Partition Functions for BPS States of the Non-Critical $E_8$ String

J.A. Minahan, D. Nemeschansky

*Physics Department, U.S.C. University Park, Los Angeles, CA 90089*

and

N.P. Warner

*Institute for Theoretical Physics University of California, Santa Barbara, CA 93106-4030*

We consider the BPS states of the $E_8$ non-critical string wound around one of the circles of a toroidal compactification to four dimensions. These states are indexed by their momenta and winding numbers. We find explicit expressions, $G_n$, for the momentum partition functions for the states with winding number $n$. The $G_n$ are given in terms of modular forms. We give a simple algorithm for generating the $G_n$, and we show that they satisfy a recurrence relation that is reminiscent of the holomorphic anomaly equations of Kodaira-Spencer theory.

*On leave from Physics Department, U.S.C., University Park, Los Angeles, CA 90089*
1. Introduction

One of the surprises in the recent past is the existence of six-dimensional, $E_8$ non-critical strings in heterotic string theory \cite{1}. These occur when a heterotic string is compactified on a $K3$ manifold with a small $E_8$ instanton. As the instanton shrinks to zero size, the tension of a non-critical string vanishes. One of the intriguing features of such strings is that they are consistent string theories that are apparently decoupled from gravity.

If one compactifies the six-dimensional theory on a circle, then one finds BPS states that correspond to momentum and winding states of the non-critical string around the compactified dimension \cite{2,3}. After a second compactification on another circle, we are left with an $N = 2$ supersymmetric theory in four dimensions, where the pre-potential has an instanton expansion whose coefficients basically count the net number of BPS states of the five-dimensional theory.

The six-dimensional heterotic theory is dual to an $F$-theory compactification on a Calabi-Yau manifold that admits an elliptic fibration. The tensionless string limit is reached when a del Pezzo 4-cycle shrinks to zero size \cite{4,5}. In \cite{6} it was shown how to describe the pre-potential of the non-critical string as a function of the string tension in terms of a Seiberg-Witten theory. In \cite{7} this was extended to incorporate other physical moduli.

In this paper, we investigate this instanton expansion as a function of two moduli: $t_S$ and $t_E$. The former is a product of the string tension and a compactification radius, and so indexes the winding states around a circle. The parameter $t_E$ is a background geometric factor that indexes the momentum states on the same circle. As an expansion in $t_S$, we find that each term of the pre-potential is an almost modular function in $t_E$. We explicitly compute the first few functions in the expansion. We then show that these functions satisfy a “modular anomaly” recurrence relation which is similar to the non-holomorphic recurrence relation of Kodaira-Spencer theory \cite{8}. Using the recurrence relation we show how to generate all functions in the instanton expansion. We also have enough information about the structure of these functions to compute the BPS degeneracies in the asymptotic limit.

In section 2 we review some relevant facts from \cite{2,3,6} and \cite{7}. In section 3 we discuss the instanton expansion and its relation to the counting of BPS states. In section 4 we discuss and then prove the recurrence relation. We also show that the recurrence relation is very useful in computing the entire instanton expansion. In section 5 we compute the asymptotic BPS state degeneracy.
2. The quantum effective action

2.1. The IIA description of the non-critical $E_n$ string

We first briefly review some of the key elements of [3,6]. One considers a compactification of the IIA theory on an elliptically fibered Calabi-Yau 3-fold in which a 4-cycle is collapsing. The magnetic non-critical string may be thought of as coming from a five-brane wrapping this cycle, and the electric excitations of the string come from membranes wrapping the 2-cycles within the 4-cycle. One obtains the $E_n$ non-critical string if the collapsing 4-cycle is a del Pezzo surface, $B_n$, obtained by blowing-up $n$ points in $\mathbb{CP}_2$ [4,5].

In [3] the foregoing is realized explicitly by using a Calabi-Yau 3-fold, $X_{F_1}$, that is an elliptic fibration over the Hirzebruch surface $F_1$. Since $X_{F_1}$ is also a $K3$ fibration, this compactification of the IIA theory is dual to the heterotic string compactified on $K3 \times T_2$. Specifically, the corresponding heterotic string compactification has an $E_8 \times E_8$ instanton embedding with $n_1 = 11, n_2 = 13$. The manifold $X_{F_1}$ has three Kähler moduli, $t_D, t_E$ and $t_F$, corresponding to the volumes of the base of $F_1$, the elliptic fiber of $X_{F_1}$, and the fiber of $F_1$ respectively. The modulus, $t_D$, is also the scale of the canonical divisor of the del Pezzo surface, and the elliptic fiber of the del Pezzo is that of $X_{F_1}$. At $t_D = 0$ only a 2-cycle collapses, but for the whole del Pezzo to vanish one must pass through a flop transition ($Im(t_D) < 0$) to a point where $t_D + t_E = 0$.

Since $t_D$ is the scale of the base of the $K3$ fibration, it follows that it must be related to the heterotic dilaton. The other two moduli, $t_E$ and $t_F$, are related to the moduli, $T = B + iR_5R_6$ and $U = e^{i\alpha}R_5/R_6$ of the torus in the heterotic compactification. In [3,7] it was shown that $S, T$ and $U$ are related to the complexified Kähler moduli according to $t_E = U, t_F = T - U$ and $t_D = S + aT + bU$ for some undetermined constants $a, b$. The point where $t_D$ vanishes corresponds to an $SU(2)$ gauge symmetry enhancement at strong coupling in the heterotic string.

It is convenient to introduce a reparametrization of the moduli, replacing $t_D$ and $t_F$ by $t_S \equiv t_D + t_E$ and $t'_F = t_F + t_B$. (Note that both these new moduli are linear in the heterotic dilaton.) From the point of view of the non-critical string, the interesting moduli are $t_E$ and $t_S$. Of the three moduli, only $t'_F$ involves the tension of the fundamental heterotic string [4], and so to decouple the BPS states of fundamental heterotic string, we will consider the limit in which $t'_F \rightarrow \infty$. The parameter $t_S$ determines the scale of the 4-cycle and hence the tension of the non-critical string. In terms of the electric BPS excitations of the non-critical string (membranes wrapping rational curves), the degrees $d_D$ and $d_E$ of the rational curve represent the winding number and momentum respectively of the non-critical string around one of the circles of the torus [3].
2.2. Consistent truncations

It is fairly evident that all the physics of the non-critical string should be captured by the structure of the vanishing del Pezzo surface, and not so much by the details of the geometry of the 3-fold in which the del Pezzo surface lives. In the nearly tensionless limit, one should be able to decouple the non-critical string from ancilliary degrees of freedom like gravity. This decoupling should be accomplished by finding a way to abstract the del Pezzo surface from the space in which it is buried. It was one of key ideas in [6] that one can see how to do this in a consistent manner by passing to a closed sub-monodromy problem. That is, one identifies a basis of cycles that close (decouple from all the others) under monodromies on a special subset of moduli. Two such closed sub-monodromy problems were identified in [6], and both of them were on the other side of the flop transition ($Im(t_D) < 0$). One finds that in the basis where the scalar fields in the vector multiplets are given by the parameters introduced above, $t_E, t_S = t_E + t_D$ and $t_F' = t_F + t_B$, three of the periods depend classically upon $t_S$ alone, and one further period is $t_E$ itself.

It was further proposed in [6] that such closed sub-monodromy problems could be modeled by considering a compactification of the IIB string on a non-compact Calabi-Yau 3-fold based upon the vanishing del Pezzo surface. This non-compact Calabi-Yau manifold would thus characterize the non-critical string decoupled from the rest of the original string theory. This proposal closely parallels the way in which the quantum effective action of gauge theories (decoupled from the rest of string theory) can be obtained from ALE fibrations [10]. This idea was implemented and tested in [6], and further tested in [7], for the three period sub-monodromy problem that involves $t_S$ alone. Our purpose here is to generalize this to include $t_E$.

In a Calabi-Yau compactification of the IIB theory, the electric and magnetic BPS states are on the same footing – they come from wrapping 3-branes around $A$ or $B$-cycles of the Calabi-Yau manifold. As in [10], one can re-cast this wrapping of 3-branes in terms of wrapping a non-critical string around a Riemann surface. This is done by decomposing the 3-cycles into 2-cycles fibered over some curve in a base. The holomorphic 3-form can then be integrated to yield some form of Seiberg-Witten differential. Thus the original pre-potential can be obtained from integrals of a meromorphic differential on a Riemann surface. For the non-critical string, such a Seiberg-Witten effective action encodes all the counting of rational curves in the del Pezzo surface [6]. In [7] this was done for sub-monodromy problem involving $t_S$ alone, which yields information about the BPS states with $d_D = d_E$. These states were focussed on because they were, in a sense, the most
stringy: they are the states that become massless when the non-critical string becomes
tensionless. The reduction of the problem from a 3-fold to a torus was also critical to [7] in
that it enabled the simple generalization to include Wilson lines, and led to a conjecture
as to how to include $t_E$.

The starting point of [6,7] was to consider the IIB string theory compactified on the
on the non-compact Calabi-Yau 3-fold defined by

$$w^2 = z_1^3 + z_2^6 + z_3^6 - \frac{1}{z_4^6} - \psi w z_2 z_3 z_4 . \quad (2.1)$$

The modulus $\psi$ determines the string tension. As described in [3], the holomorphic 3-
form on this manifold has three periods corresponding to terms in the BPS mass formula.
With suitable normalization, the period integrals can be identified with $1, t_S$ and $\partial F/\partial t_S$. This gives the truncation of the non-critical string to the sector in which $t_S$ is the only
parameter, and in which the BPS states have $d_E = d_D$.

To get the states of the string with independent winding number and momentum ($d_D$ and
$d_E$) we want the slightly less stringent truncation described in [6]: one in which the
period integrals are $1, t_S, t_E$ and $\partial F/\partial t_S$. The relevant manifold was proposed in [7]:

$$w^2 = z_1^3 + z_2^6 + z_3^6 - \frac{1}{z_4^6} + \psi^2 (1 + k^2) (z_1 z_2 z_3 z_4)^2 + k^2 \psi^4 z_1 (z_2 z_3 z_4)^4 . \quad (2.2)$$

The moduli are $\psi$ and $k$, and (2.2) reduces to (2.1) in the limit $k \to 0$ (one also has to
shift $w$ and re-scale $\psi$). It is convenient to think of $k$ as the modulus of a set of Jacobi
elliptic functions, and so we parametrize $k$ in terms of another variable, $\tau$, via

$$k \equiv \vartheta^2_2(0|\tau)/\vartheta^2_4(0|\tau) . \quad (2.3)$$

The holomorphic 3-form can be represented as

$$\Omega = \frac{\psi}{2 \omega_2} \frac{z_4 \, dz_1 \, dz_2 \, dz_3 \, w}{w} , \quad (2.4)$$

where the (constant) normalization factor $\frac{\psi}{\omega_2}$ will be defined below. To isolate the relevant
periods one now follows the approach of [3]: Go to a patch with $z_4 = 1$, and make a change
of variables $z_1 = \zeta z_2^2$. Considered as a function of $z_2$, $\Omega$ has branch cuts. These cuts
disappear when $z_3^6 = 1$. Integrate $z_2$ around a circle around all these cuts. Let $\xi = \psi z_3$, and then the period integrals reduce to:

$$\frac{1}{2 \omega_2} \int \frac{d\xi \, d\zeta}{\sqrt{\zeta^3 + 1 - (1 + k^2) \zeta^2 \xi^2 + k^2 \zeta^4}} . \quad (2.5)$$
There are two types of period integral:

(i) Integrate $\xi$ between roots of $\xi^6 = \psi^6$

(ii) Take $\zeta = x\xi^2$ and integrate $\xi$ around a circle of large radius.

Doing the latter first, one is left with a standard elliptic integral in $x$:

$$\frac{1}{\omega_2} \int \frac{dx}{\sqrt{(x^3 - (1 + k^2)x^2 + k^2x)}} = \frac{1}{\omega_2} \int \frac{dt}{\sqrt{(1 - t^2)(1 - k^2t^2)}}, \quad (2.6)$$

where $x = 1/t^2$.

Shifting $x$, one can recast the elliptic integral in standard Weierstrass form $\int dx/y$ with $y^2 = 4x^3 - g_2x - g_3$ and

$$g_2 = \frac{4}{3} (1 - k^2 + k^4), \quad g_3 = \frac{4}{27} (1 + k^2)(2k^2 - 1)(k^2 - 2). \quad (2.7)$$

The periods of the torus defined by $x$ and $y$ are denoted by $\omega_1$ and $\omega_2$. This defines $\omega_2$ in (2.4), and with this normalization the periods (2.6) are 1 and $\tau = \omega_1/\omega_2$. The parametrization (2.3) gives the relationship between $k$ and the parameter $\tau$ introduced here. The parameter $\tau$ is to be identified with the Kähler modulus $t_E$.

Returning to the other period integrals, we first recast them into a form similar to that of [7]. Make a change of variables $\zeta = \frac{1}{4} x/\xi^{10}$, $u = -2\xi^6$. The integral becomes

$$\frac{i}{12\omega_2} \int dx du / y \text{ where}$$

$$y^2 = x^3 + (1 + k^2)u^2x^2 + k^2u^4x - 2u^5. \quad (2.8)$$

This can easily be recast into standard Weierstrass form $y^2 = 4x^3 - \tilde{g}_2x - \tilde{g}_3$ with:

$$\tilde{g}_2 = \frac{4}{3} (1 - k^2 + k^4) u^4 = g_2 \ u^4, \quad \tilde{g}_3 = \frac{4}{27} (1 + k^2)(2k^2 - 1)(k^2 - 2) \ u^6 - 8 \ u^5 = g_3 - 8 \ u^5. \quad (2.9)$$

Let $\tilde{\omega}_i$ be the periods of this torus, and $\bar{\tau} = \tilde{\omega}_1/\tilde{\omega}_2$. The periods we then seek are the indefinite integrals $\int du \bar{\omega}_i/\omega_2$ where $u = -2\psi^6$.

Introduce the Eisenstein functions:

$$E_2(\tau) \equiv 1 - 24 \sum_{n=1}^{\infty} \sigma_1(n) \ q^n, \quad E_4(\tau) \equiv 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) \ q^n, \quad E_6(\tau) \equiv 1 - 504 \sum_{n=1}^{\infty} \sigma_5(n) \ q^n, \quad (2.10)$$
where $q = e^{2\pi i \tau}$ and $\sigma_p(n)$ is the sum of the $p^{th}$ powers of the divisors of $n$. One should recall that $E_4$ and $E_6$ transform as modular functions of weight 4 and 6 respectively, and that

$$E_2\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^2 \left(\frac{E_2(\tau)}{c} + \frac{6}{\pi i} \frac{c}{c\tau + d}\right). \quad (2.11)$$

From (2.9) and the standard relationship between the periods of a torus and the Eisenstein functions one finds:

$$\tilde{\omega}_2 = \frac{\omega_2}{u} \left(\frac{E_4(\tilde{\tau})}{E_4(\tau)}\right)^{1/4}, \quad \frac{E_3^4(\tilde{\tau})}{E_6^2(\tilde{\tau})} = \frac{E_3^4(\tau)}{(E_6(\tau) - v)^2}, \quad (2.12)$$

where $v = \frac{1}{2\tau}(\frac{\pi}{\omega_2})^6 \frac{1}{u}$. The second equation in (2.12) may be used to determine $\tilde{\tau}$ as a function of $\tau$ and $u$. This can then be substituted into the first equation to give $\tilde{\omega}_2/\omega_2$ in terms of $\tau$ and $u$. One then gets the other period integral from $\tilde{\omega}_1/\omega_2 = \tilde{\tau}\tilde{\omega}_2/\omega_2$.

Therefore, the other two periods of $\Omega$ are:

$$\varphi = \frac{1}{2\pi i} \int \frac{dv}{v} \left(\frac{E_4(\tilde{\tau})}{E_4(\tau)}\right)^{1/4}; \quad \varphi_D = \frac{1}{2\pi i} \int \frac{dv}{v} \tilde{\tau} \left(\frac{E_4(\tilde{\tau})}{E_4(\tau)}\right)^{1/4}, \quad (2.13)$$

where $\tau$ is fixed, and $\tilde{\tau}$ is obtained from (2.12).

3. Curve counting and the instanton expansion

To count the BPS states we want the Taylor expansion of the periods (2.13) about $v = 0$. To evolve these series it is useful to recall that:

$$E_2 = q \frac{d}{dq} \log(\Delta); \quad q \frac{d}{dq} E_2 = \frac{1}{12} (E_2^2 - E_4);$$

$$q \frac{d}{dq} E_4 = \frac{1}{3} (E_2 E_4 - E_6); \quad q \frac{d}{dq} E_6 = \frac{1}{2} (E_2 E_6 - E_4^2), \quad (3.1)$$

where $\Delta = \eta^{24} = q \prod_{n=1}^{\infty} (1-q^n)^{24} = \frac{1}{1728} (E_4^3 - E_6^2)$. Also recall that the modular invariant $j$ is given by $j = E_4^3/\Delta$. From (2.12) one obtains:

$$\tilde{\tau} = \tau + \sum_{j=1}^{\infty} a_j v^j, \quad (3.2)$$

and the $a_j$ can be obtained using (3.1). The first few terms are:

$$a_1 = 2E_4/\Delta, \quad a_2 = -\frac{1}{3\Delta^2} E_4 \left(5E_6 + E_2 E_4\right),$$

$$a_3 = -\frac{1}{\Delta^3} E_4 \left(\frac{31}{54} E_4^3 + \frac{40}{27} E_6^2 + \frac{5}{9} E_2 E_4 E_6 + \frac{1}{18} E_2^2 E_4^2\right). \quad (3.3)$$
Note that the second equation in (2.12) is invariant under the combined modular transformations:
\[ \tau \rightarrow \frac{a\tau + b}{c\tau + d}, \quad \tilde{\tau} \rightarrow \frac{a\tilde{\tau} + b}{c\tilde{\tau} + d}, \quad v \rightarrow (c\tau + d)^6 v. \] (3.4)
The fact that one must simultaneously transform \( \tau \) and \( \tilde{\tau} \) follows from requiring an expansion of the form (3.2). It is also easy to verify that the function
\[ \varphi_D - \tau \varphi = \frac{1}{2\pi i} \int \frac{dv}{v} (\tilde{\tau} - \tau) \left( \frac{E_4(\tilde{\tau})}{E_4(\tau)} \right)^{1/4} \equiv \sum_{j=1}^{\infty} b_j v^j, \] (3.5)
considered as a function of \( \tau \) and \( v \), transforms as a modular function of weight \(-2\) under (3.4). Thus upon substituting (3.2) into (3.3), and expanding in a power series is \( v \), the coefficient, \( b_j \), of \( v^j \) must be an exactly modular function of weight \(-2 - 6j\). One also sees from (2.12) that the only pole in \( \varphi_D - \tau \varphi \) is at \( \tau = i\infty \), and from the examining the form of \( a_j \), one sees that the \( b_j \) must be of the form \( P_{6j-2}(E_4, E_6)/\Delta^j \), where \( P_{6j-2}(E_4, E_6) \) is a polynomial in \( E_4 \) and \( E_6 \) alone, such that it has modular weight \( 6j - 2 \).

Up to “constants” of integration, \( \delta(\tau) \) and \( \delta_D(\tau) \), (which could be arbitrary functions of \( \tau \)), the periods \( \varphi \) and \( \varphi_D \) are \( t_S \) and \( \partial F / \partial t_S \):
\[ t_S = \varphi - \delta(\tau); \quad \partial F / \partial t_S = \varphi_D - \delta_D(\tau). \] (3.6)
Let \( q_S = e^{2\pi i t_S} \) and recall \( q = e^{2\pi i \tau} = e^{2\pi i t_E} \). Define \( C_{t_S t_S t_S} = \partial^3 F / \partial t_S^3 \), and recall that it has an instanton expansion:
\[ C_{t_S t_S t_S} = \sum_{n_1, n_2} (-1)^{n_1+1} N_{n_1, n_2} \frac{n_1^3 q_S^{n_1} q_S^{n_2}}{1 - q_S^{n_1} q_S^{n_2}}, \] (3.7)
where \( N_{n_1, n_2} \) is the number of rational curves with \( d_D = n_1 \) and \( d_E = n_1 + n_2 \).

To develop the instanton expansion we therefore have to substitute (3.2) into the expression for \( \varphi \), expand the series in \( v \). One then inverts this series to obtain a series for \( v \) in powers of \( e^{2\pi i \varphi} \), with coefficients that are functions of \( \tau \). This is then substituted into the series expansion for \( C_{t_S t_S t_S} \). The result is a series of the form:
\[ C_{t_S t_S t_S} = \sum_{n=1}^{\infty} \tilde{F}_n(\tau) e^{2\pi i n \varphi} = \sum_{n=1}^{\infty} F_n(\tau) q_S^n, \] (3.8)
where \( F_n(\tau) = \tilde{F}_n(\tau) e^{2\pi i n \delta(\tau)} \). It is trivial to see that \( \tilde{F}_1(\tau) = a_1(\tau) \), where \( a_1 \) is given in (3.3). The first term in this series (3.8) was computed in [3], and is given by:
\[ F_1(\tau) = \frac{E_4}{q^{1/2} \eta^{12}}. \] (3.9)
(The factor in the denominator is \( q^{1/2} \) and not \( q^{-1/2} \), since this is the coefficient of \( q_S = q e^{2\pi i t_D} \).) This fixes \( \delta(\tau) = \frac{1}{2\pi i} \log(-\frac{1}{2} q^{-1/2} \eta^{12}) \).

Using Mathematica\textsuperscript{TM}, one can easily compute the \( F_n \) to fairly high order \((n = 12)\). Defining \( G_n = q^{n/2} F_n \), one finds that \( G_n \) has the form of

\[
G_n = Q_{6n-2}(E_2, E_4, E_6) / \Delta^{n/2}
\]  

(3.10)

where \( Q_{6n-2} \) is a polynomial of degree \( 6n - 2 \) (where \( E_2, E_4 \) and \( E_6 \) are given weights 2, 4 and 6 respectively). Because of the presence of \( E_2 \) and the half powers of \( \Delta \), the functions \( G_n \) are not quite modular functions of weight \(-2\). The first few \( G_n \) are:

\[
G_1 = \frac{E_4}{\Delta^{1/2}}, \quad G_2 = \frac{E_4 (E_2 E_4 + 2 E_6)}{3 \Delta},
\]

\[
G_3 = \frac{1}{576 \Delta^2} E_4 \left( 54 E_2^2 E_4^2 + 109 E_4^3 + 216 E_2 E_4 E_6 + 197 E_6^2 \right),
\]

\[
G_4 = \frac{1}{972 \Delta^2} E_4 \left( 24 E_2^3 E_4^3 + 109 E_2 E_4^4 + 144 E_2^2 E_4^2 E_6 + 272 E_4^3 E_6 + 269 E_2 E_4 E_6^2 + 154 E_6^3 \right)
\]

\[
G_5 = \frac{1}{2985984 \Delta^{5/2}} E_4 \left( 18750 E_2^4 E_4^4 + 136250 E_2^2 E_4 E_6^2 + 116769 E_4^6 + 150000 E_2^3 E_4^3 E_6 + 653000 E_2 E_4^4 E_6 + 426250 E_2^2 E_4^2 E_6^2 + 772460 E_4^3 E_6^2 + 505000 E_2 E_4 E_6^3 + 207505 E_6^4 \right)
\]

\[
G_6 = \frac{1}{74649600 \Delta^3} E_4 \left( 116640 E_2^5 E_4^5 + 1177200 E_3^3 E_4^6 + 2398867 E_2 E_4^7 + 1166400 E_2^4 E_4^4 E_6 + 8229600 E_2^2 E_4^5 E_6 + 6703718 E_4^6 E_6 + 4460400 E_2^3 E_4^3 E_6^2 + 18894730 E_2 E_4^4 E_6^2 + 8100000 E_2^2 E_4^2 E_6^3 + 14280020 E_4^3 E_6^3 + 6922915 E_2 E_4 E_6^4 + 2199110 E_6^5 \right).
\]

(3.11)

These functions are completely consistent with the numbers generated in [3].

The first function in (3.11) has the simple interpretation as the \( E_8 \) root lattice partition function multiplied by the partition function of four bosonic oscillators coming from the space-time [3]. The second function is almost as simple: it can be rewritten as \( G_1 G'_1 \), where \( G'_1 = q^{4 \Delta q} G_1 \). Since this partition function represents momentum excitations of a doubly wound string, one naively expects a tensor product, or \( G_2^2 \). However, since it is a bound state, it cannot be a simple tensor product. The derivative with respect to \( \tau \) pulls
down a hamiltonian and thus removes one of the two independent translation invariances of the doubly wound state. Unfortunately the higher $G_n$ do not appear to admit such a simple interpretation. However, as we will see in the next section, the $G_n$ are indeed related to one another.

4. Modular properties and a recurrence relation

There is no a priori reason to expect the non-critical string to exhibit $T$-duality, but the $G_n$ are almost modular functions (of weight $-2$), and so the spectrum of BPS states is almost invariant under $\tau \to \frac{a\tau + b}{c\tau + d}$. There are three places in which the BPS spectrum fails to be modular invariant: (i) the bare factors of $q$ in $F_n = q^{n/2}G_n$, (ii) the odd powers of $\Delta^{1/2}$ in $G_n$, and (ii) the anomalous modular behaviour of $E_2$. The first problem has a trivial cure: one simply redefines $t_S$. The second problem means that one really has only a subgroup of the modular groups as a symmetry. Alternatively, one can work with the full modular group, and remember to make appropriate changes of sign in the odd powers of $\Delta^{1/2}$. The anomalous behaviour of $E_2$ also has a cure, but at the cost of holomorphy. That is, the function

$$\hat{E}_2(\tau) = E_2(\tau) - \frac{3}{\pi} \frac{1}{Im(\tau)},$$

satisfies

$$\hat{E}_2\left(\frac{a\tau + b}{c\tau + d}\right) = \left(c\tau + d\right)^2 \hat{E}_2(\tau).$$

Define $\hat{G}_n$ to be $G_n/n^3$ but with $E_2$ replaced by $\hat{E}_2$, and introduce:

$$\hat{G}(\sigma, \tau) = \sum_{n=1}^{\infty} \hat{G}_n(\tau) e^{2\pi i n \sigma}. \quad (4.3)$$

One can recover the instanton expansion from this by taking three derivatives with respect to $\sigma$, setting $\sigma = t_S + \frac{1}{2} \tau$, and by sending $\bar{\tau} \to \infty$ while holding $\tau$ fixed. The function $\hat{G}$ is also modular invariant. To this extent the non-critical string exhibits a $T$-duality.

The function $\hat{G}$ exhibits another remarkable property: it satisfies a recurrence relation that is reminiscent of the holomorphic anomaly equation in Kodaira-Spencer theory [8]. This follows from a “modular anomaly” recurrence relation that is satisfied by the $F_n$. Let $f_n = F_n/n^3$, and view it as a function of the variables $E_2, E_4$ and $E_6$, then we will show that

$$\frac{\partial f_n}{\partial E_2} = \frac{1}{24} \sum_{m=1}^{n-1} m(n-m) f_m f_{n-m}. \quad (4.4)$$
If one replaces $E_2$ by $\hat{E}_2$, one has $\frac{\partial}{\partial \tau} = -\frac{3i}{2\pi (Im(\tau))^2} \frac{\partial}{\partial E_2}$. Hence it follows that $\hat{G}$ satisfies:

$$\frac{\partial \hat{G}}{\partial \bar{\tau}} = -\frac{i}{16\pi (Im(\tau))^2} \frac{\partial \hat{G}}{\partial \sigma} \frac{\partial \hat{G}}{\partial \sigma}.$$  \hspace{1cm} (4.5)

This equation is almost exactly of the form of the holomorphic anomaly equation of [8], although the physical origins of the problem are rather different. In [8] the holomorphic anomaly was used to relate partition functions on higher genus Riemann surfaces to those of lower genus. In our case, the anomaly equations are used to relate multi-instanton expansions to lower instanton terms. This probably explains one key difference between the anomaly equations. In [8], the anomaly equation for $F_g$ where $g$ is the genus, contains the piece $\partial^2 F_{g-1}$. Such a term arises from pinching off a handle on the Riemann surface. It is hard to imagine an analogous process for a multi-instanton.

Still, the similarities are striking. The instanton expansion consists of maps onto a target space that is a torus with modulus $\tau$. The factor of $1/(Im(\tau))^2$ in (1.3) has the interpretation of a metric $g^{\sigma\sigma}$ on the torus [8], while the fact that the differential equation involves a derivative with respect to $\bar{\tau}$ on the left and derivatives with respect to $\sigma$ on the right is consistent with the form of the classical intersection form: Since $\varphi_D = \tau \varphi$ it follows that $C_{\tau t s t s} = 1$. There is no corresponding equation for $\partial_{\bar{\varphi}} \hat{G}$ since the classical (non-instanton) part of $C_{t s t s}$ is zero. Given this close correspondance, it is tempting to conjecture that $e^{2\pi i t_s}$ might be a loop expansion parameter for the non-critical string.

The recurrence relation in (1.4) turns out to be a powerful tool in computing the $G_n$ of the previous section. Given the lower $G_m$, the recurrence relation determines $G_n$ ($n > m$) up to a piece $E_4K_n/\Delta^{n/2}$, where $K_n$ is a modular form of weight 6($n - 1$). The space of such forms has dimension $[(n + 1)/2]$, thus to completely determine $G_n$, we need to compute $[(n + 1)/2]$ coefficients. We actually have more than enough information to do this, since we know that the $q$ expansion of $G_n$ has the form $G_n = q^{-n/2} + O(q^{n/2})$.\textsuperscript{1} Hence, all we need to do is adjust the coefficients in $K_n$, such that leading term in $G_n$ is $q^{-n/2}$ and the rest of the negative powers in $G_n$ have coefficients that are zero. A useful check is that the nonnegative powers up to the $q^{n/2}$ term also have zeros for coefficients. Using Mathematica\textsuperscript{TM}, we can easily generate the $K_n$, the first 12 of which are given by

\textsuperscript{1} This follows from (3.7) and the fact that $n_1 \geq 1$ $n_2 \geq 0$ and $N_{1,0} = 1
\[ K_1 = 1, \quad K_2 = \frac{2E_6}{3}, \quad K_3 = \frac{1}{576} (109E_4^3 + 197E_6^2), \quad K_4 = \frac{E_6}{486} (136E_4^3 + 77E_6^2), \]
\[ K_5 = \frac{1}{2985984} \left( 116769E_4^6 + 772460E_4^3E_6^2 + 207505E_6^4 \right), \]
\[ K_6 = \frac{E_6}{37324800} \left( 3351859E_4^6 + 7140010E_4^3E_6^2 + 1099555E_6^4 \right), \]
\[ K_7 = \frac{1}{386983526400} \left( 3214033725E_4^9 + 46377519701E_4^6E_6^2 + 47881472765E_4^3E_6^4 \right. \]
\[ + 4721253846E_6^6 \right), \]
\[ K_8 = \frac{E_6}{111106598400} \left( 2874313704E_4^9 + 13496101157E_4^6E_6^2 + 8126197310E_4^3E_6^4 \right. \]
\[ + 551920565E_6^6 \right), \]
\[ K_9 = \frac{1}{1213580338790400} \left( 2168558256025E_4^{12} + 54762568177568E_4^9E_6^2 \right. \]
\[ + 125727877850316E_4^6E_6^4 + 49166052530000E_4^3E_6^6 + 242308666145E_6^8 \right), \]
\[ K_{10} = \frac{E_6}{88470006697820160} \left( 621851537315031E_4^{12} + 5138509650200980E_4^9E_6^2 \right. \]
\[ + 6932453167897530E_4^6E_6^4 + 1889989579331700E_4^3E_6^6 + 70283214345575E_6^8 \right), \]
\[ K_{11} = \frac{1}{4246560321495367680000} \left( 1644909843291474375E_4^{15} + 64293839773877897511E_4^{12}E_6^2 \right. \]
\[ + 261044867981347260580E_4^9E_6^4 + 230239896247913645940E_4^6E_6^6 \right. \]
\[ + 46023695034064491975E_4^3E_6^8 + 1331341121000896775E_6^{10} \right), \]
\[ K_{12} = \frac{E_6}{5352435405218119680000} \left( 9853164552615074200E_4^{15} + 126458286011220239911E_4^{12}E_6^2 \right. \]
\[ + 303024347024677902580E_4^9E_6^4 + 187415787390550146890E_4^6E_6^6 \right. \]
\[ + 28525664977566703100E_4^3E_6^8 + 657760456052320775E_6^{10} \right), \]
\[ (4.6) \]

The coefficients of these polynomials are always positive. At this time, we know of no other system where these particular modular forms appear naturally.

We conclude by proving the recurrence relation (4.4). This relation is equivalent to:
\[ \frac{\partial}{\partial E_2} (\varphi_D - \tau \varphi) = \frac{2\pi i}{24} \frac{\partial}{\partial \varphi} (\varphi_D - \tau \varphi)^2, \]
\[ (4.7) \]
where \( \varphi_D - \tau \varphi \) is viewed as a function of \( \varphi, E_2, E_4 \) and \( E_6 \). Now recall that the coefficients \( b_j \) in (3.5) are independent of \( E_2 \), and so \( \varphi_D - \tau \varphi \) inherits its \( E_2 \) dependence only through
the implicit dependence of $v$ on $\varphi$, $E_2$, $E_4$ and $E_6$. As a result, (4.7), and hence (4.4), is equivalent to

$$\frac{\partial v}{\partial E_2} = \frac{2\pi i}{12} (\varphi D - \tau \varphi) \frac{\partial v}{\partial \varphi} \quad \text{or} \quad (\varphi D - \tau \varphi) = \frac{12}{2\pi i} \frac{\partial_{E_2} v}{\partial \varphi}.$$  \hspace{1cm} (4.8)

Differentiating the first equation in (2.13) with respect to both $\varphi$ and $E_2$ yields

$$2\pi i = \frac{\partial \varphi}{\partial E_2} v \left( \frac{E_4(\bar{\tau})}{E_4(\tau)} \right)^{1/4},$$

$$0 = \frac{\partial_{E_2} v}{v} \left( \frac{E_4(\bar{\tau})}{E_4(\tau)} \right)^{1/4} + \int^v \frac{dv}{v} \frac{\partial_{E_2} \left( \frac{E_4(\bar{\tau})}{E_4(\tau)} \right)^{1/4}}{v}, \quad \text{where the } E_2 \text{ derivative acting on the integrand is for the explicit } E_2 \text{ dependence (i.e. not the implicit dependence in } v). \text{ From these two equations one obtains:}

$$\frac{\partial_{E_2} v}{\partial \varphi} = -\frac{1}{2\pi i} \int^v \frac{dv}{v} \frac{\partial_{E_2} \left( \frac{E_4(\bar{\tau})}{E_4(\tau)} \right)^{1/4}}{v}. \quad \text{(4.10)}$$

Using this in (4.8) one sees that the recurrence relation is equivalent to:

$$\int^v \frac{dv}{v} \left( \frac{E_4(\bar{\tau})}{E_4(\tau)} \right)^{1/4} (\bar{\tau} - \tau) = -\frac{12}{2\pi i} \int^v \frac{dv}{v} \frac{\partial_{E_2} \left( \frac{E_4(\bar{\tau})}{E_4(\tau)} \right)^{1/4}}{v}, \quad \text{(4.11)}$$

or equating integrands, one has

$$\left( \frac{E_4(\bar{\tau})}{E_4(\tau)} \right)^{1/4} (\bar{\tau} - \tau) = -\frac{12}{2\pi i} \partial_{E_2} \left( \frac{E_4(\bar{\tau})}{E_4(\tau)} \right)^{1/4}. \quad \text{(4.12)}$$

Since $\left( \frac{E_4(\bar{\tau})}{E_4(\tau)} \right)^{1/4} (\bar{\tau} - \tau)$ is a modular function of weight $-2$, it cannot have any explicit $E_2$ dependence. Using this fact in the right hand side of (4.12) means that (4.12) is equivalent to:

$$\partial_{E_2} (\bar{\tau} - \tau) = -\frac{2\pi i}{12} (\bar{\tau} - \tau)^2. \quad \text{(4.13)}$$

Thus proving (4.13) is equivalent to establishing the recurrence relation. Let $H = \bar{\tau} - \tau$, then under the modular transformation (3.4),

$$H = (\bar{\tau} - \tau) \rightarrow \frac{(\bar{\tau} - \tau)}{(c\tau + d)(c\bar{\tau} + d)} = \frac{H/(c\tau + d)^2}{1 + cH/(c\tau + d)}. \quad \text{(4.14)}$$

Let $h$ be defined by

$$H = \frac{h}{1 + \frac{2\pi i}{12} E_2 h}. \quad \text{(4.15)}$$

Then (4.14) and (4.2) implies that $h \rightarrow (c\tau + d)^{-2} h$, i.e. it is a modular function of weight $-2$. It therefore cannot have any explicit $E_2$ dependence, and so from (4.13) one has:

$$\partial_{E_2} H = -\frac{2\pi i}{12} \frac{h^2}{(1 + \frac{2\pi i}{12} E_2 h)^2} = -\frac{2\pi i}{12} H^2. \quad \text{(4.16)}$$

This establishes (4.13), and hence proves the recurrence relation.
5. Asymptotic degeneracies of BPS states

An important question is the asymptotic behaviour for the degeneracies of BPS states, \( N_{n_1,n_2} \), as \( n_1 \) and \( n_2 \) approach large values in (3.7). By knowing the behaviour of the degeneracies one can compare the entropy with that of other interesting physical systems.

The contributions of multicoverings becomes negligible in the asymptotic limit, so to a very good approximation

\[
N_{n_1,n_2} = (-1)^{n_1+1} F_{n_1,n_2} / n_1^3
\]  
(5.1)

where \( F_{n_1,n_2} = G_{n_1,n_2+n_1/2} \) is the coefficient of \( q^{n_2} \) in \( F_{n_1} \). The coefficient \( G_{n,m} \) is given by

\[
G_{n,m} = \frac{1}{2\pi i} \oint \frac{dw}{w^{m+1}} G_n(\tau),
\]  
(5.2)

where \( w = e^{2\pi i \tau} \), and the integral is taken around any circle centered on the origin in the \( w \)-plane. We now follow the method in [11] of making a saddle point approximation of this integral for large values of \( m \). To do this we first recall the normalization requirement that \( G_n \sim 1.q^{-n/2} \) as \( q \to 0 \). To get a well-behaved and consistent approximation, one first makes a modular inversion, and considers the limit in which \( \tau \) is small. Recall that \( E_4 \) and \( E_6 \) are modular forms of weight 4 and 6, \( E_2 \) transforms as in (4.2), and that \( \Delta^{1/2}(-1/\tau) = -\tau^6 \Delta^{1/2}(\tau) \). If one keeps only the leading terms in \( e^{-2\pi i / \tau} \), one can neglect the anomalous modular pieces of the transformation of \( E_2 \), and so with the exception of the sign in the the transformation of \( \Delta^{1/2} \), \( G_n \) to leading order behaves like a modular function of weight \(-2\). Keeping only the leading term in the \( q \)-expansion of \( G_n \) then yields:

\[
G_{n,m} \approx \frac{1}{2\pi i} \oint \frac{dw}{w^{m+1}} (-1)^n \left( -\frac{2\pi i}{\log w} \right)^2 \exp \left( \frac{-4\pi^2 n}{2 \log w} \right).
\]  
(5.3)

The saddle point is at \( \tau = i \sqrt{\frac{m}{2}} \), which is indeed small as \( m \) becomes large. Continuing in the usual way with the standard saddle point approximation, we find

\[
G_{n,m} \approx -(-1)^n \frac{n^{5/4}}{(2m)^{7/4}} \exp \left( 2\pi \sqrt{4nm} \right).
\]  
(5.4)

Hence, the asymptotic approximation for \( N_{n_1,n_2} \) is

\[
N_{n_1,n_2} \approx (n_1^2 + 2n_1 n_2)^{-7/4} \exp \left( 2\pi \sqrt{n_1^2 + 2n_1 n_2} \right),
\]  
(5.5)

which is valid for both \( n_1 \) and \( n_2 \) large.
We can also consider the limit where \( n_1 \) is large and \( n_2 \) is zero. From (5.5), the naive behaviour is

\[
N_{n_1,0} \sim (n_1)^{-7/2} \exp(2\pi n_1) .
\]  

(5.6)

Let us compare this with the exact asymptotic behaviour. We can project onto the \( n_2 = 0 \) states by letting \( \tau \rightarrow i\infty \). This reduces the model to the case considered in [6] and [7]. Now the instanton expansion is given by

\[
\varphi_D = -\frac{1}{2} \varphi^2 + \frac{1}{2} \varphi + \frac{5}{12} - \sum_{n=1} C_n \frac{e^{2\pi in\varphi}}{4\pi^2 n^2} , \quad \partial^3 \varphi_D = \sum_{n=1} 2\pi i n C_n e^{2\pi in\varphi} .
\]  

(5.7)

For the asymptotic expansion we can again ignore the multi-coverings. Thus the degeneracy \( d_n \) is

\[
d_n \approx -(-1)^n \frac{C_n}{n^3} .
\]  

(5.8)

The instanton expansion diverges at the conifold point \( u = 1 \). Near this point \( \varphi \) and \( \varphi_D \) are approximately [6]:

\[
\varphi = c - \frac{i}{2\pi} (\varphi_D - 1) \log(u - 1) + ... \\
\varphi_D = 1 + \frac{1}{2\pi} (u - 1) .
\]  

(5.9)

So to a good approximation we have

\[
\varphi_D = 1 + 2\pi i (\varphi - c)/\log(\varphi - c) + ... \\
\partial^3 \varphi_D = \frac{2\pi i}{((\varphi - c) \log(\varphi - c))^2} + ...
\]  

(5.10)

Hence, when \( \varphi \) is close to \( c \), we have

\[
\sum_n 2\pi i n^4 (-1)^{n+1} d_n e^{2\pi in\varphi} \approx \int dn 2\pi i n^4 (-1)^{n+1} d_n e^{2\pi in\varphi} \approx \frac{2\pi i}{(\varphi - c) \log(\varphi - c))^2} .
\]  

(5.11)

If we use the ansatz that \( d_n \sim n^a (\log n)^b e^{-2\pi i c n} \), then a simple saddle point calculation yields

\[
d_n \approx (2\pi)^{3/2} n^{-3} (\log n)^{-2} (-1)^n e^{-2\pi i c n + 1} .
\]  

(5.12)

To find \( c \) note that we have to match the behaviour of \( \varphi \) and \( \varphi_D \) at the conifold point with their behaviour at the \( E_8 \) point \( u = 0 \). The behaviour is matched by noting that [6]

\[
\varphi_D = \kappa \int \frac{du}{u} \left( \xi F_0(u) + \frac{1}{\xi} F_1(u) \right) \\
\varphi = \kappa \int \frac{du}{u} \left( e^{2\pi i/3} \xi F_0(u) + e^{-2\pi i/3} \frac{1}{\xi} F_1(u) \right) ,
\]  

(5.13)
where \( \kappa = i \frac{3^{1/4}}{(4\pi^{3/2})} \) and \( \xi = -i \frac{3^{1/4} \Gamma(1/3)^3}{(2^{2/3} \pi^{3/2})} \) and

\[
F_0(u) = u^{1/6} 2F_1\left(\frac{1}{6}, \frac{1}{6}; \frac{1}{3}; u\right) \quad F_1(u) = u^{5/6} 2F_1\left(\frac{5}{6}, \frac{5}{6}; \frac{5}{3}; u\right).
\]

To find the integration constants, we know that as \( u \to 0, \varphi \to 0 \) and \( \varphi_D \to 0 \). Hence, the integrals in (5.13) are integrated from \( u = 0 \) to \( u \). Therefore, \( c \) is given by

\[
c = \kappa \int_0^1 \frac{du}{u} \left( e^{2\pi i/3} \xi F_0(u) + \frac{e^{-2\pi i/3}}{\xi} F_1(u) \right) \\
= -\frac{1}{2} + i \left( \frac{9\Gamma(1/3)^3}{2^{5/3} \pi^{3}} \right) 3F_2\left(\frac{1}{6}, \frac{1}{6}, \frac{1}{3}, \frac{7}{6}; 1 - \frac{\sqrt{3}}{2} \right),
\]

where \( 3F_2 \) is a generalized hypergeometric function evaluated at \( u = 1 \). Hence we find that

\[
d_n \approx (2\pi)^{3/2} n^{-3}(\log n)^{-2} \exp\left((2\pi - \alpha)n - 1\right).
\]

where \( \alpha \) is found from (5.15) and is approximately \( \alpha = 0.451967 \). Comparing this with (5.16), we see that the naive result is not such a terrible approximation.

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