Integrable Chiral Theories in 2 + 1 Dimensions

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Abstract

Following a recent proposal for integrable theories in higher dimensions based on zero curvature, new Lorentz invariant submodels of the principal chiral model in 2 + 1 dimensions are found. They have infinite local conserved currents, which are explicitly given for the $su(2)$ case. The construction works for any Lie algebra and in any dimension, and it is given explicitly also for $su(3)$. We comment on the application to supersymmetric chiral models.
1 Introduction

In this paper, we obtain Lorentz invariant submodels of the principal chiral model in $2 + 1$ dimensions, and we find explicitly infinitely many conserved currents, following a new generalized zero curvature approach in higher dimensions [1], thus contributing further to its understanding.

A systematic generalization of the extremely useful zero curvature formulation for integrability to dimensions higher than two is a longstanding difficult problem, especially for models with Lorentz invariance [2]. Success has been achieved so far for selfdual cases [3] and other situations which are effectively two dimensional [4], or which are not Lorentz invariant. The latter is in a strict sense the case for the extensively studied Modified Chiral Model [5], since this model fixes a direction in three dimensional spacetime [6]. Recently one of us with O.Alvarez and L.Ferreira [1] proposed a geometric approach for $d$ dimensions based on the interplay of connections (functionals of the fields) up to rank $d - 1$, where the generalized zero curvature conditions become local field equations with the appropriate algebraic structure. For $2 + 1$ dimension it was shown that for any vector $A_\mu$ and antisymmetric tensor $B_{\mu\nu}$ in a non-semisimple Lie algebra, the flatness of $A$ and constant covariance of the dual of $B$, $D\tilde{B} = 0$, are sufficient conditions for generalized zero curvature and the associated surface independence. Once accommodated into the scheme by a proper choice of the algebra and its representations, one can systematically analyze the integrability properties of the particular models. Many of the powerful tools of the zero curvature approach to obtain conserved charges and solutions can thereby be used in higher dimensions.

The $CP^1$ example was worked out this way in [1] for $A$ in $sl(2)$ and $\tilde{B}$ in an abelian subalgebra, with interesting results. It was shown that the local zero curvature conditions of the approach only imply the equations of the model for a special representation spin $j = 1$ and that the resulting conserved currents just correspond to the isometries of the model, which is therefore not integrable. On the other hand, requiring that the equations of motion are equivalent to the zero curvature conditions for any spin $j$, as one would like for integrability, selects an interesting submodel [7] with infinitely many (enumerated by $j$) new conserved currents.

In this paper we elaborate further the $CP^1$ case, obtaining general expressions for the currents in Section 2, and then we apply the method to the Principal Chiral model for $su(2)$ in Section 3. The formulation is even simpler in this case, and it leads in fact to a new class of models with infinitely many conserved charges which are explicitly given. This confirms further the suggestion in [1] of the equivalence of zero curvature and equations of motion, implemented by representation independence, as a constructive criterion of integrability, as we discuss in Section 3.2. In this section we also deal with the solutions of the submodels, working out in detail the static case. In section 3.3 we remark on the generalization to the supersymmetric chiral model, and in Section 4 we argue that the result holds also for higher algebras and we present in detail the $su(3)$ case. In the last Section 5, we summarize our results and give some general remarks on the nature of higher dimensional integrability, as well as on applications and possible future developments.
2 The \( \mathbb{C}P^1 \) model

For future reference, and in order to demonstrate the method, we will first review the case of the \( \mathbb{C}P^1 \) model, at the same time improving the result of \([1]\) by giving explicit expressions for the conserved currents (first found in \([8]\)), and improving also the results of \([8]\) by finding the conserved currents in a simple and straightforward way.

The \( \mathbb{C}P^1 \) model is defined by the equation of motion

\[
(1 + |u|^2)\partial^2 u = 2u^*\partial u\partial^\mu u \tag{2.1}
\]

In order to represent the equations of motion in the form of zero curvature conditions we follow \([1]\), and define the fields

\[
A_\mu = \frac{1}{(1 + |u|^2)} \left\{-i\partial_\mu u T_+ - i\partial_\mu u^* T_- + (u\partial_\mu u^* - u^*\partial_\mu u) T_3 \right\}
\]

\[
\tilde{B}^{(1)}_\mu = \frac{1}{(1 + |u|^2)} \left\{\partial_\mu u P_{+1}^{(1)} - \partial_\mu u^* P_{-1}^{(1)} \right\} \tag{2.2}
\]

where \( T_\pm, T_3 \) are the generators of the \( sl(2) \) algebra

\[
[T_+, T_-] = 2T_3 \quad [T_3, T_\pm] = \pm T_\pm \tag{2.3}
\]

and \( P_{+1}^{(1)} \) and \( P_{-1}^{(1)} \) (together with \( P_0^{(1)} \)) transform under the triplet representation of \( sl(2) \). In general, we consider the commutation relations associated to a generic spin-\( j \) representation of \( sl(2) \)

\[
[T_3, P_m^{(j)}] = mP_m^{(j)} \\
[T_\pm, P_m^{(j)}] = \sqrt{j(j+1) - m(m\pm 1)} P_{m\pm 1}^{(j)} \\
[P_m^{(j)}, P_{m'}^{(j)}] = 0 \tag{2.4}
\]

Then, equation (2.1) is equivalent to

\[
D_\mu \tilde{B}^{(1)\mu} = 0 \tag{2.5}
\]

where \( D_\mu = \partial_\mu \cdot [A_\mu, \cdot] \). The field \( A_\mu \) is a flat potential, i.e. \( F_{\mu\nu} = 0 \), and so it can be put in the form \( A_\mu = -\partial_\mu WW^{-1} \) where \( W \) is the group element

\[
W = \frac{1}{\sqrt{1 + |u|^2}} \begin{pmatrix} 1 & iu \\ iu^* & 1 \end{pmatrix} \tag{2.6}
\]

In order for equation (2.3) to be equivalent to the equations of motion for the \( \mathbb{C}P^1 \) model, it is important that \( \tilde{B}^{(1)}_m \) takes its value in the \( spin \ 1 \) representation. In fact, we may define a field \( \tilde{B}^{(j)}_m \) for any integer \( spin \ j \), by taking

\[
\tilde{B}^{(j)}_\mu = \frac{1}{(1 + |u|^2)} \left\{\partial_\mu u P_{+1}^{(j)} - \partial_\mu u^* P_{-1}^{(j)} \right\} \tag{2.7}
\]
In this case, the equation $D_\mu \tilde{B}^{(j)\mu} = 0$ becomes

$$0 = \left\{ \sqrt{j(j+1)} - 2 \left( -i\partial_\mu u \partial_\mu u P_{j+2}^{(j)} + i\partial_\mu u^* \partial^\mu u^* P_{j-2}^{(j)} \right) 
+ \left( (1 + |u|^2) \partial^2 u - 2u^* \partial_\mu u \partial^\mu u \right) P_{j+1}^{(j)} 
+ \left( (1 + |u|^2) \partial^2 u^* - 2u \partial_\mu u^* \partial^\mu u^* \right) P_{j-1}^{(j)} \right\}$$

(2.8)

We see that the solution of this equation is in fact independent of $j$ for $j > 1$, and that the equation is equivalent to the equation of motion of the $CP^1$ model, in addition to a new condition (integrability condition) $\partial_\mu u \partial^\mu u = 0$. Actually, this integrability condition corresponds to a known $CP^1$ submodel [7]

$$\partial^2 u = 0 \quad \partial_\mu u \partial^\mu u = 0$$

(2.9)

2.1 Conserved currents

Following again [1], we can construct an infinite number of conserved quantities in a simple way. They are of the form

$$J_\mu^{(j)} = W^{-1} \tilde{B}_\mu^{(j)} W = \sum_{m=-j}^j J_\mu^{(j,m)} P_{m}^{(j)}$$

(2.10)

In order to evaluate this expression, we use the known expression (see for example [9]) for the adjoint action of a generic group element

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

(2.11)

which is given by

$$gP_{m}^{(j)} g^{-1} = \sum_{m=-j}^j A_{mn} P_{m}^{(j)}$$

(2.12)

with

$$A_{mn} = [(j + m)!(j - m)!(j + n)!(j - n)!!]^1 \times \sum_k \frac{a^{j + m - k} b^k c^{k + n - m} d^{j - n - k}}{(j + m - k)! k! (k + n - m)! (j - n - k)!}$$

(2.13)

In our case, we replace $g$ in this expression by

$$W^{-1} = \frac{1}{\sqrt{1 + |u|^2}} \begin{pmatrix} 1 & -iu \\ -iu^* & 1 \end{pmatrix}$$

(2.14)

and we obtain expressions for the currents

$$J_\mu^{(j,m)} = \frac{1}{(1 + |u|^2)} \left\{ \partial_\mu u A_{m,1} - \partial_\mu u^* A_{m,-1} \right\}$$

(2.15)
with
\[
A_{m,\pm 1} = [(j + m)!(j - m)!(j + 1)!(j - 1)!]^{1/2} \frac{1}{(1 + |u|^2)^j} \times \sum_{k=m\pm 1}^{j+1} \frac{(-1)^k(-i)^{\pm 1-m}|u|^{2(k+1-m)}}{(j + m - k)!k!(k \pm 1 - m)!(j \mp 1 - k)!} (2.16)
\]

Using these expressions, we can now calculate the currents given in [8] in a simple way:

- \( m = 0 \)
\[
J_{\mu}^{(j,0)} = -i\sqrt{j(j+1)} \frac{(u \partial_\mu u^* - u^* \partial_\mu u)}{(1 + |u|^2)^{j+1}} \sum_{k=0}^{j-1} \gamma_k^{(j)} |u|^{2k} \] (2.17)

- \( m > 0 \)
\[
J_{\mu}^{(j,m)} = \sqrt{\frac{(j + m)!}{j(j+1)(j-m)!}} \frac{(-iu)^{m-1}}{(1 + |u|^2)^{j+1}} \times \sum_{k=0}^{j-m} \left\{ \alpha_k^{(j,m)} |u|^{2k} \partial_\mu u + (-1)^{j-m} \alpha_{j-m-k}^{(j,m)} |u|^{2k} u^2 \partial_\mu u^* \right\} \] (2.18)

- \( m < 0 \)
\[
J_{\mu}^{(j,-m)} = (-1)^m J_{\mu}^{(j,m)} \] (2.19)

where the coefficients are
\[
\gamma_k^{(j)} = (-1)^k \frac{1}{j} \binom{j}{k} \binom{j+1}{k+1}
\]
\[
\alpha_k^{(j,m)} = (-1)^k \frac{n!}{(m+n-1)!} \binom{j-m}{k} \binom{j+1}{k} \]

3 The chiral model

The pure chiral model is defined by the equations
\[
\partial_\mu A_\mu = 0 \quad A_\mu \equiv G^{-1} \partial_\mu G \] (3.20)

We take \( G \in SU(2) \), but we will show in Section [4] that the construction works also in the case of higher groups. We observe that \( A_\mu \) is a flat potential by construction, so we have the condition \( F_{\mu\nu} = 0 \). We can express in a simple way the chiral model in the general formalism of zero curvature. We consider
\[
A_\mu = A_\mu^i T_i \quad \tilde{B}_\mu = \tilde{B}_\mu^j P_j \equiv A_\mu^j P_j \] (3.21)

where \( \{T_i\}, \ i = 1, 2, 3 \) are the generators of the algebra \( su(2) \) and we have the commutation relations
\[
[T_i, T_j] = i\epsilon_{ijk} T_k
\]
\[
[T_i, P_j] = i\epsilon_{ijk} P_k
\]
\[
[P_i, P_j] = 0 \] (3.22)
Using the fact that, by construction, \([A_\mu, \tilde{B}_\mu] = 0\), it is now easy to see that the equations of motion of the chiral model can be represented by the zero curvature conditions

\[ F_{\mu\nu} = 0 \quad D_\mu \tilde{B}_\mu = 0 \tag{3.23} \]

where \(D_\mu\) is defined after eq. (2.5). If we make a change of basis \(\{T_i\} \rightarrow \{T_\pm = T_1 \pm iT_2, \; T_3\}\) we obtain the commutation relations of \(sl(2)\)

\[ [T_+, T_-] = 2T_3 \quad [T_3, T_\pm] = \pm T_\pm \tag{3.24} \]

If we make also the change of basis

\[ P_{\pm 1}^{(1)} = \mp (P_1 \pm iP_2) \]
\[ P_0^{(1)} = \sqrt{2}P_3 \tag{3.25} \]

then the new generators \(P_{\pm 1}^{(1)}\) satisfy the standard commutation relations of \(sl(2)\) representations (2.4) for spin \(j = 1\).

In this new basis

\[ A_\mu = A^-_\mu T_- + A^3_\mu T_3 + A^+_\mu T_+ \]
\[ \tilde{B}_\mu = A^-_\mu P_{-1}^{(1)} + \frac{1}{\sqrt{2}}A^3_\mu P_0^{(1)} - A^+_\mu P_{+1}^{(1)} \tag{3.26} \]

We may define \(\tilde{B}_{\mu}^{(j=1)} = \tilde{B}_\mu\), and we wish to generalize this to an infinite number of values of \(j\), in such a way that the solution of the equation \(D_\mu \tilde{B}_\mu^{(j)} = 0\) is independent of \(j\). We find that if we define, for any integer value of \(j\), the potential

\[ \tilde{B}_\mu^{(j)} = A^-_\mu P_{-1}^{(j)} + \frac{1}{\sqrt{j(j+1)}}A^3_\mu P_0^{(j)} - A^+_\mu P_{+1}^{(j)} \tag{3.27} \]

Then, the condition \(D_\mu \tilde{B}_\mu^{(j)} = 0\) reads

\[ D_\mu \tilde{B}_\mu^{(j)} = \partial^\mu A^-_\mu P_{-1}^{(j)} + \frac{1}{\sqrt{j(j+1)}}\partial^\mu A^3_\mu P_0^{(j)} - \partial^\mu A^+_\mu P_{+1}^{(j)} \]
\[ + \sqrt{j(j+1)} - 2 \left( A^-_\mu A^-_\mu P_{-2}^{(j)} - A^+_\mu A^+_\mu P_{+2}^{(j)} \right) = 0 \tag{3.28} \]

and we see that with this choice, the solutions of the equations are indeed independent of \(j\) for \(j > 1\). In fact, the zero curvature equations \(F_{\mu\nu} = 0, \; D_\mu \tilde{B}_\mu^{(j)} = 0\) for all \(j\) are equivalent to the equations of motion for a submodel of the chiral model

\[ \partial_\mu A^\mu = 0 \quad A^\mu_\pm A^\mu_\pm = 0 \tag{3.29} \]

### 3.1 Conserved currents

Any element of \(SU(2)\) can be parametrized by \(u \in \mathbb{C}, \; \theta \in \mathbb{R}\) and written in the form

\[ G = \frac{1}{(1 + |u|^2)^{\frac{1}{2}}} \begin{pmatrix} e^{i\theta} & u \\ -u^* & e^{-i\theta} \end{pmatrix} \tag{3.30} \]
With this form, the expression for the adjoint action

\[ GP_n^{(j)} G^{-1} = \sum_{m=-j}^{j} A_{mn} P_m^{(j)} \]  

is now given by

\[ A_{mn} = [(j + m)!(j - m)! (j + n)!(j - n)!)^{\frac{1}{2}} \frac{e^{i\theta(m+n)}}{(1 + |u|^2)^j} \times \sum_k (-1)^{k+n-m} u^m u^* u^{2(k-m)} |u|^2 \]  

As in the previous section we can obtain the conserved currents in the form

\[ J_\mu^{(j)} = G \tilde{B}_\mu^{(j)} G^{-1} = \sum_{m=-j}^{j} J_{\mu}^{(j,m)} P_m^{(j)} \]  

In the case of the chiral model this means

\[ J_{\mu}^{(j,m)} = -A_{m1} A_{\mu}^* + \frac{1}{\sqrt{j(j+1)}} A_{m0} A_{\mu}^3 + A_{m,-1} A_{\mu}^* \]  

On the other hand, using (3.30), we have

\[ A_{\mu} = \frac{1}{(1 + |u|^2)^{j+1}} \left\{ \left[ (u \partial_\mu u^* - u^* \partial_\mu u) + 2i \partial_\mu \theta \right] T_3 + \left[ e^{-i\theta} (\partial_\mu u + iu \partial_\mu \theta) \right] T_+ + \left[ -e^{i\theta} (\partial_\mu u^* - iu^* \partial_\mu \theta) \right] T_- \right\} \]  

Putting all together, we find explicit expressions for the conserved currents:

- \( m = 0 \)

\[ J_{\mu}^{(j,0)} = \frac{(j+1)^{\frac{3}{2}}}{(1 + |u|^2)^j} \sum_{k=0}^{j} \left\{ (u \partial_\mu u^* - u^* \partial_\mu u) \left( 1 - \frac{(j-k)(j+1)}{k+1} \right) \right\} \lambda_k^2 |u|^{2k} \]  

- \( m > 0 \)

\[ J_{\mu}^{(j,m)} = -e^{im\theta} u^{m-1} \sum_{k=0}^{j-m} \left\{ \left( \frac{k+m}{j-k+1} + \frac{|u|^2}{j+1} \right) \partial_\mu u \right. \]

\[ + \left( \frac{k+m}{j-k+1} - \frac{2}{j+1} - \frac{2}{j+1} \right) u \partial_\mu \theta \]

\[ + \left( \frac{j-k}{k+m+1} - \frac{1}{j+1} \right) u^2 \partial_\mu u^* \right\} \beta_k^{(j,m)} |u|^{2k} \]
\[ m < 0 \]

\[ J^{(j,m)}_\mu = (-1)^m J^{(j,-m)}_\mu \]  

(3.38)

where the new coefficients are

\[ \lambda_k^j = (-1)^k \binom{j}{k} \]

\[ \beta_k^{(j,m)} = (-1)^k \sqrt{\frac{j+1}{j} \left[ \frac{(j+m)!}{j!} \frac{(j-m)!}{j!} \right]^{\frac{3}{2}} \binom{j}{k} \binom{j}{k+m} } \]

### 3.2 Solutions

The constraints on the phase space of the theory, due to the infinite number of conserved currents, gives to the theory some integrability properties that should manifest themselves in the existence of non-trivial solutions.

Not much is known about analytic solutions or integrability properties in dimensions greater than 2, apart from the fact that, due to theorems like Derrick’s or Coleman-Mandula [10], it has to be different from the well studied properties of integrable theories in \( d = 2 \). Therefore, the study of integrable models in \( 2 + 1 \) dimensions is relevant, not only for the models themselves, but for a deeper understanding of integrability.

The construction of general solutions can be attempted by the dressing-like and other methods proposed in the generalized zero curvature approach, which we are analyzing at present. In this paper, our purpose is to illustrate and check the consistency of the method, and we will restrict ourselves to discuss the static solutions, including the well known ones of the submodel of \( CP^1 \). Of course, one can obtain time dependent solutions if we make a boost, because of the Lorentz invariant nature of the models, and one can also use the time-independent solutions as seeds for dressing-like methods.

We can easily verify that any solution of the Belavin-Polyakov equations (or equivalently, the static Cauchy-Riemann equations) are static solutions for the \( CP^1 \) submodel. In particular, an interesting solution is the baby-Skyrmion [1]

\[ u = x + iy \]  

(3.39)

More generally, many solutions for the \( CP^1 \) submodel are well studied [7, 11]. The general solution of the equations

\[ \partial^2 u = 0 \quad \partial^\mu u \partial_\mu u = 0 \]  

(3.40)

is obtained by solving the equation

\[ x_+ - tf(u) + x_- f(u)^2 = g(u) \]  

(3.41)

where \( f \) and \( g \) are two complex-analytic functions, and \( x_\pm = \frac{1}{2}(x \pm iy) \). If we take \( f = 0 \), then the equation imply that \( u \) is an analytic function of \( x_+ \), and thus the static solutions corresponds to the case \( f = 0 \). The next simple case is \( f(u) \) constant, \( f(u) = k \). In this case

\[ u = u(x_+ - kt + k^2 x_-) \]  

(3.42)

These solutions are the boosted versions of the static ones [4].

\[ ^4 \text{In both cases, to ensure finite energy, } u \text{ must be a rational function} \]
As we see, the variety of solutions is enormous. Some have been investigated by numerical and approximate methods, like the so called geodesic approximation, with complex but interesting results [7]. Exact time-dependent solutions, which are at present under investigation, are very promising.

Turning now to the static solutions of the integrable chiral submodel, the equations of motion are

$$\partial_i A^i = 0 \quad , \quad A_0 = A_a^0 T_a = 0$$  \hspace{1cm} (3.43)

$$A_i^+ A_i^- = 0 \Rightarrow A_i^\pm = 0 \quad i = 1, 2$$ \hspace{1cm} (3.44)

which implies

$$\partial_i A_i^3 = 0 \quad A_i^\pm = 0$$  \hspace{1cm} (3.45)

Using the expression (3.35) for $A_i^\pm$, we find

$$\partial_i u + i u \partial_i \theta = 0 \Rightarrow u = k e^{-i\varphi}$$ \hspace{1cm} (3.46)

and

$$A_i^3 = \frac{1}{(1 + |k|^2)} \{ (u \partial_i u^* - u^* \partial_i u) + 2i \partial_i \theta \} = 2i \partial_i \theta$$ \hspace{1cm} (3.47)

so the equations of motion reduces to $u = k e^{-i\varphi}$ and

$$\nabla^2 \theta = 0$$ \hspace{1cm} (3.48)

Thus, the static solutions of the chiral submodel correspond to the solutions of this equation. Note that the only solutions of this equation which are regular on the whole plane (including infinity) are constants, which in this model implies that there are no non-trivial static solutions with finite energy.

It is interesting to note that the case $k = 1$ (or any constant phase), $u = e^{i\varphi}$, is a solution of the $CP^1$ model if

$$\nabla^2 \varphi = 0$$ \hspace{1cm} (3.49)

which is the same equation as before. This is not surprising because there is a known relation between chiral and $CP^1$ models. The chiral model for $SU(2)$ is equivalent to the $O(4)$ model, and the only static solutions of the $O(4)$ model are embeddings of the $O(3)$ model which is equivalent to $CP^1$.

3.3 The supersymmetric chiral model

The approach used in this paper is easily generalized to the supersymmetric chiral model. The SUSY chiral model for the group $SU(2)$ is defined by the equations of motion

$$\mathcal{D}_\alpha A^\alpha = 0 \quad \mathcal{A}_\alpha = \mathcal{G}^{-1} \mathcal{D}_\alpha \mathcal{G}$$ \hspace{1cm} (3.50)

where $\mathcal{G}$ is an $SU(2)$-valued superfield. We use the conventions of [12]: the Grassman coordinates are Majorana spinors $\theta^\alpha$, $\alpha = \pm$, which form the vector representation under the Lorentz group $SL(2, \mathbb{R})$, while the space-time coordinates are described by a symmetric, second-rank tensor $x^{\alpha\beta} = (x^{++}, x^{+-}, x^{-+}, x^{--})$, and $\mathcal{D}_\alpha = \frac{\partial}{\partial \theta^\alpha} + i \theta^\beta \partial_\alpha \beta$, $\partial_\alpha \beta x^{\sigma\tau} = \delta^{(\alpha}_(\sigma \delta)^{\tau)}$.
Defining $T_i$, $P_j$ and $\bar{B}_\alpha$ like in equations (3.21-3.22), we find that the equations
\[ F_{\alpha\beta} = 0, \quad D_\alpha \bar{B}^\alpha = 0, \] (3.51)
with $D_\alpha \cdot = D_\alpha \cdot + [A_\alpha, \cdot]$, are equivalent to the equations of motion (3.50). For a spin-$j$ representation ($j \in \mathbb{Z}$) we define $\bar{B}_\alpha^{(j)}$ as in eq. (3.27), and we find again that the equations
\[ D_\alpha \bar{B}^{\alpha (j)} = 0 \] (3.52)
are equivalent to a submodel of the SUSY chiral model, with equations of motion
\[ D_\alpha A^\alpha = 0, \quad A^\alpha A^\pm = 0 \] (3.53)
where the $\pm$ are algebra indices, not to be confused with the spinor indices. The expressions for the conserved currents given in equations (3.36-3.38) can also be applied directly in the supersymmetric case.

4 The $SU(3)$ case

The methods described in the previous section can be generalized to higher groups. As an example, we consider the generalization to $SU(3)$, but it will be clear that the construction can be generalized to any group.

Consider first the chiral model for $SU(3)$. The equation of motion is
\[ \partial_\mu A^\mu = 0, \quad A^\mu = G^{-1} \partial^\mu G \] (4.54)
with $G \in SU(3)$. The potential $A_\mu$ is flat, so the corresponding curvature is zero, $F_{\mu\nu} = 0$. We consider
\[ A_\mu = A'_\mu T_i, \quad \bar{B}_\mu = \bar{B}^j_\mu P_j \] (4.55)
where $T_i$ are the generators of $su(3)$, and $P_j$ transform under the adjoint representation of $su(3)$, such that the total set of commutation relations are:
\[ [T^a, T^b] = f^{ab}_c T^c, \quad [T^a, P^b] = f^{ab}_c P^c, \quad [P^a, P^b] = 0 \] (4.56)
where $f^{ab}_c$ are the structure constants of $su(3)$. By construction we have $[A_\mu, \bar{B}^\mu] = 0$, and it is clear that the zero curvature conditions
\[ D_\mu \bar{B}^\mu = 0, \quad F_{\mu\nu} = 0 \] (4.57)
are equivalent to the equations of motion. The conserved currents which correspond to these zero curvature conditions are defined by
\[ J^\mu = G \bar{B}^\mu G^{-1} = \partial^\mu GG^{-1} \] (4.58)
However, the conservation equation for $J^\mu$, $\partial_\mu J^\mu = 0$, is completely equivalent to the equations of motion, $\partial_\mu A^\mu = 0$, and we do not find any new conserved currents.
4.1 An integrable submodel of the su(3) chiral model

In order to find an integrable submodel of the su(3) chiral model, we will define fields $\tilde{B}_\mu^\Lambda$ for an infinite set of highest weight representations of su(3), with highest weight $\Lambda$, such that the conditions

$$D_\mu \tilde{B}_\mu^\Lambda = 0 \quad F_{\mu\nu} = 0$$

are equivalent to the equation of motion of the submodel. As in the previous sections, we will do this by defining

$$\tilde{B}_\Lambda = A_\mu^i P_i^\Lambda$$

where $P_i^\Lambda$ belongs to a certain subset of the generators of the representation.

As a generalization of su(2) representations with integer spin, we consider su(3) representations with a highest weight of the form $\Lambda = n (\alpha_1 + \alpha_2)$, where $\alpha_i$ are the simple roots. As an illustration we show in figure 1 the weight diagram for the case $n = 2$, which is the $\{27\}$ of su(3), but the construction given below works for general $n$.

Before we proceed to the definition of the field $\tilde{B}_\mu^\Lambda$, we must define the subset $P_i^\Lambda$ of the generators, and we will show that the chosen subset have certain nice commutation relations with the generators of the su(3) algebra.

It is convenient to make a transformation to a basis with generators $H_i = \alpha_i \cdot H, E_\alpha$, and commutation-relations:

$$[H_i, E_\beta] = \alpha_i \cdot \beta E_\beta$$
$$[E_\alpha, E_{-\alpha}] = \alpha \cdot H$$
where $\Delta_R$ is the set of roots. We choose to take $E_\alpha^\dagger = E_{-\alpha}$. We take a basis for the representation of the form: $P_{(\lambda, j, m)}$, where $\lambda$ is a weight and $(j, m)$ are the $su(2)$ quantum numbers corresponding to a decomposition of the representation in terms of representations of the $su(2)$-subalgebra generated by $\{(\alpha_1 + \alpha_2) \cdot H, E_{\pm(\alpha_1+\alpha_2)}\}$. Define:

$$P_1 = \alpha_1 \cdot P = \frac{1}{2}(x P_{(0,0,0)} + y P_{(0,1,0)})$$

$$P_2 = \alpha_2 \cdot P = \frac{1}{2}(x P_{(0,0,0)} - y P_{(0,1,0)})$$

$$P_{\pm(\alpha_1+\alpha_2)} = \pm \frac{1}{2} x [E_{\pm(\alpha_1+\alpha_2)}, P_{(0,1,0)}]$$

$$P_{\pm\alpha_i} = \pm \frac{1}{2} [E_{\pm\alpha_i}, P_i]$$

where $x$ and $y$ are constants to be defined later. It is easy to show the following relations:

$$[E_\alpha, P_{-\alpha}] + [E_{-\alpha}, P_\alpha] = 0$$

$$[E_{\pm\alpha_i}, P_i] + [H, P_{\pm\alpha_i}] = 0$$

$$[E_{\pm(\alpha_1+\alpha_2)}, (\alpha_1 + \alpha_2) \cdot P] + [(\alpha_1 + \alpha_2) \cdot H, P_{\pm(\alpha_1+\alpha_2)}] = 0$$

$$[E_{\pm(\alpha_1+\alpha_2)}, (\alpha_1 - \alpha_2) \cdot P] + [(\alpha_1 - \alpha_2) \cdot H, P_{\pm(\alpha_1+\alpha_2)}] = 0$$

where the last relation is a consequence of the fact that $(\alpha_1 - \alpha_2) \cdot P$ is an $su(2)$-singlet.

Using some algebra, it is possible to show that

$$[E_{\alpha_1+\alpha_2}, [E_{\alpha_1+\alpha_2}, P_{-\alpha_1}] + [E_{-\alpha_1}, P_{\alpha_1+\alpha_2}]] = 0,$$

which implies

$$[E_{\alpha_1+\alpha_2}, P_{-\alpha_1}] + [E_{-\alpha_1}, P_{\alpha_1+\alpha_2}] \propto P_{(\alpha_2, \frac{1}{2}, \frac{1}{2})}.$$ (4.65)

Inserting the definitions (4.62) into this equation, we find that the left hand side is a sum of two terms proportional to respectively $x$ and $y$, and we can therefore choose these constants in such a way that

$$[E_{\alpha_1+\alpha_2}, P_{-\alpha_1}] + [E_{-\alpha_1}, P_{\alpha_1+\alpha_2}] = 0$$

(4.66)

Furthermore, we find that

$$0 = [E_{-(\alpha_1+\alpha_2)}, [E_{\alpha_1+\alpha_2}, P_{-\alpha_1}] + [E_{-\alpha_1}, P_{\alpha_1+\alpha_2}]]$$

$$= [E_{-(\alpha_1+\alpha_2)}, P_{\alpha_2}] + [E_{\alpha_2}, P_{-(\alpha_1+\alpha_2)}]$$

(4.67)

These two relations, together with $E_\alpha^\dagger = E_{-\alpha}$, show that

$$[E_{\pm(\alpha_1+\alpha_2)}, P_{\pm\alpha_i}] + [E_{\mp\alpha_i}, P_{\pm(\alpha_1+\alpha_2)}] = 0$$

(4.68)

The generators $P_\alpha$ and $P_i$ are the subset of generators that we where looking for. Indeed, writing $A_\mu$ in the form $A_\mu = (A^\alpha)_\mu E_\alpha + (A^i)_\mu H_i$, we define:

$$(\tilde{B}_\lambda)_\mu = (A^\alpha)_\mu P_\alpha + (A^i)_\mu P_i$$

(4.69)
Using the commutation relations (4.63) and (4.68) we find that the equations

\[ D_\mu \tilde{E}^\mu = 0, \quad F_{\mu\nu} = 0 \] (4.70)

are equivalent to the equations of motion of the chiral model, in addition to the constraints

\[
\begin{align*}
(A^1)_\mu (A^{\pm \alpha_2})^\mu &= 0 \\
(A^2)_\mu (A^{\pm \alpha_1})^\mu &= 0 \\
(A^{\alpha_1})_\mu (A^{\alpha_2})^\mu &= 0 \\
(A^{\alpha})_\mu (A^{\beta})^\mu &= 0 \quad \text{for} \ (\alpha + \beta) \in \Delta_w \setminus \Delta_R
\end{align*}
\] (4.71)

where \( \Delta_w \) is the set of weights. We see that the constraints are independent of the choice of highest weight \( \Lambda \), and so this set of constraints defines an integrable submodel of the chiral \( SU(3) \) model, in which the zero curvature conditions give rise to an infinite number of conserved currents.

5 Conclusions

Our results confirm the validity, and illustrate the simplicity, of the new higher dimensional zero curvature approach to investigate integrability and construct Lorentz invariant models with infinitely many conserved charges. This is the main conclusion here, in addition to the particular interest of the cases considered; the Principal chiral model and the \( O(3) \) model has both been extensively studied.

In order to compare our results with the results available in the literature, we have in this paper focused on 2+1 dimensions. Notice, however, that the dimensionality of space-time is never used in our construction, and in fact all the results in this paper are immediately applicable in any number of dimensions.

Integrability in two dimensions is quite well understood at present, but it has many peculiar features which cannot be transferred to higher dimensions. It is therefore of great importance to improve our understanding of the nature of integrability beyond two dimensions. For this, it will be very useful to have a better understanding of the solutions of these models, including time-dependent solutions which can not be obtained by boosting static solutions. A general analysis of these solutions, as well as a study of several interesting theories in 3+1 dimensions, is in progress.

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