1. Introduction

The (standard) Stokes theorem is one of central points of (multivariable) analysis on manifolds (see [1] for an excellent introduction). Low-dimensional versions of this theorem, known as the (proper) Stokes theorem, in dimensions 1–2, and the Gauss theorem, in dimensions 2–3, respectively, are well-known and very useful, e.g. in classical electrodynamics. In fact, it is difficult to imagine lectures on classical electrodynamics without heavy use of the Stokes theorem. The standard Stokes theorem is also being called the Abelian Stokes theorem, as it applies to ordinary (i.e. Abelian) differential forms. Classical electrodynamics is an Abelian gauge theory (gauge fields are Abelian forms), therefore its integral formulas are governed by the Abelian Stokes theorem. But a lot of interesting and important physical phenomena is described by non-Abelian gauge theories. Hence it would be very interesting and also fruitful to have at our disposal a non-Abelian version of the Stokes theorem. Since non-Abelian differential forms need different treatment, one is forced to use a more sophisticated formalism to deal with this new situation. The aim of this chapter is to present a version of the non-Abelian Stokes theorem in the framework of the path-integral formalism [2].

2. From Stokes theorem to Stokes theorem

The (Abelian) Stokes theorem says that we can convert an integral around a closed curve $\mathcal{C}$ bounding some surface $\mathcal{S}$ into an integral defined on this surface. Namely,

$$\oint_{\mathcal{C}} \vec{A} \cdot d\vec{s} = \int_{\mathcal{S}} \operatorname{curl}\vec{A} \cdot \vec{n} d\sigma,$$

(2.1)

where the curve $\mathcal{C}$ is the boundary of the surface $\mathcal{S}$, i.e. $\mathcal{C} = \partial \mathcal{S}$, $\vec{A}$ is a vector field, e.g. the vector potential of electromagnetic field, and $\vec{n}$ is a unit outward normal at
the area element \( d\sigma \). More generally, in any dimension,
\[
\int_{\partial\mathcal{M}} \omega = \int_{\mathcal{M}} d\omega,
\tag{2.2}
\]
where \( \mathcal{M} \) is a \( d \)-dimensional manifold, \( \partial\mathcal{M} \) its \((d - 1)\)-dimensional boundary, \( \omega \) is a \((d - 1)\)-form, and \( d\omega \) is its differential, a \( d \)-form. We can also rewrite Eq. 2.1 in the spirit of Eq. 2.2, i.e.
\[
\int_{\partial S} A_i dx^i = \frac{1}{2} \int_S \left( \partial_i A_j - \partial_j A_i \right) dx^i \wedge dx^j,
\tag{2.3}
\]
where \( A_i \) \((i = 1, 2, 3)\) are components of \( \vec{A} \), and the Einstein summation convention should be applied.

In electrodynamics, we define the stress tensor of electromagnetic field
\[
F_{ij} = \partial_i A_j - \partial_j A_i,
\tag{2.4}
\]
and the magnetic induction, its dual, as
\[
B_k = \frac{1}{2} \varepsilon_{ijk} F_{ij},
\]
where \( \varepsilon_{ijk} \) is the totally antisymmetric (pseudo-)tensor. RHS of Eq. 2.3 represents then the magnetic flux. In turn, in geometry \( \vec{A} \) plays the role of connection (it defines the parallel transport around \( C \)), and \( F \) is the curvature of this connection.

Unfortunately, it is not possible to mechanically generalize the Abelian Stokes theorem (Eq. 2.3) to the non-Abelian one. In the non-Abelian case one faces a qualitatively different situation because the integrands assume values in a Lie algebra \( \mathfrak{g} \) rather than in the field of complex numbers \( \mathbb{C} \). The picture simplifies significantly if one switches from the infinitesimal language to a global one. Therefore let us consider the holonomy around a closed curve \( C \),
\[
\text{Hol}_C(\vec{A}) = \exp \left( i \oint_C A_i dx^i \right).
\tag{2.5}
\]
The holonomy, Eq. 2.5, represents a parallel-transport operator around \( C \) assuming values in the Abelian Lie group \( U(1) \), the gauge group of electromagnetic interactions. Interestingly, the holonomy has a physical meaning (it is a gauge-invariant object playing the role of the phase which can be observed in the Aharonov-Bohm experiment), whereas \( \vec{A} \) has not.

If \( C = \partial S \) in Eq. 2.5 we obtain a global version of the Abelian Stokes theorem
\[
\exp \left( i \oint_{\partial S = C} A_i(x) dx^i \right) = \exp \left( \frac{i}{2} \int_S F_{ij}(x) dx^i \wedge dx^j \right),
\tag{2.6}
\]
which is rather a trivial generalization of Eq. 2.3. But nevertheless Eq. 2.6 is a good starting point for our further discussion concerning the non-Abelian Stokes theorem.
For our further convenience let us formulate an auxiliary “Schrödinger problem” governing LHS of Eq. 2.6,

\[ i \frac{d\psi}{d\tau} = -\dot{x}^i A_i \psi, \]

which expresses the fact that the “wave function” \( \psi \) should be covariantly constant along \( \mathcal{C} \),

\[ D_\tau \psi \equiv \left( \frac{d}{d\tau} - i \dot{x}^i A_i \right) \psi = 0, \]

where \( D_\tau \) is the absolute covariant derivative, \( \tau \) is a parameter on the curve \( \mathcal{C} \) which is analytically defined by \( x^i(\tau) \), and the dot means differentiation with respect to the “time” \( \tau \).

Now, we would like to remind the reader of the form of the (“classical”) operator version of the non-Abelian Stokes theorem [3]. The assumed conventions are as follows. The non-Abelian curvature or the field strength is defined by

\[ F_{ij} = \partial_i A_j - \partial_j A_i - i [A_i, A_j], \]

where the connection or the gauge potential assuming values in an irreducible representation \( R \) of the Lie algebra \( g \) is of the form

\[ A_i = A_i^a T^a, \quad T^a \dagger = T^a, \]

where the Hermitian generators \( T^a = T^a_s, \quad r, s = 1, 2, \ldots, \dim R \) fulfil the commutation relation

\[ [T^a, T^b] = i f^{abc} T^c. \]

The non-Abelian generalization of Eq. 2.6 reads

\[ \mathcal{P} \exp \left( i \oint_{\partial S - \mathcal{C}} A_i(x) dx^i \right) = \mathcal{P} \exp \left( \frac{i}{2} \oint_S F_{ij}(x) dx^i \wedge dx^j \right), \]

where \( \mathcal{P} \) denotes path ordering, and \( \mathcal{P} \) some “surface ordering” (see Ref. 3, for details). Here, \( F_{ij}(x) \) is a “path-dependent curvature” defined by the formula

\[ F_{ij}(x) \equiv U^{-1}(x, O) F_{ij}(x) U(x, O), \]

where \( U(x, O) \) is a parallel-transport operator along the path \( \ell \) in the surface \( S \) joining the base point \( O \) of \( \partial S \) with the point \( x \), i.e.

\[ U(x, O) = \mathcal{P} \exp \left( i \int_{\ell} A_i(y) dy^i \right). \]

In the same manner as quantum mechanics, initially formulated in the operator language and next reformulated into the path-integral one, we can translate the operator form of the non-Abelian Stokes theorem into the path-integral one. It appears that the key object of the approach is a two-dimensional topological quantum field theory (with a large “topological” symmetry) in an external gauge field. In order to formulate the non-Abelian Stokes theorem in the path-integral language we will make the following three steps:
1. We will derive a holomorphic path-integral representation for the parallel-transport operator calculating the transition amplitude between one-particle states of some auxiliary Hilbert-space problem (a path-integral counterpart of LHS in Eq. 2.8);

2. We will quantize a two-dimensional topological quantum field theory in an external gauge field \( A \) (a counterpart of RHS in Eq. 2.8);

3. We will apply the Abelian Stokes theorem.

3. The parallel-transport operator

First of all, let us derive the path-integral expression for the parallel-transport operator \( U \) along an arc of the curve \( C \). To this end, we should consider the non-Abelian formula (differential equation) analogous to the Abelian Eq. 2.7,

\[
i \frac{d\psi_{rs}}{d\tau} = -\dot{x}^i(\tau)A^a_i(x(\tau))T^a_{rs}\psi_s,
\]

where \( \psi \) is a “wave function” in the irreducible representation \( R \) of the gauge Lie group \( G \), which is to be parallelly transported along the arc of \( C \) parametrized by \( x^i(\tau) \), \( \tau' \leq \tau \leq \tau'' \). Formally, Eq. 3.1 can be instantaneously integrated out yielding

\[
\psi_r(x'') = U_{rs}(x'', x')\psi_s(x'),
\]

where \( x'' = x(\tau'') \), \( x' = x(\tau') \), and

\[
U(x'', x') \equiv U(\tau'', \tau') = \text{P exp} \left( i \int_{\tau'}^{\tau''} \dot{x}^i(\tau)A^a_i(x(\tau)) d\tau \right).
\]

Let us now consider the following auxiliary mechanical problem with the classical Lagrangian

\[
L(\bar{\psi}, \psi) = i\bar{\psi}D_\tau\psi,
\]

where now \( D_\tau \equiv \frac{d}{d\tau} - i\ddot{x}^i A^a_i T^a \). The equation of motion following from Eq. 3.3 reproduces Eq. 3.1, and yields the classical Hamiltonian

\[
H = i\ddot{x}^i(\tau)A^a_i(x(\tau))T^a_{rs}\bar{\pi}_r\pi_s = -\dot{x}^i(\tau)A^a_i(x(\tau))T^a_{rs}\bar{\psi}_r\psi_s.
\]

The corresponding auxiliary quantum-mechanical problem is given, according to Eq. 3.4, by the Schrödinger equation

\[
i\frac{d}{d\tau}|\Phi\rangle = \hat{H}(\tau)|\Phi\rangle,
\]

\[
\hat{H}(\tau) = -\dot{x}^i(\tau)A^a_i(x(\tau))T^a_{rs}\hat{a}_r\hat{a}_s
\]
\[ H_{rs}^a(\tau)\hat{a}_r^\dagger\hat{a}_s \]
\[ \equiv -\dot{x}^i(t)A_i^a(x(t))\hat{T}^a, \]  
(3.5)

where the creation and anihilation operators satisfy the standard commutation (−) or anticommutation (+) relations
\[ [\hat{a}_r, \hat{a}_s^\dagger]_\mp = \delta_{rs}, \quad [\hat{a}_r^\dagger, \hat{a}_s^\dagger]_\mp = [\hat{a}_r, \hat{a}_s]_\mp = 0. \]
(3.6)

It can be easily checked by direct computation that we have obtained a realization of the Lie algebra \( \mathfrak{g} \) in a Hilbert space,

\[ [\hat{T}^a, \hat{T}^b]_\mp = if^{abc}\hat{T}^c, \]
(3.7)

where \( \hat{T}^a = T_{rs}^a\hat{a}_r^\dagger\hat{a}_s \). For the irreducible representation \( R \) the second-order Casimir operator \( C_2 \) is proportional to the identity operator 1, which in turn, is equal to the number operator \( \hat{N} \) in our Fock representation, i.e. if \( T^a \rightarrow \hat{T}^a \), then 1 \( \rightarrow \hat{N} = \delta_{rs}\hat{a}_r^\dagger\hat{a}_s \). Thus, by virtue of Eq. 3.7, we obtain an important for our further considerations constant of motion \( \hat{N} \),

\[ [\hat{N}, \hat{H}]_\mp = 0. \]
(3.8)

Let us now derive the holomorphic path-integral representation for the kernel,  

\[ U(\bar{a}, a; \tau'', \tau') = \langle \bar{a} | P \exp \left\{ -i \int_{\tau'}^{\tau''} \hat{H}(\tau) d\tau \right\} a \rangle, \]

of the evolution operator \( U \). Literally repeating the standard textbook procedure [4] (it should be noted that our approach is in the spirit of the “physical approach” to the index theorem, see e.g. [5]), we obtain

\[ U(\bar{a}, a; \tau'', \tau') = \int \exp \left\{ \bar{a}(\tau'')a(\tau'') + i \int_{\tau}^{\tau''} L(\bar{a}(\tau), a(\tau)) d\tau \right\} D\bar{a}Da, \]
(3.9)

where \( L(\bar{a}(\tau), a(\tau)) \) is the classical Lagrangian of the form given in Eq. 3.3, the imposed boundary conditions are: \( \bar{a}(\tau'') = \bar{a}, a(\tau') = a \), and \( D\bar{a}Da \) is a functional “measure”. Depending on the statistics (Eq. 3.6) there are the two (\( \mp \)) possibilities

\[ a_r\bar{a}_s \mp \bar{a}_r a_s = \bar{a}_r a_s \mp a_r\bar{a}_s = a_r a_s \mp a_s a_r = 0, \]

equivalent as far as one-particle subspace of the Fock space is concerned, which takes place in our further considerations.

Let us confine our attention to the one-particle subspace of the Fock space. As the number operator \( \hat{N} \) is conserved by virtue of Eq. 3.8, if we start from the one-particle subspace of the Fock space, we shall remain in this subspace during all the evolution. The transition amplitude \( U_{rs}(\tau'', \tau') \) between the one-particle
states $|1_r⟩ = \hat{a}_r^+ |0⟩$ and $|1_s⟩ = \hat{a}_s^+ |0⟩$ is given by the following scalar product in the holomorphic representation

$$U_{rs}(\tau'', \tau') = \int U(\bar{a}, \alpha; \tau'', \tau')e^{-\bar{a}a}e^{-\bar{a}a_\tau\bar{a}_s}d\bar{a}d\alpha.$$ (3.10)

One can easily check that Eq. 3.10 represents the object we are looking for. Namely, from the Schrödinger equation (Eq. 3.5) it follows that for the general one-particle state $\alpha_r \hat{a}_r^+ |0⟩$ (summation after repeating indices) we have

$$i \frac{d}{d\tau} (\alpha_p \hat{a}_p^+ |0⟩) = H_{rs}(\tau) \hat{a}_r^+ \hat{a}_s \alpha_p \hat{a}_p^+ |0⟩ = H_{rs}(\tau) \alpha_s (\hat{a}_r^+ |0⟩).$$ (3.11)

Using the property of linear independence of Fock-space vectors in Eq. 3.11, and comparing Eq. 3.11 to Eq. 3.1, we can see that Eq. 3.10 really represents the matrix elements of the parallel-transport operator. According to Eq. 3.9, we can finally put Eq. 3.10 in the following path-integral form:

$$U_{rs}(\tau'', \tau') = \int \exp \left\{ -\bar{a}(\tau')a(\tau') + i \int_{\tau'}^{\tau''} L(\bar{a}(\tau), a(\tau))d\tau \right\} a_r(\tau'')a_s(\tau')D\bar{a}Da,$$ (3.12)

where the free boundary conditions are imposed. (One can also rewrite Eq. 3.12 in the symmetrized form [4].) For closed paths, $x(t') = x(t'') = x$, Eq. 3.12 gives the holonomy operator $U_{rs}(x)$. Since $U_{rr}$ is the famous Wilson loop, it seems that this formula could have some independent applications. Interestingly enough, the Wilson loop, which is supposed to describe a quark-antiquark interaction, is represented by a “true” quark and antiquark field, $a$ and $\bar{a}$, respectively. So, the mathematical trick can be interpreted physically. Obviously, the “full” trace of the kernel in Eq. 3.9 is obtained by imposing (anti-)periodic boundary conditions in the case of (anti-)commuting fields, and integrating with respect to all the variables without the boundary term. Analogously, one can also derive the parallel-transport operator (a generalization of the one just considered) for symmetric $n$-tensors (bosonic $n$-particle states) and for $n$-forms (fermionic $n$-particle states).

4. The non-Abelian Stokes theorem

Let us now define a (boson or fermion) Euclidean two-dimensional topological field theory of the fields $\bar{\psi}, \psi$ in the irreducible representation $\mathcal{R}$ of the Lie algebra $\mathfrak{g}$ on the compact surface $\mathcal{S}$, $\dim \mathcal{S} = 2$, $\partial \mathcal{S} \neq \emptyset$, $\mathcal{S} \subset \mathcal{M}$, $\dim \mathcal{M} = d$, in the external gauge field $A$ by the classical action

$$S_{cl} = \int \left( iD_i \bar{\psi} D_j \psi + \frac{1}{2} \bar{\psi} F_{ij} \psi \right) dx^i \wedge dx^j, \quad i, j = 1, \ldots, d,$$ (4.1a)

or in the parametrization $x^i(\sigma^1, \sigma^2)$, $\tau' \leq \sigma^1 \equiv \tau \leq \tau''$, $0 \leq \sigma^2 \leq 1$, by the action

$$S_{cl} = \int_{\mathcal{S}} \mathcal{L}_{cl}(\bar{\psi}, \psi) d^2\sigma$$
\[
= \int_S \varepsilon^{AB} \left( iD_A \bar{\psi} D_B \psi + \frac{1}{2} \bar{\psi} F_{AB} \psi \right) d^2 \sigma, \quad A, B = 1, 2, \tag{4.1b}
\]
where
\[
D_A = \partial_A x^i D_i, \quad F_{AB} = \partial_A x^i \partial_B x^j F_{ij}.
\]
The described theory possesses the following “topological” gauge symmetry:
\[
\delta \psi(x) = \theta(x), \quad \delta \bar{\psi}(x) = \bar{\theta}(x), \tag{4.2}
\]
where \(\theta(x)\) and \(\bar{\theta}(x)\) are arbitrary except at the boundary \(\partial S\) where they vanish.

The origin of the symmetry (Eq. 4.2) will become clear when we convert the action (Eq. 4.1) into a line integral. Integrating by parts in Eq. 4.1 and using the Abelian Stokes theorem we obtain
\[
S_{cl} = i \oint_{\partial S} \bar{\psi} D_F \psi dx^i, \tag{4.3a}
\]
or in the parametrization
\[
S_{cl} = i \oint_{\partial S} \bar{\psi} D_F \psi d\tau. \tag{4.3b}
\]
To covariantly quantize the theory we shall introduce the BRS operator \(s\). According to the form of the gauge symmetry (Eq. 4.2) the operator \(s\) is easily defined by
\[
s \psi = \phi, \quad s \bar{\psi} = \bar{\chi}, \quad s \phi = 0, \quad s \bar{\chi} = 0,
\]
\[
s \bar{\phi} = \bar{\beta}, \quad s \chi = \beta, \quad s \bar{\beta} = 0, \quad s \beta = 0,
\]
where \(\phi\) and \(\bar{\chi}\) are ghost fields in the representation \(R\), associated to \(\theta\) and \(\bar{\theta}\), respectively, \(\bar{\phi}\) and \(\chi\) are the corresponding antighosts, and \(\bar{\beta}, \beta\) are Lagrange multipliers. All the fields possess a suitable Grassmann parity correlated with the parity of \(\bar{\psi}\) and \(\psi\). Obviously \(s^2 = 0\), and we can gauge fix the action in Eq. 4.1 in a BRS-invariant manner by simply adding the following \(s\)-exact term:
\[
S' = s \left( \int_S \left( \phi \Delta \psi \pm \bar{\psi} \Delta \chi \right) d^2 \sigma \right) \tag{4.4}
\]
\[
= \int_S \left( \bar{\beta} \Delta \psi \pm \bar{\phi} \Delta \phi \pm \bar{\chi} \Delta \chi \pm \bar{\psi} \Delta \beta \right) d^2 \sigma.
\]
The upper (lower) sign stands for the fields \(\bar{\psi}, \psi\) of boson (fermion) statistics. Integration after the ghost fields yields only some numerical factor and the quantum action
\[
S = S_{cl} + \int_S \left( \bar{\beta} \Delta \psi + \bar{\psi} \Delta \beta \right) d^2 \sigma. \tag{4.5}
\]
If necessary, one can insert \(\sqrt{g}\) into the second term, which is equivalent to change of variables. Thus the partition function is given by
\[
Z = \int e^{iS} D\bar{\psi} D\psi D\bar{\beta} D\beta, \tag{4.6a}
\]
with the boundary conditions: \(\bar{\beta}|_{\partial S} = \beta|_{\partial S} = 0\).
One can observe that the job the fields $\beta$ and $\bar{\beta}$ are supposed to do consists in eliminating a redundant integration inside $S$. The gauge-fixing condition following from Eq. 4.5 imposes the following constraints

$$\Delta \psi = 0, \quad \Delta \bar{\psi} = 0.$$

Since values of the fields $\psi$ and $\bar{\psi}$ are fixed on the boundary $\partial S = \mathcal{C}$, we deal with two well-defined 2-dimensional Dirichlet problems. The solutions of the Dirichlet problems fix values of $\psi$ and $\bar{\psi}$ inside $S$. Another, more singular gauge-fixing, is proposed in [2].

Accordingly, we can rewrite Eq. 4.6a in the form

$$Z = \int e^{iS_{cl}} \left( D\bar{\psi} D\psi \right) \big|_{\partial S}, \quad (4.6b)$$

where the integration is confined to the boundary $\partial S$. One can say that, in a sense, we have **BRS-quantized** the Abelian Stokes theorem passing from the theorem formulated for the classical action to the theorem formulated for the partition function (path integral). One can observe that by virtue of the Abelian Stokes theorem for a closed curve $\mathcal{C}$, $\mathcal{C} = \partial S$, Eq. 4.6b is essentially equivalent to Eq. 3.12, modulo some boundary terms. It appears that this “quantized” Abelian Stokes theorem is a prototype of our main theorem.

At present, we are prepared to formulate a holomorphic path-integral version of the non-Abelian Stokes theorem. Strictly speaking, a particular version of this theorem (actually, the best-known one) that is applicable to the case of the Lie algebra valued $1(2)$-form, i.e. connection (curvature) on 1(2)-dimensional space.

In the parametrized form the theorem reads

$$\int \exp \left[ -\bar{a}(\tau') a(\tau') + i \int_{\tau'}^{\tau''} L(\bar{a}(\tau), a(\tau)) d\tau \right] a_r(\tau'') \bar{a}_s(\tau') D\bar{a} Da$$

$$= \int \exp \left[ -\bar{a}(\tau', 0) a(\tau', 0) + i \int_0^1 \int_{\tau'}^{\tau''} \mathcal{L}_{cl}(\bar{a}, a) d\tau d\sigma^2 \right] a_r(\tau'', 0) \bar{a}_s(\tau', 0) D\bar{a} Da, \quad (4.7)$$

where $L(a, \bar{a})$ and $\mathcal{L}_{cl}(\bar{a}, a)$ are defined by Eq. 3.3 and Eq. 4.1b respectively. The measure on both sides of Eq. 4.7 is the same, i.e. it is concentrated on the boundary $\partial S$ as in Eq. 4.6b, and the imposed boundary conditions are free. Eq. 4.7 can also be put in a parametrization-free form using Eq. 4.1a and Eq. 4.3a. By virtue of our earlier analysis, the proof of the Eq. 4.7 is an immediate consequence of the Abelian Stokes theorem applied to $S_{cl}$, whereas it follows from the holomorphic path-integral representation of the parallel-transport operator that the LHS of Eq. 4.7 really represents the LHS of the “operator” version of the non-Abelian Stokes theorem (Eq. 2.8). It should be noted that the surface integral on the RHS of Eq. 4.7 depends on the curvature $F$ as well as on the connection $A$ entering the covariant
derivatives, which is reminiscent of the path dependence of the curvature $F$. 

5. Final remarks

In this final section we shall discuss the two issues: further generalizations of the presented version of the non-Abelian Stokes theorem and its possible physical applications.

The proposed theorem is a very particular, though seemingly the most important, non-Abelian version of the Stokes theorem. It connects a differential 1-form in dimension 1 and 2-form in dimension 2. The forms are of a very particular shape, namely, the connection 1-form and the curvature 2-form. Thus, possible generalizations should concern arbitrary differential forms in arbitrary dimensions. Since there might be a lot of variants of such a theorem depending on a particular mathematical and/or physical application, we will only confine ourselves to presenting a general recipe.

The general idea is very simple. First of all, we should construct a topological field theory of auxiliary topological fields on $\partial M$, the boundary of the manifold $M$, in the external gauge field we are interested in. Next, we should quantize the theory, i.e. build the partition function in the form of a path-integral, where auxiliary topological fields are properly integrated out. Applying the Abelian Stokes theorem to the (effective) action (in the exponent of the integrand) we obtain the “RHS” of the non-Abelian Stokes theorem. If we also wish to extend the functional measure to the whole $M$ we should additionally quantize the theory to eliminate the redundant functional integration inside $M$.

The main source of applications of the non-Abelian Stokes theorem is coming from topological field theory of Chern-Simons type. Path-integral procedure gives the possibility of obtaining skein relations for knot and link invariants. In particular, it appears that only the path-integral version of the non-Abelian Stokes theorem permits us to nonperturbatively and covariantly generalize the method of obtaining topological invariants [6].

As a by-product of our approach we have computed the parallel-transport operator $U$ in the holomorphic path-integral representation (see Eq. 3.12). In this way, we have solved the problem of saturation of Lie-algebra indices in the generators $T^a$. This issue appears, for example, in the context of equation of motion for Chern-Simons theory in the presence of Wilson lines (see Ref. 7, where an interesting connection with the Borel-Weil-Bott theorem and quantum groups has also been suggested). Our approach enables us to write those equations in terms of $\bar{a}$ and $a$ purely classically. Incidentally, in the presence of Chern-Simons interactions the auxiliary fields $\bar{a}$ and $a$ acquire fractional statistics, which could be detected by braiding. To determine the braiding matrix one should, in turn, find the so-called
monodromy matrix, e.g. making use of non-Abelian Stokes theorem.

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ADDENDUM

The non-Abelian Stokes theorem was first worked out in M. B. Halpern, *Phys. Rev. D19* (1979) 517, where it was also used to develop field strength and dual variable formulations of gauge theory. As an extension to this development, the theorem was also used on the lattice by Batrouni a few years later.