A mathematical approach regarding a better geometry of the root fillet of symmetric and asymmetric gears with the main scope of increasing the fatigue strength of gear teeth and avoiding the occurrence of cracks

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Abstract. The main topic of the present paper consists of two main ideas: on one side, there is presented a mathematical approach on fatigue strength of a gear tooth and, on the other side, there is applied this mathematical approach on a particular case regarding a better fillet geometry of symmetric and asymmetric gears. In this mathematical approach, there is illustrated the planar curves theory and their planar contact. Then, there will be presented some theory regarding the gear failure and the appearance of cracks that generates tooth base fatigue. In the end, there will be presented some graphical results using Matlab programming language.

1. Mathematical approach

1.1. The starting point

The starting point of this paper is the main theme of another paper. The authors proposed a circular fillet root radius. The present paper presents also a circular fillet root radius but in a better variant. Here is taken into consideration and it is assumed that the geometry of a particular part has influence on the part strength. In this paper, there is presented that geometry influence upon tooth base fatigue depends on the planar curves contact. For example, in [1] and [2] there can be seen a circle that is in contact with the involute curve. It is said that that curve has an 1-degree contact with the root circle (more information regarding planar curves contact will be presented in the next sub-heading). The present paper propose another circle that, by definition, has a 2-degree contact with the involute curve of the tooth. This circle is called the osculating circle and it is unique in a particular point of the involute curve. It is assumed that the greater the degree contact, the greater the fatigue strength at the tooth base.

1.2. Curves theory regarding the present paper topic. The osculating circle

Definition 1.1. Two simple curves \((y_1)\) and \((y_2)\) \(\in C^n\), \(n \geq 1\), have a contact \(\geq m, m \leq n\), in a common point \(M_0 \in (y_1) \cap (y_2)\), if there are admitted the parametric representations

\[(y_1): \vec{r} = \vec{r}_1(t) = x_1(t)\hat{i} + y_1(t)\hat{j}, t \in I\]

and

\[(y_2): \vec{r} = \vec{r}_2(\bar{t}) = x_2(\bar{t})\hat{i} + y_2(\bar{t})\hat{j}, \bar{t} \in J\]
so that, if \( t = t_0 \) and \( \ell = \ell_0 \) are the parameters of \( M_0 \) on these two curves, then (equation (1))

\[
\vec{r}_1(t_0) = \vec{r}_2(\ell_0), \quad \vec{r}'_1(t_0) = \vec{r}'_2(\ell_0), \quad \ldots, \quad \vec{r}^{(m)}_1(t_0) = \vec{r}^{(m)}_2(\ell_0). \tag{1}
\]

If two curves have a \( m \geq 2 \) degree contact in a particular point, then those two curves have in that specific point the same tangent, the same normal and the same curvature. If two curves have a 1-degree contact, then those curves have the same normal and the same tangent.

Regarding the definition and the observation in the related definition, it can be said that a \( m \geq 2 \) degree contact gives a better “continuity” of two connected curves than a \( m=1 \) degree contact. This is the assumption regarding the possibility of cracks appearing, because the smaller degree contact between two curves, in engineering approach, the bigger the possibility of cracks occurrence (this inferior degree contact between two curves can be seen as an stress concentration factor).

1.3. Finding the osculating circle in this case

**Definition 1.2.** Let there be \( (\gamma) \) a simple curve \( \epsilon C^n, n \geq 2 \) and the point \( M_0 \epsilon (\gamma) \). It is called osculating circle of that curve in \( M_0 \), a circle that has in this point a \( m \geq 2 \) degree contact with \( (\gamma) \).

**Theorem 1.1.** In every non-inflexion point \( M_0(t_0) \) of any simple curve \( C^n, n \geq 2 \),

\[
(\gamma): \quad \vec{r} = \vec{r}_1(t) = x_1(t)i + y_1(t)j, \quad tel
\]

there exists and it is unique an osculating circle, having its center in \( C \):

\[
C(x(t_0) = \frac{y'(t_0)}{x'(t_0)y''(t_0) - y'(t_0)x''(t_0)}, \quad y(t_0) = \frac{x'(t_0)}{x'(t_0)y''(t_0) - y'(t_0)x''(t_0)}
\]

and its radius \( R = \frac{1}{|\kappa(t_0)|} \), where \( \kappa(t_0) \) is the curvature of \( (\gamma) \) in \( M_0 \).

To determine the center coordinates of this osculating circle, there are needed the first and the second derivative of each function. According to Litvin [3] and [5], the equations of the involute curve are, equation (2):

\[
\begin{align*}
(x(t) &= r_b (\sin t - t \cos t) \\
y(t) &= r_b (\cos t + t \sin t)
\end{align*}
\]

So:

\[
x'(t) = r_b t \sin t \quad and \quad x''(t) = r_b (\sin t + t \cos t) \tag{3}
\]

Also:

\[
y'(t) = r_b t \cos t \quad and \quad y''(t) = r_b (\cos t - t \sin t) \tag{4}
\]

Here, \( r_b \) is the radius of the base circle of the gear. If we consider a pinion having, for instance, \( r_b = 15 \) mm and a \( r_a = 22 \) mm, we obtain the following graphical representation of the involute curve:

The parameter \( t \) varies between 0 and the value of the radical \( \sqrt{(r_M/r_b)^2 - 1} \), where \( r_M \) is the value of the corresponding radius of any point that lies on the involute curve between the first point of the involute curve and the last one. The lowest value of \( r_M = r_b \) and the highest one is \( r_M = r_a \), where \( r_a \) is the radius of the addendum circle. In our case, for the pre-established values of each variable, the parameter \( t \) varies between 0 and 1.07 (after the calculation of the radical).
If there is substituted \( t \) with 0 in (3) and (4), we obtain:

\[
x'(0) = 0, x''(0) = 0, y'(0) = 0 \text{ and } y''(0) = r_b.
\]

This means that we cannot find the center of the osculating circle (see above the formula of the center of the osculating circle- in both fractions there will be obtained 0 as denominator). This means that in the point belonging to the involute where the parameter \( t = 0 \) is an inflexion point. And this means that the Y-axis is tangent to the involute curve in \( t = 0 \). To find the osculating circle with respect to the technical requirements, we fix another point in the neighborhood of the point where \( t = 0 \). There will be selected a very close point to the previous one. We consider the point on the involute curve where \( t = 0.05 \). In this point, the values of the derivatives are as follows:

\[
x'(0.05) = 0.0375r_b, x''(0.05) = 1.4995r_b, y'(0.05) \doteq 0.75r_b \text{ and } y''(0.05) = 14.94r_b.
\]

Calculating the values of \( x(0.05) \) and \( y(0.05) \), we obtain:

\[
x(0.05) = 0.0000417r_b \text{ and } y(0.05) = 1.0012r_b.
\]

Having all the necessary values, there can be calculated the coordinates of the center of the osculating circle, assuming that there is used the same value for \( r_b \) and that value is 15 mm. Let us consider that the center of the osculating circle is called \( C \). Its coordinates are \( C(-1.2284; 15.0197) \), according to the theorem 1.1. Then, it is calculated the radius of this circle and its value is 1.23 mm (also according to the theorem 1.1).

It can be easily proved that the center of the osculating circle lies on the normal to the involute curve in the point \( M_0 \), where \( t = 0.05 \). The coordinates of the point \( M \) are determined if there were calculated the values of \( x(0.05) \) and \( y(0.05) \). So:

\[
\begin{align*}
\{ x(0.05) & = 0.0006255 \text{mm} \\
 & y(0.05) = 15.0187 \text{mm} 
\end{align*}
\]

\[ (5) \]
There is needed to determine the tangent and normal equations in the point $M_0$. The canonic equation of the tangent is equation (6):

$$
\frac{x-x_0}{x'(t_0)} = \frac{y-y_0}{y'(t_0)} \iff \frac{x-x_0}{y'(t_0)} = \frac{y-y_0}{0.000654} = \frac{y-15.1018}{0.75}.
$$

(6)

and the canonic equation of the normal is equation (7):

$$
\frac{x-x_0}{-y'(t_0)} = \frac{y-y_0}{x'(t_0)} \iff \frac{x-x_0}{x'(t_0)} = \frac{y-y_0}{0.000654} = \frac{y-15.018}{0.75}.
$$

(7)

There was written the equation of the normal with the scope of finding the parametric equations of the osculating circle. It is necessary this because there will be presented the graphical illustration of both involute curve and osculating circle using Matlab programming language. So, there is needed to establish the variation interval of the parameter $\theta$ of the osculating circle. This parameter represents the angle between the radius which passes through the point $M_0$ (which coincides with the normal in that point) and the X-axis. So, equation (8):

$$
(7) \Rightarrow y = -0.000872x + 15.018000545.
$$

(8)

Because the slope of the straight line whose equation was described in (8) is negative, that means that the angle must be calculated like this, equation (9):

$$
m = -0.000872 = -\tan(\pi - \theta) \iff \theta = 179.95^{\circ}.
$$

(9)

It could be easily found that the interval variation of the parameter $\theta$ is $\left[\frac{3\pi}{2}, 2\pi - 0.000872\right]$. If we convert 0.000872 radians in hexadecimal degrees, we found the value $0.05^{\circ}$. This value is obtained after the calculation $\pi - \theta$. There is needed an suggestive illustration of the found circle, not the entire circle. There is needed to be shown the contact between the osculating circle and the involute curve. This is going to be presented using Matlab graphics.

The graphical illustration of the present osculating circle is presented below, figure 2.

![Figure 2. The resulted osculating circle.](image-url)
The entire needed geometry of both curves, the circle and the involute respectively, is presented below, figure 3.

![The resulted osculating circle and the involute curve](image)

**Figure 3.** The resulted circle and the involute curve.

2. Fatigue of Gears

2.1. Introduction
When it’s about gears failure, there is no specific clue regarding the reason why that gear failed. After a proper examination and a professional investigation [6], there can be identified a sum of different specific reasons. A proper analysis of a gear failure begins with classification of the failure by type or by mode. A mode of a gear failure represents a particular type of failure that has its own characteristics. After the mechanism of failure is discovered, there is needed to find the primary cause of the failure. If it is understood the mechanism of failure, there is a big step toward the minimizing the specter of possible clues regarding the causes of failure appearances. Below, in table 1, there is presented the failure modes of gears.

| Failure mode   | Type of failure                      |
|----------------|--------------------------------------|
| Fatigue        | Tooth bending, surface contact (pitting), rolling contact or thermal fatigue |
| Impact         | Tooth bending, tooth shear           |
| Wear           | Abrasive, adhesive                   |
| Stress rupture | Internal, external                   |
2.2. Fatigue
Fatigue failure results from cracking initiation and that crack grows bigger. This failure depends on the following aspects:
(1) The number of repetitions of a given stress range,
(2) Doesn’t appear below the value of fatigue limit,
(3) The presence of notches, grooves, surface discontinuities or surface imperfections lowers the stress amplitude that can be withstood for a specific number of stress cycles,
(4) The increasing the average tensile stress of the loading cycle.

There are three stages of a fatigue failure: the initiation of the crack, the propagation of the crack and the final rupture. When it is about a professional investigation regarding part failures, the attention is devoted to the first stage to answer the question: Why did it started at this point? And this is the answer the present paper is trying to figure out. It was assumed since the beginning of this paper that the degree contact between the curves that define the transversal profile of the part has a significant impact upon the probability of cracks appearances. It was again mentioned above that the greater the degree contact, the smaller the probability of cracks occurrence. It was also assumed that a 2-degree contact between the osculating circle and the involute curve lowers the possibility of cracks occurrences than in the case when there is a 1-degree contact between the root circle and the involute curve.

2.3. Causes of Gear Failures
Generally, the cause of failure is found in the mating or matching part. It is often difficult to separate cause from mode. In many cases, a single cause can generate multiple modes, depending on the applied forces. Also, one single mode could have been initiated by one or more causes. In a professional investigation, there are a lot of steps to go through. The most successful professionals in this area suggest that there are 9 fundamental steps that need to be covered. The steps are:
(1) Understanding the main scope of the investigation;
(2) A clear understanding regarding the occurred failure;
(3) A very clear identification of all causes that produced the failure;
(4) Clear evaluation of each cause;
(5) Identification of the major cause or the identification of the most probable cause;
(6) Identification of the most useful actions that help the investigator to determine the major cause;
(7) Detailed evaluation of each action;
(8) Selection of the most suitable action for a certain case;
(9) Evaluation of the effectiveness of the chosen action.

In detail, the steps presented above concretize in specific actions. It all starts with a visual examination and then the history of the working part is taken into consideration. Then, the gear is examined from different perspectives: there are needed a metallographic analysis, a chemical analysis, a physical analysis, samples collection, photos to samples, detection of manufacturing defects, detection of design errors, detection of surface defects, X-rays examination for a better expertise, simulation tests of the part, etc. The present paper is focused on finding stress concentration factors. And that factors occurs when there exists planar curves that have a $m \leq 1$ degree contact. It is a mathematical approach combined with fatigue parts theory. Beyond the geometrical approach, there were identified four major causes that produce gears fatigue:
(1) Deformation;
(2) Remaining stress;
(3) Corrosion;
(4) Wear.
3. Conclusions
The present paper topic is focused on gears fatigue principles with a different perspective. The paper approach is a mathematical one regarding planar curves contacts. A \( m \leq 1 \) degree contact between two curves can be considered as a stress concentration factor. This general principle is applied in the present paper on a particular case regarding gear root radius. The starting point of the present paper is another paper [1] in which there was proposed a similar root geometry. The fundamental difference between that approach and the current one consists in the effectiveness of the found fillet geometry of each case. In the first case, there was found a root circle that has a 1 – degree contact and the current paper found a unique circle, called the osculating circle, that has, by definition, a 2 – degree contact with the involute curve of the tooth. It was mentioned twice within this content that the greater degree contact between curves (in our case, the root circle and the involute curve), the less possibilities of cracks occurrence and a better effectiveness regarding the avoidance of cracks appearance. Within this paper were illustrated, using Matlab programming language, different important geometries regarding the approached topic. There was presented the geometry of the involute curve and the main necessary theoretical principles regarding the geometry of the tooth and the geometry of the found circle. Because there was found that, in the point where the involute curve parameter is 0, there is an inflexion point, we focused on finding another point in the immediate neighborhood of this point with respect to the theorem 1.1. That point being so close to the previous one, the functionality of gear is not affected and the technical requirements are still respected. The found geometry was illustrated with different graphics using Matlab programming language.

References
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