Anisotropic Keldysh interaction

Andrei Galiautdinov
Department of Physics and Astronomy, University of Georgia, Athens, Georgia 30602, USA

(Dated: April 3, 2019)

We generalize the classic calculations by Rytova and Keldysh of screened Coulomb interaction in semiconductor thin films to systems with anisotropic permittivity tensor. Explicit asymptotic expressions for electrostatic potential energy of interaction in the weakly anisotropic case are found in closed analytical form. The case of strong in-plane anisotropy, however, requires evaluation of the inverse Fourier transform of $1/(Ak_x^2 + Bk_y^2 + C\sqrt{k_x^2 + k_y^2})$, which, at present, remains unresolved.

1. INTRODUCTION

The important role played by the dielectric screening in determining excitonic properties of various two-dimensional semiconductor heterostructures has been the subject of numerous investigations over the last several decades (for recent studies see, for example, [1–13]). A particularly interesting direction of current experimental research involves perovskite chalcogenide films whose in-plane dielectric anisotropy gives rise to some rather unusual optical behavior [14, 15]. Past theoretical work on two-dimensional dielectric screening involved various ab initio calculations [1, 2], the use of the nonlinear Thomas-Fermi model [4], [7], the modified Mott-Wannier approach [8], the transfer matrix method [13], as well as various approaches based on effective mass approximation [6, 10]. Here we pursue what is likely the simplest possibility — generalization to anisotropic films of classic calculations by Rytova [16] and Keldysh [17, 18]. The motivation for this approach is rather obvious: we want to get a better sense of how the famous isotropic form of screened electrostatic interaction energy,

$$V(\rho) = (\pi q q'/\epsilon d)\left[H_0(\rho/\rho_0) - Y_0(\rho/\rho_0)\right],$$

is modified under the minimal number of microscopic assumptions. In what follows, we bring our analytical calculation to its logical conclusion in the weakly anisotropic case only. Interested readers are invited to improve on that calculation by exploring the strongly anisotropic scenario.

2. GENERAL CONSIDERATIONS

The electrostatic potential energy of interaction between the charges $q$ and $q'$ located at $(\rho, z)$ and $(0, z')$ ($z > z'$, $\rho = (x, y)$) inside an anisotropic semiconductor film is given by (see Appendix for derivation; compare with [16, 17])

$$V(\rho, z, z') = \int \frac{d^2k}{(2\pi)^2} e^{i\mathbf{k}\cdot\mathbf{\rho}} \mathcal{V}(\mathbf{k}, z, z'),$$

with

$$\mathcal{V}(\mathbf{k}, z, z') = \frac{4\pi q q'}{\epsilon_{2z}} \frac{\cosh[k(\frac{d}{2} - z) + \tilde{\eta}_3] \cosh[k(\frac{d}{2} + z') + \tilde{\eta}_1]}{k \sinh[kd + \tilde{\eta}_1 + \tilde{\eta}_3]},$$

where

$$\tilde{\eta}_{1,3} = \frac{1}{2} \ln\left(\frac{\epsilon_{2z}k + \epsilon_{1,3}k}{\epsilon_{2z}k - \epsilon_{1,3}k}\right), \quad \tilde{k} = \sqrt{\frac{\epsilon_{2z}k_x^2 + \epsilon_{2y}k_y^2}{\epsilon_{2z}}}, \quad k = \sqrt{k_x^2 + k_y^2},$$

and the axes of the coordinate system $(x, y, z)$ coincide with the principal axes of the film’s dielectric permittivity tensor. In the most interesting for practical applications scenario, $\epsilon_{1,3} \ll \epsilon_{2z}, \epsilon_{2z}, \epsilon_{2z}$. For distances $|z - z'| \sim d \ll \rho,$

*Electronic address: ag@physast.uga.edu
the main contribution to the integral in (2) comes from \( k \) satisfying \( k \rho \ll 1 \). Under these conditions, \( \tilde{k}d \ll 1 \), \( \tilde{\eta}_{1,3} \approx \epsilon_{1,3}k/(\epsilon_{2,3}k) \), and, with the dependence on \( z \) and \( z' \) disappearing, we get the two-dimensional form of the interaction,

\[
V(\rho) = \frac{4\pi q q'}{(2\pi)^2 d} \int \frac{dk_x dk_y e^{i(k_x x + k_y y)}}{\epsilon_{2x} k_x^2 + \epsilon_{2y} k_y^2 + (\epsilon_1 + \epsilon_3) \sqrt{k_x^2 + k_y^2}}.
\]

The problem thus reduces to the calculation of the two-dimensional Fourier integral,

\[
F(x, y) = \int \frac{dk_x dk_y}{(2\pi)^2} \frac{e^{i(k_x x + k_y y)}}{Ak_x^2 + Bk_y^2 + C \sqrt{k_x^2 + k_y^2}},
\]

with \( A, B, C > 0 \). To that end, working in polar coordinates, we write,

\[
V(\rho) = \frac{qq'}{\pi d} \int_0^\infty \int_0^{2\pi} dt d\theta e^{it \cos \theta} e^{it \cos \theta} \left[ \epsilon_{2x} \cos^2(\theta + \alpha) + \epsilon_{2y} \sin^2(\theta + \alpha) \right] + (\epsilon_1 + \epsilon_3) \frac{\rho}{\tilde{\rho}_0},
\]

where \( \alpha \) is the angle between the position vector \( \rho \) and the positive \( x \)-axis, as shown in Fig. 1. Then,

\[
V(\rho) = \frac{qq'}{\pi d} \int_0^\infty \int_0^{2\pi} dt d\theta \left[ \epsilon_{2x} \cos^2(\theta + \alpha) + \epsilon_{2y} \{1 - \cos^2(\theta + \alpha)\} \right] + (\epsilon_1 + \epsilon_3) \frac{\rho}{\tilde{\rho}_0},
\]

\[
= \frac{qq'}{\pi d} \int_0^\infty \int_0^{2\pi} dt d\theta \left[ \epsilon_{2x} \cos^2(\theta + \alpha) + \epsilon_{2y} \{1 - \cos^2(\theta + \alpha)\} \right] + (\epsilon_1 + \epsilon_3) \frac{\rho}{\tilde{\rho}_0},
\]

where

\[
I(t, \alpha, a, b) = \int_0^{2\pi} d\theta e^{it \cos \theta} / \{1 - \epsilon^2(t, a, b) \cos^2(\theta + \alpha)\},
\]

and

\[
\epsilon^2(t, a, b) = \frac{at}{t + b}, \quad a \equiv 1 - \frac{\epsilon_{2x}}{\epsilon_{2y}}, \quad b \equiv \frac{\rho}{\rho_0}, \quad \rho_0 = \frac{\epsilon_{2y} d}{\epsilon_1 + \epsilon_3}.
\]
A typical graph of the function $I(t, \alpha, a, b)$ is shown in Fig. 2.

If we assume that $\epsilon_2y \geq \epsilon_2x > 0$, then, since $t \geq 0$, we have $1 > \epsilon^2(t) \geq 0$. Taking into account the well-known Fourier series expansion [19],

$$\frac{1}{1 - \epsilon^2 \cos^2 \chi} = \frac{1}{\sqrt{1 - \epsilon^2}} \left[ 1 + 2 \sum_{n=1}^{\infty} \left( \frac{\epsilon}{1 + \sqrt{1 - \epsilon^2}} \right)^{2n} \cos(2n\chi) \right],$$

and using the fact that

$$\int_0^{2\pi} d\theta e^{it \cos \theta} \sin(2n\theta) = 0,$$

we get for the $\theta$-integral in (8) the asymptotic series,

$$I(t, \alpha, a, b) \equiv \int_0^{2\pi} d\theta e^{it \cos \theta} \frac{1}{1 - \epsilon^2(t) \cos^2(\theta + \alpha)}$$

$$= \frac{1}{\sqrt{1 - \epsilon^2(t)}} \left[ \int_0^{2\pi} d\theta e^{it \cos \theta} + 2 \sum_{n=1}^{\infty} \left( \frac{\epsilon(t)}{1 + \sqrt{1 - \epsilon^2(t)}} \right)^2 \left( \int_0^{2\pi} d\theta e^{it \cos \theta} \cos(2n\theta) \right) \cos(2n\alpha) \right]$$

$$= \frac{2\pi}{\sqrt{1 - \epsilon^2(t)}} \left[ J_0(t) + 2 \left( \frac{\epsilon(t)}{1 + \sqrt{1 - \epsilon^2(t)}} \right)^2 tJ_0(t) - 2J_1(t) \cos(2\alpha) \right]$$

$$+ 2 \left( \frac{\epsilon(t)}{1 + \sqrt{1 - \epsilon^2(t)}} \right)^4 t \left( t^2 - 24 \right) J_0(t) - 8 \left( t^2 - 6 \right) J_1(t) \cos(4\alpha)$$

$$+ 2 \left( \frac{\epsilon(t)}{1 + \sqrt{1 - \epsilon^2(t)}} \right)^6 t \left( t^4 - 144t^2 + 1920 \right) J_0(t) - 6 \left( 3t^4 - 128t^2 + 640 \right) J_1(t) \cos(6\alpha) + \ldots ,$$

where $J$ are the Bessel functions of the first kind.
3. WEAK ANISOTROPY

Treating $\varepsilon^2(t,a,b)$, with $a \ll 1$, as a small parameter in (13), we get in lowest order,

$$I(t) \approx 2\pi \left[ J_0(t) + \frac{\varepsilon^2(t)}{2} J_0 + \frac{\varepsilon^2(t)}{2} \left( J_0(t) - \frac{2J_1(t)}{t} \right) \cos(2\alpha) \right],$$

(14)

and, thus,

$$V(b,\alpha) = \frac{2qq'}{\epsilon_{2y}d} \int_0^\infty dt \left\{ J_0(t) + \frac{a}{2} \left[ \frac{tJ_0(t)}{t+b} + \frac{tJ_0(t)-2J_1(t)}{(t+b)^2} \cos(2\alpha) \right] \right\}.$$  

(15)

Performing the remaining $t$-integration, we find,

$$V(b,\alpha) = V_0(b) + V_1(b,\alpha),$$

(16)

where

$$V_0(b,\alpha) = \frac{\pi qq'}{\epsilon_{2y}d} [H_0(b) - Y_0(b)]$$

(17)

is the standard Keldysh-Rytova result, and

$$V_1(b,\alpha) = \frac{a qq'}{\epsilon_{2y}d} \left\{ \frac{\cos(2\alpha)}{b^2} \left[ 2b^3 - \pi b \left( (b^2 - 2) H_1(b) + bH_0(b) \right) + \pi b^2 Y_0(b) + \pi (b^2 - 2) bY_1(b) - 4 \right] 
- \pi bH_1(b) + \pi H_0(b) + 2b + \pi bY_1(b) - \pi Y_0(b) \right\}$$

(18)

is the linear correction whose graph is shown in Fig. 3. (In the above, various $H_i$ and $Y_i$ denote the Struve and Neumann functions, respectively.)

![Graph of $V_1$ in units of $qq'/\epsilon_{2y}d$, with $a = 0.1$, in the weakly anisotropic case, Eq. (18).](image)

FIG. 3: Graph of $V_1$ in units of $qq'/\epsilon_{2y}d$, with $a = 0.1$, in the weakly anisotropic case, Eq. (18).

For $b \ll 1$, or $d \ll \rho \ll \epsilon_{2y}d/\epsilon_1 + \epsilon_2$, we get

$$V_0 = \frac{2qq'}{\epsilon_{2y}d} \left( \log \left( \frac{2}{b} \right) - \gamma \right) = \frac{2qq'}{\epsilon_{2y}d} \left[ \ln \left( \frac{2\epsilon_{2y} d}{\epsilon_1 + \epsilon_3 \rho} \right) - \gamma \right],$$

(19)

$$V_1 = \frac{a qq'}{\epsilon_{2y}d} \left( \log \left( \frac{2}{b} \right) - \gamma - 1 - \frac{\cos(2\alpha)}{2} \right) = \frac{qq'}{\epsilon_{2y}d} \left( 1 - \frac{\epsilon_{2y}}{\epsilon_{2y}} \right) \left[ \ln \left( \frac{2\epsilon_{2y} d}{\epsilon_1 + \epsilon_3 \rho} \right) - \gamma - 1 - \frac{\cos(2\alpha)}{2} \right],$$

(20)
where $\gamma \approx 0.577216$ is the Euler constant. Since $\int_0^{2\pi} \cos(2\alpha) d\alpha = 0$, the excitonic ground state energy in this case experiences a simple first order shift,

$$\Delta E_0 \sim a \int_0^{\infty} \left( \log \left( \frac{2}{b} \right) - \gamma - 1 \right) |\psi_0(b)|^2 b \, db,$$

(21)

where $\psi_0(b)$ is the unperturbed axially symmetric ground state wave function. Also, for $b \gg 1$, or $\rho \gg \epsilon_2 d/ (\epsilon_1 + \epsilon_3)$, Eq. (18) reproduces the standard Coulomb asymptotics,

$$V(\rho) = \frac{2qq'}{\epsilon_1 + \epsilon_3 \rho}.$$  

(22)

4. SUMMARY

The classic Keldysh-Rytova formula for screened Coulomb interaction in semiconductor thin films has been generalized by taking into account the anisotropy of the layer’s dielectric permittivity tensor. The Fourier image of the anisotropic potential in momentum space, as well as the linear correction to the isotropic potential in real space, have been worked out in closed analytical form. The case of strong in-plane anisotropy, however, remains unresolved due to the appearance of the function $I(t, \alpha, a, b)$ (see Eq. (8); compare with Eq. (13)), whose explicit analytical expression is not known.

APPENDIX: Momentum space representation

Following [16] and [17], we consider a geometry in which the anisotropic semiconductor film occupies the region of space $-d/2 \leq z \leq d/2$, as shown in Fig. 1. The half-space $z < -d/2$ (the substrate) is filled with an isotropic medium whose dielectric constant is $\epsilon_1$, while the half-space $z > d/2$ with an isotropic medium whose dielectric constant is $\epsilon_3$.

We are assuming that the axes of the coordinate system $(x, y, z)$ coincide with the principal axes of the film’s dielectric permittivity tensor. The electrostatic potential at point $r \equiv (\rho, z) = (x, y, z)$ due to charge $q'$ located at $r' = (0, 0, z')$ satisfies (in regions 3, 2, and 1, respectively) the following system of equations:

$$\left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \phi_3(r, r') = 0,$$

(23)

$$\left( \epsilon_2 x \frac{\partial^2}{\partial x^2} + \epsilon_2 \frac{\partial^2}{\partial y^2} + \epsilon_3 \frac{\partial^2}{\partial z^2} \right) \phi_2(r, r') = -4\pi q' \delta(r-r'),$$

(24)

$$\left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \phi_1(r, r') = 0,$$

(25)

with the boundary conditions at the interfaces,

$$(\phi_3 - \phi_2)_{z=\frac{d}{2}} = 0,$$

(26)

$$(\phi_2 - \phi_1)_{z=-\frac{d}{2}} = 0,$$

(27)

and the boundary conditions at the two infinities,

$$\phi_3|_{z \to +\infty} = 0, \quad \phi_1|_{z \to -\infty} = 0.$$  

(28)

Fourier transforming,

$$\phi_{3,2,1}(r, r') = \int \frac{d^2k}{(2\pi)^2} e^{ik\cdot\rho} \phi_{3,2,1}(k, z, z'),$$

(29)
and substituting into (23), (24), and (25), we get the following equations for the corresponding Fourier components,

\[
\frac{\partial^2}{\partial z^2} - k^2 \phi_3(k, z, z') = 0, \tag{30}
\]

\[
\frac{\partial^2}{\partial z^2} - \tilde{k}^2 \phi_2(\tilde{k}, z, z') = -\frac{4\pi q'}{\epsilon_{2z}} \delta(z - z'), \tag{31}
\]

\[
\frac{\partial^2}{\partial z^2} - k^2 \phi_1(k, z, z') = 0, \tag{32}
\]

where

\[
k \equiv \sqrt{k_x^2 + k_y^2}, \quad \tilde{k} \equiv \sqrt{\frac{\epsilon_{2x}k_x^2 + \epsilon_{2y}k_y^2}{\epsilon_{2z}}}. \tag{33}
\]

Conditions at infinity, (28), combined with Eqs. (30) and (32) give

\[
\phi_3(k) = A_3(z') e^{-kz}, \quad \frac{\partial \phi_3(k)}{\partial z} = -kA_3(z') e^{-kz}, \tag{34}
\]

\[
\phi_1(k) = A_1(z') e^{kz}, \quad \frac{\partial \phi_1(k)}{\partial z} = kA_1(z') e^{kz}, \tag{35}
\]

while Eq. (31) gives, for \( z \neq z' \),

\[
\phi_2(\tilde{k}, z \neq z') = A_2(z') \tilde{e}^{\tilde{k}z} + B_2(z') \tilde{e}^{-\tilde{k}z}, \tag{36}
\]

and, at \( z = z' \), the jump discontinuity in the \( z \)-derivative,

\[
\frac{\partial \phi_2(\tilde{k})}{\partial z} \bigg|_{z=z'-0}^{z=z'+0} = -\frac{4\pi q'}{\epsilon_{2z}}. \tag{37}
\]

Imposing the boundary conditions at the \( z = +d/2 \) interface,

\[
A_3(z') e^{-kd/2} = A_2(z') \tilde{e}^{kd/2} + B_2(z') e^{-kd/2}, \tag{38}
\]

\[
-\epsilon_3kA_3(z') e^{-kd/2} = \epsilon_{2z} \tilde{k} \left[ A_2(z') \tilde{e}^{kd/2} - B_2(z') e^{-kd/2} \right], \tag{39}
\]

we get

\[
\phi_2(\tilde{k}) \big|_{z > z'} = A_2(z') \left[ e^{\tilde{k}z} + e^{2\tilde{q}_3 + \tilde{k}(d-z)} \right], \tag{40}
\]
where

\[ \tilde{\eta}_3 = \frac{1}{2} \ln \left( \frac{\epsilon_{2z} \tilde{k} + \epsilon_3 k}{\epsilon_{2z} \tilde{k} - \epsilon_3 k} \right). \]  

(41)

Similarly, imposing the boundary conditions at the \( z = -d/2 \) interface, we get

\[ \tilde{A}_2(z') e^{-\tilde{k}d/2} + \tilde{B}_2(z') e^{\tilde{k}d/2} = A_1(z') e^{-kd/2}, \]  

(42)

\[ \epsilon_{2z} \tilde{k} \left[ \tilde{A}_2(z') e^{-\tilde{k}d/2} - \tilde{B}_2(z') e^{\tilde{k}d/2} \right] = \epsilon_1 k A_1(z') e^{-kd/2}, \]  

(43)

and, after defining

\[ \tilde{\eta}_1 = \frac{1}{2} \ln \left( \frac{\epsilon_{2z} \tilde{k} + \epsilon_1 k}{\epsilon_{2z} \tilde{k} - \epsilon_1 k} \right), \]  

(44)

find

\[ \phi_2(\tilde{k})|_{z < z'} = \tilde{A}_2(z') \left[ e^{\tilde{k}z} + e^{-2\tilde{\eta}_1 - \tilde{k}(d+z)} \right]. \]  

(45)

Now, for \( z = z' \), Eqs. (37), (40), and (45) give

\[ \tilde{A}_2(z') \left[ e^{\tilde{k}z'} + e^{2\tilde{\eta}_1 + \tilde{k}(d-z')} \right] - \tilde{A}_2(z') \left[ e^{\tilde{k}z'} + e^{-2\tilde{\eta}_1 - \tilde{k}(d+z')} \right] = 0, \]  

(46)

\[ \tilde{k} A_2(z') \left[ e^{\tilde{k}z'} - e^{2\tilde{\eta}_1 + \tilde{k}(d-z')} \right] - \tilde{k} \tilde{A}_2(z') \left[ e^{\tilde{k}z'} - e^{-2\tilde{\eta}_1 - \tilde{k}(d+z')} \right] = \frac{4\pi q'}{\epsilon_{2z}}, \]  

(47)

resulting in

\[ \tilde{A}_2(z') = A_2(z') \frac{e^{\tilde{k}z'} + e^{2\tilde{\eta}_1 + \tilde{k}(d-z')}}{e^{\tilde{k}z'} + e^{-2\tilde{\eta}_1 - \tilde{k}(d+z')}}, \]  

(48)

and

\[ A_2(z') = \frac{4\pi q' e^{-\tilde{k}d/2 - \tilde{\eta}_3}}{\epsilon_{2z} \tilde{k}} \frac{\cosh \left[ \tilde{k}(d/2 + z') + \tilde{\eta}_1 \right]}{2 \sinh \left[ kd + \tilde{\eta}_1 + \tilde{\eta}_3 \right]}. \]  

(49)

Taking into account

\[ e^{\tilde{k}z} + e^{2\tilde{\eta}_3 + \tilde{k}(d-z)} = 2 e^{\tilde{k}d/2 + \tilde{\eta}_3} \cosh \left[ \tilde{k}(d/2 - z) + \tilde{\eta}_3 \right], \]  

(50)

we get

\[ \phi_2(\tilde{k}, z > z') = \frac{4\pi q'}{\epsilon_{2z}} \frac{\cosh \left[ \tilde{k} \left( \frac{d}{2} - z \right) + \tilde{\eta}_3 \right] \cosh \left[ \tilde{k} \left( \frac{d}{2} + z' \right) + \tilde{\eta}_1 \right]}{\tilde{k} \sinh \left[ kd + \tilde{\eta}_1 + \tilde{\eta}_3 \right]}. \]  

(51)

For \( z < z' \), a similar calculation results in

\[ \phi_2(\tilde{k}, z < z') = \frac{4\pi q'}{\epsilon_{2z}} \frac{\cosh \left[ \tilde{k} \left( \frac{d}{2} - z' \right) + \tilde{\eta}_3 \right] \cosh \left[ \tilde{k} \left( \frac{d}{2} + z \right) + \tilde{\eta}_1 \right]}{\tilde{k} \sinh \left[ kd + \tilde{\eta}_1 + \tilde{\eta}_3 \right]}. \]  

(52)

[1] P. Cudazzo, C. Attaccalite, I. V. Tokatly, and A. Rubio, “Strong Charge-Transfer Excitonic Effects and the Bose-Einstein Exciton Condensate in Graphane,” Phys. Rev. Lett. 104, 226804 (2010).
[2] P. Cudazzo, I. V. Tokatly, and A. Rubio, “Dielectric screening in two-dimensional insulators: Implications for exciton and impurity states in graphane,” Phys. Rev. B 84, 085406 (2011).

[3] A. Chernikov, T. C. Berkelbach, H. M. Hill, A. Rigosi, Y. Li, O. B. Aslan, D. R. Reichman, M. S. Hybertsen, and T. F. Heinz, “Exciton Binding Energy and Nonhydrogenic Rydberg Series in Monolayer WS$_2$,” Phys. Rev. Lett. 113, 076802 (2014).

[4] T. Low, R. Roldán, H. Wang, F. Xia, P. Avouris, L. M. Moreno, and F. Guinea, “Plasmons and Screening in Monolayer and Multilayer Black Phosphorus,” Phys. Rev. Lett. 113, 106802 (2014).

[5] X. Wang, A. M. Jones, K. L. Seyler, V. Tran, Y. Jia, H. Zhao, H. Wang, L. Yang, X. Xu and F. Xia, “Highly anisotropic and robust excitons in monolayer black phosphorus,” Nature Nanotechnology 10, 517 (2015).

[6] A. Chaves, T. Low, P. Avouris, D. Cahkri, and F. M. Peeters, “Anisotropic exciton Stark shift in black phosphorus,” Phys. Rev. B 92, 155311 (2015).

[7] S. Latini, T. Olsen, and K. S. Thygesen, “Excitons in van der Waals heterostructures: The important role of dielectric screening,” Phys. Rev. B 92, 245123 (2015).

[8] T. G. Pedersen, S. Latini, K. S. Thygesen, H. Mera and B. K. Nikolić, “Exciton ionization in multilayer transition-metal dichalcogenides,” New J. Phys. 18 (2016).

[9] M. L. Trolle, T. G. Pedersen and V. Véniard, “Model dielectric function for 2D semiconductors including substrate screening,” Nature Sci. Rep. 7, 39844 (2017).

[10] A. Hichri, I. B. Amara, S. Ayari and S. Jaziri, “Dielectric environment and/or random disorder effects on free, charged and localized excitonic states in monolayer WS$_2$,” J. Phys.: Condens. Matter 29, 435305 (2017).

[11] M. Szyniszewski, E. Mostaani, N. D. Drummond, and V. I. Falko, “Binding energies of trions and biexcitons in two-dimensional semiconductors from diffusion quantum Monte Carlo calculations,” Phys. Rev. B 95, 081301(R) (2017).

[12] E. Mostaani, M. Szyniszewski, C. H. Price, R. Maezono, M. Danovich, R. J. Hunt, N. D. Drummond, V. I. Falko, “Diffusion quantum Monte Carlo study of excitonic complexes in two-dimensional transition-metal dichalcogenides,” Phys. Rev. B 96, 075431 (2017).

[13] L. S. R. Cavalcante, A. Chaves, B. Van Duppen, F. M. Peeters, and D. R. Reichman, “Electrostatics of electron-hole interactions in van der Waals heterostructures,” Phys. Rev. B 97, 125427 (2018).

[14] Sh. Niu, G. Joe, H. Zhao, Y. Zhou, T. Orvis, H. Huyan, J. Salman, K. Mahalingam, B. Urwin, J. Wu, Y. Liu, T. E. Tiwald, S. B. Cronin, B. M. Howe, M. Mecklenburg, R. Haiges, D. J. Singh, H. Wang, M. A. Kats and J. Ravichandran, “Giant optical anisotropy in a quasi-one-dimensional crystal,” Nature Photonics 12, 392 (2018).

[15] Sh. Niu, H. Zhao, Y. Zhou, H. Huyan, B. Zhao, J. Wu, S. B. Cronin, H. Wang, and J. Ravichandran, “Mid-wave and Long-Wave Infrared Linear Dichroism in a Hexagonal Perovskite Chalcogenide,” Chem. Mater. 30 (15), 4897 (2018).

[16] N. S. Rytova, “Screened potential of a point charge in a thin film,” Vestn. Mosk. Univ. Fiz. Astron. 3, 30 (1967).

[17] L. V. Keldysh, “Coulomb interaction in thin semiconductor and semimetal films,” Pis’ma Zh. Eksp. Teor. Fiz. 29, 716 (1979) [JETP Lett. 29, 658 (1979)].

[18] L. V. Keldysh, “Excitons in Semiconductor-Dielectric Nanostructures,” Phys. Stat. Sol. (a) 164, 3 (1997).

[19] S. G. Mikhlin, Integral Equations, Second Revised Edition (Pergamon Press, 1964).