A note on Mountford and Sweet’s extension of
Kuczek’s argument to non-nearest neighbours
contact processes

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Abstract

An elementary proof of the i.i.d. nature of the growth of the right endpoint
is presented. A related large deviations result for the density of oriented
percolation is also given.

1 Introduction

The central limit theorem for the right endpoint of the nearest neighbours (one-
dimensional) contact process was established in Galves and Presutti [9]. Shortly
afterwards an alternative proof was given in Kuczek [10], the seminal idea there was
the existence of space-time points (termed break points) at which the behaviour of
the right endpoint stochastically replicates. The extension of this result for sym-
metric non-nearest neighbours contact processes is studied in Mountford and Sweet
[12]. The key to the extension is Theorem 3 there, it states that the right endpoint
of the process started from a single point is not overtaken from the right endpoint of
the process started from all points to its left for all times with positive probability.

In Section 2 we focus on giving a short and complete proof of this theorem that is
then shown to be sufficient for obtaining an elementary proof of the i.i.d. behaviour of
the right endpoint by a simple restart argument and the definition of break points in
[12]. We also note that the proof of this theorem given here relies on firstly showing

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the following consequence of the shape theorem. The contact processes started from all sites and from any finite set agree on this set for all times with positive probability, this notably extents for the process on $\mathbb{Z}^d$. It is worth stressing that no block construction arguments are needed for the proofs of these results. While on the other hand, we should emphasize that Lemma 6 in [12] is the necessary result for establishing the extension of the central limit theorem in this case, since the ideas devised here seem to be insufficient for giving an alternative proof of this.

In Section 3 we observe that a simple consequence of the result of Durrett and Schonmann [7] for oriented percolation is a sharpened large deviations result than the one that the block construction in [12] builds upon, and remark on that the corresponding large deviations result for contact processes can be obtained in a simple manner.

Finally we note that the approach and results in [12] are affirmed to be valid for all finite range contact process. This however is not evidenced there, in particular we notice that remarks pertaining to the extension in this regard of all arguments relying on self-duality, extensive in the proof of Theorem 3 in [12], are sine qua non.

2 Contact processes

The contact process on a graph $G = (V, E)$ is a continuous time Markov process $\xi_t$ whose state space is the set of subsets of $V$. Regarding each site in $\xi_t$ as occupied by a particle and all other sites as vacant, the process at rate $\mu$ evolves according to the following local prescription: (i) Particles die at rate 1. (ii) A particle at site $x$ gives birth to new ones at each site $y$ such that $xy \in E$ at rate $\mu$. (iii) There is at most one particle per site, i.e. particles being born at a site that is occupied coalesce for all subsequent times. Thus $\xi_t$ can be thought of as the particles descending from the sites in $\xi_0$. The contact process was first introduced in Harris [8] and has been greatly studied since then; an up-to-date account of main results and proofs can be found in Liggett [11]. Let us denote by $\mu_c(G)$ the critical value of the contact process on $G$, that is $\mu_c(G) = \inf\{\mu : P(\xi_t \neq \emptyset, \text{ for all } t > 0)\}$, where $\xi_t$ is the contact process on $G$ started from any $\xi_0$ finite, $\xi_0 \subset V$. We note that throughout the proofs of this section we make extensive use of the construction of contact processes from the
graphical representation, the reader is then assumed to be familiar with that and standard corresponding terminology (see [3] or [11]).

We will consider the collection of graphs $Z_M$, $M \geq 1$, where $M$ is a finite integer and $Z_M$ is the graph with set of vertices the integers, $\mathbb{Z}$, for which pairs of sites at Euclidean distance not greater than $M$ are connected by an edge. We shall also consider the related collection of graphs $Z^*_M$, $M \geq 1$, where $Z^*_M$ is the subgraph of $Z_M$ with set of vertices $Z^* := \{0, -1, \ldots\}$ obtained by retaining only edges connecting sites in $Z^*$.

Firstly, the shape theorem for contact processes on $Z^*_M$, $M \geq 1$, is stated, the result is a consequence of Durrett and Schonmann [6]. Let us denote by $1(\cdot)$ the indicator function.

**Theorem 1.** Let $\hat{\xi}^F_t$ and $\hat{\xi}^Z_t$ denote the contact processes on $Z^*_M$, $M \geq 1$, at rate $\mu$ started from $Z^*$ and $F$ respectively, let also $I_t = \inf \xi^F_t$. For all $M$, if $\mu > \mu_c(Z^*_M)$ and $F$ is finite then there is an $a > 0$ such that the set of sites $y$ such that $y \geq \inf_{s \leq t} I_s$ and $1(y \in \hat{\xi}^F_t) = 1(y \in \hat{\xi}^Z_t)$ contains $[-at, 0] \cap Z^*$ eventually, almost surely on $\{\hat{\xi}^F_t \neq \emptyset, \text{ for all } t\}$.

**Proof.** For $F = \{x\}$ this follows from the renormalized bond construction and the arguments of section 6 in [6]. The extension to all finite sets $F$ is then immediate by additivity.

The foregoing shape theorem plays a pivotal role in establishing the next result that will be central in the proof of the main theorem of this section, viz. Theorem [4]. We believe this to be of independent interest (see also Remark [1]).

**Proposition 2.** Let $\hat{\xi}^Z_t$ and $\hat{\xi}^F_t$ denote the contact processes on $Z^*_M$, $M \geq 1$, at rate $\mu$ started from $Z^*$ and $F$ respectively. For all $M$, if $\mu > \mu_c(Z^*_M)$ and $F$ is finite then $\{\hat{\xi}^Z_t \cap F = \hat{\xi}^F_t \cap F, \text{ for all } t\}$ has positive probability.

**Proof.** Fix $M$ and $F$ finite. Let $\mu > \mu_c(Z^*_M)$ and consider the processes $\hat{\xi}^Z_t$ and $\hat{\xi}^F_t$ constructed by the same graphical representation. Let $B_n$ denote the event $\{\hat{\xi}^Z_s \cap F = \hat{\xi}^F_s \cap F, \text{ for all } s \geq n\}$, for all integer $n \geq 0$.

We give some notation. A realization of the graphical representation is typically denoted by $\omega$ and, we write that for all $\omega \in E_1$, $\omega \in E_2$ a.e. for denoting that
\[ P(\{ \omega : \omega \in E_1, \omega \not\in E_2 \}) = 0, \] where a.e. is an abbreviation for "almost everywhere" (on \( E_1 \)).

Theorem 1 states that for all \( \omega \in \{ \hat{\xi}^F \neq \emptyset, \text{ for all } t \} \) there is an \( s_0 \) such that \( \omega \in \{ \hat{\xi}^F_s \cap [-as,0] = \hat{\xi}^F_s \cap [-as,0], \text{ for all } s \geq s_0 \} \) a.e.. Thus also, since \([-as,0] \supset F \) for all \( s \) sufficiently large, for all \( \omega \in \{ \hat{\xi}^F_t \neq \emptyset, \text{ for all } t \} \) there is an \( s_1 \) such that \( \omega \in \{ \hat{\xi}^F_t, \text{ for all } t \} \) a.e.. Hence
\[
\mathbb{P}(\bigcup_{n \geq 0} B_n) = \mathbb{P}(\hat{\xi}^F_t \neq \emptyset, \text{ for all } t) > 0, \] where the right side is strictly positive because \( \mu > \mu_c(Z^-_M) \). From this we have (e.g. by contradiction) that there is \( n_0 \) for which \( \mathbb{P}(B_{n_0}) > 0 \). We show that the last conclusion implies that \( \mathbb{P}(B_0) > 0 \), this completes the proof.

Let \( B'_{n_0} \) denote the event such that \( \omega' \in B'_{n_0} \) if and only if there exists \( \omega \in B_{n_0} \) such that \( \omega \) and \( \omega' \) are identical realizations except perhaps from any \( \delta \)-symbols (death events) in \( F \times (0,n_0] \). Further, let \( D \) denote the event that no \( \delta \)-symbols exist in \( F \times (0,n_0] \). By independence of the Poisson processes in the graphical representation and then because \( B'_{n_0} \supseteq B_{n_0} \), we have that
\[
\mathbb{P}(B'_{n_0} \cap D) = \mathbb{P}(B'_{n_0}) \mathbb{P}(D) \geq \mathbb{P}(B_{n_0}) e^{-|F|n_0} > 0,
\]
where \( |F| \) denotes the cardinality of \( F \), because \( B_0 \supseteq B'_{n_0} \cap D \) the proof is completed from the last display. To prove that \( B_0 \supseteq B'_{n_0} \cap D \), note that if \( \omega \) and \( \omega' \) are identical except that \( \omega' \) does not contain any \( \delta \)-symbols that possibly exist for \( \omega \) on \( F \times (0,n_0] \), then \( \omega \in B_{n_0} \) implies that \( \omega' \in B_{n_0} \) and indeed \( \omega' \in B_0 \). \( \square \)

**Remark 1.** The arguments of the preceding proof readily apply in order to obtain the analogue of Proposition 2 for contact processes on \( \mathbb{Z}^d \) by use of the corresponding shape theorem (see [1], [3]).

**Remark 2.** By an argument along the lines of that in the final paragraph of the preceding proof, the non-sequitur concluding sentence of the proof of Theorem 3 in [12] can be plausibly proved.

The next statement is the other ingredient we shall need in our proof. It is a consequence of the comparison result in Beuzuidenhout and Grimmett [1], we also note that the result first appeared in the nearest neighbours case in Durrett and Griffeath [5] (see (b) in Section 2).
Theorem 3. For all $M$, $\mu_c(Z_M) = \mu_c(Z_M^*)$.

We are now ready to state and prove the main result of this section.

Theorem 4. Let $\xi_0^t$ and $\xi_t^{Z^*}$ denote the contact processes on $Z_M$, $M \geq 1$, at rate $\mu$ started from $\{0\}$ and $Z^*$ respectively; let also $r_t = \sup \xi_0^t$ and $R_t = \sup \xi_t^{Z^*}$. For all $M$, if $\mu > \mu_c(Z_M)$ then \{ $r_t = R_t$, for all $t$ \} has positive probability.

Proof. Fix $M$ and let $\mu > \mu_c(Z_M)$. Let $\xi_t^{M}$ and $\xi_t^{Z^* \setminus M}$ be the contact process on $Z_M$ at rate $\mu$ started from $M$, $M := \{0, -1, \ldots, -M\}$, and $Z^* \setminus M$ respectively. Consider $\xi_t^Z$, $\xi_t^M$ and $\xi_t^{Z^* \setminus M}$ constructed by the same graphical representation, and let \( G \) denote the event \{ $\xi_t^M \cap M \supseteq \xi_t^{Z^* \setminus M} \cap M$, for all $t$ \}. Letting also $r_t^M = \sup \xi_t^M$, by additivity we have that,

\[
G = \{ r_t^M = R_t, \text{ for all } t \}. \tag{2.1}
\]

Furthermore let $\hat{\xi}_t^M$ and $\xi_t^{Z^* \setminus M}$ be the contact process on $Z_M^*$ at rate $\mu$ started from $M$ and $Z^* \setminus M$ respectively, also constructed by the same graphical representation by neglecting arrows from $x$ to $y$ such that $x \in M$ and $y \in \{1, 2 \ldots \}$ for all times. By coupling we have that $G = \{ \xi_t^M \cap M \supseteq \xi_t^{Z^* \setminus M} \cap M$, for all $t$ \} and, by monotonicity,

\[
G \supseteq \{ \hat{\xi}_t^M \cap M \supseteq \xi_t^{Z^* \setminus M} \cap M, \text{ for all } t \}. \tag{2.2}
\]

However, since Theorem 3 gives that $\mu > \mu_c(Z_M^*)$, Proposition 2 for $F = M$ and additivity give that the event on the right side of (2.2) has positive probability. This, from (2.2) and (2.1) give that $P(r_t^M = R_t, \text{ for all } t) > 0$.

To complete the proof from the last conclusion, consider $\xi_0^t$ and $\xi_t^{Z^*}$ constructed by the same graphical representation, the result then follows easily by considering the event \{ $\xi_0^t \supseteq [-M, 0]$ $\cap \{ \xi_t^0 \cap \{0\} \neq \emptyset$ and $R_s \leq 0,$ for all $s \in (0, 1]$, \} from monotonicity and the Markov property.

The final result of this section addresses the i.i.d. nature of the growth of the right endpoint, this is the corresponding extension of the first part of the Theorem in Kuczek \[10\].

Theorem 5. Let $\xi_0^t$ denote the contact processes on $Z_M$, $M \geq 1$, at rate $\mu$ started from $\{0\}$, let also $r_t = \sup \xi_0^t$. For all $M$, if $\mu > \mu_c(Z_M)$ then \{ $\xi_t^0 \neq \emptyset$, for all $t$ \} there are strictly increasing random (but not stopping) times $\psi_k, k \geq 0$, such that $(r_{\psi_n} - r_{\psi_{n-1}}, \psi_n - \psi_{n-1})_{n \geq 1}$ are i.i.d.
Proof. Fix $M$ and let $\mu > \mu_c(Z_M)$. Consider the graphical representation for contact processes at rate $\mu$ on $Z_M$. Given a space-time point $x \times s$, let $\xi_{t+s}^{x,s}, t \geq 0$, denote the process started from $\{y : y \leq x\}$ at time $s$ and let also $R_{t+s}^{x,s} = \sup \xi_{t+s}^{x,s}$; furthermore let $\xi_{t+s}^{x,s}, t \geq 0$, denote the process started from $\{x\}$ at time $s$, and let also $r_{t+s}^{x,s} = \sup \xi_{t+s}^{x,s}$. We write that $x \times s$ a.s. finite, which implies that $r_{t}^{x,s}$ is a.s. finite, which implies that $r_{\sigma_n}^{x,s}$ is a.s. finite because $|r_{\tau_l}|$ is bounded above in distribution by the number of events by time $t$ of a Poisson process at rate $M\mu$. This completes the proof.

\[\square\]
3 Large deviations

We consider 1-dependent oriented site percolation with density at least 1 − \(\epsilon\), that is, letting \(\mathcal{L} = \{(y, n) \in \mathbb{Z}^2 : y + n \text{ is even, } n \geq 0\}\), a collection of random variables \(w(y, n) \in \{0, 1\}\) such that \((y, n) \in \mathcal{L}\) and \(n \geq 1\), which satisfies the property that \(P(w(y, n + 1) = 0 \text{ for all } 1 \leq i \leq I\{|w(y, m), \text{ for all } m \leq n\}) \leq \epsilon^I\), where \(|y_i - y_{i+1}| > 2\) for all \(1 \leq i \leq I\) and \(1 \leq i' \leq I\). Given a realization of 1-dependent site percolation we write \((x, 0) \rightarrow (y, n)\), if there exists \(x := y_0, \ldots, y_n := y\) such that \(|y_i - y_{i-1}| = 1\) and \(w(y, i) = 1\) for all \(1 \leq i \leq n\). Let \(2\mathbb{Z} = \{x : (x, 0) \in \mathcal{L}\}\), for any given \(A \subseteq 2\mathbb{Z}\), define \(W^A_n = \{y : (x, 0) \rightarrow (y, n) \text{ for some } x \in A\}\). Let also \(2\mathbb{Z} + 1 = \{x : (x, 1) \in \mathcal{L}\}\), and define \(X(n)\) to be \(X(n) = 2\mathbb{Z}\) for even \(n\), and \(X(n) = 2\mathbb{Z} + 1\) for odd \(n\). Subsequently \(C\) and \(\gamma\) will represent positive, finite constants.

The next lemma is used in the proof of the main result of this section below. It is a consequence of the result of Durrett and Schonmann [7].

**Lemma 7.** For all \(\rho < 1\) there is \(\epsilon > 0\) such that for any \(n \geq 1\) and \(Y, Y \subset X(n)\), the probability of \(\left\{ \sum_{y \in Y} 1(y \in W^Y_n) < \rho|Y| \right\}\) is bounded by \(Ce^{-\gamma|Y|}\).

**Proof.** We first consider standard independent bond percolation process, \(B_n\), where \(B_n \subset X(n)\), and let \(p_c\) denote its critical value, for definitions see [11, 2], the next lemma is proved immediately afterwards.

**Lemma 8.** Let \(B^2\mathbb{Z}_n\) be independent bond percolation process with parameter \(p > p_c\) started from \(2\mathbb{Z}\). For all \(p' < p\) and any \(n \geq 1\) and \(Y, Y \subset X(n)\), the probability of \(\left\{ \sum_{y \in Y} 1(y \in B^2\mathbb{Z}_n) < p'|Y| \right\}\) is bounded by \(Ce^{-\gamma|Y|}\).

The proof then follows because we can choose \(\epsilon > 0\) sufficiently small such that \(W^Y_n\) stochastically dominates \(B^2\mathbb{Z}_n\) with parameter \(p\) arbitrarily close to 1, which comes by combining Theorem B24 and Theorem B26 in [11].

**proof of Lemma 8.** Let \(p > p_c\), let also \(\tilde{B}_n\) be independent bond percolation process with parameter \(p\) started from \(\tilde{B}_0\) which is distributed according to the upper invariant measure of the process. By monotonicity we easily have \(B^2\mathbb{Z}_n\) stochastically dominates \(\tilde{B}_n\). From this, the proof follows by the invariance of \((\tilde{B}_n)\) and the analogue of Theorem 1 in [7] in this case.
We now state and prove the main result of the section.

**Proposition 9.** For all $\rho < 1$ and all $\beta < 1$ there is $\epsilon > 0$ such that for any $n \geq 1$ and $Y, Y \subset X(n) \cap [−\beta n, \beta n]$, the probability of $\left\{ \sum_{y \in Y} 1(y \in W_n^0) < \rho |Y|, W_n^0 \neq \emptyset \right\}$ is bounded by $Ce^{-\gamma n} + Ce^{-\gamma |Y|}$.

**Proof.** Let $\tau = \inf \{n : W_n^0 = \emptyset\}$, let also $R_n = \sup W_n^0$ and $L_n = \inf W_n^0$. The following sequence of lemmas are known results, we refer to [2] and [11] for proofs.

**Lemma 10.** On $\{\tau = \infty\}$, $W_n^0 = W_{n-2}^{2Z} \cap [L_n, R_n]$.

**Lemma 11.** There is $\epsilon > 0$ such that for any $n \geq 1$ the probability of $\{n \leq \tau < \infty\}$ is bounded by $Ce^{-\gamma n}$.

**Lemma 12.** For all $\beta < 1$ there is $\epsilon > 0$ such that for any $n \geq 1$ the probability of $\{[L_n, R_n] \subseteq [−\beta n, \beta n], \tau = \infty\}$ is bounded by $Ce^{-\gamma n}$.

Choose $\epsilon > 0$ sufficiently small such that Lemmas [7][11] and [12] are all satisfied. By simple set theory, Lemma [11] and Lemma [12] give that it is sufficient to prove that the probability of $\left\{ \sum_{y \in Y} 1(y \in W_n^0) < \rho |Y| \right\}$ on $\{[L_n, R_n] \supseteq [−\beta n, \beta n]\} \cap \{\tau = \infty\}$ is bounded by $Ce^{-\gamma |Y|}$, this however follows from Lemma [7] by use of Lemma [10].

We finally give a consequence of the last result. The argument is from the proof of Lemma 3 in [12].

**Corollary 13.** For all $\rho < 1$ and $\beta < 1$ there is $\epsilon > 0$ such that for any $n \geq 1$ and $b \in (0, \beta]$, the probability that there exists a sequence $(y_k^b)_{k=1}^{bn}$ of consecutive points in $X(n) \cap [−\beta n, \beta n]$ such that $\sum_{k=1}^{bn} 1(y_k \in W_n^0) < \rho bn$ and $W_n^0 \neq \emptyset$, is bounded by $Ce^{-\gamma bn}$, where $C, \gamma > 0$ are independent of $n$ and $b$.

**Proof.** Since the number of $(y_k^b)_{k=1}^{bn}$ considered is of polynomial order in both $n$ and $b$, the proof follows from Proposition [9].

**Remark 3.** The last corollary implies the corresponding statement for contact processes by use of the comparison result in [9], and the argument in the proof of Proposition 3.3 in [13]. Alternatively this can be obtained by appealing to the proof of Corollary 4 in [12].
References

[1] Bezuidenhout, C.E. and Grimmett, G.R. (1990). The critical contact process dies out. *Ann. Probab.* 18 1462-1482.

[2] Durrett, R. (1984). Oriented percolation in two dimensions. *Ann. Probab.* 12 999-1040.

[3] Durrett, R. (1991). The Contact Process, 1974-1989. *Lectures in Applied Math., AMS* 27 1-18.

[4] Durrett, R. (1995). *Ten lectures on particle systems.* Lecture Notes in Math. 1608, Springer-Verlag, New York.

[5] Durrett, R. and Griffeath, D. (1983). Supercritical contact processes on $\mathbb{Z}$. *Ann. Probab.* 11 1-15.

[6] Durrett, R. and Schonmann, R. H. (1987). Stochastic growth models. In *Percolation Theory and Ergodic Theory of Infinite Particle Systems* 85-119. Springer, New York.

[7] Durrett, R. and Schonmann, R. H. (1988). Large deviations for the contact process and two dimensional percolation. *Probab. Th. Rel. Fields* 77, 583-603.

[8] Harris, T.E. (1974). Contact interactions on a lattice. *Ann. Probab.* 2 969-988.

[9] Galves, A. and Presutti, E. (1987). Edge fluctuations for the one-dimensional supercritical contact process. *Ann. Probab.* 15, 1131-1145.

[10] Kuczek, T. (1989). The Central Limit Theorem for the Right Edge of Supercritical Oriented Percolation. *Ann. Prob.,* 17, 1322-1332.

[11] Liggett, T. (1999). *Stochastic Interacting Systems: Contact, Voter and Exclusion Processes.* Springer, New York.

[12] Mountford, T. and Sweet, T. (2000). An Extension of Kuczek’s Argument to Non nearest Neighbor Contact Processes. *J. Theoret. Probab.* 13 1061-1081.

[13] Tzioufas, A. (2010). On the growth of one dimensional reverse immunization contact processes. *J. of App. Probab.* 48(3) (to appear)