Symmetries of Surface Singularities

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1 Introduction

The study of reductive group actions on a normal surface singularity $X$ is facilitated by the fact that the group $\text{Aut}_X$ of automorphisms of $X$ has a maximal reductive algebraic subgroup $G$ which contains every reductive algebraic subgroup of $\text{Aut}_X$ up to conjugation. If $X$ is not weighted homogeneous then this maximal group $G$ is finite (Scheja, Wiebe). It has been determined for cusp singularities by Wall. On the other hand, if $X$ is weighted homogeneous but not a cyclic quotient singularity then the connected component $G_1$ of the unit coincides with the $\mathbb{C}^*$ defining the weighted homogeneous structure (Scheja, Wiebe and Wahl). Thus the main interest lies in the finite group $G/G_1$. Not much is known about $G/G_1$. Ganter has given a bound on its order valid for Gorenstein singularities which are not log-canonical. Aumann-Körber has determined $G/G_1$ for all quotient singularities.

We propose to study $G/G_1$ through the action of $G$ on the minimal good resolution $\tilde{X}$ of $X$. If $X$ is weighted homogeneous but not a cyclic quotient singularity let $E_0$ be the central curve of the exceptional divisor of $\tilde{X}$. We show that the natural homomorphism $G \to \text{Aut} E_0$ has kernel $\mathbb{C}^*$ and finite image. In particular, this reproves the result of Scheja, Wiebe and Wahl mentioned above. Moreover, it allows to view $G/G_1$ as a subgroup of $\text{Aut} E_0$. For simple elliptic singularities it equals $(\mathbb{Z}_b \times \mathbb{Z}_b) \rtimes \text{Aut}_0 E_0$ where $-b$ is the self intersection number of $E_0$, $\mathbb{Z}_b \times \mathbb{Z}_b$ is the group of $b$-torsion points of the elliptic curve $E_0$ acting by translations, and $\text{Aut}_0 E_0$ is the group of automorphisms fixing the zero element of $E_0$. If $E_0$ is rational then $G/G_1$ is the group of automorphisms of $E_0$ which permute the intersection points with the branches of the exceptional divisor while preserving the Seifert invariants of these branches. When there are exactly three branches we conclude that $G/G_1$ is isomorphic to the group of automorphisms of the weighted resolution graph. This applies to all non-cyclic quotient singularities as well as to triangle singularities. We also investigate whether the maximal reductive automorphism group is a direct product $G \simeq G_1 \times G/G_1$. This is the case, for instance, if the central curve $E_0$ is rational of even self intersection number or if $X$ is Gorenstein such that its nowhere zero 2-form $\omega$ has degree $\pm 1$. In the latter case there is a “natural” section $G/G_1 \hookrightarrow G$ of $G \to G/G_1$ given by
the group of automorphisms in $G$ which fix $\omega$. For a simple elliptic singularity one has $G \simeq G_1 \times G/G_1$ if and only if $-E_0 \cdot E_0 = 1$.

In the weighted homogeneous case, we show that $G/G_1$ acts faithfully on the homology $H_1(L, \mathbb{Z})$ of the link of $X$. For a hypersurface in $\mathbb{C}^3$, not an $A_k$-singularity, defined by a weighted homogeneous polynomial $f$ this will be rephrased in terms of the group $H$ of linear right equivalences of $f$. Namely, with $F$ denoting the Milnor fibre of $f$, the group $H$ acts faithfully on the Milnor lattice $H_2(F, \mathbb{Z})$ and intersects the monodromy group in the cyclic group generated by the monodromy operator.

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2 The action on the central curve

Throughout this paper let $X = (X,0)$ denote a normal surface singularity with local ring $(\mathcal{O}_X, m_X)$. We are interested in algebraic subgroups $G$ of the group $\text{Aut} X$ of automorphisms of $X$. This means that $G \leq \text{Aut} X$ is an abstract subgroup equipped with the structure of an algebraic group such that the natural representations of $G$ on all higher cotangent spaces $m_X^k/m_X^{k+1}$ of $X$ are rational. There is an abundance of unipotent algebraic subgroups of $\text{Aut} X$, see [M4, Theorem 5]. But the situation becomes simpler if one restricts to reductive groups:

**Theorem 1.** There is a maximal reductive algebraic subgroup $G \leq \text{Aut} X$ containing every reductive algebraic subgroup of $\text{Aut} X$ up to conjugation.

**Proof.** This follows from [HM, Theorem 1] since a normal surface singularity (just as any isolated singularity) can be defined by polynomials, see [A, Theorem 3.8]. If one wants to avoid the use of [HM, Theorem 1], which depends on a deep result of Popescu and Rotthaus, one can argue more elementary as follows: Let $\tilde{X}$ be the algebroid space corresponding to the formal completion $\tilde{\mathcal{O}}_X$ of $\mathcal{O}_X$. By [M3, Satz 4] the group of automorphisms of $\tilde{X}$ contains a maximal reductive subgroup $G$. It is enough to make $G$ act on $X$ itself, with the same representation on the cotangent space, [HM, Lemma]. If $G$ is finite then [HM, Remark on p. 184] applies. Otherwise, $G$ contains some $\mathbb{C}^*$. It follows from a result of Scheja and Wiebe [SW1, 3.1] that in this case the normal surface $X$ in fact admits a good $\mathbb{C}^*$-action, i. e., $X$ is weighted homogeneous. Then [M3, Satz 5] applies. $\square$

From now on, $G$ will always denote the maximal reductive automorphism group of the normal surface singularity $X$. As mentioned in the preceding proof, $G$
is finite if $X$ is not weighted homogeneous. Otherwise, we may assume that the connected component $G_1$ of the unit contains at least the $\mathbb{C}^*$ defining the weighted homogeneous structure. In fact, Scheja and Wiebe [SW2, section 3] and Wahl [W3, 3.6.2] showed that $G_1 = \mathbb{C}^*$ unless $X$ is a cyclic quotient singularity. We shall reprove this below with a different method which, at the same time, will yield information on the finite group $G/G_1$.

Let $(\tilde{X}, E) \to (X, 0)$ be the minimal good resolution with exceptional divisor $E$. There is a natural homomorphism $\text{Aut}(X, 0) \to \text{Aut}(\tilde{X}, E)$ obtained from the universal property of the minimal good resolution. Here $\text{Aut}(\tilde{X}, E)$ denotes the group of germs of automorphisms of $\tilde{X}$ along $E$. In the weighted homogeneous case, the minimal good resolution was described by Orlik and Wagreich [OW, Theorem 2.3.1]: If $X$ is not a cyclic quotient singularity then $E$ has a component $E_0$ which is uniquely determined by the property that $E_0$ has positive genus $g$ or intersects at least three other components of $E$. The exceptional divisor is star shaped with central curve $E_0$ and a certain number $r$ of branches of rational curves. Here $r \geq 3$ if $E_0$ is rational and $r \geq 0$ otherwise. In deriving this from [OW] one has to be aware of a result of Brieskorn [B, Korollar 2.12]: If $X$ can be resolved by a chain of rational curves then $X$ is a cyclic quotient singularity. We obtain a natural homomorphism $\text{Aut} X \to \text{Aut} E_0$. Recall that $\text{Aut} E_0$ is finite if $E_0$ has genus $g \geq 2$ and $\text{Aut} E_0 = \text{PSL}(2, \mathbb{C})$ if $E_0$ is rational. Finally, if $E_0$ is an elliptic curve then $\text{Aut} E_0 = E_0 \rtimes \text{Aut}_0 E_0$. Here the Abelian group $E_0$ acts on itself by translations and the group $\text{Aut}_0 E_0$ of automorphisms fixing the zero element is cyclic of order 6, 4 or 2 if the $j$-invariant of $E_0$ is 0, 1 or else.

**Theorem 2.** Suppose that $X$ is weighted homogeneous but not a cyclic quotient singularity. Then the restriction $\rho : G \to \text{Aut} E_0$ of the natural map described above has kernel $\mathbb{C}^*$ and finite image. Hence $G_1 = \mathbb{C}^*$ and $\rho$ embeds $G/G_1$ into $\text{Aut} E_0$.

**Proof.** Let $X \subset (\mathbb{C}^n, 0)$ be a minimal embedding of the singularity $X$ in a smooth germ. By [M3, Satz 6] the action of $G$ on $X$ can be extended to an action on $(\mathbb{C}^n, 0)$ which is linear in suitable coordinates. Moreover, we may assume that $X$ is defined by equations which are weighted homogeneous polynomials in the chosen coordinates. Hence the action on the analytic space germ $X$ is induced from a global rational action on the affine algebraic variety $X$. We claim that the action on the germ $(\tilde{X}, E)$ is induced from a global rational action $G \times \tilde{X} \to \tilde{X}$ on the non-singular surface $\tilde{X}$. In fact, one obtains a $G$-equivariant good resolution $X' \to X$ by successively blowing up in $G$-invariant centres and normalizing. By the universal properties of blowing up and normalization, the action $G \times X' \to X'$ is rational. The minimal good resolution is obtained from $X'$ by blowing down $G$-invariant systems of exceptional curves of the first kind. This gives the claim. We conclude that $G \times E_0 \to E_0$ is rational.

The closed normal subgroup $K = \ker \rho \leq G$ is reductive. Consider the representation of $K$ on the tangent space $T_p \tilde{X}$ at some point $p \in E_0$. By Cartan’s
Uniqueness Theorem \([K]\) it is faithful. Since \(K\) is trivial on the subspace \(T_pE_0\) and the representation is completely reducible we see that \(K\) has a faithful one dimensional representation. Thus \(K\) is a subgroup of \(C^*\). It will follow that \(K = C^*\) if we show that \(\text{im } \rho\) is finite. This is obvious if \(E_0\) has genus \(g \geq 2\). But also for \(g = 1\) since \(C^*\) and hence every connected reductive group can only act trivially on an elliptic curve. In the remaining case \(g = 0\) consider the \(r\) intersection points of \(E_0\) with other components of \(E\). They have to be permuted by any element of \(\text{im } \rho\). We obtain a homomorphism \(\text{im } \rho \to \text{Sym}_r\) to the symmetric group which is injective since \(r \geq 3\) and since automorphisms of the projective line have at most two fixed points. 
\[\square\]

Remarks. (i) Let \(\Gamma\) denote the weighted dual graph defined by the minimal good resolution and \(\text{Aut}\Gamma\) its group of automorphisms. Let \(r\) be the number of branches emanating from the central curve of genus \(g\). We have a natural homomorphism \(\text{im } \rho \to \text{Aut}\Gamma\). A non-trivial automorphism of \(E_0\) has at most \(2g + 2\) fixed points, \([FK, V.1.1]\). Thus, if \(r > 2g + 2\) then \(G/G_1\) embeds into \(\text{Aut}\Gamma\). This is always the case if \(g = 0\) since \(r \geq 3\). The latter result was previously obtained (with a different proof) by Aumann-Körber \([A-K, 3.9]\).

(ii) If \(G/G_1 \to \text{Aut}\Gamma\) is injective we obtain sufficient conditions in terms of the weighted resolution graph for \(G/G_1\) to be trivial or cyclic. (Recall \([FK, III.7.7]\) that a finite group of automorphisms of a smooth complete curve fixing a point is cyclic.)

(iii) The exact sequence \(1 \to G_1 \to G \to G/G_1 \to 1\) splits if and only if \(G \simeq G_1 \times G/G_1\) is a direct product over \(G_1\). This follows from the fact \([W3, Proposition 3.10]\) that \(G_1 = C^*\) is central in \(G\).

(iv) Let \(X = \mathbb{C}^2/H\) be a cyclic quotient singularity where the group \(H\) is generated by \(\begin{pmatrix} \zeta & 0 \\ 0 & \zeta q \end{pmatrix}\) with \(\zeta^n = 1\) and \((q, n) = 1\). Then \(G_1 = \text{GL}(2, \mathbb{C})/H\) if \(q = 1\) and \(G_1 = C^* \times C^*\) else, see \([W3, 3.6.2]\). Also in this case, there is a natural homomorphism \(G/G_1 \to \text{Aut } \Gamma\) which, in fact, is bijective, \([A-K, 3.11, 3.12]\).

(v) Suppose that \(X\) is not necessarily weighted homogeneous but the exceptional divisor \(E\) has a component \(E_1\) which is fixed by every automorphism of \((X, E)\). (This is, e.g., the case for determinantal rational singularities of multiplicity \(m \geq 3\) since their exceptional divisor has a unique component \(E_1\) of self intersection number \(-m\), \([W2, Theorem 3.4]\).) Then there is again a natural homomorphism \(G \to \text{Aut } E_1\). The proof of Theorem 2 shows that its kernel is cyclic if \(X\) is assumed to be not weighted homogeneous.

(vi) If \(X\) is rational or minimally elliptic but not weighted homogeneous and not a cusp singularity then the kernel \(K\) of the natural homomorphism \(G \to \text{Aut } \Gamma\) is cyclic. In fact, by \([L, Lemma 1.3]\) and \([L, Proposition 3.5]\) all components of \(E\) are rational but \(\Gamma\) is not a chain nor a cycle. (Note that, besides the cusp singularities, also simple elliptic and cyclic quotient singularities are excluded because
they are weighted homogeneous.) Hence there is a component $E_1$ intersecting at least three other components. As $E_1$ is rational $K$ acts trivially on $E_1$. The proof of Theorem 2 shows that $K$ is cyclic. The special case of non weighted homogeneous exceptional unimodal singularities will be discussed in more detail in Example 3 of section 4. For cusp singularities the kernel of $G \to \text{Aut}\, \Gamma$ is Abelian but not necessarily cyclic. \[W7\].

(vii) When we are considering the maximal reductive automorphism group $G$ of some weighted homogeneous singularity $X$ we may choose coordinates such that, at the same time, $G$ acts linearly on the ambient space and $X$ is defined by weighted homogeneous polynomials, see the beginning of the proof of Theorem 2. This does not mean that if we start with a weighted homogeneous polynomial defining some $X$ then $G$ will be linear in the given coordinates. For instance, let $X \subseteq (\mathbb{C}^3, 0)$ be defined by $f = x_1^3 - 2x_3x_2^2 - x_3^3$ which is weighted homogeneous of weights 4, 3, 6 and degree 12. The $\mathbb{C}^*$-action is given by $t \cdot (x_1, x_2, x_3) = (t^4x_1, t^3x_2, t^6x_3)$. Suppose that $G \subseteq \text{Aut}(\mathbb{C}^3, 0)$ contains this $\mathbb{C}^*$ and defines a maximal reductive subgroup of $\text{Aut}\, X$. Since $\mathbb{C}^*$ is central in $G$, see Remark (iii) above, it is easily seen that $G \cap \text{GL}(3, \mathbb{C}) = \mathbb{C}^*$. But $G$ is larger than $\mathbb{C}^*$ since $X$ is isomorphic to the singularity $E_6$ defined by $g = x_1^3 + x_2^4 - x_3^2$ whose maximal reductive automorphism group clearly contains $\mathbb{C}^* \times \mathbb{Z}_2$ where $\mathbb{Z}_2$ acts by $(x_1, x_2, x_3) \mapsto (x_1, x_2, -x_3)$. It follows from Remark (i) that, in fact, $\mathbb{C}^* \times \mathbb{Z}_2$ is the maximal reductive automorphism group of $E_6$ and hence of $X$.

3 The finite group $G/G_1$

In the weighted homogeneous case, we are going to study the finite group $G/G_1$ viewed as a subgroup of the automorphism group of the central curve via the natural homomorphism $\rho$. First consider singularities $X$ such that the exceptional divisor is irreducible, $E = E_0$, of genus $g$. Let $N \to E_0$ be the normal bundle of $E_0 \subseteq \hat{X}$ with zero-section $E_0 \subseteq N$. It follows from work of Grauert [G2, §4] (see [W1, 6.2]) that the germs $(\hat{X}, E_0)$ and $(N, E_0)$ are isomorphic if the self intersection number satisfies $E_0 \cdot E_0 < 4 - 4g$. We determine $G$ for these singularities:

**Theorem 3.** Let $\pi : N \to E_0$ be a negative line bundle on a smooth complete curve $E_0$ of genus $g \geq 1$, and let $X$ be the singularity obtained by contracting the zero-section $E_0 \subseteq N$.

(i) Let $\tilde{G}$ be the group of automorphisms $g$ of the manifold $N$ which restrict to an automorphism $g_0$ of the zero-section such that $g_0 \circ \pi = \pi \circ g$. Then $\tilde{G}$ is the maximal reductive automorphism group of $X$.

(ii) Let $g = 1$ and $-b = E_0 \cdot E_0$. Then the natural map $\rho : G \to \text{Aut}\, E_0$ has image $(\mathbb{Z}_b \times \mathbb{Z}_b) \times \text{Aut}_0 E_0$ where $\mathbb{Z}_b \times \mathbb{Z}_b$ is the group of $b$-torsion points of the Abelian group $E_0$ and $\text{Aut}_0 E_0$ is the group of automorphisms fixing the zero element. The exact sequence $1 \to G_1 \to G \to G/G_1 \to 1$ splits if and only if $b = 1$. 

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Proof. Let $\tilde{\rho} : \tilde{G} \to \text{Aut } E_0$ be the restriction map. It has kernel $\mathbb{C}^*$. Let $g = 1$. We may assume that $N = \mathcal{O}(-b \cdot p_0)$ for some $p_0 \in E_0$. Further we may assume that $p_0$ is the zero element 0 of the group $E_0$. We claim that $\phi \in \text{Aut } E_0$ belongs to $\text{im } \tilde{\rho}$ if and only if $p = \phi(0)$ is a $b$-torsion point of $E_0$. In fact, for $\psi = \phi^{-1}$ let $\psi_* : \mathcal{O}(-b \cdot p) \to \mathcal{O}(-b \cdot 0)$ be the induced map satisfying $\psi \circ \pi = \pi \circ \psi_*$ (where $\pi$ denotes both bundle projections). If $\phi = g_0 = \tilde{\rho}(g)$ for some $g \in \tilde{G}$ then $g \circ \psi_* : \mathcal{O}(-b \cdot p) \to \mathcal{O}(-b \cdot 0)$ is an isomorphism of line bundles. Conversely, if $\Psi : \mathcal{O}(-b \cdot p) \to \mathcal{O}(-b \cdot 0)$ is a line bundle isomorphism then $\Psi \circ \phi_*$ is an element of $\tilde{G}$ restricting to $\phi$. Now observe that the line bundles $\mathcal{O}(-b \cdot p)$ and $\mathcal{O}(-b \cdot 0)$ are isomorphic if and only if the divisors $-b \cdot p$ and $-b \cdot 0$ are linearly equivalent if and only if $bp = 0$ in the group $E_0$. This proves the claim. It follows from $\text{Aut } E_0 = E_0 \times \text{Aut}_0 E_0$ that $\text{im } \tilde{\rho} = (\mathbb{Z}_b \times \mathbb{Z}_b) \times \text{Aut}_0 E_0$. Since $\tilde{\rho}$ has finite image also for $g \geq 2$ we conclude that $\tilde{G}$ is reductive in any case.

Now $\tilde{G}$ induces a reductive algebraic subgroup of $\text{Aut } X$. Hence $\tilde{G}$ is contained in a maximal reductive subgroup $G$ viewed as a group of automorphisms of the germ $(N,E_0)$. As both $\tilde{G}$ and $G$ are one dimensional they are equal if they have the same image in $\text{Aut } E_0$. But every $g \in G$ induces a $\tilde{\rho} \in \tilde{G}$ with the same restriction to $E_0$ since $N \to E_0$ is the normal bundle of $E_0 \subseteq N$.

Let us return to the case $g = 1$. If $b = 1$ then the divisor $-0$ defining $N$ is invariant under $\text{im } \rho = \text{Aut } E_0$. This yields a section $\text{im } \rho \to \tilde{G} = G$. To prove the non-existence of such a section for $b \geq 2$ we write $E_0 = \mathbb{C}/\Lambda$ where the lattice $\Lambda$ is generated by the primitive periods $\omega_1, \omega_2$ with $\text{Im } \omega_2/\omega_1 > 0$. Let $\phi : E_0 \to E_0$ be the translation given by $z \mapsto z + \alpha$ with $\alpha = \omega_1/b$. Set $\alpha_0 = -(b+1)/2b \cdot \omega_1$ and $\alpha_k = \alpha_0 + k \omega$ for $k \in \mathbb{Z}$. Then the divisor $D_1 = \sum_{k=1}^b \alpha_k$ on $E_0$ is $\phi$-invariant and $\phi$ induces an automorphism of $\mathcal{O}(-D_1)$ over $\phi$. It is given by

$$s_1(z) \mapsto s_1(z + \alpha)$$

with $s_1$ denoting a rational section of $\mathcal{O}(-D_1)$. Because $\alpha_1 + \ldots + \alpha_b = 0$ in $\mathbb{C}$ the divisors $D_1$ and $D_0 = b \cdot 0$ are linearly equivalent, say $D_1 - D_0 = (h_1)$ for some rational function $h_1$ on $E_0$. Using a rational section $s_0$ of $N = \mathcal{O}(-D_0)$ one defines a line bundle isomorphism $N \to \mathcal{O}(-D_1)$ by

$$s_0(z) \mapsto h_1(z)s_1(z).$$

We obtain an automorphism $\tilde{\phi} \in \tilde{G} = G$ of $N$ over $\phi$ given by

$$s_0(z) \mapsto \frac{h_1(z)}{h_1(z + \alpha)}s_0(z + \alpha).$$

Now let $\beta = \omega_2/b$, $\psi$ the corresponding translation, $\beta_0 = -(b+1)/2b \cdot \omega_2$, $\beta_k = \beta_0 + k\beta$, $D_2 = \sum_{k=1}^b \beta_k$, $D_2 - D_0 = (h_2)$ and $\tilde{\psi} \in G$ the induced map. Then $\tilde{\psi}^{-1} \tilde{\phi}^{-1} \tilde{\psi} \tilde{\phi} \in G$ is given by

$$s_0(z) \mapsto \frac{h(z)(h(z + \alpha + \beta)}{h(z + \alpha)h(z + \beta)}s_0(z).$$
where \( h = h_1/h_2 \). As this commutator is contained in the kernel of \( \rho \) the function
\[
h(z)h(z + \alpha + \beta)h(z + \alpha)^{-1}h(z + \beta)^{-1}
\]
is constant, say equal to \( \lambda \in \mathbb{C}^* \). If we assume that \( \rho \) admits a section, hence that \( G = \mathbb{C}^* \times B \) with a subgroup \( B \), see Remark (iii) of section 2, then the commutator must be contained in \( B \) and \( \lambda = 1 \). We are going to show that \( \lambda = \exp(2\pi i/b) \), hence \( b = 1 \).

Let \( \zeta \) and \( \sigma \) be Weierstrass’ \( \zeta \)- and \( \sigma \)-function. One has
\[
\sigma(z + \omega_i) = -\exp(\eta_i(z + \omega_i/2)) \cdot \sigma(z)
\]
with \( \eta_i = 2\zeta(\omega_i/2) \) for \( i = 1,2 \), see [HC, II.1.13, Satz 3]. The elliptic function \( h \) has \( \alpha_1, \ldots, \alpha_b \) and \( \beta_1, \ldots, \beta_b \) as complete systems of zeros and poles. Since \( \alpha_1 + \cdots + \alpha_b = \beta_1 + \cdots + \beta_b \) it follows from [HC, II.1.14, Satz 1] that
\[
h(z) = \prod_{k=1}^{b} \frac{\sigma(z - \alpha_k)}{\sigma(z - \beta_k)}
\]
up to a constant. One calculates
\[
\frac{h(z)h(z + \alpha + \beta)}{h(z + \alpha)h(z + \beta)} = \frac{\sigma(z - \alpha_b)\sigma(z - \beta_0)\sigma(z + \beta - \alpha_0)\sigma(z + \alpha - \beta_b)}{\sigma(z - \alpha_0)\sigma(z - \beta_b)\sigma(z + \beta - \alpha_b)\sigma(z + \alpha - \beta_0)}
\]
\[
= \exp(\eta_1 \beta - \eta_2 \alpha)
\]
\[
= \exp((\eta_1 \omega_2 - \eta_2 \omega_1)/b).
\]
Legendre’s relation \( \eta_1 \omega_2 - \eta_2 \omega_1 = 2\pi i \), see [HC, II.1.11], gives the claim. \( \square \)

Let \( X \) be weighted homogeneous but not a cyclic quotient singularity. For \( i = 1, \ldots, r \) let \( E_{ij}, j = 1, \ldots, l_i \), be the curves on the \( i \)-th branch of the exceptional divisor of the minimal good resolution. Assume that \( E_{ij} \) intersects the central curve \( E_0 \), say in the point \( p_i \), and that \( E_{ij} \) intersects \( E_{i,j+1} \). Let \( -b = E_0 \cdot E_0 \) and \(-b_{ij} = E_{ij} \cdot E_{ij} \) denote the self intersection numbers. Finally, consider the Hirzebruch-Jung continued fractions
\[
\alpha_i/\beta_i = b_{i1} - 1/(b_{i2} - 1/(\cdots - 1/b_{i,l_i})\ldots)
\]
with coprime positive integers \( \beta_i < \alpha_i \).

**Theorem 4.** Let \( X \) be weighted homogeneous but not a cyclic quotient singularity and suppose that the central curve is rational.

(i) Then the image of \( \rho : G \to \text{Aut} E_0 \) equals the group \( A \) of automorphisms of \( E_0 \) which permute the points \( p_1, \ldots, p_r \) while preserving the pairs \( (\alpha_i, \beta_i) \).

(ii) If \( b \) is even or if \( A \) is cyclic or if \( A \) is dihedral of order \( 2q \) with \( q \) odd then the maximal reductive automorphism group is a direct product \( G \simeq \mathbb{C}^* \times A \).

**Proof.** Obviously, \( \text{im} \rho \subseteq A \). For the other inclusion we use Pinkham’s description [P, Theorem 5.1] of the affine coordinate ring of a weighted homogeneous surface.
Let the divisor $D_0$ correspond to the conormal bundle of $E_0 \subseteq \tilde{X}$. Consider the divisor
\[
D = D_0 - \sum_{i=1}^{r} \frac{\beta_i}{\alpha_i} \cdot p_i
\]
with rational coefficients. Then the coordinate ring of the affine algebraic variety $X$ is isomorphic, as a graded ring, to $\bigoplus_{k=0}^{\infty} L(kD)$ where $L(kD)$ denotes the space of rational functions $f$ on $E_0$ with $(f) \geq -kD$. Now recall the classification of the finite subgroups of $\text{PSL}(2, \mathbb{C})$ and their orbits on the projective line, [1, Chapter I, §6]. The cyclic groups have a fixed point. The dihedral group of order $2q$ has orbits of length $2$ and $q$. And for the tetrahedral, the octahedral and the icosahedral group the greatest common divisor of the lengths of the orbits is $2$. Thus, if $E_0$ is rational and one of the hypotheses of (ii) is fulfilled then there exists on $E_0$ an $A$-invariant divisor $D_1$ of degree $b$. The sum in the definition of $D$ is $A$-invariant. Since $E_0$ is rational the two divisors $D_0, D_1$ of the same degree $b$ are linearly equivalent. Hence we may replace $D_0$ by $D_1$ and assume that $D$ itself is $A$-invariant. Now there is an obvious action of $A$ on $\bigoplus_{k=0}^{\infty} L(kD)$. We obtain a homomorphism $A \to \text{Aut} X : \phi \mapsto \phi'$ whose image may be assumed to lie in $G$. One checks that $\rho(\phi') = \phi$ for all $\phi \in A$. Hence $A = \text{im} \rho$ and the exact sequence $1 \to \mathbb{C}^* \to G \to A \to 1$ splits. Then $G \simeq \mathbb{C}^* \times A$, see Remark (iii) of section 2. To prove (i) in the cases not covered by (ii) apply (ii) to the cyclic groups generated by the elements of $A$. \hfill \Box

**Corollary.** Let $X$ be weighted homogeneous such that the exceptional divisor consists of exactly three branches emanating from a rational central curve. Then $G \simeq \mathbb{C}^* \times \text{Aut} \Gamma$ where $\Gamma$ denotes the weighted dual resolution graph.

**Proof.** Recall from Remark (i) of section 2 the injection $A \to \text{Aut} \Gamma$. In the present case it is an isomorphism since three points on the projective line have no moduli. Part (ii) of the Theorem applies as $\text{Aut} \Gamma$ is trivial or cyclic of order $2$ or isomorphic to the symmetric group $S_3$, i.e., dihedral of order $6$. \hfill \Box

**Remarks.** (i) The Corollary applies to all non-cyclic quotient singularities. In this special case the result was previously obtained by Aumann-Körber [A-K, 3.12]. She uses the fact [W3, 3.6.3] that for a quotient singularity $\mathbb{C}^2/H$ the maximal reductive automorphism group is $\text{N}(H)/H$ where $\text{N}(H)$ denotes the normalizer of $H$ in $\text{GL}(2, \mathbb{C})$. Then she explicitly computes this normalizer for every $H$. The Corollary applies, as well, to triangle singularities, see Example 2 below.

(ii) The proof of Theorem 4 shows $G \simeq \mathbb{C}^* \times A$ if $g$ is arbitrary and the divisor $\sum_{i=1}^{r} p_i$ corresponds to the conormal bundle. Such singularities (with $g = 1$) have been considered by Tomaru [I].

We end this section by showing that every finite group appears as $G/G_1$ for a suitable $X$. 

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Theorem 5. Let $E_0$ be an arbitrary smooth complete curve and let $A \leq \text{Aut } E_0$ be an arbitrary finite subgroup. Then there is a weighted homogeneous normal surface singularity $X$ with central curve $E_0$ and maximal reductive automorphism group $G \simeq \mathbb{C}^* \times A$.

Remark. Hurwitz showed that every finite group can be realized as a subgroup of $\text{Aut } E_0$ for some smooth complete curve $E_0$, see [G3].

Proof of the Theorem. We first show that there exist finitely many $A$-orbits $B_1, \ldots, B_k \subseteq E_0$ such that $A$ consists of exactly those $\phi \in \text{Aut } E_0$ with $\phi(B_j) = B_j$ for all $j = 1, \ldots, k$. Take some $A$-orbits $B_1, \ldots, B_l$ and let $A_1$ be the group of $\phi \in \text{Aut } E_0$ with $\phi(B_j) = B_j$ for $j = 1, \ldots, l$. If the number of elements of $B_1 \cup \ldots \cup B_l$ is sufficiently big (say, $\geq 3$ if $E_0$ has genus $g = 0$ and $\geq 5$ if $g = 1$) then $A_1$ is finite. We have $A \subseteq A_1$. If the inclusion is strict then a generic $A_1$-orbit (not containing any of the finitely many points which are fixed by some element of $A_1$) has more elements than $A$. Choose an $A_1$-orbit $B_{l+1}$ contained in a generic $A_1$-orbit. Then the group $A_2$ of elements of $A_1$ which moreover map $B_{l+1}$ onto itself satisfies $A \subseteq A_2 \subsetneq A_1$. After finitely many steps we arrive at the claim. Now attach pairs $(\alpha_i, \beta_i)$ of coprime positive integers $\beta_i < \alpha_i$ to the points $p_1, \ldots, p_r$ of $B_1 \cup \ldots \cup B_k$ in such a way that two points get the same label if and only if they belong to the same orbit. Then $A$ is the group of automorphisms of $E_0$ which permute the points $p_1, \ldots, p_r$ while preserving the pairs $(\alpha_i, \beta_i)$. Moreover, choose an $A$-invariant divisor $D_0$ on $E_0$ of degree $b > \sum_{i=1}^r \beta_i/\alpha_i$. Then Pinkham’s construction yields a weighted homogeneous $X$ with the given data and such that its maximal reductive automorphism group is isomorphic to $\mathbb{C}^* \times A$, see the proof of Theorem 4.

\[\square\]

4 Examples

Catanese [3, section 2] has studied involutions of rational double points. We extend this to all weighted homogeneous singularities with rational central curve and three branches. To determine the occurring quotients we need two Lemmas. As usual, the weight $−2$ is omitted in the resolution graphs.

Lemma 1. Let $X = X_{b,1}$ be the cyclic quotient singularity with weighted graph

\[
\begin{array}{c}
\bullet \\
-b
\end{array}
\]

Let $\sigma$ be an involution of $X$ and consider its action on the minimal resolution $\tilde{X}$ and on the exceptional curve $E$.

(i) If $\sigma$ fixes $E$ pointwise then $X/\sigma$ has weighted graph

\[
\begin{array}{c}
\bullet \\
-2b
\end{array}
\]

Otherwise there are the following cases:
If $b$ is odd then one of the two fixed points of $\sigma$ on $E$ is an isolated fixed point on $\tilde{X}$ and the other is not. Then $X/\sigma$ has weighted graph

$$-(b+1)/2$$

(iii) If $b$ is even then either both fixed points of $\sigma$ on $E$ are isolated fixed points on $\tilde{X}$ or both are not. The corresponding weighted graphs are

$$-(b+2)/2 \quad \text{and} \quad -b/2$$

Proof. Let $q$ be a fixed point of $\sigma$ lying on $E$. If $q$ is an isolated fixed point it gives rise to an $A_1$-singularity in $X/\sigma$ which can be resolved by inserting a rational $(-2)$-curve. Otherwise, $q$ is mapped onto a smooth point of $X/\sigma$. A resolution of $X/\sigma$ clearly resolves $X/\sigma$. Thus, according to whether there are two or one or no isolated fixed points lying on $E$ the quotient $X/\sigma$ has weighted graph

$$-c \quad \text{or} \quad -c \quad \text{or} \quad -c$$

with some $c$. To determine $c$ and to prove the other assertions write $X = X_{b,1} = \mathbb{C}^2/H$ where $H$ is generated by $\left( \begin{array}{cc} \zeta & 0 \\ 0 & \zeta \end{array} \right)$ and $\zeta$ is a primitive $b$-th root of unity.

As mentioned in Remark (iii) of section 2 the maximal reductive automorphism group of $X$ is $\text{GL}(2, \mathbb{C})/H$. Hence $X/\sigma \simeq \mathbb{C}^2/\Sigma$ with some group $\Sigma \leq \text{GL}(2, \mathbb{C})$ containing $H$ as a subgroup of index two. Clearly $\Sigma$ is Abelian. So we may assume that it consists of diagonal matrices.

Consider first the case that $\Sigma$ is cyclic, say generated by $\tau$. We may assume that $\tau^2 = \left( \begin{array}{cc} \zeta & 0 \\ 0 & \zeta \end{array} \right)$. If $\tau = \left( \begin{array}{cc} \eta & 0 \\ 0 & \eta \end{array} \right)$ with a $2b$-th root of unity $\eta$ then $X/\sigma \simeq \mathbb{C}^2/\tau = X_{2b,1}$ with graph as in (i). On $\tilde{X} = \{ (z_1, z_2, (w_1 : w_2)) \in \mathbb{C}^2 \times \mathbb{P}_1, \ z_1 w_2^b = z_2 w_1^b \}$ we have $\sigma(z, w) = (-z, w)$ and $\sigma$ fixes $E = 0 \times \mathbb{P}_1$ pointwise. The next possibility is $\tau = \left( \begin{array}{cc} \eta & 0 \\ 0 & -\eta \end{array} \right) = \left( \begin{array}{cc} \eta & 0 \\ 0 & \eta^{b+1} \end{array} \right)$ where again $\eta$ is a $2b$-th root of unity. If $b$ is even then $2b$ and $b+1$ are coprime and $X/\sigma \simeq \mathbb{C}^2/\tau = X_{2b,b+1}$. The continued fraction expansion of $2b/(b+1)$ shows that the graph is the first one in (iii). If $b$ is odd then $b$ and $(b+1)/2$ are coprime and $\tau^b = \left( \begin{array}{cc} -1 & 0 \\ 0 & 1 \end{array} \right)$. Since $\tau$ acts on $\mathbb{C}^2/\tau^b \simeq \mathbb{C}^2$ by $\left( \begin{array}{cc} \eta^2 & 0 \\ 0 & \eta^{b+1} \end{array} \right) = \left( \begin{array}{cc} \zeta & 0 \\ 0 & \zeta^{(b+1)/2} \end{array} \right)$ we see $X/\sigma \simeq X_{b,(b+1)/2}$ with graph as in (ii). In these two cases the assertion on the fixed points is obvious from the graphs.

Now consider the case that $\Sigma$ is not cyclic. Then $\Sigma = H \times \langle \tau \rangle$ with some involution $\tau \in \text{GL}(2, \mathbb{C})$, and $b$ must be even. As $\left( \begin{array}{cc} -1 & 0 \\ 0 & -1 \end{array} \right) \in H$ we have
\[ \tau = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \] without loss of generality and \( \Sigma \) contains the reflection group \( T \) generated by \( \tau \) and \( -\tau \). The generator of \( H \) acts on \( \mathbb{C}^2/T \simeq \mathbb{C}^2 \) by \( \begin{pmatrix} \xi^2 & 0 \\ 0 & \xi^2 \end{pmatrix} \). Hence \( X/\sigma \simeq X_{6/2,1} \) and the graph is the second one in (iii). Moreover, \( \sigma(z, w) = (z, -w) \) on \( \tilde{X} \).

**Remark.** The statement of the Lemma is true also for \( b = 1 \), i.e., if \( X = (\mathbb{C}^2, 0) \) and \( \tilde{X} \) is the blow up of 0. For \( b = 2 \) it appears in [S, p. 80].

**Lemma 2.** Let \( X \) be weighted homogeneous but not a cyclic quotient singularity. Let \( \phi \) be an automorphism of \( X \) of finite order and consider its action on the minimal good resolution \( \tilde{X} \). Suppose that \( \phi \) fixes the intersection point \( p \) of two components \( E_1, E_2 \) of the exceptional divisor \( E \) and that \( p \) is not an isolated fixed point of \( \phi \) on \( \tilde{X} \). Then near \( p \) the fixed point locus coincides with \( E_1 \) or \( E_2 \).

**Proof.** Near \( p \) the automorphism \( \phi \) can be linearized. In suitable local coordinates \( x, y \) it is given by \( \phi(x, y) = (x, \lambda y) \) with some \( \lambda \neq 1 \). As \( E \) is star shaped it is not possible that \( \phi \) interchanges \( E_1 \) and \( E_2 \). Hence they are left invariant. It is easily seen that a smooth \( \phi \)-invariant curve through \( p \) different from the fixed point locus \( \{ y = 0 \} \) must be tangent to \( \{ x = 0 \} \). Since \( E_1 \) and \( E_2 \) intersect transversely the Lemma is proven.

**Proposition 1.** Let \( X \) be weighted homogeneous such that the exceptional divisor consists of exactly three branches emanating from a rational central curve.

(ii) If \( \text{Aut} \Gamma \) is trivial then \( X \) has (up to conjugation) exactly one involution, namely the one contained in \( \mathbb{C}^* \).

(ii) If \( \text{Aut} \Gamma \) is not trivial then besides \( \sigma_1 \in \mathbb{C}^* \) there are (up to conjugation) exactly two more involutions \( \sigma_2 \) and \( \sigma_3 = \sigma_1 \sigma_2 \) in \( \text{Aut} X \). They can be distinguished by the property that for one of them, say \( \sigma_2 \), the quotient \( X/\sigma_2 \) is a cyclic quotient singularity whereas \( X/\sigma_3 \) is not. Here we agree that the smooth germ \((\mathbb{C}^2, 0)\) is called a cyclic quotient singularity, too.

**Proof.** (i) is clear from the Corollary of Theorem 4. In (ii) we have \( G \simeq \mathbb{C}^* \times \text{Aut} \Gamma \) with \( \text{Aut} \Gamma = \mathbb{Z}_2 \) or \( S_3 \). Since \( S_3 \) has, up to conjugation, only one involution there are only three involutions to consider in \( G \). We may assume that \( \sigma_2 \) and \( \sigma_3 \) interchange the second and third branch which therefore get identified in the quotient.

Both involutions fix the intersection point of \( E_0 \) with the first branch and have a second fixed point \( q \) on the projective line \( E_0 \). In suitable local coordinates \( x, y \) around \( q \) with \( E_0 = \{ x = 0 \} \) we have \( \sigma_1(x, y) = (-x, y) \), \( \sigma_2(x, y) = (x, -y) \) and \( \sigma_3(x, y) = (-x, -y) \). Hence \( q \) is mapped onto a smooth point in \( \tilde{X}/\sigma_2 \), but in the resolution of \( \tilde{X}/\sigma_3 \) there appears a new branch consisting of a single \((-2)\)-curve. The only isolated fixed points possibly lying on the first branch are intersection points of components plus, maybe, one point on the curve at the end of the branch. Thus, after resolving the \( A_1 \)-singularities appearing in the quotient
we are left with a chain of rational curves. We conclude that \(X/\sigma_2\) is a cyclic quotient singularity. And it will follow that \(X/\sigma_3\) has a star shaped graph with three branches (and hence is not a cyclic quotient singularity) if we can show that the chain of rational curves arising from the first branch of \(X\) cannot be blown down completely to a smooth point. From Lemma 1 one sees that there is only one possible way for a \((-1)\)-curve to appear. Namely, if the branch contains a \((-2)\)-curve \(E_i\), not fixed pointwise, such that the two fixed points on \(E_i\) are not isolated fixed points of the involution on \(\tilde{X}\). But then Lemma 2 shows that the neighbouring components have to be fixed pointwise. Using Lemma 1 again we see that the configuration

\[
\begin{array}{ccc}
-b_{i-1} & -b_{i+1} & \\
\end{array}
\]

produces

\[
\begin{array}{ccc}
-2b_{i-1} & -1 & -2b_{i+1} & \\
\end{array}
\]

in the quotient, hence

\[
\begin{array}{ccc}
-(2b_{i-1} - 1) & -(2b_{i+1} - 1) & \\
\end{array}
\]

after blowing down the \((-1)\)-curve. Consequently, the blowing down does not create new \((-1)\)-curves. This implies the claim. \(\blacksquare\)

**Example 1.** Proposition 1 applies to each non-cyclic quotient singularity \(X\). As can be seen from \([R]\) the exceptional divisor has a branch consisting of a single \((-2)\)-curve. It follows from Lemma 1 that this branch disappears in the quotient with respect to the involution \(\sigma_1 \in \mathbb{C}^*\). As in the proof of Proposition 1 one then shows that \(X/\sigma_1\) is a cyclic quotient singularity. Of course, it is a simple task to determine it explicitly in each case. From \([R]\) one sees that \(\text{Aut } \Gamma\) is non-trivial if and only if \(X\) is dihedral or \(X = T_m\) is tetrahedral with \(m \equiv 1\) or \(5\) mod \(6\). In these cases let \(\sigma_3\) be the involution for which \(X/\sigma_3\) is not a cyclic quotient singularity. If \(X\) is dihedral then \(X/\sigma_3\) is dihedral again. And for \(X = T_m\), \(m \equiv 1\) or \(5\) mod \(6\), one obtains \(T_m/\sigma_3 \simeq O_m\), an octahedral singularity. More precisely, if \(m = 6(b - 2) + 1\) then \(T_m\) has graph

\[
\begin{array}{ccc}
-b & \\
\end{array}
\]

Using Lemmas 1 and 2 one sees that \(T_m/\sigma_3\) has graph

\[
\begin{array}{ccc}
-4 & -(b + 1)/2 & \\
\end{array}
\]
if $b$ is odd, but

$$-(b + 2)/2$$

if $b$ is even. For $m = 6(b - 2) + 5$ the graphs of $T_m$ and $O_m \simeq T_m/\sigma_3$ are

and

if $b$ is odd, but

$$-(3 - (b + 2)/2)$$

if $b$ is even.

Before discussing the next examples we make a digression.

**Lemma 3.** For a normal surface singularity $X$ let

$$V(X) = \Gamma(X - 0, \Omega^2)/L^2(X - 0)$$

be the vector space of 2-forms defined on a deleted neighbourhood of the singular point modulo the subspace of square integrable forms. Let $H \leq \text{Aut } X$ be a finite subgroup and $\pi : X \to X/H$ the quotient map. Then the pullback of forms induces an injection $V(X/H) \hookrightarrow V(X)^H$. Here the upper index $H$ denotes the subspace of $H$-invariants. For the geometric genus $p_g(X) = \dim V(X)$, see \[L2, Theorem 3.4\], this yields

$$p_g(X/H) \leq p_g(X)$$

with strict inequality if the representation of $H$ on $V(X)$ is not trivial. If $\pi$ is unramified outside the singular point then $V(X/H) = V(X)^H$. 

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Proof. For \( \alpha \in \Gamma(X/H - 0, \Omega^2) \) the pullback \( \pi^* \alpha \in \Gamma(X - 0, \Omega^2) \) is \( H \)-invariant. And it is square integrable if and only if \( \alpha \) is so. If \( \pi \) is unramified outside 0 then every \( H \)-invariant form \( \alpha' \) on \( X - 0 \) induces a form \( \alpha \) on \( X/H - 0 \) with \( \alpha' = \pi^* \alpha \). \( \square \)

Let \( X \) be weighted homogeneous but not a cyclic quotient singularity. Suppose that \( X \) is Gorenstein with nowhere zero 2-form \( \omega \in \Gamma(X - 0, \Omega^2) \). We may assume [G1, p. 56] that \( \omega \) is \( G \)-equivariant: \( g^* \omega = \chi(g) \cdot \omega \) for all \( g \in G \) with some character \( \chi : G \to \mathbb{C}^* \). Let \( \overline{G} \) be the kernel of \( \chi \). Ganter [G1, Lemma 8.3] has shown that \( X \to X/\bar{G} \) is unramified outside the singular point. In particular, \( \omega \) induces on \( X/\bar{G} - 0 \) a nowhere zero 2-form and \( X/\bar{G} \) is Gorenstein. There is an integer \( \varepsilon \) such that \( \chi(t) = t^{-\varepsilon} \) for all \( t \in \mathbb{C}^* = G_1 \). It is known [W4, Corollary 3.3] that \( \varepsilon < 0 \) if and only if \( \varepsilon = -1 \) if and only if \( X \) is a rational double point. And \( \varepsilon = 0 \) if and only if \( X \) is simple elliptic. It may be mentioned at this place that Ganter [G1, Theorem 8.5] has obtained the bound

\[
|G/G_1| \leq 42 \cdot (-P_X \cdot P_X)/\varepsilon
\]

if \( X \) is not log-canonical. The invariant \( -P_X \cdot P_X \) can be calculated (using the notation of section 3) as

\[
-P_X \cdot P_X = \frac{(2g - 2 + r - \sum_{i=1}^{\ell} 1/\alpha_i)^2}{b - \sum_{i=1}^{\ell} \beta_i/\alpha_i},
\]

see [W3, Theorem 3.2]. If we now assume that \( \varepsilon = \pm 1 \) then \( G = \mathbb{C}^* \times \bar{G} \). Thus we obtain a “natural” section \( G/G_1 \hookrightarrow G \to G/G_1 \).

Example 2. Consider the triangle singularities \( X = D_{p,q,r} \) with graph

\[
\begin{array}{ccc}
-p & & -1 \\
\text{--} & \text{--} & \text{--} \\
-q & & -r
\end{array}
\]

It is known that they are minimally elliptic (i. e., Gorenstein with \( p_g(X) = 1 \), see [3] and have \( \varepsilon = 1 \), [W4]. Hence \( G = \mathbb{C}^* \times \bar{G} \). As \( V(X) \) is spanned by the nowhere zero form \( \omega \) we see that \( X/H \) is minimally elliptic for \( H \subseteq G \) but \( p_g(X/H) = 0 \) (i. e., \( X/H \) is rational) if \( H \not\subseteq \bar{G} \). Let us determine \( X/\bar{G} \) in case \( G \cong \text{Aut} \Gamma \) is not trivial. First take \( X = D_{p,q,q} \) with \( p \neq q \). Then \( G \) is generated by the involution \( \sigma_3 \) of Proposition 1. Hence \( D_{p,q,q}/G \cong D_{2p,2q} \) by an application of Lemmas 1 and 2. For \( X = D_{p,p,p} \) we have \( \bar{G} \cong S_3 \). Let \( H \leq G \) be the cyclic subgroup of order three. As \( H \) acts freely on \( \bar{X} - E \) the two fixed points lying on \( E_0 \) must be isolated. Lemma 4 below shows \( D_{p,p,p}/H \cong D_{3,3,3} \). Then \( D_{p,p,p}/G \cong D_{3,3,3}/\sigma_3 \cong D_{2,3,2p} \).
Lemma 4. Let $X = X_{b,1}$ with weighted graph

\[ -b \]

Let $\sigma$ be an automorphism of $X$ of order three and consider its action on $\tilde{X}$ and on $E$. If $\sigma$ fixes $E$ pointwise then $X/\sigma$ has weighted graph

\[ -3b \]

Otherwise there are the following possibilities:

\[ -(b + 3)/3 \quad -3 \quad \text{or} \quad -b/3 \]

if $b \equiv 0 \mod 3$,

\[ -3 \quad -(b + 2)/3 \quad -3 \quad \text{or} \quad -(b + 2)/3 \]

if $b \equiv 1 \mod 3$, and

\[ -(b + 4)/3 \quad \text{or} \quad -(b + 1)/3 \quad -3 \]

if $b \equiv 2 \mod 3$. The number of isolated fixed points lying on $E$ can be seen from the graphs.

Proof. Similar to the proof of Lemma 1. \qed

Example 2 (continued). Consider the fourteen triangle singularities which can be embedded as a hypersurface in $(\mathbb{C}^3,0)$, see \cite[Theorem 3.13 and section V]{L3}. In eleven cases only one of the three weights of the $\mathbb{C}^*$-action is odd so that the involution $\sigma_1 \in \mathbb{C}^*$ is a reflection. By \cite[4.2]{M2} the quotient $X/\sigma_1$ will be smooth or an isolated hypersurface singularity, hence (as it is rational) a rational double point. For $D_{3,3,6} = Q_{12}$ and $D_{3,4,5} = S_{12}$ the $\mathbb{C}^*$-action has exactly two odd weights. The quotient $D_{3,3,6}/\sigma_1$ is a rational triple point with graph

\[ -3 \]

And $D_{3,4,5}/\sigma_1$ coincides with the icosahedral quotient singularity $I_{13}$ of graph

\[ -3 \]
This seems to be a quite interesting singularity: It plays an exceptional role in Manetti’s [M1] study of smooth curves on rational surface singularities. Finally, for \( X = D_{3,3,5} = Z_{13} \) all three weights are odd. Hence \( X \to X/\sigma_1 \) is unramified outside the singular point. In fact, it is the canonical Gorenstein cover [W2, section 4] of the rational quadruple point \( X/\sigma_1 \) with graph

![Graph](image)

**Example 3.** Consider the fourteen exceptional families \( f_a = f_0 + a \cdot g, \ a \in \mathbb{C} \), of unimodal singularities. [AGV, part II]. Here the \( f_0 \) are weighted homogeneous and define exactly the fourteen triangle singularities \( X_0 \) which can be embedded as a hypersurface in \((\mathbb{C}^3,0)\). For \( a \neq 0 \) the \( f_a \) are semi-weighted homogeneous and define a singularity \( X_1 \) whose analytic type is independent of \( a \). The family \( f_a \) is topologically trivial. Hence \( X_0 \) and \( X_1 \) have the same resolution graph \( \Gamma \), [N, Theorem 2]. We claim that the maximal reductive automorphism group of \( X_1 \) is \( \mathbb{Z}_2 \times \text{Aut}(\Gamma) \). For \( i = 0, 1 \) let \( H_i^* \) be the group of \( \phi \in \text{GL}(3, \mathbb{C}) \) with \( \phi f_i = c \cdot f_i \) for some \( c \in \mathbb{C}^* \). By looking at the polynomial \( f_0 \) one sees that there is a subgroup \( B \subseteq \text{GL}(3, \mathbb{C}) \), isomorphic to \( \text{Aut}(\Gamma) \), centralized by \( \mathbb{C}^* \), intersecting \( \mathbb{C}^* \) trivially and such that \( \mathbb{C}^* \times B \subseteq H_0^* \). It follows from Proposition 2 in section 5 below that \( \mathbb{C}^* \times B = H_0^* \). Looking at \( g \), which is in fact a monomial of (weighted) degree \( \deg g = \deg f_0 + 2 \), one sees that the subgroup \( \mathbb{Z}_2 \times B \) is contained in \( H_1^* \). Now let \( G \) be a maximal reductive automorphism group of \( X_1 \) acting on \((\mathbb{C}^3,0)\) by contact equivalences of \( f_1 \) and containing \( \mathbb{Z}_2 \times B \). For each \( \phi \in G \) there is a unit \( u \), say with constant term \( c \), such that

\[
f_1 \circ \phi = u \cdot f_1 = c \cdot f_0 + c \cdot g + \text{ terms of degree } \geq d + 3
\]  

where \( d = \deg f_0 \). Here we have used that the weights \( w_i \) are \( \geq 3 \). Let \( \phi^i \) be the components of \( \phi \) and write \( \phi^i = \sum \phi_{ij} \) where \( \deg \phi_{ij} = w_i + j \). It follows from \( f_1 \circ \phi = u \cdot f_1 \) via [GHP, Theorem 2.1] that in the sum for \( \phi^i \) only indices \( j \geq 0 \) occur. Writing \( \phi_j = (\phi_{1j}, \phi_{2j}, \phi_{3j}) \) we obtain

\[
f_1 \circ \phi = f_0 \circ \phi_0 + (\partial_x f_0 \circ \phi_0) \cdot \phi_1 + \text{ terms of degree } \geq d + 2
\]  

and \( f_0 \circ \phi_0 = c \cdot f_0 \). In most of the fourteen cases (namely, if \( \text{Aut}(\Gamma) = 1 \)) the only monomial of degree \( w_i \) is \( x_i \). But also in the remaining cases one easily sees that \( f_0 \circ \phi_0 = c \cdot f_0 \) forces \( \phi_0 \) to be linear. Therefore \( \phi \mapsto \phi_0 \) defines a homomorphism \( G \to H_0^* = \mathbb{C}^* \times B \). It is not clear that \( \phi_0 \) is the linear part of \( \phi \). But the usual
linearization trick produces an analytic map germ $\psi$ with $\psi_0 = 1$ and $\phi_0 \circ \psi = \psi \circ \phi$ for all $\phi \in G$. It follows from $\psi_0 = 1$ that the linear part of $\psi$ is triangular with trivial diagonal (if the weights are suitably ordered) and hence that $\psi$ is invertible. Consequently $G$ is mapped isomorphically onto its image in $\mathbb{C}^* \times B$. It remains to show that this image is contained in $\mathbb{Z}_2 \times B$. As $G$ contains $B$ it is enough to consider $\phi \in G$ with $\phi_0 \in \mathbb{C}^*$, say $\phi_0(x) = t \cdot x = (t^{w_1}x_1, t^{w_2}x_2, t^{w_3}x_3)$. Comparing (1) and (2) once more we obtain

$$0 = (\partial_x f_0(t \cdot x)) \cdot \phi_1 = \sum_i t^{d-w_i} \cdot \phi_{i1} \cdot \partial_{x_i} f_0.$$}

Since the partials of $f_0$ form a regular sequence in $O_3$ we conclude that the $\phi_{i1}$ are contained in the Jacobian ideal $j(f_0)$. In each of the fourteen cases, the degree of every partial is larger than $w_i + 1$ for all $i$. Hence $\phi_1 = 0$. Now we have a more precise version of (2):

$$f_1 \circ \phi = f_0(t \cdot x) + g(t \cdot x) + (\partial_x f_0(t \cdot x)) \cdot \phi_2 + \text{terms of degree } \geq d + 3.$$}

We obtain $c \cdot g = t^{d+2} \cdot g + (\partial_x f_0(t \cdot x)) \cdot \phi_2$. The second summand must vanish because otherwise $g \in j(f_0)$ which is not the case. Then $c \cdot g = t^{d+2} \cdot g$ and $c \cdot f_0 = f_0(t \cdot x) = t^d \cdot f_0$ imply $t = \pm 1$ proving the claim.

**Example 4.** Consider the quadrilateral singularities $X$ with graph

```
-\nu

-\nu

-\nu

-\nu
```

Of course, the analytic type depends on the cross ratio of the four intersection points on the central curve $E_0 \simeq \mathbb{P}_1$. The symmetric group $S_4$ acts on $\mathbb{C} - \{0,1\}$ via the cross ratio. The subgroup $\mathbb{Z}_2 \times \mathbb{Z}_2$ consisting of 1 and the three products of two disjoint transpositions acts trivially. The quotient map is

$$j : \mathbb{C} - \{0,1\} \rightarrow \mathbb{C} : \lambda \mapsto \frac{4(\lambda^2 - \lambda + 1)^3}{27\lambda^2(\lambda - 1)^2}.$$}

In particular, if the four intersection points on the central curve have cross ratio $\lambda$ (with respect to some numbering) then $j(X) = j(\lambda)$ is an invariant of the singularity (independent of the numbering). The generic orbit of $S_4$ on $\mathbb{C} - \{0,1\}$ has cardinality six. There are two exceptional orbits: One of them, corresponding to $j = 0$, consists of the two solutions of $\lambda^2 - \lambda + 1 = 0$, and the other, corresponding to $j = 1$, consists of $-1, 2$ and $1/2$. These well known facts are
enough to determine, in each case, the group $A \simeq G/G_1$ of automorphisms of $\mathbb{P}_1$ which permute the intersection points while preserving the self intersection numbers. If $p = q = r = s$ then $A$ coincides with the alternating group $A_4$ for $j(X) = 0$. For $j(X) = 1$ it is dihedral of order 8 and for $j(X) \neq 0,1$ it is $\mathbb{Z}_2 \times \mathbb{Z}_2$. If $p = q = r \neq s$ then $A \simeq \mathbb{Z}_3$ for $j(X) = 0$, $A \simeq \mathbb{Z}_2$ generated by a transposition for $j(X) = 1$, and $A = 1$ for $j(X) \neq 0,1$. For the remaining cases let $\lambda$ be the cross ratio of the four intersection points $p_1, \ldots, p_4$ (in this order) and suppose that the curves intersecting in $p_1$ and $p_2$ have equal self intersection number $p = q$ whereas $r$ and $s$ are different from $p$. There is an automorphism of $\mathbb{P}_1$ interchanging $p_1$ and $p_2$ while fixing $p_3$ and $p_4$ if and only if $\lambda = -1$. If $r \neq s$ we see $A \simeq \mathbb{Z}_2$ for $\lambda = -1$ and $A = 1$ else. If $r = s$ and $\lambda = -1$ then $A \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$ is generated by the two transpositions interchanging $p_1$ and $p_2$, respectively $p_3$ and $p_4$. Finally, if $r = s$ and $\lambda \neq -1$ then $A \simeq \mathbb{Z}_2$ is generated by the product of those two transpositions. The quadrilateral singularities are minimally elliptic with $\varepsilon = 1$, [L3] and [W4]. Hence $G = G_1 \times \tilde{G}$ is a direct product, $X/H$ is rational for finite subgroups $H \leq G$ with $H \not\subset \tilde{G}$ and $X/H$ is minimally elliptic if $H \subset \tilde{G}$. The reader is invited to determine the quotient $X/\tilde{G}$. In each case, it is a triangle or a quadrilateral singularity.

5 The action on the homology of the link

The link of a weighted homogeneous surface singularity $(X,0)$ is a deformation retract of $X - 0$ with $X$ denoting the corresponding affine algebraic variety. The group $G$ acts on $X - 0$. Since the connected subgroup $G_1$ acts trivially on integral homology there is an action of $G/G_1$ on $H_1(X - 0, \mathbb{Z})$.

**Theorem 6.** Let $X$ be weighted homogeneous. Then the group $G/G_1$ acts faithfully on $H_1(X - 0, \mathbb{Z})$.

**Proof.** In the sequel all homology and cohomology modules are with integral coefficients. We need to recall how the homology of the link is expressed in terms of resolution data, see e. g. [LW, section 4]. Let $s$ be the number of components of the exceptional divisor $E$ of the minimal good resolution $\tilde{X}$ and let $g$ be the genus of the central curve $E_0$. (We suppose that $X$ is not a cyclic quotient singularity. For those the proof is left to the reader.) Since $E$ is a deformation retract of $\tilde{X}$ we have $H_2(\tilde{X}) \simeq H_2(E) \simeq \mathbb{Z}^s$ and $H_1(\tilde{X}) \simeq H_1(E) \simeq H_1(E_0) \simeq \mathbb{Z}^{2g}$. Write $L = \tilde{X} - E \simeq X - 0$. Then Lefschetz Duality gives $H_1(\tilde{X}, L) \simeq H^1(E) = 0$ and $H_2(\tilde{X}, L) \simeq H^2(E) \simeq H_2(E)'$ with the prime denoting the dual $\mathbb{Z}$-module. Observe that all identifications are equivariant with respect to $G/G_1$. The exact homology sequence of the pair $(\tilde{X}, L)$ comes down to

$H_2(E) \xrightarrow{j} H_2(E)' \rightarrow H_1(L) \rightarrow H_1(E_0) \rightarrow 0$

yielding the exact sequence

$0 \rightarrow \text{coker } j \rightarrow H_1(L) \rightarrow H_1(E_0) \rightarrow 0$.
Here \( j \) denotes the adjoint of the intersection product:

\[
j(a) : b \mapsto a \cdot b \quad \text{for} \quad a, b \in H_2(E).
\]

As the intersection matrix is negative definite the discriminant group \( \text{coker } j \) is torsion and hence equals the torsion subgroup \( H_1(L)_t \) of \( H_1(L) \). Recall from [FK, V.3.1] that \( \text{Aut } E_0 \) acts faithfully on \( H_1(E_0) \) for \( g \geq 2 \). This is not true for \( g = 1 \) but then it is easily seen that the group \( \text{Aut}_{p_0} E_0 \) of automorphisms fixing a point \( p_0 \in E_0 \) acts faithfully on \( H_1(E_0) \). Now consider the natural homomorphism \( G/G_1 \rightarrow \text{Aut } \Gamma \) and let \( \phi \in G/G_1 \) be trivial on \( H_1(L) \). We claim that triviality on \( H_1(L)_t \) forces \( \phi \) to act trivially on \( \Gamma \). Accepting this for the moment, we are done in case \( g = 0 \) since then \( G/G_1 \rightarrow \text{Aut } \Gamma \) is injective, see Remark (i) of section 2. For \( g \geq 1 \) we conclude that \( \phi \), viewed as an automorphism of the central curve \( E_0 \), has to fix the \( r \) intersection points of \( E_0 \) with other components of \( E \). As \( \phi \) is trivial on the free part \( H_1(E_0) \) of \( H_1(L) \), too, we obtain \( \phi = 1 \) in all cases except for simple elliptic \( X \). Then, by Theorem 3 we have \( G/G_1 \simeq (\mathbb{Z}_b \times \mathbb{Z}_b) \rtimes \text{Aut}_0 E_0 \) with \(-b = E_0 \cdot E_0\). It is known [LW, pp. 282 - 283] that \( \mathbb{Z}_b \times \mathbb{Z}_b \) is mapped isomorphically onto the group \( \text{Hom}(\mathbb{Z}^2, \mathbb{Z}_b) \) of automorphisms of \( H_1(L) \) which act trivially on both \( H_1(L)_t \simeq \mathbb{Z}_b \) and \( H_1(E_0) \simeq \mathbb{Z}_b^2 \). As \( \text{Aut}_0 E_0 \) acts faithfully on \( H_1(E_0) \) we conclude that \( G/G_1 \) acts faithfully on \( H_1(L) \).

Let us prove the claim. For \( i = 1, \ldots, r \) let \( E_{ij}, j = 1, \ldots, i_l \), be the curves on the \( i \)-th branch of the exceptional divisor (counted beginning at the centre), and \(-b_{ij} = E_{ij} \cdot E_{ij} \) as in section 3. Moreover, write \( E_{00} = E_0 \) and \(-b = E_0 \cdot E_0\). Let \( \lambda_{ij} \) be the basis of \( H_2(E)' \) dual to the basis \( E_{ij} \) of \( H_2(E) \). The action of \( G/G_1 \) on \( H_2(E) \) is given by permutation of the curves: \( \phi E_{ij} = E_{\phi(i,j)} \). Thus we have

\[
(\phi \lambda_{ij})(a) = a_{\phi(i,j)} \quad \text{for all } \quad a = \sum a_{ij} E_{ij} \in H_2(E).
\]

Triviality of \( \phi \) on \( H_1(L)_t = \text{coker } j \) means

\[
\phi \lambda - \lambda \in \text{im } j \quad \text{for all } \quad \lambda \in H_2(E)'.
\]

Now assume that \( \phi \) is not trivial on \( \Gamma \). Then one may assume that \( \phi E_{11} = E_{21} \).

There is \( x = \sum x_{ij} E_{ij} \in H_2(E) \) such that \( \phi \lambda_{11} - \lambda_{11} = j(x) \), i.e., \( x \cdot a = a_{21} - a_{11} \) for all \( a \in H_2(E) \). This gives \( x \cdot E_{11} = -1, x \cdot E_{21} = 1, \) and \( x \cdot E_{ij} = 0 \) for all other \((i,j)\). By looking at the curves on the \( i \)-th branch one obtains \( x_{i,i_l-1} = b_{i,i_l} \cdot x_{i,i_l} \), then \( x_{i,i_l-2} = b_{i,i_l-1} \cdot x_{i,i_l-1} - x_{i,i_l} = (b_{i,i_l-1} - 1/b_{i,i_l}) \cdot x_{i,i_l-1} \) and so on up to

\[
x_{00} = \alpha_i/\beta_i \cdot x_{i1} + \gamma_i
\]

where \( \alpha_i/\beta_i \) is the Hirzebruch-Jung continued fraction of the \( i \)-th branch and \( \gamma_i = -1, 1 \) or 0 according to \( i = 1, 2 \) or else. Finally, \( x \cdot E_0 = 0 \) implies

\[
0 = -b \cdot x_{00} + \sum_{i=1}^r x_{i1} = \left( \sum_{i=1}^r \beta_i/\alpha_i - b \right) \cdot x_{00}
\]
because \((\alpha_1, \beta_1) = (\alpha_2, \beta_2)\). The intersection matrix being negative definite we have \(b > \sum_{i=1}^3 \beta_i/\alpha_i\) and \(x_{00} = 0\). Then \(1 = \alpha_1/\beta_1 \cdot x_{11}\) with \(\alpha_1/\beta_1 > 1\) and \(x_{11} \in \mathbb{Z}\) yields a contradiction proving the claim. \(\square\)

Remark. The proof shows that \(G/G_1\) even acts faithfully on the torsion subgroup of \(H_1(X - 0, \mathbb{Z})\) if \(r > 2g + 2\).

Consider a weighted homogeneous normal surface singularity \(X\) which can be embedded as a hypersurface in \((\mathbb{C}^3, 0)\) but which is not an \(A_k\)-singularity. We may choose coordinates such that its maximal reductive automorphism group \(G\) acts linearly on \(\mathbb{C}^3\) and such that \(\mathbb{C}^* = G_1 \leq G\) acts diagonally. Then \(X\) is defined by a weighted homogeneous polynomial \(f\), say of degree \(d\).

**Proposition 2.** In this situation \(G\) equals the group \(H^*\) of \(\phi \in \text{GL}(3, \mathbb{C})\) with \(\phi f = c \cdot f\) for some \(c \in \mathbb{C}^*\). The subgroup \(H\) of \(\phi \in \text{GL}(3, \mathbb{C})\) with \(\phi f = f\) is finite. The intersection \(H \cap G_1 = H \cap \mathbb{C}^*\) is the cyclic group \(\mathbb{Z}_d\) of \(d\)-th roots of unity and \(G/G_1 \simeq H/\mathbb{Z}_d\).

**Proof.** First look at \(H\). This is an algebraic subgroup of \(\text{GL}(3, \mathbb{C})\). So one has to show that its Lie algebra \(\mathfrak{h}\) consisting of all derivations \(D = \sum \lambda_i \partial_{x_i}\) with linear forms \(\lambda_i\) such that \(Df = 0\) is reduced to 0. For \(f \in m^3\) this is shown by a standard argument which can be found at several places in the literature, e. g. in [OS]. Now suppose that \(f \notin m^3\). Since \(f\) does not define an \(A_k\)-singularity we may assume \(f = x_1^3 + g\) with \(g \in m^3\). The argument just mentioned shows that every \(D \in \mathfrak{h}\) is of form \(D = x_1 \sum_{i>1} a_i \partial_{x_i}\) with \(a_i \in \mathbb{C}\). But then \(Df = 0\) clearly implies \(D = 0\). Now turn to \(H^*\). This is also an algebraic subgroup being the image under the projection \(\text{GL}(3, \mathbb{C}) \times \mathbb{C}^* \to \text{GL}(3, \mathbb{C})\) of the algebraic group of all pairs \((\phi, c)\) with \(\phi f = c \cdot f\). Its Lie algebra consisting of all \(D = \sum \lambda_i \partial_{x_i}\) such that \(Df \in \mathbb{C} \cdot f\) is one dimensional spanned by the Euler derivation. As \(\mathbb{C}^* \subseteq H^*\) this implies that \(H^*\) is reductive. Because \(G\) is a group of linear contact equivalences of \(f\) and because \(\mathbb{C}^*\) is central in \(G\) one has \(G \subseteq H^*\). But then \(G = H^*\) by maximality. The remaining assertions are obvious. \(\square\)

**Example 5.** Let \(X\) be defined by \(f = x_1^d + x_2^d + x_3^d\) with \(d \geq 3\) and let \(H, H^*\) be as above. Since the three weights of the \(\mathbb{C}^*\)-action are equal to 1 any automorphism of \((\mathbb{C}^3, 0)\) commuting with \(\mathbb{C}^*\) is linear in the given coordinates. This shows that \(H^*\) is the maximal reductive automorphism group of \(X\), i. e., the coordinates are well chosen, compare Remark (vii) of section 2. We clearly have \(\mathbb{Z}_d^3 \rtimes S_3 \subseteq H\).

To prove equality take \(\phi \in \text{GL}(3, \mathbb{C})\) with \(\phi f = f\). The ideal \(((x_1 x_2 x_3)^{d-2})\) generated by the Hessian of \(f\) is \(\phi\)-stable. Consequently \((x_1 x_2 x_3)\) is \(\phi\)-stable and, after permutation, the ideals \((x_1), (x_2)\) and \((x_3)\) must be \(\phi\)-stable. This shows \(H = \mathbb{Z}_d^3 \rtimes S_3\). We conclude that \(G/G_1 \simeq H/\mathbb{Z}_d\) has order \(6d^2\). When \(d\) is a multiple of 3 we obtain examples where \(G \to G/G_1\) does not admit a section. In fact, then the third root of unity \(\zeta \in \mathbb{C}^* \leq G\) is contained in the commutator subgroup of \(G\), namely \(\zeta = \phi \sigma^{-1}\phi^{-1}\sigma^{-1}\) with \(\phi(x_1, x_2, x_3) = (\zeta^2 x_1, \zeta x_2, x_3)\) and \(\sigma(x_1, x_2, x_3) = (x_2, x_3, x_1)\). Clearly this prevents \(G\) from being a direct product
of $\mathbb{C}^*$ and some subgroup. (Note that for $d = 3$ we are discussing the simple elliptic singularity obtained by contracting the zero-section of a line bundle of degree $-3$ on the elliptic curve of $j$-invariant 0.) On the contrary, if 3 does not divide $d$ then $G$ is a direct product over $\mathbb{C}^*$. To prove this consider the normal subgroup $N = \mathbb{Z}_d^3 \cap \text{SL}(3, \mathbb{C})$ of $\mathbb{Z}_d^3$. It has trivial intersection with the center $\mathbb{Z}_d$ of $H$ because $d$ is not a multiple of 3. Then $|N| = d^2$ implies $\mathbb{Z}_d^3 = \mathbb{Z}_d \times N$. As $N$ is $S_3$-invariant we conclude $H = \mathbb{Z}_d \times B$ for $B = N \rtimes S_3$ and then $G = \mathbb{C}^* \times B$.

Returning to the general situation as described in Proposition 2, choose an $H$-invariant Hermitean inner product on $\mathbb{C}^3$ and let $\bar{B}_\varepsilon$ be the corresponding closed ball of small radius $\varepsilon$. One has the Milnor fibration

$$f^{-1}(\bar{D}_\delta - 0) \cap \bar{B}_\varepsilon \rightarrow \bar{D}_\delta - 0$$

where $\bar{D}_\delta \subseteq \mathbb{C}$ is a small closed disc. Then clearly the group $H$ of right equivalences of $f$ acts on the Milnor fibre $F$. Observe that by an equivariant version of the Ehresmann Fibration Theorem any two Milnor fibres are $H$-equivariantly diffeomorphic. Moreover, let $M = H_2(F, \mathbb{Z})$ equipped with the intersection form be the Milnor lattice, $O(M)$ its group of isometries, and $W \leq O(M)$ the monodromy group.

**Theorem 7.** The homomorphism $H \rightarrow O(M)$ is injective and $H \cap W \simeq \mathbb{Z}_d$, generated by the monodromy operator.

**Proof.** Consider the Jacobian ideal $j(f)$ and $U = \mathcal{O}_3/j(f)$. Clearly $H$ acts on $U$. Wall [W], see also [OS], has constructed an isomorphism $H_2(F, \mathbb{C}) \simeq U' \otimes \Lambda^3 \mathbb{C}^3$ of $H$-modules. Let $\eta \in H$ be trivial on $M = H_2(F, \mathbb{Z})$. As the basis element 1 of $U$ is an eigenvector of eigenvalue 1 we conclude that det $\eta = 1$ and hence that $\eta$ must be trivial on $U$. Consequently $\eta x_i \equiv x_i \mod j(f)$ for the coordinate functions $x_i$. Because $X$ is not an $A_k$-singularity we may assume that $x_1$ is the only linear form contained in $j(f)$. Hence det $\eta = 1$ implies

$$\eta = \begin{pmatrix} 1 & * & * \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Thus $\eta$, having finite order, must be trivial.

As $X$ is defined by a weighted homogeneous polynomial of degree $d$ the $d$-th root of unity in $H$ induces a monodromy diffeomorphism $F \rightarrow F$. Therefore the monodromy operator has order $d$ and is contained in $H$ as acting on $M$. To show $H \cap W \subseteq \mathbb{Z}_d$ consider the exact homology sequence of the pair $(F, \partial F)$. By Lefschetz Duality it looks as follows:

$$H_2(F, \mathbb{Z}) \xrightarrow{j} H_2(F, \mathbb{Z})' \rightarrow H_1(\partial F, \mathbb{Z}) \rightarrow 0$$

where $j$ is the adjoint of the intersection form. The monodromy group $W$ is generated by Picard-Lefschetz transformations. These are reflections at hyperplanes orthogonal to the vanishing cycles. In particular, $W$ acts trivially on
\(M/M'. \) Therefore \(H \cap W\) acts trivially on \(H_1(\partial F, \mathbb{Z})\). Because the fibre bundle \(f^{-1}(\tilde{D}_\delta) \cap S_\varepsilon \to \tilde{D}_\delta\) is trivial and the link \(L = X \cap S_\varepsilon\) is a deformation retract of \(X - 0\) there are \(H\)-equivariant isomorphisms \(H_1(\partial F, \mathbb{Z}) \simeq H_1(L, \mathbb{Z}) \simeq H_1(X - 0, \mathbb{Z})\). Then Theorem 6 implies that any \(\eta \in H \cap W\) must be contained in \(H \cap G_1 = \mathbb{Z}_d\).

\[\square\]

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