Nonexistence of invariant manifolds in fractional-order dynamical systems

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Abstract Invariant manifolds are important sets arising in the stability theory of dynamical systems. In this article, we take a brief review of invariant sets. We provide some results regarding the existence of invariant lines and parabolas in planar polynomial systems. We provide the conditions for the invariance of linear subspaces in fractional-order systems. Further, we provide an important result showing the nonexistence of invariant manifolds (other than linear subspaces) in fractional-order systems.

Keywords Invariant manifold · Separatrix · Stability · Tangency condition · Caputo fractional derivative

1 Introduction

Dynamical systems [1–4] is a trending branch of mathematics playing a vital role in the mathematical analysis as well as in the applied sciences [5–8]. Chaos theory and fractals [9–14] are the sub-branches of this theory which have attracted the attention of scientists as well as layman. The applications of dynamical systems are found in arts [15,16] and social sciences [17,18] also. The theoretical results such as Hartman–Grobman theorem [4], Stable manifold theorem [4] and Poincare–Bendixson theorem [4] made substantial contributions to the mathematical analysis.

In the dynamical systems theory, an invariant manifold is a set \( M \) such that every point in \( M \) is mapped to some point in the same set \( M \) under the evolution of the system. Hadamard [19], Liapunov [20] and Perron [21] proposed pioneering results in the theory of invariant manifolds. Existence and smoothness of invariant manifolds are discussed in [22]. This theory has applications in a variety of fields in science and engineering. Gorban and Karlin [23] employed this theory to reduce the description of equations arising in chemical kinetics. Beigie et al. [24] used invariant manifolds in the study of chaotic advection. Control of chaos is achieved by using invariant manifolds by Chen et al. [25]. The globally convergent, reduced-order observers are designed for general nonlinear systems in [26]. In [27], the dynamics of a nonlinear rotor is investigated using the invariant manifolds. Roussel and Fraser [28] utilized this theory to simplify the metabolic models arising in biochemistry.

Fractional calculus deals with the differentiation and integration of arbitrary order [29–36]. The fractional derivative operators are non-local and hence very useful in modeling the memory in the natural systems [37–43]. The existence and uniqueness of the solution of fractional-order initial value problems are discussed in...
Note that \( \Phi_0 = I \), the identity map.
For any \( t, s \in I(X_0) \),
\[
\Phi_t \circ \Phi_s = \Phi_{t+s}.
\]  
(3)

This is called semi-group property of the flow.

**Definition 2.2** [4] The set \( S \subseteq \mathbb{R}^n \) is said to be invariant under the flow \( \Phi_t : \mathbb{R}^n \to \mathbb{R}^n \) of system (1) if \( \Phi_t(S) \subseteq S \), \( \forall t \in I(X_0) \).

**Definition 2.3** [4] A steady-state solution of (1) is called an equilibrium point. Thus, \( X_s \) is an equilibrium point of (1) if \( f(X_s) = \mathbf{0} \).

For the classification of equilibrium points, the readers are referred to [1,4].

**Definition 2.4** [4] Let \( E \) be an open subset of \( \mathbb{R}^n \) and let \( f \in C^1(E) \). The global stable manifold of the system (1) corresponding to an equilibrium \( X_s \) is defined as
\[
S = \{ c \in \mathbb{R}^n : \lim_{t \to -\infty} \Phi_t(c) = X_s, \ c \in E \}. 
\]  
(4)

Note that \( S \) is invariant under \( \Phi_t \).

**Definition 2.5** [4] The homoclinic orbit is an invariant set which is a closed-loop passing through a saddle equilibrium. Such a loop is contained in the intersection of stable and unstable manifolds of a single equilibrium point, i.e., homoclinic orbit approaches to the single equilibrium point as \( t \to \pm \infty \).

**Definition 2.6** [4] The heteroclinic orbit connects different equilibrium points. It approaches different equilibrium points as \( t \to \pm \infty \).

**Note**

1. Every solution curve of (1) is an invariant set.
2. In particular, if \( f(X) = AX \), where \( A \) is a square matrix, then the eigenvectors of \( A \) (straight lines) are invariant sets.
3. If \( u \pm iv \) are complex eigenvalues of \( A \) and if \( W \) is a (complex) eigenvector corresponding to \( u + iv \), then the linear subspace spanned by \( \text{Re}(W) \) and \( \text{Im}(W) \) is invariant under \( \Phi_t \).
4. If \( f \) is nonlinear, then we can have some other invariant sets, e.g., curve and surface (manifolds).

**Definition 2.7** [1] Separatrix \( \mathcal{S} \) is an invariant manifold such that the qualitative properties of solutions change at \( \mathcal{S} \).

The (global) stable and unstable manifolds of saddle equilibrium are examples of separatrices.
Definition 2.8 [33] Let $\alpha \geq 0$ ($\alpha \in \mathbb{R}$). Then, Riemann–Liouville (RL) fractional integral of a function $f \in C[0, b]$, $b > 0$ of order $\alpha$ is defined as
\[
0_{0}^{\alpha} I f(t) = \frac{1}{\Gamma(\alpha)} \int_{0}^{t} (t - \tau)^{\alpha-1} f(\tau) \, d\tau.
\] (5)

Definition 2.9 [33] The Caputo fractional derivative of order $\alpha > 0$, $n - 1 < \alpha \leq n$, $n \in \mathbb{N}$ is defined for $f \in C^n[0, b]$, $b > 0$ as
\[
C_{0}^{\alpha} D f(t) = \begin{cases} 
\frac{1}{\Gamma(n-\alpha)} \int_{0}^{t} (t - \tau)^{n-\alpha-1} f^{(n)}(\tau) \, d\tau, & \text{if } n - 1 < \alpha < n \\
\frac{d^n}{dt^n} f(t), & \text{if } \alpha = n.
\end{cases}
\] (6)

Note that $C_{0}^{\alpha} D f = c$, where $c$ is a constant.

We cannot have flow for this system in classical sense.

However, we define the function $\phi_{t} : E \to \mathbb{R}^{n}$ in a similar ways, i.e., $\phi_{t}(X_0)$ is the solution of FDE (7) with initial condition $X(0) = X_0$. This $\phi_{t}$ does not satisfy the property (3) of $\Phi_{t}$.

Theorem 2.1 [67] The solution of non-homogeneous fractional-order differential equation
\[
C_{0}^{\alpha} D_{x} x(t) + \lambda x(t) = g(t), \quad 0 < \alpha < 1,
\] (8)
is given by
\[
x(t) = \int_{0}^{t} \tau^{\alpha-1} E_{\alpha,\alpha}(-\lambda \tau^{\alpha}) g(t - \tau) \, d\tau + x(0) E_{\alpha,-\lambda^{\alpha}},
\] (9)
where $E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^{k}}{\Gamma(\alpha k + \beta)}$ and $E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^{k}}{\Gamma(\alpha k + \beta)}$, $z \in \mathbb{C}$, ($\alpha > 0$, $\beta > 0$) are Mittag–Leffler functions [33].

Remark 2.1 If a manifold is given by the equation
\[
y = h(x), \quad x \in \mathbb{R}^{n}, \quad y \in \mathbb{R}^{m}
\] (10)
and the system of differential equations is given by
\[
\dot{x} = f(x, y) \quad (11)
\]
then the condition
\[
D_{h}(x) \dot{x} = \dot{y} \Rightarrow D_{h}(x) f(x, h(x)) = g(x, h(x)),
\] (12)
(where $D_{h}$ is the Jacobian of the function $h$) is necessary and sufficient to show the invariance of (10) under the flow of (11). This condition is known as tangency condition [10].

3 Some invariant manifolds of planar quadratic systems

In this section, we provide some necessary and sufficient conditions to exist the invariant lines and invariant parabolas for polynomial systems with classical derivatives
\[
\dot{x} = P_{n}(x, y), \quad \dot{y} = Q_{n}(x, y),
\] (13)
where $P_{n}(x, y)$ and $Q_{n}(x, y)$ are polynomials of degree $n$.

3.1 Necessary and sufficient conditions for the existence of invariant straight lines

Theorem 3.1 Consider planar polynomial system of degree $n$
\[
\dot{x} = \sum_{i,j=0}^{n} a_{i,j} x^{i} y^{j} \quad (14)
\]
\[
\dot{y} = \sum_{i,j=0}^{n} b_{i,j} x^{i} y^{j}
\]
with $a_{0,0} = b_{0,0} = 0$.

1. There exist infinitely many invariant straight lines $y = mx$ to the system (14) if
\[
b_{k,0} = a_{0,k} = 0 \quad \text{and} \quad b_{k-j,j} - a_{k-(j-1),j-1} = 0, \quad 1 \leq j \leq k
\] (15)
for all $k = 1, 2, \ldots, n$.

2. Consider any set of distinct values $i_1, i_2, \ldots, i_l$ from $\{1, 2, \ldots, n\}$, where $1 \leq l \leq n$ and
\[
b_{k,0} = a_{0,k} = b_{k-j,j} - a_{k-(j-1),j-1} = 0, \quad 1 \leq j \leq k
\] (16)
where $k \in \{1, 2, \ldots, n\} - \{i_1, i_2, \ldots, i_l\}$ and $1 \leq j \leq k$. The values of $m$ obtained from the system of $l$ equations,
\[
b_{i_p,0} + \sum_{j=1}^{i_p} (b_{i_p-j,j} - a_{i_p-(j-1),j-1}) m^{j} = -a_{0,i_p} m^{i_p+1} = 0,
\] (17)
where \( 1 \leq p \leq l \), will give the invariant lines \( y = mx \).

**Proof** Consider the equation of line
\[
y = mx.
\]
Differentiating (18), we get \( \dot{y} = m \dot{x} \).

Therefore, the tangency condition implies that
\[
\sum_{i,j=0}^{n} (b_{i,j} - ma_{i,j})m^{i+j}x^{i+j} = 0 \quad \forall x \in \mathbb{R}.
\]

This holds if and only if
\[
\sum_{j=0}^{k} (b_{k-j,j} - ma_{k-j,j})m^{j} = 0
\]
for each \( k = 0, 1, 2, \ldots, n \). This expression can be written as a polynomial in \( m \) as below:
\[
0 = \sum_{j=0}^{k} b_{k-j,j}m^{j} - \sum_{j=0}^{k} a_{k-j,j}m^{j+1}
\]
\[
= b_{k,0} + \sum_{j=1}^{k} b_{k-j,j}m^{j} - \sum_{j=1}^{k+1} a_{k-(j-1),j-1}m^{j}.
\]

Therefore, the tangency condition becomes
\[
b_{k,0} + \sum_{j=1}^{k} (b_{k-j,j} - a_{k-(j-1),j-1})m^{j} - a_{0,k}m^{k+1} = 0
\]
\[(19)\]
for each \( k = 0, 1, 2, \ldots, n \).

**Case 1:** If \( b_{k,0} = a_{0,k} = 0 \) and \( b_{k-j,j} - a_{k-(j-1),j-1} = 0 \), \( 1 \leq j \leq k \) for each \( k = 1, 2, \ldots, n \), then the tangency condition (19) is satisfied by any \( m \in \mathbb{R} \).

This proves the statement 1.

**Case 2:** Now, instead of equating all the coefficients of all the powers of \( m \) in (19) to zero, we solve some of equations (19) for \( m \) and proceed as in Case 1 for other equations. For \( 1 \leq l \leq n \), if we solve any \( l \) equations (19) for \( m \) and equate coefficients of powers of \( m \) to zero, in the remaining equations, then we obtain the statement 2.

Note that the statement 2 provides
\[
\binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^{n} - 1
\]
ways to find invariant straight lines for the system (14).

\[\square\]

**Corollary 1** Consider the planar polynomial system (14) of degree \( n \). Then, the lines \( x = k \) (respectively, \( y = l \)) are invariant under the flow of system (14) if and only if
\[
\sum_{i,j=0}^{n} a_{i,j}k^{i}y^{j} = 0 \quad \forall y \in \mathbb{R}
\]
\[(20)\]
(respectively, \( \sum_{i,j=0}^{n} b_{i,j}x^{i}l^{j} = 0 \quad \forall x \in \mathbb{R} \)).

where \( k \) and \( l \) are real constants.

Theorem 3.1 is illustrated for \( n = 2 \) in the following theorem.

**Theorem 3.2** Consider the planar quadratic system
\[
\dot{x} = a_{1}x + a_{2}y + a_{3}x^{2} + a_{4}y^{2} + a_{5}xy
\]
\[
\dot{y} = b_{1}x + b_{2}y + b_{3}x^{2} + b_{4}y^{2} + b_{5}xy
\]
\[(21)\]

1. The line (18) is invariant under the flow of given system (21), if and only if
   (a) \( a_{2} = 0, b_{1} = 0 \) and \( a_{1} = b_{2} \). In this case, the real values of \( m \) obtained from the cubic equation
   \[
a_{4}m^{3} + (a_{5} - b_{4})m^{2} + (a_{3} - b_{5})m - b_{3} = 0
   \]
   will give the invariant lines (18).

   OR

   (b) \( a_{4} = 0, b_{3} = 0, a_{5} = b_{4}, a_{3} = b_{5} \) and \( (b_{2} - a_{1})^{2} + 4b_{1}a_{2} \geq 0 \). In this case, the real values of \( m \) obtained from the quadratic equation
   \[
b_{1} + (b_{2} - a_{1})m - a_{2}m^{2} = 0
   \]
   provide the invariant lines.

   OR

   (c) the coefficients in the following equations
   \[
a_{2}m^{2} + (a_{1} - b_{2})m - b_{1} = 0
   \]
   \[
a_{4}m^{3} + (a_{5} - b_{4})m^{2} + (a_{3} - b_{5})m - b_{3} = 0
   \]
   are not all zero. In this case, the real values of \( m \) satisfying (22) and (23) simultaneously provide the invariant lines (18).
(d) the coefficients in (22) and (23) are all zero. In this case, there exist infinitely many invariant straight lines (18) for all \( m \in \mathbb{R} \).

2. The line \( x = k \) (respectively, \( y = l \)) is invariant under the flow of given system (21), if \( a_1 k + a_2 y + a_3 k^2 + a_4 k^3 + a_5 k^4 = 0 \), for all \( y \in \mathbb{R} \) (respectively, \( b_1 x + b_2 l + b_3 x^2 + b_4 l^2 + b_5 x l = 0 \), for all \( x \in \mathbb{R} \)).

**Remark 3.1** Theorem 3.1 corresponds to the equilibrium point 0 of system (14). If \((x_0, y_0)\) is any other equilibrium, then this result can be extended to obtain invariant lines of the form \((y - y_0) = m(x - x_0)\).

In Table 1, we provide examples supporting to Theorem 3.2. The corresponding vector fields are sketched in Fig. 1.

### 3.2 Invariant parabolas

**Theorem 3.3** Consider the planar quadratic system (21). The parabola \( y = mx^2 \) is invariant if and only if

1. \( b_1 = 0, b_4 = 2a_5, a_4 = 0 \) and
2. One of the following conditions (a), (b), (c) and (d) hold:

   (a) \( b_3 = 0, b_2 = 2a_1, b_5 \neq 2a_3 \) and \( a_2 \neq 0 \). (In this case \( m = \frac{b_5 - 2a_3}{2a_1} \)).

   (b) \( b_5 = 2a_3, a_2 = 0, b_3 \neq 0 \) and \( b_2 \neq 2a_1 \). (In this case \( m = \frac{b_3}{b_5 - 2a_3} \)).

   (c) \( b_3 \neq 0, b_2 \neq 2a_1, b_5 \neq 2a_3, a_2 \neq 0 \) and \( 2a_2 b_3 + b_2 b_5 - 2b_2 a_3 - 2a_1 b_5 + 4a_1 a_3 = 0 \). (In this case \( m = \frac{-b_3}{2a_2 - b_5} = \frac{b_3 - 2a_3}{2a_2} \)).

   (d) \( b_3 = 0, b_2 = 2a_1, b_5 = 2a_3 \) and \( a_2 = 0 \). (In this case, \( m \) is any real number.)

**Proof** The tangency condition shows that the parabola \( y = mx^2 \)

is invariant if and only if

\[
 b_1 x + (b_2 m + b_3 - 2a_1 m)x^2 + (b_5 - 2a_2 m - 2a_3)mx^3 + (b_4 - 2a_5)m^2 x^4 - 2a_4 m^3 x^5 = 0 \quad \forall x \in \mathbb{R}.
\]

This holds if and only if

\[
 b_1 = 0, \quad b_4 = 2a_5, \quad a_4 = 0, \quad (25)
\]

\[
 b_2 m + b_3 - 2a_1 m = 0 \quad (26)
\]

\[
 b_5 - 2a_2 m - 2a_3 = 0. \quad (27)
\]

From (26), we have

\[
 m = \frac{-b_3}{b_2 - 2a_1} \quad (28)
\]

and from (27), we have

\[
 m = \frac{b_5 - 2a_3}{2a_2}. \quad (29)
\]
Therefore, the parabola (24) is invariant under the flow of system (21) if and only if one of the conditions (a), (b), (c) and (d) holds along with the condition (25).

In Table 2, we provide examples supporting to Theorem 3.3. The corresponding vector fields for the systems given in the examples (viii) and (ix) are sketched in Fig. 3.

Table 1 Examples supporting to Theorem 3.2

| Ex. no. | Planar quadratic system | Related condition in Theorem 3.2 | Invariant straight lines |
|---------|-------------------------|----------------------------------|--------------------------|
| i       | \( \dot{x} = x - 4x^2 + 2y^2 + 10xy \) | 1(a) | \( y = 0, y = -4x \) |
|         | \( \dot{y} = y + 4y^2 + 4xy \)                           |                      |
| ii      | \( \dot{x} = -x + y - x^2 + 3xy \) | 1(b) | \( y = 4x \) and \( y = -2x \) |
|         | \( \dot{y} = 8x + y + 3y^2 - xy \)                                   |                      |
| iii     | \( \dot{x} = 3x - y - 6x^2 + y^2 + 2xy \) | 1(c) | \( y = 2x \) and \( y = 3x \) |
|         | \( \dot{y} = 6x - 2y - 18x^2 + 4y^2 + 3xy \)                                  |                      |
| iv      | \( \dot{x} = 3x - 6x^2 + 2xy \) | 1(d) | \( y = mx, \forall m \in \mathbb{R} \) |
|         | \( \dot{y} = 3y + 2y^2 - 6xy \)                                   |                      |
| v       | \( \dot{x} = 2x^2 - 4x + 2 \) | Remark 2 | \( x = 1, y = 3x - 2 \) and \( y = -x + 2 \) |
|         | \( \dot{y} = 6x - 2y - 3x^2 + y^2 - 2 \)                                   |                      |

The fractional-order systems are generalizations of classical systems. In this section, we show that these systems cannot have invariant manifolds other than the linear subspaces of \( \mathbb{R}^n \).

4.1 Invariant subspaces of fractional-order systems

**Theorem 4.1** The conditions \( \sum_{i=1}^{n} a_i f_i(x_1, x_2, \ldots, x_n) = 0 \) for the existence of invariant linear subspaces of the fractional-order systems \( C_0 D_\alpha^t x_i = f_i(x_1, x_2, \ldots, x_n), 1 \leq i \leq n, 0 < \alpha < 1 \) are the same as their classical counterparts.

**Proof** Consider the fractional-order system

\[
C_0 D_\alpha^t x_i = f_i(x_1, x_2, \ldots, x_n),
\]

\( 1 \leq i \leq n, 0 < \alpha < 1 \) and its classical counterpart

\[
\dot{x}_i = f_i(x_1, x_2, \ldots, x_n), \quad 1 \leq i \leq n.
\]

The tangency condition shows that the linear subspace

\[
S = \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : \sum_{i=1}^{n} a_i x_i = c, a_i \in \mathbb{R} \}
\]
Ex. no. Planar quadratic system Related condition in Theorem 3.3 Invariant parabolas

vi \[ \dot{x} = -2x + y + 3x^2 - xy \] (1) and 2(a) \[ y = -\frac{1}{2}x^2 \]

vii \[ \dot{x} = 3x - x^2 + 2xy \] (1) and 2(b) \[ y = \frac{5}{8}x^2 \]

viii \[ \dot{x} = x - y + 2x^2 + xy \] (1) and 2(c) \[ y = x^2 \]

ix \[ \dot{x} = -x + 2x^2 - 3xy \] (1) and 2(d) \[ y = mx^2, \forall m \in \mathbb{R} \]

of \( \mathbb{R}^n \) is invariant under system (30) if
\[ \sum_{i=1}^{n} a_i D_t^\alpha x_i = 0 \]
and invariant under system (31) if
\[ \sum_{i=1}^{n} a_i \dot{x}_i = 0. \]

This proves theorem.

This shows that Theorems 3.1 and 3.2 of classical system hold for fractional-order system (30) also.

Example 4.1 Consider a fractional-order planar quadratic system
\[ C_0D_t^\alpha x = x - y + 2x^2 - xy \]
\[ C_0D_t^\alpha y = -9x + y - y^2 + 2xy. \]

This system satisfies the condition 1(b) of Theorem 3.2.

4.2 Nonexistence of the invariant curves, with curvature > 0 for fractional-order systems

The general solution of the initial value problem
\[ C_0D_t^\alpha X(t) = AX(t), \ X(0) = X_0 \]
is
\[ X(t) = E_\alpha (At^\alpha)X_0. \] (38)

If \( X_0 \) is on an eigenvector of \( A \), then \( AX_0 = \lambda X_0 \), where \( \lambda \) is the corresponding eigenvalue.
\[ \Rightarrow X(t) = E_\alpha (\lambda t^\alpha)X_0. \] (39)

This is on the same eigenvector, because \( E_\alpha (\lambda t^\alpha) \) is a number for any \( t > 0 \).

The solution trajectory of (37) starting on eigenvector is a straight line and is invariant under \( \phi_t = E_\alpha (At^\alpha) \).
In the following theorem, we show that the solution curve whose initial point is not on any of the eigenvectors is not invariant under the map \( \phi_t \).

Theorem 4.2 The solution curves of linear FDEs
\[ C_0D_t^\alpha X(t) = AX(t), \ (0 < \alpha < 1) \] (40)
for which the initial point \( X_0 \) is not on an eigenvector of \( n \times n \) matrix \( A \) are not invariant under \( \phi_t = E_\alpha (At^\alpha) \).

Proof Assume that the initial point \( X_0 \) is not on any eigenvector of \( A \).

The vector \( AX_0 \) is not collinear with the vector \( X_0 \). Define \( T = \{ E_\alpha (At^\alpha)X_0 | t \geq 0 \} \), the set of all points on the solution trajectory starting at \( X_0 \). We show that \( T \) is not invariant under \( \phi_t \).
In this case, if \( Y_0 = E_\alpha (At^\alpha)X_0, \ t_k > 0 \) is any point in \( T \), then
\[ \phi_t(Y_0) \]
\[ = E_\alpha (At^\alpha)E_\alpha (At^\alpha)X_0 \]
\[ = \sum_{k=0}^{\infty} A^{k+1} \Gamma(\alpha k + 1) \sum_{j=0}^{\infty} A^{j} t^\alpha_\gamma X_0 \]
\[ = \sum_{k=0}^{\infty} \sum_{l=0}^{k} A^{k+1} \Gamma(\alpha k + 1) \Gamma(\alpha l + 1) X_0 \]
Fig. 3 Invariant parabolas $y = mx^2$

$$y = x^2$$ is invariant parabola

$$y = mx^2, \forall m \in \mathbb{R}$$ are invariant parabolas

Fig. 4 Invariant lines of system (36)

Comparing the coefficients of like powers of $A$ in (41) and (42), we get few terms as

$$s^\alpha = t_\alpha^2 t_\alpha^\alpha (1 + t_\alpha^{-\alpha}) = t_\alpha^\alpha (1 + t_\alpha^\alpha)$$  \hspace{1cm} (43)

and

$$\left(\frac{1 + t_\alpha^{-2\alpha}}{\Gamma(1 + 2\alpha)} + \frac{t_\alpha^{\alpha}}{\Gamma(1 + \alpha)^2}\right) (t_\alpha^{2\alpha}) = \frac{s^{2\alpha}}{\Gamma(1 + 2\alpha)}.$$  \hspace{1cm} (44)

Using (43) in (44), we get

$$t_\alpha^{2\alpha} + 1 + \frac{\Gamma(1 + 2\alpha)}{(\Gamma(1 + \alpha))^2} t_\alpha^{\alpha} = 1 + t_\alpha^{2\alpha} + 2 t_\alpha^{\alpha}.$$  \hspace{1cm} \therefore t_\alpha^{\alpha} = 0.

This is not possible because $t_\alpha \neq 0$ and $0 < \alpha < 1$. This contradiction shows that (42) must be wrong.

$\therefore \phi_t(Y_0) \notin T$. 

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Example 4.2  Consider the planar fractional-order system
\[ \frac{^0D_t^\alpha X(t)}{\alpha} = \begin{bmatrix} 1 & 3 \\ -3 & 1 \end{bmatrix} X(t), \quad X(0) = X_0. \] (45)
Its solution with \( X_0 = [1, 1]^T \) is given by

\[ X(t) = \begin{bmatrix} \text{Re}[E_{0.7}((1 + 3i)^{0.7})] + \text{Im}[E_{0.7}((1 + 3i)^{0.7})] \\ -\text{Im}[E_{0.7}((1 + 3i)^{0.7})] + \text{Re}[E_{0.7}((1 + 3i)^{0.7})] \end{bmatrix} \] (46)

In Fig. 5, we sketch the solution trajectory (46) (Blue color) of system (45) and another solution trajectories with initial conditions at various points \( X(t_1) \) on (46). It can be checked that the trajectories follow different paths.

Note  It can be easily checked that the tangency condition used for nonlinear case in classical sense will not provide any invariant curves for fractional-order case. For example, as in Theorem 3.3, consider fractional-order system

\[ \frac{^0D_t^\alpha x}{\alpha} = a_1 x + a_2 y + a_3 x^2 + a_4 y^2 + a_5 x y, \] (47)

and a parabola \( y = mx^2 \).

Operating \( \frac{^0D_t^\alpha y}{\alpha} \) on both the sides, we get

\[ \frac{^0D_t^\alpha y}{\alpha} = m \frac{^0D_t^\alpha x}{\alpha} x^2 \] (48)

Note that, unlike in classical case, \( \frac{^0D_t^\alpha x}{\alpha} x^2 \) cannot be written in terms of \( \frac{^0D_t^\alpha x}{\alpha} \). The generalized Leibniz rule [33] gives

\[ \frac{^0D_t^\alpha x(t)}{\alpha} = \left[ \frac{^0D_t^\alpha x}{\alpha} + \frac{t^{-\alpha}}{\Gamma(1-\alpha)} x(0) \right] x(t) + \sum_{k=1}^{\infty} \binom{\alpha}{k} \left( \frac{t^{-\alpha}}{\Gamma(1-\alpha)} \right)^k x^{(k)}(t) \] (49)

\[ - \frac{t^{-\alpha}}{\Gamma(1-\alpha)} x(0)^2. \]

\( \therefore \) (48) becomes

\[ a_4 m^3 x^5 + (a_5 - b_4)m^2 x^4 + (a_2m + a_3 - b_5)mx^3 \]

\[ + (a_1m - b_2m - b_3)x^2 + \left( \frac{mx(0)}{\Gamma(1-\alpha)} - b_1 \right) x \]

\[ + m \sum_{k=1}^{\infty} \binom{\alpha}{k} \left( \frac{t^{-\alpha}}{\Gamma(1-\alpha)} \right)^k x^{(k)}(t) \] (50)

This does not provide any nonzero value of \( m \), because of the term \( m \sum_{k=1}^{\infty} \binom{\alpha}{k} \left( \frac{t^{-\alpha}}{\Gamma(1-\alpha)} \right)^k x^{(k)}(t) \) involved in (50).

\( \Rightarrow \) any invariant parabola of the form \( y = mx^2 \) for fractional-order system (47). The similar computations can be used to show that \( \not\exists \) any invariant manifold (except linear subspace of \( \mathbb{R}^n \)) for fractional-order system (47).

Theorem 4.3  Consider fractional-order system

\[ \frac{^0D_t^\alpha X}{\alpha} = f(X), \quad 0 < \alpha < 1 \] (51)

where \( f \in C^1(E) \) and \( E \) is an open set in \( \mathbb{R}^n \).

Suppose \( X_0 = 0 \) is an equilibrium of (51).

Let \( \phi_t(X_0) \) be the solution of (51) with initial condition \( X(0) = X_0 \).

Any curve \( g(x, y) = c \) which is not a straight line (i.e., curvature \( > 0 \)), cannot be invariant under \( \phi_t \).

Proof  If \( S : g(x, y) = c \) is an invariant set under \( \phi_t \), then “any solution of (51) starting on \( S \) will stay on \( S \) for all the time.”

Let \( X(t) \) be the solution of (51) with \( X(0) = X_0 \) on the curve \( g(x, y) = c \).

Let \( Y_0 = X(t_0), t_0 > 0 \) be any point on this solution curve, i.e., on \( g(x, y) = c \).

If the solution of (51) starting at \( Y_0 \) follows the same path on \( g(x, y) = c \), then it contradicts Theorem 4.2, because the linear system (40) is a particular case of (51).

\( \therefore g(x, y) = c \) cannot be invariant under \( \phi_t \), if \( g \) is not a straight line. \( \square \)
5 Comments on the invariant manifolds in fractional-order systems presented in the literature

In [55], Cong et al. considered planar fractional-order nonlinear system. They linked the solution of this system with its linearization by defining the Lyapunov–Perron operator. The local stable manifold of the nonlinear system is then defined by using the set of all converging solutions obtained by using the unique fixed point of this operator. These results are generalized by these authors to the higher-dimensional spaces in [59]. Daftardar-Gejji and Deshpande [57,58] constructed the local stable manifold for such systems by using the matrix variate Mittag–Leffler function. They further showed that the local stable set forms a Lipschitz graph over the stable subspace of corresponding linearized system. To achieve the same goal, Sayevand and Pichaghchi [56] applied the method of successive approximations and the fractional Hartman–Grobman theorem. Ma and Li [60] proposed the generalization to the classical method and proposed the center manifolds for fractional-order systems. In [61], Wang et al. discussed the existence of the center manifold by constructing Lyapunov–Perron operator and using the asymptotic properties of Mittag–Leffler function.

It is clear from discussion in Sect. 4 that the local invariant manifolds in fractional-order systems developed in [55–61] do not satisfy the definition of invariant sets. In fact, it is not verified in any of these papers that whether the invariant manifolds \( S \) obtained are satisfying the following properties:

If \( X_0 \) is any initial condition, sufficiently close to equilibrium \( X_e \), then the solution \( \phi_t(X_0) \) of given system starting at \( X_0 \)

1. converges to \( X_e \) as \( t \to \infty \) if \( S \) is stable manifold and as \( t \to -\infty \) if \( S \) is unstable manifold

2. stays on \( S \) for all the time.

For example, the local stable manifold given in the paper [55] does not follow this property as explained below:

**Example 5.1** Consider

\[
\frac{\partial}{\partial t} D_\alpha^x x = x - y^2,
\]

\[
\frac{\partial}{\partial t} D_\alpha^y y = -y.
\]

For \( \alpha = 0.5 \), the exact solution is given as [55]

\[
x(t) = c_1 E_{\frac{1}{2}}(\sqrt{t})
\]

\[
y(t) = c_2 E_{\frac{1}{2}}(-\sqrt{t}).
\]

Also the local stable manifold [55] \( S \) is given by

\[
x = y^2 - \int_0^t (t-s)^{-1/2} E_{\frac{1}{2}}(\frac{\sqrt{-s}}{2}) (E_{\frac{1}{2}}(-\sqrt{t}))^2 ds
\]

\[
y(t) = c_2 E_{\frac{1}{2}}(-\sqrt{t}). \quad (53)
\]

If \((x(t), y(t))\) is solution of (52) with initial condition \((x(0), y(0)) = (c_1, c_2)\) on \( S \), then \( c_1 = -0.27324c_2^2 \).

In Fig. 6, we can see that the solution (53) starting on \( S \), in any small neighborhood of origin, e.g., at \((-0.27324 \times 10^{-20}, 10^{-10})\), does not tend toward origin as \( t \to \infty \).

\[
\therefore S \text{ cannot be a local stable manifold.}
\]

Now, we prove that neither the parabola \( x = my^2 \) nor the parabola \( y = mx^2 \) is invariant under the flow of system (52).

1. Consider the parabola \( x = my^2 \). Its differentiation of order \( \alpha \) gives

\[
\frac{\partial}{\partial t} D_\alpha^x x = m \frac{\partial}{\partial t} D_\alpha^y y = \frac{\partial}{\partial t} D_\alpha^y y = -y.
\]

By using generalized Leibniz rule for Caputo fractional derivative of order \( 0 < \alpha < 1 \) and substituting \( x = my^2 \), we obtain

\[
(2m - 1)y^2 - \frac{m y(0) t^{-\alpha}}{\Gamma(1 - \alpha)} [y(t) - y(0)]
\]

\[
- m \sum_{k=1}^{\infty} \binom{\alpha}{k} (0 t^{k-\alpha}) y^{(k)}(t) = 0.
\]
This holds for all $t$ if and only if $2m - 1 = 0$ and $m = 0$, which is inconsistent.  
$\Rightarrow$ There does not exist any $m \in \mathbb{R}$ such that $x = my^2$ is invariant under the flow of system (52).  
(II): Consider $y = mx^2$.  
In this case, the tangency condition gives  
$$m^2x^4 - mx^2 - mx - \frac{m x(0) t^{-\alpha}}{\Gamma(1 - \alpha)} [x(t) - x(0)] - m \sum_{k=1}^{\infty} \left( \frac{\alpha}{k} \right) (\alpha t^{k-\alpha} x(t)) x^{(k)}(t) = 0.$$  
Using the similar arguments, we can easily check that there does not exist any nonzero $m \in \mathbb{R}$ such that $y = mx^2$ is invariant under the flow of system (52).

6 Conclusion

We used tangency condition to propose the necessary and sufficient conditions for the existence of invariant straight lines and parabolas in the planar polynomial systems of ordinary differential equations. Further, we proved that the conditions for the invariance of linear subspaces in fractional-order systems are the same as their classical counterparts. An ample number of examples are provided to support the results.

Important contribution of this work is the result showing the nonexistence of invariant manifolds (except linear subspaces) in fractional-order systems. In particular, we have shown that any curve with curvature $> 0$ cannot be invariant under the flow of fractional-order system.

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Compliance with ethical standards

Conflict of interest The authors declare that they have no conflict of interest.

Appendix

Literature review

Consider a planar polynomial vector field (13).

The second part of Hilbert’s sixteenth problem [68] is related to the number of limit cycles in the polynomial system (13). The literature review of the planar quadratic system is taken by Coppel [69]. In [70], authors studied the classification of phase portraits of a quadratic system in a region surrounded by separatrix cycle.

In [71], Ye proposed the following conjecture:

**Conjecture 1** When $n$ is odd, the system (13) has at most $M_n = 2n + 2$ invariant lines; when $n$ is even, the system (13) has at most $M_n = 2n + 1$ invariant straight lines.

For $n = 2, 3$ and 4, this conjecture is proved by Sokulski [72]. However, the conjecture is false [73] if $n > 4$. It should be noted that the system (13) can have infinitely many invariant straight lines (see Example (iv)).

Artes [73] proposed the following important result:

**Theorem 6.1** Assume that the polynomial differential system (13) of degree $n$ has finitely many invariant straight lines. Then, the following statements hold for the system (13).

1. Either all the points on an invariant line are equilibrium or the line contains no more than $n$ equilibrium points.
2. No more than $n$ invariant straight lines can be parallel.
3. The set of all invariant straight lines through a single point cannot have more than $n + 1$ different slopes.
4. Either it has infinitely many finite equilibrium points, or it has at most $n^2$ finite equilibrium points.

**Theorem 6.2** There exists invariant parabola $x = my^2$ for the system (21) if and only if

1. $a_2 = 0, a_3 = 2b_5, b_3 = 0$ and
2. One of the conditions (a), (b), (c) and (d) hold.

(a) $a_4 = 0, a_1 = 2b_2, a_5 \neq 2b_4$ and $b_1 \neq 0$. (In this case $m = \frac{a_5 - 2b_4}{2b_1})$
(b) $a_5 = 2b_4, b_1 = 0, a_4 \neq 0$ and $a_1 \neq 2b_2$. (In this case $m = \frac{-a_4}{2b_1 - 2b_2})$
(c) $a_4 \neq 0, a_1 \neq 2b_2, a_5 \neq 2b_4, b_1 \neq 0$ and $a_1a_5 - 2a_1b_4 - 2b_2a_5 + 4b_2b_4 + 2a_4b_1 = 0$. (In this case $m = \frac{a_5 - 2b_4}{2b_1} = \frac{-a_4}{a_1 - 2b_2})$
(d) \( a_4 = 0, a_1 = 2b_2, a_5 = 2b_4 \) and \( b_1 = 0. \) (In this case, \( x = my^2, \forall \in \mathbb{R}.) \)

**Example 6.1** Consider
\[
\begin{align*}
\dot{x} &= 2x - 2x^2 + y^2 + 6xy \\
\dot{y} &= -y + 3y^2 - xy.
\end{align*}
\]
(56)

This system satisfies the conditions 1 and 2(b) of Theorem 6.2, and the invariant parabola is \( x = -\frac{1}{4}y^2. \)

In the following example, we can see that the general parabola is invariant under the flow.

**Example 6.2** Consider the following system
\[
\begin{align*}
\dot{x} &= \frac{1}{2}x - \frac{5}{2}y + 2\sqrt{2}x^2 + 2\sqrt{2}y^2 \\
\dot{y} &= -\frac{5}{2}x + \frac{1}{2}y + \frac{3}{\sqrt{2}}x^2 + \frac{7}{\sqrt{2}}y^2 - \sqrt{2}xy.
\end{align*}
\]
(57)

Here, \( 5x^2 - 10xy + 5y^2 - 8\sqrt{2}x - 8\sqrt{2}y = 0 \) is invariant under the flow of this system (see Fig. 7b).

Some other invariant curves

The Hamiltonian \( H(x, y) \) of a Hamiltonian system, passing through a saddle equilibrium point, works as separatrix [4].

**Example 6.3** Consider a planar quadratic system
\[
\begin{align*}
\dot{x} &= y - \sqrt{2}xy \\
\dot{y} &= \frac{1}{2}(2x + \sqrt{2}x^2 + \sqrt{2}y^2).
\end{align*}
\]
(58)

This system is a Hamiltonian system, and the Hamiltonian is given by
\[
H(x, y) = -\frac{1}{2}x^2 + \frac{1}{2}y^2 - \frac{1}{3\sqrt{2}}x^3 - \frac{1}{\sqrt{2}}xy^2.
\]

Here, \(-\frac{1}{2}x^2 + \frac{1}{2}y^2 - \frac{1}{3\sqrt{2}}x^3 - \frac{1}{\sqrt{2}}xy^2 = 0\) gives separatrix for this system and it is shown in Fig. 8.

**Theorem 6.3** The cubic curve \( y = x^3 + mx^2 + ux \) is invariant under planar quadratic system (21) if and only if

\[
\begin{align*}
a_4 &= a_2 = b_4 = a_5 = 0, \quad b_5 = 3a_3, \\
6a_1^3 + 2a_2^3b_1 - 11a_1^2b_2 - b_2^3 - a_3b_2b_3 &= 0.
\end{align*}
\]

\[ \square \]
Example 6.4 The planar quadratic system
\[
\begin{align*}
\dot{x} &= x + x^2 \\
\dot{y} &= x + 2y + 2x^2 + 3xy,
\end{align*}
\]
(59)
has invariant curve, viz. \( y = x^3 + x^2 - x \) (see Fig. 9).

Theorem 6.4 The curves \( y = mx^k \), (for any \( m \in \mathbb{R} \) and \( k > 0 \)) are invariant under the flow of planar quadratic system (21) if and only if
\[
b_1 = b_3 = a_4 = a_2 = 0, \quad b_4 = ka_5,
\]
\[
b_5 = ka_3 \quad \text{and} \quad b_2 = ka_1.
\]

The system (21) can have invariant curves other than polynomial curves also. The following theorem provides conditions for the existence of an exponential curve as an invariant.

Theorem 6.5 The system (21) has exponential curve \( y = me^x \) for all \( m \in \mathbb{R} \), as an invariant curve if
\[
a_3 = a_4 = a_5 = b_1 = b_2 = b_3 = 0,
\]
\[
b_4 = a_2 \quad \text{and} \quad b_5 = a_1.
\]

Example 6.5 Consider the planar quadratic system
\[
\begin{align*}
\dot{x} &= -2x + 3y \\
\dot{y} &= 3y^2 - 2xy.
\end{align*}
\]
(60)
It can be verified that the curve \( y = me^x \) (for all \( m \in \mathbb{R} \)) is invariant under the flow of system (60).

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