MONOTONICITY OF PRINCIPAL EIGENVALUE FOR ELLIPTIC OPERATORS WITH INCOMPRESSIBLE FLOW: A FUNCTIONAL APPROACH

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Abstract. We establish the monotonicity of the principal eigenvalue $\lambda_1(A)$, as a function of the advection amplitude $A$, for the elliptic operator $L_A = -\text{div}(a(x)\nabla) + AV \cdot \nabla + c(x)$ with incompressible flow $V$, subject to Dirichlet, Robin and Neumann boundary conditions. As a consequence, the limit of $\lambda_1(A)$ as $A \to \infty$ always exists and is finite for Robin boundary conditions. These results answer some open questions raised by Berestycki, Hamel and Nadirashvili [4]. Our method relies upon some functional which is associated with principal eigenfunctions for operator $L_A$ and its adjoint operator. As a byproduct of the approach, a new min-max characterization of $\lambda_1(A)$ is given.

1. Introduction

There have been extensive studies on the reaction-diffusion equations of the form

\begin{equation}
    w_t = \text{div}(a(x)\nabla w) - AV \cdot \nabla w + wf(x, w),
\end{equation}

which model various physical, chemical, and biological processes: On unbounded domains [16, 37], compact manifolds [10], and bounded domains with appropriate boundary conditions [1, 4, 7, 24]. Let $\Omega$ be a bounded region of $\mathbb{R}^N$ with smooth boundary $\partial \Omega$, and $n(x)$ be the outward unit normal vector at $x \in \partial \Omega$. Consider equation (1) defined on $\Omega$ and suppose that $w$ satisfies $bw + (1 - b)[a(x)\nabla w] \cdot n = 0$ on $\partial \Omega$. The stability of steady state $w \equiv 0$ is determined by the sign of the principal eigenvalue, denoted as $\lambda_1(A)$, for the linear eigenvalue problem

\begin{equation}
    L_Au := -\text{div}(a(x)\nabla u) + AV \cdot \nabla u + c(x)u = \lambda_1(A)u,
\end{equation}

subject to boundary conditions $bu + (1 - b)[a(x)\nabla u] \cdot n = 0$ on $\partial \Omega$, where $c(x) = -f(x, 0)$, and parameter $b \in [0, 1]$.
Of particular interest is the dependence of the principal eigenvalue \( \lambda_1(A) \) on the advection amplitude \( A \). If vector field \( \mathbf{V} \) is incompressible, i.e., \( \text{div}\mathbf{V} = 0 \) in \( \Omega \), Berestycki et al. investigated in [3] the asymptotic behavior of \( \lambda_1(A) \) as \( A \) approaches infinity, and they identified a direct link between the limit of \( \lambda_1(A) \) and the first integral set of \( \mathbf{V} \), defined as

\[
\mathcal{I}_b = \begin{cases} 
\{ \varphi \in H^1(\Omega) : \varphi \neq 0, \mathbf{V} \cdot \nabla \varphi = 0 \text{ a.e. in } \Omega \}, & 0 \leq b < 1 \\
\{ \varphi \in H^1_0(\Omega) : \varphi \neq 0, \mathbf{V} \cdot \nabla \varphi = 0 \text{ a.e. in } \Omega \}, & b = 1.
\end{cases}
\]

More precisely, Berestycki et al. showed in [4] that for the operator \( L_A \) defined on \( \Omega \) with Dirichlet \( (b = 1) \) or Neumann \( (b = 0) \) boundary conditions, \( \lambda_1(A) \) stays bounded as \( A \to +\infty \) if and only if \( \mathcal{I}_1 \neq \emptyset \) or \( \mathcal{I}_0 \neq \emptyset \), respectively. Furthermore, they proved that for any \( A \geq 0 \),

\[
\lambda_1(0) \leq \lambda_1(A) \leq \lim_{A \to +\infty} \lambda_1(A) = \inf_{\omega \in \mathcal{I}_0 \text{ or } \mathcal{I}_1} \frac{\int_{\Omega} \nabla \omega \cdot [a(x) \nabla \omega] \, dx + \int_{\Omega} c(x) \omega^2 \, dx}{\int_{\Omega} \omega^2 \, dx}.
\]

That is, \( \lambda_1(A) \) attains its minimum at \( A = 0 \) and its maximum at \( A = \infty \). As mentioned in [3], \( \lambda_1(A) \) is a nondecreasing function of \( |A| \) if \( \mathbf{V} \) is an incompressible gradient flow. Nevertheless, this monotonicity property has remained open for a general incompressible flow \( \mathbf{V} \).

The primary goal of this paper is to answer the above open question affirmatively. To this end, we shall focus on the following eigenvalue problem with a general incompressible flow \( \mathbf{V} \), subject to general boundary conditions:

\[
L_A u_A = -\text{div}(a(x) \nabla u_A) + A\mathbf{V} \cdot \nabla u_A + c(x) u_A = \lambda_1(A) u_A \quad \text{in } \Omega,
\]

\[
u_A > 0 \quad \text{in } \Omega,
\]

\[
b u_A + (1 - b) [a(x) \nabla u_A] \cdot \mathbf{n} = 0 \quad \text{on } \partial \Omega.
\]

Throughout this paper we always assume that \( c \in C^\alpha(\Omega) \) and the diffusion matrix \( a(x) \) is symmetric and uniformly elliptic \( C^{1,\alpha}(\Omega) \) matrix field satisfying

\[
\exists 0 < \gamma_1 < \gamma_2, \text{ such that } \gamma_1 |\xi|^2 \leq \xi^T a(x) \xi \leq \gamma_2 |\xi|^2, \forall x \in \Omega, \forall \xi \in \mathbb{R}^N,
\]

for some constant \( \alpha \in (0, 1) \). Furthermore, we always assume that the vector field \( \mathbf{V} \in C^1(\Omega) \) satisfying \( \text{div}\mathbf{V} = 0 \) in \( \Omega \), whereas an additional assumption stating that \( \mathbf{V} \cdot \mathbf{n} = 0 \) on \( \partial \Omega \) is always assumed for the case of \( 0 \leq b < 1 \). Under these assumptions the Krein-Rutman Theorem guarantees the existence of the principle eigenvalue \( \lambda_1(A) \) and it can be easily shown that \( \lambda_1(A) \) is symmetric in \( A \). Therefore, throughout this paper we shall assume \( A \geq 0 \).

Our first result can be stated as follows.

**Theorem 1.1.** Let \( L_A \) be the elliptic operator defined by (3) and \( \lambda_1(A) \) be its principle eigenvalue. Then the following statements hold:

(i) If \( u_0 \notin \mathcal{I}_0 \), then \( \frac{\partial}{\partial A} \lambda_1(A) > 0 \) for every \( A > 0 \);

(ii) If \( u_0 \in \mathcal{I}_0 \), then \( \lambda_1(A) \equiv \lambda_1(0) \) for every \( A > 0 \).

Here \( u_0 \) is the principal eigenfunction of \( L_0 \) satisfying

\[
\begin{cases}
- \text{div}(a(x) \nabla u_0) + c(x) u_0 = \lambda_1(0) u_0 \quad \text{in } \Omega, \\
u_0 > 0 \quad \text{in } \Omega, \\
b u_0 + (1 - b) [a(x) \nabla u_0] \cdot \mathbf{n} = 0 \quad \text{on } \partial \Omega.
\end{cases}
\]

Theorem [1] implies that the strict monotonocity of \( \lambda_1(A) \) with respect to the advection amplitude \( A \) relies on \( u_0 \), the principal eigenfunction of operator \( L_0 \). Interpreting this in the context of convection-enhanced diffusion, Theorem [3] suggests that larger advection amplitude generally produces faster mixing for reaction-diffusion-advection equation (1) as
long as \( u_0 \notin I_b \). In this sense, Theorem 1.1 seems to refine the well-known statement that mixing by an incompressible flow enhances diffusion in various contexts \( \text{[10, 16, 18, 19, 21, 22, 31, 37, 38]} \).

Our next result, as a corollary of Theorem 1.1 provides the boundedness and asymptotic behavior of \( \lambda_1(A) \) for Robin boundary conditions, consistent with the main result in \( \text{[4]} \) for Neumann boundary conditions.

**Theorem 1.2.** If \( 0 \leq b < 1 \), the limit \( \lim_{A \to +\infty} \lambda_1(A) \) always exists, is finite and satisfies

\[
\lambda_1(A) \leq \inf_{\omega \in I_b} \frac{b}{1-b} \int_{\Omega} \omega^2 \, dS_x + \int_{\Omega} \nabla \omega \cdot [a(x) \nabla \omega] \, dx + \int_{\Omega} c(x) \omega^2 \, dx
\]

In particular, the principal eigenvalues \( \lambda_1(A) \) of \( \text{[3]} \) are uniformly bounded.

The proof of the boundedness for \( \lambda_1(A) \) in Theorem 1.2 is essentially due to Berestycki et al. \( \text{[4]} \). Nevertheless, the existence of the limit \( \lim_{A \to +\infty} \lambda_1(A) \) for Robin boundary conditions appears to be new.

The proof of Theorem 1.1 relies heavily on properties of certain functional. Set \( L := -\text{div}(a(x) \nabla) + V \cdot \nabla + c(x) \), with adjoint operator \( L^* := -\text{div}(a(x) \nabla) - V \cdot \nabla + c(x) \), in view of \( \text{div} V = 0 \) in \( \Omega \) and particularly \( V \cdot n = 0 \) on \( \partial \Omega \) for case \( 0 \leq b < 1 \). By \( u, v \) we further denote the normalized principal eigenfunctions corresponding to \( L \) and \( L^* \), respectively. In terms of operator \( L \) and \( u, v \), we now introduce functional \( J \),

\[
J(\omega) = \int_{\Omega} uv \left( \frac{L \omega}{\omega} \right) \, dx,
\]

which is well defined on the cone

\[
\mathbb{S}_b = \left\{ \varphi \in C^2(\Omega) \cap C^1(\overline{\Omega}) : \varphi > 0 \text{ in } \Omega, \, b \varphi + (1-b)[a(x)\nabla \varphi] \cdot n = 0 \text{ on } \partial \Omega \}, \text{ for } 0 \leq b < 1
\]

\[
\mathbb{S}_b = \left\{ \varphi \in C^2(\Omega) \cap C^1(\overline{\Omega}) : \varphi > 0 \text{ in } \Omega, \, \varphi = 0 \text{ on } \partial \Omega, \, \nabla \varphi \cdot n < 0 \text{ on } \partial \Omega \}, \text{ for } b = 1
\]

A direct observation from the definition of functional \( J \) leads to \( J(u) = \lambda_1 \) and a far less obvious result (see Lemma \( \text{2.1} \)) says that functional \( J \) attains its maximum at the principal eigenfunction \( u \) and its scalar multiples. This is crucial to the proof of Theorem 1.1 and it also allows us to explore a new min-max characterization of the principal eigenvalue.

The characterization of the principal eigenvalue has always been an interesting and active topic, and we refer to Donsker and Varadhan, Nussbaum and Pinchover for some earlier works \( \text{[13, 15, 29]} \). Employing the maximum principle, Protter and Weinberger \( \text{[30]} \) established a classical characterization of the principal eigenvalue for general second order elliptic operators \( P \), given by the min-max formula

\[
\lambda_1 = \sup_{\omega > 0} \inf_{x \in \Omega} \left[ \frac{P \omega(x)}{\omega(x)} \right].
\]

This characterization is valid for general elliptic operators in both bounded and unbounded domains \( \text{[29, 30]} \). As a byproduct of properties of functional \( J \), we have the following characterization for \( \lambda_1 \):

**Theorem 1.3.** For elliptic operator \( L \) with an incompressible flow \( V \) subject to general boundary conditions with \( 0 \leq b \leq 1 \), the principal eigenvalue \( \lambda_1 \) can be characterized as

\[
\lambda_1 = \inf_{p \in \mathbb{S}_b, \int_{\Omega} p^2 = 1} \sup_{\omega \in \mathbb{S}_b} \int_{\Omega} p^2(x) \left( \frac{L \omega}{\omega} \right) \, dx.
\]
This min-max formula may not be valid for general second elliptic operators, and it reduces to the classical Rayleigh-Ritz formula when $V = 0$, by treating $p^2 dx$ as some probability measure; See Remark 2 for details. Different from the formula (5), the min-max characterization in Theorem 1.3 relies on the properties of functional $J$. They however may be connected via a min-max theorem in [32]. Via functional $J$ we observe that the min-max formula attains the extremum when $p^2 = uv$.

The rest of this paper is organized as follows: In Section 2, we shall give some properties of functional $J$. Section 3 is devoted to the proof of Theorems 1.1 and 1.2. In Section 4 we establish the new min-max characterization of the principal eigenvalue. Finally, the implications of our method/results and some open questions will be discussed in Section 5.

2. Properties of functional $J$

We shall present some properties of functional $J$ in this section, which are crucial to the proofs of main results in this paper. Before proceeding further, we point out again that throughout this paper, $u$ and $v$ are the principal eigenfunctions corresponding to $L$ and $L^*$, respectively, with general boundary conditions. Due to the slight difference between the definitions of functional $J$ in the cases of $0 \leq b < 1$ and $b = 1$, we divide this section into two subsections.

2.1. Neumann and Robin boundary conditions: $0 \leq b < 1$. Recalling the regularity requirements of coefficients $c$, $V$ and matrix field $a(x)$, Sobolev embedding theorem implies that $u, v \in C^{2,\alpha} (\Omega)$ and $u, v \in S_b$ for $0 \leq b < 1$. We emphasize here that the constant $b$ is confined to $0 \leq b < 1$ unless otherwise specified, and the incompressible flow $V$ satisfies $\text{div} V = 0$ in $\Omega$ with $V \cdot n = 0$ on $\partial \Omega$ in this subsection. Also, the eigenfunctions can be normalized as $\int_{\Omega} u^2 dx = 1$ and $\int_{\Omega} uv dx = 1$. We now recall the functional associated to operator $L$ with Neumann or Robin boundary conditions, defined on $S_b$ as in Section 1,

(7) \[ J(\omega) = \int_{\Omega} uv \left( \frac{L\omega}{\omega} \right) dx, \quad \omega \in S_b. \]

For any $\omega \in S_b$, a simple but useful observation from (7) leads to

\[
J(\omega) = -\int_{\Omega} uv \left[ \text{div} \left( \frac{a(x) \nabla \omega}{\omega} \right) \right] dx + \int_{\Omega} uv \left[ \nabla \cdot \frac{\nabla \omega}{\omega} \right] dx + \int_{\Omega} uv c dx
\]

\[
- \int_{\partial \Omega} uv \left[ a(x) \nabla \omega \right] \cdot n dS_x + \int_{\Omega} \nabla \left( \frac{uv}{\omega} \right) \cdot \left[ a(x) \nabla \omega \right] dx
\]

\[
+ \int_{\Omega} uv V \cdot \nabla \log \omega dx + \int_{\Omega} uv c dx,
\]

\[
(8) \quad = -\int_{\partial \Omega} uv \left[ a(x) \nabla \log \omega \right] \cdot n dS_x - \int_{\Omega} uv \left\{ \left( \nabla \log \omega \right) \cdot \left[ a(x) \nabla \log \omega \right] \right\} dx
\]

\[
+ \int_{\Omega} \left[ uv V + a(x) \nabla (uv) \right] \cdot \nabla \log \omega dx + \int_{\Omega} uv c dx.
\]

By equality (8), we show that the principal eigenfunction $u$ is a critical point of $J$.

**Proposition 1.** $J'(u) \varphi = 0$ for all $\varphi \in \tilde{S}_b \triangleq \left\{ \varphi \in C^2(\Omega) \cap C^1(\bar{\Omega}) : b\varphi + (1 - b) \left[ a(x) \nabla \varphi \right] \cdot n = 0 \text{ on } \partial \Omega \right\}$. 

Proof. Using equality (8), the Fréchet derivation \( J'(\omega) \) of \( \omega \in \mathbb{S}_b \) can be written as

\[
J'(\omega)\varphi = -\int_{\partial\Omega} uv \left[ a(x)\nabla \left( \frac{\varphi}{u} \right) \right] \cdot ndS_x - 2\int_{\Omega} uv \left\{ (\nabla \log u) \cdot \left[ a(x)\nabla \left( \frac{\varphi}{u} \right) \right] \right\} dx \\
+ \int_{\Omega} \left[ uvV + a(x)\nabla(auv) \right] \cdot \nabla \left( \frac{\varphi}{u} \right) dx,
\]

for all \( \varphi \in \mathbb{S}_b \). By the boundary conditions of \( u \) and \( v \), a direct calculation via integration by parts gives

\[
J'(u)\varphi = -\int_{\partial\Omega} uv \left[ a(x)\nabla \left( \frac{\varphi}{u} \right) \right] \cdot ndS_x - 2\int_{\Omega} uv \left\{ (\nabla \log u) \cdot \left[ a(x)\nabla \left( \frac{\varphi}{u} \right) \right] \right\} dx \\
+ \int_{\Omega} \left[ uvV + a(x)\nabla(auv) \right] \cdot \nabla \left( \frac{\varphi}{u} \right) dx
\]

Next we establish a crucial property of functional \( J \).}

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\textbf{Lemma 2.1.} For any \( \omega \in \mathbb{S}_b \), the following formula holds:

\[
J(u) = J(\omega) + \int_{\Omega} uv \left\{ \left[ \nabla \log \left( \frac{\omega}{u} \right) \right] \cdot \left[ a(x)\nabla \log \left( \frac{\omega}{u} \right) \right] \right\} dx.
\]
Lemma implies that are well defined. Moreover, the adjoint operator of $V$ shall be defined on $S_1$. Perhaps worth pointing out that in this case, the functional $J$ shall be defined on $\tilde{S}_b$ for any $\omega \in S_b$. Choosing $\varphi = u \log \left( \frac{\omega}{u} \right)$ in equality (9), by Proposition 1 we have

$$J(u) - J(\omega) = \int_{\partial \Omega} uv \left\{ \nabla \log \left( \frac{\omega}{u} \right) \cdot [a(x) \nabla \log \left( \frac{\omega}{u} \right)] \right\} \, dx - J' (u) \varphi$$

$$= \int_{\Omega} uv \left\{ \nabla \log \left( \frac{\omega}{u} \right) \cdot [a(x) \nabla \log \left( \frac{\omega}{u} \right)] \right\} \, dx.$$

The assertion of Lemma 2.1 thus follows.

The following result is an immediate consequence of Lemma 2.1.

**Corollary 1.**

$$\int_{\Omega} vLudx - \int_{\Omega} uLvdx = \int_{\Omega} uv \left\{ \nabla \log \left( \frac{v}{u} \right) \cdot [a(x) \nabla \log \left( \frac{v}{u} \right)] \right\} \, dx.$$

**Proof.** A simple observation leads to

$$\int_{\Omega} uLvdx = \int_{\Omega} uv \left( \frac{Lv}{v} \right) \, dx = J(v),$$

and analogously $\int_{\Omega} vLudx = J(u)$. Hence Corollary 1 follows from Lemma 2.1.

**2.2. Dirichlet boundary conditions:** $b = 1$. The case of Dirichlet boundary conditions is slightly different from the Neumann or Robin boundary conditions, as noted in [4]. It is perhaps worth pointing out that in this case, the functional $J$ shall be defined on $S_1$ and the extra assumption $V \cdot n = 0$ on $\partial \Omega$ is not needed for further discussions. Hopf Boundary Lemma implies that $\nabla u \cdot n < 0$ and $\nabla v \cdot n < 0$ on $\partial \Omega$, and thus $u, v \in S_1$ so that $J(u), J(v)$ are well defined. Moreover, the adjoint operator of $L$ subject to Dirichlet boundary conditions
can be written as \( L^* = -\text{div}(A(x)\nabla) - V \cdot \nabla + c(x) \) without \( V \cdot n = 0 \) on \( \partial \Omega \), due to \( u = 0 \) on \( \partial \Omega \). Thanks to \( \nabla \omega \cdot n < 0 \) on \( \partial \Omega \), we have \( \frac{\partial \omega}{\partial n} = 0 \) on \( \partial \Omega \) to get \( \int_{\partial \Omega} \omega v [a(x)\nabla \log \omega] \cdot ndS_x = 0 \) in equality (3).

With the same argument as in the Neumann or Robin boundary conditions, getting rid of all boundary integrals, we can show that the principal eigenfunction \( u \) is still a critical point of \( J \) in this case, i.e., \( J'(u)\phi = 0 \) for all \( \phi \in S_1 \). Based on this fact, the formula in Lemma 2.1 remains true. As the proof is similar, thus it is omitted. Therefore, the properties of functional \( J \) listed in subsection 2.1 hold for all \( 0 \leq b \leq 1 \).

3. Monotonicity and boundedness of principal eigenvalue

Recall that \( L_A = -\text{div}(a(x)\nabla) + AV \cdot \nabla + c(x) \) and its adjoint operator \( L^*_A = -\text{div}(a(x)\nabla) - AV \cdot \nabla + c(x) \). Here we emphasize that throughout this paper, \( V \) satisfies \( \text{div}V = 0 \) in \( \Omega \) and an additional assumption \( V \cdot n = 0 \) on \( \partial \Omega \) is also needed for \( 0 \leq b < 1 \) (see Remark 1 below). For all \( A \geq 0 \), there exists a unique principal eigenvalue \( \lambda_1(A) \) for eigenvalue problem (3), and a unique (up to multiplication) eigenfunction \( u_A \) satisfying problem (3). We also denote the principal eigenfunction of \( L^*_A \) by some normalized positive function \( v_A \) and write the functional related with problem (3) as

\[
J_A(\omega) = \int \omega v_A \left( \frac{L_A \omega}{\omega} \right) dx, \quad \omega \in S_b.
\]

Our first goal of this section is to show Theorem 1.1.

**Proof of Theorem 1.1**

Firstly, if \( u_0 \in \mathcal{I}_b \), then for every \( A > 0 \), \( u_0 \) satisfies

\[
\begin{cases}
- \text{div}(a(x)\nabla u_0) + AV \cdot \nabla u_0 + c(x)u_0 = \lambda_1(0)u_0 & \text{in } \Omega, \\
u_0 > 0 & \text{in } \Omega, \\
bu_0 + (1 - b)[a(x)\nabla u_0] \cdot n = 0 & \text{on } \partial \Omega.
\end{cases}
\]

Hence, \( \lambda_1(A) = \lambda_1(0) \) for all \( A > 0 \). This proves part (i).

For the proof of part (ii), we assume that \( u_0 \notin \mathcal{I}_b \). We normalize \( u_A \) and \( v_A \) such that \( \int_{\Omega} u_A^2 dx = \int_{\Omega} u_A v_A dx = 1 \).

Differentiate equation (3) with respect to \( A \) and denote \( \frac{\partial u_A}{\partial A} = u'_A \) for the sake of brevity, we obtain

\[
\begin{cases}
- \text{div} \left[ a(x)\nabla u'_A \right] + AV \cdot \nabla u'_A + V \cdot \nabla u_A + c(x)u'_A = \frac{\partial \lambda_1}{\partial A}(A)u_A + \lambda_1(A)u'_A & \text{in } \Omega, \\
bu'_A + (1 - b)[a(x)\nabla u'_A] \cdot n = 0 & \text{on } \partial \Omega, \\
\int_{\Omega} u'_A u_A dx = 0.
\end{cases}
\]

Multiply (10) by \( v_A \) and integrate the result in \( \Omega \), together with the definition of \( v_A \) we have

\[
\frac{\partial \lambda_1}{\partial A}(A) = \int_\Omega v_A V \cdot \nabla u_A dx.
\]

Observe that \( u_0 = v_0 \) for \( A = 0 \). This leads to

\[
\frac{\partial \lambda_1}{\partial A}(0) = \frac{1}{2} \int_\Omega V \cdot \nabla u_0^2 dx = 0.
\]

Here we used that \( V \) is divergence free together with \( V \cdot n = 0 \) on \( \partial \Omega \) for \( 0 \leq b < 1 \) and \( u_0 = 0 \) on \( \partial \Omega \) for \( b = 1 \).

**Claim**: For each \( A > 0 \), \( \frac{\partial \lambda_1}{\partial A}(A) \geq 0 \), and either \( \frac{\partial \lambda_1}{\partial A}(A) > 0 \), or \( \lambda_1(A) = \lambda_1(0) \).
To establish this assertion, it is illuminating to consider the special case of \( A = 1 \). Recall the definition of \( L_1 \) and \( L^*_1 \) to rewrite equality (11) as

\[
\frac{\partial \lambda_1}{\partial A}(1) = \frac{1}{2} \int_\Omega v_1(L_1 - L^*_1)u_1 \, dx = \frac{1}{2} \left[ \int_\Omega v_1 L_1 u_1 \, dx - \int_\Omega u_1 L_1 v_1 \, dx \right].
\]

A direct application of Corollary 1 and positive definiteness of \( L \) yields

\[
\frac{\partial \lambda_1}{\partial A}(1) = \frac{1}{2} \int_\Omega u_1 v_1 \left\{ \left[ \nabla \log \left( \frac{v_1}{u_1} \right) \right] \cdot \left[ a(x) \nabla \log \left( \frac{v_1}{u_1} \right) \right] \right\} \, dx \geq 0,
\]

and \( \frac{\partial \lambda_1}{\partial A}(1) = 0 \) if and only if \( u_1 = cv_1 \) for some \( c > 0 \). By \( \int_\Omega u_1^2 = 1 \) and \( \int_\Omega u_1 v_1 = 1 \), we see that \( c = 1 \) and \( u_1 = v_1 \). Furthermore, if \( u_1 = v_1 \), thus \( L_1 u_1 = L^*_1 u_1 = \lambda_1(1) u_1 \) and hence \( V \cdot \nabla u_1 = 0 \), which further implies that

\[
\begin{cases}
-\div(a(x)\nabla u_1) + c(x)u_1 = \lambda_1(1)u_1 & \text{in } \Omega, \\
u_1 > 0 & \text{in } \Omega, \\
b u_1 + (1-b)[a(x)\nabla u_1] \cdot n = 0 & \text{on } \partial \Omega.
\end{cases}
\]

Hence, \( \lambda_1(1) = \lambda_1(0) \). In summary, \( \frac{\partial \lambda_1}{\partial A}(1) \geq 0 \), and either \( \frac{\partial \lambda_1}{\partial A}(1) > 0 \), or \( \lambda_1(1) = \lambda_1(0) \).

We now proceed to consider the general case of \( A > 0 \). Rewrite the operator \( L_A \) as

\[
L_A = A \left( -\div(a(x)\nabla) + V \cdot \nabla + c(x) \right) + (1-A)(-\div(a(x)\nabla) + c(x)) = AL_1 + (1-A)L_0
\]

and define a new elliptic operator \( L_B \) by

\[
L_B := BL_A + (1-B)L_0.
\]

It is easy to verify that \( L_B = ABL_1 + (1-AB)L_0 = L_{AB} \). Set \( r_1(B) \) as the principal eigenvalue of \( L_B \). A natural fact is that \( r_1(B) = \lambda_1(AB) \). Similar to the above discussion for \( B = 1 \), it follows that \( \frac{\partial r_1}{\partial B}(1) \geq 0 \), and either \( \frac{\partial r_1}{\partial B}(1) > 0 \), or \( r_1(1) = r_1(0) \). In view of \( \frac{\partial r_1}{\partial B}(1) = A \frac{\partial \lambda_1}{\partial A}(A) \), the Claim is proved.

Before proceeding further to show \( \frac{\partial \lambda_1}{\partial A}(A) > 0 \) for all \( A > 0 \), let us calculate \( \frac{\partial^2 \lambda_1}{\partial A^2}(0) \) first.

Differentiate equation (10) with respect to \( A \) again, and applying the notation \( \frac{\partial^2 u_A}{\partial A^2} = u''_A \) for brevity arrives at

\[
(12) \quad \begin{cases}
-\div \left[ a(x) \nabla u'_{A} \right] + A V \cdot \nabla u''_{A} + 2 V \cdot \nabla u''_{A} + c(x)u''_{A} \\
\frac{\partial^2 \lambda_1}{\partial A^2}(A) u_A + 2 \frac{\partial \lambda_1}{\partial A}(A) u'_A + \lambda_1(A) u''_{A} & \text{in } \Omega, \\
b u''_{A} + (1-b)[a(x)\nabla u'_{A}] \cdot n = 0 & \text{on } \partial \Omega.
\end{cases}
\]

Setting \( A = 0 \) in (12) and multiplying it by \( u_0 \) and integrating the result in \( \Omega \), it follows from \( \frac{\partial \lambda_1}{\partial A}(0) = 0 \) that

\[
\frac{\partial^2 \lambda_1}{\partial A^2}(0) = 2 \int_\Omega u_0 V \cdot \nabla u_0' \, dx.
\]

On the other hand, multiplying equation (10) by \( u'_0 \) and setting \( A = 0 \), we have

\[
\frac{b}{1-b} \int_{\partial \Omega} (u'_0)^2 \, dS_x + \int_\Omega \nabla u_0' \cdot \left[ a(x) \nabla u_0' \right] \, dx - \int_\Omega u_0 V \cdot \nabla u_0' \, dx + \int_\Omega c(x)(u'_0)^2 \, dx
\]

\[
= \lambda_1(0) \int_\Omega (u'_0)^2 \, dx,
\]
which in turn implies that
\begin{equation}
\frac{1}{2} \frac{\partial^2 \lambda_1}{\partial A^2}(0) = \frac{b}{1-b} \int_{\partial \Omega} (u'_0)^2 dS_x + \int_{\Omega} \nabla_x u'_0 \cdot \left[ a(x) \nabla_x u'_0 \right] dx + \int_{\Omega} c(x) (u'_0)^2 dx
- \lambda_1(0) \int_{\Omega} (u'_0)^2 dx.
\end{equation}

We are now in a position to prove Theorem 1.1. According to the above Claim, it suffices to prove that \( \lambda_1(A) > \lambda_1(0) \) for every \( A > 0 \). If \( \lambda_1(\hat{A}) = \lambda_1(0) \) for some \( \hat{A} > 0 \), since \( \frac{\partial \lambda_1}{\partial A}(A) \geq 0 \), \( \lambda_1(A) \equiv \lambda_1(0) \) for \( A \in [0, \hat{A}] \). Thus \( \frac{\partial^2 \lambda_1}{\partial A^2}(0) = 0 \). By (13) we have
\[ \lambda_1(0) = \frac{b}{1-b} \int_{\partial \Omega} (u'_0)^2 dS_x + \int_{\Omega} \nabla_x u'_0 \cdot \left[ a(x) \nabla_x u'_0 \right] dx + \int_{\Omega} c(x) (u'_0)^2 dx \]
so the variational argument of principal eigenvalue \( \lambda_1(0) \) implies that \( u'_0 = cu_0 \) for some constant \( c \). Setting \( A = 0 \) and then substituting equality \( u'_0 = cu_0 \) into equation (10), we can conclude that \( V \cdot \nabla u_0 \equiv 0 \) in \( \Omega \), which is a contradiction. This completes the proof. \( \square \)

We now proceed to prove Theorem 1.2.

**Proof of Theorem 1.2** It suffices to establish the following result:

**Claim 1.** Assume that \( \mathcal{I}_b \neq \emptyset \). Then \( \lambda_1(A) \) is uniformly bounded and
\[ \lambda_1(A) \leq \frac{b}{1-b} \frac{\int_{\partial \Omega} \omega^2 dS_x + \int_{\Omega} \nabla \omega \cdot [a(x) \nabla \omega] dx + \int_{\Omega} c(x) \omega^2 dx}{\int_{\Omega} \omega^2 dx}, \quad \forall A \geq 0. \]

The idea of the proof for Claim 1 comes from Theorem 2.2 in [4] and we shall sketch the proof for the sake of completeness. Note that \( u_A > 0 \) in \( \Omega \) by Hopf Boundary Lemma for case of \( 0 \leq b < 1 \). Choose any function \( \omega \in \mathcal{I}_b \) and multiply the equation of \( u_A \) by \( \frac{\omega^2}{u_A} \), then integration by parts implies that
\begin{equation}
\frac{b}{1-b} \int_{\Omega} \omega^2 dS_x + \int_{\Omega} \nabla \left( \frac{\omega^2}{u_A} \right) \cdot \left[ a(x) \nabla u_A \right] dx + A \int_{\Omega} \omega^2 V \cdot \nabla \log u_A dx + \int_{\Omega} c \omega^2 dx
= \lambda_1(A) \int_{\Omega} \omega^2 dx.
\end{equation}

An interesting observation, in analogy with the proof of Theorem 2.2 in [4], gives that
\[ \int_{\Omega} \omega^2 V \cdot \nabla \log u_A dx = 0 \] and \( \int_{\Omega} \nabla \left( \frac{\omega^2}{u_A} \right) \cdot \left[ a(x) \nabla u_A \right] dx \leq \int_{\Omega} \nabla \omega \cdot [a(x) \nabla \omega] dx, \]
which leads to Claim 1 by combining equality (14) and \( \mathcal{I}_b \neq \emptyset \).

It turns out that \( \mathcal{I}_b \neq \emptyset \) always holds for \( 0 \leq b < 1 \), since it at least follows that \( c \in \mathcal{I}_b \) for any constant \( c \). Together with Claim 1, the monotonicity of \( \lambda_1(A) \) in Theorem 1.1 readily implies that the limit of \( \lim_{A \to \infty} \lambda_1(A) \) always exists and is finite. The proof of Theorem 1.2 is complete. \( \square \)

**Remark 1.** (Necessity of the assumption \( V \cdot n = 0 \) on \( \partial \Omega \)): We now remark that the additional assumption \( V \cdot n = 0 \) on \( \partial \Omega \) is necessary for \( 0 \leq b < 1 \), while not necessary for \( b = 1 \), corresponding to zero Dirichlet boundary condition.
• For $b = 1$, zero Dirichlet boundary condition implies $u_A = v_A = 0$ on $\partial \Omega$ and the adjoint operator of $L_A$ can be written as $L_A^* = -\text{div}(a(x)\nabla) - A\nabla \cdot \nabla + c(x)$ without the additional assumption, whence Theorem 1.1 remains true as the properties of $J_A$ in Section 2 hold without this assumption as stated in subsection 2.2.

• For $0 \leq b < 1$, Theorem 1.1 may fail without the assumption $\nabla \cdot \mathbf{n} = 0$ on $\partial \Omega$. Consider the same example as in Remark 2.5 of [4], for some constant. If Theorem 1.1 holds, since $\lambda_1(A) = c(0)$ by treating $\mathbf{V} = -\nabla(-x)$. Assume further that $c'(x) \geq 0$ and $c(x) \not\equiv$ constant. If Theorem 1.1 holds, since $\lambda_1(0) \geq \min_{x \in [0,1]} c(x) = c(0)$, we have $\lambda_1(A) \equiv c(0)$, and thus $\varphi'_0 = 0$ according to part (ii) in Theorem 1.1 which contradicts to $c(x) \not\equiv$ constant.

4. Min-Max characterization of principal eigenvalue

In this section we focus on a new min-max characterization of the principal eigenvalue for elliptic operator $L = -\text{div}(a(x)\nabla) + A\nabla \cdot \nabla + c(x)$ with incompressible flow and general boundary conditions. To state our main result, some preparations are needed. In this connection, in view of the classical min-max characterization of principal eigenvalue [30],

$$\lambda_1 = \sup_{\omega \in S_b} \inf_{x \in \Omega} \left[ \frac{L\omega(x)}{\omega(x)} \right] = \inf_{\omega \in S_b} \sup_{x \in \Omega} \left[ \frac{L\omega(x)}{\omega(x)} \right]$$

together with the facts

$$\inf_{p \in S_b, f_\Omega \geq 1} \int_{\Omega} p^2(x) \left( \frac{L\omega}{\omega} \right) dx = \inf_{x \in \Omega} \left[ \frac{L\omega(x)}{\omega(x)} \right],$$

and

$$\sup_{p \in S_b, f_\Omega \geq 1} \int_{\Omega} p^2(x) \left( \frac{L\omega}{\omega} \right) dx = \sup_{x \in \Omega} \left[ \frac{L\omega(x)}{\omega(x)} \right],$$

it is straightforward to derive the following min-max characterization of $\lambda_1$:

$$\lambda_1 = \sup_{\omega \in S_b} \inf_{p \in S_b, f_\Omega \geq 1} \int_{\Omega} p^2(x) \left( \frac{L\omega}{\omega} \right) dx$$

$$= \inf_{\omega \in S_b} \sup_{p \in S_b, f_\Omega \geq 1} \int_{\Omega} p^2(x) \left( \frac{L\omega}{\omega} \right) dx. \tag{15}$$

However, the min-max characterization in Theorem 1.3 is somewhat different. The following result is the key of the proof of Theorem 1.3

Lemma 4.1.

$$\sup_{\omega \in S_b} J(\omega) = J(u) = \lambda_1.$$

Furthermore, if $J(\omega_0) = \sup_{\omega \in S_b} J(\omega)$ for some $\omega_0 \in S_b$, then $\omega_0 = cu$ for some constant $c > 0$. 

Lemma 4.1 is a direct consequence of Lemma 2.1 by recalling the positive definiteness of \(a(x)\). With the help of Lemma 4.1, Theorem 1.3 can be proved in straightforward manner as follows.

**Proof of Theorem 1.3** We first choose \(p^2 = uv\) and apply Lemma 4.1 to obtain that
\[
\lambda_1 = \sup_{\omega \in S_b} \int_{\Omega} uv \left( \frac{L\omega}{\omega} \right) \, dx \geq \inf_{p \in S_b} \sup_{p^2 = 1} \int_{\Omega} p^2(x) \left( \frac{L\omega}{\omega} \right) \, dx.
\]
On the other hand, for any \(p \in S_b\) satisfying \(\int_{\Omega} p^2 = 1\), it is easy to see that
\[
\lambda_1 = \int_{\Omega} p^2(x) \left( \frac{L u}{u} \right) \, dx \leq \sup_{\omega \in S_b} \int_{\Omega} p^2(x) \left( \frac{L\omega}{\omega} \right) \, dx,
\]
which implies that
\[
\lambda_1 \leq \inf_{p \in S_b} \sup_{p^2 = 1} \int_{\Omega} p^2(x) \left( \frac{L\omega}{\omega} \right) \, dx.
\]
Hence equality (6) holds. The proof of Theorem 1.3 is now complete. \(\square\)

**Remark 2.** (Reduce to the classical Rayleigh-Ritz formula): The classical Rayleigh-Ritz formula is actually implicitly contained in the min-max formula in Theorem 1.3 if \(L\) is self-adjoint, i.e., \(V = 0\). It can be deduced from an important result in [14]. More specifically, viewing \(\mu = p^2 dx\) as a positive measure satisfying the mild assumption \(\mu \ll \lambda\) for the Borel measure \(\lambda\) and noting that \(\frac{\mu}{dx} = p^2\), Theorem 4 in [14] leads to
\[
\sup_{\omega \in S_b} \int_{\Omega} p^2(x) \left( \frac{L\omega}{\omega} \right) \, dx = \langle Lp, p \rangle,
\]
which reduces the formula in Theorem 1.3 to the classical Rayleigh-Ritz formula.

5. **Discussions and open questions**

In many physical and biological systems, the effect of incompressible flow \(V\) on the speed of traveling fronts of equation (1) remains an important area of active research [3, 6, 20, 25, 26, 27, 28, 34], with particular interest on the minimal speed \(c^*\). The minimal speed \(c^*_V\) can be enhanced by the introduction of incompressible flows [3, 10, 16, 36, 37], while general compressible flows may decrease \(c^*_V\); See Theorem 2.8 of [23]. In this connection, many works focus on the case of the shear flow \(V = \alpha(x_2, \ldots, x_N) e_1\), where \(\alpha \neq 0\) is zero-average, in a straight cylinder \(\Omega = \mathbb{R} \times D\) with bounded domain \(D \subset \mathbb{R}^{N-1}\) along the direction \(e_1\). Examples are known for which the minimal speed \(c^*_{AV}\) in the presence of a shear flow \(V\), is asymptotically linear in \(A\) [20]. Furthermore, \(c^*_{AV}\) is increasing in \(A\), \(c^*_{AV}/A\) is decreasing in \(A\), as well as \(c^*_{AV}/A \to \rho > 0\) as \(A \to +\infty\) [23]. The monotonicity of \(c^*_{AV}\) and \(c^*_{AV}/A\) however remains open for general incompressible flow \(V\); See Remark 1.9 in [3] and Remark 1.6 in [20] for details. Our preliminary studies suggest that the monotonicity of \(c^*_{AV}/A\) holds for general incompressible flow \(V\). We hope to report it in forthcoming work.

We now turn to consider operator \(L_A\) with gradient flow \(V_1 = \nabla m\) for some \(m \in C^2(\Omega)\), where the principal eigenvalue \(\lambda_1(A)\), in analogy with equation (1.2) in [4], can be written as
\[
\lambda_1(A) = \inf_{\omega \in H^1(\Omega) \setminus \{0\}} \frac{\frac{b}{2} \int_{\Omega} \omega^2 dS_x + \int_{\Omega} \nabla \omega \cdot [a(x) \nabla \omega] dx + \int_{\Omega} \left( \frac{A^2}{4} |V_1|^2 - \frac{A}{2} \text{div} V_1 + c(x) \right) \omega^2 dx}{\int_{\Omega} \omega^2 dx},
\]
which implies the monotonicity of \(\lambda_1(A)\) if \(V_1\) is incompressible satisfying \(\text{div} V_1 = 0\). This result can be covered by Theorem 1.1 with the extra assumption \(V_1 \cdot n = 0\) on \(\partial \Omega\). However, if
the gradient flow $\mathbf{V}_1 = \nabla m$ is incompressible and satisfies $\mathbf{V}_1 \cdot \mathbf{n} = 0$ on $\partial \Omega$, the only possibility is $m = \text{constant}$. Hence we may ask naturally: When does the monotonicity property remain true for gradient flow? Understanding the monotonicity of $\lambda_1(A)$ with general flows seems to be more difficult.

Another open question is to determine the limit value of $\lambda_1(A)$ for incompressible flow $\mathbf{V}$ with Robin boundary conditions as $A \rightarrow +\infty$, though the existence of the limit has been shown in Theorem 1.2. The results for Dirichlet and Neumann boundary conditions in [4] show that the limit of $\lambda_1(A)$ can be determined by the variational principle (2). In view of Theorem 1.2, it seems plausible to conjecture that for $0 \leq b < 1$,

$$\lim_{A \rightarrow +\infty} \lambda_1(A) = \inf_{\omega \in \mathcal{M}} \frac{b}{1-b} \int_{\partial \Omega} \omega^2 dS_x + \int_{\Omega} \nabla \omega \cdot [a(x) \nabla \omega] dx + \int_{\Omega} c(x) \omega^2 dx,$$

which would reduce to the results in [4] for the case $b = 0$. The limit value of $\lambda_1(A)$ with the gradient flow $\mathbf{V}_1 = \nabla m$ has been established by Chen and Lou [8] for Neumann boundary conditions, which can be stated as

$$\lim_{A \rightarrow +\infty} \lambda_1(A) = \min_{\mathcal{M}} c,$$

with the set $\mathcal{M}$ consisting of all points of local maximum of $m$. Hence a natural question arises: Does the limit of $\lambda_1(A)$ exist as $A \rightarrow +\infty$ for general flows under proper boundary conditions? If it exists, what is the limit value?

There are a substantial body of literatures concerning the asymptotic behavior of the principal eigenvalue of elliptic operators for small diffusion rates; See [9, 11, 12, 17, 35]. For the principal eigenvalue of operator $L_D = -D \Delta + \mathbf{V} \cdot \nabla + c(x)$, Chen and Lou [9] investigated its asymptotic behavior as $D \rightarrow 0$ when $\mathbf{V}$ is a gradient flow. Much less seems to be known when $\mathbf{V}$ is a general incompressible flow; See [2, 33].

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