Bending of light and inhomogeneous Picard–Fuchs equation

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Abstract
The bending of light rays by gravitational sources is one of the first evidence of general relativity. When the gravitational source is a stationary massive object such as a black hole, the bending angle has an integral representation, from which various series expansions up to a finite order in terms of the parameters of orbit and the background spacetime has been derived. However, it has not been clear that it has any analytic expansion. In this paper, we show that such an analytic expansion can be obtained for the case of a Schwarzschild black hole by solving an inhomogeneous Picard–Fuchs equation, which has been applied to compute effective superpotentials on D-branes in the Calabi–Yau manifolds. From the analytic expression of the bending angle, the full order expansions in both weak and strong deflection limits are obtained. We show that the result can be obtained by the direct integration approach as well. We also discuss how the charge of the gravitational source affects the bending angle and show that a similar analytic expression can be obtained for the extremal Reissner–Nordström spacetime.

Keywords: gravitational lensing, black hole, Picard–Fuchs equation

1. Introduction

The general theory of relativity was proposed by Einstein in 1915. One of the important predictions is the bending of light rays in the presence of gravitational fields. In particular, deflection by astrophysical gravitational sources such as stars, black holes, or galaxies has been studied both theoretically and observationally.

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In the weak deflection limit, the deflection angle $\alpha$ can be approximated by

$$\alpha = \frac{4M}{b},$$

where $M$ is the mass of the lensing object and $b$ is the impact parameter of the light trajectory\(^3\). In the weak deflection limit, $b \gg M$ is assumed, which is satisfied for the lensing by a star such as the Sun. Note that although deflection of light can be derived even in Newtonian gravity, the bending angle in general relativity is almost twice as much as that in Newton’s theory. This prediction was confirmed observationally in 1919 during a total solar eclipse [1].

When the lensing objects are very massive, it can give rise to a deviation from the above expression. The extension to include the higher order corrections in terms of the power series expansion with $M/b$ has been studied [2, 3]. Also, generalizations to the cases with nonvanishing spin and electric charge are found for example in reference [4].

On the other hand, in the strong deflection limit, i.e. the impact parameter and the Schwarzschild radius are comparable, the light ray can wind around the object arbitrary times producing an infinite number of images, called relativistic images [5]. This behavior is related to the circular orbit of photons, whose coordinate radius is $r = 3M$ for the Schwarzschild space-time. Analytically, the existence of the relativistic images can be understood by observing the logarithmic divergence of $\alpha$ when the impact parameter $b$ approaches the critical value [6–8].

The analytic approximation derived in these papers has been used to determine the positions and magnifications of the relativistic images, while there exist numerical studies as well for example reference [9]. The strong deflection expansion of $\alpha$ beyond the leading divergence for the Schwarzschild case was performed in reference [3]. For the cases with spin and/or electric charge, only numerical calculations or analytic approximation up to the leading logarithmic and the sub-leading constant terms using expressions with various elliptic integrals can be found in the strong field limit [8, 10–12].

The strong deflection limit of the bending angle has been of interest also because it is relevant to the optical appearance, or the shadow of a black hole [13], which has been recently observed by the Event Horizon Telescope [14].

In deriving the results mentioned above, the starting point is usually expressions of the deflection angle $\alpha$ (more precisely, $\Theta := \alpha + \pi$) in terms of the standard elliptic integrals, and the parameters of these integrals such as the modulus are complicated functions of the parameters of the background spacetime and the trajectory. Therefore we cannot read off from these expressions arbitrarily higher order terms in both weak and strong deflection limits although the first few terms have been obtained as mentioned above.

One of the aims of this paper is to obtain the full order expansion in terms of $M/b$ of the deflection angle for the Schwarzschild case. Our strategy is to consider a differential equation satisfied by $\Theta$ as a function of $M/b$, which can be seen as an inhomogeneous Picard–Fuchs equation. Picard–Fuchs equations are differential equations with respect to the moduli of algebraic manifolds satisfied by the periods integrals. The notion of Picard–Fuchs equations has been used in physics. For example, it was applied to study the dependence of some Feynman integrals on the external variables [15]. Also, it has been applied to analyze the dependence of D-brane superpotentials on the complex moduli in various Calabi–Yau manifolds in the context of mirror symmetry [16–20], and in these cases, the differential equations arise with inhomogeneous terms.

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\(^3\)We use the geometrized units where the speed of light $c$ and the gravitational constant $G$ are set equal to unity, so that the mass $M$ in the above expression should be understood as a typical length scale $M = MG/c^2$. 2
The differential equation for $\Theta$, which is derived in section 3 after introducing our notation in section 2, turns out to be a hypergeometric one with an inhomogeneous term. The explicit solution is given in terms of hypergeometric functions, from which the coefficients of arbitrarily higher order terms in the weak deflection limit can be easily read off. By using analytic continuation formulae for the hypergeometric functions and the inhomogeneous Picard–Fuchs equation, the strong deflection expansion is also derived. The result is completely consistent with the previous results in references [2, 3]. In section 4, the same result is rederived directly performing the defining integral of $\Theta$ with the help of analytic continuation to confirm our result. Furthermore, we consider a generalization of the method used in section 3 to the Reissner–Nordström black hole and show that for the extremally-charged case $\Theta$ is given in a similar expression to the uncharged case in section 5. Finally, we conclude this paper with some discussions.

2. Bending of light in the Schwarzschild geometry

The action for a massless particle is given by

$$ S = \frac{N}{2} \int d\tau g_{\mu\nu}(x(\tau)) \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau}, $$

where $N$ is a Lagrangian multiplier field (we will set $N = 1$ later). We consider the Schwarzschild geometry

$$ ds^2 = - \left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2, $$

and equation (2) is written as

$$ S = \frac{N}{2} \int d\tau \left[ - \left(1 - \frac{2M}{r}\right) \left(\frac{dr}{d\tau}\right)^2 + \left(1 - \frac{2M}{r}\right)^{-1} \left(\frac{dr}{d\tau}\right)^2 + r^2 \left(\frac{d\theta}{d\tau}\right)^2 \right]. $$

The variation with respect to $N$ and setting $N = 1$ leads to the null condition,

$$ - \left(1 - \frac{2M}{r}\right) \left(\frac{dr}{d\tau}\right)^2 + \left(1 - \frac{2M}{r}\right)^{-1} \left(\frac{dr}{d\tau}\right)^2 + r^2 \sin^2 \theta \left(\frac{d\theta}{d\tau}\right)^2 = 0. $$

The variation with respect to $t, \theta, \phi$ leads to

$$ \left(1 - \frac{2M}{r}\right) \frac{dt}{d\tau} = \varepsilon, \quad r^2 \frac{d\theta}{d\tau} = L, \quad r^2 \sin^2 \theta \frac{d\phi}{d\tau} = m, $$

where $\varepsilon, L, m$ are constants. By choosing a suitable coordinate system, we can set $\phi = 0$ ($m = 0$) and inserting equation (6) into equation (5), we find

$$ \frac{1}{L^2} \left(\frac{dr}{d\tau}\right)^2 + \left(1 - \frac{2M}{r}\right) \frac{1}{r^2} = \frac{1}{b^2}, $$

where $b$ is a constant.
where \( b = L/\varepsilon \). Eliminating \( \tau \) by using the second equation in equation (6), we obtain

\[
\frac{1}{r^4} \left( \frac{dr}{d\theta} \right)^2 + \left( 1 - \frac{2M}{r} \right) \frac{1}{r^2} = \frac{1}{b^2}. \tag{8}
\]

By setting the variable \( x \) as

\[
x = \frac{1}{r}, \tag{9}
\]

we find

\[
\left( \frac{dx}{d\theta} \right)^2 = -(1 - 2Mx)x^2 + \frac{1}{b^2}. \tag{10}
\]

Note that at a far distant region (\( \equiv x \ll 1 \)) where the gravitational field of the source can be negligible, the solution of the equation can be approximated by \( r \sin \theta = b \). Therefore \( b \) is the impact parameter of the photon trajectory as can be seen in figure 1. Equation (10) is the first order differential equation and we can easily obtain the integral form of the deflection angle \( \Theta \) as

\[
\Theta = 2 \int_{x_0}^{x_0} \frac{dx}{\sqrt{1/b^2 - x^2 + 2Mx^3}}. \tag{11}
\]

where \( x_0 \) is the turning point of the orbit and given by one of the roots of the equation \( 1/b^2 - x^2 + 2Mx^3 = 0 \), which reduces to \( x = 1/b \) when \( M = 0 \).

We can perturbatively obtain the power series expansion

\[
\Theta = \pi + \frac{4M}{b} + \frac{15\pi}{4} \left( \frac{M}{b} \right)^2 + \frac{128}{3} \left( \frac{M}{b} \right)^3 + \frac{3465\pi}{64} \left( \frac{M}{b} \right)^4 + O((M/b)^5), \tag{12}
\]

which was obtained in reference [2]. Our aim in this paper is to obtain the full order series expansion. We make the variable of integration dimensionless by defining \( t = bx \) and set the constants \( \alpha, \beta \) and \( \gamma \), which are also dimensionless, so that

\[
1 - t^2 + \frac{2M}{b} t^3 = (1 - \alpha t)(1 - \beta t)(1 - \gamma t). \tag{13}
\]

The photon can reach the observer, which is assumed to be at infinity, when the impact parameter is sufficiently large, i.e. \( 2M/b \leq (4/27)^{1/2} \). In this case, the above polynomial has two

\footnote{In reference [2], the explicit coefficients are given up to the order of \((M/b)^5\).}
positive roots and one negative root so we can take the constants so that \( \gamma < 0 < \beta < \alpha \) without loss of generality. Then we get

\[
\Theta = 2 \int_0^{1/\alpha} \frac{dt}{(1 - \alpha t)^{2}(1 - \beta t)^{2}(1 - \gamma t)^{2}}. \tag{14}
\]

Changing the variable to \( s = \alpha t \), we have

\[
\Theta = \frac{2}{\alpha} \int_0^1 \frac{ds}{(1 - s)^{2}(1 - \beta \alpha s)^{2}(1 - \gamma \alpha s)^{2}}. \tag{15}
\]

Taking the power series and making the integration, we get

\[
\Theta = \frac{2}{\alpha} \sum_{m,n=0}^{\infty} \frac{(1/2)_m(1/2)_n}{m!n!} \frac{\Gamma(m + n + 1)\Gamma(1/2)}{\Gamma(m + n + \frac{1}{2})} \left( \frac{\beta}{\alpha} \right)^m \left( \frac{\gamma}{\alpha} \right)^n
\]

\[
= \frac{4}{\alpha} \text{F}_1 \left( 1, \frac{1}{2}, \frac{1}{2}; \frac{3}{2}; \frac{\beta}{\alpha}, \frac{\gamma}{\alpha} \right), \tag{16}
\]

where \( \text{F}_1 \) is the Appell function [21] and the Pochhammer symbol is defined as \((x)_n := \Gamma(x + n)/\Gamma(x)\). However, \( \alpha, \beta, \) and \( \gamma \) are complicated functions of \( M/b \) so this expression is not adequate to obtain the series expansion in terms of \( M/b \). In the next section, we try a different approach.

3. Inhomogeneous Picard–Fuchs equation

The integral (11) can be written as an incomplete elliptic integral. It is related to the algebraic curve

\[
y^2 = 1 - t^2 + \frac{2M}{b} t^3, \tag{17}
\]

which is the defining equation of an algebraic torus in the complex \( t \)--\( y \) plane. Holomorphic one-form on the torus can be defined as

\[
\omega = \oint \frac{dy \wedge dt}{2\pi i} \frac{2}{y^2 - (1 - t^2 + \frac{2M}{b} t^3)}
\]

\[
= \frac{dt}{\sqrt{1 - t^2 + \frac{2M}{b} t^3}}. \tag{18}
\]

It is known that the holomorphic one-form \( \omega \) satisfies the Picard–Fuchs equation with respect to the moduli \( 2M/b \). Let the Picard–Fuchs operator be \( D(\partial_z) \) (the definition of \( z \) is given soon). Then the holomorphic one-form \( \omega \) satisfies

\[
D(\partial_z) \omega = -d\beta, \tag{19}
\]

where \( \beta \) is a zero form. In fact, the Picard–Fuchs operator and the zero-form \( \beta \) can be obtained by this requirement. Taking the cyclic integral of this identity, the Picard–Fuchs equation can be obtained,

\[
D(\partial_z) \oint \omega = 0. \tag{20}
\]
The deflection angle (11) can be considered as the integral with the boundary \( t = 0 \). Therefore acting \( D(\partial_t) \) will lead to the inhomogeneous Picard–Fuchs equation,

\[
D(\partial_t) \int \omega = \beta(t = 0).
\]

(21)

Now let us define the dimensionless moduli parameter \( z \)

\[
M/b = 1/3^z z^2,
\]

(22)

and the differential operator \( \theta = 2d/\pi \).

We find that the deflection angle satisfies the following inhomogeneous Picard–Fuchs equation:

\[
P\Theta = 1/3^z z^2,
\]

(23)

where the operator \( P \) is given by

\[
P = \partial^2 - z \left( \theta + \frac{1}{6} \right) \left( \theta + \frac{5}{6} \right).
\]

(24)

It is easy to get the following special solution of this equation:

\[
\frac{4}{3z^3} \frac{\gamma}{\pi} \arcsin^3 z [\left[ 2/3, 1, 4/3 \right]_{3/2, 3/2; z},
\]

(25)

Two independent solutions of the homogeneous equation are hypergeometric functions \( \_2F_1[1/6, 5/6; 1; z] \) and \( \_2F_1[1/6, 5/6; 1; 1-z] \), the latter of which contains the logarithmic singularity around \( z = 0 \). The general solution is then given by

\[
\Theta = c_1 \cdot \_2F_1[1/6, 5/6; 1; z] + c_2 \cdot \_2F_1[1/6, 5/6; 1; z] + \frac{4}{3z^3} \frac{\gamma}{\pi} \arcsin^3 z [\left[ 2/3, 1, 4/3 \right]_{3/2, 3/2; z},
\]

(26)

where \( c_1 \) and \( c_2 \) are integration constants. Since the orbit becomes a straight line in the weak deflection limit \( z \to 0 \), i.e. \( \Theta(z = 0) = \pi \), these constants are fixed uniquely as \( c_1 = \pi \), \( c_2 = 0 \).

Thus, we obtain the following result:

\[
\Theta = \pi \_2F_1[1/6, 5/6; 1; z] + \frac{4}{3z^3} \frac{\gamma}{\pi} \arcsin^3 z [\left[ 2/3, 1, 4/3 \right]_{3/2, 3/2; z},
\]

(27)

which can be expressed by the original variables as

\[
\Theta = \pi \_2F_1[1/6, 5/6; 1; \frac{27M^2}{b^2}] + \frac{4M}{b} \_2F_1[2/3, 1, 4/3; 3/2, 3/2; \frac{27M^2}{b^2}],
\]

(28)

\[
= \pi \sum_{n=0}^{\infty} \frac{(1/6)_n(5/6)_n}{(n!)^2} \left( \frac{27M^2}{b^2} \right)^n + \frac{4M}{b} \sum_{n=0}^{\infty} \frac{(2/3)_n(4/3)_n}{((3/2)_n)^2} \left( \frac{27M^2}{b^2} \right)^n.
\]

(29)

This is the complete expression for the deflection angle in the weak deflection limit. It is easy to check that the first few terms agree with equation (12).
To understand the strong deflection limit of this solution, we need analytic continuation formulae for hypergeometric functions. For \( \mathfrak{z}F_{1} \), such an identity is a classical result [21],

\[
\mathfrak{z}F_{1} [a, b; a + b; z] = \frac{\Gamma(a + b)}{\Gamma(a) \Gamma(b)} \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(a + b)_n} \left[ k_n - \log(1 - z) \right] (1 - z)^n, \tag{30}
\]

where \( k_n = 2\psi(n + 1) - \psi(n + a) - \psi(n + b) \) with \( \psi(z) = \Gamma'(z)/\Gamma(z) \) being the digamma function. For \( \mathfrak{z}F_{2} \), we can use the following identity given in reference [22] \(^5\)

\[
\frac{\Gamma(a_1) \Gamma(a_2) \Gamma(a_3)}{\Gamma(b_1) \Gamma(b_2)} \mathfrak{z}F_{2} \left[ \begin{array}{c} a_1, a_2, a_3 \\ b_1, b_2 \end{array} ; z \right] = \sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n}{(n!)^2} \left\{ \sum_{k=0}^{n} \frac{(-1)_k}{(a_1)_k (a_2)_k} A_k^{(2)} (\psi(n - k + 1) + \psi(n + 1) \\
- \psi(n + a_1) - \psi(n + a_2) - \log(1 - z)) \right. \\
+ (-1)^n n! \sum_{k=n+1}^{\infty} \frac{k - n - 1}{(a_1)_k (a_2)_k} A_k^{(2)} \left. \right\} (1 - z)^n, \tag{31}
\]

where \( A_k^{(2)} = (b_2 - a_3)_k (b_1 - a_3)_k / k! \). Note that this formula is valid only when \( \mathfrak{z}F_{2} \) is the so-called zero-balanced series, meaning that the parameters satisfy \( a_1 + a_2 + a_3 = b_1 + b_2 \).

Although these formulae are enough to understand the strong deflection limit, the inhomogeneous Picard–Fuchs equation (23) helps us simplify the expansion about \( w := 1 - z \) as we show now. Observe that \( \Theta \) takes the following form:

\[
\Theta(w) = p(w) + q(w) \log w, \tag{32}
\]

where \( p(w) \) and \( q(w) \) have power series expansions around \( w = 0 \). Since the inhomogeneous term in equation (23) is free of logarithmic singularity \( \log w, q(w) \) must be a homogeneous solution which is regular at \( w = 0 \). As a result, \( q(w) \) is proportional to \( \mathfrak{z}F_{1}[1/6, 5/6; 1; w] \). By inserting equations (30) and (31) into the expression (28) and picking up \( (n = 0) \)-term, we can identify the proportionality constant to conclude

\[
q(w) = -\mathfrak{z}F_{1} \left[ \begin{array}{c} 1/6, 5/6, 1 \\ 1 \end{array} ; w \right], \tag{33}
\]

\( p(w) \) is determined by solving the inhomogeneous hypergeometric equation assuming a power series expansion around \( w = 0 \), but it is more convenient to define \( \bar{p}(w) \) as follows:

\[
\Theta(w) = \bar{p}(w) + 2\pi \mathfrak{z}F_{1} \left[ \begin{array}{c} 1/6, 5/6, 1 \\ 1 \end{array} ; 1 - w \right], \tag{34}
\]

which is equivalent to \( p(w) = \bar{p}(w) + r(w) \), where

\[
r(w) = \sum_{n=0}^{\infty} \frac{\left( \frac{1}{2} \right)_n \left( \frac{5}{6} \right)_n}{(n!)^2} \left[ 2\psi(n + 1) - \psi(n + 1/6) - \psi(n + 5/6) \right] w^n. \tag{35}
\]

\(^5\) \(-\psi(n + a_1) \) is missing in equation (5.1) in this reference article.
Then, \( \bar{p}(w) \) is the solution of \( P(\bar{p}) = \sqrt{1 - w/3\sqrt{3}} \) with a power series expansion \( \bar{p}(w) = \sum \bar{p}_n w^n \), of which coefficients obey the following recurrence relation:

\[
n^2 \bar{p}_n = \left( n - \frac{1}{6} \right) \left( n - \frac{5}{6} \right) \bar{p}_{n-1} + \frac{1}{3\sqrt{3}} \left( \frac{j}{n-1} \right)!, \quad n \geq 1. \tag{36}
\]

The general solution for \( n \geq 1 \) is given by

\[
\bar{p}_n = \left( \frac{j}{n} \right) \frac{(2j)^n}{(n!)^2} \bar{p}_0 + \frac{(2j)^n}{(n!)^2} \sum_{j=1}^{n-1} \frac{j! \left( \frac{j}{n} \right)}{j! \left( \frac{j}{n} \right)} \sum_{j=1}^{n-1} \left( \frac{2}{j} - \frac{1}{j-1/6} - \frac{1}{j-5/6} \right) w^n. \tag{37}
\]

where \( \bar{p}_0 \) is an arbitrary constant. The first term gives a contribution \( \bar{p}_0 F_1[1/6, 5/6; 1; w] \).

Similarly, by using the identity \( \psi(x+1) = \psi(x) + \frac{1}{x} \) we can subtract the hypergeometric function from \( r(w) \) as

\[
r(w) = (2\psi(1) - \psi(1/6) - \psi(5/6)) F_1 \left[ \frac{1}{6}, \frac{5}{6}; 1; w \right]
\]

\[
+ \sum_{n=1}^{\infty} \frac{(2j)^n}{(n!)^2} \sum_{j=1}^{n-1} \left( \frac{2}{j} - \frac{1}{j-1/6} - \frac{1}{j-5/6} \right) u^n. \tag{38}
\]

The digamma function at rational numbers can be expressed in terms of elementary functions [21] so that the coefficient of the hypergeometric function is given by

\[
2\psi(1) - \psi(1/6) - \psi(5/6) = \log 432. \tag{39}
\]

Combining above results, we finally obtain

\[
\Theta = \left( \bar{p}_0 + \log \frac{432}{w} \right) F_1 \left[ \frac{1}{6}, \frac{5}{6}; 1; w \right]
\]

\[
+ \sum_{n=1}^{\infty} \frac{(2j)^n}{(n!)^2} \sum_{j=1}^{n-1} \left( \frac{2}{j} - \frac{1}{j-1/6} - \frac{1}{j-5/6} \right) \left( \frac{j-1}{j} \right) \left( \frac{1}{j} \right) \left( \frac{2}{j} \right) \left( \frac{2}{j} \right) \right] u^n. \tag{40}
\]

The remaining task is to determine \( \bar{p}_0 \). From the analytic continuation formulae for \( \psi \) given above, one of the expressions for \( \bar{p}_0 \) is found to be

\[
\bar{p}_0 = -\log 4 - \frac{3}{2} + \frac{9}{64} F_3 \left[ 1, 1, 3/2, 3/2 \right] \frac{1}{5/3, 2, 7/3}. \tag{41}
\]

Another expression can be derived by using the method of variation of constants to obtain an integral expression for \( \Theta \) in terms of the homogeneous solutions. After some calculations we found

\[
\bar{p}_0 = \frac{2\pi}{3\sqrt{3}} \int_0^1 \frac{dz}{\sqrt{z}} F_1 \left[ \frac{1}{6}, \frac{5}{6}; 1; z \right] = -\frac{4\pi}{3\sqrt{3}} F_2 \left[ \frac{1}{6}, 1/2, 5/6; 1, 1/3, 2 \right]. \tag{42}
\]

In fact, one can directly expand the original expression of \( \Theta \) to obtain

\[
\bar{p}_0 = \log(7 - 4\sqrt{3}). \tag{43}
\]

The equality of equations (42) and (43) can be proved by using Watson’s formula [23] with the help of contiguous relations [24] for \( \psi R^2 \). In reference [3], the strong deflection limit
of $\hat{\alpha} := \Theta - \pi$ was considered and the first few coefficients in $b'$-expansion was obtained, where $b' = 1 - \sqrt{1 - w}$. From our result equation (40), the expansion given in reference [3] is completely recovered.

4. Direct evaluation of the integral via analytic continuation

In the previous section, we have obtained the deflection angle by using the inhomogeneous Picard–Fuchs equation. The next natural question is whether we can obtain the result by direct integration. The method we apply below has been used in references [19, 20] to compute the superpotentials on the D-branes.

First, we note that the integral can be written as

$$\Theta = \int \frac{dt}{C \left(1 - t^2 + \frac{2M_b}{b} t^3\right)^{3/2}},$$

where $C$ is the path starting from 0, encircling around the root, and coming back to 0 (figure 2). Then we use the following representation:

$$\frac{1}{\left(1 - t^2 + \frac{2M_b}{b} t^3\right)^{3/2}} = \int \frac{ds}{2\pi i} \Gamma(-s) \left(\frac{2M_b}{b}\right)^s \frac{\Gamma \left(s + \frac{1}{2}\right)}{\Gamma \left(\frac{3}{2}\right)} (1 - t^2)^{-s-1/2},$$

where the $s$-integral takes the residues at non-negative integers. The original integrand has a cut structure at the roots of $1 - t^2 + \frac{2M_b}{b} t^3$ but now the right-hand side of the above expression has a cut structure at $t = 1$. Therefore the $t$-integral can be evaluated by two times the line integral from 0 to 1, namely

$$\Theta = 2 \int \frac{ds}{2\pi i} \Gamma(-s) \left(\frac{2M_b}{b}\right)^s \frac{\Gamma \left(s + \frac{1}{2}\right)}{\Gamma \left(\frac{3}{2}\right)} \int_0^1 dt t^3 (1 - t^2)^{-s-1/2}. $$

The $t$-integral can be evaluated by using the beta function and we find

$$\Theta = 2 \int \frac{ds}{2\pi i} \Gamma(-s) \left(\frac{2M_b}{b}\right)^s \frac{\Gamma \left(s + \frac{1}{2}\right)}{\Gamma \left(\frac{3}{2}\right) \Gamma \left(\frac{3s}{2} + 1\right)} \cos \pi s.$$ (47)

Summing up the residues at non-negative integers, we obtain

$$\Theta = \pi \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n}{(1)_n(1) \frac{2n}{2}!} \left(\frac{2M_b}{b}\right)^n.$$ (48)
To see this result is equivalent to equation (28), we divide the series into even and odd order terms,

$$\Theta = \pi \sum_{n=0}^{\infty} \left[ \frac{1}{n!} \left( \frac{2M}{b} \right)^{2n} + \frac{1}{\pi b n!} \left( \frac{2M}{b} \right)^{2n+1} \right].$$

Then various Pochhammer symbols can be factorized, for example, $(\frac{1}{2})_{3n} = 3^{3n} (\frac{1}{2})_n (\frac{5}{2})_n$. Finally we obtain

$$\Theta = \pi \sum_{n=0}^{\infty} \left[ \frac{1}{n!} \left( \frac{2M}{b} \right)^{2n} + \frac{4M}{\pi b n!} \left( \frac{2M}{b} \right)^{2n} \right] + \frac{4M}{b} \frac{2F_1}{b^2} \left[ \frac{2/3, 1, 4/3, 27M^2}{3/2, 3/2, b^2} \right],$$

which is identical to equation (28).

5. Photon trajectories for extremal Reissner–Nordström geometry

We have discussed the bending angle for Schwarzschild geometry. As we easily expected, the bending angle grows as the mass $M$ increases with the impact parameter $b$ kept fixed. The next question we consider is the effect of the charge on the bending angles. If the spherical object has electric charge, the spacetime metric is described by the Reissner–Nordström solution. We can apply our direct integration method even for this geometry but it will lead to an expansion with two variables. However, such an expression will not be so illuminating. Instead, we apply the method used in section 3 to the charged case.

The Reissner–Nordström metric is given by

$$ds^2 = - \left( 1 - \frac{2M}{r} + \frac{Q^2}{r^2} \right) dt^2 + \left( 1 - \frac{2M}{r} + \frac{Q^2}{r^2} \right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2. \quad (49)$$

where $M$ is the mass of the star and $Q$ is the charge. In this case, photon trajectories are described by

$$\left( \frac{dr}{d\theta} \right)^2 + \left( 1 - \frac{2M}{r} + \frac{Q^2}{r^2} \right) \frac{1}{r^2} = \frac{1}{b^2}. \quad (50)$$

Introducing dimensionless parameters $u = 2M/r$ and $q = Q/(2M)$, this equation can be reduced to the following elliptic form:

$$\left( \frac{du}{d\theta} \right)^2 = -q^2 u^4 + u^3 - u^2 + \frac{4M^2}{b^2}. \quad (51)$$

Before integrating this equation, we have to specify the parameter region of our interest. In order that the photon can reach the observer at infinity without crossing the event horizon, the parameters must obey

$$\frac{4M^2}{b^2} < f(u), \quad f(u) := q^2 u^4 - u^3 + u^2. \quad (52)$$

where $u_{\pm} = (3 \pm \sqrt{9 - 32q^2})/(8q^2)$ are the positions of extrema of $f(u)$. In this case, one of the roots $u_i (i = 1, 2, 3, 4)$ of $4M^2/b^2 = f(u)$ is negative and the others are positive so that
we can assume \( u_1 < 0 < u_2 < u_3 < u_4 \) without loss of generality. With this convention, integration of the differential equation gives the integral expression of the deflection angle \( \Theta \),

\[
\Theta = 2 \int_0^{u_2} \frac{du}{\sqrt{-q^2 u^4 + u^3 - u^2 + 4M^2/b^2}} = 2 \int_0^{u_2} \frac{du}{\sqrt{q^2(u - u_1)(u_2 - u)(u_3 - u)(u_4 - u)}}. \tag{53}
\]

In addition to the parameter \( M^2/b^2 \), \( \Theta \) depends on the squared background charge \( q^2 \). As a result, the Picard–Fuchs equation now becomes a partial differential equation that contains a term proportional to \( \partial_q \Theta \). Interestingly, the coefficient of the additional term is proportional to \( q^2(q^2 - 1/4) \), meaning the Picard–Fuchs equation reduces to ordinary differential equations when the background spacetime is the Schwarzschild one or the extremal Reissner–Nordström one. Hereafter we focus on the latter case, namely \( q^2 = 1/4 \). In this case, the differential equation again turns out to be an inhomogeneous hypergeometric equation,

\[
\left[ x(1-x) \frac{d^2}{dx^2} + (1-2x) \frac{d}{dx} - \frac{3}{16} \right] \Theta = \frac{1}{4\sqrt{x}}, \tag{54}
\]

where the independent variable is differently normalized, \( x = 16M^2/b^2 \). The solution satisfying the boundary condition \( \Theta(x = 0) = \pi \) is uniquely identified as

\[
\Theta = \pi_2 F_1 \left[ \frac{1}{4}, \frac{3}{4}; 1; x \right] + \sqrt{x} F_2 \left[ \frac{3}{4}, 1, \frac{5}{4}, \frac{3}{2}, \frac{3}{2}; x \right], \tag{55}
\]

or in terms of the original variables,

\[
\Theta = \pi_2 F_1 \left[ \frac{1}{4}, \frac{3}{4}; 1; \frac{16M^2}{b^2} \right] + \frac{4M}{b} F_2 \left[ \frac{3}{4}, 1, \frac{5}{4}, \frac{16M^2}{b^2} \right]. \tag{56}
\]

Explicit coefficients up to the 4th order terms are given by

\[
\Theta = \pi + \frac{4M}{b} + \frac{3\pi M^2}{b^2} + \frac{80M^3}{b^3} + \frac{105\pi M^4}{b^4} + O(M/b^5), \tag{57}
\]

which is consistent with the previous result in reference [4].

As in the case of Schwarzschild spacetime, the strong deflection limit of \( \Theta \) can be derived. We here show only the result,

\[
\Theta = \sqrt{2} F_1 \left[ \frac{1}{4}, \frac{3}{4}; 1; y \right] \log \left( \frac{64(3 - 2\sqrt{y})}{y} \right) + \sum_{n=1}^{\infty} \left( \frac{\frac{j}{2} \frac{j}{2}}{(n!)^2} \sum_{j=1}^{n} \left\{ \sqrt{2} \left( \frac{2}{j} - \frac{1}{j - 1/4} - \frac{1}{j - 3/4} \right) + \frac{4}{3} \frac{j}{2} \frac{j}{2} \left( \frac{1}{2} \right)_{j-1} \right\} y^n \right), \tag{58}
\]

where \( y = 1 - x \). The leading logarithmic divergence and the next-leading constant terms are now obvious,

\[
\Theta = -\sqrt{2} \log y + \sqrt{2} \log \left( 64(3 - 2\sqrt{y}) \right) + o(1), \quad y \to 0, \tag{59}
\]
which is in agreement with the known results for the Reissner–Nordström spacetimes [8, 10, 11]. Although our result equation (58) is limited to the extremally charged case, we obtained the full order analytic expression.

6. Summary and discussions

So far we have derived exact and explicit expressions of the bending angles of photon trajectories in Schwarzschild and extremal Reissner–Nordström spacetimes in terms of the impact parameter by solving inhomogeneous Picard–Fuchs equations. The weak deflection expansion for the former case has been derived also by directly integrating the initial expression with the help of analytic continuation. Our results are generalizations of previously obtained expansions of the bending angles [3, 4, 8, 10, 11], and are confirmed to be consistent with them. Both weak and strong deflection expansions for the bending angles now become available up to an arbitrary order. The method used here can give another analytical tool to study similar problems in other geometries such as Reissner–Nordström spacetime for an arbitrary value.

Deriving the inhomogeneous Picard–Fuchs equation itself for the case of Reissner–Nordström spacetime with an arbitrary value of charge is straightforward except for the difference that there appear several partial differential equations in two variables associated with the impact parameter and charge. The direct integration method given in section 4 is also necessary in this case although the inhomogeneous Picard–Fuchs equations were sufficient in the cases of Schwarzschild and extremal Reissner–Nordström spacetimes since they took the well-known hypergeometric differential equations. This problem will be treated in future work.

The extension of our method to rotating black holes should be a more difficult task due to the loss of spherical symmetry. However, formal application of our method would be possible for the cases restricted to the equatorial plane because the geodesic equations then reduce to a single equation similar to equation (8), which leads to an integral expression of the bending angle.

By comparing the coefficients of the power series for the deflection angles equations (28) and (56), it can be seen that the bending angle for the Schwarzschild spacetime is larger than that for the extremal Reissner–Nordström spacetime, for every fixed value of $M/b$. This is due to the repulsive effect caused by electric charge as shown numerically in reference [12].

It is known that there are several transformation formulae for Appell functions [25–28] but we could not get an adequate transformation to relate the two expressions (16) and (28). If such a formula exists, it may be useful to reduce the integral expression of the bending angle in more complicated cases. For example, the bending angle on the equatorial plane in Kerr–Newman spacetime has been obtained in terms of the standard elliptic integrals [12].

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Data availability statement

No new data were created or analysed in this study.
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