Vacuum local and global electromagnetic self-energies for a point-like and an extended field source

Roberto Passante, Lucia Rizzuto, and Salvatore Spagnolo
Dipartimento di Fisica dell’Università degli Studi di Palermo and CNISM, Via Archirafi 36, I-90123 Palermo, Italy

We consider vacuum electric and magnetic energy densities (or equivalently field fluctuations) in the space around a point-like or an extended ground-state field source, and discuss the problem of their singular behavior at the source’s position. We show that the assumption of point-like source leads to a divergence of the renormalized energy densities at the position of the source and analyze in detail the structure of such singularity in terms of a delta function and its derivatives. We then consider the case of an extended source, smeared out over a finite volume and described by an appropriate form factor, and show that in this case all divergences in the local energy densities are removed. Our results for the structure of the divergences in the field energy densities also resolve an apparent inconsistency between the vacuum expectation value of the field Hamiltonian and of the field energy densities integrated over all space.

PACS numbers: 12.20.Ds, 03.70.+k

I. INTRODUCTION

Vacuum fluctuations and the existence of the zero-point energy of the electromagnetic field are a remarkable prediction of quantum electrodynamics [1]. They have been extensively investigated in the literature, especially in relation with Casimir and Casimir-Polder forces, which are long-range electromagnetic interactions between neutral macroscopic objects (metals or dielectrics), between atoms and surfaces or among neutral atoms or molecules [2–4]. The Casimir effect, in particular, is related to the modification of the zero-point energy as a consequence of a change of the boundary conditions on the electromagnetic field [5].

Despite of its oddity, the idea of a force generated by a change of vacuum fluctuations in the presence of boundary conditions has stimulated great interest in the literature, both from theoretical and experimental point of view [6]. These purely quantum effects have now been experimentally confirmed with remarkable accuracy [6–8]. However, there are many aspects concerning with vacuum fluctuations and vacuum energy that are still unclear.

A controversial issue concerns with the appearance of surface divergences (and their cut-off dependence) in the calculation of field energy densities, in the presence of metallic boundary conditions [9–13]. The physical origin of these divergences has been recently questioned in the literature and the possibility of removing them through a suitable regularization procedure has been discussed in the case of a scalar field [11–13]. Generally speaking, the appearance of surface divergences in the field energy density is not surprising: boundary conditions are in fact a convenient oversimplification of the interaction between matter and fluctuating fields. An ideal boundary constrains all field modes, at any wavelength and this gives rise to singular energy densities on the boundary. For example, it is well-known that, in the case of a perfectly reflecting plate, renormalized electric and magnetic fluctuations or energy densities (i.e. after subtraction of the homogeneous energy density existing also in absence of the boundary) \( \langle E^2 \rangle \) and \( \langle B^2 \rangle \), diverge at the vacuum-conductor interface. These divergences can be removed by introducing an appropriate exponential upper cut-off in the frequency integrations [12,14]. It has been also shown that they disappear in the case of a boundary with a fluctuating position [11], and it has been guessed that imperfect boundaries could remove them [15].

Recently, the electromagnetic field fluctuations near a dielectric-vacuum interface have been investigated [12,16]. The structure of the surface divergences in the limit of a perfect conductor has been discussed in detail [14]. Also, local energy densities and surface divergences have been explored near fluctuating boundaries [11] or in the vicinity of a soft wall modeled by a potential which is monomial in \( z \), \( z \) being the distance from the wall [17].

In spite of the increasing interest on the subject, many topics are not yet well understood. For example, the presence of divergences in the field energy density presents serious problems when the coupling to gravity is considered, because considerable gravitational effects should be observed [13,18]. In fact, even if considering real surfaces might eliminate the divergences associated to idealized boundaries, energy densities at surfaces and their effects could be large and should be carefully considered [13,20]. Also, it should be stressed that electromagnetic energy densities can be experimentally investigated through the Casimir-Polder interaction energy with appropriate electrically and magnetically polarizable bodies [21].

All the considerations above make relevant investigating the structure and the physical origin of divergences of the vacuum energy densities in the presence of boundaries or polarizable bodies. In [14] we have investigated the origin and structure of surface divergence at the interface between an ideal conductor and the vacuum space.
exploiting an appropriate limiting procedure from a dielectric to a metal; this has also allowed us to show that surface divergences in the electromagnetic energy density are essential in order to have consistency between integrated local energy densities and global field self-energies.

It is thus natural investigating the case of renormalized vacuum electromagnetic energies near an atom or a polarizable body, and this will be the main subject of this paper.

In this paper we study the electromagnetic energy densities (or, equivalently, the field fluctuations) surrounding a dressed point-like and a dressed extended source of the electromagnetic field, whose properties are given by its dynamical polarizability and its form factor; in particular, we shall concentrate on their singular behavior at the source’s position. Compared to previous works on the subject mentioned above, in the present case we have a dynamical source of the electromagnetic field in place of a boundary condition. Some aspects of the vacuum fluctuations of the electromagnetic field in presence of a point-like source have been already explored in the literature (see for example [22] and references therein), as well as its connection with Casimir-Polder interactions between atoms and/or macroscopic bodies [23], but its singular behavior was not considered. It is known that, under appropriate conditions (in the so-called far zone), both the renormalized electric and magnetic energy densities surrounding a point-like ground-state atom are proportional to $1/r^7$, $r$ being the distance from the atom. They clearly diverge at $r = 0$, as well as their sum and their integral over all space. On the other hand, the renormalized total field energy (expectation value of the field Hamiltonian, after subtraction of the zero-point terms) can be shown to be zero (see Sec. II). This shows an inconsistency between local and global self-energies, analogous to the case of the ideal metallic boundary condition discussed in [14], where it is shown that only a careful consideration of surface divergences at the metal-vacuum interface permits to restore a full consistency between the two approaches. This gives a strong indication that also in the case of a point-like atom there should be extra singular terms at the atomic position. It has been argued that the singular behavior arises from the oversimplified assumption of point-like source and dipole approximation [24]. In some sense, point-like sources are conceptually similar to ideal boundaries, so singular energy densities at the position of the source are to be expected.

These considerations suggest that divergences could be removed with a suitable regularization procedure involving the use of an appropriate upper-frequency cut-off in the calculations. In this paper we shall explore an alternative procedure to treat singularities in the energy densities at the position of a field source, showing the presence of extra singular terms localized at $r = 0$, in addition to the $1/r^7$ term. This will allow us to prove that there are not inconsistencies between global and local field self-energies, provided the energy-density divergences localized at the position of the source are properly included. We shall also consider an extended source of finite dimension interacting with electromagnetic field in the vacuum state. The source is modeled as a collection of elementary neutral sources, smeared out over a finite volume with a density described by a function $\rho(r)$, whose Fourier transform gives a form factor for the source. This form factor, which appears in the matter-field interaction Hamiltonian, plays the role of a regularization factor in the source-field interaction, and it provides a finite length-scale for the source-field coupling which cuts off the contribution of high frequency modes. This ensures that the local energy densities of the electromagnetic field are well-defined and finite everywhere, thus eliminating the problem of divergences present in the case of the point-like source. Global and local self-energies are fully consistent in this case, even for a vanishing size of the source. The case of a point-like source can be then obtained by an appropriate limit procedure. This result also indicates that the procedure adopted is equivalent to consider an ideal point-like source together with a suitable physical cutoff as a model to describe realistic situations [19].

The paper is organized as follows. In Sec. II we consider a point-like source in its dressed ground-state interacting with the quantum electromagnetic field in the vacuum state. We assume the source located at $r = 0$ and investigate the local and global (i.e. integrated over all space) properties of the electric and magnetic energy densities in the vacuum space surrounding the source. We show that the assumption of a point-like source leads to divergences in the local electromagnetic energy densities, and obtain their explicit mathematical expression. We show that a singular behavior at $r = 0$ is at the origin of the apparent discrepancy between the total electromagnetic self-energy obtained as the expectation value of the field Hamiltonian and as a space integral of the field energy density. We find that such inconsistence is completely removed if the singularity at the source position $r = 0$ is correctly taken into account, by virtue of a subtle cancelation between diverging quantities at $r = 0$. In Sec. III we consider the case of an extended source with a finite size and analyze in detail the behavior of (local and global) field energy densities in the space around the source. We discuss how the proposed model of extended source eliminates divergences and singularities in the electric and magnetic energy densities, and also solve the mentioned inconsistencies in the calculation of the total electromagnetic energy for any nonvanishing size of the source. Sec. IV is finally devoted to some concluding remarks.

II. ELECTRIC AND MAGNETIC ENERGY DENSITY AROUND A POINT-LIKE SOURCE

In order to discuss the problem of the divergences of the (renormalized) electromagnetic energy densities, we first consider the case of a point-like atom in its ground-
state interacting with the electromagnetic field in the vacuum state. We suppose the atom located at \( \mathbf{r} = 0 \) and we describe the atom-field system adopting an effective Hamiltonian obtained from the multipolar form of the atom-field Hamiltonian in dipole approximation through an appropriate transformation \([21]\):

\[
H = H_A + H_F - \frac{1}{2} \sum_{k_j} \alpha(k) \mathbf{d}_{k_j}(0) \cdot \mathbf{d}(0)
\]

(we are working in the Coulomb gauge). \( H_A \) is the atomic Hamiltonian and \( H_F \) is the electromagnetic field Hamiltonian; \( \mathbf{d}(r) \) is the transverse displacement field, whose Fourier components are \( \mathbf{d}_{k_j}(r) \): outside the atom, it coincides with the total (transverse plus longitudinal) electric field. Introducing the usual annihilation and creation bosonic operators \( a_{k_j} \) and \( a_{k_j}^\dagger \), the Hamiltonian of the radiation field assumes the well-known expression

\[
H_F = \frac{\hbar}{8\pi} \int (\mathbf{d}^2(r) + \mathbf{b}^2(r))d^3r = \sum_{k_j} \hbar \omega_{k_j} a_{k_j}^\dagger a_{k_j} \tag{2}
\]

(in the last equality the zero-point energy has been subtracted), with

\[
\mathbf{d}(r) = i \sum_{k_j} \left( \frac{2\pi\hbar\omega_k}{V} \right)^{1/2} \left( \hat{e}_{k_j}^* a_{k_j} e^{i\mathbf{k}_j \cdot \mathbf{r}} - \hat{e}_{k_j} a_{k_j}^\dagger e^{-i\mathbf{k}_j \cdot \mathbf{r}} \right) \tag{3}
\]

\[
\mathbf{b}(r) = i \sum_{k_j} \left( \frac{2\pi\hbar\omega_k}{V} \right)^{1/2} \left( \hat{b}_{k_j} a_{k_j} e^{i\mathbf{k}_j \cdot \mathbf{r}} - \hat{b}_{k_j}^* a_{k_j}^\dagger e^{-i\mathbf{k}_j \cdot \mathbf{r}} \right) \tag{4}
\]

where \( \hat{e}_{k_j} \) are the polarization vectors and \( \hat{b}_{k_j} = \hat{e}_{k_j} \times \mathbf{k} \). Finally, the atomic isotropic dynamical polarizability \( \alpha(k) \) is given by

\[
\alpha(k) = \frac{2}{3} \sum_m \frac{E_{mg}\mu_{mg}^2}{E_{mg}^2 - (\hbar c k)^2}, \tag{5}
\]

where \( \mu_{mg} \) is the matrix element of the dipole moment operator between the atomic state \( m \) and the ground state \( g \), and \( E_{mg} = E_m - E_g \). We are interested in investigating local and global properties of the energy density of the virtual electromagnetic field surrounding the point-like source in its dressed ground state. In particular, we shall calculate the quantum average of the electric and magnetic energy densities on the dressed ground state, that is the lowest-energy eigenstate of the Hamiltonian \([\mathbf{1}]\). Using Rayleigh-Schrödinger perturbation theory, the dressed ground-state is easily obtained at first order in \( \alpha(k) \)

\[
\langle \tilde{0} | H_{\text{el}}(\mathbf{r}) | \tilde{0} \rangle = \frac{\alpha(k)}{4\pi^3} \int_0^\infty dk \int_0^\infty dk' \times Q_E(k,k',r) \frac{k^2 k'^3}{k + k'}
\]

\[
\times Q_M(k,k',r) \frac{k^2 k'^3}{k + k'}, \tag{9}
\]

where \( Q_E(k,k',r) \) and \( Q_M(k,k',r) \) are expressed in terms of spherical Bessel functions as

\[
Q_E(k,k',r) = j_0(kr)j_0(k'r) - j_0(kr)j_1(k'r), \tag{10}
\]

\[
Q_M(k,k',r) = 2j_1(kr)j_1(k'r). \tag{11}
\]
Using the relation
\[ \frac{1}{k + k'} = \int_0^\infty e^{-(k+k')\eta} d\eta \] (13)
to decouple k and k' integrations in (9) and (10), after integration over k, k' we get
\[ \langle \tilde{0} | H_{el}(r) | \tilde{0} \rangle = \frac{4\alpha \hbar c}{\pi^3} \int_0^\infty d\eta \frac{3\eta^4 - 2r^2\eta^2 + 3\eta^4}{(r^2 + \eta^2)^6} \] (14)
\[ \langle \tilde{0} | H_m(r) | \tilde{0} \rangle = -\frac{4\alpha \hbar c}{\pi^3} \int_0^\infty d\eta \frac{8r^2\eta^2}{(r^2 + \eta^2)^6}. \] (15)

Integration on \( \eta \) finally gives the well-known results (valid for \( r \neq 0 \))
\[ \langle \tilde{0} | H_{el}(r) | \tilde{0} \rangle = \frac{23}{(4\pi)^2} \frac{\alpha \hbar c}{r^7}, \] (16)
\[ \langle \tilde{0} | H_m(r) | \tilde{0} \rangle = -\frac{7}{(4\pi)^2} \frac{\alpha \hbar c}{r^7}. \] (17)

The expressions above describe the local energy densities of the electromagnetic field surrounding the point-like source. They are well defined everywhere, except at the origin \( r = 0 \), where they diverge. Before exploring in detail this singularity and show the existence of other singular terms at \( r = 0 \), it is of interest to discuss some global (integral) property of the renormalized electromagnetic energy surrounding the dressed source. First, using (2) and (6), we immediately obtain
\[ \langle \tilde{0} | H_F | \tilde{0} \rangle = \langle \tilde{0} | \sum_{k_j} \hat{\omega}_k a_k^\dagger a_{k_j} | \tilde{0} \rangle = 0. \] (18)

On the other hand, equations (16) and (17) imply
\[ \langle \tilde{0} | (H_{el}(r) + H_m(r)) | \tilde{0} \rangle = \frac{1}{(4\pi)^2} \frac{\alpha \hbar c}{r^7}, \] (19)
which diverges when integrated over all space because of the singular behavior of (16) and (17) at \( r = 0 \). This is at variance with the result (15), because \( H_F = \int d^3r (H_{el}(r) + H_m(r)) \). Thus there is a discrepancy between the value of the total electromagnetic field energy calculated as the expectation value of the field Hamiltonian \( H_F \) and as a space integral of the electromagnetic energy density (19). It appears that the average field energy cannot be obtained from the spatial integration of (19); also, the latter is divergent at the position of the atom. The mathematical origin of this difficulty is that the energy densities as given in (16) and (17) have a singularity at \( r = 0 \); we shall show in the next part of this Section that other singular terms are present and that their existence solves this (apparent) problem. This singular behavior indeed prevents from exchanging the \( r \) and \( \eta \) integrations in the calculation of the total electric and magnetic energy densities [24]. To strengthen this observation, we note that if we first perform integration over \( r \) and then that over \( \eta \) (that is in the opposite order of the \( r \) integral of (19)), the spatial integral of the total (electric plus magnetic) energy density correctly vanishes, as expected from (15). In fact, from eqs. (16) and (17) after integrations on \( k \) and \( k' \), or from (14) and (15), we have
\[
\int d^3r \langle \tilde{0} | H_{el}(r) | \tilde{0} \rangle = \frac{16\hbar c\alpha}{\pi^2} \int_0^\infty d\eta \int_0^\infty drr'^2 \times \frac{3\eta^4 - 2r^2\eta^2 + 3\eta^4}{(r^2 + \eta^2)^6} = \frac{\hbar c\alpha}{\pi^2} \int_0^\infty d\eta \int_0^\infty drr'^2 \times \frac{8r^2\eta^2}{(r^2 + \eta^2)^6} = -\frac{\hbar c\alpha}{\pi^2} \int_0^\infty d\eta \int_0^\infty drr'^2 \frac{3\pi}{4\eta^6},
\] (20)
from which we immediately obtain
\[ \int d^3r \langle \tilde{0} | (H_{el}(r) + H_{el}(r)) | \tilde{0} \rangle = 0, \] (22)
because the electric and magnetic contributions cancel each other, even if they are individually divergent. We can physically understand this result: the exponential function introduced in (13) plays the role of a cut-off function in the frequency integrations in (9) and (10), and it removes the singular behavior of energy densities at \( r = 0 \) before the integration over \( \eta \) is done. Exchanging the order of the integrations in \( \eta \) and \( r \) is somehow equivalent to remove the cut-off in the calculation, and this leads to the diverging result, essentially for the same reason that an ideal conducting boundary leads to diverging energy densities at the boundary-vacuum interface [14].

From the previous considerations, we are led to conclude that the total field energy cannot be obtained by spatial integration of the energy density, without an appropriate prescription for dealing with the singularity at the origin. This observation suggests to investigate in more detail the singular behavior of the electromagnetic energy density. We will now show that if the singularity at \( r = 0 \) is correctly evaluated and taken into account, the spatial integral of \( \langle \tilde{0} | H_{el}(r) + H_{el}(r) | \tilde{0} \rangle \) vanishes, consistently with (15). In order to do that, we first consider the electric and magnetic energy densities, following a procedure similar to that used in [24]. It involves the use of an exponential cut-off in the integrals (9) and (10), and letting the cut-off frequency to go to infinity after the frequency integrations (details are given in the Appendix A). After some algebra we get
\[
\langle \tilde{0} | H_{el}(r) | \tilde{0} \rangle = \frac{\hbar c\alpha}{(4\pi)^2} \left\{ \frac{23}{r^7} - \frac{23}{r^6} \delta(r) + \frac{10}{r^5} \delta'(r) - \frac{7}{3r^4} \delta''(r) + \frac{1}{3r^3} \delta'''(r) + \frac{1}{15r^2} \delta^{(iv)}(r) \right\}, \] (23)
\[
\langle \tilde{0} | H_m(r) | \tilde{0} \rangle = -\frac{\hbar c\alpha}{(4\pi)^2} \left\{ \frac{7}{r^7} - \frac{7}{r^6} \delta(r) + \frac{2}{r^5} \delta'(r) + \frac{1}{3r^4} \delta''(r) - \frac{1}{3r^3} \delta'''(r) - \frac{1}{15r^2} \delta^{(iv)}(r) \right\}, \] (24)
where the superscript to the delta function indicates the order of its derivative with respect to \( r \). The first term on the right-hand side of (23) and (24) coincides with (16) and (17), respectively. The other terms, involving a delta function and its higher-order derivatives, take into account the singularity of the electric and magnetic energy densities at \( r = 0 \). This singularity was not included in the calculation yielding (16) and (17). Our procedure outlined in the Appendix A of first introducing an exponential cut-off function, and taking the limit of the cut-off frequency to infinity after having done the frequency integrals, has thus allowed us to obtain the correct and complete expression of the energy-density singularity at \( r = 0 \).

We can now easily evaluate the total energy of the field. Summing (23) and (24), and integrating over all space using the properties of the distributional derivatives of \( \delta \)-functions, we finally obtain

\[
\int d^3 r (\tilde{\rho}(r) - \rho_{\text{el}}(r) + \rho_{\text{mag}}(r)) \delta(r) = \frac{\hbar c \omega}{4 \pi} \int_0^\infty dr \left\{ \frac{16}{r^5} - \frac{16}{3r^4} - \frac{48}{15r^3} + \frac{28}{15r^2} \right\} \delta(r) = 0
\]  

(25)

(cancellation of terms diverging for \( r \to 0 \) should be noted). As mentioned, this shows that, by a proper account of the divergences at the atomic position, the total field self-energy vanishes, consistently with the global self-energy as obtained in (13). This means that the total (diverging) electromagnetic energy for \( r > 0 \) is exactly canceled by the electromagnetic energy stored at \( r = 0 \), as clearly shown by eq. (25). This property is not obtained if the calculation of integrated energy density is performed without a careful consideration of the singularity at \( r = 0 \), and shows that the renormalized field energy vanishes by virtue of an intriguing cancellation among diverging quantities. This fully restores consistency between the renormalized expectation value of \( H_R \) on the dressed ground-state and the integrated energy density over all space. Although the total renormalized self-energy vanishes, its electric and magnetic components individually are however divergent at the origin for a point-like source.

We wish to stress that a similar discrepancy has been discussed in the case of quantum fields with a boundary condition, for example for the electromagnetic field in presence of a perfectly conducting plate [11][14]; it has been shown that the existence of a singular surface energy density is necessary to remove the inconsistency between the total field energy and the integrated energy density [14]. We finally observe that the exponential cut-off we have introduced in the previous calculations is just a mathematical procedure that allows us to obtain the correct form of the divergences of energy densities through a limit procedure. In the next Section we shall discuss how the assumption of a model of extended source may provide a natural physical cutoff, which removes all divergences in the energy densities of electromagnetic field.

### III. ENERGY DENSITIES AND FLUCTUATIONS OF THE ELECTROMAGNETIC FIELD NEAR AN EXTENDED SOURCE

We now focus on the second main point of this paper, specifically the local and global electromagnetic energy densities in the case of a specific model of an extended source. We consider a neutral source of finite dimension, that we model as a collection of neutral point-like sources, smeared-out over a finite volume, with density \( \rho(r) \). This introduces a form-factor in the frequency integrations. Specific examples of form factors for transitions between levels of the hydrogen atom can be found in [26]. We assume the function \( \rho(r) \) normalized, so that

\[
\int d^3 r \rho(r) = 1.
\]

A simple generalization of Hamiltonian (11) to the case of an isotropic source of finite dimensions is

\[
H = H_A + \frac{1}{8\pi} \int d^3 r (\mathbf{d}^2(r) + \mathbf{b}^2(r)) - \frac{1}{2} \sum_{k,j} \alpha(k) \left( \int d^3 r \rho(r) d_kj(r) \times \int d^3 r' \rho(r') d_{k'j'}(r') \right).
\]

(26)

This Hamiltonian reduces to (11) for a point-like source when \( \rho(r) \to \delta(r) \). In (26) the interaction with the field of an elementary source at \( r' \) also depends on the presence of the elementary source at \( r'' \).

The density function \( \rho(r) \) appearing in the interaction Hamiltonian, acts as a regularizing factor of the source-field interaction: thus our extended-source model allows to remove all divergences in the renormalized values of electric and magnetic energy densities, yielding also consistency between integrated averaged energy densities and average of the field Hamiltonian. Our model also permits to obtain the case of the point-like source discussed in the previous Section as a limit case.

In order to evaluate the electromagnetic energy density around the dressed source, we follow the same procedure of the previous Section. We first calculate the dressed ground state of Hamiltonian (26). Straightforward perturbation theory up to first order in the atomic polarizability yields

\[
\langle 0 | = \langle g, 0_kj \rangle - \frac{\pi}{V} \sum_{k,j} \alpha(k) \frac{(kk')^{1/2}}{k + k'} \hat{e}_{kj} \cdot \hat{e}_{k'j'} \times \int d^3 r d^3 r' \rho(r) \rho(r') e^{-i(k \cdot r + k' \cdot r')} \langle g, 1_{k1} 1_{k'j'} \rangle.
\]

(27)

This expression can be used to evaluate the quantum averages \( \langle \mathcal{H}_{el}(r) \rangle \) and \( \langle \mathcal{H}_{mag}(r) \rangle \). At lowest order in \( \alpha(k) \) and subtracting the spatially homogeneous zero-point contributions (present also in absence of the source), we
get

\[ \langle \bar{0} | \mathcal{H}_{el}(\mathbf{R}) | \bar{0} \rangle_{e.s.} = \frac{\hbar}{8\pi} \left( \frac{\pi}{2} \right)^2 \sum_{k j} \sum_{k' j'} (k k')^{1/2} \]

\[ \times (\hat{\epsilon}_{k j} \cdot \hat{\epsilon}_{k' j'})^2 (\alpha(k) + \alpha(k')) \bar{e}^{i(\mathbf{k} + \mathbf{k}') \cdot \mathbf{R}} \]

\[ \times \int d^3 r \rho(r) e^{-ik \cdot r} \int d^3 r' \rho(r') e^{-ik' \cdot r'} + c.c. \] (28)

for the renormalized electric energy density, and

\[ \langle \bar{0} | \mathcal{H}_{m}(\mathbf{R}) | \bar{0} \rangle_{e.s.} = \frac{\hbar}{8\pi} \left( \frac{\pi}{2} \right)^2 \sum_{k j} \sum_{k' j'} (k k')^{1/2} \]

\[ \times (\hat{\epsilon}_{k j} \cdot \hat{\epsilon}_{k' j'}) (\hat{b}_{k j} \cdot \hat{b}_{k' j'}) (\alpha(k) + \alpha(k')) \bar{e}^{i(\mathbf{k} + \mathbf{k}') \cdot \mathbf{R}} \]

\[ \times \int d^3 r \rho(r) e^{-ik \cdot r} \int d^3 r' \rho(r') e^{-ik' \cdot r'} + c.c. \] (29)

for the renormalized magnetic energy density (subscript e.s. indicates extended source). Comparison with (17) and (18) allows us to write these quantities in the more compact form

\[ \langle \bar{0} | \mathcal{H}_{el}(\mathbf{R}) | \bar{0} \rangle_{e.s.} = \frac{\hbar}{8\pi} \int d^3 r \rho(r) \int d^3 r' \rho(r') \]

\[ \times \left( \langle \bar{0} | \mathbf{d}(\mathbf{R} - \mathbf{r}) \mathbf{d}(\mathbf{R} - \mathbf{r}') \right) \rho(r)_{p.s.} \] (30)

\[ \langle \bar{0} | \mathcal{H}_{m}(\mathbf{R}) | \bar{0} \rangle_{e.s.} = \frac{\hbar}{8\pi} \int d^3 r \rho(r) \int d^3 r' \rho(r') \]

\[ \times \left( \langle \bar{0} | \mathbf{b}(\mathbf{R} - \mathbf{r}) \mathbf{b}(\mathbf{R} - \mathbf{r}') \right) \rho(r)_{p.s.} \] (31)

(where the subscript p.s. indicates point-like source), which clearly shows the presence of interference between the contributions of the elementary sources which form the extended source.

Assuming \( \rho(r) \) with a spherical symmetry, the form factor depends only on \( k \),

\[ \int d^3 r \rho(r) e^{-ik \cdot r} = \rho(k). \] (32)

Substituting (32) in (30) and (31), in the continuum limit we obtain

\[ \langle \bar{0} | \mathcal{H}_{el}(\mathbf{R}) | \bar{0} \rangle_{e.s.} = \frac{\hbar c}{16\pi^3} \int_0^\infty dk \int_0^\infty dk' (\alpha(k) + \alpha(k')) \rho(k) \rho(k') Q_E(k, k', R) \]

\[ \bar{e}^{i(k^3 k'^3)} \frac{1}{k + k'} + c.c. \] (33)

and

\[ \langle \bar{0} | \mathcal{H}_{m}(\mathbf{R}) | \bar{0} \rangle_{e.s.} = -\frac{\hbar c}{16\pi^3} \int_0^\infty dk \int_0^\infty dk' (\alpha(k) + \alpha(k')) \rho(k) \rho(k') Q_M(k, k', R) \]

\[ \bar{e}^{i(k^3 k'^3)} \frac{1}{k + k'} + c.c. \] (34)

These expressions show how the renormalized electric and magnetic energy densities explicitly depend on the structure of the source through its form factor \( \rho(k) \). In the limit \( \rho(r) \to \delta(r) \), they reduce to the expressions for a point-like source obtained in the previous Section.

The assumption of an extended source naturally introduces a physical cutoff (at a frequency related to the size of the source), that eliminates the divergences in the field energy densities discussed in the previous Section. In fact, it is easy to see that the functions inside the integrals (33) and (34) behave as \( \alpha(k) \rho(k) k \) for large \( k \). Thus choosing an appropriate form factor, decreasing as \( 1/k^\alpha \) (\( \alpha > 2 \)), the integrals on the right-hand side of (33) and (34) converge. Furthermore, because the functions \( Q_E(M)(k, k', R) \) are continuous everywhere, both the electric and magnetic energy densities are well-defined everywhere. This also makes convergent their integral over all space. In other words, the singularities present of the local energy densities in the case of point-like source are eliminated in our case of an extended source.

Finally, we may calculate explicitly the spatial integral of the electromagnetic energy density. Now the integral over \( r \) can be safely exchanged with the integrals over \( k \) and \( k' \) due to the regularization introduced by the form factor; being

\[ \int d^3 r \left( Q_E(k, k', r) - Q_M(k, k', r) \right) = 0, \] (35)

we immediately get

\[ \int d^3 r \left( \mathcal{H}_{el} + \mathcal{H}_{m} \right) | \bar{0} \rangle_{e.s.} = 0, \] (36)

as expected from eq. (18), which is valid also for the extended source. This result confirms that because of the assumption of a source of finite dimension, there is not discrepancy between the total electromagnetic energy and the integrated local energy densities for any nonvanishing size of the source, making the procedure adopted fully equivalent to the introduction of a suitable cutoff function.

IV. CONCLUSIONS

In this paper we have analyzed the renormalized vacuum electric and magnetic energy densities (or equivalently field fluctuations) near a field source, for example an atom or an electrically polarizable body, using an effective Hamiltonian describing the atom-field interaction. The importance of considering these local energy densities is related to many factors, for example they are related to Casimir-Polder forces on polarizable bodies and they should also act as a source term for gravity. We have first concentrated our interest on the structure of the divergences of the energy densities at the source position \( \mathbf{r} = \mathbf{0} \) in the case of a point-like source. We have found that the expressions of the electric and magnetic energy densities contain terms proportional to the Dirac delta
function and its derivatives evaluated at \( r = 0 \), which contain a finite amount of energy. We have shown that these singular terms in the energy densities are essential in order to have consistency between the renormalized vacuum expectation values of the field Hamiltonian and of the field energy density integrated over all space. We have also considered a model of an extended source of the electromagnetic field, which is smeared out over a finite volume of space, and shown that in this case the renormalized field energy densities are finite at any point of space for any nonvanishing size of the source. Relation of our results with recent works about surface divergences of renormalized field energy density at the interface between a conducting plate and the vacuum has been also discussed in detail.

Acknowledgments

The authors wish to thank F. Persico and N. Bartolo for interesting discussions on the subject of this paper. Financial support by the Julian Schwinger Foundation, by Ministero dell’Istruzione, dell’Università e della Ricerca and by Comitato Regionale di Ricerche Nucleari e di Struttura della Materia is gratefully acknowledged. The authors acknowledge support from the ESF Research Networking Program CASIMIR.

Appendix A: Calculation of \( \langle H_{el}(r) \rangle \) and \( \langle H_{m}(r) \rangle \)

In this Appendix we outline the procedure that leads to the expressions [23] and [24] giving the energy-density singularity at \( r = 0 \). For simplicity, we first focus on the magnetic energy density \( \langle H_{m}(r) \rangle \), given in [10]. We need to calculate the following integral

\[
I(r) = \int_{0}^{\infty} dk \int_{0}^{\infty} dk' j_1(kr) j_1(k' r) \frac{k^3 k'^3}{k + k'}.
\]  

(A1)

In order to evaluate it, we introduce an exponential cut-off so that the integral becomes

\[
I(r) = \int_{0}^{\infty} dk \int_{0}^{\infty} dk' j_1(kr) j_1(k' r) \frac{k^3 k'^3}{k + k'} e^{-\gamma(k + k')}
\]

with \( \gamma > 0 \) and where we have used the relation [13]. This integral can be easily evaluated; after some algebra we obtain

\[
I(r) = 64r^2 \int_{0}^{\infty} \frac{d\eta}{(r^2 + (\gamma + \eta)^2)^2} = \frac{32}{5} \frac{r^2}{(r^2 + \gamma^2)^2} - \frac{4}{15} \frac{r^2}{(r^2 + \gamma^2)^3}
\]

(A3)

where \( \mu = \gamma + \eta \). We now consider the limit \( \gamma \to 0 \), equivalent to removing the exponential cut-off, after the frequency integrals. Using the Lorentzian representation of the Dirac delta function, it is easy to see that the terms appearing in the expression [A3] lead to the Dirac delta function and its derivatives when \( \gamma \to 0 \). Thus, we obtain

\[
I(r) = \frac{\pi}{8} \left( \frac{7}{r^2} - \frac{7}{r^4} \delta(r) + \frac{2}{r^3} \delta'(r) + \frac{1}{3r^4} \delta''(r) - \frac{1}{3r^3} \delta'''(r) \right),
\]

(A4)

where the superscript to the delta function indicates the order of its derivative with respect to \( r \). Substituting in [10] and using [12], we finally obtain expression [24]. A similar procedure leads to expression [23] for the electric energy density.

[1] P.W. Milonni, *The Quantum Vacuum: An Introduction to Quantum Electrodynamics*, Academic Press, San Diego 1994.
[2] H.B.G. Casimir, Proc. K. Ned. Akad. Wet. B 51 (1948) 793.
[3] H.B.G. Casimir and D. Polder, Phys. Rev. 73, 360 (1948).
[4] R. Messina, R. Passante, L. Rizzuto, S. Spagnolo, and R. Vasilie, J. Phys. A 41, 164031 (2008).
[5] V.M. Mostepanenko and N.N. Trunov, *The Casimir Effect and its Applications*, Clarendon Press, Oxford, 1997.
[6] M. Bordag, G.L. Klimchitskaya, U. Mohideen, and V.M. Mostepanenko, *Advances in the Casimir Effect*, Oxford Science Publication (2009), and references therein.
[7] C.I. Sukenik, M.G. Boshier, D. Cho, V. Sandoghdar, and E.A. Hinds, Phys. Rev. Lett. 70, 560 (1993).
[8] S.K. Lamoreaux, Phys. Rev. Lett. 78, 5 (1997).
[9] D. Deutsch and P. Candelas, Phys. Rev. D 20, 3063 (1979).
[10] K.A. Milton, I. Cavero-Pelaez, and J. Wagner, J. Phys. A 39 6543 (2006).
[11] L.H. Ford and N.F. Svaiter, Phys. Rev. D 58, 065007
[12] M.J. Pfenning, Phys. Rev. A 62, 045018 (2000).
[13] K.A. Milton, *Casimir Physics*, edited by D. Dalvit, P. Milonni, D. Roberts, and F. de Rosa, Springer, Heidelberg, 2011, p. 39, and references therein.
[14] N. Bartolo and R. Passante, Phys. Rev. A 86, 012122 (2012).
[15] V. Sopova and L.H. Ford, Phys. Rev. D 66, 045026 (2002).
[16] A.D. Helfer and A.S.I. Lang, J. Phys. A 32, 1937 (1999).
[17] K.A. Milton, Phys. Rev. D 84, 065028 (2011).
[18] F.D. Mazzitelli, J.P. Nery, and A. Satz, Phys. Rev. D 84, 125008 (2011).
[19] R. Estrada, S.A. Fulling, and F.D. Mera, arXiv:1207.7013v1.
[20] R. Estrada, S.A. Fulling, L. Kaplan, K. Kirsten, Z. Liu, and K.A. Milton, J. Phys. A 41, 164055 (2008).
[21] R. Passante, E.A. Power, and T. Thirunamachandran, Phys. Lett. A 249, 77 (1998).
[22] G. Compagno, G.M. Palma, R. Passante, and F. Persico, J. Phys. B 28, 1105 (1995).
[23] R. Passante and E.A. Power, Phys. Rev. A 35, 188 (1987).
[24] G. Compagno, R. Passante, and F. Persico, Physica Scripta T21, 33 (1988).
[25] G. Compagno, R. Passante, and F. Persico, Phys. Rev. A 38, 600 (1988).
[26] H.E. Moses, Phys. Rev. A 8, 1710 (1973).