DIFFUSION-DRIVEN BLOW-UP FOR A NON-LOCAL FISHER-KPP TYPE MODEL

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Abstract. The purpose of the current paper is to unveil the key mechanism which is responsible for the occurrence of Turing-type instability for a non-local Fisher-KPP type model. In particular, we prove that the solution of the considered non-local Fisher-KPP equation in the neighbourhood of a constant stationary solution, is destabilized via a diffusion-driven blow-up. It is also shown that the observed diffusion-driven blow-up is complete, whilst its blow-up rate is completely classified. Finally, the detected diffusion-driven instability results in the formation of unstable blow-up patterns, which are also identified through the determination of the blow-up profile of the solution.

1. Introduction

The mathematical model. In as early as 1952, A. Turing in his seminal paper [T52] attempted, by using reaction-diffusion systems, to model the phenomenon of morphogenesis, the regeneration of tissue structures in hydra, an animal of a few millimeters in length made up of approximately 100,000 cells. Further observations on the morphogenesis in hydra led to the assumption of the existence of two chemical substances (morphogens), a slowly diffusing (short-range) activator and a rapidly diffusing (long-range) inhibitor. A. Turing, in [T52], indicates that although diffusion has a smoothing and trivializing effect on a single chemical, for the case of the interaction of two or more chemicals different diffusion rates could force the uniform steady states of the corresponding reaction-diffusion systems to become unstable and to lead to non-homogeneous distributions of such reactants. Since then, such a phenomenon is now known as Turing-type instability or diffusion-driven instability (DDI) and although it has been first specified in [R40].

The main purpose of the current paper is the investigation of the occurrence of a Turing-type or DDI instability for the following non-local Fisher-KPP type model

\begin{equation}
  u_t - \Delta u = |u|^{p-1}u \left( 1 - \sigma \int_{\Omega} |u|^{\beta-1}u \, dx \right), \quad x \in \Omega, \ t > 0, \tag{1.1}
\end{equation}

\begin{equation}
  \frac{\partial u}{\partial \nu} = 0, \quad x \in \partial \Omega, \ t > 0, \tag{1.2}
\end{equation}

\begin{equation}
  u(x, 0) = u_0(x) \geq 0, \quad x \in \Omega. \tag{1.3}
\end{equation}
Our motivation to investigate the possible Turing-type instability of the above model stems from the fact that the non-local Fisher-KPP equation (1.1) arises as a mathematical model in several research areas. Specifically, in ecology, and for \( p = \beta = 1 \), it describes non-local competition effects, thanks to the term \( \int_{\Omega} u \, dx \) (the average of population density over the domain of interaction \( \Omega \)), between the individuals of a biological population, as it is explained in [FGS99, LL97]. Equation (1.1) was also proposed, cf. [GP07, GVA06], as a simple model of adaptive dynamics, where now the variable \( x \) represents a phenotypical trait of a given population. The individuals of such a population with trait \( x \) face competition from all their counterparts which does not depend on the trait itself. Other types of non-local terms may arise, see [CD05, PS05] for dispersal by jumps rather than by the Brownian motion. Note also that the imposed Neumann type boundary condition (1.2) describes the fact that population does not actually interact with its external environment.

Here \( \sigma > 0 \) stands for a (non-local) parameter measuring the magnitude of the non-local term interaction, whilst \( \Omega \) is assumed to be a bounded domain in \( \mathbb{R}^N, N \geq 3 \), with boundary of class \( C^{2r} \) for some \( r \in (0, 1) \). We also consider \( u_0 \in L^\beta(\Omega) \setminus \{0\} \) and the involved exponents \( p, \beta \) are set to satisfy
\[
\beta > 1.
\]
Equation (1.1) is actually a non-local version of the well known Fisher-KPP equation was first introduced, in its scalar form,
\[
\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u^p(1 - u)
\]
by Fisher [Fish] and Kolmogorov, Petrovskii, Piskunov [KPP], both in 1937, in the context of population dynamics. Here \( u \) represents the population density and the reaction term in (1.1) is considered to be the reproduction rate of the population. When \( p = 1 \), this reproduction rate is proportional to the population density \( u \) and to the available resources \( (1 - u) \). While, when \( p = 2 \) the model actually takes into account the addition of sexual reproduction with the reproduction rate to be proportional to the square of the population density, see [VV, VP, V1, V2]. Later, in 1938, Zeldovich and Frank-Kamenetskii [ZFK] came up with equation (1.5) in combustion theory where now \( u \) stands for the temperature of the combustive mixture.

In the literature, far more cases of non-local problems are encountered where the non-local terms induced by an integral of the solution over the domain of interaction \( \Omega \), see [KTz, KS18, SI, QS] and the references there in; however a non-local reaction term close to the one of (1.1) is particularly considered in [BDS1, HY, SJM, SK]. Notably, in [BDS1] the authors considered a non-local parabolic reaction-diffusion of the form of the form
\[
\frac{\partial u}{\partial t} = \Delta u + u^p - \frac{1}{|\Omega|} \int_{\Omega} u^p \, dx.
\]
proved the occurrence of finite-time blow-up for (1.6) even for $p > 1$ and initial data satisfying an energy inequality, utilizing a gamma convergence argument in order to get appropriate lower bounds for the considered Lyapunov functional. Thanks to the negative sign of the non-local reaction term included in (1.1) and (1.6) maximum principle fails and thus comparison methods are not applicable, cf. [QS, Proposition 52.24]. Furthermore, reaction-diffusion equation (1.6) leads to a conservation of the total mass, which is a key property for the investigation of its dynamics; it also admits a Lyapunov functional a helpful tool for the derivation of a priori estimates of the solution. However, equation (1.1) lacks these two key features although the associated total mass is bounded, a crucial property which is used for the investigation of its dynamics.

Now regarding the non-local reaction-diffusion equation (1.1) there are some already existing results in the literature. More precisely, the authors in [BChL] proved that the problem (1.1)-(1.3) for $\beta = 1$ admits global-in-time solutions for $N = 1, 2$ with any $1 \leq p < 2$ or $N \geq 3$ when $1 \leq p < 1 + 2/N$. Moreover, in [BChL] the asymptotic convergence towards the solution of the heat equation is also proved. Some more existence results were shown for the whole space case, i.e. when $\Omega = \mathbb{R}^N$ as well as for different boundary conditions in [B, BCh]; we refer the interested reader to these works for more references about this kind of problems. Finally, in [LLC] the authors considered (1.1) on $\mathbb{R}$ and studied the wave fronts of the corresponding nonlinear non-local bistable reaction-diffusion equation.

Main results. In the current subsection the main results of our work related with the occurrence of a Turing-type instability for model are demonstrated. First, it is worth noting, that due to the power non-linearity and thanks to condition (1.4), if a Turing-type (or (DDI)) instability occurs for the solution of non-local problem (1.1)-(1.3), then it should lead to the non existence of global-in-time solutions. More precisely, such an instability would be exhibited in the form of a diffusion-driven blow-up (DDBU), cf. [FN, HY].

In this work we restrict ourselves to the radial symmetric case, i.e. when $\Omega = B_1$ where

$$B_1 = B_1(0) := \{x \in \mathbb{R}^N : |x| < 1\},$$

denotes the unit sphere in $\mathbb{R}^N$. Then the solution of (2.3)-(2.5) is radial symmetric, that is $u(x,t) = u(r,t)$ for $0 \leq r = |x| < 1$ and so problem (2.3)-(2.5) is reduced to

$$u_t - \Delta_r u = K(t)u^p, \quad 0 < r < 1, \quad 0 < t < T,$$

(1.7)

$$u_r(0, t) = u_r(1, t) = 0, \quad 0 < t < T,$$

(1.8)

$$u(r, 0) = u_0(r), \quad 0 < r < 1,$$

(1.9)

where $T > 0$ is the maximal existence time for the solution, $\Delta_r := \partial^2_r + \frac{N-1}{r} \partial_r$ and

$$K(t) \equiv 1 - \sigma \int_{B_1(0)} u^\beta \, dx,$$

where the absolute values have been dropped, since the solution of problem (1.7)-(1.9) is positive for initial data of the form (1.10)-(1.11) due to Lemma 2.1.

Next we consider, as in [HY, KS16], spiky initial data of the form

$$u_0(r) = \lambda \phi_\delta(r), \quad \text{for} \quad 0 < \lambda << 1,$$

(1.10)
where
\[
\phi_{\delta}(r) = \begin{cases} 
  r^{-a}, & \delta \leq r \leq 1, \\
  \delta^{-a} \left( 1 + \frac{a}{2} \right) - \frac{a}{2} \delta^{-(a+2)} r^2, & 0 \leq r < \delta,
\end{cases}
\] (1.11)

for \( a := \frac{2}{p-1} \) and \( \delta \in (0,1) \). Taking into account that \( u'_0(r) < 0 \), then \( \max_{r \in [0,1]} u = u_0(0) \), and by maximum principle, since \( K(t) > 0 \) by Lemma 2.2 (i), we also deduce that \( u_r(r,t) < 0 \), hence \( \|u(\cdot,t)\|_\infty = u(0,t) \). 

Henceforth, we will denote by \( T_\delta \) the maximum existence time of solution of (1.7)-(1.9). In the sequel we prove that initial data of the form (1.10)-(1.11) can lead to finite-time blow-up for the solution of problem (1.7)-(1.9), i.e. to the occurrence of \( T_\delta < +\infty \) such that
\[
\|u(\cdot,t)\|_\infty = u(0,t) \rightarrow +\infty \quad \text{as} \quad t \rightarrow T_\delta.
\] (1.12)

Our first main results is stated as follows:

**Theorem 1.1.** Let \( \Omega = B_1 \subset \mathbb{R}^N \) with \( N \geq 3 \), \( p > \frac{N}{N-2} \) and (1.4) hold. Then there is a \( \lambda_0 > 0 \) provided with the following property: any \( 0 < \lambda \leq \lambda_0 \) admits \( 0 < \delta_0 = \delta_0(\lambda) < 1 \) such that any solution of problem (1.7)-(1.9) with initial data of the form (1.10)-(1.11) satisfying Lemma 2.2 (i) and \( 0 < \delta \leq \delta_0 \) blows up in finite time, i.e. \( T_\delta < +\infty \).

**Remark 1.1.** Theorem 1.1 guarantees the occurrence of a diffusion-induced blow-up. Namely it can be easily seen that any spatial homogeneous solution of (1.7)-(1.9) initiating close to \( u_\infty \equiv 1 \), which actually solves the IVP
\[
\frac{dU}{dt} = U^p \left( 1 - U^\beta \right), \quad t > 0, \quad U(0) = U_0,
\]
is stable and it converges to the steady state solution \( u_\infty \). Otherwise, Theorem 1.1 states that such a solution destabilizes once diffusion enters into the equation.

It is known, see for example [QS, Proposition 52.24], that the maximum principle is not applicable for the non-local problem (1.10)-(1.11) and hence comparison techniques fail. Therefore, our main strategy to overcome this obstacle is to derive a lower estimate of the non-local term \( K(t) \) and then deal with a local problem for which comparison techniques work. Although a lower estimate of \( K(t) \) is provided by Lemma 2.2, such an estimate is not uniform in time and thus an alternative approach should be applied to derive a uniform lower bound. To that end we will follow an approach used in [HY, KS16, KS18], and which was actually inspired by ideas in [FM85]. The methods, though, need to be modified appropriately so we can tackle the non-local problem (1.7)-(1.9) which includes a very different non-local term than the ones considered in [HY, KS16, KS18]. It is worth pointing out that the underlying method can be also implemented to predict diffusion-driven blow-up (DDBU) even in the case of an isotropically evolving domain \( \Omega(t), t > 0 \), for more details see [KBM].

Next, the form of the DDBU provided by Theorem 1.1 is further investigated. As a complementary result it is then proven, see Corollary 3.1 that as soon as the solution of problem (1.7)-(1.9) blows up in finite time \( T_\delta < \infty \), then it immediately becomes unbounded along the whole domain \( \Omega \) at any subsequent time. In other words, the
observed Turing-type instability is quite severe so it destroys all the occurring instability patterns once the blow-up time is exceeded.

Our next main result identifying the blow-up (Turing-type instability) rate is presented as follows:

**Theorem 1.2.** Let $N \geq 3$ with $p > \frac{N}{N-2}$ and (1.4) hold. Then the blow-up rate of the diffusion-induced blowing solution predicted by Theorem 1.1 is determined by

$$\|u(\cdot, t)\|_\infty \approx (T_\delta - t)^{-\frac{1}{p-1}}, \quad t \uparrow T_\delta. \quad (1.13)$$

The paper is organized as follows. Section 2 introduces some first results that we are going to use throughout this work. Section 3 contains the proof of our main blow-up Theorem 1.1 and that of the completeness of blow-up given by Corollary 3.1. Section 4 discusses the exact blow-up rate of our solution provided by Theorem 1.2. In section 5 we also identify the blow-up profile of solution $u$ and we thus determine the form of Turing instability patterns occurring as a consequence of the diffusion-driven instability.

2. Preparatory results

In the current subsection we present some key properties for the $u(x, t)$ solution of (1.1)-(1.3) We first point that the existence of a unique classical local-in-time solution of the non-local problem (1.1)-(1.3) can be established by using results existing in [QS] (see Remark 5.11 and Example 5.13) and in [S].

Henceforth, we use the notation $C$ and $C_i, i = 1, \ldots$, to denote positive constants.

Next we provide a result that establishing the positivity of solutions of (1.1)-(1.3) once non-negative initial data are considered.

**Lemma 2.1.** Let consider initial date $u_0 \in L^{\beta_0}(\Omega)$ with $\beta_0 = \max\{\beta, 2\}$, $u_0(x) \geq 0$ in $\Omega$, then

$$u(x, t) \geq 0, \quad \text{for any} \quad (x, t) \in \overline{Q_T},$$

where $Q_T := \Omega \times (0, T)$ and $T \in (0, \infty]$ stands for the maximum existence time of a classical solution of (1.1)-(1.3).

**Proof.** Set $u^- := -\min\{u, 0\} \geq 0$, then by the assumption on the initial data we have $u_0^- = 0$ and thus

$$\int_{\Omega} (u_0^-)^2 \, dx = 0. \quad (2.1)$$

Next by testing (1.1) by $u^-$ we derive

$$\frac{1}{2} \frac{d}{dt} \int_{\Omega} (u^-)^2 \, dx = -\int_{\Omega} |\nabla u^-|^2 \, dx + \int_{\Omega} |u|^{p-1}(u^-)^2 \, dx \left(1 - \sigma \int_{\Omega} |u|^\beta \, dx\right) \leq \int_{\Omega} |u|^{p-1}(u^-)^2 \, dx \left(1 + \sigma \int_{\Omega} |u|^{\beta} \, dx\right) \leq C(T) \int_{\Omega} (u^-)^2 \, dx, \quad (2.2)$$
where

\[ C(T) := \left[ M^{p-1}(T) \left( 1 + \sigma M^\beta(T) \right) \right] < \infty, \]

since \( M(T) := \max_{(x,t) \in Q_T} |u(x,t)| < +\infty \) for a classical solution of (1.1)-(1.3).

Inequality (2.2) by virtue of (2.1) entails \( \int_\Omega (u^-)^2 \, dx = 0 \), and thus \( u(x,t) \geq 0 \) in \( Q_T \).

Due to the above positivity result, henceforth we focus on the investigation of the problem

\[ u_t - \Delta u = u^p \left( 1 - \sigma \int_\Omega u^\beta \, dx \right), \quad \text{in} \quad Q_T := \Omega \times (0, T), \quad (2.3) \]

\[ \frac{\partial u}{\partial \nu} = 0, \quad \text{on} \quad \Gamma_T := \partial \Omega \times (0, T), \quad (2.4) \]

\[ u(x, 0) = u_0(x) \geq 0, \quad x \in \Omega. \quad (2.5) \]

The next lemma clarifies on the evolution of the norm \( m_\beta(t) := \int_\Omega u^\beta(x,t) \, dx \), along a nontrivial solution of (2.3)-(2.5).

**Lemma 2.2.** Let \( u \) be a solution of (2.3)-(2.5) with \( u_0 \in L^\beta(\Omega) \). If \( \beta > 1 \) there holds

(i) \( 0 < m_\beta(0) \leq 1/\sigma \) implies \( m_\beta(t) < 1/\sigma \) for all \( t \in (0, T] \), and

(ii) \( m_\beta(0) \geq 1/\sigma \) implies \( m_\beta(t) < m_\beta(0) \) for all \( t \in (0, T] \).

**Proof.** A direct calculation and by virtue of (2.3) implies

\[ m'_\beta(t) = -4\beta \left( 1 - \beta \right) \int_\Omega \left| \nabla u^\beta/2 \right|^2 \, dx \beta \left( 1 - \sigma m_\beta(t) \right) m_{p+\beta-1}(t) \]

\[ < \beta \left( 1 - \sigma m_\beta(t) \right) m_{p+\beta-1}(t), \quad \text{for any} \quad 0 < t < T, \quad (2.6) \]

using also the fact \( \beta > 1 \). Under the assumption \( m_\beta(0) \leq 1/\sigma \), by (2.6) we infer that there cannot be time \( t_\sigma > 0 \) such that \( m_\beta(t_\sigma) = 1/\sigma \) and \( m'_\beta(t_\sigma) \geq 0 \). Thus \( m_\beta(t) \leq 1/\sigma \) for all \( t \in (0, T] \), and in fact strict inequality follows. Namely, if \( m_\beta(t_\sigma) = 1/\sigma \) for some \( t_\sigma \in (0, T) \), then \( m'_\beta(t_\sigma) < 0 \), due to (2.6), which infers that \( m_\beta(t) \) would have thus exceeded \( 1/\sigma \) at some previous time \( t' \in (0, t_\sigma) \), leading to a contradiction. Then an identical argument to (i) implies (ii). \( \square \)

**Remark 2.1.** An immediate consequence of Lemma 2.2 (i) is a lower estimate of the average of solution \( u \) over domain \( \Omega \). Indeed, under the assumption \( 0 < m_\beta(0) \leq 1/\sigma \), which actually guarantees that

\[ K(t) \geq 0, \quad \text{for any} \quad 0 < t < T, \]
then averaging \(2.3\) over \(\Omega\) entails
\[
\frac{d}{dt} \int_{\Omega} u \, dx \geq 0, \quad \text{for any} \quad 0 < t < T
\]
in conjunction with Lemma \(2.4\), which finally implies
\[
\bar{\Omega}(t) := \int_{\Omega} u \, dx \geq 0, \quad \text{for any} \quad 0 < t < T, \quad (2.7)
\]
since \(u_0 \in L^\beta(\Omega) \setminus \{0\}\).

The global existence of positive classical solutions was proven in \([BCLL, B]\), yet for the sake of completeness we state these results in the sequel.

**Theorem 2.1.** \([BCLL, B]\) Let \(\beta \geq 1\), and assume that \(u_0\) is non-negative and \(u_0 \in L^k(\Omega)\) for \(1 < k < +\infty\). Assume further that \(p\) satisfies
\[
1 < p < 1 + \left(1 - \frac{2}{q}\right) \beta,
\]
where
\[
q = \begin{cases} 
\frac{2N}{N-2}, & N \geq 3, \\
2 < p < +\infty, & N = 2, \\
\infty, & N = 1,
\end{cases}
\]
then there exists a unique non-negative classical global-in-time solution to \((2.3)-(2.5)\).

**Remark 2.2.** Note that for \(N \geq 3\) Theorem 2.1 guarantees the existence of global-in-time solutions of problem \((1.7)-(1.9)\) in the range \(1 < p < 1 + \frac{2}{N} \beta\), for any \(\beta \geq 1\). In particular, choosing \(\beta > \frac{N}{N-2}\) we have global-in-time solutions for \(1 < p < \frac{N}{N-2}\), while on the other hand, if \(p > \frac{N}{N-2}\) then Theorem 1.1 establishes finite-time blow-up. Consequently for the specific choice \(\beta > \frac{N}{N-2}\) Theorems 2.1 and 1.1 provide an optimal result regarding the long-time behaviour of the solution to \((1.7)-(1.9)\), although is still unclear what happens in the critical case \(p = \beta = \frac{N}{N-2}\). Nevertheless, for \(\beta < \frac{N}{N-2}\) our approach still works but leaves a gap between global existence and blow-up for the solution \(u\) in the interval \(p \in (1 + \frac{2}{N} \beta, \frac{N}{N-2})\).

We also have the following result describing the asymptotic behaviour of the solution in the case \(\beta = 1\).

**Theorem 2.2.** \([BCLL, B]\) Let \(u(x,t)\) be a non-negative classical solution obtained from Theorem 2.1, \(v\) be the solution to the heat equation with Neumann boundary condition
and initial data \( \int_{\Omega} v_0(x) dx = m_0 \), then,

\[
\|u(\cdot, t) - v(\cdot, t) - (1 - m_0)\|_{L^2(\Omega)} \leq C_1 e^{-C_2 t},
\]

(2.8)

where \( C_1, C_2 \) are constants depending on the initial mass \( m_0 \) and \( \|u_0\|_{L^2(\Omega)} \).

3. Blow-up results

3.1. Diffusion-driven blow-up. The current subsection is devoted to the proof of the occurrence of a **diffusion-driven blow-up** (DDBU) for the solution of problem (1.7)-(1.9) under spiky initial data of the form (1.10)-(1.11). Accordingly a method, previously used in \[HY\], \[KS16\], \[KS18\], will be implemented; however in order to proceed further we first need to establish some auxiliary results.

The first presented result describes the key properties of this kind of initial data.

**Lemma 3.1** (\( \phi_\delta \)-properties). Let \( p > \frac{N}{N-2} \) with \( N \geq 3 \), then the function \( \phi_\delta \) defined in (1.11) satisfies the following:

i) There holds that

\[
\Delta_r \phi_\delta \geq -Na\phi_\delta^p,
\]

(3.1)

in a weak sense for any \( \delta \in (0, 1) \).

ii) If \( \zeta > 0 \) and \( N > a\zeta \) then

\[
\frac{1}{|B_1|} \int_{B_1} \phi_\delta^\zeta(x) dx := \int_{B_1(0)} \phi_\delta^\zeta(x) dx = N \omega_N \int_0^1 r^{N-1} \phi_\delta^\zeta(r) dr = \frac{N}{N-a\zeta} + O(\delta^{N-a\zeta}), \quad \text{as} \quad \delta \downarrow 0,
\]

(3.2)

where \( \omega_N := |B_1| = \pi^{N/2}\Gamma(N/2) \) is the volume of the unit ball in \( \mathbb{R}^N \).

iii) Set

\[
\alpha_1 = \sup_{0 < \delta < 1} \frac{1}{\phi_\delta^p} \left( \int_{B_1(0)} \phi_\delta^p dx \right),
\]

\[
\alpha_2 = \inf_{0 < \delta < 1} \frac{1}{\phi_\delta^p} \left( \int_{B_1(0)} \phi_\delta^p dx \right),
\]

for \( \mu := \frac{p\ell}{k-1} > p \), and some \( 0 < k < p \) such that \( N > \frac{2p}{k-1} \); the parameter \( \ell \) is also chosen so that

\[
k - 1 < \ell < \frac{N(p-1)}{2p},
\]

and thus \( \alpha_1, \alpha_2 < \infty \) thanks to (3.2) and under the assumption

\[
p > \frac{N}{N-2},
\]

(3.3)
Consider also
\[ d = d(\lambda) := \lambda - \sigma 2^{\beta(\mu+1)/p} a_1^{\beta/p} \Lambda_1^{\beta/p} \lambda^{\beta+1}, \]
which is a positive constant for \(0 < \lambda \ll 1\) since \(0 < \Lambda_1 := \sup_{0<\delta<1} \phi_{\delta} < \infty\) thanks to (3.3).

Then there exists some \(\lambda_0 := \lambda_0(\sigma, \delta) > 0\), such that for \(0 < \lambda \leq \lambda_0\) there holds:
\[ \Delta_r u_0 + d_0 \lambda^{-1} u_0^p \geq 2u_0^p, \quad (3.4) \]
where \(d_0 = \inf_{0<\lambda<\lambda_0} d(\lambda) > 0\).

Proof. For i) and ii) see [HY, KS16, KS18]. For iii) we calculate
\[ \Delta_r u_0 + d_0 \lambda^{-1} u_0^p = \lambda \Delta_r \phi_{\delta} + d \lambda^{p-1} \phi_{\delta}^p, \]
so it is enough to prove that
\[ \lambda \Delta_r \phi_{\delta} + d \lambda^{p-1} \phi_{\delta}^p \geq 2 \lambda \phi_{\delta}^p. \]
The latter by virtue of (3.1) reduces to
\[ -\lambda N a \phi_{\delta}^p + d_0 \lambda^{p-1} \phi_{\delta}^p \geq 2 \lambda \phi_{\delta}^p, \]
or equivalently to
\[ d_0 \lambda^{p-2} \geq 2 \lambda^{p-1} + Na, \]
which is finally true for \(0 < \lambda \leq \lambda_0\) by taking \(0 < \lambda_0 \ll 1\). \(\square\)

Given \(0 < \delta < 1\), let \(T_{\delta} > 0\) be the maximal existence time for the solution to (1.7)-(1.9) with initial data \(u_0 = \lambda \phi_{\delta}\). Henceforth we consider \(0 < \lambda < \lambda_0\) so that Lemma 3.1 is valid.

The next result provides a useful point estimate for any \(0 < r < 1\) of the solution \(u(r, t)\) in terms of its average over \(B_1\).

Lemma 3.2. For any \(0 < r \leq 1\) there holds
\[ r^N u(r, t) \leq \overline{u}(t) := \int_{B_1(0)} u(x, t) \, dx = N \int_0^1 y^{N-1} u(y, t) \, dy \quad (3.5) \]
and
\[ u_r \left( \frac{3}{4}, t \right) \leq -c, \quad 0 \leq t < T_{\delta}. \quad (3.6) \]

Proof. We first define the operator
\[ \mathcal{H}[w] := w_t - w_{rr} - \frac{N-1}{r} w_r - pu^{p-1} K(t) w. \]
with $w = r^{N-1}u_r$ and then we note that
\begin{align*}
\mathcal{H}[w] &= 0, \quad 0 < r < 1, \quad 0 < t < T_\delta, \quad (3.7) \\
w(r,t) &= 0, \quad r = 0, 1, \quad 0 < t < T_\delta, \quad (3.8) \\
w(r,0) &< 0, \quad 0 < r < 1. \quad (3.9)
\end{align*}
The maximum principle implies that $w \leq 0$, thus $u_r \leq 0$ for $(r,t) \in (0,1) \times (0,T_\delta)$ and so
\[
\overline{r} := N \int_0^1 y^{N-1}u(y,t) \, dy \geq N \int_0^r y^{N-1}u(y,t) \, d\sigma \geq Nu(r,t) \int_0^r y^{N-1} \, d\sigma = u(r,t) r^N,
\]
for any $0 < r < 1$ and $0 < t < T_\delta$, recalling that $\omega_N = |B_1(0)|$.

By virtue of Lemma 2.2 and for a classical solution $u$ of (1.7)-(1.9), we obtain that the term $pK(t)u^{p-1}$, that is the coefficient of the linear term in $\mathcal{H}[w]$, is uniformly bounded in $\frac{1}{2} < r < 1$, $0 < t < T_\delta$ for all $0 < \delta < \delta_0$. Furthermore we have $pK(t)u^{p-1}w \leq 0$ due to Lemma 2.1 and Lemma 2.2. Next we compare $w$ with the solution of
\begin{align*}
\theta_t - \theta_{rr} - \frac{N-1}{r} \theta_r &= 0, \quad \frac{1}{2} < r < 1, \quad 0 < t < T_\delta, \quad (3.10) \\
\theta(r,t) &= 0, \quad r = \frac{1}{2}, 1, \quad 0 < t < T_\delta, \quad (3.11) \\
\theta(r,0) &= w(r,0) < 0, \quad \frac{1}{2} < r < 1, \quad (3.12)
\end{align*}
to obtain that $w \leq \theta \leq 0$ in $(\frac{1}{2}, 1) \times (0,T_\delta)$ in conjunction with maximum principle.

In particular we have
\[
u_r \left(\frac{3}{4}, t\right) \leq \left(\frac{4}{3}\right)^{N-1} \theta \left(\frac{3}{4}, t\right) \leq -c, \quad 0 < t < T_\delta
\]
where $c$ is independent of $0 < \delta < \delta_0$.

\[\square\]

Next we prove an essential two-side $L^p$-estimate for the solution of (1.7)-(1.9), inspired by an analogous result holding for the shadow system of Gierer-Meinhard model, see also [KS16 Proposition 8.1] or [KS18 Chapter 5, Proposition 5.3].

**Proposition 3.1.** There exist $0 < \delta_0 < 1$ and $0 < t_0 \leq 1$ independent of any $0 < \delta \leq \delta_0$, such that the following estimate holds
\[
\frac{1}{2} A_2 \overline{\mu}^t \, dx \leq \int_{B_1(0)} u^p \, dx \leq 2A_1 \overline{\mu}^t \, dx, \quad \text{for any} \quad t \in (0, \min\{t_0, T_\delta\}), \quad (3.13)
\]
where $\mu$ provided by Lemma 3.1.
The constants $A_1$ and $A_2$ in (3.13) are given as follows

$$A_1 = \sup_{0<\delta<1} \frac{1}{u_0} \left( \int_{B_1(0)} u_0^p \, dx \right) = \lambda^{\rho-\mu} \alpha_1,$$

$$A_2 = \inf_{0<\delta<1} \frac{1}{u_0} \left( \int_{B_1(0)} u_0^p \, dx \right) = \lambda^{\rho-\mu} \alpha_2,$$

and $0 < A_1, A_2 < \infty$ due to Lemma 3.1.

**Proof.** For any $0 < \delta < \delta_0$, consider $[0, t_0(\delta)]$ to be the maximal time interval for which (3.13) holds. Obviously there holds $0 < t_0(\delta) \leq T_\delta$ for each $0 < \delta < \delta_0$. In case $t_0 \geq 1$ there is nothing to prove since the statement (3.13) automatically holds by simply choosing $t_0 = 1$. So in the following we now assume that $t_0 \leq 1$.

Integration of (1.7) over $B_1$, by virtue of (3.13), entails

$$\frac{d\overline{\mu}}{dt} = \int_{\Omega} u^p \, dx \left( 1 - \sigma \int_{\Omega} u^\beta \, dx \right) \leq \int_{\Omega} u^p \, dx \leq 2A_1 \overline{\mu},$$

and thus,

$$\overline{\mu} \leq \left[ \frac{1}{u_0^{1-\mu} - 2A_1(\mu - 1)t} \right]^{\frac{1}{\mu-1}}, \quad \text{for any} \quad t \in (0, t_0).$$

(3.15)

It can be also verified that

$$\left[ \frac{1}{u_0^{1-\mu} - 2A_1(\mu - 1)t} \right]^{\frac{1}{\mu-1}} \leq 2\overline{\mu}_0,$$

provided that

$$t \leq \min \left\{ \frac{2^{\mu - 1}}{2^\mu A_1(\mu - 1)\overline{\mu}_0^{\mu-1}}, \frac{\overline{\mu}_0^{\mu-1}}{2A_1(\mu - 1)} \right\}.$$

Consequently, we deduce that

$$\overline{\mu}(t) \leq 2\overline{\mu}_0 \leq 2\Lambda := 2 \sup_{\delta \in (0, \delta_0)} \overline{\mu}_0,$$

(3.16)

when $0 < t < t_2 := \min \{t_0, t_1\}$, and for

$$t_1 := \left\{ \frac{2^{\mu - 1}}{2^\mu A_1(\mu - 1)\Lambda^{\mu-1}}, \frac{\Lambda^{\mu-1}}{2A_1(\mu - 1)} \right\},$$

(3.17)

which is independent of $0 < \delta < \delta_0$.

Next, for given $\varepsilon > 0$, we define the auxiliary function

$$\chi := r^{N-1} u_r + \varepsilon r^N \frac{u^k}{\overline{\mu}}.$$
where the exponents $k$ and $\ell$ are defined in Lemma 3.1.

It is readily seen that

$$
\mathcal{H} \left[ r^{N-1} u_r \right] = 0,
$$

(3.18)

while by straightforward calculations we derive

$$
\mathcal{H} \left[ \frac{\varepsilon r^N u^k}{u^l} \right] = 2k(2N-1)\varepsilon r^{N-1} u_r^{k-1} + \frac{k\varepsilon r^N u^{p+1} - \ell\varepsilon r^N u^k}{u^{l+1}} K(t) \int_{B_1(0)} u^p \, dx
$$

$$
- 2kN\varepsilon r^{N-1} u_r^{k-1} + \frac{k(k-1)\varepsilon r^N u^{k-1}}{u^l} K(t)
$$

$$
\leq - 2k\varepsilon r^{N-1} u_r^{k-1} + \frac{\varepsilon r^N u^k}{u^l} K(t) \int_{B_1(0)} u^p \, dx - \frac{(p-k)\varepsilon r^N u^{p-1}}{u^l} K(t)
$$

$$
= - 2k\varepsilon u^{k-1} - 2k\varepsilon r^N u^{2k-1} - \frac{\varepsilon r^N u^k}{u^2} \int_{B_1(0)} u^p \, dx + (p-k)u^{p-1}\varepsilon r^N K(t)
$$

$$
= - 2k\varepsilon u^{k-1} - \frac{\varepsilon r^N u^k}{u^2} \left[ 2k\varepsilon u^{k-1} - K(t) \int_{B_1(0)} u^p \, dx - (p-k)u^{p-1}\varepsilon r^N K(t) \right]
$$

$$
= - 2k\varepsilon u^{k-1} + I.
$$

(3.19)

Next we show that

$$
I := 2k\varepsilon u^{k-1} - K(t) \int_{B_1(0)} u^p \, dx - K(t)(p-k)u^{p-1}\varepsilon r^N K(t) \leq 0.
$$

Using (1.4) in conjunction with Jensen’s inequality and (3.13), (3.16) we immediately derive

$$
K(t) \geq 1 - \sigma \left( \int_{B_1(0)} u^p \, dx \right)^{\beta/p}
$$

$$
\geq 1 - \sigma 2^{\beta(\mu+1)/p} A_1^{\beta/p} A_1^{\beta\mu/p}
$$

$$
\geq 1 - \sigma 2^{\beta(\mu+1)/p} A_1^{\beta/p} A_1^{\beta\mu/p} \chi^2 := D,
$$

for any $0 < t < \min\{t_0, t_1\}$, (3.20)

recalling that $0 < A_1 = \sup_{0<\delta<1} \bar{\phi}_\delta < \infty$.

By virtue of (3.20) and considering $0 < \lambda < \lambda_0$ (taking $\lambda_0$ smaller if it is necessary) we deduce that

$$
D := 1 - \sigma 2^{\beta(\mu+1)/p} A_1^{\beta/p} A_1^{\beta\mu/p} \chi^2
$$
is a positive constant depending on \( u_0 \) but not on \( 0 < \delta < \delta_0 \). Now combining (3.19) with (3.13), (3.16) and (3.20) we deduce

\[
I \leq 2k\varepsilon u^{k-1} - \frac{A_2D\ell}{2}u^{\mu+\ell-1} - D(p-k)u^{p-1}u^k
\leq 2k\varepsilon u^{k-1} - D(p-k)(2\Lambda)^{\ell}u^{p-1}, \quad \text{for any } 0 < t < t_0.
\] (3.21)

Then (3.21) in conjunction with Young’s inequality leads to \( I \leq 0 \), by also choosing \( \varepsilon \) sufficiently small independent of \( 0 < \delta < \delta_0 \).

Finally combining (3.18) with (3.19) we obtain

\[
\mathcal{H}[\chi] \leq -2k\varepsilon u^{k-1} \varphi \quad \text{in} \quad \left(0, \frac{3}{4}\right) \times (0, t_2),
\]

for any \( \varepsilon \) sufficiently small and independent of \( 0 < \delta < \delta_0 \).

Next we have \( \chi(0, t) = 0 \), whilst at \( r = \frac{3}{4} \) due to Lemma 3.2 and (3.16) there holds

\[
\chi\left(\frac{3}{4}\right) = \left(\frac{3}{4}\right)^{N-1} u_{\frac{3}{4}} \left(\frac{3}{4}, t\right) + \varepsilon \left(\frac{3}{4}\right)^{N} \frac{u^{k}(\frac{3}{4}, t)}{u^{\ell}}
\leq -c \left(\frac{3}{4}\right)^{N-1} + \varepsilon \left(\frac{3}{4}\right)^{N-kN} u^{k-\ell}(t)
\leq -c \left(\frac{3}{4}\right)^{N-1} + \varepsilon \left(\frac{3}{4}\right)^{N-kN} (2\Lambda)^{k-\ell} < 0,
\] (3.22)

for \( \varepsilon \) sufficiently small.

Subsequently for \( t = 0, 0 < r < \delta \) and fixed \( \lambda \) we calculate,

\[
\chi(r, 0) = r^{N-1} \left[ \lambda \phi_{\delta}'(r) + \varepsilon r \lambda^{k-\ell} \frac{\phi_{\delta}^k}{\phi_{\delta}} \right]
\leq r^{N} \left[ -\frac{\lambda a}{\delta^{a+2}} + \varepsilon \lambda^{k-\ell} \frac{(1 + \frac{a}{2})^k}{\delta^{ak} \left(\frac{N}{N-a\beta} + O(\delta^{N-a\beta})\right)} \right]
\leq r^{N} \left[ -\frac{\lambda a}{\delta^{a+2}} + \varepsilon \lambda^{k-\ell} \frac{(1 + \frac{a}{2})^k}{\delta^{ak}} \right] < 0,
\]

since \( a + 2 = ap > ak \) and for \( \varepsilon \) small enough and independent of \( 0 < \delta < \delta_0 \).
On the other hand, for \( t = 0 \), \( \delta < r < \frac{3}{4} \) and fixed \( \lambda \) we have,

\[
\chi(r, 0) = r^{N-1} \left[ -\frac{\lambda a}{r^{a+1}} + \varepsilon r^{k-\ell} \frac{1}{r^{ak} \left( \frac{N}{N-a_\beta} + O(\delta^{N-a_\beta}) \right)} \right]
\]

\[
\leq r^{N-1} \left[ -\frac{\lambda a}{r^{a+1}} + \varepsilon r^{k-\ell} \frac{1}{r^{ak-1}} \right] < 0,
\]

since \( a + 1 > ak - 1 \) and again taking \( \varepsilon \) small enough and still independent of \( 0 < \delta < \delta_0 \).

Outlining we have

\[
\mathcal{H}[\chi] \leq -\frac{2k\varepsilon u^{k-1}}{u} \chi, \quad \text{for} \quad 0 < r < \frac{3}{4} \quad \text{and} \quad 0 < t < t_2,
\]

\[
\chi \leq 0, \quad \text{for} \quad r = 0, \frac{3}{4} \quad \text{and} \quad 0 < t < t_2,
\]

\[
\chi \leq 0, \quad \text{for} \quad 0 < r < \frac{3}{4} \quad \text{and} \quad t = 0,
\]

hence maximum principle entails

\[
ur \leq \varepsilon r \frac{u^k}{u},
\]

which by integrating over \( r \in (0, \frac{3}{4}) \) leads to

\[
u(r, t) \leq \left[ \frac{2\mu^k}{\varepsilon(k-1)} \right]^{\frac{1}{k-1}} r^{\frac{2}{k-1}} \quad \text{for} \quad 0 < r < \frac{3}{4} \quad \text{and} \quad 0 < t < t_2. \tag{3.23}
\]

Hence (3.23) implies that for any \( 0 < r < \frac{3}{4} \)

\[
\frac{1}{|B_1|} \int_{B_r(0)} u^p \, dx \leq N \left[ \frac{2}{\varepsilon(k-1)} \right]^{\frac{1}{k-1}} r^{N-\frac{2p}{k-1}} \frac{u^p}{N - \frac{2p}{k-1} \mu^p}, \quad 0 < t < t_2,
\]

and by choosing \( r \) sufficiently small, recalling that \( N > \frac{2p}{k-1} \), we end up with the following estimate

\[
\frac{1}{|B_1|} \int_{B_r(0)} u^p \, dx \leq \frac{A_2}{8 \mu^p}, \tag{3.24}
\]

for all \( 0 < t < t_2 \) since \( \mu = \frac{p\ell}{k-1} \).
Next we set \( \psi := \frac{u}{\nu} \), for \( \nu := \frac{u}{p} = \frac{t}{k-1} > 1 \), then we can easily check that \( \psi \) satisfies the following non-local equation

\[
\psi_t = \Delta \psi + \left( \frac{K(t)}{\nu} \frac{u^p}{\nu^{p+1}} - \frac{\nu}{\nu^{p+1}} \int_{B_1(0)} u^p \, dx \right).
\]

We easily observe that by virtue of Lemma 3.2 and relations (2.7), (3.13) and (3.16) the terms

\[
\frac{K(t)}{\nu} u^p, \quad \text{and} \quad \frac{\nu}{\nu^{p+1}} \int_{B_1(0)} u^p \, dx,
\]

are uniformly bounded in \([B_1(0) \setminus B_r(0)] \times (0, \min \{t_0(\delta), t_1\})\).

Then standard parabolic regularity theory, [LSU], guarantees the existence of a time \( t_3 > 0 \) independent of \( 0 < \delta < \delta_0 \) such that

\[
\left| \frac{1}{|B_1|} \int_{B_1 \setminus B_r(0)} \frac{u^p}{\nu^p} \, dx - \frac{1}{|B_1|} \int_{B_1 \setminus B_r(0)} \frac{u_0^p}{\nu_0^p} \, dx \right| < \frac{A_2}{8}, \tag{3.25}
\]

for \( 0 \leq t \leq \min \{t_0(\delta), t_2, t_3\} \).

Considering that for some \( \delta_1 \in (0, \delta_0) \) there holds that \( t_0(\delta_1) \leq \min \{t_2, t_3, T_{\delta_1}\} \) then by virtue of (3.24) and (3.25)

\[
\left| \frac{1}{|B_1|} \int_{B_1} \frac{u^p}{\nu^p} \, dx - \frac{1}{|B_1|} \int_{B_1} \frac{u_0^p}{\nu_0^p} \, dx \right|
\leq \left| \frac{1}{|B_1|} \int_{B_r(0)} \frac{u^p}{\nu^p} \, dx - \frac{1}{|B_1|} \int_{B_r(0)} \frac{u_0^p}{\nu_0^p} \, dx \right|
\leq \left| \frac{1}{|B_1|} \int_{B_1 \setminus B_r(0)} \frac{u^p}{\nu^p} \, dx - \frac{1}{|B_1|} \int_{B_1 \setminus B_r(0)} \frac{u_0^p}{\nu_0^p} \, dx \right|
\leq \frac{3A_2}{8},
\]

and thus

\[
\frac{11A_1}{8} \leq \frac{1}{|B_1|} \int_{B_1} \frac{u^p}{\nu^p} \, dx \leq \frac{5A_2}{8} \quad \text{for any} \quad 0 < t < \min \{t_1, t_0(\delta_1)\}. \tag{3.26}
\]

We can then use continuity arguments in conjunction with (3.26) and the fact that \( 0 < t_0(\delta_1) < T_{\delta_1} \) to extend the validity of (3.13) beyond \( t_0(\delta_1) \), which actually contradicts the definition of \( t_0(\delta_1) \).

Eventually we obtain that (3.13) as well as all the preceding estimations are valid for any \( 0 < t < \min \{\tilde{t}_0, T_{\delta}\} \) for \( \tilde{t}_0 = \min \{t_2, t_3\} \). This competes the proof of Proposition. \( \square \)

We now are ready to prove Theorem 1.1, our main result in the current subsection.
Proof of Theorem 1.1. We can easily see that if the key estimate (3.20) derived in the proof of Proposition 3.1 is used then
\[ u_t = \Delta r u + K(t) u^p \geq \Delta u + Du^p \quad \text{in} \quad B_1 \times (0, \min\{t_0, \delta\}), \]
where recall that the constant $D$ depends on $u_0$ but not on $0 < \delta < \delta_0$. Thus by comparison principle
\[ u(x, t) \geq \tilde{u}(x, t) \quad \text{in} \quad \bar{B}_1 \times [0, \min\{t_0, \delta\}] \quad (3.27) \]
where $\tilde{u}$ solves the following problem
\[ \tilde{u}_t = \Delta r \tilde{u} + D \tilde{u}^p \quad \text{in} \quad B_1 \times (0, \min\{t_0, \delta\}), \quad (3.28) \]
\[ \frac{\partial \tilde{u}}{\partial v} = 0, \quad \text{on} \quad \partial B_1 \times (0, \min\{t_0, \delta\}), \quad (3.29) \]
\[ \tilde{u}(x, 0) = u_0(x), \quad \text{in} \quad B_1. \quad (3.30) \]
Consider now the auxiliary function $h = \tilde{u}_t - \tilde{u}^p$ then by straightforward calculations we deduce
\[ h_t = \Delta r h + p(p - 1)\tilde{u}^{p-2} |\nabla \tilde{u}|^2 + Dp\tilde{u}^{p-1} \geq \Delta r h + Dp\tilde{u}^{p-1} \quad \text{in} \quad B_1 \times (0, \min\{t_0, \delta\}) \]
and $\frac{\partial h}{\partial v} = 0$ on $\partial B_1 \times (0, \min\{t_0, \delta\})$. Additionally, by virtue of (3.24), we have for the initial condition
\[ h(x, 0) = \Delta r \tilde{u}(x, 0) + D \tilde{u}^p(x, 0) - \tilde{u}^p(x, 0) = \Delta_r u_0 + (D - 1)u_0^p \geq u_0^p, \quad \text{in} \quad B_1. \]
Therefore maximum principle entails that $h > 0$ in $\bar{B}_1 \times [0, \min\{t_0, \delta\}]$ and that is
\[ \tilde{u}_t > \tilde{u}^p \quad \text{in} \quad \bar{B}_1 \times [0, \min\{t_0, \delta\}]. \]
Integrating we derive
\[ \tilde{u}(r, t) \geq \left( \frac{1}{u_0^{p-1}(u)} - (p - 1)t \right)^{-\frac{1}{p-1}} \quad \text{in} \quad \bar{B}_1 \times [0, \min\{t_0, \delta\}] \]
which for $r = 0$ reads
\[ \tilde{u}(0, t) \geq \left( \frac{1}{u_0^{p-1}(0)} - (p - 1)t \right)^{-\frac{1}{p-1}} = \left\{ \frac{\delta^{a(p-1)}}{[\lambda \left(1 + \frac{a}{2}\right)] - (p - 1)t} \right\}^{-\frac{1}{p-1}} \]
which entails finite time blow-up for $\tilde{u}$, i.e.
\[ \|\tilde{u}(\cdot, t)\|_\infty = \tilde{u}(0, t) \to \infty \quad \text{as} \quad t \to \tilde{T}_\delta = \frac{1}{p - 1} \left[ \lambda \left(1 + \frac{a}{2}\right) \right]^{-1-p} \delta^2, \]
and consequently finite-time blow-up for the solution $u$ of (1.7)-(1.9) at time $T_\delta \leq \tilde{T}_\delta$ due to (3.27). Note also that $T_\delta \to 0$ as $\delta \to 0$ and thus the proof is complete. \qed
Remark 3.1. The finite-time blow-up predicted by Theorem 1.1 is actually a single-point blow-up, i.e. the solution \( u(r, t) \) of (1.7)-(1.9) blows up only at the origin \( r = 0 \). Indeed, by virtue of (3.13) and (3.16) we derive the following estimate

\[
\int_{B_1(0)} u(x, t) \, dx = N \int_0^1 r^{N-1} u(r, t) \, dr \leq C < \infty, \quad \text{for any } 0 < t < T_\delta,
\]

which in conjunction with (3.5) implies that the blow-up set of \( u \)

\[
S = \left\{ r_0 \in [0, 1] : \text{there exists } r_n \to r_0 \text{ and } t_n \to T_\delta : \lim_{n \to +\infty} u(r_n, t_n) = +\infty \right\} = \{0\}.
\]

3.2. Complete blow-up. Interestingly the finite-time blow-up predicted by Theorem 1.1 for the solution \( u \) of (1.7)-(1.9) is complete, roughly speaking there holds \( u(x, t) = +\infty \) for any \( x \in B_1 \) and \( t > T_\delta \). Before proving the latter result we need to provide an auxiliary result inspired by [BC], see also [QS, Theorem 27.2] and for which we will need some preliminary concepts.

Now set \( f_k(V) := \min\{V^p, k\}, V \geq 0, k = 1, 2, \ldots \) and let \( \tilde{u}_k \) be the solution of problem

\[
V_t = \Delta_r V + f_k(V), \quad \text{in} \quad B_1 \times (0, \infty),
\]

\[
\frac{\partial V}{\partial \nu} = 0, \quad \text{on} \quad \partial B_1 \times (0, \infty),
\]

\[
V(x, 0) = u_0(x), \quad \text{in} \quad B_1.
\]

It is easily seen that \( \tilde{u}_k \) is globally defined and \( \tilde{u}_{k+1} \geq \tilde{u}_k \). Moreover \( \tilde{u}_k \) solves the integral equation

\[
\tilde{u}_k(x, t) = \int_{B_1} G(x, y, t) u_0(x) \, dy + \int_0^t \int_{B_1} G(x, y, t-s) f_k(\tilde{u}_k(y, s)) \, dy \, ds, \quad (3.31)
\]

for any \( x \in B_1, t > 0 \), where \( G \) stands for the Dirichlet heat kernel in \( B_1 \). Now since \( G > 0 \) and \( \tilde{u}_{k+1} \geq \tilde{u}_k \) if we pass to the limit into (3.31) we derive

\[
\bar{u}(x, t) = \int_{B_1} G(x, y, t) u_0(x) \, dy + \int_0^t \int_{B_1} G(x, y, t-s) \bar{u}^p(y, s) \, dy \, ds, \quad x \in B_1, t > 0,
\]

for \( \bar{u}(x, t) := \lim_{k \to \infty} \tilde{u}_k(x, t) \) and where the double integral might be infinite. Clearly \( \bar{u}(\cdot, t) = u(\cdot, t) \) for \( t < \tilde{T}_\delta \) and if we set

\[
T^c = T^c(u_0) := \inf \left\{ t \geq \tilde{T}_\delta : \bar{u}(x, t) = +\infty \quad \text{for all} \quad x \in B_1 \right\},
\]

then there holds \( T^c(u_0) \geq \tilde{T}_\delta \). Now we can provide a more rigorous definition of the complete blow-up.
Definition 3.1. We say that the solution of problem (3.28)-(3.30) blows up completely if
$$T^c(u_0) = \tilde{T}_\delta.$$ 

Theorem 3.1. If \( N \geq 3 \) and \( 1 < p < p_S := \frac{N+2}{N-2} \) then solution of problem (3.28)-(3.30) exhibits a complete blow-up at \( T_\delta \).

Proof. For reader’s convenience we split the proof in several steps.

Step 1: We claim that \( \tilde{u}_t \geq 0 \). Indeed, if we set \( z = \tilde{u}_t \) then \( z \), thanks to (3.4) satisfies
$$z_t = \Delta_r z + Dp\tilde{u}^{p-1}z, \quad \text{in} \quad B_1 \times \left(0, \tilde{T}_\delta\right),$$
$$\frac{\partial z}{\partial \nu} = 0, \quad \text{on} \quad \partial B_1 \times \left(0, \tilde{T}_\delta\right),$$
$$z(x,0) = \tilde{u}_t(x,0) = \Delta_r u_0(x) + Du_0^p(x) \geq 2u_0(x) \geq 0, \quad \text{in} \quad B_1,$$
and thus maximum principle infers our claim.

Step 2: In the current step we will prove that \( ||\tilde{u}^p(\cdot, t)||_1 \rightarrow +\infty \) as \( t \rightarrow \tilde{T}_\delta - \).
Note that since \( \tilde{u} \geq 0 \) and \( \tilde{u}_t \geq 0 \) then the function \( g : t \mapsto ||\tilde{u}^p(\cdot, t)||_1 \) is nondecreasing. Assume by contrary that \( g \) is bounded then the \( L^k - L^\ell \) estimates entail
$$\|e^{-tA}f\|_k \leq Cq(t)^{-\frac{N}{2}(\frac{1}{r}-\frac{1}{k})}e^{-\mu_2t} \|f\|_\ell, \quad 1 \leq \ell \leq k \leq \infty,$$ (3.32)
for \( t \geq 0 \) and any \( f \in L^\ell(\Omega) \), where
$$0 < q(t) = \min\{t, 1\} \leq 1,$$
and the operator \( A \) used in (3.32) denotes \( -\Delta \) provided with the Neumann boundary condition, and therefore, \( \mu_2 \) is nothing but the second eigenvalue of \( A \), see also [H, Ro].

Now by virtue of the variation-of-parameters formula we deduce
$$\|\tilde{u}(t)\|_k \leq C \left(\|u_0\|_k + \int_0^t e^{-\mu_2s}q(s)^{-\frac{N}{2}(\frac{1}{r}-\frac{1}{k})}||\tilde{u}^p(s)||_\ell ds\right), \quad \text{for any} \quad 0 < t < \tilde{T}_\delta,$$ (3.33)
where integrability near \( s = t \) of the integrand terms appeared in (3.33) is ensured under the condition
$$\frac{N}{2} \left(\frac{1}{\ell} - \frac{1}{k}\right) < 1.$$ (3.34)

Now for \( \ell = 1 \) (3.33) in conjunction with our assumption gives
$$\|\tilde{u}(t)\|_k \leq C \left(\|u_0\|_k + \int_0^t e^{-\mu_2s}q(s)^{-\frac{N}{2}(\frac{1}{r}-\frac{1}{k})}||\tilde{u}^p(s)||_1 ds\right) \leq C(\tilde{T}_\delta), \quad \text{for any} \quad 0 < t < \tilde{T}_\delta,$$ (3.35)
provided that
$$\frac{N}{2} \left(\frac{1}{18} - \frac{1}{k}\right) < 1.$$ (3.36)
It is known, see [BP, W], that for \( N \geq 3 \) and \( k > \frac{N(p-1)}{2} \) the \( L^k \)-norm of the solution of

\[
\dot{\xi}_t = \Delta \xi + D\xi^p \quad \text{in} \quad B_1 \times (0, \min\{t_0, T_\delta\}),
\]

\[
\xi = 0, \quad \text{on} \quad \partial B_1 \times (0, \min\{t_0, T_\delta\}),
\]

\[
\xi(x,0) = u_0(x), \quad \text{in} \quad B_1,
\]

blows up in finite time, and thus by comparison arguments we also derive that

\[
\|\tilde{u}(t)\|_k \to +\infty \quad \text{as} \quad t \to \tilde{T}_\delta, \quad \text{for any} \quad k > \frac{N(p-1)}{2}. \tag{3.37}
\]

Since \( 1 < p < p_S \) we can always find an exponent \( k \) so that both (3.36) and (3.37) hold true, and thus we arrive at a contradiction due to (3.35).

**Step 3:** Consider \( \varepsilon \in (0,1) \), then

\[
\int_0^1 r^{N-1} \tilde{u}^p(r,t) \, dr = \int_0^{\varepsilon} r^{N-1} \tilde{u}^p(r,t) \, dr + \int_{\varepsilon}^{1-\varepsilon} r^{N-1} \tilde{u}^p(r,t) \, dr + \int_{1-\varepsilon}^1 r^{N-1} \tilde{u}^p(r,t) \, dr
\]

\[
= I_1(\varepsilon) + I_2(\varepsilon) + I_3(\varepsilon).
\]

For \( I_1(\varepsilon) \) under the change of variable \( r = \frac{\varepsilon^N}{1-2\varepsilon} \) we derive

\[
I_1(\varepsilon) = \frac{\varepsilon^N}{(1-2\varepsilon)^N} \int_{\varepsilon}^{1-\varepsilon} (R-\varepsilon)^{N-1} \tilde{u}^p(R,t) \, dR \leq \frac{\varepsilon^N}{(1-2\varepsilon)^N} I_2(\varepsilon).
\]

An estimate for \( I_3(\varepsilon) \) is obtained as follows

\[
I_3(\varepsilon) \leq \tilde{u}^p(1,t) \left( \frac{1-(1-\varepsilon)^N}{N} \right) \leq \tilde{u}^p(1-\varepsilon, t) \left( \frac{(1-\varepsilon)^N - \varepsilon^N}{N} \right) \leq I_2(\varepsilon),
\]

provided that \( \varepsilon \) is chosen small enough so that \( 1 + \varepsilon^N < 2(1-\varepsilon)^N \), where also the fact that \( \tilde{u}_r \leq 0 \) for \( r \in (0,1) \) has been taken into account. Consequently

\[
\int_{\varepsilon}^{1-\varepsilon} r^{N-1} \tilde{u}^p(r,t) \, dr \geq C(\varepsilon) \int_0^1 r^{N-1} \tilde{u}^p(r,t) \, dr \tag{3.38}
\]

for \( C(\varepsilon) := 2 + \frac{\varepsilon^N}{(1-2\varepsilon)^N} \).

Set \( v(r) := \lim_{t \to \tilde{T}_\delta} \tilde{u}(r,t) \) for any \( r \in (0,1) \) then

\[
\int_{\varepsilon}^{1-\varepsilon} r^{N-1} v^p(r) \, dr = \lim_{t \to \tilde{T}_\delta} \int_{\varepsilon}^{1-\varepsilon} r^{N-1} \tilde{u}^p(r,t) \, dr \geq \liminf_{t \to \tilde{T}_\delta} C(\varepsilon) \int_0^1 r^{N-1} \tilde{u}^p(r,t) \, dr = \infty, \tag{3.39}
\]

where it has been successively used the monotone convergence of \( \tilde{u} \) towards \( v \), relation (3.38) and Step 2.

**Step 4:** Fix now some \( r \in (0,1) \) and take some \( t > \tilde{T}_\delta \). Then we can find \( \varepsilon > 0 \) sufficiently
small such that $t - \tilde{T}_\delta \geq 2\varepsilon$ and $\varepsilon < r < 1 - \varepsilon$. Then by virtue of (3.31) and in conjunction with $f_k(\tilde{u}_k(R, s)) \geq f_k(\tilde{u}_k(R, \tilde{T}_\delta))$ for $s \geq \tilde{T}_\delta$ we have

$$
\tilde{u}_k(r, t) \geq N \omega_N \int_0^t \int_0^1 R^{N-1} G(r, R, t - s) f_k(\tilde{u}_k(R, s)) dR ds
$$

$$
\geq N \omega_N \tilde{C}(\varepsilon) \int_{t-2\varepsilon}^{t-\varepsilon} \int_{\varepsilon}^{1-\varepsilon} R^{N-1} f_k(\tilde{u}_k(R, s)) dR ds
$$

$$
\geq N \omega_N \tilde{C}(\varepsilon) \int_{t-2\varepsilon}^{t-\varepsilon} \int_{\varepsilon}^{1-\varepsilon} R^{N-1} f_k(\tilde{u}_k(R, \tilde{T}_\delta)) dR ds
$$

$$
\geq N \omega_N \varepsilon \tilde{C}(\varepsilon) \int_{\varepsilon}^{1-\varepsilon} R^{N-1} f_k(\tilde{u}_k(R, \tilde{T}_\delta)) dR,
$$

where

$$
\tilde{C}(\varepsilon) := \inf \{ G(r, R, s) : \varepsilon < r, R < 1 - \varepsilon, s \in (\varepsilon, 2\varepsilon) \} > 0.
$$

Passing to the limit as $k \to \infty$ into (3.40) then due to (3.39) we deduce

$$
\tilde{u}(x, t) \geq N \omega_N \varepsilon \tilde{C}(\varepsilon) \lim_{k \to \infty} \int_{\varepsilon}^{1-\varepsilon} R^{N-1} f_k(\tilde{u}_k(R, \tilde{T}_\delta)) dR
$$

$$
\geq N \omega_N \varepsilon \tilde{C}(\varepsilon) \int_{\varepsilon}^{1-\varepsilon} R^{N-1} v^p(R) dR = \infty,
$$

which proves the assertion. \qed

**Corollary 3.1.** Let $N \geq 3$ with $\frac{N}{N-2} < p < p_S$. Then the solution $u$ of (1.7) - (1.9) blows up completely.

**Proof.** The proof is an immediate consequence of Theorem 3.1 and relation (3.27). \qed

**Remark 3.2.** Corollary 3.1 actually means that the diffusion-driven instability stated by Theorem 1.1 is quite severe and thus any Turing (instability) pattern is destroyed once we exceed the blow-up time.

### 4. Blow-up rate and blow-up patterns

Our aim in the current section is to determine the form the diffusion-driven blow-up provided by Theorem 1.1. We first provide some estimates of the blow-up rate of the blowing up solution $u$.

**Proof of Theorem 1.2.** We first observe that due to Lemma 2.2 and (3.20) there holds

$$
D := 1 - \sigma 2^{\beta(a+1)/p} a_1^\beta/p \lambda_1^\beta/p \lambda^\beta < K(t) < 1 < \infty, \text{ for any } 0 < t < T_\delta.
$$

(4.1)
Consider now \( \Phi \) satisfying
\[
\Phi_t = \Delta \Phi + C \Phi^p, \quad \text{in} \quad B_1 \times (0, T_\delta),
\]
\[
\frac{\partial \Phi}{\partial \nu} = 0, \quad \text{on} \quad \partial B_1 \times (0, T_\delta),
\]
\[
\Phi(x, 0) = u_0(x), \quad \text{in} \quad B_1,
\]
then via comparison principle we derive \( u \leq \Phi \) in \( \bar{B}_1 \times [0, T_\delta] \).

Yet it is known, see [QS, Theorem 44.6], that
\[
|\Phi(x, t)| \leq C_\eta |x|^{-\frac{2}{p-1} - \eta}, \quad \text{for some} \quad \eta > 0,
\]
and thus
\[
|u(x, t)| \leq C_\eta |x|^{-\frac{2}{p-1} - \eta} \quad \text{in} \quad B_1 \times (0, T_\delta), \tag{4.2}
\]
which by virtue of (4.1), (4.2) and using also standard parabolic estimates entails that
\[
u \in \mathcal{BUC}^\tau \left( \{ \rho_0 < |x| < 1 - \rho_0 \} \times \left( \frac{T_\delta}{2}, T_\delta \right) \right), \tag{4.3}
\]
for some \( \tau \in (0, 1) \) and each \( 0 < \rho_0 < 1 \), where \( \mathcal{BUC}^\tau(M) \) denotes in general the Banach space of all bounded and uniform \( \tau \)-Hölder continuous functions \( h : M \subset \mathbb{R}^N \to \mathbb{R} \), see also [QS].

Consequently (4.3) infers that \( \lim_{t \to T_\delta} u(x, t) \) exists and it is finite for all \( x \in B_1 \setminus \{0\} \).

Recalling that \( N > \frac{2p}{p-1} \) (or equivalently \( p > \frac{N}{N-2}, \quad N > 2 \) ) then by using (4.1), (4.2) and in view of dominated convergence theorem we derive
\[
\lim_{t \to T_\delta} K(t) = \omega \in (0, +\infty). \tag{4.4}
\]

Applying now Theorem 44.3(ii) in [QS], then in conjunction with (4.4) we can find a constant \( C_u > 0 \) such that
\[
\|u(\cdot, t)\|_\infty \leq C_u (T_\delta - t)^{-\frac{1}{p-1}} \quad \text{in} \quad (0, T_\delta). \tag{4.5}
\]

On the other hand, setting \( N(t) := \|u(\cdot, t)\|_\infty = u(0, t) \) then \( N(t) \) is differentiable for almost every \( t \in (0, T_\delta) \), in view of [FM85], and it also satisfies
\[
\frac{dN}{dt} \leq K(t)N^p(t).
\]

Now since \( K(t) \in C([0, T_\delta]) \) is bounded in any time interval \([0, t], \ t < T_\delta\), then upon integration we obtain
\[
\|u(\cdot, t)\|_\infty \geq C_l (T_\delta - t)^{-\frac{1}{p-1}} \quad \text{in} \quad (0, T_\delta), \tag{4.6}
\]
for some positive constant \( C_l \) and thus the proof is complete. \( \Box \)
Remark 4.1. Condition (2.2) implies that the diffusion-induced blow-up stated in Theorem 1.1 is of type I, i.e. the blow-up mechanism is controlled by the ODE part of (1.7).

Next we identify the blow-up (Turing instability) pattern of the DDBU solution predicted by Theorem 1.1.

Note that (4.2) provides a rough form of the blow-up pattern for $u$. Nonetheless, due to (4.1) the non-local problem (1.7)-(1.9) can be tackled as the corresponding local one for which the following more accurate asymptotic blow-up profile, \([MZ]\), is available

$$
\lim_{t \to T_{max}} u(|x|, t) \sim C \left[ \frac{\log |x|}{|x|^2} \right] \quad \text{for} \quad |x| \ll 1.
$$

(4.7)

The latter relation provides the form of the blow-up profile of $u$. Therefore (4.7), in the biological context, actually identifies the form of the developing patterns, which are induced as the result of the DDI. The numerical verification of those instability patterns together with the investigation of the dynamics of non-local problem (1.1)-(1.3) on an evolving domain $\Omega(t), t > 0$ is the main topic of a forthcoming paper.

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