On the convergence of fixed point iterations for the moving geometry in a fluid-structure interaction problem

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Abstract
In this paper a fluid-structure interaction problem for the incompressible Newtonian fluid is studied. We prove the convergence of an iterative process with respect to the computational domain geometry. In our previous works on numerical approximation of similar problems we refer this approach as the global iterative method [1, 2]. This iterative approach can be understood as a linearization of the so-called geometric nonlinearity of the underlying model. The proof of the convergence is based on the Banach fixed point argument, where the contractivity of the corresponding mapping is shown due to the continuous dependence of the weak solution on the given domain deformation. This estimate is obtained by remapping the problem onto a fixed domain and using appropriate divergence-free test functions involving the difference of two solutions.

keywords: fluid-structure interaction, fixed point method, continuous dependence on data, uniqueness, Banach fixed point theorem, hemodynamics, incompressible Newtonian fluid

1 Introduction
Mathematical analysis and numerical simulation of fluid-structure interaction (FSI) problems is an intensively studied part of the computational fluid dynamics. In FSI problems the computational domain deforms under the fluid
forces. Thus, the domain deformation, governed by a structure equation, depends on the solution of the fluid equation. In the literature this dependence is referred to as the geometric nonlinearity. One possible strategy to find a solution to such coupled problem is a linearization of the problem with respect to the geometric nonlinearity. The problem is solved at first for a known domain, deformed according to a given, sufficiently smooth deformation function. The second step is to prove the existence of a fixed point for the mapping between the domain deformation and the solution.

In our previous joined works on this topic we proved the existence and uniqueness of the weak solution for an approximation of the fully coupled fluid-structure interaction problem for the Navier-Stokes equations and a parabolic (viscoelastic) equation for the boundary deformation, see [3, 4]. In these works a construction of the unknown domain through the iterative process with respect to the domain geometry has been proposed. For a semi-pervious approximation of the coupling condition, the convergence of this iterative process has been shown in [3] using the Banach fixed point argument. However, it was not possible to prove the convergence of this iterative process for the fully coupled original problem due to the lack of regularity of the domain deformation.

Further results on the existence of a weak solution to the fully coupled FSI problem for the Newtonian fluid and a viscoelastic/elastic plate in 2D and 3D have been obtained by Grandmont et al. [5, 6]. In the case of a two-dimensional fluid flow and one-dimensional structure the authors obtained in [5, 6] the existence of a weak solution for the same regularity of the domain deformation as in [3] using the Schauder fixed point argument, without giving any details on the construction of the domain deformation and without providing the uniqueness of the solution. Using a similar fixed point procedure for the geometry the existence of weak solution for a fluid-structure interaction problem for a generalized power-law shear-dependent fluid including the Newtonian case has been shown in our recent joint works [7, 8], see also [9].

The continuous dependence of the weak solution on the initial data and the uniqueness have been shown for the incompressible Newtonian fluid by Padula et al. in [10]. They considered a transformation of the solution between two domains with different deformations that preserves the solenoidal property of the solution. For further works on the mathematical analysis of similar FSI problems for Newtonian as well as non-Newtonian fluids we refer also to [9, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20] and the references therein.
In this work we will deal with the geometric nonlinearity and study the iterative process with respect to the domain geometry for the fully coupled FSI problem presented in the following section.

1.1 Mathematical model

We consider a fluid flow problem in a two-dimensional channel representing a longitudinal cut of a vessel. The fluid flow is governed by the momentum and the continuity equation, written in Cartesian coordinates for the sake of simplicity,

\[
\rho \partial_t \mathbf{v} + \rho (v \cdot \nabla) \mathbf{v} - \text{div}[2\mu e(\mathbf{v})] + \nabla p = 0, \quad \text{div} \mathbf{v} = 0. \tag{1}
\]

Here \( \rho \) denotes the constant density of the fluid, \( \mathbf{v} = (v_1, v_2)^T \) the velocity vector, \( p \) the pressure and \( e(\mathbf{v}) = \frac{1}{2}(\nabla \mathbf{v} + (\nabla \mathbf{v})^T) \) the symmetric deformation tensor. We consider the fluid viscosity \( \mu \) to be constant.

The radial vessel wall deformation \( \eta \) is modelled using the one-dimensional viscoelastic string model. The deformation \( \eta \) is a function of longitudinal variable \( x_1 \) and time \( t \) governed by the following structure equation

\[
E \rho \left[ \frac{\partial^2 \eta}{\partial t^2} - a \frac{\partial^2 \eta}{\partial x_1^2} + b \eta + c \frac{\partial^5 \eta}{\partial t \partial x_1^4} - a \frac{\partial^2 R_0}{\partial x_1^2} \right] (x_1, t) = -\mathbf{T}_f n \cdot e_2 - P_w (x_0(x_1) + \eta(x_1, t), t),
\]

\[ x_1 \in (0, L), t \in (0, T). \]

Here \( E = \rho_w h \sqrt{1 + (\partial x_1 R_0)^2} \), where \( \rho_w \) is the density of the wall tissue and \( h \) its thickness. We assume that \( E \) is bounded and \( a, b, c \) are positive constants describing the mechanical properties of the vessel wall tissue. Further, \( R_0(x_1) \) stands for the reference radius of the cylinder (vessel), \( P_w \) the external pressure acting on the deformable vessel wall, \( \mathbf{n} \) the normal vector to the moving vessel wall and \( \mathbf{T}_f = 2\mu e(\mathbf{v}) - p \mathbf{I} \) the Cauchy stress tensor. Similar models for the wall deformation have been considered in [5, 20], see also [21, 22, 23].

The moving wall deforms under the fluid action, thus the computational domain for the fluid is determined by the unknown deformation \( \eta \), i.e.,

\[
\Omega(\eta(t)) \equiv \{(x_1, x_2); 0 < x_1 < L, 0 < x_2 < R_0(x_1) + \eta(x_1, t)\}, \quad 0 < t < T,
\]
where $R_0(x_1)$ describes the reference position of the deformable wall. The boundary of $\Omega(\eta(t))$ consists of one moving part $\Gamma_w(t)$,

$$\Gamma_w(t) \equiv \{(x_1, x_2); \ x_2 = R_0(x_1) + \eta(x_1, t), \ x_1 \in (0, L)\},$$

and three fixed boundaries $\Gamma_{in}$, $\Gamma_{out}$ and $\Gamma_c$ denoting the inflow, the outflow and the bottom boundary, respectively, see Fig. 1.1.

The fluid problem (1) and the structure equation (2) are coupled through the natural kinematic condition describing the continuity of the velocities on the moving interface $\Gamma_w(t)$,

$$v(x_1, R_0(x_1) + \eta(x_1, t), t) = (0, \partial_\eta(x_1, t)).$$ (3)

Moreover, the external force causing the wall deformation arises due to the fluid stress. Thus, the second coupling condition, expressed by the right-hand side of (2), has dynamical character and can be understood as the continuity of stresses.

We complete the system (1)–(3) with Neumann-type boundary conditions with kinematic pressure at the inflow $\Gamma_{in}$ and outflow boundary $\Gamma_{out}$,

$$\left(2\mu \frac{\partial v_1}{\partial x_1} - p + P_{in} - \frac{\rho}{2} |v_1|^2\right)(0, x_2, t) = 0, \quad v_2(0, x_2, t) = 0, \quad (4)$$

$$\left(2\mu \frac{\partial v_1}{\partial x_1} - p + P_{out} - \frac{\rho}{2} |v_1|^2\right)(L, x_2, t) = 0, \quad v_2(L, x_2, t) = 0. \quad (5)$$

Here $P_{in}$, $P_{out}$ are some given pressures and $0 < x_2 < R_0(\tilde{x}_1) + \eta(\tilde{x}_1, t)$ with $\tilde{x}_1 = 0, L$. On the bottom boundary $\Gamma_c$ the flow symmetry is considered,

$$v_2(x_1, 0, t) = 0, \quad \mu \frac{\partial v_1}{\partial x_2}(x_1, 0, t) = 0, \quad 0 < x_1 < L, \quad 0 < t < T. \quad (6)$$
We equip equation (2) with the following clamped boundary conditions,
\[ \eta(0, t) = \eta(L, t) = 0, \quad \eta_x(0, t) = \eta_x(L, t) = 0. \] (7)
The initial conditions for the fluid (1) and structure (2) equations read as follows,
\[ \mathbf{v}(x_1, x_2, 0) = 0 \quad \text{for any } 0 < x_1 < L, \]
\[ 0 < x_2 < R_0(x_1), \]
\[ \eta(x_1, 0) = \eta_x(x_1, 0) = \eta_t(x_1, 0) = 0 \quad \text{for any } 0 < x_1 < L. \] (8)
The problem (1)–(8) represents a mathematical model for the flow of an incompressible Newtonian fluid in a deformable domain, where the deformation of the upper wall obeys the viscoelastic string model. Such a fully coupled fluid-structure interaction problem can be used in hemodynamics to model the blood flow in (large) elastic vessels. The geometry of computational domain \( \Omega(\eta) \) depends on unknown solution \( \eta \).

As already mentioned, one possible and commonly used way to show the existence result for such fully coupled FSI problems is to linearize the problem with respect to the geometric nonlinearity and consequently to find a fixed point of this nonlinearity. At the beginning, a smooth enough domain deformation function \( \delta \) is given. The computational domain is then approximated by this function, \( \Omega(\eta) \approx \Omega(\delta) \). Next, existence of a weak solution \((\mathbf{v}, \eta)\) defined on \( \Omega(\delta) \) is proven. Finally, a fixed point of the mapping \( \mathcal{F}(\delta) = \eta \) defined by the weak formulation on \( \Omega(\delta) \) is found. By applying the Schauder fixed point theorem for the mapping \( \mathcal{F} \) it can be shown that at least one \( \eta^* \) s.t. \( \eta^* = \mathcal{F}(\eta^*) \) exists. Furthermore, if the assumptions of the Banach fixed point theorem are satisfied, the uniqueness of the fixed point and the convergence of the iterative process \( \mathcal{F}(\eta^{k-1}) = \eta^k \) can be obtained additionally. The Banach fixed point theorem is based on the contractivity of the mapping \( \mathcal{F} \), which in our case follows from the continuous dependence of the weak solution \((\mathbf{v}, \eta)\) on the given domain deformation function \( \delta \).

Let us note that in [3] the convergence of the iterative process described above and the uniqueness of the fixed point have been obtained only for a semi-pervious and pseudo-compressible approximation of the original fluid-structure interaction problem. The right-hand side of structure equation (2) has been replaced by \( \kappa(v_2 - \eta_t) \) for a fixed \( \kappa \in \mathbb{R} \). This implies more regularity for \( \eta \) than in the original case (2). The continuous dependence on data and
consequently the contractivity of the fixed point mapping has been shown in [3] only for non-solenoidal solutions and fixed $\kappa$. We would like to point out here, that by letting $\kappa \to \infty$ the original coupled problem with conditions (2), (3) can be obtained. However, existence of the weak solution for this case has been studied in [3] only for the known domain $\Omega(\delta)$.

A significant difficulty in the proof of the continuous dependence on data for incompressible fluids lies in the requirement of divergence-free test functions involving the difference of two solutions defined in two different domains. One has to test with transformed solutions, as in [10]. As a consequence, the matrix of the transformation between two domains is reflected in the terms to be estimated. This leads to technical difficulties when fluid is considered to be non-Newtonian, e.g., with shear-dependent viscosity obeying the power-law $\mu(|e(v)|) = \mu(1 + |e(v)|^2)^{\frac{p-2}{2}}, p > 1$. The application of the Banach fixed point argument for FSI problems for power-law fluids is up to authors knowledge an open problem.

The main goal of this paper is to show the convergence of the above described iterative process for the Newtonian fluid. Here we prove the essential estimate for the continuous dependence on data and then we apply the Banach fixed point theorem to show the convergence of this iterative method, which has not yet been done for the fully coupled FSI problems up to our knowledge. As a consequence we also get uniqueness of the weak solution. We refer to this iterative process as the global iterative method with respect to the domain deformation and use it in our numerical approximation of the proposed problem, see, e.g., [1, 2]. In these numerical studies the convergence of these iterations has been observed experimentally. The present paper provides the first theoretical proof of this convergence for incompressible Newtonian fluids.

The paper is organized as follows. In Section 2 we introduce the linearization of the problem (1)–(8) with respect to the so-called geometric nonlinearity and present known results on the existence of weak solutions on a given deformable domain $\Omega(\delta)$. In Section 3 we derive the estimate on continuous dependence of the weak solution on the boundary pressure and the given deformation. This estimate is formulated in Theorem 3.1. Finally, in Section 4 we apply the Banach fixed point theorem and prove the convergence of the global iterative method.
2 Linearization with respect to domain geometry \( \Omega(h) \)

In order to linearize the problem with respect to the geometric nonlinearity we assume to have a given deformation function \( \delta(x_1, t) \) in the space \( H^1(0, T; H_0^2(0, L)) \cap W^{1,\infty}(0, T; L^2(0, L)) \). We denote \( h := R_0 + \delta, \ R_0(x_1) \in C^2[0, L] \), and assume

\[
0 < \alpha \leq h(x_1, t) \leq \alpha^{-1}, \quad \left| \frac{\partial h(x_1, t)}{\partial x_1} \right| + \int_0^T \left| \frac{\partial h(x_1, t)}{\partial t} \right|^2 dt \leq K
\]

for given constants \( \alpha > 0, K > 0 \). We consider a time dependent domain, which deforms according to the given function \( \delta \), i.e. we approximate \( \Omega(\eta) \approx \Omega(\delta) =: \Omega(h) \). Due to this approximation the geometric coupling in the original fluid-structure interaction problem (1)–(8) has been decoupled.

2.1 Existence of a weak solution

The existence of a weak solution of the problem (1)–(8) defined on the given domain with deformation \( \delta \), i.e. on \( \Omega(h) \) has been shown in [7, Theorem 5.1] using so called \((\kappa, \varepsilon)\) - approximation, see also \[\text{[A]}\].

More precisely, for boundary pressures \( q_{\text{in}}, q_{\text{out}} \in L^2(0, T; L^2(0, 1)) \) and \( q_w \in L^2(0, T; L^2(0, L)) \) there exists a weak solution \((u, \eta)\), where \( u \) denotes the transformed fluid velocity \( v \) into the rectangular domain \( D \equiv (0, L) \times (0, 1) \), i.e.,

\[
u(y_1, y_2, t) \stackrel{\text{def}}{=} v(y_1, h(y_1, t)y_2, t), \quad y_1 = x_1, y_2 = \frac{x_2}{h(x_1, t)}, \ y \in D, x \in \Omega(h),
\]

with the following properties:

i) \((u, \eta) \in [L^2(0, T; V_{\text{div}})] \times H^1(0, T; H_0^2(0, L))] \cap [L^\infty(0, T; L^2(D))] \times W^{1,\infty}(0, T; L^2(0, L))\),

ii) the distributive (transformed) time derivative \( \tilde{\partial}_t(hu) \)

\[
\tilde{\partial}_t(hu) = \frac{\partial(hu)}{\partial t} - h \frac{\partial}{\partial t} \frac{\partial(y_2 u)}{\partial y_2}
\]

belongs to \( L^2(0, T; X^*) \oplus L^{4/3}((0, T) \times D) \) and

\[
\int_0^T \int_D \left\{ hu \cdot \frac{\partial \psi}{\partial t} + \frac{\partial h}{\partial t} \frac{\partial(y_2 u)}{\partial y_2} \cdot \psi \right\} dydt = -\int_0^T \left\langle \tilde{\partial}_t(hu), \psi \right\rangle_x dt,
\]

In view of the transformation onto \( D \) we have \( q_{\text{in}}(0, y_2, t) = P_{\text{in}}(0, x_2, t)/\rho, \ q_{\text{out}}(L, y_2, t) = P_{\text{out}}(L, x_2, t)/\rho, \ q_w(y_1, 1, t) = P_w(x_1, h(x_1, t), t)/\rho. \)
where

\[ X = \{ \psi \in V_{\text{div}}; \psi_2|_{S_w} \in H^2_0(0, L) \}, \quad (10) \]

\[ V_{\text{div}} \equiv \{ w \in W^{1,2}(D) : \text{div}_h w = 0 \text{ a.e. on } D, \]
\[ w_1 = 0 \text{ on } S_w \text{ and } w_2 = 0 \text{ on } S_{\text{in}} \cup S_{\text{out}} \cup S_c \}, \quad (11) \]

\[ S_w = \{(y_1, 1) : 0 < y_1 < L\}, \quad S_{\text{in}} = \{(0, y_2) : 0 < y_2 < 1\}, \]
\[ S_{\text{out}} = \{(L, y_2) : 0 < y_2 < 1\}, \quad S_c = \{(y_1, 0) : 0 < y_1 < L\}, \]

iii) \( u \) satisfies the transformed divergence-free condition
\[
\text{div}_h u \overset{\text{def}}{=} \frac{\partial u_1}{\partial y_1} - \frac{y_2}{h} \frac{\partial h}{\partial y_1} \frac{\partial u_1}{\partial y_2} + \frac{1}{h} \frac{\partial u_2}{\partial y_2} = 0 \text{ a.e. on } D, \quad (12)
\]

iv) \( u_2(y_1, 1, t) = \partial_t \eta(y_1, t) \) for a.e. \( y_1 \in (0, L), \ t \in (0, T) \),
and the following integral identity holds:
\[
0 = \int_0^T \langle \hat{\partial}_t(hu), \psi \rangle_X + \langle \partial_t \eta, \xi \rangle dt + \int_0^T \left\{ \langle (u, \psi) \rangle + b_h(u, u, \psi) \right\} dt
\]
\[
+ \int_0^1 h(L)q_{\text{out}}(y_2)\psi_1(L, y_2) - h(0)q_{\text{in}}(y_2, t)\psi_1(0, y_2) \ dy_2
\]
\[
+ \int_0^L \left( q_w + \frac{1}{2} \frac{\partial h}{\partial t} u_2 \right) \psi_2(y_1, 1)
\]
\[
+ c \frac{\partial \eta}{\partial t} \frac{\partial^2 \xi}{\partial y_1^2} \frac{\partial y_2}{\partial y_1} + a \frac{\partial \eta}{\partial y_1} \frac{\partial \xi}{\partial y_1} + \frac{\partial^2 R_0}{\partial y_1^2} \xi + b \eta \xi \ dy_1 \right\} dt
\]

for every test function

\[ \psi \in H^1(0, T; V_{\text{div}}), \ \psi_2|_{S_w} \in H^1(0, T; H^2_0(0, L)), \ \psi(y, 0) = 0, \quad (14) \]
\[ \xi(x_1, t) = E \psi_2(y_1, 1, t), \ E := E\rho. \]

In (13) the following notations for the transformed viscous and convective terms have been used:

\[ ((u, \psi))_h := \frac{\mu}{\rho} \int_D h \hat{\partial}_h (u) : \hat{\partial}_h (\psi) dy, \quad (\hat{\partial}_h(u))_{ij} = \frac{1}{2} (\hat{\partial}_i(u_j) + \hat{\partial}_j(u_i)), \quad (15) \]

where \( \hat{\partial}_1 = \left( \frac{\partial}{\partial y_1} - \frac{y_2}{h} \frac{\partial h}{\partial y_1} \frac{\partial}{\partial y_2} \right), \ \hat{\partial}_2 = \frac{1}{h} \frac{\partial}{\partial y_2}. \]
\[ b_h(u, z, \psi) := \int_D \left( hu_1 \left( \frac{\partial z}{\partial y_1} - \frac{y_2}{h} \frac{\partial h}{\partial y_1} \frac{\partial z}{\partial y_2} \right) + u_2 \frac{\partial z}{\partial y_2} \right) \cdot \psi \, dy \]

\[ - \frac{1}{2} \int_0^1 R_0 u_1 z_1 \psi_1 (L, y_2) \, dy_2 + \frac{1}{2} \int_0^1 R_0 u_1 z_1 \psi_1 (0, y_2) \, dy_2 \]

\[ - \frac{1}{2} \int_0^L u_2 z_2 \psi_2 (y_1, 1) \, dy_1. \]  

Using integration by parts the following property of the trilinear form can be proven, see [7, Lemma 3.7],

\[ b_h(u, z, \psi) = \int_D \frac{1}{2} B_h(u, z, \psi) - \frac{1}{2} B_h(u, \psi, z) \, dy \]

where \[ B_h(u, z, \psi) := \left( hu_1 \left( \frac{\partial z}{\partial y_1} - \frac{y_2}{h} \frac{\partial h}{\partial y_1} \frac{\partial z}{\partial y_2} \right) + u_2 \frac{\partial z}{\partial y_2} \right) \cdot \psi. \]

Moreover, for the distributive time derivative of our weak solution \( u \) described above holds:

\[ \int_0^T \langle \partial_t (h u), u \rangle_X \, dt = \frac{1}{2} \int_D |u|^2(T) h(T) - \frac{1}{2} \int_0^T \int_0^L \partial_t h |u_2|^2 \, dy_1 \, dt, \]

see [7] Section 4.2.1 and Appendix A], compare the analogous property in [5, p. 381].

3 Continuous dependence on the domain deformation

In this section we prove, that the weak solution is continuously dependent on the boundary pressure data \( q_{in/out/w} \) as well as on the domain deformation \( h \). The last dependence will be very useful to show the contractivity of the fixed point mapping defined by the iterative process with respect to the domain, that will be studied in Section 4.

To obtain the continuous dependence estimate we use test functions involving the difference of two solutions \( u^1, u^2 \). Unfortunately, due to the dependency of the divergence operator on \( h \), cf. (12), the difference \( u^1 - u^2 \) is not divergence-free. More precisely, neither \( \text{div}_{h^1}(u^1 - u^2) = 0 \) nor \( \text{div}_{h^2}(u^1 - u^2) = 0 \) in general. This is related to the deformability of the
domain and the fact that $\mathbf{v}^1$ is divergence-free in $\Omega(h^1)$ and $\mathbf{v}^2$ is divergence-free in $\Omega(h^2)$, i.e., with respect to another coordinates. In order to design admissible test functions, see Section 3.1, the first solution is transformed into the definition domain of the second solution. A similar technique has been previously used in [10] to prove the continuous dependence of the weak solution on the initial data for similar problems in two dimensions using the Eulerian coordinates. In comparison to [10], our problem is also remapped to the fixed reference domain $D$. After rewriting the weak formulation by means of the admissible test functions, see Section 3.1, multiple error terms arise. Their estimates can be found in Section 3.3. In Section 3.4 we present the final estimate, which yields the main result of Section 3, the continuous dependence of the weak solution on the domain deformation $h$ and boundary pressures $q_{in}, q_{out}, q_w$ formulated in the following theorem:

**Theorem** (Continuous dependence on data). Let $(\mathbf{u}^1, \eta^1)$, $(\mathbf{u}^2, \eta^2)$ be two weak solutions of the initial boundary value problem (1)–(8) transformed to the fixed rectangular domain $D$ satisfying (13). Let the corresponding domain deformation be given by some functions $h^1, h^2$ satisfying (9). Let the transformed boundary pressures $q_{in/out/w}^1$ and $q_{in/out/w}^2$ belong to $L^2(0,T; L^2(S_{in/out/s}))$, respectively. Then for almost all $t \in (0,T)$ it holds:

$$
\frac{\alpha}{2} \| \mathbb{R} \mathbf{u}^1 - \mathbf{u}^2 \|^2_{L^2(D)}(t) + \frac{\alpha \mu}{2 \rho} \tilde{K}_o \int_0^t \| \mathbb{R} \mathbf{u}^1 - \mathbf{u}^2 \|^2_{W^{1,2}(D)} ds
$$

$$
+ E \| \eta^1 - \eta^2 \|^2_{L^2(0,L)}(t) + \frac{bE}{2} \| \eta^1 - \eta^2 \|^2_{L^2(0,L)}(t) + \frac{aE}{2} \| \eta_{y_1}^1 - \eta_{y_1}^2 \|^2_{L^2(0,L)}(t)
$$

$$
+ cE \int_0^t \| \eta^1 - \eta^2 \|^2_{H^2(0,L)} ds
$$

$$
\leq C_T \left( \int_0^t \| q_{in}^1 - q_{in}^2 \|^2_{L^2(0,L)} + \| q_{out}^1 - q_{out}^2 \|^2_{L^2(0,L)} + \| q_w^1 - q_w^2 \|^2_{L^2(0,1)} ds
$$

$$
+ \omega(t) \left[ \| h^1 - h^2 \|^2_{W^{1,2}(0,T; L^2(0,L))} + \| h^1 - h^2 \|^2_{L^2(0,T; H^2(0,L))} \right] \right),
$$

where

$$
\omega(t) = \int_0^t \| \mathbf{u}^1 \|^2_{W^{1,2}(D)} + \| \mathbf{u}^2 \|^2_{W^{1,2}(D)} + \| h^1_t \|^2_{W^{1,2}(0,L)} + \| h^2_t \|^2_{W^{1,2}(0,L)} + \| q_w \|^2_{L^2(S_w)} ds
$$

and $\omega(t) \downarrow 0$ for $t \downarrow 0$. 

10
Here $\tilde{c}_{K_o}$ is the coercivity constant of the viscous form coming from the Korn inequality, $\alpha, K$ are given by (13), $\mu, \rho, E, a, b, c$ are given by the physical model and $C_7$ is a constant depending on $\alpha, \alpha^{-1}, K, \tilde{c}_{K_o}^{-1}$ and on the norms $\|h^i|_{L^\infty(0,T;H^2(\Omega))}, \|\mathbf{u}^i|_{L^\infty(0,T;L^2(D))}$, $i = 1, 2$. The matrix $R$ in the above estimate arises due to the transformation of weak solution from $\Omega(h^1)$ to $\Omega(h^2)$.

3.1 Admissible test functions

To prove the above theorem let us assume $(\mathbf{u}^1, \eta^1), (\mathbf{u}^2, \eta^2)$ are two weak solutions of our problem transformed onto a fixed reference domain $D$, both satisfying the weak formulation (13) with $h^1, h^2$, respectively.

Now we provide some preliminaries concerning admissible test functions (14), involving a difference of the two weak solutions. Let us denote the matrix of the transformation of variables between $\Omega(h^1)$ and $\Omega(h^2)$ by $J = \frac{dX}{dx}$, $X \in \Omega(h^1)$, $x \in \Omega(h^2)$ and the inverse matrix by $J^{-1}$. In the reference coordinates $y \in D$ they read,

$$J = \begin{bmatrix} 1 & 0 \\ y_2 h_2 \partial_{y_1}(\frac{h_1}{h^2}) & \frac{h_1}{h^2} \end{bmatrix}, \quad J^{-1} = \begin{bmatrix} 1 & 0 \\ y_2 h_1 \partial_{y_1}(\frac{h_2}{h^2}) & \frac{h_2}{h^1} \end{bmatrix}.$$  

Let $\mathbb{R} := JJ^{-1}$, where $J = \det J$ is the determinant of the transformation matrix $J$. We have $J = h^1/h^2$ and

$$\mathbb{R} = J J^{-1} = \begin{bmatrix} \frac{h^1}{h^2} & 0 \\ -y_2 h_2 \partial_{y_1}(\frac{h_1}{h^2}) & 1 \end{bmatrix}, \quad \mathbb{R}^{-1} = \begin{bmatrix} \frac{h_2}{h^1} & 0 \\ y_2 h_1 \partial_{y_1}(\frac{h_2}{h^2}) & 1 \end{bmatrix}.$$  

Next step is to design admissible divergence-free test function according to (14), that contain a difference of the two weak solutions. To this end we consider two sets of test functions for the fluid velocity

$$\psi^1 = \mathbf{u}^1 - \mathbb{R}^{-1} \mathbf{u}^2, \quad \psi^2 = \mathbb{R} \mathbf{u}^1 - \mathbf{u}^2,$$

i.e.,

$$\psi^1 = \mathbb{R}^{-1} \psi^2, \quad \psi^2 = \mathbb{R} \psi^1.$$  

It holds that $\text{div}_{h^2} \mathbb{R} \mathbf{u}^1 = 0$. Indeed, since $\mathbb{R} \mathbf{u}^1 = \begin{bmatrix} \frac{h_1}{h^2} u_1^1, -y_2 h_2 \partial_{y_1}(\frac{h_1}{h^2}) u_1^1 \end{bmatrix}$ +
To this end, the test functions $\psi_i$ for both $i \in \mathbb{R}^3$. Indeed, we cannot assert that the right-hand side of (21) is not yet defined through (13) as it is the case of the second component of the velocity cf. (12). Thus we get $\text{div}_h R\mathbf{u}^1 = J\text{div}_h \mathbf{u}^1 \equiv 0$. Analogously one can show $\text{div}_h R^{-1}\mathbf{u}^2 = J^{-1}\text{div}_h \mathbf{u}^2 = 0$. Thus

$$\text{div}_h \psi^1 = \text{div}_h \psi^2 = 0,$$

i.e., $\psi^i$ is solenoidal with respect to $\Omega(h^i)$ for $i = 1, 2$. Moreover, since the first component of the velocity $u_1^1|_{S_w} = 0$ we have $R\mathbf{u}^1|_{S_w} = [0, u_2^1]^T$. Similarly $R^{-1}\mathbf{u}^2|_{S_w} = [0, u_2^2]^T$. Thus the trace of $\psi^i$ on $S_w$ is equal to $[0, u_2^1 - u_2^2]^T$ for both $i = 1, 2$, and due to $u_2^i|_{S_w} = \eta^i$ a.e., we obtain $\psi^i|_{S_w} = \eta_1 - \eta_2$. Consequently, the test functions $(\psi^1, \xi), (\psi^2, \xi)$ from (20) with $\xi = E\eta_1 - E\eta_2^2$ are admissible.

### 3.2 Subtraction of weak formulations

In order to obtain the final estimate let us first pay attention to the time derivative terms in the weak formulation (13). Since $\psi^2 = R\mathbf{u}^1 - \mathbf{u}^2$ and due to the linearity of $\partial_t^h$ we can write:

$$\left< \partial_t^h (h^2 \psi^2), \psi^2 \right>_X = \left< \partial_t^h (h^2 R\mathbf{u}^1), \psi^2 \right>_X - \left< \partial_t^h (h^2 \mathbf{u}^2), \psi^2 \right>_X. \quad (21)$$

The term on the left-hand side can be rewritten using the property (18) of the distributive time derivative. Note that due to the coupling between the velocity $\mathbf{u}^1$ and the domain deformation $h^1$ the first functional on the right-hand side of (21) is not yet defined through (13) as it is the case of the second term. Indeed, we cannot assert that $R\mathbf{u}^1$ is a weak solution associated with $h^2$. Therefore, we have to investigate the object $\partial_t^h (h^2 R\mathbf{u}^1)$. To this end we derive the following equality

$$\partial_t^h (h^2 R\mathbf{u}^1) = J^{-1} \partial_t^h (h^1 \mathbf{u}^1) + E_1, \quad (22)$$
where the additional error term $\mathbb{E}_1$ reads

$$
\mathbb{E}_1 = \left[ \left( \frac{h_1}{h^2} - \frac{h^2}{h_1} \right) y_2 h^2 \left( \frac{\partial u_1^2}{\partial y_2} - y_2 \frac{\partial u_1^1}{\partial y_2} \right) + y_2 u_1^1 \left[ \partial_t h^2 E - h^2 \partial_t E \right] \right]
$$

(23)

with $E := \frac{h_1^2 h^2 - h^2 h_1^2}{h^2} = h^2 \partial_{y_1} \left( \frac{h_1}{h^2} \right)$. To show (22), (23) tedious but straightforward manipulations have been performed as follows.

For the first component of $h^2 \mathbb{R} \mathbf{u}^1$ we have

$$
\partial_t^h \left( h^2 \mathbb{R} \mathbf{u}^1 \right)_1 = \partial_t \left( h^2 u_1^1 \right) - \frac{1}{h^2} h^2 \partial_{y_2} \left( y_2 h^2 u_1^1 \right)
$$

$$
= \partial_t \left( h^2 u_1^1 \right) + \partial_{y_2} \left( y_2 h^2 u_1^1 \right) \left[ \frac{h_1}{h^2} - \frac{h^2}{h_1} \right],
$$

compare the definition of the distributive time derivative in Section 2.1.

Since $(\mathbb{R} \mathbf{u}^1)_2 = u_2^1 - y_2 E u_1^1$, we can write for the second component

$$
\partial_t^h \left( h^2 \mathbb{R} \mathbf{u}^1 \right)_2 = \partial_t \left( h^2 u_2^1 \right) - \partial_t \left( y_2 h^2 E u_1^1 \right)
$$

$$
= \partial_t \left( h^2 u_2^1 \right) - \partial_t \left( y_2 h^2 E u_1^1 \right) + \frac{1}{h^2} \partial_t \left( y_2 \partial_{y_2} \left( y_2 h^2 E u_1^1 \right) \right)
$$

$$
= \partial_t \left( h^2 u_2^1 \right) - y_2 E \partial_t \left( h^2 u_1^1 \right) - y_2 u_1^1 \left( h^2 \partial_t E - E \partial_t h^2 \right).
$$

Using the fact that $\partial_t \left( h^2 u_2^1 \right) = \frac{h_1^2}{h^2} \partial_t \left( h^1 u_1^1 \right) + y_2 h^2 \partial_{y_2} u_1^1 \left[ \frac{h_1}{h^2} - \frac{h^2}{h_1} \right]$, $i = 1, 2$, and replacing $-y_2 E$ by $y_2 \frac{h_1}{h^2} \partial_{y_1} \left( \frac{h^2}{h_1} \right)$ in the second term on the right-hand side of last equation, we finally arrive at

$$
\partial_t^h \left( h^2 \mathbb{R} \mathbf{u}^1 \right)_2 = \frac{h^2}{h^2} \partial_t \left( h^1 u_1^1 \right) + y_2 h^2 \partial_{y_1} \left( \frac{h^2}{h^2} \right) \partial_t \left( h^1 u_1^1 \right)
$$

$$
+ \left[ \frac{h_1}{h^2} - \frac{h^2}{h_1} \right] y_2 h^2 \left( \partial_{y_2} u_1^2 - y_2 E \frac{\partial u_1^1}{\partial y_2} \right) + y_2 u_1^1 \left[ \partial_t h^2 E - h^2 \partial_t E \right].
$$

Thus (22), (23) hold true, see also the definition of the matrix $\mathbb{J}^{-1}$ above.

Finally, using (18) and (22), (23) and $\mathbb{R} \mathbf{u}^1 = \mathbf{u}^2$ we obtain from (21)

$$
\int_0^T \left< \mathbb{J}^{-1} \partial_t \mathbf{u}^1, \mathbb{R} \mathbf{u}^1 \right>_X - \left< \partial_t \mathbf{u}^2, \mathbf{u}^2 \right>_X + \int_D \mathbb{E}_1 \cdot \mathbf{u}^2 \, dy \, dt
$$

$$
= \frac{1}{2} \int_D \mathbf{u}^2 \mathbf{u}^2 \mathbf{T} \, dy - \frac{1}{2} \int_0^T \int_D \partial_t \mathbf{u}^2 \mathbf{u}^2 \, dy \, dt.
$$

13
Replacing $\langle J^{-1} \partial_t h^1 (u^1), \mathbb{R} \psi^1 \rangle_X$ by $\langle R^T J^{-1} \partial_t h^1 (u^1), \psi^1 \rangle_X$ we obtain

$$
\frac{1}{2} \int_D |\psi^2|^2(T)^h(T)dy dt = \int_0^T \langle R^T J^{-1} \partial_t h^1 (u^1), \psi^1 \rangle_X - \langle \partial_t h^2 (u^2), \psi^2 \rangle_X dt + \frac{1}{2} \int_0^T \int_L \partial_t h^2 |\psi_2|^2dy_1 + \int_D E_1 \cdot \mathbf{\psi}^2 dy dt.
$$

In order to obtain the continuous dependence estimate, one needs to replace the pairings on the right-hand side of (24) according to the associated weak formulations. However, concerning the first pairing, the weak formulation for $u^1, \eta^1$ has to be multiplied with the matrix $R^T J^{-1} = h^1 h^2 J^{-T} J^{-1}$. Note that $\|R^T J^{-1}\|_\infty$ is bounded from above by some constant dependent on $\alpha, K$, cf. (9). Moreover this matrix is symmetric and positive definite with (strongly) positive eigenvalues. Thus the existence of a weak solution $u^1, \eta^1$ for the transformed weak formulation associated with pairing $\langle R^T J^{-1} \partial_t h^1 (u^1), \psi^1 \rangle_X$ can be derived using the same technique as for the original weak formulation associated with $\langle \partial_t h^1 (u^1), \psi^1 \rangle_X$. Indeed, for the sake of brevity we point out here only the coercivity of the transformed viscous form $\int_D h^1 R^T J^{-1} \hat{e}_h^1 (u^1) \hat{e}_h^1 (u^1) dy \equiv \int_D h^1 \hat{e}_h^1 (u^1) R^T J^{-1} \hat{e}_h^1 (u^1) dy$. After the orthogonal diagonalization of $R^T J^{-1}$ with eigenvalues $\lambda_{1,2} > 0$, this viscous form can be bounded from below with $\int_D \alpha \lambda_{\min} |\hat{e}_h^1 (u^1)|^2 dy$, consequently the generalized Korn inequality, cf. [24], can be applied, see also lines following (38).

Now we pay attention about the viscous, convective and boundary terms coming from the replacement of the first pairing on the right-hand side of (24) according the associated transformed weak formulation. In analogy to (22) we rewrite them by means of the deformation function $h^2$.

**Viscous terms:**

The following lemma will be useful to handle the viscous terms.

**Lemma 3.1.** For the deformation tensor $\hat{e}_h$, see (13), it holds

$$
\hat{e}_h^1 (v) = \hat{e}_h^2 (v) - [\mathbb{E}(v) + \mathbb{E}(v)^T],
$$

(25)
where \( \mathbb{E}(v) = \mathbb{E}_2(v) F_{h^1} + \mathbb{E}_2(v) \mathbb{E}_3 + \nabla v \mathbb{E}_3 \) (26)

and

\[
\mathbb{E}_2(v) = \begin{bmatrix}
v_1 \frac{E}{h^2} + \frac{h^1 - h^2}{h^2} \partial y_1 v_1 & \frac{h^1 - h^2}{h^2} \partial y_1 v_1 \\
y_2 \partial y_1 (Ev_1) & -E \partial y_2 (y_2 v_1) \end{bmatrix},
E = h^2 \partial y_1 \left( \frac{h^1}{h^2} \right),
\tag{27}
\]

\[
\mathbb{E}_3 = \frac{1}{2} \begin{bmatrix}
0 & \frac{1}{h^2} \\
y_2 \frac{E}{h^2} & \frac{1}{h^2} - \frac{1}{h^2} \end{bmatrix},
F_{h^1} = \frac{1}{2} \begin{bmatrix}
-\frac{y_2}{h^2} \partial y_1 h^1 & 0 \\
\frac{y_2}{h^2} \partial y_1 h^1 & 0 \end{bmatrix}.
\tag{28}
\]

**Proof.** Taking into account the definition [19] of the transformation matrix \( R \), one can easily obtain

\[
\nabla (R v) = \nabla v + \left[ \partial y_1 \left( \frac{h^1}{h^2} \right) v_1 + \frac{h^1 - h^2}{h^2} \partial y_1 v_1, -y_2 \partial y_1 \left( h^2 \partial y_1 \left( \frac{h^1}{h^2} \right) \right) v_1 -h^2 \partial y_1 \left( \frac{h^1}{h^2} \right) \partial y_2 v_1 \right] = \nabla v + \mathbb{E}_2(v).
\tag{29}
\]

Moreover, it is easy to show that

\[
\hat{e}_h(v) = \nabla v F_h + (\nabla v F_h)^T,
\tag{30}
\]

see, e.g., [7, Proof of Lemma 3.4]. Therefore we can write

\[
\hat{e}_h^2(R v) = \nabla (R v) F_{h^2} + (\nabla R v F_{h^2})^T = (\nabla v + \mathbb{E}_2(v)) F_{h^2} + F_{h^2}^T (\nabla v^T + \mathbb{E}_2(v)^T).
\]

One can verify that \( F_{h^2} = F_{h^1} + \mathbb{E}_3 \). Inserting this into above equality we obtain

\[
\hat{e}_h^2(R v) = \hat{e}_h^1(v) + \mathbb{E}_2(v) F_{h^1} + F_{h^1}^T \mathbb{E}_2(v)^T + \mathbb{E}_2(v) \mathbb{E}_3 + \mathbb{E}_3^T \mathbb{E}_2(v)^T + \nabla v \mathbb{E}_3 + \mathbb{E}_3^T \nabla v^T,
\]

which proves (25). \( \square \)

Now, for the difference of the viscous terms on the right-hand side of (24) using \( R = (I + \mathbb{E}_R) \) where \( (\mathbb{E}_R)_{11} = \frac{h^1 - h^2}{h^2}, (\mathbb{E}_R)_{21} = -y_2 h^2 \partial y_1 \left( \frac{h^1}{h^2} \right), (\mathbb{E}_R)_{12} = -y_2 h^2 \partial y_1 \left( \frac{h^1}{h^2} \right) \).
\((\mathbb{E}_R)_{22} = 0\) (see (19)) and Lemma 3.1, one has

\[
\frac{\mu}{\rho} \int_0^T \int_D h^1 \mathbb{R} \mathcal{J}^{-1} \hat{e}_{h^1}(u^1) : \hat{e}_{h^1}(\psi^1) - h^2 \hat{e}_{h^2}(u^2) : \hat{e}_{h^2}(\psi^2) \, dy \, dt
\]

(31)

\[
= \frac{\mu}{\rho} \int_0^T \int_D h^2 \mathbb{R} \mathcal{J}^{-1} \hat{e}_{h^1}(u^1) : \hat{e}_{h^1}(\psi^1) - h^2 \hat{e}_{h^2}(u^2) : \hat{e}_{h^2}(\psi^2) \, dy \, dt
\]

\[
= \frac{\mu}{\rho} \int_0^T \int_D h^2 \left[ \hat{e}_{h^1}(u^1) : \hat{e}_{h^1}(\psi^1) + \mathbb{E}_R \hat{e}_{h^1}(u^1) : \mathbb{E}_R \hat{e}_{h^1}(\psi^1) - \hat{e}_{h^2}(u^2) : \hat{e}_{h^2}(\psi^2) \right] \, dy \, dt
\]

(25)

\[
= \frac{\mu}{\rho} \int_0^T \int_D h^2 \left[ \hat{e}_{h^2}(\mathbb{R} u^1) - \mathbb{E}(u^1) - \mathbb{E}(u^1)^T : \hat{e}_{h^2}(\mathbb{R} \psi^1) - \mathbb{E}(\psi^1) - \mathbb{E}(\psi^1)^T + (\mathbb{R}^T \mathbb{E}_R + \mathbb{E}_R^T) \hat{e}_{h^1}(u^1) : \hat{e}_{h^1}(\psi^1) - \hat{e}_{h^2}(u^2) : \hat{e}_{h^2}(\psi^2) \right] \, dy \, dt
\]

\[
= \frac{\mu}{\rho} \int_0^T \int_D h^2 \left[ \hat{e}_{h^2}(\psi^2) : \hat{e}_{h^2}(\psi^2) - \left( \mathbb{E}(u^1) + \mathbb{E}(u^1)^T \right) : \hat{e}_{h^2}(\psi^2) := I_1 \right]
\]

\[
- \hat{e}_{h^1}(u^1) : (\mathbb{E}(\psi^1) + \mathbb{E}(\psi^1)^T) + \left( \mathbb{R}^T \mathbb{E}_R + \mathbb{E}_R^T \right) \hat{e}_{h^1}(u^1) : \hat{e}_{h^1}(\psi^1) := I_2 + I_3 \, dy \, dt.
\]

**Convection terms:**

Recalling (17), we first rewrite the trilinear term using matrix the \(F_h\) in (28) as

\[
B_{h}(u, z, \psi) = h(u \cdot F_h^T \nabla)z \cdot \psi.
\]

Thus for the corresponding convective term from the first pairing on the right-hand side of (24) holds due to the fact that \(\mathbb{R} \psi^1 = \psi^2, h^1 \mathcal{J}^{-1} = h^2 \mathbb{R}\)
and $F_h^{-1}F_h = \frac{h^2}{h^2}R^2$:

$$
\int_D \mathbb{J}^{-1}B_h^1(\mathbf{u}^1, \mathbf{u}^1, \psi^1)dy = \int_D h^1\mathbb{J}^{-1}(\mathbf{u}^1 \cdot F_h^T \nabla)\mathbf{u}^1 \cdot \psi^1 dy
= \int_D h^2\mathbb{R}(\mathbf{u}^1 \cdot F_h^T \nabla)\mathbf{u}^1 \cdot \psi^2 dy
= \int_D h^2\mathbb{R}(\mathbb{R} \mathbf{u}^1 \cdot F_h^T \nabla)\mathbf{u}^1 \cdot \psi^2 dy
= \int_D J^{-1}h^2(\mathbb{R} \mathbf{u}^1 \cdot F_h^T \nabla) (\mathbb{R} \mathbf{u}^1) \cdot \psi^2 - J^{-1}h^2(\mathbb{R} \mathbf{u}^1 \cdot F_h^T \nabla) (\mathbb{R} \mathbf{u}^1) \cdot \psi^2 dy,
$$

where we recall $J = \frac{h^1}{h^2}$, see (19), and $T_h(\mathbb{R}, \mathbf{u}, \mathbf{v}, \psi) := h(\mathbb{R} \mathbf{u} \cdot F_h^T \nabla) (\mathbb{R} \mathbf{v} \cdot \psi$. Analogously for the second part of the convective term one can obtain

$$
\int_D \mathbb{J}^{-1}B_h^1(\mathbf{u}^1, \psi^1, \mathbf{u}^1) = \int_D J^{-1}B_h^2(\mathbb{R} \mathbf{u}^1, \psi^2, \mathbb{R} \mathbf{u}^1) - J^{-1}T_h^2(\mathbb{R} \mathbf{u}^1, \psi^1, \mathbb{R} \mathbf{u}^1)dy.
$$

Using the equalities above and the trilinearity of $B_h^2$, one gets for the difference

\begin{equation}
\int_D \mathbb{J}^{-1}B_h^1(\mathbf{u}^1, \mathbf{u}^1, \psi^1) - B_h^2(\mathbf{u}^2, \mathbf{u}^2, \psi^2)dy = \int_D B_h^2(\mathbb{R} \mathbf{u}^1, \mathbb{R} \mathbf{u}^1, \psi^2) - B_h^2(\mathbf{u}^2, \mathbf{u}^2, \psi^2) + \frac{h^2-h^1}{h^2}B_h^2(\mathbb{R} \mathbf{u}^1, \mathbb{R} \mathbf{u}^1, \psi^2)
- \frac{h^2-h^1}{h^2}T_h^2(\mathbb{R} \mathbf{u}^1, \mathbf{u}^1, \psi^2) dy
= \int_D B_h^2(\psi^2, \mathbb{R} \mathbf{u}^1, \psi^2) + B_h^2(\mathbf{u}^2, \psi^2, \psi^2) + \frac{h^2-h^1}{h^2}B_h^2(\mathbb{R} \mathbf{u}^1, \mathbb{R} \mathbf{u}^1, \psi^2)
- \frac{h^2-h^1}{h^2}T_h^2(\mathbb{R} \mathbf{u}^1, \mathbf{u}^1, \psi^2) dy,
\end{equation}

Analogously one can obtain for the difference of the second part of convective term

\begin{equation}
\int_D \mathbb{J}^{-1}B_h^1(\mathbf{u}^1, \psi^1, \mathbf{u}^1) - B_h^2(\mathbf{u}^2, \psi^2, \mathbf{u}^2)dy = \int_D B_h^2(\psi^2, \psi^2, \mathbb{R} \mathbf{u}^1) + B_h^2(\mathbf{u}^2, \psi^2, \psi^2) + \frac{h^2-h^1}{h^2}B_h^2(\mathbb{R} \mathbf{u}^1, \psi^2, \mathbb{R} \mathbf{u}^1)
- \frac{h^2-h^1}{h^2}T_h^2(\mathbb{R} \mathbf{u}^1, \psi^1, \mathbb{R} \mathbf{u}^1) dy.
\end{equation}
The difference of the convective terms on the right-hand side of (24) can be obtained by subtracting (32) and (33). Using (17) we can finally write

\[
\int_0^T \frac{1}{2} \int_D \mathbb{J}^{-1}[B_h^1(u^1, u^1, \psi^1) - B_h^1(u^1, \psi^1, u^1)]dy - b_{h^2}(u^2, u^2, \psi^2)dt \tag{34}
\]

\[
= \int_0^T b_{h^2}(\psi^2, \mathbb{R}u^1, \psi^2) + \frac{h^2-h^1}{h^1}b_{h^2}(\mathbb{R}u^1, \mathbb{R}u^1, \psi^2)
\]

\[
- \frac{1}{2} \int_D \frac{h^2}{h^1} [T_h^2(\mathbb{R}u^1, u^1, \psi^2) - T_h^2(\mathbb{R}u^1, \psi^1, u^1)]dy dt.
\]

**Boundary terms:**

Note that due to the boundary conditions on \( S_w \), the first component \( (\psi^1(y_1, 1, t))_1 = (u^1(y_1, 1, t))_1 = 0 \), \( i = 1, 2 \). Recalling (20), i.e., \((\psi^2)_2 = (\mathbb{R}\psi^1)_2 = \mathbb{R}_{22}\psi^1_2 \) and \( \mathbb{R}_{22} = 1 \), \( \mathbb{J}_{22}^{-1} = \frac{h^2}{h^1} \), for the difference of boundary terms we can write

\[
\frac{1}{2} \int_0^T \int_0^L \left( \partial_t h^1 \mathbb{R}_{22} \mathbb{J}_{22}^{-1} u^1_2 \psi^1_2 - \partial_t h^2 u^1_2 \psi^2_2 \right)(y_1, 1, t)dy dt \tag{35}
\]

\[
= \int_0^T \int_0^L \partial_t h^2 \left[ \mathbb{J}_{22}^{-1} u^1_2 \mathbb{R}_{22} \psi^2_2 - u^2_2 \psi^1_2 \right] + \mathbb{J}_{22}^{-1} u^1_2 \mathbb{R}_{22} \psi^1_2 \partial_t [h^1 - h^2](y_1, 1, t)dt dy dt
\]

\[
= \frac{1}{2} \int_0^T \int_0^L \partial_t h^2 \left[ \left( \frac{h^2}{h^1} u^1_2 - u^2_2 \right) \psi^2_2 \right] + \frac{h^2}{h^1} u^1_2 \psi^2_2 \partial_t [h^1 - h^2](y_1, 1, t)dy dt
\]

\[
= \frac{1}{2} \int_0^T \int_0^L \partial_t h^2 |\psi^2_2|^2 + \left( \frac{h^2}{h^1} \partial_t [h^1 - h^2] + \frac{h^2-h^1}{h^1-h^2} \right) u^1_2 \psi^2_2(y_1, 1, t)dy dt.
\]

Let us mention that the first term on the right-hand side of (35) vanishes in the sum with the boundary term arising on the right-hand side of (24). Moreover, the pressure terms on \( S_w \) can be simplified to

\[
\int_0^T \int_0^L (\mathbb{R}_{22}\mathbb{J}_{22}^{-1} q^1_w \psi^1_2 - q^2_w \psi^2_2)(y_1, 1, t)dy dt \tag{36}
\]

\[
= \int_0^T \int_0^L \left( \frac{h^2}{h^1} q^1_w - q^2_w \right) \psi^2_2(y_1, 1, t)dy dt = \int_0^T \int_0^L (q^1_w - q^2_w) \psi^2_2 + \frac{h^2-h^1}{h^1} q^1_w \psi^2_2(y_1, 1, t).
\]

Concerning the boundary pressure terms on \( S_{\text{in/out}} \) note that the second component of the velocity is zero, thus \((\psi^2)_1 = (\mathbb{R}\psi^1)_1 = \mathbb{R}_{11}\psi^1_1 \), and \( \mathbb{J}_{11}^{-1} = \)
1. Thus

\[
\int_0^T \int_0^1 (q_m^1 h^1 \mathbb{R}_1^T \mathbb{R}_1^{11} \psi_2^1 - q_m^2 h^2 \psi_2^1)(0, y_2, t) dy_2 dt = \int_0^T \int_0^1 \left[ \int_0^1 [h^1(q_m^1 - q_m^2) + (h^1 - h^2) q_m^2] \psi_2^1(0, y_2, t) dy_2 dt. \right.
\]

In the same way the corresponding difference involving \( q_{out} \) can be obtained.

Finally, after replacing the difference of pairings on the right-hand side of (24) with differences of viscous terms, convective and boundary terms according to (31), (34) and (35)–(37), respectively, the resulting equation reads

\[
\frac{1}{2} \int_D |\psi^2|^2(T) h^2(T) dy + \frac{\mu}{\rho} \int_0^T \int_D h^2 \hat{e}_{h^2}(\psi^2) : \hat{e}_{h^2}(\psi^2) dy dt
\]

\[
+ \frac{1}{2E} \int_0^T \int_D \left( aE \frac{1}{2} |\partial_y \bar{\eta}|^2(T) + \frac{bE}{2} |\bar{\eta}|^2(T) + \frac{c}{E} \int_0^T \int_0^L |\partial_y^2 \xi|^2 dy_1 dt \right)
\]

\[
= \int_0^T \int_D \mathbb{E}_1 \cdot \psi^2 + \frac{\mu}{\rho} h^2(I_1 + I_2) - I_3 dy dt
\]

\[- \int_0^T \int_D b_{h^2}(\psi^2, \mathbb{R} u^1, \psi^2) + h^2 - h^3 b_{h^2}(\mathbb{R} u^1, \mathbb{R} u^1, \psi^2) dt
\]

\[+ \int_0^T \int_D \left( h^2 \left[ T_{h^2}(\mathbb{R} u^1, u^1, \psi^2) - T_{h^2}(\mathbb{R} u^1, \psi^1, \mathbb{R} u^1) \right] dy dt, \right.
\]

where the error terms \( \mathbb{E}_1, I_1, I_2, I_3 \) are defined in (23), (26), (27), (28), (31), respectively, and \( T_h(u, v, \psi) := h(u \cdot F_h^T \nabla)(v \cdot \psi). \)

To derive an appropriate estimate we apply Korn’s first inequality with variable coefficients, see, e.g., (25), (24) in the second term on the left-hand side of the above equality. Indeed, using (30) we can write

\[
\frac{\mu}{\rho} \int_0^T \int_D h^2 |\hat{e}_{h^2}(\psi^2)|^2 dy dt = \frac{\mu}{\rho} \int_0^T \int_D h^2 |\nabla \psi^2 F_{h^2} + (\nabla \psi^2 F_{h^2})^T|^2,
\]

19
where $F_h = F_h^2(y) : \Omega \to \mathbb{R}^{2 \times 2}$ is a continuous mapping with $\text{det}(F^2_h) = \frac{1}{h^2} > \alpha > 0$, cf. (28). Consequently, Corollary 4.1 in [24] implies the existence of a positive constant $c_{Ko}$ such that

$$\frac{\mu}{\rho} \int_D h^2|\hat{e}^2_h(\psi^2)|^2 dy \geq \frac{\mu}{\rho} c_{Ko} \alpha \int_D |\nabla \psi^2|^2 dy, \forall \psi^2 \in V_{\text{div}}.$$

Next we will estimate the terms on the right-hand side of (38) in an appropriate way to apply the Gronwall lemma.

### 3.3 Estimate of the right-hand side of (38)

In what follows we use the notation $\bar{h} = h^1 - h^2$. We also express the terms with $E = h^2 \partial_y (\frac{h^1}{h^2})$ by means of the differences $\bar{h}$,

$$h^2 E = h^2 \partial_y h^1 - h^1 \partial_y h^2 = h^1 \partial_y \bar{h} - \partial_y h^1 \bar{h}. \tag{39}$$

Before we start presenting the estimates, we introduce the following useful lemma.

**Lemma 3.2.** For all $v$ in the space $V_{\text{div}}$ (defined in (11)), it holds

$$\text{ess sup}_{0 \leq y_1 \leq L} \int_0^1 |v|^2 dy_2 \leq c \|v\|_{L^2(D)} \|\nabla v\|_{L^2(D)}.$$

**Proof.** We can write

$$\int_0^1 |v(y_1, y_2, t)|^2 dy_2 = \int_0^1 |v(0, y_2, t)|^2 dy_2 + \int_0^{y_1} \frac{\partial}{\partial \theta} \left( \int_0^1 |v(\theta, y_2, t)|^2 dy_2 \right) d\theta$$

$$= \int_0^1 |v(0, y_2, t)|^2 dy_2 + 2 \int_0^{y_1} \int_0^1 |v(\theta, y_2, t)| \left| \frac{\partial v}{\partial y_1} (\theta, y_2, t) \right| dy_2 d\theta$$

$$\leq \|v\|^2_{L^2(S_w)} + 2 \int_D |v||\nabla v|dy.$$

Applying the Hölder inequality and using a trace estimate in the first term on the right-hand side of the last inequality, i.e. $\|v\|_{L^2(\partial D)} \leq c \|v\|_{L^2(D)} \|\nabla v\|_{L^2(D)}$, cf., e.g., [7, Lemma 3.2], we get from above

$$\int_0^1 |v(y_1, y_2, t)|^2 dy_2 \leq c \|v\|_{L^2(D)} \|\nabla v\|_{L^2(D)},$$

which is valid for each $y_1 \in [0, L]$. Taking the essential supremum of $\int_0^1 |v|^2 dy_2$ over all $0 \leq y_1 \leq L$ yields the statement of this lemma. \qed
From now on we use following abbreviations for norms of functions defined on \((0, L): \| \cdot \|_{\infty} := \| \cdot \|_{L^{\infty}(0, L)}, \| \cdot \|_{1, \infty} := \| \cdot \|_{W^{1, \infty}(0, L)}.\) Moreover, \(C_{K, \alpha} := C(K, \alpha, \alpha^{-1})\) denotes a constant depending on the given constants \(K, \alpha\) from \(\S\).

**Time derivative error term**

Let us first estimate the error term \(\int_{0}^{T} \int_{D} E_{1} \cdot \psi^{2} dy dt\) on the right-hand side of (38). We first rewrite the components of \(E_{1}\) given by (23) in terms of the differences of \(\tilde{h}, \tilde{h}_{t}, \tilde{h}_{ty}\), see also (39).

\[
\frac{h_{t}^{1} - h_{t}^{2}}{h^{1}} = \frac{\tilde{h}_{t}}{h^{1}} - \frac{h_{t}^{2}}{h^{1}h^{2}} \tilde{h},
\]

\[
h^{2} \partial_{t} E = -\tilde{h}\left(h_{t}^{1} + \frac{h_{t}^{2}}{h^{2}} h_{ty}^{1}\right) + \tilde{h}_{ty} \left(h_{t}^{1} + \frac{h_{t}^{2}}{h^{2}} h_{t}^{1}\right) - \tilde{h}_{t} h_{t}^{1} + \tilde{h}_{ty} h^{1}.
\]

We first concentrate on those terms from \(E_{1}\), which do not involve \(\tilde{h}_{ty}\) (see the last term of \(h^{2} \partial_{t} E\)). We rewrite \(E_{1} = \bar{E}_{1} - y_{2} u_{1}^{1} h_{ty} h^{1}\) and estimate the term \(\int_{0}^{T} \int_{D} \bar{E}_{1} \cdot \psi^{2} dy dt\) first. We have

\[
|\bar{E}_{1}| \leq C_{K, \alpha} \left(\left|u^{1} + \nabla u^{1}\right| \|\tilde{h}_{t}\| + \left|u^{1} + \nabla u^{1}\right| \left[\|\tilde{h}_{t}\| + \|\tilde{h}_{ty}\|\right] \left[\|h_{t}^{1}\| + \|h_{ty}^{1}\| + \|h_{t}^{2}\|\right]\right).
\]

The term containing \(\bar{E}_{1}^{a}\) is now estimated by Lemma 3.2 applied to \(|\psi^{2}|\) as follows:

\[
\int_{0}^{T} \int_{D} \bar{E}_{1}^{a} \cdot \psi^{2} dy dt \leq C_{K, \alpha} \int_{0}^{T} \int_{0}^{L} |\tilde{h}_{t}| \int_{0}^{1} |u^{1} + \nabla u^{1}| \|\psi^{2}\| dy dy_{1} dt
\]

\[
\leq C_{K, \alpha} \int_{0}^{T} \left(\text{ess sup}_{0 \leq y \leq L} \int_{0}^{1} |\psi^{2}| \right)^{\frac{1}{2}} \int_{0}^{L} |\tilde{h}_{t}| \left(\int_{0}^{1} |u^{1} + \nabla u^{1}|^{2} dy_{2}\right)^{\frac{1}{2}} dy_{1} dt
\]

\[
\leq C_{K, \alpha} \int_{0}^{T} \left[\|\psi^{2}\|_{L^{2}(D)}^{\frac{1}{2}} \|\nabla \psi^{2}\|_{L^{2}(D)}^{\frac{1}{2}} \|\tilde{h}_{t}\|_{L^{2}(0, L)} \|u^{1}\|_{W^{1, 2}(D)}\right] dt.
\]

Employing \(\|\psi^{2}\|_{L^{2}(D)}^{\frac{1}{2}} \|\nabla \psi^{2}\|_{L^{2}(D)}^{\frac{1}{2}} \leq c\|\psi^{2}\|_{W^{1, 2}(D)}\) and using the Hölder and Young inequalities we get

\[
\int_{0}^{T} \int_{D} \bar{E}_{1}^{a} \cdot \psi^{2} dy dt
\]

\[
\leq \varepsilon \int_{0}^{T} \|\psi^{2}\|_{W^{1, 2}(D)}^{2} dt + C_{\varepsilon} C_{K, \alpha}^{2} \|\tilde{h}_{t}\|_{L^{2}(0, L)}^{2} \int_{0}^{T} \|u^{1}\|_{W^{1, 2}(D)}^{2} dt.
\]
Using the Hölder and the Young inequalities it follows for the second term,
\[
\int_0^T \int_D \overline{E}^h_1 \cdot \psi^2 \, dy \, dt \\
\leq C_{K,\alpha} \int_0^T \| u^1 \|_{W^{1,2}(D)} \| \psi^2 \|_{L^2(D)} \| \tilde{h} \|_{1,\infty} (\| h^1_t \|_{1,\infty} + \| h^2_t \|_{\infty}) \, dt \\
\leq \frac{C^2_{K,\alpha}}{2} \| \tilde{h} \|^2_{L^\infty(0,T;W^{1,\infty}(0,L)))} \omega_h(T) + \frac{1}{2} \int_0^T \| u^1 \|^2_{W^{1,2}(D)} \| \psi^2 \|^2_{L^2(D)} \, dt,
\]
where \( \int_0^T \| h^1_t \|_{1,\infty} + \| h^2_t \|_{\infty} \, dt =: \omega_h(T) \). Hence we can summarize that
\[
\int_0^T \int_D \overline{E} \cdot \psi^2 \, dy \, dt \\
\leq \varepsilon \int_0^T \| \psi^2 \|^2_{W^{1,2}(D)} \, dt + \frac{1}{2} \int_0^T \| u^1 \|^2_{W^{1,2}(D)} \| \psi^2 \|^2_{L^2(D)} \, dt \\
+ C^2_{K,\alpha} \left( C_\varepsilon \| \tilde{h} \|^2_{W^{1,\infty}(0,T;L^\infty(0,L)))} \int_0^T \| u^1 \|^2_{W^{1,2}(D)} \, dt \right)^{2} + \| \tilde{h} \|^2_{L^\infty(0,T;W^{1,\infty}(0,L)))} \omega_h(T) \).
\]

It remains to show the estimate of the term \( -\int_0^T \int_D y^2 u^1 \tilde{h} y u^1 \psi^2 \, dy \, dt \).

By integration by parts with respect to \( y^1 \) and due to the zero boundary conditions for \( \psi^2 \) on \( S_m \cup S_{out} \) we can rewrite this term as
\[
-\int_0^T \int_D y^2 u^1 \tilde{h} y u^1 \psi^2 \, dy \, dt = \int_0^T \int_D y^2 \tilde{h}_{y^1} (h^1 \psi^2 u^1 + h^1 \partial_{y^1} \partial_{y^1} u^1 + h^1 \psi^2 \partial_{y^1} u^1) \, dy \, dt.
\]

Thus, by (39) we get,
\[
-\int_0^T \int_D y^2 u^1 \tilde{h} y u^1 \psi^2 \, dy \, dt \leq C_{K,\alpha} \int_0^T \int_D | \tilde{h}_{y^1} | | \nabla \psi^2 | | u^1 | + | \tilde{h}_{y^1} | | \psi^2 | | u^1 | + | \nabla u^1 | \, dy \, dt.
\]

Analogously as in term \( \overline{E}^h_1 \) above, we apply now Lemma 3.2 to \( | u^1 | \) in the first term and to \( | \psi^2 | \) in the second term on the right-hand side. Consequently applying the Hölder and Young inequalities we get
\[
-\int_0^T \int_D y^2 u^1 \tilde{h} y u^1 \psi^2 \, dy \, dt \leq 2C_{K,\alpha} \int_0^T \| \tilde{h} \|_{L^2(D)} \| u^1 \|_{W^{1,2}(D)} \| \psi^2 \|_{W^{1,2}(D)} \, dt \tag{41}
\leq \varepsilon \int_0^T \| \psi^2 \|^2_{W^{1,2}(D)} \, dt + C_\varepsilon C^2_{K,\alpha} \| \tilde{h} \|_{L^\infty(0,T;L^2(0,L)))} \int_0^T \| u^1 \|^2_{W^{1,2}(D)} \, dt.
\]

\footnote{Let us point out that \( \omega_h(T) \downarrow 0 \) as \( T \downarrow 0 \). This will be essential in Section 3 in order to show the contractivity.}

22
Hence, next we estimate

$$\int_0^T \int_D \mathbb{E}_1 \cdot \psi^2 \, dy \, dt \leq 2 \varepsilon \int_0^T \| \psi^2 \|^2_{W^{1,2}(D)} \, dt + \frac{1}{2} \int_0^T \| u^1 \|^2_{W^{1,2}(D)} \| \psi^2 \|^2_{L^2(D)} \, dt$$

$$+ C_{K,\alpha} \left( 2 C_{\varepsilon} \| \bar{h} \|^2_{W^{1,\infty}(0,T;L^2(0,L))} \int_0^T \| u^1 \|^2_{W^{1,2}(D)} \, dt + \| \bar{h} \|^2_{L^\infty(0,T;W^{1,\infty}(0,L))} \omega_h(T) \right),$$

where $\omega_h(T) := \int_0^T \| h_i^1 \|^2_{1,\infty} + \| h_i^2 \|^2_{1,\infty} \, dt$ and $C_{\varepsilon} = C(\varepsilon^{-1})$.

**Viscous terms**

Let us start with the estimate of term containing $I_1$, cf. (31). We have

$$\frac{\mu}{\rho} \int_0^T \int_D h^2 I_1 \, dy \, dt \leq \frac{\mu \alpha^{-1}}{\rho} \int_0^T \| \mathbb{E}(u^1) \|_{L^2(D)} \| F_{hi} \| \| \nabla \psi^2 \|_{L^2(D)} \, dt.$$  

Note that by (31) one has $\| F_{hi} \|_\infty \leq C_{K,\alpha}$, $i = 1, 2$.

Next, we estimate $\mathbb{E}(u^1) = \mathbb{E}_2(u^1)(F_{hi} + \mathbb{E}_3) + \nabla u^1 \mathbb{E}_3$, cf. (26). The term $\mathbb{E}_3$ given by (28) can be bounded in view of (39) by $\| \mathbb{E}_3 \| \leq C_{K,\alpha} \| \bar{h} \|_{1,\infty}$. Hence,

$$\| \mathbb{E}(u^1) \|_{L^2(D)} \leq C_{K,\alpha} \left( \| \mathbb{E}_2(u^1) \|_{L^2(D)} (\| \bar{h} \|_{1,\infty} + 1) + \| \nabla u^1 \|_{L^2(D)} \| \bar{h} \|_{1,\infty} \right).$$

In the next step we concentrate on the estimate of $\| \mathbb{E}_2(u^1) \|_{L^2(D)}$. Using (39) one can easily verify that $\partial y_1 E = \frac{1}{\hbar^2} \left( h^1 \bar{h} y_1 y_1 - h^1 y_1 y_1 \bar{h} - \frac{h^2}{\hbar^2} (h^1 \bar{h} y_1 - h^1 \bar{h}) \right)$.

Therefore, for the matrix $\mathbb{E}_2(u^1)$ in (27) it holds

$$\| \mathbb{E}_2(u^1) \| \leq C_{K,\alpha} \left( \| \bar{h} \| + \| \bar{h} y_1 \| \right) \left( \| u^1 \| + \| \nabla u^1 \| \right)$$

$$+ \left( \| \bar{h} \|_{1,\infty} \right) \left( \| \bar{h} y_1 y_1 \| + \| \bar{h} y_1 \| \right) \| u^1 \|.$$  

Let us point out that the second derivatives $\partial^2_{y_1} \bar{h}, \partial^2_{y_1} h^1$ only belong to $L^2(0,L)$. Therefore we use Lemma 3.2 to estimate the corresponding terms in the expression for $\| \mathbb{E}_2(u^1) \|_{L^2(D)}$ in the following way

$$\int_D \| \bar{h} \|^2_{H^1(0,L)} | u^1 |^2 \leq \| \bar{h} \|^2 \int_0^L | h^1_{y_1 y_1} |^2 \, dy_1 \sup_{0 \leq y_1 \leq L} \int_0^1 | u^1 |^2 \, dy_2,$$

$$\leq c_1 \| \bar{h} \|_{1,\infty}^2 \| h^1 \|_{H^2(0,L)}^2 \| \nabla u^1 \|_{L^2(D)}^2.$$
Analogously,
\[
\int_D |\tilde{h}_{y_1 y_1}|^2 |h|^2 |u|^2 dy \leq c_2 \alpha^{-1} \|\tilde{h}\|_{H^2(0,L)}^2 \|\nabla u\|_{L^2(D)}^2.
\]

Consequently, taking into account (44) and estimates above we obtain
\[
\|\mathbb{E}_2(u)\|_{L^2(D)} \leq C_{K,\alpha} \left[ \|\tilde{h}\|_{1,\infty} + \|\tilde{h}\|_{\infty} \|h^1\|_{H^2(0,L)} + \|\tilde{h}\|_{H^2(0,L)} \right] \|\nabla u\|_{L^2(D)}. (45)
\]

Finally, using (43), Young’s inequality and the fact that \(\|\tilde{h}\|_{1,\infty} \leq K\), see (3), we obtain
\[
\frac{\mu}{\rho} \int_0^T \int_D h^2 I_1 dy dt \leq \frac{C_{K,\alpha} \mu}{\rho} \int_0^T \left[ \|\tilde{h}\|_{1,\infty} (1 + \|h^1\|_{H^2(0,L)}) + \|\tilde{h}\|_{H^2(0,L)} \right] \|\nabla u\|_{L^2(D)} dt
\leq \varepsilon \int_0^T \|\nabla \psi^2\|_{L^2(D)} dt + C_{\varepsilon} C_{K,\alpha}^2 \|u\|_{L^2(0,T;W^{1,2}(D))}^2 + \left( \|\tilde{h}\|_{L^2(0,T;W^{1,2}(0,L))}^2 + \|\tilde{h}\|_{L^2(0,T;H^2(0,L))}^2 \right),
\]

where \(C_{\varepsilon} = C(\varepsilon^{-1}, \frac{\mu}{\rho})\) and
\[
C_h := 1 + \|h^1\|_{L^\infty(0,T;H^2(0,L))} + \|h^2\|_{L^\infty(0,T;H^2(0,L))}
\]

is bounded.

Now we proceed with the estimate of term containing \(I_2\) in (31),
\[
\frac{\mu}{\rho} \int_0^T \int_D h^2 I_2 dy dt \leq \frac{\mu \alpha^{-1}}{\rho} \int_0^T \|F_{h^1}\|_{\infty} \|\nabla u\|_{L^2(D)} \|\mathbb{E}(\psi^1)\|_{L^2(D)} dt.
\]

We recall that \(\mathbb{E}(\psi^1) = \mathbb{E}_2(\psi^1)(F_{h^1} + \mathbb{E}_3) + \nabla \psi^1 \mathbb{E}_3\). Analogously as in (43) we can bound
\[
\|\mathbb{E}(\psi^1)\|_{L^2(D)} \leq C_{K,\alpha} \left( \|\mathbb{E}_2(\psi^1)\|_{L^2(D)} (\|\tilde{h}\|_{1,\infty} + 1) + \|\nabla \psi^1\|_{L^2(D)} \right) \|\tilde{h}\|_{1,\infty}. (48)
\]

Repeating similar estimates as in (44), (45) we obtain
\[
\|\mathbb{E}_2(\psi^1)\|_{L^2(D)} \leq C_{K,\alpha} \left[ \|\tilde{h}\|_{1,\infty} (1 + \|h^1\|_{H^2(0,L)}) + \|\tilde{h}\|_{H^2(0,L)} \right] \|\nabla \psi^1\|_{L^2(D)}. (49)
\]

24
In order to replace the norm of $\nabla$ with $\bar{\psi}$, using (51), (47), and by applying Young's inequality, we get
\[
\psi_t \leq C_{K,\alpha} \mu \int_0^T \left[ \|\bar{h}\|_{1,\infty}(1 + \|h^1\|_{H^2(0,L)}) + \|\bar{h}\|_{H^2(0,L)} \right] \|\psi^1\|_{L^2(D)} dt.
\] (50)

In order to replace the norm of $\nabla \psi^1$ by the norm of $\nabla \psi^2$ we now use the relation $\psi^1 = \mathbb{R}^{-1} \psi^2$. One can verify that
\[
\|\nabla (\mathbb{R}^{-1} \psi^2)\| \leq C_{K,\alpha} \|\nabla \psi^2\| + \left| \partial_{y_1} \left( \frac{h^2}{h^2} \right) (1 - y_2 h^1 - h^1) - y_2 h^1 \partial_{y_1} \left( \frac{h^2}{h^2} \right) \right| |\psi_1^2| \leq C_{K,\alpha} \left( \|\nabla \psi^2\| + |\psi_1^2| (1 + |\partial_{y_1} h^1| + |\partial_{y_1} h^2|) \right),
\]

thus
\[
\|\nabla \psi^1\|_{L^2(D)} \leq C_{K,\alpha} \left( \|\psi^2\|_{W^1,2(D)}(1 + \|h^1\|_{H^2(0,L)} + \|h^2\|_{H^2(0,L)}) \right). \] (51)

Inserting (51) into (50) we finally obtain an analogous estimate to (46), i.e.,
\[
\frac{\mu}{\rho} \int_0^T \int_D h^2 I_2 dy dt \leq \varepsilon \int_0^T \|\psi^2\|_{W^1,2(D)}^2 dt + \bar{C}_\varepsilon C_{K,\alpha}^2 C_{\mu}^2 \|\mu\|_{L^2(0,T;W^1,2(D))}^2 (\|\bar{h}\|^2_{L^\infty(0,T;W^1,\infty(0,L))} C_{\mu}^2 + \|\bar{h}\|^2_{L^\infty(0,T;H^2(0,L))})
\] (52)

with $\bar{C}_\varepsilon = \bar{C} \left( \varepsilon^{-1}, \frac{\mu}{\rho} \right)$.

It remains to show the estimate of the term containing $I_3$, cf. (31). Since
\[
E_{\mathbb{R}} = \begin{bmatrix} \frac{\bar{h}}{h^2} & 0 \\ -y_2 E & 0 \end{bmatrix} \text{ and } \mathbb{R}^T E_{\mathbb{R}} + E_{\mathbb{R}}^T = \begin{bmatrix} \frac{\bar{h}^2 + h^1}{h^2} + y_2^2 E^2 & -y_2 E \\ -y_2 E & 0 \end{bmatrix},
\]

with (39) we have $\|\mathbb{R}^T E_{\mathbb{R}} + E_{\mathbb{R}}\|_{\infty} \leq C_{K,\alpha} \|\bar{h}\|_{1,\infty}$. Thus
\[
\frac{\mu}{\rho} \int_0^T \int_D h^2 I_3 dy dt \leq \frac{\mu}{\rho} \int_0^T C_{K,\alpha} \|\bar{h}\|_{1,\infty}^2 \|F_{h^1}\|_{L^\infty(0,T;L^2(D))}^2 \|\nabla u^1\|_{L^2(D)} \|\nabla \psi^1\|_{L^2(D)} dt.
\]

Using (51), (47), and by applying Young's inequality, we get
\[
\frac{\mu}{\rho} \int_0^T \int_D h^2 I_3 dy dt \leq \frac{\mu}{\rho} \int_0^T C_{K,\alpha}^2 C_{\mu}^2 \|\bar{h}\|_{1,\infty}^2 \|\nabla u^1\|_{L^2(D)} \|\psi^2\|_{W^1,2(D)} dt \quad (53)
\]
\[
\leq \varepsilon \int_0^T \|\psi^2\|_{W^1,2(D)}^2 dt + \bar{C}_\varepsilon C_{K,\alpha}^8 C_{\mu}^2 \|\mu\|_{L^2(0,T;W^1,2(D))}^2 \|\bar{h}\|_{L^\infty(0,T;W^1,\infty(0,L))}^2.
\]
where $\tilde{C}_\varepsilon = \bar{C}(\varepsilon^{-1}, \frac{\varepsilon}{\rho})$. Summarizing the above estimates of viscous error terms (46), (52), (53) we finally get

$$\frac{H}{\rho} \int_0^T \int_D h^2 I_1 + h^2 I_2 - h^2 I_3 dydt \leq 3\varepsilon \int_0^T \|\psi^2\|_{W^{1,2}(D)} dt + C_\varepsilon (C_{K,\alpha}^2 + C_{K,\alpha}^8) C_h^2$$

$$\left(\|\tilde{h}\|_{L^\infty(0,T; W^{1,2}(0,L))} + \|\tilde{h}\|_{L^\infty(0,T; H^2(0,L))}\right) \|u^1\|_{L^2(0,T; W^{1,2}(D))}^2,$$

where $C_\varepsilon = C(\varepsilon^{-1}, \frac{\varepsilon}{\rho})$ and $C_h$ is given by (47).

**Convective terms**

We follow with the estimate terms coming from the difference of convective terms, see (34).

**I)** Recalling (17), (9) and using $\|R\|_\infty \leq C_{K,\alpha}$, cf. (19), for the first term on the right-hand side of (34) it holds

$$\int_0^T |b_{h^2}(\psi^2, R u^1, \psi^2)| dt \leq \frac{1}{2} \int_0^T |B_{h^2}(\psi^2, R u^1, \psi^2)| + |B_{h^2}(\psi^2, \psi^2, R u^1)| dt$$

$$\leq C_{K,\alpha} \int_0^T \int_D |\psi^2|^2 |\nabla u^1| + |\psi^2| |\nabla \psi^2| |u^1| dy dt.$$

Further, applying the Hölder inequality, the Sobolev interpolation inequality in two dimensions: $\|\varphi\|_{L^4(D)} \leq C_{\varphi} \|\nabla \varphi\|_{L^2(D)}^{1/2} \|\varphi\|_{L^2(D)}^{1/2}$, and Young’s inequality

$$ab \leq \varepsilon a^p + C_\varepsilon b^{\frac{p}{p-1}}, \quad C_\varepsilon = C(\varepsilon^{-1}), \quad 1 < p < \infty \quad (55)$$

for $p = 2, 4$ we finally get

$$\int_0^T |b_{h^2}(\psi^2, R u^1, \psi^2)| dt \leq C_{K,\alpha} \int_0^T \|\psi^2\|_{L^2} \|\nabla \psi^2\|_{L^2} \|u^1\|_{L^2}$$

$$+ \|\psi^2\|_{L^2} \|\nabla \psi^2\|_{L^2}^{3/2} \|u^1\|_{L^2}^{1/2} \|\nabla u^1\|_{L^2}^{1/2} dt$$

$$\leq 2\varepsilon \int_0^T \|\nabla \psi^2\|_{L^2}^2 dt + C_{\varepsilon} C_{K,\alpha} \int_0^T \|\nabla u^1\|_{L^2} \|\psi^2\|_{L^2}^2 dt.$$

Here we have used the abbreviation $\|\cdot\|_{L^2} := \|\cdot\|_{L^2(D)}$. Note that $C_{\varepsilon}$ is a constant depending on $\varepsilon^{-1}$ and on the norm $\|u^1\|_{L^\infty(0,T; L^2(D))}^2$. 

26
With the assistance of (17), (9) and recalling \( \|R\|_\infty \leq C_{K,\alpha} \) we can write for the second term on the right-hand side of (34)

\[
\int_0^T \left[ \frac{h_2-h_1}{h_1^2} b_{h,2}(R u^1, R u^1, \psi^2) \right] \, dt \leq C_{K,\alpha} \int_0^T \|\tilde{h}\|_\infty \int_D \|u^1\| \|\nabla u^1\| \|\psi^2\| + \|u^1\|^2 \|\nabla \psi^2\| \, dy dt. \tag{57}
\]

Now, using Hölder and Sobolev inequalities as above we obtain

\[
\int_0^T \left[ \frac{h_2-h_1}{h_1^2} b_{h,2}(R u^1, R u^1, \psi^2) \right] \, dt \leq 
C_{K,\alpha} \int_0^T \|\tilde{h}\|_\infty \left( \|\nabla u^1\| \|\nabla u^1\| L^2 \|\psi^2\| \|\nabla \psi^2\| + \|u^1\| \|\nabla u^1\| \|\nabla \psi^2\| L^2 \right) \, dt. \tag{58}
\]

With additional use of Young’s inequality \( ab \leq \frac{1}{p} a^p + \frac{p-1}{p} b^{p-1} \), \( 1 < p < \infty \)
with \( p = 4 \) the first term on the right-hand side of (58) can be bounded from above by

\[
\frac{1}{4} \int_0^T \|\psi^2\|^2 L^2 \|\nabla u^1\|^2 L^2 \, dt 
+ \frac{3}{4} C_{K,\alpha} \|\tilde{h}\|_{L^\infty((0,T) \times (0, L))} \|u^1\|_{L^{2/(1-2/3)}(0,T;L^2(D))} \|\nabla \psi^2\|_{L^2} \|\nabla u^1\|_{L^{4/3}} L^2 \, dt.
\]

Consequently, by applying Young’s inequality \( (55) \) with \( p = 3 \) we get for this term

\[
\frac{1}{4} \int_0^T \|\nabla \psi^2\|^2 L^2 \|\nabla u^1\|^2 L^2 \, dt + \varepsilon \int_0^T \|\nabla \psi^2\|^2 L^2 \, dt 
+ \frac{3}{4} C_{\varepsilon} C_{K,\alpha} \|\tilde{h}\|^2_{L^\infty((0,T) \times (0, L))} \|u^1\|_{L^{2/(1-2/3)}(0,T;L^2(D))} \int_0^T \|\nabla u^1\|^2 L^2 \, dt.
\]

The second right-hand side term in (58) can be bounded using Young’s inequality \( (55) \) with \( p = 2 \) by

\[
\varepsilon \int_0^T \|\nabla \psi^2\|^2 L^2 \, dt + C_{\varepsilon} C_{K,\alpha} \|\tilde{h}\|^2_{L^\infty((0,T) \times (0, L))} \|u^1\|^2_{L^{2/(1-2/3)}(0,T;L^2(D))} \int_0^T \|\nabla u^1\|^2 L^2 \, dt.
\]

We finally have

\[
\int_0^T \left[ \frac{h_2-h_1}{h_1^2} b_{h,2}(R u^1, R u^1, \psi^2) \right] \, dt \leq \frac{1}{4} \int_0^T \|\psi^2\|^2 L^2 \|\nabla u^1\|^2 L^2 \, dt 
+ 2\varepsilon \int_0^T \|\nabla \psi^2\|^2 L^2 \, dt + C_{\varepsilon} C_{K,\alpha} \|\tilde{h}\|^2_{L^\infty((0,T) \times (0, L))} \int_0^T \|\nabla u^1\|^2 L^2 \, dt, \tag{59}
\]

\[27]
we obtain where $C$ is a constant depending on $\varepsilon^{-1}$ and on the norm $\|u^1\|_{L^\infty(0,T;L^2(D))}$.

III. Now we estimate the third term on the right-hand side of (34). Let us remark that with the assistance of (39) one can easily show that $|\nabla R_{1,j}| \leq C_{K,\alpha}(\bar{h}(1 + |h_{1y_1}|) + |\bar{h}_{y_1}| + |\bar{h}_{y_1y_1}|)$, compare e.g. the lines above (44). Thus

$$\int_0^T \left| \frac{h^2}{\mu} T h^3 \left( \mathbb{R} u^1, u^1, \psi^2 \right) \right| dt \leq C_{K,\alpha} \int_0^T \int_D |u^1|^2 |\psi^2| \left( \|\bar{h}\|_{L^\infty(1,\infty)}(1 + |h_{1y_1}|) + |\bar{h}_{y_1}| \right).$$

We firstly apply the Hölder inequality with respect to the integral $\int_0^1 dy_2$ on the right-hand side,

$$\int_0^T \left| \frac{h^2}{\mu} T h^3 \left( \mathbb{R} u^1, u^1, \psi^2 \right) \right| dt \leq \int_0^T \int_0^L \left( \|\bar{h}\|_{L^\infty(1,\infty)}(1 + |h_{1y_1}|) + |\bar{h}_{y_1}| \right) \|u^1\|_{L^2(0,1)} \|u^1\|_{L^4(0,1)} \|\psi^2\|_{L^4(0,1)} dy_1 dt,$$

next we apply Lemma 3.2 to $\|u^1\|_{L^2(0,1)}$ and the Hölder inequality with respect to the integral $\int_0^L dy_1$,

$$\int_0^T \left| \frac{h^2}{\mu} T h^3 \left( \mathbb{R} u^1, u^1, \psi^2 \right) \right| dt \leq \int_0^T \int_0^L \left( \|\bar{h}\|_{L^\infty(1,\infty)}(1 + \|h^1\|_{H^2(0,L)} + \|\bar{h}\|_{H^2(0,L)}) \|\nabla u^1\|_{L^2(0,1)} \|u^1\|_{L^4(0,1)} \|\psi^2\|_{L^4(0,1)} dy_1 dt,$$

where $\| \cdot \|_{L^r} := \| \cdot \|_{L^r(D)}$, $r = 2, 4$. Finally, using the Sobolev interpolations as in the first item I), Young’s inequalities with $p = 4$ and (55) with $p = 3$, we obtain

$$\int_0^T \left| \frac{h^2}{\mu} T h^3 \left( \mathbb{R} u^1, u^1, \psi^2 \right) \right| dt \leq \frac{1}{4} \int_0^T \|\nabla u^1\|^2_{L^2(D)} dt + \int_0^T \|\nabla \psi^2\|^2_{L^2(D)} dt + C_\varepsilon^{HI} C_{K,\alpha} \left( C_\varepsilon^2 \|\bar{h}\|^2_{L^\infty(0,T;W^{1,\infty}(0,L))} + \|\bar{h}\|^2_{L^\infty(0,T;H^2(0,L))} \right) \int_0^T \|\nabla u^1\|_{L^2(D)} dt,$$
where \( C^{III}_{\varepsilon} \) is a constant depending on \( \varepsilon^{-1} \) and on the norm of \( u^1 \) in \( L^\infty(0,T;L^2(D)) \). We recall \( 1 + \|h^1\|_{L^\infty(0,T;H^2(0,L))} \leq C_h \) given by (47).

Thus, since \( C_h \geq 1 \) we can summarize the estimate of the third term on the right-hand side of (34),

\[
\int_0^T \left| \frac{h^2}{h^1} T_{h^2}(\mathbb{R} u^1, \psi^1, \mathbb{R} u^1) \right| dt \leq \frac{1}{4} \int_0^T \|\psi^2\|^2_{L^2} \|\nabla u^1\|^2_{L^2} dt + \varepsilon \int_0^T \|\nabla \psi^2\|^2_{L^2} dt
\]

(61) \( + C^{III}_{\varepsilon} C^2_{K,\alpha} C^2_h \left[ \|\delta h\|_{L^\infty(0,T;W^1,\infty(0,L))} + \|\delta h\|_{L^\infty(0,T;H^2(0,L))} \right] \int_0^T \|\nabla u^1\|^2_{L^2(D)} dt.
\]

IV) To the estimate the fourth term on the right-hand side of (34) we proceed analogously as in item III to come to

\[
\int_0^T \left| \frac{h^2}{h^1} T_{h^2}(\mathbb{R} u^1, \psi^1, \mathbb{R} u^1) \right| dt \leq C_{K,\alpha}\|u^1\|_{L^\infty(0,T;L^2(D))},
\]

(62) \( \left( \|\delta h\|_{L^\infty(0,T;W^1,\infty(0,L))} (1 + \|h^1\|_{L^\infty(0,T;H^2(0,L))}) + \|\delta h\|_{L^\infty(0,T;H^2(0,L))} \right) \int_0^T \|\nabla u^1\|^2_{L^2(D)} \|\psi^1\|^2_{L^2(D)} \|\nabla \psi^1\|^2_{L^2(D)} dt,
\]

cf. (60). Now replacing the norms of \( \psi^1 \) by norms of \( \psi^2 \) cf. (20) and (51) we get constant \( C^2_{K,\alpha} \) on the right-hand side of (62), however we for the sake of simplicity we denote it again as \( C_{K,\alpha} \). Moreover additional factor \( (1 + \|h^1\|_{L^\infty(0,T;H^2(0,L))} + \|h^2\|_{L^\infty(0,T;H^2(0,L))})^{1/2} \) bounded by \( C^2_h \), cf. (17), appears. Thus using Young’s inequalities as above we finally come to

\[
\int_0^T \left| \frac{h^2}{h^1} T_{h^2}(\mathbb{R} u^1, \psi^1, \mathbb{R} u^1) \right| dt \leq \frac{1}{4} \int_0^T \|\psi^2\|^2_{L^2} \|\nabla u^1\|^2_{L^2} dt + \varepsilon \int_0^T \|\nabla \psi^2\|^2_{W^{1,2}} dt
\]

(63) \( + C^{IV}_{\varepsilon} C^2_{K,\alpha} C^2_h \left[ \|\delta h\|_{L^\infty(0,T;W^1,\infty(0,L))} + \|\delta h\|_{L^\infty(0,T;H^2(0,L))} \right] \int_0^T \|\nabla u^1\|^2_{L^2(D)} dt.
\]

where \( C^{IV}_{\varepsilon} \) is a constant depending on \( \varepsilon^{-1} \) and on the norm of \( u^1 \) in \( L^\infty(0,T;L^2(D)) \), \( C_h \geq 1 \) is given by (47).

Summarizing the above estimates of convective error terms (56), (59), (61) and (63) we finally get for the difference of convective terms (34)
\[
\int_0^T b_{h_2}(\psi^2, \mathbb{R}u^1, \psi^2) + b_{h_1}^2 h_1 b_{h_2}(\mathbb{R}u^1, \mathbb{R}u^1, \psi^2)
- \frac{1}{2} \int_D \frac{h_2}{h_1} \left[ T_{h_2}(\mathbb{R}u^1, u^1, \psi^2) - T_{h_2}(\mathbb{R}u^1, \psi^1, \mathbb{R}u^1) \right] dy dt.
\]

\[
\leq \left( \frac{3}{4} + C^2_{\epsilon} C_{K, \alpha} \right) \int_0^T \| \psi^2 \|_{L^2}^2 \| \nabla u^1 \|_{L^2}^2 dt + 6 \varepsilon \int_0^T \| \psi^2 \|_{W^{1,2}}^2 dt
+ C^2_{\epsilon} C_{\epsilon \int} C^2_{h} + C^2_{\epsilon \int} C^3_{h} \left[ \| \bar{h} \|_{L^2(0,T; H^1(\Omega))} + \| \bar{h} \|_{L^2(0,T; H^2(\Omega))} \right].
\]

\[
\int_0^T \| \nabla u^1 \|_{L^2(D)} + \| \nabla u^1 \|_{L^2(D)}^2 dt.
\]

**Boundary terms**

It remains to estimate the boundary terms in (38), cf. (35), (36), (37). This can be done with the use of the following trace inequality for \(v \in V_{\text{div}},\)

\[
\| v \|_{L^r(\partial D)} \leq c \| v \|_{L^2(D)}^{\frac{1}{r}} \| \nabla v \|_{L^2(D)}^{\frac{r-1}{r}}, 0 < r < \infty,
\]

cf. [1] Lemma 3.2. As first we demonstrate the estimate of pressure terms containing \(q_{\text{in}}\). The pressure terms on the outflow \(S_{\text{out}}\) and the bottom boundary \(S_w\) can be bounded analogously. We denote \(q_{\text{in}}^1 - q_{\text{in}}^2 =: \bar{q}_{\text{in}}\). We have

\[
\int_0^T \int_0^1 |q_{\text{in}}^1 - q_{\text{in}}^2| |h_1^1| \psi_1^2(0, y_2, t)|dy_2 dt \leq \int_0^T \alpha^{-1} \| \bar{q}_{\text{in}} \|_{L^2(S_{\text{in}})} \| \psi^2 \|_{W^{1,2}(D)} dt
\]

\[
\leq \varepsilon \int_0^T \| \psi^2 \|_{W^{1,2}(D)} dt + c \alpha^{-2} \int_0^T \| \bar{q}_{\text{in}} \|_{L^2(S_{\text{in}})}^2 dt.
\]

The remaining boundary terms on \(S_w\) containing \(h_1^1 - h_2^1\) and \(h_1^2 - h_2^2\) can be estimated as follows. By the Hölder inequality we have

\[
\int_0^T \int_0^1 \frac{h_2^1}{h_1^1} |h_1^1 - h_2^1| u_2^1 \psi_2^2 |(y_1, 1, t)| + \frac{h_2^1 - h_1^1}{h_1^1} |q_{\text{in}}^1 + \frac{h_2^1}{2} u_2^1 \psi_2^2 |(y_1, 1, t)| dy_1 dt
\]

\[
\leq \int_0^T \alpha^{-2} \| \bar{h}_t \|_{L^2(0,L)} \| u^1 \|_{L^4(S_w)} \| \psi^2 \|_{L^4(S_w)} + \alpha^{-1} \| q_{\text{in}}^1 \|_{L^2(0,L)} \| \bar{h} \|_{L^\infty(0,L)} \| \psi^2 \|_{L^2(S_w)} + \alpha^{-1} \| \bar{h} \|_{L^\infty(0,L)} \| h_1^2 \|_{L^2(0,L)} \| u^1 \|_{L^4(S_w)} \| \psi^2 \|_{L^4(S_w)} dt.
\]
Using the trace argument mentioned above, \(|v|_{L^r(S_w)} \leq c\|\nabla v\|_{L^2(D)}, \ r = 2, 4\) and applying the Young inequality (55) with \(p = 2\) we obtain for the remaining boundary terms on \(S_w\)

\[
\leq 3\varepsilon \int_0^T \|\psi^2\|_{W^{1,2}(D)}^2 dt + C\varepsilon C_{K,\alpha} \|\tilde{h}_t\|_{L^\infty(0,T;L^2(0,L))}^2 \int_0^T \|\nabla u^1\|_{L^2(D)}^2 dt,
\]

\[
+ C\varepsilon C_{K,\alpha} \|\tilde{h}\|_{L^\infty(0,T;L^\infty(0,L))}^2 (1 + c_h^2) \int_0^T \|\nabla u^1\|_{L^2(D)} + \|q_w^1\|_{L^2(0,L)}^2 dt.
\]

where \(c_h := \|\tilde{h}_t^2\|_{L^\infty(0,T;L^2(0,L))}\).

Thus, collecting all estimates of the boundary terms we obtain

\[
\int_0^T \int_0^1 -(q_{1n}^1 - q_{2n}^1)h_1^1 \psi_1^2(0, y_2, t) + (q_{out}^1 - q_{out}^2)h_1^1 \psi_1^2(L, y_2, t) dy_2 dt 
\]

\[
- \int_0^T \int_0^L (q_w^2 - q_w^-)\psi_2^2(y_1, 1, t) + (h_1^2 - h_2)u_1 \psi_2^2(y_1, 1, t)
\]

\[
+ \frac{h_1^2 - h_2}{h_1^2}(q_w^1 + \frac{h_1^2}{2}u_2^1)\psi_2^2(y_1, 1, t) dy_1 dt
\]

\[
\leq 6\varepsilon \int_0^T \|\psi^2\|_{W^{1,2}(D)}^2 dt + \tilde{C}_\varepsilon C_{K,\alpha} \int_0^T \|\tilde{q}_{in}\|_{L^2(S_{in})}^2 + \|\tilde{q}_{out}\|_{L^2(S_{out})}^2 + \|q_w\|_{L^2(0,L)}^2 dt
\]

\[
+ \tilde{C}_\varepsilon C_{K,\alpha} \tilde{c}_h (\|\tilde{h}\|_{W^{1,\infty}(0,T;L^2(0,L))} + \|\tilde{h}\|_{L^\infty(0,T;L^\infty(0,L))}) \int_0^T \|\nabla u^1\|_{L^2(D)} + \|q_w^1\|_{L^2(0,L)}^2 dt,
\]

where \(Q := (0, T) \times (0, L), \tilde{C}_\varepsilon\) depends on \(\varepsilon^{-1}\) and \(\tilde{c}_h := 1 + \|\tilde{h}_t^2\|_{L^\infty(0,T;L^2(0,L))}\) is bounded.

### 3.4 Final estimate

Let us summarize the above estimates of the right-hand side of (38). Note that the constants \(C_{e,IV}, C_h\) and \(c_h\) in the estimates (42), (53), (64) depend on the positive powers of the norms \(|u|_{L^\infty(0,T;L^2(D))}, \|h^1\|_{L^\infty(0,T;L^2(D))}, \|h^2\|_{L^\infty(0,T;H^2(0,L))}, i = 1, 2\) cf. (17) and \(|h^2|_{W^{1,\infty}(0,T;L^2(0,L))}\), respectively.
Applying the Korn inequality we finally get from (38):

\[ g(T) + \frac{\alpha \mu}{\rho} \tilde{c}_{Ko} \int_0^T \| \psi^2 \|^2_{W^{1,2}(D)} + \frac{c}{E} \| \partial_{y_1}^2 \xi \|^2_{L^2(0,L)} dt \]

\[ \leq 17 \varepsilon \int_0^T \| \psi^2 \|^2_{W^{1,2}(D)} + C_1 \int_0^T \| u^1 \|^2_{W^{1,2}(D)} \| \psi^2 \|^2_{L^2(D)} dt + \psi(T), \]

where

\[ g(t) := \frac{\alpha}{2} \| \psi^2 \|^2_{L^2(D)}(t) + \frac{1}{2E} \| \xi \|^2_{L^2(0,L)}(t) \]

\[ + \frac{bE}{2} \| \tilde{\eta} \|^2_{L^2(0,L)}(t) + \frac{aE}{2} \| \partial_{y_1} \tilde{\eta} \|^2_{L^2(0,L)}(t), \]

\[ \psi(t) := C_2 \int_0^t \| \tilde{q}_{in} \|^2_{L^2(S_{in})} + \| \tilde{q}_{out} \|^2_{L^2(S_{out})} + \| \tilde{q}_w \|^2_{L^2(S_w)} ds \]

\[ + C_3 \omega(t) \left[ \| \tilde{h} \|^2_{L^2((0,t),W^{1,\infty}(0,L))} + \| \tilde{h}_t \|^2_{L^2((0,t),H^2(0,L))} + \| \tilde{h}_{tt} \|^2_{W^{1,\infty}(0,t;L^2(0,L))} \right], \]

\[ \omega(t) := \int_0^t \| u^1 \|^2_{W^{1,2}(D)} + \| u^1 \|^2_{W^{1,2}(D)} + \| h_t \|^2_{L^2(1,\infty)} + \| h_{tt} \|^2_{L^2(0,L)} + \| q_w \|^2_{L^2(S_w)} ds, \]

\[ \omega(t) \downarrow 0 \text{ as } t \downarrow 0, \text{ and} \]

\[ C_1 = 2 + C_1^2 C_{K,\alpha}^2, \quad C_2 = C_2 C_{K,\alpha}^2, \]

\[ C_3 = C_3^2 [1 + C_3 + C_3^2 (1 + C_6^2 C_{K,\alpha}^2 + C_{I^2} + C_6^2 C_{I^2}^2 + C_6^3 C_{I^3}^2 + C_6 C_{I^3}] , \]

here \( C_{I^2} \) denotes a constant depending on \( \varepsilon^{-1} \) and \( C_{K,\alpha} \).

Now, we chose an \( \varepsilon \) small enough such that \( \frac{\alpha \mu}{\rho} \tilde{c}_{Ko} - 17 \varepsilon \geq \frac{\alpha \mu}{2\rho} \tilde{c}_{Ko} \), i.e. \( \varepsilon \leq \frac{\alpha \mu}{34\rho} \tilde{c}_{Ko} \). Thus, the constants of type \( C_{I^2} \) (and consequently the constants \( C_1, C_1, C_3 \)) depend on \( (\frac{\alpha \mu}{\rho} \tilde{c}_{Ko})^{-1} \). By this choice of \( \varepsilon \), the inequality (66) yields

\[ g(T) + \frac{\alpha \mu}{2\rho} \tilde{c}_{Ko} \int_0^T \| \psi^2 \|^2_{W^{1,2}(D)} + \frac{c}{E} \| \partial_{y_1}^2 \xi \|^2_{L^2(0,L)} dt \]

\[ \leq C_1 \int_0^T \| u^1 \|^2_{W^{1,2}(D)} \| \psi^2 \|^2_{L^2(D)} dt + \psi(T). \]

Moreover, (66), (67) are also valid for all fixed \( t, 0 \leq t \leq T \). Therefore after replacing \( T \) by \( t \) and omitting positive terms in (67) we obtain

\[ g(t) \leq \psi(t) + C_4 \int_0^t \| u^1 \|^2_{W^{1,2}(D)}(s) g(s) ds, \quad C_4 = 2C_1 \alpha^{-1}. \]

Since \( \| u^1 \|^2_{W^{1,2}(D)} \geq 0 \) is an integrable in time and \( \psi \geq 0 \) is a continuous non-decreasing function of time, the Gronwall lemma, see e.g., [26] Lemma
Theorem 3.1

Let \((h_t)\) be two weak solutions of the initial boundary value problem (7)–(8) transformed to the rectangular fixed rectangular domain \(D\) satisfying (13). Let the corresponding domain deformation be given by some functions \(h^1, h^2\) satisfying (20). Let the transformed boundary pressures \(q^1_{in/out/w}\) and \(q^2_{in/out/w}\) belong to \(L^2(0,T; L^2(S_{in/out/w}))\), respectively. Then for almost
all \( t \in (0, T) \) it holds:

\[
\frac{\alpha}{2} \| \mathbb{R} \mathbf{u}^1 - \mathbf{u}^2 \|^2_{L^2(D)}(t) + \frac{\alpha \mu}{2 \rho} \tilde{c}_{K_0} \int_0^t \| \mathbb{R} \mathbf{u}^1 - \mathbf{u}^2 \|^2_{W^{1,2}(D)} ds
\]

\[
+ \frac{E}{2} \| \eta^1 - \eta^2 \|^2_{L^2(\Omega)}(t) + \frac{bE}{2} \| \eta^1 - \eta^2 \|^2_{L^2(\Omega)}(t) + \frac{aE}{2} \| \eta^1_{\eta_1} - \eta^2_{\eta_1} \|^2_{L^2(\Omega)}(t)
\]

\[
+ cE \int_0^t \| \eta^1 - \eta^2 \|^2_{H^2(\Omega)} ds
\]

\[
\leq C_7 \left( \int_0^t \| q_{in}^1 - q_{in}^2 \|^2_{L^2(\Omega)} + \| q_{out}^1 - q_{out}^2 \|^2_{L^2(\Omega)} + \| q_w^1 - q_w^2 \|^2_{L^2(\Omega)} ds 
\right.
\]

\[
\left. + \omega(t) \left( \| h^1 - h^2 \|^2_{W^{1,\infty}(0,T;L^2(\Omega))} + \| h^2 - h^2 \|^2_{L^2(0,T;L^2(\Omega))} \right) \right)
\]

where

\[
\omega(t) = \int_0^t \| \mathbf{u}^1 \|^2_{W^{1,2}(D)} + \| \mathbf{u}^1 \|^2_{W^{1,2}(D)} + \| h^1 \|^2_{W^{1,\infty}(0,T;L^2(\Omega))} + \| h^2 \|^2_{L^2(0,T;L^2(\Omega))} + \| q_w^1 \|^2_{L^2(S_w)} ds
\]

and \( \omega(t) \downarrow 0 \) for \( t \downarrow 0 \).

Here \( \tilde{c}_{K_0} \) is the coercivity constant of the viscous form coming from the Korn inequality, \( \alpha, K \) are given by (A), \( \mu, \rho, E, a, b, c \) are given by the physical model and \( C_7 \equiv \left( C_2 + C_3 \right) c_0 \) (see the proof above) is a constant depending on \( \alpha, \alpha^{-1}, K, \tilde{c}_{K_0} \) and on the norms \( \| h^1 \|^2_{L^\infty(0,T;H^2(\Omega))} \), \( \| h^2 \|^2_{W^{1,\infty}(0,T;L^2(\Omega))} \), \( \| \mathbf{u}^1 \|^2_{L^\infty(0,T;L^2(\Omega))} \), \( i = 1, 2 \).

Let us note that the reason for presence of matrix \( \mathbb{R} = J\mathbb{J}^{-1} \), cf. (19), in the above estimate lies in the fact that the domain of definition for solutions \( \mathbf{v}^1 \) and \( \mathbf{v}^2 \) differs due to the coupling with the domain deformation. We recall that \( \int_{\Omega(h^1)} v^1(X, t) dX = \int_{\Omega(h^2)} v^1(\Phi(x), t) J dx = \int_{\Omega(h^2)} v^1(x, t) J dx \), where \( \Phi : \Omega(h^2) \rightarrow \Omega(h^1) \) and \( J = \text{det } \mathbb{J} \), \( \mathbb{J} = \frac{dX}{dx} \).

**Remark 3.1.**

Since \( \mathbf{u}^1 - \mathbf{u}^2 = \mathbb{R} \mathbf{u}^1 - \mathbf{u}^2 + \mathbb{E}_{\mathbb{R}} \mathbf{u}^1 \), see the lines before (31), we can write

\[
\| \mathbf{u}^1 - \mathbf{u}^2 \|^2_{W^{1,2}(D)} \leq \| \mathbb{R} \mathbf{u}^1 - \mathbf{u}^2 \|^2_{W^{1,2}(D)} + \| \mathbb{E}_{\mathbb{R}} \mathbf{u}^1 \|^2_{W^{1,2}(D)}.
\]

Since the components of the matrix \( \mathbb{E}_{\mathbb{R}} \) contain \( \hat{h} \) and \( E \), see (39), with the

34
assistance of (9) and Lemma 3.2 we finally get
\[
\|\mathbf{u}^1 - \mathbf{u}^2\|_{L^2(D)} \leq \|\mathbb{R}\mathbf{u}^1 - \mathbf{u}^2\|_{L^2(D)} + C_{K,\alpha} \|\bar{h}\|_{W^{1,\infty}(0,L)} \|\mathbf{u}^1\|_{L^2(D)} \|h^1\|_{W^{1,\infty}(0,L)} ,
\]
\[
\|\mathbf{u}^1 - \mathbf{u}^2\|_{W^{1,2}(D)} \leq \|\mathbb{R}\mathbf{u}^1 - \mathbf{u}^2\|_{W^{1,2}(D)} + C_{K,\alpha} \|\bar{h}\|_{W^{1,\infty}(0,L)} \|\mathbf{u}^1\|_{W^{1,2}(D)} + C_{K,\alpha} \|\bar{h}\|_{H^2(0,L)} \|\mathbf{h}^1\|_{H^2(0,L)} \|\mathbf{u}^1\|_{W^{1,2}(D)}. 
\]

Hence for \(h^1 \to h^2\) in \(W^{1,\infty}(0,T;L^2(0,L))\cap H^1(0,T;H_0^1(0,L))\) and \(g^1_{in/out/w} \to g^2_{in/out/w}\) in \(L^2(0,T;L^2(\partial D))\) the above estimates and (70) imply that \(\|\mathbf{u}^1 - \mathbf{u}^2\|_{L^2(0,T;W^{1,2}(D))\cap L^\infty(0,T;L^2(D))} \to 0\). Thus the weak solution is continuously dependent on the domain deformation and the boundary pressure.

\[\Box\]

**Corollary 3.1 (Uniqueness of the weak solution).**

If the boundary conditions for the pressure as well as the boundary deformation coincide for both solutions, i.e., \(q^1_{in} = q^2_{in}, q^1_{out} = q^2_{out}, q^1_w = q^2_w, h^1 = h^2 := h\), then there exists a unique weak solution to the problem (7)-(8) on \(\Omega(h)\), i.e., to the problem linearized with respect to the geometry in the sense of (9)-(18).

**Proof.** The proof is a consequence of Theorem 3.1. Note that for \(h^1 = h^2\) we have \(\mathbb{R} = 1\) in (70). \[\Box\]

### 4 Fixed point procedure with respect to the geometry

In the previous section we have proven uniqueness of the weak solution \((\mathbf{u}, \eta)\) defined on \(\Omega(h)\) given by some sufficiently smooth functions \(\delta, R_0, h := R_0 + \delta\) satisfying (9). This allows us to consider the following fixed point procedure with respect to the geometry of the computational domain. We consider a mapping \(\mathcal{F}\) that is well defined through the weak formulation on \(\Omega(h)\) (or its transformed form (13)). It maps the given function \(\delta\) to the unique weak solution \(\eta, \mathcal{F}(\delta) = \eta\). Once there exists a fixed point of this mapping, the original fluid-structure interaction problem (11)-(18) is solved. Let us point out that the Schauder fixed point argument yields the existence (but not uniqueness) of the weak solution to similar problems, see [5, 6, 10, 7, 8]. Uniqueness for similar problems for Newtonian fluids has been shown e.g. in

\[\text{with analogous consideration as estimates in (44), (45).}\]
by deriving an additional estimate on the continuous dependence on the initial data.

In the following procedure we apply the Banach fixed point theorem and obtain a unique fixed point \( \eta^* = F(\eta^*) \). Using this fixed point argument we prove at the same time the convergence of the iterative process \( \eta^k = F(\eta^{k-1}) \), i.e. the convergence of the global iterative method with respect to the domain geometry. Additionally, Theorem 3.1 implies uniqueness of \( u \), thus we finally obtain uniqueness of the (complete) weak solution \((u, \eta)\) obtained by this iterative procedure.

We consider following iterative process with respect to \( \eta \):

Let \( \delta = \eta^k - \eta^{k-1} \), \( k = 1, 2, \ldots \), with given \( \eta^0 \), e.g., \( \eta^0 = 0 \). Let \((u^k, \eta^k)\) be the weak solution obtained on \( \Omega(h) = \Omega(R_0 + \eta^{k-1}) \), i.e., \((u^k, \eta^k)\) satisfies (13) for \( h = R_0 + \eta^{k-1} \).

We define the space \( Z := H^1(0,T; H^2_0(0,L)) \cap W^{1,\infty}(0,T; L^2(D)) \) and the mapping

\[
F : B_\alpha \subset Z \rightarrow Z, \quad F(\eta^{k-1}) = \eta^k.
\] (71)

Here \( B_\alpha \) is a ball in space \( Z \) chosen such that the necessary properties of the domain deformation from (9), avoiding the contact of the deform wall with the fixed bottom boundary \( \text{[ footnote]} \) are guaranteed. Let us specify \( B_\alpha \) in more details. Let \( 0 < \alpha < 1 \), \( R_0 \in C^2[0,L], 0 < R_{\text{min}} \leq R_0 \leq R_{\text{max}}, \{R_{\text{min}}, R_{\text{max}}\} \in \mathbb{R}^+ \) be given. Moreover let \( \alpha \) be such that

\[
\alpha < \min\{R_{\text{min}}, (R_{\text{min}} + R_{\text{max}})^{-1}\}. \quad (72)
\]

We define

\[
B_\alpha = \{ \eta \in Z \text{ s.t. } \eta(y_1,0) = \eta_t(y_1,0) = \eta_{y_1}(y_1,0) = 0, \|\eta\|_Z \leq R_{\text{min}} - \alpha \}. \quad (73)
\]

In what follows we show that each \( \eta \in B_\alpha \) satisfies properties (2) for sufficiently small time \( T \) and small boundary pressures. Indeed, due to \( Z \subset C(0,T; C^1[0,L]) \) and the zero initial conditions there exists a time \( T_\alpha \) such that it holds for each \( \eta \in B_\alpha \):

\[
\max_{0 \leq t \leq T_\alpha, 0 \leq y_1 \leq L} |\eta(y_1, t)| = \|\eta\|_{C([0,T_\alpha] \times [0,L])} \leq R_{\text{min}} - \alpha.
\]

\[4\text{The condition } \alpha \leq R_0 + \eta \leq \alpha^{-1} \text{ in (9) avoids the contact of the moving wall with the fixed bottom boundary for } \alpha > 0.\]
Then the condition $\alpha \leq R_0 + \eta \leq \alpha^{-1}$ is satisfied for each $\eta \in B_{\alpha}$ and for all $t \leq T_\alpha$, where $\alpha$ is given by property (72).

Moreover since $\eta(y_1,0) = \eta_{y_1}(y_1,0) = 0 \ \forall y_1 \in [0,L]$, there exists a maximal time $T_K$ such that

$$\int_0^{T_K} |\eta(y_1,s)|^2 ds + |\eta_{y_1}(y_1,t)|^2 \leq K \ \forall y_1 \in [0,L], \ \forall t \in [0,T_K].$$

Thus, each function from $B_\alpha$ fulfills both properties (9) for all $y_1 \in [0,L]$ and sufficiently small time $\bar{T} = \min\{T_\alpha, T_K\}$ and it is an admissible function for the domain deformation in view of our approximation.

In order to apply the Banach fixed point theorem to our mapping $F$ we have to verify that this mapping is “onto”, i.e., $F(B_\alpha) \subseteq B_{\alpha}$. Indeed for $\eta^{k-1} \in B_{\alpha}$ the weak solution obtained using $h = R_0 + \eta^{k-1}$ satisfies the following a priori estimate shown in [7], Section 5, cf. (5.1) and (4.7),

$$\|u^k\|_{L^\infty(0,T;L^2(\partial D))}^2 + \|u^k\|_{L^2(0,T;W^{1,2}(\partial D))}^2 + \|\eta^k\|_{L^\infty(0,T;H^1(0,L))}^2 + \|\eta^k_t\|_{L^2(0,T;L^2(0,L))}^2 + \|\eta^k_{y_1}\|_{L^2(0,T;H^2(0,L))}^2 \leq C(\alpha, \alpha^{-1})(1 + H\epsilon^H) \left( T\|R_0\|_{C^2[0,L]}^2 + \int_0^T \|q_{\partial D}\|_{L^2(\partial D)}^2 dt \right), \quad (74)$$

where $H \leq \int_0^T |\eta^k_{y_1}| + |\eta^k_{y_1}|^2 dt$ for all $y_1 \in [0,L]$ and $\|q_{\partial D}\|_{L^2(\partial D)}^2$ represents the sum of the norms of the boundary pressures, i.e., $\|q_{\partial D}\|_{L^2(\partial D)}^2 := \|q_{\partial D}\|_{L^2(S_{\alpha}^i)}^2 + \|q_{\partial D}\|_{L^2(S_{out})}^2 + \|q_{\partial D}\|_{L^2(S_{out})}^2$. Note, that due to (9) which is fulfilled by all $\eta^k$, $k = 1, 2, \ldots$, for $\bar{T}$ specified above, $H$ is also uniformly bounded with $H \leq \sqrt{TK} + \bar{K}$ for a final time $T \leq \bar{T}$. Thus the right-hand side of (74) can be bounded uniformly in $k$. Moreover, (74) implies that

$$\|\eta^k\|_Z^2 \leq C(\alpha, \alpha^{-1})(1 + (\sqrt{TK} + \bar{K})e^{(\sqrt{TK} + \bar{K})}) \left( T\|R_0\|_{C^2[0,L]}^2 + \int_0^T \|q_{\partial D}\|_{L^2(\partial D)}^2 dt \right)$$

for $T \leq \bar{T}$. Therefore, for given $\alpha$, $K$ and sufficiently small time $T := T_{\text{max}} \leq \bar{T}$ we get $\|\eta^k\|_Z \leq R_{\text{min}} - \alpha$. Consequently, $F(B_{\alpha}) \subseteq B_{\alpha}$ for small enough final time $T_{\text{max}}$.

The next step is to verify the contractivity of the mapping $F$, i.e., to show for $\delta^1 \neq \delta^2$, $\delta^1 := \eta^1, k, \delta^2 := \eta^2, k$ and $F(\delta^1) = \eta^1, k$, $F(\delta^2) = \eta^2, k$ that

$$\|F(\delta^1) - F(\delta^2)\|_Z \leq q\|\delta^1 - \delta^2\|_Z, \quad \text{where} \quad q < 1. \quad (75)$$
Theorem 3.1 from the previous section provides the essential estimate to prove the contractivity. Indeed, (70) implies for $q^1_{in} = q^2_{in}$, $q^1_{out} = q^2_{out}$, $q_w = q^2_w$ that

$$
\|\eta^{1,k} - \eta^{2,k}\|_{W^{1,\infty}(0,T;L^2(D)) \cap H^1(0,T;H^2(D))} \\
\leq C(E, a, b, c) C_7 \omega(t) \|\eta^{1,k-1} - \eta^{2,k-1}\|_{W^{1,\infty}(0,T;L^2(D)) \cap H^1(0,T;H^2(D))}^2
$$

where $\omega(t) \downarrow 0$ as $t \downarrow 0$. The constant $C(E, a, b, c)$ depends only on physical data, whereas the constant $C_7$ depends on $K, \alpha, \alpha^{-1}$, on $C_h$, cf. (17), on $\tilde{c}_h$, cf. (55) and on $C^{IV}_h$, thus $C_7$ contains positive powers of $\|\eta^{i,k-1}\|_{L^{\infty}(0,T;H^2(0,L))}$, $\|\eta^{i,k-1}\|_{W^{1,\infty}(0,T;L^2(D))}$, $\|\eta^{i,k}\|_{L^{\infty}(0,T;L^2(D))}$, $i = 1, 2$. Estimate (74) and analogous consideration as above imply $\|\bar{u}^{1,k}\|_{L^{\infty}(0,T;L^2(D))} \leq R_{\min} - \alpha$ for sufficiently small $T \leq T_{\max}$ as well as $C_h \leq 1 + 2(R_{\min} - \alpha)$ and $\tilde{c}_h \leq (1 + (R_{\min} - \alpha)$. Thus $C_7$ can be bounded from above with some constant independent of $k$.

In view of this considerations (75) holds with $q = C(E, a, b, c) C_7 \omega(t)$ for sufficiently small $t \leq T_{\max}$ such that $\omega(t) < \frac{1}{C(E,a,b,c)C_7}$. We have shown, that the mapping $\mathcal{F}$ is a contraction.

Finally, the Banach fixed point theorem, see, e.g., [27, Theorem 1.5], implies that for sufficiently small time and $0 < \alpha < 1$ satisfying (72), there exists one and only one fixed point $\eta^*$ of the mapping $\mathcal{F}$ defined in (71), $\eta^* = \mathcal{F}(\eta^*)$. Moreover, the iterative process $\eta^k = \mathcal{F}(\eta^{k-1})$ converges, i.e., $\eta^k \to \eta^*$ in $\mathcal{Z}$. The continuous dependence of the fluid velocity $u$ on the domain deformation (Theorem 3.1) and Corollary 3.1 applied for $h^1 = R_0 + \eta^*$, $h^2 = R_0 + \eta^k$ imply moreover the convergence of $u^k \to u^*$ in $L^2(0,T;W^{1,2}(D)) \cap L^{\infty}(0,T;L^2(D))$ and the uniqueness of the weak solution $(u^*, \eta^*)$ defined on $\Omega(\eta^*)$.

Thus, we have proven that the global iterative method with respect to the domain deformation, used in our numerical approximation of the problem (11)-(18), see [11, 12, 28], converges at least for sufficiently small time. In this construction of the solution the contact of the moving wall with the fixed bottom boundary is avoided.

Concluding remarks

In [11, 12] we have studied the existence of weak solution to the similar fluid-structure interaction problem for shear-dependent fluids with viscosity obeying the power-law $\mu(\|e(v)\|) = \mu(1 + |e(v)|^2)^{\frac{p-2}{2}}$ for $p \geq 2$. In that case the
corresponding space of weak velocities is \( L^p(0, T; W^{1,p}(D)) \). Therefore, the norms \( \| h^1 - h^2 \| \) on the right-hand side of estimate (70) would appear with powers 2 and \( p' < 2 \), where \( \frac{1}{p} + \frac{1}{p'} = 1 \). This leads to a difficult problem, since such an estimate would only imply \( \| F(\delta^1) - F(\delta^2) \|_Z \leq q\| \delta^1 - \delta^2 \|_Z^{p'/2} \), \( q < 1 \), compare (75). This estimate does not imply the Cauchy-property of the sequence \( \eta^k \), that is essential for the limiting process \( k \to \infty \) in the proof the Banach fixed point theorem and consequently for the existence of a fixed point.

Another challenge is to show the continuous dependence of the weak solution on initial data for a shear-dependent power-law fluid and to generalize the result for Newtonian fluids from [10]. In this case, again, due to the \( L^p \)-structure of the fluid velocity, the estimate of the viscous term may cause a difficulty, since except of the \( p \)-th power of the gradients of the weak solution it also contains the term \( \partial_{y^1}^2 (\eta^1) \), cf. (27), that is only bounded in \( L^2(0, L) \). Thus, more regularity of the weak solution would be necessary for the corresponding estimates. In this case the so called weak - strong uniqueness would be an interesting tool to study such fluid-structure interaction problems for power-law fluids. This may be a goal of our future work.

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A Compactness argument for the \((\kappa, \varepsilon)\)-approximate weak solution

The existence of a weak solution to the problem (1)–(8) on the deformable domain \( \Omega(h) \) moving according to the a priori given function \( h(y_1, t) \), cf. (9), for power-law fluids has been studied in our previous work [7] by transformation to the fixed rectangle domain \( D \). See also [3] for similar result for Newtonian fluid. We have applied the artificial compressibility method in order to handle the divergence-free condition and regularized it using the parabolic equation for pressure \( \varepsilon \partial_t p - \varepsilon \Delta p + \text{div}\mathbf{v} = 0 \). Moreover, the fluid-structure coupling condition on the moving boundary (2), (3) has been approximated by introduction of the semi-pervious boundary, see [7] (2.6)–(2.7) for more details.
Condition (3) has been replaced by \( \kappa(\partial_t \eta - v_2|_{\Gamma_w}) \), where \( \kappa \) is a penalization parameter; the original coupling condition is satisfied for \( \kappa = \infty \).

Taking \( \kappa = \varepsilon^{-1} \), the passage to the limit in the \((\kappa, \varepsilon)\)-approximate weak solution \((u_\kappa, q_\kappa, \sigma_\kappa), u_\kappa := u_{\kappa,\varepsilon}, q_\kappa := q_{\kappa,\varepsilon}, \sigma_\kappa := \partial_t \eta_{\kappa,\varepsilon} \) for \( \varepsilon \to 0, \kappa \to \infty \) has been performed at once. Here \( q_{\kappa,\varepsilon}(y, t) = p_{\kappa,\varepsilon}(x, t)/\rho \) denotes the pressure transformed to fixed domain \( D \). For the limiting process in our \((\kappa, \varepsilon)\)-approximate weak formulation the strong convergence \( u_\kappa \to u \) is necessary.

Since the a priori estimates for \( \partial_t u_\kappa \) depend on \( \varepsilon, \kappa \), the classical Lions-Aubin compactness argument cannot be applied. The strong convergence for similar artificial compressibility approximation on fixed domains has been shown in [29, Ch. III., Th. 8.1] by the compactness argument involving fractional time derivatives [29, Ch. III., Th. 2.2] and the Fourier transform. However, due to difficulties related to the moving domain and transformed differential operators depending on time, this approach seems to be inappropriate for deformable domains.

In our approach we used a compactness criterion based on integral equicontinuity in time [30, Lemma 1.9]. The integral equicontinuity estimates have been shown in our previous works, see [7, (5.4)], [3, (8.4)]. However, some terms containing the velocity divergence has not been treated carefully in the mention works, the dependence of equicontinuity estimates on \( \varepsilon \) has been ignored.

In the following lines we present a corrected proof of those integral equicontinuity estimate for \( u_\kappa \), which is independent on \( \varepsilon \). To this end we however shall simplify our artificial compressibility condition and consider only elliptic equation for pressure,

\[-\varepsilon \Delta p_\kappa + \text{div} v_\kappa = 0 \text{ in } \Omega(h), \quad \nabla p_\kappa \cdot n = 0 \text{ on } \partial\Omega(h), \quad \int_{\Omega(h)} p_\kappa = 0\]

in our \((\kappa, \varepsilon)\)-approximation of (1)–(8). Note, that the proof of existence of weak solution for fixed \( \varepsilon, \kappa \) with only the elliptic (instead of parabolic) compressibility approximation can be obtained analogously as the proof in [7, Section 4], compare also [3], using the same techniques. Here the coercivity of the bilinear form arising in the pressure equation is now guaranteed due to the average condition \( \int_{\Omega(h)} p_\kappa = 0 \) (Poincaré-Wirtinger inequality).

Repeating the estimation process from [7] we obtain the first a priori estimates for \( u_\kappa(y, t) = v_\kappa(x, t) \) and \( \eta_\kappa(x_1, t) \). For the pressure one gets now \( p_\kappa(x, t) = q_\kappa(y, t) \in L^2(0, T; H^1(D)) \). These estimates are, similarly as in
independent on $\kappa, \varepsilon$. Using the pressure equation (i.e. considering zero test function for $u_\kappa, \eta_\kappa$) we obtain by letting $\kappa \to \infty$, that the weak limit of $u_\kappa$ is divergence free almost everywhere, compare [7, Section 5].

Now we concentrate on the proof of the compactness of $u_\kappa$. In what follows we show the integral equicontinuity for $u_\kappa$, which is independent on $\varepsilon$.

Note, that terms coming from the divergence of the velocity and the pressure gradient are bounded by a priori estimates with $c/\sqrt{\varepsilon}$ for some constant $c$, i.e., not uniformly in $\varepsilon$. The crucial idea in our new proof is to eliminate these terms by choosing appropriate test functions involving the so called Piola transformation of the solution.

The corresponding $(\kappa, \varepsilon)$-approximate weak formulation of the coupled problem with elliptic compressibility approximation reads as follows:

\[
\int_0^T -\langle \partial_t (hu_\kappa), \psi \rangle \, dt = (76)
\]

\[
\int_0^T \left\{ \int_D -\frac{\partial h}{\partial t} \frac{\partial (y_2 u_\kappa)}{\partial y_2} \cdot \psi - h q_\kappa \text{div} h \psi + h \text{div} h u_\kappa \phi \, dy \\
+ ((u_\kappa, \psi))_h + b_h(u_\kappa, u_\kappa, \psi) + \varepsilon a_1(q_\kappa, \phi) \\
+ \int_0^1 h(L)q_{\text{out}}(y_2, t)\psi_1(L, y_2, t) - h(0)q_{\text{in}}(y_2, t)\psi_1(0, y_2, t) \, dy_2 \\
+ \int_0^L \left( q_w + \frac{1}{2} \frac{\partial h}{\partial t} u_{2, \kappa} + \kappa(u_{2, \kappa} - \sigma_\kappa) \right) \psi_2(y_1, 1, t) \, dy_1 \\
+ \int_0^L \frac{\partial \sigma_\kappa}{\partial t} \xi + c \frac{\partial^2 \sigma_\kappa}{\partial y_1^2} \frac{\partial^2 \xi}{\partial y_1^2} + a \frac{\partial}{\partial y_1} \left( \int_0^t \sigma_\kappa(y_1, s) \, ds \right) \frac{\partial \xi}{\partial y_1}(y_1, t) \\
- a \frac{\partial^2 R_0}{\partial y_1^2} \xi + b \int_0^t \sigma_\kappa(y_1, s) \, ds \xi(y_1, t) + \frac{\kappa}{E} (\sigma_\kappa - u_{2, \kappa}) \xi \, dy_1 \right\} \, dt
\]

for every $(\psi, \phi, \xi) \in H^1_0(0, T; V) \times L^2(0, T; H^1(D)) \times L^2(0, T; H^2_0(0, L))$, where $V \equiv \{ W^{1,2}(D); \ w_1 = 0 \text{ on } S_w, \ w_2 = 0 \text{ on } S_{\text{in}} \cup S_{\text{out}} \cup S_c \}$, cf. [15], [16] and the definition of the form $a_1(\cdot, \cdot)$ in [7] (2.10).

Let us first specify appropriate test functions in order to obtain the integral equicontinuity estimate. By similar procedure as in [7] p. 221-222, i.e. considering test functions $\chi_\delta(s)(\psi, \phi, \xi)$, where $\chi_\delta(s)$ is a smooth approximation of the characteristic function of the interval $(t, t + \tau)$, $\psi = \psi(y, s)$ is now a time-dependent function, $\phi(y, \xi(y_1))$ are time-independent, and letting the
smoothness parameter $\delta \to 0$ we obtain from (76)

\[- \int_0^{T-\tau} \int_D [(h_u\kappa\psi)(y, t + \tau) - (h_u\kappa\psi)(y, t)] dy \]

\[+ \int_0^L [\sigma_\kappa(y_1, t + \tau) - \sigma_\kappa(y_1, t)] \xi(y_1) dy_1 dt \]

\[= \int_0^{T-\tau} \int_t^{t+\tau} \left\{ h(y_1, s) \text{div}_{h(s)} u_\kappa(y, s) \phi(y) - (hq_\kappa)(y, s) \text{div}_{h(s)} \psi(y, s) 
-h u_\kappa(y, s) \cdot \partial_t \psi(y, s) + \ldots \right\} dy ds dt. \quad (77)\]

Note that in our case the operator of divergence depends on $h(y_1, t)$, thus on time. In order to commute the integral $\int_t^{t+\tau} ds$ with the div$_h$ - operator and to eliminate the divergence terms on the right-hand side of (77) we rewrite them in terms of the time independent div - operator using the Piola transformation.

We consider the Piola transformation $v_P(y, t) : D \times [0, T] \to \mathbb{R}^2$ of our velocity field $v : \Omega(h) \times [0, T] \to \mathbb{R}^2$, compare e.g., [31, Chapter II]

\[v_P(y, t) := \det(\nabla L(y, t)) \nabla L^{-1}(y, t) v(x, t), \]

where the mapping $L(y, t) \overset{\text{def}}{=} (y_1, y_2 h(y_1, t)) = x(t) \in \Omega(h)$ describes the transformation of variables between $\Omega(h)$ and $D$,

\[\nabla L = \frac{dx}{dy} = \begin{bmatrix} 1 & 0 \\ -\frac{y_2}{h} \partial_{y_1} h & 1 \end{bmatrix} = : J, \quad \nabla L^{-1} = \begin{bmatrix} 1 & 0 \\ \frac{y_2}{h} \partial_{y_1} h & -1 \end{bmatrix} = : J^{-1}, \quad (78)\]

compare also (19) for $h^1 \equiv h, h^2 \equiv 1$. Replacing $v(x, t)$ by $u(y, t)$ we rewrite the Piola transformation as

\[v_P(y, t) = J J^{-1}(y, t) u(y, t) = \mathbb{R}(y, t) u(y, t), \]

where $\mathbb{R} := J J^{-1}$ and $J := \det(\nabla L) = h$. Note that since $J^{-1} = J^{-1} \text{cof} J^T$ we have $\mathbb{R} = \text{cof} J^T$. Now let us denote the divergence operators with respect to fixed and moving coordinates as $\text{div} u(y, t) := \partial_{y_1} u_1 + \partial_{y_2} u_2$, $\text{div}_x u(x, t) := \partial_x v_1 + \partial_{x_2} v_2$, respectively. From the Piola identity: $\text{div}(\text{cof} \nabla L) = 0$ and the transformation of the differential operator it follows that

\[\text{div} v_P(y, t) = J(y, t) \text{div}_x u(x, t), \]

\[\text{div} v_P(y, t) = \text{div}(\mathbb{R} u(y, t)) = 0 + \langle \mathbb{R}^T \nabla_y \rangle \cdot u(y, t) = \langle \mathbb{R}^T \nabla L^T \nabla_x \rangle \cdot v(x, t) = J(y, t) \text{div}_x u(x, t). \]
i.e., divergence of the velocity with respect to time dependent coordinates \( x(t) \)
can be expressed by means of the divergence in time-independent coordinates \( y \) of its Piola transformation. Thus,

\[
J(y, t) \text{div}_x \mathbf{v}(x, t) \left[ \equiv h(y_1, t) \text{div}_{h(y_1, t)} \mathbf{u}(y, t) \right] = \text{div} (\mathbb{R}(y, t) \mathbf{u}(y, t)). \tag{79}
\]

With the assistance of (79) we obtain for the divergence terms on the right-hand side of (77),

\[
\int_{T-\tau}^T \int_D \left[ \text{div} \left( \int_t^{t+\tau} (\mathbb{R}u_\kappa)(y, s) ds \right) \phi(y) - \int_t^{t+\tau} q_\kappa(s) \text{div} (\mathbb{R}\psi)(y, s) ds \right] dy \, dt.
\tag{80}
\]

Now, let us consider the following test functions

\[
\psi(y, s) = \mathbb{R}^{-1}(y, s) \left[ (\mathbb{R}u_\kappa)(y, t + \tau) - (\mathbb{R}u_\kappa)(y, t) \right], \tag{81}
\]

\[
\phi(y; t) = q_\kappa(y, t) - q_\kappa(y, t + \tau),
\]

\[
\xi(y_1; t) = \sigma_\kappa(y_1, t + \tau) - \sigma_\kappa(y_1, t),
\]

where \( s \in [t, t + \tau] \) and \( t \in [0, T - \tau] \) is fixed. Let us insert the test functions (81) into (80). Using the following notations

\[
U(t) := \int_t^{t+\tau} \mathbb{R}(s) u_\kappa(s) ds, \quad V(t) := \int_t^{t+\tau} q_\kappa(s) ds,
\]

(80) can be rewritten as

\[
- \int_0^{T-\tau} \int_D \left[ V'(t) \text{div} U(t) + V(t) \text{div} U'(t) \right] dy \, dt
= \int_D V(0) \text{div} U(0) - V(T - \tau) \text{div} U(T - \tau) \, dy
= \int_D \int_0^\tau q_\kappa ds \int_0^\tau \text{div}(\mathbb{R}u_\kappa) ds - \int_0^T q_\kappa ds \int_{T-\tau}^T \text{div}(\mathbb{R}u_\kappa) ds dy.
\]

According to the above considerations using the test functions (81) we finally
obtain from (77),
\[
\int_0^T \int_D \left[(h u_\kappa)(t + \tau) \cdot \mathbb{R}^{-1}(t + \tau) - (h u_\kappa)(t) \cdot \mathbb{R}^{-1}(t)\right] \left[(\mathbb{R} u)(t + \tau) - (\mathbb{R} u)(t)\right] dy \\
+ \int_0^L |\sigma_\kappa(t + \tau) - \sigma_\kappa(t)|^2 dy_1 dt =
\]
\[
- \int_D \int_0^\tau q_\kappa(s) ds \int_0^T h(s) \text{div}_h u_\kappa(s) ds + \int_0^\tau q_\kappa(s) ds \int_0^T h(s) \text{div}_h u_\kappa(s) ds dy \\
+ \int_0^{T-\tau} \int_0^{T+\tau} \int_D h(u)(s) \cdot (\partial_t \mathbb{R}^{-1}(s)) [\left(\mathbb{R} u\right)(t + \tau) - (\mathbb{R} u)(t)] dy ds dt + \ldots
\]
(82)

In what follows we show, that the terms I, II on the right-hand side of (82) can be estimated with \(\tau C_{K,\alpha}\) independently on \(\varepsilon\), where \(C_{K,\alpha}\) is some constant dependent on \(\alpha, K\), cf. (9). We will demonstrate this estimate for term I, term II can be estimated analogously. Using the equation for pressure, i.e., inserting test functions \((0, \phi, 0)\) into (76), integrating it over \(\int_0^\tau ds\) instead of \(\int_0^T dt\), where \(\tau < T\) is fixed, and using time-independent test function \(\phi(y; \tau) = \int_0^\tau q_\kappa(y, s) ds\) we obtain
\[
I = \varepsilon \int_0^\tau a_1(q_\kappa(y, s), \phi(y; \tau)) ds.
\]
From the definition of \(a_1(\cdot, \cdot)\), see [7, (2.10)] and [9] it follows
\[
I \leq C_{K,\alpha} \int_0^\tau \|\sqrt{\varepsilon} \nabla q_\kappa(s)\|_{L^2(D)} ds \left\| \int_0^\tau \sqrt{\varepsilon} \nabla q_\kappa(s) ds \right\|_{L^2(D)}.
\]
Since by the Hölder inequality we have
\[
\int_0^\tau \|\sqrt{\varepsilon} \nabla q_\kappa\|_{L^2(D)} ds \leq \tau^{\frac{1}{2}} \|\sqrt{\varepsilon} q_\kappa\|_{L^2(0, T; H^1(D))},
\]
\[
\left\| \int_0^\tau \sqrt{\varepsilon} \nabla q_\kappa(s) ds \right\|_{L^2(D)}^2 \leq \tau \int_D \int_0^\tau |\sqrt{\varepsilon} \nabla q_\kappa|^2(y, s) ds dy \leq \tau \|\sqrt{\varepsilon} q_\kappa\|_{L^2(0, T; H^1(D))}^2,
\]
we finally obtain
\[
I \leq \tau C_{K,\alpha} \|\sqrt{\varepsilon} q_\kappa\|_{L^2(0, T; H^1(D))}^2 \leq c\tau.
\]
(83)
Now we rewrite the first term on the left-hand side of (82) by means of square of time differences. To this end we first rewrite it as

\[
\int_0^{T-\tau} \int_D \left[ \mathbb{R}^{-T} h \mathbf{u}_\kappa(t + \tau) - \mathbb{R}^{-T} h \mathbf{u}_\kappa(t) \right] \cdot \left[ \mathbb{R} \mathbf{u}_\kappa(t + \tau) - \mathbb{R} \mathbf{u}_\kappa(t) \right] dy \, dt \quad (84)
\]

Here we have used \( \mathbb{R}^{-T} = h^{-1} \mathbb{J}^T \).

Further, since \( \mathbb{J}^T \mathbf{u}_\kappa : \mathbb{R} \mathbf{u}_\kappa = \mathbb{R}^T \mathbb{J}^T \mathbf{u}_\kappa : \mathbf{u}_\kappa = h |\mathbf{u}_\kappa|^2 \), we can rewrite equality (84) in terms of square of the time difference of \( \sqrt{h} \mathbf{u}_\kappa \) and an additional term IV as follows.

\[
\int_0^{T-\tau} \int_D \left[ (\mathbb{J}^T \mathbf{u}_\kappa)(t + \tau) - (\mathbb{J}^T \mathbf{u}_\kappa)(t) \right] \cdot \left[ (\mathbb{R} \mathbf{u}_\kappa)(t + \tau) - (\mathbb{R} \mathbf{u}_\kappa)(t) \right] dy \, dt \quad (85)
\]

where

IV := \( \int_0^{T-\tau} \int_D 2 (\sqrt{h} \mathbf{u}_\kappa)(t + \tau) \cdot (\sqrt{h} \mathbf{u}_\kappa)(t) \)

\(- (\mathbb{J}^T \mathbf{u}_\kappa)(t) \cdot (\mathbb{R} \mathbf{u}_\kappa)(t + \tau) - (\mathbb{J}^T \mathbf{u}_\kappa)(t + \tau) \cdot (\mathbb{R} \mathbf{u}_\kappa)(t) dy \, dt.
\)

Due to \( \mathbb{R} \mathbf{u}_\kappa = (\sqrt{h} \mathbb{J}^{-1})(\sqrt{h} \mathbf{u}_\kappa) \), the term IV can be rewritten as

IV = \( \int_0^{T-\tau} \int_D (\mathbb{J}^T \mathbf{u}_\kappa)(t + \tau) \cdot \left[ (\sqrt{h} \mathbb{J}^{-1})(t + \tau) - (\sqrt{h} \mathbb{J}^{-1})(t) \right] (\sqrt{h} \mathbf{u}_\kappa)(t) \)

\( + (\mathbb{J}^T \mathbf{u}_\kappa)(t) \cdot \left[ (\sqrt{h} \mathbb{J}^{-1})(t) - (\sqrt{h} \mathbb{J}^{-1})(t + \tau) \right] (\sqrt{h} \mathbf{u}_\kappa)(t + \tau) dy \, dt \)

= \( \int_0^{T-\tau} \int_D (\mathbb{J}^T \mathbf{u}_\kappa)(t + \tau) \cdot \partial_t \mathbb{M}(\sqrt{h} \mathbf{u}_\kappa)(t) - (\mathbb{J}^T \mathbf{u}_\kappa)(t) \cdot \partial_t \mathbb{M}(\sqrt{h} \mathbf{u}_\kappa)(t + \tau) dy \, dt, \)

where \( \mathbb{M} := \int_t^{t+\tau} \sqrt{h}(s) \mathbb{J}^{-1}(s) ds. \)

Finally, taking into account (84) and (85) we obtain from (82),

\[
\int_0^{T-\tau} \int_D \left[ (\sqrt{h} \mathbf{u}_\kappa)(t + \tau) - (\sqrt{h} \mathbf{u}_\kappa)(t) \right]^2 dy + \int_0^L [\sigma_{\kappa}(t + \tau) - \sigma_{\kappa}(t)]^2 dy \, dt
\]

= I + II + III + IV + \ldots \quad (86)

In previous lines we have already shown that \( I + II \leq c \tau \) independently on \( \varepsilon \), cf. (83). Now we estimate the additional terms III, IV arising from
the (weak) time derivative of \((h\mathbf{u}_\kappa), \text{ cf. (78), (85)}\) with \(c\tau\) (independently on \(\varepsilon\)) as well. Indeed, due to the assumptions on \(h\), \((77)\), it is obvious that \(|J| \leq C_{K,\alpha}\), moreover

\[
|\partial_t M| \leq \int_t^{t+\tau} |\partial_t (\sqrt{h}\mathbb{J}^{-1}(s))| ds \leq C_{K,\alpha} \int_t^{t+\tau} |h_t(s)| + |h_{ty_1}(s)| ds,
\]

cf. (78). Therefore

\[
IV \leq C_{K,\alpha} \int_0^{T-\tau} \int_t^{t+\tau} \|h_t(s)\|_{W^{1,\infty}(0,L)} ds \|\mathbf{u}_\kappa(t)\|_{L^2(D)} \|\mathbf{u}_\kappa(t+\tau)\|_{L^2(D)} dt \leq C_{K,\alpha} \|\mathbf{u}_\kappa\|_{L^2(0,T;L^2(D))}^2 \int_0^{T-\tau} \|h_t\|_{W^{1,\infty}(0,L)} ds \leq \tau C_{K,\alpha} \|\mathbf{u}_\kappa\|_{L^2(0,T;L^2(D))} \|h_t\|_{L^2(0,T;W^{1,\infty}(0,L))} \leq c\tau.
\]

Here we have used the fact that \(||[\varphi]\|_{L^2(0,T-\tau)} \leq \|\varphi\|_{L^2(0,T)}, [\varphi]_\tau(t) := \frac{1}{\tau} \int_t^{t+\tau} \varphi(s) ds\), which can be proven by the Hölder inequality and the Fubini theorem as follows.

\[
||[\varphi]\|_{L^2(0,T-\tau)}^2 \leq \frac{1}{\tau} \int_0^{T-\tau} \int_t^{t+\tau} \varphi^2(s) ds dt = \frac{1}{\tau} \int_0^{T-\tau} \int_t^{t+\tau} 1_{(t, t+\tau)}(s) \varphi^2(s) ds dt = \frac{1}{\tau} \int_0^{T-\tau} \varphi^2(s) \int_0^{T-\tau} 1_{(s-\tau, s)}(t) dt ds \leq \|\varphi\|_{L^2(0,T)}^2. \quad (88)
\]

Analogously we can estimate term III using \(|\mathbb{R}^{-T}| \leq C_{K,\alpha}\) and \(|\partial_t \mathbb{R}^{-T}| \leq C_{K,\alpha} \|h_t(s)\|_{W^{1,\infty}(0,L)}\) as follows.

\[
III \leq \int_0^{T-\tau} \int_D \int_t^{t+\tau} |(\partial_t \mathbb{R}^{-T} h \mathbf{u}_\kappa)(s)| \cdot \left( |\mathbb{R} \mathbf{u}_\kappa(t + \tau)| + |\mathbb{R} \mathbf{u}_\kappa(t)| \right) ds dy dt \leq C_{K,\alpha} \int_0^{T-\tau} \int_t^{t+\tau} \|h_t(s)\|_{W^{1,\infty}(0,L)} \|\mathbf{u}_\kappa(s)\|_{L^2(D)} ds \|\mathbf{u}_\kappa(t + \tau)| + |\mathbf{u}(t)|_{L^2(D)} dt \\
\leq \tau C_{K,\alpha} \|\mathbf{u}_\kappa\|_{L^2(0,T;L^2(D))} \int_0^{T-\tau} \|h_t\|_{W^{1,\infty}(0,L)} \|\mathbf{u}_\kappa\|_{L^2(D)} dt \\
\leq \tau C_{K,\alpha} \|\mathbf{u}_\kappa\|_{L^2(0,T;L^2(D))} \|h_t\|_{L^2(0,T;W^{1,\infty}(0,L))} \leq c\tau.
\]

Estimates of remaining terms on the right-hand side of (85) can be done analogously as in [7 Section 5.1], cf. also [3 Section 8], however some further
estimates has to be done due to new test functions containing $\mathbb{R}$, cf. (81). In the viscous term (15), taking into account (9), we additionally obtain terms of type

$$
\frac{\mu}{\rho} C_{K,\alpha} \int_0^{T-\tau} \int_t^{t+\tau} \int_D |\nabla u_{\kappa}(y, s)| |\partial_y h(y_1, \theta_1)| |u_{\kappa}(y, \theta_2)| dy ds dt,
$$

where $\theta_1 := t, t + \tau,$ or $s$, and $\theta_2 := t,$ or $t + \tau$. Rewriting the integral $\int_D dy = \int_0^L \int_0^1 dy dy_1$, applying the Hölder inequality and Lemma 3.2 we can bound the above terms by

$$
\leq \frac{\mu}{\rho} C_{K,\alpha} \int_0^{T-\tau} \int_t^{t+\tau} \|\nabla u_{\kappa}(\theta_2)\|_{L^2(D)} \|h(\theta_1)\|_{H^2(0,L)} \|\nabla u_{\kappa}(s)\|_{L^2(D)}.
$$

Applying the Hölder inequality for time integrals and the property (88) we finally get for (90)

$$
\leq \tau \frac{\mu}{\rho} C_{K,\alpha} \|h(\theta_1)\|_{L^\infty(0,T;H^2(0,L))} \|u_{\kappa}\|_{L^2(0,T;W^{1,2}(D))} \leq c\tau.
$$

The convective term (16) can be estimated using the property (17) by $c\tau$ applying similar technique using Lemma 3.2. Let us also recall, that the boundary term containing $\kappa$ is bounded by $c\tau$ as well, where $c$ is some constant independent on $\kappa$, for more details see [7, Section 5.1], [3, Section 8].

Finally, due to estimates (83), (87), (89) and above described estimates of remaining terms on the right-hand side of (86), we can conclude that

$$
\int_0^{T-\tau} \int_D \left[ (\sqrt{h} u_{\kappa})(t + \tau) - (\sqrt{h} u_{\kappa})(t) \right]^2 dy + \int_0^L [\sigma_{\kappa}(t+\tau) - \sigma_{\kappa}(t)]^2 dy_1 dt \leq c\tau
$$

with some constant $c$ dependent on $\alpha, K, T$, but independent on $\varepsilon$ and $\kappa$. The integral equicontinuity estimate (91) together with the compactness argument [30, Lemma 1.9] provide us the strong convergences $u_{\kappa} \rightarrow u$ in $L^r((0,T) \times D), 1 \leq r < 4$ and $\sigma_{\kappa} = \partial_\eta \eta_{\kappa} \rightarrow \sigma = \partial_\eta \eta$ in $L^2((0,T) \times (0,L)), 1 \leq s < 6$, see [7, p. 223] or [3, p. 36] for more details. Finally, the limiting process for $\kappa \rightarrow \infty$ in (16) with the test function $\xi(y_1,t) = E\psi_2(y_1,1,t) \in H_0^2(0,L)$ completes the proof of Theorem 5.1 [7]. \[\square\]
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