Geodesic behaviour around cosmological milestones

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Abstract. In this paper we provide a thorough classification of Friedman-Lemaître-Robertson-Walker (FLRW) cosmological models in terms of the strong or weak character of their singularities according to the usual definitions. The classification refers to a generalised Puiseux power expansion of the scale factor of the model around a singular event.

1. Introduction
In the last few years the number of discussed matter contents for the different ages of our universe has increased. Besides the original dust, radiation and cosmological constant terms, researchers have come to include other contributions to the right-hand side of Einstein equations, being quintessence and phantom energy some of the most usual. The consideration of these new terms has been motivated by the attempts of explaining the experimentally inferred accelerated expansion of the universe [1].

The inclusion of these new matter contents, which do not satisfy some of the classical energy conditions, has as a consequence the appearance of new types of singularities which did not come up in former models, such as big rip [2] and sudden singularities [3], and other non-singular features, such as bounces or extremality events. All of these have been comprised under the name “cosmological milestones” by Cattoën and Visser [4] together with the classical big bag and big crunch singularities.

In that paper the authors perform a classification of FLRW cosmological models in terms of the first coefficients of a generalised Puiseux expansion of the scale factor in time coordinate around one of these cosmological milestones. These coefficients are used to determine violations of energy conditions and appearance of polynomial scalar curvature singularities and derivative curvature singularities. Generically the classification depends on just the first three exponents of the Puiseux expansion.

What we would like to do now is to complete the classification by considering other definitions of singularities. Geodesic incompleteness is commonly accepted as an indicator of the existence of a singularity in a space-time [5] and may happen even in cases where there is no polynomial scalar curvature singularity.

Furthermore, it has been shown that even in the cases where a causal geodesic is incomplete, this does not mean that finite objects are necessarily crushed on approaching the singularity. These are considered weak singularities. In our classification we take into account this fact with the most common definitions of strong singularities.
This involves calculation of causal geodesics in the corresponding space-times. This topic is reviewed in section 2. In section 3 geodesic equations are solved for the power expansion of the scale factor and the differentiability of the geodesics is analysed. Finally, in section 4 the strength of the singularities, if any, is discussed in relation to the values of the exponents of the power expansion. The conclusions are summarized in section 5. More details about this issue may be found in [6].

2. Geodesic equations

Generally calculation of geodesics in a space-time is a cumbersome task, since it requires solving a system of four ordinary quasilinear differential equations. Geodesics are parametrised by their proper time,

\[
\frac{d\tau}{2} = -g_{ij}dx^i dx^j, \tag{1}
\]

where \(g_{ij}\) are the components of the metric tensor of the space-time in the coordinate chart provided by \(\{x_0, x_1, x_2, x_3\}\). Proper time is defined up to a change of scale and origin, \(\tilde{\tau} = a\tau + b\), and hence it is also called affine parameter. We denote by a dot derivatives with respect to proper time.

A geodesic is said to be complete if it can defined for all values of \(\tau\). On the contrary, it is said to be incomplete in the past (future) if it can be extended just to a value \(\tau_0\) instead of \(-\infty\) (\(\infty\)).

Geodesics are defined as the curves \(\Gamma\) on the space-time for which the length functional,

\[
L[\Gamma] = \int_{s_0}^{s_1} ds, \quad ds^2 = -d\tau^2 = g_{ij}dx^i dx^j, \tag{2}
\]

has a extremum.

The corresponding Euler-Lagrange equations,

\[
\ddot{x}^i + \Gamma^i_{jk}\dot{x}^j \dot{x}^k = 0, \tag{3}
\]

may be written in terms of the Christoffel symbols for the metric tensor,

\[
\Gamma^i_{jk} = \frac{1}{2}g^{il} \left\{ g_{lj,k} + g_{lk,j} - g_{jk,l} \right\}, \tag{4}
\]

plus an additional equation,

\[
\delta = -g_{ij}\ddot{x}^i \dot{x}^j, \tag{5}
\]

which simply states that we are using proper time as parameter. The constant \(\delta\) takes value one for timelike, zero for lightlike and minus one for spacelike geodesics.

We use spherical coordinates, \(\{t, r, \theta, \phi\}\), with the usual ranges and \(t\) is coordinate time. Geodesics are therefore described providing \((t(\tau), r(\tau), \theta(\tau), \phi(\tau))\).

In the case of FLRW cosmological models,

\[
ds^2 = -dt^2 + a^2(t) \left\{ f^2(r)dr^2 + r^2 \left( d\theta^2 + \sin^2 \theta d\phi^2 \right) \right\}
\]

\[
f^2(r) = \frac{1}{1-kr^2}, \quad k = 0, \pm 1, \tag{6}
\]

the large number of isometries allows us a quick integration of geodesic equations.

Since the space-time is homogeneous and isotropic, geodesics are straight lines and hence we may restrict the discussion to a plane \(\theta = \pi/2, \phi = \text{const}\), choosing as origin of coordinates one of the points of the geodesic.
Furthermore, changing the radial coordinate to
\[
R = \begin{cases} 
\arcsinh r & k = -1 \\
r & k = 0 \\
\arcsin r & k = 1 
\end{cases}, \quad ds^2 = -dt^2 + a^2(t) \left\{ dR^2 + \frac{\sinh^2 R}{R^2} \left( d\theta^2 + \sin^2 \theta d\phi^2 \right) \right\}, \tag{7}
\]
it is easy to check that \( \partial_R = \partial_r / f(r) \) is another generator of isometries and therefore
\[
P = a^2(t) f(r) \dot{r} \tag{8}
\]
is conserved along geodesics.

We are left then with just one equation,
\[
\dot{t}^2 = \delta + \frac{P^2}{a^2(t)} \tag{9}
\]
and the quadrature
\[
\dot{r} = \frac{P}{a^2(t) f(r)}, \tag{10}
\]
which may be integrated after solving the equation in \( t \) and therefore need not be considered here.

At this point it is clear that all information about the geodesics is encoded in the scale factor \( a(t) \).

We consider just future-pointing geodesics, \( \dot{t} > 0 \).

Without losing much generality, we assume that the expansion,
\[
a(t) = c_0 |t - t_0|^{\eta_0} + c_1 |t - t_1|^{\eta_1} + \cdots, \tag{11}
\]
where the exponents \( \eta_i \) are real and ordered,
\[
\eta_0 < \eta_1 < \cdots
\]
is valid close to a cosmological milestone at \( t_0 \). The coefficient \( c_0 \) must be positive in order to have a positive scale factor.

We consider just singularities in the past, \( t > t_0 \), in order to avoid signs and absolute values. Since equations are time-reversal symmetric, no information is lost with this restriction.

At lowest order, \( \eta_0 \), in the flat universe case, \( k = 0 \), the model behaves like a perfect fluid of density \( \rho \) and pressure \( p \) with a linear equation of state,
\[
p = w \rho, \quad w = -1 + \frac{2}{3} \eta_0. \tag{12}
\]

Also at lowest order, three different behaviours are possible for the scale factor at \( t_0 \): zero, finite and divergent:
- \( \eta_0 > 0 \): the scale factor vanishes at \( t_0 \) and generically we have a big bang or big crunch singularity.
- \( \eta_0 = 0 \): the scale factor is finite at \( t_0 \). If \( a(t) \) is analytical, the event at \( t_0 \) is regular. Otherwise a sudden singularity comes up \([3, 7]\).
- \( \eta_0 < 0 \): the scale factor diverges at \( t_0 \) and a big rip singularity comes up.

Since completeness of just causal geodesics is required for the analysis of singularities, we focus only on lightlike and timelike geodesics.
Table 1. Derivatives of lightlike geodesics at $t_0$.

| $\eta_0$ | $\eta_1$ | $i$ | $\dot{i}$ | $\ddot{i}$ | $t^n$ |
|----------|----------|-----|---------|---------|-----|
| $(0, \infty)$ | $(\eta_0, \infty)$ | $\infty$ | $\infty$ | $\infty$ | $\infty$ |
| 0 | $(0, 1)$ | finite | $\infty$ | $\infty$ | $\infty$ |
| | $(1, 2)$ | finite | finite | $\infty$ | $\infty$ |
| | $(2, 3)$ | finite | finite | finite | $\infty$ |
| $(-1/2, 0)$ | $(\eta_0, \infty)$ | finite | $\infty$ | $\infty$ | $\infty$ |
| $-1/2$ | $(-1/2, 0)$ | finite | finite | $\infty$ | $\infty$ |
| | $(0, 1/2)$ | finite | finite | finite | $\infty$ |
| $(-2/3, -1/2)$ | $(\eta_0, \infty)$ | finite | finite | $\infty$ | $\infty$ |
| $-2/3$ | $(-2/3, -1/3)$ | finite | finite | finite | $\infty$ |
| $(1-n, 2-n)$ | $(\eta_0, \infty)$ | finite | finite | finite | $\infty$ |
| $(-\infty, -1)$ | $(\eta_0, \infty)$ | / | / | / | / |

3. Geodesic completeness of causal geodesics

Lightlike geodesic equations are straightforwardly integrated,

$$a(t)\dot{i} = P \Rightarrow \int_{t_0}^t a(t') dt' = P(\tau - \tau_0).$$

At lowest order,

$$a(t) \simeq c_0 |t - t_0|^{\eta_0} \Rightarrow t \simeq t_0 + \left\{ \begin{array}{ll}
\left\{ \frac{(1+\eta_0)P}{c_0} \right\}^{1/(1+\eta_0)} (\tau - \tau_0)^{1/(1+\eta_0)} & \eta_0 \neq -1 \\
Ce^{P\tau/c_0} & \eta_0 = -1
\end{array} \right..$$

Since generically $t$ behaves as a power $1/(1 + \eta_0)$ of proper time, different levels of regularity appear depending on the value of $\eta_0$.

It is worth mentioning that for $\eta_0$ lower or equal than minus one, lightlike geodesics do not reach the cosmological milestone at $t_0$, since it would take them an infinite proper time to reach it. They therefore do not see the singularity. This limiting case, which corresponds to a model with $w = -5/3$ when the universe is flat, has been named superphantom and considered in [8].

Results on the differentiability of lightlike geodesics at $\eta_0$ are consigned in table 1.

For the values of $\eta_0$ for which the class of differentiability increases by one, it is necessary to consider further terms of the Puiseux expansion.

We notice that, as $\eta_0$ decreases, the class of differentiability increases.

Timelike geodesics may be analysed similarly, though in this case geodesic equations cannot be solved analytically.

At lowest order,

$$\dot{i} = \sqrt{1 + \frac{P^2}{a^2}} \simeq \sqrt{1 + \frac{P^2}{c_0^2} (t - t_0)^{-2\eta_0}},$$

the geodesic is singular at $t = t_0$ for $\eta_0 > 0$, since $\dot{i}$ blows up.

However, the derivative is well defined for negative (big rip) $\eta_0$. Near $t_0$ we may write

$$\dot{i} \simeq 1 + \frac{P^2}{2c_0^2} (t - t_0)^{-2\eta_0},$$
Table 2. Derivatives of timelike geodesics at $t_0$.

| $\eta_0$ | $\eta_1$ | $i$ | $i'$ | $i''$ | $i^{(n)}$ |
|----------|----------|-----|------|-------|-----------|
| $(0, \infty)$ | $\eta_0, \infty$ | $\infty$ | $\infty$ | $\infty$ | $\infty$ |
| 0 | $(0, 1)$ | finite | $\infty$ | $\infty$ | $\infty$ |
| | $(1, 2)$ | finite | finite | $\infty$ | $\infty$ |
| | $(2, 3)$ | finite | finite | finite | $\infty$ |
| $(-1/2, 0)$ | $\eta_0, \infty$ | finite | $\infty$ | $\infty$ | $\infty$ |
| $-1/2$ | $(-1/2, 1/2)$ | finite | finite | $\infty$ | $\infty$ |
| | $(1/2, 3/2)$ | finite | finite | finite | $\infty$ |
| $(-1, -1/2)$ | $\eta_0, \infty$ | finite | finite | $\infty$ | $\infty$ |
| $-1$ | $(-1, 0)$ | finite | finite | finite | $\infty$ |
| $(1-n^2, 2-n^2)$ | $\eta_0, \infty$ | finite | finite | finite | $\infty$ |

and similar expressions for higher derivatives,

$$t^{(n)} \sim (t - t_0)^{-2\eta_0 - n + 1}.$$ (17)

Again, as it is shown in table 2, the class of differentiability of timelike geodesics increases as $\eta_0$ decreases. However, every timelike geodesic reaches the cosmological milestone at $t_0$ and there are no curves with all finite derivatives there.

4. Strength of singularities

In the previous section we have shown that qualitatively the strength of singularities at $t_0$ decreases with $\eta_0$, since the class of differentiability increases. We proceed now to check this qualitative statement with the most usual definitions of strong singularities.

Roughly speaking, a singularity is considered strong if tidal forces are capable of disrupting a finite object falling into it [9].

This concept has been developed in several ways. For Tipler [10] the finite volume is spanned by three Jacobi fields that form an orthonormal basis with the velocity $u$ of the geodesic. The singularity if strong if the volume tends to zero on approaching the singularity.

Another definition is due to Krółak [11], for which the singularity is strong if the derivative of the volume is negative close to the singularity. Obviously, if a singularity is strong according to Tipler’s definition, it is strong according to Krółak’s, but not conversely.

These definitions are meant to be used for gravitational collapse and therefore do not consider the possibility of big rip singularities. But these may be included in the framework just reversing a sign.

Both definitions have been written in an amenable form by Clarke and Krółak [12] in terms of integrals of Riemann components along the geodesics.

In our case the situation is even much simpler since the space-time is conformally flat and the Weyl tensor vanishes.

For instance, a lightlike geodesic meets a strong singularity, according to Tipler’s definition, at proper time $\tau_0$ if and only if

$$\int_0^\tau d\tau' \int_0^{\tau'} d\tau'' R_{ij} u^i u^j$$ (18)

diverges as $\tau$ tends to $\tau_0$.
Table 3. Degree of singularity of null geodesics around $t_0$.

| $\eta_0$          | $\eta_1$          | $k$   | $c_0$       | Tipler    | Królik    |
|-------------------|-------------------|-------|-------------|-----------|-----------|
| $(-\infty, -1]$   | $(\eta_0, \infty)$| $0, \pm 1$ | $(0, \infty)$ | Regular  | Regular   |
| $(-1, 0)$         |                   |       |             | Strong    | Strong    |
| $0$               | $(0, 1)$          |       |             | Weak      | Strong    |
|                   | $(\eta_0, \infty)$|       |             | Weak      | Weak      |
| $(0, 1)$          |                   | $0, 1$ |             | Strong    | Strong    |
| $(1, \infty)$     | $-1$              |       | $(0, 1) \cup (1, \infty)$ | Strong    | Strong    |
| $(1, 3)$          |                   | $1$   |             | Weak      | Strong    |
| $(3, \infty)$     | $(\eta_0, \infty)$|       |             | Weak      | Weak      |
| $(1, \infty)$     |                   | $0, \pm 1$ | $(0, \infty)$ | Strong    | Strong    |

And a lightlike geodesic meets a strong singularity at proper time $\tau_0$ if and only if

$$\int_0^\tau d\tau' R_{ij} u^i u^j$$

diverges as $\tau$ tends to $\tau_0$.

In our case, the velocity of the geodesic is

$$R_{ij} u^i u^j = 2P^2 \left( \frac{a^2 + k}{a^2} - \frac{a''}{a^3} \right) \simeq \frac{2P^2 \eta_0}{c_0^2 |t - t_0|^{2(\eta_0 + 1)}} + \frac{2kP^2}{c_0^4 |t - t_0|^4\eta_0} + \cdots,$$

and at lowest order there are two cases, depending on whether the curvature term dominates over the first term.

The results according to both definitions are summarized in table 3.

As we see, besides the models with $\eta_0 \leq -1$, for which no lightlike geodesic reaches the cosmological milestone at $t_0$, there are only two cases without strong singularities: some of the cases with $\eta_0 = 0$, which are named sudden singularities, some of which had already been studied in [13]; and some of the cases with $\eta_0 = 1, c_0 = 1, k = -1$, which are at first order Milne universe, which is Minkowski empty space in other coordinates.

Again, there are limiting cases which require resorting to further terms in the expansion.

As it was pointed out, it is explicitly checked that Królik’s definition includes more cases than Tipler’s.

The analysis of the strength of singularities of timelike geodesics is somewhat more involved, since there are no both necessary and sufficient conditions for the appearance of strong singularities:

According to Tipler’s definition, a timelike geodesic meets a strong singularity at proper time $\tau_0$ if

$$\int_0^\tau d\tau' \int_0^\tau d\tau'' R_{ij} u^i u^j$$

diverges as $\tau$ tends to $\tau_0$.

With Królik’s definition, a timelike geodesic meets a strong singularity at proper time $\tau_0$ if

$$\int_0^\tau d\tau' R_{ij} u^i u^j$$

and at lowest order there are two cases, depending on whether the curvature term dominates over the first term.
Table 4. Degree of singularity of the fluid congruence of timelike geodesics around $t_0$.

| $\eta_0$ | $\eta_1$ | Tipler | Królik |
|----------|----------|--------|--------|
| $(-\infty, 0)$ | $(\eta_0, \infty)$ | Strong | Strong |
| 0 | (0, 1) | Weak | Strong |
| | [1, $\infty$) | Weak | Complete |
| (0, 1) | $(\eta_0, \infty)$ | Strong | Strong |
| 1 | (1, 2] | Weak | Strong |
| | (2, $\infty$) | Weak | Weak |
| (1, $\infty$) | $(\eta_0, \infty)$ | Strong | Strong |

diverges as $\tau$ tends to $\tau_0$.

Necessary conditions are slightly different. With Tipler’s definition [12], if a timelike geodesic meets a strong singularity, then

$$I_j^i(\tau) = \int_0^\tau d\tau' \int_0^{\tau'} d\tau'' |R_{kjl}^iu^k u^l|,$$

(23)
diverges as $\tau$ tends to $\tau_0$ for some $i, j$, where the components are referred to a parallely transported orthonormal frame.

With Królik’s definition, if a timelike geodesic meets a strong singularity, then

$$I_j^i(\tau) = \int_0^\tau d\tau' |R_{kjl}^iu^k u^l|,$$

(24)
diverges as $\tau$ tends to $\tau_0$ for some $i, j$.

Fortunately, this set of sufficient and necessary conditions is accurate enough to allow a thorough classification of the singularities of timelike geodesics in FLRW models.

There are two sets of timelike geodesics with different behaviour:

Timelike geodesics with $P = 0$, for which the time coordinate is essentially proper time,

$$t - t_0 = \tau - \tau_0, \quad r = r_0,$$

(25)
form the congruence of fluid worldlines, since the coordinates are comoving for the perfect fluid, $u = \partial_t$, and do not suffer therefore any problems of differentiability. On applying the conditions for the appearance of strong singularities, we reach the results of table 4.

Along timelike geodesics with radial velocity, $P \neq 0$, strong singularities appear in more cases, as we show in the results comprised in table 5.

As we see, the results are essentially the same as for lightlike geodesics, with a difference of behaviour for models with coefficient $\eta_0$ lower or equal than minus one. Timelike geodesics do reach the cosmological milestone at $t_0$ in the form of a strong singularity. These models are lightlike geodesically complete, though they are timelike geodesically incomplete. In fact, table 5 provides the classification of FLRW cosmological models according to the strength of their singularities.

5. Conclusions

In this paper we have provided a complete classification of FLRW cosmological models according to the strength of their singularities in terms of a generalised Puiseux expansion of the scale factor in coordinate time around cosmological milestones.
Table 5. Degree of singularity of timelike geodesics around $t_0$.

| $\eta_0$ | $\eta_1$ | $k$ | $c_0$ | Tipler | Królak |
|----------|----------|-----|------|-------|--------|
| $(-\infty, 0)$ | $(\eta_0, \infty)$ | 0, $\pm 1$ | $(0, \infty)$ | Strong | Strong |
| 0 | $(0, 1)$ | | | Weak | Strong |
| 1 | $(1, \infty)$ | | | Weak | Weak |
| $(0, 1)$ | $(\eta_0, \infty)$ | | | Strong | Strong |
| 1 | $(1, \infty)$ | 0, 1 | | Strong | Strong |
| | $(1, \infty)$ | $-1$ | $(0, 1) \cup (1, \infty)$ | Strong | Strong |
| | $(1, 3)$ | 1 | | Weak | Strong |
| | $(3, \infty)$ | | | Weak | Weak |
| $(1, \infty)$ | $(\eta_0, \infty)$ | 0, $\pm 1$ | $(0, \infty)$ | Strong | Strong |

Though the velocity of the geodesic is finite at big rips, this does not prevent the appearance of strong singularities, except for two groups of models: those which behave at lowest order as Milne universe and those with sudden singularities. However, lightlike geodesics do not reach the big rip singularities for exponents $\eta_0$ lower or equal than minus one.

The results of Cattoën and Visser in [4] are more restrictive, since they do not deal with the strength of the curvature singularities. For them models with $\eta_0 = 0$, $\eta_1 \geq 2$ or $\eta_1 = 1$, $\eta_2 \geq 2$, and those with $\eta_0 = 1$, $k = -1$, $c_0 = 1$, $\eta_1 \geq 3$ are free of polynomial scalar curvature singularities. These results coincide with ours for Milne-like models with Królak’s definition for strong singularities, but include more models in the case of sudden singularities.

Their results for derivative curvature singularities are even more restrictive. The only models which are free of such singularities are those with $\eta_0 = 0$ and natural exponents $\eta_i$, $i \geq 1$ and those with $\eta_0 = 1$, $k = -1$, $c_0 = 1$ and natural exponents $\eta_i \geq 3$, $i \geq 1$.

This is not surprising, since derivatives of the curvature tensor do not appear in our equations.

However, the main difference between both results arises from the fact that curvature singularities do not see that lightlike geodesics do not reach the singularity in finite proper time.

Though this classification of singularities conveys the idea of ubiquitous singularities in FLRW cosmological models, it is worthwhile mentioning that singularities mostly appear in models with vanishing, divergent or non-smooth scale factors.

A similar scenario appeared in inhomogeneous scalar field Abelian diagonal $G_2$ models [14], where singularity-free cosmological models formed an open set.

Acknowledgments

L.F.-J. is supported by the Spanish Ministry of Education and Science Project FIS-2005-05198. R.L. is supported by the University of the Basque Country through research grant UPV00172.310-14456/2002 and by the Spanish Ministry of Education and Culture through the RYC program, and research grants FIS2004-01626 and FIS2005-01181.

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