Coulomb oscillations as a remedy for the helium atom

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Abstract

The largest failure of the old, Bohr-Sommerfeld quantum theory was with the helium atom. It brought about the theory’s demise. I show that this failure does not originate, as commonly believed, with the orbit concept per se. Instead, it was caused by the wrong choice of orbits, compounded by ignorance of the exclusion principle. Choosing semiclassical electron oscillations through the He nucleus, I calculate a singlet ground-state energy that rivals in accuracy with quantum-mechanical results. The same method reveals Bohr’s historic energy value as the forbidden triplet ground state—a result beyond the reach of quantum mechanics. At the qualitative level, the concept of Coulomb oscillations visually explains the major features in the He double spectrum in terms of crossed or parallel orbit orientation.

Keywords: Atomic electron structure, Semiclassical theories, Helium spectrum, History of quantum theory
I. INTRODUCTION

The helium atom broke the back of the old quantum theory. The approach that had started so promising with the Bohr model of 1913 was finally abandoned a decade later. Already in his seminal trilogy, Bohr extended his model of the hydrogen atom to $^He$ (Bohr 1913) such that two electrons would orbit—at diametrically opposite positions—a nuclear double charge on one circle (see Fig. 1). The calculated ground-state energy was of the right order of magnitude—5% off the measured value. Although an encouraging result, it was a far cry from the “spectroscopic” accuracy of the energy terms that the Bohr model gave for the (one-electron) hydrogen atom. Therefore efforts were soon made by a host of researchers—Sommerfeld, Landé, Kramers and Bohr, Van Vleck, Pauli and Born, Heisenberg and Born (Mehra and Rechenberg 1982a)—with improved models of the $^He$ atom. Among the numerous attempts were a distinction of separate (inner and outer) coplanar electron orbits, then elliptical orbits in tilted planes subject to various phase relations. Later, perturbation theory from celestial mechanics was employed. However, despite increasing efforts and sophistication, those calculations did not converge toward the experimental value. In fact, the most extensive methods, carried out by Kramers and by Van Vleck (Mehra and Rechenberg 1982a), gave results that disagreed about as much (+5%) from the experimental ground-state energy as Bohr’s original, “two-seat-roundabout” model (-5%). Much worse results were obtained by Born and Heisenberg for the excited states of $^He$. In despair, the old quantum theory of orbit quantization was forgone in the mid 1920s and a search for new principles ensued. This led to Heisenberg’s discovery of matrix mechanics. Among the early triumphs of the new quantum mechanics were easy calculations of the ground-state energy of $^He$ with perturbation theory (relative deviation $\Delta \approx 5\%$) and variational techniques ($\Delta < 2\%$), and Heisenberg’s explanation of the origin of the $^He$ singlet and triplet spectra (Mehra and Rechenberg 1982b).
The helium atom was not the only failure of the old quantum theory. Two other short-comings were the theory’s failure to give the correct multiplet structure of the hydrogen atom (neglecting spin) and the stability of the hydrogen molecule ion, $H_2^+$. However, as I have recently shown (Bucher 2005), both these dilemmas can be resolved, within the framework of semiclassical orbit quantization, by including oscillations of the electron through the atomic nucleus (molecular nuclei). I call them Coulomb oscillations. After those successes it is reasonable to extend the Coulomb-oscillator concept to the $He$ atom—a two-electron system. It is then found that the calculated energy for the singlet ground state is comparable with that from quantum mechanics. The same method yields the energy of the forbidden triplet ground state—a result outside the scope of quantum mechanics. The Coulomb oscillations also provide, through orbit visualization, a simple explanation of the symmetry splitting and the energy distinctions of the excited states in the $He$ spectrum.

II. BOHR’S HELIUM MODEL

In Bohr’s historic model of the $He$ atom, the centripetal force on each electron is

$$\frac{mv^2}{r} = \frac{Ze^2}{r^2} - \frac{e^2}{(2r)^2} \equiv \frac{Z'e^2}{r^2}. \quad (1)$$

The electron-electron repulsion can be formally combined with the nuclear charge (number) $Z$ such as if each electron were subject to only a central
force from an effective nuclear charge,

\[ Z' = Z - \frac{1}{4}. \]  

The orbit is then quantized by the requirement that each electron possesses an angular momentum

\[ mvr = n\hbar, \]  

where \( \hbar = \hbar/2\pi \) is the reduced Planck constant and \( n \) a quantum number. Solving Eqs. (1) and (3) for \( v^2 \), and equating, yields the quantized orbit radius,

\[ r_n = \frac{n^2}{Z' r_B}. \]  

Here the Bohr radius, \( r_B = \hbar^2/me^2 = 0.53 \times 10^{-10} \text{m} \), serves as a universal atomic length unit. The orbit energy of both electrons,

\[ E = 2 \times \frac{1}{2}mv^2 - \frac{2Ze^2}{r} + \frac{e^2}{2r} = mv^2 - \frac{2Z'e^2}{r}, \]  

becomes quantized, after insertion of \( v^2 \) and \( r_n \) from Eqs. (3) and (4),

\[ E_n = -\frac{2Z'^2}{n^2}R_y, \]  

expressed, for convenience, in terms of the Rydberg energy unit, \( R_y = me^4/2\hbar^2 = 13.6 \text{ eV} \).

From the formalism above one can see that Bohr’s “two-seat-roundabout” model of the \( He \) atom is merely a \( Z' \)-scaled version, Eq. (2), of his model of the \( H \) atom. The ground-state energy of the \( He \) atom is then

\[ E_1 = -2Z'^2R_y. \]
III. COUPLED COULOMB OSCILLATORS

First assume that both electrons swing in opposite phase along the same axis through the nucleus, always at mirror positions, $x_1 = -x$ and $x_2 = x$, with respect to the nucleus at $x = 0$ (see Fig. 2). I call this oscillation mode "contralinear." The atom’s total energy $E_{-xx}$ at any mirror position $x$ of the electrons must equal the potential energy of both electrons at their turning points, $-B$ and $B$,

$$E_{-xx} = 2 \times \frac{1}{2} m u^2 - \frac{2Ze^2}{|x|} + \frac{e^2}{|2x|} = -\frac{2Ze^2}{B} + \frac{e^2}{2B} \equiv -\frac{2Z'_{-xx}e^2}{B} \quad (8)$$

with

$$Z'_{-xx} = Z' = Z - 0.25 \quad (9)$$

from Eq. (2). Here $u$ denotes each electron’s speed in the contralinear oscillation,

$$u(x) = \sqrt{\frac{e^2}{m} \sqrt{\frac{2Z}{|x|} - \frac{1}{|2x|} - \frac{2Z}{B} + \frac{1}{2B}}} \quad (10)$$
and, after simplification,

\[ u(x) = \sqrt{\frac{2Z'_xx e^2}{m}} \sqrt{\frac{1}{|x|} - \frac{1}{B}}. \]  

(11)

The action integral over an oscillation cycle must, by Sommerfeld’s quantization condition, be an integer multiple of Planck’s constant,

\[ A_{-xx} = m \int u(x)dx = 4m \int_0^B u(x)dx = nh, \]  

(12)

with \( n = 1 \) for each electron in the \( He \) ground state. The integral in Eq. (12) can be solved analytically (Bucher 2005). This then gives the oscillation amplitude,

\[ B_n = \frac{2n^2}{Z'_{-xx}} r_B, \]  

(13)

and, after insertion into Eq. (8), the same ground-state energy as in Eq. (7),

\[ E_{-xx} = -2Z'^2_{-xx} R_y. \]  

(14)

The contralinear Coulomb oscillator is thus energetically equivalent to Bohr’s two-seat-roundabout model of the \( He \) atom: In both cases the electron-electron distance \(-xx\) is twice the (varying or, respectively, constant) electron-nucleus distance,

\[ -xx = 2x. \]  

(15)

Let us next explore synchronous Coulomb oscillations of two electrons through the \( He \) nucleus in perpendicular directions and with equal amplitude \( C \) (dashed in Fig. 2). Again, the atom’s total energy \( E_{xy} \) for electron positions mirrored off the \( y = x \) diagonal, \( \mathbf{r}_1 = (x, 0) \) and \( \mathbf{r}_2 = (0, y) \), must equal the potential energy of both electrons at their turning points, \( x = C \) and \( y = C \),

\[ E_{xy} = 2 \times \frac{1}{2} m w^2 - \frac{2Ze^2}{|y|} + \frac{e^2}{|y|\sqrt{2}} = \frac{2Ze^2}{C} + \frac{e^2}{C\sqrt{2}} \equiv \frac{2Z'_{xy} e^2}{C}, \]  

(16)

with a new effective nuclear charge number,

\[ Z'_{xy} = Z - \frac{\sqrt{2}}{4} \approx Z - 0.35. \]  

(17)
Its lesser amount, $Z'_{xy} < Z'_{xx}$, is a consequence of the closer electron-electron distance,
\[ \overline{xy} = x\sqrt{2}. \] (18)

compared to Eq. (15). Since Eq. (16) scales, via the effective charge, with Eq. (8), the ground-state energy of this crossed-synchronous Coulomb oscillation is analogous to Eq. (7),
\[ E_{xy} = -2Z_{xy}^2 R_y. \] (19)

Fig. 3. Cross-delayed Coulomb oscillations of two electrons through a nucleus. The (dashed) distance-line between the electrons is tangent to an asteroid curve. The third quadrant serves in assessing the average electron-electron distance.

Lastly, consider crossed Coulomb oscillations that are out of phase by a quarter period (see Fig. 3). When now one electron is at a turning point (say, $x = D$), then the other electron traverses the nucleus ($y = 0$) in the perpendicular ($y$) direction. I denote this “cross-delayed” oscillation mode by the suffix $x/y$. The previous method of equating the total energy at any position with the potential energy of both electrons at the turning points, Eqs. (8) and (16), no longer suffices. However, averaged over a period, we can still
employ the two-body notion of an effective nuclear charge, $Z'_{x/y}$, acting exclusively on each electron as if it comprised both the true nuclear attraction and the repulsion from the other electron. With respect to the energy, the time-average distance from the fixed nucleus to an electron in Coulomb oscillation is half the latter’s amplitude (Bucher, Elm and Siemens 1998; Pauling and Wilson 1935). For a pair of contralinear oscillators, the average electron-electron distance is, by Eq. (15), twice that value, $\langle -xx \rangle = 2 \times B/2 = B$, but for cross-synchronous oscillators less than the amplitude, $\langle xy \rangle = \sqrt{2} \times C/2 = C/\sqrt{2}$, Eq. (18). For cross-delayed oscillators, the distance-line between the electrons (dashed in Fig. 3) is tangent to an asteroid curve. The average electron-electron distance of cross-delayed oscillators, $\langle x/y \rangle$, must be shorter than the amplitude, $D$, but longer than the closest separation, indicated in the third quadrant of Fig. 3. It seems safe to assume that $D/\sqrt{2} < \langle x/y \rangle < D$. Therefore the average of the bracketing effective charges may serve as a reasonable approximation for the effective nuclear charge in cross-delayed Coulomb oscillations,

$$Z'_{x/y} \approx \frac{Z'_{-xx} + Z'_{xy}}{2} = Z - \frac{1 + \sqrt{2}}{8} \approx Z - 0.30, \quad (20)$$

and accordingly for the energy,

$$E_{x/y} \approx -2Z'^2_{x/y}R_y. \quad (21)$$

**IV. GROUND-STATE RESULTS**

Table I shows for several nuclear charges $Z$—and Fig. 4 for the special case of He—experimental values and calculations of the ground-state energy. [A simple formula for the ground-state ionization potential of He and two-electron ions has recently appeared in these pages (Elo 2007).] The calculated energies $E_{xy}$ of the cross-synchronous oscillator deviate from experiment about as much as the $E_{-xx}$ values of the contralinear oscillator, $|E_{xy} - E_{expt}|/|E_{expt}| = \Delta_{xy} \approx \Delta_{-xx} = |E_{-xx} - E_{expt}|/|E_{expt}|$, although in opposite directions, $E_{-xx} < E_{expt} < E_{xy}$. Interestingly, these values are similar (or equal) to historical results from the old quantum theory: The energy $E_{xy}(He) = -5.42 \ R_y$ is close to the values obtained with enormous efforts by Kramers ($-5.52 \ R_y$) and by Van Vleck ($-5.53 \ R_y$) (Mehra and Rechenberg
1982a) and $E_{-xx}$ agrees, as noted, with $E_1$ of Bohr’s two-seat-roundabout model.

The energy of the cross-delayed Coulomb oscillations is found to be remarkably close to the experimental data, $E_{x/y} \approx E_{\text{expt}}$. Table I shows deviations of 0.5% for $He$, 0.2% for higher-$Z$ ions, and about 6% for the hydride ion, $H^-$. The reason for the large deviation of $H^-$ is the relatively stronger electron-electron interaction which gives rise to a radius-dependent screening of the nuclear charge, $Z - s(r)$, instead of a constant screening factor $s$ as in all the other cases (Bethe and Salpeter 1957).

Two instructive comparisons are provided by quantum-mechanical expressions. First-order perturbation theory (Pauling and Wilson 1935) gives

$$E_{QM}^{\text{prt}} = -(2Z^2 - \frac{5}{4}Z)R_y.$$  \hfill (22)
The perturbation values $E_{\text{prt}}^{QM}$ are about 5% less binding (higher level) than the experimental values $E_{\text{expt}}$ but slightly more binding than those of the cross-synchronous Coulomb oscillator, $E_{xy}$. A one-parameter variational treatment (Pauling and Wilson 1935) gives

$$E_{QM}^{\text{var}} = -2(Z_{QM}^{\text{var}})^2 R_y,$$  \hspace{1cm} (23)

with

$$Z_{QM}^{\text{var}} = Z - \frac{5}{16} \approx Z - 0.31.$$  \hspace{1cm} (24)

Table I shows that the variational values $E_{QM}^{\text{var}}$ deviate slightly more from experiment than those of the cross-delayed Coulomb oscillator, $E_{x/y}$.

V. INTERPRETATION

To understand the results, a reflection on the abilities and deficiencies of the Bohr-Sommerfeld quantum theory is in order. The triple success of the orbit-based, old quantum theory is its correct description of the $H$ atom (and one-electron ions) concerning (1) the energy levels $E_n$, (2) the orbital angular momenta $L_{nl}$—if corrected as $L_{nl}^2 = l(l+1)\hbar^2$ and with the Coulomb oscillator included—and (3) the orbits’ space quantization—with $(L_{nl})_z = m_l \hbar$. These three achievements are succinctly represented by the corresponding (principal, angular and magnetic) quantum numbers: $n, l$, and $m_l$.

The shortcomings of the old quantum theory lie in its neglect of three particle properties: their spin, their wave nature and their quantum statistics. Inclusion of spin—expressed by the spin quantum number $s$—is necessary to account for the total (instead of only orbital) angular momentum. The wave nature—expressed in Schrödinger’s formulation of quantum mechanics by the wavefunction $\psi$—is necessary to account for the particles’ spatial (probability) distribution, $\psi\psi^* = |\psi|^2$. Quantum statistics becomes relevant for systems with identical (indistinguishable) particles—like the two electrons about the nucleus in the $He$ atom.

In the framework of quantum mechanics, including Fermi-Dirac quantum statistics and the related Pauli exclusion principle, the $He$ atom is described in terms of both spatial wavefunctions $\psi(r_i, r_j)$ and spin wavefunctions $\alpha(i)\beta(j)$ which depend on the electrons’ position and spin orientation, respectively, and whose product is antisymmetric with respect to an
interchange of the electrons, $i \leftrightarrow j$ (Bethe and Salpeter 1957). More specifically, an interchange of electrons in corresponding linear combinations of such wavefunctions must leave the spatial part the same, but change the sign of the spin part, and vice versa. In the first case, the combination of spatial wave functions is said to be symmetric and the spin-wave combination antisymmetric. The opposite holds in the second case. In accordance with the quantum-mechanical multiplicity of angular momentum, the combination of spatial wavefunctions,

$$\psi_{ab}(r_1, r_2) = \frac{1}{\sqrt{2}} [\phi_a(r_1)\phi_b(r_2) \pm \phi_a(r_2)\phi_b(r_1)],$$

are then symmetric (+) if the electrons’ spin is opposite (singlet state) and antisymmetric (-) when their spins are parallel (triplet state). (Disregarding spin-orbit coupling, the terminology “singlet” and “triplet” refers to the non-splitting and threefold splitting, respectively, of such levels in a magnetic field.) Here $\phi_a$ and $\phi_b$ are one-electron wavefunctions. Optical transitions occur only between states of the same symmetry. Hence the observed $He$ double spectrum, arising from two sets of either singlet or triplet energy levels, as was first explained by Heisenberg.

The probability density distribution $|\psi_{ab}|^2$ of the $He$ atom contains, by Eq. (25), both atomic orbital density terms $|\phi_a|^2$ and $|\phi_b|^2$, and so-called “interference terms,” $\phi_a\phi_b$. The latter give rise to increased or decreased electron density in the overlap region of the $\phi_a$ and $\phi_b$ orbitals according to symmetry (+) and antisymmetry (−) of $\psi_{ab}$, respectively. In this way the ± symmetry distinction corresponds to spatially contracted or expanded states of the $He$ atom.

Since (Fermi-Dirac) quantum statistics requires that all spatial wavefunctions of the triplet state be antisymmetric, this rules out, by Eq. (25), the $1^3S$ ground state, $\phi_{100,100} = 0$. With that symmetry property inherent in the formalism, quantum mechanics cannot give the energy of the (not naturally occurring) $1^3S$ state of $He$. The old quantum theory, in contrast, does not account for quantum statistics at all. It does therefore yield symmetry-forbidden quantum states which have to be ruled out “by hand.”

The spatially anti-symmetric $1^3S$ state is semiclassically represented by contralinear Coulomb oscillations ($-xx$)—an association borne out by a unifying pattern that comprises both the ground state and the excited states in the $He$ spectrum, shown below. The energy of the $1^3S$ state of $He$ is thereby
$E_{-xx} = -6.125R_y$—Bohr’s historic value for the two-seat-roundabout model of $He$.

How realistic is the attribution of the energy level $E_{-xx}$ to the forbidden triplet ground state? I offer two plausibility arguments: Qualitatively, it would extend the pattern of a higher singlet than triplet level from all the excited states to the observed and forbidden ground state, $E_{\text{exp}} > E_{-xx}$ (see Figs. 4 and 5). How accurate, then, is its quantitative value, $E_{-xx} = -6.125R_y$?

Recall that both the Bohr model and the simple Coulomb oscillator (Bucher 2005) give, like quantum mechanics, the correct (non-relativistic) energy of the hydrogen atom. The reason why these energy calculations are successful is that the representation of such periodic processes in phase space encloses the same volume $(nh)$, despite the different (and unrealistic) geometries of the semiclassical orbits compared to the quantum-mechanical orbitals. Likewise it can be expected of the extensions of both the Bohr model and the Coulomb oscillator to the $He$ atom, that they give (at least) the correct ground-state energy: For once, Bohr’s two-seat-roundabout model, and equivalently the contralinear Coulomb oscillations ($-xx$), are reduced, via scaling, to the (correctly solvable) one-electron problem. Furthermore, they represent the largest (average) electron-electron separation which yields the lowest energy (ground state). Nevertheless, the semiclassical $1^3S$ state must be discarded because of its forbidden quantum symmetry, or equivalently, its violation of the exclusion principle—a constraint beyond the old quantum theory.

The experimentally observed ground-state of the $He$ atom is the remaining $1^1S$ singlet state. By Eq. (25), its wavefunction $\phi_{100,100}^{+}$ is spatially symmetric. Semiclassically, it is represented by crossed Coulomb oscillations which possess higher (fourfold) symmetry than the (twofold) mirror symmetry of the linear oscillations. When the cross-oscillations are synchronous ($xy$), then the lesser electron separation, Eq. (18), causes too high an energy level, $E_{xy} > E_{\text{exp}}$. The larger average electron separation of the cross-delayed oscillations, indicated in Fig. 3, lowers the energy level accordingly, $E_{x/y} \approx E_{\text{exp}}$. Incidentally, those two types of cross-oscillations can be regarded as the semiclassical equivalent of the quantum-mechanical perturbative and, respectively, variational treatment which similarly differ in the average electron separation (Bethe and Salpeter 1957). Such correspondence is reflected in comparable energy levels, $E_{xy} \approx E_{\text{pr}}^{\text{QM}}$ and $E_{x/y} \approx E_{\text{var}}^{\text{QM}}$ (see...
Fig. 5. Experimental ionization potentials $IP$ (left scale) or energy levels $E$ (right scale) of helium in $R_y$ units. The dotted horizontal lines represent the energy levels of hydrogen.
VI. EXCITED STATES

All excited states of the \( \text{He} \) atom consist of one electron in the \( 1s \) ground state and the other electron in an \( nl \) excited state. [The lowest doubly excited state \((2s, 2s)\) has already a higher energy—above \(2 \times (-1R_y)\)—than the one-electron ion \( \text{He}^+ (-4R_y) \) and is therefore unstable.] Figure 5 shows the double scheme of the \( \text{He} \) energy levels in terms of ionization potentials \( (IP) \) with spectroscopic notation \( S, P, D \) for angular quantum numbers \( l = 0, 1, 2 \). The energy level \( E_{nl} \) of the \( \text{He} \) atom (marked on the right scale of Fig. 5) is related to the ionization potential (left scale) as

\[
E_{nl} = -4R_y - IP_{nl},
\]

(26)
due to its atomic ground-state and excited-state contributions.

At first glance the level scheme of excited \( \text{He} \) states in Fig. 5 may look complicated. However, one quickly finds that many terms fall at the (dotted) hydrogen levels. In these cases the excited electron experiences an effective nuclear charge that results from the double charge of the \( \text{He} \) nucleus and maximal shielding by the \( 1s \) electron,

\[
Z_H = +2 + (-1) = +1,
\]

(27)

that is, effectively a hydrogen nucleus.

The exceptions fall in three groups: the triplet \( ^3S \) states, the singlet \( ^1S \) states, and the triplet \( ^3P \) states (see Fig. 5). Common to all exceptions is that those excited terms fall beneath the associated hydrogen levels, \( E_n(H) = -1R_y/n^2 \). The reason is partial shielding, caused by penetration of the excited electron into the range of the \( 1s \) electron orbit. The semiclassical orbits in Fig. 6 provide a visual assessment, and qualitative explanation of the amount of partial shielding for the three groups of exceptions. As in the case of the \( \text{He} \) ground state, we maintain the association of parallel and crossed Coulomb oscillations with the excited triplet and, respectively, singlet states. If the excited orbit is an \( nl \) Sommerfeld ellipse, then the same association is extended to the orientation of its major axis relative to the \( 1s \) Coulomb oscillator.
Fig. 6. One-electron Coulomb oscillators and Sommerfeld ellipses about a He nucleus, drawn to scale and arranged for easy comparison with Fig. 5. The 1s orbits are of a helium ion, He⁺. All other orbits are of a hydrogen atom, H, as if the nucleus was maximally shielded by 1s.

An example of the first group of exceptions is the 2³S triplet state, illustrated in Fig. 6(c) by parallel Coulomb oscillators. The short and fast Coulomb oscillation of the 1s ground-state electron is indicated by the bold line; the (∼8× as) long and (∼1/16 as) slow oscillation of the (shielded) 2s electron by the thin line. Outside the range of the 1s electron, the 2s electron experiences an electric potential like from a hydrogen nucleus, Z_H. Inside the 1s range, it experiences only partial shielding which results in stronger
binding (lower energy level). The same mechanism of 1s-range penetration and correspondingly lower energy levels also holds for the parallel Coulomb oscillators with higher quantum number \( n \) (not shown in Fig. 6), although to a lesser degree: The \( 2n^2 \)-times longer amplitudes of the \( n \)th (screened) Coulomb oscillator, compared to 1s, causes a relatively lesser lack of screening and leads to \( n^3 \)S levels of lesser depth beneath the \( n \)th hydrogen levels (see 3\(^3\)S and 4\(^3\)S in Fig. 5).

An example of the second group—the singlet 1S states—is 2\(^1\)S, illustrated in Fig. 6(c) by crossed Coulomb oscillators. When now the 2s electron oscillates perpendicularly to the 1s electron, it is less inside the (dashed) 1s range. Accordingly, the perpendicularly oscillating 1s electron causes a lesser lack of shielding as in the previous case of parallel 1s oscillations, \( E(n^3 \text{S}) < E(n^1 \text{S}) < E(n \text{H}) \).

Examples of the third group—the triplet 3P states—are 2\(^3\)P and 3\(^3\)P, illustrated in Fig. 6(b,c), respectively. Passage of the 2p and 3p Sommerfeld orbits through the (bold) parallel 1s range is marginal, causing only a small lack of screening and level-lowering.

No passage of the 2p and 3p Sommerfeld orbits occurs through the crossed (dashed) 1s range in Fig. 6(b,c). As a consequence of maximal screening, the energies of the associated 2\(^1\)P and 3\(^1\)P singlet states then fall (almost) at the hydrogen levels in Fig. 5. Finally, the 3d orbit in Fig. 6(c) passes far from the parallel and crossed 1s range. This results in the 3\(^3\)D and, respectively, 3\(^1\)D terms at the \( E_3(\text{H}) \) level.

VII. CONCLUSION

The concept of the Coulomb oscillator overcomes impasses of the old quantum theory not only with one-electron systems—orbital angular-momentum hierarchy in the \( H \) atom, stability of the \( H_2^+ \) molecule ion—but also with the \( He \) atom. Treating the motion of its \textit{two} electrons as synchronous permits a reduction to a one-electron problem. Basic distinctions arise for parallel and crossed Coulomb oscillations. They are associated with the symmetry distinctions of Fermi-Dirac quantum statistics. With regard to the average electron-electron distance, combined with screening effects, the parallel and crossed Coulomb oscillations emulate quantum-mechanical symmetry constraints and accordingly yield similar ground-state and excited-state energy levels of the \( He \) double spectrum.
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Table I. Ground-state energy of $^7$He and two-electron ions: Experimental values $E_{\text{exp}}$ (Bethe and Salpeter 1957) and calculations with contralinear ($-xx$), cross-synchronous ($xy$), and cross-delayed ($x/y$) Coulomb oscillators, and one-parameter QM variation. Relative deviations are denoted by $\Delta$.

| Ion    | Z  | $E_{\text{exp}}$ ($R_y$) | $E_{xx}$ ($R_y$) | $\Delta_{xx}$ [%] | $E_{xy}$ ($R_y$) | $E_{x/y}$ ($R_y$) | $\Delta_{x/y}$ [%] | $E_{\text{var}}^{\text{QM}}$ ($R_y$) |
|--------|----|--------------------------|------------------|-------------------|-----------------|------------------|-------------------|-------------------------|
| $H^-$  | 1  | -1.06                    | -1.13            | 7.6               | -0.84           | -0.99            | 5.7               | -0.95                   |
| $^8$He | 2  | -5.81                    | -6.13            | 5.5               | -5.42           | -5.77            | 0.6               | -5.70                   |
| $^7$Li$^+$ | 3 | -14.56                   | -15.13           | 3.9               | -14.01          | -14.57           | 0.0               | -14.45                  |
| $^8$Be$^{2+}$ | 4 | -27.31                   | -28.13           | 3.0               | -26.59          | -27.36           | 0.2               | -27.20                  |
| $^7$B$^{3+}$ | 5 | -44.06                   | -45.13           | 2.4               | -43.18          | -44.16           | 0.2               | -43.95                  |
| $^8$C$^{4+}$ | 6 | -64.82                   | -66.13           | 2.0               | -63.76          | -64.94           | 0.2               | -64.70                  |
| $^8$N$^{5+}$ | 7 | -89.58                   | -91.13           | 1.8               | -88.35          | -89.74           | 0.2               | -89.45                  |
| $^8$O$^{6+}$ | 8 | -118.34                  | -120.13          | 1.5               | -116.94         | -118.54          | 0.2               | -118.20                 |
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