MAX-PROJECTIVE MODULES
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Abstract. A right \( R \)-module \( M \) is called \textit{max-projective} provided that each homomorphism \( f : M \to R/I \) where \( I \) is any maximal right ideal, factors through the canonical projection \( \pi : R \to R/I \). We call a ring \( R \) right almost-\( QF \) (resp. right max-\( QF \)) if every injective right \( R \)-module is \( R \)-projective (resp. max-projective). This paper attempts to understand the class of right almost-\( QF \) (resp. right max-\( QF \)) rings. Among other results, we prove that a right Hereditary right Noetherian ring \( R \) is right almost-\( QF \) if and only if \( R \) is right max-\( QF \) if and only if \( R = S \times T \), where \( S \) is semisimple Artinian and \( T \) is right small. A right Hereditary ring is max-\( QF \) if and only if every injective simple right \( R \)-module is projective. Furthermore, a commutative Noetherian ring \( R \) is almost-\( QF \) if and only if \( R \) is max-\( QF \) if and only if \( R = A \times B \), where \( A \) is \( QF \) and \( B \) is a small ring.

1. Introduction and Preliminaries

Throughout, \( R \) will denote an associative ring with identity, and modules will be unital right \( R \)-modules, unless otherwise stated. Let \( M \) and \( N \) be \( R \)-modules. \( M \) is called \( N \)-projective (projective relative to \( N \)) if every \( R \)-homomorphism from \( M \) into an image of \( N \) can be lifted to an \( R \)-homomorphism from \( M \) into \( N \). \( M \) is called \( R \)-projective if it is projective relative to the right \( R \)-module \( R \). The module \( M \) is called projective if \( M \) is \( N \)-projective, for every \( R \)-module \( N \). A right \( R \)-module \( M \) is called \textit{max-projective} provided that each homomorphism \( f : M \to R/I \) where \( I \) is any maximal right ideal, factors through the canonical projection \( \pi : R \to R/I \). This notion properly generalizes the notions \( R \)-projective and rad-projective modules studied in [3].

Characterizing rings by projectivity of some classes of their modules is a classical problem in ring and module theory. A result of Bass [6, Theorem 28.4] states that a ring \( R \) is right perfect if and only if each flat right \( R \)-module is projective. On the other hand, the ring \( R \) is \( QF \) if and only if each injective right \( R \)-module is projective, [14]. Recently, the notion of \( R \)-projectivity and its generalizations are considered in [1–3, 5, 24]. The rings whose flat right \( R \)-modules are \( R \)-projective and max-projective are characterized in [4,5] and [8], respectively.

We call a ring \( R \) right almost-\( QF \) (resp. right max-\( QF \)) in case all injective right \( R \)-modules are \( R \)-projective (resp. max-projective). Right almost \( QF \)-rings are max-\( QF \). The ring of integers is almost-\( QF \), since \( \text{Hom}(E, \mathbb{Z}/n\mathbb{Z}) = 0 \) for each injective \( \mathbb{Z} \)-module \( E \).

In this paper, we investigate some properties of max-projective \( R \)-modules, and give some characterizations of almost-\( QF \) and max-\( QF \) rings.

We organize the paper as follows. In Section 2, some properties of max-projective \( R \)-modules are investigated. We obtain that \( R \)-projectivity and max-projectivity coincide over the ring of integers and over right perfect rings. Characterizations of semiperfect, perfect and \( QF \) rings in terms of max-projectivity are given. As an application, we show that a ring

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$R$ is right (semi)perfect if and only if every (finitely generated) right $R$-module has a max-projective cover if and only if every (simple) semisimple right $R$-module has a max-projective cover. By [1, Lemma 2.1] any finitely generated $R$-projective right $R$-module is projective. This result is not true when $R$-projectivity is replaced with max-projectivity. We prove that if $R$ is either a semiperfect or nonsingular self-injective ring, then finitely generated max-projective right $R$-modules are projective. We show that any max-projective right $R$-module of finite length is projective.

In Section 3, we give some characterizations of almost-$QF$ and max-$QF$ rings. Every right small ring is right max-$QF$, while a right small ring is right almost-$QF$ provided $R$ is right Hereditary or right Noetherian. A right Hereditary right Noetherian ring $R$ is right almost-$QF$ if and only if $R$ is right max-$QF$ if and only if $R = S \times T$, where $S$ is a semisimple Artinian and $T$ is a right small ring. A right Hereditary ring $R$ is right max-$QF$ if and only if every simple injective right $R$-module is projective. A commutative Noetherian ring $R$ is almost-$QF$ if and only if $R$ is max-$QF$ if and only if $R = A \times B$, where $A$ is $QF$ and $B$ is a small ring. A right Noetherian local ring is almost-$QF$ if and only if $R$ is $QF$ or right small.

As usual, we denote by $\text{Mod}-R$ the category of right $R$-modules. For a module $M$, $E(M)$, $Z(M)$, $\text{Rad}(M)$ and $\text{Soc}(M)$ denote the the injective hull, singular submodule, Jacobson radical and socle of $M$, respectively. The notation $K \ll M$ means that $K$ is a superfluous submodule of $M$ in the sense that $K + L \neq M$ for any proper submodule $L$ of $M$.

## 2. Max-projective modules

**Definition 1.** A right $R$-module $M$ is said to be max-projective if for every epimorphism $f : R \to R/I$ with $I$ is a maximal right ideal of $R$, and every homomorphism $g : M \to R/I$, there exists a homomorphism $h : M \to R$ such that $fh = g$.

**Example 1.**

(a) Every projective $R$-module is max-projective.

(b) The $\mathbb{Z}$-module $\mathbb{Q}$ is max-projective, since $\text{Hom}(\mathbb{Q}, \mathbb{Z}_p) = 0$ for each simple $\mathbb{Z}$-module $\mathbb{Z}_p$.

(c) Every simple max-projective $R$-module is projective. For if $S$ is a simple right $R$-module and $1_S : S \to S$ is the identity map, then by max-projectivity of $S$ there is a homomorphism $f : S \to R$ such that $\pi f = 1_S$, where $\pi : R \to S$ is the natural epimorphism. Then $R \cong K \oplus S$, and so $S$ is projective.

(d) Any $R$-module $M$ with $\text{Rad}(M) = M$ is max-projective, since $M$ has no simple factors.

Given modules $M$ and $N$, $M$ is said to be $N$-subprojective if for every homomorphism $f : M \to N$ and for every epimorphism $g : B \to N$, there exists a homomorphism $h : M \to B$ such that $gh = f$ (see [16]).

**Lemma 1.** For an $R$-module $M$, the following are equivalent.

1. $M$ is max-projective.
2. $M$ is $S$-subprojective for each simple $R$-module $S$.
3. For every epimorphism $f : N \to S$ with $S$ simple, and homomorphism $g : M \to S$, there exists a homomorphism $h : M \to N$ such that $fh = g$.

**Proof.** (2) $\iff$ (3) By definition. (3) $\implies$ (1) is clear.

(1) $\implies$ (3) Let $f : N \to S$ be an epimorphism with $S$ is simple $R$-module and $g : M \to S$ a homomorphism. Since $S$ is simple, there exists an epimorphism $\pi : R \to S$. By the hypothesis there exists a homomorphism $h : M \to R$ such that $\pi h = g$. Since $R$ is projective,
there exists a homomorphism \( h' : R \to N \) such that \( fh' = \pi \). Then \( f(h'h) = \pi h = g \), and so \( M \) is max-projective.

We need the following result in the sequel.

**Lemma 2.** The following conditions are true.

1. A direct sum \( \oplus_{i \in I} A_i \) of modules is max-projective (resp. R-projective) if and only if each \( A_i \) is max-projective (resp. R-projective).
2. If \( 0 \to A \to B \to C \to 0 \) is an exact sequence and \( M \) is B-projective, then \( M \) is projective relative to both \( A \) and \( C \).

**Proof.** (1) Since it is similar to the one provided in [6, Proposition 16.10] for R-projective modules, the proof is omitted for max-projective modules.

(2) is clear by [6, Proposition 16.12].

**Corollary 1.** For a ring \( R \), the following are equivalent.

1. \( R \) is semisimple.
2. Every right \( R \)-module is max-projective.
3. Every finitely generated right \( R \)-module is max-projective.
4. Every cyclic right \( R \)-module is max-projective.
5. Every simple right \( R \)-module is max-projective.

**Proof.** (1) \( \Rightarrow \) (2) \( \Rightarrow \) (3) \( \Rightarrow \) (4) \( \Rightarrow \) (5) are clear.

(5) \( \Rightarrow \) (1) Example 1(c) and the hypothesis implies that each simple right \( R \)-module is projective. Thus \( R \) is semisimple.

In [3], the module \( M \) is called rad-projective if, for any epimorphism \( \sigma : R \to K \) where \( K \) is an image of \( R/J(R) \) and any homomorphism \( f : M \to K \), there exists a homomorphism \( g : M \to R \) such that \( f = \sigma g \). We have the following implications:

projective \( \Rightarrow \) R-projective \( \Rightarrow \) rad-projective \( \Rightarrow \) max-projective

**Proposition 1.** Let \( R \) be a semilocal ring and \( M \) an \( R \)-module. Then the following are equivalent.

1. \( M \) is rad-projective.
2. \( M \) is max-projective.
3. Every homomorphism \( f : M \to R/J(R) \) can be lifted to a homomorphism \( g : M \to R \).

**Proof.** (1) \( \Rightarrow \) (2) Clear. (3) \( \Rightarrow \) (1) By [2, Proposition 3.14].

(2) \( \Rightarrow \) (3) Since \( R/J(R) \) is semisimple, \( R/J(R) = \oplus_{i=1}^{n} K_i \), with each \( K_i \) simple as an \( R \)-module. Let \( \pi_i : \oplus_{i=1}^{n} K_i \to K_i \), and \( \pi : R \to \oplus_{i=1}^{n} K_i \). Set \( h := \pi_i \pi \). By the hypothesis, there exists a homomorphism \( g : M \to R \) such that \( hg = \pi_i f \). Since \( R/J(R) \) is semisimple, \( \pi_i \) splits and there exists a homomorphism \( \varepsilon_i : K_i \to \oplus_{i=1}^{n} K_i \) such that \( \varepsilon_i \pi_i = 1_{R/J(R)} \). Then, \( \pi g = \varepsilon_i hg = \varepsilon_i \pi_i f = f \).

In the next Proposition we provide a sufficient condition for an \( R \)-module to be max-projective. We establish a converse in the case of self-injective rings.

**Proposition 2.** If \( M \) is a right \( R \)-module such that \( \text{Ext}_R^1(M, I) = 0 \) for every maximal right ideal \( I \) of \( R \), then \( M \) is max-projective. The converse is true when \( R \) is a right self-injective ring.
Proof. By applying $\text{Hom}(M, -)$ to the short exact sequence $0 \to I \to R \to R/I \to 0$, with $I$ being a maximal right ideal of $R$, we obtain the following exact sequence:

$0 \to \text{Hom}(M, I) \to \text{Hom}(M, R) \to \text{Hom}(M, R/I) \to \text{Ext}_R^1(M, I) \to \text{Ext}_R^1(M, R) \to \ldots$

If $\text{Ext}_R^1(M, I) = 0$ for every maximal right ideal $I$ of $R$, it follows that $M$ is max-projective. Conversely, since $R$ is right self injective, $\text{Ext}_R^1(M, R) = 0$. If $M$ is a max-projective right $R$-module, then the map $\text{Hom}(M, R) \to \text{Hom}(M, R/I)$ is onto, and so $\text{Ext}_R^1(M, I) = 0$ for any maximal right ideal $I$ of $R$.

Proposition 3. Let $0 \to A \to B \to C \to 0$ be a short exact sequence. If $M$ is $A$-subprojective and $C$-subprojective, then $M$ is $B$-subprojective.

Proof. Let $\gamma : F \to B$ be an epimorphism with $F$ projective. Then using the pullback diagram of $\gamma : F \to B$ and $\beta : A \to B$, and applying $\text{Hom}(M, -)$, we get a commutative diagram with exact rows and columns:

\[
\begin{array}{ccccccccc}
0 & \to & \text{Hom}(M, K) & \to & \text{Hom}(M, X) & \to & \text{Hom}(M, A) & \to & 0 \\
0 & \to & \text{Hom}(M, K) & \to & \text{Hom}(M, F) & \to & \text{Hom}(M, B) & \to & 0 \\
& & \text{Hom}(M, C) & \to & \text{Hom}(M, C) & & & & \\
\end{array}
\]

Since $M$ is $A$-subprojective and $C$-subprojective, $\theta$ and $\phi$ are epic. Hence, $\gamma^*$ is epic by [6, Five Lemma 3.15].

Proposition 4. Let $M$ be an $R$-module. $M$ is max-projective if and only if $M$ is $N$-subprojective for any $R$-module $N$ with composition length $\text{cl}(N) < \infty$.

Proof. Let $M$ be a max-projective $R$-module and $N$ be an $R$-module with $\text{cl}(N) = n$. Then there exists a composition series $0 = S_0 \subset S_1 \ldots \subset S_n = N$ with each composition factor $S_{i+1}/S_i$ simple. Consider the short exact sequence $0 \to S_1 \to S_2 \to S_2/S_1 \to 0$. Since $M$ is max-projective, by Lemma 1, $M$ is $S_1$-subprojective and $S_2/S_1$-subprojective. So, by Proposition 3, $M$ is $S_2$-subprojective. Continuing in this way, $M$ is $S_i$-subprojective for each $0 \leq i \leq n$. Hence, $M$ is $N$-subprojective. Conversely, since each simple right $R$-module has finite length, $M$ is max-projective by Lemma 1.

Corollary 2. A $\mathbb{Z}$-module $M$ is max-projective if and only if $M$ is $\mathbb{Z}$-projective.

Proof. By the Fundamental Theorem of Abelian Groups, a cyclic $\mathbb{Z}$-module $M$ is isomorphic either to \( \mathbb{Z} \) or to a finite direct sum of $\mathbb{Z}$-modules of finite length. Now the proof is clear by Proposition 4.

Corollary 3. Let $M$ be an $R$-module with finite composition length. If $M$ is max-projective, then it is projective.
Proof. Let \( f : R^n \to M \) be an epimorphism. The module \( M \) is \( M \)-subprojective by Proposition 4. That is, there is a homomorphism \( g : M \to R^n \) such that \( 1_M = fg \). Thus the map \( f \) splits, and so \( M \) is projective. \( \square \)

Submodules of max-projective \( R \)-modules need not be max-projective. Consider the ring \( R = \mathbb{Z}/p^2 \mathbb{Z} \), for some prime integer \( p \). \( R \) is max-projective, whereas the simple ideal \( pR \) is not max-projective, since the epimorphism \( R \to pR \to 0 \) does not split.

Recall that a ring \( R \) is called right \( V \)-ring (resp. right \( GV \)-ring) if all simple (resp. all singular simple) right \( R \)-modules are injective.

**Proposition 5.** Consider the following conditions for a ring \( R \):

1. \( R \) is a right \( GV \)-ring.
2. Submodules of max-projective right \( R \)-modules are max-projective.
3. Submodules of projective right \( R \)-modules are max-projective.
4. Every right ideal of \( R \) is max-projective.

Then, (1) \( \Rightarrow \) (2) \( \Rightarrow \) (3) \( \Rightarrow \) (4). Also, if \( R \) is a right self injective ring, then (4) \( \Rightarrow \) (1).

**Proof.** (1) \( \Rightarrow \) (2) Let \( N \) be a submodule of a max-projective right \( R \)-module \( M \). Consider the following diagram:

\[
\begin{array}{c}
0 \rightarrow N \rightarrow M \\
R \rightarrow S \rightarrow 0
\end{array}
\]

where \( S \) is a simple right \( R \)-module, \( i : N \to M \) is the inclusion map and \( \pi : R \to S \) is the canonical quotient map. Since the simple module \( S \) is either projective or singular, the former implies \( \pi : R \to S \) splits and there exists a homomorphism \( \varepsilon : S \to R \) such that \( \varepsilon \pi = 1_R \). In the latter one, \( S \) is singular, so it is injective by the hypothesis. Thus, there is a homomorphism \( g : M \to S \) such that \( gi = f \). Since \( M \) is max-projective, there is a homomorphism \( h : M \to R \) such that \( \pi h = g \). Hence, \( \pi(hi) = gi = f \). In either case, there exists a homomorphism from \( N \) to \( R \) that makes the diagram commute. This implies that \( N \) is max-projective.

(2) \( \Rightarrow \) (3) \( \Rightarrow \) (4) Clear. (4) \( \Rightarrow \) (1) Let \( I \) be a right ideal of \( R \) and \( J \) a maximal right ideal of \( R \).

Consider the following diagram:

\[
\begin{array}{c}
0 \rightarrow I \rightarrow R \\
R \rightarrow R/J \rightarrow 0
\end{array}
\]

where \( R/J \) is a simple right \( R \)-module, \( i : I \to R \) is the inclusion map and \( \pi : R \to R/J \) is the canonical quotient map. Since \( I \) is max-projective, there is a homomorphism \( h : I \to R \) such that \( \pi h = f \). Since \( R \) is injective, there exists a homomorphism \( \lambda : R \to R \) such that \( \lambda i = h \). Now the map \( \beta = \pi \lambda : R \to R/J \) satisfies \( \beta i = \pi \lambda i = \pi h = f \), as required. \( \square \)

**Proposition 6.** Let \( R \) be a commutative or semilocal ring. Then pure submodules of max-projective \( R \)-modules are max-projective.

**Proof.** Let \( M \) be a max-projective (right) module and \( N \) a pure submodule of \( M \). Let \( S \) be a simple (right) module and \( f : N \to S \) be a homomorphism. Since \( S \) is pure-injective and
N is a pure submodule of \( M \), there is \( g : M \to S \) such that \( gi = f \), where \( i : N \to M \) is the inclusion map. By max-projectivity of \( M \), there is a homomorphism \( h : M \to R \) such that \( g = \pi h \), where \( \pi : R \to S \) is the natural epimorphism. Now we have \( f = gi = \pi hi \), i.e. \( hi : N \to R \) lifts \( f \). This proves that \( N \) is max-projective.

**Lemma 3.** Let \( R \) be a ring and \( \tau \) be a preradical with \( \tau(R) = 0 \). If \( M \) is a max-projective \( R \)-module, then \( M/\tau(M) \) is max-projective.

**Proof.** Let \( M \) be a max-projective \( R \)-module and \( f : M/\tau(M) \to S \) a homomorphism with \( S \) simple \( R \)-module. Consider the following diagram:

\[
\begin{array}{ccc}
M & \overset{\pi}{\longrightarrow} & M/\tau(M) \\
\downarrow{f} & & \downarrow{f} \\
R & \overset{\eta}{\longrightarrow} & S \\
\end{array}
\]

Since \( M \) is max-projective, there exists a homomorphism \( g : M \to R \) such that \( f\pi = \eta g \). Since \( g(\tau(M)) \subseteq \tau(R) = 0 \), \( \tau(M) \subseteq \ker(g) \), and so there exists a homomorphism \( h : M/\tau(M) \to R \) such that \( h\pi = g \). Now, since \( \eta h\pi = \eta g = f\pi \) and \( \pi \) is an epimorphism, \( \eta h \equiv f \), and so \( M/\tau(M) \) is max-projective.

**Remark 1.** Recall that any finitely generated \( R \)-projective module is projective, [1, Lemma 2.1]. This is not true for max-projective modules in general. Let \( R \) be a right \( V \)-ring which is not right semihereditary. Then \( R \) has a finitely generated right ideal which is not projective. By Proposition 5, each right ideal of \( R \) is max-projective.

**Proposition 7.** Let \( R \) be a right nonsingular right self-injective ring. Every finitely generated max-projective right \( R \)-module is projective.

**Proof.** Let \( M \) be a finitely generated max-projective right \( R \)-module. As \( R \) is a right nonsingular ring, by Lemma 3, \( M/Z(M) \) is max-projective. Since \( M/Z(M) \) is finitely generated, there exists an epimorphism \( f : F \to M/Z(M) \) such that \( F \) is finitely generated free. This means \( \ker(f) \) is closed in \( F \). By the injectivity of \( F \), \( \ker(f) \) is a direct summand of \( F \), and so \( M/Z(M) \) is projective. Then, \( M = Z(M) \oplus K \) for some projective submodule \( K \) of \( M \). We claim that \( Z(M) = 0 \). Assume to the contrary that \( Z(M) \neq 0 \). Since, \( Z(M) \) is a finitely generated submodule of \( M \), there exists a nonzero epimorphism \( g : Z(M) \to S \) for some simple right \( R \)-module \( S \). Then, by Lemma 2, \( Z(M) \) is max-projective, and so there exists a nonzero homomorphism \( h : Z(M) \to R \) such that \( \pi h = g \), where \( \pi : R \to S \) is the natural epimorphism. But then \( h(Z(M)) \subseteq Z(R) = 0 \), a contradiction. Thus we must have \( Z(M) = 0 \), whence \( M \) is projective.

A ring \( R \) is called **right max-ring** if every nonzero right \( R \)-module \( M \) has a maximal submodule i.e. \( \text{Rad}(M) \neq M \).

**Proposition 8.** The following conditions are true.

1. Over a semiperfect ring \( R \), every max-projective right \( R \)-module with small radical is projective.

2. A ring \( R \) is right perfect if and only if \( R \) is semilocal and every max-projective right \( R \)-module is projective.

**Proof.** (1) Let \( M \) be a max-projective right \( R \)-module with \( \text{Rad}(M) \ll M \). Since \( R \) is semilocal, \( M \) is rad-projective by Proposition 1. Hence \( M \) is projective by [2, Theorem 4.7].
(2) Since over a right perfect ring $R$ every right $R$-module has small radical, it follows from (1) that every max-projective right $R$-module is projective. Conversely, assume that $R$ is semilocal and every max-projective right $R$-module is projective. Let $M$ be a nonzero right $R$-module. We claim that $\text{Rad}(M) \neq M$. Assume to the contrary that $M$ has no maximal submodule, i.e. $\text{Rad}(M) = M$. Since $\text{Hom}(M, S) = 0$ for any simple right $R$-module, $M$ is max-projective. Thus $M$ is projective, by the hypothesis. Since projective modules have maximal submodules, this is a contradiction. Hence, every right $R$-module has a maximal submodule. Since $R$ is semilocal, $R$ is right perfect by [6, Theorem 28.4]. □

Recall that if $R$ is a right perfect ring, every $R$-projective right $R$-module is projective, [24]. Thus the following result follows from Proposition 8(2).

**Corollary 4.** Let $R$ be a right perfect ring and $M$ be a right $R$-module. Then the following are equivalent.

1. $M$ is projective.
2. $M$ is $R$-projective.
3. $M$ is max-projective.

The following Remark is an example of a right nonperfect ring $R$ such that every max-projective module is $R$-projective.

**Remark 2.** Let $K$ be a field, and $R$ the subalgebra of $K^\omega$ consisting of all eventually constant sequences in $K^\omega$. For each $i < \omega$, we let $e_i$ be the idempotent in $K^\omega$ whose $i$th component is 1 and all the other components are 0. Notice that $\{e_i : i < \omega\}$ a set of pairwise orthogonal idempotents in $R$, so $R$ is not perfect, [25, Lemma 2.3]. By [25, Lemma 2.3 and Lemma 2.4], $R/\text{Soc}(R)$ is simple $R$-module and a module $M$ is $R$-projective if and only if it is projective with respect to the projection $\pi : R \to R/\text{Soc}(R)$. Thus, an $R$-module $M$ is max-projective if and only if $M$ is $R$-projective.

The following Corollary follows from [25, Theorem 3.3] and Remark 2.

**Corollary 5.** Let $K$ be a field of cardinality $\leq 2^\omega$ and $R$ the subalgebra of $K^\omega$ consisting of all eventually constant sequences in $K^\omega$. Assume Gödel’s Axiom of Constructibility ($V = L$). Then all max-projective $R$-modules are projective.

**Lemma 4.** If $M_R$ is max-projective and $\bar{R} = R/J(R)$, then $(M/\text{Rad}(M))_{\bar{R}}$ is max-projective.

**Proof.** Let $\pi : \bar{R}_R \to K_{\bar{R}}$ be an $\bar{R}$-epimorphism with $K_{\bar{R}}$ simple $\bar{R}$-module. Consider the following diagram:

$$
\begin{array}{ccc}
M & \xrightarrow{\eta} & M/\text{Rad}(M) \\
\bar{R}_R & \xrightarrow{\pi} & K_{\bar{R}} \\
\downarrow f & & \downarrow 0 \\
0 & & 0
\end{array}
$$

Since $M$ is max-projective, there exists a homomorphism $\lambda : M_R \to \bar{R}_R$ such that $\pi \lambda = f \eta$. Since $\lambda(\text{Rad}(M)) \subseteq \text{Rad}(R/J(R)) = 0$, $\text{Rad}(M) \subseteq \ker(\lambda)$, and so there exists a homomorphism $\delta : (M/\text{Rad}(M))_{\bar{R}} \to \bar{R}_R$ such that $\delta \eta = \lambda$. Now, since $\pi \delta \eta = \pi \lambda = f \eta$ and $\eta$ is an epimorphism, $\pi \delta = f$, and so $(M/\text{Rad}(M))_{\bar{R}}$ is a max-projective $\bar{R}$-module. □

It is well-known that a ring $R$ is semiperfect if and only if every simple $R$-module has a projective cover. In the next Proposition, we extend this result by replacing projective
covers with max-projective covers. Let \( R \) be a ring and \( \Omega \) be a class of right \( R \)-modules which is closed under isomorphisms. A homomorphism \( f : P \rightarrow M \) is called an \( \Omega \)-cover of the right \( R \)-module \( M \), if \( P \in \Omega \) and \( f \) is an epimorphism with small kernel. That is to say, if \( \Omega \) is the class of max-projective right \( R \)-modules, the homomorphism \( f : P \rightarrow M \) is called max-projective cover of \( M \). With the help of an argument similar to the one provided in [3, Theorem 18], we can establish the next Proposition.

**Proposition 9.** For a ring \( R \), the following are equivalent.

1. \( R \) is semiperfect.
2. Every finitely generated right \( R \)-module has a max-projective cover.
3. Every cyclic right \( R \)-module has a max-projective cover.
4. Every simple right \( R \)-module has a max-projective cover.

**Proof.** (1) \( \Rightarrow \) (2) \( \Rightarrow \) (3) \( \Rightarrow \) (4) are clear.

(4) \( \Rightarrow \) (1) We first show that \( \bar{R} = R/J(R) \) is a semisimple ring. Let \( S \) be a simple right \( \bar{R} \)-module. By the hypothesis \( S_R \) has a max-projective cover \( P_R \), say \( f : P \rightarrow S \) with \( \text{Rad}(P) = \text{Ker}(f) \ll P \). Since \( S \) is simple and \( P/\text{Rad}(P) \cong S \), \( P/\text{Rad}(P) \) is a simple \( \bar{R} \)-module. So, \( (P/\text{Rad}(P))_{\bar{R}} \) is max-projective by Lemma 4, whence \( (P/\text{Rad}(P))_{\bar{R}} \) is projective. Consider the map \( \bar{f} : P/\text{Rad}(P) \rightarrow S \). This map induces an isomorphism. Since \( P/\text{Rad}(P) \) is projective \( \bar{R} \)-module, \( P/\text{Rad}(P) \) is the projective cover of \( S_R \). Hence, \( \bar{R} \) is a semiperfect ring. Therefore, \( \bar{R} \) is semisimple as an \( \bar{R} \)-module, and hence semisimple as an \( R \)-module. Write \( \bar{R} = R/J(R) = \oplus_{i=1}^{n} K_i \), with each \( K_i \) simple as a right \( R \)-module, and let \( L_i \) be a max-projective cover of \( K_i \), \( 1 \leq i \leq n \), as right \( R \)-modules. Now, in order to prove that \( R \) is a semiperfect ring, it is enough to show that each \( L_i \), \( 1 \leq i \leq n \), is projective as a right \( R \)-module. Clearly, \( L = \oplus_{i=1}^{n} L_i \), as a right \( R \)-module, is a max-projective cover of \( \bar{R} \). Consider the diagram

\[
\begin{array}{ccc}
L_R & \xrightarrow{2g} & R_R \\
\downarrow{f} & & \downarrow{\pi} \\
\bar{R} & \xrightarrow{\pi} & 0
\end{array}
\]

with \( f \) being the max-projective cover of \( \bar{R} \), and \( \pi \) the canonical \( R \)-epimorphism. By the max-projectivity of \( L_R \), \( f \) can be lifted to a map \( g : L_R \rightarrow \bar{R} \) such that \( \pi g = f \). Since \( \bar{R} = \text{Im}(g) + J(R) \) and \( J(R) \ll R \) we infer that \( \bar{R} = \text{Im}(g) \) and \( g \) is onto. By the projectivity of \( R \), the map \( g \) splits and \( L_R = \text{Ker}(g) \oplus A \) for a submodule \( A \) of \( L_R \). Since \( \text{Ker}(g) \subseteq \text{Ker}(f) \ll L_R \), \( \text{Ker}(g) = 0 \) and \( L_R \cong R_R \) is projective. Therefore, each \( L_i \), \( 1 \leq i \leq n \), is projective as a right \( R \)-module, and \( R \) is semiperfect.

**Proposition 10.** For a ring \( R \), the following conditions are equivalent.

1. \( R \) is right perfect.
2. Every right \( R \)-module has a max-projective cover.
3. Every semisimple right \( R \)-module has a max-projective cover.

**Proof.** (1) \( \Rightarrow \) (2) \( \Rightarrow \) (3) are clear.

(3) \( \Rightarrow \) (1) By Proposition 9, \( R \) is a semiperfect ring. Let \( M \) be a semisimple right \( R \)-module and \( f : P \rightarrow M \) be a max-projective cover of \( M \). Since \( \text{Rad}(P) = \text{Ker}(f) \ll P \), \( P \) is projective by Proposition 8(1). Thus every semisimple right \( R \)-module has a projective cover, and so \( R \) is right perfect.
Let $R$ be any ring and $M$ be an $R$-module. A submodule $N$ of $M$ is called radical submodule if $N$ has no maximal submodules, i.e. $N = \text{Rad}(N)$. By $P(M)$ we denote the sum of all radical submodules of a module $M$. For any module $M$, $P(M)$ is the largest radical submodule of $M$, and so $\text{Rad}(P(M)) = P(M)$. Moreover, $P$ is an idempotent radical with $P(M) \subseteq \text{Rad}(M)$ and $P(M/P(M)) = 0$, (see [7]).

In [11, Lemma 1], the authors prove that over a right nonsingular right $V$-ring, max-projective right $R$-modules are nonsingular. Regarding the converse of this fact, we have the following.

**Proposition 11.** If every max-projective right $R$-module is nonsingular, then $R$ is right nonsingular and right max-ring.

**Proof.** Clearly the ring $R$ is right nonsingular. If $R$ is right $V$-ring, then $\text{Rad}(M) = 0$ for any right $R$-module $M$. Thus $R$ is a max-ring. Suppose $R$ is not right $V$-ring and let $S$ be a noninjective simple right $R$-module. We shall first see that $P(E(S)) = 0$. Suppose $\text{Rad}(P(E(S))) = P(E(S)) \neq 0$. Then $P(E(S))/S$ is singular. Furthermore, since $\text{Rad}(P(E(S))/S) = P(E(S))/S$, $P(E(S))/S$ is max-projective. This contradicts with the hypothesis. Therefore, for every simple right $R$-module $S$, $P(E(S)) = 0$. Let $M$ be a nonzero right $R$-module. We claim that $\text{Rad}(M) \neq M$. Assume to the contrary that $\text{Rad}(M) = M$. Let $0 \neq x \in M$ and $K$ be a maximal submodule of $xR$. Then the simple right $R$-module $S = xR/K$ is noninjective, because $S$ small. Now, the obvious map $xR \to E(S)$ extends to a nonzero map $f : M \to E(S)$. Since $P(\text{Im}(f)) \subseteq P(E(S)) = 0$, $P(M/\text{Ker}(f)) = 0$. This contradicts with $P(M) = M$. Hence $\text{Rad}(M) \neq M$ for every right $R$-module $M$, and so $R$ is a right max-ring. □

**Corollary 6.** For a ring $R$, the following are equivalent.

1. $R$ is semilocal and every max-projective right $R$-module is nonsingular.
2. $R$ is right perfect and right nonsingular.

### 3. Almost-$QF$ and Max-$QF$ Rings

Recall that a ring $R$ is $QF$ if and only if every injective (right) $R$-module is projective (see, [14]). We slightly weaken this condition and obtain the following definition.

**Definition 2.** A ring $R$ is called right *almost-$QF$* if every injective right $R$-module is $R$-projective. We call $R$ *right max-$QF$*, if every injective right $R$-module is max-projective. Left almost-$QF$ and left max-$QF$ rings are defined similarly.

Clearly, we have the following inclusion relationship:

$$
\{\text{QF rings}\} \subseteq \{\text{right almost-QF rings}\} \subseteq \{\text{right max-QF rings}\}.
$$

**Example 2.** The ring of integers $\mathbb{Z}$, is a right almost-$QF$ but not $QF$: For every injective $\mathbb{Z}$-module $E$, we have $\text{Rad}(E) = E$. Thus $\text{Hom}(E, \mathbb{Z}/n\mathbb{Z}) = 0$, for each cyclic $\mathbb{Z}$-module $\mathbb{Z}/n\mathbb{Z}$. This means that each injective $\mathbb{Z}$-module is $\mathbb{Z}$-projective, and so $\mathbb{Z}$ is almost-$QF$.

**Remark 3.** Sandomierski [24] proved that if $R$ is a right perfect ring, then every $R$-projective right module is projective. Thus a ring $R$ is right perfect and right almost-$QF$ if and only if $R$ is $QF$. 
Proposition 12. Let $R$ and $S$ be Morita equivalent rings. Then, $R$ is right almost-QF if and only if $S$ is right almost-QF.

Proof. An $R$-module $M$ is $R$-projective if and only if $M$ is $N$-projective for any finitely generated projective $R$-module $N$. Now, by [6, propositions 21.6 and 21.8], since injectivity, relative projectivity and being finitely generated are preserved by Morita equivalence, the proof is clear. □

Lemma 5. Let $R_1$ and $R_2$ be rings. Then $R = R_1 \times R_2$ is right almost-QF (resp. right max-QF) if and only if $R_1$ and $R_2$ are both right almost-QF (resp. right max-QF).

Proof. Let $M$ be an injective right $R_1$-module. Then $M$ is an injective right $R$-module, as well as an $R$-projective module by the hypothesis. Hence, by Lemma 2, $M$ is $R_1$-projective, and so $R_1$ is right almost-QF. Similarly, $R_2$ is right almost-QF. Conversely, let $M$ be an injective right $R$-module. Since we have the decomposition $M = MR_1 \oplus MR_2$, $MR_1$ is an injective right $R$-module, whence it is an injective right $R_1$-module. On the other hand, since $(MR_2)R_1 = 0$, $MR_2$ is an $R_1$-module, and so it is an injective $R_1$-module. This means that $MR_1$ and $MR_2$ are $R_1$-projective by the hypothesis. Then, by Lemma 2, $M = MR_1 \oplus MR_2$ is $R_1$-projective. Similarly, $M$ is $R_2$-projective. Therefore, $M$ is $R$-projective. Since it is similar to the one provided for almost-QF rings, the proof is omitted for max-QF rings. □

Proposition 13. Let $R$ be a right Hereditary ring and $E$ be an indecomposable injective right $R$-module. Then the following are equivalent.

1. $E$ is $R$-projective.
2. $E$ is max-projective.
3. Either $\text{Rad}(E) = E$ or $E$ is projective.

Proof. (1) $\Rightarrow$ (2) Clear.

(2) $\Rightarrow$ (3) Assume that $\text{Rad}(E) \neq E$. Then $E$ has a simple factor module isomorphic to $R/I$. Let $f : E \rightarrow R/I$ be a nonzero homomorphism. Since $E$ is max-projective, there exists a homomorphism $g : E \rightarrow R$ such that $\text{Im}(g) \neq 0$. By the fact that $R$ is right Hereditary, $\text{Im}(g)$ is projective, whence $E \cong \text{Im}(g) \oplus K$ for some right $R$-module $K$. Since $E$ is indecomposable, either $K = 0$ or $\text{Im}(g) = 0$, where the latter case implies that $g = 0$ which is a contradiction.

In the former case $K = 0$, implying that $E$ is projective.

(3) $\Rightarrow$ (1) Conversely, if $E$ is projective then $E$ is clearly $R$-projective. Now suppose $\text{Rad}(E) = E$ and let $f : E \rightarrow R/I$ be a homomorphism. Then $f(E) = f(\text{Rad}(E)) \subseteq \text{Rad}(R/I) \ll R/I$. Moreover $f(E)$ is a direct summand of $R/I$ since $R$ is right Hereditary. Therefore $f(E) = 0$, and so $f$ can be lifted to $R$.

Lemma 6. (See [22, 3.3]) For a ring $R$ the following are equivalent.

1. $R$ is a right small ring.
2. $\text{Rad}(E) = E$ for every injective right $R$-module $E$.
3. $\text{Rad}(E(R)) = E(R)$.

Corollary 7. If $R$ is a right Hereditary right small ring, then $R$ is right almost-QF.

Proposition 14. If $R$ is a right semihereditary right small ring, then $\text{Hom}(E, R) = 0$, for any injective right $R$-module $E$. In particular, $R$ is right almost-QF if and only if $\text{Hom}(E, R/I) = 0$ for any right ideal $I$ of $R$. 
Proof. Let $E$ be an injective right $R$-module and $f \in \Hom(E, R)$. Then $f(E) = f(\Rad(E)) \subseteq J(R)$. Since $R$ is right semihereditary, $f(E)$ is absolutely pure. This means that $R/f(E)$ is flat by [18, Corollary 4.86]. Then, by [18, §4 Exercise 20], $f(E) = 0$, i.e. $\Hom(E, R) = 0$. Hence, the rest is clear. \hfill $\square$

Recall that by Example 1(d), any right small ring $R$ is right max-QF. Moreover, if $R$ is right Noetherian, we have the following.

**Proposition 15.** If $R$ is a right Noetherian and right small ring, then $R$ is right almost-QF.

**Proof.** Let $E$ be an injective right $R$-module. Then, by Lemma 6, $\Rad(E) = E$. Now let $f : E \to R/I$ be a homomorphism for any right ideal $I$ of $R$. This implies that $f(E) \subseteq R/I$ and since $\Rad(f(E)) = f(E)$, we have $\Rad(E) = f(E)$. By the right Noetherian assumption, $R/I$ is a Noetherian right $R$-module and its submodule $f(E)$ is finitely generated, i.e. $\Rad(f(E)) \neq f(E)$. Also since $\Rad(f(E)) = f(E)$, this means that $f(E) = 0$, whence $f : E \to R/I$ can be lifted to $R$. Consequently, $E$ is $R$-projective. \hfill $\square$

**Theorem 1.** Let $R$ be a right Hereditary and right Noetherian ring. Then the following are equivalent.

1. $R$ is right almost-QF.
2. $R$ is right max-QF.
3. Every injective right $R$-module $E$ has a decomposition $E = A \oplus B$ where $\Rad(A) = A$ and $B$ is projective and semisimple.
4. $R = S \times T$, where $S$ is a semisimple Artinian ring and $T$ is a right small ring.

**Proof.** (1) $\Rightarrow$ (2) Clear.
(2) $\Rightarrow$ (3) Let $E$ be an injective right $R$-module. Then $E$ has an indecomposable decomposition $E = \bigoplus_{i \in \Gamma} A_i$ where $A_i$'s are either projective or $\Rad(A_i) = A_i$ by Proposition 13. Let $\Lambda = \{ j \in \Gamma : A_j \text{ is projective} \}$. So the decomposition of $E$ can be written as $E = (\bigoplus_{j \in \Lambda} A_j) \oplus (\bigoplus_{i \in \Gamma - \Lambda} A_i)$. We claim that each $A_j$ is simple for $j \in \Lambda$. Since $A_j$ is projective for $j \in \Lambda$, $\Rad(A_j) \neq A_j$. So there exists a simple factor $B_j$ of $A_j$ i.e. $B_j \cong A_j/N \cong R/I$ for some maximal submodule $N$ of $A_j$ and for some maximal right ideal $I$ of $R$. Since $B_j$ is injective, by (2), the following diagram commutes.

\[
\begin{array}{ccc}
R & \xrightarrow{h} & R/I \\
\downarrow{g} & & \downarrow{f} \\
B_j & & 0
\end{array}
\]

With the Hereditary assumption on $R$, $\text{Im}(g) \cong B_j$ is projective and so $A_j \cong N \oplus B_j$. However $A_j$ is indecomposable, whence $N = 0$. Consequently, each $A_j$ is simple for $j \in \Lambda$.

(3) $\Rightarrow$ (1) Let $E$ be an injective right $R$-module. By the assumption, $E = A \oplus B$ where $\Rad(A) = A$ and $B$ is semisimple and projective. Since $B$ is $R$-projective, we only need to show that $A$ is $R$-projective. By the Noetherian assumption, the injective $R$-module $A$ has a decomposition $A = \bigoplus_{i \in \Gamma} A_i$ where each $A_i$ is indecomposable injective with $\Rad(A_i) = A_i$. Proposition 13 implies that each $A_i$ is $R$-projective, whence $A$ is $R$-projective by Lemma 2. Therefore, $M = A \oplus B$ is $R$-projective by Lemma 2.

(2) $\Rightarrow$ (4) Let $S$ be the sum of minimal injective right ideals of $R$. Then $S$ is injective since $R$ is right Noetherian. Thus we have the decomposition $R = S \oplus T$ for some right ideal $T$ of $R$ such that $\text{Soc}(S) = S$ and $T$ has no simple injective submodule. If $f : S \to T$ is a nonzero
homomorphism, then \( f(Soc(S)) = f(S) \subseteq Soc(T) \), where \( f(S) \) is injective by the Hereditary assumption, and so \( Soc(T) \) contains a semisimple injective direct summand \( f(S) \). This means that \( f(S) = 0 \), a contradiction. Thus, we have \( \text{Hom}(S, T) = 0 \), and so \( S \) is a two sided ideal.

On the other hand, if \( g : T \to S \) is a nonzero homomorphism, then \( T/\text{Ker}(g) \cong \text{Im}(g) \subseteq S \), and so \( T/\text{Ker}(g) \) is projective by Hereditary assumption. Also since \( S \) is a semisimple injective \( R \)-module, \( T/\text{Ker}(g) \) is semisimple injective, whence \( K/\text{Ker}(g) \) is semisimple injective for any maximal submodule \( K/\text{Ker}(g) \) of \( T/\text{Ker}(g) \). This implies that \( T/\text{Ker}(g) \cong K/\text{Ker}(g) \oplus T/K \).

Then the simple \( R \)-module \( T/K \) is injective and projective, and so \( T \) contains an isomorphic copy of a simple injective \( R \)-module \( T/K \), yielding a contradiction. Therefore, \( \text{Hom}(T, S) = 0 \), and so \( T \) is a two sided ideal. Consequently, \( R = S \oplus T \) is a ring decomposition. Now let \( E(T) \) be the injective hull of \( T \) as an \( R \)-module. The injective hull \( E(T) \) is also a \( T \)-module by the fact that \( E(T)S = 0 \). We claim that \( \text{Rad}(E(T)) = E(T) \). Suppose the contrary and let \( K \) be a maximal submodule of \( E(T) \). Then \( E(T)/K \) is injective by the Hereditary assumption and it is max-projective by (2). Since \( E(T)/K \) is a simple right \( R \)-module, it is isomorphic to \( R/I \) for some maximal right ideal \( I \) of \( R \), and so \( R/I \) is injective. Then, the isomorphism \( \alpha : E(T)/K \to R/I \) lifts to \( \beta : E(T)/K \to R \) i.e. the following diagram commutes.

\[
\begin{array}{ccc}
E(T)/K & \xrightarrow{\beta} & R/I \\
\downarrow & & \downarrow \\
R \xrightarrow{h} & & R/I \\
\end{array}
\]

Since \( \beta \) is monic and \( E(T)/K \) injective, \( U = \beta(E(T)/K) \) is a direct summand of \( R \). It is easy to see that \( U \) is also a right \( T \)-module and so \( U \subseteq T \). On the other hand, since \( U \) is minimal and injective, \( U \) is also contained in \( S \), a contradiction. So we must have \( \text{Rad}(E(T)) = E(T) \), whence \( T \ll E(T) \) by Lemma 6. This proves (4).

(4) \( \Rightarrow \) (1) Clear, by Lemma 5 and Proposition 15.

\[ \square \]

**Theorem 2.** Let \( R \) be a right Hereditary ring. Then the following are equivalent.

1. \( R \) is right max-QF.
2. Every simple injective right \( R \)-module is projective.
3. Every singular injective right \( R \)-modules is \( R \)-projective.
4. Every singular injective right \( R \)-modules is max-projective.
5. \( \text{Rad}(E) = E \) for every singular injective right \( R \)-module \( E \).
6. Every injective right \( R \)-module \( E \) can be decomposed as \( E = Z(E) \oplus F \) with \( \text{Rad}(Z(E)) = Z(E) \).

**Proof.** (1) \( \Rightarrow \) (4), (3) \( \Rightarrow \) (4) and (6) \( \Rightarrow \) (5) are clear.

(4) \( \Rightarrow \) (2) Let \( S \) be a simple injective right \( R \)-module. We claim that \( S \) is projective. Assume that \( S \) is not projective. Then it is singular and injective. This implies, by our hypothesis that \( S \) is max-projective, hence \( S \) is projective, this is a contradiction. The conclusion now follows.

(2) \( \Rightarrow \) (1) Let \( E \) be an injective right \( R \)-module and \( f : E \to S \) with \( S \) is a simple right \( R \)-module. If \( f = 0 \), there is nothing to prove. We may assume that \( f \) is a nonzero homomorphism, and so \( f \) is an epimorphism. Since \( R \) is right Hereditary, \( S \) is injective, and so by (2), \( S \) is projective. Hence, the natural epimorphism \( \pi : R \to S \) splits, i.e. there exists a homomorphism \( \eta : S \to R \) such that \( \pi \eta = 1_S \). Then, \( \pi \eta f = f \), and so \( E \) is max-projective.
(4) \(\Rightarrow\) (5) Let \(E\) be a singular injective right \(R\)-module. Assume to the contrary that \(E\) has a maximal submodule \(K\) such that \(E/K \cong R/I\) for some maximal right ideal \(I\) of \(R\). So, there is a nonzero homomorphism \(f : E \to R/I\), and by (4), there exists a nonzero homomorphism \(g : E \to R/I\) such that \(\pi g = f\), where \(\pi : R \to R/I\) is the canonical epimorphism. Since \(E\) is singular, \(\text{Im}(g)\) is singular. Moreover, \(\text{Im}(g) \subseteq R\), and so \(\text{Im}(g)\) is nonsingular. This implies that \(g(E) = 0\), yielding a contradiction.

(5) \(\Rightarrow\) (6) Let \(E\) be an injective right \(R\)-module. Since \(R\) is a right nonsingular ring, \(Z(E)\) is a closed submodule of \(E\), and so \(E = Z(E) \oplus F\) for some submodule \(F\) of \(E\). Then, by (5), \(\text{Rad}(Z(E)) = Z(E)\).

(5) \(\Rightarrow\) (3) Let \(E\) be a singular injective right \(R\)-module. This implies, by our hypothesis, that \(\text{Rad}(E) = E\). Let \(f : E \to R/I\) be homomorphism for some right ideal \(I\) of \(R\). Since \(\text{Rad}(E) = E\) and \(\text{Rad}(R/I) \neq R/I\), \(f : E \to R/I\) is not an epimorphism. By the right Hereditary assumption, \(f(E)\) is injective, and so \(f(E) \cong R/I\). This means, \(f(E) = 0\), whence \(\text{Hom}(E, R/I) = 0\) for each right ideal \(I\) of \(R\). Therefore, \(E\) is \(R\)-projective.

\[\square\]

**Proposition 16.** Let \(R\) be a local right max-QF ring. Then \(R\) is either right self-injective or right small.

**Proof.** Let \(J\) be the unique maximal right ideal of \(R\) and \(E\) be the injective hull of the ring \(R\). Assume first that \(R\) is not a small ring i.e. \(\text{Rad}(E) \neq E\). Then \(E\) has a maximal submodule \(K\) such that \(E/K \cong R/J\) and denote this isomorphism by \(f\). Consider the composition \(f\pi\) where \(\pi : E \to E/K\) is the canonical projection. Since \(R\) is right max-QF, there is a nonzero homomorphism \(g : E \to R\) such that

\[
\begin{array}{c}
E \\
\downarrow g \\
\downarrow f\pi \\
R \\
\downarrow h \\
R/J \\
\downarrow 0
\end{array}
\]

commutes. Furthermore, \(h\) is a small epimorphism and \(f\pi\) is an epimorphism, which means \(g : E \to R\) is also an epimorphism and splits. Thus, \(E \cong R \oplus T\) for some \(T\). Hence, \(R\) is a right self injective ring.

\[\square\]

**Corollary 8.** Let \(R\) be a commutative semiperfect ring. If \(R\) is max-QF, then \(R = S \times T\) where \(S\) is self-injective and \(T\) is small.

**Proof.** Let \(R\) be a commutative semiperfect ring, then by [19, Theorem 23.11], \(R = R_1 \times \ldots \times R_n\), where \(R_i\) is a local ring \((1 \leq i \leq n)\). Hence, by Lemma 5 and Proposition 16, \(R\) can be written as a direct product of local max-QF rings and every local max-QF ring either self-injective or small.

\[\square\]

**Corollary 9.** Let \(R\) be a right Noetherian local ring. Then the following are equivalent.

1. \(R\) is right almost-QF.
2. \(R\) is right max-QF.
3. \(R\) is QF or right small.

**Proof.** (1) \(\Rightarrow\) (2) Clear. (3) \(\Rightarrow\) (1) Follows from Proposition 15.

(2) \(\Rightarrow\) (3) Clear by Proposition 16, since right Noetherian right self-injective rings are QF.

\[\square\]
We do not know whether every right chain ring is almost-$QF$. But the following result will imply that each right chain ring with $P(R) = 0$ is right almost $QF$.

**Proposition 17.** Let $R$ be a right chain ring and $J = J(R)$. Then $P(R) = \bigcap_{n \geq 1} J^n$.

**Proof.** Assume first that $J^n = 0$ for some $n \in \mathbb{Z}^+$. Then $\bigcap_{n \geq 1} J^n = 0$, and so, by [13, Proposition 5.3(b)], $R$ is a right Noetherian ring with $P(R) = 0$. On the other hand if we suppose that $J^n \neq 0$ for all $n \in \mathbb{Z}^+$, then, by [13, Proposition 5.2(d)], $A = \bigcap_{n \geq 1} J^n$ is a completely prime ideal. Let us now look at the case $J \neq AJ$. Then $\frac{A}{J}$ is simple right $R$-module and $AJ \ll A$. Let $a \in A \setminus AJ$. If we have $A = aR + AJ$, then $A = aR$, whence either $A = J(R)$ or $A = 0$, by [13, Proposition 5.2(f)]. If $A = \bigcap_{n \geq 1} J^n = 0$, then $R$ is a right Noetherian ring with $P(R) = \bigcap_{n \geq 1} J^n = 0$. Otherwise, if $A = J(R) = \bigcap_{n \geq 1} J^n$, then $J = J^2$, but since $A \neq AJ$, this is not the case. If we look at the case $A = AJ$, then $A \subseteq P(R)$. Since $P(R) = P^2(R)$, $P(R)$ is a completely prime ideal of $R$, and so, by [13, Lemma 5.1], $P(R) \subseteq A$. Hence, $P(R) = A = \bigcap_{n \geq 1} J^n$. □

**Corollary 10.** Let $R$ be a right chain ring. Then $R/P(R)$ is a right almost-$QF$ ring.

**Proof.** Since $P(R)$ is an ideal of $R$, and every factor ring of a right chain ring is a right chain ring, without loss of generality we may assume that $P(R) = 0$. Then by Proposition 17 and [13, Proposition 5.3], $R$ is a right Noetherian ring. We have two cases for $J = J(R)$: if $J$ is nilpotent, then $R$ is Artinian. This implies that $R$ is right self-injective by [13, Lemma 5.4] which then yields, $R$ is $QF$. So now assume that $J$ is not nilpotent. Then $R$ is a domain by [13, Proposition 5.2(d)], whence $R$ is right small. So, $R$ is right almost-$QF$ by Proposition 15. Thus in any case $R$ is right almost-$QF$. □

We shall characterize commutative Noetherian max-$QF$ rings.

**Proposition 18.** (See [20]) Let $R$ be a commutative Noetherian ring, $P$ be a prime ideal of $R$, $E = E(R/P)$, and $A_i = \{x \in E : P^i x = 0\}$. Then:

(1) $A_i$ is a submodule of $E$, $A_i \subseteq A_{i+1}$, and $E = \bigcup A_i$.

(2) If $P$ is a maximal ideal of $R$, then $A_i \subseteq E(R/P)$ is a finitely generated $R$-module for every integer $i$.

(3) $E(R/P)$ is Artinian.

**Lemma 7.** Let $R$ be a commutative Noetherian ring, and let $E = E(R/Q)$ for a maximal ideal $Q$ of $R$. The following are equivalent.

(1) $E$ is $R$-projective.

(2) $E$ is max-projective.

(3) $\text{Rad}(E) = E$ or $E$ is projective, local and isomorphic to an ideal of $R$.

**Proof.** (1) ⇒ (2) is clear.

(2) ⇒ (3) Assume that $\text{Rad}(E) \neq E$. Since $R$ is commutative, $\text{Rad}(E) = \bigcap_{i \in \Lambda} IE$, where $\Lambda$ is the set of all maximal ideals of $R$, [6, Exercises 15.5]. Now we will see that $IE = E$ for any maximal ideal $I$ distinct from $Q$. Let $I$ be a maximal ideal distinct from $Q$. The fact $I + Q = R$ implies $I + Q^n = R$ for any $n \in \mathbb{N}$. Let $x \in E$. Then $Q^n x = 0$ for some $n \in \mathbb{N}$, by Proposition 18. We have $1 = y + z$, where $y \in I$, $z \in Q^n$, and then $x = yx \in IE$. Hence, $\text{Rad}(E) = \bigcap_{i \in \Lambda} IE = QE \neq E$. Since $R$ is commutative and $(E/QE)Q = 0$, $E/QE$ is a semisimple $R/Q$-module, and so $E/QE$ semisimple as an $R$-module. Then $E/QE$ is finitely generated by Artinianity of $E$, and hence $QE + K = E$ for some finitely generated...
submodule $K$ of $E$. Since $K$ is finitely generated, $K$ is a submodule of $A_n$ for some $n$, by Proposition 18. Thus $Q^aK = 0$. Since $QE + K = E$, $Q^{a+1}E = Q^aE$, implying $Q^aE \subseteq P(E)$. On the other hand, $Q^2E + QK = QE$, and so $Q^2E + K = E$. Continuing in this manner $Q^nE + K = E$, whence $E/Q^nE$ is finitely generated. Since $R$ is Noetherian, $P(E/Q^nE) = 0$ and so $P(E) = Q^nE$. Since $E/P(E)$ is finitely generated, $E/P(E)$ has finite composition length by Proposition 18(3). By max-projectivity of $E$ and Lemma 3, $E/P(E)$ is max-projective. Thus $E/P(E)$ is projective by Corollary 3. Then, $E = P(E) \oplus L$ for some projective submodule $L$ of $E$. Since $E$ is indecomposable and $P(E) \neq E$, $E = L$. Therefore $E$ is projective. Furthermore, since $E$ is indecomposable, the endomorphism ring of $E$ is local by [13, Lemma 2.25]. By [26, Theorem 4.2], $E$ is a local module, so it is cyclic and $R \cong E \oplus I$ for some ideal $I$ of $R$. Hence $E$ is isomorphic to an ideal of $R$. This proves (3).

(3) $\Rightarrow$ (1) is obvious. \hfill $\square$

**Lemma 8.** (See [17, 9.7]) Suppose $R$ commutative Noetherian or semilocal right Noetherian ring and $\{M_i\}_{i \in I}$ be a class of right $R$-modules. Then $\text{Rad}(\prod_{i \in I} M_i) = \prod_{i \in I} \text{Rad}(M_i)$.

**Lemma 9.** Let $R$ be a commutative Noetherian ring. Then the following are equivalent.

1. $R$ is a small ring, i.e., $R \ll E(R)$.
2. $\text{Rad}(E(S)) = E(S)$ for each simple $R$-module $S$.

**Proof.** (1) $\Rightarrow$ (2): Clear by Lemma 6.

(2) $\Rightarrow$ (1): Let $\Delta$ be a complete set of representatives of simple $R$-modules. Then $C = \bigoplus_{S \in \Delta} E(S)$ is an injective cogenerator. Then, for some index set $I$, the injective hull $E(R)$ of $R$ is a direct summand of $C^I$. By Lemma 8, $\text{Rad}(C^I) = C^I$. Since $E(R)$ is a direct summand of $C^I$, we have $\text{Rad}(E(R)) = E(R)$. Thus $R$ is a small ring by Lemma 6. \hfill $\square$

**Theorem 3.** Let $R$ be a commutative Noetherian ring. Then the following are equivalent.

1. $R$ is almost-$QF$.
2. $R$ is max-$QF$.
3. $R = A \times B$, where $A$ is $QF$ and $B$ is small.

**Proof.** (1) $\Rightarrow$ (2) is clear.

(2) $\Rightarrow$ (3) First suppose that $\text{Rad}(E(S)) = E(S)$ for all simple $R$-module $S$. Then $R$ is a small ring by Lemma 9. On the other hand, if $\text{Rad}(E(S)) \neq E(S)$ for some simple $R$-module $S$, then $E(S)$ is isomorphic to a direct summand of $R$ by Lemma 7. Let $X$ be sum of minimal ideals $U$ of $R$ with $\text{Rad}(E(U)) \neq E(U)$. Then $E(U)$ is isomorphic to an ideal of $R$. Thus without loss of generality we can assume that $E(U)$ is an ideal of $R$. Since $R$ is Noetherian, $X$ is finitely generated, and so $A = E(X) = E(U_1) \oplus \cdots E(U_n)$ where each $E(U_i)$ is an ideal of $R$. Thus $R = A \oplus B$ for some ideal $B$ of $R$. Now $A$ is injective and Noetherian, so $A$ is a $QF$ ring. On the other hand, let $V$ be a simple $B$-module, then $V$ is a simple $R$-module. Let $E(V)$ be the injective hull of $V$. As $V$ is a $B$-module, $VA = 0$. If $\text{Rad}(E(V)) \neq E(V)$, then this would imply $V \subseteq A$, by the same arguments above. Thus $\text{Rad}(E(V)) = E(V)$, and so $B$ is a small ring by Lemma 9.

(3) $\Rightarrow$ (1) Clear, by Proposition 15 and Lemma 5. \hfill $\square$

**Proposition 19.** Let $R$ be a semiperfect ring. Then the following are equivalent.

1. $R$ is right almost-$QF$ and direct sum of small right $R$-modules is small.
2. $R$ is right max-$QF$ and direct sum of small right $R$-modules is small.
3. $R$ is right almost-$QF$ and $\text{Rad}(Q) \ll Q$ for each injective right $R$-module $Q$. 

(4) $R$ is right max-$QF$ and $\text{Rad}(Q) \ll Q$ for each injective right $R$-module $Q$.

(5) $R$ is $QF$.

Proof. (1) $\Rightarrow$ (2) and (3) $\Rightarrow$ (4) Clear. (2) $\Rightarrow$ (3) and (3) $\Rightarrow$ (1) By [23, Lemma 9].

(4) $\Rightarrow$ (5) Let $M$ be an injective right $R$-module. Since $M$ is max-projective with $\text{Rad}(M) \ll M$, by Proposition 8(1), $M$ is projective. Hence $R$ is $QF$.

(5) $\Rightarrow$ (3) Let $M$ be an injective right $R$-module. By the hypothesis, $M$ is projective. Since $R$ is right Artinian, every right $R$-module has a small radical, whence $\text{Rad}(M) \ll M$. □

In [10], a submodule $N$ of a right $R$-module $M$ is called coneat in $M$ if $\text{Hom}(M, S) \rightarrow \text{Hom}(N, S)$ is epic for every simple right $R$-module $S$. In [9], $N$ is called s-pure in $M$ if $N \otimes S \rightarrow M \otimes S$ is monic for every simple left $R$-module $S$. $M$ is absolutely coneat (resp., absolutely s-pure) if $M$ is coneat (resp., s-pure) in every extension of it. If $R$ is commutative, then s-pure short exact sequences coincide with coneat short exact sequences, [15, Proposition 3.1].

Proposition 20. Consider the following conditions for a ring $R$:

(1) $R$ is right max-$QF$.

(2) Every absolutely coneat right $R$-module is max-projective.

(3) Every absolutely s-pure right $R$-module is max-projective.

(4) Every absolutely pure right $R$-module is max-projective.

Then (3) $\Rightarrow$ (4) $\Rightarrow$ (1) $\Rightarrow$ (2). Also, if $R$ is a commutative ring, then (2) $\Rightarrow$ (3).

Proof. (3) $\Rightarrow$ (4) $\Rightarrow$ (1) Clear.

(1) $\Rightarrow$ (2) Let $M$ be an absolutely coneat right $R$-module. Consider the following diagram:

$$
\begin{array}{ccc}
0 & \rightarrow & M \\
& & \downarrow{f} \\
& & E(M) \\
& \pi \downarrow & \\
R & \rightarrow & S \\
& \rightarrow & 0
\end{array}
$$

where $S$ is a simple right $R$-module, $i : M \rightarrow E(M)$ is the inclusion map and $\pi : R \rightarrow S$ is the canonical quotient map. Since $M$ coneat in $E(M)$, there is a homomorphism $g : E(M) \rightarrow S$ such that $gi = f$. Also, by (1), there exists a homomorphism $h : E(M) \rightarrow R$ such that $\pi h = g$. Hence, $(\pi h)i = gi = f$.

(2) $\Rightarrow$ (3) Let $M$ be an absolutely s-pure right $R$-module. Then $M$ is s-pure in $E(M)$. Since $R$ is commutative, $M$ is coneat in $E(M)$. Hence, $M$ is max-projective by (2). □

In [21, Lemma 1.16], it was shown that for a projective module $M$, if $M = P + K$, where $P$ is a summand of $M$ and $K \subseteq M$, then there exists a submodule $Q \subseteq K$ with $M = P \oplus Q$. By using the same method in the proof of [5, Theorem 2.8], one can prove the following result.

Proposition 21. A ring $R$ is right almost-$QF$ if and only if for every injective right $R$-module $E$, if $E = P + L$, where $P$ is a finitely generated projective summand of $E$ and $L \subseteq E$, then $E = P \oplus K$ for some $K \subseteq L$.

Let $R$ be a ring and $\Omega$ be a class of $R$-modules which is closed under isomorphic copies. Following Enochs, a homomorphism $\varphi : G \rightarrow M$ with $G \in \Omega$ is called an $\Omega$-precover of the $R$-module $M$ if for each homomorphism $\psi : H \rightarrow M$ with $H \in \Omega$, there exists $\lambda : H \rightarrow G$ such that $\varphi \lambda = \psi$.

Lemma 10. Let $R$ be a right self-injective ring. Then the following are equivalent.
(1) $R$ is right almost-QF.

(2) Every finitely generated right $R$-module has an injective precover which is $R$-projective.

(3) Every cyclic right $R$-module has an injective precover which is $R$-projective.

Proof. (1) $\implies$ (2) Let $M$ be a finitely generated right $R$-module and $g : R^n \to M$ be an epimorphism. For any homomorphism $f : E \to M$ with $E$ is injective, there exists $h : E \to R^n$ such that $gh = f$. Since $R^n$ is injective, $g$ is an injective precover of $M$.

(2) $\implies$ (3) Clear.

(3) Suppose that every right ideal of $R$ is copure-injective. First, by induction, we show that every submodule of $R^n$ is copure-injective. The case $n = 1$ follows by the hypothesis. Now suppose that $n > 1$ and every submodule of $R^{n-1}$ is copure-injective. Let $N$ be a submodule of $R^n$, and consider the exact sequence $0 \to N \to R^n \to N/(N \cap R^{n-1}) \to 0$. By induction hypothesis, $N \cap R^{n-1}$ is copure-injective, and $N/(N \cap R^{n-1}) \cong (N + R^{n-1})/R^{n-1} \subseteq R^n/R^{n-1} \cong R$ is also copure-injective. Therefore, for any injective right $R$-module $E$, consider the exact sequence $\Ext^1_R(E, N \cap R^{n-1}) \to \Ext^1_R(E, N) \to \Ext^1_R(E, N/(N \cap R^{n-1}))$. Since $\Ext^1_R(E, N \cap R^{n-1}) = \Ext^1_R(E, N/(N \cap R^{n-1})) = 0$, we have $\Ext^1_R(E, N) = 0$. Therefore, $N$ is copure-injective. Now if $M$ is a submodule of a finitely generated projective right $R$-module $P$, then there is $n \geq 1$ such that $M \subseteq P \subseteq R^n$. By the above observation, $M$ is also copure-injective. (3) $\implies$ (2) is clear. (2) $\implies$ (1) by Proposition 2. \qed

Proposition 22. Let $R$ be a ring. Then the followings are equivalent.

(1) $R$ is right almost-QF and $R_R$ is copure-injective.

(2) Every right ideal of $R$ is copure-injective.

(3) Every submodule of a finitely generated projective right $R$-module is copure-injective.

Proof. (1) $\implies$ (2) Let $E$ be an injective right $R$-module and $I$ be a right ideal of $R$. By applying $\Hom(E, -)$ to the short exact sequence $0 \to I \to R \to R/I \to 0$, we obtain the following exact sequence: $0 \to \Hom(E, I) \to \Hom(E, R) \to \Hom(E, R/I) \to \Ext^1_R(E, I) \to \Ext^1_R(E, R) \to \ldots$. Since $R_R$ is copure-injective, $\Ext^1_R(E, R) = 0$. Then the map $\Hom(E, R) \to \Hom(E, R/I)$ is onto since $E$ is $R$-projective. Hence, $\Ext^1_R(E, I) = 0$ for any injective $R$-module $E$.

(2) $\implies$ (3) Suppose that every right ideal of $R$ is copure-injective. First, by induction, we show that every submodule of $R^n$ is copure-injective. The case $n = 1$ follows by the hypothesis. Now suppose that $n > 1$ and every submodule of $R^{n-1}$ is copure-injective. Let $N$ be a submodule of $R^n$, and consider the exact sequence $0 \to N \cap R^{n-1} \to N \to N/(N \cap R^{n-1}) \to 0$. By induction hypothesis, $N \cap R^{n-1}$ is copure-injective, and $N/(N \cap R^{n-1}) \cong (N + R^{n-1})/R^{n-1} \subseteq R^n/R^{n-1} \cong R$ is also copure-injective. Therefore, for any injective right $R$-module $E$, consider the exact sequence $\Ext^1_R(E, N \cap R^{n-1}) \to \Ext^1_R(E, N) \to \Ext^1_R(E, N/(N \cap R^{n-1}))$. Since $\Ext^1_R(E, N \cap R^{n-1}) = \Ext^1_R(E, N/(N \cap R^{n-1})) = 0$, we have $\Ext^1_R(E, N) = 0$. Therefore, $N$ is copure-injective. Now if $M$ is a submodule of a finitely generated projective right $R$-module $P$, then there is $n \geq 1$ such that $M \subseteq P \subseteq R^n$. By the above observation, $M$ is also copure-injective. (3) $\implies$ (2) is clear. (2) $\implies$ (1) by Proposition 2. \qed

Proposition 23. Let $R$ be a ring. Then the followings are equivalent.

(1) $R$ is semisimple.

(2) $R$ is right almost-QF right V-ring.

(3) $R$ is right almost-QF and every submodule of an $R$-projective right module is $R$-projective.

(4) $R$ is right self-injective and every submodule of an $R$-projective right module is $R$-projective.
Proof. (1) ⇒ (2), (1) ⇒ (3) and (1) ⇒ (4) are clear.

(2) ⇒ (1) Since \( R \) is a right \( V \)-ring, every simple right \( R \)-module is injective. By the hypothesis, every simple right \( R \)-module is \( R \)-projective, whence projective.

(4) ⇒ (1) Let \( M \) be a cyclic right \( R \)-module and \( I \) a right ideal of \( R \). Consider the following diagram:

\[
\begin{array}{ccc}
0 & \longrightarrow & I \\
& & \downarrow i \\
& & R \\
& & \downarrow f \\
R & \longrightarrow & M & \longrightarrow 0
\end{array}
\]

where \( i : I \to R \) is the inclusion map and \( \pi : R \to M \) is the canonical quotient map. Since \( I \) is \( R \)-projective there exists \( h : I \to R \) such that \( \pi h = f \). By the injectivity of \( R \), there exists \( \lambda : R \to R \) such that \( \lambda i = h \). Then \((\pi \lambda) i = \pi h = f \), and \( \pi \lambda : R \to M \) is the required map.

(3) ⇒ (1) Since every simple right \( R \)-module can be embedded in an injective \( R \)-module, every simple right \( R \)-module is \( R \)-projective, and so every simple right \( R \)-module is projective. Hence, \( R \) is semisimple. \( \square \)

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