Research Article

Life-Span of Classical Solutions to Hyperbolic Inverse Mean Curvature Flow

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In this paper, we investigate the life-span of classical solutions to hyperbolic inverse mean curvature flow. Under the condition that the curve can be expressed in the form of a graph, we derive a hyperbolic Monge–Ampère equation which can be reduced to a quasilinear hyperbolic system in terms of Riemann invariants. By the theory on the local solution for the Cauchy problem of the quasilinear hyperbolic system, we discuss life-span of classical solutions to the Cauchy problem of hyperbolic inverse mean curvature.

1. Introduction

In this paper, we study hyperbolic inverse mean curvature flow (HIMCF):

\[
\begin{align*}
\frac{\partial^2 \gamma}{\partial t^2} &= -k^{-1} \vec{N} + k^{-1} (k^{-1})_s \vec{T}, \quad \forall (z,t) \in S^1 \times [0,T), \\
y(z,0) &= y_0(z), \\
\frac{\partial y}{\partial t}(z,0) &= y_1(z), \\
\left\{ y_{1z}, \vec{N}_0 = 0 \right\},
\end{align*}
\]

(1)

where \( k \) denotes the mean curvature of the curve \( \gamma \) and \( \vec{N} \) and \( \vec{T} \) are, respectively, the unit inner normal and tangent vectors of the curve \( \gamma (z, t) \). \( y_0 \) stands for a smooth strictly convex closed curve, \( y_1 \) denotes the initial velocity of \( y_0 \), and \( \vec{N}_0 \) is the unit inner normal vector of \( y_0 \).

Definition 1 (see [1]). A flow is called a normal preserving flow if the normal vector of the curve is independent of time \( \frac{d \vec{N}}{dt} = 0 \), i.e., \( \left\{ y_{1z}, \vec{N} \right\} = 0 \) holds for all time.

It is well known that inverse mean curvature flow is an important method to derive the energy estimates in general relativity; for example, Huisken and Ilmanen developed a theory of weak solutions of the inverse mean curvature flow and used it to prove successfully Riemannian Penrose inequality which plays an important role in general relativity (see [2]). In [3], Urbas proved that, for inverse mean curvature flow, the surfaces stay strictly convex and smooth for all time. Furthermore, the surfaces become more and more spherical in the process. Similar results have also been obtained for star-shaped initial data with the positive mean curvature of the surface, see Urbas [3] and Gerhardt [4]. In [2], Huisken and Ilmanen proved a sharp lower bound of mean curvature, from which they proved that if the initial surface is the boundary of a strictly star-shaped domain and has nonnegative mean curvature, a smooth solution of the inverse mean flow will exist for all time and converge to a manifold.

The hyperbolic mean curvature flow, i.e., the hyperbolic version of mean curvature flow, has been introduced by some authors, see Gurtin and Podio-Guidugli [5], He et al. [6], Kong et al. [7, 8], Lefloch and Smoczyk [9], Notz [10], Rotstein et al. [11], and Wang [12, 13]. In fact, Yau in [14] has suggested the following equation related to a vibrating membrane or the motion of a surface:
where $H$ is the mean curvature and $\mathbf{N}$ is the unit inner normal vector of the surface and pointed out that very little is known about the global time behavior of the hypersurfaces. Recently, in [1], Chou and Wo proposed a new hyperbolic curvature flow for convex hypersurfaces. This flow is most suited when the Gauss curvature is involved. The equation satisfied by the graph of the hypersurface under this flow gives rise to a new class of fully nonlinear Euclidean invariant hyperbolic equations. Finally, they consider the expanding Gauss curvature flow driving by negative powers $-k^\theta$ of the Gauss curvature. In the special case $\beta = 1$, the expanding flow becomes, once written in terms of the support function $S(\theta, t)$, a linear problem:

\[
\begin{align*}
S_{tt} &= S_{\theta\theta} + S, \\
S(0) &= \phi, \\
S_0(0) &= \psi,
\end{align*}
\]

where $S(\theta, t) = \{y(\theta, t), (\cos \theta, \sin \theta)\}$, and the normal angle $\theta$ is determined modulo $2\pi$. They get the following result.

**Proposition 1** (see [1]). Consider (3) where the initial values are smooth and satisfy $\phi_{\theta\theta} + \phi > 0$ and $\psi_{\theta\theta} + \psi > 0$. Then, the flow remains smooth and expands to infinity like a circle.

Motivated by the inverse mean curvature flow and the local solution theory of the Cauchy problem for the quasilinear hyperbolic system (see [15]), we will focus on life span (the maximum existence time of unique local classical solutions) of classical solutions to the Cauchy problem for the hyperbolic inverse mean curvature flow. Our first result is the following local existence theorem for initial value problem (1).

**Theorem 1** (local existences and uniqueness). Suppose that $\gamma_0$ is a smooth, strictly convex curve, and $\gamma_1: \mathbb{S}^1 \rightarrow \mathbb{R}^2$ is a smooth vector function. Then, there exist positive $T$ and a family of strictly convex closed curves $\gamma(\cdot, t)$ with $t \in [0, T)$ such that $\gamma(\cdot, t)$ satisfies (1).

The following theorem concerns the life-span of local (in space) smooth solutions of flow (1) that can be written as convex graphs over an interval $R \subset \mathbb{R}$ in the form

\[
\gamma(t, z) = (x, u(t, x)), \text{ for some } u: [0, T) \times R \rightarrow \mathbb{R}. \quad (G).
\]

Writing the initial conditions as

\[
\begin{align*}
\gamma(0, z) &= (x, u_0(z)), \\
\frac{\partial \gamma}{\partial t} &= (\gamma_t, u_t),
\end{align*}
\]

for some $f \in C^2(\mathbb{R})$ and $g \in C^1(\mathbb{R})$, we obtain the following result.

**Theorem 2.** Assume that $\gamma_0$ is a strictly convex closed curve and $\gamma_1: \mathbb{S}^1 \rightarrow \mathbb{R}^2$ is a smooth vector function on $\mathbb{S}^1$. Then, a lower bound $\delta_*$ for the maximum time $T$ of existence of a solution of (1) in the form $G$ is given by

\[
\delta_* = \min \left\{ \frac{\|\phi\|}{A_0 + A_1\|\phi\|^2} : A_0, A_1 > 0 \right\},
\]

where $\phi = (\phi_1, \ldots, \phi_3)^T = ((1 + f^2 - g'f)'/f''', ((-1 - f^2 - g'f)'/f''), f, g, f')$, $\phi(x) = \phi(x)/dx$, and $A_0, A_1 > 0$ are the constant coefficients of the Gauss curvature. In the special case $\beta = 1$, the expanding Gauss curvature flow is

\[
\delta_0 = \left( \frac{\|\phi\|}{A_0 + A_1\|\phi\|^2} \right)_{A_0 > 0, A_1 > 0}.
\]

The paper is organized as follows. In Section 2, we derive a hyperbolic Monge–Ampère equation by the hyperbolic inverse mean curvature flow and give the short-time existence theorem, i.e., Theorem 1. In Section 3, we reduce the hyperbolic Monge–Ampère equation to a quasilinear hyperbolic system in Riemann invariants. Section 4 is devoted to prove the main result, i.e., Theorem 2, by the local solution of theory for the Cauchy problem of the quasilinear hyperbolic system.

### 2. Hyperbolic Monge–Ampère Equation

Suppose for $x \in (a, b)$ and $t \in (t_0, t_1)$, the curve $y(z, t)$ can be expressed in the form of a graph $(x, u(x, t))$, and we have

\[
\begin{align*}
y_t &= x_t (1, u_x) + (0, u_t), \\
y_{tt} &= x_{tt} (1, u_x) + (0, u_{xx}x_t^2 + 2u_{xt}x_t + u_{tt}).
\end{align*}
\]

Taking the inner product with the choice $\bar{N} = (-u_x, 1)/(\sqrt{1 + u_x^2})$ and $\bar{T} = (1, u_x)/(\sqrt{1 + u_x^2})$, respectively, yields

\[
\begin{align*}
u_{tt} + 2x_t u_{x_t} + x_t^2 u_{xx} = \sqrt{1 + u_x^2} (k^{-1}), \\
x_{tt} = \frac{(k^{-1} + u_x)k^{-1}}{\sqrt{1 + u_x^2}}.
\end{align*}
\]

It would be convenient to write the expression of the curvature $k$ for graphs:

\[
k = \frac{u_{xx}}{(1 + u_x^2)^{3/2}}.
\]

Flow (1) is a normal preserving flow. First, we need to compute $\partial_t \left\{ \gamma, \mathbf{N} \right\}$. Using the identities

\[
\begin{align*}
\frac{\partial^2 X}{\partial t^2} &= H \mathbf{N},
\end{align*}
\]
\[ \vec{N}_t = \left[ \vec{N}, \vec{T} \right] \vec{T} = - \left[ \vec{N}, \vec{T} \right] \vec{T} \]
\[ = -|y_2|^{-1} \{ \vec{N}, y_{t2} \} + \partial_t \{ |y_2|^{-1} \{ \vec{N}, y_{x2} \} \} \vec{T} \]
\[ = -|y_2|^{-1} \{ \vec{N}, y_{t2} \} \vec{T}, \]
\[ \langle y_{t2}, \vec{N} \rangle = -(k^{-1}) z + k^{-1}(k^{-1}) \langle \vec{T}, z, \vec{N} \rangle \]
\[ = -(k^{-1}) z + k^{-1}(k^{-1})(k|y_2|) = 0, \] 
(12)

one computes
\[ \frac{\partial}{\partial t} \langle y_{t2}, \vec{N} \rangle = \langle y_{t2}, \vec{N} \rangle + \langle y_{x2}, \vec{N} \rangle \]
\[ = -|y_2|^{-1} \langle y_{t2}, \vec{T} \rangle \langle y_{t2}, \vec{N} \rangle \]
\[ = -\langle y_{t2}, \vec{T} \rangle \langle y_{t2}, \vec{N} \rangle. \]
(13)

This is an ODE of the form dy/dt = by. Clearly, (1) is a normal preserving flow. Then, having observed that also the tangent vector is a constant, by using it, we can easily to show the identity
\[ u_{xx}x_t + u_{xt} = 0. \] 
(14)

By \( x_t = -(u_{st}/u_{xx}) \), the normal preserving flow is reducible, and the curvature \( k \) for graphs \( u_{xx}/(1 + u_x^2)^{3/2} \), graph form of flow (1), satisfies the following equation:
\[ u_{xx}u_{tt} - u_t^2 = -(1 + u_x^2)^2, \] 
(15)

This is a fully nonlinear hyperbolic Monge–Amphère equation as long as the curve is uniformly convex, i.e., flow (1) reduces to
\[ \begin{cases} 
  u_{xx}u_{tt} - u_t^2 = -(1 + u_x^2)^2, \\
  u(x, 0) = f(x), \\
  u_t(x, 0) = g(x). 
\end{cases} \] 
(16)

Remark 1. We have expressed flow (1) as the equation of support function in (3). When the curve \( y \) is represented as a graph and described by the support function simultaneously, the following relations hold:
\[ S = \frac{u - xu_x}{\sqrt{1 + u_x^2}}, \]
\[ \tan \theta = \frac{1}{u_x}, \]
\[ S_\theta = \frac{-x - uu_x}{\sqrt{1 + u_x^2}}. \]
(17)

For an unknown function \( h = h(x, t) \) defined on \( (x, t) \in \mathbb{R}^2 \), Monge–Amphère equation is
\[ A + Bh_t + Ch_{xt} + Dh_{xx} + E(h_{xx}h_{tt} - h_{xt}^2) = 0, \] 
(18)

where the coefficients \( A, B, C, D, \) and \( E \) depend on \( t, x, h, h_t, h_x \). We say that (18) is a hyperbolic Monge–Amphère equation if
\[ \Delta^2 (t, x, h, h_t, h_x) \equiv C^2 - 4BD + 4AE > 0, \]
\[ h_{xx} + B(t, x, h, h_t, h_x) \neq 0. \] 
(19)

Initial conditions for the Cauchy problem on the Ox axis are
\[ h(0, x) = h_0(x), \]
\[ h_1(0, x) = h_1(x), \] 
(20)

where \( h_0 \in C^3(\mathbb{R}^1) \) and \( h_1 \in C^2\mathbb{R}^1 \). Suppose \( h_0 \) and \( h_1 \) satisfy the next two conditions. First, the axis Ox is free, i.e.,
\[ h''_0 + B(0, x, h_0(x), h_1(x), h_0'(y)) \neq 0. \] 
(21)

Secondly, on the axis Ox, the hyperbolic condition
\[ \Delta^2(0, x, h_0(y), h_1(y), h_0'(y)) > 0, \] 
(22)

holds.

It is easy to prove that (15) is a hyperbolic Monge–Amphère equation, where
\[ A = (1 + u_x^2)^2, \]
\[ B = 0, \]
\[ C = 0, \]
\[ D = 0, \]
\[ E = 1. \] 
(23)

In fact,
\[ \Delta^2(t, x, u, u_t, u_x) = C^2 - 4BD + 4AE \]
\[ = 0^2 - 4 \cdot 0^2 + 4(1 + u_x^2) = 4(1 + u_x^2)^2 > 0, \]
\[ u_{xx} + B(t, x, u, u_t, u_x) = u_{xx} + 0 = k\sqrt{(1 + u_x^2)^2} \neq 0. \] 
(24)

Furthermore, we suppose that \( g(x) \) is the third continuous differential function and \( f(x) \) is second continuous differential function; then, initial conditions satisfy
\[ \Delta^2(0, x, f, g, f') = 4\left(1 + f'^2\right)^2 > 0, \]
\[ u_{xx}(x, 0) + B(0, x, f, g, f') = f'' + 0 = k_0\sqrt{(1 + f'^2)^3} \neq 0, \] 
(25)

in which \( f' = df/dx \) and \( f'' = d^2f/dx^2 \).

Hence, by the standard theory of hyperbolic equations (see [16]), we have local existences and uniqueness theorem (i.e., Theorem 1).

Remark 2. In fact, the derivation of (16) and Theorem 1 can also be obtained by a direct application of the arguments therein (cf. Section 1 in [1]). When establishing the local
solvability for the normal preserving flow, the authors in [1] suppose the initial curve \( \gamma_0 \in H^k(S^1) \) and \( \gamma_i(0) \in H^{k+1}(S^1) \), \( k > 5/2 \); however, we just provide that \( \gamma_0 \) and \( \gamma_i(0) \) are smooth.

### 3. Systems in Riemann Invariants

This section is concerned with the reduction of (15). Let \( u(x, t) \) be a \( C^3 \) solution of equation (15) in some domain \( \bar{D} \). Suppose
\[
\left. u_{xx} \right|_{x=0} \neq 0, \quad \forall \ (x, t) \in \bar{D}.
\]

Condition (26) means that vertical lines \( t = \text{const} \) are free. By definition, put
\[
r = \frac{C + \Delta - 2u_{xt}}{2u_{xx}}, \quad s = \frac{-1 - u_{xx}^2}{2u_{xx}}.
\]

The functions \( r \) and \( s \) are tangents of angles of inclinations of characteristics of equation (15), and we always call them Riemann invariants of (15). Let \( p = u_t \) and \( q = u_x \), then
\[
\begin{align*}
r_t + sr_x &= (r-s)q, \\
s_t + rs_x &= (r-s)q, \\
&u_t + ru_x = p + rq, \\
p_t + sp_x &= s(1 + q^2), \\
q_t + rq_x &= 1 + q^2.
\end{align*}
\]

**Theorem 3.** Let \( u(x, t) \) be a \( C^3 \) solution of (15). Suppose (26) and (15) are satisfied by \( u \). Then, the set of functions \( r, s, u, p, q \), where \( r \) and \( s \) are obtained from (27), \( p = u_t \), and \( q = u_x \), is a \( C^4 \) solution of the system of five equations (28) in \( \bar{D} \).

**Theorem 4.** Let \( (r, s, u, p, q) \) be a \( C^4 \)–solution of (28) in the domain \( \bar{D} \), satisfying the initial value condition
\[
\begin{align*}
u(0, x) &= f(x), \\
p(0, x) &= g(x), \\
q(0, x) &= f'(x), \\
r(0, x) &= \frac{1 + f'^2(x) - g'(x)}{f''(x)}, \\
s(0, x) &= \frac{-1 - f'^2(x) - g'(x)}{f''(x)}.
\end{align*}
\]

Let \( \bar{D} \) be a domain in which this solution is defined. Suppose
\[
\Delta (t, x, u(t, x), u_x(t, x), u_{xx}(t, x)) > 0,
\]
holds. Then, \( u(x, t) \) be a \( C^3 \) solution of (15) in the domain \( \bar{D} \), and furthermore, \( u_t = p, \ u_x = q, \) and (26) holds.

Let \( u = (u_1, u_2, u_3, u_4, u_5)^T = (r, s, u, p, q)^T \),
\[
\Lambda = \begin{pmatrix}
\lambda_1 & 0 & 0 & 0 & 0 \\
0 & \lambda_2 & 0 & 0 & 0 \\
0 & 0 & \lambda_3 & 0 & 0 \\
0 & 0 & 0 & \lambda_4 & 0 \\
0 & 0 & 0 & 0 & \lambda_5
\end{pmatrix}
\]
\[
\mu = (\mu_1, \ldots, \mu_5)^T = ((r-s)q, (r-s)q, p + rq, s(1 + q^2), 1 + q^2)^T.
\]

Then, we reduce equation (28) to the following Cauchy problem for a quasilinear hyperbolic system in terms of Riemann invariants:
\[
\begin{pmatrix}
\sum_{j=1}^5 \delta_{ij} \frac{\partial u_j}{\partial t} + \lambda_i \frac{\partial u_j}{\partial x}
\end{pmatrix} = \mu_i, \quad i = 1, \ldots, 5,
\]

where \( \phi = (\phi_1, \ldots, \phi_5)^T = ((1 + f'^2 - g')/f''', (-1 - f'^2 - g')/f'''', f, g, f')^T. \)

Throughout this paper, we shall use the following notation: the absolute value of any vector \( u = (u_1, \ldots, u_5) \) is defined as
\[
|u| = \max_i |u_i|,
\]
and the \( C^0 \) norm of a vector function \( u = u(t, x) = (u_1(t, x), \ldots, u_5(t, x)) \) on a bounded domain \( R \) is defined as
\[
\|u\| = \sup_{(t, x) \in R} |u(t, x)|.
\]

Firstly, suppose \( \phi = (\phi_1, \ldots, \phi_5)^T \) is bounded on \( R \) and \( \phi \in C^1(R) \). Take a suitable positive constant \( \Omega \) such that
\[
\|\phi\| < \Omega.
\]

For the continuous differentiable vector function \( u(t, x) \), we may define
\[
\|u\|_1 = \|u\| + \|\frac{\partial u}{\partial t}\| + \|\frac{\partial u}{\partial x}\|.
\]

Furthermore, for any finite set of functions set \( \Gamma \subset C^1(R) \) or \( \Gamma \subset C^1(\mathbb{R} \times [0, \delta]) \), and we can similarly define its norm \( \|\|_1, \|\| \).
such that the

satisfies (16).

By the standard hyperbolic

that

The following is about the proof of Theorem 2.

First, we consider quasilinear hyperbolic system (28) to be linear. Take a suitable positive constant $\Omega_1$ such that

$$\Omega_1 > \|\phi\| + (1 + \Omega)\|\dot{\phi}\| + \Omega^3 + 2\Omega^2 + \Omega + 1.$$  (37)

Here, $\Omega_1$ is a positive constant derived from second a priori estimate, $\phi = d\phi/dx$. Suppose the following function set

$$\sum(\delta, \Omega, \Omega_1) = \{y(t, x) | y = (y_1, \ldots, y_5)^T \in C^1[\delta], \|y\| \leq \Omega, \|y\| \leq \Omega_1\},$$  (38)

and for each $y \in \sum(\delta, \Omega, \Omega_1)$, by the linear hyperbolic system,

$$\sum_{l=1}^{5} \sigma_{l} \left( \frac{\partial u_l}{\partial t} + \tilde{\lambda}_l(t, x) \frac{\partial u_l}{\partial x} \right) = \tilde{\mu}_l(t, x), \quad l = 1, \ldots, 5,$$  (39)

and the initial conditions $u_0(0, x) = \phi(x), x \in R^1, l = 1, \ldots, 5$, in which

$$\tilde{\lambda}_l(t, x) = \lambda_l(t, x, n(t, x)), \quad \tilde{\mu}_l(t, x) = \mu_l(t, x, n(t, x)), \quad \text{i.e.,}$$

$$\tilde{\mu} = \mu(t, x, n(t, x)) = \left( (n_1 - n_3), \quad (n_1 - n_2), \quad n_3, \quad n_4 \right)^T.$$  (40)

Defining the function set $\Gamma^* = \{\lambda_l, (\partial \lambda_l/\partial x), (\partial \lambda_l/\partial u_l), \mu_l, (\partial \mu_l/\partial x), (\partial \mu_l/\partial u_l), l, j = 1, \ldots, 5\}$, we get the $C^0$ norm estimates of the set $\Gamma^*$:

$$\max_{l=1, \ldots, 5} |\lambda_l(t, x, v)| = \max_{l=1, \ldots, 5} |\lambda_l(t, x, v)| \leq \Omega,$$

$$\max_{l=1, \ldots, 5} \frac{\partial \lambda_l}{\partial x} (t, x, v) = 0,$$

$$\max_{l=1, \ldots, 5} \frac{\partial \mu_l}{\partial x} (t, x, v) = 1,$$

$$\max_{l=1, \ldots, 5} |\mu_l(t, x, v)| \leq \max \left\{ 2\Omega^2, 2\Omega^2, \Omega + \Omega^3, (1 + \Omega^3)\Omega, 1 + \Omega^2 \right\}$$

$$\leq \Omega^3 + 2\Omega^2 + \Omega + 1,$$

$$\max_{l=1, \ldots, 5} \frac{\partial \mu_l}{\partial u_j} (t, x, v) = 0,$$

$$\max_{l=1, \ldots, 5} \frac{\partial \mu_l}{\partial u_j} (t, x, v) \leq \max \left\{ 2\Omega^2, 1 + \Omega^3, 2\Omega^2 + 2\Omega + 1 \right\}$$

$$\leq 2\Omega^2 + 2\Omega + 1.$$  (41)

Hence, the $C^0$ norm of the set $\Gamma^*$ can be bounded as follows:

$$\|\Gamma^*\| = \max \left\{ \Omega, 0, 1, \Omega^3 + 2\Omega^2 + \Omega + 1, 2\Omega^2 + 2\Omega + 1 \right\}.$$  (42)

Let $\Gamma_2[y] = \left[ \tilde{\lambda}_l(t, x, \dot{\lambda}(t, x), \mu_l(t, x, \dot{\mu}(t, x)) \right]$; by

$$\frac{\partial \lambda_l}{\partial x} (t, x, y) = \frac{\partial \lambda_l}{\partial x} (t, x, y) + \sum_{l=1}^{5} \frac{\partial \lambda_l}{\partial u_j} (t, x, y) \frac{\partial y_j}{\partial x}, \quad l = 1, \ldots, 5,$$  (43)

$$\frac{\partial \mu_l}{\partial x} (t, x, y) = \frac{\partial \mu_l}{\partial x} (t, x, y) + \sum_{l=1}^{5} \frac{\partial \mu_l}{\partial u_j} (t, x, y) \frac{\partial y_j}{\partial x}, \quad l = 1, \ldots, 5,$$  (44)

we have

$$\|\partial \lambda_l/\partial x\| \leq \Omega_1,$$

$$\|\partial \mu_l/\partial x\| \leq \Omega_1 \max \left\{ 4\Omega, 2\Omega + 1, 3\Omega^2 + 1 \right\} \leq \left( 3\Omega^2 + 4\Omega + 1 \right)\Omega_1.$$  (45)
Therefore, we can bound the $C^0$ norm of the set $\Gamma_2[v]$ as follows:

$$\|\Gamma_2[v]\| \leq \max\{\Omega, \Omega_1, \Omega^3 + 2\Omega^2 + \Omega + 1, (3\Omega^2 + 4\Omega + 1)\Omega_1\}. \quad (46)$$

In order to prove Theorem 2, we will get some priori estimates for the solutions of Cauchy problems for linear hyperbolic systems. These estimates will be useful in proving the existence and uniqueness of solutions for the initial value problem of quasilinear hyperbolic systems.

Firstly, by the first associated integral relations of initial value problem (39),

$$\begin{align*}
\begin{cases}
\frac{d\omega_l}{dt} = \lambda_l(t, \omega_l(t, t, x)), \\
\omega_l(t, t, x) = x,
\end{cases}
\end{align*} \quad (47)$$

Hereafter, the argument of each function in the $l$-th integrand is $(\tau, \xi) = (t, \omega_l(t, t, x))$,

$$\begin{align*}
\frac{d\omega_l}{dt} &= \lambda_l(t, \omega_l(t, x)), \\
\omega_l(t; t, x) &= x,
\end{align*} \quad (48)$$

and $d/dt$ means the directional derivative along the $l$-th characteristic curve, i.e.,

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \lambda_l \frac{\partial}{\partial \xi}. \quad (49)$$

Moreover,

$$u_k^0 = \sum_{l,j=1}^n \delta^{kj}(t, t) \delta_{lj}(0, \xi_j(t, x)) \phi_l(\xi_j(t, x)), \quad k = 1, 2, \ldots, 5, \quad (50)$$

where $(\delta^{kj}(x, t))$ is the inverse matrix of $(\delta_{lj}(t, x))$, and each $\delta^{kj}(x, t)$ is also a $C^1$ function of $(t, x)$. By

$$\begin{align*}
\max_{k=1,\ldots,5} |u_k^0(t, x)| &\leq \|\phi\|, \\
\max_{l=1,\ldots,5} |\mu_l(t, x, v)| &\leq \Omega^3 + 2\Omega^2 + \Omega + 1,
\end{align*} \quad (51)$$

if $0 \leq t \leq \delta$, we have

$$\|w\| \leq \|\phi\| + (\Omega^3 + 2\Omega^2 + \Omega + 1)\delta. \quad (52)$$

This is first a priori estimate for the solution of Cauchy problem.

We are going to construct second a priori estimate for the first derivatives of the solutions of Cauchy problem. Define the function set

$$p = (p_i(t, x)) = \left(\frac{\partial u_i(t, x)}{\partial t}\right),$$

$$w = (w_i(t, x)) = \left(\frac{\partial u_i(t, x)}{\partial x}\right), \quad i = 1, \ldots, 5, \quad (53)$$

By the second associated integral,

$$\begin{align*}
u_k(t, x) &= u_k^0(t, x) + \int_0^t \mu_k(t, \omega_k(t, t, x))dt, \\
p_k(t, x) &= p_k^0(t, x) - \lambda_k \int_0^t p_k(t, \omega_k(t, t, x))dt.
\end{align*} \quad (54)$$

On $R(\delta)$, we have

$$\max_{k=1,\ldots,5} |u_k^0(t, x)| \leq \|\phi\|, \quad (55)$$

Let

$$\begin{align*}
w(t) &= \sup_{k=1,\ldots,5} \{|u_k(t, x)|, (t, x) \in R(t)\}, \\
p(t) &= \sup_{k=1,\ldots,5} \{|p_k(t, x)|, (t, x) \in R(t)\},
\end{align*} \quad (56)$$

obviously; we have

$$\|w\| = \sup_{0 \leq t \leq \delta} w(t), \quad (57)$$

$$\|p\| = \sup_{0 \leq t \leq \delta} p(t). \quad (58)$$

By (54), if $0 \leq t \leq \delta$,

$$w(t) \leq \|\phi\| + \left(3\Omega^3 + 4\Omega + 1\right)\Omega_1\delta + \Omega_1 \int_0^t w(t) dt. \quad (59)$$

By Gronwall inequality,

$$w(t) \leq \|\phi\| + \left(3\Omega^3 + 4\Omega + 1\right)\Omega_1\delta \left[\Omega_1\delta \right]^{t_0}. \quad (60)$$

we have

$$\|w\| \leq \left[\|\phi\| + \left(3\Omega^3 + 4\Omega + 1\right)\Omega_1\delta \right]^{t_0}. \quad (61)$$

By (62), we have
\[ p(t) \leq \Omega ||\phi|| + \Omega^3 + 2\Omega^2 + \Omega + 1 + \Omega + 1 + (3\Omega^2 + 4\Omega + 1)\Omega_0, \delta \\
+ \Omega_1 \int_0^t \left[ ||\phi|| + (3\Omega^2 + 4\Omega + 1)\Omega_1, \delta \right] e^{\Omega_1, t} dt \\
= \Omega^3 + 2\Omega^2 + \Omega + 1 + \Omega \left[ ||\phi|| + (3\Omega^2 + 4\Omega + 1)\Omega_1, \delta \right] e^{\Omega_1, t}, \] (64)
i.e.,
\[ ||p|| \leq \Omega^3 + 2\Omega^2 + \Omega + 1 + \Omega \left[ ||\phi|| + (3\Omega^2 + 4\Omega + 1)\Omega_1, \delta \right] e^{\Omega_1, t}. \] (65)

By the definition of \( ||u|| \),
\[ ||u|| \leq ||\phi|| + (\Omega^3 + 2\Omega^2 + \Omega + 1) \left[ 1 + \delta \right] (\Omega + 1) \\
\cdot \left[ ||\phi|| + (3\Omega^2 + 4\Omega + 1)\Omega_1, \delta \right] e^{\Omega_1, t}. \] (66)

Without loss generality, we assume \( \delta_0 \leq \left( \Omega_1/t \right) \), and we restrict to \( \delta \in (0, \delta_0) \); then,
\[ e^{\Omega_1, \delta} < e^1 < 3. \] (67)

By Taylor’s expansion,
\[ e^{\Omega_1, \delta} < 1 + \Omega_1, \delta e^{\Omega_1, \delta} < 1 + 3\Omega_1, \delta. \] (68)

Plugging (68) into (66) and simplifying the second order term of \( \delta \),
\[ ||u|| \leq ||\phi|| + (\Omega + 1)||\phi|| + \left[ \Omega^3 + 2\Omega^2 + \Omega + 1 \right] (1 + \delta) \\
+ (\Omega + 1)(12\Omega^2 + 16\Omega + 4 + 3||\phi||) \Omega_1, \delta. \] (69)

This is second estimate for the first derivatives of the solution of the Cauchy problem.

Next, we are going to estimate the modulus of continuity of the first-order derivatives of the solution. Let \( \psi(t, x) \) be a function on the bounded domain \( \mathbb{R} \). The module of continuity of \( \psi(t, x) \) is defined by the following nonnegative function:
\[ \rho(\eta) = \rho(\eta | \psi) = \sup_{t', t'' \in [\xi, \xi']} \left| \psi(t', x') - \psi(t'', x'') \right|, \]
\[ 0 \leq \eta < \infty. \] (70)

Similarly, we can define the module of continuity of a function with any number of independent variables. Also, the modulus of continuity of a vector function \( \psi = (\psi_i(t, x)) (i = 1, \ldots, n) \) or a set of functions \( \Gamma = \{ \psi_i \} \) is defined as follows:
\[ \rho(\eta | \psi) = \max_{i=1, \ldots, n} \rho(\eta | \psi_i), \]
\[ \rho(\eta | \Gamma) = \max_{\psi \in \Gamma} \rho(\eta | \psi). \] (71)

It is easy to see that the modulus of continuity possesses the properties stated in Lemmas 1 and 2.

**Lemma 1** (see [15]). It holds that
\[ \rho(\eta_2) \geq \rho(\eta_1) \geq 0, \quad \text{if} \quad \eta_2 \geq \eta_1 \geq 0, \] (72)
\[ \rho(\eta_1 + \eta_2) \leq \rho(\eta_1) + \rho(\eta_2). \]

\( f \) is a continuous function on \([t_1, t_2] \subset \mathbb{R} \) iff
\[ \rho(\eta) \longrightarrow 0 \quad \text{as} \quad \eta \longrightarrow 0. \] (73)

\( f \) is Hölder continuous on \([t_1, t_2] \subset \mathbb{R} \) with exponent \( \alpha(0 < \alpha \leq 1) \) iff there exists a constant \( L \) such that
\[ \rho(\eta) \leq L\eta^\alpha, \] (74)
for any natural number \( N \) and any positive constant \( C \), we have
\[ \rho(N\eta) \leq N\rho(\eta), \]
\[ \rho(C\eta) \leq (\lfloor C \rfloor + 1)\rho(\eta), \] (75)
where \( \lfloor C \rfloor \) denotes the integer part of \( C \).

**Lemma 2** (see [15]). Assuming that the right hand side in each of the following formulas makes sense, then
\[ \rho(\eta | \varphi \pm \psi) \leq \rho(\eta | \varphi) + \rho(\eta | \psi); \]
\[ \rho(\eta | \varphi \psi) \leq ||\varphi|| \rho(\eta | \psi) + ||\psi|| \rho(\eta | \varphi); \]
\[ \text{if} \quad |\psi| \geq a = \text{constant} > 0, \]
then
\[ \rho(\eta | \varphi \psi) \leq \frac{||\varphi||}{a^2} \rho(\eta | \psi) + \frac{1}{a} \rho(\eta | \varphi); \] (78)

if \( \varphi = \varphi(t) \), then
\[ \rho(\eta | \varphi(t)) \leq ||\varphi||; \quad \varphi = \frac{d\varphi}{dt}, \]
\[ \rho(\eta | \varphi(\psi)) \leq \rho\left( \rho(\eta | \psi) \right); \]

letting
\[ \psi(t) = \int_{\alpha(t)}^{b(t)} \varphi(t, t) dt, \] (80)

we have
\[ \rho(\eta | \psi(t)) \leq ||\varphi|| [a + \rho(\eta | b)] + \sup_t \int_{\alpha(t)}^{b(t)} \rho(\eta | \varphi(t)) dt, \] (81)
where the integrand \( \rho(\eta | \varphi(t)) \) is the modulus of continuity of \( \varphi \) with respect to \( t \) (\( t \) is regarded as a parameter).

By the ordinary equation which \( \xi = \omega(\tau, t, x) \) satisfies
\[ \frac{\partial \xi}{\partial \tau} = \lambda_i(\tau, \xi(\tau, t, x)), \] (82)
and the initial conditions
\[ \tau = t: \omega_i(t, t, x) = x. \] (85)

It is easy to see that
\[
\begin{align*}
\frac{\partial \omega_i}{\partial x} (\tau; t, x) &= \exp \left\{ \int_t^\tau \frac{\partial \lambda_i}{\partial x} (\tau_1, \omega_i (\tau_1; t, x)) \, d\tau_1 \right\}, \\
\frac{\partial \omega_i}{\partial t} (\tau; t, x) &= -\lambda_i \exp \left\{ \int_t^\tau \frac{\partial \lambda_i}{\partial x} (\tau_1, \omega_i (\tau_1; t, x)) \, d\tau_1 \right\}.
\end{align*}
\]

Hence,
\[
\begin{align*}
\frac{\partial \omega_i}{\partial x} &\leq \exp \left( \left| \frac{\partial \lambda_i}{\partial x} \right| | t \right) \leq 1 + \frac{1}{\delta} \left| \frac{\partial \lambda_i}{\partial x} \right| \delta, & l = 1, \ldots, 5, \\
\frac{\partial \omega_i}{\partial t} &\leq \frac{1}{\delta} \left| \frac{\partial \lambda_i}{\partial x} \right| \delta, & l = 1, \ldots, 5.
\end{align*}
\]

Since \( \delta < \delta_0 \leq 1/\Omega_1 \),
\[
\begin{align*}
\left| \frac{\partial \omega}{\partial x} \right| &\leq 1 + 3\Omega_1 \delta, \\
\left| \frac{\partial \omega}{\partial t} \right| &\leq (1 + 3\Omega_1 \delta) \Omega.
\end{align*}
\]

By (55) and (57) and Lemma 1, on \( R(\delta)(0 \leq \delta \leq \delta_0) \), we have
\[
\begin{align*}
\rho(a | u_k (t, x)) &= \rho(\eta | \phi(\omega_k (0; t, x))) \leq (1 + \Omega_1 (1 + 3\Omega_1 \delta) \rho(\eta | \phi), \\
\rho(\eta | p_k (t, x)) &= \rho(\eta | \mu_k - \lambda_k \phi(\omega_k (0; t, x))) \\
&\leq \rho(\eta | \mu_k) + \| \lambda_k \rho(\eta | \phi(\omega_k (0; t, x))) \| + \| \rho(\eta | \lambda_k). 
\end{align*}
\]

Then,
\[
\begin{align*}
\rho(\eta | p_k (t, x)) &\leq \Omega (1 + \Omega_1 (1 + 3\Omega_1 \delta) \rho(\eta | \phi_k) + (1 + \| \phi \|) \chi(\eta), \\
\rho(\eta | p_k (t, x)) &\leq \Omega (1 + \Omega_1 (1 + 3\Omega_1 \delta) \rho(\eta | \phi) + (1 + \| \phi \|) \chi(\eta).
\end{align*}
\]

where
\[
\chi(\eta) = \max\{ \rho(\eta | \mu_k), \rho(\eta | \lambda_k) \}.
\]

Hence, we have
\[
\rho(\eta | w^0) \leq (1 + \Omega_1 (1 + 3\Omega_1 \delta) \rho(\eta | \phi),
\]
\[
\rho(\eta | p^0) \leq \Omega (1 + \Omega_1 (1 + 3\Omega_1 \delta) \rho(\eta | \phi) + (1 + \| \phi \|) \chi(\eta).
\]

Furthermore,
\[
\rho(\eta | w^0) + \rho(\eta | p^0) \leq (1 + \Omega_1 (1 + 3\Omega_1 \delta) \rho(\eta | \phi) + (1 + \| \phi \|) \chi(\eta).
\]

Define
\[
\rho(t, \eta | w) = \sup_{k=1, \ldots, 5} \left| w_k (t', x') - w_k (t'', x'') \right|.
\]

Then, it is obvious that, on \( R(\delta)(0 \leq \delta \leq \delta_0) \),
\[
\rho(\eta | w) = \sup_{0 \leq \delta \leq \delta} \rho(t, \eta | w) = \rho(\delta, \eta | w),
\]
\[
\rho(\eta | p) = \sup_{0 \leq \delta \leq \delta} \rho(t, \eta | p) = \rho(\delta, \eta | p).
\]

By the second integral relations (54) and (56), we will estimate \( \rho(t, \eta | w) \) and \( \rho(t, \eta | p) \) for \( 0 \leq t \leq \delta \). By Lemmas 1 and 2 (and (94), for \( 0 \leq t \leq \delta \) (note that here and in the following, \( \tau (\tau \leq t) \) is regarded as a parameter), we get
\[
\rho(\lambda, \eta | \tilde{\omega}_k (\tau, \omega_k (\tau; t, x))) \leq 4 (1 + \Omega) \rho(\tau, \eta | \tilde{\omega}_k (\tau, \xi)),
\]
and as \( \tau \leq t \leq \delta \), we have
\[
\begin{align*}
\rho(\tau, \eta | \tilde{\omega}_k (\tau, \xi)) &= \rho\left( \tau, \eta | \frac{\partial \mu_k (\tau, \xi)}{\partial x} \rho\left( \tau, \eta | \frac{\partial \lambda_k (\tau, \xi)}{\partial x} w_k (\tau, \xi) \right) \right) \\
&\leq \rho\left( \tau, \eta | \frac{\partial \mu_k (\tau, \xi)}{\partial x} + \frac{\partial \lambda_k (\tau, \xi)}{\partial x} \rho(\tau, \eta | w_k (\tau, \xi)) \right) \\
&+ \| \tilde{w}_k \| \rho(\tau, \eta | \frac{\partial \lambda_k (\tau, \xi)}{\partial x}) \\
&\leq \Omega_1 \rho(\tau, \eta | w) + (1 + \| \omega \|) \chi(\eta) \\
&\leq \Omega_1 \rho(\tau, \eta | w) + (4 | \phi | + 12\Omega^2 + 16\Omega + 5) \chi(\eta),
\end{align*}
\]
where
\[
\chi(\eta) = \max_{k=1, \ldots, 5} \rho\left( \tau, \eta | \frac{\partial \mu_k (\tau, \xi)}{\partial x} \rho\left( \tau, \eta | \frac{\partial \lambda_k (\tau, \xi)}{\partial x} \right) \right).
\]

Hence, we have
\[
\rho(\lambda, \eta | \tilde{\omega}_k (\tau, \omega_k (\tau; t, x))) \leq (1 + \Omega_1 | \Omega_1 \rho(\tau, \eta | w) \\
+ (4 | \phi | + 12\Omega^2 + 16\Omega + 5) \chi(\eta).
\]

By
\[
\tilde{\omega}_k = \frac{\partial \mu_k}{\partial x} - \frac{\partial \lambda_k}{\partial x} w_k, \quad k = 1, \ldots, 5,
\]
we obtain
\[
\| \tilde{\omega}_k \| \leq \frac{\partial \mu_k}{\partial x} + \frac{\partial \lambda_k}{\partial x} \| w_k \| \leq (15\Omega^2 + 20\Omega + 5 + 4 | \phi |) \Omega_1.
\]
\[ \| \psi \| \leq \left( 15 \Omega^2 + 20 \Omega + 5 + 4 \| \phi \| \right) \Omega_1. \]

By the properties of the continuity modulus,
\[ \rho \left( \eta | \int_a^{b(t)} \psi(r, t) \, dr \right) \leq \| \psi \| [\rho(\eta | a) + \rho(\eta | b)] + \sup_{t} \int_a^{b(t)} \rho(\eta | \psi(r, t)) \, dr, \]
we have
\[ \rho \left( \int_0^T \| \bar{\eta}_k (\tau, \omega_k (\tau; x)) \| \, d\tau \right) \leq \| \bar{\eta}_k \| \eta \]
\[ + \int_0^T \rho(\tau, \eta | \bar{\eta}_k (\tau, \omega_k (\tau; x))) \, d\tau \]
\[ \leq \left( 15 \Omega^2 + 20 \Omega + 5 + 4 \| \phi \| \right) \Omega_1 \eta + 4(1 + \Omega) \Omega_1 \]
\[ \cdot \int_0^T \rho(\tau, \eta | \omega) \, d\tau + 4(1 + \Omega)(4 \| \phi \| + 12 \Omega^2 + 16 \Omega + 5) \delta \chi_1(\eta). \]
\[ \tag{105} \]
Hence, putting together (104) and (93),
\[ \rho(t, \eta | p_k) + \rho(t, \eta | w_k) \leq \left( 1 + \| \lambda \| \right) \rho \left( t, \eta | p_k \right) + \rho \left( t, \eta | w_k \right) \]
\[ + (1 + \| \lambda \|) \rho \left( t, \eta | \int_0^T \bar{\eta}_k \, d\tau \right) + \int_0^T \rho(\tau, \eta | \omega) \, d\tau \]
\[ \leq \left( 1 + \| \phi \| + (1 + \Omega + \Omega_1) \right) \left( 15 \Omega^2 + 20 \Omega + 5 + 4 \| \phi \| \right) \chi(\eta) + 4(1 + \Omega)^2 (4 \| \phi \| + 12 \Omega^2 + 16 \Omega + 5) \delta \chi_1(\eta) \]
\[ + 4(1 + \Omega)^2 \Omega_1 \int_0^T \rho(\tau, \eta | \omega) \, d\tau + \int_0^T \rho(\tau, \eta | \rho) \, d\tau. \]
\[ \tag{106} \]
Denote
\[ \rho(t, \eta) = \rho(t, \eta | p) + \rho(t, \eta | w), \]
and by Gronwall inequality and \( t \leq \delta < \delta_0 < (1/\Omega) \), we have
\[ \rho(t, \eta) \leq e^{4(1+\Omega)^2} \left( \left( 1 + \| \phi \| + (1 + \Omega + \Omega_1) \right) \left( 15 \Omega^2 + 20 \Omega + 5 + 4 \| \phi \| \right) \chi(\eta) + 4(1 + \Omega)^2 \left( \rho(\eta | \phi) + (4 \| \phi \| + 12 \Omega^2 + 16 \Omega + 5) \delta \chi_1(\eta) \right) \right). \]
\[ \tag{108} \]
By Cauchy problems of the linear hyperbolic system, there exists a unique continuous differential solution \( u = (u(t, x)) \) of (39) on \( R(\delta) \). Denote
\[ u = T v. \]
\[ \tag{109} \]
Obviously, the fixed point of the operator \( T \) is the solution of the initial value problem. Therefore, it is sufficient to prove that the operator \( T \) has a unique fixed point, provided that \( \delta > 0 \) is suitably small. We first prove that there exists a positive number \( \delta_1 \leq \delta_0 \) depending only on the norm \( \| T^* \| \) and \( \| \phi \| _1 \):
\[ \delta_1 = \delta_1(\| T^* \|, \| \phi \| _1), \]
\[ \tag{110} \]
such that, for any \( \delta \leq \delta_1 \), the operator \( T \) maps \( \sum (\delta | \Omega, O_1) \) to itself. To do so, we use first estimate (52) and second estimate (69) to bound \( \| u \| \) and \( \| u \|_1 \). It follows that, for \( u = T v \), we have on \( R(\delta) \)
\[ \| u \| \leq \| \phi \| + \left( \Omega^2 + \Omega^2 + \Omega + 1 \right) \delta. \]
\[ \tag{111} \]
Take \( \Omega = 2 \| \phi \| \),
\[ \delta = \frac{\| \phi \|}{8 \| \phi \|^3 + 8 \| \phi \| ^2 + 2 \| \phi \| + 1}. \]
\[ \tag{112} \]
we have
\[ \| u \| \leq \Omega. \]
\[ \tag{113} \]
By (69),
\[ \text{and therefore, the operator } T \text{ maps } \sum (\delta | \Omega, O_1) \text{ to itself.} \]

Now, we will use third estimate (108) to prove the following Lemma.
Lemma 3. There exists a positive number \( \delta_2 = \delta_2(\|\phi\|, \|\psi\|) \leq \delta_1 \) and a nonnegative function \( \Omega_2(\eta) \) (\( 0 < \eta < \infty \)) such that

\[
\lim_{\eta \to 0} \Omega_2(\eta) = 0,
\]

(119) for any \( \delta \leq \delta_2 \), if \( \nu \in \sum(\delta \mid \Omega, \Omega_1) \) and if

\[
\rho(\eta \mid \frac{\partial \nu}{\partial x}) + \rho(\eta \mid \frac{\partial \nu}{\partial x}) \leq \Omega_2(\eta),
\]

(120) then for \( u = \Omega_1 \), we have \( \rho(\eta \mid p) + \rho(\eta \mid w) \leq \Omega_2(\eta) \).

Proof. Let

\[
\overline{\chi}(\eta) = \max_{i=1, \ldots, 5} \{ \rho(w_i \mid \lambda_i), \rho(\eta \mid \mu_i), \eta \}.
\]

(121) By the definition of the modulus of continuity and the construction of \( \Omega_1 \), we have

\[
\overline{\chi}(\eta) \leq \max \{ \Omega_1, \eta \} = \left( 3 \Omega^2 + 4 \Omega + 1 \right) \Omega_1 \eta, \eta = \left( 3 \Omega^2 + 4 \Omega + 1 \right) \Omega_1 \eta.
\]

(122) where

\[
\chi_1(\eta) = \max_{k=1, \ldots, 5} \left\{ \rho(\tau, \eta \mid \frac{\partial \nu_k}{\partial x}), \rho(\tau, \eta \mid \frac{\partial \nu_k}{\partial x}) \right\}.
\]

(123) By

\[
\rho(\tau, \eta \mid \frac{\partial \nu_j}{\partial x}) \leq \max \left\{ \rho(\tau, \eta \mid \frac{\partial \nu_1}{\partial x}), \rho(\tau, \eta \mid \frac{\partial \nu_2}{\partial x}) \right\} \leq \rho(\tau, \eta \mid \frac{\partial \nu}{\partial x}),
\]

(124) we have

\[
\chi_1(\eta) \leq \left( 3 \Omega^2 + 4 \Omega + 1 \right) \rho(\eta \mid \frac{\partial \nu}{\partial x}).
\]

(125) Denote

\[
\Omega_2(\eta) = 2e^{4(1+\Omega)^2} \left\{ 4(1 + \Omega)^2 \rho(\eta \mid \phi) + \left( 3 \Omega^2 + 4 \Omega + 1 \right) \times \left[ 1 + \|\phi\| + (1 + \Omega_1 + \Omega_1)(15 \Omega^2 + 20 \Omega + 5 + 4\|\phi\|) \right] \right\} \Omega_1 \eta,
\]

(126) and then we take

\[
\delta_2 = \frac{1}{4(1 + \Omega)^2 \left( 4\|\phi\| + 12 \Omega^2 + 16 \Omega + 5 \right) \left( 3 \Omega^2 + 4 \Omega + 1 \right)}
\]

(127) then

\[
\delta_2 = \frac{1}{C_0 + C_1 \|\phi\|}
\]

(129) Then, if \( 0 < \delta_2 \), Lemma 3, (1) and (2), holds. The proof is completed.
Take $\delta_3 = \min\{\delta_1, \delta_2\}$, and we introduce the function set
\[
\sum (\delta_3) = \sum \left| \Omega, \Omega_1, \Omega_2 (\eta) \right|
\]
\[
= \left\{ v(t, x) | v \in \sum (\delta_3) | \Omega, \Omega_1, \rho \left( \eta | \frac{\partial v}{\partial x} \right) \right\} + \rho \left( \eta | \frac{\partial v}{\partial x} \right) \leq \Omega_2 (\eta), \right\}
\]
then the operator $T$ maps $\sum (\delta_3)$ to itself. It is easy to see that $T$ is compact on $C^1[R(\delta)]$ and closed with respect to the $C^0$ norm. We now prove that there exists a positive number $\delta_* = \delta_* \left( \| \phi \|, \| \phi \| \right) \leq \delta_3$ such that, on $R(\delta_*)$, the operator $T$ is a contraction operator with respect to the $C^0$ norm.

Firstly, under the $C^1$ norm, the set $\sum (\delta_3)$ is bounded and equicontinuous; hence, it is precompact. Assume
\[
\lim_{n \to \infty} v_n = v,
\]
then
\[
\rho (\eta | v) = \lim_{n \to \infty} \rho (\eta | v_n) \leq \Omega_2 (\eta), \| v \| \leq \Omega, \| v \| \leq \Omega_1.
\]

Hence, the set is compact in $C^1[R(\delta)]$. Moreover, $\sum (\delta_3)$ is closed with respect to the $C^1$ norm (because $\| v \| \leq \| v \|$), the convergence under the $C^1$ norm provides the convergence under $C^0$ norm; hence, it is complete with respect to $C^0$ norm). Hence, we only need to prove that there exists $\delta_* \leq \delta_3$ such that the operator $T$ is a contraction operator in $R(\delta_*)$ with respect to the $C^0$ norm. If we can show this, the existence of a unique fixed point for $T$ will be proved.

Let $v^{(1)}, v^{(2)} \in \sum (\delta_3)$; then, $u^{(1)} = Tv^{(1)}, u^{(2)} = Tv^{(2)} \in \sum (\delta_3)$. Assume $v_* = v^{(1)} - v^{(2)}$, $u_* = u^{(1)} - u^{(2)}$, and
\[
\begin{aligned}
\tilde{\lambda}^{(a)}_1 (t, x) &= \tilde{\lambda}^{(a)}_1 (t, x, \psi^{(a)} (t, x), \\
\tilde{\lambda}^{(a)}_2 (t, x) &= \tilde{\lambda}^{(a)}_2 (t, x, \psi^{(a)} (t, x), \\
&= \alpha, 1, 2.
\end{aligned}
\]

Then, we have
\[
\frac{\partial u_*}{\partial t} + \tilde{\lambda}^{(1)}_k (t, x) \frac{\partial u_*}{\partial x} = \tilde{h}_k (t, x),
\]
\[
t = 0: u_* = 0,
\]
\[
k = 1, \ldots, 5,
\]
where $\tilde{h}_k = - \tilde{\lambda}^{(1)}_k - \tilde{\lambda}^{(2)}_k (\partial u^{(2)}_k / \partial x) + (\mu^{(1)} - \mu^{(2)}_k), k = 1, \ldots, 5$.

Now, notice that, on $R(\delta)$, we have
\[
\| h \| \leq \Omega_1 + 3\Omega_2 + 4\Omega + 1 \| v_* \|.
\]

By the estimate in (52), it follows from (134) that, on $R(\delta)$,
\[
\| u_* \| \leq \left( \Omega_1 + 3\Omega_2 + 4\Omega + 1 \right) \| \psi_* \|.
\]

Take
\[
\delta_* = \frac{1}{3(\Omega_1 + 3\Omega_2^2 + 4\Omega + 1)}
\]
\[
\Omega_1 = 2\| \phi \| + 2\Omega^2 + 4\Omega + 2 \| (\psi \| + 1) (\Omega + 1)
\]
\[
= (4\| \phi \| + 2)\| \psi \| + (16\| \phi \|^3 + 16\| \phi \|^2 + 6\| \phi \| + 2).
\]

Let $D_0 (\| \phi \|) = 48\| \phi \|^3 + 84\| \phi \|^2 + 42\| \phi \| + 9$ and $D_1 (\| \phi \|) = 6(2\| \phi \| + 1)$; then, we have
\[
\delta_* = \frac{1}{D_0 + D_1 \| \phi \|}
\]
if we choose $\delta_* = \min \{\delta_3, \delta_4\}$, and the operator $T$ is a contraction operator on $R(\delta_*)$.

From the above, we have $\delta_* = \min \{\delta_3, \delta_4\}$, and by
\[
D_0 (\| \phi \|) < C_0 (\| \phi \|), D_1 (\| \phi \|) < C_1 (\| \phi \|),
\]
we get $\delta_* < \delta_4$. Hence, the lower bound of the local solution of the Cauchy problem $\delta_*$ is
\[
\delta_* = \min \left\{ \frac{\| \phi \| + A_0 + A_1 \| \phi \|}{A_0 - \| \phi \| B_0 \| \phi \| + B_1 \| \phi \|^2 + C_0 + C_1 \| \phi \|} \right\}
\]
\[
\frac{1}{D_0 + D_1 \| \phi \|}.
\]

The proof of Theorem 2 is completed. □

Data Availability

There are no data in this paper.

Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this paper.

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