A mirror duality for families of $K3$ surfaces associated to bimodular singularities

Makiko Mase

Key Words: $K3$ surfaces, toric varieties, Picard lattices
AMS MSC2010: 14J28 14M25 14C22

Abstract

Ebeling and Ploog [4] studied a duality of bimodular singularities which is part of the Berglund–Hübsch mirror symmetry. Mase and Ueda [7] showed that this duality leads to a polytope mirror symmetry of families of $K3$ surfaces. We discuss in this article how this symmetry extends to a symmetry between lattices.

1 Introduction

Bimodular singularities classified by Arnold [1] have a duality studied by Ebeling and Ploog [4] analogous to Arnold’s strange duality for unimodular singularities. Namely, a pair $((B, f), (B', f'))$ of singularities $B, B'$ in $C^3$ together with appropriate defining polynomials $f, f'$ is dual if the matrices $A_f, A_{f'}$ of exponents of $f$ and $f'$ are transpose to each other. Moreover, in some cases, such polynomials are compactified as anticanonical members of 3-dimensional weighted projective spaces whose general members are Gorenstein $K3$. The strange duality for unimodular singularities is related with the polytope mirror symmetry for families of $K3$ surfaces that are obtained by compactifying the singularities by Kobayashi [6] in a certain sense. In the study of bimodular singularities, Mase and Ueda [7] extend the duality by Ebeling and Ploog to a polytope mirror symmetry of families of $K3$ surfaces. More precisely, the following statement is shown:

Theorem [7] Let $((B, f), (B', f'))$ be a dual pair in the sense of [4] of singularities $B$ and $B'$ together with their defining polynomials $f$ and $f'$ that are respectively compactified into polynomials $F$ and $F'$ as in [4]. Then, there exists a reflexive polytope $\Delta$ such that $\Delta_F \subset \Delta$ and $\Delta_{F'} \subset \Delta^*$. Here, $\Delta_F$ and $\Delta_{F'}$ are respectively the Newton polytopes of $F$ and of $F'$, and $\Delta^*$ is the polar dual to $\Delta$.

In this article, we consider whether or not it is possible to extend the duality obtained in [7] further to the lattice mirror symmetry of families of $K3$ surfaces. More precisely, our problem is stated as follows:

Problem Let $\Delta$ be a reflexive polytope as in [7]. Does there exist general members $S \in F_\Delta$ and $S' \in F_{\Delta^*}$ such that an isometry $T(S) \simeq \text{Pic}(S') \oplus U$ holds.
Here, $F_\Delta$ and $F_{\Delta^*}$ are families of $K3$ surfaces associated to the polytopes $\Delta$ and $\Delta^*$, $\tilde{S}$ denotes the minimal model of $S$, Pic($\tilde{S}$) and $T(\tilde{S})$ are respectively the Picard and transcendental lattice of $\tilde{S}$.

The problem is answered in Theorem 3.2 together with an explicit description in Proposition 3.1 of the Picard lattices Pic($\Delta$) and Pic($\Delta^*$) defined in section 3, with ranks $\rho(\Delta)$ and $\rho(\Delta^*)$, of the minimal model of appropriate general members in the families. The main result of this article is summarized here. In the sequel, the names of singularities follow Arnold [1], and singularities in a same row of Table 1 are dual to each other in the sense of [4].

**Proposition 3.1** and **Theorem 3.2** Let $\Delta$ be the reflexive polytope obtained in [7]. For the following transpose-dual pairs, the polar duality extends to a lattice mirror symmetry between the families $F_\Delta$ and $F_{\Delta^*}$, where the Picard lattices are given in Table 1. Here we use the notation $C_8^5 := \left( \begin{array}{cc} -4 & 1 \\ 1 & -2 \end{array} \right)$.

| Singularity | Pic($\Delta$) | $\rho(\Delta)$ | Pic($\Delta^*$) | $\rho(\Delta^*)$ | Singularity |
|-------------|---------------|-----------------|-----------------|------------------|-------------|
| $Q_{12}$    | $U \oplus E_6 \oplus E_8$ | 16              | $U \oplus A_2$  | $E_{18}$         |
| $Z_{1,0}$   | $U \oplus E_7 \oplus E_8$ | 17              | $U \oplus A_1$  | $E_{19}$         |
| $E_{20}$    | $U \oplus E_8^{18}$       | 18              | $U \oplus A_1$  | $E_{20}$         |
| $Q_{2,0}$   | $U \oplus A_6 \oplus E_8$ | 16              | $U \oplus C_8^5$| $Z_{17}$         |
| $E_{25}$    | $U \oplus E_7 \oplus E_8$ | 17              | $U \oplus A_1$  | $Z_{19}$         |
| $Q_{18}$    | $U \oplus E_6 \oplus E_8$ | 16              | $U \oplus A_2$  | $E_{30}$         |

Table 1: Picard lattices for lattice mirror symmetric pairs

Section 2 is to define the polytope- and lattice- mirror theories in subsection 2.3 and to define the transpose duality following [4] in subsection 2.4 based on a brief introduction to lattice theory in subsection 2.1 and of toric geometry in subsection 2.2 where several formulas and results are stated without proof.

The main theorem of this article is stated in section 3 following auxiliary results. The facts introduced in the previous section are used in their proof.

Denote by $\Delta_B$ the reflexive polytope obtained in [7] for a singularity $B$. As is seen in Table 1 there are isometric Picard lattices Pic($\Delta_{Q_{12}}$) $\simeq$ Pic($\Delta_{Q_{18}}$), and Pic($\Delta_{Z_{1,0}}$) $\simeq$ Pic($\Delta_{E_{25}}$). We consider and affirmatively answer in Proposition 4.1 the following question as an application in section 4:

**Problem** Are the families $F_{\Delta_{Q_{12}}}$ (resp. $F_{\Delta_{Z_{1,0}}}$) and $F_{\Delta_{Q_{18}}}$ (resp. $F_{\Delta_{E_{25}}}$) essentially the same in the sense that general members in these families are birationally equivalent?

**Acknowledgement**

The author thanks to the referee for his helpful comments particularly about the proof of the key Proposition 3.1 in the original manuscript.
2 Preliminary

We start with having a consensus as to Gorenstein K3 and K3 surfaces.

Definition 2.1 A compact complex connected 2-dimensional algebraic variety \( S \) with at most ADE singularities is called Gorenstein K3 if (i) \( K_S \sim 0 \); and (ii) \( H^3(S, \mathcal{O}_S) = 0 \). If a Gorenstein K3 surface \( S \) is nonsingular, \( S \) is simply called a K3 surface.

2.1 Brief lattice theory

A lattice is a non-degenerate finitely-generated free \( \mathbb{Z} \)-module with a symmetric bilinear form called an intersection pairing. The discriminant group of a lattice \( L \) is defined by \( A_L := L^*/L \), which is finitely-generated and abelian, where \( L^* := \text{Hom}(L, \mathbb{Z}) \) is dual to \( L \). It is known that the order \( |A_L| \) of the discriminant group is equal to the determinant of any intersection matrix of \( L \). Let us recall a standard lattice theory by Nikulin [8].

Corollary 2.1 (Corollary 1.13.5-(1) [8]) If an even lattice \( L \) of signature \((t_+, t_-) \) satisfies \( t_+ \geq 1, t_- \geq 1 \), and \( t_+ + t_- \geq 3 + \text{length} \ A_L \), then, there exists a lattice \( T \) such that \( L \simeq U \oplus T \), where \( U \) is the hyperbolic lattice of rank 2.

In particular, if an even lattice \( L \) is of \( \text{rk} \ L > 12, t_+ \geq 1, \) and \( t_- \geq 1 \), then, there exists a lattice \( T \) such that \( L \simeq U \oplus T \).

Suppose \( L \) is a sublattice of a lattice \( L' \) with inclusion \( i : L \hookrightarrow L' \). Denote by \( L_i \) the orthogonal complement of \( L \) in \( L' \). The embedding \( i \) is called primitive, and \( L \) called a primitive sublattice of \( L' \) if the finite abelian group \( L'/i(L) \) is torsion-free. In other words, if there is no overlattice that is an intermediate lattice between \( L \) and \( L' \) of rank equal to the rank of \( L \). Note that if a direct sum \( L_1 \oplus L_2 \) of lattices is a sublattice of \( L' \), then \( (L_1 \oplus L_2)^*_L \simeq (L_1^*_L) \oplus (L_2^*_L) \).

For a K3 surface \( S \), it is known that \( H^2(S, \mathbb{Z}) \) with the intersection pairing is isometric to the K3 lattice \( \Lambda_{K3} \) that is even unimodular of rank 22 and signature (3, 19), being \( U^{8,3} \oplus E_8^{1,1} \), where \( E_8 \) is the negative-definite even unimodular lattice of rank 8. There is a standard exact sequence \( 0 \rightarrow H^1(S, \mathcal{O}_S^\circ) \rightarrow H^2(S, \mathbb{Z}) \rightarrow 0 \) of cohomologies so \( H^1(S, \mathcal{O}_S^\circ) \) is inherited a lattice structure from \( H^2(S, \mathbb{Z}) \). Define the Picard lattice \( \text{Pic}(S) \) of a K3 surface \( S \) as the group \( c_1(H^1(S)) \cap H^2(S, \mathbb{Z}) \) with the lattice structure. The rank of \( \text{Pic}(S) \) is called the Picard number, denoted by \( \rho(S) \). The Picard lattice is hyperbolic since a K3 surface is complex and algebraic, and is known to be a primitive sublattice of \( \Lambda_{K3} \) under a marking \( H^2(S, \mathbb{Z}) \simeq \Lambda_{K3} \).

If an even hyperbolic lattice \( L \) of rank \( \text{rk} \ L \leq 20 \) has \( |A_L| \) being square-free, \( L \) is a primitive sublattice of \( \Lambda_{K3} \). Indeed, if so, for any lattice \( L \subset L'' \subset \Lambda_{K3} \) the general relation \( |A_L| = [L'' : L]^2 |A_{L''}| \) implies \( [L'' : L] = 1 \) thus \( L'' \simeq L \). Hence there is no overlattice of \( L \). Moreover, by surjectivity of the period mapping [2], there exists a K3 surface \( S \) such that \( \text{Pic}(S) \simeq L \). Let \( M \subset \Lambda_{K3} \) be a hyperbolic sublattice. A K3 surface is \( M \)-polarised [3] if there exists a marking \( \phi \) such that all divisors in \( \phi^{-1}(C_M^{pol}) \) are ample, where \( C_M^{pol} \) is the positive cone in \( M_\mathbb{R} \) minus \( \bigcup_{d \in \Delta_M} H_d \), \( \Delta_M = \{ d \in M | d.d = -2 \} \), and \( H_d = \{ x \in \mathbb{P}(\Lambda_{K3}) | x.d = 0 \text{ for all } d \in \Delta_M \} \).

Nishiyama [9] gives the orthogonal complements of primitive sublattices of type ADE of \( E_8 \) in possible cases.
Lemma 2.1 (Lemma 4.3 [9]) There exist primitive embeddings of lattices of type ADE into \( E_8 \) with orthogonal complements given as follows. All the notation follows Bourbaki except \( C^6_8 := \left( \begin{array}{rr} -4 & 1 \\ 1 & -2 \end{array} \right) \).

\[
\begin{align*}
(A_1)_{E_8} \simeq E_7 & \quad (A_2)_{E_8} \simeq E_6 & \quad (A_3)_{E_8} \simeq A_2 & \quad (A_4)_{E_8} \simeq A_1 \\
(A_5)_{E_8} \simeq A_1 \oplus A_2 & \quad (A_6)_{E_8} \simeq C^6_8 & \quad (A_7)_{E_8} \simeq \left( -8 \right) & \quad (A_8)_{E_8} \simeq \left( -4 \right)
\end{align*}
\]

2.2 Brief toric geometry

Here we summarize toric divisors and \( \Delta \)-regularity. Let \( M \) be a rank-3 lattice with the standard basis \( \{ e_1, e_2, e_3 \} \), \( N \) be its dual, and \( (, ,) : M \times N \to \mathbb{Z} \) be the natural pairing. From now on, a polytope means a 3-dimensional convex hull of finitely-many points in \( \mathbb{Z}^3 \) embedded into \( \mathbb{R}^3 \), namely, integral, and the origin is the only lattice point in the interior of it.

Let \( \mathbb{P}_\Delta \) be the toric variety defined by a polytope \( \Delta \) in \( M \otimes \mathbb{Z} \), to which one can associate a fan \( \Sigma_\Delta \) whose one-dimensional cones, called \( \textit{one-simplices} \), are generated by primitive lattice vectors each of whose end-point is an intersection point of \( N \) and an edge of its polar dual \( \Delta^* \) defined by

\[
\Delta^* := \{ y \in N \otimes \mathbb{R} \mid (x, y) \geq -1 \text{ for all } x \in \Delta \}.
\]

Let \( \widetilde{\mathbb{P}_\Delta} \) be the toric resolution of singularities in \( \mathbb{P}_\Delta \). A \textit{toric divisor} is a divisor admitting the torus action, identified with the closure of the torus-action orbit of a one-simplex. Let \( \text{Div}_\mathbb{P}(\widetilde{\mathbb{P}_\Delta}) \) be the set of all toric divisors \( D_i = \text{orb}(R_{z_0} v_i), i = 1, \ldots, s \) on \( \mathbb{P}_\Delta \), where \( v_i \) is a primitive lattice vector, then, \(-K_{\mathbb{P}_\Delta} = \sum_{i=1}^{s} D_i \).

By a standard exact sequence and a commutative diagram \([10]\)

\[
\begin{array}{cccc}
0 & \to & M & \to & \text{Div}_\mathbb{P}(\widetilde{\mathbb{P}_\Delta}) & \to & \text{Pic}(\widetilde{\mathbb{P}_\Delta}) & \to & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & M & \to & \bigoplus_{i=1}^{s} \mathbb{Z}D_i & \to & A_2(\widetilde{\mathbb{P}_\Delta}) & \to & 0
\end{array}
\]

there is a system of linear equations among toric divisors

\[
\sum_{i=1}^{s} (e_j, v_i)D_i = 0, \quad j = 1, 2, 3.
\]

Thus the solution set of the linear system is generated by \((s-3)\) elements corresponding to divisors which generate the Picard group \( \text{Pic}(\widetilde{\mathbb{P}_\Delta}) \) of \( \widetilde{\mathbb{P}_\Delta} \).

**Definition 2.2** A polytope is \textit{reflexive} if its polar dual is also integral. ■

The importance of that we consider reflexive polytopes is by the following:

**Theorem 2.1** c.f. [7] The following conditions are equivalent:

1. A polytope \( \Delta \) is reflexive.
2. The toric 3-fold \( \mathbb{P}_\Delta \) is Fano, in particular, general anticanonical members of \( \mathbb{P}_\Delta \) are Gorenstein K3. ■

4
For a reflexive polytope $\Delta$, denote by $F_\Delta$ the family of hypersurfaces parametrised by the complete anticanonical linear system of $\mathbb{P}_\Delta$. Note that general members in $F_\Delta$ are Gorenstein $K3$ due to the previous theorem so that they are birationally equivalent to $K3$ surfaces by the existence of crepant resolution. Thus we call the family $F_\Delta$ a family of $K3$ surfaces.

We recall from §3 of [7] the notion of $\Delta$-regularity.

**Definition 2.3** Let $F$ be a Laurent polynomial defining a hypersurface $Z_F$, whose Newton polytope is a polytope $\Delta$. The hypersurface $Z_F$ is called $\Delta$-regular if for every face $\Gamma$ of $\Delta$, the corresponding affine stratum $Z_{F,\Gamma}$ of $Z_F$ is either empty or a smooth subvariety of codimension $1$ in the torus $\mathbb{T}_\Gamma$ that is contained in the affine variety associated to $\Gamma$.

It is shown in [7] that $\Delta$-regularity is a general condition, and singularities of all $\Delta$-regular members are simultaneously resolved by a toric desingularization of $\mathbb{P}_\Delta$. From now on, suppose a polytope $\Delta$ is reflexive and $S$ is a $\Delta$-regular member whose minimal model $\tilde{S}$ is obtained by a toric resolution.

**Definition 2.4** For a restriction $r : \mathbb{P}_\Delta \to \tilde{S}$, let $r_* : H^{1,1}(\mathbb{P}_\Delta) \to H^{1,1}(\tilde{S})$ be the induced mapping. Define a lattice $L_D(\tilde{S}) := r_*(H^{1,1}(\mathbb{P}_\Delta)) \cap H^2(\tilde{S}, \mathbb{Z})$ of the intersection of the image of $r_*$ and $H^2(\tilde{S}, \mathbb{Z})$, and its orthogonal complement $L_0(\tilde{S}) := L_D(\tilde{S}) \perp_{H^2(\tilde{S}, \mathbb{Z})}$ in $H^2(\tilde{S}, \mathbb{Z})$.

It is known [7] that $\rho(\tilde{S})$ and $\text{rk } L_0(\tilde{S})$ only depend on the number of lattice points in edges of $\Delta$ and $\Delta^\ast$. Thus we define the Picard number $\rho(\Delta) := \rho(\tilde{S})$, and the rank $\text{rk } L_{0,\Delta} := \text{rk } L_0(\tilde{S})$ associated to $\Delta$. More precisely, denote by $\Gamma^\ast$ in $\Delta^\ast$ the dual face to a face $\Gamma$ of $\Delta$, and $l^\ast(\Gamma)$ is the number of lattice points in the interior of $\Gamma$, and $\Delta^{[1]}$ the set of edges in $\Delta$. Let $s$ be the number of one-simplices of $\Sigma_\Delta$. Then

$$\text{rk } L_{0,\Delta} = \sum_{\Gamma \in \Delta^{[1]}} l^\ast(\Gamma) l^\ast(\Gamma^\ast) = \text{rk } L_{0,\Delta^\ast},$$  

$$\rho(\Delta) = s - 3 + \text{rk } L_{0,\Delta},$$  

$$\rho(\Delta) + \rho(\Delta^\ast) = 20 + \text{rk } L_{0,\Delta^\ast}.$$  

If $l^\ast(\Gamma) = n_{\Gamma}$ and $l^\ast(\Gamma) = m_{\Gamma}$ for an edge $\Gamma$ of $\Delta$, there is a singularity of type $\Delta_{m_{\Gamma} + 1}$ with multiplicity $m_{\Gamma} + 1$ on an affine variety associated to $\Gamma$.

As we will see later, we only need formulas when $\text{rk } L_{0,\Delta} = 0$ for the intersection numbers of the divisors $\{r_\ast D_i\}$ in $H^2(\tilde{S}, \mathbb{Z})$ given as follows.

$$r_\ast D_i^2 = r_\ast D_i r_\ast D_i = \begin{cases} 2l^\ast(v_i^\ast) - 2 & \text{if } v_i \text{ is a vertex of } \Delta^\ast, \\ -2 & \text{otherwise} \end{cases}$$

If end-points of $v_i$ and $v_j$ are on the edge $\Gamma_{ij}^\ast$ of $\Delta^\ast$, then

$$r_\ast D_i r_\ast D_j = \begin{cases} 1 & \text{if } v_i \text{ and } v_j \text{ are next to each other,} \\ l^\ast(\Gamma_{ij}) + 1 & \text{if } l^\ast(\Gamma_{ij}) = 0, \text{ and } v_i, v_j \text{ are both vertices,} \\ 0 & \text{otherwise}. \end{cases}$$

The Picard lattices of the minimal models of any $\Delta$-regular members, which are generated by components of restricted toric divisors, are isometric.
the Picard lattice $\text{Pic}(\Delta)$ of the family $F_\Delta$ as the Picard lattice of the minimal model of a $\Delta$-regular member with rank $\rho(\Delta)$. The orthogonal complement $T(\Delta) = \text{Pic}(\Delta)^{\perp}_{\Lambda K_3}$ is called the transcendental lattice of $F_\Delta$.

2.3 Mirrors

We define the polytope- and lattice-mirror theories.

2.3.1 Polytope Mirror

We focus on polytopes that “represent” the anticanonical members in a toric variety as is seen in subsection 2.2.

**Definition 2.5** A pair $(\Delta_1, \Delta_2)$ of reflexive polytopes or a pair $(F_{\Delta_1}, F_{\Delta_2})$ of families of $K3$ surfaces associated to $\Delta_1$ and $\Delta_2$ is called polytope mirror symmetric if an isometry $\Delta_1 \simeq \Delta_2^*$ holds. ■

2.3.2 Lattice Mirror

For a $K3$ surface $S$, $\text{Pic}(S)$ is the Picard lattice, and $T(S) = \text{Pic}(S)^{\perp}_{\Lambda K_3}$ is the transcendental lattice. A mirror for family of $M$-polarised $K3$ surfaces is defined when $M$ is a sublattice of $\Lambda K_3$ in general [3]. Here, we deal with the most strict case, namely, mirror for $K3$ surfaces with Picard lattice as their polarisation.

**Definition 2.6**

(1) A pair $(S, S')$ of $K3$ surfaces is called lattice mirror symmetric if an isometry $T(S) \simeq \text{Pic}(S') \oplus U$ holds.

(2) A pair $(F, F')$ of families whose general members are Gorenstein $K3$ surfaces is lattice mirror symmetric if there exist general members $S \in F$ and $S' \in F'$ pair of whose minimal models is lattice mirror symmetric. ■

Note that a lattice mirror pair $(S, S')$ of $K3$ surfaces satisfies, by definition, $\text{rk Pic}(S') + 2 = \text{rk } T(S) = 22 - \text{rk Pic}(S)$, thus

$$\rho(S') + \rho(S) = 20.$$  

(7)

2.4 Bimodular singularities and the transpose duality

Being classified by Arnold [1] in 1970’s, bimodular singularities have two specific classes: quadrilateral and exceptional. Quadrilateral bimodular singularities are 6 in number with exceptional divisor of type $I_0^*$, whilst exceptional are 14 in number with exceptional divisor of type $II^*$, $III^*$ or $IV^*$ in Kodaira’s notation.

A non-degenerate polynomial $f$ in three variables is called invertible if $f$ has three terms $f = \sum_{j=1}^{3} x^{a_{1j}} y^{a_{2j}} z^{a_{3j}}$ such that its matrix $A_f := (a_{ij})_{1 \leq i,j \leq 3}$ of exponents is invertible in $GL_3(\mathbb{Q})$.

**Definition 2.7** c.f. [4] Let $B = (0, (f = 0))$ and $B' = (0, (f' = 0))$ be germs of singularities in $\mathbb{C}^3$. A pair $(B, B')$ of singularities is called transpose dual if the following three conditions are satisfied.

(1) Defining polynomials $f$ and $f'$ are invertible.

(2) Matrices $A_f$ and $A_{f'}$ of exponents of $f$ and $f'$ are transpose to each other.
(3) \(f\) (resp. \(f'\)) is compactified to a four-term polynomial \(F\) (resp. \(F'\)) in \(-K_P(a)\) (resp. \(-K_P(b)\)), where \(P(a)\) (resp. \(P(b)\)) is the 3-dimensional weighted projective space whose general members are Gorenstein K3 with weight \(a\) (resp. \(b\)) out of the list of 95 weights classified by [12] [5] [11].

Condition (1) and (2) is said that they are Berglund–Hübsch mirror symmetric. Ebeling-Ploog [4] show that there are 16 transpose-dual pairs among quadrilateral and exceptional bimodular singularities, exceptional unimodular singularities, and the singularities \(E_{25}, E_{30}, X_{2,0},\) and \(Z_{2,0}\).

### 3 The transpose dual and the lattice mirror

For a transpose-dual pair \((B, B')\) of defining polynomial \(f\) (resp. \(f'\)) being compactified to a polynomial \(F\) (resp. \(F'\)), consider the Newton polytope \(\Delta_F\) (resp. \(\Delta_{F'}\)) of \(F\) (resp. \(F'\)) all of whose corresponding monomials are fixed by an automorphism action on \((F = 0)\) (resp. \((F' = 0)\)).

Remark 1 A compactified member \(F\) to \(f\) does not always define a Gorenstein K3 surface because \(\Delta_F\) may not be reflexive.

However, the Newton polytopes are extended to be reflexive and dual.

**Theorem 3.1** [7] For each transpose-dual pair \((B, B')\), there exists a reflexive polytope \(\Delta\) such that \(\Delta_F \subset \Delta\) and \(\Delta_{F'} \subset \Delta^*.\)

Computing Picard lattices is generally difficult, but it seems possible for \(\Delta\)-regular members by subsection 2.2. Let us reformulate our problem.

**Problem** For a polytope \(\Delta\) obtained in [7], is a pair \((\tilde{S}, \tilde{S}')\) of minimal models of \(\Delta\)-regular \(S \in \mathcal{F}_\Delta\) and \(\Delta^*\)-regular \(S' \in \mathcal{F}_{\Delta^*}\) lattice mirror symmetric?

First we study the rank \(\text{rk} L_{0,\Delta}\).

**Lemma 3.1** The list of \(\text{rk} L_{0,\Delta}\) for the reflexive polytope \(\Delta\) obtained in [7] is given in Table 2.

| transpose-dual pair | \(\text{rk} L_{0,\Delta}\) | transpose-dual pair | \(\text{rk} L_{0,\Delta}\) |
|---------------------|--------------------------|---------------------|--------------------------|
| \((Q_{12}, E_{18})\) | 0                        | \((Z_{17}, Q_{2,0})\) | 2                        |
| \((Z_{1,0}, E_{19})\) | 0                        | \((U_{1,0}, U_{1,0})\) | 2                        |
| \((E_{20}, E_{20})\) | 0                        | \((U_{16}, U_{16})\) | 2                        |
| \((Q_{12}, Z_{17})\) | 0                        | \((Q_{17}, Z_{2,0})\) | 2                        |
| \((E_{25}, Z_{19})\) | 0                        | \((W_{1,0}, W_{1,0})\) | 3                        |
| \((Q_{18}, E_{20})\) | 0                        | \((W_{17}, S_{1,0})\) | 5                        |
| \((Z_{1,0}, Z_{1,0})\) | 1                        | \((W_{18}, W_{18})\) | 6                        |
| \((Z_{13}, J_{2,0})\) | 2                        | \((S_{17}, X_{2,0})\) | 6                        |

Table 2: \(\text{rk} L_{0,\Delta}\)

**Proof.** The assertion follows from direct and case-by-case computation by formula (2) in subsection 2.2.
1. \((Q_{12}, E_{18})\) The polytope \(\Delta\) is given in subsection 4.7 of [7]. There is no contribution to \(\text{rk}\ L_{0,\Delta}\) since \(l^*(\Gamma)\) or \(l^*(\Gamma^*)\) is zero for an edge \(\Gamma\). Thus, \(\text{rk}\ L_{0,\Delta} = 0\) by formula (2).

Similar for the cases \((Z_{1,0}, E_{19}), (E_{20}, E_{20}), (Q_{2,0}, Z_{17}), (E_{25}, Z_{19}), (Q_{18}, E_{30})\).

2. \((Z_{13}, J_{3,0})\) The polytope \(\Delta\) is given in subsection 4.1 of [7]. The only contribution to \(\text{rk}\ L_{0,\Delta}\) is by an edge \(\Gamma\) between vertices \((0, 0, 1)\) and \((-2, -6, -9)\), whose dual \(\Gamma^*\) is between \((8, -1, -1)\) and \((-1, 2, -1)\) so \(l^*(\Gamma) = 1\) and \(l^*(\Gamma^*) = 2\).

Thus, \(\text{rk}\ L_{0,\Delta} = 1 \times 2 = 2\) by formula (2).

Similar for the cases \((Z_{1,0}, Z_{1,0}), (Z_{17}, Q_{2,0}), (U_{1,0}, U_{1,0}), (U_{16}, U_{16}), (Q_{17}, Z_{2,0}), (W_{1,0}, W_{1,0}), (W_{17}, S_{1,0}), (W_{18}, W_{18}), (S_{17}, X_{2,0})\).

\textbf{Corollary 3.1} No \(\Delta\)- and \(\Delta^*\)-regular members for transpose-dual pairs \((Z_{13}, J_{3,0}), (Z_{1,0}, Z_{1,0}), (Z_{17}, Q_{2,0}), (U_{1,0}, U_{1,0}), (U_{16}, U_{16}), (Q_{17}, Z_{2,0}), (W_{1,0}, W_{1,0}), (W_{17}, S_{1,0}), (W_{18}, W_{18}), (S_{17}, X_{2,0})\) admit a lattice mirror symmetry.

\textbf{Proof.} For each \(\Delta\) associated to the presented pairs, by formula (4),

\[\rho(\Delta^*) + \rho(\Delta) = 20 + \text{rk}\ L_{0,\Delta} > 20\]

since \(\text{rk}\ L_{0,\Delta} > 0\) by Lemma 3.1. Thus, the equation (7) does not hold. Therefore, \(\Delta\)- and \(\Delta^*\)-regular members do not admit a lattice mirror symmetry.

\textbf{Corollary 3.2} The restriction mapping \(r_* : \text{Pic}(\tilde{\mathcal{P}}_\Delta) \to \text{Pic}(\tilde{S})\), for \(\Delta\)-regular \(S \in \mathcal{F}_\Delta\) is surjective for the transpose-dual pairs \((Q_{12}, E_{18}), (Z_{1,0}, E_{19}), (E_{20}, E_{20}), (Q_{2,0}, Z_{17}), (E_{25}, Z_{19}), (Q_{18}, E_{30})\).

\textbf{Proof.} By Lemma 3.1 \(\text{rk}\ L_{0,\Delta} = 0\) for each case. By definition, \(\text{rk}\ L_{0,\Delta}\) is equal to the rank of the orthogonal complement of \(r_*(H^1(\tilde{\mathcal{P}}_\Delta))\) in \(H^2(\tilde{S}, \mathbb{Z})\). Thus, that \(\text{rk}\ L_{0,\Delta} = 0\) means that \(r_*\) is surjective.

By Corollary 3.1, we may only focus on the transpose-dual pairs appearing in Corollary 3.2 whose statement means moreover that \(\text{Pic}(\Delta)\) is generated by restricted toric divisors generating \(\text{Pic}(\tilde{\mathcal{P}}_\Delta)\), and analogous to \(\Delta^*\).

Let \(A_L\) denote the discriminant group, \(q_L\) the quadratic form, and \(\text{discr}\ L\) the discriminant of a lattice \(L\). If \(p = \text{discr}\ L\) is prime, then \(A_L \simeq \mathbb{Z}/p\mathbb{Z}\). Before stating our main results, note a fact in Proposition 1.6.1 in [8]. Suppose that lattices \(S\) and \(T\) are primitively embedded into the \(K3\) lattice \(\Lambda_{K3}\). If \(A_S \simeq A_T\) and \(q_S = -q_T\), then, it is determined that the orthogonal complement \(S^\perp_{\Lambda_{K3}}\) in \(\Lambda_{K3}\) is \(T\). And \(q_S = -q_T\) if and only if \(\text{discr}\ S = -\text{discr}\ T\).

\textbf{Proposition 3.1} The Picard lattice \(\text{Pic}(\Delta^*)\) for \(\Delta\) in Corollary 3.2 is as in Table 8 where singularities in a row are transpose-dual. In each case, one gets

\[\text{discr}\ \text{Pic}(\Delta) = -\text{discr}(U \oplus \text{Pic}(\Delta^*))\], and \(A_{\text{Pic}(\Delta)} \simeq A_{U \oplus \text{Pic}(\Delta^*)}\).
Denote $C^0_S := \begin{pmatrix} -4 & 1 \\ 1 & -2 \end{pmatrix}$.

| Singularity | $\rho(\Delta^*)$ | $\text{Pic}(\Delta^*)$ | Singularity |
|-------------|------------------|------------------------|-------------|
| $Q_{12}$    | 4                | $U \oplus A_2$         | $E_{18}$    |
| $Z_{1,0}$   | 3                | $U \oplus A_1$         | $E_{19}$    |
| $E_{20}$    | 2                | $U$                    | $E_{20}$    |
| $Q_{2,0}$   | 4                | $U \oplus C^0_S$       | $Z_{17}$    |
| $E_{25}$    | 3                | $U \oplus A_1$         | $Z_{19}$    |
| $Q_{18}$    | 4                | $U \oplus A_2$         | $E_{30}$    |

Table 3: Pic($\Delta^*$) for $\Delta$ in Corollary 3.2.

**Proof.**

1. $Q_{12}$ and $E_{18}$ The polytope $\Delta$ is given in subsection 4.7 of [1] and the associated toric 3-fold has 19 toric divisors $D_i$ corresponding to the one-simplices generated by vectors

\[
\begin{align*}
    m_1 &= (1, 2, 2) & m_2 &= (0, 1, 1) & m_3 &= (-8, -11, -9) \\
    m_4 &= (1, -2, 0) & m_5 &= (1, 1, 0) & m_6 &= (-4, -5, -4) \\
    m_7 &= (-7, -10, -8) & m_8 &= (-6, -9, -7) & m_9 &= (-5, -8, -6) \\
    m_{10} &= (-4, -7, -5) & m_{11} &= (-3, -6, -4) & m_{12} &= (-2, -5, -3) \\
    m_{13} &= (-1, -4, -2) & m_{14} &= (0, -3, -1) & m_{15} &= (1, 0, 1) \\
    m_{16} &= (-5, -7, -6) & m_{17} &= (-2, -3, -3) & m_{18} &= (1, -1, 0) \\
    m_{19} &= (1, 0, 0)
\end{align*}
\]

and by solving the linear system (11) : $\sum_{i=1}^{19} (e_j, m_i)D_i = 0$ ($j = 1, 2, 3$), we get linear relations among toric divisors

\[
\begin{align*}
    D_1 & \sim -9D_4 + 3D_5 - D_7 - 2D_8 - 3D_9 - 4D_{10} - 5D_{11} - 6D_{12} - 7D_{13} - 8D_{14} \\
        & \quad - 5D_{15} + D_{16} + 2D_{17} - 5D_{18} - D_{19}, \\
    D_2 & \sim D_3 + 10D_4 - 2D_5 + 2D_7 + 3D_8 + 4D_9 + 5D_{10} + 6D_{11} + 7D_{12} + 8D_{13} \\
        & \quad + 9D_{14} + 5D_{15} - D_{17} + 6D_{18} + 2D_{19}, \\
    D_6 & \sim -2D_3 - 2D_4 + D_5 - 2D_7 - 2D_8 - 2D_9 - 2D_{10} - 2D_{11} - 2D_{12} - 2D_{13} \\
        & \quad - 2D_{14} - D_{15} - D_{16} - D_{18}.
\end{align*}
\]

So the set $\{D_i \mid i \neq 1, 2, 6\}$ of toric divisors is linearly independent. Let $L$ be the lattice generated by the set $\{r_i D_i \mid i \neq 1, 2, 6\}$ of their restrictions to a $\Delta$-regular member. We shall check that $L$ is primitively embedded into the $K3$ lattice to show that $L$ is indeed the Picard lattice of the family $\mathcal{F}_\Delta$. By computer calculation with formulas (5) and (6), the determinant of an intersection matrix
of $L$ is $-3$ since this matrix is given by

$$
\begin{pmatrix}
-2 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}
$$

Since the discriminant of $L$ is $-3$ that is square-free, there exists no overlattice of $L$ ; indeed, if $H \subseteq \Lambda_{K3}$ were an overlattice of $L$, then, by the standard relation $-3 = [H : L]^2 \text{discr } H$, we get $[H : L] = 1$ and $\text{discr } H = -3$ so that $L \cong H$. Hence, $L$ is primitively embedded into the $K3$ lattice. By construction, $L$ is indeed the Picard lattice of the family $F_\Delta$.

The dual polytope $\Delta^*$ associates a toric 3-fold with 7 toric divisors $D'_i$ corresponding to the one-simplices generated by vectors

$$
\begin{align*}
v_1 &= (1, 1, -2) & v_2 &= (1, -2, 1) & v_3 &= (2, -3, 2) & v_4 &= (-1, 0, 1) \\
v_5 &= (-1, 0, 0) & v_6 &= (1, 0, -1) & v_7 &= (1, -1, 0)
\end{align*}
$$

and by solving the linear system $$: \sum_{i=1}^{7} (e_j, m_i)D'_i = 0 \quad (j = 1, 2, 3)$$, we get linear relations among toric divisors

$$D'_1 \sim 2D'_2 + 3D'_3 + D'_7, \quad D'_2 \sim 3D'_2 + 4D'_3 + D'_6 + 2D'_7, \quad D'_5 \sim D'_3.$$ 

Thus the set $\{ D'_i | i \neq 1, 4, 5 \}$ of toric divisors is linearly independent. Let $L'$ be the lattice generated by the set $\{ r_iD'_i | i \neq 1, 4, 5 \}$ of their restrictions to a $\Delta^*$-regular member. By formulas (2) and (3), an intersection matrix associated to $L'$ is given by

$$
\begin{pmatrix}
-2 & 1 & 0 & 1 \\
1 & 0 & 0 & 0 \\
0 & 0 & -2 & 1 \\
1 & 0 & 1 & -2
\end{pmatrix}
$$

that is equivalent to $U \oplus A_2$ by re-taking the generators as

$$\{ r_iD'_2 + r_iD'_3, r_iD'_2, r_iD'_6, r_iD'_3 + r_iD'_7 \}.$$ 

Since the discriminant of $L'$ is $-3$ that is square-free, there exists no overlattice of $L'$ ; indeed, if $H' \subseteq \Lambda_{K3}$ were an overlattice of $L'$, then, by the standard relation $-3 = [H' : L']^2 \text{discr } H'$, we get $[H' : L'] = 1$ and $\text{discr } H' = -3$ so that $H' \cong L'$. Hence, $L'$ is primitively embedded into the $K3$ lattice. By
construction, \( L' \) is indeed the Picard lattice of the family \( F_\Delta \). Therefore \( \text{Pic}(\Delta^*) = L' = (\mathbb{Z}^4, U \oplus A_3) \).

Similarly, the lattice \( U \oplus \text{Pic}(\Delta^*) \) is also primitively embedded into the \( K3 \) lattice since it is of signature \((2, 4)\) and is of discriminant 3. Besides, \( \text{discr}(U \oplus \text{Pic}(\Delta^*)) = 3 = -\text{discr}(\text{Pic}(\Delta)) \), and moreover, \( A_{\text{Pic}(\Delta)} \simeq A_{U \oplus \text{Pic}(\Delta^*)} \simeq \mathbb{Z}/3\mathbb{Z} \).

### 2. \( Z_{1,0} \) and \( E_{19} \)

The polytope \( \Delta \) is given in subsection 4.8 of [7] and the associated toric 3-fold has 20 toric divisors \( D_i \) corresponding to

\[
\begin{align*}
m_1 &= (1, -1, 2) & m_2 &= (0, -1, 1) & m_3 &= (0, 0, 1) \\
m_4 &= (4, 2, -1) & m_5 &= (-6, 2, -11) & m_6 &= (2, 0, 1) \\
m_7 &= (3, 1, 0) & m_8 &= (-2, 0, -3) & m_9 &= (-4, 1, -7) \\
m_{10} &= (2, 1, 0) & m_{11} &= (-3, 1, -5) & m_{12} &= (3, 2, -2) \\
m_{13} &= (2, 2, -3) & m_{14} &= (1, 2, -4) & m_{15} &= (0, 2, -5) \\
m_{16} &= (-1, 2, -6) & m_{17} &= (-2, 2, -7) & m_{18} &= (-3, 2, -8) \\
m_{19} &= (-4, 2, -9) & m_{20} &= (-5, 2, -10) \\
\end{align*}
\]

and \( \{ D_i | i \neq 1, 2, 3 \} \) is linearly independent by system (1). The lattice \( L := (r_{D_i} | i \neq 1, 2, 3)\mathbb{Z} \) has \( \text{rk} L = 17 \), and \( \text{discr} L = 2 \) by an explicit calculation of its intersection matrix using formulas \([5]\) and \([6]\). Since the discriminant of \( L \) is square-free, \( L \) is primitively embedded into the \( K3 \) lattice, and thus \( L = \text{Pic}(\Delta) \).

The dual polytope \( \Delta^* \) associates a toric 3-fold with 6 toric divisors \( D'_i \) corresponding to

\[
\begin{align*}
v_1 &= (1, -3, -1) & v_2 &= (2, 0, -1) & v_3 &= (1, 0, -1) \\
v_4 &= (0, -1, -1) & v_5 &= (-1, 2, 1) & v_6 &= (0, 1, 0) \\
\end{align*}
\]

and \( \{ D'_i | i \neq 1, 2, 5 \} \) is linearly independent by system (1). By re-taking the generators as \( \{ r_{D'_i} + r_{D'_4}, r_{D'_4}, r_{D'_5} - r_{D'_3} \} \), the lattice \( L' := (r_{D'_i} | i \neq 1, 2, 5)\mathbb{Z} \) has an intersection matrix \( U \oplus A_1 \). Since \( \text{rk} L' = 3 \), and \( \text{discr} L' = 2 \), which is square-free, \( L' \) is primitively embedded into the \( K3 \) lattice, and thus \( \text{Pic}(\Delta^*) = L' = (\mathbb{Z}^3, U \oplus A_1) \).

Similarly \( U \oplus \text{Pic}(\Delta^*) \) is primitively embedded into the \( K3 \) lattice. Besides, \( \text{discr}(U \oplus \text{Pic}(\Delta^*)) = -2 = -\text{discr} \text{Pic}(\Delta) \), and moreover, \( A_{\text{Pic}(\Delta)} \simeq A_{U \oplus \text{Pic}(\Delta^*)} \simeq \mathbb{Z}/2\mathbb{Z} \).

### 3. \( E_{20} \) and \( E_{20} \)

The polytope \( \Delta \) is given in subsection 4.9 of [7] and the associated toric 3-fold has 21 toric divisors \( D_i \) corresponding to

\[
\begin{align*}
m_1 &= (-1, -1, 2) & m_2 &= (-1, -1, -1) & m_3 &= (-1, 11, 2) \\
m_4 &= (1, -1, 0) & m_5 &= (-1, -1, 1) & m_6 &= (-1, -1, 0) \\
m_7 &= (-1, 3, 0) & m_8 &= (-1, 7, 1) & m_9 &= (0, -1, 1) \\
m_{10} &= (0, 5, 1) & m_{11} &= (-1, 0, 2) & m_{12} &= (-1, 1, 2) \\
m_{13} &= (-1, 2, 2) & m_{14} &= (-1, 3, 2) & m_{15} &= (-1, 4, 2) \\
m_{16} &= (-1, 5, 2) & m_{17} &= (-1, 6, 2) & m_{18} &= (-1, 7, 2) \\
m_{19} &= (-1, 8, 2) & m_{20} &= (-1, 9, 2) & m_{21} &= (-1, 10, 2) \\
\end{align*}
\]

and \( \{ D_i | i \neq 10, 13, 14 \} \) is linearly independent by system (1). The lattice \( L := (r_{D_i} | i \neq 10, 13, 14)\mathbb{Z} \) has \( \text{rk} L = 18 \), and \( \text{discr} L = -1 \) by an explicit calculation of its intersection matrix using formulas \([5]\) and \([6]\). As it being unimodular, \( L \) is primitively embedded into the \( K3 \) lattice, and thus \( L = \text{Pic}(\Delta) \).
The dual polytope $\Delta^*$ associates a toric 3-fold with 5 toric divisors $D'_i$ corresponding to

$$v_1 = (0, 1, 0) \quad v_2 = (1, 0, 0) \quad v_3 = (-1, 0, -1)$$
$$v_4 = (-2, -1, 4) \quad v_5 = (-1, 0, 2)$$

and $\{D'_i | i \neq 1, 2, 3\}$ is linearly independent by system $\mathbb{I}$. By an intersection matrix computed by formulas $\mathbb{F}$ and $\mathbb{G}$, the lattice $L' := \langle r_s D'_i | i \neq 1, 2, 3 \rangle \mathbb{Z}$ has $\text{discr } L' = -1$, $\text{rk } L' = 2$, and $L'$ is even. By the classification of even unimodular lattices, $L'$ is isometric to $U$, which is primitively embedded into the $K3$ lattice, and thus $\text{Pic}(\Delta^*) = L' = \langle \mathbb{Z}^2, U \rangle$.

Similarly, $U \oplus \text{Pic}(\Delta^*)$ is primitively embedded into the $K3$ lattice. Besides, $\text{discr}(U \oplus \text{Pic}(\Delta^*)) = 1 = -\text{discr } \text{Pic}(\Delta)$, and moreover, $\text{A}_{\text{Pic}(\Delta)} \simeq \text{A}_{U \oplus \text{Pic}(\Delta^*)} \simeq \{0\}$.

4. $Q_{2,0}$ and $Z_{17}$ The polytope $\Delta$ is given in subsection 4.10 of $\mathbb{H}$ and the associated toric 3-fold has 19 toric divisors $D_i$ corresponding to

$$m_1 = (0, 1, 1) \quad m_2 = (1, 2, 2) \quad m_3 = (1, 1, 2)$$
$$m_4 = (0, -1, 0) \quad m_5 = (-6, -7, -9) \quad m_6 = (1, 0, -2)$$
$$m_7 = (1, 0, 1) \quad m_8 = (1, 1, 0) \quad m_9 = (-2, -3, -3)$$
$$m_{10} = (-4, -5, -6) \quad m_{11} = (1, 0, -1) \quad m_{12} = (1, 0, 0)$$
$$m_{13} = (-5, -6, -8) \quad m_{14} = (-4, -5, -7) \quad m_{15} = (-3, -4, -6)$$
$$m_{16} = (-2, -3, -5) \quad m_{17} = (-1, -2, -4) \quad m_{18} = (0, -1, -3)$$
$$m_{19} = (-3, -3, -4)$$

and $\{D_i | i \neq 1, 2, 3\}$ is linearly independent by system $\mathbb{I}$. The lattice $L := \langle r_s D_i | i \neq 1, 2, 3 \rangle \mathbb{Z}$ has $\text{rk } L = 16$, and $\text{discr } L = -7$ by an explicit calculation of its intersection matrix using formulas $\mathbb{F}$ and $\mathbb{G}$. Since the discriminant of $L$ is square-free, $L$ is primitively embedded into the $K3$ lattice, and thus $L = \text{Pic}(\Delta)$.

The dual polytope $\Delta^*$ associates a toric 3-fold with 7 toric divisors $D'_i$ corresponding to

$$v_1 = (1, -2, 1) \quad v_2 = (-1, 1, 0) \quad v_3 = (2, 1, -2) \quad v_4 = (1, 0, -1)$$
$$v_5 = (0, 1, -1) \quad v_6 = (-1, 0, 0) \quad v_7 = (1, -1, 0)$$

and $\{D'_i | i \neq 1, 2, 6\}$ is linearly independent by system $\mathbb{I}$. By re-taking the generators as $\{r_s D'_3, r_s D'_4 + r_s D'_4, 2r_s D'_4 + r_s D'_4 - r_s D'_3, r_s D'_4 - r_s D'_3\}$, the lattice $L' := \langle r_s D'_i | i \neq 1, 2, 6 \rangle$ has an intersection matrix $U \oplus C_6^8$. Since $\text{discr } L' = -7$ is square-free, $L'$ is primitively embedded into the $K3$ lattice, and thus $\text{Pic}(\Delta^*) = L' = \langle \mathbb{Z}^4, U \oplus C_6^8 \rangle$.

Similarly $U \oplus \text{Pic}(\Delta^*)$ is primitively embedded into the $K3$ lattice. Besides, $\text{discr}(U \oplus \text{Pic}(\Delta^*)) = 7 = -\text{discr } \text{Pic}(\Delta)$, and moreover, $\text{A}_{\text{Pic}(\Delta)} \simeq \text{A}_{U \oplus \text{Pic}(\Delta^*)} \simeq \mathbb{Z}/7\mathbb{Z}$.
5. $E_{25}$ and $Z_{19}$ The polytope $\Delta$ is given in subsection 4.11 of [7] and the associated toric 3-fold has 20 toric divisors $D_i$ corresponding to

\[
m_1 = (-1, 2, 0) \quad m_2 = (-1, -1, 9) \quad m_3 = (-1, -1, -1) \\
m_4 = (-1, 2, -1) \quad m_5 = (1, -1, -1) \quad m_6 = (-1, 1, 3) \\
m_7 = (-1, 0, 6) \quad m_8 = (-1, -1, 8) \quad m_9 = (-1, -1, 7) \\
m_{10} = (-1, -1, 6) \quad m_{11} = (-1, -1, 5) \quad m_{12} = (-1, -1, 4) \\
m_{13} = (-1, -1, 3) \quad m_{14} = (-1, -1, 2) \quad m_{15} = (-1, -1, 1) \\
m_{16} = (-1, -1, 0) \quad m_{17} = (-1, 0, -1) \quad m_{18} = (-1, 1, -1) \\
m_{19} = (0, -1, 4) \quad m_{20} = (0, -1, -1)
\]

and $\{D_i \mid i \neq 1, 4, 5\}$ is linearly independent by system [1]. The lattice $L := \langle r, D_i \mid i \neq 1, 4, 5 \rangle\mathbb{Z}$ has $rk L = 17$, and $disr L = 2$ by an explicit calculation of its intersection matrix using formulas [5] and [9]. Since the discriminant of $L$ is square-free, $L$ is primitively embedded into the $K3$ lattice, and thus $L = \text{Pic}(\Delta)$.

The dual polytope $\Delta^*$ associates a toric 3-fold with 6 toric divisors $D'_i$ corresponding to

\[
v_1 = (0, 1, 0) \quad v_2 = (0, 0, 1) \quad v_3 = (-3, -2, 0) \\
v_4 = (-5, -3, -1) \quad v_5 = (1, 0, 0) \quad v_6 = (-1, -1, 0)
\]

and $\{D'_i \mid i \neq 1, 2, 3\}$ is linearly independent by system [1]. By re-taking the generators as $\{r, D'_i, r, D'_5 - 8r, D'_4, r, D'_6 - r, D'_1\}$, the lattice $L' := \langle r, D'_i \mid i \neq 1, 2, 3 \rangle\mathbb{Z}$ has an intersection matrix $U \oplus A_1$. Since $disr L' = 2$ is square-free, $L'$ is primitively embedded into the $K3$ lattice, and thus $\text{Pic}(\Delta^*) = L' = (\mathbb{Z}^3, U \oplus A_1)$.

Similarly, $U \oplus \text{Pic}(\Delta^*)$ is primitively embedded into the $K3$ lattice. Besides, $disr(U \oplus \text{Pic}(\Delta^*)) = -2 = -\text{disr Pic}(\Delta)$, and moreover, $A_{\text{Pic}(\Delta)} \simeq A_{U \oplus \text{Pic}(\Delta^*)} \simeq \mathbb{Z}/2\mathbb{Z}$.

6. $Q_{18}$ and $E_{30}$ The polytope $\Delta$ is given in subsection 4.13 of [7] and the associated toric 3-fold has 19 toric divisors $D_i$ corresponding to

\[
m_1 = (1, -1, -1) \quad m_2 = (-1, -1, -1) \quad m_3 = (-1, -1, 8) \\
m_4 = (1, -1, 0) \quad m_5 = (-1, 2, -1) \quad m_6 = (0, -1, -1) \\
m_7 = (-1, -1, 0) \quad m_8 = (-1, -1, 1) \quad m_9 = (-1, -1, 2) \\
m_{10} = (-1, -1, 3) \quad m_{11} = (-1, -1, 4) \quad m_{12} = (-1, -1, 5) \\
m_{13} = (-1, -1, 6) \quad m_{14} = (-1, -1, 7) \quad m_{15} = (0, -1, 4) \\
m_{16} = (-1, 0, -1) \quad m_{17} = (-1, 1, -1) \quad m_{18} = (-1, 0, 5) \\
m_{19} = (-1, 0, 2)
\]

and $\{D_i \mid i \neq 1, 4, 5\}$ is linearly independent by system [1]. The lattice $L := \langle r, D_i \mid i \neq 1, 4, 5 \rangle\mathbb{Z}$ has $rk L = 16$, and $disr L = -3$ by an explicit calculation of its intersection matrix using formulas [5] and [9]. Since the discriminant of $L$ is square-free, $L$ is primitively embedded into the $K3$ lattice, and thus $L = \text{Pic}(\Delta)$.

The dual polytope $\Delta^*$ associates a toric 3-fold with 7 toric divisors $D'_i$ corresponding to

\[
v_1 = (0, 0, 1) \quad v_2 = (1, 0, 0) \quad v_3 = (-4, -3, -1) \quad v_4 = (-3, -2, 0) \\
v_5 = (0, 1, 0) \quad v_6 = (-2, -1, 0) \quad v_7 = (-1, 0, 0)
\]
and \( \{D' \mid i \neq 1, 2, 5\} \) is linearly independent by system \((1)\). By re-taking the generators as \( \{r_s D'_1, r_s D'_2, r_s D'_6, r_s D'_7, r_s D'_8\} \), the lattice \( L' := \langle \ast D'_1, D'_2, D'_6, D'_7, D'_8 \rangle \) has an intersection matrix \( U \oplus A_2 \). Since \( \text{discr} L' = 3 \) is square-free, \( L' \) is primitively embedded into the \( K3 \) lattice, and thus \( \text{Pic}(\Delta^*) = L' = \langle \mathbb{Z}^4, U \oplus A_2 \rangle \).

Similarly, \( U \oplus \text{Pic}(\Delta^*) \) is primitively embedded into the \( K3 \) lattice. Besides, \( \text{discr}(U \oplus \text{Pic}(\Delta^*)) = 3 = -\text{discr}(\text{Pic}(\Delta)) \), and moreover, \( A_{\text{Pic}(\Delta)} \simeq A_{U \oplus \text{Pic}(\Delta^*)} \simeq \mathbb{Z}/3\mathbb{Z} \). \( \square \)

**Remark 2** The choice of \( \Delta \) is not actually unique. However, for any possible reflexive polytopes \( \Delta \) for transpose-dual pairs outside Table 3, we have a relation \( \rho(\Delta) + \rho(\Delta^*) \neq 20 \).

**Theorem 3.2** The families \( \mathcal{F}_\Delta \) and \( \mathcal{F}_{\Delta^*} \) are lattice mirror symmetric for polytopes \( \Delta \) in Corollary 3, that is, \( \text{Pic}(\Delta) \simeq (U \oplus \text{Pic}(\Delta^*))_{A_{K3}}^\perp \). Explicitly, the lattices \( \text{Pic}(\Delta) \) are given as in the Table 4.

| Singularity | \( \text{Pic}(\Delta) \) | \( \rho(\Delta) \) | \( \rho(\Delta^*) \) | \( \text{Pic}(\Delta^*) \) | Singularity |
|-------------|-----------------|--------------|-----------------|-----------------|-------------|
| \( Q_{12} \) | \( U \oplus E_6 \oplus E_8 \) | 16 | 4 | \( U \oplus A_2 \) | \( E_{18} \) |
| \( Z_{1,0} \) | \( U \oplus E_7 \oplus E_8 \) | 17 | 3 | \( U \oplus A_1 \) | \( E_{19} \) |
| \( E_{20} \) | \( U \oplus E_6^{\oplus 2} \) | 18 | 2 | \( U \) | \( E_{20} \) |
| \( Q_{2,0} \) | \( U \oplus A_6 \oplus E_8 \) | 16 | 4 | \( U \oplus C_8 \) | \( Z_{17} \) |
| \( E_{25} \) | \( U \oplus E_7 \oplus E_8 \) | 17 | 3 | \( U \oplus A_1 \) | \( Z_{19} \) |
| \( Q_{18} \) | \( U \oplus E_6 \oplus E_8 \) | 16 | 4 | \( U \oplus A_2 \) | \( E_{30} \) |

Table 4: Picard lattices for lattice mirror symmetric pairs

**Proof.** For a lattice \( L \), denote by \( A_L \) the discriminant group of \( L \), and \( q_L \) the quadratic form of \( L \). We see in Proposition 3.1 that

\[
A_{\text{Pic}(\Delta)} \simeq A_{U \oplus \text{Pic}(\Delta^*)}, \quad \text{and} \quad q_{\text{Pic}(\Delta)} = -q_{U \oplus \text{Pic}(\Delta^*)}
\]

for each case. Thus by Proposition 1.6.1 in [5], there is an isometry

\[
\text{Pic}(\Delta) \simeq (U \oplus \text{Pic}(\Delta^*))_{A_{K3}}^\perp.
\]

So \( \mathcal{F}_\Delta \) and \( \mathcal{F}_{\Delta^*} \) are lattice mirror symmetric. We shall determine \( \text{Pic}(\Delta) \).

1. \( Q_{12} \) and \( E_{18} \) By Corollary 2.1

\[
\text{Pic}(\Delta) \simeq (U \oplus U \oplus A_2)_{A_{K3}}^\perp \simeq U \oplus (A_2)_E E_8 \oplus E_8 \simeq U \oplus E_6 \oplus E_8.
\]

2. \( Z_{1,0} \) and \( E_{19} \) By Corollary 2.1

\[
\text{Pic}(\Delta) \simeq (U \oplus U \oplus A_1)_{A_{K3}}^\perp \simeq U \oplus (A_1)_E E_8 \oplus E_8 \simeq U \oplus E_7 \oplus E_8.
\]

3. \( E_{20} \) and \( E_{20} \) Since \( A_{K3} \simeq U^{\oplus 3} \oplus E_8^{\oplus 2} \)

\[
\text{Pic}(\Delta) \simeq (U \oplus U)_{A_{K3}}^\perp \simeq U \oplus E_8^{\oplus 2}.
\]

4. \( Q_{2,0} \) and \( Z_{17} \) By Corollary 2.1

\[
\text{Pic}(\Delta) \simeq (U \oplus U \oplus C_8)_{A_{K3}}^\perp \simeq U \oplus (C_8)_E E_8 \oplus E_8 \simeq U \oplus A_6 \oplus E_8.
\]
5. $E_{25}$ and $Z_{19}$ By Corollary 2.1

$$\text{Pic}(\Delta) \simeq (U \oplus U \oplus A_1)_{\mathbb{K}}^1 \simeq U \oplus (A_1)_{E_8} \oplus E_8 \simeq U \oplus E_7 \oplus E_8.$$  

6. $Q_{18}$ and $E_{30}$ By Corollary 2.1

$$\text{Pic}(\Delta) \simeq (U \oplus U \oplus A_1)_{\mathbb{K}}^1 \simeq U \oplus (A_1)_{E_8} \oplus E_8 \simeq U \oplus E_6 \oplus E_8.$$  

Thus the assertions are verified. □

4 Application

Denote by $\Delta_B$ the reflexive polytope obtained in [7] for a singularity of type $B$. As is seen in Table 3, there are isometric Picard lattices $\text{Pic}(\Delta_{Q_{12}}) \simeq \text{Pic}(\Delta_{Q_{18}})$ and $\text{Pic}(\Delta_{Z_{1,0}}) \simeq \text{Pic}(\Delta_{E_{25}})$. Families $F_{\Delta_B}$ and $F_{\Delta_D}$ are said to be essentially the same if general members in these families are birationally equivalent. Not only the Picard lattices are isometric, but also we have

Proposition 4.1 The families $F_{\Delta_{Q_{12}}}$ (resp. $F_{\Delta_{Z_{1,0}}}$) and $F_{\Delta_{Q_{18}}}$ (resp. $F_{\Delta_{E_{25}}}$) are essentially the same.

Proof. It is directly shown that the polytopes $\Delta_{Q_{12}}$ (resp. $\Delta_{Z_{1,0}}$) and $\Delta_{Q_{18}}$ (resp. $\Delta_{E_{25}}$) are isometric. Indeed, define an invertible matrix in $GL_3(\mathbb{Z})$ as

$$M_1 = \begin{pmatrix} -1 & 0 & 1 \\ 1 & 1 & -2 \\ 2 & -3 & 2 \end{pmatrix} \quad \text{resp.} \quad M_2 = \begin{pmatrix} -1 & 2 & 1 \\ 1 & -3 & -1 \\ 2 & 0 & -1 \end{pmatrix}$$

and one obtains an isometry

$$m M_1 = m'(\text{resp.} \quad m M_2 = m')$$

that sends $m \in \Delta_{Q_{18}}$ to $m' \in \Delta_{Q_{12}}$ (resp. $m \in \Delta_{E_{25}}$ to $m' \in \Delta_{Z_{1,0}}$). Therefore, there exists an explicit projective transformation that maps each Laurent polynomial in $|-K_P(\Delta_{Q_{12}})|$ (resp. $|-K_P(\Delta_{Z_{1,0}})|$) to a Laurent polynomial in $|-K_P(\Delta_{Q_{18}})|$ (resp. $|-K_P(\Delta_{E_{25}})|$). This mapping also birationally sends a general member in $F_{\Delta_{Q_{12}}}$ (resp. $F_{\Delta_{Z_{1,0}}}$) to a general member in $F_{\Delta_{Q_{18}}}$ (resp. $F_{\Delta_{E_{25}}}$). Thus the statement is proved. □

We conclude our study to remark that general members in compactifications of non-equivalent singularities can be transformed via a reflexive polytope.

References

[1] Arnol’d, V. I., Critical points of smooth functions and their normal forms, Russian Math. Surveys 30, 1–75 (1975).
[2] Barth, W.P. and Hulek, K. and Peters, C.A.M. and Van de Ven, A., Compact Complex Surfaces, Second ed, Springer (2004).
[3] Dolgachev, I., Mirror symmetry for lattice polarized $K3$ surfaces, J. Math. Sci. 81, 2599–2630 (1996).
[4] Ebeling, W. and Ploog, D., A geometric construction of Coxeter-Dynkin diagrams of bimodal singularities, Manuscripta Math. 140, 195–212 (2013).

[5] Iano-Fletcher, A. R., Working with weighted complete intersections, in Explicit Birational Geometry of 3-folds, Alessio Corti and Miles Reid (eds.) London Mathematical Society Lecture Note Series 281, 101–173 (2000).

[6] Kobayashi, M., Duality of weights, mirror symmetry and Arnold’s strange duality, Tokyo J. Math., 31, 225–251 (2008).

[7] Mase, M. and Ueda, K., A note on bimodal singularities and mirror symmetry, Manuscripta Math.(online 31 August 2014) 146, 153–177 (2015).

[8] Nikulin, V.V., Integral symmetric bilinear forms and some of their applications, Math. USSR-Izv. 14, 103–167 (1980).

[9] Nishiyama, K., The Jacobian fibrations on some K3 surfaces and their Mordell-Weil groups, Japan J. Math. 22, 293–347 (1996).

[10] Oda, T., Torus Embeddings and Applications, Springer, Tata Institute of Fundamental Research Lectures 57 (1978).

[11] Reid, M., Canonical 3-folds, in Journeés de géométrie algébrique d’Angers, edited by A. Beauville, Sijthoff and Noordhoff, Alphen, 273–310 (1980).

[12] Yonemura, T., Hypersurface simple K3 singularities, Tôhoku Math. J. 42, 351–380 (1990).

Makiko Mase

e-mail: mtmase@arion.ocn.ne.jp

Department of Mathematics and Information Sciences, Tokyo Metropolitan University
192-0397 1-1 Minami Osawa, Hachioji-shi, Tokyo, Japan.

Osaka City University Advanced Mathematical Institute
558-8585 3-3-138 Sugimoto-cho, Sumiyoshi-ku, Osaka, Japan.