GUARANTEED CONSTRAINT SATISFACTION IN KOOPMAN-BASED OPTIMAL CONTROL

MANUEL SCHALLER\textsuperscript{1}, KARL WORTHMANN\textsuperscript{1} FRIEDRICH PHILIPP\textsuperscript{1}, SEBASTIAN PEITZ\textsuperscript{2} AND FELIKS NÜSKE\textsuperscript{2,3}

Abstract. We present an approach for guaranteed constraint satisfaction by means of data-based optimal control, where the model is unknown and has to be obtained from measurement data. To this end, we utilize the Koopman framework and an eDMD-based bilinear surrogate modeling approach for control systems to show an error bound on predicted observables, i.e., functions of the state. This result is then applied to the constraints of the optimal control problem to show that satisfaction of tightened constraints in the purely data-based surrogate model implies constraint satisfaction for the original system.

Keywords. Approximation error, control of constrained systems, data-based control, eDMD, finite-data error quantization, Koopman operator, MPC, nonlinear predictive control

Notation: Let $\mathbb{N}_0$ and $\mathbb{R}$ denote the natural (including zero) and the real numbers, respectively. Further, let $C^k([0,T],\mathbb{R})$, $k \in \mathbb{N}_0$, and $L^\infty([0,T],\mathbb{R})$ be the spaces of $k$-times continuously differentiable and of Lebesgue-measurable, essentially bounded functions, respectively.

1. Introduction and outline

Let the system dynamics be governed by the nonlinear control-affine differential equation
\begin{equation}
\dot{x}(t) = f(x(t)) + \sum_{i=1}^{n_c} g_i(x(t))u_i(t)
\end{equation}
with initial condition $x(0) = x^0$, locally Lipschitz-continuous vector fields $f, g_1, \ldots, g_{n_c} : \mathbb{R}^n \to \mathbb{R}^n$ and control function $u \in L^\infty([0,T],\mathbb{R}^{n_c})$ such that existence and uniqueness is guaranteed on some maximal interval of existence. Moreover, we impose control constraints, i.e., $u(t) \in \mathbb{U}$ for some compact, nonempty set $\mathbb{U} \subset \mathbb{R}^{n_c}$. Further, let state constraints represented by the inequality constraints $h_j(x) \leq 0$, $j \in \{1, \ldots, p\}$, with functions $h_1, \ldots, h_p \in C^2(\mathbb{R}^n,\mathbb{R})$ and a (finite) prediction/optimization horizon $T \in (0, \infty)$ be given.

\textsuperscript{1}Technische Universität Ilmenau, Institute of Mathematics, Germany (e-mail: {friedrich.philipp,manuel.schaller,karl.worthmann}@tu-ilmenau.de).
\textsuperscript{2}Paderborn University, Department of Computer Science, Germany, (e-mail: sebastian.peitz@upb.de).
\textsuperscript{3}Max Planck Institute for Dynamics of Complex Technical Systems, Magdeburg, Germany, (e-mail: nueske@mpi-magdeburg.mpg.de).

Acknowledgments: F. Philipp was funded by the Carl Zeiss Foundation within the project DeepTurb—Deep Learning in und von Turbulenz. M. Schaller was partially funded by the German Research Foundation (DFG; grant WO 2056/7-1, project number 430154635). K. Worthmann gratefully acknowledges funding by the German Research Foundation (DFG; grant WO 2056/6-1, project number 406141926).
For initial value $x^0 \in \mathbb{R}^n$, a control function $u$ is said to be \textit{admissible} if and only if $u \in U_T(x^0)$, where

$$U_T(x^0) \triangleq \left\{ u \in L^\infty([0,T], \mathbb{R}^n_c) \mid \exists! x(\cdot; x^0, u) \text{ on } [0,T] \right\}.$$ 

Then, for continuous stage cost $\ell : \mathbb{R}^n \times \mathbb{R}^n_c \rightarrow \mathbb{R}$ suitably encoding some control objective, we might consider the Optimal Control Problem

$$(\text{OCP}) \quad \text{Minimize}_{u \in U_T(x^0)} \int_0^T \ell(x^0, u(t)) \, dt$$

subject to the system dynamics (1), the initial condition $x(0; x^0, u) = x^0$ and, for $j \in \{1, \ldots, p\}$, the state constraints

$$h_j(x(t; x^0, u)) \leq 0 \quad \forall t \in [0,T].$$

We call a pair $(u, x(\cdot; x^0, u))$ \textit{feasible} if $u \in U_T(x^0)$ and the state constraints (2) are satisfied for all $j \in \{1, \ldots, p\}$.

Optimal control problems are widely applied in practice to determine optimal solutions to mathematically-described decision problems, see, e.g., the textbook [25] and [36, Section 8] for numerical solution methods. Moreover, OCPs also play a predominant role in optimization-based control techniques like Model Predictive Control where the (infinite-dimensional) optimization Problem (OCP) on an infinite-time horizon, i.e., $T = \infty$, is approximately solved by solving (OCP) at each instant with the current state as initial value, see, e.g., the monographs [36, 6] and [10] w.r.t. MPC for continuous-time systems.

While optimal control based on models derived from first principles is nowadays well established, data-driven control design is becoming more and more popular – in particular, if only measurement data is available. However, \textit{guaranteed} satisfaction of the state constraints (2) for all $i \in \{1, \ldots, p\}$ remains challenging. In particular, there is lack in rigorous error estimates depending on the amount of employed data points.

1.1. \textbf{Constrained data-based control with guarantees.} In this paper, we propose an approach based on recently-derived error estimates [32] for control-affine systems governed by ordinary (and/or stochastic) differential equations to successfully accomplish this task. To this end, we proceed as follows.

Firstly, we briefly recap the Koopman framework in the introductory paragraph of Section 2 to reformulate the state constraints (2), $i \in \{1, \ldots, p\}$, using the infinite-dimensional Koopman operator $\mathcal{K}_u^t$, $t \in [0,T]$. This yields the equality

$$(\mathcal{K}_u^t h_j)(x^0) = h_j(x(t; x^0, u))$$

and, thus, allows to evaluate the imposed state constraints as illustrated in Figure 1. The constraint function $h_j$ plays the role of a so-called \textit{observable}.

Secondly, in Subsection 2.1 we recall the extended Dynamic Mode Decomposition (eDMD) – a widely applied numerical scheme – to derive a data-based finite-dimensional approximation $\tilde{\mathcal{K}}_u^t$ of the Koopman semigroup $\mathcal{K}_u^t$ governing the dynamics of the surrogate observable

$$\tilde{h}_j(t) = \tilde{h}_j(t; x^0, u) := (\tilde{\mathcal{K}}_u^t h_j)(x^0)$$

to approximate $h_j(x(t; x^0, u)) = (\mathcal{K}_u^t h_j)(x^0)$. Here, we make two errors:
GUARANTEED CONSTRAINT SATISFACTION IN KOOPMAN-BASED OPTIMAL CONTROL

Initial value $x^0 \in X$ \hspace{1cm} ODE-flow \hspace{1cm} $x(t; x^0, u)$ \hspace{1cm} eval. \hspace{0.5cm} $h_i(x(t; x^0, u))$

Observable $h_i \in L^2(X)$ \hspace{1cm} Koopman-flow \hspace{1cm} $\mathcal{K}^t u h_i$ \hspace{1cm} eval. \hspace{0.5cm} $(\mathcal{K}^t u h_i)(x^0)$

**Figure 1.** Schematic sketch of the Koopman framework.

- One for the projection on the finite dictionary, which can be estimated, e.g., based on techniques well-known for finite-element methods, see, e.g., [4, 35]. This error decays with increasing size of the dictionary.
- The other error can be estimated for uncontrolled ordinary differential equations depending on the number $m$ of the identically-and-independently distributed (i.i.d.) data points using [39]. The respective error estimates are generalized to the control setting in [32] using ideas from [33] such that a bilinear surrogate model is obtained. This error decays with an increasing number of data points.

If the data is drawn i.i.d. and hence subject to randomness, it is clear that the approximation error can only be assessed via probabilistic estimates, that is, for a tolerance $\varepsilon > 0$ and a probability level $\delta \in (0, 1)$ there is a minimal amount of data such that

$$P \left( |h_j(x(t; x^0, u)) - \tilde{h}_j(t; x^0, u)| \leq \varepsilon \right) \geq 1 - \delta.$$  

In other words, if we compute a control $u$ such that the propagated observable along [4] satisfies the tightened constraints

$$\tilde{h}_j(t; x^0, u) \leq -\varepsilon $$

then by means of the above error estimate, the state constraint [2] is ensured with probability at least $1 - \delta$. Hence, feasibility of the pair $(u, x(t; x^0, u))$ is guaranteed with probability $1 - \delta$. Let us highlight that, contrary to the popular DMD with control approach proposed by [21], which yields linear surrogate models of the form $Ax + Bu$, i.e., the control enters linearly, numerical simulation studies indicate that bilinear surrogate models are better suited if control and state are coupled, see Subsection 3.3 and [32, Section 4] and also the preliminary studies presented in [14, 33, 5]. Another key feature of the bilinear approach is that the state-space dimension is not augmented by the number of control inputs, which counteracts the curse of dimensionality in comparison to the more widespread approach introduced in [21].

Furthermore, the idea of using the constraint function as an observable is inspired by that pursued in [29, Section 2.5] and not considered as the genuine contribution; that is rather the combination of this trick with the recently-derived error estimates to ensure satisfaction of the state constraints.

1.2. Chance constraints, optimal control, and MPC. The concept of constraint satisfaction with a certain probability as imposed by the inequality constraint [5] is often referred to as chance constraints [8], see also the more recent textbook [1] and the references therein. The usage of chance constraints in optimization is well-established

---

1In an ergodic setting, e.g., for stochastic differential equations, we refer to [32, Section 2].

2The authors choose a Lyapunov function as observable to prove a stability result of the learned system assuming a quantitative error estimate and a particular robustness.
in economics \cite{9}, has attracted considerable interest \cite{13} and is also applied in optimal path planning \cite{2} MPC, see, e.g., \cite{26}, but also more recent work on stochastic MPC by \cite{7, 30, 12}.

In this paper, we approach the problem from a different perspective. In the cited references, chance constraints were employed to properly take uncertainty into account. A key feature is that a violation has to be acceptable as long as it doesn’t occur too often, e.g., rotations of a wind turbine at high speed as considered in \cite{23}. Here, chance constraints are motivated by the error estimates for data-based surrogate modeling of nonlinear systems, which are naturally probabilistic.

2. BILINEAR eDMD-BASED APPROXIMATION OF CONTROL SYSTEMS

Instead of first propagating the state dynamics forward in time and, then, evaluating the state constraint \( h_j \) for some \( j \in \{1, \ldots, p\} \), the Koopman framework allows to propagate the observable \( h_j \) forward in time using the Koopman operator \( \mathcal{K}_u^t \) or, equivalently, the non-autonomous generator \( L^u \) – both depending on the chosen control function \( u \) – and, then, evaluate it such that the corresponding Koopman operator satisfies the equality \eqref{eq:koopman} and, thus, allows to evaluate the imposed state constraints, see Figure \ref{fig:koopman}

In this part, we will present eDMD as a popular approach to obtain data-based approximations of the unknown Koopman operator \( \mathcal{K}_u^t \) and its generator \( L^u \).

2.1. eDMD for autonomous systems. Extended Dynamic Mode Decomposition is a very powerful data-based framework to describe the dynamics of observable functions by means of the corresponding Koopman theory \cite{20}, see \cite{6, 31, 37}. Whereas most approximation results are of asymptotic nature, i.e., in the infinite data limit, recently the preprints \cite{39} and \cite{24} provided quantitative bounds for autonomous systems.

In this part we introduce the data-driven finite-dimensional approximation of the Koopman generator and operator for autonomous systems, i.e., setting \( u \equiv \bar{u} \in U \) in cf. \eqref{eq:koopman}, by means of eDMD, see, e.g., \cite{35, 15}. To this end, we assume that

\[ X = \{ x \in \mathbb{R}^d : h_i(x) \leq 0 \text{ for all } i = 1, \ldots, p \} \]

is compact and consider the ODE with state \( x(t) \in X \)

\[ \dot{x}(t) = f(x(t)), \]

where \( f : X \to \mathbb{R}^d \) is locally Lipschitz continuous. For initial datum \( x^0 \), the Koopman operator semigroup acting on bounded measurable functions \( \varphi \in L^2(X) \) is defined by

\[ (\mathcal{K}^t \varphi)(x^0) = \varphi(x(t)) \]

and the generator \( L : D(L) \subset L^2(X) \to L^2(X) \) is deduced via the derivative of each orbit map at zero, i.e.,

\[ L \varphi := \lim_{t \to 0} \frac{\mathcal{K}^t \varphi - \varphi}{t} \]

and hence, for \( h \in L^2(X), h = \mathcal{K}^t \varphi \in L^2(X) \) solves the Cauchy problem

\[ \dot{h} = Lh, \quad h(0) = \varphi. \]

In order to define the Koopman operator semigroup for all times \( t \geq 0 \), one usually assumes invariance of the compact set \( X \), which is needed to render \( h = \mathcal{K}^t \varphi \in L^2(X) \) for all \( t \geq 0 \) and \( \varphi \in L^2(X) \). However, first, our approach presented below will approximate the generator, and thus we only need our state to be contained in \( X \) up to any arbitrary small time \( t > 0 \) to be able to define the generator as in \eqref{eq:koopman}. Then, in order to obtain
error estimates for arbitrary long time horizons when going to a control setting, we have to ensure that the state trajectories remain in the set \( X \) by means of our chosen control function. Besides a controlled forward-invariance of the set \( X \) this can be ensured by choosing an initial condition contained in a suitable sub-level set of the optimal value function of a respective optimal control problem, see, e.g., [3] or [11] for an illustrative application of such a technique in showing recursive stabil ity of Model Predictive Control (MPC) without stabilizing terminal constraints for discrete- and continuous-time systems, respectively.

For a fixed dictionary of linearly independent observables \( \psi_1, \ldots, \psi_N \in D(\mathcal{L}) \) we consider the finite dimensional subspace
\[
\mathcal{V} := \text{span}\{\{\psi_j\}_{j=1}^N\} \subset D(\mathcal{L})
\]
and denote by \( P_V \) denote the orthogonal projection onto \( V \). Then, we denote the Galerkin projection of the Koopman generator by
\[
\mathcal{L}_V := P_V \mathcal{L} |_{\mathcal{V}}.
\]

Along the lines of [19], we have the representation \( \mathcal{L}_V = C^{-1}A \), where \( C, A \in \mathbb{R}^{N \times N} \) with
\[
C_{i,j} = \langle \psi_i, \psi_j \rangle_{L^2(\mathcal{X})} \quad \text{and} \quad A_{i,j} = \langle \psi_i, \mathcal{L}\psi_j \rangle_{L^2(\mathcal{X})}.
\]
Consider data points \( x_0, \ldots, x_{m-1} \in \mathcal{X} \) and the matrices
\[
\Psi(X) := \left( \begin{array}{c} \psi_1(x_0) \\ \vdots \\ \psi_N(x_0) \\ \psi_1(x_{m-1}) \\ \vdots \\ \psi_N(x_{m-1}) \end{array} \right),
\]
\[
\mathcal{L}\Psi(X) := \left( \begin{array}{c} (\mathcal{L}\psi_1)(x_0) \\ \vdots \\ (\mathcal{L}\psi_N)(x_0) \\ (\mathcal{L}\psi_1)(x_{m-1}) \\ \vdots \\ (\mathcal{L}\psi_N)(x_{m-1}) \end{array} \right).
\]
Then, defining matrices \( \tilde{C}_m, \tilde{A}_m \in \mathbb{R}^{N \times N} \) by
\[
\tilde{C}_m = \frac{1}{m} \Psi(X) \Psi(X)^\top \quad \text{and} \quad \tilde{A}_m = \frac{1}{m} \Psi(X) \mathcal{L}\Psi(X)^\top,
\]
an empirical, i.e., purely data-based estimator for the Galerkin projection \( \mathcal{L}_V \) is given by
\[
\tilde{\mathcal{L}}_m = \tilde{C}_m^{-1} \tilde{A}_m.
\]

2.2. Building a bilinear surrogate control system. In the context of data-based approximation of control systems, a lot of research has been invested over the past years, beginning with the popular DMD with control [34], which was later used in Model Predictive Control (MPC), see, e.g., [21]. Another popular method is to use a coordinate transformation into Koopman eigenfunctions [17] or a component-wise Taylor series expansion [28]. In [27], the prediction error of the method proposed in [34] was estimated using the convergence result of [22]. However, the result is of purely asymptotic nature, i.e., it does not state a convergence rate in terms of data points.

We briefly sketch the main steps of the bilinear surrogate modeling approach presented in [33], for which a finite-data error estimate was given in [32]. Considering a control \( u \in L^\infty([0, T], \mathbb{R}^{n_c}) \), it turns out that by control affinity of the system, also the Koopman generators are control affine, cf. [33]. Thus, setting
\[
\mathcal{L}^u(t) = \mathcal{L}^0 + \sum_{i=0}^{n_c} u_i(t) (\mathcal{L}^{c_i} - \mathcal{L}^0),
\]
where $\mathcal{L}^\bar{u}$ is the Koopman generator for the autonomous system with constant control $\bar{u} \in \{0, e_1, \ldots, e_{n_c}\}$, we can describe the time evolution of an observable function $h$ via
\[
\dot{h} = \mathcal{L}^\bar{u}(t)h.
\]
In view of (10), it is intuitively clear that we can quantitatively estimate the error for approximations of the controlled dynamics if we have access to quantitative error bounds for approximations of the autonomous generators $\mathcal{L}^{e_i}$, $i = 0, \ldots, n_c$, where we set $e_0 := 0$ for brevity of notation.

Analogously to Subsection 2.1 for a given control $u \in L^\infty([0, T]; \mathbb{R}^{n_c})$ we denote the projection of (10) onto a finite dictionary $V$ by
\[
\mathcal{L}^u_V(t) := \mathcal{L}^0_V + \sum_{i=1}^{n_c} u_i(t) (\mathcal{L}^i_V - \mathcal{L}^0_V),
\]
and the corresponding approximation by means of eDMD using $m$ data points by
\[
\tilde{\mathcal{L}}^u_m(t) := \tilde{\mathcal{L}}^0_m + \sum_{i=1}^{n_c} u_i(t) (\tilde{\mathcal{L}}^i_m - \tilde{\mathcal{L}}^0_m).
\]

Example 1. We briefly present a numerical example with a Duffing oscillator, cf. [32, Section 4.2.1] for more details, using the bilinear surrogate model approach to showcase its superior performance to the standard approach of eDMDc [34, 21], whenever state and control are coupled. Consider the dynamics
\[
\frac{d}{dt} x = \begin{pmatrix} x_2 \\ -\delta x_2 - \alpha x_1 - 2\beta x_1^3 u \end{pmatrix}, \quad x(0) = x^0,
\]
where we set $\alpha = -1, \beta = 1, \delta = 0$ and the initial state $x^0 = (1, 1)^T$. As the dictionary $\{\psi_j\}_{j=1}^N$, we choose monomials with maximal degree five. Figure 2 shows the prediction accuracy for $m = 100$ and the excellent agreement is observed for the bilinear surrogate model. In particular the relative error is below 0.1 percent for almost 3 seconds, whereas the eDMDc approach has a large error of approximately 10 percent from the start and becomes unstable within the first second.

3. Finite-data error bounds Koopman-based control

In our work [32], we provided first quantitative error bounds for approximating control systems by means of bilinear eDMD-based surrogate modeling of control systems as described in Subsection 2.2. Our results were formulated for the broad class of Stochastic Differential Equations (SDEs) from which, the ODE-dynamics (11) can be obtained as a particular case. In this context, we also obtained new results considering either i.i.d. sampling or ergodic sampling. In the following, we sketch the main results of our analysis for ODE-systems subject to i.i.d. sampling and briefly comment on the available extensions in Subsection 3.3.

3.1. Existing results for i.i.d. sampling of ODE systems. Thus, we now assume that we have sampled data points for each autonomous system with constant control $e_i$, $i = 0, \ldots, n_c$, and formed the respective data-based estimators of the generators. Now we state the main assumption that is central to obtain a quantitative error bound.

Assumption 2. Assume that the data obtained for each autonomous system with constant control $u \equiv e_i$, $i = 0, \ldots, n_c$, is sampled i.i.d. from the Lebesgue measure and contained in the compact subset $X$. 
Under this assumption we can provide an error bound on the individual generators corresponding to the autonomous systems.

**Theorem 3.** Let Assumption 2 hold. Then, for any desired error bound $\varepsilon > 0$ and probabilistic tolerance $\delta \in (0, 1)$, there is a number of data points $m_0$ such that for any $m \geq m_0$, we have the estimate

$$P\left( \|L^{e_i}_{V} - \tilde{L}^{e_i}_{m}\|_F \leq \varepsilon \right) \geq 1 - \delta.$$

for all $i = 0, \ldots, n_c$.

This can now be put together as an error bound for the control system, exploiting control affinity.

**Corollary 4.** Let Assumption 2 hold. Then, for any desired error bound $\tilde{\varepsilon} > 0$ and probabilistic tolerance $\tilde{\delta} \in (0, 1)$, prediction horizon $T > 0$, and control function $u \in L^\infty(0, T; \mathbb{R}^{n_c})$ there is a number of data points $m_0$ such that for any $m \geq m_0$, we have the estimate

$$\operatorname{ess inf}_{t \in [0, T]} P\left( \|L^u_{V}(t) - \tilde{L}^u_{m}(t)\|_F \leq \tilde{\varepsilon} \right) \geq 1 - \tilde{\delta}.$$

Having a bound on the nonautonomous generator at hand, we can derive a bound on resulting trajectories of observables by means of Gronwall’s inequality.

**Corollary 5.** Let Assumption 2 hold. Let $T, \varepsilon > 0$ and $\delta \in (0, 1)$, $z^0 \in \mathbb{R}^N$ and $u \in L^\infty(0, T; \mathbb{R}^{n_c})$ such that the state response along the dynamics \( (1) \) is contained in $X$. Then there is a number of data points $m_0$ such that for any $m \geq m_0$, the solutions $z, \tilde{z}$ of

\[
\begin{align*}
\dot{z}(t) &= L^u_{V}(t)z \\
\dot{\tilde{z}}(t) &= \tilde{L}^u_{m}(t)\tilde{z}
\end{align*}
\]

\[
\begin{align*}
z(0) &= z^0 \\
\tilde{z}(0) &= \tilde{z}^0
\end{align*}
\]
satisfy
\[
\min_{t \in [0,T]} \mathbb{P}\left( \| z(t) - \tilde{z}(t) \|_2 \leq \varepsilon \right) \geq 1 - \delta.
\]

3.2. Guaranteed constraint satisfaction. If the dictionary consisting of the constraint functionals \( V = \{ h_i \}_{i=1}^p \) form a Koopman invariant subspace, the previous method directly yields an estimate for the error in the constraint functionals.

If this is not the case, one further has to analyze the error resulting from projection onto the dictionary (3). To this end, we choose a dictionary that consists of our constraint functionals \( h_i, \ i = 1, \ldots, p \), as introduced in Section 11 and is further enriched by finite elements, i.e.,
\[
V_{\Delta x} = \text{span}\{ \{ h_i \}_{i=1}^p, \{ \psi_i \}_{i=1}^l \}
\]
where \( \psi_i, i = 1, \ldots, l \), are the usual linear hat functions defined on a regular uniform triangulation of \( X \) with meshsize \( \Delta x > 0 \) (e.g., the incircle diameter of each cell) and nodes \( x_j, j = 1, \ldots, l \), i.e., we have \( \psi_i(x_j) = \delta_{ij} \), where the latter is the usual Kronecker symbol. As usual in finite-element theory, we assume that \( X \) has a locally Lipschitz boundary. Note that the size of the dictionary is given by \( \text{dim} V_{\Delta x} = \Delta x^d + p \). For details on finite elements, we refer to, e.g., (4) \cite{35}.

In the following, we will not explicitly use that we included the state constraint functions \( h_i \) in the dictionary. However, including them has the advantage that in case \( V_{\Delta x} \) is a Koopman invariant subspace, the projection error vanishes and we directly obtain estimates on the evaluated constraints in terms of the data.

We now provide a novel approximation result that does not only consider the projected system, but gives an estimate for the full error, similar to the one obtained in \cite{39} Proposition 5.1] for autonomous systems.

**Theorem 6.** Let \( h \in C^2(X, \mathbb{R}) \), a probabilistic tolerance \( \varepsilon > 0 \), a probability level \( \delta \in (0,1) \) and a control \( u \in L^\infty(0,T; \mathbb{R}^{nu}) \) be given such that the state response of (1) is contained in \( X \). Then, there is a mesh size \( \Delta x > 0 \) and a minimal amount of data \( m \in \mathbb{N} \), such that the data-based prediction
\[
\frac{d}{dt} \tilde{h}_{m,\Delta x}(t) = \mathcal{L}^h_{\Delta x}(t) \tilde{h}_{m,\Delta x}(t)
\]
with initial state \( \tilde{h}_{m,\Delta x}(0) = P_{V_{\Delta x}} h \) satisfies
\[
\mathbb{P}\left( \| h(x(t;x^0,u)) - \tilde{h}_{m,\Delta x}(x^0) \| \leq \varepsilon \right) \geq 1 - \delta.
\]

**Proof.** Denote by \( h_{\Delta x} \) the solution of
\[
h_{\Delta x} = \mathcal{L}^h_{\Delta x}(t) h_{\Delta x}
\]
with initial condition \( h_{\Delta x}(0) = P_{V_{\Delta x}} h \), i.e., the Galerkin projection of the Koopman dynamics onto the finite element enriched space of observables \( V_{\Delta x} \). Using finite element convergence results for transport equations \cite{35} Section 14.3], we obtain that \( |h_{\Delta x}(x^0) - \tilde{h}(x(t;x^0,u))| \leq c(h)\Delta x \). Hence, we conclude
\[
\| \tilde{h}_{m,\Delta x}(t) - h(x(t;x^0,u)) \| \leq \| \tilde{h}_{m,\Delta x}(t) - h_{\Delta x}(t) \| + \| h_{\Delta x}(t) - h(x(t;x^0,u)) \|
\leq c(h) \left( \frac{1}{\sqrt{m}} \Delta x^d + p + \Delta x \right),
\]
where the first bound follows from Corollary 5 resp. \cite{32} Corollary 18] and the second from finite element theory for transport equations \cite{35} Section 14.3].
This result can now be used to obtain constraint satisfaction.

**Corollary 7** (Chance constraint satisfaction). For any probabilistic tolerance \(\varepsilon > 0\), probability level \(\delta \in (0,1)\) and control \(u \in L^\infty(0,T;\mathbb{R}^n_c)\) such that the state response of (1) is contained in \(X\), there is a mesh size \(\Delta x = \mathcal{O}(\varepsilon)\) and a minimal amount of data \(m = \mathcal{O}(\varepsilon^{-d+2})\), such that if the data-based prediction

\[
\frac{d}{dt} \tilde{h}_{i,m,\Delta x}(t) = \tilde{L}_{\Psi_k}(t)\tilde{h}_{i,m,h}(t)
\]

with initial state \(\tilde{h}_{i,m,\Delta x}(0) = P_{\Delta,\Delta}h_i\) satisfies

\[
\tilde{h}_{i,m,\Delta x}(t) \leq -\varepsilon
\]

then we have that the state response of (1) satisfies

\[
h_i(x(t,x^0,u)) \leq 0
\]

with probability at least \(1 - \delta\).

**Proof.** The proof follows by applying Theorem 6 to each constraint function \(h_i\). \(\square\)

In the previous result we have the data requirements \(m = \mathcal{O}(\varepsilon^{-d+2})\). This effect when approximating functions on a \(d\)-dimensional domain is widely known as the curse of dimensionality, e.g., in numerical methods for the Hamilton-Jacobi-Bellman equation. We further refer the reader to the discussion in [39], where the readers compare the data efficiency of eDMD for system identification with other methods. We stress, that in particular due to this exponential scaling with the state dimension the bilinear approach has severe advantages in terms of data efficiency, as the state is not augmented by the control.

### 3.3. Extension to the SDE case and ergodic sampling.

In our work [32] we considered the more general case of nonlinear stochastic differential equations of the form

\[
(SDE) \quad dX_t = F(X_t)\,dt + \sigma(X_t)\,dW_t,
\]

where \(X_t \in X \subset \mathbb{R}^d\) is the state, \(F: X \rightarrow \mathbb{R}^d\) is the drift vector field, \(\sigma: X \rightarrow \mathbb{R}^{d \times d}\) is the diffusion matrix field, and \(W_t\) is a \(d\)-dimensional Brownian motion. For SDEs, the Koopman semigroup can then be defined by means of the expected value of observables along the flow and the functional analytic framework for analyzing the semigroup and its generator is very similar to the ODE-case.

A central result in our work [32] was an error estimate for data that is obtained from one trajectory and hence is not necessarily i.i.d. This result is of particular interest in applications, where sampling one long trajectory might be easier than obtaining i.i.d. samples. In order to still capture the structural properties of the dynamics by observing only one trajectory, we make use of the concept of ergodicity. This means that there is a a measure \(\mu\) with positive density that is invariant under the action of the Koopman semigroup, i.e.,

\[
\int_X K^t f\,d\mu = \int_X f\,d\mu
\]

for all bounded measurable functions \(f\) and all \(t \geq 0\). Using this invariant measure, assuming exponential stability of the Koopman semigroup on the orthogonal complement of the constant functions our main results Theorem 6 and Corollaries 4 resp. 5 also hold true for the case of ergodic sampling, that is, obtaining data points from one (possibly long) trajectory.
4. Conclusion and outlook

Motivated by data-based surrogate modeling for optimal control problems with state constraints, we derived quantitative error estimates for eDMD-approximations of control systems. In this context, we showed that constraints can be satisfied for the original, possibly unknown system, if a tightened constraint for the data-based surrogate model holds.

Future work considers the refinement of the results in an optimal control context, where as of now, we only guaranteed constraint satisfaction. Here, suboptimality estimates explicitly depending on both data and dictionary size have to be derived. Moreover, a sensitivity analysis of the OCP could reveal robustness of optimal solutions with respect to approximation errors, that can be further exploited by numerical techniques, cf. [16]. Moreover, the results in this work enable the development of data-driven MPC techniques using surrogate models.

References

[1] J. R. Birge and F. Louveaux. Introduction to stochastic programming. Springer Science & Business Media, 2011.
[2] L. Blackmore, M. Ono, and B. C. Williams. Chance-constrained optimal path planning with obstacles. *IEEE Transactions on Robotics*, 27(6):1080–1094, 2011.
[3] A. Boccia, L. Grüne, and K. Worthmann. Stability and feasibility of state constrained MPC without stabilizing terminal constraints. *Systems & Control Letters*, 72(8):14–21, 2014.
[4] D. Braess. *Finite Elements*. Cambridge University Press, 1997.
[5] D. Bruder, X. Fu, and R. Vasudevan. Advantages of bilinear Koopman realizations for the modeling and control of systems with unknown dynamics. *IEEE Robotics and Automation Letters*, 6(3):4369–4376, 2021.
[6] S. L. Brunton, M. Budišić, E. Kaiser, and J. N. Kutz. Modern Koopman theory for dynamical systems, 2021. Preprint available at: arXiv:2102.12086.
[7] M. Cannon, B. Kouvaritakis, S. V. Raković, and Q. Cheng. Stochastic tubes in model predictive control with probabilistic constraints. *IEEE Transactions on Automatic Control*, 56(1):194–200, 2010.
[8] A. Charnes and W. Cooper. Chance constraints and normal deviates. *Journal of the American statistical association*, 57(297):134–148, 1962.
[9] A. Charnes and W. W. Cooper. Chance-constrained programming. *Management science*, 6(1):73–79, 1959.
[10] J.-M. Coron, L. Grüne, and K. Worthmann. Model predictive control, cost controllability, and homogeneity. *SIAM Journal on Control and Optimization*, 58(5):2979–2996, 2020.
[11] W. Esterhuizen, K. Worthmann, and S. Streif. Recursive feasibility of continuous-time model predictive control without stabilising constraints. *IEEE Control Systems Letters*, 5(1):265–270, 2020.
[12] M. Farina, L. Giuliani, and R. Scattolini. Stochastic linear model predictive control with chance constraints—a review. *Journal of Process Control*, 44:53–67, 2016.
[13] A. Geletu, M. Klüppel, H. Zhang, and P. Li. Advances and applications of chance-constrained approaches to systems optimisation under uncertainty. *International Journal of Systems Science*, 44(7):1299–1322, 2013.
[14] D. Goswami and D. A. Paley. Global bilinearization and controllability of control-affine nonlinear systems: a Koopman spectral approach. In 2017 IEEE 56th Annual Conference on Decision and Control (CDC), pages 6107–6112, 2017.
[15] L. Grüne and J. Pannek. Nonlinear model predictive control. In *Nonlinear model predictive control*, pages 45–69. Springer, 2017.
[16] L. Grüne, M. Schaller, and A. Schiela. Efficient Model Predictive Control for parabolic PDEs with goal oriented error estimation. To appear in *SIAM Journal of Scientific Computing*. Preprint available at: arXiv:2007.14446, 2020.
[17] E. Kaiser, J. N. Kutz, and S. L. Brunton. Data-driven discovery of Koopman eigenfunctions for control. *Machine Learning: Science and Technology*, 2:035023, 2021.
[18] S. Klus, F. Nüske, P. Kolter, H. Wu, I. Kevrekidis, C. Schütte, and F. Noé. Data-Driven Model Reduction and Transfer Operator Approximation. *Journal of Nonlinear Science*, 28(3):985–1010, 2018.

[19] S. Klus, F. Nüske, S. Peitz, J.-H. Niemann, C. Clementi, and C. Schütte. Data-driven approximation of the Koopman generator: Model reduction, system identification, and control. *Physica D*, 406:132416, 2020.

[20] B. O. Koopman. Hamiltonian Systems and Transformations in Hilbert Space. *Proceedings of the National Academy of Sciences*, 17(5):315–318, 1931.

[21] M. Korda and I. Mezić. Linear predictors for nonlinear dynamical systems: Koopman operator meets model predictive control. *Automatica*, 93:149–160, 2018.

[22] M. Korda and I. Mezić. On Convergence of Extended Dynamic Mode Decomposition to the Koopman Operator. *Journal of Nonlinear Science*, 28(2):687–710, 2018.

[23] B. Kouvaritakis and M. Cannon. Stochastic model predictive control., 2015.

[24] A. J. Kurdila and P. Bobade. Koopman theory and linear approximation spaces, 2018. Preprint available at: arXiv:1811.10809.

[25] E. B. Lee and L. Markus. Foundations of optimal control theory. Technical report, Minnesota Univ Minneapolis Center For Control Sciences, 1967.

[26] P. Li, M. Wendt, and G. Wozny. A probabilistically constrained model predictive controller. *Automatica*, 38(7):1171–1176, 2002.

[27] Q. Lu, S. Shin, and V. M. Zavala. Characterizing the predictive accuracy of dynamic mode decomposition for data-driven control. *IFAC-PapersOnLine*, 53(2):11289–11294, 2020.

[28] G. Mamakoukas, M. L. Castano, X. Tan, and T. D. Murphey. Derivative-Based Koopman Operators for Real-Time Control of Robotic Systems. *IEEE Transactions on Robotics*, 2021.

[29] A. Mauroy, Y. Susuki, and I. Mezić. *Koopman operator in systems and control*. Springer, 2020.

[30] A. Mesbah, S. Streif, R. Findeisen, and R. D. Braatz. Stochastic nonlinear model predictive control with probabilistic constraints. In *2014 American control conference*, pages 2413–2419. IEEE, 2014.

[31] I. Mezić. Spectral Properties of Dynamical Systems, Model Reduction and Decompositions. *Nonlinear Dynamics*, 41:309–325, 2005.

[32] F. Nüske, S. Peitz, F. Philipp, M. Schaller, and K. Worthmann. Finite-data error bounds for koopman-based prediction and control, 2021. Preprint available at: arXiv:2108.07102.

[33] S. Peitz, S. E. Otto, and C. W. Rowley. Data-driven model predictive control using interpolated Koopman generators. *SIAM Journal on Applied Dynamical Systems*, 19(3):2162–2193, 2020.

[34] J. L. Proctor, S. L. Brunton, and J. N. Kutz. Dynamic mode decomposition with control. *SIAM Journal on Applied Dynamical Systems*, 15(1):142–161, 2016.

[35] A. Quarteroni and A. Valli. *Numerical approximation of partial differential equations*, volume 23. Springer Science & Business Media, 2008.

[36] J. B. Rawlings, D. Q. Mayne, and M. Diehl. *Model predictive control: theory, computation, and design*, volume 2. Nob Hill Publishing Madison, 2017.

[37] C. W. Rowley, I. Mezić, S. Bagheri, P. Schlatter, and D. S. Henningson. Spectral analysis of nonlinear flows. *Journal of Fluid Mechanics*, 641:115–127, 2009.

[38] M. O. Williams, I. G. Kevrekidis, and C. W. Rowley. A data-driven approximation of the Koopman operator: Extending dynamic mode decomposition. *Journal of Nonlinear Science*, 25(6):1307–1346, 2015.

[39] C. Zhang and E. Zuazua. A quantitative analysis of Koopman operator methods for system identification and predictions. Preprint available at: hal-03278445, 2021.