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Dynamical properties of bosons in an optical lattice with a synthetic magnetic field

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Abstract. We study the dynamical properties of bosons in an optical lattice subject to a synthetic magnetic field at zero temperature. First, we consider a superfluid regime where the nearest-neighbor hopping is much larger than the on-site repulsion and a large number of bosons are occupied in each site. Then, the dynamics is well described by the discrete nonlinear Schrödinger equation. Second, we study the dynamics of hard-core bosons as a limit of the strong interparticle interaction by solving the truncated Green's functions for the boson and density operators. We discuss the evolution of the density and the vorticity, which are clearly distinct between two regimes.

1. Introduction
Ultracold atoms in an optical lattice provide an ideal testing ground for many-body effects associated with the model Hamiltonian in conventional condensed matter systems [1]. Recently, synthesis of an (artificial) gauge field was realized by using the Raman coupling between internal atomic states to imprint the required geometric phases, opening a door for more wide ranging studies associated with an artificial magnetic field that gives rise to a orbital motion of neutral atoms [2, 3]. Much recent interest has been directed towards understanding the cold atom systems with combining synthetic magnetic fields and optical lattices because of interesting physical effects expected in the presence of a lattice [4].

In this work, we study the dynamical properties of bosons in a two-dimensional (2D) optical lattice subject to a synthetic magnetic field at zero temperature. Study of nonequilibrium dynamics given by the Bose-Hubbard Hamiltonian has been one of the challenging problem, because its quantum many-body character makes the problem very hard. When the system is in a superfluid regime, the dynamics can be well described by a discrete nonlinear Schrödinger equation [5], which we use in this work. We extend the discussion to the strongly-interacting limit, i.e., hard-core bosons, by solving the truncated Green’s functions for the boson and density operators. We discuss the motion of the boson density and the vorticity with the typical spatial period determined by the strength of the magnetic field, where significant difference emerges from the superfluid bosons and the hard-core bosons.

2. Formulation
2.1. Hamiltonian
We consider a system of $N$-bosons put on the sites of a two-dimensional square lattice with the size $V = L \times L$. The two-dimensional Bose-Hubbard model subject to an uniform abelian gauge
potential is described by the Hamiltonian

\[ \hat{H} = \frac{i}{2} \sum_{\langle ij \rangle} \left[ \hat{a}_i^\dagger \hat{a}_j e^{i A_{ij}^+} + \text{h.c.} \right] + \frac{U}{2} \sum_i \hat{n}_i (\hat{n}_i - 1) - \sum_i \mu_i \hat{n}_i. \]  

(1)

The operator \( \hat{a}_i \) destroys (creates) a boson on the lattice site \( i = (i_x, i_y) \), \( U (\geq 0) \) describes the on-site repulsion, and \( t \) is the hopping amplitude between nearest neighbors \( \langle ij \rangle \). The chemical potential \( \mu_i \) is site-dependent to express an additional external potential of the system. The Hamiltonian conserves the total number of bosons, \( \hat{N} = \sum_i \hat{n}_i = \sum_i \hat{a}_i^\dagger \hat{a}_i \). Throughout this work, we consider an uniform system without additional trap potentials such as a harmonic one. The field \( A_{ij} \) describes the imposed gauge potential, defined by \( A_{ij} = \int_{r_j} A \cdot dr \). All of the physics of the system governed by the Hamiltonian (1) is gauge-invariant. Hence, its properties depend only on the fluxes through plaquettes

\[ \Phi = \int_{\text{plaq}} dS \cdot (\nabla \times A) = \sum_{i,j \in a} A_{ij} = 2\pi f \]  

(2)

where \( a \) labels the plaquette and the sum represents the directed sum of the gauge fields around that plaquette. In the following, we use the vector potential in the symmetric gauge \[ \mathbf{A} = (-By/2, Bx/2, 0) \] with the magnetic field \( \mathbf{B} = (0, 0, B) = (0, 0, 2\pi f/d^2) \) \( (d \) is a typical lattice spacing) in the direction perpendicular to the lattice plane.

2.2. Boson model

First, we derive the equation of motion for the usual boson annihilation/creation operators with the commutation relation \( [\hat{a}_i, \hat{a}_j^\dagger] = \delta_{ij} \). The Heisenberg equation \( i\hbar (d\hat{a}_i/dt) = [\hat{a}_i, \hat{H}] \) gives

\[ i\hbar \frac{d\hat{a}_i}{dt} = -t \sum_j a_j e^{i A_{ij}^+} - \mu_i \hat{a}_i + U \hat{n}_i \hat{a}_i, \]  

(3)

where the sum is taken only the nearest neighbors of the \( i \)-site. Provided the system is deep in superfluid phase \((tN > 2U)\), the operator can be replaced by the averaged value \( \langle a_i \rangle = \psi_i \), and thus the equation of motion reduces to the classical equation of motion, known as the discrete nonlinear Schrödinger equation (DNLSE) [5]

\[ i\hbar \frac{d\psi_i}{dt} = -t \sum_j \psi_j e^{i A_{ij}^+} - \mu_i \psi_i + U n_i \psi_i, \]  

(4)

where \( n_i = \langle \hat{a}_i^\dagger \hat{a}_i \rangle = |\psi_i|^2 \) is the mean particle number at \( i \)-site. Since the complex classical field can be represented by \( \psi_i = \sqrt{n_i} e^{i \phi_i} \), the model is referred to as the “fuzzy XY spin model” with varying amplitude of the XY spin vector. The DNLSE is a well known equation in a literature of nonlinear lattice system, where the properties of topological solitons have been thoroughly studied [6]. Equation (4) is an extended DNLSE including the gauge field.

2.3. Hard core boson model

Next, we consider a strongly interacting regime by taking the hard-core limit \((U \rightarrow \infty)\), where double occupancy at the same site is excluded. We introduce the hard-core boson destruction operator as \( \hat{\phi}_i \), which satisfies the following canonical-(anti)commutation relations \( [\hat{\phi}_i, \hat{\phi}_j^\dagger] = 0 \) for the different sites \( i \neq j \), and \( \{ \hat{\phi}_i, \hat{\phi}_j \} = 1 \) for the same site. Thus, the eigenvalue of the
hard-core boson number operator \( \hat{n}_i = \hat{\phi}_i \hat{\phi}_i \) becomes 0 or 1, and the eigenstate is denoted as |\( n_i = 0 \rangle \) and |\( n_i = 1 \rangle \). Then, the Hamiltonian \( \hat{H} \) is written as

\[
\hat{H}_{hc} = -\frac{t}{2} \sum_{\langle i,j \rangle} (\hat{\phi}_i \hat{\phi}_j e^{iA_{ij}} + \text{h.c.}) - \sum_i \mu_i \hat{\phi}_i^\dagger \hat{\phi}_i,
\]

and the resulting Heisenberg equation is

\[
i \hbar \frac{d\hat{\phi}_i}{dt} = -i \sum_j' [(1 - 2\hat{n}_i)\hat{\phi}_j e^{iA_{ij}}] + \mu_i \hat{\phi}_i.
\]

It is interesting that the nonlinear contribution of Eq. (6) arises from the hopping term due to the hard core nature. The evolution of \( \hat{n}_i \) in the right-hand side of Eq. (6) is governed by

\[
i \hbar \frac{d\hat{n}_i}{dt} = -\frac{t}{2} \sum_{j,j'} (\hat{\phi}_i \hat{\phi}_j e^{iA_{ij}} - \hat{\phi}_j \hat{\phi}_i e^{iA_{ji}} - \text{h.c.}) - \mu_i \hat{\phi}_i.
\]

Here, the sum is taken in such a way that the direction \( i \rightarrow j \) is opposite to \( i \rightarrow j' \).

We solve Eqs. (6) and (7) simultaneously to simulate the dynamics of the hard-core bosons. To this end, the operators are replaced by the averaged c-number values \( \langle \hat{\phi}_i \rangle \simeq \phi_i \) and \( \langle \hat{n}_i \rangle \simeq n_i \), the equations of motion being reduced to the classical ones. This truncation of Eqs. (6) and (7) allows us to represent the classical field of hard-core bosons by \( \phi_i = \sin \lambda_i \cos \lambda_i e^{i\theta_i} \) and \( n_i = (1 + \cos 2\lambda_i)/2 \) with the angle of the CP\(^1\) variables \( w_i = (\cos \lambda_i e^{i\theta_i}, \sin \lambda_i e^{i\theta_i^2}) \), where \( \theta_i = \theta_i^1 - \theta_i^2 \).

3. Results

We calculate the dynamical equations Eq. (4) for the boson model and the classical version of Eqs. (6) and (7) for the hard-core boson model by using the Crank-Nicholson algorithm. The calculation system consists of 60 \times 60 sites and the periodic boundary condition. The initial condition of the simulations was chosen to be the uniform density \( n_i = 0.6 \) and the phase \( \theta_i = 0 \). We give a magnetic field \( f = 1/2 \) suddenly at \( t = 0 \) to cause the nonequilibrium dynamics. To characterize the dynamical evolution of the system, we monitor the time development of the mean density \( n_i \) and the vorticity

\[
m_i = \frac{1}{2\pi} \sum_{i,j \in a} \text{mod}(\theta_i - \theta_j, 2\pi) ,
\]

where \( \text{mod} \) restricts the range \([−\pi, \pi]\).

Figure 1 shows the results of the numerical simulations. Figure 1(a) represents the spatial pattern of the vorticity Eq. (8), where one can see the emergence of \( 4 \times 4 \) periodicity with the \( 2 \times 2 \) sublattice structures, one of which exhibits current-flowing motion within this cell but the other makes no motion. We find that for \( f = p/q \) the pattern has \( 2q \times 2q \) periodicity.

Although these periodic structure of vorticity appears in both models, the motion of the density at each site is significantly different. Figures 1(b)-(d) show the oscillation of the population at a typical site \((i,j) = (2,4)\) for the boson model Eq. (4) for several values of on-site repulsion \( U \). For small \( U \), where the particles are easy to move to different sites, the density repeats to flow-in (out) from (to) the nearest-neighbor sites with relatively large amplitude exceeding unity. As \( U \) is increased, the amplitude of the population oscillation is suppressed. This is due to the effect of the self-trapping, where the particle numbers are locked at each site.
Figure 1. The figure (a) represents a typical snapshot of the spatial distribution of the vorticity with $4 \times 4$ periodicity for $f = 1/2$, which can be seen in both two models. Black, gray and white correspond to $m_i = +1, 0$ and $-1$, respectively, and the direction of the current is indicated by arrows. The oscillation of the number density $n_i$ at the typical site $i = (2, 4)$ (marked by a circle in (a)) is shown for the boson model (b) $U/2t = 1$, (c) $U/2t = 10$, (d) $U/2t = 100$, and for the hard-core boson model (e).

by the randomization of the relative phases [7]. This gives an indication that the system enters an insulating-like phase.

On the other hands, the dynamics obtained by the hard-core boson model (truncation of Eqs. (6) and (7)) shows different behavior from the simple boson model in the large $U$ limit. As shown in Fig. 1(e), the time period of the density oscillation becomes longer than that of the boson model. Also, the density does not exceed unity. Therefore, the truncated dynamical equations of Eqs. (6) and (7) can correctly capture the hard-core nature, where the double occupancy at the same site is prohibited.

4. Conclusion
We represents the preliminary results of the boson dynamics in an optical lattice subject to an uniform synthetic magnetic field. We solve the dynamical equations to reveal the response for a sudden turn on of a magnetic field, finding significant difference between superfluid bosons and hard-core bosons. For future plan, we will study the various nonlinear dynamics of vortices and transport properties under a synthetic magnetic field.

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