Generalised ‘Arcsine’ laws for run-and-tumble particle in one dimension

Prashant Singh and Anupam Kundu

International Centre for Theoretical Sciences, TIFR, Bengaluru 560089, India
E-mail: prashant.singh@icts.res.in

Received 12 April 2019
Accepted for publication 10 June 2019
Published 16 August 2019

Online at stacks.iop.org/JSTAT/2019/083205
https://doi.org/10.1088/1742-5468/ab3283

Abstract. The ‘Arcsine’ laws of Brownian particles in one dimension describe distributions of three quantities: the time $t_m$ to reach maximum position, the time $t_r$ spent on the positive side and the time $t_{\ell}$ of the last visit to the origin. Interestingly, the cumulative distribution of all three quantities are the same and given by Arcsine function. In this paper, we study distribution of these three times $t_m$, $t_r$ and $t_{\ell}$ in the context of single run-and-tumble particle in one dimension, which is a simple non-Markovian process. We compute exact distributions of these three quantities for arbitrary time and find that all three distributions have a delta function part and a non-delta function part. Interestingly, we find that the distributions of $t_m$ and $t_r$ are identical (reminiscent of the Brownian particle case) when the initial velocities of the particle are chosen with equal probability. On the other hand, for $t_{\ell}$, only the non-delta function part is the same as the other two. In addition, we find explicit expressions of the joint distributions of the maximum displacement and the time at which this maxima occurs. We verify all our analytical results through numerical simulations.

Keywords: active matter, Brownian motion
1. Introduction

The ‘Arcsine’ laws provide statistical characterisation of certain temporal behaviours of Brownian motion or more generally of discrete random processes [1, 2]. These laws describe the probability densities of the following three quantities measured along a trajectory over a time interval $t$: (i) the time $t_m$ to reach the maximum of the process, (ii) the residence time $t_r$ spent on the positive (or the negative) semi axis and (iii) the last time $t_\ell$ the process changes sign (or crosses the origin). Remarkably, for a one dimensional Brownian process $B(\tau)$ starting from the origin, the cumulative probabilities of the above mentioned three observables are same and it is given by $F_c(\tau, t) = \text{Prob.}(t_c \leq \tau) = (2/\pi)\text{Arcsine}(\sqrt{\tau/t})$ where $t_c$ representing $t_m$, $t_r$ or $t_\ell$. These
results are often known as 1st, 2nd and 3rd ‘Arcsine’ laws, respectively, for obvious reason. The fact that the corresponding probability density function
\[ P_{\text{Br}}(t_a, t) = \frac{1}{\pi t_a} \frac{1}{t - t_a}, \]
has counterintuitive (integrable) divergences at the edges \( t_a = 0 \) and \( t_a = t \), has attracted a lot of general interests in these laws over the years. Since the discovery of these laws, these three quantities have been studied, either together or individually in various contexts starting from simple and constrained Brownian motion [3, 4], queuing process [5], random acceleration process, [6, 7], fractional Brownian motion [8], renewal processes [9–11], convex hull problems [12, 13], vicious walkers [14], Bessel Process [15] to general Markovian [16, 17] and non Markovian [18, 19] processes. The ‘Arcsine’ laws and related distributions have also been explored in other contexts like, finance [20–22], diffusion in disordered potential [23, 24], conductance in disordered materials [25, 26], chaotic dynamical systems [27], quantum chaotic scattering [28], partial melting of polymers [29] and, quite recently in stochastic thermodynamics [30].

In this paper, we study the three observables, namely, \( t_m \) (the time to reach maxima), \( t_r \) (the residence time) and \( t_f \) (the time of the last visit to the origin) for a single run and tumble process. This process describes the motion of a particle in one dimension subject to telegraphic noise. The Langevin equation of motion of a run and tumble particle (RTP) is given by
\[ \frac{dx}{dt} = v\sigma(t), \]
where \( x \) is the position of the particle at time \( t \) and \( v(>0) \) is the speed. In contrast to the Brownian motion the noise \( \sigma(t) \) takes only \( \pm 1 \) values and changes sign with rate \( \gamma \). The noise \( \sigma(t) \) at different times are exponentially correlated
\[ <\sigma(t_1)\sigma(t_2)> = v^2 e^{-\gamma|t_1-t_2|} \]
which makes the process \( x(t) \) non-Markovian. However, in the limit \( v \rightarrow \infty \) and \( \gamma \rightarrow \infty \) limit keeping \( \frac{v^2}{\gamma} \) finite, the noise \( \eta(t) = v\sigma(t) \) becomes delta correlated
\[ <\eta(t_1)\eta(t_2)> = 2D\delta(t_1-t_2) \] with \( D = \frac{v^2}{2\gamma} \) and the process reduces to the usual Brownian motion. In figure 1 we show a typical trajectory \( x(t) \) of a RTP particle and in the inset we show a realization of the noise \( \sigma(t) \). The three observables \( t_m, t_r \) and \( t_f \) have also been indicated in the figure 1 along with their definition given in the caption. In the random walk literature the RTP process has been studied extensively in the past and is known as persistent random walk or telegrapher’s equation [31, 32]. Recently, it has received renewed interest because of its application in biological system or, more generally in active systems. In many theoretical studies of active matter or collective behavior of many active agents, often RTPs are used as microscopic constituents [33, 34]. For example \( E. \text{Coli} \) bacteria runs for some time along a straight line and then tumbles to randomly choose a new direction of run [35]. In the last few years there has been a lot of interests in studying RTPs at the individual level as they show interesting phenomena like, accumulation near boundaries [36–38], clustering [39–41], passive to active transition [42, 43], climbing against the hill [44], Kramer’s escape problem [45] etc.

While the single Brownian motion has been studied quite extensively over many years and a huge number of exact results related to various statistical properties of it
have been proved, very less amount of study has been performed for RTPs or in general for non-Markovian processes except for a few [6, 8, 17–19]. In an attempt towards this direction, we here study the probability distribution functions \( P_M(t_m, t) \), \( P_R(t_r, t) \) and \( P_L(t_\ell, t) \) of \( t_m \), \( t_r \) and \( t_\ell \), respectively, measured over RTP trajectories of duration \( t \) which start at the origin. We find that the distributions of all the three quantities have delta function parts which appear either at both edges or only at \( t = 0 \) and, have a regular non-delta function part over \( 0 < t < t \). Remarkably, we find that the non-delta function parts of the three distributions \( P_M(t_m, t) \), \( P_R(t_r, t) \) and \( P_L(t_\ell, t) \) have identical functional form when the initial velocity directions \( \pm \) are chosen with equal probability. More explicitly, we find

\[
P_M(t_m, t) = \frac{h(t)}{2} \left[ \delta(t_m) + \delta(t_m - t) \right] + \frac{\gamma}{2} h(t_m) h(t - t_m)
\]

\[
P_R(t_r, t) = \frac{h(t)}{2} \left[ \delta(t_r) + \delta(t_r - t) \right] + \frac{\gamma}{2} h(t_r) h(t - t_r)
\]

\[
P_L(t_\ell, t) = h(t) \delta(t_\ell) + \frac{\gamma}{2} h(t_\ell) h(t - t_\ell), \quad \text{where,}
\]

\[
h(t) = e^{-\gamma t} \left[ I_0(\gamma t) + I_1(\gamma t) \right],
\]

and, \( I_0(z) \) and \( I_1(z) \) are modified Bessel functions of the first kind. Note that the distributions \( P_M(t_m, t) \) and \( P_R(t_r, t) \) are identical whereas \( P_L(t_\ell, t) \) is little different. It has delta function only at \( t_\ell = 0 \). Also note that the above distributions are completely

**Figure 1.** A typical trajectory \( x(\tau) \) of duration \( t \) of the RTP starting from the origin. The maximum displacement \( M \) occurs at time \( t_m (\leq t) \) within time interval \([0, t]\). The time duration \( t_r (\leq t) \) represents the total amount of time the RTP spends on the \( x \geq 0 \) region over the duration \( t \). The time \( t_\ell (\leq t) \) represents the time at which the RTP crosses the origin (from below in the particular trajectory shown) for the last time and after that in the remaining time \( t - t_\ell \) it did not cross the origin again. Inset: shows a typical realization of the telegraphic noise \( \sigma(\tau) \) over interval \([0, t]\).
independent of the speed $v$. This is due to the fact that by definition the values of the observables $t_m$, $t_r$ and $t_\ell$ measured along a trajectory $\{x(\tau); 0 \leq \tau \leq t\}$ do not change under rescaling of the position $x$ with $v$ (see equation (2)). Finally, observe that in the $\gamma \rightarrow \infty$ limit all the three distributions approach to the distribution $P_{Br}(t_a, t)$ obtained for a Brownian motion. This can be seen using the large argument asymptotic of the Bessel functions $I_\nu(z) \sim \frac{e^z}{\sqrt{2\pi z}} (1 + O(z^{-1}))$ for all $\nu$.

The results in equation (3) are valid for symmetric initial condition case where the initial values of $\sigma$ (velocity direction) are chosen with equal probability. However, for asymmetric initial conditions in which the directions $+$ and $-$ are chosen with probabilities $a$ and $b$, respectively, such that $a + b = 1$, one can straightforwardly extend our calculations and obtain the distributions $P_M(t_m, t)$, $P_R(t_r, t)$ and $P_L(t_\ell, t)$. Explicit expressions and derivations of these probabilities are given in appendices F and G.

In addition to the above results, we show (again for $a = b = 1/2$) that the joint distribution of $M$ and $t_m$ is given by

$$\mathcal{P}(M, t_m, t) = \frac{h(t)}{2} \frac{\delta(M) \delta(t_m) - \delta(t_m - t) + h(t - t_m)}{2} \frac{d}{dM} \left[ \Theta(vt - M) \ g(M, t) \right],$$

where, $g(M, t) = e^{-\gamma t} \left[ I_0 \left( \frac{\gamma}{v} \sqrt{v^2 t^2 - M^2} \right) + \sqrt{\frac{vt - M}{vt + M}} I_1 \left( \frac{\gamma}{v} \sqrt{v^2 t^2 - M^2} \right) \right],$ (5)

and $h(t)$ is given in equation (4). From this joint distribution, we obtain the marginal distribution $\mathcal{P}_M(M, t)$ of the maximum displacement $M$ in time $t$ in addition to the marginal distribution $P_M(t_m, t)$ mentioned before. We show that the cumulative probability $Q(M, t) = \int_0^M dM' \ \mathcal{P}_M(M', t)$ is given by

$$Q(M, t) = 1 - \frac{1}{2} g(M, t) \Theta(vt - M) - \frac{\gamma}{2} \int_0^t dt' \ \Theta(vt' - M) \ h(t - t') \ g(M, t').$$

To prove these results, an essential quantity is the propagator of the RTP in presence of an absorbing wall for arbitrary initial and final conditions. We start our calculations by computing these propagators in section 2, which is followed by the computation of the corresponding survival probabilities in the section 3. The results obtained in these sections are used in the later sections to obtain the distributions of $t_m$, $t_r$ and $t_\ell$. In section 4, we study at the joint distribution of $M$ and $t_m$ and in section 4.1 we study the marginal distribution of $t_m$. In the last two sections 5 and 6, we obtain the distributions of $t_r$ and $t_\ell$ respectively. Finally, in section 7 we conclude the paper. Some details of the calculations are put in the appendices.

2. Propagator of a RTP in presence of an absorbing barrier at $x = M$

We start by computing the propagators of the RTP in presence of an absorbing boundary. Let $P_\pm(x, t | 0, \alpha, 0)$ represent the probability densities that the RTP, starting at the origin with velocity direction $\alpha \in \{1, -1\}$, reaches the position $x$ at time $t$ with velocity direction $\pm 1$ in presence of an absorbing barrier at $x = M > 0$. This particular choice of the position of the absorbing barrier will be useful in later sections where we compute

https://doi.org/10.1088/1742-5468/ab3283
the joint distribution of the maximum displacement and the time at which the maximum occurs. Starting from the Langevin equation (2), it is easy to show [36, 46] that these propagators satisfy the following forward master equations

\[
\partial_t P_{\pm}(x,t|0,\alpha,0) = -v\partial_x P_{\pm}(x,t|0,\alpha,0) - \gamma P_{\pm}(x,t|0,\alpha,0) + \gamma P_{\pm}(x,t|0,\alpha,0),
\]

\[
\partial_x P_{\pm}(x,t|0,\alpha,0) = v\partial_x P_{\pm}(x,t|0,\alpha,0) + \gamma P_{\pm}(x,t|0,\alpha,0) - \gamma P_{\pm}(x,t|0,\alpha,0),
\]

where the problem of finding the boundary condition at the origin stood as follows. For finite and distribution of the maximum displacement \(t_m\), it is easy to show [36, 46] that the maximum occurs. Starting from the Langevin equation (2), it is easy to show [36, 46] that these propagators satisfy the following forward master equations

\[
P_{\pm}(x \rightarrow -\infty, t|0,\pm,0) = 0, \quad P_{\pm}(x \rightarrow M, t|0,\pm,0) = 0
\]

and, the initial conditions \(P_{\pm}(x,0|0,\alpha,0) = a \delta_{\alpha,\pm}\delta(x)\) and \(P_{\pm}(x,0|0,\alpha,0) = b \delta_{\alpha,-}\delta(x)\) such that \(a + b = 1\). The master equations in equation (7) are also known as Telegrapher’s equations and have been studied extensively in various contexts like electromagnetic transmission, chromatography, fluorescence spectroscopy etc (see [31] for a review). Different choices for the values of \(a\) and \(b\) correspond to different initial velocity distributions. For example, \(a = 1\) and \(b = 0\) implies that the particle starts with velocity \(+v\) initially while \(a = 0\) and \(b = 1\) means that the particle starts with velocity \(-v\) initially. The boundary conditions stated above can be understood as follows. For finite \(t\), the particle, starting from the origin, can never reach \(x \rightarrow -\infty\) in finite duration \(t\) which implies \(P_{\pm}(x \rightarrow -\infty, t|0,\pm,0) = 0\). To see how the boundary condition at \(x = M\) appear, we follow [46] where the problem of finding the propagator of a single RTP in presence of an absorbing boundary (at the origin) has recently been considered. In this paper, the final result of the total probability \(P(x,t|0,0) = a[P_{+}(x,t|0,+,0) + P_{-}(x,t|0,+,0)] + b[P_{+}(x,t|0,+,0) + P_{-}(x,t|0,+,0)]\) at the wall has been presented explicitly. We require expressions of the individual probabilities \(P_{\pm}(x,t|0,+,0), P_{\pm}(x,t|0,+,0)\) near the absorbing wall to compute the joint distribution of the maximum displacement \(M\) and \(t_m\), for which we feel it is convenient to re-derive the solutions of the master equation (7) in our setting i.e. in presence of an absorbing boundary at \(x = M\).

Coming back to the boundary condition at \(x = M\), let us write the evolution of the second equation in equation (7) from time \(t\) to \(t + dt\) at \(x = M\) as \(P_{\pm}(M, t + dt|0,\alpha,0) = (1 - \gamma dt)P_{\pm}(M + vdt, t|0,\alpha,0) + \gamma dt P_{\pm}(M, t|0,\alpha,0).\) Since there can not be any particle at \(x > M\) for \(t > 0\) due to the presence of an absorbing boundary at \(x = M\), we have \(P_{\pm}(M + vdt, t|0,\alpha,0) = 0\). Now taking \(dt \rightarrow 0\) limit, we have \(P_{\pm}(M, t|0,\alpha,0) = 0\). So finally the boundary conditions with which the equation (7) have to be solved are \(P_{\pm}(x \rightarrow -\infty, t|0,\pm,0) = 0\) at \(x \rightarrow -\infty\) and \(P_{\pm}(M, t|0,\alpha,0) = 0\) at \(x = M\). It turns out that these boundary conditions are enough to find the propagator \(P_{\pm}(x,t|0,\alpha,0)\) uniquely [46].

For clarity and compactness of the presentation, we prefer to give the details of the derivation of the propagators in the appendix A. We here instead present only the final results. We find the following explicit expressions for the propagators for the four possible combinations of the initial and final velocity directions,

\[
P_{\pm}(x,t|0,+,0) = \frac{\gamma}{2v}[\mathcal{I}(|x|, v, \gamma, t) - \mathcal{I}(2M - x, v, \gamma, t)],
\]

\[
P_{\pm}(x,t|0,+,0) = \frac{\gamma}{2v}[\mathcal{J}(|x|, v, \gamma, t) - \mathcal{J}(2M - x, v, \gamma, t)] - \Theta(x) \frac{d\mathcal{I}(x,v,\gamma,t)}{dx}
\]
\[ P_-(x, t|0, -0, 0) = \frac{\gamma}{2v} [J(|x|, v, \gamma, t) - J(2M - x, v, \gamma, t)] - \Theta(-x) \frac{dI(|x|, v, \gamma, t)}{dx} \]  
(10)

\[ P_+(x, t|0, -0, 0) = \frac{\gamma}{2v} [I(|x|, v, \gamma, t) - I(2M - x, v, \gamma, t)] + T(2M - x, v, \gamma, t) \]  
(11)

where the functions \( I(g, v, \gamma, t) \), \( J(\alpha, g, v, \gamma, t) \) and \( T(g, v, \gamma, t) \) (defined for \( g > 0 \)) are given by:

\[ I(g, v, \gamma, t) = \Theta(vt - g) e^{-\gamma t} I_0 \left( \frac{\sqrt{v^2 t^2 - g^2}}{v} \right), \]  
(12)

\[ J(g, v, \gamma, t) = \Theta(vt - g) e^{-\gamma t} \sqrt{\frac{vt - g}{vt + g}} I_1 \left( \frac{\sqrt{v^2 t^2 - g^2}}{v} \right), \]  
(13)

\[ T(g, v, \gamma, t) = \Theta(vt - g) e^{-\gamma t} \frac{g}{vt + g} \left[ \frac{\gamma g I_0 \left( \frac{\sqrt{v^2 t^2 - g^2}}{v} \right)}{\sqrt{\frac{vt - g}{vt + g}} I_1 \left( \frac{\sqrt{v^2 t^2 - g^2}}{v} \right)} \right]. \]  
(14)

Observe that the term \( \frac{dI}{dx} \) would produce some delta function terms like \( e^{-\gamma t} \delta(x \pm vt) \). These terms arise from the events in which the RTP has not changed its direction till time \( t \) and such events occur with probability \( e^{-\gamma t} \). From these expressions, we can obtain the total probability density \( P(x, t|\alpha, 0) = a[P_+(x, t|0, +, 0) + P_-(x, t|0, +, 0)] + b[P_+(x, t|0, -, 0) + P_-(x, t|0, -, 0)] \) of finding the particle at \( x \) in time \( t \) and check that at \( x = M \) it matches with the result given in [46]. Using the large \( z \) asymptotic of \( I_0(z) \simeq e^z/\sqrt{2\pi z} \) for both \( v = 0 \) and \( 1 \), one can show that in the limit \( v \to \infty \) and \( \gamma \to \infty \) keeping \( v^2/(2\gamma) = D \) fixed, the individual probability densities in equations (8)–(11) reduce to

\[ P_{\alpha}(x, t|0, \alpha', 0) \xrightarrow{v^2(2\gamma) = D} \frac{1}{\sqrt{4\pi D t}} \left[ \exp \left( -\frac{x^2}{4Dt} \right) - \exp \left( -\frac{(2M - x)^2}{4Dt} \right) \right], \]  
(15)

for \( \alpha = \pm 1 \), \( \alpha' = \pm 1 \) and for \( x < M \). This is the propagator of a Brownian walker in presence of absorbing boundary at the origin, with the initial and final positions are \( y_0 = M \) and \( y_t = M - x \), respectively. Hence the total probability \( P(x, t|0, 0) = a[P_+(x, t|0, +, 0) + P_-(x, t|0, +, 0)] + b[P_+(x, t|0, -, 0) + P_-(x, t|0, -, 0)] \) with \( a + b = 1 \) for \( x < M \), reduces to the Brownian motion result with an absorbing boundary at \( x = M \), as it should. One obtains the same Brownian motion result in the large \( t \) asymptotic as well. On the other hand, for finite \( \gamma \), the effect of the activity will be stronger and to observe these effects, we plot the probability densities in equations (8)–(11) as functions of \( x \) for \( \gamma = 0.5 \) at different times in figures 2 and 3. In these figures we observe excellent agreement with the simulation data (filled circles).
3. Survival probability of the RTP in presence of the absorbing barrier at $x = M$

Here we look at the survival probability of the RTP in presence of the absorbing barrier at $x = M > 0$. In principle one can compute the survival probabilities by integrating out final positions over $(-\infty, M)$ in the propagators obtained in the previous section. However, it seems more convenient and illustrative to solve the backward master equation to get the survival probabilities. The survival probability with an absorbing barrier at $x = 0$ for a run-and-tumble particle has recently been studied in [36]. We mainly use results from [36] after appropriately modifying for our case. However, for self-containedness of the paper we here briefly outline the derivation of the survival probabilities. At this point we also would like to mention that a generalisation of the survival probability problem has been recently addressed in the context of RTP with resetting [47].
Let $S_{\pm}(x, t)$ represent the survival probability of the RTP, starting from the origin with velocity direction $\pm 1$. Note that in our problem, the absorbing barrier is at $M \neq 0$ and the particle starts at $x \leq M$. The backward master equations for the survival probability $S_{\pm}(x, t)$ are given by [36]

$$\partial_t S_{+}(x, t) = v \partial_x S_{+}(x, t) - \gamma S_{+}(x, t) + \gamma S_{-}(x, t),$$

(16)

$$\partial_t S_{-}(x, t) = -v \partial_x S_{-}(x, t) + \gamma S_{+}(x, t) - \gamma S_{-}(x, t).$$

(17)

We solve the above equations with initial condition $S_{\pm}(x, 0; M) = 1$ and the boundary conditions $S_{\pm}(x \to -\infty; t; M) = 1$ and $S_{\pm}(M^{-}; t; M) = 0$. The first boundary condition takes into account of the fact that the particle, regardless of its initial velocity direction, will always survive over any finite time if it starts initially from $x \to -\infty$. On the other hand if the particle starts at $x = M$ with positive velocity then the particle will immediately get absorbed. With these conditions one can solve equations (16) and (17) to find [36]

$$S_{+}(x, t) = 1 + v \frac{d}{dM} \int_0^t d\tau \Theta \left( \tau - \frac{M - x}{v} \right) e^{-\gamma \tau} I_0 \left( \frac{\gamma}{v} \sqrt{v^2 \tau^2 - (M - x)^2} \right),$$

(18)

$$S_{-}(x, t) = 1 - \int_0^t d\tau \Theta \left( \tau - \frac{M - x}{v} \right) \frac{ve^{-\gamma \tau}}{v \tau + (M - x)} \left[ I_0 \left( \frac{\gamma}{v} \sqrt{v^2 \tau^2 - (M - x)^2} \right) \right] + \sqrt{\frac{v \tau - (M - x)}{v \tau + (M - x)}} I_1 \left( \frac{\gamma}{v} \sqrt{v^2 \tau^2 - (M - x)^2} \right).$$

(19)

The results in equations (18) and (19) are plotted in figure 4 and compared with the numerical simulations, in which one starts with a large number of independent RTPs initially. We observe excellent agreement. We notice that both $S_{\pm}$ remains 1 till the time $t_b = \frac{M - x}{v}$. This is because the RTPs, starting from $x < M$, reach the absorbing barrier for the first time at least after time $t_b$. Before this time the RTPs do not feel the presence of the absorbing barrier. Once the RTPs reach (they can reach only with +ve velocity) the barrier at $t_b$, in the next instant some fraction of the RTP change their velocity direction from + to − and the rest of the RTPs do not change. The fraction of particles which do not change their velocity direction get absorbed at the barrier at $t_b$ time. As a result $S_{+}(x, t_b)$ drops suddenly as manifested by a discontinuity in the figure 4(a). No sudden drop occurs $S_{-}(x, t)$ because no particles with −ve velocity can reach the barrier at $t_b$. However, since the overall number of particles decrease, the $S_{-}(x, t)$ also starts decreasing from value 1 after time $t_b$ but continuously. As done in the previous section, using the large argument asymptotic behaviour of $I_0(z)$, one can show that in the $\gamma \to \infty$ and $v \to \infty$ limit keeping $v^2/(2\gamma) = D$ fixed, one indeed recovers the survival probability of a Brownian particle from the absorbing wall at $M$ when started from $x$:

$$S_{\pm}(x, t) \xrightarrow{\gamma \to \infty, v \to \infty} \text{erf} \left( \frac{M - x}{\sqrt{4Dt}} \right) \text{ where, } \text{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z du \ e^{-u^2}. \quad (20)$$

https://doi.org/10.1088/1742-5468/ab3283
3.1. Cumulative probability $Q(M, t)$ of maximum displacement $M$ in absence of any absorbing barrier

It is easy to see that the distribution of the maximum $M$ over the duration $t$ in absence of any absorbing barrier is related to the survival probability of the RTP in presence of an absorbing barrier at $M$. The cumulative probability $Q(M, t)$ of the maximum displacement made by the RTP in time duration $t$ (given that it started from the origin) is exactly same as the survival probability of the particle till time $t$ in presence of an absorbing barrier at $M$ and starting from $x = 0$. In both the probabilities, exactly the same ensemble of paths contribute which, starting from $x = 0$ stays below the level $x = M$ throughout the course of evolution till time $t$. For a discussion on the connection between $Q(M, t)$ and survival probability in the context of Brownian motion see $[48]$. This connection allows us to write,

$$Q(M, t) = \text{Prob.} (x_{\text{max}} \leq M, t) = [S_+(0, t) + S_-(0, t)]/2,$$

where

$$S_+(0, t) = 1 + v \frac{d}{dM} \int_0^t d\tau \, \Theta(v\tau - M)e^{-\gamma\tau}I_0 \left( \frac{\gamma}{v} \sqrt{v^2\tau^2 - M^2} \right),$$

$$S_-(0, t) = 1 - \int_0^t d\tau \, \Theta(v\tau - M) \frac{ue^{-\gamma\tau}}{v\tau + M} \left[ \frac{\gamma M}{v}I_0 \left( \frac{\gamma}{v} \sqrt{v^2\tau^2 - M^2} \right) + \sqrt{\frac{v\tau - M}{v\tau + M}} I_1 \left( \frac{\gamma}{v} \sqrt{v^2\tau^2 - M^2} \right) \right],$$

where the factor $1/2$ is due to the choice $a = b = 1/2$. These expressions agree with the results obtained in $[49]$ where the maximum displacement made by a persistent Brownian walker was computed. From equation (20), one can easily see that in the diffusive asymptotic limit one correctly recovers the Brownian result. In figure 4(b), we compare our theoretical cumulative distribution $Q(M, t)$ against numerical simulation and we observe nice agreement. The discontinuous jumps at $M = 0$ and $M = vt$ are due to the presence of delta functions. Other parameters used in both the plots are $\gamma = 2.5, v = 2$.

**Figure 4.** (a) Comparison of the survival probabilities $S_{\pm}$ given in equations (18) and (19) with simulation results (filled circles). We have chosen the initial position at $x = 1$ and the absorbing barrier at $M = 3$. (b) Numerical verification of analytical result of the cumulative distribution $Q(M, t) = [S_+(0, t) + S_-(0, t)]/2$ of the maximum displacement $M$ made by the particle in time $t = 3$, starting from the origin. The discontinuous jumps at $M = 0$ and $M = vt$ are due to the presence of delta functions. Other parameters used in both the plots are $\gamma = 2.5, v = 2$. 

https://doi.org/10.1088/1742-5468/ab3283
indicate the presence of delta functions in the corresponding probability distribution function $P_M(M, t)$ which we will see in the next section.

4. Joint probability distribution $P(M, t_m, t)$ of $M$ and $t_m$ and the marginal distributions

Let us now focus on the joint distribution $P(M, t_m, t)$ of the maximum $M$ and the time $t_m$ at which this maximum occurs within the time interval $[0, t]$. To compute this distribution, we first break the trajectory $x(\tau)$ into two parts (see figure 5). The first part is $\{x(\tau); 0 \leq \tau \leq t_m\}$ and the second part is $\{x(\tau); t_m \leq \tau \leq t\}$. In the first part $0 \leq \tau \leq t_m$, the trajectory of the RTP is always below $M$ i.e. $x(\tau) < M$ and reaches $M$ exactly at $t_m$ (with velocity $+v$). Similarly, in the second part, the trajectory, starting from $M$ also stays below $M$ throughout in the remaining duration $t_m < \tau \leq t$; however, the final position can be anywhere below $M$. The probability weight in the first part can be obtained from the propagator with an absorbing barrier at $M$. Since in the variables $(x, \sigma)$, the RTP process $\{x(\tau), \sigma(\tau)\}$ is Markovian, the probability weight of the paths in the second part is independent of that of the first part and is proportional to the survival probability of the RTP starting from $M$ in presence of a barrier at $M$. It is intuitively clear that in the first interval of time, the RTP can not reach $M$ (at time $t_m$) with negative velocity while staying completely below $M$ over the full duration $0 \leq \tau \leq t_m$. Similarly, in the second interval, the RTP can not survive from the absorbing wall at $M$ if it starts with a positive velocity at $x = M$. From the expressions of the propagators in equations (8)–(11) and the survival probabilities in equations (18) and (19), it is easy to observe that if one puts $x = M$ then $P_-(M, t_m|0, \pm, 0) = 0$ and $S_+(M, t) = 0$. To proceed at this point, we follow the procedure given in [4]. We use the propagators $P_\pm(M - \epsilon, t_m|0, 0)$ that the RTP, starting at the origin, reach $M - \epsilon$.
(for $\epsilon > 0$) with velocity $\pm v$ at time $t_m$ and, the probabilities $S_{\pm}(M - \epsilon, t - t_m)$ that the RTP, starting from position $M - \epsilon$ with velocity $\pm v$ survives over the duration $t - t_m$. Later at some appropriate step, we take the $\epsilon \to 0$ to get the final answers. In terms of these propagators and survival probabilities, we can write
\[ p_{\pm}(M - \epsilon, t_m|0,0) S_{\pm}(M - \epsilon, t - t_m) + p_{\pm}(M - \epsilon, t_m|0,0) S_{\pm}(M - \epsilon, t - t_m) \]
\[ = \lim_{\epsilon \to 0} \frac{p_{\pm}(M - \epsilon, t_m|0,0) S_{\pm}(M - \epsilon, t - t_m) + p_{\pm}(M - \epsilon, t_m|0,0) S_{\pm}(M - \epsilon, t - t_m)}{Z(\epsilon, t)} , \]  
(23)
for $0 < M < vt$ and $0 < t_m < t$ where $Z(\epsilon, t)$ is a normalization factor and the propagators $p_{\pm}(M - \epsilon, t_m|0,0)$ each are the sum of two terms corresponding to two initial velocities (occurring with probability 1/2 each) i.e. $p_{\pm}(M - \epsilon, t_m|0,0) = \frac{1}{2} [p_{\pm}(M - \epsilon, t_m|0,+,0) + p_{\pm}(M - \epsilon, t_m|0,-,0)]$. We emphasise that the propagators in equations (8)–(11) were derived for biased initial conditions with $(a,b) = (1,0)$ or $(0,1)$. However while writing $p_{\pm}$, we consider $a = b = \frac{1}{2}$. To take the $\epsilon \to 0$ limit, we first expand each terms separately in $\epsilon$. We find
\[ S_{-}(M - \epsilon, t - t_m) \approx \Theta(t - t_m) h(t - t_m) + O(\epsilon) , \]
\[ S_{+}(M - \epsilon, t - t_m) \approx \frac{2}{v} h(t - t_m) + O(\epsilon^2) , \]
\[ h(t) = e^{-\gamma t}[I_0(\gamma t) + I_1(\gamma t)] , \]  
(24)
\[ p_{+}(M - \epsilon, t_m|0,0) \approx -\frac{d}{dM} \left[ \Theta(vt_m - M) g(M, t_m) + O(\epsilon) \right] , \]
\[ p_{-}(M - \epsilon, t_m|0,0) \approx -\frac{2}{v} \frac{d}{dM} \left[ \Theta(vt_m - M) g(M, t_m) + O(\epsilon^2) \right] \]  
(25)
where,
\[ g(M, t) = e^{-\gamma t} \left[ I_0 \left( \frac{2}{v} \sqrt{vt^2 - M^2} \right) + \sqrt{vt^2 - M} \right] \]  
(26)
Note that, $h(t)$ is the survival probability of the RTP in presence of an absorbing barrier at the origin while starting from the origin itself [46]. Inserting the small $\epsilon$ expansions from equations (24) and (25) into (23) one can get the joint distribution of $M$ and $t_m$. Note, however, that the above procedure can provide the distribution only for $0 < M < vt$ and $0 < t_m < t$. The contributions from events $M = 0$ (or equivalently $t_m = 0$) and $t_m = t$ have to be added separately. Contribution to the $M = 0$ (or $t_m = 0$) comes from trajectories which starting from the origin goes to the negative $x$ and never visits the positive $x$ region within time $t$. Probability with which such events occur is exactly the probability that the RTP, starting from the origin, survives from an absorbing barrier at the origin till time $t$. This survival probability can be easily computed similarly as done in section 3 and is given exactly by $h(t)$ (see also [46]). On the other hand the contributions to $t_m = t$ event come from trajectories which starting from the origin reach the maximum displacement $M$ at time $t_m = t$. The contribution of these paths to the joint distribution $\mathcal{P}(M, t_m, t)$ is given by
\[ \mathcal{P}(M, t_m, t) = \delta(t_m - t) \left[ \frac{P_+(M, t|0,+,0) + P_+(M, t|0,-,0)}{2} \right] \]  
(27)
where $\frac{P_+(M, t|0,+,0) + P_+(M, t|0,-,0)}{2}$ is the probability density to find the particle at time $t$ in position $M$. Inserting the explicit forms of these propagators from equations (9) and (11), respectively, we get
\[ \text{https://doi.org/10.1088/1742-5468/ab3283} \]
Generalised ‘Arcsine’ laws for run-and-tumble particle in one dimension

\[
\mathcal{P}(M, t_m, t) \overset{\text{contribution from paths with } t_m=t}{=} \delta(t_m - t) \frac{1}{2} \left[ \mathcal{T}(M, v, \gamma, t) - \frac{d\mathcal{L}(M, v, \gamma, t)}{dM} \right]
\]

\[
= -\frac{d}{dM} [\Theta(vt - M) \ g(M, t)],
\]

where \(g(M, t)\) is given in equation (26). Hence, adding all the three contributions, as mentioned above, one gets

\[
\mathcal{P}(M, t_m, t) = \frac{h(t)}{2} \delta(M) \delta(t_m) - \frac{\delta(t_m - t)}{2} \frac{d}{dM} [\Theta(vt - M) \ g(M, t)]
\]

\[
- \frac{\gamma \epsilon}{2v \mathcal{Z}(\epsilon, t)} \frac{d}{dM} [\Theta(vt_m - M)g(M, t_m)],
\]

where \(h(t)\) and \(g(M, t)\) are given in equations (24) and (26). The factor 1/2 in the first term comes from the probability with which the RTP, starting from the origin, chooses –ve velocity initially. The factor \(\mathcal{Z}(\epsilon, t)\) is a normalisation constant and has to be chosen in such a way that,

\[
\int_0^t dt_m \int_0^{vt_m} dM \mathcal{P}(M, t_m, t) = 1.
\]

The contribution from the first term in equation (29) to the total unit probability is \(h(t)/2\) as can be easily seen

\[
\text{Contribution from the first term} = \int_0^t dt_m \int_0^{vt_m} dM \frac{h(t)}{2} \delta(M) \delta(t_m) = \frac{h(t)}{2}.
\]

It is also easy to show that the contribution from second term (see appendix B) is

\[
\text{Contribution from the second term} = -\int_0^t dt_m \int_0^{vt_m} dM \frac{\delta(t_m - t)}{2} \frac{d}{dM} [\Theta(vt - M) \ g(M, t)] = \frac{h(t)}{2},
\]

where we have used the explicit expression of \(g(M, t)\) given in equation (26). Similarly, one can show that the contribution from the third term is

\[
\text{Contribution from the third term} = -\frac{\gamma \epsilon}{2v \mathcal{Z}(\epsilon, t)} \int_0^t dt_m h(t - t_m) \int_0^{vt_m} dM \frac{d}{dM} [\Theta(vt_m - M)g(M, t_m)]
\]

\[
= \frac{\epsilon}{2v \mathcal{Z}(\epsilon, t)} 2[1 - h(t)],
\]

where the identity \(\int_0^t dt' h(t - t') h(t') = 2[1 - h(t)]\) (see appendix C for the proof) has been used. Adding all the contributions and substituting in equation (30), we get \(\mathcal{Z}(\epsilon, t) = \frac{\epsilon}{\gamma \epsilon} \), inserting which in equation (29), we finally obtain the joint distribution of \(M\) and \(t_m\):

\[
\mathcal{P}(M, t_m, t) = \frac{h(t)}{2} \delta(M) \delta(t_m) - \frac{\delta(t_m - t)}{2} + \frac{\gamma h(t - t_m)}{2} \frac{d}{dM} [\Theta(vt_m - M) \ g(M, t_m)].
\]

This is one of our main results. Although this result is derived for \(a = b = 1/2\) but for other choices of \(a\) and \(b\) (such that \(a + b = 1\)), once can easily extend the above calculation (see appendix F).

https://doi.org/10.1088/1742-5468/ab3283
Integrating out the variables \( t_m \) or \( M \) in the above joint distribution in equation (34), one can get the marginal distributions \( P_M(M, t) \) and \( P_M(t_m, t) \), respectively. Let us first look at the marginal distribution of the maximum displacement \( M \). It turns out convenient to compute the cumulative distribution

\[
Q(M, t) = \int_0^M dM' \ P_M(M', t) = \int_0^M dM' \int_0^t dt_m \ P(M, t_m, t)
\]

(See also before equation (18)). Performing the integrals along with some straightforward manipulations (see appendix B), we find

\[
Q(M, t) = 1 - \frac{1}{2} g(M, t) \Theta(vt - M) - \frac{\gamma}{2} \int_0^t dt' \ (vt' - M) \ h(t - t') \ g(M, t'),
\]

which can be shown to agree with our earlier result for \( Q(M, t) \) given in equations (21) and (22).

### 4.1. Marginal probability distribution of \( t_m \)

We now look at the other marginal distribution \( P_M(t_m, t) = \int_0^{vt_m} dM \ P(M, t_m, t) \) of \( t_m \). It is easy to perform the integral over \( M \) as follows

\[
P_M(t_m, t) = \int_0^{vt_m} dM \ P(M, t_m, t),
\]

\[
= \int_0^t dM \left[ \frac{h(t)}{2} \delta(M) \delta(t_m - t) - \frac{\gamma}{2} \int_0^t dt' \ \Theta(vt' - M) \ h(t - t') \ g(M, t') \right],
\]

\[
= \frac{h(t)}{2} \delta(t_m) + \frac{\delta(t_m - t) + \gamma h(t - t_m)}{2} h(t_m),
\]

\[
= \frac{h(t)}{2} \left[ \delta(t_m) + \delta(t - t_m) \right] + \frac{\gamma}{2} h(t - t_m) h(t_m),
\]

(36)

where to go from the second line to the third line, we have used

\[
\int_0^{vt_m} dM \frac{d}{dM} \left[ \Theta(vt_m - M) g(M, t_m) \right] = -h(t_m)
\]

with \( h(t) \) is given in equation (24). This is our second main result. Let us try to understand the different terms appearing in the above equation starting with the delta function terms. As mentioned in the previous section, one can identify that the term proportional to \( \delta(t_m) \) gets contributions from trajectories which, starting from the origin have gone to the negative side and never have crossed the origin again within the interval \([0, t]\). On the other hand, the term proportional to \( \delta(t_m - t) \) gets contributions from the trajectories which reach their maxima at the final time \( t \) (see figure 6(a)). Interestingly, both these events occur with probability proportional to \( h(t) \). For the first term it is easy to understand. For the second term, this can be seen by shifting the origin from \( x = 0 \) to the position of the maxima \( x = M \) for each trajectory \( \{x(\tau); 0 \leq \tau \leq t\} \) before performing the integration over \( M \). More precisely, if one makes the following variable transformations

\[
\bar{x} = M - x, \ \bar{\tau} = t - \tau \quad \text{and} \quad \bar{\sigma} = \sigma \quad \text{(as done in [6])},
\]

then in the new coordinates the RTP, starting from \( \bar{x} = 0 \) reaches some position above it at time \( t \) while staying above the (new) origin throughout. It is easy to check that the path probabilities of the original

https://doi.org/10.1088/1742-5468/ab3283
process and the transformed process are same. Hence, the survival problem for the original trajectories, starting from some position $x(0) < M$, with the boundary condition $S_+(M, t) = 0$ at the absorbing wall placed at $x = M$, is same as the survival problem for the transformed trajectories, starting from $\bar{x}(0) > 0$ with the boundary condition $S_-(0, t) = 0$ at the absorbing wall placed at $\bar{x} = 0$. As a result the survival probability of the transformed trajectories is exactly $h(t)$.

We now focus on the non-delta function part. This term is proportional to the product of $h(t_m)$ and $h(t - t_m)$, which are survival probabilities in the first and second durations $t_m$ and $t - t_m$, respectively. The trajectories that contribute to $t_m$ have their maximum displacements at $t_m$. Since the process $\{x(\tau), \sigma(\tau)\}$ is Markovian, one can break the whole trajectory into two parts (see figure 6(b)) as done in the previous section. Following [6], we consider the variable transformations $\bar{x} = M - x$, $\bar{\tau} = t_m - \tau$ in the first part of duration $t_m$ and $\bar{\tau} = \tau - t_m$ in the second part of duration $(t - t_m)$. Under these variable transformations one can see that the contribution from the trajectories in the first and second parts are again survival probabilities $h(t_m)$ and $h(t - t_m)$, respectively.

In figure 7 we plot the theoretical result for $P(t_m, t)$ in equation (36) along with the same obtained from numerical simulation. We observe that both the delta function (see inset) and the non-delta function parts agree nicely with simulation data. The distribution $P_M(t_m, t)$ for RTP has interesting feature similar to what one observes for the Brownian particle case. In both cases, the distribution is peaked near $t_m = 0$ and $t_m = t$ (see equation (1)). However for RTP, the distribution near the edges do not diverge as in the Brownian particle case,—instead there are delta functions. It is easy to show that with increasing $\gamma$ the distribution in equation (36) indeed approaches the Brownian motion result in equation (1).

Note that the distribution $P_M(t_m, t)$ in equation (36) is computed for the symmetric initial conditions whereby particle starts with $\pm v$ with equal probability. As mentioned earlier, one can however extend this calculation for general initial conditions as shown in appendix F where we have provided an explicit expression for $P_M(t_m, t)$. 

---

**Figure 6.** (a) An example of the trajectories which contribute to the event $t_m = t$ i.e. the maximum occurs at time $t$. (b) A typical trajectory which contribute to the non-delta function part of the distribution $P(M, t_m, t)$. The trajectory divided into two parts, from 0 to $t_m$ (red part) and from $t_m$ to $t$ (blue part) (c) Illustration of how the trajectory in (b) look like after the transformations $\bar{x} = M - x$, $\bar{\tau} = t_m - \tau$ in the first (red) part and $\bar{x} = M - x$, $\bar{\tau} = \tau - t_m$ in the second (blue) part.
We now study the time that the RTP spends at $x > 0$ region over the duration $t$. This time, denoted by $t_r$, is called the residence time and is defined by $t_r = \int_0^t d\tau \Theta(x(\tau))$. The question of the distribution of $t_r$ has been studied for single Brownian particle in one dimension [3–5]. Intuitively one would naively expect that over a duration $t$, the particle starting at the origin, would spend half of its time in the $x > 0$ region and the remaining half in the $x < 0$ region. However, it is known that the probability distribution of $t_r$ (given in equation (1)) is peaked (divergence) near $t_r = 0$ and $t_r = t$. This implies that the Brownian particle, once gone to the $+ve$ (or $-ve$) side is reluctant to come back. This behaviour exhibits a counter-intuitive aspect of the Brownian motion. In this section, we investigate if an RTP also shows such behaviour in one dimension.

We want to compute the distribution $P_R(t_r, t)$ of $t_r$ for a given evolution time $t$ when the RTP starts from the origin. To do so we consider the following individual distributions $P^\pm(t_r, t, x_0)$ of $t_r$ corresponding to the initial velocities $\pm v$ of the particle starting at $x_0$, such that $P_R(t_r, t) = [P^+_R(t_r, t, 0) + P^-_R(t_r, t, 0)]/2$. The factor 1/2 arises from the probability 1/2 with which the initial velocity directions are chosen. Let $Q^\pm(p, x, t) = \langle e^{-pt}\xi(\pm) \rangle = \int_0^\infty dt_r e^{-pt} P^\pm_R(t_r, t, x_0)$ represent the Laplace transformation of $P^\pm_R(t_r, t, x_0)$ with respect to $t_r$. One can show that $Q^\pm(p, t)$ satisfies the following backward master equations

$$\partial_t Q^+(p, x, t) = -v \partial_x Q^+(p, x, t) - \gamma Q^+(p, x, t) + \gamma Q^-(p, x, t) - pU(x_0)Q^+(p, x, t),$$

$$\partial_t Q^-(p, x, t) = v \partial_x Q^-(p, x, t) + \gamma Q^+(p, x, t) - \gamma Q^-(p, x, t) - pU(x_0)Q^-(p, x, t),$$

where $U(x_0) = \Theta(x_0)$. In appendix D we provide a detailed derivation of the above equations for a general functional $Y[x(\tau)] = \int_0^\infty d\tau U[x(\tau)]$ of the trajectory $x(\tau)$.

We now have to solve the above coupled partial differential equations with appropriate initial and boundary conditions. At $t = 0$ the residence time $t_r = 0$, implying $Q^\pm(p, x_0, t = 0) = 1$. One the other hand at any finite $t$, we have the following boundary conditions:
• If \( x_0 \to -\infty \), the particle always stays at \( x < 0 \) region over a finite duration \( t \) implying \( t_r = 0 \). Hence, \( Q^\pm(p, x_0 \to -\infty, t) = 1 \).

• If \( x_0 \to \infty \), the particle is not able to visit the origin in finite time \( t \) at \( x > 0 \) implying \( t_r = t \). Hence, \( Q^\pm(p, x_0 \to +\infty, t) = e^{-pt} \).

To proceed further, it seems convenient to take another Laplace transform with respect to \( t \): \( \tilde{Q}^\pm(p, x_0, s) = \int_0^\infty dt e^{-st}Q^\pm(p, x_0, t) \). The master equations now become coupled linear first order differential equations, which can be solved with the above mentioned boundary and initial conditions. For better presentation of our results in the main text, here also we prefer to move the details of the calculation in appendix \( \text{E} \) while here, present only the expressions of \( \tilde{Q}^\pm(p, x_0, s) \) with \( x_0 = 0 \):

\[
\tilde{Q}^+(p, 0, s) = -\frac{1}{2\gamma} \left( 1 + \frac{\lambda_0}{s} - \frac{\lambda_p}{s + p} - \frac{\lambda_0\lambda_p}{s(s + p)} \right),
\]

\[
\tilde{Q}^-(p, 0, s) = -\frac{1}{2\gamma} \left( 1 - \frac{\lambda_0}{s} + \frac{\lambda_p}{s + p} - \frac{\lambda_0\lambda_p}{s(s + p)} \right),
\]

where \( \lambda_p = \sqrt{(s + p)(2\gamma + s + p)} \) and \( \lambda_0 = \lambda_{p=0} \). Now to get \( P_R(t_r, t) \), one needs to perform the inverse Laplace transform with respect to \( s \) and then to \( p \) of \( \tilde{Q}(p, s) = \tilde{Q}^+(p, 0, s) + \tilde{Q}^-(p, 0, s) = -\frac{1}{2\gamma} \left( 1 - \frac{\lambda_0\lambda_p}{s(s + p)} \right) \). This can be done by carrying out the appropriate Bromwich integrals. We finally obtain the following explicit expression for the distribution of \( t_r \) spent by a RTP, starting form the origin, on the positive side over the duration \( t \):

\[
P_R(t_r, t) = \frac{h(t)}{2} \left[ \delta(t_r) + \delta(t - t_r) \right] + \frac{\gamma}{2} h(t_r) h(t - t_r),
\]

where \( h(t) \) is given in equation (24). Note that this distribution is identical to that of \( t_m \) in equation (36) as one observes in the context of Brownian motions. The terms involving delta-functions can be interpreted in a similar way as done in the previous section. They get contributions from trajectories which starting from the origin stay either to the –ve side or the +ve side throughout the entire duration of evolution \( t \) and probabilities of such events are \( h(t)/2 \) each. We observe that, similar to \( P_M(t_m, t) \), the non-delta-function part of \( P_R(t_r, t) \) is also proportional to the product of \( h(t_m) \) and \( h(t - t_m) \). Unlike the case of \( t_m \), we in this case do not find any obvious connections to survival events associated to \( h(t_m) \) and \( h(t - t_m) \). The expression in equation (41) is verified numerically in the figure 8 where we observe very good agreement.

Once again note that the expression for \( P_R(t_r, t) \) in equation (41) is derived for symmetric initial conditions whereby particle starts with \( \pm v \) with equal probability. In this case also one can easily extend the above calculation for general initial conditions as shown in appendix \( \text{G} \) where we provide an explicit expression for \( P_R(t_r, t) \).
Generalised ‘Arcsine’ laws for run-and-tumble particle in one dimension

In this section we study the distribution $P_{L}(t_{\ell}, t)$ of the time $t_{\ell}$ at which the RTP crosses the origin for the last time (see figure 1) given that it started from the origin choosing $+ \text{ or } -$ direction of the initial velocity with equal probability. Let $F(a, t) = \text{Prob} [t_{\ell} \leq a, t]$ be the cumulative distribution of $t_{\ell}$ such that $P_{L}(t_{\ell}, t) = (\partial F_{L}(a, t)/\partial a)_{a=t_{\ell}}$. It is easy to argue that

$$F_{L}(a, t) = 2 \sum_{\alpha=\pm} \int_{0}^{\infty} dy \ [\text{prob. density to reach } y \text{ with direction } \alpha \text{ at time } a] \times [\text{prob. that the particle, starting from } y \text{ with direction } \alpha \text{ never comes back to the origin in the remaining time } (t - a)]$$

$$= 2 \int_{0}^{\infty} dy \ [P(y, +, a|0, 0) S_{+}(y, t - a) + P(y, -, a|0, 0) S_{-}(y, t - a)]$$

(42)

6. Last passage time distribution $P_{L}(t_{\ell}, t)$

Figure 8. Comparison of analytical expression of $P_{R}(t_{r}, t)$ given in equation (41) with numerical simulations for $\gamma = 1.5$ and $t = 5$. The solid blue line is the analytic result and the black dots are obtained in simulations by averaging over $10^6$ realisations. The inset shows the delta functions at $t_{r} = 0$ and $t_{r} = t$.

Figure 9. Comparison of analytical expression of $P_{L}(t_{\ell}, t)$ given in equation (47) with numerical simulations for $\gamma = 1.5$ and $t = 5$. The solid red line is the analytic result and the black dots are obtained in simulations by averaging over $10^6$ realisations. The inset shows the delta function at $t_{\ell} = 0$.
where \( P(y, \pm, a|0, 0) = \frac{[P(y, \pm, a|0, +, 0) + P(y, \pm, a|0, -0)]}{2} \). Here \( P(y, \alpha', t|0, \alpha, 0) \) is the probability density with which the RTP, starting from the origin with direction \( \alpha \in [+,-] \), reach position \( y \) with direction \( \alpha' \in [+,-] \) at time \( a \). \( S_\pm(y, t - a) \) is probability that in the remaining time \( (t - a) \), the particle, starting from \( y \) with direction \( \pm \) did never come back to the origin. This probabilities are basically the survival probabilities as introduced in section 3 with the only difference is that, now the absorbing barrier is at the origin instead of at \( x = M \). They also satisfy the same backward master equations (16) and (17). The factor 2 in equation (42) arises because the RTP can cross the origin (for the last time at \( t_\ell < a \)) from below or from above and these contributions are equal as can easily be seen by transforming \( y \rightarrow -y \) and interchanging + and − directions. Note that, the propagator \( P(y, \alpha', t|0, \alpha, 0) \) is different from the propagator \( P_{\alpha'}(x,t|0,\alpha,0) \) discussed in section 2, although they satisfy same master equation (7). The differences are in the boundary conditions. The former one corresponds to the full space problem without any absorbing barrier while the later one corresponds to the case with an absorbing barrier. The upper limit of the integration in equation (42) is \( x = a \) because that is the maximum distance the RTP can move in time \( a \) in the positive direction.

Now taking derivative with respect to \( a \) on both sides of equation (42) and evaluating at \( a = t_\ell \) we get

\[
P_L(t_\ell, t) = 2v \left[ S_+(t_\ell, t - t_\ell) P(t_\ell, +, t_\ell) + S_-(t_\ell, t - t_\ell) P(t_\ell, -, t_\ell) \right] \\
+ 2 \int_0^{vt_\ell} dy [S_+(y, t - t_\ell) \partial_t P(y, +, t_\ell) + S_-(y, t - t_\ell) \partial_t P(y, -, t_\ell) \\
- P(y, +, t_\ell) \partial_t S_+(y, t - t_\ell) - P(y, -, t_\ell) \partial_t S_-(y, t - t_\ell)],
\]

where we have used short hand notation \( P(y, \pm, t) \) for \( P(y, \pm, t|0,0) \) by simply \( P(y, \pm, t) \).

Using equations (7), (16) and (17), one can simplify the above equation considerably and we finally get

\[
P(t_\ell, t) = 2v S_+(0, t - t_\ell) P(0, +, t_\ell|0,0),
\]

where we have used the following facts: \( P(v t_\ell, -, t_\ell|0,0) = 0 \) (because the RTP can not reach the position \( v t_\ell \) at time \( t_\ell \) with −ve velocity) and \( S_-(0, t - t_\ell) = 0 \) (because the absorbing boundary is at the origin). The survival probability can be obtained from equation (19) after putting \( x = M \), performing the integral over \( \tau \) and finally changing − to +. One gets,

\[
S_+(0, t - t_\ell) = e^{-\gamma(t-t_\ell)} [I_0(\gamma(t - t_\ell)) + I_1(\gamma(t - t_\ell))].
\]

To obtain the \( P(0, +, t_\ell) \) we follow the procedure given in [36] and get

\[
P(0, +, t_\ell) = \frac{\delta(t_\ell)}{2v} + \frac{\gamma}{4v} [I_0(\gamma t_\ell) + I_1(\gamma t_\ell)].
\]

Inserting, these two expressions in equation (43), we get the following final expression for \( P(t_\ell, t) \):

\[
P_L(t_\ell, t) = \delta(t_\ell) h(t) + \frac{\gamma}{2} h(t) h(t - t_\ell),
\]

where \( h(t) \) is given in equation (24). In figure 9, we plot our results for \( P_L(t_\ell, t) \) and compare with numerical simulations. Note that the distribution of \( P_L(t_\ell, t) \) is almost
Generalised ‘Arcsine’ laws for run-and-tumble particle in one dimension

same as the other two distributions \( P_M(t_m, t) \) and \( P_R(t_r, t) \) except that the distribution of last passage time (to the origin) has one delta function at \( t\ell = 0 \). Presence of this \( \delta \)-function at \( t\ell = 0 \) stems from the same reasons with which they appear in the cases of \( t_m \) and \( t_r \). This represents the contributions from those trajectories which starting from the origin, go in either +ve or −ve direction and stays in that region for the entire duration of evolution \( t \).

One interesting point to note is that although the result in equation (44) is derived for the symmetric initial condition case \( a = b = 1/2 \), it turns out that it is valid for arbitrary initial conditions as well. This can be easily verified from the straightforward generalisation of equation (42).

7. Conclusion

In this paper, we mainly have investigated the distributions of three quantities in the context of an RTP moving in one dimension. The three quantities are, respectively, the time \( t_m \) at which maximum displacement \( M \) occurs, the time \( t_r \) the RTP spends on the +ve side and the time \( t\ell \) at which the RTP crosses the origin for the last time. These quantities have been well studied in the context of Brownian motion and other stochastic process, but not well studied for non-Markovian processes except for a few [6, 8, 17–19]. In the context of random acceleration process [6] the distribution of \( t_m \) has been computed however the distribution of other two observables have not been computed. In particular computation of \( P_R(t_r, t) \) for random acceleration process is still an open problem. In the context of Brownian motion, interestingly the distribution of all the three quantities turn out to be identical and their cumulative distributions are given by arcsine function. In this paper we have computed the distributions of \( t_m, t_r \) and \( t\ell \) for an RTP particle starting at the origin \( x = 0 \) with velocities +\( v \) (or −\( v \)) chosen with probability \( a \) (or \( b = 1 - a \)). We have shown that these distributions have delta function parts at the edges (\( t\ell = 0 \) and \( t_r = t \)) and a non delta function part. The distributions of \( t_m \) and \( t_r \) have delta functions at both the edges of the support \([0, t]\) whereas the distribution of \( t\ell \) has delta function only at one edge \( t\ell = 0 \). Interestingly, we have found that the non-delta function parts of all the three distributions are identical for \( a = b = 1/2 \) and are explicitly given by modified Bessel functions. In addition, we also compute the joint distribution \( P(M, t_m, t) \) of \( M \) and \( t_m \), which provides us an explicit expression of the marginal distribution of \( M \).

The fact that for symmetric initial condition \( a = b = 1/2 \) these distributions are similar (at least the non-delta function parts) is reminiscent of the fact that in the context of random walks or Brownian motion \( B(t) \) the distributions of the three times are identical [1]. In the Brownian motion case one can in fact establish that the stochastic times \( t_m \) and \( t_r \) measured over a time duration \( t \), are equivalent. This is done based on the following symmetry properties of Brownian motion: the reflection principle, inversion symmetry \([B(t) \rightarrow -B(t)]\) and time reversal symmetry \((\hat{B}(u) = B(t - u) - B(t)\) is also a Brownian process). Using these properties one can show that the stochastic process \( X(t) = M(t) - B(t) \) generated by subtracting \( B(t) \) from its current maximum process \( M(t) \) is a reflected Brownian motion. For RTP with symmetric initial
condition, while the inversion symmetry is present but the reflection symmetry and the
time reversal symmetry are not present in the true sense. As a result it is not obvious
if an equivalence between \( t_m \) and \( t_\ell \) can be established by extending the arguments of
Brownian motion straightforwardly. However, we feel that for RTP the above men-
tioned symmetry properties are valid approximately which can possibly explain the
similarity between the distributions of \( t_m \) and \( t_\ell \).

We believe that our results for non-interacting RTPs in one dimension will be infor-
mative for addressing the study of \( t_m, t_r \) and \( t_\ell \) for other active particle models. For
example, often in practical situations the tumbling rate of the bacteria depends on the
concentration of the food, which itself can evolve stochastically. Such processes can be
described by a RTP whose tumbling rate is coupled to another process. In such cases
an interesting statistics to know is the amount of time the bacteria spends in a region
of specified food concentration.

Acknowledgment

PS and AK would like to acknowledge fruitful discussions with Kanaya Malakar,
Arghya Das and Urna Basu. AK would like to acknowledge support from DST under
the Grant No. ECR/2017/000634 and from the Indo-French Centre for the promotion
of advanced research (IFCPAR) under Project No. 5604-2.

Appendix A. The propagator with an absorbing barrier at \( x = M \)

We start from the coupled differential equations in equation (7). We first take Laplace
transforms with respect to the time variable:

\[
\bar{P}_\pm(x, s) = \int_0^\infty dt e^{-st} P_\pm(x, t).
\]

The differential equations in equation (7), now becomes

\[
\left(v \partial_x + \gamma + s\right)\bar{P}_+(x, s) = \gamma \bar{P}_-(x, s) + a\delta(x),
\]

\[
\left(v \partial_x - \gamma - s\right)\bar{P}_-(x, s) = -\gamma \bar{P}_+(x, s) + b\delta(x),
\]

where the coefficients \( a \) and \( b \) of the delta functions appear from the initial conditions
\( P_+(x, 0|0, \alpha, 0) = a \delta_{\alpha,+}\delta(x) \) and \( P_-(x, 0|0, \xi, 0) = b \delta_{\alpha,-}\delta(x) \). For \( x > 0 \), we can write equations (A.2) and (A.3) without \( \delta(x) \) terms in the RHS,

\[
L_+ \bar{P}_+(x, s) = \gamma \bar{P}_-(x, s), \quad \text{with } L_+ = \left(v \partial_x + \gamma + s\right),
\]

and,

\[
L_- \bar{P}_-(x, s) = -\gamma \bar{P}_+(x, s), \quad \text{with } L_- = \left(v \partial_x - \gamma - s\right).
\]

Applying \( L_- \) on both sides of equation (A.4) and \( L_+ \) on both sides of equation (A.5), we
can write the closed form equations for \( \bar{P}_\pm \) in the following way:
\[ (v \partial_x - \gamma - s) (v \partial_x + \gamma + s) \bar{P}_+(x, s) = -\gamma^2 \bar{P}_+(x, s), \]  
(A.6)

\[ (v \partial_x + \gamma + s) (v \partial_x - \gamma - s) \bar{P}_-(x, s) = -\gamma^2 \bar{P}_-(x, s). \]  
(A.7)

These two equations are ordinary second-order differential equations and we try the solution \( \bar{P}_+ \sim e^{\beta x} \) and put this in equation (A.6). We then get \((\beta - \gamma - s)(\beta + \gamma + s) = -\gamma^2 \) which gives \( \beta = \pm \frac{1}{2} \sqrt{s(2\gamma + s)} = \pm \lambda(s) \) where \( \lambda(s) = \frac{1}{a} \sqrt{s(2\gamma + s)} \). For \( x > 0^+ \), we have \( \bar{P}_+ = Ae^{\lambda x} + Be^{-\lambda x} \) where \( x \) can go up to \( M \). To get \( \bar{P}_- \), we can put \( \bar{P}_+ \) in equation (A.4) and get \( \bar{P}_- = A(\nu \lambda + \gamma + s)e^{\lambda x} + B(-\nu \lambda + \gamma + s)e^{-\lambda x} \). Similarly, one can also get the solutions of equations (A.2) and (A.3) for \( x < 0^- \) region. However, here we will have to consider only positive \( \lambda \) so that \( \bar{P}_\pm \) remain finite as \( x \to -\infty \). We can write the solution of equations (A.2) and (A.3) as shown below:

\[ \bar{P}_+(x, s) = \begin{cases} Ae^{\lambda(s)x} + Be^{-\lambda(s)x}, & \text{if } 0 < x < M \\ Ce^{\lambda(s)x}, & \text{if } -\infty < x < 0 \end{cases} \]  
(A.8)

\[ \bar{P}_-(x, s) = \frac{1}{\gamma} \begin{cases} A(v(\lambda(s) + \gamma + s)e^{\lambda(s)x} + B(-v\lambda(s) + \gamma + s)e^{-\lambda(s)x}, & \text{if } 0 < x < M \\ C(v\lambda(s) + \gamma + s)e^{\lambda(s)x}, & \text{if } -\infty < x < 0. \end{cases} \]  
(A.9)

The constants \( A, B \) and \( C \) are still unknown and to determine them we first integrate both sides of equations (A.8) and (A.9) from \( x = -\epsilon \) to \( x = +\epsilon \) with \( \epsilon \to 0^+ \). We get the following discontinuity equations

\[ \bar{P}_+(x \to 0^+, s) - \bar{P}_+(x \to 0^-, s) = \frac{a}{v}, \]  
(A.10)

\[ \bar{P}_-(x \to 0^+, s) - \bar{P}_-(x \to 0^-, s) = \frac{b}{v}. \]  
(A.11)

We also note that the boundary condition \( \bar{P}_-(M, t) = 0 \) now reads

\[ \bar{P}_-(M, s) = 0. \]  
(A.12)

Inserting the solutions from equations (A.8)–(A.12), one gets linear equations for the constants \( A, B \) and \( C \) solving which one finally gets these constants in terms of \( s, \gamma, \nu \) for different choices of \( a \) and \( b \).

For example, for \( a = 1 \) and \( b = 0 \) i.e. for + initial condition, one obtains

\[ A(s) = -\frac{s + \gamma - v\lambda(s)}{2\nu^2 \lambda(s)} e^{-2\nu M \lambda(s)}, \]  
(A.13)

\[ B(s) = \frac{s + \gamma + v\lambda(s)}{2\nu^2 \lambda(s)}, \]  
and,  
(A.14)

\[ C(s) = \frac{s + \gamma - v\lambda(s)}{2\nu^2 \lambda(s)} \left(1 - e^{-2\nu M \lambda(s)}\right). \]  
(A.15)

Using these expressions of the constants in equations (A.8) and (A.9)
Generalised ‘Arcsine’ laws for run-and-tumble particle in one dimension

\[ \tilde{P}_+(x, s|0, +, 0) = \frac{1}{2v^2 \lambda(s)} \begin{cases} (s + \gamma + v\lambda(s)) e^{-\lambda(s)x} & \text{if } 0 < x < M \\ (s + \gamma - v\lambda(s)) e^{\lambda(s)x} - (s + \gamma - v\lambda(s)) e^{\lambda(s)x-2M} & \text{if } -\infty < x < 0 \end{cases}, \]

\[ \tilde{P}_-(x, s|0, +, 0) = \frac{\gamma}{2v^2 \lambda(s)} e^{\lambda(s)|x|} - e^{\lambda(s)(x-2M)}. \]

Similarly, for \( a = 0 \) and \( b = 1 \) i.e. for \( - \) initial condition, one obtains

\[ \tilde{P}_+(x, s|0, -, 0) = \frac{\gamma}{2v^2 \lambda(s)} \left( e^{-\lambda(s)|x|} - \frac{s + \gamma - v\lambda(s)}{s + \gamma + v\lambda(s)} \right) e^{\lambda(s)(x-2M)} \]

\[ \tilde{P}_-(x, s|0, -, 0) = \frac{1}{2v^2 \lambda(s)} \begin{cases} (s + \gamma - v\lambda(s)) e^{-\lambda(s)x} - (s + \gamma - v\lambda(s)) e^{\lambda(s)(x-2M)} & \text{if } 0 < x < M \\ (s + \gamma + v\lambda(s)) e^{\lambda(s)x} - (s + \gamma + v\lambda(s)) e^{\lambda(s)(x-2M)} & \text{if } -\infty < x < 0. \end{cases} \]

Now in order to get the solutions in time domain, one need to perform inverse Laplace transforms \( \tilde{P}_\pm(x, s|0, \pm, 0) \). Looking at the expressions in equations (A.16)–(A.19), we need to know the following inverse Laplace transforms

\[ L_{s \rightarrow t}^{-1} \left[ \frac{1}{\lambda(s)} e^{-g\lambda(s)} \right] = ve^{-\gamma t} I_0 \left( \frac{\gamma}{v} \sqrt{vt^2 - g^2} \right) \Theta(vt - g) \]

\[ L_{s \rightarrow t}^{-1} \left[ \frac{s + \gamma + \alpha v\lambda(s)}{\lambda(s)} e^{-\lambda(s)g} \right] = \gamma ve^{-\gamma t} \left[ \frac{vt - g}{vt + g} \right] I_1 \left( \frac{\gamma}{v} \sqrt{vt^2 - g^2} \right) \Theta(vt - g) - \frac{1 + \alpha}{2} v^2 \frac{dt}{dg} \left[ \Theta(vt - g)I_0 \left( \frac{\gamma}{v} \sqrt{vt^2 - g^2} \right) \right], \] for \( \alpha = \pm 1, \)

\[ L_{s \rightarrow t}^{-1} \left[ (s + \gamma - v\lambda(s)) e^{-\lambda(s)g} \right] = \frac{\gamma v}{vt - g} \left[ \frac{\gamma g}{v} I_0 \left( \frac{\gamma}{v} \sqrt{vt^2 - g^2} \right) \right] + \sqrt{\frac{vt - g}{vt + g}} I_1 \left( \frac{\gamma}{v} \sqrt{vt^2 - g^2} \right) \Theta(vt - g). \]

Using these results and performing some simplifications, one arrives at the expressions of the propagators given in equations (8)–(11).

**Appendix B. Cumulative distribution \( Q(M, t) \)**

The cumulative distribution for \( M \) is defined as,

\[ Q(M, t) = \int_0^M \text{d}M' \mathcal{P}(M', t) = \int_0^M \text{d}M' \int_0^t \text{d}t_m \mathcal{P}(M', t_m, t), \]

where \( \mathcal{P}(M, t_m, t) \) is given by equation (34). One can easily integrate equation (34) with \( t_m \) to get \( \mathcal{P}(M, t) \).
Generalised ‘Arcsine’ laws for run-and-tumble particle in one dimension

\[ \mathcal{P}(M, t) = \int_0^t dt_m \mathcal{P}(M', t_m, t) \]

\[ = \frac{h(t)}{2} \delta(M) - \frac{1}{2} \frac{d}{dM} [\Theta (vt - M) g(M, t)] \]

\[ - \frac{\gamma}{2} \int_0^t dt_m h(t - t_m) \frac{d}{dM} [\Theta (vt_m - M) g(M, t_m)] . \]  

The contribution of first term in the R.H.S of equations (B.2) to (B.1) is just \( \frac{h(t)}{2} \). The contribution from the second term is given by,

\[ \frac{1}{2} \int_0^M dM' \frac{d}{dM'} [\Theta (vt - M') g(M', t)] = \frac{1}{2} [\Theta (vt - M') g(M', t)] \bigg|_0^M , \]

\[ = \Theta (vt - M) \frac{g(M, t)}{2} - \frac{h(t)}{2} . \]  

Similarly one can get the contribution of the third term in the R.H.S of equations (B.2) to (B.1).

\[ \frac{\gamma}{2} \int_0^M dM' \int_0^t dt_m h(t - t_m) \frac{d}{dM} [\Theta (vt_m - M) g(M, t_m)] \]

\[ = \frac{\gamma}{2} \int_0^t dt_m h(t - t_m) [\Theta (vt_m - M') g(M', t_m)] \bigg|_0^M , \]

\[ = \frac{\gamma}{2} \int_0^t dt_m h(t - t_m) [\Theta (vt_m - M) g(M, t_m) - h(t_m)] , \]

\[ = \frac{\gamma}{2} \int_0^t dt_m [\Theta (vt_m - M) h(t - t_m) g(M, t_m) - 1 - h(t)] , \]

where we have used equation (C.1) to go from second line to third line. Adding all these contributions, we finally get \( Q(M, t_m) \),

\[ Q(M, t) = 1 - \frac{1}{2} g(M, t) \Theta (vt - M) - \frac{\gamma}{2} \int_0^t dt' \Theta (vt' - M) h(t - t') g(M, t') . \]

Appendix C. Proof of the identity

Here we prove the identity

\[ \gamma \int_0^t dt' h(t - t') h(t') \overset{?}{=} 2[1 - h(t)] \]  

where \( h(t) = e^{-\gamma t} [I_0(\gamma t) + I_1(\gamma t)] \). The best way to prove this identity is to take the Laplace transform of the left hand side with respect to the variable \( t \). Since the left hand side is in the convolution form, we have

\[ L_{t \rightarrow s} \left[ \gamma \int_0^t dt' h(t - t') h(t') \right] = \gamma \hat{h}(s)^2 \]  

where \( \hat{h}(s) = \mathcal{L}[h(t)] \) is the Laplace transform of \( h(t) \). Since \( \hat{h}(s) \) is known, the identity follows.
where \( \tilde{h}(s) \) is the Laplace transform of \( h(t) \) and is given by
\[
\tilde{h}(s) = L_{t \to s}[h(t)] = \frac{1}{\gamma} \left( \frac{\sqrt{s(2\gamma + s)}}{s} - 1 \right). \tag{C.3}
\]

Using this result in equation (C.2) and performing some algebraic simplifications, we get
\[
L_{t \to s} \left[ \gamma \int_0^t dt' h(t-t')h(t') \right] = 2 \left[ \frac{1}{s} - \frac{1}{\gamma} \left( \frac{\sqrt{s(2\gamma + s)}}{s} - 1 \right) \right]. \tag{C.4}
\]

Finally performing inverse Laplace transform on both sides with respect to \( s \) one proves the identity in equation (C.1).

**Appendix D. Backward master equation for \( Q^\sigma(p, x_0, t) \)**

Here we provide a derivation of the Backward master equations in (37) and (38) for a general functional \( Y[x(t)] \)
\[
Y[x(t)] = \int_0^t d\tau U[x(\tau)], \tag{D.1}
\]
of the trajectory \( \{x(\tau); 0 \leq \tau \leq t\} \) of the RTP starting from \( x_0 \) at \( \tau = 0 \). Here we mainly follow the general steps done for Brownian motion in [3]. It is clear that \( Y \) is a random variable. Let \( P^\sigma(Y, x_0, t) \) denote the probability density for \( Y \) where \( \sigma = \pm 1 \) is the direction of the initial velocity. We define the characteristic function:
\[
Q^\sigma(p, x_0, t) = \int_0^{\infty} dY e^{-pY} P^\sigma(Y, x_0, t) = \langle e^{-pY} \rangle_{(x_0, \sigma)}, \tag{D.2}
\]
where \( \langle ... \rangle_{(x_0, \sigma)} \) denotes the average with initial position \( x_0 \) and velocity orientation \( \sigma \). Following the definition in equation (D.2), we write for a small time interval \( \Delta t \)
\[
Q^\sigma(p, x_0, t + \Delta t) = \langle e^{-p \int_{t}^{t+\Delta t} d\tau U[x(\tau)]} \rangle_{(x_0, \sigma)} = \langle e^{-p \int_{0}^{\Delta t} d\tau U[x(\tau)]} e^{-p \int_{t}^{t+\Delta t} d\tau U[x(\tau)]} \rangle_{(x_0, \sigma)} = (1 - pU(x_0)\Delta t) \langle e^{-p \int_{0}^{\Delta t} d\tau U[x(\tau)]} \rangle_{(x_0, \sigma)} \tag{D.3}
\]
where \( (x_0, \sigma) \) is the initial configuration. In time interval \( \Delta t \) the state \( (x_0, \sigma) \) of the RTP can change to \( (x_0 + \sigma v \Delta t, \sigma) \) with probability \( 1 - \gamma \Delta t \) and to \( (x_0, -\sigma) \) with probability \( \gamma \Delta t \). We can write equation (D.3) as following:
\[
Q^\sigma(p, x_0, t + \Delta t) = (1 - pU(x_0)\Delta t) \left[ (1 - \gamma \Delta t) \langle e^{-p \int_{0}^{\Delta t} d\tau U[x(\tau)]} \rangle_{(x_0 + \sigma v \Delta t, \sigma)} \right. \tag{D.4}
\]
\[
+ \gamma \Delta t \langle e^{-p \int_{0}^{\Delta t} d\tau U[x(\tau)]} \rangle_{(x_0 - \sigma v \Delta t, -\sigma)} \right]. \tag{D.5}
\]
\[
= (1 - pU(x_0)\Delta t) \left[ (1 - \gamma \Delta t) Q^\sigma(p, x_0 + \sigma v \Delta t, t) + \gamma \Delta t Q^{-\sigma}(p, x_0, t) \right]. \tag{D.6}
\]
Taking the \( \Delta t \to 0 \) limit we obtain

https://doi.org/10.1088/1742-5468/ab3283
\[ \partial_t Q^\sigma(p, x_0, t) = \sigma v \partial_x Q^\sigma(p, x_0, t) - \gamma Q^\sigma(p, x_0, t) + \gamma Q^{-\sigma}(p, x_0, t) - pU(x_0)Q^\sigma(p, x_0, t). \] (D.7)

If we choose \( U(x) = \Theta(x) \) then \( Y \) represents the residence time \( t \) the particle on the positive semi axis and for this choice one obtains the differential equations in equations (37) and (38).

**Appendix E. Probability distribution \( P_R(t, t) \)**

In this appendix, we find the solutions of equations (37) and (38) to obtain \( P_R(t, t) \). We take the Laplace transformation of \( Q^\pm(p, x_0, t) \) in the time \( t \) as \( \tilde{Q}^\pm(p, x_0, s) = \int_0^\infty dt e^{-st}Q^\pm(p, x_0, t) \). As a result equations (37) and (38) now become

\[
[v\partial_x - \gamma - pU(x_0) - s] \tilde{Q}^+(p, x_0, s) = -\gamma \tilde{Q}^-(p, x_0, s) - 1, \quad \text{(E.1)}
\]

\[
[v\partial_x + \gamma + pU(x_0) + s] \tilde{Q}^-(p, x_0, s) = \gamma \tilde{Q}^+(p, x_0, s) + 1. \quad \text{(E.2)}
\]

The boundary conditions for \( Q^\pm(p, x_0, s) \) (discussed in section 5) can now be translated for \( \tilde{Q}^\pm(p, x_0, s) \) as \( \tilde{Q}^\pm(p, x_0 \to -\infty, s) = \frac{1}{s} \) and \( \tilde{Q}^\pm(p, x_0 \to \infty, s) = \frac{1}{sx^p} \). Let us first solve the equations (E.1) and (E.2) for \( x_0 > 0 \) region where we put \( U(x_0) = 1 \) and we have,

\[
[v\partial_x - \gamma - p - s] \tilde{Q}^+(p, x_0, s) = -\gamma \tilde{Q}^-(p, x_0, s) - 1, \quad \text{(E.3)}
\]

\[
[v\partial_x + \gamma + p + s] \tilde{Q}^-(p, x_0, s) = \gamma \tilde{Q}^+(p, x_0, s) + 1. \quad \text{(E.4)}
\]

These equations can be made homogenous by shift \( Q^\pm(p, x_0, s) = \frac{1}{s} + Z^\pm(p, x_0, s) \) to get,

\[
[v\partial_x - \gamma - p - s] Z^+(p, x_0, s) = -\gamma Z^-(p, x_0, s), \quad \text{(E.5)}
\]

\[
[v\partial_x + \gamma + p + s] Z^-(p, x_0, s) = \gamma Z^+(p, x_0, s). \quad \text{(E.6)}
\]

The boundary condition of \( Q^\pm(p, x_0 \to \infty, s) = \frac{1}{s} \) now becomes \( Z^\pm(p, x_0 \to -\infty, s) = 0 \).

Following the same procedure as done in appendix A, we find that the solutions are given by

\[
\tilde{Q}^+_+(p, x_0, s) = \frac{1}{s + p} + A(s, p)e^{-\frac{\lambda_p(s)x_0}{v}}, \quad \text{for, } x_0 > 0, \quad \text{(E.7)}
\]

\[
\tilde{Q}^-_+(p, x_0, s) = \frac{1}{s + p} + \frac{A(s, p)}{\gamma} (s + p + \gamma + \lambda_p) e^{-\frac{\lambda_p(s)x_0}{v}}, \quad \text{(E.8)}
\]

where \( \lambda_p(s) = \sqrt{(s + p)(2\gamma + s + p)} \). The solutions obtained in equations (E.9) and (E.10) are true for \( x_0 < 0 \) region. For \( x_0 < 0 \), one can follow the same procedure. Only difference is that now \( U(x_0) = 0 \). We find

\[
\tilde{Q}^+_+(p, x_0, s) = \frac{1}{s} + B(s, p)e^{\frac{\lambda_p(s)x_0}{v}}, \quad \text{(E.9)}
\]

https://doi.org/10.1088/1742-5468/ab3283
The constants $A(s, p)$ and $B(s, p)$ in the expressions for $\hat{Q}_\pm(p, x_0, s)$ are unknown constants which can be computed from the continuity of $Q^\pm(p, x_0, s)$ at $x = 0$, i.e. $\hat{Q}^\pm(x_0 \to 0) = \hat{Q}^\pm(x_0 \to 0_\pm)$. After some simple manipulations, we find

$$A(s, p) = \frac{p(\lambda_0(s) - s)}{s(s + p)(p + \lambda_0(s) + \lambda_p(s))}. \quad (E.11)$$

$$B(s, p) = -\frac{p(s + p + \lambda_p(s))}{s(s + p)(p + \lambda_0(s) + \lambda_p(s))}. \quad (E.12)$$

For $x_0 = 0$ the solutions simplifies further and we have

$$\hat{Q}^+(p, 0, s) = -\frac{1}{2\gamma} \left( 1 + \frac{\lambda_0(s)}{s} - \frac{\lambda_p(s)}{s + p} - \frac{\lambda_0\lambda_p(s)}{s(s + p)} \right), \quad (E.13)$$

$$\hat{Q}^-(p, 0, s) = -\frac{1}{2\gamma} \left( 1 - \frac{\lambda_0(s)}{s} + \frac{\lambda_p(s)}{s + p} - \frac{\lambda_0\lambda_p(s)}{s(s + p)} \right). \quad (E.14)$$

If the particle starts with $\pm v$ velocity with equal probability $1/2$, the total probability is given by $P_R(t_r, t) = \frac{P^{\hat{Q}^+(p, 0, s)}(t_r, t) + P^{\hat{Q}^-(p, 0, s)}(t_r, t)}{2}$. Accordingly, $\hat{Q}(p, 0, s) = \int_0^\infty dt_e e^{-st} \int_0^{t_r} dt_r e^{-pt_r} P_R(t_r, t) = \frac{\hat{Q}^+(p, 0, s) + \hat{Q}^-(p, 0, s)}{2}$ which, after some simplifications, is given by

$$\hat{Q}(p, 0, s) = \hat{Q}^+(p, s) + \hat{Q}^-(s, p) = -\frac{1}{2\gamma} \left( 1 - \frac{\lambda_0(s)\lambda_p(s)}{s(s + p)} \right). \quad (E.15)$$

Now to get $P_R(t_r, t)$ we have to perform inverse Laplace transform of $\hat{Q}(p, s)$ with respect to $s$ and $p$ both i.e.

$$P_R(t_r, t) = L_{s \to t}^{-1} L_{p \to t_r}^{-1} \left[ -\frac{1}{2\gamma} \left( 1 - \frac{\lambda_0(s)\lambda_p(s)}{s(s + p)} \right) \right]. \quad (E.16)$$

Using the following inverse Laplace transform result,

$$L_{p \to t_r}^{-1} \left[ \frac{\lambda_p(s)}{s + p} \right] = \delta(t_r) + \gamma e^{-st_r} e^{-\gamma t_r} [I_0(\gamma t_r) + I_1(\gamma t_r)] \quad (E.17)$$

we get the final solution as given in equation (41) of the main text.

**Appendix F. Probability distribution for time $t_m$ to reach maximum $M$ for general initial condition**

In section 4.1 we computed the distribution for the time to reach maximum $t_m$ for a RTP that starts from $x = 0$ with $\pm v$ with equal probability i.e. $a = b = 1/2$. Here we
Figure F1. (a) Plot of the $P_M^\text{as}(t_m,t)$ in equation (F.6) for various combinations of $(a,b)$. Note that we have not shown delta-function part here. (b) Numerical verification of the same for $a = 0.25, b = 0.75$. The inset shows the comparison of the delta-function parts. The Other parameters used in both the plots are $\gamma = 1.5$ and $t = 5$.

derive the distribution $P_M^\text{as}(t_m,t)$ of $t_m$ for arbitrary $a$ and $b$. (‘as’ in the superscript is to represent assymmetric initial conditions). We follow the same procedure as done in section 2. Using the expressions for $P_\pm(M - \epsilon, t = 0, \pm, 0)$ obtained in section 2, we first compute $p_\pm(M - \epsilon, t_m|0,0) = aP_\pm(M - \epsilon, t_m|0,+,0) + bP_\pm(M - \epsilon, t_m|0,-,0)$ for general initial conditions. After some simplifications, the expressions for $p_\pm(M - \epsilon, t_m|0,0)$ read as,

$$
\begin{align*}
p_+(M - \epsilon, t_m|0,0) &\approx -\frac{d}{dM} \left[ \Theta(\epsilon) \frac{\epsilon}{\sqrt{v^2 t^2 - M^2}} g^\text{as}(M, t_m) \right] + O(\epsilon), \\
p_-(M - \epsilon, t_m|0,0) &\approx -\frac{\epsilon}{\sqrt{v^2 t^2 - M^2}} \left[ \Theta(\epsilon) g^\text{as}(M, t_m) \right] + O(\epsilon^2),
\end{align*}
$$

(F.1)

where the function $g(M, t)$ in equation (26) is now changed to

$$
g^\text{as}(M, t) = e^{-\gamma t} \left[ aI_0 \left( \frac{\gamma}{v} \sqrt{v^2 t^2 - M^2} \right) + b \sqrt{\frac{vt}{v+M}} I_1 \left( \frac{\gamma}{v} \sqrt{v^2 t^2 - M^2} \right) \right].
$$

(F.2)

Using these expressions in equation (23), we get the part of the joint distribution $P^\text{as}(M, t_m, t)$ of $M$ and $t_m$ for $0 < M < vt$ and $0 < t_m < t$.

We now have to add the contribution from the events $M = 0$ (or equivalently $t_m = 0$) and $t_m = t$. As discussed in the symmetric case, this contribution from the events $t_m = 0$ comes from those paths which initially moves with $-v$ and survives the origin till time $t$. Hence,

$$
P^\text{as}(M, t_m, t) = \delta(t_m) \delta(M) bh(t).
$$

(F.3)

On the other hand the contributions to $t_m = t$ event come from trajectories which starting from the origin reach the maximum displacement $M$ at time $t_m = t$. The contribution is obtained by generalising equation (27) straightforwardly for general initial conditions as

$$
P^\text{as}(M, t_m, t) = \delta(t_m - t) [aP_+(M, t|0,+,0) + bP_+(M, t|0,-,0)].
$$

(F.4)
Inserting the explicit forms of these propagator $P_+(M, t|0, ±, 0)$ from equations (9) and (11), and simplifying we get

$$
\frac{\partial^{as}(M, t_m, t)}{\partial t_m | t_m = t} = -\frac{d}{dM}[\Theta(\nu t - M) \; g^{as}(M, t)].
$$  

(F.5)

Now putting all the three contributions together, performing some simplifications, fixing the normalisation and integrating over $M$, we finally get,

$$
P^{as}_M(t_m, t) = \int_0^{vt} dM \; \mathcal{P}^{as}(M, t_m, t),
$$

where $h^{as}(t) = e^{-\gamma t}[aI_0(\gamma t) + bI_1(\gamma t)]$. Note that for $a = b = \frac{1}{2}$, $h^{as}(t) = \frac{h(t)}{2}$ and equation (F.6) reduces to the symmetric case distribution in equation (36). However for $(a \neq b)$, $P^{as}_M(t_m, t)$ is different from equation (36). In figure F1(a), we have plotted our analytical result for various combinations of $a$ and $b$ and in figure F1(b) we have verified it with numerical simulations for a particular choice of $a$. We observe excellent agreement between the two.

**Appendix G. Probability distribution for residence time $t_r$ for general initial condition**

In this section, we extend the computation of the residence time distribution $P_R(t_r, t)$ in section 5 from symmetric initial condition to asymmetric initial condition for the velocity. For this we assume that the particle starts with $+v(-v)$ velocity with probability $a(b)$ such that $a + b = 1$. Let us denote the Laplace transform of $P^{as}_R(t_r, t)$ with respect to $t$ and $t_r$ by $\hat{Q}^{as}(p, s)$ where ‘$as$’ in the superscript once again represents asymmetric initial condition. Using the equations (39) and (40), and performing some simplifications we get

$$
\hat{Q}^{as}(p, s) = a\hat{Q}_+(p, s) + b\hat{Q}_-(p, s),
$$

$$
= -\frac{1}{2\gamma} \left(1 - \frac{\lambda_0 \lambda_p}{s(s + p)} \right) - \frac{a - b}{2\gamma} \left(\frac{\lambda_0}{s} - \frac{\lambda_p}{s + p} \right).
$$  

(G.1)

To get the distribution $P^{as}_R(t_r, t)$ in the time domain, we have to do two Laplace transformations: one from $s \rightarrow t$ and the other from $p \rightarrow t_r$. Looking at the expression in equation (G.1), one notes that the first term in R.H.S is identical to the symmetric case whose Laplace transformation is given in equation (41). The Laplace transformation of the second term in R.H.S of equation (G.1) is given in equation (E.17). One can therefore write the distribution for $t_r$ for general initial condition as

$$
P^{as}_R(t_r, t) = h(t) \left[b\delta(t_r) + a\delta(t_r - t)\right] + \frac{\gamma}{2} h(t_r) h(t - t_r).
$$  

(G.2)

Note that $P^{as}_R(t_r, t)$ in equation (G.2) reduces to $P_R(t_r, t)$ in equation (41) for $a = b = \frac{1}{2}$. Also note that the asymmetry in the initial condition only alters the strengths of the delta-functions of $P_R(t_r, t)$ but leaves the non-delta function part unchanged. These
weights of the delta-functions can be understood in the same way as for the symmetric case. In figure G1, we verify our analytical result with numerical simulation and find excellent agreement with it.

References

[1] Feller W 1968 An Introduction to Probability Theory and its Applications (New York: Wiley)
[2] Lévy P 1939 Sur certains processus stochastiques homogènes Compos. Math. 7 283–339
[3] Majumdar S N 2005 Brownian functionals in physics and computer science Curr. Sci. 89 2076
[4] Majumdar S N, Randon-Furling J, Kearney M J and Marc Y 2008 On the time to reach maximum for a variety of constrained Brownian motions J. Phys. A: Math. Theor. 41 305005
[5] Randon-Furling J and Majumdar S N 2007 Distribution of the time at which the deviation of a Brownian motion is maximum before its first-passage time J. Stat. Mech. P10008
[6] Majumdar S N, Rosso A and Zoia A 2010 Time to reach the maximum for a random acceleration process J. Phys. A: Math. Theor. 43 115001
[7] Bouchet H, Bouetou T, Burkhardt T W, Rosso A, Zoia A and Crepin K T 2016 Occupation time statistics of the random acceleration model J. Stat. Mech. 053213
[8] Sadhu T, Delorne M and Wiese K J 2018 Generalized arcsine laws for fractional Brownian motion Phys. Rev. Lett. 120 040603
[9] Baldassarri A, Bouchaud J-P, Dornic I and Godrèche C 1999 Statistics of persistent events: an exactly solvable model Phys. Rev. E 59 R20
[10] Godrèche C and Luck J M 2001 Statistics of the occupation time of renewal processes J. Stat. Phys. 104 489–524
[11] Burov S and Barkai E 2011 Residence time statistics for N renewal processes Phys. Rev. Lett. 107 170601
[12] Randon-Furling J, Majumdar S N and Comtet A A 2009 Convex hull of N plana Brownian motions: application to ecology Phys. Rev. Lett. 103 140602
[13] Majumdar S N, Comtet A and Randon-Furling J 2010 Random convex hulls and extreme value statistics J. Stat. Phys. 138 955
[14] Rambeau J and Schehr G 2011 Distribution of the time at which N vicious walkers reach their maximal height Phys. Rev. E 83 061146
[15] Schehr G and Le Doussal P 2010 Extreme value statistics from the real space renormalization group: Brownian motion, Bessel processes and continuous time random walks J. Stat. Mech. P01009
[16] Dhar A and Majumdar S N 1999 Residence time distribution for a class of Gaussian Markov processes Phys. Rev. E 59 6413–8

Figure G1. Comparison of analytical expression of $P_R^a(t_\tau, t)$ given in equation (G.2) with numerical simulations for $a = 0.25, b = 0.75, \gamma = 1.5$ and $t = 5$. The solid red line is the analytic result and the black dots are obtained in simulations by averaging over $10^7$ realisations. The inset shows the delta function at $t_\tau = 0$ and $t_\tau = t$. Note that the non-delta function part is identical to that in figure 8 for symmetric case but the delta-function parts are different.
Generalised ‘Arcsine’ laws for run-and-tumble particle in one dimension

[17] Kasahara Y 1977 Limit theorems of occupation times for Markov processes Publ. RIMS, Kyoto Univ. 12 801–18
[18] Majumdar S N and Dean D S 2002 Exact occupation time distribution in a non-Markovian sequence and its relation to spin glass models Phys. Rev. E 66 041102
[19] Lamperti J 1958 An occupation time theorem for a class of stochastic processes Trans. Am. Math. Soc. 88 380–7
[20] Charles D and Rosemarie W 1980 The arcsine law and the treasury bill futures market Financ. Anal. J. 36 71–4
[21] Shiryaev A N 2002 Quickest detection problems in the technical analysis of the financial data Mathematical Finance Bachelier Congress 2000 (New York: Springer) p 487
[22] Majumdar S N and Bouchaud J-P 2008 Optimal time to Sell a Stock in black-Scholes model Quant. Finance 8 753
[23] Majumdar S N and Comtet A 2002 Local and the occupation time of a particle diffusing in a random medium Phys. Rev. Lett. 89 060601
[24] Sabhapandit S, Majumdar S N and Comtet A 2006 Statistical properties of functionals of the paths of a particle diffusing in a one-dimensional random potential Phys. Rev. E 73 051102
[25] Nazarov Y V 1994 Limits of universality in disordered conductors Phys. Rev. Lett. 73 134
[26] Beenakker C W J 1997 Random-matrix theory of quantum transport Rev. Mod. Phys. 69 731
[27] Akimoto A 2008 Generalized arcsine law and stable law in an infinite measure dynamical system J. Stat. Phys. 132 171
[28] Mejia-Monasterio C, Oshanin G and Schehr G 2011 Symmetry breaking between statistically equivalent, independent channels in few-channel chaotic scattering Phys. Rev. E 84 035203
[29] Oshanin G and Redner S 2009 Helix or coil? Fate of a melting heteropolymer Europhys. Lett. 85 10008
[30] Barato A C, Roldán E, Martínez I A and Pigolotti S 2018 Arcsine laws in stochastic thermodynamics Phys. Rev. Lett. 121 090601
[31] Weiss G H 2002 Some applications of persistent random walks and the telegrapher’s equation Physica A 311 381–410
[32] Masoliver J and Lindenberg K 2017 Continuous time persistent random walk: a review and some generalizations Eur. Phys. J. B 90 107
[33] Tailleur J and Cates M E 2008 Statistical mechanics of interacting run-and-tumble bacteria Phys. Rev. Lett. 100 218103
[34] Solon A P, Cates M E and Tailleur J 2015 Active Brownian particles and run-and-tumble particles: a comparative study Eur. Phys. J. Spec. Top. 224 1231–62
[35] Berg H C 2003 E. coli in Motion (New York: Springer)
[36] Malakar K, Jenseena V, Kundu A, Kumar K V, Sabhapandit S, Majumdar S N, Redner S and Dhar A 2018 Steady state, relaxation and first-passage properties of a run-and-tumble particle in one-dimension J. Stat. Mech. 043215
[37] Tailleur J and Cates M E 2009 Sedimentation, trapping, and rectification of dilute bacteria Europhys. Lett. 86 60002
[38] Dhar A, Kundu A, Majumdar S N, Sabhapandit S and Schehr G 2019 Run-and-tumble particle in one-dimensional confining potential: steady state, relaxation and first passage properties Phys. Rev. E 99 032132
[39] Slowman A B, Evans M R and Blythe R A 2016 Jamming and attraction of interacting run-and-tumble random walkers Phys. Rev. Lett. 116 218101
[40] Slowman A B, Evans M R and Blythe R A 2017 Exact solution of two interacting run-and-tumble random walkers with finite tumble duration J. Phys. A: Math. Theor. 50 375601
[41] Peruani F, Starruß J, Jakovljevic V, Søegaard-Andersen L, Deutsch A and Bär M 2012 Collective motion and nonequilibrium cluster formation in colonies of gliding bacteria Phys. Rev. Lett. 108 098102
[42] Takatori S C, De Dier R, Vermaut J and Brady J F 2016 Acoustic trapping of active matter Nat. Commun. 7 10694
[43] Malakar K, Das A, Kundu A, Kumar K V and Dhar A 2019 Exact steady state of active Brownian particles in a 2D harmonic trap (arXiv:1902.04171)
[44] Dauchot O and Démery V 2019 Dynamics of a self-propelled particle in a harmonic trap Phys. Rev. Lett. 122 068002
[45] Demaere T and Maes C 2018 Active processes in one dimension Phys. Rev. E 97 032604
[46] Doussal P L, Majumdar S N and Schehr G 2019 Non-crossing run-and-tumble particles on a line (arXiv:1902.06176v1)

https://doi.org/10.1088/1742-5468/ab3283
Evans M R and Majumdar S N 2018 Run and tumble particle under resetting: a renewal approach *J. Phys. A: Math. Theor.* **51** 475003

Majumdar S N 2010 Universal first-passage properties of discrete-time random walks and Lévy flights on a line: statistics of the global maximum and records *Physica A* **389** 4299–316

Masoliver J and Weiss G H 1993 On the maximum displacement of a one dimensional diffusion process described by the telegrapher’s noise *Physica A* **195** 93–100

https://doi.org/10.1088/1742-5468/ab3283