TOLEDO INVARIANT OF LATTICES IN $SU(2,1)$ VIA SYMMETRIC SQUARE

INKANG KIM AND GENKAI ZHANG

Abstract. In this paper, we address the issue of quaternionic Toledo invariant to study the character variety of two dimensional complex hyperbolic uniform lattices into $SU(4,2)$. We construct four distinct representations to prove that the character variety contains at least four distinct components. We also address the holomorphic horizontal liftability to various period domains of $SU(4,2)$.

1. Introduction

After Weil’s local rigidity theorem of uniform lattices in semisimple Lie groups, there have been many generalizations in different contexts. Due to Margulis’ superrigidity and Corelette’s theorem, lattices in higher rank semisimple Lie groups and in quaternionic, octonionic hyperbolic groups are very rigid. Hence it is only meaningful to study the embedding of uniform lattices in real and complex hyperbolic spaces into bigger Lie groups.

Several studies have been done for complex hyperbolic lattices in different Lie groups. In terms of maximal representations, Burger and Iozzi studied the representations of a lattice in $SU(1, p)$ with values in a Hermitian Lie group $G$ [1, 2]. Koziarz and Maubon [3] studied the similar representations in rank 2 Hermitian Lie groups. Pozzetti [4] deals with maximal representations of complex hyperbolic lattices in $SU(m, n)$. Recently Oscar-Garcia and Toledo [5] proved a global rigidity of complex hyperbolic lattices in quaternionic hyperbolic spaces. More precisely, they defined the Toledo invariant $c(\rho)$ of a complex

12000 Mathematics Subject Classification. 51M10, 57S25.

Key words and phrases. Quaternionic structure, Toledo invariant, character variety, symmetric square.

Research partially supported by STINT-NRF grant (2011-0031291). Research by G. Zhang is supported partially by the Swedish Science Council (VR). I. Kim gratefully acknowledges the partial support of grant (NRF-2014R1A2A2A01005574) and a warm support of Chalmers University of Technology during his stay.
hyperbolic lattice $\Gamma$ under the representation $\rho : \Gamma \to PSp(n, 1)$ by

$$\int_M f_\rho^* \omega \wedge \omega_0^{n-2}$$

where $f_\rho$ is a descended map to $M = \Gamma \setminus PSU(n, 1)/S(U(n) \times U(1))$ from a $\rho$-equivariant map from $H^m_{\mathbb{C}}$ to $H^m_{\mathbb{H}}$, $\omega$ is the quaternionic Kähler form on $H^m_{\mathbb{H}}$ and $\omega_0$ is the complex Kähler form on $M$. They showed that this invariant $c(\rho)$ satisfies Milnor-Wood inequality and the maximality holds if and only if the representation stabilizes a copy of $H^m_{\mathbb{C}}$ inside $H^m_{\mathbb{H}}$. Such a use of Toledo invariant goes back to Toledo [14] where he proves that a maximal representation from a surface group into $SU(1, q)$ fixes a complex geodesic. Hernandez [6] also studied maximal representations from a surface group into $SU(2, q)$ and showed that the image must stabilize a symmetric space associated to the group $SU(2, 2)$.

In this paper we attempt to generalize their result to different quaternionic Kähler manifolds. The first goal would be to prove a similar result in

$$\Gamma \subset SU(n, 1) \subset SU(2n, 2)$$

using Toledo invariant

$$c(\rho) = \int_M f_\rho^* \omega^n$$

for $n$ even where $\omega$ is the quaternionic Kähler 4-form on the associated symmetric space of $SU(2n, 2)$. This Toledo invariant is constant on each connected component of the character variety $\chi(\Gamma, SU(2n, 2))$. Hence it can be used to distinguish different components of the character variety.

As a starting point, we consider the simplest case

$$\Gamma \subset SU(2, 1) \subset SU(4, 2).$$

This case is interesting since the symmetric space of $SU(4, 2)$ has both Hermitian and quaternionic structures and it is worth to study the interplay between them. We will consider several different embeddings coming from the natural holomorphic, totally real and symmetric square representations, and obtain

**Theorem 1.1.** There are at least 4 distinct connected components in $\chi(\Gamma, SU(4, 2))$ where $\Gamma \subset SU(2, 1)$ is a uniform lattice.

Here the group $SU(4, 2)$ acts on $\text{Hom}(\Gamma, SU(4, 2))$ via conjugation on the target group and the character variety is defined by

$$\chi(\Gamma, SU(4, 2)) = \text{Hom}(\Gamma, SU(4, 2))//SU(4, 2)$$
in the sense of geometric invariant theory.

This is one of the first examples known in higher dimensional complex hyperbolic lattices. For different examples of character variety $\chi(\Gamma, SU(2,1))$, see [15]. It is known in surface group case that there are $6(g - 1) + 1$ distinct components in $\chi(\pi_1(S), PSU(2,1))$ [5, 16]. Indeed, in [5], a discrete faithful representation into $SU(2,1)$ is constructed that on each component of $S \setminus \Sigma_0$, where $\Sigma_0$ is a set of disjoint simple closed geodesics, the representation stabilizes either a complex line or a totally real plane. Then the Toledo invariants are maximal on pieces contained in complex line, are zero on pieces contained in totally real plane. Hence one can realize any even integer between $\chi(S)$ and $-\chi(S)$. This implies that there are $6(g - 1) + 1$ distinct components in $\chi(\pi_1(S), PSU(2,1))$.

To prove the global rigidity, the common technique known so far is to consider a holomorphic horizontal lifting of a $\rho$-equivariant map to a proper period domain (or twistor space) where one can do complex geometry. It was successful in the case that Oscar-Garcia and Toledo considered in [4]. But in general, for higher rank case, there even does not exist a horizontal holomorphic lifting. At the last section of this paper, we give two cases where there exists or does not exist a holomorphic horizontal lifting of a symmetric square representation.

**Theorem 1.2.** Let $\iota : B \to X$ be a totally geodesic map inducing the symmetric square representation where $B = SU(2,1)/S(U(2) \times U(1))$ and $X = SU(4,2)/S(U(4) \times U(2))$. Then it lifts to a holomorphic horizontal map to the period domain $D = SU(4,2)/S(U(3) \times U(1) \times U(2))$.

See Section 3 for the definition of the symmetric square representation.

We thank D. Toledo for numerous discussions and suggestions for various period domains for liftability problem. We also thank B. Klinger for a suggestion for possible different period domains. Lastly, we thank Mathematics department at Stanford University where the first author spent a sabbatical year and the second author visited in June 2014 while part of this paper was written.

2. Quaternionic structure of $SU(2n,2)$

2.1. Quaternionic Kähler manifold in general. A Riemannian manifold $M$ of real dimension $4n$ is quaternionic Kähler if its holonomy group is contained in $Sp(n)Sp(1)$. We denote by $\mathcal{P}_M$ the canonical $Sp(n)Sp(1)$-reduction of the principal bundle of orthogonal frames of $M$, and by $\mathcal{E}_M$ the canonical three-dimensional parallel subbundle
of \( \text{End}(TM) \). Since \( Sp(n)Sp(1) \)-module \( \wedge^4(\mathbb{R}^{4n})^* \) admits a unique trivial submodule of rank 1, any quaternionic Kähler manifold \( M \) admits a nonzero closed 4-form \( \omega \), canonical up to homothety. In [12], it is proved that the form \( \omega \) (properly normalized) is the Chern-Weil form of the first Pontryagin class \( p_1(E_M) \in H^4(M, \mathbb{Z}) \).

Let \( N \) be a smooth closed manifold and \( \rho : \pi_1(N) \to G \) a representation into a quaternionic Kähler group \( G \), i.e., the associated symmetric space is a quaternionic Kähler noncompact irreducible symmetric space \( X \). Choose any \( \rho \)-equivariant smooth map \( \phi : \tilde{N} \to X \). The pullback \( \phi^*E_X \) descends to a bundle over \( N \), still denoted \( \phi^*E_X \). By the functoriality of characteristic classes, the 4-form \( \phi^*\omega \) represents the Pontryagin class \( p_1(\phi^*E_X) \in H^4(N, \mathbb{Z}) \). As \( X \) is contractible, any two \( \rho \)-equivariant maps give rise to the same class depending only on \( \rho \).

Then by the integrality of the Pontryagin class, the quaternionic Toledo invariant \( c(\rho) = \int_N \phi^*\omega^2 \) is constant on each connected component of the character variety.

2.2. Kähler and Quaternionic structures of \( SU(2n, 2)/S(U(2n) \times U(2)) \). Let \( G = SU(p, q) \), \( p \geq q \), be in its standard realization as linear transformations on \( \mathbb{C}^{p+q} = \mathbb{C}^p \oplus \mathbb{C}^q \) preserving the indefinite Hermitian form of signature \( (p, q) \). We shall later specify \( G \) to the case \( SU(2n, 2) \) or \( SU(n, 1) \). Let \( \mathcal{X} \) be the Hermitian symmetric space \( \mathcal{X} = G/K, K = S(U(p) \times U(q)) \). We recall briefly [13] the Harish-Chandra realization of the symmetric space \( \mathcal{X} \) into \( M_{p \times q} \) which might be useful in understanding various totally geodesic embeddings in our present paper. Fix \( V_0^+ = \mathbb{C}^p, V_0^- = \mathbb{C}^q \) mutually orthogonal subspaces of \( \mathbb{C}^{p+q} \) which are positive and negative definite respectively with respect to the Hermitian form \( h^\mathbb{C} \). Fix orthonormal basis \( \{e_1, \ldots, e_p\}, \{e_{p+1}, e_{p+q}\} \) of \( V_0^+, V_0^- \) respectively. Then \( G \) acts on the set \( \mathcal{X} \) of \( q \)-dimensional negative definite subspaces. Any other \( q \)-dimensional negative definite subspace \( V^- \) is a graph of a unique linear map \( A_{p \times q} = (z_{ij}) \) from \( V_0^- \) so that

\[
\sum_{i=1}^p e_i z_{ij} + e_{p+j}, \quad j = 1, \ldots, q
\]

form a basis of \( V^- \). Hence \( \mathcal{X} \) is identified with

\[
\{Z \in M_{p \times q} : I_q - Z^t Z > 0\}.
\]

The center of maximal compact subgroup \( K \) of is parameterized by the center of \( U(p) \) and it defines a complex Kähler structure. To be more precise let \( \mathfrak{g} \) be the Lie algebra of \( G \), and \( \mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} \) its Cartan decomposition, where \( \mathfrak{k} \) is the Lie algebra of \( K \), with \( \mathfrak{p} \) consisting of
matrices of the form
\[
\begin{pmatrix}
0 & A \\
A^* & 0
\end{pmatrix}, \ A \in M_{p \times q}.
\]
The real tangent space at 0 = eK of \( \mathcal{X} = G/K \) is identified with \( p \) with a complex structure \( J \) on \( T_0 \mathcal{X} \) is
\[
J \begin{pmatrix}
0 & A \\
A^* & 0
\end{pmatrix} = \begin{pmatrix} 0 & iA \\
-iA^* & 0 \end{pmatrix}.
\]
The Kähler metric on \( T_0 \mathcal{X} \) is
\[
g_0(X, Y) = 2 \text{Tr}(YX) = 4 \text{ReTr}(B^*A), \text{ for } X = \begin{pmatrix} 0 & A \\
A^* & 0 \end{pmatrix}, Y = \begin{pmatrix} 0 & B \\
B^* & 0 \end{pmatrix}.
\]
The corresponding complex Kähler form is
\[
\Omega_0(X, Y) = g_0(JX, Y).
\]
Let now \( G = SU(2n, 2) \). The second factor \( U(2) \) of \( K \) defines a quaternionic structure as follows. The holomorphic tangent space of \( \mathcal{X} \) at 0 is identified with \( \begin{pmatrix} 0 & M_{2n \times 2} \\
0 & 0 \end{pmatrix} \) where \( M_{2n \times 2} \) denotes \( 2n \times 2 \) complex matrix. The real tangent space will be parametrized and identified by the holomorph tangent space. The three elements of \( SU(2) \)
\[
\begin{pmatrix} i & 0 \\
0 & -i \end{pmatrix}, \begin{pmatrix} 0 & 1 \\
-1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & i \\
i & 0 \end{pmatrix}
\]
acts on the tangent space as the quaternionic multiplication by \( i, j, k \) as follows. The adjoint action of \( \begin{pmatrix} i & 0 \\
0 & -i \end{pmatrix} \) is
\[
\begin{pmatrix} I_{2n} & 0 \\
0 & (-i)I_{2n} \end{pmatrix}\begin{pmatrix}
x_1 & y_1 \\
x_2 & y_2 \\
\vdots & \vdots \\
x_{2n} & y_{2n}
\end{pmatrix}\begin{pmatrix} I_{2n} & 0 \\
0 & (i)I_{2n} \end{pmatrix} = \begin{pmatrix}
x_1i & y_1(-i) \\
x_2i & y_2(-i) \\
\vdots & \vdots \\
x_{2n}i & y_{2n}(-i)
\end{pmatrix}.
\]
Hence if we identify a matrix \( (x, y) \in M_{2n \times 2} = \mathbb{C}^{2n} \times \mathbb{C}^{2n} \) with a quaternionic vector \( q \in \mathbb{H}^{2n} \), with \( \mathbb{H} = \mathbb{C} + \mathbb{C}j \) being the quaternionic
number, by
\[ X = (x, y) \leftrightarrow qX = (x_1 + y_1j, x_2 + y_2j, \ldots, x_{2n} + y_{2n}j), \]
the multiplication by \( i \) on the right becomes
\[ (x_1i + y_1(-i)j, x_2i + y_2(-i)j, \ldots, x_{2n}i + y_{2n}(-i)j) = (x_1 + y_1j, x_2 + y_2j, \ldots, x_{2n} + y_{2n}j)i = q_Xi, \]
i.e., the adjoint action of \( \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \) is just the multiplication by \( i \) on the right. It is easy to check that the adjoint action of the other two elements correspond to the multiplication by \( j \) and \( k \) on the right. When no confusion would arise we shall just write the identification \( Z \rightarrow q_Z \) as \( q_Z = Z \).

The quaternionic parallel closed nondegenerate Kähler 4-form, at the origin is given by
\[ \omega = \omega_i \wedge \omega_i + \omega_j \wedge \omega_j + \omega_k \wedge \omega_k \]
where
\[ \omega_i(X, Y) = \text{Re}(q_X \cdot \bar{q}_Y i), \omega_j(X, Y) = \text{Re}(q_X \cdot \bar{q}_Y j), \omega_k(X, Y) = \text{Re}(q_X \cdot \bar{q}_Y k), \]
and \( p \cdot \bar{q} = \sum_{m=1}^{2n} p_m \bar{q}_m \) is the standard quaternionic Hermitian form on \( \mathbb{H}^{2n} \).

Then it is easy to check that this \( \omega \) and \( \Omega_0^2 \), where \( \Omega_0 \) is the complex Kähler form on \( \mathcal{X} \) defined above, are linearly independent on \( H^4(M, \mathbb{R}) \) where \( M = \Gamma \backslash \mathcal{X} \).

2.3. Totally geodesic embeddings of the unit ball \( B = SU(n, 1)/U(n) \) in \( X \). The complex hyperbolic space \( H^n_\mathbb{C} \), i.e. the symmetric space \( SU(n, 1)/U(n) \), will be realized as the unit ball \( B \) in \( \mathbb{C}^n \) as in §2.2.

A natural holomorphic embedding of \( H^n_\mathbb{C} = B = \{(z_1, \cdots, z_n) \in \mathbb{C}^n : \sum |z_i|^2 < 1 \} \) into \( \mathcal{X} \) is given by
\[ \rho : (z_1, \cdots, z_n) \mapsto Z = \begin{pmatrix} z_1I_2 \\ z_2I_2 \\ \vdots \\ z_nI_2 \end{pmatrix} \]
which seems to give rise to the maximal Toledo invariant of \( \omega \). On this holomorphic embedding, the tangent vectors to the image are
\[ X = (x_1, x_1j, \cdots, x_n, x_nj) \in \mathbb{H}^{2n}, \quad x_k = a_k + ib_k \in \mathbb{C}, \quad k = 1, \ldots, n \]
on which the form \( \omega_j \) and \( \omega_k \) and
\[ \omega(X, Y, Z, W) = \omega_i \wedge \omega_i(X, Y, Z, W) = \text{Re}(iX \cdot \bar{Y})\text{Re}(iZ \cdot \bar{W}) - \text{Re}(iX \cdot \bar{Z})\text{Re}(iY \cdot \bar{W}) + \text{Re}(iX \cdot \bar{W})\text{Re}(iY \cdot \bar{Z}) \]
\[ \begin{align*} &\text{= Re}(2i \sum_{i=1}^{n} x_i \bar{y}_i) \text{Re}(2i \sum_{i=1}^{n} z_i \bar{w}_i) - \text{Re}(2i \sum_{i=1}^{n} x_i \bar{z}_i) \text{Re}(2i \sum_{i=1}^{n} y_i \bar{w}_i) \\
&\quad + \text{Re}(2i \sum_{i=1}^{n} x_i \bar{w}_i) \text{Re}(2i \sum_{i=1}^{n} y_i \bar{z}_i). \end{align*} \]

But when we write \( X = \begin{pmatrix} 0 & A \\ A^* & 0 \end{pmatrix}, Y = \begin{pmatrix} 0 & B \\ B^* & 0 \end{pmatrix} \)

\[ \Omega_0(X, Y) = g_0(JX, Y) = 4 \text{Re} \text{ Tr}(iB^*A) = 4 \text{Re}(2i \sum_{i=1}^{n} x_i \bar{y}_i). \]

Hence

\[ \Omega_0^2(X, Y, Z, W) = 16 \omega(X, Y, Z, W) \]

for tangent vectors \( X, Y, Z, W \) at the image of the natural holomorphic embedding of \( H^n_C \). In other words,

\[ (2) \quad \rho^* \Omega_0^2 = 16 \rho^* \omega, \quad \rho^* \omega = \frac{1}{16} \rho^* \Omega_0^2 \]

for the natural holomorphic embedding \( \rho \) of \( H^n_C \) into \( \mathcal{X} \).

On the other hand, another natural embedding

\[ SU(n, 1) \subset Sp(n, 1) \subset SU(2n, 2) \]

gives rise to a totally real embedding

\[ (z_1, \cdots, z_n) \mapsto Z = \begin{pmatrix} \begin{pmatrix} z_1 & 0 \\ 0 & \bar{z}_1 \end{pmatrix} \\ \vdots \\ \begin{pmatrix} z_n & 0 \\ 0 & \bar{z}_n \end{pmatrix} \end{pmatrix} \]

whose Toledo invariant of \( \Omega_0 \) is zero. Contrary to \( SU(1, 1) \) case, this totally real embedding is locally rigid for \( n > 1 \), see [7]. On this totally real embedding, the tangent vectors are \( X = (x_1, \bar{x}_1 j, \cdots, x_n, \bar{x}_n j) \in \mathbb{H}^{2n}, x_i \in \mathbb{C}, \) and

\[ \omega(X, Y, Z, W) = \omega_i \wedge \omega_i(X, Y, Z, W) = 0. \]

Hence the Toledo invariant of \( \omega \) also vanishes. These two special embeddings suggest that the Toledo invariant of \( \omega \) is maximal on holomorphic embedding and zero on totally real embedding.

**Warning:** If we identify the holomorphic tangent space of \( \mathcal{X} \) with \( \mathbb{H}^{2n} \) by \( (x_1 + jy_1, \cdots, x_{2n} + jy_{2n}) \), \( \omega \) vanishes on holomorphic embedding and \( 16 \omega = \Omega_0^2 \) on totally real embedding. Hence the convention determines which one has a maximal Toledo invariant. In [3], it seems
that they use a different convention from ours. Nevertheless we stick
to our convention in this paper.

3. **Symmetric square representation of SU(2, 1) and
related 4-forms**

Denote $V = \mathbb{C}^{2+1} = \mathbb{C}^2 + \mathbb{C}e_3$ the space $\mathbb{C}^3$
equipped with the
Hermitian metric with signature $(2, 1)$ and $B = SU(2, 1)/U(1)$ as in
§2.2. Recall that it is also identified as the open domain in $\mathbb{P}^2$ of lines
$\mathbb{C}(z + e_3)$ with negative metric, i.e. $|z| = |(z_1, z_2)| < 1$.

Let $W = V^2$ be the symmetric square of $V$. Then $W$ is equipped with
the square of the Hermitian metric of $V$ and $W = \mathbb{C}^4 + \mathbb{C}^2 = ((\mathbb{C}^2)^2 +
\mathbb{C}e_3^2) \oplus (\mathbb{C}^2 \otimes e_3)$ is of signature $(4, 2)$. Here $e_i \otimes e_j = \frac{1}{2}(e_i \otimes e_j + e_j \otimes e_i)$.

We fix an orthonormal basis $\{E_1, E_2, E_3, E_4, E_5, E_6\}$ of $W$ with

$$E_j = e_j^2, E_4 = \frac{1}{\sqrt{2}}(e_1 \otimes e_2 + e_2 \otimes e_1) = \sqrt{2}e_1 \otimes e_2,$$

$$E_{4+i} = \frac{1}{\sqrt{2}}(e_3 \otimes e_i + e_i \otimes e_3) = \sqrt{2}e_3 \otimes e_i, j = 1, 2, 3, i = 1, 2.$$

The square of the defining representation of $H = SU(2, 1)$ defines a
representation

$$\iota : H \to G = SU(4, 2), g \mapsto \otimes^2 g$$

As in §2.2 the symmetric space $X$ of $SU(4, 2)$ will be realized as
the open domain of Grassmannian manifold $Gr(2, W)$ 2-dimensional
complex subspaces in $W$ with negative metric, and is further identified
with the space of $4 \times 2$ matrices $Z$ with matrix norm $\|Z\| < 1$ under
the identification

$$\{Zx \oplus x; x \in \mathbb{C}^2\} \mapsto Z.$$  

Recall also the normalization of the Kähler metric on $B$ and on $X$

$$g_B(u, v) = 4\text{Re}(u_1 \bar{v}_1 + u_2 \bar{v}_2), g_X(u, v) = 4\text{Re} \text{Tr} v^*u$$

where the real tangent space of $B$ and $X$ at $z = 0$ and $Z = 0$ are identified with $\mathbb{C}^2$ and $M_{4 \times 2}$; the respective Kähler forms are $\Omega_B(u, v) = g_B(iu, v)$ and $\Omega_X = g_X(iu, v)$.

The representation $\iota : H \to G$ induces a totally geodesic mapping
(with the same notation) $\iota : B \to X$. In terms of the above identifi-
cation of $B$ and $X$ as submanifolds of projective and Grassmannian
manifolds the map $\iota$ is

$$\iota(l) = l \otimes l^\perp$$

where $l^\perp$ is the orthogonal complement of $l$ in $V$ and $l \otimes l^\perp$ is the
subspace of vectors $u \otimes v + v \otimes u, u \in l, v \in l^\perp$. We find now the map
$\iota_*$ at $z = 0 \in B$. 

Fixing the reference line \( C e_3 \in \mathbb{P}^2 \) and the plane \( \mathbb{C}^2 \odot e_3 \in \text{Gr}(2, W) \) corresponding to the point \( 0 \in B \) and \( 0 \in X \), the map \( \iota \) is
\[
\iota : \exp(tX) \cdot (C e_3) \mapsto (\exp(tX) \cdot (C e_3)) \odot (\exp(tX) \cdot (C^2)),
\]
where
\[
X = \begin{pmatrix}
0 & 0 & a_1 \\
0 & 0 & a_2 \\
\bar{a}_1 & \bar{a}_2 & 0
\end{pmatrix} \in \mathfrak{p}
\]
and \( \mathfrak{su}(2, 1) = \mathfrak{k} + \mathfrak{p} \). Thus \( \iota^*_\ast (X) \) is the linear transformation
\[
\iota^*_\ast (X) : \mathbb{C}^2 \to \mathbb{C}^4,
\]
\[
\mathbb{C}\{E_5, E_6\} \mapsto \mathbb{C}\{(Xe_3) \odot e_1 + e_3 \odot (Xe_1), (Xe_3) \odot e_2 + e_3 \odot (Xe_2)\}.
\]
Note that \( Xe_1 = \bar{a}_1 e_3, \ Xe_2 = \bar{a}_2 e_3, \ Xe_3 = a_1 e_1 + a_2 e_2 \) and
\[
(Xe_3) \odot e_1 + e_3 \odot (Xe_1) = (a_1 e_1 + a_2 e_2) \odot e_1 + e_3 \odot \bar{a}_1 e_3
= a_1 e_1 \odot e_1 + a_2 e_2 \odot e_1 + \bar{a}_1 e_3 \odot e_3 = a_1 E_1 + \bar{a}_1 E_3 + \frac{a_2}{\sqrt{2}} E_4.
\]
A similar calculation for the second factor shows that, under the basis \( \{E_j\} \), \( \iota^*_\ast (X) \) corresponds to the \( 4 \times 2 \) matrix
\[
\begin{bmatrix}
a_1 & 0 \\
0 & a_2 \\
\bar{a}_1 & \bar{a}_2 \\
\frac{a_2}{\sqrt{2}} & \frac{a_1}{\sqrt{2}}
\end{bmatrix} = T_a,
\]
Taking the basis vectors \( X = \begin{pmatrix}
0 & 0 & 1 \\
0 & 0 & 0 \\
1 & 0 & 0
\end{pmatrix}, Y = \begin{pmatrix}
0 & 0 & i \\
0 & 0 & 0 \\
-i & 0 & 0
\end{pmatrix}, Z = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & 0
\end{pmatrix}, W = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & i \\
0 & -i & 0
\end{pmatrix} \) we find the corresponding images in \( \mathbb{H}^4 \) under \( \iota_\ast \)
\[
\iota_\ast (X) = (1, 0, 1, \frac{1}{\sqrt{2}} i), \ \iota_\ast (Y) = (i, 0, -i, \frac{k}{\sqrt{2}}),
\]
\[
\iota_\ast (Z) = (0, j, j, \frac{1}{\sqrt{2}}), \ \iota_\ast (W) = (0, k, -k, \frac{i}{\sqrt{2}})
\]
and that
\[
\omega(\iota_\ast (X), \iota_\ast (Y), \iota_\ast (Z), \iota_\ast (W)) = \frac{11}{4}.
\]
Namely
\[
\iota^* \omega = \frac{11}{64} \Omega^2_B
\]
where \( \Omega_B \) is the Kähler form on \( B \).

4. Character variety \( \chi(\Gamma, SU(4, 2)) \)

**Theorem 4.1.** There are at least 4 distinct connected components in \( \chi(\Gamma, SU(4, 2)) \) where \( \Gamma \subset SU(2, 1) \) is a uniform lattice.

**Proof.** Let

\[
X = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad W =
\]

\[
\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & -i & 0 \end{pmatrix}
\]

be the standard basis of \( T_\mathfrak{g}B = \mathfrak{p} \) such that

\[
\Omega^2_B(X, Y, Z, W) = \Omega_B(X, Y)\Omega_B(Z, W) = 4\text{Tr}(YJX)\text{Tr}(WJZ) = 4 \cdot 4 = 16.
\]

Consider first the holomorphic embedding \( \rho \) in \( \mathfrak{g}^2 \). The images of the above vectors under \( \rho^* \), written as block 3 \( \times \) 3-matrix with each entry being 2 \( \times \) 2 matrix, are

\[
\rho^*(X) = \begin{pmatrix} 0 & 0 & I_2 \\ 0 & 0 & 0 \\ I_2 & 0 & 0 \end{pmatrix}, \quad \rho^*(Y) = \begin{pmatrix} 0 & 0 & iI_2 \\ 0 & 0 & 0 \\ -iI_2 & 0 & 0 \end{pmatrix},
\]

\[
\rho^*(Z) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & I_2 \\ 0 & I_2 & 0 \end{pmatrix}, \quad \rho^*(W) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & iI_2 \\ 0 & -iI_2 & 0 \end{pmatrix},
\]

which correspond to

\[
\rho^*(X) = (1, j, 0, 0), \quad \rho^*(Y) = (i, ij, 0, 0),
\]

\[
\rho^*(Z) = (0, 0, 1, j), \quad \rho^*(W) = (0, 0, i, ij)
\]

in \( \mathbb{H}^2 \) coordinates, see Section 2. Then by Equation (1)

\[
\rho^*\omega(X, Y, Z, W) = 4, \quad \text{i.e.} \quad \rho^*\omega = \frac{1}{4}\Omega^2_B,
\]

whereas for the square representation \( \iota \), by Equation (4)

\[
\iota^*\omega(X, Y, Z, W) = \frac{11}{64}, \quad \text{i.e.} \quad \iota^*\omega = \frac{11}{64}\Omega^2_B.
\]

For the totally real embedding \( \hat{\iota} \), the pull-back form vanishes. This implies that the quaternionic Toledo invaraints are

\[
\int_{\Gamma \backslash H_\mathbb{C}^2} \rho^*\omega = \frac{1}{4} \int_{\Gamma \backslash H_\mathbb{C}^2} \Omega^2_B = \frac{1}{4} \text{vol}(\Gamma \backslash H_\mathbb{C}^2),
\]

\[
\int_{\Gamma \backslash H_\mathbb{C}^2} \iota^*\omega = \frac{11}{64} \int_{\Gamma \backslash H_\mathbb{C}^2} \Omega^2_B = \frac{11}{64} \text{vol}(\Gamma \backslash H_\mathbb{C}^2), \quad 0
\]
The last representation with a different Toledo invariant is given by the embedding \( \phi : (z_1, \cdots, z_n) \rightarrow ((z_1, 0), \cdots, (z_n, 0)) \) which produces that

\[
\phi^* \omega = \frac{1}{16} \Omega_B^2.
\]

Since the quaternionic Toledo invariant is constant on each connected component, we are done.

Note that for a lattice \( \Gamma \subset SU(2,1) \), the holomorphic embedding \( \rho \) corresponds to the diagonal embedding \( \gamma \rightarrow (\gamma, \gamma) \in SU(2,1) \times SU(2,1) \subset SU(4,2) \), and the totally real embedding to \( \gamma \rightarrow (\gamma, \overline{\gamma}) \) whereas the last example in the previous theorem corresponds to the embedding \( \gamma \rightarrow (\gamma, id) \in SU(2,1) \times SU(2,1) \subset SU(4,2) \).

In this direction, Toledo constructed the following examples [15]. There are examples of two complex hyperbolic surfaces \( X = \Gamma \backslash H^2_\mathbb{C} \) and \( Y = \Gamma' \backslash H^2_\mathbb{C} \) with a surjective holomorphic map \( f : X \rightarrow Y \) with \( 0 < \deg(f) < \frac{\text{vol}(X)}{\text{vol}(Y)} \), which induces a group homomorphism \( f_* : \Gamma \rightarrow \Gamma' \).

See also [3, 10] for the constructions of various subgroups \( \Gamma' \subset \Gamma \) of finite index. (The volumes \( \text{vol}(X) \) and \( \text{vol}(Y) \) can be further computed by using the Chern-Gauss-Bonnet theorem for orbifolds.) Consider the following representation

\[
\Gamma \xrightarrow{f} \Gamma' \xrightarrow{\phi} SU(4,2),
\]

where \( \phi \) is the restriction of the holomorphic embedding (5) above. Then the quaternionic Toledo invariant of this representation is

\[
\int_X f^*(\phi^* \omega) = \int_X f^* \left( \frac{1}{16} \Omega_B^2 \right) = \frac{1}{16} \deg(f) \text{vol}(Y) < \frac{1}{16} \text{vol}(X),
\]

with \( \frac{1}{16} \text{vol}(X) \) being the smallest among the Toledo invariants in Theorem 4.1 except zero case. We obtain thus an improvement of Theorem 4.1 in this case, viz

**Proposition 4.2.** Let \( \Gamma \subset \Gamma' \) be as above. There exist at least 5 distinct components in \( \chi(\Gamma, SU(4,2)) \).

Some versions of local rigidity for the representations in some of the components above have been studied in [7, 8].

5. **Holomorphic Lifting to Various Period Domains**

In [4] the authors study some holomorphic liftings of mappings from the complex hyperbolic ball to quaternionic hyperbolic ball to holomorphic mapping to the (pseudo-Hermitian) twister space, which enable
them to apply a variant of Schwarz lemma and to prove local rigidity theorems. Following a suggestion of Toledo we shall study holomorphic liftings in our context.

5.1. Non-lifting property. Let $D = SU(4,2)/S(U(4) \times U(1) \times U(1))$ be a twistor space. We shall realize it as an open subset in a homogeneous flag manifold. Let $W^*$ be the dual space equipped with the $G$-invariant metric of signature $(4,2)$, and let $\{\epsilon_j\}$ be the dual basis of $\{E_j\}$. Let $D_1$ be the set of orthogonal pairs $(l, \lambda)$ in $P(W) \times P(W^*)$, i.e., satisfying $\epsilon(e) = 0$ for all $(e, \epsilon) \in l \times \lambda$. As a homogeneous manifold of $SU(4,2)$ $D = SU(4,2)/S(U(4) \times U(1) \times U(1))$ can be realized as the subset $(l, \lambda)$ such that $l$ and $\lambda$ are negative definite. Indeed, first it is elementary to see that $SU(4,2)$ acts transitively on the subset of lines. Second we need to check that a stabilizer of $(l, \lambda)$ is $S(U(4) \times U(2))$ and a stabilizer in $S(U(4) \times U(2))$ of the pair $(l, \ker \lambda)$ of subspaces in $W$, equivalently the pair $(l, \lambda)$ in $P(W) \times P(W^*)$, is exactly $U(1) \times U(1)$. Hence as a differentiable manifold $D$ has such a realization.

Then as an open set in $D_1 \subset P(W) \times P(W^*)$, $D$ is equipped with the corresponding complex structure.

In general if a homogeneous manifold $G/(L \times U(1))$ has $U(1)$ factor in the stabilizer, it inherits a complex structure as follows. Let $u(1) = \mathbb{R}iH$ and consider the root space decomposition of $g^\mathbb{C}$ under the action of $H = \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & -1 \end{pmatrix}$. Set $\mathfrak{b}$ to be the Borel subalgebra consisting of zero and negative eigenspaces. The positive eigenspace constitutes the holomorphic tangent space as an open set $G/(L \times U(1)) \subset G^\mathbb{C}/B$.

We find the holomorphic tangent space of $D$ in this context. To find a realization of the complex tangent space we fix the pair $(\mathbb{C}E_6, \mathbb{C}e_5)$ as a base point of $D \subset D_1$. The space $D_1$ is a complex homogeneous space of $SL(6,\mathbb{C})$, $D_1 = SL(6,\mathbb{C})/B$, where $B$ is the Borel subgroup whose Lie algebra consists of elements in $\mathfrak{sl}(6,\mathbb{C})$ of the special form.

To justify this, note that $B$ is equal to the stabilizer of $(\mathbb{C}E_6, \mathbb{C}e_5)$. Hence $B$ should have the block matrix of form,

$$
\begin{pmatrix}
* & * & 0 \\
0 & * & 0 \\
* & * & * 
\end{pmatrix},
$$


the size of the matrix being \((4 + 1 + 1) \times (4 + 1 + 1)\). Alternatively \(\mathfrak{b}\) consists of non-positive roots spaces of \(H\), i.e. eigenspaces of \(\text{ad}(H)\) of non-positive eigenvalues, where

\[
H = \begin{pmatrix}
0 & 0 \\
0 & 1 & 0 \\
0 & 0 & -1
\end{pmatrix}.
\]

Thus holomorphic tangent space \(\mathfrak{n}\) consists of elements of \(\mathfrak{g}^C\) of the form, the size of the matrix being the same as above,

\[
\begin{pmatrix}
0 & 0 & * \\
* & 0 & * \\
0 & 0 & 0
\end{pmatrix}.
\]

**Lemma 5.1.** The above quadratic map \(\iota : B \to \mathcal{X}\) does not lift to a holomorphic horizontal mapping into \(D = SU(4, 2)/S(U(4) \times U(1) \times U(1))\).

**Proof.** Suppose \(F\) is a holomorphic horizontal lifting. The complexification of \(F_*\), still denoted by \(F_*\), maps \(\mathfrak{b}^+\), the holomorphic tangent space of \(B\) to holomorphic tangent space \(\mathfrak{n}\) (up to changing of base point under \(\text{SU}(2)\)-action). In particular the image of \(\iota_*\) of \(\mathfrak{b}^+\) is contained in \(\pi_* \mathfrak{n}\) where \(\pi : D \to \mathcal{X}\) is the natural projection. In particular \(\iota_* \mathfrak{b}^+\) is a subspace of \(\pi_* \mathfrak{n}\). Using the above formula for \(\mathfrak{n}\) we find that elements in \(\pi_* \mathfrak{n} \supset \iota_* \mathfrak{b}^+\) are of the form

\[
\begin{pmatrix}
0 & 0 & * \\
* & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}.
\]

However our computations above show that for

\[
S = \begin{pmatrix}
0 & 0 & a_1 \\
0 & 0 & a_2 \\
0 & 0 & 0
\end{pmatrix} = \frac{1}{2} \begin{pmatrix}
0 & 0 & a_1 \\
0 & 0 & a_2 \\
\bar{a}_1 & \bar{a}_2 & 0
\end{pmatrix}

- \sqrt{-1} \begin{pmatrix}
0 & 0 & ia_1 \\
0 & 0 & ia_2 \\
-i\bar{a}_1 & -i\bar{a}_2 & 0
\end{pmatrix} \in \mathfrak{b}^+
\]

its image \(\iota_*(S)\) is

\[
\iota_*(S) = \begin{bmatrix}
0 & U \\
V & 0
\end{bmatrix}
\]
where
\[ U = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \\ \frac{a_1}{\sqrt{2}} & \frac{a_2}{\sqrt{2}} \end{pmatrix}, \quad V = \begin{pmatrix} 0 & 0 & a_1 & 0 \\ 0 & 0 & a_2 & 0 \end{pmatrix}. \]

This is a contradiction to the form of \( \pi_*(n) \). \( \square \)

We may construct similarly the twistor cover \( SU(2m, 2)/S(U(2m) \times U(1) \times U(1)) \) of \( \mathcal{X} = SU(2m, 2)/S(U(2m) \times U(2)) \) as above and consider the question of holomorphic lifting of maps from \( B \) to \( \mathcal{X} \). The above proof leads to a simple necessary condition for the existence.

**Corollary 5.2.** Given a representation \( \rho : \Gamma \subset SU(n, 1) \rightarrow SU(2m, 2) \), with a \( \rho \)-equivariant map \( f \) on the associated symmetric spaces \( B = SU(n, 1)/S(U(n) \times U(1)) \), \( \mathcal{X} = SU(2m, 2)/S(U(2m) \times U(2)) \) and a fixed base point \( o = [K] \in SU(n, 1)/S(U(n) \times U(1)) \), let
\[
Df_o \begin{pmatrix} X \star \quad 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & U \\ U^* & 0 \end{pmatrix}
\]
be a differential map at the base point, where \( X \in \mathbb{C}^n, U = (U_1, U_2) \in M_{2m \times 2} \). For \( f \) to have a holomorphic lift to the twistor space, every component of \( U_1 \) is an conjugate \( \mathbb{C} \)-linear in \( X \), and every component of \( U_2 \) is a \( \mathbb{C} \)-linear in \( X \). Here we regard \( Df_o \) as a map from \( \mathbb{C}^n \) to \( M_{2m \times 2} = \mathbb{C}^{4m} \).

**Proof.** Note that \( Df_o \) is a real linear map between real tangent spaces \( T_o B \) and \( T_{f(o)} \mathcal{X} \). For \( X = (z_1, \cdots, z_n) \) and \( z_i = x_i + iy_i \), let \( X = (x_1, \cdots, x_n, y_1, \cdots, y_n) = (x, y) \), with the same notation, be the corresponding coordinates in \( \mathbb{R}^{2n} \). Then \( iX \) corresponds to
\[ iX = (-y_1, \cdots, -y_n, x_1, \cdots, x_n) = (-y, x) \]
as usual. For \( f \) to lift to the holomorphic map to the twistor space, the equation (6) should read
\[
Df_o(X - \sqrt{-1}iX) = (U'_1, U'_2) = (0, U'_2).
\]
Hence from \( U'_1 = 0 \), we get
\[
\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} - \sqrt{-1} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} -y \\ x \end{pmatrix} = 0.
\]
It is
\[
\begin{pmatrix} A \, x + B \, y \\ C \, x + D \, y \end{pmatrix} + \begin{pmatrix} -C \, y + D \, x \\ -A \, y + B \, x \end{pmatrix} = 0.
\]
From this we get
\[ A = -D, \quad B = C. \]
This exactly implies that every component function of $U_1$ is conjugate $\mathbb{C}$-linear in $X = (z_1, \cdots, z_n)$ variables. Using the equation for $U^*$, a similar calculation shows that every component function of $U_2$ is $\mathbb{C}$-linear in $z_i$ variables for $f$ to have a holomorphic lift to the twistor space.

When $\rho$ is reductive, we can choose $f$ to be harmonic. Hence if we write $Df = (f_1, \cdots, f_{2m}; g_1, \cdots, g_{2m})$, $f_i$ are anti-holomorphic and $g_i$ are holomorphic for $f$ to have a holomorphic lift to the twistor space.

5.2. Lifting to a different period domain. We consider a different map from $B$ to $D = SU(4, 2)/S(U(3) \times U(1) \times U(2))$ and show that there exists a holomorphic horizontal lifting of $\iota$.

Let $f$ associate $S^2L^\perp, L^2, L \odot L^\perp$ to a negative line $L$ in $V$. Then $S^2L^\perp$ is a positive 3-dimensional space in $W$, $L^2$ is a positive line in $W$, and $L \odot L^\perp$ is a negative plane in $W$. Hence their stabilizers are $U(3), U(1)$ and $U(2)$ respectively. Therefore

$$f : B \rightarrow D = SU(4, 2)/S(U(3) \times U(1) \times U(2)).$$

Since

$$\iota(L) = (L \odot L^\perp, (L \odot L^\perp)^\perp),$$

$$f(L) = ((S^2L^\perp, L^2), \iota(L))$$ is a lifting of $\iota$ to $D$.

We claim that $f$ is holomorphic with respect to a complex structure on $D$ induced by $S(U(3) \times U(1))$. In the explicit coordinates, if $L = \mathbb{C}e_3$, then $L^\perp = \langle e_1, e_2 \rangle$ and

$$S^2L^\perp = \langle e_1^2, e_2^2, e_1 \odot e_2 \rangle = \langle E_1, E_2, E_4 \rangle, \quad L^2 = \langle e_3^2 \rangle = \langle E_3 \rangle,$$

$$L \odot L^\perp = \langle e_1 \odot e_3, e_2 \odot e_3 \rangle = \langle E_5, E_6 \rangle.$$}

The center of the stabilizer $U(3)$ of $S^2L^\perp$ is generated by the diagonal matrix, under the above decomposition of $W = S^2L^\perp + L^2 + L \odot L^\perp = \mathbb{C}^3 + \mathbb{C} + \mathbb{C}^2$.

$$\text{diag}(1, 1, 0, 1, 0, 0)$$

This means that under the adjoint action of an element in the center of $U(3) \times U(1) \times U(2)$

$$\text{diag}(1, 1, -3, 1, 0, 0),$$
the positive eigenspace constitutes the holomorphic tangent space of $D$:

$$
\begin{pmatrix}
0 & 0 & 0 & * & * \\
0 & 0 & 0 & * & * \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & * & * \\
0 & 0 & * & 0 & 0 \\
0 & 0 & * & 0 & 0
\end{pmatrix}.
$$

Hence the claim follows from the fact that the holomorphic tangent vector in $B$

$$
S = \begin{pmatrix}
0 & 0 & a_1 \\
0 & 0 & a_2 \\
0 & 0 & 0
\end{pmatrix} = \frac{1}{2} \begin{pmatrix}
0 & 0 & a_1 \\
0 & 0 & a_2 \\
\bar{a}_1 & \bar{a}_2 & 0
\end{pmatrix} - \sqrt{-1} \begin{pmatrix}
0 & 0 & ia_1 \\
0 & 0 & ia_2 \\
-i\bar{a}_1 & -i\bar{a}_2 & 0
\end{pmatrix} \in \mathfrak{b}^+_
$$

is mapped to $\iota_*(S)$

$$
\iota_*(S) = \begin{bmatrix}
0 & U \\
V & 0
\end{bmatrix}
$$

as in the proof of Lemma 5.1 where

$$
U = \begin{pmatrix}
a_1 & 0 \\
0 & a_2 \\
\frac{a_1}{\sqrt{2}} & \frac{a_2}{\sqrt{2}}
\end{pmatrix},
V = \begin{pmatrix}
0 & 0 & a_1 & 0 \\
0 & 0 & a_2 & 0
\end{pmatrix}.
$$

Now we show the horizontality, i.e., the image lies in the form $L \odot L^\perp$. For any smooth curve in $B$, denote it by $L(t) = (v_0 + w(t))$ where $w(t) \subset v_0^\perp$, a differentiable family of lines, such that $w(0) = 0, w'(0) \in v_0^\perp$. Then we can write $L(t)^\perp = \langle v(t) \rangle^\perp$ where $v(0) = v_0, v'(0) = w'(0) \in v_0^\perp$.

Since $L(t) \odot L(t)^\perp$ is already horizontal, it suffices to show the horizontality of $L(t)^2$ and $S^2(L(t)^\perp)$. But

$$
L(t)^2 = \langle (v_0 + w(t)) \odot (v_0 + w(t)) \rangle = \langle v_0^2 + v_0 \odot w(t) + w(t)^2 \rangle.
$$

Hence

$$
\frac{d}{dt} \big|_{t=0} L(t)^2 = v_0 \odot w'(0) \in L(0) \odot L(0)^\perp.
$$

Similar calculation shows that

$$
\frac{d}{dt} \big|_{t=0} S^2(L(t)^\perp) = \frac{d}{dt} \big|_{t=0} \langle v(t)^\perp \odot v(t)^\perp \rangle
$$

$$
= \langle v'(0)^\perp \odot v_0^\perp \rangle \subset \langle v_0 \odot v_0^\perp \rangle \subset L(0) \odot L(0)^\perp,
$$

completing the proof.
TOLEDO INVARIANT OF LATTICE IN SU(2,1)

REFERENCES

[1] M. Burger and A. Iozzi, Bounded cohomology and representation varieties of lattices in $PU(1,n)$, preprint announcement, 2000.
[2] M. Burger and A. Iozzi, A measurable Cartan theorem andnd applications to deformation rigidity in complex hyperbolic geometry, Pure Appl. Math. Q., 4(1, Special Issue: In honor of Grigory Margulis. Part 2): 181-2-2, 2008.
[3] P. Deligne and G. Mostow, Monodromy of hypergeometric functions and non-lattice integral monodromy groups, Publ. Math. IHES, 63 (1986), 5-90.
[4] O. García-Prada and D. Toledo, A Milnor-Wood inequality for complex hyperbolic lattices in quaternionic space, Geom & Topology, 15 (2011), no. 2, 1013-1027.
[5] W. Goldman, M. Kapovich and B. Leeb, Complex hyperbolic manifolds homotopy equivalent to a Riemann surface, Comm. Anal. Geom., 9 (2001), 61-95.
[6] L. Hernández, Maximal representations of surface groups in bounded symmetric domains, Trans. Amer. Math. Soc., 324 (1) (1991), 405-420.
[7] I. Kim, B. Klingler and P. Pansu, Local quaternionic rigidity for complex hyperbolic lattices, Journal of the Institute of Mathematics of Jussieu. 11 (2012), no 1, 133-159.
[8] B. Klingler, Local rigidity for complex hyperbolic lattices and Hodge theory, Invent. Math., 184 (2011), no.3, 455-498.
[9] V. Koziarz and J. Maubon, Representations of complex hyperbolic lattices into rank 2 classical Lie groups of Hermitian type, Geom. Dedicata., 137 (2008), 85-111.
[10] G. Mostow, Monodromy of hypergeometric functions and nonlattice integral monodromy. Publ. Math. IHES, 63 (1986), 589.
[11] M. B. Pozzetti, Maximal representations of complex hyperbolic lattices into $SU(m,n)$, preprint.
[12] S. Salamon, Quaternionic Kähler manifolds, Invent. Math., 67 (1) (1982), 143-171.
[13] I. Satake, Algebraic structures of symmetric domains, Kano Memorial Lectures 4, Iwanami Shoten, Tokyo, Princeton University Press, Princeton. NJ, 1980.
[14] D. Toledo, Representations of surface groups in complex hyperbolic space, J. Differential Geom., 29 (1) (1989), 125-133.
[15] D. Toledo, Maps between complex hyperbolic surfaces, Special volume dedicated to the memory of Hanna Miriam Sandler. Geom. Dedicata, 97 (2003), 115-128.
[16] E. Xia, The moduli of flat $PU(2,1)$ structures on Riemann surfaces, Pacific J. Math., 195 (2000) 231-256.

School of Mathematics, KIAS, Heogiro 85, Dongdaemun-gu Seoul, 130-722, Republic of Korea
E-mail address: inkang@kias.re.kr

Mathematical Sciences, Chalmers University of Technology and Mathematical Sciences, Göteborg University, SE-412 96 Göteborg, Sweden
E-mail address: genkai@chalmers.se