REMARKS ON MOTIVES OF MODULI SPACES OF RANK 2 VECTOR BUNDLES ON CURVES

KYOUNG-SEOG LEE

ABSTRACT. Let $C$ be an algebraic curve of genus $g \geq 2$ and $M_L$ be the moduli space of rank 2 stable vector bundles on $C$ whose determinants are isomorphic to a fixed line bundle $L$ of degree 1 on $C$. In [2], S. del Bano studied motives of moduli spaces of rank 2 vector bundles on $C$ and computed the motive of $M_L$. In this note, we prove that his result gives an interesting decomposition of the motive of $M_L$. This motivic decomposition is compatible with a conjecture of M. S. Narasimhan which predicts semi-orthogonal decomposition of derived category of the moduli space.

1. Introduction

Let $C$ be a smooth projective curve of genus $g \geq 2$ over $\mathbb{C}$ and $M_L$ be the moduli space of rank 2 stable vector bundles on $C$ whose determinants are isomorphic to a fixed line bundle $L$ of degree 1 on $C$. Let $E$ be the Poincaré bundle on $C \times M_L$. We denote by $D(C)$ (resp. $D(M_L)$) the bounded derived category of coherent sheaves on $C$ (resp. $M_L$). Then $E$ defines a functor $\Phi_E : D(C) \to D(M_L)$ by $\Phi_E(F) := R_{p_{M_L}^{*}}(L_{p_{C}^{*}}(F) \otimes^L E)$, where $F$ is an element in $D(C)$ and $p_{C}$ (resp. $p_{M_L}$) is the projection map from $C \times M_L$ to $C$ (resp. $M_L$). It was proved that $\Phi_E$ is fully faithful for every smooth projective curve of genus greater than or equal to 2 in [1, 2, 8, 9]. Moreover the results in [8] imply that there is the following semi-orthogonal decomposition.

$$D(M_L) = \biglangle D(pt), D(pt), D(C), \cdots , D(C^{g-1}) \bigrangle \perp \biglangle D(pt), D(pt), D(C) \bigrangle$$

It is an interesting and important task to understand the semi-orthogonal component $\biglangle D(pt), D(pt), D(C) \bigrangle$. It was conjectured by M. S. Narasimhan that the derived category of $M_L$ has a semi-orthogonal decomposition consisting of two copies of the derived category of point, two copies of the derived category of $C$, · · · , two copies of the derived category of $C^{g-2}$ and one copy of the derived category of $C^{g-1}$, where $C^{(n)}$ denotes the $n$-th symmetric power of $C$.

Conjecture 1.1. The derived category of $M_L$ has the following semi-orthogonal decomposition

$$D(M_L) = \langle D(pt), D(pt), D(C), D(C), \cdots , D(C^{g-2}), D(C^{(g-2)}), D(C^{(g-1)}) \rangle.$$ 

Using del Bano’s work(cf. [2]) we found that the motive of $M_L$ has the following interesting decomposition. Assuming some conjectures about derived categories and motives (cf. [10]), the following motivic decomposition is compatible with the above conjecture.

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Theorem 1.2. In any semisimple category of motives, there is the following isomorphism.

\[ h(M_L) \cong \bigoplus_{k=0}^{g-2} h(C^{(k)}) \otimes (L^\otimes k \oplus L^\otimes 3g-3-2k) \oplus h(C^{(g-1)}) \otimes L^\otimes g-1. \]

The author was informed that P. Belmans, S. Galkin and S. Mukhopadhyay obtained the conjecture 1.1 and a similar stronger formula in the Grothendieck group of integral K-motives independently.

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Notation. We will follow the notations in [2].

2. Motivic decomposition

We refer [1, 2] for notations and backgrounds about motives. In [1], Bano proved the following motivic decomposition.

Theorem 2.1. Let \( C^{(n)} \) be the \( n \)-th symmetric power of \( C \). Then there is the following decomposition.

\[ h(C^{(n)}) = \bigoplus_{a+b+c=n} 1^\otimes a \otimes \lambda^b h^1(C) \otimes L^\otimes c \]

where \( a, b, c \) are nonnegative integers.

In [2], Bano proved the following motivic decomposition using works of Thaddeus.

Theorem 2.2. [2, Corollary 2.7] In any semisimple category of motives, there is the following isomorphism.

\[ h(M_L) \cong \bigoplus_{k=0}^{g} \lambda^k h^1 C \otimes (1 \oplus L \oplus \cdots \oplus L^\otimes g-k-1) \otimes (1 \oplus L^\otimes 2 \oplus \cdots \oplus L^\otimes 2g-2k-2) \otimes L^\otimes k. \]

The main observation of this note is the following.

Theorem 2.3. In any semisimple category of motives, there is the following isomorphism.

\[ h(M_L) \cong \bigoplus_{k=0}^{g-2} h(C^{(k)}) \otimes (L^\otimes k \oplus L^\otimes 3g-3-2k) \oplus h(C^{(g-1)}) \otimes L^\otimes g-1. \]

Proof. It is enough to compare

\[ \bigoplus_{k=0}^{g} \lambda^k h^1 C \otimes (1 \oplus L \oplus \cdots \oplus L^\otimes g-k-1) \otimes (1 \oplus L^\otimes 2 \oplus \cdots \oplus L^\otimes 2g-2k-2) \otimes L^\otimes k \]
Then it is enough to compare the coefficients of $\lambda^i h^1 C$ for both sides. It is easy to see that the coefficient of $\lambda^i h^1 C$ is

$$
\bigoplus_{k=0}^{g-2} \left( \bigoplus_{a+c=k-i} (1^{\otimes a} \otimes \lambda^i h^1 C \otimes L \circ k) \otimes (L \otimes k \otimes L \otimes 3g-3-2k) \otimes h(C^{g-1}) \otimes L \otimes g^{-1} \right)
$$

and

$$
\bigoplus_{k=0}^{g-2} \left( \bigoplus_{a+c=k-i} (1^{\otimes a} \otimes \lambda^i h^1 C \otimes L \circ k) \otimes (L \otimes k \otimes L \otimes 3g-3-2k) \otimes (1^{\otimes a} \otimes \lambda^i h^1 C \otimes L \circ k) \otimes L \otimes g^{-1} \right)
$$

Therefore we get the desired result.

**Remark 2.4.** The above motivic decomposition implies an interesting decomposition of Hodge diamond of $M_L$.

It was also suggested by M. S. Narasimhan that the conjectural semi-orthogonal decomposition of derived category of $M_L$ has a meaning in the homological mirror symmetry. Assuming some conjectures, it seems to imply an interesting decomposition of the (Karoubi completed) Fukaya category of $M_L$. We have the following vague conjecture.

**Conjecture 2.5.** The Fukaya category of $M_L$ has the following orthogonal decomposition.

$$\text{Fuk}(M_L) = \langle \text{Fuk}(pt), \text{Fuk}(pt), \text{Fuk}(C), \text{Fuk}(C), \cdots, \text{Fuk}(C^{g-2}), \text{Fuk}(C^{g-2}), \text{Fuk}(C^{g-1}) \rangle$$
This conjecture is compatible with results and conjectures in [5, 7, 11].

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Center for Geometry and Physics, Institute for Basic Science (IBS), Pohang 37673, Republic of Korea
E-mail address: kyoungseog02@gmail.com