Spin Dynamics Of $qqq$ Wave Function On Light Front In High Momentum Limit Of QCD : Role Of $qqq$ Force

A.N.Mitra *
244 Tagore Park, Delhi-110009, India

Abstract

The contribution of a spin-rich $qqq$ force (in conjunction with pairwise $qq$ forces) to the analytical structure of the $qqq$ wave function is worked out in the high momentum regime of QCD where the confining interaction may be ignored, so that the dominant effect is Coulombic. A distinctive feature of this study is that the spin-rich $qqq$ force is generated by a $ggg$ vertex (a genuine part of the QCD Lagrangian) wherein the 3 radiating gluon lines end on as many quark lines, giving rise to a (Mercedes-Benz type) $Y$-shaped diagram. The dynamics is that of a Salpeter-like equation (3D support for the kernel) formulated covariantly on the light front, a la Markov-Yukawa Transversality Principle (MYTP) which warrants a 2-way interconnection between the 3D and 4D Bethe-Salpeter (BSE) forms for 2 as well as 3 fermion quarks. With these ingredients, the differential equation for the 3D wave function $\phi$ receives well-defined contributions from the $qq$ and $qqq$ forces. In particular a negative eigenvalue of the spin operator $i\sigma_1 \cdot \sigma_2 \times \sigma_3$ which is an integral part of the $qqq$ force, causes a characteristic singularity in the differential equation, signalling the dynamical effect of a spin-rich $qqq$ force not yet considered in the literature. The potentially crucial role of this interesting effect vis-a-vis the so-called ‘spin anomaly’ of the proton, is a subject of considerable physical interest.

*Email: (1)ganmitra@nde.vsnl.net.in; (2)anmitra@physics.du.ac.in
1 Introduction

The concept of a fundamental 3-body force (on par with a 2-body force) is hard to realize in physics, leaving aside certain ad hoc representations of higher order effects, for example those of $\Delta, N^*$ resonances in hadron physics. At the deeper quark-gluon level on the other hand, a truly 3-body $qqq$ force shows up as a folding of a $ggg$ vertex (a genuine part of the gluon Lagrangian in QCD) with 3 distinct $\bar{q}gq$ vertices, so as to form a Y-shaped diagram (see fig 1 below). Indeed a 3-body $qqq$ force of this type, albeit for ‘scalar’ gluons, was first suggested by Ernest Ma [1], when QCD was still in its infancy. [A similar representation is also possible for $NNN$ interaction via $\rho\rho\rho$ or $\sigma\sigma\sigma$ vertices, but was never in fashion in the literature [2]]. We note in passing that a $Y$-shaped (Mercedes-Benz type) picture [3] was once considered in the context of a preon model for quarks and leptons.

In the context of QCD as a Yang-Mills field, a $ggg$ vertex has a momentum representation of the form [4]

$$W_{ggg} = -ig_s f_{abc}[(k_1 - k_2)_{\lambda}\delta_{\mu\nu} + (k_2 - k_3)_{\mu}\delta_{\nu\lambda} + (k_3 - k_1)_{\nu}\delta_{\lambda\mu}]$$

(1.1)

where the 4-momenta emanating from the $ggg$ vertex satisfy $k_1 + k_2 + k_3 = 0$, and $f_{abc}$ is the color factor. When this vertex is folded into 3 $\bar{q}gq$ vertices of the respective forms $g_s\bar{u}(p'_1)\gamma_{\mu}\lambda^a{(p_1)}u(p_1)$, and two similar terms, the resultant $qqq$ interaction matrix (suppressing the Dirac spinors for the 3 quarks) becomes

$$V_{qqq} = \frac{g_s^4}{2^3}[i\gamma^{(1)}(k_2 - k_3)\gamma^{(2)}\gamma^{(3)} + \{2\} + \{3\}]{\lambda_1}{\lambda_2}{\lambda_3}/\{k_{1}^2k_{2}^2k_{3}^2\}$$

(1.2)

where $k_i = p_i - p'_i$; $\lambda_i$ are the color matrices which get contracted into the corresponding scalar triple products in an obvious notation. [Note that the flavour indices are absent here since the quark gluon interaction is flavour blind].

This interaction will be considered in conjunction with 3 pairs of $qq$ forces within the framework of a Bethe-Salpeter type dynamics to be specified below. Before proceeding in this direction, it is in order to explain a possible motivation behind the use of a direct $qqq$ force with such a rich spin dependence. Apart from the intrinsic beauty of this term, the immediate provocation for its use comes from the issue of ”proton spin” which, after making headlines about two decades ago, has come to the fore once again,
thanks to the progress of experimental techniques in polarized deep inelastic scattering off polarized protons, and their variations thereof, which allow for an experimental determination of certain key QCD parameters by relating them to certain observable quantities emanating from external probes: (see a recent review [5] for references and other details). On the other hand it is also of considerable theoretical interest to determine these very quantities directly from the intrinsic premises of QCD provided one has a "good" qqq wave function to play with. Such a plea would have sounded rather utopian in the early days of QCD when phenomenology was the order of the day. Today however many aspects of QCD are understood well enough to make such studies worthwhile by hindsight, with possible ramifications beyond their educational value. The role of the direct 3-body force may be seen in this light, while working for simplicity in the experimentally accessible regime of valence quarks. To that end this paper is specifically concerned with the effect of the 3-body force (1.2) on the analytical structure of the qqq wave function, while the formalism dealing with the contributions of the various operators (iγµγ5, 2-gluon effects, etc) to the proton spin is reserved for a subsequent paper.

1.1 Theoretical Ingredients

In the valence quark regime, we need to consider a qqq system governed by pairwise qq forces as well as a direct 3-quark force of the type (1.2). A further simplification occurs in the high momentum regime where the effect of confining forces may be neglected, so that only coulombic forces need be kept track of. We shall take the dynamics of a qqq hadron in the high−momentum regime to be governed by a Bethe-Salpeter Equation (BSE) whose kernel is a sum of three pairs of (coulombic) qq forces plus a single 3-body term (1.2). Unfortunately the 4D form of the BSE is too general to be of practical value for qqq dynamics, so a better option is the Salpeter Equation [6] representing its instantaneous form. And, except for its lack of covariance, the Salpeter equation has the remarkable property of 3D-4D interlinkage, a feature that had been present all along in the original formulation itself [6], but had somehow remained hidden from view in the literature, until clarified [7] in the context of a comprehensive two−tier BSE formalism developed independently in a covariant manner [8]. (A practical significance of the ‘two-tier’ formalism is that the reduced 3D form is ideal for the determination of hadron mass spectra of both qq and qqq types [9, 10], while the
reconstructed 4D form is convenient for the evaluation of transition amplitudes [9] via Feynman techniques for loop diagrams; see also Munz et al ref.(9) . A covariant formulation of the BSE is centered around the hadron 4-momentum $P_\mu$ in accordance with the Markov-Yukawa Transversality Principle (MYTP) [11,12], which was shown to be a ‘gauge principle’ in disguise [13]. It ensures that the interactions among the constituents be transverse to the direction of $P_\mu$. In the high momentum regime to be considered here, the confining interaction has been ignored for simplicity, which leaves the 3D form of the BS dynamics inadequate for mass spectral determination, yet its dynamical on the spin-structure of the wave function should be realistic enough for dealing with the hadron spin in the high momentum limit.

Now to another vital element of the theory: Although the Salpeter Equation, as the instant form of BSE, admits of a covariant formulation in the rest frame of the total hadron 4-momentum $P_\mu$, it suffers from certain ill-defined 4D loop integrals due to a ‘Lorentz-mismatch’ among the rest-frames of the participating hadronic composites, resulting in time-like momentum components in the exponential/gaussian factors associated with their vertex functions. This is especially true of triangle loops, such as applicable to the pion form factor and $\rho - \pi\pi$ coupling [14] where this disease causes unwarranted ”complexities” in the amplitudes, while two-quark loops just escape this pathology. For a possible remedy against this disease, without losing the benefit of an MYTP-based 3D-4D interconnection, a promising candidate is the light-front approach of Dirac [15] by virtue of its bigger $(7)$ stability group compared with 6 for the instant form theory. Its basic simplicity was first noticed by Weinberg [16] who formulated the infinite momentum frame towards the same end. A covariant generalization of MYTP on the light front, requiring the use of two null 4-vectors $n_\mu$ and $\tilde{n}_\mu$ that satisfy $n^2 = \tilde{n}^2 = 0$ and $n.\tilde{n} = 1$, cures the ‘Lorentz mismatch’ disease noted above [17]. The remaining problem of $n$-dependence of transition amplitudes is solved through a simple ansatz of ‘Lorentz completion’ [17,18], so as to yield a Lorentz-invariant pion e.m. form factor in accord with experiment. A similar approach had been considered by Carbonell et al [19] in the context of the Kadychevsky-Karmanov formalism [20,21], except for their missing out on the second (dual) null-vector $\tilde{n}$ which happens to be crucial for recovering a Lorentz-invariant structure in a more natural way.
1.2 Plan of the Paper

The plan of the paper is based on an interlinked 3D-4D BSE formalism characterized by a Lorentz-covariant 3D support for its kernel a la MYTP [11,12], adapted to the light front (LF) [17]. The full structure of the kernel is a sum of 3 pairs of coulombic \( qq \) forces plus a \( qqq \) force, Eq.(1.2), whose 3D support is implied by the fact that all internal momenta \( q \) be \textit{transverse} to \( P_\mu \), viz., \( \hat{q}_\mu = q_\mu - q.P_\mu/P^2 \). However the propagators are left untouched in their standard 4D forms. [The light front formulation requires a \textit{collinear} frame [17] which is further specified in Section 2]. The strategy now lies in a step-wise reduction of the (Salpeter-like) 4D Master Equation involving the actual (4D) fermionic wave function \( \Psi \). Step A consists in expressing \( \Psi \) in terms of an auxiliary (bosonic) 4D quantity \( \Phi \) satisfying an equivalent (bosonic) 4D (Salpeter-like) equation. Step B involves a ”Gordon reduction” to eliminate the \( \gamma_\mu \) matrices in favour of \( \sigma_{\mu\nu} \) matrices. Finally Step C consists in a 3D reduction of this ‘bosonized’ Master Equation, and a subsequent \textit{reconstruction} of the original \( \Psi \) in 4D form by a suitable reversal of steps. In the process a 3D scalar wave function \( \phi \) is introduced which not only facilitates an explicit solution of the 3D (albeit fully covariant) Salpeter-like equation but is also a key component of the (reconstructed) 4D fermionic wave function \( \Psi \). To facilitate the process of 3D-4D interlinkage, a Green’s function approach is employed a la [22], from which it is straightforward to derive the corresponding wave functions via the appropriate \textit{‘pole’} limits. The entire exercise involves a close correspondence between the (earlier) instantaneous form [22] and the (later) LF form [17], so as to project only the latter by making free use of the results of the former. A new element will be a generalization of the earlier formalism [22] so as to include the effect of the 3-body force, Eq (1.2), on the structure of the \( qqq \) Green’s function which will require a more elaborate strategy, keeping in view the relative strengths of \( qq \) and \( qqq \) forces.

After a short account of the correspondence between the instant and LF forms of the dynamics, \textbf{Section 2} is devoted to Steps A and B for converting the Master Equation for the actual 4D \( qqq \) wave function \( \Psi \) (fermionic) to an equivalent 4D (bosonic) form involving \( \Phi \), with all \( \gamma \)-matrices eliminated in favour of Pauli matrices, by exploiting the effective 3D support for the \( qq \) and \( qqq \) kernels on the light-front, as described above. \textbf{Section 3} describes the first part of Step C, viz., the use of \textit{Green’s-} function method for the 3D reduction of the Master Equation for the 4D Green’s function
corresponding to $\Phi$, resulting in an integral equation for the 3D Green’s function $\hat{G}$ corresponding to $\phi$, specialized to the large momentum regime on the light-front. The second part of Step C, namely its reconstruction back to the 4D quantity $G$ a la [22], leading to a formal connection between $\Phi$ and $\phi$, is the subject of Section 4, now with the added complexity behind the inclusion of the 3-body $qqq$ force along with the (dominant) pairwise $qq$ forces. [ The subsequent reconstruction of the 4D fermionic form $\Psi$ follows from the results of Section 2]. As a further aid to the understanding of the 3D-4D interconnection, Appendix A describes the complete procedure for a prototype $qq$ subsystem, whose results are freely used for justifying several steps in Sections 3 and 4. In Section 5, the full 3D BSE for $\phi$ is set up in a simplified form designed to check for the dynamical effect of spin present in both the $qq$ and $qqq$ forces. on the structure of the differential equation for $\Phi$ (the dominant effect being of the latter ! ). Section 6 is devoted to a short critique of the role of the $V_{qqq}$ term on the analytic structure of the $qqq$ wave function, while the derivation of the requisite QCD parameters for the baryon from the forward scattering amplitude off the $i\gamma_\mu\gamma_5$ operator, via the quark constituents, as well as a more elaborate 2-gluon contribution to the proton spin, is reserved for a subsequent paper.

2 Salpeter Eq on LF : From $\Psi$ To $\Phi$

2.1 Master Eq : Instant Vs LF Forms of Dynamics

Since the first step in the formulation of a covariant Salpeter-like equation on the light front (LF) is to establish a correspondence between the instant and LF forms of the dynamics, we first recall some definitions [17] for the LF quantities $p_\pm = p_0 \pm p_3$ defined covariantly as $p_+ = n.p\sqrt{2}$ and $p_- = -\bar{n}.p\sqrt{2}$, while the perpendicular components continue to be denoted by $p_\perp$ in both notations. Now for a typical internal momentum $q_\mu$, the parallel component $P.qP/\tilde{P}^2$ of the instant form translates in the LF form as $q_{3\mu} = zP_nn_\mu$, where $P_n = P.\bar{n}$, and $z = n.q/n.P$. As a check, $\hat{q}^2 = q_\perp^2 + z^2 M^2$ which shows that $zM$ plays the role of the third component of $\hat{q}$ on LF. Next, we collect some of the more important definitions / results of the LF formalism [17]

\begin{align}
q_\perp &= q - q_n n; \hat{q} = q_\perp + zP_nn; z = q.n/P.n; q_n = q.\bar{n}; \\
P_n &= P.\bar{n}; P.q = P_n q_n + P.nq_n; \hat{q} . \bar{n} = P_\perp q_\perp = 0; \tag{2.1}
\end{align}
\[ P \hat{q} = P_n q.n; \hat{q}^2 = q_{\perp}^2 + M^2 z^2; P^2 = -M^2 \]

For a \textit{qqq} baryon, there are two internal momenta, each separately satisfying the relations (2.1). Note that for any 4-vector \( A \), \( A.n \) and \(-A_n\) correspond to \( 1/\sqrt{2} \) times the usual light front quantities \( A_\pm = A_0 \pm A_\ perpendicular \). But since a \textit{physical} amplitude must not depend on the orientation \( n \), a simple device termed Lorentz \textit{completion} via the \textit{collinear} trick [17] yields a Lorentz-invariant amplitude for a transition process with \textit{three} external lines \( P, P', P'' \) as follows. Since collinearity implies \( P_\perp P'_\perp = 0 \), the 4-scalar product \( P.P' = P.n P'_n + P'_n P.n + P_\perp P'_\perp \) simplifies to \( P.n P'_n + P'_n P.n \). Then ' Lorentz completion ' simply amounts to \textit{reversing} the last step via the 'zero' quantity \( P_\perp P'_\perp \), so as to recover the Lorentz invariant quantity \( P.P' \) at the end! And as a practical simplification, one does not even have to use the \( n, \tilde{n} \) symbols; it suffices to use the more familiar light-front components \( A_\pm \) for the covariant quantities \( \sqrt{2}[A.n, -A_n] \) respectively. For ready reference, the precise correspondence between the instant and LF definitions of the 'parallel (z)' and 'time-like (0)' components of the various 4-momenta for a \textit{qqq} baryon ( \( i = 1,2,3 \)) [22, 23]:

\[
p_{iz}; p_{i0} = \frac{M p_{i+}}{P_+}, \frac{M p_{i-}}{2P_-}; \hat{p}_i \equiv \{ p_{i\perp}, p_{iz} \} \quad \text{(2.2)}
\]

The last part of Eq.(2.2) defines a covariant 3-vector on the LF that will frequently appear in the reduced 3D BSE for the \textit{qqq} proton. Our goal is to write down (and solve) the Master equation for three fermion quarks complete with all internal d.o.f.'s, in the presence of both \( qq \) and direct \textit{qqq} forces which reads [8]:

\[
\Psi(p_1 p_2 p_3) = \sum_1^3 S_F(p_1)S_F(p_2)g_s^2 \int \frac{d^4q_1 d^4q_2}{(2\pi)^4} \gamma^{(1)}_\mu \gamma^{(2)}_\nu D_{\mu\nu}(k_{12})\Psi(p_1', p_2', p_3) + S_F(p_1)S_F(p_2)S_F(p_3) \int \frac{d^4q_1 d^4q_2 d^4q_3}{(2\pi)^8} V_{qqq} \Psi(p_1' p_2' p_3') \quad \text{(2.3)}
\]

where the definitions for the various momenta, and the phase conventions for the quark propagators are those of [8], while the direct 3-quark interaction \( V_{qqq} \) in the last term is \textit{new}, and given by (1.2). The central problem is now the reduction of this Master equation (2.3) through the three steps (A, B, C) vide Sect (1.2), as outlined below.
2.2 Reduction of Master Eq from $\Psi$ to $\Phi$

Since permutation symmetry plays a crucial role for a $qqq$ hadron for fermion quarks, we define at the outset a pair of internal variables $(\xi; \eta)$ with, say, the index #3 as basis, as [22]

$$\sqrt{2}\xi_3 = p_1 - p_2; \quad \sqrt{6}\eta_3 = -2p_3 + p_1 + p_2; \quad P = p_1 + p_2 + p_3 \quad (2.4)$$

where the time-like and space-like parts of each are given by (2.2), and the corresponding 3-vector defined as $\hat{p}_i \equiv \{p_{i1}, p_{i2}\}$. Two identical sets of momentum pairs $\xi_1, \eta_1$ and $\xi_2, \eta_2$ are similarly defined, but can be expressed in terms of the set (2.4) via permutation symmetry. Now Step A, which is designed to dispose of the fermion d.o.f.'s for a $qqq$ system, consists in defining an auxiliary scalar function $\Phi$ related to the actual BS wave function $\Psi$ by [8]

$$\Psi = \Pi_{123} S^{-1}_{F_i} (-p_i) \Phi(p_1 p_2 p_3) W(P) \quad (2.5)$$

Here the quantity $W(P)$ is independent of the internal momenta but includes the spin-cum-flavour wave functions $\chi, \phi$ of the 3 quarks involved (see [?]):

$$W(P) = [\chi' \phi' + \chi'' \phi''] / \sqrt{2} \quad (2.6)$$

where $\phi', \phi''$ are the standard flavour functions of mixed symmetry [23] [not to be confused with the 3D wave function $\phi$ !], and $\chi', \chi''$ are the corresponding relativistic spin functions. The latter may be defined either in terms of the quark # indices as in Eqs (1.2) or (2.3), or sometimes more conveniently in a common Dirac matrix space as [24-26]

$$|\chi' >; |\chi'' >= \left[ \frac{M - i\gamma_i P}{2M} [i\gamma_5; i\gamma_\mu / \sqrt{3}] C / \sqrt{2} \right]_{\beta \gamma} \otimes [1; \gamma_5 \hat{\gamma}_\mu u(P)]_\alpha \quad (2.7)$$

where the first factor is the $\beta \gamma$-element of a 4 x 4 matrix in the joint spin space of the quark #s 1, 2 [26], and the second factor the $\alpha$ element of a 4 x 1 spinor in the spin space of quark # 3; $C$ is a charge conjugation matrix with the properties [27]

$$-\hat{\gamma}_\mu = C^{-1} \gamma_\mu C; \quad \hat{\gamma}_5 = C^{-1} \gamma_5 C;$$

and $\hat{\gamma}_\mu$ is the component of $\gamma_\mu$ orthogonal to $P_\mu$. Finally, the representations of the flavour functions $\phi', \phi''$ satisfy the following relations in the "3" basis [28]

$$< \phi''|1; \bar{\tau}^{(3)}|\phi'' >= < \phi'|1; -\frac{1}{3} \bar{\tau}^{(3)}|\phi' >$$
### 2.3 Gordon Reduction on $V_{qq3}$ & $V_{qqq}$

At this stage we indicate the effect of Gordon reduction on the pairwise kernels $V(\hat{\xi},\hat{\eta})$ and the 3-body kernel $V_{qqq}$, following the original treatment of [29], (also quoted in [8]). The strategy lies in a close scrutiny of Eqs.(2.3-2.5) with a view to eliminate the Dirac matrices in favour of the Pauli matrices $\sigma_{\mu\nu}$. To that end, we express the Dirac propagators $S_F(p_i)$ as

$$iS_F(p_i) = (m_q - i\gamma^{(i)}_p . p_i)/\Delta_i; \quad \Delta_i = m_q^2 + p_i^2$$

Now employing the notation

$$V^{(i)}_\mu = (m_q - i\gamma^{(i)}_p . p_i)\gamma^{(i)}_\mu$$

the result of Gordon reduction is expressed by the formula [29, 8]

$$V^{(1)}_1 V^{(2)}_2 = - M_{12}^2 - (\hat{q} + \hat{q}')^2 + i(\hat{q} + \hat{q}') \times (\hat{\sigma}_1 + \hat{\sigma}_2) \times (\hat{q} - \hat{q}') - \sigma^{(1)}_{ij} \sigma^{(2)}_{ik} (q - q')_j (q - q')_k$$

where we have used a mixed vector-tensor notation for the various 3-vectors on the light-front in the sense of (2.2) and a short-hand notation $q$ for $q_{12}$. It is now possible to identify the pairwise kernel arising from the first group of terms on the rhs of Eq.(2.3) after expressing it in LF notation of Eq.(2.2) as follows:

$$V_{qq3} \equiv V(\hat{\xi}_3, \hat{\xi}_3') = g^2 s V^{(1)} V^{(2)} \times \frac{-4}{3} \times \frac{1}{(\hat{q}_{12} - \hat{q}'_{12})^2}$$

where we have included a net color factor of $(-4/3)$ arising from a folding of the factor $F_{12}$ for each $qq$ pair in a $3^*$ state, into $\lambda_3/2$ for the spectator quark # 3 in a 3 state; the corresponding gluon propagator has been taken on the light-front, in the notation of (2.2). Similarly the 3-body kernel $V_{qqq}$ in Eq.(2.3) may be identified as a sum of 3 terms arising from a folding of (1.2) with the 3 Dirac factors $(m_q - i\gamma^{(i)}_p . p_i)$ in the second term of (2.3).

$$V_{qqq} = \sum_{123} (-4/3) g^4 s [V^{(1)}_1 (\hat{k}_2 - \hat{k}_3) \times V^{(2)}_2 V^{(3)} + \{2\} + \{3\}]/\{k_1^2 k_2^2 k_3^2\}$$

using the notation $V^{(i)}_\mu$ of (2.8). The net effect of the (antisymmetric) combination of the color matrices $\lambda$ is represented by the factor $(-4/3)$ sitting in front; and $k_i = p_i - p'_i$. The ‘bosonized’ form of Eq. (2.3) is now

$$\Phi(p_1 p_2 p_3) = \sum_{123} \frac{1}{\Delta_1 \Delta_2} \int \frac{d^4 q_{12}' }{(2\pi)^4} V(\hat{\xi}_3, \hat{\xi}_3') \Phi(p'_1, p'_2, p_3)$$

$$+ \frac{1}{\Delta_1 \Delta_2 \Delta_3} \int \frac{d^4 q_{12}' d^4 p'_3}{(2\pi)^8} V_{qqq} \Phi(p'_1 p'_2 p'_3)$$

(2.12)
in terms of the structures (2.10) and (2.11) for the 2- and 3-body kernels respectively. And the Gordon reduction formula (2.9) may be repeatedly used to further simplify the structures of $V_{qq}$ [Eq. (2.10)] and $V_{qqq}$ [Eq. (2.12)], which are collected as follows.

### 2.3.1 Further Reduction of $V_{qq3}$ & $V_{qqq}$

The 2-body and 3-body interactions are pictorially represented by fig 1(a) where the vertices of the triangle $ABC$ stand for the position vectors $(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3)$ of the 3 quarks in configuration space, while the centroid $O$ has the position vector $\mathbf{R} = ((\mathbf{r}_1 + \mathbf{r}_2 + \mathbf{r}_3)/3$. The corresponding momenta of the gluon lines connecting them for pairwise $qq$ interactions are precisely the quantities $q_{ij} - q'_{ij}$ that appear as arguments in Eq. (2.10), while the momenta associated with the gluon lines connecting all the 3 quarks together, a la the $Y$-shaped fig. (1b), are the quantities $\hat{k}_i$ corresponding to the vectors $(\mathbf{r}_i - \mathbf{R})$. In the notation of Eq.(2.4), the former momenta are precisely $\hat{\xi}_i$, while the latter are the dual variables $\hat{\eta}_i$. [This interpretation will prove useful in the subsequent analysis].

The pairwise terms $V_{qq}$ were earlier considered in detail [8]. The ‘convective’ terms are of the form $(p_1 + p_2).(p'_1 + p'_2)$ which simplifies to $-M^2_{12} \approx -4M^2/9$, plus smaller terms. The spin-orbit terms are unimportant for the ground state of the $qqq$ proton, but the spin-spin terms survive for its ground state $S = 1/2$; their angular average works out as:

$$\frac{2}{3} \sum_{ij} \hat{\sigma}_i . \hat{\sigma}_j = \frac{1}{3} |\Sigma^2 - 9|$$
where we have introduced the total spin operator $\Sigma$ for the proton state, with $\Sigma^2 = 3$. Collecting these results the sum of the 3 pairwise interactions (2.10) work out as

$$V_{qq3} = g_s^2 \frac{8}{27} \times \left[ \sum \frac{4M^2 p_{3z}}{(q_{12} - q_{12}')}^2 - 3p_{3z}(\Sigma^2 - 9) \right] (2.13)$$

We now consider the more interesting term $V_{qqq}$, Eq.(2.12). The numerator can be broken up into 3 distinct groups: a) convective; b) spin-orbit; c) spin-spin. Using the notation $2\bar{p}_i = p_i + p'_i$, the convective part vanishes (due to antisymmetry with too few d.o.f.’s):

$$\sum 4[\bar{p}_1.\bar{p}_2][2\bar{p}_3.(k_1 - k_2)] \Rightarrow 0$$

The spin-orbit part comes from two groups of terms, one of which vanishes:

$$\sum 2\bar{p}_1.(k_1 - k_2)[\hat{\sigma}_3.(\hat{p}_3 \times \hat{k}_3)] \Rightarrow 0$$

where a mixed notation is used for the Pauli matrices, light-faced tensor for the 4D and hatted vector for the 3D forms. The other group gives rise to a non-zero contribution as

$$\sum 4\bar{p}_1.\bar{p}_2[i\sigma^{(3)}_{\mu\nu}\hat{k}_{3\mu}(k_{1\mu} - k_{2\mu})] \approx (-4iM^2/9)\sum \hat{\sigma}_i.\hat{\xi}_3 \times \hat{\eta}_3 (2.14)$$

in the notation of Eq.(2.4). Note the overall antisymmetry of the last factor (independent of the index 3).

Finally the (totally antisymmetric) spin-spin-spin contribution $SSS$ is defined as:

$$SSS = \sum_{123} [i\sigma^{(1)}_{\mu\nu}\hat{k}_{1\nu}(\hat{k}_{2\mu} - \hat{k}_{3\mu})i\sigma^{(2)}_{\lambda\rho}\hat{k}_{2\rho}i\sigma^{(3)}_{\lambda\sigma}\hat{k}_{3\sigma}] (2.15)$$

The simplification of this expression (which is elementary) is helped by the identity $\sigma^{(i)}_{\mu\nu} = \epsilon_{\mu\nu\alpha}\hat{\sigma}_\alpha^{(i)}$. The next step is angular averaging over each of the independent 3-vectors involved in each of the three terms above, namely, $<\hat{k}_i\hat{k}_j> = \hat{k}^2\delta_{ij}/3$. The resulting expression for $SSS$ is

$$SSS = \frac{2i}{9} \left\{ \hat{k}_1^2\hat{k}_2^2\hat{k}_3^2 \right\} \hat{\sigma}_1.\hat{\sigma}_2 \times \hat{\sigma}_3 \sum \frac{1}{\hat{k}_i^2}$$

Substituting for $SSS$ in Eq (2.11) and ignoring the spin-orbit terms (for the ground state of the proton) the net 3-body force is

$$V_{qqq} = -\frac{8ig_s^4}{27} (\hat{\sigma}_1.\hat{\sigma}_2 \times \hat{\sigma}_3) \sum \frac{1}{\hat{k}_i^2} (2.16)$$
3 3D Reduction By Green’s Function Method

We are now in a position to implement Step C of our program (establishing a 3D-4D interconnection between the corresponding wave functions), for which it is convenient to employ the Green’s function approach [22]. Calling the 4D Green’s functions associated with Ψ and Φ by $G_F$ and $G_S$ respectively, the connection between them analogously to Eq.(2.5) may be written as

$$G_F(\xi \eta; \xi' \eta') = W(P) \otimes \Pi_{123} S_{F,1}^{-1} (-p_i) G_S(\xi \eta; \xi' \eta') \Pi_{123} S_{F,1}^{-1} (-p'_i) \bar{W}(P')$$ (3.1)

where we have indicated the 4-momentum arguments of the Green’s functions involved, in a common $S_3$ basis $(\xi, \eta)$, and expressed the spin-flavour dependence of $G_F$ as a matrix product implied by the notation $W(P) \otimes \bar{W}(P')$.

Now to show the 3D-4D interconnection it is enough to work at the level of the ‘scalar’ Green’s function $G_S$ (relabelled as $\hat{G}$ for simplicity), since it is trivial to include the spin-flavour d.o.f.’s in a matrix notation via (2.5) later. The main steps for the scalar $qqq$ Green’s function are shown next, while Appendix A sketches the corresponding process for a typical $qq$ subsystem, from 3D reduction to 4D reconstruction, so as to serve as a simpler prototype for the actual $qqq$ case.

3.1 Reduction from 4D $G(\xi \eta; \xi' \eta')$ to 3D $\hat{G}(\hat{\xi} \hat{\eta}; \hat{\xi}' \hat{\eta}')$

We shall now outline a Green’s function method to establish a 3D-4D inter-linkage between the 4D $\Phi$ and the 3D $\phi$, following a generalization of the procedure developed some time ago [22], so as to include the effect of the direct 3-body force $V_{qqq}$. To that end we shall rederive the $qq$ denominator functions $D_{ij}$ of [22] arising from single integrations over $d\xi_{i0}$, to show that they are exactly proportional to the 3D denominator function $D_{123}$ arising from a double integration over $d\xi_{0} d\eta_{0}$ in the direct 3-body term to be considered here. [This is a big improvement over the earlier derivation [22] where this property was missing, and helps pave the way for putting together the effects of both 2- and 3-body forces within a common dynamical framework].

The full 4D Green’s function for ‘scalar’ quarks is $G(\xi \eta; \xi' \eta')$ (taking out the c.m $\delta$-fn), while its 3D counterpart $\hat{G}$ is [22]

$$\hat{G}(\hat{\xi} \hat{\eta}; \hat{\xi}' \hat{\eta}') = \int d\xi_{0} d\eta_{0} d\xi'_{0} d\eta'_{0} G(\xi \eta; \xi' \eta')$$ (3.2)
where the time-like subscript ‘0’ should be read as the corresponding LF component in the sense of Eq. (2.2). Note that both $G$ and $\hat{G}$ are $S_3$-symmetric, since the combination $d\xi_0 d\eta_0$ has this property. But the two hybrid Green’s functions defined as

$$
\hat{G}_{3\xi}(\xi_3 \eta_3; \xi_3' \eta_3') = \int d\xi_0 d\xi_0' G(\xi \eta; \xi' \eta');
\hat{G}_{3\eta}(\xi_3 \eta_3; \xi_3' \eta_3') = \int d\eta_0 d\eta_0' G(\xi \eta; \xi' \eta')
$$

are not $S_3$-symmetric (hence they are indexed), since the integration now involves only one of the two $\xi, \eta$ variables. Next, the 4D $G(\xi \eta; \xi' \eta')$ satisfies a BSE of the form

$$
i(2\pi)^4 G(\xi \eta; \xi' \eta') = \sum_{123} \int \frac{d^4\xi''}{4\Delta_1 \Delta_2} V(\hat{\xi}_3, \hat{\xi}_3'') G(\xi_3'' \eta_3; \xi_3' \eta_3')
+ \int \frac{d^4\xi'' d^4\eta''}{9(2\pi)^4 \Delta_1 \Delta_2 \Delta_3} V_{qqq} G(\xi_3'' \eta_3; \xi_3' \eta_3')
$$

which is analogous to Eq. (2.12) for the corresponding wave functions, except for a change of variables wherein the factors $1/4$ and $1/9$ in the first and second groups of terms stem from the relations (2.4) among the corresponding variables. It is understood that both these types of interaction have covariant 3D support, and they incorporate the effect of ‘Gordon reduction’ as outlined in Sect 2.3 above. The 3D reduction of (3.4) is now achieved by integrating it a la (3.2), leading to a structure of the form :

$$
(2\pi)^3 \hat{G}(\hat{\xi}; \hat{\xi}') = \sum_{123} \frac{1}{2\sqrt{2} D_{12}} \int d^3\hat{\xi}_3'' V(\hat{\xi}_3, \hat{\xi}_3'') \hat{G}(\hat{\xi}_3'' \eta_3; \hat{\xi}_3' \eta_3')
+ \frac{1}{3\sqrt{3}(2\pi)^3 D_{123}} \int d^3\xi'' d^3\eta'' V_{qqq} \hat{G}(\xi_3'' \eta_3; \xi_3' \eta_3')
$$

To explain the structure of certain factors, Eq (3.5) shows that we have two types of 3D denominator functions $D_{ij}$ and $D_{123}$, associated with pairwise 2-body and direct 3-body forces respectively. It is easily shown that they are simply related to each other. To that end, Appendix A already shows the structure of $D_{ij}$ :

$$
D_{12} = \frac{MD_{12+}}{P_+}; \quad D_{12+} = 2M [\omega_{1+}^2 p_{2+} + \omega_{2+}^2 p_{1+} - P_{12-} - p_{1+} + p_{2+}]
$$
where \( \omega_{1\perp}^2 \) equals \( m_q^2 + p_{1\perp}^2 \). Using the on-shellness (\( \Delta_3 = 0 \)) of the spectator (#3) then gives \( P_{12-} = P_- - P_{3-} \) which reduces further to \( P_- - \omega_{3\perp}^2/p_{3+} \). Its substitution back in (3.6) leads to the \( S_3 \) symmetric result:

\[
p_{3+} D_{12+} \equiv \frac{P^2}{2M^2} D_{123} = 2 \sum_{123} \{p_{2+} p_{3+} \omega_{1\perp}^2\} - 2p_{1+} p_{2+} p_{3+} P_-
\]

(3.7)

Here we have anticipated the structure of \( D_{123} \) associated with the \( V_{qqq} \) term whose formal definition is

\[
\frac{1}{D_{123}} = \int \frac{P^2 dq_{12-} dp_{3-}}{4M^2(2\pi)^2 \Delta_1 \Delta_2 \Delta_3}
\]

(3.8)

where the (double) contour integrations in the indicated variables leads to the desired result. The resultant 3D BSE for \( \hat{G} \) is now

\[
(2\pi)^3 D_{123} \hat{G}(\hat{\xi}; \hat{\eta}, \hat{\xi}', \hat{\eta}') = \sum_{123} P_{12} \int d^3 \hat{x}_3' V(\hat{\xi}_3, \hat{\eta}_3) \hat{G}(\hat{\xi}_3\hat{\eta}_3; \hat{\xi}_3\hat{\eta}_3')
\]

\[
+ \frac{1}{3\sqrt{3}(2\pi)^3} \int d^3 \hat{x}_3'' d^3 \hat{y}_3'' V_{qqq} \hat{G}(\hat{\xi}_3\hat{\eta}_3'; \hat{\xi}_3\hat{\eta}_3) \quad (3.9)
\]

The structure of this equation reveals some interesting symmetries, when (2.13) and (2.16) are substituted for \( V_{qq} \) and \( V_{qqq} \) respectively. Namely, in the first group of terms the \( \eta \) variable is the spectator since the integration is only over the \( \xi \) variable, while in the second group of terms, their relative roles are interchanged with the \( \xi \) variable effectively a spectator! This fact is not immediately apparent because of the double integration involved, but a little reflection shows that the absence of the \( \xi \) variable in (2.16) effectively ensures its spectator status in the argument of the corresponding Green’s function. We shall see this feature more clearly in the coordinate space representation of the \( \phi \) equation (corresponding to Eq (3.9) for \( \hat{G} \), to be considered in Section 5. But first we turn our attention to the reconstruction of the 4D Green’s function from the 3D form \( \hat{G} \) satisfying (3.9). To that end we note that the \( qq \) and \( qqq \) group of terms in (3.9) need different strategies and are best handled separately, one at a time, (temporarily) ignoring the presence of the other.
4 Reconstruction of 4D Wave Function

4.1 Reconstruction of $G$ with $qq$ Forces Only

Considering first the $qq$ group, we proceed exactly as in (A.10) of Appendix A for the $qq$ system. Namely, first express $\tilde{G}_{3\eta}$, eq.(3.3), in terms of the 3D $\hat{G}$, using the notation of (2.2) and the result of (3.7):

$$\tilde{G}_{3\eta}(\xi_3^3 \eta_3^3; \xi_3^3 \eta_3^3) = \frac{D_{123}}{2i\pi p_{3z}} \hat{G}(\hat{\xi}_3^3 \hat{\eta}_3^3) \frac{D_{123}'}{2i\pi p_{3z}'} \Delta_1 \Delta_2 \Delta_1' \Delta_2'$$ (4.1)

In a similar way the fully 4D $G$ function is expressible in terms of the hybrid function $\tilde{G}_{3\xi}$ as

$$G(\xi \eta; \xi' \eta') = \sum_{123} \frac{D_{123}}{2i\pi p_{3z}} \tilde{G}_{3\xi}(\xi_3^3 \eta_3^3; \xi_3^3 \eta_3^3) \frac{D_{123}'}{2i\pi p_{3z}'} \Delta_1 \Delta_2 \Delta_1' \Delta_2'$$ (4.2)

Now since the $\tilde{G}_{3\xi}$ function is not determined from $qqq$ dynamics alone, we invoke an ansatz similar to, but more symmetrical than, [22] :

$$\tilde{G}_{3\xi}(\xi_3^3 \eta_3^3; \xi_3^3 \eta_3^3) = \hat{G}(\hat{\xi}_3^3 \hat{\eta}_3^3) F(p_3, p_3')$$ (4.3)

where the balance of the $p_3$ (spectator) dependence is in the (as yet unknown) $F$ function, subject to an explicit self-consistency check. To that end, try the (symmetrical) Lorentz-invariant form

$$F(p_3, p_3') = \frac{A_3}{\Delta_3} \delta(\Delta_3 - \Delta_3')$$ (4.4)

and integrate both sides of (4.4) w.r.t. $dp_{30}dp_{30}'$, in the notation of (2.2), to show that the consistency check is met with the mass shell value of the spectator momentum :

$$A_3 = \frac{4p_{3z}p_{3z}'}{2i\pi}$$

Substitution from $A_3$ in (4.4) then gives the symmetrical form

$$F(p_3, p_3') = \frac{4p_{3z}p_{3z}'}{2i\pi \Delta_3} \delta(\Delta_3 - \Delta_3')$$

whence the 4D $G$-fn in terms of $\hat{G}$ via the sequence (4.3) and (4.2), works out as

$$G(\xi \eta; \xi' \eta') = \sum_{123} \frac{D_{123}}{\Delta_1 \Delta_2} \hat{G}(\hat{\xi}_3^3 \hat{\eta}_3^3) \frac{D_{123}'}{\Delta_1' \Delta_2'} \delta(\Delta_3 - \Delta_3') \frac{(2\pi i)^5 \sqrt{\Delta_3 \Delta_3'}}{(2\pi i)^5 \sqrt{\Delta_3 \Delta_3'}}$$ (4.5)
4.2 Reconstruction of $G$ with $qqq$ Forces Only

Next we consider only the $qqq$ group of terms for a reconstruction from $\hat{G}$ (Eq (3.5)) to $G$ (Eq (3.4)). This case is more akin to the $qq$ case of Appendix A in the sense that if we use the collective indices $(\xi, \eta) \equiv \rho$ and $(\xi', \eta') \equiv \rho'$ for the initial and final state arguments of the $G$-function we can define two kinds of $\tilde{G}$ as follows.

$$\tilde{G}(\hat{\rho}; \hat{\rho}') = \int d\rho_0 G(\rho; \rho'); \quad \tilde{G}(\rho; \hat{\rho}') = \int d\rho'_0 G(\rho; \rho')$$  \hspace{1cm} (4.6)

where the integrals on the RHS are each of the double integral type, so that the definitions of the denominator functions involved are given by Eq.(3.8). Thus, following (A. 9), $\tilde{G}$ and $\hat{G}$ are connected as follows

$$\tilde{G}(\rho, \hat{\rho}') = \frac{D_{123}(\hat{\rho})}{(2i\pi)^2 \Delta_1 \Delta_2 \Delta_3} \hat{G}(\hat{\rho}; \hat{\rho}')$$  \hspace{1cm} (4.7)

together with a second one with the roles of $\rho, \rho'$ interchanged. Continuing exactly as in Appendix A, $G$ of Eq (3.4) gets expressed in terms of $\tilde{G}$ on the RHS, since the interaction $V_{qqq}$ does not involve the variables $\rho_0'$. Thence another application of Eq.(4.7) for expressing $\tilde{G}(\rho; \hat{\rho}')$ in terms of $\hat{G}$ finally yields the desired connection:

$$G(\rho; \rho') = \frac{D_{123}(\hat{\rho})}{(2i\pi)^2 \Delta_1 \Delta_2 \Delta_3} \tilde{G}(\hat{\rho}; \hat{\rho}') \frac{D_{123}(\hat{\rho}')}{(2i\pi)^2 \Delta'_1 \Delta'_2 \Delta'_3}$$  \hspace{1cm} (4.8)

where the symbol $\rho$ stands collectively for $(\xi, \eta)$, as does $\rho'$. Note that in this case the reconstruction of $G$ in terms of $(the~fully~reduced)\tilde{G}$ is unique, and does not require any extra ansatz like (4.4).

4.3 Putting Both $V_{qq}$ & $V_{qqq}$ Together

We have now two distinct types of 3D-4D interconnections valid for pure $qq$ and pure $qqq$ forces respectively. But when both types of forces are present, an exact interconnection is very difficult to derive. We shall therefore strive for an approximate but sufficiently realistic solution based on the relative strengths of the two forces, namely, an overwhelming preponderance of $qq$ over $qqq$ forces. To give effect to this strategy, we first note that the 3D level does not involve any approximation since Eq. (3.9) treats both types of interaction on par. It is only at the 4D level of reconstruction that an
approximation is necessary for putting the two forces together. Now taking account of the dominance of $qq$ over $qqq$ forces, the simplest choice is to prefer the structure of Eq (4.5) over that of Eq. (4.8). And although the ratio of the two 3D-4D interconversion jackets in (4.5) over (4.8) is a singular quantity

$$\delta(\Delta_3 - \Delta_3') \times (2\pi i)\sqrt{\Delta_3\Delta_3'}$$

it causes no harm for physical amplitudes since such singular quantities get ironed out on integration over the various momenta \[23\]. With this understanding, Eq.(4.5) represents the reconstruction of the full 4D Green’s function $G(\xi;\eta)\xi\eta')$ in terms of the 3D quantity $\hat{G}(\hat{\xi};\hat{\eta}')$ where the latter now satisfies Eq (3.9) which includes the effect of both $qq$ and $qqq$ forces.

### 4.4 3D-4D Interconnection for Wave Functions

Finally the spectral representations of $G(\xi;\eta')$ and $\hat{G}(\hat{\xi};\hat{\eta})$ for the $qqq$ system, exactly on the lines of (A.12-A.13) for a $qq$ subsystem near a bound state pole $P^2 = -M^2$, are

$$G(\xi;\eta') = \sum_n \Phi_n(\xi,\eta)\Phi_n^*(\xi',\eta')/(P^2 + M^2); (4.9)$$

and a similar equation for $\hat{G}$ vis-a-vis $\phi$. These give the connection between the 4D wave function $\Phi$ which satisfies Eq (2.12), and the 3D wave function $\phi$ corresponding to the Green’s function $\hat{G}$ which satisfies (3.9). For purposes of evaluating transition amplitudes via Feynman diagrams, it is convenient to index $\Phi$ as $\Phi_1 + \Phi_2 + \Phi_3$, and a corresponding indexing for the associated vertices as $V = V_1 + V_2 + V_3$, as in ref \[22\], so as to keep track of which quark is involved in which vertex.

$$\Phi_3(\xi,\eta) \equiv \frac{V_3}{\Delta_1\Delta_2\Delta_3} = \frac{D_{123}}{\Delta_1\Delta_2}\phi(\xi,\eta)\frac{\sqrt{\delta(\Delta_3)}}{(2\pi i)^{5/2}\sqrt{\Delta_3}} (4.10)$$

The $\delta$-function in eq.(4.7) has nothing to do with any connectedness problem; see ref.\[22\] for detailed reasons. Finally, the use of Eq.(2.5) with (4.10) yields an explicit structure for the actual (fermionic) wave function $\Psi$ as

$$\Psi(\xi,\eta) = \Pi_{123}S_F(p_i)D_{123} \sum_{123} \left[\phi(\xi,\eta)\frac{\sqrt{\delta(\Delta_3)\Delta_3}}{(2\pi i)^{5/2}}\right] \times W(P) (4.11)$$
5  Complete $\phi$ Equation In Coordinate Space

Our final task is to set up (and solve) the 3D BSE for $\phi$ which may be inferred from (3.9) by making use of a spectral representation similar to (4.9), and going to the pole $P^2 = -M^2$:

$$(2\pi)^3 D_{123} \phi(\hat{\xi}, \hat{\eta}) = \sum_{123} \frac{P_{3z}}{\sqrt{2}} \int d^3 \xi_3 V_{qq3} \phi(\hat{\xi}_3, \hat{\eta}_3) + \frac{1}{3\sqrt{3}(2\pi)^3} \int d^3 \xi'' V_{qqq} \phi(\hat{\xi''}_3, \hat{\eta''}_3)$$  \hspace{1cm} (5.1)

where $V_{qq3}$ is given by (2.13); $V_{qqq}$ by (2.16); and $D_{123}$ by (3.7). To transform this equation in coordinate space, define the combinations analogous to (2.4) as

$$\sqrt{2}s_3 = r_1 - r_2; \quad \sqrt{3}t_3 = -2r_3 + r_1 + r_2$$  \hspace{1cm} (5.2)

Then a Fourier transform to the coordinate space representation gives for the pairwise terms $V_{qq3}$ the coulombic structure

$$V^{(2)}(s) = -\frac{2g^4}{9} \phi(s, t)$$

which multiplies the coordinate space wave function $\phi(s, t)$ ( $S_3$ symmetric in its arguments). Similarly, the ($S_3$ symmetric) double Fourier transform of $V_{qqq}$ is defined as

$$V^{(3)}(s, t) = \frac{-4ig^4}{9} \int d^3 \xi' \exp[i\hat{\xi'} \cdot s] \times \sum_{123} \frac{1}{(\hat{\eta}_3 - \hat{\eta}_3')^2} \left[ \hat{\sigma}_1.(\hat{\sigma}_2 \times \hat{\sigma}_3) \right]$$  \hspace{1cm} (5.3)

The integration in (5.3) involves a 3D $\delta$-function $\delta^3(\hat{s}_3)$; and the double integration in (5.4) additionally involves the factor $1/|t_3|$ due to the $\eta$ integration. The $\delta$ function being singular may be ‘regularized’ by using a differential representation (acting on the scalar function $\phi(s, t)$), with $|\hat{s}_3| = |s_3|:

$$\frac{\delta^3(\hat{s}_3)}{(2\pi)^3} \phi(s, t) = -\frac{1}{4\pi|s_3|} [\partial^2_{s_3}] \phi(s, t)$$  \hspace{1cm} (5.5)

The result of integration in (5.3) is then expressed as

$$V^{(2)}(s) = \sum \left[ \frac{8M^2 \alpha_s p_{3z}}{27|s_3|} + \frac{\alpha_s (\Sigma^2 - 9)}{27|s_3|} p_{3z} \partial^2_{s_3} \right]$$  \hspace{1cm} (5.6)
Next the 3-body term term (5.4) integrates, with the help of (5.5), to

\[
4\alpha_s^2 \frac{i\hat{\sigma}_1.(\hat{\sigma}_2 \times \hat{\sigma}_3)}{27\sqrt{3}} \sum \frac{1}{s_3 t_3} \partial^2_{s_3}
\]

Now the product \(1/s_3 t_3\) can be approximately replaced by the \(S_3\) symmetric expression \(2/[s_3^2 + t_3^2]\) which can be taken out of the summation sign (with index ‘3’ dropped) to give \(\sum \partial^2_{s_3}\) which in turn equals \((3/2)(\partial^2_s + \partial^2_t)\), an \(S_3\) symmetric sum with index ‘3’ dropped again. This finally leads from (5.4) to the \(S_3\) symmetric form of the 3-body term in coordinate space:

\[
V^{(3)}(s, t) = 4\alpha_s^2 \frac{i\hat{\sigma}_1.(\hat{\sigma}_2 \times \hat{\sigma}_3)}{9\sqrt{3}(s^2 + t^2)} [\partial^2_s + \partial^2_t]
\]

(5.7)

For further manipulations it is convenient to replace the antisymmetric spin operator \(A = [i\hat{\sigma}_1.(\hat{\sigma}_2 \times \hat{\sigma}_3)]\) by one of its possible eigenvalues as follows: Squaring this quantity yields in a simple way

\[
A^2 = -A - 15 + \Sigma^2; \quad \Sigma \equiv \hat{\sigma}_1 + \hat{\sigma}_2 + \hat{\sigma}_3
\]

from which one obtains the successive results

\[
A^3 + A^2 + 15A = \Sigma^2A; \quad A^2(A + 1)^2 = [\Sigma^2 - 15]^2
\]

The last equation yields the four solutions

\[
A = -\frac{1}{2} \pm \sqrt{15 - \Sigma^2 - 1/4}; \quad A = -\frac{1}{2} \pm \sqrt{\Sigma^2 - 15 - 1/4}
\]

(5.8)

whose substitution in Eqs (5.4) and (5.7) summarises the full content of the total spin effect of the \(V_{qqq}\) term. Finally, the denominator term \(D_{123}\) is a differential operator in coordinate space:

\[
D_{123}/4 = \sum_{123} \left[(-m_q^2 + \partial^2_{\perp 3})\partial_{1z}\partial_{2z}\partial_{3z} - iM\partial_{1z}\partial_{2z}\partial_{3z}\right]
\]

(5.9)

Thus \(\phi\) satisfies the following equation in coordinate space:

\[
D_{123}\phi(s, t) = [V^{(2)}(s) + V^{(3)}(s, t)]\phi(s, t)
\]

(5.10)

where the various operators are given by (5.6) - (5.9).
5.1 Simplified Form of the $\phi$ Equation (5.10)

Since this paper is intended as a preliminary mathematical framework for the dynamical effect of the spin terms, especially the spin-rich $V_{qqq}$ term on the spatial structure of the proton wave function, pending a full-fledged study of the proton spin anomaly, we shall at this stage merely outline a qualitative procedure to determine the nature of the solution of Eq. (5.10), with particular reference to the possible role of the eigenvalues (5.8) of $A$, on the solution of this differential equation. To that end, we shall make some drastic simplifications, starting with the differential operator (5.9) whose principal terms (up to second order), in momentum space, work out as

$$D_{123} \approx \frac{4M^2}{9} (\xi^2 + \eta^2) + \left[ \frac{2M^2}{3} - 2m_q^2 \right] (\xi^2 + \eta^2) + \frac{4M^2}{3} (m_q^2 - M^2/9)$$

Further simplification arises with the 'special' value $M = 3m_q$ which then yields a simple yet transparent expression in coordinate space:

$$D_{123} \approx \frac{4M^2}{9} [-\partial_s^2 - \partial_t^2]$$  \hspace{1cm} (5.11)

an operator with full rotational invariance and $S_3$ symmetry in the 3D space (on light-front) defined by Eq. (2.2). In a similar vein, the 2-body terms (5.6) can be simplified by the replacements $p_{iz} \approx M/3$, so as to yield

$$V^{(2)}(s, t) = \frac{8\sqrt{2}M^3\al_s}{27\sqrt{s^2 + t^2}} + \frac{M\al_s(\Sigma^2 - 9)\sqrt{s^2 + t^2}}{54\sqrt{s^2 + t^2}} [\partial_s^2 + \partial_t^2]$$  \hspace{1cm} (5.12)

where we have made a further simplification based on certain standard inequalities [31]

$$\sum_{123} \frac{1}{|s_3|} \approx \frac{3\sqrt{2}}{\sqrt{s^2 + t^2}}$$

And the 3-body term (5.7) may be written more compactly in terms of of the operator $A$ with eigenvalues (5.8) as

$$V^{(3)}(s, t) = \al_s^2 \frac{4A}{9\sqrt{3}(s^2 + t^2)} [\partial_s^2 + \partial_t^2]$$  \hspace{1cm} (5.13)

One has now a differential equation for (5.10), with the simplified operators (5.11-13) which exhibit a 6D symmetry. Taking $R^2 = s^2 + t^2$, the 6D Laplacian for the ground state of the proton (with all angular d.o.f.'s dropped)
takes the simpler form
\[
\left[4M^2/9 + \frac{M\alpha_s(\Sigma^2 - 9)}{27R\sqrt{2}} + \frac{4A\alpha_s^2}{9R^2\sqrt{3}}\right](\partial_R^2 + \frac{5}{R}\partial_R)\phi(R) + \frac{8M^3\alpha_s\sqrt{2}}{27R}\phi(R) = 0
\]

which on rescaling with the dimensionless variable \(X = MR\) leads to the simpler form
\[
\left[1 + \frac{\alpha_s(\Sigma^2 - 9)\sqrt{2}}{24X} + \frac{A\alpha_s^2}{\sqrt{3}X^2}\right](\partial_X^2 + \frac{5}{X}\partial_X)\phi + \frac{2\alpha_s\sqrt{2}}{3X}\phi = 0
\]  \hspace{1cm} (5.14)

### 5.2 Spin Effects of \(V_{qq}\) & \(V_{qqq}\) Terms on \(\phi\) Singularity

The first thing to notice from the \(\phi\) Equation, (5.14), is that the spin-dependent parts of both \(V_{qq}\) and \(V_{qqq}\) appear in the multiplying factor with the 6D Laplacian acting on \(\phi\), and are proportional to \(\alpha_s\) and \(\alpha_s^2\) respectively, with the \(V_{qq}\) term having a milder singularity \((\sim X^{-1})\) than the \(V_{qqq}\) term \((\sim X^{-2})\). Further information on the singularity of the differential equation (at points other than \(R = 0\)) hinges on the nature of the eigenvalues of the spin operator \(A\) which appears in above multiplying factor. Now these eigenvalues are given by (5.8), two of which are complex for the state of the proton \((\Sigma^2 = 3)\), but the other two are real. Of the real solutions the positive eigenvalue does not yield a zero in this multiplying factor but the negative eigenvalue \((\approx -4)\) does give a zero for a real value of \(R\) at a point \(X_0^2\) between \(X^2 = 0\) and \(X^2 = \infty\):

\[
X_0^2 = \frac{\alpha_s\sqrt{2}}{4}X_0 + \frac{A\alpha_s^2}{\sqrt{3}} = 0; \quad X_0 = +\frac{\alpha_s\sqrt{2}}{8} \pm \sqrt{[\alpha_s^2/32 - \frac{A\alpha_s^2}{\sqrt{3}}]}
\]  \hspace{1cm} (5.15)

where the value of \(\Sigma^2 = 3\) for the proton state has been substituted. Indeed this zero (for \(A \approx -4\)) is a key element of the dynamical effect of the spin-rich 3-body force term, although the spin effect of the 2-body \(V_{qq}\) term is marginal.

To study this effect more quantitatively, we seek an approximate solution of Eq. (5.14) in the neighbourhood of the zero at (5.15) for which a crude numerical estimate suggests the following location. Taking a 3-flavour structure \(\alpha_s = 2\pi/[9 \ln M/\Lambda]\), with \(\Lambda \approx 150 MeV\), one finds \(\alpha_s \approx 0.39\), whence

\[
X_0 \approx 0.0689 \pm X_1; \quad X_1 \equiv 0.5841
\]
This shows that the $qq$ term gives $\sim 10\%$ shift around the central value of $X_1$, which corresponds to $R \approx 0.12 fm$, and is almost entirely due to the spin effect of the $V_{qqq}$ term. Therefore in the spirit of this qualitative investigation, it makes sense to drop this 10\% effect, in which case Eq (5.14) simplifies to

$$
[1 - \frac{X_1^2}{X^2}](\partial_X^2 + \frac{5}{X}\partial_X)\phi + \frac{2\alpha_s\sqrt{2}}{3X}\phi = 0; \quad X_1 = 0.584
$$

(5.16)

To bring it nearer to a standard form, transform to the independent variable $z$ according to $zX_0^2 = X_0^2 - X^2$, which yields

$$
z(1 - z)\partial_z^2\phi - 3z\partial_z\phi - \beta\sqrt{1 - z}\phi = 0; \quad \beta \equiv \frac{\alpha_s}{3\sqrt{2}}X_1 \approx 0.058
$$

(5.17)

This equation is almost (not quite) of the hypergeometric form but it can be reduced to a standard one (with singularities located at $z = 0, 1, \infty$) [32] by exploiting the smallness ($\beta = 0.058$) of the last term to replace it with a constant (with value corresponding to $z = 1/2$):

$$
z(1 - z)\partial_z^2\phi - 3z\partial_z\phi - \frac{\beta}{\sqrt{2}}\phi = 0
$$

(5.18)

An equivalent equation may be obtained with the transformation $z = 1 - x$:

$$
x(1 - x)\partial_x^2\phi + 3(1 - x)\partial_x\phi - \frac{\beta}{\sqrt{2}}\phi = 0
$$

(5.19)

where $x = 1$ corresponds to the location $X = X_1$ of the singularity, as indicated in Eq. (5.16). [Note that this singularity corresponds to the negative eigenvalue of $A$, thus reflecting the dynamical effect of the spin-rich $V_{qqq}$ term, while a positive or complex eigenvalue of $A$ would result in the disappearance of this singularity.] The solution of Eq.(5.19) is then given by [32]

$$
\phi = F(a, b| 3|x); \quad a + b = 2; \quad ab = \frac{\beta}{\sqrt{2}}
$$

(5.20)

with the entire machinery of hypergeometric functions available [32] for exploring its properties according to need. The scaled variable $x$ is related to the 6D distance $R$ by $MR = X_1x$ where $X_1 \approx 1.5\alpha_s$ is directly related to the location of the singularity induced by the spin structure of the Y-shaped 3-body force. And the reconstruction of the full Bethe Salpeter wave function $\Psi$ as a sum of 3 pieces $\Psi_i$ is now only a matter of substituting (5.20) in (4.11), whence the corresponding vertex functions $V_i$ are immediately identified [22,23] for specific transition amplitudes a la Feynman diagrams.
6  Retrospect : Critique Of The 3-Body Force

In retrospect, we have considered, in conjunction with pairwise \( qq \) forces, the effect of a new type of 3-body force \( V_{qqq} \), rich in spin content, on the analytical structure of the \( qqq \) wave function in the high momentum regime of QCD where the confining interaction is unimportant, rendering the dominant force Coulombic. As to the anatomy of this (spin-rich) \( V_{qqq} \), we have taken it to be generated by a \( ggg \) vertex (a genuine part of the QCD Lagrangian) wherein the 3 radiating gluon lines end on as many quark lines, giving rise to a (Mercedes-Benz type) \( Y \)-shaped diagram, a la fig 1. From a physical point of view it is natural to expect that such a spin-rich structure should play a potentially crucial role in the so-called ‘spin anomaly’ of the proton, a subject that seems once again to have raised its head in the context of new polarized beam techniques now available [5] for resolving the issue experimentally. With that end in view, our strategy has been to determine the dynamical effect of the spin dependence of \( V_{qq} \) and \( V_{qqq} \) forces on the analytical structure of the internal 3D wave function \( \phi \). [It is emphasized that this effect must be carefully distinguished from the ” kinematical ” effect of spin, which manifests in several other ways, namely through the presence of various \( \gamma \) matrices that appear in equations like (4.11) connecting \( \phi \) to the full 4D wave function \( \Psi \)]. Indeed we found in Section 5 that while the spin-dynamical effect of 2-body forces is marginal, that of the spin-rich 3-body force is quite pronounced, and shows up through the possible ” eigenvalues ”, Eq (5.7), of the spin operator \((i\sigma_1 \cdot \sigma_2 \times \sigma_3)\) which is a part of \( V_{qqq} \): only a negative eigenvalue (there is only one !), induces a singularity in the differential equation for \( \phi \), but not others. The resulting dynamical effect is expressed by a hypergeometric function, Eq.(5.20), which gets folded into the 4D BS amplitude \( \Psi \) via Eq.(4.11), thus fulfilling the original (limited) motivation behind this study.

Unfortunately, the dynamical framework underlying the contents of this paper has had a long history born out of the author’s long involvement with the so-called Bethe-Salpeter Equation (BSE), often with extended periods of stalemate (and frustration !), but mostly centred around the quest for a satisfactory definition of probability within the BSE framework. Eventually it became possible to settle for a toned down version of BSE, that of a Salpeter-like equation (3D support for the kernel), which is amenable to a probability interpretation at the 3D level. And its 4D features, although present in the original formulation itself [6], had for decades remained hidden from view, but
were finally dug out [7] in the context of an independent approach designed to explore a 3D-4D interconnection between the corresponding BS amplitudes when the kernel has a 3D support [8]. Further, the lack of covariance in the original formulation [6] was subsequently remedied via a special (instantaneous) frame of reference in which the composite hadron of 4-momentum $P_\mu$ is at rest [33]. This result turned out to be in conformity with the Markov-Yukawa Transversality Principle (MYTP) [11, 12], which happens to be a ‘gauge principle’ in disguise [13]: It ensures that the interactions among the constituents be transverse to the direction of $P_\mu$. Subsequent refinements have been mostly technical, especially the use of Dirac’s light-front formulation so as to extend the dynamical range of validity of the BS framework by overcoming certain practical problems like ‘Lorentz mismatch disease’ [17] associated with different vertices of a given Feynman diagram.

It therefore looked worthwhile to employ this old-fashioned formalism in the present context of a new kind of 3-body force $V_{qqq}$, with enough details put in for a reasonably self-contained description without having to dig frequently into the original sources. This has also given an opportunity to make several refinements, especially the derivation of a common denominator function associated with both the $qq$ and $qqq$ forces and extending the earlier formalism to accommodate the $V_{qqq}$ force, for which this theoretical framework seems to be well suited.

The next part of this programme involves the derivation of the requisite QCD parameters for the baryon spin, for which some key ingredients are i) the forward scattering amplitude off the $i\gamma_\mu\gamma_5$ operator, inserted at the individual quark lines, and ii) a more elaborate 2-gluon contribution to the proton spin, which is reserved for a subsequent paper within the same formalism.

## 7 Acknowledgements

Several colleagues have contributed to the evolution of the BS formalism, with and without active authorships, and the present paper also shares the general acknowledgement. Yet two items stand out in the specific context of this paper: First, the analysis in Section 5 owes its origin to a simple-minded methodology due to his late Father, Jatindranath Mitra. Second, the pedagogical techniques of $qqq$ symmetry with several d.o.f.,s, which have played a key role in this paper, are due to the late Mario Verde as described
Appendix A : $qq$ Subsystem Formalism

Using the correspondence (2.2) of the text, most of the covariant instant form results of [22] may be taken over to the present light-front situation [7]. In this Appendix, we outline the structure of the 4D / 3D interconnection between the corresponding Green’s functions for the $qq$ sub-system as a prototype for the actual $qqq$ system considered in Section 2 of the text. The $qq$ Green’s functions satisfy the respective equations [22]:

$$(2\pi)^4iG(q, q'; P) = \frac{1}{\Delta_1\Delta_2} \int d^4q''V(\hat{q}, q'')G(q'', q'; P); \quad (A.1)$$

$$\hat{G}(\hat{q}, \hat{q}') = \int dq_0dq'_0G(q, q'; P) \quad (A.2)$$

Integrating both sides of (A.1) gives via (A.2), the 3D BSE for a bound state (no inhomogeneous term!):

$$(2\pi)^3D(\hat{q})\hat{G}(\hat{q}, \hat{q}') = \int d^3\hat{q}''V(\hat{q}, \hat{q}'')\hat{G}(\hat{q}'', \hat{q}') \quad (A.3)$$

where the 3D denominator function $D(\hat{q})$ is defined as

$$\frac{2i\pi}{D(\hat{q})} = \int \frac{dq_0}{\Delta_1\Delta_2} \quad (A.4)$$

leading (for general unequal mass kinematics) to [8]

$$D(\hat{q}) = \frac{M}{P_+}D_+(\hat{q}); \quad D_+(\hat{q}) = 2P_+[\hat{q}^2 - \frac{\lambda(M^2, m_1^2, m_2^2)}{4M^2}] \quad (A.5)$$

where $\lambda$ is the triangle function of its arguments. Now define the hybrid Green’s functions [22]:

$$\tilde{G}(\hat{q}, q') = \int dq_0G(q, q'; P); \quad \tilde{G}(q, \hat{q}') = \int dq_0'G(q, q'; P) \quad (A.6)$$
Using (A.6) on the RHS of (A.1) gives

\[(2\pi)^4 i G(q, q'; P) = \frac{1}{\Delta_1 \Delta_2} \int d^3 \tilde{q}'' V(\tilde{q}, \tilde{q}'') \tilde{G}(\tilde{q}'', q') \] (A.7)

Integrating (A.1) w.r.t. \( dq_0' \) only, and using (2.7) again, gives

\[(2\pi)^4 i \tilde{G}(q, \tilde{q}') = \frac{1}{\Delta_1 \Delta_2} \int d^3 \tilde{q}'' V(\tilde{q}, \tilde{q}'') \tilde{G}(\tilde{q}'', \tilde{q}') \] (A.8)

From (A.8) and (A.3), \( \tilde{G} \) and \( \hat{G} \) are connected as:

\[\tilde{G}(q, \tilde{q}') = \frac{D(\tilde{q})}{2i\pi \Delta_1 \Delta_2} \hat{G}(\tilde{q}, \tilde{q}') \] (A.9)

Interchanging \( q \) and \( q' \) in (A.9) gives the dual result

\[\tilde{G}(\tilde{q}, q') = \frac{D(\tilde{q}')}{2i\pi \Delta'_1 \Delta'_2} \hat{G}(\tilde{q}, \tilde{q}') \] (A.10)

Substitution in (A.7) gives the desired 3D-4D interconnection

\[G(q, q'; P) = \frac{D(\tilde{q})}{2i\pi \Delta_1 \Delta_2} \hat{G}(\tilde{q}, \tilde{q}'; P) \frac{D(\tilde{q}')}{2i\pi \Delta'_1 \Delta'_2} \] (A.11)

Next, spectral representations for the 4D and 3D G-fns in (A.11) give [22]

\[G(q, q'; P) = \sum_n \Phi_n(q; P) \Phi_n^*(q'; P)/(P^2 + M^2); \] (A.12)

\[\tilde{G}(\tilde{q}, \tilde{q}') = \sum_n \phi_n(\tilde{q}) \phi_n^*(\tilde{q}')/(P^2 + M^2) \] (A.13)

where \( \Phi_n \) and \( \phi_n \) are 4D and 3D wave functions, so that their interconnection (valid for any bound state \( n \)), is expressed by:

\[\Gamma(\tilde{q}) \equiv \Delta_1 \Delta_2 \Phi(q; P) = \frac{D(\tilde{q})\phi(\tilde{q})}{2i\pi} \] (A.14)

which tells us that the covariant vertex function \( \Gamma \) on the light-front is a function of \( \tilde{q} \) only [22] via its definition (2.2). This derivation is a prototype for the \( qqq \) case of text.

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