THE (NON-UNIFORM) HRUSHOVSKI-LANG-WEIL ESTIMATES

K. V. SHUDDHODAN

Abstract. In this article, we obtain a non-uniform version of Hrushovski’s generalization of the Lang-Weil estimates using ℓ-adic methods and without recourse to alterations. Our method also implies that the associated generating function is rational. Along the way we obtain a bound for the local terms for a class of non-proper correspondence which could be of independent interest.

Contents

1. Introduction 2
2. Notations and conventions 5
3. Outline of the proof 7
4. Varshavsky’s trace formula 10
  4.1. Correspondences and cohomological correspondences 10
  4.2. The Lefschetz-Verdier trace formula 13
  4.3. Trace maps 14
  4.4. Locally contracting correspondences 16
  4.5. Generalities on Cohomology with support 20
5. Preliminary reductions 20
  5.1. Reduction to $X$ smooth and $c_2$ generically étale 20
  5.2. A bound on the global terms 27
6. Local terms along a locally invariant subset 28
  6.1. Essentially proper correspondences 29
  6.2. Theorem 1.0.5 implies Theorem 1.0.3, (1) and (3) 31
  6.3. The pairing 32
  6.4. Trace along the graphs of Frobenius 34
  6.5. Local terms using the pairing (6.3.9) 36
  6.6. Action of partial Frobenius on the pairing 39
  6.7. Proof of Theorem 6.1.7 43
  6.8. Proof of Theorem 1.0.3, (2) 43
Appendix A. A cohomological correspondence of the intersection complex 43
  A.1. Basic properties of $IC_X$ 43
  A.2. Decomposition in the derived category and a Lemma 45
  A.3. A pullback map on $IC_X$ 46
References 51

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1. Introduction

Let $k$ be an algebraically closed field of characteristic $p > 0$. For any $p$-primary number $q = p^n$, and any scheme $X/k$, by $X^{(q)}$ we mean the base change of $X/k$ along the $n$th iterate of the absolute Frobenius of $k$. Let $F^{(q)}_{X/k}$ be the induced relative Frobenius morphism from $X$ to $X^{(q)}$.

Now suppose $X$ is a closed subvariety of $\mathbb{A}_k^n$. Consider a correspondence between $X$ and $X^{(q)}$ given by a closed subvariety $C$ of $X \times_k X^{(q)}$. Let $c_1$ and $c_2$ denote the projections from $C$ to $X$ and $X^{(q)}$ respectively. Suppose that $c_1$ and $c_2$ are dominant, and at least one of them is quasi-finite.

Note that $X$ and $X^{(q)}$ have natural compactifications (say $\overline{X}$ and $\overline{X^{(q)}}$ respectively) inside $\mathbb{P}^n_k$. Similarly $C$ has a natural compactification (say $\overline{C}$) inside $\mathbb{P}^n_k \times \mathbb{P}^n_k$ which in turn can be embedded inside $\mathbb{P}^{n^2 + 2n}_k$ (the Segre embedding). Let $\Delta^{(q)}_{X/k}$ be the graph of $F^{(q)}_{X/k}$ considered as a subscheme of $X \times_k X^{(q)}$.

Combining techniques from model theory and intersection theory, Hrushovski proved the following generalisation of the Lang-Weil estimates ([LW54], Theorem 1).

**Theorem 1.0.1** (Hrushovski-Lang-Weil estimates). ([Hru12], Theorem 1.1 (1))

Let $n, d_1$ and $d_2$ be nonnegative integers. There exists an integer $M(n, d_1, d_2)$ depending only on $n, d_1$ and $d_2$ satisfying the following properties.

1. For any choice of $p, q, k, n, X$ and $C$ as above with $\deg(\overline{X}) \leq d_1$, $\deg(\overline{C}) \leq d_2$ and $q \geq M(n, d_1, d_2)$, the schematic intersection $C \cap \Delta^{(q)}_{X/k}$ is finite.
2. Moreover we have the following bound for the number of points in the intersection

$$
\# C \cap \Delta^{(q)}_{X/k}(k) - \frac{\delta}{\delta'} q^{\dim(X)} \leq M(n, d_1, d_2) q^{\dim(X) - \frac{1}{2}}.
$$

Here $\delta$ and $\delta'$ are the degree and the inseparable degree of $c_1$ and $c_2$ respectively.

Now suppose that $k$ is an algebraic closure of a finite field $\mathbb{F}_q$. For any scheme $X/k$ defined over $\mathbb{F}_q$, let $F^{(q)}_{X/\mathbb{F}_q} : X \rightarrow X$ be the geometric Frobenius with respect to $\mathbb{F}_q$. Let $\Delta^{(n)}$ be the graph of $F^{(q)}_{X/\mathbb{F}_q}$, considered as a closed subscheme of $X \times_k X$. Recently Varshavsky ([Var18]) gave a geometric proof of the following corollary to Hrushovski’s result.

**Corollary 1.0.2.** ([Hru12], Corollary 1.2, [Var18], Theorem 0.1)

Let $c : C \rightarrow X \times_k X$ be a morphism of schemes finite type over $k$ such that,

1. $C$ and $X$ are irreducible.
2. $c_1$ and $c_2$ are dominant.
3. $X$ is defined over $\mathbb{F}_q$.

Then for $n$ sufficiently large, $c^{-1}(\Delta^{(n)})$ is non-empty.

Corollary 1.0.2 has applications to algebraic dynamics ([Fak03], [Ame11]), group theory ([BS05]) and algebraic geometry ([EM10], [ESB16]).

This article aims to prove a non-uniform avatar of Theorem 1.0.1 using geometric methods. Our methods also imply that generating function keeping track of the fixed points is rational. More precisely we obtain the following result.
Theorem 1.0.3. Let $c : C \to X \times_k X$ be a morphism of schemes finite type over $k$. Suppose that $X$ is defined over $\mathbb{F}_q$ and that $c_2$ is quasi-finite. Then

1. there exists an integer $N$ such that, $\text{Fix}(c^{(n)}) := c^{-1}(\Delta^{(n)})$ is finite (over $k$) for every $n \geq N$. Further there exists a real number $M$ such that for any $n \gg 0$, $\#\text{Fix}(c^{(n)})(k) \leq M q^{n\dim(C)}$.
2. The formal series $Z(c, t) := \sum_{n \geq N} \#\text{Fix}(c^{(n)})(k) t^n \in \mathbb{Z}[t] \subset \mathbb{Q}((t))$ is a rational function in $t$, that is it belongs to $\mathbb{Q}(t)$.
3. Moreover when $C$ and $X$ are irreducible with $c_1$ and $c_2$ dominant, there exists a real number $M'$ such that for any $n \gg 0$,

\[
\left| \#\text{Fix}(c^{(n)})(k) - \frac{\delta}{\delta'} q^{n\dim(X)} \right| \leq M' q^{n\dim(X) - \frac{1}{2}},
\]

where $\delta$ and $\delta'$ are the degree and inseparable degree of $c_1$ and $c_2$ respectively.

Remark 1.0.4. The inequality (1.0.2) under the assumptions of Corollary 1.0.2 implies that

\[
\lim_{n \to \infty} \frac{\#\text{Fix}(c^{(n)})(k)}{q^{n\dim(X)}} = \frac{\delta}{\delta'},
\]

and hence $\#\text{Fix}(c^{(n)}) = c^{-1}(\Delta^{(n)})$ is nonempty for $n$ sufficiently large (see Corollary 1.0.2).

Our proof of Theorem 1.0.3 is closely related to the circle of ideas around Deligne’s conjecture on the Lefschetz-Verdier trace formula for non-proper varieties over an algebraic closure of a finite field. The conjecture was first verified by Pink ([Pin92]) assuming the resolution of singularities and later by Fujiwara ([Fuj97]) unconditionally, following an idea of Gabber. Subsequently Varshavsky obtained an effective generalization of Fujiwara’s trace formula ([Var07]). The key notion in both Fujiwara and Varshavsky’s approach is that of a contracting correspondence, which ensures vanishing of local terms along the boundary of a compactification.

The connection between Theorem 1.0.3 and Deligne’s conjecture was already observed by Hrushovski ([Hru12], Section 1.1). However as noted there, the non-properness of $c_1$ (crucial to make sense of Deligne’s conjecture) rules out a ‘direct’ argument. This connection was reestablished in the recent proof of Corollary 1.0.2 by Varshavsky ([Var18]).

Indeed, as a first step in the proof of Corollary 1.0.2 in [Var18] it is shown that (at the cost of shrinking $X$) one can assume the natural compactification of $c$ is locally invariant along the boundary. That the boundary can be made only locally invariant and not globally, is a manifestation of $c_1$ not being proper. Then using a construction of Pink ([Pin92]), Varshavsky obtains a trace formula which in turn implies an asymptotic growth of the form (1.0.3) for a modified correspondence, which is sufficient to show the desired nonemptiness.

In this article, we attempt to compute $\text{Fix}(c^{(n)})$ directly using the Lefschetz-Verdier trace formula. In particular we do not use alterations or resolution of singularities unlike Hrushovski (in [Hru12]) and Varshavsky (in [Var18]). As mentioned earlier the non-properness of $c_1$ immediately leads to technical difficulties, most important of which is the

\footnote{Though one is tempted to find a middle path combining the methods of this article with those in [Var18], it appears that a plausible proof of uniformity using this approach leads to many difficulties, primarily among them being the lack of a suitable norm on correspondences of $\ell$-adic sheaves.}
possible nonvanishing of local terms along the boundary. Hence an important step for us in the proof of Theorem 1.0.3 is the following estimate (Theorem 1.0.5) for these local terms, which could be of independent interest. We now describe the various terms appearing in Theorem 1.0.5.

As before let \( k \) be an algebraic closure of a finite field \( \mathbb{F}_q \). Let \( c: C \to X \times_k X \) be a correspondence (define over \( \mathbb{F}_q \)), with \( C \) and \( X \) proper over \( k \). Let \( c_1 \) and \( c_2 \) be the induced maps from \( C \) to \( X \). Denote by \( c^{(n)} \) the correspondence \( (F^n_{X/\mathbb{F}_q} \circ c_1, c_2): C \to X \times_k X \).

Let \( Z \subseteq X \) be a closed subset of \( X \) defined over \( \mathbb{F}_q \), which is locally \( c \)-invariant over \( \mathbb{F}_q \). That is there exists a cover of \( Z \) by open sets \( U_i \) of \( X \), defined over \( \mathbb{F}_q \), such that \( c_2^{-1}(U_i \cap Z) \cap c_1^{-1}(U_i) \subseteq c_1^{-1}(Z) \).

Let \( \mathcal{F}_0 \in D_{\leq w}^b(X_0, \overline{\mathbb{Q}_\ell}) \) be a mixed sheaf of weight less than or equal to \( w \) on \( X_0 \) (the chosen model of \( X \) over \( \mathbb{F}_q \)). Assume that \( \mathcal{F}_0|_{Z_0} \) belongs to \( \nu D^{\geq w}(Z_0, \overline{\mathbb{Q}_\ell}) \). Let \( \mathcal{F} \) be the base change of \( \mathcal{F}_0 \) to \( k \).

Let \( u \) be an element in \( \text{Hom}_{D^b_c(X_0, \overline{\mathbb{Q}_\ell})}(c^1_! \mathcal{F}, c^1_1 \mathcal{F}) \) (a cohomological correspondence of \( \mathcal{F} \) lifting \( c \)). Then for any \( n \geq 1 \) we have a cohomological correspondence \( u^{(n)} \) of \( \mathcal{F} \) lifting \( c^{(n)} \), given by the natural structure of a Weil sheaf on \( \mathcal{F} \).

Moreover fix a field isomorphism (say \( \tau \)) of \( \overline{\mathbb{Q}_\ell} \) with \( \mathbb{C} \).

Since \( \text{Fix}(c^{(n)}) \) is proper, and \( Z \) a locally \( c^{(n)} \)-invariant closed subset, we can make sense of the local terms \( \text{LT}(u^{(n)}|_Z) \) (see Lemma 6.1.3 and Section 4.4.3). In this setting we obtain the following bound on the local terms.

**Theorem 1.0.5.** For any \( \epsilon > 0 \), there exists a natural number \( N(\epsilon) \) and a positive real number \( M(\tau) \), such that for any \( n \geq N(\epsilon) \),

\[
\text{LT}(u^{(n)}|_Z) \leq M(\tau) q^{n \left( \frac{\nu_{\mathbb{Q}_\ell}}{2} + \dim(Z) \right)} + \epsilon.
\]

Here the norm on the left is with respect to the chosen isomorphism \( \tau \).

More generally we prove such a bound for a class of correspondences, which we call *essentially proper* over \( \mathbb{F}_q \) (see Definition 6.1.1).

Note that if \( Z \) were \( c \)-invariant (and not just locally), Theorem 1.0.5 would be an immediate consequence of the Lefschetz-Verdier trace formula. The main difficulty here is that even though the local terms along a locally invariant subset are defined, they do not correspond to a *global* term in a natural fashion.

Before we give a brief outline of the proof, we describe the notations and conventions followed in this article.

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many suggestions and corrections which have improved the exposition and readability of the article.

2. Notations and conventions

2.0.1. All the schemes appearing in this article are assumed to be separated over \( \mathbb{Z} \). For any scheme \( X \), \( \pi_0(X) \) denotes the set of its connected components. A variety over a field \( k \) is a geometrically integral scheme of finite type over \( k \). For any integral scheme \( X/k \), by \( k(X) \) we mean the function field of \( X \).

2.0.2. Let \( k \) be either a finite field or an algebraically closed field. For a scheme \( X \) of finite type over \( k \), by \( \mathcal{D}^b_c(X, \overline{\mathbb{Q}}_\ell) \) we mean the bounded derived category of \( \overline{\mathbb{Q}}_\ell \) sheaves with constructible cohomology ([Del80], Section 1.1.2-3, [BS14], Section 6.6). When we say sheaves on \( X \), we mean objects in \( \mathcal{D}^b_c(X, \overline{\mathbb{Q}}_\ell) \). For any two sheaves \( \mathcal{F} \) and \( \mathcal{G} \) on \( X \), by \( \text{Hom}(\mathcal{F}, \mathcal{G}) \), we mean \( \text{Hom}_{\mathcal{D}^b_c(X, \overline{\mathbb{Q}}_\ell)}(\mathcal{F}, \mathcal{G}) \). The constant sheaf with coefficients in \( \mathbb{Q}_\ell \) on \( X \) will be denoted by \( \mathbb{Q}_\ell \).

The usual \( t \)-structure on \( \mathcal{D}^b(X, \overline{\mathbb{Q}}_\ell) \) will be denoted by \( (\mathcal{D}^\leq 0, \mathcal{D}^\geq 0) \). The perverse \( t \)-structure will be denoted by \( (p\mathcal{D}^\leq 0, p\mathcal{D}^\geq 0) \). We use \( p\text{H}^i \) to denote the corresponding perverse cohomology functors. For any variety \( X/k \) (possibly non-normal) we denote by

\[
(2.0.1) \quad \text{IC}_X := j_!(\overline{\mathbb{Q}}_\ell[\dim(X)]),
\]

where \( j: X^{\text{reg}} \hookrightarrow X \) is the inclusion of the regular locus of \( X \). Here \( j_* \) is the intermediate extension functor ([BBDG18], Définition 1.4.22).

2.0.3. We are mostly interested in the triangulated versions of the sheaf operations, so we will denote them without the usual decorations of ‘R or ‘L’. For example, the derived direct image functors will be denoted by \( f_* \). The only exception to this is the derived (local) internal Hom functor, which will be denoted by \( \mathcal{R}\text{Hom} \).

For a morphism of schemes \( f: X \to Y \), finite type over \( k \), we have adjoint pairs \( (f!, f^!) \) and \( (f^*, f_*). \) Moreover when \( f \) is proper we have an adjoint triple \( (f^*, f_! = f_* f^!), f^!) \). For an embedding \( f: Y \hookrightarrow X \) and any \( \mathcal{F} \in \mathcal{D}^b_c(X, \overline{\mathbb{Q}}_\ell) \) we write \( \mathcal{F}|_Y \) instead of \( f^!\mathcal{F} \). We also follow a similar convention for a morphism of sheaves.

For any two sheaves \( \mathcal{F} \) and \( \mathcal{G} \) on \( X \), and a morphism \( u \) from \( \mathcal{F} \) to \( \mathcal{G} \), \( f_* u \) will denote the induced morphism from \( f_* \mathcal{F} \) to \( f_* \mathcal{G} \). We will also follow a similar convention for the other functors.

2.0.4. When \( k \) is an algebraically closed field, we identify \( \mathcal{D}^b_c(\text{Spec}(k), \overline{\mathbb{Q}}_\ell) \) with the bounded derived category of finite dimensional \( \overline{\mathbb{Q}}_\ell \) vector spaces.

2.0.5. For any scheme \( X \) of finite type over \( k \), let \( \pi_X: X \to \text{Spec}(k) \) denote the structural morphism. Let \( K_X := \pi_X^! \overline{\mathbb{Q}}_\ell \) be the dualizing complex of \( X \). We denote by \( \mathbb{D}_X \) the Verdier duality functor. For a morphism of sheaves \( u: \mathcal{F} \to \mathcal{G} \), we denote by \( u^\vee \) the induced morphism from \( \mathbb{D}_X \mathcal{G} \) to \( \mathbb{D}_X \mathcal{F} \).
2.0.6. For schemes $X_1$ and $X_2$ over $k$, let $\text{pr}_1$ and $\text{pr}_2$ denote the projections from $X_1 \times_k X_2$ onto $X_1$ and $X_2$ respectively. Given any morphism $c: C \to X_1 \times_k X_2$, by $c_1$ and $c_2$ we mean $\text{pr}_1 \circ c$ and $\text{pr}_2 \circ c$ respectively.

Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be sheaves on $X_1$ and $X_2$ respectively. Denote by $\mathcal{F}_1 \boxtimes \mathcal{F}_2$ the object $\text{pr}_1^! \mathcal{F}_1 \otimes \text{pr}_2^! \mathcal{F}_2$ in $D^b_c(X_1 \times_k X_2, \mathcal{Q}_\ell)$. There is a canonical isomorphism ([Ill77], (1.7.6) and (2.2.4))

\[
\mathbb{D}_{X_1} \mathcal{F}_1 \boxtimes \mathbb{D}_{X_2} \mathcal{F}_2 \simeq \mathbb{D}_{X_1 \times_k X_2} (\mathcal{F}_1 \boxtimes \mathcal{F}_2).
\]

2.0.7. Consider a Cartesian (upto nilpotents) diagram of schemes (finite type over $k$)

\[
\begin{array}{ccc}
X' & \xrightarrow{g'} & X \\
\downarrow f' & & \downarrow f \\
Y' & \xrightarrow{g} & Y
\end{array}
\]

We shall make repeated use of the proper base change theorem ([SGA5], Exposé VI, 2.2.3) either in the form $g^* f_! \simeq f'_! g^*$ or its Verdier dual $g'^! f_* \simeq g^! f'_*$. Moreover there is also a natural transformation of functors $f'^! g^* \to g^! f'_*$. In any case such morphisms will be simply denoted by (BC).

2.0.8. Let $X/k$ be a finite type scheme, and $j : X \hookrightarrow \overline{X}$ a compactification. Let $\mathcal{F}$ and $\mathcal{G}$ be sheaves on $X$. Then there exists a natural commutative diagram

\[
\begin{array}{ccc}
j_! \mathcal{F} \otimes j_* \mathcal{G} & \xrightarrow{\simeq} & j_!(\mathcal{F} \otimes \mathcal{G}) \\
\downarrow & & \downarrow \\
j_* \mathcal{F} \otimes j_* \mathcal{G} & \xrightarrow{} & j_*(\mathcal{F} \otimes \mathcal{G})
\end{array}
\]

Here the isomorphism along the top row comes from the projection formula ([SGA4], XVII, (5.2.9)). The map on the bottom row is immediate from the adjunction between $(j^*, j_*)$. The vertical maps are the usual forget support maps. Applying $\text{Hom}(\_ , \mathcal{Q}_\ell)$ we get

\[
\begin{array}{ccc}
H^0_c(X, \mathcal{F}) \otimes H^0(X, \mathcal{G}) & \xrightarrow{} & H^0_c(X, \mathcal{F} \otimes \mathcal{G}) \\
\downarrow & & \downarrow \\
H^0(X, \mathcal{F}) \otimes H^0(X, \mathcal{G}) & \xrightarrow{} & H^0(X, \mathcal{F} \otimes \mathcal{G})
\end{array}
\]

Here the vertical maps are forget support maps, and the horizontal ones are cup product maps.

2.0.9. Let $f : X \to Y$ be a morphism of schemes finite type over $k$. We denote by $t_{f'} : f^! \mathcal{F} \otimes f'^! \mathcal{G} \to f'^!(\mathcal{F} \otimes \mathcal{G})$, the natural map adjoint to the composition $f_!(f^! \mathcal{F} \otimes f'^! \mathcal{G}) \simeq f_! f^! \mathcal{F} \otimes \mathcal{G} \to \mathcal{F} \otimes \mathcal{G}$. Here the first isomorphism is due to the projection formula, and the second arrow is due to the adjoint pair $(f_!, f^!)$.
2.0.10. For a closed subscheme $Z$ of $X$, let $\mathcal{I}_Z$ denote its ideal sheaf. By $Z_{\text{red}}$ we mean the reduced closed subscheme underlying $Z$. By $Z_d$, $d \geq 1$, we mean the closed subscheme of $X$ defined by the ideal sheaf $\mathcal{I}_Z^d$. In particular $Z_1 = Z$, and $Z_r$ is a closed subscheme of $Z_s$, whenever $r \leq s$. For a morphism of schemes $f: Y \to X$, by $f_{\text{red}}$ we mean the induced morphism from $Y_{\text{red}}$ to $X_{\text{red}}$.

2.0.11. Let $k_0$ be an arbitrary finite field. Let $k$ be an algebraic closure of $k_0$. Objects over $k_0$ will be denoted by a subscript 0 (for example $X_0$, $f_0$, $\mathcal{F}_0$, etc.). The corresponding object over $k$ will be denoted without a subscript, for example $X$, $f$, $\mathcal{F}$, etc. For a scheme $X/k$ defined over $k_0$, we denote by $F_{X/k_0}$ the geometric Frobenius morphism of $X$ (with respect to $k_0$ and the chosen model of $X$ over $k_0$). By $F_{X/k_0}^0$ we mean the identity map on $X$.

We shall denote the graphs of $F_{X/k_0}^n$ by $\Delta^{(n)}_X$ (or $\Delta^{(n)}$ if there is no scope of confusion) considered as subschemes of $X \times_k X$. $\Delta_0$ will be simply denoted by $\Delta_X$ (or $\Delta$).

2.0.12. Let $c: C \to X \times_k X$ be a morphism of schemes finite type over $k$. Suppose $X$ is defined over $k_0$. Then for any $n \geq 0$, we denote by $c^{(n)}: C \to X \times_k X$ the morphism induced by $(F_{X/k_0}^n, c_1, c_2)$.

Let $\mathcal{F}$ be a Weil sheaf on $X$, that is a sheaf $\mathcal{F}$ and an isomorphism

$$F_{\mathcal{F}}: F_{X/k_0}^* \mathcal{F} \simeq \mathcal{F}. \tag{2.0.6}$$

Let $F_{\mathcal{F}}^{ns}$ denote the induced isomorphism from $F_{X/k_0}^* \mathcal{F}$ to $\mathcal{F}$ obtained by iterating (2.0.6). For any sheaf $\mathcal{G}$ on $X$ and any element $u$ in $\text{Hom}(c_1^* \mathcal{F}, c_2^! \mathcal{G})$, we denote by $u^{(n)}$ the element in $\text{Hom}(c_1^{(n)*} \mathcal{F}, c_2^! \mathcal{G})$ obtained as follows

$$u^{(n)}: c_1^{(n)*} F_{X/k_0}^{ns} \mathcal{F} \xrightarrow{=} c_1^{n*} \mathcal{F} \xrightarrow{u} c_2^! \mathcal{G}. \tag{2.0.7}$$

2.0.13. For a scheme $X_0$ of finite type over $k_0$, we also have the category of mixed sheaves $D_{m}^b(X_0, \overline{\mathbb{Q}}_\ell)$ ([BBDG18], Section 5.1.6). As in [BBDG18] we denote by $D_{\leq w}^b(X_0, \overline{\mathbb{Q}}_\ell)$ the full subcategory of objects in $D_{m}^b(X_0, \overline{\mathbb{Q}}_\ell)$ of weight less than or equal to $w$. Note that $\text{IC}_X$ as defined in (2.0.1) is pure of weight $\text{dim}(X)$ ([BBDG18], Corollaire 5.3.2).

3. Outline of the proof

Our proof can be divided into three distinct steps, which are logically independent of each other. We describe these now.

1. Preliminary reductions. As a first step towards proving Theorem 1.0.3, we show that we can reduce to the situation of Theorem 1.0.3, (3) (Lemma 5.1.2). In addition we can also assume that $c_2$ is generically étale (Proposition 5.1.9). The idea here is simple, we use the absolute Frobenius to get rid of generic inseparability. Since the desired property is generic, we work over function fields (Lemma 5.1.6), and then spread it out.

Next, we would like to show that there exists a compactification of $c$ that leaves the boundary locally invariant (see Definition 4.4.6). This will guarantee that the boundary becomes contracting (see Definition 4.4.10) after twisting the correspondence by a high enough power of the Frobenius. This, in turn, allows us to use a result of Varshavsky on
the decomposition of local terms in the form of Corollary 4.4.16. As shown by Varshavsky ([Var18], Section 2) the local invariance along the boundary can always be achieved at the cost of shrinking $X$. In conclusion, Lemma 5.1.2 and Proposition 5.1.10 allow us to assume that

(a) $X$ is smooth and quasi-projective over $k$.
(b) $c$ is proper and defined over $\mathbb{F}_q$.
(c) $c_2$ is étale.
(d) There exists a compactification of $c$ that leaves the boundary locally invariant over $\mathbb{F}_q$.

2. Constructing a cohomological correspondence. We intend to prove Theorem 1.0.3 using the Lefschetz-Verdier trace formula (see Theorem 4.3.5). The formula takes as input a proper correspondence and a cohomological correspondence lifting it, and produces equality between the associated local and global terms.

In the setting of Theorem 1.0.3 we do have a proper correspondence (constructed in Step 1), $c : \overline{C} \to \overline{X} \times_k \overline{X}$ (denoted by $\overline{c}$). Let $j : X \hookrightarrow \overline{X}$ be the open immersion. On $X$ we have the following natural cohomological correspondence

\begin{equation}
(3.0.1) \quad c_1^! \mathbb{Q}_\ell \cong \mathbb{Q}_\ell \cong c_2^! \mathbb{Q}_\ell ,
\end{equation}

lifting $c$. The second isomorphism in (3.0.1) is a consequence of $c_2$ being étale.

However (3.0.1) does not extend in a natural way to a cohomological correspondence lifting $\overline{c}$. In fact if either $c_1$ (or $c_2$) were proper, (3.0.1) can be extended to a cohomological correspondence of $j_! \mathbb{Q}_\ell$ (or $j_* \mathbb{Q}_\ell$), but as mentioned earlier we cannot ensure this in general. What does happen in general, is that (3.0.1) can be extended to a morphism

\begin{equation}
(3.0.2) \quad \overline{c}_1 j_! \mathbb{Q}_\ell \to \overline{c}_2 j_* \mathbb{Q}_\ell .
\end{equation}

This duality motivated us to consider the intermediate extension $\mathcal{IC}_{\overline{X}}$. Constructing pullback maps for the intersection complex has been considered by various authors ([BBFGK95], [Web99], [Web04], [HS06]). However none of these constructions are functorial for general maps. For our purposes this lack of functoriality is not an issue. Using the results obtained in Appendix A, we construct a cohomological correspondence (Corollary 5.2.2)

\begin{equation}
(3.0.3) \quad u : \overline{c}_1^! \mathcal{IC}_{\overline{X}} \to \overline{c}_2^! \mathcal{IC}_{\overline{X}} ,
\end{equation}

which restricts to (3.0.1) on $X$ (upto shift by $d := \text{dim}(X)$). Moreover our construction of (3.0.3) is such that the linear map $\overline{c}_2 \overline{c}_1^! : H^d(\overline{X}, \mathcal{IC}_{\overline{X}}) \to H^d(\overline{X}, \mathcal{IC}_{\overline{X}})$ induced by (3.0.3) is multiplication by the generic degree of $c_1$ (Proposition A.3.8). This can be seen as a manifestation of $\overline{c}_1$ and $\overline{c}_2$ being dominant, and hence their images intersecting the smooth locus of $\overline{X}$.

Purity of $\mathcal{IC}_{\overline{X}}$ then ensures that the global term associated to (3.0.3), produces the correct leading and error terms in the estimate (1.0.2) (Proposition 5.2.3).

Finally using Varshavsky’s result on decomposition of local terms for a contracting correspondence (in the form of Corollary 4.4.16), we reduce the problem to computing the naive local terms of $u^{(n)}$ on $X$, and the local terms $\text{LT}(u^{(n)}|_Z)$, where $Z$ is the closed complement of
3. Local terms along a locally invariant subset. We observe that (see Diagram 6.1.3) local invariance of a closed subset \( Z \) over \( \mathbb{F}_q \), gives rise to a correspondence \( c : C \to Z \times_k Z \), and a diagram of the form

\[
\begin{array}{ccc}
C & \xrightarrow{c} & Z \times_k Z \\
\downarrow{\tau_U} & & \\
U & \xrightarrow{j} & Z \times_k Z
\end{array}
\]

such that

(a) \( Z \) is proper.
(b) \( \tau_U \) is proper.
(c) \( U = \bigcup_{i=1}^r U_i \times_k U_i \), where \( U_i \)'s are open subsets of \( Z \) defined over \( \mathbb{F}_q \) and cover \( Z \).

Such correspondences \( c : C \to Z \times_k Z \) are defined to be essentially proper over \( \mathbb{F}_q \) (Definition 6.1.1). It is easy to see that the non-proper correspondences which come from locally invariant subsets are essentially proper (Lemma 6.1.4).

Note that (b) implies \( U \) contains \( \Delta^{(n)} \) as a closed subscheme for all \( n \geq 0 \), and is stable under the partial Frobenius, \( F_l := F_{Z/\mathbb{F}_q} \times 1_Z \). Since \( \Delta^{(n)} \) are closed subschemes of \( U \), \( \text{Fix}(c^{(n)}) \) is necessarily proper over \( k \). Thus given a cohomological correspondence \( u \) of a Weil sheaf \( F \) on \( X \) lifting \( c \), we can make sense of \( \text{LT}(u^{(n)}) \). The goal of Section 6 is to prove Theorem 6.1.7 on the growth of these local terms with respect to \( n \), which implies Theorem 1.0.5, and hence Theorem 1.0.3 (see Section 6.2).

In general, there are two equivalent ways to compute local terms. One via Varshavsky’s recipe (see Section 4.3), and the other via a pairing defined by Illusie. In any case one constructs a trace map

\[
(3.0.5) \quad \mathcal{T}\tau_c : \text{Hom}(c^{(n)}(c), F) \to H^0(\text{Fix}(c), K_{\text{Fix}(c)}).
\]

The local terms are then obtained by composing (3.0.5) with the adjunction \( H^0(\text{Fix}(c), K_{\text{Fix}(c)}) \to \mathbb{Q}_{\ell} \), provided \( \text{Fix}(c) \) is proper over \( k \).

The local terms of \( u^{(n)} \) are obtained by twisting the correspondence and applying (3.0.5), whose target now becomes \( H^0(\text{Fix}(c^{(n)}), K_{\text{Fix}(c^{(n)})}) \). Since there are no natural maps in general between \( \text{Fix}(c) \) and \( \text{Fix}(c^{(n)}) \), it is not possible to compare these maps. To be able to compare these maps we need a common ‘target’ for them.

A source of this problem is that the trace map (and hence the Lefschetz-Verdier trace formula) is adapted to the diagonal. This is unlike the Lefschetz trace formula for smooth projective varieties which allows for the intersection between arbitrary cohomology classes in the right degree. In Section 6.3 we give an alternate description of the trace map which has this additional flexibility. In the notation of (3.0.4), we define a pairing.
\( \Phi : \text{Hom}(c_1^*F, c_2^*F) \otimes_{\mathbb{Q}_l} H^0_c(U, j^*(F \boxtimes \mathbb{D}_Z F)) \rightarrow H^0(\mathcal{Z} \times_k \mathcal{Z}, K_{\mathcal{Z}\times_k \mathcal{Z}}), \)

and hence for any cohomological correspondence \( u \), a linear functional

\[
(3.0.7) \quad \Phi_u(\beta) := \text{Tr}_{\mathcal{Z}\times_k \mathcal{Z}}(\Phi(u \otimes \beta)),
\]

on \( H^0_c(U, j^*(F \boxtimes \mathbb{D}_Z F)) \). Here \( \text{Tr}_{\mathcal{Z}\times_k \mathcal{Z}} \) is the natural trace map on \( H^0(\mathcal{Z} \times_k \mathcal{Z}, K_{\mathcal{Z}\times_k \mathcal{Z}}) \).

Further using the fact that \( \Delta^{(n)} \) are closed subschemes of \( U \), the Weil sheaf \( F \) and the evaluation map (4.3.8), we define cohomology classes \([\Delta^{(n)}]\) in \( H^0_c(U, j^*(F \boxtimes \mathbb{D}_Z F)) \) which satisfy (see Proposition 6.5.1)

\[
(3.0.8) \quad \Phi_u([\Delta^{(n)}]) = \text{LT}(u^{(n)}).
\]

The equality (3.0.8) can be seen as a trace formula in this non-proper setting, where the object on the left is thought of as a global term.

Recall that the partial Frobenius \( F_l \) acts on \( U \), and since \( F \) has the structure of a Weil sheaf, it acts on \( H^0_c(U, j^*(F \boxtimes \mathbb{D}_Z F)) \) too. It essentially follows from the definition of \([\Delta^{(n)}]\) that (see Lemma 6.6.1)

\[
(3.0.9) \quad (F_l^*)^n([\Delta]) = [\Delta^{(n)}],
\]

for any \( n \geq 0 \).

When \( F \) comes from a mixed sheaf \( F_0 \in D^b_w(Z_0, \mathbb{Q}_l) \) (on the chosen model \( Z_0 \) of \( Z \)), the action of \( F_l^* \) on \( H^0_c(U, j^*(F \boxtimes \mathbb{D}_Z F)) \) is easy to understand using the almost product structure of \( U \). The key point here is that as far as the partial Frobenius is concerned, \( F_0 \boxtimes \mathbb{D}_{Z_0} F_0 \) behaves as a mixed sheaf of weight less than or equal to \( w \), on a variety of dimension \( \frac{\dim(U)}{2} \). Thus the weights of \( F_l^* \) on \( H^0_c(U, j^*(F \boxtimes \mathbb{D}_Z F)) \) are bounded above by \( w + a + \dim(Z) \) (see Lemma 6.6.2).

Having bounded the weights of the partial Frobenius, Theorem 6.1.7 follows immediately by combining (3.0.8) and (3.0.9) using a linear algebra argument (Lemma 6.6.3).

**4. Varshavsky’s Trace Formula**

In this section we recall the formalism behind Varshavsky’s trace formula ([Var07].) Since we will be using the Lefschetz-Verdier trace formula ([Ill77], Corollary 4.7) we also recall the same.

**4.1. Correspondences and cohomological correspondences.** In this section \( k \) will denote an arbitrary algebraically closed field.

**Definition 4.1.1** (Correspondence). A correspondence from a scheme \( X_1 \) to \( X_2 \) is a morphism of schemes \( c : C \rightarrow X_1 \times_k X_2 \). We will denote this by \([c] \in (C, c_1, c_2)\). When there is no risk of confusion we will also denote this simply by \( c \).
4.1.1. The natural isomorphism $c_{tr} : \text{Spec}(k) \to \text{Spec}(k) \times_k \text{Spec}(k)$ is a self-correspondence of $\text{Spec}(k)$, denoted by $[c_{tr}] = (\text{Spec}(k), 1_{\text{Spec}(k)}, 1_{\text{Spec}(k)})$.

**Definition 4.1.2** (Morphism of correspondences). Let $[c] = (C, c_1, c_2)$ be a correspondence from $X_1$ to $X_2$ and let $[b] = (B, b_1, b_2)$ be a correspondence from $Y_1$ to $Y_2$. A morphism of $[c]$ to $[b]$ consists of a triple of morphisms $[f] := (f_1, f^#, f_2)$ which fit into a commutative diagram

$$
\begin{array}{ccc}
X_1 & \xrightarrow{c_1} & C & \xrightarrow{c_2} & X_2 \\
| & \downarrow{f_1} & | & \downarrow{f^#} & | \\
Y_1 & \leftarrow{b_1} & B & \leftarrow{b_2} & Y_2
\end{array}
$$

4.1.2. Let $c : C \to X_1 \times_k X_2$ be a correspondence from $X_1$ to $X_2$. Then $[\pi]_c := (\pi_{X_1}, \pi_{C}, \pi_{X_2})$ is a morphism from $[c]$ to $[c_{tr}]$ called the structural morphism of $[c]$.

4.1.3. We say a morphism of correspondences $[f] = (f_1, f^#, f_2)$ is proper (respectively an open immersion, respectively a closed immersion) if each of the $f_1$, $f^#$ and $f_2$ is proper (respectively an open immersion, respectively a closed immersion). We say a correspondence $[c]$ is proper, if $C$, $X_1$ and $X_2$ are proper over $k$.

**Definition 4.1.3** (Compactification of correspondences). A compactification of a correspondence $c : C \to X_1 \times_k X_2$, is an open immersion $[j] = (j_1, j^#, j_2)$ of $[c]$ into a correspondence $\overline{c} : \overline{C} \to \overline{X}_1 \times_k \overline{X}_2$, such that $[\overline{c}]$ is proper and $j_1, j^#, j_2$ are dominant.

We have the following Lemma.

**Lemma 4.1.4.** Let

$$
(4.1.1) \quad \begin{array}{ccc}
C & \xrightarrow{j_C} & \overline{C} \\
| & \downarrow{f} & | \\
X & \xrightarrow{j} & \overline{X}
\end{array}
$$

be a commutative diagram such that $f$ is a proper morphism. Suppose that $j$ and $j_C$ are open immersions. If $j_C$ has a dense image, then (4.1.1) is necessarily Cartesian.

**Proof.** Let $j'$ be the induced morphism from $C$ to $\overline{f}^{-1}(X)$. Since $j_C$ is a dense open immersion, so is $j'$. Moreover since $f$ is proper (and our schemes are assumed to be separated), $j'$ is also proper [EGAII, Corollaire 5.4.3] and hence is an isomorphism. Thus (4.1.1) is necessarily Cartesian.

The following corollary is an immediate consequence of Lemma 4.1.4 and will be used later.

**Corollary 4.1.5.** Let $\overline{c} : \overline{C} \to \overline{X} \times_k \overline{X}$ be a compactification of a correspondence $c : C \to X \times_k X$. If $c$ is proper then $\overline{c}^{-1}(X \times_X X) = C$.

**Definition 4.1.6** (Restriction of a correspondence to an open subscheme). Let $[c] = (C, c_1, c_2)$ be a correspondence from $X$ to itself. Let $U \subseteq X$ be an open subscheme. Then the restriction of $c$ to $U$ is the correspondence, $[c]|_U := (c_1^{-1}(U) \cap c_2^{-1}(U), c_1|_{c_1^{-1}(U) \cap c_2^{-1}(U)}, c_2|_{c_1^{-1}(U) \cap c_2^{-1}(U)})$ from $U$ to itself. We shall also denote this correspondence by $c|_U$. 11
Similarly if \( W \subseteq C \) is an open subscheme of \( C \), the restriction of \( c \) to \( W \) is the correspondence \([c]|_W := (W, c_1|_W, c_2|_W)\). As before \([c]|_W \) shall denote the induced morphism from \( W \) to \( X \times_k X \), and also the correspondence \([c]|_W\).

**Definition 4.1.7** (Cohomological correspondence). Let \([c] = (C, c_1, c_2)\) be a correspondence from \( X_1 \) to \( X_2 \). Let \( \mathcal{F}_1 \) and \( \mathcal{F}_2 \) be sheaves on \( X_1 \) and \( X_2 \) respectively. A cohomological correspondence from \( \mathcal{F}_1 \) to \( \mathcal{F}_2 \) lifting \( c \) is an element of \( \text{Hom}(c_2!c_1^!, \mathcal{F}_1, \mathcal{F}_2) \).

4.1.4. **Restriction of cohomological correspondence to an open subscheme.** Let \( c: C \to X \times_k X \) be a self-correspondence of \( X \). Let \( C^0 \subseteq C \) and \( X_i^0 \subseteq X \), \( i = 1, 2 \) be open subschemes. Suppose that \( c \) induces a correspondence \( c^0: C^0 \to X_1^0 \times_k X_2^0 \). Thus we have a commutative diagram

\[
\begin{array}{ccc}
C^0 & \longrightarrow & C \\
\downarrow^{c_0} & & \downarrow^{c_2} \\
X_2^0 & \longrightarrow & X_2
\end{array}
\]

For any sheaf \( \mathcal{F} \) on \( C \), there exists a natural adjunction morphism \( \mathcal{F} \to c_2^!c_2!\mathcal{F} \). Restricting the above morphism to \( C^0 \), and using the adjunction between \( c_2^! \) and \( c_2! \), we get a morphism

\[
(4.1.2) \quad \text{BC}(\mathcal{F}): c_2^!(\mathcal{F}|_{C^0}) \to (c_2!\mathcal{F})|_{X_2^0}.
\]

Let \( \mathcal{F}_1 \) and \( \mathcal{F}_2 \) be sheaves on \( X_1 \) and \( X_2 \) respectively. Let \( u \in \text{Hom}(c_2!c_1^!, \mathcal{F}_1, \mathcal{F}_2) \) be a cohomological correspondence from \( \mathcal{F}_1 \) to \( \mathcal{F}_2 \) lifting \( c \). Then we can restrict \( u \) to give a cohomological correspondence from \( \mathcal{F}_1|_{X_2^0} \) to \( \mathcal{F}_2|_{X_2^0} \) lifting \( c_0 \) as follows,

\[
u^0: c_2^0c_1^0(\mathcal{F}_1|_{X_2^0}) \simeq c_2^0(c_1^0\mathcal{F}_1|_{C^0}) \xrightarrow{\text{BC}(c_1^0, \mathcal{F}_1)} (c_2^!c_1^!\mathcal{F}_1)|_{X_2^0} \xrightarrow{u|_{X_2^0}} \mathcal{F}_2|_{X_2^0}.
\]

In particular, for any open subscheme \( U \subseteq X \) we have a cohomological correspondence \( u|_U \) lifting \( c|_U \) (see Definition 4.1.6).

4.1.5. **Action of a correspondence on cohomology.** Let \( c: C \to X_1 \times_k X_2 \) be a correspondence. Let \( u \) be a cohomological correspondence from \( \mathcal{F}_1 \) to \( \mathcal{F}_2 \) lifting \( c \). Suppose that \( c_1 \) is a proper morphism. Consider the following sequence of morphisms

\[
\pi_{X_1}|_{\mathcal{F}_1} \xrightarrow{(A)} \pi_{X_1|_{c_1^!c_1^!\mathcal{F}_1}} \simeq \pi_{C_1!c_1^!\mathcal{F}_1} \xrightarrow{\pi_{C_1!}u} \pi_{C_2!c_2^!\mathcal{F}_2} \simeq \pi_{X_2|_{c_2^!c_2^!\mathcal{F}_2}} \xrightarrow{(A)} \pi_{X_2|_{\mathcal{F}_2}}.
\]

Here the second isomorphism follows from the properness of \( c_1 \). The morphisms \( (A) \) are adjunctions. In particular when \( X_1 = X_2 \) and \( \mathcal{F}_1 = \mathcal{F}_2 = \mathcal{F} \), we get an endomorphism \( R\Gamma_c(u) \) of the perfect complex \( R\Gamma_c(X, \mathcal{F}) \).

Assume that there exists an open subscheme \( X_1^0 \subseteq X_1 \), such that \( \mathcal{F}_1 \) is supported on \( X_1^0 \). Suppose that \( c_1|_{c_1^{-1}(X_1^0)}: C^0 := c_1^{-1}(X_1^0) \to X_1^0 \) is proper. Let \( c_1^0 \) and \( c_2^0 \) be the induced morphism from \( C^0 \) to \( X_1^0 \) and \( X_2^0 \) respectively.

Note that we have isomorphisms

\[
(4.1.3) \quad R\Gamma_c(X_1, \mathcal{F}_1) \simeq R\Gamma_c(X_1^0, \mathcal{F}_1|_{X_1^0}) = \pi_{X_1|_{\mathcal{F}_1}}(\mathcal{F}_1|_{X_0})
\]

and
(4.1.4) \[ \pi_{\mathbb{C}^0} c_2^0 (\mathcal{F}_2) \simeq \pi_{X_2} c_2^0 (\mathcal{F}_2). \]

Since \( c_1^0 \) is proper one also has

(4.1.5) \[ \pi_{X_1} c_1^0 c_1^0 (\mathcal{F}_1 | x_0) \simeq \pi_{X_1} c_1^0 (\mathcal{F}_1 | x_0^0). \]

Further, there are morphisms induced by adjunction

(4.1.6) \[ \pi_{X_1} (\mathcal{F}_1 | x_0) \to c_1^0 c_1^0 \pi_{X_1} (\mathcal{F}_1 | x_0), \]

and

(4.1.7) \[ \pi_{X_2} c_2^0 c_0^0 (\mathcal{F}_2) \to \pi_{X_2} (\mathcal{F}_2). \]

Further we have a correspondence \([c^0] := (C^0, c_1^0 | c_0^0, c_2^0 | c_0^0)\) between \( X_1^0 \) and \( X_2 \), and a cohomological correspondence \( u^0 \) between \( \mathcal{F}_1 | x_0^0 \) and \( \mathcal{F}_2 \) lifting \( [c^0] \) (see Section 4.1.4). Applying \( \pi_{c_0^0} \) to \( u^0 \), and using (4.1.5) and (4.1.4) we get a morphism

(4.1.8) \[ \pi_{X_1} c_1^0 c_1^0 (\mathcal{F}_1 | x_0) \to \pi_{X_2} c_2^0 c_2^0 (\mathcal{F}_2). \]

Combining (4.1.3), (4.1.6), (4.1.7) and (4.1.8) we get a morphism \( R \Gamma_c (u) : R \Gamma_c (X_1, \mathcal{F}_1) \to R \Gamma_c (X_2, \mathcal{F}_2) \). In particular if \( X_1 = X_2 = X \) and \( \mathcal{F}_1 = \mathcal{F}_2 = \mathcal{F} \), we get an endomorphism \( R \Gamma_c (u) \) of the perfect complex \( R \Gamma_c (X, \mathcal{F}) \).

4.2. The Lefschetz-Verdier trace formula. In this section, we describe a recipe to obtain the local and global terms in the Lefschetz-Verdier trace formula. Let \( k \) be an arbitrary algebraically closed field.

4.2.1. Scheme of fixed points. Let \( c : C \to X \times_k X \) be a correspondence. The scheme of fixed points of the correspondence \( c \) is the closed subscheme \( \operatorname{Fix}(c) := C \times_{X \times_k X} X \) of \( C \). Here \( X \) is looked at as a scheme over \( X \times_k X \) via the diagonal embedding \( \Delta \).

Let \( \Delta' \) denote the embedding of \( \operatorname{Fix}(c) \) inside \( C \). Let \( c' \) be the restriction of \( c \) to \( \operatorname{Fix}(c) \). Thus we have a Cartesian diagram

(4.2.1)

\[
\begin{array}{ccc}
\operatorname{Fix}(c) & \xrightarrow{c'} & X \\
\downarrow{\Delta'} & & \downarrow{\Delta} \\
C & \xrightarrow{c} & X \times_k X
\end{array}
\]

4.2.2. Let \( c : C \to X \times_k X \) be a correspondence from \( X \) to itself. Let \( u \) be a cohomological correspondence of a sheaf \( \mathcal{F} \) to itself lifting \( c \). Further assume that \( c_2 \) is quasi-finite. Proper base change implies that for any closed point \( x \in X \), the stalk at \( x \) of \( c_2^0 c_1^0 \mathcal{F} \) is isomorphic to \( \bigoplus_{c_2 (y) = x} \mathcal{F}_{c_1 (y)} \).

Hence \( u_x \) induces a morphism \( \bigoplus_{c_2 (y) = x} \mathcal{F}_{c_1 (y)} \to \mathcal{F}_x \). In particular for any closed point \( y \in \operatorname{Fix}(c) \) we have an induced endomorphism (denoted by \( u_y \)) of \( \mathcal{F}_{c (y)} \). Here \( c' \) is the map induced from \( \operatorname{Fix}(c) \) to \( X \) (see Diagram 4.2.1).
Definition 4.2.1 (Naive local term). Using the assumptions and notations in Section 4.2.2, for any closed point \( y \in \text{Fix}(c) \), we define the naive local term at \( y \) to be the trace of the endomorphism \( u_y \).

The Lefschetz-Verdier trace formula can be viewed as a consequence of the commutativity of certain trace maps with proper push forward. Now we describe these trace maps.

4.3. Trace maps. [Var07, Section 1.2]

Let \( c: C \to X \times_k X \) be a correspondence from \( X \) to itself. Let \( \mathcal{F} \) be a sheaf on \( X \). Let \( b: B \to X \times_k X \) be another correspondence. Consider the Cartesian diagram

\[
\begin{array}{ccc}
A & \xrightarrow{c'} & B \\
\downarrow{b'} & & \downarrow{b} \\
C & \xrightarrow{c} & X \times X
\end{array}
\]

Further, assume that we have a map

\[
ev_{\mathcal{F}, b}: \mathbb{D}_X \mathcal{F} \boxtimes \mathcal{F} \to b_* K_B.
\]

One has the natural evaluation map \( \mathbb{D}_X \mathcal{F} \otimes \mathcal{F} \to K_X \). Since \( \Delta^*(\mathbb{D}_X \mathcal{F} \boxtimes \mathcal{F}) \simeq \mathbb{D}_X \mathcal{F} \otimes \mathcal{F} \), by adjunction one gets a morphism

\[
ev_{\mathcal{F}}: \mathbb{D}_X \mathcal{F} \boxtimes \mathcal{F} \to \Delta_* K_X.
\]

Base change applied to the Cartesian diagram 4.3.1 implies

\[
c'(b_* K_B) \xrightarrow{\simeq} (BC) b'_* c'' K_B \xrightarrow{\simeq} b'_* K_A.
\]

Thus combining (4.3.8) and (4.3.4) we get a morphism

\[
c'(\mathbb{D}_X \mathcal{F} \boxtimes \mathcal{F}) \xrightarrow{c' ev_{\mathcal{F}, b}} c' b_* K_B \xrightarrow{(4.3.4)} b'_* K_A.
\]

In [Ill77] (see Sections 3.1.1 and 3.2.1), Illusie obtained a canonical isomorphism

\[
\mathcal{R} \mathcal{H} \mathcal{O} \mathcal{M}(c_1^* \mathcal{F}, c_2^* \mathcal{F}) \simeq c^!(\mathbb{D}_X \mathcal{F} \boxtimes \mathcal{F}).
\]

Combining (4.3.6) and (4.3.5) we get a morphism

\[
\mathcal{T} \mathcal{R} \mathcal{B}_B: \mathcal{R} \mathcal{H} \mathcal{O} \mathcal{M}(c_1^* \mathcal{F}, c_2^* \mathcal{F}) \to b'_* K_A.
\]

Example 4.3.1. The simplest case where one can use the above formalism is when \( B = X \) and \( b \) is the diagonal morphism. Indeed in that case one has the natural evaluation map \( \mathbb{D}_X \mathcal{F} \otimes \mathcal{F} \to K_X \). Since \( \Delta^*(\mathbb{D}_X \mathcal{F} \boxtimes \mathcal{F}) \simeq \mathbb{D}_X \mathcal{F} \otimes \mathcal{F} \), by adjunction one gets a morphism

\[
ev_{\mathcal{F}}: \mathbb{D}_X \mathcal{F} \boxtimes \mathcal{F} \to \Delta_* K_X.
\]
Note that in this case $A$ is $\text{Fix}(c)$. Moreover applying $H^0(C, \quad )$ to (4.3.7) one obtains the trace map
\begin{equation}
(4.3.9) \quad T_{r_c} : \text{Hom}(c_2 c_1^* F, F) \to H^0(\text{Fix}(c), K_{\text{Fix}(c)}).
\end{equation}

For an open subset $\beta$ of $\text{Fix}(c)$, let $j_\beta$ denote the inclusion of $\beta$ into $\text{Fix}(c)$. The natural adjunction morphism $K_{\text{Fix}(c)} \to j_\beta^* j_\beta^! K_{\text{Fix}(c)}$ induces a morphism
\begin{equation}
(4.3.10) \quad \text{Res}_\beta : H^0(\text{Fix}(c), K_{\text{Fix}(c)}) \to H^0(\beta, K_\beta).
\end{equation}

Let $T_{r_\beta} := \text{Res}_\beta \circ T_{r_c}$. If $\beta$ is proper over $k$, then the adjunction $\pi_\beta^! \pi_\beta^! Q_\ell \to Q_\ell$ gives rise to a morphism $\pi_\beta^! : H^0(\beta, K_\beta) \to Q_\ell$. Thus we get a morphism
\begin{equation}
(4.3.11) \quad LT_\beta := \pi_\beta^! \circ T_{r_\beta} : \text{Hom}(c_2 c_1^* F, F) \to Q_\ell.
\end{equation}

In particular if $\text{Fix}(c)$ is proper over $k$ we get a morphism
\begin{equation}
(4.3.12) \quad LT : \text{Hom}(c_2 c_1^* F, F) \to Q_\ell.
\end{equation}

**Definition 4.3.2 (Local term).** For any proper connected component $\beta$ of $\text{Fix}(c)$, and any cohomological correspondence $u$ lifting $c$, the local term at $\beta$ is defined to be $LT_\beta(u)$. Moreover if $\text{Fix}(c)$ is proper over $k$ we define the local term of $u$ to be $LT(u)$.

Clearly when $\text{Fix}(c)$ is proper
\begin{equation}
(4.3.13) \quad LT(u) = \sum_{\beta \in \pi_0(\text{Fix}(c))} LT_\beta(u).
\end{equation}

**Remark 4.3.3.** Our definition of a local term is the one in [Var07], Section 1.2. It is compatible with the definition in [Ill77], Section 4.2.5 (see [Var07], Appendix A).

**Example 4.3.4.** The following example will be relevant to us in the proof of Theorem 1.0.1. Let $k$ be an algebraic closure of $\mathbb{F}_q$. Let $c : C \to X \times_k X$ be a correspondence defined over $\mathbb{F}_q$. Let $F$ be a Weil sheaf on $X$. Hence $F$ comes equipped with an isomorphism
\begin{equation}
(4.3.14) \quad F_F : F_{X/\mathbb{F}_q}^* F \simeq F.
\end{equation}

Dualizing (4.3.14) we get an isomorphism
\begin{equation}
(4.3.15) \quad F_F^! : D_X F \simeq F_{X/\mathbb{F}_q}^! D_X F.
\end{equation}

For each $n$ we have a Cartesian diagram
By definition, the composition of the arrows in the lower row is $c^{(n)}: C \to X \times_k X$ given by $(F^n_{X/F_q} \circ c_1, c_2)$. Denote the composition of the arrows in the topmost row by $c^{(n)'}$. Moreover, since $!$-pullback commutes with external products ([Ill77], (1.7.3)) we have an isomorphism

\[(F^n_{X/F_q} \times 1_X)^! \simeq F^n_{X/F_q} \otimes X \otimes F \simeq \Delta^{(n)}_* K_X,\]

induced by (4.3.15). Further by applying the functor $(F^n_{X/F_q} \times 1_X)^!$ to the evaluation map (4.3.8), and using base change along the right Cartesian square in (6.4.1) we get a morphism

\[(F^n_{X/F_q} \times 1_X)^! \to \Delta_!^{(n)} K_X.\]

Thus combining (4.3.17) and (4.3.18) we get for each $n \geq 0$, morphism

\[ev^{(n)}_F: \Delta_! K_X \to \Delta_!^{(n)} K_X.\]

Thus the above formalism gives rise to

\[(4.3.20) \quad Tr_{c}^{(n)}: \text{Hom}(c^{(n)}_{2!} F, F) \to H^0(\text{Fix}(c^{(n)}), K_{\text{Fix}(c^{(n)})})\]

Now we are in a position to state the Lefschetz-Verdier trace formula.

**Theorem 4.3.5.** ([Ill77], [Var07])

Let $c: C \to X \times_k X$ be a correspondence with $C$ and $X$ proper over $k$. Then for any cohomological correspondence $u$ from $F$ to itself lifting $c$

\[\text{Tr}(R\Gamma_c(u)) = \sum_{\beta \in \pi_0(\text{Fix}(c))} LT_{\beta}(u).\]

Here $Tr(R\Gamma_c(u))$ is the trace of the endomorphism $R\Gamma_c(u)$ of the perfect complex (of $\mathbb{Q}_\ell$ vector spaces) $R\Gamma_c(X, F)$ induced by $u$ (see Section 4.1.5).

### 4.4. Locally contracting correspondences.

As before let $k$ be an arbitrary algebraically closed field. Let $c: X \to Y$ be a morphism of schemes finite type over $k$.

**Definition 4.4.1** (Ramification along a closed subscheme). For a reduced closed subscheme $Z \subseteq Y$ its **ramification along** $c$ is the smallest positive integer $n$ such that, $c^{-1}(Z) \subseteq (c^{-1}(Z)_{\text{red}})^n$. We denote this by $\text{Ram}(Z, c)$.

**Definition 4.4.2** (Ramification degree of a morphism). If $c$ is quasi-finite, then $\text{ram}(c)$ is defined as the maximum of $\text{ram}(y, c)$, as $y$ varies over all the closed points of $Y$. We denote this by $\text{ram}(c)$.
Remark 4.4.3. Note that our notation for ramification along a closed subscheme differs from the one in [Var07] to avoid any possibility of confusion with the notation for the ramification degree of a morphism.

Now let $c : C \to X \times_k X$ be a self-correspondence of $X$.

Definition 4.4.4 (Invariant closed subset). A closed subset $Z \subseteq X$ is said to be $c$-invariant if $c_1(c_2^{-1}(Z))$ is set theoretically contained in $Z$.

Remark 4.4.5. Suppose $Z \subseteq X$ be an invariant closed subset of $X$. Then we define the restriction of $[c]$ to $Z$ by $[c]|_Z := (c_2^{-1}(Z))_{\text{red}}, c_{1,Z}, c_{2,Z}$). Here $c_{1,Z}$ and $c_{2,Z}$ are the maps induced by $c_1$ and $c_2$ respectively. Note that $[c]|_Z$ is a self correspondence of $Z$. We shall also denote this by $c|_Z$.

Definition 4.4.6 (Locally invariant closed subset). A closed subset $Z \subseteq X$ is said to be locally $c$-invariant if for each $x \in Z$ there exists an open neighbourhood $U$ of $x$ in $X$ such that $Z \cap U$ is $c|_U$-invariant (see Definition 4.1.6).

4.4.1. If $c_2$ is quasi-finite then any closed point of $X$ is locally $c$-invariant (see [Var07] Example 1.5.2).

Definition 4.4.7 (Invariant in a neighbourhood of fixed points). A closed subset $Z \subseteq X$ is said to be invariant in a neighbourhood of fixed points if there exists an open neighbourhood $W$ of $\text{Fix}(c)$ in $C$ such that $Z$ is $c|_W$-invariant (see Definition 4.1.6).

4.4.2. Restriction of cohomological correspondence to an invariant subset. ([Var07], 1.5.6 (a)) Let $c : C \to X \times_k X$ be a correspondence, and $u$ a cohomological correspondence of $F$ lifting $c$. Let $Z \subseteq X$ be a closed subset of $X$ left invariant by $c$. Then we have a diagram

\[
\begin{array}{ccc}
Z & \xrightarrow{c_1,Z} & c_2^{-1}(Z)_{\text{red}} \xrightarrow{c_{2,Z}} Z \\
\downarrow i & & \downarrow i_c \\
X & \xrightarrow{c_1} & C \xrightarrow{c_2} X
\end{array}
\]

where the right-hand square is Cartesian up to nilpotents. Thus as in Section 2.0.7, one has a natural transformation of functors,

\[
i^*_C c_1^i \to c_{2,Z}^i.
\]

The cohomological correspondence $u$ is given by a map $c_1^i F \to c_2^i F$. Applying $i^*_C$ to this, we get a map from $c_{1,Z}^i F \simeq i_C^* c_1^i F \to i_C^* c_2^i F$, which using (4.4.2) gives a map

\[
u|_Z : c_{1,Z}^i(F|_Z) \to c_{2,Z}^i(F|_Z).
\]

Note that $u|_Z$ is a cohomological correspondence of $F|_Z$ lifting a self correspondence of $Z$ given by the top row in Diagram (4.4.1).

Remark 4.4.8. (Compare [Var18], 1.9, (b)) Suppose $X$ is defined over a subfield $k'$ of $k$. A closed subset $Z \subseteq X$ is said to be locally $c$-invariant over $k'$ if the open neighbourhoods $U$ in Definition 4.4.6 can be chosen to be defined over $k'$. 

17
4.4.3. **Local term along a subscheme invariant in a neighbourhood of fixed points.** (Compare [Var07, 1.5.6 (b)])

Suppose \( c : C \to X \times_k X \) is a correspondence such that \( \text{Fix}(c) \) is proper over \( k \). Let \( u \) be a cohomological correspondence of \( F \) lifting \( c \). Let \( Z \) be a closed subset of \( X \) which is invariant in a neighbourhood of fixed points (see Definition 4.4.7). Let \( W \) be an open neighbourhood of \( \text{Fix}(c) \) in \( C \) such that \( c|_W \) leaves \( Z \) invariant. Then combining Definition 4.1.6 and Remark 4.4.5 we can make sense of a self correspondence \( c|_W \cdot Z \) of \( Z \) as the restriction of \( c|_W \) to \( Z \).

Moreover there exists a cohomological correspondence (say \( u|_W \)) of \( F \) lifting \( c|_W \) (see Section 4.1.4). Since \( Z \) is \( c|_W \) invariant, we can restrict \( u|_W \) along \( Z \) to get a cohomological correspondence (say \( u|_W \cdot Z \)) of \( F|_Z \) (see Section 4.4.2) lifting \( c|_W \cdot Z \).

Let \( \beta \) be any connected component of \( \text{Fix}(c|_W \cdot Z) \). Since \( W \) was chosen to be a neighbourhood of fixed points and \( \text{Fix}(c) \) was assumed to be proper, \( \beta \) is necessarily proper over \( k \). Thus we can make sense of the local term (say \( \text{LT}_\beta(u|_W \cdot Z) \in \mathcal{O}_\beta \)) of \( u|_W \cdot Z \) at \( \beta \). We also have a well defined local term \( \text{LT}(u|_W \cdot Z) \) (see Definition 4.3.2). Since the local term at a connected component of the scheme of fixed points, depends only on an open neighbourhood of the connected component, \( \text{LT}_\beta(u|_W \cdot Z) \) (and hence \( \text{LT}(u|_W \cdot Z) \) ) are independent of the chosen open neighbourhood \( W \) of \( \text{Fix}(c) \). Hence we can remove \( W \) from the notation and simply denote it by \( \text{LT}_\beta(u|_Z) \) (and \( \text{LT}(u|_Z) \)).

**Definition 4.4.9.** A closed subscheme \( Z \) is said to be **stabilized by** \( c \) if \( c^{-1}_2(Z) \) is a closed subscheme of \( c^{-1}_1(Z) \).

Recall that for a closed subscheme \( Z \) of \( X \), \( Z_k \) is the closed subscheme of \( X \) defined by the ideal \( T_Z^k \).

**Definition 4.4.10.** \( c \) is said to be **contracting near a closed subscheme** \( Z \subseteq X \) if \( c \) stabilizes \( Z \), and \( c^{-1}_2(Z_{n+1}) \) is a closed subscheme of \( c^{-1}_1(Z_n) \) for some \( n \geq 1 \).

We have the following local variant of Defintion 4.4.10

**Definition 4.4.11.** \( c \) is said to be **locally contracting near a closed subscheme** \( Z \subseteq X \) if for each \( x \in Z \) there exists an open neighbourhood \( U \) of \( x \) in \( X \) such that \( c|_U \) (see Definition 4.1.6) is contracting near \( Z \cap U \).

**Remark 4.4.12.** (Compare [Var18, 1.9 (c)]) Suppose \( X \) is defined over a subfield \( k' \) of \( k \). \( c \) is said to be locally contracting near a closed subscheme \( Z \subseteq X \) over \( k' \) if the open neighbourhod of \( k \) in the open neighbourhod of \( k' \) in \( \text{Definiton 4.4.11} \) can be chosen to be defined over \( k' \).

**Definition 4.4.13.** \( c \) is said to be contracting near closed subscheme \( Z \subseteq X \) in a **neighbourhood of fixed points** if there exists an open subscheme \( W \) of \( C \) containing \( \text{Fix}(c) \) such that \( c|_W \) is contracting near \( Z \) (see Definiton 4.1.6).

We will need the following key result in the form of Corollary 4.4.16.

**Theorem 4.4.14.** ([Var07, Theorem 2.1.3])

Let \( c : C \to X \times_k X \) be a correspondence contracting near a closed subscheme \( Z \subseteq X \) in a neighbourhood of fixed points, and let \( \beta \) be an open connected subset of \( \text{Fix}(c) \) such that \( c'(\beta) \cap Z \neq \emptyset \) (see Section 4.2.1). Then

1. \( \beta \) is contained set-theoretically in \( c^{-1}(Z) \), hence \( \beta \) is an open connected subset of \( \text{Fix}(c) \cdot W ) \) (see Section 4.4.3). Here \( W \) is an open neighbourhood of \( \text{Fix}(c) \) in \( C \) such that \( c|_W \) leaves \( Z \) invariant.

18
For every cohomological correspondence \( u \) from \( \mathcal{F} \) to itself lifting \( c \), one has \( \text{Tr}_\beta(u) = \text{Tr}_{\beta(u|_Z)} \) (see Section 4.3). In particular if \( \beta \) is proper over \( k \) then, \( LT_\beta(u) = LT_\beta(u|_Z) \) (see (4.3.11) and Section 4.4.3).

**Remark 4.4.15.** The theorem above holds over arbitrary algebraically closed fields and \( c \) need not to be proper.

**Corollary 4.4.16.** Let \( c: C \to X \times_k X \) be a correspondence such that \( \text{Fix}(c) \) is proper over \( k \). Let \( u \) be a cohomological correspondence of \( \mathcal{F} \) lifting \( c \). Suppose the following conditions are satisfied.

(a) There exists an open subset \( U \subseteq X \) such that \( c_2|_{c_1^{-1}(U) \cap c_2^{-1}(U)} \) is quasi-finite.
(b) \( c|_U \) is contracting near every closed point of \( U \) in a neighbourhood of fixed points.
(c) \( c \) is contracting near the closed subscheme \( Z := (X \setminus U)_{\text{red}} \) in a neighbourhood of fixed points.

Then \( \text{Fix}(c|_U) \) is finite over \( k \). Moreover

\[
\text{LT}(u) = \sum_{\beta \in \pi_0(\text{Fix}(c|_U))} \text{Tr}(u_\beta) + \text{LT}(u|_Z).
\]

Here \( \text{Tr}(u_\beta) \) is the naive local term at \( \beta \) (see Definition 4.2.1), which is well defined by assumption (a).

**Proof.** We have a Cartesian diagram

\[
\begin{array}{ccc}
\text{Fix}(c) & \xrightarrow{c'} & X \\
\downarrow{\Delta'} & & \downarrow{\Delta} \\
C & \xrightarrow{c} & X \times X
\end{array}
\]

Let \( W \) be an open neighbourhood of \( \text{Fix}(c) \) such that \( c|_W \) leaves \( Z \) invariant (such a \( W \) exists by assumption (c)).

Let \( \beta \in \pi_0(\text{Fix}(c)) \) be such that \( c'(\beta) \cap Z \neq \emptyset \), then condition (c) and Theorem 4.4.14 together imply that \( \beta \in \pi_0(\text{Fix}(c|_{W,Z})) \), and that \( LT_\beta(u) = LT_\beta(u|_Z) \). Thus (4.3.13) implies that

\[
\text{LT}(u) = \sum_{\beta \in \pi_0(\text{Fix}(c|_U))} \text{LT}_\beta(u|_U) + \text{LT}(u|_Z).
\]

Hence it remains to show that \( \text{Fix}(c|_U) \) is finite over \( k \), and that for any \( \beta \in \pi_0(\text{Fix}(c|_U)) \), \( \text{LT}_\beta(u|_U) = \text{Tr}(u_\beta) \).

Replacing \( X \) by \( U \), \( c \) by \( c|_U \) and \( u \) by \( u|_U \), we can assume that \( c_2 \) is quasi-finite, and \( c \) is contracting near every closed point of \( U \) in a neighbourhood of fixed points.

Let \( x \in c'(\beta) \subseteq U \) be a closed point. Since \( c \) is contracting near every closed point of \( U \) in a neighbourhood of fixed points Theorem 4.4.14, (1) implies that \( \beta \) is a connected component of \( \text{Fix}(c|_{\{x\}}) \). Since \( c_2 \) is quasi-finite, \( \beta \) is necessarily a point. Hence \( \text{Fix}(c) \) is finite over \( k \).

Moreover Theorem 4.4.14, (2) implies that the local term \( \text{LT}_\beta(u) \) at such a \( \beta \) equals \( \text{LT}_\beta(u|_{\{x\}}) \) which in turn equals the naive local term at \( \beta \) (see [Var07], Section 1.5.7). \( \square \)
4.5. **Generalities on Cohomology with support.** Let $Y$ be any closed subscheme of $X \times_k X$ contained in $U$. As before $Z$ is the complement of $U$ in $X$, with the reduced structure. Let $i_{Y,U}$ and $i_Y$ denote the inclusion of $Y$ into $U$ and $X \times_k X$ respectively. Thus one has a diagram

\[
Y \xrightarrow{i_{Y,U}} U \xrightarrow{j} X \times_k X \xleftarrow{i} Z.
\]

The composite $j \circ i_{Y,U}$ equals $i_Y$ by definition and $Y \cap Z = \emptyset$. For any sheaf $\mathcal{G}$ on $X \times_k X$, by $H^p_Y(X \times_k X, \mathcal{G})$ we mean the cohomology $H^p(Y, i_Y^* \mathcal{G})$. Similarly for any sheaf $\mathcal{G}$ on $U$ by $H^p_Y(U, \mathcal{G})$ we mean the cohomology $H^p(Y, i_Y^* \mathcal{U}, \mathcal{G})$. We have a natural isomorphism

\[
H^0_Y(X \times_k X, \mathcal{G}) \cong \text{Hom}(i_Y^* \mathbb{Q}_\ell, \mathcal{G}).
\]

Since $\text{Hom}(i_Y^*, i_*) \cong 0$, applying the cohomological functor $\text{Hom}(i_Y^* \mathbb{Q}_\ell, \mathcal{G})$ to the triangle

\[
j_* j^* \mathcal{G} \longrightarrow \mathcal{G} \longrightarrow i_* i^* \mathcal{G} \cong 1,
\]

and using (4.5.2) we obtain an natural isomorphism

\[
H^0_Y(X \times_k X, \mathcal{G}) \cong H^0_Y(X \times_k X, j_! j^* \mathcal{G}).
\]

5. **Preliminary reductions**

In this section, we carry out some preliminary reductions towards the proof of Theorem 1.0.3. The crucial one is the reduction to the locally invariant boundary (see Definition 4.4.6) case using [Var18], Section 2.

5.1. **Reduction to $X$ smooth and $c_2$ generically étale.**

Assume $\dim(C) = 0$. The claim (1) and (3) in Theorem 1.0.3 are trivially true when $\dim(C) = 0$. Thus we only need to verify the rationality claim in (2). To do so we can assume $C$ is connected, and hence is a closed point $c$.

If $c_2(c)$ is not in the Frobenius orbit of $c_1(c)$, then $\text{Fix}(c^{(n)})$ is empty for all $n$, and the rationality is trivial. Else $\text{Fix}(c^{(n)}) = 1$, for all $n \equiv r \mod s$, for some integers $r$ and $s$. This implies the rationality of $Z(c, t)$ in this case. In particular Theorem 1.0.3 is true when $\dim(X) = 0$.

Before we carry out the necessary reductions we record the following simple observation which shall be repeatedly used in this Section.

**Remark 5.1.1.** Since Theorem 1.0.3 is true when $\dim(X) = 0$, inducting on $\dim(X)$, to prove Theorem 1.0.3 for $c : C \to X \times_k X$, we can throw away an arbitrary closed subset of $X$ (defined over $\mathbb{F}_p$) of strictly smaller dimension. In particular we may assume that $X$ is affine (and hence quasi-projective over $k$).

One can also induct on the dimension of $C$, and hence to prove Theorem 1.0.3 for $c : C \to X \times_k X$, we can throw away an arbitrary closed subset of $C$ of strictly smaller dimension. As above we may also assume that $C$ is quasi-projective over $k$. 

20
Lemma 5.1.2. It suffices to prove Theorem 1.0.3 under the following additional assumptions,

\begin{enumerate}
\item $c$ is defined over $\mathbb{F}_q$.
\item $\dim(C) = \dim(X)$.
\item $C$ and $X$ are irreducible and quasi-projective.
\item $c_1$ and $c_2$ are dominant.
\item $c$ is proper.
\item $X$ is smooth.
\end{enumerate}

Proof. Reduction (1):

Assume that $c$ is defined over $\mathbb{F}_q$. Any natural number $n$ can be written (uniquely) as $n = qs + r$, for $r \in \{0, 1, 2, \ldots s - 1\}$. Clearly the assertion for $c : C \to X \times_k X$ with respect to $\mathbb{F}_q$, is equivalent to the assertion for $c^{(i)} : C \to X \times_k X, \ 0 \leq i \leq s - 1$ with respect to $\mathbb{F}_q$. Thus one can assume $c$ is defined over $\mathbb{F}_q$. Henceforth we shall assume that $c$ is defined over $\mathbb{F}_q$.

Reduction (2):

Clearly $\dim(C) \leq \dim(X)$. If $\dim(C) < \dim(X)$. There exists a non-empty open subset $U$ of $X$ such that $c^{-1}(U \times_k U) = \emptyset$. Hence Theorem 1.0.3 is trivially true for $c^{-1}(U \times_k U) \to U \times_k U$. Thus the reduction now follows from Remark 5.1.1. Henceforth we shall assume that $\dim(C) = \dim(X)$.

Reduction (3):

Let $X_1, X_2, \ldots, X_t$ be the finitely many irreducible components of $X$ (over $k$). Let $\mathbb{F}_q^s$ be a finite sub extension of $k$ containing $\mathbb{F}_q$, over which $X_i$ and all the irreducible components of $c_1^{-1}(X_i) \cap c_2^{-1}(X_j)$ are defined for $1 \leq i, j \leq t$.

We now have two possible sub-cases.

Subcase 1: $s = 1$

This means all the $X_i$’s and all the components of $c_1^{-1}(X_i) \cap c_2^{-1}(X_j)$ are defined over $\mathbb{F}_q$ for $1 \leq i, j \leq t$. Let $C_{iv}$ be the distinct irreducible components of $c_1^{-1}(X_i) \cap c_2^{-1}(X_j)$, for $1 \leq i \leq t$ with $i'$ running over a finite indexing set. Then $c$ restricts to morphisms $c_{iv'} : C_{iv'} \to X_i \times_k X_i$, which are quasi-finite along second projection, and are defined over $\mathbb{F}_q$.

It then follows from Remark 5.1.1 that Theorem 1.0.3 holds for the correspondence $c$ if it holds for the finitely many correspondences $c_{iv'}$.

Subcase 2: $s > 1$

When $s > 1$ we can argue as in Reduction (1) to reduce to the case $s = 1$. Indeed consider the correspondences $c^{(n')} : C \to X \times_k X, \ 0 \leq n' \leq s - 1$. Since the geometric Frobenius $F_{X/\mathbb{F}_q}$ permutes the irreducible components of $X$, for any two indices $i$ and $j$, any irreducible component of $(c_1^{(n')})^{-1}(X_i) \cap (c_2^{(n')})^{-1}(X_j)$ is an irreducible component of $c_1^{-1}(X_i) \cap c_2^{-1}(X_j)$ for some indices $i'$ and $j'$. Moreover Theorem 1.0.3 for $c : C \to X \times_k X$ with respect to $\mathbb{F}_q$, is equivalent to the assertion for $c^{(n')} : C \to X \times_k X, \ 0 \leq i \leq s - 1$ with respect to $\mathbb{F}_q^s$. Thus one can assume $s = 1$, in which case we are reduced to Subcase 1.

Henceforth we shall assume that $C$ and $X$ are irreducible.

Reduction (4):

If either $c_1$ or $c_2$ is not dominant, then there exists a non-empty open subset $U$ of $X$ such that $c^{-1}(U \times_k U) = \emptyset$. Hence Theorem 1.0.3 is trivially true for $c^{-1}(U \times_k U) \to U \times_k U$. Thus the reduction now follows from Remark 5.1.1.
Henceforth we shall assume that $c_1$ and $c_2$ are dominant.

Reduction (5):
Since $C$ and $X$ are quasi-projective, we can factor $c$ as

$$
\begin{array}{ccc}
C & \xrightarrow{c} & X \times X \\
\downarrow j_C & & \downarrow \pi \\
\end{array}
$$

Here $\overline{C}$ is a quasi-projective variety, $j_C$ a dense open immersion and $\pi$ a proper (even projective) morphism. Let $\overline{c}_1$ and $\overline{c}_2$ be the induced projections onto $X$ from $\overline{C}$. By assumption $c_2$ is dominant map between varieties of the same dimension, and hence so is $\overline{c}_2$. Thus there exists a non-empty open subset $U$ of $X$ such that, $c_2^{-1}(U) \to U$ is a finite morphism. Finally using Remark 5.1.1 we first shrink $X$ to $U$, and then extend $C$ to $c_2^{-1}(U \times_k U)$.

Henceforth we shall assume that $c$ is proper.

Reduction (6):
This is an immediate consequence of Remark 5.1.1.

\[ \square \]

Remark 5.1.3. We note that under the reductions guaranteed by Lemma 5.1.2, the bound in Theorem 1.0.3 (1) is a consequence of the bound in (3). Thus it suffices to prove the bound in (3).

As a next step towards proving Theorem 1.0.3 (in addition to the above reductions) we reduce to the case when $c_2$ is generically étale. For any scheme $X$ over $\mathbb{F}_p$, we denote the absolute Frobenius morphism by $F_X$. Let $k$ be an arbitrary field of characteristic $p > 0$. For any $p$-primary number $q = p^r$, and any scheme $X/k$, by $X^{(q)}$ we mean the base change $k \times_k X$, where $k$ is considered as a $k$-scheme via the $r$th iterate of the absolute Frobenius of $k$. Note that $X^{(q)}$ naturally gets a structure of a $k$-scheme via the first projection. We call $X^{(q)}$ the $q$th Frobenius twist of $X$ with respect to $k$. Let $F_{X/k}^{(q)}$ be the induced relative Frobenius morphism from $X \to X^{(q)}$ (with respect to $k$). Note that $F_{X/k}^{(q)}$ is a morphism over $k$.

We will need the following simple Lemmas which we state without proof.

Lemma 5.1.4. Let $X$ be a scheme of finite type over a field $k$. Then the natural inclusion of subschemes, $((X_{\text{red}})^{(q)})_{\text{red}} \subseteq (X^{(q)})_{\text{red}}$ induced by the inclusion $X_{\text{red}} \subseteq X$ is an isomorphism.

Lemma 5.1.5. Let $X = \text{Spec}(A)$ be étale over a field $k$ of characteristic $p > 0$. Then the relative Frobenius $F_{X/k}^{(q)} : X \to X^{(q)}$ is an isomorphism for all $p$-primary numbers $q$.

The following lemma is a local form of our desired result.

Lemma 5.1.6. Let $L/K$ be a finite extension of fields of characteristic $p > 0$. Suppose that the inseparable degree of $L/K$ is $p^n$. Then $(\text{Spec}(L)^{(p^n)})_{\text{red}}$ is connected and étale over Spec $(K)$ of rank $[L : K]/p^n$. Here $\text{Spec}(L)^{(p^n)}$ is the Frobenius twist of Spec $(L)$, considered as $K$-scheme.

Proof. Let $K'$ be the separable closure of $K$ in $L$. Thus we have a commutative diagram with Cartesian squares
(5.1.1) \[
\begin{array}{ccc}
\text{Spec} (L)(p^n) & \longrightarrow & \text{Spec} (L) \\
\downarrow & & \downarrow \\
\text{Spec} (K')(p^n) & \longrightarrow & \text{Spec} (K') \\
\downarrow & & \downarrow \\
\text{Spec} (K) & \overset{F^n_{\text{Spec}(K)}}{\longrightarrow} & \text{Spec} (K)
\end{array}
\]

Here \( \text{Spec} (K')(p^n) \) is the \( p^n \) Frobenius twist of \( \text{Spec} (K') \) with respect to \( K \). Lemma 5.1.5 implies that \( \text{Spec} (K')(p^n) \) is isomorphic to \( \text{Spec} (K') \) under the relative Frobenius (with respect to \( \text{Spec} (K) \)). Thus the lower square in Diagram 5.1.1 is isomorphic to

(5.1.2) \[
\begin{array}{ccc}
\text{Spec} (K') & \longrightarrow & \text{Spec} (K') \\
\downarrow & & \downarrow \\
\text{Spec} (K) & \overset{F^n_{\text{Spec}(K')}}{\longrightarrow} & \text{Spec} (K)
\end{array}
\]

Since the inseparable degrees of \( L/K \) and \( L/K' \) are the same, transitivity of base change applied to the Diagram 5.1.1 implies that it suffices to prove the lemma for \( L/K' \). Thus we are reduced to proving that if \( L/K \) is a purely inseparable extension of degree \( p^n \), then the natural morphism \( (\text{Spec} (L)(p^n))_{\text{red}} \to \text{Spec} (K) \) is an isomorphism. This is immediate.

\[\square\]

Corollary 5.1.7. Let \( L/K \) be a finite extension of fields of characteristic \( p > 0 \). Suppose that the inseparable degree of \( L/K \) is \( p^n \). Then \( (\text{Spec} (L)(p^n))_{\text{red}} \) is connected and étale over \( \text{Spec} (K) \) of rank \( [L : K]/p^n \) for all \( m \geq n \).

Proof. Lemma 5.1.4 implies that

(5.1.3) \[
(\text{Spec} (L)(p^m))_{\text{red}} \simeq (((\text{Spec} (L)(p^n))_{\text{red}})(p^{m-n}))_{\text{red}}.
\]

Here all the Frobenius twists are with respect to \( K \). Lemma 5.1.6 implies that \( (\text{Spec} (L)(p^n))_{\text{red}} \) is connected and étale over \( \text{Spec} (K) \) of rank \( [L : K]/p^n \). The corollary is then a consequence of Lemma 5.1.5.

\[\square\]

We will need the following global variant of Corollary 5.1.7.

Lemma 5.1.8. Let \( k \) be a field of characteristic \( p > 0 \). Let \( f : Y \to X \) be a dominant morphism between varieties (over \( k \)) of dimension \( d \). Let \( \delta := [k(Y) : k(X)] \) induced by \( f \) at the level of generic points. Let \( p^n \) be the inseparable degree of \( k(Y)/k(X) \). Let \( F_X : X \to X \) be the absolute Frobenius of \( X \) with respect to \( \mathbb{F}_p \). For every \( m \geq 1 \), consider the Cartesian diagram
Then

1. the map $F_X^{m'}$ is an universal homeomorphism for any $m \geq 1$.
2. $\dim(Y \times_{X,F_X^m} X) = d$.
3. $f^{(m)}$ is dominant for any $m \geq 1$.
4. For any $m \geq n$, $(f^{(m)})_{\text{red}}$ is generically étale of degree $\delta/p^n$.

**Proof.** The morphism $F_X : X \to X$ being a universal homeomorphism immediately implies part (1) and (2) of the above proposition. Since $f$ is assumed to be dominant, $f^{(m)}$ is necessarily dominant and thus (3) holds. Hence it remains to show (4).

Note that (2) and (3) together then imply that the natural map

\[(5.1.4) \quad \text{Spec}(k((Y \times_{X,F_X^m} X)_{\text{red}})) \to ((Y \times_{X,F_X^m} X) \times_{X} \text{Spec}(k(X)))_{\text{red}}\]

is an isomorphism. Similarly since $X$ and $Y$ are varieties of the same dimension, the dominance of $f$ implies that the natural map

\[(5.1.5) \quad \text{Spec}(k(Y)) \to Y \times_{X} \text{Spec}(k(X)),\]

is an isomorphism.

Let $j_{\eta_X} : \text{Spec}(k(X)) \to X$ be the inclusion of the generic point of $X$ into $X$. Note that the morphism induced by the absolute Frobenius $F_X$ at the level of the generic point corresponds to the Frobenius of $k(X)$. Thus one has a commutative diagram

\[(5.1.6) \quad \begin{array}{ccc}
X & \xrightarrow{F_X^m} & X \\
\text{Spec}(k(X)) & \xrightarrow{F_X^m} & \text{Spec}(k(X)) \\
j_{\eta_X} & & j_{\eta_X}
\end{array}\]

Thus the isomorphism (5.1.5) and transitivity of base change together imply that

\[(5.1.7) \quad ((Y \times_{X,F_X^m} X) \times_{X} \text{Spec}(k(X)))_{\text{red}} \simeq \text{Spec}(k(Y))^{(p^m)},\]

where the Frobenius twist on the right is with respect to the $k(X)$-scheme structure via $f$. Combining the isomorphisms (5.1.4) and (5.1.7) we get,

\[(5.1.8) \quad \text{Spec}(k((Y \times_{X,F_X^m} X)_{\text{red}})) \simeq \text{Spec}(k(Y))^{(p^m)},\]

as $\text{Spec}(k(X))$ schemes. Thus (4) now follows from Corollary 5.1.7.

\[
\square
\]

**Proposition 5.1.9.** In addition to the reductions above, It suffices to prove Theorem 1.0.3 under the assumption that $c_2$ is generically étale.
Proof. If $c_2$ is generically étale we are done. Hence assume that $c_2$ is not generically étale, and let $\delta' = p^n$ be the inseparable degree of $k(X)/k(C)$ induced by $c_2$. This in particular implies $d = \dim(X) > 0$.

Let $F_{X/\mathbb{F}_q}$ be the geometric Frobenius of $X$ with respect to $\mathbb{F}_q$. Let $r$ be the smallest integer such that $q^r \geq p^n$. Consider the Cartesian diagram

\begin{align*}
C \times_{X \times_k X} (X \times_k X) & \ar{r} \ar{d} & X \times_k X \ar{r}{pr_2} \ar{d}{1 \times F_{X/\mathbb{F}_q}} & X \ar{d}{F_{X/\mathbb{F}_q}} \\
C & \ar{r} & X \times_k X \ar{r}{pr_2} & X
\end{align*}

Let $C' := (C \times_{X \times_k X} (X \times_k X))_{\text{red}}$ and $c' : C' \subseteq X \times_k X$ be the induced correspondence. Let $F'$ be the induced map from $C'$ to $C$. Since $F_{X/\mathbb{F}_q}$ is a universal homeomorphism, $C'$ is necessarily irreducible.

By definition $c'_1 = c_1 \circ F$. Hence

\begin{align*}
\text{deg}(c'_1) = \delta \text{deg}(F').
\end{align*}

We claim that

1. $c'_2$ is generically étale of degree $\frac{\text{deg}(c_2)}{p}$.
2. To prove Theorem 1.0.3 for the correspondence $c : C \subseteq X \times_k X$, it suffices to prove Theorem 1.0.3 for the correspondence $c' : C' \subseteq X \times_k X$.

(1) and (2) together imply the Proposition. We now prove these claims.

Proof of claim 1. First, note that $c'$ is also defined over $\mathbb{F}_q$. Further transitivity of base change applied to the Diagram 5.1.9 implies that the following diagram is Cartesian upto nilpotents

\begin{align*}
C' \ar{r}{F'} \ar{d}{c_2} & C \ar{d}{c_2} \\
X \ar{r}{F_{X/\mathbb{F}_q}} & X
\end{align*}

Let

\begin{align*}
C_0 \times_{X_0 \times_{\mathbb{F}_q} X_0} (X_0 \times_{\mathbb{F}_q} X_0) & \ar{r} \ar{d} & X_0 \times_{\mathbb{F}_q} X_0 \ar{r}{pr_2} \ar{d}{1 \times F_{X_0/\mathbb{F}_p}^{[r]}} & X_0 \ar{d}{F_{X_0/\mathbb{F}_p}^{[r]}} \\
C_0 & \ar{r} & X_0 \times_{\mathbb{F}_q} X_0 \ar{r}{pr_2} & X_0
\end{align*}

be a model for the Diagram 5.1.9 over $\mathbb{F}_q$. Here as before $F_{X_0}$ is the absolute Frobenius of $X_0$ with respect to $\mathbb{F}_p$. By assumption

\begin{align*}
[F_q : \mathbb{F}_p]^r \geq n.
\end{align*}
Let $C'_0 := (C_0 \times_{X_0 \times_{\mathbb{F}_q} X_0} (X_0 \times_{\mathbb{F}_q} X_0))_{\text{red}}$. Since $\mathbb{F}_q$ is perfect, $C'_0$ is a model of $C'$ over $\mathbb{F}_q$. Thus the morphism $c'_0 : C'_0 \subseteq X_0 \times_{\mathbb{F}_q} X_0$ is a model for $c'$. Let $(c_2)_0$ and $(c'_2)_0$ be the morphism induced from $C_0$ and $C'_0$ respectively, to $X_0$ under the second projection. As before we get a diagram which is Cartesian upto nilpotents

\begin{equation}
\begin{CD}
C' @> F' >> C \\
@V(c'_2)_0 VV @VV(c_2)_0 V \\
X_0 @>> F_{X_0/\mathbb{F}_q} & X
\end{CD}
\end{equation}

which is a model for the Diagram 5.1.11 over $\mathbb{F}_q$.

The inequality (5.1.13) and Lemma 5.1.8 together imply that $(c'_2)_0$ is generically étale of degree $\frac{\deg(c_2')}{\delta}$, and hence so is $c'_2$.

**Proof of claim (2).** Since $\deg(c'_2) = \frac{\deg(c_2)}{\delta}$, the commutative diagram (5.1.11) implies that $\deg(F') = q^{d_1}$. Thus the equality (5.1.10) implies that the generic degree $\delta_1'$ of $c'_1$ is $\delta q^{d}/p^n$.

Note that $F'$ induces a bijection (also denoted by $F'$) between $|C'| = C'(k)$ and $|C| = C(k)$. We claim that this bijection restricts to a set theoretic bijection between $\text{Fix}(c^{(n')})_1(k) \subseteq C'(k)$ and $\text{Fix}(c^{(n')})(k) \subset C(k)$ for all positive integers $n'$.

Indeed let $\alpha \in \text{Fix}(c^{(n')})(k) \subset C(k)$. Then $F'_{X/\mathbb{F}_q} \circ c_1 \circ F'(\alpha) = c'_2(\alpha)$. Hence the commutative diagram (5.1.11) implies that $F'_{X/\mathbb{F}_q} \circ F'_{X/\mathbb{F}_q} \circ c_1 \circ F'(\alpha) = F'_{X/\mathbb{F}_q} \circ c'_2(\alpha) = c_2 \circ F'(\alpha)$. Thus $F'(\alpha) \in \text{Fix}(c^{(n')})(k)$. The converse is also true since $F'_{X/\mathbb{F}_q}$ is a homeomorphism. Now suppose Theorem 1.0.3 has been established for $c'$. This clearly implies (1) and (2) in Theorem 1.0.3 for $c$. Thus we only need to establish (3) for $c$ assuming the same for $c'$.

The degree of $c'_1$ is $\delta q^{d}/p^n$, and $c'_2$ is generically étale. Thus there exist integers $N'$ and $M'$ such that for any integer $n' \geq N'$, $\text{Fix}(c^{(n')})$ is finite (over $k$), and

\begin{equation}
|\text{Fix}(c^{(n')})(k)| - \delta q^{d}/p^n q^{n'd} \leq M' q^{n'(d-\frac{1}{2})}.
\end{equation}

Moreover since $\text{Fix}(c^{(n')})(k)$ and $\text{Fix}(c^{(n'+r)})(k)$ are set theoretically bijective, the inequality (5.1.15) can be rewritten as

\begin{equation}
|\text{Fix}(c^{(n'+r)})(k)| - \delta q^{(n'+r)d} \leq M q^{(n'+r)(d-\frac{1}{2})},
\end{equation}

where $M = \frac{M'}{q^{d(d-\frac{1}{2})}}$. Note that $\delta$ is the degree of $c_1$, and $\delta' = p^n$ the inseparable degree of $c_2$. Thus Theorem 1.0.3. (3) holds true for $c$ with $N$ and $M$ equal to $N' + r$ and $\frac{M'}{q^{d(d-\frac{1}{2})}}$ respectively.

Finally, the key reduction is analogous to the one carried out in [Var18]. Indeed using Varshavsky’s construction ([Var18], Corollary 2.4 and Claim 5.2, Step 4) we reduce to the case of locally invariant boundary. The key construction being [Var18], Proposition 2.3 which shows
Proposition 5.1.10. It suffices to prove Theorem 1.0.3 under the assumption that \( c : C \to X \times_k X \) has a compactification \( \overline{c} : \overline{C} \to \overline{X} \times_k \overline{X} \) (over \( \mathbb{F}_q \)) with \( \overline{C} \) and \( \overline{X} \) projective and such that the boundary is locally invariant over \( \mathbb{F}_q \).

Remark 5.1.11. We note that since by assumption \( c \) is proper, Corollary 4.1.5 implies that \( \overline{c}^{-1}(X \times_k X) = C \). In particular if \( c_2 \) is étale, so is \( \overline{c}_2 \) when restricted to \( C \).

5.2. A bound on the global terms. Let \( k \) be an algebraic closure of \( \mathbb{F}_q \). Let \( c : C \to X \times_k X \) be a correspondence of projective varieties\(^2\). Further assume that

1. \( c \) is defined over \( \mathbb{F}_q \) (and we choose a model).
2. \( \dim(C) = \dim(X) = d \).
3. \( c_1 \) and \( c_2 \) are dominant.
4. \( c_2 \) is generically étale.

Let \( j : U \hookrightarrow X \) be a non-empty smooth open subscheme defined over \( \mathbb{F}_q \) such that, \( c_2|_{c_1^{-1}(U) \cap c_2^{-1}(U)} \) is étale. Recall that we are in such a set up (see Remark 5.1.11).

Let \( C' := c_1^{-1}(U) \cap c_2^{-1}(U) \). By definition \( [c]|_U \) is the correspondence \( c' : C' \to U \times_k U \) induced by \( c \). Further since \( c_2|_{C'} \) is assumed to be étale, \( C' \) is smooth over \( k \).

There is a natural cohomological correspondence of \( \mathbb{Q}_l[d]|_U \) lifting \( c' \) given by

\[
(5.2.1) \quad c'_1 \mathbb{Q}_l[d] \xrightarrow{\sim} \mathbb{Q}_l[d] \xrightarrow{\sim} (S) \xrightarrow{\sim} \mathbb{Q}_l \xrightarrow{\sim} d_{C'} \mathbb{Q}_l[d] \xrightarrow{\sim} c'_1 \mathbb{Q}_l[d] \xrightarrow{\sim} c'_1 \mathbb{Q}_l[d].
\]

Here the isomorphisms \((S)\) come from the smoothness of \( C' \). The morphism \( c'_2 \) is dual to the natural isomorphism \( c'_2 : c'_2 \mathbb{Q}_l[d] \simeq \mathbb{Q}_l[d] \).

Since the correspondence \( c|_U \) is quasi-finite along the second projection, we can make sense of the naive local term along closed points \( y \) in \( \text{Fix}(c|_U) \) (see Definition 4.2.1).

Lemma 5.2.1. For the cohomological correspondence \( u' \) in (5.2.1), and any closed point \( y \) in \( \text{Fix}(c|_U) \), the naive local term \( \text{Tr}(u'_y) = 1 \).

Proof. The map \( c'_2 c'_1 \mathbb{Q}_l[d] \to \mathbb{Q}_l[d] \) induced by (5.2.1) via adjunction between \((c'_2, c'_1)\) under the isomorphism (5.2.1), corresponds to the natural adjunction map \( c'_2 c'_1 \mathbb{Q}_l[d] \to \mathbb{Q}_l[d] \). Since \( c'_2 \) is étale, this coincides with the trace map for quasi-finite flat morphism [FK88, Chapter II, Lemma 1.1] and the result follows.

\( \square \)

In this set up we have the following Corollary to Proposition A.3.2.

Corollary 5.2.2. There exists a cohomological correspondence (defined over \( \mathbb{F}_q \))

\[
(5.2.2) \quad u : c'_1 \text{IC}_X \to c'_2 \text{IC}_X
\]

lifting \( c \) such that, \( u|_U \) (see Section 4.1.4) is the correspondence (5.2.1).

\(^2\)In this section \( C \) and \( X \) are projective varieties. We deliberately avoid the ‘bar’ decoration since we only need to deal with the compactified correspondence in this section.
Proof. Corollary A.3.4 applied to \( c_1 : C \to X \) gives us a map (defined over \( \mathbb{F}_q \))

\[
(5.2.3) \quad c_1^* : c_1^* \mathrm{IC}_X \to \mathrm{IC}_C.
\]

Corollary A.3.5 applied to \( c_2 : C \to X \) gives us a map (defined over \( \mathbb{F}_q \))

\[
(5.2.4) \quad c_2^\dagger : \mathrm{IC}_C \to c_2^\dagger \mathrm{IC}_X.
\]

We define \( u : c_1^* \mathrm{IC}_X \to c_2^\dagger \mathrm{IC}_X \) to be the composition of (5.2.3) and (5.2.4). Clearly \( u \) is also defined over \( \mathbb{F}_q \). Since \( C' \) is smooth, the compatibility with (5.2.1) is an immediate consequence of the compatibility in Corollaries A.3.4 and A.3.5.

\[\square\]

We now show that (5.2.2) produces the 'correct' leading term and error term in (1.0.2). Let \( \tau : \overline{\mathbb{Q}_p} \hookrightarrow \mathbb{C} \) be an embedding. Let \( \delta \) be the generic degree of \( c_1 \).

Proposition 5.2.3. There exists a positive real number \( M_{\text{glo}}(\tau) \) such that for any \( n \geq 0 \)

\[
(5.2.5) \quad |\mathrm{Tr}(R\Gamma(u^{(n)})) - \delta q^{nd}| \leq M_{\text{glo}}(\tau) q^{n(\frac{d}{2})}.
\]

Here \( \mathrm{Tr}(R\Gamma(u^{(n)})) \) is the trace of the endomorphism \( R\Gamma(u^{(n)}) \), of the perfect complex (of \( \overline{\mathbb{Q}_p} \) vector spaces) \( R\Gamma(X, \mathrm{IC}_X) \), induced by the cohomological correspondence (5.2.2).

Proof. Since \( H^i(X, \mathrm{IC}_X) \) vanishes for \( i \notin [-d, d] \) ([BBDG18], Section 4.2.4), by the definition of \( \mathrm{Tr}(R\Gamma(u^{(n)})) \) we have an equality

\[
(5.2.6) \quad \mathrm{Tr}(R\Gamma(u^{(n)})) = \sum_{i=-d}^{d} (-1)^i \mathrm{Tr}((c_{2*}c_1^*) \circ F_{X/\mathbb{F}_q}^{\text{nis}}) : H^i(X, \mathrm{IC}_X) \to H^i(X, \mathrm{IC}_X)).
\]

Here \( c_1^* \) and \( c_{2*} \) are the pullback and pushforward induced by (5.2.3) and (5.2.4) respectively. Since \( c_1, c_2 \) and \( u \) are defined over \( \mathbb{F}_q \), \( c_{2*}c_1^* \) commutes with \( F_{X/\mathbb{F}_q}^{\text{nis}} \). Thus the bound (5.2.5) is an immediate consequence of Proposition A.3.8, and purity of \( \mathrm{IC}_X \) ([BBDG18], Théorème 5.4.10).

\[\square\]

6. LOCAL TERMS ALONG A LOCALLY INVARIANT SUBSET

This section aims to prove Theorem 1.0.5, and along the way obtain a rationality statement. As mentioned in the introduction we prove a similar bound for the larger class of essentially proper correspondences. We begin by defining them.

Let \( k \) be an algebraic closure of the finite field \( \mathbb{F}_q \). Through this section we will be working over schemes of finite type over the field \( k \). Some of the results can be generalized in an obvious way to arbitrary algebraically closed fields. But we do not need these generalisations for our purposes, hence do not pursue them here.
6.1. Essentially proper correspondences. Let \( X/k \) be a proper scheme defined over \( \mathbb{F}_q \). Let \( c: C \rightarrow X \times_k X \) be an arbitrary correspondence.

**Definition 6.1.1** (Essentially proper correspondence). We say \( c \) is an essentially proper correspondence over \( \mathbb{F}_q \), if it can be factored as

\[
\begin{array}{ccc}
C & \xrightarrow{c} & X \\
\downarrow{\varpi_U} & & \downarrow{\pi_U} \\
U & \xrightarrow{i} & X \times_k X
\end{array}
\]

such that

(a) \( \varpi_U \) is proper.

(b) \( U = \bigcup_{i=1}^{r} U_i \times_k U_i \), where \( U_i \)'s are open subsets of \( X \) defined over \( \mathbb{F}_q \) and cover \( X \).

**Remark 6.1.2.** Proper correspondences are trivially essentially proper (over any finite sub-field containing \( \mathbb{F}_q \)).

The following is our motivation for studying essentially proper correspondences.

Let \( \varpi: \mathcal{X} \rightarrow \mathcal{X} \times_k \mathcal{X} \) be a proper correspondence. Let \( Z \subseteq \mathcal{X} \) be a closed subset of \( \mathcal{X} \) defined over \( \mathbb{F}_q \), which is locally \( \varpi \)-invariant over \( \mathbb{F}_q \) (see Remark 4.4.8). Thus there exists a finite collection of open subsets \( \{U_i\} \), \( 1 \leq i \leq r \) of \( \mathcal{X} \) defined over \( \mathbb{F}_q \), which cover \( Z \), and set theoretic inclusions

\[
\varpi^{1}_{2}(U_i \cap Z) \cap \varpi^{1}(U_i) \subseteq \varpi^{1}(Z),
\]

for every \( i \).

Let \( U := \mathcal{X} \setminus Z \). Note that \( U \) is also defined over \( \mathbb{F}_q \). Clearly \( U \) and \( U_i \)'s together form an open cover of \( \mathcal{X} \). Let \( W := \varpi^{-1}((U \times_k U) \cup (\cup_{i=1}^{r}(U_i \times_k U_i))) \).

**Lemma 6.1.3.** \( W \) is a neighbourhood of fixed points of \( \varpi^{(n)} \) such that, \( Z \) is \( \varpi^{(n)}|_W \)-invariant for any \( n \geq 0 \).

**Proof.** Since \( ((U \times_k U) \cup (\cup_{i=1}^{r}(U_i \times_k U_i))) \) is an open neighbourhood of \( \Delta^{(n)}_{\mathcal{X}} \), \( W \) by definition is an open neighbourhood of \( \text{Fix}(\varpi^{(n)}) \).

Now we show that \( Z \) is \( \varpi^{(n)}|_W \)-invariant for every \( n \geq 0 \). Let \( x \in W \) be such that \( \varpi_2(x) \in Z \). We need to show that \( F^\varpi_{\mathcal{X}/\mathbb{F}_q}(\varpi_1(x)) \in Z \), for every \( n \geq 0 \). Since \( Z \) is defined over \( \mathbb{F}_q \) it suffices to prove that \( \varpi_1(X) \in Z \).

Since \( x \in W = \varpi^{-1}((U \times_k U) \cup (\cup_{i=1}^{r}(U_i \times_k U_i))) \), \( \varpi_2(x) \in Z \) implies that \( x \in \varpi^{-1}(U_i \times U_i) \) for some index \( i \). Thus \( \varpi_2(x) \in U_i \cap Z \) and \( \varpi_1(x) \in U_i \). (6.1.2) then implies that \( \varpi_1(x) \in Z \).

In light of Lemma 6.1.3 our discussion in Section 4.4.3 applies to \( Z \), and we can make sense of the correspondences \( \varpi|_{W,Z} \). Consider the Cartesian diagram

\[
\begin{array}{ccc}
W \cap \varpi^{1}_{2}(Z) \cap \varpi^{1}_{2}(Z) & \xrightarrow{\varpi^{1}_{2}} & \varpi^{1}_{2}(Z) \cap \varpi^{1}_{2}(Z) \\
\downarrow{\cup_i((U_i \cap Z) \times_k (U_i \cap Z))} & & \downarrow{\cup_i((U_i \cap Z) \times_k (U_i \cap Z))} \\
\varpi^{1}_{1}(Z) \times_k \varpi^{1}_{1}(Z) & \xrightarrow{\varpi^{1}_{1}} & \varpi^{1}_{1}(Z) \times_k \varpi^{1}_{1}(Z)
\end{array}
\]

29
6.1.1. The correspondence $\overline{c}|_{W,Z}$ is then by definition the map from $W \cap \overline{c}_1^{-1}(Z) \cap \overline{c}_2^{-1}(Z)$ to $Z \times_k Z$. The Cartesian diagram (6.1.3) trivially implies the following Lemma.

**Lemma 6.1.4.** $\overline{c}|_{W,Z}$ is an essentially proper correspondence over $\mathbb{F}_q$.

Moreover since $Z$ is defined over $\mathbb{F}_q$, we can make sense of $\overline{c}^{(n)}|_{W,Z}$, and clearly we have

\[(6.1.4) \quad \overline{c}^{(n)}|_{W,Z} = \overline{c}|_{W,Z},\]

as correspondences from $W \cap \overline{c}_1^{-1}(Z) \cap \overline{c}_2^{-1}(Z)$ to $Z \times_k Z$.

6.1.2. Now suppose we are given a cohomological correspondence $u$ of a Weil sheaf $F$ on $X$ lifting the correspondence $\overline{c}$. Then Lemma 6.1.3 and the discussion in Section 4.4.3 implies that there exists a cohomological correspondence $u^{(n)}|_{W,Z}$ of $F|_{Z}$ lifting the correspondence $c^{(n)}|_{W,Z}$. Here $F|_{Z}$ is given the natural Weil structure coming from restricting the one on $F$.

In particular we can calculate $\text{LT}(u^{(n)}|_{Z})$ using $W$. Also note that analogous to (6.1.4) we have

\[(6.1.5) \quad u^{(n)}|_{W,Z} = (u|_{W,Z})^{(n)}\]

as cohomological correspondences of $F|_{Z}$ lifting $\overline{c}^{(n)}|_{W,Z} = \overline{c}|_{W,Z}$. In particular

\[(6.1.6) \quad \text{LT}(u^{(n)}|_{Z}) = \text{LT}((u|_{W,Z})^{(n)}).\]

Having motivated essentially proper correspondences, we now establish some basic properties of essentially proper correspondences over $\mathbb{F}_q$.

As before let $c: C \to X \times_k X$ be an essentially proper correspondence over $\mathbb{F}_q$. Recall that $X$ is assumed to be proper and defined over $\mathbb{F}_q$. In particular we have a diagram (6.1.1).

**Lemma 6.1.5.** $c^{(n)}$ is essentially proper over $\mathbb{F}_q$ for all $n \geq 1$.

**Proof.** Let $F_i := F_{X/\mathbb{F}_q} \times 1_X$ be the partial Frobenius on $X \times_k X$. The diagram (6.1.1) can be enlarged to the following diagram

\[(6.1.7) \quad \begin{array}{ccc}
C & \xrightarrow{c} & \overline{c}_U \\
\downarrow & & \downarrow \\
U \times_k X & \xrightarrow{\pi_U \times 1_X} & X \times_k X \\
\downarrow & & \downarrow \\
U \times_k X & \xrightarrow{F^n|U} & F^n_{X/\mathbb{F}_q} \times 1_X \\
\downarrow & & \downarrow \\
U \times_k X & \xrightarrow{\pi_U \times 1_X} & X \times_k X
\end{array}\]

The square is Cartesian because of the condition (b) in Definition 6.1.1. Thus the outer square in (6.1.7) implies that $c^{(n)}$ is also essentially proper over $\mathbb{F}_q$.

\[\square\]

The following Lemma is analogous to Lemma 6.1.3.

**Lemma 6.1.6.** For any $n \geq 0$, the graphs of $F^n_{X/\mathbb{F}_q}$ and their transpose are contained in $U$ (see Diagram 6.1.1). Thus $\text{Fix}(c^{(n)})$ is proper over $k$.  

Lemma 6.1.6 implies that, if we are given a cohomological correspondence \( u \) of a Weil sheaf \( F \) on \( X \) lifting the possibly non-proper but essentially proper correspondence \( c \), we can make sense of the local terms \( \mathrm{LT}(u^n) \) (Definition 4.3.2).

Now suppose \( \mathcal{F}_0 \in D^b_{\leq 0}(X_0, \overline{\mathbb{Q}}_\ell) \) is a mixed sheaf of weight less than or equal to \( 0 \) on \( X_0 \). Suppose that \( \mathcal{F}_0 \) is in \( \mathcal{P}D_{\leq 0}(X_0, \overline{\mathbb{Q}}_\ell) \). Here \( X_0 \) is the chosen model of \( X \) over \( \mathbb{F}_q \). Let \( u \) be a cohomological correspondence of \( \mathcal{F} \) lifting \( c \), an essentially proper correspondence over \( \mathbb{F}_q \).

Fix a field isomorphism (say \( \tau \)) of \( \overline{\mathbb{Q}}_\ell \) with \( \mathbb{C} \).

This section aims to obtain the following estimate for the local terms.

**Theorem 6.1.7.** For any \( \epsilon > 0 \), there exists a natural number \( N(\epsilon) \) and a positive real number \( M(\tau) \) such that, for any \( n \geq N(\epsilon) \),

\[
\left| \mathrm{LT}(u^n) \right| \leq M(\tau)q^{n(\ell n + n + \dim(X)) + \epsilon}.
\]

Here the norm on the left is with respect to the chosen isomorphism \( \tau \).

**Theorem 6.1.7** follows easily from the Lefschetz-Verdier trace formula if one assumes \( c \) is proper (and not just essentially proper). In light of Lemma 6.1.4 and the equality \( (6.1.6) \), Theorem 1.0.5 is an immediate consequence of Theorem 6.1.7 (with \( Z \) playing the role of \( X \) above).

Now we show that Theorem 1.0.5 combined with the results of the earlier sections implies (1) and (3) in Theorem 1.0.3.

### 6.2. Theorem 1.0.5 implies Theorem 1.0.3, (1) and (3).

**Proof.** Recall that we have a correspondence \( c: C \to X \times_k X \), together with a compactification \( \overline{c}: \overline{C} \subseteq \overline{X} \times_k \overline{X} \) of \( c \). Lemma 5.1.2, Proposition 5.1.9 and Proposition 5.1.10 imply that we can assume the following

(a) \( X \) is smooth over \( k \).
(b) \( \overline{c}_2 \) restricted to \( \overline{c}^{-1}(X \times_k X) = C \) is étale.
(c) \( \overline{c} \) leaves \( Z := \overline{X} \setminus X \) locally invariant (see Definition 4.4.6) over \( \mathbb{F}_q \).

Since \( Z \) is defined over \( \mathbb{F}_q \), (c) implies that \( Z \) is also locally \( \overline{c}(n) \)-invariant over \( \mathbb{F}_q \) for all \( n \geq 1 \). Thus \([\text{Var07}], 2.2.4\) implies that there exists an integer \( N' \) such that for all \( n \geq N' \), \( \overline{c}(n) \) is contracting in a neighbourhood of fixed points around \( Z \) (see Definition 4.4.13) and \( \overline{c}(n)|_X \) is contracting near every closed point of \( X \) in a neighbourhood of fixed points.

Let \( u: \overline{c}_1^* \text{IC}_{\overline{X}} \to \overline{c}_2^* \text{IC}_{\overline{X}} \) be the cohomological correspondence defined in Corollary 5.2.2. Since \( \text{Fix}(\overline{c}(n)) \) is proper over \( k \), we can apply Corollary 4.4.16 to the cohomological correspondence \( u^{(n)} \) for any \( n \geq N' \).

Corollary 4.4.16 implies that for \( n \geq N' \), \( \text{Fix}(c^{(n)}) \) is finite. Moreover we have for any \( n \geq N' \),
\[(6.2.1) \quad \text{LT}(u(n)) = \sum_{\beta \in \text{Fix}(c(n)|_X)} \text{Tr}(u^{(n)}_\beta) + \text{LT}(u(n)|_Z).\]

Since \(c^{-1}(X \times X) = C\) and \(u|_X\) is the correspondence \((5.2.1)\), Lemma \(5.2.1\) implies that

\[(6.2.2) \quad \sum_{\beta \in \text{Fix}(c(n)|_X)} \text{Tr}(u^{(n)}_\beta) = \#\text{Fix}(c(n))(k).\]

Since the restriction of \(\text{IC}_X\) to \(Z\) is of weight less than or equal to \(\dim(X)\) \(([BBDG18], 5.1.14 (i))\) and belongs to \(pD^{\leq -1}\) \(([BBDG18], \text{Corollaire 1.4.24})\), Theorem \(1.0.5\) implies that there exists a positive integer \(N''\) such that for all \(n \geq N''\),

\[(6.2.3) \quad |\text{LT}(u^{(n)}|_Z)| \leq M_{\text{loc}}(\tau)q^{n(\dim(X) - \frac{1}{2})}\]

for some positive real number \(M_{\text{loc}}(\tau)\) (see Remark \(6.2.1\)).

Finally note that Lefschetz-Verdier trace formula (Corollary \(4.3.5\)) and Proposition \(5.2.3\) together imply that

\[(6.2.4) \quad |\text{LT}(u^{(n)}) - \delta q^{n(\dim(X))}| \leq M_{\text{glo}}(\tau)q^{n(\dim(X) - \frac{1}{2})},\]

for some positive real number \(M_{\text{glo}}(\tau)\) (depending on \(\tau\)).

Thus combining \((6.2.1)\), \((6.2.2)\), \((6.2.3)\) and \((6.2.4)\) we obtain the bound

\[(6.2.5) \quad |\#\text{Fix}(c(n))(k) - \delta q^{n(\dim(X))}| \leq M(\tau)q^{n(\dim(X) - \frac{1}{2})},\]

for all \(n \geq N = \max\{N', N''\}\) with \(M(\tau) = M_{\text{loc}}(\tau) + M_{\text{glo}}(\tau)\).

\[\square\]

**Remark 6.2.1.** The bound \((6.2.3)\) is a consequence of both the dimension and perversity dropping when restricted to the boundary. The bound in \((6.1.8)\) has an error term of \(\epsilon\) and if we did not have the perversity drop, Theorem \(1.0.5\) would give us a weaker bound with the local term growing as \(q^{n(\dim(X) - 1/2 + \epsilon)}\), which is clearly insufficient for our purposes.

The rest of the article is devoted to the proof of Theorem \(6.1.7\).

### 6.3. The pairing

Let \(c : C \rightarrow X \times_k X\) be an essentially proper correspondence over \(\mathbb{F}_q\), with the choice of a factorization as in \((6.1.1)\). Let \(Z := (X \times_k X \setminus U)_{\text{red}}\) the complimentary closed subscheme and \(i : Z \hookrightarrow X \times_k X\) be the corresponding closed immersion.

Choose an arbitrary compactification \(\overline{c} : \overline{C} \rightarrow X \times_k X\) of \(c\). Let \(\partial \overline{C}\) be the reduced complement of \(C\) in \(\overline{C}\). Let \(i_{\partial \overline{C}}\) and \(\overline{c}_Z\) be the induced maps from \(\partial \overline{C}\) to \(\overline{C}\) and \(Z\) respectively.

Thus we have a diagram

\[(6.3.1) \quad \begin{array}{ccc}
C & \xrightarrow{j_C} & \overline{C} \\
\pi_U \downarrow & & \downarrow i_{\partial \overline{C}} \\
U & \xrightarrow{j} & X \times_k X \\
\pi_Z & & \downarrow \tau \\
& & Z
\end{array}\]

where both the squares are Cartesian (upto nilpotents) as a consequence of Lemma 4.1.4. On \( C \) we have an exact triangle

\[
\begin{array}{c}
j_C \bar{Q}_\ell \longrightarrow j_C^* \bar{Q}_\ell \longrightarrow i_{\partial C}^* i_{\partial C}^* j_C \bar{Q}_\ell \longrightarrow +1
\end{array}
\]

Pushing forward this triangle to \( X \times_k X \) via \( c_* = c_! \), we obtain an exact triangle

\[
\begin{array}{c}
c_! \bar{Q}_\ell \longrightarrow c_* \bar{Q}_\ell \longrightarrow i_* c_\partial Z \partial i_* \longrightarrow j_C \bar{Q}_\ell \longrightarrow +1
\end{array}
\]

on \( X \times_k X \).

Clearly \( \text{Hom}(i_* , j_* ) \simeq 0 \). Thus applying the cohomological functor \( \text{Hom} ( , j_* j^*(\mathbb{D}_X F \boxtimes F)) \) to the triangle (6.3.3), we get an isomorphism

\[
\text{Hom}(c_! Q_\ell , j_* j^*(\mathbb{D}_X F \boxtimes F)) \simeq \text{Hom}(c_* \bar{Q}_\ell , j_* j^*(\mathbb{D}_X F \boxtimes F)).
\]

The natural map \( F \boxtimes \mathbb{D}_X F \rightarrow j_* j^*(\mathbb{D}_X F \boxtimes F) \) combined with the isomorphisms (4.3.6) and (6.3.4) gives a natural map

\[
\text{Hom}(c_1^* F , c_2^! F) \rightarrow \text{Hom}(c_* \bar{Q}_\ell , j_* j^*(\mathbb{D}_X F \boxtimes F)).
\]

By duality and properness of \( X \times_k X \), we have isomorphisms

\[
\text{Hom}(j_* j^*(\mathbb{D}_X F \boxtimes F), K_{X \times_k X}) \simeq \text{Hom}(\bar{Q}_\ell , j_* j^*(\mathbb{D}_X F \boxtimes F)) \simeq H^0_c(U , j^*(F \boxtimes \mathbb{D}_X F)).
\]

Also, there exists a pairing

\[
\text{Hom}(c_* \bar{Q}_\ell , K_{X \times_k X}) \rightarrow \text{Hom}(\bar{Q}_\ell , K_{X \times_k X}) \simeq H^0(X \times_k X , K_{X \times_k X}).
\]

Thus combining (6.3.5), (6.3.6), (6.3.8) and (6.3.7) we get a natural pairing

\[
\Phi: \text{Hom}(c_1^* F , c_2^! F) \otimes_{\bar{Q}_{\ell}} H^0_c(U , j^*(F \boxtimes \mathbb{D}_X F)) \rightarrow H^0(X \times_k X , K_{X \times_k X}).
\]

In particular if we fix a cohomological correspondence \( u \) of \( F \) lifting \( c \), we get a linear functional on \( H^0_c(U , j^*(F \boxtimes \mathbb{D}_X F)) \) given by

\[
\Phi_u(\beta) := \text{Tr}_{X \times_k X}(\Phi(u \otimes \beta)),
\]

where \( \text{Tr}_{X \times_k X} \) is the natural trace map on \( H^0(X \times_k X , K_{X \times_k X}) \) (recall that \( X \) is proper over \( k \)).
6.4. **Trace along the graphs of Frobenius.** Let $k$ be an algebraic closure of $\mathbb{F}_q$. Let $c : C \to X \times_k X$ be a correspondence defined over $\mathbb{F}_q$. Let $\mathcal{F}$ be a Weil sheaf on $X$. Let $u$ be a cohomological correspondence of $\mathcal{F}$ lifting $c$. For each $n$ we have a Cartesian diagram

\[
\begin{array}{ccc}
\text{Fix}(c^{(n)}) & \xrightarrow{c^{(n)\prime}} & X \\
\Delta^{(n)\prime} & \nearrow & \downarrow \Delta \\
C & \xrightarrow{c} & X \times_k X \times_k X
\end{array}
\]

We briefly recall the map (see 4.3.20)

\[
\mathcal{T}r_{c}^{(n)}(u) : \text{Hom}(c_2c^\ast \mathcal{F}, \mathcal{F}) \to H^0(\text{Fix}(c^{(n)}), K_{\text{Fix}(c^{(n)})})
\]
constructed in the Section 4.3 here. Let $u$ be a cohomological correspondence of $\mathcal{F}$ lifting $c$. Thus $u$ corresponds to a map

\[
c_1 \overline{\mathbb{Q}_\ell} \to (\mathbb{D}_X \mathcal{F} \boxtimes \mathcal{F}),
\]
or equivalently to

\[
\overline{\mathbb{Q}_\ell} \to c_1^\prime (\mathbb{D}_X \mathcal{F} \boxtimes \mathcal{F}),
\]

Composing (6.4.3) and (4.3.19) we get a map

\[
c_1 \overline{\mathbb{Q}_\ell} \to \Delta_\ast^{(n)} K_X.
\]

Applying $c_1^\prime$ to (4.3.17) and composing with (6.4.4) we get a map (denoted by $u^{(n)}$)

\[
u^{(n)} : \overline{\mathbb{Q}_\ell} \to c_1^{(n)\prime} (\mathbb{D}_X \mathcal{F} \boxtimes \mathcal{F}),
\]

Moreover, we have natural isomorphisms

\[
\text{Hom}(c_1 \overline{\mathbb{Q}_\ell}, \Delta_\ast^{(n)} K_X) \simeq \text{Hom}(\overline{\mathbb{Q}_\ell}, c_1^\prime \Delta_\ast^{(n)} K_X),
\]
and

\[
\text{Hom}(\overline{\mathbb{Q}_\ell}, c_1^\prime \Delta_\ast^{(n)} K_X) \simeq \text{Hom}(\overline{\mathbb{Q}_\ell}, \Delta_\ast^{(n)\prime} K_{\text{Fix}(c^{(n)})}) \simeq H^0(\text{Fix}(c^{(n)}), K_{\text{Fix}(c^{(n)})}).
\]

Here (6.4.7) comes from adjunction, and (6.4.8) from base change along the left inner square of (6.4.1). Combining (6.4.5), (6.4.7) and (6.4.8) we get an element in $H^0(\text{Fix}(c^{(n)}), K_{\text{Fix}(c^{(n)})})$, which we call $\mathcal{T}r_{c}^{(n)}(u)$.

The maps $\mathcal{T}r_{c}^{(n)}$ are similar to $\mathcal{T}r_c$ in (4.3.9), but adapted to the graphs of Frobenius. In fact these two trace maps are compatible in an obvious way.
Lemma 6.4.1. $\mathcal{T} r_c(u^{(n)}) = \mathcal{T} r'_{c}(u)$.

Proof. For ease of writing we set $\mathcal{G} = \mathbb{D}_{X} \mathcal{F} \mathcal{F}$. Let us recall Varshavsky's recipe (see Section 4.3) to compute $\mathcal{T} r_c(u^{(n)})$. First we apply $c^{(n)}$ to (4.3.8) to get a map

\begin{equation}
6.4.9 \quad c^{(n)}(|\mathcal{F}| : c^{(n)}(\mathcal{G}) \rightarrow c^{(n)} \Delta_{*} K_{X}.
\end{equation}

Composing (6.4.9) with the map $u^{(n)}$ in (4.4.6), we get a map

\begin{equation}
6.4.10 \quad c^{(n)}(|\mathcal{F}| : c^{(n)}(\mathcal{G}) \rightarrow c^{(n)} \Delta_{*} K_{X}.
\end{equation}

Base change applied to the right hand square of (6.4.1) gives us a natural isomorphism

\begin{equation}
6.4.11 \quad \text{Hom}(\overline{\mathbb{Q}}_{\ell}, c^{(n)} \Delta_{*} K_{X}) \simeq \text{Hom}(\overline{\mathbb{Q}}_{\ell}, c^{(n)} \Delta_{*} K_{X}).
\end{equation}

Finally note that (6.4.8) gives an isomorphism of $\text{Hom}(\overline{\mathbb{Q}}_{\ell}, c^{(n)} \Delta_{*} K_{X})$ with $H^{0}(\text{Fix}(c^{(n)}), K_{\text{Fix}(c^{(n)})})$. The element in $H^{0}(\text{Fix}(c^{(n)}), K_{\text{Fix}(c^{(n)})}$ corresponding to (6.4.10) via the isomorphisms (6.4.11) and (6.4.8) is by definition $\mathcal{T} r_c(u^{(n)})$. Thus $\mathcal{T} r_c(u^{(n)})$ apriori occurs as an element in $\text{Hom}(\overline{\mathbb{Q}}_{\ell}, c^{(n)} \Delta_{*} K_{X})$ (say $u_{\Delta^{(n)}}$), which is then considered as an element in $H^{0}(\text{Fix}(c^{(n)}), K_{\text{Fix}(c^{(n)})}$ via the isomorphisms (6.4.11) and (6.4.8).

On the other hand $\mathcal{T} r'_{c}(u)$ naturally exists as an element in $\text{Hom}(c^{(n)} \overline{\mathbb{Q}}_{\ell}, \Delta_{*} K_{X})$ (say $u_{\Delta^{(n)}}''$), which is then considered as an element in $H^{0}(\text{Fix}(c^{(n)}), K_{\text{Fix}(c^{(n)})}$ via the isomorphisms (6.4.7) and (6.4.8).

To compare these elements we consider the following commutative diagram

\begin{equation}
6.4.12 \quad u_{\Delta^{(n)}} \in \text{Hom}(\overline{\mathbb{Q}}_{\ell}, c^{(n)} \Delta_{*} K_{X}) \xrightarrow{(6.4.11)} \text{Hom}(\overline{\mathbb{Q}}_{\ell}, c^{(n)} \Delta_{*} K_{X}) \xrightarrow{\cong (6.4.7)} H^{0}(\text{Fix}(c^{(n)}), K_{\text{Fix}(c^{(n)})}).
\end{equation}

Our discussion above implies that the image of $u_{\Delta^{(n)}}$ along the top row is $\mathcal{T} r_c(u^{(n)})$, while the image of $u_{\Delta^{(n)}}''$ under the maps (6.4.7) and (6.4.8) in (6.4.12) is $\mathcal{T} r'_{c}(u^{(n)})$. Thus it suffices to show $\psi(u_{\Delta^{(n)}}) = \psi^{-1}(u_{\Delta^{(n)}}'')$ as elements in $\text{Hom}(c^{(n)} \overline{\mathbb{Q}}_{\ell}, (F_{X/\mathbb{F}_{q}} \times 1_{X})^{(n)} \Delta_{*} K_{X})$.

Consider the diagram
Here $\alpha$, $\beta$ and $\gamma$ are induced by adjunction between $(c_1, c')$, $(BC)$ denotes the map induced by base change along the right square in (6.4.1), and $F_{\text{End}(\mathcal{F})}^{-1}$ is inverse to the isomorphism (4.3.17). By functoriality of adjunction $c_1 c' \to 1$, the squares commute.

Note that by definition of $u^{(n)}$ (see (6.4.6)), the composition $c_1 c' F_{\text{End}(\mathcal{F})}^{-1} \circ c_1 u$ is $c_1 u^{(n)}$. Thus by definition of the adjunction map $\psi$ in (6.4.12),

$$
\psi(u^{(n)}) = \gamma \circ c_1 (c^{(n)} \circ \text{ev}_\mathcal{F}) \circ c_1 u^{(n)}.
$$

On the other hand by definition the composition $\alpha \circ c_1 u$ is the map in (6.4.3), and the composition $BC \circ (F^n_{X/\mathbb{F}_q} \times 1_X)\circ c_1 u$ is the map in (4.3.19). Hence $u^{(n)}_\alpha$ is the composition of these two maps. Thus we have $\psi_u^{(n)} = (F^n_{X/\mathbb{F}_q} \times 1_X) \circ c_1 u^{(n)}$, which by commutativity of (6.4.13) equals $\gamma \circ c_1 (c^{(n)} \circ \text{ev}_\mathcal{F}) \circ c_1 u^{(n)}$, and hence equals $\psi(u^{(n)})$ by (6.4.14).

---

6.5. Local terms using the pairing (6.3.9). In this section we shall use the pairing (6.3.9) to give a formula for computing the local terms of a cohomological correspondence lifting an essentially proper correspondence. Through out this section we fix an essentially proper correspondence (defined over $\mathbb{F}_q$), $c : C \to X \times_k X$ together with a diagram as in (6.1.1).

Now consider the following natural map

$$
H^0_{\Delta^{(n)}}(X \times_k X, \mathcal{F} \boxtimes \mathbb{D}_X \mathcal{F}) \simeq H^0_{\Delta^{(n)}}(X \times_k X, j^! j^*(\mathcal{F} \boxtimes \mathbb{D}_X \mathcal{F})) \to H^0_c(U, j^*(\mathcal{F} \boxtimes \mathbb{D}_X \mathcal{F})),
$$

here the first isomorphism follows from Lemma 6.1.6 and the isomorphism (4.5.4), and the second map is the forget supports map.

6.5.1. The cohomology classes. Let $[\Delta^{(n)}] \in H^0_{\Delta^{(n)}}(X \times_k X, \mathcal{F} \boxtimes \mathbb{D}_X \mathcal{F})$ be the cohomology class obtained by taking the Verdier dual of (4.3.19). By abuse of notation we shall also denote the image of $[\Delta^{(n)}]$ under the morphism (6.5.1) by $[\Delta^{(n)}]$. All these cohomology classes now live in the same space $H^0_c(U, j^*(\mathcal{F} \boxtimes \mathbb{D}_X \mathcal{F})).$

Let $u$ be a cohomological correspondence of $\mathcal{F}$ lifting $c$. Recall that $u$ induces a linear function $\Phi_u$ on $H^0_c(U, j^*(\mathcal{F} \boxtimes \mathbb{D}_X \mathcal{F}))$ (see (6.3.10)). Moreover by Lemma 6.1.6 since $\text{Fix}(c^{(n)})$ is proper we can make sense of the local terms $LT(u^{(n)})$ (see Definition 4.3.2).

**Proposition 6.5.1.** For any $n \geq 0$, $\Phi_u([\Delta^{(n)}]) = LT(u^{(n)})$. 


Remark 6.5.2. For $n \geq 1$, both $u^{(n)}$ and the cohomology classes $[\Delta^{(n)}]$ depend on the choice of the Weil structure on $F$.

Proof. Recall that (see (6.3.10) and (4.3.12)),

\begin{equation}
\Phi_u([\Delta^{(n)}]) = \Tr_{X \times_k X}(\Phi(u \otimes [\Delta^{(n)}])),
\end{equation}

and

\begin{equation}
\LT(u^{(n)}) = \Tr_{\Fix(c^{(n)})}(\TT_{c^{(n)}}(u^{(n)})).
\end{equation}

Moreover Lemma 6.4.1 implies that it suffices to show

\begin{equation}
\Tr_{X \times_k X}(\Phi(u \otimes [\Delta^{(n)}])) = \Tr_{\Fix(c^{(n)})}(\TT_{c^{(n)}}(u^{(n)})).
\end{equation}

The trace maps $\Tr_{X \times_k X}$ and $\Tr_{\Fix(c^{(n)})}$ by their definitions are compatible with the natural push forward map

\begin{equation}
\psi^{(n)} : H^0(\Fix(c^{(n)}), K_{\Fix(c^{(n)})}) \to H^0(X \times_k X, K_{X \times_k X}),
\end{equation}

Thus it suffices to show that the image of $\TT_{c^{(n)}}(u) \in H^0(\Fix(c^{(n)}), K_{\Fix(c^{(n)})})$ under (6.5.5) is $\Phi(u \otimes [\Delta^{(n)}])$.

Unwinding the definitions of $\Phi$ and $\TT_{c^{(n)}}$, one observes that to prove the Proposition it suffices to show that the following diagram is commutative

\begin{equation}
\begin{array}{ccc}
\Hom(c_\ast \Q_{\ell}, K_{X \times_k X}) & \leftarrow & \Hom(c_\ast \Q_{\ell}, \Delta^{(n)}_\ast K_X) \\
\downarrow \cong & & \downarrow \cong \\
\Hom(\Q_{\ell}, K_{X \times_k X}) & \cong & \Hom(c_\ast \Q_{\ell}, \Delta^{(n)}_\ast K_X) \\
\downarrow \cong & & \downarrow \cong \\
H^0(X \times_k X, K_{X \times_k X}) & \cong & \Hom(\Q_{\ell}, c_\ast \Delta^{(n)}_\ast K_X) \\
\downarrow \cong & & \downarrow \cong \\
H^0(\Fix(c^{(n)}), K_{\Fix(c^{(n)})}) & \leftarrow & \Hom(\Q_{\ell}, \Delta^{(n)}_\ast K_{\Fix(c^{(n)})})
\end{array}
\end{equation}

Here the maps (2) and (3) are induced by natural maps $\Delta^{(n)}_\ast K_X \to K_{X \times_k X}$ and $\Q_{\ell} \to c_\ast \Q_{\ell}$ respectively. The map in (1) is induced by the natural map $c_\ast \Q_{\ell} \to c_\ast \Q_{\ell}$. (1) is an isomorphism follows from applying the cohomological functor $\text{Hom}(\ast, \Delta^{(n)}_\ast K_X)$ to the triangle (6.3.3) and noting that $\text{Hom}(i_\ast, \Delta^{(n)}_\ast)$ vanishes, thanks to Lemma 6.1.6.

Our strategy is simple. We begin with an element $\gamma \in H^0(\Fix(c^{(n)}), K_{\Fix(c^{(n)})})$, then using the isomorphisms above we construct a map $\gamma : c_\ast \Q_{\ell} \to \Delta^{(n)}_\ast K_X$, and its unique lift $\gamma_\ast : c_\ast \Q_{\ell} \to \Delta^{(n)}_\ast K_X$. Finally we show that $\psi^{(n)}(\gamma) \in H^0(X \times_k X, K_{X \times_k X})$ corresponds to the composition
\[(6.5.7) \quad \bar{Q}_\ell \xrightarrow{\gamma} \Delta^{(n)} K_X \xrightarrow{} K_{X \times X}, \]

and thus establishing the commutativity of (6.5.6).

We have a Cartesian diagram

\[(6.5.8) \quad \text{Fix}(c^{(n)}) \xrightarrow{\tilde{c}^{(n)'}} X \xrightarrow{} X \times X. \]

Since \text{Fix}(c^{(n)}) is proper over \(k\) (Lemma 6.1.3), the natural maps

\[(6.5.9) \quad c_! \Delta^{(n)'} \rightarrow c_* \Delta^{(n)'} \]

and

\[(6.5.10) \quad \Delta^*_s \tilde{c}^{(n)'} \rightarrow \Delta^*_s \tilde{c}^{(n)'} \]

are isomorphisms. Moreover commutativity of the diagram (6.5.8) implies that the maps

\[(6.5.11) \quad c_! \Delta^{(n)'} \rightarrow \Delta^*_s \tilde{c}^{(n)'} \]

and

\[(6.5.12) \quad c_* \Delta^{(n)'} \rightarrow \Delta^*_s \tilde{c}^{(n)'} \]

are isomorphisms.

We have a natural adjunction map

\[(6.5.13) \quad \alpha_{n} : \tilde{c}^{(n)} \text{Fix}(c^{(n)}) \rightarrow K_X. \]

Now suppose we have a global section of \(K_{\text{Fix}(c^{(n)})}\) (say \(\gamma\)). Note that the isomorphisms on the right hand side of (6.5.6) imply that the element in \(\text{Hom}(c_! \bar{Q}_\ell, \Delta^{(n)} K_X)\) (say \(\gamma_!\)) corresponding to \(\gamma\) is obtained by composing

\[(6.5.14) \quad \gamma_! : c_! \bar{Q}_\ell \xrightarrow{c \gamma} c_! \Delta^{(n)'} K_{\text{Fix}(c^{(n)})} \xrightarrow{\Delta^*_s \tilde{c}^{(n)'}} K_{\text{Fix}(c^{(n)})} \xrightarrow{\Delta^*_s \alpha_{n}} \Delta^{(n)} K_X. \]

But the isomorphism (1) in Diagram 6.5.6 (yet to be shown commutative) implies any such \(\gamma_!\) has an unique lift (say \(\gamma_*\)) into an element in \(\text{Hom}(c_* \bar{Q}_\ell, \Delta^{(n)} K_X)\). The commutative diagram (6.5.15) allows us to construct a (and hence the only) lift of \(\gamma_!\). Indeed given any such \(\gamma\) we have a diagram

38
The left hand square in (6.5.15) is commutative by functoriality of the map \( c_1 \to c_\ast \), and that the right-hand square is commutative follows from the commutativity of the Diagram (6.5.8).

The composite arrow from \( c_\ast Q_\ell \) to \( \Delta_\ast^{(n)} K_X \) in (6.5.15) is the unique lift \( \gamma_\ast \) of \( \gamma_! \). Since applying global sections functor to the map \( c_\ast \gamma \) in (6.5.15) gives the element \( \gamma \) in \( H^0(\text{Fix}(c^{(n)}, K_{\text{Fix}(c^{(n)})})) \), the element \( \psi^{(n)}(\gamma) \) in \( H^0(X \times_k X, K_{X \times_k X}) \) corresponds to the composition

(6.5.16) \[
\overline{Q}_\ell \longrightarrow c_\ast \overline{Q}_\ell \xrightarrow{\gamma_\ast} \Delta_\ast^{(n)} K_X
\]
as desired. \( \square \)

6.6. Action of partial Frobenius on the pairing. Recall that we were interested in understanding the local terms \( \text{LT}(u^{(n)}) \) of a cohomological correspondence of a mixed sheaf, lifting a correspondence which is essentially proper over \( \mathbb{F}_q \).

So far from the geometric side, we have only used the properness of \( X \) (and \( \text{Fix}(c^{(n)}) \)), and the fact that there exists an open \( U \hookrightarrow X \times_k X \) containing all the graphs of Frobenius. Also we have only required a Weil structure on \( F \) so far. To use Proposition 6.5.1 to bound the local terms we will need further information on the geometry of \( U \), and the Weil structure on \( F \).

Now we use the fact that \( U \) is stable under the partial Frobenius \( F_1 := F_{X/\mathbb{F}_q} \times 1_X \), that is \( F_1^{-1}(U) = U \). As before \( F_1^n \) is to be understood as the identity morphism of \( X \times_k X \).

More generally let \( \mathcal{P} \) be the set whose elements are open subsets of \( X \times_k X \) of the form \( \cup_i U_i \times_k U_i \), with \( U_i \) open in \( X \) and defined over \( \mathbb{F}_q \). We do not require that \( \{U_i\} \)'s cover \( X \). Clearly \( \mathcal{P} \) is stable under finite unions and intersections.

Let \( V \) be an arbitrary element in \( \mathcal{P} \). Then \( V \) is also stable under the partial Frobenius, and for any integer \( n \geq 0 \) we have a Cartesian diagram

(6.6.1) \[
\begin{array}{ccc}
V & \xrightarrow{j_V^n} & X \times_k X \\
F_1^n \downarrow & & \downarrow F_1^n \\
V & \xrightarrow{j_V} & X \times_k X
\end{array}
\]

Here \( j_V \) is the open immersion of \( V \) into \( X \times_k X \).

As before let \( F_\mathcal{F} \) be the structure of a Weil sheaf on \( \mathcal{F} \) via (4.3.14). The diagram (6.6.1) and functoriality of the adjunction \( j_V \downarrow j_V^* \to 1 \), implies that there exists a commutative diagram,
where the isomorphisms (1) and (2) are induced by the Weil structure of $\mathcal{F}$, and (BC) arises from base change.

Since $F^n_l$ is finite (and hence proper) taking global sections along the top row of (6.6.2) we have an induced action of $F^n_l$ on compactly supported cohomology

\[(6.6.3)\]

$$F^n_l^*: H^i_c(V, j^*_V(\mathcal{F} \boxtimes \mathbb{D}_X \mathcal{F})) \rightarrow H^i_c(V, j^*_V(\mathcal{F} \boxtimes \mathbb{D}_X \mathcal{F})).$$

In particular since $U \in \mathcal{P}$, for any integer $n \geq 0$ we have an action of $F^n_l^*$ on $H^0(U, j^*(\mathcal{F} \boxtimes \mathbb{D}_X \mathcal{F}))$.

For any sheaf $\mathcal{H}$ on $X \times_k X$, and any closed subset $Y \hookrightarrow X \times_k X$ there is a pullback map

\[(6.6.4)\]

$$H^0_Y(X \times_k X, \mathcal{H}) \rightarrow H^0_{(F^n_l)^{-1}(Y)}(X \times_k X, F^n_l^* \mathcal{H}),$$

which fits into a commutative diagram

\[(6.6.5)\]

$$\begin{array}{ccc}
H^0_Y(X \times_k X, \mathcal{H}) & \xrightarrow{(6.6.4)} & H^0_{(F^n_l)^{-1}(Y)}(X \times_k X, F^n_l^* \mathcal{H}) \\
\downarrow & & \downarrow \\
H^0(X \times_k X, \mathcal{H}) & \rightarrow & H^0(X \times_k X, F^n_l^* \mathcal{H})
\end{array}$$

The vertical arrows in (6.6.5) are induced by the forget supports map, and the arrow in the bottom row is induced by pullback.

Recall that given a structure of a Weil sheaf on $\mathcal{F}$ we had defined cohomology classes $[\Delta^{(n)}]$ in $H^0_c(U, j^*(\mathcal{F} \boxtimes \mathbb{D}_X \mathcal{F}))$ (see (6.5.1)).

**Lemma 6.6.1.** For any $n \geq 0$, $(F^n_l)^*([\Delta]) = [\Delta^{(n)}]$.

**Proof.** First note that as closed subschemes of $X \times_k X$, $(F^n_l)^{-1}(\Delta) = \Delta^{(n)}$. Since $F^n_l^* = (F^n_l)^*$ as endomorphisms of $H^0_c(U, j^*(\mathcal{F} \boxtimes \mathbb{D}_X \mathcal{F}))$, it suffices to prove $F^n_l^*([\Delta]) = [\Delta^{(n)}]$. For ease of writing let $\mathcal{G} := \mathcal{F} \boxtimes \mathbb{D}_X \mathcal{F}$. Consider the diagram of cohomology groups
The isomorphisms along the vertical arrows of (6.6.6) are induced by the Weil structure (see (6.6.2)). The arrow from $H^0_c(U, j^*G)$ to $H^0_c(U, F^{i*}j^*G)$ is induced by pullback, and the arrow from $H^0_{\Delta(n)}(X \times_k X, F^i_{l*}j^*G)$ to $H^0_{\Delta(n)}(X \times_k X, F^i_{l*}j^*G)$ is the composition of base change and forget supports map.

That the square on the upper-left corner of Diagram 6.6.6 commutes is a consequence of the outer commutative square in (6.6.2). The square in the upper-right corner of (6.6.6) commutes by functoriality of the forget supports map applied to $F^i_{l*}j^*G \to j^*G$. The square in the bottom-left corner commutes by the definition of pullback map (6.6.4). The square in the bottom-right corner commutes as a consequence of the commutative diagram (6.6.5). Thus the Diagram 6.6.6 is commutative.

Recall that the cohomology classes $[\Delta^{(n)}]$ begin their life in $H^0_{\Delta(n)}(X \times_k X, G)$. Thus to prove the Lemma it suffices to show that image of the class $[\Delta] \in H^0_{\Delta}(X \times_k X, G)$ along the left column is $[\Delta^{(n)}] \in H^0_{\Delta(n)}(X \times_k X, G)$. This is immediate from the definition of $[\Delta^{(n)}]$ (see (6.5.1)).

Now suppose $F$ arises from a mixed sheaf $F_0$ on $X_0$ (the chosen model of $X$) of weight less than or equal to $w$. Further assume that $F_0$ is in $pD^{\leq a}(X_0, \underline{\mathbb{Q}_l})$. Thus $F$ comes equipped with a canonical structure of a Weil sheaf.

**Lemma 6.6.2.** For any $V \in \mathcal{P}$ and any integer $n$, the eigenvalues of $F_i^*$ acting on $H^i_c(V, j^*_V(\mathcal{F} \boxtimes \mathcal{D}_X \mathcal{F}))$ are Weil numbers of weight less than or equal to $w + a + \dim(X)$.

**Proof.** If $V = V' \cup V''$, where $V'$ and $V''$ are elements in $\mathcal{P}$, the Mayer-Vietoris sequence implies that the Lemma is true for $V$, if it is true for $V'$, $V''$ and $V' \cap V''$.

Now suppose $V = (U_1 \times U_1) \cup (\cup_{i=2}^r(U_i \times_k U_i))$ is an union of $r \geq 2$ open sets of the form $U_i \times_k U_i$. Note that the intersection $(U_1 \times U_1) \cap (\cup_{i=2}^r(U_i \times_k U_i)) = (\cup_{i=2}^r(U_i \times_k U_i))$, is an union of at most $r - 1$ open sets of the form $U_i \times_k U_i$. Thus Mayer-Vietoris allows us to reduce to the case $r = 1$.

Hence suppose $V = U \times_k U$ for an open subset $U$ of $X$ defined over $\mathbb{F}_q$. Then Künneth formula and Poincare duality imply that for any integer $n$

$$H^i_c(U \times_k U, j^*_V(\mathcal{F} \boxtimes \mathcal{D}_X \mathcal{F})) \simeq \bigoplus_i (H^i_c(U, \mathcal{F}|_U) \otimes_{\mathbb{Q}_l} (H^{i-n}(U, \mathcal{F}|_U))^\vee),$$

and that the action of $F_i^*$ on $H^i_c(U \times_k U, j^*_V(\mathcal{F} \boxtimes \mathcal{D}_X \mathcal{F}))$ under the isomorphism (6.6.7) corresponds to $F_i^*(U \times_k U, j^*_V(\mathcal{F} \boxtimes \mathcal{D}_X \mathcal{F}))$ on each factor $H^i_c(U, \mathcal{F}|_U) \otimes_{\mathbb{Q}_l} (H^{i-n}(U, \mathcal{F}|_U))^\vee$. Here $F_i^*(U \times_k U)$ is the Frobenius pullback on $H^i_c(U, \mathcal{F}|_U)$.
Since $F$ comes from a mixed sheaf $F_0$, the eigenvalues of Frobenius on $H^i_c(U, F|_U)$ are Weil numbers ([Del80], Théorème 1.1). Moreover since $F_0$ is of weight less than or equal to $w$, the Frobenius weights on $H^i_c(U, F|_U)$ are bounded above by $w + i$ ([BBDG18], 5.1.14 (i)). Also $F|_U$ continues to be in $\mathcal{F}D_{\leq a}$ ([BBDG18], Proposition 1.4.16 (i)). Thus $H^i_c(U, F|_U)$ vanishes for $i > \dim(X) + a$ ([BBDG18], Section 4.2.4).

Hence (6.6.7) implies that the eigenvalues of $F_i^*$ acting on any $H^n_c(V, j^*_V(F \boxtimes \mathbb{D}_X F))$ are Weil numbers of weight less than or equal to $w + a + \dim(X)$.

We will need the following elementary linear algebra lemma, whose proof is carried out for the sake of completion.

**Lemma 6.6.3.** Let $V$ be a finite dimensional complex vector space. Let $T : V \to V$ be a linear map, and $w \in \mathbb{R}_{\geq 1}$, a positive real number such that all the eigenvalues $\alpha$ of $T$ satisfy $|\alpha| \leq w$.

Let $\Phi$ be a linear functional on $V$. Then for any $v \in V$

1. there exists a positive real number $M_v$ such that, for any $n \geq 0$

$$|\Phi(T^n v)| \leq M_v n^{\dim(V)} w^n. \tag{6.6.8}$$

2. The power series $\sum_{n \geq 1} \Phi(T^n v) t^n \in \mathbb{C}[t] \subseteq \mathbb{C}((t))$, is a rational function of $t$, that is it belongs to $\mathbb{C}(t) \subseteq \mathbb{C}((t))$.

**Proof.** If $T$ is semi-simple, choosing a basis in which $T$ is diagonal we have

$$|\Phi(T^n v)| \leq M'_v w^n, \tag{6.6.9}$$

for some constant $M'_v$ possibly depending on $v$. Also (2) is immediate, by choosing a basis in which $T$ is diagonal.

In general $T$ can be written as a sum of two commuting operators $T^{ss}$ and $T^{nil}$,

$$T = T^{ss} + T^{nil}, \tag{6.6.10}$$

with $T^{ss}$ semi-simple and $T^{nil}$ nilpotent. Thus for any positive integer $n$

$$T^n = \sum_{i=0}^{\min\{n, \dim(V)\}} \binom{n}{i} (T^{ss})^{n-i} (T^{nil})^i. \tag{6.6.11}$$

Since the set of eigenvalues of $T$ and $T^{ss}$ are the same, combining (6.6.9) and (6.6.11) we get the necessary bound as shown below

$$|\Phi(T^n v)| \leq \sum_{i=0}^{\min\{n, \dim(V)\}} \binom{n}{i} M'_{(T^{nil})^i v} w^{n-i} \leq M_v n^{\dim(V)} w^n, \tag{6.6.12}$$

with $M_v := (\dim(V) + 1) \max_{0 \leq i \leq \dim(V)} M'_{(T^{nil})^i v}$.

(2) is also immediate in the non-semi-simple case as a consequence of the equality (6.6.11), and the result in the semi-simple case. \qed
6.7. **Proof of Theorem 6.1.7.** Combining Proposition 6.5.1, Lemma 6.6.1, Lemma 6.6.2 and Lemma 6.6.3, (1) we have for all \( n \geq 0 \),

\[
|\text{LT}(u^{(n)})| \leq M(\tau)n^rq^{n\left(\frac{\ell w + a + \dim(X)}{2}\right)},
\]

for some positive real number \( M(\tau) \). Here \( r = \dim_{\mathbb{Q}_l}(H^0_c(U, j^*(\mathcal{F} \boxtimes \mathbb{D}_X \mathcal{F}))) \).

Given any \( \epsilon > 0 \), let \( N(\epsilon) \) be the smallest positive integer \( n \) such that \( q^{n\epsilon} \geq n^r \). Then (6.7.1) implies that for any \( n \geq N(\epsilon) \),

\[
|\text{LT}(u^{(n)})| \leq M(\tau)q^{n\left(\frac{\ell w + a + \dim(X)}{2}\right)+\epsilon},
\]

as desired.

**Remark 6.7.1.** The proof of Lemma 6.6.3 allows us to explicitly determine \( M(\tau) \) in (6.7.2). Indeed if one knew that the action of the partial Frobenius on \( H^0_c(U, j^*(\mathcal{F} \boxtimes \mathbb{D}_X \mathcal{F})) \) was semi-simple then one could have used (6.6.9) instead to obtain a sharper bound as compared to (6.7.2). Moreover as observed there, the constant \( M(\tau) \) would then only depend on the class \([\Delta]\).

6.8. **Proof of Theorem 1.0.3, (2).** We proceed as in the proof of Theorem 1.0.3, (1) and (3) in Section 6.2 and conclude that (using the same notations)

\[
\#\text{Fix}(c^{(n)}(k)) = \text{LT}(u^{(n)}) - \text{LT}(u^{(n)}|_Z).
\]

The result now follows from combining the Lefschetz-Verdier trace formula, Proposition 6.5.1, Lemma 6.6.3, (2), and a standard Hankel determinant argument (see for example [Del74], Lemme 1.7).

**APPENDIX A. A COHOMOLOGICAL CORRESPONDENCE OF THE INTERSECTION COMPLEX**

In this section, we will use geometric semi-simplicity of pure perverse sheaves to construct a cohomological correspondence. A similar result was earlier obtained in an unpublished note of Hanamura-Saito [HS06, Theorem 2]. For the sake of completeness we present an alternative approach here. Throughout this section let \( k_0 \) be an arbitrary finite field. Let \( k \) be an algebraic closure of \( k_0 \).

A.1. **Basic properties of IC\(_X\).** In this section we summarize a few basic properties of IC\(_X\) which will be used later. Recall for any variety \( X/k \) of dimension \( d \) there is a natural map

\[
\mathbb{Q}_l[d] \to \text{IC}_X,
\]

which is an isomorphism on the regular locus of \( X \). An analogous map exists for varieties over \( k_0 \) also.

We can complete (A.1.1) into a triangle

\[
\mathbb{Q}_l[d] \to \text{IC}_X \to \mathcal{F} \to \mathcal{F}^+.
\]
Clearly $\mathcal{F}$ is supported on the singular locus of $X$. Since the singular locus is of dimension at most $d - 1$, the restriction of $\mathcal{O}_d$ to the singular locus is in $\mathcal{O}_d^{\leq -1}$ ([BBDG18], (4.0.1)). Further $\mathrm{IC}_X$ by construction, when restricted to the singular locus belongs to $\mathcal{O}_d^{\leq -1}$. Thus $\mathcal{F}$ is in $\mathcal{O}_d^{\leq -1}$, and has dimension of support at most $d - 1$. Hence $H^i_c(X, \mathcal{F})$ vanishes for $i \geq d - 1$ ([BBDG18], Section 4.2.4). Thus the natural map

$$(A.1.3) \quad H_c^{2d}(X, \mathcal{O}_d) \to H_c^d(X, \mathrm{IC}_X),$$

is an isomorphism.

Dualizing ($A.1.1$) gives a natural map

$$(A.1.4) \quad \mathrm{IC}_X \to K_X[-d],$$

which is also an isomorphism on the regular locus. Dualizing ($A.1.3$) implies that the natural map

$$(A.1.5) \quad H^{-d}(X, \mathrm{IC}_X) \to H^{-2d}(X, K_X),$$

induced by ($A.1.4$) is an isomorphism.

For any non-empty open subset $j : U \hookrightarrow X$, functoriality of the adjunction $j_*j^* \to 1$ applied to ($A.1.1$) and ($A.1.4$) gives a commutative diagram

$$(A.1.6) \quad \begin{array}{ccc}
H^0_c(U, K_U) & \xrightarrow{\cong} & H^0_c(X, K_X) \\
\downarrow & & \downarrow \cong \\
H^d_c(U, \mathrm{IC}_U) & \xrightarrow{\cong} & H^d_c(X, \mathrm{IC}_X) \\
\cong (A.1.3) & & \cong (A.1.3) \\
H^2d_c(U, \overline{Q}_\ell) & \xrightarrow{\cong} & H^2d_c(X, \overline{Q}_\ell) \\
\end{array}$$

Here Tr$_X$ is the natural trace map on $H^0_c(X, K_X)$. Since $X$ is assumed to be a variety and $U$ is a nonempty open subset of $X$, the top and bottom rows of ($A.1.6$) are isomorphisms.

If $U$ is contained in the regular locus of $X$ then the arrows in the left column of ($A.1.6$) are trivially isomorphisms, and thus the natural map

$$(A.1.7) \quad H^d_c(X, \mathrm{IC}_X) \to H^0_c(X, K_X),$$

is also an isomorphism.

By abuse of notation, we will also call the composite map

$$(A.1.8) \quad H^2d_c(X, \overline{Q}_\ell) \to \overline{Q}_\ell$$

in ($A.1.6$) as Tr$_X$. Note that this coincides with the usual trace map when $X$ is regular.

We can also dualize the diagram ($A.1.6$)
Also \((A.1.7)\) dualizes to give a natural isomorphism

\[(A.1.10)\]

\[H^0(X, \QL) \simeq H^{-d}(X, \IC_X).\]

A.2. Decomposition in the derived category and a Lemma. Let \(f_0 : Y_0 \to X_0\) be a projective morphism of schemes of finite type over \(k_0\). Let \(\eta \in H^2(Y_0, \QL(1))\) be the Chern class of a relatively ample line bundle on \(Y_0\). Then \(\eta\) defines a map (in \(D^b_c(Y_0, \QL)\))

\[(A.2.1)\]

\[\eta : \QL \to \QL(1)[2].\]

Tensoring \((A.2.1)\) with \(\IC_{Y_0}\), we get a map

\[(A.2.2)\]

\[\eta_{\IC_{Y_0}} : \IC_{Y_0} \to \IC_{Y_0}(1)[2].\]

Thus for any integer \(i\), \((A.2.2)\) induces maps on the perverse cohomologies

\[(A.2.3)\]

\[\eta^i : pH^{-i}(f_0^*\IC_{Y_0}) \to pH^i(f_0^*\IC_{Y_0})(i).\]

Purity of \(\IC_{Y_0}\) ([BBDG18], Corollaire 5.3.2) and the relative hard Lefschetz ([BBDG18], Théorème 5.4.10) imply that the maps in \((A.2.3)\) are isomorphisms. Using these isomorphisms Deligne obtained a canonical self dual isomorphism over \(k_0\) ([Del94], Section 3)\(^3\)

\[(A.2.5)\]

\[f_0^*\IC_{Y_0} \simeq \oplus_i pH^i(f_0^*\IC_{Y_0})[-i].\]

A.2.1. A Lemma. Let \(j_0 : U_0 \hookrightarrow X_0\) be an open immersion. Let \(K_0\) be a perverse sheaf on \(X_0\). The adjoint triple \((j_!, j^*, j_*)\) gives rise to a commutative diagram of perverse sheaves on \(X_0\).

\[f_*\IC_Y \simeq \oplus_i pH^i(f_*\IC_Y)[-i].\]

It is unclear how to descend this isomorphism to a finite subfield of \(k\). I thank the referee for pointing this out.

\[^3\text{If one assume } f_0 \text{ is only proper, purity of } \IC_{Y_0} \text{ [BBDG18, Théorème 5.3.8] together with [Del80, Proposition 6.2.6] implies that } f_0^*\IC_{Y_0} \text{ is pure and hence by [BBDG18, Théorème 5.4.5] there exists a non-canonical isomorphism over } k.\]
Lemma A.2.1. If in addition, $K_0$ is pure then, $\tilde{\psi}_0$ is an isomorphism.

Proof. To check that $\tilde{\psi}_0$ is an isomorphism we can work geometrically. Thus we have a diagram analogous to Diagram A.2.6 but over $k$

\[
\begin{array}{c}
\text{image}(\phi) \xrightarrow{\tilde{\psi}} j_0^*j_0^*K_0 := \text{image}(\psi_0 \circ \phi) .
\end{array}
\]

and we need to show that $\tilde{\psi}$ is an isomorphism.

Since $K_0$ is assumed to be pure, it is geometrically semi-simple ([BBDG18, Théorème 5.3.8]). Thus we can assume that $K$ is a simple perverse sheaf.

If $j^*K$ is 0, then so are $\text{image}(\phi)$ and $j_0^*j^*K$, and $\tilde{\psi}$ is trivially an isomorphism. Else $\text{image}(\phi)$ is necessarily nonzero (since its restriction to $U$ is nonzero), and by the simplicity of $K$ is necessarily equal to $K$. Thus $\text{image}(\phi)$ is also simple, and hence the surjection $\tilde{\psi}$ is necessarily an isomorphism.

□

Lemma A.2.1 implies that when $K_0$ is a pure perverse sheaf on $X_0$, there is a natural injection of perverse sheaves

\[
(A.2.8) \quad j_0^*j_0^*K_0 \rightarrow K_0,
\]

whose restriction to $U_0$ is the natural isomorphism

\[
(A.2.9) \quad j_0^*j_0^*K_0 \cong j_0^*K_0.
\]

A.3. A pullback map on $\text{IC}_X$. Let $f: Y \to X$ be a dominant and projective morphism of varieties of the same dimension $d$ over $k$. Let $f_0: Y_0 \to X_0$ be a choice of model (of $f: Y \to X$) over a finite sub field $k_0$ of $k$.

Let $j_0: U_0 \hookrightarrow X_0$ be a non-empty regular open subset of $X_0$. We have a Cartesian diagram

\[
(A.3.1) \quad f_0^{-1}(U_0) \xrightarrow{j_0} Y_0 . \]

By proper base change
\( (A.3.2) \)
\[
j_0^* f_{0*} IC_{Y_0} \approx f_0'^* IC_{f_0^{-1}(U_0)}.
\]

On \( f_0^{-1}(U_0) \) (A.1.1) implies that there exists a natural map 
\( (A.3.3) \)
\[
\mathbb{Q}_\ell[d] \to IC_{f_0^{-1}(U_0)}.
\]

Since \( U_0 \) is regular, we have morphisms
\( (A.3.4) \)
\[
j_0^* IC_{X_0} \xrightarrow{\cong} \mathbb{Q}_\ell[d] \xrightarrow{(A)} f_{0*}(\mathbb{Q}_\ell[d]) \xrightarrow{(A.3.3)} f_{0*} IC_{f_0^{-1}(U_0)} \xrightarrow{\cong (BC)} j_0^* f_{0*} IC_{Y_0}.
\]

Here the morphism \( (A) \) is induced by adjunction. Denote by \( f_{0*} \) the composite map
\( (A.3.5) \)
\[
f_{0*} : j_0^* IC_{X_0} \to j_0^* f_{0*} IC_{Y_0},
\]
in (A.3.4), and by \( f' \) the corresponding map
\( (A.3.6) \)
\[
f' : j^* IC_X \to j^* f_* IC_Y,
\]
obtained by base change of (A.3.5) to \( k \). By construction of (A.3.6), we have a commutative diagram of sheaves on \( U \)
\( (A.3.7) \)
\[
\begin{array}{ccc}
\mathbb{Q}_\ell[d] & \xrightarrow{\cong} & j^* IC_X \\
\xrightarrow{(A)} & & \xrightarrow{(A.3.6)} \\
f'_* \mathbb{Q}_\ell[d] & \xrightarrow{\cong (BC)} & j'_* f_* IC_Y \\
\xrightarrow{(A.1.1)} & & \xrightarrow{(BC)} \\
f'_* IC_{f^{-1}(U)} & &
\end{array}
\]
where \( (A) \) is induced by adjunction.

**Remark A.3.1.** The \( f_{0*} \) in (A.3.5) is not to be confused with the functor \( f_0' \). Whenever either makes an appearance it will be clear from the context which one we mean.

Having made these choices there is a canonical map as shown in the following lemma.

**Proposition A.3.2.** There is a map \( f_* : IC_X \to f_* IC_Y \) (defined over \( k_0 \)) such that when restricted to \( U \) it is the map in (A.3.6).

**Proof.** We begin by choosing a decomposition
\( (A.3.8) \)
\[
\phi_0 : f_{0*} IC_{Y_0} \cong \oplus_i p^H_i(f_{0*} IC_{Y_0})[-i].
\]

Let \( \phi \) be the corresponding isomorphism over \( k \). Let
\( (A.3.9) \)
\[
\pi_0 : \oplus_i p^H_i(f_{0*} IC_{Y_0})[-i] \to p^H_0(f_{0*} IC_{Y_0})
\]
be the projection to the zeroth graded piece. Let $\pi$ be the corresponding projection over $k$. Let

$$u_0: \oplus_i^0 f_0^* IC_{Y_0} \to \oplus_i^0 f_0^* IC_{Y_0}[-i]$$

be the inclusion of the zeroth graded piece, and denote by $u$ the corresponding inclusion over $k$.

Restricting (A.3.8) to $U_0$, and composing with (A.3.5) gives us maps

$$f_i^0: j^* IC_{X_0} \to j^* P^i(f_0^* IC_{Y_0})[-i],$$

by projecting on each of the summand. Let $f_i^0$ be the corresponding maps over $k$.

Since $j_0^* IC_{X_0}$ and $j_0^* P^i(f_0^* IC_{Y_0})$ are perverse sheaves on $U_0$, the maps $f_i^0$ are all zero for $i > 0$ ([BBDG18], Corollaire 2.1.21). Further since $P^i(f_0^* IC_{Y_0})$ is pure of weight $i$, the morphisms $f_i^0$ are all zero for $i < 0$ ([BBDG18], Proposition 5.1.15 (iii)). Thus

$$f_i^0 = (j^* \phi^{-1}) \circ (j^* u) \circ f_i^0.$$

Note that the maps on either side of the equality (A.3.12) are defined over $k_0$, but the equality is over $k$.

The map $f_i^0$ is a map of pure perverse sheaves over $k_0$, thus applying Lemma A.2.1 in the form of (A.2.8), we get a natural map

$$f_i^0: IC_{X_0} \to P^0(f_0^* IC_{Y_0})$$

of perverse sheaves on $X_0$, which restricts to $f_i^0$ over $U_0$.

We define

$$f_0^*: IC_{X_0} \to f_0^* IC_{Y_0}.$$

Let $f_*$ be the corresponding map over $k$. Since by construction $f_0^*$ restricts to $f_0^*$ over $U_0$, (A.3.12) implies that $f_*$ restricts to $f_*$ over $U$ as desired.

We also have the dual map.

**Corollary A.3.3.** There is natural map

$$f_*^\vee: f_* IC_{Y} \to IC_{X}$$

(defined over $k_0$) such that when restricted $U$ it is dual to the map (A.3.6).

We can also apply adjunction to $f_*$ to obtain the following.

**Corollary A.3.4.** There is a natural map

$$f^*: f^* IC_{X} \to IC_{Y},$$

which when restricted to $f^{-1}(U)$ is the map in (A.1.1).
Dualizing we also have the following result.

**Corollary A.3.5.** There is a natural map

\[(A.3.17) \quad f^i : IC_Y \to f^i IC_X,\]

which when restricted to \(f^{-1}(U)\) is dual to the map in \((A.1.1)\).

**Remark A.3.6.** Note that compatibility of \(f_*|_U\) with \((A.3.6)\) is independent of the choice of an isomorphism in \((A.2.5)\). Since the argument only depends on vanishing of \(f_*^i\) for \(i \neq 0\), and all the choices lead to this vanishing.

In the rest of this section, we assume that \(X_0\) (and hence \(Y_0\)) is projective over \(k_0\).

The morphism of sheaves

\[(A.3.18) \quad f_* : IC_X \to f_* IC_Y,\]

constructed in Proposition A.3.2 induces a linear map

\[(A.3.19) \quad f^* : H^i(X, IC_X) \to H^i(Y, IC_Y).\]

By properness of \(f\) we also have an action on the compactly supported cohomology

\[(A.3.20) \quad f^*_c : H^i_c(U, IC_U) \to H^i_c(f^{-1}(U), IC_{f^{-1}(U)}).\]

Moreover, we have a diagram of cohomology groups

\[(A.3.21) \quad \begin{array}{ccc}
H^{i+d}_c(U, \mathbb{Q}_\ell) & \xrightarrow{\approx} & H^i_c(U, IC_U) \\
\downarrow f^* & & \downarrow (A.3.20) f^*_c \\
H^{i+d}(f^{-1}(U), \mathbb{Q}_\ell) & \xrightarrow{\approx} & H^i(f^{-1}(U), IC_{f^{-1}(U)})
\end{array} \quad \text{ and } \quad \begin{array}{ccc}
H^i(X, IC_X) & \xrightarrow{\approx} & H^i(Y, IC_Y) \\
\downarrow (A.3.19) f^* & & \downarrow f^*
\end{array}
\]

The square in the left of the Diagram A.3.21 is commutative as a result of \((A.3.7)\). The square on the right commutes by the functoriality of the adjunction \(j j^* \to 1\). When \(i = d\), the Diagram A.1.6 implies that all the row maps in the Diagram A.3.21 are isomorphisms.

We also have a commutative diagram of usual cohomology groups

\[(A.3.22) \quad \begin{array}{ccc}
\mathbb{Q}_\ell & \xrightarrow{\approx} & H^{2d}(X, \mathbb{Q}_\ell) \\
\downarrow \text{deg}(f) & & \downarrow f^* \\
\mathbb{Q}_\ell & \xrightarrow{\approx} & H^{2d}(Y, \mathbb{Q}_\ell)
\end{array} \quad \begin{array}{ccc}
H^{2d}(X, \mathbb{Q}_\ell) & \xrightarrow{\approx} & H^{2d}_c(U, \mathbb{Q}_\ell) \\
\downarrow f^* & & \downarrow f^*
\end{array} \quad \begin{array}{ccc}
\mathbb{Q}_\ell & \xrightarrow{\approx} & H^{2d}(Y, \mathbb{Q}_\ell) \\
\downarrow \text{Tr}_X & & \downarrow f^* \\
\mathbb{Q}_\ell & \xrightarrow{\approx} & H^{2d}(f^{-1}(U), \mathbb{Q}_\ell)
\end{array}
\]

Here \(\text{Tr}_X\) and \(\text{Tr}_Y\) are the trace maps on the top cohomology \((A.1.8)\). Finally combining the Diagrams A.3.21 and A.3.22 we have the following result.
Lemma A.3.7. The following diagram is commutative

\[ \begin{array}{cccccc}
\mathbb{Q}_\ell & \xrightarrow{\cong} & H^{2d}(X, \mathbb{Q}_\ell) & \xrightarrow{\cong} & H^d(X, \text{IC}_X) \\
\text{deg}(f) & \downarrow & f^* & \downarrow & f^* \\
\mathbb{Q}_\ell & \xleftarrow{\cong} & H^{2d}(Y, \mathbb{Q}_\ell) & \xleftarrow{\cong} & H^d(Y, \text{IC}_Y)
\end{array} \] (A.3.23)

We can also dualize the arguments above. The morphism (A.3.15) induces a pushforward on the intersection cohomology groups

\[ f_* : H^i(Y, \text{IC}_Y) \rightarrow H^i(X, \text{IC}_X). \] (A.3.24)

As in Lemma A.3.7 we would like to understand this action when \( i = d \). Note that by construction

\[ f_* : H^d(Y, \text{IC}_Y) \rightarrow H^d(X, \text{IC}_X). \] (A.3.25)

is dual to the pullback map

\[ f^* : H^{-d}(X, \text{IC}_X) \rightarrow H^{-d}(Y, \text{IC}_Y), \] (A.3.26)

induced by (A.3.18). Moreover restricting (A.3.18) to \( U \), and we obtain pullback maps

\[ f^* : H^i(U, \text{IC}_U) \rightarrow H^i(f^{-1}(U), \text{IC}_{f^{-1}(U)}). \] (A.3.27)

As before the commutative diagram (A.3.7) and functoriality of the adjunction \( 1 \rightarrow j_*j^* \) gives rise to a commutative diagram

\[ \begin{array}{cccccc}
H^{i+d}(X, \mathbb{Q}_\ell) & \xrightarrow{\cong} & H^{i+d}(U, \mathbb{Q}_\ell) & \xrightarrow{\cong} & H^i(U, \text{IC}_U) & \xleftarrow{\cong} & H^i(X, \text{IC}_X) \\
\downarrow f^* & & \downarrow f^* & & \downarrow f_* & & \downarrow f^* \\
H^{i+d}(Y, \mathbb{Q}_\ell) & \xrightarrow{\cong} & H^{i+d}(f^{-1}(U), \mathbb{Q}_\ell) & \xrightarrow{\cong} & H^i(f^{-1}(U), \text{IC}_{f^{-1}(U)}) & \xrightarrow{\cong} & H^i(Y, \text{IC}_Y)
\end{array} \] (A.3.28)

In particular when \( i = -d \), the Diagram A.1.9 implies that all the row maps in the Diagram A.3.28 are isomorphisms. Thus we get a commutative diagram

\[ \begin{array}{cccccc}
\mathbb{Q}_\ell & \xrightarrow{\cong} & H^0(X, \mathbb{Q}_\ell) & \xrightarrow{\cong} & H^{-d}(X, \text{IC}_X) \\
\downarrow f^* & & \downarrow f^* & & \downarrow f^* \\
\mathbb{Q}_\ell & \xrightarrow{\cong} & H^0(Y, \mathbb{Q}_\ell) & \xrightarrow{\cong} & H^{-d}(Y, \text{IC}_Y)
\end{array} \] (A.3.29)

Dualizing the Diagram A.3.29 we get a commutative diagram
Here \( f_* : H^0(Y, K_Y) \rightarrow H^0(X, K_X) \) is induced by the adjunction \( f_* f^! \rightarrow 1 \) and is dual to the pullback map \( f^* : H^0(X, \overline{\mathbb{Q}}_\ell) \rightarrow H^0(Y, \overline{\mathbb{Q}}_\ell) \). Thus combining Diagrams (A.3.30) and (A.1.6) we have a commutative diagram

\[
\begin{array}{c}
\text{(A.3.30)} \\
\begin{array}{c}
\overline{\mathbb{Q}}_\ell \\
\text{Tr}_X
\end{array}
\begin{array}{c}
\overset{\cong}{\rightarrow}
\begin{array}{c}
H^0(X, K_X) \\
\text{Tr}_X
\end{array}
\begin{array}{c}
\overset{\cong}{\rightarrow}
\begin{array}{c}
H^d(X, IC_X) \\
\text{Tr}_X
\end{array}
\end{array}
\begin{array}{c}
\overset{f_*}{\rightarrow}
\begin{array}{c}
H^d(Y, IC_Y) \\
\text{Tr}_Y
\end{array}
\end{array}
\begin{array}{c}
\overset{\cong}{\rightarrow}
\begin{array}{c}
\overline{\mathbb{Q}}_\ell \\
\text{Tr}_Y
\end{array}
\end{array}
\begin{array}{c}
\overset{f_*}{\rightarrow}
\begin{array}{c}
H^0(Y, K_Y) \\
\text{Tr}_Y
\end{array}
\end{array}
\begin{array}{c}
\overset{\cong}{\rightarrow}
\begin{array}{c}
H^d(Y, IC_Y) \\
\text{Tr}_Y
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\]

The Diagrams (A.3.23) and (A.3.31) immediately imply the following proposition.

**Proposition A.3.8.** Let \( f_0, g_0 \) be two dominant morphisms from \( X_0 \) to \( Y_0 \). Let \( f^* : H^d(X, IC_X) \rightarrow H^d(Y, IC_Y) \) and \( g_* : H^d(Y, IC_Y) \rightarrow H^d(X, IC_X) \) be the linear maps as defined in (A.3.19) and (A.3.25). Then

\[
\begin{array}{c}
\text{(A.3.32)} \\
g_* \circ f^* = \deg(f),
\end{array}
\]

as endomorphisms of the one dimensional \( \overline{\mathbb{Q}}_\ell \) vector space \( H^d(X, IC_X) \).

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