A criterion of the existence of an embedding of a monothetic monoid into a topological group

Using properties of unitary Cauchy filters on monothetic monoids, we prove a criterion of the existence of an embedding of such a monoid into a topological group. The proof of the sufficiency is constructive: under the corresponding assumptions, we are building a dense embedding of a given monothetic monoid into a monothetic group.

Keywords: monothetic monoid, Cauchy filter, completion

MSC: 22A15, 54D35

Introduction

The problem of embedding a topological semigroup into a topological group was investigated by a number of authors. A list of papers published before 1990 is contained in [9]. In particular, a necessary and sufficient condition of the existence of such an embedding is proved in [5]. However, it is technical and very cumbersome. A detailed study of the case that a right reversible cancellative topological semigroup embeds as an open subsemigroup in a topological group, is contained in [10]. For commutative semigroups, another condition of the existence of such an embedding has been earlier found in [11]. Amongst of papers published later, we mention the paper [8] where the case of locally compact semigroups is considered. There are also many papers on embeddings of topological semigroups into Lie groups (see, for example, [9]).

In this paper, we prove a necessary and sufficient condition under which a monothetic monoid can be embedded into a topological group. Unlike the cited above papers, our construction does not use the group of quotients of the given monoid. The required group is a unitary extension of this monoid, i.e. such its extension where all its unitary Cauchy filters converge. The concept of such a filter was defined and studied in [1] and [2]. It is a generalization of the concept of a fundamental sequence of reals. The construction of such an extension has been proposed in [3].

In the first section, we study properties of unitary Cauchy filters on a monothetic monoid whose topology satisfies some additional conditions. In the second one, we use the proved properties of these filters in order to show that the mentioned above extension is the required topological group. This group is monothetic, and this embedding is dense.

Unfortunately, we could not write a completely self-contained paper: the number of places where it was necessarily to refer to previous papers [1], [2], and [3] of this series was too large.

*Corresponding Author: Boris G. Averbukh: Bauman Moscow State Technical University, Department of mathematics, Moscow, Russian Federation; E-mail: averbuch@gmx.de
1 C-filters on monothetic monoids

In this preparatory section, we study the topology of a monothetic monoid satisfying some additional requirements and prove properties of C-filters on it. We only consider Hausdorff topological groups and semigroups and use multiplicative notations for them.

A) Let $\mathcal{S}$ be a Hausdorff topological monoid. Its topology is said to be non-viscous (see [1]), if, for any neighborhood $U$ of the neutral element $1$ of $\mathcal{S}$ and for any $a \in \mathcal{S}$, there exists a neighborhood $V$ of $a$ such that, for any elements $a_1$, $a_2 \in V$, if the equation $a_1x = a_2$ or the equation $xa_1 = a_2$ has a solution, then this solution belongs to $U$.

**Proposition 1.1.** If $\mathcal{S}$ is Hausdorff and compact and, for any $a \in \mathcal{S}$, $x = 1$ is the only solution of each of the equations $ax = a$ and $xa = a$, then the topology of $\mathcal{S}$ is non-viscous.

**Proof.** Consider equations of the form $ax = a$. Suppose that there exist $a \in \mathcal{S}$ and a neighborhood $U$ of 1 such that, for any neighborhood $V$ of $a$, there exist $a_1$, $a_2 \in V$ such that the equation $a_1x = a_2$ has a solution which lies outside $U$. Then there exist nets $\{a_1a\}$, $\{a_2a\}$ convergent to $a$ and a net $\{xa\}$ lying outside $U$ such that $a_1xa = a_2a$ for all $a$. If $x$ is a cluster point of $\{xa\}$, then $x \neq 1$ and $ax = a$. \hfill $\square$

It is evident that the topology of a monothetic group and the relative topology of any submonoid of a monoid with a non-viscous topology are non-viscous, too.

The product of any family of topological monoids has a non-viscous Tychonoff product topology if and only if the topology of each factor is non-viscous. Indeed, denote the given family by $\{\mathcal{S}_a\}_{a \in A}$ and its product by $\mathcal{S}$. For any $a \in A$, there exists a standard algebraic and topological embedding of $\mathcal{S}_a$ into $\mathcal{S}$. Therefore, if the topology of $\mathcal{S}$ is non-viscous, then it is also true for $\mathcal{S}_a$. Conversely, let the topologies of all $\mathcal{S}_a$ be non-viscous and $U$ an arbitrary neighborhood of the neutral element $1 \in \mathcal{S}$. For any $a \in A$, there exists a neighborhood $W_a$ of the neutral element $1_a \in \mathcal{S}_a$ such that $W_a \neq \mathcal{S}_a$ only for a finite number of $a$ and $W = \prod W_a \subset U$. Let now $a = \{a_a\}$ be an arbitrary point of $\mathcal{S}$ and, for any $a \in A$, $V_a$ be a neighborhood of $a_a \in \mathcal{S}_a$ such that $x \in W_a$ for any solution $x$ of the equation $a_1x = a_2$ or the equation $xa_1 = a_2$ with coefficients $a_1$, $a_2$ from $V_a$. If $W_a = \mathcal{S}_a$, then we take $V_a = \mathcal{S}_a$. Then $V = \prod V_a$ is a neighborhood of $a$ in $\mathcal{S}$, and if $a_1$, $a_2$ are coefficients from $V$ and $x$ is a solution of one of the equations $a_1x = a_2$ or $xa_1 = a_2$, then $x \in W \subset U$.

$(\mathbb{R}, +)$ with the topology of the Sorgenfrey line (= the right half-open topology whose a base at each point $a$ consists of intervals $[a; a+\varepsilon]$ with $\varepsilon > 0$) is an example of a cancellative topological monoid whose topology is not non-viscous.

For commutative cancellative monoids, one can give another interpretation of this property. Let $\mathcal{S}$ be such a monoid and $\mathcal{B}$ a subset of $\mathcal{S} \times \mathcal{S}$ consisting of pairs $(a_1, a_2)$ for which the equation above has a solution. It is evident that this solution is unique. If $(a'_1, a'_2)$ and $(a''_1, a''_2)$ are such pairs and $x_1$ and $x_2$ are the corresponding solutions, then $a'_1a''_1x_1 = a'_2a''_2$. It means that $\mathcal{B}$ is a submonoid of $\mathcal{S} \times \mathcal{S}$ containing its diagonal. Consider now the map $\phi: \mathcal{B} \to \mathcal{S}$ taking any element $(a_1, a_2) \in \mathcal{B}$ into the unique solution $x$ of the corresponding equation $a_1x = a_2$. It also follows from the formula above that $\phi$ is a homomorphism which takes the diagonal of $\mathcal{S} \times \mathcal{S}$ into the point 1. The property of the topology of $\mathcal{S}$ to be non-viscous means that $\phi$ is continuous at any point of the diagonal.

The following statement shows that the concept of a non-viscous topology can be useful.

**Proposition 1.2.** Any topological monoid with a non-viscous topology which is algebraically a group, is a topological group.

**Proof.** Denote again the considered monoid by $\mathcal{S}$ and show that the mapping $g \to g^{-1}$ sending any $g \in \mathcal{S}$ to its inverse is continuous on $\mathcal{S}$. Let $g_0$ be an arbitrary element of $\mathcal{S}$ and $W$ an arbitrary neighborhood of $g_0^{-1}$. Find a neighborhood $U$ of 1 such that $g_0^{-1}U \subset W$ and a neighborhood $V$ of $g_0$ such that the requirement of the definition of a non-viscous topology is satisfied for $a = g$ and for this $U$. For any $g \in V$, $g_0g^{-1}$ is a solution of the equation $xg = g_0$ and lies in $U$. Hence, $g^{-1} = g_0^{-1}(g_0g^{-1}) \in W$. \hfill $\square$

Let now $\Gamma$ be a cancellative monothetic monoid with a generator $p$, i.e. $p$ is its element such that the set $\{p^n : n \in \mathbb{N}\}$ is dense in $\Gamma$ (see [4]). Such a monoid is commutative. In the following, we always assume that its topology $\tau$ satisfies the following conditions:

(i) the underlying space of $\Gamma$ is $T_3$.
(ii) this topology is non-viscous.

Consider now another property of this topology. For an arbitrary commutative topological monoid \( S \), let \( B \) be a base of its topology at its identity. It was proved in [7] that the family of all sets of the form \( gU \) with \( g \in S \), \( U \in B \) forms a base of a topology on \( S \), and the multiplication in this monoid is continuous in this topology. We will call this topology the GKO-topology corresponding to the initial topology of \( S \). It follows from the continuity of the multiplication in \( S \) that its GKO-topology is finer than the initial one.

Returning to the monothetic monoid \( \Gamma \), denote by \( \Theta \) its submonoid \( \{ p^n : n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\} \} \), by \( \tau \) the restriction of the topology \( \tau \) on this submonoid and by \( \Gamma \) its embedding into \( \Gamma \). When talking about elements from \( \Theta \), we will often omit the symbol \( j \) and write, for example, \( p \in \Gamma \). However, if \( \mathcal{F} \) is a filter on \( \Theta \), then \( \mathcal{F} \) and the filter on \( \Gamma \) with a base \( \{ j(F) : F \in \mathcal{F} \} \) (we denote it by \( j(F) \)) are different filters.

**Proposition 1.3.** The topology \( \tau \) coincides with the corresponding to it GKO-topology on \( \Theta \).

**Proof.** Check that the topology \( \tau \) is finer than its GKO-topology, i.e. for any neighborhood \( U \) of \( 1 \) in \( \Theta \) and for any \( n \in \mathbb{N}_0 \), there exists a neighborhood \( V \) of \( p^n \) in \( \Theta \) such that \( V \subset p^n U \). Indeed, let \( V' \) be a neighborhood of \( p^n \) which only contains elements of the form \( p^k \) with \( k \geq n \), and \( V'' \) its neighborhood such that \( x \in U \) for any solution of the equation \( p^m x = p^{m'} \) with \( p^m, p^{m'} \in V'' \). It exists since \( \tau \) is non-viscous. Set \( V = V' \cap V'' \). Then \( p^{-n} U \subset V \) for any \( p^{-n} U \).

**Corollary 1.4.** All translations in \( \Theta \) are injective open maps.

**Proof.** Let \( U \) run a base of the topology \( \tau \) at the point 1 and \( a \in \Theta \). The translation \( x \rightarrow ax \) in \( \Theta \) is injective since \( \tau \) is cancellative, and takes the base \( \{ xU \} \) at the point \( ax \).

**Corollary 1.5.** For any subset \( B \subset \Theta \) and for any open subset \( A \subset \Theta \), the subset \( AB \) is open in \( \Theta \).

**Proof.** It is evident.

For any \( n \in \mathbb{N} \), denote \( Q_n = \{ 1, p, \ldots, p^n-1 \} \).

**Proposition 1.6.** For any closed in \( \Theta \) set \( F \subset \Theta \), the set \( p^n F \cup Q_n \) is closed in \( \Theta \).

**Proof.** By Corollary 1.4, the set \( p^n(\Theta \setminus F) = p^n \Theta \setminus p^n F \) is open, and the set \( \Theta \setminus p^n F \cup p^n F \cup Q_n \) is closed in \( \Theta \).

**B) Here, we consider \( C \)-filters on \( \Theta \). We refer the reader to papers [1], [2] and [3] containing all necessary definitions and proofs of properties of such filters for arbitrary Hausdorff topological monoids. The contents of papers [1] and [2] are also summarized in the first section of paper [3]. We retain the main notations used in these papers. Quote, nevertheless, the basic definitions.

A net \( S = \{ s_a \}_{a \in A} \) in a Hausdorff topological monoid \( \mathcal{S} \) is called a \( C \)-net if, for each neighborhood \( U \) of \( 1 \), there exists \( a_0 \in A \) and, for each \( a \geq a_0 \), there exists \( a'_0 \in A \) such that \( s_{a'a'} \in U \) for all \( a' \geq a'_0 \).

Here, the hat on top denotes the topological closure (in \( \mathcal{S} \)) and, in the following, in \( \Gamma \). Omitting this accent, we obtain a definition of a strict \( C \)-net. It is a generalization of the concept of a fundamental sequence of reals for topological monoids.

Filters corresponding to (strict) \( C \)-nets are called (strict) \( C \)-filters. Their direct definition is:

A filter \( \mathcal{F} \) on \( \mathcal{S} \) is called a \( C \)-filter if the set \( M_U(\mathcal{F}) = \{ g \in \mathcal{S} : Ug^{-1} \in \mathcal{F} \} \) belongs to \( \mathcal{F} \) for any neighborhood \( U \) of \( 1 \). To define a strict \( C \)-filter, it is necessary to consider sets \( M_1 U = \{ g \in \mathcal{S} : Ug^{-1} \in \mathcal{F} \} \). If \( \mathcal{S} \) is commutative, it is sufficient to consider \( gU \) instead of \( UgU \).

For given \( C \)-filters \( \mathcal{F}_1, \mathcal{F}_2 \), we write \( \mathcal{F}_1 \preceq \mathcal{F}_2 \) if \( M_U(\mathcal{F}_2) \subset \mathcal{F}_1 \) for any neighborhood \( U \) of \( 1 \). It is proved in [1] that it is a quasi-order relation. For the corresponding \( C \)-nets \( S_1, S_2, S_1 \preceq S_2 \) means that, for any \( U \), the set of all \( s_a \) from \( S_1 \) for which elements of \( S_2 \) eventually lie in \( U s_a U \), contains all sufficiently remote elements of this net. \( S_1 \preceq S_2 \) are said to be equivalent (we write \( S_1 = S_2 \)) if \( \mathcal{F}_1 \approx \mathcal{F}_2 \) and \( \mathcal{F}_2 \simeq \mathcal{F}_1 \). The notations \( S_1 \preceq S_2 \) \( (S_1 = S_2) \) and \( S_1 \approx S_2 \) have a similar meaning for strict \( C \)-filters and strict \( C \)-nets.

By Proposition 2.3 from [1], for any \( C \)-filter \( \mathcal{F} \), the intersection of all \( C \)-filters which are equivalent to it, is a \( C \)-filter, too. It is called the least one in this class (or \( C \)-least) and is denoted by \( \mathcal{F}_{\text{lst}} \). For any point \( x \in \mathcal{S} \), \( \hat{x} \) denotes the filter consisting of all subsets containing \( x \). It is a strict \( C \)-filter.

Some of \( C \)-filters on \( \Theta \) converge in \( \Theta \). For these filters and for some others, their images under the map \( j \) converge in \( \Gamma \), and such ones can exist that their images under this map diverge.

**Proposition 1.7.** Let \( \mathcal{F} \) be a filter on \( \Theta \). The filter \( j(\mathcal{F}) \) is a \( C \)-filter on \( \Gamma \) if and only if \( \mathcal{F} \) is a \( C \)-filter. For any \( C \)-filters \( \mathcal{F}_1 \) and \( \mathcal{F}_2 \) on \( \Theta \), \( j(\mathcal{F}_1) \approx j(\mathcal{F}_2) \) if \( (j(\mathcal{F}_1) = j(\mathcal{F}_2)) \) holds if and only if \( \mathcal{F}_1 \approx \mathcal{F}_2 \) (respectively, \( \mathcal{F}_1 \preceq \mathcal{F}_2 \)).

**Proof.** These statements follow immediately from Proposition 2.1 from [2] since \( \Theta \) is dense in \( \Gamma \).
Proposition 1.8. Any $c$-least $C$-filter on $\Theta$ has a base consisting of closed sets.

Proof. Let $F$ be such a filter. It was proved in [1] (see Lemma 2.10) that it is generated by sets of the form $M_U = M_U(\mathcal{F})$ and $xU$ where the line on top denotes the topological closure in $(\Theta, \tau)$, $U$ runs any base of $\tau$ at the point 1, and $x$ runs (for any $U$) an arbitrary set $M_U$ with $U' \prec U$. This inequality means that there exists a neighborhood $O$ of 1 such that $UO \subset U$, and it implies that $M_U \subset M_U$.

Denote by $M'_U$ the set $\cap_{U'<U} M_{U'}$. Then $M'_U \supset M_U$, and, therefore, $M'_U \in \mathcal{F}$ for any $U$. Moreover, if $U$, $U'$ are neighborhoods of 1 such that $U' \prec U$, then $M'_{U'} \subset M_U$, and that’s why the sets $M'_U$ and above sets of the form $xU$ generate $\mathcal{F}$, too. Therefore, we only need to prove that all $M'_U$ are closed in $\Theta$. Let $x \in \overline{M'_U}$ for some $U$. Then $xO_1 \cap M'_U \neq \emptyset$ for any neighborhood $O_1$ of 1 by Proposition 1.3. Show that $x \in M_{U'}$ for any $U'$ with $U \prec U'$. Denote by $O$ a neighborhood of 1 such that $UO \subset U'$ and assume that $O_1^U \subset O$. Set $U'' = UO_1$. Then $U \prec U''$ and $xO_1 \cap M_{U''} \neq \emptyset$. Therefore, there exists $y \in xO_1$ such that $yU'' \in \mathcal{F}$. It implies that $\overline{xO_1U''} = xO_1^U \in \mathcal{F}$, $xU''$ belongs to $\mathcal{F}$, too, and $x \in M_{U''}$.

\[\square\]

Corollary 1.9. A $C$-filter on $\Theta$ converges to $x \in \Theta$ if and only if it is equivalent to $x$.

Proof. It is straightforward that the following properties are equivalent: a) a $C$-filter $F$ converges to $x$; b) the corresponding $c$-least $C$-filter $F_{\text{lst}}$ converges to $x$ (see Corollary 1.10 from [1]); c) any closed set from $F_{\text{lst}}$ contains $x$; d) any set from $F_{\text{lst}}$ contains $x$ (see the previous Proposition); e) $F_{\text{lst}} \subset \dot{x}$; f) $F_{\text{lst}} = \dot{x}$ (see Proposition 2.2 from [1]); g) $F = \dot{x}$.

We will say that a $C$-filter $F$ on $\Theta$ is trivial if there exists $x \in \Theta$ such that $F \subset \dot{x}$. By Propositions 2.2 and 1.9 from [1], in this case, the filters $F$ and $\dot{x}$ are equivalent, and $F$ converges to $x$. A $C$-filter on $\Theta$ is trivial if and only if there exists $x \in \Theta$ such that members of the corresponding net $S(F)$ are frequently equal to $x$. Such nets are said to be trivial, too.

Remark 1.10. For convergent strict $C$-filters on $\Theta$, the notions $=_{\Theta}$ and $\approx_{\Theta}$ do not coincide. Indeed, by Propositions 1.12 and 1.14 below, for any $x \in \Theta$, there exists a non-trivial strict $C$-net $S_x$ converging to $x$. The filters $\mathcal{F}(S_x)$ and $\dot{x}$ are equivalent as $C$-filters but not as strict ones since $\mathcal{F} \cup \dot{x}$ does not take place. In particular, it implies that, by Propositions 2.2 and 2.3 from [1], the intersection of these filters is a $C$-filter converging to $x$ but not a strict one.

A strict $C$-filter $F$ on $\Theta$ converges to $x \in \Theta$ if and only if $x \in \dot{x}$. Indeed, it is straightforward that the following properties are equivalent: a) $F$ converges to $x$; b) any neighborhood of $x$ belongs to $F$; c) the set $xU$ belongs to $F$ for any neighborhood $U$ of 1 in $\Theta$; d) $x \in M_1(\mathcal{F})$ for any such $U$; e) $M_1(\mathcal{F}) \in \mathcal{F}$ for any such $U$; f) $x \in \dot{x}$.

Lemma 1.11. Let $S = \{p^m\}_{a \in A}$ with $m_a \in \mathbb{N}_0$ for all $a$ be a non-trivial $C$-net in $\Theta$. Then, for any $m \in \mathbb{N}_0$, there exists $a(m) \in A$ such that $m_a \geq m$ for any $a \geq a(m)$.

Proof. For any $k = 0, \ldots, m - 1$, there exists $a_k$ such that $m_a \neq k$ for all $a \geq a_k$. Otherwise, $\emptyset \subset \dot{x}$ for some $x = 1, \ldots, p^{m-1}$. We can now take as $a(m)$ any arbitrary $a$ with $a \geq a_k$ for all these $k$.

\[\square\]

Proposition 1.12. Any non-trivial $C$-filter on $\Theta$ is strict. If $F_1, F_2$ are non-trivial $C$-filters on $\Theta$ such that $F_1 \supsetneq F_2$ ($F_1 = F_2$), then $F_1 \supsetneq F_2$ (respectively, $F_1 = \dot{F}_2$).

Proof. Let $F$ be such a filter, $U$ an arbitrary neighborhood of 1 in $\Theta$, and $x = p^n \in M_U = M_U(\mathcal{F})$. As above, denote $Q_n = \{1, p, \ldots, p^{n-1}\}$. Since the set $xU \cup Q_n$ is closed by Proposition 1.6 and contains $xU$, it then contains $xU$. Therefore, the inclusion $xU \in \mathcal{F}$ implies $xU \cup Q_n \in \mathcal{F}$. For any $k = 0, \ldots, n - 1$, it follows from $r \subset p^k$ that there exists a set $F_k \in \mathcal{F}$ such that $p^n, F_k \cap F_k$. Set $F = (xU \cup Q_n) \cap F_0 \cap \cdots \cap F_{n-1}$. This set belongs to $F$ and lies in $xU$. It implies $xU \in F$.

Let now $V$ be an arbitrary neighborhood of 1 in $\Theta$ and $U$ its neighborhood such that $U \subset V$. If $x \in M_U$, then it follows from $xU \in F$ that $xV \in F$ and $x \in M_U(\mathcal{F})$, i.e., $M_U(\mathcal{F}) \subset M_U(\mathcal{F})$, $M_U(\mathcal{F}) \in F$, and $F$ is a strict $C$-filter.

If $F_1 \supsetneq F_2$ for non-trivial $C$-filters $F_1$ and $F_2$, then $M_U(F_2) \in F_1$. Therefore, $M_U(F_2) \in F_1$ for any neighborhood $V$ of 1 and $F_1 \supsetneq F_2$.

\[\square\]

Corollary 1.13. Each equivalence class of $C$-filters on $\Theta$ contains a strict $C$-filter.

Proof. The topology $\tau$ is $T_1$, therefore, by Corollary 1.10 from [1], either all $C$-filters from this class converge to the same $x \in \Theta$ or all diverge in $\Theta$. In the first case, they are equivalent to the strict $C$-filter $\dot{x}$, in the second one, they are non-trivial and that’s why strict ones.

\[\square\]

Proposition 1.14. (i) For any element $a \in \Gamma$, there exists a non-trivial $C$-filter $F$ on $\Theta$ such that $a = \lim j(F)$.
(ii) If $\mathcal{F}_1$, $\mathcal{F}_2$ are non-trivial $C$-filters on $\Theta$ (they are strict ones) such that $\lim j(\mathcal{F}_1) = \lim j(\mathcal{F}_2)$ (the condition "non-trivial" is automatically satisfied if this limit does not belong to $\Theta$), then $\mathcal{F}_1 = \sqsupseteq \mathcal{F}_2$.

Proof. (i) Let $a \not\in \Theta$ or $a = 1$ and $(n_a)_{a \in A}$ be a net in $\mathbb{N}$ such that the net $D = \{p^\mu\}_{a \in A}$ converges to $a$. Prove that $D$ is a $C$-net. Denote by $U$ an arbitrary neighborhood of $1$ in $\Gamma$ and find a neighborhood $V$ of $a$ possessing, for this $U$, the property from the above definition of a non-viscous topology. There exists $a_0 \in A$ such that $p^\mu u \in V$ for all $\mu \geq a_0$. For each $\mu \geq a_0$, there exists $a'_0 \geq a_0$ such that $n_\mu > n_\mu$ if $a' \geq a'_0$. Then $p^{\mu u - n_\mu' u} \in U$ and $p^{\mu u'} \in p^\mu U$ for such $a$ and $a'$, i.e. $D$ is a strict $C$-net. It is non-trivial since it cannot cluster to any $p^\mu u$ with $n \in \mathbb{N}$. To find a required filter for $a = p^\mu u$ with $n \in \mathbb{N}$, it suffices to move the net $D$ corresponding to $a = 1$ by means of the translation $x \rightarrow p^\mu x$.

(ii) Denote $a = \lim j(\mathcal{F}_1)$ and consider nets $S(\mathcal{F}_1)$, $S(\mathcal{F}_2)$ corresponding to the given filters. For the same $U$ and $V$, if $p^\mu m$ is an element of the first net lying in $V$, then, by Lemma 1.11, elements $p^\mu m$ of the second one eventually satisfy the conditions $m \geq m_0$ and $p^\mu m \in V$. Therefore, $p^\mu m \in p^\mu m U$ and $S(\mathcal{F}_1) \subseteq S(\mathcal{F}_2)$. □

Also hereinafter, it will be sometimes more convenient for us to consider nets instead of filters. In particular, for any element $t$ from $\Gamma \setminus \Theta$, we will use the net $s(t) = \{s_\delta \mid \delta \in M(t)\}$ in $\Theta$ constructed as follows. The $\Delta(t)$ is the set of all couples $(W, p^\mu)$ each of which is composed of a neighborhood $W$ of $t$ and of an element of the form $p^\mu y$ lying in $W$. We set $\delta_1 = (W_1, p^\mu_1) \leq \delta_2 = (W_2, p^\mu_1)$ if $W_1 \supset W_2$. It is easy to see that $(\Delta(t), \leq)$ is a directed set. For $\delta = (W, p^\mu)$, denote $s_\delta = p^\mu W$. It is straightforward that $t = \lim j(s(t))$ and $j(s(t))$ does not have other cluster points. For any $\delta = (W, p^\mu)$, denote $m(\delta) = \min \{k \in \mathbb{N} : p^k \in W\}$. It only depends on $W$, and we write sometimes $m(W)$. This $m(\delta)$ is an increasing and unbounded function on $\Delta(t)$: $\delta \leq \delta_2$ implies $m(\delta_1) \leq m(\delta_2)$ and, for any $m \in \mathbb{N}$, there exists $s_\delta \in \Delta(t)$ such that $m(\delta) \leq m$. Let $\mathcal{F} = j(S(t))$ be a filter on $\Theta$ corresponding to the net $s(t)$. It is generated by all sets of the form $\{s_\delta : \delta \leq s_\delta\}$ with $s_\delta \in \Delta(t)$. For $s_\delta_0 = (W_0, p^\mu_0)$, it is the set all $p^\mu u$ lying in $W_0$, or equivalently, the set $W_0 \cap \Theta$.

Lemma 1.15. For any $t \in \Gamma \setminus \Theta$, $s(t)$ is a strict $C$-net, and $j(s(t))$ is a $C$-least strict $C$-filter.

Proof. Let $U$ be an arbitrary neighborhood of the neutral element $1$ in $\Theta$ and $U'$ its neighborhood in $\Gamma$ such that $U' \cap \Theta = U$. There exists a neighborhood $V$ of $t$ such that $x \in U'$ for any solution $x$ of the equation $a_1 x = a_2$ with coefficients $a_1, a_2 \in V$. Denote $\delta_0 = (V, p^m(V))$ and consider an arbitrary $\delta = (W, p^m) \geq \delta_0$. Let now $\delta_0' = (W_0, p^m_0)$ satisfy the conditions $\delta_0' \leq \delta$ and $m(\delta_0') \geq m$. For $\delta' = (W', p^m') \geq \delta_0'$, we have $W' \subset W_0 \subset W \subset V$ and $m' \geq m(\delta_0') \geq m$. Therefore, $p^m, p^m \in V, p^m \leq m \in U$ and $s_\delta = p^m \in p^m U = s_\delta U$ for all $\delta' \geq \delta_0$, i.e. $s(t)$ is a strict $C$-net.

Let now $\mathcal{F}$ be a $C$-filter on $\Theta$ such that $j(\mathcal{F})$ converges to the same $t$. Then any neighborhood $W$ of $t$ belongs to $j(\mathcal{F})$. Therefore, the set $W \cap \Theta$ belongs to $\mathcal{F}$ for any $W$ and $\mathcal{F} \supseteq j(S(t))$. □

Proposition 1.16. Any non-trivial $C$-least $C$-filter on $\Theta$ has a base consisting of open sets.

Proof. Let $\mathcal{F}$ be such a filter. By Proposition 1.12, it is strict. Similarly to Lemma 2.10 from [1], one can prove that it is generated by families of sets of the form $M_\mu^\alpha = M_\mu^\alpha(\mathcal{F})$ and $x U$ where $U$ runs a base of neighborhoods of $1$ in $\Theta$, and $x$ runs, for any $U$, an arbitrary set $M_\mu^\alpha$ with $U \prec x U$. This inequality means that there exists a neighborhood $O$ of $1$ in $\Theta$ such that $U' O \subset U$.

All the sets $x U$ are open, therefore, it suffices to prove that the interior of the set $M_\mu^\alpha$ in $\Theta$ belongs to $\mathcal{F}$ for any $U$. For given $U$, let $V$ be a neighborhood of $1$ in $\Theta$ such that $p^m \in V$ follows from $p^m \in V$ and $m \geq m_0 \geq m$. Take an arbitrary $x = p^k u \in M_\mu^\alpha$. Then $x V \in \mathcal{F}$. Denote by $S = S(\mathcal{F}) = (x_a)_{a \in A}$ the strict $C$-net in $\Theta$ corresponding to $\mathcal{F}$ and by $y = p^\mu$, $n \geq k$, an arbitrary element from $x V$. There exists $a_0 \in A$ such that $a \geq a_0$ implies $x_a = p^m u_{x_{\mu}}$ and $m_a \geq m$. For such $a$, we have $p^m_{x_{\mu} - x_{\mu} - x_{\mu}} \in V$ and $p^m_{x_{\mu} - U} \in U$, i.e. $x_a \in y U$. Hence, $y U \in \mathcal{F}$, $y \in M_\mu^\alpha$ and $x V \subset M_\mu^\alpha$ for any $x \in M_\mu^\alpha$, i.e. $V M_\mu^\alpha \subset M_\mu^\alpha$. In particular, it means that $M_\mu^\alpha$ lies inside $M_\mu^\alpha$ and that’s why the interior of $M_\mu^\alpha$ in $\Theta$ belongs to $\mathcal{F}$.

In the following, we will also use the next lemma.

Lemma 1.17. For any $C$-filter $\mathcal{F}$ on $\Theta$ and for any neighborhood $U$ of $1$ (in the topology $\tau$), there exists an open (in this topology) $V \in \mathcal{F}$ such that $x \in U$ for any $a_1, a_2 \in V$ and for any solution $x$ of the equation $a_1 x = a_2$ lying in $\Theta$.

Proof. If $\mathcal{F}$ converges to a point $a$, then we can take as $V$ a neighborhood of $a$ possessing this property. It exists since the topology $\tau$ is non-viscous.

Let now $\mathcal{F}$ diverge in $\Theta$. Since any $C$-filter contains the corresponding $C$-least one, then we may assume that $\mathcal{F}$ is itself a non-trivial $C$-least $C$-filter. For given $U$, denote by $U'$ a neighborhood of $1$ in $\Theta$ such that $y \in U$
for any $x_1, x_2 \in U'$ and any solution $y$ of the equation $x_1 y = x_2$. Let $S(\mathcal{F}) = \{s_\alpha\}_{\alpha \in A}$ be the strict $C$-net in $\Theta$ corresponding to $\mathcal{F}$. Then there exists $a_0 \in A$ such that, for any $\alpha \geq a_0$, there exists $a'_\alpha$ such that $s_{a'\alpha} \in s_\alpha U'$ for any $a' \geq a'_\alpha$. Fix any such $a$ and take arbitrary $a_1 = s_{a'\alpha}$, $a_2 = s_{a'\alpha}$ with $a_1, a_2' \geq a'_\alpha$. Denote by $l, m, n$ naturals such that $s_\alpha = p^l$, $a_1 = p^m$ and $a_2 = p^n$. By Lemma 1.11, we may assume that $m, n \geq l$. If $m > n$, then the equation $a_1 x = a_2$ does not have solutions lying in $\Theta$. If $n > m$, then $x = p^{m-n}$ is the only solution. In this case, we have $p^m \in p^l U'$, $p^n \in p^l U'$ and $p^{m-n}$ is also a solution of the equation $a_1 x = a_2$ with $a_1' = p^{m-l}$ and $a_2' = p^{n-l}$ from $U'$. It implies that $x \in U$. Now, the set $F = \{s_\alpha\}_{\alpha \geq a_0}$ belongs to $\mathcal{F}$, and an arbitrary open member of $\mathcal{F}$ lying in $F$ is suitable as $V$. It exists by the previous proposition.

**Proposition 1.18.** For any non-trivial $C$-least $C$-filter $\mathcal{F}$ on $\Theta$ and for any $F_1 \in \mathcal{F}$, there exist an open $F_2 \in \mathcal{F}$ and a neighborhood $O$ of 1 in $\Theta$ such that $F_2 O \subset F_1$.

**Proof.** Consider the same family of generators of the filter $\mathcal{F}$ as in the proof of Proposition 1.16. If $F_1 = xU$ with $x \in M^1_{U'}$, where $U' \prec\prec U$ and $O$ is a neighborhood of 1 such that $U' O \subset U$, then we can set $F_2 = xU'$. In general case, $F_1$ contains the intersection of a finite number of sets of the form $M^1_{U'}$ and $xU$ where $U$ runs a base of neighborhoods of 1 in $\Theta$, and $x$ runs, for any $U$, an arbitrary set $M^1_{U'}$ with $U' \prec\prec U$. If $U_1 \subset U_2$, then $M^1_{U_1} \subset M^1_{U_2}$. Therefore, we can assume that not more than one set of the form $M^1_{U'}$ takes part in this intersection, and it has the form $M^1_{U_0} \cap x_1 U_1 \cap \cdots \cap x_n U_n$ where $U_0, \ldots, U_n, U'_1, \ldots, U'_n$ are neighborhoods of 1 in $\Theta$ and $x_k \in M^1_{U_k'}$ with $U_k' \prec\prec U_k$ for $k = 1, \ldots, n$. For each of these $k$, let $O_k$ be a neighborhood of 1 such that $U'_k O_k \subset U_k$, and denote by $V$ a neighborhood of 1 such that $V M^1_{U_0} \subset U_0$ (see the proof of Proposition 1.16 again), and by $O_k$ its neighborhood such that $\cap O_k \subset V$. Set now $O = O_0 \cap O_1 \cap \cdots \cap O_n$ and $F_2 = O M^1_{U'} \cap x'_1 U'_1 \cap \cdots \cap x_n U'_n \in \mathcal{F}$. Then it is easy to see that $F_2$ is open and $F_1 \subset F_2 O$.

**C.** In this section, we prove the main lemma of this paper.

**Lemma 1.19.** For any divergent strict $C$-net $\mathcal{F}$ in $\Gamma$, there exists a strict $C$-net $S$ in $\Theta$ such that $j(S) \subset \mathcal{F}$.

**Proof.** Denote $\mathcal{F} = \{t_\sigma\}_{\sigma \in \Sigma}$. We may assume that $t_\sigma \notin j(\Theta)$ for all $\sigma$ since, otherwise, there would exist a finer than $\mathcal{F}$ net $\mathcal{G}$ which would be included under the map $j$ of a net $\mathcal{G}$ in $\Theta$. By Proposition 2.2 from [1], this net $\mathcal{G}$ would be an equivalent to $\mathcal{F}$ net in $\Gamma$, and $S$ would be a $C$-net by Proposition 1.7. If this net would be trivial, its members would be frequently equal to some $x \in \Theta$, then it would be also true for $\mathcal{G}$ and $\mathcal{F}$. However, this is impossible since $\mathcal{F}$ is divergent and can not cluster by Corollary 1.10 from [1]. Hence, $S$ would be the required strict $C$-net by Proposition 1.12 above.

It follows now from this assumption that, for any $\sigma$, there exists a strict $C$-net $S_\sigma = S(t_\sigma) = \{s_{\sigma\delta}\}_{\delta \in \Delta_\sigma}$ in $\Theta$ constructed as it was described before Lemma 1.13 and such that $\lim j(s_{\sigma\delta}) = t_\sigma$. The directed set $\Delta_\sigma$ is the set of all couples $(W, p^0)$ each of which consists of a neighborhood $W$ of $t_\sigma$ and of an element of the form $p^m$ lying in $W$. Observe that any $S_\sigma$ is non-trivial, and, therefore, the statement of Lemma 1.11 is true for it.

Using these nets $S_\sigma$, we will construct the required strict $C$-net $S$. For that, consider the set $F$ of all maps $f : \Sigma \to \bigcup_{\sigma \in \Sigma} \Delta_\sigma$ such that $f(\sigma) \in \Delta_\sigma$ for any $\sigma$. Denote by $B$ the set of all couples of the form $(\sigma, f)$ with $\sigma \in \Sigma$, $f \in F$. For couples $\beta_0 = (\sigma_0, f_0), \beta_1 = (\sigma_1, f_1), \beta_2 = (\sigma_2, f_2)$, set $\beta_0 \leq \beta_2$ if $\sigma_0 \leq \sigma_2$ and $f_1(\sigma) \leq f_2(\sigma)$ for all $\sigma \in \Sigma$. Then it is straightforward that $(B, \leq)$ is a directed set. For any $\beta = (\sigma, f) \in B$, set $s_{\beta} = s_{\sigma f(\sigma)}$. We will prove that $\{s_\beta\}_{\beta \in B}$ is the required net $S$.

**Remark 1.20.** We used a well-known construction here. For example, see exercise 1.6.B from [6].

First, show that $S$ is a strict $C$-net. For any neighborhood $U$ of 1 in $\Theta$, we need to find $\beta_0 = (\sigma_0, f_0) \in B$ such that, for any $\beta \geq \beta_0$, there exists $\beta_0' \in S_{\sigma_0} U$ for any $\beta' \geq \beta_0'$. Denote by $U_1$ a neighborhood of 1 in $\Theta$ such that $U_1' \subset U$. Since $\Theta$ is a $T_1$-space, its topology has a base consisting of canonically open sets, i.e., open sets which coincide with the interiors of their closures in this space. Therefore, we may assume that $U$ and $U_1$ are canonically open. Set $U_1' = 1^1 U_1$. Here, the symbol int and the hat on top denote the interior and the closure of a considered set in $\Gamma$. $U_1'$ is a neighborhood of 1 in $\Gamma$ containing $U_1 = U_1' \cap \Theta$ as a dense subset. For any $\sigma \in \Sigma$, let $V_\sigma$ be a neighborhood of $t_\sigma$ such that $x \in U'_{\sigma}$ for any solution of the equation $a_1 x = a_2$ with coefficients from $V_\sigma$. Set $f_0(\sigma) = (V_\sigma, p^k)$ where $k_0 = \min \{k \in N_0 : p^k \in V_\sigma\}$. It is a function from $F$. Choose $\sigma_0$ so that, for any $\sigma \geq \sigma_0$, there exists $\sigma_0'$ such that $t_{\sigma'} = u_{\sigma\sigma'} t_\sigma$ with $u_{\sigma\sigma'} \in U_1$ for any $\sigma \geq \sigma_0$. It is possible since $\mathcal{F}$ is a strict $C$-net in $\Gamma$.

Let now $\beta = (\sigma, f) \geq \beta_0 = (\sigma_0, f_0)$. It follows from $f \geq f_0$ that $s_\beta = s_{\sigma f(\sigma)} \in V_\sigma$. All members of $S_\sigma$ have the form $p^m$ with $m \in N_0$, and these exponents $m$ form an increasing and unbounded function on $\Delta_\sigma$. The
2 Properties of a weakly unitary completion of the monoid $\Theta$. The main theorems

A) In this section, we prove our first theorem.

Before starting, recall the necessary definitions and some results of papers [1], [2], and [3]. A Hausdorff topological monoid is said to be weakly unitarily complete if all its strict $C$-filters converge. Let $X$ be a Hausdorff monoid, $\gamma$ a weakly unitarily complete monoid and $f : X \to \gamma$ an algebraic and topological embedding. The couple $(f, \gamma)$ is called a weakly unitary completion of $X$ if $\gamma$ properly contains no weakly unitarily complete submonoid containing $f(X)$.

For an arbitrary commutative Hausdorff monoid on a $T_1$-underlying space, a construction of its weakly unitary completion was described in paragraph 3 of paper [3]. For the monoid $\Theta$, it looks as follows. Denote by $\bar{\Theta}$ the set of all equivalence classes of $C$-filters on $\Theta$. It was proved above that each of them contains a strict $C$-filter. In place of classes of $C$-filters, one can consider the corresponding to them $C$-least $C$-filters. For an arbitrary $C$-filter $F$ on $\Theta$, we denote by $[F]$ the corresponding point from $\bar{\Theta}$. There exists a canonical embedding $i$ of $\Theta$ into $\bar{\Theta}$: for any $x \in \Theta$, $i(x)$ is the class of the filter $x$.

Since $\Theta$ is commutative, the product of $C$-filters on $\Theta$ is such a filter, too (see Corollary 2.6 from [2]). For arbitrary such filters $F_1, F_2$, denote by $[F_1] \cdot [F_2]$ the class of the filter $F_1 \cdot F_2$. This is a commutative multiplication in $\bar{\Theta}$ (see [2], Proposition 2.9). Its neutral element $1_\Theta$ is equal to $i(1)$, and $i$ is an identity preserving map. Show that it is an algebraic embedding of monoids. Indeed, it is evident that the inclusion $(x_1x_2) \subset (x_1)_{\text{int}}(x_2)_{\text{int}}$ is true for any $x_1, x_2 \in \Theta$. By Proposition 2.2 from [1], these $C$-filters are equivalent, and their corresponding $C$-least filters coincide. Therefore, $i(x_1x_2) = i(x_1) \cdot i(x_2)$ for any $x_1, x_2 \in \Theta$.

The underlying set of $\bar{\Theta}$ endowed with the family $\Sigma$ of all $C$-filters on this monoid forms a Cauchy space whose convergence structure defines a Hausdorff topology on $\Theta$ (see [1], Theorem 3.2). Since $\tau| T_3$, then this so-called unitary topology is finer than $\tau|$. There exists a family $\tilde{\Sigma}$ of filters on the set $\bar{\Theta}$ such that this set endowed with this family forms the Wyler completion of $(\Theta, \Sigma)$. Topological properties of this space are studied in [2]. In particular, the convergence structure of the space $(\Theta, \tilde{\Sigma})$ defines a Hausdorff topology on the set $\tilde{\Theta}$ which is said to be natural. For any filter $F \in \Sigma$, the filter $i(F)$ belongs to $\tilde{\Sigma}$ and converges to the point $[F]$ in the natural topology, and each point
of $\Theta$ is the limit of such a filter. In the couple of topologies (the unitary one, the natural one), the map $i$ is a homeomorphism of $\Theta$ onto an open subspace of $\tilde{\Theta}$ (see Theorem 1.9 from [2]).

In the following proof, we use also some of the ideas of the proofs of Theorems 3.2 and 3.7 from [3] but not these theorems themselves since we are going to get a stronger result. In particular, we construct another topology on $\tilde{\Theta}$ which we denote, as in [3], by $\tau_c$, and use a family $\{\mathcal{O}_i\}_{i \in \tilde{\Theta}}$ of filters of open subsets of $\Theta$ for that.

Let $\mathcal{F}$ be a strict $C$-filter on $\Theta$ belonging to an equivalence class of $C$-filters $\bar{x}$. If it converges to a point $x_0 \in \Theta$, then we denote by $\mathcal{O}_{\mathcal{F}}$ the set of all neighborhoods of this point $x_0$ in the space $\Theta$. If $\mathcal{F}$ diverges in $\Theta$ (it is non-trivial in this case), then $\mathcal{O}_{\mathcal{F}}$ is the set of open subsets of $\Theta$ belonging to the corresponding $C$-least $\mathcal{F}_{\text{lst}}$. It is also divergent, non-trivial and strictly by Proposition 1.12. In any case, $\mathcal{O}_{\mathcal{F}}$ lies in $\mathcal{F}$ and only depends on $\bar{x}$. Therefore, we also denote it by $\mathcal{O}_x$.

Show that $\mathcal{O}_{\tilde{x}_1} = \mathcal{O}_{\tilde{x}_2}$ implies $\tilde{x}_1 = \tilde{x}_2$. Let $\mathcal{F}_1$, $\mathcal{F}_2$ be strict $C$-filters from classes $\tilde{x}_1$, respectively, $\tilde{x}_2$. Consider two cases. First, let $\mathcal{F}_1$ converge to a point $x \in \Theta$. Then $\mathcal{O}_{\tilde{x}_1} = \mathcal{O}_{\tilde{x}_2}$ means that $\mathcal{F}_2$ converges to the same point, and it involves $\tilde{x}_1 = \tilde{x}_2$ by Corollary 1.9. If both these filters diverge in $\Theta$, then, by Proposition 1.16, non-trivial $C$-least $\mathcal{F}_{\text{lst}}(\mathcal{F}_1)_{\text{lst}}$ and $\mathcal{F}_{\text{lst}}(\mathcal{F}_2)_{\text{lst}}$ have the same bases, and $\tilde{x}_1 = \tilde{x}_2$ again.

Define now the topology $\tau_c$. Similarly to section 3.A) from [3], for any non-empty open $V \subset \Theta$, set

$$V' = \{x \in \tilde{\Theta} : V \in \mathcal{O}_x\}.$$  

It is easy to check that $V'_1 \cap V'_2 = (V_1 \cap V_2)'$ for arbitrary open $V_1, V_2 \subset \Theta$, and it means that the family of all sets of the form $V'$ forms a base of a topology on $\tilde{\Theta}$ which we denote by $\tau_c$.

We could define this topology differently. By Corollary 1.9, any convergent to $x \in \Theta$ $C$-filter is equivalent to $\bar{x}$. Therefore, $\tilde{\Theta}$ is the union of its subset $i(\Theta)$ and its subset $Y$ consisting of classes of divergent $C$-filters on $\Theta$ or, it is equivalent, of $C$-least $\mathcal{F}'$'s on this monoid. All these filters are strict and have bases consisting of closed sets (Propositions 1.12 and 1.8 above). The latter property means that we are in the situation which was described in section 2 C) of paper [3]. Therefore, we can define a topology on the set $i(\Theta) \cup Y$ as follows. For any open $V \subset \Theta$, we denote by $V^*$ the set $i(V) \cup \{\mathcal{F} \in Y : V \in \mathcal{F}\}$. For $C$-least divergent $\mathcal{F}$ on $\Theta$ and open subsets $V \subset \Theta$, the requirements $V \in \mathcal{O}_{\mathcal{F}}$ and $V \in \mathcal{F}$ coincide. Moreover, the inclusions $x \in V$ and $V \in \mathcal{O}_i(x)$ are equivalent for any $x$ and for any open $V$ from $\Theta$. Hence, these different definitions of sets $V^*$ lead to the same result. and, for $X = \Theta$, the space $(\tilde{\Theta}, \tau_c)$ coincides with the space $\nu X$ which was defined in section 2.C) of paper [3]. In the following, we will also use the denotation $\nu \Theta$ for this space.

The latter description of the topology $\tau_c$ is more suitable for the characterization of closed sets. As in [3], for each closed $Z \subset \Theta$, denote $Z_* = \tilde{\Theta} \setminus (\Theta \setminus Z)^*$. This closed in the topology $\tau_c$ subset consists of $i(Z)$ and of all point $[\mathcal{F}] \in Y$ such that $Z \cap F \neq \emptyset$ for any $F \in \mathcal{F}_{\text{lst}}$. Each closed in this topology subset of $\tilde{\Theta}$ is an intersection of subsets of the form $Z_*$. The equality $i(H) = (\hat{H})$ holds for any subset $H \subset \Theta$. The accent tilde on the left side denotes the closure in the topology $\tau_c$, and, as above, the line on top on the right side denotes the closure in the topology $\tau_i$ on $\Theta$.

Show now that the topology $\tau_c$ is coarser than the natural topology or coincides with it. It involves, in particular, that $i$ is a dense embedding of $\Theta$ into $\tilde{\Theta}$ endowed with the topology $\tau_c$, any filter of the form $i(\mathcal{F})$ where $\mathcal{F}$ is a $C$-filter on $\Theta$, converges in this topology, and any point from $\tilde{\Theta}$ is the limit of such a filter. Indeed, let $\bar{x} \in \Theta$ be an arbitrary point and $V'$ its neighborhood in the topology $\tau_c$ from the base above. First, assume that there exists $x \in \Theta$ such that $\bar{x} = i(x)$. Then $\bar{x} \in i(V) \subset V'$, and $i(V)$ is a neighborhood of $\bar{x}$ in the natural topology lying in $V'$. Consider the second case. If $\bar{x} = [\mathcal{F}] \in \Theta \setminus i(\Theta)$, then $V \in \mathcal{F}$. By Lemma 1.10 from [2], the set $\{[\mathcal{F}] \cup i(V)$$\}$ is a neighborhood of the point $[\mathcal{F}]$ in the natural topology on $\tilde{\Theta}$. It is evident that this neighborhood lies in $V'$.

Now, we can prove our first theorem.

**Theorem 2.1.** $(\tilde{\Theta}, \cdot, \tau_c)$ is a complete abelian topological group and, together with the map $i$, a weakly unitary completion of the monoid $\Theta$.

**Proof.** We prove this result in a sequence of several steps.
The map $i$ is an algebraic and a topological embedding. It is a topological embedding of the monoid $\Theta$ into the monoid $(\hat{\Theta}, \bullet, \tau_c)$ since $V' \cap i(\Theta) = i(V)$ for any open $V \subset \Theta$, i.e., the restriction of the topology $\tau_c$ onto $i(\Theta)$ coincides with its initial topology $\tau$. It is also noted above that this embedding is an algebraic one.

The topology $\tau_c$ is $T_3$. First, check that it is $T_1$. Let $x_1, x_2$ be different points of $\hat{\Theta}$ and $F_1, F_2$ the corresponding $C$-least $C$-filters. If the first of them converges in $\Theta$ to a point $x_1$ and the second one converges to another point or diverges, then there exists a neighborhood $V$ of $x_1$ which does not belong to $F_2$. Therefore, $x_1$ belongs, and $x_2$ does not belong to $V'$.

If $F_2$ diverges in $\Theta$ and that’s why does not cluster there (see Corollary 1.10 from [1]), while $F_2$ converges to a point $x \in \Theta$, then there exists an open set $V \in F_1$ which does not contain $x$. Otherwise, by Proposition 1.16, all members of $F_1$ would contain $x$. Therefore, $x_1$ belongs, and $x_2$ does not belong to $V'$ again.

Finally, if both the filters $F_1$ and $F_2$ diverge in $\Theta$, then there exists an open $V \in F_1$ which does not belong to $F_2$. Otherwise, again by Proposition 1.16, we would have $F_1 \subset F_2$, and it would imply to $F_1 = F_2$ (see Proposition 2.2 from [1]) and $x_1 = x_2$. For this $V$, $x_1 \in V'$ and $x_2 \notin V'$.

Let now $x_0$ be an arbitrary point of $\hat{\Theta}$ and $V'$ its arbitrary neighborhood in the space $(\hat{\Theta}, \tau_c)$ belonging to the base above and corresponding to an open $V \subset \Theta$. Find a neighborhood $W'$ of this point such that $W' \subset V'$.

First, let the $C$-least $C$ filter $F_0$ on $\Theta$ corresponding to $x_0$ converge to $x_0 \in \Theta$. Since the topology $\tau_c$ on $\Theta$ is $T_3$, there exist a neighborhood $W$ of $x_0$ and a neighborhood $U$ of 1 in $\Theta$ such that $UW \subset V$. Then $W'$ is a neighborhood of $x_0$ in the space $(\hat{\Theta}, \tau_c)$. Show that its closure in $(\hat{\Theta}, \tau_c)$ lies in $V'$ and start with the following remark. For any $\hat{x} \in \hat{\Theta}$, the family $O_{\hat{x}}$ is a filter in the set of open subsets of $\Theta$. Its open subsets $W$ and $\Theta \setminus W$ have an empty intersection and can not belong the same filter $O_{\hat{x}}$, i.e., $W' \setminus (\Theta \setminus W)' = \emptyset$. It implies that $W'$ lies in the closed in the topology $\tau_c$ subset $\hat{\Theta} \setminus (\Theta \setminus W)' = (W')$, which consists of $i(W) \subset i(V) \subset V'$ and of all points $\hat{x}$ from $\Theta$ corresponding to $C$-least strict $C$-filters possessing the property that intersections of all their members with $W$ are non-empty. If $\Sigma$ is one of these filters, then $M_{\Sigma} \cap W \neq \emptyset$, and that’s why there exists $y \in W$ such that $y \in U$. It implies that $UW \in \Sigma$, $V \in \Sigma$, and the point $\hat{x}$ corresponding to $\Sigma$ lies in $V'$.

Let now $F_0$ diverge. It is a strict non-trivial $C$-least $C$-filter, and $V \in F_0$. By Propositions 1.8, 1.16 and 1.18, there exist an open $W \in F_0$ and a neighborhood $U$ of 1 in $\Theta$ such that $UW \subset V$. Then the previous arguments show that this $W$ is required.

The multiplication $\bullet$ is continuous in the topology $\tau_c$. For any $\hat{x}_1, \hat{x}_2 \in \hat{\Theta}$ and for an arbitrary $W \in O_{\hat{x}_1} \setminus \hat{x}_2$, we have to find $V_1 \in O_{\hat{x}_1}$ and $V_2 \in O_{\hat{x}_2}$ such that $V_1 \cap V_2 \cap W$ imply $W \in O_{\hat{x}_1} \setminus \hat{x}_2$ for any $\hat{x}_1, \hat{x}_2 \in \hat{\Theta}$.

Let $F_1, F_2$ be strict $C$-filters from classes $\hat{x}_1$, respectively, $\hat{x}_2$. Then the class $\hat{x}_1 \setminus \hat{x}_2$ contains the strict $C$-filter $\Sigma = F_1 \setminus F_2$. First, we show that, for any $W \in O_{\hat{x}_1}$, there exists $W_0 \in O_{\hat{x}_1}$ and a neighborhood $U$ of 1 in $\Theta$ such that $W_0 \cup U \subset W$. Indeed, if $\Sigma$ converges to some point and $W$ is a neighborhood of this point, then the existence of these $W_0$ and $U$ follows from the fact that the underlying space of $\Theta$ is $T_3$. If $\Sigma$ diverges and $W$ belongs to the corresponding $C$-least non-trivial $C$-filter $\Sigma_{\text{last}}$, then it follows from Propositions 1.8, 1.16 and 1.18.

Find now the sets $V_1$ and $V_2$. There exist sets $V_1' \in F_1, V_2' \in F_2$ such that $V_1' V_2' \subset W_1$. If the filter $F_1$ diverges in $\Theta$, then, for any neighborhood $U_1$ of 1, the set $V_1' U_1$ is open in $\Theta$ and belongs to $(F_1)_{\text{last}}$. Indeed, $(F_1)_{\text{last}}$ diverges, too, and that’s why both these filters are non-trivial. Now, it follows from Proposition 1.12 that $F_1 \Rightarrow (F_1)_{\text{last}}, M_{(F_1)_{\text{last}}}, M_{(F_1)_{\text{last}}} \cap (F_1)_{\text{last}} \neq \emptyset$, there exists a point $a \in V_1'$ such that $a \in (F_1)_{\text{last}}$ and $V_1' U_1 \in (F_1)_{\text{last}}$. Therefore, $V_1' \subset O_{\hat{x}_1}$ for any $U_1$. Choose $U_1$ so that $U_1 \subset U$, and set $V_1 = V_1' U_1$. If the filter $F_1$ converges to a point $x \in \Theta$, then $x \in V_1$ and $V_1 = V_1' U_1$ is a neighborhood of $x$ and converges to $O_{\hat{x}_1}$.

The choice of $V_2$ is similar. In both these cases, $V_1' V_2' \subset (V_1' V_2' U) \subset W_1 U$ and $(V_1' V_2' U) \subset W$. Let now $F_1, F_2$ be strict $C$-filters corresponding to points $\hat{x}_1, \hat{x}_2 \in \hat{\Theta}, V_1 \in O_{\hat{x}_1}, V_2 \in O_{\hat{x}_2}$ imply $V_1' V_2' \subset \emptyset$. If $\Sigma$ converges to $x$, then $x \in V_1 V_2$ and $x U$ is a neighborhood of $x$. Since $x U \subset V_1 V_2 U \subset W$ and $W$ is open, then $W$ is a neighborhood of $x$, too, and, therefore, $W \in O_{\hat{x}_1} \setminus \hat{x}_2$. If $\Sigma$ diverges in $\Theta$, then, as above, $V_1 V_2 \subset \emptyset$ implies $V_1' V_2' \subset \emptyset$ and $W \in O_{\hat{x}_1} \setminus \hat{x}_2$ again.

Thus, $(\hat{\Theta}, \bullet, \tau_c)$ is a topological monoid on a $T_3$-underlying space. Since $i$ is a dense algebraic and topological embedding, then it is monothetic with a generator $i(p)$. 


The topology $\tau_c$ is non-viscous. Take an arbitrary neighborhood of the neutral element $1_\Theta$. We may assume that it has the form $U'$ where $U$ is a neighborhood of the neutral element $1_\Theta$. Denote by $U'$ a neighborhood of $1_\Theta$ such that the set $U'U'$ is contained in $U$. Let $\hat{a}$ be an arbitrary point of $\Theta$ and $f$ the corresponding $C$-least $C$-filter on $\Theta$. Find an open subset $V \subset \Theta$ such that the set $V'$ is a neighborhood of $\hat{a}$ satisfying the requirement of the definition of a non-viscous topology from the beginning of section 1A. If $f$ diverges in $\Theta$, then $V$ is an open member of $f$ from lemma 1.17 corresponding to $U'$ in place of $U$. If $f$ converges to a point $x \in \Theta$, then $V$ is a neighborhood of this $x$ corresponding to $U'$ by the definition of a non-viscous topology. In both these cases, $V \in \mathcal{O}_x$.

Consider now a solution $x$ of the equation $a_1x = a_2$ in $(\hat{\Theta}, \bullet)$ with $a_1$, $a_2 \in V'$. Denote by $\mathcal{F}_1$ and $\mathcal{G}$ arbitrary strict $C$-filters from the classes $a_1$ and $x$, respectively, and set $\mathcal{F}_2 = \mathcal{F}_1 \mathcal{G}$. It is a strict $C$-filter from the class $a_2$. It follows from $a_1$, $a_2 \in V'$ that $V$ belongs to the corresponding filters $\mathcal{O}_{x_1} \subset \mathcal{F}_1$ and $\mathcal{O}_{x_2} \subset \mathcal{F}_2$. Since it belongs to the product of the filters $\mathcal{F}_1$ and $\mathcal{G}$, there exist members $F_1$ and $G$ of these filters, respectively, such that $F_1$ and $F_1 \mathcal{G}$ are subsets of $V$. It follows now from the choice of $U'$ and $V$ that $G \subset U'$ and so $U' \in \mathcal{G}$. As above, if $\mathcal{G}$ diverges in $\Theta$, then $U'$ belongs to the corresponding $C$-least $C$-filter $\mathcal{G}_{int}$, $U \in \mathcal{O}_\Theta$, and $x \in U'$. If $\mathcal{G}$ converges to $g \in \Theta$, then $g$ belongs to the closure of $U'$ and therefore it lies in $U$. Hence, $U \in \mathcal{O}_\Theta$, and $x \in U'$ again.

The monoid $(\hat{\Theta}, \bullet)$ is cancellative. Let $\hat{x}_1$, $\hat{x}_2$, $\hat{x}_3 \in \hat{\Theta}$ and $\hat{x}_2 \bullet \hat{x}_3 = \hat{x}_1 \bullet \hat{x}_3$. Denote by $\mathcal{S}_1 = \{s_{1a}\}_{a \in A}$, $\mathcal{S}_2 = \{s_{2p}\}_{p \in B}$ and $\mathcal{S}_3 = \{s_{3y}\}_{y \in C}$ strict $C$-nets in $\Theta$ such that $i(\mathcal{S}_1)$ converges to $\hat{x}_1$, $i(\mathcal{S}_2)$ converges to $\hat{x}_2$, and $i(\mathcal{S}_3)$ converges to $\hat{x}_3$. All members of these nets have the form $p^m$ with $m \in \mathbb{N}$. Since $\tau_c$ is non-viscous, for any neighborhood $U$ of $1_\Theta$ in $\Theta$, there is a neighborhood $U'$ of the point $\hat{x}_3 \bullet \hat{x}_2 \in \hat{\Theta}$ such that the pair $U'$, $U'$ satisfies the requirement of the definition of a non-viscous topology. There exist $a_0 \in A$, $\beta_0 \in B$, $\gamma_0 \in C$ such that, for $a \geq a_0$, $\beta \geq \beta_0$, $\gamma \geq \gamma_0$, the elements $i(s_{1as_{2p}})$ and $i(s_{1as_{3y}})$ belong to $V'$. Both these elements have the form $i(p)^m$, $m \in \mathbb{N}$, therefore, for any such $a$, $\beta$ and $\gamma$, one of these elements can be obtained from the other by multiplication with a factor of the same form. This factor lies in $U' \cap i(\Theta) = i(U)$. Since $i$ is an algebraic embedding, it means that there exists $u_{a\beta\gamma} \in U$ such that $s_{1as_{2\beta}} = s_{1as_{3\gamma}}u_{a\beta\gamma}$ or $s_{1as_{2\beta}}u_{a\beta\gamma} = s_{1as_{3\gamma}}$. $\Theta$ is cancellative, and we may cancel the first factors. Since $U$ is arbitrary, the topology $\tau_c$ is Hausdorff, and the multiplication $\bullet$ is continuous, the obtained formula implies that the nets $i(\mathcal{S}_2)$ and $i(\mathcal{S}_3)$ have equal limits $\hat{x}_2$ and $\hat{x}_3$.

Thus, $(\hat{\Theta}, \bullet, \tau_c)$ possesses the same properties which were assumed for $\Gamma$. Therefore, all statements which were proved in section 1 for $\Gamma$, remain valid for this monoid. In particular, it is true (with the replacement of $j$ by $i$) for Proposition 1.7 and Lemma 1.19.

The monoid $(\hat{\Theta}, \bullet, \tau_c)$ is weakly unitarily complete. Indeed, let $\mathcal{T} = \{t_\alpha\}_{\alpha \in \Sigma}$ be its strict $C$-net. By Lemma 1.19, there exists a strict $C$-net $\mathcal{S}$ in $\Theta$ such that $i(\mathcal{S}) \succeq \Theta$. It was noted above that the $C$-net $i(\mathcal{S})$ converges for any $C$-net $\mathcal{S}$ in $\Theta$. Therefore, the net $\mathcal{T}$ converges by Corollary 1.10 from [1].

$(\hat{\Theta}, \bullet, \tau_c)$ is a complete topological group. First, prove that all elements from $i(\Theta)$ are invertible, and the function $i(p)^n \to i(p)^n$ is continuous on $i(\Theta)$. Let a strict $C$-net $\mathcal{D} = \{p_{\alpha}^n\}_{\alpha \in A}$ in $\Theta$ converge to $1_\Theta$. Then the net $\mathcal{D}' = \{p_{\alpha}^{n+1}\}_{\alpha \in A}$ is a strict $C$-net, too, the net $i(\mathcal{D}')$ converges in $(\hat{\Theta}, \tau_c)$, and $i(p) \cdot q = 1_\Theta$ holds for its limit $q$, i.e. $q = i(p)^{-1}$. It means that $i(p)^n$ exists for any $n \in \mathbb{N}$. To prove the continuity of the function $i(p)^n \to i(p)^n$ on $i(\Theta)$, take an arbitrary $n \in \mathbb{N}$ and denote by $W$ an arbitrary neighborhood of $i(p)^n$ and by $U$ a neighborhood of $1_\Theta$ such that $i(p)^n \bullet U \subset W$. Find a neighborhood $V$ of $a = i(p)^n$ corresponding to $U$ for this point $a$ in the definition of a non-viscous topology. It exists since the topology $\tau_c$ is non-viscous. If $i(p)^n \in V$ for some $m \in \mathbb{N}$, then $i(p)^{m+n} \in U$ and $i(p)^m \in W$.

Show now that all elements of $(\hat{\Theta}, \bullet)$ are invertible. Let $\mathcal{D}_a = \{p_{\alpha}^n\}_{\alpha \in A}$ be a $C$-net in $\Theta$ such that the net $i(\mathcal{D}_a)$ converges to a given $a \in \hat{\Theta}$. Then it follows from the proved continuity of the function $i(p)^n \to i(p)^n$ that $i(D_{a})^{-1} = \{i(p)^{-n}\}_{\alpha \in A}$ is a strict $C$-net in $(\hat{\Theta}, \bullet, \tau_c)$, and it is evident that $a \bullet h = 1$ for its limit $h$. Since $(\hat{\Theta}, \bullet)$ is cancellative, there exists an only $h$ with this property.

Thus, $(\hat{\Theta}, \bullet, \tau_c)$ is algebraically a group. Therefore, it is a topological group by Proposition 1.2. Note also that the continuity of the function $x \to x^{-1}$ at any point $x \in \hat{\Theta}$ can be proved similarly to the above argument for the case $x \in i(\Theta)$.

Any Cauchy filter in $(\hat{\Theta}, \bullet, \tau_c)$ converges since Cauchy filters in commutative topological groups are strict $C$-filters. □
B) In this section, we prove the main theorems of this paper.

**Theorem 2.2.** A monothetic monoid can be (algebraically and topologically) embedded into a topological group if and only if it is cancellative and its topology is $T_3$ and non-viscous. If these conditions are satisfied, then the constructed above weakly unitary completion $(\bar{\Theta}, \cdot, \tau_c)$ of the submonoid $\Theta = \{p^n : n \in \mathbb{N}_0\}$ generated by the generator $p$ of the initial monoid is such a topological group, the natural embedding of this monoid into this group is dense, and this group is monothetic.

**Proof.** The necessity is evident. To prove the sufficiency, show, keeping the previous notation, that the map $i$ can be continued up to an algebraic and topological dense embedding $\phi: \Gamma \rightarrow (\bar{\Theta}, \cdot, \tau_c)$.

Let $g$ be an arbitrary element from $\Gamma$. By Corollary 1.9 and Proposition 1.14, there exists a unique $C$-least $C$-filter $\mathcal{F}$ on $\Theta$ such that $j(\mathcal{F})$ converges to $g$. Denote by $\phi(g)$ the point $[\mathcal{F}]$ of $\bar{\Theta}$ corresponding to $\mathcal{F}$. It is evident that this map $\phi: \Gamma \rightarrow \bar{\Theta}$ coincides with $i$ on $\Theta$. It is injective since $g_1 \neq g_2$ implies $\mathcal{F}_1 \neq \mathcal{F}_2$, but it is not necessarily surjective since some $C$-filters of the form $j(\mathcal{F})$ can diverge in $\Gamma$. It is an algebraic embedding of monoids since the multiplication $\cdot$ corresponds to the multiplication of $C$-filters and the limit of the product of convergent filters in $\Gamma$ is equal to the product of their limits.

It now suffices to show that $\phi$ is a continuous open map of $\Gamma$ onto $\phi(\Gamma)$. Since $\Gamma$ is a $T_3$-space, there exists its base $\mathcal{B}$ consisting of canonically open sets, i.e. such that $U = \text{Int} \bar{U}$ for any $U \in \mathcal{B}$, where, as above, the hat on top and the symbol $\text{Int}$ denote the closure and the interior of the corresponding subset of $\Gamma$. The intersections of the form $U' = U \cap \Theta$ with $U \in \mathcal{U}$ form a base of $\Theta$, and the corresponding sets of the form $U''$ form a base in $(\bar{\Theta}, \cdot, \tau_c)$. We will only consider these bases.

Proof that $\phi(U) = \phi(\Gamma) \cap U''$ and $\phi^{-1}(U'') = U$ for any $U \in \mathcal{U}$. Let $g \in U \in \mathcal{U}$. For $g \in U'$, the corresponding filter $\mathcal{F}$ is $g_{\text{lst}}$, and it implies that $U' \in \mathcal{F}_\tau$ and $\phi(g) \in U''$. Conversely, if $g \notin \Theta$ and $\phi(g) \in U''$, then $U' \in \mathcal{F}_\tau$, and it means that $g \in U' \subset U$ since $\mathcal{F}_\tau$ consists of neighborhoods of $g$ in $\Theta$.

Let now $g \in U \cap U'$. Then $U' \in \mathcal{F}$ since $\mathcal{F}$ is a filter on $\Theta$ and $j(\mathcal{F})$ converges to $g$. Since $\mathcal{F}$ diverges and $U'$ is open in $\Theta$, then $U' \in \mathcal{F}_\tau$ and $\phi(g) = [\mathcal{F}] \in U''$. Conversely, if $g \notin \Theta$ and $\phi(g) \in U''$, then $U' \in \mathcal{F}_\tau \cap \mathcal{F}$.

As it was showed in Lemma 1.15, the filter $\mathcal{F}$ coincides with the filter $\mathcal{F}(\mathcal{F}(g))$ (see text before the proof of this lemma), and there exists a neighborhood $W$ of $g$ such that $W' = W \cap \Theta$ is contained in $U'$. $W'$ is dense in $W$ and $U'$ is dense in $U$, and it implies that $\bar{W'} = \bar{W}$ and $\bar{U'} = \bar{U}$. Now, the following inclusions are true: $W \subset \text{Int} W' \subset \text{Int} \bar{U} = U$, i.e. $g \in U$. It completes the proof. □

**Corollary 2.3.** Let $\mathcal{S}$ be the product of any family of monothetic monoids endowed with the Tychonoff product topology. If there exists an algebraic and topological embedding of $\mathcal{S}$ into a topological group, then $\mathcal{S}$ and all its factors are cancellative and have $T_3$ non-viscous topologies. If $\mathcal{S}$ is cancellative and has a $T_3$ non-viscous topology, then there exists its dense algebraic and topological embedding into a topological group.

**Proof.** If there exists an algebraic and topological embedding of $\mathcal{S}$ into a topological group, then $\mathcal{S}$ is cancellative and has a $T_3$ non-viscous topology. Therefore, each factor of $\mathcal{S}$ has these properties, too. If $\mathcal{S}$ has these properties, then each its factor has them, too, and there exists its dense algebraic and topological embedding into a topological group. The product of these embeddings is a required embedding of $\mathcal{S}$. □

Show now that the correspondence $\Gamma \rightarrow (\bar{\Theta}, \cdot, \tau_c)$ possesses the following universal property.

**Theorem 2.4.** Under the same notation and assumptions about $\gamma$ as above, let $\gamma$ be an arbitrary complete commutative topological group and $f: \Theta \rightarrow \mathcal{S}$ such that $f: \Gamma \rightarrow \mathcal{S}$ be an algebraic and topological embedding of the monoid $\Theta$ (respectively, of the monoid $\Gamma$). Then there exists a continuous homomorphism $\tilde{f}$ of the topological group $(\bar{\Theta}, \cdot, \tau_c)$ into $\mathcal{S}$ such that $f = \tilde{f} \circ i$ (respectively, $f = \tilde{f} \circ \phi$ where $\phi$ is the constructed above embedding of $\Gamma$ into $(\bar{\Theta}, \cdot, \tau_c)$).

**Proof.** First, consider the case when $f$ is a map of $\Theta$, and define the map $\tilde{f}$. Let $\tilde{x}$ be a point of $\bar{\Theta}$, $[\mathcal{F}]$ the corresponding to it equivalence class of $C$-filters on $\Theta$, and $\mathcal{F} \in [\mathcal{F}]$. Then $f([\mathcal{F}])$ is a $C$-filter on $\Theta$, i.e. a Cauchy filter of its standard uniformity. Hence, this filter converges, and we denote its limit by $\tilde{f}(x)$. This limit does not depend on the choice of $\mathcal{F}$ in $[\mathcal{F}]$ since equivalent $C$-filters have equal limits (see Corollary 1.10 from [1]) and equivalent images under homomorphisms (Proposition 2.1 from [2]). If $x \in i(\Theta)$, i.e. $\mathcal{F}$ is equivalent to the filter $\tilde{x}$ for some $x \in \Theta$, then $\tilde{f}(\mathcal{F}) = f(x)$ and $f = \tilde{f} \circ i$.

Show that $\tilde{f}$ is a homomorphism. Consider arbitrary $\tilde{x}_1, \tilde{x}_2 \in \bar{\Theta}$ and denote by $\mathcal{F}_1 = \{F_1\}_{a \in A}$, $\mathcal{F}_2 = \{F_2\}_{b \in B}$ the corresponding $C$-filters on $\Theta$. Then the corresponding to $\tilde{x}_1 \cdot \tilde{x}_2$ $C$-filter $\mathcal{F}_1 \cdot \mathcal{F}_2$ is generated by all sets of the form $F_1 \cdot F_2$ with $a \in A, b \in B$. Since $f$ is a homomorphism, the filter $f([\mathcal{F}_1 \cdot \mathcal{F}_2])$ is generated by
sets \( f(F_{1a}F_{2β}) = f(F_{1a} \ast f(F_{2β}) \), \( a \in A, β \in B \). Here, \( * \) denotes the multiplication in \( G \). Therefore, \( f(3_13_2) = f(3_1) \ast f(3_2) \) and \( \tilde{f}(x_1 \ast x_2) = \tilde{f}(x_1) \ast \tilde{f}(x_2) \).

Prove that \( \tilde{f} \) is continuous. For any \( \tilde{x} \in \hat{Θ} \), let \( U \) be an arbitrary neighborhood of the point \( \tilde{f}(\tilde{x}) \) and \( V \) its neighborhood such that \( \tilde{V} \subseteq U \). Here, the hat on top denotes the closure in \( G \). Show that the open subset \( W = f^{-1}(V) \) is a neighborhood of \( \tilde{x} \). Indeed, if \( \tilde{x} = i(x) \), then \( f(x) \in V \) implies \( \tilde{x} \in f^{-1}(V) \). For \( \tilde{x} \in \hat{Θ} \setminus i(Θ) \), if \( 3 \) is the corresponding to \( \tilde{x} \) least divergent \( C \)-filter on \( Θ \), then \( \tilde{f}(\tilde{x}) \in V \) implies \( V \in f(3) \) and \( f^{-1}(V) \in 3 \). Therefore, \( \tilde{x} \in f^{-1}(V) \) holds again. Check now that \( \tilde{f}(W) \subseteq U \). For any point \( \tilde{y} \in W \cap i(Θ) \), the inclusion \( \tilde{f}(\tilde{y}) \subseteq U \) follows from \( i^{-1}(\tilde{y}) \in f^{-1}(V) \). If \( \tilde{y} \in \hat{Θ} \setminus i(Θ) \) and \( 3' \) is the corresponding to it \( C \)-filter, then \( \tilde{y} \in f^{-1}(V) \) implies \( f^{-1}(V) \in 3' \) and \( V \in f(3') \). Hence, \( \tilde{f}(\tilde{y}) = \lim f(3') \in \tilde{V} \subseteq U \).

Let now \( f \) be an algebraic and topological embedding of \( Γ \). Then \( f \circ j \) is an embedding of \( Θ \), and there exists a continuous homomorphism \( \tilde{f} : (\hat{Θ}, \mathbin{\bullet}, τ_c) \to G \) such that \( f \circ j = \tilde{f} \circ ϕ \circ j \). The continuous maps \( f \) and \( \tilde{f} \circ ϕ \) coincide on \( Γ \) since they coincide on its dense subset \( j(Θ) \).

### References

[1] Averbukh B.G., *On unitary Cauchy filters on topological monoids*, Top. Algebra Appl., 1 (2013), 46-59.

[2] Averbukh B.G., *On finest unitary extensions of topological monoids*, Top. Algebra Appl., 3 (2015), 1-10.

[3] Averbukh B.G., *On unitary extensions and unitary completions of topological monoids*, Top. Algebra Appl., 4 (2016), 18-30.

[4] Carruth J.H., Hildebrant J.A., Koch R.J., *The theory of topological semigroups*, Pure and Applied Mathematics, Marcel Dekker, Inc., 1983, vi + 244 pp., ISBN 0-8-247-1795-3.

[5] Christoph F.T. Jr., *Free topological semigroups and embedding topological semigroups in topological groups*, Pacific Journal of Mathematics, 34, No. 2 (1970), 343-353.

[6] Engelking R., *General topology. Rev. and compl. ed.*. Sigma Series in Pure Mathematics, 6., Berlin: Heldermann Verlag, 1989, viii + 529 pp., ISBN 3-88538-006-4, Zbl 0684.54001.

[7] Gelbaum B., Kalisch G.K., Olmsted J.M.H., *On the embedding of topological semigroups and integral domains*, Proc. Amer. Math. Soc. 2 (1951), 807-821.

[8] Lau K.-S., Lawson J.D., Zeng W.-B., *Embedding Locally Compact Semigroups into Groups*, Semigroup Forum 56 (1998), 151-156.

[9] Lawson J.D., *Embedding semigroups into Lie groups*, The Analytical and Topological Theory of Semigroups: Trends and Developments, editors Hofmann K.H., Lawson J.D., Pym J.S., Walter de Gruyter, 1990, 398 pp.

[10] McKilligan S.A., *Embedding topological semigroups in topological groups*, Proc. Edinburgh Math. Soc. 17 (1970/71), 127-138.

[11] Rothman N.J., *Embedding of topological semigroups*, Math. Ann. 139 (1960), 197-203.