Anomalous diffusion as a signature of collapsing phase in two dimensional self-gravitating systems

Mickaël Antoni
Max-Planck-Institut für Physik Komplexer Systeme, Bayreuther Str. 40, D-01187 Dresden
(Germany)

Alessandro Torcini*
Dipartimento di Energetica "S. Stecco", Università di Firenze, via S. Marta, 3, I-50139 Firenze
(Italy)
(March 12, 2018)

Abstract

A two dimensional self-gravitating Hamiltonian model made by $N$ fully-coupled classical particles exhibits a transition from a collapsing phase (CP) at low energy to a homogeneous phase (HP) at high energy. From a dynamical point of view, the two phases are characterized by two distinct single-particle motions: namely, superdiffusive in the CP and ballistic in the HP. Anomalous diffusion is observed up to a time $\tau$ that increases linearly with $N$. Therefore, the finite particle number acts like a white noise source for the system, inhibiting anomalous transport at longer times.

PACS numbers: 05.45.+b, 05.40+j, 05.70.Fh, 64.60.Cn

*INFM, Firenze (Italy)
In the past years the thermodynamical properties of gravitational models have been studied in detail from a theoretical and computational point of view. In particular, it has been shown that at low energy the gravitational forces give rise to a collapsing phase (CP), identified by the presence of a single cluster of particles floating in a dilute homogeneous background. At high energy a homogeneous phase (HP) is recovered: the cluster disappears and the particles move almost freely. In the transition region the system is characterized (in the microcanonical ensemble) by a negative specific heat: the corresponding instability (termed "gravo-thermal catastrophe") is of extreme relevance for astrophysics (see [5] for more details). This apparent thermodynamical inconsistency has been solved by Hertel and Thirring in Ref. [1], where they demonstrated the non-equivalence of canonical and microcanonical ensemble in proximity of the transition region. These theoretical results have been successfully confirmed by numerical investigations of self-gravitating non-singular systems with short range interaction [2,3].

More recently, in one dimensional lattices of fully and nearest-neighbour coupled symplectic maps with an attractive interaction it has been noticed that clustering phenomena are associated with anomalous diffusion (in particular, with subdiffusive motion), at least for short times [6]. Anomalous diffusion can be defined through the time dependence of the single particle mean square displacement (MSQD) \( <r^2(t)> \), that typically reads as

\[
<r^2(t)> \propto t^\alpha
\]

where the average \(< \cdot >\) is performed over different time origins and over all the particles of the system. The transport is anomalous when \(\alpha \neq 1\): superdiffusive for \(1 < \alpha < 2\) and subdiffusive for \(0 < \alpha < 1\) [7,8]. Anomalous transport has been revealed in dissipative and Hamiltonian models [10] as well as in experimental measurements [9]. However, the main part of literature focuses on systems with few degrees of freedom (namely, one or two) and only few studies have been devoted to extended models with \(N \gg 1\) [1].

In this Rapid Communication, the thermodynamical and dynamical properties of a 2-D Hamiltonian system, constituted of \(N\) particles interacting via a long range attractive
potential, are analyzed. In particular, we observe a transition from CP to HP associated
to a dynamical transition from anomalous to ballistic transport. Finite \( N \) effects induce
a crossover from anomalous to normal diffusion at long times. In the limit \( N \to \infty \), the
transport mechanism remains anomalous for any time and reduces to that of a single particle
in an "egg-crate" potential [10].

We consider a system of \( N \) identical fully coupled particles with unitary mass evolving
in a 2 dimensional periodic cell described by the Hamiltonian [11,12]

\[
H = K + V = \sum_{i=1}^{N} \frac{p_{x,i}^2 + p_{y,i}^2}{2} \quad + \quad \frac{1}{2N} \sum_{i,j}^{N} \left[ 3 - \cos(x_i - x_j) - \cos(y_i - y_j) - \cos(x_i - x_j) \cos(y_i - y_j) \right],
\]

(2)

where \((x_i, p_{x,i})\) and \((y_i, p_{y,i})\) are the two pairs of conjugate variables with \((x_i, y_i) \in [-\pi, \pi] \times [-\pi, \pi]\), \( K \) and \( V \) are the kinetic and potential energy, respectively. The potential part corresponds to the first three terms of the Fourier expansion of a 2-D attractive
potential of the kind \( V(r) \propto \log |r| \). Such type of interaction arises in self-gravitating 2-
d gases [1–3] as well as in point vortices model for 2-d turbulence [13]. Due to the long
range interaction among all the particles, this model can be described in terms of meanfield
variables. In particular, the potential energy can be rewritten as \( V = \frac{1}{2} \sum_{i=1}^{N} V_i \), with

\[
V_i = 3 - \mathbf{M}_x \cos(x_i - \phi_x) - \mathbf{M}_y \cos(y_i - \phi_y) - \frac{1}{2} \left[ \mathbf{M}_{xy}^+ \cos(x_i + y_i - \phi_{xy}^+) + \mathbf{M}_{xy}^- \cos(x_i - y_i - \phi_{xy}^-) \right],
\]

(3)

where \( \mathbf{M}_z = \langle \cos(z) \rangle_N, \langle \sin(z) \rangle_N \rangle = \mathbf{M}_z \exp[i \phi_z] \) represents four two-dimensional
meanfield vectors with \( z = x, y, x \pm y \) and \( \langle .. \rangle_N \) denotes the average over \( N \). However,
the single particle potentials \( V_i \) are non-autonomous since the meanfield quantities \( \mathbf{M}_z \) and
\( \phi_z \) are defined through the instantaneous values of the particles coordinates. The motion of
each particle is therefore determined self-consistently by an attractive and non-autonomous
force field that is itself uniquely determined through the motion of all the particles. The
self-gravitating nature of the model is due to this effective force acting among the particles.
Since \( V \) is invariant under the transformations \( x \leftrightarrow -x, y \leftrightarrow -y \) and \( x \leftrightarrow y \), it turns out
that in the "meanfield limit" (i.e. for $N \to \infty$ with $U = H/N$ constant) $M_x = M_y = M$ and $M^+_x = M^-_y = P$. Moreover, in this limit and assuming $\phi_z = 0$ the single potential $V_i$ turns out to be an egg-crate potential similar to that studied in \cite{10}. This periodic potential is characterized in each elementary cell by a minimum ($V_m = 3 - 2M - P$), 4 maxima ($V_M = 3 + 2M - P$) and 4 saddle points ($V_s = 3 + P$).

For specific energy $U$ smaller than a critical value $U_c \simeq 2$, we observe that the particles are mainly in a clustered state. Above $U_c$ a HP is recovered. Following Refs. \cite{6,12}, the degree of clustering of the particles can be characterized through the time averages $<M_z>_t$ \cite{14,15}. When at each time the particles have almost the same position $<M_z>_t$ are $O(1)$, while for a HP their values vanishes as $1/\sqrt{N}$ \cite{12}. Fig. 1 shows that, for $U \to 0$, the average quantities $M, P$ tend to one. This indicates that the particles are almost all trapped in a potential well of depth $\simeq (V_s - V_m)$ forming a compact cluster. For increasing energy $U$, the kinetic contribution becomes more relevant and the average number of particles trapped in the potential well reduces. As a consequence the value of $<M_z>_t$ decreases together with $(V_s - V_m)$. For $U \geq U_c$, the system is no more clustered and the particles can move almost freely. Moreover, due to finite $N$ effects $<M_z>_t$ is not exactly zero, but $O(1/\sqrt{N})$.

In Fig. 2 the temperature $T = <K>_t/N$ is reported as a function of $U$. Above $U_c$, $T$ increases linearly with $U$ indicating that the system behaves like a free particle gas. In the CP, the tendency of the system to collapse is balanced by the increase of the kinetic energy \cite{2}. This competition leads initially (for $0 < U < 1.8$) to a steady increase of $T$, followed (for $1.8 < U < U_c$) by a rapid decay of $T$. This yields a negative specific heat as illustrated in the inset of Fig. 2. These results are in full agreement with theoretical predictions based on the analysis of a simple classical cell model \cite{1} and with numerical findings \cite{2,3} for short ranged attractive potentials. The phenomenon of negative specific heat can be explained within a microcanonical approach with an heuristic argument \cite{1}: approaching the transition, a small increase of $U$ leads to a significative reduction of the number of collapsed particles (as confirmed from the drop exhibited by $M$ and $P$ for $U > 1.8$); as a consequence the value of $V$ grows and, due to energy conservation, the system becomes cooler.
Our data confirm also another important prediction of Hertel and Thirring [1]: the non-equivalence of canonical and microcanonical ensemble nearby the transition region. In the inset of Fig. 2 are reported the microcanonical findings, obtained via standard molecular dynamics (MD) simulations, and the theoretical canonical results, derived in the mean-field limit [16,17]. These two sets of data coincide everywhere, except in the energy interval $1.6 < U < 2.0$. The discrepancy is due to the impossibility of the canonical ensemble to exhibit a negative specific heat, a prohibition that does not hold for the microcanonical ensemble. Our theoretical estimation of the Helmholtz free energy $F = F(T)$ reveals that usually $F$ has an unique minimum. For $T < 0.5$ the minimum $F_C$ corresponds to non zero values of $M$ and $P$ (i.e. to the CP), while for $T > 0.55$ the minimum $F_H$ is associated to $M = P = 0$ (i.e. to the HP). In the region $0.5 < T < 0.55$, both minima $F_C$ and $F_H$ coexist as local minima of the free energy. However, for $T < T_c = 0.54$ the CP is observed because $F_C < F_H$, while for $T > T_c$ the HP prevails since $F_H < F_C$. At $T = T_c$ the two minima are equivalent and a jump in energy from $U(T_c^-) \approx 1.6$ to $U(T_c^+) \approx 2.0$ is observed. This picture suggests that this transition can be considered as a first order transition [2].

Let us now investigate if the observed thermodynamical transition has any consequence on the dynamical behaviour of the system. In order to characterize the single particle dynamics, we consider the MSQD $< r^2(t) >$. As shown in Fig. 3, in the CP the diffusion is anomalous for times shorter than a crossover time $\tau$, while for longer times the Einstein law is recovered $< r^2(t) > \propto 4Dt$ (where $D$ is the diffusion coefficient). A similar behaviour for the MSQD has been already observed for a system of $N$ coupled symplectic maps in [3], but with $\alpha < 1$. However, in the present case $\tau$ increases linearly with $N$ [15] indicating that in the mean-field limit the asymptotic dynamical regime will be superdiffusive [18].

The observed dynamical behaviour can be explained noticing that in the mean-field limit each particle $i$ will see essentially the same constant 2-D egg-crate potential $V_i$. Moreover, it has been shown in Ref. [10] that a single particle moving in a egg-crate potential with an energy between $V_s$ and $V_M$ exhibits superdiffusion. This is due to the fact that the particle moves for long times almost freely along the channels of the potential and episodically is
trapped for a while in the potential well. In phase space, the superdiffusive phenomenon can be explained by a trapping mechanism in a hierarchy of cantori around a cylindrical KAM-surface [7]. Therefore, in our model (2) for \( N \to \infty \) anomalous transport is due only to the fraction of particles that can move along the channels.

For finite \( N \), the potential \( V_i \) seen by the particle \( i \) will fluctuate in time. Hence, particles having an energy close to \( V_S \) have the possibility to be trapped in the potential well as well as to escape from it. As a consequence, for sufficiently long time scales each particle can experience free and localized motions. The fluctuations of the potential \( V_i \) reflect themselves on the structure of the phase space, introducing a white noise that destroys the self-similar structure of the island chains and of the cantori below a certain cut-off size. Being the self-similarity no more complete, one expect that on long time scales normal diffusion will be recovered [19].

As pointed out in Refs. [20,21], if white noise is added to a dynamical system exhibiting superdiffusive behaviour, \( D \) (measured in the limit \( t \to \infty \)) is inversely proportional to the noise amplitude. Therefore, we expect that in our model the value of \( D \) will increase with \( N \). That is indeed the case, and we observe a power-law dependence of the type \( D \propto N^\gamma \).

For example, considering systems with \( 100 \leq N \leq 10,000 \) we have found for \( U = 1.48 \) and \( U = 2.00 \) a \( \gamma \)-value equal to \( 0.7 \pm 0.1 \) and \( 1.0 \pm 0.1 \), respectively. The \( N \)-dependence of the diffusion coefficient can be explained noticing that \( D \propto \tau^{\alpha-1} \) [22]. This result coincide with that found theoretically in Ref. [20] and confirmed numerically by considering as dynamical models two very simple noisy maps. For subdiffusive motion \( D \) is inversely proportional to \( \tau \) (as found in [19]), while for superdiffusive motion (\( \alpha > 1 \)) a direct proportionality is expected [20]. As already reported \( \tau \propto N \), therefore we will have that \( \gamma = \alpha - 1 \). Assuming for \( \alpha \) the corresponding asymptotic values [23], we can estimate as theoretical values \( \gamma \simeq 0.64 \) and \( \simeq 0.9 \) for \( U = 1.48 \) and 2.00, respectively. In view of all the present limitations, these values can be considered consistent with the numerical findings.

As a final point, we examine the dependence of the asymptotic \( \alpha \)-values from the energy \( U \) of the system. A transition from anomalous diffusion to ballistic motion (\( \alpha = 2 \)) at
$U \simeq U_c$ is evident from Fig. 1, where the $\alpha$-values, obtained for $N = 4,000$, are reported. In particular, for $0.4 \leq U \leq 2.0$, we observe an increase of $\alpha$ from $1.3 \pm 0.1$ to $1.9 \pm 0.1$. This phenomenon is a consequence of the flattening of the single particle potential (i.e. of the reduction of $V_M - V_m$) observed for growing $U$. The decrease of the average number of particles trapped in the cluster, and the consequent increase of those moving freely, naturally drives the diffusion mechanism toward a ballistic behaviour. Moreover, for $U > U_c$, the potential $V_i$ fluctuates with typical amplitude $O(1/\sqrt{N})$ around a constant value and a ballistic motion is expected for all the particles. For small energies ($U < 0.3$) the MSQD seems to saturate to a constant value, indicating that all the particles are always clustered. However, we believe that in these cases our observation time was not sufficient to detect particles escaping from the potential well.

In conclusion, we have shown for the first time a thermodynamical transition associated to a dynamical transition from anomalous to ballistic transport. Moreover, the transport in our $N$-body system can be interpreted in terms of a noisy single particle motion in a 2-D Hamiltonian egg-crate potential. The asymptotic dynamics of the model is strongly influenced by the order in which the two limits $N \to \infty$ and $t \to \infty$ are taken. Indeed, if the limit $N \to \infty$ is performed before the limit $t \to \infty$ the diffusion will be always anomalous. Otherwise normal diffusion is recovered for sufficiently long times.

As noticed in [1], the dimensionality of the system should not affect the main characteristics of the observed thermodynamical transition. We expect that the same should be true also for the corresponding dynamical behaviours. Moreover, anomalous diffusion should be observable for atomic clusters [2], turbulent vortices [3] (for which indeed has been already observed [4]) and gravitational systems [5]. All the cited systems share as a common aspect to exhibit a clustered phase.

As a final remark, we claim that the inclusion of other terms of the Fourier expansion in the expression of the gravitational potential, will not affect the main results here presented. However, as far as transport properties are concerned, we believe the range of the force to play an important role. In particular, for short-ranged interactions we expect a more chaotic
behaviour of the particles. This may prevent the anomalous diffusion regime to persist in the thermodynamical limit [19]. Future work will be devoted to the study of these fundamental points [15].

We would like to thank L. Casetti, R. Livi, P. Grassberger, P. Grigolini, H. Kantz, J. Klafter, R. MacKay, S. Ruffo, A. Seyfried and G. Zumofen for useful suggestions and the University of Wuppertal for the kind hospitality. This work was also partially supported through the European contract N. ERBCHRX-CT94-0460.
REFERENCES

[1] P. Hertel and W. Thirring, Ann. of Physics, 63, 520 (1970).

[2] A. Compagner, C. Bruin and A. Roesle, Phys. Rev. A, 39, 5989 (1989).

[3] H.A. Posch, H. Narnhofer and W. Thirring, Phys. Rev. A 42, 1880 (1990).

[4] P.A. Martin and J. Piasecki, J. Stat. Phys. 84, 837 (1996).

[5] D. Lynden-Bell and R. Woo, Mon. Notic. Roy. Astron. Soc. 138, 495 (1968).

[6] K. Kaneko and T. Konishi, Physica D 71, 146 (1994).

[7] T. Geisel, in Levy flights and related topics in physics , eds. M.F. Schlesinger et al. (Springer, Berlin, 1995) p. 153.

[8] P. Grigolini, in Chaos: the interplay between stochastic and deterministic behaviour , eds. P. Garbaczewski et al. (Lecture Notes in Physics, Springer, Berlin, 1995) 457 p. 101.

[9] T.H. Solomon, E.R. Weeks and H.L. Swinney, Phys. Rev. Lett. 71, 3975 (1993).

[10] T. Geisel, A. Zacherl and G. Radons, Phys. Rev. Lett. 58, 1100 (1987) and Z. Phys. B, Cond. Matter, 71, 117 (1988); D.K. Chaikovsky and G.M. Zalavsky, Chaos, 1, 463 (1991); J. Klafter and G. Zumofen, Phys. Rev. E, 49, 4873 (1994).

[11] The present model can be considered as an extention to 2-D of the model recently introduced in [12].

[12] M. Antoni and S. Ruffo, Phys. Rev. E 52, 2361 (1995).

[13] U. Frisch, ”Turbulence” (Cambridge University Press,1995); T.S. Lundgren and Y.B. Pointin, J. Stat. Phys. 17 (1977) 323.

[14] For finite values of $N$, the equalities $< M_x >_t = < M_y >_t = M$ and $< M_{+xy}^+ >_t = < M_{-xy}^- >_t = P$ are not expected to hold a priori, even in the limit $t \to \infty$. However,
numerical tests indicate that the values of $< M_z >_t$ (averaged for a sufficiently long time) do not depend remarkably from $N$. Apart exactly at the transition point $U = U_c$, where metastable states have been observed [15]. This justifies the identification of the time averages with the mean field asymptotic values.

[15] A. Torcini and M. Antoni, in preparation.

[16] The MD simulations have been performed within the microcanonical ensemble adopting a 4th-order symplectic integrator [17] with an integration time step $dt = 0.3$. The energy is conserved with a relative precision of $10^{-8}$ for any considered $N$. The canonical results have been obtained theoretically in the limit $N \to \infty$ from a straightforward estimation of the partition function obtained applying the Hubbard-Stratonovich trick, analogously to what made in [12] (more details will be reported in [15]).

[17] R. I. McLachlan and P. Atela, Nonlinearity 5, 541 (1992).

[18] In order to give an unambiguous definition of $\tau$, we have considered the local slope of $\ln[< r^2(t) >]$ as a function of $\ln(t)$. The crossover time is defined as the time for which this local slope becomes smaller than a given threshold $\beta$. We have verified the linear dependence of $\tau$ versus $N$ considering two $\beta$-values (namely, $\beta = 1.1$ and 1.2) and two energy values (namely, $U = 1.48$ and 2.00). The dependence of $\tau$ on $U$ is not monotonous, however we observe that for $N = 4,000 \ 20,000 < \tau < 100,000$ choosing $\beta = 1.1$.

[19] K. Kaneko and T. Konishi, Phys. Rev. A 40, R6130 (1989).

[20] E. Floriani, R. Mannella, and P. Grigolini, Phys. Rev. E 52, 5910 (1995).

[21] R. Bettin, R. Mannella, B.J. West and P. Grigolini, Phys. Rev. E 51, 212 (1995).

[22] We have verified that the velocity autocorrelation function (VACF) shows a long-time tail decaying as $\propto t^\beta$, where $\beta \simeq \alpha - 2$: a results fully consistent with the observed time dependence of the MSQD [8]. However, for finite $N$ the VACF after a time $t_V$
exponentially vanishes. It is reasonable to expect that $t_V \propto \tau$, by making this assumption the reported relation between $D$ and $\tau$ is easily derived [15].

[23] For the present model (2) a dependence of $\alpha$ on $N$ has been noticed. However, a saturation to an asymptotic value is clearly observed for $N > 4,000$ [15].

[24] "Clusters of atoms and molecules I", ed. H. Haberland (Springer, Berlin, 1995); P. Lebastie and R.L. Whetten, Phys. Rev. Lett. 65 (1990) 1567.
FIGURES

FIG. 1. Time averages of $M$ and $P$ as a function of $U$. The solid curves refer to the theoretical estimation (i.e. to canonical results) and the symbols to the MD findings (i.e. to microcanonical results). The exponents $\alpha$, defined in eq. (1), are also reported (triangles). The MD data have been obtained with $N = 4,000$ (apart few points with $N = 10,000$) and averaged over a total integration time ranging from $t = 1,2 \times 10^6$ to $t = 2,4 \times 10^6$, with a time step $dt = 0.3$. The $\alpha$-values have been estimated in the time interval $150 < t < 10,000$ for any reported $U$.

FIG. 2. Temperature $T$ as a function of the specific energy $U$. The solid line corresponds to the analytical estimation (canonical ensemble) and the triangles to the simulations results (microcanonical ensemble). In the inset, an enlargement of the transition region is reported: the solid (resp. dashed) curves refer to the principal (resp. relative) minimum of $F(T)$. The parameters for the MD simulations are the same as in Fig. 1.

FIG. 3. Mean square displacement $\langle r^2(t) \rangle$ as a function of time in a log-log plot. The cross-over time $\tau$ from anomalous to normal diffusion is also reported. The data refer to $U = 1.1$ and $N = 4,000$, the total integration time is $t = 1.2 \times 10^6$. In the inset, the logarithm of the cross-over time $\tau$ is reported as a function of $\ln(N)$ for $U = 1.48$. The reported $\tau$-values (circles) have been estimated adopting a threshold $\beta = 1.1$ (for more details see [17]). The solid line represents a best linear fit to the data and its slope is $0.95 \pm 0.08$. 

12
Fig. 1 – Antoni et al.
Fig. 2 – antoni et al.
\[
\ln(N) \sim 7
\]

\[
\ln(\tau) \sim 8
\]

\[
\alpha \sim 1
\]

\[
\alpha \sim 1.6
\]

Fig. 3 – antoni et al.