The gradient flow at higher orders in perturbation theory

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Various results for higher-order perturbative calculations in the gradient-flow formalism are reviewed, including the gradient-flow beta function and the small-flow-time expansion of the hadronic vacuum polarization and the energy-momentum tensor. In addition, the strategy of regions is outlined in order to obtain systematic expansions of gradient-flow integrals, for example at large and small flow times.
1. Introduction

Quantum Chromodynamics (QCD) is a theory with many different facets. So far, quantitative phenomenological results have been obtained mostly in either the strong-coupling regime using lattice regularizations or in the weak-coupling regime where perturbation theory is applicable. Both calculational approaches are highly evolved in themselves. Cross-fertilization is often hindered by the inherently different treatment of ultraviolet divergences in these two calculational approaches.

The gradient-flow formalism (GFF) may provide an excellent opportunity to change this situation. It represents a UV regularization scheme which can be implemented both on the lattice and in perturbation theory. This contribution reviews a number of concrete examples where such a cross-fertilization could be achieved, and where the perturbative calculations have been performed beyond next-to-leading order in perturbation theory. Furthermore, the application of the strategy of regions to gradient-flow integrals is presented, which allows to obtain systematic expansions in dimensionless parameters.

2. The perturbative gradient flow

In the GFF, one defines flowed fields $B_a^\mu = B_a^\mu(t)$ and $\chi = \chi(t)$ as solutions of the equations \[1, 2\] (see also \[3, 4\])

\[
\begin{align*}
\partial_t B_a^\mu &= D_{\mu}^{ab} G_{\nu}^{ab} + \kappa D_{\mu}^{ab} \partial_{\nu} B_{\nu}^{b}, \\
\partial_t \chi &= \Delta \chi - \kappa \partial_{\mu} B_{\mu}^{aT} a \chi, \\
\partial_t \bar{\chi} &= \bar{\chi} \Delta + \kappa \bar{\chi} \partial_{\mu} B_{\mu}^{aT} a. 
\end{align*}
\]

The initial conditions supplementing these differential equations establish the contact to regular QCD:

\[
B_a^\mu(t = 0) = A_a^\mu, \quad \chi(t = 0) = \psi, 
\]

where $A_a^\mu$ and $\psi$ are the regular gluon and quark fields, respectively, and

\[
\begin{align*}
D_{\mu}^{ab} &= \delta^{ab} \partial_{\mu} - f^{abc} B_{\mu}^{c}, \\
\Delta &= (\partial_{\mu} + B_{\mu}^{aT}) (\partial_{\mu} + B_{\mu}^{bT}), \\
G_{\mu\nu}^{a} &= \partial_{\mu} B_{\nu}^{a} - \partial_{\nu} B_{\mu}^{a} + f^{abc} B_{\mu}^{b} B_{\nu}^{c}. 
\end{align*}
\]

The arbitrary parameter $\kappa$ will be set equal to one in the following.

Our practical implementation of the GFF in perturbation theory follows the strategy developed in Ref.\[5\]. It leads to Feynman rules resembling those of regular QCD, but supplemented by flow-time dependent exponentials in the propagators. In addition, the flow equations are reflected through so-called flow lines which couple to the flowed quarks and gluons via flowed vertices. The latter involve integrations over finite intervals of the flow-time variables.

A systematic method how to handle the corresponding Feynman diagrams and integrals through three-loop level has been introduced in Ref.\[6\]. It is based on \texttt{qgraf}\[7\] for the generation of Feynman diagrams, \texttt{FORM}\[8–10\] for the algebraic manipulation of the resulting amplitudes, \texttt{Kira+FireFly}\[11–15\] for the reduction of the Feynman integrals to master integrals, and \texttt{q2e/exp}\[16\] for interfacing all of these programs. The master integrals can be calculated by following the method outline in Ref.\[17\], for example.
3. Gluon condensate, quark condensates, and gradient-flow beta function

3.1 Gluon condensate and gradient-flow beta function

The first quantity considered at finite flow time was the gluon condensate in massless QCD [1]. Its perturbative expansion reads

\[
\langle G_{\mu\nu}^a(t)G_{\mu\nu}^a(t) \rangle = 3 \frac{\alpha_s(\mu)}{\pi t^2} \left[ 1 + \frac{\alpha_s(\mu)}{4\pi} \left( e_{10} + 4\beta_0 l_{\mu\nu} \right) \right]
+ \left( \frac{\alpha_s(\mu)}{4\pi} \right)^2 \left( e_{20} + 8 \left( e_{10} \beta_0 + 2\beta_1 \right) l_{\mu\nu} + 16\beta_0^2 l_{\mu\nu}^2 \right) + \cdots = \frac{3\bar{\alpha}_s(t)}{\pi t^2},
\]

where \( l_{\mu\nu} = \ln 2\mu^2 t + \gamma_E \) with Euler’s constant \( \gamma_E = 0.577 \ldots \), and \( \alpha_s(\mu) \) is the strong coupling in the \( \overline{\text{MS}} \) scheme which obeys

\[
\mu^2 \frac{d}{d\mu^2} \frac{\alpha_s(\mu)}{\pi} = \beta(\alpha_s(\mu)) , \quad \text{with} \quad \beta(\alpha_s) = -\sum_{n=0}^{\infty} \left( \frac{\alpha_s}{\pi} \right)^{2n} \beta_n. \quad (5)
\]

In QCD, the first three coefficients of the \( \beta \) function in the \( \overline{\text{MS}} \) scheme read

\[
\beta_0 = \frac{11}{4} - \frac{n_f}{6} , \quad \beta_1 = \frac{51}{8} - \frac{19}{24} n_f , \\
\beta_2 = \frac{2857}{192} - \frac{5033}{1152} n_f + \frac{325}{3456} n_f^2 \approx 22.3 - 4.37 n_f + 0.094 n_f^2. \quad (6)
\]

Setting \( t \) to some multiple of \( \mu^2 \), i.e. \( t = \mu^2/\rho \), acting with \( \mu^2 d/d\mu^2 \) on Eq. (4), and iteratively replacing \( \alpha_s \) by \( \bar{\alpha}_s \) according to Eq. (4), one finds the evolution of the gradient-flow coupling:

\[
\mu^2 \frac{d}{d\mu^2} \frac{\bar{\alpha}_s(\mu)}{\pi} = \bar{\beta}(\bar{\alpha}_s(\mu)) , \quad \text{with} \quad \bar{\beta}(\bar{\alpha}_s) = -\sum_{n=0}^{\infty} \left( \frac{\bar{\alpha}_s}{\pi} \right)^{2n} \bar{\beta}_n. \quad (7)
\]

The first two coefficients are universal, i.e. \( \bar{\beta}_0 = \beta_0 \) and \( \bar{\beta}_1 = \beta_1 \), while

\[
\bar{\beta}_2 = \beta_2 - \frac{1}{4} e_{10} \beta_1 + \frac{1}{16} \left( e_{20} - e_{10}^2 \right) \beta_0 = -59.1 - 0.536 n_f + 0.304 n_f^2 - 0.0030 n_f^3. \quad (8)
\]

Note that the \( \rho \)-dependence drops out of the \( \bar{\beta} \) function. The difference to the \( \overline{\text{MS}} \) value of Eq. (6) is remarkable. It is illustrated in dependence of \( n_f \) in Fig. 1 (a). The impact on the QCD \( \beta \) function is shown in Fig. 1 (b). One immediately notices that the perturbative convergence is significantly worse in the gradient-flow scheme than in the \( \overline{\text{MS}} \) scheme (see also Refs. [18–21]). It would be interesting to understand the source of this behavior in order to allow for a precise independent lattice determination of \( \alpha_s(M_Z) \) through the GFF.

3.2 Quark mass effects

So far, we have considered the case of massless quarks. For the gluon condensate, quark mass effects occur only at next-to-leading order (NLO) through the single Feynman diagram shown in Fig. 2 (a). They can be taken into account quite easily by using the well-known one-loop expression
for the two-point function with external gluons. The result has been expressed in terms of a one-dimensional integral [17]. At higher orders in perturbation theory, approximate results of the mass effects could be obtained using the so-called strategy of regions [23]. To illustrate its application in the GFF, let us consider the simpler case of the quark condensate, where mass effects occur already at leading order (LO), see Fig. 2 (b). The exact mass dependence leads to an incomplete $\Gamma$ function in this case [2]:

$$S(t) \equiv \langle \bar{\chi}(t)\chi(t) \rangle = -\frac{3m}{8\pi^2t} f(m^2, 2t),$$

with

$$f(m^2, t) \equiv (4\pi t)^{D/2} t^{-1} \int_k \frac{e^{-t k^2}}{k^2 + m^2} = 1 - m^2 e^{m^2 t} \Gamma(0, m^2 t),$$

$$\int_k \equiv \int_k \frac{d^D k}{(2\pi)^D}, \quad \Gamma(s, x) = \int_x^\infty du u^{s-1} e^{-u},$$

Figure 1: (a) Three-loop coefficient of the $\beta$ function in the $\overline{\text{MS}}$ (solid-red) and the gradient-flow scheme (dashed-blue). (b) $\beta$ function in the $\overline{\text{MS}}$ scheme and the gradient-flow scheme for $n_f = 0$ ($a_s \equiv \alpha_s/\pi$).

Figure 2: Leading-order contributions for the quark mass effects to the gluon and the quark condensate. Diagrams produced with FeynGame [22].
where we have dropped terms that vanish as $\epsilon \to 0$. Assume that we would like to solve the momentum integral in Eq. (10) as an expansion around $m^2 \ll 1/t$. Obviously, simply interchanging the expansion with the integration, which corresponds to assuming $k^2 \gg m^2$ in the integrand, leads to IR-divergent integrals:

$$f^{(i)}(m^2, t) = (4\pi t)^{D/2} t^{-1} \sum_{n=1}^{\infty} \frac{(-m^2)^{n-1}}{n!} \int_k \frac{e^{-tk^2}}{k^{2n}} = \sum_{n=1}^{\infty} \frac{(-m^2)^{n-1}}{n!} \frac{\Gamma(D/2-n)}{\Gamma(D/2)} =$$

$$= 1 + m^2 t \left( \frac{1}{\epsilon} + 1 \right) e^{m^2 t} - (m^2 t)^2 - \frac{3}{4} (m^2 t)^2 + \ldots \;.$$  

(11)

On the other hand, considering the region $k^2 \sim m^2$, it follows that $tk^2 \ll 1$, so we can expand the exponential:

$$f^{(ii)}(m^2, t) = (4\pi t)^{D/2} t^{-1} \sum_{n=0}^{\infty} \frac{(-t)^n}{n!} \int_k \frac{k^{2n}}{k^2 + m^2} = m^2 t \left[ -\frac{1}{\epsilon} - 1 + \gamma_E + \ln m^2 \right] e^{m^2 t}. \;$$

(12)

We recall that, despite the fact that the expansion of the integrand is justified only in the respective region, the momentum integral can be taken over all values of $k$, because all complementary regions will combine to scale-less integrals which are discarded in dimensional regularization. Combining the two regions, the $1/\epsilon$ pole cancels and one finds

$$f(m^2, t) \overset{m^2 \ll 1/t}{\sim} f^{(i)}(m^2, t) + f^{(ii)}(m^2, t) =$$

$$= 1 + m^2 t e^{m^2 t} \left( \ln m^2 t + \gamma_E \right) - (m^2 t)^2 - \frac{3}{4} (m^2 t)^3 + O((m^2 t)^4),$$

(13)

which agrees with the asymptotic expansion of the explicit expression given in Eq. (10).

Now let us consider the opposite case: $m^2 \gg 1/t$. We have again two regions, the first one leading to

$$f(m^2, t) \overset{m^2 \gg k^2}{\sim} \hat{f}^{(i)}(m^2, t) = (4\pi t)^{D/2} t^{-1} \sum_{n=0}^{\infty} \frac{(-1)^n}{m^{2(n+1)}} \int_k k^{2n} e^{-k^2 t} =$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n!} (m^2 t)^{-n}. \;$$

(14)

The second region is given again by $k^2 \sim m^2$, which means that $k^2 t \gg 1$. Its contribution vanishes, because the Taylor series of $e^{-1/x}$ around $x = 0$ is identical to zero. Therefore,

$$f(m^2, t) \overset{m^2 \gg 1/t}{\sim} = \hat{f}^{(ii)}(m^2, t) = \frac{1}{m^2 t} - \frac{2}{(m^2 t)^2} + \frac{6}{(m^2 t)^3} + \ldots \;.$$  

(15)

which again agrees with the Taylor series of Eq. (10) around $1/(m^2 t) = 0$.

Of course, our presentation here is only a sketch of the general idea. At higher orders, one needs to take into account integrations over flow-time parameters. In the small-$t$ limit, all flow-time integration variables are bound to be small as well, and the extension to higher loop order is straightforward. In the large-$t$ limit, however, integration over flow-time parameters extends over “large” and “small” regions, and the expansion of the integrand becomes non-trivial. A general treatment of the strategy of regions for flow-time integrals at higher orders is thus ongoing work and will be presented elsewhere.
4. Hadronic vacuum polarization

Consider the operator product expansion of the correlator of two vector currents $j_\mu(x)$ in $n_f$-flavor QCD with a single massive quark flavor [24]:

$$\Pi_{\mu\nu}(Q) \equiv \int d^4x \, e^{iQx} \langle T j_\mu(x) j_\nu(0) \rangle = (-\delta_{\mu\nu} + Q_\mu Q_\nu / Q^2) \Pi(Q^2)$$

$$\Pi(Q^2) \approx \sum_\infty C^{(0),B}(Q) + m_B^2 C^{(2),B}(Q) + \sum_n C^n_B(Q)(O_n(x = 0)),$$

(16)

This form reflects the fact that, up to mass dimension two, only the trivial operators $\mathbb{1}$ and $m_B^2 \mathbb{1}$ contribute, where $m_B$ is the bare quark mass. At mass dimension four, one has the following set of physical operators (the space-time argument is suppressed in most of what follows):

$$O_1 = \frac{1}{\delta^B} F^a_{\mu\nu} F^a_{\mu\nu}, \quad O_2 = \sum_{q=1}^{n_f} \bar{\psi}_q \gamma_\mu \gamma_\nu \psi_q, \quad O_3 = m_B^4 \mathbb{1},$$

(17)

where

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + f^{abc} A^b_\mu A^c_\nu,$$

$$D_\mu = \partial_\mu - igA_\mu + 2A_B^a T^a.$$

(18)

After renormalization of $m_B$ and the bare coupling $g_B$, the operator matrix elements on the r.h.s. of Eq. (16) are still divergent. The divergences can be absorbed into the bare coefficient functions with the help of operator renormalization:

$$\sum_n C^n_B O_n \equiv \sum_n C_n Z_{nm} O_m \equiv \sum_n C_n O^R_m.$$

(19)

The fact that the classical mass dimension of the operators is the same as that of the Lagrangian allows one to express the renormalization matrix $Z$ in terms of the QCD $\beta$ function, the quark mass anomalous dimension, and the anomalous dimension of the vacuum energy to all orders [25].

The operator product expansion of Eq. (16) represents a factorization into long- and short-distance effects. The former are contained in the matrix elements $\langle O_n \rangle$ and their evaluation requires non-perturbative methods such as lattice QCD. The latter are in the coefficient function whose perturbative expressions are known through next-to-next-to-leading order (NNLO) and beyond (see Refs. [26–28], for example). A precise prediction of $\Pi_{\mu\nu}(Q)$ requires full control of the matching between the two, which is notoriously difficult due to the different regularization and renormalization schemes.

The small-flow-time expansion (SFTX) provides a potential solution to this problem by unifying the renormalization scheme for both the coefficient functions and the operators [5]. The spectrum of possible applications is enormous (see Refs. [29–36], for example). The idea is to define flowed operators $\tilde{O}_n(t)$ by replacing the regular by flowed fields in Eq. (17), and then expressing them in terms of regular operators in the limit $t \to 0$:

$$\tilde{O}_n(t) \approx \zeta_n^{(0),B}(t) + \zeta_n^{(2),B}(t)m_B^2 + \sum_m \zeta_{nm}(t)O_m = \zeta_n^{(0)}(t) + \zeta_n^{(2)}(t)m^2 + \sum_{m,k} \zeta_{nk}(t)O^R_k.$$

(20)

Here, the symbol $\tilde{\cdot}$ denotes that the relation holds only asymptotically for $t \to 0$. The coefficients $\zeta_n^{(0)}(t)$, $\zeta_n^{(2)}(t)$, and $\zeta_{nm}(t)$ are UV finite. They have been calculated in Ref. [33] through NNLO.
QCD. While the $\zeta_{nm}(t)$ depend only logarithmically on $t$, $\zeta_n^{(0)}(t)$ and $\zeta_n^{(2)}(t)$ behave as $1/t^2$ and $1/t$ as $t \to 0$. In fact, they simply correspond to the first two terms in the Taylor expansion of the vevs around $m = 0$:

$$\zeta_n^{(0)}(t) + \zeta_n^{(2)}(t)m^2 = \langle O_n(t) \rangle_{m=0} + m^2 \frac{d}{dm^2}\langle O_n(t) \rangle_{m=0}. \tag{21}$$

Inverting Eq. (20) and inserting it into Eq. (16) leads to

$$\Pi(Q) \sim C^{(0)}(Q) + m^2 C^{(2)}(Q) + \sum_n \tilde{C}_n(t) \tilde{O}_n(t), \tag{22}$$

with

$$\tilde{C}_n(t) = \sum_m C_m \xi^{-1}(t)_{mn}, \quad \tilde{O}_n(t) = \tilde{O}_n(t) - \xi_n^{(0)}(t) - m^2 \xi_n^{(2)}(t). \tag{23}$$

Note that power divergences in the limit $t \to 0$ cancel in the combination $\tilde{O}_n(t)$. A precise lattice determination of the $\langle \tilde{O}(t) \rangle$ could thus open the way towards a novel calculation of the vacuum polarization, and thus independent input for the lattice determination of hadronic contributions to low-energy observables such as the muon anomalous magnetic moment.

5. Energy-momentum tensor

Dropping terms that vanish either under a BRST transformation or by equations of motion, the energy-momentum tensor of QCD takes the form of an operator product expansion similar to Eq. (16):

$$T_{\mu\nu} = \sum_{n=1}^{4} C_n O_{n,\mu\nu}, \quad \tag{24}$$

where

$$O_{1,\mu\nu} = \frac{1}{g_B^2} G_{\mu\rho} G_{\nu\sigma}, \quad O_{2,\mu\nu} = \frac{\delta_{\mu\nu}}{g_B^2} G_{\rho\sigma} G_{\rho\sigma}, \quad O_3 = \bar{\psi} \left( \gamma_\mu \overleftrightarrow{D}_\nu + \gamma_\nu \overleftrightarrow{D}_\mu \right) \psi, \quad O_4 = \delta_{\mu\nu} m_B \bar{\psi} \psi. \tag{25}$$

However, as opposed to Eq. (16), the “Wilson coefficients” $C_n$ in this case are given by simple numerical constants to all orders in perturbation theory:

$$C_1 = 1, \quad C_2 = -\frac{1}{4}, \quad C_3 = \frac{1}{4}, \quad C_4 = 0. \tag{26}$$

Furthermore, due to the Ward-Takahashi identities among the $Z_{nm}$, the energy-momentum tensor is finite, in the sense that the coefficients $C_n$ are not renormalized, i.e. $C_n = C_n^B$.

Using the SFTX, we write the operators as

$$O_n = \sum_{m=1}^{4} \xi_n^{B,-1}(t)_{nm} \tilde{O}_m(t) \tag{27}$$

and insert this into Eq. (24) to obtain [29, 30]

$$T_{\mu\nu} = \sum_{n=1}^{4} c_n(t) \tilde{O}_{n,\mu\nu}, \quad c_n(t) = \sum_{m=1}^{4} C_m \xi_n^{B,-1}(t) = \sum_{m=1}^{4} C_m \xi_n^{-1}(t). \tag{28}$$

The coefficients $c_n(t)$ are finite even without operator renormalization. They have been evaluated through NNLO [29, 30, 37] and used to study thermodynamics of QCD [32, 38–41].
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