LIMIT THEOREMS FOR ITERATES OF THE SZÁSZ–MIRAKYAN OPERATOR IN PROBABILISTIC VIEW

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Abstract. The Szász–Mirakyan operator is known as a positive linear operator which uniformly approximates a certain class of continuous functions on the half line. The purpose of the present paper is to find out limiting behaviors of the iterates of the Szász–Mirakyan operator in a probabilistic point of view. We show that the iterates of the Szász–Mirakyan operator uniformly converges to a continuous semigroup generated by a second order degenerate differential operator. A probabilistic interpretation of the convergence in terms of a discrete Markov chain constructed from the iterates and a limiting diffusion process on the half line is captured as well.

1. Introduction and main results

For \( n \in \mathbb{N} \), we define a linear operator \( B_n \) acting on \( C([0,1]) \), the Banach space of all continuous functions on \([0,1]\), by

\[
B_n f(x) := \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f \left( \frac{k}{n} \right), \quad f \in C([0,1]), \ x \in [0,1].
\] (1.1)

The operator \( B_n \) was originally introduced by Bernstein in [Ber12]. Therefore, it is called the Bernstein operator and \( B_n f \) the Bernstein polynomial after his name. The Bernstein operator is famous for its remarkable property that \( B_n f \) approximates continuous functions uniformly in \( x \in [0,1] \).

**Proposition 1.1** (cf. [Ber12]). We have

\[
\lim_{n \to \infty} \max_{x \in [0,1]} |B_n f(x) - f(x)| = 0, \quad f \in C([0,1]).
\]

It turns out that Proposition 1.1 provides a constructive proof of the celebrated Weierstrass approximation theorem in a probabilistic way. Indeed, the proof is an immediate consequence of the weak law of large numbers and is referred to as an exercise of interest in many textbooks of probability theory (cf. [Kle08, Example 5.15]). We also refer to e.g., [Bus17] for basic facts related to the Bernstein operator.

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In focusing on iterates $B^k_n$ of $B_n$ itself $k$ times, it is interesting to reveal the limiting behaviors of $B^k_nf$ as $n \to \infty$ and/or $k \to \infty$. Kelisky and Rivlin first studied such kind of limit theorems for $B^k_n$ in [KR67] and obtained

$$\lim_{k \to \infty} \max_{x \in [0,1]} |B^k_n f(x) - \{f(0) + (f(1) - f(0))x\}| = 0, \quad n \in \mathbb{N}, \ f \in C([0,1]),$$

which means that $B^k_nf$ converges uniformly to a linear function which interpolates between $f(0)$ and $f(1)$ as $k \to \infty$ with fixed $n$. We also refer to [Jac09] for Kelisky–Rivlin type theorems for various kinds of positive linear operators.

Next, let us consider the case where both $k$ and $n$ tend to infinity and the ratio $k/n$ tends to some constant $t > 0$. We fix $n \in \mathbb{N}$. For $x \in [0,1]$, let $S_n = S_n(x)$ denote a random variable given by

$$\mathbb{P}(S_n(x) = k) = \binom{n}{k} x^k (1-x)^{n-k}, \quad k = 0, 1, 2, \ldots, n,$$

and $G_n = G_n(x)$ be a random function on $[0,1]$ defined by

$$G_n(x) := \frac{1}{n} S_n(x), \quad x \in [0,1].$$

Moreover, let $\{G^k_n\}_{k=1}^\infty$ be a sequence of independent copies of $G_n$. For $k \in \mathbb{N}$, we define a random function $H^k_n : [0,1] \to \mathbb{R}$ by $H^k_n := G^k_n \circ G^{k-1}_n \circ \cdots \circ G^1_n$. Then, it follows from the definition of $B_n$ that

$$\mathbb{E}[f(H^k_n(x))] = B^k_n f(x), \quad x \in [0,1], \ k = 1, 2, \ldots \quad (1.2)$$

Then, for every $x \in [0,1]$, we easily observe that the sequence $\{H^k_n(x)\}_{k=1}^\infty$ is a time-homogeneous Markov chain with values in $\{i/n \mid i = 0, 1, 2, \ldots, n\}$ whose one-step transition probability is given by

$$\mathbb{P} \left( H^k_{n+1}(x) = \frac{i}{n} \bigg| H^k_n(x) = \frac{j}{n} \right) = \binom{n}{j} \left( \frac{i}{n} \right)^j \left( 1 - \frac{i}{n} \right)^{n-j}, \quad i, j = 0, 1, 2, \ldots, n.$$

This describes the simplest stochastic model in mathematical biology, called the Wright–Fisher model for population genetics. We refer to e.g., [EK80, Chapter 10] for more details in a probabilistic point of view. Besides, Konstantopoulos, Yuan and Zazanis showed in [KYZ18] that the random curve defined by $t \mapsto H^k_n(x) \circ t \geq 0$, weakly converges to a certain diffusion process with values in $[0,1]$. The precise statement is the following.

**Proposition 1.2** (cf. [KYZ18, Theorem 3]). For $x \in [0,1]$, let $(X_t(x))_{t \geq 0}$ be the diffusion process which solves the stochastic differential equation

$$dX_t(x) = \sqrt{X_t(x)(1-X_t(x))} \, dW_t, \quad X_0(x) = x,$$
where \((W_t)_{t \geq 0}\) is a one-dimensional standard Brownian motion. Then, for every \(f \in C([0, 1])\) and \(t \geq 0\), we have

\[
\lim_{n \to \infty} \max_{x \in [0, 1]} |B_n^{[nt]} f(x) - \mathbb{E}[f(X_t(x))]| = 0.
\]

Since the Markov chain \(\{H_n^k(x)\}_{k=1}^\infty\) is absorbing and it reaches the states 0 or 1 within finite times with probability one, the limiting process \((X_t(x))_{t \geq 0}\) is also absorbed at the boundary of \([0, 1]\) (cf. [KYZ18, Lemma 1]). The stochastic process \((X_t(x))_{t \geq 0}\) is called the Wright–Fisher diffusion, which is the most fundamental model in population genetics used so as to approximate the discrete Markov chain \(\{H_n^k(x)\}_{k=1}^\infty\). See e.g., [Fel50] for an early work on this topic.

We note that Proposition 1.2 can be also proved by employing functional analytic techniques such as the celebrated Trotter’s approximation theorem (cf. [Tro58], see also [Kur69]). On the other hand, in [KYZ18], the authors apply usual techniques in stochastic calculus in order to establish limit theorems for the iterates of the Bernstein operator including Proposition 1.2. Hence, they can reveal remarkable relations between the limiting behaviors of iterates of the Bernstein operator and the stochastic phenomena behind the operator approximating continuous functions, which motivates a further study of interest.

So far, a large amount of generalizations of the Bernstein operator have been investigated extensively in various settings. There seems to be several directions to generalize the classical Bernstein operator (1.1) in view of approximations of continuous functions. Among them, it should be most natural to consider generalizations of (1.1) to the infinite interval cases such as \([0, \infty)\) and \(\mathbb{R} = (-\infty, \infty)\). We refer to e.g., [AC02], for a generalization of (1.1) to the case of \([0, \infty)\).

Let \(C([0, \infty))\) be the linear space consisting of continuous functions defined on \([0, \infty)\). The present paper focuses on the so-called Szász–Mirakyan operator on \([0, \infty)\), which was originally introduced by Mirakyan [Mir41] and Szász [Sza50] independently as a generalization of (1.1) to the case of \([0, \infty)\).

**Definition 1.3** (Szász–Mirakyan operator). For \(n \in \mathbb{N}\), we define the Szász–Mirakyan operator \(P_n\) acting on \(C([0, \infty))\) by

\[
P_n f(x) := \sum_{k=0}^\infty e^{-nx} \frac{(nx)^k}{k!} f \left( \frac{k}{n} \right), \quad f \in C([0, \infty)), \quad x \in [0, \infty).
\]

(1.3)

Compared with the case of uniform approximations of continuous functions on \([0, 1]\), it is harder to treat the infinite interval cases since continuous functions on \([0, \infty)\) such as polynomial functions are not always bounded. Therefore, we need to restrict ourselves to some function spaces weighted by bounded functions so as to make the approximation of continuous functions by \(P_n\) go well (see Proposition 2.3).
Before stating our main results, we need to fix some notations on several function spaces on \([0, \infty)\). We denote by \(C_b([0, \infty))\) the linear space of all bounded continuous functions on \([0, \infty)\), which is a Banach space with respect to the usual norm \(\|f\|_\infty = \sup_{x \in [0, \infty)} |f(x)|\). Let \(C_\infty([0, \infty))\) be the linear space of all continuous functions vanishing at infinity, which is also a Banach subspace of \(C_b([0, \infty))\). For a weight function \(w \in C([0, \infty))\) satisfying \(w(x) > 0, \ x \in [0, \infty)\), we put \(C^w_\infty([0, \infty)) := \{ f \in C([0, \infty)) \mid wf \in C_\infty([0, \infty)) \}\). It also becomes a Banach space when we endow it with the weighted norm
\[
\|f\|_{\infty, w} := \sup_{x \in [0, \infty)} |w(x) f(x)|, \quad f \in C^w_\infty([0, \infty)).
\]
For \(1 \leq r \leq \infty\), we denote by \(C^r([0, \infty))\) the linear space of all functions on \([0, \infty)\) having a continuous \(r\)-th derivative. We also put \(C^r_\infty([0, \infty)) := C_\infty([0, \infty)) \cap C^r([0, \infty))\). By \(\text{Lip}([0, \infty))\), we mean the set of all Lipschitz continuous functions on \([0, \infty)\). We also put
\[
\text{Lip}(f) := \sup_{x, y \in [0, \infty), x \neq y} \frac{|f(x) - f(y)|}{|x - y|}, \quad f \in \text{Lip}([0, \infty)).
\]
For \(\alpha \geq 1\), we define
\[
w_\alpha(x) := \frac{1}{1 + x^\alpha}, \quad x \in [0, \infty).
\] (1.4)
The first main result of the present paper is the Voronovskya-type theorem for the sequence \(\{n(P_n - I)\}_{n=1}^\infty\) on the function space weighted by \(w_\alpha\), where \(I\) stands for the identity operator.

**Theorem 1.4.** Let \(\alpha \geq 1\). For every \(f \in C^2_\infty([0, \infty))\), we have
\[
\lim_{n \to \infty} \|n(P_n f - f) - A f\|_{\infty, w_\alpha} = 0,
\]
where \(A\) is the degenerate differential operator defined by
\[
A f(x) := \begin{cases} 
\frac{x}{2} f''(x) & \text{if } x > 0 \\
0 & \text{if } x = 0
\end{cases} \quad \text{for } f \in D(A), \quad \text{where the domain } D(A) \text{ of } A \text{ is given by}
\]
\[
D(A) = \left\{ f \in C^w_\infty([0, \infty)) \cap C^2([0, \infty)) \mid \lim_{x \to 0+} x f''(x) = \lim_{x \to \infty} w_\alpha(x)(x f''(x)) = 0 \right\}. \quad (1.6)
\]
Moreover, for \(f \in C^2_\infty([0, \infty))\) with \(f'' \in \text{Lip}([0, \infty))\) and \(\alpha > 3/2\), we have
\[
\|n(P_n f - f) - A f\|_{\infty, w_\alpha} \leq \frac{1}{6 \sqrt{n}} M_\alpha \text{Lip}(f''), \quad n \in \mathbb{N},
\]
where
\[
M_\alpha := \frac{3^{3/4}(2\alpha - 3)}{2\alpha} \left( \frac{3}{2\alpha - 3} \right)^{3/2\alpha} + 4\alpha - 3 \left( \frac{3}{4\alpha - 3} \right)^{3/4\alpha}.
\]
This theorem tells us a rate of convergence of \( \{n(P_n - I)\}_{n=1}^\infty \) to the differential operator \( \mathcal{A} \) with respect to the weighted norm. Due to the degeneracy of \( \mathcal{A} \) at the boundary of \([0, \infty)\), we need to take care of the boundary condition for \( \mathcal{A} \), which is occasionally called the Wentzell-type condition.

Based on Theorem 1.4, we also establish the uniform convergence of the iterates of the Szász–Mirakyan operator to the \( C_0 \)-semigroup generated by \( \mathcal{A} \) with respect to the weighted norm, together with its rate of convergence. The following is the second main result of the present paper.

**Theorem 1.5.** For \( \alpha > 1 \), \( f \in C^{w_\alpha}_{\infty}([0, \infty)) \) and \( t \geq 0 \), we have

\[
\lim_{n \to \infty} \|P_n^{[nt]} f - P_t f\|_{\infty,w_\alpha} = 0,
\]

where \( (P_t)_{t \geq 0} \) is a contraction \( C_0 \)-semigroup on \( C^{w_\alpha}_{\infty}([0, \infty)) \) generated by \( (\mathcal{A}, D(\mathcal{A})) \) defined by (1.5) and (1.6). Furthermore, let \( D_0 \) be a subspace of \( C^{w_\alpha}_{\infty}([0, \infty)) \) given by

\[
D_0 := \{ f \in C^{w_\alpha}_{\infty}([0, \infty)) \mid f'' \in \text{Lip}([0, \infty)), P_t f \in C^{w_\alpha}_{\infty}([0, \infty)) \text{ and } (P_t f)'' \in \text{Lip}([0, \infty)) \}.
\]

If \( f \in D_0 \) and \( \alpha > 3/2 \), we have

\[
\|P_n^{[nt]} f - P_t f\|_{\infty,w_\alpha} \leq \left( \sqrt{\frac{T}{n}} + \frac{1}{n} \right) \left( \|A f\|_{\infty,w_\alpha} + \frac{1}{6\sqrt{n}} M \alpha \text{Lip}(f'') \right) + \int_0^t \frac{1}{6\sqrt{n}} M \alpha \text{Lip}((P_s f)''') \, ds.
\]

We should emphasize that our arguments in the proofs basically rely on the probabilistic approaches. In particular, the proof of Theorem 1.5 is given by the combination of the probabilistic approaches with some results in functional analysis such as Trotter’s approximation theorem (cf. [Tro58, Kur69]) and a result on its rate of convergence (cf. [CT08, CT10]). On the other hand, one wonders if or not some probabilistic interpretation of the \( C_0 \)-semigroup \( (P_t)_{t \geq 0} \) can be given, since its infinitesimal generator \( \mathcal{A} \) is a second order differential operator and it may highly relate to a continuous stochastic process called a diffusion process. After the proofs of main results, we consider such a probabilistic interpretation and obtain a relation between \( (P_t)_{t \geq 0} \) and a diffusion semigroup on \( C^{w_\alpha}_{\infty}([0, \infty)) \). Furthermore, we are going to describe this relation in terms of a time-homogeneous Markov chain constructed by the iterates of the Szász–Mirakyan operator and a diffusion process captured as its scaling limit. See Theorem 4.1.

The rest of the present paper is organized as follows: In Section 2, we review several known facts about the approximation properties of the Szász–Mirakyan operator. We show in Section 3 our two main results, Theorems 1.4 and 1.5. Section 4 discusses an interesting interpretation of main theorems from probabilistic perspectives. In particular,
we show in Theorem 4.1 that the $C_0$-semigroup obtained as a limit of the iterates of (1.3) coincides with a certain diffusion semigroup. In Section 5, we give not only a conclusion of the present paper but also further possible directions of this study.

2. Approximation properties of the Szász–Mirakyan operator

We start with the approximation properties of the Szász–Mirakyan operator $P_n$ defined by (1.3). Suppose that \( \{Y_i = Y_i(x)\}_{i=1}^{\infty} \) is a sequence of independent and identically distributed Poisson random variables with the parameter $x \in [0, \infty)$, that is,

\[
P(Y(x) = k) = e^{-x} \frac{x^k}{k!}, \quad k = 0, 1, 2, \ldots.
\]

Put $T_n(x) := Y_1(x) + Y_2(x) + \cdots + Y_n(x)$ for $n \in \mathbb{N}$. Then, the distribution of $T_n(x)$ is also Poisson with the parameter $nx$ by reproductive property. Hence, we can express (1.3) as

\[
P_n f(x) = \mathbb{E}\left[f\left(\frac{1}{n}T_n(x)\right)\right], \quad n \in \mathbb{N}, f \in C([0, \infty)), x \in [0, \infty).
\]

(2.1)

It is convenient to compute several moments of $T_n(x)$ for later use. They are given by

\[
\begin{align*}
\mathbb{E}[T_n(x)] &= nx, \\
\mathbb{E}[T_n(x)^2] &= (nx)^2 + nx, \\
\mathbb{E}[T_n(x)^3] &= (nx)^3 + 3(nx)^2 + nx, \\
\mathbb{E}[T_n(x)^4] &= (nx)^4 + 6(nx)^3 + 7(nx)^2 + nx.
\end{align*}
\]

(2.2)

We now restrict ourselves to the subspace $C_\infty([0, \infty))$. Then, we have the following.

**Proposition 2.1.** If $f \in C_\infty([0, \infty))$, then $P_n f \in C_\infty([0, \infty))$ and $\|P_n f\|_\infty \leq \|f\|_\infty$.

Moreover, with the aid of the Poisson distribution, Szász showed in [Sza50] that the similar approximation property to Proposition 1.1 holds for (1.3) as well.

**Proposition 2.2** (cf. [Sza50, Theorem 3]). For every $f \in C_\infty([0, \infty))$, we have

\[
\lim_{n \to \infty} \|P_n f - f\|_\infty = 0.
\]

We next give an approximation property of the Szász–Mirakyan operator as a linear operator acting on the weighted function space. Let $\alpha \geq 1$ and $w_\alpha$ the weight function defined by (1.4). Note that $C_\infty([0, \infty))$ is dense in $C_{w_\alpha}^{\infty}([0, \infty))$ for all $\alpha > 1$. Therefore, every dense subspace of $C_\infty([0, \infty))$ is also dense in $C_{w_\alpha}^{\infty}([0, \infty))$.

Then, we also have the following.

**Proposition 2.3** (cf. [Alt10, Theorem 6.17]). If $\alpha \geq 1$, $n \in \mathbb{N}$ and $f \in C_{w_\alpha}^{\infty}([0, \infty))$, then we have $P_n f \in C_{w_\alpha}^{\infty}([0, \infty))$ and $\|P_n f\|_{\infty, w_\alpha} \leq \|f\|_{\infty, w_\alpha}$. Moreover, we have

\[
\lim_{n \to \infty} \|P_n f - f\|_{\infty, w_\alpha} = 0, \quad f \in C_{w_\alpha}^{\infty}([0, \infty)).
\]
In fact, the latter assertion is an easy consequence of the Korovkin-type theorem. Let $0 < \lambda_1 < \lambda_2 < \lambda_3$ and $f_{\lambda_k}(x) = \exp(-\lambda_k x)$, $k = 1, 2, 3$, $x \in [0, \infty)$. Then, one has
\[
P_n f_{\lambda_k}(x) = \exp\left\{-nx \left(1 - \exp\left(-\frac{\lambda_k}{n}\right)\right)\right\}, \quad i = 1, 2, 3, \quad x \in [0, \infty),
\]
and $\|P_n f_{\lambda_k} - f_{\lambda_k}\|_{\infty, w_\alpha} \to 0$ as $n \to \infty$ holds for $k = 1, 2, 3$. Since \{f_{\lambda_1}, f_{\lambda_2}, f_{\lambda_3}\} forms a Korovkin set of $C_{\infty}^{w_\alpha}([0, \infty))$, we apply Proposition B.4 to conclude $\|P_n f - f\|_{\infty, w_\alpha} \to 0$ as $n \to \infty$ for every $f \in C_{\infty}^{w_\alpha}([0, \infty))$. The reader may consult [Alt10], which is a concise survey on Korovkin-type theorems.

3. Proofs of Theorems 1.4 and 1.5

In this section, we aim to give the proofs of Theorems 1.4 and 1.5 by putting an emphasis of the probabilistic representation (2.1) of the Szász–Mirakyan operator. At the beginning, we demonstrate the proof of Theorem 1.4.

Proof of Theorem 1.4. It follows from (2.2) that
\[
E \left(1_n T_n(x) - x\right) = 0, \quad E \left(\left(\frac{1}{n} T_n(x) - x\right)^2\right) = \frac{x}{n}.
\]

By applying the Taylor formula with the integral remainder, one has
\[
P_n f(x) - f(x)
= E \left[f \left(\frac{1}{n} T_n(x)\right) - f(x)\right]
= E \left[f'(x) \left(\frac{1}{n} T_n(x) - x\right) + \frac{1}{n} T_n(x) - t\right] f''(t) dt
= E \left[\int_x^{T_n(x)/n} \left(\frac{1}{n} T_n(x) - t\right) f''(t) dt + \frac{x}{2n} f''(x), \quad x \in [0, \infty), \right.
\]
where we used
\[
E \left[\int_x^{T_n(x)/n} \left(\frac{1}{n} T_n(x) - t\right) dt\right] = E \left[\frac{1}{2} \left(\frac{1}{n} T_n(x) - x\right)^2\right] = \frac{x}{2n}
\]
for the fourth line. Therefore, (3.1) leads to
\[
n (P_n f(x) - f(x)) - Af(x)
= nE \left[\int_x^{T_n(x)/n} \left(\frac{1}{n} T_n(x) - t\right) (f''(t) - f''(x)) dt\right] =: nE[I_n(x)]
\]
for \( n \in \mathbb{N} \) and \( x \in [0, \infty) \). Since \( f'' \in C([0, \infty)) \) is uniformly continuous, for \( \varepsilon > 0 \), we can find a sufficiently small \( \delta > 0 \) such that \( 0 < |x - y| < \delta \) implies \(|f''(x) - f''(y)| < \varepsilon\). Hence, one has

\[
\mathbb{E} \left[ I_n(x) : \left| \frac{1}{n} T_n(x) - x \right| < \delta \right] \leq \varepsilon \mathbb{E} \left[ \int_x^{T_n(x)/n} \left| \frac{1}{n} T_n(x) - t \right| \ dt : \left| \frac{1}{n} T_n(x) - x \right| < \delta \right] \leq \frac{\varepsilon x}{2n}. \tag{3.3}
\]

On the other hand, it follows from the Schwarz inequality that

\[
\mathbb{E} \left[ I_n(x) : \left| \frac{1}{n} T_n(x) - x \right| \geq \delta \right] \leq 2\|f''\|_\infty \mathbb{E} \left[ \int_x^{T_n(x)/n} \left| \frac{1}{n} T_n(x) - t \right| \ dt : \left| \frac{1}{n} T_n(x) - x \right| \geq \delta \right] \leq 2\|f''\|_\infty \mathbb{E} \left[ \left( \frac{1}{n} T_n(x) - x \right)^4 \right]^{1/2} \mathbb{P} \left( \left| \frac{1}{n} T_n(x) - x \right| \geq \delta \right)^{1/2}. \tag{3.4}
\]

By using (2.2), we see that

\[
\mathbb{E} \left[ \left( \frac{1}{n} T_n(x) - x \right)^4 \right]^{1/2} = \left( \frac{3x^2}{n^2} + \frac{x}{n^3} \right)^{1/2} \leq \frac{\sqrt{3}x}{n} + \frac{\sqrt{x}}{n\sqrt{n}}. \tag{3.5}
\]

Moreover, by applying [Can19, Theorem 1], we have

\[
\mathbb{P} \left( \left| \frac{1}{n} T_n(x) - x \right| \geq \delta \right) \leq 2 \exp \left( -\frac{n\delta^2}{2(x + \delta)} \right). \tag{3.6}
\]

Therefore, by combining (3.2) with (3.3), (3.4), (3.5), and (3.6), we obtain

\[
w_\alpha(x) \ |n (\mathcal{P}_n f(x) - f(x)) - Af(x) | \leq \frac{\varepsilon}{2} x w_\alpha(x) + \sqrt{2} \|f''\|_\infty \left( \sqrt{3} x w_\alpha(x) + \frac{1}{\sqrt{n}} \sqrt{x} w_\alpha(x) \right) \exp \left( -\frac{n\delta^2}{2(x + \delta)} \right)
\]

for \( n \in \mathbb{N} \) and \( x \in [0, \infty) \). Here, we note that

\[
\sup_{x \in [0, \infty)} x w_\alpha(x) = \frac{\alpha - 1}{\alpha} \left( \frac{1}{\alpha - 1} \right)^{1/\alpha}, \quad \sup_{x \in [0, \infty)} \sqrt{x} w_\alpha(x) = \frac{2\alpha - 1}{2\alpha} \left( \frac{1}{2\alpha - 1} \right)^{1/2\alpha} < \infty.
\]
Furthermore, it follows from a direct calculus that the function $xw_\alpha(x) \exp(-\frac{n\delta^2}{4(x+\delta)})$ has a unique maximizer $x_\ast \in (0, \infty)$. Therefore, we have

$$
\|n(P_n f - f) - Af\|_{\infty, w_\alpha} \leq \frac{\varepsilon(\alpha - 1)}{2\alpha} \left( \frac{1}{\alpha - 1} \right)^{1/\alpha} + \sqrt{2} \|f''\|_{\infty} \left( \sqrt{3x_\ast w_\alpha(x_\ast)} \exp \left( -\frac{n\delta^2}{2(x_\ast + \delta)} \right) \right)
\leq \frac{\varepsilon(\alpha - 1)}{2\alpha} \left( \frac{1}{\alpha - 1} \right)^{1/\alpha}
$$

as $n \to \infty$. Since $\varepsilon > 0$ is arbitrary, we obtain the desired convergence by letting $\varepsilon \downarrow 0$.

Next, suppose that $f''$ is Lipschitz and $\alpha > 3/2$. Then, it follows from (3.5) that

$$
\|n(P_n f(x) - f(x)) - Af(x)\|_{\infty, w_\alpha} \leq \frac{1}{6} \text{Lip}(f'') \mathbb{E} \left[ \int_{x \land T_n(x)/n} \frac{1}{n} T_n(x) - t \left| t - x \right| dt \right]
\leq \frac{1}{6} \text{Lip}(f'') \mathbb{E} \left[ \frac{1}{n} T_n(x) - x \right]^3
\leq \frac{1}{6} \text{Lip}(f'') \mathbb{E} \left[ \frac{1}{n} T_n(x) - x \right]^{3/4}
\leq \frac{1}{6} \text{Lip}(f'') \left( \frac{3x^2}{n^2} + \frac{x}{n^3} \right)^{3/4} \leq \frac{1}{6} \text{Lip}(f'') \left( \frac{x}{\sqrt{n}} \right) + \frac{x^3}{\sqrt{n}}.
$$

By noting

$$
\sup_{x \in [0, \infty)} (3^{3/4} x^{3/2} + x^{3/4}) w_\alpha(x)
= \frac{3^{3/4}(2\alpha - 3)}{2\alpha} \left( \frac{3}{2\alpha - 3} \right)^{3/2\alpha} + \frac{4\alpha - 3}{4\alpha} \left( \frac{3}{4\alpha - 3} \right)^{3/4\alpha} < \infty,
$$

we obtain

$$
\|n(P_n f - f) - Af\|_{\infty, w_\alpha} \leq \frac{1}{6\sqrt{n}} M_\alpha \text{Lip}(f''), \quad n \in \mathbb{N},
$$

which is the desired estimate. \hfill \Box

As a combination of Theorem 1.4, Trotter’s approximation theorem and a technique demonstrated in [AC02], we can give the proof of Theorem 1.5. For more details on Trotter’s approximation theorem, see Appendix A.

Proof of Theorem 1.5 Let $\alpha > 1$. Recall that $C^2_\infty([0, \infty)) \subset D(A)$ is dense in $C_\infty([0, \infty))$ by applying the (generalization of) Stone–Weierstrass theorem. Hence, it is also dense in
Moreover, by virtue of the proof of [AC97, Lemma 4.2], we can show that

\((I - A)(C^2_\infty([0, \infty)))\) is dense in \(C^\infty([0, \infty))\) and so is in \(C^{w_a}_\infty([0, \infty))\).

We now consider the linear operator \(A_*\) defined by

\[ A_*f := \lim_{n \to \infty} n(P_n f - f), \quad f \in D(A_*), \]

where the domain \(D(A_*)\) of \(A_*\) is given by

\[ D(A_*):= \left\{ f \in C^{w_a}_\infty([0, \infty)) \mid \lim_{n \to \infty} n(P_n f - f) \text{ exists in } C^{w_a}_\infty([0, \infty)) \right\}. \]

Thanks to \(C^2_\infty([0, \infty)) \subset D(A_*),\) we know that \(D(A_*)\) is dense in \(C^{w_a}_\infty([0, \infty))\). Moreover, \((I - A)(C^2_\infty([0, \infty)))\) coincides with \((I - A_*)(C^2_\infty([0, \infty)))\) and it is dense in \(C^{w_a}_\infty([0, \infty))\). Therefore, Trotter’s approximation theorem (see Theorem A.1) implies that the closure \((\overline{A_*}, D(\overline{A_*}))\) of \((A_*, D(A_*))\) generates a \(C_0\)-semigroup \((S_t)_{t \geq 0}\) on \(C^{w_a}_\infty([0, \infty))\) and we obtain

\[ \lim_{n \to \infty} \|P_n^{\text{int}} f - S_t f\|_{\infty, w_a} = 0 \]

for every \(f \in C^{w_a}_\infty([0, \infty))\) and \(t \geq 0\). On the other hand, if \(f \in C^2_\infty([0, \infty))\), we then see that the \(\overline{A_*} f = A f\) and therefore it turns out that

\[ (I - \overline{A_*})(C^2_\infty([0, \infty))) = (I - A)(C^2_\infty([0, \infty))) \]

is dense in \(C^2_\infty([0, \infty))\). Particularly, the linear operators \(I - \overline{A_*}\) and \(I - A\) are invertible and \(C^2_\infty([0, \infty))\) is a core for both \(\overline{A_*}\) and \(A\). Thus, we obtain \(D(\overline{A_*}) = D(A)\) and \(\overline{A_*} = A\). This concludes \(S_t = T_t\) for \(t \geq 0\).

The latter part is readily obtained from Proposition A.2 once we put two semi-norms \(\varphi_n\) and \(\psi_n\) as

\[ \varphi_n(f) := \|A f\|_{\infty, w_a} + \frac{1}{6\sqrt{n}} M_\alpha \text{Lip}(f''), \quad \psi_n(f) := \frac{1}{6\sqrt{n}} M_\alpha \text{Lip}(f''), \]

for \(f \in C^2_\infty([0, \infty))\).

In [AC02], Altomare and Carbone dealt with a degenerate differential operator \(A f(x) = a(x) f''(x)\) on the weighted function space \(C^w_\infty([0, \infty)), \alpha \geq 1\), and show that it generates a \(C_0\)-semigroup of positive linear operators, where the continuous function \(a(x)\) satisfies

\[ \lim_{x \to 0^+} a(x) f''(x) = \lim_{x \to \infty} w_\alpha(x)(a(x)f''(x)) = 0. \]

Moreover, they apply the result to deduce that the iterates of the (generalized) Szász–Mirakyan operator uniformly converges to the \(C_0\)-semigroup in a purely functional-analytic approach. However, our limit theorems together with its rate of convergence are based on not only a functional-analytic view but also a probabilistic one. Therefore, we come to capture an interesting characterization of the limiting semigroup in terms of a certain diffusion process, as in the next section.
4. A probabilistic interpretation of Theorem 1.5

As is seen in (1.2), the iterate of the Bernstein operator is represented as the expectation of a certain time-homogeneous Markov chain. Thanks to the representation, we obtain a characterization of the convergence of the iterates of the Bernstein operator in terms of the diffusion process given by the weak limit of the Markov chain (see Proposition 1.2).

We also give such a characterization of Theorem 1.5 by using a certain discrete Markov chain constructed by the iterates of the Szász–Mirakyan operator. Let \( G_n : [0, \infty] \to [0, \infty] \) be a random function defined by

\[
G_n(x) := \frac{1}{n} T_n(x), \quad x \in [0, \infty),
\]

and \( \{G_n^{k}\}_{k=1}^{\infty} \) a sequence of independent copies of \( G_n \). For \( k \in \mathbb{N} \), we also define a random function \( H_n^k : [0, \infty) \to [0, \infty) \) by

\[
H_n^k(x) := (G_n \circ G_n^{k-1} \circ \cdots \circ G_n^1)(x), \quad x \in [0, \infty).
\]

Then, by the definition of \( P_n \), we see that

\[
\mathbb{E}[f(H_n^k(x))] = P_n^k f(x), \quad x \in [0, \infty). \tag{4.1}
\]

For every \( x \in [0, \infty) \), we observe that the sequence \( \{H_n^k(x)\}_{k=1}^{\infty} \) is a time-homogeneous Markov chain with values in \( \{i/n \mid i = 0, 1, 2, \ldots\} \) with the one-step transition probability

\[
p\left( \frac{i}{n}, \frac{j}{n} \right) = \mathbb{P}\left( H_{n+1}^k(x) = \frac{j}{n} \mid H_n^k(x) = \frac{i}{n} \right) = e^{-\frac{ij}{j!}} i, j = 0, 1, 2, \ldots.
\]

We note that, since \( p(0,0) = 1 \) and \( p(0,i/n) = 0 \) for \( i = 1, 2, \ldots \), the state 0 is an absorbing state of \( \{H_n^k(x)\}_{k=1}^{\infty} \). By using the representation (4.1), Theorem 1.5 reads

\[
\lim_{n \to \infty} \sup_{x \in [0, \infty)} w_\alpha(x) \mathbb{E}[f(H_n^{\text{int}}(x)) - P_t f(x)] = 0
\]

for \( \alpha > 1, f \in C^w_\infty([0, \infty)) \) and \( t \geq 0 \). We now give another representation of the limiting semigroup \( (P_t)_{t \geq 0} \) in terms of a certain diffusion process.

**Theorem 4.1.** The \( C_0 \)-semigroup \( (P_t)_{t \geq 0} \) acting on \( C^w_\infty([0, \infty)) \) is represented as

\[
P_t f(x) = \mathbb{E}\left[ f(Y_t(x)) \right], \quad x \in [0, \infty), \ t \geq 0,
\]

where \( (Y_t(x))_{t \geq 0} \) is a diffusion process which is the unique strong solution to the stochastic differential equation

\[
dY_t(x) = \sqrt{Y_t(x)} \, dW_t, \quad Y_0(x) = x \in [0, \infty), \tag{4.2}
\]

with \( (W_t)_{t \geq 0} \) being a one-dimensional standard Brownian motion.
Proof. We first verify that the linear operator $A$ given by (1.5) satisfies the positive maximal principle, that is, for $f \in C^2_c([0, \infty))$, the condition $f(x_0) = \sup_{x \in [0, \infty)} f(x) \geq 0$ for some $x_0 \in [0, \infty)$ implies $Af(x_0) = (x_0/2)f''(x_0) \leq 0$. Hence, by applying the Hille–Yosida theorem (cf. [Kal02, Theorem 19.11]), we know that the closure of $(A, C^2([0, \infty)))$ generates a unique Feller semigroup $(P_t)_{t \geq 0}$ on $C_\infty([0, \infty))$.

On the other hand, we easily have

$$
\begin{align*}
&n\mathbb{E}[(H_n^{k+1}(x) - H_n^k(x))^2 | H_n^k(x) = y] = n\mathbb{E}[(G_n(y) - y)^2] = y, \\
&n\mathbb{E}[(H_n^{k+1}(x) - H_n^k(x)) | H_n^k(x) = y] = n\mathbb{E}[(G_n(y) - y)] = 0.
\end{align*}
$$

(4.3) (4.4)

Furthermore, for $\varepsilon > 0$, $R > 0$ and $0 \leq y < R$, one has

$$
\mathbb{P}(|H_n^{k+1}(x) - H_n^k(x)| > \varepsilon | H_n^k(x) = y) = \mathbb{P}(|G_n(y) - y| > \varepsilon) \leq 2 \exp \left( -\frac{n\delta^2}{2(y + \delta)} \right) < 2 \exp \left( -\frac{n\delta^2}{2(R + \delta)} \right)
$$

(4.5)

by using (3.6). Hence, it follows from (4.3), (4.4) and (4.5) that $(H_n^{[nt]}(x))_{t \geq 0}$, $n = 1, 2, \ldots$, converges weakly to the diffusion process $(Y_t(x))_{t \geq 0}$ which solves (4.2) as $n \to \infty$. Here, we have applied the convergence criteria given in [SV79] Theorem 11.2.3. In particular, we conclude that

$$
\lim_{n \to \infty} \sup_{x \in [0, \infty)} |\mathbb{E}[f(H_n^{[nt]}(x))] - \mathbb{E}[f(Y_t(x))]| = 0
$$

for $f \in C_\infty([0, \infty))$, $t \geq 0$, and

$$
P_tf(x) = \mathbb{E}[f(Y_t(x))], \quad f \in C_\infty([0, \infty)), \; t \geq 0, \; x \in [0, \infty).
$$

(4.6)

Since $C_\infty([0, \infty))$ is dense in $C^w_\infty([0, \infty))$ and each $P_t$, $t \geq 0$, is bounded, the representation (4.6) can be extended to $C^w_\infty([0, \infty))$ as well. □

5. Conclusions and further directions

Throughout the present paper, we have discussed limit theorems for the iterates of the Szász–Mirakyan operator $\mathcal{P}_n$ and have shown that the iterates of $\mathcal{P}_n$ uniformly converges to the diffusion semigroup generated by a second order degenerate differential operator. In fact, there are a few papers discussing limit theorems for the iterates of $\mathcal{P}_n$ being essentially same as our main theorems (see e.g., [AC02]). Nonetheless, we cannot find any references in which the explicit characterization of the limiting semigroup $(P_t)_{t \geq 0}$ in terms of a diffusion process is obtained. The present paper makes a significant contribution to probability theory as well as approximation theory in that we find a stochastic phenomenon behind a positive linear operator approximating certain continuous functions on the half line.
As pointed out in [KZ70], a positive linear operator represented as the expectation of a certain random variable like (1.1) and (1.3) may uniformly approximate certain continuous functions. Indeed, yet another example of such a linear operator was introduced in [Bas57]. For \( n \in \mathbb{N} \), a linear operator \( V_n \) acting on \( C([0, \infty)) \) defined by

\[
V_n f(x) := \sum_{k=0}^{\infty} \binom{n+k-1}{k} \frac{x^k}{(x+1)^{n+k}} f \left( \frac{k}{n} \right), \quad f \in C([0, \infty)), \ x \in [0, \infty),
\]

is called the Baskakov operator. As is easily seen, this operator is defined in terms of the negative binomial distribution. We also refer to [Bec78] for the approximation property of \( V_n \). By heuristic calculation, we observe that the linear operator given by \( Af(x) = (x(x+1)/2)f''(x) \) appears as the limit of \( \{n(V_n - I)\}_{n=1}^{\infty} \). We then expect a new example of limit theorems which captures another kind of diffusion process as the limit. Moreover, by highly generalizing these linear operators, we can define positive linear operators approximating certain class of continuous functions on a locally compact \( E \subset \mathbb{R} \) associated with probability measures on \( E \). The approximating properties and the convergences of the iterates of the generalized operators to some diffusion semigroups from probabilistic perspectives are to be discussed in the forthcoming paper.

On the other hand, some papers concern with multidimensional generalizations of the classical Bernstein operator and discuss their limit theorems, to say nothing of the approximating properties. See e.g., [CT08, Theorem 2.3] for a limit theorem for iterates of the multidimensional Bernstein operator on a simplex. Despite such developments, as far as we know, multidimensional generalizations have not been discussed sufficiently except for the case of the Bernstein operator. Therefore, it should be an interesting problem to define multidimensional generalizations of existing positive linear operators such as Szász–Mirakyan and Baskakov operators and to investigate their fundamental properties.

Furthermore, it is also intriguing to consider the infinite-dimensional generalizations. In fact, there exists an infinite-dimensional analogue of the multidimensional Wright–Fisher diffusion, which is known as the (measure-valued) Fleming–Viot process. See e.g., [EK93] for more details. Since the Wright–Fisher diffusion is viewed as the limit of the iterates of the Bernstein operator, we expect that there is an “operator” on the space of probability measures whose “iterates” converges to the diffusion semigroup corresponding to the Fleming–Viot process in some sense. If we find such “operators” in several infinite-dimensional settings, we can also construct several measure-valued diffusion processes through some limit theorems for their iterates.

**Appendix A. Trotter’s approximation theorem and its rate of convergence**

Trotter’s approximation theorem provides a sufficient condition that the iterates of a bounded linear operator acting on a Banach space converges to a \( C_0 \)-semigroup. We give
the statement of Trotter’s approximation theorem when the iteration of a linear operator enjoys the contraction property.

**Proposition A.1** (cf. [Tro58, Theorem 5.1], [Kur69, Theorem 2.13]). Let $(\mathcal{U}, \| \cdot \|_\mathcal{U})$ be a Banach space. Suppose that a sequence $\{T_n\}_{n=1}^\infty$ of bounded linear operators on $\mathcal{U}$ satisfies that $\|T_n\| \leq 1$ for $n \in \mathbb{N}$. Put $A_n := n(T_n - I)$, $n \in \mathbb{N}$. We define a linear operator $A$ by the closure of the limit of $A_n$. If the domain $D(A)$ is dense in $\mathcal{U}$ and the range of $\lambda - A$ is dense in $\mathcal{U}$ for some $\lambda > 0$, then there exists a $C_0$-semigroup $(T_t)_{t \geq 0}$ acting on $\mathcal{U}$ satisfying $\|T_t\| \leq 1$, $t \geq 0$, and

$$
\lim_{n \to \infty} \|T_n^{[nt]} f - T_t f\|_\mathcal{U} = 0, \quad t \geq 0, \quad f \in \mathcal{U}.
$$

(A.1)

On the other hand, as is easily seen, Proposition A.1 does not imply any quantitative estimates of (A.1). Campiti and Tacelli established in [CT08] a refinement of Proposition A.1 by giving the rate of convergence of (A.1).

**Proposition A.2** (cf. [CT08, Theorem 1.1], see also [CT10]). Suppose that $\mathcal{U}$, $\{T_n\}_{n=1}^\infty$, $\{A_n\}_{n=1}^\infty$ and $(T_t)_{t \geq 0}$ are as in Proposition A.1. Let $D$ be a dense subspace of $\mathcal{U}$. We assume that

$$
\|A_n f\|_\mathcal{U} \leq \varphi_n(f), \quad \|A_n f - A f\|_\mathcal{U} \leq \psi_n(f), \quad f \in D,
$$

where $\varphi_n, \psi_n : D \to [0, \infty)$ are semi-norms with $\psi_n(f) \to 0$ as $n \to \infty$ for $f \in D$. Then, for $t \geq 0$ and $f \in \{g \in D \mid T_t g \in D, t \geq 0\}$, we have

$$
\|T_n^{[nt]} f - T_t f\|_\mathcal{U} = \sqrt{\frac{t}{n} \varphi_n(f) + \frac{1}{n} \varphi_n(f) + \int_0^t \psi_n(T_s f) \, ds}, \quad n \in \mathbb{N}.
$$

(A.2)

Indeed, we have used Proposition A.2 in order to deduce the rate of convergence of the iterates of the Szász–Mirakyan operator (see Theorem 1.5).

**Appendix B. Korovkin-type theorems**

It is well-known that the Korovkin-type theorems provide a quite powerful sufficient condition to deduce that a sequence of positive linear operators acting on some function spaces converges strongly to the identity operator. Korovkin’s first theorem, which was first discovered by Korovkin himself in [Kor53], is stated as follows:

**Proposition B.1** (Korovkin’s first theorem). Let $e_0(x) \equiv 1$, $e_1(x) = x$ and $e_2(x) = x^2$ for $x \in [0, 1]$. If a sequence $\{L_n\}_{n=1}^\infty$ of positive linear operators acting on $C([0, 1])$ satisfies

$$
\lim_{n \to \infty} \|L_n e_i - e_i\|_\infty = 0, \quad i = 0, 1, 2,
$$

then, it holds that

$$
\lim_{n \to \infty} \|L_n f - f\|_\infty = 0, \quad f \in C([0, 1]).
$$
So far, a number of generalizations of Proposition B.1 have been established in various settings. We refer to [Alt10] and [AC94] for good surveys of this topic. In the sequel, \((\mathcal{X}, \| \cdot \|_\mathcal{X})\) is used in order to represent Banach spaces of some continuous functions on a locally compact space \(E \subset \mathbb{R}\). We give a general formulation of the Korovkin-type theorem.

**Definition B.2 (Korovkin set).** A subset \(\mathcal{K}\) of \(\mathcal{X}\) is called a Korovkin set of \(\mathcal{X}\) if for every sequence \(\{L_n\}_{n=1}^{\infty}\) of positive linear operators acting on \(\mathcal{X}\) satisfying \(\sup_{n \in \mathbb{N}} \|L_n\| < \infty\) and \(\lim_{n \to \infty} \|L_n f - f\|_\mathcal{X} = 0\), \(f \in \mathcal{K}\), then, it holds that \(\lim_{n \to \infty} \|L_n f - f\|_\mathcal{X} = 0\), \(f \in \mathcal{X}\).

Namely, Korovkin-type results aim to find out which functions form Korovkin sets of a given function space. Note that, in other words, Korovkin’s first theorem asserts that the set \(\{e_0, e_1, e_2\} \subset C([0, 1])\) is a Korovkin set of \(C([0, 1])\). We next give several examples of Korovkin sets of \(C_\infty([0, \infty))\).

**Proposition B.3 (cf. [Alt10, Corollary 6.7]).** Suppose that \(0 < \lambda_1 < \lambda_2 < \lambda_3\). Then, \(\{f_{\lambda_1}, f_{\lambda_2}, f_{\lambda_3}\}\) is a Korovkin set of \(C_\infty([0, \infty))\), where \(f_{\lambda_k}(x) := \exp(-\lambda_k x)\) for \(k = 1, 2, 3\).

Moreover, we obtain an easy way to find out Korovkin sets of \(C^w_\infty([0, \infty))\) when the weight function \(w\) is supposed to be bounded.

**Proposition B.4 (cf. [Alt10, Proposition 6.16]).** Let \(w \in C_b([0, \infty))\). Then, every Korovkin set of \(C_\infty([0, \infty))\) is also a Korovkin set of \(C^w_\infty([0, \infty))\). In particular, the set \(\{f_{\lambda_1}, f_{\lambda_2}, f_{\lambda_3}\}\) as in Proposition B.3 is a Korovkin set of \(C^w_\infty(E[0, \infty))\).

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