Two–center quantum MICZ–Kepler system and the Zeeman effect in the charge-dyon system

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Abstract

The quantum two-center MICZ–Kepler system is considered in the limit when one of the interaction centers is situated at infinity, which leads to homogeneous electric and magnetic fields appearing in the system. The emerging system admits separation of variables in the Schrödinger equation and is integrable at the classical level. The first order corrections to the unperturbed spectrum of the ordinary MICZ–Kepler system are calculated. Particularly, the linear Zeeman-effect and effects of MICZ-terms are analyzed. The possible realizations of the system in some quantum dots are considered.

1 Introduction

The MICZ–Kepler system describes an electrically charged scalar particle moving in the field of static Dirac dyon(s), i.e. a particle carrying both electrical and magnetic charges. This system was originally proposed by Zwanziger

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and McIntosh and Cisneros independently $^2$. In their papers the corresponding system was obtained from the ordinary Coulomb problem by replacing the interaction center by a dyon which has, beside the scalar potential, the vector-potential as well, and by introducing into the Hamiltonian the additional centrifugal-like term, $s^2/2\mu r^2$, where $s = eg$ is the monopole number (we use $c = \hbar = 1$ units), $e$ is the electric charge of the probe particle, $g$ is the magnetic charge of the static dyon and $\mu$ is the mass of the probe particle. According to the Dirac quantization rule the monopole number can take integer or half–integer values: $s = 0, \pm 1/2, \pm 1...$ The corresponding system is described by the following Hamiltonian:

$$H_{MICZ} = \frac{1}{2\mu} (p - eA_s)^2 + \frac{s^2}{2\mu r^2} + \frac{eq}{r},$$  \hspace{1cm} (1)

where

$$\text{rot} A_g = \frac{g r}{r^3}. $$  \hspace{1cm} (2)

Due to this additional potential term the emerging system inherits all distinguished features of the underlying Coulomb system. For instance, the MICZ-Kepler system, beside the general rotational symmetry generated by the algebra of angular momentum components ($so(3)$), has also a higher $so(4)$-hidden symmetry which is connected to the conservation of the Laplace–Runge–Lentz vector. The expressions for these conserved quantities are very similar to those of the Coulomb problem

$$J = r \times (p - eA) + s \frac{r}{r}, \quad I = \frac{1}{\mu} (p - eA) \times J - eq \frac{r}{r}. $$  \hspace{1cm} (3)

As one can see, only the additional spin-like term appears in the expression for the orbital momentum. The hidden symmetry of the MICZ–Kepler system leads to the possibility of separation of variables in several coordinate systems, e. g. in spherical and parabolic ones $^3$. The origin of the additional potential term (MICZ–term) could be interpreted as the interaction energy between the spherically symmetric magnetic field of Dirac monopole and the magnetic moment of the probe particle, $M = \frac{eq}{2\mu c}J$, where $J$ is given by Eq. (3). However, additional part of the angular momentum vector proportional to the monopole number $s$ should be rather assigned to the monopole electromagnetic field by itself but not to the particle. Thus, such a clear and natural, at the first glance, explanation of the origin of MICZ-term is not quite correct. On the other hand, this term can be obtained both on classical and quantum level when one reduce the four–dimensional isotropic oscillator with the aid of so–called Kustaanheimo–Stiefel transformations $^4$.

The shape of the classical trajectories of the MICZ–Kepler system coincides with these of the ordinary Coulomb one, however in contrast to the latter the orbital plane are non orthogonal to the angular momentum vector. The angle between $J$ and orbital plane normal satisfy the relation:

$$\cos \chi = \frac{s}{|J|}. $$  \hspace{1cm} (4)
which immediately followed from Eq. (3). The properties of the solutions of the quantum mechanical MICZ–Kepler problem undergo minor modifications with respect to those of the underlying Coulomb one. Namely, it has the same spectrum, given in terms of the same quantum numbers. The only difference is in the range of the possible values of angular momentum. The presence of the monopole with monopole number \( s \) shifts upward the allowed values of orbital and magnetic quantum numbers \( j \) and \( m: j = |s|, |s| + 1, |s| + 2, \ldots \) There is also more general statement concerning the incorporation of Dirac monopoles in spherically symmetric mechanical systems. Any spherically symmetric system (without monopoles) defined on the conformally flat space with metric \( G(r) \) will preserve its spectral properties except the possible range of angular momentum values mentioned above, if adding the monopole potential will accompany with the following modification of potential term [5, 6, 7, 8]:

\[
U(r) \rightarrow U(r) + \frac{s^2}{2G(r)\mu r^2}
\]  

(5)

Another difference between Coulomb systems and their MICZ–counterparts manifests itself in the behavior in external fields. So, one can observe the modification of the selection rules in the dipole transitions [9, 10]. The investigations of the generalizations of the MICZ-Kepler system to the higher dimensions and/or curved spaces as well revealed their close similarity to the underlying Coulomb systems [7, 11, 12, 13]. Multi–center generalization of MICZ–Kepler systems has been found recently in Ref. [7, 14]. The Hamiltonian which describes the charged particle moving in the electromagnetic field of \( n \) Dirac dyons fixed in the Euclidean space points with radius-vectors \( \mathbf{a}_i, i = 1, \ldots, n \) reads:

\[
\mathcal{H} = \frac{1}{2\mu} \left( \mathbf{p} - e \sum_{i=1}^n \mathbf{A}_{g_i}(\mathbf{r} - \mathbf{a}_i) \right)^2 + e \sum_{i=1}^n \frac{q_i}{|\mathbf{r} - \mathbf{a}_i|} + \frac{e^2}{2m} \left( \sum_{i=1}^n \frac{q_i}{|\mathbf{r} - \mathbf{a}_i|} \right)^2, \quad \text{rot}\mathbf{A}_{g_i}(\mathbf{r}) = \frac{g_i\mathbf{r}}{r^3}
\]  

(6)

where \( q_i (g_i) \) are the electric(magnetic) charges of the the \( i \)-th dyon. The last term in the expression generalizes the MICZ-term for the multi-center situation. In full analogy with the corresponding pure Coulomb system two–center case is integrable on the classical level [7, 13], i.e. it allows separation of variables in the Hamilton–Jacobi equation in elliptic coordinates. The limiting case of the two–center MICZ–Kepler system when one of the dyons is situated at infinity which results in the homogeneous electric and magnetic fields, is also integrable on classical level. The MICZ-term corresponding to this dyon results in its turn in the quadratic potential. Corresponding Hamilton–Jacobi equation become separable in parabolic coordinates. Because of presence of homogeneous electric and magnetic fields imposed to the one–center MICZ–Kepler (charge–dyon) system one can call it MICZ–Kepler–Zeeman–Stark system. In this paper we consider some quantum mechanical issues of the MICZ–Kepler–Zeeman–Stark system. In addition to the pure stark effect in charge–dyon system considered earlier in Refs. [13, 10], we present perturbative corrections to the spectrum for the charge–dyon system corresponding to the homogeneous magnetic field(linear Zeeman effect).

It is also noteworthy that multi–center MICZ–Kepler system allows \( \mathcal{N} = 4 \) supersymmetric extension when magnetic and electric charges of all dyons satisfy the trivial Dirac–Schwinger–Zwanziger condition

\[
g_i q_j - g_j q_i = 0
\]  

(7)

The corresponding \( \mathcal{N} = 4 \) supersymmetric mechanical systems were constructed in Refs. [16, 17] and with a little different approach leading to the MICZ–Kepler system on the three–dimensional sphere in Ref. [18].

The paper is organized as follows. In the second section we derive the Schrödinger equation for the MICZ–Kepler–Zeeman–Stark system in parabolic coordinates. The solution of the unperturbed problem is presented and calculated the first and second order corrections in \( B \) and \( E \). The average dipole and magnetic moments acquired the system are also calculated. In the third section the related quantum dot models are discussed. Some final remarks are presented in the conclusion. The appendix contains some technical points.

2 Two–center MICZ–Kepler system and quantum MICZ–Kepler–Stark–Zeeman System

The Hamiltonian of the two center MICZ–Kepler system on Euclidean space has the form [14]

\[
\mathcal{H} = \frac{1}{2} \left( \mathbf{p} - e\mathbf{A}_{g_1}(\mathbf{r} - \mathbf{r}_1) - e\mathbf{A}_{g_2}(\mathbf{r} - \mathbf{r}_2) \right)^2 + \frac{1}{2} \left( \frac{s_1}{|\mathbf{r} - \mathbf{r}_1|^2} + \frac{s_2}{|\mathbf{r} - \mathbf{r}_2|^2} \right)^2 + e \left( \frac{q_1}{|\mathbf{r} - \mathbf{r}_1|} + \frac{q_2}{|\mathbf{r} - \mathbf{r}_2|} \right),
\]  

(8)

\[\text{For a unified superfield formulation of N=4 off-shell supermultiplets in one space-time dimension see } [19]. \text{ For earlier work see } [20].\]
Seeking for the eigenfunctions in the form $m$ where $r$ then the corresponding Hamiltonian in spherical coordinates reads

$$H = \frac{1}{2} \left( \frac{p_r^2}{r^2} + \frac{p_\theta^2}{r^2 \sin^2 \theta} + \frac{(p_\varphi - s \cos \theta - \frac{1}{r} e B r^2 \sin^2 \theta)^2}{r^2 \sin^2 \theta} \right) + \frac{1}{2} \left( \frac{s_1}{r_1} + \frac{s_2}{r_2} \right)^2 + e q_1 + e q_2,$$

where $r_{1,2} = \sqrt{x^2 + y^2 + (z \pm a)^2}$. As it was shown in Ref. [14] this system on the classical level admits separation of variables in elliptic coordinates, which leads to the integrable system of Hamilton–Jacobi equations. The two–center MICZ–Kepler system has an important limiting case, when one of the interaction centers is situated at infinity. The field of such a dyon results in the homogeneous electric and magnetic fields being parallel to each other. If one takes for the vector potential of the homogeneous magnetic field the following gauge:

$$A_r = A_\theta = 0, \quad A_\varphi = \frac{1}{2} B r^2 \sin^2 \theta,$$

then the corresponding Hamiltonian in spherical coordinates reads

$$H = \frac{1}{2} \left( \frac{p_r^2}{r^2} + \frac{p_\theta^2}{r^2 \sin^2 \theta} + \frac{(p_\varphi - s \cos \theta - \frac{1}{r} e B r^2 \sin^2 \theta)^2}{r^2 \sin^2 \theta} \right) + \frac{1}{2} \left( \frac{s}{r} + e B z \right)^2 + e q \frac{r}{r} - e E z,$$

where $B$ and $E$ stand for the modulus of the magnetic and electric field respectively which are pointed in the $z$ direction. The classical case is characterized by the separation of variables in the Hamilton–Jacobi equation in parabolic coordinates given by the following relations:

$$\xi = r + z, \quad \eta = r - z, \quad \varphi = \arctan \frac{y}{x}. \quad (14)$$

The same feature holds in the quantum case for the Schrödinger equation for the Hamiltonian $H$ as well. The quantum Hamiltonian resulting from Eq. (13) reads

$$\mathcal{H} = -\frac{1}{2} \Delta + i \left( \frac{s \cos \theta}{r^2 \sin^2 \theta} + \omega_B \right) \frac{\partial}{\partial \varphi} + \frac{s^2}{2 r^2 \sin^2 \theta} + \frac{\omega_B^2}{2} \left( \rho^2 + 4z^2 \right) + 3 s \omega_B \cos \theta + \frac{e q}{r} - e E z,$$

where $\omega_B = \frac{e B}{\hbar}$ is the cyclotron frequency in our units ($\hbar = \mu = 1$) and $\rho^2 = r^2 \sin^2 \theta = x^2 + y^2$. Being rewritten in the parabolic coordinates given by Eqs. (14), the corresponding Schrödinger equation takes the form

$$\frac{4}{\xi + \eta} \left( \frac{\partial}{\partial \xi} \left( \xi \frac{\partial \Psi}{\partial \xi} \right) + \frac{\partial}{\partial \eta} \left( \eta \frac{\partial \Psi}{\partial \eta} \right) \right) + \frac{1}{\eta \xi} \left( \frac{\partial^2}{\partial \varphi^2} - s^2 \right) \Psi + 2 i e \left( \frac{s}{\xi + \eta} \left( \frac{1}{\xi} - \frac{1}{\eta} \right) - \omega_B \right) \frac{\partial \Psi}{\partial \varphi} + \left( 2 E - \frac{\omega_B^2 (\xi^2 + \eta^2)}{\xi + \eta} - 6 s \omega_B \frac{\xi - \eta}{\xi + \eta} - 4 e q \frac{\xi^2 - \eta^2}{\xi + \eta} + e E \frac{2 \xi^2 - \eta^2}{\xi + \eta} \right) \Psi = 0.$$

Hereafter in order to deal with discrete spectrum we suppose the electric charge $e$ of the probe particle to be negative. Seeking for the eigenfunctions in the form

$$\Psi = f_1(\xi) f_2(\eta) e^{i m \varphi}, \quad (17)$$

where $m$ is the magnetic quantum number, and dividing the equation on $\frac{4 f_1 f_2}{\xi + \eta}$, one obtains

$$\frac{d}{d\xi} \left( \xi \frac{df_1}{d\xi} \right) f_1 + \frac{d}{d\eta} \left( \eta \frac{df_2}{d\eta} \right) f_2 + U(\xi) + V(\eta) = -|e| q \quad (18)$$
where

\[ U(\xi) = -\frac{s_+^2}{4\xi} - \frac{\omega^2}{4} \xi^3 + \frac{e\mathcal{E}}{4} \xi^2 + W_\xi, \]

\[ V(\eta) = -\frac{s_-^2}{4\eta} - \frac{\omega^2}{4} \eta^3 - \frac{e\mathcal{E}}{4} \eta^2 + W_\eta, \]

and the following notations are introduced \( s_\pm = m \pm s, W_\pm = 1/4(2E + 2m\omega_B \pm 6s\omega_B) \). Separating the variables \( \xi \) and \( \eta \) one obtains the system of equation for the functions \( f_1 \) and \( f_2 \)

\[ \frac{d}{d\xi} \left( \xi \frac{df_1}{d\xi} \right) + \left( \frac{1}{2} E\xi - \frac{s_+^2}{4\xi} + |e|q\beta_1 \right) f_1 = 0, \]

\[ \frac{d}{d\eta} \left( \eta \frac{df_2}{d\eta} \right) + \left( \frac{1}{2} E\eta - \frac{s_-^2}{4\eta} + |e|q\beta_2 \right) f_2 = 0, \]

where the separation constants \( \beta_1 \) and \( \beta_2 \) satisfy the condition

\[ \beta_1 + \beta_2 = 1 \]

### 2.1 One-center MICZ–Kepler system in parabolic coordinates

Note that the case of the one-center MICZ-Kepler (\( B = 0, \mathcal{E} = 0 \)) system completely coincides with the quantum-mechanical problem of the hydrogen atom.\(^2\) It leads to the same spectrum. The only difference consists in the allowed values of magnetic quantum number, which are just shifted by \( s \). Now one can regard the terms corresponding to the homogeneous electric and magnetic fields as perturbations with respect to the integrable case of the one center MICZ-Kepler Hamiltonian which leads to the following equations: \[^3\]

\[ \frac{d}{d\xi} \left( \xi \frac{df_1}{d\xi} \right) + \left( \frac{1}{2} E\xi - \frac{s_+^2}{4\xi} + |e|q\beta_1 \right) f_1 = 0, \]

\[ \frac{d}{d\eta} \left( \eta \frac{df_2}{d\eta} \right) + \left( \frac{1}{2} E\eta - \frac{s_-^2}{4\eta} + |e|q\beta_2 \right) f_2 = 0, \]

which are almost identical to those for the hydrogen atom in parabolic coordinates (see for example \[^22\]). We consider only the discrete spectrum of the unperturbed problem (\( E < 0 \)). In this case it is convenient to pass to the new variables \( n = 1/\sqrt{-2E}, \rho_1 = \xi/n \) and \( \rho_2 = \eta/n \). Let us write down the equation for the function \( f_1 \) in these coordinates

\[ \frac{d^2 f_1}{d\rho_1^2} + \frac{1}{\rho_1} \frac{df_1}{d\rho_1} + \left( -\frac{1}{4} + \frac{1}{\rho_1} \left( n_1 + \frac{1}{2} (|s_+| + 1) - \frac{s_+^2}{4\rho_1^2} \right) \right) f_1 = 0, \]

where we introduce the notations and put for simplicity \( |e|q = 1 \) (natural units)

\[ n_1 = n/\beta_1 - \frac{1}{2} (|s_+| + 1), \quad n_2 = n/\beta_2 - \frac{1}{2} (|s_-| + 1). \]

The corresponding solution for \( f_2 \) can be obtained by replacing \( \beta_1 \) by \( \beta_2 \) and \( s_+ \) by \( s_- \). Taking into account the long- and short-distance asymptotic behavior of the solutions one can seek the function \( f_1 \) in the following form: \[^{22}\]

\[ f_1(\rho_1) = e^{-\rho_1/2} \rho_1^{\frac{|s_+|}{2}} F_1(\rho_1), \]

which leads to the equation for the confluent hypergeometric function for \( F_1(\rho_1) \)

\[ \rho_1 \frac{d^2 F_1}{d\rho_1^2} + (|s_+| + 1 - \rho_1) \frac{dF_1}{d\rho_1} + n_1 F_1 = 0. \]

Thus, the functions \( F_1 \) and \( F_2 \) which satisfy the finiteness condition read

\[ F_1(\rho_1) = F(-n_1, |s_+| + 1; \rho_1), \quad F_2(\rho_1) = F(-n_2, |s_-| + 1; \rho_2), \]

\(^2\) Related supersymmetric systems with Dirac monopoles as well as the details of the Hamiltonian reduction technique have been considered in \[^{21}\].
where \( n_1 \) and \( n_2 \) must be non-negative integers. So, from Eq. (24) one obtains that the principal quantum number for the one-center MICZ-Kepler system is

\[
n = n_1 + n_2 + \frac{|m + s| + |m - s|}{2} + 1, \tag{28}
\]

Thus, for the unperturbed part of the Hamiltonian (13), each stationary state of the discrete spectrum in parabolic coordinates is characterized by three integer quantum numbers, i.e. the parabolic quantum numbers \( n_1 \) and \( n_2 \) and the magnetic quantum number \( m \). The normalized eigenfunctions are

\[
\Psi_{n_1,n_2,m}(\xi,\eta,\varphi) = \frac{1}{n^2 \sqrt{\pi}} f_{n_1,m+s}(\xi/n) f_{n_2,m-s}(\xi/n) e^{im\varphi}, \tag{29}
\]

where

\[
f_{k,p}(\rho) = \frac{1}{|p|!} \sqrt{\frac{(k+|p|)!}{k!}} e^{-\rho^2/2} \rho^{p/2} F(-k,|p|+1;\rho). \tag{30}
\]

### 2.2 The perturbative first order corrections and linear Zeeman splitting

One can regard all terms, corresponding to magnetic and electric field as small perturbations. Thus, we have the unperturbed Hamiltonian of the one-center MICZ–Kepler system with exactly known stationary state eigenfunctions (29) and the perturbation given by the operator

\[
\mathcal{W} = |e|\mathcal{E}z - \frac{3s|e|B}{2} \cos \theta + \frac{e^2B^2}{8} (\rho^2 + 4z^2) = \frac{|e|\mathcal{E}}{2} (\xi - \eta) - \frac{3s|e|B \xi - \eta}{2} \frac{e^2B^2}{8} \xi^3 + \frac{\eta^3}{\xi + \eta}. \tag{31}
\]

There is also a constant shift in the energy level, caused by the magnetic field and proportional to the magnetic quantum number \( m \), i.e. \(-m\omega_B\). So, the first order correction to the stationary unperturbed hydrogen–like spectrum (32)

\[
E_{n_1,n_2,m}^{(0)} = -\frac{1}{2} \frac{1}{(n_1 + n_2 + 1 + \frac{|m+s|+|m-s|}{2})^2}, \tag{32}
\]

reads

\[
\Delta E_{n_1,n_2,m}^{(1)} = \langle n_1, n_2, m | \mathcal{W} | n_1, n_2, m \rangle, \tag{33}
\]

Thus

\[
E_{n_1,n_2,m}^{(1)} = E_{n_1,n_2,m}^{(0)} - m\omega_B + \Delta E_{n_1,n_2,m}^{(1)}, \tag{34}
\]

where

\[
\Delta E_{n_1,n_2,m}^{(1)} = \frac{|e|\mathcal{E}}{4} I_2^{\perp} - \frac{3s|e|B}{4n} I_4^{\perp} + \frac{ne^2B^2}{8} I_3^{\perp}, \tag{35}
\]

and \( I_k^{\perp} \) stand for the integrals of hypergeometric functions presented in Appendix. Here we restrict ourselves with the first-order corrections in \( \mathcal{E} \) and \( B \) (we will suppose \( \mathcal{E} \) and \( B \) to be of the same order). Thus,

\[
E_{n_1,n_2,m}^{(1)} = -\frac{1}{2n^2} \frac{3}{2} |e|\mathcal{E}(mn_+ - \frac{ms}{3}) - \frac{1}{2} |e|B\left(\frac{3s}{n} - m\right) + O(B^2), \tag{36}
\]

where

\[
n_+ = \frac{|m + s| - |m - s|}{2} + n_1 - n_2. \tag{37}
\]

Putting \( B = 0 \) one obtains the well known result of purely Stark–effect in the charge–dyon system [10] [15]. The third term in the expression corresponds to the linear Zeeman–effect. As usual, in this approximation magnetic field removes the degeneracy with respect to the magnetic quantum number \( m \) and induce the magnetic moment in the system. The average magnitude of induced magnetic moment could be obtained by taking the derivative of Eq. (35) with respect to \( B \). However, the term \( \frac{3s|e|B}{2} \cos \theta \) in the operator (31) actually comprise of two parts of different origin. The first
is \( \frac{seB}{2} \cos \theta \) which is originated from the magnetic field in the kinetic term. The second part is \( s|e|B \cos \theta \) coming from the additional potential term (MICZ–term). Indeed, the MICZ–term in this system just has the coefficient which is coincide with magnetic field by magnitude but physically is not identical with it. Thus, in order to calculate the induced magnetic moment of the charge–dyon system in the external magnetic field one must subtract from \( E_{n_1n_2m}^{(1)} \) the impact of the MICZ–term prior to taking the derivative:

\[
\mathcal{M}_z = -\frac{\partial}{\partial B} \left( E_{n_1n_2m}^{(1)} - s|e|B(n_1, n_2, m|\cos \theta|n_1, n_2, m) \right) = \frac{1}{2}|e|(sn_- - m) \tag{38}
\]

where \( \mu_B = \frac{1}{2}|e| \) is the Bohr magneton in our units. Taking into account above mentioned properties of the linear in \( B \) corrections one can write down the spectrum in the following form:

\[
E_{n_1n_2m}^{(1)} = E_{n_1n_2m}^{(0)} - \frac{s|e|n_- B}{n} = \mathcal{M}_z B + \frac{s|e|n_- B}{n}. \tag{39}
\]

The last term originated from the MICZ-term and

\[
\mathcal{M}_z = -\mu_B \frac{m}{1 + |s|}. \tag{41}
\]

Thus, the linear Zeeman–effect in the charge–dyon system removes the \((2|s| + 1)\)-fold degeneracy by \( m \) of the ground state. The induced magnetic momentum for positive values of \( m \) is always negative and goes to zero at \(|s| \to \infty \). At \( s = 0 \) one get the expected value \( \mathcal{M}_z = -\mu_B m \) corresponding to the ordinary Zeeman splitting in the hydrogen atom. Hence, for the ground state of the system under consideration one can write

\[
E_{00m}^{(0)} = -\frac{1}{2(1 + |s|)^2} + m|e|s\text{gn} \left( |s| + \frac{3}{2} \right) \mathcal{E} + \frac{1}{2}|e| \frac{m}{1 + |s|} B + |e| \frac{m|s|}{1 + |s|} B. \tag{42}
\]

As one can see, the sign of the linear Zeeman effect depends only on the sign of the magnetic quantum number \( m \), whereas the linear Stark effect depends on the relative sign of \( m \) and \( s \). If we suggest that coefficient in the oscillatory potential \( \frac{e^2}{2}(\rho^2 + 4z^2) \) in Eq. (31) is independent of \( B \) and is of the same order with \( \mathcal{E} \) and \( B \) then according to the formula (38) we will get the corresponding first order correction to the ground state energy in the following form (see Appendix)

\[
\Delta \hat{E}_0^{1} = \frac{\omega^2}{2} (1 + |s|)((m^2 + s^2)(|s| + 6) + |s|(2m^2 + 11) + 6). \tag{43}
\]

### 3 Additional potential term and related quantum dot models

It is obviously seen that one can add to the Hamiltonian (13) an additional potential term of the form

\[
\mathcal{U} = \frac{\omega^2}{2}(\rho^2 + 4z^2) = \frac{\omega^2}{2}(\xi^3 + \eta^3) \tag{44}
\]

without breaking its classical integrability and separation of variables in the quantum case in parabolic coordinates. Such kind of potential can originate in the model of a cylindrical quantum dot, where the influence of the dot boundary is described by the confining potential of the parabolic type (oscillator potential, see also [28]) with special rate of radial and transverse frequencies. Moreover, if we put \( s = 0 \) the emerging system could be identified with the model of a charged particle moving in the axially symmetric quantum dot in the field of Coulomb center and homogeneous
electric and magnetic field pointing along the dot symmetry axis\textsuperscript{3} The corresponding confining potential takes the form
\[ U_C = \frac{1}{2} \left( \omega^2 \rho^2 + \tilde{\omega}^2 z^2 \right), \quad \tilde{\omega} = 2 \sqrt{\omega^2 + \omega_B^2}. \] (45)

This model is integrable at the classical level \cite{14, 7} and leads to the separable Schrödinger equation. One can also consider a little different context of that issue. Let us suppose we have a particle moving in the field of Coulomb center (or the relative motion of two electrically charged particles) in the cylindrical quantum dot with an axially symmetric oscillatory confining potential with arbitrary frequencies $\omega_\rho$ and $\omega_z$. The Schrödinger equation for such a system does not admit separation of variables, except for the case $\omega_z = 4 \omega_\rho$. Let us assume also that there are homogeneous electric and magnetic fields both pointing along the $z$-axis
\[ \mathcal{H} = \frac{1}{2} \left( p_\rho^2 + p_z^2 + \left( \frac{p_\rho - \frac{e B}{\rho^2}}{2} \right)^2 \right) + \frac{e q}{\sqrt{\rho^2 + z^2}} - e E_\rho + \frac{1}{2} \left( \omega_\rho^2 \rho^2 + \omega_z^2 z^2 \right). \] (46)

However, if the magnetic field magnitude $B$ takes the value $\frac{1}{2} \sqrt{\omega_z^2 - 4 \omega_\rho^2}$ the variables separate in parabolic coordinate. So, at the classical level one obtains an integrable system, which can be a subject for the perturbation theory in the quantum case. The quantum dots models with Dirac monopole inside have not only academic interest. Though, no elementary particle carrying magnetic charge has been discovered up to now, there are very promising theoretical evidences of the magnetic monopoles existence as emergent particles, i.e., as the quasi-particles in some strongly correlated many–body systems. It was shown in Ref. \cite{25} that such kind of magnetic monopoles do emerge in so-called spin-ice materials, the exotic class of magnets in which local magnetic moments (spins) are residing on the sites of pyrochlore lattice (corner shared tetrahedra) and are constrained to point along their local Ising axes. The spins interact to each other via nearest neighbor exchange and long–range dipole–dipole interaction. Thus, the model discussing in this section can be realized to certain extent in the quantum dots prepared from spin-ice compound, such as Ho\textsubscript{2}Ti\textsubscript{2}O\textsubscript{7}, Dy\textsubscript{2}Ti\textsubscript{2}O\textsubscript{7}, etc. etc. (for review of spin-ice see \cite{26}).

In order to apply perturbative results from previous section to the quantum dots model discussing here the parameters of confining potentials must be small (more precisely, the corresponding correction $\Delta E_{n_1 n_2 \mu}$ must be much smaller than the distance between corresponding neighbor levels of unperturbed problem), otherwise only the numerical calculations are relevant. The corresponding correction to the ground state energy has the same form as Eq. (45) with $\omega_0 = \omega_B + \omega$. In this case, in order for the perturbative calculation to be relevant, this correction must be much smaller than the ground state energy, which leads to
\[ \omega_0 \ll \left( (1 + |s|)^3((m^2 + s^2)(|s| + 6) + |s|(2m^2 + 11) + 6) \right)^{-1/2}. \] (47)

A similar model of the two–electron quantum dot subjected to the external homogeneous electric field was considered numerically in Ref. \cite{27}.

4 Conclusion

In this paper we considered the quantum mechanical two–center MICZ–Kepler system in the limit when one of the dyons is situated at infinity. The electro–magnetic field of such a dyon results in the homogeneous electric and magnetic fields parallel to each other. At the same time, the specific additional potential term (MICZ–term) transforms into oscillator potential in the direction of the external fields and potential proportional to $\cos \theta$. Thus, the system considered in the present paper concerns the “monopolic” generalization of the hydrogen atom (Kepler problem) subjected simultaneously to the constant uniform electric and magnetic fields. We separated the variables in the corresponding Schrödinger equation in the parabolic coordinates and analyzed the simplest case of the emergent MICZ–Kepler–Stark–Zeeman system when external electric and magnetic fields can be regarded as small perturbations. The separation of variables is possible in virtue of the additional potential term. As a starting point we considered the Hamiltonian of one–center MICZ–Kepler system with well known exact solution, whereas all other terms were treated as perturbations. The exact wave functions in parabolic coordinates were used to develop first order perturbative calculations. We obtained linear in $E$ and $B$ corrections to the unperturbed spectrum of the ordinary MICZ–Kepler problem. The Stark–effect in the charge–dyon system was calculated within the perturbation theory up to second order earlier in series of paper \cite{13, 10}. Here we investigated the first order corrections corresponding to magnetic field

\textsuperscript{3} In the noncommutative framework the Coulombic monopole has been considered in \cite{24}.
effect (linear Zeeman effect). We found the average magnetic moment acquired by the charge–dyon system in this case.

For the ground state the induced magnetic moment is even function of the monopole number $s$, $\mathcal{M}_z(-s) = \mathcal{M}_z(s)$ and vanishes at $s \to \pm \infty$. It is also non–analytic at $s = 0$. As usually, magnetic field removes the degeneracy of the ground state. The impact of the additional potential, so–called MICZ–term, in the first order has much in common with that of the magnetic field. The sign of the corresponding correction depends only on the sign of magnetic quantum number $m$. We also analyzed some condensed matter models which can be relevant in the context of the quantum mechanical system considered in the papers [14] [27]. The recent results concerning possibility of formation of the magnetic monopoles as quasi–particles in some strongly correlated spin systems, so–called spin–ice materials make the attempts of understanding the behavior of the various (quantum–)mechanical systems at the monopole background very important and promising for future experiments. Unfortunately, in order to obtain reliable results in the quantum dots models with monopoles in the external fields one can not restrict himself with integrable cases and/or first order perturbative calculations. One of the reasons for it is the essential role of the confining potentials which, generally speaking, can not be regarded as perturbation. However, the second order corrections in the MICZ–Kepler–

Stark–Zeeman system can lead to new interesting features, for instance, to the interplay between electric and magnetic properties of the charge–dyon system. Namely, the correction to the spectrum proportional to $EB$ which appears in the second order perturbative calculations means that average dipole(magnetic) moment depend on magnetic(electric) field. Another important issue is to understand spin effect and spin–orbital coupling in the systems with monopoles within the quantum dots models. We intend to consider all these questions in the forthcoming papers.

5 Appendix

Here we calculate the general integrals which emerge in the perturbation theory calculations for the quantum mechanical systems in parabolic coordinates

$$I^k_\pm = \int_0^\infty \int_0^\infty (\rho_1^k \pm \rho_2^k) f_{n_1,m+s}^2(\rho_1) f_{n_2,m-s}^2(\rho_2) d\rho_1 d\rho_2$$

Let us plug into the integral the expressions for functions $f_{n,p}$ from Eq. (30). Then

$$I^k_\pm = C_{n,m+s}^2 C_{n,m-s}^2 \left( J_{n_1,+,p_1}^k, J_{n_2,\mp}^k \right),$$

where normalization constant square is

$$C_{n,m\pm s}^2 = \frac{(|m \pm s| + n)!}{n!(|m \pm s|)!^2}$$

and

$$J_{n,\pm}^k = \int_0^\infty e^{-\rho |m \pm s| + k} F^2(-n, |m \pm s| + 1; \rho) d\rho$$

The details of calculations of the more general integral can be found in [22].

$$\int_0^\infty e^{-k z} z^{\nu-1} F^2(-n, \gamma; k z) dz =$$

$$\frac{\Gamma(\nu)n!}{k^\nu \gamma(\gamma - 1)\ldots(\gamma + n - 1)} \left( 1 + \sum_{p=0}^{n-1} \frac{n(n-1)\ldots(n-p)(\gamma - \nu - p - 1)(\gamma - \nu - p)\ldots(\gamma - \nu + p)}{(p+1)!^2(\gamma + 1)\ldots(\gamma + p)} \right)$$

Applying this formula to Eq. (51) one obtains

$$J_{n,\pm}^0 = \frac{n!(|m \pm s|)!^2}{(|m \pm s| + n)!} = \frac{1}{C_{n,m\pm s}^2}$$

$$J_{n,\pm}^k = \frac{n!(|m \pm s| + k)!|m \pm s|!}{(|m \pm s| + n)!} \left( 1 + \sum_{p=0}^{k-1} \frac{n!(|m \pm s|)!^2 (k - p + q) \prod_{q=0}^{2p+1} (k - p - q)}{(p+1)!^2(n - p - 1)!(|m \pm s| + p + 1)!} \right), \quad k > 0$$

Thus

$$I^k_\pm = \frac{(|m \pm s| + k)!}{|m \pm s|!} \left( 1 + \sum_{p=0}^{k-1} \frac{n!(|m \pm s|)!^2 (k - p + q) \prod_{q=0}^{2p+1} (k - p - q)}{(p+1)!^2(n - p - 1)!(|m \pm s| + p + 1)!} \right) \pm$$

$$\frac{(|m \pm s| + k)!}{|m \pm s|!} \left( 1 + \sum_{p=0}^{k-1} \frac{n!(|m \pm s|)!^2 (k - p + q) \prod_{q=0}^{2p+1} (k - p - q)}{(p+1)!^2(n - p - 1)!(|m \pm s| + p + 1)!} \right)$$
Particularly
\[ I_+^1 = (|m + s| + 1) \left( 1 + \frac{2n_1}{|m + s| + 1} \right) + (|m - s| + 1) \left( 1 + \frac{2n_2}{|m - s| + 1} \right) = 2n \]  
\[ I_- = |m + s| - |m - s| + 2(n_1 - n_2) = 2n_- \]  
\[ I_-^2 = 6 \left( n_{m-} - \frac{m_s}{3} \right) \]  
\[ I_+^3 = (|m + s| + 3)(|m + s| + 2)(|m + s| + 1) + (|m - s| + 3)(|m - s| + 2)(|m - s| + 1) + 12(n_1(|m + s| + 3)(|m + s| + 2) + n_2(|m - s| + 3)(|m - s| + 2)) + 30(n_1(n_1 - 1)(|m - s| + 1) + n_2(n_2 - 1)(|m - s| + 1)) + 20(n_1(n_1 - 1)(n_1 - 2) + n_2(n_2 - 1)(n_2 - 2)) \]

6 Acknowledgements

We are indebted to Armen Nersessian for valuable ideas and important comments. We also express our gratitude to Armen Yeranyan, Hayk Sarkisyan and Pasquale Onorato for useful discussions and interest toward this work. V. O. thanks the INFN-Laboratori Nazionali di Frascati where the essential part of the work was done for warm hospitality. This work was partially supported by grants NFSAT-CRDF UC-06/07, ANSEF-1386-PS, INTAS under contract 05-7928, and by the European Community Human Potential Program under contract MRTN-CT-2004-005104 “Constituents, fundamental forces and symmetries of the universe”.

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