A second order differential equation for the relativistic description of electrons and photons

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Abstract

A new relativistic description of quantum electrodynamics is presented. Guideline of the theory is the Klein-Gordon equation, which is reformulated to consider spin effects. This is achieved by a representation of relativistic vectors with a space-time algebra made up of Pauli matrices and hyperbolic numbers. The algebra is used to construct the differential operator of the electron as well as the photon wave equation. The properties of free electron and photon states related to this wave equation are investigated. Interactions are introduced as usual with the minimal substitution of the momentum operators. It can be shown that the new wave equation is equivalent to the quadratic form of the Dirac equation. Furthermore, the Maxwell equations can be derived from the corresponding wave equation for photons.

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1. Introduction

The Dirac equation [1] is considered as the fundamental equation for the description of relativistic particles. Theoretical calculations within quantum electrodynamics agree with experimental results to highest precision. In addition, the Dirac equation is the basis of the theory of electroweak interactions, quantum chromodynamics and quantum hadrodynamics. In all cases, the combination of the relativistic dispersion relation with the Pauli spin matrices provides a natural explanation for, e.g., energy spectra, polarization observables, and cross sections. Nevertheless, one has to ask whether there exist other possibilities to introduce spin in relativistic quantum physics.

The Klein-Gordon equation seems to be the most natural starting point for the description of quantum mechanical wave phenomena, but spin is not considered in this equation. The present work wants to show that there exists a modification...
of the Klein-Gordon equation, which includes the relativistic spin effects. The equation can be obtained using an algebraic representation of the relativistic vector space, where the basis vectors are given by the elements of a Clifford algebra. The importance of such Clifford algebras in physical applications has been investigated by Hestenes [2]. In the last years this approach has become more and more popular in the description of physical processes [3]. In the present study these concepts will be applied. However, a new relativistic algebra is introduced, which represents a modification of the quaternion formalism. The quaternion algebra is altered with the help of the hyperbolic numbers, a number system which has a long history [4, 5] but is rarely used in physical applications (see e.g. [6, 7, 8, 9, 10]).

The representation matrices of the modified quaternions form the basis vectors of the relativistic vector space. Using these vectors it is possible to define a differential operator which is identical to the differential operator of the classical wave equation but transforms formally like a spin operator. The operator will be applied for the definition of the electron wave equation, which is called quantum wave equation.

The Dirac equation is a first order differential equation, whereas the Maxwell equations correspond to a second order differential equation for the photon field. The initial motivation of this paper was the question whether one can find a differential operator which forms the basis for the description of the fundamental charged fermion field as well as for the corresponding gauge field. On the level of quantum electrodynamics such a unification is possible, since from the quantum wave equation for photons, constructed with the same differential operator, the Maxwell equations can be derived.

In a theory for free fields the spin structure of the quantum wave equation is not important. In this case the differential operator is identical to the differential operator of the classical wave equation, i.e. the mass operator of the Poincaré group. Therefore, free plane wave states of electrons and photons are investigated within this group. In particular, the investigation of the spin is needed for the construction of the theory.

The introduction of interactions can be done with the conventional method of minimal substitution, leading to a Lagrangian which is invariant under gauge transformations. This opens the possibility to compare the formalism with the Dirac theory. It can be shown that the quantum wave equation corresponds to the quadratic form of the Dirac equation. Furthermore, the inhomogeneous terms of the Maxwell equations are given in the correct form. This connection supports the assumption that the calculation of physical processes will give similar results as in the conventional theories.

The organisation of the paper is as follows. In Sections 2 and 3 hyperbolic numbers and relativistic vectors including their Lorentz and Poincaré transfor-
mation properties are investigated. In Section 4 the quantum wave equation for electrons is introduced. Plane wave states for electrons are investigated in Section 5. The wave equation can be used also for anticommuting field operators. This will be shown in Sections 6 and 7. In Section 9 the Maxwell equations are derived starting from the quantum wave equation for photons. The photon plane wave states are investigated in Section 10. Interactions between electrons and photons are introduced and the equations of motion of quantum electrodynamics are calculated in Section 11. In Section 12 the connection with the Dirac theory is discussed.

2. Complex hyperbolic numbers

In the present investigation hyperbolic numbers are used for the mathematical formalation of the electron and the photon wave equation. Since these numbers are rarely used in physical applications a brief introduction of this number system is given.

In combination with the complex numbers the hyperbolic numbers \( x \in H \) are defined as

\[
x = x_0 + jx_1, \quad x_0, x_1 \in C,
\]

where the hyperbolic unit \( j \) has the property

\[
j^2 = 1.
\]

This leads to the following rules for the multiplication and addition of two hyperbolic numbers \( x = x_0 + jx_1 \) and \( y = y_0 + jy_1 \)

\[
x + y = (x_0 + y_0) + j(x_1 + y_1), \quad xy = (x_0y_0 + x_1y_1) + j(x_0y_1 + x_1y_0).
\]

Since there exist non-zero elements which have no inverse these numbers form a commutative ring. The hyperbolic unit \( j \) provides a relation between the hyperbolic sine, cosine and the exponential function

\[
cosh \phi + j \sinh \phi = e^{j\phi},
\]

which can be derived in the same way as the corresponding relation for the complex numbers.

Two conjugations will be used. The conventional complex conjugation changes the sign of the complex unit \( i \) but leaves the hyperbolic unit \( j \) unchanged

\[
x^* = x_0^* + jx_1^*.
\]

In addition, a hyperbolic conjugation will be introduced which changes only the sign of the hyperbolic unit

\[
x^- = x_0 - jx_1.
\]
The properties and definitions of the hyperbolic numbers presented here are sufficient for the following investigations. More informations about the hyperbolic number system can be found e.g. in the Refs. [3, 4, 5, 6, 7, 8, 9, 10].

3. Relativistic vectors and the spin group

3.1. Relativistic vectors

The relevance of Clifford algebras for the mathematical description of physical theories has been investigated by Hestenes [2]. He reinterpreted the elements of the Pauli or the Dirac algebra as the basis vectors of a vector space. In the same way the present study will be based on such a geometric algebra. However, a new relativistic algebra is introduced, which represents a modification of Hamilton’s quaternions.

A contravariant Lorentz vector with the coordinates $x^\mu = (x^0, x^i) \in \mathbb{C}^4$ can be expressed as follows

$$X = x^\mu \sigma_\mu.$$  

(7)

In contrast to the formalism used in the context of the $SU(2)$ group, the basis vectors $\sigma_\mu$ are made up of the unity and the elements of the Pauli algebra multiplicated by the hyperbolic unit $j$

$$\sigma_\mu = (1, j\sigma_i).$$  

(8)

Two other notations for the vector $X$ to be used in the following are

$$X = x^0 + jx = x^0 1 + jx^i \sigma_i.$$  

(9)

This means the elements of the Pauli algebra will be included in the following in the three-dimensional vectors $x = x^i \sigma_i$. This notation has become popular in the physical applications of Clifford algebras. The Pauli algebra is characterized by its multiplication rules, which can be written as

$$\sigma_i \sigma_j = \delta_{ij} 1 + i\epsilon_{ijk} \sigma_k.$$  

(10)

Using the Pauli matrices as the explicit representation of $\sigma_i$, the vector $X$ can be expressed in terms of a $2 \times 2$ matrix according to

$$X = \begin{pmatrix} x^0 + jx^3 & jx^1 - ijx^2 \\ jx^1 + ijx^2 & x^0 - jx^3 \end{pmatrix}.$$  

(11)

The formalism is not restricted to four-dimensional vectors. Adding a vector which is multiplicated by the factor $ij$ an eight-dimensional multivector can be constructed

$$Z = X + ijY.$$  

(12)
$X$ denotes the vector contribution, whereas $Y$ is interpretated as a pseudovector. This interpretation follows from a comparison with the Maxwell equations, which are given in Section 9. The eight complex coordinates of $Z$ in Eq. (12) are the maximum number of coordinates that can be placed in a $2 \times 2$ matrix. This means, comparing with the sixteen-dimensional complex vector space of the $4 \times 4$ Dirac matrices, the new relativistic formalism halves the number of dimensions arising from the mathematical structure of the vector space. In the following only the real vector coordinates of the multivector will be considered, i.e. the investigation is restricted to the four-dimensional Minkowski space.

A scalar product between two vectors can be defined using the trace of the matrix $\bar{X}Y$

$$\langle X \vert Y \rangle = \frac{1}{2} Tr (\bar{X}Y).$$  \hfill (13)

The symbol $\bar{X} = X^\dagger$ denotes transposition, complex and hyperbolic conjugation of the matrix. $\bar{X}Y$ corresponds to a matrix multiplication of the two $2 \times 2$ matrices $\bar{X}$ and $Y$. As stated above, the Pauli matrices can be considered as the basis vectors of the relativistic vector space $\sigma_\mu \equiv e_\mu$. These basis vectors form a non-cartesian orthogonal basis with respect to the scalar product defined in Eq. (13)

$$\langle e_\mu \vert e_\nu \rangle = g_{\mu\nu},$$  \hfill (14)

where the metric tensor $g_{\mu\nu}$ is a diagonal $4 \times 4$ matrix with the matrix elements

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$  \hfill (15)

The metric tensor can be used as usual for raising and lowering the indices. Eq. (13) can now be expressed in the conventional notation. One finds

$$\langle X \vert Y \rangle = \langle e_\mu \vert x^\mu y^\nu \vert e_\nu \rangle = x_\mu y^\mu,$$  \hfill (16)

and the infinitesimal distance corresponds to

$$ds^2 = \langle dX \vert dX \rangle = dx_\mu dx^\mu.$$  \hfill (17)

As an example the energy-momentum vector of a free classical pointlike particle, moving with the velocity $\mathbf{v}$ relative to the observer, is expressed in terms of the matrix algebra. The relativistic momentum vector for this particle can be written as

$$P = \frac{E}{c} + j \mathbf{p} = mc \exp(j\xi),$$  \hfill (18)
with $c$ denoting the velocity of light, $\xi$ the rapidity, $E$ the energy and $p$ the momentum of the particle. The rapidity is defined as

$$\tanh \xi = \frac{v}{c} = \frac{pc}{E},$$

(19)

where $\xi = |\xi|$ and $p = |p|$. Rapidity and momentum point into the same direction $n = v/|v|$ as the velocity.

In quantum mechanics energy and momentum are substituted by differential operators. With $\nabla = \partial^{\mu}\sigma_{\mu}$ the momentum operator is then given by

$$P = i\hbar \nabla.$$  

(20)

This operator forms the basis of the wave equation with spin, which will be introduced in Section 4. In the following $c$ and $\hbar$ will be set equal to one.

3.2. Lorentz and Poincaré Transformations

In analogy to the relation between $SU(2)$ and $SO(3)$ the transformation properties of the vectors defined in the last subsection give a relation between $SO(3,1)$ and a spin group defined as an extension of the unitary group $SU(2)$. In the following rotations and boosts will be investigated. The rotation parameters $\theta$ are defined with the conventions of Ref. [11]. The rotation of a vector has the form

$$X \mapsto X' = RX R^\dagger,$$

$$R = \exp \left(-i\theta/2\right),$$

(21)

where $X = x^{\mu}\sigma_{\mu}$. In addition, a vector can be boosted to a different system. The boost parameters $\xi$ are chosen to make the considered vector describe an object moving into the positive direction for positive values of $\xi$. In many investigations a different sign convention is used. For the boosts one finds the transformation rule

$$X \mapsto X' = BX B^\dagger,$$

$$B = \exp \left(j\xi/2\right).$$

(22)

The dagger in the above equations includes only a hermitian conjugation and not a hyperbolic conjugation. For the boost transformation one finds the relation $B^\dagger = B$, whereas the inverse of the boost operator corresponds to $B^{-1} = \bar{B}$. The explicit matrix representations of the boost matrices $B$ are

$$B_x = \begin{pmatrix} \cosh \xi^1/2 & j \sinh \xi^1/2 \\ j \sinh \xi^1/2 & \cosh \xi^1/2 \end{pmatrix}.$$  

(23)
for a boost in the direction of the \( x \)-axis and
\[
B_y = \begin{pmatrix}
\cosh \xi^2/2 & -ij \sinh \xi^2/2 \\
ij \sinh \xi^2/2 & \cosh \xi^2/2
\end{pmatrix}, \quad B_z = \begin{pmatrix}
e^{j\xi^3/2} & 0 \\
0 & e^{-j\xi^3/2}
\end{pmatrix}
\] (24)
for boosts along the \( y \)- and the \( z \)-axis.

To proof that these transformation matrices are a representation of the Lorentz group the corresponding Lie algebra has to be investigated. Before doing this, the three-dimensional dot and cross products will be introduced, which are given for the three-dimensional vectors \( \mathbf{x} \) and \( \mathbf{y} \) as
\[
\mathbf{x} \cdot \mathbf{y} = Re \{ \mathbf{x} \mathbf{y} \} = x_i y^i,
\]
\[
\mathbf{x} \times \mathbf{y} = Im \{ \mathbf{x} \mathbf{y} \} = \epsilon^{ijk} x_i y_j \sigma_k.
\] (25)

If boosts and rotations are combined as follows
\[
X \mapsto X' = \mathbf{L} \mathbf{X} \mathbf{L}^\dagger,
\]
\[
\mathbf{L} = \exp (-i (\mathbf{J} \cdot \theta + \mathbf{K} \cdot \xi)),
\] (26)
the infinitesimal generators of these transformations can be identified with
\[
J_i = \sigma_i/2, \quad K_i = i j \sigma_i.
\] (27)

With the commutation relations of the Pauli matrices one can derive that the generators satisfy the Lie algebra of the Lorentz group \( SO(3,1) \)
\[
[J_i, J_j] = i \epsilon_{ijk} J^k, \]
\[
[K_i, J_j] = i \epsilon_{ijk} K^k, \]
\[
[K_i, K_j] = -i \epsilon_{ijk} J^k.
\] (28)

Therefore, the matrices \( R \) and \( B \) given in Eqs. (21) and (22) can be recognized as the transformation matrices of the covering, i.e. the spin group of \( SO(3,1) \).

It is possible to express the Lorentz transformations in the conventional tensor formalism. Using the generators of the spin \( s = 1/2 \) representation given in Eq. (27) the relativistic generalization of the spin angular momentum operator can be defined as
\[
S_{ij} = \epsilon_{ijk} J^k, \quad S_{0i} = -S_{i0} = K_i
\] (29)
and the Lorentz transformations given in Eq. (26) can be formulated according to
\[
\mathbf{L} = \exp (-\frac{i}{2} S_{\mu\nu} \omega^{\mu\nu}).
\] (30)

The boost parameters \( \omega^{\mu\nu} \) are given as \( \omega^{ij} = \epsilon^{ijk} \theta_k \) and \( \omega^{i0} = \xi^i \).
The relativistic orbital angular momentum can be introduced in terms of vector operators for position and momentum obeying the following commutation relations
\[
[X^\mu, X^\nu] = 0, \quad [P^\mu, P^\nu] = 0, \quad [P^\mu, X^\nu] = ig^{\mu\nu}.
\]  
(31)
In the following the convention is used that big letters denote operators and small letters numbers. The relativistic orbital angular momentum is defined as
\[
L^\mu = X^\mu P^\nu - X^\nu P^\mu.
\]
If the operators \(J_i\) and \(K_i\) are given according to
\[
J_i = \frac{1}{2} \epsilon_{ijk} L^{jk}, \quad K_i = L_{0i} = -L_{i0},
\]
(32)
these generators satisfy the Lie algebra of the Poincaré group, i.e. beside the relations given in Eq. (28) one finds the following commutation relations
\[
[J_i, P_0] = 0, \quad [J_i, P_j] = i\epsilon_{ijk} P^k, \quad [K_i, P_0] = -iP_i, \quad [K_i, P_j] = -i\delta_{ij} P_0.
\]
(33)

The Lorentz transformations corresponding to Eq. (30) can now be expressed as
\[
L = \exp \left( -\frac{i}{2} L^{\mu\nu} \omega_{\mu\nu} \right),
\]
(34)
The boost parameters \(\omega_{\mu\nu}\) are defined as \(\omega_{ij} = \epsilon_{ijk} \theta_k\) and \(\omega_{i0} = \xi_i\). This transformation is acting on relativistic Hilbert space functions, which will not be specified here further. To complete the transformation properties the translations are introduced by
\[
T = \exp \left( -iP_\mu a^\mu \right).
\]
(35)

Some remarks on the conventions should be made here. In the \(SO(3)\) subspace of the Lorentz group, which is indicated by the roman indices, there is no difference between upper and lower components, i.e. \(x^i = x_i\). The relativistic contravariant vector is then given as \(x^\mu = (x^0, x^i)\) and the covariant vector as \(x_\mu = (x_0, -x_i)\). A similar convention is made for tensors. One finds e.g. for the zero-components of the orbital angular momentum \(L^{0\mu} = (0, L^{0i})\) and \(L_{0\mu} = (0, -L_{0i})\), where \(L^{0i} = L_{0i}\). For the relativistic spin matrices \(\sigma_\mu\) a special notation is used in which \(\sigma_i\) furthermore corresponds to a Pauli matrix.

4. The quantum wave equation for electrons

The Dirac equation is accepted as the fundamental relativistic equation for the description of fermionic particles. In the present work a relativistic wave equation
will be introduced, which is closely related to the classical wave equation. The new equation will be called quantum wave equation. The differential operator of the quantum wave equation is formed by the momentum operator $P$, which is dual in the sense that a scalar is obtained if the operator product is inserted between two dual spinor functions. Therefore, the following ansatz is made for the quantum wave equation

$$P \bar{P} \psi(x) = m^2 \psi(x),$$  \hspace{1cm} (36)

where $P = i \nabla$. The differential operator $P \bar{P}$ can be replaced by $PP^\dagger$ since the momentum operator is hermitian. In the following investigations there are no differences between these two choices even if interactions are introduced. The wave function $\psi(x)$ has the general structure

$$\psi(x) = \varphi(x) + j \chi(x),$$ \hspace{1cm} (37)

where $\varphi(x)$ and $\chi(x)$ are two-component spinor functions. They depend on the four space-time coordinates $x^\mu$. The transformation properties of the operator $\bar{P}$ can be deduced by a hermitian and hyperbolic conjugation of the corresponding equations given in the last section.

In order to clarify the structure of the quantum wave equation, some explicit details are presented. The Pauli matrices in Eq. (36) become apparent if one inserts Def. (7)

$$\sigma_\mu \bar{\sigma}_\nu P^\mu P^\nu \psi(x) = m^2 \psi(x).$$ \hspace{1cm} (38)

The tensor $\sigma_\mu \bar{\sigma}_\nu$ represents the spin structure which is acting on the spinor function. The explicit form is obtained by a matrix multiplication of the $2 \times 2$ basis matrices. The tensor can be separated into a symmetric and an antisymmetric contribution

$$\sigma_\mu \bar{\sigma}_\nu = g_\mu \nu - i \sigma_\mu \nu,$$ \hspace{1cm} (39)

where $g_\mu \nu$ corresponds to the metric tensor and the antisymmetric part is given by

$$\sigma_\mu \nu = \begin{pmatrix}
0 & -ij \sigma_1 & -ij \sigma_2 & -ij \sigma_3 \\
ij \sigma_1 & 0 & \sigma_3 & -\sigma_2 \\
ij \sigma_2 & -\sigma_3 & 0 & \sigma_1 \\
ij \sigma_3 & \sigma_2 & -\sigma_1 & 0
\end{pmatrix}.$$ \hspace{1cm} (40)

The antisymmetric contribution $\sigma_\mu \nu$ is directly related to the relativistic generalization of the spin angular momentum operator. Using the generators of the spin $s = \frac{1}{2}$ representation given in Eq. (29) one finds

$$S_\mu \nu = \frac{\sigma_\mu \nu}{2}.$$ \hspace{1cm} (41)
Since $P\mu P\nu$ is symmetric and $\sigma_{\mu\nu} = -\sigma_{\nu\mu}$, the operator $P\bar{P}$ is equivalent to $P\bar{P} = P_\mu P^\mu$.

At this point the particular form of the differential operator seems to be without any effect. However, the spin information which is included in the differential operator of the quantum wave equation becomes essential if the momentum operators are replaced by covariant derivates. The influence of this spin structure can be illustrated by the following example. Coordinate and momentum vector satisfy the relation

$$X\bar{P} = X_\mu P^\mu - iS_{\mu\nu}L^{\mu\nu},$$

where $L^{\mu\nu} = X_\mu P^\nu - X^\nu P_\mu$ corresponds to the relativistic orbital angular momentum.

5. Electron plane wave states

In this section the solutions of the quantum wave equation for electrons will be studied in the free non-interacting case. Due to the simplification of the differential operator $P\bar{P} = P_\mu P^\mu$, which is identical with the mass operator of the Poincaré group, the solutions will be expressed in terms of the corresponding plane wave representations. The section is separated into three parts. The first part describes how the plane wave states can be generated. The second part gives additional information on the relativistic bracket notation, which will be applied in the following. The third section investigates the connection between spin and Pauli-Lubanski vector.

5.1. Induced representation method

The irreducible representations of the Poincaré group were investigated by Wigner [12]. He found that the plane wave states are labelled by the mass $m$ and the spin $s$. In the present work these states will be generated with the induced representation method, which is described e.g. in Ref. [11]. In this method a state vector is defined within the little group of the Poincaré group, i.e. the subgroup that leaves a particular standard vector invariant. An arbitrary state is then generated with the remaining transformations which were not considered in the little group. In the following the transformation rules of Section 3.2 will be applied.

For electrons ($m^2 > 0$) one can choose the standard vector $p^\mu_i = (m,0,0,0)$. The little group of this standard vector is $SO(3)$. The explicit representation of the spin $s = \frac{1}{2}$ states is given by the Pauli spinor, which will be denoted by $|\lambda\rangle = \chi_\lambda$. The polarization is chosen along the z-axis. For the description of the mass quantum number $m$ a ket $|P_i\rangle$ is introduced. By doing this, the translations, as one group of the remaining transformations, are taken into account. The properties of the momentum kets will be investigated separately in
the next section. Note, that though the kets are represented by \( |P_t\rangle = |p_\mu \sigma_\mu \rangle \), they correspond to usual Hilbert space elements which depend only on the four-momentum \( p_\mu \). One therefore starts with the following state, which corresponds to a \((m,s)\)-representation of the Poincaré group

\[
|\lambda\rangle \otimes |P_t\rangle = |\lambda P_t\rangle.
\]  

(43)

Now, the boosts, as the last group of the remaining transformations, are acting on this state according to

\[
D(B)|\lambda P_t\rangle = B\chi \lambda |B P_t B^\dagger\rangle,
\]

(44)

where \( B \) has been defined in Eq. (22). Since the boost transforms from the rest frame to a particular frame, in which the state is described by the momentum \( P = B P_t B^\dagger \), the boost parameters can be identified with the rapidity. With this information it is possible to calculate the explicit form of the relativistic spinor. In analogy to the Dirac formalism one can introduce the notation

\[
u(p,\lambda) = B\chi \lambda.
\]

(45)

Explicitly the boost matrix can be written as

\[
B = \exp\left( j\xi /2 \right) = \cosh\xi /2 + j n \sinh\xi /2,
\]

(46)

where the rapidity \( \xi \) satisfies the following relations

\[
\cosh\xi /2 = \sqrt{p^0 + m \over 2m}, \quad \sinh\xi /2 = \sqrt{p^0 - m \over 2m}.
\]

(47)

Inserting these results into Eq. (45) the spinor is given by

\[
u(p,\lambda) = \sqrt{p^0 + m \over 2m} \left( 1 + {j p \over p^0 + m} \right) \chi \lambda.
\]

(48)

The antiparticle spinor is constructed in analogy to the Dirac theory, where upper and lower components are interchanged compared to the particle spinor. In the formalism presented here, this can be done by multiplying the particle spinor by the hyperbolic unit \( j \), i.e. \( v(p,\lambda) = j u(p,\lambda) \). Therefore, one can write

\[
v(p,\lambda) = \sqrt{p^0 + m \over 2m} \left( {p \over p^0 + m} + j \right) \chi \lambda.
\]

(49)

The normalization and the completeness of the plane wave states can be calculated as follows

\[
\langle u(p,\lambda)|u(p,\lambda')\rangle = +\delta_{\lambda\lambda'},
\]

\[
\langle v(p,\lambda)|v(p,\lambda')\rangle = -\delta_{\lambda\lambda'}.
\]  

(50)
To explain this notation the scalar product of two plane wave spinors is determined explicitly
\[
\langle u(p, \lambda) | u(p, \lambda') \rangle = \bar{u}(p, \lambda) u(p, \lambda') = \chi^{\dagger}_\lambda \bar{B} B \chi_{\lambda'} = \delta_{\lambda \lambda'}.
\] (51)

The antiparticle spinor is normalized to $-1$, which is due do the additional $j$-factor and its hyperbolic conjugated counterpart $j^- = -j$. Note, that there is no explicit orthogonality between particle and antiparticle spinor. However, if the spinors appear in combination with the momentum kets of the states, orthogonality of particle and antiparticle states will be restored. This can be checked with the relations given in the following sections.

For the completeness one finds
\[
\sum_{\lambda} | u(p, \lambda) \rangle \langle u(p, \lambda) | = - \sum_{\lambda} | v(p, \lambda) \rangle \langle v(p, \lambda) | = 1.
\] (52)

Together with bras and kets for the momentum these expressions correspond to negative and positive energy projectors, respectively. In contrast to the appropriate expressions in the Dirac theory, they have a simple matrix structure and no momentum dependence.

The transformation properties of the states are
\[
D(L) | u(p, \lambda) P \rangle = (Lu(p, \lambda)) | LPL^\dagger \rangle, \\
D(T) | u(p, \lambda) P \rangle = u(p, \lambda) \exp \left( -ip_m a^m \right) | P \rangle,
\] (53)
where the Lorentz transformations are performed with the matrix $L$ of Eq. (26).

The notation presented here was introduced in analogy to the Dirac formalism and it therefore adopted the expression $u(p, \lambda)$. In the following the abstract notation $| q \lambda P \rangle$ will be used, where a quantum number denoting a particle with $q = 1$ and an antiparticle with $q = -1$ is introduced. The explicit form of these states is
\[
| + \lambda P \rangle = | u(p, \lambda) P \rangle, \quad | - \lambda P \rangle = | v(p, \lambda) -P \rangle.
\] (54)

Again the antiparticle state is introduced by definition. A proper derivation of these states requires the investigation of the discrete Lorentz transformations.

5.2. The relativistic $| P \rangle$ and $| X \rangle$ representations

In the last subsection the kets $| P \rangle$ and $| X \rangle$ for relativistic Hilbert space elements related to a $(m^2 > 0)$ representation of the Poincaré group were introduced. In the following the normalization and completeness of the states will be summarized, where the description follows the covariant conventions given e.g. in Refs. [11], [14]. The spin degrees of freedom will be considered separately in the next section.
The normalization and completeness of the states is expressed in an abstract notation. The normalization is given by

\[ \langle P | P' \rangle = \delta(P - P'), \quad \langle X | X' \rangle = \delta(X - X'), \]  

(55)

and the completeness can be written as

\[ \int dP | P \rangle \langle P | = 1, \quad \int dX | X \rangle \langle X | = 1. \]  

(56)

The dependence on the mass \( m \) is suppressed in the bras and kets. The explicit covariant representation of the momentum delta function is given \([11, 14]\) according to

\[ \delta(P - P') = \frac{1}{(2\pi)^3 2p^0} \delta^3(p - p'), \]  

(57)

where \( p^0 = \sqrt{p^\mu p^\mu + m^2} \) is fixed by the mass shell condition. The integration over \( dP \) corresponds to

\[ \int dP = \int d^3p \frac{1}{(2\pi)^3 2p^0}, \]  

(58)

with \( d^3p = dp^1 dp^2 dp^3 \).

Using these representations the abstract completeness and normalization of Eqs. (55) and (56) are consistent in momentum space. In coordinate space the situation is more complicated. For the unity expressed in terms of bras and kets one can define

\[ \int dX | X \rangle \langle X | = \int d^3x | X \rangle \underleftrightarrow{P^0} \langle X |, \]  

(59)

where the abbreviation

\[ A \underleftrightarrow{P^\mu} B = A(P^\mu B) - (P^\mu A)B \]  

(60)

has been introduced. To investigate the properties of the delta function in coordinate space the representation of the momentum ket is needed

\[ \langle X | P \rangle = \exp(-ip_\mu x^\mu). \]  

(61)

With this expression the covariant delta function can be calculated using the completeness of the momentum kets

\[ \delta(X - X') = \int dP \langle X | P \rangle \langle P | X' \rangle = \int \frac{d^3p}{(2\pi)^3 2p^0} \exp(-ip_\mu (x^\mu - x'^\mu)). \]  

(62)

The delta function is therefore equivalent to the positive frequency part \( \Delta_+ (x - x') \) of the invariant function \( i\Delta(x - x') \) given in Ref. \([13]\). This reflects the fact that
these considerations are restricted to positive energy states. The action of the
delta function applied to the state \( \langle X | P \rangle \) is as follows

\[
\int dX \delta(X' - X) \langle X | P \rangle = \int dP' \int dX' \langle X' | P' \rangle \langle P' | X \rangle \langle X | P \rangle
\]

\[
= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2p^0} \int d^3x e^{-ip_\mu(x'^\mu - x_\mu)} P^0 e^{-ip_\mu x^\mu} = \langle X' | P \rangle.
\]

This shows that the explicit representations are consistent with the abstract com-
pleteness given in Eq. (56).

Using the relations given above one can transform a momentum dependent
function \( \psi(p) = \langle P | \psi \rangle \), which is given on the mass shell, into the coordinate
space according to

\[
\psi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2p^0} e^{-ip_\mu x^\mu} \psi(p),
\]

whereas in the opposite direction the transformation has the form

\[
\psi(p) = \int d^3x e^{i(p_\mu x^\mu - p^0 \partial^0)} \psi(x).
\]

In the next sections the notation presented here will be applied e.g. in the second
quantization of the electron field on the equal-time plane.

5.3. The Pauli-Lubanski vector and spin operators

To show that the plane wave states derived in the first part of this section
correspond to an irreducible representation of the Poincaré group, the connection
of these states with the second Casimir operator, the Pauli-Lubanski vector, will
be investigated. Since the spin \( s = \frac{1}{2} \) representation is considered, the spin
angular momentum operators of Eq. (29) will be used for the definition of the
Pauli-Lubanski vector

\[
W^\mu = \frac{1}{2} \epsilon^{\mu\rho\sigma\nu} S_{\rho\sigma} P_\nu = \tilde{S}^{\mu\nu} P_\nu,
\]

where \( \tilde{S}^{\mu\nu} \) is the dual tensor of the relativistic angular momentum tensor. It is
interesting that the dual tensor can also be calculated using the relation \( \tilde{S}^{\mu\nu} =
-ij S^{\mu\nu} \), which can be derived from Eqs. (10) and (11). In the notation of Eq. (7)
the Pauli-Lubanski vector has the form

\[
W = -J \cdot P - j(J P^0 + K \times P),
\]

where in the expression \( J = J^i \sigma_i \) the Pauli matrices in \( J^i = \sigma^i/2 \) are coordinates
of the Pauli matrices \( \sigma_i \) in the basis.
With the Pauli-Lubanski vector the relativistic spin operators can be defined, where the investigation follows the methods given in [15, 16]. One chooses a set of four orthogonal vectors $\mathbf{n}^{(\nu)}$ satisfying the relation

$$n_{\mu}^{(\mu)}n_{\nu}^{(\nu)} = g_{\mu\nu}. \tag{68}$$

Using these vectors the spin operators are defined according to

$$S^i = \frac{1}{m} W^i n^{(i)} \tag{69}.$$

If the plane wave states derived in the first part of this section shall be eigenstates of the spin operators, a particular set of orthogonal vectors $n^{(\mu)} = (n^{0(\nu)}, n^{k(\nu)})$ has to be introduced

$$n^{\mu(0)} = \left( \frac{P^0}{m}, \frac{P^k}{m} \right), \quad n^{\mu(i)} = \left( \frac{P^i}{m}, \delta^{ki} + \frac{P^k P^i}{m(P^0 + m)} \right). \tag{70}$$

In the rest frame, these vectors reduce to the canonical orthogonal system. The three spin operators can be regarded as the relativistic generalization of the non-relativistic spin operators. They can be expressed as $(S = S^i \sigma_i)$

$$S = \frac{1}{m} \left( J P^0 + K \times P - (J \cdot P) \frac{P}{P^0 + m} \right). \tag{71}$$

For the relativistic spin operators one finds $S^2 = s(s + 1) = -W^\mu W_\mu / m^2$. The operators satisfy the commutation relations of the little group $SO(3)$. The third component of the spin vector can be used to characterize the polarization.

The spin operators were constructed in that way, that they coincide with boosted generators $J^i$, where the boost matrices are acting on the coordinates of $\mathbf{J} = J^i \sigma_i$

$$S^i = B J^i \bar{B}. \tag{72}$$

As well as in the derivation of the plane wave states, the boost parameters in $B$ have to be identified with the rapidity of the state vector. From the above equation follows that under arbitrary Lorentz transformations the spin operators have to transform according to

$$S^i \mapsto S'^i = L S^i \bar{L}, \tag{73}$$

where $L$ corresponds to the Lorentz transformation matrix given in Eq. (26). From Eq. (72) one can deduce that it is sufficient to define the spin in the rest frame of the state, according to the non-relativistic description with the non-relativistic spin operators $J^i = \sigma^i/2$, whereas the boost operators $K^i = ij J^j$ are not used to
characterize the single-particle state. Vector products of the form $A \bar{B}$, with two arbitrary vectors $A$ and $B$, transform in the same way as the spin operators

$$(A \bar{B}) \rightarrow (A \bar{B})' = L(A \bar{B}) \bar{L}. \quad (74)$$

Therefore, $P \bar{P}$ was chosen as the differential operator of the quantum wave equation. This guarantees that the operator, which is acting between two spinor functions, shows the correct transformation property.

Now, the properties of the positive energy states can be summarized. The plane wave states for positive energies are eigenstates of the four operators $\{P_\mu P^\mu, P^\mu, W_\mu W^\mu, S^3\}$ and satisfy the relations

$$P_\mu P^\mu | + \lambda P \rangle = m^2 | + \lambda P \rangle, \quad P^\mu | + \lambda P \rangle = p^\mu | + \lambda P \rangle,$$

$$W_\mu W^\mu | + \lambda P \rangle = -m^2 s(s + 1) | + \lambda P \rangle, \quad S^3 | + \lambda P \rangle = \lambda | + \lambda P \rangle. \quad (75)$$

Together with the negative energy states one finds the following normalization

$$\langle q | \lambda P | q' \lambda' P' \rangle = \delta_{qq'} \delta_{\lambda\lambda'} \delta(P - P'). \quad (76)$$

and completeness

$$\sum_\lambda \int dP | + \lambda P \rangle \langle + \lambda P | = 1, \quad \sum_\lambda \int dP | - \lambda P \rangle \langle - \lambda P | = 1, \quad (77)$$

where the completeness is restricted to the subspaces of positive and negative energy, respectively. With these states the solution of the quantum wave equation $\psi(x)$ can be expressed as the following plane wave expansion

$$\psi(x) = \sum_\lambda \int dP \left( \langle X | + \lambda P \rangle \langle + \lambda P | \psi \rangle + \langle X | - \lambda P \rangle \langle - \lambda P | \psi \rangle \right)$$

$$= \sum_\lambda \int \frac{d^3p}{(2\pi)^3 2\hbar} \left( u(p, \lambda) e^{-i p\cdot x} b(p, \lambda) + v(p, \lambda) e^{i p\cdot x} \bar{d}(p, \lambda) \right). \quad (78)$$

At this point the solution is understood as an expansion of a single particle wave function.

6. Variational principle and current conservation

A second order differential equation is expected to be unsuitable for the description of fermionic quantum fields. Therefore, some quantum field theoretical
aspects will be investigated in the next two sections. In this section a Lagrangian for a free non-interacting theory will be introduced which leads to the quantum wave equation using standard variational techniques. The Lagrange equations, currents, and conserved quantities will be derived.

The initial point of this investigation is a Lagrangian of the form

\[ L(x) = \bar{\psi}(x) P \bar{P} \psi(x) - m^2 \bar{\psi}(x) \psi(x) , \]

(79)

where both momentum vectors are acting to the right side. The Lagrangian has the property of being invariant under hermitian including hyperbolic conjugation, i.e. \( L(x) = \bar{L}(x) \). Note, that the wave functions are two-component spinor functions and a product of two spinor function \( \bar{\psi} \psi \equiv \bar{\psi}_i \psi_i \) implies a contraction of the two-component field functions. The same convention is used in expressions like the following

\[ \frac{\partial L}{\partial (\partial_\mu \psi)} \delta (\partial_\mu \psi) = \frac{\partial L}{\partial (\partial_\mu \bar{\psi})} \delta (\partial_\mu \bar{\psi}) . \]

(80)

The equations of motion, currents, and conserved quantities can be derived as usual from a variational principle which is applied to the action. The investigation presented here follows the common treatment of this topic, which can be found e.g. in Ref. [14]. Therefore, only the main points shall be discussed briefly. The action is defined as

\[ S = \int d^4 x L(x) , \]

(81)

where \( d^4 x = dx^0 dx^1 dx^2 dx^3 \). \( L(x) \) is a function of \( L(\psi, \bar{\psi}, \partial_\mu \psi, \partial_\mu \bar{\psi}, x^\mu) \). The variation of this Lagrangian is then given by

\[ \delta L = \frac{\partial L}{\partial \psi} \delta \psi + \frac{\partial L}{\partial (\partial_\mu \psi)} \delta (\partial_\mu \psi) + \delta (\partial_\mu \bar{\psi}) \frac{\partial L}{\partial (\partial_\mu \bar{\psi})} + (\partial_\mu L) \delta x^\mu , \]

(82)

where the variation of the coordinates and the variation of the wave function is defined as

\[ x^\mu \rightarrow x'^\mu = x^\mu + \delta x^\mu , \]

\[ \psi(x) \rightarrow \psi'(x) = \psi(x) + \delta \psi(x) . \]

(83)

Both variations vanish on a boundary \( \partial R \). Using \( \delta L \) the variation of the action \( \delta S \) can be calculated. Applying the principle of least action, \( \delta S = 0 \), to this expression one obtains the Lagrange equations

\[ \frac{\partial L}{\partial \psi} - \partial_\mu \frac{\partial L}{\partial (\partial_\mu \psi)} = 0 , \quad \frac{\partial L}{\partial \bar{\psi}} - \partial_\mu \frac{\partial L}{\partial (\partial_\mu \bar{\psi})} = 0 . \]

(84)

From the surface terms of \( \delta S \) one can derive the conserved current, which has the form

\[ J^\mu = \frac{\partial L}{\partial (\partial_\mu \psi)} \Delta \psi + \Delta \bar{\psi} \frac{\partial L}{\partial (\partial_\mu \bar{\psi})} - \theta^\mu \delta x^\nu . \]

(85)
\[ \Delta \psi \text{ corresponds to the total variation } \psi'(x') = \psi(x) + \Delta \psi(x) \text{ and } \theta^{\mu\nu} \text{ denotes the energy momentum tensor defined as } \]
\[ \theta^{\mu\nu} = \frac{\partial L}{\partial (\partial_{\mu} \psi)} (\partial_{\nu} \psi) + (\partial_{\nu} \bar{\psi}) \frac{\partial L}{\partial (\partial_{\mu} \bar{\psi})} - g^{\mu\nu} L. \quad (86) \]

The quantum wave equation can now be derived with the help of the Lagrange equations (84), where it is useful to rewrite the Lagrangian in analogy to Eq. (38) to separate the dynamical variables from the basis matrices.

The currents for global phase transformations and translations can be calculated with the definition of the current given in Eq. (85). A global phase transformation of the wave function is expressed as

\[ \psi(x) \mapsto \psi'(x) = e^{-i\Lambda} \psi(x). \quad (87) \]

This leads to the total field variations

\[ \Delta \psi = -i\Lambda \psi, \quad \Delta \bar{\psi} = i\bar{\psi} \Lambda \quad (88) \]

and to \( \delta x^\nu = 0 \). Using the relation \( PP = P_\mu P^\mu \) the current can be calculated as

\[ J^\mu(x) = \bar{\psi}(x) P^\mu \psi(x). \quad (89) \]

The following sections will show that if interactions are considered the spin structure becomes important and the appropriate expression for the current will be more complicated.

With the formalism developed in Section 5.2 one can relate the charge, defined on the equal-time plane, with a scalar product in the relativistic Hilbert space

\[ \langle \psi | \phi \rangle = \int dX \langle \psi | X \rangle \langle X | \phi \rangle = \int d^3x \bar{\psi}(x) P^0 \phi(x). \quad (90) \]

This yields the relation

\[ Q = \langle \psi | \psi \rangle = \int dX \langle \psi | X \rangle \langle X | \psi \rangle = \int d^3x J^0(x). \quad (91) \]

The four-momentum of a classical field configuration is related to the translations, which are characterized by the following variations

\[ \delta x^\mu = \epsilon^\mu, \quad \Delta \psi = 0. \quad (92) \]

The conserved current corresponds to the energy-momentum tensor, which has in the case of free electrons the structure

\[ \theta^{\mu\nu}(x) = -(P^\mu \bar{\psi}(x))(P^\nu \psi(x)) - (P^\nu \bar{\psi}(x))(P^\mu \psi(x)) - g^{\mu\nu} L(x), \quad (93) \]
where $\theta^{\mu\nu} = \bar{\theta}^{\mu\nu} = \theta^{\nu\mu}$. With this expression one can define the four-momentum of the field configuration according to

$$P^{\mu} = \int d^3x \theta^{0\mu}(x).$$  \hspace{1cm} (94)

A certain amount of energy is able to create electron-positron pairs, i.e. to bring an electron from a negative energy state to a positive energy state. The negative energy spectrum, which is naturally included in a relativistic treatment of physics, has the consequence that a relativistic quantum theory can only be constructed within a many-particle framework. One possibility to go in this direction is to quantize the fields, i.e. to consider them as operator valued quantum fields which are acting between many-particle states. This procedure is called second quantization and will be performed in the next section.

7. Canonical quantization of free electrons

In contrast to common belief the canonical quantization is straightforward in a theory of non-interacting electrons based on the Klein-Gordon equation. The standard techniques of the canonical quantization [13] and the partial wave expansion derived in the foregoing sections will be used to demonstrate this.

The Lagrangian given in Eq. (79) is equivalent to the Lagrangian of the Klein-Gordon equation

$$L = \bar{\psi}(x) P^{\mu} P_{\mu} \psi(x) - m^2 \bar{\psi}(x) \psi(x).$$  \hspace{1cm} (95)

For the second quantization the conjugated momentum will be introduced as usual

$$\pi(x) = \frac{\partial L}{\partial (\partial_{\partial_{0}} \bar{\psi}(x))} = \partial_{0} \bar{\psi}(x).$$  \hspace{1cm} (96)

The field quantization follows from implying equal-time anticommutation relations on the field variables

$$\left\{ \bar{\psi}_i(x), \pi_j(y) \right\}_{x^0 = y^0} = +i \delta_{ij} \delta^3(x - y),$$  \hspace{1cm} (97)

$$\left\{ \bar{\psi}_i(x), \bar{\psi}_j(y) \right\}_{x^0 = y^0} = -i \delta_{ij} \delta^3(x - y).$$

Subscripts where introduced to indicate the two-component structure of the fields.

One can show that these relations are leading to the Heisenberg equations of motion

$$[\hat{P}^{\mu}, \psi(x)] = -i \partial^{\mu} \psi(x), \quad [\hat{P}^{\mu}, \bar{\psi}(x)] = -i \partial^{\mu} \bar{\psi}(x),$$  \hspace{1cm} (98)

where the momentum of the field configuration $P^{\mu}$ given in Eq. (94) has been changed to a many-particle operator $\hat{P}^{\mu}$ including the operator valued fields $\psi(x)$ and $\bar{\psi}(x)$. 

19
From the plane wave expansion of $\psi$ given in Eq. (79) one can project out the amplitudes $b(P, \lambda)$ and $d(P, \lambda)$ according to

$$
\begin{align*}
    b(p, \lambda) &\equiv \langle + \lambda P | \psi \rangle = \int dX \langle + \lambda P | X \rangle \langle X | \psi \rangle \\
    &= \int d^3x e^{ip_\mu x^\mu} \bar{u}(p, \lambda) \not{p} \psi(x), \\
    d(p, \lambda) &\equiv \langle \psi | - \lambda P \rangle = \int dX \langle \psi | X \rangle \langle X | - \lambda P \rangle \\
    &= \int d^3x \bar{\psi}(x) \not{P} \psi(p, \lambda) e^{ip_\mu x^\mu}.
\end{align*}
$$

(99)

The two-component spinor functions $\bar{u}(p, \lambda)$ and $\psi(x)$ have to be contracted according to the convention introduced in the last section. Using the anticommutation relations for the fermion fields one can derive the anticommutators

$$
\{b(p, \lambda), \bar{b}(p', \lambda')\} = \{d(p, \lambda), \bar{d}(p', \lambda')\} = \delta_{\lambda \lambda'} \delta(P - P').
$$

(100)

The many-particle operators $\hat{Q}$ and $\hat{P}^\mu$ given in Eqs. (91) and (94) can be expressed in terms of the operator valued amplitudes $b(P, \lambda)$ and $d(P, \lambda)$ according to

$$
\begin{align*}
    \hat{Q} &= \int d^3x : J^0(x) : \\
    &= \sum_\lambda \int dP (\bar{b}(p, \lambda)b(p, \lambda) - \bar{d}(p, \lambda)d(p, \lambda))
\end{align*}
$$

(101)

and

$$
\begin{align*}
    \hat{P}^\mu &= \int d^3x : \theta^{0\mu}(x) : \\
    &= \sum_\lambda \int dP p^\mu (\bar{b}(p, \lambda)b(p, \lambda) + \bar{d}(p, \lambda)d(p, \lambda)).
\end{align*}
$$

(102)

The Feynman propagator, which is defined as the time ordered vacuum expectation value of the field operators, can be calculated with the results of the preceding equations. Adding a small imaginary part to the denominator one finds

$$
\begin{align*}
    iS_F(x, y)_{ij} = \langle 0 | T(\psi_i(x) \bar{\psi}_j(y)) | 0 \rangle = i\delta_{ij} \int \frac{d^4p}{(2\pi)^4} \frac{e^{-ip_\mu(x^\mu - y^\mu)}}{p_\mu p^\mu - m^2 + i\epsilon},
\end{align*}
$$

(103)

where the $p^0$ components in the integrand are not fixed by the mass shell condition, i.e. $p^0 \neq \sqrt{p^2 + m^2}$. From the above expression follows that the differential
operator of the quantum wave equation acting on the Feynman propagator is equal to the four-dimensional delta function

\[(P\bar{P} - m^2)S_F(x, y) = \delta^4(x - y),\]  

where the matrix indices are not indicated.

The consistence of the above equations is not self-evident if one uses anticommuting fields. The key point, which is responsible for these results, is the negative normalization and completeness of the antiparticle spinors given in Eqs. (50) and (52). The negative normalization is also part of the Dirac theory. It is therefore essentially the same mechanism which provides a consistent treatment of second quantization in the formalism presented in this paper.

The main intention of this brief discussion was to show that a second order differential equation for fermions is in accordance with basic quantum field theoretical considerations. In the following, the fields will be considered again as single-particle wave functions.

8. Charged massless fermions

In the preceding sections massive electrons have been studied. With regard to the discussion of photons, the \(m^2 = 0\) representation of the Poincaré group will be investigated in the following. The discussion is restricted to charged fermions, i.e. one can expect that electrons at very high energy can be described by these states. Furthermore, the free non-interacting electron field may obey this equation. The quantum wave equation for massless fermions has the form

\[P\bar{P}\psi(x) = 0.\]  

Again, the spinor will be defined within the little group of a standard vector. Since massless particles are moving with the velocity of light they have no rest frame. Therefore, the standard frame will be defined as the system in which the momentum is directed along the polarization axis. If the particle is polarized along the z-axis the positive-energy standard vector is given as

\[p^\mu = (|p^3|, 0, 0, h|p^3|),\]  

where \(h = p^3/|p^3|\) denotes the helicity of the particle.

To obtain an arbitrary momentum vector \(p^\mu\) one has to boost perpendicular to the z-axis

\[p^\mu = (|p^3| \cosh \xi, |p^3| n^1 \sinh \xi, |p^3| n^2 \sinh \xi, h|p^3|).\]
The unit vector $\mathbf{n}_\perp = (n^1, n^2, 0)$ characterizes the direction of the boost. With the perpendicular momentum vector $p_{\perp} = (p^1, p^2, 0)$ the rapidity is defined by the relation
\[
\tanh \xi = \frac{p_{\perp}}{p^0},
\]
(108)
where $p_{\perp} = |p_{\perp}|$. Using $\xi = \mathbf{n}_\perp \cdot \xi$ the boost can be written again in the form $B = \exp (j \xi / 2)$. The components of the Pauli-Lubanski vector in the standard frame of Eq. (106) are
\[
W^0 = -h|P^3|J^3, \\
W^1 = -|P^3|(J^1 + hK^2), \\
W^2 = -|P^3|(J^2 - hK^1), \\
W^3 = -|P^3|J^3.
\]
(109)
The operators $W^0$ and $W^3$ are linear dependent and will be represented in the following by $J^3$. The three generators $J^3$, $W^1$ and $W^2$ satisfy the Lie algebra of the Euclidean group in two dimensions
\[
[J^3, W^1] = iW^2, \quad [J^3, W^2] = -iW^1, \quad [W^1, W^2] = 0,
\]
(110)
which defines the little group of the $m^2 = 0$ representation. To derive the above equations the angular momentum operators have to be defined with the orbital angular momentum operators of Eq. (32).

For the definition of the spin one has to insert the generators $J^i = \sigma^i/2$ and $K^i = ij\sigma^i/2$ into Eq. (109). Then one finds in the standard frame and therefore in all frames
\[
W_\mu W^\mu = 0,
\]
(111)
i.e. the spin is given in the degenerate spin $s = 0$ representation of $E_2$. The basis vectors are chosen as eigenvectors of $J^3$ with the eigenvalues $\lambda = \pm 1/2$. Therefore, one can adopt Def. (43) with the Pauli spinor $|\lambda\rangle$ and the standard momentum $p^\mu_i$ of Eq. (108). A general basis vector is obtained using Eq. (44) with the boost parameters defined above. This leads to the spinor
\[
u(p, \lambda) = \sqrt{p^0 + |p^3|} \left( 1 + \frac{j p_{\perp}}{p^0 + |p^3|} \right) \chi_\lambda.
\]
(112)
The antiparticle spinor is obtained as in Eq. (49), i.e. the above expression is multiplied by the hyperbolic unit $j$. This ensures that the second quantization for charged anticommuting fields can be performed consistently due to the negative normalization of the spinor
\[
u(p, \lambda) = \sqrt{p^0 + |p^3|} \left( \frac{p_{\perp}}{p^0 + |p^3|} + j \right) \chi_\lambda.
\]
(113)
The transformation properties of the basis vectors are the same as in Eq. (53).

It remains to define the spin operator. A set of four orthogonal vectors \( n_\mu^{(0)} = (n^0, n^k) \) can be introduced

\[
n_\mu^{(0)} = \left( \frac{P^0}{|P^3|}, \frac{P^k}{|P^3|} \right), \quad n_\mu^{(i)} = \left( \frac{P^i}{|P^3|}, \delta^{ki} + \frac{P^k P^i}{|P^3|^2 (P^0 + |P^3|)} \right). \tag{114}
\]

For the degenerate representation of \( E_2 \) one operator is sufficient to characterize the state vectors \([11]\). This operator is chosen as follows

\[
S^3 = \frac{1}{|P^3|} W_\mu n_\mu^{(3)}. \tag{115}
\]

The spin operator corresponds again to a boosted generator \( J^3 \). Explicitly written one finds

\[
S^3 = \frac{1}{|P^3|} (J^3 P^0 + K^1 P^2 - K^2 P^1). \tag{116}
\]

The results can be summarized now. If one uses the notation \( | + \lambda P \rangle = | u(p, \lambda) P \rangle \) the particle kets are characterized by the operators \( \{ P_\mu P^\mu, W_\mu W^\mu, P_\mu, S^3 \} \) and satisfy the relations

\[
\begin{align*}
P_\mu P^\mu | + \lambda P \rangle &= 0, \\
P^\mu | + \lambda P \rangle &= p^\mu | + \lambda P \rangle, \\
W_\mu W^\mu | + \lambda P \rangle &= 0, \\
S^3 | + \lambda P \rangle &= \lambda | + \lambda P \rangle. \tag{117}
\end{align*}
\]

The plane wave expansion is formally equivalent to Eq. (78), but the spinors \( u(p, \lambda) \) and \( v(p, \lambda) \) have to be replaced with the specific \( m^2 = 0 \) form given in Eqs. (112) and (113). The states could also be characterized by the helicity, whereas in the description presented here the helicity is fixed and the two polarizations are used to characterize the two possible states. Changing the chosen helicity leads to an interchange of the two polarizations.

9. The quantum wave equation for photons

The motivation for this work was the question whether one can find a differential operator which forms the basis for the description of the fundamental fermion field as well as for the corresponding gauge field, i.e. whether there exists an universal equation of the form

\[
D \phi(x) = m_\phi^2 \phi(x), \tag{118}
\]

where \( D \) is a suitable differential operator, \( \phi(x) \) denotes the considered quantum field, and \( m_\phi^2 \geq 0 \) corresponds to the mass of the particle. This means, for
fundamental particles the information about the particle spin should be included only in the field $\phi(x)$ and not in the differential operator of the wave equation.

In quantum electrodynamics, i.e. more precisely $\phi(x) \in \{\psi(x), A(x)\}$, this unification is possible, where $D = P\bar{P}$ has to be chosen as the underlying differential operator. It will be shown that the Maxwell equations can be derived starting from this unified wave equation. For free photon fields one therefore begins with

$$P\bar{P}A(x) = 0,$$

where $A(x) = A^0(x) + jA(x)$ is a vector field and $m_A^2 = 0$. With the electromagnetic fields

$$E^i(x) = -\partial^0 A^i(x) - \partial^i A^0(x), \quad B^i(x) = \epsilon^{ijk}\partial_j A_k(x),$$

one can show that Eq. (119) can be expressed in terms of these fields according to

$$P\bar{P}A(x) = -\nabla \cdot E(x) - \partial^0 C(x)
+ ij \nabla \cdot B(x)
- j(\nabla \times B(x) - \partial^0 E(x) - \nabla C(x))
- i(\nabla \times E(x) + \partial^0 B(x)) = 0.
$$

The calculation has to be done in a specific order. In the first step the expression $\bar{P}A(x)$ has to be evaluated. In the second step the operator $P$ is acting on the electromagnetic fields. Calculating $P\bar{P} = P_\mu P^\mu$ is leading to the classical wave equation for the vector potential.

In the quantum wave equation the four homogeneous Maxwell equations are included. The straightforward derivation provides two additional terms including the field

$$C(x) = \partial_\mu A^\mu(x).$$

In the Lorentz gauge this term vanishes and the correct form of the Maxwell equations is restored. This indicates that the gauge should not be chosen arbitrary, i.e. the vector potential should be perpendicular to the momentum operator in the relativistic sense

$$P_\mu A^\mu(x) = 0.$$

Otherwise, the six-component tensor structure of the electromagnetic field would be modified and extended to a seventh component, which can not be incorporated into an antisymmetric second rank tensor.

In this formalism gauge invariance is not satisfied as naturally as in the conventional formulation of the Maxwell equations based on the antisymmetric tensor
$F^{\mu\nu}$. The gauge transformation of the vector potential $A(x) = A^\mu(x)\sigma_\mu$ is given as usual according to

$$A(x) \mapsto A'(x) = A(x) + \frac{1}{e} \nabla \Lambda(x),$$

(124)

where $\Lambda(x)$ is a scalar field. From the electromagnetic fields $E$ and $B$ one knows that they are invariant under this transformation. Comparing with Eq. (121) one observes that it is only the field $C(x)$ which is required to be invariant. Since this term is supposed to be zero and the gauge condition should not be modified by the gauge transformation the scalar field has to fulfil the relation

$$P\bar{P}\Lambda(x) = 0.$$  

(125)

Therefore, gauge invariance is achieved in the present formalism by restricting on a certain class of gauge functions $\Lambda(x)$.

10. Photon plane wave states

In this section a plane wave expansion for free photon fields will be derived, where the techniques developed for massless fermions will be applied. The transformation properties of the vector components $A^\mu(x)$ can be understood in terms of $4 \times 4$ transformation matrices acting on four-component vectors

$$X^\mu \mapsto X'^\mu = (L)^{\mu}_{\nu} X^\nu,$$

$$L = \exp (-i(J \cdot \theta + K \cdot \xi)).$$

(126)

For the generators $J^i$ and $K^i$ only the third components are displayed

$$(J^3)^{\mu}_{\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad (K^3)^{\mu}_{\nu} = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}.$$ 

(127)

Now, one can use essentially the results of Section 8. The standard frame is again given as the system in which the momentum is directed along the polarization axis. The Pauli-Lubanski vector in the standard frame is given by Eq. (109), where the generators now have to be replaced by the generators of Eq. (126).

One can construct two eigenstates $|\lambda\rangle$ of $J^3$ with the eigenvalues $\lambda = \pm 1$

$|\pm\rangle = \epsilon^\mu_{\pm} \sigma_\mu = \frac{\mp \epsilon^1_\mu - i \epsilon^2_\mu}{\sqrt{2}} \sigma_\mu.$

(128)
These vectors are eigenstates of $J^3$, i.e. $(J^3)_{\lambda\nu}^\mu e^\nu_\lambda = \lambda e^\mu_\lambda$. $e^\mu_1$ and $e^\mu_2$ are unit vectors along the x- and y-axis. In the standard frame the eigenstates satisfy the relation

$$W_\mu W^\mu|\lambda\rangle = (W_\mu W^\mu)^{\alpha\beta} e^{\beta}_\lambda e^{\alpha}_\lambda = 0. \quad (129)$$

The basis vectors of the irreducible representation are defined according to Eq. (43) and boosted perpendicular to the polarization axis to the system, where the state is described by the momentum $P$

$$D(B)|\lambda P_1\rangle = ((B)^{\mu\nu} e^\nu_\lambda) \sigma_\mu |BP_1B^1\rangle. \quad (130)$$

Introducing the abbreviation $e^\mu(p,\lambda) = (B)^{\mu\nu} e^\nu_\lambda$ the polarization vectors are given as

$$e^\mu(p,\lambda) = \left(\frac{p^\lambda}{|p^\lambda|}, e^1_\lambda + \frac{p^1 p^\lambda}{|p^\lambda||p^\mu + |p^\mu|}}, \right), \quad (131)$$

where $p^\pm = (\mp p^1 - ip^2)/\sqrt{2}$. The transformation rules for the plane wave states are

$$D(L)|e(p,\lambda) P\rangle = ((L)^{\mu\nu} e^\nu_\lambda(p,\lambda)) \sigma_\mu |LPL^1\rangle,$$

$$D(T)|e(p,\lambda) P\rangle = e^\mu(p,\lambda) \sigma_\mu \exp (-ip_\mu a^\mu)|P\rangle. \quad (132)$$

The plane wave states can be written in the notation

$$|+\lambda P\rangle = |e(p,\lambda) P\rangle, \quad |-\lambda P\rangle = |e^*(p,\lambda) -P\rangle. \quad (133)$$

The spin operator $S^3$ is defined as in Eq. (115) with the appropriate generators for $J^i$ and $K^i$. The defining relations for the states are formal identical to Eq. (117) except the different eigenvalues for the polarization. For photons one finds $\lambda = \pm 1$.

The plane wave expansion for the free photon field is given by

$$A(x) = \sum_\lambda \int dP \left( \langle X | +\lambda P\rangle \langle +\lambda P | A \rangle + \langle X | -\lambda P\rangle \langle -\lambda P | A \rangle \right) \quad (134)$$

$$= \sum_\lambda \int \frac{d^3p}{(2\pi)^{3/2}p^0} \left( e^{\mu}(p,\lambda)e^{-ip_\mu x^\mu} a(p,\lambda) + e^{\mu*(p,\lambda)}e^{ip_\mu x^\mu} a^*(p,\lambda) \right) \sigma_\mu. \quad (134)$$

The components of the electromagnetic fields can be obtained from $A(x) = A^{\mu}(x)\sigma^\mu$. Since the solution of the quantum wave equation is expanded in terms of the physical photon states, which are defined by the representations of the Poincaré group, the Lorentz gauge condition is satisfied. Therefore, the $C(x)$ contributions discussed in the last section vanish and the original form of the Maxwell equations is restored.
11. Interactions

As well as in the Dirac theory the Lagrange function of the new formalism should be invariant under local gauge transformations. This has the consequence that a gauge field, which is identified with the photon field, has to be introduced by a substitution of the momentum operators. The free theory then changes into a theory which couples the electron and photon fields in the interaction terms of the Lagrangian.

The gauge field is introduced by replacing the momentum operators according to

\[ P^\mu \mapsto P^\mu - eA^\mu(x), \quad (135) \]

where the charge \( e < 0 \) corresponds to the negative charge of the electron. This minimal substitution of the momentum operators in the Lagrangian of Eq. (79) is leading to

\[ \mathcal{L}(x) = \bar{\psi}(x)(P - eA(x))(\bar{P} - e\bar{A}(x))\psi(x) \]
\[ -m^2\bar{\psi}(x)\psi(x) + \frac{1}{2}Tr(\bar{A}(x)P\bar{P}A(x)) , \quad (136) \]

where an additional term for the photon field has been added to the Lagrangian.

The Lagrangian (136) is comparable to the QED Lagrangian for scalar particles since a seagull term appears which is not present in the Dirac theory. If the gauge field transforms according to Eq. (124), and Eq. (125) is satisfied, this Lagrangian is invariant under a local gauge transformation of the form

\[ \psi(x) \mapsto \psi'(x) = e^{-i\Lambda(x)}\psi(x). \quad (137) \]

Starting from the above Lagrangian the equation of motion for the fermion field can be calculated using Eq. (84)

\[ (P - eA(x))(\bar{P} - e\bar{A}(x))\psi(x) = m^2\psi(x). \quad (138) \]

For the calculation one should separate the basis matrices from the vector components. The \( \sigma_\mu\sigma_\nu \)-terms, which are included in the Lagrangian, are given explicitly in Eqs. (39) and (40).

In the same way the equation of motion for the photons can be derived with the Lagrange equation

\[ \frac{\partial \mathcal{L}}{\partial A^\mu} - \partial^\nu \frac{\partial \mathcal{L}}{\partial (\partial^\nu A^\mu)} = 0. \quad (139) \]

This Lagrange equation has been simply adopted from the Maxwell theory. As well as in the description of the electron field the spin matrices are considered as
basis vectors which have no further effect on the dynamical variables. Using
the above equation a straightforward calculation is leading to

$$P \bar{P} A(x) = -J(x).$$

(140)

The current \(J(x) = J^{\mu}(x)\sigma_\mu\) includes the photon field and the spin matrices \(\sigma_\mu \bar{\sigma}_\nu\) between the spinor functions. Using Eq. (39) one can separate the current into a
spin dependent and a spin independent contribution

$$J^{\mu}(x) = I^{\mu}(x) + K^{\mu}(x),$$

(141)

where the contribution \(I^{\mu}(x)\) is identical to the electromagnetic current for scalar
particles except that two-component spinor functions are used

$$I^{\mu}(x) = -e \bar{\psi}(x)(\vec{P}^{\mu} - e\vec{A}^{\mu}(x))\psi(x) + e \bar{\psi}(x)(\vec{P}^{\mu} + e\vec{A}^{\mu}(x))\psi(x).$$

(142)

The spin dependent part can be expressed with the spin operators \(\sigma^{\mu \nu}\), which are
related to the generators of the covering group of \(SO(3,1)\) according to Eq. (41)

$$K^{\mu}(x) = -ie \bar{\psi}(x)(\vec{P}_\nu - eA_\nu(x))\sigma^{\nu \mu}\psi(x) + ie \bar{\psi}(x)\sigma^{\mu \nu}(\vec{P}_\nu + eA_\nu(x))\psi(x).$$

(143)

It is the current given in Eq. (141) which is conserved when the electromagnetic
field is present.

Comparing Eq. (140) with the Maxwell equations (121) one finds, assuming
\(C(x) = 0\), that the inhomogeneous terms are given in the correct form. Eqs. (138)
and (140) are the new equations of motion of quantum electrodynamics. These
equations are very complicated coupled differential equations and they can be
solved only in a suitable approximation scheme. To show that these equations
provide a reasonable basis for calculating electromagnetic processes, the connection
of the electron wave equation (138) with the Dirac equation will be outlined
in the following.

12. Quantum wave equation and Dirac equation

The Dirac theory is able to explain experimental data with highest accuracy.
The quantum wave equation should therefore be in agreement with the Dirac
equation. One can show that in the case of electromagnetic interactions the
differential operator of the quantum wave equation is equivalent to the quadratic
form of the Dirac differential operator, from which one knows that the energy
spectrum of hydrogen like systems is exactly the same as for the Dirac equation
[17, 18].

To show this relationship, the quantum wave equation (138) will be considered
in detail. A short calculation is leading to

$$((P^0 - eA^0)^2 - (\mathbf{P} - e\mathbf{A})^2 - j[\%P^0 - eA^0, \mathbf{P} - e\mathbf{A}] - m^2)\psi(x) = 0,$$

(144)
where the mass term is now on the left side of the equation. The second term of the equation still includes the Pauli matrices. One can evaluate this expression according to
\[
(P - eA)^2 = (P - eA) \cdot (P - eA) - eB,
\]
where one should compare the notation with Eq. (25). \( B \) corresponds to the magnetic field given in Eq. (120). The commutator, which is proportional to the hyperbolic unit, can be calculated according to
\[
[P^0 - eA^0, P - eA] = ieE,
\]
with the electric field \( E \). Inserting these results into Eq. (144) gives
\[
((P - eA)_\mu(P - eA)^\mu - eE^i i\sigma_i + eB^i\sigma_i - m^2) \psi(x) = 0.
\]
Here, the Pauli matrices \( \sigma_i \) are written explicitly for a better comparison with the Dirac equation. It is possible to express Eq. (147) completely in the relativistic tensor formalism if Pauli matrices and electromagnetic fields are expressed with the antisymmetric tensor \( \sigma_{\mu\nu} \) given in Eq. (10) and \( F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \)
\[
((P - eA)_\mu(P - eA)^\mu - \frac{e}{2}\sigma_{\mu\nu} F^{\mu\nu} - m^2) \psi(x) = 0.
\]
This equation is formal identical to the quadratic form of the Dirac equation which can be derived with the Dirac formalism. The Dirac equation is given by
\[
(\gamma_\mu P^\mu - e\gamma_\mu A^\mu(x) - m)\psi(x) = 0
\]
with the Dirac matrices \( \gamma_\mu \). The quadratic form can be found if one multiplies the Dirac equation by the operator \( \gamma_\mu P^\mu - e\gamma_\mu A^\mu(x) + m \). This yields
\[
((\gamma_\mu P^\mu - e\gamma_\mu A^\mu)^2 - m^2)\psi(x) =
\]
\[
((P - eA)_\mu(P - eA)^\mu - \frac{i}{2}\sigma_{\mu\nu}[P^\mu - eA^\mu, P^\nu - eA^\nu] - m^2)\psi(x) =
\]
\[
((P - eA)_\mu(P - eA)^\mu - \frac{e}{2}\sigma_{\mu\nu} F^{\mu\nu} - m^2)\psi(x) = 0.
\]
The two wave equations of Eqs. (148) and (150) have the same form. However, there are two differences: The first difference is given in the structure of the spinors \( \psi(x) \). In the case of the quantum wave equation \( \psi(x) \) has a two-component structure, whereas in the Dirac equation \( \psi(x) \) corresponds to a four-component spinor
\[
\text{Quantum wave equation : } \psi(x) = \varphi(x) + j\chi(x),
\]
\[
\text{Dirac equation : } \psi(x) = \begin{pmatrix} \varphi(x) \\ \chi(x) \end{pmatrix}.
\]
The other difference is the spin tensor $\sigma_{\mu\nu}$. In the Dirac theory this term is defined according to $\sigma_{\mu\nu} = i/2 [\gamma_{\mu}, \gamma_{\nu}]$. With this tensor one is able to express Eq. (150) according to

\[ \left( (P - eA)_{\mu}(P - eA)^\mu - eE^i i\alpha_i + eB^i \sigma_i - m^2 \right) \psi(x) = 0. \]  

(152)

Comparing this equation with Eq. (147) one observes that in both cases the term including the electric field is the only term which couples the upper and the lower component of the spinor. In the quantum wave equation the coupling term is proportional to $j\sigma_i$, in the quadratic Dirac equation the term corresponds to $\alpha_i = \gamma_0 \gamma_i = \gamma_5 \sigma_i$. One can show, using the Dirac representation of $\gamma_5$, that $j$ and $\gamma_5$ have the same effect on the spinor, an interchange between upper and lower components

\[ j\psi(x) = \chi(x) + j\varphi(x), \]  

(153)

\[ \gamma_5 \psi(x) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \varphi(x) \\ \chi(x) \end{pmatrix} = \begin{pmatrix} \chi(x) \\ \varphi(x) \end{pmatrix}. \]

One therefore finds in both cases the same two coupled differential equations. In the quantum wave equation the terms proportional to the hyperbolic unit belong to one differential equation, the other terms to the second equation. In the quadratic Dirac equation the differential equations are separated by the component structure.

13. Summary and Conclusions

In this work a new formulation of quantum electrodynamics was presented. The motivation for the investigation was the assumption that fundamental free quantum fields should be described by a unified differential operator, which does not depend on any particle properties. In the case of electrons and photons it was shown that, concerning the spin structure, this unification is possible. The wave equation constructed with this differential operator has been denoted by quantum wave equation. It is leading to the same coupled differential equations for electrons as the quadratic form of the Dirac equation. Furthermore, the quantum wave equation for photons is equivalent to the four Maxwell equations.

These results can be obtained if the basis vectors of the relativistic vector coordinates are represented by a relativistic matrix algebra which includes the unit matrix and the Pauli matrices multiplied by the hyperbolic unit. Relativistic vectors based on this matrix algebra were investigated. Their behaviour under Lorentz transformations was studied and a relativistic spin group was constructed in analogy to the non-relativistic treatment. The quantum wave equation for photons shows that vector fields appear in combination with the same matrix algebra.
Therefore, it seems to be a general property of quantum physics that vector coordinates have to be considered together with the corresponding algebra for the basis vectors.

It was shown that the differential operator of the quantum wave equation transforms like a spin operator. However, for a non-interacting system the spin structure of the differential operator is not important and the operator is reduced to the mass operator of the Poincaré group. Therefore, the properties of the free electron field have been investigated within this group and the plane wave representation was generated explicitly. It was shown that a boosted Pauli matrix can be related to the Pauli-Lubanski vector. This offers the possibility to define the spin in analogy to the non-relativistic treatment. Former problems in the second quantization of anticommuting Klein-Gordon fields disappear if the negative energy contribution is multiplicated by the hyperbolic unit. A free massless charged fermion field has been investigated, where the plane wave expansion of the field can be constructed in analogy to the massive electron field.

The quantum wave equation for photons is leading to the Maxwell equations including two additional terms. These terms vanish in the Lorentz gauge. A plane wave expansion for the photon field was derived, which removes the additional terms and stays in close analogy to the description of the electrons.

Though the spin structure is not important in a non-interacting system, the inclusion of interactions requires the given form of the quantum wave equation. A new Lagrange function of quantum electrodynamics can be constructed using the concept of minimal substitution. The equations of motion for the electron and photon fields were derived and the relation of the electron equation to the quadratic form of the Dirac equation was shown. The equation of motion for the photons is equivalent to the inhomogeneous Maxwell equations. Since the new formalism can be related with the Dirac and the Maxwell equations one can expect that explicit calculations of observables will lead to results which should be close to the results of the conventional formalism.

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