PERIODIC SOLUTIONS TO NON-AUTONOMOUS EVOLUTION EQUATIONS WITH MULTI-DELAYS

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Abstract. In this paper, we provide some sufficient conditions for the existence, uniqueness and asymptotic stability of time $\omega$-periodic mild solutions for a class of non-autonomous evolution equation with multi-delays. This work not only extend the autonomous evolution equation with multi-delays studied in [37] to non-autonomous cases, but also greatly weaken the condition presented in [37] even for the case $a(t) \equiv a$ by establishing a general abstract framework to find time $\omega$-periodic mild solutions for non-autonomous evolution equation with multi-delays. Finally, one illustrating example is supplied.

1. Introduction and main results. The theory of partial differential equations with delays has extensive physical background and realistic mathematical model, hence it has been considerably developed and the numerous properties of their solutions have been studied, see [9, 14, 15, 16, 21, 23, 24, 27, 33, 35, 38, 39] and references therein. The problems concerning periodic solutions of partial differential equations with delays are an important area of investigation in recent years and the existence or attractivity of periodic solutions for evolution equations with delays have been considered by several authors, see [5, 6, 7, 18, 19, 20, 23, 35, 36, 37] and references listed therein for more comments and citations. Most of these results are established by applying semigroup theory [7, 19, 20, 35, 36], corresponding fixed point theorems [5, 6, 7, 20, 23, 35], coincidence degree theory [18], Lyapunov functionals combined with the prior bound theory [37] and so on. In 2006, Liu and Li [23] obtained the existence of periodic solutions for a class of parabolic evolution equations with delay by utilizing Schaefer type theorem, which extend the corresponding results of Burton and Zhang [6]. Recently, by using the method of constructing some suitable Lyapunov functionals and establishing the prior bound for all possible periodic solutions, Zhu, Liu and Li [37] investigated the existence,
uniqueness and global attractivity of time periodic solutions for the following one-dimensional parabolic autonomous evolution equation with multi-delays

\[
\begin{aligned}
\frac{\partial}{\partial t} u(x, t) &= \frac{\partial^2}{\partial x^2} u(x, t) + au(x, t) + g(x, t) \\
&\quad + f(u(x, t - \tau_1), \ldots, u(x, t - \tau_n)), \quad (x, t) \in (0, 1) \times \mathbb{R}, \\
&\quad u(0, t) = u(1, t) = 0, \quad t \in \mathbb{R},
\end{aligned}
\tag{1}
\]

where \(a \in \mathbb{R}, f : \mathbb{R}^n \to \mathbb{R} \) is locally Lipschitz continuous, \( g : [0, 1] \times \mathbb{R} \to \mathbb{R} \) is Hölder continuous and \( g(x, t) \) is \( \omega \)-periodic in \( t \), \( \tau_1, \tau_2, \ldots, \tau_n \) are positive constants. This type of equations is usually used to model some process of biology. For example, Eq. (1) is called Hematopoiesis model \([27]\) for \( n = 1 \) and \( f(u) = \frac{\alpha u}{1 - au} \) with \( \alpha, m > 0 \); Eq. (1) is named blood cell production model \([34]\) for \( n = 1 \) and \( f(u) = e^{-ku} \) with \( k > 0 \); Eq. (1) is called Nicholson’s blowflies model \([17, 28]\) for \( n = 1 \) and \( f(u) = u e^{-ku} \) with \( k > 0 \). One of the main results in \([37]\) is the following theorem:

**Theorem A.** Assume that there exist some positive constants \( \beta_1, \beta_2, \ldots, \beta_n \) and \( K \) satisfying following conditions

(A1) The integer \( n \leq 3 \);

(A2) \( |f(\xi_1, \xi_2, \ldots, \xi_n) + g(x, t)| \leq \sum_{k=1}^{n} \beta_k |\eta_k| + K \) for \( x \in [0, 1] \) and \((\xi_1, \xi_2, \ldots, \xi_n) \in \mathbb{R}^n;\)

(A3) \(|a| + 2 + \pi < \beta_k < \pi^2 + 1.

Then Eq. (1) admits a nontrivial \( \omega \)-periodic solution.

On the other hand, the dynamical characteristics (including stable, unstable, attract, oscillatory and chaotic behavior) of differential equations have become a subject of intense research activities. For the details of this field, we refer the reader to the monographs of Burton \([5]\), Hale \([18]\) and the papers of Caicedo, Cuevasa, Mophoub and N’Guérékata \([7]\), Chen and Guo \([8]\), Li and Wang \([21]\) and Wang, Liu and Li \([33]\). Especially, by adding the following condition

(A4) \( |f(\eta_1, \eta_2, \ldots, \eta_n) - f(\xi_1, \xi_2, \ldots, \xi_n)| \leq \sum_{k=1}^{n} \beta_k |\eta_k - \xi_k| \) for \( x \in [0, 1], (\eta_1, \eta_2, \ldots, \eta_n) \in \mathbb{R}^n \) and \((\xi_1, \xi_2, \ldots, \xi_n) \in \mathbb{R}^n;\)

Zhu, Liu and Li \([37]\) also obtained the following uniqueness and asymptotically stable result of \( \omega \)-periodic solution for Eq. (1):

**Theorem B.** Assume that the conditions (A1), (A2), (A3) and (A4) are satisfied. Then Eq. (1) admits a unique nontrivial \( \omega \)-periodic solution and it is asymptotically stable.

In Eq. (1), \( a \in \mathbb{R} \) is a constat and independent the time variable \( t \), which means that the equation under consideration is autonomous. However, when treating some parabolic evolution equations, it is usually assumed that the partial differential operators depend on time \( t \) on account of this class of operators appears frequently in the applications, for the details please see Amann \([4]\), Chen et al. \([9, 10, 11, 12, 13]\), Fu \([15]\), Fu and Zhang \([16]\), Liang, Liu and Xiao \([22]\), Ouyang \([24]\), Tanabe \([29]\), Wang, Ezzinbi and Zhu \([32]\), Wang and Zhu \([31]\) and Zhu, Liu and Wu \([38, 39]\). Therefore, it is interesting and significant to investigate the existence, uniqueness and global attractivity of time periodic solutions for non-autonomous evolution equation with delays, i.e., the differential operators in the main parts of the considered equations are dependent of time variable \( t \).
Inspired by above-mentioned aspects, in this paper, we will investigate the existence, uniqueness and asymptotic stability of time $\omega$-periodic mild solutions for the following non-autonomous evolution equation with multi-delays of the form

$$\begin{cases}
\frac{\partial}{\partial t}u(x,t) = \frac{\partial^2}{\partial x^2}u(x,t) + a(t)u(x,t) + g(x,t) \\
\quad + f(u(x,t - \tau_1), \cdots, u(x,t - \tau_n)), \\
(\tau_1, \tau_2, \cdots, \tau_n) > 0,
\end{cases}$$

where $a(t) : \mathbb{R} \to \mathbb{R}$ is a continuously differentiable function and it is $\omega$-periodic in $t$, $f : \mathbb{R}^n \to \mathbb{R}$ is a continuous nonlinear function, $g : [0,1] \times \mathbb{R} \to \mathbb{R}$ is continuous and $g(t,x)$ is $\omega$-periodic in $t$, $\tau_1, \tau_2, \cdots, \tau_n$ are positive constants. Our main purpose in this paper is to extend the results of [37] to the non-autonomous case and essentially weaken the conditions presented in [37]. We provide sufficient conditions for the existence, uniqueness and asymptotic stability of time $\omega$-periodic mild solutions for the non-autonomous evolution equation with multi-delays (2) using Schauder Fixed Point Theorem and the theory of linear evolution family.

For the sake of convenience, we denote by

$$a_{\text{max}} := \max_{t \in [0,\pi]} a(t).$$

Then the main results of this paper are as follows.

**Theorem 1.1.** Assume that $f : \mathbb{R}^n \to \mathbb{R}$ is continuous, $g : [0,1] \times \mathbb{R} \to \mathbb{R}$ is continuous and $g(t,x)$ is $\omega$-periodic in $t$. If the condition (A2) and the following condition

$$\text{(A3)' } a_{\text{max}} + \sum_{k=1}^{n} \beta_k < \pi^2,$$

hold, then non-autonomous evolution equation with multi-delays (2) has at least one time $\omega$-periodic mild solution $u \in C([0,1] \times \mathbb{R}, \mathbb{R})$.

**Theorem 1.2.** Assume that $f : \mathbb{R}^n \to \mathbb{R}$ is continuous, $g : [0,1] \times \mathbb{R} \to \mathbb{R}$ is continuous and $g(t,x)$ is $\omega$-periodic in $t$. If the conditions (A3)' and (A4) hold, then non-autonomous evolution equation with multi-delays (2) has a unique time $\omega$-periodic mild solution $u \in C([0,1] \times \mathbb{R}, \mathbb{R})$.

Furthermore, if we strengthening the condition (A3)' in Theorem 1.2 as

$$\text{(A3)'' } a_{\text{max}} + \sum_{k=1}^{n} \beta_k e^{(\pi^2-a_{\text{max}})\tau_k} < \pi^2,$$

then we will obtain the following asymptotic stability result of the time $\omega$-periodic mild solution.

**Theorem 1.3.** Assume that $f : \mathbb{R}^n \to \mathbb{R}$ is continuous, $g : [0,1] \times \mathbb{R} \to \mathbb{R}$ is continuous and $g(t,x)$ is $\omega$-periodic in $t$. If the conditions (A3)'' and (A4) hold, then non-autonomous evolution equation with multi-delays (2) has a unique time $\omega$-periodic mild solution $u \in C([0,1] \times \mathbb{R}, \mathbb{R})$ which is globally asymptotically stable and exponentially attracts the other solutions.

**Remark 1.** We observed that the authors required that $n \leq 3$ in [37], which means that there are at most three delays in nonlinear term and is a strong restriction. In this paper, we completely delete this condition by using a completely different method with which used in [37].
Remark 2. For the case that \( a(t) \equiv a \) for \( t \in \mathbb{R} \), i.e., the case of autonomous evolution equation with multi-delays (1), from (3) we know that \( a_{\text{max}} = a \). By a simple calculation one gets that the condition (A3) \( \Rightarrow (A3)' \), which means that the condition (A3)' weakens the condition (A3) greatly. Therefore, the results of this paper not only extend Eq. (1) to non-autonomous evolution equation with multi-delays (2), but also greatly weaken the conditions to the main results Theorems A and B in [37] even for the case \( a(t) \equiv a \).

The discussions of this paper are based on a frame of abstract infinite-dimensional space. By define a linear evolution operator \( A(t) \), which generates a linear evolution family \( \{ U(t, s) : t \geq s \} \) in abstract space \( E = L^2((0, 1), \mathbb{R}) \), we can transfer the non-autonomous evolution equation with multi-delays (2) into the abstract form for a class of non-autonomous evolution equation with multi-delays in the frame of abstract space \( E \), and then applying Schauder Fixed Point Theorem, the theory of nonlinear analysis and linear evolution family to discuss the existence and uniqueness of \( \omega \)-periodic mild solutions for abstract non-autonomous evolution equation. This is in turn the time \( \omega \)-periodic mild solution of non-autonomous evolution equation with multi-delays (2). Furthermore, based on the uniqueness of time \( \omega \)-periodic mild solution combined with an integral inequality of Bellman type with delays, we obtained the global asymptotic stability of time \( \omega \)-periodic mild solution for non-autonomous evolution equation with multi-delays (2) by using the exponentially decay property of linear evolution family \( \{ U(t, s) : t \geq s \} \), which can be find in Section 2 inequality (9).

The remainder of this work is arranged as follows. In section 2 we present some necessary notations, definitions, and preliminary facts which will be used throughout this paper as well as a general framework of linear evolution family. Especially, the non-autonomous evolution equation with multi-delays (2) is transformed into an abstract non-autonomous evolution equation with multi-delays in abstract space. Moreover, the existence and uniqueness result of \( \omega \)-periodic mild solution for the corresponding linear non-autonomous evolution equation is established, which plays an important role in the proof of our main results. In section 3, we give the proof of our main results. Finally, as an illustration of the developed theory, we apply it to one example of hematopoiesis model to one-dimensional reaction-diffusion equation with variable coefficient and delay in Section 4.

2. Preliminaries. In this section, we introduce some notations, definitions, and preliminary facts which will be used throughout this paper. Let \( E = L^2((0, 1), \mathbb{R}) \) be a real Hilbert space with the \( L^2 \)-norm \( \| \cdot \|_2 \) defined by

\[
\| u \|_2 = \left( \int_0^1 |u(x)|^2 dx \right)^{\frac{1}{2}}, \quad \forall \ u \in L^2((0, 1), \mathbb{R})
\]

and inner product \( \langle \cdot, \cdot \rangle \) defined by

\[
\langle u, v \rangle = \int_0^1 u(x)v(x) dx, \quad \forall \ u, v \in L^2((0, 1), \mathbb{R}).
\]

Denote by \( \mathcal{L}(E) \) be the Banach space of all linear and bounded operators in \( E \) endowed with the topology defined by the operator norm. Consider the operator \( B \) on \( E \) defined by

\[
Bu := \frac{\partial^2}{\partial x^2} u, \quad u \in D(B),
\]
where
\[ D(B) := \{ u \in L^2(0, 1; \mathbb{R}), u'' \in L^2((0, 1), \mathbb{R}), u(0) = u(1) = 0 \}. \]  

(5)

It is well known from Pazy [25] and Pruss [26] that \( B \) generates a compact and analytic \( C_0\)-semigroup in \( E \), \( B \) has a discrete spectrum, and its eigenvalues are \(-n^2 \pi^2, n \in \mathbb{N}^+\) with the corresponding normalized eigenvectors \( v_n(x) = \sqrt{2} \sin(n \pi x)\).

Define the operator \( A(t) \) on \( E \) by
\[ A(t)u = Bu + a(t)u \]

with domain
\[ D(A(t)) = D(B), \quad t \in \mathbb{R}. \]

(7)

It follows from [25, Lemma 6.1 in Chapter 7] that \( A(t) \) satisfy the following condition

\[(AT_1) \quad \text{For each } t \in \mathbb{R}, A(t) \text{ is a closed linear operator on } E \text{ and there exist constants } \lambda_0 \geq 0, \theta \in (\frac{\pi}{2}, \pi), M_1 \geq 0 \text{ such that } \Sigma_\theta \cup \{0\} \subset \rho(A(t) - \lambda_0) \text{ and for all } \lambda \in \Sigma_\theta \cup \{0\} \text{ and } t \in [0, a],
\]
\[ \|R(\lambda, A(t) - \lambda_0)\|_{\mathcal{L}(E)} \leq \frac{M_1}{1 + |\lambda|}. \]

Furthermore, by again [25, Lemma 6.1 in Chapter 7] together with continuously differentiability of coefficient \( a(t) \) one know that the following condition

\[(AT_2) \quad \text{There exist constants } M_2 > 0 \text{ and } \theta, \beta \in (0, 1] \text{ with } \theta + \beta > 1 \text{ such that for all } \lambda \in \Sigma_\theta \text{ and } t \geq s,
\]
\[ \|(A(t) - \lambda_0)R(\lambda, A(t) - \lambda_0)[R(\lambda_0, A(t)) - R(\lambda_0, A(s))]\|_{\mathcal{L}(E)} \leq \frac{M_2|t - s|^{\theta}}{|\lambda|^\beta}, \]

where
\[ \Sigma_\theta = \{ \lambda \in \mathbb{C} \setminus \{0\} : |\lambda| \leq \theta \}, \]

is satisfied. Conditions \((AT_1)\) and \((AT_2)\), which are initiated by Acquistapace and Terreni in [2] and Acquistapace in [1] for \( \lambda_0 = 0 \), are well understood and widely used in the literature. Under the above conditions \((AT_1)\) and \((AT_2)\), the family \( \{A(t) : t \in \mathbb{R}\} \) generates a unique linear evolution system, or called linear evolution family, \( \{U(t, s) : t \geq s\} \). Moreover, by an obvious rescaling from [1, Theorem 2.3] and [2, Theorem 2.1] combined with the Acquistapace and Terreni conditions \((AT_1)\) and \((AT_2)\) one gets the following properties for the family of linear operator \( \{U(t, s) : t \geq s\} \).

**Lemma 2.1.** The family of the linear operator \( \{U(t, s) : t \geq s\} \) satisfies the following properties:

(i) \( U(t, r)U(r, s) = U(t, s), U(t, t) = I \) for \( s \leq r \leq t \);

(ii) The map \( (t, s) \mapsto U(t, s)x \) is continuous for all \( x \in E \) and \( t \geq s \);

(iii) \( U(\cdot, s) \in C^1((s, \infty), \mathcal{L}(E)), \frac{dU(t, s)}{dt} = A(t)U(t, s) \) for \( t > s \), and \( \|A^k(t)U(t, s)\|_2 \leq M_3(t - s)^{-k} \) for \( 0 < t - s \leq 1 \) and \( k = 0, 1 \), where \( M_3 > 0 \) is a constant;

(iv) \( \frac{dU(t, s)}{ds} = -U(t, s)A(s)x \) for \( t > s \) and \( x \in D(A(s)) \) with \( A(s)x \in D(A(s)) \).

Therefore, from the above discussion we know that the family \( \{A(t) : t \geq s\} \) defined by (6) generates a strongly continuous evolution family \( \{U(t, s) : t \geq s\} \) defined by

\[ U(t, s)u = \sum_{n=1}^{\infty} e^{-n^2 \pi^2(t-s)+\int_{t}^{s} a(\tau)\,d\tau} \langle u, v_n \rangle v_n, \quad t \geq s, u, v_n \in E. \]  

(8)
A direct calculation gives
\[ \|U(t, s)\|_{L(E)} \leq e^{-\left(\pi^2 - a_{\max}\right)(t-s)}, \quad t \geq s, \quad (9) \]
where \( a_{\max} \) is defined by (3). By (6) and the fact that the function \( a(t) : \mathbb{R} \to \mathbb{R} \) is \( \omega \)-periodic in \( t \) for \( \omega > 0 \), we know that the linear operator \( A(t) \) is also \( \omega \)-periodic in \( t \), i.e., \( A(t + \omega) = A(t) \) for all \( t \in \mathbb{R} \). From which one gets that the evolution family \( \{U(t, s) : t \geq s\} \) is \( \omega \)-periodic in \( t \) and \( s \) for \( \omega > 0 \) with \( t \geq s \), i.e.,
\[ U(t + \omega, s + \omega) = U(t, s) \quad \text{for} \quad t \geq s \text{ and } \omega > 0. \quad (10) \]

**Definition 2.2.** An evolution family \( \{U(t, s) : t \geq s\} \) is said to be compact if for all \( t > s \), \( U(t, s) \) is continuous and maps bounded subsets of \( E \) into precompact subsets of \( E \).

**Lemma 2.3.** ([14]) For each \( t \in [0, a] \) and some \( \lambda \in \rho(A(t)) \), if the resolvent \( R(\lambda, A(t)) \) is a compact operator, then \( U(t, s) \) is a compact operator whenever \( t > s \).

**Lemma 2.4.** Let \( \{U(t, s) : t \geq s\} \) be a compact evolution family on \( E \). Then for each \( s \in \mathbb{R} \), the function \( t \mapsto U(t, s) \) is continuous by operator norm for \( t > s \).

Furthermore, note from [19] that, for each \( t, s \in \mathbb{R} \) with \( t > s \), the evolution family \( U(t, s) \) defined by (8) is a nuclear operator. This fact combined with Lemma 2.3 one gets that the evolution family \( U(t, s) \) defined by (8) is compact for every \( t > s \).

Let \( u(t) = u(\cdot, t), \ f(u(t - \tau_1), \cdots, u(t - \tau_n)) = f(u(\cdot, t - \tau_1), \cdots, u(\cdot, t - \tau_n)), \ g(t) = g(\cdot, t) \). Then the non-autonomous evolution equation with multi-delays (2) can be rewritten into the abstract form of non-autonomous evolution equation with multi-delays
\[ u'(t) = A(t)u(t) + f(u(t - \tau_1), \cdots, u(t - \tau_n)) + g(t), \quad t \in \mathbb{R} \quad (11) \]
in Hilbert space \( E = L^2((0, 1), \mathbb{R}) \), where \( A(t) \) is the linear operator defined by (6). Therefore, in order to investigate non-autonomous evolution equation with multi-delays (2), we only need to study the abstract non-autonomous evolution equation with multi-delays (11).

Denote by
\[ C_{\omega}(\mathbb{R}, E) = \{u \mid u : \mathbb{R} \to E \text{ is continuous and } u(t + \omega) = u(t) \text{ for every } t \in \mathbb{R}\}. \]
Then it is easy to verify that \( C_{\omega}(\mathbb{R}, E) \) is a Banach space endowed with the norm
\[ \|u\|_C = \max_{t \in [0, \omega]} \|u(t)\|_2, \quad \forall \ u \in C_{\omega}(\mathbb{R}, E). \]
In what follows, we discuss the existence and uniqueness of \( \omega \)-periodic mid solutions for the following linear non-autonomous evolution equation
\[ u'(t) = A(t)u(t) + h(t), \quad t \in \mathbb{R}, \quad (12) \]
where \( h \in C_{\omega}(\mathbb{R}, E) \).

**Lemma 2.5.** Assume that \( a_{\max} < \pi^2 \). Then for every \( h \in C_{\omega}(\mathbb{R}, E) \), the linear non-autonomous evolution equation (12) exists a unique \( \omega \)-periodic mild solution \( u \in C_{\omega}(\mathbb{R}, E) \) and restricted it on \([0, \omega]\) is given by
\[ u(t) = U(t, 0) \left( I - U(\omega, 0) \right)^{-1} \int_0^\omega U(\omega, s)h(s)ds \]
\[ + \int_0^t U(t, s)h(s)ds, \quad t \in [0, \omega]. \tag{13} \]

**Proof.** From [25, Chapter 5, Theorem 7.1], we know that for any \( u_0 \in E \), if the linear function \( h \in L^1(\mathbb{R}_+, E) \), then the initial value problem (IVP) of linear non-autonomous evolution equation
\[
\begin{cases}
  u'(t) - A(t)u(t) = h(t), & t \in \mathbb{R}_+, \\
  u(0) = u_0
\end{cases}
\tag{14}
\]
exists a unique mild solution \( u \in C(\mathbb{R}_+, E) \) expressed by
\[
u(t) = U(t, 0)u_0 + \int_0^t U(t, s)h(s)ds, \quad t \in \mathbb{R}_+. \tag{15}\]
Therefore, by (15) one gets that
\[
u(\omega) = U(\omega, 0)u_0 + \int_0^\omega U(\omega, s)h(s)ds. \tag{16}\]
From (9) combined with the fact \( a_{\text{max}} < \pi^2 \) one gets that
\[
\|U(\omega, 0)\|_2 \leq e^{-(\pi^2 - a_{\text{max}})\omega} < 1.
\]
Therefore, by operator spectrum theorem we know that the operator \( I - U(\omega, 0) \) has a bounded inverse operator \( \left( I - U(\omega, 0) \right)^{-1} \). Hence, there exists a unique initial value
\[
u_0 = \left( I - U(\omega, 0) \right)^{-1} \int_0^\omega U(\omega, s)h(s)ds \tag{17}\]
such that the unique mild solution \( u \) of IVP (14) expressed by (15) satisfies the periodic boundary condition \( u(0) = u_0 = u(\omega) \). Therefore, from (10), (15), (16), (17), Lemma 2.1 (i) and the fact that \( h(t) = h(t + \omega) \) for \( t \in \mathbb{R} \), we get that for every \( t \in \mathbb{R}_+ \)
\[
u(t + \omega) = U(t + \omega, 0)u_0 + \int_0^\omega U(t + \omega, s)h(s)ds + \int_0^{t+\omega} U(t + \omega, s)h(s)ds
\]
\[
= U(t + \omega, \omega) \left[ U(\omega, 0)u_0 + \int_0^\omega U(\omega, s)h(s)ds \right]
\]
\[
+ \int_0^t U(t + \omega, s + \omega)h(s + \omega)ds
\]
\[
= U(t, 0)u(\omega) + \int_0^t U(t, s)h(s)ds
\]
\[
= u(t).
\]
Therefore, the \( \omega \)-periodic extension of \( u \) on \( \mathbb{R} \) is a unique \( \omega \)-periodic mild solution of linear non-autonomous evolution equation (12). Combining (15) and (17), we get that the mild solution \( u \) of linear non-autonomous evolution equation (12) satisfies (13) on \( [0, \omega] \) and it can be extend to \( \mathbb{R} \).
the proof of Lemma 2.5. \hfill \Box

By above discuss, we can give the definition of mild solutions for non-autonomous evolution equation with multi-delays (11).

**Definition 2.6.** A continuous function \( u \in C_\omega(\mathbb{R}, E) \) is called a \( \omega \)-periodic solution of nonlinear non-autonomous evolution equation (11) if restricted it on \([0, \omega] \) satisfies the following integral equation

\[
\begin{align*}
  u(t) &= U(t, 0) \left( I - U(\omega, 0) \right)^{-1} \int_0^\omega U(\omega, s) [f(u(s - \tau_1), \cdots, u(s - \tau_n)) + g(s)] ds \\
  &\quad + \int_0^t U(t, s) [f(u(s - \tau_1), \cdots, u(s - \tau_n)) + g(s)] ds.
\end{align*}
\]

3. **Proof of the main results.** In this section, we give the proofs of Theorem 1.1, Theorem 1.2 and Theorem 1.3.

**Proof of Theorem 1.1.** By the discussions in Section 2, we know that the non-autonomous evolution equation with multi-delays (2) can be transformed into the abstract form of non-autonomous evolution equation with multi-delays (11) in infinite-dimensional space \( E = L^2([0, 1], \mathbb{R}) \). In what follows, we prove the existence of time \( \omega \)-periodic mild solutions for abstract non-autonomous evolution equation with multi-delays (11).

Consider the operator \( \mathcal{P} \) on \( C([0, \omega], E) \) defined by

\[
\begin{align*}
  (\mathcal{P}u)(t) &= U(t, 0) \left( I - U(\omega, 0) \right)^{-1} \int_0^\omega U(\omega, s) [f(u(s - \tau_1), \cdots, u(s - \tau_n)) + g(s)] ds \\
  &\quad + \int_0^t U(t, s) [f(u(s - \tau_1), \cdots, u(s - \tau_n)) + g(s)] ds, \quad t \in [0, \omega].
\end{align*}
\]

By Definition 2.6 and Lemma 2.5 one can easily see that the operator \( \mathcal{P} \) can be extended to \( C_\omega(\mathbb{R}, E) \) and the time \( \omega \)-periodic mild solutions of abstract delay evolution equation (11) is equivalent to the fixed point of operator \( \mathcal{P} \) defined by (18). In the following, we will prove the operator \( \mathcal{P} \) has at least one fixed point by applying the Schauder Fixed Point Theorem.

Denote

\[
\Omega_R = \{ u \in C_\omega(\mathbb{R}, E) : \| u \|_C \leq R \},
\]

then \( \Omega_R \) is a closed ball in \( C_\omega(\mathbb{R}, E) \) with center \( \theta \) and radius \( R \). Firstly, we prove that there exists a constant \( R \) big enough such that the operator \( \mathcal{P} \) maps \( \Omega_R \) to \( \Omega_R \). By the condition (A2) we know that for any \( u \in C_\omega(\mathbb{R}, E) \)

\[
\| f(u(t - \tau_1), \cdots, u(t - \tau_n)) + g(t) \|_2 \leq \sum_{k=1}^n \beta_k \| u(t - \tau_k) \|_2 + K, \quad t \in \mathbb{R}.
\]

Furthermore, by (9) and the condition (A3)' one gets that

\[
\| U(\omega, 0) \|_2 \leq e^{-\left( \pi^2 - a_{\max} \right) \omega} < 1.
\]

(20) combined with Neumann expression, we know that \( \left( I - U(\omega, 0) \right)^{-1} \) can be expressed by

\[
\left( I - U(\omega, 0) \right)^{-1} = \sum_{n=0}^{\infty} U^n(\omega, 0).
\]

Inversely, we can verify directly that the function \( u \in C_\omega(\mathbb{R}, E) \) given by (13) is a mild solution of linear non-autonomous evolution equation (12). This completes the proof of Lemma 2.5.
Therefore, from equality (21) combined with (9) one gets that
\[
\left\| \left( I - U(\omega, 0) \right)^{-1} \right\|_2 = \left\| \sum_{n=0}^{\infty} U^n(\omega, 0) \right\|_2 \leq \sum_{n=0}^{\infty} e^{-(\pi^2 - a_{\text{max}})n\omega} = \frac{1}{1 - e^{-(\pi^2 - a_{\text{max}})\omega}}.
\]  
(22)

From the condition (A3)', we can choosing
\[
R \geq \frac{K}{\pi^2 - a_{\text{max}} - \sum_{k=1}^{n} \beta_k}.
\]  
(23)

Then for any \( u \in \Omega_R \) and \( t \in \mathbb{R} \), by (9), (18), (19), (22), (23), the conditions (A2) and (A3)', we have
\[
\left\| (Pu)(t) \right\|_2 \leq \frac{e^{-(\pi^2 - a_{\text{max}})t}}{1 - e^{-(\pi^2 - a_{\text{max}})\omega}} \int_{0}^{\omega} e^{-(\pi^2 - a_{\text{max}})(\omega - s)} \left[ \sum_{k=1}^{n} \beta_k \left\| u(s - \tau_k) \right\|_2 + K \right] ds
\]
\[
+ \int_{0}^{t} e^{-(\pi^2 - a_{\text{max}})(t - s)} \left[ \sum_{k=1}^{n} \beta_k \left\| u(s - \tau_k) \right\|_2 + K \right] ds
\]
\[
\leq \left[ \frac{e^{-(\pi^2 - a_{\text{max}})t}}{\pi^2 - a_{\text{max}}} + \frac{1 - e^{-(\pi^2 - a_{\text{max}})t}}{\pi^2 - a_{\text{max}}} \right] \cdot \left[ \sum_{k=1}^{n} \beta_k \left\| u \right\|_C + K \right]
\]
\[
\leq \frac{1}{\pi^2 - a_{\text{max}}} \left( R \sum_{k=1}^{n} \beta_k + K \right)
\]
\[
\leq R,
\]
which means that
\[
\left\| Pu \right\|_C = \max_{t \in [0, \omega]} \left\| (P)(t) \right\|_2 \leq R.
\]  
(24)

From (24) one gets that \( Pu \in \Omega_R \). Therefore, we proved that the operator \( P \) defined by (18) maps \( \Omega_R \) to \( \Omega_R \).

Secondly, we prove \( P : \Omega_R \to \Omega_R \) is a continuous operator. To this end, let \( \{u_n\}_{n=1}^{\infty} \subset \Omega_R \) be a sequence such that \( \lim_{n \to \infty} u_n = u \) in \( \Omega_R \). By the continuity of the nonlinear function \( f \), we have
\[
\lim_{n \to \infty} \sum_{j=1}^{n} f(u_n(s - \tau_j)), \ldots, u_n(s - \tau_n)) = f(u(s - \tau_1), \ldots, u(s - \tau_n)), \quad \forall \ s \in \mathbb{R}.
\]  
(25)

By the condition (A2) combined with (19) we know that for every \( s \in \mathbb{R} \)
\[
\left\| \sum_{j=1}^{n} f(u_n(s - \tau_j)) + g(s) - \sum_{j=1}^{n} f(u(s - \tau_j)) + g(s) \right\|_2 \leq 2 \left( R \sum_{k=1}^{n} \beta_k + K \right).
\]  
(26)
Therefore, by (9), (18), (22), (25), (26) and the Lebesgue dominated convergence theorem, we get that
\[
\|(P_{u_n})(t) - (P_{u})(t)\|
\leq \frac{1}{1 - e^{-((\sigma^2 - a_{\max})\omega)}} \int_{0}^{\omega} \|[f(u_n(s - \tau_1), \ldots, u_n(s - \tau_n)) + g(s)] - [f(u(s - \tau_1), \ldots, u(s - \tau_n)) + g(s)]\|_{2}ds
\]
\[+ \int_{0}^{t} \|[f(u_n(s - \tau_1), \ldots, u_n(s - \tau_n)) + g(s)] - [f(u(s - \tau_1), \ldots, u(s - \tau_n)) + g(s)]\|_{2}ds
\]
\[\rightarrow 0 \quad \text{as} \quad n \rightarrow \infty,
\]
which means that
\[
\|(P_{u_n}) - (P_{u})\|_{C} = \max_{t \in [0, \omega]} \|(P_{u_n})(t) - (P_{u})(t)\|_{2} \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty. \quad (27)
\]
Therefore, by (27) we know that \(P : \Omega_R \rightarrow \Omega_R\) is a continuous operator.

Next, we demonstrate that \(P : \Omega_R \rightarrow \Omega_R\) is a compact operator. To prove this, we first show that \(\{(P_{u})(t) : u \in \Omega_R\}\) is relatively compact in \(E\) for every \(t \in \mathbb{R}\).
From the periodicity of the operator \((P_{u})(t)\) for \(t \in \mathbb{R}\) and \(u \in \Omega_R\), we only need to prove that \(\{(P_{u})(t) : u \in \Omega_R\}\) is relatively compact in \(E\) for \(0 \leq t \leq \omega\). It is easy to know from Lemma 2.1 and (18) that for every \(u \in \Omega_R\),
\[
(P_{u})(0) = \left(I - U(\omega, 0)\right)^{-1} \int_{0}^{\omega} U(\omega, s)[f(u(s - \tau_1), \ldots, u(s - \tau_n)) + g(s)]ds. \quad (28)
\]
For any \(0 < \epsilon < \omega\) and \(u \in \Omega_R\), we define the operator \(P_{\epsilon}^\ast\) by
\[
(P_{\epsilon}^\ast u)(0) = U(\omega, \omega - \frac{\epsilon}{2})U(\omega - \frac{\epsilon}{2}, \omega - \epsilon)\left(I - U(\omega, 0)\right)^{-1}
\]
\[\cdot \int_{0}^{\omega - \epsilon} U(\omega - \epsilon, s)[f(u(s - \tau_1), \ldots, u(s - \tau_n)) + g(s)]ds. \quad (29)
\]
Since \(U(\omega, \omega - \frac{\epsilon}{2}) \in \mathcal{L}(E)\) and \(U(\omega - \frac{\epsilon}{2}, \omega - \epsilon)\) is compact in \(E\), the set \(\{(P_{\epsilon}^\ast u)(0) : u \in \Omega_R\}\) is relatively compact in \(E\) for every \(\epsilon \in (0, \omega)\). Moreover, for every \(u \in \Omega_R\), by (9), (18), (19), (22), (28), (29), the conditions (A2), (A3)' and Lemma 2.1, we get that
\[
\|(P_{u})(0) - (P_{\epsilon}^\ast u)(0)\|_{2}
\leq \left\|U(\omega, \omega - \frac{\epsilon}{2})U(\omega - \frac{\epsilon}{2}, \omega - \epsilon)\left(I - U(\omega, 0)\right)^{-1}
\right.
\]
\[\cdot \int_{0}^{\omega - \epsilon} U(\omega - \epsilon, s)[f(u(s - \tau_1), \ldots, u(s - \tau_n)) + g(s)]ds
\]
\[\left. - \left(I - U(\omega, 0)\right)^{-1} \int_{0}^{\omega - \epsilon} U(\omega, s)[f(u(s - \tau_1), \ldots, u(s - \tau_n)) + g(s)]ds\right\|_{2}
\]
Hence, we have proved that there exists relatively compact set \( \{ \mathcal{P}^t u(0) : u \in \Omega_R \} \) arbitrarily close to the set \( \{ \mathcal{P}u(0) : u \in \Omega_R \} \), this means that the set \( \{ \mathcal{P}u(0) : u \in \Omega_R \} \) is relatively compact in \( E \). Let \( 0 < t \leq \omega \) be given, \( 0 < \epsilon < t \) and \( u \in \Omega_R \), we define the operator \( \mathcal{P}^t u \) by

\[
(\mathcal{P}^t u)(t) = U(t, 0)(\mathcal{P}u)(0) + U(t, t - \frac{\epsilon}{2})U(t - \frac{\epsilon}{2}, t - \epsilon) \cdot \int_{t - \epsilon}^{t - \epsilon} U(t, s)[f(u(s - \tau_1), \cdots, u(s - \tau_n)) + g(s)]ds.
\]

Since \( U(t, t - \frac{\epsilon}{2}) \in L(E), U(t - \frac{\epsilon}{2}, t - \epsilon) \) and \( U(t, 0) \) are compact in \( E \), the set \( \{ \mathcal{P}u(0) : u \in \Omega_R \} \) is relatively compact in \( E \), the set \( \{ \mathcal{P}^t u(t) : u \in \Omega_R \} \) is relatively compact in \( E \) for every \( \epsilon \in (0, t) \) and \( 0 < t \leq \omega \). Furthermore, for every \( u \in \Omega_R \), by (9), (18), (19), (22), (28), (30), the conditions (A2), (A3)', and Lemma 2.1, we get that

\[
\| (\mathcal{P}u)(t) - (\mathcal{P}^t u)(t) \|_2 = \left\| \int_{t - \epsilon}^{t} U(t, s)[f(u(s - \tau_1), \cdots, u(s - \tau_n)) + g(s)]ds \right\|_2
\leq \int_{t - \epsilon}^{t} e^{-(\pi^2 - a_{\text{max}})(t - s)} \left[ \sum_{k=1}^{n} \beta_k \|u(s - \tau_k)\|_2 + K \right]ds
\leq \frac{1 - e^{-(\pi^2 - a_{\text{max}})\epsilon}}{(\pi^2 - a_{\text{max}})(1 - e^{-(\pi^2 - a_{\text{max}})\omega})} \left( R \sum_{k=1}^{n} \beta_k + K \right)
\rightarrow 0 \quad \text{as} \quad \epsilon \rightarrow 0.
\]

Therefore, we have proved that there exists relatively compact set \( \{ (\mathcal{P}^t u)(t) : u \in \Omega_R \} \) arbitrarily close to the set \( \{ (\mathcal{P}u)(t) : u \in \Omega_R \} \) in \( E \) for \( 0 < t \leq \omega \). Hence, the set \( \{ (\mathcal{P}u)(t) : u \in \Omega_R \} \) is also relatively compact in \( E \) for \( 0 < t \leq \omega \), which combined with the fact that the set \( \{ (\mathcal{P}u)(0) : u \in \Omega_R \} \) is relatively compact in \( E \) we get the relatively compactness of the set \( \{ (\mathcal{P}u)(t) : u \in \Omega_R \} \) in \( E \) for \( 0 \leq t \leq \omega \).

In the following, we prove that \( \mathcal{P} : \Omega_R \rightarrow \Omega_R \) is an equicontinuous operator. For any \( u \in \Omega_R \) and \( t_1, t_2 \in \mathbb{R} \) with \( t' < t'' \), we get form (9), (18), (19), the conditions (A2) and (A3)' that

\[
\| (\mathcal{P}u)(t'') - (\mathcal{P}u)(t') \|_2 \leq \left\| (U(t'', 0) - U(t', 0))(I - U(\omega, 0))^{-1} \int_{\omega}^{t''} U(\omega, s) \cdot [f(u(s - \tau_1), \cdots, u(s - \tau_n)) + g(s)]ds \right\|_2
\]
\[ + \left( R \sum_{k=1}^{n} \beta_k + K \right) \int_{0}^{t'} \| U(t'', s) - U(t', s) \|_{2} ds \]
\[ + \left( R \sum_{k=1}^{n} \beta_k + K \right) \int_{t'}^{t''} e^{-\left( \pi^2 - a_{\text{max}} \right) (t'' - s)} ds \]
\[ := J_1 + J_2 + J_3, \]

where

\[ J_1 = \left\| (U(t'', 0) - U(t', 0)) \left( I - U(\omega, 0) \right)^{-1} \right\|_2 \]
\[ \cdot \int_{0}^{\omega} U(\omega, s) [f(u(s - \tau_1), \ldots, u(s - \tau_n)) + g(s)] ds \right\|_{2}, \tag{31} \]
\[ J_2 = \left( R \sum_{k=1}^{n} \beta_k + K \right) \int_{0}^{t'} \| U(t'', s) - U(t', s) \|_{2} ds, \tag{32} \]
\[ J_3 = \left( R \sum_{k=1}^{n} \beta_k + K \right) \int_{t'}^{t''} e^{-\left( \pi^2 - a_{\text{max}} \right) (t'' - s)} ds. \tag{33} \]

Therefore, we only need to check \( J_i \) tend to 0 independently of \( u \in \Omega_R \) when \( t'' - t' \to 0 \) for \( i = 1, 2, 3 \).

For \( J_1 \), by (9), (18), (19), (22), (35), the conditions (A2) and (A3)', we get that

\[ \left\| (I - U(\omega, 0))^{-1} \int_{0}^{\omega} U(\omega, s) [f(u(s - \tau_1), \ldots, u(s - \tau_n)) + g(s)] ds \right\|_{2} \]
\[ \leq \frac{1}{1 - e^{-\left( \pi^2 - a_{\text{max}} \right) \omega}} \int_{0}^{\omega} e^{-\left( \pi^2 - a_{\text{max}} \right) (\omega - s)} \left[ \sum_{k=1}^{n} \beta_k \| u(s - \tau_k) \|_{2} + K \right] ds \]
\[ \leq \frac{R \sum_{k=1}^{n} \beta_k + K}{\pi^2 - a_{\text{max}}}. \tag{34} \]

Therefore, from Lemma 2.1 (ii), Lemma 2.4, (31) and (34) one can easily to get that \( J_1 \to 0 \) as \( t'' - t' \to 0 \).

For \( t' = 0, 0 < t'' \leq a \), it is easy to see that \( J_2 = 0 \). For \( 0 < t' < a \) and arbitrary \( 0 < \delta < t' \), by Lemmas 2.1 and 2.4, the assumptions (A2), (A3)', (35) and the arbitrariness of \( \delta \), we get that

\[ J_2 \leq \left( R \sum_{k=1}^{n} \beta_k + K \right) \int_{0}^{t' - \delta} \| U(t'', s) - U(t', s) \|_{\mathcal{L}(E)} ds \]
\[ + \left( R \sum_{k=1}^{n} \beta_k + K \right) \int_{t' - \delta}^{t'} \left[ e^{-\left( \pi^2 - a_{\text{max}} \right) (t'' - s)} + e^{-\left( \pi^2 - a_{\text{max}} \right) (t' - s)} \right] ds \]
\[ \leq \sup_{s \in [0, t' - \delta]} \| U(t'', s) - U(t', s) \|_{\mathcal{L}(E)} \cdot \left( R \sum_{k=1}^{n} \beta_k + K \right) (t' - \delta) \]
\[ + 2 \left( R \sum_{k=1}^{n} \beta_k + K \right) \delta \]

\[ \rightarrow 0 \quad \text{as} \quad t'' - t' \rightarrow 0 \quad \text{and} \quad \delta \rightarrow 0. \]

For \( J_3 \), we can get by direct calculus that

\[ J_3 \leq \frac{R \sum_{k=1}^{n} \beta_k + K}{\pi^2 - a_{\max}} \left[ 1 - e^{-(\pi^2 - a_{\max})(t'' - t')} \right] \]

\[ \rightarrow 0 \quad \text{as} \quad t'' - t' \rightarrow 0. \]

As a result, \( \|(P u)(t'') - (P u)(t')\|_2 \) tends to zero independently of \( u \in \Omega_R \) as \( t'' - t' \to 0 \), which means that the operator \( P : \Omega_R \to \Omega_R \) is an equicontinuous operator. Hence by the Arzela-Ascoli theorem one gets that \( \{u \in \Omega_R \} \) is relatively compact. Therefore, the continuity of operator \( P \) combined with relatively compactness of the set \( \{u \in \Omega_R \} \) imply that \( P : \Omega_R \to \Omega_R \) is a completely continuous operator. Hence, by Schauder Fixed Point Theorem we obtain that \( P \) has at least one fixed point \( u \in \Omega_R \), which is just a time \( \omega \)-periodic mild solution of abstract non-autonomous evolution equation with multi-delays (11). This means that the non-autonomous evolution equation with multi-delays (2) has at least one time \( \omega \)-periodic mild solution \( u \in C([0,1] \times \mathbb{R}, \mathbb{R}) \). This completes the proof of Theorem 1.1.

Next, we prove the uniqueness of time \( \omega \)-periodic mild solutions for non-autonomous evolution equation with multi-delays (2).

**Proof of Theorem 1.2.** By proof of Theorem 1.1 we know that non-autonomous evolution equation with multi-delays (2) can be transformed into the abstract form of non-autonomous evolution equation with multi-delays (11) in infinite-dimensional space \( E = L^2([0,1], \mathbb{R}) \) and the time \( \omega \)-periodic mild solutions of abstract non-autonomous evolution equation with multi-delays (11) is equivalent to the fixed point of operator \( P \) defined by (18), which maps \( C_{\omega}(\mathbb{R}, E) \) to \( C_{\omega}(\mathbb{R}, E) \). By the condition (A4) we know that for every \( t \in \mathbb{R} \) and \( u, v \in C_{\omega}(\mathbb{R}, E) \)

\[ \|f(u(t - \tau_1), \ldots, u(t - \tau_n)) - f(v(t - \tau_1), \ldots, v(t - \tau_n))\|_2 \]

\[ \leq \sum_{k=1}^{n} \beta_k \|u(t - \tau_k) - v(t - \tau_k)\|_2. \quad (35) \]

Therefore, for any \( u, v \in C_{\omega}(\mathbb{R}, E) \), by (9), (18), (22), (35), the conditions (A4) and (A3)', we get that

\[ \|(P u)(t) - (P v)(t)\|_2 \]

\[ \leq e^{-(\pi^2 - a_{\max}) t} \int_0^{\omega} e^{-(\pi^2 - a_{\max})(\omega - s)} \left( \sum_{k=1}^{n} \beta_k \|u(s - \tau_k) - v(s - \tau_k)\|_2 \right) ds \]

\[ + \int_0^{t} e^{-(\pi^2 - a_{\max})(t - s)} \left( \sum_{k=1}^{n} \beta_k \|u(s - \tau_k) - v(s - \tau_k)\|_2 \right) ds \]
\[
\begin{align*}
&\leq \left[ \frac{e^{-(\pi^2-a_{\text{max}})t}}{\pi^2-a_{\text{max}}} + \frac{1-e^{-(\pi^2-a_{\text{max}})t}}{\pi^2-a_{\text{max}}} \right] \cdot \left( \sum_{k=1}^{n} \beta_k \|u-v\|_C \right) \\
&= \frac{\sum_{k=1}^{n} \beta_k}{\pi^2-a_{\text{max}}} \|u-v\|_C < \|u-v\|_C,
\end{align*}
\]
which means that,
\[
\|\mathcal{P}u - \mathcal{P}v\|_C = \max_{t \in [0,\omega]} \|(\mathcal{P}u)(t) - (\mathcal{P}v)(t)\|_2 < \|u-v\|_C.
\]

Hence, \( \mathcal{P} : C_{\omega}(\mathbb{R}, E) \rightarrow C_{\omega}(\mathbb{R}, E) \) is a contraction operator, and therefore \( \mathcal{P} \) has a unique fixed point \( u \in C_{\omega}(\mathbb{R}, E) \), which is in turn the unique time \( \omega \)-periodic mild solution of the abstract non-autonomous evolution equation with multi-delays (11). This means that the non-autonomous evolution equation with multi-delays (2) exists a unique time \( \omega \)-periodic mild solution \( u \in C([0, 1] \times \mathbb{R}, \mathbb{R}) \). This completes the proof of Theorem 1.2. \( \square \)

In order to prove the asymptotic stability of time \( \omega \)-periodic mild solution for non-autonomous evolution equation with multi-delays (2), i.e., Theorem 1.3, we also need the following Bellman type inequality with delays, which can be find in [20, Lemma 4.1].

**Lemma 3.1.** Denote \( r = \max\{\tau_1, \tau_2, \ldots, \tau_n\} \). Let \( \varphi \in C([-r, \infty), \mathbb{R}_+) \). If there exist positive constants \( b_1, b_2, \ldots, b_n \) such that \( \varphi \) satisfy the integral inequality
\[
\varphi(t) \leq \varphi(0) + \sum_{k=1}^{n} b_k \int_{0}^{t} \varphi(s-\tau_k) ds, \quad t \geq 0.
\]
Then for every \( t \geq 0 \),
\[
\varphi(t) \leq \|\varphi\|_{C[-r,0]} e^{\left( \sum_{k=1}^{n} b_k \right) t},
\]
where \( \|\varphi\|_{C[-r,0]} = \max_{t \in [-r,0]} |\varphi(t)| \).

**Proof of Theorem 1.3.** One can easily see that (A3)” = (A3)’. Therefore, By Theorem 1.2 we know that the non-autonomous evolution equation with multi-delays (2) has a unique time \( \omega \)-periodic mild solution \( u \in C([0, 1] \times \mathbb{R}, \mathbb{R}) \). Let \( \bar{\varpi} \in C([0, 1] \times \mathbb{R}, \mathbb{R}) \) be another time \( \omega \)-periodic mild solution of non-autonomous evolution equation with multi-delays (2). By the fact that the non-autonomous evolution equation with multi-delays (2) can be transformed into the abstract form of non-autonomous evolution equation with multi-delays (11) in infinite-dimensional space \( E = L^2([0, 1], \mathbb{R}) \) combined with Definition 2.6, we know that the functions \( u(t) = u(\cdot, t) \) and \( \bar{\varpi}(t) = \bar{\varpi}(\cdot, t) \) can be expressed by
\[
u(t) = U(t, 0)^{-1} \int_{0}^{\omega} U(\omega, s)[f(u(s-\tau_1), \ldots, u(s-\tau_n)) + g(s)] ds \\
+ \int_{0}^{t} U(t, s)[f(u(s-\tau_1), \ldots, u(s-\tau_n)) + g(s)] ds
\]
\[
= U(t, 0)u(0) + \int_{0}^{t} U(t, s)[f(u(s-\tau_1), \ldots, u(s-\tau_n)) + g(s)] ds, \quad t \in \mathbb{R} \quad (36)
\]
and

$$\overline{u}(t) = U(t,0)\left(I - U(\omega,0)\right)^{-1}\int_0^\omega U(\omega,s)[f(\overline{u}(s),\cdots, \overline{u}(s - \tau_n)) + g(s)]ds$$

$$+ \int_0^t U(t,s)[f(\overline{u}(s - \tau_1),\cdots, \overline{u}(s - \tau_n)) + g(s)]ds$$

$$= U(t,0)\overline{u}(0) + \int_0^t U(t,s)[f(\overline{u}(s - \tau_1),\cdots, \overline{u}(s - \tau_n)) + g(s)]ds, \quad t \in \mathbb{R}, \quad (37)$$

where

$$u(0) = \left(I - U(\omega,0)\right)^{-1}\int_0^\omega U(\omega,s)[f(u(s - \tau_1),\cdots, u(s - \tau_n)) + g(s)]ds,$$

$$\overline{u}(0) = \left(I - U(\omega,0)\right)^{-1}\int_0^\omega U(\omega,s)[f(\overline{u}(s - \tau_1),\cdots, \overline{u}(s - \tau_n)) + g(s)]ds.$$

Therefore, by (9), (22), (35), (36), (37), the conditions (A3)$^\prime$ and (A4), we get that

$$\|u(t) - \overline{u}(t)\|_2 \leq e^{-(\pi^2 - a_{\max})t}\|u(0) - \overline{u}(0)\|_2$$

$$+ \int_0^t e^{-(\pi^2 - a_{\max})(t-s)}\left(\sum_{k=1}^n \beta_k \|u(s - \tau_k) - \overline{u}(s - \tau_k)\|_2\right)ds$$

$$= e^{-(\pi^2 - a_{\max})t}\|u(0) - \overline{u}(0)\|_2 + e^{-(\pi^2 - a_{\max})t}\sum_{k=1}^n \beta_k e^{(\pi^2 - a_{\max})\tau_k}$$

$$\cdot \int_0^t e^{(\pi^2 - a_{\max})(s - \tau_k)}\|u(s - \tau_k) - \overline{u}(s - \tau_k)\|_2ds. \quad (38)$$

From (38) we get that

$$e^{(\pi^2 - a_{\max})t}\|u(t) - \overline{u}(t)\|_2 \leq \|u(0) - \overline{u}(0)\|_2 + \sum_{k=1}^n \beta_k e^{(\pi^2 - a_{\max})\tau_k}$$

$$\cdot \int_0^t e^{(\pi^2 - a_{\max})(s - \tau_k)}\|u(s - \tau_k) - \overline{u}(s - \tau_k)\|_2ds. \quad (39)$$

Setting

$$\varphi(t) = e^{(\pi^2 - a_{\max})t}\|u(t) - \overline{u}(t)\|_2, \quad t \in [-r, \infty).$$

Then from (39) we get that

$$\varphi(t) \leq \varphi(0) + \sum_{k=1}^n \beta_k e^{(\pi^2 - a_{\max})\tau_k} \int_0^t \varphi(s - \tau_k)ds, \quad t \geq 0. \quad (40)$$

Therefore, by (40) and Lemma 3.1, we know that for every $t \geq 0$,

$$\varphi(t) = e^{(\pi^2 - a_{\max})t}\|u(t) - \overline{u}(t)\|_2$$

$$\leq \max_{t \in [-r,0]} e^{(\pi^2 - a_{\max})t}\|u(t) - \kappa(t)\|_2e^{\left[\sum_{k=1}^n \beta_k e^{(\pi^2 - a_{\max})\tau_k}\right]t}, \quad (41)$$
Therefore, from (42) and (43) we get that 

\[ \| u(t) - \overline{u}(t) \|_2 \leq \max_{t \in [-\tau, 0]} e^{(\pi^2 - a_{\max})t} \| u(t) - \kappa(t) \|_2 e \left( \sum_{k=1}^{n} \beta_k e^{(\pi^2 - a_{\max})\tau_k - (\pi^2 - a_{\max})} \right) t. \]  

(42)

By the condition (A3)'', we know that 

\[ \sum_{k=1}^{n} \beta_k e^{(\pi^2 - a_{\max})\tau_k - (\pi^2 - a_{\max})} < 0. \]  

(43)

Therefore, from (42) and (43) we get that 

\[ \left( \int_0^1 |u(t, x) - \overline{u}(t, x)|^2 dx \right)^{\frac{1}{2}} = \| u(t) - \overline{u}(t) \|_2 \to 0 \quad \text{as} \quad t \to +\infty. \]

Hence, the time \( \omega \)-periodic mild solution \( u \) of non-autonomous evolution equation with multi-delays (2) is globally asymptotically stable and it exponentially attracts the other solutions. This completes the proof of Theorem 1.3.

□

4. An example. In this section, we present an example to illustrate the applicability of our main results. Consider the following hematopoiesis model of one-dimensional reaction-diffusion equation with variable coefficient and delay 

\[ \begin{cases} \frac{\partial}{\partial t} u(x, t) = \frac{\partial^2}{\partial x^2} u(x, t) + a(t) u(x, t) + g(x, t) + f(u(x, t - \tau)), & (x, t) \in (0, 1) \times \mathbb{R}, \\ u(0, t) = u(1, t) = 0, & t \in \mathbb{R}, \end{cases} \]  

(44)

where \( u(t) \) denotes the density of mature cells in blood circulation, \( a(t) : \mathbb{R} \to \mathbb{R} \) is a continuously differentiable function which means the rate of cells lost from the circulation and it is \( \omega \)-periodic in \( t \) for \( \omega > 0 \), \( g : [0, 1] \times \mathbb{R} \to \mathbb{R} \) is continuous and \( g(x, t) \) is \( \omega \)-periodic in \( t \), the flux \( f(u(x, t - \tau)) = \frac{\alpha u(x, t - \tau)}{1 + u(x, t - \tau)} \) of cells into the circulation from the stem cell compartment depends on \( u(x, t - \tau) \) at time \( t - \tau \) with \( \alpha > 0 \), and \( \tau > 0 \) is the time delay between the production of immature cells in the bone marrow and their maturation for release in circulating bloodstreams.

**Theorem 4.1.** If the coefficient \( a(t) \) satisfies the following condition 

\[ \max_{t \in [0, \pi]} a(t) < \pi^2 - \alpha, \]  

(45)

then the reaction-diffusion equation with delay (44) has a unique time \( \omega \)-periodic mild solution \( u \in C([0, 1] \times \mathbb{R}, \mathbb{R}) \).

**Proof.** By the definition of nonlinear function \( f(u(x, t - \tau)) = \frac{\alpha u(x, t - \tau)}{1 + u(x, t - \tau)} \) combined with a simple calculation one gets that the condition (A4) hold with \( \beta_k = \alpha \) and \( k = 1 \). This fact combined with (45) we know that the condition (A3)' is satisfied. Therefore, all the conditions of Theorem 1.2 are satisfied. Hence, from Theorem 1.2 we know that the reaction-diffusion equation with delay (44) exists a unique time \( \omega \)-periodic mild solution \( u \in C([0, 1] \times \mathbb{R}, \mathbb{R}) \). This completes the proof of Theorem 4.1.

□

Furthermore, if we strengthening the condition (45), we have the following asymptotic stability result of time \( \omega \)-periodic mild solution for reaction-diffusion equation with delay (44).
Theorem 4.2. If the coefficient $a(t)$ satisfies the following condition
\[
\max_{t \in [0, \pi]} a(t) < \pi^2 - \alpha e^{\pi^2 - \max_{t \in [0, \pi]} a(t)} \tau, \tag{46}
\]
then the reaction-diffusion equation with delay (44) has a unique time $\omega$-periodic mild solution $u \in C([0, 1] \times \mathbb{R}, \mathbb{R})$ which is globally asymptotically stable and exponentially attracts the other solutions.

Proof. By the proof of Theorem 4.1 we know that the condition (A4) hold with $\beta_k = \alpha$ and $k = 1$. Furthermore, from the condition (46) one gets that the condition (A3)$^\prime$ is satisfied. Therefore, all the conditions of Theorem 1.3 are satisfied. Hence, from Theorem 1.3 we know that the reaction-diffusion equation with delay (44) exists a unique time $\omega$-periodic mild solution $u \in C([0, 1] \times \mathbb{R}, \mathbb{R})$ which is globally asymptotically stable and exponentially attracts the other solutions. This completes the proof of Theorem 4.2.

Remark 3. The conclusions of Theorem 4.1 and Theorem 4.2 are also valid when $f(u(x, t - \tau)) = u(x, t - \tau)e^{-ku(x, t - \tau)}$ with $k > 0$ by replacing the conditions (45) and (46) with the following conditions
\[
\max_{t \in [0, \pi]} a(t) < \pi^2 - 1 \tag{47}
\]
and
\[
\max_{t \in [0, \pi]} a(t) < \pi^2 - e^{\pi^2 - \max_{t \in [0, \pi]} a(t)} \tau, \tag{48}
\]
respectively.

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