SLOW CONVERGENCE TO EQUILIBRIUM FOR AN EVOLUTIONARY EPIDEMIOLOGY INTEGRO-DIFFERENTIAL SYSTEM

JEAN-BAPTISTE BURIE*
Univ. Bordeaux, IMB, UMR 5251, F-33400 Talence, France

RAMSÈS DJIDJOU-DEMASSE
MIVEGEC, IRD, CNRS, Univ. Montpellier, Montpellier, France

ARNAUD DUCROT
Normandie Univ, UNIHAVRE, LMAH
FR-CNRS-3335, ISCN, 76600 Le Havre, France

ABSTRACT. This work is devoted to the study of an integro-differential system of equations modelling the genetic adaptation of a pathogen by taking into account both mutation and selection processes. Using the variance of the dispersion in the phenotype trait space as a small parameter we provide a complete picture of the dynamical behaviour of the solutions of the problem. We show that the dynamics exhibits two main and long regimes — those durations are estimated — before the solution finally reaches its long time configuration, the endemic equilibrium. The analysis provided in this work rigorously explains and justifies the complex behaviour observed through numerical simulations of the system.

1. Introduction. In this note, we extend previous results obtained in [3, 10] on the dynamical behaviour of a nonlocal problem arising in evolutionary plant epidemiology.

The model we consider here reads as

\[
\begin{aligned}
\frac{dU(t)}{dt} &= \Lambda - \mu_v U(t) - U(t) \int_{\mathbb{R}^N} \beta(y)L[v(t, \cdot)](y)dy, \quad t > 0, \\
\frac{\partial v(t, x)}{\partial t} &= \beta(x)U(t)L[v(t, \cdot)](x) - \mu_v v(t, x), \quad t > 0, \quad x \in \mathbb{R}^N,
\end{aligned}
\]

(1)

wherein \( L \) denotes the integral operator

\[
L[v(t, \cdot)](x) = \int_{\mathbb{R}^N} J(x - y) v(t, y) dy.
\]

(2)

This model, which has been studied in [3], has been proposed in a more complex biological setting in [15]. Here, contrary to the classical adaptive dynamics approach [9, 13, 18], the evolution and the epidemiological processes are not decoupled, in concordance with the population genetics approach proposed in [7, 8, 17].

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* Corresponding author: Jean-Baptiste Burie.
Model (1)-(2) describes the evolution in time $t$ of a spatially homogeneous population of healthy and infected plants of density $U(t)$ and $v(t, x)$ respectively. The variable $x \in \mathbb{R}^N$, where $N$ is some positive integer, describes the phenotypic trait of a pathogen variant on which depends the infection efficiency $\beta(x)$ and the spores production rate $r(x)$, where $r$ and $\beta$ are non negative functions. Also, this model is a simplified version of the more complex one in [10] in which the density of spores produced by the pathogen was explicitly taken into account as a state variable. The mutations from a strain $x$ to a strain $y$ of the pathogen population are described by the kernel $J$ in the integral operator (2). The remaining parameters of the model are the constant flux $\Lambda > 0$ of healthy plant density, and the constant mortality rates $\mu_u > 0$ and $\mu_v > 0$ of the healthy and infected plant densities respectively.

We investigate the dynamical properties of the solutions of this model in some specific cases. In particular, as in [3, 10], we first assume that the mutations are localised in the phenotypic space, which leads to introduce a small parameter $0 < \varepsilon < 1$ and write the mutation kernel $J = J_\varepsilon$ as follows

$$J_\varepsilon(x) = \frac{1}{\varepsilon^N} J \left( \frac{x}{\varepsilon} \right).$$

Next, we define the so-called fitness function $\Psi$ by

$$\Psi(x) = \beta(x) r(x).$$

Then, as $\varepsilon$ tends to 0, it was proved in [10] (for a more detailed model) that there exists an unique endemic steady state if and only if the following threshold condition is satisfied

$$R_0 := \frac{\Lambda}{\mu_u \mu_v} \sup_{x \in \mathbb{R}^N} \Psi(x) > 1.$$

It was also shown that the endemic state will concentrate, as $\varepsilon \to 0$, on a trait $x$ where $\Psi(x)$ reaches its maximum, and typically on a single trait.

Such concentration properties of steady state solutions for nonlocal mutation selection models have already been the object of study. For example, in [2, 4, 5, 6, 14], this study has been achieved using the spectral properties of some nonlocal operators. In [1] (see also references therein), a nonlocal reaction-diffusion model is considered, and the steady state solutions are described in terms of the underlying Schrödinger spectral elements. Another approach has been introduced in [9]: assuming that mutations are localised in the phenotypic space according to a small parameter $\varepsilon$, a change of time scale leads to a limit Hamilton-Jacobi equation as $\varepsilon \to 0$. Such an approach was successfully applied in e.g. [20, 21, 22].

Following the classical adaptive dynamics theory [9, 13, 18], these traits where the fitness function reaches its maximum were called Evolutionary Attractors (EA for short). Additionally in [10] conditions were given to identify the “strongest” strain in the case of multiple EAs. This “strongest” strain was then called the Globally Stable Evolutionary Attractor (GSEA for short). These conditions were related to the spectral properties of some nonlocal integral operator depending on $J_\varepsilon$ and $\Psi$.

Next, in [3] we investigated the asymptotic behaviour of the solutions of Problem (1)-(2). We considered the case where $\Psi(x)$ reaches its maximum for two distinct EAs. We proved that, after a small transition time and for an initial pathogen population far from the GSEA, the solution remains close to the “weakest” EA and far from the GSEA of the system during a long transition time.
In the current note, we shall complete these results by describing in details the dynamics of the solution towards the asymptotic equilibrium when $\varepsilon$ is small.

To illustrate the results we will prove, we now show some numerical simulations of the model in the case $N = 1$. We also explain the possible biological applications of this study which concerns the durability of disease resistant varieties [11, 12, 16, 24].

Thus, we consider two pathogen populations $P_1$ and $P_2$ associated respectively to fitness functions $\Psi_1$ and $\Psi_2$, each one having a unique peak at EAs $x = x_1$ and $x_2$ respectively, corresponding to their adaptation to a resistant host $H_2$ and a non resistant one $H_2$. We assume that at time $t = 0$ the pathogen population $P_2$ is well adapted to its host $H_2$ by choosing an initial condition $v_0$ concentrated around $x_2$ and such that $v_0(x_1) = 0$. Next, the resistant host variety $H_1$ is introduced in the environment with a proportion $(1 - \chi)$ whereas $\chi \in (0, 1)$ is the proportion of the non resistant host in the total population. The global fitness of the environment is a double peaked function $\Psi$ given by

$$\Psi(x) = \chi \Psi_1(x) + (1 - \chi)\Psi_2(x), \forall x \in \mathbb{R}^N.$$  

In the presence of the resistant variety, the pathogen population will bypass this resistance after some transient time that we wish to maximize. In this context, we impose $x_1$ to be the only GSEA by enforcing $\Psi^{(n)}(x_1) = \Psi^{(n)}(x_2)$ for $n = 0, 1, 2, 3$ and $\Psi^{(4)}(x_1) \neq \Psi^{(4)}(x_2)$. This choice is related to the spectral properties of the nonlocal operator $L_\varepsilon$ defined below in (4). This configuration allows us to satisfy the spectral gap assumption described in Assumption 2 below.

For the numerical simulations, we thus set

$$\Psi_1(x) = \begin{cases} -\frac{1}{2} \sin \left(2\sqrt{2}(x - 2)\right) & \text{if } x \in \left[2 \frac{\pi}{2\sqrt{2}}, 2\right], \\ 0 & \text{else,} \end{cases}$$

$$\Psi_2(x) = \begin{cases} 2(x - 2)(3 - x) & \text{if } x \in [2, 3], \\ 0 & \text{else,} \end{cases}$$

and $\chi = \frac{1}{2}, x_1 = 2 - \frac{\pi}{4\sqrt{2}}, x_2 = \frac{5}{2}.$

Other parameters in the model (1) were set to $\Lambda = 10$, $\mu = 1$, $\mu_v = 1.05$, and $\beta_1 \equiv \beta_2 \equiv 1$, so that $r_1 = \Psi_1$ and $r_2 = \Psi_2$.

The fitness function and the initial condition $v_0$ are displayed on Figure 1 together with the time evolution of the scaled infected population $\sqrt{\varepsilon}v(t, x)$. Here $\sqrt{\varepsilon}$ is a scaling parameter such that $\sup_{x \in \mathbb{R}, t > t^*} \{\sqrt{\varepsilon}v(t, x)\}$ (with $t^* > 0$ a small transient time) converges to a constant as $\varepsilon$ tends to 0. This scaling form is further detailed after our main theorem, namely Theorem 2.2 below. The simulation shows that the pathogen population indeed remains concentrated around the EA $x_2$ during a long transient time before shifting and concentrating around the GSEA $x_1$.

Next, in Figure 2 is displayed the time evolution of the scaled infected population $\sqrt{\varepsilon}v(t, x)$ at the EAs $x = x_1$ and $x = x_2$ for two values of $\varepsilon$. We observe that for a smaller value of $\varepsilon$, both solutions $v(t, x_1)$ and $v(t, x_2)$ are shifted to the right and that the duration of the coexistence regime increases.

Finally, in Figure 3 we investigate the dependence with respect to $\varepsilon$ of the position of this coexistence regime. To achieve this, we define the “coexistence time” $t_{\varepsilon,c} > t^*$ (with $t^*$ a small transient time) as the time where the curves $t \to \sqrt{\varepsilon}v(t, x_1)$ and $t \to \sqrt{\varepsilon}v(t, x_2)$ cross. We obtain that $t_{\varepsilon,c} \sim c\varepsilon^{-3.4}$. 

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Figure 1. Fitness function $\Psi$ and density of infected population at time $t = 0$. (Left) Time evolution of the infected population $\sqrt{\varepsilon}v(t, x)$ for $\varepsilon = 0.02$. (Right)

Figure 2. Slow convergence of the infected population at $x = x_1$ and $x = x_2$ towards the asymptotic configuration for $\varepsilon = 0.02 \times 2^{1/3}$ (Left) and $\varepsilon = 0.02$ (Right)

Figure 3. Linear dependence in log-log scale with respect to $\varepsilon$ of the value of the “coexistence time” $t_{\varepsilon,c}$ for which $v(t_{\varepsilon,c}, x_1) = v(t_{\varepsilon,c}, x_2)$. The slope of the corresponding line is approximatively $-3.4$. 
This note is organized as follows. In Section 2 we first state our main assumptions and recall some known and useful results for the solutions of Model (1)-(2) obtained in our previous works [10, 3]. Next, we state the main result of this manuscript, Theorem 2.2, which provides a complete picture of the dynamics of the solutions as well as estimations of the durations of the various transient regimes. In Section 3, we briefly recall some preliminary results extracted from [10, 3] that will be needed to prove Theorem 2.2 in Section 4.

2. Assumptions and main results. In this section we present the main set of assumptions that will be used in this work and we state our main results related to the description of the dynamical behaviour of (1)-(2) when $\varepsilon \ll 1$.

As in [3], our analysis relies on a symmetric reformulation of Problem (1)-(2). To that aim we introduce the following functions

$$\tilde{v}(t,x) := \sqrt{\frac{r(x)}{\beta(x)}} v(t,x), \tilde{\beta}(x) := \sqrt{\frac{\beta(x)}{r(x)}},$$

and $\Theta(x) = \sqrt{\beta(x)r(x)}$ is the square root of the fitness function $\Psi(x)$.

Then, by straightforward computations, system (1)-(2) becomes

$$\begin{aligned}
\frac{dU(t)}{dt} &= \Lambda - \mu_u U(t) - U(t) \int_{\mathbb{R}^N} \tilde{\beta}(x) \Theta(x) J_\varepsilon \ast (\Theta \tilde{v}(t,\cdot))(x) dx, \\
\frac{\partial \tilde{v}}{\partial t}(t,x) &= U(t) \Theta(x) J_\varepsilon \ast (\Theta(\cdot) \tilde{v}(t,\cdot))(x) - \mu_v \tilde{v}(t,x),
\end{aligned}$$

wherein the symbol $\ast$ denotes the convolution product in $\mathbb{R}^N$, i.e.

$$J_\varepsilon \ast (\Theta(\cdot) \tilde{v}(t,\cdot))(x) = \int_{\mathbb{R}^N} J_\varepsilon(x-y) \Theta(y) \tilde{v}(t,y) dy.$$

For convenience we omit the tildes so that setting also $\mu_u = 1$, $\mu_v = \mu > 0$ the above system then rewrites as follows, for $t > 0$ and $x \in \mathbb{R}^N$,

$$\begin{aligned}
\frac{dU(t)}{dt} &= \Lambda - U(t) - U(t) \int_{\mathbb{R}^N} \beta(x) \Theta(x) J_\varepsilon \ast (\Theta v(t,\cdot))(x) dx, \\
\frac{\partial v}{\partial t}(t,x) &= U(t) \Theta(x) J_\varepsilon \ast (\Theta(\cdot)v(t,\cdot))(x) - \mu v(t,x).
\end{aligned}$$

(3)

To handle Problem (3) and continue the analysis provided in [3], we make use of the same set of assumptions starting with

**Assumption 1.** We assume that

(i) $\Lambda$ and $\mu$ are positive constants.

(ii) The positive function $\beta$ is assumed to be bounded and to belong to $L^2(\mathbb{R}^N) \cap C^0(\mathbb{R}^N)$ while the function $\Theta$ is positive and continuous on $\mathbb{R}^N$ with $\Theta(x) \to 0$ as $|x| \to \infty$.

(iii) The mutation kernel $J \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ satisfies $\int_{\mathbb{R}^N} J(x) dx = 1$, $J(x) > 0$ and $J(-x) = J(x)$ a.e. in $\mathbb{R}^N$.

(iv) There exist two points $x_1, x_2$ in $\mathbb{R}^N$ such that $x_1 \neq x_2$ and

$$\{x_1, x_2\} = \{x \in \mathbb{R}^N, \Theta(x) = \|\Theta\|_\infty\},$$

where $\|\Theta\|_\infty = \sup_{x \in \mathbb{R}^N} \Theta(x)$. 

Remark 1. The positivity of \( J \) in Assumption 1 (iii) can be relaxed. For example, it is sufficient to assume that there exists \( R > 0 \) such that \( J(x) > 0 \) for all \( |x| < R \), this will guarantee the irreducibility of the operator \( L \) which is needed hereafter. More precisely, the following condition can be found in [19] or [23] (for a general kernel operator):

Let \( \mu \) denote the Lebesgue measure of \( \mathbb{R}^N \). The positive kernel operator \( L \) defined as \( L(f)(x) = \int_{\mathbb{R}^N} \Theta(x) \Theta(y) J(x - y) f(y) \mu(dy) \) with \( J \geq 0 \) is irreducible on \( L^p(\mathbb{R}^N) \) for \( p \in [1, \infty) \) if and only if, for any measurable set \( Y \subset \mathbb{R}^N \) such that \( \mu(Y) > 0 \) and \( \mu(C_Y) > 0 \) (where \( C_Y \) is the complement of \( Y \) in \( \mathbb{R}^N \)), one has

\[
\int_Y \mu(dx) \int_{C_Y} \Theta(x) \Theta(y) J(x - y) \mu(dy) > 0.
\]

Note that in the above assumption, the point (iv) assumes the fitness function \( \Theta \) (or \( \Psi = \Theta^2 \)) exhibits two peaks at its maximal level. This situation corresponds to the configuration considered in the numerical experiments presented in the introduction.

Next, using the above assumption, the linear operator \( L = L_\epsilon \), defined on \( L^p(\mathbb{R}^N) \) for any \( p \in [1, \infty) \) by

\[
L_\epsilon[\varphi](x) = \int_{\mathbb{R}^N} \Theta(x) J_\epsilon(x - y) \Theta(y) \varphi(y) dy,
\]

is a bounded positive, compact, irreducible operator that is furthermore self-adjoint operator when considered as acting on \( L^2(\mathbb{R}^N) \). Therefore, for each \( \epsilon > 0 \), this operator \( L_\epsilon \) admits a spectral decomposition with positive eigenvalues \( (\lambda_n = \lambda_n^\epsilon)_{n \geq 1} \) such that

\[
\lambda_1 > \lambda_2 \geq \lambda_3 \geq \cdots \geq \lambda_n \quad \text{with} \quad \lambda_n \to 0 \quad \text{as} \quad n \to +\infty,
\]

the corresponding set of eigenvectors \( \{ \varphi_n^\epsilon \}_{n \geq 1} \) form an Hilbert basis of \( L^2(\mathbb{R}^N) \), and the principal eigenvector \( \varphi_1^\epsilon \) is positive on \( \mathbb{R}^N \). Here the eigenvectors are normalized in \( L^2(\mathbb{R}^N) \) and, due to Assumption 1 (iii), they all belong to \( \cap_{p \in [1, \infty)} L^p(\mathbb{R}^N) \).

Keeping in mind the above notations, we present our next set of assumptions that are related to the asymptotic behaviour of some spectral elements of the linear operator \( L_\epsilon \) (see (4)) when \( \epsilon \ll 1 \). More specifically we assume that

Assumption 2 (Spectral gap). Let Assumption 1 be satisfied. We assume that the following properties hold true:

(i) The first and second spectral gaps, denoted by \( G(\epsilon) \) and \( H(\epsilon) \) respectively and defined by

\[
G(\epsilon) := \lambda_1^\epsilon - \lambda_2^\epsilon \quad \text{and} \quad H(\epsilon) := \lambda_2^\epsilon - \lambda_3^\epsilon,
\]

satisfy for \( \epsilon \ll 1 \)

\[
H(\epsilon) = h(\epsilon) \epsilon \quad \text{with} \quad h(\epsilon) \to h > 0 \quad \text{and} \quad G(\epsilon) \ll \frac{\epsilon}{|\ln \epsilon|}.
\]

(ii) The two first eigenvectors \( \varphi_1^\epsilon \) and \( \varphi_2^\epsilon \) satisfy the following concentration properties: for each \( \rho > 0 \) one has

\[
\int_{\|x-x_1\| \geq \rho} \varphi_1^\epsilon(x) dx = C(\rho, \epsilon) = o(\epsilon^\infty) \quad \text{as} \quad \epsilon \to 0,
\]

where \( C(\rho, \epsilon) = o(\epsilon^\infty) \) as \( \epsilon \to 0 \) means

\[
\forall k \in \mathbb{N}, \exists c_{k,\rho} > 0, \quad C(\rho, \epsilon) \leq c_{k,\rho} \epsilon^k \quad \text{as} \quad \epsilon \to 0,
\]
and for all continuous and bounded function \( f : \mathbb{R}^N \to \mathbb{R} \) one has
\[
\lim_{\varepsilon \to 0} \int_{\mathbb{R}^N} \frac{\varphi_2^\varepsilon(x)f(x)dx}{\int_{\mathbb{R}^N} \varphi_2^\varepsilon(y)dy} = f(x_2).
\]

Herein the points \( x_i, i = 1, 2, \) are defined in Assumption 1 (iv).

The above set of assumptions has been used in [3]. As explained in Remark 2.5 of the aforementioned paper, these assumptions are satisfied, in the one dimension phenotype space case (namely \( N = 1 \)), when the fitness function \( \Theta \) is smooth, the kernel function \( J \) has a rather fast decay at infinity and, finally, when two fitness peaks, \( x_1 \) and \( x_2, \) satisfy \( \Theta''(x_1) = \Theta''(x_2) < 0 \) while there exists a higher order derivative, say \( k \geq 3, \) such that \( \Theta^{(k)}(x_1) \neq \Theta^{(k)}(x_2) \). In that situation, that corresponds to the above configuration used for our numerical experiments, the above set of assumptions is satisfied and \( G(\varepsilon) = O(\varepsilon^3) \). Note that the difference \( \Theta^{(k)}(x_1) \neq \Theta^{(k)}(x_2) \) means that \( x_1 \) and \( x_2 \) are distinguishable without explicitly saying which one is the “strongest” peak.

Note also that the existence of a principal eigenpair \((\lambda_1^\varepsilon, \varphi_1^\varepsilon)\) for the operator \( L_\varepsilon \) is strongly related to the unique endemic equilibrium of Model (1)-(2) (see [10]). Indeed, an endemic equilibrium \((U_1^\varepsilon, v_1^\varepsilon)\) is such that: (i) from the second equation in system (3), \((U_1^\varepsilon, v_1^\varepsilon)\) satisfies
\[
L_\varepsilon[v_1^\varepsilon] = \frac{\mu}{U_1^\varepsilon}v_1^\varepsilon;
\]
so that \((\frac{\mu}{U_1^\varepsilon}, v_1^\varepsilon)\) corresponds to the principal eigenpair of the operator \( L_\varepsilon \) i.e. \( U_1^\varepsilon = \frac{\mu}{\lambda_1^\varepsilon} \) and \( v_1^\varepsilon = c_1^\varepsilon \varphi_1^\varepsilon, \) where \( c_1^\varepsilon \) is a positive constant since \( \varphi_1^\varepsilon \) is positive. (ii) from the first equation in system (3), we recover that \( c_1^\varepsilon \) is given by
\[
c_1^\varepsilon = \frac{1}{\lambda_1^\varepsilon \int_{\mathbb{R}^N} \beta \varphi_1^\varepsilon dx} \left( \frac{\Lambda}{\mu} \lambda_1^\varepsilon - 1 \right),
\]
and \( c_1^\varepsilon > 0 \) if and only if \( \frac{\Lambda}{\mu} \lambda_1^\varepsilon > 1. \)

So, the first concentration property in Assumption 2 (ii) states that, when \( \varepsilon \ll 1, \) the endemic equilibrium of Model (1)-(2), when it exists, is basically monomorphic and concentrates on a single trait value \( x_1, \) namely the GSEA.

In addition to the above assumptions we assume that the initial data \((U_0, v_0)\) for System (3) satisfies the following properties.

**Assumption 3 (Initial data).** Assume that the initial data \((U_0, v_0) \in [0, \infty) \times (C_0^0(\mathbb{R}^N) \cap L_2^2(\mathbb{R}^N))\) is such that there exists \( \rho > 0 \) such that \( v_0 \equiv 0 \) on \( B(x_1, \rho), \) and \( v_0(x_2) > 0. \) Here \( B(x_1, \rho) \) denotes the open ball of radius \( \rho \) centred at \( x_1 \) and \( C_0^0(\mathbb{R}^N) \) the set of the continuous and bounded functions on \( \mathbb{R}^N. \)

Assumption 3 above means that the initial infected population is far from the GSEA \( x_1, \) while it displays a non negligible contribution on \( x_2. \) What we have in mind, as in the numerical experiments presented in the introduction, corresponds to an initial distribution of the infected with phenotypes concentrated around \( x_2, \) so that the initial infected population is mostly well adapted to this peak of the fitness.

To state our main result let us recall that, under the above sets of assumptions, one has (see [3] for more details)
\[
\lim_{\varepsilon \to 0} \lambda_1^\varepsilon = \lim_{\varepsilon \to 0} \lambda_2^\varepsilon = \|\Theta\|_\infty^2.
\]
Now we introduce the quantities \( R_{\varepsilon,0,i} \), \( i = 1, 2 \) and \( R_0 \) defined by
\[
R_{\varepsilon,0,i} = \frac{\Lambda \lambda_i}{\mu} \quad \text{and} \quad R_0 = \lim_{\varepsilon \to 0} R_{\varepsilon,0,i} = \frac{\Lambda \|\Theta\|_{\infty}^2}{\mu},
\]
and we define, for \( i = 1, 2 \) and all \( \varepsilon \) small enough the positive quantities
\[
\beta_{\varepsilon i} = \int_{\mathbb{R}^N} \beta(x) \varphi_i^\varepsilon(x) dx,
\]
as well as
\[
V_{\varepsilon i} = \frac{R_{\varepsilon,0,i} - 1}{\beta_{\varepsilon i} \lambda_i}.
\]
Next the following results have been obtained in our previous work [3]

**Theorem 2.1.** Let Assumption 1, 2 and 3 be satisfied. Assume furthermore that \( R_0 > 1 \). If we denote by \((U^\varepsilon, v^\varepsilon)\) the solution of \((3)\), then the following properties hold true

(i) For any \( q > 0 \) there exist some positive constants \( \ell \) and \( C \) such that for all \( 0 < \varepsilon < 1 \) one has
\[
\|v^\varepsilon(t,.) - V_{\varepsilon i}^\varepsilon \varphi_i^\varepsilon(.)\|_{L^2(\mathbb{R}^N)} \leq C \varepsilon^q, \quad \forall t \in \left[ \ell \frac{\ln \varepsilon}{\varepsilon}, \frac{1}{G(\varepsilon)} - \frac{\ln \varepsilon}{\varepsilon} \right].
\]

(ii) For each given \( \varepsilon > 0 \) small enough, one has
\[
\lim_{t \to \infty} v^\varepsilon(t,x) = V_{\varepsilon i}^\varepsilon \varphi_i^\varepsilon(x),
\]
wherein the above convergence holds in \( L^2(\mathbb{R}^N) \).

Set for \( i = 1, 2 \)
\[
w_{\varepsilon i}^\varepsilon(t) = \int_{\mathbb{R}^N} v^\varepsilon(t,x) \varphi_i^\varepsilon(x) dx,
\]
and
\[
V^0 := \lim_{\varepsilon \to 0} \beta_{\varepsilon i} V_{\varepsilon i}^\varepsilon = \frac{R_0 - 1}{\|\Theta\|_{\infty}^2}.
\]
As a special case, the above theorem states that
\[
\beta_2^\varepsilon w_2^\varepsilon(t) \to V^0 \quad \text{as} \quad \varepsilon \to 0,
\]
uniformly for \( t \in \left[ \ell \frac{\ln \varepsilon}{\varepsilon}, \frac{1}{G(\varepsilon)} - \frac{\ln \varepsilon}{\varepsilon} \right] \), for some constant \( \ell > 0 \), while for any \( \varepsilon > 0 \) small enough one has
\[
\beta_2^\varepsilon w_2^\varepsilon(t) \to 0 \quad \text{as} \quad t \to \infty.
\]
This allows us to define, for each \( \alpha \in (0, 1) \) and all \( \varepsilon \) small enough, the time \( t_{\varepsilon, \alpha} \) given by
\[
t_{\varepsilon, \alpha} = \inf \left\{ t \geq G(\varepsilon)^{-1} - \frac{\ln \varepsilon}{\varepsilon}, \beta_2^\varepsilon w_2^\varepsilon(t) = \alpha V^0 \right\}.
\]
This time roughly corresponds to the first decrease of the concentration of the infection around \( x_2 \). It will play an important role in our arguments as well as in the description of the global behaviour of the infection. As it will been seen in our main result below, this time corresponds to the moment where the infection shifts its concentration from \( x_2 \) to the \( x_1 \), the GSEA.

Keeping in mind the above behaviour, we are now able to state our main result that roughly speaking provides a complete picture of the solution when \( \varepsilon \ll 1 \).
Theorem 2.2. Let Assumptions 1, 2 and 3 be satisfied. Assume furthermore that \( R_0 > 1 \) and let us denote by \((U^\varepsilon, v^\varepsilon)\) the solution of (3). Let \( \alpha \in (0, 1) \) be given and fixed and consider, for each \( \varepsilon \) small enough the time \( t_{\varepsilon, \alpha} \) defined above in (8).

Then there exists \( k > 0 \) such that

\[
\|v^\varepsilon(t, \cdot) - w_1^\varepsilon(t)\varphi_1^\varepsilon(\cdot) - w_2^\varepsilon(t)\varphi_2^\varepsilon(\cdot)\|_{L^2(\mathbb{R}^N)} \to 0,
\]

and

\[
U^\varepsilon(t) \to U^0 := \frac{\mu}{\|\Theta\|_{L^\infty}}.
\]

uniformly for \( t \geq k\frac{\ln \varepsilon}{\varepsilon} \) as \( \varepsilon \to 0 \).

Moreover the components \( w_i^\varepsilon \) satisfy, uniformly for \( t \geq k\frac{\ln \varepsilon}{\varepsilon} \)

\[
\beta_1^\varepsilon w_1^\varepsilon(t) + \beta_2^\varepsilon w_2^\varepsilon(t) \to V^0 \text{ as } \varepsilon \to 0, \quad (11)
\]

\[
\beta_1^\varepsilon w_1^\varepsilon(t) - \frac{(1 - \alpha)V^0}{1 - \alpha + \varepsilon^{-G(\varepsilon)}\frac{\mu}{\|\Theta\|_{L^\infty}}(t-t_{\varepsilon, \alpha})} \to 0 \text{ as } \varepsilon \to 0. \quad (12)
\]

and

\[
\beta_2^\varepsilon w_2^\varepsilon(t) - \frac{\alpha V^0}{\alpha + (1 - \alpha)e^{G(\varepsilon)}\frac{\mu}{\|\Theta\|_{L^\infty}}(t-t_{\varepsilon, \alpha})} \to 0 \text{ as } \varepsilon \to 0. \quad (13)
\]

Furthermore the time \( t_{\varepsilon, \alpha} \) satisfies

\[
t_{\varepsilon, \alpha} \gg \frac{1}{G(\varepsilon)} \text{ as } \varepsilon \to 0. \quad (14)
\]

Note that the parameters \( \beta_1^\varepsilon \) and \( \beta_2^\varepsilon \) are both positive and tend to 0 as \( \varepsilon \to 0 \).

In the above results, they act as normalisation parameters to rescale both functions \( w_i^\varepsilon \), for \( i = 1, 2 \). Let us recall that they both satisfy the following estimates (see [3]) that will also be used in the sequel

\[
\varepsilon^{N/2} = O(\beta_1^\varepsilon) \text{ and } \beta_1^\varepsilon = o(1) \text{ as } \varepsilon \to 0. \quad (15)
\]

Note also that the above result provides a complete picture of the solution \((U^\varepsilon, v^\varepsilon)\) when \( \varepsilon \ll 1 \). This description includes the three regimes exhibited by the function \( v^\varepsilon \) observed in the numerical experiments above and that roughly corresponds to:

1. A long regime of length larger than \( G(\varepsilon)^{-1} \) where \( v^\varepsilon \) is highly concentrated around \( x_2 \);
2. A long transition behaviour with a length of order \( G(\varepsilon)^{-1} \) during which the two EAs, namely \( x_1 \) and \( x_2 \) coexist at a non negligible level;
3. The solution reaches the endemic equilibrium, which is highly concentrated in the GSEA \( x_1 \).

As an application of the above result, we consider the case where \( N = 1 \) and, as in [10], we furthermore assume that there exist \( \gamma_0 \in (0, 1) \) and \( M > 0 \) such that

\[
J(x) \leq M \exp(-|x|^\gamma), \quad \forall x \in \mathbb{R},
\]

and for notational simplicity, we assume that \( \int_\mathbb{R} x^2 J(x)dx = 1 \). Also we assume that \( \Theta \) is smooth, namely of the class \( C^\infty \) and

\[
\Theta''(x_1) = \Theta''(x_2) < 0,
\]

while there exists \( k \geq 3 \) such that \( \Theta^{(k)}(x_1) \neq \Theta^{(k)}(x_2) \). As already mentioned above, in this context, we know from the detailed spectral analysis of operator \( L_\varepsilon \) provided in [10] that Assumption 2 is satisfied with \( G(\varepsilon) = O(\varepsilon^3) \) (see above and Remark 2.5 in [3]).
Next let us set
\[ m = \sqrt{-\Theta''(x_1)} > 0. \]
Then, according to the construction of quasi-modes in [10], one obtains an asymptotic expansion for the two first eigenvectors \( \varphi^\varepsilon_i \), \( i = 1, 2 \) that reads as follows. Consider the function
\[
\psi_0(x) = \exp \left( -\frac{mx^2}{2} \right).
\]
Then one has for all \( \varepsilon < < 1 \)
\[
\gamma^\varepsilon_i \varphi^\varepsilon_i = \varepsilon^{-\frac{1}{4}} \psi_0 \left( \frac{x - x_i}{\sqrt{\varepsilon}} \right) + O \left( \sqrt{\varepsilon} \right) \quad \text{in} \quad L^2(\mathbb{R}),
\]
wherein we have set
\[
\gamma^\varepsilon_i := \|\psi_0\|_{L^2(\mathbb{R})} + O \left( \sqrt{\varepsilon} \right) \to \|\psi_0\|_{L^2(\mathbb{R})} \quad \text{as} \quad \varepsilon \to 0.
\]
Using the above asymptotic expansion for \( \varphi^\varepsilon_i \) one may firstly strengthen the estimates given in (15) as follows. Indeed for \( i = 1, 2 \) one has
\[
\gamma^\varepsilon_i \beta^\varepsilon_i \varepsilon^{\frac{1}{4}} = \int_{\mathbb{R}} \beta(x) \left[ \frac{1}{\sqrt{\varepsilon}} \psi_0 \left( \frac{x - x_i}{\sqrt{\varepsilon}} \right) \right] \, dx + O \left( \varepsilon^{\frac{1}{4}} \right),
\]
from which one obtains
\[
\lim_{\varepsilon \to 0} \frac{\beta^\varepsilon_i}{\varepsilon^{\frac{1}{4}}} = \beta(x_i) \frac{\|\psi_0\|_{L^1(\mathbb{R})}}{\|\psi_0\|_{L^2(\mathbb{R})}},
\]
that is
\[
\beta^\varepsilon_i = \beta(x_i) \frac{\|\psi_0\|_{L^1(\mathbb{R})}}{\|\psi_0\|_{L^2(\mathbb{R})}} \varepsilon^{\frac{1}{4}} + o \left( \varepsilon^{\frac{1}{4}} \right).
\]
Finally fix \( \alpha \in (0, 1) \) and set
\[
\Pi^\varepsilon_1(t) := \frac{(1 - \alpha)V^0}{1 - \alpha + \alpha e^{-G(\varepsilon)\frac{t}{2}(t, t, \alpha)}},
\]
\[
\Pi^\varepsilon_2(t) := \frac{\alpha V^0}{\alpha + (1 - \alpha)e^{-G(\varepsilon)\frac{t}{2}(t, t, \alpha)}}.
\]
Then, within this framework Theorem 2.2 rewrites as follows
\[
\varepsilon^{\frac{1}{4}} v^\varepsilon(t, x) \sim \frac{\|\psi_0\|_{L^2(\mathbb{R})}}{\|\psi_0\|_{L^1(\mathbb{R})}} \sum_{i=1}^{2} \Pi^\varepsilon_i(t) \frac{1}{\beta(x_i)} \varepsilon^{\frac{1}{4}} \psi_0 \left( \frac{x - x_i}{\sqrt{\varepsilon}} \right) \quad \text{in} \quad L^2(\mathbb{R}).
\]
The above computations roughly explain the scaling form used for Figure 1 and Figure 2. Indeed, since
\[
\psi_0 \left( \frac{x_1 - x_2}{\sqrt{\varepsilon}} \right) = o \left( \varepsilon^{-\infty} \right),
\]
one obtains from the above asymptotic – at least formally – that
\[
\sqrt{\varepsilon} v(t, x_i) \sim \frac{\|\psi_0\|_{L^2(\mathbb{R})}}{\beta(x_i)\|\psi_0\|_{L^1(\mathbb{R})}} \Pi^\varepsilon_i(t), \quad i = 1, 2.
\]
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3. Preliminaries. In this section we recall some results obtained in [3] that are needed for the proofs of Theorem 2.2. We refer the reader to [3] for more details and for the proofs of these results.

Here recall that, for $i = 1, 2$, $w^\varepsilon_i$ and $\beta^\varepsilon_i$ are defined by

$$w^\varepsilon_i(t) = \int_{\mathbb{R}^N} v^\varepsilon(t, x) \varphi^\varepsilon_i(x) dx$$

and

$$\beta^\varepsilon_i = \int_{\mathbb{R}^N} \beta(x) \varphi^\varepsilon_i(x) dx.$$

Next recall that $w^\varepsilon_1$ and $w^\varepsilon_2$ are linked through the following relation.

**Lemma 3.1.** For any $m, n \in \{1, 2\}$ the following identity holds true

$$\forall t, s \geq 0, \quad w^\varepsilon_m(t) = w^\varepsilon_m(s) \frac{w^\varepsilon_n(t)}{w^\varepsilon_n(s)} \left( e^{-\frac{\lambda^\varepsilon_m - \lambda^\varepsilon_n}{\lambda^\varepsilon_n} \mu (t-s)} \right).$$

The next lemma was extensively used in [3].

**Lemma 3.2.** Let $\Lambda > 0$, $\lambda > 0$ and $\mu > 0$ be three given parameters. Consider the system

$$\begin{align*}
V'(t) &= \Lambda - V(t) - \lambda w(t) V(t), \\
w'(t) &= w(t) (\lambda V(t) - \mu).
\end{align*}$$

Then the following properties hold true:

(i) If $R_0 := \frac{\lambda \Lambda}{\mu} \leq 1$ then the above system has a unique bounded and non negative complete orbit $(V(t), w(t)) \equiv (\Lambda, 0)$ for all $t \in \mathbb{R}$.

(ii) Assume $R_0 > 1$. Let $(V(t), w(t))_{t \in \mathbb{R}}$ be a bounded non negative complete orbit. Then one has

$$\inf_{t \in \mathbb{R}} w(t) > 0 \implies (V(t), w(t)) \equiv \left( \frac{\mu}{\lambda}, \frac{R_0 - 1}{\lambda} \right), \forall t \in \mathbb{R}.$$

(iii) Assume $R_0 > 1$. Let $(V(t), w(t))_{t \geq 0}$ be a bounded and non negative orbit. Then one has

$$\exists \delta_0 \geq 0, \ w(t_0) > 0 \implies \lim_{t \to \infty} (V(t), w(t)) = \left( \frac{\mu}{\lambda}, \frac{R_0 - 1}{\lambda} \right).$$

Before recalling important results obtained in [3] let us introduce, for $k > 0$ and $\varepsilon > 0$ the time $t^\varepsilon_k$ defined by

$$t^\varepsilon_k = \frac{k}{2} \ln \left( \frac{1}{\varepsilon} \right).$$

Now recall the following uniform estimation result

**Lemma 3.3.** Let us consider the function

$$Q = Q^\varepsilon := U^\varepsilon + \beta^\varepsilon_1 w_1^\varepsilon + \frac{1}{2} \beta^\varepsilon_2 w_2^\varepsilon.$$

Then there exist positive constants $c$ and $k_1$ such that for all $\varepsilon > 0$ small enough the following upper bound holds true

$$0 < Q(t) \leq c, \forall t \geq t^\varepsilon_{k_1}.$$

We now recall the following estimates for $w_1 = w_1^\varepsilon$ and for

$$P(t) = P^\varepsilon(t) = \sum_{n=3}^{+\infty} \lambda^\varepsilon_n \beta^\varepsilon_n w_n^\varepsilon(t),$$

(17)
wherein we have set, for any $n \geq 3$,

$$w_n^\varepsilon(t) = \int_{\mathbb{R}^N} v^\varepsilon(t,x) \varphi_n^\varepsilon(x) \, dx$$
and

$$\beta_n^\varepsilon = \int_{\mathbb{R}^N} \beta(x) \varphi_n^\varepsilon(x) \, dx.$$

**Remark 2.** Note that with these notations, Lemma 3.1 holds true for any $m, n \geq 1$ whenever $w_n^\varepsilon(0) \neq 0$.

Then the following lemma holds true.

**Lemma 3.4.** Let $k_1$ be defined in the previous lemma. The following estimates hold true for $\varepsilon$ small enough

$$\forall t \in [t_k^\varepsilon, G(\varepsilon)^{-1}], \quad w_1^\varepsilon(t) \leq c\varepsilon^{-N} C(\rho, \varepsilon) = o(\varepsilon^\infty), \quad (18)$$
and

$$\forall t \geq t_k^\varepsilon, \quad \|P^\varepsilon(t)\| \leq c\varepsilon^{-N} \exp(-c\varepsilon t), \quad (19)$$
wherein $c$ is a positive constant independent of $\varepsilon < 1$.

As consequence of Lemma 4.7 in [3] we have a strong persistence property for $\beta_2^\varepsilon w_2^\varepsilon$ stated in the following lemma.

**Lemma 3.5.** There exists $k > 1$ large enough and $\eta > 0$ such that for any $\varepsilon << 1$ small enough one has

$$\beta_2^\varepsilon w_2^\varepsilon(t) \geq \eta \text{ for } t \in [t_k^\varepsilon, G(\varepsilon)^{-1}].$$

We complete this section by describing the following behaviour for the function $w_2^\varepsilon$, that was obtained in [3] using the above persistence property.

**Proposition 1.** Assume $\mathcal{R}_0 > 1$. There exists $k > 0$ large enough such that for any $\delta > 0$ there exists $\varepsilon_0 > 0$ small enough such that for all $\varepsilon \in (0, \varepsilon_0)$ one has

$$|\beta_2^\varepsilon w_2^\varepsilon(t) - \beta_2^\varepsilon V_2^\varepsilon| \leq \delta, \quad \forall t \in \left[t_k^\varepsilon, \frac{1}{G(\varepsilon)} - t_1^\varepsilon\right],$$
where $V_2^\varepsilon$ is defined in (6).

4. **Proof of Theorem 2.2.** In this section we complete the proof of Theorem 2.2. We start by checking (9) and this follows from Lemma 3.1 together with Remark 2 above.

**Proof of (9).** Note that since $\{\varphi_n^\varepsilon\}_{n \geq 1}$ forms an Hilbert basis, one has

$$v^\varepsilon(t,x) = \sum_{i=1}^{2} w_i^\varepsilon(t) \varphi_i^\varepsilon(x) + R^\varepsilon(t,x),$$
with

$$R^\varepsilon(t,x) = \sum_{n \geq 3} w_n^\varepsilon(t) \varphi_n^\varepsilon(x).$$

Hence, using Lemma 3.1 and Remark 2, we get

$$\|R^\varepsilon(t, \cdot)\|^2_{L^2(\mathbb{R}^N)} = \sum_{m \geq 3} |w_m^\varepsilon(t)|^2$$

$$= \sum_{n \geq 3} |w_n^\varepsilon(0)|^2 \left|\frac{w_2^\varepsilon(t)}{w_2^\varepsilon(0)}\right|^{\frac{2\lambda_m}{\lambda_2}} \exp \left(-2 \frac{\lambda_m}{\lambda_2} t\right).$$
However from Lemma 3.3, one gets, for any $\varepsilon$ small enough and for any $t \geq t^*_k$, 
\[
\|R^\varepsilon(t, \cdot)\|_{L^2(\mathbb{R}^N)}^2 \leq \left( \sum_{n \geq 3} |w^\varepsilon_n(0)|^2 \right) \frac{c}{\beta^2_w w^\varepsilon_2(0)} \exp \left(-2 \frac{\lambda^C_3 - \lambda^C_2}{\lambda^C_2} t \right).
\]

Therefore, because of (15) and Assumption 2 (i) (for the second spectral gap $H(\varepsilon)$), there exists some constant $C > 0$ such that for all $\varepsilon > 0$ small enough and $t \geq t^*_k$, one has 
\[
\|R^\varepsilon(t, \cdot)\|_{L^2(\mathbb{R}^N)}^2 \leq \|v_0\|_{L^2(\mathbb{R}^N)}^2 C \varepsilon^{-2N} \exp (-C \varepsilon t),
\]
and the result follows. This implies (9). \(\square\)

Throughout this section, for notation simplicity, we set
\[
v^\varepsilon_n(t) = \beta^\varepsilon_n w^\varepsilon_n(t), \quad \forall n \geq 1. \tag{20}
\]

Next, similarly as in [3], projecting System (3) on the Hilbert basis $\{\varphi^\varepsilon_n\}_{n \geq 1}$ ensures that $(U^\varepsilon, v^\varepsilon_n)$ becomes a solution of the following infinite system of ODEs
\[
\begin{cases}
\dot{U}^\varepsilon(t) = \Lambda - U^\varepsilon(t) - U^\varepsilon(t) \sum_{n=1}^{\infty} \lambda^\varepsilon_n v^\varepsilon_n(t), \\
\dot{v}^\varepsilon_n(t) = v^\varepsilon_n(t) (U^\varepsilon(t) \lambda^\varepsilon_n - \mu), \quad n \geq 1,
\end{cases}
\]

In this section we will extensively use the quantity $\mathcal{R}_0$ and $V^0$ defined respectively in (5) and (7). And, throughout this section we assume that $\mathcal{R}_0 > 1$.

Let $\alpha \in (0, 1)$ be given and fixed. Observe that as a consequence of Theorem 2.1, we have for any $\varepsilon > 0$,
\[
\lim_{t \to +\infty} v^\varepsilon_n(t) = 0,
\]
while Proposition 1 implies that
\[
\lim_{\varepsilon \to 0} v^\varepsilon_2(G(\varepsilon)^{-1} - t^*_1) = V^0,
\]
so that we are able to define for all $\varepsilon$ small enough the time $t_{\varepsilon, \alpha}$ defined in (8). Note that this definition implies that, for all $\varepsilon$, one has
\[
v^\varepsilon_2(t) \geq \alpha V^0, \quad \forall t \in [G(\varepsilon)^{-1} - t^*_1, t_{\varepsilon, \alpha}]. \tag{21}
\]

Next we claim that the following lemma holds true

**Lemma 4.1.** There exists $\eta_1 > 0$ such that for all $\varepsilon$ small enough one has
\[
v^\varepsilon_2(t_{\varepsilon, \alpha}) = \alpha V^0 \quad \text{and} \quad v^\varepsilon_1(t_{\varepsilon, \alpha}) \geq \eta_1.
\]

**Proof.** The first statement in the above lemma follows from the definition of $t_{\varepsilon, \alpha}$ in (8). To prove the second statement for $v^\varepsilon_1$, we assume by contradiction that there exists a sequence $(\varepsilon_n)_{n \geq 0}$ with $\varepsilon_n \to 0$ as $n \to \infty$ such that
\[
\lim_{n \to \infty} v^\varepsilon_1(t_{\varepsilon_n, \alpha}) = 0.
\]

Let us define, for each $n \geq 0$ and for $t \geq -t_{\varepsilon_n, \alpha}$, the function
\[
Y_n(t) = \begin{pmatrix}
U^{\varepsilon_n}(t + t_{\varepsilon_n, \alpha}) \\
v^\varepsilon_1(t + t_{\varepsilon_n, \alpha}) \\
v^\varepsilon_2(t + t_{\varepsilon_n, \alpha})
\end{pmatrix}
\]

Because of Lemma 3.3, the above sequence of functions is uniformly bounded for \( t \geq t_{k_1}^{\varepsilon} - t_{\varepsilon,n} \). Then, using Arzelà-Ascoli theorem and since \( t_{k_1}^{\varepsilon} - t_{\varepsilon,n} \to -\infty \) as \( n \to \infty \), possibly up to a subsequence, \( Y_n(t) \) converges locally uniformly towards a complete orbit \( Y(t) = (V(t), w_1(t), w_2(t)) \) of the following system

\[
\begin{cases}
 V' = \Lambda - V - \|\Theta\|_\infty^2 (w_1 + w_2)V \\
 w_1' = w_1 (\|\Theta\|_\infty^2 V - \mu) \\
 w_2' = w_2 (\|\Theta\|_\infty^2 V - \mu)
\end{cases}, t \in \mathbb{R}, \tag{22}
\]

together with the conditions

\( V(0) \geq 0, w_1(0) = 0 \) and \( w_2(0) = \alpha V^0 > 0 \).

In particular, we first obtain that \( w_1 \equiv 0, \) and \( w_2(t) > 0 \) for all \( t \in \mathbb{R} \). And as \( R_0 > 1 \) then Lemma 3.2 \((iii)\) implies

\[
\lim_{t \to \infty} (V(t), w_2(t)) = \left( \frac{\mu}{\|\Theta\|_\infty^2}, \frac{R_0 - 1}{\|\Theta\|_\infty^2} \right). \tag{23}
\]

Next due to Lemma 3.5 and the definition of \( t_{\varepsilon,n} \) (see (8) and (21)), one has

\[
\min \{v_2^{\varepsilon_n}(t), t_{k_1}^{\varepsilon_n} \leq t \leq t_{\varepsilon,n}\} \geq \min \{\alpha, \eta\}, \forall n > 1.
\]

Next, since

\[
t_{k_1}^{\varepsilon_n} - t_{\varepsilon,n} \leq t_{k_1}^{\varepsilon_n} + t_{k_1}^{\varepsilon_n} - \frac{1}{G(\varepsilon_n)} \to -\infty
\]

then we have

\[
w_2(t) \geq \min \{\alpha, \eta\}, \forall t \leq 0. \tag{24}
\]

And, coupling \( w_2 > 0 \) together with (23) and (24) ensures that

\[
\inf_{t \in \mathbb{R}} w_2(t) > 0.
\]

Finally Lemma 3.2 \((ii)\) applies and implies \( w_2 \equiv V^0 \), a contradiction with the normalisation condition \( w_2(0) = \alpha V^0 < V^0 \). This completes the proof of Lemma 4.1. \( \square \)

The next lemma gives an estimate of the value of \( t_{\varepsilon,n} \) for a given and fixed \( \alpha \in (0, 1) \) when \( \varepsilon \ll 1 \).

**Lemma 4.2.** Let \( \alpha \in (0, 1) \) be given and fixed. Then one has

\[
t_{\varepsilon,n} = G(\varepsilon)^{-1} + \frac{\lambda_2}{\mu G(\varepsilon)} \left[ \ln \frac{v_2^{\varepsilon}(0)}{v_1^{\varepsilon}(0)} + O(1) \right] \text{ for } \varepsilon \ll 1, \tag{25}
\]

and

\[
t_{\varepsilon,n} - G(\varepsilon)^{-1} \sim \frac{\|\Theta\|_\infty^2}{\mu G(\varepsilon)} \ln \frac{1}{v_1^{\varepsilon}(0)} \text{ as } \varepsilon \to 0. \tag{26}
\]

Before proving this result, let us recall (see Lemma 4.3 in [3]) that both quantities \( v_1^{\varepsilon}(0) \) and \( v_2^{\varepsilon}(0) \) tend to 0 as \( \varepsilon \to 0 \). Moreover, in addition to (15), the following estimates hold true

\[
\varepsilon^N = O(v_2^{\varepsilon}(0)), \ v_2^{\varepsilon}(0) = o(1) \text{ and } v_1^{\varepsilon}(0) = o(\varepsilon^\infty) \text{ as } \varepsilon \to 0. \tag{27}
\]

The latter estimates allow us to conclude that (14) in Theorem 2.2 holds true.
Proof. We use the identity linking $v_1$ to $v_2$ given by Lemma 3.1 with $m = 1, n = 2$, $t = t_{\varepsilon, \alpha}$ and $s = G(\varepsilon)^{-1}$, that is

$$v_1^\varepsilon(t_{\varepsilon, \alpha}) = v_1^\varepsilon(G(\varepsilon)^{-1}) \left( \frac{v_2^\varepsilon(t_{\varepsilon, \alpha})}{v_2^\varepsilon(G(\varepsilon)^{-1})} \right)^{\frac{\lambda_1^\varepsilon}{\lambda_2^\varepsilon}} e^{-\frac{\lambda_2^\varepsilon - \lambda_1^\varepsilon}{\lambda_2^\varepsilon} \mu(t_{\varepsilon, \alpha} - G(\varepsilon)^{-1})},$$

and also between with $s = 0$ and $t = G(\varepsilon)^{-1}$, that reads as

$$v_2^\varepsilon(G(\varepsilon)^{-1}) = v_1^\varepsilon(0) \left( \frac{v_2^\varepsilon(G(\varepsilon)^{-1})}{v_2^\varepsilon(0)} \right)^{\frac{\lambda_1^\varepsilon}{\lambda_2^\varepsilon}} e^{\frac{\mu}{\lambda_2^\varepsilon}}.$$

This also use to eliminate $v_1^\varepsilon(G(\varepsilon)^{-1})$ from the first identity. Indeed, recalling that $v_2^\varepsilon(t_{\varepsilon, \alpha}) = \alpha V^0$, this reads as follows

$$v_1^\varepsilon(t_{\varepsilon, \alpha}) = v_1^\varepsilon(0) e^{\frac{\mu}{\lambda_2^\varepsilon}} \left( \frac{\alpha V^0}{v_2^\varepsilon(0)} \right)^{\frac{\lambda_1^\varepsilon}{\lambda_2^\varepsilon}} e^{-\frac{\lambda_2^\varepsilon - \lambda_1^\varepsilon}{\lambda_2^\varepsilon} \mu(t_{\varepsilon, \alpha} - G(\varepsilon)^{-1})}.$$

The above formula allows to compute the value of $t_{\varepsilon, \alpha}$ as follows

$$t_{\varepsilon, \alpha} = G(\varepsilon)^{-1} + \frac{\lambda_2^\varepsilon}{\mu G(\varepsilon)} \left[ -\ln v_1^\varepsilon(0) + \frac{\lambda_1^\varepsilon}{\lambda_2^\varepsilon} \ln v_2^\varepsilon(0) + \ln v_1^\varepsilon(t_{\varepsilon, \alpha}) - \frac{\mu}{\lambda_2^\varepsilon} - \frac{\lambda_1^\varepsilon}{\lambda_2^\varepsilon} \ln \alpha V^0 \right].$$

Now using Lemma 3.3 and 4.1, one knows that $\ln v_1^\varepsilon(t_{\varepsilon, \alpha}) = O(1)$ as $\varepsilon \ll 1$. Hence we get

$$t_{\varepsilon, \alpha} = G(\varepsilon)^{-1} + \frac{\lambda_2^\varepsilon}{\mu G(\varepsilon)} \left[ \ln \frac{v_2^\varepsilon(0)}{v_1^\varepsilon(0)} \frac{\lambda_1^\varepsilon}{\lambda_2^\varepsilon} + O(1) \right] \text{ for } \varepsilon \ll 1.$$

Hence (25) follows. To check (26), it is sufficient to remember that, due to (27), one has

$$\frac{v_1^\varepsilon(0)}{v_2^\varepsilon(0)} = o(\varepsilon^\infty) \to 0 \text{ as } \varepsilon \to 0.$$

The next lemma shows that once the component $v_1$ is uniformly away – with respect to $\varepsilon \ll 1$ – from 0 at $t = t_{\varepsilon, \alpha}$, it remains so for all greater values of time $t$.

**Lemma 4.3.** Let $\alpha \in (0, 1)$ be given. There exists $\eta_1^\varepsilon > 0$ such that for all $\varepsilon > 0$ small enough the following persistence property holds true for $v_1^\varepsilon$

$$v_1^\varepsilon(t) \geq \eta_1^\varepsilon, \quad \forall t \geq t_{\varepsilon, \alpha}. \quad (28)$$

Furthermore, there exists some constant $C > 1$ such that for all $\varepsilon$ small enough it holds that:

(i) for all $t \in [t_{\varepsilon, \alpha}, t_{\varepsilon, \alpha}]$, one has

$$C^{-1} e^{-G(\varepsilon) \frac{\lambda_1^\varepsilon}{\lambda_2^\varepsilon} (t-t_{\varepsilon, \alpha})} \leq v_1^\varepsilon(t) \leq C e^{-G(\varepsilon) \frac{\lambda_1^\varepsilon}{\lambda_2^\varepsilon} (t-t_{\varepsilon, \alpha})}, \quad (29)$$

(ii) for all $t \geq t_{\varepsilon, \alpha}$, one has

$$C^{-1} e^{-G(\varepsilon) \frac{\lambda_1^\varepsilon}{\lambda_2^\varepsilon} (t-t_{\varepsilon, \alpha})} \leq v_2^\varepsilon(t) \leq C e^{-G(\varepsilon) \frac{\lambda_1^\varepsilon}{\lambda_2^\varepsilon} (t-t_{\varepsilon, \alpha})}. \quad (30)$$
Proof. The proof of the first part of this lemma makes use a Lyapunov-like argument. To that aim set \( g(x) = x - 1 - \ln x \) and define for \( t > 0 \) the function
\[
\mathcal{V}(t) = U_1^\varepsilon g \left( \frac{U_1^\varepsilon(t)}{U_1^\varepsilon(t)} \right) + \beta_1^\varepsilon V_1^\varepsilon g \left( \frac{v_1^\varepsilon(t)}{\beta_1^\varepsilon V_1^\varepsilon} \right) + v_2^\varepsilon(t). \tag{31}
\]
Here, recall that \( V_1^\varepsilon \) is defined in (6) and that we have set
\[
U_1^\varepsilon = \frac{\mu}{\lambda_1^\varepsilon}.
\]

Then we claim that

Claim 1. Setting
\[
P_\varepsilon(t) = \sum_{n=1}^{\infty} \lambda_n^\varepsilon v_n^\varepsilon(t),
\]
for any \( t > 0 \), the following inequality holds true
\[
\mathcal{V}'(t) \leq - \frac{(U_1^\varepsilon(t) - U_1^\varepsilon)^2}{U_1^\varepsilon(t)} - \mu \frac{\lambda_1^\varepsilon - \lambda_2^\varepsilon}{\lambda_1^\varepsilon} v_2^\varepsilon(t) + \frac{U_1^\varepsilon(t) - U_1^\varepsilon}{U_1^\varepsilon(t)} P_\varepsilon(t) + S_\varepsilon(t). \tag{32}
\]

Before proving this claim, recall that since we use the notation (20), namely \( v_n^\varepsilon = \beta_n^\varepsilon w_n^\varepsilon \), the function \( P_\varepsilon \) defined above corresponds those defined in (17).

Proof of Claim 1. Using the definitions of \( U_1^\varepsilon \) and \( V_1^\varepsilon \), straightforward algebra leads to
\[
\mathcal{V}'(t) = - \frac{(U_1^\varepsilon(t) - U_1^\varepsilon)^2}{U_1^\varepsilon(t)} - \mu \frac{\lambda_1^\varepsilon - \lambda_2^\varepsilon}{\lambda_1^\varepsilon} v_2^\varepsilon(t) + \frac{U_1^\varepsilon(t) - U_1^\varepsilon}{U_1^\varepsilon(t)} P_\varepsilon(t) + S_\varepsilon(t),
\]
wherein we have set
\[
S_\varepsilon(t) = 2\mu V_1^\varepsilon - \frac{\mu^2 V_1^\varepsilon}{\lambda_1^\varepsilon} \frac{1}{U_1^\varepsilon(t)} - \lambda_1^\varepsilon V_1^\varepsilon U_1^\varepsilon(t),
\]
and \( S_\varepsilon(t) \leq 0 \) thanks to the inequality \( a + b \geq 2\sqrt{ab} \), which proves (32).

To complete the proof of (28), let us recall that (19) ensures that there exists some constant \( c > 0 \) such that
\[
\forall t \geq t_{\varepsilon, \alpha} = \frac{k_1}{\varepsilon} \ln \frac{1}{\varepsilon}, \quad |P_\varepsilon(t)| \leq \frac{c}{\varepsilon^N} e^{-c t}.
\]
Thus using the inequality (32) in the claim above, one obtains that for all \( t \geq t_{\varepsilon, \alpha} \geq G(\varepsilon)^{-1} \)
\[
\mathcal{V}'(t) \leq \frac{c}{\varepsilon^N} e^{-c t},
\]
which provides by integrating between \( t_{\varepsilon, \alpha} \) and \( t \geq t_{\varepsilon, \alpha} \)
\[
\mathcal{V}(t) \leq \mathcal{V}(t_{\varepsilon, \alpha}) + \frac{1}{\varepsilon^{N+1}} e^{-c t_{\varepsilon, \alpha}}, \forall t \geq t_{\varepsilon, \alpha}.
\]
Next due to the definition of \( t_{\varepsilon, \alpha} \) in (8), one knows that \( \mathcal{V}(t_{\varepsilon, \alpha}) \) is uniformly bounded with respect to \( \varepsilon \). Moreover, since \( e^{-c t_{\varepsilon, \alpha}} = o(\varepsilon^{\infty}) \) for \( \varepsilon \ll 1 \), the quantity \( \mathcal{V}(t) \) is also uniformly bounded with respect to \( \varepsilon \) and with respect to \( t \geq t_{\varepsilon, \alpha} \). Recalling the definition of \( \mathcal{V} \) (using the function \( g \)) in (31), (28) follows.

Now in order to prove (30) and (29), we apply Lemma 3.1 with \( m = 2, n = 1, s = t_{\varepsilon, \alpha} \) and \( t > 0 \). That yields
\[
v_2^\varepsilon(t) = v_2^\varepsilon(t_{\varepsilon, \alpha}) \left( \frac{v_1^\varepsilon(t)}{v_1^\varepsilon(t_{\varepsilon, \alpha})} \right)^{\lambda_2^\varepsilon/\lambda_1^\varepsilon} e^{-G(\varepsilon)^{\frac{\mu}{\lambda_1^\varepsilon}}(t-t_{\varepsilon, \alpha})}. \tag{33}
\]
Next, since $v_1^\varepsilon$ is bounded above for $t \geq t_{\varepsilon,\alpha}$ (see Lemma 3.3) and below away from zero due to (28), there exists a constant $C > 1$ such that for all $\varepsilon$ small enough and all $t \geq t_{\varepsilon,\alpha}$
\[ C^{-1}e^{-G(\varepsilon)\frac{\mu}{\Theta}(t-t_{\varepsilon,\alpha})} \leq v_2^\varepsilon(t) \leq Ce^{-G(\varepsilon)\frac{\mu}{\Theta}(t-t_{\varepsilon,\alpha})}, \forall t \geq t_{\varepsilon,\alpha}, \]
and (30) follows.

We now come back to (33) and prove (29). To that aim recall that there exists a constant $C > 1$ such that for all $\varepsilon$ small enough one has
\[ C^{-1} \leq v_1^\varepsilon(t_{\varepsilon,\alpha}) \leq C, \]
\[ C^{-1} \leq v_2^\varepsilon(t) \leq C, \forall t \in [t_{\varepsilon,k}^\varepsilon, t_{\varepsilon,\alpha}]. \]
As a consequence of (33), there exists some constant, still denoted by $C > 1$ such that for all $\varepsilon < 1$, for all $t \in [t_{\varepsilon,k}^\varepsilon, t_{\varepsilon,\alpha}]$, one has
\[ C^{-1}e^{G(\varepsilon)\frac{\mu}{\Theta}(t-t_{\varepsilon,\alpha})} \leq v_1^\varepsilon(t) \leq Ce^{G(\varepsilon)\frac{\mu}{\Theta}(t-t_{\varepsilon,\alpha})}. \]
This completes the proof of (29) and thus the proof of the lemma.

Next we are able to prove the following result, that roughly speaking states that the functions $U^\varepsilon$ and $v_1^\varepsilon + v_2^\varepsilon$ are mostly constant for $\varepsilon \ll 1$.

**Proposition 2.** The following limits hold true
\[ U^\varepsilon(t) \to \frac{\mu}{\|\Theta\|_\infty^2} \] (34)
and
\[ v_1^\varepsilon(t) + v_2^\varepsilon(t) \to V^0, \] (35)
as $\varepsilon \to 0$, uniformly for $t \in \mathbb{R}$. Here recall that $t_k^\varepsilon$ is defined in (16) while the constant $k$ is provided by Lemma 3.5.

Note that the above proposition implies (10) and (11) in Theorem 2.2.

**Proof.** First note that (34) directly follows from (35). Hence, in this proof we focus upon proving (35).

In order to prove (35) we argue by contradiction by assuming that there exist some constant $\delta > 0$, a sequence $(\varepsilon_n)_{n \geq 0}$ that tends to 0 as $n$ tends to infinity and a sequence $(t_n)_{n \geq 0}$ with $t_n \geq t_{k+1}^\varepsilon$ for all $n \geq 0$ such that
\[ |v_1^\varepsilon_n(t_n) + v_2^\varepsilon_n(t_n) - V^0| \geq \delta. \]
Let us define for each $n \geq 0$ and for $t \geq -t_n$ the function $Y_n(t)$ by
\[ Y_n(t) = \begin{pmatrix} U^\varepsilon_n(t + t_n) \\ v_1^\varepsilon_n(t + t_n) \\ v_2^\varepsilon_n(t + t_n) \end{pmatrix}. \]
Since this sequence of functions is uniformly bounded, using Azelà-Ascoli theorem one may assume, possibly along a subsequence, that $Y_n(t)$ converges as $n \to \infty$, locally uniformly for $t \in \mathbb{R}$, to some function $t \mapsto Y(t)$ given by
\[ t \to Y(t) = \begin{pmatrix} V(t) \\ w_1(t) \\ w_2(t) \end{pmatrix}. \]
that becomes a complete orbit of the system of equations (22). This function also inherits from the condition
\[ |w_1(0) + w_2(0) - V_0| \geq \delta. \] (36)

Let us consider the function \( w(t) = w_1(t) + w_2(t) \) with \( t \in \mathbb{R} \). Next adding up the last two equations in (22) yields
\[
\begin{cases}
V' = A - V\lambda_0 w V, \\
w' = \mu (w_0 V - \mu), \quad t \in \mathbb{R}
\end{cases}
\] (37)
while (36) ensures that \( w(0) \neq V_0 \).

Moreover, the following property also holds
\[ \exists \eta_S > 0, \forall n \gg 1, \forall t \in [t^n_k, \infty), \ v^n_2(t) + v^n_2(t) \geq \eta_S. \] (38)
Indeed for all \( n \), due to Lemma 3.5, one has \( v^n_2(t) \geq \eta > 0 \) for all \( t \in [t^n_k, t^n_{\varepsilon_n, \alpha}] \) whereas, due to Lemma 4.3, \( v^n_1(t) \geq \eta^*_1 > 0 \) for all \( t \geq t^n_{\varepsilon_n, \alpha} \), so that (38) holds true.

Next, (38) rewrites as, for all \( n \) large enough,
\[ v^n_1(t + t_n) + v^n_2(t + t_n) \geq \eta_S > 0 \]
for all \( t \geq t^n_k - t_n \). And, since \( t^n_k - t_n \geq t^n_k - t^n_{k+1} \to -\infty \) as \( n \to \infty \), we conclude that
\[ w(t) \geq \eta_S > 0, \ \forall t \in \mathbb{R}. \] (39)

As a consequence, \( t \mapsto (V(t), w(t)) \) becomes a non negative complete orbit of (37) satisfying (39), that ensures that \( (V(t), w(t)) \equiv (\frac{A}{\lambda_0}, V^0) \), a contradiction with \( w(0) \neq V^0 \). This completes the proof of the proposition.

We are now able to conclude the proof of Theorem 2.2. To that aim it remains to complete the proof for (12) and (13). Since these proofs are similar we only prove (12).

Proof of (12). Coming back to (33), we have, for all \( \varepsilon \ll 1 \) and \( t > 0 \),
\[ v^n_2(t) = v^n_2(t, \varepsilon, \alpha) \left( \frac{v^n_1(t)}{v^n_1(t, \varepsilon, \alpha)} \right) ^{\lambda^*_1 / \lambda^*_1} e^{-G(\varepsilon) \frac{\varepsilon}{\lambda^*_1} (t-t_{\varepsilon, \alpha})}. \]
Next we set
\[ R^\varepsilon(t) := v^n_1(t) + v^n_2(t) - V^0. \]
And, recalling the definition of \( t_{\varepsilon, \alpha} \) in (8), we have \( v^n_2(t, \varepsilon, \alpha) = \alpha V^0 \). So that we infer from the above equality that
\[ V^0 - v^n_1(t) + R^n(t) = \alpha V^0 \left( \frac{v^n_1(t)}{v^n_1(t, \varepsilon, \alpha)} \right) ^{1 - G(\varepsilon) / \lambda^*_1} e^{-G(\varepsilon) \frac{\varepsilon}{\lambda^*_1} (t-t_{\varepsilon, \alpha})}, \] (40)
Let us denote by \( K = K(\alpha) \) the quantity
\[ K(\alpha) = \frac{\alpha}{1 - \alpha}. \]
Then (40) rewrites as
\[ V^0 - v^n_1(t) + R^n(t) = K v^n_1(t) e^{-G(\varepsilon) \frac{\varepsilon}{\lambda^*_1} (t-t_{\varepsilon, \alpha})} - W^n(\varepsilon(\varepsilon)), \]
with
\[ W^n(\varepsilon(\varepsilon)) = K v^n_1(t) e^{-G(\varepsilon) \frac{\varepsilon}{\lambda^*_1} (t-t_{\varepsilon, \alpha})} \left( 1 - \frac{(1 - \alpha) V^0}{(v^n_1(t, \varepsilon, \alpha)) ^{1 - G(\varepsilon) / \lambda^*_1}} (v^n_1(t)) ^{-G(\varepsilon) / \lambda^*_1} \right). \]
Now to complete the proof of the theorem, let us show from the above formula that
\[ v_1^\varepsilon(t) = \frac{V_0 + R_\varepsilon(t) + W_\varepsilon(t)}{1 + Ke^{-G(\varepsilon)K_1(t-t_{\varepsilon,\alpha})}}. \]
This yields
\[ \Delta_\varepsilon(t) := v_1^\varepsilon(t) - \frac{V_0}{1 + Ke^{-G(\varepsilon)K_1(t-t_{\varepsilon,\alpha})}} \to 0, \]
as \( \varepsilon \to 0 \) uniformly for \( t \geq t_k^\varepsilon \).
To that aim we split our estimates into two different regions: for \( t \geq \hat{T}_\varepsilon \) for a suitable \( \hat{T}_\varepsilon \leq t_{\varepsilon,\alpha} \) and finally for \( t \in [t_k^\varepsilon, \hat{T}_\varepsilon] \).
To perform our analysis, note that one also has for all \( t \geq t_{\varepsilon,\alpha} \)
\[ |\Delta_\varepsilon(t)| \leq |R_\varepsilon(t)| + |W_\varepsilon(t)|. \]
Next, set
\[ C > \text{sup} |\Delta_\varepsilon(t)| \leq \text{sup} |R_\varepsilon(t)| + \text{sup} |W_\varepsilon(t)|. \]
Furthermore (29) ensures that there exists some constant \( C > 0 \) such that for all \( \varepsilon \) small enough one has
\[ v_1^\varepsilon(t)e^{-G(\varepsilon)K_1(t-t_{\varepsilon,\alpha})} \leq C, \forall t \geq t_k^\varepsilon. \]
Next, set
\[ I_\varepsilon(t) := G(\varepsilon) \ln v_1^\varepsilon(t). \]
And, using (29) and since \( v_1^\varepsilon(t) \) is uniformly bounded and away from 0 for \( t \geq t_{\varepsilon,\alpha} \), one obtains that
\[ \lim_{\varepsilon \to 0} \sup_{t \geq t_{\varepsilon,\alpha}} |I_\varepsilon(t)| = 0. \]
Furthermore (29) ensures that there exists some constant \( C > 0 \) such that, for all \( \varepsilon \) small enough, it holds that
\[ |I_\varepsilon(t)| \leq G(\varepsilon)(C + G(\varepsilon)t), \forall t \in [0, t_{\varepsilon,\alpha} - t_k^\varepsilon]. \]
Let \( (T_\varepsilon) \) be a family of positive times such that \( T_\varepsilon \to \infty \) as \( \varepsilon \to 0 \) and
\[ G(\varepsilon)^{-1} \ll T_\varepsilon \ll \min \left( G(\varepsilon)^{-2}, t_{\varepsilon,\alpha} - t_k^\varepsilon \right) \text{ as } \varepsilon \to 0. \]
Recall that the latter condition is feasible since \( G(\varepsilon)^{-1} \ll t_{\varepsilon,\alpha} \), (see Lemma 4.2).
Then one has
\[ \lim_{\varepsilon \to 0} \sup_{t \in [t_{\varepsilon,\alpha} - T_\varepsilon, t_{\varepsilon,\alpha}]} |I_\varepsilon(t)| = 0. \]
Therefore one gets \( (v_1^\varepsilon(t))^{-G(\varepsilon)/K_1} \to 1 \) uniformly for \( t \geq T_\varepsilon \) as \( \varepsilon \to 0 \) and
\[ \lim_{\varepsilon \to 0} \sup_{t \geq t_{\varepsilon,\alpha} - T_\varepsilon} |W_\varepsilon(t)| = 0. \]
And using (41) and setting \( \hat{T}_\varepsilon := t_{\varepsilon,\alpha} - T_\varepsilon \), this ensures that
\[ \lim_{\varepsilon \to 0} \sup_{t \geq \hat{T}_\varepsilon} |\Delta_\varepsilon(t)| = 0. \]
We now estimate \( \Delta_\varepsilon(t) \) for \( t \in [t_k^\varepsilon, \hat{T}_\varepsilon] \). To do so, note that one also has
\[ \sup_{t \in [t_k^\varepsilon, t_{\varepsilon,\alpha} - T_\varepsilon]} |\Delta_\varepsilon(t)| \leq \sup_{t \in [t_k^\varepsilon, t_{\varepsilon,\alpha} - T_\varepsilon]} v_1^\varepsilon(t) + \sup_{t \in [t_k^\varepsilon, t_{\varepsilon,\alpha} - T_\varepsilon]} \frac{V_0}{1 + Ke^{-G(\varepsilon)K_1(t-t_{\varepsilon,\alpha})}}. \]
And, because of the choice \( T \varepsilon \) in (42) and using (29), one obtains that

\[
\sup_{t \in [t_\varepsilon, t_{\varepsilon, a} - T \varepsilon]} |\Delta^\varepsilon(t)| = O \left( e^{-\frac{\varepsilon}{T \varepsilon} G(\varepsilon) T \varepsilon} \right) \to 0 \text{ as } \varepsilon \to 0.
\]

This proves that \( \Delta^\varepsilon(t) \to 0 \) uniformly for \( t \geq t_\varepsilon \) as \( \varepsilon \to 0 \). This completes the proof of (12) and thus the one of Theorem 2.2. \( \square \)

REFERENCES

[1] M. Alfaro and M. Veruete, Evolutionary branching via replicator mutator equations, *Journal of Dynamics and Differential Equations*, (2018), 1–24.

[2] O. Bonnefon, J. Coville and G. Legendre, Concentration phenomenon in some non-local equation, *Discrete Continuous Dynam. Systems - B*, 22 (2017), 763–781.

[3] J.-B. Burie, R. Djidjou-Demasse and A. Ducrot, Asymptotic and transient behaviour for a nonlocal problem arising in population genetics, *European Journal of Applied Mathematics*, 2018, 1–27.

[4] Å. Calsina, S. Cuadrado, L. Desvillettes and G. Raoul, Asymptotics of steady states of a selection-mutation equation for small mutation rate, *Proc. Math. Roy. Soc. Edinb.*, 143 (2013), 1123–1146.

[5] Å. Calsina and S. Cuadrado, Stationary solutions of a selection mutation model: The pure mutation case, *Mathematical Models and Methods in Applied Sciences*, 15 (2005), 1091–1117.

[6] S. Cuadrado, Equilibria of a predator prey model of phenotype evolution, *J. Math. Anal. Appl.*, 354 (2009), 286–294.

[7] T. Day and S. Gandon, Applying population-genetic models in theoretical evolutionary epidemiology, *Ecology Letters*, 10 (2007), 876–888.

[8] T. Day and S. R. Proulx, A general theory for the evolutionary dynamics of virulence, *The American Naturalist*, 163 (2004), E40–E63.

[9] O. Diekmann, P. E. Jabin, S. Mischler and B. Perthame, The dynamics of adaptation: An illuminating example and a Hamilton-Jacobi approach, *Theor. Popul. Biol.*, 67 (2005), 257–271.

[10] R. Djidjou-Demasse, A. Ducrot and F. Fabre, Steady state concentration for a phenotypic structured problem modeling the evolutionary epidemiology of spore producing pathogens, *Mathematical Models and Methods in Applied Sciences*, 27 (2017), 385–426.

[11] F. Fabre, E. Rousseau, L. Mailleret and B. Moury, Durable strategies to deploy plant resistance in agricultural landscapes, *New Phytologist*, 193 (2012), 1064–1075.

[12] F. Fabre, E. Rousseau, L. Mailleret and B. Moury, Epidemiological and evolutionary management of plant resistance: optimizing the deployment of cultivar mixtures in time and space in agricultural landscapes, *Evol. Appl.*, 8 (2015), 919–932.

[13] S. A. Geritz, J. A. Metz, É. Kisdi and G. Meszéna, Dynamics of adaptation and evolutionary branching, *Phys. Rev. Lett.*, 78 (1997), 2024–2027.

[14] Q. Griette, Singular measure traveling waves in an epidemiological model with continuous phenotypes, *Transactions of the American Mathematical Society*, 371 (2019), 4411–4458.

[15] G. L. Iacono, F. van den Bosch and N. Paveley, The evolution of plant pathogens in response to host resistance: factors affecting the gain from deployment of qualitative and quantitative resistance, *J. Theo. Biol.*, 304 (2012), 152–163.

[16] C. Lannou, Variation and selection of quantitative traits in plant pathogens, *Annu. Rev. Phytopath.*, 50 (2012), 319–338.

[17] S. Lion and S. Gandon, Spatial evolutionary epidemiology of spreading epidemics, *Proc. R. Soc. B*, 283 (2016).

[18] J. A. J. Metz, S. A. H. Geritz, G. Meszéna, F. J. A. Jacobs and J. S. van Heerwaarden, Adaptive dynamics, a geometrical study of the consequences of nearly faithful reproduction, in *Stochastic and Spatial Structures of Dynamical System* (eds. S. J. van Strien, S. M. Verduyn Lunel), North-Holland, Amsterdam, 45 (1996), 183–231.

[19] P. Meyer-Nieberg, *Banach Lattices*, Springer-Verlag Berlin Heidelberg, 1991.

[20] S. Mirrahimi, B. Perthame, E. Bouin and P. Millien, Population formulation of adaptive meso-evolution: Theory and numerics, in *The Mathematics of Darwins Legacy*, Springer Basel, Basel, (2011), 159–174.
[21] S. Mirrahimi, B. Perthame and J. Y. Wakano, Evolution of species trait through resource competition, *J. Math. Biol.*, 64 (2012), 1189–1223.

[22] S. Nordmann, B. Perthame and C. Taing, Dynamics of concentration in a population model structured by age and a phenotypical trait, *Acta Appl. Math.*, 155 (2018), 197–225.

[23] M. Zerner, Quelques propriétés spectrales des opérateurs positifs, *Journal of Functional Analysis*, 72 (1987), 381–417.

[24] J. Zhan, P. H. Thrall, J. Papaï, L. Xie and J. J. Burdon, Playing on a pathogen’s weakness: Using evolution to guide sustainable plant disease control strategies, *Annu. Rev. Phytopathol.*, 53 (2015), 19–43.

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E-mail address: jean-baptiste.burie@u-bordeaux.fr
E-mail address: ramses.djidjouedemasse@ird.fr
E-mail address: arnaud.ducrot@univ-lehavre.fr