Effective wall-laws for the Stokes equations over curved rough boundaries

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Abstract

We derive effective wall-laws for Stokes systems with inhomogeneous boundary conditions in three dimensional bounded domains with curved rough boundaries. No-slip boundary condition is given on the locally periodic rough boundary parts with characteristic roughness size $\varepsilon$ and boundary data is assumed to be supported in the nonoscillatory smooth boundary.

Based on the analysis of a boundary layer cell problem depending on geometry of the fictitious boundary and roughness shape, boundary layer approximations are constructed using orthogonal tangential vectors and normal vector on the fictitious boundary, which have $O(\varepsilon^{3/2})$-order in $L^2$-norm and $O(\varepsilon)$-order in energy-norm. Then, a Navier wall-law with error estimates of $O(\varepsilon^{3/2})$-order in $L^2$-norm and $O(\varepsilon)$-order in $W^{1,1}$-norm is obtained, which is proved to be irrespective of the choice of the orthogonal tangent vectors.

Keywords: Stokes equations; rough boundary; wall-laws; homogenization

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1 Introduction

Rough boundary problems have many practical applications in aerodynamics, electromagnetism, hydrodynamics and hemodynamics, etc. Direct numerical computation around rough boundaries is usually out of reach for the time being since the problems have both macroscopic and microscopic scales and hence need lots of computational burden. Therefore one usually changes the boundary condition on rough boundary with a new boundary condition on a regularized fictitious boundary close to the rough boundary, that is so-called a wall-law. For viscous fluid flows, no-slip boundary condition at the rough wall is replaced by a type of Navier slip boundary condition, Navier wall-law, at the fictitious boundary. The derivation of Navier wall-laws are also important for shape optimization of roughness for better drag reduction since the procedure of shape optimization for drag reduction requires to know a priori the Navier’s coefficient in the slip boundary condition.

In this article we study effective wall-laws for the Stokes system

$$
\begin{align*}
-\Delta u^\varepsilon + \nabla p^\varepsilon &= f & \text{in } \Omega^\varepsilon, \\
\text{div } u^\varepsilon &= 0 & \text{in } \Omega^\varepsilon, \\
u^\varepsilon &= \psi & \text{in } \partial \Omega^\varepsilon,
\end{align*}
$$

(1.1)

where $\Omega^\varepsilon \subset \mathbb{R}^3$ is a bounded and simply connected domain and its sufficiently smooth boundary $\partial \Omega^\varepsilon$ consists of a nonoscillatory part and a rough part formed by locally periodic microscopic roughness of characteristic size $O(\varepsilon)$. Boundary data is assumed to be supported in the nonoscillatory part of $\partial \Omega^\varepsilon$.

There is a number of papers dealing with effective wall-laws for Stokes and Navier-Stokes equations, see e.g. [2] [3] [5] [7] [9] [11] [20] [21] [22] [23] [25] and the references therein for the case of periodic roughness and [11] [19] for the case of nonperiodic random roughness. Moreover, one can
find results concerning explicit or implicit wall-laws for Poisson equations, see, e.g. [1, 6, 8, 12, 13, 14, 15, 16, 25, 26, 29]. Here, we note that the Poisson equations describe simplified flows which are uniform in longitudinal direction. The main techniques used to derive effective wall-laws are domain decomposition and multiscale asymptotic expansions.

Most rough boundaries we meet in reality are curved boundaries, and for practical applications, results of flat rough boundaries may be applied to curved rough boundary problems with small curvature to some extent. However, if the curvature is considerably large, for more accurate analysis near the rough surfaces and for determination of micro-roughness shape giving better performance of drag reduction, the curved rough boundary must be considered as it is. We note that most of above mentioned references concern the flat rough boundaries, while for references dealing with wall-laws for curved rough boundaries, we refer to e.g. [1, 25, 26, 29]. In a pioneering work [1], a first order wall-law for the Poisson equation in a ring with many small holes near the outer boundary was obtained using domain decomposition techniques. Later, in [25, 26], first and second order wall-laws for Poisson equations in general two-dimensional annular domains with curved rough boundaries were obtained by combining techniques of domain decomposition and two-scale asymptotic expansions. We note that two-dimensional problems correspond to the case where longitudinal grooves form rough surfaces.

Wall-laws for multi-dimensional Poisson problem over curved rough boundary were obtained in [29]. More precisely, for Poisson problem with homogeneous Dirichlet boundary condition on curved compact boundary with locally periodic roughness on it the authors constructed suitable approximations of $O(\varepsilon^{3/2})$-order in $L^2$-norm and $O(\varepsilon)$-order in energy norm based on analysis of a boundary layer cell problem. Then, a wall-law with the same order of error estimates as the approximations in interior domains was derived.

We refer to a review article [27] for more details on the derivation and analysis of wall laws of fluid flows.

Motivated by [29], in this article we address derivation of wall-laws for inhomogeneous boundary value problem for the Stokes systems (1.1) over curved rough boundaries. We note that the system (1.1) may be used to analyze exterior fluid flows, if the boundary has two components and boundary data is supported only in outer non-oscillatory boundary part. Furthermore, if the rough boundary part and non-oscillatory boundary part are adjacent, the system (1.1) may be used for local analysis of fluid flows near a curved rough surface that can be a part of boundary of any type of objects.

To achieve our goal, first, we analyse a boundary layer cell problem depending on the geometry of the fictitious boundary and roughness shape, that is elliptic in the sense of Agmon, Douglis and Nirenberg, see (BL)1, in subsection 3.2, by using technique of Fourier series expansion. Then we construct boundary layer approximations of $O(\varepsilon^{3/2})$-order in $L^2$-norm and $O(\varepsilon)$-order in energy norm using the orthogonal tangential vectors and normal vector on the fictitious surface. Using these approximations we obtain an effective Navier wall-law which is shown to be irrespective of the choice of the orthogonal tangential vector fields and has error of $O(\varepsilon^{3/2})$-order in $L^2$-norm and $O(\varepsilon)$-order in $W^{1,1}$-norm. The main theorems of the paper are Theorems 3.20, 3.22 and 3.24.

Dealing with the Stokes system, we are encountered with additional difficulties compared to the Poisson problem, which are mainly related to the difference between vectorial and scalar case as well as the structural complicatedness of the Stokes system over Poisson equation. The main difference from the case of scalar Poisson equation is that for the construction of the approximations and wall-laws we need to consider curvilinear systems of tangential vectors and normal vector on the fictitious surface; we should take careful observations of dependence of approximations on the local curvilinear system. Moreover, due to inhomogeneous boundary condition, we need some sophisticated techniques using cut-off functions in construction and estimates of boundary layer approximations so that artificial vertical layer flows around the nonoscillatory boundary part could not be generated thus ensuring the required approximation order near the edge between nonoscillatory and oscillating parts of boundary.

For simplicity we consider the case of spacial dimension $n = 3$, but the result of the paper can be directly extended to the case of $n > 3$ without essential change.
This paper is organized as follows. In Section 2 we describe the rough domain considered in the paper and give the main notations. Section 3, the major part of the paper, consists of several subsections. Estimates of Dirichlet wall-law are given in subsection 3.1 and a boundary layer cell problem is analyzed in subsection 3.2. Subsections 3.3 and 3.4 concern the construction of local and global boundary layer correctors, respectively. In subsection 3.5 boundary layer approximations are constructed and an effective Navier wall-law with higher order error estimates is derived. Finally, in Appendix we give a refined analysis for divergence problem ensuring the estimate constant for a solution of divergence equation in our rough domain being independent of micro-roughness size \( \varepsilon \).

## 2 Domains with rough boundaries and main notations

We give description on the rough domain and notations. The domain \( \Omega^\varepsilon \subset \mathbb{R}^3 \) is bounded with its boundary \( \partial \Omega^\varepsilon \) consisting of rough part \( \Gamma_0 \) and nonoscillatory smooth part \( \Gamma_1 \), i.e.,

\[
\partial \Omega^\varepsilon = \Gamma_0 \cup \Gamma_1, \quad \Gamma_0 \cap \Gamma_1 = \emptyset,
\]

where \( \Gamma_1 \) is closed and \( \Gamma_0 \) consists of finite locally \( \varepsilon \)-periodic oscillating parts with microscopic size \( \varepsilon \). The domain \( \Omega^\varepsilon \) is divided into \( \Omega \) and \( \Omega^\varepsilon \setminus \Omega \) by an open and nonoscillatory sufficiently smooth surface \( \Gamma \) (fictitious boundary) such that \( \Gamma \) is at the distance of \( O(\varepsilon) \) from \( \Gamma_0 \) and \( \partial \Omega = \Gamma \cup \Gamma_1 \), \( \Gamma \cap \Gamma_1 = \emptyset \). If \( \Gamma \) and \( \Gamma_1 \) are adjacent, we assume \( \partial \Omega \in C^{0,1} \).

Denoting by \( \nu(x') \) the outward normal vector for \( \Omega \) at \( x' \in \Gamma \), suppose that there is some positive \( \delta = O(1) \) such that the mapping

\[
\Upsilon : \Gamma \times (-\delta, \delta) \to \mathbb{R}^3, \quad \Upsilon(x', t) = x' + t\nu(x')
\]

diffeomorphism. Moreover, suppose that there are some bounded open sets \( U_i, i = 1, \ldots, N, \) of \( \mathbb{R}^2 \), \( V_i, i = 1, \ldots, N, \) of \( \Gamma \) and diffeomorphism \( \varphi_i : U_i \to V_i, i = 1, \ldots, N, \) such that \( \{ \varphi_i, U_i, V_i \}_{i=1}^N \) is a chart of \( \Gamma \) and the rough surface \( \Gamma_0 \) is expressed by

\[
\begin{align*}
\Gamma_0 &= \{ \Upsilon(x', \gamma^\varepsilon(x')) : x' \in \Gamma \}, \\
\gamma^\varepsilon(x') &= \varepsilon \gamma_1(\varphi_i^{-1}(x'), \varphi_i^{-1}(x')) / \varepsilon, \quad x' \in V_i, i = 1, \ldots, N,
\end{align*}
\]

where \( \gamma_1 \geq 0 \) defined in \( U_i \times \mathbb{R}^2 \) is \((1, 1)\)-periodic with respect to the second variable and may take multi-values. In this sense the rough boundary part \( \Gamma_0 \) is locally \( \varepsilon \)-periodic. In addition, let

\[
|\gamma^\varepsilon(x')| \leq \varepsilon M < \frac{\delta}{2}, \quad x' \in \Gamma,
\]

and put

\[
\begin{align*}
\Gamma_\delta := \Upsilon(\Gamma \times (-\delta, \delta)), \\
\Gamma^\varepsilon_\delta := \Gamma_\delta \cap \Omega^\varepsilon, \\
\Gamma^\varepsilon_\delta,i := \Upsilon(V_i \times (-\delta, \delta)), \quad \Gamma^\varepsilon_\delta,i \cap \Omega^\varepsilon = \emptyset, \quad i = 1, \ldots, N.
\end{align*}
\]

Obviously, \( \Gamma^\varepsilon_\delta = \bigcup_{i=1}^N \Gamma^\varepsilon_\delta,i, \Gamma^\varepsilon_\delta \supset \Omega^\varepsilon \setminus \Omega \).

It is natural to assume that \( \Omega^\varepsilon \) can be expressed as a type of domain

\[
\Omega^\varepsilon = \bigcup_{j=1}^M G^{(j)}(x), \quad G^{(j)}(x) = G_0^{(j)} \cup \bigcup_{k=1}^{m_j} G_k^{(j)}, \quad j = 1, \ldots, M,
\]

where \( M \sim O(1), |G_0^{(j)}| \sim O(1), |G_k^{(j)}| \sim O(\varepsilon^3) \) and \( m_j \sim O(1/k^2), k = 1, \ldots, m_j, j = 1, \ldots, M, \) and

\[
G_0^{(j)} \cap G_k^{(j)} \neq \emptyset, G_0^{(j)} \cap G_k^{(j)} = \emptyset, k \neq l, 1 \leq k, l \leq m_j,
\]

for each \( j \in \{1, \ldots, M\} \). We assume further that each \( G_k^{(j)} \), \( k = 0, \ldots, m_j, j = 1, \ldots, M \), is a star-shaped domain with respect to some ball of radius \( R_k^{(j)} \) and \( \frac{\delta(G_k^{(j)})}{R_k^{(j)}} \sim O(1) \), where \( \delta(G_k^{(j)}) \) is
It is well known, cf. e.g. [17], that the system (1.1) has a unique solution.

Consider the approximation of the system (1.1) as

$$\text{3.1 Dirichlet wall-law}$$

Let us assume for the data of (1.1) that

$$\Omega \subset \Omega_{\varepsilon}, \quad \Gamma' \equiv \Gamma$$

and $\Omega_{0}$ is a sufficiently smooth domain satisfying

$$\Omega_{0} \subset \Omega, \quad \Gamma_{1} \cap \Gamma \subset \partial \Omega_{0}$$

and $\Omega_{0}$ has thickness of size $O(\varepsilon^{2})$.

Then, obviously, $|\Omega_{0} \setminus \Omega| \leq O(\varepsilon^{3})$. Let $\Gamma' \equiv \Upsilon(\Gamma' \times (-\delta, \delta))$.

As usual, $\mathbb{N}$ is the set of all natural numbers, $\mathbb{N}_{0} = \mathbb{N} \cup \{0\}$, and $\mathbb{Z}$ is the set of all integers.

For a domain $G \subset \mathbb{R}^{n}$ its closure is denoted by $\overline{G}$ and its boundary by $\partial G$. We do not distinguish between spaces of scalar- and vector-, or even tensor-valued functions as long as no confusion arises.

For Lebesgue, Sobolev spaces on a domain or boundary we use standard notations $L^{r}, W^{k,r}, W_{0}^{k,r}$, $1 \leq r \leq \infty$, $k \in \mathbb{Z}$, respectively. We use notation $L_{m}^{2}(\Gamma) := \{ \varphi \in L^{2}(\Gamma) : \int_{\Gamma} \varphi(x) \, dx = 0 \}$. Let $H^{1} = W^{1,2}, H_{0}^{1} = W_{0}^{1,2}$ and $H^{-1}$ the dual of $H_{0}^{1}$. The closures in $H^{1}(\Gamma)$ and $L^{r}(\Gamma)$ of the set $\{ u \in C_{0}^{\infty}(\Gamma) : \text{div} \, u = 0 \}$ are denoted by $H_{0}^{1}(\Gamma)$ and $L^{2}(\Gamma)$, respectively. The notation $A \lesssim B$ ($A \gtrsim B$) implies $A \leq cB$ ($A \geq cB$) with constant $C$ independent of $\varepsilon$.

### 3 Effective wall-laws for the Stokes system

Let us assume for the data of (1.1) that

$$f \in L^{2}(\Omega_{\varepsilon}), \psi \in W^{2-1/q,q}(\partial \Omega_{\varepsilon}), q \geq 2, \quad \text{supp} \psi \subset \Gamma_{1}, \int_{\Gamma_{1}} \psi \cdot \nu \, dx = 0. \quad (3.1)$$

It is well known, cf. e.g. [17], that the system (1.1) has a unique solution $\{ u^{\varepsilon}, p^{\varepsilon} \}$ satisfying $u^{\varepsilon} \in W^{2,q}(\Omega_{\varepsilon})$, $p^{\varepsilon} \in W^{1,q}(\Omega_{\varepsilon})$ and

$$\| u^{\varepsilon} \|_{W^{2,q}(\Omega_{\varepsilon})} + \| p^{\varepsilon} \|_{W^{1,q}(\Omega_{\varepsilon})} \leq C(\Omega_{\varepsilon})(\| f \|_{L^{q}(\Omega_{\varepsilon})} + \| \psi \|_{W^{2-1/q,q}(\Gamma_{1})}). \quad (3.2)$$

#### 3.1 Dirichlet wall-law

Consider the approximation of the system (1.1) as

$$-\Delta u + \nabla p = f \quad \text{in} \ \Omega_{0},$$

$$\text{div} \ u = 0 \quad \text{in} \ \Omega_{0},$$

$$u = \psi \quad \text{in} \ \Gamma_{1},$$

$$u = 0 \quad \text{in} \ \partial \Omega_{0} \setminus \Gamma_{1}. \quad (3.3)$$

The system (3.3) has a unique solution $\{ u, p \} \in W^{2,q}(\Omega_{0}) \times W^{1,q}(\Omega_{0})$ such that

$$\| u \|_{W^{2,q}(\Omega_{0})} + \| p \|_{W^{1,q}(\Omega_{0})} \leq C(\Omega_{0})(\| f \|_{L^{q}(\Omega_{0})} + \| \psi \|_{W^{2-1/q,q}(\Gamma_{1})}). \quad (3.4)$$
Lemma 3.1 If $\varphi \in H^1(\Omega^\varepsilon \setminus \Omega)$, $\varphi|_{\Gamma_0} = 0$, then
\[ \|\varphi\|_{L^2(\partial\Omega_0 \setminus \Gamma_1)} + \|\varphi\|_{L^2(\Gamma_{1})} \lesssim \varepsilon^{1/2} \|\nabla\varphi\|_{L^2(\Omega^\varepsilon \setminus \Omega)}. \] (3.5)

In addition, if $\varphi \in H^2(\Omega^\varepsilon)$, then
\[ \|\varphi\|_{L^2(\partial\Omega_0 \setminus \Gamma_1)} + \|\varphi\|_{L^2(\Gamma_{1})} \lesssim \varepsilon \|\varphi\|_{H^2(\Omega^\varepsilon)}. \] (3.6)

Proof. For $x' \in \Gamma$ let $l(x')$ and $l_0(x')$ be the distances from $x'$ to the intersection points of the outer normal line at $x'$ for $\Omega$ with $\partial\Omega_0 \setminus \Gamma_1$ and $\Gamma_0$, respectively. Let $\tilde{\varphi}$ be extension of $\varphi$ by 0 to $\Gamma_0 \setminus \Gamma_3$. Then we get by $\varphi|_{\Gamma_0} = 0$ and Minkowski’s inequality that
\[ \|\varphi\|_{L^2(\partial\Omega_0 \setminus \Gamma_1)} = \left( \int_{\Gamma} \left( \int_{l_0(x')}^{l(x')} \frac{\partial\varphi}{\partial x_3}(x', x_3) \, dx_3 \right)^2 \, dx' \right)^{1/2} \leq \left( \int_{\Gamma} \left( \int_{l_0(x')}^{l(x')} \frac{\partial\varphi}{\partial x_3}(x', x_3) \, dx_3 \right)^2 \, dx' \right)^{1/2} \lesssim \int_0^{M \varepsilon} \left( \int_{\Gamma} \frac{\partial\varphi}{\partial x_3}(x', x_3)^2 \, dx' \right)^{1/2} \, dx_3. \] (3.7)

Then, by Hölder inequality one gets
\[ \|\varphi\|_{L^2(\partial\Omega_0 \setminus \Gamma_1)} \lesssim \varepsilon^{1/2} \|\partial\varphi/\partial x_3 (x', x_3)\|_{L^2(0, M \varepsilon; L^2(\Gamma))} \lesssim \varepsilon^{1/2} \|\nabla\varphi\|_{L^2(\Omega^\varepsilon \setminus \Omega)}. \]

The estimate of $\|\varphi\|_{L^2(\Gamma)}$ can be obtained in the same way. Thus, \((3.5)\) is proved.

Let us prove \((3.6)\). Let $\tilde{\varphi} \in H^2(\Omega^\varepsilon \cup \Gamma_3)$ be an extension of $\varphi$ satisfying
\[ \|\partial\varphi/\partial x_3\|_{H^1(\Gamma_1)} \leq C \|\partial\varphi/\partial x_3\|_{H^1(\Omega^\varepsilon)}. \] (3.8)

with constant $C > 0$ independent of $\varepsilon$. The existence of such an extension is guaranteed by Sobolev extension theorem. Note that, due to the continuous embedding
\[ H^1((-\delta, M \varepsilon); L^2(\Gamma)) \to L^\infty((-\delta, M \varepsilon); L^2(\Gamma)) \]

with an embedding constant independent of $\varepsilon$ and \((3.8)\), one has
\[ \|\partial\tilde{\varphi}/\partial x_3\|_{L^\infty(0, M \varepsilon; L^2(\Gamma))} \lesssim \|\partial\tilde{\varphi}/\partial x_3\|_{H^1((-\delta, M \varepsilon); L^2(\Gamma))} \lesssim \|\varphi\|_{H^2(\Omega^\varepsilon)}. \]

Consequently, we can proceed in \((3.7)\) as
\[ \int_0^{M \varepsilon} \left( \int_{\Gamma} \left( \frac{\partial\varphi}{\partial x_3}(x', x_3)^2 \, dx' \right)^{1/2} \, dx_3 \right)^{1/2} \leq \int_0^{M \varepsilon} \left( \int_{\Gamma} \frac{\partial\tilde{\varphi}}{\partial x_3}(x', x_3)^2 \, dx' \right)^{1/2} \, dx_3 \lesssim \varepsilon \|\varphi\|_{H^2(\Omega^\varepsilon)}. \]

Combining this inequality with \((3.7)\) yields the required estimate for $\|\varphi\|_{L^2(\partial\Omega_0 \setminus \Gamma_1)}$ in \((3.6)\). The estimate of $\|\varphi\|_{L^2(\Gamma)}$ in \((3.6)\) can be obtained in the same way.

Thus, the proof of the lemma is complete. \(\Box\)

We have the following theorem on the error estimate for the zeroth order approximation system \((3.3)\).

Theorem 3.2 Let $u^\varepsilon$ be the solution to \((1.1)\) and let $\tilde{u}, \tilde{\varphi}$ be extensions of $u, p$ by 0 to $\Omega^\varepsilon$, respectively. Then,
\[ \|\nabla(u^\varepsilon - \tilde{u})\|_{L^2(\Omega^\varepsilon)} \lesssim \varepsilon^{1/2} (\|f\|_{L^2(\Omega^\varepsilon)} + \|\psi\|_{H^{1/2}(\Gamma_1)}), \]
\[ \|u^\varepsilon - \tilde{u}\|_{L^2(\Omega^\varepsilon)} \lesssim \varepsilon (\|f\|_{L^2(\Omega^\varepsilon)} + \|\psi\|_{H^{1/2}(\Gamma_1)}). \]
Proof. Let \( v = u^r - \bar{u} \) and \( s = p^r - \bar{p} \). Then, obviously, \( v \in H^1_{0,\sigma}(\Omega^r) \) and for any \( \varphi \in H^1_{0,\sigma}(\Omega^r) \) we have using integration by parts that
\[
(\nabla v, \nabla \varphi)_{L^2(\Omega^r)} = (-\Delta v + \nabla s, \varphi)_{\Omega^r \setminus \Omega_0} = \int_{\partial \Omega_0 \cap \Gamma_1} \frac{\partial u}{\partial \nu} - pv \cdot \varphi \, dx,
\]
where \((\cdot, \cdot)\) denotes either \( L^2\)-scalar product or duality paring between \( H^{-1} \) and \( H^1_0 \). By Poincaré’s inequality one has
\[
| (f, \varphi)_{\Omega^r \setminus \Omega_0} | \leq \| f \|_{L^2(\Omega^r \setminus \Omega_0)} \| \varphi \|_{L^2(\Omega^r \setminus \Omega_0)} \lesssim \| f \|_{L^2(\Omega^r \setminus \Omega_0)} \| \nabla \varphi \|_{L^2(\Omega^r \setminus \Omega_0)}
\]
and, by Lemma 3.1 and (3.4),
\[
| \int_{\partial \Omega_0 \cap \Gamma_1} \frac{\partial u}{\partial \nu} - pv \cdot \varphi \, dx | \leq \| \frac{\partial u}{\partial \nu} - pv \|_{L^2(\partial \Omega_0 \cap \Gamma_1)} \| \varphi \|_{L^2(\partial \Omega_0 \cap \Gamma_1)} \lesssim \epsilon \| f \|_{L^2(\Omega^r)} + \| \psi \|_{H^{3/2}(\Gamma_1)} \| \nabla \varphi \|_{L^2(\Omega^r \setminus \Omega_0)}
\]
Therefore, the first inequality of the theorem is proved.

Let us prove the second inequality of the theorem. Let \( A \) be the Stokes operator in \( L^2_\sigma(\Omega^r) \), i.e.,
\[
D(A) = H^2(\Omega^r \cap H^1_{0,\sigma}(\Omega^r)), \quad A \varphi := -\Delta \varphi,
\]
where \( P \) is the Helmholtz projection of \( L^2(\Omega^r) \) onto \( L^2_\sigma(\Omega^r) \). It is well known that
\[
\| \varphi \|_{H^2(\Omega^r)} \lesssim \| A \varphi \|_{L^2(\Omega^r)} \lesssim \| \varphi \|_{H^2(\Omega^r)}, \quad \forall \varphi \in D(A),
\]
Therefore, in view of (3.10), (3.11), we get
\[
| (v, A \varphi)_{L^2(\Omega^r)} | \lesssim \epsilon (\| f \|_{L^2(\Omega^r)} + \| \psi \|_{H^{3/2}(\Gamma_1)}) \| A \varphi \|_{L^2(\Omega^r)},
\]
which yields the second inequality of the theorem since the range of \( A \) is \( L^2_\sigma(\Omega^r) \).

The proof of the theorem is complete.

\[ \square \]

3.2 Boundary layer analysis

In order to derive a wall-law of higher order approximation for (1.1) we analyze the boundary layer near the rough boundary.

For \( x \in \Gamma_\delta \), let
\[
x = \Phi_i(x) := \Upsilon(\delta(i'(x'), x_3) = \varphi_i(x') + x_3\nu(\varphi_i(x'))), \quad x = (x', x_3) \in U_i \times (-\delta, \delta),
\]
(see Section 2 for \( \varphi_i \)).

Based on the expression of gradient \( \nabla_x \), divergence \( \text{div}_x \) and Laplacian \( \Delta_x \) with respect to the coordinate \( x \), that is,
\[
\nabla_x g = (D_x \Phi_i^{-1})^T (\nabla_x g(\varphi_i(x'))) \circ \Phi_i^{-1} = (D_x \Phi_i)^{-T} (\nabla_x g(\varphi_i(x'))) \circ \Phi_i^{-1},
\]
\[
\text{div}_x h = \text{div}_x ((D_x \Phi_i)^{-1} h(\varphi_i(x'))) \circ \Phi_i^{-1},
\]
\[
\Delta_x g = \text{div}_x ((D_x \Phi_i)^{-1} (D_x \Phi_i)^{-T} \nabla_x g(\varphi_i(x'))) \circ \Phi_i^{-1},
\]
(3.12)
where $D_x \Phi_i$ is Jacobian matrix for $\Phi$, we introduce matrices $A_i(x), B_i(x)$ as

$$B_i(x') := (D_x \Phi_i(x',0))^{-T} = (D_x \varphi_i(x'), \nu(\varphi_i(x')))^{-T}, \quad A_i(x') := B_i(x')^T B_i(x').$$  

(3.13)

Note that

$$A_i(x') = \begin{pmatrix} (D_x \varphi_i(x')^T D_x \varphi_i(x'))^{-1} & 0 \\ 0 & 1 \end{pmatrix}, \quad x' \in U_i.$$

Then we formulate the boundary layer cell problem $(BL)^i_{\lambda, x'}$ with parameter $x' \in U_i$ and $\lambda \in \mathbb{R}^3$:

$$(BL)^i_{\lambda, x'} : \left\{ \begin{array}{l}
-\text{div}_x(A_i(x') \nabla y \varphi(x', y)) + B_i(x') \nabla y \omega(x', y) = 0, \quad \text{in} \ Z_{BL} \setminus S, \\
\text{div}_y(B_i(x')^T \varphi(x', y)) = 0, \quad \text{in} \ Z_{BL} \setminus S, \\
\beta(x', y) = s = 0, \quad \text{in} \ Z_{BL} \setminus S, \\
\beta(x', y, y_3) \text{ is periodic with respect to } y', \quad \text{on} \ \partial Z_{BL} \setminus \{ y_3 = \gamma_i(x', y') \}.
\end{array} \right.$$

Here $Z_{BL}$ denotes the semi-infinite cylinder $\{ (y', y_3) \in \mathbb{R}^3 : y' \in (0,1) \times (0,1), y_3 < \gamma_i(x', y') \}$, and $|\varphi|_S := \lim_{\mu \to 0} |\varphi(x + \mu c)| - |\varphi(x - \mu c)|$ is the jump at $S = (0,1) \times (0,1) \times \{0\}$. In this subsection the unit vector in the direction of $y_l$-axis is denoted by $e_l$ for $l = 1 \sim 3$.

We assume w.l.o.g. that $D \varphi_i(x') \in C^1(\hat{U}_i), \ (D \varphi_i(x'))^{-1} \in C^1(\hat{V}_i)$.

Define the space $V$ by

$$V := \{ v \in L^2_{\text{loc}}(Z_{BL}) : \nabla v \in L^2(Z_{BL}), \text{div}_y(B_i(x')^T v) = 0, \quad \nu(\cdot, \gamma_i(\cdot, \cdot)) = 0 \text{ (in a trace sense), } \nu \text{ is } (1,1)\text{-periodic w.r.t. } y' \}$$

endowed with norm $\| v \|_V := \| \nabla v \|_{L^2(Z_{BL})}$. Then $V$ is a Banach space.

Testing $(BL)^i_{\lambda, x'}$ formally with $\varphi \in V$, one gets the equality

$$(B_i(x') \nabla y \varphi(x', y), B_i(x') \nabla y \varphi)_{Z_{BL}} = - \int_S \lambda \cdot \varphi \ dy'.$$

(3.14)

**Definition 3.3** A function $\beta = \beta_i(x', y, \lambda) \in V$ is called a solution to $(BL)^i_{\lambda, x'}$ if it satisfies (3.14) for all $\varphi \in V$.

**Theorem 3.4** There exists a unique solution to the problem $(BL)^i_{\lambda, x'}$ in the sense of Definition 3.3.

**Proof.** Note that, by Poincaré’s inequality,

$$\left| \int_S \lambda \cdot \varphi \ dy' \right| \leq |\lambda| \| \varphi \|_{L^2(S)} \leq c |\lambda| \| \nabla \varphi \|_{L^2(Z_{BL}^i)} \leq c |\lambda| \| \varphi \|_V,$$

where and in what follows $Z_{BL}^i = \{ y \in Z_{BL} : y_3 > 0 \}$. Thus, by Lax-Milgram’s lemma we get the conclusion.

We give a variation of De-Rham’s lemma without proof, that can be easily proved using standard techniques.

**Lemma 3.5** Let a matrix $B$ be nonsingular and suppose that $h \in H^{-1}(Z_{BL})$ satisfies

$$\langle h, v \rangle_{H^{-1}, H^1} = 0$$

for all $v \in H^1_0(Z_{BL})$ with $\text{div}(B^T v) = 0$. Then

$$h = B \nabla \varphi$$

(3.15)

with some unique $\varphi \in L^2_{(m)}(Z_{BL})$. 

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Remark 3.6 If $\beta$ is a solution to $(\mathbb{B}L)_{\lambda,x'}^i$ in the sense of Definition 3.2, then it follows by Lemma 3.7 and integration by parts that there is some $\omega = \omega_3(x', \cdot, \lambda) \in L^2_{(m)}(Z_{BL})$ such that $(\beta, \omega)$ solves the first equation of $(\mathbb{B}L)_{\lambda,x'}^i$. On the other hand, one can easily verify that the first and second equations of $(\mathbb{B}L)_{\lambda,x'}^i$ form an elliptic system in the sense of Agmon, Douglis and Nirenberg. Then, by the interior regularity for solutions to ADN elliptic systems (cf. [3], Theorem 10.3), we get

$$\{\beta, \omega\} \in (V \cap C^\infty(Z_{BL} \setminus S))^3 \times (L^2_{(m)}(Z_{BL}) \cap C^\infty(Z_{BL} \setminus S)).$$

(3.16)

Moreover, it follows that $(\beta, \omega)$ implies the fourth equation of $(\mathbb{B}L)_{\lambda,x'}^i$, that is, the jump condition. In fact, since

$$\text{div}_y(-A_i(x')\nabla_y\beta(x', y) + B_i(x')^T\omega(x', y)) = 0, \quad \text{in } Z_{BL} \setminus S,$$

and

$$-A_i(x')\nabla_y\beta(x', y) + B_i(x')^T\omega(x', y) \in L^2(Z_{BL}),$$

one gets that for any Lipschitz subdomain $G$ of $Z_{BL} \setminus S$

$$(-A_i(x')\nabla_y\beta(x', y', y_3) + B_i(x')^T\omega(x', y', y_3)) \cdot \mathbf{n} \in H^{-1/2}_\loc(\partial G)$$

where $\mathbf{n}$ is the outward normal vector at the boundary $\partial G$, see [32] or [17]. Now, testing the first equation of $(\mathbb{B}L)_{\lambda,x'}^i$ with $\varphi \in V \cap C^\infty_0(Z_{BL})$ yields

$$\langle [\frac{\partial\beta}{\partial y_3} - \omega_3(\varphi(x'))]_\nu, \varphi \rangle_{H^{-1/2}(S), H^{1/2}(S)} = \int_S \lambda \cdot \varphi \, dy'$$

implying the fourth equation of $(\mathbb{B}L)_{\lambda,x'}^i$.

Furthermore, testing the first equation of $(\mathbb{B}L)_{\lambda,x'}^i$ with suitable functions, one can easily see that $\omega$ is $(1,1)$-periodic with respect to $y'$ provided its trace has a meaning.

Henceforth we shall call $\{\beta, \omega\}$ a solution to $(\mathbb{B}L)_{\lambda,x'}^i$ as well.

Remark 3.7 Given $b < 0$, let $(\beta^{(b)}, \omega^{(b)})$ be the unique solution to the problem $(\mathbb{B}L)_{\lambda,x'}^i$ with jump conditions at $S$ replaced by the ones at $S(b) := \{y_3 = b\}$. Let us denote by $\{\beta^{(0)}, \omega^{(0)}\}$ the unique solution to $(\mathbb{B}L)_{\lambda,x'}^i$ with jump conditions at $S$. Then, it is easily checked that

$$\{\beta^{(b)}(x', y), \omega^{(b)}(x', y)\} = \begin{cases} 
\{\beta^{(0)}(x', y), \omega^{(0)}(x', y)\} & y_3 > 0, \\
\{\lambda y_3 + \beta^{(0)}(x', y), \omega^{(0)}(x', y)\} & b < y_3 \leq 0, \\
\{b \lambda + \beta^{(0)}(x', y), \omega^{(0)}(x', y)\} & y_3 \leq b.
\end{cases}$$

The next theorem shows behavior of solutions to $(\mathbb{B}L)_{\lambda,x'}^i$ near the interface $S$ and for $y_3 \to -\infty$.

Theorem 3.8 Let $\{\beta = \beta_i(x', \cdot, \lambda), \omega = \omega_i(x', \cdot, \lambda)\}$ be the solution to $(\mathbb{B}L)_{\lambda,x'}^i$. Then, there exist a constant $\alpha_i = \alpha_i(x') > 0$ and constant vector $c_i^m = c_i^m(x', \lambda)$ depending on $\gamma_i$ and $\Gamma$ satisfying

$$\forall k \in \mathbb{N}_0, \ \forall \bar{m}, \bar{l} \in \mathbb{N}_0, \ \forall (x', y) \in U_i \times \{y \in Z_{BL} : y_3 < 0\};$$

$$|D_{x}^k D_{y}^l \tilde{\beta}_i(x', y, y_3, \lambda)| + |D_{x}^m D_{y}^l \tilde{\omega}_i(x', y, y_3, \lambda)| \lesssim e^{\alpha_i y_3},$$

and

$$\forall k \in \mathbb{N}_0, \ \forall \bar{m} \in \mathbb{N}_0^3, \ \forall x' \in U_i, \ \forall r \in (1, \infty);$$

$$\|D_{x}^k D_{y}^l \tilde{\beta}_i(x', y, y_3, \lambda)\|_{W^{1,r}(Z_{BL}^+) \setminus C^\infty} + \|D_{x}^m D_{y}^l \tilde{\omega}_i(x', y, y_3, \lambda)\|_{L^r(Z_{BL}^+) \setminus C} \leq C,$$

(3.18)

where

$$\tilde{\beta}_i(x', y, \lambda) \equiv \beta_i(x', y, \lambda) - c_i^m(x', \lambda),$$

(3.19)

and $C$ depends on the boundedness constants of $D\tilde{\varphi}_1, D\tilde{\varphi}_1^{-1}$ and $\tilde{I}$ and $r$. 

8
Proof. We rely on Fourier expansion techniques. We shall write $A = A_1, B = B_1$ for simplicity.
Let $A = (a_{jl})_{j,l=1,2}$. Due to Remark 3.7 we may assume w.l.o.g. that $\gamma_i(x',y') > 1$ for all $(x',y') \in U_1 \times Z'$ where $Z' = (0,1) \times (0,1)$. In view of the definition of the solution to (3.22) and Remark 3.6 we get that

$$\beta(x',\cdot,\cdot) \in C((-\infty,1], L^2(Z')), \quad \omega(x',\cdot,\cdot) \in L^2(-\infty,1; L^2(Z'))$$

and that $\beta, \omega$ as functions of $y'$ belong to $C_\text{per}^\infty(Z')$, where $Z' = (0,1) \times (0,1)$ and $C_\text{per}^\infty(Z')$ is the subspace of $C^\infty(Z')$ formed by all (1,1)-periodic functions. Hence we have Fourier expansions of $\beta, \omega$ such that

$$\beta(x',y,\cdot) = \sum_{m \in \mathbb{Z}^2} c_m(x',y,\cdot)e^{2\pi i m \cdot y'}, \quad \omega(x',y,\cdot) = \sum_{m \in \mathbb{Z}^2} d_m(x',y,\cdot)e^{2\pi i m \cdot y'},$$

where Fourier coefficients $c_m = (c_{m,1}, c_{m,2}, c_{m,3}), d_m = (m_1, m_2)$ are vector and scalar functions in $y_3$, respectively. Then, using $a_{3j} = a_{33} = 0$ ($j = 1,2), a_{33} = 1$ we get for $y_3 \in (-\infty,1) \setminus \{0\}$ that

$$\text{div}(A \nabla_y \beta) = \sum_{m \in \mathbb{Z}^2} \left( \frac{\partial}{\partial y_3} c_m - 4\pi^2 \xi_m c_m \right) e^{2\pi i m \cdot y'},$$

where $\xi_m(x') = \sum_{1 \leq j \leq 2} a_{jl}(x') m_j m_l$. By positivity of the matrix $A$ there is some $\alpha_i = \alpha_i(x') > 0$ satisfying

$$\xi_m(x') \geq \frac{\alpha_i^2(x')}{\pi^2} |m|^2, \quad \forall m \in \mathbb{Z}^2.$$ 

(3.21)

Here, without loss of generality we may regard $\alpha_i(x')$ as a continuous function in $x'$ since $a_{jl}(x'), j,l = 1,2,$ is continuous in $x'$. Moreover, we have for $y_3 \in (-\infty,1) \setminus \{0\}$

$$B \nabla_y \omega = \sum_{m \in \mathbb{Z}^2} B \left( \frac{2\pi i d_m m}{d_m} \right) e^{2\pi i m \cdot y'}.$$ 

On the other hand, $\text{div}(B^T \beta) = 0$ implies $\frac{d}{dy_3} c_m \cdot \nu(\varphi_i(x')) + ((D \varphi_i)^T c_m) \cdot 2\pi i m = 0$ for all $m \in \mathbb{N}_0^2$. Thus, we get the following system of ordinary equations for each given $m \in \mathbb{Z}^2$:

$$\begin{cases}
\frac{d^2}{dy_3^2} c_m - 4\pi^2 \xi_m c_m - B \left( \frac{2\pi i d_m m}{d_m} \right) = 0, & \text{for } y_3 \in (-\infty,1) \setminus \{0\} \\
\frac{d}{dy_3} c_m \cdot \nu(\varphi_i(x')) + ((D \varphi_i)^T c_m) \cdot 2\pi i m = 0, & \text{for } y_3 \in (-\infty,1) \setminus \{0\}.
\end{cases}$$

(3.22)

In particular, for $m = (0,0)$ we have

$$\frac{d^2}{dy_3^2} c_{(0,0)} - \frac{d}{dy_3} d_{(0,0)} \nu = 0, \quad \frac{d}{dy_3} c_{(0,0)} \cdot \nu = 0, \quad \text{for } y_3 \in (-\infty,1) \setminus \{0\},$$

(3.23)

yielding

$$c_{(0,0)}(y_3) \equiv c^{bl}_{(0,0)}(x',\cdot,\cdot) = \text{const}, \quad d_{(0,0)}(y_3) \equiv 0, \quad \forall y_3 < 0,$$

(3.24)

in view of $\nabla_y \beta, \omega \in L^2(Z' \times (-\infty,0))$.

The solution $\{c_{m}, d_{m}\}, |m| \geq 1$, to (3.22) is found as

$$c_{m}(x',y,\cdot) = \left( c_{m}^0 - 2\pi \sqrt{\xi_m} y_3 \frac{\partial}{\partial y_3} B \left( \frac{im}{\sqrt{\xi_m}} \right) e^{2\pi \sqrt{\xi_m} y_3} \right),$$

$$d_{m}(x',y,\cdot) = -4\pi \sqrt{\xi_m} d_m e^{2\pi \sqrt{\xi_m} y_3}, \quad \forall y_3 \in (-\infty,0),$$

(3.25)

with

$$c_{m}^0 := \lim_{y_3 \to 0} c_{m}(x',y_3,\cdot), \quad d_m^0 := c_{m}^0 \cdot B \left( \frac{im}{\sqrt{\xi_m}} \right) = c_{m}^0 \cdot \nu(\varphi_i(x')) + ((D \varphi_i)^T c_m^0) \cdot \frac{im}{\sqrt{\xi_m}}.$$
Now let us determine \( \{ \beta, \omega \} \) for \( y_3 > 0 \). From the jump condition \( [\frac{\partial \varphi}{\partial y_3} - \omega \nu]_S = \lambda \) we get that

\[
[\frac{dc_m}{dy_3} - d_m \nu(\varphi_1(x'))]_S = \begin{cases} 
0 & \text{for } m \neq (0,0), \\
\lambda & \text{for } m = (0,0).
\end{cases}
\]  
(3.26)

By (3.23), for \( y_3 \in (0,1) \) we have

\[
d_{(0,0)} = \text{const}, \quad c_{(0,0)}(y_3) = (d_{(0,0)} \nu(\varphi_1(x')) + \lambda)y_3 + c^{bl}(x', \lambda).
\]  
(3.27)

Moreover, in view of (3.26) we have (3.25) for \( |m| \geq 1 \), \( y_3 \in (0,1) \) as well.

Thus, we get that \( c_m, d_m \) are defined for all \( y_3 \in (-\infty, 1) \) and

\[
c_m(x', y_3, \lambda) = \left( c_m^1 - 2\pi \sqrt{\xi_m}(y_3 - 1)d_m^1 \left( \frac{m}{\sqrt{\xi_m}} \right) \right) e^{2\pi \sqrt{\xi_m}(y_3 - 1)},
\]

\[
d_m(x', y_3, \lambda) = -4\pi \sqrt{\xi_m}d_m^1 e^{2\pi \sqrt{\xi_m}(y_3 - 1)}, \quad \forall y_3 \in (-\infty, 1),
\]  
(3.28)

where \( c_m^1 = c_m(x', 1, \lambda) \) and \( d_m^1 = \frac{c_m^1}{1 - 2\pi \sqrt{\xi_m} B \left( \frac{m}{\sqrt{\xi_m}} \right)^2} \).

By (3.28), for all \( y_3 < 1 \) we have

\[
|c_m(x', y_3, \lambda)| + |y_3 - 1||d_m(x', y_3, \lambda)| \leq c |c_m^1| e^{2\pi \sqrt{\xi_m}(y_3 - 1)},
\]  
(3.29)

where the constant \( c > 0 \) depends on the boundedness constants of \( D\varphi_i, D\varphi_i^{-1} \). Moreover, \( \beta \) is continuous at \( y_3 = 1 \) in the norm of \( L^2(Z') \)

\[
\sum_{m \in \mathbb{Z}^2} |c_m^1|^2 = \| \beta(x', \cdot, 1, \lambda) \|_{L^2(Z')}^2.
\]

Hence, in view of (3.23), (3.24), (3.25), we get that \( \beta - c_{(0,0)}, \omega - d_{(0,0)} \) are infinitely differentiable in \( U_1 \times Z_{BL} \) and by Parceval’s equality that

\[
|\beta(x', y', y_3, \lambda) - c^{bl}(x', \lambda)| + |y_3 - 1||\omega(x', y', y_3, \lambda)|
\]  
(3.30)

\[
= (\sum_{|m| \geq 1} |c_m(x', y_3, \lambda)|^2)^{1/2} + |y_3 - 1|(\sum_{|m| \geq 1} |d_m(x', y_3, \lambda)|^2)^{1/2}
\]  
(3.31)

\[
\leq c \|\beta(x', \cdot, 1, \lambda)\|_{L^2(Z')}^2 e^{\alpha(x')(y_3 - 1)}, \quad \forall (x', y', y_3) \in U_1 \times Z' \times (-\infty, 0),
\]

and, in view of (3.29), (3.27), that

\[
|\beta(x', y', y_3, \lambda) - c^{bl}(x', \lambda)| - (d_{(0,0)} \nu(\varphi_i(x')) + \lambda)y_3| + |y_3 - 1||\omega(x', y', y_3, \lambda) - d_{(0,0)}|
\]  
(3.32)

\[
\leq c \|\beta(x', \cdot, 1, \lambda)\|_{L^2(Z')}^2 e^{\alpha(x')(y_3 - 1)}, \quad \forall (x', y', y_3) \in U_1 \times Z' \times [0, 1).
\]

In the same way, using the expression (3.23), (3.24), we get for \( \tilde{m} \in \mathbb{N}_0^3 \) with \( |	ilde{m}| \geq 1 \) that

\[
|y_3 - 1|^{|\tilde{m}|} D_{\tilde{m}}^\beta(\beta(x', y', y_3, \lambda) - c^{bl}(x', \lambda)) + |y_3 - 1|^{|\tilde{m}|+1} D_{\bar{m}}^\omega(\varphi_i(x'), y, \lambda)| \leq \epsilon c e^{\alpha(x')y_3},
\]  
(3.33)

\[
\forall (x', y', y_3) \in U_1 \times Z' \times (-\infty, 0),
\]

and that

\[
|y_3 - 1|^{|\tilde{m}|} D_{\tilde{m}}^\beta(\beta(x', y', y_3, \lambda) - c^{bl}(x', \lambda)) - (d_{(0,0)} \nu(\varphi_i(x')) + \lambda)y_3|
\]  
(3.34)

\[
+ |y_3 - 1|^{|\tilde{m}|+1} D_{\bar{m}}^\omega(\varphi_i(x'), y, y_3, \lambda) - d_{(0,0)}|)
\]  
(3.35)

\[
\leq c \|\beta(x', \cdot, 1, \lambda)\|_{L^2(Z')}^2 e^{\alpha(x')(y_3 - 1)}, \quad \forall (x', y', y_3) \in U_1 \times Z' \times [0, 1),
\]
with constant $c > 0$ depending on the boundedness constants of $D\varphi_1, D\varphi_1^{-1}$ and $\vec{l}$. In particular, (3.30) $\sim$ (3.33) imply that

$$|D_3^{\vec{F}}(x', y', \lambda)| + |D_3^{\vec{F}}(x', y, \lambda)| \leq ce^{-\alpha_1(x')}|y_3|, \quad \forall (x', y', y_3) \in U_i \times Z' \times (-\infty, 1/2].$$

Thus, (3.17) for $|\vec{m}| = |\vec{k}| = 0$ is proved.

In order to prove (3.18) for $|\vec{m}| = |\vec{k}| = 0$, consider a smooth domain $Z_0$ expressed by Fig 3.2 (b). By the above proved regularity, periodicity and continuity at $y_3 = 0$ of $\beta$, it follows that the trace of $\beta$ on $\partial Z_0$ belongs to $W^{1-1/r, r}(\partial Z_0)$ for any $r \in (1, \infty)$. Therefore, by well-known theory of existence of solutions to inhomogeneous boundary value problems to ADN elliptic systems, see [4],

$$\{\beta, \omega\} \in W^{1, r}((y \in Z_{BL} : 1/2 < y_3 < \gamma_1(x', y')) \times L^r((y \in Z_{BL} : 1/2 < y_3 < \gamma_1(x', y'))), \forall r \in (1, \infty).$$

Hence, (3.18) for $|\vec{m}| = |\vec{k}| = 0$ holds true.

Now, let us show for all $\vec{k} \in \mathbb{N}_0^d$ with $|\vec{k}| \geq 1$ and $\vec{l} \in \mathbb{N}_0^3$ with $|\vec{l}| \geq 1$ that

$$|D_3^{\vec{F}}(x', y, \lambda)| + |D_3^{\vec{F}}(x', y, \lambda)| \leq ce^{\alpha_2(x')|y_3|}, \quad \forall (x', y', y_3) \in U_i \times Z' \times (-\infty, 0).$$

Differentiating the variational equation (3.14) in $x_3$, we get a new variational equation with the unknown $D_3\beta$ and additional external force terms which are exponentially decreasing. More precisely, we get

$$(B_3 \nabla_{x_3} \nabla_{x_3} \beta, B_3 \nabla_{x_3} \varphi) = -(B_3 \nabla_{x_3} \beta, D_3 B_3 \nabla_{x_3} \varphi) - (D_3 B_3 \nabla_{x_3} \beta, B_3 \nabla_{x_3} \varphi), \quad \forall \varphi \in V.$$
where \( \lambda = \lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3 \). Hence, by already proved conclusion of the theorem for \(|\vec{m}| = 0\) we get the conclusion for \(|\vec{m}| = 1\). For the case \(|\vec{m}| > 1\) \((3.17)\) is proved since \(\frac{\partial^2 \beta^i}{\partial \lambda^l \partial \lambda^k} = \frac{\partial^2 \beta^i}{\partial \lambda^k \partial \lambda^l} = 0, l, h = 1 \sim 3\).

The proof of the theorem is complete. \(\square\)

**Remark 3.9** From Theorem 3.8 and its proof one can infer the following facts:

(i) If \(\gamma_i(x', y') \geq d \geq 0\),

\[
\forall \vec{k} \in N_0^3, \forall \vec{l}, \vec{m} \in N_0^3 (|\vec{l}| \geq 1); \quad |D_{\lambda}^\vec{m} D_y^\vec{l} \beta_i(x', y, \lambda)| \lesssim e^{\alpha_i(x') y_3}, \quad (x', y) \in U_i \times (\{y \in Z_{BL} : y_3 < d/2\} \setminus S).
\]

In particular, for \(|\vec{m}| > 1\)

\[
D_{\lambda}^\vec{m} D_y^\vec{l} \beta_i(x', y, \lambda) = 0, \quad D_{\lambda}^\vec{m} D_y^\vec{l} \omega_3(x', y, \lambda) = 0, \quad (x', y) \in U_i \times (\{y \in Z_{BL} : y_3 < d/2\} \setminus S).
\]

(ii) In view of \(A_i, B_i \subset C^\infty(U_i) \cap C^1(\bar{U}_i)\), it follows that

\[
c_i^l(x', \lambda) = \int_S \beta_i(x', y', 0, \lambda) dy' \in C^\infty(U_i) \cap C^1(\bar{U}_i), \quad i = 1, \ldots, N.
\]

(iii) All the constants have the order of \(O(\lambda)\) by the linearity of the problem \((\mathcal{BL})_{\lambda, x'}\).

(iv) It follows that

\[
\int_{Z'} \beta_i(x', y', y_3, \lambda) \cdot \nu(\varphi_i(x')) dy' = 0, \quad \forall x' \in U_i, \forall y_3 \leq 0,
\]

by integrating \(\text{div} (B^T \beta) = 0\) in the domain \(Z_{BL}^+ \) and \(Z_{BL} \setminus Z_{BL}^+ \) in view of the jump condition \(\beta |_{Z'} = 0\) and \(B_{c3} = \nu(\varphi_i(x'))\). In particular, for any fixed \(\lambda \in \mathbb{R}^3\)

\[
c_i^l(x', \lambda) \cdot \nu(\varphi_i(x')) = 0, \quad x' \in U_i, i = 1, \ldots, N. \quad (3.38)
\]

The next lemma shows additional properties of the solution to \((\mathcal{BL})_{\lambda, x'}\).

**Lemma 3.10** For \(i = 1, \ldots, N\) let \(c_i^l(x', e_l) = (c_{i1}, c_{i2}, c_{i3})^T\) be the constant vector for \(\lambda = e_l, l = 1 \sim 3\) in Theorem 3.8. Then, \(c_{ik} = c_{ik} \) for \(l, k = 1 \sim 3\) and the matrix

\[
\tilde{C}_{ik}^l := \begin{pmatrix}
c_{i1} & c_{i2} \\
c_{i2} & c_{i3}
\end{pmatrix}
\]

is negatively definite.

**Proof.** Let \(\beta_i^l = (\beta_{i1}^l, \beta_{i2}^l, \beta_{i3}^l), l = 1, 2\). We get by the definition of solution to \((\mathcal{BL})_{\lambda, x'}^i\) that for \(l, k = 1 \sim 3\)

\[
c_{ik} = \int_{Z'} \beta_i^l \cdot e_k dy' = -(B_i \nabla_y \beta_i^k, B_i \nabla_y \beta_i^l)_{Z_{BL}}
\]

and, consequently, \(c_{ik} = c_{ik} \).

By the linearity of \((\mathcal{BL})_{\lambda, x'}^i\) with respect to \(\lambda\), for \(\lambda = (\lambda_1, \lambda_2, 0)^T\) one gets \(\beta_i(x', y, \lambda) = \lambda_1 \beta_{i1}^l + \lambda_2 \beta_{i2}^l\), and

\[
\tilde{C}_{ik}^l \lambda \lambda = \sum_{l, k=1}^3 c_{lk} \lambda_1 \lambda_2 \lambda_k = \int_{Z'} \sum_{l, k=1}^3 \beta_{i1}^l \lambda_1 \lambda_2 \lambda_k dy' = \int_{Z'} \beta_i \lambda dy' = -\|B_i \nabla_y \beta_i\|^2_{L^2(Z_{BL})} \leq 0.
\]

It follows by uniqueness of solution to \((\mathcal{BL})_{\lambda, x'}^i\) that the equality in the above inequality holds if and only if \(\lambda = 0\). Therefore the matrix \(\tilde{C}_{ik}^l\) is negatively definite.

Thus, the proof is complete. \(\square\)
3.3 Local boundary layer corrector

Using the result of boundary layer analysis, we construct a local boundary layer corrector in $\Gamma^\varepsilon_{\delta,i}$ for $i = 1, \ldots, N$. Let a three-dimensional vector field $\lambda = (\lambda_1, \lambda_2, \lambda_3) \in C^\infty(\Gamma^\varepsilon)$ on $\Gamma$ be given.

Define the function $\tilde{\beta}^{\varepsilon,\lambda}_i : \Gamma^\varepsilon_{\delta,i} \rightarrow \mathbb{R}^3$ by

$$
\tilde{\beta}^{\varepsilon,\lambda}_i(x) = \tilde{\beta}^{\varepsilon,\lambda}_i(\Phi_i(x)) := \tilde{\beta}_i(x', \frac{x}{\varepsilon}, \lambda \circ \varphi_i(x')),
$$

(3.39)

where $\tilde{\beta}_i$ is defined by (3.19) using the solution $\beta_i(x', \cdot, \lambda \circ \varphi_i(x'))$ to $(\mathbb{H} L)_{\lambda \circ \varphi_i(x'), x'}$. We also define $\tilde{\omega}^{\varepsilon,\lambda}_i : \Gamma^\varepsilon_{\delta,i} \rightarrow \mathbb{R}^3$, $\hat{c}^{b,\lambda}_i : \Gamma^\varepsilon_{\delta,i} \rightarrow \mathbb{R}^3$ by

$$
\tilde{\omega}^{\varepsilon,\lambda}_i(x) = \tilde{\omega}^{\varepsilon,\lambda}_i(\Phi_i(x)) := \omega_i(x', \frac{x}{\varepsilon}, \lambda \circ \varphi_i(x')),
$$

(3.40)

$$
\hat{c}^{b,\lambda}_i(x) := c^b_i(x', \lambda \circ \varphi_i(x')).
$$

(3.41)

Then, $\hat{c}^{b,\lambda}_i \in C^\infty(\Gamma^\varepsilon_{\delta,i} \cap \Omega)^3 \cap C^1(\bar{\Gamma}^\varepsilon_{\delta,i} \cap \bar{\Omega})^3$, by Remark (ii) and $\hat{c}^{b,\lambda}_i$ is tangential on $\Gamma$ by (3.38).

**Lemma 3.11** Let $\rho(x) = d(x, \Gamma)$ denote the distance from $x$ to $\Gamma$. For all $i = 1, \ldots, N$ we have:

(i) $|D^2_{x \varepsilon} \tilde{\beta}^{\varepsilon,\lambda}_i(x) + |D^2_{x \varepsilon} \tilde{\omega}^{\varepsilon,\lambda}_i(x)| \leq \varepsilon^{-|k|} e^{-\alpha_i(x')/\varepsilon}, \quad \forall x \in \Gamma^\varepsilon_{\delta,i} \cap \Omega, \ v \in \mathbb{N}_0$

$\varepsilon \|\nabla \tilde{\beta}^{\varepsilon,\lambda}_i(x)\|_{L^r(\Gamma^\varepsilon_{\delta,i} \cap \Omega)} + \varepsilon \|\tilde{\omega}^{\varepsilon,\lambda}_i(x)\|_{L^r(\Gamma^\varepsilon_{\delta,i} \cap \Omega)} \leq \varepsilon^{1/r}, \quad \forall r \in (1, \infty).$

(ii) $| - \varepsilon \Delta \tilde{\beta}^{\varepsilon,\lambda}_i(x) + \nabla \tilde{\omega}^{\varepsilon,\lambda}_i(x)| \leq \varepsilon^{-\alpha_i(x')/2\varepsilon}, \quad \forall x \in \Gamma^\varepsilon_{\delta,i} \cap \Omega$

$\| - \varepsilon \Delta \tilde{\beta}^{\varepsilon,\lambda}_i(x) + \nabla \tilde{\omega}^{\varepsilon,\lambda}_i(x)\|_{L^r(\Gamma^\varepsilon_{\delta,i} \cap \Omega)} \leq \varepsilon^{1/r}, \quad \forall r \in (1, \infty).$

(iii) $\|\text{div} \tilde{\beta}^{\varepsilon,\lambda}_i(x)\|_{L^r(\Gamma^\varepsilon_{\delta,i} \cap \Omega)} \leq \varepsilon^{-\alpha_i(x')/\varepsilon}, \quad \forall x \in \Gamma^\varepsilon_{\delta,i} \cap \Omega$

$\|\text{div} \tilde{\beta}^{\varepsilon,\lambda}_i(x)\|_{L^r(\Gamma^\varepsilon_{\delta,i} \cap \Omega)} \leq \varepsilon^{1/r}, \quad \forall r \in (1, \infty).$

(iv) $|\varepsilon \frac{\partial}{\partial x^j} \tilde{\beta}^{\varepsilon,\lambda}_i(x) - \tilde{\omega}^{\varepsilon,\lambda}_i(x) \mu(x)| = \lambda(x), \quad x \in V_i$

(v) $\tilde{\beta}^{\varepsilon,\lambda}_i(x) = -\hat{c}^{b,\lambda}_i(x), \quad x \in \Gamma^\varepsilon_{\delta,i} \cap \Gamma_0.$

**Proof.** Fix any $i \in \{1, \ldots, N\}$.

- **Proof of (i).**

  By chain rule, for $j = 1$ we get

$$
\frac{\partial}{\partial x^j} \tilde{\beta}^{\varepsilon,\lambda}_i(x) = \sum_{k=1}^2 \frac{\partial}{\partial x^k} \tilde{\beta}_i(\xi_k, \frac{x}{\varepsilon}, \lambda \circ \varphi_i(x')) |_{\xi = x', \lambda \circ \varphi_i(x')} \Phi_i^{-1} \cdot \frac{\partial (\Phi_i^{-1})}{\partial x^j}
$$

$$
+ \varepsilon \sum_{l=1}^3 \frac{\partial}{\partial x^l} \tilde{\beta}_i(x', \xi, \lambda \circ \varphi_i(x')) |_{\xi = x', \lambda \circ \varphi_i(x')} \Phi_i^{-1} \cdot \frac{\partial (\Phi_i^{-1})}{\partial x^j}
$$

$$
+ \sum_{m=1}^3 \frac{\partial}{\partial x^m} \tilde{\beta}_i(x', \xi, \mu) |_{\mu = \lambda \circ \varphi_i(x')} \Phi_i^{-1} \cdot \frac{\partial (\Phi_i^{-1})}{\partial x^j}.
$$

(3.42)

Also, we have the expression of the first derivatives of $\tilde{\omega}^{\varepsilon,\lambda}_i(x)$ in a similar form. By Theorem (3.8) for $x \in \Gamma^\varepsilon_{\delta,i} \cap \Omega$ both the moduli of the first and third term in the right-hand side of (3.42) are estimated by $e^{-\alpha_i(x')/\varepsilon}$ and the modulus of the second term by $\varepsilon^{-1} e^{-\alpha_i(x')/\varepsilon}$. Note that $\rho(x) = x_3$. Therefore the first estimate of (i) for $|\tilde{k}| = 1$ is proved. The first estimate of (i) for the cases $|\tilde{k}| > 1$ can be obtained by differentiating (3.42) repeatedly.

The second estimate of (i) follows by (3.42) and (3.11) of Theorem (3.8) in view of

$$
\|h(\frac{x}{\varepsilon})\|_{L^r(\Gamma^\varepsilon_{\delta,i} \cap \Omega)} \leq \varepsilon^{1/r}, \quad \forall r \in [1, \infty),
$$

(3.43)
for any \( h \in L^r(\Omega^r) \).

- **Proof of (ii):**

  Note that

  \[
  (D_x \Phi_i)^{-T} = (D_x \Phi_i(x', 0))^{-T} + R(x', x_3) = B_i(x') + R(x', x_3),
  \]

  \[
  (D_x \Phi_i)^{-1}(D_x \Phi_i)^{-T} = A_i(x') + S(x', x_3),
  \]

  where matrices \( R(x', x_3) \) and \( S(x', x_3) \) satisfy

  \[
  \|R(x', x_3)\|_{\infty} + \|S(x', x_3)\|_{\infty} \lesssim |x_3|, \quad \forall x_3 \in (-\delta, \delta),
  \]

  see (3.13). Therefore, in view of the expression of \( \Delta_x \), see (3.12), one can get for \( x \in \Gamma^\delta_{i, \delta} \) that

  \[
  \Delta_x \tilde{\beta}_i^{\epsilon, \lambda}(x) = \sum_{l=1}^3 A_i(x') \frac{\partial^2}{\partial x_l \partial x_k} \tilde{\beta}_l(x', x_3, \lambda \circ \varphi_i(x')) \Phi_i^{-1}
\]

  as in the proof of Theorem 5.1 in \([29]\). By Theorem 3.8 and

  \[
  |x_3| e^{-\alpha_{i}(x') |x_3| / \epsilon} \lesssim e^{-\alpha_{i}(x') |x_3| / 2\epsilon}, \forall x_3 \in \mathbb{R},
  \]

  the moduli of the second and third terms in the right-hand side of (3.44) are estimated by \( O(\epsilon)^{-1} e^{-\alpha_{i}(x') \rho(x') / 2\epsilon} \) for \( x \in \Gamma^\delta_{i, \delta} \cap \Omega \), while their \( L^r(\Gamma^\delta_{i, \delta} \setminus \Omega) \)-norms are estimated by \( O(\epsilon)^{-1 + 1/r} \) in view of (3.43).

  Now let us expand the first term in the right-hand side of (3.45). Direct calculation yields

  \[
  \sum_{l=1}^3 A_i(x') \frac{\partial^2}{\partial x_l \partial x_k} \tilde{\beta}_l(x', x_3, \lambda \circ \varphi_i(x'))
  \]

  By Theorem 3.8 and (3.37) from the second to fourth terms in the right-hand side of (3.46) are estimated by \( \epsilon^{-1} e^{-\alpha_{i}(x') \rho(x') / \epsilon} \) for \( x \in \Gamma^\delta_{i, \delta} \cap \Omega \) and have \( L^r(\Gamma^\delta_{i, \delta} \setminus \Omega) \)-norm equal to \( O(\epsilon^{1/r}) \). Thus, for \( x \in \Gamma^\delta_{i, \delta} \cap \Omega \) we have

  \[
  \epsilon \Delta_x \tilde{\beta}_i^{\epsilon, \lambda}(x) = \frac{1}{\epsilon} \sum_{l=1}^3 A_i(x') \frac{\partial^2}{\partial x_l \partial x_k} \tilde{\beta}_l(x', x_3, \lambda \circ \varphi_i(x')) \Phi_i^{-1} + R_1
  \]

  where \( |R_1(x)| \lesssim O(e^{-\alpha_{i}(x') \rho(x') / 2\epsilon}) \) for \( x \in \Gamma^\delta_{i, \delta} \cap \Omega \) and \( \|R_1\|_{L^r(\Gamma^\delta_{i, \delta} \setminus \Omega)} \lesssim \epsilon^{1/r} \).

  On the other hand, it follows from (3.12), (3.44) that

  \[
  \nabla_x \tilde{\omega}_i^{\epsilon, \lambda}(x) = (B_i(x') + R(x', x_3)) \nabla_x \omega_i(x', x_3, \lambda \circ \varphi_i(x')) \Phi_i^{-1},
  \]

  where

  \[
  \nabla_x \omega_i(x', x_3, \lambda \circ \varphi_i(x'))
  \]

  \[
  = \left( \nabla_{x'} \omega_i(x', x_3, \lambda \circ \varphi_i(x')) \Big|_{\xi = x'} + \frac{1}{\epsilon} \nabla_{x_3} \omega_i(x', x_3, \lambda \circ \varphi_i(x')) \Big|_{\xi = x_3} + \sum_{m=1}^3 \omega_i^m(x', x_3, x_3) \nabla_{x'} (\lambda_m \circ \varphi_i) \right).
  \]
Hence we get that
\[
\nabla_x \tilde{\omega}_1^{\varepsilon,\lambda}(x) = \left( \frac{B_i'(x, x_3, x_3)}{\varepsilon} \nabla_x \omega_i(x', x_3, x_3, \lambda \circ \varphi_i(x')) \right)_{\xi = \frac{x'}{x}} + B'_i(x') \nabla_x \omega_i(x', \xi, \lambda \circ \varphi_i(x')) \bigg|_{\xi = x'} + \sum_{m=1}^3 \omega^m(x', \xi) B'_i(x') \nabla_x \omega_i(x', \xi, \lambda \circ \varphi_i(x')) + R(x', x_3) \nabla_x \omega_i(x', \xi, \lambda \circ \varphi_i(x')) \bigg) \circ \Phi_i^{-1}.
\]

By Theorem 3.8, the sum from the second to fourth terms in the bracket of the right-hand side of (3.49) equals \(O(e^{-\alpha_1(x')/\varepsilon})\) for \(x \in \Gamma_{\theta, 3} \cap \Omega\) and have \(L^r(\Gamma_{\theta, 3} \setminus \Omega)\)-norms equal to \(O(\varepsilon)^{1/r}\).

Now, subtracting (3.49) from (3.47) yields the conclusion of (ii).

- Proof of (iii):
Using the fact that divergence of a vector field is independent of the choice of orthogonal coordinate system, we get by Theorem 3.8 that
\[
\text{div}_x \tilde{\beta}_1^{\varepsilon,\lambda}(x) = \text{div}_x \tilde{\beta}_i(x', \xi, \lambda \circ \varphi_i(x')) \circ \Phi_i^{-1}
\]

\[
= \sum_{k=1}^3 \frac{\partial}{\partial x_k} \tilde{\beta}_i(x', \xi, \lambda \circ \varphi_i(x'))_{\xi = x'} + \sum_{k,m=1}^3 \omega^m(x', \xi) \frac{\partial}{\partial x_k} \tilde{\beta}_i(x', \xi, \lambda \circ \varphi_i(x'))_{\xi = x'}
\]

\[
= \sum_{k=1}^3 \frac{\partial}{\partial x_k} \tilde{\beta}_i(x', \xi, \lambda \circ \varphi_i(x'))_{\xi = x'} + \sum_{k,m=1}^3 \frac{\partial}{\partial x_k} \tilde{\beta}_i(x', \xi, \lambda \circ \varphi_i(x'))_{\xi = x'}
\]

\[
= \frac{\partial}{\partial x_k} \tilde{\beta}_i(x', \xi, \lambda \circ \varphi_i(x'))_{\xi = x'} + \sum_{k,m=1}^3 \frac{\partial}{\partial x_k} \tilde{\beta}_i(x', \xi, \lambda \circ \varphi_i(x'))_{\xi = x'}
\]

where \((\tilde{\beta}_1, \tilde{\beta}_2, \tilde{\beta}_3) \equiv \tilde{\beta}_i\). Thus, by the same argument as in the proof of (ii) we get the conclusion.

- Proof of (iv):
For \(x \in V_i\), we have
\[
\left[ \varepsilon \frac{\partial \tilde{\beta}_i^{\varepsilon,\lambda}(x)}{\partial x_k} - \tilde{\omega}_i^{\varepsilon,\lambda}(x) \nu(x) \right] \nabla_x \omega_i(x', \xi, \lambda \circ \varphi_i(x'))_{\xi = x'} = \lambda \circ \varphi_i(x') \circ \Phi_i^{-1} = \lambda(x).
\]

- Proof of (v): (v) is obvious from definition of \(\tilde{\beta}_i^{\varepsilon,\lambda}\).

\[
\]

### 3.4 Global boundary layer corrector

In this subsection, a global boundary layer corrector is constructed using cut-off functions for \(\Gamma\) and local boundary layer correctors \(\tilde{\beta}_i^{\varepsilon,\lambda}, \tilde{\omega}_i^{\varepsilon,\lambda}, i = 1, \ldots, N\). Let \(\psi_i \in C^\infty(\tilde{\Gamma}), i = 1, \ldots, N\), be cut-off functions such that
\[
\psi_i \in C^\infty(\tilde{\Gamma}), \quad \text{supp} \psi_i \subset \tilde{V}_i, \quad i = 1, \ldots, N, \quad \sum_{i=1}^N \psi_i(x') = 1, \quad x' \in \Gamma,
\]

where \(\{V_i\}_{i=1}^N\) is the open covering of \(\Gamma\) introduced in Section 2. Let
\[
\alpha := \min_{i=1, \ldots, N} \min_{x' \in \tilde{U}_i} \alpha_i(x')/2.
\]

Let \(\tilde{\psi}_i(x) = \psi_i(x')\) for \(x \in \Gamma_{\delta}, x = \Upsilon(x', x_3), (x', x_3) \in \Gamma \times (-\delta, \delta)\).

Given a three-dimensional vector field \(\lambda \in C^\infty(\Gamma)^3\) on \(\Gamma\), a global boundary layer corrector \(\{\beta^{\varepsilon,\lambda}, \omega^{\varepsilon,\lambda}\}\) on \(\Gamma_{\delta}^3\) is defined as
\[
\beta^{\varepsilon,\lambda}(x) := \sum_{i=1}^N \tilde{\psi}_i(x) \tilde{\beta}_i^{\varepsilon,\lambda}(x), \quad \omega^{\varepsilon,\lambda}(x) := \sum_{i=1}^N \tilde{\psi}_i(x) \tilde{\omega}_i^{\varepsilon,\lambda}(x), \quad x \in \Gamma_{\delta}^3,
\]
Lemma 3.12 (i) It holds
\[ \frac{\partial}{\partial r} \beta_{\epsilon, \lambda}(x) - \omega_{\epsilon, \lambda}(x) \nu(x) \mid_\Gamma = \lambda. \]

Proof. - Proof of (i):

Note that
\[ D^{\beta}_{\epsilon, \lambda}(x) = \sum_{i=1}^N \sum_{K_1+K_2 = K} D^{K_1} \beta_i \psi_i(x) D^{K_2} \beta_{\epsilon, \lambda}(x), \]
\[ D^{\omega}_{\epsilon, \lambda}(x) = \sum_{i=1}^N \sum_{K_1+K_2 = K} D^{K_1} \omega_i \psi_i(x) D^{K_2} \omega_{\epsilon, \lambda}(x). \]

Hence, by Lemma 3.11 we get the conclusion (i).

- Proof of (ii):

Direct calculations yield that
\[ -\epsilon \Delta \beta_{\epsilon, \lambda}(x) + \nabla \omega_{\epsilon, \lambda}(x) = \sum_{i=1}^N \psi_i(x) (-\epsilon \Delta \beta_{\epsilon, \lambda}(x) + \nabla \omega_{\epsilon, \lambda}(x)) + \text{derivatives of } \beta_{\epsilon, \lambda} \text{ up to first order multiplied by } \epsilon \]
\[ + \text{zeroth derivative terms of } \omega_{\epsilon, \lambda}. \]

Thus, from Lemma 3.11 (ii) we get the conclusion (ii).

- Proof of (iii):

The conclusion (iii) follows directly from Lemma 3.11 (iii) since
\[ \text{div } \beta_{\epsilon, \lambda}(x) = \sum_{i=1}^N (\psi_i(x) \text{div } \tilde{\beta}_i - \text{zeroth derivative terms of } \beta_{\epsilon, \lambda}. \]

- Proof of (iv):

The conclusion (iv) is obvious since \( \tilde{\psi}_i(x), x \in \Gamma_\delta \), depends only on tangential variables of \( \Gamma \) and hence \( \frac{\partial}{\partial \nu} \tilde{\psi}_i(x) = 0 \).
3.5 Construction of first order approximations and Navier wall-laws

The global boundary layer corrector constructed above rapidly decreases with exponential decay rate going from $\Gamma$ to the interior of $\Omega$. Using the corrector we construct higher order approximations for the real solution $u^\varepsilon$. Then, we derive an effective Navier wall-law.

Let us fix a vector field $\lambda^{(l)} \in C^\infty(\Gamma)^3, l = 1 \sim 3,$ on $\Gamma$ with

$$|\lambda^{(l)}(x')| = 1, \lambda^{(l)}(x') \perp \lambda^{(k)}(x') \ (l \neq k), \quad x' \in \Gamma,$$

and $\lambda^{(3)}(x') = \nu(x').$

For $x \in \Gamma_3^\varepsilon, \ x = \Upsilon(x', x_3),$ let $\lambda(x) \equiv \lambda(x'),$

$$\beta^{\varepsilon, l}(x) \equiv \beta^{\varepsilon, \lambda^{(l)}(x)}, \omega^{\varepsilon, l}(x) \equiv \omega^{\varepsilon, \lambda^{(l)}(x)}, \quad l = 1 \sim 3,$$

(3.52)

see (3.50), and let

$$c_{lk}(x') := c^{bl,(l)}(x') \cdot \lambda^{(k)}(x'), \quad x' \in \Gamma, \ l, k = 1 \sim 3,$$

(3.53)

see (3.51). Note that

$$c_{13}(x) = 0, \quad l = 1 \sim 3,$$

since $c^{bl,(l)}(x'), l = 1 \sim 3,$ is tangential on $\Gamma$.

Now, define $2 \times 2$ matrix $c^{bl}(x')$ by

$$c^{bl}(x') = \begin{pmatrix} c_{11}(x') & c_{12}(x') \\ c_{21}(x') & c_{22}(x') \end{pmatrix}, \quad x' \in \Gamma.$$

(3.54)

Then, by Lemma 3.10 the matrix $c^{bl}(x')$ for all $x' \in \Gamma$ is negatively definite and $c^{bl} \in C^1(\Gamma)$. The extension of $c^{bl}$ by zero matrix on $\Gamma_1$ is denoted again by $c^{bl}$.

Let us take a function $\Psi \in C^1(\Gamma)$ and its extension $\tilde{\Psi} \in W^{1,\infty}(\Gamma_3^\varepsilon)$ such that

$$\tilde{\Psi}(x') \equiv 1 \quad \text{for } x' \in \Gamma, \quad \tilde{\Psi}(x) \equiv 1 \quad \text{for } x \in \Gamma_3^\varepsilon$$

if $\Gamma$ and $\Gamma_1$ are components of $\partial \Omega$. If $\Gamma$ and $\Gamma_1$ are adjacent, then we take a function $\Psi \in C^1(\Gamma)$ satisfying

$$\begin{cases} 0 \leq \Psi(x') \leq 1 & x' \in \Gamma, \quad \Psi(x') \equiv 1 \quad \text{for } x' \in \Gamma', \\
\Psi(x') \sim \varepsilon^{-1}d(x', \Gamma \cap \Gamma_1), & |D_x\Psi(x')| \lesssim \varepsilon^{-1} \quad \text{for } x' \in \Gamma \setminus \Gamma'.
\end{cases}$$

In order to take a suitable extension $\tilde{\Psi}$ of $\Psi$ onto $\Gamma_3^\varepsilon,$ let us choose a domain $D \subset \Gamma_3^\varepsilon$ such that $D \subset \Gamma_3^\varepsilon \cap \Omega$ and

$$\partial D \cap \Gamma = \Gamma', \quad \text{dist}(x, \partial \Gamma_3^\varepsilon \setminus \Gamma_0) \gtrsim \varepsilon + kx_3^{1/4} \quad \text{for } x = \Upsilon(x', x_3) \in \partial D \cap \Omega$$

with some constant $k > 0.$ Then we choose a function $\hat{\Psi}$ such that

$$\hat{\Psi}(x) = 0 \quad \text{for } x \in (\Gamma_3^\varepsilon \cap \Omega) \setminus D, \quad |\nabla \hat{\Psi}(x)| \leq \frac{K}{\varepsilon + x_3^{1/4}} \quad \text{for } x \in \Gamma_3^\varepsilon \cap \Omega,$$

and $\hat{\Psi}(x) \equiv \Psi(x')$ for $x = \Upsilon(x', x_3) \in \Omega^\varepsilon \setminus \Omega.$ Here the constant $K$ depends on $\delta, \Gamma.$ Obviously,

$$\|\nabla \hat{\Psi}\|_{L^\infty(\Omega^\varepsilon \setminus \Omega)} \lesssim \frac{1}{\varepsilon}, \quad \nabla \hat{\Psi} \equiv 0 \quad \text{in } (\Omega^\varepsilon \setminus \Omega) \cap \Gamma_3^\varepsilon.$$

Lemma 3.13 Let $q > 2$ if $\Gamma_0$ and $\Gamma_1$ are components of $\partial \Omega^\varepsilon$ (equivalently, $\Gamma$ and $\Gamma_1$ are components of $\partial \Omega$) and $q > 3$ if $\Gamma_0$ and $\Gamma_1$ are adjacent (equivalently, $\Gamma$ and $\Gamma_1$ are adjacent). Then, for all $v \in W^{1,q}(\Gamma_3^\varepsilon)$ the following inequality holds:

$$\|\nabla (\hat{\Psi} v)\|_{L^2(\Gamma_3^\varepsilon)} \lesssim \|v\|_{W^{1,q}(\Gamma_3^\varepsilon)}.$$

(3.55)
Proof. The proof for the case where \( \Gamma_0 \) and \( \Gamma_1 \) are components of \( \partial \Omega \) is trivial.

Let \( \Gamma_0 \) and \( \Gamma_1 \) be adjacent. Then, in view of the construction of \( \tilde{\Psi} \) we get that
\[
\|\nabla (\tilde{\Psi} v)\|_{L^2(\Gamma_0')} \leq \|\tilde{\Psi} \nabla v\|_{L^2(\Gamma_0')} + \|\nabla \tilde{\Psi} v\|_{L^2(\partial \Omega \setminus \Gamma_0')} + \|\nabla \tilde{\Psi} v\|_{L^2(\partial \Omega \setminus \partial \Omega)}
\]
(3.56)
where \( \tilde{e} = \frac{1}{|\Omega|} \int_{\Omega \setminus \Omega} \omega \, dx \) and \( \omega := \Omega \setminus (\Omega \cup \Gamma_0') \). Note that \( |\omega| = O(\varepsilon^2) \). The second term in the right-hand side of (3.56) is estimated as
\[
\|\chi_{\Psi} v\|_{L^2(\Gamma_0')} \leq \|\chi_{\Psi} \|_{L^2(\partial \Omega \setminus \partial \Omega)} \leq \|\tilde{\Psi} v\|_{L^2(\partial \Omega \setminus \partial \Omega)}
\]
using Hölder's inequality and Sobolev embedding theorem, and the third term as
\[
\|\nabla \tilde{\Psi} v\|_{L^2(\partial \Omega \setminus \partial \Omega)} \leq \|\nabla \tilde{\Psi} v\|_{L^2(\Omega \setminus \Omega)} \leq \|\tilde{\Psi} v\|_{L^2(\Omega \setminus \Omega)}
\]
using Poincaré's inequality. Finally, the fourth term in the right-hand side of (3.56) is estimated as
\[
\|\tilde{\Psi} v\|_{L^2(\partial \Omega \setminus \partial \Omega)} \leq \|\tilde{\Psi} v\|_{L^2(\Omega \setminus \Omega)} \leq \|\tilde{\Psi} v\|_{L^2(\Omega \setminus \Omega)}
\]
with the help of Sobolev embedding \( W^{1,q}(\Gamma_0') \hookrightarrow L^\infty(\Gamma_0') \) due to \( q > 3 \).

Thus (3.55) is proved. \( \square \)

Now, let \( \frac{\partial u}{\partial \nu} \tilde{\Psi} \in W^{1,q}(\Omega') \) be respectively some extensions of \( \frac{\partial u}{\partial \nu}\Gamma, p|\Gamma \) given by a linear bounded extension operator from \( W^{1-1/q,q}(\Gamma) \) to \( W^{1,q}(\Gamma_0') \) such that
\[
\tilde{\Psi} = 0, \quad \frac{\partial u}{\partial \nu} \tilde{\Psi} = 0, \quad x \in \Omega \setminus \Gamma_0',
\]
and
\[
\|\tilde{\Psi}\|_{W^{1,q}(\Gamma_0')} \leq \|u\|_{W^{1,q}(\Omega')} + \|p\|_{W^{1,q}(\Omega)}.
\]
(3.57)
The existence of such extension operator can be shown by Sobolev extension theorem using the assumption on \( \Gamma \). In the sequel, we use the notation
\[
\chi_{\ell}(x) := \Psi(x) \frac{\partial u_{\ell}}{\partial \nu}(x) \cdot \lambda^{(l)}(x'), l \in [1,2], \quad \chi_{3}(x) := -\Psi(x) \tilde{\Psi}(x), \quad x \in \Gamma_0',
\]
(3.58)
where \( \frac{\partial u_{\ell}}{\partial \nu} \) denotes the \( \nu \)-directional derivative of
\[
u_{\ell}(x) = u_{\ell}(\Omega(x', x_3)) := u(x) - u_{\ell}(x) \nu(x'), \quad u_{\ell}(x) := u_{\ell}(x) \cdot \nu(x').
\]
Note that \( \frac{\partial u_{\ell}(x)}{\partial \nu} \) on \( \Gamma' \) is tangential on \( \Gamma' \) since \( \frac{\partial u}{\partial \nu} \cdot \nu = \frac{\partial u}{\partial \nu} - u_{\nu} = 0 \) in view of the solenoidal condition for \( u \) and \( u|_{\Gamma'} = 0 \). We put \( \chi := \sum_{l=1}^{3} \chi_{l} \lambda^{(l)} \).

We construct a correction \( \eta^{\ell} \) rapidly oscillating in a neighborhood of \( \Gamma \) by
\[
\eta^{\ell}(x) := \varepsilon \sum_{l=1}^{3} \beta^{\ell,l}(x) \chi_{l}(x), x \in \Gamma_0',
\]
(3.59)
Note that the function \( \eta^{\ell} \) after extended by 0 to \( \Omega' \) belongs to \( W^{1,q}(\Omega') \). Moreover, if \( q \geq 2 \) is given as in Lemma 3.13 then by (3.57) and Sobolev embedding theorem one has
\[
\|\chi_{l}\|_{W^{1,q}(\Gamma_0')} \leq \|u\|_{W^{2,q}(\Omega')} + \|p\|_{W^{1,q}(\Omega)}, \quad l = 1 \sim 3.
\]
(3.60)
In order to construct a non-oscillating correction, consider the following problem:
\[
-\Delta \eta + \nabla \zeta = 0 \quad \text{in } \Omega,
\]
\[
\text{div } \eta = 0 \quad \text{in } \Omega,
\]
\[
\eta_{\tau} = \Psi c^{\ell} \frac{\partial u_{\ell}}{\partial \nu} \quad \text{on } \Gamma,
\]
\[
\eta_{\nu} = 0 \quad \text{on } \Gamma,
\]
\[
\eta = 0 \quad \text{on } \Gamma_0
\]
\[
\eta = 0 \quad \text{on } \Gamma_1.
\]
(3.61)
where and in what follows
\[ c_{lk} \frac{\partial u_x}{\partial \nu} = \sum_{l, k=1}^2 c_{lk}(\frac{\partial u_x}{\partial \nu}) k \lambda^{(l)}, \frac{\partial u_x}{\partial \nu} \cdot \lambda^{(k)}. \] (3.62)

The system (3.61) has a unique weak solution \{\eta, \zeta\} \in W^{1,2}(\Omega) \times L^2_{(m)}(\Omega) since the boundary data for \eta belongs to \(H^{1/2}(\partial \Omega)\), and, in view of (3.61), (3.59), respectively, are independent of the choice of orthogonal tangent vector fields on \(\eta\), provided \(q\) is given as in Lemma 3.11. Let us construct a correction \(\bar{\eta}^\sigma\) non-oscillating in a neighborhood of \(\Gamma\) by

\[ \bar{\eta}^\sigma(x) = \begin{cases} \varepsilon \eta(x), & \text{for } x \in \Omega, \\ \varepsilon \Psi(x) c^{bl}(x) \frac{\partial u_x}{\partial \nu}, & \text{for } x \in \Omega^c \setminus \Omega. \end{cases} \] (3.64)

Note that \(\text{div} (\bar{\eta}^\sigma + \eta^\sigma) \neq 0\) in general, and \((\bar{\eta}^\sigma + \eta^\sigma)|_{\partial \Omega} = 0\) in view of Lemma 3.11 (v) and Remark 3.9 (iv).

**Lemma 3.14** The vector \(c^{bl} \frac{\partial u_x}{\partial \nu}\) on \(\Gamma\) defined by (3.62) and the approximations \(\bar{\eta}^\sigma\) and \(\eta^\sigma\) defined by (3.64), (3.59), respectively, are independent of the choice of orthogonal tangent vector fields on \(\Gamma\).

**Proof:** Let \{\(\lambda^{(1)}, \lambda^{(2)}\)\} and \{\(\xi^{(1)}, \xi^{(2)}\)\} be different curvilinear systems of orthogonal tangential vector fields on \(\Gamma\). For \(x' \in \Gamma\) we denote the rotational matrix from \{\(\xi^{(1)}(x'), \xi^{(2)}(x')\)\} to \{\(\lambda^{(1)}(x'), \lambda^{(2)}(x')\)\} by \(N = \begin{pmatrix} n_1 & n_2 \\ -n_2 & n_1 \end{pmatrix}\), where \(n_1^2 + n_2^2 = 1\), i.e.,

\[ \Lambda = N \Xi, \quad \Lambda := (\lambda^{(1)}(x'), \lambda^{(2)}(x'))^T, \quad \Xi := (\xi^{(1)}(x'), \xi^{(2)}(x'))^T. \]

Let \(C = (c_{ij})\) and \(D = (d_{ij})\) denote the \(2 \times 2\) matrices defined by (3.59) corresponding to \{\(\lambda^{(1)}, \lambda^{(2)}\)\} and \{\(\xi^{(1)}, \xi^{(2)}\)\}, respectively. When \(A_j, B_j, j = 1, 2\), are two dimensional vectors, we use short notation

\[ \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} : \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} := \begin{pmatrix} A_1 \cdot B_1 + A_1 \cdot B_2 \\ A_2 \cdot B_1 + A_2 \cdot B_2 \end{pmatrix}. \]

Then, it is easily seen that

\[ N \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} : \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} = N \left[ \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} : \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} \right]. \]

When \(x' = \varphi_i(x')\), let

\[ C^{bl}(\Lambda)(x') := \begin{pmatrix} c^{bl}(x', \lambda^{(1)} \circ \varphi_i(x')) \\ c^{bl}(x', \lambda^{(2)} \circ \varphi_i(x')) \end{pmatrix}, \quad C^{bl}(\Xi)(x') := \begin{pmatrix} c^{bl}(x', \xi^{(1)} \circ \varphi_i(x')) \\ c^{bl}(x', \xi^{(2)} \circ \varphi_i(x')) \end{pmatrix}. \]

Note that \(C^{bl}(\Lambda) = NC^{bl}(\Xi)\) holds by the linearity of \(c^{bl}(x', \lambda)\) w.r.t. \(\lambda\). Hence, in view of \(NTN = I\), we have

\[ \sum_{l, k=1}^2 c_{lk}(\frac{\partial u_x}{\partial \nu}) \cdot \lambda^{(k)} \lambda^{(l)} = \Lambda^T \left[ C^{bl}(\Lambda) : \begin{pmatrix} \lambda^{(1)}(\frac{\partial u_x}{\partial \nu}) \cdot \lambda^{(1)} \\ \lambda^{(2)}(\frac{\partial u_x}{\partial \nu}) \cdot \lambda^{(2)} \end{pmatrix} \right] = \Xi^T N \left[ C^{bl}(\Xi) : \begin{pmatrix} \lambda^{(1)}(\frac{\partial u_x}{\partial \nu}) \cdot \lambda^{(1)} \\ \lambda^{(2)}(\frac{\partial u_x}{\partial \nu}) \cdot \lambda^{(2)} \end{pmatrix} \right] = \Xi^T \left[ C^{bl}(\Xi) : \begin{pmatrix} \lambda^{(1)}(\frac{\partial u_x}{\partial \nu}) \cdot \lambda^{(1)} \\ \lambda^{(2)}(\frac{\partial u_x}{\partial \nu}) \cdot \lambda^{(2)} \end{pmatrix} \right]. \]
Here, the first component of the vector \[ C^{bl}(\Xi) : \left( \frac{\lambda^{(1)}}{\lambda^{(2)}} \right) \] is calculated as

\[
\left[ C^{bl}(\Xi) : \left( \frac{\lambda^{(1)}}{\lambda^{(2)}} \right) \right]_1 = c^{bl}(x', \xi) \cdot \left[ \left( n_1 \xi^{(1)} + n_2 \xi^{(2)} \right) \left( n_1 \frac{\partial u}{\partial \nu} \cdot \xi^{(1)} + n_2 \frac{\partial u}{\partial \nu} \cdot \xi^{(2)} \right) \right] + \left( n_1 d_{11} + n_2 d_{12} \right) \left( n_1 \frac{\partial u}{\partial \nu} \cdot \xi^{(1)} + n_2 \frac{\partial u}{\partial \nu} \cdot \xi^{(2)} \right)
\]

In the same way, one can check that the second component of \[ C^{bl}(\Xi) : \left( \frac{\lambda^{(1)}}{\lambda^{(2)}} \right) \] is equal to \[ d_{21} \frac{\partial u}{\partial \nu} \cdot \xi^{(1)} \] and \[ d_{22} \frac{\partial u}{\partial \nu} \cdot \xi^{(2)} \]. Thus \[ c^{bl} \frac{\partial u}{\partial \nu} \] and hence \[ \tilde{\eta}^c \] are irrespective of the choice of tangential vectors.

Next, in order to get the conclusion for \[ \eta^c \], it is enough to prove

\[
\sum_{i=1}^{2} \beta_i^{c,\lambda}(x') \left( \frac{\partial u}{\partial \nu} \right)^{(i)} \cdot \xi^{(i)} = \sum_{i=1}^{2} \beta_i^{c,\lambda}(x') \left( \frac{\partial u}{\partial \nu} \right)^{(i)} \cdot \xi^{(i)}
\]

for all \[ i = 1, \ldots, N \] in view of the construction of \[ \eta^c \] (see \[ (3.59), (3.58), (3.52), (3.50) \] and \[ (3.39) \]). This equality follows directly by inserting \[ \lambda^{(1)} = n_1 \xi^{(1)} + n_2 \xi^{(2)}, \lambda^{(2)} = -n_2 \xi^{(1)} + n_1 \xi^{(2)} \] in view of the linearity of \[ \beta_i^{c,\lambda} \] w.r.t. \[ \lambda \] and \[ n_1^2 + n_2^2 = 1 \].

Thus, the proof of the lemma is complete. \[ \square \]

**Lemma 3.15** Assume for \( q \) the same as in Lemma 3.13 Then, for the function \( \tilde{\eta}^c \) defined by 3.64 it holds

\[
\| \nabla \tilde{\eta}^c \|_{L^2(\Omega^c)} \lesssim \varepsilon \| u \|_{W^{2,q}(\Omega)}.
\]

**Proof.** For a vector \( v \in \mathbb{R}^3 \), let \( v_j \) denote the \( j \)-th component of \( v \). For any \( \varphi \in \mathcal{D}(\Omega^c), j = 1 \sim 3, \) using integration by parts we have

\[
\langle \tilde{\eta}^c_j, \varphi \rangle_{\mathcal{D}'(\Omega^c)} = - \langle \eta^c_j, \text{div} \varphi \rangle_{L^2(\Omega^c)}
\]

\[
= - \varepsilon (\eta^c_j, \text{div} \varphi)_{L^2(\Omega^c)} - \varepsilon \left( \tilde{\Psi}_j^{bl} \frac{\partial u}{\partial \nu}, \text{div} \varphi \right)_{L^2(\Omega^c \setminus \Gamma)}
\]

\[
= \varepsilon (\nabla \eta^c_j, \varphi)_{L^2(\Omega^c)} - \varepsilon \int_{\partial \Omega} \eta^c_j \varphi \, dx + \varepsilon (\tilde{\Psi}_j^{bl} \frac{\partial u}{\partial \nu})_{j} \varphi \|_{L^2(\Omega^c \setminus \Gamma)}
\]

Here, in view of \( \varphi|_{\partial \Omega^c} = 0 \) and boundary condition in 3.61,

\[
- \int_{\partial (\Omega^c \setminus \Omega)} \left( \tilde{\Psi}_j^{bl} \frac{\partial u}{\partial \nu} \right)_j \varphi \, dx = \int_{\partial \Omega} \left( \Psi_j^{bl} \frac{\partial u}{\partial \nu} \right)_j \varphi \, dx = \int_{\partial \Omega} \left( \eta^c_j \right)_j \varphi \, dx = \int_{\partial \Omega} \eta^c_j \varphi \, dx.
\]

Hence, by 3.63, 3.64 we have

\[
\langle \nabla \tilde{\eta}^c_j, \varphi \rangle_{\mathcal{D}'(\Omega^c)} = \varepsilon \left( \varepsilon \| \nabla \eta^c_j \|_{L^2(\Omega^c)} + \| \nabla \left( \tilde{\Psi}_j^{bl} \frac{\partial u}{\partial \nu} \right) \|_{L^2(\Omega^c \setminus \Gamma)} \right)
\]

\[
\leq \varepsilon (\| \nabla \eta^c_j \|_{L^2(\Omega^c)} + \| \nabla \left( \tilde{\Psi}_j^{bl} \frac{\partial u}{\partial \nu} \right) \|_{L^2(\Omega^c \setminus \Gamma)}) \| \varphi \|_{L^2(\Omega)}
\]

\[
\lesssim \varepsilon \| u \|_{W^{2,q}(\Omega)}
\]

for all \( j = 1 \sim 3 \), yielding

\[
\nabla \tilde{\eta}^c \in L^2(\Omega^c), \quad \| \nabla \tilde{\eta}^c \|_{L^2(\Omega^c)} \lesssim \varepsilon \| u \|_{W^{2,q}(\Omega)}
\]

by denseness argument. Thus the lemma is proved. \[ \square \]
Lemma 3.16 Let \( r \geq 2, r' = \frac{r-1}{r-1} \) and let \( |g(x', x_3)| \preceq e^{\alpha x_3/\varepsilon} \) in the curvilinear coordinate system \((x', x_3)\) in \(\Gamma_3^\varepsilon\). Then, there holds the following:

(i) If \( h \in W^{1,2}(\Omega') \), \( h|_{\partial \Omega' \cap \Gamma_3^\varepsilon} = 0 \), then

\[
\|gh\|_{L^{r'}(\Gamma_3^\varepsilon)} \preceq \varepsilon^{3/2-1/r}\|h\|_{W^{1,2}(\Gamma_3^\varepsilon)}.
\]

(ii) If \( h \in H^2(\Omega') \cap H_0^1(\Omega') \) where \( \Omega' = \{ x \in \Omega : d(x, \Gamma) > M\varepsilon \} \), then

\[
\|gh\|_{L^{r'}(\Gamma_3^\varepsilon \cap \Omega')} \preceq \varepsilon^{2-1/r}\|h\|_{H^2(\Gamma_3^\varepsilon \cap \Omega')},
\]
\[
\|g\nabla h\|_{L^{r'}(\Gamma_3^\varepsilon \cap \Omega')} \preceq \varepsilon^{1-1/r}\|h\|_{H^2(\Gamma_3^\varepsilon \cap \Omega')}.
\]

Proof. - Proof of (i):

By (74) of [29], one has

\[
\int_{\Gamma'} |h(x', x_3)|^{r'} \, dx' \preceq (M\varepsilon - x_3)^{r'/2}\|\nabla h\|_{L^2(\Gamma_3^\varepsilon)}^{r'/2}
\]

leading to

\[
\left( \int_{\Gamma_3^\varepsilon} |gh(x', x_3)|^{r'} \, dx' \, dx_3 \right)^{1/r'} \preceq \left( \int_{-\delta}^{M\varepsilon} e^{\alpha r\varepsilon/\varepsilon} \left( \int_{\Gamma'} |h(x', x_3)|^{r'} \, dx' \, dx_3 \right)^{1/r'} \right)^{1/r'}
\]
\[
\preceq \left( \int_{-\delta}^{M\varepsilon} e^{\alpha r\varepsilon/\varepsilon} (M\varepsilon - x_3)^{r'/2} \, dx_3 \right)^{1/r'} \|\nabla h\|_{L^2(\Gamma_3^\varepsilon)}.
\]

Here,

\[
\int_{-\delta}^{M\varepsilon} e^{\alpha r\varepsilon/\varepsilon} (M\varepsilon - x_3)^{r'/2} \, dx_3 \preceq e^{r'/2+1} \int_{0}^{\infty} e^{-\alpha r' y} y^{r'/2} \, dy,
\]

which completes the proof of (i).

- Proof of (ii):

See page 498. of [29]. \(\square\)

Lemma 3.17 Let \( h \in L^r(\Gamma_3^\varepsilon), r \geq 1 \), satisfy \( |h(x)| \preceq e^{-\alpha d(x, \Gamma)/\varepsilon}, x \in \Gamma_3^\varepsilon \). Then,

\[
\|h\|_{L^r(\Gamma_3^\varepsilon)} \preceq \varepsilon^{1/r}.
\]

Lemma 3.18 Assume for \( q \) the same as in Lemma 3.13 Then, for function \( \eta^r \) defined by (3.59), it holds

\[
(\nabla \eta^r, \nabla \varphi)_{\Omega'} = -\int_{\Gamma'} \Psi \left( \frac{\partial u}{\partial \nu} - p\nu \right) \cdot \varphi \, ds + O(\varepsilon)(\|u\|_{W^{2,q}(\Omega)} + \|p\|_{W^{1,q}(\Omega)})\|\nabla \varphi\|_{L^2(\Omega')} \tag{3.65}
\]

for all \( \varphi \in H^1_{0,q}(\Omega') \).

Proof. Given any \( \varphi \in H^1_{0,q}(\Omega') \), one has

\[
(\nabla \eta^r, \nabla \varphi)_{\Omega'} = \varepsilon \sum_{i=1}^{3} \int_{\Gamma_3^\varepsilon} \nabla \beta^r \cdot \nabla \varphi dx
\]
\[
= \varepsilon \sum_{i=1}^{3} \int_{\Gamma_3^\varepsilon} \chi_i \nabla \beta^r \cdot \nabla \varphi dx + \varepsilon \sum_{i=1}^{3} \int_{\Gamma_3^\varepsilon} \beta^r \cdot (\nabla \chi_i \cdot \nabla \varphi) dx \tag{3.66}
\]
\[
= (i) + (ii),
\]

where (ii) is estimated by

\[
|(ii)| \preceq \varepsilon \sup_{1 \leq i \leq 3} \|\beta^r \nabla \chi_i\|_2 \|\nabla \varphi\|_{L^2(\Omega')}
\]
\[
\preceq \varepsilon \sup_{1 \leq i \leq 3} \|\beta^r\|_{L^\infty(\Gamma_3^\varepsilon)} \|\nabla \chi_i\|_2 \|\nabla \varphi\|_{L^2(\Omega')}
\]
\[
\preceq \varepsilon (\|u\|_{W^{2,q}(\Omega)} + \|p\|_{W^{1,q}(\Omega)})\|\nabla \varphi\|_{L^2(\Omega')} \tag{3.67}
\]

for all \( \varphi \in H^1_{0,q}(\Omega') \).
using (3.60) and (3.11). On the other hand, we have

\[ (i) = \varepsilon \sum_{i=1}^{3} \int_{\Gamma} \chi_{i} \nabla \beta^{\varepsilon,l} \cdot \varphi \, d\Gamma - \varepsilon \sum_{i=1}^{3} \int_{\Gamma} \chi_{i} \Delta \beta^{\varepsilon,l} \cdot \varphi \, d\Gamma \]

\[ = -\varepsilon \sum_{i=1}^{3} \left( \int_{\Gamma} \chi_{i} \left[ \frac{\partial \beta^{\varepsilon,l}}{\partial n} \right] \cdot \varphi \, d\Gamma + \int_{\Gamma} \chi_{i} \Delta \beta^{\varepsilon,l} \cdot \varphi \, d\Gamma \right) \]

where

\[ \varepsilon \int_{\Gamma} \nabla \beta^{\varepsilon,l} \cdot \nabla \chi_{l} \varphi \, d\Gamma \leq \| \nabla \chi_{l} \|_{L^{2}(\Gamma)} \| \varepsilon \nabla \beta^{\varepsilon,l} \varphi \|_{L^{2}(\Gamma)} \]

\[ \leq \| \nabla \chi_{l} \|_{L^{2}(\Gamma)} \| \varepsilon \nabla \beta^{\varepsilon,l} \varphi \|_{L^{2}(\Gamma)} + \| \varepsilon \nabla \beta^{\varepsilon,l} \|_{L^{2}(\Gamma)} \| \varphi \|_{L^{2}(\Gamma)} \]  

(3.69)

by Lemma 3.16 (i), (3.60), Lemma 3.12 (i) and Poincaré’s inequality. On the other hand, by Lemma 3.12 (iv) and Poincaré’s inequality. The proof of the lemma is complete.

Here we used that

\[ \frac{\partial u}{\partial \nu} \cdot \varphi = \frac{\partial u}{\partial \nu} - \frac{\partial u}{\partial \nu} \nu \cdot u \cdot \varphi = \frac{\partial u}{\partial \nu} \cdot \varphi \]

since \( \frac{\partial u}{\partial \nu} |_{\Gamma^{\prime}} = 0 \) in view of \( \text{div} \, u = 0, u \mid_{\Gamma^{\prime}} = 0 \). The last two terms in the right-hand side of (3.70) are shown to be equal to

\[ O(\varepsilon)(\| u \|_{W^{2,q}(\Omega)} + ||p||_{W^{1,q}(\Omega)}) \| \nabla \varphi \|_{L^{2}(\Omega')} \]

by Lemma 3.12 (i), (ii), Lemma 3.16 (i) and (3.60).

Let \( \Gamma \setminus \Gamma' \) be nontrivial (\( q > 3 \) in this case). Since the width of two-dimensional annular disc \( \Gamma \setminus \Gamma' \) is \( O(\varepsilon) \) and \( \varphi = 0 \) on its outer boundary \( \Gamma \cap \Gamma_1 \), we get by Poincaré’s inequality \( \| \varphi \|_{L^{2}(\Gamma \setminus \Gamma')} \lesssim \varepsilon \| \nabla \varphi \|_{L^{2}(\Omega')} \). Then, by complex interpolation we get that

\[ \| \varphi \|_{L^{2}(\Gamma \setminus \Gamma')} \lesssim \varepsilon^{1/2} \| \varphi \|_{H^{1/2}(\Gamma \setminus \Gamma')} \lesssim \varepsilon^{1/2} \| \nabla \varphi \|_{L^{2}(\Omega')} \]  

(3.71)

Consequently, by Sobolev embedding theorem in view of \( q > 3 \) we have

\[ \left| \int_{\Gamma \setminus \Gamma'} \nabla \varphi \right| \leq \varepsilon \| \nabla \varphi \|_{L^{2}(\Omega')} \]

(3.72)

The proof of the lemma is complete.

\[ \square \]

Lemma 3.19 Assume for \( q \) as in Lemma 3.13. Then the inhomogeneous boundary value problem

\[ -\Delta w^{\varepsilon} + \nabla r^{\varepsilon} = 0 \quad \text{in} \quad \Omega^{\varepsilon}, \]

\[ \text{div} \, w^{\varepsilon} = -\text{div} \,( \eta^{\varepsilon} + \eta^{\varepsilon}) \quad \text{in} \quad \Omega^{\varepsilon}, \]

\[ w^{\varepsilon} = 0 \quad \text{on} \quad \partial \Omega^{\varepsilon}, \]

has a unique weak solution \( \{ w^{\varepsilon}, r^{\varepsilon} \} \in H^{1}_{0}(\Omega^{\varepsilon}) \times L^{2}_{(m)}(\Omega^{\varepsilon}) \) such that

\[ \| \nabla w^{\varepsilon} \|_{L^{2}(\Omega')} + ||r^{\varepsilon}||_{L^{2}(\Omega')} \lesssim \varepsilon \| u \|_{W^{2,q}(\Omega)} + ||p||_{W^{1,q}(\Omega)} \]  

(3.73)
Proof. First of all, we remark that for every \( g \in L^2_{(m)}(\Omega^\varepsilon) \) the divergence problem

\[
\text{div } \psi = g \quad \text{in } \Omega^\varepsilon, \quad \psi|_{\partial \Omega^\varepsilon} = 0,
\]

has a solution \( \psi \in W^{1,2}_0(\Omega^\varepsilon) \) satisfying the estimate

\[
\| \psi \|_{W^{1,2}_0(\Omega^\varepsilon)} \leq C\| g \|_{L^2(\Omega^\varepsilon)},
\]

where the constant \( C \) is independent of \( \varepsilon \). This fact follows by Appendix. Lemma A.1 and Lemma A.2 using the assumption \((2.2)\) on \( \Omega^\varepsilon \) that \( \Omega^\varepsilon \) can be expressed by sum of several rough domains \( G^{(j)}_{\varepsilon} \) where \( G^{(j)}_{\varepsilon}, j = 1, \ldots, m \), is again a sum of one “main” macroscopic star-shaped domain and many microscopic \( \mathcal{O}(\varepsilon) \)-size star-shaped domains, i.e.,

\[
G^{(j)}_{\varepsilon} = G^{(j)}_{0} \cup \bigcup_{k=1}^{m_j} G^{(j)}_{k}, \quad G^{(j)}_{0} \cap G^{(j)}_{k} \neq \emptyset, G^{(j)}_{k} \cap G^{(j)}_{l} = \emptyset, k \neq l, k, l = 1, \ldots, m_j.
\]

Let \( \psi \in W^{1,2}_0(\Omega^\varepsilon) \) be such that \( \text{div } \psi = -\text{div } (\tilde{\eta}^\varepsilon + \eta^\varepsilon) \) and \( \| \psi \|_{W^{1,2}_0(\Omega^\varepsilon)} \lesssim \| \text{div } (\tilde{\eta}^\varepsilon + \eta^\varepsilon) \|_{L^2(\Omega^\varepsilon)} \). Then, it is standard to show the existence of unique weak solution \( \{w^\varepsilon, r^\varepsilon\} \) to the problem \((3.72)\) such that

\[
\| \nabla w^\varepsilon \|_{L^2(\Omega^\varepsilon)} + \| r^\varepsilon \|_{L^2(\Omega^\varepsilon)} \lesssim \| \nabla \psi \|_{L^2(\Omega^\varepsilon)} \lesssim \| \text{div } (\tilde{\eta}^\varepsilon + \eta^\varepsilon) \|_{L^2(\Omega^\varepsilon)}.
\]

By the way, we get

\[
\| \text{div } \tilde{\eta}^\varepsilon \|_{L^2(\Omega^\varepsilon)} \leq \varepsilon \| \tilde{\psi} \|_{H^1(\Omega^\varepsilon \setminus \Gamma_1)} \lesssim \varepsilon (\| u \|_{W^{2,q}(\Omega^\varepsilon)} + \| p \|_{W^{1,q}(\Omega^\varepsilon)})
\]

from \((3.51), (3.60)\). Moreover, we have

\[
\| \text{div } \eta^\varepsilon \|_{L^2(\Omega^\varepsilon)} \leq \varepsilon \sum_{i=1}^{3} (\| \text{div } \beta^\varepsilon \cdot X^i \|_{L^2(\Gamma_1)} + \| \beta^\varepsilon \cdot X^i \|_{L^2(\Gamma_1^\varepsilon)})
\]

\[
\leq \varepsilon \sum_{i=1}^{3} (\| \text{div } \beta^\varepsilon \|_{L^\infty(\Gamma_1^\varepsilon \setminus \Gamma_0)} + \| \beta^\varepsilon \|_{L^2(\Gamma_1^\varepsilon \setminus \Gamma_0)} + \| \beta^\varepsilon \|_{L^\infty(\Gamma_1^\varepsilon \setminus \Gamma_0)} + \| \beta^\varepsilon \|_{L^2(\Gamma_1^\varepsilon \setminus \Gamma_0)}) (\| u \|_{W^{2,q}(\Omega^\varepsilon)} + \| p \|_{W^{1,q}(\Omega^\varepsilon)})
\]

\[
\lesssim \varepsilon \| u \|_{W^{2,q}(\Omega^\varepsilon)} + \| p \|_{W^{1,q}(\Omega^\varepsilon)}
\]

from \((3.59), (3.60)\), Lemma \((3.12)\), and \((3.60)\).

Thus, the proof comes to end.

Now we can prove the following theorem on the error estimates of first order approximation for \( u^\varepsilon \).

Theorem 3.20 Assume for \( q \) the same as in Lemma \((3.13)\). Let \( f \in L^q(\Omega^\varepsilon), \psi \in W^{2-1/q,q}(\partial \Omega^\varepsilon) \), \text{supp } \psi \subset \Gamma_1 \) and let \( u \in W^{2,q}(\Omega_0) \) be the solution to \((3.3)\). Then, the estimates

\[
\| \nabla (u^\varepsilon - (\tilde{u} + \tilde{\eta}^\varepsilon + \eta^\varepsilon)) \|_{L^2(\Omega^\varepsilon)} \lesssim \varepsilon (\| f \|_{L^q(\Omega^\varepsilon)} + \| \psi \|_{W^{2-1/q,q}(\Gamma_1)})
\]

and

\[
\| u^\varepsilon - (\tilde{u} + \tilde{\eta}^\varepsilon + \eta^\varepsilon) \|_{L^2(\Omega^\varepsilon)} \lesssim \varepsilon^{3/2} (\| f \|_{L^q(\Omega^\varepsilon)} + \| \psi \|_{W^{2-1/q,q}(\Gamma_1)})
\]

hold true.

Proof. Let

\[
v^\varepsilon := u^\varepsilon - (\tilde{u} + \tilde{\eta}^\varepsilon + \eta^\varepsilon + w^\varepsilon),
\]

where \( w^\varepsilon \) is the solution to the system \((3.72)\) and \( \tilde{\eta}^\varepsilon, \eta^\varepsilon \) are defined by \((3.61), (3.59)\), respectively. Then, we have \( v^\varepsilon \in H^1_{1,\sigma}(\Omega^\varepsilon) \) since \( \text{div } v^\varepsilon = 0 \) in \( \Omega^\varepsilon \) and \( v^\varepsilon = 0 \) on \( \partial \Omega^\varepsilon \).

For any \( \varphi \in H^1_{1,\sigma}(\Omega^\varepsilon) \) we have

\[
(\nabla v^\varepsilon, \nabla \varphi)_{\Omega^\varepsilon} = (\nabla (u^\varepsilon - (\tilde{u} + \tilde{\eta}^\varepsilon + \eta^\varepsilon + w^\varepsilon)), \nabla \varphi)_{\Omega^\varepsilon}
\]

\[
= -\int_{\Gamma_1^\varepsilon} \frac{\partial v^\varepsilon}{\partial n} \cdot \varphi \ ds - (\nabla \tilde{\eta}^\varepsilon, \nabla \varphi)_{\Omega^\varepsilon} - (\nabla \eta^\varepsilon, \nabla \varphi)_{\Omega^\varepsilon}
\]

\[
- (\nabla w^\varepsilon, \nabla \varphi)_{\Omega^\varepsilon} + (f, \varphi)_{\Omega^\varepsilon} + (\nabla u, \nabla \varphi)_{\Omega_0 \setminus \Omega}.
\]

\[
(\nabla v^\varepsilon, \nabla \varphi)_{\Omega^\varepsilon} = -\int_{\Gamma_1^\varepsilon} \frac{\partial v^\varepsilon}{\partial n} \cdot \varphi \ ds - (\nabla \tilde{\eta}^\varepsilon, \nabla \varphi)_{\Omega^\varepsilon} - (\nabla \eta^\varepsilon, \nabla \varphi)_{\Omega^\varepsilon}
\]

\[
- (\nabla w^\varepsilon, \nabla \varphi)_{\Omega^\varepsilon} + (f, \varphi)_{\Omega^\varepsilon} + (\nabla u, \nabla \varphi)_{\Omega_0 \setminus \Omega}.
\]
By Lemma 3.18, Lemma 3.15, (3.73) and (3.74), the sum of the first four terms in the right-hand side of (3.78) is equal to

$$
\int_{\Gamma} (\Psi - 1)(\frac{\partial u}{\partial \nu} - p\nu) \cdot \varphi \, dx + O(\varepsilon)(\|f\|_{L^q(\Omega')} + \|\psi\|_{W^{2-1/q,q}(\Gamma_1)})\|\nabla \varphi\|_{L^2(\Omega')}.
$$

Moreover, by Poincaré's inequality the fifth term in the right-hand side of (3.78) is estimated by

$$
|\langle f, \varphi \rangle_{\Omega'\setminus\Omega}| \leq \|f\|_{L^2(\Omega'\setminus\Omega)}\|\varphi\|_{L^2(\Omega'\setminus\Omega)} \lesssim \varepsilon \|f\|_{L^2(\Omega'\setminus\Omega)}\|\nabla \varphi\|_{L^2(\Omega')},
$$

and the sixth term by

$$
\|\nabla u\|_{L^2(\Omega'\setminus\Omega)} \leq \|\nabla u\|_{L^q(\Omega'\setminus\Omega)}\Omega' \setminus \Omega)^{1/3} \lesssim \varepsilon \|u\|_{W^{2-1/q,q}(\Omega)},
$$

$$
\lesssim \varepsilon (\|f\|_{L^2(\Omega')} + \|\psi\|_{W^{2-1/q,q}(\Gamma_1)}).
$$

using Sobolev embedding theorem and $|\Omega' \setminus \Omega| \leq O(\varepsilon^3)$. Therefore, if we prove

$$
\int_{\Gamma} (\Psi - 1)(\frac{\partial u}{\partial \nu} - p\nu) \cdot \varphi \, dx' \leq \varepsilon (\|f\|_{L^q(\Omega')} + \|\psi\|_{W^{2-1/q,q}(\Gamma_1)}),
$$

(3.79)

then we get

$$
|\langle \nabla v^\varepsilon, \nabla \varphi \rangle_{\Omega'}| \lesssim \varepsilon (\|f\|_{L^q(\Omega')} + \|\psi\|_{W^{2-1/q,q}(\Gamma_1)})\|\nabla \varphi\|_{L^2(\Omega')}, \quad \forall \varphi \in H^1_{0,\sigma}(\Omega'),
$$

and hence (3.75) by Lemma 3.19. We need to consider only the case where $\Gamma$ and $\Gamma_1$ are adjacent. Note that $q > 3$ in this case. It follows from the construction of $\Psi$ and (3.71) that

$$
\left| \int_{\Gamma} (\Psi - 1)(\frac{\partial u}{\partial \nu} - p\nu) \cdot \varphi \, dx' \right| \lesssim \|\frac{\partial u}{\partial \nu} - p\nu\|_{L^2(\Gamma' \setminus \Gamma')} \|\varphi\|_{L^2(\Gamma' \setminus \Gamma')} \lesssim \varepsilon^{1/2}\|\frac{\partial u}{\partial \nu} - p\nu\|_{L^2(\Gamma' \setminus \Gamma')} \|\nabla \varphi\|_{L^2(\Omega')},
$$

where

$$
\|\frac{\partial u}{\partial \nu} - p\nu\|_{L^2(\Gamma' \setminus \Gamma')} \leq \|\frac{\partial u}{\partial \nu} - p\nu\|_{L^\infty(\Gamma')}\Gamma' \setminus \Gamma'^{1/2} \lesssim \varepsilon^{1/2}(\|u\|_{W^{2,q}(\Omega)} + \|p\|_{W^{1,q}(\Omega)}).
$$

Hence, we get (3.79) due to (3.4) and (3.75) is proved.

Next, in order to prove (3.76), we use the idea of [29]. Let

$$
z : = u^\varepsilon - (\bar{u} + \bar{\eta}^\varepsilon + \eta^\varepsilon), \quad \Omega' := \{x \in \Omega : \rho(x) > M\varepsilon\}.
$$

Then, by Poincaré’s inequality and already proved (3.75) we have

$$
\|z\|_{L^2(\Omega' \setminus \Omega)} \lesssim \varepsilon \|\nabla z\|_{L^2(\Omega' \setminus \Omega)} \lesssim \varepsilon^2 (\|f\|_{L^q(\Omega')} + \|\psi\|_{W^{2-1/q,q}(\Gamma_1)}). \tag{3.80}
$$

Therefore, for the proof of (3.76) we only need to estimate $\|z\|_{L^2(\Omega')}$. Let $w \in H^2(\Omega') \cap H^1_{0,\sigma}(\Omega')$ be the unique solution to the Stokes problem

$$
-\Delta w + \nabla s = z \quad \text{in} \ \Omega',
$$

$$
div w = 0 \quad \text{in} \ \Omega',
$$

$$
w = 0 \quad \text{on} \ \partial \Omega'. \tag{3.81}
$$

Then, one has $\|w\|_{H^2(\Omega')} + \|s\|_{H^1(\Omega')} \lesssim \|z\|_{L^2(\Omega')}$ and

$$
\|z\|_{L^2(\Omega')}^2 = \int_{\Omega'} (-\Delta w + \nabla s) \cdot z \, dx = \int_{\Omega'} (\nabla w \cdot \nabla z - s \div z) \, dx - \int_{\partial \Omega'} (\frac{\partial w}{\partial \nu} - s) \cdot z \, ds. \tag{3.82}
$$
Note that the estimate (3.5) in Lemma 3.1 holds for $\Omega'$ as well. Hence, it follows from $z|_{\partial \Omega'} = 0$ and (3.75) that

$$\|z\|_{L^2(\partial \Omega')} \lesssim \varepsilon^{1/2}\|\nabla z\|_{L^2(\Omega' \setminus \Omega')} \lesssim \varepsilon^{3/2}(\|f\|_{L^q(\Omega')} + \|\psi\|_{W^{2-1/q,q}(\Gamma_1)}).$$

Therefore the second term in the right-hand side of (3.82) is estimated by

$$\left| \int_{\partial \Omega'} (\partial_w - sv) \cdot z \, ds \right| \lesssim \|\nabla w - s\|_{H^1(\Omega')} \|z\|_{L^2(\partial \Omega')} \lesssim \|z\|_{L^2(\Omega')} \|z\|_{L^2(\partial \Omega')} \lesssim \varepsilon^{3/2}\|z\|_{L^2(\partial \Omega')} (\|f\|_{L^q(\Omega')} + \|\psi\|_{W^{2-1/q,q}(\Gamma_1)}).$$

Let us get estimate of the first term in the right-hand side of (3.82). Obviously, we have

$$\int_{\Omega'} (\nabla w \cdot \nabla z - s \text{div } z) \, dx = - \int_{\Omega'} (\nabla w \cdot \nabla \eta^\varepsilon - s \text{div } \eta^\varepsilon) \, dx.$$

where

$$\int_{\Omega'} \nabla w \cdot \nabla \eta^\varepsilon \, dx = \varepsilon \sum_{l=1}^3 \int_{\Omega'} \nabla w \cdot \nabla (\beta^{\varepsilon,l} \chi_l) \, dx$$

$$= \varepsilon \sum_{l=1}^3 \int_{\Omega'} (\nabla w \cdot \nabla \chi_l) \cdot \beta^{\varepsilon,l} \, dx + \nabla w \cdot \nabla \beta^{\varepsilon,l} \chi_l) \, dx$$

$$= \sum_{l=1}^3 \int_{\Omega'} \nabla w \cdot \nabla \chi_l) \cdot \beta^{\varepsilon,l} \, dx$$

$$+ \sum_{l=1}^3 \int_{\Omega'} \nabla w \cdot \nabla \chi_l) \cdot \beta^{\varepsilon,l} \, dx$$

$$(3.83)$$

By Lemma 3.12, $\beta^{\varepsilon,l} - \varepsilon \Delta \beta^{\varepsilon,l} + \beta^{\varepsilon,l} \chi_l \cdot \nabla \beta^{\varepsilon,l} \, dx + \int_{\Omega'} \nabla (w \cdot (\varepsilon \beta^{\varepsilon,l} \chi_l) \, dx)$$

By Lemma 3.12 (ii), (3.60) and (3.4) we have

$$\|\nabla w \cdot \nabla \eta^\varepsilon \| \lesssim \varepsilon^{3/2} \sum_{l=1}^3 (\|\chi_l\|_{L^2} + \|\chi_l\|_{L^2(\Omega')}) \|w\|_{H^2(\Omega')}$$

$$\lesssim \varepsilon^{3/2}(\|u\|_{W^{2,q}(\Omega')} + \|p\|_{W^{1,q}(\Omega')}) \|z\|_{L^2(\Omega')}$$

$$\lesssim \varepsilon^{3/2}(\|f\|_{L^q(\Omega')} + \|\psi\|_{W^{2-1/q,q}(\Gamma_1)}) \|z\|_{L^2(\Omega')}.$$ 

In the same way, using decay estimate of $\text{div } \beta^{\varepsilon,l}$, $l = 1 \sim 3$, given by Lemma 3.12 (iii), the integral of $\text{div } \eta^\varepsilon$ can be estimated with the same order of $O(\varepsilon^{3/2})$.

Thus we have

$$\|z\|_{L^2(\Omega')} \lesssim \varepsilon^{3/2}(\|f\|_{L^q(\Omega')} + \|\psi\|_{W^{2-1/q,q}(\Gamma_1)}),$$

and hence (3.76).

Now, let us construct an effective Navier wall-law for the Stokes system (1.1) as follows:

$$-\Delta u^{\text{eff}} + \nabla p^{\text{eff}} = f \quad \text{in } \Omega,$$

$$\text{div } u^{\text{eff}} = 0 \quad \text{in } \Omega,$$

$$u^{\text{eff}} = \varepsilon \Psi(c^{bl}(x')) \frac{\partial u^{\text{eff}}}{\partial \nu} \quad \text{on } \Gamma,$$

$$u^{\text{eff}} = 0 \quad \text{on } \Gamma,$$

$$u^{\text{eff}} = \psi \quad \text{on } \Gamma_1.$$

(3.84)

Remark 3.21 The Navier wall-law of (3.84) is irrespective of the choice of curvilinear systems of orthogonal tangential vectors on $\Gamma$ due to Lemma 3.14.

Since the matrix $c^{bl}(x')$ is negatively definite and $\Psi(x') \geq 0$ for all $x' \in \Gamma$, the problem (3.84) is well-posed and has a weak solution $u^{\text{eff}} \in H^1(\Omega)$ by Lax-Milgram’s lemma.
Theorem 3.22 Assume for \( q \) the same as in Lemma 3.13. Let \( u \) be the solution to (3.3), and let \( \tilde{\eta} \) be defined by (3.64). Then,

\[
\| u^{\varepsilon ff} - u - \tilde{\eta} \|_{H^1(\Omega)} \lesssim \varepsilon (\| f \|_{L^2(\Omega)} + \| \psi \|_{W^{2-1/q,q}(\Gamma_1)}) ,
\]

\[
\| u^{\varepsilon ff} - u - \tilde{\eta} \|_{L^2(\Omega)} \lesssim \varepsilon^{3/2} (\| f \|_{L^3(\Omega')} + \| \psi \|_{W^{2-1/q,q}(\Gamma_1)}).
\]

Proof. Let \( v := u^{\varepsilon ff} - u - \tilde{\eta} \). Then, \( v \) solves the system

\[
\begin{align*}
-\Delta v + \nabla s &= 0 & x & \in \Omega, \\
\text{div} v &= 0 & x & \in \Omega, \\
v_\tau &= \phi & x' & \in \Gamma, \\
v_\nu &= -u_\nu \chi_{\Gamma \setminus \Gamma'} & x' & \in \Gamma, \\
v &= 0 & x' & \in \Gamma_1,
\end{align*}
\]

(3.85)

where \( \phi = \varepsilon \Psi(x') e^{bl}(x')(\partial v + \partial_\nu s) - u_\nu \chi_{\Gamma \setminus \Gamma'} \), \( \chi_{\Gamma \setminus \Gamma'} \) is the characteristic function of \( \Gamma \setminus \Gamma' \).

For the associate pressure \( s \) we may assume without loss of generality that \( s \in L^2_{(m)}(\Omega) \). Then,

\[
\| s \|_{L^2_{(m)}(\Omega)} \lesssim \| \nabla v \|_{L^2(\Omega)}.
\]

In fact, given any \( h \in L^2_{(m)}(\Omega) \) there is some \( \varphi \in H^1_0(\Omega) \) such that \( \text{div} \varphi = h \), \( \| \varphi \|_{H^1_0(\Omega)} \leq c(\| h \|_{L^2_{(m)}(\Omega)} \). Hence,

\[
(s, h)_\Omega = (s, \text{div} \varphi)_\Omega = (\nabla s, \nabla \varphi)_\Omega
\]

and \( |(s, h)| \leq \| \nabla v \|_2 \| \nabla \varphi \|_2 \lesssim \| \nabla v \|_2 \| h \|_{L^2_{(m)}(\Omega)} \) implying \( \| s \|_{L^2_{(m)}(\Omega)} \lesssim \| \nabla v \|_2 \).

Since \( \nabla v, s \in L^2(\Omega) \) and \( \text{div} (\nabla v - sI) = 0 \), we get that

\[
\left( \frac{\partial v}{\partial \nu} - s \nu \right)_\Gamma \in H^{-\frac{1}{2}}(\Gamma), \quad \| \frac{\partial v}{\partial \nu} - s \nu \|_{H^{-\frac{1}{2}}(\Gamma)} \lesssim \| \nabla v - sI \|_{L^2(\Omega)} \lesssim \| \nabla v \|_{L^2(\Omega)},
\]

(3.86)

see [17], Ch. 3, Theorem 2.2; cf. also [32]. By the same reasoning, for the solution \( \{ \eta, \zeta \} \) to (3.61) we have

\[
\left( \frac{\partial \eta}{\partial \nu} - \zeta \nu \right)_\Gamma \in H^{-\frac{1}{2}}(\Gamma), \quad \| \frac{\partial \eta}{\partial \nu} - \zeta \nu \|_{H^{-\frac{1}{2}}(\Gamma)} \lesssim \| \nabla \eta - \zeta I \|_{L^2(\Omega)} \lesssim \| \nabla \eta \|_{L^2(\Omega)}.
\]

(3.87)

Since the matrix \( \Psi(x') e^{bl}(x') \) for any \( x' \in \Gamma \) is invertible, it follows from the boundary condition of (3.85) that

\[
\frac{\partial v}{\partial \nu} = (\varepsilon \Psi e^{bl})^{-1} (v_\tau + u_\tau \chi_{\Gamma \setminus \Gamma'}) - \frac{\partial \eta}{\partial \nu} \text{ on } \Gamma.
\]

Hence, by testing (3.85) with \( v \) in view of (3.86) and negativity of \( e^{bl} \), we have

\[
\| \nabla v \|_{L^2(\Omega)}^2 = \langle v, \frac{\partial v}{\partial \nu} - (s v) \rangle_{H^\frac{1}{2}(\Gamma), H^{-\frac{1}{2}}(\Gamma)} = \langle v_\tau, \frac{\partial v}{\partial \nu} - (s + \frac{\partial \eta}{\partial \nu}) \rangle_{H^\frac{1}{2}(\Gamma), H^{-\frac{1}{2}}(\Gamma)} = \int_\Gamma v_\tau \cdot (\varepsilon \Psi e^{bl})^{-1} (v_\tau + u_\tau \chi_{\Gamma \setminus \Gamma'}) dx' - \langle v_\tau, \frac{\partial \eta}{\partial \nu} \rangle_{H^\frac{1}{2}(\Gamma), H^{-\frac{1}{2}}(\Gamma)} \quad (3.88)
\]

\[
\leq \int_{\Gamma \setminus \Gamma'} v_\tau \cdot (\varepsilon \Psi e^{bl})^{-1} u_\tau dx' - \langle v_\tau, \frac{\partial \eta}{\partial \nu} \rangle_{H^\frac{1}{2}(\Gamma), H^{-\frac{1}{2}}(\Gamma)} \]

\[
= \int_{\Gamma \setminus \Gamma'} v_\tau \cdot (\varepsilon \Psi e^{bl})^{-1} u_\tau dx' - \langle v_\tau, \frac{\partial \eta}{\partial \nu} \rangle_{H^\frac{1}{2}(\Gamma), H^{-\frac{1}{2}}(\Gamma)}
\]

\[
= \int_{\Gamma \setminus \Gamma'} v_\tau \cdot (\varepsilon \Psi e^{bl})^{-1} u_\tau dx' - \langle v_\tau, \frac{\partial \eta}{\partial \nu} \rangle_{H^\frac{1}{2}(\Gamma), H^{-\frac{1}{2}}(\Gamma)}.
\]
Since $u$ vanishes at the boundary of $\Omega_0$ and the thickness of the annular disc $\Omega_0 \setminus \Omega$ is $O(\varepsilon^2)$, it follows that
\[
\|u\|_{L^2(\Gamma')} \lesssim \varepsilon \|\nabla u\|_{L^2(\Omega_0 \setminus \Omega)} \lesssim \varepsilon \|\nabla u\|_{L^6(\Omega_0 \setminus \Omega)} \|\Omega_0 \setminus \Omega\|^{1/3} \lesssim \varepsilon^2 \|\nabla^2 u\|_{L^2(\Omega_0)} \tag{3.89}
\]
and, by trace theorem,
\[
\|u\|_{H^{3/2}(\Gamma')} \lesssim \|\nabla^2 u\|_{L^2(\Omega_0)}.
\]
Therefore, it follows by complex interpolation $H^\theta(\Gamma \setminus \Gamma') = [L^2(\Gamma \setminus \Gamma'), H^{3/2}(\Gamma \setminus \Gamma')]_{\theta/2}$ for $1 \leq \theta \leq 3/2$ that
\[
\|u\|_{H^\theta(\Gamma \setminus \Gamma')} \lesssim \|u\|_{L^2(\Gamma')} \|u\|_{H^{3/2}(\Gamma \setminus \Gamma')} \lesssim \varepsilon^{2-\theta/3} \|\nabla^2 u\|_{L^2(\Omega_0)}.
\]
Consequently, the third term in the right-hand side of (3.88) is estimated using (3.88) as
\[
\left| (u_{\nu'}, \frac{\partial u_{\nu'}}{\partial \nu} - s + \frac{\partial u_{\nu'}}{\partial \nu} \cdot \nu) H^{\frac{1}{2}}(\Gamma_1 \setminus \Gamma_1'), H^{\frac{1}{2}}(\Gamma_1 \setminus \Gamma_1') \right| \lesssim \|u_{\nu'}\|_{H^{\frac{1}{2}}(\Gamma_1 \setminus \Gamma_1')} \left( \|\frac{\partial u_{\nu'}}{\partial \nu}\|_{H^{\frac{1}{2}}(\Gamma_1 \setminus \Gamma_1')} \right)
\lesssim \|u\|_{H^{\frac{1}{2}}(\Gamma_1 \setminus \Gamma_1')} \|\nabla u\|_{L^2(\Omega_0)} \lesssim \varepsilon \|u\|_{W^{2,2}(\Omega_0)} \|\nabla u\|_{L^2(\Omega_0)}.
\]
On the other hand, due to the construction of $\Psi$, the first term in the right-hand side of (3.88) is estimated as
\[
\left| \int_{\Gamma \setminus \Gamma'} v_{\tau'} \cdot (\varepsilon \Psi c^{bl})^{-1} u_{\tau'} \, dx' \right| \lesssim \int_{\Gamma \setminus \Gamma'} |d(x', \Gamma \cap \Gamma_1)^{-1} v_{\tau} \cdot u_{\tau'}| \, dx'
\tag{3.92}
\]
Note that
\[
\|x^{-\gamma}_h\|_{L^2(\mathbb{R}^2_+)} \leq c(\gamma) \|h\|_{H^\gamma(\mathbb{R}^d_+), \gamma \in [0, 1] \setminus \{\frac{1}{2}\}},
\]
by [24], Ch.1, Theorem 11.3. This inequality can be extended to the region where $\mathbb{R}^d_+$ is replaced by $\Gamma \setminus \Gamma'$ using diffeomorphism $\varphi_i$ between $U_i \subset \mathbb{R}^2$ and $V_i \subset \Gamma$ for $i = 1, \ldots, N$. In particular, for $\gamma \in [0, 1] \setminus \{\frac{1}{2}\}$
\[
\|d(x', \Gamma \cap \Gamma_1)^{-\gamma} z\|_{L^2(\Gamma')} \leq c(\gamma, \Gamma \setminus \Gamma') \|z\|_{H^{\gamma, \gamma^2}(\Gamma')} \quad \forall z \in H^{\gamma, \gamma^2}(\Gamma \setminus \Gamma'),
\]
where $H^{\gamma, \gamma^2}(\Gamma \setminus \Gamma')$ is the closure in $H^\gamma(\Gamma \setminus \Gamma')$ of the set of all smooth functions vanishing on $\Gamma_1 \setminus \Gamma_1$. This inequality together with (3.90) with $\theta = 3/4$ yields
\[
\int_{\Gamma \setminus \Gamma'} |d(x', \Gamma \cap \Gamma_1)^{-1} v_{\tau} \cdot u_{\tau'}| \, ds \leq \|d(x', \Gamma \cap \Gamma_1)^{-3/4} u_{\tau'}\|_{L^2(\Gamma \cap \Gamma')} \int_{\Gamma \setminus \Gamma'} |d(x', \Gamma \cap \Gamma_1)^{-1/4} v_{\tau} \|_{L^2(\Gamma \cap \Gamma')} \lesssim \|u_{\tau'}\|_{H^{3/4}(\Gamma_1 \setminus \Gamma_1')} \|v_{\tau}\|_{H^{1/4}(\Gamma')} \lesssim \varepsilon \|u\|_{W^{2,2}(\Omega_0)} \|\nabla^2 u\|_{L^2(\Omega_0)}.
\]
Therefore, from (3.82) it follows that
\[
\left| \int_{\Gamma \setminus \Gamma'} v_{\tau} \cdot (\varepsilon \Psi c^{bl})^{-1} u_{\tau} \, dx' \right| \lesssim \varepsilon \|u\|_{W^{2,2}(\Omega_0)} \|\nabla u\|_{L^2(\Omega)}.
\tag{3.93}
\]
Note that
\[
\langle v_{\tau}, \frac{\partial u_{\tau'}}{\partial \nu} \rangle_{H^{\frac{1}{2}}(\Gamma), H^{-\frac{1}{2}}(\Gamma)} = \langle \frac{\partial u_{\tau'}}{\partial \nu}, \eta_{\tau'} \rangle_{H^\frac{1}{2}(\Gamma), H^{-\frac{1}{2}}(\Gamma)} \quad \text{since } v \cdot \nu|_{\Gamma} = 0, \eta \cdot \nu|_{\Gamma} = 0.
\]
Therefore, by (3.88), (3.63) and (4.4) we get that
\[
\left| \langle v_{\tau}, \frac{\partial u_{\tau'}}{\partial \nu} \rangle_{H^{\frac{1}{2}}(\Gamma), H^{-\frac{1}{2}}(\Gamma)} \right| \leq \varepsilon \|\nabla u\|_{2} \|\eta\|_{H^1(\Omega)}
\lesssim \varepsilon \|\nabla^2 u\|_{L^2(\Omega')} + \|\psi\|_{W^{2-1/\alpha, \alpha}(\Gamma_1)}.
\tag{3.94}
\]
Thus, it follows from (3.63), (3.4) and (3.95) that
\[ \| \nabla v \|_2 \lesssim \varepsilon (\| f \|_{L^2(\Omega')} + \| \psi \|_{W^{2-1/q, q}(\Gamma_{1})}), \] (3.95)
which proves the first inequality of the theorem.

Next, let us prove the second inequality of the theorem. Notice that (3.71) holds for \( v \) as well.

Then, we get from the boundary condition on \( v \), uniform negativity of matrix \( \partial^b(x') \) with respect to \( x' \in \Gamma', \) and \( \varepsilon \)
invariance with respect to scaling transforms, and hence we have (3.97).

Therefore, by (3.63), (3.4) and (3.95) we have
\[ \| v \|_{L^2(\partial \Omega)} \lesssim \varepsilon^3 (\| f \|_{L^2(\Omega')} + \| \psi \|_{W^{2-1/q, q}(\Gamma_{1})})^2 + \varepsilon^2 \| \nabla v \|_{L^2(\Omega)} (\| f \|_{L^2(\Omega')} + \| \psi \|_{W^{2-1/q, q}(\Gamma_{1})}). \] (3.96)

Thus, if we prove
\[ \| v \|_{L^2(\Omega)} \lesssim \varepsilon \| v \|_{L^2(\partial \Omega)} + \varepsilon^{1/2} \| \nabla v \|_{L^2(\Omega)}, \] (3.97)
then, in view of (3.96) and the first inequality already proved, we have the second inequality of the theorem.

When \( \Gamma_1 \) is a component of \( \partial \Omega \), (3.97) is obvious from the property \( \| v \|_{L^2(\Omega)} \lesssim \| v \|_{L^2(\partial \Omega)} \) for a very weak solution to the Stokes system in \( \Omega \) of \( C^2 \)-class (see e.g. [13]). But, we can not claim \( \| v \|_{L^2(\Omega)} \lesssim \| v \|_{L^2(\partial \Omega)} \) when \( \Gamma_1 \) and \( \Gamma \) are adjacent, since \( \Omega \) is then \( C^{0,1} \)-domain. In that case, let us choose a smooth subdomain \( \Omega' \subset \Omega \) which is obtained by cutting off a very small tube \( \Omega \setminus \Omega' \) from \( \Omega \) such that
\[ \Omega \setminus \Omega' \subset \{ x \in \Omega : d(x, \Gamma \cap \Gamma_1) \leq \varepsilon^{1/2} \}. \]
Then, \( |\Omega \setminus \Omega'| \sim \varepsilon^3 \). Note that, the estimate constant \( C \) in the Sobolev inequality \( \| \phi \|_6 \leq C \| \nabla \phi \|_2 \) is invariant with respect to scaling transforms, and hence we have \( \| v \|_{L^6(\Omega \setminus \Omega')} \leq \varepsilon \| \nabla v \|_{L^2(\Omega \setminus \Omega')} \) with constant \( c \) independent of \( \varepsilon \). Moreover, notice the inequality \( \| v \|_{L^2(\Omega')} \leq (\text{constant}) \| v \|_{L^2(\partial \Omega')} \) known for very weak solutions to Stokes equations. Therefore, by Lemma (3.1) and (3.96) we get that
\[ \| v \|_{L^2(\Omega')} \lesssim \| v \|_{L^2(\partial \Omega)} + \| \nabla v \|_{L^2(\partial \Omega')} \]
using that \( \Omega \) is a Lipschitz domain. Consequently, we have
\[ \| v \|_{L^2(\Omega')} \lesssim \| v \|_{L^2(\partial \Omega \setminus \partial \Omega')} + \| \nabla v \|_{L^2(\Omega')} + \varepsilon \| \nabla v \|_{L^2(\Omega')} \]
and hence (3.97).

The proof of the theorem is complete. \( \Box \)
Lemma 3.23 Let $q > 3$. For $\eta^c$ defined by (3.39) there hold

$$
\| \eta^c \|_{L^2(\Omega)} \lesssim \varepsilon^{3/2}(\| f \|_{L^q(\Omega^c)} + \| \psi \|_{W^{2-1/q,q}(\Gamma_1)})
$$

and

$$
\| \nabla \eta^c \|_{L^1(\Omega)} \lesssim \varepsilon (\| f \|_{L^q(\Omega^c)} + \| \psi \|_{W^{2-1/q,q}(\Gamma_1)}).
$$

Proof. Due to embedding $W^{1,q}(\Omega) \subset L^\infty(\Omega)$, (3.60) and Lemma 3.17, we have

$$
\beta \varepsilon, i,l \leq \beta \varepsilon, l(\Omega) \| \chi_l \|_{L^\infty(\Omega)}, \| \beta \varepsilon, l(\Omega) \| \chi_l \|_{W^{1,q}(\Omega)} \lesssim \varepsilon^{1/2}(\| f \|_{L^q(\Omega^c)} + \| \psi \|_{W^{2-1/q,q}(\Gamma_1)}), l = 1 \sim 3.
$$

Hence, (3.98) is proved in view of the construction of $\eta^c$, see (3.59).

By Theorem 3.22 and Lemma 3.23 we get the following theorem showing the error estimate for the obtained wall-law (3.84).

Theorem 3.24 Let $f \in L^q(\Omega^c)$, $\psi \in W^{2-1/q,q}(\partial \Omega^c)$, $q > 3$, and let $u^c$ and $u^{eff}$ be the solutions to the systems (1.1) and (3.84), respectively. Then,

$$
\| u^c - u^{eff} \|_{L^2(\Omega)} \lesssim \varepsilon^{3/2}(\| f \|_{L^q(\Omega^c)} + \| \psi \|_{W^{2-1/q,q}(\Gamma_1)}),
$$

$$
\| \nabla (u^c - u^{eff}) \|_{L^1(\Omega)} \lesssim \varepsilon (\| f \|_{L^q(\Omega^c)} + \| \psi \|_{W^{2-1/q,q}(\Gamma_1)}).
$$

Remark 3.25 As seen above, the Navier-wall law derived in this work is independent of the choice of the orthogonal tangent vectors; it depends only on the geometry of the fictitious boundary $\Gamma$ and roughness shape since the matrix $c^{bl}$ in (3.54) is constructed using boundary layers near the rough surface, which are determined by the boundary layer cell problems ($BL_{\lambda, x'}$).

It will be shown in the forthcoming papers [30, 31] that the results of boundary layer analysis given in §3.2 are still fundamental for derivation of effective wall-laws for Navier-Stokes equations over curved rough boundaries as well as for fluid flows around rotating bodies. For these problems, $c^{bl}$, constructed in (3.54), will also be shown to be useful coefficient matrix to be involved in the effective wall-laws.

Remark 3.26 The result of the paper can be directly extended to the case of spacial dimension $n > 3$ without any nontrivial changes.
A Estimate for Divergence Problem $\text{div} \, u = f$

Divergence problem is one of the fundamental problems in the study of Navier-Stokes equations. In some references rigorous estimates for some solutions of the divergence problem is known, see e.g. \cite{17}, Ch.III, Section 3. Unfortunately, however, the results of \cite{17} do not guarantee that for our domain $\Omega^\varepsilon$ given by (2.2) the estimate constants for solutions to divergence equation do not depend on the microscopic size $\varepsilon$. Therefore, in this appendix, we give a refined analysis for the dependence of the estimate constant for solution to the divergence problem in some specific domains.

Lemma A.1 Let a simply connected and bounded domain $G$ of $\mathbb{R}^n, n \geq 2$, be expressed as

$$G = G_0 \cup \bigcup_{k=1}^m G_k, \quad G_0 \cap G_k \neq \emptyset, \quad G_k \cap G_l = \emptyset (k \neq l), \quad k, l = 1, \ldots, m,$$

where $G_k, k = 0, \ldots, m$, has cone-property and star-shaped with respect to some balls $B(x_k, R_k)$ of radius $R_k$, and $\frac{\text{diam}(G_0)}{R_0} + \frac{\text{diam}(G_k)}{R_k} + \frac{|G_k|}{|\partial G_k|} < l$ with some constant $l > 0$ for all $k \in \{1, \ldots, m\}$.

If $f \in L^q(G), 1 < q < \infty$, $\int_G f(x) \, dx = 0$, then the divergence problem

$$\text{div} \, u = f \quad \text{in} \, G, \quad u|_{\partial G} = 0,$$

(A.1)

has a solution $u \in W^{1,q}_0(G)$ satisfying

$$\|u\|_{W^{1,q}_0(G)} \leq C \|f\|_{L^q(G)}$$

(A.2)

with constant $C = C(n, q, l) > 0$ independent of $m$ and $\text{diam}(G_k), k = 0, \ldots, m$.

\textbf{Proof:} Since the existence for the problem (A.1) is already well-known, see e.g. \cite{17}, Ch.III, Theorem 3.1, we shall show that the constant $C$ in (A.2) is irrespective of $m$ and $\text{diam}(G_k), k = 0, \ldots, m$, and depends only on $n, q$ and $l$. For $k = 1, \ldots, m$, let us define $f_k$ on $G_k$ by

$$f_k(x) = \begin{cases} f(x) & \text{for } x \in G_k \setminus G_0, \\ f(x) - a_k & \text{for } x \in G_k \cap G_0, \end{cases}$$

where $a_k = \int_{G_k \setminus G_0} f(x) \, dx$, and let $f_0 := f - \sum_{k=1}^m f_k$. Obviously, $\text{supp} \, f_k \subset G_k$, $\int_{G_k} f_k \, dx = 0$ for all $k \in \{0, \ldots, m\}$ and, denoting the extension by 0 of $f_k$ to $G$ again by $f_k$, we have $f = \sum_{k=0}^m f_k$. Then, for $k = 1, \ldots, m$ using Hölder inequality and $(a + b)^q \leq c(q)(a^q + b^q)$ for $a, b \geq 0$ we get that

$$\int_{G_k} |f_k|^q \, dx = \int_{G_k \setminus G_0} |f(x)|^q \, dx + \int_{G_k \cap G_0} |f(x) - a_k|^q \, dx$$

$$\leq \int_{G_k \setminus G_0} |f(x)|^q \, dx + c(q) \int_{G_k} |f(x)|^q \, dx + |a_k|^q |G_k \cap G_0|$$

$$= c(q) \left( \int_{G_k} |f(x)|^q \, dx + \int_{G_k} f(x)^q \, dx \right) |G_k \cap G_0|^{1-q}$$

$$\leq c(q)(1 + l^{q-1}) \int_{G_k} |f(x)|^q \, dx,$$

(A.3)

Using (A.3) we get that

$$\int_{G_0} |f_0(x)|^q \, dx = \int_{G_0 \setminus \bigcup_{k=1}^m G_k} |f(x)|^q \, dx + \sum_{k=1}^m \int_{G_k \cap G_0} |f(x) + f_k|^q \, dx$$

$$\leq \int_{G_0 \setminus \bigcup_{k=1}^m G_k} |f(x)|^q \, dx + c(q) \sum_{k=1}^m \left( \int_{G_k \cap G_0} |f(x)|^q + |f_k(x)|^q \right) \, dx$$

$$\leq c(q)(1 + c(q) + l^{q-1}) \sum_{k=1}^m \int_{G_k} |f(x)|^q \, dx.$$

(A.4)
On the other hand, in view of the assumption \( \frac{\text{diam}(G_k)}{R_k} < l \), it follows by [17], Ch.III, Theorem 3.1 that for \( k = 0, \ldots, m \) the problem
\[
\text{div } u_k = f_k \quad \text{in } G_k, \quad u_k|_{\partial G_k} = 0,
\]
has a solution \( u_k \in W^{1,q}_0(G_k) \) such that
\[
\|u_k\|_{W^{1,q}_0(G_k)} \leq c_0(n, q, l) \|f_k\|_{L^q(G_k)} \tag{A.5}
\]
with constant \( c_0(n, q, l) \) independent of \( k \). Thus, denoting extension by 0 of \( u_k \) to \( G \) again by \( u_k \), we get by cone-property of \( G \) that \( u := \sum_{k=0}^{m} u_k \in W^{1,q}_0(\Omega) \) and by [A.3]-[A.5] that
\[
\|u\|_{W^{1,q}_0(\Omega)} = \|u_0\|_{W^{1,q}_0(\Omega \setminus \bigcup_{k=1}^{m} G_k)} + \sum_{k=1}^{m} \|u_k\|_{W^{1,q}_0(G_k)} \\
\leq \|u_0\|_{W^{1,q}_0(\Omega \setminus \bigcup_{k=1}^{m} G_k)} + \bar{c}(q) \sum_{k=1}^{m} (\|u_0\|_{W^{1,q}_0(G_k \cap G_0)} + \|u_k\|_{W^{1,q}_0(G_k)}) \\
\leq \bar{c}(q) c_0(n, q, l) \left( \|u_0\|_{L^q(G_0)} + \sum_{k=1}^{m} \|f_k\|_{L^q(G_k)} \right) \\
\leq 2\bar{c}(q) c_0(n, q, l)^q \left( 1 + \bar{c}(q) + l^{q-1} \right) \|f\|_{L^q(G)}.
\]

Thus, [A.2] is proved with \( C = c_0(n, q, l)(2\bar{c}(q)(1 + \bar{c}(q) + l^{q-1}))^{1/q} \). □

In the next lemma we consider more general setting for the divergence problem.

**Lemma A.2** Let a simply connected and bounded domain \( G \) of \( \mathbb{R}^n, n \geq 2 \), be expressed as
\[
G = \bigcup_{j=1}^{N} G^{(j)}, \quad G^{(j)} \cap G^{(j+1)} \neq \emptyset, \quad j = 1, \ldots, N - 1,
\]
where \( G^{(j)}, j = 1, \ldots, N \), are simply connected domains with cone-property. Moreover, suppose that for each \( j = 1, \ldots, N \) the divergence problem (A.1) in \( G^{(j)} \) has a solution \( u \in W^{1,q}_0(G^{(j)}) \) satisfying (A.2) with constant \( C_j > 0 \) bounded by linear combination of \( c(q)C_1 \frac{\min\{|(G^{(j)}|)/(|G^{(j+1)}|)\}^{1/4}}{|(G^{(j)} \cap G^{(j+1)})|^{1/4}} \) and \( c(q)C_j \frac{\min\{|(G^{(j)}|)/(|G^{(j+1)}|)\}^{1/4}}{|(G^{(j)} \cap G^{(j+1)})|^{1/4}} \), \( j = 2, \ldots, N - 1 \).

**Proof:** First, construct functions \( f_j \in L^q(G), j = 1, \ldots, N \), such that
\[
\text{supp } f_j \subset G^{(j)}, \quad \int_{G^{(j)}} f_j \, dx = 0, \quad f(x) = \sum_{j=1}^{N} f^{(j)}(x), x \in G. \tag{A.6}
\]
Put
\[
f_1(x) = \begin{cases} 
  f(x) & \text{for } x \in G^{(1)} \setminus G^{(2)} \\
  f(x) - a_1 & \text{for } x \in G^{(1)} \cap G^{(2)} 
\end{cases}
\]
and for \( j = 2, \ldots, N \)
\[
f_j(x) = \begin{cases} 
  a_{j-1} & \text{for } x \in (G^{(j)} \cap G^{(j-1)}) \setminus \bigcup_{i=1}^{j-2} G^{(i)} \\
  f(x) & \text{for } x \in G^{(j)} \setminus (G^{(j+1)} \cup \bigcup_{i=1}^{j-1} G^{(i)}) \\
  f(x) - a_j & \text{for } x \in (G^{(j)} \cap G^{(j+1)}) \setminus \bigcup_{i=1}^{j-1} G^{(i)} \\
  0 & \text{for } x \in G^{(j)} \cap \bigcup_{i=1}^{j-2} G^{(i)} 
\end{cases} \tag{A.8}
\]
with
\[
a_j = \frac{\int_{G^{(j)} \setminus G^{(j+1)}} f(x) \, dx}{|(G^{(j)} \cap G^{(j+1)}) \setminus \bigcup_{i=1}^{j-1} G^{(i)}|}.
\]

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Here and in what follows, $G^{(0)} = G^{(N+1)} = \emptyset$ and hence in (A.8) we neglect the cases where $G^{(0)}$ appears for $j = 2$ or $G^{(N+1)}$ appears for $j = N$.

Denote the extension by 0 of $f_j$, $j = 1, \ldots, N$, to $G$ again by $f_j$. Then $f_j$, $j = 1, \ldots, N$, satisfy (A.6). In fact, it is clear that \( \int_{G^{(i)}} f_1 \, dx = 0 \), and for $j = 2, \ldots, N$ we have

\[
\int_{G^{(i)}} f_j \, dx = a_j - 1 \sum_{i=1}^{j-2} \left( |G^{(j)} \setminus G^{(j-1)}| - a_j \right) |G^{(j)} \cap G^{(j+1)}| \int_{G^{(i)}} f(x) \, dx
\]

\[
= \sum_{i=1}^{j-1} |G^{(j)} \setminus G^{(i+1)}| f(x) \, dx - \int_{G^{(j)}} f(x) \, dx + \int_{G^{(i)}} f(x) \, dx
\]

\[
= 0.
\]

Moreover, \( \sum_{j=1}^N f_j = f \) can be easily checked in view of the recursive construction of $f_j$, $j = 1, \ldots, N$.

Now, let us get estimate of $\| f_j \|_{L^q(G^{(j)})}$, $j = 1, \ldots, N$. In view of \( \int_B f(x) \, dx = \int_{G \setminus B} f(x) \, dx \) for any measurable set $B \subset G$, we get

\[
\| f_1 \|_{L^q(G^{(1)})}^q \leq \bar{c}(q) \left( \int_{G^{(1)}} |f(x)|^q \, dx + \frac{\| f_1 \|_{L^q(G^{(2)})}^q}{|G^{(1)} \cap G^{(2)}|} \right)
\]

\[
\leq \bar{c}(q) \left( \int_{G^{(1)}} |f(x)|^q \, dx + \frac{\min\{ \| f_1 \|_{L^q(G^{(1)})}^q, \| f_1 \|_{L^q(G^{(2)})}^q \}}{|G^{(1)} \cap G^{(2)}|} \right)
\]

\[
\leq \bar{c}(q) \left( 1 + \frac{\min\{ \| f_1 \|_{L^q(G^{(1)})}^q, \| f_1 \|_{L^q(G^{(2)})}^q \}}{|G^{(1)} \cap G^{(2)}|} \right) \| f \|_{L^q(G)}^q
\]

using the same technique of Lemma A.1. In the same way, for $j = 2, \ldots, N - 1$ we get that

\[
\| f_j \|_{L^q(G^{(j)})}^q \leq \bar{c}(q) \left( 1 + \frac{\min\{ \| f_{j-1} \|_{L^q(G^{(j)})}^q, \| f_{j+1} \|_{L^q(G^{(j)})}^q \}}{|G^{(j-1)} \cap G^{(j)}|} \right) \| f \|_{L^q(G)}^q
\]

and for $j = N$

\[
\| f_N \|_{L^q(G^{(N)})}^q \leq \bar{c}(q) \left( 1 + \frac{\min\{ \| f_{N-1} \|_{L^q(G^{(N)})}^q, \| f_N \|_{L^q(G^{(N)})}^q \}}{|G^{(N-1)} \cap G^{(N)}|} \right) \| f \|_{L^q(G)}^q.
\]

Thus, by the assumption of the lemma, for each $j = 1, \ldots, N$ the divergence problem (A.1) in $G^{(j)}$ with $f_j$ in the right-hand side has a solution $u_j \in W_0^{1,q}(G^{(j)})$ such that

\[
\| u_j \|_{W_0^{1,q}(G^{(j)})} \leq C_j \tilde{C}_j
\]

where

\[
\tilde{C}_j \leq \bar{c}(q)^{1/q} \left( 1 + \frac{\min\{ \| f_{j-1} \|_{L^q(G^{(j)})}^q, \| f_{j+1} \|_{L^q(G^{(j)})}^q \}}{|G^{(j-1)} \cap G^{(j)}|} \right) \right)^{1-1/q} + \left( \frac{\min\{ \| f_{j-1} \|_{L^q(G^{(j)})}^q, \| f_{j+1} \|_{L^q(G^{(j)})}^q \}}{|G^{(j-1)} \cap G^{(j+1)}|} \right)^{1-1/q};
\]

when $j = 1$ the second term in the bracket of the right-hand side is neglected and when $j = N$ the third term is neglected.

Obviously, $u = \sum_{j=1}^N u_j$ solves (A.1) with right-hand side $f$ and the estimate (A.2) holds with constant $C$ bounded by a sum of $\bar{c}(q) C \frac{\min\{ \| f_{j-1} \|_{L^q(G^{(j)})}^q, \| f_{j+1} \|_{L^q(G^{(j)})}^q \}}{|G^{(j-1)} \cap G^{(j)}|}^{1-1/q}$.

\[ \square \]

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