Fermion Masses, Neutrino Oscillations, and Proton Decay in the Light of SuperKamiokande

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Abstract

Within the framework of unified gauge models, interactions responsible for neutrino masses can also provide mechanisms for nucleon instability. We discuss their implications concretely in the light of recent results on neutrino oscillation from the SuperKamiokande collaboration. We construct a predictive SO(10)-based framework that describes the masses and mixing of all quarks and leptons. An overconstrained global fit is obtained, that makes five successful predictions for quarks and charged leptons. The same description provides agreement with the SuperK results on atmospheric neutrinos and supports a small-angle MSW mechanism. We find that current limits on nucleon stability put significant stress on the framework. Further, a distinctive feature of the SO(10) model developed here is the likely prominence of the $\mu^+K^0$ mode in addition to the $\overline{\nu}K^+$ mode of proton decay. Thus improved searches in these channels for proton decay will either turn up events, or force us outside this circle of ideas.

I. INTRODUCTION

Recent SuperKamiokande observations on atmospheric neutrinos establish the oscillation of $\nu_\mu$ to $\nu_\tau$ with a mass splitting $\delta m^2 \sim (10^{-2} \text{ to } 10^{-3}) \text{ eV}^2$ and an oscillation angle $\sin^2 2\theta_{\mu\tau} \sim (0.82 - 1.0)$. To be more precise, the observations do not directly exclude oscillation into some other $\nu_X$, as long as it is not to $\nu_e$, but Occam's razor and the framework adopted in this paper suggest $X = \tau$, and we shall assume that in what follows, without further comment. These observations clearly require new physics beyond what is usually contemplated in the Standard Model.

As we shall presently discuss, a tau neutrino mass consistent with the observed oscillations fits extremely naturally into the framework supplied by unified gauge theories of the
strong, weak, and electromagnetic interactions \cite{2,3} that includes the symmetry structure
\( G_{224} = SU(2)_L \times SU(2)_R \times SU(4)_C \), the minimal such symmetry being \( SO(10) \). In this
framework, the low-energy degrees of freedom need be no larger than the Standard Model.
Neutrino masses appear as effective nonrenormalizable (dimension 5) operators.

Unified gauge theories were already very impressive on other grounds. They combine
the scattered multiplets of the Standard Model (five per family) into a significantly smaller
number (two for \( SU(5) \), one for \( SO(10) \)). They rationalize the otherwise bizarre-looking
hypercharge assignments in the Standard Model \cite{2,3}. Finally, especially in their supersym-
metric version \cite{5}, they account quantitatively for the relative values of the strong, weak,
and electromagnetic couplings \cite{6,7}.

This last feat is accomplished by renormalizing the separate couplings down from a single
common value at a unification scale, taking into account the effects of vacuum polarization
due to virtual particles, down to the much lower mass scales at which they are observed
experimentally. A by-product of this overconstrained, and singularly successful, calculation,
is to identify the mass scale at which the unified symmetry is broken, to be \( M_U \sim 2 \times 10^{16} \)
GeV \cite{7}.

This value is interesting in several respects. First, from data and concepts purely internal
to gauge theories of particle interactions, it brings us to the threshold of the fundamental
scale of quantum gravity, namely the Planck mass \( 2 \times 10^{18} \) GeV (in rational units). Reading
it the other way, by demanding unification, allowing for both the classical power-law running
of the gravitational coupling and the quantum logarithmic running of gauge couplings, we
obtain a roughly accurate calculation of the observed strength of gravity.

Second, it sets the scale for phenomena directly associated with unification but for-
bidden in the Standard Model, notably nucleon decay and neutrino masses. Prior to the
SuperKamiokande observations, the main phenomenological virtues of the large value of the
unification mass scale were its negative implications. It explained why nucleon decay is
rare, and neutrino masses are small, although both are almost inevitable consequences of
unification. Now the scale can also be positively identified, at least semi-quantitatively.

Indeed, any unification based on \( G_{224} \) \cite{2} requires the existence of right-handed neutrinos
\( \nu^R \). When \( G_{224} \) is embedded in \( SO(10) \), \( \nu^R \) fills out, together with the 15 left-handed quark
and lepton fields in each Standard Model family, the 16 dimensional spinor representation
of \( SO(10) \). The \( \nu^R \) are Standard Model singlets, so that they can, and generically will,
acquire large Majorana masses at the scale where unified \( SO(10) \) symmetry breaks to the
Standard Model \( SU(3) \times SU(2) \times U(1) \). The ordinary, left-handed neutrinos couple to these
\( \nu^R \) much as ordinary quarks and leptons couple to their right-handed partners, through
\( SU(2) \times U(1) \) non-singlet Higgs fields. For the quarks and leptons, condensation of those
Higgs fields transforms such interactions directly into mass terms. For neutrinos the effect
of this condensation is slightly more involved. As mentioned, the \( \nu^R \) have an independent,
and much larger, source of mass. As a result, through the “see-saw” mechanism \cite{8}, the
effective masses for the left-handed neutrinos, acquired through their virtual transitions into
\( \nu^R \) and back, are predictably tiny.

In this paper we do two things. First, we flesh out with quantitative detail the rough
picture just sketched. Its most straightforward embodiment leads to the hierarchical pattern
\( m_{\nu_e} << m_{\nu_\mu} << m_{\nu_\tau} \), and to a value of \( m_{\nu_\tau} \) very consistent with the SuperK observations,
interpreted as \( \nu_{\mu} - \nu_\tau \) oscillations. We demonstrate that the large mixing angle observed
does not force us to swerve from this straightforward direction. Indeed, it can arise rather plausibly in the context of ideas which have been applied successfully to understanding quark masses and mixings. Motivated by the success of the circle of ideas mentioned above, we insist that the pattern of neutrino masses and mixings should be discussed together with those of the quarks and charged leptons, and not in isolation. A simple and predictive \( SO(10) \)–based mass structure that describes the observed masses and mixings of all fermions including those of the neutrinos will be presented and analyzed. We thus demonstrate by example how the large \( \nu_\mu - \nu_\tau \) oscillation angle can be obtained quite naturally along with a large hierarchy in the\( \nu_\mu - \nu_\tau \) masses. (This is in contrast to several recent attempts where such a large oscillation angle is explained as a consequence of the near degeneracy of the \( \nu_\mu - \nu_\tau \) system. The bulk of the papers in Ref. \[9,10\] belong to this category.) In this scheme, \( \nu_e - \nu_\mu \) oscillation drives the small angle MSW explanation \[11\] of the solar neutrino puzzle.

Second, we revisit a previously noted link between neutrino masses and nucleon decay, in the framework of supersymmetric \( SO(10) \) unified models \[12\]. Previously, motivated in part by possible cosmological indications for a hot dark matter component, we used numerical estimates for \( m_{\nu_\tau} \sim 1 \) eV, considerably larger than are now favored by the SuperK result (\( \sim 1/20 \) eV). This amounts to an increase in the Majorana mass of \( \nu_\tau^R \) compared to previous work, and correspondingly an increase in the strength of the neutrino mass related \( d = 5 \) proton decay rate. Another important change is caused by the large \( \nu_\mu - \nu_\tau \) oscillation angle suggested by the SuperK result. With hierarchical neutrino masses, we will argue, their result strongly suggests substantial mixing in the charged lepton (\( \mu - \tau \)) sector. That, in turn, affects the strength and the flavor structure not only of the neutrino related, but also of the standard \( d = 5 \) proton decay operators induced by the exchange of color triplet partners of the electroweak Higgs doublets \[13\]. These adjustments in our expectations for proton decay turn out to be quite significant quantitatively. They considerably heighten the tension around nucleon decay: either it is accessible, or the framework fails.

As an accompanying major result, we observe that, in contrast to the minimal supersymmetric \( SU(5) \) model for which the charged lepton mode \( p \to \mu^+ K^0 \) has a negligible branching ratio (\( \sim 10^{-3} \), see text), for the \( SO(10) \) model developed here the corresponding branching ratio should typically lie in the range of 20 to even 50\%, if the relevant hadronic matrix elements have comparable magnitudes. This becomes possible because of the presence of the new \( d = 5 \) proton decay operators mentioned above, which are related to neutrino masses. Thus the \( \mu^+ K^0 \) mode, if observed, would provide an indication in favor of an \( SO(10) \) model of the masses and mixings of all fermions including neutrinos, as developed here.

This paper is organized as follows. In Sec. II, we discuss the scale of new physics implied by the SuperK observations. In Sec. III we describe a caricature model that accommodates large neutrino oscillation angle as suggested by SuperK without assuming neutrino mass degeneracy. Sec. IV is devoted to a more ambitious \( SO(10) \) model that accounts for the masses of second and third generation quarks and leptons including the large neutrino oscillation angle. In Sec. V we suggest, by way of example, a predictive way to incorporate the first family fermions into the \( SO(10) \) scheme that retains the success of Sec. IV, leading to a total of eight successful predictions for the masses and the mixings of the fermions including the neutrinos, and supports a small angle MSW mechanism. In Sec. VI we discuss the issue of proton decay in the context of neutrino masses. Four Appendices (A,B,C and D) contain relevant technical details of our proton decay calculations including unification scale
threshold corrections to $\alpha_3(m_Z)$. Finally, a summary of our results and some concluding remarks are given in Sec. VII.

II. $M_{\nu_\tau}$ AND THE UNIFICATION SCALE

Using the degrees of freedom of the Standard Model, small Majorana masses for neutrinos arise from dimension-5 operators in the form

$$\mathcal{L} = \lambda_{ij} \frac{L_i L_j \phi \phi}{M} + \text{h.c.}$$  \hspace{1cm} (1)

where $L_i = (\nu_i, \ell_i)^T$ denote the lepton doublets and $\phi = (\phi^+, \phi^0)^T$ the Higgs doublet of the Standard Model. Interpreting the SuperK result as a measure of $m_{\nu_\tau} = (1/30 - 1/10)$ eV, momentarily ignoring mixing, and using $\langle \phi^0 \rangle = 246$ GeV, we find $M/\lambda_{33} = (6 - 18) \times 10^{14}$ GeV.

According to the seesaw mechanism, and again putting off the question of mixing, the tau neutrino mass is given as

$$m_{\nu_\tau} = \frac{(m^D_{\nu_\tau})^2}{M^R_{\tau}}$$  \hspace{1cm} (2)

where $m^D$ is the Dirac mass that the neutrino would acquire in the absence of the large Majorana mass of the right-handed neutrino, and $M^R$ is the value of this Majorana mass.

If one assumes, within $SO(10)$, that the Dirac masses of the third family are dominated by a contribution from a fundamental Higgs $\mathbf{10_H}$ condensate then one obtains the relation $m_\tau(M_U) = m_b(M_U)$ between the masses of the tau lepton and bottom quark at the unification scale, which is known to be successful [16]. This suggests that the third family fermions get their masses primarily from a $\mathbf{10_H}$ condensate through a Yukawa coupling $\lambda_{16}^{16} \mathbf{16} \mathbf{10_H}$.

(16i, $i = 1 - 3$ denotes the three generations of fermions.) This hypothesis entails the relation

$$m^D_{\nu_\tau}(M_U) \approx m_t(M_U) \approx (100 - 120) \text{ GeV}.$$  \hspace{1cm} (3)

(This numerical estimate is valid for most values of the MSSM parameters, so long as $\tan \beta$ is neither too large ($\geq 30$) nor too small ($\leq 2$).) Combining this with the seesaw formula and the SuperK measurement, one obtains $M^R_{\tau} \approx (1 - 3) \times 10^{14}$ GeV.

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1Preliminary results of this investigation were announced at summer conferences [14].

2As is well known, for very low values of $\tan \beta < 2$, which corresponds to a large $\nu_\tau$ Yukawa coupling, renormalization group effects lead to significant increase in $m_{\nu_\tau}$, by more than 40%, as the running momentum ($\mu$) is lowered from $M_U$ to 1 GeV [17]. Such effects however become progressively smaller ($\leq 12\%$) for $\tan \beta \geq 3$. Barring very low values of $\tan \beta \leq 2$ which seem to be disfavored (see remarks later), such renormalization effects on $m_{\nu_\tau}$ are thus small. Henceforth we neglect them.
In $SO(10)$, the Majorana masses of right-handed neutrinos can be generated using either the five-index self-dual antisymmetric tensor $\mathbf{126}_H$, or a bilinear product of the spinorial Higgs $\mathbf{16}_H$. The relevant interactions are the renormalizable interaction $\tilde{f}_{ij}\mathbf{16}_i\mathbf{16}_j\mathbf{126}_H$ or the effective nonrenormalizable interaction $f_{ij}\mathbf{16}_i\mathbf{16}_j\mathbf{16}_H\mathbf{16}_H/M$, respectively. In terms of the $SU(2)_L \times SU(2)_R \times SU(4)_C$ subgroup the relevant multiplets transform as $(1,3,10)_H$ or $(1,2,4)_H$ respectively. If the $\mathbf{126}_H$ is used to induce the Majorana mass of $\nu^R_\tau$, then with $\langle \mathbf{126}_H \rangle \approx M_U$ we require the Yukawa coupling $f_{33} \sim 10^{-2}$. Considering that it was constructed by balancing quantities of vastly different magnitudes, the nearness of this coupling to the ‘natural’ value unity is encouraging.

Still more interesting is the situation that arises if we employ the $\mathbf{16}_H$. In this case, as mentioned just above, we require an effective nonrenormalizable interaction. Such an interaction could well arise through the exchange of superheavy states associated with quantum gravity. Then using $\langle \mathbf{16}_H \rangle \sim M_U \approx 2 \times 10^{16}$ GeV and $M \approx M_{\text{Planck}} \approx 2 \times 10^{18}$ GeV we find

$$M^R_\tau \approx f_{33} \frac{\langle \mathbf{16}_H \rangle^2}{M} \approx (f_{33}) \times 2 \times 10^{14} \text{ GeV}.$$  

This is nicely consistent with the required value of $M^R_\tau$, for $f_{33}$ close to unity.

Many of the considerations that follow do not depend on which of these alternatives, $\mathbf{16}_H$ or $\mathbf{126}_H$, is chosen. Motivated partly by the foregoing numerology, and partly by some suggestions from higher symmetry schemes and string theory [18], we will mainly discuss models in which the pair $(\mathbf{16}_H, \mathbf{16}_H)$ is used to break $(B-L)$. Let us note, however, that the $\mathbf{126}_H$ does have some advantages. Specifically, its couplings $\tilde{f}_{ij}$ are renormalizable, and its vacuum expectation value violates $(B-L)$ by two units, and thus conserves an R parity automatically [19]. The latter property is important for eliminating catastrophic – dimension 4 – sources of proton decay. With the $SO(10)$ spinor condensates, we must postulate a suitable $Z_2$ symmetry for this purpose separately.

### III. LARGE \((\nu_\mu - \nu_\tau)\) OSCILLATION ANGLE WITH HIERARCHICAL MASSES

Based on its measurements of atmospheric cosmic ray neutrino oscillations, the SuperK group estimates a large oscillation angle $\sin^2 2\theta_{\mu\tau}^\text{osc} = (0.82 - 1)$ [1]. If we compare this to the analogous angle for quarks, $\sin^2 2\theta_{cb} \approx 4|V_{cb}|^2 \approx 6 \times 10^{-3}$, a challenge arises. How are we to understand the enormous difference between these two mixings, in a framework where quarks and leptons are unified?

One widely considered possibility is to propose that the two relevant neutrino flavor eigenstates are nearly degenerate [9,10]. Then a small perturbation that lifts this degeneracy will induce maximal mixing. Such behavior is familiar from the $K^0 - \bar{K}^0$ system. But while there is a well-established fundamental symmetry (CPT) which guarantees exact degeneracy

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3The effects of neutrino mixing and of possible choice of $M = M_{\text{string}} \approx 4 \times 10^{17}$ GeV (instead of $M = M_{\text{Planck}}$) on $M^R_\tau$ are discussed later.
of $K^0$ and $\bar{K}^0$, no such symmetry is known to operate for the different flavors in the quark–lepton system. In the context of the seesaw mechanism, such degeneracy is awkward to accommodate, at best. Furthermore, if $\nu_\mu$ and $\nu_\tau$ are nearly mass degenerate with a common mass of $\geq 1/30$ eV, then there is no simple way to address the solar neutrino puzzle through $\nu_e - \nu_\mu$ or $\nu_e - \nu_\tau$ oscillation. The possibilities appear bizarre – either $\nu_e$ is nearly degenerate with $\nu_\mu$ and $\nu_\tau$, or there is a fourth “sterile” neutrino with the right mass and mixing parameters. In light of all this, it seems reasonable to consider alternatives sympathetically.

We will argue now that large $\nu_\mu - \nu_\tau$ oscillation angle can in fact arise, without requiring the neutrinos to be nearly degenerate, along the lines of some not entirely unsuccessful attempts to relate mixing angles and hierarchical masses in the quark sector.

In the two sections following this one we shall analyze a complete model for the quark and lepton masses that is both overconstrained and phenomenologically successful (though, to be sure, its construction involves considerable guesswork). Since that analysis becomes rather complicated and perhaps intimidating, we shall first, in this section, highlight some salient features in a caricature model.

A leading idea in many attempts to understand large mixing in the quark sector is to utilize the properties of special matrices, whose form might be constrained by simple symmetry requirements (e.g., requiring symmetry or antisymmetry) and selection rules. Such constraints can easily arise, as we shall see, from the group theory of unification, given specific choices for the Higgs fields whose condensation generates the masses.

The first attempts along these lines, which remain quite intriguing and instructive, utilized a $2 \times 2$ system of the form \[ M_f = \begin{pmatrix} 0 & a_f \\ a_f & b_f \end{pmatrix}, \quad f = (u, d) \] (5) with $a_f \ll b_f$. Symmetric matrices arise, for example, in $SO(10)$ (or left–right symmetric models [21]) where the matter fields are $16$s if the relevant Higgs fields are $10$s. The vanishing of the $(1,1)$ entry can be ensured by a suitable flavor symmetry that distinguishes the two relevant families. The eigenvalues of this matrix are $(m_1, m_2)_f \simeq (a^2/b, b)_f$. Thus the off–diagonal element $a_f$ is nearly the geometrical mean of the two eigenvalues. For convenience in discussion, we will refer to mass matrices of this form as type A. For type A mass matrices, the mixing angles in each sector, up and down, are given by

$$\tan \theta_f = \sqrt{\frac{m_1}{m_2}}.$$ (6)

To obtain the observable mixing angle one must of course combine the mixing angles of the up and the down sectors. In a world with two flavors, this leads to the well–known formula for the Cabibbo angle [20]:

$$\theta_C \simeq \sqrt{\frac{m_d}{m_s}} - e^{i \phi} \sqrt{\frac{m_u}{m_c}}.$$ (7)

Using $\sqrt{m_d/m_s} \simeq 0.22$ and $\sqrt{m_u/m_c} \simeq 0.06$, we see that Eq. (7) works within 30% for any value of the phase $\phi$, and perfectly for a value of the phase parameter $\phi$ around $\pi/2$.  

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A notable feature of the type A pattern, which remains valid even if the requirement of symmetry is relaxed, is that it can support a strong hierarchy of eigenvalues by means of a much weaker hierarchy of matrix elements. For example, with \((a/b)_f = 1/10\), one obtains a large hierarchy \((m_1/m_2)_f \sim 1/100\).

Now let us consider the implications of adopting the type A pattern for the \(\mu - \tau\) sector, including the \(2 \times 2\) Dirac mass matrices of the charged leptons \((\mu - \tau)\) and the neutrinos \((\nu_\mu - \nu_\tau)\), and the Majorana mass matrix of the right-handed neutrinos \((\nu_\mu^R - \nu_\tau^R)\). Including the first family generally will not much affect the discussion of the \(\mu - \tau\) sector, as we shall see.

The three matrices in the \(\mu - \tau\) leptonic sector have the form

\[
M_{\ell,\nu}^D = \begin{pmatrix} 0 & d_{\mu\tau} \\ d_{\mu\tau} & d_{\tau\tau} \end{pmatrix} ; \quad M_{\nu}^R = \begin{pmatrix} 0 & y \\ y & 1 \end{pmatrix} M_R
\]

with the understanding that in each case the \((\tau\tau)\) entry dominates. One finds easily that the physical mass matrix for the light left-handed neutrinos, \(M_{\nu}^{\text{light}} = -M_{\nu}^D(M_{\nu}^D)^{-1}(M_{\nu}^D)^T\), takes a similar form, viz.

\[
M_{\nu}^{\text{light}} = \frac{1}{M_R y^2} \begin{pmatrix} 0 & -d_{\mu\tau}^2 y \\ -d_{\mu\tau}^2 y & \{d_{\mu\tau}^2 - 2d_{\tau\tau}d_{\mu\tau}y\} \end{pmatrix}^\nu.
\]

Denoting the two eigenvalues as \(m_{\nu_2}\) and \(m_{\nu_3}\), this yields:

\[
\frac{m_{\nu_2}}{m_{\nu_3}} \approx \frac{-y^2}{(1 - 2y \frac{d_{\mu\tau}^2}{d_{\mu\tau}^2})^2} ; \quad \tan \theta_{\mu\tau}^\nu = \sqrt{\frac{m_{\nu_2}}{m_{\nu_3}}} , \quad \tan \theta_{\mu\tau}^\nu = \sqrt{\frac{m_{\mu}}{m_{\tau}}}.
\]

Observe that the square–root formula holds for the mixing angles in all sectors, including the light neutrinos.

To illustrate the possibilities, let us consider the phenomenologically interesting hierarchy ratio \(m_{\nu_\mu}/m_{\nu_\tau} \approx 1/10\). With this ratio, one can accommodate \(m_{\nu_\mu} \approx .03\) eV as suggested by atmospheric neutrino oscillations and a value of \(m_{\nu_\mu}\) consistent with the small-angle MSW solution (with very small \(m_{\nu_\mu}\)) \([22]\). This ratio is achieved for \((d_{\mu\tau}^\nu/d_{\tau\tau}^\nu, y) = (1/5, 1/13)\).

Combining the contributions from the charged lepton sector and from the neutrino sector, the physical oscillation angle following from Eq. (10) is

\[
\theta_{\mu\tau}^{\text{osc}} \approx \sqrt{\frac{m_{\mu}}{m_{\tau}}} - \epsilon^{m_{\mu}/m_{\nu_\tau}}.
\]

Although the mixing angle is not large in either sector \((\theta_{\mu\tau}^\nu \approx \sqrt{m_{\mu}/m_{\tau}} \approx 0.25 \approx 14^\circ}\) and \(\theta_{\mu\tau}^\nu \approx \sqrt{m_{\nu_\mu}/m_{\nu_\tau}} \approx \sqrt{1/10} \approx 0.31 \approx 18^\circ\), if \(\eta \approx \pi\) one obtains a near-maximal value for the physical mixing parameter. Actually the small angle approximation used in Eq. (11) is too crude; the precise expression is given by

\[
\sin^2 2\theta_{\mu\tau} = \frac{4 \left( \sqrt{\frac{m_{\mu}}{m_{\tau}}} \pm \sqrt{\frac{m_{\mu}}{m_{\nu_\mu}}} \right)^2 \left( 1 \mp \sqrt{\frac{m_{\mu}}{m_{\tau}}} \sqrt{\frac{m_{\mu}}{m_{\nu_\mu}}} \right)^2}{\left( 1 + \frac{m_{\mu}}{m_{\tau}} \right)^2 \left( 1 + \left| \frac{m_{\mu}}{m_{\nu_\mu}} \right| \right)^2},
\]

where the \(\pm\) corresponds to \(\eta = (\pi, 0)\). For \(\eta = \pi\) and \(m_{\nu_\mu}/m_{\nu_\tau} = 1/10\), \(\sin^2 2\theta_{\mu\tau}^{\text{osc}} \approx 0.79\). This example shows that large oscillation angles can arise in a simple hierarchical model, without any extreme adjustment of parameters. As shown below, even larger oscillation angles can arise quite naturally in more realistic models which can account also for \(V_{cb}\).
IV. A MORE AMBITIOUS $SO(10)$ MODEL

The model discussed in the previous section is not adequate for a unified treatment of the masses and mixings of the second and the third family fermions. Indeed, the square-root formula for $V_{cb}$ reads

$$|V_{cb}| \simeq \sqrt{\frac{m_c}{m_t}} - e^{i\chi} \sqrt{\frac{m_s}{m_b}},$$

so that with $\sqrt{m_s/m_b} \simeq 0.17$ and $\sqrt{m_c/m_t} \simeq 0.06$, one cannot obtain observed value $V_{cb} \simeq 0.04 \pm 0.003$ for any value of $\chi$. Thus the simplest symmetrical type A mass matrices (Eq. (5)) cannot adequately describe the hierarchical masses and the mixings of the quarks.

We now propose to study in detail a concrete proposal for asymmetric type A mass matrices that can be obtained within $SO(10)$, and predicts correlations among the quark-lepton and up-down mass matrices that are phenomenologically acceptable. Before plunging into the analysis, let us briefly summarize its main conclusions. In this section, we shall temporarily ignore the first family.

The four $2 \times 2$ Dirac mass matrices ($U, D, L,$ and $N$) will be generated via four Yukawa couplings. The Majorana matrix of the right-handed neutrinos involves two additional Yukawa couplings. Thus there are seven parameters (six Yukawa couplings plus one ratio of vacuum expectation values) to describe 10 observables (eight masses – $(m_c, m_t, m_s, m_b, m_\mu, m_\tau, m_{\nu_2}, m_{\nu_3})$ – and two mixing angles – $\theta^\text{osc}_{\mu\tau}$ and $V_{cb}$) for the $\mu$ and the $\tau$ families. The system is overconstrained, and predicts three relations among observables. Two of these concern the charged fermion sector, and they are reasonably well satisfied. The third prediction concerns the neutrino sector: the neutrino oscillation angle will be predicted as a function of the mass ratio $m_{\nu_2}/m_{\nu_3}$. For $m_{\nu_2}/m_{\nu_3} = (1/10 - 1/30)$, consistent with mass determinations from the SuperK atmospheric neutrino data and the small angle MSW solution of the solar neutrino puzzle, a large $(\nu_\mu - \nu_\tau)$ oscillation angle is obtained, as indicated by the SuperK atmospheric data.

In forming hypotheses for the form of the couplings responsible for the mass matrices, we are guided by several considerations. We assume an underlying $SO(10)$ unification symmetry, and that unified symmetry breaking occurs through a minimal system of low dimensional Higgs multiplets – specifically, $\langle 45_H \rangle$, one pair of $\langle 16_H \rangle$ and $\langle \overline{16}_H \rangle$. A single $10_H$ is employed for electroweak symmetry breaking. The hierarchical masses will be assumed to arise from type A mass matrices, as in Eq. (5), generalized to asymmetric form. In addition, we must of course satisfy the broad phenomenological constraints $m_b(M_U) \simeq m_\tau(M_U)$, $m_s(M_U) \neq m_\mu(M_U)$, and $V_{CKM} \neq 1$. Combining these considerations with a restriction to low dimensional operators ($d \leq 5$), we are led to suggest the following set of Yukawa couplings [23]:

$$L_{\text{Yukawa}} = h_{33} 16_3 16_3 10_H + \frac{a_{23}}{M} 16_2 16_3 10_H 45_H + \frac{g_{23}}{M} 16_2 16_3 16_H 16_H + h_{23} 16_2 16_3 10_H.$$  

Here $M$, a scale associated with the effective non–renormalizable interactions, could plausibly lie somewhere between the unification scale $M_U$ and $M_{\text{Planck}}$. For example, the $a_{23}$
term might result from integrating out a $16 + \overline{16}$ superfield pair with mass of order $M$; or $M$ might be identified as $M_{\text{Planck}}$ itself if the nonrenormalizable interactions are associated with gravity. A mass matrix of type $A$ results if the first term, $h_{33} \langle \mathbf{10}_H \rangle$, is dominant. This ensures $m_\nu(M_U) \simeq m_\tau(M_U)$ and $m_\nu(M_U) \simeq m_\nu(M_U)$. The remaining terms, responsible for off-diagonal mixings, must be smaller by about one order of magnitude.

In more detail, our rationale for favoring the particular off–diagonal couplings $a_{23}$ and $g_{23}$ is the following. If the fermions acquired mass only through $10$s, one would obtain too much symmetry between down quark and charged lepton masses, and in particular the bad relation $m_\nu(M_U) = m_\mu(M_U)$. To avoid this, while eschewing proliferation of extraneous Higgs multiplets, the simplest possibility is to bring in the vacuum expectation value of the $(1,1,15)$ component (under $G_{224}$) in the $45_H$ that is proportional to $(B - L)$. A vacuum expectation value of this form figures prominently in the Dimpoulos-Wilczek mechanism for doublet–triplet splitting in $SO(10)$ [24], as we shall discuss at length in connection with proton decay. Now if one restricts to $d = 4$ and $d = 5$ operators in the superpotential, the only relevant $SO(10)$ invariant effective coupling that involves the $45_H$ is our $a_{3j}16_i16_j10_H45_H/M$. In this term, only the $120$ in the decomposition $10_H \times 45_H (= 10 + 120 + 320)$ can contribute to fermion mass matrix, and its contribution is antisymmetric in $(i, j)$. These two couplings still do not distinguish up and down quark mass patterns, and so if they were the whole story one would have a trivial, identity, Cabibbo-Kobayashi-Maskawa (CKM) matrix. The effective couplings $g_{ij}16_i16_j16_H16_H/M$ remedy this problem, without requiring addition to our small set of Higgs multiplets. When a vacuum expectation value of order the unification scale for the $SU(3) \times SU(2) \times U(1)$ singlet component of $16_H$ and a vacuum expectation value of order the electroweak scale for the $SU(2) \times U(1)$ breaking component of $16_H$ are inserted, this term contributes to quark and lepton mass matrices. To be more precise, since the electroweak doublet contained in $16_H$ has the quantum numbers of a down–type Higgs doublet, it contributes to the down quark and charged lepton mass matrices, but not to the up sector. The resulting up–down asymmetry generates non–zero CKM mixing angles.

With these four effective Yukawa couplings, the Dirac mass matrices of quarks and leptons of the second and the third families at the unification scale take the form:

$$U = \begin{pmatrix} 0 & \epsilon + \sigma \\ -\epsilon + \sigma & 1 \end{pmatrix} m_U, \quad D = \begin{pmatrix} 0 & \epsilon + \eta \\ -\epsilon + \eta & 1 \end{pmatrix} m_D, \quad N = \begin{pmatrix} 0 & -3\epsilon + \sigma \\ 3\epsilon + \sigma & 1 \end{pmatrix} m_U, \quad L = \begin{pmatrix} 0 & -3\epsilon + \eta \\ 3\epsilon + \eta & 1 \end{pmatrix} m_D . \quad (15)$$

Here the matrices are multiplied by left–handed fermion fields from the left and by anti–fermion fields from the right. $(U, D)$ stand for the mass matrices of up and down quarks, while $(N, L)$ are the Dirac mass matrices of the neutrinos and the charged leptons.

The entries $(1, \epsilon, \sigma)$ arise respectively from the $h_{33}, a_{23}$ and $h_{23}$ terms in Eq. (14), while $\eta$ entering into $D$ and $L$ receives contributions from both $g_{23}$ and $h_{23}$; thus $\eta \neq \sigma$. Note the quark–lepton correlations between $(U, N)$ as well as $(D, L)$, and the up–down correlation between $(U, D)$ as well as $(N, L)$. These correlations arise because of the symmetry structure of $SO(10)$. The relative factor of $-3$ between quarks and leptons involving the $\epsilon$ entry reflects the fact that $\langle 45_H \rangle \propto (B - L)$, while the antisymmetry in this entry arises from the $SO(10)$ structure as explained above.

Assuming $\epsilon, \eta, \sigma \ll 1$, we obtain at the unification scale:
One virtue of the asymmetric nature of the mass matrices (Eq. (15)) is worth noting. The simple expression $|\sigma - \eta|$ for $|V_{cb}|$ actually stands for $\left| \sqrt{m_s/m_b \left( \frac{\eta + \epsilon}{\eta - \epsilon} \right)^{1/2}} - \sqrt{m_c/m_t \left( \frac{\sigma + \epsilon}{\sigma - \epsilon} \right)^{1/2}} \right|$. Comparing with Eq. (13), which was based on symmetric type A mass matrices and which led to too big a value of $V_{cb}$, we see that the square–root mass ratios are multiplied by the respective asymmetry factors $(\frac{\eta + \epsilon}{\eta - \epsilon})^{1/2}$ and $(\frac{\sigma + \epsilon}{\sigma - \epsilon})^{1/2}$ (see Eq. (15)). For $\epsilon$ relatively negative compared to $\eta$ and $\sigma$ (as suggested by considerations of the leptonic mixings, see below), these factors provide the desired lowering of $V_{cb}$ compared to Eq. (13).

In Eq. (16) all the mass and mixing angle parameters are to be identified with those at the unification scale. One can evaluate $\sigma, \eta, \epsilon$ in terms of the observed masses and mixing (extrapolated to the unification scale):

$$\begin{align*}
\sigma &\simeq \frac{V_{cb}^2 + m_s/m_b - m_c/m_t}{(2V_{cb})}, \\
\eta &\simeq \frac{-V_{cb}^2 + m_s/m_b - m_c/m_t}{(2V_{cb})}, \\
\epsilon^2 &\simeq \{\sigma^2 + m_c/m_t\} \simeq (\eta^2 + m_\mu/m_\tau)/9. \\
\end{align*}$$

This leads to the sum rule:

$$\frac{m_s}{m_b} \simeq \frac{m_c}{m_t} - \frac{5}{4} V_{cb}^2 \pm V_{cb} \left[ \frac{9}{16} V_{cb}^2 + \frac{1}{2} \frac{m_\mu}{m_\tau} - \frac{9}{2} \frac{m_c}{m_t} \right]^{1/2}.$$  (18)

A word of explanation is needed here. The parameters $\sigma, \eta$ are in general complex. In Eqs. (16)-(18) therefore, one should interpret the mass ratios and $V_{cb}$ to be also complex. Let us denote $m_c/m_t = \eta_{ct} |m_c/m_t|, m_s/m_b = \eta_{sb} m_s/m_b, V_{cb} = \eta_{cb} |V_{cb}|$, with $|\eta_{ij}| = 1$. Noting that each term on the RHS of Eq. (18) is small compared to $m_s/m_b \simeq (1/30 - 1/50)$ (e.g: $m_c/m_t \approx 1/300$, $|V_{cb}|m_{12}/(2m_\tau)|^{1/2} \approx 1/140$), we see that essentially only one choice of the phase factors can possibly allow the sum rule to work so that the RHS is maximized – i.e., $\eta_{ct} = -1, \eta_{\mu\tau} = +1$ and $\eta_{sb} = -1$ with $\{\pm V_{cb} \}$ having a negative sign. Thus, all phases are constrained to be near the CP conserving limit (0 or $\pi$). For simplicity, we will take the $\eta_{ij}$'s to be real. As long as the phases are small ($\leq 10^9$, say), they would not of course affect our results.

The relatively simple pattern shown in Eq. (15) provides a reasonable fit to all the masses and the mixing of the quarks and the charged leptons in the second and the third families. For example, if we take as input $m_t^{\text{physical}} = 174$ GeV, $m_c(m_t) = 1.37$ GeV, $V_{cb} = 0.045$, in agreement with the values advocated in Ref. [23,26], and the known $\mu$ and $\tau$ lepton masses, then we obtain the predictions

$$\begin{align*}
m_\mu(m_b) &\simeq 4.9 \text{ GeV}, \\
m_s(1 \text{ GeV}) &\simeq 116 \text{ MeV}. \\
\end{align*}$$

In quoting the numbers in Eq. (19), we have extrapolated the GUT scale values down to low energies using the beta functions of the minimal supersymmetric extension of the Standard
Model (MSSM), assuming $\alpha_s(M_Z) = 0.118$, an effective SUSY threshold of 500 GeV and $\tan\beta = 5$. Our results depend only weakly on these input choices, so long as $\tan\beta$ is neither too large ($\geq 30$) nor too small ($\leq 2$).

The mass of the strange quark is somewhat low compared to the central value advocated in Ref. [25], but is closer in value to the recent lattice determinations [27]. The $b$–quark mass prediction is also in reasonable agreement with determination from $\Upsilon$ spectroscopy. In this regard, it should be mentioned that exact $b$–$\tau$ unification in supersymmetric unified models, without taking finite threshold effects due to the gluino into account, would yield a $b$–quark mass which is about 10–20% above the experimental value for a wide range of the parameter $\tan\beta$ [28]. In our case, $b$ and $\tau$ masses are not exactly equal at $M_U$, $m_b$ is about 8% lower than $m_\tau$ (see Eq. (16)). This difference, which has its origin in the $(B - L)$ generator associated with the off–diagonal entry $\epsilon$ in Eq. (15), leads to better agreement with the experimental value of $m_b$.

The parameters $\sigma, \eta, \epsilon, m_U$ and $m_D$ are found to be

$$\sigma \simeq -0.110 \eta_{cb}, \eta \simeq -0.151 \eta_{cb}, \epsilon \simeq 0.095 \eta_k,$$

$$m_U \simeq m_t(M_U) \simeq (100 - 120) \text{ GeV}, \quad m_D \simeq m_b(M_U) \simeq 1.5 \text{ GeV}.$$  \hspace{1cm} (20)

($\eta_k = \pm 1$ is the phase of $\epsilon$.) In addition to the two predictions in Eq. (19), the Dirac masses of the neutrinos and the left–handed mixing angles in the $\mu$–$\tau$ sector are determined to be

$$m_{\nu_{\mu}}^D(M_U) \simeq m_t(M_U) \simeq 100 - 120 \text{ GeV},$$

$$m_{\nu_{\tau}}^D(M_U) \simeq (9\epsilon^2 - \sigma^2)m_U \simeq 8 \text{ GeV},$$

$$\theta_{\mu\tau}^\ell \simeq -3\epsilon + \eta \simeq -0.437\eta_k \quad (\text{for } \eta_{cb}/\eta_k = +1).$$ \hspace{1cm} (21)

$\eta_{cb}/\eta_k = +1$ is required to maximize $\theta_{\mu\tau}^\ell$. Note that the Dirac mass of $\nu_\mu$ is quite different from $m_c(M_U) \simeq 300 \text{ MeV}$. This is contrary to what is often adopted in $SO(10)$. The difference arises because of the type A pattern of the mass matrices and the factor of $-3$ associated with the $(B - L)$ generator that goes into the $\nu_\mu$ Dirac mass [29].

Given the bizarre pattern of quark and lepton masses and mixing, we regard the overall fit to all of them, good to within 10%, using the pattern shown in Eq. (15), as reason to take this pattern seriously.

Note that owing to the asymmetric mass matrix, the square–root formula for the mixing angle $\theta_{\mu\tau}^\ell$ receives a correction given by the factor $[(-3\epsilon + \eta)/(3\epsilon + \eta)]^{1/2}$. For the values of $\eta$ and $\epsilon$ determined in Eq. (21), this factor is about 1.8 for $\eta_{cb}/\eta_k = +1$.

Although all the entries for the Dirac mass matrix are now fixed, to obtain the parameters for the light neutrinos one needs to specify the Majorana mass matrix. Unfortunately, here there is much less information to guide our hypotheses. For concreteness, let us imagine that this too takes the type A form:

$$M_\nu^R = \begin{pmatrix} 0 & y \\ y & 1 \end{pmatrix} M_R,$$ \hspace{1cm} (22)

where we allow $y = \eta_y |y|$ to have either sign, i.e., $\eta_y = \pm 1$. Note that Majorana mass matrices are constrained to be symmetric by Lorentz invariance. The seesaw mass matrix $(-N(M_\nu^R)^{-1}N^T)$ for the light ($\nu_\mu - \nu_\tau$) system is then
\[ M^\text{light}_\nu = \begin{pmatrix} 0 & A \\ A & B \end{pmatrix} \frac{m^2_U}{M_R}, \]  

where \( A \simeq (\sigma^2 - 9\epsilon^2)/y \) and \( B \simeq -(\sigma + 3\epsilon)(\sigma + 3\epsilon - 2y)/y^2 \). With \( A \ll B \), this yields

\[ m_{\nu_3} \simeq B \frac{m^2_U}{M_R}; \quad m_{\nu_2} \simeq -\frac{A^2}{B^2}; \quad \tan \theta^\nu_{\mu \tau} = \sqrt{\frac{m_{\nu_2}}{m_{\nu_3}}}. \]  

Correspondingly,

\[ y\eta_\ell \simeq \pm \sqrt{\frac{m_{\nu_2}}{m_{\nu_3}}} (3|\epsilon| - |\sigma|) \]

\[ \frac{3|\epsilon| + |\sigma| \pm 2\sqrt{m_{\mu_\ell}/m_{\nu_3}}}{3}, \]

where \( \eta_\ell = \pm 1 \) and we have used the fact that \( \epsilon \) and \( \sigma \) are relatively negative for \( \eta_{\nu_\beta}/\eta_\tau = +1 \) (See Eq. (21)). For a given choice of the sign of \( y \) relative to that of \( \epsilon \), and for a given mass ratio \( m_{\nu_2}/m_{\nu_3} \), we can now determine \( y\eta_\ell \) using Eq. (25) and the values of \( \epsilon \) and \( \sigma \) obtained in Eq. (20). Corresponding to \( m_{\nu_2}/m_{\nu_3} = (1/10, 1/15, 1/20, 1/25, 1/30) \), we calculate \((y\eta)\) = (0.0543, 0.0500, 0.0468, 0.0444, 0.0424) and \((y\eta)\) = (0.2359, 0.3781, 0.7681, 8.4095, 1.0519), where the subscripts \( \pm \) correspond to \( y\eta_\ell = \pm 1 \). The case of \( y\eta_\ell = +1 \) typically requires smaller values of \( |y| \) than for the case of \( y\eta_\ell = -1 \). The former \((y\eta_\ell = +1) \) is more in accord with the idea of the type A matrix with flavor symmetries being the origin of hierarchical masses, than the latter \((y\eta_\ell = -1) \).

We obtain for the neutrino oscillation angle:

\[ \theta^\nu_{\mu \tau} \simeq \theta^\ell_{\mu \tau} - \theta^\nu_{\mu \tau} \simeq 0.437 \pm \sqrt{\frac{m_{\nu_2}}{m_{\nu_3}}}. \]  

If \( \eta_\beta/\eta_\tau = +1 \), the relative + sign should be chosen, and for \( \eta_\beta/\eta_\tau = -1 \) the relative - sign should be chosen. Taking the relative + sign, i.e., \( \eta_\beta\eta_\tau = +1 \) (in accord with the smaller values of \( y \leq 1/10 \)), and using the more precise expression given in Eq. (12) (and replacing \( \sqrt{m_{\mu}/m_{\tau}} \) by the numerical value of the charged lepton mixing angle \( \simeq 0.437 \)), we obtain

\[ \sin^2 2\theta^\nu_{\mu \tau} = (0.96, 0.91, 0.86, 0.83, 0.81) \]

for \( m_{\nu_2}/m_{\nu_3} = (1/10, 1/15, 1/20, 1/25, 1/30) \).

As previously advertized, we see that one can derive rather plausibly a large \( \nu_\mu - \nu_\tau \) oscillation angle \( \sin^2 2\theta^\nu_{\mu \tau} \geq 0.8 \), together with an understanding of hierarchical masses and mixing of the quarks and the charged leptons, while maintaining a large hierarchy in the seesaw derived masses \( (m_{\nu_2}/m_{\nu_3} = 1/10 - 1/30) \) of \( \nu_\mu \) and \( \nu_\tau \), all within a unified framework including both quarks and leptons. In the example exhibited here, the mixing angles for the mass eigenstates of neither the neutrinos nor the charged leptons are really large, \( \theta^\ell_{\mu \tau} \simeq 0.437 \simeq 23^0 \) and \( \theta^\nu_{\mu \tau} \simeq (0.18 - 0.31) \approx (10 - 18)^0 \), yet the oscillation angle obtained by combining the two is near-maximal. This contrasts with most previous work, in which a large oscillation angle is obtained either entirely from the neutrino sector (with nearly degenerate neutrinos) or entirely from the charged lepton sector. In our case, the mass eigenstates of the neutrinos and the charged leptons are approximately also the respective gauge eigenstates.
V. INCLUSION OF THE FIRST FAMILY: \( \nu_E - \nu_\mu \) OSCILLATION

There are several alternative ways to include the first family, without upsetting the successful predictions of the 2-3 sector. In the absence of a deeper understanding, the theoretical uncertainties in analyzing the masses and mixings of the first family are much greater than for the heavier families, simply because the masses of the first family are so small, that relatively tiny perturbations can significantly affect their values. With this warning, we will now briefly consider, as an illustrative example and “proof of principle”, a minimal extension to the first family, inspired by the type A pattern, and also a noteworthy variant.

The 3 \times 3 Dirac mass matrices in the four sectors take the form:

\[
U = \begin{pmatrix} 0 & \epsilon' & 0 \\ -\epsilon' & 0 & \epsilon + \sigma \\ 0 & -\epsilon + \sigma & 1 \end{pmatrix} m_U, \quad D = \begin{pmatrix} 0 & \epsilon' + \eta' & 0 \\ -\epsilon' + \eta' & 0 & \epsilon + \eta \\ 0 & -\epsilon + \eta & 1 \end{pmatrix} m_D,
\]

\[
N = \begin{pmatrix} 0 & -3\epsilon' & 0 \\ 3\epsilon' & 0 & -3\epsilon + \sigma \\ 0 & 3\epsilon + \sigma & 1 \end{pmatrix} m_U, \quad L = \begin{pmatrix} 0 & -3\epsilon' + \eta' & 0 \\ 3\epsilon' + \eta' & 0 & -3\epsilon + \eta \\ 0 & 3\epsilon + \eta & 1 \end{pmatrix} m_D. \quad (28)
\]

At the level of underlying primary couplings, the \( \epsilon' \) term is very similar to the \( \epsilon \) term, arising through a Yukawa coupling \( a_{12}16_i16_245_H10_H \). The \( \eta' \) term arises through the coupling \( g_{12}16_i16_216_H16_H \). With \( \epsilon, \sigma, \eta, m_U \) and \( m_D \) determined essentially by considerations of the second and the third families (Eq. (20)), we now have just two new parameters in Eq. (28) – i.e. \( \epsilon' \) and \( \eta' \) – which describe five new observables in the quark and charged lepton sector: \( m_u, m_d, m_e, \theta_C \) and \( V_{ub} \). Thus with \( m_u \approx 1.5 \text{ MeV} \) (at \( M_U \)) and \( m_e/m_\mu \) taken as inputs to fix \( \epsilon' \) and \( \eta' \), one can calculate the other three observables. In addition, \( m^{D}_{\nu_c} \) will also be determined.

At the outset, it is worth noting that this specific pattern (Eq. (28)) is consistent with the empirical unification scale relations: \( m_d \approx 3m_e, m_s \approx m_\mu/3 \) \cite{ref20} and \( m_b \approx m_\tau \), which in turn imply \( m_dm_am_b \approx m_em_\mu m_\tau \), and thus \( \text{Det}(D) \approx \text{Det}(L) \). Eq. (28) obeys this relation to a good approximation, for the following reason. From \( \text{Det}(U) = \epsilon'^2m^3_U \), one obtains \( \epsilon' \approx \sqrt{m_u/m_e(m_e/m_t)} \approx 2 \times 10^{-4} \). Using \( \text{Det}(L) = (9\epsilon'^2 - \eta'^2)m^3_D = m_em_\mu m_\tau \), and neglecting the \( \epsilon'^2 \) term (justified \textit{a posteriori}), one obtains \( |\eta'| \approx \sqrt{m_e/m_\mu(m_\mu/m_\tau)} \approx 4.4 \times 10^{-3} \). Since \( |\epsilon'| \ll |\eta'| \), one gets \( \text{Det}(D) \approx \text{Det}(L) \), as described.

Combining the two predictions for the second and the third families given by Eq. (19), we are thus led to a total of five predictions for observable parameters of the quark and charged lepton system of the three families.

\[
m_b(m_b) \approx 4.9 \text{ GeV}
\]

\[
m_s(1 \text{ GeV}) \approx 116 \text{ MeV}
\]

\[
m_d(1 \text{ GeV}) \approx 8 \text{ MeV}
\]

\[
\theta_C \approx \sqrt{m_d/m_s} - \epsilon^{i\phi} \sqrt{m_u/m_c}
\]

\[
|V_{ub}/V_{cb}| \approx \sqrt{m_u/m_c} \approx 0.07 . \quad (29)
\]
Further, the Dirac masses and mixings of the neutrinos and the mixings of the charged leptons also get determined. Including those for the $\mu - \tau$ families listed in Eq. (21) we obtain:

\[
m_D^{\nu_\tau}(M_U) \approx 100 - 120 \text{ GeV}; \quad m_D^{\nu_\mu}(M_U) \approx 8 \text{ GeV}, \quad \theta_{\mu\tau}^\ell \approx -0.437 \eta_e
\]

\[
m_D^{\nu_e}(M_U) \approx \frac{9\epsilon^2}{(9\epsilon^2 - \sigma^2)} m_U \approx 0.4 \text{ MeV}
\]

\[
\theta_{\mu\mu}^\ell \approx \left[ \frac{\eta' - 3\epsilon'}{\eta' + 3\epsilon'} \right]^{1/2} \sqrt{m_e/m_\mu} \approx 0.85 \sqrt{m_e/m_\mu} \approx 0.06
\]

\[
\theta_{\mu\tau}^\ell \approx \frac{1}{0.85} \sqrt{m_e/m_\tau}(m_\mu/m_\tau) \approx 0.0012.
\] (30)

In evaluating $\theta_{\mu\mu}^\ell$, we have assumed $\epsilon'$ and $\eta'$ to be relatively positive.

Note that the first five predictions in Eq. (29) pertaining to observed parameters in the quark system are fairly successful. Considering the bizarre pattern of the masses and mixings of the fermions in the three families (recall comments on $V_{cb}$, $m_b/m_\tau$, $m_s/m_\mu$ and $m_d/m_e$), we feel that the success of the mass pattern exhibited by Eq. (28) is rather remarkable. It is worth stressing that (a) the asymmetric mass pattern induced by the $a_{03}$ coupling (see Eq. (14)), (b) its contribution being proportional to $B - L$ and (c) the $g_{23}$ coupling leading to non–trivial CKM matrix have all played essential roles in leading to a successful mass pattern with a minimal Higgs system – i.e., involving a single $45_H$, $16_H$ and $10_H$ – two of which are needed for unification scale symmetry breaking of $SO(10)$ anyhow. This is one reason for taking patterns like Eq. (28) seriously as a guide for considerations on proton decay. A particularly interesting variant is obtained in the limit $\epsilon' \to 0$, as we will discuss at the end of this section.

Turning to the Majorana mass matrix, one could in general introduce (1,1), (1,2) and (1,3) entries involving the first family. We choose the (1,2) element to be zero, both because by itself (i.e., with (1,2) $\neq 0$, but (1,1) = (1,3) = 0) it would upset the success of the $\nu_\mu - \nu_\tau$ sector obtained above, and also for economy in parameters. We therefore consider the following pattern:

\[
M^R_\nu = \begin{pmatrix} x & 0 & z \\ 0 & 0 & y \\ z & y & 1 \end{pmatrix} M_R.
\] (31)

Consistent with our presumption that hierarchical masses have their origin in the type A pattern, we will assume that $x \sim z^2$ and $z \leq y < 1$ and that the zero in the (2,2) entry only reflects that this entry is small compared to $y^2$.

Note that all entries in the Majorana matrix arise through $f_{ij} \mathbf{16}_j \mathbf{16}_i \mathbf{10}_H \mathbf{10}_H$, which however do not contribute to the Dirac mass matrices. As we have stressed before, they therefore need not have the same flavor structure as the Dirac mass matrices of Eq. (28). Given the Dirac matrix $N$, we can work out the light neutrino mass matrix after the seesaw diagonalization. It is given by:

\[
M^\text{light}_\nu = \frac{m^2_U}{M_R \times y^2} x.
\] (32)
where we have inserted the mass of $\nu_2 \times$ figures prominently only in the mass of $x$ proton lifetime, see later.) Second, the discussion of the $\mu \epsilon z$, of the (2 $\nu$ charged lepton mixing) is tiny, but still significant for the small angle $M_{SW}$ explanation of angle arising from the light neutrino mass matrix itself (as opposed to the contribution from $M$ and $m$ small.  

$m$ justified, that of $\nu_3 \times$ can also be anticipated. We should add that for $\nu_e - \nu_\tau$ mixing is also small, since $\epsilon' \approx 2 \times 10^{-4}$ is extremely small.

Note that the $3 \times 3$ Majorana mass matrix $M^R_\nu$ introduces four new parameters: $x, y, z$ and $M_R$, of which the magnitude of $M_R \sim$ (few to 10) $\times 10^{14}$ GeV can quite plausibly be justified in the context of supersymmetric unification (see Sec. II). There are altogether six observables in the three neutrino system: the three masses and the three oscillation angles. While these depend on parameters of both the Majorana and the the Dirac mass matrices, since the latter are fully determined from the quark and the charged lepton system, one can expect at least two predictions for the neutrino system. These can be taken to be $\theta^{osc}_{\mu\tau}$ and, for example, $m_{\nu_e}$. In addition, to the extent that the magnitude of $M_R$ can plausibly be justified, that of $m_{\nu_e} \sim (1/10 - 1/30) eV$ can also be anticipated. We should add that for the limiting case of $\epsilon' = 0$ (a parameter of the Dirac sector), one would obtain predictions for $\theta^{osc}_{\mu\tau}$ and $\theta^{osc}_{e\tau}$ as well, regardless of the values of $x, y, z$ (see below).

As an example, let us take $x = 10^{-4}, y = 0.05$ and $z = -0.002$ along with the “central values” of $\sigma, \eta, \epsilon$ given in Eq. (21) and $\epsilon' = 2 \times 10^{-4}, \eta' = 4.4 \times 10^{-3}$ (from a fit to the $u$-quark and electron masses). Let us also put, in accord with discussions in Sec. II, $M_R \approx (5 - 15) \times 10^{14}$ GeV. The mass eigenvalues and the mixing angles in the neutrino sector are then:

$$m_{\nu_e} \approx (1/10 - 1/30) eV$$
$$m_{\nu_\mu} \approx 6.7 \times 10^{-3}(1 - 1/2) eV$$
$$m_{\nu_\tau} \approx 2 \times 10^{-6}(1 - 1/2) eV$$
$$\theta^{\nu}_{\mu\tau} \approx 0.25, \theta^{\nu}_{e\mu} \approx 0.009, \theta^{\nu}_{e\tau} \approx 0.0034 .$$

Combining with the mixing angles in the charged lepton sector, the oscillation angles, including that for $\nu_\mu - \nu_\tau$ sector given before, are given by (setting the relative phases to be 0 or $\pi$):

$$\begin{pmatrix}
9\epsilon'^2(x - z^2) & 3\epsilon'y\{-3\epsilon'z + (-3\epsilon + \sigma)x\} & 3\epsilon'(xy - (3\epsilon + \sigma)(x - z^2)) \\
3\epsilon'y\{-3\epsilon'z + (-3\epsilon + \sigma)x\} & -9\epsilon'^2y^2 & (3\epsilon + \sigma)y\{3\epsilon'z - (-3\epsilon + \sigma)x\} \\
3\epsilon'(xy - (3\epsilon + \sigma)(x - z^2)) & (3\epsilon + \sigma)y\{3\epsilon'z - (-3\epsilon + \sigma)x\} & (3\epsilon + \sigma)\{-2xy + (3\epsilon + \sigma)(x - z^2)\}
\end{pmatrix}$$

Three features arising from Eq. (30) are especially noteworthy. First, that the parameter $x$ figures prominently only in the mass of $\nu_e$, which is given by

$$m_{\nu_e} \approx \frac{81\epsilon'^4}{xy^4(9\epsilon^2 - \sigma^2)^2}\frac{m^4_{\nu_\mu}}{M^R} \approx \frac{10^{-10}}{x} \text{ eV} ,$$

where we have inserted the mass of $\nu_\tau \simeq 1/20$ eV. For $x = (10^{-5} - 10^{-4}), m_{\nu_e} \approx (2 \times 10^{-5} - 2 \times 10^{-6})$ eV. ($x$ should be less than $10^{-4}$, otherwise one would run into a conflict with proton lifetime, see later.) Second, the discussion of the $\mu - \tau$ sector in Sec. IV is corrected by terms that are proportional to $\epsilon'$, which is tiny, or by terms proportional to $z$. As long as $z^2 \leq x/3$ (from the (3,3) entry) and $z < 10^6x$ (from the (2,3) entry), the discussion of the (2,3) sector will remain essentially unaltered. And third, that the $\nu_e - \nu_\mu$ mixing angle arising from the light neutrino mass matrix itself (as opposed to the contribution from charged lepton mixing) is tiny, but still significant for the small angle MSW explanation of the solar neutrino puzzle. The $\nu_e - \nu_\tau$ mixing is also small, since $\epsilon' \approx 2 \times 10^{-4}$ is extremely small.
\[
\begin{align*}
\theta_{\mu e}^{\text{osc}} & \simeq 0.437 + \sqrt{m_{\nu_2}/m_{\nu_3}} \\
\theta_{e\mu} \simeq \theta_{e\mu} - \theta_{e\mu}^{\nu} & \simeq 0.06 \pm 0.009 \\
\theta_{e\tau} & \simeq \theta_{e\tau} - \theta_{e\tau}^{\nu} \simeq 5.5 \times 10^{-4} \pm 0.0034 .
\end{align*}
\] (35)

Though the mixing angles involving the first family are small, \(\theta_{e\mu}^{\nu}\) does play an important role for solar neutrinos. Combining \(\theta_{e\mu}^{\nu}\) with the corresponding mixing angle from the charged lepton sector, \(\theta_{e\mu}^{\nu} \simeq 0.85 \sqrt{m_e/m_\mu} \simeq 0.06\), where the factor 0.85 is a correction factor from the asymmetric \(e^l\) term in Eq. (28), we find that \(\theta_{e\mu}^{\nu} \simeq 0.05\) if the two contributions subtract. This is just about the right value to explain the solar neutrino data via the small angle MSW mechanism [22]. Note that without the (1,3) entry \(z\), this is just about the right value to explain the solar neutrino data via the small angle MSW mechanism [22].

Note that without the (1,3) entry \(z\) in Eq. (31), \(\theta_{e\mu}^{\nu}\) would have been less than 0.002, which would have resulted in \(\theta_{e\mu}^{\nu}\) being larger than the required value for MSW by about 15-20%. This is a possible motivation to introduce the (1,3) entry \(z\) into Eq. (31). Since the discussion of the (2-3) sector is little affected so long as \(z < 10^3 x\) and \(z^2 \leq x/3\), we will allow a range \(|z| = (0.002 - 0.006)\), corresponding to \(x = (10^{-5} - 10^{-4})\). This will be significant for the discussion of proton decay, especially for final states involving charged leptons.

Another consideration for the choice of the various entries in \(M_\nu^{R}\) is the generation of cosmological baryon asymmetry. Note that for \(x = 10^{-5} - 10^{-4}\), the mass of \(\nu_\tau^{R}\) is in the range \(5 \times (10^9 - 10^{11})\) GeV, which is of the right magnitude for producing \(\nu_\tau^{R}\) following reheating and inducing lepton asymmetry in \(\nu_\tau^{R}\) decay, that is subsequently converted to baryon asymmetry via the electroweak sphalerons [31]. For the lepton number violating decay of \(\nu_\tau^{R}\) to be out of equilibrium, it is necessary that the mixing of \(\nu_\tau^{R}\) with \(\nu_\tau^{R}\) be less than a few times \(10^{-3}\). Otherwise the decay \(\nu_\tau^{R} \rightarrow H^0 + \nu_\tau^{L}\), which can proceed by utilizing the large (order one) Dirac Yukawa coupling \(h_{33}\) of \(\nu_\tau^{R}\) will bring \(\nu_\tau^{R}\) into equilibrium, inhibiting the generation of lepton asymmetry. The choice \(z \simeq 0.002\) is consistent with this constraint.

While we cannot claim that our several choices have been unique, we have demonstrated that a rather simple pattern for the four Dirac mass matrices, motivated and constrained by the group structure of \(SO(10)\), is consistent within 10 to 20% with all observed masses and mixing of the quarks and the charged leptons. This fit is significantly overconstrained, as we have discussed. The same pattern, supplemented with a similar structure for the Majorana mass matrix, quite plausibly accommodates both the SuperKamiokande result with the large \(\nu_\mu - \nu_\tau\) oscillation angle required for the atmospheric neutrinos and a small \(\nu_e - \nu_\mu\) oscillation angle relevant for theories of the solar neutrino deficit.

**The \(e' = 0\) Limit: A special case**

Before turning to proton decay, it is worth noting that much of our discussion of fermion masses and mixings, including those of the neutrinos, is essentially unaltered if we go to the limit \(e' \rightarrow 0\) of Eq. (28). This limit clearly involves:

\[
\begin{align*}
m_u & = 0, \ \theta_C \simeq \sqrt{m_d/m_s} \\
|V_{ub}| & \simeq \sqrt{\frac{\eta - \epsilon}{\eta + \epsilon}} \sqrt{m_d/m_b(m_s/m_b)} \simeq (2.1)(0.039)(0.023) \simeq 0.0019 \\
m_{\nu_e} & = 0, \ \theta_{e\mu}^{\nu} = \theta_{e\tau}^{\nu} = 0.
\end{align*}
\] (36)
All other predictions will remain unaltered. Now, among the observed quantities in the list above, $\theta_C \simeq \sqrt{m_d/m_s}$ is indeed a good result. Considering that $m_u/m_t \approx 10^{-5}$, $m_u = 0$ is also a pretty good result. There are of course, plausible small corrections arising from higher dimensional operators (for example) involving Planck scale physics which could induce a small value for $m_u$. The question arises: Does such a small correction contributing to $m_u$ arise primarily through the $(1,2)$ entry (i.e., $\epsilon' \neq 0$) in $U$ and $N$ or does it arise primarily through the $(1,1)$ entry – to be called $\delta \neq 0$ of $U$ and $N$? In case of the former (with $\delta = 0$), one would get $\epsilon' \approx 2 \times 10^{-4}$, as noted above; while for the latter (with $\epsilon' = 0, \delta \neq 0$), one would obtain $\delta \simeq m_u/m_t \approx 10^{-5}$. Note that putting $\epsilon' = 0$ does not matter for $D$ and $L$ because $\eta' \gg \epsilon'$. The two cases therefore yield essentially identical results as regard fermion masses and mixings. Differences of about 20% between their predictions can arise only as regards $\theta_{\text{osc}}$ for which the former can fare better (depending upon the choice of phases) than the latter; while the latter $(\delta \neq 0, \epsilon' = 0)$ fares better than the former as regards $\theta_C$. One possible way to distinguish these two closely related variants for fermion masses and mixings will be a more precise determination of $V_{ub}$. The former predicts it to be near $\sqrt{m_u/m_c} V_{cb} \simeq (0.07)(0.04) \simeq 0.0028$, while the latter is somewhat smaller ($V_{ub} \simeq 0.0019$). We will refer to these two variants as cases I and II respectively.

\begin{align*}
\text{Case I : } & \epsilon' \approx 2 \times 10^{-4}, \delta = 0 \\
\text{Case II : } & \delta \approx 10^{-5}, \epsilon' = 0
\end{align*} \hspace{1cm} (37)

As we will see, however, proton decay amplitudes differ significantly between the two cases. Typically they are larger for case I (by a factor of 2-3) than for case II. Not knowing the precise origin of the small corrections contributing to $m_u$, we will consider cases I and II as equally viable guides for forming expectations for proton decay. We will in fact derive the proton decay amplitudes with $\epsilon' \neq 0$ (case I). Given that $\delta \ll \epsilon'$, we obtain the results for case II simply by setting $\epsilon' \approx 0$.

VI. NEUTRINO MASSES AND PROTON DECAY

In an earlier paper \[12\] we pointed out that the theory of neutrino masses can significantly affect expectations for proton decay, as regards both its rate and, especially, its branching ratios. This happens because in supersymmetric unified theories a new set of color triplet fields is needed to generate heavy Majorana masses for the right-handed (RH) neutrinos, as required for the seesaw mechanism. Exchange of its superpartners generates new dimension 5 operators, that appear in addition to the “standard” $d = 5$ operators in the effective Lagrangian. The standard operators arise through the exchange of color triplets that are related to the electroweak doublets, such as those appearing in the $5 + \overline{5}$ of $SU(5)$ or the $10$ of $SO(10)$ \[82,33\].

The standard $d = 5$ operators are estimated to dominate over the gauge boson mediated $d = 6$ proton decay operators. Furthermore, owing to a combination of color antisymmetry, Bose symmetry of the superfields, and hierarchical Yukawa couplings of the fermions, they predict that the dominant modes are $\overline{\nu}_\mu K^+$ and $\overline{\nu}_\mu \pi^+$ modes, while $e^+\pi^0$ and $e^+K^0$ and even $\mu^+\pi^0$ and $\mu^+K^0$ modes are highly suppressed, at least for small and moderate values of $\tan\beta \ (\leq 15)$ \[82,33\].
In Ref. [12] we argued that for neutrino masses in a plausible range the new $d = 5$ operators could compete favorably with the standard ones, and induce potentially observable, though not yet excluded, proton decay rates. Within the new contribution charged lepton decay mode amplitudes, including those for $\mu^+K^0, \mu^+\pi^0$ and possibly $e^+K^0$ and $e^+\pi^0$, were generally found to be significant relative to the $\pi K^+$ and $\pi\pi^+$ modes, even for low $\tan\beta$. Thus charged lepton modes of nucleon decay potentially furnish an independent signature for the interactions that play a central role in the theory of neutrino masses.

As mentioned earlier, in our previous quantitative work we had to hypothesize values for the neutrino masses. Now, after the SuperK observations, we can revisit the quantitative aspect with some crucial experimental information reliably in hand. Changes are induced not only for the new $d = 5$ operators that are directly related to neutrino masses, but also for the standard $d = 5$ operators.

To address these issues in detail, we must specify the nature of the color triplet Higgsino couplings. This issue is closely tied to the mechanism of doublet–triplet splitting in SO(10), to which we now turn.

A. Natural doublet-triplet splitting in SO(10)

In supersymmetric SO(10), a natural doublet–triplet splitting can be achieved by coupling the adjoint Higgs $45_H$ to a $10_H$ and a $10'_H$, with $45_H$ acquiring a unification–scale VEV in the $B − L$ direction [24]: $\langle 45_H \rangle = (a, a, a, 0, 0) \times \tau_2$ with $a \sim M_U$. As discussed in Section II, to generate CKM mixing for fermions we require an $\langle 16_H \rangle_d$ that acquires an electroweak scale vacuum expectation value. To insure accurate gauge coupling unification, the effective low energy theory should not contain split multiplets beyond those of MSSM. Thus the MSSM Higgs doublets must be linear combinations of the $SU(2)_L$ doublets in $10_H$ and $16_H$. A simple set of superpotential terms that ensures this and incorporates doublet–triplet splitting is:

$$W_H = \lambda 10_H 45_H 10'_H + M_{10} 10_H^2 + \lambda' \overline{16_H} 16_H 10_H + M_{16} 16_H \overline{16_H}.$$  \hspace{1cm} (38)

A complete superpotential for $45_H, 16_H, \overline{16_H}, 10_H, 10'_H$ and possibly other fields, which ensure that $45_H, 16_H$ and $\overline{16_H}$ acquire unification scale VEVs with $\langle 45_H \rangle$ being along the $(B − L)$ direction, that exactly two Higgs doublets ($H_u, H_d$) remain light, with $H_d$ being a linear combination of $(10_H)_d$ and $(16_H)_d$, and that there are no unwanted pseudoGoldstone bosons, can be constructed [35, 39]. The various possibilities generate different predictions for threshold corrections in the unification of gauge couplings, for example. As we will explain, such differences will have implications for proton decay rate, and we will allow for such effects.

The Higgs doublet and the color triplet mass matrices following from Eq. (38) are, in $SU(5)$ notation,

\[\text{It is intriguing that the Higgs fields used in [37] can be neatly incorporated into a single adjoint representation of E7.}\]
With the vacuum expectation value \( \langle 45_H \rangle \) in the \( B-L \) direction it does not contribute to the doublet matrix, so one pair of Higgs doublet remains light, while all triplets acquire unification scale masses. The light MSSM Higgs doublets are

\[
H_u = 10_u, \quad H_d = \cos \gamma 10_d + \sin \gamma 16_d ,
\]

with \( \tan \gamma \equiv \lambda' \langle 10_\mathbb{H} \rangle /M_{16} \). Consequently, \( \langle 10_d \rangle = \cos \gamma v_d, \langle 16_d \rangle = \sin \gamma v_d \), with \( \langle H_d \rangle = v_d \) and \( \langle 16_d \rangle \) and \( \langle 10_d \rangle \) denoting the electroweak VEVs of those multiplets. Note that the \( H_u \) is purely in \( 10_\mathbb{H} \) and that \( \langle 10_d \rangle^2 + \langle 16_d \rangle^2 = v_d^2 \).

This pattern of gauge symmetry breaking, while motivated on separate grounds, nicely harmonizes with the fermion mass pattern advocated in Secs. IV and V. Specifically, the \( g_{ij} \) terms will contribute to the down–flavored fermion masses, while there are no analogous terms in the up–flavor sector. We also note the relation \( \tan \gamma \tan \beta \approx m_l /m_u \approx 60 \). Since \( \tan \beta \) is separately observable, the angle \( \gamma \), which will prove relevant for proton decay, is thereby anchored.

**B. Baryon number violation**

By combining Eqs. (38)-(40) with the Yukawa couplings from Eq. (14), we can now obtain the effective baryon number violating superpotential. Denote the color triplet and anti–triplet in \( 10_\mathbb{H} \) as \( (H_C, \overline{H}_C) \), in \( 10_{\mathbb{H}}^a \) as \( (H'_C, \overline{H'}_C) \) and in \( (16_\mathbb{H}, 16_\mathbb{H}) \) as \( (H_C, \overline{H}_C) \). The relevant Yukawa couplings of these color triplets, extracted from Eq. (14) are:

(i) \( h_{ij} (\frac{1}{2} Q_i Q_j H_C + Q_i L_j \overline{H}_C) \),

(ii) \( -a_{ij} [\frac{1}{2} Q_i Q_j H_C (B-L) q_j + Q_i L_j \overline{H}_C (B-L) l_j] \),

(iii) \( g_{ij} \langle 16_\mathbb{H} \rangle /M Q_i L_j \overline{H}_C \),

(iv) \( f_{ij} \langle 16_\mathbb{H} \rangle /M Q_i q_j H_C \).

Note that for the \( a_{ij} \) coupling there are two contractions: \( 10_\mathbb{H} \times 45_\mathbb{H} \supset 10 + 120 \). Whereas the antisymmetric \( 120 \) contributes to fermion masses (for \( \langle 45_\mathbb{H} \rangle \propto B-L \)), as in Eq. (15), it is the symmetric \( 10 \) that leads to the Yukawa couplings of \( H_C \) and \( \overline{H}_C \). Although the \( a_{ij} \) term can arise through quantum gravity or stringy effects, for concreteness, we have assumed that it arises by integrating out a pair of superheavy \( 16 + \overline{16} \) states which couple through the renormalizable interactions \( W \supset (16_2 16_2) 10_\mathbb{H} + (16_3 16_3) 45_\mathbb{H} + M' 16_\mathbb{H} \). This is why the \( (B-L) \) generator appears in the \( a_{ij} \) couplings involving \( H_C, \overline{H}_C \). The strength of this coupling is then fixed in terms of the corresponding doublet coupling. Similarly, in the \( f_{ij} \) coupling, there are two \( SO(10) \) contractions, and we shall assume both to have comparable strength.

Baryon number violating operators of the type \( QQQL/M \) are the ones that connect to ordinary quarks and leptons by wino dressing, and which generally dominate (but see below).

---

5Here we have absorbed the factor \( \langle 45_\mathbb{H} \rangle /M \) into \( a_{ij} \) and the factor \( \langle 16_\mathbb{H} \rangle /M \) into \( g_{ij} \). We use the same notation \((a_{ij} \) and \( g_{ij} \)) for these redefined quantities. As for \( f_{ij} \), we define \( \hat{f}_{ij} \equiv f_{ij} \langle 16_\mathbb{H} \rangle /M \).

6A variant is to interchange the indices 2 and 3 in this renormalizable interaction.
Integrating out the color triplet fields, one arrives at the following effective superpotential terms involving these operators:

\[
W_{\text{eff}}^{(L)} = M_{\text{eff}}^{-1} \left\{ (u^T \hat{H} d') \left\{ u^T \hat{H} V' \ell - d'^T \hat{H} V' \nu' \right\} + 3(u^T \hat{H} d') \left\{ u^T \hat{A} V' \ell - d'^T \hat{A} V' \nu' \right\} \\
- (u^T \hat{A} d') \left\{ u^T \hat{H} V' \ell - d'^T \hat{H} V' \nu' \right\} - 3(u^T \hat{A} d') \left\{ u^T \hat{A} V' \ell - d'^T \hat{A} V' \nu' \right\} \\
- \tan \gamma (u^T \hat{H} d') \left\{ u^T \hat{G} V' \ell - d'^T \hat{G} V' \nu' \right\} + \tan \gamma (u^T \hat{A} d') \left\{ u^T \hat{G} V' \ell - d'^T \hat{G} V' \nu' \right\} \right\} + M_{16}^{-1} (u^T \hat{F} d') \left\{ u^T \hat{G} V' \ell - d'^T \hat{G} V' \nu' \right\}.
\]

(41)

Here \( M_{\text{eff}} = (\lambda a)^2/M_{10} \). \( u \) and \( \ell \) denote the column matrices of the physical left–handed up quark and charged lepton superfields in the supersymmetric basis (i.e., a basis in which neutral gaugino interactions are flavor diagonal). The \( d' \) and \( \nu' \) fields are related to the physical down quark and light neutrino fields by the CKM matrices for quarks and leptons: \( d' = V_{\text{CKM}} d \) and \( \nu' = V_{\text{CKM}}' \nu \), while \( V' = V_\nu V_\ell \), where \( V_\nu \) and \( V_\ell \) diagonalize respectively the left–handed up quark and charged lepton mass matrices: \( u^{(g)} = V_\nu u^{(m)} \), where \( (g) \) and \( (m) \) denote the gauge and mass eigenstates. In writing Eq. (41), the color indices \( (\alpha, \beta, \gamma) \) on quark fields are suppressed and use is made of the fact that \( (u^T \hat{H} d') \) and \( (u^T \hat{G} V' \ell - d'^T \hat{G} V' \nu') \), which holds because of antisymmetry under the interchange \( \alpha \leftrightarrow \beta \) and because \( \hat{H} \) is symmetric. The superscript \( (L) \) on \( W_{\text{eff}}^{(L)} \) signifies that all the fields in \( W_{\text{eff}}^{(L)} \) belong to \( SU(2)_L \) doublets. We comment later on the contributions from \( W_{\text{eff}}^{(R)} \), involving \( RRRR \) operators of the form \( u^c u^c d^c e^c \), involving \( SU(2)_L \) singlets, which can be important in certain range of supersymmetric parameter space.

The \( 3 \times 3 \) matrices \((\hat{H}, \hat{A}, \hat{G}, \hat{F})\) operate in the family space and are related to the Yukawa coupling matrices given by the elements \((h_{ij}, a_{ij}, g_{ij}, f_{ij})\) respectively, as follows:

\[
(\hat{H}, \hat{A}, \hat{G}, \hat{F}) = V_u^T (h, a, g, f) V_u.
\]

(42)

The matrices \((h, a, g, f)\) are related to those appearing in Eqs. (28),(31) and are given by:

\[
h = h_{33} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \sigma \\ 0 & \sigma & 1 \end{pmatrix}, \quad a = h_{33} \begin{pmatrix} 0 & \epsilon' & 0 \\ \epsilon' & 0 & \epsilon \\ 0 & \epsilon & 0 \end{pmatrix},
\]

\[
g = \frac{h_{33}}{\tan \gamma} \begin{pmatrix} 0 & \eta' & 0 \\ \eta' & 0 & \eta - \sigma \\ 0 & \eta - \sigma & 0 \end{pmatrix}, \quad f = \tilde{f}_{33} \begin{pmatrix} x & 0 & z \\ 0 & y & z \\ z & y & 1 \end{pmatrix},
\]

(43)

where \( \tilde{f}_{ij} = f_{ij} v_R/M \). Values of \( \sigma, \eta, \epsilon, \eta', \epsilon' \) and \( y \) have been obtained in Sec. IV-V from considerations of fermion masses and mixings. From proton decay constraints and the neutrino sector we have estimated \( x \sim (10^{-5} - 10^{-4}) \) and \( z \sim (0.002 - 0.006) \). \( V_u \) can be worked out using Eq. (28) and (20), and is found to be close to an identity matrix. Its largest off–diagonal entry is the \((1,2)\) element \( \simeq \epsilon'/(\epsilon^2 - \sigma^2) \simeq -0.06 \). However the matrix \( V_\ell \) that diagonalizes the charged lepton mass matrix \( L \) is far from trivial, and so \( V' = V_u^T V_\ell \) picks up a substantial \((2,3)\) element \( \simeq -3\epsilon + \eta \simeq -0.437 \), reflecting sizable \( \mu - \tau \) mixing. The numerical values of the matrices \( V_u, V_\ell \) and \( V' \) are given in Appendix A, as are the matrices \((\hat{H}, \hat{A}, \hat{G}, \hat{F})\).
Even if one is skeptical of our particular pattern of fermion mass matrices, it is difficult to avoid the general conclusion\footnote{Barring near-complete accidental cancellation between diagonal and off–diagonal contributions to $m_{\nu_{\mu}}$.} that if neutrino masses are hierarchical, a large $\nu_\mu - \nu_\tau$ oscillation angle (the SuperK result) requires a sizable $\mu - \tau$ mixing angle (say $\geq 0.3$). This has significant implications for proton decay, as we now discuss.

After wino dressing, which dominates over gluino dressing for $\tan \beta \leq 20$, each of the seven terms in Eq. (41) leads to twelve four fermion operators for proton decay into $\nu + X$, a subset of which was exhibited in Ref. \cite{12}. In Appendix B.1 we give the complete expression for the neutrino as well as the charged lepton decay modes of the proton. Representative contributions to the proton decay amplitudes are analyzed in more detail for the dominant modes in Appendix B.2. We now briefly summarize the results of a lengthy investigation of the net effect of all these terms.

To evaluate the strength of each term in Eq. (41), we need $h_{33}$ and $\hat{f}_{33}$ (see Eq. (43)) at $M_U$. $h_{33}$ is determined using $h_{33}v_u \approx m_t(M_U) \approx 100 - 120$ GeV, which yields $h_{33} \approx 1/2$. $\hat{f}_{33}$ can be determined as follows. Using Eq. (23) for the mass matrix of the light $\nu_\mu - \nu_\tau$ sector, we have

$$m_{\nu_3} \approx B m^2_{U}/M_R,$$

where

$$B = -(\sigma + 3\epsilon)(\sigma + 3\epsilon - 2y) \approx 5.$$  

Here we have put $\sigma$ and $\epsilon$ from Eq. (20) and used $y = 0.047$, corresponding to $m_{\nu_2}/m_{\nu_3} = 1/15$. Putting $m_{\nu_3} = (1/20$ eV)$\zeta$, where $\zeta = 2$ to $1/2$, corresponding to SuperK results, and $m_U \approx m_t(M_U)$, we get $M_R \approx 10^{15}$ GeV/$\zeta$. Using $M_R = f_{33}v^2_R/M$ (see Eq. (4)), with $v_R = \langle 16_H \rangle = (2 \times 10^{16})$ GeV$\kappa_R$ and $\kappa_R \approx 1/2$ to 2, we find

$$\hat{f}_{33} = f_{33}v_R/M \approx (1/20)(1/\kappa_R\zeta).$$

We will use $\hat{f}_{33} \approx 1/20$ with the understanding that it is uncertain by a factor of 2-3 either way. Note that $\hat{f}_{33}$ is considerably larger, by about a factor of 200–700, than the value estimated in Ref. \cite{12}. This results partly from lowering of $m_{\nu_\tau}$ from a few eV (used in Ref. \cite{12}) to about 1/20 eV (the SuperK value) and in part from interplay between the mixings in the Dirac and the Majorana mass matrices via the seesaw mechanism. The latter has the net effect of enhancing $M_R \approx B m^2_{U}/m_{\nu_3}$, for a given $m_{\nu_3}$, precisely by a factor of $B \approx 5$, compared to what it would be without mixing. (Compare $M_R \approx 10^{15}$ GeV with its counterpart in Sec. II where, by ignoring mixing, we got $m^2_R = (1 - 3) \times 10^{14}$ GeV.) The two effects together, – i.e., lowering of $m_{\nu_3}$ and increase through $B$ – significantly enhance $\hat{f}_{33} = [B m^2_{U}/m_{\nu_\tau}]/v_R$.

It is interesting to note that the larger value of $M_R$ arising because of mixing has a further implication. Using $M_R \approx 10^{15}$ GeV (corresponding to $\zeta \approx 1$), $v_R = 2 \times 10^{16}$ GeV (for $\kappa_R \approx 1$) and $f_{33} \approx 1$, one obtains: $M = f_{33}v^2_R/M_R \approx 4 \times 10^{17}$ GeV, which is the perturbative string scale \cite{10}.
Note that the net value of $B$ (see Eq. (45)) depends on the parameters of the Majorana mass matrix as well as on $\sigma$ and $\epsilon$ from the Dirac mass matrix of the neutrinos, which in turn are determined within $SO(10)$ by the masses and mixings of quarks and charged leptons. That is why our expectations for proton decay are significantly affected by our understanding of the masses and mixings of quarks and charged leptons.

The full set of contributions to the proton decay amplitudes (from all the operators in Eq. (41) involving all possible combinations of family indices) was obtained numerically using Mathematica and is listed in Appendix B.1. Calculations of a few representative (dominant) contributions to the amplitudes are exhibited in detail in Appendix B.2, where estimates of the amplitudes allowing for uncertainties in the relative phases of different contributions are presented. A general discussion of proton decay rate is given in Appendix C. Based on the result of these appendices, we now discuss the general constraint on the proton decay amplitude and thereby on the mass scale $M_{\text{eff}}$ and $M_{16} \tan \gamma$ corresponding to the numerical estimate of the amplitude given in Appendix B. These constraints arise from existing lower limits on the proton lifetime.

C. Constraints on proton decay amplitudes from the proton lifetime

In Appendix C we show that with a certain (apparently) reasonable choice of supersymmetric spectrum, the lifetime of the proton decaying into neutrinos is:

$$\Gamma^{-1}(p \to \overline{\nu}_\tau K^+) \approx (2.2 \times 10^{31}) \text{ yrs} \times \left(\frac{0.67}{A_S} \right)^2 \left[\frac{0.006 \text{ GeV}^3}{\beta_H} \right]^2 \left[\frac{(1/6)}{(m_\overline{W}/m_\overline{q})} \right]^2 \left[\frac{m_\overline{q}}{1 \text{ TeV}} \right]^2 \left[\frac{2 \times 10^{-24} \text{ GeV}^{-1}}{A(\overline{\nu})} \right]^2. \quad (47)$$

Here $\hat{A}(\overline{\nu}) = A(\overline{\nu})/(2 \overline{f})$, where $A(\overline{\nu})$ is the strength of the four fermion proton decay amplitude and $\overline{f}$ is the average wino–dressing function (see Eq. (93) in Appendix B and Eq. (111)-(114) in Appendix C). The quantity $A(\overline{\nu})$ is simply the product of all the vertex factors in the wino–dressed Higgsino exchange diagram divided by the effective mass of the relevant color triplet Higgsino. For normalization purpose we can define $A(\overline{\nu})^{SU(5)} = (\lambda_c \lambda_s \theta_C^2)/M_{H_C}$, where $\lambda_i$ stand for the Yukawa couplings of the quarks at $M_U$. (For clarity of discussions, only the second generation contribution is kept here.) The quantity $A(\overline{\nu})$ is the full amplitude, including the loop factor associated with the wino dressing. If we substitute $A(\overline{\nu})^{SU(5)}$ defined above into Eq. (47), we will reproduce the results given in Ref. [34]. (We have allowed for a factor of 4 enhancement in the lifetime relative to [34], corresponding to an apparent slip by a factor of $\frac{1}{2}$ in going from Eq. (3.7) to Eq. (3.8) of that paper.)

Note that in writing Eq. (47), the short distance renormalization $A_S$ of the $d = 5$ operator in going from $M_U$ to $M_{\text{SUSY}}$ (see Ref. [34]) as well as the running factor to go from $M_{\text{SUSY}}$ to 1 GeV have been included. $A_S$ has a central value of about 0.67, which we shall adopt even for the $SO(10)$ model.

We have discussed in the appendix that typically proton would decay dominantly into $\overline{\nu}_\tau K^+$ and $\overline{\nu}_\mu K^+$ modes.\footnote{In some cases the former supercedes the latter by factors of 2-5} $\mu^+ K^0$ is also a likely prominent mode of the proton decay (see Sec. VI H) with a typical
in the rate, while the converse is true in some other cases (see Table. 1, Appendix C). We see from Eq. (47) that with $A(\nu_i) \approx 2 \times 10^{-24}$ GeV$^{-1}$, a reasonable “central value” for the partial lifetime $\Gamma^{-1}(p \to \nu_i K^+)^{-1}$ is $2.2 \times 10^{31}$ yrs., which corresponds to the hadronic matrix element $\beta_H = 0.006$ GeV$^3$ (this is the central value quoted in lattice calculations [41]), $(m_{\nu}/m_\mu) \approx 1/6$ and $m_\mu \approx 1$ TeV. Now, adding the $\nu_\mu K^+$ mode with a branching ratio $R$ ($R_{\text{average}} \approx 0.3$, say), this is however $25(1 + R)$ times smaller than the empirical lower limit [12]

$$\Gamma^{-1}(p \to \nu K^+)_\text{expt} \geq 5.6 \times 10^{32} \text{ yr.}$$

Thus, if the parameters have nearly their central values, the amplitude for the dominant of the two modes must satisfy the bound: $A(\nu_i K^+) \leq 2 \times 10^{-24}$ GeV$^{-1}/\sqrt{1 + R} \approx 4 \times 10^{-24}$ GeV$^{-1}/\sqrt{1 + R}$, where $\ell = \mu$ or $\tau$. Allowing that both $\beta_H$ and the ratio of superpartner masses $(m_{\nu}/m_\mu)$ might well be smaller by factor of 2 (say) than the value quoted above, and that $m_\mu$ could be (say) 1.4 TeV rather than 1 TeV, the theoretical value of the lifetime could plausibly increase by a factor of 32 compared to the “central value” $2.2 \times 10^{31}$ yr. This saturation of all uncertainties in the parameters, all in the direction so as to extend proton lifetime, strains credulity. If, nevertheless, one allows for such a variation in the parameters, the bound on the amplitude mentioned above will be relaxed by a factor of 5 to 6. Thus the empirical limit on proton lifetime leads to the rather conservative bound:

$$A(\nu_i) \leq 2.3 \times 10^{-24} \text{ GeV}^{-1}/\sqrt{1 + R}.$$  

(49)

This limit should be satisfied for every neutrino flavor $\nu_i$ (assuming $\nu_\mu K^+$ and $\nu_\tau K^+$ are comparable). For squark masses not exceeding about 1.5 TeV, we take Eq. (49) as a strict upper bound. It should however be noted that the lifetime depends quartically on the squark mass, so increasing $m_\tilde{q}$ by a factor of 2 to about 3 TeV, holding $m_{\tilde{W}}$ fixed, would lengthen proton lifetime by a factor of 16. If true, this would relax the bound on $A(\nu_i)$ quoted in Eq. (49) by as much as a factor of 4 to $A(\nu_i) \leq 9 \times 10^{-24}$ GeV$^{-1}/\sqrt{1 + R}$. Such a heavy spectrum ($m_\tilde{q} \sim 3$ TeV) for all three generations of squarks would however require severe fine-tuning of parameters in order to keep the vacuum expectation value of the light Higgs field at the electroweak scale. We shall assume a lighter squark spectrum ($m_\tilde{q} \sim 1.5$ TeV), for which there is no need for such an adjustment of parameters. In this case, the bound in Eq. (49) will have to be satisfied.

D. Constraints on $M_{\text{eff}}$ and proton decay via standard operators

In Appendix B, Eqs. (101) and (102) (or Table 1), we show that within the concrete $SO(10)$ model, the dominant $\nu_\tau K^+$ or $\nu_\mu K^+$ decay amplitude from the standard $d = 5$ operator for cases I and II are given by:

$$A(\nu K^+)_\text{std} \simeq \left[\frac{2\hat{h}_{\tilde{\nu} W}^2}{M_{\text{eff}}} \right] \frac{1}{2} \begin{pmatrix} 2.8 \times 10^{-5} \\
1.2 \times 10^{-5} \end{pmatrix} \left(\frac{1}{2} \epsilon_{\alpha \beta \gamma}(d^\alpha u^\beta)(s^\gamma \nu_3) \right).$$

(50)

branching ratio $\sim 20-30\%$. This will however not alter our discussion here appreciably.
There is an analogous expression for the new neutrino mass–related $d = 5$ operator, that will be discussed in the next subsection (E). The upper and lower entries in Eq. (50) correspond to cases I and II respectively.

We now compare the upper limit (Eq. (49)) on the amplitude from proton decay searches against theoretical expectations based on the concrete SO(10) model. This leads to constraints on $M_{\text{eff}}$ from the standard $d = 5$ operator (and on $M_{16}\tan\gamma$ from the new operator). The upper bound (Eq. (49)) on $\hat{A} (\nu_i)$ applies to the net amplitude, which is given by the sum of the contributions from the standard (Eq. (50)) and the new operators (Eq. (51) below). For the sake of clarity, we will derive constraints on $M_{\text{eff}}$ from the standard (respectively, the new) operator dominates. Indeed, near-complete accidental cancellation between the two contributions is unlikely to occur for both $\nu_\tau K^+$ and $\nu_\mu K^+$ modes.

Using the net contribution from the standard operators to the amplitude given by Eq. (50), and the definition of $\hat{A} (\nu_i)$, we obtain (putting $h_{33} \simeq 1/2$)

$$\hat{A} (\nu_\tau)_{\text{std}} \approx \frac{1}{M_{\text{eff}}} \left[ \frac{7}{3} \right] \times 10^{-6} (1/2 \text{ to } 3/2).$$

Comparing with the empirical upper limit on $\hat{A} (\nu_i)$ (Eq. (49)) obtained as above, we get:

$$M_{\text{eff}} \geq \left[ \frac{4}{1.7} \right] \times 10^{18} \text{ GeV} \ (1/2 \text{ to } 3/2).$$

Thus we see that $M_{\text{eff}}$ has to be rather large compared to the MSSM unification scale of $2 \times 10^{16}$ GeV in order that the standard operators may not run into conflict with the observed limits on proton lifetime. In effect, this reflects a net enhancement – by almost two orders of magnitude – of the standard $d = 5$ proton decay operators for realistic SO(10), compared to those in minimal SU(5), with low $\tan\beta \leq 3$ (see discussion in Appendix C). In the latter case, one need only require that the color triplet mass exceed about $(2 - 3) \times 10^{16}$ GeV.

Now, as mentioned in Appendix C, there are theoretically attractive mechanisms whereby the mass of $10'_H$, denoted by $M_{10}$ (see Eq. (38)), can be suppressed relative to the unification scale $M_U$. In this case, $M_{\text{eff}} \equiv (\lambda a)^2 / M_{10}$ can be larger than $\lambda a \sim M_U$. Very large values of $M_{\text{eff}} \gg M_U$ could however lead to large positive corrections to $\alpha_3 (m_Z)$, just from the doublet–triplet mechanism, above and beyond the value expected on the basis of simple coupling unification. For the doublet–triplet splitting mechanism described by Eq. (39) the shift in $\alpha_3 (m_Z)$ from this sector alone is found to be (see Appendix D)

$$\Delta \alpha_3 (m_Z) |_{\text{DT}} = \frac{[\alpha_3 (m_Z)]^2}{2\pi} \frac{9}{7} \ln \left( \frac{M_{\text{eff}} \cos \gamma}{M_U} \right).$$

This generalizes the expression given in Ref. [12][13], where the MSSM Higgs doublets were assumed to be contained entirely in $10'_H$ and not in $16_H$, corresponding to $\cos \gamma = 1$. The argument of the logarithm in Eq. (53) is simply the ratio: (product of the three color triplet masses)/ (product of the two superheavy doublet masses $\times M_U$). From the determinant of Eq. (39) we see that the product of the color triplet masses is equal to $M_{\text{eff}} M_{10} M_{16}$, while the two heavy doublets have masses given by $M_{10}$ and $M_{16}/\cos \gamma$. Note that the second heavy doublet has a mass larger than $M_{16}$. This is the reason for the presence of the $\cos \gamma$ factor in Eq. (53).
To evaluate the RHS of Eq. (53), we use the lower bound of $M_{\text{eff}}$ that is suggested by proton lifetime constraints, viz., $M_{\text{eff}} \geq (2 - 6) \times 10^{18}$ GeV or $(0.8 - 2.5) \times 10^{18}$ GeV (see Eq. (52)), and the MSSM unification scale of $M_U \simeq 2 \times 10^{16}$ GeV. We should also specify the value of $\cos \gamma$. It is obtained in terms of $\tan \beta$ as follows. From $m_t \simeq h_{33} \langle 10_{\text{u}} \rangle_u = h_{33} v_u$ and $m_b \simeq h_{33} \langle 10_{\text{d}} \rangle_d = h_{33} \cos \gamma v_d$, we have $m_t/m_b \simeq (v_u/v_d)/(1/\cos \gamma)$. Inserting $m_t/m_b \simeq 60$, we thus obtain $\cos \gamma \simeq (\tan \beta/60)$. (This can also be expressed as $\tan \beta \tan \gamma \simeq m_t/m_b$, for which is valid for $\tan \gamma \geq 3$.) We see that $\cos \gamma$ is a small number ($\approx 1/30 - 1/6$) for small and moderate values of $\tan \beta$ ($\approx 2 - 10$).

Now, let us recall that, in the absence of unification–scale threshold and Planck–scale effects, the MSSM value of $\alpha_3(m_Z)$ in the $\overline{\text{MS}}$ scheme, obtained by assuming gauge coupling unification, is given by $\alpha_3^0(m_Z)|_{\text{MSSM}} = 0.125 - 0.13 \pm 0.003 \pm 0.005$. This is about 5-10% higher than the observed value: $\alpha_3(m_Z) = 0.118 \pm 0.003 \pm 0.005$. Substituting the central value, $\alpha_3(M_Z) = 0.118$, and the MSSM unification scale of $M_U = 2 \times 10^{16}$ GeV, one obtains for $\cos \gamma = 1/20$: $\Delta \alpha_3(m_Z)|_{\text{DT}} \simeq 0.0046 - 0.0077$, for $M_{\text{eff}} \approx (2 - 6) \times 10^{18}$ GeV. Thus the constraint on $M_{\text{eff}}$ from proton lifetime for case I amounts to having in MSSM a net value $\alpha_3(m_Z)|_{\text{net}} = \alpha_3(M_Z)|_{\text{MSSM}} + \Delta \alpha_3(m_Z)|_{\text{DT}} + \Delta'_3 \simeq (0.132 - 0.135) + \Delta'_3$, for $\cos \gamma = 1/20$, where $\Delta'_3$ denotes other unification scale threshold and Planck scale effects evaluated at the electroweak scale. Including $\Delta \alpha_3(m_Z)|_{\text{DT}}$ and $\Delta'_3$, the net theoretical value of $\alpha_3(m_Z)$ is given by $\alpha_3(m_Z)|_{\text{net}} = \alpha_3(M_Z)|_{\text{MSSM}} + \Delta \alpha_3(m_Z)|_{\text{DT}} + \Delta'_3$. Since $\alpha_3(m_Z)|_{\text{MSSM}}$ is higher than the observed value, and $\Delta \alpha_3(m_Z)|_{\text{DT}}$ is relatively large (depending on $M_{\text{eff}} \cos \gamma$) and is positive (see below), one would need a net appropriately large negative value of $\Delta'_3$ so that $\alpha_3(m_Z)|_{\text{net}}$ may agree with the observed value. By varying the parameter $T = M_{\text{eff}} \cos \gamma/(2 \times 10^{18}$ GeV) appropriately so as to include the range $M_{\text{eff}} \geq (1 - 6) \times 10^{18}$ GeV, obtained from proton lifetime constraint, we can evaluate $\Delta \alpha_3(m_Z)|_{\text{net}}$ and thereby assess the needed value of $\Delta'_3 = \alpha_3(m_Z)_{\text{obs}} - \hat{\alpha}_3(m_Z)$, where $\hat{\alpha}_3(m_Z) \equiv \alpha_3(m_Z)|_{\text{MSSM}} + \Delta \alpha_3(m_Z)|_{\text{DT}}$. This is shown in the Table below.

| $T$  | 1/60 | 1/40 | 1/30 | 1/20 | 1/10 | 1/3 |
|------|------|------|------|------|------|------|
| $\Delta \alpha_3(m_Z)|_{\text{DT}}$ | 0.0014 | 0.0026 | 0.0034 | 0.0046 | 0.0066 | 0.0100 |
| $\alpha_3(m_Z)$ | 0.1284 | 0.1296 | 0.1304 | 0.1316 | 0.1336 | 0.1370 |
| $\Delta'_3$ | -8.8% | -10% | -10.5% | -11.5% | -13.2% | -16.1% |

Here $\Delta'_3 = \Delta'_3 = \alpha_3(m_Z)_{\text{obs}} - (\alpha_3(m_Z)_{\text{obs}} - \hat{\alpha}_3(m_Z))/\alpha_3(m_Z)_{\text{obs}}$. In above, we have used a reasonable lower limit on $\alpha_3(m_Z)|_{\text{MSSM}} = 0.127$ for $m_{\tilde{q}} \leq 1$ TeV [42] and have used the central value for $\alpha_3(m_Z)_{\text{obs}} = 0.118$. Allowing for $\alpha_3(m_Z)_{\text{obs}} = 0.118 \pm 0.003$ would amount to adding nearly $\pm 2\%$ change to $\Delta'_3$. For concreteness, we will quote results for central value of $\alpha_3(m_Z)_{\text{obs}}$, but bear in mind the possibility of the $\pm 2\%$ change in $\Delta'_3$. Note that the variation of the parameter $T$ used above include proton lifetime constraints (Eq. (48)) of $M_{\text{eff}} \geq (2 - 6) \times 10^{18}$ GeV for case I, with $\cos \gamma \approx 1/60$ to 1/9, and $M_{\text{eff}} \geq (0.8 - 2.4) \times 10^{18}$ GeV for case II, with $\cos \gamma \approx 1/60$ to 1/4.

The Table shows that if unification should hold, one must assume, for the case of MSSM embedded in $SO(10)$, that other unification scale threshold and Planck scale effects (excluding $\Delta \alpha_3(m_Z)|_{\text{DT}}$), denoted by $\Delta'_3$, provide a substantial negative contribution to $\alpha_3(m_Z)$, typically varying between $-8\%$ to as much as $-16\%$, as evaluated at the electroweak scale. Before discussing the feasibility of such large negative correction, it should be emphasized that the presence of $\cos \gamma$ inside the logarithm of Eq. (53) has played a
significant role in diminishing the positive threshold correction to $\alpha_3(m_Z)$. In its absence, with $M_{\text{eff}} \geq (1 - 2) \times 10^{18}$ GeV, negative threshold corrections exceeding even (18-20)% would have been required from the other unification scale threshold and Planck scale effects.

We now show that it is reasonable to expect the net threshold correction from other effects, denoted by $\Delta_\gamma$, evaluated at the electroweak scale, to be negative in sign, and to have magnitude no more than about 6 to 10%, if one confines oneself to low dimensional Higgs multiplets such as $45,16,\overline{10}$ and $10$ (as we do). First, let us note that since the coupling and the correction get enhanced by nearly the same factor ($\approx 3$ for $\alpha_3$) as they run from $M_U$ to $m_Z$, a -10% correction to $\alpha_3$ at $m_Z$ roughly corresponds to a -3.3% correction at the unification scale. The precise value of the correction depends on the details of the model, including the nature of the Higgs system that causes mass splittings within complete $SU(5)$ multiplets. Fortunately, as previously observed by several authors (see e.g. [42], [44]), for Higgs multiplets that are not too large (such as $45$ and $16$ of $SO(10)$ as in our model) and with the split masses ($M^\alpha$) of the sub-multiplets belonging to a given $SO(10)$ multiplet being within a factor of $\lesssim 2 - 10$ of $M_U$ either way, the net threshold correction to $\alpha_3(m_U)$, evaluated at the unification scale is typically no more than about 1 to 2%, and the corrections from a given sub-multiplet can be either positive or negative depending upon whether $M^\alpha$ is greater or smaller than $M_U$, and also on the $SU(3) \times SU(2) \times U(1)$ quantum numbers of the sub-multiplet.

To get a feel for the magnitude of the correction, consider (for illustration only) the case of minimal supersymmetric $SU(5)$ (with $24,5,\overline{5}$ of Higgs). In this case, owing to the color triplets ($H_3, H_\overline{3}$) in $(5, \overline{5})$, which are separated from the doublets (albeit by fine-tuning) and acquire unification scale masses, $\alpha_3(m_U)$ receives a correction $\Delta \alpha_3(M_U)_{H_3} \equiv \alpha_3(M_U)(\epsilon_3)_{H_3}$ at the unification scale, where [42] $(\epsilon_3)_{H_3} = [3\alpha_{\text{unif}} \ln(M_{H_3}/M_U)]/5\pi \approx +(0.5 - 1.7)%$ for $M_{H_3}/M_U \approx 2$ to 10, and $\alpha_{\text{unif}} \approx 0.04$. Note this correction is expected to be positive because proton lifetime constraint suggests $M_{H_3} > M_U$, and even for the extreme value of the mass ratio of 10, it is less than 2%.\(^9\)

In the context of the minimal Higgs system (like ours) that utilizes the VEVs of just the $45,16$ and $\overline{10}$ of Higgs to break $SO(10)$ to the standard model gauge symmetry, we can identify two (rather definite) sources of negative contribution to $\alpha_3(m_Z)$. These arise from

\(^9\)Note that the threshold corrections owing to doublet–triplet splitting in $SO(10)$ (discussed in Sec. VI.A) is a special case, because the combination $M_{\text{eff}} \cos \gamma = (\lambda a)^2 \cos \gamma / M_{10}$, that enters into the logarithm in Eq. (53), does not represent the masses of the color triplets (which is of order $\lambda a$). This arises because of the $\cos \gamma$ factor in the logarithm, and also because $M_{10}$ representing the mass of the complete multiplet of $10'$ (see VI.A) can be much smaller than $M_U$; (as needed from proton decay constraint, Eq. (52)). This is the reason that for $M_{\text{eff}} \cos \gamma \geq 2 \times 10^{18}$ GeV, $\Delta \alpha_3(m_Z)_{DT}$ can lead to large positive corrections to $\alpha_3$ at $m_Z$.

\(^{10}\)One can obtain large negative contribution to $\alpha_3(m_Z)$ in $SU(5)$ by introducing large Higgs multiplets, such as $75$ and $50$ [42] which are used in the missing partner mechanism for doublet–triplet splitting in $SU(5)$. This however is a special case associated with large multiplets and does not apply to lower dimensional multiplets like $45,16,10$ of $SO(10)$. Large multiplets such as $75$ of $SU(5)$ do not “explain” the observed unification of couplings, but rather accommodate it.
mass-splittings within (a) the gauge multiplets, and (b) the Higgs sub–multiplets $45_H$.

First consider the gauge multiplets. Representing the VEVs of $16_H$ and $45_H$ by $c$ and $a$ respectively (see Appendix D), one finds:

\[ \Delta \alpha_3(m_Z)_{\text{gauge}} \simeq (+0.002, -0.00026, -0.0015, -0.0028, -0.0031, -0.0030, -0.0018, -0.001) \]

for

\[ p = (0.1, 0.2, 0.3, 0.5, 0.7, 1, 2, 3), \]

where $p \equiv 2c/a$. Thus the contribution can be positive for sufficiently small $p$, but for most of its range, $p \geq 0.2$, the contribution is negative, with $(\delta_3')_{\text{gauge}} \equiv \Delta \alpha_3(m_Z)_{\text{gauge}}/\alpha_3(m_Z)_{\text{obs}}$ varying from $-1.3$ to about $-2.6\%$, for $p$ varying from $0.3$ to $2.0$. Note that the maximum of $-2.6\%$ evaluated at $m_Z$ corresponds to a correction of about $-0.8\%$ at the unification scale, in accord with expectations.

The contribution to $\alpha_3(m_Z)$ from the splitting of $45_H$ has been evaluated in Appendix D (see Eq. (119)). As shown there, if one ignores the coupling of $45_H$ to other multiplets, the allowed superpotential has a simple form: $W = M_1(45_H)^2 + \kappa(45_H)^2/M$. One can then argue that the masses of the submultiplets are characterized by $M_1 \approx \kappa(M_0^2/M) \sim 10^{-2}M_U$, if $\kappa \sim 1$ and $M \sim M_{\text{Planck}}$. One then obtains $\Delta \alpha_3(m_Z)_{45_H} \simeq -0.0045$.

Thus, at least in the simplest approximation, the contribution from $45_H$ to $\alpha_3(m_Z)$ is negative and is about $-4\%$ at the electroweak scale. Of course, the above superpotential is a simplification because one needs to couple $45_H$ to other fields that acquire unification scale VEVs, like $16_H, \overline{16}_H$, together with possible additional vector–like pairs ($16_V, \overline{16}_V$) without VEVs to avoid uneaten pseudo-Goldstone bosons. But such couplings do not alter the basic feature that $M_1 \ll M_U$ (see [37]).

Thus we see that with the minimal Higgs system (i.e., with just $45_H, 16_H$ and $\overline{16}_H$ acquiring VEVs), there are good reasons for the unification scale threshold correction to $\alpha_3(m_Z)$ from our “other effects” denoted by $\Delta_3'$, to receive negative contributions from the gauge sector of $-1.3$ to about $-2.6\%$ (depending on $p$), and plausibly about $-3$ to $-4\%$ from the Higgs multiplet $45_H$ at the electroweak scale. These two together can easily combine to yield about $-4.5$ to $-6.6\%$ correction to $\alpha_3(m_Z)$.

Having identified two rather definite sources of negative threshold corrections to $\alpha_3(m_Z)$, the remaining other contributions (excluding $\Delta \alpha_3(m_Z)_{DT}$) arise through splittings within $16_H, \overline{16}_H, 16_V$ and $\overline{16}_V$ (which would be induced through $< 45_H >$). Contributions to each of these multiplets can be either positive or negative depending on the parameters in $W$. Since the induced splittings within these multiplets will be relatively small, as also their sizes (for mass ratios $\leq 2$), the magnitudes of these corrections from each of these multiplets to $\alpha_3(m_Z)$ can be estimated to be less than about $1\%$. Thus the combined correction from $16_H, \overline{16}_H, 16_V, \overline{16}_V$ is expected to be less than about $2$ to $4\%$ (positive or negative) at the electroweak scale.

Finally, it is worth noting that with only $45_H, 16_H$ and $\overline{16}_H$ acquiring VEVs (as in our model), contribution to $\alpha_3(m_Z)$ from Planck scale physics through effective operators $F_{\mu\nu}F^{\mu\nu}45_H/M$ will vanish because of antisymmetry in the $SO(10)$ contraction.

Thus, at least for the minimal Higgs system utilizing the VEVs of only $45_H, 16_H, \overline{16}_H$, we see that the net “other” threshold correction to $\alpha_3(m_Z)$ from unification and Planck scale effects, excluding $\Delta \alpha_3(m_Z)_{DT}$ but including those from (a) the gauge sector ($-1$ to $-2.6\%$), (b) from $45_H$ ($-3$ to $-4\%$), (c) collectively from $16_H, \overline{16}_H, 16_V, \overline{16}_V$ ($\pm 2$ to $4\%$),
and (d) from Planck scale effects (≈ 0%), is very likely negative. But after adding all these, it seems plausible to assume that the magnitude of this net other contribution is not more than about 8 to 10%, at the electroweak scale.

This assumption in turn implies (see the Table above) that $T \equiv M_{\text{eff}} \cos \gamma / (2 \times 10^{18} \text{ GeV})$ is bounded from above by about 1/30, or conservatively by 1/20. If we add a +1.5% correction to the entries for $\delta'_3$ in the Table to allow for $\alpha_3(m_Z) = 0.118 + 0.0017$, then $\delta'_3$ (reduced as above) is ≤ 10%. This in turn leads to the upper bound:

$$\frac{M_{\text{eff}} \cos \gamma}{10^{18} \text{ GeV}} \leq 1/10$$

(54)

For a given $\cos \gamma \simeq \tan \beta/60$, this upper bound yields an upper limit on $M_{\text{eff}}$ and thereby an upper limit on proton lifetime. In particular, if we assume $\tan \beta \geq 2$, and thus $\cos \gamma \geq 1/30$, we obtain $M_{\text{eff}} \leq 3 \times 10^{18} \text{ GeV}$. Putting $\tan \beta = (2, 3, 6, 8, 10)$ and thus $\cos \gamma = (1/30, 1/20, 1/10, 1/7.5, 1/6)$, the upper bound given above yields $(M_{\text{eff}}/10^{18} \text{ GeV}) \leq (3, 2, 1, 0.75, 0.6)$. Combining this upper limit on $M_{\text{eff}}$ (which holds for both cases I and II) arising from the threshold corrections to $\alpha_3(m_Z)$ with the lower limits on $(M_{\text{eff}}/10^{18} \text{ GeV}) \geq (2 - 6)/(32 to 1/2)$ for case I and $\geq (0.8 - 2.4)$ for case II, given by the proton lifetime constraint (Eq. (52)), we see that the two limits would be in conflict with each other if $\cos \gamma > (1/20, 1/7.5)$ – i.e., if $\tan \beta > (3, 8)$ – for cases (I, II). These considerations based on $\Delta \alpha_3(m_Z)$ and the experimental limit on proton lifetime suggest that rather small values of $\tan \beta$ -i.e. $\tan \beta \leq 3$ (case I) and $\tan \beta \leq 8$ (case II) are favored (but see below). In either case, for the MSSM, assuming $\tan \beta \geq 2$, by demanding accurate coupling unification we obtain an upper limit on $M_{\text{eff}}$ given by

$$M_{\text{eff}} \leq 3 \times 10^{18} \text{ GeV}$$

(55)

On the other hand, the proton lifetime constraint (Eq. (52)) implies that $M_{\text{eff}}$ must exceed $(2 to 6) \times 10^{18} \text{ GeV}$ for case I and $(0.85 to 2.5) \times 10^{18} \text{ GeV}$ for case II, where the range in $M_{\text{eff}}$ corresponds to the uncertainty factor $B = (1/2 to 3/2)$ in the amplitude (see Eq. (51) and Appendix B) in a correlated manner. For instance, for $M_{\text{eff}} \leq 3 \times 10^{18} \text{ GeV}$ (rather than $6 \times 10^{18} \text{ GeV}$) $B$ can vary only between $(1/2 to 3/4)$ for case I, while for case II, $B$ can still vary between $(1/2 to 3/2)$. Using this range and $M_{\text{eff}} \leq 3 \times 10^{18} \text{ GeV}$, we can obtain a lower limit for the proton decay amplitude (given by Eq. (51)):

$$\hat{A}(\overline{\nu}_\tau K^+)_{\text{std}} \geq \left[ \frac{(7 \times 10^{-24} \text{ GeV}^{-1})(1/6 to 1/4)}{(3 \times 10^{-24} \text{ GeV}^{-1})(1/6 to 1/2)} \right]$$

(56)

Substituting into Eq. (47) and adding the contribution form the second competing mode $\overline{\nu}_\mu K^+$ with a typical branching ratio $R \approx 0.3$, we obtain

$$\Gamma^{-1}(\overline{\nu} K^+)_{\text{std}} \leq \left[ \frac{(3 \times 10^{31} \text{ yrs.})(1.6 to 0.7)}{(6.8 \times 10^{31} \text{ yrs.})(4 to 0.44)} \right] (32 to 1/32)$$

(57)

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11If $\tan \beta \leq 1.5$, the top Yukawa coupling will blow up before reaching $\mu = M_U$. Within the standard supergravity spectrum of supersymmetry breaking, there are indications that $\tan \beta \geq 2$ from LEP constraints.
Here the uncertainty (32 to 1/32) corresponds to the uncertainty in $\beta_H, (m_\tilde{W}/m_\tilde{q})$ and $m_\tilde{q}$, by factors of 2, 2, and $\sqrt{2}$ respectively, either way, around the “central” values reflected in Eq. (47). Thus we find that for MSSM the inverse partial proton decay rate should satisfy:

$$\Gamma^{-1}(p \to \bar{\nu}K^+)_{\text{std}} \leq \left[ 3 \times 10^{31.7} \text{ yrs.} \right]$$

$$\leq \left[ 1.5 \times 10^{33} \text{ yrs.} \right] \quad (\text{MSSM})$$

The upper limit in Eq. (58) essentially reflects the upper limit on $M_{\text{eff}}$, while the remaining uncertainties of matrix elements and spectrum are reflected in the exponents. We stress that the upper limit on lifetime exhibited in Eq. (58) has arisen by first taking an upper limit of 10% on the net unification scale threshold correction to $-\alpha_3$ (excluding the contribution from the doublet–triplet splitting), which appears to us generous. Furthermore, the amplitude limit exhibited in Eq. (56) is obtained only if the uncertainties in the amplitude, $\beta_H, (m_\tilde{W}/m_\tilde{q})$ and $m_\tilde{q}$ all go in the same direction to about their extreme values to extend proton lifetime, so it is a generous limit.

Before discussing the contributions of the new operator to proton decay, we wish to note an interesting possibility, that can be relevant especially if $\tan \beta$ is large. For large values of $\tan \beta \approx 20 - 30$, corresponding to $\cos \gamma \approx 1/3$ to 1/2, with $M_{\text{eff}} \geq (2 - 6) \times 10^{18}$ GeV (which is needed for case I to satisfy the proton lifetime constraint), one would have $T = M_{\text{eff}} \cos \gamma / (2 \times 10^{18}$ GeV) $\geq 1/3$ to 3/2. In this case, in the context of MSSM, a negative threshold correction ($\delta'$) exceeding 16 to 20% in magnitude would be required from other sources (see the Table given above for $\delta'_3$ as a function of $T$). As we said before, although such a large negative correction, together with a matching positive one, is in principle possible (see e.g. Ref. [43]), it diminishes the luster of the observed agreement of simple coupling unification, by making it appear somewhat fortuitous. In this connection it is noteworthy that extra vector–like matter – specifically a $16 + \overline{16}$ as proposed in the so-called ESSM (Extended Supersymmetric Standard Model) – at the TeV scale could greatly ease this problem, and thereby allow large values of $\tan \beta$, while leaving our discussion of ordinary fermion masses essentially unaltered. In this case, $\alpha_{\text{unif}}$ is raised to nearly 0.25 to 0.3, compared to 0.04 in the MSSM. Owing to increased two–loop effects, the scale of unification $M_U$ is raised to (1–2)$\times 10^{17}$ GeV, while $\alpha_3(m_Z)|_{\text{ESSM}}$ is lowered to about 0.112 – 0.118. With increased $M_U$ the correction $\Delta \alpha_3(M_Z)|_{DT}$ is also lowered. As a result, even for $\tan \beta \approx 20 - 30$, i.e., $\cos \gamma \approx (1/3 - 1/2)$, and $M_{\text{eff}} \approx 6 \times 10^{18}$ GeV, one obtains for the case of ESSM, the net value of $\alpha_3(m_Z) = \alpha_3(m_Z)|_{\text{ESSM}} + \Delta \alpha_3(m_Z)|_{DT} + \Delta'_3 \approx (0.123 - 0.124 + \Delta'_3)$. Thus for ESSM embedded in $SO(10)$, unification scale threshold corrections from “other” sources to $\alpha_3(m_Z)$ (represented by $\Delta'_3/\alpha_3(m_Z)$), though negative, need be no more than 5% in magnitude, even in the rather extreme case $\tan \beta = 30$, with $M_{\text{eff}} = 6 \times 10^{18}$ GeV. As shown above, for the minimal Higgs system a net negative threshold correction to $\alpha_3(m_Z)$ of this magnitude is not unexpected. As an added feature, we note that since in the ESSM case $M_{\text{eff}}$ can be rather large, $\approx (4–8) \times 10^{18}$ GeV, without requiring large negative threshold corrections from other sources, we can be compatible with the observed limit on proton lifetime for the central values of uncertainties in the parameters describing the spectrum of the supersymmetric particles and the relevant matrix elements.

To be specific, for $M_{\text{eff}} \approx 6 \times 10^{18}$ GeV and $\tan \beta = 30$, which corresponds to $-\delta'_3 = 4$.
to 5%), we obtain (compare with Eq. (58)):

\[ \Gamma^{-1}(\mathcal{W}K^+)_{\text{std}} \approx \begin{cases} 1.2 \times 10^{32} \text{ yrs.} & (1.6 \text{ to } 0.7) \times 32 \text{ to } 1/32, \\ 2.7 \times 10^{32} \text{ yrs.} & (4 \text{ to } 0.44) \times (32 \text{ to } 1/32), \end{cases} \text{ ESSM} \quad (59) \]

where the last factor arises from allowing \((m_{\tilde{W}}/m_{\tilde{q}}), \beta_H\) to vary by factor of \(2, \sqrt{2}\) respectively about their “central” values. The upper and lower entries correspond to cases I and II respectively. Note that allowing for a modest factor of 1 to 4 jointly from the two brackets for case I (and 1 to 2 for case II), which correspond to nearly central values of the parameters mentioned above, keeps the theoretical value of proton lifetime within the experimental limit. On the other hand, allowing for a factor of 20 jointly from the two brackets (for either case I or case II), the proton lifetime in the ESSM case typically lies in the range \((1 - 5) \times 10^{33} \text{ yrs}, which should be accessible.

E. Constraint on \(M_{16} \tan \gamma\) and proton decay via the new operator

The decay amplitude for the new operator for the leading mode (which in this case is \(\nu_{\mu}K^+\)) is given by

\[
A(\nu_{\mu}K^+)_{\text{new}} \simeq \left( \frac{\hat{f}_{33} h_{33}}{M_{16} \tan \gamma} \right) \left( (3 \times 10^{-6})(1/2 \text{ to } 2) \right) \left[ f(t, d) + f(t, l) \right] \times \\
\epsilon_{\alpha \beta \gamma} (d^\alpha u^\beta)(s^\gamma \nu_3), \quad (60)
\]

as shown in Appendix C. This is true (approximately) for both cases I and II.

From the definition of \(\hat{A}(\nu)\) we obtain:

\[
\hat{A}(\nu_{\mu})_{\text{new}} \approx \left[ \frac{\hat{f}_{33} h_{33}}{M_{16} \tan \gamma} \right] \left( (3 \times 10^{-6})(1/2 \text{ to } 2) \right). \quad (61)
\]

The first bracket \(P \equiv [\hat{f}_{33} h_{33}/(M_{16} \tan \gamma)]\) may be evaluated by using results of Sec. VI A and B, as follows. Putting \(f_{33} = f_{33} v_R/M\) (see Eq. (46)), \(M_{16} \tan \gamma = \lambda' v_R\) (see Eq. (40), and \(h_{33} \simeq 1/2\), we obtain:

\[
P = (f_{33}/M)(1/2\lambda'). \quad (62)
\]

Here \(M\) stands for the mass scale that characterizes the strength of the effective non–renormalizable operators (see Sec. III). Using \(M_R \equiv f_{33} v_R^2/M\) and the result that \(M_R \approx B(m_{\tilde{U}}^2/m_{\nu_3}) \simeq 5[m_{\tilde{U}}^2(M_U)/(1/20 \text{ eV } \zeta) \simeq 10^{15} \text{ GeV}/\zeta\) (see Eqs. (44) and (45)), we find\(^{12}\)

\[
f_{33}/M = M_R/v_R^2 \approx (\kappa_R^2 \zeta)^{-1}[4 \times 10^{17} \text{ GeV}]^{-1} \quad (63)
\]

\(^{12}\)We note that the \(SO(10)\) contraction for the \(f_{ij}\) coupling contributing to neutrino masses is not the only one that contributes to proton decay. We assume that the two possible contractions are comparable in strength.
Here $\kappa_R$ and $\zeta$ denote uncertainties in $v_R$ and $m_{\nu_3}$—i.e., $v_R \equiv \kappa_R(2 \times 10^{16}$ GeV) and $m_{\nu_3} \equiv \zeta(1/20)$ eV. A priori, we expect $v_R \sim M_U$ and thus $\kappa_R \approx (1/2 - 3)$. Since $\mathbf{16}_H \rightarrow SO(10)$ to $SU(5)$, $v_R = \mathbf{16}_H \rightarrow$ may in fact be somewhat larger than $M_U$—that is, $\kappa_R > 1$. From the SuperK results, we have $\zeta \approx 1.5 - 1/5$. In Sec. III, we have discussed the appropriateness of the characteristic mass $M$ being either the Planck or the string scale, subject to the presumption that the effective coupling $f_{33}$ for the third family is nearly maximal ($\sim 1$). Given some uncertainty in this regard, a good choice appears to be $M = 10^{18}$ GeV, which is intermediate between $M_{\text{Planck}}$ and $M_{\text{string}}$. We expect that such an intermediate value of $M$ should represent fairly well either choice $M = M_{\text{Planck}}$ or $M = M_{\text{string}}$, given that we allow a range in $\kappa_R^2 \zeta$ and thereby in $f_{33}$. Setting $M = 10^{18}$ GeV, Eq. (63) yields: $f_{33} \approx 2.5/(\kappa_R^2 \zeta)$. Noting that values of $f_{33} \gg 1$ are implausible, we thus expect $\kappa_R^2 \zeta \geq 1$ in accord with the remark mentioned above (rather than < 1). Under the presumption that $f_{33}$ is maximal, it therefore seems quite reasonable to assume that $\kappa_R^2 \zeta \approx (1$ to 5) $\approx 2.5(1/2$ to 2), which corresponds to $f_{33} \approx (2$ to 1/2). Substituting this range for $\kappa_R^2 \zeta$ into Eq. (63) and (62), we obtain

$$P \approx (5 \times 10^{-19} \text{ GeV}^{-1})(1/2$ to 2)$/$x$

Now, on the one hand, the validity of the perturbative treatment requires $x' \leq 1$. One the other hand, given that $M_{16} \tan \gamma = \lambda' v_R$ and that from considerations of standard operators for proton decay, we have the constraint that $\tan \beta < (3, 8)$ and thus $\tan \gamma > (20, 7)$ for cases (I, II) in MSSM, the choice of $x' < 1$ will tend to make $M_{16} \ll v_R$, which is implausible. We therefore take $x' \approx 1$. Substituting this into $P$ and in turn into Eq. (60), we obtain

$$A(\overline{\nu}_\mu K^+)_{\text{new}} \approx (1.5 \times 10^{-24} \text{ GeV}^{-1})(1/4$ to 1.3)

(64)

Here the upper end of the uncertainty has been restricted to conform to the limit on proton lifetime, see Eq. (49).

Comparing with Eq. (56) we see that the contributions of the new and the standard operators (for $M_{\text{eff}} \approx (2 - 3) \times 10^{18}$ GeV) to the proton decay amplitude are comparable to one another. Since there is no reason to expect near cancellation between them (especially for both $\overline{\nu}_\mu K^+$ and $\overline{\nu}K^+$ modes), we expect the net amplitude (standard + new) to be in the range exhibited for either one. Assuming that the new operator dominates and substituting Eq. (64) into Eq. (47), we obtain:

$$\Gamma^{-1}(\overline{\nu}K^+)_{\text{new}} \approx (3 \times 10^{31} \text{ yrs})[16 \text{ to } 1/1.7][32 \text{ to } 1/32]

(65)

In this estimate we have included the contribution of the $\overline{\nu}_\mu K^+$ mode with a typical branching ratio $R \approx 0.4$ (see Appendix C). Here the second factor, inside the square bracket, reflects the uncertainties in the amplitude (see Eq. (64)), while the last factor corresponds to varying $\beta_H, (m_{\overline{\mu}/m_\tilde{q}})$ and $m_\tilde{q}$ around the central values reflected in Eq. (47).

With a net factor of even 20 to 100 arising jointly from the square and the curly brackets, i.e., without going to extreme ends of all parameters, the new operators related to neutrino masses lead by themselves to proton decay lifetimes $\approx (0.6 - 3) \times 10^{33}$ yrs. Thus if the new operators were the only source of proton decay—i.e., if the standard $d = 5$ operators were somehow absent—one could be comfortably compatible with existing limits, but optimistic regarding future observation. We now briefly elaborate on this possibility.
F. The neutrino mass related operator as the sole source of proton decay

As we have seen, straightforward minimal embedding of the MSSM in $SO(10)$, with informed hypotheses about the fermion-Higgs couplings, leads to contributions from the standard $d = 5$ operators are disturbingly large, if the relevant parameters have nearly their central values and/or if $M_{\text{eff}} < 2 \times 10^{18}$ GeV. This is especially true for case I in the MSSM (see Eq. (57)-(58)). One is therefore motivated to wonder whether the standard operators might not be present. This possibility is realized, if the higher gauge symmetry is $G_{224}$ (or $G_{2113} = SU(2)_L \times I_3 R \times (B-L) \times SU(3)_c$) rather than $SO(10)$. Such gauge symmetries have been proposed to appear in solutions of string theory (see Ref. [48] for $G_{224}$ and Ref. [52] for $G_{2113}$). Such possibilities retain some attractive features of $SO(10)$, notably the unification of quark-lepton families into single multiplets including the right-handed neutrino $\nu_R$ as indicated for the neutrino seesaw. One sacrifices a simple group-theoretic explanation for the observed unification of coupling, but if the models derive from an underlying string theory, one might still expect such unification at the string scale. Plausible mechanisms to reconcile the string and the MSSM unification scale have been proposed [50]. The color triplets related to the electroweak doublets, which generate the standard $d = 5$ proton decay operators, need not exist. The standard source of $d = 5$ operators can be absent in such models [51].

It is possible that after projecting out appropriate fields the couplings of those that remain reflect the original higher symmetry\textsuperscript{13}. This is what is believed to occur for the gauge couplings, as previously mentioned. If it also occurs for the Higgs superpotential couplings, our considerations on fermion and neutrino masses in Sec. II-IV and their relationships to proton decay will remain valid.

G. Baryon number violation from the $RRRR$ operator

So far we have focused on the charged wino dressing of the effective superpotential $W_{\text{eff}}^{(L)}$ of Eq. (41). For small values of $\tan \beta$ and $\mu$, this gives the dominant contribution to the proton decay amplitude. However, if $\mu \tan \beta$ is large, ($\geq 2$ TeV), dressing of the effective baryon number violating operator involving only the right-handed fields, $u^c u^c d^c e^c$, by the charged Higgsino can become important [53-55]. The ratio of this amplitude to the usual wino contribution scales as $\tan \beta (\mu/m_{\tilde{W}})$ (for $m_{\tilde{q}} \gg m_{\tilde{W}}$). In minimal supersymmetric $SU(5)$, the Higgsino dressing becomes more important than the wino dressing when $\tan \beta \geq 9 (m_{\tilde{W}}/\mu)$ [55]. This estimate takes into account the differences in the renormalization of the ($RRRR$) compared to the ($LLLL$) operator owing to running from $M_U$ to $M_{\text{SUSY}}$, and the difference between their strong matrix elements. We now show that, by contrast, in the $SO(10)$ model the $RRRR$ operator is strongly suppressed relative to the $LLLL$ operator, and that they will be comparable only if $(\mu \tan \beta/m_{\tilde{W}}) \approx 500$.

\textsuperscript{13}For example, in a class of string solutions leading to $G_{2113}$, the cubic level top and $\nu_\tau$ Yukawa couplings are claimed to be equal at the string scale despite $SU(4)_C$ breaking [52].
In the $SO(10)$ model, once the quark and lepton masses and mixings are fixed, the strength of the $RRRR$ operator is determined by the corresponding effective superpotential (analogous to $W^{(LL)}_{\text{eff}}$) which is given by:

$$
W^{(R)}_{\text{eff}} = -M_{\text{eff}}^{-1}[2(u^c T H \tilde{V}^T e^c)(u^c T H K d^c) + (u^c T \tilde{H} \tilde{V}^T e^c)(u^c T \tilde{A} K d^c) - 3(u^c T \tilde{A} \tilde{V}^T e^c)(u^c T \tilde{H} K d^c) - 3(u^c T \tilde{A} \tilde{V}^T e^c)(u^c T \tilde{A} K d^c) - \tan \gamma(u^c T \tilde{H} \tilde{V}^T e^c)(u^c T \tilde{G} K d^c) + 3(u^c T \tilde{A} \tilde{V}^T e^c)(u^c T \tilde{G} K d^c)] + M_{16}^{-1}(u^c T \tilde{F} \tilde{V}^T e^c)(u^c T \tilde{G} K d^c).
$$

(66)

Here

$$
(\tilde{H}, \tilde{A}, \tilde{G}, \tilde{F}) \equiv \tilde{V}_u^T (h, a, g, f) \tilde{V}_d,
$$

(67)

with $\tilde{V}_u$ the unitary matrix that rotates the $u^c$ fields $u^{c(g)} = \tilde{V}_u u^{c(m)}$ from the gauge basis to the mass basis, $\tilde{V}_d$ is the unitary matrix that similarly rotates the right–handed electron field, and $\tilde{V}^T \equiv \tilde{V}_u^T \tilde{V}_d$. $K$ is the right–handed analog of $V_{CKM}$, $K \equiv \tilde{V}_d^T \tilde{V}_d$.

The contribution to proton decay amplitude from Eq. (66) is estimated as follows. Since a charged Higgsino is involved in the dressing, the internal scalars have to be from the third generation (other diagrams will be suppressed by small Yukawa couplings). This uniquely picks out the $\tilde{t}_R$ and $\tilde{\tau}_R$ as the internal scalars. The external quark fields are then fixed to be $u$ and either a $d$ or an $s$. This suggests that the combination of indices $(ij)(kl)$ in Eq. (66) must be $(33)(11)$, $(33)(12)$, $(13)(32)$ or $(13)(31)$. Among these four, the combination $(33)(12)$ can proceed without utilizing any of the right handed mixing angles. (Though the right–handed mixing angles in the $(23)$ sector of both $u$ and $d$ are $\sim 0.2$, not terribly small.) We find the amplitude, after Higgsino dressing, from this dominant contribution to be

$$
\hat{A}[(u^c s^c)^\dagger (d v^c)] \approx M_{\text{eff}}^{-1} h_{33}^2 q^2 V_{id} + (M_{16} \tan \gamma)^{-1} h_{33}^2 f_{33} q^2 V_{id} \approx 6.6 \times 10^{-6} / M_{\text{eff}} + 1.3 \times 10^{-24} \text{GeV}^{-1}(1/2 \text{ to } 2).
$$

(68)

(69)

Other contributions are not much bigger. In going from Eq. (68) to (69), we have used $h_{33}^2 \approx 1/4, q^2 \approx 4.4 \times 10^{-3}$ (see Sec. IV), $V_{id} \approx 6 \times 10^{-3}$ and $P \equiv f_{33} h_{33} / M_{16} \tan \gamma \approx 5 \times 10^{-19} \text{GeV}^{-1}(1/2 \text{ to } 2)$ (see the discussion following Eq. (63)). It is understood that the relative sign (phase) of the two terms in Eq. (68) and (69) is arbitrary. The full amplitude $A[(u^c s^c)^\dagger (d v^c)]$ will be obtained by multiplying the above expression by a loop function analogous to the function for the wino dressing (see Eq. (92)). There are two differences in this function however: (i) The factor $(\alpha_2/4\pi)$ will be replaced by $(\lambda_1 \lambda_\tau)/(16\pi^2)$, where the Yukawa couplings are to be evaluated at the momentum scale $M_{\text{SUSY}}$. (ii) The mass parameter $m_{\tilde{w}}$ is replaced by $\mu$. The presence of $\lambda_\tau$ brings in a $\tan \beta$ dependence in the Higgsino dressing relative to the wino dressing. (In minimal supersymmetric $SU(5)$, the wino dressing has a $\tan \beta$ dependence, but the Higgsino dressing will have a $(\tan \beta)^2$ dependence.)

Comparing the first term of Eq. (69) with the standard $LLLL$ amplitude (see Eq. (51)) and making the replacements as above, we obtain: $\hat{A}_{RRRR(\text{std})}/\hat{A}_{LLLL(\text{std})} \approx [h_t h_\tau \mu/(g_2^2 m_{\tilde{w}})] V_{ts} \times \{6.6 \times 10^{-6}/[7, 3 \times 10^{-6}](1/2) \approx (\mu \tan \beta/m_{\tilde{w}}) V_{ts}[1/100, 1/48],$ where we have put $h_t(m_t) \approx 1$ and $h_\tau(m_{\tilde{\tau}}) \approx m_{\tilde{\tau}}/v_{\tilde{\tau}} \approx \tan \beta/100$. Substituting $V_{ts} \approx 1/25$, we thus see that even for relatively large values of $(\mu \tan \beta/m_{\tilde{w}}) \approx 100$, the $RRRR$ amplitude
is smaller than the $LLLL$ amplitude by a factor of $25-12$. Thus, unlike the case of $SU(5)$, in the $SO(10)$ model developed here, the $RRRR$ operator can safely be ignored compared to the $LLLL$ operator, as long as $(\mu \tan \beta/m_{\tilde{W}}) \leq 200$ (say).

The charged Higgsino dressing results in decay modes of the proton containing final state neutrinos, not charged leptons. Apart from the direct charged Higgsino exchange, there are diagrams involving $\tilde{t}_L - \tilde{t}_R$ mixing and $\tilde{\tau}_L - \tilde{\tau}_R$ mixing (the latter being proportional to $\tan \beta$) followed by the exchange of charged wino. This contribution is smaller than the direct charged Higgsino exchange by roughly a factor $v_u/M_{\text{SUSY}}$. Similar arguments apply to mixed contributions involving charged Higgsino–wino mixing.

**H. Charged lepton decay mode**

In minimal supersymmetric $SU(5)$ and many of its variants, charged lepton decay of the proton is suppressed as long as $\tan \beta \leq 20$ or so. This is because of a GIM-type cancellation in the wino dressing diagrams, which brings in a suppression factor in this amplitude, proportional to the small $u$–quark mass. Gluino dressing can lead to charged lepton decay of the proton, but owing to flavor conservation of the primary gluino vertex and the flavor antisymmetry of the effective superpotential, this contribution is suppressed for small values of $\tan \beta$ ($\leq 20$). For large values of $\tan \beta$ ($\geq 20$), flavor mixing in the up–squark sector becomes significant and the gluino graph begins to be important. However, for minimal supersymmetric $SU(5)$, such large values of $\tan \beta$ ($\geq 20$) are highly disfavored owing to limits on proton lifetime (see for example Ref. [34]). Even if we ignore this difficulty, for large $\tan \beta \geq 20$ dressing of the $RRRR$ operator by charged Higgsino becomes the dominant source of proton decay in $SU(5)$ (if $\mu/m_{\tilde{W}} \geq 1$), and the neutrino modes dominate anyhow. So in minimal supersymmetric $SU(5)$, and many of its variants, charged lepton decay mode of the proton is highly suppressed relative to the neutrino modes, for all values of $\tan \beta$.

The situation is different in our $SO(10)$ model, for two reasons. First, the contribution to $p \rightarrow \mu^+ K^0$ arising from the standard $d = 5$ wino dressing diagram in this model is not small. Since we have a realistic spectrum of quark and lepton masses, especially with $m_s \neq m_\mu$ and $m_d \neq m_e$ at $M_U$, the wino contribution does not experience a GIM cancellation. Second, the new $d = 5$ operator related to neutrino masses lends comparable strength to the neutrino and the charged lepton modes.

In evaluating the expected branching ratio for the $\mu^+K^0$ mode – i.e., $B(\mu^+K^0) \equiv \Gamma(\mu^+K^0)/[\Gamma(\tau \tau K^+ + \Gamma(\mu \mu K^+) + \Gamma(\mu^+K^0)]$ – let us concentrate on case I. Very similar results hold for case II. The amplitudes for $p \rightarrow \mu^+K^0$ arising from the standard and the new operators for case I are given by (see Appendix B, Eq. (107) and (109) or Table 1):

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14The main reason why the standard $RRRR$ operator is suppressed, relative to the $LLLL$ operator, in $SO(10)$ but not in $SU(5)$, is simply that the standard $LLLL$ operator is effectively enhanced by about two orders of magnitude in $SO(10)$, for $M_{\text{eff}} = M_C$. See the remarks following Eq. (114) in Appendix C.
\[ \hat{A}(\mu^+ K^0)_{\text{std}} \approx \left( \frac{h_{33}^2}{M_{\text{eff}}} \right) (3 \times 10^{-6})(1/2 \text{ to } 2) \approx (0.30)(1/2 \text{ to } 2) \times 10^{-24} \text{GeV}^{-1} \]  

(70)

\[ \hat{A}(\mu^+ K^0)_{\text{new}} \approx P(10^{-6})(1/3 \text{ to } 2) \approx (0.5)(1/6 \text{ to } 4) \times 10^{-24} \text{GeV}^{-1} \]  

(71)

In Eq. (70) we have inserted \( h_{33}^2 \approx 1/4 \) and an average value of \( M_{\text{eff}} \approx 2.5 \times 10^{18} \text{ GeV} \) (see discussions following Eq. (55), showing \( P \approx 2(2-3) \times 10^{18} \text{ GeV for case I} \). In Eq. (71), we have inserted the value of \( P \equiv f_{33} h_{33}/(M_{16} \tan \gamma) \approx (5 \times 10^{-19} \text{GeV}^{-1})(1/2 \text{ to } 2) \), obtained in Sec. VI E. The standard and the new amplitudes for the \( \nu_e K^+ \) and the \( \nu_\mu K^+ \) modes (for \( M_{\text{eff}} \approx 2.5 \times 10^{18} \text{ GeV} \) for case I are given by (see Eq. (101), (102), (105) and (106) and Table 1 in Appendix B):

\[ \hat{A} \left( \bar{\nu}_e K^+ \right)_{\text{std}} \approx \left\{ \begin{array}{c} (1.8)(0.8 - 1.2) \\ (1)(0.8 - 2) \end{array} \right\} \times 10^{-24} \text{GeV}^{-1} \]  

(72)

\[ \hat{A} \left( \bar{\nu}_\mu K^+ \right)_{\text{new}} \approx \left\{ \begin{array}{c} (0.75)(1/4 - 2.8) \\ (1.5)(1/4 - 1.3) \end{array} \right\} \times 10^{-24} \text{GeV}^{-1} \]  

(73)

Note that the upper ends of the ranges shown for the different amplitudes have been restricted to conform with the limit on proton lifetime (see Eq. (49)). Assuming that the relevant matrix elements for the \( \mu^+ K^0 \) and \( \bar{\nu} K^+ \) modes are comparable (but see below), it may be inferred from the amplitudes noted above (or the discussion in Appendix B) that the standard operators by themselves lead to a branching ratio for the \( \mu^+ K^0 \) mode in the range \( B(\mu^+ K^0)_{\text{std}} \approx 1 \text{ to } 10\% \), while the new operators (related to neutrino masses) by themselves can lead to \( B(\mu^+ K^0)_{\text{new}} \) from a few up to 40 or 50\%.

With contributions from both the standard and the new operators present, and comparable in magnitude, one must of course add the two contributions allowing for interference between them. Adding the contributions of the standard and the new operators to the amplitudes for all three decay modes \( (\bar{\nu}_e K^+, \bar{\nu}_\mu K^+, \mu^+ K^0) \) we estimate the “maximum” and the “minimum” of the sum of the two contributions (standard and new) to each of these modes, by allowing for the uncertainties in their amplitudes reflected in Eq. (70)-(73). We thereby estimate the following range for \( B(\mu^+ K^0) \):

\[ B(\mu^+ K^0)_{\text{std+new}} \approx \{1 \text{ to } (50 - 60)\%\} \rho \]  

(74)

where \( \rho \) denotes the ratio of the squares of relevant matrix elements for the \( \mu^+ K^0 \) and \( \bar{\nu} K^+ \) modes.

If one uses the chiral Lagrangian method as a guide \[38,34\], one would obtain \( \rho = [1 - (m_p/m_B)(D - F)^2] \times [2/3(m_p/m_B)Dq + m_p/(3m_B)(D + 3F)]^{-2} \), where \( D \approx 0.76, F \approx 0.48 \) and \( m_B \approx 1150 \text{ MeV} \), and \( q \equiv C(\text{usd}v)/C(\text{uds}v) \). The entity \( q \) denotes the ratio of the

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15 In estimating the upper and the lower ends of the sum \{\( \Gamma(\bar{\nu}_e K^+) + \Gamma(\nu_\mu K^+) \)\} and thereby the expected range of \( B(\mu^+ K^0) \), we stipulate that the maximum (or minimum) of \( \hat{A}(\bar{\nu}_e K^+) \) is accompanied by a median rather than the maximum (or minimum) value of \( \hat{A}(\nu_\mu K^+) \) and vice versa. This is because the various terms contributing to the two amplitudes are unlikely to be entirely constructive (or destructive) simultaneously for both of them. Thus the logical maximum and minimum of \( B(\mu^+ K^0) \) lie beyond the stipulated range quoted above.
coefficients of the $d = 5$ amplitudes leading to spinor contractions $(us)(dv)$ and $(ud)(sv)$ respectively (see Appendix B). In the $SO(10)$ model (developed here), we find typically $q \approx 0.2$ to 0.6. Thus the chiral Lagrangian method (CLM) would suggest $\rho \approx 1/4$ to 1/5. It should however be noted that CLM leads to matrix elements that are inconsistent with the results of lattice calculations [41] by factors of 1.5 to 4, for proton decaying either into $(\ell^+\pi^0$ and $\pi^+K^0)$ or $\ell^+K^0$ or both for all values of $\beta_H \simeq (0.006 \text{ GeV}^3)(1/2$ to $3)$ (see the discussion following Eq. (116) for the definition of $\beta_H$.) For instance, for $\beta_H \approx 0.003 \text{ GeV}^3$, the $p \to \pi^0$ and $p \to \pi^+$ amplitudes are larger by a factor $\approx (1.5 - 2)$ for CLM relative to the lattice calculation, while that for $p \to K^0$ is smaller by a similar factor for CLM relative to the lattice value [41]. Lattice calculations of $\nu K^+$ are apparently not available at present. Evidently, one cannot regard the estimate of $\rho$ based on the chiral Lagrangian method as entirely reliable. Even a modest 25 to 30% correction to the matrix element of $\nu K^+$ and $\mu^+K^0$ (compared to CLM values) can alter the estimate of $\rho$ by more than a factor of 2.

In the absence – presumably temporary – of a reliable calculation, one should remain open to the possibility of larger values, say $\rho \approx 1/2$ to 1. Clearly, it is only such larger values of $\rho$, together with the estimate of $B(\mu^+K^0)$ presented in Eq. (74), that permit optimism for discovery of the $\mu^+K^0$ mode.

Returning to the estimate of Eq.(74), we find that typically for a large range of parameters, which correspond to the uncertainties in the amplitudes exhibited in Eq. (72)-(73), the branching ratio $B(\mu^+K^0)$ can lie in the range of 20 to 30% (if $\rho \approx 1$). Thus we see that the $\mu^+K^0$ mode is likely to be prominent in the $SO(10)$ model presented here, and if $\rho \approx 1$ it can even become a dominant mode. This contrasts sharply with the minimal $SU(5)$ model, in which the $\mu^+K^0$ is expected to have a branching ratio of only about $10^{-3}$. In the $SO(10)$ model, the standard operator by itself gives a branching ratio that is at least an order of magnitude larger than the $SU(5)$ value, while the potential prominence of the $\mu^+K^0$ mode arises only through the new operator related to neutrino masses. Thus the $\mu^+K^0$ mode of the proton decay serves as a signature for the sort of $SO(10)$ model of fermion masses and mixings explored here.

Because of the hierarchical nature of the Yukawa couplings in the Dirac as well as the Majorana sectors, proton decay into $e^+\pi^0$ and $e^+\pi^0$ are highly suppressed in our scenario compared to the $\pi K^+$ or the $\mu^+K^0$.

VII. SUMMARY AND CONCLUDING REMARKS

One major goal of this paper has been to understand the masses and mixings of the neutrinos, suggested by the atmospheric and the solar neutrino anomalies, in conjunction with those of the quarks and charged leptons. Adopting familiar ideas of generating eigenvalues through off–diagonal mixings, we find that the bizarre pattern of masses and mixings

\footnote{The chiral Lagrangian method has played a role in Ref. [34] and thus in our discussion in Appendix and Sec. VI on proton decay rate. We note that corrections of 20 to 30% in the amplitude would not however alter the total proton decay rate by more than a factor of 1.5 to atmost 2.}
observed in the charged fermion sector can be adequately described (with ~ 10% accuracy) within an economical $SO(10)$ framework. A concrete proposal was presented that provides five successful predictions for the masses and mixings in the quark and the charged lepton systems. The same description goes extremely well with a value of $m_{\nu_\tau} \sim 1/20$ eV as well as with a large $\nu_\mu - \nu_\tau$ oscillation angle ($\sin^2 2\theta_{\mu\tau}^{\text{osc}} \simeq 0.82 - 0.96$), despite highly non–degenerate masses for the light neutrinos. Both these features are in good agreement with the SuperK results on atmospheric neutrinos.

The other major goal of this paper has been to revisit the previously noted link between neutrino masses and nucleon decay \[12\] in the light of the SuperKamiokande result. We find that the mass of $\nu_\tau$ ($\sim 1/20$ eV), together with the large $\nu_\mu - \nu_\tau$ oscillation angle implied by the SuperK result, suggest a significant enhancement in the standard as well as in the new (neutrino mass related) $d = 5$ proton decay operators, relative to previous estimates, including those of Ref. \[12\]. There are many uncertainties in the prediction for the proton decay rate, including ones arising from uncertainties in the SUSY spectrum, from the hadronic matrix elements, from the relative phases of the many different contributions (see the discussions in Appendix B), and from the allowed extent of unification scale threshold corrections to $\alpha_3(m_Z)$. Nevertheless, following a detailed analysis of threshold corrections for the minimal Higgs system $(45_H, 16_H, \bar{16}_H)$ used in our work (see Sec. VI D and Appendix D) we found that the standard operator contributions severely constrain the underlying model.

Specifically, we found that for MSSM embedded in $SO(10)$, the standard operators, with generous allowance for the uncertainties, lead to lifetime estimate $\Gamma^{-1}(p \rightarrow \bar{\nu} K^+)_{\text{std}}^{MSSM} \leq [1.5, 7] \times 10^{33}$ yrs corresponding to cases (I,II) (see Sec. V, Eq. (37) for the origin of these two cases). In the process, by combining constraints from the observed limit on proton lifetime together with a reasonable upper limit on threshold corrections, we found that $\tan \beta$ must be rather small ($\leq 3, 8$) for cases (I, II), in MSSM. For larger values of $\tan \beta \sim 20$, one must turn to the Extended Supersymmetric Standard Model (ESSM) embedded in $SO(10)$. This allows for two extra (vector-like) families at the TeV scale, and has been motivated on other grounds \[46\]. In this framework, the standard $d = 5$ operators still represent perfectly viable sources for proton decay. We find typically $\Gamma^{-1}(p \rightarrow \bar{\nu} K^+)_{\text{std}}^{ESSM} \leq (1 - 5) \times 10^{33}$ yrs for $\tan \beta \geq 20$ (Sec. VI.D).

As observed in our earlier work \[12\], one very important consequence of quark–lepton unification is the likely existence of new $d = 5$ proton decay operators that are related to neutrino masses. Here we have shown that the “observed” mass $\nu_\tau \sim 1/20$ eV and large $\nu_\mu - \nu_\tau$ oscillation angle (which go well with theoretical expectations in $SO(10)$) enhance these operators relative to our previous estimates \[12\]. As a result, the standard and the new operators appear to make comparable contributions (see remarks following Eq. (64)). We have remarked in Sec. VLF that in some string-inspired models leading to $G_{224}$ or $G_{2113}$ symmetries (rather than intact $SO(10)$), the color triplets related to electroweak doublets get projected out of the spectrum altogether, and thus the standard $d = 5$ operators do not contribute. In that case, the new $d = 5$ operators related to neutrino masses will survive and dominate. We found (Sec. VLE) that (given the SuperK result) these new operators by themselves lead to proton lifetime $\Gamma^{-1}(p \rightarrow \bar{\nu} K^+)_{\text{new}}$ with a “central value” of about $3 \times 10^{31}$ yrs, and a range that is perfectly compatible with the observed limit. Assuming rather generous uncertainties (i.e., those in $(m_{\tilde{W}}/m_{\tilde{q}})$, $m_{\tilde{q}}$, the matrix element $\beta_H$), and the
amplitude, the new operators by themselves lead to a proton lifetime \( \leq (1 - 6) \times 10^{33} \text{ yrs.} \) Thus for the MSSM (or ESSM) embedded in \( SO(10) \), we expect the proton lifetime to be shorter than about \( 10^{34} \text{ yrs.} \)

A distinctive feature of the \( SO(10) \) framework discussed here is the potential prominence of the charged lepton mode \( p \rightarrow \mu^+ K^0 \). In minimal \( SU(5) \), this mode becomes prominent only for \( \tan \beta \geq 20 \), when gluino dressing becomes significant. But such large values of \( \tan \beta \) are highly disfavored in \( SU(5) \), owing to observed limit on proton lifetime. Furthermore, for such large values of \( \tan \beta \), the Higgsino dressing of the baryon number violating \( RRRR \) operator becomes dominant, leading anyhow to dominance of the neutrino mode. Thus in conventional \( SU(5) \)-like supersymmetric unified models, charged lepton decay of the proton is (even relatively!) scarce. In the \( SO(10) \) framework, on the contrary, the decay \( p \rightarrow \mu^+ K^0 \) competes favorably with the neutrino mode. This becomes possible primarily because of enhancement of the new neutrino–mass related \( d = 5 \) operators, in which the amplitudes for \( \mu^+ K^0 \) and \( \tau K^+ \) modes are comparable. Thus observation of the \( \mu^+ K^0 \) decay mode of the proton would be very encouraging for the circle of ideas discussed here. Owing to the hierarchical Yukawa couplings of the Majorana neutrinos suggested by the solar and the atmospheric neutrino data, the modes \( p \rightarrow e^+ K^0 \) and \( p \rightarrow e^+ \pi^0 \) are predicted to be highly suppressed relative the \( p \rightarrow \pi K^+ \).

While we focused on a specific \( SO(10) \) example, several of our results are likely to be more general. In any scenario with just three light neutrinos, a simultaneous resolution of the solar and atmospheric neutrino anomalies argues for hierarchical neutrino masses. In a framework that also unifies quarks and leptons (with modest mixing angles), a simple way to generate the near-maximal neutrino oscillation angle needed for the atmospheric neutrino anomaly is to attribute it partly to the \((\mu - \tau)\) sector and partly to the \((\nu_\mu - \nu_\tau)\) sector. The precise way this division is made is model dependent. As one goes beyond minimal supersymmetric \( SU(5) \) to a unified framework where small neutrino mass is a compelling feature, the predicted lifetime of the proton tends to decrease. In such unified models there are various factors contributing to shorten the lifetime (see Appendix C) including (i) an enhanced coupling of the muon to the color triplets (relative to the coupling of the strange quark), (ii) an enhanced up–quark coupling to the color triplet which scales as \( \sqrt{m_c m_t} \) rather than \( m_c \), and (iii) the presence of new operators related to neutrino mass.

Proton decay has been anticipated for quite some time in the context of unified theories. Recent data from SuperK on neutrino mass makes the case for observable proton decay still more compelling. With improved searches, especially for the \( \tau K^+ \) and \( \mu^+ K^0 \) modes, either proton decay will be revealed, or some promising and otherwise remarkably successful ideas on unification will be called into question seriously.

**Acknowledgments:** K.S.B is supported by funds from the Oklahoma State University. The research of J.C.P. has been supported in part by NSF Grant No. Phy-9119745 and by a Distinguished Research Fellowship awarded by the University of Maryland. F.W is supported by DOE grant No. DE-FG02-90ER-40542.
APPENDIX A

Numerical values of various matrices

Here we give the numerical values of the matrices $V_u$, $V_\ell$ and $V' = V_u^\dagger V_\ell$, where $V_u$ diagonalises the up quark matrix $U$ of Eq. (28): $u^{(g)} = V_u u^{(m)}$ with $(g)$ and $(m)$ denoting the gauge and mass eigenstates, $V_\ell$ diagonalises the charged lepton mass matrix. We also give the numerical values of the matrices $(\hat{H}, \hat{A}, \hat{G}, \hat{F})$ as well as their matrix product with $V'$, these are relevant for proton decay amplitude calculations. From the fit to the fermion masses discussed in Sec. IV-V, we have determined the (approximate) values of the parameters $(\sigma, \eta, \epsilon, \epsilon', \eta'; \ell)$. For numerical purposes we shall choose their “central values”:

\[\begin{align*}
\sigma &= -0.1096 \, \eta_{cb}, \quad \eta = -0.1507 \, \eta_{cb}, \quad \epsilon = 0.0954 \, \eta_e, \\
\epsilon' &= 1.76 \times 10^{-4} \, \eta_e', \quad \eta' = 4.14 \times 10^{-3} \, \eta_{h'}. 
\end{align*}\]  

(75)

With these values, the matrix $V_u$ is given by:

\[V_u \simeq \begin{pmatrix}
1 & -\epsilon' \sigma^{-1} & \epsilon'(-\epsilon + \sigma) \\
-\epsilon' \sigma^{-1} & 1 & \epsilon + \sigma \\
-\epsilon' \sigma^{-1} & \epsilon + \sigma & 1
\end{pmatrix}.\]  

(76)

An analogous expression for $V_\ell$ is obtained by the replacement $\sigma \to \eta, \epsilon \to -3\epsilon$ and $\epsilon' \to -3\epsilon'$ in the first part of Eq. (76). The numerical values of $V_\ell$ and $V' = V_u^\dagger V_\ell$ are:

\[V_\ell \simeq \begin{pmatrix}
1 & 0.07 \eta_{h'} & 5.6 \times 10^{-4} \eta_{cb} \eta_{h'} \\
-0.07 \eta_{h'} & 0.905 & -0.4369 \eta_{cb} \\
-0.031 \eta_{cb} \eta_{h'} & 0.4369 \eta_{cb} & 0.905
\end{pmatrix},\]  

(77)

\[V' \simeq \begin{pmatrix}
1 & 0.052 \eta_{h'} + 0.07 \eta_{h'} & -0.026 \eta_{cb} \eta_{h'} \\
-(0.052 \eta_{h'} + 0.07 \eta_{h'}) & 0.911 & -0.4227 \eta_{cb} \\
-0.030 \eta_{cb} \eta_{h'} & 0.4227 \eta_{cb} & 0.911
\end{pmatrix}.\]  

(78)

Numerical values of the matrices $(\hat{H}, \hat{A}, \hat{G}, \hat{F})$ and their products with $V'$ are:

\[\hat{H} \approx h_{33} \begin{pmatrix}
-1.1 \times 10^{-5} & -1.8 \times 10^{-4} \eta_{h'} & -5.8 \times 10^{-3} \eta_{cb} \eta_{h'} \\
-1.8 \times 10^{-4} \eta_{h'} & -2.9 \times 10^{-3} & -0.0954 \eta_{cb} \\
-5.8 \times 10^{-3} \eta_{cb} \eta_{h'} & -0.0954 \eta_{cb} & 1
\end{pmatrix},\]  

(79)

\[\hat{H}V' \approx h_{33} \begin{pmatrix}
1.8 \times 10^{-4} \eta_{h'} \eta_{h'} & -2.6 \times 10^{-3} \eta_{h'} & -5.8 \times 10^{-3} \eta_{cb} \eta_{h'} \\
3.0 \times 10^{-3} \eta_{h'} & -0.043 & -0.0954 \eta_{cb} \\
2.9 \times 10^{-3} \eta_{cb} \eta_{h'} & 0.3273 \eta_{cb} & 1
\end{pmatrix},\]  

(78)

39
\[
A \simeq h_{33} \left( \begin{array}{ccc}
3.1 \times 10^{-5} & 3.4 \times 10^{-4} \eta_{\ell'} & 5.8 \times 10^{-3} \eta_{cb} \eta_{\ell'} \\
3.4 \times 10^{-4} \eta_{\ell'} & 2.7 \times 10^{-3} & 0.0954 \eta_{cb} \\
5.8 \times 10^{-3} \eta_{cb} \eta_{\ell'} & 0.0954 \eta_{cb} & -2.7 \times 10^{-3}
\end{array} \right),
\]
\[
\hat{A}V' \simeq h_{33} \left( \begin{array}{ccc}
-1.9 \times 10^{-4} \eta_{\ell'} \eta_{\ell'} + 3.5 \times 10^{-5} & 2.8 \times 10^{-3} \eta_{\ell'} & 5.8 \times 10^{-3} \eta_{cb} \eta_{\ell'} \\
3.0 \times 10^{-3} \eta_{\ell'} + 5.6 \times 10^{-4} \eta_{\ell'} & 0.043 & 0.0954 \eta_{cb} \\
-6.7 \times 10^{-3} \eta_{cb} \eta_{\ell'} + 8.6 \times 10^{-4} \eta_{cb} \eta_{\ell'} & 0.0954 \eta_{cb} & -0.043
\end{array} \right),
\]
\[
\tan \gamma \hat{G} \simeq h_{33} \left( \begin{array}{ccc}
5.4 \times 10^{-4} \eta_{\ell'} \eta_{\ell'} & 4.1 \times 10^{-3} \eta_{\ell'} & -2.5 \times 10^{-3} \eta_{cb} \eta_{\ell'} \\
4.1 \times 10^{-3} \eta_{\ell'} & -1.2 \times 10^{-3} & -0.0411 \eta_{cb} \\
-2.5 \times 10^{-3} \eta_{cb} \eta_{\ell'} & -0.0411 \eta_{cb} & 1.2 \times 10^{-3}
\end{array} \right),
\]
\[
\tan \gamma \hat{GV}' \simeq h_{33} \left( \begin{array}{ccc}
\{3.7 \times 10^{-4} \eta_{\ell'} \eta_{\ell'} \} & \{-1.1 \times 10^{-3} \eta_{\ell'} \} & \{-2.5 \times 10^{-3} \eta_{cb} \eta_{\ell'} \} \\
-2.9 \times 10^{-4} & +4.1 \times 10^{-3} \eta_{\ell'} & -1.8 \times 10^{-3} \eta_{cb} \eta_{\ell'} \\
5.4 \times 10^{-3} \eta_{\ell'} & -0.019 & -0.0411 \eta_{cb}
\end{array} \right) \tag{80}
\]
\[
\hat{F} \simeq \hat{f}_{33} \left( \begin{array}{ccc}
x + 7.3 \times 10^{-7} + 1.0 \times 10^{-4} \eta_{cb} y & 1.7 \times 10^{-3} \eta_{\ell'} \eta_{cb} y - 0.06 \eta_{\ell'} x & 0.06 \eta_{\ell'} y \\
1.7 \times 10^{-3} \eta_{\ell'} \eta_{cb} y - 0.06 \eta_{\ell'} x & 0.0284 \eta_{cb} y & y + 0.0142 \eta_{cb}
\end{array} \right) \tag{81}
\]

**APPENDIX B**

**B.1. Numerical evaluation of the full amplitudes for proton decay**

In converting the superpotential given in Eq. (41) into a proton decay amplitude, the first step is to dress two of the sfermions among the four superfields in each term of Eq. (41) *i.e.*, to convert them into fermions. The dominant contribution to the amplitude arises from the dressing involving the charged wino. Here we list the 12 terms that arise in dressing each one of the 7 terms of Eq. (41) with a final state neutrino. The four fermion amplitude after wino dressing of a generic superpotential term

\[
W = M_{\text{eff}}^{-1} (u^T F d') \{ u^T G \ell - d^T G \nu' \} \tag{82}
\]
is

\[
A(p \rightarrow \bar{\nu} X) = M_{\text{eff}}^{-1} \epsilon_{\alpha \beta \gamma} \times \left[ F_{11} G_{21}(u^\alpha d^\beta) (s^\gamma \nu_1') (f(c,l) + f(u,d)) \\
- F_{22} G_{11}(u^\alpha s^\beta) (s^\gamma \nu_1') (f(c,l) + f(u,d)) - F_{12} G_{11}(u^\alpha d^\beta) (s^\gamma \nu_1') (f(c,l) + f(u,d)) \\
+ F_{12} G_{21}(u^\alpha s^\beta) (s^\gamma \nu_1') (f(c,l) + f(d,c)) - F_{13} G_{11}(u^\alpha d^\beta) (b^\gamma \nu_1') (f(t,l) + f(u,d)) \\
+ F_{13} G_{21}(u^\alpha b^\beta) (s^\gamma \nu_1') (f(c,l) + f(d,t)) + F_{13} G_{31}(u^\alpha b^\beta) (b^\gamma \nu_1') (f(t,l) + f(d,t)) \\
- F_{23} G_{11}(u^\alpha b^\beta) (s^\gamma \nu_1') (f(c,l) + f(t,d)) - F_{23} G_{11}(u^\alpha s^\beta) (b^\gamma \nu_1') (f(c,d) + f(t,l)) \\
- F_{33} G_{11}(u^\alpha b^\beta) (b^\gamma \nu_1') (f(t,l) + f(d,t)) + F_{11} G_{31}(u^\alpha d^\beta) (b^\gamma \nu_1') (f(t,l) + f(u,d)) \\
+ F_{12} G_{31}(u^\alpha s^\beta) (b^\gamma \nu_1') (f(t,l) + f(d,c)) \right] \left( \frac{\alpha_2}{4\pi} \right) \tag{83}
\]
The dressing functions $f(a,b)$ are defined later in Eq. (93). In writing Eq. (83), use has been made of the symmetric nature of $F$ ($F_{ij} = F_{ji}$). As described in the text, the notation $d' = V_{CKM}d$ and $\nu' = V_{CKM}^\ell \nu$ has been adopted.

A simpler expression can be obtained for proton decay amplitude with a charged lepton in the final state (for the same generic superpotential term as in Eq. (82)):

$$
\hat{A}(p \to \ell^+ X) = M_{e\ell}^1 \epsilon_{\alpha\beta\gamma} \times [(u^\alpha d^\beta)(u^\gamma \ell^-)[V_{cd}(G_{11}F_{21} - G_{21}F_{11}) + V_{td}(G_{11}F_{31} - G_{31}F_{11}) + (u^\alpha s^\beta)(u^\gamma \ell^-)[V_{cs}(G_{11}F_{21} - G_{21}F_{11}) + V_{ts}(G_{11}F_{31} - G_{31}F_{11})].
$$

The amplitude with a hat is the full four–Fermion amplitude divided by the loop function $(2f)$, defined in Eq. (110) and (111) of Appendix C. Note that the down type quark fields appearing above (without primes) are the physical ones.

With the numerical values of the Yukawa coupling matrices relevant for the color triplet exchange, we can use Eqs. (83) and (84) to compute the decay amplitude for any given channel. Since there are seven terms in Eq. (83), for the neutrino mode, there will be a total of $7 \times 12 = 84$ terms to be summed. This is most efficiently done numerically using Mathematica. We now present some of the dominant amplitudes. For these estimates we define $V_{cd} = 0.22\eta_{cd}, V_{td} = 0.006\eta_{td}, V_{ts} = 0.04\eta_{ts}$ and $V_{ud} = V_{cs} = 1$. The $\eta_{ij}$ are the unknown phase factors, they can also be used to vary the central values of the CKM mixing angles adopted. We drop terms which are smaller by more than an order of magnitude compared to the leading term in each set.

$$
\hat{A}[(ud)(\nu\tau)] \approx M_{e\ell}^1 h_{33}^2 [1.9 \times 10^{-5} \eta_{cd} \eta_{ts} \eta_{\ell^+} - 9.0 \times 10^{-6} \eta_{cd} \eta_{cb} \eta_{\ell^+} - 6.1 \times 10^{-6} \eta_{td} \eta_{ts} \eta_{cb} \eta_{\ell^+} + 4.8 \times 10^{-6} \eta_{ts} - 2.5 \times 10^{-6} \eta_{cb} + 2.1 \times 10^{-6} \eta_{td} \eta_{h^+} + 3.0 \times 10^{-6} \eta_{cd} \eta_{ts} \eta_{h^+} + 2.2 \times 10^{-6} \eta_{cd} \eta_{cb} \eta_{h^+}] + (M_{16} \tan \gamma)^{-1} h_{33} h_{33} [3.1 \times 10^{-7} \eta_{cd} \eta_{ts} \eta_{\ell^+} + 6.0 \times 10^{-7} \eta_{td} \eta_{ts} \eta_{cb} \eta_{\ell^+} + 4.3 \times 10^{-7} \eta_{cd} \eta_{ts} \eta_{cb} \eta_{h^+} + 1.5 \times 10^{-7} \eta_{cd} \eta_{h^+} + 2.3 \times 10^{-7} \eta_{cd} \eta_{ts} \eta_{h^+} - 0.0411 \eta_{cb} x + 3.1 \times 10^{-5} \eta_{cd} \eta_{\ell^+} y + 2.2 \times 10^{-5} \eta_{cd} \eta_{ts} \eta_{cb} \eta_{\ell^+} y + 1.1 \times 10^{-5} \eta_{cd} \eta_{cb} \eta_{h^+} y + 1.6 \times 10^{-5} \eta_{cd} \eta_{ts} \eta_{cb} \eta_{h^+} y + z \{-2.47 \times 10^{-4} \eta_{cd} \eta_{cb} \eta_{h^+} y - 1.28 \times 10^{-4} \eta_{cd} + 9.94 \times 10^{-5} \eta_{ts} \eta_{cb} \eta_{\ell^+} + 7.23 \times 10^{-5} \eta_{ts} \eta_{cb} \eta_{h^+} \}].
$$

$$
\hat{A}[(ud)(d\nu\tau)] \approx M_{e\ell}^1 h_{33}^2 [2.9 \times 10^{-6} \eta_{cd} \eta_{td} \eta_{\ell^+} - 2.0 \times 10^{-6} \eta_{cd} \eta_{cb} \eta_{\ell^+}] + (M_{16} \tan \gamma)^{-1} h_{33} h_{33} [4.7 \times 10^{-8} \eta_{cd} \eta_{td} \eta_{cb} \eta_{\ell^+} + 6.9 \times 10^{-8} \eta_{cd} \eta_{cb} \eta_{\ell^+} + 1.8 \times 10^{-8} \eta_{cb} \eta_{h^+} + 1.3 \times 10^{-8} \eta_{td} + 6.5 \times 10^{-8} \eta_{cd} \eta_{h^+} + 6.8 \times 10^{-8} \eta_{cd} \eta_{td} \eta_{h^+} - 1.8 \times 10^{-8} \eta_{cd} \eta_{cb} \eta_{h^+} - 9.0 \times 10^{-3} \eta_{cd} \eta_{cb} \eta_{x} - 6.8 \times 10^{-6} \eta_{\ell^+} y + 3.3 \times 10^{-6} \eta_{cd} \eta_{td} \eta_{cb} \eta_{\ell^+} y + 4.8 \times 10^{-6} \eta_{cd} \eta_{td} \eta_{cb} \eta_{h^+} y + 2.5 \times 10^{-6} \eta_{cd} \eta_{h^+} y + z \{-5.42 \times 10^{-5} \eta_{cd} \eta_{td} \eta_{cb} - 2.82 \times 10^{-5} + 1.49 \times 10^{-5} \eta_{cd} \eta_{cb} \eta_{h^+} + 1.08 \times \eta_{cd} \eta_{cb} \eta_{h^+} \}].
$$

(85)
\[ A[(ud)(s\nu_\mu)] \simeq M_{\text{eff}}^{-1} h^2_{33} \left[ -3.5 \times 10^{-6} \eta_{td} \eta_{ts} \eta_{\nu} + 8.5 \times 10^{-6} \eta_{cd} \eta_{hs} \eta_{cb} \eta_{\nu} \right. \\
\left. - 3.4 \times 10^{-6} \eta_{cd} \eta_{\nu} + 2.1 \times 10^{-6} \eta_{ts} \eta_{cb} - 4.7 \times 10^{-6} \eta_{td} \eta_{hs} \eta_{\nu} \right. \\
\left. - 7.0 \times 10^{-6} \eta_{cd} \eta_{hs} \eta_{cb} \eta_{\nu} + 5.1 \times 10^{-6} \eta_{cd} \eta_{\nu} - 2.1 \times 10^{-6} \eta_{cd} \eta_{hs} \eta_{\nu} \right. \\
\left. - 1.9 \times 10^{-6} \eta_{cd} \eta_{hs} \eta_{cb} \eta_{\nu} \right] + (M_{16} \tan \gamma)^{-1} f_{33} h_{33} [2.6 \times 10^{-7} \eta_{cd} \eta_{hs} \eta_{\nu} \\
+ 1.4 \times 10^{-7} \eta_{cd} \eta_{hs} \eta_{cb} \eta_{\nu} - 9.9 \times 10^{-7} \eta_{cd} \eta_{hs} \eta_{\nu} \right. \\
\left. - 3.5 \times 10^{-7} \eta_{cd} \eta_{hs} \eta_{cb} \eta_{\nu} + 5.2 \times 10^{-5} \eta_{cd} \eta_{hs} \eta_{cb} \eta_{\nu} - 1.8 \times 10^{-7} \eta_{cd} \eta_{\nu} \right. \\
\left. - 1.4 \times 10^{-5} \eta_{cd} \eta_{hs} \eta_{cb} \eta_{\nu} - 1.6 \times 10^{-5} \eta_{cd} \eta_{hs} \eta_{cb} \eta_{\nu} - 1.8 \times 10^{-2} \eta_{cd} \eta_{hs} \eta_{cb} \eta_{\nu} \right. \\
\left. + 9.2 \times 10^{-6} \eta_{cd} \eta_{hs} \eta_{cb} \eta_{\nu} - 1.4 \times 10^{-8} \eta_{cd} \eta_{hs} \eta_{cb} \eta_{\nu} + 2.5 \times 10^{-6} \eta_{ts} \eta_{\nu} \right. \\
\left. - 2.5 \times 10^{-7} \eta_{cd} \eta_{hs} \eta_{cb} \eta_{\nu} + 3.6 \times 10^{-5} \eta_{cd} \eta_{hs} \eta_{cb} \eta_{\nu} + 2.6 \times 10^{-5} \eta_{cd} \eta_{hs} \eta_{cb} \eta_{\nu} \right. \\
\left. - 1.0 \times 10^{-5} \eta_{cd} \eta_{hs} \eta_{cb} \eta_{\nu} - 7.1 \times 10^{-6} \eta_{cd} \eta_{hs} \eta_{cb} \eta_{\nu} \eta_{\nu} \right. \\
\left. + z \{-9.86 \times 10^{-6} \eta_{cd} \eta_{hs} \eta_{cb} - 1.11 \times 10^{-6} \eta_{td} \right. \\
\left. - 5.79 \times 10^{-5} \eta_{cd} \eta_{cb} + 4.20 \times 10^{-5} \eta_{ts} \eta_{\nu} \} \right]
\tag{87}

Similarly, the amplitudes \( A[(us)(d\nu_\mu)], A[(us)(d\nu_\mu)] \) and \( A[(ud)(d\nu_\mu)] \) can be computed. We do not display these results here, since the amplitudes for these operators are always somewhat smaller than the ones displayed. Furthermore, the matrix element for the former two turn out to be suppressed by about a factor of 3.

Turning now to the charged lepton decay amplitude,
\[ A[(us)(u\mu^-)] \simeq M_{\text{eff}}^{-1} h^2_{33} V_{cs} \left[ -8.4 \times 10^{-7} - 2.1 \times 10^{-6} \eta_{c} \eta_{\nu} \right] \\
+ V_{ts} \left[ 5.2 \times 10^{-7} - 4.8 \times 10^{-5} \eta_{cb} \eta_{c} \eta_{\nu} \right] \\
+ \left( M_{16} \tan \gamma \right)^{-1} f_{33} h_{33} \left\{ V_{cs} \left[ -5.0 \times 10^{-8} \eta_{c} \eta_{\nu} + 1.8 \times 10^{-2} \eta_{c} \right] \\
- 7.1 \times 10^{-6} \eta_{cb} \eta_{c} \eta_{\nu} + V_{ts} \left[ 9.3 \times 10^{-7} \eta_{cb} - 3.6 \times 10^{-5} \eta_{cb} \eta_{c} \eta_{\nu} \right] \\
- 0.0411 \eta_{cb} \eta_{c} + 6.4 \times 10^{-5} \eta_{c} + 2.5 \times 10^{-4} \eta_{cb} \eta_{c} \right\} \\
+ z \left\{ 4.2 \times 10^{-5} \eta_{ts} \eta_{c} - 1.59 \times 10^{-5} \eta_{c} \right\} \\
- 1.66 \times 10^{-4} \eta_{ts} \eta_{\nu} - 5.88 \times 10^{-5} \eta_{c} \eta_{\nu} \right] \right] \right] .
\tag{88}

\[ A[(us)(ue^-)] \simeq h^2_{33} M_{\text{eff}}^{-1} \left\{ V_{cs} \left[ 8.4 \times 10^{-8} \eta_{\nu} + 1.5 \times 10^{-7} \eta_{c} \right] \\
+ V_{ts} \left[ 8.3 \times 10^{-7} \eta_{cb} \eta_{c} - 6.9 \times 10^{-6} \eta_{cb} \eta_{\nu} + 3.3 \times 10^{-6} \eta_{cb} \eta_{c} \eta_{\nu} \right] \\
+ \left( M_{16} \tan \gamma \right)^{-1} f_{33} h_{33} \left\{ V_{cs} \left[ 3.5 \times 10^{-9} \eta_{c} - 1.7 \times 10^{-4} \eta_{c} \right] \\
+ 5.3 \times 10^{-3} \eta_{c} \eta_{c} + 5.0 \times 10^{-7} \eta_{cb} \eta_{c} \eta_{\nu} \right\} + V_{ts} \left[ 8.3 \times 10^{-9} \eta_{cb} \eta_{c} \right] \\
+ 3.1 \times 10^{-7} \eta_{cb} \eta_{\nu} + 2.5 \times 10^{-7} \eta_{cb} \eta_{c} + 3.7 \times 10^{-4} \eta_{cb} \eta_{c} \eta_{\nu} \right\} \\
+ 2.9 \times 10^{-3} \eta_{cb} \eta_{c} \eta_{\nu} + 2.2 \times 10^{-5} \eta_{c} \eta_{\nu} + 1.8 \times 10^{-5} \eta_{c} \eta_{\nu} \right\} \\
+ z \left\{ -1.44 \times 10^{-5} \eta_{ts} \eta_{c} \eta_{\nu} \right\} \right] .
\tag{89}

The decay amplitude for \( p \to \pi^0 \mu^+ \) can be obtained from Eq. (88) by the replacement \( V_{cs} \to V_{cd} \) and \( V_{ts} \to V_{td} \). The amplitude for \( p \to e^+ \pi^0 \) may be obtained from Eq. (89) by the replacement: \( V_{cs} \to V_{cd}, V_{ts} \to V_{td} \).
B.2 Representative contributions to proton decay amplitudes and their estimated magnitudes:

In this subsection, we exhibit in detail the evaluation of a few representative (dominant) contributions to proton decay amplitudes. The full set of contributions are listed in the preceding subsection. We also estimate the magnitudes of the full amplitudes allowing for uncertainty in the relative phase of the different contributions.

To obtain some of the leading terms in the results exhibited above, consider the part containing the $\nu'$ field – in the first term of $W^{(L)}$ (Eq. (41)), which we specify by subscript $I\nu'$.

$$W^{(L)}_{I\nu'} = -M_{\text{eff}} \epsilon_{\alpha\beta\gamma} \hat{H}_{ij}(\hat{H}V')_{kl}(u_i^\alpha d_j^\beta)(d_k^\gamma \nu'_l) .$$

Here $(\alpha, \beta, \gamma)$ denote color indices. The transpose symbols on the relevant fields are dropped henceforth. We will first consider case I discussed in Sec. V for which $\epsilon' \neq 0$. The first and the second terms in Eq. (82) arise by making the choices (A and B) and (C and D) respectively, for the indices as given below:

$$\begin{align*}
(ij)(kl) &= (21)(23) \quad \text{(Choice A)} \\
&= (22)(13) \quad \text{(Choice B)} \\
&= (23)(13) \quad \text{(Choice C)} \\
&= (21)(33) \quad \text{(Choice D)}
\end{align*}$$

Note that A and B are related by the interchange of $j \leftrightarrow k$, and similarly C and D.

Choice A: $(ij)(kl) = (21)(23)$: Starting with the operator $(u_2^\alpha d_1^\beta)(d_2^\gamma \nu_3)$ in the superpotential, two fields need to be utilized for wino dressing. Which ones remain external gets determined as follows. The field $d_2^\gamma$ must remain external, yielding a strange quark $s^\gamma$, accompanied by the CKM factor $V_{cs} \simeq 1$.17 (If $\tilde{d}_2^\gamma$ were dressed instead, it would yield an external charm quark, which is kinematically disallowed.) Thus both $u_2^\alpha$ and $d_1^\beta$ must be utilized for dressing, which, after conversion of $\tilde{c}$ and $\tilde{d}'$ at the wino vertex, respectively, yield $V_{cd}d_2^\alpha \equiv (\eta_{cd}\theta_C)d_2^\alpha$ and $V_{ub}u_2^\beta$, where $\theta_C \simeq 0.22$. As discussed before, $\eta_{cd} = +1$. $\eta_{ud}$ and $\eta_{us}$ are chosen to be +1 by convention. Thus, the corresponding contribution to the four–fermion proton decay operator leading to $\nu_3$ emission is given by

$$A^{(1)}_{I\nu'} \simeq \left[ -\hat{f}(c, d) \right] \left( \theta_C \eta_{cd} \epsilon_{\alpha\beta\gamma} \hat{H}_{21}(\hat{H}V')_{23}(d_2^\alpha u_2^\beta)(s^\gamma \nu_3) \right)$$

$$\simeq - \left[ \frac{h_{23}^2}{M_{\text{eff}}} \right] \hat{f}(c, d)(3.8 \times 10^{-6})\eta_{cd}\eta_{ub}\epsilon_{\alpha\beta\gamma}(d_2^\alpha u_2^\beta)(s^\gamma \nu_3)$$

where

\text{17If } d_2^\gamma \text{ is used to yield an external } d-\text{quark, and } \tilde{u}_3 \text{ (after dressing) an } s-\text{quark, one would obtain the four–fermion operator } (su)(d\nu_3). \text{ It turns out that the matrix element of this operator is suppressed by about a factor of 2-3 compared to that of } (du)(s\nu_3). \text{ See [34].}
\[
\dot{f}(a, b) = \left( \frac{\alpha_2}{4\pi} \right) \frac{m_{\nu}^2}{m_a^2 - m_b^2} \left( \frac{m_a^2}{m_a^2 - m_{\nu}^2} \ln \frac{m_{\nu}^2}{m_{W}^2} - [a \leftrightarrow b] \right) \equiv \left( \frac{\alpha_2}{4\pi} \right) f(a, b). 
\]  

(93)

In getting Eq. (92), we have used the numerical values of the elements of \( \hat{H} \) and \( \hat{H}V' \) given in Appendix A (Eq. (78)). The phase factors \( \eta_\ell' \) and \( \eta_{cb} \) are ±1. Likewise, confining still to the index combination A, the contributions from the remaining five standard operators of \( W^{(1)} \) labeled by the subscripts (II-VI) are found to be:

\[
(A_{\text{II},\ell'}^{(1)}, A_{\text{III},\ell'}^{(1)}, A_{\text{IV},\ell'}^{(1)}, A_{\text{V},\ell'}^{(1)}, A_{\text{VI},\ell'}^{(1)}) \simeq (-3, +2, -6, -0.4, -0.8)A_{\ell'}^{(1)} \tag{94}
\]

Combining the contributions (92) and (94), the total contribution of the terms containing \( \nu' \) in the first six operators (I to VI) in \( W^L \) is given by

\[
A_{\text{I to VI},\ell'}^{(1)} \simeq \left[ \frac{h_{333}^2 \dot{f}(c, d)}{M_{\text{eff}}} \right] \eta_{cd} \eta_{\ell'} \eta_{cb} (2.74 \times 10^{-5}) \epsilon_{\alpha\beta\gamma}(d^\alpha u^\beta)(s^\gamma \nu_3). \tag{95}
\]

Next consider choice B. In this case, since one starts with the operator \( \epsilon_{\alpha\beta\gamma}(u_3^\alpha d_3^\beta)(d_3^\gamma \nu_3) \), using the dressing as in case A, one gets an extra minus sign compared to A, owing to the color factor. Following the same procedure as above, one obtains

\[
B_{\text{I to VI},\ell'}^{(1)} \simeq \left[ \frac{h_{333}^2 \dot{f}(c, d)}{M_{\text{eff}}} \right] \eta_{cd} \eta_{\ell'} \eta_{cb} (-1.82 \times 10^{-5}) \epsilon_{\alpha\beta\gamma}(d^\alpha u^\beta)(s^\gamma \nu_3). \tag{96}
\]

Adding Eq. (95) and (96), the total contribution of terms containing \( \nu' \) in the first six operators of \( W^{(L)} \) for the index combinations A and B, leading to \( \nu_3 K^+ \) emission, is given by:

\[
(A^{(1)} + B^{(1)})_{\text{I to VI},\ell'} \simeq \left[ \frac{h_{333}^2 \dot{f}(c, d)}{M_{\text{eff}}} \right] \eta_{cd} \eta_{\ell'} \eta_{cb} (0.9 \times 10^{-5}) \epsilon_{\alpha\beta\gamma}(d^\alpha u^\beta)(s^\gamma \nu_3). \tag{97}
\]

This checks with the second term in Eq. (85) obtained numerically, up to an overall sign, which is due to the difference in the sign of Eq. (96) compared to Eq. (41). Likewise, the corresponding contributions for the index combinations C and D from the first six operators in \( W^{(L)} \) is found to be:

\[
(C^{(1)} + D^{(1)})_{\text{I to VI}} \simeq \left[ \frac{h_{333}^2 \dot{f}(c, d)}{M_{\text{eff}}} \right] \eta_{cd} \eta_{\ell'} \eta_{s} (-1.9 \times 10^{-5}) \epsilon_{\alpha\beta\gamma}(d^\alpha u^\beta)(s^\gamma \nu_3) \tag{98}
\]

This is the first term of Eq. (85) – up to an overall sign. Note that Eqs. (97)-(98) add constructively because \( \eta_{s} = -\eta_{cb} \). We still have to include (i) the contribution that arises by interchanging \( i \leftrightarrow j \) while keeping \( (k, l) \) fixed (see Eq. (83)) and (ii) that from the terms containing charged lepton field \( \ell \) in \( W^{(L)} \). The latter contribute to neutrino emission by dressing the charged slepton fields \( \tilde{\ell} \). Including contributions from both (i) and (ii), one can verify that the net contribution of each kind – such as A, B, C, D listed above – is obtained by simply making the following substitution in Eq. (97)-(98), etc:

\[
\dot{f}(c, d) \rightarrow \dot{f}(c, d) + \dot{f}(c, \ell). \tag{99}
\]
In arriving at this result, use has been made of the symmetric nature of the matrices $H, A, G, F$, as well as color antisymmetry. Thus, adding Eq. (97)-(98), and making the above substitutions, the net contribution to $\bar{\nu}_3 K^+$ decay mode from the first six terms of $W^{(L)}$, for the index combination $[A+B+C+D + (i \leftrightarrow j)]$ is given by:

$$[A + B + C + D + (i \leftrightarrow j)]_{VI} \approx \left[ \frac{h_{33}^2 \left[ \hat{f}(c, d) + \hat{f}(c, \ell) \right]}{M_{\text{eff}}} \right] \eta_{cd}\eta_{cb} \times (2.8 \times 10^{-5}) \epsilon_{\alpha\beta\gamma} (d^\alpha u^\beta)(s^\gamma \nu_3). \tag{100}$$

Having shown how the first two terms of the first bracket of Eq. (85) arise, let us now consider all the other terms in this bracket, which are proportional to $M_{\text{eff}}^1$ and estimate their net magnitude. It is clear that the joint contribution of the first two terms (Eq. (85)) dominates over the other terms in the same bracket by factors of 5 to 10. Although the relative signs (phases) of these subleading terms are uncertain (since we do not know the $\eta_{ij}$), we can see that their combined magnitude even if they all add constructively, is less than about $(1.5 \times 10^{-5})$, which is about half of that given by the first two terms of Eq. (85). Not knowing the relative signs (phases) we conclude that the net contribution from the standard operator (first six terms of $W$) is given by: $h_{33}^2 [2 \hat{f}(c, d)/M_{\text{eff}}] (2.8 \times 10^{-5})(1/2$ to $3/2)$.

Turning to case II ($\epsilon' = 0, \delta \neq 0$, see Sec. V), the net contribution from the standard operator to the same amplitude is obtained simply by setting $\eta_{\nu'} = 0$ in the first square bracket in Eq. (85). Since $\delta \approx 10^{-5} \simeq \epsilon'/20$, contributions proportional to $\delta$ can safely be neglected compared to the other terms. Noting that $\eta_{ts} = -\eta_{cb}$, the said contribution is thus given by $[(4.8 + 2.5) \times 10^{-6}\eta_{ts} + (3 + 2.2) \times 10^{-6}\eta_{ts}\eta_{kd}\eta_{\nu'}] = (0.73 + 0.55\eta_{cd}\eta_{\nu'})\eta_{ts} \times 10^{-5}$. Allowing for unknown relative phase as above, this contribution will vary in magnitude between $(1.28$ to $0.18)) \times 10^{-5}$, which may be represented approximately by $(0.7 \times 10^{-5})(1 \pm 0.7)$.

Thus we estimate that the contribution of the standard operator to the $\bar{\nu}_\tau K^+$ amplitude for the cases I and II are given by:

$$A(\bar{\nu}_\tau K^+)_{\text{std}} \approx \left[ \frac{h_{33}^2 \hat{f}(c, d)}{M_{\text{eff}}} \right] \epsilon_{\alpha\beta\gamma}(d^\alpha u^\beta)(s^\gamma \nu_3) \times \left\{ \begin{array}{ll}
2.8 \times 10^{-5} & (\frac{1}{2} \text{ to } \frac{3}{2}) - \text{Case I} \\
0.7 \times 10^{-5} & (1 \pm 0.7) - \text{Case II} 
\end{array} \right\}. \tag{101}$$

Note that the standard amplitude for case II ($\epsilon' = 0$) is typically smaller than that for case I ($\epsilon' \neq 0$) by a factor of 3.5 to 4. Of course, if there is a near cancellation for case II for the $\bar{\nu}_\tau K^+$ mode, so that the curly bracket in Eq. (101) (for case II) is as small as $0.2 \times 10^{-5}$, then contributions from the new operator (related to neutrino masses) to $\bar{\nu}_\tau K^+$ as well as contributions of the standard and the new operators to the $\bar{\nu}_\mu K^+$ mode would turn out to be far more important, as discussed below.

**Contributions to the $\bar{\nu}_\mu K^+$ mode from the standard operator (First six terms in $W$)**

---

18The reader will note that these unknown phases (or signs) are just the analogs of those which enter into $\tilde{t}$ versus $\tilde{c}$ contributions in minimal $SU(5)$, which is represented by a parameter $y_{tK}$ etc, see Ref. [34].
Contributions to the $\nabla_\mu K^+$ mode from the standard operator for the cases I and II can be estimated by using Eq. (87) in a manner similar to that noted above. Collecting terms proportional to $\eta_{\nu'}$ and those independent of it from the first square bracket of Eq. (87), and using $\eta_{ls} = -\eta_{cb} = \pm 1$, the said contribution is given by: \[\{(-1.24\eta_{cd} - 0.35\eta_{sd}\eta_{ls} - 
abla 0.21 - 0.19)\eta_{\nu'}) + \{1.2\eta_{cd}\eta_{\nu'} - 0.21 - 0.47\eta_{sd}\eta_{cb}\eta_{\nu'}\}\times 10^{-5}\]. Allowing for uncertainty of relative phases, a reasonable estimate of the magnitude of this contribution is thus given by $[1.5(1 \pm 0.5) \times 10^{-5}]$ for case I ($\epsilon' \neq 0$) and by $[1.2(1 \pm 0.5) \times 10^{-5}]$ for case II ($\epsilon' = 0$). We thus obtain:

$$A(\nabla_\mu K^+)|_{std} \approx \left[\frac{\hat{F}_{33}^2 f(c, d)}{M_{\text{eff}}}\right] \left[\frac{1.5}{1.2}\right] (1 \pm 0.5) \times 10^{-5} \epsilon_{\alpha\beta\gamma}(d^{\alpha}u^{\beta})(s^{\gamma}\nu_\mu).$$  \hspace{1cm} (102)

The upper and lower entries refer to cases I and II respectively. We observe that the amplitudes for $\nabla_\mu K^+$ and $\nabla_\tau K^+$ modes have very similar contributions from the standard operators; the former is expected to be leading for case II and the latter for case I.

Combining the rates for the two modes, we would expect that the total proton decay rate, represented to a good approximation by the sum $[\Gamma(\nabla_\tau K^+ + \Gamma(\nabla_\mu K^+))$, would typically be proportional (for central values of the amplitudes) to $[(2.8)^2 + (1.5)^2] \approx 10$ for case I, and to $[(0.7)^2 + (1.2)^2] \approx 2$ for case II. In short, in so far as contributions from the standard operator is concerned, proton decay rate would typically be suppressed by about a factor of 5 for case II compared to case I.

$$\frac{\Gamma(\nabla_\tau K^+ + \Gamma(\nabla_\mu K^+))_{\text{Case II}}}{\Gamma(\nabla_\tau K^+ + \Gamma(\nabla_\mu K^+))_{\text{Case I}}} \approx \frac{1}{5}.$$  \hspace{1cm} (103)

This is an interesting and significant effect on proton decay, having its origin entirely in the fermion mass matrix.

** Contribution to $\nabla_\tau K^+$ and $\nabla_\mu K^+$ mode from the new operator related to neutrino masses (the seventh term in $W^{(L)}$):**

Confining first to the part containing $\nu'$ in the seventh term of $W^{(L)}$ (see Eq. (41)), we have:

$$W^{(L)}_{\nu, \nu'} = -\frac{\epsilon_{\alpha\beta\gamma}}{M_{16}} \hat{F}_{ij}(G\nu')(kl)(u_i^\alpha d_j^\beta)(d_k^\gamma\nu_\mu').$$  \hspace{1cm} (104)

Since $\hat{F}_{33}$ is the leading element of $\hat{F}$, first consider the following combination of indices:

**Choice E: (ij)(kl) = (33)(13);** Starting with the operator $(u_3^\alpha d_3^\beta)(d_1^\gamma\nu_3)$, it is clear that $d_3^\beta$ must be external. This yields $V_{ls}s^{\beta}$ (if $d_3$ is dressed, it would lead to external top, which is forbidden); thus $\tilde{u}_3^\alpha$ and $\tilde{d}_1^\gamma$ must be dressed to respectively yield $V_{td}d^{\alpha}$ and $u^{\gamma}$. Thus one finds:

$$E_{L, \nu'}^{(1)} \approx \frac{\hat{F}_{33}(G\nu')(13)}{M_{16}} V_{td}V_{ts}f(t, d)\epsilon_{\alpha\beta\gamma}(d^{\alpha}u^{\beta})(s^{\gamma}\nu_3)$$

$$\approx \left(\frac{\hat{F}_{33}\hat{F}_{33}}{M_{16} \tan \gamma}\right) \left[-(6\eta_{cb}\eta_{\nu'} + 4.3\eta_{cs}\eta_{\nu'}\eta_{td}\eta_{ls} \times 10^{-7}] \times \right.$$

$$\zeta_{KM} f(t, d)\epsilon_{\alpha\beta\gamma}(d^{\alpha}u^{\beta})(s^{\gamma}\nu_3).$$  \hspace{1cm} (105)

Here, we have used values of $\hat{F}_{ij}$ and $(G\nu')(kl)\tan \gamma$ listed in Appendix A (Eq. (80)-(81)), and have put $V_{td} = 0.006\eta_{td}$, $V_{ts} = 0.04\eta_{ts}$. The factor $\zeta_{KM} = 1/2$ to 1 denote the
combined uncertainty in the two CKM elements. Note that the two terms in Eq. (104) check with the second and the fourth terms in Eq. (85) proportional to \((M_{16} \tan \gamma)^{-1}\). Some of the other leading contributions arise by choosing \((ij) = (23)\) or \((32)\), varying \((kl)\). These are proportional to \(y\) and is given by the seventh through the eleventh terms in the second bracket of Eq. (85). Note that the tenth and the eleventh terms subtract (because \(\eta_{ts} \eta_{cb} = -1\), and thus have a combined magnitude \(\approx 0.5 \times 10^{-5}y\), while the seventh and the eighth terms subtract to have a net magnitude of \(0.9 \times 10^{-5}y\). Thus, allowing for unknown phases of each of these contributions and also of the 10th term, we estimate that the total contribution from the \(y\) dependent terms is \(\approx (1.3 \pm 1) \times 10^{-5}y \approx (0.7 \pm 0.5) \times 10^{-6}\), where we have put \(y \approx 1/20\) (see Eq. (25)). Now the \(y\)-independent terms in the second square bracket of Eq. (85), including that exhibited in Eq. (104), are each individually smaller by factors of 1.5 to 2.5 compared to \(0\) except that the term proportional to \(x\) is nearly \(4 \times 10^{-6}(1/5\) to 1\) for \(x \approx 10^{-4}(1/5\) to 1\). Although the relative phases (signs) of the various contributions, which depend on \(\eta_{ij}\), are not known, allowing for possible cancellation between some of the smaller terms, it seems most plausible (as a conservative estimate) that the second square bracket in Eq. (85) has a net magnitude \(\approx (1.5 \times 10^{-6})(1/2.5\) to 2.5\) for case I. Estimating similarly, one obtains a very similar magnitude for case II as well. Now, including the contributions from interchange of \(i \leftrightarrow j\), keeping \((k, l)\) fixed, as appropriate, and that involving charged lepton dressing, as before, the net contribution from the new operator (the seventh term in \(W^{(L)}\), related to the neutrino masses is thus given by:

\[
A(\bar{\tau} K^+)_{\text{new}} \approx \left(\frac{\hat{h}_{33}}{M_{16} \tan \gamma}\right) \left[(1.5 \times 10^{-6})(1/2.5\) to 2.5\)] \[f(t, d) + f(t, l)\] \[\epsilon_{\alpha\beta\gamma}(d^\alpha u^\beta)(s^\gamma)\nu_2\] (105)

for case I as well as case II.

Using Eq. (87), the contribution of the new operator to the \(\tau \mu K^+\) mode for both cases I and II is estimated to be:

\[
\hat{A}(\bar{\tau} \mu K^+)_{\text{new}} \approx \left(\frac{\hat{h}_{33}}{M_{16} \tan \gamma}\right) \left[(3 \times 10^{-6})(1/2\) to 2\)] \[\times[f(t, d) + f(t, l)]\epsilon_{\alpha\beta\gamma}(d^\alpha u^\beta)(s^\gamma)\nu_\mu\] (106)

Here we have used \(x \approx 10^{-4}(1/5\) – 1\) and \(z \approx (1/400\) – 1/200\), in accord with discussions in Sec. V. Thus, for the new operator, the \(\tau \mu K^+\) mode would typically supercede the \(\tau \tau K^+\) mode, and the amplitudes for either mode remain essentially the same for both cases I and II. It is worth noting that the main source of the relative enhancement of the \(\tau \mu K^+\) amplitude is due to the fact that the dominant \(y\)-independent terms add for \(\tau \mu K^+\) (see Eq. 88), and they subtract for \(\tau \tau K^+\) (see Eq. (85)), if one uses \(\eta_{ts} = -\eta_{cb}\).

**Charged lepton decay modes:**

The amplitude for proton decay into charged leptons, \(p \to \ell^+ X\), is given in Eq. (84) for a generic superpotential of the form Eq. (82). The number of terms are fewer compared to the amplitude for neutrino mode, so they can be summed easily. The full result is given in Eqs. (88)-(89) for the muonic and electronic modes. We will briefly describe the origin of the dominant terms in the amplitude involving charged muons (Eq. (88)).
The standard $d = 5$ amplitude receives two contributions, one from the exchange of $\tilde{c} - \tilde{d}'$ squarks (the term proportional to $V_{cs}$ in Eq. (88)), and one from the exchange of $\tilde{u} d'$ (the $V_{ts}$ term). They are comparable numerically. Note that unlike in minimal $SU(5)$ models, where these contributions undergo a unitarity cancellation, here the amplitude is not much suppressed compared to the neutrino mode.

Following the same procedure as discussed in the other cases, the contribution of the standard operator to the $\mu^+ K^0$ mode (given in Eq. (88)) is estimated to be

$$A(\mu^+ K^0)_{\text{std}} \simeq \frac{h^2_{33} 2 f(a, b)}{M_{\text{eff}}} \left[ 3 \times 10^{-6} (1/2 \text{ to } 2) - \text{Case I} \right] \epsilon_{\alpha \beta \gamma} (u^\alpha s^\beta)(u^\gamma \mu^-) . \quad (107)$$

Thus the standard amplitudes for the $\mu^+ K^0$ mode are nearly the same for both cases I and II. They are typically smaller however than those for both $\overline{\nu}_\tau K^+$ and $\overline{\nu}_\mu K^+$ modes by a factor of 3 to 10 (compare Eq. (107) with Eq. (101) and (102)):

$$A(\mu^+ K^0)_{\text{std}} / A(\overline{\nu}_\ell K^+)_{\text{std}} \simeq 1/3 - 1/10, \ \ell = \mu, \tau . \quad (108)$$

We see that for the standard operators, the charged lepton ($\mu^+$) mode, though relatively enhanced compared to minimal $SU(5)$ is still expected to be suppressed in the rate, even in the $SO(10)$ model presented here, by one to two orders of magnitude, compared to the neutrino modes.

The contributions of the new operator (related to neutrino masses) to $\mu^+ K^0$ may also be estimated using Eq. (88). Suppressing relative phases, it is found to be:

$$A(\mu^+ K^0)_{\text{new}} \approx \left( \frac{\hat{f}_{33} h_{33}}{M_{16} \tan \gamma} \right) \left[ 1.8 \times 10^{-6} (x/10^{-4}) + (2 \text{ to } 10) \times 10^{-7} \right]$$

$$\approx \left( \frac{\hat{f}_{33} h_{33}}{M_{16} \tan \gamma} \right) (10^{-6})(1/3 \text{ to } 2) \text{ (for I and II)} \quad (109)$$

In above, we have used $z \approx 1/300$ (in Step I) and $x \approx (1/3 - 1) \times 10^{-4}$ (in Step 2) in accord with discussions in Sec. V. Thus, confining to the contributions of only the new operator, we would typically expect: $A(\mu^+ K^0)_{\text{new}} / A(\overline{\nu}_\ell K^+)_{\text{new}} \approx 1 - 1/3$ for $\ell = \mu, \tau$. (Compare Eq. (109) with Eq. (105) and (106).) This means that if somehow the standard operators were absent (or highly suppressed) and the new $d = 5$ operators were the only (dominant) source of proton decay (a possibility discussed in Sec. VI.F), the charged lepton $\mu^+ K^0$ could be comparable to the neutrino mode, having a branching ratio of nearly few to even 50%. As discussed in the text (Sec. VI), this is also found to be the case by including contributions from the standard as well as the new operators.

The estimates of the proton decay amplitudes arising from the standard and the new (neutrino-related) $d = 5$ operators for the three relevant modes ($\overline{\nu}_\tau K^+, \overline{\nu}_\mu K^+, \mu^+ K^0$) are summarized in Table 1.

| Mode          | (Case I)$_{\text{std}} \ (\epsilon' \neq 0)$ | (Case II)$_{\text{std}} \ (\epsilon' = 0)$ | (Case I)$_{\text{new}} \ (\epsilon' \neq 0)$ | (Case II)$_{\text{new}} \ (\epsilon' = 0)$ |
|---------------|-----------------------------------------------|-----------------------------------------------|-----------------------------------------------|-----------------------------------------------|
| $\overline{\nu}_\tau K^+$ | $2.8 \times 10^{-9}(1 \pm 0.5)$               | $0.7 \times 10^{-9}(1 \pm 0.7)$               | $1.5 \times 10^{-6}(1/2.5 - 2.5)$            | $2 \times 10^{-6}(1/2.5 - 2.5)$              |
| $\overline{\nu}_\mu K^+$   | $1.5 \times 10^{-5}(1 \pm 0.5)$               | $1.2 \times 10^{-5}(1 \pm 0.5)$               | $3 \times 10^{-6}(1/2 - 2)$                  | $3 \times 10^{-6}(1/2 - 2)$                  |
| $\mu^+ K^0$           | $3 \times 10^{-6}(1/2 - 2)$                   | $(2 \pm 0.8) \times 10^{-6}$                 | $10^{-6}(1/3 - 2)$                           | $10^{-6}(1/3 - 2)$                           |
The magnitudes for the standard and the new operators can be obtained by multiplying the respective entries shown in the Table by the factor $X \equiv \hat{h}_{33}^2 2 f(c,d)/M_{\text{eff}}$ and $Y \equiv \hat{f}_{33} \hat{h}_{33}^2 2 f(c,d)/(M_{16} \tan \gamma)$ respectively (see the text for definitions, in particular Sec. VI E, where one derives $P \equiv \hat{f}_{33} \hat{h}_{33}^2 / (M_{16} \tan \gamma) \approx 5 \times 10^{-19}$GeV$^{-1}$ (1/2 to 2), and Eqs. (101) and (105) as examples).

**APPENDIX C. CONTRIBUTION TO PROTON DECAY RATE FROM THE $D = 5$ OPERATOR**

In this Appendix we give a general discussion of the numerical value for the $d = 5$ proton decay amplitude.

Define

\[ \hat{A}(\nu_i) = A_5 (p \rightarrow \nu K^+)/ (2 \mathcal{F}) . \]  

(110)

$A_5$ denotes the strength of the respective four–fermion proton decay amplitudes arising from the $d = 5$ operators in $W$. One representative contribution to $A_5$ is given by Eq. (97). The net value of $A_5$ includes contributions from all the $d = 5$ operators, allowing for all possible combinations of the indices $(ij)(kl)$. It also includes the wino dressing factor and the relevant CKM mixing factors at all vertices. The net values are listed in Appendix B. The quantity $\mathcal{F}$ is the average of the two relevant dressing functions. For instance, for the amplitude in Eq. (100):

\[ \mathcal{F} \equiv \left[ \hat{f}(c,d) + \hat{f}(c,l) \right] / 2 . \]  

(111)

Assuming approximate degeneracy of the sfermions (for simplicity, see however, remarks later), $\hat{f}$ is the same for all contributions to the amplitude. Then $\hat{A}$ defined as above is just the net strength of the corresponding $d = 5$ operator in the superpotential, multiplied by the relevant CKM factors which occur at the vertices involving the color triplet and the wino exchanges. Since $\mathcal{F}$ has dimension of (mass)$^{-1}$, $\hat{A}$ has dimension 5.

For comparison purposes, it is useful to note that for minimal SUSY $SU(5)$ involving exchange of color triplet $H_C$ between $(\tilde{c}s)$ and $(d\nu_\mu)$ pairs, the $d = 5$ operator has strength $= (h_{22}^u h_{12}^d / M_{H_C}) \approx (m_c m_s \sin \theta_C / (v_u v_d)) / M_{H_C} \approx (m_c m_s \sin \theta_C / v_u^2) (\tan \beta / M_{H_C}) \approx 9 \times 10^{-8} (\tan \beta / M_{H_C})$, where $\tan \beta = v_u / v_d$ and we have put $v_u = 174$ GeV and the extrapolated values of the fermion masses at the unification scale – i.e., $m_c \approx 300$ MeV, and $m_s \approx 40$ MeV. Multiplying further by an additional factor of $\sin \theta_C$ due to conversion of $\tilde{c}$ to $d$ at the wino vertex, for the case of $SU(5)$,

\[ \hat{A}_{\tilde{c}d}(SU(5)) \approx (1.9 \times 10^{-8}) (\tan \beta / M_{H_C}) , \]  

(112)

corresponding to exchange of the pair $(\tilde{c}, \tilde{d})$. There is a similar contribution involving the exchange of the pair $(\tilde{t}, \tilde{d})$ with the substitution $m_c \sin \theta_C \rightarrow m_t V_{td}$.

In calculating the proton decay rate, we will assume the following spectrum of supersymmetric particles as a guide: Squarks are nearly degenerate, with masses $m_{\tilde{q}} \approx 1$ TeV (1 to 1.5) and wino is lighter than the squarks – i.e., $m_{\tilde{W}} / m_{\tilde{q}} \approx 1/6$ (1/2 to 2). Consistent with light gaugino masses, we will furthermore assume that $m_{1/2}$ (the common gaugino mass)
is small compared to $m_0$ (the common scalar mass). In this case, starting with universal masses for the scalars at the unification scale, we would expect the slepton masses to be nearly degenerate with the squark masses at the electroweak scale. The two $f$-functions that enter into the amplitude are then nearly equal (e.g. in Eq. (99)), $f(c, d) \approx f(c, l)$ and $\hat{f} \approx (m_\tilde{W}/m_{s_0}^2)(\alpha_2/4\pi)$. A SUSY spectrum as described, i.e., $m_\tilde{W} \ll m_\tilde{q} \approx m_\tilde{f} \sim 1$ TeV is needed anyway, a posteriori, because without it SUSY grand unified models based on $SU(5)$ or $SO(10)$ are likely to run into conflict with the experimental limits on proton lifetime. A spectrum of this type is plausible in several scenarios of SUSY breaking.

With the strength of $(du)(sv_\tau)$ operator being given by $A_\tau = \hat{A}(2T)$, as in Eq. (109), the inverse decay rate for $p \to \tau K^+$ is given by

$$\Gamma^{-1}(p \to \tau K^+) \approx (2.2 \times 10^{31}) \text{ yrs} \times \left(\frac{.67}{A_S}\right)^2 \left[\frac{0.066 \text{ GeV}^3}{\beta_H}\right]^2 \left[\frac{(1/6)}{(m_\tilde{W}/m_\tilde{q})}\right]^2 \left[\frac{m_\tilde{q}}{1 \text{ TeV}}\right]^2 \left[\frac{2\times 10^{-24} \text{ GeV}^{-1}}{A(\tau)}\right]^2.$$

(113)

Here $\beta_H$ denotes the hadronic matrix element defined by $\beta_H = \epsilon_{\alpha\beta\gamma}(0)(d^0_i u_\gamma)u_\beta^i|p, \vec{k}\rangle$. While the range $\beta_H = (0.003 - 0.03)$ GeV$^3$ has been used in the past [13], given that one lattice calculations yield $\beta_H = (5.6 \pm 0.5) \times 10^{-3}$ GeV$^3$ [11], we will take as a plausible range: $\beta_H = (0.006 \text{ GeV}^3)(1/2$ to 2). $A_S$ stands for the short distance renormalization factor of the $d = 5$ operator. In minimal $SU(5)$ it has a central value of 0.67. Although this factor is slightly different in the $SO(10)$ model, we shall adopt $A_S = 0.67$ for the $SO(10)$ model as well.

Note that the familiar factors that appear in the expression for proton lifetime — i.e., $M_{H_C^*}, (1 + y_{t\tau})$ representing the interference between the $\tilde{t}$ and $\tilde{c}$ contributions and tan $\beta$ — are all effectively contained in $\hat{A}(\tau)$. In fact, the analog of $M_{H_C}$ for $SU(5)$ is given in our case by two mass scales: $M_{\text{eff}}$ and $M_{16}$, representing contributions of the standard and the neutrino mass-related operators, respectively. The analog of $(1 + y_{t\tau})$ is reflected by the uncertainty in the net value of the many terms in Eq. (85), which depends on their unknown relative phases (see discussion in Appendix B, where a reasonable estimate of the uncertainty owing to the phases is given (Eq. (101) and (105)).

In minimal SUSY $SU(5)$, $\hat{A}$ is proportional to $(\sin 2\beta)^{-1} = 1/2(\tan \beta + 1/\tan \beta)$, and thus approximately to tan $\beta$ for tan $\beta \gtrsim 3$ or so. Corresponding to a realistic treatment of fermion masses as in Sec. IV and V, this approximate proportionality to tan $\beta$ does not however hold for $SO(10)$. The reason is this: If the fermions acquire masses only through the $10_H$ in $SO(10)$, as is well known, the up and the down quark Yukawa couplings will be

\footnote{Allowing for sleptons to be lighter than the squarks by a factor of 2 to 3 while keeping the squark masses to be about 1 TeV would amount to increasing $\hat{f}$ by about 50%, compared to its value for the case of nearly degenerate masses $(m_\tilde{q} \approx m_\tilde{f})$, and thereby enhancing the rate by a factor of 2. This possibility is of course not presently excluded.}

\footnote{For example, models of SUSY breaking based on contributions from a family universal anomalous $U(1) D$-term, superposed with subdominant dilaton $F$-term contributions [17] would lead to such a spectrum.}
equal. This would give the familiar \( t - b - \tau - \nu^D \) unification. By itself, it would also lead to a large value of \( \tan \beta = m_t/m_b \simeq 60 \), and correspondingly to a large enhancement in proton decay amplitude. Furthermore, it would also lead to the bad relations: \( m_c/m_s = m_t/m_b \) and \( V_{CKM} = 1 \). However, in the presence of additional Higgs multiplets contributing to fermion masses, such as \( 16_H \), which (a) distinguish between the up and the down sectors and (b) correct the bad relation mentioned above (see Sec. IV and V), \( \tan \beta \) can get lowered considerably – for instance to values like 10 - 20. Now, with the contributions from (a) and (b) correct the bad relation mentioned above (see Sec. IV and V), \( \tan \beta \) disappears. In this case it is more useful to write \( A \) or \( \hat{A} \) simply in terms of the relevant Yukawa couplings. That is what we have done in writing \( W \) (see Eq. (40)) in terms of the products of Yukawa coupling matrices like \((\hat{H})(\hat{H}^\prime)\) etc, and likewise for the amplitude.

It is instructive to compare a typical amplitude for \( SO(10) \) to the corresponding one for \( SU(5) \). This ratio is given by:

\[
\frac{\hat{A}(\bar{\nu}_\mu K^+)^{SO(10)}}{A(\bar{\nu}_\mu K^+)^{SU(5)}} \approx \frac{\hat{h}_{33}^2}{M_{\text{eff}}} \frac{2 \times 10^{-5}}{1.9 \times 10^{-8} (\tan \beta / M_{H_c})} \approx (m_{H_c}/M_{\text{eff}})(88/3 \tan \beta).
\]

(114)

We have put \( h_{33}^2 \approx 1/4 \) in going from the first line to the second in Eq. (114). Thus we see that \( M_{\text{eff}} \) has to be \((88 - 53)\) times larger (for \( \tan \beta = 3 \) to 5) so that the \( SO(10) \) amplitude may be comparable to that of \( SU(5) \). In other words, if \( M_{\text{eff}} \) were equal to \( M_{H_c} \), one would have a net enhancement by about two orders of magnitude of the \( d = 5 \) amplitude for proton decay in \( SO(10) \) compared to that of \( SU(5) \). This large enhancement of the amplitude in \( SO(10) \) – apart from the factor of \( M_{H_c}/M_{\text{eff}} \) – has come about due to a combination of several factors: (i) the large off–diagonal coupling of the up–type quarks with the color triplets which scale as \( \sqrt{m_c m_t} \) in the \( SO(10) \) model in contrast to the diagonal \( m_c \) in minimal \( SU(5) \), and (ii) the larger muon Yukawa coupling to color triplets (relative to the strange quark Yukawa in \( SU(5) \)).

\( SO(10) \), however, has a possible source of suppression of the \( d = 5 \) amplitude, because of the nature of the doublet–triplet splitting mechanism in it. The suppression would arise if the mass \( M_{10} \) of \( 10_H' \) is considerably smaller than the scale of \( \lambda \langle 45_H \rangle = \lambda a \), as discussed in Section VI.A. In this case, \( M_{\text{eff}} \equiv (\lambda a)^2/M_{10} \) can far exceed \( \lambda a \). Since \( \langle 45_H \rangle \) breaks \( SO(10) \) to \( SU(3)_C \times SU(2)_L \times I_{3R} \times (B - L) \), one of course naturally expects \( \lambda a \) to be nearly equal to or somewhat larger than the unification scale \( M_U \approx 2 \times 10^{16} \) GeV. But \( M_{10} \) can in general be one or even two orders of magnitude smaller than \( M_U \). For instance, \( 10_H' \) may carry a charge that would forbid a mass term \( (10_H')^2 \) at the renormalizable level. Such a mass term could still effectively arise through nonrenormalizable operators by utilizing the VEVs of certain fields \( \phi_i \) which do not conserve the respective charge. In this case, the mass term \( M_{10} \) may be suppressed by relevant powers of \( \langle \phi_i \rangle / M \) where one may expect \( \langle \phi_i \rangle / M \sim 1/10 \). If \( M_{10} \sim (1/10 - 1/100)\lambda a \), \( M_{\text{eff}} \) can exceed \( M_U \) and thus \( M_{H_c} \) by one or two orders of magnitude. The significance of such a large value of \( M_{\text{eff}} \) for coupling unification is discussed in the text.

More precise constraints on \( M_{\text{eff}} \) and \( M_{16} \tan \gamma \) are obtained directly by using limits on proton lifetime. They are discussed in the text in Sec. VI.
APPENDIX D: THRESHOLD CORRECTIONS TO $\alpha_3(M_Z)$ DUE TO LOW DIMENSIONAL MULTIPLETS OF $SO(10)$

Given $\alpha_1$ and $\alpha_2$ at $M_Z$, and the MSSM spectrum, the hypothesis of grand unification allows us to predict $\alpha_3(M_Z)$. This prediction, however, receives so-called threshold corrections owing to mass-splittings, at the unification scale $M_U$, between submultiplets belonging to complete $SO(10)$ multiplets. Such mass splittings are of course induced when $SO(10)$ breaks spontaneously to lower symmetries using Higgs multiplets such as $45_H$ and $16_H$.

Denoting the one-loop threshold corrections to $\alpha^{-1}_i(m_Z)$ by $-\Delta_i$, so that $\alpha^{-1}_i(m_Z) = \alpha^{-1}_i(M_Z) - \Delta_i$, we obtain: $\Delta_i = \sum_{\alpha} (b^\alpha_i)/\alpha_{U} \ln(M_U/m_\alpha)$ where $b_i = (33/5, 1, -3)$ yield the familiar one-loop $\beta$ functions for the evolution of the three gauge couplings ($i = 1, 2, 3$) for the MSSM spectrum, and $b^\alpha_i$ is the contribution to the evolution from the $\alpha$th sub-multiplet with mass $m_\alpha$. Using straightforward algebra, the threshold correction to $\alpha_3(m_Z)$ is thus found to be:

$$\Delta\alpha_3(m_Z) = [\alpha_3(m_Z)]^2 \left( \frac{5}{7}\Delta_1 - \frac{12}{7}\Delta_2 + \Delta_3 \right)$$

For illustration only, consider the contributions to $\Delta\alpha_3(m_Z)$ due to a splitting between the doublet and the triplet members in a $5 + \overline{5}$ of $SU(5)$ (or equivalently a $10$ of $SO(10)$) with masses $m_2$ and $m_3$ respectively, which are both of order $M_U$. In this case, one obtains:

$$\Delta_1 = (1/2\pi)(3/5)\ln(m_2/M_U) + (1/2\pi)(2/5)\ln(m_3/M_U)$$
$$\Delta_2 = (1/2\pi)(1)\ln(m_2/M_U), \quad \Delta_3 = (1/2\pi)(1)\ln(m_3/M_U)$$

Using Eq. (116), one then obtains:

$$\Delta\alpha_3(m_Z)_{DT} = [\alpha_3(m_Z)]^2(9/7)(1/2\pi)\ln(m_3/m_2)$$

Using the doublet–triplet splitting mechanism exhibited in Eq. (39) and the fact that the MSSM evolution includes the two light Higgs doublets, one obtains the corresponding formula for $\Delta\alpha_3(m_Z)_{DT}$ exhibited in the text (Eq. (53)), where $(m_3/m_2)$ is replaced by $(M_{eff}\cos\gamma/M_U)$ as explained there.

The threshold corrections owing to mass-splittings within the gauge multiplet and the Higgs multiplet $45_H$ can be computed in the same manner. First consider the gauge multiplet.

The masses of the gauge particles acquiring superheavy masses are given by:

$$M^2(3, 1, 2/3) = 4g^2(c^2 + a^2), \quad M^2(3, 2, 1/6) = g^2(4c^2 + a^2), \quad M^2(1, 1, \pm 1) = 4g^2c^2, \quad M^2(3, 2, -5/6) = g^2a^2,$$

where the numbers in parentheses denote $SU(3) \times SU(2) \times U(1)$ quantum numbers, and $g$ is the unified gauge coupling. Such a splitting leads to a correction in $\alpha_3$, which at the electroweak scale is given by:

$$\Delta\alpha_3(m_Z)_{gauge} = -\frac{\alpha_3(m_Z)^2}{70\pi} [75\ln(4 + p^2) - 105\ln(1 + p^2) + 30\ln p^2]$$

where $p = 2c/a$. Eq. (118) includes contributions from the gauge particles as well as from the corresponding gauginos and respective Higgsinos which together make 4–component Dirac particles. Numerical values of this correction are presented in the text.

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The contribution to $\alpha_3(m_Z)$ from the splitting of $45_H$ is found to be:

$$\Delta \alpha_3(m_Z)_{45H} = \frac{\alpha_3(m_Z)^2}{14\pi} [6\ln\left(\frac{M(1,1,\pm1)}{M_U}\right) - 24\ln\left(\frac{M(1,3,0)}{M_U}\right) + 21\ln\left(\frac{M(8,1,0)}{M_U}\right)]$$  \hspace{1cm} (119)$$

Here the numbers (a,b,c) in $M$ denote the quantum numbers of the respective sub-multiplet in $45_H$ under $SU(3) \times SU(2) \times U(1)$. Within the simplest model consisting of a $45_H$ only (i.e., if we ignore its couplings to other multiplets), one can argue that the masses of the sub-multiplets are about two (to one) order of magnitude lower than $M_U$. This is because with only a $45_H$, the effective $SO(10)$–invariant superpotential including self-couplings has the form $W_{45_H} = M_1(45_H)^2 + \kappa(45_H)^4/M$ where $M$ characterizes the effect of quantum gravity. Note no cubic term $(45_H)^3$ is allowed. Setting $F_{45_H} = M_1 < 45_H > + \kappa < 45_H >^3 / M = 0$, we get $M_1 \sim \kappa < 45_H >^2 / M \approx \kappa M_U^2 / M$, where we put $< 45_H > \sim M_U$. Putting $\kappa \sim 1$ and $M \approx M_{Pl} = 2 \times 10^{18}$ GeV, we get $M_1 \sim 10^{-2} M_U$. With the masses of the sub-multiplets given by $M_1 \approx 10^{-2} M_U$, Eq. (119) yields $\Delta \alpha_3(m_Z) \approx -0.0045$.

Note that if one had introduced larger Higgs multiplets like $126_H$ or even $54$ of $SO(10)$, one would have encountered substantially larger threshold correction to $\alpha_3(m_Z)$. For example, with just a $54$, which contains a $(6,1,4/3) + (6,1,-4/3)$ and a $(8,1,0)$ sub-multiplets, the threshold correction from the color (sextets, octet) to $\alpha_3(m_Z)$ is given by $\Delta \alpha_3(m_Z)_{6+8} = [(51,21)/(14\pi)] \alpha_3(m_Z)^3 \ln(M_{6,8}/M_U)$. This yields a correction to $\alpha_3(m_Z)$ exceeding about 14% (of either sign) if $M_{6,8}/M_U \sim 1/2$ or 2. It therefore appears that $SO(10)$ multiplets with dimension larger than 45 are highly disfavored, if the observed unification of gauge couplings is not to be regarded as a coincidence.
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