Exact gap statistics for the random average process on a ring with a tracer

J Cividini, A Kundu, Satya N Majumdar and D Mukamel

1 Department of Physics of Complex Systems, Weizmann Institute of Science, Rehovot 76100, Israel
2 International center for theoretical sciences, TIFR, Bangalore—560012, India
3 LPTMS, CNRS, University Paris-Sud, Université Paris-Saclay, F-91405 Orsay, France

E-mail: anupam.kundu@icts.res.in

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Abstract
We study the statistics of the gaps in the random average process on a ring with particles hopping symmetrically, except one tracer particle which could be driven. These particles hop either to the left or to the right by a random fraction $\eta$ of the space available till next particle in the respective directions. The random fraction $\eta \in [0, 1)$ is chosen from a distribution $R(\eta)$. For a non-driven tracer, when $R(\eta)$ satisfies a necessary and sufficient condition, the stationary joint distribution of the gaps between successive particles takes a universal form that is factorized except for a global constraint. Some interesting explicit forms of $R(\eta)$ are found which satisfy this condition. In the case of a driven tracer, the system reaches a current-carrying steady state where such factorization does not hold. Analytical progress has been made in the thermodynamic limit, where we computed the single site distribution inside the bulk. We have also computed the two point gap–gap correlation exactly in that limit. Numerical simulations support our analytical results.

Keywords: stochastic particle dynamics, exact results, correlation functions

(Some figures may appear in colour only in the online journal)

1. Introduction

Interacting particle problems have been widely studied in statistical mechanics as examples of many-body systems that usually exhibit rich types of behavior [1]. For equilibrium systems
all the stationary properties are in principle known through the Gibbs–Boltzmann distribution, and the dynamics close to equilibrium can be described in very general terms [2]. Out of equilibrium, studying such interacting many-particle systems analytically in dimensions other than one is usually difficult. In one dimension, one of the most studied examples is the simple exclusion process (SEP), where each lattice site is occupied by one hardcore particle or it is empty. In every small time interval $dt$, each particle moves to the neighboring site on the right (left) with probability $pdt$ ($qdt$) iff the target site is empty. A large number of results are known for this system (see [3, 4] and references therein) and these are obtained using sophisticated analytical tools such as Bethe ansatz, matrix product ansatz or mappings to growth models [4, 5]. Another widely studied model of interacting particle systems is the random average process (RAP). This process was first introduced by Ferrari and Fontes [6]. In the RAP particles move on a one-dimensional continuous line in contrast to SEP where hardcore particles move on a lattice. Each particle moves to the right (left) by a random fraction of the space available till the nearest particle on the right (left) with some rate. Thus the jumps in one direction are a random fraction $\eta$ of the gap to the nearest particle in that direction where the random number $\eta \in [0, 1)$ is chosen from some distribution $R(\eta)$.

Many different results have been reported for this model in the literature. For example, dynamical properties like the diffusion coefficient, the variance of the tracer position and the pair correlation between the positions of two particles have been studied on an infinite line [7–9]. In particular, it has been shown that the variance of the tagged particle position in the steady state grows at late times as $\sim t$ for the asymmetric case (different jumping rates to the right or to the left), whereas for the symmetric case it grows as $\sim \sqrt{t}$. For the fully asymmetric case, this result was first derived by Krug and Garcia [7] using a heuristic hydrodynamic description. Later, the variance as well as other correlation functions, both for the symmetric and asymmetric cases, were computed rigorously by Rajesh and Majumdar [9]. The RAP model has also been shown to be linked to the porous medium equation [10] and appears in a variety of problems like the force propagation in granular media [11, 12], in models of mass transport [7, 12], models of voting systems [6, 13], models of wealth distribution [14] and in the generalized Hammersley process [15]. The RAP can be shown to be equivalent, up to an overall translation, to mass transfer (MT) models [7] when one identifies the gaps or the inter-particle spacings in the RAP picture with the masses. In terms of the MT picture, the statistics of masses/gaps has been well studied. For uniform distributions $R(\eta)$, invariant measures of the masses have been computed for a general class of RAP (or equivalently MT) models [12] and for a totally asymmetric version of the model [7] defined on an infinite line with different dynamics. For a parallel updating scheme a mean-field calculation has been shown to be exact for certain parameters [16] and a matrix product ansatz has been developed [17]. Finally, it has been shown that a condensation transition occurs in the related MT model if one imposes a cutoff on the transferred mass (i.e. on the amount of jump made by the particles in RAP picture) [18].

Recently there has been considerable interest in studying the motion of a special driven tagged particle in the presence of other non-driven interacting particles. Such a special particle is called the tracer particle. In experimental studies, driven tracers in quiescent media have been used to probe rheological properties of complex media such as DNA [19], polymers [20], granular media [21, 22] or colloidal crystals [23]. Some practical examples of biased tracers are a charged impurity being driven by applied electric field or a colloidal particle being pulled by optical tweezers in the presence of other colloid particles performing random motion. On the theoretical side, problems with driven tracers have been studied in the context of the SEP where both particle number conservation and non-conservation (absorption/desorption) have been considered [24–27]. In the presence of a driven tracer on an infinite
line, the particle density profile is inhomogeneous around the tracer and in the absence of absorption or desorption, the current flowing across the system vanishes in the large time limit, as does the velocity of the tracer \([24-26]\). Other quantities like the mean and the variance of the position of the tracer have also been studied \([28]\). In this contribution we derive new results for RAP concerning the statistics of the gaps between particles on a ring in the presence of a tracer particle which may be driven.

More precisely we consider \(N\) particles moving according to RAP on a ring of size \(L\). In a small time duration \(dt\), any particle, except the tracer particle, jumps to the right or left by a random fraction of the space available up to the neighboring particle on the right or left, respectively, with equal probability \(dt/2\). The tracer particle moves to the right or left by a random fraction of the space available in the respective direction with probability \(pd\) and \(qd\), respectively, see figure 1. In this paper we study the statistics of the gaps between neighboring particles in the stationary state (SS) for the following two cases: (i) \(p = q\) (ii) \(p \neq q\). Note that the dynamics of the gaps does not satisfy detailed balance even in the \(p = q\) case, always out of equilibrium. Using a mapping from the RAP model of particles to an equivalent (except for a global translation) MT model, we find various interesting exact results related to gap statistics. A summary of these results is given below:

- When \(p = q = 1/2\) i.e. the tracer particle is not driven and moving identically as other particles. In this case, for a large class of jump distributions \(R(\eta)\) that satisfy some necessary and sufficient condition, we find that the stationary joint distribution of the gaps \(\{g_i, i = 1, 2, \ldots, N\}\) takes the following universal form:

\[
P_{N,L}(g_1, g_2, \ldots, g_N) = \frac{1}{Z_{N,L}(\beta)} \prod_{i=1}^{N} g_i^{\beta-1} \delta \left( \sum_{i=1}^{N} g_i - L \right),
\]

where \(Z_{N,L}(\beta)\) is the normalization constant and the delta function represents the global constraint due to total mass conservation in the MT picture. The parameter \(\beta > 0\) is given by \(\beta = \frac{\mu_k}{\mu_k} - 1\), where \(\mu_k = \int_0^L \eta^k R(\eta)d\eta\) is the \(k\)th moment of the jump distribution \(R(\eta)\). Interestingly, this factorized form for the joint probability distribution function (JPDM) in (1) still holds even when the hopping rates of the tracer particle are \(p = q \neq 1/2\), i.e. the tracer particle is still moving symmetrically but differently from the other particles. Similar factorized joint distributions with power law weight functions...
have also been obtained in the context of the $q$-model of force fluctuations in granular media [11] as well as in a totally asymmetric version of the RAP on an infinite line with parallel dynamics [16].

For arbitrary jump distribution $R(y)$ we find that the average mass profile is given by
\[ \langle g_i \rangle = \omega_0 = L/N \] and the two point gap–gap (mass–mass) correlation
\[ d_{i,j} = \left\langle g_i g_j \right\rangle - \left\langle g_i \right\rangle \left\langle g_j \right\rangle \]
is expressed in terms of $\mu_1$ and $\mu_2$ as
\[ d_{i,j} = \left( \frac{\mu_2 \omega_0^2}{\mu_1 + (N-1)(\mu_1 - \mu_2)} \right) [N \delta_{i,j} - 1]. \] (2)

- In the second case when $p \neq q$ i.e. the tracer particle is driven, the factorization form of the joint distribution of the gaps does not hold. In this case the SS has a global current associated with the non-zero mean velocity of the tracer, which supports an inhomogeneous mean mass profile in the MT picture. We compute this average mass
\[ m_i = \left\langle g_i \right\rangle \] and it is given by
\[ m_i = \frac{\omega_0}{(p + q)(N-1) + 1} [2(p - q)i + (2qN - 2p + 1)]. \] (3)

We also compute the pair correlation
\[ d_{i,j} = \left\langle g_i g_j \right\rangle - \left\langle g_i \right\rangle \left\langle g_j \right\rangle \]
in the steady state. In the thermodynamic limit i.e. for both, $L \to \infty$ and $N \to \infty$ limit, while keeping the mean gap density $\omega_0 = L/N$ fixed, we find that the average mass $m_i$, variance $d_{i,i}$ and the correlation $d_{i,j}$ scale as
\[ m_i = \omega_0 \mathcal{M} \left( \frac{i}{N} \right) + o(1/N), \] (4)
\[ d_{i,j} = \omega_0^2 \frac{1}{N} \mathcal{D} \left( \frac{i}{N}, \frac{j}{N} \right) + o(1/N), \quad i \neq j, \] (5)
\[ d_{i,i} = \omega_0^2 \left[ \mathcal{C}_0 \left( \frac{i}{N} \right) + \frac{1}{N} \mathcal{C}_1 \left( \frac{i}{N} \right) + \frac{1}{N} \mathcal{D} \left( \frac{i}{N}, \frac{i}{N} \right) + o(1/N) \right], \] (6)

where $o(\ell)$ represents order smaller that $\ell$. We find explicit expressions of the scaling functions $\mathcal{M}(x)$, $\mathcal{C}_0(x)$ and $\mathcal{C}_1(x)$ in (40), (43) and (47), respectively. The explicit expression of $\mathcal{D}(x, y)$ is given in (52) for $q = 0$ whereas the expression for general $q$ can be obtained following the analysis given in appendix C.

The paper is organized as follows. In section 2 the model and basic equations are written. We then study the RAP without driven tracer in section 3, where we focus on factorized stationary distributions with a global constraint and on the jump distributions for which they occur. In section 4 we study the influence of a driven tracer. For this case, we obtain the single site gap distribution, mean gap and two-point correlations of the gaps in the thermodynamic limit. Finally, section 5 concludes the paper. Some details are given in appendices A–C.

2. Model and basic equations

We consider $N$ particles moving on a ring of size $L$ (see figure 1) which are labeled as $i = 1, ..., N$. Without any loss of generality, we consider the first particle as the tracer particle which may be driven. The positions of the particles at time $t$ are denoted by $x_i(t)$ for $i = 1, ..., N$. The dynamics of the particles are given as follows. In an infinitesimal time interval $t$ to $t + dt$, any
particle (say $i$th) other than the tracer, jumps from $x_i(t)$, either to the right or to the left with probability $dr/2$ and with probability $(1 - dr)$ it stays at $x_i(t)$. The tracer particle jumps from $x_i(t)$ to the right with probability $pdt$, to the left with probability $qdt$ and does not jump with probability $(1 - (p + q)dt)$. The jump length, either to the right or to the left, made by any particle is a random fraction of the space available between the particle and its neighboring particle to the right or to the left, respectively. For example, the $i$th particle jumps by an amount $\eta_i^r[x_i(t) - x_{i-1}(t)]$ to the right and by an amount $\eta_i^l[x_{i+1}(t) - x_i(t)]$ to the left. The random variables $\eta_i^r, \eta_i^l$ are independently chosen from the interval $[0, 1)$ and each is distributed according to the same distribution $R(\eta)$. For most of our calculations in this paper we consider arbitrary $R(\eta)$ unless otherwise specified. Following the dynamics of the particles for the homogeneous case from [9], one can write down the stochastic evolution equations for the locations of the particles as:

\[
x_i(t + dt) = \begin{cases} 
  x_i(t) + \eta_i^r(x_{i+1}(t) - x_i(t)), & \text{with Prob. } R(\eta_i^r)d\eta_i^r \frac{dr}{2}, \\
  x_i(t) + \eta_i^l(x_{i-1}(t) - x_i(t)), & \text{with Prob. } R(\eta_i^l)d\eta_i^l \frac{dr}{2}, \\
  x_i(t), & \text{with Prob. } 1 - dr,
\end{cases}
\]

and

\[
x_i(t + dt) = \begin{cases} 
  x_i(t) + \eta_i^r(x_{i+1}(t) - x_i(t)), & \text{with Prob. } R(\eta_i^r)d\eta_i^r pdt, \\
  x_i(t) + \eta_i^l(x_{i-1}(t) - x_i(t)), & \text{with Prob. } R(\eta_i^l)d\eta_i^l qdt, \\
  x_i(t), & \text{with Prob. } 1 - (p + q)dt,
\end{cases}
\]

where we introduced two auxiliary particles with positions $x_{N+1}(t) \equiv x_1(t) + L$ and $x_0(t) \equiv x_N(t) - L$ for all $t$ to impose periodicity in the problem. Note that the dynamics is invariant under simultaneous dilation of $L$ and of all the positions. In this paper, the results are however stated for general $L$.

Since we are interested in the statistics of the gaps $g_i = x_{i+1} - x_i$, $i = 1, 2, \ldots, N$, it is convenient to work with an equivalent and appropriate MT model with $N$ sites. The MT model is defined as follows. Corresponding to the $N$ particles in the RAP, we consider a periodic one-dimensional lattice of $N$ sites with a mass at each site. Particles from the RAP picture are mapped to the links between lattice sites in the MT picture. For example, the $i$th particle in RAP corresponds to the link between sites $(i - 1)$ and $i$ in MT model whereas the mass at site $i$ is equal to the gap $g_i = x_{i+1} - x_i$ between the $i$th and $(i + 1)$th particle in RAP.

As a result for each configuration $X(t) = [x_1(t), x_2(t), \ldots, x_N(t)]$ of the positions of the particles in the RAP we have a unique mass configuration $G(t) = [g_1(t), g_2(t), \ldots, g_N(t)]$ in the MT model where $g_i = x_{i+1} - x_i$, $i = 1, 2, \ldots, N$. The opposite is however not true, as the instantaneous configuration of the masses leaves freedom for a global translation of the system in the RAP. Such a mapping from particle model to mass model has been considered in other contexts like from the exclusion process to the zero range process [29].

Concerning the dynamics, a hop of the $i$th particle by $\eta(x_{i+1} - x_i)$ towards the $(i + 1)$th particle in the RAP corresponds to a transfer of mass $\eta g_i \equiv \eta(x_{i+1} - x_i)$ from $i$th site to the $(i - 1)$th site in the MT model whereas a hop of the $i$th particle by $\eta(x_i - x_{i-1})$ towards the $(i - 1)$th particle in the RAP corresponds to a transfer of mass $\eta g_{i-1} \equiv \eta(x_i - x_{i-1})$ from $(i - 1)$th site to the $i$th site. More precisely, the updating rules for the configurations in the MT model are given by:

\[
g_i(t + dt) = g_i(t) + \sigma_{r,i}^{i+1} \eta_{i+1} g_{i+1}(t) + \sigma_{l,i}^{i-1} \eta_{i-1} g_{i-1}(t) - (\sigma_{r,i}^{i} + \sigma_{l,i}^{i}) \eta_i g_i(t),
\]
where the $\eta$ variables are independent and identically distributed according to $R(\eta)$ and $\sigma_{r,l}$ are 1 with probability $\frac{df}{2}$ and 0 otherwise except for $\sigma^N_r$ and $\sigma^1_r$. The random variable $\sigma^N_r$ is 1 with probability $qdt$ and 0 with probability $1 - qdt$. Similarly, $\sigma^1_r$ is 1 with probability $pdt$ and 0 with probability $1 - pdt$. The periodicity is imposed by $g_{N+1}(t) = g_1(t)$ and $g_0(t) = g_N(t)$, for all $t$.

**Steady state master equation:** The dynamics described in (9) will take the system eventually to an SS. If $P_{N,L}(G)$ represents the steady state joint probability distribution of the configuration $G = (g_1, g_2, \ldots, g_N)$, then it satisfies the following master equation:

$$(N + p + q - 1)P_{N,L}(G) = \sum_{i=1}^{N} \int_0^\infty \int_0^\infty \int_0^{N+1} \int_0^1 d\eta R(\eta) P_{N,L}(G'_{i+1})$$

$$\times \left[(1/2 + (q - 1/2)\delta_{i,N}) \delta(g_i - g'_i + \eta g'_i) \delta(g_{i+1} - g'_{i+1} - \eta g'_{i+1})
+ (1/2 + (p - 1/2)\delta_{i,N}) \delta(g_i - g'_i - \eta g'_i) \delta(g_{i+1} - g'_{i+1} + \eta g'_{i+1})\right],$$

(10)

where we used the shorthand notation $G'_{i+1} = (g_1, \ldots, g_{i-1}, g'_i, g'_{i+1}, g_{i+2}, \ldots, g_N)$. The conservation of the number of particles follows naturally from the fact that the lattice size is fixed. Moreover, the dynamics clearly conserves the total mass i.e. we have $\sum_{i=1}^{N} g_i(t) = L$, for all times. The steady state distribution $P_{N,L}(G)$ is never an equilibrium distribution even when $p = q$.

This is because the Kolmogorov criterion is not satisfied, no matter what the distribution $R(\eta)$ is or the hopping rates $p$ and $q$ are. To find whether a system satisfies detailed balance or not is not always easy without knowing a priori the steady state itself. However, there exists a beautiful theorem due to Kolmogorov [30], which provides a way to check the validity of the detailed balance condition without knowing the steady state. It states that if one considers a loop of configurations in the configuration space, then the product of the clockwise rates must be equal to the product of the anticlockwise rates, for all possible loops in the configuration space. This condition is particularly useful when the detailed balance actually does not hold: if one can find just one loop (could be any) for which the clockwise and the anticlockwise products are not equal, detailed balance is guaranteed to fail. In our case, it is easy to find such a loop. For example, let us consider a two-site system in the MT model with uniform $R(\eta) = 1$ and $p = q = \frac{1}{2}$. In this case, the rate of the transition $(g, L - g) \rightarrow (g', L - g')$ is $\frac{1}{2}$ if $g > g'$ and $\frac{1}{L - g}$ if $g < g'$. For a cycle $(g, L - g) \rightarrow (g + a, L - g - a) \rightarrow (g + a + b, L - g - a - b) \rightarrow (g, L - g)$ with $a, b > 0$, the product of the clockwise (forward) rates is $1/(L - g)$ and $1/((L - g)(g + a + b))$, while that of the anticlockwise (backward) rates is $1/(L - g)(g + a + b)$ and $1/((L - g)(g + a))$. For then the two products to be equal, we need $g + a = L - g - a$ which is clearly not possible.

### 3. Unbiased tracer case: $p = q$

In this case the tracer particle is not driven and all the particles are moving symmetrically. In the large time limit, the inter-particle spacings or gaps will reach an SS. Understanding completely the statistics of the gaps in SS would require us to find the JPDF of all the gaps $g_i; i = 1, 2, \ldots, N$. Clearly, this JPDF could be different for different choices of jump distributions $R(\eta)$. Finding this JPDF for arbitrary $R(\eta)$ is generally a hard task. We ask a relatively simpler question: for what choices of the jump distribution does the stationary JPDF have a factorized form.
except for a global mass conservation condition represented by the delta function? Here 

\[ Z_{NL} = \int_{G \in [0,\infty]^N} W_{NL}(G) dG \]

ensures normalization. Analogous questions have also been asked in other contexts, for example in mass transport models \cite{7,12,16}, in zero range processes \cite{31,32} and in finite range processes \cite{33}. In the following subsection we will see that there exists a large class of \( R(\eta) \) for which the above form for JPDF is true. Moreover we will see that the weights \( w_i(g) \) corresponding to individual sites are \( i \) independent and the weight functions have universal form.

\subsection*{3.1. Determination of the weight functions \( w_i(g) \)}

To obtain the weight functions \( w_i(g) \), we first insert \( P_{NL}(G) \) from (11) in the SS master equation (10) and then integrate over all \( g_j \) except \( j = i \). We get

\[ 2 \left[ 1 + (p - 1/2)(\delta_{i1} + \delta_{iN}) \right] W^{(1)}_{NL}(g_i) \]

\[ = \int_0^\infty d\eta_i \int_0^\infty d\eta' \int_0^1 d\eta \, R(\eta) \left[ 1/2 + (p - 1/2)\delta_{i1} \right] \times [\delta(g_i - g_i' + \eta g') + \delta(g_i - g_i' - \eta g_{i-1})] W^{(2)}_{NL}(g_{i-1}, g_i') \]

\[ + \int_0^\infty d\eta_i \int_0^\infty d\eta' \int_0^1 d\eta \, R(\eta) \left[ 1/2 + (p - 1/2)\delta_{iN} \right] \times [\delta(g_i - g_i' + \eta g') + \delta(g_i - g_i' - \eta g_{i+1})] W^{(2)}_{NL}(g_i', g_{i+1}), \] (12)

where

\[ W^{(1)}_{NL}(g_i) = \left( \prod_{j=i}^N \int_0^\infty d\eta \right) W_{NL}(G), \quad \text{and} \quad W^{(2)}_{NL}(g_{i-1}, g_i') = \left( \prod_{j=i}^N \int_0^\infty d\eta \right) W_{NL}(G). \] (13)

We define the Laplace transform of any function \( f(x) \) as \( \tilde{f}(s) = \int_0^\infty f(x)e^{-sx} \, dx \) where \( s \) is the Laplace conjugate of \( x \). Taking the Laplace transform over \( L \) as well as over \( g_i \) on both sides of (12) we get

\[ 2 \frac{\tilde{w}_i(s + s')}{\tilde{w}_i(s)} \]

\[ = \int_0^1 d\eta \, R(\eta) \frac{\tilde{w}_i(s + (1 - \eta)s')}{\tilde{w}_i(s)} \]

\[ + \frac{\tilde{w}_i(s + s')}{\tilde{w}_i(s)} \int_0^1 d\eta \, R(\eta) \left[ \frac{1/2 + (p - 1/2)\delta_{i1}}{1 + (p - 1/2)(\delta_{i1} + \delta_{iN})} \tilde{w}_i(s + \eta s') \right] \]

\[ + \frac{1/2 + (p - 1/2)\delta_{iN}}{1 + (p - 1/2)(\delta_{i1} + \delta_{iN})} \tilde{w}_{i+1}(s + \eta s'), \] (14)

where \( s \) is the Laplace conjugate of \( L \) and \( s' \) is the Laplace conjugate of \( g_i \). While deriving the above equation we have assumed that the Laplace transform of \( w_i(g) \), defined by \( \tilde{w}_i(s) = \int_0^\infty d\eta \, e^{-\eta w_i(g)} \), exists. Equation (14) provides the condition satisfied by \( \tilde{w}_i(s) \), in order to get a factorized JPDF as in (11). To find the solution for \( \tilde{w}_i(s) \), let us expand both sides of (14) in powers of \( s' \) and equate coefficients of each power on both sides. One can easily see that at order \( s'^0 \), condition (14) is automatically satisfied because of the normalization \( \int_0^1 d\eta R(\eta) = 1 \). At order \( s'^1 \), we get
The general solution of the above equation is 
\[ \frac{\hat{w}(s)}{w(s)} = \frac{A(s) + B(s)}{s} \]
where, \( A(s) \) and \( B(s) \) are \( s \) dependent constants. From the boundary conditions at \( i = 1 \) and \( i = N \), we determine 
\( A(s) = 0 \). Hence we find that \( \hat{w}(s) \) is independent of site index \( i \). To proceed further, we now look at the expansion of \(14\) at order \( s^2 \) and get
\[ (\mu_2 - \mu_1) \frac{\hat{w}''(s)}{\hat{w}(s)} + \mu_1 \left( \frac{\hat{w}'(s)}{\hat{w}(s)} \right)^2 = 0. \] (16)

The above equation can be easily solved to get
\[ \hat{w}(s) = A_0 (s + B_0)^{-\beta}, \]
where, \( \beta = \frac{\mu_1 - \mu_2}{\mu_2} \), (17)
and \( A_0, B_0 \) are constants. Taking inverse Laplace transform we get 
\[ w(g) = A_0 \Gamma[1 - \beta]^{-1} e^{-B_0 g} \]
where \( \Gamma[x] \) is the Gamma function. Inserting this form of \( w(g) \) in \(11\) and absorbing \( A_0, B_0 \) and the gamma function in the normalization constant \( Z_{N,L} \) we arrive at the result
\[ P_{N,L}(g_1, g_2, \ldots, g_N) = \frac{1}{Z_{N,L}^{\beta}} \prod_{i=1}^{N} g_i^{\beta - 1} \delta \left( \sum_{i=1}^{N} g_i - L \right), \] (18)
as stated in \((1)\). The normalization constant \( Z_{N,L} \) can be computed to give
\[ Z_{N,L} = Z_{N,L}^{\beta} = \frac{L^{\beta N - 1} \Gamma[\beta]}{\Gamma[\beta N]} . \] (19)

A striking property of the SS \((18)\) is that it does not depend on the value of \( p \). This can be understood by the fact that even if the SS \((18)\) is not strictly an equilibrium state, compensations of the currents still occur independently for each pair of neighboring sites. Mathematically, we see that if \( \hat{w}_i(s) \) is taken independent of \( i \), equation \((14)\) becomes independent of \( p \) and equivalent to the condition we would obtain by imposing pairwise cancellations between nearest neighbors. This independence of the SS on the microscopic rates is similar in spirit to the equilibrium situation, where the stationary invariant distribution remains the same independently of the individual rates as long as detailed balance is maintained.

3.2. Jumping distributions that yield \((18)\)

In the previous section we have seen that if there exists a jump distribution \( R(\eta) \) for which JPDF \( P_{N,L}(G) \) is in the form \((11)\) then, \( w_i(g_i) \) should always be equal to \( g_i^{\beta - 1} \) for all \( i \) where \( \beta = \frac{\mu_1 - \mu_2}{\mu_2} \). Now we assume \((18)\) to be true and find what conditions \( R(\eta) \) should satisfy. To get that condition we start with \((12)\) which implies \((14)\). Using now
\[ \int_0^\infty dg e^{-g} g^{\beta - 1} = \Gamma[\beta] s^{-\beta} \]
in \((14)\) we find
\[ \frac{2}{(1 + u)^\beta} = \int_0^1 d\eta \frac{R(\eta)}{(1 + (1 - \eta)u)^\beta} + \frac{1}{(1 + u)^\beta} \int_0^1 d\eta \frac{R(\eta)}{(1 + \eta u)^\beta}, \] (20)
for all \( u = (s'/s) \geq 0 \). This is a necessary condition that \( R(\eta) \) should satisfy to get a factorized JPDF in SS. For distributions \( R(\eta) \) satisfying \((20)\) such that the system is ergodic, we expect the SS to be unique. Consequently the condition \((20)\) is sufficient to ensure that the steady state joint gap distribution is given by \((18)\). Analogous conditions satisfied by hopping
rates, have been obtained in other mass transport models [31, 34] and in finite range process [33].

We next find some solutions of (20) for particular values of $\beta$, as well as a family of solutions that covers the whole $\beta$ range.

- Let us first consider the $\beta = 0$ case. Formally, $\beta = 0$ corresponds to $\mu_1 = \mu_2$, i.e. $R(\eta) = \delta(\eta - 1)$. For this choice of $R(\eta)$ system breaks ergodicity and the SS is a trivial one with all the particles at a single point. As a result, $P_{N,L}(G) = (1/N) \sum_{j=1}^{N} \prod_{s \in j} \delta(g) \delta(\sum_{s \in j} g_s - L)$.

- A very special case is $\beta = 1$, for which the stationary distribution (18) is uniform over the allowed configurations. For this case some simple examples of the solutions are $R(\eta) = \delta(\eta - 1/2)$, $R(\eta) = \frac{\Gamma[1 + 2\alpha]}{\Gamma[\alpha] \Gamma[\alpha + 1]} \eta^{\alpha - 1}(1 - \eta)^\alpha$; $\alpha \geq 0$ which can be easily verified by directly inserting them in (20) with $\beta = 1$. Interestingly, one can find all the possible solutions of (20) in this particular case. Defining $v = \frac{\mu_1^2}{1 + \mu_2}$ and $R(\eta) = (1 - \eta)f(\eta)$, elementary manipulations show that for $\beta = 1$, equation (20) is equivalent to

$$
\int_0^1 \mathrm{d} \eta \frac{(1 - \eta) \eta^2 (2\eta - 1)}{1 + v\eta (1 - \eta)} [f(\eta) - f(1 - \eta)] = 0, \quad \forall \ v > 0. \quad (21)
$$

The factor multiplying $f(\eta) - f(1 - \eta)$ on the rhs is always positive, which implies that $f$ is symmetric with respect to $1/2$. Hence all possible solutions are of the form $R(\eta) = (1 - \eta)f(\eta)$ with $f(\eta) = f(1 - \eta)$ for all $\eta \in (0, 1)$. The converse is shown to be true iff the integral of $f$ is equal to 2. Therefore, the set of solutions for $\beta = 1$ are

$$
R(\eta) = (1 - \eta)f(\eta), \quad \text{with}, \quad f(\eta) = f(1 - \eta), \quad \text{and} \quad \int_0^1 f(\eta) \mathrm{d} \eta = 2. \quad (22)
$$

- General $\beta$: One can find a simple family of solutions that spans the whole range of $\beta > 0$ values

$$
R(\eta) = 2\beta (1 - \eta)^{2\beta - 1}, \quad \beta > 0. \quad (23)
$$

In particular, taking $\beta = \frac{1}{2}$ gives the uniform distribution.

- The situation is quite interesting for $\beta \to \infty$. One can check that the following scaling form $R(\eta) = \beta e^{-\psi(\beta \eta)}$ solves (20) for $\beta \to \infty$, where $\psi(y)$ is a real, positive function for $y \geq 0$ and its Laplace transform $\hat{\psi}$ satisfies

$$
\int_0^\infty \mathrm{d} y \psi(y) \left( e^{-y(1+u)} + e^{-y u} \right) = \frac{\hat{\psi}(1 + u) + \hat{\psi} \left( \frac{1}{1 + u} \right)}{1 + u} = 2, \quad \text{for} \quad u \geq 0. \quad (24)
$$

Note that $\hat{\psi}(1) = 1$. There exists an infinite number of solutions of the above equation. For example, one class of solutions can be chosen as $\hat{\psi}(s) = \frac{H_s(\psi)}{H_{m+1}(\psi)}$ where, both $H_s(\psi)$ and $H_{m+1}(\psi)$, are polynomials of same order, say $m$. There are $2(m + 1)$ constants corresponding to $m$ coefficients of different powers of $s$ associated to the two polynomials among which only $m + 1$ are independent, as they are linked by (24). Now these $m + 1$ coefficients have to be chosen such that $\hat{\psi}(1) = 1$ and $\psi(y)$ is real and positive for $y \geq 0$. The simplest of them is $\hat{\psi}(s) = \frac{s}{s + 1}$ which gives the solution

$$
\psi(y) = \frac{y}{y + 1}. \quad (25)
$$


\[ R(\eta) = 2\beta e^{-2\beta\eta} \quad \text{for} \quad \beta \to \infty. \] (25)

Some other examples of solutions in \( \hat{\phi}(s) = \frac{h(\alpha)}{k(\alpha)} \) form are given in appendix A.

The above analysis suggests that there are possibly infinitely many solutions of (20) for any \( \beta > 0 \). Although we found all the solutions for \( \beta = 0 \) and \( \beta = 1 \), and a somewhat simpler characterization of them for large \( \beta \), a proper characterization of all the possible solutions for arbitrary \( \beta \) seems difficult. We are however able to make a few general remarks about the properties of the solutions of (20).

- If \( R_1(\eta) \) and \( R_2(\eta) \) both are solutions of (20) corresponding to the same \( \beta \), then any normalized linear combination of them, i.e. \( \tilde{R}(\eta) = rR_1(\eta) + (1-r)R_2(\eta) \) for \( 0 \leq r \leq 1 \), is also a solution.

- Secondly, the system is invariant by dilation of time. Replacing \( \eta \) by \( \frac{\eta}{\eta} \) is equivalent to replacing \( R(\eta) \) by \( \tilde{R}(\eta) \) for \( 0 \leq r \leq 1 \). Indeed, with this of \( R(\eta) \), a chosen particle hops with probability \( 1 - r \) only, and nothing happens with probability \( r \).

Clearly, the values of \( \mu_1 \) and \( \mu_2 \) depend on the value of \( r \) however \( r \) should not appear in the stationary distribution. The stationary distribution therefore cannot depend on \( \mu_1 \) and \( \mu_2 \) independently, but only through a particular combination in which \( r \) cancels, i.e. \( \beta = \frac{\mu_1 - \mu_2}{\mu_2} \). All the solutions given above are understood up to a dilation of time.

Equation (20), necessary for getting the SS (18), is quite robust, as the same equation can be obtained in the case where all the particles are identical but asymmetrically moving. In a much more general case where each particle has its own hopping rates \( p_i + r \) and \( p_i - r \) to the right and to the left, respectively, it can be shown that one still obtains (20) as the condition for getting the factorized SS (18). But when all the particles have arbitrary jumping rates, there is possibly no jump distribution \( R(\eta) \) available for which such factorized states exist and this most general case has not been studied in the literature.

### 3.3. Single mass distribution

In this subsection we compute the marginal mass distribution and compare it to numerically obtained distributions for different jump distributions \( R(\eta) \). The marginal distribution \( P_{N,L}(g) \) can be readily computed form \( P_{N,L}(g) \) by integrating out all \( g_i \) except \( g_i \). For factorized SSs we get from (18) that

\[
P_{N,L}(g) = \frac{1}{L} \frac{\Gamma[\beta N]}{\Gamma[\beta] \Gamma[\beta(N-1)]} \left( \frac{g}{L} \right)^{\beta-1} \left( 1 - \frac{g}{L} \right)^{\beta(N-1)-1}, \quad 0 \leq g \leq L, \quad 1 \leq i \leq N.
\] (26)

The corresponding moments of the above distribution for arbitrary \( N \) and \( L \) are

\[
\langle g^k \rangle = \frac{L^k \Gamma[\beta + k] \Gamma[\beta N]}{\Gamma[\beta] \Gamma[\beta N + k]}, \quad \forall k \geq 0.
\] (27)

In the limit \( L \to \infty \) and \( N \to \infty \) with fixed density \( \omega_0 = \frac{L}{N} \) the marginal mass distribution in (26) becomes

\[
\lim_{N \to \infty} P_{\omega N,L}(g) = \frac{\beta^\beta}{\omega_0 \Gamma[\beta]} \left( \frac{g}{\omega_0} \right)^{\beta-1} e^{-\frac{g^{\beta}}{\omega_0}},
\] (28)

which matches with the previously known result [12] for the uniform case \( \beta = \frac{1}{2} \). A similar Gamma distribution for the inter-particle spacings has been obtained by Zielen and Schadschneider in the context of totally asymmetric RAP with parallel update on an infinite...
line [16]. As in our case, their stationary distribution of the gaps also depends on the jump distribution $R(\eta)$. Different curves correspond to different forms of jump distribution $R(\eta)$. Here ‘PL’ corresponds to power law: $R(\eta) = 2/\beta (1 - \eta)^{b-1}$ for $\beta = 0.2, 1.0$ and 5.0. ‘Sin’ represents the law $R(\eta) = \pi (1 - \eta) \sin(\pi \eta)$ for which $\beta = 1$. ‘Exp’ represents the exponential law: $R(\eta) = 2/\beta e^{-2 \beta \eta}$ for $\beta = 30$. Finally, ‘AR’ corresponds to $R(\eta) = 6\eta (1 - \eta)$ for which $\beta = 2/3$. Although the distribution form ‘AR’ does not satisfy (20), we see that the numerically obtained marginal distribution still matches with the analytical expression in (26) with $\beta = 2/3$.

### 3.4. Variance and correlations of the gaps for arbitrary jump distribution $R(\eta)$

For arbitrary jump distributions that do not satisfy (20), the near factorization of the stationary JPDF for the gaps does not hold in general and finding exact expression of JPDF is somewhat difficult. However, one can compute the mean, variance and correlations of the gaps in the SS for arbitrary $R(\eta)$. Multiplying both sides of the SS master equation (10) by $g_i$ and then integrating over all $g_j$, one can easily find that the average mass profile $m_i = \langle g_i \rangle$ satisfies...
\[ m_{i+1} - 2m_i + m_{i-1} = (2p - 1)(\delta_{i,1} - \delta_{i,N})(m_1 - m_N), \quad \text{for } i = 1, 2, \ldots, N. \]  

(29)

This equation can be easily solved to get \( m_i = \omega_0 = L/N \) for all \( i \). Similarly multiplying both sides of (10) by \((g_i - \omega_0)(g_j - \omega_0)\) and then integrating over all \( g_i \)s one obtains the equation satisfied by the correlation function \( \mu_{ij} = \{g_i g_j\} - \omega_0^2 \). The equation is given by

\[
\begin{align*}
\mu_1(4d_{ij} - d_{i,j+1} - d_{i,j-1} - d_{i+1,j} - d_{i-1,j}) &= \mu_2(2\delta_{j,i} - 2\delta_{i+1,j} + 2\delta_{i-1,j} + d_{i,j}) + \mu_1(\delta_{i,N} - \delta_{i,1})(2p - 1)(d_{j,1} - d_{j,N}) + \mu_1(\delta_{j,N} - \delta_{j,1})(2p - 1)(d_{1,i} - d_{N,i}) \\
+ \mu_2(\delta_{i,N} - \delta_{i,1})(\delta_{j,N} - \delta_{j,1})(2p - 1)(d_{N,N} + d_{1,1}) - 2\mu_2 \omega_0^2 (\delta_{j,i+1} - 2\delta_{j,i} + \delta_{j,i-1}) \\
+ 2(2p - 1)\omega_0^2 \mu_2 (\delta_{i,N} - \delta_{i,1})(\delta_{j,N} - \delta_{j,1}).
\end{align*}
\]

(30)

One can try to solve this equation numerically only to observe that \( d_{ij} = D_1 \delta_{ij} + D_2 \) where \( D_1 \) and \( D_2 \) are constants that depend on \( \mu_1, \mu_2, \omega_0 \) and \( N \) but not on \( p \). Using this form of \( d_{ij} \) in (30) one can solve for \( D_1 \) and \( D_2 \) to get

\[
d_{ij} = \left( \frac{\mu_2 \omega_0^2}{\mu_1 + (N - 1)\mu_1 - \mu_2} \right) [N\delta_{ij} - 1], \quad \text{for } p = q,
\]

(31)

as stated in the introduction. It can be checked that in the case of factorized SS, correlations computed from the JPDF (18) consistently yield the same expression.

Using the above knowledge of the fluctuations of the gaps one can study the motion of the CM which is defined as \( X_{cm}(t) = \frac{1}{N} \sum_{i=1}^{N} x_i(t) \). From the evolution of \( x_i(t) \)s in (7) and (8), one can write the evolution of \( X_{cm}(t) = X_{cm}(t + dt) - X_{cm}(t) = \Delta X_{cm}(t) = \frac{1}{N} \sum_{i=1}^{N} \psi_i g_i(t) \), where the \( \psi_i \)s are stationary random variables independent of the \( g_i \) and uncorrelated to order \( dt \). As a result, the distribution of \( X_{cm}(t) \) in the large \( t \) limit can be computed to be a Gaussian distribution with variance \( \sim 2D_{cm}t \) where the diffusion coefficient is given by \( D_{cm} = \frac{1}{N} \omega_0^2 \frac{\mu_1 \mu_2}{\mu_1 - \mu_2} \), and this prediction agrees with the numerics for large enough values of \( p \).

For small \( p \) the motion of the tracer is so slow that we expect anticorrelations between the increments to appear on timescales of order \( p^{-1} \), which is shown in figure 3. As the particles stay ordered, the center of mass cannot diffuse faster than the tracer. The tracer hops at rate \( p \) of an average distance \( \mu_1 \omega_0 \) with probability 1/2 to the right or to the left, so that the diffusion coefficient of the center of mass is expected to behave like \( \frac{1}{2} p \mu_1 \omega_0 \).

4. Biased tracer case: \( p \neq q \)

In this section we consider the \( p \neq q \) case where the tracer particle is driven. As a result, in the long time limit the system will reach a current carrying steady state which will support an inhomogeneous mass/gap density profile in contrast to the \( p = q \) case. As in the unbiased case here too we are interested in the statistics of the gaps between neighboring particles in the steady state. The joint probability distribution of the gaps \( P_{g,j}\{ \mathbf{G} \} \) in this case will in general not have a factorized form as it has in the \( p = q \) case for some choices of \( \mathbf{G}(t) \).

Computation of \( n \) point joint distributions naturally involves \( (n + 1) \) point joint distributions, which makes it difficult to determine the JPDF exactly. However, we will later see that the correlations among different sites decrease to zero as \( \sim 1/N \) in \( N \to \infty \) limit. Hence, let us proceed with the assumption that the joint probability distribution has the following form
in the large L and N limit keeping the mass density \( \omega_0 = L/N \) fixed. Inserting this form of JPDF in (10) we find

\[
P_{N,L}(g_1, g_2, \ldots, g_N) \approx \prod_{i=1}^{N} P_i(g_i, \omega_0) \left[ 1 + O\left( \frac{1}{N} \right) \right] \delta \left( \sum_{i=1}^{N} g_i - L \right),
\]

(32)

where the explicit \( \omega_0 \) dependence of \( P_i(g_i, \omega_0) \) has been omitted for convenience. Taking the Laplace transform of both sides with respect to \( L \) and \( g \), we get

\[
2 \frac{\tilde{P}_i(s + s')}{\tilde{P}_i(s)} \approx \int_0^1 \text{d} \eta R(\eta) \left[ \frac{\tilde{P}_i(s + s' - \eta s')}{\tilde{P}_i(s)} + \frac{\tilde{P}_i(s + s')}{\tilde{P}_i(s)} \right] + O(1/N),
\]

for \( 1 < i < N \)

(34)
in the thermodynamic limit, where the notation $\tilde{P}_i(s) = \int_0^\infty dg \ e^{-s\omega_0} P_i(g)$ and the approximation: $P_{i+1}(g) \approx P_i(g) + O(1/N)$ have been used. In the above equation the argument $s$ in $\tilde{P}_i$ appears while taking the Laplace transform with respect to $L$ whereas $s'$ appears while taking the Laplace transform with respect to $g_i$. Note that (34) is quite similar to (14). Hence following the same procedure as performed in section 3.1 one can find the solution of (34) as

$$\tilde{P}_i(s) = (sh_i + 1)^{-\beta}, \quad \text{where} \quad \beta = \frac{\mu_1 - \mu_2}{\mu_2},$$

and $h_i$ is a site index $i$ dependent constant. Inverting the Laplace transform of $\tilde{P}_i(s)$ we get exactly the same distribution (28) except $\omega_0$ is now replaced by $h_i \beta$. This suggests $h_i$ has to be related to the local average gap $\langle g_i \rangle$. Taking the first derivative of $\tilde{P}_i(s)$ in (35) with respect to $s$ and evaluating it at $s = 0$, one can show that $m_i = \langle g_i \rangle = \int dg \ g \ P_i(g) = h_i \beta$. Hence we have

$$P_i(g) = \frac{\beta^\beta}{m_i \Gamma(\beta)} \left( \frac{g}{m_i} \right)^{\beta-1} e^{-\frac{g}{m_i}}.$$  

The question now is: how does the average gap $m_i$ depend on $i$? In the next section we explicitly compute this dependence.

4.1. Average gap/mass profile

To compute the average mass $m_i = \langle g_i \rangle$ at site $i$, we follow the same procedure as in the $p = q$ case in section 3.4. Multiplying both sides of the SS master equation (10) by $g_i$ and integrating over all masses $g_j, \ j = 1, 2, \ldots, N$ we get the following equation:

$$m_{i+1} - 2m_i + m_{i-1} = (\delta_{i,1} - \delta_{i,N})(2p - 1)m_1 - (2q - 1)m_N, \quad \text{for} \quad i = 1, 2, \ldots, N.$$  

Note that this equation reduces to (29) for $p = q$. The solution of the above equation can be easily obtained as
which agrees very nicely with the numerical results shown in figure 4. Using this expression of $m_i$, we compare our theoretical prediction (36) to the numerical measurements for two choices of $R(\eta)$ in figure 5. We find quite a good match as long as $i$ is not too close to the tracer in the direction towards which it is driven, for example, towards $i = 2, 3, 4, \ldots$ for $p > q$.

From the value of the masses the stationary velocity $v^\text{st}$ of the tracer particle can be computed. Taking the average over $\eta$ variables in (8) and using (38), one gets

$$v^\text{st} = \lim_{\Delta t \to 0} \frac{x_i(t + \Delta t) - x_i(t)}{\Delta t} = \mu_1(p m_1 - q m_N) = \omega_0 \frac{\mu_1(p - q)}{(p + q)(N - 1) + 1}.$$

Note that the velocity vanishes in the thermodynamic limit as $\sim 1/N$. In this limit the expression of $m_i$ takes the following scaling form.
In section 4 we have seen that to the leading order in the thermodynamic limit the joint probability distribution of the gaps $P_{NL}(G)$ can be approximated by a factorized form (32). However, for finite size systems there will be corrections to this factorized form and those corrections arise from the correlation among different gaps. In this section we study the correlations of the gaps for $pq 
eq 1$. We denote the connected two-point correlation function by $d_{ij}$, which by definition is symmetric $d_{ij} = d_{ji}$. Also due to the mass conservation property $\sum_{i=1}^{N} g_i = L$, the correlation $d_{ij}$ satisfies $\sum_{i=1}^{N} d_{ij} = 0$ for all values of $j$. To find the equations satisfied by $d_{ij}$ in the SS, we multiply both sides of equation (10) by $\langle g_i - \langle g_i \rangle \rangle \langle g_j - \langle g_j \rangle \rangle$ and then integrating both sides over all the gaps/masses we get

$$m_i = \omega_0 \mathcal{M} \left( \frac{x}{N} \right) \text{ where } \mathcal{M}(x) = \frac{2}{p+q} \left[ (p-q)x + q \right], \quad x \in [0, 1). \quad (40)$$

### 4.2. Fluctuations and correlations of the masses

In section 4 we have seen that to the leading order in the thermodynamic limit the joint probability distribution of the gaps $P_{NL}(G)$ can be approximated by a factorized form (32). However, for finite size systems there will be corrections to this factorized form and those corrections arise from the correlation among different gaps. In this section we study the correlations of the gaps for $pq \neq 1$. We denote the connected two-point correlation function by $d_{ij} = \langle g_i g_j \rangle - \langle g_i \rangle \langle g_j \rangle$ which by definition is symmetric $d_{ij} = d_{ji}$. Also due to the mass conservation property $\sum_{i=1}^{N} g_i = L$, the correlation $d_{ij}$ satisfies $\sum_{i=1}^{N} d_{ij} = 0$ for all values of $j$. To find the equations satisfied by $d_{ij}$ in the SS, we multiply both sides of equation (10) by $\langle g_i - \langle g_i \rangle \rangle \langle g_j - \langle g_j \rangle \rangle$ and then integrating both sides over all the gaps/masses we get
$\mu_{ij}(4d_{ij} - d_{i,j+1} - d_{i,j-1} - d_{i+1,j} - d_{i-1,j})$

$= \mu_2(\delta_{j,N} - \delta_{j,1})(d_{i,1}(2p - 1) - (2q - 1)d_{j,N})$

$+ \mu_1(\delta_{j,1} - \delta_{j,N})(d_{i,1}(2p - 1) - (2q - 1)d_{j,1})$

$+ \mu_2(\delta_{j,1} - \delta_{j,N})(m_i^2 + m_j^2)$

$+ \mu_2(\delta_{j,N} - \delta_{j,1})(m_i^2(2q - 1) + (2p - 1)m_j^2)$,

\[(41)\]

where $m_i$ is given in (38). One now needs to solve this equation for $d_{ij}$. Before discussing the solution of equation \[(41)\], let us present our simulation results. Numerical measurements of the two-point correlation function are plotted in figures 6 and 7. In figure 6 we plot the scaled stationary mass variance $\omega_0^2 d_{ij}$ as a function of the rescaled coordinate $x = \frac{i}{N+1}$ for $j = N/4$, $N/2$ and $3N/4$ with $N = 24$, 48, 96 and $L = 1$. The circles are numerically obtained and the black solid line is the theoretical expression \[(52)\]. The first particle is the tracer particle which has $q = 0$. Jump distribution is uniform $R(q) = 1$.

\[\text{Figure 7. Rescaled stationary non-diagonal mass–mass correlations } \omega_0^2 N d_{ij} \text{ as a function of the rescaled coordinate } x = \frac{i}{N+1} \text{ for } j = N/4, \ N/2 \text{ and } 3 \ N/4 \text{ with } N = 24, \ 48, \ 96 \text{ and } L = 1. \text{ The circles are numerically obtained and the black solid line is the theoretical expression } (52). \text{ The first particle is the tracer particle which has } q = 0. \text{ Jump distribution is uniform } R(q) = 1.\]
\[
\begin{align*}
\text{J. Phys. A: Math. Theor.} & \text{ 49 (2016) 085002 J Cividini et al}
\end{align*}
\]

\[
d_{i,j} = \frac{\omega_0^2}{N} D\left(\frac{i}{N}, \frac{j}{N}\right) + o(N^{-1}) \quad \text{for } i \neq j,
\]

\[
d_{i,i} = \frac{\omega_0^2}{N} C_i\left(\frac{i}{N}\right) + \frac{\omega_0^2}{N} D\left(\frac{i}{N}, \frac{i}{N}\right) + o(N^{-1}),
\]

(42)

where \( o(\ell) \) represents terms of orders smaller than \( \ell \). We thus look for solution of \((41)\) in the forms \((42)\). We insert the scaling form of \( m_i \) from \((40)\) and of \( d_{ij} \) from \((42)\) in \((41)\). Expanding both sides of \((41)\) as powers of \(1/N\) keeping \(\omega_0 = L/N\) fixed and equating terms of same power, we find that order \(N^{-1}\) and \(N^{-2}\) give

\[
C_0(x) = \frac{\mu_2}{\mu_1 - \mu_2} M^2(x) = \frac{\mu_2}{\mu_1 - \mu_2} \frac{4}{(p + q)^2} (p - q)x + q^2,
\]

(43)

whereas order \(N^{-3}\) yields the following differential equation:

\[
(\partial_x^2 + \partial_y^2) D(x, y) = -\frac{\mu_2}{\mu_1 - \mu_2} \delta(x - y) \frac{d^2 M(x)^2}{dx^2}
\]

\[= -\frac{8\mu_2}{\mu_1 - \mu_2} \left(\frac{p - q}{p + q}\right)^2 \delta(x - y)[1 - \delta(x - 1)],
\]

(44)

with boundary conditions

\[
p \ D(x, y)|_{x = 0} = q \ D(x, y)|_{x = 1},
\]

\[
p \ D(x, y)|_{y = 0} = q \ D(x, y)|_{y = 1},
\]

\[
\partial_x D(x, y)|_{x = 0} = \partial_x D(x, y)|_{x = 1},
\]

\[
\partial_y D(x, y)|_{y = 0} = \partial_y D(x, y)|_{y = 1}.
\]

(45)

The details of the derivation of the above equations are given in appendix B. Equations \((44)\) and \((45)\) have to be supplemented by the vanishing integral condition

\[
C_0(x) + \int_{y=0}^{1} D(x, y) dy = 0,
\]

(46)

which comes directly from the fact that the sum of \(d_{ij}\) on a full row vanishes due to mass conservation.

Note that the leading term \(C_0(x) = \frac{\mu_2}{\mu_1 - \mu_2} M(x)^2\) for the diagonal correlation \(d_{ii}\) in \((42)\) can also be obtained directly from the distribution \((36)\). It can also be obtained by writing the discrete equation \((41)\) for \(i = j\) and neglecting any non-diagonal correlation (as they are order \(O(1/N)\) smaller than the diagonal correlation) in thermodynamic limit. This way of obtaining the continuum description also provides information about the terms at the next order \(\sim 1/N\) in the second line of \((42)\), (see appendix B)

\[
C_0(x) = \frac{\mu_2}{\mu_1 - \mu_2} D(x, x).
\]

(47)

In figure 6 we compare the theoretical solution for \(d_{ij}\) at leading order i.e. \(C_0(x)\) from \((43)\), with the numerical results and observe very nice agreement. In the inset of the same figure we verify \((47)\) numerically.

We now turn our attention back to solving the differential equation \((44)\) satisfied by the scaling function \(D(x, y)\) associated to non-diagonal correlation. Equation \((44)\) can be interpreted as a Poisson equation where \(D(x, y)\) is the electric potential created inside the square domain \((x, y) \in [0,1]^2\) by the charge distribution along the diagonal. Similar interpretations have been used in \([36]\) in the context of long range density–density correlation in a diffusive
lattice gas with simple exclusion interaction. The boundary conditions (45) on the derivatives being the same at 0 and 1 implies that the total flux of the electric field exiting the square domain vanishes, so that the charge inside the square must vanish as well. One can directly observe this fact by integrating the charge distribution on the right-hand side of (44) over the unit square domain.

In general, to solve an inhomogeneous differential equations one usually defines a Green’s function as the solution of $\nabla^2 G(x, y|x_0, y_0) = \delta(x - x_0)\delta(y - y_0)$ with boundary conditions the same as those of the original inhomogeneous differential equation. However we cannot use this Green’s function to compute $D(x, y)$ because the rhs would correspond to the potential created by a net unit charge, which has to be compensated in order to ensure that the Green’s function is well-defined. We choose the following charge distribution inside the square $(x, y) \in [0, 1]^2$

$$\nabla^2 G_{q/p}(x, y|x_0, y_0) = \delta(x - x_0)\delta(y - y_0) - \frac{4}{(p + q)^2}(px + q(1 - x))(py + q(1 - y)), \quad (48)$$

with boundary conditions (45). The purpose of choosing this particular charge distribution will become clear in appendix C, where we provide the details of the solution $G_{q/p}(x, y|x_0, y_0)$ of the above equation for arbitrary $p$ and $q$. For simplicity, here we restrict ourselves to the $q = 0$ case for which the Green’s function reads

$$G_0(x, y|x_0, y_0) = 2\pi \sum_{m=1}^{\infty} \left[ \frac{\sin(2\pi my)\cos(2\pi mx)}{\pi^2 m^2} - \frac{y\cos(2\pi my)\cos(2\pi mx_0) + (1 - y_0)\sin(2\pi my)\sin(2\pi mx_0)}{\pi^2 m^2} \right]$$

$$+ 4 \sum_{m=1}^{\infty} F_{2\pi m}(y|y_0) \left[ x\cos(2\pi mx)\cos(2\pi mx_0) + (1 - x_0)\sin(2\pi mx)\sin(2\pi mx_0) \right]$$

$$+ 16\pi \sum_{m=1}^{\infty} m \sin(2\pi mx)\cos(2\pi mx_0) \Sigma_{2\pi m}(y|y_0), \quad (49)$$

where

$$F_a(y|y_0) = \frac{1}{a(1 - \cosh[a])} \begin{cases} \cosh[a(1 - y_0)] \sinh[ay], & y \leq y_0 \\ \sinh[a(y - y_0)] + \sinh[ay_0] \cosh[a(1 - y)], & y > y_0 \end{cases}$$

and

$$\Sigma_a(y|y_0) = \int_0^1 dz \ F_a(y|z)F_a(z|y). \quad (50)$$

Once we know the Green function $G_0(x, y|x_0, y_0)$, the general solution $D(x, y)$ is given by the sum of a particular solution expressed in terms of the Green’s function and a homogeneous solution of the equation $V_0 xy$, where $V_0$ is an integration constant determined using the integral condition (46). The complete solution of (44)–(46) for $q = 0$ is finally given by
\[
D(x, y) = \frac{16\mu_2}{3(\mu_1 - \mu_2)} xy - \frac{8\mu_2}{\mu_1 - \mu_2} \int_0^1 dx_0 \int_0^1 dy_0 G_0(x, y|x_0, y_0) \\
\times \delta(x_0 - y_0)[1 - \delta(x_0 - 1)],
\]  

where the first term is the homogeneous part and the second term is the particular solution. In figure 7 we compare the above theoretical solution (solid black line) for \(D(x, y)\) along with the Green’s function in (49) with the numerical results (circles). Nice agreement between the two provides verification of our analytical result.

Once again using the knowledge of the gap fluctuations one could intend to study the distribution of the center of mass position \(X_{cm}(t)\) as was done in the unbiased case. However the predicted diffusion coefficient does not match with the numerical simulation results. This discrepancy can be attributed to strong correlations among increments \(X_{cm}(t)\) at different times, as evidenced in figure 3.

5. Conclusion

In this paper we have studied the statistics of the gaps between neighboring particles moving according to the RAP on a ring. The particles hop symmetrically on both sides except for a tracer particle which may be driven. Taking advantage of the mapping between the RAP and an MT model, equivalent up to a global shift, we have obtained some exact results. In this mapping the masses correspond to the gaps between successive particles of the RAP, and an global constraint therefore enforces that the sum of the masses is constant and is equal to the length of the ring.

In the non-driven tracer case \((p = q)\) we found that when the jump distribution \(R(\eta)\) satisfies a necessary and sufficient condition (20) the stationary joint distributions of the gaps takes a universal form (18) parametrized by a positive constant \(\beta = \frac{\mu_1 - \mu_2}{\mu_2}\). We showed that there exists an infinite number of solutions of (20) for \(R(\eta)\). We have found a quite large number of solutions however those do not include all the possible solutions. The calculation and qualitative results are very similar to those obtained by Coppersmith et al [11] in the context of the \(q\)-model of force fluctuations in granular media and to those obtained by Zielen and Schadschneider [16] for the totally asymmetric RAP with parallel update on an infinite line. In their cases the factorized stationary distributions depend only on one parameter, which is however different from ours. As mentioned in the introduction, these distributions are never equilibrium ones, cancellations in the master equation still occur between nearest neighbors only and there is no large-scale mass current. From this joint stationary distribution the single site marginal distribution of gaps/masses has been computed, which also has universal form. Interestingly, we numerically observe that, for some \(R(\eta)\)s which do not satisfy (20), the single site mass/gap distribution can also be described quite well by this universal form.

In the case with a driven tracer the stationary distribution of the masses is not in the form of a factorized joint distribution. In the thermodynamic limit progress has however been made using the fact that the correlations between different masses become negligible for large systems. Besides the average mass profile, which is easily obtained for any size, we computed the two-point gap correlations in the thermodynamic limit. In the same limit we also have computed approximate single site mass distribution of the gaps.

There are several other future extensions of our work. For example, we have already mentioned the problem of characterizing all the solutions \(R(\eta)\)s of (20) properly. For those jump distributions \(R(\eta)\) that do not satisfy (20), it would be interesting to look for other exact forms of the SS such as matrix products [17] or cluster-factorized states [33]. Similarly,
computing the distribution of $X_{\text{cm}}(t)$ for the driven tracer case remains an interesting future problem, and solving this problem probably requires studying the whole history of the system.

A different possibility is to vary the nature of the drive. Note that the $p = q = 0$ case of this paper corresponds to a closed segment with fixed walls at $x = 0$ and $x = L$. For this case a large class of fraction distributions $R(t)$ have been presented for which the SS factorizes. It would be interesting to design natural open boundaries for the RAP in order to study, say, its conduction properties, and in particular the influence of $\beta$. Another possible extension would be to introduce a second driven tracer. In this case an important question is to find the nature and the magnitude of the interaction between the two tracers.

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Appendix A. Some examples of $\hat{\phi}(s)$ in the rational form

Here we present some solutions of (24) in $\hat{\phi}(s) = \frac{\hat{f}_{\beta}(s)}{\hat{f}_{\beta}(s)}$ form for large $\beta$:

$$\hat{\phi}(s) = \frac{2 + es}{1 + es + s^2}, \quad e \geq 2,$$

$$\hat{\phi}(s) = \frac{s\sqrt{a^2 - 4b + 4} - 3as - 2(b - 1)s^2 - 4}{s^3\sqrt{a^2 - 4b + 4} - a(s^2 + 2)s - 2bs^2 - 2},$$

$$a \leq 4, \quad b + 3 < 2a, \quad \text{and} \quad a > 4, \quad 4b \leq (4 + a^2). \quad (A1)$$

Appendix B. Derivation of (44)

To derive the differential equation (44), we first assume the following expansions of the correlation in $1/N$ for large $L$ and $N$ keeping $L/N = \omega_0$ fixed:

$$d_{i,j} = \omega_0^2 \left[ \frac{1}{N} D\left( \frac{i}{N}, \frac{j}{N} \right) + \frac{1}{N^2} D\left( \frac{i}{N}, \frac{j}{N} \right) + o(N^{-2}) \right] \quad (B1)$$

$$d_{i,i} = \omega_0^2 \left[ C_0\left( \frac{i}{N} \right) + \frac{1}{N} \left\{ C_1\left( \frac{i}{N} \right) + D\left( \frac{i}{N}, \frac{i}{N} \right) \right\} \right.$$

$$+ \left. \frac{1}{N^2} \left\{ C_2\left( \frac{i}{N} \right) + D\left( \frac{i}{N}, \frac{i}{N} \right) \right\} + o(N^{-2}) \right]. \quad (B2)$$

where $C_0(x), C_2(x)$ and $D(x, y)$ are scaling functions at sub-leading order.

In the second step, focusing only on the bulk equations (i.e. leaving the boundary equations) we write (41) in the following discrete Laplace equation form:
\[(\Delta d)_{i,j} = \delta_{i,j} A(\{d_{i,j}\}) + \delta_{i+1,j} B_1(\{d_{i,j}\}) + \delta_{i+1,j} B_{-1}(\{d_{i,j}\}), \quad \text{where,} \quad (B3)\]

\[(\Delta d)_{i,j} = \mu_1 (d_{i,j+1} + d_{i,j-1} + d_{i+1,j} + d_{i-1,j} - 4d_{i,j}),\]

\[A(\{d_{i,j}\}) = -\mu_2 (d_{i+1,j+1} + 2d_{i,j} + d_{i-1,j-1}) - \mu_2 (m_{i+1}^2 + 2m_i^2 + m_{i-1}^2)\]

\[B_1(\{d_{i,j}\}) = \mu_2 (d_{i,j} + d_{i+1,j+1}) + \mu_2 (m_i^2 + m_{i+1}^2)\]

\[B_{-1}(\{d_{i,j}\}) = \mu_2 (d_{i,j} + d_{i-1,j-1}) + \mu_2 (m_i^2 + m_{i-1}^2). \quad (B4)\]

We now put the scaling forms (B1) and (B2) of \(d_{i,j}\), and the scaling form (40) of \(m_i\) in the above equation. Equating the terms at order \(N^{-1}\), one obtains

\[C_0(x) = \frac{\mu_2}{\mu_1 - \mu_2} \mathcal{M}^2(x), \quad (B5)\]

which has been verified numerically in figure 6. After imposing (B5) we can check that terms of order \(N^{-2}\) automatically vanish. The order \(N^{-3}\) of (B3) gives a Laplacian of \(\mathcal{D}\), while diagonal terms on the left and right-hand sides combine to give a source term

\[(\partial^2_x + \partial^2_y) \mathcal{D}(x, y) = -\frac{\mu_2}{\mu_1} \delta(x - y) \left[ (\mathcal{M}^2)''(x) + C_0''(x) \right]\]

\[= -\frac{\mu_2}{\mu_1 - \mu_2} \delta(x - y) (\mathcal{M}^2)''(x), \quad (B6)\]

where \(f''\) is the derivative of the function \(f\). While taking the double derivative of \(\mathcal{M}(x)\) one should keep in mind that the function changes discontinuously when going from \(x = 1^-\) (which is actually \(x = 0^-\) on a ring) to \(x = 0^+\). This discontinuity in \(\mathcal{M}(x)\) can be easily visualized if we consider middle particle (\(N/2\)th particle) as the driven tracer particle instead of the 1st particle in RAP. In this case the discontinuity in the average gap/mass profile is at the middle point. For example with \(q = 0\) and \(p = \frac{1}{2}\) one can show that

\[m_i \equiv \begin{cases} \frac{2L}{N(N+1)} & i + 1 - \frac{N}{2} < i \leq N/2, \\ \frac{2L}{N(N+1)} & i + 1 - \frac{N}{2} \leq i \leq N, \end{cases} \quad (B7)\]

which in the continuum limit would imply \(\mathcal{M}(x) = 2x - \operatorname{sgn}(x - 1/2)\). Hence for \(q = 0\) the source term of the differential equation (B6) is given by \(-\frac{8\mu_2}{\mu_1 - \mu_2} \delta(x - y) [1 - \delta(x - 1)]\) where other terms that may appear while taking derivative of \(\mathcal{M}(x)\) have been omitted as they do not contribute to \(\mathcal{D}(x, y)\). Similarly for arbitrary \(p\) and \(q\) one can show

\[(\partial^2_x + \partial^2_y) \mathcal{D}(x, y) = -\frac{8\mu_2}{\mu_1 - \mu_2} \left( \frac{p - q}{p + q} \right)^2 \delta(x - y) [1 - \delta(x - 1)], \quad (B8)\]

To obtain the boundary conditions of (B8), let us now look at the following boundary equations

\[(2p + 3)d_{1,j} = d_{2,j} + d_{2,j-1} + d_{N,j+1} + 2q d_{N,j}, \quad 3 \leq j \leq N - 1, \quad (B9)\]

\[(2q + 3)d_{N,j} = d_{N,j-1} + d_{N,j+1} + d_{N-1,j} + 2p d_{N,j}, \quad 2 \leq j \leq N - 2. \quad (B10)\]

Again putting the scaling form in (B1) and equating terms at orders \(N^{-k}\) for \(k = 1\) and 2 to zero, one gets
While obtaining the above relations we look at terms until order \( N^{-2} \) because had we included these boundary equations in (B3) they would appear with Kronecker deltas \( \delta_{i,1} \) or \( \delta_{i,N} \) which would provide an extra \( \frac{1}{N} \) in the continuum limit. Similarly one can check that, given (B5), the remaining equations (for some specific points at boundaries) are automatically satisfied up to order \( N^{-3} \).

To prove (47) for \( \mathcal{C}_i(x) \) which provides the next order term for \( d_{ij} \) in (B2), we start with the discrete equation for \( d_{ij} \) from (41). Putting the scaling forms for \( m_i \) and \( d_{ij} \) and using

\[
\mathcal{C}_0(x) = \frac{\mu_1}{\mu_1 - \mu_2} \mathcal{M}^0(x) \quad \text{one finds}
\]

\[
\mathcal{C}_0(x) = \frac{\mu_1}{\mu_1 - \mu_2} \mathcal{D}(x, x)
\]  

at order \( \mathcal{O}(1/N) \). In the inset of figure 6 we provide numerical verification of this relation.

**Appendix C. Derivation of the Green function** \( \mathcal{G}_{q/p(x,y|x_0,y_0)} \) in (49)

In this appendix we compute the Green’s function that appears in the determination of the off-diagonal correlations (49). The Laplacian with boundary conditions (B11), is not a self-adjoint operator. Hence the solution can not be expanded in the usual \( \sin \) and \( \cos \) basis. Here we extend the method used by Bodineau et al ([37], appendix A) to our two-dimensional case and consider the following basis functions

\[
f^{(1)}_k(x) = 2(px + q(1 - x)) \cos(2\pi kx), \quad f^{(2)}_k(x) = \sin(2\pi kx)
\]

\[
g^{(1)}_k(x_0) = \cos(2\pi kx_0), \quad g^{(2)}_k(x_0) = 2(p(1 - x_0) + qx_0) \sin(2\pi kx_0).
\]  

(C1)

The functions \( f^{(1)}_k(x) \) and \( f^{(2)}_k(x) \) are independent, each of them satisfies the boundary conditions (B11) at fixed \( y \), and their second derivatives can be expressed as

\[
(f^{(1)}_k)^{''} = -(2\pi k)^2 f^{(1)}_k - 8(p - q)\pi k f^{(2)}_k,
\]

\[
(f^{(2)}_k)^{''} = -(2\pi k)^2 f^{(2)}_k.
\]  

(C2)

In terms of these functions one can check that for \( (x, x_0) \in [0, 1]^2 \) delta function \( \delta(x - x_0) \) can be expanded as

\[
\delta(x - x_0) = \frac{2}{p + q} \sum_{k=0}^{\infty} [a^{(1)}_k f^{(1)}_k(x) g^{(2)}_k(x_0) + a^{(2)}_k f^{(2)}_k(x) g^{(1)}_k(x_0)],
\]  

(C3)

where

\[
a^{(1)}_k = 1 - \frac{\delta_{k,0}}{2}, \quad a^{(2)}_k = 1.
\]  

(C4)

We may therefore look for a solution of the form

\[
\mathcal{G}_{q/p}(x, y|x_0, y_0) = \frac{4}{(p + q)^2} \sum_{(i,j) \neq (0,0)} \sum_{i,j=1}^{\infty} [c^{(i)}_i a^{(i)}_k f^{(i)}_k(x) g^{(j)}_k(x_0) a^{(j)}_l f^{(j)}_l(y) g^{(j)}_l(y_0)],
\]  

(C5)
where the $c_{k,l}^{ij}$ are the unknowns to be determined. Note that in the above expansion the $(0, 0)$ term is not there. Since the contribution of such term on lhs is zero, we have to avoid the presence of such term on rhs too. That is done by conveniently adding $-\frac{q}{(p+q)\pi} [px + q (1 - x)] \left[ py + q (1 - y) \right]$ to the delta source in (48). Putting the form (C5) in the equation for the Green function (48), we obtain an algebraic equation for each value of $(k, l) \neq (0, 0)$ which after solving provide

$$c_{k,l}^{1,1} = -\frac{1}{4 \pi^2 (k^2 + l^2)},$$

$$c_{k,l}^{1,2} = -\frac{1}{4 \pi^2 (k^2 + l^2)} + 2(p - q) \frac{a_{l}^{(1)} g_{l}^{(1)}(y_0)}{a_{l}^{(2)} g_{l}^{(2)}(y_0)} \frac{l}{4 \pi (k^2 + l^2)^2},$$

$$c_{k,l}^{2,1} = -\frac{1}{4 \pi^2 (k^2 + l^2)} + 2(p - q) \frac{a_{k}^{(1)} g_{k}^{(1)}(x_0)}{a_{k}^{(2)} g_{k}^{(2)}(x_0)} \frac{k}{4 \pi (k^2 + l^2)^2},$$

$$c_{k,l}^{2,2} = -\frac{1}{4 \pi^2 (k^2 + l^2)} + 2(p - q) \frac{a_{k}^{(1)} g_{k}^{(1)}(x_0)}{a_{k}^{(2)} g_{k}^{(2)}(x_0)} \frac{k}{4 \pi (k^2 + l^2)^2} + 2(p - q) \frac{a_{l}^{(1)} g_{l}^{(1)}(y_0)}{a_{l}^{(2)} g_{l}^{(2)}(y_0)} \frac{l}{4 \pi (k^2 + l^2)^2} - 8(p - q)^2 \frac{a_{l}^{(1)} g_{l}^{(1)}(y_0)}{a_{l}^{(2)} g_{l}^{(2)}(y_0)} \frac{a_{l}^{(1)} g_{l}^{(1)}(y_0)}{a_{l}^{(2)} g_{l}^{(2)}(y_0)} \frac{kl}{a_{l}^{(2)} g_{l}^{(2)}(y_0) a_{l}^{(2)} g_{l}^{(2)}(x_0) 4 \pi (k^2 + l^2)^3}. \tag{C6}$$

Note that Green function $\mathcal{G}_{q/p}(x, y|x_0, y_0)$ in (C5) involves double infinite sum. However, we can consider the following expansion of the Green function which involves one infinite sum:

$$\mathcal{G}_{q/p}(x, y|x_0, y_0) = \frac{4}{(p + q)\pi} \sum_{j=0}^{\infty} \sum_{i=1}^{2} \left[ c_{i,j}^{(1)} a_{i}^{(1)} f_{i}^{(1)}(x) g_{j}^{(1)}(y_0) a_{j}^{(1)} f_{j}^{(1)}(y) g_{i}^{(1)}(y_0) \right]$$

$$+ \frac{2}{p + q} \sum_{k=0}^{\infty} \sum_{l=1}^{2} \left[ a_{k}^{(1)} f_{k}^{(1)}(x) g_{l}^{(1)}(x_0) F_{2,2k}^{(1)}(y)|y_0) \right], \tag{C7}$$

where the coefficients $a(k)^{(i)}$ and $c_{k,l}^{ij}$ are given in (C4) and (C6) respectively. The functions $F_{2,2k}^{(1)}(y)|y_0)$ for $k \geq 1$ are unknown functions to be determined. Putting this expansion of Green function in (48) we find that the functions have to satisfy

$$F_{2,2(k)}^{(1)}(y)|y_0)'' = (2\pi k)^2 F_{2,2(k)}^{(1)}(y)|y_0) = \delta(y - y_0),$$

$$F_{2,2(k)}^{(2)}(y)|y_0)'' = (2\pi k)^2 F_{2,2(k)}^{(2)}(y)|y_0) = \delta(y - y_0) + 8(p - q)^2 \pi k A_{l}^{(1)} g_{l}^{(1)}(y_0) A_{l}^{(2)} g_{l}^{(2)}(y_0) F_{2,2l}^{(1)}(y)|y_0). \tag{C8}$$

Solving equation (C8) and simplifying gives

$$F_{2,2(k)}^{(1)}(y)|y_0) = F_{2,2k}^{(1)}(y)|y_0),$$

$$F_{2,2(k)}^{(2)}(y)|y_0) = F_{2,2k}^{(2)}(y)|y_0) + 8(p - q)^2 \pi k A_{l}^{(1)} g_{l}^{(1)}(y_0) A_{l}^{(2)} g_{l}^{(2)}(y_0) \Sigma_{2,2k}(y)|y_0), \tag{C9}$$

where $F_{2}(y)|y_0)$ and $\Sigma_{2}(y)|y_0)$ are given in (50) and (51) respectively. For $q = 0$, putting these results in the expansion (C7) gives the solution (49).
From (C5) one can also obtain (C7) by using the following identity for $k > 0$

$$F_{2m}(y|y_0) = \sum_{m=0}^{\infty} \left( \frac{m \sin(2\pi my) \cos(2\pi my_0)}{\pi^2 (k^2 + m^2)^2} - \frac{(1 - y_0) \sin(2\pi my) \sin(2\pi my_0) + a_{ym} \cos(2\pi my) \cos(2\pi my_0)}{\pi^2 (k^2 + m^2)} \right).$$

(C10)

References

[1] Spohn H 1991 *Large Scale Dynamics of Interacting Particles* (Berlin: Springer)
[2] Bertini L, Sole A D, Gabrielli D, Jona-Lasinio G and Landim C 2015 Macroscopic fluctuation theory *Rev. Mod. Phys.* 87 593–636
[3] Derrida B 2007 Non-equilibrium steady states: fluctuations and large deviations of the density and of the current *J. Stat. Mech.* P07023
[4] Chou T, Mallick K and Zia R K P 2011 Non-equilibrium statistical mechanics: from a paradigmatic model to biological transport *Rep. Prog. Phys.* 74 116601
[5] Lazarescu A 2015 The physicist’s companion to current fluctuations: one-dimensional bulk-driven lattice gases *J. Phys. A: Math. Theor.* 48 503001
[6] Ferrari P A and Fontes L R G 1998 Fluctuations of a surface submitted to a random average process *Electron. J. Probab.* 3 1–34
[7] Krug J and Garcia J 2000 Asymmetric particle systems on $\mathbb{R}$ *J. Stat. Phys.* 99 31–55
[8] Schütz G M 2000 Exact tracer diffusion coefficient in the asymmetric random average process *J. Stat. Phys.* 99 1045–9
[9] Rajesh R and Majumdar S N 2001 Exact tagged particle correlations in the random average process *Phys. Rev. E* 64 036103
[10] Feng S, Iscoe I and Seppäläinen T 1996 A class of stochastic evolutions that scale to the porous medium equation *J. Stat. Phys.* 85 513–7
[11] Coppersmith S N, Liu C H, Majumdar S, Narayan O and Witten T A 1996 Model for force fluctuations in bead packs *Phys. Rev. E* 53 4673–85
[12] Rajesh R and Majumdar S N 2000 Conserved mass models and particle systems in one-dimension *J. Stat. Phys.* 99 943–65
[13] Melzak Z A 1976 *Companion to Concrete Mathematics* (Mathematical Ideas, Modeling and Applications vol 2) (New York: Wiley)
[14] Ispolatov S, Krapivsky P L and Redner S 1998 Wealth distributions in asset exchange models *Eur. Phys. J. B* 2 267–76
[15] Aldous D and Diaconis P 1995 Hammersley’s interacting particle process and longest increasing subsequences *Probab. Theory Relat. Fields* 103 199–215
[16] Zielen F and Schadschneider A 2002 Exact mean-field solutions of the asymmetric random average process *J. Stat. Phys.* 106 173–85
[17] Zielen F and Schadschneider A 2003 Matrix product approach for the asymmetric random average process *J. Phys. A: Math. Gen.* 36 3709–23
[18] Zielen F and Schadschneider A 2002 Broken ergodicity in a stochastic model with condensation *Phys. Rev. Lett.* 89 090601
[19] Gutsche C, Kremer F, Krüger M, Rauscher M, Weeber R and Harting J 2008 Colloids dragged through a polymer solution: experiment, theory, and simulation *J. Chem. Phys.* 129 084902
[20] Krüger M and Rauscher M 2009 Diffusion of a sphere in a dilute solution of polymer coils *J. Chem. Phys.* 131 094902
[21] Candelier R and Dauchot O 2010 Journey of an intruder through the fluidization and jamming transitions of a dense granular media *Phys. Rev. E* 81 011304
[22] Pesic J, Terdik J Z, Xu X, Tian Y, Lopez A, Rice S A, Dinner A R and Scherer N F 2012 Structural responses of quasi-two-dimensional colloidal fluids to excitations elicited by nonequilibrium perturbations *Phys. Rev. E* 86 031403
[23] Dullens R P A and Bechinger C 2011 Shear thinning and local melting of colloidal crystals *Phys. Rev. Lett.* 107 138301
[24] Burlatsky S F, Oshanin G S, Mogutov A V and Moreau M 1992 Directed walk in a one-dimensional lattice gas Phys. Lett. A 166 230–4
[25] Burlatsky S F, Oshanin G, Moreau M and Reinhardt W P 1996 Motion of a driven tracer particle in a one-dimensional symmetric lattice gas Phys. Rev. E 54 3165–72
[26] Landim C, Olla S and Volchan S B 1998 Driven tracer particle in one-dimensional symmetric simple exclusion Commun. Math. Phys. 192 287–307
[27] Bénichou O, Cazabat A M, Lemarchand A, Moreau M and Oshanin G 1999 Biased diffusion in a one-dimensional adsorbed monolayer J. Stat. Phys. 97 351–71
[28] Benichou O, Illien P, Mejia-Monasterio C and Oshanin G 2013 A biased intruder in a dense quiescent medium: looking beyond the force–velocity relation J. Stat. Mech. P05008
[29] Evans M R and Hanney T 2005 Nonequilibrium statistical mechanics of the zero-range process and related models J. Phys. A: Math. Gen. 38 R195–239
[30] Kolmogorov A N 1936 Zur theorie der markoffschen ketten Math. Ann. Bd. 112 155
[31] Evans M R, Majumdar S N and Zia R K P 2004 Factorized steady states in mass transport models J. Phys. A: Math. Gen. 37 L275
[32] Zia R K P, Evans M R and Majumdar S N 2004 Construction of the factorized steady state distribution in models of mass transport J. Stat. Mech. L10001
[33] Chatterjee A, Pradhan P and Mohanty P K 2015 Cluster-factorized steady states in finite range processes Phys. Rev. E 92 032103
[34] Majumdar S N 2010 Real-space condensation in stochastic mass transport models Les Houches Lecture Notes for the Summer School on Exact Methods in Low-dimensional Statistical Physics and Quantum Computing (Oxford: Oxford University Press)
[35] Rajesh R and Majumdar S N 2001 Exact phase diagram of a model with aggregation and chipping Phys. Rev. E 63 036114
[36] Sadhu T, Majumdar S N and Mukamel D 2014 Long-range correlations in a locally driven exclusion process J. Stat. Phys. 90 012109
[37] Bodineau T, Derrida B and Lebowitz J L 2010 A diffusive system driven by a battery or by a smoothly varying field J. Stat. Phys. 140 648