Triaxial Bright Solitons in Bose-Condensed Atomic Vapors

Luca Salasnich

*Istituto Nazionale per la Fisica della Materia, Unità di Milano,
Dipartimento di Fisica, Università di Milano,
Via Celoria 16, 20133 Milano, Italy
E-Mail: salasnich@mi.infm.it

Abstract

The properties of triaxial bright solitons (TBSs) made of attractive Bose-Einstein condensed atoms under transverse anisotropic harmonic confinement are investigated by using a variational approach. We show that these metastable TBSs change their shape from cigar-like to disc-like by increasing the inter-atomic strength. Moreover, we find that the collective oscillations of a TBS strongly depend on the anisotropy parameter of the external potential. We calculate in detail the properties of TBS close to the collapse, which sets up at a critical value of the inter-atomic strength. This critical strength is maximal in the isotropic (axially symmetric) case and slightly reduces by increasing the anisotropy. Finally, we investigate the formation of multiple TBSs via modulational instability induced by a sudden change of the scattering length from positive to negative.
Bright solitons (BSs) have been recently obtained in two experiments with Bose-Einstein condensates (BECs) of $^7$Li dilute vapors at ultra-low temperatures [1,2]. In these experiments the attractive BEC is confined in two directions by a transverse cylindrical isotropic harmonic potential and travels in the third direction, the longitudinal cylindrical axis, without a relevant spreading. The theoretical analysis of these BS configurations has been developed by various authors [3-8]. It has been shown that the confinement in two directions is necessary to create single [3-6] or multiple [7,8] metastable BSs, which collapse above a critical number of particles [5,6,8]. In the previous theoretical investigations only an isotropic transverse cylindrical harmonic confinement has been studied. Under these conditions the BS is cigar-shaped, it becomes less cigar-shaped by increasing the inter-atomic strength and it acquires a quasi-spherical shape only very close to the critical inter-atomic strength of the collapse [5,6].

In this paper we consider the case of an attractive BEC under *anisotropic* harmonic confinement. This condensate BS is triaxial and we find that it can be cigar-shaped or disc-shaped by changing the inter-atomic strength. The triaxial bright soliton (TBS) collapses at a critical inter-atomic strength that decreases by increasing the anisotropy parameter of the transverse harmonic potential. By using a variational approach we calculate the collective oscillations of the TBS. We find that close to the collapse the longitudinal collective frequency reaches its maximum value and then quickly goes to zero. The maximum value of the longitudinal collective frequency strongly depends on the anisotropy parameter. We calculate also the number of TBSs which can be obtained, starting with a repulsive and axially uniform BEC, by suddenly changing the scattering length from positive to negative. This mechanism, known as modulational instability [7,8], has been experimentally applied to get a soliton train of axially symmetric BSs [2].

Nowadays BECs are routinely created and trapped in magnetic or optical traps. Such traps can be modeled with good accuracy with harmonic potentials. Here we study an attractive BEC with $N$ atoms confined in the cylindrical transverse direction by an anisotropic harmonic potential $U(\mathbf{r}) = (m/2)(\omega_1^2 x^2 + \omega_2^2 y^2)$, where $m$ is the mass of a Bose-condensed
atom and $\omega_1$ and $\omega_2$ are the two frequencies of the transverse confinement. The ratio
$\lambda = \omega_2/\omega_1$ is the anisotropy parameter of the harmonic potential. Note that there is no
confinement along the $z$ axis.

The scaled Lagrangian density of this attractive BEC is given by
\begin{equation}
\mathcal{L} = i \psi^* \partial_t \psi + \frac{1}{2} \psi^* \nabla^2 \psi - U|\psi|^2 - 2\pi \frac{a_s}{a_H}|\psi|^4, \tag{1}
\end{equation}
where $\psi(\mathbf{r}, t)$ is the macroscopic wave function of the BEC normalized to $N$, $a_s$ is the s-
wave scattering length ($a_s < 0$ with attractive atoms) and $a_H = (\hbar/(m\omega_H))^{1/2}$ is the average
harmonic length of the trapping potential with $\omega_H = (\omega_1\omega_2)^{1/2}$. Note that in Eq. (1) lengths
are in units of $a_H$, energies are in units of $\hbar\omega_H$, and the time is in units of $\omega_H^{-1}$. In the rest
of the paper we use these scaled variables.

The Euler-Lagrange equation obtained from the Lagrangian density $\mathcal{L}$ is the familiar
3D time-dependent Gross-Pitaevskii equation (GPE), which describes very accurately a
pure BEC of Bosonic vapors [9]. By using the 3D GPE one finds that the ground-state of
an attractive BEC is a collapsed state of zero radius and infinite negative energy but, as
previously stressed, metastable states can be obtained, up to a critical number of particles,
under external confinement in two or three directions [5,6]. The stationary metastable states
of an attractive BEC under confinement in two directions are referred as BSs because they
are self-confined in the third direction and can propagate without spreading along that
direction [10]. For $\lambda = 1$ one has axial BSs, whose statical and dynamical properties have
been analyzed in detail in previous papers [5-8]. For $\lambda \neq 1$ one gets TBSs, whose properties
are investigated in this paper.

Instead of numerically solving the full 3D GPE we investigate the properties of the TBS
by using a variational approach [11], which is computationally much less demanding. Our
variational ansatz for the macroscopic wave function of the attractive BEC is the following
\begin{equation}
\psi(\mathbf{r}, t) = \prod_{k=1}^{3} \frac{N^{1/6}}{(\pi \sigma_k(t)^2)^{1/4}} \exp \left\{-\frac{x_k^2}{2\sigma_k(t)^2} + i\beta_k(t)x_k^2\right\} \tag{2}
\end{equation}
with $\mathbf{r} = (x_1, x_2, x_3) = (x, y, z)$. $\sigma_k(t)$ and $\beta_k(t)$ are the time-dependent variational parameters. The $\sigma_k(t)$ are the widths of the TBS in the three axial directions. Note that in order
to describe the time evolution of the variational wave function, the phase factor $i\beta_k(t)x_k^2$ is needed.

By using this Gaussian trial wave function, after spatial integration of the Lagrangian density $L$ of Eq. (1), one finds a new effective Lagrangian $L$ that has $\sigma_k(t)$ and $\beta_k(t)$ as generalized coordinates. The three Euler-Lagrange equations of $L$ with respect to $\sigma_k(t)$ are given by

$$\ddot{\sigma}_1 + \lambda^2_{1}\sigma_1 = \frac{1}{\sigma_1^3} - \sqrt{\frac{2}{\pi}} \frac{\gamma}{\sigma_1^2 \sigma_2 \sigma_3},$$
$$\ddot{\sigma}_2 + \lambda^2_{2}\sigma_2 = \frac{1}{\sigma_2^3} - \sqrt{\frac{2}{\pi}} \frac{\gamma}{\sigma_1 \sigma_2^2 \sigma_3},$$
$$\ddot{\sigma}_3 = \frac{1}{\sigma_3^3} - \sqrt{\frac{2}{\pi}} \frac{\gamma}{\sigma_1 \sigma_2 \sigma_3^2},$$

where $\gamma = N|a_s|/a_H$ is the scaled inter-atomic strength and $\lambda_1 = \omega_1/\omega_H = 1/\sqrt{\lambda}$ and $\lambda_2 = \omega_2/\omega_H = \sqrt{\lambda}$ are the scaled frequencies of the transverse harmonic potential. The three Euler-Lagrange equations of $L$ with respect to $\beta_k(t)$ give instead the following expressions:

$$\beta_k = -\dot{\sigma}_k/(2\sigma_k),$$
with $k = 1, 2, 3$. Thus the time dependence of $\beta_k(t)$ is fully determined by that of $\sigma_k(t)$.

The stationary metastable states of Eqs. (3), namely the TBS static configurations, are found by setting $\dot{\sigma}_k = 0$. The resulting algebraic equations can be easily solved numerically. The solutions correspond to metastable states if they are local minima of the effective potential energy

$$W(\sigma_1, \sigma_2, \sigma_3) = \frac{1}{2} \left( \frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2} + \frac{1}{\sigma_3^2} \right)$$
$$+ \frac{1}{2} \left( \lambda^2_{1}\sigma_1^2 + \lambda^2_{2}\sigma_2^2 \right) - \sqrt{\frac{2}{\pi}} \frac{\gamma}{\sigma_1 \sigma_2 \sigma_3}$$

associated to the Newtonian Eqs. (3). In Figure 1 we plot the three widths $\sigma_1$, $\sigma_2$ and $\sigma_3$ of the metastable TBS as a function of the inter-atomic strength $\gamma$. Note that the transverse widths $\sigma_1$ and $\sigma_2$ do not change very much with respect to the non-interacting values $\sigma_1 = 1/\sqrt{\lambda_1} = \lambda^{1/4}$ and $\sigma_2 = 1/\sqrt{\lambda_2} = 1/\lambda^{1/4}$, while the longitudinal width $\sigma_3$ is divergent for $\gamma = 0$ and approaches one of the two transverse widths for $\gamma$ close to the critical
value \( \gamma_c \). For \( \gamma > \gamma_c \) there are no more metastable solutions: one has the so-called collapse of the attractive BEC.

It is important to observe that, by using the Gaussian variational approach, the critical value \( \gamma_c \) of the collapse is slightly overestimated with respect to the “exact” numerical calculations. For instance, with \( \lambda = 1 \) the “exact” result gives \( \gamma_c = 0.676 \) [12] while our variational method predicts \( \gamma_c = 0.778 \). Moreover, in the case of asymmetric harmonic trap along the three axes, it has been shown that the relative error of \( \gamma_c \) by using the variational approach is always within 10% [12].

Figure 1 shows that for small values of \( \gamma \) the TBS is highly cigar-shaped but near \( \gamma_c \) the TBS becomes disk-shaped due to the enormous compression along the longitudinal \( z \) axis. In particular, close to the collapse with \( \lambda = 1 \) the system is spherical-shaped but choosing a large anisotropy (\( \lambda \gg 1 \) or \( \lambda \ll 1 \)) it is strongly disk-shaped. Note that, due to the symmetry of the problem, one has \( \sigma_1(\lambda) = \sigma_2(\lambda^{-1}) \) and \( \sigma_3(\lambda) = \sigma_3(\lambda^{-1}) \).

The Gaussian approximation of the TBS wave function can be used to study also the collective oscillations the TBS. The diagonalization of the Hessian matrix \( \frac{\partial^2 W}{\partial \sigma_k \partial \sigma_l} \) of the effective potential energy \( W \) of Eq. (4) gives three frequencies \( \Omega_1, \Omega_2 \) and \( \Omega_3 \) of collective excitations around the TBS solution. The \( 3 \times 3 \) Hessian matrix can be numerically diagonalized by choosing, for fixed \( \gamma \) and \( \lambda \), the widths \( \sigma_k \) that satisfy the Eqs. (3) with \( \ddot{\sigma}_k = 0 \).

In Figure 2 we plot such frequencies. Only for \( \gamma = 0 \) the frequencies \( \Omega_1, \Omega_2 \) and \( \Omega_3 \) can be interpreted as collective oscillations along \( x \) axis, \( y \) axis and \( z \) axis, respectively; nevertheless the mixing angle remains quite small also for finite values of \( \gamma \) so they can be associated to the motion along the three axes. The transverse frequencies \( \Omega_1 \) and \( \Omega_2 \) are practically constant with respect to \( \gamma \) and equal to the non-interacting values \( \Omega_1 = 2\lambda_1 = 2/\sqrt{\lambda} \) and \( \Omega_2 = 2\lambda_2 = 2\sqrt{\lambda}; \) only very close to \( \gamma_c \) the two transverse collective frequencies suddenly grows. Instead, the longitudinal frequency \( \Omega_3 \) is zero for \( \gamma = 0 \), it increases with \( \gamma \) but close to \( \gamma_c \) it falls down to zero. Note that, again due to the symmetry of the problem, one has \( \Omega_1(\lambda) = \Omega_2(\lambda^{-1}) \) and \( \Omega_3(\lambda) = \Omega_3(\lambda^{-1}) \). The maximum value \( \Omega_3^{(m)} \) of the frequency \( \Omega_3 \) is
plotted in Figure 3 (dot-dashed line) as a function of $\lambda$. In Figure 3 is also shown (full line) the graph of $\gamma_c(\lambda)$. Both $\Omega_3^{(m)}$ and $\gamma_c$ slowly decreases by increasing the anisotropy (note the logarithmic scale in the $\lambda$ axis of Figure 3).

In a recent experiment [2] a train of bright solitons has been obtained with an initially attractive and axially uniform BEC by a sudden change of the scattering length $a_s$ from positive (repulsive) to negative (attractive), induced by an external magnetic field (Feshbach resonance [13]). The formation of this multi-soliton configuration can be explained as due to the modulational instability of the time-dependent wave function of the BEC, driven by imaginary Bogoliubov excitations [7,8]. An analytical formula for the number $N_s$ of bright solitons generated with a quasi-1D condensate via modulational instability has been derived by Al Khawaja et al. [7] and by Salasnich, Parola and Reatto [8]. The formula has been generalized by Salasnich [8] to the case of a 3D BEC under isotropic transverse confinement. Here we predict that the anisotropy of the transverse harmonic confinement strongly affects the number $N_s$ of bright solitons.

Let us consider a BEC made of $N$ atoms with positive scattering length ($a_s > 0$) trapped by a harmonic potential in the transverse direction ($x, y$) and by a box potential of length $L$ in the longitudinal axial direction ($z$). In this way the stationary BEC is uniform along the $z$ axis (axially uniform) and it can be modeled by the following variational wave function

$$
\psi(r) = n^{1/2} \frac{1}{(\pi \sigma_1 \sigma_2)^{1/2}} \exp \left\{ - \left( \frac{x^2}{2\sigma_1^2} + \frac{y^2}{2\sigma_2^2} \right) \right\},
$$

(5)

where $n = N/L$ is the axial density of the BEC and the two variational parameters $\sigma_1$ and $\sigma_2$ are the width of the BEC along $x$ and $y$ axes. It is straightforward to derive the energy of this BEC from the Lagrangian density of Eq. (1) by setting $\partial_t \psi = 0$, inserting the Eq. (5) into Eq. (1) and integrating the lagrangian density over $x$ and $y$. In this way one finds for the energy the following formula

$$
E = \frac{g n^2}{2\sigma_1 \sigma_2} + \frac{1}{4} \left( \frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2} + \lambda_1^2 \sigma_1^2 + \lambda_2^2 \sigma_2^2 \right) n,
$$

(6)

where $g = 2a_s/a_H$. This energy depends on the two variational parameters $\sigma_1$ and $\sigma_2$. The minimization of the energy $E$ with respect to $\sigma_1$ and $\sigma_2$ gives the equations
\begin{align}
\lambda_1^2 \sigma_1^4 &= 1 + gn \frac{\sigma_1}{\sigma_2}, \\
\lambda_2^2 \sigma_2^4 &= 1 + gn \frac{\sigma_2}{\sigma_1}.
\end{align}

(7)

From these nontrivial equations one numerically finds, for a fixed inter-atomic strength \( g \), the transverse widths \( \sigma_k \) of the Bose gas as a function of the axial density \( n \). Having derived the functions \( \sigma_k(n) \), one can numerically determine the chemical potential \( \mu = \partial E/\partial n \). From the chemical potential \( \mu \) one gets the sound velocity \( c \) in the longitudinal axial direction, which satisfies the equation \( c = (n \partial \mu/\partial n)^{1/2} = (n \partial^2 E/\partial n^2)^{1/2} \). It is important to stress that the energy \( E \), the chemical potential \( \mu \) and the sound velocity \( c \) obtained with Eqs. (6-7) give precisely the 1D GPE results in the weak-coupling 1D limit (\( \sigma_1 = \lambda^{1/4} \) and \( \sigma_2 = 1/\lambda^{1/4} \)) while they have the correct (Thomas-Fermi) 3D GPE behaviour in the strong-coupling 3D limit (\( \sigma_1^4 = \lambda^2 gn \) and \( \sigma_2^2 = gn/\lambda^2 \)) (see also [14] for the \( \lambda = 1 \) case).

The sound velocity is useful to calculate the Bogoliubov excitations \( \epsilon_k = [(k^2/2)(k^2/2 + 2c^2)]^{1/2} \) of the axially uniform BEC. By suddenly changing the scattering length \( a_s \) to a negative value, the excitations frequencies corresponding to \( k < k_c = 2|c| \) become imaginary and, as a result, small perturbations grow exponentially in time. The maximum rate of growth is at \( k_0 = k_c/\sqrt{2} \) and the wavelength of this mode is \( \lambda_0 = 2\pi/k_0 \). The ratio \( L/\lambda_0 \) gives an estimate of the number \( N_s \) of TBSs which are generated: \( N_s = |c| L/(\sqrt{2}\pi) \), where \( |c| \) is now a function of \( n|g| = 2\gamma/L \) with \( \gamma = N|a_s|/a_H \). Note that the criterion of modulational instability we use has been recently re-derived with a sophisticated time-dependent analysis [15].

In Figure 4 we plot \( N_s/L \) as a function of \( \gamma/L \) for different values of the anisotropy parameter \( \lambda \). For small values of \( \gamma/L \) we are in the weak-coupling quasi-1D regime but approaching \( \gamma/L = 1/2 \) our procedure gives a divergent number of solitons because one of the two transverse widths shrinks to zero for a large attractive inter-atomic interaction. Figure 4 shows that the number \( N_s \) of TBSs generated via modulational instability increases with the anisotropy parameter \( \lambda \). Moreover, as expected, we find \( N_s(\lambda) = N_s(\lambda^{-1}) \).

In conclusion, we have analyzed metastable states (triaxial bright solitons) and collective oscillations of an attractive Bose condensate under anisotropic harmonic confinement.
We have predicted for the triaxial bright soliton a transition from a cigar-like shape to a disk-like shape by increasing the inter-atomic strength, namely the number of particles or the (negative) scattering length, for instance by using Feshbach resonances. We have also shown that the number $N_s$ of triaxial bright solitons generated with a sudden change of the scattering length from positive to negative grows with the anisotropy of the transverse confinement. Moreover $N_s$ diverges at a critical inter-atomic strength, where one of the two transverse widths of the soliton train shrinks to zero.
REFERENCES

[1] L. Khaykovich et al., Science 296, 1290 (2002).

[2] K.E. Strecker et al., Nature 417, 150 (2002).

[3] W.P. Reinhardt and C.W. Clark, J. Phys. B 30, L785 (1997); L.D. Carr, C.W. Clark, and W.P. Reinhardt, Phys. Rev. A 62, 063611 (2000).

[4] Th. Busch and J.R. Anglin, Phys. Rev. Lett. 87 010401 (2001).

[5] V.M. Perez-Garcia, H. Michinel, and H. Herrero, Phys. Rev. A 57, 3837 (1998).

[6] L. Salasnich, A. Parola, and L. Reatto, Phys. Rev. A 65, 043614 (2002); L. Salasnich, Laser Phys. 12, 198 (2002); L. Salasnich, A. Parola, and L. Reatto, Phys. Rev. A 66, 043603 (2002).

[7] U. Al Khawaja, H.T.C. Stoof, R.G. Hulet, K.E. Strecker, and G.B. Partridge, Phys. Rev. Lett. 89, 200404 (2002).

[8] L. Salasnich, A. Parola, and L. Reatto, Phys. Rev. Lett. 91, 080405 (2003); L. Salasnich, Laser Phys. 13, 543 (2003).

[9] C.J. Pethick and H. Smith, Bose Einstein Condensation in Dilute Gases (Cambridge Univ. Press, Cambridge, 2001).

[10] Strictly speaking a bright soliton requires additional properties, like collisional stability. For details see P.G. Drazin and R.S. Johnson, Solitons: An Introduction (Cambridge University Press, Cambridge, 1988).

[11] V.M. Perez-Garcia et al., Phys. Rev. Lett. 77, 5320 (1996); L. Salasnich, Int. J. Mod. Phys. B 14, 1 (2000).

[12] A. Gammal, L. Tomio, and T. Federico, Phys. Rev. A 66, 043619 (2002).

[13] S. Inouye et al., Nature 392, 151 (1998); J. Stenger et al., Phys. Rev. Lett. 82, 2422
(1999).

[14] L. Salasnich, A. Parola, and L. Reatto, Phys. Rev. A 69, 045601 (2004); L. Salasnich, A. Parola, and L. Reatto, Phys. Rev. A 70, 013606 (2004).

[15] Z. Rapti, P.G. Kevrekidis, A. Smerzi, A.R. Bishop, Phys. Rev. E 69, 017601 (2004).
FIG. 1: Widths $\sigma_k$ of the TBS as a function of the inter-atomic strength $\gamma = N|a_s|/a_H$, where $\lambda = \omega_2/\omega_1 = \lambda_2/\lambda_1$ and $a_H = (\hbar/(ma_0\omega_H)^{1/2}$ with $\omega_H = (\omega_1\omega_2)^{1/2}$. Dotted line: $\sigma_1$, dashed line: $\sigma_2$, solid line: $\sigma_3$. Lengths are in units of the harmonic length $a_H$ of the external transverse potential.
FIG. 2: Collective frequencies $\Omega_k$ of the TBS as a function of the inter-atomic strength $\gamma = N|a_s|/a_H$ for some values of the anisotropy parameter $\lambda = \omega_2/\omega_1 = \lambda_2/\lambda_1$. Frequencies are in units of the harmonic frequency $\omega_H = (\omega_1\omega_2)^{1/2}$ of the external transverse potential.
FIG. 3: Properties of the TBS at the collapse as a function of the anisotropy parameter \( \lambda = \omega_2/\omega_1 = \lambda_2/\lambda_1 \). The solid line is the critical inter-atomic strength \( \gamma_c \). The dot-dashed line is the maximum value \( \Omega_3^{(m)} \) of the collective frequency \( \Omega_3 \) along the longitudinal \( z \) axis. Units as in Fig. 2.
FIG. 4: Number $N_s$ of bright solitons generated via modulational instability as a function of the inter-atomic strength $\gamma = N|a_s|/a_H$. $L$ is the initial axial length of the BEC with $N$ atoms. $\lambda = \omega_2/\omega_1 = \lambda_2/\lambda_1$ is the anisotropy parameter of the transverse harmonic confinement. Units as in Fig. 1.