Efficient primal-dual fixed point algorithm with dynamic stepsize for convex problems with applications to imaging restoration

Meng Wen 1,2, Shigang Yue 4, Yuchao Tang 3, Jigen Peng 1,2

1. School of Mathematics and Statistics, Xi’an Jiaotong University, Xi’an 710049, P.R. China
2. Beijing Center for Mathematics and Information Interdisciplinary Sciences, Beijing, P.R. China
3. Department of Mathematics, NanChang University, Nanchang 330031, P.R. China
4. School of Computer Science, University of Lincoln, LN6 7TS, UK

Abstract In this paper we consider the problem of finding the minimization of the sum of a convex function and the composition of another convex function with a continuous linear operator from the view of fixed point algorithms based on proximity operators. We design a primal-dual fixed point algorithm with dynamic stepsize based on the proximity operator (PDFP 2 O DS n for a n ∈ (0, 1)) and obtain a scheme with a closed-form solution for each iteration. Based on Modified Mann iteration and the firmly nonexpansive properties of the proximity operator, we achieve the convergence of the proposed PDFP 2 O DS n algorithm. Moreover, under some stronger assumptions, we can prove the global linear convergence of the proposed algorithm. We also give the connection of the proposed algorithm with other existing first-order methods and fixed point algorithms FP 2 O (Micchelli et al 2011 Inverse Problems 27 45009-38), PDFP 2 O (Chen et al 2013 Inverse Problems 29). Finally, we illustrate the efficiency of PDFP 2 O DS n through some numerical examples on the CT image reconstruction problem. Generally speaking, our method PDFP 2 O DS is comparable with other state-of-the-art methods in numerical performance, while it has some advantages on parameter selection in real applications and converges faster than PDFP 2 O.

* Corresponding author.
E-mail address: wen5495688@163.com
Keywords: fixed point algorithm; convex separable minimization; proximity operator; duality

MR(2000) Subject Classification 47H09, 90C25,

1 Introduction

The purpose of this paper is to designing and discussing an efficient algorithmic framework with dynamic stepsize for minimizing the sum of a convex function and the composition of another convex function with a continuous linear operator, i.e.

$$\min (f_1 \circ D)(x) + f_2(x),$$

where $f_1 \in \Gamma_0(\mathbb{R}^m)$, $f_2 \in \Gamma_0(\mathbb{R}^n)$, and $f_2$ is differentiable on $\mathbb{R}^n$ with a $1/\beta$-Lipschitz continuous gradient for some $\beta \in (0, +\infty)$ and $D : \mathbb{R}^n \to \mathbb{R}^m$ a linear transform. This parameter $\beta$ is related to the convergence conditions of algorithms presented in the following section. Here and in what follows, for a real Hilbert space $\mathcal{H}$, $\Gamma_0(\mathcal{H})$ denotes the collection of all proper lower semi-continuous convex functions from $\mathcal{H}$ to $(-\infty, +\infty]$. Despite its simplicity, many problems in image processing can be translated into the form of (1.1). For example, the following variational sparse recovery models are often considered in image restoration and medical image reconstruction:

$$\min \frac{1}{2} \|Ax - b\|_2^2 + \lambda \psi(Dx),$$

where $\| \cdot \|_2$ denotes the usual Euclidean norm for a vector, $A \in \mathbb{R}^{p \times n}$ describes a blur operator, $b \in \mathbb{R}^p$ represents the blurred and noisy image and $\lambda > 0$ is the regularization parameter in the context of deblurring and denoising of images. The class of regularizers (1.2) includes a plethora of methods, depending on the choice of the function $\psi$ and of matrix $D$. Our motivation for studying this class of penalty functions arises from sparsity inducing regularization methods which consider $\psi$ to be either the $l_1$ norm or a mixed $l_1 - l_2$ norm. When $D$ is the identity matrix, the latter case corresponds to the well-known Group Lasso method [15], for which well studied optimization techniques are available. Other choices of the matrix $D$ give rise to different kinds of Group Lasso with overlapping groups [16-17], which have proved to be effective in modeling
structured sparse regression problems. Problem (1.2) can be expressed in the form of (1.1) by setting $f_1 = \lambda \psi$, $f_2 = \frac{1}{2} \| Ax - b \|^2_2$. One of the main difficulties in solving it is that $\psi$ are non-differentiable. The case often occurs in many problems we are interested in.

For problem (1.1), Peijun Chen, Jianguo Huang and Xiaoqun Zhang proposed a primal-dual fixed point algorithm (PDFP2O) in [1], i.e.

$$
\begin{align*}
  v_{n+1} &= (I - \text{prox}_{\frac{\lambda}{2} f_1})(D(x_n - \gamma \nabla f_2(x_n)) + (I - \lambda DD^T)v_n), \\
  x_{n+1} &= x_n - \gamma \nabla f_2(x_n) - \lambda D^T v_{n+1},
\end{align*}
$$

(1.3)

where $0 < \lambda \leq 1/\lambda_{\max}(DD^T)$, $0 < \gamma < 2\beta$, and the operator $\text{prox}_f$ is defined by

$$
\text{prox}_f : \mathcal{H} \to \mathcal{H}
$$

$$
x \mapsto \arg \min_{y \in \mathcal{H}} f(y) + \frac{1}{2} \| x - y \|^2_2,
$$

called the proximity operator of $f$. Note that this type of splitting method was originally studied in [1,8] and the notion of proximity operators was first introduced by Moreau in [9] as a generalization of projection operators. For general $D$ and $f_2$, each step of the proposed algorithm is explicit when $\text{prox}_{\frac{\lambda}{2} f_1}$ is easy to compute. However, the proximity operators for the general form $f = f_1 \circ D$ as in (1.1) do not have an explicit expression, leading to the numerical solution of a difficult subproblem. In fact for $\lambda \psi = \mu \| \cdot \|$, the subproblem of (1.2) is

$$
\min \frac{1}{2} \| x - b \|^2_2 + \mu \| Dx \|,
$$

(1.4)

where $A \in \mathbb{R}^{p \times n}$ describes a blur operator, $b \in \mathbb{R}^p$ denotes a corrupted image to be denoised.

The obvious advantage of the algorithm (PDFP2O) proposed by Chen et al [1] for problem (1.1) is that it is very easy for parallel implementation. However, in this paper we aim to provide a more general iteration in which the coefficient $\gamma$ is made iteration-dependent to solve the general problem (1.1), errors are allowed in the evaluation of the operators $\text{prox}_{\frac{\lambda}{2} f_1}$ and $\nabla f_2$, and a relaxation sequence $\lambda_n$ is introduced. The errors allow for some tolerance in the numerical implementation of the algorithm, while the flexibility introduced by the iteration-dependent parameters $\gamma_n$ and $\lambda_n$ can be used
to improve its convergence pattern. In addition, we will reformulate our fixed point
type of methods and show their connections with some existing first-order methods
and primal-dual fixed point algorithm for (1.1) and (1.2).

The rest of this paper is organized as follows. In the next section, we recall the
primal-dual fixed point algorithm (PDFP^2O) and some related works and then deduce
the proposed PDFP^2O_{DS} algorithm and its extension PDFP^2O_{DS_n} from our intuitions.
In section 3, we first deduce PDFP^2O_{DS_n} again in the setting of fixed point iteration; we
then establish its convergence under a general setting and the convergence rate under
some stronger assumptions on \nabla f_2 and D. In section 4, we give the equivalent form
of PDFP^2O_{DS}, and the relationships and differences with other first-order algorithms.
In the final section, we show the numerical performance and efficiency of PDFP^2O_{DS_n}
through some examples on the CT image reconstruction problem and compare their
performances to the ones of some iterative schemes recently introduced in the literature.

2 Fixed Point Algorithms Based on Proximity Operators

Similar to the proximity algorithms (FP^2O) for Image Models: Denoising proposed by
Micchelli et al [8], Andreas Argyriou et al proposed an algorithm called IFP^2O in [10]
to solve
\[
\min (f_1 \circ D)(x) + \frac{1}{2} x^T Q x - b^T x,
\]
where \( x \in \mathbb{R}^n, Q \in M_n, \) with \( M_n \) being the collection of all symmetric positive definite
\( n \times n \) matrices, \( b \in \mathbb{R}^n \). Define
\[
H(v) = (I - \text{prox}_{\frac{1}{\lambda}})(DQ^{-1}b + (I - \lambda DQ^{-1}D^T)v) \text{ for all } v \in \mathbb{R}^m.
\]
Then, the corresponding algorithm is given below, called algorithm 1, which can be
viewed as a fixed point algorithm based on the inverse matrix and proximity operator (IF
P^2O). Here \( H_\kappa \) is the \( \kappa \)-averaged operator of \( H \), i.e. \( H_\kappa = \kappa I + (1 - \kappa)H \) for \( \kappa \in (0, 1) \); see definition 3.3 in the following section, the matrix \( Q \) is assumed to be invertible
and the inverse can be easily calculated, which is unfortunately not the case in most
of the applications in imaging science. Moreover, there is no theoretical guarantee of convergence if the linear system is only solved approximately.

**Algorithm 1** FP^2O based on inverse matrix, IFP^2O [10].

1. Choose \( v_0 \in \mathbb{R}^m, 0 < \lambda \leq 2/\lambda_{\text{max}}(DQ^{-1}D^T), \kappa \in (0,1). \)
2. calculate \( v^* \), which is the fixed point of \( H \), with iteration \( v_{n+1} = H_\kappa(v_n) \).
3. \( x^* = Q^{-1}(b - \lambda D^Tv^*) \).

The authors in [10] combined a proximal forward-backward splitting (PFBS) algorithm proposed by Combettes and Wajs [2] and FP^2O for solving problem (1.3), for which we call PFBS_FP^2O (cf algorithm 2 below). Precisely speaking, at step k in PFBS, after one forward iteration \( x_{n+1/2} = x_n - \gamma \nabla f_2(x_n) \), we need to solve for \( x_{n+1} = \text{prox}_{\gamma f_1 \circ D}(x_{n+1/2}) \). FP^2O is then used to solve this subproblem, i.e. the fixed point \( v_{n+1}^* \) of \( H_{x_{n+1/2}} \) is obtained by the fixed iteration form \( v_{k+1} = (H_{x_{n+1/2}})_\kappa(v_k) \), where

\[
H_{x_{n+1/2}}(v) = (I - \text{prox}_{\gamma f_1 \circ D})(Dx_{n+1/2} + (I - \lambda DD^T)v) \text{ for all } v \in \mathbb{R}^m. \tag{2.1}
\]

Then \( x_{n+1} \) is given by setting \( x_{n+1} = x_{n+1/2} - \lambda D^Tv_{n+1}^* \). The acceleration combining with the Nesterov method [11-14] was also considered in [10]. But the algorithm 2 involves inner and outer iterations, and it is often problematic to set the appropriate inner stopping conditions to balance computational time and precision.

**Algorithm 2** Proximal forward-backward splitting based on FP^2O, PFBS_FP^2O [10].

1. Choose \( x_0 \in \mathbb{R}^n, 0 < \gamma < 2\beta \).
2. for \( k = 0, 1, 2, \ldots \)
   - \( x_{n+1/2} = x_n - \gamma \nabla f_2(x_n) \),
   - calculate the fixed point \( v_{n+1}^* \) of \( H_{x_{n+1/2}} \), with iteration \( v_{n+1} = (H_{x_{n+1/2}})_\kappa(v_k) \),
   - \( x_{n+1} = x_{n+1/2} - \lambda D^Tv_{n+1}^* \).
   end for

Further, the authors in [1] suppose \( \kappa = 0 \) in FP^2O, the idea is to take the numerical solution \( v_n \) of the fixed point of \( H_{x_{(n-1)+1/2}} \) as the initial value, and only perform one
iteration for solving the fixed point of $H_{x_{n+1/2}}$; then they obtained the iteration scheme (1.4), i.e.

$$
\begin{align*}
  v_{n+1} &= (I - \text{prox}_{\frac{\lambda}{2} f_1})(D(x_n - \gamma \nabla f_2(x_n)) + (I - \lambda DD^T)v_n), \\
  x_{n+1} &= x_n - \gamma \nabla f_2(x_n) - \lambda D^Tv_{n+1}.
\end{align*}
$$

Then, the corresponding algorithm is given below, called algorithm 3. Since $v$ is actually the dual variable of the primal-dual form related to (1.1), so algorithm 3 can be viewed as a primal-dual fixed point algorithm based on the proximity operator (PDFP$^2O$).

**Algorithm 3** Primal-dual fixed point algorithm based on proximity operator, PDFP$^2O$ [1].

**Initialization:** Choose $x_0 \in \mathbb{R}^n$, $v_0 \in \mathbb{R}^m$, $0 < \lambda \leq 1/\lambda_{\text{max}}(DD^T)$, $0 < \gamma < 2\beta$.

**Iterations ($n \geq 0$):** Update $x_n$, $v_n$, $x_{n+1/2}$ as follows

$$
\begin{align*}
  x_{n+1/2} &= x_n - \gamma \nabla f_2(x_n), \\
  v_{n+1} &= (I - \text{prox}_{\frac{\lambda}{2} f_1})(Dx_{n+1/2} + (I - \lambda DD^T)v_n), \\
  x_{n+1} &= x_{n+1/2} - \lambda D^Tv_{n+1}.
\end{align*}
$$

Moreover, borrowing the fixed point formulation of PDFP$^2O$, the authors in [1] introduce a relaxation parameter $\kappa \in [0, 1)$ to obtain algorithm 4, which is exactly a Picard method with parameters. If $\kappa = 0$, then PDFP$^2O_\kappa$ reduces to PDFP$^2O$. 

---

6
Algorithm 4 PDFP$^2O_\kappa$ [1].

Initialization: Choose $x_0 \in \mathbb{R}^n$, $v_0 \in \mathbb{R}^m$, $0 < \lambda \leq 1/\lambda_{\text{max}}(DD^T)$, $0 < \gamma < 2\beta$, $\kappa \in [0, 1)$.

Iterations ($n \geq 0$): Update $x_n$, $v_n$, $x_{n+\frac{1}{2}}$ as follows

\begin{align*}
    x_n &= x - \gamma \nabla f_2(x_n), \\
    \bar{v}_{n+1} &= (I - \text{prox}_{\frac{\gamma}{n} f_1})(Dx + \frac{1}{2} + (I - \lambda DD^T)v_n), \\
    \bar{x}_{n+1} &= x + \frac{1}{2} - \lambda D^T \bar{v}_{n+1}, \\
    v_{n+1} &= \kappa v_n + (1 - \kappa) \bar{v}_{n+1}, \\
    x_{n+1} &= \kappa x_n + (1 - \kappa) \bar{x}_{n+1}.
\end{align*}

The fixed point characterization provided by Peijun Chen et al [1] suggests solving Problem (1.1) via the fixed point iteration scheme (1.3) for a suitable value of the parameter $\gamma$, $\lambda$. This iteration, which is referred to as a primal-dual fixed point algorithm for convex separable minimization with applications to image restoration. A very natural idea is to provide a more general iteration in which the coefficient $\gamma$ is made iteration-dependent to solve the general problem (1.1), then we can obtain the following iteration scheme:

\begin{align*}
    v_{n+1} &= (I - \text{prox}_{\frac{\gamma}{n} f_1})(Dx_n - \gamma_n \nabla f_2(x_n)) + (I - \lambda_n DD^T)v_n, \\
    x_{n+1} &= x_n - \gamma_n \nabla f_2(x_n) - \lambda_n D^T v_{n+1},
\end{align*}

(2.2)

which produces our proposed method algorithm 5, described below. This algorithm can also be deduced from the fixed point formulation, whose detail we will give in the following section. On the other hand, since the parameter $\gamma_n$ and $\lambda_n$ are dynamic, so we call our method a primal-dual fixed point algorithm based on proximity operator with dynamic stepsize, and abbreviate it as PDFP$^2O_{DS}$. If $\gamma_n \equiv \gamma$, $\lambda_n \equiv \lambda$ then form (2.2) is equivalent to form (1.3). So PDFP$^2O$ can be seen as a special case of PDFP$^2O_{DS}$. Moreover, PFEP and FP$^2O$ are also the special case of PDFP$^2O_{DS}$. We will show the connection to this algorithm and other ones in section 4.
Algorithm 5 Primal-dual fixed point algorithm based on proximity operator with dynamic stepsize PDFP$^2O_{DS}$

Initialization: Choose $x_0 \in \mathbb{R}^n$, $v_0 \in \mathbb{R}^m$, $0 < \liminf_{n \to \infty} \gamma_n \leq \limsup_{n \to \infty} \gamma_n < 2\beta$, $0 < \liminf_{n \to \infty} \lambda_n \leq \limsup_{n \to \infty} \lambda_n \leq 1/\lambda_{\text{max}}(DD^T)$.

Iterations ($n \geq 0$): Update $x_n$, $v_n$, $y_n$ as follows

\[
\begin{align*}
  z_{n+1} &= x_n - \gamma_n \nabla f_2(x_n), \\
  v_{n+1} &= (I - \text{prox}_{\frac{\gamma_n}{\lambda_n}f_1})(Dz_{n+1} + (I - \lambda_n DD^T)v_n), \\
  x_{n+1} &= z_{n+1} - \lambda_n D^T v_{n+1}.
\end{align*}
\]

Borrowing the fixed point formulation of PDFP$^2O_{DS}$, we can introduce a relaxation parameter $\alpha_n \subset (0, 1)$ to obtain algorithm 6, which is exactly a Mann method with parameters. The rule for parameter selection will be illustrated in section 3. Our theoretical analysis for PDFP$^2O_{DS_n}$ given in the following section is mainly based on this fixed point setting.

Algorithm 6 PDFP$^2O_{DS_n}$

Initialization: Choose $x_0 \in \mathbb{R}^n$, $v_0 \in \mathbb{R}^m$, $0 < \liminf_{n \to \infty} \gamma_n \leq \limsup_{n \to \infty} \gamma_n < 2\beta$, $0 < \liminf_{n \to \infty} \lambda_n \leq \limsup_{n \to \infty} \lambda_n \leq 1/\lambda_{\text{max}}(DD^T)$, $\alpha_n \subset (0, 1)$.

Iterations ($n \geq 0$): Update $x_n$, $v_n$, $y_n$ as follows

\[
\begin{align*}
  z_{n+1} &= x_n - \gamma_n \nabla f_2(x_n), \\
  \tilde{v}_{n+1} &= (I - \text{prox}_{\frac{\gamma_n}{\lambda_n}f_1})(Dz_{n+1} + (I - \lambda_n DD^T)v_n), \\
  \tilde{x}_{n+1} &= z_{n+1} - \lambda_n D^T \tilde{v}_{n+1}, \\
  v_{n+1} &= \alpha_n v_n + (1 - \alpha_n)\tilde{v}_{n+1}, \\
  x_{n+1} &= \alpha_n x_n + (1 - \alpha_n)\tilde{x}_{n+1}.
\end{align*}
\]
3 Convergence analysis

3.1 General convergence

First of all, let us mention some related definitions and lemmas for later requirements. We always assume that problem (1.1) has at least one solution. As shown in [2], if the objective function \((f_1 \circ D)(x) + f_2(x)\) is coercive, i.e.

\[
\lim_{\|x\| \to +\infty} \left( (f_1 \circ D)(x) + f_2(x) \right) = +\infty,
\]

then the existence of solution can be ensured for (1.1).

**Definition 3.1.** (Subdifferential [3]). Let \(f\) be a function in \(\Gamma_0(H)\). The subdifferential of \(f\) is the set-valued operator \(\partial f : H \to 2^H\), the value of which at \(x \in H\) is

\[
\partial f(x) = \{ v \in H | \langle v, y - x \rangle + f(x) \leq f(y) \text{ for all } y \in H^2 \},
\]

where \(\langle \cdot, \cdot \rangle\) denotes the inner-product over \(H\).

**Definition 3.2.** (Nonexpansive operators and firmly nonexpansive operators [3]). An operator \(T : H \to H\) is nonexpansive if and only if it satisfies

\[
\|Tx - Ty\|_2 \leq \|x - y\|_2 \text{ for all } (x,y) \in H^2.
\]

\(T\) is firmly nonexpansive if and only if it satisfies one of the following equivalent conditions:

(i) \(\|Tx - Ty\|_2^2 \leq \langle Tx - Ty, x - y \rangle\) for all \((x,y) \in H^2\).

(ii) \(\|Tx - Ty\|_2^2 = \|x - y\|_2^2 - \|(I - T)x - (I - T)y\|_2^2\) for all \((x,y) \in H^2\).

It is easy to show from the above definitions that a firmly nonexpansive operator \(T\) is nonexpansive.

**Lemma 3.1.** Suppose \(f \in \Gamma_0(\mathbb{R}^m)\) and \(x \in \mathbb{R}^m\). Then there holds

\[
y \in \partial f(x) \iff x = \text{prox}_f(x + y).
\]  

(3.1)

Furthermore, if \(f\) has \(1/\beta\)-Lipschitz continuous gradient, then

\[
\langle \nabla f(x) - \nabla f(y), x - y \rangle \geq \beta \|\nabla f(x) - \nabla f(y)\|^2 \text{ for all } (x,y) \in \mathbb{R}^m.
\]  

(3.2)
Proof. The first result is nothing but proposition 2.6 of [4]. If \( f \) has \( 1/\beta \)-Lipschitz continuous gradient, we have from [2] that \( \beta \nabla f \) is firmly nonexpansive, which implies (3.2) readily.

\[ \square \]

**Lemma 3.2.** (Lemma 2.4 of [2]). Let \( f \) be a function in \( \Gamma_0(\mathbb{R}^m) \). Then \( \text{prox}_f \) and \( I - \text{prox}_f \) are both firmly nonexpansive operators.

**Lemma 3.3.** (The Resolvent Identity [5,6]). For \( \lambda > 0 \) and \( \nu > 0 \) and \( x \in E \),

\[ J_{\lambda}x = J_{\nu}\left(\frac{\nu}{\lambda} + (1 - \frac{\nu}{\lambda})J_{\lambda}x\right). \]

**Lemma 3.4.** ([7]). Let \( H \) be a real Hilbert space with inner product \( \langle \cdot, \cdot \rangle \) and norm \( \| \cdot \| \), then

\[ \forall x, y \in H, \forall \alpha \in [0, 1], \|\alpha x + (1 - \alpha)y\|^2 = \alpha\|x\|^2 + (1 - \alpha)\|y\|^2 - \alpha(1 - \alpha)\|x - y\|^2. \]

**Lemma 3.5.** ([7]). Let \( C \) be a nonempty closed convex subset of \( H \), \( T : C \to C \) is a nonexpansive mapping, and \( \text{Fix}(T) \neq \emptyset \). Then the mapping \( I - T \) is demiclosed at zero, that is \( x_n \rightharpoonup x \) and \( \|x_n - Tx_n\| \to 0 \), then \( x = Tx \).

The following lemmas are obtained from the reference [1].

From reference [1], we know that for any two positive numbers \( \lambda \) and \( \gamma \), define \( T_1 : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^m \) as

\[ T_1(v, x) = (I - \text{prox}_{\frac{\gamma}{\lambda}f_1})(D(x - \gamma\nabla f_2(x))) + (I - \lambda DD^T)v \]  

(3.3)

and \( T_2 : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^m \) as

\[ T_2(v, x) = x - \gamma\nabla f_2(x) - \lambda D^T \circ T_1. \]  

(3.4)

Denote

\[ T(v, x) = (T_1(v, x), T_2(v, x)). \]  

(3.5)

**Lemma 3.6.** Let \( \lambda \) and \( \gamma \) be two positive numbers. Suppose that \( \hat{x} \) is a solution of (1.1). Then there exists \( \hat{\nu} \in \mathbb{R}^m \) such that

\[
\begin{align*}
\hat{\nu} &= T_1(\hat{\nu}, \hat{x}), \\
\hat{x} &= T_2(\hat{\nu}, \hat{x}).
\end{align*}
\]
In other words, \( \hat{u} = (\hat{v}, \hat{x}) \) is a fixed point of \( T \). Conversely, if \( \hat{u} \in \mathbb{R}^m \times \mathbb{R}^n \) is a fixed point of \( T \), with \( \hat{u} = (\hat{v}, \hat{x}) \), \( \hat{v} \in \mathbb{R}^m \), \( \hat{x} \in \mathbb{R}^n \) then \( \hat{x} \) is a solution of (1.1).

Denote

\[
g(x) = x - \gamma \nabla f_2(x), \quad \text{for all } x \in \mathbb{R}^n. \tag{3.6}
\]

\[
M = I - \lambda DD^T. \tag{3.7}
\]

When \( 0 < \lambda \leq 1/\lambda_{\text{max}}(DD^T) \), \( M \) is a symmetric positive semi-definite matrix, so we can define the semi-norm

\[
\|V\|_M = \sqrt{(v, Mv)}, \quad \text{for all } v \in \mathbb{R}^m. \tag{3.8}
\]

For an element \( u = (v, x) \in \mathbb{R}^m \times \mathbb{R}^n \), with \( v \in \mathbb{R}^m \) and \( x \in \mathbb{R}^n \), let

\[
\|u\|_\lambda = \sqrt{\|x\|_2^2 + \lambda \|v\|_2^2}. \tag{3.9}
\]

We can easily see that \( \| \cdot \|_\lambda \) is a norm over the produce space \( \mathbb{R}^m \times \mathbb{R}^n \) whenever \( \lambda > 0 \).

According to the definitions in (3.3)-(3.5), the component form of \( u_{n+1} = T(u_n) \) can be expressed as

\[
\begin{align*}
\{v_{n+1} = T_1(v_n, x_n) &= (I - \text{prox}_{\frac{2}{\lambda} f_1})(D(x_n - \gamma \nabla f_2(x_n)) + (I - \lambda DD^T)v_n), \\
\{x_{n+1} = T_2(v_n, x_n) &= x_n - \gamma \nabla f_2(x_n) - \lambda DD^T \circ T_1(v_n, x_n) \\
&= x_n - \gamma \nabla f_2(x_n) - \lambda DT_n v_{n+1}.
\end{align*}
\]

Therefore, the iteration \( u_{n+1} = T(u_n) \) is equivalent to (1.3).

**Lemma 3.7.** If \( 0 < \gamma < 2\beta, 0 < \lambda \leq 1/\lambda_{\text{max}}(DD^T) \), then \( T \) is nonexpansive under the norm \( \| \cdot \|_\lambda \).

**Lemma 3.8.** Suppose \( 0 < \gamma < 2\beta, 0 < \lambda \leq 1/\lambda_{\text{max}}(DD^T) \). Let \( u_n = (v_n, x_n) \) be the sequence generated by PDFP\( ^2O \). Then the sequence \( \{u_n\} \) converges to a fixed point of \( T \), and the sequence \( \{x_n\} \) converges to a solution of problem (1.1).

Now, we are ready to discuss the convergence of PDFP\( ^2O_{DS_n} \). To this end, let

\[
0 < \liminf_{n \to \infty} \gamma_n \leq \limsup_{n \to \infty} \gamma_n < 2\beta, 0 < \liminf_{n \to \infty} \lambda_n \leq \limsup_{n \to \infty} \lambda_n \leq 1/\lambda_{\text{max}}(DD^T),
\]

define \( T^n_1 : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^m \) as

\[
T^n_1(v, x) = (I - \text{prox}_{\frac{2}{\lambda_n} f_1})(D(x - \gamma_n \nabla f_2(x)) + (I - \lambda_n DD^T)v) \tag{3.10}
\]

11
\[ T_2^n(v, x) = x - \gamma_n \nabla f_2(x) - \lambda_n D^T \circ T_1^n. \]  

(3.11)

Denote
\[ T^n(v, x) = (T^n_1(v, x), T^n_2(v, x)). \]  

(3.12)

In the following, we will show the algorithm PDFP\(^2\)O\(_{DS_n}\) is a modified Mann iterative method related to the operator \(S^n\).

**Theorem 3.1.** Suppose \(0 < \liminf_{n \to \infty} \alpha_n \leq \limsup_{n \to \infty} \alpha_n < 1\). Set \(S^n = \alpha_n I + (1 - \alpha_n)T^n\). Then the sequence \(u_n\) of \(S^n\) is exactly the one obtained by the algorithm PDFP\(^2\)O\(_{DS_n}\).

**Proof.** According to the definitions in (3.10)-(3.12), the component form of \(u_{n+1} = T^n(u_n)\) can be expressed as

\[
\begin{aligned}
v_{n+1} &= T^n_1(v_n, x_n) = (I - \text{prox}_{\frac{\gamma_n}{\lambda_n} f_1})(D(x_n - \gamma_n \nabla f_2(x_n)) + (I - \lambda_n D^T)v_n), \\
x_{n+1} &= T^n_2(v_n, x_n) = x_n - \gamma_n \nabla f_2(x_n) - \lambda_n D^T \circ T^n_1(v, x_n) \\
&= x_n - \gamma_n \nabla f_2(x_n) - \lambda_n D^T v_{n+1}.
\end{aligned}
\]

Therefore, the iteration \(u_{n+1} = T^n(u_n)\) is equivalent to (2.2). Employing the similar argument, we can obtain the conclusion for general \(S^n\) with \(0 < \liminf_{n \to \infty} \alpha_n \leq \limsup_{n \to \infty} \alpha_n < 1\). \(\Box\)

**Remark 3.1.** From the last result, we find out that algorithm PDFP\(^2\)O\(_{DS_n}\) can also be obtained in the setting of fixed point iteration immediately.

**Theorem 3.2.** Let \(T^n, T\) be defined by 3.12, 3.5 respectively, suppose \(0 < \liminf_{n \to \infty} \gamma_n \leq \limsup_{n \to \infty} \gamma_n < 2\beta, 0 < \liminf_{n \to \infty} \lambda_n \leq \limsup_{n \to \infty} \lambda_n \leq 1/\lambda_{\text{max}}(D^T)\), if for any bounded sequence \(\{u_n\} \subset \mathbb{R}^m \times \mathbb{R}^n\),

\[
\lim_{n \to \infty} \|u_n - T^n(u_n)\|_\lambda = 0,
\]

then there exists a subsequence \(\{u_{n_k}\} \subset \{u_n\}\) such that \(\lim_{n_k \to \infty} \|u_{n_k} - T(u_{n_k})\|_\lambda = 0\).
Proof. Since the sequence $\gamma_n$ is bounded, there exists a subsequence $\gamma_{n_k} \subset \gamma_n$ such that $\gamma_{n_k} \to \gamma$ with $\gamma \in (0, 2\beta)$. Since the sequence $\lambda_n$ is bounded, there exists a subsequence $\lambda_{n_k} \subset \lambda_n$ such that $\lambda_{n_k} \to \lambda$ with $\lambda \in (0, 1/\lambda_{\text{max}}(DD^T))$. Let $T$ be defined by 3.5, since $\gamma \in (0, 2\beta)$ and $\lambda \in (0, 1/\lambda_{\text{max}}(DD^T))$, so $T$ is a nonexpansive mapping under the norm $\| \cdot \|_\lambda$. Since sequence $\{u_n\}$ is bounded and $\lim_{n \to \infty} \|u_n - T^n(u_n)\|_\lambda = 0$. We can know that

$$\|u_{n_k} - T(u_{n_k})\|_\lambda \leq \|u_{n_k} - T^{n_k}(u_{n_k})\|_\lambda + \|T^{n_k}u_{n_k} - T(u_{n_k})\|_\lambda. \quad (3.13)$$

From (3.9) we know

$$\|T^{n_k}u_{n_k} - T(u_{n_k})\|_\lambda^2 = \|T^{n_k}_1u_{n_k} - T_1(u_{n_k})\|^2 + \lambda\|T^{n_k}_2u_{n_k} - T_2(u_{n_k})\|^2. \quad (3.14)$$

By lemma 3.2, $I - \text{prox}_{\frac{\gamma}{\lambda} f_1}$ is a firmly nonexpansive operator. So

$$\|T^{n_k}_1u_{n_k} - T_1(u_{n_k})\| = \|(I - \text{prox}_{\frac{\gamma}{\lambda_{n_k}} f_1})(D(x_{n_k} - \gamma_{n_k} \nabla f_2(x_{n_k}))$$

$$+ (I - \lambda_{n_k} DD^T)v_{n_k}) - (I - \text{prox}_{\frac{\gamma}{\lambda} f_1})(D(x_{n_k} - \gamma \nabla f_2(x_{n_k}))$$

$$+ (I - \lambda DD^T)v_{n_k})\|$$

$$= \|\text{prox}_{\frac{\gamma_{n_k}}{\lambda_{n_k}} f_1}(D(x_{n_k} - \gamma_{n_k} \nabla f_2(x_{n_k}))$$

$$+ (I - \lambda_{n_k} DD^T)v_{n_k}) - \text{prox}_{\frac{\gamma}{\lambda} f_1}(D(x_{n_k} - \gamma \nabla f_2(x_{n_k}))$$

$$+ (I - \lambda DD^T)v_{n_k})\|$$

$$\leq \|\text{prox}_{\frac{\gamma_{n_k}}{\lambda_{n_k}} f_1}(D(x_{n_k} - \gamma_{n_k} \nabla f_2(x_{n_k}))$$

$$+ (I - \lambda_{n_k} DD^T)v_{n_k}) - \text{prox}_{\frac{\gamma}{\lambda} f_1}(D(x_{n_k} - \gamma \nabla f_2(x_{n_k}))$$

$$+ (I - \lambda DD^T)v_{n_k})\|$$

$$+ \|\text{prox}_{\frac{\gamma}{\lambda} f_1}(D(x_{n_k} - \gamma \nabla f_2(x_{n_k}))$$

$$+ (I - \lambda DD^T)v_{n_k}) - (I - \text{prox}_{\frac{\gamma_{n_k}}{\lambda_{n_k}} f_1})(D(x_{n_k} - \gamma_{n_k} \nabla f_2(x_{n_k}))$$

$$+ (I - \lambda_{n_k} DD^T)v_{n_k})\|$$

$$\leq \|\text{prox}_{\frac{\gamma_{n_k}}{\lambda_{n_k}} f_1}(D(x_{n_k} - \gamma_{n_k} \nabla f_2(x_{n_k}))$$

$$+ (I - \lambda_{n_k} DD^T)v_{n_k}) - \text{prox}_{\frac{\gamma}{\lambda} f_1}(D(x_{n_k} - \gamma \nabla f_2(x_{n_k}))$$

$$+ (I - \lambda DD^T)v_{n_k})\| + \|\text{prox}_{\frac{\gamma}{\lambda} f_1}(D(x_{n_k} - \gamma \nabla f_2(x_{n_k}))$$

$$+ (I - \lambda DD^T)v_{n_k}) - \text{prox}_{\frac{\gamma_{n_k}}{\lambda_{n_k}} f_1}(D(x_{n_k} - \gamma_{n_k} \nabla f_2(x_{n_k}))$$

$$+ (I - \lambda_{n_k} DD^T)v_{n_k})\|.$$
\[(I - \lambda DD^T)v_{n_k})\].

Let \(z_{n_k} = D(x_{n_k} - \gamma_{n_k} \nabla f_2(x_{n_k})) + (I - \lambda_{n_k} DD^T)v_{n_k}\). Since \(J_{\lambda f_1} = (I + \lambda f_1)^{-1} = \text{prox}_{\lambda f_1}\) and by lemma 3.3, we can know \(\text{prox}_{\nu f_1}x = \text{prox}_{\mu f_1}(\frac{\nu}{\mu}x + (1 - \frac{\nu}{\mu})\text{prox}_{\nu f_1}x)\), so

\[
\|\text{prox}_{\frac{\mu}{\nu}f_1}(z_{n_k}) - \text{prox}_{\frac{\nu}{\mu}f_1}(z_{n_k})\| = \|\text{prox}_{\frac{\mu}{\nu}f_1}(z_{n_k}) - \text{prox}_{\frac{\nu}{\mu}f_1}(z_{n_k})\|
\]

\[
\leq \left|1 - \frac{\gamma}{\lambda}/\lambda_{n_k}\right|\|\text{prox}_{\frac{\mu}{\nu}f_1}z_{n_k} - z_{n_k}\|
\]

\[
= |1 - \frac{\gamma}{\lambda}/\lambda_{n_k}|\|\text{prox}_{\frac{\mu}{\nu}f_1}z_{n_k} - z_{n_k}\|. \tag{3.16}
\]

On the other hand

\[
\|\text{prox}_{\frac{\mu}{\nu}f_1}(D(x_{n_k} - \gamma_{n_k} \nabla f_2(x_{n_k})) + (I - \lambda_{n_k} DD^T)v_{n_k})
\]

\[
- \text{prox}_{\frac{\nu}{\mu}f_1}(D(x_{n_k} - \gamma \nabla f_2(x_{n_k})) + (I - \lambda DD^T)v_{n_k})\|
\]

\[
\leq \|(\gamma - \gamma_{n_k})D\nabla f_2(x_{n_k}) + (\lambda - \lambda_{n_k})DD^Tv_{n_k}\|
\]

\[
\leq |\gamma - \gamma_{n_k}|\|D\nabla f_2x_{n_k}\| + |\lambda - \lambda_{n_k}|\|DD^Tv_{n_k}\|. \tag{3.17}
\]

Put (3.16) and (3.17) into (3.15), we can know

\[
\|T_{1}^{n_k}u_{n_k} - T_{1}(u_{n_k})\| \leq |1 - \frac{\gamma}{\lambda}/\lambda_{n_k}|\|\text{prox}_{\frac{\mu}{\nu}f_1}z_{n_k} - z_{n_k}\|
\]

\[
+ |\gamma - \gamma_{n_k}|\|D\nabla f_2x_{n_k}\| + |\lambda - \lambda_{n_k}|\|DD^Tv_{n_k}\|. \tag{3.18}
\]

Since \(\gamma_{n_k} \to \gamma\) and \(\lambda_{n_k} \to \lambda\), from (3.18) we can know

\[
\|T_{1}^{n_k}u_{n_k} - T_{1}(u_{n_k})\| \to 0. \tag{3.19}
\]

It follows from (3.11) that

\[
\|T_{2}^{n_k}u_{n_k} - T_{2}(u_{n_k})\| = \|x_{n_k} - \gamma_{n_k} \nabla f_2(x_{n_k}) - \lambda_{n_k} DT_{1}^{n_k}
\]

\[
- x_{n_k} - \gamma \nabla f_2(x_{n_k}) + \lambda DT_{1}\|
\]

\[14\]
\[ \leq |\gamma - \gamma_{n_k}|\|\nabla f_{2x_{n_k}}\| + |\lambda - \lambda_{n_k}|\|D^T T_{1n_k}\| \\
+ \|\lambda D^T\|\|T_{1n_k} - T_1\|. \] (3.20)

Since \(\gamma_{n_k} \to \gamma\) and \(\lambda_{n_k} \to \lambda\), from (3.19) we can know
\[ \|T_{2n_k} u_{n_k} - T_2(u_{n_k})\| \to 0. \] (3.21)

Put (3.19) and (3.21) into (3.14), we can know
\[ \|T_{n_k} u_{n_k} - T(u_{n_k})\|_\lambda^2 \to 0. \] (3.22)

Put (3.22) into (3.13), we can know
\[ \|u_{n_k} - T(u_{n_k})\|_\lambda \to 0. \] (3.23)

\[ \square \]

**Theorem 3.3.** Let \(T^n, T\) be defined by 3.12, 3.5 respectively, suppose \(0 < \liminf_{n \to \infty} \gamma_n \leq \limsup_{n \to \infty} \gamma_n < 2\beta, 0 < \liminf_{n \to \infty} \lambda_n \leq \limsup_{n \to \infty} \lambda_n \leq 1/\lambda_{\text{max}}(DD^T)\), let \(u_n\) be sequence defined by PDFP\(^2\)O\(_D\)S\(_n\), that is:
\[ u_{n+1} = S^n(u_n) = \alpha_n u_n + (1 - \alpha_n)T^m u_n, \] (3.24)

where \(\alpha_n\) satisfy
\[ 0 < \liminf_{n \to \infty} \alpha_n \leq \limsup_{n \to \infty} \alpha_n < 1. \] (3.25)

Then the sequence \(\{u_n\}\) defined by (3.24) converges to a fixed point of \(T\), and the sequence \(\{x_n\}\) converges to a solution of problem (1.1).

**Proof.** Let \(\hat{u} = (\hat{v}, \hat{x}) \in \mathbb{R}^m \times \mathbb{R}^n\) be a fixed point of \(T\). From (3.24) and lemma 3.4, we have
\[ \|u_{n+1} - \hat{u}\|_\lambda^2 = \|(1 - \alpha_n)u_n + \alpha_n T^m u_n - \hat{u}\|_\lambda^2 = \alpha_n \|u_n - \hat{u}\|_\lambda^2 + (1 - \alpha_n)\|T^m u_n - \hat{u}\|_\lambda^2 - \alpha_n(1 - \alpha_n)\|u_n - T^m u_n\|_\lambda^2. \] (3.26)

Since the sequence \(\lambda_n\) is bounded, there exists a convergent subsequence converges to \(\lambda\), without loss of generality, we may assume that the convergent subsequence is \(\lambda_n\).
itself, then we have \( \lambda_n \to \lambda \). That is, \( \exists N_0 \in \mathbb{N} \) such that \( \lambda_n \leq \lambda \). So by the similar proof of theorem 3.3 in [1], for \( \forall n \geq N_0 \), we have

\[
\| T^n u_n - \hat{u} \|^2_{\lambda} = \| T^n u_n - T^n \hat{u} \|^2_{\lambda} \\
\leq \| u_n - \hat{u} \|^2_{\lambda} + \lambda_n \| v_n - \hat{v} \|^2 + (\lambda - \lambda_n) \| T_1^n (u_n) - \hat{v} \|^2 \\
\leq \| u_n - \hat{u} \|^2_{\lambda}. \quad (3.27)
\]

Substituting (3.27) into (3.26), we obtain

\[
\| u_{n+1} - \hat{u} \|^2_{\lambda} \leq \| u_n - \hat{u} \|^2_{\lambda} - \alpha_n (1 - \alpha_n) \| u_n - T^n u_n \|^2_{\lambda}. \quad (3.28)
\]

Which implies that

\[
\| u_{n+1} - \hat{u} \|_{\lambda} \leq \| u_n - \hat{u} \|_{\lambda},
\]

this implies that sequence \( u_n \) is a Fejér monotone sequence, and \( \lim_{n \to \infty} \| u_{n+1} - \hat{u} \|_{\lambda} \) exists.

Since the sequence \( \alpha_n \) satisfies (3.25), there exists \( \bar{a}, \underline{a} \in (0, 1) \) such that \( \underline{a} < \alpha_n < \bar{a} \). So by (3.28), we know

\[
\underline{a} (1 - \bar{a}) \| u_n - T^n u_n \|^2_{\lambda} \leq \alpha_n (1 - \alpha_n) \| u_n - T^n u_n \|^2_{\lambda} \\
\leq \| u_n - \hat{u} \|^2_{\lambda} - \| u_{n+1} - \hat{u} \|^2_{\lambda}, \quad (3.29)
\]

Let \( n \to \infty \) in (3.29), we have

\[
\| u_n - T^n u_n \|_{\lambda} \to 0. \quad (3.30)
\]

Since the sequence \( u_n \) is bounded and there exists a convergent subsequence \( u_{n_j} \) such that

\[
u_{n_j} \to \tilde{u}, \quad (3.31)\]

for some \( \tilde{u} \in \mathbb{R}^m \times \mathbb{R}^n \). From Theorem 3.2 and (3.30), we have

\[
\| u_{n_j} - T u_{n_j} \|_{\lambda} \to 0.
\]

16
By Lemma 3.5, we know $\tilde{u} \in \text{Fix}(T)$. Moreover, we know that $\|u_n - \tilde{u}\|_\lambda$ is non-increasing for any fixed point $\hat{u}$ of $T$. In particular, by choosing $\hat{u} = \tilde{u}$, we have $\|u_n - \tilde{u}\|_\lambda$ is non-increasing. Combining this and (3.31) yields

$$u_n \to \tilde{u}.$$  

Writing $\tilde{u} = (\tilde{v}, \tilde{x})$ with $\tilde{v} \in \mathbb{R}^m, \tilde{x} \in \mathbb{R}^n$, we find from Lemma 3.6 that $\tilde{x}$ is the solution of problem (1.1). 

\[\square\]

### 3.2 Linear convergence rate for special cases

In this section, we will give some stronger theoretical results about the convergence rate in some special cases. For this, we present the following condition.

**Condition 3.1.** For any two real numbers $\lambda$ and $\gamma$ satisfying that $0 < \gamma < 2\beta$ and $0 < \lambda \leq 1/\lambda_{\text{max}}(DD^T)$, there exist $\mu, \nu \in [0, 1)$ such that $\|I - \lambda DD^T\|_2 \leq \mu^2$ and

$$\|g(x) - g(y)\|_2 \leq \nu\|x - y\|_2, \quad \text{for all } x, y \in \mathbb{R}^n.$$ 

**Remark 3.2.** If $D$ has full row rank, $f_2$ is strongly convex, i.e. there exists some $\sigma > 0$ such that

$$\langle \nabla f_2(x) - \nabla f_2(y), x - y \rangle \geq \sigma\|x - y\|_2^2, \quad \text{for all } x, y \in \mathbb{R}^n, \quad (3.33)$$

then this condition can be satisfied. In fact, when $D$ has a full row rank, we can choose

$$\mu^2 = 1 - \lambda \lambda_{\text{min}}(DD^T)$$

where $\lambda_{\text{min}}(DD^T)$ denotes the smallest eigenvalue of $DD^T$. In this case, $\mu^2$ takes its minimum

$$(\mu^2)_{\text{min}} = 1 - \frac{\lambda_{\text{min}}(DD^T)}{\lambda_{\text{max}}(DD^T)}$$

at $\lambda = 1/\lambda_{\text{max}}(DD^T)$. On the other hand, since $f_2$ have $1/\beta$-Lipschitz continuous gradient and is strongly convex, it follows from proof in [1] we know

$$\|g(x) - g(y)\|_2^2 \leq (1 - \frac{\gamma \sigma (2\beta - \gamma)}{\beta})\|x - y\|_2^2. \quad (3.29)$$

17
Hence we can choose
\[ \nu^2 = 1 - \left( \frac{\gamma \sigma (2 \beta - \gamma)}{\beta} \right). \]
In particular, if we choose \( \beta = \gamma \), then \( \nu^2 \) takes its minimum in the present form:
\[ \nu^2 = 1 - \sigma \gamma. \]

Despite most of our interesting problems not belonging to these special cases, and there will be more efficient algorithms if condition 3.1 is satisfied, the following results still have some theoretical values where the best performance of PDFP\(^2\) can be achieved. First of all, we show that \( S \) is contractive under condition 3.1.

**Theorem 3.4.** Assume condition 3.1 holds true. Let the operator \( T \) be given in (3.5) and \( S = \alpha_n I + (1 - \alpha_n)T \) for \( 0 < \lim \inf_{n \to \infty} \alpha_n \leq \lim \sup_{n \to \infty} \alpha_n < 1 \). Then \( S \) is contractive under the norm \( \| \cdot \|_\lambda \).

**Proof.** Let \( \eta = \max\{\mu, \nu\} \). It is clear that \( 0 \leq \eta \leq 1 \). Then, owing to the condition 3.1 and the proof of Theorem 3.6 of [1], for all \( u_1 = (v_1, x_1), u_2 = (v_2, x_2) \in \mathbb{R}^m \times \mathbb{R}^n \), there holds
\[ \|T(u_1) - T(u_2)\|_\lambda \leq \eta \|u_1 - u_2\|_\lambda, \]
then
\[ \|S(u_1) - S(u_2)\|_\lambda \leq \alpha_n \|u_1 - u_2\|_\lambda + (1 - \alpha_n)\|T(u_1) - T(u_2)\|_\lambda \leq \theta_{\alpha_n} \|u_1 - u_2\|_\lambda, \]
with \( \theta_{\alpha_n} = \alpha_n + (1 - \alpha_n) \eta \in (0, 1) \). So, operator \( S \) is contractive. By the Banach contraction mapping theorem, it has a unique fixed point, denoted by \( \bar{u} = (\bar{v}, \bar{x}) \). It is obvious that \( S \) has the same fixed points as \( T \), so \( \bar{x} \) is the unique solution of problem (1.1) from lemma 3.6.

Now, we are ready to analyze the convergence rate of PDFP\(^2\)\(O_{DS_n}\).

**Theorem 3.5.** Assume condition 3.1 holds true. Let the operator \( T \) be given in (3.5) and \( T^n \) be defined as 3.12 with \( \emptyset \neq Fix(T) = \cap_{n=1}^\infty Fix(T^n) \). For any \( u_0 \in \mathbb{R}^m \times \mathbb{R}^n \), the sequence \( u_n \) be a sequence obtained by algorithm PDFP\(^2\)\(O_{DS_n}\), and \( 0 < \lim \inf_{n \to \infty} \alpha_n \leq \).
lim sup\(\alpha_n < 1\). Then the sequence \(\{u_n\}\) must converge to the unique fixed point \(\bar{u} = (\bar{v}, \bar{x})\) \(\in \mathbb{R}^m \times \mathbb{R}^n\) of \(T\) with \(\bar{x}\) being the unique solution of problem (1.1). Furthermore, there holds the estimate
\[
\|x_n - \bar{x}\|_2 \leq \frac{d(\theta_{\alpha_n})^n}{1 - \theta_{\alpha_n}},
\]
where \(d = \|u_1 - u_0\|_\lambda\), \(\theta_{\alpha_n} = \alpha_n + (1 - \alpha_n)\eta \in (0, 1)\) and \(\eta = \max\{\mu, \nu\}\) with \(\mu\) and \(\nu\) given in condition 3.1.

Proof. From Theorem 3.3, we can know that the sequence \(\{u_n\}\) converges to \(\bar{u}\). On the other hand, it follows from theorem 3.4 that
\[
\|u_{n+1} - u_n\|_\lambda \leq \|u_n - u_{n-1}\|_\lambda \leq \cdots \leq (\theta_{\alpha_n})^n \|u_1 - u_0\|_\lambda = d(\theta_{\alpha_n})^n.
\]
So for all \(0 < l \in \mathbb{N}\),
\[
\|u_{n+l} - u_n\|_\lambda \leq \sum_{i=1}^{l} \|u_{n+i} - u_{n+i-1}\|_\lambda = d(\theta_{\alpha_n})^n \sum_{i=1}^{l} (\theta_{\alpha_n})^{i-1} \leq \frac{d(\theta_{\alpha_n})^n}{1 - \theta_{\alpha_n}},
\]
which immediately implies
\[
\|x_n - \bar{x}\|_2 \leq \|u_n - \bar{u}\|_\lambda \leq \frac{d(\theta_{\alpha_n})^n}{1 - \theta_{\alpha_n}},
\]
by letting \(l \to +\infty\). The desired estimate (3.29) is then obtained. \(\Box\)

Remark 3.3. Since sequence \(\alpha_n\) satisfy \(0 < \lim\inf_{n \to \infty} \alpha_n \leq \lim\sup_{n \to \infty} \alpha_n < 1\), then exists \(\underline{a}, \bar{a} \in (0, 1)\) such that \(\underline{a} < \alpha_n < \bar{a}\). So we have \(\underline{a} + (1 - \bar{a})\eta < \alpha_n + (1 - \alpha_n)\eta\). In particular, if we choose \(\theta_{\alpha_n} = \underline{a} + (1 - \bar{a})\eta = \theta_a\), then we obtain
\[
\|x_n - \bar{x}\|_2 \leq \frac{d(\theta_a)^n}{1 - \theta_a}.
\]
(3.35)

It will follow that our scheme shows an \(o\left(\frac{d(\theta_a)^n}{1 - \theta_a}\right)\) convergence to the optimum for the variable \(x_n\), which is an optimal rate.

4 Connections to other algorithms

We will further investigate the proposed algorithm PDFP\(^2\)O\(_{DS}\) from the perspective of primal-dual forms and establish the connections to other existing methods.
4.1 Primal-dual and proximal point algorithms

For problem (1.1), we can write its primal-dual form using the Fenchel duality [18] as

$$\min_x \max_y G(x, y) := \langle Dx, v \rangle - f_1^*(v) + f_2(x),$$

(4.1)

where $f_1^*$ is the convex conjugate function of $f_1$ defined by

$$f_1^*(v) = \sup_{w \in \mathbb{R}^m} \langle v, w \rangle - f_1(w).$$

By introducing a new intermediate variable $y_{n+1}$, equations (2.2) are reformulated as

$$\left\{ \begin{array}{l}
y_{n+1} = x_n - \gamma_n \nabla f_2(x_n) - \lambda_n D^T v_n, \\
v_{n+1} = (I - \text{prox}_{\frac{\lambda_n}{\gamma_n} f_1})(Dy_{n+1} + v_n), \\
x_{n+1} = x_n - \gamma_n \nabla f_2(x_n) - \lambda_n D^T v_{n+1},
\end{array} \right. \quad (4.2)$$

According to Moreau decomposition (see equation (2.21) in [2]), for all $v \in \mathbb{R}^m$, we have

$$v = v_\oplus + v_\ominus,$$

where $v_\oplus = \text{prox}_{\frac{\lambda_n}{\gamma_n} f_1} v$, $v_\ominus = \frac{\lambda_n}{\gamma_n} \text{prox}_{\frac{\lambda_n}{\gamma_n} f_1} \left( \frac{\lambda_n}{\gamma_n} v \right)$, from which we know

$$(I - \text{prox}_{\frac{\lambda_n}{\gamma_n} f_1})(Dy_{n+1} + v_n) = \frac{\lambda_n}{\gamma_n} \text{prox}_{\frac{\lambda_n}{\gamma_n} f_1} \left( \frac{\lambda_n}{\gamma_n} Dy_{n+1} + \frac{\lambda_n}{\gamma_n} v_n \right).$$

Let $\bar{v}_n = \frac{\lambda_n}{\gamma_n} v_n$. Then (4.2) can be reformulated as

$$\left\{ \begin{array}{l}
y_{n+1} = x_n - \gamma_n \nabla f_2(x_n) - \lambda_n D^T \bar{v}_n, \\
v_{n+1} = \text{prox}_{\frac{\lambda_n}{\gamma_n} f_1} \left( \frac{\lambda_n}{\gamma_n} Dy_{n+1} + \bar{v}_n \right), \\
x_{n+1} = x_n - \gamma_n \nabla f_2(x_n) - \lambda_n D^T \bar{v}_{n+1}.
\end{array} \right. \quad (4.3)$$

For terms of the saddle point formulation (4.1), with the same idea in [1](4.1 Primal-dual and proximal point algorithms), the iterations (4.3) can be expressed as

$$\left\{ \begin{array}{l}
\bar{v}_{n+1} = \arg \max_{v \in \mathbb{R}^m} G(x_{n+1}, v) - \frac{\gamma_n}{2\lambda_n} \| \bar{v}_n - \bar{v}_n \|^2 M_n, \\
x_{n+1} = x_n - \gamma_n \nabla_x G(x_n, \bar{v}_{n+1}),
\end{array} \right. \quad (4.4)$$

where $M_n = I - \lambda_n DD^T.$

20
Table 1. Comparison between CP ($\theta_n = 1$) and PDFP$^{2O_{DS}}$.

|            | CP ($\theta_n = 1$)                                                                 | PDFP$^{2O_{DS}}$                                                                 |
|------------|-----------------------------------------------------------------------------------|-----------------------------------------------------------------------------------|
| Form       | $\bar{v}_{n+1} = (I + \sigma_n \partial f_1^*)^{-1}(\bar{v}_n + \sigma_n Dy_{n+1})$ | $\bar{v}_{n+1} = (I + \frac{\lambda_{n}}{\gamma_n} \partial f_1^*)^{-1}(\bar{v}_n + \frac{\lambda_{n}}{\gamma_n} Dy_{n+1})$ |
|            | $x_{n+1} = (I + \tau_n \nabla f_2^{-1})(x_n - \tau_n DT \bar{v}_{n+1})$           | $x_{n+1} = x_n - \gamma_n \nabla f_2^{-1}(x_n - \gamma_n DT \bar{v}_{n+1})$      |
|            | $y_{n+1} = 2x_{n+1} - x_n$                                                        | $y_{n+1} = x_{n+1} - \gamma_n \nabla f_2^{-1}(x_n + 1 - \gamma_n DT \bar{v}_{n+1})$ |
| Convergence| $0 < \lim \inf_{n \to \infty} \sigma_n \tau_n \leq \lim \sup_{n \to \infty} \sigma_n \tau_n < 1/\lambda_{\text{max}}(DD^T)$ | $0 < \lim \inf_{n \to \infty} \gamma_n \leq \lim \sup_{n \to \infty} \gamma_n < 2\beta$ |
| Relation   | $\sigma_n = \frac{\lambda_{n}}{\gamma_n}$, $\tau_n = \gamma_n$                | $\sigma_n = \frac{\lambda_{n}}{\gamma_n}$, $\tau_n = \gamma_n$                |

This leads to a close connection with a class of primal-dual method studied in [19-22]. For example, in [19], Chambolle and Pock proposed the following scheme for solving (4.1):

$$\begin{align*}
\bar{v}_{n+1} &= (I + \sigma_n \partial f_1^*)^{-1}(\bar{v}_n + \sigma_n Dy_{n+1}) \quad (4.5a) \\
x_{n+1} &= (I + \tau_n \nabla f_2^{-1})(x_n - \tau_n DT \bar{v}_{n+1}) \quad (4.5b) \\
y_{n+1} &= \theta_n x_{n+1} - x_n \quad (4.5c)
\end{align*}$$

where $\sigma_0, \tau_0 > 0$, $\theta_n \in [0, 1]$ is a variable relaxation parameter. For $\sigma_n = \sigma$, $\tau_n = \tau$ and $\theta_n \equiv 0$, we can obtain the classical Arrow-Hurwicz-Uzawa (AHU) method in [23]. The convergence of AHU with very small step length is shown in [20]. Under some assumptions on $f_1$ or strong convexity of $f_2$, global convergence of the primal-dual gap can also be shown with specific chosen adaptive steplength [19].

According to equation (4.3), using the relation $\text{prox}_{\frac{\lambda_{n}}{\gamma_n} f_1^*} = (I + \frac{\lambda_{n}}{\gamma_n} \partial f_1^*)^{-1}$, and
changing the order of these equations, we know that PDFP$^2O_{DS}$ is equivalent to
\begin{align*}
\begin{cases}
\bar{v}_{n+1} = (I + \frac{\lambda_n}{\gamma_n} \partial f_1^*)^{-1}(\bar{v}_n + \frac{\lambda_n}{\gamma_n} Dy_n), & (4.6a) \\
x_{n+1} = x_n - \gamma_n \nabla f_2(x_n) - \gamma_n D^T \bar{v}_{n+1}, & (4.6b) \\
y_{n+1} = x_{n+1} - \gamma_n \nabla f_2(x_{n+1}) - \gamma_n D^T \bar{v}_{n+1}. & (4.6c)
\end{cases}
\end{align*}

Let $\sigma_n = \frac{\lambda_n}{\gamma_n}, \tau_n = \gamma_n$ ($n \in \mathbb{N}$), then we can see that equations (4.5b) and (4.5c) are approximated by two explicit steps (4.6b)-(4.6c). In summary, we list the comparisons of CP for $\theta_n \equiv 1$ with the fixed step length and PDFP$^2O_{DS}$ in table 1.

### 4.2 Splitting type of methods

There are other types of methods which are designed to solve problem (1.1) based on the notion of an augmented Lagrangian. For simplicity, we only study the connections and differences in alternating split Bregman (ASB), split inexact Uzawa (SIU) and PDFP$^2O_{DS}$, for $f_2(x) = \frac{1}{2} \|Ax - b\|^2_2$.

ASB present by Goldstein and Osher [24] can be described as follows:
\begin{align*}
\begin{cases}
x_{n+1} = (A^T A + \nu_n D^T D)^{-1}(A^T b + \nu_n D^T (d_n - v_n)), & (4.7a) \\
d_{n+1} = \text{prox}_{\frac{1}{\nu_n} f_1}(Dx_{n+1} + v_n), & (4.7b) \\
v_{n+1} = v_n - (d_{n+1} - Dx_{n+1}), & (4.7c)
\end{cases}
\end{align*}

where $\lim_{n \to \infty} \nu_n > 0$ is a dynamic parameter. The explicit SIU method proposed in the literature [22] can be described as
\begin{align*}
\begin{cases}
x_{n+1} = x_n - \delta_n A^T (Ax_n - b) - \delta_n \nu_n D^T (Dx_n - d_n + v_n), & (4.8a) \\
d_{n+1} = \text{prox}_{\frac{1}{\nu_n} f_1}(Dx_{n+1} + v_n), & (4.8b) \\
v_{n+1} = v_n - (d_{n+1} - Dx_{n+1}), & (4.8c)
\end{cases}
\end{align*}

where $\lim_{n \to \infty} \delta_n > 0$ is a dynamic parameter.

From (4.2a) and (4.2c), we can find out a relation between $y_n$ and $x_n$, given by
\begin{equation*}
x_n = y_n - \lambda_n D^T (v_n - v_{n+1}).
\end{equation*}

Then eliminating $x_n$, PDFP$^2O_{DS}$ can be expressed as
\begin{align*}
\begin{cases}
y_{n+1} = y_n - \lambda_n D^T (2v_n - v_{n-1}) - \gamma_n \nabla f_2(y_n - \lambda_n D^T (v_n - v_{n-1})), & (4.9a) \\
v_{n+1} = (I - \text{prox}_{\frac{1}{\nu_n} f_1})(Dy_{n+1} + v_n). & (4.9b)
\end{cases}
\end{align*}
By introducing the splitting variable \( d_{n+1} \) in (4.9b), (4.9) can be further expressed as

Table 2 The comparisons among ASB, SIU and PDFP\(^2\)O\(_{DS}\).

| Method | Form | Convergence |
|--------|------|-------------|
| **ASB** | \[ x_{n+1} = (A^T A + \nu_n D^T D)^{-1}(A^T b + \nu_n D^T (d_n - v_n)) \] | \( \lim \inf_{n \to \infty} \nu_n > 0 \) |
| | \[ d_{n+1} = \text{prox}_{\frac{1}{\nu_n} f_1} (D x_{n+1} + v_n) \] | |
| | \[ v_{n+1} = v_n - (d_{n+1} - D x_{n+1}) \] | |
| **SIU** | \[ x_{n+1} = x_n - \delta_n A^T (A x_n - b) - \delta_n \nu_n D^T (D x_n - d_n + v_n) \] | \( \lim \inf_{n \to \infty} \nu_n > 0 \) |
| | \[ d_{n+1} = \text{prox}_{\frac{1}{\nu_n} f_1} (D x_{n+1} + v_n) \] | \( 0 < \lim \inf_{n \to \infty} \delta_n \leq \lim \sup_{n \to \infty} \delta_n \leq 1/\lambda_{\max}(A^T A + D D^T) \) |
| | \[ v_{n+1} = v_n - (d_{n+1} - D x_{n+1}) \] | |
| **PDFP\(^2\)O\(_{DS}\)** | \[ x_{n+1} = x_n - \delta_n A^T (A x_n - b) - \delta_n \nu_n D^T (D x_n - d_n + v_n) \] | \( 0 < \lim \inf_{n \to \infty} \delta_n \leq \lim \sup_{n \to \infty} \delta_n < 2/\lambda_{\max}(A^T A) \) |
| | \[ -\delta_n^2 \nu_n A^T A D^T (d_n - D x_n) \] | \( 0 < \lim \inf_{n \to \infty} \delta_n \nu_n \leq \lim \sup_{n \to \infty} \delta_n \nu_n \leq 1/\lambda_{\max}(D D^T) \) |
| | \[ d_{n+1} = \text{prox}_{\frac{1}{\nu_n} f_1} (D y_{n+1} + v_n), \] | |
| | \[ v_{n+1} = v_n - (d_{n+1} - D y_{n+1}) \] | |

\[
\begin{aligned}
y_{n+1} & = y_n - \lambda_n D^T (D y_n - d_n + v_n) - \gamma_n \nabla f_2 (y_n - \lambda_n D^T (D y_n - d_n)), \\
d_{n+1} & = \text{prox}_{\frac{1}{\nu_n} f_1} (D y_{n+1} + v_n), \\
v_{n+1} & = v_n - (d_{n+1} - D y_{n+1}).
\end{aligned}
\]

For \( f_2(x) = \frac{1}{2} \| Ax - b \|_2^2 \), \( \nabla f_2(x) = A^T (Ax - b) \). By changing the order and letting
\( \gamma_n = \delta_n, \lambda_n = \delta_n \nu_n (\forall n \in \mathbb{N}), \) (4.10) becomes

\[
\begin{cases}
    y_{n+1} = y_n - \delta_n A^T (Ay_n - b) - \delta_n \nu_n D^T (Dy_n - d_n + v_n) \\
    -\delta_n^2 \nu_n A^T AD^T (d_n - By_n), \\
    d_{n+1} = \text{prox}_{\frac{1}{\delta_n} f_1} (Dy_{n+1} + v_n), \\
    v_{n+1} = v_n - (d_{n+1} - Dy_{n+1}).
\end{cases}
\] (4.11a)

We can easily see that equation (4.7a) in ASB is approximated by (4.10a). Although it seems that PDFP\(^2\)O\(_{DS}\) requires more computation in (4.10a) than SIU in (4.8a), PDFP\(^2\)O\(_{DS}\) has the same computation cost as that of SIU if the iterations are implemented cleverly. For the reason of comparison, we can change the variable \( y_n \) to \( x_n \) in (4.10). Table 2 gives the summarized comparisons among ASB, SIU and PDFP\(^2\)O\(_{DS}\). We note that the only difference of SIU and PDFP\(^2\)O\(_{DS}\) is in the first step. As two algorithms converge, the algorithm PDFP\(^2\)O\(_{DS}\) behaves asymptotically the same as SIU since \( d_n - Dx_n \) converges to 0. The parameters \( \delta_n \) and \( \nu_n \) satisfy respectively different conditions to ensure the convergence.

5 Numerical experiments

In this section, we compare our proposed algorithm with the state-of-the-art methods of PDFP\(^2\)O in the CT image reconstruction problem. The test image is the standard benchmark Shepp-Logan phantom (see Figure 2) with size of 256 \( \times \) 256 and the pixels values vary from 0 to 1. All experiments were performed under Windows 7 and MATLAB (R2009a) running on a desktop with an Intel Core 2 Quad cpu and 2GB of RAM.

We use the toolbox of AIRTools to create 2D tomography test problems. In the experiment setting, the projection angle is chosen from 0 to 175 degrees in increments of 10 degrees and the number of parallel rays in each angle is \( p = 362 \). We add Gaussian white noise \( e \) of relative magnitude \( \| e \| / \| Ax_{true} \| = 0.01. \)
The performances were evaluated in terms of the mean signal-to-noise ratio (SNR) and the relative error (RelErr). The definitions of SNR and RelErr are given as follows:

\[
SNR = 20 \log_{10} \left( \frac{\|x_{\text{true}}\|}{\|x - x_{\text{true}}\|} \right),
\]

and

\[
RelErr = \frac{\|x - x_{\text{true}}\|^2}{\|x_{\text{true}}\|^2},
\]

where \(x\) and \(x_{\text{true}}\) are the reconstructed image and original image, respectively.

We follow the paper of [1] to choose the parameters for the PDFP^2O. That is, the \(\gamma = 2/\beta\), where \(\beta\) is the Lipschitz constant, and \(\lambda = 1/8\). For our proposed algorithm, we choose the dynamic stepsize \(\gamma_n\) as follows:

\[
\gamma_n = \frac{f_2(x_n)}{\|\nabla f_2(x_n)\|^2},
\]

where \(f_2(x_n) = \|Ax_n - b\|^2\).

We tested anisotropic total variation and isotropic total variation regularization term and found the performance of anisotropic total variation slightly better than isotropic total variation. Therefore, we only present results using anisotropic total variation here. The reconstructed image is shown in Figure 2. As we can see, both the algorithms achieve the good performance to reconstruct the original image.
Figure 2: The image reconstructed by the PDFP\textsuperscript{2}O and PDFP\textsuperscript{2}O\textsubscript{DS}. Their SNR are 23.43 and 23.42 (db), respectively.

![Figure 2](image1.png)

Figure 3: The comparison of SNR and RelErr between PDFP\textsuperscript{2}O and PDFP\textsuperscript{2}O\textsubscript{DS}

![Figure 3](image2.png)

We can see from Figure 3 that the proposed algorithm perform better than the PDFP\textsuperscript{2}O. Since the dynamic stepsize was introduced in PDFP\textsuperscript{2}O\textsubscript{DS}, it converges faster than the original with constant stepsize. The more details of the choice of parameters $\gamma_n$ and $\lambda_n$ can be found in [25].

Acknowledgements

This work was supported by the National Natural Science Foundation of China (11131006, 41390450, 91330204, 11401293), the National Basic Research Program of China (2013CB 329404), the Natural Science Foundations of Jiangxi Province (CA20110...
References

[1] Chen P J, Huang J G and Zhang X Q 2013 A primal-dual fixed point algorithm for convex separable minimization with applications to image restoration Inverse Problems 29 025011-33.

[2] Combettes P L and Wajs V R 2005 Signal recovery by proximal forward-backward splitting Multiscale Model. Simul. 4 1168-200.

[3] Rudin L I, Osher S and Fatemi E 1992 Nonlinear total variation based noise removal algorithms Physica D 60 259-68.

[4] Micchelli C A, Shen L and Xu Y 2011 Proximity algorithms for image models: denoising Inverse Problems 27 45009-38.

[5] Bruck R E and Passty G B 1979 Almost convergence of the infinite product of resolvents in Banach spaces Nonlinear Anal. 3 279-282.

[6] Bruck R E and Reich S 1977 Nonexpansive projections and resolvents in Banach spaces Houston J. Math. 3 459-470.

[7] Bauschke HH, Combettes PL. Convex Analysis and Monotone Operator Theory in Hilbert Spaces[M]. Springer, London, 2011.

[8] Micchelli C A, Shen L and Xu Y 2011 Proximity algorithms for image models: denoising Inverse Problems 27 45009-38.

[9] Moreau J-J 1962 Fonctions convexes duales et points proximaux dans un espace hilbertien C. R. Acad. Sci., Paris I 255 2897-99.

[10] Argyriou A, Micchelli C A, Pontil M, Shen L and Xu Y 2011 Efficient first order methods for linear composite regularizers arXiv:1104-1436.

[11] Goldstein T, ODonoghue B and Setzer S 2012 Fast alternating direction methods UCLA CAM Report (12-35)
[12] Nesterov Y 1983 A method of solving a convex programming problem with convergence rate $O(1/k^2)$ Sov. Math.-Dokl. 27 372-6.

[13] Tseng P 2008 On accelerated proximal gradient methods for convex-concave optimization Preprint (pages.cs.wisc.edu/~brecht/cs72bdocs/Tseng.APG.pdf)

[14] Tseng P 2010 Approximation accuracy, gradient methods, and error bound for structured convex optimization Math. Program. 125 263-95.

[15] Yuan M, Lin Y 2006 Model selection and estimation in regression with grouped variables. Journal of the Royal Statistical Society, Series B 68(1)49-67.

[16] Jenatton R, Audibert J Y and Bach F 2009 Structured variable selection with sparsity-inducing norms. arXiv:0904.3523v2.

[17] Zhao P, Rocha G and Yu B 2009 Grouped and hierarchical model selection through composite absolute penalties. Annals of Statistics, 37(6A):3468-3497.

[18] Rockafellar R T 1970 Convex Analysis (Princeton, NJ: Princeton University Press)

[19] Chambolle A and Pock T 2011 A first-order primal-dual algorithm for convex problems with applications to imaging J. Math. Imaging Vis. 40 120-45.

[20] Esser E, Zhang X and Chan T F 2010 A general framework for a class of first order primal-dual algorithms for convex optimization in imaging science SIAM J. Imaging Sci. 3 1015-46.

[21] He B and Yuan X 2012 Convergence analysis of primal-dual algorithms for a saddle-point problem: from contraction perspective SIAM J. Imaging Sci. 5 119-49.

[22] Zhang X, Burger M and Osher S 2011 A unified primal-dual algorithm framework based on Bregman iteration J. Sci. Comput. 46 20-46.

[23] Arrow K J, Hurwicz L and Uzawa H 1958 Studies in Linear and Non-linear Programming (Stanford: Stanford University Press)
[24] Goldstein T and Osher S 2009 The split Bregman method for l1 regularized problems SIAM J. Imaging Sci. 2 323-43.

[25] G. López, V. Martín-Márquez, F H Wang and H K Xu 2012 Solving the split feasibility problem without prior knowledge of matrix norms Inverse Problems 28 (2012) 085004 (18pp)
