Dual double EPW-sextics and their periods

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1 Introduction

Eisenbud, Popescu and Walter have constructed certain singular sextic hypersurfaces (EPW-sextics) in $\mathbb{P}^5$ (Example (9.3) of [4]) which come provided with a natural double cover: we have shown [13] that the generic such double cover is a deformation of the Hilbert square of a $K3$ and that the family of double EPW-sextics is a locally versal family of projective deformations of $(K3)^2$. Thus the family of double EPW-sextics is similar to the family of Fano varieties of lines on a cubic 4-fold (see [2]), with the following difference: the Plücker ample divisor on the Fano variety of lines has square 6 for the Beauville-Bogomolov quadratic form (see [1, 2]) while the natural polarization of a double EPW-sextic has square 2 (see [13]). Let $Y \subset \mathbb{P}^5$ be a generic EPW-sextic: we proved in [13] that the dual $Y^\vee \subset (\mathbb{P}^5)^\vee$ is another generic EPW-sextic. Thus we may associate to the natural double cover $X$ of $Y$ a “dual” variety $X^\vee$ namely the natural double cover of $Y^\vee$. This construction defines a (rational) involution on the moduli space of double EPW-sextics. In [13] we showed that a generic EPW-sextic is not self-dual and hence the involution on the moduli space of double EPW-sextics is not the identity; in this paper we determine the relation between the periods of a double EPW-sextic and its dual. Before stating the result we recall the definition of EPW-sextics. Let $V$ be a 6-dimensional $\mathbb{C}$-vector space. Choose an isomorphism $\text{vol}: \wedge^6 V \xrightarrow{\sim} \mathbb{C}$ and let $\omega$ be the symplectic form on $\wedge^3 V$ defined by wedge product followed by $\text{vol}$. Let $P(V)$ be the projective space of 1-dimensional sub vector spaces of $V$; then $\omega$ gives $\wedge^3 V \otimes O_P(V)$ the structure of a symplectic vector-bundle of rank 20. Let $F$ be the sub-vector-bundle of $\wedge^3 V \otimes O_P(V)$ whose fiber $F_v$ over $[v] \in P(V)$ consists of tensors divisible by $v$:

$$F_v := \{v \wedge w | w \in \wedge^2 V\}. \quad (1.0.1)$$

As is easily checked $F$ is a Lagrangian sub-bundle of $\wedge^3 V \otimes O_P(V)$. Let $LG(\wedge^3 V)$ be the symplectic Grassmannian parametrizing $\omega$-Lagrangian subspaces of $\wedge^3 V$. For $A \in LG(\wedge^3 V)$ we let $\lambda_A$ be the composition

$$F \rightarrow \wedge^3 V \otimes O_P(V) \rightarrow (\wedge^3 V/A) \otimes O_P(V) \quad (1.0.2)$$

and $Y_A \subset P(V)$ be the zero-scheme of $\det(\lambda_A)$; unless $\det(\lambda_A)$ is identically zero

$Y_A$ is a sextic hypersurface because $\det F \cong O_P(V)(-6)$. An EPW-sextic is a

\footnote{If $A$ is the fiber of $F$ over a fixed point then $\det(\lambda_A)$ is identically zero.}
hypersurface in \( \mathbb{P}(V) \) which is equal to \( Y_A \) for some \( A \in \mathbb{L}(\Lambda^3 V) \). In [13] we described explicitly the non-empty Zariski-open \( \mathbb{L}(\Lambda^3 V)^0 \subset \mathbb{L}(\Lambda^3 V) \) parametrizing \( A \) such that \( Y_A \) has no singularities other than those forced by its description as a degeneracy locus; for \( A \in \mathbb{L}(\Lambda^3 V)^0 \) the singular locus of \( Y_A \) is a smooth degree-40 irreducible surface and at a singular point \( Y_A \) is locally (in the classical topology) isomorphic to the product of \( \text{sing}(Y_A) \) and an \( A_1 \)-singularity. If \( A \in \mathbb{L}(\Lambda^3 V)^0 \) there is a natural double cover \( f_A : X_A \to Y_A \) ramified only over \( \text{sing}(Y_A) \), with \( X_A \) smooth. As shown in [13] the 4-fold \( X_A \) is a deformation of \( (K3)^2 \) - the Hilbert square of a K3. Let \( L_A \) be the line-bundle on \( X_A \) defined by

\[
L_A := f_A^* \mathcal{O}_{Y_A}(1);
\]

then \( c_1(L_A) \) has square 2 for the Beauville-Bogomolov quadratic form. The family of \( (X_A, L_A) \) for \( A \) varying in \( \mathbb{L}(\Lambda^3 V)^0 \) is a locally complete family of polarized deformations of \( (K3)^2 \). Now we recall the duality map. Let \( \text{vol}^\vee \) be a trivialization of \( \Lambda^6 V^\vee \) and \( \omega^\vee \) be the symplectic form on \( \Lambda^3 V^\vee \) given by wedge-product followed by \( \text{vol}^\vee \); let \( \mathbb{L}(\Lambda^3 V^\vee) \) be the symplectic Grassmannian parametrizing \( \omega^\vee \)-Lagrangian subspaces of \( \Lambda^3 V^\vee \). For \( A \in \mathbb{L}(\Lambda^3 V) \) the annihilator \( A^\perp \subset \Lambda^3 V^\vee \) belongs to \( \mathbb{L}(\Lambda^3 V^\vee) \). Thus we have an isomorphism

\[
\delta : \mathbb{L}(\Lambda^3 V) \xrightarrow{\sim} \mathbb{L}(\Lambda^3 V^\vee)
\]

Let

\[
\mathbb{L}(\Lambda^3 V)^{00} := \mathbb{L}(\Lambda^3 V)^0 \cap \delta^{-1} \mathbb{L}(\Lambda^3 V^\vee)^0.
\]

Thus \( \mathbb{L}(\Lambda^3 V)^{00} \) is open dense in \( \mathbb{L}(\Lambda^3 V) \). Assume that \( A \in \mathbb{L}(\Lambda^3 V)^{00} \). By Proposition (3.1) of [13] \( Y_A^\perp \) is \( Y_A^\vee \) (the dual of \( Y_A \)); we let

\[
X_A^\vee \to Y_A^\vee = Y_A^\perp
\]

be the natural double cover and we call \( X_A^\vee \) the dual of \( X_A \). We will show how to obtain the Hodge structure on \( H^2(X_A^\vee) \) from the Hodge structure on \( H^2(X_A) \). First we describe the relevant period space. Let \( \Lambda \) be the even lattice

\[
\Lambda := U^3 \oplus (-E_8)^2 \oplus (-2)
\]

where \( U \) is the hyperbolic plane and \((-2)\) is the lattice generated by a single element of square \(-2\); we will denote by \((\cdot, \cdot)\) the symmetric bilinear form on \( \Lambda \). Let \( U \) be one of the hyperbolic planes appearing in Decomposition [13.0.7] and choose

\[
u \in U \text{ of square } 2.
\]

Let \( e_1 \in U \) be a generator of \( u^\perp \cap U \) and \( e_2 \) be a generator of the direct summand \((-2)\) appearing in Decomposition [13.0.7]. Then

\[
\Lambda := u^\perp = U^3 \oplus (-E_8)^2 \oplus \mathbb{Z} e_1 \oplus \mathbb{Z} e_2 \cong U^3 \oplus (-E_8)^2 \oplus (-2)^2.
\]

The period domain of interest to us is

\[
\mathcal{D}_2 := \{ [v] \in \mathbb{P}(\Lambda \otimes \mathbb{C}) | (v, v) = 0, \ (v, \overline{v}) > 0 \}.
\]

Let \( \text{Stab}(u) < O(\Lambda) \) be the subgroup of isometries fixing \( u \) and \( \rho : \text{Stab}(u) \to O(\Lambda) \) be the restriction map - this makes sense because \( \Lambda = u^\perp \). Let

\[
\Gamma := \text{Im}(\rho) < O(\Lambda).
\]

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The period moduli space is the quotient
\[ \mathcal{Q}_2 := \Gamma \backslash \mathcal{D}_2. \]

We will prove in Subsection (2.2) that
\[ [O(\Lambda) : \Gamma] = 2. \]

An explicit \( r \in (O(\Lambda) \backslash \Gamma) \) is the isometry which interchanges \( e_1 \) with \( e_2 \) and fixes the elements of \( \{e_1, e_2\}^\perp \), see (2.2.12). By (1.0.13) the involution \( r \) descends to a non-trivial involution
\[ \tau : \mathcal{Q}_2 \to \mathcal{Q}_2. \]

Let \( A \in \mathcal{L}_G(\wedge^3 V)_0^0 \): the isomorphism class of the Hodge structure on \( c_1(L_A)^\perp \subset H^2(X_A) \) is a point of \( \mathcal{Q}_2 \) (see Section (2)) that we denote by \( \mathcal{P}(A) \) - this is the period point of \( X_A \). The global period map is defined by
\[ \mathcal{L}_G(\wedge^3 V)^0_0 \overset{\mathcal{P}}{\rightarrow} \mathcal{Q}_2 \quad A \mapsto \mathcal{P}(A). \]

Theorem 1.1. Keep notation as above. If \( A \in \mathcal{L}_G(\wedge^3 V)^0_0 \) then
\[ \mathcal{P}(A^\perp) = \tau \circ \mathcal{P}(A). \]

The theorem may be stated informally as follows: the polarized Hodge structures on \( c_1(L_A)^\perp \perp \subset H^2(X_A) \) are always isomorphic while in general there is no Hodge isometry between \( H^2(X_A) \) and \( H^2(X_A^\perp) \).

There are three steps in the proof of the above theorem. First of all there exists a rational Hodge isometry
\[ H^2(X_A) \supset c_1(L_A)^\perp \cong c_1(L_A^\perp)^\perp \subset H^2(X_A^\perp). \]

In fact let \( \tilde{Y}_A, \tilde{Y}_A^\perp \) be the desingularizations of \( Y_A, Y_A^\perp \) respectively - they are obtained by blowing up \( \text{sing}(Y_A) \) and \( \text{sing}(Y_A^\perp) \) respectively. From the equality \( Y_A^\vee = Y_A^\perp \) one gets that
\[ \tilde{Y}_A \cong \tilde{Y}_A^\perp. \]

Now \( H^4(Y_A) \) is a subgroup of \( \text{Sym}^2 H^2(X_A) \) and similarly for \( H^4(Y_A^\perp) \); this fact together with (1.0.18) gives (1.0.17) after some work. It follows from (1.0.17) that locally (in the classical topology) there exists \( g \in O(\Lambda \otimes \mathbb{Q}) \) relating the periods of \( X_A \) and \( X_A^\perp \); this means that we have
\[ \mathcal{P}_\Phi(A^\perp) = g \circ \mathcal{P}_\Phi(A) \]
where \( \mathcal{P}_\Psi, \mathcal{P}_\Phi \) are local liftings of the global period maps (\( \Psi, \Phi \) are markings of the weight-2 cohomology). The second step consists in showing that (1.0.16) holds for \( A \) belonging to a certain locally closed codimension-1 submanifold \( \Delta^0_0(V) \subset \mathcal{L}_G(\wedge^3 V)^0_0 \). If \( A \in \Delta^0_0(V) \) then \( X_A \) is isomorphic to a moduli space of sheaves on a \( K3 \) surface \( S \subset \mathbb{P}^6 \) of degree 10. On the other hand there is a map
Let \( f : S^{[2]} \to Y_{A^+} \) of degree 2 which contracts a certain plane \( P \subset S^{[2]} \) and is finite over \( Y_{A^+} \setminus f(P) \). One shows that \( \text{mult}_f(P)Y_{A^+} = 3 \) and hence \( A^+ \not\in LG(\wedge^3 V^*)^0 \), i.e. \( A \not\in LG(\wedge^3 V^*)^0 \). Morally, \( X_{A^+} \) is the singular symplectic 4-fold obtained from \( S^{[2]} \) by contracting the plane \( P \). We do not prove a precise version of this; however we do prove that the local period map of \( LG(\wedge^3 V^*)^0 \) extends across \( A^+ \) and that its value is given by the periods of \( S^{[2]} \). This will allow us to check that (1.0.10) holds for \( A \in \Delta^0(V) \). Since the local period map sends \( \Delta^0(V) \) to the intersection of \( D_2 \) and a hyperplane \( \zeta^\perp \) it will follow that the rational isometry \( g \) of (1.0.19) is (with suitable markings \( \Psi, \Phi \)) either \( \sigma \) or \( r \circ r_\zeta \) where \( r_\zeta \) is the reflection in the span of \( \zeta \). We rule out this second possibility by a monodromy argument: this is the third step in the proof of Theorem [1.1].

The paper is organized as follows. In Section (2) we recall the definition of local and global period maps and we prove (1.0.13). In the next section we prove the results on \( \Delta^0 \) and we prove (1.0.14). In the final section we prove the existence of a rational Hodge isometry (1.0.17) and we give the monodromy argument.

**Notation:** \( V \) will always be a complex vector-space of dimension 6.

## 2 The period map and the lattice \( \Lambda \)

### 2.1 The period map

Let \( X \) be a deformation of \((K3)^{[2]}\). Let \( (\cdot, \cdot)_X \) be the Beauville-Bogomolov symmetric bilinear form on \( H^2(X) \). The restriction of \( (\cdot, \cdot)_X \) to \( H^2(X;\mathbb{Z}) \) is a non-degenerate integral symmetric form; thus \( H^2(X;\mathbb{Z}) \) is a lattice. As is well-known \( H^2(X;\mathbb{Z}) \) and \( \tilde{\Lambda} \) are isometric. Now assume that we are given a holomorphic line-bundle \( L \) on \( X \) such that \( (c_1(L), c_1(L))_X = 2 \). Since \( O(\tilde{\Lambda}) \) acts transitively on the set of vectors of square 2 there exists an isometry

\[
\psi : H^2(X;\mathbb{Z}) \xrightarrow{\sim} \tilde{\Lambda}
\]

such that \( \psi(c_1(L)) = u \) where \( u \) is as in (1.0.8); this is a marking of \((X, L)\). Let \( \psi_C : H^2(X;\mathbb{C}) \xrightarrow{\sim} \tilde{\Lambda} \otimes \mathbb{C} \) be the map obtained from \( \psi \) by extension of scalars. Let \( \sigma \) be a generator of \( H^{2,0}(X) \); then

\[
(c_1(L), \sigma)_X = 0 = (\sigma, \sigma)_X, \quad (\sigma, \overline{\sigma})_X > 0.
\]

Thus

\[
\mathcal{P}_\psi(X, L) := \psi_C(H^{2,0}(X)) \in D_2.
\]

This is the **local period point of** \((X, L)\) **associated to** \( \psi \) (or periods of \((X, L)\)). Any other marking of \((X, L)\) is given by \( \gamma \circ \psi \) where \( \gamma \in \Gamma \); thus the \( \Gamma \)-orbit of \( \mathcal{P}_\psi(X, L) \) is a well-defined point of \( Q_2 \); this is the **global period point of** \((X, L)\), we denote it by \( \mathcal{P}(X, L) \). If \( A \in LG(\wedge^3 V^*)^0 \) we let \( \mathcal{P}(A) := \mathcal{P}(X_A, L_A) \).

Later on we will study global and local period maps for certain families of double EPW-sextics. Let

\[
\pi : \mathcal{X} \to T
\]

be a proper submersive map between complex manifolds such that each fiber is a deformation of \((K3)^{[2]}\). For \( t \in T \) we let \( X_t := \pi^{-1}(t) \). We assume that we
are given a (holomorphic) line-bundle $L$ on $X$ such that $c_1(L|_{X_t})$ has square 2 for every $t \in T$; we let $L_t := L|_{X_t}$. The global period map of $(X, L)$ is given by

$$T \xrightarrow{\mathcal{P}} \mathbb{Q}_2 \quad t \mapsto \mathcal{P}(t) := \mathcal{P}(X_t, L_t) \quad (2.1.5)$$

Griffiths proved that $\mathcal{P}$ is a holomorphic map. Let $\pi: X \to LG(\wedge^3 V)^0$ be the tautological family of double EPW-sextics and $L$ be the tautological relatively ample line-bundle on $X$ which restricts to $L_A$ (see (1.0.3)) on $X_A$; then $\mathcal{P}$ is the period map of (1.0.15), in particular the map of (1.0.15) is holomorphic. Let’s go back to a general family (2.1.4): it is not always possible to lift the global period map $\mathcal{P}$ to a map $T \to \mathbb{D}^2$, in fact a necessary and sufficient condition is that $R^2\pi_*\mathbb{Z}$ is trivial. Suppose that $R^2\pi_*\mathbb{Z}$ is trivial. Then there exists a trivialization $\Psi: R^2\pi_*\mathbb{Z} \xrightarrow{\sim} T \times \tilde{\Lambda}$ sending the section corresponding to $c_1(L)$ to the section given by $u$ - this is a marking of $(X, L)$. Given such a marking we let $\Psi_t: H^2(X_t; \mathbb{Z}) \xrightarrow{\sim} \tilde{\Lambda}$ be the fiber of $\Psi$ over $t$. The local period map of $(X, L)$ associated to $\Psi$ is given by

$$T \xrightarrow{\mathcal{P}_\Psi} \mathbb{D}_2 \quad t \mapsto \mathcal{P}_\Psi(t) := \mathcal{P}_{\Psi_t}(X_t, L_t). \quad (2.1.6)$$

Griffiths proved that $\mathcal{P}_\Psi$ is holomorphic.

### 2.2 Proof of (1.0.13)

We start by recalling the definition of discriminant group and discriminant quadratic form of an even lattice $(L, (\cdot, \cdot)_L)$, i.e. a free finitely generated abelian group $L$ equipped with a symmetric integral even non-degenerate bilinear form $(\cdot, \cdot)_L$. We follow [10]. The bilinear form $(\cdot, \cdot)_L$ extends to a $\mathbb{Q}$-valued bilinear form on $L \otimes \mathbb{Q}$; abusing notation we denote by $(\cdot, \cdot)_L$ the extended form. Let $L^\vee := Hom(L, \mathbb{Z})$; by non-degeneracy of $(\cdot, \cdot)_L$ we have a natural chain of inclusions

$$L \subset L^\vee \subset L \otimes \mathbb{Q}. \quad (2.2.1)$$

The discriminant group of $L$ is $A_L := L^\vee / L$; it comes provided with the discriminant bilinear-form

$$A_L \times A_L \xrightarrow{b_L} \mathbb{Q}/\mathbb{Z} \quad ([\alpha], [\beta]) \mapsto [(\alpha, \beta)_L] \quad (2.2.2)$$

and the discriminant quadratic-form

$$A_L \xrightarrow{q_L} \mathbb{Q}/2\mathbb{Z} \quad [\alpha] \mapsto [(\alpha, \alpha)_L]. \quad (2.2.3)$$

The formula

$$q_L(\alpha + \beta) \equiv q_L([\alpha]) + q_L([\beta]) + 2b_L([\alpha], [\beta]) \pmod{2\mathbb{Z}} \quad (2.2.4)$$

shows that $q_L$ determines uniquely $b_L$. An index-$i$ overlattice of $L$ consists of a lattice $M$ and an inclusion of lattices $L \subset M$ (the restriction of $(\cdot, \cdot)_M$ to $L$ is equal to $(\cdot, \cdot)_L$) of index $i$. Two overlattices $M_1 \supset L \subset M_2$ are equivalent if there
exists an isometry $M_1 \sim_{\sim} M_2$ which is the identity on $L$, i.e. if the inclusions $M_i \hookrightarrow L$ for $i = 1, 2$ have the same image. Suppose that $L \subset M$ is an index-$i$ overlattice of $L$; the inclusion $M \subset L$ defines an inclusion $M/L \subset A_L$ with image a subgroup of cardinality $i$ which is $q_L$-isotropic. This construction defines a one-to-one correspondence between the set of equivalence classes of index-$i$ overlattices of $L$ and the set of $q_L$-isotropic subgroups of $A_L$ of cardinality $i$. The correspondence is equivariant for the natural actions of $O(L)$ on both sets.

Now consider the lattice $L = \mathbb{Z}u \oplus \Lambda$ where $u$ and $\Lambda$ are as in Section (1): we will describe the discriminant group and discriminant form of $L$. Let $e_1, e_2 \in \Lambda$ be as in Section (1). A straightforward computation gives that

$$A_L = \mathbb{Z}/(2) \left[ \frac{u}{2} \right] \oplus \mathbb{Z}/(2) \left[ \frac{e_1}{2} \right] \oplus \mathbb{Z}/(2) \left[ \frac{e_2}{2} \right] \quad (2.2.5)$$

and that

$$q_L \left( x \left[ \frac{u}{2} \right] + y_1 \left[ \frac{e_1}{2} \right] + y_2 \left[ \frac{e_2}{2} \right] \right) \equiv \frac{1}{2} (x^2 - y_1^2 - y_2^2) \pmod{2\mathbb{Z}}. \quad (2.2.6)$$

The set $I \subset A_L$ of non-zero isotropic vectors is given by

$$I = \left\{ \left[ \frac{u}{2} \right] + \left[ \frac{e_1}{2} \right], \left[ \frac{u}{2} \right] + \left[ \frac{e_2}{2} \right] \right\}. \quad (2.2.7)$$

The group $O(L)$ acts naturally on $A_L$ and hence also on $I$; thus we have a homomorphism

$$O(L) \rightarrow Aut(I) \cong \mathbb{Z}/(2). \quad (2.2.8)$$

The overlattice $\tilde{\Lambda} \supset L$ is of index 2 because $L$ is the kernel of the surjection

$$\tilde{\Lambda} \twoheadrightarrow \mathbb{Z}/(2) \quad (2.2.9)$$

The correspondence described above defines an $O(L)$-equivariant bijective map between $I$ and the set of equivalence classes of index-2 overlattices of $L$; thus

$$\text{Im}(O(\tilde{\Lambda}) \rightarrow O(L)) = \ker(\epsilon). \quad (2.2.10)$$

The subgroup of $O(L)$ consisting of isometries which are the identity on $\mathbb{Z}u$ is naturally identified with $O(\Lambda)$; thus $O(\Lambda) < O(L)$. By (2.2.11) and the definition of $\Gamma$ (see (1.0.11)) we get that

$$\Gamma = \ker(\epsilon|_{O(\Lambda)}). \quad (2.2.11)$$

Let $r \in O(\Lambda)$ be the involution characterized by the following properties:

$$r(e_1) = e_2, \quad r(e_2) = e_1, \quad r|_{\{e_1, e_2\}^\perp} = \text{identity}. \quad (2.2.12)$$

Then $\epsilon(r)$ is the non-trivial permutation of $I$ and hence $\epsilon|_{O(\Lambda)}$ is surjective. Thus (1.0.13) follows from (2.2.11).

### 3 Explicit dual couples

Let $F_5^3 \subset P^6$ be the intersection of $Gr(2, \mathbb{C}^5) \subset P(\wedge^2 \mathbb{C}^5) \cong P^9$ with a transversal 6-dimensional linear subspace of $P^9$. Let $T \subset |O_{ps}(2)|$ be the open dense subset
parametrizing quadrics which are transversal to $F^3_3$. For $t \in T$ let $Q_t$ be the quadric corresponding to $t$ and 

$$S_t := F^3_3 \cap Q_t. \quad (3.0.1)$$

Then $S_t$ is a degree-10 linearly normal $K3$ surface; in fact the generic such surface is projectively equivalent to $S_t$ for some $t \in T$ by Mukai [8]. Let $D_t$ be the hyperplane divisor class on $S_t$ and $M_t$ be the moduli space of $D_t$-semistable rank-2 sheaves $\mathcal{F}$ on $S_t$ with $c_1(\mathcal{F}) = c_1(D_t)$ and $c_2(\mathcal{F}) = 5$. Suppose that

for all divisors $E$ on $S_t$ we have $E \cdot D_t \equiv 0 \pmod{10}$. \quad (3.0.2)

Then (see Section (5) of [13]) there exists $A_t \in LG(\wedge^3 V)^0$ such that

$$X_{A_t} \cong M_t. \quad (3.0.3)$$

Furthermore $Y_{A_t}$ is explicitly described as follows. Let $\Sigma_t$ be the divisor on $|I_S(2)|$ parametrizing singular quadrics, since all quadrics parametrized by $|I_{F^3_3}(2)|$ are singular we have

$$\Sigma_t = |I_{F^3_3}(2)| + \Sigma'_t. \quad (3.0.4)$$

Then

$$Y_{A_t} \cong \Sigma'_t. \quad (3.0.5)$$

(Regularly $\Sigma'_t$ is a degree-6 divisor; in the above equation we are implicitly stating that $\Sigma'_t$ is a reduced divisor and hence we may view it as a degree-6 hypersurface.) The set of $t$ for which (3.0.2) holds is the complement of a countable union of proper algebraic subvarieties of $T$ however a straightforward argument shows that there is an open dense $T'' \subset T$ such that for $t \in T''$ there exists $A_t \in LG(\wedge^3 V)^0$ for which both (3.0.3) and (3.0.5) hold; we give the argument in Subsection (3.1). We let $\Delta^0(V) \subset LG(\wedge^3 V)^0$ be the set of $A$ such that $X_A$ is isomorphic to $M_t$ for some $t \in T''$; this is a locally closed subset of $LG(\wedge^3 V)^0$. Computing the periods of $X_A$ for $A \in \Delta^0(V)$ we will show that $\Delta^0(V)$ has codimension 1 in $LG(\wedge^3 V)^0$. Let $A \in \Delta^0(V)$, thus $X_A \cong M_t$ for some $t \in T$: the dual $Y_A^*$ is described as follows. The Hilbert scheme $S^*_t$ contains a copy of $\mathbb{P}^2$, call it $P_t$, parametrizing $Z \subset S_t$ which span a line contained in $F^3_3$. Let $S^*_t \to N_t$ be the contraction of $P_t$ - thus $N_t$ is a singular symplectic variety. There is an involution on $N_t$ whose quotient is isomorphic to $Y_A^*$. From this it will follow that $A \not\in LG(\wedge^3 V)^0$ and hence apparently it will not make sense to compute $\mathcal{P}(A^\perp)$. The main point of this subsection is to prove that the period map extends across $A^\perp$, in fact the local period map extends and its value at $A^\perp$ is given by the periods of $(S^*_t, D_t)$.  

### 3.1 The locus $\Delta^0(V)$

We recall that the Mukai vector $v(\mathcal{F})$ of a sheaf $\mathcal{F}$ on $S_t$ is

$$v(\mathcal{F}) := ch(\mathcal{F}) \sqrt{Td(S_t)} = ch(\mathcal{F})(1 + \eta_t) \in H^*(S_t; \mathbb{Z}), \quad (3.1.1)$$

where $\eta_t \in H^4(S_t; \mathbb{Z})$ is the orientation class. If $[\mathcal{F}] \in M_t$ (we let $[\mathcal{F}] \in M_t$ be the equivalence class of the semistable sheaf $\mathcal{F}$ - we recall that if $\mathcal{F}$ is stable this
is the same as the isomorphism class of $\mathcal{F}$ the class $\vartheta(\mathcal{F})$ is independent of $\mathcal{F}$, we denote it by $\vartheta_1$; explicitly
\begin{equation}
\vartheta_1 = 2 + c_1(D_t) + 2\eta_t.
\end{equation}

**Proposition 3.1.** Keep notation as above. There is an open dense subset $T''' \subset T$ such that the following holds. Let $t \in T'''$; then there exists $A_t \in \mathbb{G}^{(3)}(V) \setminus \{0\}$ such that both (3.0.3) and (3.0.4) hold. In particular we have a canonical identification $\mathbb{P}(V) \cong [I_S(2)]$.

**Proof.** By Maruyama [7] there exists a projective map $\rho: \mathcal{M} \to T$ with (schematic) fiber $M_t$ over $t \in T$. Let $t \in T$; we say that $S_t$ is unsuitable if there exists a divisor class $C$ on $S_t$ such that
\begin{equation}
\int_{S_t} c_1(C) \wedge c_1(D_t) = 0, \quad -10 \leq \int_{S_t} c_1(C)^2 < 0.
\end{equation}
The set of unsuitable $t$ is a proper closed subset of $T$, thus the complement $T'$ is an open dense subset of $T$. Let $\mathcal{M}' := \rho^{-1}(T')$ and $\rho': \mathcal{M}' \to T'$ be the restriction of $\rho$. It is known that if $t \in T'$ then every sheaf parametrized by $M_t$ is slope-stable and $M_t$ is a smooth 4-dimensional scheme (Main Theorem (0.1.2) and Proposition (2.1) of [11] - notice that $t \in T_1$ if and only if $D_t$ is $|\vartheta_1|$-generic).

Let’s show that there is an open dense $T'' \subset T'$ such that for $t \in T''$ the Mukai reflection (see (4.2.2) of [12]) is a regular involution $\phi_t$ on $M_t$ - we recall that in general the Mukai reflection acts on the derived category of coherent sheaves on $S_t$. In order for $\phi_t$ to be a regular involution on $M_t$ it suffices that for all $[\mathcal{F}] \in M_t$ the following hold:
\begin{enumerate}
\item[(a)] $h^0(\mathcal{F}) = \chi(\mathcal{F}) = 4$,
\item[(b)] $\mathcal{F}$ is globally generated away from (at most) a zero-dimensional subset of $S_t$,
\item[(c)] the kernel of the evaluation map $H^0(\mathcal{F}) \otimes \mathcal{O}_{S_t} \to \mathcal{F}$, call it $\mathcal{E}$, is a $D_t$-slope-stable sheaf.
\end{enumerate}

If (a)-(b)-(c) are satisfied for all $[\mathcal{F}] \in M_t$ then the generic sheaf $\mathcal{F}$ parametrized by $M_t$ is globally generated and for such a sheaf $\phi_t([\mathcal{F}] = [\mathcal{E}])$ where $\mathcal{E}$ is as in (c) above. Let $T'' \subset T'$ be the set of $t$ such that (a)-(b)-(c) hold for every $[\mathcal{F}] \in M_t$. Let’s prove that $T''$ is open. First we show that the subset $T_a \subset T$ of $t \in T'$ such that (a) holds for every $[\mathcal{F}] \in M_t$ is open. Let $t \in T'$: if $[\mathcal{F}] \in M_t$ then by stability $h^2(\mathcal{F}) = 0$ and hence $h^0(\mathcal{F}) \geq 4$, thus by the upper-semicontinuity Theorem the set of $[\mathcal{F}] \in M_t$ that violate (a) is closed. By properness of the map $\rho': \mathcal{M}' \to T'$ it follows that $(T' \setminus T_a)$ is closed i.e. $T_a$ is open. A similar argument shows that the set of $t \in T_a$ such that (b)-(c) hold for every $[\mathcal{F}] \in M_t$ is open; thus $T''$ is open. Since $T'''$ contains the subset of $t$ such that (3.0.2) holds it is dense in $T$. Let $T''' \subset T''$ be the set of $t$ such that $S_t$ contains no effective non-zero divisor $E$ with $E \cdot D_t \leq 5$. (3.1.4)

Thus $T'''$ is open and dense in $T$. We claim that if $t \in T'''$ then
\begin{equation}
M_t/\langle \phi_t \rangle \cong \Sigma'_t \subset [I_S(2)] \cong \mathbb{P}^5.
\end{equation}

More precisely Proposition (5.1) of [12] holds for $M_t = \mathcal{M}(\vartheta_1)$. In order to prove this it suffices to show that for every $t \in T'''$ the following holds:
(1) If \([\mathcal{F}] \in M_t\) and \(\sigma \in H^0(\mathcal{F}) = \text{Hom}(\mathcal{O}_{S_t}, \mathcal{F})\) is non-zero then the quotient \(\mathcal{F}/\text{Im}(\sigma)\) is locally-free in codimension 1 (\(\sigma\) has isolated zeroes) - this is used in the proof of Lemma (5.4) of [12].

(2) If \(\mathcal{G}\) is a globally generated rank-2 vector bundle on \(S_t\) with \(\det \mathcal{G} \cong \mathcal{O}_{S_t}(D_t)\) and \(c_2(\mathcal{G}) = 5\) then \(\mathcal{G}\) is \(D_t\)-slope stable - this is used in the proof of Lemma (5.7) of [12].

Let’s show that (1) above holds. Suppose that \(\sigma\) does not have isolated zeroes: then it vanishes along a non-zero effective divisor \(E\) and by slope stability of \(\mathcal{F}\) we have \(E \cdot D_t < \text{slope}(\mathcal{F}) = 5\) contradicting \((3.1.4)\). Let’s show that (2) above holds. Suppose that \(\mathcal{G}\) is not \(D_t\)-slope stable; since \(\mathcal{G}\) has rank 2 there is a destabilizing sequence \(\mathcal{G} \to I_Z \otimes \mathcal{O}_{S_t}(E)\) where \(Z \subset S_t\) is zero-dimensional and \(E\) is a divisor with \(E \cdot D_t \leq \text{slope}(\mathcal{F}) = 5\). Since \(\mathcal{G}\) is globally generated there is a non-zero section of \(I_Z \otimes \mathcal{O}_{S_t}(E)\) and hence \(E\) is effective; by \((3.1.4)\) we get that \(E = 0\). Thus we have an exact sequence

\[
0 \rightarrow \mathcal{O}_{S_t}(D_t) \rightarrow \mathcal{G} \rightarrow I_Z \rightarrow 0. \tag{3.1.6}
\]

Since \(\ell(Z) = c_2(\mathcal{G}) = 5\) the zero-dimensional subscheme \(Z \subset S_t\) is non-empty and hence \(h^0(I_Z) = 0\); this contradicts the hypothesis that \(\mathcal{G}\) is globally generated. We have proved that \((3.1.5)\) holds for \(t \in T''\). By Theorem (1.1) of [13] we get that there exists \(A_t \in \mathbb{L}G(\wedge^3 V)^0\) such that both \((3.0.3)\) and \((3.0.5)\) hold.

The proof of the above proposition together with Claim (5.18) of [12] gives the following result.

**Corollary 3.2.** Let \(t \in T'''\) and let \(A_t \in \mathbb{L}G(\wedge^3 V)^0\) be such that both \((3.0.3)\) and \((7.0.2)\) hold. The map \(f_{A_t} : X_{A_t} \to Y_{A_t}\) is identified with the quotient map \(M_t \to M_t/\langle \phi_t \rangle\).

Let \(\Delta^0(V) \subset \mathbb{L}G(\wedge^3 V)^0\) be the locus of \(A\) such that \(X_A\) is isomorphic to \(M_t\) for some \(t \in T''\). Our next task is to show that \(\Delta^0(V)\) is a locally closed subset of codimension 1. First we recall how one describes \(H^2(M_t)\) for \(t \in T'\). Let \(u, w \in H^*(S_t)\) and let \(u_q, w_q\) be the degree-\(q\) components of \(u, w\) respectively. One sets \(w^\vee := u_0 - u_2 + u_4\) and

\[
\langle u, w \rangle := \int_{S_t} (u_2 \wedge w_2 - u_0 w_4 - u_4 w_0) = -\int_{S_t} u^\vee \wedge w. \tag{3.1.7}
\]

This is Mukai’s bilinear symmetric form. One defines a positive\(^2\) weight-two Hodge structure on \(H^*(S_t)\) by defining the Hodge filtration as

\[
F^1H^*(S_t) := H^0(S_t) \oplus F^1H^2(S_t) \oplus H^4(S_t), \quad F^2H^*(S_t) := F^2H^2(S_t). \tag{3.1.8}
\]

Let \(v_t\) be the Mukai vector \((3.1.2)\), then \(v_t\) is integral of type \((1, 1)\) and hence \(v_t^\perp\) is an integral Hodge substructure of \(H^*(S_t)\) and furthermore the restriction of Mukai’s bilinear symmetric form to \(v_t^\perp\) is integral. Mukai defined (see [9] [11]) a map

\[
\theta_t : v_t^\perp \longrightarrow H^2(M_t) \tag{3.1.9}
\]

\(^2\)This means that \(h^{p,q} = 0\) if \(p < 0\).
by taking Künneth components of the Chern character of a tautological sheaf on \( S_t \times M_t \) (if such a sheaf does not exists one considers a quasi-tautological sheaf). In [11] we proved that \( \theta_t \) is an isomorphism of Hodge structures and an isometry of lattices - of course the bilinear form on \( v_t^\perp \) is the restriction of the Mukai pairing. Now assume that \( t \in T'' \) and let \( A_t \) be as in Proposition (3.1) and \( L_{A_t} \) be as in (1.0.3); then by Corollary (3.2) and Proposition (5.1) of [12] we have
\[
c_1(L_{A_t}) = \theta_t(\eta_t - 1). \tag{3.1.10}
\]

We let \( h_t := \theta_t(\eta_t - 1) \).

**Proposition 3.3.** Keep notation as above. Then \( \Delta^0(V) \) is a \( \text{PGL}(V) \)-invariant subset of \( \text{LG}(\wedge^3 V)^0 \) which is locally (in the classical topology) a codimension 1 submanifold.

**Proof.** The subset \( \Delta^0(V) \) is \( \text{PGL}(V) \)-invariant by definition. Let \( \pi : X \to \text{LG}(\wedge^3 V)^0 \) be the tautological family of double EPW-sextics and \( L \) be the tautological relatively ample line-bundle on \( X \); thus the restriction of \( L \) to \( X_A \) is isomorphic to \( L_A \). Let \( A_p \in \Delta^0(V) \). Thus there exists \( p \in T'' \) such that \( X_{A_p} \cong M_p \). Let \( U \subset \text{LG}(\wedge^3 V)^0 \) be a small open ball containing \( A_p \). Let \( \mathcal{X}_U := \pi^{-1}(U) \) and \( L_U := L|_{\mathcal{X}_U} \). Since \( U \) is contractible there is a marking \( \Psi \) of \( (\mathcal{X}_U, L_U) \); let \( P_\Psi : U \to D_2 \) be the corresponding local period map. We notice that \( (5+2c_1(D_p)+5\eta_p) \in V_t^\perp \) and hence \( \theta_p(5+2c_1(D_p)+5\eta_p) \in H^2(M_p) \). Furthermore since \( h_t - 1, 5 + 2 c_1(D_p) + 5 \eta_p - 5 \eta_p \) is a flat section of \( R^2\rho_t^*\mathbb{Z} \) and for all \( t \in T'' \) we have
\[
(\theta_t(5+2c_1(D_t) + 5\eta_t), H^{2,0}(S_t)|_{M_t} = 0 \tag{3.1.13}
\]

because \( \theta_t(5+2c_1(D_t) + 5\eta_t) \in H^{1,1}(M_t) \); the above equality gives (3.1.12). Next we prove that
\[
P_\Psi(\Delta^0(V) \cap U) \supset P_\Psi(U) \cap \Psi_p \circ \theta_p(5 + 2c_1(D_p) + 5\eta_p)^\perp. \tag{3.1.14}
\]

First we notice that
\[
\{v_p, \eta_p - 1, 5 + 2c_1(D_p) + 5\eta_p\}^\perp = H^2(S_p|_{prim}) \tag{3.1.15}
\]

where the primitive cohomology \( H^2(S_p|_{prim}) \subset H^2(S) \) is the orthogonal to \( c_1(D_p) \). Since \( h_p = \theta_p(\eta_p - 1) \) we get that
\[
H^2(M_p) \supset \{h_p, \theta_p(5 + 2c_1(D_p) + 5\eta_p)^\perp = \theta_p(H^2(S_p)|_{prim}). \tag{3.1.16}
\]

Let \( K_{10} \) be the period space for \( K3 \) surfaces with a polarization of degree 10; Equality (3.1.16) defines an isomorphism
\[
D_2 \cap \Psi_p \circ \theta_p(5 + 2c_1(D_p) + 5\eta_p)^\perp \sim K_{10} \tag{3.1.17}
\]
which is compatible with local period maps defined by the family \( \rho''' : M''' \to T' \) and the family \( \zeta : S''' \to T''' \) with fiber \( S_t \) over \( t \in T''' \). Let \( S'''_U := \zeta^{-1}(U) \). Since \( S''' \) contains the generic K3 of degree 10 the local period map of the family \( S'''_U \to U \) is a submersive map from \( U \) to an open ball in \( K_{10} \); since (3.1.17) is an isomorphism this proves (3.1.14). We also get that \( P_\phi \) is submersive and hence in order to show that \( \Delta^0(V) \cap U \) is a codimension 1 submanifold it suffices to prove that

\[
D_2 \cap \Psi_p \circ \theta_p(5 + 2c_1(D_p) + 5\eta_p)_{\perp}^{\perp}
\]

is smooth. The period domain \( D_2 \) is an open subset of the quadric of isotropic lines for the non-degenerate quadratic form \( (\cdot, \cdot)_{\Lambda^3 \mathbb{C}} \) and hence if \( \Psi_p \circ \theta_p(5 + 2c_1(D_p) + 5\eta_p) \) is not isotropic then (3.1.18) is smooth. Since

\[
\left( \Psi_p \circ \theta_p(5 + 2c_1(D_p) + 5\eta_p), \Psi_p \circ \theta_p(5 + 2c_1(D_p) + 5\eta_p) \right) = \langle 5 + 2c_1(D_p) + 5\eta_p, 5 + 2c_1(D_p) + 5\eta_p \rangle = -10
\]

the intersection (3.1.18) is indeed smooth. \( \square \)

### 3.2 The dual of \( Y_A \) for \( A \in \Delta^0(V) \)

If \( A \in \mathcal{LG}(\wedge^3 V)^{\geq 0} \) then \( Y_A^\vee = Y_A^{\perp} \) by Proposition (3.1) of [13]. As we will see \( \Delta^0(V) \cap \mathcal{LG}(\wedge^3 V)^{00} = \emptyset \) and hence in order to show that \( Y_A^\vee = Y_A^{\perp} \) for \( A \in \Delta^0(V) \) we need to improve on the result of [13].

**Proposition 3.4.** Let \( A \in \mathcal{LG}(\wedge^3 V) \) and \( \mathbb{P}(W) \in \mathbb{P}(V^\vee) \). Then \( \mathbb{P}(W) \in Y_A^{\perp} \) if and only if

\[
\wedge^3 W \cap A \neq \emptyset.
\]

**Proof.** Let \( \phi \in V^\vee \) be a linear function such that \( W = \ker(\phi) \); then

\[
F_\phi := \{ \phi \wedge \psi \mid \psi \in \wedge^2 V^\vee \} = \left( \wedge^3 W \right)^{\perp}.
\]

By definition \( \mathbb{P}(W) \in Y_A^{\perp} \) if and only if

\[
\{ 0 \} \neq F_\phi \cap A^{\perp} = \left( \wedge^3 W + A \right)^{\perp}.
\]

Since \( 10 = \dim(\wedge^3 W) = \dim A \) and \( \dim(\wedge^3 V) = 20 \) we get that (3.2.3) holds if and only if (3.2.4) holds. \( \square \)

**Corollary 3.5.** Let \( A \in \mathcal{LG}(\wedge^3 V)^{0} \). Then \( Y_A^{\perp} \) is a hypersurface and \( Y_A^\vee = Y_A^{\perp} \).

**Proof.** Let \( \mathbb{P}(W) \in Y_A^{\perp} \); by Proposition (3.2) this is equivalent to (3.2.4). Let \( 0 \neq \alpha \in (\wedge^3 W \cap A) \); since \( \dim W = 5 \) there exists \( v \in W \) such that \( \alpha \) is divisible by \( v \) and hence \( [v] \in Y_A \). We decompose \( W = C v \oplus W_0 \) and write \( \alpha = v \wedge w \) where \( w \in \wedge^2 W_0 \); by Definition (2.5) of [13] the rank of \( v \) is 4, thus \( W = \text{span}(v, w) \). Furthermore if \( \dim(F_{[v]} \cap A) = 1 \) then \( Y_A \) is smooth at \( [v] \) and \( \mathbb{P}(W) \) is the projective tangent space to \( Y_A \) at \( [v] \) - see the proof of Proposition (3.1) of [13]. By Proposition (2.8) of [13] we know that \( \dim(F_{[v]} \cap A) = 1 \) unless \( [v] \in \text{sing}(Y_A) \) and in this case \( \dim(F_{[v]} \cap A) = 2 \). Since \( \text{sing}(Y_A) \) is a surface a straightforward dimension count gives that \( \mathbb{P}(W) \notin Y_A^{\perp} \) for generic \( \mathbb{P}(W) \in \mathbb{P}(V^\vee) \), thus \( Y_A^{\perp} \) is a hypersurface. The same dimension count gives that the generic \( \mathbb{P}(W) \in Y_A^{\perp} \) is tangent to \( Y_A \) at one of its smooth points; this proves the corollary. \( \square \)
We will describe explicitly \( Y_A^V = Y_A \) for \( A \in \Delta^0(V) \); essentially we will give a refinement of Proposition (5.20) and Corollary (5.21) of [12]. Let \( t \in T''' \) and \( S_t \) be the K3 surface corresponding to \( t \); in order to simplify notation we temporarily drop the subscript \( t \). Let \( R \) be the Fano variety of lines on \( F_3^5 \). If \( [\ell] \in R \) then \( \ell \not\subset S \) by (3.1.4) and hence \( \ell \cap Q \) is a 0-dimensional scheme of length 2 contained in \( S \): thus we have a regular map

\[
R \rightarrow S^{[2]} \quad \ell \mapsto \ell \cap Q. \tag{3.2.4}
\]

Let \( P \subset S^{[2]} \) be the image of the above map: then (3.2.4) defines a regular map \( R \rightarrow P \) with inverse given by

\[
P \rightarrow R \quad [Z] \mapsto \text{span}(Z) \tag{3.2.5}
\]

and hence \( P \) is isomorphic to \( R \). It is known [5] that \( R \cong \mathbb{P}^2 \), thus \( P \cong \mathbb{P}^2 \).

Since \( S^{[2]} \) is a symplectic variety it follows that we can contract \( P \):

\[
c : S^{[2]} \rightarrow N. \tag{3.2.6}
\]

A priori \( N \) is a complex space, we will show soon that it is projective. Let \( p := c(P) \); thus \( p \) is the unique singular point of \( N \). On \( S^{[2]} \) there is an interesting map to \( |I_S(2)|^V \), see (4.3) of [12]; we recall the definition. The K3 surface \( S \) is cut out by quadrics and it contains no lines by (3.1.4); thus we have a regular map

\[
S^{[2]} \rightarrow |I_S(2)|^V \cong \mathbb{P}^5 \quad [Z] \mapsto \{ Q_\lambda \in |I_S(2)| \mid \text{span}(Z) \subset Q_\lambda \}. \tag{3.2.7}
\]

Let \( W \subset |I_S(2)|^V \) be the image of the above map; thus defines a map \( f : S^{[2]} \rightarrow W \). If \( [Z] \in P \) then \( f([Z]) = |I_{F_3^5}(2)| \) hence \( f \) is constant on \( P \); we will see that the point

\[
f(P) = |I_{F_3^5}(2)| \in W \tag{3.2.8}
\]

is quite special. Since \( f \) is constant on \( P \) and \( N \) is normal the map \( f \) descends to a regular map

\[
\overline{f} : N \rightarrow W. \tag{3.2.9}
\]

**Lemma 3.6.** Keep notation as above. There exist a non trivial involution \( \phi : N \rightarrow N \) and a birational morphism \( \epsilon : N/\langle \phi \rangle \rightarrow W \) with finite fibers such that \( \overline{f} \) is the composition

\[
N \xrightarrow{\pi} N/\langle \phi \rangle \xrightarrow{\epsilon} W. \tag{3.2.10}
\]

In particular \( N \) is projective and \( \deg W = 6 \).

**Proof.** Let’s show that \( \overline{f} \) has finite fibers of cardinality at most 2 and that the generic fiber has cardinality 2. The fiber of \( \overline{f} \) over \( |I_{F_3^5}(2)| \) in \( |I_S(2)|^V \) consists of the unique singular point \( p \) of \( N \). Now let\( \Lambda \in (W \setminus \{ |I_{F_3^5}(2)| \}) \), i.e.

\[
\Lambda = f([Z]), \quad [Z] \notin P. \tag{3.2.11}
\]

Then \( \overline{f}^{-1}(\Lambda) = f^{-1}(\Lambda) \). Let \( \Lambda_0 := \Lambda \cap |I_{F_3^5}(2)| \) and choose \( \lambda_0 \in (\Lambda \setminus |I_{F_3^5}(2)|) \). One has

\[
\bigcap_{\lambda \in \Lambda_0} Q_\lambda = F_3^5 \cup A_\Lambda \tag{3.2.12}
\]
where $A_\Lambda$ is a plane such that $A_\Lambda \cap F_3 = C_\Lambda$ is a conic. (See the proof of Lemma (4.20) of [12].) We claim that $Q_{\lambda_0} \supset A_\Lambda$: in fact if $Q_{\lambda_0} \supset A_\Lambda$ then $C_\Lambda \subset S$ contradicting (3.1.3). Thus $Q_{\lambda_0} \cap A_\Lambda$ is a conic $C'_\Lambda$. By (3.2.11) the line $\text{span}(Z)$ is contained in $C'_\Lambda$. Thus $C'_\Lambda$ is degenerate and $f^{-1}(\Lambda)$ consists of the set of lines contained in $C'_\Lambda$. This shows that $f^{-1}(\Lambda) = \overline{f^{-1}(\Lambda)}$ has cardinality at most 2. It also follows easily that the generic fiber of $\overline{f}$ consists of 2 distinct points. Since $N$ is normal there is a regular covering involution $\overline{\varphi}$ such that $\overline{f}$ factors through the quotient map $N \to \overline{N}(\overline{\varphi})$. Since $\overline{f}$ has finite fibers so does $\epsilon$, since the generic fiber of $\overline{f}$ consists of 2 points the map $\epsilon$ is birational. The line-bundle $\overline{f}^*\mathcal{O}_W(1)$ is ample because $\overline{f}: N \to W$ has finite fibers, thus $N$ is projective. We know (see (4.3) of [12]) that

$$\int_{S^{[2]}} c_1(f^*\mathcal{O}_W(1)) = 12. \quad (3.2.13)$$

Since $f: S^{[2]} \to W$ has generic fiber of cardinality 2 we get that $\deg W = 6$. □

We will show that the map $\epsilon$ of Lemma (3.6) is an isomorphism. Since $(S^{[2]} \setminus P) \cong (N \setminus \{p\})$ the involution $\overline{\varphi}$ defines a birational involution $\phi: S^{[2]} \dasharrow S^{[2]}$. (This is the birational involution of Proposition (4.21) of [12].) Let

$$\beta: \text{Bl}_P(S^{[2]}) \to S^{[2]} \quad (3.2.15)$$

be the blow-up of $P$: since $P$ is Lagrangian the symplectic form on $S^{[2]}$ induces an isomorphism $N_{P/S^{[2]}} \cong \Omega^1_P$, and hence the exceptional divisor of $\beta$ is identified with the incidence variety $\Gamma \subset P \times P'$ and the restriction of $\beta$ to the exceptional divisor is identified with the projection $\Gamma \to P$. We abuse notation and view $\Gamma$ as the exceptional divisor in $\text{Bl}_P(S^{[2]})$.

**Lemma 3.7.** The map $\phi$ of (3.2.14) is not regular along $P$. There is a regular involution $\overline{\varphi}: \text{Bl}_P(S^{[2]}) \to \text{Bl}_P(S^{[2]})$ which is equal to $\phi$ on $(S^{[2]} \setminus P) \subset \text{Bl}_P(S^{[2]})$. There is an identification $P \cong P'$ such that $\overline{\varphi}|_\Gamma$ is induced by the involution on $P \times P'$ which interchanges the factors.

**Proof.** The eigenspaces of the isometry $H^2(\phi)$ on $H^2(S^{[2]})$ induced by $\phi$ are given by (see (4.3) of [12])

$$H^2(\phi)_+ = \mathbb{C}f^*\mathcal{O}_W(1), \quad H^2(\phi)_- = f^*\mathcal{O}_W(1)^\perp. \quad (3.2.16)$$

Suppose $\phi$ is regular along $P$: if $D$ is an ample divisor on $S^{[2]}$ then $c_1(D + \phi^*D)$ is an ample $\phi$-invariant class, this contradicts (3.2.10) because $f^*\mathcal{O}_W(1)$ is not ample. Let $\psi: S^{[2]} \dasharrow X$ be the flop of $P$: thus $\psi$ is the inverse of the blowup $\text{Bl}_P(S^{[2]}) \to S^{[2]}$ followed by the morphism $\text{Bl}_P(S^{[2]}) \to X$ which contracts $\Gamma$ along the “other” fibration $\Gamma \to P'$. In particular $X$ contains $P'$. Let $\ell \subset P'$ and $\ell' \subset P'$ be lines. The isometry $H^2(\psi)$ identifies $(\ell')^\perp$ with $\ell^\perp$. In fact we have contractions $c: S^{[2]} \to N$ and $c^\prime: X \to N$ which give identifications $\ell^\perp = H^2(N) = (\ell')^\perp$. In particular $H^2(\psi)$ sends a nef divisor class in $(\ell')^\perp$ to a nef divisor class. On the other hand $H^2(\psi)$ maps the half-space $\ell_{>0}$ to the half-space $\ell_{<0}$. By (3.2.10) we get that $(\psi \circ \phi)^*$ maps an ample divisor to an ample
divisor: since \((\psi \circ \phi)\) defines a regular map between the complements of subsets of codimension 2 it follows that \((\psi \circ \phi)\) is regular and hence an isomorphism. It follows also that \(\phi\) induces a regular involution \(\tilde{\phi}: Bl_P(S^{[2]}) \rightarrow Bl_P(S^{[2]}).\)

Let’s show that the restriction of \(\tilde{\phi}\) to \(\Gamma\) is as stated. Any automorphism of \(\Gamma\) sends the projection \(\Gamma \rightarrow P\) to itself composed with an automorphism of \(P\) or to the projection \(\Gamma \rightarrow P^\vee\) composed with an automorphism of \(P^\vee\). Since \(\phi\) is not regular the latter holds and it follows that \(\phi|_\Gamma\) is as stated.

**Corollary 3.8.** The map \(\epsilon: N/\langle \phi \rangle \rightarrow W\) of Lemma 3.6 is an isomorphism.

**Proof.** The quotient \(Bl_P(S^{[2]})/\langle \phi \rangle\) is a projective birational model of \(N/\langle \phi \rangle\) and hence it is birational to \(W\) by Lemma 5.6. The Kodaira dimension of \(Bl_P(S^{[2]})/\langle \phi \rangle\) is 0 hence also the Kodaira dimension of \(W\) is 0. By Lemma 3.10 we know that \(\deg W = 6\) and hence by adjunction \(W\) is smooth in codimension 1. Thus \(W\) is normal: since \(\epsilon\) is birational with finite fibers we get that \(\epsilon\) is an isomorphism.

Now we reintroduce the subscript \(t\); thus we have \(S_t, f_t, N_t, W_t\) etc.

**Proposition 3.9.** Keep notation as above. Let \(A \in \Delta^0(V)\) and let \(t \in T^m\) such that \(X_A \cong M_t\). Thus by Proposition 3.7 and Corollary 3.5 we have \(Y_{A^\perp} \subset \{I_{S_t}(2)\}^{\vee}\). Then

\[
Y_{A^\perp} = W_t. \tag{3.2.17}
\]

**Proof.** Proposition (5.20) of 12 gives that the reduced scheme \((Y_{A^\perp})_{\text{red}}\) is equal to \(W_t\) (the hypothesis of that Proposition is that \(\Phi(h)\) holds, however the same proof goes through because all that is needed is the validity of \(\Phi(0,1,5)\)). Now \(Y_{A^\perp}\) is a degree-6 hypersurface because \(Y_{A^\perp} \neq P(V^\vee)\) and on the other hand \(W_t\) is a degree-6 hypersurface by Lemma 3.6 and hence from \((Y_{A^\perp})_{\text{red}} = W_t\) we get that \(Y_{A^\perp}\) is reduced and equal to \(W_t\).

Let \(A \in \Delta^0(V)\) and let \(t \in T^m\) such that \(X_A \cong M_t\). Then \(|I_{F_2}(2)| \in W_t\) and \(\epsilon\) is as stated. By the above proposition \(|I_{F_2}(2)| \in Y_{A^\perp}\); we denote this point by \(q_{A^\perp}\).

**Proposition 3.10.** Let \(A \in \Delta^0(V)\) and let \(t \in T^m\) such that \(X_A \cong M_t\). Then

\[
\text{mult}_{q_{A^\perp}} Y_{A^\perp} = 3. \tag{3.2.18}
\]

Let \(y \in (Y_{A^\perp} \setminus \{q_{A^\perp}\})\) and hence \(f_t^{-1}(y) = f_t^{-1}(y)\) consists of two points or of one point.

(a) If \(f_t^{-1}(y)\) consists of two points then \(Y_{A^\perp}\) is smooth at \(y\).

(b) If \(f_t^{-1}(y)\) consists of a single point then the analytic germ \((Y_{A^\perp}, y)\) is isomorphic to the product of a smooth 2-dimensional germ and the germ of an \(A_1\)-singularity.

**Proof.** By Proposition (6.2) of 13 \(Y_{A^\perp}\) has multiplicity 3 in \(|I_{F_2}(2)|\); this proves 3.2.13. In order to prove (a)-(b) we notice that \(Y_{A^\perp} = W_t\) and by Proposition 3.8 the map \(\overline{f}_t: N_t \rightarrow W_t\) is identified with the quotient map \(N_t \rightarrow N_t/\langle \phi \rangle\). Since \(p_t = \overline{f}_t^{-1}(q_A)\) is the unique singular point of \(N_t\) this gives Item (a). In order to prove Item (b) we notice that \((N_t \setminus \{p_t\}) = (S_t^{[2]} \setminus P_1)\) and that the
quotient map \((N_i \setminus \{p_i\}) \to (N_i \setminus \{p_i\})/\phi\) is identified with the quotient map \((S_i^{[2]} \setminus P_i) \to (S_i^{[2]} \setminus P_i)/\phi\). By Proposition (4.21) of [12] the restriction of \(\phi\) to \((S_i^{[2]} \setminus P_i)\) is an anti-symplectic involution and hence its fixed point set is a Lagrangian surface; this proves (b).

Let \(A \in \Delta^0(V)\); then by Proposition 3.12 the sextic \(Y_A\) has a point of multiplicity 3 and hence \(A^\perp \not\in \mathbb{LG}(V^\vee)^0\) because if \(B \in \mathbb{LG}(V^\vee)^0\) then \(Y_B\) has points of multiplicity at most 2. Thus

\[
\Delta^0(V) \cap \mathbb{LG}(V^\vee)^0 = \emptyset. \quad (3.2.19)
\]

Let \(\Delta^0_0(V) \subset \Delta^0(V)\) be the set of points which are smooth points of the projective variety \((\mathbb{LG}(V^\vee)^0 \setminus \mathbb{LG}(\wedge^3 V)^0)\). By Proposition (3.21) we know that \(\Delta^0(V)\) is locally (in the classical topology) a codimension 1 submanifold of \(\mathbb{LG}(\wedge^3 V)\); since \(\Delta^0(V) \cap \mathbb{LG}(\wedge^3 V)^0 = \emptyset\) it follows that \(\Delta^0_0(V)\) is open dense in \(\Delta^0(V)\). We let \(T_s \subset T''\) be the set of \(t\) such that \(M_t \cong X_A\) for some \(A \in \Delta^0_0(V)\); since \(\Delta^0_0(V)\) is open dense in \(\Delta^0(V)\) also \(T_s\) is open dense in \(T''\). Let

\[
\mathbb{LG}(\wedge^3 V)^0_0 := \mathbb{LG}(\wedge^3 V)^0 \cup \Delta^0_0(V). \quad (3.2.20)
\]

**Proposition 3.11.** Keep notation as above. Then \(\mathbb{LG}(\wedge^3 V)^0_0\) is open in \(\mathbb{LG}(\wedge^3 V)^0\) (for the classical topology) and \(\Delta^0_0(V)\) is a non-empty codimension-1 submanifold of \(\mathbb{LG}(\wedge^3 V)^0_0\).

**Proof.** Let \(A \in \Delta^0_0(V)\). By definition of \(\Delta^0_0(V)\) there exists an open \(U \subset \mathbb{LG}(\wedge^3 V)^0\) such that \(U \cap \Delta^0_0(V) = U \cap \mathbb{LG}(\wedge^3 V)^0_0\) and hence \(U \subset \mathbb{LG}(\wedge^3 V)^0_0\); this proves that \(\mathbb{LG}(\wedge^3 V)^0_0\) is open. We have already proved that \(\Delta^0_0(V)\) is a non-empty locally closed codimension-1 submanifold of \(\mathbb{LG}(\wedge^3 V)^0_0\); this gives the second statement of the proposition.

We let

\[
\Delta^\infty(V^\vee) := \delta(\Delta^0_0(V)), \quad (3.2.21)
\]
\[
\mathbb{LG}(\wedge^3 V)^\infty_0 := \delta(\mathbb{LG}(\wedge^3 V)^0_0). \quad (3.2.22)
\]
\[
\mathbb{LG}(\wedge^3 V)^\infty := \mathbb{LG}(\wedge^3 V)^0_0 \cup \Delta^0_0(V). \quad (3.2.23)
\]

Of course every definition above has a “dual” definition obtained by substituting \(V^\vee\) to \(V\); thus we have \(\Delta^0_0(V^\vee) \subset \mathbb{LG}(\wedge^3 V^\vee)^0_0\), \(\Delta^\infty(V^\vee) \subset \mathbb{LG}(\wedge^3 V^\vee)^\infty\) etc. Let

\[
\mathbb{LG}(\wedge^3 V)^\perp := \mathbb{LG}(\wedge^3 V^\vee)^0_0 \cup \mathbb{LG}(\wedge^3 V^\vee)^\infty_0 = \mathbb{LG}(\wedge^3 V^\vee)^0_0 \cup \Delta^0_0(V^\vee) \cup \Delta^\infty(V^\vee). \quad (3.2.24)
\]

Let \(\mathcal{Y}(V^\vee) \subset \mathbb{LG}(\wedge^3 V)^\perp \times \mathbb{P}(V^\vee)\) be the tautological EPW-sextic; thus \(\mathcal{Y}(V^\vee) \cap \{B\} \times \mathbb{P}(V^\vee) = Y_B\). If \(B \in \Delta^\infty(V^\vee)\) then by Proposition 3.10 there is a unique point \(q_B \in Y_B\) of multiplicity strictly greater than 2; let

\[
\mathcal{Q}(V^\vee) := \{(B, q_B) | B \in \Delta^\infty(V^\vee)\}, \quad (3.2.25)
\]
\[
\mathcal{Y}(V^\vee)^\perp := \mathcal{Y}(V^\vee) \setminus \mathcal{Q}(V^\vee). \quad (3.2.26)
\]

**Proposition 3.12.** There exists a double cover \(f: \mathcal{X}(V^\vee) \to \mathcal{Y}(V^\vee)^\perp\) with the following properties.
(1) Let \( \pi: X(V^\vee) \to LG(\wedge^3 V^\vee)^2 \) be the composition of \( f \) and the projection \( Y(V^\vee)^4 \to LG(\wedge^3 V^\vee)^4 \); then \( \pi \) is a submersion of smooth manifolds.

(2) Let \( B \in LG(\wedge^3 V^\vee)^0 \); then \( \pi^{-1}(B) \cong X_B \) and the map \( \pi^{-1}(B) \to Y_B \) defined by \( f \) is isomorphic to the natural double cover \( X_B \to Y_B \).

(3) Let \( B \in \Delta^\infty_a(V^\vee) \) and \( A := B^\perp \). Since \( A \in \Delta^*_0(V) \) there exists \( t \in T_6 \) such that \( X_A \cong M_t \) and \( Y_A \cong \Sigma_t \). Then \( \pi^{-1}(B) \cong (S^2_t \setminus \{ P_1 \}) \) and the map \( \pi^{-1}(B) \to (Y_B \setminus \{ q_B \}) \) defined by \( f \) is isomorphic to the double cover \( (S^2_t \setminus \{ P_1 \}) \to (W_t \setminus \{ T_{F_3}(2) \}) \) given by the restriction of \( \text{(3.2.7)} \). (Recall that \( (Y_B \setminus \{ q_B \}) \cong (W_t \setminus \{ T_{F_3}(2) \}) \) by Proposition \( \text{(3.4)} \)).

Proof. Let \( B \in LG(\wedge^3 V^\vee)^3 \). Then \( Y_B \neq \mathbb{P}(V^\vee) \) by Corollary \( \text{(3.3)} \); thus the map \( \lambda_B \) defined in Section \( \text{(1)} \) (with \( V^\vee \) replacing \( V \)) is non-zero and \( Y_B \) is the zero-scheme of \( \text{det}(\lambda_B) \). Since \( Y_B \) is a Lagrangian degeneracy locus there exists locally in \( \mathbb{P}(V^\vee) \) a symmetric map of vector-bundles giving a resolution of \( \text{coker}(\lambda_B) \), i.e. we can cover \( \mathbb{P}(V^\vee) \) by open sets \( U \) such that on each \( U \) we have a locally-free resolution

\[
0 \to E_U \xrightarrow{\alpha_U} E^\vee_U \longrightarrow \text{coker}(\lambda_B)|_U \to 0 \tag{3.2.27}
\]

where \( \alpha_U \) is a symmetric map of vector-bundles. Furthermore if \( F \hookrightarrow \wedge^3 V^\vee \otimes \mathcal{O}_{\mathbb{P}(V^\vee)} \) is the Lagrangian sub-vector-bundle defined in Section \( \text{(1)} \) (with \( V^\vee \) replacing \( V \)) then there is an isomorphism \( F_p \cap B \cong \ker(\alpha_p) \) for all \( p \in U \).

Thus if \( \dim(F_p \cap B) \geq r \) then \( \text{mult}_p(Y_B) \geq r \). Let \( p \neq q_B \). Then \( \text{mult}_p(Y_B) \leq 2 \) by Proposition \( \text{(3.10)} \) and hence \( \dim(F_p \cap B) \leq 2 \). Furthermore one of the following holds:

1. If \( \dim(F_p \cap B) = 1 \) then locally around \( p \) we have \( \text{coker}(\lambda_B) \cong i_* \mathcal{O}_{Y_B} \)
where \( i: Y_B \to \mathbb{P}(V^\vee) \) is the inclusion.

2. If \( \dim(F_p \cap B) = 2 \) there exist an open (in the classical topology) \( U \subset \mathbb{P}(V^\vee) \) containing \( p \), functions \( x, y, z \in Hol(U) \) vanishing at \( p \) with linearly independent differentials and an exact sequence

\[
0 \to \mathcal{O}^\vee_U \xrightarrow{M} \mathcal{O}^\vee_U \longrightarrow \text{coker}(\lambda_B)|_U \to 0 \tag{3.2.28}
\]

where \( M \) is the map defined by the matrix

\[
\begin{pmatrix}
x \\
y \\
z
\end{pmatrix}.
\tag{3.2.29}
\]

In particular we see that there exists a sheaf \( \zeta_B \) on \( (Y_B \setminus \{ q_B \}) \) such that outside \( q_B \) we have \( \text{coker}(\lambda_B) = i^0_\zeta_B \) where \( i^0: (Y_B \setminus \{ q_B \}) \to (\mathbb{P}(V^\vee) \setminus \{ q_B \}) \) is the inclusion.

From the local description of \( \text{coker}(\lambda_B) \) given above we also get a canonical isomorphism of sheaves on \( (\mathbb{P}(V^\vee) \setminus \{ q_B \}) \):

\[
\text{Ext}^1(\text{coker}(\lambda_B), \mathcal{O}_{\mathbb{P}(V^\vee)})|_{\mathbb{P}(V^\vee) \setminus \{ q_B \}} \cong i^0_\zeta_B(\zeta^\vee_B \otimes N_{Y_B/\mathbb{P}(V^\vee)}) \tag{3.30}
\]

(See Proposition (4.3) of \( \text{[14]} \).) Let \( B^\vee \subset \wedge^3 V^\vee \) be a Lagrangian subspace complementary to \( B \); thus we have a direct sum decomposition

\[
\wedge^3 V = B \oplus B^\vee. \tag{3.31}
\]
Then $\lambda_B$ can be identified with the map of vector-bundles $\lambda_B : B^\vee \otimes O_{\mathcal{P}(V^\vee)} \to B^\vee \otimes O_{\mathcal{P}(V^\vee)}$. Let $\mu_B : F \to B \otimes O_{\mathcal{P}(V^\vee)}$ be the “other” map associated to Decomposition (3.2.31). The diagram

$$
\begin{array}{ccc}
F & \xrightarrow{\lambda_B} & B^\vee \otimes O_{\mathcal{P}(V^\vee)} \\
\mu_B & \downarrow & \mu_B' \\
B \otimes O_{\mathcal{P}(V^\vee)} & \xrightarrow{\lambda_B'} & F^\vee
\end{array}
$$

(3.2.32)

is commutative because $F (\mu_B \lambda_B) : (B \otimes B^\vee) \otimes O_{\mathcal{P}(V^\vee)}$ is a Lagrangian embedding. The map $\lambda_B$ is an injection of sheaves because $Y_B \neq \mathbb{P}(V^\vee)$ and hence also $\lambda_B'$ is an injection of sheaves. Thus there is a unique $\beta_B : coker(\lambda_B) \to Ext^1(coker(\lambda_B), O_{\mathcal{P}(V^\vee)})$ making the following diagram commutative with exact horizontal sequences:

$$
\begin{array}{cccc}
0 & \to & F & \xrightarrow{\lambda_B} & B^\vee \otimes O_{\mathcal{P}(V^\vee)} & \to & coker(\lambda_B) & \to & 0 \\
\mu_B & \downarrow & \mu_B' & \downarrow & \beta_B \\
0 & \to & B \otimes O_{\mathcal{P}(V^\vee)} & \xrightarrow{\lambda_B'} & F^\vee & \to & Ext^1(coker(\lambda_B), O_{\mathcal{P}(V^\vee)}) & \to & 0
\end{array}
$$

(3.2.33)

By Isomorphism (3.2.31) we get that the restriction of $\beta_B$ to $(\mathbb{P}(V^\vee) \setminus \{q_B\})$ defines a map of sheaves on $(Y_B \setminus \{q_B\})$.

$$
\zeta_B \to \zeta_B'(6).
$$

(3.2.34)

Since $F (\mu_B \lambda_B) : (B \otimes B^\vee) \otimes O_{\mathcal{P}(V^\vee)}$ is an injection of vector-bundles the above map is an isomorphism - this follows from Claim (4.5) of [13]. Let $\xi_B := \zeta_B(-3)$; then $\xi_B$ defines an isomorphism $\xi_B \to \xi_B'$ which is symmetric and hence it gives $O_{(Y_B \setminus \{q_B\})} \otimes \xi_B$ the structure of a commutative finite $O_{(Y_B \setminus \{q_B\})}$-algebra. Let $X_B := Spec(O_{(Y_B \setminus \{q_B\})} \otimes \xi_B)$ and $f_B : X_B \to (Y_B \setminus \{q_B\})$ be the structure map; clearly $f_B$ is finite of degree 2. The above construction is the analogue of the construction of the natural double cover $X_B \to Y_B$ for $B \in \mathbb{L}(\wedge^3 V^\vee)$. Thus we have a double cover $f : \mathcal{X}(V^\vee) \to \mathcal{X}(V^\vee)$ such that Item (2) holds and such that for $B \in \Delta_{\infty}(V^\vee)$ we have $\pi^{-1}(B) = X_B$ and the map $\pi^{-1}(B) \to (Y_B \setminus \{q_B\})$ is the structure map $f_B$ defined above. It remains to prove that Items (1) and (3) hold. Let

$$
C_B := \{p \in (sing Y_B \setminus q_B) \mid dim(F_p \cap B) = 1\}
$$

(3.2.35)

$$
D_B := \{p \in (sing Y_B \setminus q_B) \mid dim(F_p \cap B) = 2\}.
$$

(3.2.36)

Thus $(sing Y_B \setminus q_B) = C_B \bigcup D_B$. The local description given above of $coker(\lambda_B)$ near $p \in (Y_B \setminus q_B)$ shows that both $C_B$ and $D_B$ are smooth and closed (in $(Y_B \setminus q_B)$). Let $p \in (Y_B \setminus q_B)$: the map $f_B$ behaves differently depending on whether $p \in C_B$ or $p \in D_B$: in fact (see [13])

(a) If $p \in C_B$ then $f_B$ is unramified over $p$.

(b) If $p \in D_B$ then $f_B$ is ramified over $p$ and $X_B$ is smooth at $f_B^{-1}(p)$.

Let’s prove that

$$
X_B \cong (S_t^{[2]} \setminus P_t).
$$

(3.2.37)
Let $B' \in L\mathcal{G}(\wedge^3 V^\vee)^0$ and define $D_{B'}$ as above. Then $D_{B'}$ is not empty in fact it is a surface; it follows that $D_B \neq \emptyset$ and by Item $(\beta)$ above we get that the étale covering

$$(X_B \setminus f_B^{-1}(\text{sing}Y_B)) \to (Y_B \setminus \text{sing}Y_B)$$

(3.2.38)

is not trivial. On the other hand let $t \in T_\ast$ such that $X_{A^\ast} \cong M_t$ and $Y_{A^\ast} \cong \Sigma_t^\ast$. Let $f_t : S_t^2 \to Y_B$ be the map defined by (3.2.7); then $f_t$ defines an étale double cover

$$(S_t^2 \setminus f_t^{-1}(\text{sing}Y_B)) \to (Y_B \setminus \text{sing}Y_B).$$

(3.2.39)

Now $S_t^2$ is simply connected and $f_t^{-1}(\text{sing}Y_B)$ has codimension 2 in $S_t^2$ hence $(S_t^2 \setminus f_t^{-1}(\text{sing}Y_B))$ is simply connected. Thus $\pi_1(Y_B \setminus \text{sing}Y_B) \cong \mathbb{Z}/(2)$ and hence there is a unique non-trivial double cover of $(Y_B \setminus \text{sing}Y_B)$ and it is given by $\Sigma_t^\ast$. Since $\Sigma_t^\ast$ is a non-trivial double cover it follows that $C_B = \emptyset$ and that $(\ref{eq:non-trivial-double-cover})$ holds. This proves Items (1)-(3) of the proposition. \hfill \square

### 3.3 Extension of the local period map across $\Delta^\infty_{\ast}(V^\vee)$

We will prove that the local period map extends across $\Delta^\infty_{\ast}(V^\vee)$. Let $\pi^0 : \mathcal{X}(V^\vee)^0 \to L\mathcal{G}(\wedge^3 V^\vee)^0$ be the tautological family of double EPW-sextics i.e. the restriction of the map $\pi$ of Proposition (3.12) to $\pi^{-1}(L\mathcal{G}(\wedge^3 V^\vee)^0)$. Since $\pi^0$ is proper it defines the variation of Hodge structures

$$(R^2\pi^0_{\ast}\mathbb{Z}, F^p)$$

(3.3.1)

where $F^0 \supset F^1 \supset F^2$ is the Hodge filtration of $(R^2\pi^0_{\ast}\mathbb{Z}) \otimes O_{L\mathcal{G}(\wedge^3 V^\vee)^0}$ i.e. the fiber of $F^p$ over $A$ is

$$F^p_A = F^p H^2(X_A) := \bigoplus_{p' \geq p} H^{p' - 2p'}(X_A).$$

(3.3.2)

Furthermore we have the symmetric section

$$\mathbb{B} \in H^0((R^2\pi^0_{\ast}\mathbb{Q}) \otimes (R^2\pi^0_{\ast}\mathbb{Q}))$$

(3.3.3)

which gives the Beauville-Bogomolov bilinear form on $H^2(X_A)$ for every $A \in L\mathcal{G}(\wedge^3 V^\vee)^0$.

**Proposition 3.13.** There exist

1. a local system $\mathbb{H}(V^\vee)$ on $L\mathcal{G}(\wedge^3 V^\vee)^0$,
2. a decreasing filtration of $\mathbb{H}(V^\vee) \otimes O_{L\mathcal{G}(\wedge^3 V^\vee)^0}$ by holomorphic sub-bundles $\mathbb{F}^0 \supset \mathbb{F}^1 \supset \mathbb{F}^2$,
3. a symmetric section $\mathbb{B} \in H^0((\mathbb{H}(V^\vee) \otimes (\mathbb{H}(V^\vee) \otimes \mathbb{Q}))$

such that the following hold:

- $(\mathbb{H}(V^\vee), \mathbb{F}^p)$ extends the variation of Hodge structures (3.3.1).
- Let $B \in \Delta^\infty_{\ast}(V^\vee)$ and $t \in T_\ast$ such that $M_t \cong X_{B^\ast}$; then there exists an isomorphism of Hodge structures

$$\mathbb{H}(V^\vee)|_B, \mathbb{F}^p_B) \cong H^2(S_t^2).$$

(3.3.4)
(c) The restriction of $\overline{B}$ to $L\mathbb{G}(\wedge^3 V^\vee)_0$ is equal to $B$.

(d) Let $B \in \Delta^\infty(V^\vee)$ and $t \in T_*$ such that $M_t \cong X_{B^1}$; then Isomorphism 3.3.4 is an isometry between $(H(V^\vee)_B, \overline{B}_B)$ and $H^2(S_2^1)$ equipped with the Beauville-Bogomolov symmetric bilinear form.

Proof. Let $U \subset L\mathbb{G}(\wedge^3 V^\vee)^\mathbb{C}$ be an open ball. We assume that $U$ is small: then there exists a hyperplane $H \subset \mathbb{P}(V^\vee)$ such that $q_B \not\in H$ for all $B \in U \cap \Delta^\infty(V^\vee)$.

Let $\pi$ and $f$ be as in Proposition 3.1.2; we let

\[
\mathcal{X}(V^\vee)_U := \pi^{-1}(U),
\]

\[
Z(U, H) := \mathcal{X}(V^\vee)_U \cap f^{-1}H.
\]

Let $\rho: Z(U, H) \to U$ be given by the restriction of $f$. By our choice of $H$ the map $\rho$ is proper submersive with fibers smooth 3-folds. Thus we have a variation of Hodge structures $(R^2\rho_*\mathbb{Z}, F^p)$; we denote it by $(\mathbb{H}(U, H), F^p(U, H))$. Let $B \in (U \setminus \Delta^\infty)$; then $\rho^{-1}(B)$ is an ample divisor on $X_B$ and hence we have a canonical isomorphism

\[
(\mathbb{H}(U, H), F^p(U, H))|_{(U \setminus \Delta^\infty)} \cong (R^2\rho_*\mathbb{Z}, F^p)|_{(U \setminus \Delta^\infty)}. \tag{3.3.7}
\]

This shows that $(\mathbb{H}(U, H), F^p(U, H))$ does not depend on the choice of $H$ and that the collection of $(\mathbb{H}(U, H), F^p(U, H))$ gives an extension $(\mathbb{H}(V^\vee), F^p)$ of the variation of Hodge structures 3.3.3. Now let’s prove Item (b). Let $B \in (U \cap \Delta^\infty(V^\vee))$ and let $t$ be as in Item (b). Let $f_t: S_2^1 \to W_t = Y_B$ be the map given by (3.2.7). Then $\rho^{-1}(B) = f_t^{-1}H$ and since $f_t$ is semi-small the restriction map $H^2(S_2^1) \to H^2(f_t^{-1}H)$ is an isomorphism of (integral) Hodge structures; this proves Item (b) because $(\mathbb{H}(V^\vee), F^p)$ is isomorphic to the Hodge structure on $H^2(f_t^{-1}H)$ by definition. We define $\overline{B}$ as follows. Let $U$ and $\rho: Z(U, H) \to U$ be as above; let $Z_B := \rho^{-1}(B)$. For $B \in U$ we have the Lefschetz decomposition

\[
\mathbb{H}(V^\vee)_B = H^2(Z_B; \mathbb{Q}) = \mathbb{Q}c_1(L_B)|_{Z_B} \oplus H^2(Z_B; \mathbb{Q})_{prim}, \tag{3.3.8}
\]

where $L_B$ is the tautological ample line-bundle on $X_B$. We let $\overline{B}(U, H)_B$ be the symmetric bilinear form on $\mathbb{H}(V^\vee)_B$ characterized by the following requirements:

1. Decomposition 3.3.8 is orthogonal for $\overline{B}(U, H)_B$.

2. $\overline{B}_B(c_1(L_B)|_{Z_B}, c_1(L_B)|_{Z_B}) = 2$.

3. If $x, y \in H^2(Z_B; \mathbb{Q})_{prim}$ then $\overline{B}(U, H)_B(x, y) = \frac{1}{2} \int_{Z_B} c_1(L_B) \wedge x \wedge y$.

Since the Lefschetz decomposition is flat for the Gauss-Manin connection we have a well-defined section $\overline{B}(U, H)_B \in H^0(\overline{H}(V^\vee)|_U)$ with value $\overline{B}(U, H)_B$ at $B \in U$. Let $B \in (U \setminus \Delta^\infty(V^\vee)$ and let $\iota: Z_B \hookrightarrow X_B$ be the inclusion. If $\xi_1, \xi_2, \xi_3 \in H^2(X_B)$ then

\[
\int_{Z_B} \iota^* \xi_1 \wedge \iota^* \xi_2 \wedge \iota^* \xi_3 = \int_{X_B} c_1(L_B) \wedge \xi_1 \wedge \xi_2 \wedge \xi_3 = \frac{1}{2} \sum_{\sigma \in S_3} (c_1(L_B), \xi_{\sigma(1)}) \cdot (\xi_{\sigma(2)}, \xi_{\sigma(3)}). \tag{3.3.9}
\]

It follows from this that $\overline{B}(U, H)$ does not depend on $H$ and that the collection of $\overline{B}(U, H)$’s defines an extension of $B$. Item (d) holds because Formula 3.3.9 holds if we replace $X_B$ by $S_2^1$. \qed
The map $\delta$ defines an isomorphism
\[
\begin{align*}
\Lambda LG(\wedge^3 V) & \xrightarrow{\sim} \Lambda LG(\wedge^3 V) \\
A & \mapsto \delta(A) = A^\perp.
\end{align*}
\] (3.3.10)

From now on we will denote by $\delta$ what is actually the restriction of $\delta$ to $\Lambda LG(\wedge^3 V)$. Let $U \subset \Lambda LG(\wedge^3 V)$ be an open set: then we have two local systems, namely
\[
\mathbb{H}(V)|_U, \quad \mathbb{H}(V^\vee)|_{\delta(U)}.
\] (3.3.11)

**Definition 3.14.** Assume that both local systems $\mathbb{H}(V)|_U$ are trivial. A marking of $\Lambda LG(\wedge^3 V)$ is an isomorphism $\Psi: \mathbb{H}(V)|_U \xrightarrow{\sim} U \times \Lambda$ such that:

1. $\bar{\Psi}_A(x, y) = (\Psi(x), \Psi(y))$ for every $A \in U$ and $x, y \in \mathbb{H}(V)_A$, and
2. $\Psi$ sends the flat section $A \mapsto c_1(L_A)|_{\mathbb{H}(U,H)}$ (notation as in the proof of Proposition (3.13)) to $U \times u$ where $u$ is given by (1.0.8).

We define similarly a marking $\Psi$ of $\mathbb{H}(V^\vee)|_{\delta(U)}$.

Of course if $U$ is a small open ball then both local systems $\mathbb{H}(V)|_U$ are trivial. Let $\Phi$ and $\Psi$ be markings of $\mathbb{H}(V)|_U$ and $\mathbb{H}(V^\vee)|_{\delta(U)}$ respectively. Then we have holomorphic local period maps
\[
\begin{align*}
U & \xrightarrow{\mathcal{P}_\Phi} D_2 \\
A & \mapsto \Phi_C(F^2_A) \\
\delta(U) & \xrightarrow{\mathcal{P}_\Psi} D_2 \\
B & \mapsto \Phi_C(F^2_B).
\end{align*}
\] (3.3.12)

The restrictions of $\mathcal{P}_\Phi$ and $\mathcal{P}_\Psi$ to $U \setminus \Delta_0^\infty(V)$ and $\delta(U) \setminus \Delta_0^\infty(V^\vee)$ respectively are local period maps for the families of double EPW-sextics parametrized by $\Lambda LG(\wedge^3 V)^0$ and $\Lambda LG(\wedge^3 V)^0$ respectively. We will be interested in comparing $\mathcal{P}_\Phi(A)$ and $\mathcal{P}_\Phi(A^\perp)$ for $A \in \Delta_0^\infty(V)$ - thus $A^\perp \not\in \Lambda LG(\wedge^3 V)^0$. Let $t \in T$ be such that $M_t \cong X_A$. By (3.3.3) the marking $\Phi$ defines a marking $\mathcal{P}_\Phi(A^\perp)$ of $(S_t^{[2]}, f_t^*\mathcal{O}_{Y_B}(1))$ and
\[
\mathcal{P}_\Phi(A^\perp) = \mathcal{P}_{\Phi_{A^\perp}}(S_t^{[2]}, f_t^*\mathcal{O}_{Y_B}(1)).
\] (3.3.13)

### 4 Proof of Theorem (1.1)

If $A \in \Lambda LG(\wedge^3 V)^0$, we have smooth double covers $f_A: X_A \to Y_A$ and $f_{A^\perp}: X_{A^\perp} \to Y_{A^\perp}$: we will show that $X_A$ and $X_{A^\perp}$ are “isogenous”. Given a smooth open $U \subset \Lambda LG(\wedge^3 V)$ we may consider markings $\Psi$ and $\Phi$ of $\mathbb{H}(V)|_U$ and $\mathbb{H}(V^\vee)|_{\delta(U)}$ respectively and the associated local period maps $\mathcal{P}_\Phi$ and $\mathcal{P}_\Psi$. We will show that locally near $\Delta_0^0(V)$ we may choose $\Psi$ and $\Phi$ so that $\mathcal{P}_\Phi \circ \delta$ is either $r \circ \mathcal{P}$ or the composition of $r \circ \mathcal{P}$ with a certain specific reflection. In the final subsection we will rule out the latter case by considering the monodromy action; by analytic continuation this will prove Theorem (1.1).

#### 4.1 Isogeny between $X_A$ and $X_{A^\perp}$

Let $A \in \Lambda LG(\wedge^3 V)^0$ and set $L_A := f_A^*\mathcal{O}_{Y_A}(1)$, $L_{A^\perp} := f_{A^\perp}^*\mathcal{O}_{Y_{A^\perp}}(1)$. Let
\[
\begin{align*}
H^2(X_A)_{prim} := c_1(L_A)^\perp & \subset H^2(X_A), \\
H^2(X_{A^\perp})_{prim} := c_1(L_{A^\perp})^\perp & \subset H^2(X_{A^\perp}).
\end{align*}
\] (4.1.1) (4.1.2)
**Proposition 4.1.** Let \( A \in \mathcal{L}(\wedge^3 V)^0 \). There exists an isomorphism of rational Hodge structures

\[
g_A : H^2(X_A)_{\text{prim}} \xrightarrow{\sim} H^2(X_{A^\perp})_{\text{prim}}
\]

well-defined up to \( \pm 1 \) and such that for \( \gamma \in H^2(X_A)_{\text{prim}} \)

\[
(\gamma, \gamma)_{X_A} = (g_A(\gamma), g_A(\gamma))_{X_{A^\perp}}.
\]

**Proof.** We recall that \( Y_{A^\perp} = Y_A^\vee \), see Corollary (3.3). Let \( \Gamma_A \subset Y_A \times Y_{A^\perp} \) be the closure of the Gauss maps:

\[
\Gamma_A := \{(p, T_p Y_A) \mid p \in Y_A^{\text{sm}} \} = \{(T_q Y_{A^\perp}, q) \mid q \in Y_{A^\perp}^{\text{sm}} \}.
\]

Since \( A \in \mathcal{L}(\wedge^3 V)^0 \) the germ of \( Y_A \) at each of its singular points is isomorphic to the product of \((\mathbb{C}^2, 0)\) and an \( A_1 \)-singularity (see Proposition (2.8) of [14]) and similarly for \( Y_{A^\perp} \). Thus the projection \( \Gamma_A \to Y_A \) is identified with the blow-up \( \widetilde{Y_A} \to Y_A \) of \( \text{sing}(Y_A) \). Similarly the projection \( \Gamma_A \to Y_{A^\perp} \) is identified with the blow-up \( \widetilde{Y}_{A^\perp} \to Y_{A^\perp} \) of \( \text{sing}(Y_{A^\perp}) \). Thus \( \Gamma_A \) defines an isomorphism \( \widetilde{Y_A} \xrightarrow{\sim} \widetilde{Y}_{A^\perp} \) and hence it gives an isomorphism of integral Hodge structures

\[
H^4(\widetilde{Y_A}) \xrightarrow{\sim} H^4(\widetilde{Y}_{A^\perp}).
\]

The cohomology groups \( H^4(\widetilde{Y_A}) \) and \( H^4(X_A) \) are related as follows. Let \( \phi_A : X_A \to X_A \) be the involution covering \( f_A : X_A \to Y_A \). Let \( F_A \subset X_A \) be the fixed locus of \( \phi_A \); this is a Lagrangian smooth surface in \( X_A \) because \( \phi_A \) is anti-symplectic. Let \( \widetilde{X_A} \to X_A \) be the blow-up of \( F_A \); we have an isomorphism of integral Hodge structures

\[
H^4(\widetilde{X_A}) \cong H^4(X_A) \oplus H^2(F_A)(-1).
\]

The involution \( \phi_A \) lifts to an involution \( \widetilde{\phi_A} : \widetilde{X_A} \to \widetilde{X_A} \) and \( \widetilde{Y_A} \cong \widetilde{X_A}/\langle \widetilde{\phi_A} \rangle \). Thus we have an isomorphism of rational Hodge structures

\[
H^4(\widetilde{Y_A}) \cong H^4(\widetilde{X_A})/(\widetilde{\phi_A}) \cong H^4(X_A)^{\langle \phi_A \rangle} \oplus H^2(F_A)(-1).
\]

Since \( X_A \) is a deformation of \((K3)^2\) we have an isomorphism of rational Hodge Structures \( \text{Sym}^2 H^2(X_A) \xrightarrow{\sim} H^4(X_A) \) defined by cup-product. The action of \( \phi_A \) on \( H^2(X_A) \) has \((+1)\)-eigenspace generated by \( c_1(L_A) \) and \((-1)\)-eigenspace equal to \( H^2(X_A)_{\text{prim}} \) thus we get an isomorphism of rational Hodge structures

\[
H^4(X_A)^{\langle \phi_A \rangle} = \mathbb{C}c_1(L_A)^2 \oplus \text{Sym}^2 H^2(X_A)_{\text{prim}}.
\]

The right-hand side of the above equality contains a rational \((2, 2)\) class \( q_A^\vee \) defined by “inverting” the Beauville-Bogomolov bilinear form (see Section (3) of [14]); let \( W_A := H^2(X_A)_{\text{prim}} \cap (q_A^\vee)\perp \). One has a decomposition of rational Hodge Structures (Claim (3.1) of [14])

\[
\mathbb{C}c_1(L_A)^2 \oplus \mathbb{C}q_A^\vee \oplus W_A.
\]

Of course we have analogous notions for \( X_{A^\perp} \) and hence (4.1.6) defines an isomorphism of rational H.S.’s

\[
\mathbb{C}c_1(L_A)^2 \oplus \mathbb{C}q_A^\vee \oplus W_A \otimes H^2(F_A)(-1) \cong \mathbb{C}c_1(L_{A^\perp})^2 \oplus \mathbb{C}q_A^\perp \oplus W_{A^\perp} \otimes H^2(F_{A^\perp})(-1).
\]
If $A$ is very general (outside a countable union of proper analytic subsets of $\mathbb{L}_G(\Lambda^3 V^0) \cap \delta^{-1}(\Lambda^3 V^0)$) then $W_A$ and $W_{A^\perp}$ are both indecomposable rational Hodge Structures - see Section (3) of [14]: since they contain the non-zero components $H^{4,0}(X_A)$ and $H^{4,0}(X_{A^\perp})$ the above isomorphism defines an isomorphism of rational H.S.'s

$$h_A: W_A \sim W_{A^\perp}. \quad (4.1.12)$$

Let $Q_A \subset H^2(X_A)_{prim}$ and $Q_{A^\perp} \subset H^2(X_{A^\perp})_{prim}$ be the cones of isotropic classes (with respect to the Beauville-Bogomolov bilinear form). Let

$$H^2(X_A)_{prim} \xrightarrow{\nu_A} Sym^2 H^2(X_A)_{prim} \xrightarrow{\alpha^2}$$

be the (affine) Veronese map; we define similarly $\nu_{A^\perp}$. Then

$$W_A = \text{span}(\nu_A(Q_A)), \quad W_{A^\perp} = \text{span}(\nu_{A^\perp}(Q_{A^\perp})). \quad (4.1.14)$$

We claim that

$$h_A(\nu_A(Q_A)) = \nu_{A^\perp}(Q_{A^\perp}). \quad (4.1.15)$$

In fact let $U \subset \mathbb{L}_G(\Lambda^3 V^0) \cap \delta^{-1}(\Lambda^3 V^0)$ be an open ball containing $A$. The Gauss-Manin connection gives identifications

$$H^2(X_B) \cong H^2(X_A), \quad H^2(X_{B^\perp}) \cong H^2(X_{A^\perp}) \quad (4.1.16)$$

for every $B \in U$. Gauss-Manin gives also identifications $W_B \cong W_A$ and $W_{B^\perp} \cong W_{A^\perp}$ respectively sending $\nu_B(Q_B)$ to $\nu_A(Q_A)$ and $\nu_{B^\perp}(Q_{B^\perp})$ to $\nu_{A^\perp}(Q_{A^\perp})$ respectively. The isomorphism $h_B$ is flat for the Gauss-Manin connection hence it is identified with a (constant) map $h: W_A \to W_{A^\perp}$. For $B \in U$ let $\sigma_B, \sigma_{B^\perp}$ be symplectic forms on $X_B$ and $X_{B^\perp}$ respectively. Since $h_B$ is an isomorphism of H.S.'s we have

$$h[\sigma_B^2] = [\sigma_{B^\perp}^2]. \quad (4.1.17)$$

Now $\sigma_B \in Q_A$ and $\sigma_{B^\perp} \in Q_{A^\perp}$ - here we make the identifications (4.1.16) - and as $B$ varies in $U$ both $[\sigma_B]$ and $[\sigma_{B^\perp}]$ fill out non-empty open (in the classical topology) subsets of $\mathbb{P}(Q_A)$ and $\mathbb{P}(Q_{A^\perp})$ respectively. Since $\mathbb{P}(Q_A)$ and $\mathbb{P}(Q_{A^\perp})$ are non singular quadrics any non-empty open subset is Zariski-dense and hence (4.1.17) proves (4.1.15). It follows from (4.1.15) that there exists a linear map (4.1.13) well-defined up to $\pm 1$ such that for $\alpha \in Q_A$ we have $h_{A^\perp}(\alpha^2) = [\alpha^2].$ By (4.1.15) we have

$$\frac{g_A(\alpha)}{\alpha^2} \in Q_{A^\perp} \text{ if and only if } \alpha \in Q_A. \quad (4.1.18)$$

The rationality of $g_A$ follows from the fact that $h_A$ is defined over $\mathbb{Q}$, and $\nu_A, \nu_{A^\perp}$ give bijective maps between $\mathbb{P}(Q_A)(\mathbb{Q})$, $\nu_A(\mathbb{P}(Q_A))(\mathbb{Q})$, $\nu_{A^\perp}(\mathbb{P}(Q_{A^\perp}))(\mathbb{Q})$, $\nu_{A^\perp}(\mathbb{P}(Q_{A^\perp}))(\mathbb{Q})$ respectively. Finally let’s prove that Equation (4.1.14) holds.

First we show that

$$(\alpha, \beta)_{X_A} = (g_A(\alpha), g_A(\beta))_{X_{A^\perp}}, \quad \alpha, \beta \in Q_A. \quad (4.1.19)$$

Since (4.1.14) respects the intersection forms we have

$$\int_{X_A} \alpha^2 \wedge \beta^2 = \int_{X_{A^\perp}} h_A(\alpha^2) \wedge h_A(\beta^2) = \int_{X_{A^\perp}} g_A(\alpha^2) \wedge g_A(\beta^2). \quad (4.1.20)$$
On the other hand $\alpha, \beta$ are isotropic and by (4.1.15) $g_A(\alpha), g_A(\beta)$ are isotropic as well; thus

$$\int_{X_A} \alpha^2 \wedge \beta^2 = 2(\alpha, \beta)_X^2$$

(4.1.21)

$$\int_{X_A} g_A(\alpha) \wedge g_A(\beta)^2 = 2(g_A(\alpha), g_A(\beta))_{X_A^+}^2$$

(4.1.22)

(4.1.23)

(See Section (2) of [13].) Equations (4.1.20), (4.1.21) and (4.1.22) prove that either (4.1.19) holds or else

$$\int_{X_A} (\alpha, \beta)_{X_A} = -(g_A(\alpha), g_A(\beta))_{X_A^+}, \quad \alpha, \beta \in Q_A.$$  

(4.1.24)

Assume that (4.1.24) holds. Let $\sigma_A \in H^{2,0}(X_A)$ be the class of a symplectic form on $X_A$; since $g_A$ is an isomorphism of Hodge structures $g_A(\sigma_A)$ is represented by a symplectic form on $X_{A^+}$. Then

$$(\sigma_A, \gamma_A, X_A) = -(g_A(\sigma_A), g_A(\sigma_A))_{X_A^+}.$$  

(4.1.25)

This is absurd because the Beaville-Bogomolov $(\cdot, \cdot)_X$ of an irreducible symplectic manifold $X$ has the property that $(\cdot, \sigma)_X > 0$ for every symplectic form $\sigma$ on $X$. This finishes the proof of Equation (4.1.19). Now let $\gamma \in H^2(X_A)_{\text{prim}}$, then $\gamma = \alpha + \beta$ for certain $\alpha, \beta \in Q_A$; by Equations (4.1.18) and (4.1.19) we get

$$(\gamma, \gamma)_X = 2(\alpha, \beta)_X + 2(g_A(\alpha), g_A(\beta))_{X_A^+} = (g_A(\gamma), g_A(\gamma))_{X_A^+}.$$  

(4.1.26)

This proves (4.1.19).

Proposition (4.1) implies that locally there exists $g \in O(\Lambda \otimes \mathbb{Q})$ which relates the periods of $X_{A^+}$ to those of $X_A$ and furthermore $g$ normalizes the subgroup of monodromy operators in $O(\Lambda)$. We introduce some notation to formalize this observation. Let $U \subset LG(\wedge^3 V)^t$ be a small open ball and let $\Psi, \Phi$ be markings of $\mathbb{H}(V)|_{\mathcal{U}}$ and $\mathbb{H}(V^\vee)|_{\mathcal{U}}$ respectively - see Definition (3.14). Let $\overline{A} \in \mathcal{U}$ be a reference point: the monodromy representation of $\pi_1(LG(\wedge^3 V)^t, \overline{A})$ on $\mathbb{H}(V)$ determines via $\Psi_{\overline{A}}$ a monodromy representation $\pi_1(LG(\wedge^3 V)^t, \overline{A}) \to \text{Stab}(u)$ where $\text{Stab}(u) < O(\Lambda)$ is the subgroup fixing the element $u$ given by (4.1.8). An element of the image of the monodromy representation is a $\Phi_{\Lambda^t}$-monodromy operator. Similarly we have a monodromy representation of $\pi_1(LG(\wedge^3 V^\vee), \overline{A})$ on $\mathbb{H}(V^\vee)$; this determines via $\Phi_{\overline{A}^t}$ a monodromy representation $\pi_1(LG(\wedge^3 V^\vee)^t, \overline{A}^t) \to \text{Stab}(u)$. An element of the image of this second monodromy representation is a $\Phi_{\Lambda^t}$-monodromy operator. Here and in the rest of the paper we will adopt the following conventions. First we view both $O(\Lambda)$ and $O(\Lambda \otimes \mathbb{Q})$ as subgroups of $O(\Lambda \otimes \mathbb{C})$. Secondly if $\gamma \in O(\Lambda \otimes \mathbb{C})$ we denote by $\overline{\gamma} \in O(\Lambda \otimes \mathbb{C})$ the isometry which fixes $u$ and equals $\gamma$ on $\Lambda$. As a rule letters decorated by a tilde denote elements of $O(\Lambda \otimes \mathbb{C})$, letters with no tilde denote elements of $O(\Lambda \otimes \mathbb{Q})$.

**Corollary 4.2.** Let $U \subset LG(\wedge^3 V)^t$ be a small open ball and let $\Psi, \Phi$ be markings of $\mathbb{H}(V)|_{\mathcal{U}}$ and $\mathbb{H}(V^\vee)|_{\mathcal{U}}$ respectively. Let $\mathcal{P}_\Psi, \mathcal{P}_\Phi$ be the local period maps (3.3.13). There exists $g \in O(\Lambda \otimes \mathbb{Q})$ well-determined up to $\pm 1$ such that

$$\mathcal{P}_\Phi(A^t) = g \circ \mathcal{P}_\Psi(A)$$  

(4.1.27)
for all \( A \in \mathcal{U} \). Let \( \tilde{\gamma} \in \text{Stab}(u) \) be a \( \Psi_{A^+} \)-monodromy operator; then \( \tilde{g} \circ \tilde{\gamma} \circ \tilde{g}^{-1} \) is a \( \Phi_{A^+} \)-monodromy operator, in particular \( g \circ \gamma \circ g^{-1} \in O(\Lambda) \).

**Proof.** Equation (4.1.27) holds on \( \mathcal{U} \cap LG(\wedge^3 V)^0 \) by Proposition 4.1 and flatness of \( g_A \); by continuity Equation (4.1.27) holds on all of \( \mathcal{U} \). The statement about monodromy operators holds by flatness of \( g_A \). \( \square \)

### 4.2 Restriction to \( \Delta_0^0(V) \) of the local period maps

Our next task is to analyze the restriction to \( \Delta_0^0(V) \) of the local period maps \( \mathcal{P}_\Psi \) and \( \mathcal{P}_\Phi \).

**Proposition 4.3.** Let \( \mathcal{U} \subset LG(\wedge^3 V)^4 \) be a small open ball. There is a choice of markings \( \Psi \) and \( \Phi \) of \( \mathbb{H}(V)|_U \) and \( \mathbb{H}(V')|_{\delta(U)} \) respectively such that

\[
\mathcal{P}_\Psi(\Delta_0^0(V) \cap \mathcal{U}) = (e_1 + 2e_2)^\perp \cap \mathcal{P}_\Psi(\mathcal{U})
\]

(4.2.1)

where \( e_1, e_2 \in \Lambda \) are as in Section 4.1. Furthermore

\[
\mathcal{P}_\Phi(\Lambda^\perp) = r \circ \mathcal{P}_\Psi(A), \quad A \in \Delta_0^0(V)
\]

(4.2.2)

where \( r \) is the involution defined by \( \Lambda^\perp \).

**Proof.** First we embed the lattice \( \tilde{\Lambda} \) in a unimodular lattice as follows. Let \( \hat{\Lambda} := U^4 \oplus (-E_8)^2 \). Let \( U_1 < \hat{\Lambda} \) be one of the hyperbolic lattices, let \( z \in U_1 \) be a vector of square 2 and \( e_2 \) be a generator of \( z^\perp \cap U_1 \). Then we have an isomorphism

\[
z^\perp \cong \hat{\Lambda}
\]

(4.2.3)

and we can choose it so that it matches the present \( e_2 \) with the vector \( e_2 \) appearing in (1.0.9); we fix such an isomorphism once for all. Let \( u, e_1 \in \tilde{\Lambda} = z^\perp \) be as in Section 4.1; then \( \{(u \pm e_1)/2, (z \pm e_2)/2\} \subset \hat{\Lambda} \). Furthermore the sublattices \( (u + e_1)/2, (u - e_1)/2 \) and \( (z + e_2)/2, (z - e_2)/2 \) are orthogonal hyperbolic planes. Thus we have an orthogonal decomposition

\[
\hat{\Lambda} = ((u + e_1)/2, (u - e_1)/2) \oplus ((z + e_2)/2, (z - e_2)/2) \oplus U^2 \oplus (-E_8)^2.
\]

(4.2.4)

Now we pass to the geometry. Let \( T_* \) be as in Subsection 4.2. Let \( A \in \Delta_0^0(V) \cap \mathcal{U} \); by definition there exists \( t \in T_* \) such that

\[
M_t \cong X_A.
\]

(4.2.5)

Since \( \mathcal{U} \) is a small open ball there exists a small open ball \( \mathcal{V} \subset T_* \) such that if \( t \in \mathcal{V} \) then (4.2.5) holds for some \( A \in \Delta_0^0(V) \cap \mathcal{U} \) and conversely if \( A \in \Delta_0^0(V) \cap \mathcal{U} \) then there exists \( t \in \mathcal{V} \) such that (4.2.5) holds. Let \( \kappa: S \to T_* \) be the tautological family of K3 surfaces parametrized by \( T_* \). Let \( t \in T_* \). Let \( S_t = \kappa^{-1}(t), D_t \) etc. be as in Subsection 4.1 and \( A_t \in \Delta_0^0(V) \cap \mathcal{U} \) such that \( M_t \cong X_{A_t} \). Let \( f_t: M_t \to Y_{A_t} \) be the double cover. Let \( v_t, w_t \in H^*(S_t) \) be given by

\[
v_t := 2 + c_1(D_t) + 2\eta_t, \quad w_t := 1 - \eta_t
\]

(4.2.6)

where \( \eta_t \in H^4(S_t; \mathbb{Z}) \) is the orientation class. The Mukai map

\[
\theta_{v_t}: v_t^\perp \to H^2(M_t)
\]

(4.2.7)
We define a trivialization $\gamma$ is an isomorphism of polarized Hodge structures and an isometry. Let $w$ be an isometry of Hodge structures - see Subsection (3.1). Furthermore one has

$$c_1(f^*_t L_A) = \theta_v(\eta_t - 1) = \theta_v(-w^\gamma_t). \tag{4.2.8}$$

(See the line preceding (5.1.7) for the definition of $w^\gamma_t$.) The local system $R^2 \kappa_* \mathbb{Z}|_\mathcal{V}$ is trivial because $\mathcal{V}$ is a small open ball. Thus there exist sections $\alpha, \beta \in \Gamma(R^2 \kappa_* \mathbb{Z}|_\mathcal{V})$ such that $c_1(D_t) = \alpha_t + 5\beta_t$ for all $t \in \mathcal{V}$. We define a trivialization

$$R^2 \kappa_* \mathbb{Z}|_\mathcal{V} = (R^0 \kappa_* \mathbb{Z} \oplus R^2 \kappa_* \mathbb{Z} \oplus R^4 \kappa_* \mathbb{Z})|_\mathcal{V} \xrightarrow{\Upsilon} \mathcal{V} \times \Lambda \tag{4.2.9}$$

as follows. For $t \in \mathcal{V}$ let

$$
\Upsilon_t(1) := -u/2 + e_1/2 - z + e_2 \\
\Upsilon_t(\eta_t) := u/2 + e_1/2 - z + e_2 \\
\Upsilon_t(\alpha_t) := -2e_1 + 5z/2 - 3e_2/2 \\
\Upsilon_t(\beta_t) := z/2 - e_2/2 \tag{4.2.10}
$$

and let

$$\Upsilon_t|_{\{1, \eta_t, \alpha_t, \beta_t\}} : \{1, \eta_t, \alpha_t, \beta_t\}^\perp \xrightarrow{\sim} \{(u \pm e_1)/2, (z \pm e_2)/2\}^\perp \tag{4.2.11}$$

be an arbitrary isometry - notice that $\{1, \eta_t, \alpha_t, \beta_t\}^\perp \cong U^2 \oplus (-E_8)^2$ is isometric to $\{(u \pm e_1)/2, (z \pm e_2)/2\}^\perp$ by (4.2.4). A straightforward computation shows that $\Upsilon_t$ is an isometry. The trivialization (4.2.9) is defined to have value $\Upsilon_t$ at $t \in \mathcal{V}$. Now notice that $\Upsilon_t(\nu_t) = z$ and hence we have an isometry $\Upsilon_t \circ \theta_v^{-1} : H^2(M_t) \xrightarrow{\sim} z^\perp = \Lambda$. Since $\Upsilon_t(-w^\gamma_t) = u$ Equation (4.2.8) gives that

$$\Upsilon_t \circ \theta_v^{-1}(c_1(f^*_t L_A)) = u. \tag{4.2.12}$$

Hence $\Upsilon_t \circ \theta_v^{-1}$ defines a marking of $(M_t, f^*_t L_A)$ for every $t \in \mathcal{V}$; since $\mathbb{H}(V)|_U$ is trivial there exists a marking $\Psi : \mathbb{H}(V)|_U \to U \times \Lambda$ such that

$$\Phi_{A_1} = \Upsilon_t \circ \theta_v^{-1}, \quad t \in \mathcal{V}. \tag{4.2.13}$$

Equation (4.2.11) follows from Equation (3.1.11) and the equality

$$\Phi_{A_1}(\theta_v(5 + 2c_1(D_t) + 5\eta_t)) = \Upsilon_t(5 + 2c_1(D_t) + 5\eta_t) = e_1 + 2e_2. \tag{4.2.14}$$

Next we define a marking $\Phi$ for $\mathbb{H}(V^\vee)|_{\delta(U)}$. By (3.3.1) this will be equivalent to a marking of $S^{[2]}_1$, hence we first recall the description of $H^2(S^{[2]}_1)$. We notice that $w_t$ is the Mukai vector (see (3.1.1)) of any ideal sheaf $I_Z$ where $[Z] \in S^{[2]}_1$; Mukai’s map

$$\theta_{w_t} : w_t^\perp \to H^2(S^{[2]}_1) \tag{4.2.15}$$

is an isomorphism of polarized Hodge structures and an isometry. Let $g_t : S^{[2]}_1 \to Y_{A^+_t}$ be the map defined by (3.2.7); by Subsection (5.3) of (12)

$$c_1(g^*_t \mathcal{O}_{Y_{A^+_t}}(1)) = \theta_{w_t}(-2 + c_1(D_t) - 2\eta_t) = \theta_{w_t}(-w^\gamma_t). \tag{4.2.16}$$

We define a trivialization

$$R^2 \kappa_* \mathbb{Z}|_\mathcal{V} = (R^0 \kappa_* \mathbb{Z} \oplus R^2 \kappa_* \mathbb{Z} \oplus R^4 \kappa_* \mathbb{Z})|_\mathcal{V} \xrightarrow{\Theta} \mathcal{V} \times \Lambda \tag{4.2.17}$$

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as follows. For $t \in \mathcal{V}$ let
\[
\Theta_t(1) := u - e_1 + \frac{z}{2} - \frac{e_2}{2} \\
\Theta_t(\eta_t) := u - e_1 - \frac{z}{2} - \frac{e_2}{2} \\
\Theta_t(\alpha_t) := 5u/2 - 3e_1/2 - 2e_2 \\
\Theta_t(\beta_t) := u/2 - e_1/2
\]  
and let the restriction of $\Theta_t$ to $\{1, \eta, \alpha, \beta\}$ be equal to the restriction of $\Upsilon_t$. A straightforward computation shows that $\Theta_t$ is an isometry. The trivialization (4.2.17) is defined to have value $\Theta_t$ at $t$ \in \mathcal{V}$. Now notice that $\Theta_t(w_t) = z$ and hence we have an isometry $\Theta_t : H^2(S_t[2]) \overset{\sim}{\rightarrow} z \perp \Lambda$. Since $\Theta_t(-v_t^\vee) = u$ Equation (4.2.16) gives that
\[
\Theta_t \circ \theta^{-1}_{w_t}(c_1(g^*_t \mathcal{O}_Y_{A_t^t})) = u.
\] Hence $\Theta_t \circ \theta^{-1}_{w_t}$ defines a marking of $(S_t[2], g^*_t \mathcal{O}_Y_{A_t^t}(1))$ for every $t \in \mathcal{V}$; by (3.3.13) and triviality of $\mathbb{H}(V^\vee)|_{\delta(U)}$ there exists a marking $\Phi : \mathbb{H}(V^\vee)|_{\delta(U)} \rightarrow U \times \tilde{\Lambda}$ such that
\[
\Phi_{A_t} = \Theta_t \circ \theta^{-1}_{w_t}, \quad t \in \mathcal{V}.
\]

Now let’s prove (4.2.2). Equation (4.2.2) is equivalent to
\[
\mathcal{P}_{\Theta_t \circ \theta^{-1}_{w_t}}(S_t[2], g^*_t \mathcal{O}_Y_{A_t^t}(1)) = r \circ \mathcal{P}_{\Upsilon_t \circ \theta^{-1}_{w_t}}(M_t, f_t^* LA), \quad t \in \mathcal{V}
\] by Equation (3.3.13). Since $\theta_{w_t}$ and $\theta_{w_t}$ are isomorphism of Hodge structures the above equation may be rewritten as
\[
\Theta_t(H^{2,0}(S_t)) = r \circ \Upsilon_t(H^{2,0}(S_t)), \quad t \in \mathcal{V}.
\] For $t \in \mathcal{V}$ let
\[
H^*(S_t; \mathbb{Q}) \overset{\Xi_t}{\rightarrow} H^*(S_t; \mathbb{Q}) \\
\gamma \quad \mapsto \quad -\gamma^\vee + \frac{1}{2}(\gamma^\vee, v_t + w_t)(v_t + w_t)
\] i.e. the composition of the isometry $\gamma \mapsto \gamma^\vee$ and the reflection which is $+(+1)$ on $\mathbb{Q}(v_t + w_t)$ and $(-1)$ on $(v_t + w_t)^\perp$; thus $\Xi_t$ is a rational (not integral !) Hodge isometry. A straightforward computation shows that $\Theta_t = r \circ \Upsilon_t \circ \Xi_t$.

Thus (4.2.2) follows at once from $\Xi_t(H^{2,0}(S)) = H^{2,0}(S)$. \hfill \Box

If $\gamma \in \Lambda \otimes \mathbb{Q}$ is non-isotropic we let
\[
\Lambda \otimes \mathbb{Q} \overset{\gamma}{\rightarrow} x \mapsto -x + \frac{2}{(\gamma, \gamma)}(x, \gamma)
\] be the reflection with $(+1)$-eigenspace $\mathbb{Q}\gamma$ and $(-1)$-eigenspace $\gamma^\perp$. Let
\[
\zeta := e_1 + 2e_2.
\]

**Corollary 4.4.** Keep notation and assumptions of Proposition (4.3). Let $\Psi, \Phi$ be the markings of Proposition (4.3). Then:

1. $\mathcal{P}_{\Psi}(A^\perp) = r \circ \mathcal{P}_{\psi}(A)$ for all $A \in \mathcal{U}$ or
(2) $\mathcal{P}_\Phi(A^\perp) = r \circ r_\zeta \circ \mathcal{P}_\Psi(A)$ for all $A \in \mathcal{U}$.

Proof. By Corollary (4.2) there exists $g \in O(\Lambda \otimes \mathbb{Q})$ such that $\mathcal{P}_\Phi(A^\perp) = g \circ \mathcal{P}_\Psi(A)$ for all $A \in \mathcal{U}$. By Proposition (4.3) $r^{-1} \circ g$ fixes the points of $\zeta^\perp \cap D_2$: it follows that $r^{-1} \circ g$ fixes $\zeta^\perp \subset \mathbb{P}(\Lambda \otimes \mathbb{C})$. Since $r \in O(\Lambda)$ we have $r^{-1} \circ g \in O(\Lambda \otimes \mathbb{Q})$: thus we get that that

$$r^{-1} \circ g|_{\zeta^\perp} = \pm Id_{\zeta^\perp}. \tag{4.2.26}$$

Since $\zeta$ is non-isotropic (in fact $(\zeta, \zeta) = -10$) and $r^{-1} \circ g \in O(\Lambda \otimes \mathbb{Q})$ we get that $r^{-1} \circ g(\zeta) = \pm \zeta$. It follows that $r^{-1} \circ g = \pm Id$ or $r^{-1} \circ g = \pm r_\zeta$. Since $-Id$ acts trivially on $D_2 \subset \mathbb{P}(\Lambda)$ the corollary follows. \square

4.3 The proof

We will apply the monodromy statement of Corollary (4.2) in order to show that Item (2) of Corollary (4.4) can not hold. We will use the following result.

Claim 4.5. Let

$$\xi = a_1e_1 + a_2e_2 + \nu \in \Lambda \tag{4.3.1}$$

be a $(-2)$-vector i.e. $(\xi, \xi) = -2$ and assume that $r_\zeta \circ r_\xi \circ r_\zeta \in O(\Lambda)$. Then

$$(\xi, \zeta) \equiv 0 \pmod{5}. \tag{4.3.2}$$

Proof. A tedious straightforward computation gives that

$$r_\zeta \circ r_\xi \circ r_\zeta(e_1) = \frac{1}{25}(18a_1^2 - 48a_1a_2 + 32a_2^2 - 25)e_1 -$$

$$- \frac{1}{25}(24a_1^2 - 14a_1a_2 - 24a_2^2)e_2 - \frac{2}{5}(3a_1 - 4a_2)\nu. \tag{4.3.3}$$

Thus $18a_1^2 - 48a_1a_2 + 32a_2^2 \equiv 0 \pmod{25}$. Since

$$18a_1^2 - 48a_1a_2 + 32a_2^2 = 2(3a_1 - 4a_2)^2 \tag{4.3.4}$$

we get that $3a_1 - 4a_2 \equiv 0 \pmod{5}$. This proves (4.3.2) because

$$(\xi, \zeta) = -2a_1 - 4a_2 \equiv 3a_1 - 4a_2 \pmod{5}. \tag{4.3.5}$$

\square

Proposition 4.6. Keep notation and assumptions of Proposition (4.3). Let $\Psi$, $\Phi$ be the markings of Proposition (4.3) and $A \in \Delta^0_\Lambda(V)$. There exists a $(-2)$-vector $\xi \in \Lambda$ such that $-r_\xi$ is a $\Psi_{\lambda A}$-monodromy operator and

$$(\xi, \zeta) \not\equiv 0 \pmod{5}. \tag{4.3.6}$$

We grant the above proposition for the moment being and we proceed to prove Theorem (1.1). Let notation and assumptions be as in Proposition (4.3) and $\Psi$, $\Phi$ be the markings of Proposition (4.3). Then either (1) or (2) of Corollary (4.3) holds. Suppose that (2) holds; we will arrive at a contradiction. Let $\xi$ be as in Proposition (4.3). by Corollary (4.2) we have $-r_\zeta \circ r_\xi \circ r_\zeta \circ r \in O(\Lambda)$. Since $r \in O(\Lambda)$ we get that $r_\zeta \circ r_\xi \circ r_\zeta \circ r \in O(\Lambda)$: this contradicts Claim (4.5).
because of (4.3.6). Thus (1) of Corollary (4.3) holds. Let \( U_{00} := U \cap LG(\Lambda^3 V)^{00} \), then \( U_{00} \) is an open (in the euclidean topology) non-empty subset of \( LG(\Lambda^3 V)^{00} \). Since (1) of Corollary (4.3) holds we have
\[
P \circ \delta|_{U_{00}} = \tau \circ P|_{U_{00}}. \tag{4.3.7}
\]
Both \( P \circ \delta \) and \( \tau \circ P \) are holomorphic maps with domain the connected manifold \( LG(\Lambda^3 V)^{00} \); by analytic continuation we get that Theorem (4.3) holds.

**Proof of Proposition 4.6** Let \( F \subset \mathbb{P}^3 \) be a smooth quartic, thus \( F \) is a K3 surface. We have a regular map
\[
F^{[2]} \to \mathbb{P}^5
\]
\[
[Z] \mapsto \text{span}(Z)
\]
and \( c_1(g^*O_{Gr(1,\mathbb{P}^3)}(1)) \) has square 2 for the Beauville-Bogomolov form. If \( F \) does not contain lines the above map is finite and hence \( g^*O_{Gr(1,\mathbb{P}^3)}(1) \) is an ample line-bundle on \( F^{[2]} \), if \( F \) contains a line then \( g^*O_{Gr(1,\mathbb{P}^3)}(1) \) is big and nef but it restricts to the trivial line-bundle on the \( \mathbb{P}^2 \) given by \( R(2) \subset F^{[2]} \).

Assume that \( F \) does not contain lines; we proved in Section (6) of [12] that
\[
(F^{[2]}, g^*O_{Gr(1,\mathbb{P}^3)}(1)) \text{ is deformation equivalent to } (M_t, f^*_t L_{A_t}). \tag{4.3.9}
\]
where \( t \in T \), i.e. there exists a polarized family of irreducible symplectic 4-folds over a connected basis with one fiber isomorphic to \( (F^{[2]}, g^*O_{Gr(1,\mathbb{P}^3)}(1)) \) and another fiber isomorphic to \( (M_t, f^*_t L_{A_t}) \). Using this result we will show that the monodromy operator on \( F \) given by a suitable \((-2)\)-class orthogonal to \( c_1(O_F(1)) \) gives rise to a \( \Psi(\mathcal{A}) \)-monodromy operator for which (4.3.8) holds. Before proving this we must dive into the details of the proof of (4.3.6). Let \( F_0 \subset \mathbb{P}^3 \) be a smooth quartic surface containing a line \( R \) and with Picard number 2, i.e. \( Pic(F_0) = \mathbb{Z}[A_0] \oplus \mathbb{Z}[R] \) where \( A_0 \) is the (hyper)plane class. The divisor \( (2A_0 - R) \) is very ample and \( c_1(2A_0 - R)^2 = 10 \); thus we have an embedding
\[
F_0 \to |2A_0 - R|^\vee \cong \mathbb{P}^5
\]
as a linearly normal K3 surface of degree 10. Let
\[
w_0 := 1 + c_1(A_0) + \eta_0, \quad v_0 := 2 + c_1(2A_0 - R) + 2\eta_0 \tag{4.3.11}
\]
where \( \eta_0 \in H^4(F_0; \mathbb{Z}) \) is the orientation class. Let \( M_{w_0} \) be the moduli space of torsion-free sheaves \( G \) on \( F_0 \) such that \( v(G) = w_0 \); every such sheaf is equal to \( I_Z \otimes O_{F_0}(A_0) \) for a unique \( [Z] \in F^{[2]}_0 \) and hence
\[
M_{w_0} \cong F^{[2]}_0. \tag{4.3.12}
\]
We let \( L_{w_0} := g^*O_{Gr(1,\mathbb{P}^3)}(1) \). Let \( M_{v_0} \) be the moduli space of \((2A_0 - R)\)-semistable sheaves \( F \) on \( F_0 \) such that \( v(F) = v_0 \). The moduli space \( M_{v_0} \) is smooth because \( (2A_0 - R) \) is \( v_0 \)-generic (see Section (6) of [12]) and hence it is a deformation of \((K3)^{[2]} \). Mukai’s map gives isometries of Hodge structures
\[
\theta_{w_0} : w_0^! \cong H^2(M_{w_0}), \quad \theta_{v_0} : v_0^! \cong H^2(M_{v_0}). \tag{4.3.13}
\]
One has (see p.1241 of [12])
\[
c_1(L_{w_0}) = \theta_{w_0}(\eta_0 - 1). \tag{4.3.14}
\]
We let $L_{v_0}$ be the line-bundle on $M_{v_0}$ such that

$$c_1(L_{v_0}) = \theta_{v_0}(\eta_0 - 1).$$

(4.3.15)

In Lemma (6.2) of [12] we considered the birational map $M_{w_0} \dashrightarrow M_{v_0}$ whose inverse

$$\varphi: M_{v_0} \dashrightarrow M_{w_0}$$

(4.3.16)

is the Mukai reflection defined by the $(-2)$-vector

$$u_0 := (1 + c_1(A_0 - R) + \eta_0)$$

(4.3.17)

(notice that $-r_{w_0}(v_0) = w_0$), see [10]. Since $M_{v_0}$ and $M_{w_0}$ are irreducible symplectic manifolds the birational map $\varphi$ induces an isomorphism of lattices $\varphi^*: H^2(M_{w_0}) \cong H^2(M_{v_0})$. By Theorem (2.9) of [16] we have

$$\varphi^* \theta_{w_0}(\alpha) = \theta_{v_0}(-r_{w_0}(v_0)),$$

(4.3.18)

in particular by (4.3.14)-(4.3.15) we have

$$\varphi^* L_{w_0} \cong L_{v_0}.$$  

(4.3.19)

The birational map $\varphi$ is the flop of

$$\Pi_{w_0} := R^{(2)} \subset F^{[2]} = M_{w_0}.$$  

(4.3.20)

It follows that $M_{v_0}$ contains $\Pi_{v_0} \cong \Pi_{w_0} \cong \Pi_{v_0}$ and from (4.3.19) we get that $L_{v_0}$ is big, nef and its restriction to $\Pi_{v_0}$ is trivial. Let $X \rightarrow B_{v_0}$ be a representative for the deformation space of $(M_{v_0}, L_{v_0})$, i.e. deformations of $M_{v_0}$ that “keep $c_1(L_{v_0})$ of type $(1, 1)$”. Similarly let $X' \rightarrow B_{w_0}$ be a representative for the deformation space of $(M_{w_0}, L_{w_0})$. We let $0 \in B_{v_0}$ and $0 \in B_{w_0}$ be the points corresponding to $(M_{v_0}, L_{v_0})$ and $(M_{w_0}, L_{w_0})$ respectively. Thus for each $q \in B_{v_0}$ the fiber $X_q$ of $X \rightarrow B_{v_0}$ over $q$ has a line-bundle $L_q$ which is a deformation of $L_{v_0}$. Similarly for each $s \in B_{w_0}$ the fiber $X'_s$ of $X' \rightarrow B_{w_0}$ over $s$ has a line-bundle $L'_s$ which is a deformation of $L_{w_0}$. We may and will assume that $B_{v_0}, B_{w_0}$ are contractible and hence Gauss-Manin gives identifications

$$H^2(X_q) \cong H^2(M_{v_0}), \quad H^2(X'_s) \cong H^2(M_{w_0}), \quad q \in B_{v_0}, \quad s \in B_{w_0}$$

(4.3.21)

which match $c_1(L_q)$ to $c_1(L_{v_0})$ and $c_1(L'_s)$ to $c_1(L_{w_0})$. Let $B(\Pi_{v_0}) \subset B_{v_0}$ be the locus parametrizing deformations $X_q$ which contain a deformation of $\Pi_{v_0}$, and similarly let $B(\Pi_{w_0}) \subset B_{w_0}$ be the locus parametrizing deformations of $X'_s$ which contain a deformation of $\Pi_{v_0}$. By a Theorem of Voisin [15] each of these loci is smooth of codimension 1. There is a natural isomorphism of germs

$$\mu: (B_{w_0}, 0) \simto (B_{v_0}, 0)$$

(4.3.22)

such that $\mu(\Pi_{w_0}) = \Pi_{v_0}$ and if $s \notin \Pi_{w_0}$ then $(X_{\mu(s)}, L_{\mu(s)}) \cong (X'_s, L'_s)$. Let $s \in B_{w_0}$ be such that $(X'_s, L'_s) \cong (F^{[2]}, g^*O_{Gr(1, P^3)}(1))$ where $F$ is a quartic containing no lines. Then $s \notin \Pi_{w_0}$ and hence

$$(X_{\mu(s)}, L_{\mu(s)}) \cong (X'_s, L'_s) \cong (F^{[2]}, g^*O_{Gr(1, P^3)}(1)).$$

(4.3.23)
On the other hand there exists $q \in B_v$ such that $(X_q, c_1(L_q)) \cong (M_t, \theta_t(\eta_t - 1))$ for $t \in T$, because the parameter space for linearly normal $K3$ surfaces of degree 10 (an open subset of the relevant Hilbert scheme) is irreducible - notice that $q \notin \Pi_w$ because $\theta_t(\eta_t - 1)$ is ample. Thus there exists $\bar{\mathbf{A}} \in \Delta^0(V)$ such that

$$(X_{\bar{\mathbf{A}}}, L_{\bar{\mathbf{A}}}) \cong (X_q, L_q).$$  \hspace{1cm} (4.3.24)$$

Let $\gamma \in H^2(F; \mathbb{Z})$ be a $(-2)$-class orthogonal to $c_1(O_F(1))$. Then $\gamma$ determines a monodromy operator on $H^2(F^{[2]}); \text{ by (4.3.24)} \text{ and (4.3.24)}, \text{ this monodromy operator can be identified with a monodromy operator on } X_{\bar{\mathbf{A}}} \text{ because polarized deformation spaces of irreducible symplectic manifolds are smooth.}$

Given the trivializations (4.3.21) the monodromy operator in question is equal to $-r \varphi^* \theta^0(\gamma^0)$ - here $\gamma^0 \in H^2(F_0)$ is the class corresponding to $\gamma$ (the second trivialization of (4.3.21) defines an identification $H^2(F) \cong H^2(F_0)$). The corresponding $\Psi_{\bar{\mathbf{A}}}$-monodromy operator is equal to $-r \zeta$ where

$$\xi := \Psi_{\bar{\mathbf{A}}} (\varphi^* \theta^0(\gamma^0)) \in \Lambda$$ \hspace{1cm} (4.3.25)

**Claim 4.7.** *Keep notation as above. Let $\gamma^0 \in c_1(A_0)^\perp \subset H^2(F_0; \mathbb{Z})$ be a $(-2)$-class such that

$$\int_{F_0} \gamma^0 \wedge c_1(R) \neq 0 \pmod{5}.$$ \hspace{1cm} (4.3.26)$$

Then

$$(\xi, \zeta) \neq 0 \pmod{5}.$$ \hspace{1cm} (4.3.27)$$

**Proof.** By (4.3.18) we must check that

$$(\varphi^* \theta^0(\gamma^0), \theta_{\tau_0}(5 + 2c_1(2A_0 - R) + 5\eta_0)) \neq 0 \pmod{5}.$$ \hspace{1cm} (4.3.28)$$

By (4.3.15) this is equivalent to

$$(-r_{\tau_0}(\gamma^0), 5 + 2c_1(2A_0 - R) + 5\eta_0) \neq 0 \pmod{5}.$$ \hspace{1cm} (4.3.29)$$

where $u_0$ is given by (4.3.17). A straightforward computation gives that

$$(-r_{\tau_0}(\gamma^0), 5 + 2c_1(2A_0 - R) + 5\eta_0) = -8 \int_{F_0} \gamma^0 \wedge c_1(R)$$ \hspace{1cm} (4.3.30)$$

and hence we get (4.3.28). \hspace{1cm} \Box$

Certainly there exists a class $\gamma^0$ satisfying the hypotheses of Claim (4.7): the vector $\xi \in \Lambda$ given by (4.3.25) satisfies the thesis of Proposition (4.6).

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