ON ASYMPTOTIC EXPANSIONS FOR THE DERIVATIVES OF THE ALTERNATING HURWITZ ZETA FUNCTIONS

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ABSTRACT. By using the Boole summation formula, we obtain asymptotic expansions for the first and higher order derivatives of the alternating Hurwitz zeta function

\[ \zeta_E(z, q) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+q)^z} \]

with respect to its first argument

\[ \zeta_E^{(m)}(z, q) \equiv \frac{\partial^m}{\partial z^m} \zeta_E(z, q). \]

1. HISTORY OF THE SUBJECT

Let \( \mathbb{N} = \{1, 2, \ldots\} \) and \( \mathbb{N}_0 = \mathbb{N} \cup \{0\} \).

The Hurwitz zeta function is defined by the following series

\[ (1.1) \quad \zeta(z, q) = \sum_{n=0}^{\infty} \frac{1}{(n+q)^z}, \quad \text{Re}(z) > 1, \quad q \neq 0, -1, -2, \ldots \]

and it can be analytically continued to the entire \( z \)-plane except for a simple pole at \( z = 1 \) with residue 1 (see [3] and [22]). Special cases of \( \zeta(z, q) \) include the Riemann zeta function

\[ \zeta(z, 1) = \zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z} \]

and

\[ \zeta \left( z, \frac{1}{2} \right) = 2^z \sum_{n=0}^{\infty} \frac{1}{(2n+1)^z} = (2^z - 1)\zeta(z). \]

It is known that ([3] p. 76, Corollary 9.6.10)]

\[ (1.2) \quad \zeta(1 - n, q) = -\frac{1}{n} B_n(q) \]

for \( n \in \mathbb{N} \), where \( B_n(q) \) denotes the \( n \)th Bernoulli polynomial which is defined as the coefficient of \( t^n \) in the generating function

\[ \frac{te^{qt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(q) \frac{t^n}{n!}. \]
In particular, \( B_n(0) = B_n \) is the \( n \)th Bernoulli number. The Bernoulli numbers and polynomials arise from Bernoulli’s calculations of power sums in 1713, that is,
\[
\sum_{j=0}^{m} j^n = \frac{B_{n+1}(m+1) - B_{n+1}}{n+1}
\]
(see [21, p. 5, (2.2)]).

The derivatives of Hurwitz zeta function \( \zeta(z, q) \) at non-positive integers have many applications in number theory and mathematical physics (see [8] and [10]). In 1986, by using Watson’s Lemma and Laplace’s method, Elizalde [6] gave an asymptotic expansion for the first derivatives
\[
\zeta'(-n, q) \equiv \frac{\partial}{\partial z} \zeta(z, q) \bigg|_{z=-n}, \quad n \in \mathbb{N}_0.
\]

It is also interesting to refer [23] and [7, 19, 20] for some researches on \( \zeta'(-n, q) \) at \( z = 0 \) and at \( -z \in \mathbb{N} \), respectively. In 1984, to study an analogue of the formula of Chowla and Selberg for real quadratic fields, Deninger [5] investigated the second derivative \( \zeta''(0, q) \). In [7] (also see [8, Chapter 2]), Elizalde further obtained some recursive formulas for the higher order derivatives, and presented explicit formulas for the case of small \( m \) and small \( -z \in \mathbb{N} \).

In this note we shall consider the alternating Hurwitz (or Hurwitz-type Euler) zeta function which is defined by the following series
\[
\zeta_E(z, q) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+q)z^n}, \quad \text{Re}(z) > 0, \quad q \neq 0, -1, -2, \ldots
\]
(see [2, (3.1)], [12, (1.3)] and [23, (1.1)]). We see that \( \zeta_E(z, q) \) can be analytically continued to the entire \( z \)-plane without any pole and it satisfies the following identities:
\[
\zeta_E(z, q + 1) + \zeta_E(z, q) = q^{-z},
\]
\[
\frac{\partial}{\partial q} \zeta_E(z, q) = -z \zeta_E(z + 1, q).
\]

In analogue with [1.2], it is known that ([23, p. 41, (3.8)] and [2, p. 520, (3.20)])
\[
\zeta_E(-n, q) = \frac{1}{2} E_n(q)
\]
for \( n \in \mathbb{N}_0 \), where \( E_n(q) \) denotes the \( n \)th Euler polynomial which is defined as the coefficient of \( t^n \) in the generating function
\[
\frac{2e^{qt}}{e^t + 1} = \sum_{n=0}^{\infty} E_n(q) \frac{t^n}{n!}.
\]
The Euler polynomials were introduced by Euler who calculated the alternating power sums, that is,
\[ \sum_{j=0}^{m} (-1)^j j^n = \frac{(-1)^m E_n(m + 1) + E_n(0)}{2} \]
(see [21, p. 5, (2.3)]).

In algebraic number theory, the zeta function \( \zeta_E(z, q) \) represents a partial zeta function of cyclotomic fields in one version of Stark’s conjectures (see [16, p. 4249, (6.13)]), and its special case, the alternative series,
\[ \zeta_E(z, 1) = \zeta_E(z) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^z}, \]
is also a particular case of Witten’s zeta functions in mathematical physics (see [17, p. 248, (3.14)]).

In the following we will denote the \( m \)th derivative of \( \zeta_E(z, q) \) with respect to its first argument by \( \zeta_E^{(m)} \) and \( \zeta_E^{(0)} \equiv \zeta_E \) and \( \zeta_E^{(1)} \equiv \zeta_E^{(1)} \) and \( \zeta_E^{(2)} \equiv \zeta_E^{(2)} \). Inspiring by the works of Elizalde [7] and Rudaz [19], using the Boole summation formula, we provide asymptotic expansions for the higher order derivatives of the alternating Hurwitz zeta function \( \zeta_E(z, q) \) (see Section 3).

2. Boole summation formula

In this section, as a preliminary we state the Boole summation formula. It will serve as a main tool for our approach. Then as an application, we shall prove an asymptotic series expansion of \( \zeta_E(z, q) \) (see Theorem 2.3). But at first, we need to recall the definition of quasi–periodic Euler functions \( \overline{E}_n(x) \), see [11] for some details on their properties and applications.

Let \( n \in \mathbb{N}_0 \). If \( 0 \leq x < 1 \), we write
\[ \overline{E}_n(x) = E_n(x), \]
where \( E_n(x) \) denotes the \( n \)th Euler polynomials. We may extend this definition to all \( x \in \mathbb{R} \) by setting
\[ \overline{E}_n(x+1) = -\overline{E}_n(x). \]
In this way, \( \overline{E}_n(x) \) becomes a quasi–periodic function on \( \mathbb{R} \), and it has the following Fourier expansion
\[ \overline{E}_n(x) = \frac{4n!}{\pi^{n+1}} \sum_{k=0}^{\infty} \frac{\sin((2k + 1)\pi x - \frac{1}{2}\pi n)}{(2k + 1)^{n+1}}, \]
where \( 0 \leq x < 1 \) if \( n \in \mathbb{N} \) and \( 0 < x < 1 \) if \( n = 0 \) (see [11, p. 805, 23.1.16]).

Now using \( \overline{E}_n(x) \) we state the Boole summation formula. It is obtained by Boole, but a similar one has been shown by Euler previously (see [4, Theorem 1.2], [15, Lemma 2.1] and [18, 24.17.1–2]).
Lemma 2.1 (Boole summation formula). Let $\alpha, \beta$ and $N$ be integers such that $\alpha < \beta$ and $N \in \mathbb{N}$. If $f^{(N+1)}(t)$ is absolutely integrable over $[\alpha, \beta]$. Then

$$2 \sum_{n=\alpha}^{\beta-1} (-1)^n f(n) = \sum_{k=0}^{N} \frac{E_k(0)}{k!} \left( (-1)^{\beta-1} f^{(k)}(\beta) + (-1)^{\alpha} f^{(k)}(\alpha) \right)$$

$$+ \frac{1}{N!} \int_{\alpha}^{\beta} E_N(-t) f^{(N+1)}(t) dt,$$

where $E_n(t), n \in \mathbb{N}_0,$ is the $n$th quasi-periodic Euler functions.

From the above lemma we get the following asymptotic expansion. It should be noted that in the sense of asymptotic expansions, the infinite series here need to be understood as a sum over $k$ terminating at some arbitrarily chosen finite $N$, together with an integral formula for the remainder term $R_{N+1}$ (see [6, (7)]).

Lemma 2.2. Let $C^\infty[0, 1]$ be the space of infinitely differentiable real-valued functions on the interval $[0, 1]$. For $f(t) \in C^\infty[0, 1]$, we have the following asymptotic expansion

$$f(0) = \frac{1}{2} \Delta(0) - \frac{1}{4} \Delta'(0) - \frac{1}{2} \sum_{k=2}^{\infty} \frac{(-1)^k}{k!} E_k(0) \Delta^{(k)}(0) + R_{N+1},$$

where

$$R_{N+1} = \frac{1}{2N!} \int_0^1 E_N(-t) f^{(N+1)}(t) dt.$$

Then noticing that $\Delta(t) = f(t+1) + f(t)$ and $E_0(0) = 1$, $E_1(0) = -\frac{1}{2}$, $E_k(0) = 0$ if any even $k \geq 2$ (e.g. [21, p. 5, Corollary 1.1]), we get

$$f(0) = \frac{1}{2} \Delta(0) - \frac{1}{4} \Delta'(0) - \frac{1}{2} \sum_{k=2}^{N} \frac{(-1)^k}{k!} E_k(0) \Delta^{(k)}(0) + R_{N+1},$$

where

$$R_{N+1} = \frac{1}{2N!} \int_0^1 E_N(-t) f^{(N+1)}(t) dt.$$
Theorem 2.3. For all $z \in \mathbb{C}$ and $q \neq 0, -1, -2, \ldots$, we have the following asymptotic expansion

\begin{equation}
\zeta_E(z, q) = \frac{1}{2} q^{-z} + \frac{1}{4} z q^{-z - 1} - \frac{1}{2} \sum_{k=2}^{\infty} \frac{1}{k!} E_k(0)(z) k q^{-z - k},
\end{equation}

as $q \to \infty$, where

\[(z)_k = z(z + 1) \cdots (z + k - 1) = \frac{\Gamma(z + k)}{\Gamma(z)}\]

is Pochhammer's symbol (rising factorial function). Here $\Gamma(z)$ is the classical Euler gamma function.

Remark 2.4. We refer to V.A. Zorich's book [24, p. 593, Definition 2] for general concept of asymptotic expansions.

Remark 2.5. In a recent article [13], by using the alternating Hurwitz zeta function (1.4), we defined modified Stieltjes constants $\tilde{\gamma}_k(q)$ from the following Taylor expansion,

\begin{equation}
\zeta_E(z, q) = \sum_{k=0}^{\infty} \frac{(-1)^k \tilde{\gamma}_k(q)}{k!} (z - 1)^k.
\end{equation}

Then by using the asymptotic expansion (2.6), we obtained an asymptotic expansion of $\tilde{\gamma}_k(q)$, see [13, Theorem 3.17] and its proof.

Theorem 2.3 implies the following known results on the special values of $\zeta_E(z, q)$ at $z = -n$ for $n \in \mathbb{N}_0$. Its proof will be given in Section 4.

Corollary 2.6 ([23, (3.8)]). For all $n \in \mathbb{N}_0$ we have

\[\zeta_E(-n, q) = \frac{1}{2} E_n(q),\]

where $E_n(q)$ denotes the nth Euler polynomials. In particular, when $n = 0$, we obtain $\zeta_E(0, q) = 1/2$.

3. Main results

In this section, we state our main results, that is, we shall provide asymptotic expansions for the first and higher order derivatives of the alternating Hurwitz zeta function $\zeta_E(z, q)$. Their proofs will be shown in Section 4.

Theorem 3.1. For all $z \in \mathbb{C}$ and $q \neq 0, -1, -2, \ldots$, we have the following asymptotic expansion

\[\zeta'_E(z, q) = \frac{1}{4} q^{-z - 1} - \zeta_E(z, q) \log q - \frac{1}{2} \sum_{k=2}^{\infty} E_k(0) \sum_{j=0}^{k-1} \frac{(z)_j}{j!(k-j)} q^{-z - k},\]

as $q \to \infty$.

Now setting $z = -n$ for integer $n \in \mathbb{N}_0$ in Theorem 3.1. Using Corollary 2.6 and

\[(n)_j = (-1)^{n-j} \frac{\Gamma(n+1)}{\Gamma(n+1-j)} = (-1)^j \frac{n!}{(n-j)!},\]

we obtain

\[\zeta'_E(-n, q) = \frac{1}{2} E_n(q).\]
we obtain the asymptotic expansion
\[
\zeta'_E(-n, q) = \frac{1}{4} q^{n-1} - \frac{1}{2} E_n(q) \log q
\]
(3.1)
\[
- \frac{1}{2} \sum_{k=2}^{\min(n,k-1)} E_k(0) \sum_{j=0}^{n} \binom{n}{j} \frac{(-1)^j}{k-j} q^{n-k},
\]
as \(q \to \infty\). In particular, setting \(n = 0\) and \(1\) in (3.1) respectively, we have the following asymptotic expansions

(3.2) \[\zeta'_E(0, q) = -\frac{1}{2} \log q + \frac{1}{4} q^{-1} - \frac{1}{2} \sum_{k=2}^{\infty} \frac{E_k(0)}{k} q^{-k},\]

(3.3) \[\zeta'_E(-1, q) = \frac{1}{4} - \frac{1}{2} \left( q - \frac{1}{2} \right) \log q + \frac{1}{2} \sum_{k=2}^{\infty} \frac{E_k(0)}{k(k-1)} q^{-(k-1)},\]
as \(q \to \infty\).

Furthermore, let \(n \geq 2\). Then it is not difficult to see that the following double sum identity holds

(3.4) \[
\sum_{k=2}^{\infty} \min(n,k-1) \sum_{j=0}^{n} b_n(k, j) = \sum_{k=2}^{n} a(k) \sum_{j=0}^{k-1} b_n(k, j) + \sum_{k=n+1}^{\infty} a(k) \sum_{j=0}^{n} b_n(k, j).
\]

Using (3.1), (3.4) and the following identity (see [19, p. 2833, (16)])

(3.5) \[
\sum_{j=0}^{n} \binom{n}{j} \frac{(-1)^j}{k-j} = \frac{(-1)^n n!}{k(k-1) \cdots (k-n)},
\]
we arrive at the following corollary.

**Corollary 3.2.** For all \(n \geq 2\) we have the following asymptotic expansion

\[
\zeta'_E(-n, q) = \frac{1}{4} q^{n-1} - \frac{1}{2} E_n(q) \log q
\]
\[
- \frac{1}{2} \sum_{k=2}^{n} E_k(0) \sum_{j=0}^{k-1} \binom{n}{j} \frac{(-1)^j}{k-j} q^{n-k}
\]
\[
+ (-1)^{n+1} \frac{n!}{2} \sum_{k=n+1}^{\infty} \frac{E_k(0)}{k(k-1) \cdots (k-n)} q^{n-k},
\]
as \(q \to \infty\).

In particular, setting \(n = 2\) and \(3\) in Corollary 3.2 we have the following expansions

(3.6) \[\zeta'_E(-2, q) = \frac{1}{4} q - \frac{1}{2} (q^2 - q) \log q - \sum_{k=3}^{\infty} \frac{E_k(0)}{k(k-1)(k-2)} q^{-(k-2)},\]
\[ \zeta_E(-3, q) = -\frac{11}{48} + \frac{1}{4} q^2 - \frac{1}{2} \left( q^3 - \frac{3}{4} q^2 + \frac{1}{4} \right) \log q \]

(3.7)

\[ + 3 \sum_{k=4}^{\infty} \frac{E_k(0)}{k(k-1)(k-2)(k-3)} q^{-(k-3)}, \]

as \( q \to \infty \), since \( E_2(0) = 0 \).

In the following, we will employ the usual convention that an empty sum is taken to be zero. For example, if \( m = 1 \), then we understand that \( \sum_{k=1}^{m-1} = 0 \).

Denote by \( \log^j q = (\log q)^j \) for \( j \geq 1 \). Using Theorem 2.3, we may represent \( \zeta_E^{(m)}(z, q), \) the higher order derivatives of the alternating Hurwitz zeta functions, in terms of the lower orders from the following asymptotic expansion.

**Theorem 3.3.** For all \( z \in \mathbb{C}, q \neq 0, -1, -2, \ldots \) and \( m \geq 2 \), we have the following asymptotic expansion

\[ \zeta_E^{(m)}(z, q) = -\sum_{j=1}^{m} \binom{m}{j} \zeta_E^{(m-j)}(z, q) \log^j q - \frac{1}{2} \sum_{k=2}^{\infty} c_{k,m}(z) q^{-z-k}, \]

as \( q \to \infty \), where

(3.8)

\[ c_{k,m}(z) = E_k(0) \left\{ \sum_{j_1=0}^{k-1} \frac{1}{k-j_1} \sum_{j_2=0}^{j_1-1} \frac{1}{j_1-j_2} \cdots \sum_{j_m=0}^{j_{m-1}-1} \frac{(z)_{j_m}}{j_m!(j_{m-1}-j_m)} \right\}. \]

If we restrict ourselves to the particular values \( z = -n, n \in \mathbb{N}_0 \), then we obtain the following result.

**Corollary 3.4.** For all \( n \in \mathbb{N}_0 \) and \( m \geq 2 \), we have the following asymptotic expansion

\[ \zeta_E^{(m)}(-n, q) = -\sum_{j=1}^{m} \binom{m}{j} \zeta_E^{(m-j)}(-n, q) \log^j q - \frac{1}{2} \sum_{k=2}^{\infty} c_{k,m}(-n) q^{-n-k}, \]

as \( q \to \infty \), where

\[ c_{k,m}(-n) = E_k(0) \left\{ \sum_{j_1=0}^{k-1} \frac{1}{k-j_1} \sum_{j_2=0}^{j_1-1} \frac{1}{j_1-j_2} \cdots \sum_{j_m=0}^{\min(n,j_{m-1}-1)} \frac{n}{j_m} \frac{(-1)^{j_m}}{j_m!(j_{m-1}-j_m)} \right\}. \]

**Remark 3.5.** We remark here that the formula in Theorem 3.1 for \( \zeta_E'(z, q) \) cannot be obtained from Theorem 3.3 simply by setting \( m = 1 \) as it contains an additional term in \( q^{-z-1} \).

4. PROOFS OF THE RESULTS

In this section, we prove Theorems 2.3, Corollary 2.6, Theorems 3.1 and 3.3 respectively.
Proof of Theorem 2.3. We give a proof of Theorem 2.3 using Proposition 2.2.

For \( t \in \mathbb{R} \), put \( f(t) = \zeta_E(z, q + t) \) and then from (1.5) we immediately have

\[
\Delta (0) = f(1) + f(0) = q^{-z},
\]

\[
\Delta'(0) = \frac{d}{dt} [f(t + 1) + f(t)]_{t=0}
\]

\[
= \frac{d}{dt} [\zeta_E(z, q + t + 1) + \zeta_E(z, q + t)]_{t=0}
\]

\[
= \frac{d}{dt} [(q + t)^{-z}]_{t=0}
\]

\[
= -zq^{-z-1}
\]

and

\[
\Delta^{(k)}(0) = (-1)^{k} (z) k q^{-z-k},
\]

where \( k \in \mathbb{N}_0 \). Substituting (4.1) and (4.2) into Proposition 2.2 we complete the proof.

Proof of Corollary 2.6. By putting \( z = -n, n \in \mathbb{N}_0 \) in Theorem 2.3 we have the following formula for \( \zeta_E(-n, q) \):

\[
\zeta_E(-n, q) = \frac{1}{2} q^n - \frac{1}{4} n q^{n-1} - \frac{1}{2} \sum_{k=2}^{\infty} \frac{1}{k!} E_k(0) (-n)_k q^{-n-k}
\]

\[
= \frac{1}{2} q^n - \frac{1}{4} n q^{n-1} + \frac{1}{2} \sum_{k=2}^{\infty} (-1)^{k+1} \binom{n}{k} E_k(0) q^{-n-k}
\]

\[
= \frac{1}{2} q^n - \frac{1}{4} n q^{n-1} + \frac{1}{2} \sum_{k=2}^{n} \binom{n}{k} E_k(0) q^{-n-k}
\]

(by using \( E_k(0) = 0 \) if any even \( k \geq 2 \))

\[
= \frac{1}{2} \sum_{k=0}^{n} \binom{n}{k} E_k(0) q^{-n-k},
\]

where we used \( E_0(0) = 1 \) and \( E_1(0) = -1/2 \). The result follows by noticing that

\[
E_n(q) = \sum_{k=0}^{n} \binom{n}{k} E_k(0) q^{-n-k}
\]

(see [1] p. 804, 23.1.7 and [2] p. 3, (1.4)).

Proof of Theorem 3.1. To prove this, we need the following lemma.

Lemma 4.1. For any nonnegative integer \( k \), denote by

\[
\binom{z}{k} = \frac{z(z-1) \cdots (z-k+1)}{k!}
\]
Then we have
\[
\frac{d}{dz} \binom{-z}{k} = \sum_{j=0}^{k-1} \frac{(-1)^{k-j}}{k-j} \binom{-z}{j}
\]
and
\[
\frac{d}{dz} \left\{ \frac{(z)_k}{k!} \right\} = \sum_{j=0}^{k-1} \frac{(z)_j}{j!(k-j)}.
\]

For the reader’s convenience, we briefly review the proof of Lemma 4.1.

**Proof of Lemma 4.1**. (4.3) follows directly from the induction. The case \( k = 1 \) is clear. It is well-known that
\[
\binom{-z}{k+1} = \binom{-z-1}{k} + \binom{-z-1}{k+1}.
\]
Hence, by differentiating it with respect to \( z \), we obtain
\[
\frac{d}{dz} \frac{-z}{k+1} = \frac{d}{dz} \left\{ \frac{-z-1}{k} + \frac{-z-1}{k+1} \right\}
= \sum_{j=0}^{k-1} \frac{(-1)^{k-j}}{k-j} \binom{-z-1}{j} + \sum_{j=0}^{k} \frac{(-1)^{k-j+1}}{k-j+1} \binom{-z-1}{j}.
\]
Since
\[
\sum_{j=0}^{k} \frac{(-1)^{k-j+1}}{k-j+1} \binom{-z-1}{j} = \sum_{j=0}^{k-1} \frac{(-1)^{k-j}}{k-j} \binom{-z-1}{j+1} + \frac{(-1)^{k+1}}{k+1},
\]
(4.5) becomes
\[
\frac{d}{dz} \frac{-z}{k+1} = \sum_{j=0}^{k-1} \frac{(-1)^{k-j}}{k-j} \left\{ \binom{-z-1}{j} + \binom{-z-1}{j+1} \right\} + \frac{(-1)^{k+1}}{k+1}
= \sum_{j=0}^{k-1} \frac{(-1)^{k-j}}{k-j} \binom{-z}{j+1} + \frac{(-1)^{k+1}}{k+1}
= \sum_{j=1}^{k} \frac{(-1)^{k-j+1}}{k-j+1} \binom{-z}{j} + \frac{(-1)^{k+1}}{k+1}
= \sum_{j=0}^{k} \frac{(-1)^{k-j+1}}{k-j+1} \binom{-z}{j}.
\]
Thus (4.3) follows.

To see (4.4), note that
\[
\binom{-z}{k} = \frac{(-1)^{k}(z)_k}{k!}
\]
and it then follows directly from the first part. □
Now we are at the position to prove Theorem 3.1. Setting
\[ f(t) = \zeta'(z, q + t) = \frac{\partial}{\partial z} \zeta_E(z, q + t). \]
From (1.5) we immediately have
\[
\Delta(t) = \frac{\partial}{\partial z} [\zeta_E(z, q + t + 1) + \zeta_E(z, q + t)]
\]
\[
= \frac{\partial}{\partial z} (q + t)^{-z}
\]
\[
= \frac{\partial}{\partial z} [e^{-z \log(q + t)}]
\]
\[
= -(q + t)^{-z} \log(q + t)
\]
and setting \( t = 0 \) in (4.7), we find that
\[
\Delta(0) = -q^{-z} \log q.
\]
By (4.7), we have
\[
\Delta'(t) = -(q + t)^{-z-1} + z(q + t)^{-z-1} \log(q + t).
\]
Setting \( t = 0 \) in (4.9), we obtain
\[
\Delta'(0) = -q^{-z-1} + zq^{-z-1} \log q.
\]
Similarly,
\[
\Delta^{(2)}(t) = (z + 1)(q + t)^{-z-2} + z(q + t)^{-z-2}
\]
\[
- z(z + 1)(q + t)^{-z-2} \log(q + t)
\]
and
\[
\Delta^{(3)}(t) = -(z + 1)(z + 2)(q + t)^{-z-3} - z(z + 2)(q + t)^{-z-3}
\]
\[
- z(z + 1)(q + t)^{-z-3} + z(z + 1)(z + 2)(q + t)^{-z-3} \log(q + t).
\]
This suggests the following general formula for positive integer \( k \geq 2 \),
\[
\Delta^{(k)}(t) = (-1)^k \sum_{j=0}^{k-1} \frac{(z)_k}{z + j}(q + t)^{-z-k}
\]
\[
+ (-1)^{k+1}(z)_k(q + t)^{-z-k} \log(q + t).
\]
It is easy to see that
\[
\sum_{j=0}^{k-1} \frac{(z)_k}{z + j} = \frac{d}{dz} [(z)_k],
\]
so that

\[
\Delta^{(k)}(t) = (-1)^k \frac{d}{dz}[(z)_k](q + t)^{-z-k} \\
+ (-1)^k(z)_k(q + t)^{-z-k}\log(q + t)
\]

(4.14)

\[
= (-1)^k k! \sum_{j=0}^{k-1} \frac{(z)_j}{j!(k-j)}(q + t)^{-z-k} \\
+ (-1)^{k+1}(z)_k(q + t)^{-z-k}\log(q + t),
\]

the last equality follows from Lemma 4.1. Setting \( t = 0 \) in (4.14), we get

(4.15) \[ \Delta^{(k)}(0) = (-1)^k k! \sum_{j=0}^{k-1} \frac{(z)_j}{j!(k-j)} q^{-z-k} + (-1)^{k+1}(z)_k q^{-z-k}\log(q). \]

From (4.8), (4.10) and (4.15), Proposition 2.2 immediately implies that

\[
\zeta_E'(z, q) = -\frac{1}{2} q^{-z} \log q + \frac{1}{4} \left( q^{-z-1} - z q^{-z-1} \log q \right) \\
- \frac{1}{2} \sum_{k=2}^{\infty} E_k(0) \sum_{j=0}^{k-1} \frac{(z)_j}{j!(k-j)} q^{-z-k} \\
+ \frac{1}{2} \sum_{k=2}^{\infty} \frac{1}{k!} E_k(0)(z)_k q^{-z-k} \log q.
\]

(4.16)

From Theorem 2.3 we have

(4.17) \[ \sum_{k=2}^{\infty} \frac{1}{k!} E_k(0)(z)_k q^{-z-k} \log q = \left( \frac{1}{2} q^{-z} + \frac{1}{4} q^{-z-1} - \zeta_E(z, q) \right) \log q. \]

Substituting (4.17) into (4.16), we obtain

\[
\zeta_E'(z, q) = \frac{1}{4} q^{-z-1} - \zeta_E(z, q) \log q - \frac{1}{2} \sum_{k=2}^{\infty} E_k(0) \sum_{j=0}^{k-1} \frac{(z)_j}{j!(k-j)} q^{-z-k},
\]

which completes the proof.

**Proof of Theorem 3.3** First, we extend the proof of Theorem 3.1 to obtain an asymptotic expansion representing the higher order derivatives of the alternating Hurwitz (or Hurwitz-type Euler) zeta function in terms of the lower orders. Recall our notation

(4.18) \[ \zeta_E^{(m)}(z, q) \equiv \frac{\partial^m}{\partial z^m} \zeta_E(z, q) \quad (m \geq 2). \]
Now, repeating, for $\zeta''_E(z, q)$, the same procedure just used for $\zeta'_E(z, q)$ in the proof of Theorem 3.1, we arrive at the following equality

$$\zeta''_E(z, q) = -\frac{1}{4}q^{-z-1}\log q - \zeta'_E(z, q)\log q$$

(4.19)

$$+ \frac{1}{2} \sum_{k=2}^{\infty} E_k(0) \sum_{j_1=0}^{k-1} \frac{(z)_{j_1}}{j_1!(k-j_1)} q^{z-k} \log q$$

$$- \frac{1}{2} \sum_{k=2}^{\infty} E_k(0) \sum_{j_1=0}^{k-1} \frac{1}{k-j_1} \sum_{j_2=0}^{j_1-1} \frac{(z)_{j_2}}{j_2!(j_1-j_2)} q^{z-k}.\)$$

In fact, the above identity (4.19) follows directly from Theorem 3.1 and Lemma 4.1. From Theorem 3.1, we have

$$\frac{1}{2} \sum_{k=2}^{\infty} E_k(0) \sum_{j_1=0}^{k-1} \frac{(z)_{j_1}}{j_1!(k-j_1)} q^{z-k} \log q$$

(4.20)

$$= \left( -\frac{1}{4}q^{-z-1} - \zeta_E(z, q)\log q - \zeta'_E(z, q) \right) \log q.\)$$

Substituting (4.20) into (4.19), we get

$$\zeta''_E(z, q) = -2\zeta'_E(z, q)\log q - \zeta_E(z, q)\log^2 q$$

(4.21)

$$- \frac{1}{2} \sum_{k=2}^{\infty} E_k(0) \sum_{j_1=0}^{k-1} \frac{1}{k-j_1} \sum_{j_2=0}^{j_1-1} \frac{(z)_{j_2}}{j_2!(j_1-j_2)} q^{z-k}.\)$$

With some additional effort, the following general recurrence can be derived, which yields an asymptotic expansion for a higher order derivative of the alternating Hurwitz zeta function in terms of the lower orders:

$$\zeta^{(m)}_E(z, q) = -\sum_{j=1}^{m} \binom{m}{j} \zeta^{(m-j)}_E(z, q)\log^j q - \frac{1}{2} \sum_{k=2}^{\infty} c_{k,m}(z) q^{-z-k},$$

where $z \in \mathbb{C}, q \neq 0, -1, -2, \ldots, m \geq 2$, and $c_{k,m}(z)$ being given by (3.8).

The proof of (4.22) proceeds by an induction on $m$. The case $m = 2$ is clear by (4.21). Also it is easily seen that

(4.23)

$$\frac{\partial}{\partial z} c_{k,m}(z) = E_k(0) \left\{ \sum_{j_1=0}^{k-1} \frac{1}{k-j_1} \sum_{j_2=0}^{j_1-1} \frac{1}{j_1-j_2} \cdots \sum_{j_{m-1}=0}^{j_{m-2}} \frac{1}{j_{m-2}-j_{m-1}} \frac{\partial}{\partial z} \left[ (z)_{j_{m-1}} \right] \right\}$$

(recall (3.8))

$$= E_k(0) \left\{ \sum_{j_1=0}^{k-1} \frac{1}{k-j_1} \sum_{j_2=0}^{j_1-1} \frac{1}{j_1-j_2} \cdots \sum_{j_{m+1}=0}^{j_{m}} \frac{1}{j_{m+1}!(j_m-j_{m+1})} (z)_{j_{m+1}} \right\}$$

(by Lemma 4.1)

$$= c_{k,m+1}(z)$$
and
\[ \frac{1}{2} \sum_{k=2}^{\infty} c_{k,m}(z) q^{-z-k} \log q = - \sum_{j=1}^{m} \binom{m}{j} \zeta_{E}^{(m-j)}(z, q) \log^{j+1} q - \zeta_{E}^{(m)}(z, q) \log q, \]

where we used the inductive hypothesis (see (4.22)). By using (4.22), (4.23) and (4.24), we obtain
\[ \zeta_{E}^{(m+1)}(z, q) = \frac{\partial}{\partial z} \zeta_{E}^{(m)}(z, q) \]

(by the inductive hypothesis (4.22))
\[ = - \sum_{j=1}^{m} \binom{m}{j} \zeta_{E}^{(m-j+1)}(z, q) \log^{j} q + \frac{1}{2} \sum_{k=2}^{\infty} c_{k,m}(z) q^{-z-k} \log q \]
\[ - \frac{1}{2} \sum_{k=2}^{\infty} \frac{\partial}{\partial z} \left[ c_{k,m}(z) \right] q^{-z-k} \]
\[ = - \sum_{j=1}^{m} \binom{m}{j} \zeta_{E}^{(m-j+1)}(z, q) \log^{j} q - \sum_{j=1}^{m} \binom{m}{j} \zeta_{E}^{(m-j)}(z, q) \log^{j+1} q \]
\[ - \zeta_{E}^{(m)}(z, q) \log q - \frac{1}{2} \sum_{k=2}^{\infty} \frac{\partial}{\partial z} \left[ c_{k,m}(z) \right] q^{-z-k} \]

(by (4.24))
\[ = - \binom{m+1}{1} \zeta_{E}^{(m)}(z, q) \log q \]
\[ - \sum_{j=1}^{m-1} \left\{ \binom{m}{j+1} + \binom{m}{j} \right\} \zeta_{E}^{(m-j)}(z, q) \log^{j+1} q \]
\[ - \binom{m+1}{m+1} \zeta_{E}(z, q) \log^{m+1} q - \frac{1}{2} \sum_{k=2}^{\infty} c_{k,m+1}(z) q^{-z-k} \]

(by (4.23))
\[ = - \sum_{j=1}^{m+1} \binom{m+1}{j} \zeta_{E}^{(m-j+1)}(z, q) \log^{j} q - \frac{1}{2} \sum_{k=2}^{\infty} c_{k,m+1}(z) q^{-z-k} \]

and this completes the inductive argument.

5. Concluding remarks

In this section, as an example, we shall write the asymptotic series for the second derivative of \( \zeta_{E}(z, q) \) at non-positive values of \( z \) explicitly.
For an integer \( n \in \mathbb{N}_0 \), it follows from Corollary 3.4 with \( m = 2 \) that

\[
\zeta_E''(-n, q) = -2\zeta_E'(-n, q) \log q - \zeta_E(-n, q) \log^2 q
\]

(5.1)

\[
- \frac{1}{2} \sum_{k=2}^{\infty} E_k(0) \sum_{j_1=0}^{k-1} \frac{1}{k-j_1} \sum_{j_2=0}^{\min(n,j_1-1)} \left( \frac{n}{j_2} \right) (-1)^{j_2} j_1 - j_2 q^{n-k}.
\]

Now setting \( n = 0, 1 \) in (5.1) respectively, we have

\[
\zeta_E''(0, q) = \frac{1}{2} \log^2 q - \frac{1}{2} q^{-1} \log q
\]

(5.2)

\[
+ 2 \sum_{k=3}^{\infty} E_k(0) \left[ \frac{\log q}{k} - \frac{1}{2} \sum_{j=1}^{k-1} \frac{1}{j(k-j)} \right] q^{-k},
\]

and

\[
\zeta_E''(-1, q) = \frac{1}{2} q \log^2 q - \frac{1}{2} \left( \log q + \frac{1}{2} \log^2 q \right)
\]

(5.3)

\[
+ \frac{1}{2} \sum_{k=3}^{\infty} E_k(0) \left[ \sum_{j=2}^{k-1} \frac{1}{(k-j)j(j-1)} - \frac{1}{k-1} - \frac{2 \log q}{k(k-1)} \right] q^{-k+1}.
\]

For \( n \geq 2 \), separating the last term in the sum (5.1), we have

\[
\zeta_E''(-n, q) = -2\zeta_E'(-n, q) \log q - \zeta_E(-n, q) \log^2 q
\]

(5.4)

\[
- \frac{1}{2} \sum_{k=2}^{n} E_k(0) \sum_{j_1=0}^{k-1} \frac{1}{k-j_1} \sum_{j_2=0}^{j_1-1} \left( \frac{n}{j_2} \right) (-1)^{j_2} j_1 - j_2 q^{n-k}
\]

\[
- \frac{1}{2} \sum_{k=n+1}^{\infty} E_k(0) \sum_{j_1=0}^{k-1} \frac{1}{k-j_1} \sum_{j_2=0}^{\min(n,j_1-1)} \left( \frac{n}{j_2} \right) (-1)^{j_2} j_1 - j_2 q^{n-k}.
\]

Thus, using (3.4) and (3.5), we can rewrite (5.4) as

\[
\zeta_E''(-n, q) = -2\zeta_E'(-n, q) \log q - \zeta_E(-n, q) \log^2 q
\]

(5.5)

\[
- \frac{1}{2} \sum_{k=2}^{n} E_k(0) \sum_{j_1=0}^{k-1} \frac{1}{k-j_1} \sum_{j_2=0}^{j_1-1} \left( \frac{n}{j_2} \right) (-1)^{j_2} j_1 - j_2 q^{n-k}
\]

\[
- \frac{1}{2} \sum_{k=n+1}^{\infty} E_k(0) \sum_{j_1=0}^{n} \frac{1}{k-j_1} \sum_{j_2=0}^{n} \left( \frac{n}{j_2} \right) (-1)^{j_2} j_1 - j_2 q^{n-k}
\]

\[
- \frac{1}{2} \sum_{k=n+1}^{\infty} E_k(0) \sum_{j_1=n+1}^{k-1} \frac{(-1)^n n!}{(k-j_1)(j_1-1) \cdots (j_1-n)} q^{n-k}.
\]
When \( n \geq 2 \), substituting the results of Corollaries 2.6 and 3.2 into (5.5) it is seen that

\[
\zeta''_E(-n, q) = \frac{1}{2} E_n(q) \log^2 q - \frac{1}{2} q^{n-1} \log q \\
+ \sum_{k=2}^n E_k(0) \sum_{j_1=0}^{k-1} \left( \frac{n}{j_1} \right) \left( \frac{(-1)^{j_1} \log q}{k-j_1} \right) - \frac{1}{2(k-j_1)} \sum_{j_2=0}^{j_1-1} \left( \frac{n}{j_2} \right) \left( \frac{(-1)^{j_2}}{j_1-j_2} \right) q^{n-k}
\]

\[
+ \sum_{k=n+1}^\infty E_k(0) \left[ \frac{(-1)^n \log q}{k(k-1)\cdots (k-n)} - \frac{1}{2} \sum_{j_1=0}^n \frac{1}{k-j_1} \sum_{j_2=0}^{j_1-1} \left( \frac{n}{j_2} \right) \left( \frac{(-1)^{j_2}}{j_1-j_2} \right) \right] q^{n-k}.
\]

Below there is a list of the first few results for \( n = 2, 3 \) from (5.6):

\[
\zeta''_E(-2, q) = \frac{1}{2} q^2 \log^2 q - \frac{1}{2} (\log q + \log^2 q) q + \frac{1}{4} \left( \log q - \frac{3}{2} \right) q^{-1}
\]

\[
+ \sum_{k=5}^\infty E_k(0) \left[ \frac{2 \log q}{k(k-1)(k-2)} - \frac{1}{2} \left( \frac{1}{k-1} - \frac{5}{2(k-2)} \right) \right] q^{-k+2}
\]

and

\[
\zeta''_E(-3, q) = \frac{1}{2} q^3 \log^2 q - \frac{1}{2} \left( \frac{3}{2} \log q^2 + \log q \right) q^2
\]

\[
+ \left( \frac{1}{8} \log^2 q + \frac{11}{24} \log q + \frac{1}{4} \right) + \frac{1}{8} \left( \frac{1}{5} \log q + \frac{1}{6} \right) q^{-2}
\]

\[
+ \sum_{k=7}^\infty E_k(0) \left[ \sum_{j=4}^{k-1} \frac{3}{(k-j)j(j-1)(j-2)(j-3)} \right. \left. \frac{6 \log q}{k(k-1)(k-2)(k-3)} - \frac{1}{2} \left( \frac{1}{k-1} - \frac{5}{2(k-2)} + \frac{11}{6(k-3)} \right) \right] q^{-k+3}.
\]

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