Oscillation Phenomena of *Rayleigh* Equation Disturbed by Superposition of Two Harmonic Excitation

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**Abstract.** The *Rayleigh* Nonlinear Oscillation Equation Phenomenon disturbed by two external harmonic forces. One of which is small, will be discussed. Using the Multiple Scales method to obtain analytical solutions, several cases of resonance are determined. Steady-state solutions and their stabilities will be investigated. Depending on the value of amplitudes and frequencies of the external forces given, for a non-trivial steady-state solution, there are two possibilities, namely a periodic solution, and a non-periodic solution that raises the beating phenomenon.

1. **Introduction**

Disturbed system phenomena are often found in everyday life considering that there is almost no ideal system without any outside force interference. Therefore, it is necessary to pay attention to the forces that affect the system, even though they are quite small. Some ways that can be done to overcome this problem is to approach the solution using analytical, numeric or frequently used methods, namely a combination of the two. One analytical method commonly used is asymptotic expansion commonly known as perturbation method and more specifically that will be used in writing this article, namely the Multiple Scales method. The method is suitable for analyzing the influence of parameters that are of little value to the system.

The equation that will be discussed in this article is *Rayleigh* equation. *Rayleigh* equation is an equation that has a stable solution at a certain limit cycle for \( t \to \infty \). One particular example of the popular *Rayleigh* equation is the *van der pol* equation. In general, a one-dimensional homogeneous *Rayleigh* equation is,

\[
\ddot{u} + F(u, \dot{u})\dot{u} + u = 0
\]

The discussion of nonlinear system solutions is widely discussed in [3] [4]. Many applications in engineering, medicine and the like are found in [5]. The discussion will begin with completing *Rayleigh* equation which has been given two different dimensionally sized external forces using Multiple Scales method. Using this method leads to different cases which will be discussed in the next section. In each case a discussion of the solution and equilibrium was carried out.

2. **Multiple Scale**

Given *Rayleigh* equation which has been done non-dimensionless process

\[
\ddot{u} + u = \epsilon\left(\ddot{u} - \frac{1}{3}\dot{u}^3\right)
\]

By adding two harmonic external forces, an equation will be obtained,

\[
\ddot{u} + u = \epsilon\left(\ddot{u} - \frac{1}{3}\dot{u}^3\right) + k_1 \cos \Omega_1 t + \theta_1 + \epsilon k_2 \cos \Omega_2 t + \theta_2
\]  

(2.1)
Using the Multiple Scales method means we assume that \( u \) is a functional solution that depends not only on \( t \) and \( \epsilon \), but also on \( \epsilon t, \epsilon^2 t, \epsilon^3 t, \ldots \) so that it can be written,

\[
u = \tilde{u}(T_0, T_1, T_2, \ldots ; \epsilon)
\]

where \( T_n = \epsilon^n t \). This assumption means that the solution can be written as a combination of several different time scales ranging from fast movements to slow movements after getting a solution in each order. For example, clockwork on a wall clock (not digital), with \( \epsilon = \frac{1}{60} \), \( T_0 \) is a time scale so that a fast solution movement can be observed properly which in this case is indicated by a second hand movement. \( T_1 \) represents the time scale in which the solution movement can be seen when observing the solution until \( t = \epsilon^{-1} \) in this case can be indicated by minute motion, as well as \( T_2 \) which shows the time scale in which the solution movement can be observed while observing the movement of the solution to \( t = \epsilon^{-2} \) in this case it can be represented by a hours motion. Thus \( T_0 \) represents the fast-moving time scale of the solution, \( T_1 \) is a time scale for measuring movements that are slower \( \epsilon^n \) times than \( T_0 \) (Nayfeh et al, 1995). By dividing the time scale \( t \) into \( T_0, T_1, T_2, \ldots \) it means changing the equation that was originally in the form of an ordinary differential equation into a partial differential equation so it is necessary to redefine \( \frac{d}{dt} \) and \( \frac{d^2}{dt^2} \). Using the chain rule is obtained,

\[
\frac{d}{dt} = D_0 + \epsilon D_1 + \epsilon^2 D_2 + \cdots
\]

\[
\frac{d^2}{dt^2} = D_0^2 + 2 \epsilon D_0 D_1 + \epsilon^2 (D_1^2 + D_0 D_2) + \cdots
\]

with \( D_n = \frac{\partial}{\partial T_n} \) dan \( D_n^2 = \frac{\partial^2}{\partial T_n^2} \). Next expand the solution \( u \),

\[
u = u_0(T_0, T_1, T_2, \ldots) + \epsilon u_1(T_0, T_1, T_2, \ldots) + \cdots \tag{2.2}
\]

Substitute the results of the expansion to Equation (2.1) and ignore the order \( \epsilon^2 \) or higher terms obtained,

\[
\text{Order } \epsilon^0 : D_0^2 u_0 + u_0 = k_1 \cos \cos (\Omega_1 T_0 + \theta_1)
\]

\[
\text{Order } \epsilon^1 : D_0^2 u_1 + u_1 = -2D_0 D_1 u_0 + D_0 u_0 - \frac{1}{3} (D_0 u_0)^3 + k_2 \cos \cos (\Omega_2 T_0 + \theta_2)
\]

The solution to the equation \( \epsilon^0 \) can be expressed as,

\[
u_0 = a \cos \cos (T_0 + \beta) + \frac{k_1}{1 - a^2} \cos \cos (\Omega_1 T_0 + \theta_1) \tag{2.3}
\]

with \( a \) and \( \beta \) functions in \( T_1 \) as well as \( \Omega_1 \neq 1 \). In other forms, the solution can be written,

\[
u_0 = \frac{a}{2} \left( e^{i(T_0 + \beta)} + e^{-i(T_0 + \beta)} \right) + \frac{k_1}{2(1 - \Omega_1^2)} \left( e^{i(\Omega_1 T_0 + \theta_1)} + e^{-i(\Omega_1 T_0)} \right)
\]

where \( A = \frac{a}{2} e^{i\beta} \) dan \( B = \frac{1}{2(1 - \Omega_1^2)} e^{i\beta} \). Substitute \( u_0 \) in the equation order \( \epsilon \),

\[
D_0^2 u_1 + u_1 = (-2\hat{A} + A) e^{iT_0} - \frac{1}{3} (A^3 e^{3iT_0} - B^3 \Omega_1^3 e^{3i\Omega_1 T_0}
- 3A^2 B \hat{A} e^{(2+\Omega_1)T_0} + 3A^2 \hat{A} e^{iT_0} - 3A^2 B \hat{A} e^{(\Omega_1 - 2)T_0} - 3A^2 B \hat{A} e^{(2\Omega_1 + 1)T_0}
+ 3A^2 B \hat{A} e^{(2\Omega_1 - 1)T_0} + 3B^2 \hat{A} e^{iT_0} + 6A^2 \hat{A} e^{iT_0} + 6AB \hat{A} \Omega_1 e^{iT_0} + 6AB \hat{A} \Omega_1^2 e^{iT_0} + \frac{k_2}{2} e^{i(\Omega_2 T_0 + \theta_2)} + cc
\tag{2.4}
\]

with \( \hat{A} \) declaring \( \frac{d\hat{A}}{dT_1} \) and \( cc \) declares complex conjugate term of the right side. Equation (2.4) contains a secular term that is a term that depends on \( T_0 \) so that if solution \( u_1 \) is applied to Equation (2.3), \( u_1 \) which should have an order smaller than \( u_0 \), when \( t \geq \epsilon^{-1} \) the order of \( u_1 \) will be equal or more height of \( u_0 \). This causes the solution Equation (2.3) to be infinite or cause the phenomenon of resonance, namely the amplitude of the solution continues to increase for \( t \to \infty \). Therefore, in order to obtain a uniform expansion of solutions, secular term must be eliminated. Secular term are indicated by a term containing \( e^{iT_0} \). To eliminate secular terms Equation (2.4), it must be,

\[
-2\hat{A} + A - A^2 \hat{A} - 2ABB \Omega_1^2 = 0
\]

Paying attention to equality (2.4), even though only a small external force is added, but if \( \Omega_2 \approx 1 \) there will be an addition to the secular term. This shows that one of the causes of the addition of the secular term is accumulation of the external force frequency to the frequency of the cause of
resonance. Because the two external forces work together on the system, there is a possibility of resonance originating from the two external forces. From Equation (2.4), the superposition of the external forces causes resonance and possible small dividers namely,
1. $\Omega_1 \approx 1$ dan $\Omega_2 \approx 1$
2. $\Omega_1 \approx 3$ dan $\Omega_2 \approx 1$
3. $3\Omega_1 \approx 1$ dan $\Omega_2 \approx 1$

For the first case because of the value of $\Omega_1 \approx \Omega_2$ then both can be written as a unit with provisions such as, Nayfeh 1995, so that the case is similar to the problem of one external force. The following will be investigated for cases $\Omega_2 \approx 1$ with $\Omega_1$ far from 3 or $\frac{1}{3}$ before discussing the combination of $\Omega_1$ and $\Omega_2$.

3. **Case $\Omega_2 \approx 1$**

3.1. **Approximate Solutions**

For $\sigma_2$ is a detuning parameter that states the closeness of $\Omega_2$ with number one. Write,

$$\Omega_2 = 1 + \epsilon \sigma_2$$

Subtitles in Equation (2.4) then there will be additional secular term so to eliminate them,

$$-2\dot{A} + A - A^2\overline{A} - 2AB\overline{B}\Omega_2^2 - i\frac{k_2}{2}e^{i(\sigma_2 T_1 + \theta_2)} = 0$$

for $A = \frac{a}{2}e^{i\beta}$ and $B = \frac{k_1}{2(1-a^2)}e^{i\theta}$ we get,

$$-\dot{a} - ai\dot{\beta} + \frac{a}{2} - \frac{a^3}{8} - aA^2 - i\frac{k_2}{2}e^{i(\sigma_2 T_1 + \theta_2 - \beta)} = 0 \quad (3.1)$$

Separating the real and imaginary parts from Equation (3.1) will be obtained,

$$\dot{a} = \frac{a}{2} - \frac{a}{8} - aA^2 + \frac{k_2}{2} \sin(\sigma_2 T_0 + \theta_2 - \beta), \quad a\beta = -\frac{k_2}{2} \cos(\sigma_2 T_1 + \theta_2 - \beta) \quad (3.2)$$

As the initial approach a uniform solution can be written,

$$u = a\cos(T_0 + \beta) + \frac{k_2}{1 - a^2} \cos(\Omega_1 T_0 + \theta_1) + O(\epsilon)$$

where $O(\epsilon)$ states the order solution $\epsilon$, $a$ and $\beta$ are obtained from system (3.2). In order not to depend explicitly on time, a new variable $\gamma = \sigma_2 T_1 + \theta_2 - \beta$ is introduced so that equation (3.2) becomes,

$$\dot{a} = \frac{a}{2} - \frac{a}{8} - aA^2 + \frac{k_2}{2} \sin(\gamma), \quad a\gamma = a\sigma_2 + \frac{k_2}{2} \cos(\gamma) \quad (3.3)$$

Completing Equation (3.3) will obtained $a$ and $\gamma$. Therefore solutions can be written as,

$$u = a\cos(\Omega_2 t - \gamma + \theta_2) + \frac{k_2}{1 - a^2} \cos(\Omega_1 T_0 + \theta_1) + O(\epsilon)$$

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3.2. **Steady State Solution**

Steady state solution are obtained when $\dot{a} = \gamma = 0$ so that they meet,

$$\frac{a}{2} - \frac{a^3}{8} - aA^2 = -\frac{k_2}{2} \sin(\gamma) \quad (3.4)$$

$$a\sigma_2 = -\frac{k_2}{2} \cos(\gamma) \quad (3.5)$$

Squaring the two fields of Equation (3.4) and (3.5) then adding them together will be obtained,

$$\left(\frac{a}{2} - \frac{a^3}{8} - aA^2\right)^2 + (a\sigma_2)^2 = \left(\frac{k_2}{2}\right)^2 \quad (3.6)$$

Equation (3.6) is commonly called the response frequency equation. Figure 1 shows the response to changes in amplitude due to changes in $k_2$ by specifying the detuning parameter and also the response to changes in amplitude due to changes in detuning parameters by setting $k_2$. Further observed, it will be known that for $0 < k_2 < 0.5284$ there are three equilibrium points in some interval and for the rest interval there is only one equilibrium point. For $0.5284 < k_2 < 1$ there will be some interval so that there is only one equilibrium point, than followed by some interval that have three equilibrium points than followed by interval with one equilibrium point depending on the selection of $k_2$. For $k_2 > 1$, it will only have one equilibrium point for each real $\sigma_2$ value with a
maximum amplitude that decreases for the larger $k_2$ as shown in Figure 1. Next, the stability of the equilibrium point will be investigated.

![Figure 1](image)

**Figure 1.** With $\Omega_1 = 2, k_1 = 1$ changes in amplitude to (a) $k_2$ with some $\sigma_2$, (b) Detuning parameters with some values of $k_2$

### 3.3. Equilibrium Points Stability

Stability analysis can be done by investigating the movement of points around the equilibrium point. For that to be defined,

$$ a = a^* + a_0, \quad \gamma = \gamma^* + \gamma_0 $$

(3.7)

with $a^*, \gamma^*$ is a small parameter and $a_0, \gamma_0$ is the equilibrium point so that it satisfies Equation (3.4) and (3.5). By substituting (3.7) to Equation (3.3) and maintaining the linearity of $a^*$ and $\gamma^*$ there will be a coefficient matrix,

$$
\begin{pmatrix}
\frac{1}{2} - \frac{3a_0^2}{8} - \Lambda^2 - \lambda & \frac{k_2}{2} \cos \gamma_0 \\
-\left(\frac{k_2}{2a_0^2} \cos \gamma_0\right) & -\left(\frac{k_2}{2a_0^2} \sin \gamma_0\right) - \lambda
\end{pmatrix} = 0
$$

(3.8)

So we get,

$$
\lambda^2 - \left(1 - 2\Lambda^2 - \frac{a_0^2}{2}\right)\lambda + \Delta = 0
$$

(3.9)

where $\Delta = \sigma^2 + \left(\frac{1}{2} - \Lambda^2 - \frac{3a_0^2}{8}\right)\left(\frac{1}{2} - \Lambda^2 - \frac{a_0^2}{8}\right)$. 

A stable periodic solution will be obtained if \( \Delta > 0 \) and \( a_0^2 > 1 - 2A^2 \) by set \( k_1, \epsilon \) and \( \Omega_1 \) so that the equilibrium stability area will not depend on the amplitude of the external force \( k_2 \) as shown in Figure 2 (a). The equilibrium points in the shading area are stable while those outside the shading area are unstable. If given \( k_1 = 1, \epsilon = 0.1, \Omega_1 = 2 \) and \( k_2 = 0.3 \) there will be a stable equilibrium point producing the periodic solution when \( |\sigma_2| < 0.0855 \) which means for a long time, the solution's amplitude will be a constant amplitude \( a_0 \). Whereas if chosen \( |\sigma_2| > 0.0855 \) there will be a beating phenomenon, namely a solution whose amplitude oscillates as shown in Figures 2 (c) and (f).

### 4. Case \( \Omega_1 \approx 3 \) and \( \Omega_2 \approx 1 \)

#### 4.1. Approximate Solution

For \( \sigma_2 \) is a detuning parameter that states the closeness of \( \Omega_2 \) with number one. Write,

\[
\Omega_1 = 3 + \epsilon \sigma_1, \quad \Omega_2 = 1 + \epsilon \sigma_2
\]

Subtitles in Equation (2.4) then there will be additional secular term so to eliminate them,

\[
-2\tilde{A} + A - A^2 \tilde{A} - 2AB \tilde{B} \Omega_1^2 + A^2 B \Omega_1 e^{\sigma_1 i\tilde{T}_1} - \frac{ik_2}{2} e^{i(\sigma_2 T_1 + \theta_2)} = 0
\]

with \( A = \frac{a}{2} e^{i\theta} \) and \( B = \frac{k_1}{2(1-B)} e^{i\theta} \) we get a new system,

\[
\begin{align*}
\dot{a} &= \frac{a}{2} - \frac{a^3}{8} - aA^2 + \frac{a^2}{4} A \cos(\sigma_1 T_1 + \theta_1 - 3\beta) + \frac{k_2}{2} \sin(\sigma_2 T_1 + \theta_2 - \beta) \\
\dot{\beta} &= \frac{a^2}{4} A \sin(\sigma_1 T_1 + \theta_1 - 3\beta) - \frac{k_2}{2} \cos(\sigma_2 T_1 + \theta_2 - \beta)
\end{align*}
\]

(4.1)
In order not to depend explicitly on time, a new variable $y = \sigma T_1 - 3\beta$ is introduced where $\sigma_1 = 3\sigma_2 = \sigma$ so that equation (4.1) becomes,

$$\dot{a} = \frac{a}{2} - \frac{a^3}{8} - a\Lambda^2 + \frac{a^2}{4}\Lambda\cos(y + \theta_1) + \frac{k_2}{2}\sin\left(\frac{1}{3}y + \theta_2\right)$$

$$\frac{a}{3}\dot{y} = \frac{a\sigma}{3} - \frac{a^2}{4}\Lambda\sin(y + \theta_1) + \frac{k_2}{2}\cos\left(\frac{1}{3}y + \theta_2\right)$$

(4.2)

Completing Equation (4.2) will be obtained $a$ and $y$. Therefore solutions can be written as,

$$u = a\cos\left(\frac{1}{3}(\Omega_1 t - y)\right) + \frac{k_1}{1-a_1^2}\cos(\Omega_1 T_0 + \theta_1) + O(\epsilon)$$

(4.3)

4.2. Steady State Solution

Steady state solution are obtained when $\dot{a} = \dot{y} = 0$ so that they meet,

$$\frac{a}{2} - \frac{a^3}{8} - a\Lambda^2 = -\left(\frac{a^2}{4}\Lambda\cos(y + \theta_1) + \frac{k_2}{2}\sin\left(\frac{1}{3}y + \theta_2\right)\right)$$

$$\frac{a\sigma}{3} = \frac{a^2}{4}\Lambda\sin(y + \theta_1) - \frac{k_2}{2}\cos\left(\frac{1}{3}y + \theta_2\right)$$

(4.4)

Squaring the two fields of Equation (4.4) then adding them together will be obtained frequency response equation,

$$\left(\frac{a}{2} - \frac{a^3}{8} - a\Lambda^2\right)^2 + \left(\frac{a\sigma}{3}\right)^2 = \left(\frac{a^2}{4}\Lambda\right)^2 + \left(\frac{k_2}{2}\right)^2 - \frac{a^2k_2}{4}\Lambda\sin\left(-\frac{2}{3}y + \theta_2 - \theta_1\right)$$

(4.5)

Figure 3. With $\epsilon = 0.1, k_2 = 0.3$ (a) Change in amplitude versus $k_1$ with several $\sigma$, (b) Change in amplitude versus detuning parameter with some value of $k_1$.

For $0 < k_1 < 2.545$ there are three equilibrium points around the detuning parameter $\sigma$ zero then for $\sigma$ which is large enough there is one equilibrium point. Whereas for $2.545 < k_1 < 3.43$ there are $\sigma$ regions around zero which have one equilibrium point, also there are regions with three equilibrium points then for $\sigma$ is large enough there is only one equilibrium point. For $k_1 > 3.43$ there is only one equilibrium point for each $\sigma$. Then the stability of the equilibrium point will be investigated.

4.3. Equilibrium Points Stability

Using the same way as the previous section applied to the system (4.2) the coefficient matrix is fulfilled,

$$\begin{vmatrix} J_1 - \lambda & J_2 \\ J_3 & J_4 - \lambda \end{vmatrix} = 0$$

(4.6)

where,

$$J_1 = \frac{1}{2} - \frac{3a\gamma}{8} - \Lambda^2 + \frac{a^2}{2}\Lambda\cos(y_0 + \theta_1); \quad J_2 = -\frac{a^2}{4}\sin(y_0 + \theta_1) + \frac{k_2}{2}\cos\left(\frac{\gamma_0}{3} + \theta_2\right)$$

$$J_3 = -\frac{3}{4}\Lambda\sin(y_0 + \theta_1) - \frac{3k_2}{2a_0}\cos\left(\frac{\gamma_0}{3} + \theta_2\right); \quad J_4 = -\frac{3a_0}{4}\Lambda\cos(y_0 + \theta_1) - \frac{k_2}{2a_0}\sin\left(\frac{\gamma_0}{3} + \theta_2\right)$$
Figure 4. With $\epsilon = 0.1$ and $k_2 = 0.3$ (a) The equilibrium curve for $k_1 = 2$ and the stability area (b) Change $a$ and $\gamma$ for $\sigma = 0.235$ (c) Change $a$ and $\gamma$ for $\sigma = 0.23$ (d) Homogeneous Solution $\sigma = 0.235$ (e) Homogeneous solution for $\sigma = 0.23$ (f) Plot solution $\sigma = 0.235$ (g) Plot the solution when $\sigma = 0.23$.

From Equation (4.7), there will be a stable equilibrium point producing the periodic solution if

$$\frac{a^2}{2} > 1 - 2A^2 \quad \text{and} \quad \Delta > 0.$$

It can be seen that the conditions obtained by the periodic solution are stable depending on the selection of $k_1$, which means that the condition of the periodic solution stability changes with changes in $k_1$. The equilibrium point in the shading area in Figure 4 (a) is stable while the other is unstable. For example for $\epsilon = 0.1$, $k_2 = 0.3$ selected $k_1 = 2$, a stable equilibrium solution for periodic solutions will be obtained when $|\sigma| < 0.233$ as Figure 4 (a). The selection of $\sigma$ outside the area will cause a homogeneous solution to produce a beating phenomenon, namely the amplitude of the periodic oscillating solution as shown in Figures 4 (d) and (f).

5. Case $3\Omega_1 \approx 1$ and $\Omega_2 \approx 1$

5.1. Approximate Solution

A For $\sigma_2$ is a detuning parameter that states the closeness of $\Omega_2$ with number one. Write,

$$3\Omega_1 = 1 + \epsilon\sigma_1; \quad \Omega_2 = 1 + \epsilon\sigma_2$$

Subtitles in Equation (2.4) then there will be additional secular term so to eliminate them,

$$-2A + A - A^2 \Theta - 2AB\Omega_2^2 + \frac{1}{3} B^3 \Omega_2^3 e^{i\sigma_1 T_1} - \frac{k_2}{2} e^{i(\sigma_2 T_1 + \theta_2)} = 0$$

with $A = \frac{a}{2} e^{i\beta}$ and $B = \frac{k_1}{2(1-a^2)} e^{i\theta}$ we get a new system,

$$\dot{a} = \frac{a}{2} - \frac{a^3}{8} - aA^2 + \frac{1}{3} A^3 \cos(\sigma_1 T_1 + 3\theta_1 - \beta) + \frac{k_2}{2} \sin(\sigma_2 T_1 + \theta_2 - \beta)$$

$$a\dot{\beta} = \frac{1}{3} A^3 \sin(\sigma_1 T_1 + 3\theta_1 - \beta) - \frac{k_2}{2} \cos(\sigma_2 T_1 + \theta_2 - \beta)$$

(5.1)
In order not to depend explicitly on time, a new variable \( \gamma = \sigma T_1 - \beta \) is introduced where \( \sigma_1 = \sigma_2 = \sigma \) so that equation (5.1) becomes,

\[
\dot{a} = \frac{a}{2} - \frac{a^3}{8} - aL^2 + \frac{1}{3}a^3 \cos(\gamma + 3 \theta_1) + \frac{k_2}{2} \sin(\gamma + \theta_2)
\]

\[
a \dot{\gamma} = a \sigma - \frac{1}{3}a^3 \sin(\gamma + 3 \theta_1) + \frac{k_2}{2} \cos(\gamma + \theta_2)
\]

(5.2)

Completing Equation (5.2) will be obtained with,

\[
\dot{J} = \text{fulfilled, 5.3.}
\]

\[
\sigma \text{ly one equilibrium point. There are also regions with three equilibrium points then for there is only one equilibrium point. Whereas for there are three equilibrium points around zero which have one equilibrium point, there are also regions with three equilibrium points then for } \sigma \text{ large enough there is only one equilibrium point. For } k_1 > 2.83 \text{ there is only one equilibrium point for each } \sigma. \text{ Then the stability of the equilibrium point will be investigated.}
\]

| \[ J_1 = \frac{1}{2} - \frac{3a_0^2}{8} - A^2; \] | \[ J_2 = \frac{k_2}{2} \cos(\gamma_0 + \theta_2) - \frac{1}{3}a^3 \sin(\gamma_0 + 3 \theta_1) \] |
|---|---|
| \[ J_3 = \frac{1}{3}a_0^3 \sin(\gamma_0 + 3 \theta_1) - \frac{k_2}{2a_0} \cos(\gamma_0 + \theta_2); \] | \[ J_4 = -\frac{1}{3}a^3 \cos(y_0 + 3 \theta_1) + \frac{k_2}{2a_0} \sin(\gamma_0 + \theta_2) \] |

So we get,

\[
\lambda^2 - (1 - 2L^2 - \frac{a_0^2}{8}) \lambda + \Delta = 0
\]

(5.6)

with \( \Delta = \left( \frac{1}{2} - \frac{3a_0^2}{8} - A^2 \right) \left( \frac{1}{2} - \frac{a_0^2}{8} - A^2 \right) + \sigma^2. \)
Figure 6. With $\epsilon = 0.1$ and $k_2 = 0.3$ (a) The equilibrium point curve for $k_1 = 2.5$ and the stability area (b) Change $a$ and $\gamma$ for $\sigma = 0.11$ (c) Change $a$ and $\gamma$ for $\sigma = 0.1$ (d) Homogeneous Solution $\sigma = 0.11$ (e) Homogeneous Solution for $\sigma = 0.1$ (f) Plot solution when $\sigma = 0.11$ (g) Plot the solution when $\sigma = 0.1$.

From Equation (5.6), a periodic solution point will be stable when $\frac{a_0^2}{2} > 1 - 2\Delta^2$ and $\Delta > 0$. It can be seen that the conditions obtained by the periodic solution are stable depending on the selection of $k_1$, which means that the condition of the periodic solution stability changes with changes in $k_1$. The shaded area in Figure 6 (a) shows the area that meets the stability requirements for $\epsilon = 0.1, k_2 = 0.3$ and $k_1 = 2.5$ is in $|\sigma| < 0.1$ in other words the equilibrium point in the area is stable while the other is unstable. The choice of $\sigma$ in the area that has a stable equilibrium point causes the solution to converge at a constant amplitude the point remains stable $a_0$. The selection of $\sigma$ outside the area will cause the amplitude of the homogeneous solution to oscillate to beating as shown in Figures 6 (b), (d) and (f).

6. Conclusion
The magnitude of the frequency and amplitude of the external force given can affect the stability of the solution. Stable solution leads to a constant amplitude and phase so that a solution is obtained in the form of a harmonic oscillation wave. Unstable solution leads to an amplitude or phase which changes over time. Solutions that have periodic amplitude fluctuating over time will give rise to a beating phenomenon.

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