LÖWNER EVOLUTION AND FINITE DIMENSIONAL REDUCTIONS OF INTEGRABLE SYSTEMS

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Dedicated to Ludwig D. Faddeev on the occasion of his 80th birthday

Abstract. The Löwner equation is known as a one-dimensional reduction of the Benney chain as well as the dispersionless KP hierarchy. We propose a reverse process showing that time splitting in the Löwner or the Löwner-Kufarev equation leads to some known integrable systems.

1. Introduction

One of the central problems in the theory of integrable systems is their finite-dimensional reductions. We start with the dispersionless Kadomtsev–Petviashvili (dKP) hierarchy as an illustration. Let \( \lambda(z,t) \) be a meromorphic function in variable \( z \) and depending on an infinite family of generalized times \( t = (t_0 = x, t_1, \ldots) \) with the expansion

\[
\lambda(z,t) = z + \sum_{n=0}^{\infty} \frac{A^n(t)}{z^{n+1}},
\]

about infinity (here \( A^n \) means index instead of exponent). The Poisson structure \( \{\cdot,\cdot\} \) is defined by

\[
\{F,G\} = \frac{\partial F}{\partial z} \frac{\partial G}{\partial x} - \frac{\partial F}{\partial x} \frac{\partial G}{\partial z}.
\]

Then the dKP hierarchy (see [6]) is an infinite number of commuting flows

\[
\frac{\partial \lambda}{\partial t_n} = \{\mathcal{L}_{n+1}, \lambda\}, \quad n \geq 0,
\]

where \( \mathcal{L}_n = \frac{1}{n} (\lambda^n)_{\geq 0}, n = 1, 2, \ldots \), denotes the polynomial part of \( \lambda^n \). In particular, \( (\lambda)_{\geq 0} = z \), \( (\lambda^2)_{\geq 0} = z^2 + 2A^0 \), \( (\lambda^3)_{\geq 0} = z^3 + 3zA^0 + 3A^1 \), etc. For \( n = 2 \) and \( s = t_1 \),

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the second equation in the hierarchy is equivalent to
\[(2)\quad A^n_x + A^{n+1}_x + nA^{n-1}_x A^0_x = 0,\]
known as the Benney moment equation. Benney [1] investigated long non-linear waves propagating on a free surface showing that the governing equations have an infinite number of conservation laws.

The compatibility condition for (1) is
\[(3)\quad \frac{\partial \mathcal{L}_m}{\partial t_n} - \frac{\partial \mathcal{L}_n}{\partial t_m} + \{\mathcal{L}_m, \mathcal{L}_n\} = 0,\]
which means that the flows (1) commute. In particular, for \(m = 3, n = 2, y = t_2\) and \(u := A^0\) we arrive at the equation
\[(4)\quad u_{ss} + \frac{\partial}{\partial x}(u_y + uu_x) = 0,\]
known as the dKP equation or the Zabolotskaya–Khokhlov equation [23].

A finite-dimensional reduction suggests that the function \(\lambda\) depends on the generalized times \(t\) via a finite number of functions \(u_k = u_k(t)\), i.e.,
\[\lambda(z, t) = \lambda(z, u(t)) = \lambda(z, u_1(t), \ldots, u_N(t)).\]

A well-known polynomial reduction was proposed by Zakharov [24]. Kodama and Gibbons [6] realized that dKP equation possesses infinitely many multi component two dimensional reductions. They presented several examples and the dependence \(\lambda\) was found in these particular cases. They considered the vector-function \(u\) satisfying a system of hydrodynamic type
\[\frac{\partial u}{\partial t_n} = \xi_n(u) \frac{\partial u}{\partial x}, \quad n > 1.\]

Gibbons and Tsarēv [4] were first who noticed that the chordal Löwner equation plays an essential role in the classification of reductions of the Benney equations. If we denote
\[u = u_0 = A^0, \ldots, u_N = A^N, \quad A^n = A^n(u), \quad n > N,\]
then we obtain \(N(N - 1)/2\) compatibility conditions for \(A^{N+1}\) and the function \(z = z(\lambda, t)\) inverse to \(\lambda\) satisfies the equations
\[(5)\quad \partial_k z = -\frac{\partial_k u}{z - \mu_k},\]
where \(\partial_k = \frac{\partial}{\partial \mu_k}\) and \(\mu_k\) are the zeros of the function \(\lambda_z\) and the values \(r^k = \lambda(\mu_k, t)\) are the Riemann invariants. Formally, the equation above is similar to the chordal Löwner equation which we will discuss in what follows. The consistency conditions for (5) are
\[(6)\quad \partial_i \mu_k = \frac{\partial_k u}{\mu_i - \mu_k}, \quad \partial_i \partial_k u = 2\frac{\partial_i u \partial_k u}{(\mu_i - \mu_k)^2},\]
which is the Gibbons–Tsarēv system.
A generalization was suggested by Mañas, Martínez Alonso and Medina [9], where the authors were looking for the function \( \lambda \) and its inverse as a solution to the system of equation

\[
\frac{\partial z}{\partial u_k} = \sum_{k=1}^{N} \frac{\eta_{ik}}{z - \mu_k},
\]

satisfying some compatibility conditions. Formally, again the above equations are of the form of multi-slit chordal Löwner equation.

Later Takebe, Teo, and Zabrodin [18] showed that the chordal and radial Löwner PDE served as consistency conditions for one-variable reductions of dispersionless KP and Toda hierarchies, respectively. In the chordal case, the function \( \lambda \) satisfying \( (1) \) depends on \( t \) via one function \( s(t) \) and

\[
\frac{\partial \lambda}{\partial s} = -k \frac{\partial \lambda}{z - \xi \partial z},
\]

with the compatibility condition of hydrodynamic type

\[
\frac{\partial s}{\partial t_n} = \chi_n \frac{\partial s}{\partial x},
\]

where \( k \) is the \( s \)-derivative of the coefficient at \( 1/z \) in the Laurent expansion of \( \lambda \), the functions \( \chi_n(s) \) are constructed in a canonical way from the Lax function, and again we see the chordal Löwner PDE. These approaches are somewhat in want of analytic background of the Löwner theory.

On the other hand, another evolution process described by Laplacian growth [5] possesses an infinite number of conservation laws, harmonic moments. Being a typical field problem the moments of the Laplacian growth bring us to the dispersionless Toda hierarchy [11]. Unlike the Laplacian growth the Löwner evolution represents another group of models, in which the evolution is governed by the infinite number of parameters, namely the controllable dynamical system, where the infinite number of degrees of freedom follows from the infinite number of driving terms. Surprisingly, the same structural background, the Virasoro algebra, appears again for this group [10].

The idea of this paper is to revisit Gibbons and Tsarëv observation and show that the chordal Löwner evolution also possesses an infinite number of conservation laws, moments. We show that the Löwner PDE is exactly the Vlasov equation under an appropriate change of variables and the Löwner ODE implies the hydrodynamic type conservation equation. We start with the Löwner evolution and splitting time we arrive at integrable chains. This approach shows universality of the Löwner equation as an attraction point for several integrable chains, this was noticed in [12].

2. VLASOV AND LÖWNER EQUATIONS, CONSERVATION LAWS

Let us consider a Löwner chain of receding domains \( \mathbb{H}_t = \mathbb{H} \setminus \gamma_t \) in the upper half-plane \( \mathbb{H} = \{ : \text{Im} z > 0 \} \) and let \( f: \mathbb{H} \to \mathbb{H}_t \) is normalized near infinity as

\[
f(z, t) = z + \frac{A^0}{z} + O \left( \frac{1}{z^2} \right),
\]
where \((-A^0(t))\) is the half-plane capacity of \(\gamma_t\). Let \(\gamma_t\) be a Jordan curve in \(\mathbb{H}\) except for an end point on the real axis \(\mathbb{R}\), \(\gamma_t\) is parameterized by \(t\). Then \(f\) satisfies the L"owner PDE

\[
(z - \xi_t) \frac{\partial f(z, t)}{\partial t} - \frac{dA^0}{dt} \frac{\partial f(z, t)}{\partial z} = 0,
\]

with a real-valued continuous driving function \(\xi_t\) and an initial condition \(f(z, 0) = f_0(z)\). For every \(t \geq 0\), the function \(f(z, t)\) has a continuous extension on the closure of \(\mathbb{H}\), and the extended function denoted also by \(f(z, t)\) satisfies equation (8) almost everywhere. The driving function \(\xi_t\) generates the growing slit \(\gamma_t\).

We will also use the two-parametric family of conformal maps

\[
g(w, t, \tau) := f^{-1}(w(z, t), \tau) = f^{-1}(f(z, t), \tau),
\]

where \(0 \leq \tau \leq t < \infty\). We also denote \(g(w, t, 0) := g(w, t)\). The function \(g\) maps the half-plane \(\mathbb{H}\) onto a subset of \(\mathbb{H}\). It satisfies the L"owner ODE for the half-plane

\[
\frac{\partial g(w, t, \tau)}{\partial t} = -\frac{dA^0/dt}{g(w, t, \tau) - \xi_t}, \quad 0 \leq \tau \leq t < \infty, \quad g(w, \tau, \tau) = w.
\]

Moreover, \(\lim_{t \to \infty} g(w, t, \tau) = \lim_{t \to \infty} f^{-1}(f(z, t), \tau) = f(z, \tau)\).

Define the time splitting as the real-valued functions \(t = t(x, s)\), a solution to the quasi-linear differential equation

\[
\xi_t \frac{\partial t}{\partial x} + \frac{\partial t}{\partial s} = 0,
\]

satisfying the asymptotic behaviour \(\lim_{s \to \infty} t(x, s) = \lim_{x \to \infty} t(x, s) < \infty\). Assume that \(\xi_t\) is a function which admits a cone of solutions to (10) with the needed asymptotic behaviour.

Now, let us consider the superposition \(f(z, t(x, s))\) and multiply both sides of (8) by \(\frac{\partial}{\partial x}\). By abuse of notation, we continue to write \(f\) for the function \(f(z, t(x, s)) = f(z, x, s)\). Then

\[
z \frac{\partial f}{\partial x} - \xi_t \frac{\partial f(z, t)}{\partial t} \frac{\partial t}{\partial x} - \frac{\partial A^0}{\partial x} \frac{\partial f}{\partial z} = 0.
\]
If we use equation \((10)\), then
\[
(11) \quad z \frac{\partial f}{\partial x} + \frac{\partial f}{\partial s} - \frac{\partial A^0}{\partial x} \frac{\partial f}{\partial z} = 0,
\]
which is the Vlasov equation, see [1, 22], describing time evolution of the distribution function of plasma consisting of charged particles with long-range interaction. In fluid descriptions of plasmas one does not consider the velocity distribution but rather the plasma moments \(A^n(t(x, s)) \equiv A^n(x, s)\).

Among solutions to the Vlasov equation \((11)\) let us choose those which provide finite integrals for the moments \(A^n\). Namely, for a given solution \(f(z, x, s)\) with the normalization \((7)\), choose a solution \(\phi(z, x, s) = \varphi(f(z, x, s))\) where \(\varphi\) is an appropriate rapidly decreasing at infinities \(z \to \pm \infty\) function, see, e.g., [16]. For example, \(\varphi(f) = \exp(-f^2)\) is appropriate. Then the moments \(A^n(x, s)\) are defined by
\[
A^n(x, s) = \int_{-\infty}^{\infty} w^n \phi(w, x, s) \, dw, \quad n \geq 1.
\]
The direct computations implies
\[
A^n_s = \int_{-\infty}^{\infty} w^n \frac{\partial \phi}{\partial s} \, dw, \quad A^n_x = \int_{-\infty}^{\infty} w^{n+1} \frac{\partial \phi}{\partial x} \, dw.
\]
Integrating by parts yields
\[
A^{n-1} = -\int_{-\infty}^{\infty} \frac{w^n}{n} \frac{\partial \phi}{\partial w} \, dw.
\]
Now we can use the Vlasov equation \((11)\) and arrive at the equation for the moments
\[
(12) \quad A^n_s + A^{n+1}_x + nA^{n-1}A^0_x = 0,
\]
which is an infinite autonomous system, known as Benney’s moment equations, see [1], which appear in long wavelength hydrodynamics of an ideal incompressible fluid of a finite depth in a gravitational field.

Following [4, 7] let us define a function \(\lambda(z, x, s)\) by the Cauchy principal value of a singular integral
\[
\lambda(z, x, s) = z + \int_{-\infty}^{\infty} \phi(w, x, s) \frac{dw}{z - w} = z + \sum_{n=0}^{\infty} \frac{A^n}{z^{n+1}}, \quad z \to \infty \text{ in } \mathbb{H},
\]
where \(z = g(w, t(x, s))\) and the coefficient \(A^0\) is the same as for \(f\). Then,
\[
\lambda_s = \frac{\partial \lambda}{\partial s} = z_s + \sum_{n=0}^{\infty} \left( \frac{A^n_s}{z^{n+1}} - \frac{(n+1)A^n z_s}{z^{n+2}} \right),
\]
\[
\lambda_x = \frac{\partial \lambda}{\partial x} = z_x + \sum_{n=0}^{\infty} \left( \frac{A^n_x}{z^{n+1}} - \frac{(n+1)A^n z_x}{z^{n+2}} \right),
\]
and
\[
\lambda_s + z \lambda_x = z_s + z \cdot z_x + A^0_x + \sum_{n=0}^{\infty} \frac{A^n_s + A^{n+1}_x - nA^{n-1}z_s - (n+1)A^n z_x}{z^{n+1}}.
\]
Making use of the moment equations we come to

\[ \lambda_s + z\lambda_x = z_s + z \cdot z_x + A_x^0 + \sum_{n=0}^{\infty} \frac{-nA_{n-1}A_x^0}{z^{n+1}} - \sum_{n=1}^{\infty} \frac{nA_{n-1}(z_s + z \cdot z_x)}{z^{n+1}} = H_A \]

\[ = A_x^0\lambda_z + (z_s + z \cdot z_x) \left( 1 - \sum_{n=1}^{\infty} \frac{nA_{n-1}}{z^{n+1}} \right) = \lambda_z \left( A_x^0 + z_s + z \cdot z_x \right). \]

The Löwner ODE (9) implies that \( A_x^0 = x(\xi_t - z) \) and the definition of the function \( t(x, s) \) yields that

(13)

\[ A_x^0 + z_s + z \cdot z_x = 0, \]

and therefore, the equality \( \lambda_s + z\lambda_x = 0 \) holds along the trajectories of the Löwner ODE (9). Equation (13) received the name the Gibbons equation in \([13]\) following the original Gibbons’ paper \([3]\).

Let us consider the map \( z(\lambda, x, s) \) which is the inverse to \( \lambda(z, x, s) \) with respect to \( \lambda \leftrightarrow z \),

(14)

\[ \lambda(z, x, s) = z + \sum_{n=0}^{\infty} \frac{A^n}{z^{n+1}}, \quad z \to \infty \text{ in } \mathbb{H}, \]

(15)

\[ z(\lambda, x, s) = \lambda - \sum_{n=0}^{\infty} \frac{H^n}{\lambda^{n+1}}. \]

Then,

\[ \sum_{n=0}^{\infty} \frac{A^n}{z^{n+1}} = \sum_{n=0}^{\infty} \frac{H^n}{\lambda^{n+1}}, \]

and

\[ \frac{\lambda}{z} \left( A^0 + \frac{A^1}{z} + \ldots \right) = H^0 + \frac{H^1}{\lambda} + \ldots \]

So \( H^0 = A^0 \). We continue by

\[ \lambda \left( \frac{\lambda}{z} - 1 \right) A^0 + \lambda^2 \left( \frac{\lambda^2}{z^2} - 1 \right) A^1 + \lambda^3 \left( \frac{\lambda^3}{z^3} - 1 \right) A^2 + \ldots = H^1 + \frac{H^2}{\lambda} + \ldots, \]

and conclude \( H^1 = A^1 \). In the same fashion we come to

\[ \lambda^2 \left( \frac{\lambda}{z} - 1 \right) A^0 + \lambda \left( \frac{\lambda^2}{z^2} - 1 \right) A^1 + \lambda^3 \left( \frac{\lambda^3}{z^3} - 1 \right) A^2 + \ldots = H^2 + \frac{H^3}{\lambda} + \ldots, \]

and \( H^2 = A^2 + (A^0)^2 \). Finally, we have

\[ \sum_{k=0}^{n} \lambda^{-k} \left( \frac{\lambda^{k+1}}{z^{k+1}} A^k - H^k \right) + \frac{\lambda^{n+1}}{z^{n+2}} \left( A^{n+1} + \frac{A^{n+2}}{z} + \ldots \right) = \frac{1}{\lambda} \left( H^{n+1} + \frac{H^{n+2}}{\lambda} + \ldots \right), \]

and the coefficient \( H^n \) is calculated as \( H^n = A^n + P(A^0, \ldots, A^{n-1}) \), where \( P(A^0, \ldots, A^{n-1}) \) is a polynomial of \( A^0, \ldots, A^{n-1}, n \geq 2 \). The first coefficients are

\[ H^0 = A^0, \quad H^1 = A^1, \quad H^2 = A^2 + (A^0)^2, \quad H^3 = A^3 + 3A^0 A^1, \]

\[ H^4 = A^4 + 4A^0 A^2 + 2(A^1)^2 + 2(A^0)^3. \]
Analogous coefficients were calculated in, e.g., [7, 19].

This way the Löwner ODE (9) becomes the conservation equation in the following sense. According to (13)

$$\frac{d}{ds} \int_{-\infty}^{\infty} z(\lambda, x, s) \, dx = - \int_{-\infty}^{\infty} \left( A_x^0 + z \cdot z_x \right) \, dx,$$

where we integrate with respect to $x \in \mathbb{R}$ in the Cauchy principal value sense. The requirements on the asymptotic behaviour of $t(x, s)$ as $x \to \pm \infty$ imply that

$$\int_{-\infty}^{\infty} \left( A_x^0 + z \cdot z_x \right) \, dx = 0,$$

Therefore

$$\frac{d}{ds} \int_{-\infty}^{\infty} z(\lambda, x, s) \, dx = 0,$$

which corresponds to the momentum conservation law. So the conserved quantities of the evolution are the moments

$$I^n = \int_{-\infty}^{\infty} H^n(x, s) \, dx, \quad n \geq 0.$$

Analogous integrals of motion were studied in the original work by Benney [1] as well as in [7, 15, 24].

The Poisson structure allows us to reformulate the Benney moment equation (12) as an evolution equation with a Hamiltonian function. The Kupershmidt-Manin Poisson structure [7, 8] starts with the operators of differentiation and multiplication to the right for the moments $A_x^n \partial_{x}$ as skew-symmetric operators with respect to the $L^2(\mathbb{R})$-paring, acting to the right by

$$\{ A_x^m, A_x^n \}(\cdot) = -mA_x^{n+m-1} \frac{\partial}{\partial x} (\cdot) - n \frac{\partial}{\partial x} (A_x^{n+m-1}(\cdot)).$$

Then for any two observables $F(A)$ and $G(A)$, the Poisson bracket can be written as

$$\{ F, G \}(A) = \sum_{m,n=0}^{\infty} \int_{-\infty}^{\infty} \frac{\delta F}{\delta A_x^m} \{ A_x^m, A_x^n \} \frac{\delta G}{\delta A_x^n} \, dx.$$

Writing $\bar{H}^k = \frac{1}{k} \int_{-\infty}^{\infty} H^k \, dx = \frac{1}{k} I^k$, we have the hierarchy of commuting flows with the Hamiltonians $\bar{H}^k$: $\{ \bar{H}^k, \bar{H}^j \} = 0$, in the form of evolution equations

$$\frac{\partial A_x^m}{\partial t_k} = \sum_{n=0}^{\infty} \{ A_x^m, A_x^n \} \frac{\delta \bar{H}^k}{\delta A_x^n},$$

so that equation (12) becomes the second equation in this hierarchy.
3. Finite-dimensional time

The Löwner PDE (16) can be generalized to the form

\[
\frac{\partial f(z,t)}{\partial t} = \sum_{k=1}^{m} \frac{\mu_k(t)}{z - \xi_k(t)} \frac{\partial A^0}{\partial t} f(z,t), \quad f(z,0) = f_0(z),
\]

with piecewise continuous coefficients \(\mu_k(t) \geq 0, k = 1, \ldots, m, \sum_{k=1}^{m} \mu_k(t) = 1\), and real-valued continuous driving functions \(\xi_1(t), \ldots, \xi_m(t)\). A solution \(f(z,t)\) to (16) maps \(\mathbb{H}\) onto \(\mathbb{H} \setminus \cup_{k=1}^{m} \gamma_k(t)\) where \(\gamma_k(t)\) are growing Jordan curves (slits) in \(\mathbb{H}\) except for their endpoints on \(\mathbb{R}\). The driving functions \(\xi_k(t)\) generate slits \(\gamma_k(t)\), and the coefficients \(\mu_k(t)\) govern the relative dynamics of the slits \(\gamma_k(t)\) with respect to each other.

Instead of (16), it is possible to consider the generalized Löwner PDE with a generalized time-vector \(t = (t_1, \ldots, t_m)\)

\[
(z - \xi_k(t)) \frac{\partial f(z,t)}{\partial t_k} = \partial A^0 \frac{\partial f(z,t)}{\partial z}, \quad f(z,0) = f_0(z), \quad k = 1, \ldots, m.
\]

In this model, \(A^0(t) = A^0(t_1, \ldots, t_m)\) is not an arbitrary function of \(t\). At every point \(t = (t_1, \ldots, t_m)\),

\[
\frac{\partial A^0}{\partial t_1} = \frac{\partial A^0}{\partial t_2} = \ldots = \frac{\partial A^0}{\partial t_m}.
\]

For every \(t = (t_1, \ldots, t_m)\), the solution \(f(z,t)\) to system (17) maps \(\mathbb{H}\) onto \(\mathbb{H} \setminus \cup_{k=1}^{m} \gamma_k(t_k)\), where \(\gamma_k(t_k)\) is an endpoint of the slit \(\gamma_k\) generated by the driving function \(\xi_k(t)\).

Similarly to the scalar \(t\), let us denote by

\[
g(w, \tau, t) := f^{-1}(f(z, \tau, t), t) = f^{-1}(f(z, \tau, t), t), \quad \tau = (\tau_1, \ldots, \tau_m),
\]

where \(0 \leq \tau_k \leq t_k < \infty\) for any \(k = 1, \ldots, m\). We also write \(g(w, 0, t) =: g(w, t)\), where \(0\) states for the null-vector. The function \(g\) maps the half-plane \(\mathbb{H}\) onto a subset of \(\mathbb{H}\). It satisfies the system of Löwner’s ODE in the half-plane

\[
\frac{\partial g(w, \tau, t)}{\partial t_k} = -\frac{\partial A^0 / \partial t_k}{g(w, \tau, t) - \xi_k(t)}, \quad 0 \leq \tau_j \leq t_j < \infty, \quad g(w, \tau, \tau) = w,
\]

where \(j = 1, \ldots, m, \quad k = 1, \ldots, m\).

Moreover, \(\lim_{t_k \to \infty} g(w, \tau, (\tau_1, \ldots, \tau_{k-1}, t_k, \tau_{k+1}, \ldots, \tau_m)) = f(z, \tau)\).

Again we can define the vector-function \(\mathbf{t} = \mathbf{t}(x,s)\), as a solution to the system of quasi-linear differential equations

\[
\xi_k(t) \frac{\partial t_k}{\partial x} + \frac{\partial t_k}{\partial s} = 0, \quad k = 1, \ldots, m,
\]

satisfying the asymptotic behaviour \(\lim_{x \to \infty} t_k(x,s) = \lim_{x \to -\infty} t_k(x,s) < \infty, \quad k = 1, \ldots, m\).

Assume that functions \(\xi_k(t_k)\) admit their cones of solutions to (19) under the necessary asymptotic behaviour.
Equation (17) implies that $f(z, t(x, s)) =: f(z, x, s)$ satisfies
\[z \frac{\partial f}{\partial t_k} \frac{\partial t_k}{\partial x} - \xi_k(t) \frac{\partial f}{\partial t_k} \frac{\partial f}{\partial x} \frac{\partial A^0}{\partial t_k} \frac{\partial f}{\partial x} \frac{\partial f}{\partial z} = 0\]
which together with (19) gives
\[z \frac{\partial f}{\partial t_k} \frac{\partial t_k}{\partial x} + \frac{\partial f}{\partial t_k} \frac{\partial t_k}{\partial s} - \frac{\partial A^0}{\partial t_k} \frac{\partial f}{\partial x} \frac{\partial f}{\partial z} = 0, \quad k = 1, \ldots, m.
\]
Summing up the latter equations for $k = 1, \ldots, m$, we obtain the Vlasov equation for $f(z, t(x, s))$
\[z \frac{\partial f(z, t(x, s))}{\partial x} + \frac{\partial f(z, t(x, s))}{\partial s} - \frac{\partial A^0}{\partial x} \frac{\partial f(z, t(x, s))}{\partial z} = 0.
\]
Similarly to the scalar case, there appear the moments $A^n(t(x, s)) =: A^n(x, s)$ satisfying equation (12), the functions $\lambda(z, x, s)$ and $z(\lambda, x, s)$ and the hierarchy of commuting flows with the Hamiltonians $\tilde{H}^k$. Equations (17) can be reduced to (16). Indeed, assume that $\mu_1(t) > 0$ and construct the following reduction
\[\frac{dt_k}{dt_1} = \frac{\mu_k}{\mu_1}, \quad t_k(0) = 0, \quad k = 2, \ldots, m.
\]
Then, after multiplying by $\frac{\partial f}{\partial t_k}$ and summing up, equations (17) for $t_1 = t$ and $f(z, t(t)) =: f(z, t)$ become
\[
\frac{\partial f(z, t(t))}{\partial t} = \frac{\partial f}{\partial t_1} + \frac{\partial f}{\partial t_2} \frac{\partial t_2}{\partial t} + \cdots + \frac{\partial f}{\partial t_m} \frac{\partial t_m}{\partial t} = \frac{1}{\mu_1} \left[ \frac{\mu_1}{z - \xi_1} \frac{\partial A^0}{\partial t_1} + \cdots + \frac{\mu_m}{z - \xi_m} \frac{\partial A^0}{\partial t_m} \right] \frac{\partial f}{\partial z}
\]
which is equivalent to (16) with $\tilde{A}^0 = A^0/\mu_1$.

4. Infinite-dimensional time

The limiting case of equation (16) as $m \to \infty$ leads to the Löwner-Kufarev type equation
\[(20) \quad \frac{\partial f}{\partial t} = \int_{\mathbb{R}} \frac{d\nu_t(\xi)}{z - \xi(t)} \frac{dA^0}{dt} \frac{\partial f}{\partial z},\]
where, for every $t \geq 0$, $d\nu_t(\xi)$ is a probability measure with a compact support $I_t \subset \mathbb{R}$. A solution $f(z, t)$ to (20) maps $\mathbb{H}$ onto $\mathbb{H} \setminus K_t$ where, in general, the omitted set $K_t \cap \mathbb{H}$ is not reduced to a countable set of slits. The set $K_t$ is generated by the measure $d\nu_t(\xi)$.

However, the domain $\mathbb{H} \setminus K_t$ is the Carathéodory kernel for the sequence of domains $\mathbb{H} \setminus \cup_{k=1}^m \gamma_k(t)$ as $m \to \infty$. Here the slits $\gamma_k$ are dense in $K_t$. In this interpretation the measure $d\nu_t(\xi)$ is represented as a limit of point mass measures with a dense set of mass points in the support of $d\nu_t$.

In this case it is impossible to generalize directly system (17) passing to an infinite set of equations corresponding to the countable set of coordinates $(t_1, t_2, \ldots)$. Let us
build a model with a successive dynamics of every slit $\gamma_1, \gamma_2, \ldots$. For $k = 1, 2, \ldots$, denote by

$$P_k(z, t_k) = \frac{\partial A^0}{\partial t_k} \frac{\partial f(z, t_k)}{\partial z} \quad \text{for} \quad T_{k-1} < t_k < T_k, \quad T_0 = 0,$$

and

$$P_k(z, t_k) = 0 \quad \text{for} \quad t_k \in \mathbb{R} \setminus (T_{k-k}, T_k), \quad k = 1, 2, \ldots.$$

Now, instead of (20), we are able to introduce a system of PDE with an infinite set of coordinates $t := (t_1, t_2, \ldots)$,

$$(21) \quad (z - \xi_k(t_k)) \frac{\partial f}{\partial t_k} = P_k(z, t_k), \quad k = 1, 2, \ldots.$$

Suppose a function $f_0(z)$ is expanded near infinity as

$$(22) \quad f_0(z) = z + \sum_{n=0}^{\infty} A^n z^{n+1},$$

and let $f_0$ serve as an initial data for the Löwner chain $f(z, t)$ governed by (20) and as an initial data for the first equation of system (21). Successively, the function $f(z, t_k)$ serves as an initial data for the $(k+1)$-th equation in (21). It is clear that the resulting chain $f(z, t) = f(z, t)$, $t = (t_1, t_2, \ldots)$, in contrast to the chain obtained from (20), is piecewise differentiable. The functions $f(z, t)$ are normalized as in (22) with $A^n = A^n(t)$. So there exist driving functions $\xi_1(t_1), \xi_2(t_2), \ldots$ such that $f(z, t)$, $t = (t_1, t_2, \ldots)$, is a solution to the infinite system of PDE (21).

Let us apply the results by Takebe, Teo and Zabrodin [18] to construct a one-variable reduction of dispersionless KP hierarchy for the system of PDE (21).

Let $g(w, t) := f^{-1}(w, t)$ be the inverse to $f(z, t)$. Then $g$ is normalized at infinity as

$$g(w, t) = w + \sum_{n=1}^{\infty} \frac{b_n(t)}{w^n}.$$ 

Denote by $\Phi_k(w, t) = [g^k(w, t)]_{\geq 0}$, $k \geq 1$, the Faber polynomials for $g(w, t)$. Let us forget for the moment the dependence on $t$ and let us write simply $g(w)$ and $\Phi_k(w)$.

The first Faber polynomials are

$$\Phi_0 = 1, \quad \Phi_1 = w, \quad \Phi_2 = w^2 - 2b_1, \quad \Phi_3 = w^3 - 3b_1w - 3b_2,$$

and the recurrence formula

$$\Phi_{n+1} = w\Phi_n - \sum_{k=1}^{n-1} b_{n-k}\Phi_k - (n + 1)b_n$$

holds for all $n \geq 1$. The Faber polynomials are related to the Grunsky coefficients which implies that

$$\log \frac{g(w) - \xi}{w} = -\sum_{n=1}^{\infty} \frac{1}{nw^n} \Phi_n(\xi).$$
Changing variables $\xi = e^u$ and differentiating both sides with respect to $u$ yields

$$\frac{1}{g(w) - \xi} = \sum_{n=1}^{\infty} \frac{1}{nw^n} \Phi'_n(\xi).$$

Returning back to equation (21) we conclude that the function $g = f^{-1}$, w.r.t. the first variable, satisfies the system of equations

$$\frac{\partial g}{\partial t_k} = -\frac{\partial A^0}{\partial t_k} \sum_{n=1}^{\infty} \frac{\Phi'_n(\xi_k, t_k)}{nw^n}, \quad T_{k-1} < t_k < T_k,$$

and

$$\frac{\partial g}{\partial t_k} = 0 \quad \text{for} \quad t_k \in \mathbb{R} \setminus (T_{k-1}, T_k), \quad k = 1, 2, \ldots.$$ 

This implies that, for all $k \geq 1$,

$$n \frac{\partial b_k}{\partial t_k} = -\frac{\partial A^0}{\partial t_k} \Phi'_k(\xi_k, t_k), \quad T_{k-1} < t_k < T_k,$$

and

$$\frac{\partial b_k}{\partial t_k} = 0, \quad t_k \notin (T_{k-1}, T_k).$$

There is an evident way to write dependence on $t$ through a single variable $t = t_1$. Set

$$\tau(t) = t_1, \quad \text{if} \quad t_1 \in (0, T_1),$$

$$\tau(t) = kt_k(t_1) \quad \text{if} \quad t_1 \in (T_{k-1}, T_k), \quad k \geq 2,$$

where

$$\frac{dt_k}{dt_1} = \Phi'_k(\xi_k, t_k) \quad \text{for} \quad t_1 \in (T_{k-1}, T_k)$$

and

$$\frac{dt_k}{dt_1} = 0 \quad \text{for} \quad t_1 \notin (T_{k-1}, T_k), \quad k \geq 2.$$

In the spirit of the results of Takebe, Teo and Zabrodin [18] we conclude that the non-intersecting intervals $(T_{k-1}, T_k)$ imply that given $f(z, \tau(t))$ as the solution to system (21) with the initial condition $f_0$, one has the Lax function $L = f(z, \tau(t))$ which solves the dKP hierarchy by

$$\frac{\partial L}{\partial t_k} = \{L_k, L\}, \quad T_{k-1} < t_k < T_k,$$

where $L_k = \frac{1}{k} [L^k]_{\geq 0}$, and the Poisson bracket is given by

$$\{F, G\} = \frac{\partial F}{\partial w} \frac{\partial G}{\partial x} - \frac{\partial F}{\partial x} \frac{\partial G}{\partial w}, \quad T_{k-1} < x := t_1 < T_k.$$

The Benney equations again can be recovered as the second equation of dKP in the following way. Set $s = t_2$. Then,

$$\frac{\partial L}{\partial s} = \{(z^2 + 2A^0), L\}, \quad T_1 < s, x < T_2.$$
where
\[ \mathcal{L} = z + \sum_{n=0}^{\infty} \frac{A^n}{z^{n+1}}. \]
Equating the coefficients in front of powers of \( z \) leads to the equations (12). The higher equations in the hierarchy lead to interesting PDEs due to the conditions on commuting flows (compatibility conditions) (3). For example, if \( n = 2 \) and \( m = 3 \) imply the dKP equation (4) (Zabolotskaya-Khokhlov equation [23]) for \( A^0 \).

5. Vlasov kinetic equation

Let us return back to the consistency conditions (6) (the Gibbons-Tsarëv system), where \( u(r) \) is a conservation law density and \( \mu_k(r) \) are the characteristic velocities of \( N \) component hydrodynamic type system
\[ r^i_s + \mu_i(r)r^i_x = 0, \quad i = 1, \ldots, N, \]
written in the Riemann invariants. System (23) is integrable by the generalized hodograph method, see [20, 21], and has infinitely many conservation laws and commuting flows
\[ r^i_y + \lambda_i(r)r^i_x = 0, \]
where \( \lambda_i \) are functions of two entries \( \mu_i \) and \( u \) only: \( \lambda_i = F(\mu_i, u) \). We will make use of the Tsarëv system
\[ \frac{\partial_i \lambda_k}{\lambda_i - \lambda_k} = \frac{\partial_i \mu_k}{\mu_i - \mu_k}, \quad i \neq k, \]
which is a direct consequence of the commutation \( (r^i_x)_y = (r^i_y)_t \). Substitution of the ansatz \( \lambda_i = F(\mu_i, u) \) into (25) yields
\[ (\mu_k - \mu_i) \frac{\partial F(\mu_i, u)}{\partial u} = \frac{F(\mu_k, u) - F(\mu_i, u)}{\mu_k - \mu_i} - \frac{\partial F(\mu_i, u)}{\partial \mu_i}, \quad i \neq k. \]
Interchanging indices and summing up both formulas we obtain
\[ (\mu_k - \mu_i) \left( \frac{\partial F(\mu_i, u)}{\partial u} + \frac{\partial F(\mu_k, u)}{\partial u} \right) = \frac{\partial F(\mu_k, u)}{\partial \mu_k} - \frac{\partial F(\mu_i, u)}{\partial \mu_i}. \]
In the limit \( \mu_k \to \mu_i \) this formula becomes
\[ 2 \frac{\partial F(\mu_i, u)}{\partial u} = \frac{\partial^2 F(\mu_i, u)}{(\partial \mu_i)^2}. \]
Then (27) reads
\[ (\mu_k - \mu_i) \left( \frac{\partial^2 F(\mu_i, u)}{(\partial \mu_i)^2} + \frac{\partial^2 F(\mu_k, u)}{(\partial \mu_k)^2} \right) = 2 \left( \frac{\partial F(\mu_k, u)}{\partial \mu_k} - \frac{\partial F(\mu_i, u)}{\partial \mu_i} \right). \]
Taking derivative of this relationship with respect to \( \mu_k \), we obtain
\[ \frac{\partial^3 F(\mu_k, u)}{(\partial \mu_k)^3} = \frac{\partial^2 F(\mu_k, u)}{(\partial \mu_k)^2} - \frac{\partial^2 F(\mu_i, u)}{(\partial \mu_i)^2}, \quad i \neq k. \]
Interchanging indices we conclude that
\[ \frac{\partial^3 F(\mu_k, u)}{(\partial \mu_k)^3} = a'(u) \]
for any index \( k \). Thus
\[ F(\mu_k, u) = \frac{1}{6} a(u)(\mu_k)^3 + b(u)(\mu_k)^2 + c(u)\mu_k + d(u), \]
where functions \( a(u), b(u), c(u), d(u) \) still have not been determined yet. However the substitution \((30)\) into \((26)\) leads to \( a(u) = \text{const}, b(u) = \text{const}, 2c'(u) = a, d'(u) = b \). So, \((30)\) admits the form
\[ F(\mu_k, u) = \frac{a}{6}(\mu_k)^3 + \frac{a}{2}u\mu_k + b[(\mu_k)^2 + u]. \]
Finally, the substitution \((30)\) into \((26)\) implies \( a = 0 \). Thus we have found the so called ‘dispersive relation’
\[ \lambda_i = (\mu_i)^2 + u. \]
Following [2] we write the Löwner system
\[ \partial_iz = \frac{\partial_iz}{\mu_i - z}. \]
Then, see \([23], [24], [31], [32]\),
\[ z_x = \sum \partial_iz \cdot r^i_x = \sum \frac{\partial_iz}{\mu_i - z}r^i_x, \]
\[ -z_s = -\sum \partial_iz \cdot r^i_s = \sum \frac{\partial_iz}{\mu_i - z}r^i_s = \sum \frac{\partial_iz}{\mu_i - z}u_x + zz_x, \]
\[ -z_y = -\sum \partial_iz \cdot r^i_y = \sum \frac{\partial_iz}{\mu_i - z}\lambda_i r^i_x = \sum \frac{\partial_iz}{\mu_i - z}[(\mu_i)^2 + u]r^i_x \]
\[ = \sum \partial_iz \cdot (\mu_i - z)r^i_x + 2z \sum \partial_iz \cdot r^i_x + (z^2 + u) \sum \frac{\partial_iz}{\mu_i - z}r^i_x \]
\[ = (v_x - zu_x) + 2zu_x + (z^2 + u)z_x = v_x + zu_x + uz_x + z^2z_x, \]
where we have introduced a new function \( v(x) \) such that \( \partial_xv = \mu_i\partial_xu \). Indeed the potential function \( v \) exists because the compatibility condition \( \partial_x(\partial_xv) = \partial_x(\partial_xv) \) leads to the identity according to the Gibbons–Tsarëv system \((6)\). Thus we reconstructed two equations
\[ z_s + \left(\frac{z^2}{2} + u\right)_x = 0, \quad z_y + \left(\frac{z^3}{3} + uz + v\right)_x = 0. \]
Their compatibility condition \((z_s)_y = (z_y)_s\) leads to the equations
\[ (34) \quad v_x + uz = 0, \quad v_s + uy + uux = 0, \]
which are equivalent to \((4)\).
Now we introduce the so called vertex equation
\[ \partial_{r(\zeta)}z(\lambda) = \partial_x\ln(z(\lambda) - z(\zeta)), \]
where \( z(\lambda) \) is just a short notation for \( z(\lambda; t_0, t_1, t_2, \ldots) \). Here we use infinitely many ‘time’ variables \( t_k \), which will be discussed below. Let us consider the formal expansion (15) as \( \zeta \to \infty \) and

\[
\partial_{t(\zeta)} = -\frac{1}{\zeta} \partial_{t_0} - \frac{1}{\zeta^2} \partial_{t_1} - \frac{1}{\zeta^3} \partial_{t_2} - \ldots
\]

Then one can obtain infinitely many equations

\[
\partial_{t_n} z(\lambda) = \partial_x \frac{\Phi_{n+1}(z(\lambda))}{n+1}, \quad n \geq 0,
\]

where \( \Phi_n \) stands for the Faber polynomials (see Section 4), or for the first polynomials,

\[
\partial_{t_0} z(\lambda) = z_x(\lambda), \quad \partial_{t_1} z(\lambda) = \left( \frac{z^2(\lambda)}{2} + H^0 \right)_x,
\]

\[
\partial_{t_2} z(\lambda) = \left( \frac{z^3(\lambda)}{3} + H^0 z(\lambda) + H^1 \right)_x, \ldots
\]

identifying \( x = t_0, s = -t_1, y = -t_2 \) as well as \( u = H^0, v = H^1 \) (see (33)). Substituting a similar formal expansion \( (\lambda \to \infty) \)

\[
z(\lambda) = \lambda - \frac{H^0}{\lambda} - \frac{H^1}{\lambda^2} - \frac{H^2}{\lambda^3} - \ldots
\]

in these generating functions of conservation laws leads to infinitely many local conservation laws. For instance

\[
(H^k)_s + \left( H^{k+1} - \frac{1}{2} \sum_{m=0}^{k-1} H^m H^{k-1-m} \right)_x = 0, \quad k = 0, 1, 2, \ldots
\]

This is nothing but the Benney hydrodynamic chain (2), written in the conservative form where all conservation law densities \( H^k \) are polynomials with respect to moments \( A^m \) as in Section 2.

Alternative expansions \( (\zeta \to 0) \)

\[
z(\zeta) = H^{-1} + \zeta H^{-2} + \zeta^2 H^{-3} + \ldots, \quad \partial_{t(\zeta)} = \partial_{t_{-1}} + \zeta \partial_{t_{-2}} + \zeta^2 \partial_{t_{-3}} + \ldots
\]

lead to another generating functions of conservation laws, for instance

\[
\partial_{t_{-1}} z(\lambda) = \partial_x \ln(z(\lambda) - H^{-1}), \quad \partial_{t_{-2}} z(\lambda) = -\partial_x \frac{H^{-2}}{z(\lambda) - H^{-1}}, \ldots
\]

Substituting expansion (37) and the expansion \( (\lambda \to 0) \)

\[
z(\lambda) = H^{-1} + \lambda H^{-2} + \lambda^2 H^{-3} + \ldots
\]

implies extra infinitely many conservation laws (cf. (38)). If for instance, we substitute the above expansion in (33), two additional conservation laws

\[
\partial_s H^{-1} = -\left( \frac{1}{2} (H^{-1})^2 + H^0 \right)_x = -\partial_x \frac{\Phi_2(H^{-1})}{2},
\]

\[
\partial_y H^{-1} = -\left( \frac{1}{3} (H^{-1})^3 + H^0 H^{-1} + H^1 \right)_x = -\partial_x \frac{\Phi_3(H^{-1})}{3}
\]
follow. Infinitely many conservative laws (cf. (38))

\[(H^{-1})_s + \left( H^0 + \frac{1}{2}(H^{-1})^2 \right)_x = 0, \]

\[(H^k)_s + \left( H^{k+1} - \frac{1}{2} \sum_{m=0}^{k-1} H^m H^{k-1-m} \right)_x = 0, \quad k = 0, 1, \ldots \]

also can be written as the modified Benney hydrodynamic chain (see details in [14])

\[B^k_s + B^{k+1}_x + \frac{1}{2} B^0 B^k_x + \frac{k + 1}{2} B^k B^0_x + k B^{k-1} \left( \frac{1}{2} B^1 - \frac{1}{8} (B^0)^2 \right)_x = 0, \]

where \( H^{-1} = B^0, H^0 = B^1, H^1 = B^2 + B^0 B^1 - \frac{1}{12} (B^0)^3, \ldots \) This Modified Benney hydrodynamic chain is related to the modified dKP equation (cf. (34))

\[H_s^{-1} + \left( H^0 + \frac{1}{2}(H^{-1})^2 \right)_x = 0, \quad H^0_s = H^0_y + \left( H^0 H^{-1} + \frac{1}{3}(H^{-1})^3 \right)_x, \]

which can be obtained from the compatibility condition \((\tilde{z})_y = (\tilde{z})_s\), where

\[\tilde{z} + \left( \frac{\tilde{z}^2}{2} + H^{-1} \tilde{z} \right)_x = 0, \quad \tilde{z}_y + \left( \frac{\tilde{z}^3}{3} + H^{-1} \tilde{z}^2 + (H^0 + (H^{-1})^2) \tilde{z} \right)_x = 0. \]

One can derive the modified Löwner equations

\[\partial_i \tilde{z} = \tilde{z} \frac{\partial_i H^{-1}}{\mu_i - \tilde{z} - H^{-1}}, \]

which are equivalent to the original Löwner equations (32) by substituting \( z = \tilde{z} + H^{-1} \) and \((\mu_i - H^{-1}) \partial_i H^{-1} = \partial_i H^0, \)

Both hydrodynamic chains have the same local Hamiltonian structure

\[A^k_x = -\frac{1}{2} [(k + m) A^{k+m-1} \partial_x + mA_x^{k+m-1}] \frac{\partial H^2}{\partial A^m}, \]

\[B^k_x = -\frac{1}{2} [(k + m) B^{k+m-1} \partial_x + mB_x^{k+m-1}] \frac{\partial H^1}{\partial B^m}, \]

where their Hamiltonian densities are

\[H^2 = A^2 + (A^0)^2, \quad H^1 = A^1 = B^2 + B^0 B^1 - \frac{1}{12} (B^0)^3. \]

Since all moments \( A^k \) can be expressed via moments \( B^0, B^1, \ldots, B^k, B^{k+1} \), the Kupershmidt–Manin Poisson brackets

\[\{ B^k, B^m \} = [(k + m) B^{k+m-1} \partial_x + mB_x^{k+m-1}] \delta(x - x'), \]

can be recalculated via moments \( A^s \). This means that the Benney hydrodynamic chain has at least two local Hamiltonian structures (see details in [14]).

As in the previous particular case, we are looking for \( N \) components commuting the hydrodynamic reductions

\[r^i_{\tau(\zeta)} = w^i(r, \zeta)r^i_x. \]
Then (35) reduces to the form

$$\partial_i z(\lambda) = \partial_i z(\zeta) \frac{\partial_i z(\zeta)}{1 - [z(\lambda) - z(\zeta)]w^i(r, \zeta)}.$$

Taking into account (32), one can obtain

$$w^i(r, \zeta) = \frac{1}{\mu_i - z(\zeta)}.$$

Using expansions (36), (39), one can expand the generating function (with respect to parameter $\zeta$ at infinity and about zero, respectively) of infinitely many higher commuting flows

$$r^i_\tau(\zeta) = \frac{1}{\mu_i - z(\zeta)} r^i_x.$$

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