Aspects of Optimality of Plans Orthogonal Through Other Factors and Related Multiway Designs

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Abstract

In a blocked main effect plan (MEP), a pair of factors is said to be orthogonal through the block factor if their totals adjusted for the block are uncorrelated, as defined in Bagchi (Technometrics 52:243–249, 2010). This concept is extended here to orthogonality through a set of other factors. We discuss the impact of such an orthogonality on the precision of the estimates as well as on the data analysis. We construct a series of plans in which every pair of factors is orthogonal through a given pair of factors. Next we construct plans orthogonal through the block factors (POTB). We construct the following POTBs for symmetrical experiments. There are an infinite series of E-optimal POTBs with two-level factors and an infinite series of universally optimal plans for three-level factors. We also construct an universally optimal POTB for an $s'(s+1)/2$ experiment on blocks of size $(s+1)/2$, where $s \equiv 3 \pmod{4}$ is a prime power. Next we study optimality aspects of the “duals” of main effect plans with desirable properties. Here by the dual of a main effect plan we mean a design in a multi-way heterogeneity setting obtained from the plan by interchanging the roles of the block factors and the treatment factors. Specifically, we take up two series of universally optimal POTBs for symmetrical experiments constructed in Morgan and Uddin (Ann Stat 24:1185–1208, 1996). We show that the duals of these plans, as multi-way designs, satisfy M-optimality. Finally, we construct another series of multiway designs, which are also duals of main effect plans, and proved their M-optimality. This result generalizes the result of Bagchi and Shah (J Stat Plan Inf 23:397–402, 1989) for a row–column set-up. It may be noted that M-optimality includes all commonly used optimality criteria like A-, D- and E-optimality.

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### 1 Introduction

Morgan and Uddin [26] have pioneered the path of deviation from the traditional condition of orthogonality among all factors, constructing plans where treatment factors are orthogonal to each other, but not necessarily orthogonal to the blocking factors. Later Mukherjee, Dey and Chatterjee [27] discussed and constructed main effect plans (MEPs) with blocks of small size with similar properties. The plans constructed in both of these papers satisfy optimality. Das and Dey [11] also constructed similar plans. Restricting to blocks of size two, Bose and Bagchi [8] provided plans satisfying similar properties, but with fewer blocks. The condition of ‘orthogonality through the block factor’ between a pair of treatment factors was formally defined in Bagchi [3], and a plan satisfying this condition was named ‘plans orthogonal through the block factor’ (POTB). By now, new classes of POTBs have been constructed by many authors. Jacroux and his co-authors [15–21] have come up with a number of such plans, mostly for two-level factors, many of them satisfying optimality properties. Other authors include Saharay and Dutta [32]. Bagchi [5] constructed plans orthogonal as well as inter-class orthogonal through the block factor for symmetric experiments.

In this paper, we define the concept of ‘orthogonality through a set of factors’. We show how such a property leads to simplicity and precision in the data analysis. Next we go for construction. We construct (a) a series of plans orthogonal through a pair of factors, (b) a few series of plans orthogonal through the block factor, satisfying optimality property. The recursive method of construction used for constructing POTBs is similar to that used in Bagchi [5], but the generated POTBs are different.

Then, we go for optimality study in a multiple heterogeneity set-up. Such a set-up occurs when an experimental unit is subjected to a number of heterogeneity directions. Optimality study in an $m$-way heterogeneity setting was initiated by Cheng [10]. He assumed a model with no interaction among the blocking factors and considered a set-up with constant number of level combinations of the blocking factors. Mukhopadhyay and Mukhopadhyay [28] assumed the same model as Cheng [10], but required a much smaller set of experimental units. Assuming that the level combinations of the block factors form an orthogonal array of strength two with variable number of symbols, they obtained optimality results very similar to those of Cheng [10]. Bagchi and Mukhopadhyay [6] considered the situations where two factor interactions are present among the block factors and obtained optimality results. Morgan [25] considered an $m$-way setting with $t$-factor interactions and generalized all the optimality results obtained so far.

The dual of an optimal block design often satisfies optimality property. In this paper, we try to see whether that happens for a main effect plan (MEP) also. In
this context, we interpret duality as an interchange between the roles of the block factors and the treatment factors. Thus, dual of a blocked MEP is a design in a multi-way setting [see Definition 6.1]. We have taken up two series of universally optimal blocked MEPs constructed in Morgan and Uddin [26] and proved that the duals of these plans are also optimal as multi-way designs, although the type of optimality is different and the competing class of designs is smaller. We have also constructed a new series of multi-way designs and proved its optimality.

In Sect. 2, we present definitions and notations for a main effect plan and also for the optimality criteria. In Sect. 3, we define ‘orthogonality through a set of factors’. We have shown that regarding inference on a factor, say $A$, one may forget all factors other than those in a set $T$ if and only if $A$ is orthogonal to all factors through $T$ [see Theorem 3.1]. In Sect. 4, we concentrate on construction. In Sect. 4.1, we construct an infinite series of plans orthogonal through a pair of factors [see Theorem 4.2]. In Sect. 4.2, we construct POTBs. Specifically, we obtain (a) an infinite series of E-optimal POTBs for two-level factors [see Theorem 4.6], (b) an infinite series of universally optimal POTBs for three-level factors [see Theorem 4.8] and (c) an infinite series of universally optimal POTBs, for an asymmetrical experiment with bigger sets of levels [see Theorem 4.9].

In Sect. 5, we describe three multiway settings and derive the information matrix (or C-matrix) of the treatment effects for a design in each of these settings [see Lemma 5.2]. We take up the designs of our interest in Sect. 6. We describe the duals of two series of plans of Morgan and Uddin [26], study the properties of these multi-way designs and then prove their optimality [see Theorem 6.1 and 6.2]. In Sect. 6.3, we construct a series of multiway designs satisfying adjusted orthogonality [see Theorem 6.3] and prove their optimality property [see Theorem 6.4].

## 2 Preliminaries

We shall consider main effect plans for an experiment in which a block factor may or may not be present.

**Notation 2.1**  
(a) The number of factors of a main effect plan $\mathcal{P}$ is denoted by $m$ and the number of runs by $n$.

(b) For $i = 1, 2, \cdots, m$, $A_i$ denotes the $i$th factor, $S_i$ the set of levels of $A_i$ and $s_i = |S_i|$. We shall view the general effect as a factor, say $A_0$, so that $s_0 = 1$. $\mathcal{F} = \{0, 1 \cdots m\}$ will denote the index for the set of all factors.

(c) The vector $x = (x_1, x_2, \cdots x_m)' \in S^m$ represents a level combination or run of $\mathcal{P}$, in which $A_i$ is at level $x_i \in S_i$, $i = 1, 2, \cdots m$.

(d) Fix $i \in \mathcal{F}$. The replication vector of $A_i$ is denoted by the $s_i \times 1$ vector $r_i$, the $p$th entry of which is the number of runs $x$ of $\mathcal{P}$ such that $x_i = p, p \in S_i$. $R_i$ denotes a diagonal matrix with diagonal entries same as those of $r_i$ in the same order.

(e) For $0 \leq i, j \leq m$, the $A_i$ versus $A_j$ incidence matrix is the $s_i \times s_j$ matrix $N_{ij}$. The $(p, q)$th entry of this matrix is $N_{ij}(p, q)$, which is the number of runs $x$ of $\mathcal{P}$ such that $x_i = p, x_j = q, p \in S_i, q \in S_j$. When $j = i, N_{ij} = R_i$. 

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**Notation 2.2** (a) $1_n$ will denote the $n \times 1$ vector of all-ones, while $J_{m \times n}$ will denote the $m \times n$ matrix of all-ones. We shall write $J_m$ for $J_{m \times m}$.
(b) $\mathcal{C}(M)$ will denote the column space of a matrix $M$.
(c) The $n \times 1$ vector of responses will be denoted by $Y$.
(d) The $s_i \times 1$ vector $\alpha^i$ will denote the vector of unknown effects of $A_i$, $1 \leq i \leq m$.

The scalar $\mu$ will denote the general effect.

For $i = 0, 1, \ldots, m$, $X_i$ is the design matrix for $A_i$. Thus, $X_i$ is the $n \times s_i$ matrix having the $(u, r)$th entry 1 if in the $u$th run the factor $A_i$ is set at level $r$ and 0 otherwise. In particular, $X_0 = 1_n$. The following relations are well known.

\[
\mathcal{C}(X_0) \subseteq \mathcal{C}(X_i), \ N_{i0} = r_i \text{ and } N_{ij} = X_i'X_j, \ i, j \in \mathcal{F}. \tag{2.2}
\]

We shall use the following notations for the sake of compactness.

**Notation 2.3** (a) For any $m \times n$ matrix $M$, $P_M$ will denote the projection operator on the column space of $M$. Thus, $P_M = MM'M^{-1}M'$, where $B^{-}$ denotes a g-inverse of $B$.
Further, $P_i$ will denote the projection operator onto the column space of $X_i, i \in \mathcal{F}$.
(b) Let $T = \{i, j, \ldots\}$ be a subset of $\mathcal{F}$.
(i) $A_T$ will denote $\{A_i, A_j, \ldots\}$.
(ii) $X_T$ will denote $[X_i \ X_j \ \ldots]$. Moreover, $X_T'X_T$ will be denoted by $N_{TT}$ (which is consistent with (2.2)).
(iii) $\alpha^T$ will denote $\begin{bmatrix} \alpha^1 \\ \alpha^2 \\ \vdots \end{bmatrix}$. $\hat{\alpha}^T$ will denote the least square estimate of $\alpha^T$.
(iv) $P_T$ will denote the projection operator onto the column space of $X_T$.
(c) Consider a pair of disjoint subsets $T$ and $U$ of $\mathcal{F}$. We define the matrix $C_{T:U}$ and the vector $Q_{T:U}$ as follows. In case $|W| = 1$, say $W = \{w\}$, then we use $w$ in place of $W$, $W = T$ or $U$.
\[
C_{T:U} = ((C_{ij:U}))_{i,j \in T}, \ C_{ij:U} = X_i'(I - P_U)X_j, \tag{2.3}
\]
\[
Q_{T:U} = ((Q_{i:j}))_{i \in T}, \ Q_{i:j} = X_i'(I - P_U)Y. \tag{2.4}
\]
(d) Sum of squares : Fix a set of factors $T$. For $i \notin T$, the sum of squares for $i$, adjusted for the set of factors $A_T$ will be denoted by $SS_{i:T}$.

The following relations are well known.

**Lemma 2.1** (a) The normal equation for the least square estimates of the vector of all effects is $X'X\hat{\alpha} = X'Y$. 

\[ \text{ Springer} \]
(b) The reduced normal equation for $\hat{\alpha}_T$ is given by $C_{T;\bar{T}}\hat{\alpha}^T = Q_{T;\bar{T}}$, where $\bar{T} = F \setminus T$. In particular, the reduced normal equation for $\alpha^i$ is

$$
C_{i;\bar{i}}\hat{\alpha}^i = Q_{i;\bar{i}}, \quad \text{where } \bar{i} = F \setminus \{i\}.
$$

(2.5)

(c) The sums of squares considered in (d) of Notation 2.3 can be expressed as

$$
SS_{i;T} = Q'_{i;T}(C_{i;T})^{-1}Q_{i;T}.
$$

In particular the sum of squares for $i$, adjusted for all the other factors is

$$
SS_{i;\bar{i}} = Q'_{i;\bar{i}}(C_{i;\bar{i}})^{-1}Q_{i;\bar{i}}.
$$

Remark 2.1 The matrix $C_{i;\bar{i}}$ in (b) of Lemma 2.1 is referred to as the “information matrix” or “C-matrix” of $i$. In order that every main effect contrast of $A_i$ is estimable, rank of $C_{i;\bar{i}}$ must be $s_i - 1$. We, therefore, consider only the plans satisfying $\text{Rank}(C_{i;\bar{i}}) = s_i - 1$, for every $i \in F$ and refer to such a plan as ‘connected’.

We need the following well-known results, which also follow from Lemma 2.1

Lemma 2.2 (a) For $i \neq j$, $i, j \in F$, $SS_{i;j}$ is the quadratic form $Y'P_UY$, where $U = (I - P_j)X_i$.

(b) More generally, for $T \subset F$, $i \notin T$, $SS_{i;T}$ is the quadratic form $Y'P_VY$, where $V = (I - P_T)X_i$. In particular, $SS_{i;\bar{i}} = Y'P_HY$, where $H = (I - P_{\bar{i}})X_i$.

(c) The so-called unadjusted sum of squares for $A_i$ is $SS_{i;0} = T'_i(R_i)^{-1}T_i - G^2/n$, where $T_i$ is the vector of raw totals for $A_i$ and $G$ is the grand total.

2.1 Reduced Normal Equation for the Contrasts

Sometimes it is useful to consider the reduced normal equation for the contrasts and the corresponding C-matrix. Towards that, we introduce the following notations.

Notation 2.4 (a) For each $i : 1 \leq i \leq m$, $O_i$ will denote an $(s_i - 1) \times s_i$ matrix such that $O_iO'_i = I_{s_i-1}$ and $O_i1_{s_i} = 0$.

(b) $Z_i = X_iO'_i$, $\gamma^i = O_i\alpha^i$.

Using this notation, the model (2.1) can also be expressed as follows ($X_0, \mu$ and $\epsilon$ are as in (2.1)).

$$
Y = X_0\mu + Z\delta + \epsilon \quad \text{where } Z = \begin{bmatrix} Z_1 & \ldots & Z_m \end{bmatrix} \text{ and } \delta = \begin{bmatrix} \gamma^1 & \ldots & \gamma^m \end{bmatrix}'.
$$

(2.6)
Eliminating $\hat{\mu}$ from the normal equations, we get the reduced normal equation for the vector of all contrasts as

$$\tilde{C}\delta = \tilde{Q}, \text{ where } \tilde{C} = Z'Z \text{ and } \tilde{Q} = Z'Y. \quad (2.7)$$

**Remark 2.2** The $C$-matrix ($\tilde{C}$) of the contrasts for a plan $\mathcal{P}$ is particularly useful when partial orthogonality holds among one or more factors of $\mathcal{P}$ [see Definition 2.4 of Bagchi [4]]. As an example, let us take $\mathcal{P}$ to be the plan $ICA(N, 3'2^p)$ of Huang, Wu, and Yen, C.H. [14]. One can see that for this plan $\tilde{C}$ is of the form

$$\tilde{C} = \begin{bmatrix} \tilde{C}_1 & 0 \\ 0 & \tilde{C}_2 \end{bmatrix},$$

where $\tilde{C}_2 = dI_p$ for a real number $d$ and $p = N - 2(l + 1)$. Further, permuting the rows and columns of $\tilde{C}_1$ by a suitable permutation matrix, we can write it as

$$M_1 = a_i I + b_i J, i = 1, 2.$$ With this expression, one can see that the set of all contrasts of all the three-level factors satisfy inter-class orthogonality, the orthogonal classes being the linear contrasts and the quadratic contrasts.

### 2.2 Plans with a Block Factor

We now assume that a block factor is present. It is convenient to view the block factor separately from the treatment factors. We, therefore, use the following notation.

**Notation 2.5**

(a) $B$ will denote the block factor.

(b) $b$ will denote the number of blocks and $k_j$ the size of the $j$th block, $1 \leq j \leq b$. So, the total number of runs is $n = \sum_{j=1}^{b} k_j$. $D_k$ will denote the diagonal matrix whose diagonal entries are $k_j$, $1 \leq j \leq b$, in that order.

(c) $L_i$ will denote the $A_i$-versus block incidence matrix.

(d) $\tau$ will denote the vector of effects of all the $m$ treatment factors.

Substituting $T = \{1, 2, \cdots, m\}$ (so that $\bar{T} = \{0, B\}$) in Lemma 2.1 (b) we obtain the reduced normal equation for $\hat{\tau}$, after eliminating the block effects as

$$C_{T:B}\hat{\tau} = Q_{T:B}, \text{ where } C_{T:B} = ((C_{i:j:B}))_{1 \leq i,j \leq m}, \ C_{i:j:B} = N_{ij} - L_i D_k^{-1} L_j' \quad (2.8)$$

and $Q_{T:B}$ can be obtained From (2.4).

Finally, using Notation 2.4, we obtain the reduced normal equations for the vector of all contrasts after eliminating the block effects.

$$\tilde{C}\delta = \tilde{Q}, \text{ where } \tilde{C} = ((\tilde{C}_{i:j}))_{1 \leq i,j \leq m}, \tilde{C}_{i:j} = O_i (N_{ij} - L_i D_k^{-1} L_j') O_j' \quad (2.9)$$

and $\tilde{Q} = ((\tilde{Q}_j))_{1 \leq i \leq m}, \ \tilde{Q}_j = O_j Q_{i:B}$, where $Q_{i:B}$ can be obtained from (2.4) by substituting $P_B = X_B D_k^{-1} P_U'$ for $P_U$. Here $X_B$ is the design matrix for the block factor.

**Note:** $\tilde{C}$, the $C$-matrix of all the contrasts is as in (2.7) if no block factor is present, while it is as in (2.9) if a block factor is present.
2.3 Optimality tools

To save space, we refer to Shah and Sinha [34] for the definition and other details of universal optimality. We discuss a class of optimality criteria, which is not so well known.

**Notation 2.6** For a real symmetric \( n \times n \) matrix \( A \),

(a) \( \mu_0(A) \leq \cdots \leq \mu_{n-1}(A) \) will denote the eigenvalues of \( A \).

(b) Consider a design \( d \). \( C_d \) will denote its C-matrix. By the vector of eigenvalues of \( d \) we shall mean the vector of positive eigenvalues of \( C_d \) in the nondecreasing order and it will be denoted by \( \mu(d) \).

**Notation 2.7** \( \mathcal{F} \) will denote the class of all non-increasing Schur convex real valued functions on \( R^+ = (0, \infty) \). For \( f \in \mathcal{F} \), \( \Psi_f : (R^+)^n \rightarrow R \) is defined by \( \Psi_f(x) = \sum_{i=1}^{n} f(x_i) \).

**Definition 2.1** (a) For \( x, y \in (R^+)^n \) \( x \) is said to be \( \Psi_f \)-better than \( y \) if

\[ \Psi_f(x) \leq \Psi_f(y). \] (2.10)

(b) Consider a class of designs \( D \) and a member \( d^* \) of \( D \). If \( \mu(d^*) \) is \( \Psi_f \)-better than \( \mu(d) \) for every \( d \in D \), then \( d^* \) is said to be \( \Psi_f \)-optimal over \( D \).

The following members of \( \mathcal{F} \) are of special interest as the corresponding \( \Psi_f \) criteria have important statistical interpretation. These are the functions \( f(u) \equiv u^{-1} \) corresponding to A-optimality and \( f(u) \equiv -\log(u) \) corresponding to D-optimality [see Shah and Sinha [34] for more details]. Another popular optimality criterion is E-optimality. It may be obtained as the limit of a class of \( \Psi_f \) criteria, but has a simpler description. We describe E-optimality in terms of the C-matrix of the contrasts, which is what we need later.

**Definition 2.2** A plan \( P^* \in \Pi \) is said to be E-optimal in for the inference on all the main effect contrasts if

\[ \mu_1(\tilde{C}(P^*)) \geq \mu_1(\tilde{C}(P)) \quad \forall P \in \Pi. \]

A powerful approach to design optimality problems is through the concept of majorization.

**Notation 2.8** For \( x = (x_1, \cdots, x_n) \in (R)^n \), \( x \uparrow = (x_1, \cdots, x_n) \) will denote the non-decreasing re-ordering of \( x \).

**Definition 2.3** (Marshall, Olkin and Arnold [24]) For \( x, y \in (R)^n \), \( x \) is said to be weakly majorized from above by \( y \) (in symbols, \( x \preceq_w y \)) if

\[ \sum_{i=1}^{k} x_{\uparrow i} \geq \sum_{i=1}^{k} y_{\uparrow i}, \quad k = 1, 2, \ldots, n, \] (2.11)
See Marshall, Olkin and Arnold [24] for a comprehensive treatment of majorization concepts and results.

Following Bagchi and Bagchi [2], we define the following.

**Definition 2.4** (a) For \( x, y \in (\mathbb{R}^n) \), \( x \) is said to be M-better than \( y \) if \( x \prec^w y \).

(b) For \( d, d' \in D \), \( d \) is said to be better than \( d' \) in the sense of majorization (in short M-better), written \( d \succ^M d' \), if \( \mu(C_d) \prec^w \mu(C_{d'}) \), equivalently \( \mu(d) \prec^w \mu(d') \). A design \( d^* \) is said to be optimal in the sense of majorization in a subclass \( D \) of \( D_{b,k,v} \) (or, in short, \( d^* \) is M-optimal in \( D \)) if it is M-better than every member of \( D \).

In view of a theorem in Tomic (1949), we have the following useful result. [See Proposition B.2 of chapter 4 of Marshall, Olkin and Arnold [24] for the theorem of Tomic (1949)].

**Theorem 2.1** A necessary and sufficient condition for a design \( d^* \in D \) to be \( \Psi_f \)-optimal over \( D \) for every \( f \in F \) is that (2.11) holds with \( x = \mu(C_{d^*}) \) and \( y = \mu(C_d) \) for any \( d \in D \).

**Sufficient conditions for M-optimality:** The following result is easy to verify. Here by \( A \geq B \) we mean that \( A - B \) is non-negative definite.

**Lemma 2.3** Suppose \( A \) and \( B \) are real symmetric matrices. If \( A \geq B \), then \( \mu(A) \prec^w \mu(B) \).

Using the fact that for an \( n \times 1 \) vector \( y \), \((1/i) \sum_{j=1}^{i} y_j^j \) is increasing in \( i \), we get another useful result.

**Lemma 2.4** Let \( m < n \). Consider \( x \in (\mathbb{R}^n)^\uparrow \), such that \( m \) of the \( x_i \)'s are \( a \) and the remaining \( n - m \) of them are \( b \), where \( a < b \). Suppose \( y \in (\mathbb{R}^m)^\uparrow \) satisfies \( \sum_{i=1}^{n} y_i = \sum_{i=1}^{m} x_i \) and \( y_m+1 \geq b \). Then, \( x \prec^w y \).

### 3 Orthogonality through other factors versus usual orthogonality

In this section, we seek the answer to the following questions. **Consider a main effect plan for \( m(\geq 3) \) factors. Fix a factor, say \( A_i \). What conditions must the design matrices satisfy so that the inference on \( A_i \) depends only on the relation of it with the factors in a certain class of factors (say \( T \)) ?** In other words, what conditions must the incidence matrices satisfy so that the inference on \( A_i \) remains same even if the factors other than those in \( T \) are ignored ?

Towards an attempt to answer these questions, we need a definition.

**Definition 3.1** (a) Consider \( T \subset \mathcal{F} \), \( i, j \notin T \). Then, the factors \( A_i \) and \( A_j \) are said to be **orthogonal through the factors in \( T \)**, denoted by \( A_i \perp_T A_j \), if
We now seek answers to the question posed above.

**Theorem 3.1** Fix \( i \in \mathcal{F} \). Partition \( \tilde{i} \) as \( \tilde{i} = S \cup T \). Then, a necessary and sufficient condition for each of the following statements is that \( A_i \perp_T A_j, \forall j \in S \).

(a) \( C_{i;\tilde{i}} = C_{i:T} \).

(b) \( SS_{i;\tilde{i}} = SS_{i:T} \) with probability 1.

[Here \( C_{i:T} \) and \( SS_{i:T} \) are as in Notation 2.3]. The proof is in Appendix A.

Extending Definition 3.1, we have the following.

**Definition 3.2** Consider a plan \( \mathcal{P} \) with \( m \) factors satisfying the following. There is a set \( T \) of size \( t \) such that for every pair \((i, j)\), \( i, j \not\in T \), \( A_i \perp_T A_j \). Then we say that \( \mathcal{P} \) is a plan orthogonal through \( t \) factors.

**Special cases**:

**Case** \( |T| = 1 \) and \( T = \{0\} \): This reduces Definition 3.1 to the usual definition of orthogonality and (3.1) to the proportional frequency condition (PFC) of Addelman [1]. We present this important condition below. Two factors \( A_i, A_j \), \( i \neq j \) are orthogonal if

\[
nN_{ij} = r_ir_j'.
\]

[Here \( N_{ij} \) and \( r_i \) are as in Notation 2.1]. This case of Theorem 3.1 is well-known.

**Case** \( |T| = 1 \), but \( T \neq \{0\} \): Let \( T = \{t\} \). This case is referred to as orthogonality between a pair of factors through a third factor \( (A_t) \) in Bagchi [4], where \( A_t \) is a treatment factor. Examples of plans satisfying the conditions of Definition 3.2 may be found in Sect. 2 of the same paper. The case when \( T = \{B\} \) is considered in Bagchi [3], where the plan \( \mathcal{P} \) of Definition 3.2 is termed a “plan orthogonal through the block factor (POTB)”. By now many POTBs are available in the literature.

**Case** \( |T| = 2 \): In this case the plan may be termed as a plan orthogonal through a pair of factors. We shall construct a series of such plans in Sect. 4.1.

### 4 Construction of plans orthogonal through one or two factors

We shall now proceed to construct main effect plans, mostly for symmetric experiments. Most of the constructions are of recursive type, in the sense that from a given initial plan we generate a plan by adding runs or blocks and/or factors. We present the necessary notation here.
Notation 4.1  
(a) $s$ is a prime power. $F$ will denote the Galois field of order $s$. $F^m$ will denote the vector space of dimension $m$ over $F$. In the plans we consider, the set of levels of each factor is $F$ unless stated otherwise.

(b) For a subset $G$ of $F^m$ and $v \in F^m$, $G + v$ will denote the following subset of $F^m$.

$$G + v = \{ g + v : g \in G \},$$

where the addition is the vector addition in $F^m$.

(c) For a subset $V$ of $F^m$, $G + V$ will denote the set $\bigcup_{v \in V} (G + v)$.

(d) Consider a plan $P_0$ for an $s^m$ experiment with set of runs $\rho_0$. For $V \subset F^m$, $P_0 + V$ will denote the plan for the same experiment as $P_0$ having $\rho_0 + V$ as the set of runs. In particular, if $V = \{u_1, u \in F\}$, then $P_0 + V$ will be denoted simply by $P_0 \oplus F$.

4.1 Plans orthogonal through a pair of factors

We shall construct a series of plans orthogonal through a pair of factors (POTP). The condition for such an orthogonality is rather strong and so a POTP is rather rare. The following result provides a direction for the search of a POTP. The proof is by straightforward verification of (3.1) with $T = \{1, 2\}$.

Theorem 4.1 Consider an $s^m$ experiment. If the incidence matrices are as shown below, then $A_i$ and $A_j$ are orthogonal through the pair $(A_1, A_2)$, for every pair $(i, j)$, $i, j > 2$. Here $c$ is an integer.

$$N_{12} = 2c(J_s - I_s), \text{ and } N_{ij} = c((s - 2)I_s + J_s), \ i \neq j, \ (i, j) \neq (1, 2). \ (4.1)$$

Example Table 1 below is a plan for a $3^4$ experiment satisfying the conditions of Theorem 4.1 with $c = 1$.

We shall now present a general construction for a POTP for a symmetric experiment, using recursive method. The proof is in Appendix B.

Theorem 4.2 Suppose $q \equiv 0 \pmod{4}$ and $q$ is the order of a Hadamard matrix. If $s \equiv 3 \pmod{4}$ is a prime power, then there exists a POTP for a $s^{2q}$ experiment on $2qs(s - 1)/2$ runs.
**Example** Taking \( s = 7 \) and \( q = 4 \), we get a POTP \( P^* \) for a \( 7^8 \) experiment on 8.7.3 runs. Here \( C_0 = \{1, 2, 4\} \). The initial plan \( P_0 = [P \ 2P \ 4P] \), where \( P \) is the \( 8 \times 8 \) array in Table 2. \( P^* \) is obtained by adding \( i \) (mod 7) to the level of every factor in every run of \( P_0 \).

### Table 2 Array \( P \)

|    | \( A_1 \) | \( A_2 \) | \( A_3 \) | \( A_4 \) | \( A_5 \) | \( A_6 \) | \( A_7 \) | \( A_8 \) |
|----|---------|---------|---------|---------|---------|---------|---------|---------|
|    | 0       | 0       | 0       | 0       | 0       | 0       | 0       | 0       |
|    | 1       | 1       | 1       | 1       | -1      | -1      | -1      | -1      |
|    | 1       | 0       | 0       | -1      | -1      | 0       | 0       | 0       |
|    | 1       | 1       | 1       | 0       | 0       | -1      | 0       | -1      |
|    | 1       | 0       | 0       | 1       | 0       | 0       | -1      | 0       |
|    | 1       | 1       | 0       | 0       | 0       | 0       | -1      | -1      |
|    | 1       | 0       | 1       | 0       | 0       | -1      | 0       | -1      |
|    | 1       | 0       | 0       | 1       | 0       | -1      | -1      | 0       |

### 4.2 Plans orthogonal through the block factor

In this section, we construct main effect plans orthogonal through the block factor. It will be convenient to express this special case of Definitions 3.1 and 3.2 using Notation 2.5. It is interesting to note that Condition (4.2) is equivalent to equation (7) of Morgan and Uddin [26] in the context of nested row–column designs.

**Definition 4.1** (a) Fix \( i \neq j, \ i,j = 1, \ldots m \). The factors \( A_i \) and \( A_j \) are said to be **orthogonal through the block factor** if

\[
N_{ij} = L_i(D_k)^{-1}L'_j.
\]  

(4.2)

(b) A plan \( P \) is said to be a **plan orthogonal through the block factor (POTB)** if every (treatment) factor is orthogonal to every other one through the block factor.

**Remark 4.1** In the rest of this section, orthogonality will mean orthogonality through the block factor.

We shall construct POTBs by using recursive method. The necessary definition is presented below.

**Definition 4.2** Consider a plan \( P_0 \) for an \( s^m \) experiment laid out on \( b \) blocks \( \beta_1, \ldots \beta_b \). For \( V \subset F^m \), consider the plan for the same experiment as \( P_0 \) with \( b|V| \) blocks \( \beta_j + v, \ v \in V, \ 1 \leq j \leq b \). Here \( \beta_j + v \) is as in Notation 4.1 (b)

This new plan is said to be **generated from \( P_0 \) by adding \( V \)** and will be denoted by \( P_0 + V \). \( V \) is said to be the **generator**. As in Notation 4.1, if \( V = \{u1_m, u \in F\} \), then \( P_0 + V \) will be denoted simply by \( P_0 \oplus F \).
In a recursive construction, we need to find a suitable initial plan $P_0$ as well as a suitable generator $V$, so that the generated plan $P_0 + V$ satisfies one or more desirable properties. We now look for such a suitable generator.

**Notation 4.2** Consider a subset $V$ of $F^m$, viewed as a set of runs for an $s^m$ experiment.

(a) For every $i$, $1 \leq i \leq m$, $V_i$ will denote the multiset of size $|V|$, consisting of the levels of $A_i$ appearing in $V$. In symbol, $V_i = \{v : v = (v_1, \ldots, v_m) \in V\}$.

(b) Similarly, for every pair $(i, j), i \neq j$, $1 \leq i \leq m$, $V_{ij}$ will denote the multiset of level combinations of $A_i$ and $A_j$ appearing in $V$. That is $V_{ij}$ is the following multiset of $|V|$ members of $F \times F$.

$$V_{ij} = \{(v_i, v_j) : v = (v_1, \ldots, v_m) \in V\}.$$

The following useful result can be verified easily.

**Lemma 4.1** If every element of $F \times F$ appears a constant number of times in $V_{ij}$, then $A_i \perp A_j$ in $P_0 + V$, no matter what $P_0$ is.

We shall now see how we can enlarge the set of factors of a given plan, while keeping the number of blocks fixed.

**Definition 4.3** (a) Consider a plan $P$ as in Notation 2.5 and an integer $t$. Let $x_{ij}$ denote the $j$th run in the $i$th block of $P$. Consider the vector $y_{ij}$ obtained by juxtaposing the vector $x_{ij}$ $t$ times. Then, the plan on $b$ blocks with $y_{ij}$ as the $j$th run in the $i$th block, $1 \leq j \leq k_i, 1 \leq i \leq b$ is said to be obtained by taking the $t$-th power of $P$. The new plan will be denoted by $P^t$. We name the factors of $P$ and its power $P^t$ as in Notation 4.3.

Note that $P^t$ may not be an useful plan as it may not have enough degrees of freedom. However, it will be used as a building block for constructing useful plans, as shown below.

**Notation 4.3** Consider a plan $P$ having a set of factors $T_0 = \{A_1, \ldots, A_m\}$. The set of factors of $P^t$ will be named as $T = \bigcup_{i=1}^t T_i$, where $T_i = \{A_{i1}, \ldots, A_{im}\}$.

Combining Definitions 4.2 and 4.3, we get a recursive construction in which factors as well as blocks are added to the initial plan.

**Definition 4.4** Consider an initial plan $P_0$ for an $s^m$ experiment laid in $b$ blocks of sizes $k_1, \ldots, k_b$. Consider a $p \times q$ array $H = (h_{ij})_{1 \leq i \leq p, 1 \leq j \leq q}$, $h_{ij} \in F$. We now obtain a plan for an $s^{mp}$ experiment on $bp$ blocks using the array $H$ as follows. We first obtain $P^H_0$ following Definition 4.3.

Let $w_i = \{h_{i1}, h_{i2}, \ldots, h_{iq}\}$, $1 \leq i \leq p$ and $W_H = \{w_i, 1 \leq i \leq p\}$.

Our required plan $P$ is $P^H_0 + W_H$ and it will be denoted by $H \hat{\bowtie} P_0$. Symbolically,

$$P = H \hat{\bowtie} P_0 = P^H_0 + W_H.$$  \hspace{1cm} (4.3)
Here the factors of $P_0$ as well as $P$ are named in accordance with Notation 4.3. Our task is to find a suitable array $H$ so that the plan $H \otimes P_0$ satisfies certain desirable properties. The natural choice would be an orthogonal array of strength two [see Rao [31]]. We shall use a slightly modified version of it, so as to accommodate a few more factors.

**Notation 4.4** An orthogonal array of $m$ rows, $N$ columns with the entries from the set of integers modulo $s$ and strength $t$ will be denote by $OA(N, m, s, t)$. The array obtained by adding a row of all zeros (in the 0th position, say) to an $OA(N, m - 1, s, 2)$ will be denoted by $Q(N, m, s)$.

We get the following result from the recursive construction described in Definition 4.4.

**Theorem 4.3** Consider a plan $P_0$ for an $s^t$ experiment on $b$ blocks. Suppose an orthogonal array $OA(N, m - 1, s, 2)$ exists. Then there exists a plan $P$ with a set of $sm^t$ factors on $bN$ blocks with the following properties. Here the names of the factors are in accordance with Notation 4.3.

(a) If $P_0$ has $b_j$ blocks of size $k_j$, then $P$ has $Nb_j$ blocks of size $k_j$, $j = 1, \ldots, b$.

(b) Consider $i \neq j$, $1 \leq i, j \leq t$. If $A_i \perp A_j$ in $P_0$, then $A_{hi} \perp A_{hj}$ for every $h$, $0 \leq h \leq m - 1$.

(c) $A_{hi} \perp A_{gj}$, $g \neq h$, $0 \leq g, h \leq m - 1$.

**Proof** By hypothesis, the array $Q(N, m, s)$ exists. The required plan $P$ is $Q(N, m, s) \otimes P_0$. Properties (a) and (b) follow from the construction while (c) follows from Lemma 4.1. □

**Remark 4.2** The method of recursive construction presented in Definition 4.4 and the subsequent result in Theorem 4.3 are also used in Bagchi [5]. As shown in this theorem, the generated plan depends on the initial plan, apart from the generator. In Bagchi [5], orthogonal as well as non-orthogonal initial plans are chosen. The generated plans are thus POTBs and interclass orthogonal plans, respectively. In this paper, we begin with POTBs as initial plans and thus generate only POTBs, even though the same generator (as in the other paper) is used. See Theorem 4.6 and also Remark 4.4.

We shall proceed to construct new POTBs satisfying optimality properties. Now, regarding comparison between plans, one may look at the plan as a whole or look at its performance regarding one or more factors. In the first approach, the plans are compared in terms of the C-matrix of all the contrasts of interest. In the second approach, one chooses a factor $A_i$ and then compare the plans in terms of $C_{ri}$ [see (b) of Lemma 2.1], using an optimality criterion. This approach has been
used in Bose and Bagchi [8], where a plan was found to be E-optimal for two factors, while universally optimal for the other two. Modifying the celebrated Theorem of Kiefer [23] to be applicable to the context of plans with a block factor, we get the following.

**Theorem 4.4** Consider a class $\Pi$ of connected plans as described in Notation 2.5 and a plan $\mathcal{P}^* \in \Pi$. Let $t_{ij} = \lfloor k_j/s_i \rfloor$, $1 \leq j \leq b, 1 \leq i \leq m$. Consider the following conditions.

(a) For a fixed $i$, $\mathcal{P}^*$ satisfies the following.

(i) Each of the $s_i$ levels of $A_i$ appears $t_{ij}$ or $t_{ij} + 1$ times in the $j$th block, $1 \leq j \leq b$.

(ii) $A_i \perp A_j, j \neq i, j = 1, \ldots, m$.

(iii) $C_{i;j}$ of $\mathcal{P}^*$ is of the form $aI_{s_i} + bJ_{s_i}$.

(b) $\tilde{C}(\mathcal{P}^*)$ is of the form $aI_v$, where $v = \sum_{i=1}^{m} (s_i - 1)$.

We have the following.

(A) The condition (a) is sufficient for universally optimality of $\mathcal{P}^*$ for the inference on $A_i$.

(B) If a plan $\mathcal{P}^* \in \Pi$ satisfies the condition (a)(i) for every factor $A_i$ and also satisfies condition (b), then $\mathcal{P}^* \in \Pi$ is universally optimal in $\Pi$ for the inference on all the main effect contrasts.

A sketch of the proof: By Theorem 3.1 Condition (a)(ii) implies that $C_{i;j} = C_{i;j} + L_i D^{-1}_i L_i'$ [recall Notations 2.1, 2.3, 2.5 and equation (2.8)]. So, Conditions (a)(i) and (a)(ii) together ensure that $\text{Tr}(C_{i;j})$ is maximized. These together with Condition (a)(iii) say that Kiefer’s [23] sufficient condition for universal optimality is satisfied. So, the statement (A) is proved.

To prove statement (B) we note that Condition (b) says that $\tilde{C}(\mathcal{P}^*)$ is completely symmetric. Also, if condition (a)(i) is satisfied for every factor $A_i$ then that together with Condition (b) implies that $\text{Tr}(\tilde{C}(\mathcal{P}))$ is maximized by $\mathcal{P}^*$. Thus, these two conditions together imply Kiefer’s [23] sufficient condition for universal optimality. \hfill \Box

**Construction 1: New POTB for two-level factors.**

We present an initial plan $\mathcal{P}_0$ for seven factors on two blocks $B_1$ and $B_2$, each of size five in Table 3 below.

**Theorem 4.5** (a) The plan $\mathcal{P}_0$ is a POTB for a $2^7$, shown in Table 3 experiment.

(b) Moreover, it is E-optimal among all plans for the same experiment on the same set-up.
Proof (a) is proved by straightforward verification of (4.2) for each pair 
\((i, j) : i \neq j, i, j = 1, \ldots, 7\).

(b) Using (2.9), the C-matrix of the contrasts of \(P_0\) is obtained as \(\tilde{C} = 4I_7\). The rest follows from Theorem 3.1 of Jacroux and Kealy-Dichone [19]. [See also Definition 2.2]. \(\Box\).

Remark 4.3 The set-up of \(P_0\) is an example of Case 3 of Jacroux and Kealy-Dichone [19]. The E-optimal plan constructed in that paper in the same set-up has at most four factors, while \(P_0\) accommodates seven factors. The model used in Jacroux and Kealy-Dichone [19] is a bit different from the one given in Notation 2.4. This is due to the fact that the rows of the matrix \(O_i\) in their model are not orthonormal. If we use their model, then \(\tilde{C}\) would be \(8I_7\).

Next we derive an infinite series of POTBs from \(P_0\).

Theorem 4.6 (a) If there exists a Hadamard matrix of order \(h \geq 2\), then there is a POTB \(P_h\) for a \(2^h\) experiment in \(2h\) blocks of size 5 each.

(b) \(P_h\) is E-optimal among all plans for the same experiment on the same set-up.

Proof Suppose the hypothesis holds. By the well-known relation between orthogonal arrays of strength two and Hadamard matrices, [see Theorem 7.5 in Hedayat, Sloane and Stufken [13], for instance], an \(OA(h - 1, h, 2, 2)\) and hence \(Q(h, h, 2)\) exists.

(a) Let \(P_h = Q(h, h, 2) \diamond P_0\), where \(P_0\) is as in Table 3 (recall Definition 4.4). Now, Theorem 4.5 and Theorem 4.3 together imply that the plan \(P_h\) is a POTB.

(b) Theorem 4.3 together with Theorem 4.5 implies that the information matrix of \(P_h\) is \(4hI_{7h}\). Hence, by the same argument as in Theorem 4.5 the E-optimality of \(P_h\) follows. \(\Box\)

Remark 4.4 Let \(\tilde{P}_0\) denote the plan obtained by deleting the factor \(A_7\) from \(P_0\) of Table 3. In Theorem 6.3 of Bagchi [5], a series of plans for two-level factors is constructed. Deleting the factors \(D_1\) and \(D_2\) from the plan \(P_{A_2}\) of that series, we get a plan with six factors, say \(P_1\), in the same set-up as \(P_0\). Comparing these two plans,
we find the following. \(^\hat{P}_0\) satisfies Condition (b) but not (a)(i) of Theorem 4.4. On the other hand, \(P_1\) satisfies Condition (a)(i), but does not satisfy (b). Thus, neither of these plans satisfies the sufficient condition for universal optimality. This is the usual story in the search for optimal designs, which is why an universally optimal design is rare.

**Construction 2: New POTB for three-level factors.**

We construct is an initial plan \(P_0\) for three factors on three blocks of sizes 4, 4, 2. See Table 4 above.

**Theorem 4.7** (a) The plan \(P_0\) is a POTB for a \(3^3\) experiment and it is universally optimal among all plans for the same experiment in the same set-up.

**Proof** Since (4.2) holds for each pair 

\[(A_i, A_j), i \neq j, \ i, j = 1, 2, 3 \]

in \(P_0\), it is a POTB by Definition 4.1. Next, we note that the condition (a)(i) of Theorem 4.4 is satisfied. Again, by (2.9) we see that \(\hat{C}\) for \(P_0\) is \(6I_6\); so that condition (b) of the same theorem is also satisfied. Hence the result follows from the same theorem.

Next we generate an infinite series of POTBs from \(P_0\).

**Theorem 4.8** (a) If an OA\((N,m-1,3,2)\) exists, then there exists a connected POTB \(P_m\) for a \(3^m\) experiment in \(3N\) blocks. Among these \(3N\) blocks, \(2N\) are of size four, while the remaining \(N\) are of size 2 each.

(b) \(P_m\) is universally optimal among all plans for the same experiment on the same set-up.

**Proof** (a) By assumption \(Q(N, m, s)\) exists [see Notation 4.4]. Let \(P_m = Q(N, m, s)\hat{P}_0\), where \(P_0\) is as in Table 4.2.3. That the block sizes are as in the statement follows from the construction. Now, Theorem 4.3 together with Theorem 4.7 implies that the plan \(P_m\) is a POTB.

(b) Theorem 4.3 and 4.7 together with (2.9) implies that the information matrix of \(P_h\) is \(6N\ell_{6m}\); The rest are as in the proof of Theorem 4.7.
4.3 An Infinite Series of Plans with for an Asymmetrical Experiment, Based on Finite Field

We continue with Notation 4.1 and add a few notations and definitions to be used throughout the rest of this paper.

**Notation 4.5** $s = ht + 1$. $F^*$ will denote the set of nonzero elements of $F$.

(a) For $A \subset F$ and $p \in F$, $pA$ will denote the set $\{pa : a \in A\}$.

(b) For an integer $h$, $I_h = \{0, 1, \ldots, h - 1\}$. Throughout the rest of this paper, addition in the suffix will be modulo $h$, whenever the suffix is in $I_h$.

(c) $C_0 = \{\alpha_1, \ldots, \alpha_t\}$ will denote the multiplicative subgroup of order $t$ of $F^*$. For $i \in I_h$, $C_i$ will denote the $i$th coset of $C_0$. The cosets are ordered so that $C_iC_j = C_{i+j}$, $i, j \in I_h$.

(d) $\bar{C}_i = C_i \cup \{0\}, i \in I_h$.

(e) Further, $F^+$ will denote the set $F \cup \{\infty\}$. The following rule will define addition in $F^+$.

$$\infty + u = u + \infty = \infty, \quad u \in F. \quad (4.4)$$

We define a class of matrices.

**Notation 4.6** For $i \in I_h$, $M_i$ will denote the following $s \times s$ matrix with rows and columns indexed by $F$. $M_i(x, y) = \begin{cases} 1 & \text{if } y - x \in C_i, \\ 0 & \text{otherwise}. \end{cases}$

$M$ will denote the following partitioned matrix of order $hs \times hs$. $M = ((M_{i-j}))_{i,j \in I_h}$.

From the definition, the following is immediate.

$$\sum_{j \in I_h} M_i = J - I. \quad (4.5)$$

**Definition 4.5** Consider a plan $P_0$ (as in Notation 2.1) in which $F$ is the set of levels of each factor and $R$ is the set of runs. For any two factors $A_i, A_j$ ($i \neq j$, $1 \leq i, j \leq m$), the multiset $D(i, j) = \{x_j - x_i : x \in R\}$ is said to be the $(A_i, A_j)$ plot difference.

The following result is easy to verify.

**Lemma 4.2** Consider a plan $P_0$ as in Definition 4.5.

(a) Fix an ordered pair $(A_i, A_j)$ of factors. If an element $u$ of $F$ appears $p$ times in the $(A_i, A_j)$ plot difference, then for the plan $P_0 \oplus F$, $N_{ij}(x, y) = p$ if $y - x = u$.

(b) Suppose $P_0$ is a plan for an asymmetric experiment and the set of levels of $A_j$ is $F^+$. Then also the statement (a) holds for $u \in F^+$.
In what follows, we shall talk of various plans for the same experiment, i.e. for the same set of factors. Such a plan can be identified by the set of runs of it. Keeping this in mind, we introduce the following notation.

**Notation 4.7**

(a) \(\bigcup\) will denote the union counting multiplicity.

(b) Let \(J\) be a set of integers. For \(j \in J\), let \(\mathcal{P}_j\) be a plan for an experiment with factors \(A_1, \ldots, A_m\) and set of runs \(R_j\). Then, \(\bigcup_{j \in J} \mathcal{P}_j = \mathcal{P}\) (say) will denote the plan for the same experiment with set of runs \(\bigcup_{j \in J} R_j\).

(c) For the plan \(\mathcal{P}_j\), the \(A_i\) versus \(A_i/\text{uni}\) incidence matrix will be denoted by \(N_{ij}^{(j)}\), while \(N_{ii}^{(j)}\) will denote the same incidence matrix for \(\mathcal{P}\).

It is easy to see that

\[
\sum_{j \in J} N_{ii}^{(j)} = N_{ii}.
\]  \(
(4.6)
\)

Making use of the properties of the cosets \(C_j\)’s and (4.6) we extend Lemma 4.2 to get an useful result.

**Notation 4.8**

(a) Fix a vector \(q = (q_1, \ldots, q_g) \in F^g\) \((g \geq 1)\), \(q \neq 0\). For \(j \in I_h\) let \(\mathcal{P}_0^j\) be the plan with set of factors \(\{A_1, \ldots, A_g\}\) and \(R_j = qC_j\) as the set of runs. Let \(\mathcal{P}_j = \mathcal{P}_0^j \oplus F\) and \(\mathcal{P} = \bigcup_{j \in J} \mathcal{P}_j\).

(b) Let \(\mathcal{P}'_j\) be the plan with the same set of factors as \(\mathcal{P}_0^j\), but \(\bar{R}_j = q\bar{C}_j\) as the set of runs. Let \(\mathcal{P}'_j = \mathcal{P}'_0^j \oplus F\) and \(\mathcal{P}' = \bigcup_{j \in J} \mathcal{P}'_j\). Further, Let \(\tilde{N}_{ij}^{(j)}\) (respectively, \(\tilde{N}_{ii}^{(j)}\)) denote the \(A_i\) versus \(A_i/\text{ uni}\) incidence matrix in \(\mathcal{P}'_j\) (respectively, in \(\mathcal{P}\)).

The next lemma relates the incidence matrices of these plans.

**Lemma 4.3**

Fix \(i \neq i', i, i' = 1, \ldots, g\). We can say the following about the \(A_i\) versus \(A_i/\text{ uni}\) incidence matrices for the plans in Notation 4.8.

(a) \(N_{ij}^{(j)} = M_{j+i}\), if \(q_{i'} - q_i \in C_i\) and \(N_{ii'} = J - I\).

(b) \(\tilde{N}_{ij}^{(j)} = M_{j+i} + I_s\), if \(q_{i'} - q_i \in C_i\) and \(\tilde{N}_{ii'} = (h - 1)I + J\).

We are now in a position to construct the required series of POTBs.

**Theorem 4.9**

Suppose \(s\) is an odd prime power. Let \(t = (s - 1)/2\). Then we have the following.

(a) There exists a POTB \(\mathcal{P}^s\) for an \(s'(s + 1)\) experiment on \(2s\) blocks of size \(t + 1\) each.
(b) \( \mathcal{P}^* \) is universally optimal for the inference on each \( s \)-level factor.

(c) If \( s \equiv 3 \pmod{4} \), then \( \mathcal{P}^* \) is universally optimal for the inference on the \( s + 1 \) -level factor too.

**Proof** Note that \( s = 2t + 1 \), so that \( h = 2 \). Let \( \beta \in C_1 \) and \( \alpha_i \)'s as in Notation 4.5 (b). Consider the following pair of \( t + 1 \times 1 \) vectors \( R_C \) and \( R_\infty \) with co-ordinates indexed by \( \{1, \ldots , t\} \cup \{\infty\} \).

\[
\begin{align*}
\text{For } 1 \leq i \leq t, \quad &R_C(i) = \alpha_i, R_\infty(i) = 0, \\
&R_C(\infty) = \beta, R_\infty(\infty) = \infty.
\end{align*}
\]

Let \( R_0 = \{pR_C : p \in \tilde{C}_0\} \) and \( R_1 = \{\beta pR_C : p \in C_0\} \cup R_\infty \). (4.7)

Consider the initial plan \( \mathcal{P}^0 \) having \( R_0 \) and \( R_1 \) as blocks. So, \( \mathcal{P}^0 \) is a plan for \( t + 1 \) factors, say \( A_1, \ldots , A_t \), and \( A_\infty \) on two blocks of size \( t + 1 \) each. The set of levels for \( A_\infty \) is \( F^+ \), while that for others is \( F \). Let \( \mathcal{P}^* = \mathcal{P}^0 \oplus F \). Thus, \( \mathcal{P}^* \) is a plan for a \( s'(s + 1) \) experiment on \( 2s \) blocks of size \( t + 1 \) each. We need to show that \( \mathcal{P}^* \) satisfies the required properties.

To prove (a), we find the incidence matrices of \( \mathcal{P}^* \). From (4.7) we see that

\[
L_i = [M_0 + I \ M_1 + I], 1 \leq i \neq j \leq t.
\]

Here \( M_0 \) and \( M_1 \) are as in Notation 4.6. Again, the same equation, in view of Lemma 4.3 (b) implies that \( N_{ij} = I_{s} + J_{s}, j \neq i, 1 \leq i \neq j \leq t \). It follows that for \( 1 \leq i \neq j \leq t \), \( L_iL'_j = (t + 1)N_{ij} \), so that \( A_i \perp A_j \) by Definition 4.1.

Next, for a fixed \( i \), suppose \( \beta - \alpha_i \in C_j \). From (4.7) using Lemma 4.3 (a) we find that

\[
N_{i\infty}^{(0)} = [M_{i+1} + I \ 0_{s}].\]

Similarly, \( N_{i\infty}^{(1)} = [M_{i+1} + 1_{s}] \). Using (4.5) and (4.6) we get

\[
N_{i\infty} = \tilde{J}_{i<s+1}.\]

Therefore, \( N_{i\infty} \) is orthogonal to \( A_i \) in the usual sense, \( 1 \leq i \leq t \). Hence \( \mathcal{P}^* \) is a POTB and the proof of (a) is complete.

Next, fix \( i : 1 \leq i \leq t \). We see that \( L_i \) is the incidence matrix of a BIBD \( d_i \) with parameters \( (v = s, b = 2s, r = s + 1, k = t + 1, \lambda = t + 1) \). Thus, the conditions (a)(i) and (a)(iii) of Theorem 4.4 are satisfied. Since \( \mathcal{P}^* \) is a POTB, Condition (a)(ii) of the same theorem is also satisfied. Hence, by the same theorem, (b) is proved for \( A_i \) for every such \( i \).

Finally, we see that \( L_\infty \) can be expressed as \( L_\infty = [L_{\infty 0} \ L_{\infty 1}] \), where \( L_{\infty 0} = [M_{1} + I \ 0_{s}] \) and \( L_{\infty 1} = [M_{0} \ 1_{s}] \). Thus, if \( s \equiv 3 \pmod{4} \), then \( L_\infty \) is the incidence matrix of a BIBD \( d_2 \) with parameters \( (v = s + 1, b = 2s, r = s, k = t + 1, \lambda = t) \). Therefore, the conditions of Theorem 4.4 (a) are satisfied for the factor \( A_\infty \) also and hence (c) is proved by the same theorem.

We illustrate the construction with the help of the smallest non-trivial member of the series. Let \( s = 7 \). Then, \( C_0 = \{1, 2, 4\} \). We take \( \beta = 3 \). We present the two initial blocks of \( \mathcal{P}^* \) in Table 5.
Remark 4.5 A special case of Lemma 2.5 of Morgan and Uddin [26] yields a main effect plan for an $s^t$ experiment on $2s$ blocks of size $t + 1$ each. In Theorem 4.9 we have added a factor with $s + 1$ levels to that plan and obtained $P^*$, which satisfies properties similar to their plan, when $s \equiv 3 \pmod{4}$.

5 Multiway designs in nearly orthogonal setting

In this section and the next, we shall consider designs in a multiway heterogeneity setting, in which an experimental unit is subjected to more than one heterogeneity directions or blocking factors. As an example, let us consider a situation where a few medicines are being tried on a group of patients. Here the patients are the experimental units. There may be variations among the patients regarding their conditions like age, period of suffering, treatment received in the past, other health problems and so on. These aspects of a patient may be viewed as heterogeneity directions or blocking factors. Thus, given the set of experimental units, the blocking factors, the sets of their levels and the occurrence of the level combinations are all specified. This information is called the “experimental setting”. The experimenter can only choose the units to which a particular treatment will be given. We shall now list a few new notation to be used for this set-up.

Notation 5.1 (a) The number of experimental units will be denoted by $n$ and the number of blocking factors by $m$, unless stated otherwise. We shall use the term ‘unit’ and not ‘experimental unit’.

(b) The blocking factors will be denoted by $B_1, \ldots, B_m$. $M$ will denote the index for the set of all blocking factors, $S_i$ the set of levels of $B_i$ and $s_i = |S_i|, i \in M$.

(c) Consider an unit $u$, in which the level of $B_i$ is $u_i, 1 \leq i \leq m$ We shall view $u$ as an $m \times 1$ column vector having $u_i$ as the $i$th entry, $1 \leq i \leq m$.

(d) For $i \neq j$, $1 \leq i, j \leq m$, the $B_i$ versus $B_j$ incidence matrix is the $s_i \times s_j$ matrix $L_{ij}$. The $(p, q)$th entry of this matrix is $L_{ij}(p, q)$, which is the number of units such that $u_i = p$ and $u_j = q, p \in S_i, q \in S_j$.

(e) $V$ will denote the treatment factor and $v$ will denote the number of treatments. $r_i$ denotes the replication number of the treatment $i, 1 \leq i \leq v$, while the $v \times 1$ vector $r$ denotes the replication vector. $D_r$ denotes the diagonal matrix with diagonal entries same as those of $r$ in the same order. $N_i$ will denote the treatment versus $B_i$ incidence matrix, $i \in M$.

| Blocks | $B_0$ | $B_1$ |
|--------|-------|-------|
| Factors $\downarrow$ | $A_1$ | $1$ | $2$ | $4$ | $0$ | $3$ | $6$ | $5$ | $0$ |
| $A_2$ | $2$ | $4$ | $1$ | $0$ | $6$ | $5$ | $3$ | $0$ |
| $A_3$ | $4$ | $1$ | $2$ | $0$ | $5$ | $3$ | $6$ | $0$ |
| $A_\infty$ | $3$ | $6$ | $5$ | $0$ | $2$ | $4$ | $1$ | $\infty$ |
(f) By an “allotted” unit, we mean an \(m + 1 \times 1\) column vector \(\delta = (u, t)\), where \(u\) is a unit (as described in (c)) and \(t\) is the treatment allotted to \(u\). An \(m\)-way design \(d\) is a set of allotted units, that is a set of units, in each of which a treatment is allotted.

(g) The \(v \times 1\) vector of unknown effects of the treatments will be denoted by \(\tau\). \(\beta^i\) will denote the \(s_i \times 1\) vector of unknown effects of \(B_i\), \(i \in \mathcal{M}\). The least square estimate of \(\alpha\) will be denoted by \(\hat{\alpha}\), \(\alpha = \tau\) or \(\beta^i\).

By the C-matrix of a multiway design \(d\) we mean the coefficient matrix of the reduced normal equation for the treatment effects. This is nothing but the matrix \(C_{V,\mathcal{M}}\) (recall (2.3), which we shall refer to as \(C_d\). We shall consider only the ‘connected’ designs, i.e. those satisfying rank of \(C_d = v - 1\).

A design is said to be equireplicate, if the replication numbers of the treatments are all equal. For an equireplicate design, the constant replication number \((r)\) is said to be the replication number of the design.

In order to study the performance of a multi-way design \(d\) we need the C-matrix \(C_d\). This matrix has been derived in Mukhopadhyay and Mukhopadhyay [28], under the assumption that the set of units form an orthogonal array of strength two with variable number of symbols. This condition is not satisfied in any of the experimental set-up we consider here and so those results are not applicable. However, these settings satisfy certain other conditions, which may be described as ‘near orthogonality’, as the incidence matrix of every pair of blocking factors is of the form \(aI + bJ\). Here \(a\) and \(b\) are integers varying from one type of setting to another.

We now describe three different types of settings. In this description, \(m\) is an integer \(\geq 3\).

**Notation 5.2**

(a) By a setting of type 1 we mean a setting with \(n = s(s - 1)\) units, in which each blocking factor has \(s\) levels. Moreover, \(L_{ii'} = I_s - J_s\), \(i \neq i'\), \(1 \leq i, i' \leq m\).

(b) A setting of type 2 has \(n = s(p + s)\) for a positive integer \(p\), \(s\) levels for each blocking factor and \(L_{ii'} = pl_s + J_s\), \(i \neq i'\), \(1 \leq i, i' \leq m\).

(c) A setting of type 3 has \(n = s(s + 1)\) and \(m + 1\) blocking factors denoted by \(B_1, \cdots, B_\infty\). \(B_\infty\) has \(s + 1\) levels, while the other ones have \(s\) levels each. The incidence matrices are as follows:

\[
\begin{align*}
L_{ii'} &= I_s + J_s, i' \neq i, 1 \leq i, i' \leq m, \\
L_{i\infty} &= J_{s \times (s + 1)}, 1 \leq i \leq m.
\end{align*}
\]

Note that \(\mathcal{M} = \{1, \cdots, m\}\), if the setting is of type 1 or type 2, while for a setting of type 3, \(\mathcal{M} = \{1, \cdots, m\} \cup \{\infty\}\).

We proceed to obtain the reduced normal equation for the treatment effects. For the sake of compactness, we shall use the following notations in the statement of Lemma 5.2.

**Notation 5.3**

(a) \(C_r = C_{VV,0}\) [see (2.3)]. Thus, \(C_r\) is the \(v \times v\) matrix \(D_r = (1/n)rr'\), where \(D_r\) and \(r\) are as in Notation 5.1 (e).
(b) \( E_i = N_i - (s_i)^{-1} r \tilde{r}'_i \)

The following properties of \( E_i \)'s are easy to verify.

**Lemma 5.1** For every \( i \in \mathcal{M} \) the following hold.

(a) \( 1' E_i = 0 \).

(b) \( E_i E_i' = N'_i N_i - (s_i)^{-1} r r' \).

**Lemma 5.2** The C-matrix of a design \( d \) in a multi-way settings of one of the three types described in Notation 5.2 is given below.

(a) If the setting is of type 1 or type 2, then the C-matrix is

\[
C_d = C_r - \frac{1}{s} \sum_{i=1}^{m} E_i E_i' + \frac{p}{su} S_B S_B', \quad \text{where} \quad S_B = \sum_{i=1}^{m} E_i, \quad u = s + mp
\]

and \( p = -1 \) if the setting is of Type 1.

(b) The C-matrix for a design in a setting is of type 3 is as given below.

\[
C_d = C_r - \frac{1}{s} \sum_{i \in \mathcal{M}} E_i E_i' + \frac{1}{su} S_B S_B',
\]

where \( u = s + m, S_B \) as in (a) above and \( \mathcal{M} \) is as stated after Notation 5.2.

The proof is in Appendix A.

6 Optimality of the Duals of POTBs

In this section, we shall study three multi-way designs, two of which are duals of existing main effect plans. So, we define what we mean by the dual of a main effect plan.

**Definition 6.1** Consider a main effect plan \( \mathcal{P} \) for an \( s_1 \times \cdots \times s_m \) experiment on \( n \) runs laid out on \( b \) blocks. By the dual \( d(\mathcal{P}) \) of \( \mathcal{P} \), we mean the following design in a multi-way heterogeneity set-up. \( d(\mathcal{P}) \) has \( b \) treatments, to be tested on \( n \) experimental units. The units are subjected to \( m \) heterogeneity directions (blocking factors), say \( B_1, \cdots, B_m \), \( B_i \) corresponding to the factor \( A_i \) of \( \mathcal{P} \), thus having \( s_i \) levels, \( 1 \leq i \leq m \). A unit \( u \) of \( d(\mathcal{P}) \) in which \( B_i \) is at level \( u_i, i \in \mathcal{M} \) corresponds to the run \((u_1, \cdots, u_m)'\) of \( \mathcal{P} \). If \( u \) is in the \( j \)th block of \( \mathcal{P} \), then \( d(\mathcal{P}) \) allocates treatment \( j \) to \( u, 1 \leq j \leq b \).
Now onwards we assume that $s$ is a prime power. We follow Notation 4.5. We need some more notation before we proceed to the description of the multi-way designs of interest.

**Notation 6.1**
(a) The set of treatments for all the designs we discuss below is given by $\{(i,x), x \in F, i \in I_h\}$. Thus, $\nu = hs$.

(b) All factors except $B_\infty$ in Setting of type 3 have $F$ as the set of levels, while for $B_\infty$ it is $F^+ = F \cup \{\infty\}$.

(c) For $i \in M$, the treatment-versus $B_i$ incidence matrices will be represented as

$$N_{ij} = \begin{cases} N_{i0} & \text{if } 1 \leq i \leq m, \\ \vdots & \\ N_{i, h-1} & \text{if } i = \infty. \end{cases}$$

Thus, for $j \in I_h$, $N_{ij}$ is of order $s \times s$, $1 \leq i \leq m$ and $N_{i, \infty}$ is of order $s \times s + 1$.

### 6.1 The designs $d_1^*$, $d_2^*$ and their properties:

In this section, we shall study the properties of two multi-way designs, which are duals of main effect plans constructed in Morgan and Uddin [26].

Corollary 2.3 of Morgan and Uddin [26] has constructed a universally optimal main effect plan, say $d_0^*$, for an $sh$ experiment on $hs$ blocks of size $t$ each (here $s$, $h$ and $t$ are as in Notation 4.5). The dual of $d_0^*$ is an $h$-way design, named $d_1^*$. We present a description of $d_1^*$ in terms of our notation.

**The set-up of $d_1^*$:** Let $\delta_i \in C_i$, $i \in I_h$. Thus, for $i \neq i'$, $\delta_i$ and $\delta_{i'}$ are from different cosets of $C_0$. Let $R_\delta$ be the $h \times 1$ vector with $\delta_i$ as the $i$th entry, $0 \leq i \leq h - 1$.

Let $R_j^1 = \{qR_\delta : q \in C_j\}$, $j \in I_h$, $R^1 = \bigcup_{j \in I_h} R_j^1$ and $U^1 = \{R + a1_h : R \in R^1, a \in F\}$. $U^1$ is the set of all units. Clearly, $R^1$ is of size $ht = s - 1$ and $U^1$ is of size $s(s - 1)$. There are $h$ blocking factors $B_0, \ldots, B_{h-1}$.

**The design $d_1^*$:** For $a \in F$ and $j \in I_h$, $d_1^*$ allocates treatment $(j, a)$ to the unit $u$ of the form $u = a1_h + R$, where $R \in R_j^1$. This is done for every $a \in F$, $j \in I_h$. We shall now obtain the properties of $d_1^*$, in terms of Notations 5.2 and 4.6.

**Lemma 6.1**
(a) $d_1^*$ is a design in a setting of type 1 with $m = h$.

(b) The treatment versus blocking factor incidence matrices of $d_1^*$ are as follows.
\[ N_i = \begin{bmatrix} M_i \\ \vdots \\ M_{i+h-1} \end{bmatrix}. \quad (6.1) \]

**Proof** For \( j \in I_h \), Let \( \tilde{R}_j^1 \) denote the set of \( h + 1 \times 1 \) vectors obtained from \( R_j^1 \) by juxtaposing 0 in the \( h + 1 \)th position of each member of \( R_j^1 \). Let \( \mathcal{P}_j^0 \) denote the plan for an \( s^{h+1} \) experiment having the set of factors \( \{A_i : i \in I_h \} \cup \{A_h\} \) and \( \tilde{R}_j^1 \) as the set of runs. Let \( \mathcal{P}_j = \mathcal{P}_j^0 \oplus F \) and \( \mathcal{P} = \bigcup_{j \in I_h} \mathcal{P}_j \). Fix \( i \neq i', 1 \leq i, i' \leq h \). Applying Lemma 4.3(a) to \( \mathcal{P} \), we get \( N_{ii'} = J - I \). But by the description of the setting, \( L_{ii'} \) of \( d_1^* \) is nothing but \( N_{ii'} \) of \( \mathcal{P} \). Thus, (a) follows from the definition of setting of type 1.

Next, to prove (b), we fix \( j \in I_h \). Applying (a) of Lemma 4.3 to \( \mathcal{P}_j \) we see that \( N_{ji} = M_{j+i} \). Now, from the description of \( d_1^* \), using Notation 6.1 (c) we see that \( N_{ij} = N_{ji} \). Hence the result follows. \( \square \)

In the study of the properties of \( d_1^* \), as well as its performance, the following \( v \times s \) matrix plays an important role.

\[ H_d = \sum_{i=1}^{m} N_i. \quad (6.2) \]

One can verify that for an equireplicate design in a setting of type 1 with replication number \( r \), the following holds.

\[ S_B' S_B = H_d H_d' - \frac{m^2 r^2}{s} J_v. \quad (6.3) \]

We collect all the information on \( d_1^* \) obtained so far. We shall use the following compact notations, apart from Notation 4.6 and (6.2).

\[ N = [ N_0 \ldots N_{h-1} ] \quad \text{and} \quad E_B = [ E_0 \ldots E_{h-1} ]. \quad (6.4) \]

Here \( E_i \)'s are as in Notation 5.3 (b). We see that \( N \) and \( E_B \) are of order \( v \times v \). The following relation holds.

\[ E_B E_B' = N N' - \frac{h r^2}{s} J_v. \quad (6.5) \]

For an integer \( t \)

\[ \text{let } K_t \text{ denote the symmetric idempotent matrix } I_t - (1/t) J_t. \quad (6.6) \]

**Lemma 6.2** \( d_1^* \) satisfies the following properties.

(a) \( d_1^* \) is equireplicate with replication number \( r = t \).

(b) \( C_{d_1} = r K_v - \frac{1}{s} M M' - \frac{1}{s(s-h)} H_d H_d' + \frac{hr^2}{s^2} J_v. \)
(c) The spectrum of $C_{d_1^*}$ is $r^{h-1}(r - \frac{1}{s-h})t^{t-1}(r - 1)^{(h-1)(s-1)}1^1$.

Proof (a) follows from (6.1). (b) follows from Lemma 5.2 (a) in view of (6.1),(6.2) and (6.4). Proof of (c) is in Appendix B.

We shall now obtain another multiway design $d_2^*$. Among the series of universally optimal main effect plans constructed in Lemma 2.6 of Morgan and Uddin [26] one is a plan, say $\rho_2$, for a $s^t$ experiment on $hs$ blocks of size $t + 1$ each (here $s$, $h$ and $t$ are as in Notation 4.5 ). We shall study the dual of $\rho_2$, named $d_2^*$. We describe it using our notation.

The set-up of $d_2^*$: Let $R_0$ be the $t \times 1$ vector with $a_i$ as the $i$th entry, $1 \leq i \leq t$, where $a_i$'s are as in Notation 4.5 (b).

Let $R_j^2 = \{qR_0 : q \in \tilde{C}_j\}$, $j \in I_h$. $U_j^2 = \{R + a1 : R \in R_j^2, a \in F\}$ and $U^2 = \bigcup_{j \in I_h} U_j^2$.

Then, $U^2$ is the set of all units. The size of $U^2$ is $s(s + h - 1)$. Note that the 0 vector is a member of $R_j^2$ for every $j \in I_h$. Thus, for every $a \in F$, the unit $a1$, is a member of $U_j^2$ for every $j \in I_h$ and so it appears $h$ times in the multiset $U^2$. These units with identical level combinations receive different treatments in $d_2^*$ as we shall see below. There are $t$ blocking factors $B_1, \ldots, B_t$.

The design $d_2^*$: $d_2^*$ is the design which allocates treatment $(j, a)$ to an unit $u = R + a1$, if $R \in R_j^2$, $j \in I_h$, $a \in F$. Let us proceed to study the properties of $d_2^*$.

Lemma 6.3 (a) $d_2^*$ is a design in a setting of type 2 with $m = t$ and $p = h - 1$.

(b) For every $i = 1, \ldots, t$, $N_i = M_0 + I$,

\[
\begin{bmatrix}
M_0 + I \\
M_1 + I \\
\vdots \\
M_{h-1} + I
\end{bmatrix}
\]

where $M_j s$ are as in Notation 4.6.

Proof For $j \in I_h$. Let $\tilde{R}_j^2$ denote the set of $h + 1 \times 1$ vectors obtained from $R_j^2$ by juxtaposing 0 in the $(t + 1)$th position of each member of $R_j^2$. Let $\tilde{P}_j^0$ denote the plan for an $s^{t+1}$ experiment having set of factors $\{A_1, \ldots, A_t\} \cup \{A_0\}$ and $\tilde{R}_j^2$ as the set of runs. Let $P_j = P_j^0 \oplus F$ and $P = \bigcup_{j \in I_h} P_j$.

Fix $i \neq i'$, $1 \leq i, i' \leq t$. Using Lemma 4.3 (b) on the plan $P$ we find that $N_{ii'} = (h - 1)L_i + J_i$. Since $L_{ii'}$ of $d_2^*$ is the same as $N_{ii'}$ of $P$ for every such $(i, i')$ the result in (a) follows from the definition of setting of type 2 [recall Notation 5.2].

Next, fix $i$, $1 \leq i \leq t$. For $j \in I_h$, consider the plan $P_j^0$. By Lemma 4.3 (b) $N_{i0}^{(j)} = M_j + I_i$. But by the description of $d_2^*$, $N_{ij} = N_{i0}^{(j)}$, $j \in I_h$. Hence the result in (b) follows in view of Notation 6.1 (c).

We present the following well-known result for ready reference.
Lemma 6.4  If $N$ is the incidence matrix of a BIBD with parameters $(v, b, r, k, \lambda)$, then the spectrum of $N'N$ is $0^{b-r}(r-\lambda)^{-1}(rk)^1$.

We can say the following about $d_2^*$. Here $U = 2s - t - 1$

Lemma 6.5  (a) $d_2^*$ is equireplicate with $r = t + 1$.

(b) The C-matrix of $C_{d_2^*} = rK_v - \frac{t}{u}(N_2'N_2 - \frac{r^2}{s}J_{svv})$, where $N_2$ is the incidence matrix of a BIBD with parameters $(v = s, b = hs, r = h(t + 1), k = t + 1, \lambda = t + 1)$.

(c) The spectrum of $C_{d_2^*}$ is $r^v s - \frac{1}{h-1}(t+1)^{s-1}0^1$.

Proof  (a) follows from Lemma 6.3. (b) follows from Lemma 5.2 (a) in view of Lemma 6.3. (c) follows from (b) in view of Lemma 6.4.

6.2 Optimality of $d_1^*$ and $d_2^*$

We define a class of competing designs in the setting of $d_1^*$, in terms of the total $(H)$ of the incidence matrices (recall (6.2)).

Definition 6.2  An $m$-way design $d$ is said to be totally binary if the entries of $H_d$ are 0 or 1. The class of all equireplicate and totally binary $m$-way designs in the setting of $d_1^*$ will be denoted by $\mathcal{D}_r^B$.

We note that $d_1^*$ is totally binary and any design in $\mathcal{D}_r^B$ shares the following properties with $d_1^*$.

Lemma 6.6  For a design $d \in \mathcal{D}_r^B$, the following hold.

(a) For every $i \in I_h$, the entries of $N_i$ are 0 or 1.

(b) By permuting the rows and columns if necessary, $H_d$ can be reduced to $H_d = 1_h \otimes (J_s - I_s)$.

(c) The spectrum of $S_B S_B'$ is $0^{s+1} h^{s-1}$. [Here $S_B$ is as in Lemma 5.2 (a)].

Proof  Fix a totally binary design $d$. Since the entries of each $N_i$ are non-negative, (a) follows from the definition of a totally binary design.

Next, let $\hat{H} = J_{hsxs} - H_d$. Since $d$ is totally binary, $\hat{H}$ is a 0, 1 matrix with exactly one entry 1 in every row and exactly $h$ entries 1 in every column. So, $\hat{H} = 1_h \otimes I_s$ up to permutation of the rows and columns. Hence (b) follows.

Finally, substituting $m = h$ and $r = t$ in (6.3) we get
So, (b) implies that \( S_B S'_B = J_h \otimes K_s \). Now (c) is immediate. \( \square \)

In view of Lemma 5.2, Lemma 6.6 implies that

\[
tr(C_d) = tr(C_{d_i}^*), \ d \in \mathcal{D}_r^B.
\] (6.8)

**Remark 6.1** Now onwards we shall assume that for every design \( d \) in \( \mathcal{D}_r^B \), the rows and columns of each \( N_i \) is permuted (if necessary) in accordance with the permutation (if any) used in Lemma 6.6.

We proceed towards obtaining optimality property of \( d_1^* \).

**Theorem 6.1** \( d_1^* \) is M-optimal in \( \mathcal{D}_r^B \), the class of all connected equireplicate and totally binary designs in the Setting of \( d_1^* \).

Towards the proof of this theorem, we fix a design \( d \in \mathcal{D}_r^B \) and study certain properties of the matrices in the expression for \( C_d \). We shall write \( C_d \) in a more compact form using (6.4).

\[
C_d = rK_v - \frac{1}{s} E_B E'_B - \frac{1}{s u} S_B S'_B, \text{ where } u = s - h.
\] (6.9)

We shall use a pair of lemmas, the proofs of which are in Appendix A.

**Lemma 6.7** The sum of \( s - 1 \) smallest positive eigen values of \( E_B E'_B \) is \( \leq s - 1 \).

**Lemma 6.8** \( \mu(C_d) \) satisfies the following.

(0) \( \mu_0(C_d) = 0 \).

(i) \( \mu_{r-i}(C_d) = r, \ 1 \leq i \leq h - 1 \).

(ii) \( \sum_{i=0}^{s-2} \mu_{r-h-i}(C) \geq (s - 1)(r - \frac{1}{s-h}) \).

**Proof of Theorem 6.1** Let us take a pair of \( v - h \times 1 \) vectors \( x \) and \( y \), where \( y_i = \mu_i(C_d) \) and \( x_i = \mu_i(C_{d_i}^*) \), \( 1 \leq i \leq v - h \). The information in Lemma 6.8 and Lemma 6.2 (c) says that the other eigenvalues of \( C_d \) are equal to the corresponding ones of \( C_{d_i}^* \). This, in view of (6.7), shows that Lemma 2.4 is applicable to the vectors \( x \) and \( y \). Using the bound obtained in Lemma 6.8 (ii) and applying Lemma 2.4 we find that \( y \) is M-worse than \( x \). Hence the proof is complete. \( \square \)

We now proceed towards studying optimality aspects of \( d_2^* \).
Theorem 6.2 $d_2^*$ is M-optimal in $\mathcal{D}_2$, the class of all connected equireplicate designs in the setting containing $d_2^*$.

Towards proving this theorem, we need a few results, the proofs of which are in Appendix A.

Lemma 6.9 Consider $p \times q$ matrices $A_1, \cdots, A_k$. Let $H = \sum_{i=1}^k A_i$. Then, the following hold.

(a) $k \sum_{i=1}^k A_i A_i' - HH' = \sum_{i,j=1}^k (A_j - A_i) (A_i - A_j)'$.

(b) $\sum_{i=1}^k A_i A_i' \geq (1/k)HH'$.

(c) Suppose each $A_i$ is an integer matrix with constant row sum $r < q$. Then, $Tr(HH') \geq k^2 pr$.

Lemma 6.10 Let $A$ be an $n \times n$ n.n.d matrix with row sum 0, rank $\leq \rho$ and trace $\geq T$. Let $C = dK_n - A$, where $d \geq T / \rho = a$ (say) and $K_n$ is as in (6.8). Let $\gamma \in (\mathbb{R}^+)^n$ be the vector given by $\gamma_0 = 0, \gamma_i = d - a$ for $1 \leq i \leq \rho, \gamma_i = d$ for $\rho + 1 \leq i \leq n - 1$. Then, $\gamma$ is M-better than $\mu(C)$.

Proof of Theorem 6.2 Let $d \in \mathcal{D}_2$. Then, $C_d$ is as given in (a) of Lemma 5.2 with $m = \tau, p = h - 1$. As $d$ is equireplicate with replication number $r$, $C_r = rK_r$. Thus, using the fact that $u = s + (h - 1)\tau$, $C_d$ can be written as

$$C_d = rK_v - \frac{1}{u} \sum_{i=1}^\tau E_i E_i' - \frac{h - 1}{su} (\tau \sum_{i=1}^\tau E_i E_i' - S_B S_B').$$

By Lemma 6.9 (a) the third term is $-\frac{h - 1}{su} \sum_{j<i} (E_j - E_i) (E_i - E_j)'$, which is $\leq 0$ by part (b) of the same Lemma. Again, by the same part of the same lemma, the second term of $C_d$ is $\leq -\frac{1}{tu} S_B S_B'$. All these together says that

$$C_d \leq C_{d_1} = rK_v - \frac{1}{tu} S_B S_B'.$$

Now, $C_{d_1}$ is of the form of $C$ in Lemma 6.10 with $n = hs = v, d = r, A = (1/tu)S_B S_B', \rho = s - 1$. Also, in view of Lemma 5.1 (a) the row sum of $A$ is 0. By Lemma 6.9 (c) and (6.3) $Tr(A) \geq (h - 1)(s - 1)(\tau + 1)u/t$. Therefore, by Lemma 6.10, $\mu(C_{d_1})$ is M-worse than $\gamma$, where $\gamma_0 = 0, \gamma_i = r - (h - 1)(\tau + 1)u/t, 1 \leq i \leq s - 1, \gamma_i = r, s \leq i \leq v - 1$. But from Lemma 6.5 we see that $\gamma = \mu(C_{d_2}^*)$. Since $C_d \leq C_{d_1}$, the result follows from Lemma 2.3.
6.3 Construction and optimality of a new multiway design

In this section we construct a multi-way design $d^n_3$ and prove its optimality property. We assume that $s \equiv 3 \pmod{4}$. We follow Notation 4.5 with the extra assumption that $h = 2$.

The construction of $d^n_3$ is based on a subset $W$ of size $w = (s - 3)/4$ of $C_0$ and an $1-1$ function $f$ from $W$ to $C_1$. The proof of the existence of $W, f$, their relevant properties and other details are in Appendix B.

We shall now construct the design $d^n_3$.

**The set of units:** Let us order the members of $W$ as $W = \{ \xi_1, \cdots \xi_w \}$. Let $\beta \in C_1$.

Let us consider three $(w + 1) \times 1$ vectors $R_W, R_f$ and $R_\infty$, the co-ordinates of which are indexed by $\{1, \cdots w\} \cup \{\infty\}$. We also define a pair of sets of vectors defined in terms of these three vectors.

\[
\begin{align*}
\text{For } 1 \leq i \leq w, & \quad R_W(i) = \xi_i, \quad R_f(i) = \beta f(\xi_i), \quad R_\infty(i) = 0, \\
R_W(\infty) = 1, & \quad R_f(\infty) = \beta, \quad R_\infty(\infty) = \infty.
\end{align*}
\]

(6.10)\[R_3^0 = \{ qR_W : q \in \tilde{C}_0 \} \text{ and } R_3^3 = \{ qR_f : q \in C_0 \} \cup R_\infty.\]

For $l = 0, 1$, let $U_3^l = \{ R + \alpha 1_{w+1} : R \in R_3^1, \alpha \in F \}$. Then, $U_3^3 = \bigsqcup_{l=0,1} U_3^l$ is the set of all units. Thus, there are $w + 1$ blocking factors $B_1, \cdots B_w$ and $B_\infty$.

The set of levels of the factor $B_\infty$ is $F^+$, while the set of levels of all others is $F$. Clearly $U_3^3$ is of size $s(s + 1)$. Moreover, for every $\alpha \in F$, the unit $\alpha 1_{w+1}$ is a member of $U_3^0$ as well as $U_3^3$ and thus appears twice in the multiset $U_3^3$. Such a pair of units receives different treatments in $d^n_3$.

**The design $d^n_3$:** $d^n_3$ is the design which allocates treatment $(l, \alpha)$ to an unit $u = R + \alpha 1_{l}$, if $R \in U_3^l, l = 0, 1, \alpha \in F$.

**Lemma 6.11** (a) $d^n_3$ is a design in a setting of Type 3 with $(s + 1)/4$ blocking factors. It is equireplicate with replication number $r = t + 1 = (s + 1)/2$.

(b) For every $i = 1, \cdots, w$, $N_i = \begin{bmatrix} M_0 + I \\ M_0 + I \end{bmatrix}$.

(c) $N_\infty = \begin{bmatrix} M_0 + I & 0_{s \times 1} \\ M_1 & 1_s \end{bmatrix}$. Here $M_0$ and $M_1$ are as in Notation 4.6 with $h = 2$.

Proof is in Appendix B.

Now, the following properties of $d^n_3$ are immediate.

**Corollary 6.1** (a) For $1 \leq i \leq w$, $N'_i$ is the incidence matrix of a BIBD with parameters $(v = s, b = 2s, r = s + 1, k = t + 1, \lambda = t + 1), 1 \leq i \leq w$.

(b) $N'_\infty$ is the incidence matrix of a BIBD with parameters $(v = s + 1, b = 2s, r = s, k = t + 1, \lambda = t)$. 
We shall now proceed to study the C-matrix of $d^*_3$. Towards that, we need the important concept of adjusted orthogonality and a related result. Following Eccleston and Russel [12] we define the following in terms of quantities in Notation 2.3.

**Definition 6.3** For three distinct factors $A_i, A_j, A_k$ (irrespective of whether they are treatment factors or block factors), we say that $A_i$ and $A_j$ are **adjusted orthogonal with respect to** $A_k$ if $\text{Cov}(Q_{i;k}, Q_{j;k}) = 0$.

If $A_k$ is equireplicate with replication number $r$, say, then

$$\text{Cov}(Q_{i;k}, Q_{j;k}) = N_{ij} - r^{-1}N_{ik}N_{kj}.$$ 

Therefore, $A_i$ and $A_j$ are adjusted orthogonal with respect to $A_k$, if and only if $N_{ik}N_{kj} = rN_{ij}$.

Shah and Eccleston [33] proved a few interesting properties of a row–column design, in which the factors row and column are adjusted orthogonal with respect to the treatment factor. Those results can be easily generalized to a multiway heterogeneity setting. Among those, we present one result, restricting to the equireplicate class.

**Lemma 6.12** Consider an equireplicate multiway design $d$. Suppose the pair of blocking factors $B_i, B_j$ satisfy $M_{ij} = J_{s \times s}$. Consider the $v \times v$ matrices $T_i = N_iN'_i$ and $T_j = N_jN'_j$. If $B_i$ and $B_j$ are adjusted orthogonal with respect to $V$, then the following hold.

Suppose $x'1_v = 0$ and $x$ is an eigenvector of $T_i$ with nonzero eigenvalue. Then $T_jx = 0$.

Proof is in Appendix A.

We are now in a position to get the crucial information on $d^*_3$.

**Theorem 6.3** (a) $C_{d^*_3} = rK_v - \frac{w}{s+w}E_1E'_1 - \frac{1}{s}E_{\infty}E'_{\infty}$.

(b) The spectrum of $C_{d^*_3}$ is $(\frac{rs}{s+w})^{-1}(\frac{r(s-1)}{s})^{v-1}$.

**Proof** (a) follows from Lemma 5.2 (b) and Lemma 6.11. So we go to the proof of (b).

By Lemma 5.1 we get $E_1E'_1 = N_1N'_1 - \frac{r^2}{s}J_v$, and $E_{\infty}E'_{\infty} = N_{\infty}(N_{\infty})' - \frac{r^2}{s+1}J_v$. By Lemma 6.4 and Corollary 6.1, we find that the spectrum of $E_1E'_1$ is $0^n r^{n-1}$ (as $r = s - t$), and the spectrum of $E_{\infty}E'_{\infty}$ is $r^n 0^{v-s}$.

Next, by definition of $M_0$ and $M_1$ given in Notation 4.6, $(M_0 + I)1_s = (t + 1)1_s$ and $M_0 + I + M_1 = J_s$. So, from Lemma 6.11 we see that $N_iN_{\infty} = (M_0 + I)'J_{s \times s+1} = (t + 1)J_{s \times s+1} = rL_{\infty}$ for $1 \leq i \leq w$. Therefore, by Definition 6.3 the factors $B_i$ and $B_{\infty}$ are adjusted orthogonal with respect to the treatment factor for $1 \leq i \leq w$. Therefore, (b) follows from (a), in view of Lemma 6.12.

Finally, we obtain the optimality result.
Theorem 6.4 $d_3^w$ is $M$-optimal in $\mathcal{D}_3$, the class of all connected equireplicate designs in the setting of $d_3^w$.

Proof Let $d \in \mathcal{D}_3$. From Lemma 5.2 (b), using the fact that $d$ is equireplicate with replication $r$, we get

$$C_d = rK_v - \frac{1}{s} \sum_{i=1}^{w} E_i E_i' + \frac{1}{su} S_B S_B',$$

where $u = s + w$ and $S_B$ is as defined in the same lemma.

By arguments similar to those used in Theorem 6.2 we see that $C_d \leq C_{d_1}$, where

$$C_{d_1} = rK_v - \frac{1}{u} \sum_{i=1}^{w} E_i E_i' - \frac{1}{s} E_\infty E_\infty'.$$

Again by Lemma 5.1 (b) and Lemma 6.9 (a) we find that $C_{d_1} \leq C_{d_2}$, where

$$C_{d_2} = rK_v - \frac{1}{wu} S_B S_B'.$$

Now, $C_{d_2}$ is of the form of $C$ in Lemma 6.10 with $n = 2s, A = (wu)^{-1} S_B S_B'$, $d = r$, $\rho = s - 1, T = wr(s - 1)/u$. So, $a = wr/u = wr/(s + w)$. Therefore, by the same lemma, $\mu(C_{d_2})$ is $M$-worse than $r$. As $C_{d_1} \leq C_{d_2}$, $\mu(C_{d_1})$ is $M$-worse than $r$. In particular, $\sum_{j=1}^{l} \mu_j(C_{d_1}) \leq l(r - a), 1 \leq l \leq s - 1$. Next, one can check that $Tr(C_{d_1}) \leq (s - 1)(r - a) + r(s - 1)$. Let $\delta$ be a $v - 1 \times 1$ vector such that $\delta_i = r - a$ if $1 \leq i \leq s - 1$ and $r(s - 1)/s$ if $s \leq i \leq v - 1$. It follows that $\mu(C_{d_1})$ is $M$-worse than $\delta$. But, by Theorem 6.3 (b) $\delta = \mu(C_{d_3})$. Since $C_d \leq C_{d_3}$ the result follows. \qed

We present the smallest member of this series in Table 6 below.

**Remark 6.2** Theorem 6.4 is an extension of the result of Bagchi and Shah [7]. Specifically, if we forget $w - 1$ of the set of blocking factors $\{B_1, \ldots, B_w\}$, then $d_3^w$ reduces to a row–column design, say $d^*$, the optimality property of which follows from the main result of Bagchi and Shah [7]. However, $d^*$ exists for every odd prime power $s$. 

\begin{table}[h]
\centering
\caption{d_3^s for $s = 11$}
\begin{tabular}{cccccccccccc}
\hline
& $B_1$ & 3 & 9 & 5 & 4 & 1 & 0 & 6 & 7 & 10 & 8 & 2 & 0 \\
& $B_2$ & 4 & 1 & 3 & 9 & 5 & 0 & 8 & 2 & 6 & 7 & 10 & 0 \\
& $B_\infty$ & 1 & 3 & 9 & 5 & 4 & 0 & 2 & 6 & 7 & 10 & 8 & $\infty$ \\
\hline
Treatment & $(0,0)$ & $(0,0)$ & $(0,0)$ & $(0,0)$ & $(0,0)$ & $(1,0)$ & $(1,0)$ & $(1,0)$ & $(1,0)$ & $(1,0)$ & $(1,0)$ & $(1,0)$ & $(1,0)$ \\
\end{tabular}
\end{table}
i.e. it exists even when $s \equiv 1 \pmod{4}$. [See Preece, Wallis and Yucas [30] and Nilsson and Cameron [29] for more details].

**Remark 6.3** In Theorem 4.9 an MEP $\mathcal{P}^*$ is constructed for an $s^{(s-1)/2}(s+1)$ experiment on $2s$ blocks of size $(s+1)/2$ each. Let $d_4$ denote the dual of $\mathcal{P}^*$, which is a multi-way design with $(s+1)/2$ blocking factors, one having $s+1$ levels and others having $s$ levels. Although $d_4$ satisfies properties similar to those of $d_3^*$ given in Theorem 6.3, the $s+1$-level factor is not adjusted orthogonal to the other factors. We, therefore, are unable to prove M-optimality of $d_4$. Whether $d_4$ satisfies any specific optimality property remains to be seen.

### 7 Discussion and concluding remarks

In this paper, we study main effect plans and their duals (as multiway designs). We begin with main effect plans: generalize the concept of orthogonality through one factor to the concept of orthogonality through a set of factors. Let us discuss this concept in a little more details. Consider a plan with $m \geq 5$ factors and look at a pair of factors $A_1, A_2$. The best scenario for them is that they are mutually orthogonal in the usual sense (that is they satisfy Addelman’s proportional frequency condition (PFC) presented in (3.2)). If they don’t, then the next best possibility is that they are orthogonal through another factor, say $A_3$. Now suppose this is satisfied, but $A_1$ is orthogonal to $A_4$ through $A_5$, not through $A_3$. In that case, the second condition does not make us happier - regarding the precision in the estimates of $A_1$ as well as simplicity in the analysis. Thus, in the absence of a completely orthogonal plan (as one can derive from an orthogonal array of strength two), the best possibility is that there is a factor, say $A_m$, such that for every pair $1 \leq i \neq j < m$, $A_i$ is orthogonal to $A_j$ through $A_m$ [see Remark 2.2 of Bagchi [4]]. The situation is even better if $A_m$ is a blocking factor (since the estimation of the block effects is not of our concern). In that case, the plan is referred to as POTB and the data analysis is very similar to the data analysis of a block design. In a situation when a POTB cannot exist, one can look for a plan orthogonal through a pair of factors. We have constructed an infinite series of such plans. We pose an open problem on the construction of such a plan after the discussion. We have also constructed a few infinite series of POTBs, each member satisfying optimality property.

In the second part of this paper, we take up optimality study in a set-up with three or more blocking factors. In the earlier works in this area, the basic assumption was orthogonality among the level combinations of the blocking factors. We have deviated from that simplifying assumption.

We have concentrated on studying the optimality aspects of a few multiway designs, which are the duals of optimal main effect plans (MEPs). Interestingly, the performances of the duals of MEPs with similar status are not so similar. For instance, the multiway designs $d_1^*$ and $d_2^*$ are duals of the MEPs $\rho_1$ and $\rho_2$, both universally optimal, as shown in Morgan and Uddin [26]. We find that both $d_1^*$ and $d_2^*$ are M-optimal, but the former in a smaller class of ‘totally binary’ designs. Whether $d_1^*$ is optimal in the general class w.r.t. any optimality criterion is an open question.
Regarding the multiway design $d^*_3$, our findings are even more striking. A class of adjusted orthogonal row–column design with optimal row design and column design has been proved to be M-optimal in Bagchi and Shah [7]. One, would, therefore expect that a multiway design with similar properties would be optimal. Now consider a multiway design $d$ which is the dual of a main effect plan $\rho$. The condition of adjusted orthogonality between a pair of blocking factors of $d$ (through the treatment factor) is equivalent to the condition of orthogonality between the corresponding pair of treatment factors of $\rho$ through the block factor. Thus, it is reasonable to expect that the dual of an optimal POTB will also be optimal as a multiway design. However, we find that the issue is not so simple, as we have found with the example of the universally optimal plan $\mathcal{P}^*$ of Theorem 4.9.

For a fixed $i$, consider the pair of factors $A_i$ and $A_\infty$ of $P^*_i$ of Remark 6.3. The factors $B_i$ and $B_\infty$ are not adjusted orthogonal. This is true for every $i = 1, \cdots, t$. We are, therefore, unable to say if $d_4$ is optimal. This fact once again highlights that the performance of a design depends very much on the kind of information we want to get from it.

We have, however, found the design $d^*_3$ fulfilling our aim of generalizing the result of Bagchi and Shah [7] to a multiway setting (with $(s + 1)/4$ blocking factors). We would like to add here that optimal or not, $d_4$ is an useful multiway design with $(s + 1)/2$ blocking factors and it exists for every $s$ which is an odd prime power.

Open Problem : Construct a main effect plan with at least 4 factors with $N_{12} = J$ such that every pair of factors is orthogonal through the pair $(A_1, A_\infty)$. This plan may be used for an experiment having $A_1$ and $A_\infty$ as blocking factors, in which case the data analysis would be just like that of a row–column design.

Appendix A : Results involving linear algebra

The proofs of the results in Section 3 :
The results in the following Lemma are well-known. [See Exercise 7 of Chapter 2 of Yanai, H., Takeuchi, K. and Takane, K. [35], for instance].

Lemma 8.1 Consider matrices $U$, $V$, $W$ with the same number of rows. Then, the following hold.

(a) Suppose $C(V) \subset C(W)$. Then

$$C(P_V U) = C(P_W U) \iff (P_W - P_V) U = 0.$$ 

(b) If $W = [U, V]$ then $P_W - P_V = P_Z$, where $Z = (I - P_V) U$.

Proof of Theorem 3.1:
Taking $W = X_i$, $U = X_S$ and $V = X_T$ and applying Lemma 8.1 we find that
\[ P_i = P_T + P_Z, \text{ where } Z = (I - P_T)X_i. \] (8.1)

**Proof of (a):** In view of (2.3), (2.5) and (8.1) we see that \( C_{i;\bar{}} = C_{i;T} - X'_i P_Z X_i \). Therefore, the required necessary and sufficient condition is that \( P_Z X_i = 0 \), which is equivalent to (3.1). Hence the result.

**Proof of (b):** In view of Lemma 2.2 the following hold.

\[
SS_{i;\bar{}} = Y' P_U Y, SS_{i;T} = Y' P_V Y,
\] (8.2)

where \( U = (I - P_i)X_i, V = (I - P_T)X_i \). (8.3)

Since the support of \( Y \) is \( R^m \), \( Y' P_U Y = Y' P_V Y \) w.p.1 if and only if \( P_U = P_V \). Thus, the required necessary and sufficient condition is that \( \mathcal{C}(U) = \mathcal{C}(V) \). Applying Lemma 8.1 we see that the required necessary and sufficient condition is

\[ (P_i - P_T)X_i = 0. \] (8.4)

But in view of (8.1) this is the same as \( P_Z X_i = 0 \), which is equivalent to (3.1). Hence the result.

The proofs of the results in Section 5:

**Proof of Lemma 5.2:** Eliminating \( \hat{\mu} \) from the normal equations [see Lemma 2.1] we get a system of equation in \( \hat{\beta}_i, i \in \mathcal{M} \) and \( \hat{\tau} \). These equations are as follows.

\[
\sum_{i,j \in \mathcal{M}} C_{i;\bar{}} \hat{\beta}_j + C_{iV;0} \hat{\tau} = Q_{i;0}, \quad i \in \mathcal{M}
\] (8.5)

and

\[
\sum_{i \in \mathcal{M}} C_{i;\bar{}} \hat{\beta}_i + C_{i\tau;0} \hat{\tau} = Q_{i;0}.
\] (8.6)

Here \( C_{i;0} \)'s and \( Q_{i;0} \)'s are as in Notation 2.3 (c).

**Proof of (a):** By the hypothesis, we find that the individual C-matrices involving only the blocking factors are as follows.

\[
C_{i;0} = \begin{cases} (p + s)K_s & \text{if } i' = i, \\ pK_s & \text{otherwise}, \end{cases} \quad i, i' \in \mathcal{M},
\] (8.7)

Here \( p = -1 \) for the type 1 setting and \( K_s \) is as in (6.6).

For a fixed \( i \), we eliminate all \( \hat{\beta}_i, i' \neq i \) from (8.5) by using (8.7). Then we get an equation involving only \( \hat{\beta}_i \) and \( \hat{\tau} \). We use this equation to eliminate all \( \hat{\beta}_i \)'s from (7.24). Then we get the reduced normal equation for \( \hat{\tau} \), in the form \( C_d \hat{\tau} = Q \), where \( C_d \) is as in the statement.

**Proof of (b):**

By the hypothesis, the individual C-matrices are as follows.

\[
C_{i;0} = \begin{cases} (s + 1)K_s & \text{if } i' = i, \\ K_s & \text{otherwise}, \end{cases} \quad 1 \leq i, i' \leq m,
\] (8.8)
Following the same procedure as in Case (a) we get the C-matrix as in the statement.

The proofs of the results in Section 6:

**Proof of Lemma 6.7:** Let \( X = \{x_1, \ldots, x_{s-1}\} \) be a set of orthonormal vectors in \( \langle 1_s \rangle^\perp \). Let \( Z = \{z_i = 1_h \otimes x_i, x_i \in X\} \). For a \( z \in Z, \) \( Nz = Hx_i \) for some \( i \). So, by Lemma 6.6 (b), \( Nz = -z \), so that \( z'N'Nz = z'z \). Let \( \bar{\mu}_1 \leq \cdots \leq \bar{\mu}_{s-1} \) be the smallest \( s-1 \) positive eigen values of \( NN' \). Then, \( \bar{\mu}_1, \ldots, \bar{\mu}_{s-1} \) are also the smallest \( s-1 \) positive eigen values of \( N'N \). Therefore,

\[
\sum_{i=1}^{s-1} \bar{\mu}_i \leq \sum_{i=1}^{s-1} (z'_iN'Nz_i)/(z'_iz_i) = (s-1).
\]

Since \( N \) is an incidence matrix, the largest eigenvalue of \( NN' \) corresponds to the eigenvector \( 1_v \). Thus, the eigenvector \( e_i \) corresponding to \( \bar{\mu}_i \) cannot be \( 1_v \), and therefore \( e'_i1_v = 0 \), for each \( i \). Hence the result follows from (6.5) and (8.10).

**Proof of Lemma 6.8:** Let \( \mathcal{N}(A) \) denote the null space of the matrix \( A \) and \( \nu(A) \) the nullity of \( A \).

By (6.9)

\[
C_d = rK_v - \frac{1}{s} E_BE_B' - \frac{1}{s(s-h)} S_B S_B'.
\]

By Lemma 5.1 (a) and the definition of \( S_B, 1'vE_B = 0 = 1'vS_B \). Thus, \( 1'vC_d = 0 \) and (0) is proved.

We now prove (i). Substituting for \( E_BE_B' \) from (6.5) and \( S_B S_B' \) from (6.6) we get the following expression for \( C_d \).

\[
C_d = rK_v - \frac{1}{s} (NN' - \frac{r(s-1)}{s} J_v) - \frac{1}{s(s-h)} (HH' - \frac{(s-1)^2}{s} J_v).
\]

By definition of \( H = H_d, \mathcal{N}(NN') \subset \mathcal{N}(HH') \). Now, let \( W = \langle 1_h \rangle^\perp \otimes 1_v \). Since \( d \) is equireplicate, \( Nw = 0, \forall w \in W \). Therefore, \( \nu(NN') = \nu(N'N) \geq h-1 \), implying \( \nu(HH') \geq h-1 \). Let \( x \in \mathcal{N}(NN') \). Since \( 1_v \) is the eigenvector of \( NN' \) corresponding to the largest eigenvalue, \( x \neq 1_v \). Therefore, \( x'1_v = 0 \). So, \( C_d x = r \). Hence (i) follows.

Next we proceed to prove (ii), which is about the next largest eigen values of \( C_d \). Let \( P = aE_BE_B' + bS_B S_B' \), \( a, b > 0 \). While proving (i), we have also proved that \( \mu_i(P) = 0, 0 \leq i \leq h-1 \). Let \( \bar{\mu}_1(T) \leq \cdots \leq \bar{\mu}_{s-1}(T) \) be the smallest \( s-1 \) positive eigen values of \( T, T = P \) or \( E_BE_B' \). Fix \( i : 1 \leq i \leq s-1 \). By an well-known result [see Corollary III.2.2 of Bhatia [9], for instance] we get

\[
\bar{\mu}_i(P) \leq a\bar{\mu}_i(E_BE_B') + bh,
\]

which is \( a\bar{\mu}_i(E_BE_B') + bh \), by Lemma 6.6 (c). So, by Lemma 6.7
\[
\sum_{i=1}^{s-1} \hat{\mu}_i(P) \le (s - 1)(a + bh).
\] (8.11)

But \( P \) becomes \( rK_v - C_d \), if we put \( a = \frac{1}{s}, b = \frac{1}{s(s-h)} \). So, the result follows from (8.11).

**The proof of Theorem 6.2**

We begin with an well-known result.

**Lemma 8.2** Suppose \( x_1, \cdots, x_n \) are real numbers with \( \sum_{i=1}^{n} x_i = a \). Then, the following hold. (a) \( \sum_{i=1}^{n} x_i^2 \ge a^2/n = \) when \( x_i = a/n, \forall i. \)

(b) In particular, if \( x_i \)'s are integers, then \( \sum_{i=1}^{n} x_i^2 \) is minimum, when \( x_i = [a/n] \) or \( [a/n] + 1. \)

**Proof of Lemma 6.9:**

(a) Follows by straightforward computation.

(b) Fix an arbitrary \( x \in \mathbb{R}^p \). For \( 1 \le i \le k \), let us write \( A'x = [y_{i1} \cdots y_{iq}]' \). Then, \( x' \sum_{i=1}^{k} A_iA'_i x = \sum_{i=1}^{k} y_i' y_i = \sum_{j=1}^{q} \sum_{i=1}^{k} x_{ij}^2 \). Now, \( \sum_{i=1}^{k} y_i' = \sum_{l=1}^{p} T(l,j)x_l = z_j \) say. Therefore, by Lemma 8.2, \( \sum_{i=1}^{p} x_{ij}^2 \ge z_j^2/k \). Since \( z_j \) is the \( j \)th entry of \( T'x \), the result follows.

(c) \( \text{Tr}(HH') = \sum_{i=1}^{p} \sum_{j=1}^{q} (H(i,j))^2 \). Again, \( \sum_{i=1}^{p} \sum_{j=1}^{q} H(i,j) = kpr \). Therefore, by Lemma 8.2, \( \sum_{i=1}^{p} \sum_{j=1}^{q} (H(i,j))^2 \) is minimum, when the following hold. \( H(i,j) = \lfloor rk/q \rfloor \) or \( \lceil rk/q \rceil \) + 1, \( 1 \le i \le p, 1 \le j \le q \). Since a sufficient condition for the above is \( A_1 = A_2 = \cdots A_k \), and \( A_1(i,j) = 0 \) or 1, \( 1 \le i \le p, 1 \le j \le q \), the result follows.

**Proof of Lemma 6.10:**

By definition of \( C \), \( h_0(C) = 0 = \gamma_0 \), \( \mu_i(C) = d - \mu_n+1-i(A) \) for \( i > 0 \). Therefore, \( \sum_{i=1}^{p} \mu_i(C) = d \rho - \text{tr}(A) \le \rho(d - a) \). Since \( (1/l) \sum_{i=1}^{l} \mu_i(C) \) is increasing in \( l \), it follows that \( \sum_{i=1}^{l} \mu_i(C) \le l(d - a), \) \( 1 \le l \le \rho \). Since \( \mu_i(C) \le d \) \( \forall i \), the result follows.

**Proof of Lemma 6.12:**

Since \( d \) is equireplicate, \( 1_v \) is an eigenvector of \( T_i \) as well as of \( T_j \) with eigenvalue \( r \), where \( r \) is the replication number (for the treatments). Again, by the hypothesis, \( T_iT_j = r^3 J_v = T_jT_i \). Thus, \( T_i \) and \( T_j \) are commuting matrices and hence there is an orthonormal basis consisting of common eigenvectors of these two matrices. Therefore, \( C(T_i) \cap C(T_i) = C(T_iT_j) = C(J_v) = \langle \{1_v\} \rangle \). Hence the result.

**Appendix B:** Results involving finite field

The proofs of the results in Section 4: We begin with the proof of Theorem 4.2.
Let $C_0$ be the set of all nonzero squares of $F$ and $C_1$ the set of all nonzero non-squares of $F$.

**Notation 9.1** Consider an $m \times n$ array $P$ with entries from $F$. For $1 \leq i, j \leq m$, $k \in F$, let

$$d^k_{ij} = |\{l : p_{jl} - p_{il} = k\}|.$$

$C_0P$ will denote the $m \times (s - 1)n/2$ array $\{cP : c \in C_0\}$.

**Lemma 9.1** Suppose $\exists$ an array $P$ as in Notation 9.1 satisfying the following.

$$\sum_{k \in C_0} d^k_{ij} = \sum_{k \in C_1} d^k_{ij} = u_{ij},$$

(say). Let $P'$ be a plan having the set of columns of the array $C_0P$ as the set of runs. Let $P^* = P \oplus F$. Then, for a pair $(i, j)$, $i \neq j$, $1 \leq i, j \leq m$, the incidence matrix $N_{ij}$ of $P^*$ satisfies the following.

$$N_{ij}(x, y) = w_{ij}s + u_{ij}J_s,$$

where $w_{ij} = (s - 1)n/2 - su_{ij}$.

**Proof** $N_{ij}(x, y) = |\{(l, \alpha, q) : q \in C_0, q(p_{jl} - p_{il}) = y - x, \alpha = x - qp_{il}, 1 \leq l \leq n\}|$, which is $= |\{(l, q) : y - x = q(p_{jl} - p_{il}), q \in C_0, 1 \leq l \leq n\}|$. So, by hypothesis, if $y - x \in C_0$, then $N_{ij}(x, y) = |\{l : p_{jl} - p_{il} \in C_0\}| = \sum_{k \in C_0} d^k_{ij} = u_{ij}$.

Similarly, if $y - x \in C_1$, then $N_{ij}(x, y) = \sum_{k \in C_1} d^k_{ij}$. Therefore, if $y = x$, then $N_{ij}(x, y) = \frac{s-1}{2}(n - 2u_{ij})$. Hence the result. \hfill \square

Using the fact that when $s \equiv 3 \pmod{4}$, $-1 \in C_1$, we can prove the following result from Lemma 9.1.

**Lemma 9.2** Suppose $s \equiv 3 \pmod{4}$ is a prime power and $n$ is a multiple of 4. Suppose there is an $m \times n$ array $P$ with entries 0, 1, $-1$ (viewed as members of $F$) satisfying the following.

(a) $p_{1,j} = 0, 1 \leq j \leq n$.

(b) For every ordered pair $(i, j)$, $p_{ij} - p_{jl} \in \{0, 1, -1\}, 1 \leq l \leq n$.

(c) For $k = 1, -1$, $d^k_{ij} = \begin{cases} n/2 & \text{if } (i, j) = (1, 2), \\ n/4 & \text{otherwise} \end{cases}$

Then the plan $P^* = C_0P \oplus F$ satisfies the conditions of Theorem 4.1 with $c = ns(s - 1)/8$. Here $C_0P \oplus F$ is to be interpreted as in Notation 4.1 (d)

**Proof of Theorem 4.2** : Let $H$ be a Hadamard matrix of order $q$. W.l.g., we assume that the first row of $H$ consists of only 1’s. Write $H = [1_q \ H']$. Consider the array
\[ P = \begin{bmatrix} 0_{1\times q} & 0_{1\times q} \\ J_{1\times q} & -J_{1\times q} \\ (\bar{H} + J_{q-1\times q})/2 & -(\bar{H} + J_{q-1\times q})/2 \\ (\bar{H} + J_{q-1\times q})/2 & (\bar{H} - J_{q-1\times q})/2 \end{bmatrix}. \]

It is easy to check that \( P \) satisfies the conditions of Theorem 9.2. Hence the proof of Theorem 4.2 is complete in view of Theorem 4.1. \(
\)

The proofs of the results in Section 6.1:

We present the result in Lemma 6.2 (c) as a theorem.

**Theorem 9.1** The spectrum of \( C_{d_1} \) is \( r^{h-1}(r - (1/(s - h)))^{s-1}(r - 1)^{(h-1)(s-1)} \).

This is the only part of this paper where matrices with complex entries occur. If \( A \) is such a matrix, then \( A^* \) will denote its conjugate transpose. To prove this theorem, we need a number of tools. We introduce some notations. Some of these were already in the main body of the paper; we have repeated them for the sake of readability.

**Notation 9.2** (0) \( s = p^m = ht + 1 \), where \( p \) is a prime, \( m, t, h \) are integers, \( m \geq 1, t \geq 2 \). \( F_p \) and \( F_s \) are finite fields of orders \( p \) and \( s \), respectively. [In this section, we use the notation \( F_s \) (rather than \( F \) like in the other sections) so as to distinguish it from the field of order \( p \)]

(i) Addition and subtraction in \( I_h = \{0, 1, \ldots, h - 1\} \) will always be modulo \( h \). \( C_i, i \in I_h \) will denote the cosets of the subgroup \( C_0 \) of order \( t \) in \( F_s^* \), ordered in such a way that \( C_iC_j = C_{i+j} \) for \( i, j \in I_h \).

The rows and columns of every \( Rs \times s \) (respectively, \( h \times h \)) matrix will be indexed by \( F_s \) (respectively, \( I_h \)). Moreover, the rows and columns of every \( Rs \times s \) matrix will be indexed by \( I_h \times F_s \).

(ii) As in Notation 4.6, \( M \) will denote the \( Rs \times s \) matrix \( ((M_{i,j}))_{i,j \in I_h} \), where for \( i \in I_h, M_i \) will denote the \( s \times s \) matrix given by \( M_i(x, y) = \begin{cases} 1 & \text{if } y - x \in C_i, \\ 0 & \text{otherwise}. \end{cases} \)

(iii) \( \eta \) and \( \omega \) are primitive \( h \)th and \( p \)th roots of unity, respectively.

(iv) Consider the function \( \text{trace} : F_s \to F_p \) defined as follows. \( \text{trace}(x) = \sum_{i=1}^{m} x_i^{\nu}, x \in F_s \). [This is \( F_p \)-linear and into \( F_p \) since \( x \to x^{\nu} \) is an automorphism of \( F_s \) and its fixed field is \( F_p \)].

(v) \( U \) and \( V \) are unitary matrices of orders \( h \) and \( s \), respectively, given as follows.

\[ U(i,j) = (1/\sqrt{h})\eta^{ij}, i, j \in I_h \quad \text{and} \quad V(x, y) = (1/\sqrt{s})\omega^{\text{trace}(xy)}, x, y \in F_s. \]

(vi) Consider the sums \( g_i \) given by \( g_i = \sum_{x \in C_i} \omega^{\text{trace}(x)}, i \in I_h \).

(vii) For \( k \in I_h, G_k \) is the \( h \times h \) matrix given by \( G_k(i, j) = g_{i-j+k}, i, j \in I_h \).

(viii) For \( k \in I_h, E_k \) is the \( s \times s \) diagonal matrix given by
\[ E_k(x, x) = \begin{cases} -t & \text{if } x = 0 \\ 1 & \text{if } x \in C_k \\ 0 & \text{otherwise} \end{cases} \]

(ix) \( W \) will denote the \( hs \times hs \) unitary matrix \( U \otimes V \).

(x) \( T \) is the \( h \times h \) diagonal matrix with the entries: \( T(l, l) = \eta^l, \ l \in I_h \).

We study the behaviour of \( M \) under the actions of \( U \) and \( V \).

**Lemma 9.3** \((I_h \otimes V)^*M(I_h \otimes V) = \sum_{k \in I_h} G_k \otimes E_k.\)

**Proof** A computation shows that we have \( V^*M_iV = \text{Diag}(\lambda_i(x), \ x \in F_x) = \sum_{k \in I_h} g_{i+k}E_k, \) where

\[ \lambda_i(x) = \begin{cases} t & \text{if } x = 0 \\ g_{i+k} & \text{if } x \in C_k, k \in I_h. \end{cases} \]

[To verify the second equality here (as well as for later use), we need to observe that \( \sum_{k \in I_h} g_{i+k} = \sum_{k \in I_h} g_k = \sum_{x \in F_x^*} \omega^{-\text{trace}(x)} = -1, \) since \( \sum_{x \in F_x} \omega^{-\text{trace}(x)} = 0. \)]

But, by the definition of \( M \), the left hand side of the statement of this lemma is \((V^*M_{i-j}V)_{i,j \in I_h} \). Hence the result follows from the definition of \( G_k \) and \( E_k \).

**Lemma 9.4** \( W^*MW \) is a diagonal matrix with the following entries. For \( i \in I_h, x \in F_x \), the \((i, x)\)th diagonal entry of \( W^*MW \) is

\[ \delta(i, x) = \begin{cases} s - 1 & \text{if } i = 0, x = 0, \\ 0 & \text{if } i \neq 0, x = 0, \\ \eta^{ik} \sum_{j \in I_h} g_j \eta^{-ij} & \text{if } x \in C_k. \end{cases} \]

**Proof** Since \( W = (I \otimes V)(U \otimes I) \), Lemma 9.3 implies that

\[ W^*MW = (U \otimes I)^* \left( \sum_{k \in I_h} G_k \otimes E_k \right) (U \otimes I) = \sum_{k \in I_h} (U^* G_k U) \otimes E_k. \]

But one can verify that

\[ U^* G_k U = \sum_{j \in I_h} g_{k-j}T_j, \ k \in I_h. \]
So, $W^*MW = \sum_{k \in I_h} \sum_{j \in I_h} g_{k-j} T^j \otimes E_k$. Now, the formulae for $T$ and $E_k$ imply the result.

\[\square\]

**Notation 9.3**  (a) $\Omega_h$ is the multiplicative group of all $h$th roots of unity.

(b) For $i \in I_h$, $\chi_i : F_s^* \to \Omega_h$ is defined by $\chi_i(x) = \eta^{-ij}$, if $x \in C_j$, $j \in I_h$.

**Lemma 9.5** $|\sum_{j \in I_h} g_{i} \eta^{-ij}|^2 = \begin{cases} 1 & \text{if } i = 0, \\ s & \text{if } 0 < i < h \end{cases}$

**Proof** Since $C_j C_k = C_{j+k}$ (where the addition in the suffix is modulo $h$), $\chi_i$ is a group homomorphism (character) on $F_s^*$. From the definition of $g_j$’s we have

\[\sum_{j \in I_h} g_{i} \eta^{-ij} = \sum_{j \in I_h} \sum_{x \in C_j} \omega^{-\text{trace}(x)} \chi_i(x) = \sum_{x \in F_s^*} \omega^{-\text{trace}(x)} \chi_i(x) = g(\chi_i),\]

which is the Gauss sum attached to the character $\chi_i$. But $|g(\chi_i)|^2 = \begin{cases} 1 & \text{if } i = 0, \\ s & \text{if } i \neq 0 \end{cases}$, by a classical result on such Gauss sums [see, for instance Chapter 10 of Ireland and Rosen [22]]. Hence the result. \[\square\]

Putting the information from Lemmas 9.4 and 9.5 together, we get the spectrum of $MM'$.

**Lemma 9.6** $W^*MM'W = D$, where the diagonal entries of the diagonal matrix $D$ are as follows.

\[|\delta(i, x)|^2 = \begin{cases} (s-1)^2 & \text{if } i = 0, x = 0 \\ 0 & \text{if } i \neq 0, x = 0 \\ 1 & \text{if } i = 0, x \neq 0, \\ s & \text{if } i \neq 0, x \neq 0. \end{cases}\]

In order to get the spectrum of $C_{d_1}'$, we need the spectrum of $HH'$.

**Lemma 9.7** $W^*HH'W$ is a diagonal matrix with the $(i, x)$th entry

\[h(s-1)^2 & \text{if } i = 0, x = 0 \\ h & \text{if } i \neq 0, x = 0 \\ 0 & \text{if } i \neq 0. \]

**Proof** Let $\triangle_h$ denote the $h \times h$ matrix having the $(0, 0)$th entry 1 and all other entries 0. $\triangle_s$ is defined in a similar manner. It is easy to verify that

\[U^*J_h U = h \triangle_h \text{ and } V^*J_s V = s \triangle_s.\]

Since $H = 1_h \otimes (J_s - I_s)$ [recall Lemma 6.6], we see that

\[W^*HH'W = h \triangle_h \otimes (s \triangle_s - I_s)^2,\]

which is a diagonal matrix with the entries as in the statement.

\[\square\]
Proof of Theorem 9.1: Lemmas 9.6 and 9.7 imply the result in view of the expression for $C_{d^l}$ in Lemma 6.2.

The proofs of the results in Section 6.3:

The following result lies at the foundation of the construction of $d^s_3$:

**Lemma 9.8** Let $s \equiv 3 \pmod 4$ be a prime power. Then there is a subset $W$ of $C_0$ and a function $f : W \to C_1$ satisfying the following.

(a) $|W| = (s - 3)/4$.

(b) For every $\xi \in W$, $(\xi - 1)(f(\xi) - 1) \in C_0$.

(c) For $\xi \neq \xi' \in W$, $(\xi - \xi')(f(\xi) - f(\xi')) \in C_0$.

**Proof** Let $W = \{x \in C_0 : 1 - x^2 \in C_0\}$, $\tilde{W} = \{x \in C_1 : 1 - x^2 \in C_1\}$. Note that $x \mapsto -x$ is a bijection from $W$ onto $(C_1 \setminus \tilde{W}) \setminus \{-1\}$. Therefore, $|W| = |C_1 \setminus \tilde{W}| - 1 = (s - 3)/2 - |\tilde{W}|$. Thus, $|W| + |\tilde{W}| = (s - 3)/2$. Also, $x \mapsto -1/x$ is a bijection from $W$ onto $\tilde{W}$. Thus, $|W| = |\tilde{W}|$, which proves (a).

Let $f : W \to \tilde{W}$ be defined by $f(x) = -1/x$, $x \in W$. Then, for $\xi \in W$, $1 - \xi^2 \in C_0$, so that $(1 - \xi)(1 - f(\xi)) = \xi^{-1}(1 - \xi^2)$ which is in $C_0$. This proves (b).

Again, for $\xi \neq \xi' \in W$, $(f(\xi) - f(\xi'))(\xi - \xi') = (\xi \xi')(\xi - \xi')^2 \in C_0$, which implies (c). □

Proof of Lemma 6.11: Recall (6.10). For $l = 0, 1$, let $\tilde{R}^3_l$ denote the set of $(w + 2) \times 1$ vectors obtained from $R^3_l$ by juxtaposing 0 in the $(w + 2)$th position of each member of $R^3_l$. Let $P^0_l$ denote the plan for an $s^{w+1}(s + 1)$ experiment with $\tilde{R}^3_l$ as the set of runs and the set of factors $\{A_1, \ldots, A_w\} \cup \{A_{\infty}\} \cup \{A_0\}$. Let $P_l = P^0_l \oplus F$ and $P = \bigcup_{l=0,1} P_l$. Needless to mention that while generating $P_l$ from $P^0_l$, we make use of (4.4).

Proof of (a): Fix $i \neq j$, $1 \leq i, j \leq w$. Let $D_l$ denote the $(A_j, A_j)$ plot difference for the plan $P^0_l$, $l = 0, 1$ [recall Definition 4.5]. From (6.10) we see that

$$D_0 = (\xi_j - \xi_i)C_0$$

and

$$D_1 = (f(\xi_j) - f(\xi_i))C_1.$$

By (c) of Lemma 9.8, $D_0 \cup D_1 = F \cup \{0\}$. Now applying Lemma 4.2 on the plan $P^0_0 \cup P^0_1$ we see that

$$L_{ij} = I_s + J_s. \quad (9.1)$$

Next fix $i, 1 \leq i \leq w$. Let $D_l$ denote the $(A_i, A_{\infty})$ plot difference for the plan $P^0_l$, $l = 0, 1$. From (6.10) and the equation next to it we see that

$$D_0 = (1 - \xi_i)C_0$$

and

$$D_1 = (1 - f(\xi_i))C_1 \cup \{\infty\},$$

so that $D_0 \cup D_1 = F^+$.

By (b) of Lemma 9.8. Thus, by Lemma 4.2 $L_{i\infty} = J_{\infty s+1}$. This relation together with (9.1) completes the proof.
**Proof of (b):** Fix \( i, 1 \leq i \leq w \). Let \( D_l \) be the \((A_0, A_i)\) plot difference for the plan \( \mathcal{P}_l^0, l = 0, 1 \). From (6.10) we see that

\[
D_0 = \xi_i \bar{C}_0 \quad \text{and} \quad D_1 = f(\xi_i) \bar{C}_1,
\]

so that \( D_0 = D_1 = \bar{C}_0 \), by the definition of \( W \). Thus, \( N^{(l)}_{0i} = M_0 + I_s, l = 0, 1 \), by Lemma 4.2. But from the description of \( d^*_3 \), in view of Notation 6.1 we see that

\[
N_i = \begin{bmatrix}
N_{0i}\\
N_{1i}
\end{bmatrix}, \quad \text{where} \quad N_{il} = N^{(l)}_{0i}, \quad i \in \{1, \cdots, w\} \cup \{\infty\}.
\]

(9.2)

Hence the result.

**Proof of (c):** Now we consider the ordered pair of factors \((A_0, A_\infty)\) of \( \mathcal{P}_l^0, l = 0, 1 \) and find the \((A_0, A_\infty)\) for each of \( \mathcal{P}_0 \) and \( \mathcal{P}_1 \). We see that \( D_0 = \bar{C}_0 \) and \( D_1 = C_1 \cup \{\infty\} \). Thus,

\[
N^{(0)}_{0,\infty} = [M_0 + I_s \ 0_{s \times 1}], \quad \text{while} \quad N^{(1)}_{0,\infty} = [M_1 \ 1_s].
\]

Now using (9.2) we get the result. \( \square \)

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**Conflict of interest** On behalf of all authors, the corresponding author states that there is no conflict of interest.

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