The inhomogeneous Sprindžhuk conjecture over a local field of positive characteristic

Arijit Ganguly and Anish Ghosh

Abstract. We prove a strengthened version of the inhomogeneous Sprindžhuk conjecture in metric Diophantine approximation, over a local field of positive characteristic. The main tool is the transference principle of Beresnevich and Velani [4] coupled with earlier work of the second named author [13] who proved the standard, i.e. homogeneous version.

1. Introduction

The context of this paper is the metric theory of Diophantine approximation over local fields of positive characteristic. In [13], the second named author proved the Sprindžhuk conjectures in this setting (in fact, also in multiplicative form), here we prove the inhomogeneous variant of the conjecture. We use the inhomogeneous transference principle of Beresnevich and Velani [4] to transfer the homogeneous result from [13] and also use a positive characteristic version of the transference principle of Bugeaud and Laurent interpolating between uniform and standard Diophantine exponents, established recently by Bugeaud and Zhang [6]. The possibility of proving the S-arithmetic and positive characteristic inhomogeneous Sprindžhuk conjectures was suggested by Beresnevich and Velani (§8.4) and the present paper realises this expectation in the positive characteristic setting.

Metric Diophantine approximation on manifolds is a subject which studies the extent to which typical Diophantine properties for Lebesgue measure on $\mathbb{R}^n$ are inherited by smooth submanifolds or other measures. The theory began with Mahler [22] who conjectured that almost every point on the Veronese curve is \textit{not very well approximable}. Mahler’s conjecture was resolved by Sprindžhuk [23, 24], who in turn made a stronger conjecture which was resolved by Kleinbock and Margulis [17] using methods from the ergodic theory of group actions on homogeneous spaces, specifically, sharp nondivergence estimates for unipotent flows on the space.

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of lattices. Subsequently, an $S$-arithmetic version of the conjectures were established by Kleinbock and Tomanov \cite{19} and a positive characteristic version was established by the second named author \cite{13}. Both the latter works used adaptations of the dynamical approach of Kleinbock and Margulis. In \cite{4}, Beresnevich and Velani proved a transference principle which allowed them to prove an inhomogeneous versions of the Baker-Sprindžuk conjectures. We refer the reader to the above papers for more details. We will recall all the relevant concepts in the function field context in the next section.

Following the work of Beresnevich and Velani, there have been several recent advances in inhomogeneous Diophantine approximation. In \cite{3}, an inhomogeneous Khintchine type theorem was established for affine subspaces, complementing the earlier work \cite{2} for nondegenerate manifolds, see also \cite{14} for more inhomogeneous results on affine subspaces. Further, an $S$-arithmetic inhomogeneous Khintchine type theorem for nondegenerate manifolds was established by Datta and the second named author \cite{8}.

1.1. The setup. We follow our paper \cite{11} in setting the notation. Let $p$ be a prime and $q := p^r$, where $r \in \mathbb{N}$, and consider the function field $\mathbb{F}_q(T)$. We define a function $| \cdot | : \mathbb{F}_q(T) \to \mathbb{R}_{\geq 0}$ as follows.

$$|0| := 0 \text{ and } \left| \frac{P}{Q} \right| := e^{\deg P - \deg Q} \text{ for all nonzero } P, Q \in \mathbb{F}_q[T].$$

Clearly $| \cdot |$ is a nontrivial, non-archimedian and discrete absolute value in $\mathbb{F}_q(T)$. This absolute value gives rise to a metric on $\mathbb{F}_q(T)$.

The completion field of $\mathbb{F}_q(T)$ is $\mathbb{F}_q((T^{-1}))$, i.e. the field of Laurent series over $\mathbb{F}_q$. The absolute value of $\mathbb{F}_q((T^{-1}))$, which we again denote by $| \cdot |$, is given as follows. Let $a \in \mathbb{F}_q((T^{-1}))$. For $a = 0$, define $|a| = 0$. If $a \neq 0$, then we can write

$$a = \sum_{k \leq k_0} a_k T^k \text{ where } k_0 \in \mathbb{Z}, a_k \in \mathbb{F}_q \text{ and } a_{k_0} \neq 0.$$  

We define $k_0$ as the degree of $a$, which will be denoted by $\deg a$, and $|a| := e^{\deg a}$. This clearly extends the absolute value $| \cdot |$ of $\mathbb{F}_q(T)$ to $\mathbb{F}_q((T^{-1}))$ and moreover, the extension remains non-archimedian and discrete. Let $\Lambda$ and $F$ denote $\mathbb{F}_q[T]$ and $\mathbb{F}_q((T^{-1}))$ respectively from now on. It is obvious that $\Lambda$ is discrete in $F$. For any $d \in \mathbb{N}$, $F^d$ is throughout assumed to be equipped with the supremum norm which is defined as follows

$$||x|| := \max_{1 \leq i \leq n} |x_i| \text{ for all } x = (x_1, x_2, ..., x_d) \in F^d,$$

and with the topology induced by this norm. Clearly $\Lambda^n$ is discrete in $F^n$. Since the topology on $F^n$ considered here is the usual product topology on $F^n$, it follows that $F^n$ is locally compact as $F$ is locally compact. Let $\lambda$ be the Haar measure on $F^d$ which takes the value 1 on the closed unit ball $||x|| = 1$.

Diophantine approximation in the positive characteristic setting consists of approximating elements in $F$ by ‘rational’ elements, i.e. those from $\mathbb{F}_q(T)$. This subject has been extensively studied, beginning with Mahler who developed Minkowski’s
geometry of numbers in function fields and continuing with Sprindžuk who, in addition to proving the analogue of Mahler’s conjectures, also proved some transference principles in the function field setting. The subject has also received considerable attention of late, we refer the reader to [10, 21] for overviews and to [11, 20, 16, 15, 12] for a necessarily incomplete set of references.

In the present paper we prove an inhomogeneous analogue of the Sprindžuk conjectures, our main result is an upper bound for inhomogeneous Diophantine exponents.

**Theorem 1.1.** Let $U \subseteq F^d$ be open and $f : U \to F^n$ be a $(C, \alpha_0)$-good map, for some $C, \alpha_0 > 0$, and assume that $(f, \lambda)$ is nonplanar. Then, for every $\theta \in F$, and $\lambda$ almost every $x \in U$,
\[
\omega(f(x), \theta) \leq 1.
\]

We also establish the corresponding lower bound.

**Theorem 1.2.** Let $U \subseteq F^d$ be open and $f : U \to F^n$ be a $(C, \alpha_0)$-good map, for some $C, \alpha_0 > 0$, and assume that $(f, \lambda)$ is nonplanar. Then, for every $\theta \in F$, and $\lambda$ almost every $x \in U$,
\[
\omega(f(x), \theta) \geq 1.
\]

Remarks:

1. The relevant definitions are made in the next section. A main example to keep in mind is the original setup of Diophantine approximation on manifolds, i.e. if $f = (f_1, \ldots, f_n)$ where the $f_i$’s are analytic and $1, f_1, \ldots, f_n$ are linearly independent over $F$, then $f$ is $(C, \alpha)$ good for some $C, \alpha$ and nonplanar. More generally, if $f$ is a smooth nondegenerate map, then it is $(C, \alpha)$-good as well as nonplanar. The notions of $(C, \alpha)$ good functions and nondegenerate maps were introduced by Kleinbock and Margulis [17].

2. The homogeneous analogue of Theorem 1.1 was proved in [13] (Theorem 3.7), the lower bound is a consequence of Dirichlet’s theorem.

3. Though we do not discuss this here, in fact Theorems 1.1 and 1.2 should hold for a wider class of measures, the so called strongly contracting measures as introduced by Beresnevich and Velani, a category which includes the important class of friendly measures introduced earlier by Kleinbock, Lindenstrauss and Weiss [18]. We have also not considered the setting of multiplicative Diophantine approximation where also, it should be possible to prove the analogue of Theorem 1.1.

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2. **Homogeneous and Inhomogeneous Diophantine exponents**

The theory of Diophantine approximation in positive characteristic begins with Dirichlet’s theorem, which we now recall.
Theorem 2.1. (Theorem 2.1 [11]) Let $m, n \in \mathbb{N}$, $k = m + n$ and 

$$a^+ := \{ t := (t_1, t_2, \ldots, t_k) \in \mathbb{Z}^k_+ : \sum_{i=1}^m t_i = \sum_{j=1}^n t_{m+j} \}.$$ 

Consider $m$ linear forms $Y_1, Y_2, \ldots, Y_m$ over $F$ in $n$ variables. Then for any $t \in a^+$, there exist solutions $q = (q_1, q_2, \ldots, q_n) \in \Lambda^n \setminus \{0\}$ and $p = (p_1, p_2, \ldots, p_m) \in \Lambda^m$ of the following system of inequalities

$$\begin{align*}
|Y_i q - p_i| &< e^{-t_i} \quad \text{for } i = 1, 2, \ldots, m, \\
|q_j| &< e^{t_{m+j}} \quad \text{for } j = 1, 2, \ldots, n.
\end{align*}$$

(2.1)

We will consider only unweighted Diophantine approximation in this paper, so $t_1 = \cdots = t_m = 1/m$ and $t_{n+1} = \cdots = t_n = 1/n$. We denote by $M_{m \times n}(F)$, the vector space of $m \times n$ matrices with entries from $F$ equipped with the supremum norm. In view of Theorem 2.1, it is natural to define exponents of Diophantine approximation as follows. Let $X \in M_{m \times n}(F)$ and $\theta \in F^m$. The inhomogeneous exponent, $\omega(X, \theta)$ of $X$, is the supremum of the real numbers $\omega$ for which, for arbitrarily large real numbers $T$, the inequalities

$$\|Xq - p - \theta\| < e^{-\frac{\omega}{m} T}, \quad \|q\| < e^T,$$

have a solution $(p, q) \in \Lambda^m \times (\Lambda^n \setminus \{0\})$. The uniform inhomogeneous exponent, $\bar{\omega}(X, \theta)$, is the supremum of the real numbers $\bar{\omega}$ for which, for all sufficiently large real numbers $T$, the inequalities

$$\|Xq - p - \theta\| < e^{-\frac{\bar{\omega}}{m} T}, \quad \|q\| < e^T,$$

have a solution $(p, q) \in \Lambda^m \times (\Lambda^n \setminus \{0\})$.

In this paper, we will adopt the point of view of Diophantine approximation of single linear forms, i.e. we will assume that $y \in F^n$ where $F^n$ is identified with $M_{1 \times n}(F)$ as opposed to simultaneous Diophantine approximation where one considers $y \in M_{n \times 1}$.

If $\theta = 0$, then the corresponding Diophantine exponent $\omega(y) := \omega(y, 0)$ (resp. $\tilde{\omega}(y)$) is called the homogeneous Diophantine exponent. By Dirichlet’s theorem stated above, $\omega(y) \geq 1$ for every $y \in F^n$. We are following the normalisation in [4] rather than the one used in [17, 13] according to which the critical exponent is $n$.

The Borel-Cantelli lemma implies that $\omega(y) = 1$ for $\lambda$ almost every $y \in F^n$. It is therefore natural to define $y \in F^n$ to be very well approximable if $\omega(y) > 1$. Sprindžuk [24] proved that for $\lambda$ a.e. $x \in F$, \n
$$f(x) := (x, x^2, \ldots, x^n)$$

is not very well approximable, thereby settling the positive characteristic analogue of Mahler’s conjecture. A special case of the theorems proved in this paper is that for almost every $x$, every $\theta \in F$, $\omega(f(x), \theta) = 1$. Following [4] we may define inhomogeneously extremal measures as follows.

Definition 2.2. Let $\mu$ be a measure supported on a subset of $F^n$. We say that $\mu$ is inhomogeneously extremal if for all $\theta \in F$, 

$$\omega(y, \theta) = 1 \text{ for } \mu \text{ a.e. } y \in F^n.$$
Then our main theorems can be restated as follows:

**Theorem 2.3.** Let \( U \subseteq F^d \) be open and \( f : U \to F^n \) be a \((C, \alpha_0)\) - good map, for some \( C, \alpha_0 > 0 \), and assume that \( f \) is nonplanar. Then \( f, \lambda \) is inhomogeneously extremal.

### 3. Good and nonplanar maps

The following is taken from §1 and 2 of [19]. For the sake of generality, we assume \( X \) is a Besicovitch metric space, \( U \subseteq X \) is open, \( \nu \) is a radon measure on \( X \), \((F, | \cdot |)\) is a valued field and \( f : X \to F \) is a given function such that \(|f|\) is measurable. For any \( B \subseteq X \), we set

\[
||f||_{\nu, B} := \sup_{x \in B \cap \text{supp} (\nu)} |f(x)|.
\]

**Definition 3.1.** For \( C, \alpha > 0 \), \( f \) is said to be \((C, \alpha)\) - good on \( U \) with respect to \( \nu \) if for every ball \( B \subseteq U \) with center in \( \text{supp} (\nu) \), one has

\[
\nu(\{ x \in B : |f(x)| < \varepsilon \}) \leq C \left( \frac{\varepsilon}{||f||_{\nu, B}} \right)^{\alpha} \nu(B).
\]

The following properties are immediate from Definition 3.1.

**Lemma 3.2.** Let \( X, U, \nu, F, f, C, \alpha \), be as given above. Then one has

1. \( f \) is \((C, \alpha)\) - good on \( U \) with respect to \( \nu \) if and only if \( |f| \) is measurable.
2. \( f \) is \((C, \alpha)\) - good on \( U \) with respect to \( \nu \) if and only if \( g \) is \((C, \alpha, c)\) - good for all \( c \in F \).
3. \( \forall i \in I, f_i \) are \((C, \alpha)\) - good on \( U \) with respect to \( \nu \) and \( \sup_{i \in I} |f_i| \) is measurable if and only if \( \sup_{i \in I} |f_i| \) is measurable.
4. \( f \) is \((C, \alpha)\) - good on \( U \) with respect to \( \nu \) if and only if \( g : V \to \mathbb{R} \) is a continuous function such that \( c_1 \leq |f| \leq c_2 \) for some \( c_1, c_2 > 0 \) and \( g \) is \((C, \alpha, c)\) - good on \( U \) with respect to \( \nu \).
5. Let \( C_2 > 1 \) and \( \alpha_2 > 0 \). \( f \) is \((C_1, \alpha_1)\) - good on \( U \) with respect to \( \nu \) and \( C_1 \leq C_2, \alpha_2 \leq \alpha_1 \) if and only if \( f \) is \((C_2, \alpha_2)\) - good on \( V \) with respect to \( \nu \).

We say a map \( f = (f_1, f_2, \ldots, f_n) \) from \( U \) to \( F^n \), where \( n \in \mathbb{N} \), is \((C, \alpha)\) - good on \( U \) with respect to \( \nu \), or simply \((f, \nu)\) is \((C, \alpha)\) - good on \( U \), if every \( F \) - linear combination of \( 1, f_1, \ldots, f_n \) is \((C, \alpha)\) - good on \( U \) with respect to \( \nu \).

**Definition 3.3.** Let \( f = (f_1, f_2, \ldots, f_n) \) be a map from \( U \) to \( F^n \), where \( n \in \mathbb{N} \). We say that \((f, \nu)\) is nonplanar at a given point \( x_0 \in U \) if for any ball \( B \) with center at \( x_0 \), the restrictions of the functions \( f_1, \ldots, f_n \) on \( B \cap \text{supp} (\nu) \) are linearly independent over \( F \). If \((f, \nu)\) is nonplanar at \( \nu \) almost every point of \( U \), then it is called nonplanar. We also simply say \( f \) is nonplanar when there is no possibility of confusion.

A typical example is provided by \( f = (f_1, f_2, \ldots, f_n) \) where \( 1, f_1, \ldots, f_n \) are smooth and linearly independent on \( U \). Such a map has been called nondegenerate by Kleinbock and Margulis.

For \( m \in \mathbb{N} \) and a ball \( B = B(x, r) \subseteq X \), where \( x \in X \) and \( r > 0 \), we shall use the notation \( 3^m B \) to denote the ball \( B(x, 3^m r) \). Finally, we will need the notion of a doubling measure.
The measure $\nu$ is said to be doubling on $U$ if there exists $D > 0$ such that for every ball $B$ with center in $\text{supp}(\nu)$ such that $2B \subseteq U$, one has
\[ \frac{\nu(2B)}{\nu(B)} \leq D. \]

4. Transference principles and lower bounds

The lower bound will follow immediately from two Diophantine transference principles. The following result was proved by Bugeaud and Zhang \cite{6} and constitutes a positive characteristic version of the transference principle of Bugeaud and Laurent \cite{5}.

**Theorem 4.1.** (Theorem 1.2, \cite{6}) Let $X \in F^m \times n$. Then for all $\theta \in F^m$, we have
\begin{equation}
\omega(X, \theta) \geq \frac{1}{\hat{\omega}(X^t)} \quad \text{and} \quad \hat{\omega}(X, \theta) \geq \frac{1}{\omega(X^t)},
\end{equation}
with equalities for almost every $\theta$.

We will also need a positive characteristic version of Dyson’s transference principle \cite{9} which can be formulated as follows.

**Theorem 4.2.** For $y \in F^n$, $$\omega(y) = 1 \text{ if and only if } \omega(y^t) = 1.$$ We omit the short proof which can be obtained by a verbatim repetition of the proof in \cite{9}, or the more recent, more general version proved in Theorem 1.7 in \cite{7}.

It is now easy to complete the proof of the lower bound Theorem 1.2.

**Proof.** Under the hypothesis of Theorem 1.2 using Theorem 3.7 of \cite{13}, we have that for $\lambda$ almost every $x$, $\omega(f(x)) = 1$. Set $y = f(x)$, then by Dyson’s transference principle, $\omega(y^t) = 1$. By Dirichlet’s theorem, $\omega(y) \geq 1$ and the trivial inequality
\[ \omega(y, \theta) \geq \hat{\omega}(y, \theta) \geq 0, \]
applied to $y^t$ and $\theta = 0$ we get that $\omega(y^t) = 1$. Finally, by (4.1), we get that $\omega(y, \theta) \geq 1$ which completes the proof.

\[ \square \]

5. The Transference principle of Beresnevich-Velani

In this section we state the inhomogeneous transference principle of Beresnevich and Velani from \cite{4} Section 5] which will allow us to convert our inhomogeneous problem to the homogeneous one. Let $(\Omega, d)$ be a locally compact metric space. Given two countable indexing sets $\mathcal{A}$ and $\mathcal{T}$, let $H$ and $I$ be two maps from $\mathcal{T} \times \mathcal{A} \times \mathbb{R}^+$ into the set of open subsets of $\Omega$ such that
\begin{equation}
H : (t, \alpha, \varepsilon) \in \mathcal{T} \times \mathcal{A} \times \mathbb{R}^+ \rightarrow H_t(\alpha, \varepsilon)
\end{equation}
and
\begin{equation}
I : (t, \alpha, \varepsilon) \in T \times A \times \mathbb{R}_+ \to I_t(\alpha, \varepsilon)
\end{equation}

Furthermore, let
\begin{equation}
H_t(\varepsilon) := \bigcup_{\alpha \in A} H_t(\alpha, \varepsilon) \quad \text{and} \quad I_t(\varepsilon) := \bigcup_{\alpha \in A} I_t(\alpha, \varepsilon).
\end{equation}

Let \( \Psi \) denote a set of functions \( \psi : T \to \mathbb{R}_+ : t \to \psi_t \). For \( \psi \in \Psi \), consider
\begin{equation}
\Lambda_H(\psi) = \limsup_{t \in T} H_t(\psi_t) \quad \text{and} \quad \Lambda_I(\psi) = \limsup_{t \in T} I_t(\psi_t).
\end{equation}

The sets associated with the map \( H \) will be called homogeneous sets and those associated with the map \( I \), inhomogeneous sets. We now come to two important properties connecting these notions.

**The intersection property.** The triple \((H, I, \Psi)\) is said to satisfy the intersection property if, for any \( \psi \in \Psi \), there exists \( \psi^* \in \Psi \) such that, for all but finitely many \( t \in T \) and all distinct \( \alpha \) and \( \alpha' \) in \( A \), we have that
\begin{equation}
I_t(\alpha, \psi_t) \cap I_t(\alpha', \psi_t) \subset H_t(\psi^*_t).
\end{equation}

**The contraction property.** Let \( \mu \) be a finite, non atomic, doubling measure supported on a bounded subset \( S \) of \( \Omega \). We say that \( \mu \) is contracting with respect to \((I, \Psi)\) if, for any \( \psi \in \Psi \), there exists \( \psi^+ \in \Psi \) and a sequence of positive numbers \( \{k_t\}_{t \in T} \) satisfying
\begin{equation}
\sum_{t \in T} k_t < \infty,
\end{equation}

such that, for all but finitely \( t \in T \) and all \( \alpha \in A \), there exists a collection \( C_{t,\alpha} \) of balls \( B \) centred at \( S \) satisfying the following conditions:
\begin{equation}
S \cap I_t(\alpha, \psi_t) \subset \bigcup_{B \in C_{t,\alpha}} B
\end{equation}

\begin{equation}
S \cap \bigcup_{B \in C_{t,\alpha}} B \subset I_t(\alpha, \psi^+_t)
\end{equation}

and
\begin{equation}
\mu(5B \cap I_t(\alpha, \psi_t)) \leq k_t \mu(5B).
\end{equation}

We are now in a position to state Theorem 5 from [4].

**Theorem 5.1.** Suppose that \((H, I, \Psi)\) satisfies the intersection property and that \( \mu \) is contracting with respect to \((I, \Psi)\). Then
\begin{equation}
\mu(\Lambda_H(\psi)) = 0 \forall \psi \in \Psi \Rightarrow \mu(\Lambda_I(\psi)) = 0 \forall \psi \in \Psi.
\end{equation}
6. Proof of Theorem 1.1

Fix \( \theta \in F \). It is enough to show that for any open ball \( V \subseteq U \) such that \( 5V \subseteq U \), \( \omega(f(x), \theta) \leq 1 \) for \( \lambda \) almost all \( x \in V \). In fact, we prove

\[
\forall \omega > 1, \lambda(\{x \in V : \omega(f(x), \theta) > \omega\}) = 0.
\]

For each \((t, \alpha) = (p, q), \varepsilon) \in \mathbb{N} \times (\Lambda \times \Lambda^n \setminus \{0\}) \times \mathbb{R}_+, \) we set

\[
I_t(\alpha, \varepsilon) \overset{\text{def}}{=} \{x \in V : |f(x) - (p - p')| \leq \varepsilon, ||q'|| \leq \varepsilon t\},
\]

and

\[
H_t(\alpha, \varepsilon) \overset{\text{def}}{=} \{x \in V : |f(x) - (p - p')| \leq \varepsilon, ||q'|| \leq \varepsilon t\}.
\]

Let \( \Psi \) denote the collection of functions \( \psi_\omega : \mathbb{N} \to \mathbb{R}, t \mapsto \frac{1}{\omega t} \), for \( \omega > 1 \). We denote the restriction of \( \lambda \) to \( V \) by \( \mu \) and thus it is supported on \( V \).

Since \( \forall \omega > 1, \lambda(\{x \in V : \omega(f(x), \theta) > \omega\}) \subseteq \Lambda I(\psi_\omega) \) so, it suffices to show that \( \lambda(\Lambda I(\psi_\omega)) = 0 \) for any \( \omega > 1 \). Theorem 3.7 in [13] implies that

\[
\forall \omega > 1, \lambda(\Lambda H(\psi_\omega)) = 0.
\]

Therefore to prove Theorem 1.1 in view of the Theorem 5.1 we only need to verify the intersection and contraction properties. These will be performed in the following two subsections.

6.1. Verification of the intersection property. Let \( t \in \mathbb{N}, \alpha = (p, q), \alpha' = (p', q') \in \Lambda \times \Lambda^n \setminus \{0\} \) with \( \alpha \neq \alpha' \) and \( \omega > 1 \). If at least one of \( ||q'|| \) and \( ||q'|| \) is \( > e^t \), then there is nothing to prove. Otherwise, the ultrametric property yields that

\[
x \in I_t(\alpha, \psi_\omega(t)) \cap I_t(\alpha', \psi_\omega(t)) \implies (f(x) \cdot (q - q') + (p - p')) \leq \max\{|f(x) \cdot q + p + \theta|, |f(x) \cdot q' + p' + \theta|\} \leq \frac{1}{\omega t}.
\]

Note that if \( q = q' \), then \( |p - p'| \leq \frac{1}{\omega t} \) and so \( p = p' \) which is impossible. Hence, it follows from (6.1) that \( I_t(\alpha, \psi_\omega(t)) \cap I_t(\alpha', \psi_\omega(t)) \subseteq H_t(\alpha - \alpha', \psi_\omega(t)) \).

6.2. Verification of the contraction property. Fix \( \alpha \in \Lambda \times \Lambda^n \setminus \{0\} \). We observe that, for any \( t \in \mathbb{N}, I_t(\alpha, \psi_\omega(t)) \subseteq I_t(\alpha, \psi_{\omega + t}(t)) \) and

\[
\mu(I_t(\alpha, \psi_{\omega + t}(t))) \leq \mu(\{x \in V : |f(x) \cdot q + \theta| < \frac{1}{e^{\frac{1}{2} + t} t}\}) \leq \frac{1}{e^{\frac{1}{2} + t} t} \mu(V),
\]

since \( f \) is \((\mathcal{C}, \alpha_0) - \text{good} \) on \( U \). The absolute constant appearing in the last inequality of the (6.2) does not depend on \( \alpha \) due to the nonplanarity of \( f \). Thus it turns out from (6.2) that, for all sufficiently large \( t \),

\[
I_t(\alpha, \psi_{\omega + t}(t)) \subseteq \frac{1}{e^{\frac{1}{2} + t} t} V \text{ for all } \alpha.
\]

For any \( t \) that satisfies (6.3) and all \( \alpha \), we now construct a collection of balls \( C_{t, \alpha} \) centered in \( V \) which makes \( (5.7), (5.8) \) hold. If \( I_t(\alpha, \psi_\omega(t)) = \emptyset \) then we set \( C_{t, \alpha} \) as the empty collection and consequently, (5.7)-(5.9) become trivial. Suppose \( I_t(\alpha, \psi_\omega(t)) \) is nonempty. Let \( x \in I_t(\alpha, \psi_\omega(t)) \). Since \( I_t(\alpha, \psi_{\omega + t}(t)) \) is open, there
exists a ball $B'(x)$ with center $x$ such that $B'(x) \subseteq I_t(\alpha, \psi_{\omega}^+(t))$. We can scale it and denote it by $B(x)$, due to (6.3), in such a way that
\begin{equation}
B(x) \subseteq I_t(\alpha, \psi_{\omega}^+(t)) \not\supseteq V \cap 5B(x).
\end{equation}
It is also clear from the construction that $5B(x) \subseteq 5V \subseteq U$. Consider
\begin{equation}
C_{t, \alpha} \overset{\text{def}}{=} \{ B(x) : x \in I_t(\alpha, \psi_{\omega}(t)) \}.
\end{equation}
The conditions (5.7) and (5.8) are obvious.

Define $F_\alpha : U \rightarrow \mathbb{R}$, $F_\alpha(x) = |f(x) \cdot q + p + \theta|$, $\forall x \in U$ and let $B \in C_{t, \alpha}$. By the last inequality given in (6.4), we see that
\begin{equation}
\sup_{x \in 5B} F_\alpha(x) \geq \frac{1}{e^{\frac{w+1}{2}nt}}. \tag{6.5}
\end{equation}
Furthermore, one has
\begin{equation}
\sup_{x \in 5B \cap I_t(\alpha, \psi_{\omega}(t))} F_\alpha(x) < \frac{1}{e^{\frac{w+1}{2}nt}} \leq \frac{1}{e^{\frac{w+1}{2}nt}} \times \frac{1}{e^{\frac{w+1}{2}nt}} \sup_{x \in 5B} F_\alpha(x) = \frac{1}{e^{\frac{w+1}{2}nt}} \sup_{x \in 5B} F_\alpha(x), \tag{6.6}
\end{equation}
due to $(6.5)$. Hence, from $(6.6)$ and the assumption that $f$ is $(\mathcal{C}, \alpha_0)$-good on $U$, it follows now that
\begin{equation}
\mu(5B \cap I_t(\alpha, \psi_{\omega}(t))) = \lambda(5B \cap I_t(\alpha, \psi_{\omega}(t))) \\
\leq \lambda \left( \left\{ x \in 5B : F_\alpha(x) < \frac{1}{e^{\frac{w+1}{2}nt}} \sup_{x \in 5B} F_\alpha(x) \right\} \right) \\
\leq \frac{\mathcal{C}}{e^{\frac{w+1}{2}nt\alpha_0}} \lambda(5B). \tag{6.7}
\end{equation}

Since $5B \cap V = V$ or $5B$, accordingly as $V \subseteq 5B$ or $5B \subseteq V$, so we have $\mu(5B) = \lambda(V)$ or $\lambda(5B)$. In the first case, we obtain $\lambda(5B) \leq \lambda(5V) = 5^\alpha \lambda(V) = 5^\alpha \mu(5B)$, and $\lambda(5B) = \mu(5B)$ in the later. Thus in either case, we see that $\lambda(5B) \leq 5^\alpha \mu(5B)$. In view of this and (6.4), the condition (6.3) of the contraction property is obvious as soon as we set
\begin{equation}
k_t \overset{\text{def}}{=} \frac{5^\alpha \mathcal{C}}{e^{\frac{w+1}{2}nt\alpha_0}}, \forall t \gg 1.
\end{equation}

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