THE SUBGROUP HOMOLOGY DECOMPOSITION FOR FUSION SYSTEMS IS SHARP

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ABSTRACT. We extend Dwyer’s sharp subgroup homology decomposition of the classifying space of a finite group to arbitrary saturated fusion systems and arbitrary Mackey functors.

1. INTRODUCTION

Let $G$ be a finite group and $p$ a prime. In the nineties there was an intense interest among topologists in reconstructing the classifying space $BG$, up to mod-$p$ cohomology, by gluing together classifying spaces of subgroups of $G$. These are so called homology decompositions and among the main contributors are Benson, Dwyer, Jackowski, McClure, Oliver and Wilkerson. See [8] for a unified treatment and references. Some homology decompositions turn out to be sharp, in the sense that the associated Bousfield–Kan mod-$p$ cohomology spectral sequence collapses onto the vertical axis. As a consequence, a sharp homology decomposition describes the mod-$p$ cohomology of $BG$ as a limit of the cohomology of the classifying spaces of certain subgroups of $G$. The systematic study of this phenomenon was initiated by Dwyer in [9] and completed by Grodal and Smith in [10].

It is well-known that the properties of $BG$ up to mod-$p$ cohomology are closely related to the $p$-fusion pattern of $G$, namely the $p$-subgroups of $G$ and conjugation maps between them. Two examples of this deep connection are the Cartan–Eilenberg stable elements theorem [4, XII.10.1] and the, now proven, Martino–Priddy conjecture [12, 13]. The notion of the $p$-fusion pattern of a finite group has been successfully extended to that of a saturated fusion system. A saturated fusion system is a category $F$ whose objects are the subgroups of a fixed finite $p$-group $S$ and whose morphisms are injective group homomorphisms between them. These morphisms must satisfy some additional conditions modelled on the $p$-fusion pattern of a finite group. By the combined work of group theorists, representation theorists and topologists, now saturated fusion systems are considered to be the right setup for studying $p$-local aspects of finite groups and related spaces. See [1] for an introduction to this subject and for precise definitions.

Saturated fusion systems have a rich homotopy theory similar to that of $p$-completed classifying spaces of finite groups. After much effort, it has been proven recently [5, 14] that each saturated fusion system $F$ has a unique classifying space $BF$. This space has homology decompositions [2] §2 [1] III.5.5, 5.6, 5.8 similar to
those for $BG$. In particular, $B\mathcal{F}$ has a subgroup homology decomposition

$$B\mathcal{F} \simeq \operatorname{hocolim} \tilde{B}.$$  

Dwyer [9, Theorem 10.3] showed that the corresponding decomposition for $BG$

$$BG \simeq \operatorname{hocolim} \tilde{B}$$

is sharp, but it has not been known whether the decomposition for $B\mathcal{F}$ is sharp or not. This problem figures in Oliver’s list of open questions [1, III.7(9)]. In this paper, we answer this question in the affirmative.

**Theorem 1.1.** Let $\mathcal{F}$ be a saturated fusion system on a finite $p$-group $S$. Then

$$\lim_{\to \infty} \mathcal{O}(\mathcal{F}^c) M^*|_{\mathcal{O}(\mathcal{F}^c)} = 0$$

for $i > 0$.

This also gives a new proof of the following theorem.

**Theorem 1.2** ([2, Theorem 5.8]). For any saturated fusion system $\mathcal{F}$ on a finite $p$-group $S$, the natural homomorphism

$$H^*(BF; \mathbb{F}_p) \to \lim_{\to \mathcal{F}} H^*(-; \mathbb{F}_p)$$

is an isomorphism, and the ring $H^*(BF; \mathbb{F}_p)$ is Noetherian.

In fact, we prove the following more general theorem, which we posed as a conjecture and proved in some special cases in our previous work [6].

**Theorem 1.3** (Subgroup sharpness for fusion systems). Let $\mathcal{F}$ be a saturated fusion system on a finite $p$-group $S$ and let $k$ be a field of characteristic $p$. Let $M = (M^*, M_*) : \mathcal{O}(\mathcal{F}) \to k$-mod be a Mackey functor over the full orbit category of $\mathcal{F}$ with values finite dimensional $k$-vector spaces. Then

$$\lim_{\to i} \mathcal{O}(\mathcal{F}^c) M^*|_{\mathcal{O}(\mathcal{F}^c)} = 0$$

for $i > 0$.

The proof of this theorem appears in Section 3. The cohomology functor $H^*(-; \mathbb{F}_p)$ restricts to a Mackey functor for $\mathcal{F}$ (see Section 2 for the relevant definitions) and so Theorem 1.1 is a special case of Theorem 1.3. The reason why we conjectured this more general version is the result [6, Theorem A] going back to Jackowski and McClure [11] that Mackey functors over certain categories have vanishing higher limits. Axioms for Mackey functors were first formulated by Dress [7], and they have been successfully employed in different areas. There exists a general theory of Mackey functors, including a parametrization and an explicit description of simple Mackey functors. See the work of Webb [15] for example.

We have adapted the theory of Mackey functors to the fusion system setting [6], and they play an essential role in the proof of Theorem 1.3. The other key ingredient in the proof is an adaptation of Oliver’s homological algebraic version [14] of Chermak’s proof [5] of the existence and uniqueness of a classifying space $B\mathcal{F}$. Theorem 1.3 is thus proven by combining two filtrations, one for the functor and one for the category on which the functor is defined: First, we filter the Mackey functor $M$ to reduce the problem to the simple Mackey functors $S_{Q,V}$ (see Proposition 2.6 and Lemma 2.8) that appear as composition factors. Then, for each simple Mackey functor $S_{Q,V}$ we construct a filtration of the category $\mathcal{O}(\mathcal{F}^c)$ based on Chermak’s ideas; Chermak’s filtration was built using the Thompson subgroups $J(P)$ of $P \leq S$.

In terms of Oliver’s interpretation of Chermak’s proof, this filtration is a correct one.
for the center functor $Z_F$, the vanishing of whose second and third derived limits over $O(F^c)$ implies the existence and uniqueness of $BF$ by obstruction theory. We get a filtration of $O(F^c)$ which is correct for the simple Mackey functor $S_{Q,V}$ at work by using an analogue $Q^*(P)$ (Definition 3.1) of the Thompson subgroup.

2. Simple Mackey functors and local vanishing results

In this section we briefly review the relevant results on higher limits from Oliver’s work [14] and basic notions about Mackey functors for fusion systems [6]. Throughout this section we denote by $k$ a commutative ring with identity (we will be mainly interested in the case where $k$ is a field of characteristic $p > 0$), by $k\text{-Mod}$ the category of $k$-modules and by $k\text{-mod}$ the category of finitely generated $k$-modules.

**Definition 2.1.** Let $F$ be a fusion system on the finite $p$-group $S$. An interval is a collection $R$ of subgroups of $S$ such that $P < Q < R$ and $P, R \in R$ imply $Q \in R$. An interval is $F$-invariant if it is closed under taking $F$-conjugacy.

For instance, the $F$-conjugacy class $R = P^F$ of the subgroup $P \leq S$ is an $F$-invariant interval. Note that every $F$-invariant interval is of the form $\mathcal{X} \setminus \mathcal{X}_0$ for $F$-invariant intervals $\mathcal{X}_0 \subseteq \mathcal{X}$ which are closed under overgroups. Now let $\mathcal{X}$ be an $F$-invariant interval which is closed under overgroups; for example, $\mathcal{X} = F^c$, the collection of all $F$-centric subgroups of $S$. The orbit category $O(\mathcal{X})$ has objects the subgroups in $\mathcal{X}$ and morphisms given by $\text{Mor}_{O(\mathcal{X})}(P,Q) = \text{Inn}(Q) \setminus \text{Hom}_F(P,Q)$.

**Definition 2.2.** Let $F$ be a fusion system on the $p$-group $S$ and let $N: O(F^c)^{op} \to k\text{-Mod}$ be a functor. If $R \subseteq F^c$ is an $F$-invariant interval define

$$N^R: O(F^c)^{op} \to k\text{-Mod}$$

as the subquotient functor of $N$ with value $N^R(P) = N(P)$ for $P \in R$ and 0 otherwise. An $F$-partition is a collection of $F$-invariant intervals $R_0, R_1, \ldots, R_n$ such that:

1. $F^c = R_0 \cup R_1 \cup \ldots \cup R_n$ is a partition of $F^c$.
2. $R_i \in R_i$, $R_j \in R_j$ and $i < j$ imply $R_i \not\subseteq R_j$.

The $F$-partitions defined above permit to reduce higher limits to simpler pieces.

**Lemma 2.3** ([14] Lemma 1.7). Let $F$ be a fusion system on the finite $p$-group $S$ and let $N: O(F^c)^{op} \to k\text{-Mod}$ be a functor. If $R_0, R_1, \ldots, R_n$ is an $F$-partition such that $\lim_{\leftarrow} N^{R_i} = 0$ for all $i$ and $* > 0$, then $\lim_{\leftarrow} N^* = 0$ for $* > 0$.

Now recall that, roughly speaking, a Mackey functor consists of a contravariant functor and a covariant functor that satisfy various compatibility conditions, including a Mackey decomposition formula. This notion extends naturally to the setup of fusion systems.

**Definition 2.4.** Let $F$ be a fusion system on a finite $p$-group $S$ and let $\mathcal{X}$ be an $F$-invariant interval which is closed under overgroups. An $\mathcal{X}$-restricted Mackey functor for $F$ over $k$ is a pair of functors

$$M = (M^*, M_*): O(\mathcal{X}) \to k\text{-Mod}$$

satisfying the following conditions.
Proposition 2.5 can be parametrized and precisely described. The category $S$ group $F$ for $Q^k$ $S$ is chosen as these are the only parts relevant to our main theorem. We start by setting $F$ we also fix an $P$ where the direct sum is taken over the $\alpha$ isomorphism $\alpha$ of the functors $Q,V$. Here $\alpha$ is an isomorphism in $O(X)$.

The full Mackey functor structure of $Q,V$ is $S$-conjugate to $S$. The simple Mackey functors in $Q,V$ as a set for each $x \in X$ as a set and its $\alpha$-module structure of $S$ runs over the double cosets $QxR$ in $P$ such that $Q \cap xR \in X$. When $X$ consists of all subgroups of $S$, we simply say that $M$ is a Mackey functor for $F$.

We denote by $\text{Mack}_k^X(F)$ the category of $X$-restricted Mackey functors for $F$ over $k$, and we simply write $\text{Mack}(F)$ for the category of Mackey functors for $F$ over $k$. The category $\text{Mack}_k^X(F)$ is an abelian category in which kernels and cokernels are constructed “objectwise”. The simple objects in the abelian category $\text{Mack}_k(F)$ can be parametrized and precisely described.

**Proposition 2.5 (\cite[Proposition 3.2]{6}).** Let $F$ be a fusion system on a finite $p$-group $S$. The simple Mackey functors in $\text{Mack}_k(F)$ are of the form $S_{Q,V}$, where $Q$ runs over the subgroups of $S$ and $V$ runs over the simple $k\text{Out}_F(Q)$-modules. Moreover, $S_{Q,V}$ and $S_{R,W}$ are isomorphic in $\text{Mack}_k(F)$ if and only if there is a $F$-isomorphism $\alpha: Q \to R$ such that $\alpha V \cong W$ as $k\text{Out}_F(R)$-modules.

Here $\alpha V = V$ as a set and its $k\text{Out}_F(R)$-module structure is obtained from the $k\text{Out}_F(Q)$-module structure of $V$ by transporting the action along the $\alpha$-isomorphism $\alpha: Q \to L$. From the above proposition, it is easy to deduce \cite[Proposition 3.3]{6} that the simple objects in $\text{Mack}_k^X(F)$ are the restrictions to $O(X)$ of the functors $S_{Q,V}$ for $Q \in X$, although we do not need this fact in this work.

The full Mackey functor structure of $S_{Q,V}$ is described in \cite[§3]{6}. Here we only describe the values $S_{Q,V}(P)$ for $P \leq S$ and its structure as an $\text{Out}_F(P)$-module as these are the only parts relevant to our main theorem. We start by setting $S_{Q,V}(L) = V$ as a set for each $F$-conjugate $L$ of $Q$. For each such subgroup $L$ we also fix an $F$-isomorphism $\alpha: Q \to L$. If $\alpha_i: Q \to L_i$ ($i = 1, 2$) are the chosen isomorphisms for subgroups $L_1$ and $L_2$ that are $F$-conjugate to $Q$ and $[\gamma]: L_1 \to L_2$ is an isomorphism in $O(F)$ we define

\begin{equation}
S_{Q,V,*}([\gamma])(v) = [\alpha_2^{-1}\gamma\alpha_1] \cdot v \text{ for } v \in V
\end{equation}

and $S_{Q,V}([\gamma]) = S_{Q,V,*}([\gamma^{-1}])$. Consequently we have $S_{Q,V}(L) \cong \alpha V$ as $k\text{Out}_F(L)$-modules.

For an arbitrary subgroup $P \leq S$ we set

\begin{equation}
S_{Q,V}(P) \cong \bigoplus_{Q \leq L \leq P} \text{tr}^{N_P(L)}_L(\alpha V),
\end{equation}

where the direct sum is taken over the $P$-conjugacy classes of the subgroups $L$ of $P$ that are $F$-conjugate to $Q$. For each of these classes a representative $L$ is chosen and an $F$-isomorphism $\alpha: Q \to L$ is also fixed. The map $\text{tr}^{N_P(L)}_L: \alpha V \to \alpha V$ is the relative trace map, where $N_P(L)$ acts on the $k\text{Out}_F(L)$-module $\alpha V$ via the map $N_P(L) \to \text{Out}_F(L)$ given by conjugation. It is easy to check that the
above description of $S_{Q,V}(P)$ does not depend on the choices of $L$ and $\alpha$ up to isomorphism.

From this description, we obtain some vanishing properties of $S_{Q,V}$ below. Combined with a correct $\mathcal{F}$-partition they produce the desired sharpness result. First we recall a group theoretic definition.

**Definition 2.6.** Let $Q \leq P$ be groups. We say that $Q$ is **centric** in $P$ if $C_P(Q) \leq Q$.

Thus, if $\mathcal{F}$ is a fusion system on a finite $p$-group $S$, the subgroup $P \leq S$ is $\mathcal{F}$-centric if and only if all its $\mathcal{F}$-conjugates are centric in $S$.

**Lemma 2.7.** Let $\mathcal{F}$ be a fusion system on a finite $p$-group $S$. Consider the simple Mackey functor $S_{Q,V}$ for $\mathcal{F}$ where $Q \leq S$ and $V$ is a simple $k\Out_\mathcal{F}(Q)$-module. Let $P \leq S$ and let $\text{tr}^{N_P(L)}_L(\alpha V)$ be a direct summand of $S_{Q,V}(P)$ as in (2) with a chosen $\mathcal{F}$-isomorphism $\alpha: Q \to L \leq P$.

(a) If $k$ has characteristic $p$ and $C_P(L) \not\leq L$, then $\text{tr}^{N_P(L)}_L(\alpha V) = 0$.

(b) If $g \in (N_S(P) \cap C_S(L)) \setminus P$ and $P$ is centric in $S$, then $[c_g] \in \Out_\mathcal{F}(P)$ is a nontrivial $p$-element that acts trivially on $\text{tr}^{N_P(L)}_L(\alpha V)$.

**Proof.** The first item follows from the transitivity of the relative trace map

$$\text{tr}^{N_P(L)}_L(\alpha V) = (\text{tr}^{N_P(L)}_L \circ \text{tr}^{P(L)}_L)(\alpha V)$$

and the fact that $\text{tr}^{P(L)}_L(\alpha V) = 0$. For the second item, as $P$ is centric in $S$, it is clear that $[c_g] \in \Out_\mathcal{F}(P)$ is a nontrivial $p$-element. Moreover, conjugation by $g$ fixes $L$ elementwise and hence, by (1), $[c_g]$ acts on $\text{tr}^{N_P(L)}_L(\alpha V)$ as

$$S_{Q,V,*}([c_g])(v) = [\alpha^{-1},c_g \alpha] \cdot v = v \quad \text{for } v \in \alpha V .$$

If $k$ is a field and $M \in \Mack_k(\mathcal{F})$ takes as values finite dimensional $k$-vector spaces, then $M$ has a composition series of finite length, which can be used to reduce the acyclicity of $M$ over $\mathcal{O}(\mathcal{F})^\circ$ to that of simple Mackey functors.

**Lemma 2.8 ([B Proposition 4.3]).** Let $\mathcal{F}$ be a fusion system on a finite $p$-group $S$ and let $k$ be a field. Consider a Mackey functor $M: \mathcal{O}(\mathcal{F}) \to k\text{-mod}$. If for each composition factor $S_{Q,V}$ of $M$, where $Q \leq S$ and $V$ is a simple $k\Out_\mathcal{F}(Q)$-module, we have $\lim_{\leftarrow i} S_{Q,V,\mathcal{O}(\mathcal{F}^i)} = 0$ for $i > 0$, then $\lim_{\leftarrow i} S_{Q,V,\mathcal{O}(\mathcal{F})}^* = 0$ for $i > 0$.

The next lemma describes local vanishing conditions for higher limits on an $\mathcal{F}$-invariant interval.

**Definition 2.9.** Let $\mathcal{F}$ be a fusion system on the finite $p$-group $S$, let $\mathcal{R} \subseteq \mathcal{F}^\circ$ be an $\mathcal{F}$-invariant interval and let $Q \leq S$ be an $\mathcal{F}$-centric subgroup.

(a) We say that $\mathcal{R}$ is $Q$-**atomic** if $\mathcal{R} = Q^\mathcal{F}$.

(b) We say that $\mathcal{R}$ is $Q$-**normal** if $Q^\mathcal{F} \subseteq \mathcal{R} \subseteq \mathcal{T}^\mathcal{F} = \{P^\mathcal{F}| P \in \mathcal{T}\}$ with

$$\mathcal{T} = \{ P \leq S | Q \leq P, \text{ and } R \in Q^\mathcal{F} \text{ and } R \leq P \text{ imply } R = Q \} .$$

**Lemma 2.10.** Let $\mathcal{F}$ be a saturated fusion system on the finite $p$-group $S$, let $\mathcal{R} \subseteq \mathcal{F}^\circ$ be an $\mathcal{F}$-invariant interval and let $Q \leq S$ be an $\mathcal{F}$-centric subgroup which is fully $\mathcal{F}$-normalized.

(a) If $\mathcal{R}$ is $Q$-atomic and $N: \mathcal{O}(\mathcal{F}^\circ)^{\text{op}} \to k\text{-Mod}$ is a functor such that $p$ divides the order of the kernel of the $\Out_\mathcal{F}(Q)$-action on $N(Q)$, then $\lim_{\leftarrow i} S_{Q,V,\mathcal{O}(\mathcal{F}^i)}^* N^\mathcal{R} = 0$. 
Proof. For part (a), we have
\[ \lim_{i} \frac{i}{\Omega(F)} (N^i | \Omega(F))^{R} = 0 \text{ for } i > 0. \]

(b) If \( R \) is \( Q \)-normal and \( N = (N^*, N_c) : \Omega(F) \rightarrow k\text{-Mod} \) is a Mackey functor for \( F \), then \( \lim_{i} \frac{i}{\Omega(F)} (N^i | \Omega(F))^{R} = 0 \text{ for } i > 0. \)

Lemma 3.2. Let \( F \) be a fusion system on a finite \( p \)-group \( S \) and let \( Q \leq S \).

The subgroup \( Q^*(P) \) satisfies properties similar to those of the Thompson subgroup \( J(P) \). The proof is immediate from the definition of \( Q^*(P) \) and hence omitted.

Definition 3.1. Let \( F \) be a fusion system on a finite \( p \)-group \( S \) and let \( Q \leq S \).

For \( P \leq S \), we define \( Q^*(P) \) to be the subgroup of \( P \) generated by all subgroups of \( P \) which are \( F \)-conjugate to \( Q \).
Proof of Theorem 1.3. Consider the simple Mackey functor $M = S_{Q,V}$, where $Q \leq S$ and $V$ is a simple $k\text{Out}_F(Q)$-module. We shall define an $F$-partition using the subgroups $Q^*(P)$: Choose inductively subgroups $X_0, X_1, \ldots, X_n \in F^c$ and $F$-invariant intervals $\emptyset = Q_{-1} \subseteq Q_0 \subseteq \cdots \subseteq Q_N = F^c$ as follows. Assume that $Q_{n-1}$ has been defined ($n \geq 0$), and that $Q_{n-1} \neq F^c$. Consider the following collections of subgroups of $S$:

$$U_1 = \{P \in F^c \mid Q^*(P)\text{ maximal}\},$$

$$U_2 = \{P \in U_1 \mid Q^*(P) \in F^c\},$$

$$U_3 = \begin{cases} \{P \in U_2 \mid |P| \text{ minimal}\}, & \text{if } U_2 \neq \emptyset, \\
\{P \in U_1 \mid |P| \text{ maximal}\}, & \text{if } U_2 = \emptyset. \end{cases}$$

Let $X_n$ be any subgroup in $U_3$ such that both $X_n$ and $Q^*(X_n)$ are fully $F$-normalized. By Lemma 3.2, the collections $U_1, U_2$ and $U_3$ are all $F$-invariant intervals. So we may take $X_n \in U_3$ which is fully $F$-normalized. By Lemma [1, I.2.6(c)], for each $F$-conjugate $Y$ of $Q^*(X_n)$ which is fully $F$-normalized, there is an $F$-morphism $\varphi: N_S(Q^*(X_n)) \to N_S(Y)$ such that $\varphi(Q^*(X_n)) = Y$. By Lemma 3.2 again, $N_S(X_n) \leq N_S(Q^*(X_n))$ and so $\varphi(N_S(X_n)) \leq N_S(\varphi(X_n))$ and $Y = Q^*(\varphi(X_n))$. Since $X_n$ is fully $F$-normalized, it follows that $\varphi(N_S(X_n)) = N_S(\varphi(X_n))$ and hence $\varphi(X_n)$ is also fully $F$-normalized. Thus $X_n := \varphi(X_n) \in U_3$ has the desired property that both $X_n$ and $Q^*(X_n)$ are fully $F$-normalized.

Let $Q_n$ be the union of $Q_{n-1}$ with the collection of all subgroups $P \leq S$ which contain some $F$-conjugate of $X_n$. Set $R_n = Q_n \setminus Q_{n-1}$. Thus the collections $Q_n$ and $R_n$ are $F$-invariant intervals and $Q_n$ is closed under overgroups too. By the definition of $U_3$ and Lemma 3.2, $X_n = Q^*(X_n)$ if $Q^*(X_n) \in F^c$, while $R_n = X_n^F$ if $Q^*(X_n) \not\in F^c$. For example, $X_0 = Q^*(S)$ and $R_0 = Q_0 = \{P \leq S \mid P \geq Q^*(S)\}$ if $Q^*(S) \not\in F^c$, while $X_0 = S$ and $R_0 = Q_0 = \{S\}$ if $Q^*(S) \in F^c$. By Lemma 2.3 it suffices to prove, for each $n$, that

$$\lim_{i \to \infty}^{F^c} (S_{Q,V}|_{O(F^c)})^{R_n} = 0 \quad \text{for all } i > 0.$$

**Case 1:** Assume that $Q^*(X_n) \not\in F^c$. Hence $R_n = X_n^F$ is $X_n$-atomic. Since $Q^*(X_n)$ is fully $F$-normalized and not $F$-centric, $C_S(Q^*(X_n)) \not\leq Q^*(X_n)$.

On the one hand, if $Q^*(X_n)$ is not centric in $X_n$, then there is an element $x \in C_{X_n}(Q^*(X_n)) \setminus Q^*(X_n)$ and we have $x \in C_{X_n}(L) \setminus L$ for any $F$-isomorphism $\alpha: Q \to L$ with $L \leq X_n$. Thus $S_{Q,V}(X_n) = 0$ by Lemma 2.7(b), so $(S_{Q,V}|_{O(F^c)})^{R_n}$ is the zero functor and we are done.

On the other hand, if $Q^*(X_n)$ is centric in $X_n$, then $C_S(Q^*(X_n)) \not\leq Q^*(X_n)$ implies that $C_S(Q^*(X_n)) \not\leq X_n$. Thus $X_n C_S(Q^*(X_n)) > X_n$ and so the normalizer $N_{X_n} C_S(Q^*(X_n))(X_n)$ properly contains $X_n$. Hence there exists $x \in N_S(X_n) \setminus X_n$ which centralizes $Q^*(X_n)$. Then, by Lemma 2.7(b), $[c] \in \text{Out}_F(X_n)$ is a nontrivial $p$-element that lies in the kernel of the action of $\text{Out}_F(X_n)$ on $S_{Q,V}(X_n)$. Thus we are done by Lemma 2.10(b).

**Case 2:** Assume that $Q^*(X_n) \in F^c$, and hence $X_n = Q^*(X_n)$. We show that $R_n$ is $X_n$-normal. Note that $X_n^F \subseteq R_n$ and set

$$T = \{P \leq S \mid X_n \leq P, \text{ and } R \in X_n^F \text{ and } R \leq P \text{ imply } R = X_n\}.$$
Then we are left with proving that \( \mathcal{R}_n \subseteq \mathcal{T}_F \). Observe now that

\[(3) \quad P \in \mathcal{R}_n \implies Q^*)(P) \text{ is the only subgroup of } P \text{ which is } F\text{-conjugate to } X_n.\]

Indeed, suppose \( P \in \mathcal{R}_n \). This means that \( P \) contains an \( F\)-conjugate \( X'_n \) of \( X_n \) and \( P \notin Q_{n-1} \). By Lemma 5.2, \( Q^*(P) \geq Q^*(X'_n) = X'_n \) and \( |Q^*(X'_n)| = |Q^*(X_n)| \).

By the definition of \( \mathcal{U}_1 \), it follows that \( Q^*(P) = Q^*(X'_n) = X'_n \), as claimed.

Let \( \mathcal{R} \) be the collection of all \( P \in \mathcal{R}_n \) such that \( P \geq X_n \). By (3), it is immediate that \( \mathcal{R} \subseteq \mathcal{T} \) and hence it is enough to show that \( \mathcal{R}_n \subseteq \mathcal{R}_F \). Let \( P \in \mathcal{R}_n \). Then \( P \) contains an \( F\)-conjugate \( X'_n \) of \( X_n \). By (3), \( X'_n = Q^*(P) \leq P \), so \( P \leq N_S(X'_n) \).

Since \( X_n \) is fully \( F\)-normalized, there is an \( F\)-morphism \( \varphi : N_S(X'_n) \to N_S(X_n) \) such that \( \varphi(X'_n) = X_n \). Thus \( \varphi(P) \geq \varphi(X'_n) = X_n \) and so \( \varphi(P) \in \mathcal{R} \), as claimed.

It follows that \( \mathcal{R}_n \) is \( X_n \)-normal and hence we are done by Lemma 2.10(b). \( \square \)

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