A simple and accurate method to calculate square roots and cube roots for Chemistry students without calculator.

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Abstract

Presented here are two simple formulae, for the calculation of square and cube roots that are frequently encountered in Chemistry courses, at school and college levels (whether undergraduate or graduate). However, tests are difficult to be designed which do not provide for a calculator and which require the calculation of these quantities. The formulae are simple interval-weighted denominator method based to get an accurate value of these quantities. This will enable students to quickly and accurately compute square roots and cube roots.

Introduction

Square roots and cube roots appear frequently in many chemical equations. A simple interval-weighted denominator method to get an accurate value of these quantities is therefore, desirable at all levels of Chemistry programs, whether undergraduate or graduate. The method described in this paper is a convenient one that can be used in an exam setting easily, without a use of a calculator. Alternatively, quicker results can be obtained with the use of a non-programmable calculator which lacks the cube root function. The method described can be adapted to school and college Chemistry curricula with equal ease. Apart from Chemistry, this formula can well be adapted in all other fields which require the use of square roots and cube roots of numbers.

Methodology

A. Determination of Square Roots (SR):

In this method, a knowledge of the perfect squares between which the desired number lies must be known. Between 1 and 1000, there are 31 perfect squares, namely, 1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144, 169, 196, 225, 256, 289, 324, 361, 400, 441, 484, 529, 576, 625, 676, 729, 784, 841, 900 and 961. In this method:

a) Determine the numbers which are perfect squares between which the given number say, X) is present. For e.g. 85, which lies between 81 and 100.

b) Write down the various parameters in the equation (i) separately.

c) Plug the parameters into equation (i) to obtain the square root of the desired number.

\[
SR = S_1 + \frac{N}{(D - 1) + A} \quad ...eq. (i)
\]

Where SR = Square root of the desired number,

S1 = Square root of the smaller perfect square in the interval in which the desired number lies,

N = X – Smaller perfect square in the interval
D = Difference between the higher and lower perfect squares of the interval where X lies, and

A = N/D

**Illustrative Examples:**

(i) Suppose we are required to calculate the square root of 85, which lies between 81 and 100. So, the different parameters according to eq. (i) are:

X = 85

S1 = 9 (the square root of 81, the smaller perfect square of the interval)

N = 85 – 81 = 4

D = 100 – 81 = 19

A = 4/19 = 0.210

Plugging all these values in eq. (i), we get

SR = 9.220 as against the actual value of 9.220 up to three decimal places, an excellent agreement.

(ii) Suppose X =62.480, which lies between 49 and 64. In this case, we have the various parameters as:

S1 = 7 (the square root of 49)

N = 62.480 – 49 =13.48

D = 64 – 49 = 15

A = 13.480/15 = 0.899

Gives the SR = 7.904 as against the actual value of 7.904.

**B. Determination of Cube Roots (CR):**

In this method, a knowledge of the perfect cubes between which the cube root of the desired number (X) lies must be known. Thus, the interval within which the number X lies is important. As an example, let us take the determination of cube roots of numbers from 1 to 1000. Beyond 1000, the numbers can be expressed in the form p × 10^q, where p is any number in between 1 to 1000 whose cube root can be determined easily by this method and q is any natural number. In this method:

(a) There are 10 perfect cubes between 1 and 1000, both numbers included. They are 1, 8, 27, 64, 125, 216, 343, 512, 729 and 1000. The two perfect cubes between which the desired number lies must be known.

(b) Write down the various parameters in eq. (ii) separately.
\[ CR = C1 + \frac{N}{B + \left(\frac{N}{n} \times D\right)} \ldots eq(ii) \]

Where \( CR \) = the desired Cube Root of the number \( X \),

\( C1 \) = the cube root of the smaller perfect cube in the interval,

\( N = X - \) smaller perfect cube in the interval,

\( B \) = Base obtained from Table 1 or eq. (iii),

\( n = \) difference between the larger perfect cube and the smaller perfect cube in the interval,

\( D = n - B \).

(c) Base “B” for the eight intervals 8-27, 27-64, 64-125, 125-216, 216-343, 343-512, 512-729 and 729-1000 is determined for each interval separately and is calculated using eq. (iii) given in Table 1.

\[ B = 3 \times C1^2 \ldots eq(iii) \]

Where \( C1 \) is the cube root of the smaller perfect cube in each interval. On putting \( C1 = 2, 3, 4, 5, 6, 7, 8 \) and 9 we get values of \( B = 12, 27, 48, 75, 108, 147, 192 \) and 243. For the 1-8 interval, however, a value of 4 instead of 3 is used.
Table 1
Value of Base for different intervals for calculation of cube roots

| S. No | Intervals | Base |
|-------|-----------|------|
| 1     | 1-8       | 4    |
| 2     | 8-27      | 12   |
| 3     | 27-64     | 27   |
| 4     | 64-125    | 48   |
| 5     | 125-216   | 75   |
| 6     | 216-343   | 108  |
| 7     | 343-512   | 147  |
| 8     | 512-729   | 192  |
| 9     | 729-1000  | 243  |

Alternatively, a mnemonic method for the students to find the base for each of the intervals is: for the 8-27 interval where the cube root of 8 is 2 and the cube root of 27 is 3, the value of $B = 27 - 8 - (2 + 3 + 2) = 12$ or $B = 19 - (2 + 3 + 2) = 12$. Similarly, for the 27-64 interval, $B = 64 - 27 - (3 + 4 + 3) = 27$; for the 64-125 interval, $B = 125 - 64 - (4 + 5 + 4) = 48$; for the 125-216 interval, $B = 216 - 125 - (5 + 6 + 5) = 75$; for the 216-343 interval, $B = 343 - 216 - (6 + 7 + 6) = 108$; for the 343-512 interval, $B = 512 - 343 - (7 + 8 + 7) = 147$; for the 512-729 interval, $B = 729 - 512 - (8 + 9 + 8) = 192$; and for the 729-1000 interval, $B = 1000 - 729 - (9 + 10 + 9) = 243$. This has been summarized in Table 1.

In function form, the quantity $B$ can be written as

$$B = \text{Floor} \left( \frac{1}{(E + 1)^{\frac{1}{3}} - (E)^{\frac{1}{3}}} \right) \quad \text{...eq(iv)}$$

Where $B$ is the floor function in $E$ and $E$ = smaller perfect cube of the interval within which $X$ lies. This function is, however, applicable for only eight intervals from 8-27 to 729-1000. For the interval 1-8, the ceiling function may be used.

(d) Plugging these various parameters in eq. (ii) gives the desired cube root of the number $X$.

Illustrative Example:
Suppose we are required to calculate the cube root of 130 which lies between 125 and 216. So, the different parameters according to eq. (ii) are:

\[ C_1 = 5 \] (The cube root of 125)

\[ N = 130 - 125 = 5. \]

\[ B = 3 \times 5^2 = 75 \]

\[ n = 216 - 125 = 91 \]

\[ D = 91 - 75 = 16 \]

Plugging all these values in eq. (ii), we get a value of 5.066 as against the actual value of 5.066, an excellent agreement.

Results And Discussion

(i) Square Roots

The square roots of a few selected numbers from 1 to 100 calculated by eq. (i) are shown in Table 2 along with the actual square roots. The square roots of the earlier numbers, 2 and 3 present in the first interval of 1-4 are not as accurate compared to the actual values. Accuracies improve from 5, where they reach 99.9% and improve to 99.99% beyond 25, an excellent result. In numbers beyond 100, accuracies of 99.999% are achieved, considered a near perfect result in a practical setting. Such accuracies obtained from this method are acceptable in a real life chemistry setting, whether at school or college.
Table 2
A comparison of the square roots obtained from eq. (i) and the actual square roots

| Number | Square Root (SR) from eq. (i) | Actual Square Root (SR) | % Accuracy w.r.t Actual SR |
|--------|------------------------------|-------------------------|---------------------------|
| 2      | 1.429                        | 1.414                   | 98.985                    |
| 3      | 1.750                        | 1.732                   | 98.964                    |
| 5      | 2.238                        | 2.236                   | 99.909                    |
| 8      | 2.833                        | 2.828                   | 99.827                    |
| 15     | 3.875                        | 3.873                   | 99.948                    |
| 20     | 4.474                        | 4.472                   | 99.965                    |
| 26     | 5.099                        | 5.099                   | 99.998                    |
| 40     | 6.325                        | 6.325                   | 99.993                    |
| 52     | 7.211                        | 7.211                   | 99.998                    |
| 60     | 7.747                        | 7.746                   | 99.992                    |
| 62.480 | 7.904                        | 7.904                   | 99.996                    |
| 67     | 8.185                        | 8.185                   | 99.999                    |
| 75     | 8.661                        | 8.660                   | 99.994                    |
| 85     | 9.220                        | 9.220                   | 99.999                    |
| 200    | 14.142                       | 14.142                  | 100.000                   |
| 300    | 17.321                       | 17.321                  | 100.000                   |

Accurate square roots of 2 and 3 can be obtained by using the square roots of 200 and 300 and dividing the result by 10, a result that gives excellent values. In cases where the obtained and actual square root values are the same, yet percentage accuracy is not 100% because the figures have been rounded off to three decimal places whereas the accuracies have been calculated on seven decimal places, a practice usually followed in Chemistry courses around the world, where rounding off is usually done at the end.

(ii) **Cube Roots**

A comparison of the values of cube roots obtained from this method for numbers in all the nine intervals from 1-8 to 729-1000 are shown in Table 3. In each interval, a number at the beginning, one in the middle and one at the end have been taken. In this table, it can be observed that even for smaller numbers in the first interval, the determination of cube root of the number near the end of the interval is fairly high (99.9% or more). The accuracy of the middle number is also fairly high in the first interval itself and is acceptable but increases tremendously (99.9% or more) from the 64-125 interval onwards.
| Number | Cube Root (CR) from eq. (ii) | Actual Cube Root (SR) | % Accuracy w.r.t. Actual CR |
|--------|-----------------------------|-----------------------|-----------------------------|
| 2      | 1.226                       | 1.260                 | 97.292                      |
| 5      | 1.700                       | 1.710                 | 99.417                      |
| 7      | 1.913                       | 1.913                 | 99.994                      |
| 9      | 2.081                       | 2.080                 | 99.963                      |
| 18     | 2.638                       | 2.621                 | 99.357                      |
| 26     | 2.966                       | 2.962                 | 99.878                      |
| 28     | 3.037                       | 3.037                 | 99.997                      |
| 46     | 3.591                       | 3.583                 | 99.771                      |
| 63     | 3.980                       | 3.979                 | 99.973                      |
| 65     | 4.021                       | 4.021                 | 100.000                     |
| 87     | 4.435                       | 4.431                 | 99.916                      |
| 124    | 4.987                       | 4.987                 | 99.991                      |
| 130    | 5.066                       | 5.066                 | 99.998                      |
| 180    | 5.650                       | 5.646                 | 99.940                      |
| 210    | 5.945                       | 5.944                 | 99.982                      |
| 217    | 6.009                       | 6.009                 | 100.000                     |
| 280    | 6.544                       | 6.542                 | 99.966                      |
| 340    | 6.980                       | 6.980                 | 99.996                      |
| 345    | 7.014                       | 7.014                 | 100.000                     |
| 450    | 7.665                       | 7.663                 | 99.977                      |
| 510    | 7.990                       | 7.990                 | 99.998                      |
| 515    | 8.016                       | 8.016                 | 100.000                     |
| 615    | 8.505                       | 8.504                 | 99.986                      |
| 720    | 8.963                       | 8.963                 | 99.996                      |
| 730    | 9.004                       | 9.004                 | 100.000                     |
| 865    | 9.529                       | 9.528                 | 99.990                      |
| 990    | 9.967                       | 9.967                 | 99.998                      |
| Number | Cube Root (CR) from eq. (ii) | Actual Cube Root (SR) | % Accuracy w.r.t. Actual CR |
|--------|-----------------------------|-----------------------|----------------------------|
| 128    | 5.039                       | 5.040                 | 99.984                     |

On the other hand, the accuracy of the cube root of the number at the beginning of an interval increases to excellent values (99.9% or more) in the second interval itself. For smaller numbers such as 2, accurate cube root values can be obtained by multiplying and dividing by a perfect cube that can take the numerator to an interval where fairly accurate values of cube root are obtainable. For e.g. 2 can multiplied and divided by 64, wherein the denominator has a cube root of 4 whereas the numerator has been converted to 128 for which a fairly accurate (99.984%) result can be obtained. This then gives the cube root of 2 as 1.260 as against the actual value of 1.260 (Table 3). Hence, as in the square roots, if smaller numbers are converted into fractional numbers with a numerator above 100 and a denominator which is a perfect cube, accurate values of cube roots can be obtained. Subsequent values are well in agreement with the actual cube root.

On the whole, these formulae can be easily adapted to chemistry curricula, both at the school and college (undergraduate and graduate) levels and may act to supplement other methods in promoting calculator-free tests.

**Conclusions**

In conclusion, simple, accurate and adaptable formulae have been described to calculate square roots and cube roots of numbers. These formulae can be used universally in a simple school or college setting in a Chemistry course, whether General or Physical Chemistry courses wherein tests are administered without the provision of calculators for the test-takers. The various parameters described in the two formulae are easy to memorize and give accurate and quick results.

**Declarations**

Conflict of Interest:

The authors declare that they have no conflict of interest.

Competing Interests:

The authors declare no competing interests.

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**Supplementary Files**

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- Formula.xlsx