BGV theorem and Geodesic deviation

Dawood Kothawala*

Department of Physics, Indian Institute of Technology Madras, Chennai 600 036

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I point out a simple expression for the “Hubble” parameter $H$ defined by Borde, Guth and Vilenkin (BGV) in their proof of past incompleteness of inflationary spacetimes. I show that $H$ is equal to the fractional rate of change of the magnitude of the Jacobi field $\xi$ of the congruence $u$ used by BGV, measured along the points of intersection of an arbitrary observer $O$ with $u$.

The BGV theorem illustrates past geodesic incompleteness of inflationary universes under very plausible assumptions, without appealing to field equations or energy conditions. Instead, the theorem uses a well motivated kinematical setup to define the “expansion rate” $H$ that an observer $O$ will associate with a given congruence in an arbitrary curved spacetime.

I here show that the BGV expression for $H$ can be cast completely in terms of the Jacobi fields (geodesic deviation) associated with the congruence.

I. REWRITING $H$

In [1], the expansion rate of a geodesic congruence $u$, as measured by an observer $O$ with four velocity $v$, is defined as

$$H = \frac{(n \cdot \nabla_v u) \Delta \lambda}{(n \cdot v) \Delta \lambda}$$

(1)

where $\lambda$ is the affine parameter along $v$, and $n$ is a unit vector orthogonal to $u$, such that $v = \gamma \left( u + v n \right)$, with $\gamma = (1 - v^2)^{-1/2}$. It is then easy to show that

$$H = n \cdot \nabla_n u$$

(2)

At this point, we note that $H$ is being measured by the observer $O$ at each of its intersections with $u$. Therefore, there is a one to one correspondence between $n$, and the Jacobi field/geodesic deviation vector $\xi$ that connects two nearby geodesics of the congruence which $O$ intersects in succession. Specifically, we can write $\xi = \xi n$.

Now, using the fact that $L_u \xi = [u, \xi] = 0$, it is easy to show that

$$\nabla_n u = \nabla_u n + n \nabla_u \ln \xi$$

(3)

which immediately yields

$$H = \nabla_u \ln \xi$$

(4)

Note that we have not assumed $v$ to be a geodesic. For sake of completeness, we also quote the expression derived in BGV. A few simple manipulations yield

$$H = -\frac{1}{\gamma^2 - 1} \left( \nabla_v \gamma + u \cdot a_{(v)} \right)$$

(5)

where $a_{(v)} = \nabla_v v$ is the acceleration of $v$. For $a_{(v)} = 0$, this gives

$$H = \nabla_v \left[ \ln \sqrt{\frac{\gamma + 1}{\gamma - 1}} \right]$$

(6)

which is the result in BGV.

II. GEODESIC INCOMPLETENESS

We now consider the average

$$H_{\text{avg}} = \frac{\int_{\tau_i}^{\tau_f} H d\tau}{\Delta \tau}$$

(7)

where $\tau$ is the proper time along $u$ (this is different from BGV, where the average is taken along $v$). This results in

$$\frac{\xi_f}{\xi_i} = \exp \left[ H_{\text{avg}} \Delta \tau \right]$$

(8)
Now suppose that $\xi$ is bounded above by, say $1/\sqrt{\Lambda}$, and also below by, say, $\ell_0$.

**Comment 1:** Though the upper bound might come from size of the observable universe, there is no reason to expect a direct connection since a bound on size does not necessarily imply that geodesics are bounded within it. A more precise characterisation, for example, in terms of causal structure, is needed for this. Here we simply assume such a bound exists. The lower bound is expected to come from quantum gravitational fluctuations, and their effects on (de-)focussing of geodesics \[2, 3\]. Assuming such bounds exist, $\xi_f < 1/\sqrt{\Lambda}$ and $\xi_i > \ell_0$, so that

$$\frac{\xi_f}{\xi_i} < \frac{1}{\sqrt{\Lambda \ell_0^2}} \quad (9)$$

This immediately implies, for $H_{\text{avg}} > 0$,

$$\Delta \tau < \frac{1}{2H_{\text{avg}}} \ln \left( \frac{1}{\Lambda \ell_0^2} \right) \quad (10)$$

The implications of the above result are:

1. For $\Lambda \ell_0^2 = 0$, the congruence would be geodesically complete. This would happen if either $\Lambda = 0$ or $\ell_0 = 0$.

2. For $\Lambda \ell_0^2 < 1$, the congruence would be geodesically incomplete and can not be extended beyond a proper time

$$\Delta \tau_{\text{max}} = -(2H_{\text{avg}})^{-1} \ln \left( \Lambda \ell_0^2 \right)$$

**Comment 2:** I must emphasise that the above comments only imply geodesic (in)completeness of the congruence $u$. The original BGV theorem still applies, of course, for the geodesic observer $v$, implying incompleteness.

### III. EVOLUTION OF H

Having expressed $H$ in terms of geodesic deviation, one can use the geodesic deviation equation

$$\nabla_u \nabla_u \xi^a = R^a_{\ ijm} u^i u^j \xi^m \quad (11)$$

to evaluate the derivative of $H$ along $u$, which is expected to be of interest as a measure of acceleration associated with the expansion of the congruence by an arbitrary observer. A straightforward computation gives

$$\frac{\nabla_u^2 \xi}{\xi} = -R_{abcd} u^a u^b u^c u^d + (\nabla_u n)^2 \quad (12)$$

or, in terms of $H$,

$$\nabla_u H = -R_{abcd} u^a u^b u^c u^d + (\nabla_u n)^2 - H^2 \quad (13)$$

**Quantum fluctuations?**

It is, of course, interesting to probe deeper the fate of geodesic incompleteness, as described by BGV theorem, in a complete theory that incorporates quantum fluctuations of matter fields as well as spacetime. In Sec. II, while discussing the issue of geodesic incompleteness of the congruence $u$, the effects of quantum fluctuations were incorporated through the (covariantly defined) lower bound $\ell_0$ on distances. The existence of a lower bound on geodesic intervals seems to be a generic consequence of combining principles of GR and quantum mechanics, and expected to be independent of any specific model/framework of quantum gravity. A mathematical formalism that incorporates this into the small structure of spacetime is presented/developed in some recent work \[2\].

It is also interesting to try to address the issue of quantum fluctuations by treating Eq. (13) as a Langevin equation sourced by a fluctuating Riemann tensor. The fluctuations might arise due to matter fields via the fields equations, or due to quantum nature of spacetime itself.

Such an analysis can be done along the lines of \[3\]. At least in linearised gravity, this would yield a specific scaling of quantum fluctuations $\Delta H$ in $H$ with $\ell_0$. Needless to say, such a scaling acquires importance not only from an observational point of view, but also in understanding better the so called cosmological constant problem.

Indeed, as is evident from the discussion in Sec. II, the parameter $\Lambda \ell_0^2$ plays an important role in the issue of geodesic (in)completeness (of the congruence).

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