Entropy functionals for nonholonomic geometric flows, quasiperiodic Ricci solitons, and emergent gravity

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Abstract

We investigate gravity models emerging from nonholonomic (subjected to non-integrable constraints) Ricci flows. Considering generalizations of G. Perelman’s entropy functionals, relativistic geometric flow equations, nonholonomic Ricci soliton and equivalent (modified) Einstein equations are derived. There are studied nonholonomic configurations which allow explicit modeling of E. Verlinde type entropic scenarios for gravity and dark matter. Using the anholonomic frame deformation method, the systems of nonlinear partial differential equations for such geometric flow evolution and/or nonlinear dynamical and gravitational systems can be decoupled and integrated in general forms. We elaborate on stationary models of emergent gravity with quasi-periodic gravitational, matter fields and dark energy/matter structure. Such configurations can not be described thermodynamically using the concept of Bekenstein-Hawking entropy if there are not involved additionally any area-entropy, holographic or duality relations. We elaborate on nonholonomic deformations of the F- and W-entropy and formulation of relativistic thermodynamic models for more general classes of physically important solutions with quasiperiodic and pattern forming structure in modified gravity theories and general relativity.

1 Introduction

The Einstein equations can be derived in thermodynamic models with area-entropy for horizons of black holes, BH, and/or making certain assumptions on elastic properties of gravity [1,2,3]. This supports the idea and indication of the emergent nature of spacetime when gravity comes from the laws of BH thermodynamics [4,5,6,7]. In the past four decades, the Bekenstein–Hawking entropy and Hawking temperature played the central role for research in various directions of modern gravity, cosmology and astrophysics. Significant advances started with the explanation of the microscopic origin of Bekenstein-Hawking entropy in string theory [8]; with a subsequent development of the (anti) de Sitter, (A)dS black hole physics and/or cosmology, conformal field theories, CFTs, and AdS/CFT, correspondence [9]; and formulating quasilocal relations when analogous laws of the thermodynamics are defined for various sorts of 'apparent' horizons [10]. Latter, it was realized that similar formulas characterize quantum entanglement [11] and explain the connectivity of the classical spacetime [12]. Recent theoretical advances reveal a deep connection between the quantum information theory and ideas on emergence of spacetime and gravity, for instance, by deriving the (linearized) Einstein equations from general quantum information principles [13,14,15,16,17,18].

E. Verlinde conjectured [3,18] that gravity links to an entropic force as a spacetime elasticity when the gravitational interactions would result from the information regarding the positions of material bodies.
His models entail the hypothesis that gravity should be viewed as an emergent phenomenon. It joins a thermal gravity–treatment to ’t Hooft’s and many other authors’ efforts involving the holographic principle in particle physics, quantum gravity, cosmological inflation, cosmology, dark energy and dark matter physics etc. The literature on such directions of modern theoretical and mathematical physics and information theory is immense, see [19, 20, 21, 22, 23] and references therein. Nevertheless, this paper deals with none of these issues, though. We just show that well known and fundamental geometric and classical physics reasoning support E. Verlinde’s ideas on the origin of gravity as an effect of the entropic force. We argue also that this can be grounded and explained in a rigorous mathematical form using relativistic (and certain noncommutative, nonholonomic, supersymmetric etc.) generalizations of the theory of Ricci flow evolution [24, 25, 26, 27].

It should be noted here that E. Verlinde works [3, 18] and a covariant formulation due to S. Hossenfelder [28] were criticized by authors of [29, 30]. Perhaps, certain technical and theoretical inconsistencies are present in all these works which are consequences of the model dependence and attempts to explain observational data with periodic up-dates and new information, for instance, on dark matter physics. Nevertheless, we consider that the most important ideas and explicit constructions of models of emergent gravity and spacetime thermodynamics have a rigorous support from the theory of geometric flow evolution and geometric methods of constructing exact solutions for systems of nonlinear partial differential equations, PDEs.

The main goal of this work (see details, proofs, more examples and applications in a partner paper [31]) is to prove that E. Verlinde conjecture that gravity can be derived from an entropic force and related spacetime elasticity can be considered as a modified relativistic version of the Poincaré conjecture when the G. Perelman W-entropy can be used deriving both gravitational equations and their thermodynamic properties. For certain well-defined conditions on geometric flows of Riemannian and Kähler metrics, the Poincaré–Thurston conjecture was proven in a rigorous mathematical form [32], see also some early works [33, 34] and reviews of geometric results in [35, 36, 37]. There were not elaborated such rigorous analytic, geometric and topological tools, for instance, for pseudo-Riemannian spaces, thermodynamic geometry and almost Kähler manifolds which are important for applications in classical and quantum gravity and modern cosmology and astrophysics. These are tasks for future research in modern geometry and mathematical physics. Nevertheless, using the anholonomic frame deformation method, AFDM, see a review of results in [38] (on MGTs and applications in modern cosmology, see [29, 40, 41]), it is possible to construct explicit solutions for important nonlinear systems of PDEs describing the statistical thermodynamics and kinetics of spacetime geometric flows and emergent gravity. Such methods and results are well-defined mathematically. In a series of our works [24, 25, 26, 27] (see also references therein), we elaborated on applications in classical and quantum physics of certain geometric methods related to entropy type functionals.

2 Nonholonomic deformations of Perelman’s F- and W-functionals

Let us consider a (pseudo) Riemannian manifold $\mathbf{V}$ of dimension $n + m$, with $n, m \geq 2$ (for GR and four dimensional, 4-d, modifications, we can consider $\dim \mathbf{V} = 4 = 2 + 2$). Such a manifold is nonholonomic if it is endowed with a non integrable $(n + m)$ splitting of dimensions into conventional horizontal, h, and vertical, v, components defined by a Whitney sum $N : \mathbf{T V} = h \mathbf{V} \oplus v \mathbf{V}$, where $\mathbf{T V}$ is the tangent bundle on $\mathbf{V}$. In 4-d, we can consider a Lorentzian manifold with local pseudo-Euclidean signature $(+ + - -)$ for a metric field $g = (h g, v g)$. Any nonlinear connection, N-connection, structure $\mathbf{N}$ defines corresponding subclasses of N-adapted (co) frames which allows, for instance, nonholonomic diadic decompositions of geometric and physical objects. Such N-adapted decompositions can be computed for a corresponding set of coefficients $N_i^a$, when $\mathbf{N} = N_i^a(u) dx^i \otimes \partial_a$. Details on geometric constructions in abstract and coefficient forms in theories with nonholonomic (modified) spacetime geometry are given in [38, 31, 24, 25, 26, 27] and references therein.

1 For a local calculus on $\mathbf{V}$, we can parameterize the coordinates in the form: $u^a = (x^i, y^i)$, (in brief, $u = (x, y)$), where indices respectively take values of type $i, j, ... = 1, 2, ..., n$ and $a, b, ... = n + 1, n + 2, ..., n + m$. In this and the partner work
There are two important linear connections determined by the same metric structure:

\[
\mathbf{g} \rightarrow \begin{cases} 
\nabla : & \nabla \mathbf{g} = 0; \quad \nabla \mathbf{T} = 0, \\
\mathbf{D} : & \mathbf{D} \mathbf{g} = 0; \quad \mathbf{hT} = 0, \quad \nu \mathbf{T} = 0.
\end{cases}
\]

The Levi–Civita, LC, connection; we follow the Einstein convention on summation on "up-low" repeating indices and use boldface symbols for spaces and geometric objects adapted to a N-connection splitting.

In our previous works, we wrote \( \hat{\mathbf{D}} \) for the canonical d-connection in [31]. The small Greek indices take typical values 1, 2, ..., \( n + m \). In 4-d, we consider that \( u^4 = y^4 = t \) is the time like coordinate; we follow the Einstein convention on summation on "up-low" repeating indices and use boldface symbols for spaces and geometric objects adapted to a N-connection splitting.
where the condition $\int_{t_1}^{t_2} \int_{\Omega} (4\pi\tau)^{-3} e^{-f} \sqrt{g} \, d^4u = 1$ is imposed on the normalizing function $f(\tau, u)$. The difference from the original Grisha Perelman F- and W-functionals \[32\] introduced for the Ricci flows of 3-d Riemannian metrics (see details in monographs \[35, 35, 37\]) is that we study geometric flows of canonical geometric data $(g(\tau), N(\tau), D(\tau))$ for nonholonomic Lorentz manifolds and various generalizations for MGTs as in \[38, 24, 25, 26, 27\]. In formulas (2) and (3), we consider the gravitational Lagrangian $\mathcal{L} = F(sR)$ as a functional of the scalar curvature for $D$, or $\mathcal{L} = R[\nabla]$ for considering as particular cases models of geometric evolution of exact solutions in GR.

Functional of type $\mathcal{F}$ and $\mathcal{W}$ were considered in our works for deriving relativistic nonlinear flow evolution equations and encoding modified gravity analogs of the Hamilton equations studied in geometric analysis and topology \[34\]. It should be noted here similar geometric flow equations related to quantum renormalization group equations were considered in physical literature \[33\] some years before classical mathematical works due to R. Hamilton. The functional $\mathcal{W}$ transforms into the standard Perelman W-entropy \[32\] (being analogous to minus entropy) for non-relativistic holonomic flows of 3-d hypersurface Riemannian metrics. It was considered as an entropy type value for formulating a statistical thermodynamics model for Ricci flows. In this and partner \[31\] papers, we work with generalized entropy functionals determined by $F(sR) + \mathcal{L}$ and $D$, respectively, instead of the Riemannian $\mathcal{R}$ and $\nabla$ used in former mathematical works. In our approach, above F- and W–functionals characterize relativistic thermodynamic models with analogous nonlinear hydrodynamic flows of families of entropic values, metrics and generalized connections, encoding interactions of gravitational and matter fields as it is motivated in \[25, 26, 27\]. We can compute relativistic entropies (2) and (3) for any 3+1 splitting with 3-d closed hypersurface fibrations $\tilde{\tau}_i$. In general, it is possible to work with any class of normalizing functions $f(\tau, u)$ which can be fixed by certain constant values or conditions specifying some systems of nonlinear PDEs. In many cases, such a function is chosen in a non–explicit form which allows us to study non–normalized geometric flows but with nonholonomic constraints which allow general decoupling and integration of respective physically important systems of nonlinear PDEs. Such generic off-diagonal solutions can be constructed in explicit form \[31, 38, 24, 25, 26, 27\] which validates our nonholonomic geometric flow entropic approach, involving metrics with pseudo-Euclidean signature even analogs of Poincaré–Thurston conjecture have not been formulated and proven for the Lorentzian spacetimes.

3 Relativistic geometric evolution and (modified) Einstein equations

Following a N-adapted variational procedure, for instance, for the functional $\mathcal{F}(\tau)$ \[2\] (see details in \[25, 26, 27, 31\] being, in principle, similar rigorous mathematical proofs in \[32, 35, 36, 37\]), we obtain a system of nonlinear PDEs that generalize the R. Hamilton equations in order to nonholonomic geometric flow evolution of canonical data $(g = \{g_{\mu\nu} = [g_{ij}, g_{ab}]\}, N = \{N^a\}, D, \mathcal{L})$,

\[
\begin{align*}
\partial_\tau g_{ij} & = -2(R_{ij} - \tau \mathcal{Y}_{ij}); \\ \partial_\tau g_{ab} & = -2(R_{ab} - \tau \mathcal{Y}_{ab}); \\ R_{ia} & = R_{ai} = 0; R_{ij} = R_{ji}; R_{ab} = R_{ba}; \partial_\tau f = -\hat{D}f + |Df|^2 - sR + \tau \mathcal{Y}_{a}^a, \nonumber
\end{align*}
\]

where $\hat{D} = D^a D_a$ and $\tau \mathcal{Y}_{a}^a$ is defined below. The conditions $R_{ia} = 0$ and $R_{ai} = 0$ for the Ricci tensor $\text{Ric}[D] = \{R_{a\beta} = [R_{ij}, R_{ia}, R_{ai}, R_{ab}]\}$ are necessary if we want to keep the metric $g(\tau)$ to be symmetric under nonholonomic Ricci flow evolution. We note that similar variational and/or geometric methods allows to derive from $W(\tau)$ \[3\] another types nonlinear evolution equations which are equivalent to (4).

For self-similar point $\tau = \tau_0$ configurations when $\partial_\tau g_{\mu\nu} = 0$, with a specific choice of the normalizing geometric flow function $f$, the equations (4) transform into relativistic nonholonomic Ricci soliton equations which are equivalent to (modified) Einstein equations in (MGT) GR for corresponding definitions of $\tau \mathcal{Y}_{a}^a$. A class of MGTs and GR can be formulated as geometric models of entropic elasticity which is similar to the idea of emergent gravity put forward by E. Verlinde \[3, 18\].

\[2\text{in our case, we use } D \text{ instead of } \nabla \text{ and so-called N-adapted differential and partial derivatives]}

4
Let us study the conditions when entropic elastic scenarios can be derived from nonholonomic Ricci solitons. We introduce three important values determined by a conventional displacement vector field $u^\alpha$, cosmological constant $\Lambda$ and some constants $\alpha, \beta, \gamma$:

$$
\varepsilon_{\alpha \beta} = D_\alpha u_\beta - D_\beta u_\alpha - \text{the elastic strain tensor} ; \phi = u / \sqrt{|\Lambda|} - \text{a dimensionless scalar} ; \\
\chi = \alpha(D_\mu u^\nu)(D_\nu u^\mu) + \beta(D_\mu u_\nu)(D^\mu u^\nu) + \gamma(D_\mu u_\nu)(D^{\mu} u^{\nu}) - \text{a general kinetic term for } u^\mu,
$$

when short hands $u := \sqrt{|u_\alpha u^\alpha|}, \varepsilon = \varepsilon^\beta_\beta$ and $n^\alpha := u^\alpha / u$ are used. For (2) and (3), and respective (4), there are considered nonholonomic distributions when corresponding total, effective gravitational, usual matter, interaction and kinetic terms Lagrangians are postulated in the form $L^{tot} = g \mathcal{L} + m \mathcal{L} + \int t \mathcal{L} + \chi \mathcal{L}$ for

$$
L = M_P^2 F( s R), \quad \int t \mathcal{L} = -\sqrt{|\Lambda|} m T_{\mu \nu} u^\mu u^\nu / u, \quad \chi \mathcal{L} = M_P^2 |\Lambda| (\chi^{3/2} + |\Lambda||u|^{2z}).
$$

In these formulas, $M_P$ is the Plank gravitational mass; $z = 1$ if we search for compatibility with [28], or $z = 2$ if we search for a limit to the standard de Sitter space solution [29, 30]. The energy-momentum tensors considered in above formulas and/or derived from respective Lagrangians in (5) are computed using variations on $g^{\mu \nu}$, for instance, when $m T_{\mu \nu} := -\frac{2}{\sqrt{|\Lambda|}} \frac{\delta \sqrt{|\Lambda|} m \mathcal{L}}{\delta g^{\mu \nu}} = \frac{2}{\sqrt{|\Lambda|}} m \mathcal{L} + g_{\mu \nu} m \mathcal{L}$. For the full system, the effective energy-momentum tensor is computed for $F T_{\beta \gamma} = \left[ \frac{1}{2} (F - \frac{\partial F}{\partial s R}) g_{\beta \gamma} - (g_{\beta \gamma} D_\alpha D^\alpha - D_\beta D_\gamma) \frac{\partial F}{\partial s R} \right] (\frac{\partial F}{\partial s R})^{-1},$

when $\delta \mathcal{L} = \left[ \frac{1}{2} \delta g_{\mu \nu} m \mathcal{L} \right] = \left[ \frac{1}{2} \delta g_{\mu \nu} \mathcal{L} \right]$. The N-adapted

Choosing $F( s R) = s R$ and the Levi-Civita connection $D = \nabla$, we obtain respective formulas for $\int t \mathcal{L}$ and $\chi \mathcal{L}$ being similar to (10)-(13) in [29, 30]. In result, the generalized source splits into three components,

$$
\mathcal{L} = \mathcal{L}^{tot} = \mathcal{L}^{int} + \mathcal{L}^{\text{kin}} + \mathcal{L}^{\text{int}},
$$

where $\mathcal{L}$ is determined in standard form by the Newton gravitational constant $G$.

We conclude that an analogous emergent gravity in an E. Verlinde sense [3, 18, 28], can be constructed for Lagrange distributions (3) and respective sources (6) introduced as generating data for the nonholonomic Hamilton equations (4) and respective relativistic Ricci solitons. Such geometric flow evolution theories and their spacetime "elastic" properties are determined by the generalized W–entropy (4).

4 Entropic geometric flows to stationary and quasi-periodic structures

Let us consider effective sources which via $N$–adapted frames can be parameterized in the form

$$
\mathbf{Y}_\mu (\tau) = e^n_\mu (\tau) e^n_\nu (\tau) \left[ \mathcal{L}^{tot} \mathbf{Y}_{\mu \nu} (\tau) \right] = \frac{1}{2} \partial_\tau g_{\mu \nu} (\tau) \mathbf{Y}_\mu (\tau) + \frac{1}{2} \partial_\tau g_{\mu \nu} (\tau) \mathbf{Y}_\nu (\tau) + \frac{1}{2} \partial_\tau g_{\mu \nu} (\tau) \mathbf{Y}_{\mu \nu} (\tau),
$$

for families of vielbein transforms $e^\mu_\nu (\tau) = e^\mu_\nu (\tau, u)$ and their duals $e^n_\nu (\tau, u)$, when $e^\mu = e^\mu_\nu du^n$. The values $[ h \mathbf{Y}(\tau, x^k), \mathbf{Y}(\tau, x^k, y^3)]$ can be fixed as generating functions for (effective) matter sources imposing nonholonomic frame constraints on stationary distributions or cosmological dynamics of (effective) matter fields. Such effective sources allow us to construct in explicit form exact stationary solutions with Killing symmetry on $\partial_4 = \partial_t$ of the system of nonlinear PDEs (4) for families of metrics and $N$-connections parameterized for 4-d configurations in the form $g(\tau) = [g_t(\tau), g_a(\tau), g_b(\tau)]$. The N-adapted coefficients of such geometric data do not depend on variable $y^3$ and can be parameterized in the form $g_t(\tau) = e^\psi (\tau, x^k), \quad g_a(\tau) = h_a(\tau, x^k, y^3), \quad N^3 = w_i(\tau, x^k, y^3), \quad N^4 = n_i(\tau, x^k, y^3).$

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3 We use a system of notation similar to [38, 24, 25, 26, 27] which is different from [28, 29, 30].
Following the geometric procedure described in Tables 1 and 2 and appendix A to [38], see also details in [31], we decouple and integrate the generalized Hamilton equations in this form:

$$ds^2 = e^{\psi(\tau)}[(dx^1)^2 + (dx^2)^2] - \frac{(\partial_3 h_4(\tau))^2}{|\int dy^3 Y(\tau)(\partial_3 h_4(\tau))|} \frac{dy^3}{Y(\tau)} \frac{\partial_3 h_4(\tau)}{dy^3} [dx^i] + h_4(\tau)|dt + (1n_k(\tau) + 42n_k(\tau) \int dy^3 \frac{(\partial_3 h_4(\tau))^2}{|\int dy^3 Y(\tau)(\partial_3 h_4(\tau))|} (h_4(\tau))^{5/2}] dx^k].$$

(8)

In this formula, $1n_k(\tau) = 1n_k(\tau, x^i)$ and $2n_k(\tau) = 2n_k(\tau, x^i)$ are integration functions, $h_4(\tau) = h_4(\tau, x^i)$ can be taken as a generating function, and $\psi(\tau) = \psi(\tau, r, \theta)$ is a solution of the 2-d Poisson equation, $\partial_1^2 \psi + \partial_2^2 \psi = 2 \ h \ Y$. Such solutions are, in general, with nontrivial nonholonomically induced torsion which allows us to extract LC-configurations by imposing additional constraints.

Let us study how quasicrystalline structures and analogous dynamic phase fluid models can be elaborated using a generic flow evolution on parameter $\tau$. A quasicrystal, QC, structure can be defined by generating functions $\overline{\eta} = \overline{\eta}(x^i, y^3, \tau)$ defined as a solution of an evolution equation with conserved dynamics,

$$\frac{\partial \overline{\eta}}{\partial \tau} = b \hat{\Delta} \left[ \frac{\delta F}{\delta \overline{\eta}} \right] = -b \hat{\Delta}(\Theta \overline{\eta} + Q\overline{\eta}^2 - \overline{\eta}^3).$$

(9)

Such evolution is considered on 3-d spacelike hypersurface $\Xi_t$ when the canonically nonholonomically deformed hypersurface Laplace operator $b \hat{\Delta} := (b D)^2 = b^{ij} \hat{D}_i \hat{D}_j$, where indices run values $i, j, ..., 1, 2, 3$. This operator is a distortion of $b \Delta := (b \nabla)^2$ constructed in 3-d Riemannian geometry. The functional $F$ in (9) is an effective free energy $F[\overline{\eta}] = \int \left[ -\frac{1}{2b} \Theta \overline{\eta} - \frac{Q}{4} \overline{\eta}^3 + \frac{1}{b^4} \right] \sqrt{b} dx^1 dx^2 dy^3$, where $b = \det |b_{ij}|$, $\delta y^3 = e^3$ and the operators $\Theta$ and $Q$ are explained in [38]. This class of nonlinear interactions is stabilized by the cubic term and the second order resonant interactions are varied by setting an observable value of $Q$, for instance, for certain quasiperiodicity of cosmological structure. The average value $<\overline{\eta}>$ is conserved for any fixed $t$ and/or $\tau_0$. Fixing a constant $\tau_0$, we generate quasiperiodic Ricci solitons determined by an effective parameter $\overline{\eta}$ of the system. We can choose $<\overline{\eta}>|_{\tau = \tau_0} = 0$ when other values are accommodated by redefining the normalization function $f$ and operator $\Theta$ and $Q$.

Off-diagonal stationary metrics of type (5) define two classes of exact solutions of (1) if

$$h_4(\tau) = (\tau) = \overline{\eta}(x^i, y^3, \tau), \quad \text{for gravitational QC configurations};$$

$$Y(\tau) = Y[\overline{\eta}(|)] = Y[\overline{\eta}(x^i, y^3, \tau)], \quad \text{for elastic QC structures for DM}.$$ 

Fixing a QC source for $Y(\tau)$, parameterized in N-adapted form (7), as a functional of a solution (9), we prescribe a quasi-periodic evolution/dynamics for effective fields $\chi$ and/or $u$ in (5). This imposes QC sources $\nabla^m Y_{\mu \nu}$ and/or $\nabla^m \chi_{\mu \nu}$ in (6). Details on generating such pattern forming, space QC and/or time and space QC structures with applications in modern cosmology can be found in [38, 30, 31]. Relativistic evolution scenarios, with supersymmetric or noncommutative variables, Finsler like generalizations are studied in [24, 25, 26, 27] and references therein. In those works, there were used different types of effective Lagrangians and nonholonomic Perelman functionals with possible dependence on time like coordinates.

5 Perelman like entropies instead of the Bekenstein–Hawking entropy

Generic off-diagonal stationary solutions with QC for the geometric flow evolution and/or Ricci soliton, or (modified) Einstein equations of type (4) are not subjected, in general, to any surface-area, horizon, or duality conditions to such configurations. The thermodynamic properties of spacetimes under geometric evolution on $\tau$, or for a fixed $\tau_0$ for relativistic dynamic gravitational and (effective) matter field equations can not be characterized by the Bekenstein-Hawking entropy and temperature. For such new classes of stationary and
cosmological solutions in MGTs and GR, we suggested \([25, 26, 27]\) to elaborate on statistical and geometric thermodynamics models using the corresponding generalizations of the Perelman F- and W-entropy.

We introduce a 3+1 decomposition with local coordinates \(u^i = x^i = (x^1, x^2, x^3)\) and \(u^i = t\), which is additional to the 2+2 decomposition, when for a \([31, 25, 26, 27]\) \(g = [g_{ij}, N^a_i] = \{b_{ij}, b_3, \bar{N}\}\), where

\[
b_{ij} = \text{diag}(b_{ij}) = (b_i = g_i, b_3 = h_3 = -\frac{(\partial_3 h_4(\tau))^2}{\int dy^3 \bar{Y}(\tau)(\partial_3 h_4(\tau)) | h_4(\tau)}
\]

on a hypersurface \(\Xi_i\) and \(\bar{N}^2 = -h_4(\tau)\) is the lapse function. The N-connection coefficients are parameterized \(N_i^3 = w_i(\tau)\) and \(N^i_4 = n_i(\tau)\).

For stationary configurations, we can characterise the geometric flows by analogous thermodynamic systems on corresponding families of 3-d closed hypersurfaces \(\Xi_i\) using \(\bar{W} = \mathcal{W}(\tau)|\Xi_i\) which is constructed using respective 3-d projections with data \((b_{ij}, b_3)\) and \(\mathcal{D} := \mathcal{D}_{\Xi_i}\) nonholonomically deformed W-entropy \(\bar{W}\), see details in section 5.1 of \([25]\) and \([31]\). In such an approach, it is considered a standard partition function \(\bar{Z} := \exp \left\{ \int_{\Xi_i} M \sqrt{|b_{ij}|} d^3x [-\bar{f} + 3] \right\}\) for the conditions stated for definitions \([2]\) and \([3]\) and when \(\int_{\Xi_i} M \sqrt{|b_{ij}|} d^3x = 1\) for a hypersurface volume element \(\sqrt{|b_{ij}|} d^3x\) and \(M = (4\pi\tau)e^{-\bar{f}}\) determined by a normalization function \(\bar{f}\). This allows us to compute all necessary thermodynamical values for the hypersurface canonical connection \(\mathcal{D}\) and/or the Levi–Civita connection \(\mathcal{N}\) as it was constructed in \([22,35]\) for the dimension \(n = 3\). This way, a statistical model can be elaborated for any \(\bar{Z}\) associated to a \(Z = \int \exp(-\beta E) d\omega(E)\) for a canonical ensemble at temperature \(\beta^{-1} = \tau\) and measure taken for a density of states \(\omega(E)\).

For \(\mathcal{N} \to \mathcal{D}\) and GR derived from Ricci soliton configurations in \([3]\), we obtain

\[
\bar{\mathcal{E}} = -\tau^2 \int_{\Xi_i} M \sqrt{|b_{ij}|} d^3x (\mathcal{R} + |\mathcal{D} f|^2 - \frac{3}{\tau}) , \quad \bar{S} = -\int_{\Xi_i} M \sqrt{|b_{ij}|} d^3x \left[ \mathcal{R} + \mathcal{D} f \right] + f - 6 ,
\]

\[
\bar{\sigma} = 2 \tau^4 \int_{\Xi_i} M \sqrt{|b_{ij}|} d^3x \left[ |\mathcal{R}_{ij} + \mathcal{D}_i \mathcal{D}_j f - \frac{1}{2\tau} b_{ij}|^2 \right].
\]

Using these thermodynamical values and the lapse function \(\bar{N}(\tau)\), we can compute the corresponding average energy, entropy and fluctuations for evolution both on redefined parameter \(\tau\) and on a time like parameter \(t\) of any family of closed hypersurfaces all determined by \(\mathcal{D}\) and/or \(\mathcal{N}\), \(\bar{\mathcal{E}}(\tau) = \int_{t_i}^{t_2} dt \bar{N}(\tau) \bar{\mathcal{E}}(\tau) , \quad \bar{S}(\tau) = \int_{t_i}^{t_2} dt \bar{N}(\tau) \bar{S}(\tau) , \quad \bar{\sigma}(\tau) = \int_{t_i}^{t_2} dt \bar{N}(\tau) \bar{\sigma}(\tau).\)

These formulas are related by distortion formulas with corresponding values determined by \(\mathcal{N}\), see details and results of such computations for metrics of type \([8]\) and various quasi-periodic, cosmological, black hole deformed structures etc. in \([31,25,26,27]\).

6 Conclusion

In this paper we elaborated on a geometric model of quasi-periodic Ricci flows as a proof of the E. Verlinde conjecture \([3,18]\) that gravity is an emergent phenomenon liking to an entropic force which may explain dark matter properties. In our approach, the spacetime elasticity results from the relativistic geometric flow evolution. Modified Einstein equations are derived from nonholonomic deformations \([25,26,27]\) of F- and W-entropies introduced by G. Perelman in order to prove the Poincaré–Thurston conjecture \([32,35,36,37]\).

Based on our proposal, we shown that entropic geometric flows may result for certain well-defined conditions in stationary and quasi-periodic structures in MGTs and GR. This allows to elaborate on dark matter and dark energy theories and to model structure formation in modern accelerating cosmology. We checked that

\[4\text{Using such constructions, the standard thermodynamical values are computed for the average energy, } \mathcal{E} := \langle E \rangle := \frac{-\partial \log Z}{\partial \beta}, \text{ the entropy } S := \beta \langle E \rangle + \log Z \text{ and the fluctuation } \sigma := \langle (E - \langle E \rangle)^2 \rangle = \partial^2 \log Z/\partial \beta^2.\]
for non-relativistic flows and zero nonholonomic torsion configurations the formulas coincides to those for the Hamilton-Perelman theory but reformulated for exact solutions on 3-d hypersurfaces in pseudo-Riemannian geometry.

The quaisi-periodic solutions for relativistic geometric flows, nonholonomic Ricci solitons and generalized gravitational field equations with quasi-periodic structure constructed and studied in this and our partner works [31, 38, 40] are described by generic off-diagonal metrics and, in principle, by nonholonomically deformed non-Riemannian or pseudo-Riemannian connections. Such configurations are not characterized, in general, by entropy-area, holographic or duality conditions to certain conformal or gauge like models. In result, we can not elaborate on thermodynamic models of such (modified) theories and exact or parametric solutions using only the concepts related to the Bekenstein-Hawking entropy. We argue that there is an alternative and more general approach when stationary and cosmological solutions in MGTs and GR can be derived and characterized using nonholonomic deformations of Perelman’s W-entropy. Such constructions are similar to the well-known results on relativistic locally anisotropic thermodynamics and kinetics [25, 41].

Finally, we note that our approach provides new geometric methods and possible applications in the theory of quantum informatics flows, quantum systems with entanglement and emergent gravity and accelerating cosmology models simulated by classical and quantum computers. Such directions will be developed in our future works.

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