^2 + r(ns + mt) + mn} \right)
\]

involves only non-negative powers of \( q. \)

To connect our results with [M2], we shall first consider minimal models with

\[
c_{2,2k+1} = 1 - \frac{6(2k - 1)^2}{(4k + 2)}, \quad k \geq 2,
\]

and the corresponding weights

\[
h_{2,2k+1}^{1,i} = \frac{(2k - 1 + 1)^2 - (2k - 1)^2}{8(2k + 1)}, \quad i = 1, ..., k.
\]

The previous theorem implies a well-known result: For \( i = 1, ..., k, \)

\[
\tilde{\text{ch}}_{c_{2,2k+1}h_{2,2k+1}^{1,i}}(q) = q^{(h_{2,2k+1}^{1,i} - c_{2,2k+1}/24)} \sum_{n \in \mathbb{Z}} (-1)^n q^{(2k-1)\eta(n)(2k+1)n^2/2} / \eta(q).
\]

(3.2)

In [M2] we have used different, but equivalent formula (cf. [IT])

\[
\tilde{\text{ch}}_{c_{2,2k+1}h_{2,2k+1}^{1,i}}(q) = q^{(h_{2,2k+1}^{1,i} - c_{2,2k+1}/24)} \prod_{n \neq \pm i \mod (2k+1)} \frac{1}{1 - q^n},
\]

obtained from (3.2) by application of the Jacobi triple product identity [A]. We have used the infinite-product expressions (3.2) in [M2] for purposes of proving one of Ramanujan’s ”Lost Notebook” formulas.
4 Modular invariance

An interesting fact about minimal models is that, once we fix the level \( c_{s,t} \), the vector space spanned by graded traces
\[
\{ \bar{\chi}_{c_{s,t}, h_{s,t}^{m,n}}(q) : 1 \leq m < s, 1 \leq n < t \}
\]
is modular invariant [CIZ], i.e., a \( SL(2, \mathbb{Z}) \)-module, where an element \( \gamma \in SL(2, \mathbb{Z}) \) acts on the modulus \( \tau \) in the standard way. The best explanation of this phenomena was provided by Zhu in his work on modular invariance of characters [Zh] (see also [Hu3]), which uses in an essential way the theory of vertex operator algebras [FHL], [FLM], [Hu2]. For purposes of this paper we do not recall any of the theory of vertex operator algebras here, but we mention that large portions of this theory have been used in [M2] and henceforth in this paper.

5 “Strange formulas”

We showed in [M2] that product of all characters of \( c_{2,2k+1} \)-minimal models can be expressed in terms of powers of quotients of two Dedekind eta functions with different moduli. One of the reasons for multiplying these \( q \)-series stems from the following observation (cf. [M2]):
\[
\sum_{i=1}^{k} h_{2,2k+1}^{1,i} = \sum_{i=1}^{k} \left( h_{2,2k+1}^{1,i} - \frac{c_{2,2k+1}}{24} \right) = \frac{2k(k-1)}{24}. \tag{5.1}
\]
Here the number \( k \) is also the number of inequivalent \( c_{2,2k+1} \)-minimal models (or the number of inequivalent irreducible modules for the vertex operator algebra \( L(c_{2,2k+1},0) \) [Wa], cf. [M2]). The rational number appearing on the right hand side of (5.1) is important because it is related to asymptotic behavior of products of characters as \( q \to 0 \). Now, let us compute a version of (5.1) for arbitrary minimal models. As we already mentioned there are in total
\[
a_{s,t} = \frac{(s-1)(t-1)}{2}
\]
inequivalent minimal models. It is easy to see that
\[
\frac{1}{2} \sum_{m=1}^{s-1} \sum_{n=1}^{t-1} \bar{h}_{s,t}^{m,n} = \frac{1}{2} \sum_{m=1}^{s-1} \sum_{n=1}^{t-1} \left( \frac{(ns - mt)^2 - (s-t)^2}{4st} - \frac{1}{24} \left( 1 - \frac{6(s-t)^2}{st} \right) \right)
\]
\[
= \frac{(s-1)(t-1)(st - s - t - 1)}{48}, \tag{5.2}
\]
where we rescale the sum (viz. the factor 1/2) because every number \( h_{s,t}^{m,n} \) appears in the summation exactly twice. Because of the identity
\[
\frac{(s-1)(t-1)(st - s - t - 1)}{48} = \frac{2a_{s,t}(a_{s,t} - 1)}{24},
\]
it follows that (5.2) depends, as in the \( c_{2,2k+1} \) case, only on the number of inequivalent minimal models. Notice that the list \( 2k(k-1), k \geq 2 \) does not correspond to any of the lists of dimensions of classical finite-dimensional Lie algebras (i.e., \( \dim(A_k), \dim(B_k), \dim(C_k) \) and \( \dim(D_k) \)). Nevertheless, if we add \( \frac{1}{24} \) contribution from each of the characters (3.2), we get a much nicer expression
\[
\frac{2k(k-1) + k}{24} = \frac{(2k-1)k}{24}, \tag{5.3}
\]
which is equal to
\[
\frac{\dim(\mathfrak{so}(2k))}{24}
\]
and appears on the Dyson-Macdonald’s list. The last expression indicates that there might be some kind of relationship between higher dimensional conformal field theory and the Virasoro minimal models.

In the next section the observation \[\text{(5.3)}\] will be used in connection with some specialized Macdonald’s identities.

6 The Main Theorem

In \[\text{(M2)}\], for every integer \(k\) we obtained a homogeneous differential equation of order \(k\) with a fundamental system of solutions formed by the characters of \(c_{2,2k+1}\)-minimal models. In our approach the crucial role will play the following holomorphic, quasimodular (normalized) Eisenstein series
\[
\tilde{G}_2(q) = -\frac{1}{12} + 2 \sum_{n \geq 1} \frac{nq^n}{1 - q^n}.
\]

Let us recall a result from \[\text{(M2)}\] (see the proof of Theorem 8.6):

**Theorem 6.1** There is a homogeneous differential equation of order \(k\) with holomorphic coefficients
\[
\left(q \frac{d}{dq}\right)^k y(\tau) + k(k - 1)\tilde{G}_2(\tau) \left(q \frac{d}{dq}\right)^{k-1} y(\tau) + \cdots = 0, \quad (6.1)
\]
with a fundamental system of solutions formed by
\[
y_i(\tau) = ch_{c_{2,2k+1},h_{1,i}^{2,2k+1}}(q), \quad i = 1, \ldots, k.
\]

Moreover,
\[
W(y_1, \ldots, y_k) = \lambda_k \eta(\tau)^{2k(k-1)},
\]
where \(\lambda_k\) is a non-zero constant which depends only on \(k\).

As we observed in the previous section, in order to achieve the right powers of the Dedekind \(\eta\)-function as in Dyson-Macdonald’s identities for \(D_k\)-series we need additional powers of the Dedekind \(\eta\)-function. These powers can be obtained by clearing out the denominator
\[
\eta(\tau)
\]
in the character formula \[\text{(3.2)}\]. The resulting expressions (i.e., the numerators in \[\text{(3.2)}\]) satisfy another linear differential equation. The following lemma incorporates this procedure:

**Lemma 6.2** After the substitution \(\tilde{y}(\tau) = y(\tau)\eta(\tau)\), the homogeneous differential equation \[\text{(6.1)}\] becomes
\[
\left(q \frac{d}{dq}\right)^k \tilde{y}(\tau) + \left(k(k - 1) + \frac{k}{2}\right)\tilde{G}_2(\tau) \left(q \frac{d}{dq}\right)^{k-1} \tilde{y}(\tau) + \cdots = 0, \quad (6.2)
\]
with a fundamental system of solutions formed by
\[
\tilde{y}_i(\tau), \quad i = 1, \ldots, k.
\]
**Proof:** Our starting point is the homogeneous differential equation of degree $k$ in Theorem 6.1. By using the logarithmic derivative formula for the Dedekind eta function [A], [M2], we easily compute

\[
\left( q \frac{d}{dq} \right) \tilde{y}(\tau) = \left( q \frac{d}{dq} \right) (\eta(\tau)y(\tau))
\]

\[
= -\frac{1}{2} \tilde{G}_2(\tau)\eta(\tau)y(\tau) + \eta(\tau) \left( q \frac{d}{dq} \right) y(\tau),
\]

hence

\[
\eta(\tau) \left( q \frac{d}{dq} \right) y(\tau) = \left( \left( q \frac{d}{dq} \right) + \frac{1}{2} \tilde{G}_2(\tau) \right) \tilde{y}(\tau).
\]

If we iterate the previous formula we obtain

\[
\eta(\tau) \left( q \frac{d}{dq} \right)^2 y(\tau) = \eta(\tau) \left( q \frac{d}{dq} \right) y'(\tau)
\]

\[
= \left( \left( q \frac{d}{dq} \right) + \frac{1}{2} \tilde{G}_2(\tau) \right) \eta(\tau)y'(\tau) = \left( \left( q \frac{d}{dq} \right) + \frac{1}{2} \tilde{G}_2(\tau) \right)^2 y(\tau).
\]

Now, by the induction, we get

\[
\left( \left( q \frac{d}{dq} \right) + \frac{1}{2} \tilde{G}_2(\tau) \right)^r \tilde{y}(\tau) = \eta(\tau)y^{(r)}(\tau).
\]

If we apply now the Leibnitz rule we get

\[
\eta(\tau)y^{(r)}(\tau) = \left( \left( q \frac{d}{dq} \right) + \frac{1}{2} \tilde{G}_2(\tau) \right)^r \tilde{y}(\tau) + \cdots,
\]

where the dots denote the terms with lower order derivatives of $\tilde{y}(\tau)$. The proof now follows after we multiply the equation (6.1) by $\eta(\tau)$ and apply (6.3) for $r = 1, \ldots, k$.

**Corollary 6.3** We have

\[
W(\tilde{y}_1(\tau), \ldots, \tilde{y}_k(\tau)) = \tilde{C}_k \eta(\tau)^{(2k-1)k},
\]

where $\tilde{C}_k$ is some non-zero constant.

Now, let us work out the Wronskian on the left hand side of equation (6.4). From (6.2) it follows that

\[
\tilde{y}_i(\tau) = q^{h_{2,2k+1}^{1,i} - \frac{c_{2,2k+1}}{24} + \frac{1}{24}} \sum_{n_i \in \mathbb{Z}} (-1)^{n_i} q^{\frac{2(k-i)+1}{2} n_i + (2k+1)n_i^2}.
\]

For every $n_i \in \mathbb{Z}$, $i = 1, \ldots, k$, let

\[
a(n_i) = h_{2,2k+1}^{1,i} - \frac{c_{2,2k+1}}{24} + \frac{1}{24} + \frac{(2(k-i)+1)n_i + (2k+1)n_i^2}{2},
\]

\[
\tilde{y}_i(\tau) = \sum_{n_i \in \mathbb{Z}} (-1)^{n_i} q^{a(n_i)}.
\]
Now we prove Theorem 1.1. 

Proof of Theorem 1.1. By (2.1)

\[ W(\tilde{y}_1(\tau), \ldots, \tilde{y}_k(\tau)) = \begin{vmatrix}
\tilde{y}_1(\tau) & \tilde{y}_2(\tau) & \cdots & \tilde{y}_k(\tau) \\
\tilde{y}_1'(\tau) & \tilde{y}_2'(\tau) & \cdots & \tilde{y}_k'(\tau) \\
\tilde{y}_1''(\tau) & \tilde{y}_2''(\tau) & \cdots & \tilde{y}_k''(\tau) \\
\vdots & \vdots & \ddots & \vdots \\
\tilde{y}_1^{(k-1)}(\tau) & \tilde{y}_2^{(k-1)}(\tau) & \cdots & \tilde{y}_k^{(k-1)}(\tau)
\end{vmatrix}
\]

= \sum_{n_1, \ldots, n_k} (-1)^{\sum_{i=1}^k n_i} \begin{vmatrix}
(-1)^n_1 a(n_1) & (-1)^n_2 a(n_2) & \cdots & (-1)^n_k a(n_k) \\
(-1)^n_1 a(n_1)q^{a(n_1)} & (-1)^n_2 a(n_2)q^{a(n_2)} & \cdots & (-1)^n_k a(n_k)q^{a(n_k)} \\
\vdots & \vdots & \ddots & \vdots \\
(-1)^n_1 a(n_1)q^{a(n_1)} & (-1)^n_2 a(n_2)q^{a(n_2)} & \cdots & (-1)^n_k a(n_k)q^{a(n_k)} \\
\end{vmatrix}

= \sum_{n_1, \ldots, n_k} (-1)^{\sum_{i=1}^k n_i} q^{\sum_{i=1}^k a(n_i)} V(a(n_1), \ldots, a(n_k))

= (-1)^{k(k-1)/2} \sum_{n_1, \ldots, n_k} (-1)^{\sum_{i=1}^k n_i} q^{\sum_{i=1}^k a(n_i)} \prod_{1 \leq i < j \leq k} (a(n_i) - a(n_j)). \quad (6.5)

Clearly,

\[ \sum_{i=1}^k a(n_i) = \frac{2k^2 - k}{24} + \sum_{i=1}^k \frac{(2(k - i) + 1) n_i + (2k + 1) n_i^2}{2} \]

and hence (cf. (1.6))

\[ L(n) = \sum_{i=1}^k a(n_i). \]

Also

\[ (-1)^{k(k-1)/2} \prod_{i<j} (a(n_i) - a(n_j)) = \prod_{i<j} (a(n_{k-i+1}) - a(n_{k-j+1})). \]

Now the formula

\[ (a(n_{k-i+1}) - a(n_{k-j+1})) \]

= \frac{1}{2(2k + 1)} (i - j + 2kn_i - 2kn_j + n_i - n_j) (i + j - 1 + 2n_i k + 2n_j k + n_i + n_j)

together with

\[ \chi_D(n) = \prod_{1 \leq i < j \leq k} ((2i - 1 + n_i(4k + 2))^2 - (2j - 1 + n_j(4k + 2))^2) \quad (6.6) \]

= \frac{1}{2^{k(k-1)}} \prod_{1 \leq i < j \leq k} (i - j + 2n_i k - 2n_j k + n_i - n_j) (1 + j - 1 + 2n_i k + 2n_j k + n_i + n_j)
7  GENERAL CASE

imply

$$C_k \eta(\tau) 2^{k^2-k} = \frac{(4k + 2)^{k(k-1)/2}}{2^{k(k-1)}} q^{\frac{2k^2-k}{2^{k^2-k}}} \sum_{n_1, \ldots, n_k} (-1)^{\sum_{i=1}^k n_i} \chi_D(n) q^L(n),$$

where the summation is over all \( k \)-tuples \( n = (n_1, \ldots, n_k) \in \mathbb{Z}^k \). The constant \( \tilde{C}_k \) is equal to

$$\frac{(4k + 2)^{k(k-1)/2}}{\prod_{1 \leq i < j \leq k} (i - j)(i + j - 1)}$$

and this proves Theorem 1.1. \( \blacksquare \)

7  General case

In this part we extend specialized Macdonald’s identities from the previous section to all \( c_{s,t} \)-minimal models. (Un)fortunately, characters for minimal models in general do not admit nice infinite product expansions. We shall rewrite first the character formula in Theorem 3.1 as a single sum (cf. [HK]):

$$\bar{c}_{c_{s,t},h_{s,t}}(q) = \sum_{r \geq 0} \chi_{2st}(r) q^{\frac{r^2}{2st}} \eta(q),$$

(7.1)

where

$$\chi_{2st}(r) = \begin{cases} 1 & \text{for } r = \pm (ns - mt) \mod 2st \\ -1 & \text{for } r = \pm (ns + mt) \mod 2st \\ 0, & \text{otherwise.} \end{cases}$$

Even though this formula is not very transparent it is the perhaps the only closed expression that covers for all the minimal models. Also, it seems that \( \chi_{2st} \) does not have some obvious arithmetic interpretation.

Now, among all the pairs \((m, n)\) we have to identify those pairs that give different \( h_{s,t}^{m,n} \) values \((\frac{(s-1)(t-1)}{2} \) in total). It is not hard to see that the first \( k \) values in the sequence

$$h_{s,t}^{1,1}, h_{s,t}^{1,2}, \ldots, h_{s,t}^{s-1,t-2}, h_{s,t}^{s-1,t-1},$$

starting with \( h_{s,t}^{1,1} \), give the wanted values. For simplicity we will enumerate these (rational) numbers by \( h_1, h_2, \ldots, h_k \).

We will need a stronger version of Theorem 6.1.

**Theorem 7.1** Let \( k \) be as above and for every \( i = 1, \ldots, k \), let

$$\bar{y}_i(\tau) = \eta(\tau) \bar{c}_{c_{s,t},h_i}(q),$$

Then

$$W(\bar{y}_1(\tau), \ldots, \bar{y}_k(\tau)) = C_{s,t} \eta(\tau)^{(2k-1)k},$$

where \( C_{s,t} \) is some constant that depends only on \( s \) and \( t \).
Proof: In order to apply Theorem 6.1 for all $c_{s,t}$-minimal models we need the following fact (observed by Feigin and Fuchs): The vacuum module $V(c_{s,t},0)$ (see [Wa, M2]), which carries a vertex operator algebra structure, contains a singular vector of the weight $(s - 1)(t - 1)$ of the form

$$(L^{(s-1)(t-1)/2}(-2) + \cdots)1 \in V(c_{s,t},0)$$

where the dots denote the lower order terms in the natural filtration of $U(\text{Vir}_{\leq 2})$ (for the notation see [M2, FF]). Now, if apply the same procedure as in [M2], we get a $k$-th order homogeneous linear differential equation satisfied by $\chi_{\nu c_{s,t},\nu h_i(q)}$, $i = 1, \ldots, k$, which is certainly a fundamental system of solutions (character for $c_{s,t}$-minimal models are always linearly independent). This differential equation is (again) of the form

$$(q \frac{d}{d\tau})^k y(\tau) + f_1(\tau) \left(q \frac{d}{d\tau}\right)^{k-1} y(\tau) + \cdots + f_k(\tau) y(\tau) = 0,$$  

(7.2)

where

$$f_1(\tau) = k(k - 1)G_2(\tau),$$

and $f_i(\tau), i \geq 2$ are some polynomials in Eisenstein series [Zh]. Now we apply Lemma 6.2 and the proof follows. Let us notice here that existence of a singular vector in $V(c_{s,t},0)$ of the weight $(s - 1)(t - 1)$ has been used in [DLM] (and assumed in [Zh]) to show that the vertex operator algebra $L(c_{s,t},0)$ satisfies the $C_2$-cofiniteness condition, which implies regularity.

Theorem 7.2 ($c_{s,t}$-denominator formula) For every $s, t, \in \mathbb{N}$, such that $2 \leq s < t$ and $(s,t) = 1$,

$$C_{s,t}\eta(\tau)^{2k^2-k} = \sum_{n \in \mathbb{N}_0^k} \left( \prod_{i=1}^k \chi_{2st}(n_i) \right) V(n_1^2, \ldots, n_k^2)q^{L(n)},$$  

(7.3)

where $n = (n_1, \ldots, n_k) \in \mathbb{N}_0^k$,

$$L(n) = \sum_{i=1}^k \frac{n_i^2}{4st}$$

and $C_{s,t}$ is some non-zero constant.

Proof: Theorem 7.1 gives an expression for $C_{s,t}\eta(\tau)^{2k(k-1)}$ as a Wronskian determinant. Therefore, we only have to evaluate the Wronskian determinant and for this we use (7.1). Notice that we can factor the constant $(\frac{1}{4st})^{k(k-1)/2}$ from the Vandermonde determinant $V(n_1^2, \ldots, n_k^2)$. The proof follows.

The constant $C_{s,t}$ can be computed explicitly by comparing the leading nonzero coefficients on both sides of (7.3). In the special case $s = 2$, $t = 2k + 1$, Theorem 7.2 implies the formula in Theorem 6.1.

Remark 1 Let $\mu(k)$ be the number of positive integer solutions $s, t$ of the equation

$$2k = (s - 1)(t - 1),$$

where $2 \leq s < t$, $s$ and $t$ are relatively prime. For instance $\mu(9) = 3$, with the solutions $s = 2, t = 19$, $s = 3, t = 10$ and $s = 4, t = 7$. Our Theorem 7.2 implies that for every integer $k \geq 2$ we have $\mu(k)$ expressions for

$$\eta(\tau)^{(2k-1)^k}.$$
8 Factorization of linear combinations of characters: \( c_{3,4} \) example

As we already noticed \( c_{2,2k+1} \)-minimal models are easier to handle because the numerator formulas for characters are various specialization of theta constants (so we can use the Jacobi triple product identity \( \text{[A]} \)). In general characters of minimal models do not rise to “nice” infinite-product expressions (at least not in a straightforward way). In spite of this, a recent work of Bytsko and Fring (see \( \text{[BF]} \) and references therein) shows that certain linear combinations of characters do have infinite product expansions (these factorization properties are useful to prove certain Rogers-Ramanujan-type formulas via dilogarithms \( \text{[BF]} \)). Unfortunately, most of these factorizations occur for some special modules among \( c_{s,t} \)-minimal models so it is not clear what to expect in general.

From our point of view it is natural to consider linear combinations of characters because of the following elementary fact: Let \( T \) be an invertible matrix with entries being numbers, and

\[
(h_1(\tau), \ldots, h_k(\tau)) = T(f_1(\tau), \ldots, f_k(\tau)),
\]

where \( (\cdot, \ldots, \cdot) \) denote a \( k \)-size column vector, then

\[
W(h_1(\tau), \ldots, h_k(\tau)) = \det(T)W(f_1(\tau), \ldots, f_k(\tau)). \tag{8.1}
\]

This simple observation and factorization properties of linear combinations can be used to prove some nontrivial modular identities (see below).

Perhaps the most famous example of factorization of linear combinations of characters occurs when the central charge is \( c_{3,4} = \frac{1}{2} \), i.e., the Ising model. This model is unitary and well-understood. In order to connect these models with infinite products we recall the definition of (normalized) Weber’s functions \( \text{[We]} \):

\[
f(\tau) = q^{\frac{1}{24}} \prod_{n \geq 0} (1 + q^{n+\frac{1}{2}}),
\]

\[
f_1(\tau) = q^{\frac{1}{24}} \prod_{n \geq 0} (1 - q^{n+\frac{1}{2}}),
\]

\[
f_2(\tau) = q^{\frac{1}{12}} \prod_{n \geq 0} (1 + q^{n+1}).
\]

Our goal is to prove the following identity.

**Proposition 8.1** We have

\[
\begin{vmatrix}
f(\tau) & f_1(\tau) & f_2(\tau) \\
f'(\tau) & f'_1(\tau) & f'_2(\tau) \\
f''(\tau) & f''_1(\tau) & f''_2(\tau)
\end{vmatrix} = 256 \eta(\tau)^{12} = 7\sqrt{\Delta}, \tag{8.2}
\]

where \( \Delta \) is the Ramanujan’s discriminant function.

**Proof:** Our Theorem \( \text{[RA]} \) (cf. formula \( \text{[MA]} \)) implies that there is a 3-rd order homogeneous linear differential equation

\[
\left(q \frac{d}{dq}\right)^3 y(q) + 6G(q) \left(q \frac{d}{dq}\right)^2 y(q) + f_2(q) \left(q \frac{d}{dq}\right) y(q) + f_3(q) y(q) = 0,
\]

with a fundamental system of solutions being

\( ch_{1/2,0}(q), \ ch_{1/2,1/2}(q), \text{ and } ch_{1/2,1/16}(q) \).
On the other hand it is known (see for instance [KR]) that
\[
\tilde{c}_3,4,h_{3,4,1}(q) = q^{\frac{1}{24}} \prod_{n \geq 0} (1 + q^{n+1}),
\]
\[
\tilde{c}_3,4,h_{3,4,2}(q) = q^{\frac{1}{24}} \prod_{n \geq 0} (1 + q^{n+\frac{3}{2}}).
\]

Because of (8.1),
\[
W(f(\tau), f_1(\tau), f_2(\tau)) = C W(\tilde{c}_1/2,0(\tau), \tilde{c}_1/2,1/2(\tau), \tilde{c}_1/2,1/16(\tau))
\]
where \(C\) is some non-zero constant. Now, Theorem 6.1 implies that
\[
W(\tilde{c}_1/2,0(\tau), \tilde{c}_1/2,1/2(\tau), \tilde{c}_1/2,1/16(\tau))
\]
is a non-zero multiple of \(\eta(\tau)^{12} = \sqrt{\Delta}\). By comparing the leading coefficients in \(W(f(\tau), f_1(\tau), f_2(\tau))\) and \(\eta(\tau)^{12}\) the proof follows.

\begin{remark}
We are confident that our Proposition 8.1 can be proven by using classical Jacobi theta function techniques (e.g. by using formulas for half-periods of Weierstrass series expressed in terms of Jacobi theta constants and their derivatives).
\end{remark}

\begin{remark}
Our Proposition 8.1 implies that various factorization of linear combinations (e.g., several formulas obtained in [BF]) can be used to derive modular identities in terms of Dedekind \(\eta\)-functions with different periods (e.g. \(\eta(m\tau), \eta(\tau/n), m, n \in \mathbb{N}\)).
\end{remark}

9 Conclusion and future work

(i) We obtained a two-parametric generalization of the Euler’s identity (1.4). In the case of \(c_{2,2k+1}\)-minimal models we gave a new proof of a series of specialized Macdonald’s identities for the affine root system of type \(A_{2k,1}^{(2)}\), \(k \geq 2\), involving powers of the Dedekind \(\eta\)-function (cf. Theorem 1.1). We stress that for different values of \(s\) and \(t\) we got different looking identities. It is an open question to relate identities in Theorem 7.2 to other specialized Macdonald’s identities.

(ii) (Work in progress) We are modifying some of techniques from this paper and [M1], [M2] for \(N = 1\) and \(N = 2\) superconformal minimal models in connection with a work of Kac and Wakimoto [KW3] on affine Lie superalgebras, and Milne’s work on sums of squares [Mi].

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