TIME DISCRETIZATION OF A NONLOCAL PHASE-FIELD SYSTEM WITH INERTIAL TERM

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Time discretizations of phase-field systems have been studied. For example, a time discretization, an error estimate for a parabolic-parabolic phase-field system have been studied by Colli–K. [Commun. Pure Appl. Anal. 18 (2019)]. Also, a time discretization and an error estimate for a simultaneous abstract evolution equation applying parabolic-hyperbolic phase field systems and the linearized equations of coupled sound and heat flow have been studied (see K. [ESAIM Math. Model. Numer. Anal. 54 (2020), Electron. J. Differential Equations 2020, Paper No. 96]). On the other hand, although existence, continuous dependence estimates, behavior of solutions to nonlocal phase-field systems with inertial terms have been studied by Grasselli–Petzeltová–Schimperna [Quart. Appl. Math. 65 (2007)], time discretizations of these systems seem to be not studied yet. In this paper we focus on employing a time discretization scheme for a nonlocal phase-field system with inertial term and establishing an error estimate for the difference between continuous and discrete solutions.

Received on March 9, 2021

AMS 2010 Subject Classification: 35A35, 35G30, 80A22

Keywords: time discretizations, nonlocal phase field systems, inertial terms, existence, error estimates

S.K. is partially supported by JSPS Research Fellowships for Young Scientists, No. 18J21006.
1. Introduction

Time discretizations of phase-field systems have been studied. For example, for the classical phase-field model proposed by Caginalp (cf. [2, 4]; one may also see the monographs [1, 5, 16])

\[
\begin{align*}
\theta_t + \varphi_t - \Delta \theta &= f &\text{in } \Omega \times (0, T), \\
\varphi_t - \Delta \varphi + \beta(\varphi) + \pi(\varphi) &= \theta &\text{in } \Omega \times (0, T),
\end{align*}
\]

(E1)

Colli–K. [3] have studied a time discretization and an error estimate, where \(\Omega\) is a domain in \(\mathbb{R}^d\) \((d \in \mathbb{N})\), \(T > 0\), \(\beta : \mathbb{R} \to \mathbb{R}\) is a maximal monotone function, \(\pi : \mathbb{R} \to \mathbb{R}\) is an anti-monotone function, \(f : \Omega \times (0, T) \to \mathbb{R}\) is a given function. Also, for a simultaneous abstract evolution equation applying the parabolic-hyperbolic phase-field system (see e.g., [6–8, 17, 18])

\[
\begin{align*}
\theta_t + \varphi_t - \Delta \theta &= f &\text{in } \Omega \times (0, T), \\
\varphi_{tt} + \varphi_t - \Delta \varphi + \beta(\varphi) + \pi(\varphi) &= \theta &\text{in } \Omega \times (0, T),
\end{align*}
\]

(E2)

a time discretization scheme has been employed and an error estimate has been derived (see [11]). Moreover, for a simultaneous abstract evolution equation applying (E2) (in the case that \(f = 0\)) and the linearized equations of coupled sound and heat flow (see e.g, Matsubara–Yokota [13])

\[
\begin{align*}
\theta_t + (\gamma - 1) \varphi_t - \sigma \Delta \theta &= 0 &\text{in } \Omega \times (0, T), \\
\varphi_{tt} - c^2 \Delta \varphi - m^2 \varphi &= -c^2 \Delta \theta &\text{in } \Omega \times (0, T),
\end{align*}
\]

(E3)

a time discretization and an error estimate have been studied, where \(c > 0\), \(\sigma > 0\), \(m \in \mathbb{R}\), \(\gamma > 1\) are constants (see [12]). On the other hand, Grasselli–Petzeltová–Schimperna [9] have derived existence, a continuous dependence estimate and behavior of solutions to the nonlocal phase-field system

\[
\begin{align*}
\theta_t + \varphi_t - \Delta \theta &= f &\text{in } \Omega \times (0, T), \\
\varphi_{tt} + \varphi_t + a(\cdot) \varphi - J * \varphi + \beta(\varphi) + \pi(\varphi) &= \theta &\text{in } \Omega \times (0, T),
\end{align*}
\]

(E4)

where \(a(x) := \int_\Omega J(x-y) \, dy\) for \(x \in \Omega\), \((J * \varphi)(x) := \int_\Omega J(x-y) \varphi(y) \, dy\) for \(x \in \Omega\), \(J : \mathbb{R}^d \to \mathbb{R}\) is a given function. However, time discretizations of (E4) seem to be not studied yet.

In this paper, for the nonlocal phase-field system with inertial term

\[
\begin{align*}
\theta_t + \varphi_t - \Delta \theta &= f &\text{in } \Omega \times (0, T), \\
\varphi_{tt} + \varphi_t + a(\cdot) \varphi - J * \varphi + \beta(\varphi) + \pi(\varphi) &= \theta &\text{in } \Omega \times (0, T), \\
\partial_\nu \theta &= 0 &\text{on } \partial \Omega \times (0, T), \\
\theta(0) = \theta_0, \varphi(0) = \varphi_0, \varphi_t(0) = v_0 &\text{in } \Omega,
\end{align*}
\]

(P)
we employ the following time discretization scheme: find \((\theta_{n+1}, \varphi_{n+1})\) such that

\[
\begin{aligned}
\frac{\theta_{n+1} - \theta_n}{h} + \frac{\varphi_{n+1} - \varphi_n}{h} - \Delta \theta_{n+1} &= f_{n+1} & \text{in } \Omega, \\
z_{n+1} + v_{n+1} + a(\cdot) \varphi_n - J(\varphi_n) + \beta(\varphi_{n+1}) + \pi(\varphi_{n+1}) &= \theta_{n+1} & \text{in } \Omega, \\
z_{n+1} &= \frac{v_{n+1} - v_n}{h}, \quad v_{n+1} = \frac{\varphi_{n+1} - \varphi_n}{h} & \text{in } \Omega, \\
\partial_{\nu} \theta_{n+1} &= 0 & \text{on } \partial \Omega
\end{aligned}
\]

for \(n = 0, \ldots, N - 1\), where \(h = \frac{T}{N}, \ N \in \mathbb{N}\) and \(f_k := \frac{1}{h} \int_{(k-1)h}^{kh} f(s) \, ds\) for \(k = 1, \ldots, N\). Here \(\Omega \subset \mathbb{R}^d (d = 1, 2, 3)\) is a bounded domain with smooth boundary \(\partial \Omega\), \(\partial_{\nu}\) denotes differentiation with respect to the outward normal of \(\partial \Omega\), \(\theta_0 : \Omega \to \mathbb{R}\), \(\varphi_0 : \Omega \to \mathbb{R}\) and \(v_0 : \Omega \to \mathbb{R}\) are given functions. Moreover, in this paper we assume that

(A1) \(J(-x) = J(x)\) for all \(x \in \mathbb{R}^d\) and \(\sup_{x \in \Omega} \int_{\Omega} |J(x - y)| \, dy < +\infty\).

(A2) \(\beta : \mathbb{R} \to \mathbb{R}\) is a single-valued maximal monotone function such that there exists a proper lower semicontinuous convex function \(\hat{\beta} : \mathbb{R} \to [0, +\infty)\) satisfying that \(\hat{\beta}(0) = 0\) and \(\beta = \partial \hat{\beta}\), where \(\partial \hat{\beta}\) is the subdifferential of \(\hat{\beta}\). Moreover, \(\beta : \mathbb{R} \to \mathbb{R}\) is local Lipschitz continuous.

(A3) \(\pi : \mathbb{R} \to \mathbb{R}\) is a Lipschitz continuous function.

(A4) \(f \in L^2(\Omega \times (0, T)), \ \theta_0 \in H^1(\Omega), \ \varphi_0, v_0 \in L^\infty(\Omega)\).

In the case that \(\beta(r) = ar^3, \ \hat{\beta}(r) = \frac{a_4}{4} r^4, \ \pi(r) = br + c\) for \(r \in \mathbb{R}\), where \(a > 0, \ b, c \in \mathbb{R}\) are some constants, the conditions (A2) and (A3) hold.

**Remark 1.1.** We see from (A2), (A4) and the definition of the subdifferential that

\[0 \leq \hat{\beta}(\varphi_0) \leq \beta(\varphi_0) \varphi_0 \in L^\infty(\Omega).\]

Let us define the Hilbert spaces

\[H := L^2(\Omega), \ V := H^1(\Omega)\]

with inner products

\[(u_1, u_2)_H := \int_\Omega u_1 u_2 \, dx \quad (u_1, u_2 \in H), \]

\[(v_1, v_2)_V := \int_\Omega \nabla v_1 \cdot \nabla v_2 \, dx + \int_\Omega v_1 v_2 \, dx \quad (v_1, v_2 \in V),\]
respectively, and with the related Hilbertian norms. Moreover, we use the notation

\[ W := \{ z \in H^2(\Omega) \mid \partial_\nu z = 0 \text{ a.e. on } \partial \Omega \}. \]

We define solutions of (P) as follows.

**Definition 1.2.** A pair \((\theta, \phi)\) with

\[
\theta \in H^1(0, T; H) \cap L^\infty(0, T; V) \cap L^2(0, T; W), \\
\phi \in W^{2,\infty}(0, T; H) \cap W^{2,2}(0, T; L^\infty(\Omega)) \cap W^{1,\infty}(0, T; L^\infty(\Omega))
\]

is called a solution of (P) if \((\theta, \phi)\) satisfies

\[
\begin{align*}
\theta_t + \phi_t - \Delta \theta &= f \quad \text{a.e. on } \Omega \times (0, T), \\
\phi_{tt} + \phi_t + a(\cdot)\phi - J^* \phi + \beta(\phi) + \pi(\phi) &= \theta \quad \text{a.e. on } \Omega \times (0, T), \\
\theta(0) &= \theta_0, \quad \phi(0) = \phi_0, \quad \phi_t(0) = \nu_0 \quad \text{a.e. on } \Omega.
\end{align*}
\]

The first main result asserts existence and uniqueness of solutions to \((P)_n\) for \(n = 0, \ldots, N - 1\).

**Theorem 1.3.** Assume that (A1)-(A4) hold. Then there exists \(h_0 \in (0, 1]\) such that for all \(h \in (0, h_0]\) there exists a unique solution \((\theta_{n+1}, \phi_{n+1})\) of \((P)_n\) satisfying

\[
\theta_{n+1} \in W, \quad \phi_{n+1} \in L^\infty(\Omega) \quad \text{for } n = 0, \ldots, N - 1.
\]

Here, setting

\[
\begin{align*}
\hat{\theta}_h(t) &:= \theta_n + \frac{\theta_{n+1} - \theta_n}{h} (t - nh), \\
\hat{\phi}_h(t) &:= \phi_n + \frac{\phi_{n+1} - \phi_n}{h} (t - nh), \\
\hat{v}_h(t) &:= \nu_n + \frac{\nu_{n+1} - \nu_n}{h} (t - nh)
\end{align*}
\]

for \(t \in [nh, (n+1)h]\), \(n = 0, \ldots, N - 1\), and

\[
\begin{align*}
\overline{\theta}_h(t) &:= \theta_{n+1}, \quad \overline{\phi}_h(t) := \phi_{n+1}, \quad \overline{\phi}_h(t) := \phi_n, \\
\overline{v}_h(t) &:= \nu_{n+1}, \quad \overline{z}_h(t) := \nu_{n+1}, \quad \overline{f}_h(t) := f_{n+1}
\end{align*}
\]

(1.4)
for $t \in (nh, (n + 1)h], n = 0, ..., N - 1$, we can rewrite $(P)_n$ as

$$
\begin{align*}
(\hat{\theta}_n)_t + (\check{\varphi}_n)_{t} - \Delta \overline{\theta}_n & = \tilde{f}_h & \text{in } \Omega \times (0,T), \\
\varphi_h + \overline{v}_h + a(\cdot) \varphi_{n} - J * \varphi_{n} + \beta(\overline{\varphi}_h) + \pi(\overline{\varphi}_n) & = \overline{\theta}_h & \text{in } \Omega \times (0,T), \\
\varphi_h = (\tilde{\varphi}_n)_t, \overline{v}_h = (\check{\varphi}_n)_t & \text{in } \Omega \times (0,T), \\
\partial_{\nu} \overline{\theta}_n & = 0 & \text{on } \partial \Omega \times (0,T), \\
\hat{\theta}_n(0) = \theta_0, \check{\varphi}_n(0) = \varphi_0, \tilde{v}_h(0) = v_0 & \text{in } \Omega.
\end{align*}
$$

We can prove the following theorem by passing to the limit in $(P)_h$ as $h \downarrow 0$ (see Section 4).

**Theorem 1.4.** Assume that (A1)-(A4) hold. Then there exists a unique solution $(\theta, \varphi)$ of $(P)$.

The following theorem is concerned with the error estimate between the solution of $(P)$ and the solution of $(P)_h$.

**Theorem 1.5.** Let $h_0$ be as in Theorem 1.3. Assume that (A1)-(A4) hold. Assume further that $f \in W^{1,1}(0,T; H)$. Then there exist constants $M > 0$ and $h_{00} \in (0, h_0)$ depending on the data such that

$$
\| \widehat{\varphi}_h - \varphi \|_{C([0,T]; H)} + \| \overline{v}_h - v \|_{L^2(0,T; H)} + \| \hat{\overline{\theta}}_n - \overline{\theta} \|_{C([0,T]; H)} + \| \hat{\theta}_n - \theta \|_{C([0,T]; H)} + \| \nabla (\overline{\theta}_n - \overline{\theta}) \|_{L^2(0,T; H)} \leq M h^{1/2}
$$

for all $h \in (0, h_{00})$, where $v = \varphi_t$.

**Remark 1.6.** From (1.1)-(1.5) we can obtain directly the following properties:

$$
\begin{align*}
\| \hat{\theta}_n \|_{L^\infty(0,T;V)} & = \max \{ \| \theta_0 \|_V, \| \overline{\theta}_n \|_{L^\infty(0,T;V)} \}, \\
\| \hat{\varphi}_n \|_{L^\infty(0,T;L^\infty(\Omega))} & = \max \{ \| \varphi_0 \|_{L^\infty(\Omega)}, \| \overline{\varphi}_n \|_{L^\infty(0,T;L^\infty(\Omega))} \}, \\
\| \hat{\varphi}_n \|_{L^\infty(0,T;L^\infty(\Omega))} & = \max \{ \| v_0 \|_{L^\infty(\Omega)}, \| \overline{v}_n \|_{L^\infty(0,T;L^\infty(\Omega))} \}, \\
\| \bar{\varphi}_n - \hat{\theta}_n \|_{L^2(0,T;H)} & = \frac{h^2}{3} \| (\hat{\theta}_n)_t \|_{L^2(0,T;H)}, \\
\| \overline{\varphi}_n - \hat{\varphi}_n \|_{L^\infty(0,T;L^\infty(\Omega))} & = h \| (\hat{\varphi}_n)_t \|_{L^\infty(0,T;L^\infty(\Omega))} = h \| \bar{\varphi}_n \|_{L^\infty(0,T;L^\infty(\Omega))}, \\
\| \overline{v}_n - \hat{v}_n \|_{L^\infty(0,T;H)} & = h \| (\hat{v}_n)_t \|_{L^\infty(0,T;H)} = h \| \bar{v}_n \|_{L^\infty(0,T;H)}, \\
h(\hat{\varphi}_n)_t & = \overline{\varphi}_n - \hat{\varphi}_n.
\end{align*}
$$
Remark 1.7. Unlike in the case of local parabolic-hyperbolic phase-field systems, we cannot establish the $L^p(0,T;V)$-estimate ($1 \leq p \leq \infty$) for $\{\varphi_h\}_h$ and cannot apply the Aubin–Lions lemma (see e.g., [15, Section 8, Corollary 4]) for $\{\varphi_h\}_h$. Thus, since $\pi : \mathbb{R} \to \mathbb{R}$ is not monotone, to obtain the strong convergence of $\{\varphi_h\}_h$ in $L^\infty(0,T;H)$, which is necessary to verify that $\pi(\varphi_h) \to \pi(\varphi)$ strongly in $L^\infty(0,T;H)$ as $h = h_j \searrow 0$ by the Lipschitz continuity of $\pi$ and the property (1.10), we will try to confirm Cauchy’s criterion for solutions of (P)$_h$ (see Lemma 3.8).

This paper is organized as follows. In Section 2 we prove existence and uniqueness of solutions to (P)$_n$ for $n = 0, \ldots, N - 1$. In Section 3 we derive a priori estimates and Cauchy’s criterion for solutions of (P)$_h$. Section 4 is devoted to the proofs of existence and uniqueness of solutions to (P) and an error estimate between the solution of (P) and the solution of (P)$_h$.

2. Existence and uniqueness for the discrete problem

In this section we will show Theorem 1.3.

Lemma 2.1. For all $g \in H$ and all $h \in (0, \frac{1}{\|\pi'\|_{L^\infty(\mathbb{R})}})$ there exists a unique solution $\varphi \in H$ of the equation

$$\varphi + h\varphi + h^2 \beta(\varphi) + h^2 \pi(\varphi) = g \quad \text{a.e. on } \Omega. \quad (2.1)$$

Proof. We set the operator $\Phi : D(\Phi) \subset H \to H$ as

$$\Phi z := h^2 \beta(z) \quad \text{for } z \in D(\Phi) := \{z \in H \mid \beta(z) \in H\}.$$

Then this operator is maximal monotone. Also, we define the operator $\Psi : H \to H$ as

$$\Psi(z) := hz + h^2 \pi(z) \quad \text{for } z \in H.$$

Then this operator is Lipschitz continuous, monotone for all $h \in (0, \frac{1}{\|\pi'\|_{L^\infty(\mathbb{R})}})$.

Thus the operator $\Phi + \Psi : D(\Phi) \subset H \to H$ is maximal monotone (see e.g., [14, Lemma IV.2.1 (p.165)]) and then for all $g \in H$ and all $h \in (0, \frac{1}{\|\pi'\|_{L^\infty(\mathbb{R})}})$ there exists a unique solution $\varphi \in D(\Phi)$ of the equation (2.1).

Proof of Theorem 1.1. We can rewrite (P)$_n$ as

$$\begin{cases}
\theta_{n+1} - h\Delta \theta_{n+1} = hf_{n+1} + \varphi_n - \varphi_{n+1} + \theta_n, \\
\varphi_{n+1} + h\varphi_{n+1} + h^2 \beta(\varphi_{n+1}) + h^2 \pi(\varphi_{n+1}) = h^2 \theta_{n+1} + \varphi_n + hv_n + h\varphi_n - h^2 a(\cdot)\varphi_n + h^2 J* \varphi_n.
\end{cases} \quad (Q)_n$$
It is enough for the proof of Theorem 1.3 to establish existence and uniqueness of solutions to (Q) in the case that \( n = 0 \). Let \( h \in (0, \frac{1}{\| \pi' \|_{L^\infty(\mathbb{R})}}) \). Then for all \( \phi \in H \) there exists a unique function \( \theta \in W \) such that

\[
\theta - h\Delta \theta = hf_1 + \phi_0 - \phi + \theta_0. \tag{2.2}
\]

Also, owing to (A1), (A4) and Lemma 2.1, for all \( \theta \in H \) there exists a unique solution \( \phi \in H \) of the equation

\[
\phi + h\phi + h^2 \beta(\phi) + h^2 \pi(\phi) = h^2 \theta + \phi_0 + hv_0 + h\phi_0 - h^2 a(\cdot) \phi_0 + h^2 J * \phi_0. \tag{2.3}
\]

Thus we can set \( \mathcal{T} : H \to H, \mathcal{U} : H \to H \) and \( \mathcal{S} : H \to H \) as

\[
\mathcal{T} \phi = \theta, \quad \mathcal{U} \phi = \phi \quad \text{for} \quad \phi, \theta \in H
\]

and

\[
\mathcal{S} = \mathcal{U} \circ \mathcal{T},
\]

respectively. Now we let \( \phi, \tilde{\phi} \in H \). Then it follows from (2.2) that

\[
\| \mathcal{T} \phi - \mathcal{U} \phi \|_H^2 + h^2 \| \nabla (\mathcal{T} \phi - \mathcal{U} \phi) \|_H^2 = - (\phi - \tilde{\phi}, \mathcal{T} \phi - \mathcal{U} \phi)_H
\]

\[
\leq \| \phi - \tilde{\phi} \|_H \| \mathcal{T} \phi - \mathcal{U} \phi \|_H
\]

and then it holds that

\[
\| \mathcal{T} \phi - \mathcal{U} \phi \|_H \leq \| \phi - \tilde{\phi} \|_H. \tag{2.4}
\]

Also we use (2.3) and (A3) to have that

\[
(1 + h) \| S \phi - S \tilde{\phi} \|_H^2 + h^2 \beta(S \phi) - \beta(S \tilde{\phi}, S \phi - S \tilde{\phi})_H
\]

\[
= h^2 (\mathcal{T} \phi - \mathcal{U} \phi, S \phi - S \tilde{\phi})_H - h^2 (\pi(S \phi) - S \tilde{\phi}, S \phi - S \tilde{\phi})_H
\]

\[
\leq h^2 \| \mathcal{T} \phi - \mathcal{U} \phi \|_H \| S \phi - S \tilde{\phi} \|_H + \| \pi' \|_{L^\infty(\mathbb{R})} h^2 \| S \phi - S \tilde{\phi} \|_H^2,
\]

whence the monotonicity of \( \beta \) leads to the inequality

\[
\| S \phi - S \tilde{\phi} \|_H \leq \frac{h^2}{1 + h - \| \pi' \|_{L^\infty(\mathbb{R})} h^2} \| \mathcal{T} \phi - \mathcal{U} \phi \|_H. \tag{2.5}
\]

Therefore combining (2.4) and (2.5) implies that

\[
\| S \phi - S \tilde{\phi} \|_H \leq \frac{h^2}{1 + h - \| \pi' \|_{L^\infty(\mathbb{R})} h^2} \| \phi - \tilde{\phi} \|_H.
\]
and hence there exists $h_{00} \in (0, \min\{1, \frac{1}{\|\pi\|_{L^\infty(\mathbb{R})}}\})$ such that

$$0 < \frac{h^2}{1 + h - \|\pi\|_{L^\infty(\mathbb{R})} h^2} < 1$$

for all $h \in (0, h_{00})$. Thus $S : H \rightarrow H$ is a contraction mapping in $H$ for all $h \in (0, h_{00})$ and then the Banach fixed-point theorem means that for all $h \in (0, h_{00})$ there exists a unique function $\phi_1 \in H$ such that $\phi_1 = S\phi_1$. Hence, for all $h \in (0, h_{00})$, by putting $\theta_1 := T\phi_1 \in W$, there exists a unique pair $(\theta_1, \phi_1) \in H \times H$ satisfying (Q)$_n$ in the case that $n = 0$. Next we verify that $\phi_1 \in L^\infty(\Omega)$. Let $h \in (0, h_{00})$. Then, noting that $g_1 := h^2\theta_1 + \phi_0 + hv_0 + h\phi_0 - h^2a(\cdot)\phi_0 + h^2J*\phi_0 \in L^\infty(\Omega)$ by $\theta_1 \in W, W \subset L^\infty(\Omega), (A4)$ and (A1), we can obtain that

$$|\phi_1(x)|^2 + h|\phi_1(x)|^2 + h^2\beta(\phi_1(x))\phi_1(x) = g_1(x)\phi_1(x) - h^2(\pi(\phi_1(x)) - \pi(0))\phi_1(x) - h^2\pi(0)\phi_1(x)$$

$$\leq \frac{1}{2}\|g_1\|^2_{L^\infty(\Omega)} + \frac{1}{2}|\phi_1(x)|^2 + h^2\|\pi'\|_{L^\infty(\mathbb{R})}\|\phi_1(x)|^2 + \frac{1}{2}h^2|\phi_1(x)|^2 + \frac{1}{2}h^2|\pi(0)|^2$$

by multiplying the second equation in (Q)$_0$ by $\phi_1(x)$ and by using the Young inequality and (A3). Therefore, by the monotonicity of $\beta$, there exists $h_0 \in (0, h_{00})$ such that for all $h \in (0, h_0)$ there exists a constant $C_1 = C_1(h) > 0$ such that $|\phi_1(x)| \leq C_1$ for a.a. $x \in \Omega$. 

3. Uniform estimates and Cauchy’s criterion

In this section we will derive a priori estimates and Cauchy’s criterion for solutions of (P)$_h$.

**Lemma 3.1.** Let $h_0$ be as in Theorem 1.3. Then there exist constants $C > 0$ and $h_1 \in (0, h_0)$ depending on the data such that

$$\|\nabla h\|^2_{L^2(0,T;H)} + \|\phi h\|^2_{L^2(0,T;H)} + \| (\theta_1)_r\|^2_{L^2(0,T;H)} + \|\overline{\theta h}\|^2_{L^2(0,T;V)} \leq C$$

for all $h \in (0, h_1)$.

**Proof.** We test the first equation in (P)$_n$ by $\theta_{n+1} - \theta_n$, integrate over $\Omega$ and use the identities $a(a - b) = \frac{1}{2}a^2 - \frac{1}{2}b^2 + \frac{1}{2}(a - b)^2 (a, b \in \mathbb{R})$ and $\nu_{n+1} = \frac{\theta_{n+1} - \theta_n}{h}$, the Young inequality to infer that

$$h\left|\frac{\theta_{n+1} - \theta_n}{h}\right|^2 + \frac{1}{2}\|\nu_{n+1}\|^2_{H} - \frac{1}{2}\|\theta_n\|^2_{V} + \frac{1}{2}\|\theta_{n+1} - \theta_n\|^2_{V}$$

$$= h\left(f_{n+1}, \frac{\theta_{n+1} - \theta_n}{h}\right)_H - h\left(v_{n+1}, \frac{\theta_{n+1} - \theta_n}{h}\right)_H + h\left(\frac{\theta_{n+1} - \theta_n}{h}\right)_H$$

$$\leq \frac{3}{2}h\|f_{n+1}\|^2_{H} + \frac{3}{2}h|\nu_{n+1}|^2_{H} + \frac{3}{2}h\|\theta_{n+1}\|^2_{V} + \frac{1}{2}h\left|\frac{\theta_{n+1} - \theta_n}{h}\right|^2_{H}. \quad (3.1)$$
TIME DISCRETIZATION OF A NONLOCAL PHASE-FIELD SYSTEM

Multiplying the second equation in (P)_n by hv_{n+1}, integrating over Ω and applying the identity a(a−b) = \frac{1}{2}a^2 - \frac{1}{2}b^2 + \frac{1}{2}(a-b)^2 (a, b \in \mathbb{R}) we see from (A1), (A3) and the Young inequality that there exists a constant C_1 > 0 such that

\[
\frac{1}{2} ||v_{n+1}||_H^2 - \frac{1}{2} ||v_n||_H^2 + \frac{1}{2} ||v_{n+1} - v_n||_H^2 + h||v_{n+1}||_H^2 + (\beta(\varphi_{n+1}), \varphi_{n+1} - \varphi_n)_H \\
= h(\varphi_{n+1}, v_{n+1})_H - h(a(\cdot)\varphi_n - J * \varphi_n, v_{n+1})_H - h(\pi(\varphi_{n+1}) - \pi(0), v_{n+1})_H \\
- h(\pi(0), v_{n+1})_H \\
\leq \frac{1}{2} h||\theta_{n+1}||_V^2 + C_1 h||\varphi_n||_H^2 + \frac{1}{2} h||\varphi_n||_H^2 \\
+ \frac{1}{2} h||\pi(0)||_H^2 \frac{1}{2} h^2 + 2h||v_{n+1}||_H^2 + \frac{1}{2} h||\varphi_{n+1}||_H^2 (3.2)
\]

for all h ∈ (0, h_0). Here it follows from (A2) and the definition of the subdifferential that

\[
(\beta(\varphi_{n+1}), \varphi_{n+1} - \varphi_n)_H \geq \int_\Omega \hat{\beta}(\varphi_{n+1}) - \int_\Omega \hat{\beta}(\varphi_n) \quad (3.3)
\]

and we have from the identities a(a−b) = \frac{1}{2}a^2 - \frac{1}{2}b^2 + \frac{1}{2}(a-b)^2 (a, b \in \mathbb{R}) and hv_{n+1} = \varphi_{n+1} - \varphi_n, the Young inequality that

\[
\frac{1}{2} ||\varphi_{n+1}||_H^2 - \frac{1}{2} ||\varphi_n||_H^2 + \frac{1}{2} ||\varphi_{n+1} - \varphi_n||_H^2 \\
= (\varphi_{n+1}, \varphi_{n+1} - \varphi_n)_H = h(\varphi_{n+1}, v_{n+1})_H \leq \frac{1}{2} h||\varphi_{n+1}||_H^2 + \frac{1}{2} h||v_{n+1}||_H^2. (3.4)
\]

Hence, owing to (3.1)-(3.4) and summing over n = 0, …, m−1 with 1 ≤ m ≤ N, it holds that

\[
\frac{1}{2} h \sum_{n=0}^{m-1} ||\frac{\varphi_{n+1} - \varphi_n}{h}||_H^2 + \frac{1}{2} ||\theta_0||_H^2 + \frac{1}{2} h^2 \sum_{n=0}^{m-1} ||\varphi_{n+1} - \varphi_n||_H^2 \\
+ \frac{1}{2} h^2 \sum_{n=0}^{m-1} ||\pi_{n+1}||_H^2 + \frac{1}{2} h \sum_{n=0}^{m-1} ||\theta_{n+1}||_V^2 + \frac{1}{2} h \sum_{n=0}^{m-1} ||\varphi_{n+1}||_H^2 \\
\leq \frac{1}{2} ||\theta_0||_V^2 + \frac{1}{2} ||v_0||_H^2 + \int_\Omega \hat{\beta}(\varphi_0) + \frac{1}{2} ||\varphi_0||_H^2 + \frac{3}{2} h \sum_{n=0}^{m-1} ||f_{n+1}||_H^2 \\
+ 4h \sum_{n=0}^{m-1} ||v_{n+1}||_H^2 + 2h \sum_{n=0}^{m-1} ||\theta_{n+1}||_V^2 \\
+ C_1 h \sum_{n=0}^{m-1} ||\varphi_n||_H^2 + \frac{1}{2} ||\pi'||_L^2 + \frac{1}{2} h \sum_{n=0}^{m-1} ||\varphi_{n+1}||_H^2 + \frac{1}{2} ||\pi(0)||_H^2 T
\]
for all $h \in (0, h_0)$ and $m = 1, \ldots, N$. Then the inequality
\[
\frac{1}{2} h \sum_{n=0}^{m-1} \left\| \frac{\theta_{n+1} - \theta_n}{h} \right\|_H^2 + \frac{1 - 4h}{2} \left\| \theta_m \right\|_V^2 + \frac{1}{2} h^2 \sum_{n=0}^{m-1} \left\| \frac{\theta_{n+1} - \theta_n}{h} \right\|_V^2 \\
+ \frac{1 - 8h}{2} \left\| v_m \right\|_H^2 + h^2 \sum_{n=0}^{m-1} \left\| z_{n+1} \right\|_H^2 + h \sum_{n=0}^{m-1} \left\| v_{n+1} \right\|_H^2 + \int_{\Omega} \beta(\phi_m) \\
+ \frac{1 - (\|\pi\|_{L^2(\mathbb{R})}^2 + 1)h}{2} \left\| \phi_m \right\|_H^2 + \frac{1}{2} h^2 \sum_{n=0}^{m-1} \left\| v_{n+1} \right\|_H^2 \\
\leq \frac{1}{2} \left\| \theta_0 \right\|_H^2 + \frac{1}{2} \left\| v_0 \right\|_H + \int_{\Omega} \beta(\phi_0) + \frac{1}{2} \left\| \phi_0 \right\|_H^2 + \frac{3}{2} h \sum_{n=0}^{m-1} \left\| f_{n+1} \right\|_H^2 \\
+ 4h \sum_{j=0}^{m-1} \left\| v_j \right\|_H^2 + 2h \sum_{j=0}^{m-1} \left\| \theta_j \right\|_V^2 + \frac{2C_1 + \|\pi\|_{L^2(\mathbb{R})}^2 + 1}{2} h \sum_{j=0}^{m-1} \left\| \phi_j \right\|_H^2 \\
+ \frac{\|\pi(0)\|_H^2 T}{2}
\]
holds for all $h \in (0, h_0)$ and $m = 1, \ldots, N$. Thus there exist constants $C_2 > 0$ and $h_1 \in (0, h_0)$ such that
\[
h \sum_{n=0}^{m-1} \left\| \frac{\theta_{n+1} - \theta_n}{h} \right\|_H^2 + \left\| \theta_m \right\|_V^2 + h^2 \sum_{n=0}^{m-1} \left\| \frac{\theta_{n+1} - \theta_n}{h} \right\|_V^2 \\
+ \left\| v_m \right\|_H^2 + h^2 \sum_{n=0}^{m-1} \left\| z_{n+1} \right\|_H^2 + h \sum_{n=0}^{m-1} \left\| v_{n+1} \right\|_H^2 + \int_{\Omega} \beta(\phi_m) \\
+ \left\| \phi_m \right\|_H^2 + h^2 \sum_{n=0}^{m-1} \left\| v_{n+1} \right\|_H^2 \\
\leq C_2 + C_2 h \sum_{j=0}^{m-1} \left\| v_j \right\|_H^2 + C_2 h \sum_{j=0}^{m-1} \left\| \theta_j \right\|_V^2 + C_2 h \sum_{j=0}^{m-1} \left\| \phi_j \right\|_H^2
\]
for all $h \in (0, h_1)$ and $m = 1, \ldots, N$. Therefore, thanks to the discrete Gronwall lemma (see e.g., [10, Prop. 2.2.1]), we can obtain that there exists a constant $C_3 > 0$ such that
\[
h \sum_{n=0}^{m-1} \left\| \frac{\theta_{n+1} - \theta_n}{h} \right\|_H^2 + \left\| \theta_m \right\|_V^2 + h^2 \sum_{n=0}^{m-1} \left\| \frac{\theta_{n+1} - \theta_n}{h} \right\|_V^2 \\
+ \left\| v_m \right\|_H^2 + h^2 \sum_{n=0}^{m-1} \left\| z_{n+1} \right\|_H^2 + h \sum_{n=0}^{m-1} \left\| v_{n+1} \right\|_H^2 + \int_{\Omega} \beta(\phi_m) \\
+ \left\| \phi_m \right\|_H^2 + h^2 \sum_{n=0}^{m-1} \left\| v_{n+1} \right\|_H^2 \leq C_3
\]
for all $h \in (0, h_1)$ and $m = 1, \ldots, N$. \hfill \Box

**Lemma 3.2.** Let $h_1$ be as in Lemma 3.1. Then there exists a constant $C > 0$ depending on the data such that

$$\|\theta_h\|_{L^2(0,T;W)} \leq C$$

for all $h \in (0, h_1)$.

**Proof.** We have from the first equation in (P)$_h$ and Lemma 3.1 that there exists a constant $C_1 > 0$ such that

$$\| - \Delta \theta_h \|_{L^2(0,T;H)} \leq C_1$$

(3.5)

for all $h \in (0, h_1)$. Thus we can prove Lemma 3.2 by Lemma 3.1, (3.5) and the elliptic regularity theory. \hfill \Box

**Lemma 3.3.** Let $h_1$ be as in Lemma 3.1. Then there exist constants $C > 0$ and $h_2 \in (0, h_1)$ depending on the data such that

$$\|\bar{v}_h\|^2_{L^\infty(\Omega \times (0,T))} + \|\bar{\theta}_h\|^2_{L^\infty(\Omega \times (0,T))} \leq C$$

for all $h \in (0, h_2)$.

**Proof.** We derive from the identities $a(a - b) = \frac{1}{2}a^2 - \frac{1}{2}b^2 + \frac{1}{2}(a - b)^2$ ($a, b \in \mathbb{R}$) and $hv_{n+1} = \varphi_{n+1} - \varphi_n$, the Young inequality that

$$\frac{1}{2}|\varphi_{n+1}(x)|^2 - \frac{1}{2}|\varphi_n(x)|^2 + \frac{1}{2}|\varphi_{n+1}(x) - \varphi_n(x)|^2$$

$$= \varphi_{n+1}(x)(\varphi_{n+1}(x) - \varphi_n(x))$$

$$= h\varphi_{n+1}(x)v_{n+1}(x)$$

$$\leq \frac{1}{2}h|\varphi_{n+1}|^2_{L^\infty(\Omega)} + \frac{1}{2}h|v_{n+1}|^2_{L^\infty(\Omega)}.$$  (3.6)

Testing the second equation in (P)$_h$ by $hv_{n+1}(x)$ and using (A1), (A3), the Young
inequality mean that there exists a constant $C_1 > 0$ such that
\[
\frac{1}{2} |v_{n+1}(x)|^2 - \frac{1}{2} |v_n(x)|^2 + \frac{1}{2} |v_{n+1}(x) - v_n(x)|^2 \\
+ \beta(\varphi_{n+1}(x))(\varphi_{n+1}(x) - \varphi_n(x)) = h(\theta_{n+1}(x) - a(x) \varphi_n(x) + (J \ast \varphi_n)(x) + \pi(0) - \pi(\varphi_{n+1}(x)) - \pi(0))v_{n+1}(x)
\leq \frac{1}{2} h \|\theta_{n+1}\|_{L^\infty(\Omega)}^2 + \frac{1}{2} h - a(\cdot) \varphi_n + J \ast \varphi_n \|_{L^\infty(\Omega)}^2 \\
+ \frac{\|\pi\|_{L^\infty(\mathbb{R})}^2}{2} h \|\varphi_{n+1}\|_{L^\infty(\Omega)}^2 + \|\pi(0)\|_{L^\infty(\mathbb{R})}^2 h + 2 h \|v_{n+1}\|_{L^\infty(\Omega)}^2 \\
\leq \frac{1}{2} h \|\theta_{n+1}\|_{L^\infty(\Omega)}^2 + C_1 h \|\varphi_n\|_{L^\infty(\Omega)}^2 \\
+ \frac{\|\pi\|_{L^\infty(\mathbb{R})}^2}{2} h \|\varphi_{n+1}\|_{L^\infty(\Omega)}^2 + \frac{\|\pi(0)\|_{L^\infty(\mathbb{R})}^2}{2} h + 2 h \|v_{n+1}\|_{L^\infty(\Omega)}^2 (3.7)
\]
for all $h \in (0, h_1)$ and a.a. $x \in \Omega$. Here the condition (A2) and the definition of the subdifferential lead to the inequality
\[
\beta(\varphi_{n+1}(x))(\varphi_{n+1}(x) - \varphi_n(x)) \geq \hat{\beta}(\varphi_{n+1}(x)) - \hat{\beta}(\varphi_n(x)). \quad (3.8)
\]
Thus it follows from (3.6)-(3.8), summing over $n = 0, \ldots, m - 1$ with $1 \leq m \leq N$ and Remark 1.1 that
\[
\frac{1}{2} |\varphi_m(x)|^2 - \frac{1}{2} |\varphi_{m-1}(x)|^2 + \hat{\beta}(\varphi_m(x)) \\
\leq \frac{1}{2} \|\varphi_0\|_{L^\infty(\Omega)}^2 + \frac{1}{2} \|v_0\|_{L^\infty(\Omega)}^2 + \|\hat{\beta}(\varphi_0)\|_{L^\infty(\Omega)} \\
+ \frac{1}{2} h \sum_{n=0}^{m-1} \|\theta_{n+1}\|_{L^\infty(\Omega)}^2 + C_1 h \sum_{n=0}^{m-1} \|\varphi_n\|_{L^\infty(\Omega)}^2 \\
+ \frac{\|\pi\|_{L^\infty(\mathbb{R})}^2}{2} + \frac{1}{2} h \sum_{n=0}^{m-1} \|\varphi_{n+1}\|_{L^\infty(\Omega)}^2 + \frac{5}{2} h \sum_{n=0}^{m-1} \|v_{n+1}\|_{L^\infty(\Omega)}^2 + \frac{\|\pi(0)\|_{L^\infty(\mathbb{R})}^2}{2} T
\]
for all $h \in (0, h_1)$, $m = 1, \ldots, N$ and a.a. $x \in \Omega$, which implies that
\[
\frac{1}{2} \|\varphi_m\|_{L^\infty(\Omega)}^2 + \frac{1}{2} \|v_m\|_{L^\infty(\Omega)}^2 \\
\leq \frac{1}{2} \|\varphi_0\|_{L^\infty(\Omega)}^2 + \frac{1}{2} \|v_0\|_{L^\infty(\Omega)}^2 + \|\hat{\beta}(\varphi_0)\|_{L^\infty(\Omega)} \\
+ \frac{1}{2} h \sum_{n=0}^{m-1} \|\theta_{n+1}\|_{L^\infty(\Omega)}^2 + C_1 h \sum_{n=0}^{m-1} \|\varphi_n\|_{L^\infty(\Omega)}^2 \\
+ \frac{\|\pi\|_{L^\infty(\mathbb{R})}^2}{2} + \frac{1}{2} h \sum_{n=0}^{m-1} \|\varphi_{n+1}\|_{L^\infty(\Omega)}^2 + \frac{5}{2} h \sum_{n=0}^{m-1} \|v_{n+1}\|_{L^\infty(\Omega)}^2 + \frac{\|\pi(0)\|_{L^\infty(\mathbb{R})}^2}{2} T
\]
for all $h \in (0, h_1)$ and $m = 1, \ldots, N$. Then the inequality

$$
\frac{1}{2} - \left( \frac{\| \pi' \|^2_{L^\infty(\mathbb{R})} + 1}{2} \right) h \| \phi_m \|^2_{L^\infty(\Omega)} + \frac{1 - 5h}{2} \| v_m \|^2_{L^\infty(\Omega)}
\leq \frac{1}{2} \| \phi_0 \|^2_{L^\infty(\Omega)} + \frac{1}{2} \| v_0 \|^2_{L^\infty(\Omega)} + \| \hat{\beta}(\phi_0) \|^2_{L^\infty(\Omega)}
+ \frac{1}{2} h \sum_{n=0}^{m-1} \| \theta_{n+1} \|^2_{L^\infty(\Omega)} + \frac{2C_1 + \| \pi' \|^2_{L^\infty(\mathbb{R})} + 1}{2} h \sum_{j=0}^{m-1} \| \phi_j \|^2_{L^\infty(\Omega)}
+ \frac{5}{2} h \sum_{j=0}^{m-1} \| v_j \|^2_{L^\infty(\Omega)} + \frac{|\pi(0)|^2}{2} T
$$

(3.9)

holds for all $h \in (0, h_1)$ and $m = 1, \ldots, N$. Here we see from the continuity of the embedding $W \hookrightarrow L^\infty(\Omega)$ and Lemma 3.2 that there exist constants $C_2, C_3 > 0$ such that

$$
h \sum_{n=0}^{N-1} \| \theta_{n+1} \|^2_{L^\infty(\Omega)} = \| \overline{\theta}_h \|^2_{L^2(0,T;L^\infty(\Omega))} \leq C_2 \| \overline{\theta}_h \|^2_{L^2(0,T;W)} \leq C_3
$$

(3.10)

for all $h \in (0, h_1)$. Therefore we have from (3.9) and (3.10) that there exist constants $C_4 > 0$ and $h_2 \in (0, h_1)$ such that

$$
\| \phi_m \|^2_{L^\infty(\Omega)} + \| v_m \|^2_{L^\infty(\Omega)}
\leq C_4 + C_4 h \sum_{j=0}^{m-1} \| \phi_j \|^2_{L^\infty(\Omega)} + C_4 h \sum_{j=0}^{m-1} \| v_j \|^2_{L^\infty(\Omega)}
$$

for all $h \in (0, h_2)$ and $m = 1, \ldots, N$. Then we can obtain that there exists a constant $C_5 > 0$ such that

$$
\| \phi_m \|^2_{L^\infty(\Omega)} + \| v_m \|^2_{L^\infty(\Omega)} \leq C_5
$$

for all $h \in (0, h_2)$ and $m = 1, \ldots, N$ by the discrete Gronwall lemma (see e.g., [10, Prop. 2.2.1]).

**Lemma 3.4.** Let $h_2$ be as in Lemma 3.3. Then there exists a constant $C > 0$ depending on the data such that

$$
\| \phi_h \|^2_{L^\infty(\Omega \times (0,T))} \leq C
$$

for all $h \in (0, h_2)$.

**Proof.** We can verify this lemma by Lemma 3.3 and (A4).
**Lemma 3.5.** Let $h_2$ be as in Lemma 3.3. Then there exists a constant $C > 0$ depending on the data such that

$$\|z_h\|^2_{L^2(0,T;L^\infty(\Omega))} \leq C$$

for all $h \in (0,h_2)$.

**Proof.** Since it follows from Lemma 3.3 and the continuity of $\beta$ that there exists a constant $C_1 > 0$ such that

$$\|\beta(\Phi_h)\|_{L^\infty(\Omega \times (0,T))} \leq C_1$$

for all $h \in (0,h_2)$, we can confirm that Lemma 3.5 holds by the second equation in (P)$_h$, (A1), (A3), Lemmas 3.3, 3.4, the continuity of the embedding $W \hookrightarrow L^\infty(\Omega)$ and Lemma 3.2.

**Lemma 3.6.** Let $h_2$ be as in Lemma 3.3. Then there exist constants $C > 0$ and $h_3 \in (0,h_2)$ depending on the data such that

$$\|z_h\|^2_{L^\infty(0,T;H)} \leq C$$

for all $h \in (0,h_3)$.

**Proof.** Since the second equation in (P)$_n$ leads to the identity

$$z_1 + hz_1 + a(\cdot)\varphi_0 - J * \varphi_0 + \beta(\varphi_1) + \pi(\varphi_1) = \theta_1,$$

it holds that

$$\|z_1\|^2_H + h\|z_1\|^2_H$$

$$= -(a(\cdot)\varphi_0 - J * \varphi_0, z_1)_H - (\beta(\varphi_1), z_1)_H - (\pi(\varphi_1), z_1)_H + (\theta_1, z_1)_H.$$

Thus we deduce from the Young inequality, (A1), the continuity of $\beta$, (A3), Lemmas 3.1, 3.3 that there exists a constant $C_1 > 0$ such that

$$\|z_1\|^2_H \leq C_1$$

(3.11)

for all $h \in (0,h_2)$. Now let $n \in \{1,\ldots,N-1\}$. Then we have from the second equation in (P)$_n$ that

$$z_{n+1} - z_n + hz_{n+1} + ha(\cdot)v_n - hJ * v_n + \beta(\varphi_{n+1}) - \beta(\varphi_n) + \pi(\varphi_{n+1}) - \pi(\varphi_n) = \theta_{n+1} - \theta_n.$$  

(3.12)
Moreover, we test (3.12) by \( z_{n+1} \), integrate over \( \Omega \), recall (A1), Lemma 3.3, the local Lipschitz continuity of \( \beta \), (A3), and use the Young inequality to infer that there exist constants \( C_2, C_3 > 0 \) such that
\[
\frac{1}{2} \| z_{n+1} \|_{H}^2 + \frac{1}{2} \| z_n \|_{H}^2 + \frac{1}{2} \| z_{n+1} - z_n \|_{H}^2 + h \| z_{n+1} \|_{H}^2
\]
\[- h (a(\cdot)v_n - J*v_n,z_{n+1} + h (\beta(\varphi_{n+1}) - \beta(\varphi_n), z_{n+1} + \Delta H + h (\theta_{n+1} - \theta_n, z_{n+1} + \Delta H
\]
\[\leq C_2 h \| v_n \|_{H} \| z_{n+1} \|_{H} + C_2 h \| v_n \|_{H} \| z_{n+1} \|_{H} + h \| \frac{\theta_{n+1} - \theta_n}{h} \|_{H} \| z_{n+1} \|_{H}
\]
\[\leq C_3 h \| z_{n+1} \|_{H} + h \| \frac{\theta_{n+1} - \theta_n}{h} \|_{H} \| z_{n+1} \|_{H}
\]
\[\leq h \| z_{n+1} \|_{H}^2 + \frac{1}{2} h \| \frac{\theta_{n+1} - \theta_n}{h} \|_{H}^2 + \frac{C_3^2}{2} h
\] (3.13)
for all \( h \in (0,h_2) \). Thus, summing (3.13) over \( n = 1, \ldots, \ell - 1 \) with \( 2 \leq \ell \leq N \), we see from (3.11) and Lemma 3.1 that there exists a constant \( C_4 > 0 \) such that
\[
\frac{1}{2} \| z_{\ell} \|_{H}^2 \leq \frac{1}{2} \| z_{1} \|_{H}^2 + h \sum_{n=1}^{\ell-1} \| z_{n+1} \|_{H}^2 + \frac{1}{2} h \sum_{n=1}^{\ell-1} \| \frac{\theta_{n+1} - \theta_n}{h} \|_{H}^2 + \frac{C_3^2}{2} h
\]
\[\leq C_4 + h \sum_{n=1}^{\ell-1} \| z_{n+1} \|_{H}^2
\]
for all \( h \in (0,h_2) \) and \( \ell = 2, \ldots, N \), whence we have from (3.11) that there exist constants \( C_5 > 0 \) and \( h_3 \in (0,h_2) \) such that
\[
\| z_m \|_{H}^2 \leq C_5 + C_5 h \sum_{j=0}^{m-1} \| z_{j} \|_{H}^2
\]
for all \( h \in (0,h_3) \) and \( m = 1, \ldots, N \). Therefore the discrete Gronwall lemma (see e.g., [10, Prop. 2.2.1]) implies that there exists a constant \( C_6 > 0 \) such that
\[
\| z_m \|_{H}^2 \leq C_6
\]
for all \( h \in (0,h_3) \) and \( m = 1, \ldots, N \).

**Lemma 3.7.** Let \( h_3 \) be as in Lemma 3.6. Then there exists a constant \( C > 0 \) depending on the data such that
\[
\| \hat{\theta}_h \|_{H^1(0,T;H) \cap L^\infty(0,T;V)} + \| \hat{\varphi}_h \|_{W^{1,\infty}(0,T;L^\infty(\Omega))}
\]
\[+ \| \hat{\psi}_h \|_{W^{1,\infty}(0,T;H) \cap W^{1,2}(0,T;L^\infty(\Omega)) \cap L^\infty(0,T;L^\infty(\Omega))} \leq C
\]
for all \( h \in (0,h_3) \).
We can obtain that
\[ \|\bar{v}_h - \bar{v}_\tau\|_{C([0,T];H)} + \|v_h - v_\tau\|_{L^2(0,T;H)} + \|\bar{\phi}_h - \bar{\phi}_\tau\|_{C([0,T];H)} \]
\[ + \|\bar{\theta}_h - \bar{\theta}_\tau\|_{C([0,T];H)} + \|\nabla(\bar{\theta}_h - \bar{\theta}_\tau)\|_{L^2(0,T;H)} \leq C(h^{1/2} + \tau^{1/2}) + C\|\bar{f}_h - \bar{f}_\tau\|_{L^2(0,T;H)} \]
for all \( h, \tau \in (0,h_3) \).

**Proof.** It holds that
\[
\frac{1}{2} \frac{d}{ds} \|\bar{v}_h(s) - \bar{v}_\tau(s)\|_H^2 = (\bar{z}_h(s) - \bar{z}_\tau(s), \bar{v}_h(s) - \bar{v}_\tau(s))_H \\
= (\bar{z}_h(s) - \bar{z}_\tau(s), \bar{v}_h(s) - \bar{v}_\tau(s))_H + (\bar{z}_h(s) - \bar{z}_\tau(s), \bar{v}_h(s) - \bar{v}_\tau(s))_H \\
+ (\bar{z}_h(s) - \bar{z}_\tau(s), \bar{v}_h(s) - \bar{v}_\tau(s))_H. \tag{3.14}
\]

Here we derive from the second equation in (P)_h that
\[
(\bar{z}_h(s) - \bar{z}_\tau(s), \bar{v}_h(s) - \bar{v}_\tau(s))_H \\
= -\|v_h(s) - v_\tau(s)\|_H^2 \\
+ (-a(\cdot)(\bar{\phi}_h(s) - \bar{\phi}_\tau(s)) + J \ast (\bar{\phi}_h(s) - \bar{\phi}_\tau(s)), \bar{v}_h(s) - \bar{v}_\tau(s))_H \\
- (\bar{\beta}(\bar{\theta}_h(s)) - \bar{\beta}(\bar{\theta}_\tau(s)), \bar{v}_h(s) - \bar{v}_\tau(s))_H \\
- (\bar{\pi}(\bar{\phi}_h(s)) - \bar{\pi}(\bar{\phi}_\tau(s)), \bar{v}_h(s) - \bar{v}_\tau(s))_H \\
+ (\bar{\theta}_h(s) - \bar{\theta}_\tau(s), \bar{v}_h(s) - \bar{v}_\tau(s))_H. \tag{3.15}
\]

The property (1.12) means that
\[
\|\bar{\phi}_h(s) - \bar{\phi}_\tau(s)\|_H^2 \\
= \|h(\bar{\phi}_h)_t(s) + \tau(\bar{\phi}_\tau)_t(s) + \bar{\phi}_h(s) - \bar{\phi}_\tau(s)\|_H^2 \\
\leq 3h^2\|((\bar{\phi}_h)_t(s))_H + 3\tau^2\|((\bar{\phi}_\tau)_t(s))_H + 3\|\bar{\phi}_h(s) - \bar{\phi}_\tau(s)\|_H^2. \tag{3.16}
\]

We can obtain that
\[
\|\bar{\phi}_h(s) - \bar{\phi}_\tau(s)\|_H^2 \\
= \|\bar{\phi}_h(s) - \bar{\phi}_h(s) + \bar{\phi}_h(s) - \bar{\phi}_\tau(s) + \bar{\phi}_\tau(s) - \bar{\phi}_\tau(s)\|_H^2 \\
\leq 3\|\bar{\phi}_h(s) - \bar{\phi}_h(s)\|_H^2 + 3\|\bar{\phi}_h(s) - \bar{\phi}_\tau(s)\|_H^2 + 3\|\bar{\phi}_\tau(s) - \bar{\phi}_\tau(s)\|_H^2 \tag{3.17}
\]
and
\[ \frac{1}{2} \frac{d}{ds} \| \hat{\phi}_h(s) - \hat{\tau}(s) \|_H^2 = (\nabla \hat{h}(s) - \nabla \tau(s), \hat{\phi}_h(s) - \hat{\tau}(s))_H. \]

(3.18)

It follows from the first equation in (P) that
\[
\frac{1}{2} \frac{d}{ds} \| \hat{\theta}_h(s) - \hat{\tau}(s) \|_H^2 \\
= - (\Delta (\bar{\theta}_h(s) - \bar{\tau}(s)), \hat{\theta}_h(s) - \hat{\tau}(s))_H - (\bar{\nu}_h(s) - \bar{\nu}_\tau(s), \hat{\theta}_h(s) - \hat{\tau}(s))_H \\
+ (\bar{f}_h(s) - \bar{f}_\tau(s), \hat{\theta}_h(s) - \hat{\tau}(s))_H \\
= - \| \nabla (\bar{\theta}_h(s) - \bar{\tau}(s)) \|_H^2 - (\Delta (\bar{\theta}_h(s) - \bar{\tau}(s)), \hat{\theta}_h(s) - \hat{\tau}(s))_H \\
- (\bar{\nu}_h(s) - \bar{\nu}_\tau(s), \hat{\theta}_h(s) - \hat{\tau}(s))_H - (\bar{\nu}_h(s) - \bar{\nu}_\tau(s), \bar{\theta}_h(s) - \bar{\tau}(s))_H \\
+ (\bar{f}_h(s) - \bar{f}_\tau(s), \hat{\theta}_h(s) - \hat{\tau}(s))_H. \\
(3.19)
\]

Therefore we have from (3.14)-(3.19), the integration over \((0, t)\), where \(t \in [0, T]\), the Schwarz inequality, the Young inequality, (A1), Lemma 3.3, the local Lipschitz continuity of \(\beta\), (A3), (1.9)-(1.11), Lemmas 3.2, 3.6 and 3.7 that there exists a constant \(C_1 > 0\) such that
\[
\| \bar{\nu}_h(t) - \bar{\nu}_\tau(t) \|_H^2 + \int_0^t \| \bar{\nu}_h(s) - \bar{\nu}_\tau(s) \|_H^2 ds + \| \hat{\phi}_h(t) - \hat{\tau}(t) \|_H^2 \\
+ \| \hat{\theta}_h(t) - \hat{\tau}(t) \|_H^2 + \int_0^t \| \nabla (\bar{\theta}_h(s) - \bar{\tau}(s)) \|_H^2 ds \\
\leq C_1 (h + \tau) + C_1 \| \bar{f}_h - \bar{f}_\tau \|_{L^2(0, T; H)} + C_1 \int_0^t \| \hat{\phi}(s) - \hat{\tau}(s) \|_H^2 ds \\
+ C_1 \int_0^t \| \hat{\theta}(s) - \hat{\tau}(s) \|_H^2 ds
\]

for all \(h, \tau \in (0, h_4)\). Then we can prove Lemma 3.8 by the Gronwall lemma. \(\square\)

4. Existence and uniqueness for (P) and an error estimate

In this section we verify Theorems 1.4 and 1.5.

Proof of Theorem 1.2. Since \(\bar{f}_h\) converges to \(f\) strongly in \(L^2(0, T; H)\) as \(h \searrow 0\) (see [3, Section 5]), we see from Lemmas 3.1-3.8, (1.9)-(1.12) that there exist some functions
\[
\theta \in H^1(0, T; H) \cap L^\infty(0, T; V) \cap L^2(0, T; W), \\
\varphi \in W^{2, \infty}(0, T; H) \cap W^{2, 2}(0, T; L^\infty(\Omega)) \cap W^{1, \infty}(0, T; L^\infty(\Omega))
\]
such that
\[
\hat{\theta}_h \to \theta \quad \text{weakly* in } H^1(0,T;H) \cap L^\infty(0,T;V), \quad (4.1)
\]
\[
\overline{\theta}_h \to \theta \quad \text{weakly* in } L^\infty(0,T;V),
\]
\[
\overline{\theta}_h \to \theta \quad \text{weakly in } L^2(0,T;W),
\]
\[
\hat{\phi}_h \to \theta \quad \text{strongly in } C([0,T];H),
\]
\[
\overline{\phi}_h \to \theta \quad \text{strongly in } L^2(0,T;V),
\]
\[
\hat{\phi}_h \to \phi \quad \text{weakly* in } W^{1,\infty}(0,T;L^\infty(\Omega)),
\]
\[
\overline{\phi}_h \to \phi \quad \text{weakly* in } L^\infty(\Omega \times (0,T)),
\]
\[
\phi_h \to \phi \quad \text{weakly* in } L^\infty(\Omega \times (0,T)),
\]
\[
\hat{\phi}_h \to \phi \quad \text{strongly in } C([0,T];H),
\]

and
\[
\hat{v}_h \to \varphi_t \quad \text{weakly* in } W^{1,\infty}(0,T;H) \cap W^{1,2}(0,T;L^\infty(\Omega)) \cap L^\infty(\Omega \times (0,T)),
\]
\[
\overline{v}_h \to \varphi_t \quad \text{weakly* in } L^\infty(\Omega \times (0,T)),
\]
\[
\hat{v}_h \to \varphi_t \quad \text{strongly in } C([0,T];H),
\]
\[
\overline{v}_h \to \varphi_t \quad \text{strongly in } L^2(0,T;H),
\]
\[
\overline{z}_h \to \varphi_{tt} \quad \text{weakly* in } L^\infty(0,T;H) \cap L^2(0,T;L^\infty(\Omega))
\]

as \( h = h_j \downarrow 0 \). Here we recall (1.10) and Lemma 3.3 to derive from (4.5) that
\[
\| \overline{\phi}_h - \phi \|_{L^\infty(0,T;H)} \leq \| \overline{\phi}_h - \hat{\phi}_h \|_{L^\infty(0,T;H)} + \| \hat{\phi}_h - \phi \|_{L^\infty(0,T;H)}
\]
\[
\leq |\Omega|^{1/2} \| \overline{\phi}_h - \hat{\phi}_h \|_{L^\infty(0,T;L^\infty(\Omega))} + \| \hat{\phi}_h - \phi \|_{L^\infty(0,T;H)}
\]
\[
= \frac{|\Omega|^{1/2}}{\sqrt{3}} \| \overline{v}_h \|_{L^\infty(0,T;L^\infty(\Omega))} + \| \hat{\phi}_h - \phi \|_{L^\infty(0,T;H)} \to 0
\]
as \( h = h_j \downarrow 0 \), and hence it holds that
\[
\overline{\phi}_h \to \varphi \quad \text{strongly in } L^\infty(0,T;H)
\]

as \( h = h_j \downarrow 0 \). Thus we infer from (1.12), (4.9) and Lemma 3.7 that
\[
\overline{\phi}_h \to \varphi \quad \text{strongly in } L^\infty(0,T;H)
\]
as \( h = h_j \downarrow 0 \). Therefore, owing to (4.1)-(4.10), (A1), Lemma 3.3, the local Lipschitz continuity of \( \beta \), and (A3), we can establish existence of solutions to (P). Moreover, we can confirm uniqueness of solutions to (P) in a similar way to the proof of Lemma 3.8.
Proof of Theorem 1.3. Since the inclusion $f \in L^2(0, T; H) \cap W^{1,1}(0, T; H)$ implies that there exists a constant $C_1 > 0$ such that
\[
\left\| \mathcal{P}_h - f \right\|_{L^2(0, T; H)} \leq C_1 h^{1/2}
\]
for all $h > 0$ (see [3, Section 5]), we can prove Theorem 1.5 by Lemma 3.8. 

Acknowledgements

The author would like to thank the anonymous referees for their reading. The author is supported by JSPS Research Fellowships for Young Scientists (No. 18J21006).

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