Semilinear evolution equations for the Anderson Hamiltonian in two and three dimensions

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Abstract

We analyze nonlinear Schrödinger and wave equations whose linear part is given by the renormalized Anderson Hamiltonian in two and three dimensional periodic domains.

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1 Introduction

The basic aim of this paper is to study the following random Cauchy problems

\[ i \partial_t u = Hu - u|u|^2, \quad u(0) = u_0 \quad (1) \]
\[ \partial_t^2 u = Hu - u^3, \quad (u, \partial_t u)|_{t=0} = (u_0, u_1) \quad (2) \]
on the \( d \)-dimensional torus \( \mathbb{T}^d \) with \( d = 2, 3 \). Here \( H \) is formally the Anderson Hamiltonian

\[ H = \Delta + \xi, \]

where \( \xi \) is a space white noise and \( \Delta \) the Laplacian with periodic boundary conditions.

The presence of white noise makes this kind of problems not well posed in classical functional spaces. Indeed it is well known that white noise has sample paths which are only distributions of regularity \(-d/2 - \varepsilon\) in Hölder-Besov spaces, where \( \varepsilon \) is an arbitrary small but non-zero constant. A sign of this difficulty is the fact that the above equations have to be properly renormalized by subtracting a formally infinite constant to the operator \( H \) in order to obtain well defined limits.

In the parabolic setting there is a, by now, well developed theory of such singular SPDEs, thanks to Hairer’s invention of the theory of regularity structures \[16\] and the parallel development of the paracontrolled approach \[14\] by Gubinelli, Imkeller and Perkowski. The first results for non parabolic evolution equations have been obtained in \[9\] where the authors manage to solve the linear and the cubic nonlinear (with a range of powers) Schrödinger equations with multiplicative noise on \( \mathbb{T}^2 \) by first applying a transform inspired by \[17\] and then using mass and energy conservation along with certain interpolation arguments. The wave equations in \( d = 2 \) with polynomial non-linearities and additive space-time white noise have been considered in \[12\]. The main difficulty is that the absence of parabolic regularization makes the control of the non-linear terms involving the singular noise contributions non-trivial.

Here we exploit the insights of \[2\] in order to identify an appropriately renormalized version of \( H \) as a self-adjoint operator on \( L^2(\mathbb{T}^d) \) and use the related spectral decomposition to give a meaning to the above equations as abstract evolution equations in Hilbert space. Our first contribution is then the study of the Anderson Hamiltonian on \( \mathbb{T}^3 \) and the derivation of some additional results when \( d = 2 \), for example the characterization of the form domain of the operator and some related functional inequalities which are needed in the abstract treatment of the evolution equations.

For the sake of the reader, and also to illustrate the proof strategy in the \( d = 3 \) case, we pursue a complete treatment of the \( d = 2 \) case showing the self-adjointness of the Hamiltonian and the convergence of suitable regularized operators in norm resolvent sense. Resolvent convergence is used in the second part to “prepare” suitable initial conditions adapted to prove convergence of approximations. We mention also the proof of a version of the classical Brezis-Gallouet inequality \[5\] for the Anderson Hamiltonian in \( d = 2 \). For \( d = 3 \) we prove that the Anderson Hamiltonian satisfies an inequality which is analogous to the classical Agmon’s inequality, see Lemma \[2.61\] These functional inequalities are
instrumental then in the second part of this work in order to control the non-linear terms of the evolution equations.

An interesting byproduct of our approach is a estimate which expresses the fact that the paraproduct is “almost” adjoint to the resonant product whose definitions we recall in the Appendix. This implies in particular that the energy norm with respect to the Anderson Hamiltonian can be estimated from below in a precise way and allows us to characterize (see Proposition 2.27) both the domain and the form domain of $H$ by using certain Sobolev norms.

The second part of our paper is concerned with the solution of the above equations with different regularities of the initial conditions and with the proof of convergence of solutions of approximate equations where the noise has been regularized and which are then classically well-posed. While the general methodology is the same adopted in [9], namely the use of conservation laws and functional inequalities to control the non-linear term, one of the main contributions of our work is to clarify the role of the spectral theory of the Anderson Hamiltonian and of relative functional spaces in the apriori control of the solutions and in the analysis of the non-linear terms. This simplifies and unifies the analysis of the $d = 2$ and $d = 3$ cases.

After these, having all the necessary Sobolev and $L^p$-estimates at our disposal along with an analogue of Brezis-Gallouet inequality and proper approximation tools; in Section 3 we move on to the study of the nonlinear Schrödinger and wave equations for the Anderson Hamiltonian (properly shifted for positivity) in dimensions 2 and 3. One important point is that by having undertaken the stochastic analysis of the Anderson Hamiltonian in the paracontrolled setting; we are now in a position to address these PDE problems by using classical techniques which sorts out the stochastic and analytical components and their interaction in a coherent and transparent way.

To officially recap, we study the PDEs (1) and (2) (with a range of powers for the non-linearity) with operator domain and finite energy data, depending on the cases and study their uniqueness.

We also work out the convergence of the solutions of regularized equations, obtained by suitable approximations of the initial data and the Gaussian white noise, to the solutions of the above PDEs:

\[ i \partial_t u_\varepsilon = H_\varepsilon u_\varepsilon - u_\varepsilon |u_\varepsilon|^2, \quad u_\varepsilon(0) = u_0^\varepsilon. \numbered{3} \]

\[ \partial_t^2 u_\varepsilon = H_\varepsilon u_\varepsilon - u_\varepsilon^3 \quad \text{on} \quad T^d, \quad (u_\varepsilon, \partial_t u_\varepsilon)|_{t=0} = (u_0^\varepsilon, u_1^\varepsilon). \numbered{4} \]

In Theorem 3.5, we establish the well posedness of (1) with operator domain data in $d = 2$. This is achieved, in part, using our version of the Brezis–Gallouet inequality for the Anderson Hamiltonian. In Theorem 3.9 we show that the solutions to the regularized equations, namely to (3), converge to that of equation (1). Observe that, in this context, establishing this convergence is important as the domain of the Anderson Hamiltonian is contained in $\mathcal{H}^1$—whereas the domain of the approximations lie in $\mathcal{H}^2$. So, there is a drop in smoothness that needs to be addressed carefully. Extension of some of these results to $d = 3$ and focusing case is possible as we prove an analogue of Agmon’s inequality in Lemma 2.61 to replace the Brezis-Gallouet inequality, please see Remark 3.10. Although
we have given the proofs for cubic nonlinearity, one can easily modify this to get a general power nonlinearity, see Remark 3.8.

As we characterize the energy domain for the Anderson Hamiltonian in Lemma 2.27, now we can also make sense of the energy solutions for the NLS, namely (1). Accordingly, in Theorem 3.11 we show the existence of solutions. Observe that, in this case we were not able to show uniqueness; but our result could be considered optimal in view of [6], as Strichartz estimates for the Anderson Hamiltonian is not known. Furthermore, as in the domain case, we show in Corollary 3.15 the convergence of the regularized solutions. One can generalize the result given in the section to the other powers of the nonlinearity similar to the domain data case.

Being able to characterize the energy domain for the Anderson Hamiltonian both in dimensions 2 and 3 enables us to also treat nonlinear stochastic wave equations in both dimensions. In subsection 3.3 we prove our results regarding the well posedness of (2) in 2 and 3 dimensions. In Theorem 3.18 we obtain the well posedness with initial data in the domain and the energy domain. Similar to the Schrödinger case we also show convergence of the regularized solutions in Theorem 3.20. Then we conclude by stating Theorem 3.21 which details the well posedness for initial data in the energy domain and the $L^2$ for (2) and whose proof follows from our earlier considerations in the same section. By our version of Agmon’s inequality and similar methods certain extensions to the different power nonlinearities is possible, please see Remark 3.22 for an elaboration on this topic.

Although we solve the PDEs with an Anderson Hamiltonian which is properly shifted to result in a positive operator, this does not cause any weaker results. As known, this shift simply causes a phase shift (i.e. multiplication by $e^{iCt}$ for some constant $C$) in the NLS case, which one can simply rotate back to the solution of the original equation. In the wave case it simply adds a lower order nonlinearity, in fact a linear term.

In the sequel, we use $\mathcal{H}$ for Sobolev spaces, $L$ for $L^p$-spaces and $\mathcal{C}$ for the Besov-Hölder spaces. As we work either on $\mathbb{T}^2$ or $\mathbb{T}^3$ and it is very clear in what setting we consider throughout the paper, we drop the domain parameter i.e. for $\mathcal{H}^2(\mathbb{T}^3)$ we simply write $\mathcal{H}^2$. We denote the Gaussian white noise by $\xi$ and enhanced noise by $\Xi$ (see Definition 2.4 and Theorem 2.38).

We reserve the letter $A$ for the Anderson Hamiltonian and we use the letter $H$ to denote the operator shifted by a specific constant $K_\Xi$. We denote by $C_\Xi$ the constants depending on certain norms (which will be clear from the context) of the (enhanced) noise. This constant can change value from line to line. We use the notation $\mathcal{X}$ for the enhanced noise space both in $d = 2$ and $d = 3$.

For the convenience of the reader we included an Appendix containing mostly classical results and the results from other papers such as [2,14,15]. In Appendix, we recall the definitions of relevant function spaces and other harmonic analysis topics such as Littlewood-Paley theory and Bony paraproducts. The result included in Proposition A.6 is new and possibly of independent interest.

After the completion of the present work we became aware of recent, still unpublished, work of C. Labbé [18] where he constructs the Anderson Hamiltonian in $d = 3$ with Dirichlet boundary conditions using regularity structures and produces some results about the law of its eigenvalues.
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2 The Anderson Hamiltonian in two and three dimensions

We resume certain concepts and definitions that we will use throughout this section. First we recall the definition of Gaussian white noise on $T^d$.

**Definition 2.1.** The Gaussian white noise $\xi$ is a family of centered Gaussian random variables $\{\xi(\phi) \mid \phi \in L^2(T^d)\}$, covariance given by

$$E(\xi(\phi)\xi(\psi)) = \langle \phi, \psi \rangle_{L^2(T^d)}.$$

To get an intuitive description, let $\hat{\xi}(k)$ be i.i.d centered complex Gaussian random variables with $\hat{\xi}(k) = \overline{\hat{\xi}(-k)}$ and covariance

$$E(\hat{\xi}(k)\hat{\xi}(l)) = \delta(k - l).$$

Formally, the Gaussian white noise, on the torus, can be thought as the following random series

$$\xi(x) = \sum_{k \in \Lambda} \hat{\xi}(k) e^{2\pi ik \cdot x},$$

where in this section we will respectively take $\Lambda$ to be $\mathbb{Z}^2$ and $\mathbb{Z}^3 \setminus \{0\}$. That is, in the 3d case, we simply take out the zero mode for ease of computations.

We also define the regularized spatial white noise as

$$\xi_\varepsilon(x) = \sum_{k \in \Lambda} m(\varepsilon k) e^{2\pi ik \cdot x} \hat{\xi}(k), \quad (5)$$

where $m$ is a smooth radial function on $\mathbb{R} \setminus \{0\}$ with compact support such that

$$\lim_{x \to 0} m(x) = 1.$$

We put

$$c_\varepsilon := \sum_{k \in \mathbb{Z}^2} \frac{|m(\varepsilon k)|^2}{1 + |k|^2} \log \left( \frac{1}{\varepsilon} \right) \sim \log \left( \frac{1}{\varepsilon} \right),$$

which will act as one of the renormalization constants in 2d case. Note that this constant depends on the choice of mollifier.

We also recall the Anderson Hamiltonian, which is formally the following operator

$$A = \Delta + \xi \quad (6)$$

where $\xi$ is the Gaussian white noise. As we have articulated in the introduction, this operator cannot be defined in $L^2(T^{2,3})$ because of the low Hölder regularity of $\xi$. The
Besov-Hölder regularity of Gaussian white noise on $\mathbb{T}^d$ is $-\frac{d}{2} - \delta$, that is $\xi \in C^{-\frac{d}{2} - \delta}$ almost surely, for any positive $\delta > 0$ \cite{14}.

Therefore, we will consider a renormalization of this operator in the context of paracontrolled distributions which one can formally write as

$$ A = \Delta + \xi - \infty $$

and to which we will give meaning as a suitable limit $\varepsilon \to 0$ of the regularized Hamiltonians

$$ A_\varepsilon = \Delta + \xi_\varepsilon - c_\varepsilon, $$

for precise constants $c_\varepsilon$.

Accordingly, in this section, we define the Anderson Hamiltonian and introduce suitable regularizations in the setting of paracontrolled distributions in two and three dimensional torus, respectively in the following subsections. Namely, we construct a suitable (dense) domain for the operator and then show closedness, symmetry, self-adjointness and norm resolvent convergence (of the regularized Hamiltonians). At the end of both 2d and 3d cases, we prove certain functional inequalities which we will use in the PDE part of the paper, namely in Section 3.

2.1 The two dimensional case

In this part, we work on the 2d torus. We follow the same line of thought as in \cite{2} with important modifications. In \cite{2} authors worked in the 2d case but our modifications will enable us to use similar proofs in Section 2.2, namely for the 3d case, and also obtain certain functional inequalities such as the Brezis-Gallouet inequality for the Anderson Hamiltonian. In this section, for paraproducts we use the notations “≺” and “≻”; please see the Appendix for precise definitions of the function spaces and concepts from harmonic analysis that will be used throughout this section.

2.1.1 Enhanced noise, the domain and the $\Gamma$-map

In order to introduce the paracontrolled ansatz (see \cite{13} Section 3) for motivation), which will enable us to define the domain of the operator, we need the following definition.

**Definition 2.2.** For $\alpha \in \mathbb{R}$, we define $\mathcal{E}^\alpha := C^\alpha \times C^{2\alpha+2}$ and $\mathcal{E}^{-\alpha}$ as the closure of the set $\{(\eta, \eta \circ (1-\Delta)^{-1}(\eta + c)) : \eta \in C^\infty(\mathbb{T}^2), c \in \mathbb{R}\}$ w.r.t. the $\mathcal{E}^\alpha$ topology, where $\mathcal{E}^\alpha = B^\alpha_{\infty\infty}$ denotes the Besov-Hölder space.

We point out that, $\mathcal{E}^{-\alpha}$ is the space of “enhanced noise”. In some sense, one needs to lift the singular term white noise into a larger space which, in some sense, also encodes the regularization. However, one needs to do this consistently; namely the lift should not depend on the mollifier. This is the content of the following result, which was proved in \cite{2} Theorem 5.1].
Theorem 2.3. For any \( \alpha < -1 \) we have
\[
\Xi^\varepsilon := (\xi^\varepsilon, \xi^\varepsilon \circ (1 - \Delta)^{-1} \xi + c^\varepsilon) \to \Xi = (\Xi_1, \Xi_2) \in \mathcal{X}^\alpha,
\]
where the convergence holds as \( \varepsilon \to 0 \) in \( L^p(\Omega; \mathcal{E}^\alpha) \) for all \( p > 1 \) and almost surely in \( \mathcal{E}^\alpha \). Moreover, the limit is independent of the mollifier and \( \Xi_1 = \xi \).

By this result, one can see that
\[
\|\xi\|_{C^{\alpha}}, \|\Xi_2\|_{C^{2\alpha+2}}, \|(1 - \Delta)^{-1}\xi\|_{C^{\alpha+2}} < \infty \text{ a.s.}
\]
by Schauder estimates.

We also recall the following definition which describes the domain of the Anderson Hamiltonian.

Definition 2.4. Assume \( -\frac{4}{7} < \alpha < -1 \) and \( -\frac{\alpha}{7} < \gamma \leq \alpha + 2 \). Then we define the space of functions paracontrolled by the enhanced noise \( \Xi \) as follows
\[
\mathcal{D}_\Xi^\gamma := \{ u \in \mathcal{H}^\gamma \text{ s.t. } u = u \prec X + B_\Xi(u) + u^\sharp, \text{ for } u^\sharp \in \mathcal{H}^2 \}
\]
where \( X = (1 - \Delta)^{-1} \xi \in \mathcal{C}^{\alpha+2} \) and
\[
B_\Xi(u) := (1 - \Delta)^{-1}(\Delta u \prec X + 2\nabla u \prec \nabla X + \xi \prec u - u \prec \Xi_2) \in \mathcal{H}^{2\gamma}.
\]
This space is equipped with the scalar product given by, \( u, w \in \mathcal{D}_\Xi^\gamma \),
\[
\langle u, w \rangle_{\mathcal{D}_\Xi^\gamma} := \langle u, w \rangle_{\mathcal{H}^\gamma} + \langle u^\sharp, w^\sharp \rangle_{\mathcal{H}^2}.
\]

We have several remarks now, that explains our modification of this definition.

Remark 2.5. For the rest of the paper, we set
\[
\mathcal{D}(A) := \mathcal{D}_\Xi^\gamma.
\]
Observe that at this point this is to uniformize notation and we are not stating equality as normed spaces, namely as a normed space we have \( \mathcal{D}(A) = (\mathcal{D}_\Xi^\gamma, \|\cdot\|_{\mathcal{D}_\Xi^\gamma}) \). But in the sequel, it will be clear after Proposition \( 2.27 \) that we also have \( (\mathcal{D}(A), \|\cdot\|_{\mathcal{D}(A)}) = (\mathcal{D}_\Xi^\gamma, \|\cdot\|_{\mathcal{D}_\Xi^\gamma}) \) where \( \|\cdot\|_{\mathcal{D}(A)} \) denotes the standard (functional analytic) domain norm.

We work out the following modification of the above ansatz (10) to fit our purposes. Assume \( u \) is of the form
\[
u = \Delta_{>N}(u \prec X + B_\Xi(u)) + u^\sharp,
\]
for \( 2/3 < \gamma < 1 \). and \( \Delta_{>N} \) denotes a frequency cut-off at \( 2^N \), more precisely,
\[
\Delta_{>N}f := \mathcal{F}^{-1} \chi_{|\cdot|>2^N} \mathcal{F} f,
\]
with \( N \in \mathbb{N} \) which will be chosen depending on the (enhanced) noise \( \Xi \). We also define
\[
B_\Xi(u) := (1 - \Delta)^{-1}(\Delta u \prec X + 2\nabla u \prec \nabla X + \xi \prec u - u \prec \Xi_2).
\]
Note that by Schauder estimates we have the following bound for \( B \),
\[
\|B_\Xi(u)\|_{\mathcal{H}^s} \lesssim s C_\Xi \|u\|_{\mathcal{H}^{s-\gamma}}, \quad s \in [0, 2\gamma].
\]
Recall that, we denote by \( C_\Xi \) a constant that depends explicitly on the norm of the realization of the enhanced noise \( \Xi \) and can change from line to line.
**Remark 2.6.** This modification changes the decomposition by a smooth function so it does not change the space. Strictly speaking, one obtains a different norm depending on $N$, which is equivalent to the $\mathcal{G}(A)$ norm above. In fact, assume that for a function $f$ and some $N \geq 1$ we have

\[ f = f \prec X + B_\Xi(f) + f_1^0, \]

and

\[ f = \Delta_{>N}(f \prec X + B_\Xi(f)) + f_2^N. \]

Then we readily have the estimate

\[
\|f_1^0\|_{\mathcal{H}^2} = \|f - f \prec X + B_\Xi(f)\|_{\mathcal{H}^2} \\
\quad = \|f - \Delta_{>N}(f \prec X + B_\Xi(f)) - \Delta_{\leq N}(f \prec X + B_\Xi(f))\|_{\mathcal{H}^2} \\
\quad \leq \|f_2^N\|_{\mathcal{H}^2} + C(N, \Xi)\|f\|_{\mathcal{H}^\gamma}
\]

and analogously $\|f_2^N\|_{\mathcal{H}^2} \leq \|f_2^N\|_{\mathcal{H}^2} + C(N, \Xi)\|f\|_{\mathcal{H}^\gamma}$. This proves the norm equivalence.

With this modification of the ansatz, we can write $u$ as a function of $u^\sharp$. In order to do so, we define the following linear map $\Gamma$

\[ \Gamma f = \Delta_{>N}(\Gamma f \prec X + B_\Xi(\Gamma f)) + f, \]

so that $u = \Gamma u^\sharp$. For $N$ large enough, depending on the realization of $\Xi$, we can show that this map exists and has useful bounds.

**Remark 2.7.** In the following, we will utilize this map $\Gamma$ to show density of the domain, symmetry and norm resolvent convergence. The key point is the map $\Gamma$ can also be defined in the 3d case and be used there in a similar manner, which we will do in the 3d section.

By these considerations, we can bound certain Sobolev norms of $u$ by that of $u^\sharp$, which is the content of the following result.

**Proposition 2.8.** We can choose $N$ large enough depending only on $C_\Xi$ and $s$ so that

\[ \|\Gamma f\|_{L^\infty} \leq 2\|f\|_{L^\infty}, \quad (13) \]

\[ \|\Gamma f\|_{\mathcal{H}^s} \leq D_\Xi\|f\|_{\mathcal{H}^s}, \quad (14) \]

for some constant $D_\Xi$ for $s \in [0, \gamma]$ and $D_\Xi = 3$ for $s \in [0, \gamma)$.

**Proof.** Let us start with proving the $L^\infty$ bound. Choose $\delta > 0$ and let $g = \Gamma f$, we have

\[
\|B_\Xi(g)\|_{\mathcal{H}^{\gamma-\delta}} \leq \|\Delta g \prec X\|_{\mathcal{H}^{\gamma-\delta-2}} + 2\|\nabla g \prec \nabla X\|_{\mathcal{H}^{\gamma-\delta-2}} \\
+ \|\xi \prec g\|_{\mathcal{H}^{\gamma-\delta-2}} + \|g \prec \Xi_2\|_{\mathcal{H}^{\gamma-\delta-2}} \\
\leq 3\|g\|_{\mathcal{H}^{-\delta}}\|X\|_{\mathcal{H}^{\gamma}} + \|\xi\|_{\mathcal{H}^{\gamma-2}}\|g\|_{\mathcal{H}^{-\delta}} \leq C_\Xi\|g\|_{\mathcal{H}^{-\delta}} \lesssim C_\Xi\|g\|_{L^\infty}
\]

by paraproduct estimates and the fact that $\|g\|_{\mathcal{H}^{-\delta}} \lesssim \|g\|_{L^\infty}$ for any small $\delta > 0$. Now we can write,

\[
\|g\|_{L^\infty} \leq \|\Delta_{>N}(\Gamma f \prec X + B_\Xi(\Gamma f))\|_{L^\infty} + \|f\|_{L^\infty} \leq 2^{(\delta-\gamma)N}\|g \prec X + B_\Xi(g)\|_{\mathcal{H}^{\gamma-\delta}} + \|f\|_{L^\infty}
\]

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\[ \leq 2^{(\delta^{-\gamma})N}(\|X\|_{\mathcal{K}^{\gamma-\delta}} + C_\delta C_\Xi)\|g\|_{L^\infty} + \|f\|_{L^\infty} \leq C_\delta C_\Xi 2^{(\delta^{-\gamma})N}\|g\|_{L^\infty} + \|f\|_{L^\infty} \]
and choose \( N \) large enough so that \( 2C_\delta C_\Xi 2^{(\delta^{-\gamma})N} \leq 1 \) which implies \( \|g\|_{L^\infty} \leq 2\|f\|_{L^\infty} \).

For the \( \mathcal{K}^s \) bound we can proceed more simply by noting that
\[ \|B_\Xi(g)\|_{\mathcal{K}^s} \leq (3\|X\|_{\mathcal{K}^\gamma} + \|\xi\|_{\mathcal{K}^{\gamma-\delta}} + \|\Xi\|_{\mathcal{K}^{\gamma-\delta}})\|g\|_{L^2} \]
and if \( s \leq \gamma \) we have
\[ \|g\|_{\mathcal{K}^s} \leq CC_\Xi 2^{(s-\gamma)N}\|g\|_{L^2} + \|f\|_{\mathcal{K}^s}. \]

If \( s = 0 \) we can choose \( N \) large enough so that \( \|g\|_{L^2} \leq 2\|f\|_{L^2} \) and as a consequence we have also
\[ \|g\|_{\mathcal{K}^s} \leq 2CC_\Xi 2^{(s-\gamma)N}\|f\|_{L^2} + \|f\|_{\mathcal{K}^s} \]
for all \( s \leq \gamma \). If \( s < \gamma \) we can have \( N \) large enough (depending on \( s \)) so that \( \|g\|_{\mathcal{K}^s} \leq 3\|f\|_{\mathcal{K}^s}. \)

**Remark 2.9.** Note that \( \mathcal{D}_\Xi^2 \) is actually independent of \( \gamma \), since for \( \gamma, \gamma' \in (2/3, 1) \) we can compute
\[ \|u\|_{\mathcal{K}^\gamma} \lesssim \|u^\delta\|_{\mathcal{K}^\gamma} \lesssim \|u\|_{\mathcal{D}_\Xi^\gamma'}. \]
and vice versa, so the \( \mathcal{D}_\Xi^2 \) and \( \mathcal{D}_\Xi^\gamma' \) norms are equivalent and we will from now on drop the \( \gamma \) and write simply \( \mathcal{D}_\Xi \). That is, we have \( \mathcal{D}(A) = \mathcal{D}_\Xi \).

As a first step we prove that the domain of \( A \), now defined to be \( \mathcal{D}(A) \), is dense in \( L^2 \).

Before that we note the following remark and then a lemma.

**Remark 2.10.** In the sequel, we put
\[ X_\varepsilon = (1 - \Delta)^{-1}\xi_\varepsilon. \]
and similar to the operator \( \Gamma \) in Lemma 2.8 we define \( \Gamma_\varepsilon \) as follows
\[ \Gamma_\varepsilon u := \Delta_N(\Gamma_\varepsilon u \prec X_\varepsilon + B_\Xi(\Gamma_\varepsilon u)) + u^\varepsilon, \]
where \( \Xi_\varepsilon \to \Xi \) in \( \mathcal{K}^\alpha \).

For the above introduced \( \Gamma_\varepsilon \) we prove the following lemma, which will be useful in the sequel.

**Lemma 2.11.** We have that \( \|\text{id} - \Gamma_\varepsilon^{-1}\|_{\mathcal{K}^{\gamma-\gamma}} \to 0. \)

**Proof.** For \( f \in \mathcal{K}^{\gamma} \), we can write, by using Proposition 2.8
\[ \|f - \Gamma_\varepsilon^{-1}(f)\|_{\mathcal{K}^{\gamma}} = \|\Gamma(f - f \prec X + B_\Xi(f)) - \Gamma(f - f \prec X_\varepsilon + B_\Xi(f))\|_{\mathcal{K}^{\gamma}} \]
\[ = \|\Gamma(f \prec (X_\varepsilon - X) + B_{\Xi_\varepsilon - \Xi}(f))\|_{\mathcal{K}^{\gamma}} \]
\[ \leq D_\Xi \|f\|_{\mathcal{K}^{\gamma}} \|\Xi_\varepsilon - \Xi\|_{\mathcal{K}^{\alpha}}, \]
which shows that \( \text{id} = \Gamma^{-1} \) converges to \( \Gamma_\varepsilon^{-1} \) in operator norm.

**Corollary 2.12.** The space \( \mathcal{D}(A) \), as defined in Definition 2.4, is dense in \( \mathcal{K}^{\gamma} \), therefore dense in \( L^2 \).
Proof. For \( f \in \mathcal{H}_\gamma \), by Lemma 2.11, simply observe that
\[
\|f - \Gamma^{-1}_\varepsilon f\|_{\mathcal{H}_\gamma} \to 0.
\]
Hence, the result. \(\square\)

We are in a position to define the operator \( A \) in \( L^2 \) on its domain \( \mathcal{D}(A) \).

**Definition 2.13.** We define the operator \( A : \mathcal{D}(A) \to L^2 \) as
\[
Au := \Delta u^\sharp + u^\flat \circ \xi + G(u),
\]
where in \( u^\sharp = u \prec \xi + \xi \prec u + u \circ \xi \) we have defined
\[
u \circ \xi := (\Delta > N (u \prec X + B\Xi(u))) \circ \xi = C_N(u, \xi, X) + u\Xi_2 + (\Delta > N B\Xi) \circ \xi + u^\flat \circ \xi \quad \text{and we put}\n\]
\[
G(u) := \Delta \leq N (u \prec \xi + u \succ \xi + u \prec \Xi_2) + \Delta > N (-B\Xi(u) - u \prec X + u \succ \Xi_2 + C_N(u, X, \xi) + B\Xi(u) \circ \xi).
\]

Here, \( C_N(u, \xi, X) := (\Delta > N (u \prec X)) \circ \xi - u(X \circ \xi) \) is the modified commutator which satisfies bounds (depending on the fixed \( N \)) as shown in Proposition A.3.

**Remark 2.14.** By using the regularities in Definition 2.4, one can easily check, through Proposition A.1, that \( Au \) is in fact in \( L^2 \). Later, respectively in Proposition 2.19 and Theorem 2.31 we obtain this operator as a norm and norm resolvent limit of \( A_\varepsilon \) which perfectly motivates the informal identity
\[
A = \Delta + \xi - \infty.
\]

In the following result, we show that the \( \mathcal{H}^2 \)-norm of \( u^\sharp \) can be bounded above by the (standard) domain norm of \( A \).

**Proposition 2.15.** There exists a constant \( C_\Xi > 0 \) depending on the enhanced noise such that
\[
\|u^\sharp\|_{\mathcal{H}^2} \leq 2\|Au\|_{L^2} + C_\Xi \|u\|_{L^2}.
\]

**Proof.** First, we note that \( \Delta u^\sharp \in L^2 \) by assumption. For the resonant term we compute
\[
\|u^\sharp \circ \xi\|_{L^2} \leq \|((\Delta_{\leq M} u^\sharp) \circ \xi)\|_{L^2} + \|((\Delta_{> M} u^\sharp) \circ \xi)\|_{L^2}
\leq C_\Xi 2^{2M} \|u^\sharp\|_{L^2} + \|\Delta_{> M} u^\sharp\|_{\mathcal{H}^{1+\delta}} \|\xi\|_{\mathcal{H}^{-1-2\delta}}
\]
for \( \delta \) sufficiently small, giving, for any \( M \geq 0 \),
\[
\|u^\sharp \circ \xi\|_{L^2} \lesssim (2^M \|u^\sharp\|_{L^2} + 2^{(\delta - 1)M} \|u^\sharp\|_{\mathcal{H}^2}),
\]
where we have used Bernstein’s inequality (Lemma A.4) and Theorem 2.3 for the noise. Using again Bernstein’s inequality for the low-frequency terms and the paraproduct estimates for the high-frequency terms, we obtain the bound
\[
\|G(u)\|_{L^2} \leq C_\Xi \|u\|_{\mathcal{H}_\gamma},
\]
for $\gamma < 1$, where the constant can be chosen as
\[ C_\Xi = C 2^{2N} (\| \xi \|_{C^\alpha} + \| \Xi_2 \|_{C^{2\alpha+2}}) \]
with $\alpha < -1$ as before.

By using these, for the $H^2$ bound, we compute
\[ \| \Delta u^\sharp \|_{L^2} \leq \| Au \|_{L^2} + \| u^\sharp \circ \xi \|_{L^2} + \| G(u) \|_{L^2}. \]

Now, as above we have
\[ \| u^\sharp \circ \xi \|_{L^2} \lesssim_\Xi (2^M \| u^\sharp \|_{L^2} + 2^{\gamma M} \| u^\sharp \|_{H^2}) \]
and
\[ \| G(u) \|_{L^2} \lesssim_\Xi \| u \|_{H^\gamma} \lesssim_\Xi \| u^\sharp \|_{H^\gamma} \approx \| \Delta_{> M} u^\sharp \|_{H^\gamma} + \| \Delta_{\leq M} u^\sharp \|_{H^\gamma} \]
and using again Bernstein’s inequality for the low-frequency part we get
\[ \| G(u) \|_{L^2} \lesssim C_\Xi (2^M \| u \|_{L^2} + 2^{\gamma M} \| u^\sharp \|_{H^2}) \]
where we have used the straightforward bound $\| u^\sharp \|_{L^2} \leq C_\Xi \| u \|_{L^2}$. Finally, choosing $M$ large enough (depending on $\Xi$), we obtain
\[ \| u^\sharp \|_{H^2} \leq 2 \| Au \|_{L^2} + C_\Xi \| u \|_{L^2}. \]

Hence, the result. \( \square \)

2.1.2 Density, symmetry, self-adjointness and convergence

In the following, we show that $A$ is a closed and symmetric operator on $\mathcal{D}(A)$. We first establish closedness.

Proposition 2.16. We have that $A$ is a closed operator on its dense domain $\mathcal{D}(A)$.

Proof. Assume $(u_n) \subset \mathcal{D}(A)$ is a sequence s.t.
\[ u_n \to u \quad \text{in} \quad L^2 \]
and
\[ Au_n \to g \quad \text{in} \quad L^2 \]
for some $g \in L^2$. Then $u_n^\sharp$ forms a Cauchy sequence in $H^2$ and thus converges to a limit that we call $u^\sharp$. Moreover $\Gamma u^\sharp = u$, so $u \in \mathcal{D}(A)$. Thus
\[ \| Au - g \|_{L^2} \leq \| Au - Au_n \|_{L^2} + \| Au_n - g \|_{L^2} \leq C_\Xi \| u^\sharp - u_n^\sharp \|_{H^2} + \| Au_n - g \|_{L^2} \]
where the second step comes from the proof of Proposition 2.15. Since both terms on the right-hand side tend to zero as $n \to \infty$ we get $Af = g$, namely that $A$ is closed. \( \square \)
Before we show the symmetry we need the following approximation result.

**Proposition 2.17.** For every $u \in \mathcal{D}(A)$ there exists a sequence $u_\varepsilon \in \mathcal{H}^2$ such that

$$||u - u_\varepsilon||_{\mathcal{H}^\gamma} + ||u^\sharp - u_\varepsilon^\sharp||_{\mathcal{H}^2} \to 0$$

as $\varepsilon \to 0$.

**Proof.** We take an arbitrary approximation $u_\varepsilon^\sharp \to u^\sharp$ in $\mathcal{H}^2$ and set $q_\varepsilon := \Gamma_\varepsilon(u_\varepsilon^\sharp)$, $g_\varepsilon := \Gamma(u_\varepsilon^\sharp)$, that is

$$q_\varepsilon = \Delta_\varepsilon N(q_\varepsilon \prec X_\varepsilon + B_\varepsilon(q_\varepsilon)) + u_\varepsilon^\sharp$$
$$g_\varepsilon = \Delta_\varepsilon N(g_\varepsilon \prec X + B_\varepsilon(g_\varepsilon)) + u_\varepsilon^\sharp.$$  

We have $||u^\sharp - u_\varepsilon^\sharp||_{\mathcal{H}^2} \to 0$ by construction. We can write

$$||u - q_\varepsilon||_{\mathcal{H}^\gamma} \leq ||u - g_\varepsilon||_{\mathcal{H}^\gamma} + ||g_\varepsilon - q_\varepsilon||_{\mathcal{H}^\gamma}$$

We have, by Proposition 2.8 that

$$||u - g_\varepsilon||_{\mathcal{H}^\gamma} = ||\Gamma(u^\sharp) - \Gamma(u_\varepsilon^\sharp)||_{\mathcal{H}^\gamma} \leq ||u^\sharp - u_\varepsilon^\sharp||_{\mathcal{H}^2} \to 0$$

Similar to Proposition 2.8 we also have that

$$||g_\varepsilon - q_\varepsilon||_{\mathcal{H}^\gamma} = ||\Gamma_\varepsilon(u^\sharp) - \Gamma_\varepsilon(u_\varepsilon^\sharp)||_{\mathcal{H}^\gamma} \leq ||u^\sharp - u_\varepsilon^\sharp||_{\mathcal{H}^2} \to 0.$$  

This settles (19).

**Remark 2.18.** Observe that, though stated generally, in Proposition 2.17 one might as well take $u_\varepsilon^\sharp = u^\sharp$ and obtain the approximations $u_\varepsilon = \Gamma_\varepsilon(u^\sharp)$ with the stated properties.

Now, we are also ready to show the norm convergence of the approximating operators.

**Proposition 2.19.** Let $u^\sharp \in \mathcal{H}^2$, $u = \Gamma(u^\sharp)$ and $u_\varepsilon = \Gamma_\varepsilon(u^\sharp)$. We have that

$$||Au - A_\varepsilon u_\varepsilon||_{L^2} \lesssim \Xi ||u^\sharp||_{\mathcal{H}^2}.$$  

Consequently, this implies that

$$||A\Gamma - A_\varepsilon \Gamma_\varepsilon||_{\mathcal{H}^2 \to L^2} \to 0.$$  

That is to say, $A_\varepsilon \Gamma_\varepsilon \to A\Gamma$ in norm.

**Proof.** By using the formula 15 we observe that all terms in $Au - A_\varepsilon u_\varepsilon$ are bilinear. For the upper bound, by addition and subtraction of cross terms, one obtains terms of the form

$$||\Xi_\varepsilon - \Xi||_{\mathcal{H}^\alpha} ||u^\sharp||_{\mathcal{H}^2} + ||u_\varepsilon - u||_{\mathcal{H}^\gamma} ||\Xi||_{\mathcal{H}^\alpha}.$$  

Now, recall that $u = \Gamma(u^\sharp), u_\varepsilon = \Gamma_\varepsilon(u^\sharp)$. Then, we obtain terms of the form

$$||\Xi_\varepsilon - \Xi||_{\mathcal{H}^\alpha} ||u^\sharp||_{\mathcal{H}^2} + ||\Gamma_\varepsilon - \Gamma||_{\mathcal{H}^2 \to \mathcal{H}^\gamma} ||u^\sharp||_{\mathcal{H}^2} ||\Xi||_{\mathcal{H}^\alpha}.$$  

By using Lemma 2.11 and the estimate in its proof, the result (20) is now immediate.
After this we immediately obtain the symmetry of the operator.

**Corollary 2.20.** Let \( u \in \mathcal{D}(A) \) and \( u_\varepsilon \in \mathcal{H}^2 \) be as in Proposition 2.17. Then, we obtain
\[
\langle A_\varepsilon u_\varepsilon, v_\varepsilon \rangle \to \langle Au, v \rangle.
\]

Consequently, we have that \( A \) is a symmetric operator on its dense domain \( \mathcal{D}(A) \).

**Proof.** This directly follows from Proposition 2.17. Using the symmetry of \( A_\varepsilon \) implies the symmetry of \( A \) through the equalities
\[
\langle u, Av \rangle = \lim_{\varepsilon \to 0} \langle u_\varepsilon, A_\varepsilon v_\varepsilon \rangle = \lim_{\varepsilon \to 0} \langle A_\varepsilon u_\varepsilon, v_\varepsilon \rangle = \langle Au, v \rangle.
\]
Hence, the result.

The next result shows that the quadratic form given by \(-A\) is, through addition of a constant, bounded from below by the \( \mathcal{H}^1 \) norm of \( u^\varepsilon \). We will later use this estimate to bound certain norms by a (conserved) energy, when we deal with the NLS and the nonlinear wave equations.

**Proposition 2.21.** There exists a constant \( C_\Xi > 0 \) such that
\[
\frac{1}{2} \langle \nabla u^\varepsilon, \nabla u^\varepsilon \rangle \leq -\langle u, Au \rangle + C_\Xi \|u\|^2_{L^2}.
\]

**Proof.** Expanding the Ansatz and integrating by parts we get
\[
\langle u, Au \rangle = \langle u, \Delta u^\varepsilon \rangle + \langle u, u^\varepsilon \circ \xi \rangle + \langle u, G(u) \rangle
\]
\[
= \langle \Delta \varepsilon (u < X), \Delta u^\varepsilon \rangle + \langle u^\varepsilon , \Delta u^\varepsilon \rangle + \langle u, u^\varepsilon \circ \xi \rangle + \langle u, G(u) \rangle + \langle \Delta \varepsilon B_\Xi (u), \Delta u^\varepsilon \rangle
\]
\[
= -\langle \Delta \varepsilon (u < \xi), u^\varepsilon \rangle + \langle \Delta \varepsilon (u < X), u^\varepsilon \rangle - \langle \nabla u^\varepsilon, \nabla u^\varepsilon \rangle + \langle u, u^\varepsilon \circ \xi \rangle
\]
\[
+ \langle u, G(u) \rangle + \langle \Delta \varepsilon B_\Xi (u), \Delta u^\varepsilon \rangle + 2\langle \Delta \varepsilon (\nabla u < \nabla X), u^\varepsilon \rangle + \langle \Delta \varepsilon B_\Xi (u), \Delta u^\varepsilon \rangle
\]
\[
= D(u, \xi, \Delta \varepsilon (u^\varepsilon) \circ \xi) + \langle u, \Delta \varepsilon (u^\varepsilon) \circ \xi \rangle + \langle \Delta \varepsilon B_\Xi (u), \Delta u^\varepsilon \rangle
\]
where
\[
D(u, \xi, \Delta \varepsilon (u^\varepsilon) \circ \xi) := \langle u, \Delta \varepsilon (u^\varepsilon) \circ \xi \rangle - \langle u < \xi, \Delta \varepsilon (u^\varepsilon) \rangle.
\]

Now fix a sufficiently small \( \delta > 0 \), then
\[
|\langle u, (\Delta \leq \varepsilon u^\varepsilon) \circ \xi \rangle| \leq 2^{2(1+2\delta)N} \|\xi\|_{\mathcal{H}^{1-\delta}} \|u^\varepsilon\|_{L^2} \|u\|_{L^2} \leq C_\Xi 2^{2(1+2\delta)N} \|u\|^2_{L^2}
\]
since from the Ansatz we have easily \( \|u^\varepsilon\|^2_{L^2} \leq C_\Xi \|u\|^2_{L^2} \). Moreover
\[
|\langle u, G(u) \rangle| \leq \|u\|^2_{L^2} + \|u^\varepsilon\|^2_{\mathcal{H}^{1-\delta}}
\]
\[
|\langle \Delta \varepsilon u < X, u^\varepsilon \rangle| + |\langle \nabla u < \nabla X, u^\varepsilon \rangle| \leq \|u\|_{\mathcal{H}^{1-\delta}} \|X\|_{\mathcal{H}^{1-\delta}} \|u^\varepsilon\|^2_{\mathcal{H}^{2\delta}} \leq C_\Xi \|u\|^2_{\mathcal{H}^{1-\delta}}
\]
\[
|\langle \Delta \varepsilon B_\Xi (u), \Delta u^\varepsilon \rangle| = |\langle \Delta \varepsilon B_\Xi (u), \Delta u^\varepsilon \rangle| \leq \|B_\Xi (u)\|_{\mathcal{H}^{2\delta}} \|u^\varepsilon\|^2_{\mathcal{H}^{2\delta}} \leq C_\Xi \|u\|^2_{\mathcal{H}^{1-\delta}}
\]
and similarly we bound the term \( \langle \Delta \varepsilon (u < X), u^\varepsilon \rangle \). By proof of Proposition 2.16 we have also
\[
|D(u, \xi, \Delta \varepsilon (u^\varepsilon))| \leq \|\xi\|_{\mathcal{H}^{1-\delta}} \|u\|_{\mathcal{H}^{(1+\delta)/2}} \|\Delta \varepsilon (u^\varepsilon)\|_{\mathcal{H}^{(1+\delta)/2}} \leq C_\Xi \|u\|^2_{\mathcal{H}^{1-\delta}}
\]

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Using that
\[ \|u^i\|_{\mathcal{H}^{1-\delta}}^2 \lesssim \|\Delta M u^i\|_{\mathcal{H}^{1-\delta}}^2 + \|\Delta_{\leq M} u^i\|_{\mathcal{H}^{1-\delta}}^2 \lesssim 2^{2M(1-\delta)} \|u\|_{L^2} + 2^{-2\delta M} \|u^i\|_{\mathcal{H}^{1}}^2 \]
and choosing \( M \) large enough we can obtain that
\[ \frac{1}{2} \langle \nabla u^i, \nabla u^i \rangle \leq -\langle u, Au \rangle + C_\Xi \|u\|_{L^2}^2. \]

Now, we are in a position to define the form domain of the operator. We first shift the operators \( A \) and \( A_\varepsilon \) by a constant to obtain a positive operator.

**Remark 2.22.** One can readily check that the preceding analysis is valid as well for the approximate Hamiltonians \( A_\varepsilon \), given by (8), simply by replacing the noise \( \Xi \) by its regularization \( \Xi_\varepsilon \). Moreover, since all the constants we obtain are polynomials in the \( \mathcal{P}^a \) norm of the noise, one sees that they can be chosen to hold uniformly in \( \varepsilon \), since \( \|\Xi_\varepsilon\|_{\mathcal{P}^a} \leq \|\Xi\|_{\mathcal{P}^a} \). In particular, the result in Proposition 2.21 is true for \( A_\varepsilon \) and \( \Xi_\varepsilon \) for the same constant \( C_\Xi \).

**Proposition 2.23.** There exists a constant \( K_\Xi \) which is independent of \( \varepsilon \) s.t.
\[ (K_\Xi - A)^{-1} : L^2 \rightarrow \mathcal{D}(H) \]  \hspace{1cm} (26)
\[ (K_\Xi - A_\varepsilon)^{-1} : L^2 \rightarrow \mathcal{H}^2 \]  \hspace{1cm} (27)
are bounded.

**Proof.** We will prove the statement for \( A \) using a generalization of Lax-Milgram, see [3]. The proof for \( A_\varepsilon \) follows the same lines with the same constant \( K_\Xi \), in virtue of Remark 2.22.

Fix the constant \( K_\Xi > C_\Xi > 0 \) (\( C_\Xi \) as in (24)) such that
\[ \|u\|_{L^2}^2 < \langle -(A - K_\Xi)u, u \rangle \ \forall u \in \mathcal{D}(A). \]
which is possible by Proposition 2.21

Define the bilinear map
\[ B : \mathcal{D}(A) \times L^2 \rightarrow \mathbb{R} \]
\[ B(u, v) := \langle -(A - K_\Xi)u, v \rangle, \]
then \( B \) is continuous, namely
\[ |B(u, v)| \lesssim \|u\|_{\mathcal{D}(A)} \|v\|_{L^2}, \ \forall u \in \mathcal{D}(A), \ v \in L^2, \]
and it is weakly coercive i.e.
\[ \|u\|_{\mathcal{D}(A)} = \|-(A - K_\Xi)u\|_{L^2} = \sup_{\|v\|_{L^2} = 1} \langle -(A - K_\Xi)u, v \rangle \text{ for any } u \in \mathcal{D}(A). \]
The last property to check is that for any \( 0 \neq v \in L^2 \),
\[ \sup_{\|u\|_{\mathcal{D}(A)} = 1} |B(u, v)| > 0. \]
Assume for the sake of contradiction that there is a $0 \neq v \in L^2$ s.t.,

$|B(u, v)| = 0, \quad \forall u \in \mathcal{D}(A),$

This means that

$\langle u, v \rangle_{\mathcal{D}(A), \mathcal{D}(A)^*} = 0$ for all $u \in \mathcal{D}(A),$

i.e. $v = 0$ in $\mathcal{D}(A)^*.$ But since $\mathcal{D}(A)$ is dense in $L^2$, this implies $v = 0$ in $L^2$ which is a contradiction. Then the Babuska-Lax-Milgram Theorem says that for any $f \in (L^2)^* = L^2$ there exists a unique $u_f \in \mathcal{D}(A)$ with

$B(u_f, v) = \langle f, v \rangle$ for all $v \in L^2$

with the bound $\|u_f\|_{\mathcal{D}(A)} \lesssim \|f\|_{L^2}.$ In other words

$(-A + K\Xi)^{-1} : L^2 \to \mathcal{D}(A)$

is bounded.

**Definition 2.24.** We define the following shifted operators

$H_\varepsilon := A_\varepsilon - K\Xi$

$H := A - K\Xi.$

**Remark 2.25.** We would like to point out that in the sequel the constant $K\Xi$ can be updated to be larger, as needed, without notice.

Also, we use the above estimates to give a characterization of the domain and the form domain in terms of standard Sobolev norms of $u^\sharp$. First, we define the form domain.

**Definition 2.26.** We set $u^\sharp = \Gamma u^\sharp$, as in Lemma [2.8]. The form domain of $H$, that we denote as $\mathcal{D}(\sqrt{-H})$, is defined as the closure of the domain under the following norm

$\|\Gamma u^\sharp\|_{\mathcal{D}(\sqrt{-H})} := \sqrt{\langle \Gamma u^\sharp, -H\Gamma u^\sharp \rangle} = \sqrt{\langle u^\sharp, -Hu^\sharp \rangle}.$

**Proposition 2.27.**

1. $\Gamma u^\sharp \in \mathcal{D}(H) \Leftrightarrow u^\sharp \in \mathcal{H}^2$, where $\Gamma$ is the map from Proposition [2.8]. More precisely, on $\mathcal{D}(H)$ we have the following norm equivalence

$\|u^\sharp\|_{\mathcal{H}^2} \lesssim \|H\Gamma u^\sharp\|_{L^2} \lesssim \|u^\sharp\|_{\mathcal{H}^2}. \quad (28)$

2. $\Gamma u^\sharp \in \mathcal{D}(\sqrt{-H}) \Leftrightarrow u^\sharp \in \mathcal{H}^1$, where the form domain of $-H$ is given by the closure of $\mathcal{D}(H)$ under the norm

$\|\Gamma u^\sharp\|_{\mathcal{D}(\sqrt{-H})} := \sqrt{\langle \Gamma u^\sharp, -H\Gamma u^\sharp \rangle}. \quad (29)$

We will see in the following pages that the operator $-H$ is self-adjoint and positive, so this is in fact a norm. Then the precise statement is that on $\mathcal{D}(H)$ the following norm equivalence holds

$\|u^\sharp\|_{\mathcal{H}^1} \lesssim \|\Gamma u^\sharp\|_{\mathcal{D}(\sqrt{-H})} \lesssim \|u^\sharp\|_{\mathcal{H}^1},$

and hence the closures with respect to the two norms coincide.
Proof. 1. The first inequality in (28) follows directly from (16) and the second by first expanding using (15) and then estimating as in the proof of Theorem 2.15.

2. In (29), the first inequality follow directly from the Proposition 2.21. For the second term, one plugs in the definition (15) and then the only non-trivial term is $\langle u^\sharp \circ \xi, u^\sharp \rangle$.

For this term, we also have
\[
|\langle u^\sharp \circ \xi, u^\sharp \rangle| \leq C_Ξ ||u^\sharp||_H^2
\]
by similar arguments as in the proof of Proposition 2.21.

In order to show self-adjointness we would like to use the following result.

**Proposition 2.28.** [20, X.1] A closed symmetric operator on a Hilbert space $H$ is self-adjoint if it has at least one real value in its resolvent set.

Now, we can show self-adjointness.

**Lemma 2.29.** The operators $H : \mathcal{D}(H) \to L^2$ and $H_\varepsilon : \mathcal{H}^2 \to L^2$ as defined in Theorem 2.15 are self-adjoint.

**Proof.** This follows from Proposition 2.28. Observe that Proposition 2.23 implies $K_\Xi$ is in the resolvent of $A$ and $A_\varepsilon$. Therefore, the result.

Next, we first show, in Theorem 2.30, the strong resolvent convergence of $H_\varepsilon$ to $H$ which we will use in the PDE part. Then in Theorem 2.31 we prove the stronger result of norm resolvent convergence. In 2d case, this result was obtained in [2, Lemma 4.15] but we give a proof in our framework which can also be applied to the 3d case, namely that, generalizes the result in the cited article to the 3d case.

**Theorem 2.30.** Recall the operators $H_\varepsilon$ and $H$, as defined in Definition 2.24. For any $f \in L^2$, we have
\[
||H^{-1} f - H_\varepsilon^{-1} f||_{L^2} \to 0
\]
as $\varepsilon \to 0$. Namely, $H_\varepsilon$ converges to $H$ in the strong resolvent sense.

**Proof.** First we take an approximation as in Proposition 2.17. Since $H_\varepsilon^{-1} : L^2 \to L^2$ is a bounded operator we have
\[
||u_\varepsilon - H_\varepsilon^{-1} Hu||_{L^2} = ||H_\varepsilon^{-1}(Hu - H_\varepsilon u_\varepsilon)||_{L^2} \lesssim ||Hu - H_\varepsilon u_\varepsilon||_{L^2}
\]
Then, by Proposition 2.19, we readily obtain
\[
||u_\varepsilon - H_\varepsilon^{-1} Hu||_{L^2} \to 0
\]
as $\varepsilon \to 0$. Now, for $u \in \mathcal{D}(H)$ we estimate
\[
||u - H_\varepsilon^{-1} Hu||_{L^2} \leq ||u - u_\varepsilon||_{L^2} + ||u_\varepsilon - H_\varepsilon^{-1} Hu||_{L^2}
\]
and obtain that $H_\varepsilon^{-1} H : \mathcal{D}(H) \subset L^2 \to L^2$ tend to the identity operator over $\mathcal{D}(H)$. Then, for any $f = Hu \in L^2$ we can write
\[
||H_\varepsilon^{-1} Hu - u||_{L^2} = ||H_\varepsilon^{-1} Hu - H^{-1} Hu||_{L^2} = ||H^{-1} f - H^{-1} f||_{L^2} \to 0
\]
for all $f \in L^2$. By the existence of $H^{-1}$, indeed for any $f \in L^2$ we can find such $u$. Hence, the result.
In the next theorem, we show that in fact the above convergence can be improved to the norm resolvent convergence in the $\mathcal{H}^\gamma$-norm.

**Theorem 2.31.** We have

$$||H^{-1} - H_\varepsilon^{-1}||_{L^2 \to \mathcal{H}^\gamma} \to 0$$

as $\varepsilon \to 0$. Namely, $H_\varepsilon$ converges to $H$ in the norm resolvent sense.

**Proof.** Recall that $\Gamma : \mathcal{H}^2 \to \mathcal{D}(H)$ and $\Gamma_\varepsilon : \mathcal{H}^2 \to \mathcal{H}^2$ in which case we have $\Gamma^{-1} : \mathcal{D}(H) \to H^2$ and $\Gamma_\varepsilon^{-1} : \mathcal{H}^2 \to \mathcal{H}^2$. Recall that in Proposition 2.19 we obtained

$$||H_\varepsilon \Gamma - H\Gamma||_{\mathcal{H}^2 \to L^2} \to 0.$$

This implies the norm resolvent convergence

$$||\Gamma_\varepsilon^{-1}H_\varepsilon^{-1} - \Gamma^{-1}H^{-1}||_{L^2 \to \mathcal{H}^2} \to 0.$$

To conclude, by using Proposition 2.8 we can write the estimate

$$||\Gamma_\varepsilon^{-1}H_\varepsilon^{-1} - \Gamma^{-1}H^{-1}||_{\mathcal{H}^\gamma} \leq 3||\Gamma_\varepsilon^{-1}H_\varepsilon^{-1} - \Gamma^{-1}H^{-1}||_{\mathcal{H}^\gamma}$$

where, as $\varepsilon \to 0$ by Lemma 2.11 we get the convergence

$$||H_\varepsilon^{-1} - H^{-1}||_{L^2 \to \mathcal{H}^\gamma} \to 0.$$

Hence, the result. \hfill \Box

As a corollary of the norm resolvent convergence we note following observation.

**Corollary 2.32** (cfr. [19], VIII.20). For any bounded continuous function $f : [-C_\Xi, \infty) \to \mathbb{C}$ we get

$$f(H_\varepsilon)g \to f(H)g \text{ in } L^2$$

for any $g \in L^2$ i.e. strong operator convergence.

### 2.1.3 Functional inequalities

In this section, we obtain certain inequalities for the Anderson Hamiltonian which will be crucial when we study the PDEs.

The first one is an $L^p$-embedding result.

**Lemma 2.33** ($L^p$ estimates). For $u \in \mathcal{D}(\sqrt{-H})$ and $p \in [1, \infty)$ we have

$$||u||_{L^p} \lesssim ||u||_{\mathcal{D}(\sqrt{-H})}. \quad (30)$$

Moreover, for $v \in \mathcal{D}(\sqrt{-H_\varepsilon}) = \mathcal{H}^1$, we have

$$||v||_{L^p} \lesssim ||u||_{\mathcal{D}(\sqrt{-H_\varepsilon})}. \quad (31)$$
Proof. For $p < \infty$ and $\delta(p) > 0$ small enough we have by Sobolev embedding and Propositions 2.8 and 2.27

$$||u||_{L^p} \lesssim ||u||_{H^{1-\delta}} \lesssim ||u||_{H^{1}} \lesssim \Xi \left|\left|u\right|\right|_{H^{1-\delta}} \lesssim \Xi \left|\left|u\right|\right|_{H^{1}} \lesssim \Xi \left|\left|u\right|\right|_{L^2} \lesssim \Xi \left|\left|u\right|\right|_{D(\sqrt{-H})}.$$  

and by Remark 2.22 the same computation works for the second inequality with constants independent of $\varepsilon$.

In light of the Proposition 2.27, the following result is an analogue of the embedding $H^2 \subset L^\infty$ in 2d.

Lemma 2.34. For $u \in D(H)$ we have

$$||u||_{L^\infty} \lesssim ||Hu||_{L^2}$$

Proof. By using $H^2 \subset L^\infty$ and Propositions 2.8 and 2.27 we have the following chain of inequalities:

$$||u||_{L^\infty} \lesssim \Xi \left|\left|u\right|\right|_{L^\infty} \lesssim \Xi \left|\left|u\right|\right|_{H^2} \lesssim \Xi \left|\left|Hu\right|\right|_{L^2}.$$  

Hence, the result.

In addition to the above result, we can also prove an inequality that, in some sense, interpolates the $L^\infty$-norm between the energy norm and the logarithm of the domain norm. Namely, we prove a version of Brezis-Gallouet inequality for the Anderson Hamiltonian. We first recall below the original version of the inequality.

Theorem 2.35. [5] Let $\Omega$ be a domain in $\mathbb{R}^2$ with compact smooth boundary. Then, for $v \in H^2(\Omega)$ we have

$$||v||_{L^\infty} \lesssim C \left(1 + \sqrt{1 + \log ||v||_{H^2}}\right).$$  

for every $v$ that satisfies $||v||_{H^1(\Omega)} \leq 1$.

Our version for the Anderson Hamiltonian is as follows.

Theorem 2.36. For $v \in D(H)$ we have

$$||v||_{L^\infty} \lesssim \Xi \left|\left|v\right|\right|_{D(\sqrt{-H})} \sqrt{(1 + \log(1 + ||\sqrt{-H}v||_{L^2}))}.$$  

As a corollary, we obtain, for $v \in D(H_\varepsilon) = H^2$,

$$||v||_{L^\infty} \lesssim ||(-H_\varepsilon)^{1/2}v||_{L^2} \sqrt{(1 + \log(1 + ||H_\varepsilon v||_{L^2}))},$$  

where the constant depends on the limiting noise $\Xi$ and can be chosen independently of $\varepsilon$.

Proof. We first observe

$$||v||_{L^\infty}^2 \leq 2||\Delta_{\leq M}v||_{L^\infty}^2 + 2||\Delta_{> M}v||_{L^\infty}^2.$$  

By Bernstein inequalities, Lemma A.3 (in $d = 2$), we can write

$$||\Delta_{\leq M}v||_{L^\infty} \lesssim \sum_{i=-1}^{2M} ||\Delta_i v||_{L^\infty} \lesssim \sum_{i=-1}^{2M} ||\Delta_i v||_{L^\infty} \lesssim \sum_{i=-1}^{2M} 2^i ||\Delta_i v||_{L^2} \lesssim 2^{M/2} ||v||_{H^1}.$$  

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Moreover one can show that for any \( \delta > 0 \) we have
\[
\| \Delta_M v \|_{L^\infty} \lesssim \| \Delta_{M+1} v \|_{\mathcal{H}^1} \lesssim 2^{M(\delta-1)} \| v \|_{\mathcal{H}^2}
\]
so
\[
\| v \|_{L^\infty}^2 \leq 2^{M+1} \| v \|_{\mathcal{H}^1}^2 + 2^{2M(\delta-1)} \| v \|_{\mathcal{H}^2}^2 \leq 2M \| v \|_{\mathcal{H}^1}^2 + 2^{2M(\delta-1)} (1 + \| v \|_{\mathcal{H}^2}^2).
\]
Choosing \( 2^{2M(\delta-1)} (1 + \| v \|_{\mathcal{H}^2}^2) = 1 \) we reobtain the usual Brezis–Gallouet inequality, in the following form
\[
\| v \|_{L^\infty} \lesssim \| v \|_{\mathcal{H}^1} \sqrt{1 + \log(1 + \| v \|_{\mathcal{H}^2})}.
\]
By using this and Propositions 2.21 and 2.8 we obtain
\[
\begin{align*}
\| v \|_{L^\infty} & \lesssim \| v^\xi \|_{L^\infty} \lesssim \| v^\xi \|_{\mathcal{H}^1} \sqrt{1 + \log(1 + \| v^\xi \|_{\mathcal{H}^2})} \\
& \lesssim \| v \|_{\mathcal{H}(H)} \sqrt{1 + \log(1 + \| v \|_{\mathcal{H}(H)})}
\end{align*}
\]
By Remark 2.22 the same estimates as for \( \mathcal{H}(H) \) are also true for \( \mathcal{H}(H_\epsilon) \), in particular the estimates in Proposition 2.21 hold with constants independent of \( \epsilon \). Hence, the result.

2.2 The three-dimensional case

In this section, we study the Anderson Hamiltonian in 3d. As in the 2d case we will perform a paracontrolled analysis of the Anderson Hamiltonian. This case is more technical since the noise term has much lower Hölder regularity of \( C^{3/2} \). So, the paracontrolled ansatz as in the 2-d case turns out to be insufficient. We follow a two step procedure for the defining the operator. As a first step, similar to [9], we perform an exponential transformation depending on the noise and as a second step we make an Ansatz for the transformed operator using paracontrolled distributions.

2.2.1 Enhanced noise in 3d

Recall that in the 2d case we needed to define the space of enhanced noise (see Def. 2.2), namely \( \mathcal{X}^\alpha \), for the renormalization. In this section, we first define this space in the 3d case.

In the results below, we prove that \( X = (-\Delta)^{-1} \xi \) can be lifted to an element \( \Xi \) in the space \( \mathcal{X}^\alpha \) of enhanced distributions such that all the stochastic terms, we will need for the ansatz in the next section, exist with correct regularities. In the following sequence of results, we construct the enhanced white noise space in 3d and prove the related approximation results. In particular, we show that the lifts \( \Xi_\epsilon \) (of the regularized noise \( \xi_\epsilon \)) converges to an element, that we denote by \( \Xi \), in \( \mathcal{X}^\alpha \).

**Definition 2.37.** For \( 0 < \alpha < \frac{1}{2} \), we define the space \( \mathcal{X}^\alpha \) to be the closure of the set
\[
\left\{ \left( \phi, \phi^\alpha, \phi^\cdot, \phi, \phi^\ast, \nabla \phi \circ \nabla \phi \right) : (a, b) \in \mathbb{R}^2, \phi \in \mathcal{C}^2(\mathbb{T}^3) \right\}
\]
with respect to the \( C^\alpha(\mathbb{T}^3) \times C^{2\alpha}(\mathbb{T}^3) \times C^{\alpha+1}(\mathbb{T}^3) \times C^{\alpha+1}(\mathbb{T}^3) \times C^{4\alpha}(\mathbb{T}^3) \times C^{2\alpha-1}(\mathbb{T}^3) \) norm. Here, we defined

\[
\begin{align*}
\phi_a^\nu & := (1 - \Delta)^{-1}(|\nabla \phi|^2 - a) \\
\phi_a & := 2(1 - \Delta)^{-1}(|\nabla \phi|^2 - a) \\
\phi_a^\nu & := (1 - \Delta)^{-1}(\nabla \phi \cdot \nabla \phi_a^\nu) \\
\phi_a & := (1 - \Delta)^{-1}(|\nabla \phi|^2 - b).
\end{align*}
\]

**Theorem 2.38.** For \( \xi_\varepsilon \) given by (5) we define

\[
\begin{align*}
X_\varepsilon^\nu & = (-\Delta)^{-1}\xi_\varepsilon \\
X_\varepsilon^\nu & = (1 - \Delta)^{-1}(|\nabla X_\varepsilon|^2 - c_1^\varepsilon) \\
X_\varepsilon^\nu & = 2(1 - \Delta)^{-1}(\nabla X_\varepsilon \cdot \nabla X_\varepsilon^\nu) \\
X_\varepsilon^\nu & = (1 - \Delta)^{-1}(\nabla X_\varepsilon \cdot \nabla X_\varepsilon^\nu) \\
X_\varepsilon^\nu & = (1 - \Delta)^{-1}(|\nabla X_\varepsilon|^2 - c_2^\varepsilon),
\end{align*}
\]

where the \( c_\varepsilon \) are diverging constants which can be chosen as

\[
\begin{align*}
c_1^\varepsilon & = \sum_{k \in \mathbb{Z}^3 \setminus \{0\}} \frac{|m(\varepsilon k)|^2}{|k|^2} \sim \frac{1}{\varepsilon} \\
c_2^\varepsilon & = \sum_{k_1, k_2 \neq 0} \frac{|m(\varepsilon k_1)|^2 |m(\varepsilon k_2)|^2}{|k_1 - k_2|^2 |k_1|^4 |k_2|^2} \sim \left( \log \frac{1}{\varepsilon} \right)^2.
\end{align*}
\]

Then the sequence \( \Xi_\varepsilon \in \mathcal{X}^\alpha \), given by

\[
\Xi_\varepsilon := (X_\varepsilon^\nu, X_\varepsilon^\nu, X_\varepsilon^\nu, X_\varepsilon^\nu, X_\varepsilon^\nu, \nabla X_\varepsilon \circ \nabla X_\varepsilon^\nu)
\]

converges a.s. to a unique limit \( \Xi \in \mathcal{X}^\alpha \), given by

\[
\Xi := (X, X^\nu, X^\nu, X^\nu, X^\nu, \nabla X \circ \nabla X^\nu), \tag{32}
\]

where

\[
\begin{align*}
X & = (-\Delta)^{-1}\xi \\
X^\nu & = (1 - \Delta)^{-1}(|\nabla X|^2) \\
X^\nu & = 2(1 - \Delta)^{-1}(\nabla X \cdot \nabla X^\nu) \\
X^\nu & = (1 - \Delta)^{-1}(\nabla X \cdot \nabla X^\nu) \\
X^\nu & = (1 - \Delta)^{-1}(|\nabla X^\nu|^2).
\end{align*}
\]

**Proof.** We omit the parts of the proof which goes in a similar way to Theorem 7.11 in [7] (see also Chapter 9 of [15]). Note that their estimates are for the parabolic case, but by using the resolvent identity

\[
\int_0^\infty e^{-t} e^{t\Delta} dt = (1 - \Delta)^{-1},
\]

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one can easily adapt their computations to our setting, essentially through multiplying by $e^{-t}$ and integrating over $t$. This, in particular, implies that the diverging constants are the same. Note that the last term in our enhanced noise is slightly different from the one in [7]. However one can easily show that the most singular part of $\nabla X \circ \nabla X^\nu$ is given by $\nabla X \circ \nabla (1 - \Delta)^{-1} \nabla X$, which is the term from [7]. In fact, we have

$$\nabla X \circ \nabla X^\nu = \nabla X \circ \nabla (1 - \Delta)^{-1} \left( \nabla X \prec \nabla X^\nu + \nabla X^\nu \prec \nabla X + \nabla X \circ \nabla X^\nu \right)$$

$$= \nabla X \circ (1 - \Delta)^{-1} \left( \nabla \left( \nabla X \prec \nabla X^\nu + \nabla X \circ \nabla X^\nu \right) \right) + \nabla^2 X^\nu \prec \nabla X$$

$$+ \nabla X \circ (1 - \Delta)^{-1} \left( \nabla X^\nu \prec \nabla^2 X \right),$$

where first expression make sense assuming the correct regularity for the other stochastic terms. For the second term, we apply the commutator Lemma A.8 (or more precisely the H" older version) and Proposition A.2. We compute

$$\nabla X \circ (1 - \Delta)^{-1} \left( \nabla X^\nu \prec \nabla^2 X \right) = \nabla X \circ \left( \nabla X^\nu \prec (1 - \Delta)^{-1} \nabla^2 X + R \left( \nabla X^\nu, \nabla^2 X \right) \right)$$

$$= \nabla X^\nu \nabla X \circ (1 - \Delta)^{-1} \nabla X + C \left( \nabla X^\nu, (1 - \Delta)^{-1} \nabla^2 X, \nabla X \right)$$

$$+ \nabla X \circ R \left( \nabla X^\nu, \nabla^2 X \right).$$

This proves that $\nabla X \circ (1 - \Delta)^{-1} \nabla X \in \mathcal{C}^{2\alpha - 1}$ which in turn implies that $\nabla X \circ \nabla X^\nu \in \mathcal{C}^{2\alpha - 1}$. Thus our result follows from Theorem 7.11 in [7].

See also Theorems 9.1 and 9.3 in [15] where a similar renormalization was performed with 1d space-time white noise with regularity that of the 3d spatial white noise. \hfill \qed

**Lemma 2.39.** Let $\alpha, X, X^\nu, X^\nu, X^\nu, X^\nu, X^\nu$ be as above, then $e^X \in \mathcal{C}^{\alpha}, e^{X^\nu} \in \mathcal{C}^{2\alpha}, e^{X^\nu} \in \mathcal{C}^{\alpha + 1}$ and

$$e^{X^\nu} \rightarrow e^X \quad \text{in } \mathcal{C}^{\alpha}$$

$$e^{X^\nu} \rightarrow e^{X^\nu} \quad \text{in } \mathcal{C}^{2\alpha}$$

$$e^{X^\nu} \rightarrow e^{X^\nu} \quad \text{in } \mathcal{C}^{\alpha + 1}.$$

**Proof.** We prove the result for $X$, the others are proved in the same way. Since $\alpha > 0$, we use the equivalent classical Hölder norms on $\mathcal{C}^{\alpha}$. One easily sees that the spaces $\mathcal{C}^{\alpha}$ are Banach Algebras, so $e^X = \sum_{n \geq 0} \frac{1}{n!} X^n \in \mathcal{C}^{\alpha}$ and since $X \rightarrow X$ in $\mathcal{C}^{\alpha}$, we can estimate

$$\left\| e^X - e^{X^\nu} \right\|_{\mathcal{C}^{\alpha}} \leq \left\| e^X \right\|_{\mathcal{C}^{\alpha}} \left\| 1 - e^{X^\nu - X} \right\|_{\mathcal{C}^{\alpha}} = \left\| e^X \right\|_{\mathcal{C}^{\alpha}} \left( \sum_{n \geq 1} \frac{1}{n!} (X - X)^n \right)_{\mathcal{C}^{\alpha}}$$

$$\leq \left\| e^X \right\|_{\mathcal{C}^{2\alpha}} \left( e \|X^\nu - X\|_{\mathcal{C}^{\alpha}} - 1 \right),$$

and conclude that $e^{X^\nu} \rightarrow e^X$ in $\mathcal{C}^{\alpha}$.

**Lemma 2.40.** For $\alpha, X, X^\nu$ as above, $W := X + X^\nu + X^\nu$ and

$$Z = (1 - \Delta)^{-1} \left(\left\| \nabla X^\nu \right\|^2 + 2 \nabla X^\nu \cdot \nabla X^\nu - X^\nu \right) + X^\nu + 2X^\nu.$$

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we have
\[ \nabla e^X \cdot \nabla e^X \in \mathcal{C}^{\alpha - 1}, \]
which implies that \( e^{2W} (1 - \Delta)Z \in \mathcal{C}^{\alpha - 1} \).

**Proof.** We use paralinearisation, see Lemma A.5, to rewrite
\[
e^X = e^X \prec X + g^2 \\
e^X \psi = e^X \prec X \psi + f^2,
\]
where \( g^\sharp \in \mathcal{C}^{2\alpha} \) and \( f^\sharp \in \mathcal{C}^{2\alpha + 1} \). Thus,
\[
\nabla e^X = (\nabla e^X) \prec X + e^X \prec \nabla X + \nabla g^\sharp \\
\text{and} \\
\nabla e^X \psi = (\nabla e^X \psi) \prec X \psi + e^X \psi \prec \nabla X \psi + \nabla f^\sharp.
\]

Note that the only problematic term in the product is
\[
\left( e^X \prec \nabla X \right) \left( e^X \psi \prec \nabla X \psi \right).
\]

More precisely, we only have to make sense of the resonant product in (33). Since the paraproducts are always defined. We compute
\[
(e^X \prec \nabla X) \circ (e^X \psi \prec \nabla X \psi)
\]
\[
= e^X \psi \left( \nabla X \psi \circ (e^X \prec \nabla X) \right) + C \left( e^X \psi, \nabla X \psi, (e^X \prec \nabla X) \right)
\]
\[
= e^X \psi + X \left( \nabla X \psi \circ \nabla X \right) + e^X \psi C \left( e^X, \nabla X, \nabla X \psi \right)
\]
\[
+ C \left( e^X \psi, \nabla X \psi, (e^X \prec \nabla X) \right).
\]

Now, since \( \nabla X \psi \circ \nabla X \) is assumed to be in \( \mathcal{C}^{2\alpha - 1} \), the above resonant product is also in \( \mathcal{C}^{2\alpha - 1} \). This finishes the proof that \( \nabla e^X \cdot \nabla e^X \psi \in \mathcal{C}^{\alpha - 1} \). Moreover, by reinserting the definitions we obtain
\[
e^{2W} (1 - \Delta)Z
\]
\[
= e^{2X + 2X \psi \prec 2X \psi} \left( \left| \nabla X \psi \right|^2 + \left| \nabla e^X \psi \right|^2 + 2 \nabla X \cdot \nabla e^X \psi + 2 \nabla X e^X \cdot \nabla X \psi - X \psi - X e^X \psi \right)
\]
\[
= e^{2X + 2X \psi \prec 2X \psi} \left( \left| \nabla X \psi \right|^2 + \left| \nabla e^X \psi \right|^2 + 2 \nabla X e^X \cdot \nabla X \psi - X \psi - X e^X \psi \right)
\]
\[
+ \frac{1}{2} e^{2X} \nabla (e^{2X}) \nabla \left( e^{2X} \psi \right)
\]
by noting the previous computations and the fact that all the terms in the first bracket have regularity at least \( 2\alpha - 1 \). We can finally conclude that
\[
\left\| e^{2W} (1 - \Delta)Z \right\|_{\mathcal{C}^{\alpha - 1}} \lesssim \left\| e^{2W} \right\|_{\mathcal{C}^{\alpha}} \left\| \Xi \right\|_{\mathcal{C}^{\alpha}}.
\]
\[
\square
\]
2.2.2 The domain, the \( \Gamma \)-map and the definition of the 3-d Hamiltonian

In this section, building on our work in Section 2.2.1, we perform the renormalization of the Anderson Hamiltonian in 3d. Recall the following quantities we introduced and justified in Section 2.2.1.

\[
X = (-\Delta)^{-1} \xi(x) \in \mathcal{C}^{1/2}
\]
\[
X^\mathcal{V} = (1 - \Delta)^{-1} : |\nabla X|^2 : \in \mathcal{C}^{3/2}
\]
\[
X^\mathcal{V} = 2(1 - \Delta)^{-1} \left( \nabla X \cdot \nabla X^\mathcal{V} \right) \in \mathcal{C}^{3/2}
\]
\[
X^\mathcal{V} = (1 - \Delta)^{-1} : |\nabla X|^2 : \in \mathcal{C}^{2}
\]

In the following we first motivate the ansatz, that we will use in 3d, through informal calculations and then conclude formally in Definition 2.42.

Initially, we make the following Ansatz for the domain of the Hamiltonian

\[
u = e^{X + X^\mathcal{V} + X^\mathcal{V}} u^b,
\]

where the form of \( u^b \) will be specified later. We begin by computing

\[
\Delta u + u\xi = e^{X + X^\mathcal{V} + X^\mathcal{V}} \left( \nabla \left( X + X^\mathcal{V} + X^\mathcal{V} \right) \right) u^b + \left| \nabla \left( X + X^\mathcal{V} + X^\mathcal{V} \right) \right|^2 u^b + \Delta u^b + 2\nabla \left( X + X^\mathcal{V} + X^\mathcal{V} \right) \nabla u^b + u^b \xi
\]
\[
= e^{X + X^\mathcal{V} + X^\mathcal{V}} \left( \Delta u^b + \left| \nabla X \right|^2 : + \left| \nabla X^\mathcal{V} \right| \right)^2 + 2\nabla \left( X + X^\mathcal{V} + X^\mathcal{V} \right) \nabla u^b + 2\nabla \left( X + X^\mathcal{V} + X^\mathcal{V} \right) \cdot \nabla u^b.
\]

Note that the regularity of \( X^\mathcal{V} \) is too low for the term \( \left| \nabla X^\mathcal{V} \right|^2 \) to be defined so we have to replace it by by its Wick ordered version, also note the appearing difference \( \left| \nabla X \right|^2 : + \left| \nabla X^\mathcal{V} \right| \). Here one sees the two divergences that arise, since we formally have

\[
\left| \nabla X \right|^2 : = \left| \nabla X \right|^2 - \infty, \quad \left| \nabla X^\mathcal{V} \right| ^2 : = \left| \nabla X^\mathcal{V} \right|^2 - \infty.
\]

However this notation is misleading since the rate of divergence is different in both cases, as we calculated as constants \( c_1^1 \) and \( c_2^1 \) in Theorem 2.38. This again suggests that, as in 2d, the renormalized Hamiltonian can be formally written in the suggestive form

\[
A = \Delta + \xi - \infty.
\]

We set

\[
Au = A(e^W u^b) = e^W (\Delta u^b + 2(1 - \Delta) \nabla \cdot \nabla u^b + (1 - \Delta) Z u^b),
\]

for functions \( u^b \) that this auto-consistently makes sense with the form of \( u^b \). For simplicity, we put

\[
W = X + X^\mathcal{V} + X^\mathcal{V}
\]
\[
\nabla W = (1 - \Delta)^{-1} \nabla W
\]
\[
Z = (1 - \Delta)^{-1} \left( \left| \nabla X^\mathcal{V} \right|^2 + 2\nabla X^\mathcal{V} \cdot \nabla X^\mathcal{V} - X^\mathcal{V} - X^\mathcal{V} \right) + X^\mathcal{V} + 2X^\mathcal{V}.
\]
As we have seen in section 3.2.1 these stochastic terms have the following regularities
\[ X, W \in C^{1/2-}, X^\mathcal{V} \in C^1, X^\mathcal{V}, X^\mathcal{W}, W, Z \in C^{3/2-} \quad \text{and} \quad X^\mathcal{V} \in C^2. \]
This suggests to make a paracontrolled ansatz for \( u^b \) in terms of \( Z \) and \( \tilde{W} \) since the products appearing are classically ill-defined. In fact, we make the following ansatz
\[ u^b = u^b : Z + \nabla u^b : \tilde{W} + B_\Xi(u^b) + u^\sharp, \tag{35} \]
with \( u^\sharp \in \mathcal{H}^2 \) and for a correction term that we denote by \( B_\Xi(u^b) \). To the correction term, we will absorb the terms which has regularity not worse than \( \mathcal{H}^2 \). Similar to the 2d case, we will introduce a frequency cut-off that will allow us to write \( u^b \) as a function of \( u^\sharp \) but for notational brevity, we will omit this for the time being.

For the remainder of this section, we define
\[ L := (1 - \Delta) \quad \text{and} \quad L^{-1} = (1 - \Delta)^{-1}. \]
Note that the ansatz (35) directly implies \( u^b \in \mathcal{H}^{3/2-} \) by the paraproduct estimates in Lemma A.1.

We want to determine the form of the correction term \( B_\Xi(u^b) \) in (35). We first compute
\[ \Delta u^b = \Delta u^b : Z + 2\nabla u^b : \nabla Z + u^b : \Delta Z + \nabla \Delta u^b : \tilde{W} + 2\nabla^2 u^b : \nabla \tilde{W} \]
\[ + \nabla u^b : \Delta \tilde{W} + \Delta B_\Xi + \Delta u^\sharp \]
\[ = \Delta u^b : Z + 2\nabla u^b : \nabla Z - u^b : (LZ - Z) + \nabla \Delta u^b : \tilde{W} + 2\nabla^2 u^b : \nabla \tilde{W} \]
\[ - \nabla u^b : (L\tilde{W} - \tilde{W}) - LB_\Xi(u^b) - B_\Xi(u^b) + \Delta u^\sharp. \]

By using the paraproduct decomposition we obtain
\[ \Delta u^b + 2L\tilde{W} \cdot \nabla u^b + LZ u^b = \Delta u^b + \tilde{G}(u^b) + 2L\tilde{W} \circ \nabla u^b + LZ \circ u^b, \tag{36} \]
where we have defined
\[ \tilde{G}(u^b) := \Delta u^b : Z + 2\nabla u^b : \nabla Z + u^b : Z + \nabla \Delta u^b : \tilde{W} + 2\nabla^2 u^b : \nabla \tilde{W} + \nabla u^b : \tilde{W} \]
\[ - LB_\Xi(u^b) - B_\Xi(u^b) + 2L\tilde{W} \cdot \nabla u^b + LZ \cdot u^b. \]
These are the “non-problematic” terms that can also be absorbed into \( B_\Xi \). We still have to take care of the resonant product \( L\tilde{W} \circ \nabla u^b \), which is not a priori defined and the other resonant product which is actually defined as is, but we shall see at a later time that it is necessary to decompose it further. To be precise, we insert the ansatz and use Proposition A.2.
\[
\begin{align*}
L\tilde{W} \circ \nabla u^b &= L\tilde{W} \circ (\nabla u^b : Z + u^b : \nabla Z + \nabla^2 u^b : \tilde{W} + \nabla u^b : \nabla \tilde{W} + \nabla B_\Xi(u^b) + \nabla u^\sharp) \\
&= \nabla u^b(L\tilde{W} \circ Z) + C(\nabla u^b, Z, L\tilde{W}) + u^b(L\tilde{W} \circ \nabla Z) + C(u^b, \nabla Z, L\tilde{W}) \\
&\quad + L\tilde{W} \circ (\nabla^2 u^b : \tilde{W}) + \nabla u^b(L\tilde{W} \circ \nabla \tilde{W}) \\
&\quad + C(\nabla u^b, \nabla \tilde{W}, L\tilde{W}) + L\tilde{W} \circ (\nabla B_\Xi(u^b) + \nabla u^\sharp).
\end{align*}
\]
In section 2.2.1, we have seen that the following stochastic terms can be defined and have regularity
\[
L\tilde{W} \circ Z \in \mathcal{C}^{1-},
\]
\[
L\tilde{W} \circ \nabla Z \in \mathcal{C}^{0-},
\]
\[
L\tilde{W} \circ \nabla \nabla \tilde{W} \in \mathcal{C}^{0-}.
\]

We furthermore expand the products appearing above as
\[
L\tilde{W} \circ \nabla u^b = \nabla u^b \prec (L\tilde{W} \circ Z) + \nabla u^b \succ (L\tilde{W} \circ Z) + C(\nabla u^b, Z, L\tilde{W}) + u^b \prec (L\tilde{W} \circ \nabla Z) + u^b \succ (L\tilde{W} \circ \nabla Z)
\]
\[
+ \nabla u^b \prec (L\tilde{W} \circ \nabla \tilde{W}) + C(\nabla u^b, \nabla \tilde{W}, L\tilde{W}) + L\tilde{W} \circ (\nabla B_{\Xi}(u^b) + \nabla u^b).
\]

For the other resonant product in (36), we do the same and get
\[
LZ \circ u^b = u^b \prec (LZ \circ Z) + u^b \succ (LZ \circ Z) + (LZ \circ O) + \nabla u^b \prec (LZ \circ \tilde{W})
\]
\[
+ \nabla u^b \succ (LZ \circ \tilde{W}) + C(\nabla u^b, \tilde{W}, LZ) + LZ \circ (B_{\Xi}(u^b) + u^b).
\]

Now we are in a position to give the precise definition of the correction term, we put
\[
B_{\Xi}(u^b) := L^{-1} \left[ \Delta u^b \prec Z + 2\nabla u^b \prec \nabla Z + u^b \prec Z + \nabla \Delta u^b \prec \tilde{W} + 2\nabla^2 u^b \prec \nabla \tilde{W}
\]
\[
- \nabla u^b \prec \tilde{W} + 2L\tilde{W} \prec \nabla u^b + LZ \prec u^b
\]
\[
+ 2\nabla u^b \prec (L\tilde{W} \circ Z) + 2\nabla u^b \succ (L\tilde{W} \circ Z)
\]
\[
+ 2u^b \prec (L\tilde{W} \circ \nabla Z) + 2u^b \succ (L\tilde{W} \circ \nabla Z) + 2\nabla u^b \prec (L\tilde{W} \circ \nabla \tilde{W})
\]
\[
+ 2\nabla u^b \succ (L\tilde{W} \circ \nabla \tilde{W}) + u^b \prec (LZ \circ Z) + u^b \succ (LZ \circ Z)
\]
\[
+ \nabla u^b \prec (LZ \circ \tilde{W}) + \nabla u^b \succ (LZ \circ \tilde{W}) \right].
\]

Using again the paraproduct estimates from Lemma A.1, one sees that the terms in the brackets are at least of regularity $\mathcal{H}^{0-}$, which implies $B_{\Xi}(u^b) \in \mathcal{H}^{2-}$. We make this precise in the following result.

**Lemma 2.41.** Let $B_{\Xi}$ be defined as above, then we have the following bounds for $\sigma < 2$ and $\varepsilon > 0$

1. $\|B_{\Xi}(v)\|_{\mathcal{H}^{\sigma}} \leq C_{\Xi}\|v\|_{\mathcal{H}^{\sigma-1/2+\varepsilon}}$
2. $\|B_{\Xi}(v)\|_{\mathcal{H}^{\sigma}} \leq C_{\Xi}\|v\|_{\mathcal{H}^{\sigma-1/2}}$

where for the the constant we can choose $C_{\Xi} = C\|\Xi\|_{\mathcal{H}^{\sigma-3/2}}$, see Definition 2.37 for the precise definition of the norm, where $C > 0$ is an independent constant.

**Proof.** This follows from the paraproduct estimates, Lemma A.1 for the first case. The second case works precisely in the same way using the paraproduct estimates for Besov-Hölder spaces and Schauder estimates, see e.g. [14].

Finally we collect everything in the following rigorous definition which describes the domain of the Anderson Hamiltonian.
Definition 2.42. Let $W, \tilde{W}, Z$ be as above. Then, for $1 < \gamma < 3/2$, we define the space
$$\mathcal{H}_\gamma^\gamma := e^W \mathcal{H}_\gamma := e^W \{ u^b \in \mathcal{H} \text{ s.t. } u^b = u^b + Z + \nabla u^b \prec \tilde{W} + B_\lambda(u^b) + u^z, \text{ for } u^z \in \mathcal{H}_\gamma \},$$
where $B_\lambda(u^b)$ is as in \eqref{eqn:Phi}. We furthermore equip the space with the scalar product given by,
$$\langle u, w \rangle_{\mathcal{H}_\gamma^\gamma} := \langle u^b, w^b \rangle_{\mathcal{H}_\gamma} + \langle u^z, w^z \rangle_{\mathcal{H}_\gamma},$$
Given $u = e^W u^b \in \mathcal{H}_\gamma^\gamma$ we define the renormalized Anderson Hamiltonian acting on $u$ in the following way
$$Au = e^W (\Delta u^z + LZ \circ u^z + 2L\tilde{W} \circ \nabla u^z + G(u^b)), \quad (38)$$
where
$$G(u^b) := B_\lambda(u^b) + 2\nabla u^b \circ (L\tilde{W} \circ Z) + 2C(\nabla u^b, Z, L\tilde{W}) + u^b \circ (L\tilde{W} \circ \nabla Z)$$
$$\quad \quad + 2\nabla u^b \circ (L\tilde{W} \circ \nabla Z) + 2C(\nabla u^b, L\tilde{W}, L\tilde{W}) + 2L\tilde{W} \circ \nabla B_\lambda(u^b)$$
and $C$ denotes the commutator from Proposition \ref{prop:commutator}. Note that this definition is equivalent to \\eqref{eqn:renormalized} by construction.
After this definition, some remarks are in order.
Remark 2.43. In view of \\eqref{eqn:renormalized}, for regularized white noise $\xi$, we set
$$A_\xi u := e^{W_\xi} (\Delta u^b + 2(1 - \Delta)\tilde{W}_\xi \cdot \nabla u^b + (1 - \Delta)Z \circ u^b)$$
$$\quad \quad \quad = \Delta u + \xi u - (c_1^2 + c_2^2) u,$$
where we have
$$W_\xi = X_\xi + X_\xi^\gamma + X_\xi^\gamma$$
$$X_\xi = (-\Delta)^{-1} \xi$$
$$X_\xi^\gamma = (1 - \Delta)^{-1} (|\nabla X_\xi|^2 - c_1^2)$$
$$X_\xi^\gamma = 2(1 - \Delta)^{-1} (\nabla X_\xi \cdot \nabla X_\xi^\gamma)$$
$$\tilde{W}_\xi = (1 - \Delta)^{-1} \nabla W_\xi$$
$$Z_\xi = (1 - \Delta)^{-1} \left( |\nabla X_\xi^\gamma|^2 - c_2^2 + |\nabla X_\xi^\gamma|^2 + 2\nabla X_\xi \cdot \nabla X_\xi^\gamma + 2\nabla X_\xi \cdot \nabla X_\xi^\gamma \cdot \nabla X_\xi^\gamma - X_\xi^\gamma - X_\xi^\gamma \right)$$
and
$$u^b := e^{-W_\xi} u.$$
Recall that the renormalization constants, from Section \ref{sec:renormalization}, are
$$c_1^2 = O(\epsilon^{-1}) \quad \text{and} \quad c_2^2 = O(\log \epsilon).$$
Observe that, now this makes the constant $c_2$ in \eqref{eqn:renormalized} precise as $c_2 = c_1^2 + c_2^2$.
Remark 2.44. As in the 2d case, the space $\mathcal{H}_\gamma^\gamma$ is independent of $\gamma$ and we will denote it simply by $\mathcal{H}_\gamma$. Moreover, one can introduce a Fourier cutoff $\Delta > N$ at level $N^\gamma$ and write
$$u^b = \Delta > N (u^b \prec Z + \nabla u^b \prec \tilde{W} + B_\lambda(u^b)) + u^z. \quad (40)$$
In this case, the space can be shown to be the same in a similar way to 2d case, please see Remark \ref{prop:fourier_cutoff}.
We set up the notation with the following remark.

**Remark 2.45.** Similar to Remark 2.5, we introduce the notation

\[ \mathcal{D}(A) := \mathcal{W}_\Xi. \]

We can generalize our framework that uses the $\Gamma$-map to 3d. This time, we define the linear map $\Gamma$ as

\[ \Gamma f = \Delta_{>N}(\Gamma f \prec Z + \nabla (\Gamma f) \prec \tilde{W} + B_\Xi(\Gamma f)) + f. \quad (41) \]

This allows us to realize $u^\flat$ as a fixed point of this map, that is $u^\flat = \Gamma u^\#$. Similar to the 2d case, for $N$ large enough, we can show this map exists and has useful bounds, and obtain the following generalization of Proposition 2.8 to 3d.

**Proposition 2.46.** We can choose $N$ large enough depending only on $\Xi$ and $s$ so that

\[ \| \Gamma f \|_{L^\infty} \leq 2 \| f \|_{L^\infty}, \quad (42) \]

\[ \| \Gamma f \|_{H^s} \leq 2 \| f \|_{H^s}, \quad (43) \]

for $s \in [0, \frac{3}{2})$.

**Proof.** With slight modifications, the proof is basically the same as in the 2d case, namely Proposition 2.8. For (42) choose again a small $\delta > 0$, then

\[
\| \Gamma f \|_{L^\infty} \leq \| f \|_{L^\infty} + \| \Delta_{>N}(\Gamma f \prec Z + \nabla (\Gamma f) \prec \tilde{W} + B_\Xi(\Gamma f)) \|_{\mathcal{E}^\delta}
\]

and

\[
\| \Gamma f \prec Z \|_{\mathcal{E}^{2\delta}} \lesssim \| \Gamma f \|_{\mathcal{E}^{-\delta}} \| Z \|_{\mathcal{E}^{3\delta}} \lesssim C_\Xi \| \Gamma f \|_{L^\infty}
\]

\[
\| \nabla (\Gamma f) \|_{\mathcal{E}^{-1-\delta}} \| \tilde{W} \|_{\mathcal{E}^{1+3\delta}} \lesssim C_\Xi \| \Gamma f \|_{L^\infty}
\]

\[
\| B_\Xi(\Gamma f) \|_{\mathcal{E}^{2\delta}} \lesssim C_\Xi \| \Gamma f \|_{\mathcal{E}^{2\delta-1/2}} \lesssim C_\Xi \| \Gamma f \|_{L^\infty},
\]

which allows us to conclude by choosing $N$ large enough depending on the norm of the enhanced noise $\Xi$. The proof of the Sobolev case is similar. \qed

**Remark 2.47.** Also in this (3d) case, similar to the remark 2.10, we define $\Gamma_\varepsilon$, using the approximations in Theorem 2.38.

By using the map $\Gamma$ we can obtain an analysis very similar to the 2d case. For starters, we state the following analogous result and conclude this section.

**Lemma 2.48.** Let $\gamma$ be as in Definition 2.44. We have that $\| \id - \Gamma_\varepsilon^{-1} \|_{\mathcal{H}_\gamma \to \mathcal{H}_\gamma} \to 0$.

**Proof.** The proof is very similar to that of Lemma 2.11. \qed

### 2.2.3 Density, symmetry and self-adjointness

First, we prove the density of the domain of $A$, as stated in Definition 2.42.

**Proposition 2.49.** Let $\beta < 1/2$, then the space $\mathcal{W}_\Xi$, as introduced in Definition 2.42 is dense in $\mathcal{H}^\beta$. Therefore, dense in $L^2$. 

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Proof. By using the multiplication estimates in Proposition [A.1] for \( f \in \mathscr{H}^\gamma \), we can write
\[
\|e^W f - e^W \Gamma_\varepsilon f\|_{\mathscr{H}^\beta} \lesssim \|e^W\|_{\mathscr{H}^\beta}\|f - \Gamma_\varepsilon^{-1} f\|_{\mathscr{H}^\gamma}
\]
taking the limit as \( \varepsilon \to 0 \) shows that any element in the space \( e^W H^\gamma \subset H^\beta \) can be approximated by elements in \( \mathscr{H}_2 \). For an arbitrary \( f \in H^\beta \) one can further approximate it by elements \( e^W (e^{-W\varepsilon} f_\varepsilon) \) where we took \( f_\varepsilon \in H^\gamma \) and \( \|f - f_\varepsilon\|_{H^\beta} \to 0 \).

The following is an analogue of Theorem [2.15] for the 3d Hamiltonian.

**Theorem 2.50.** The renormalized Anderson Hamiltonian \( A : \mathscr{D}(A) \to L^2 \) is a bounded operator and we get the following \( \mathscr{H}^2 \) bound for \( u^\varepsilon \)
\[
\|u^\varepsilon\|_{\mathscr{H}^2} \lesssim \|e^{-W} Au\|_{L^2} + C\|u^\beta\|_{L^2}.
\] (44)

**Proof.** By the definition of \( A \) we have
\[
e^{-W} Au = \Delta u^\varepsilon + LZ \circ u^\varepsilon + 2L\tilde{W} \circ \nabla u^\varepsilon + G(u^\beta),
\]
then we estimate
\[
\|LZ \circ u^\varepsilon\|_{L^2} \lesssim \|Z\|_{\mathscr{H}^{3/2-\delta}} \|u^\varepsilon\|_{\mathscr{H}^{1/2+2\delta}} \leq C_{\varepsilon, \delta} C\|u^\beta\|_{L^2} + \varepsilon \|u^\varepsilon\|_{\mathscr{H}^2}
\]
and
\[
\|L\tilde{W} \circ \nabla u^\varepsilon\|_{L^2} \lesssim \|\tilde{W}\|_{\mathscr{H}^{3/2-\delta}} \|u^\varepsilon\|_{\mathscr{H}^{3/2+2\delta}} \leq C_{\varepsilon, \delta} C\|u^\beta\|_{L^2} + \varepsilon \|u^\varepsilon\|_{\mathscr{H}^2}
\]
for any \( \varepsilon > 0 \) using Young’s inequality, Sobolev interpolation, and the straightforward bound \( \|u^\varepsilon\|_{L^2} \leq C\|u^\beta\|_{L^2} \). Moreover we bound \( G(u^\beta) \)
\[
\|G(u^\beta)\|_{L^2} \leq C\|u^\beta\|_{\mathscr{H}^{1+\delta}} \leq C_{\varepsilon, \delta} C\|u^\beta\|_{L^2} + \varepsilon \|u^\varepsilon\|_{\mathscr{H}^2},
\]
where the first estimate follows from the paraproduct estimates, Proposition [A.1], and the commutator bounds (Proposition [A.2]). This allows us to conclude
\[
\|Au\|_{L^2} = \|e^W e^{-W} Au\|_{L^2} \leq \|e^W\|_{L^\infty} \|e^{-W} Au\|_{L^2} \leq C\|u^\varepsilon\|_{\mathscr{H}^2} + \|u^\beta\|_{L^2},
\] (45)
and, in a similar manner,
\[
\|u^\varepsilon\|_{\mathscr{H}^2} \leq \|e^{-W} Au\|_{L^2} + \|LZ \circ u^\varepsilon + 2L\tilde{W} \circ \nabla u^\varepsilon\|_{L^2}
\leq \|e^{-W} Au\|_{L^2} + C\|u^\beta\|_{L^2} + \frac{1}{2} \|u^\varepsilon\|_{\mathscr{H}^2},
\]
using the above bounds. \( \square \)

**Proposition 2.51.** We have that \( A \) is a closed operator over its dense domain \( \mathscr{D}(A) \).

**Proof.** For \( u_n \in \mathscr{D}(A) \), suppose that
\[
u_n \to u \quad A u_n \to g.
\]
Then, by (II), we have that \( u_n^\varepsilon \) is a Cauchy sequence and \( \|w - u_n^\varepsilon\|_{\mathscr{H}^2} \to 0 \) for some \( w \). We observe that then \( u = e^W \Gamma(w) \), that is \( u \in \mathscr{D}(A) \). After that, writing the same estimate in the end of the proof of Proposition [2.17] concludes the proof, this time utilizing (II) instead. \( \square \)
For the domain what we know is $\mathcal{D}(A) \subset e^{W}H^\gamma$. But in the sequel we will need a precise approximation by smooth elements in $H^2$. The following Proposition establishes that.

**Proposition 2.52.** For every $u \in \mathcal{D}(A)$ there exists $u_\varepsilon \in H^2$ such that

$$||u^\flat - u_\varepsilon^\flat||_{H^\gamma} + ||u^\sharp - u_\varepsilon^\sharp||_{H^2} \to 0$$

as $\varepsilon \to 0$. For $u, v \in \mathcal{D}(A)$, with this approximation, we obtain

$$\langle A_\varepsilon u_\varepsilon, v_\varepsilon \rangle \to \langle Au, v \rangle.$$ 

Consequently, $A$ is a closed symmetric operator.

**Proof.** The proof is similar to that of 2d case, this time using Proposition 2.46. In this case, for $u^\sharp \in H^2$, we take $u^\flat = \Gamma_\varepsilon(u^\sharp)$ and $u_\varepsilon = e^{W_\varepsilon} \Gamma_\varepsilon(u^\sharp)$ for the approximations. We omit the details. □

Before we introduce the resolvent and also the form domain we need the following result.

**Proposition 2.53.** Let $W$ be as above, then there exists a constant $C_\Xi > 0$ such that

$$\|\nabla u^\flat\|^2_{L^2} \lesssim \|e^{-2W}\|_{L^\infty} (-\langle u, Au \rangle + C_\Xi \|u\|_{L^2}),$$

where $u = e^{W}u^\flat \in \mathcal{D}(A)$.

**Proof.** Using (34), we write

$$\langle u, Au \rangle = \langle e^{2W}u^\flat, \Delta u^\flat + \nabla u^\flat \nabla W + LZu^\flat \rangle$$

$$= -\langle e^{2W} \nabla u^\flat, \nabla u^\flat \rangle + \langle e^{2W}u^\flat, LZu^\flat \rangle,$$

where the gradient term disappeared because we integrated by parts. Thus

$$\|\nabla u^\flat\|^2_{L^2} \leq \|e^{-2W}\|_{L^\infty} \|e^{W} \nabla u^\flat\|_{L^2}$$

$$= \|e^{-2W}\|_{L^\infty} (\|e^{2W} u^\flat, LZu^\flat\| - \langle u, Au \rangle)$$

$$\leq \|e^{-2W}\|_{L^\infty} (\|u^\flat\|_{H^{1/2+\varepsilon}} \|e^{2W} LZu^\flat\|_{H^{-1/2-\varepsilon}} - \langle u, Au \rangle)$$

$$\leq \|e^{-2W}\|_{L^\infty} (C_\Xi \|e^{2W} \|_{q^{1/2+\varepsilon}} \|u^\flat\|^2_{H^{1/2+\varepsilon}} - \langle u, Au \rangle),$$

where we have used Lemma 2.40. Using again Sobolev interpolation and Young’s inequality we can conclude by choosing $\varepsilon > 0$ small enough and pick a proper constant $C_\Xi > 0$ for the conclusion. □

After this, we are ready to conclude the self-adjointness of the operator.

**Theorem 2.54.** The operator $H$ with domain $\mathcal{D}(H)$ is self-adjoint.

**Proof.** Choosing $C_\Xi > 0$ (using Proposition 2.53) large enough, we again want to prove that

$$(C_\Xi - A)^{-1} : \mathcal{D}(A) \to L^2$$

is bounded.
This can be done in precisely the same way as the 2d case, similar to the proof of Proposition 2.23, by applying again the Babuska-Lax-Milgram theorem to the the bilinear map

\[ B : \mathcal{D}(A) \times L^2 \rightarrow \mathbb{R} \]

\[ B(u, v) := \langle (C\Xi - A)u, v \rangle. \]

Afterwards, one concludes self-adjointness by using Proposition 2.28.

Observe that the Proposition 2.53 implies the positivity of the form for \( C\Xi - A \). Accordingly, we introduce the shifted operators.

**Definition 2.55.** For a constant \( K\Xi > C\Xi \), where \( C\Xi \) is as in the proof of Theorem 2.54, we define the following shifted operators

\[ H_\varepsilon := A_\varepsilon - K\Xi \]

\[ H := A - K\Xi \]

where, in the future, the constant \( K\Xi \) will be updated to be larger, as needed.

Now, we also define the form domain.

**Definition 2.56.** From Proposition (2.46) recall that \( u = \Gamma u^\sharp \). We define the form domain of \( H \), denoted by \( \mathcal{D}(\sqrt{-H}) \), as the closure of the domain under the following norm

\[ \| \Gamma u^\sharp \|_{\mathcal{D}(\sqrt{-H})} := \sqrt{\langle \Gamma u^\sharp, -H\Gamma u^\sharp \rangle}. \]

We furthermore have the following classification for the domain and the form domain of \( H \), this is the 3d version of Proposition 2.27.

**Proposition 2.57.** We have the following characterizations for the domain and the form domain:

1. \( \Gamma u^\sharp \in e^{-W} \mathcal{D}(H) \Leftrightarrow u^\sharp \in \mathcal{H}^2 \). More precisely, on \( \mathcal{D}(H)e^W \mathcal{H}_{\Xi} \) we have the following norm equivalence

\[ \| u^\sharp \|_{\mathcal{H}^2} \lesssim_{\Xi} \| HTu^\sharp \|_{L^2} \lesssim_{\Xi} \| u^\sharp \|_{\mathcal{H}^2}. \]

2. \( u \in \mathcal{D}(\sqrt{-H}) \Leftrightarrow e^{-W}u \in \mathcal{H}^1 \). Then the precise statement is that on \( \mathcal{D}(H) \) the following norm equivalence holds

\[ \| e^{-W}u \|_{\mathcal{H}^1} \lesssim_{\Xi} \| u \|_{\sqrt{-H}} \lesssim_{\Xi} \| e^{-W}u \|_{\mathcal{H}^1}, \]

and hence the closures with respect to the two norms coincide.

**Proof.** This follows from Theorem 2.50 and Proposition 2.53 similar to the proof of Proposition 2.27.
2.2.4 Norm resolvent convergence

In this section, we address the resolvent convergence results for the regularized operators as introduced in remark 2.43 and definition 2.55. We first address the norm convergence of approximating Hamiltonians composed with the $\Gamma$-maps.

**Proposition 2.58.** Let $u^\sharp \in \mathcal{H}^2$, $u = e^W \Gamma(u^\sharp)$, $u^\flat = \Gamma(\varepsilon u^\sharp)$ and $u_\varepsilon = e^{W_\varepsilon} u_\varepsilon^\flat$. We have that
\[
\|Hu - H_\varepsilon u_\varepsilon\|_{L^2} \lesssim \|\Xi - \Xi\|_{X_\infty} \|u^\sharp\|_{\mathcal{H}^2}.
\]
Consequently, this implies that
\[
\|He^W \Gamma - H_\varepsilon e^{W_\varepsilon} \Gamma(\varepsilon u^\sharp)\|_{\mathcal{H}^2 \rightarrow L^2} \rightarrow 0.
\]
That is to say, $H e^W \Gamma \rightarrow H_\varepsilon e^{W_\varepsilon} \Gamma(\varepsilon u^\sharp)$ in norm.

**Proof.** The proof is similar to that of Proposition 2.49. This time one uses the formula (38). Then, proceeds in the same way by using Lemma 2.48 instead. Hence, the result.

In the following results, using the techniques we have used in the 2d part, we address the notions of strong resolvent and norm resolvent convergence.

**Theorem 2.59.** For any $f \in L^2$, we have
\[
\|H^{-1} f - H_\varepsilon^{-1} f\|_{L^2} \rightarrow 0
\]
as $\varepsilon \rightarrow 0$. Namely, $H_\varepsilon$ converges to $H$ in the strong resolvent sense.

**Proof.** The proof follows the lines of the 2d case, that of Theorem 2.30. This time we take $u_\varepsilon = e^{W_\varepsilon} \Gamma(\varepsilon u^\sharp)$ and use the Proposition 2.58.

**Theorem 2.60.** Let $\beta$ be as defined in Proposition 2.49. Then, we have
\[
\|H^{-1} - H_\varepsilon^{-1}\|_{L^2 \rightarrow X_\beta} \rightarrow 0
\]
as $\varepsilon \rightarrow 0$. Namely, $H_\varepsilon$ converges to $H$ in the norm resolvent sense.

**Proof.** This proof is similar to that of Theorem 2.31. We only mention the different points.

By Proposition 2.58, we have that
\[
\|H_\varepsilon e^{W_\varepsilon} \Gamma(\varepsilon u^\sharp) - He^W \Gamma u^\sharp\|_{\mathcal{H}^2 \rightarrow L^2} \rightarrow 0.
\]
This implies
\[
\|\Gamma_\varepsilon^{-1} e^{-W_\varepsilon} H_\varepsilon^{-1} - \Gamma^{-1} e^{-W} H^{-1}\|_{L^2 \rightarrow X_\beta} \rightarrow 0.
\]
By using the same tricks as in the proof of Theorem 2.31, this time using Proposition 2.46 and Lemma 2.48, one obtains
\[
\|e^{-W_\varepsilon} H_\varepsilon^{-1} - e^{-W} H^{-1}\|_{L^2 \rightarrow H^\beta} \rightarrow 0.
\]
Observe also that $\|e^{(W-W_\varepsilon)}\|_{H^\gamma \rightarrow H^\beta} \rightarrow \text{id}$. We can write the estimate
\[
\|e^{(W-W_\varepsilon)} H_\varepsilon^{-1} - H^{-1}\|_{L^2 \rightarrow H^\beta} = \|e^W (e^{-W_\varepsilon} H_\varepsilon^{-1} - e^{-W} H^{-1})\|_{L^2 \rightarrow H^\beta} \leq \|e^W\|_{H^\beta \rightarrow H^\beta} \|e^{-W_\varepsilon} H_\varepsilon^{-1} - e^{-W} H^{-1}\|_{L^2 \rightarrow H^\beta},
\]
which gives the result.
Lastly, we give a version of Agmon’s inequality which can be seen as a 3d analogue of Theorem 2.36.

**Lemma 2.61.** For \( u \in \mathcal{D}(H) \) and \( \mathcal{D}(H_\varepsilon) \) respectively, we have the following \( L^\infty \) bounds

\[
\|u\|_{L^\infty} \lesssim \Xi \left\| H u \right\|_{L^2}^{1/2} \left\| \sqrt{-H} u \right\|_{L^2}^{1/2}
\]

\[
\|u\|_{L^\infty} \lesssim \Xi \left\| H_\varepsilon u \right\|_{L^2}^{1/2} \left\| \sqrt{-H_\varepsilon} u \right\|_{L^2}^{1/2}.
\]

**Proof.** The classical version of Agmon’s inequality [1] gives the bound

\[
\|v\|_{L^\infty} \lesssim \|v\|_{H^1}^{1/2} \|v\|_{H^2}^{1/2}.
\]

Now we compute

\[
\|u\|_{L^\infty} \lesssim \|e^W\|_{L^\infty} \|\Gamma(u^\sharp)\|_{L^\infty} \lesssim \Xi \left\| u^\sharp \right\|_{L^\infty} \lesssim \Xi \left\| u^\sharp \right\|_{H^1}^{1/2} \left\| u^\sharp \right\|_{H^2}^{1/2}
\]

\[
\lesssim \Xi \left\| Hu \right\|_{L^2}^{1/2} \left\| \sqrt{-H} \right\|_{L^2}^{1/2} u \left\|_{L^2}^{1/2},
\]

where we have used Propositions 2.46 and 2.57 in addition to Agmon’s inequality and the straightforward bound \( \|u^\sharp\|_{H^1} \lesssim \Xi \left\| u^\sharp \right\|_{H^1} \). The second inequality follows the same argument with the note that the inequality constant is independent of \( \varepsilon \).

\[\square\]

### 3 Semilinear evolution equations

To recap, in the previous section we have introduced the operators \( H \) and \( H_\varepsilon \) (Definitions 2.24 and 2.55) along with their domains \( \mathcal{D}(H), \mathcal{D}(H_\varepsilon) = \mathcal{H}^2 \) (Remarks 2.5 and 2.45) and energy domains \( \mathcal{D}(\sqrt{-H}), \mathcal{D}(\sqrt{H_\varepsilon}) = \mathcal{H}^1 \) (Definitions 2.26 and 2.56). We have also studied their resolvents and norm resolvent convergence of regularized operators (Theorems 2.31 and 2.60). At the end, we have obtained certain functional inequalities to be used in the present section.

In this part, we utilize this preceding analysis in the study of some semilinear PDEs, more precisely nonlinear Schrödinger and wave-type equations with the linear part given by the 2-d and 3-d Anderson Hamiltonian. As preliminaries, we derive and record some simple results for the corresponding linear equations as well as for certain PDEs with sufficiently nice nonlinearities.

#### 3.1 Linear equations and bounded nonlinearities

In this section, as a transition to the more sophisticated nonlinear cases; we first demonstrate the solutions to the linear evolutions and PDEs with bounded type nonlinearities. We also obtain the convergence of the solutions to the regularized equations.
3.1.1 Abstract Cauchy theory for the linear and bounded nonlinear equations

We want to apply Theorem 3.3.1 from Cazenave [8]. This proves global well-posedness of

\[\begin{align*}
    i\partial_t u &= Qu + g(u) \\
    u(0) &= u_0 \in \mathcal{D}(Q)
\end{align*}\]

in the strong sense, meaning \(u \in C(\mathbb{R}; \mathcal{D}(Q)) \cap C^1(\mathbb{R}; X)\), for a sufficiently nice nonlinearity \(g\) and self-adjoint \(Q\).

**Theorem 3.1.** Consider the abstract Cauchy problem

\[\begin{align*}
    i\partial_t u &= Qu + g(u) \\
    u(0) &= u_0
\end{align*}\] \hspace{1cm} (48)

where \(Q\) is a self-adjoint operator on a Hilbert space \(X\). Then we have the following two results for Schrödinger and wave equations respectively.

1. Assume \((Qu, u) \leq 0\) for \(u \in \mathcal{D}(Q)\) and \(g: X \to X\) is Lipschitz on bounded sets as well as \((g(x), ix)_X = 0\) for all \(x \in X\) and \(g = G'\) where \(G \in C^1(\mathcal{D}(\sqrt{-Q}))\). For the case \(Q = H\), \(X = L^2(\mathbb{T}^d)\) \(d = 2, 3\) \(u_0 \in \mathcal{D}(H)\) and \(g(u) := K\xi u + w\varphi(|u|^2)\) where \(\varphi \in C^2_b\) we get a unique global strong solution of (48)

\[u \in C([0, \infty); \mathcal{D}(H)) \cap C^1([0, \infty); L^2)\].

We can also relax this slightly if we ask for \(u_0 \in \mathcal{D}(\sqrt{-H})\). We get a unique global energy solution

\[u \in C([0, \infty); \mathcal{D}(\sqrt{-H})) \cap C^1([0, \infty); \mathcal{D}^*(\sqrt{-H}))\].

In both cases conservation of mass and energy holds for all times.

2. For the wave equation, with \(d = 2, 3\), we set

\[\begin{align*}
    Q &= i \begin{pmatrix} 0 & \mathbb{I} \\ H & 0 \end{pmatrix}, \quad \mathcal{D}(Q) = \mathcal{D}(H) \oplus \mathcal{D}(\sqrt{-H}) \\
    X &= (L^2(\mathbb{T}^d))^2, \quad g(u) = \begin{pmatrix} 0 \\ -K\xi u \end{pmatrix}.
\end{align*}\]

Then the abstract linear wave equation

\[\begin{align*}
    \frac{d}{dt} \begin{pmatrix} u \\ \partial_t u \end{pmatrix} &= \begin{pmatrix} 0 & \mathbb{I} \\ -H & 0 \end{pmatrix} \begin{pmatrix} u \\ \partial_t u \end{pmatrix} + \begin{pmatrix} 0 \\ -K\xi u \end{pmatrix} \\
    \begin{pmatrix} u \\ \partial_t u \end{pmatrix}_{t=0} &= \begin{pmatrix} u_0 \\ u_1 \end{pmatrix}
\end{align*}\]

has a unique global strong solution \((u, \partial_t u) \in C([0, \infty); \mathcal{D}(A)) \times C^1([0, \infty); L^2)\) i.e. \(u \in C([0, \infty); \mathcal{D}(H)) \cap C^1([0, \infty); \mathcal{D}(\sqrt{-H})) \cap C^2([0, \infty); L^2)\) and energy conservation holds.
2. See [20, Chapter X.13].

In this part, we demonstrate the solution to the linear Schrödinger equation

$$i \partial_t u = H u \quad \text{on } \mathbb{T}^d,$$

with domain initial data. A simple but important observation is that the Schrödinger equation conserves the $L^2$ norm. Also observe that $\partial_t u$ formally satisfies

$$i \partial_t \partial_t u = H (\partial_t u),$$

so it solves the same equation and in particular we have that $\|\partial_t u(t)\|_{L^2}$ is conserved and that

$$\|H u\|_{L^2} = \|\partial_t u(t)\|_{L^2} = \|\partial_t u(0)\|_{L^2} = \|H u(0)\|_{L^2},$$

with $f(u) := \varphi(|u|^2)$. This proves the differentiability. Lastly we prove the $L^2$ local Lipschitz property of $g$. Fix $v \in L^2$ and $u \in B_M(v)$, for some $M > 0$. Then

$$\|g(u) - g(v)\|_{L^2} \leq K\|u - v\|_{L^2} + \|v\varphi'(|v|^2) - v\varphi'(|u|^2)\|_{L^2} \leq K\|u - v\|_{L^2} + \|v\varphi' - \varphi'(u)\|_{L^2} \leq K\|u - v\|_{L^2} + \|\varphi'\|_{\infty} \|u - v\|_{L^2} + \|v\|_{L^2} \|\varphi'(u)\|_{\infty} \|u - v\|_{L^2},$$

hence $g$ is locally Lipschitz as a map from $L^2$ to $L^2$.

3. See [20] Chapter X.13].
which we will assume to be finite. Naturally, this gives the condition that the initial data have to satisfy. Therefore, we will assume \( u_0 \in \mathcal{D}(H) \) which implies \( \|Hu_0\|_{L^2} < \infty \), by Theorem 2.15. To make this precise, we write

\[
\frac{d}{dt}u(t) = -ie^{-itH}Hu_0
\]

\[
\left\| \frac{d}{dt}u(t) \right\|_{L^2} = \|Hu_0\|_{L^2} = \|Hu(t)\|_{L^2}.
\]

So \( \|\partial_t u(t)\|_{L^2} \) is conserved for the abstract solution \( u \) as defined in Section 3.1.1. For the regularized equation, the unique solution is given by

\[
u_{\varepsilon}(t) = e^{-itH_\varepsilon}u_{\varepsilon}^0 \in \mathcal{H}^2,
\]

where \( u_{\varepsilon}^0 \in \mathcal{H}^2 \) is the regularized initial datum. If we choose the regularization

\[
u_{\varepsilon}^0 := H_\varepsilon^{-1}H u_0 \in \mathcal{H}^2,
\]

then we readily have, \( H_\varepsilon u_{\varepsilon}^0 = Hu_0 \) and we get \( u_{\varepsilon}^0 \to u_0 \) in \( L^2 \), by norm resolvent convergence, namely Theorem 2.31. By [19, Theorem VIII.21], \( e^{-itH_\varepsilon} \to e^{-itH} \) strongly for any time \( t \), which implies

\[
e^{-itH}u_{\varepsilon}^0 \to e^{-itH}u_0,
\]

\[
\text{and}
\]

\[
e^{-itH_\varepsilon}H_\varepsilon u_{\varepsilon}^0 \to e^{-itH}Hu,
\]

in \( L^2 \) for any \( t \in \mathbb{R} \).

We summarize these results in the following theorem

**Theorem 3.2.** Let \( T > 0 \), \( u_0 \in \mathcal{D}(H) \). Then there exists a unique solution \( u \in C([0,T];\mathcal{D}(H)) \cap C^1([0,T];L^2) \) to the equation

\[
\begin{cases}
   i\partial_t u = Hu \\
   u(0,\cdot) = u_0
\end{cases}
\quad \text{on } [0,T] \times \mathbb{T}^d.
\]

Moreover, this agrees with the \( L^2 \)-limit of the solutions \( u_{\varepsilon} \in C([0,T];\mathcal{H}^2) \cap C^1([0,T];L^2) \) to

\[
\begin{cases}
   i\partial_t u_{\varepsilon} = H_\varepsilon u_{\varepsilon} \\
   u_{\varepsilon}(0,\cdot) = u_{\varepsilon}^0
\end{cases}
\quad \text{on } [0,T] \times \mathbb{T}^d,
\]

with the regularized data given as

\[
u_{\varepsilon}^0 := H_\varepsilon^{-1}H u_0 \in \mathcal{H}^2.
\]

One also obtains the convergence of \( \partial_t u_{\varepsilon} \) and \( H_\varepsilon u_{\varepsilon} \) to \( \partial_t u \) and \( H u \) in \( L^2 \).

**Remark 3.3.** One could even get global wellposedness for the equation with initial data in \( \mathcal{D}(\sqrt{-H}) \) or in \( L^2 \). Moreover one could have a bounded nonlinearity as in section 3.1.1.
3.1.3 The linear multiplicative wave equation

Similarly to the Schrödinger case, now we consider the linear wave equation

$$\frac{\partial^2}{\partial t^2} u = Hu$$

with initial data \((u_0, u_1) \in \mathcal{D}(H) \times \mathcal{D}(\sqrt{-H})\). For the regularized equation

$$\frac{\partial^2}{\partial t^2} u_\varepsilon = H_\varepsilon u_\varepsilon$$

\((u_\varepsilon, \partial_t u_\varepsilon)|_{t=0} = (u_\varepsilon^0, u_\varepsilon^1) \in \mathcal{H}^2 \times \mathcal{H}^1\)

a solution is given by

$$\begin{pmatrix} u_\varepsilon \\ \partial_t u_\varepsilon \end{pmatrix} = e^{itQ_\varepsilon} \begin{pmatrix} u_\varepsilon^0 \\ u_\varepsilon^1 \end{pmatrix}, \quad \text{where} \quad Q_\varepsilon = i \begin{pmatrix} 0 & \mathbb{I} \\ -H_\varepsilon & 0 \end{pmatrix}.$$

We again choose the same approximation for \(u_0\) as in the Schrödinger case

$$u_\varepsilon^0 := (-H_\varepsilon)^{-1} (-H) u_0 \in \mathcal{H}^2$$

$$u_\varepsilon^1 := (\sqrt{H_\varepsilon})^{-1} \sqrt{-H} u_1 \in \mathcal{H}^1$$

for some initial datum \(u_0 \in \mathcal{D}(H)\). Then we again have

$$u_\varepsilon^0 \to u_0 \text{ in } L^2$$

$$H_\varepsilon u_\varepsilon^0 \to Hu_0 \text{ in } L^2.$$

For the initial velocity, we also have

$$u_\varepsilon^1 \to u_1 \text{ in } L^2.$$

Then for any time \(t\) we get as in the Schrödinger case

$$\begin{pmatrix} u_\varepsilon(t) \\ \partial_t u_\varepsilon(t) \end{pmatrix} = e^{itQ_\varepsilon} \begin{pmatrix} u_\varepsilon^0 \\ u_\varepsilon^1 \end{pmatrix} \to e^{itQ} \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} \text{ in } L^2$$

and

$$\frac{d}{dt} \begin{pmatrix} u_\varepsilon(t) \\ \partial_t u_\varepsilon(t) \end{pmatrix} = it e^{itQ_\varepsilon} Q_\varepsilon \begin{pmatrix} u_\varepsilon^0 \\ u_\varepsilon^1 \end{pmatrix} \to it e^{itQ} Q \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} \text{ in } L^2.$$

Moreover, we have that the convergence of the energies, namely

$$E_\varepsilon(t) := \left\langle \begin{pmatrix} u_\varepsilon(t) \\ \partial_t u_\varepsilon(t) \end{pmatrix}, \begin{pmatrix} -H_\varepsilon & 0 \\ 0 & \mathbb{I} \end{pmatrix} \begin{pmatrix} u_\varepsilon(t) \\ \partial_t u_\varepsilon(t) \end{pmatrix} \right\rangle$$

$$\to \left\langle \begin{pmatrix} u(t) \\ \partial_t u(t) \end{pmatrix}, \begin{pmatrix} -H & 0 \\ 0 & \mathbb{I} \end{pmatrix} \begin{pmatrix} u(t) \\ \partial_t u(t) \end{pmatrix} \right\rangle = E(t)$$

for any time \(t\) and thus in particular the energy conservation passes to the limit. We record these observations in the following theorem.
Theorem 3.4. Let $T > 0$ and $(u_0, u_1) \in \mathcal{D}(H) \times \mathcal{D}(\sqrt{-H})$. Then there exists a unique solution $(u, \partial_t u) \in C([0, T]; \mathcal{D}(H) \times \mathcal{D}(\sqrt{-H})) \cap C^1([0, T]; \mathcal{D}(\sqrt{-H}) \times L^2)$ to the equation
\[
\partial_t^2 u = Hu \text{ in } (0, T) \times \mathbb{T}^d
\]
moreover it is equal to the $L^\infty((0, T); L^2(\mathbb{T}^d) \times L^2(\mathbb{T}^d))$ limit of the approximate solutions $(u_\xi, \partial_t u_\xi)$ to
\[
\partial_t^2 u_\xi = H_\xi u_\xi \text{ in } (0, T) \times \mathbb{T}^d
\]
and moreover we have the following convergences
\[
u_\xi(t) \to u(t) \text{ in } L^2
\]
\[H_\xi u_\xi(t) \to Hu(t) \text{ in } L^2
\]
\[\langle -H_\xi \rangle^{1/2} \partial_t u_\xi(t) \to \langle -H \rangle^{1/2} \partial_t u(t) \text{ in } L^2
\]
\[\partial_t^2 u_\xi(t) \to \partial_t^2 u(t) \text{ in } L^2
\]
with $(u_0, u_1)$ as above. Also, the energies converge and are conserved in time.

Proof. The computations above prove that the $L^2$ limit of the solutions we obtain is equal to the solution of the abstract Cauchy problem in Theorem 3.1 for all times. Hence, the two are equal.

3.2 Nonlinear Schrödinger equations in two dimensions

In this section, we are interested in solving the following defocussing cubic Schrödinger-type equation
\[
i\partial_t u = Hu - u|u|^2,
\]
with domain and energy space data.

Recall that, for the operator $H$ we have
\[\langle u, -Hu \rangle \geq 0 \text{ for all } u \in \mathcal{D}(H).
\]

We consider the mild formulation of
\[u(t) = e^{-itH}u_0 + i \int_0^t e^{i(s-t)H} u(s) |u(s)|^2 ds
\]
Furthermore, we introduce the energy for $u$ as
\[E(u)(t) := \frac{1}{2} \langle u, Hu \rangle + \frac{1}{4} \int |u|^4.
\]
Using the equation one sees that the energy is formally conserved in time.
3.2.1 Solution with initial conditions in $\mathcal{D}(H)$

In this section, we assume $u_0 \in \mathcal{D}(H)$. This is similar in spirit to the global strong well-posedness of the classical cubic NLS with initial data in $\mathcal{H}^2$, which was solved in $\mathbb{R}^2$. We obtain global in time strong solutions in our setting, which is the best one can hope for in view of the classical result. We regularize the initial data in the following way

$$u_0^\varepsilon = (-H\varepsilon)^{-1}(-H)u_0 \in \mathcal{D}(H\varepsilon)$$

so that by the norm resolvent convergence of $H\varepsilon$ to $H$ (by Theorem 2.31), we have

$$\lim_{\varepsilon \to 0} u_0^\varepsilon = u_0 \in L^2$$

$$H\varepsilon u_0^\varepsilon = Hu_0 \in L^2.$$ 

Note that $\mathcal{D}(H\varepsilon) = \mathcal{H}^2$ so there exists global solutions $u_\varepsilon \in C([0, T]; \mathcal{H}^2) \cap C^1([0, T]; L^2)$. This is an immediate consequence of the following result, which shows that for the operators $H\varepsilon$ and $H$, we obtain global in time strong solutions of the associated cubic NLS on $\mathbb{T}^2$.

**Theorem 3.5.** For an arbitrary time $T > 0$, there exist unique solutions $u_\varepsilon \in C([0, T]; \mathcal{H}^2) \cap C^1([0, T]; L^2)$ and $u \in C([0, T]; \mathcal{D}(H)) \cap C^1([0, T]; L^2)$ to

$$u_\varepsilon(t) = e^{-itH_\varepsilon}u_0^\varepsilon + i \int_0^t e^{-isH_\varepsilon}|u_\varepsilon|^2(t - s)ds,$$  

and

$$u(t) = e^{-itH}u_0 + i \int_0^t e^{-isH}|u|^2(t - s)ds$$  

respectively, with initial data $u_0^\varepsilon \in \mathcal{H}^2$ and $u_0 \in \mathcal{D}(H)$.

Before we prove the theorem, we need the following technical lemmas which will be used throughout the proof. The first one is a logarithmic Gronwall lemma.

**Lemma 3.6.** Let $C_2, \log C_1 \geq 1$ and $\theta(t) \geq 1$ satisfy

$$\theta(t) \leq C_1 + C_2 \int_0^t \theta(s) \log(1 + \theta(s))ds = h(t).$$

Then we have

$$h(t) \leq \exp(\log h(0)e^{C_2 t}) - 1.$$ 

**Proof.** We have that $h$ is a subsolution of the equation

$$\partial_t h(t) = C_2 \theta(t) \log(1 + \theta(t)) \leq C_2 (h(t) + 1) \log(h(t) + 1).$$

So taking $\rho(t)$ being a solution of

$$\partial_t \rho(t) = C_2(\rho(t) + 1) \log(\rho(t) + 1), \quad \rho(0) = h(0),$$

we have $\rho(t) \geq h(t)$. Indeed $\rho(0) = h(0)$ and whenever we have $\rho(t) = h(t)$ then

$$\partial_t(\rho(t) - h(t)) \geq C_2(\rho(t) + 1) \log(\rho(t) + 1) - C_2(h(t) + 1) \log(h(t) + 1) = 0.$$ 

Observe moreover that

$$\partial_t \log(\rho(t) + 1) = C_2 \log(\rho(t) + 1) \Rightarrow \log(\rho(t) + 1) = (\log h(0))e^{C_2 t}.$$ 

\qed
**Lemma 3.7.** For \( v \in C([0,T]; \mathcal{H}^2) \cap C^1([0,T]; L^2) \), \( f(v)(t) = |v(t)|^2v(t) \) is \( C^1 \) as a map from \([0,T]\) to \( L^2 \). The same is true for \( v \in C([0,T]; \mathcal{D}(H)) \cap C^1([0,T]; L^2) \).

**Proof.** We write

\[
\frac{|v(t+h)|^2v(t+h) - |v(t)|^2v(t)}{h}
\]

and add and subtract the term \( v^2(t+h)\bar{v}(t) \) which yields

\[
\frac{\bar{v}(t+h)(v(t+h))v(t+h) - v^2(t+h)\bar{v}(t) + v^2(t+h)\bar{v}(t) - \bar{v}(t)v(t)v(t)}{h}
\]

This can be rearranged as

\[
v^2(t+h)\frac{\bar{v}(t+h) - \bar{v}(t)}{h} + (\bar{v}(t)(v(t+h) + v(t))) \frac{v(t+h) - v(t)}{h}
\]

where all the terms converge individually in \( L^2 \) as \( h \to 0 \). Indeed, one can easily check that the multiplication map \( (f,g) \to f \cdot g \) defines a continuous map \( \mathcal{H}^2 \times L^2 \to L^2 \). This follows from the embedding \( \mathcal{H}^2 \hookrightarrow L^\infty \) in 2d. Since, by Lemma 2.34, we also have the embedding \( \mathcal{D}(H) \hookrightarrow L^\infty \), the same holds in this case. \( \square \)

**Proof of Theorem 3.3.** This is a fixed point argument, which is essentially the same in both cases. For fixed \( u_0 \in \mathcal{D}(H) \), we define the operator

\[
\Phi(u)(t) := e^{-itH}u_0 + i \int_0^t e^{-isH}u|u|^2(t-s)ds
\]

and claim that is in fact a contraction on \( X = C([0,T_E]; \mathcal{D}(H)) \cap C^1([0,T_E]; L^2) \), where the time \( T_E > 0 \) depends on the initial data and the energy, which is conserved. This will allow us to obtain a global in time solution.

We bound, for \( \|u\|_X \leq M \) with \( M \) chosen below, using Theorem 2.36,

\[
\|\partial_t \Phi(u)\|_{L^2}(t) \leq \|Hu_0\|_{L^2} + \int_0^t \|\partial_t (u|u|^2)(s)\|_{L^2} ds + C\|u_0\|_{L^6}^2
\]

\[
\leq \|Hu_0\|_{L^2} + \int_0^t CME(u)(s)(1 + \log(M + 1)) ds + C\|u_0\|_{L^6}^2 \leq \frac{M}{2}
\]

for \( t \leq T_E \) small enough such that

\[
\int_0^{T_E} CME(u)(s)(1 + \log(M + 1)) ds \leq \frac{M}{2} - (\|Hu_0\|_{L^2} + C\|u_0\|_{L^6}^2)
\]

and \( M \) such that \( \frac{M}{2} - (\|Hu_0\|_{L^2} + C\|u_0\|_{L^6}^2) > 0 \).

Analogously, we compute

\[
\|H\Phi(u)\|_{L^2}(t) \leq \|Hu_0\|_{L^2} + \int_0^t CME(u)(s)(1 + \log(M + 1)) ds + C\|u_0\|_{L^6}^3 \leq \frac{M}{2},
\]

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We obtain, by using Theorem 2.36 and Lemma 2.34,
\[ \partial_t \Phi(u)(t) - \partial_t \Phi(v)(t) = \int_0^t e^{-isH} \partial_t (u|u|^2 - v|v|^2)(t-s) \, ds . \]

We obtain, by using Theorem 2.36 and Lemma 2.34,
\[ \| \partial_t \Phi(u)(t) - \partial_t \Phi(v)(t) \|_{L^2} \leq \]
\[ \leq 3 \int_0^t \| u(s) \|_{L^\infty}^2 \| \partial_t u - \partial_t v \|_{L^\infty L^2} + \| \partial_t v \|_{L^\infty L^2} \| H u - H v \|_{L^\infty L^2} (\| u(s) \|_{L^\infty} + \| v(s) \|_{L^\infty}) \, ds \]
\[ \leq 3 \| u - v \|_X \int_0^t E(u)(s)(1 + \log(1 + M)) + M \sqrt{(1 + \log(M + 1))(E^{1/2}(u)(s) + E^{1/2}(v)(s))} \, ds \]
\[ < \| u - v \|_X \]
for \( t \leq T_E \) by possibly making \( T_E \) smaller depending on \( E(u) \). This gives us short time wellposedness, but since the time span depends only on the energy and the initial data, this can be iterated to yield a global strong solution. In fact, the only thing that is left to show is that \( \| H u(T_E) \|_{L^2} \) can be bounded by \( \| H u_0 \|_{L^2} \), i.e. a priori bounds. This allows us to choose a global \( M \) and then also a fixed time span \( T_E \) which immediately implies a global solution. For the solution that exists up to time \( T_E \), we have the estimate
\[ \| H u(T_E) \|_{L^2} \lesssim \| H u_0 \|_{L^2} + \| u_0 \|_{L^6}^3 + \int_0^{T_E} \| \partial_t u(|u|^2)(s) \|_{L^2} \, ds \]
\[ \lesssim \| H u_0 \|_{L^2} + \| u_0 \|_{L^6}^3 + \int_0^{T_E} \| \partial_t u(s) \|_{L^2} \| u(s) \|_{L^\infty}^2 \, ds \]
\[ \lesssim \| H u_0 \|_{L^2} + \| u_0 \|_{L^6}^3 + \int_0^{T_E} (\| H u(s) \|_{L^2} + E^{3/2}(u_0)) E(u_0)(1 + \log(1 + \| H u(s) \|_{L^2})) \, ds , \]
where we have used again Theorem 2.36 and the fact that one can estimate \( \| \partial_t u \|_{L^2} \) by \( \| H u \|_{L^2} \) using the equation. Now, we can conclude by using Lemma 3.6. This gives us a bound, by possibly taking larger constants, of the form
\[ \| H u(T_E) \|_{L^2} \lesssim C\| E^{3/2}(u_0) \| + \exp(\| E(u_0) \| T \log(C \| E^{3/2}(u_0) \| + \| H u_0 \|_{L^2})) - 1 , \quad (55) \]
where \( T \) is the maximum time of existence. Hence \( M \), and therefore \( T_E \), can be chosen globally which means that we can solve the cubic NLS on the whole interval \([0, T]\) by iterating. The proof for the regularized Hamiltonian follows the same lines with the crucial note that the inequality constant in Theorem 2.36 does not blow up, namely the constant does not depend on \( \varepsilon \) but only on \( \Xi \).

**Remark 3.8.** One sees from the proof that the same remains true for NLS with general power nonlinearity, i.e.
\[ i \partial_t u = Hu - u|u|^{p-1} , \]
with \( p \in (1, \infty) \), since all \( L^p \)-norms can be controlled by the energy. The result will also remain true in the focusing case under some suitable smallness conditions on \( u_0 \).
We will moreover prove that the approximate solutions $u_\varepsilon$, which are strong solutions of
\begin{equation}
\begin{aligned}
i \partial_t u_\varepsilon &= H_\varepsilon u_\varepsilon - u_\varepsilon |u_\varepsilon|^2, \\
u_\varepsilon(0) &= u_0^\varepsilon \in \mathcal{D}(H_\varepsilon) = \mathcal{H}^2.
\end{aligned}
\end{equation}

converges to the solution $u$ of the limiting problem. We prove the following result.

**Theorem 3.9.** Let $u_0 \in \mathcal{D}(H)$ and $T > 0$ be an arbitrary time. Solutions to the regularized equations with initial data $u_0^\varepsilon := (-H_\varepsilon)^{-1}(-H)u_0 \in \mathcal{H}^2$, (the unique global strong solutions $u_\varepsilon$ of (56)) converges to the unique global strong solutions $u \in C([0,T]; \mathcal{D}(H)) \cap C^1([0,T]; L^2)$ of
\begin{equation}
\begin{aligned}
i \partial_t u &= Hu - |u|^2u, \\
u(0) &= u_0,
\end{aligned}
\end{equation}
which is obtained in Theorem 3.3. In fact, we get the following convergence results
\begin{align*}
u_\varepsilon(t) &\to u(t) \quad \text{in } L^2 \\
H_\varepsilon u_\varepsilon(t) &\to Hu(t) \quad \text{in } L^2 \\
\partial_t u_\varepsilon(t) &\to \partial_t u(t) \quad \text{in } L^2
\end{align*}
for all $t \in [0,T]$.

**Proof.** We know that the $u_\varepsilon$ satisfy the mild formulation
\begin{equation}
u_\varepsilon(t) = e^{-iH_\varepsilon t}u_0^\varepsilon + i \int_0^t e^{-i(t-s)H_\varepsilon}u_\varepsilon(s)|u_\varepsilon(s)|^2\,ds 
\end{equation}
and $u$ satisfies
\begin{equation}
u(t) = e^{-iH t}u_0 + i \int_0^t e^{-i(t-s)H}u(s)|u(s)|^2\,ds.
\end{equation}

We compute
\begin{align*}
Hu(t) - H_\varepsilon u_\varepsilon(t) &= \\
= (e^{-itH} - e^{-itH_\varepsilon})Hu_0 + \int_0^t e^{-i(t-s)H_\varepsilon}\partial_s(u|u|^2(s))\,ds - \int_0^t e^{-i(t-s)H_\varepsilon}\partial_s(u_\varepsilon|u_\varepsilon|^2(s))\,ds \\
&= (e^{-itH} - e^{-itH_\varepsilon})Hu_0 + \int_0^t (e^{-i(t-s)H} - e^{-i(t-s)H_\varepsilon})\partial_s(u|u|^2(s))\,ds \\
&\quad - \int_0^t e^{-i(t-s)H_\varepsilon} \partial_s(u_\varepsilon|u_\varepsilon|^2(s)) - \partial_s(u|u|^2(s))\,ds \\
&\quad + \int_0^t \partial_s(u|u|^2(s)) - \partial_s(u_\varepsilon|u_\varepsilon|^2(s))\,ds + u_0|u_0|^2 - u_0^\varepsilon|u_0^\varepsilon|^2.
\end{align*}
Therefore, we have
\begin{align*}
\|Hu(t) - H_\varepsilon u_\varepsilon(t)\|_{L^2} &\lesssim \|(e^{-itH} - e^{-itH_\varepsilon})Hu_0\|_{L^2} + \|u_0|u_0|^2 - u_0^\varepsilon|u_0^\varepsilon|^2\|_{L^2} \\
&\quad + \int_0^t \|(e^{-i(t-s)H} - e^{-i(t-s)H_\varepsilon})\partial_s(u|u|^2(s))\|_{L^2} \,ds \\
&\quad + \int_0^t \|\partial_s(u|u|^2(s)) - \partial_s(u_\varepsilon|u_\varepsilon|^2(s))\|_{L^2} \,ds,
\end{align*}
where the first three terms converge to zero by norm resolvent convergence and Theorem VIII.21 in [19]. For the last term we can bound similarly to the proof of Theorem 3.3

\[
\int_0^t \left\| \partial_s (u|u|^2(s)) - \partial_s (u_\varepsilon|u_\varepsilon|^2(s)) \right\|_{L^2} ds \\
\lesssim \int_0^t \left\| \partial_s u(s) - \partial_s u_\varepsilon(s) \right\|_{L^2} \left( \|u\|_{L^\infty L^\infty}^2 + \|u_\varepsilon\|_{L^\infty L^\infty}^2 \right) \\
+ \|u(s) - u_\varepsilon(s)\|_{L^\infty} \|\partial_t u\|_{L^2} \left( \|u\|_{L^\infty L^\infty} + \|u_\varepsilon\|_{L^\infty L^\infty} \right) ds \\
\lesssim C(T, u_0, \Xi) \int_0^t \left\| \partial_s u(s) - \partial_s u_\varepsilon(s) \right\|_{L^2} + \|H u(s) - H \varepsilon u_\varepsilon(s)\|_{L^2} \\
+ \|u(s) - u_\varepsilon(s)\|_{L^2} ds + C(T, u_0, \Xi) \|\Xi - \Xi_\varepsilon\|_{\mathcal{X}^\alpha},
\]

where we have used the a priori bounds obtained in the proof of Theorem 3.3 and the bound

\[
\|u(s) - u_\varepsilon(s)\|_{L^\infty} \lesssim \|u^\varepsilon(s) - u_\varepsilon^\varepsilon(s)\|_{\mathcal{X}^2} \\
\lesssim \|H u(s) - H \varepsilon u_\varepsilon(s)\|_{L^2} + \|u(s) - u_\varepsilon(s)\|_{L^2} + C(T, u_0) \|\Xi - \Xi_\varepsilon\|_{\mathcal{X}^\alpha}
\]

which can be proved in the same way as Theorem 2.15 using Proposition 2.8 and the embedding $\mathcal{X}^2 \hookrightarrow L^\infty$.

Similarly we can bound (by $O(\varepsilon)$ we denote terms that converge to zero as $\varepsilon \to 0$)

\[
\|\partial_t u(t) - \partial_t u_\varepsilon(t)\|_{L^2} \\
\lesssim \|(e^{-\varepsilon tH} - e^{-itH}) H u_0\|_{L^2} + \|e^{-\varepsilon tH} u_0\|_{L^2} - e^{-itH} u_0\|_{L^2}^2 \|H u(t) - H \varepsilon u_\varepsilon(t)\|_{L^2} \\
+ \left\| \int_0^t e^{-isH} \partial_t (u|u|^2(t - s)) ds - \int_0^t e^{-isH} \partial_t (u_\varepsilon|u_\varepsilon|^2(t - s)) ds \right\|_{L^2} \\
\lesssim O(\varepsilon) + C(T, u_0, \Xi) \int_0^t \left\| \partial_s u(s) - \partial_s u_\varepsilon(s) \right\|_{L^2} \\
+ \|H u(s) - H \varepsilon u_\varepsilon(s)\|_{L^2} + \|u(s) - u_\varepsilon(s)\|_{L^2} ds
\]

and we have

\[
\|u(t) - u_\varepsilon(t)\|_{L^2} \\
\lesssim \|e^{-\varepsilon tH} u_0 - e^{-itH} u_0\|_{L^2} + \left\| \int_0^t e^{-i(t-s)H} (u|u|^2(s)) ds - \int_0^t e^{-i(t-s)H \varepsilon} (u_\varepsilon|u_\varepsilon|^2(s)) ds \right\|_{L^2} \\
\lesssim O(\varepsilon) + C(T, u_0, \Xi) \int_0^t \|u(s) - u_\varepsilon(s)\|_{L^2} ds.
\]

Thus, for $\phi_\varepsilon(t) := \|u(t) - u_\varepsilon(t)\|_{L^2} + \|\partial_t u(t) - \partial_t u_\varepsilon(t)\|_{L^2} + \|H u(t) - H \varepsilon u_\varepsilon(t)\|_{L^2}$ we have

\[
\phi_\varepsilon(t) \lesssim O(\varepsilon) + \int_0^t \phi_\varepsilon(s) ds
\]

and by Gronwall we can conclude that $\phi_\varepsilon(t) \to 0$ for all $t$ as $\varepsilon \to 0$.

This finishes the proof. \qed
Remark 3.10. Observe that the above also works in three dimensions. The only difference is that one uses Lemma 2.61 instead of Theorem 2.36. But note that, this gives only local in time strong solutions and as in Theorem 3.9, we also obtain the convergence of solutions to the approximated PDEs. This is due to the fact that, unlike the 2d case, one uses a polynomial type Gronwall [10], as opposed to a logarithmic Gronwall, which leads to an estimate that blows up in finite time. In fact, this can be formulated as a blow up alternative (with respect to the $L^\infty$-norm) similarly to the classical case of $H^2$-solutions [8].

3.2.2 Energy solutions

In this section we solve (50) in the energy space. For the global well-posedness of (standard) cubic NLS on the 2d torus see [4]. Note that the result we get here is somewhat weaker, since we obtain only existence and partial regularity in time. But the Strichartz estimates in the case of Anderson Hamiltonian is not known and our result is still as good as what one would get in the classical case without the use of Strichartz estimates, please see [6] and references therein for further information. In this section, we denote the dual of $D(\sqrt{-H})$ by $D(\sqrt{-H})^\ast$. So, we naturally have $D(\sqrt{-H}) \subseteq L^2 \subseteq D(\sqrt{-H})^\ast$.

Theorem 3.11. For $u_0 \in D(\sqrt{-H})$, the equation (50) has a solution $u$ such that $u \in C^{1/2}([0,T];L^2) \cap C([0,T];D(\sqrt{-H}))$.

For the initial datum $u_0$, we construct the following approximation

$$u_0^\varepsilon := (1 + \varepsilon \sqrt{-H})^{-1} u_0 \in D(H).$$

Note that by continuous functional calculus the operator $(-H)^{-1/2} : L^2 \to D(\sqrt{-H})$ is bounded and we have $u_0^\varepsilon \to u_0$ in $D(\sqrt{-H})$.

Lemma 3.12. For $u_0, u_0^\varepsilon$ as above, we have the following convergence of energies.

$$E_c(u_0^\varepsilon) := -\frac{1}{2} \langle u_0^\varepsilon, Hu_0^\varepsilon \rangle + \frac{1}{4} \int |u_0^\varepsilon|^4 \to -\frac{1}{2} \langle u_0, Hu_0 \rangle + \frac{1}{4} \int |u_0|^4 = E(u_0)$$

Proof. By the above observation the first terms converge. For the $L^4$ terms, we can conclude using Lemma 2.33 and the $D(\sqrt{-H})$ convergence.

Consider the nonlinear Schrödinger equation

$$i\partial_t u_\varepsilon = Hu_\varepsilon - u_\varepsilon |u_\varepsilon|^2 \quad (57)$$

$$u_\varepsilon(0) = u_0^\varepsilon \in D(H).$$

As we have seen in section 3.2.1, there exists a unique solution $u_\varepsilon$ to this equation in $C([0,T]; D(H)) \cap C^1([0,T];L^2)$ which conserves the energy $E(u_\varepsilon(0)) = E(u_\varepsilon(t)) := -\frac{1}{2} \langle u_\varepsilon(t), Hu_\varepsilon(t) \rangle + \frac{1}{4} \|u_\varepsilon(t)\|_{L^4}^4$. 43
Lemma 3.13 (A priori bounds). For solutions $u_\varepsilon$ to (57), we have the following uniform bounds.

\[
\|u_\varepsilon\|_{L^\infty L^2} \lesssim \|u_0\|_{L^2} \\
\|(-H)^{1/2}u_\varepsilon\|_{L^\infty L^2} \lesssim E^{1/2}(u_0) \\
\|(-H)^{-1/2}\partial_t u_\varepsilon\|_{L^\infty L^2} \lesssim E^{1/2}(u_0) + E^{3/2}(u_0)
\]

Proof. Since we have conservation of mass and energy, the first and second follow directly, using also Lemma 3.12 and the positivity of the energy. For the third bound, we use the equation and the fact that

\[
\|u_\varepsilon\|^3_{L^6} \lesssim E^{3/2}(u_0),
\]

which follows from Lemma 2.33. 

Lemma 3.14 (Compactness). Given $u_\varepsilon$ as above, we can extract a subsequence $u_{\varepsilon,n}$ and obtain a limit $u \in L^\infty([0,T]; D(\sqrt{-H}))$ s.t.

\[
u_{\varepsilon,n}(t) \to u(t) \text{ in } L^2 \\
(-H)^{1/2}u_{\varepsilon,n}(t) \to (-H)^{1/2}u(t) \text{ in } L^2 \\
\]

for all times $t \in [0,T]$.

Proof. By weak compactness in the Hilbert space $D(\sqrt{-H})$ we obtain a subsequence $u_{\varepsilon,n}$ and a limit $u$ s.t.

\[
u_{\varepsilon,n}(t) \to u(t) \text{ in } L^2, \\
u_{\varepsilon,n}(t) \to u(t) \text{ in } D(\sqrt{-H}),
\]

for a dense set of times and using the third a priori bound from Lemma 3.13 we can extend this to all times $t \in [0,T]$. In particular get the $L^\infty$ bound in time. Lastly, we can use the convergence of energies to deduce the convergence of the $D(\sqrt{-H})$ norms of $u_\varepsilon$ and thus conclude that in fact strong convergence holds.

Now we can conclude this section by proving Theorem 3.11

Proof of Theorem 3.11. We prove that the limit we obtain in the previous lemma solves the mild formulation of (50). We have by construction that the $u_\varepsilon$ solves

\[
u_\varepsilon(t) = e^{-itH}u_0^\varepsilon + i \int_0^t e^{i(s-t)H}u_\varepsilon(s)|u_\varepsilon(s)|^2ds
\]

for all $t \in [0,T]$. Now we can prove that this converges in $L^2$ as $\varepsilon \to 0$ for all times. The first term converges precisely as in the linear case from section 3.1.2. For the nonlinear term the convergence follows from the fact that $u_\varepsilon(t) \to u(t)$ strongly in $L^6$ for all times. This is due to the fact that the embedding $D(\sqrt{-H}) \hookrightarrow H^{1-\delta}$ is continuous and the embedding $H^{1-\delta} \hookrightarrow L^6$ is compact (in fact this is true for any $L^p$ with $p < \infty$).
For continuity in $\mathcal{D}(\sqrt{-H})$, we simply observe
\[
\|\sqrt{-H}u(t) - \sqrt{-H}u(s)\|_{L^2} \leq \|\sqrt{-H}u(t) - \sqrt{-H}u_{\epsilon_n}(t)\|_{L^2} + \|\sqrt{-H}u_{\epsilon_n}(t) - \sqrt{-H}u_{\epsilon_n}(s)\|_{L^2} + \|\sqrt{-H}u_{\epsilon_n}(s) - \sqrt{-H}u(s)\|_{L^2}.
\]
By using Lemma 3.14 for a given $\delta > 1$, we can choose $N$ large such that
\[
\sup_{\tau} ||\sqrt{-H}u(\tau) - \sqrt{-H}u_{\epsilon_N}(\tau)||_{L^2} < \delta/3
\]
for this chosen $N$ we can choose $\kappa > 0$ such that; $|t - s| < \kappa$ implies
\[
||\sqrt{-H}u_{\epsilon_N}(t) - \sqrt{-H}u_{\epsilon_N}(s)||_{L^2} < \delta/3.
\]
That is, we have found a $\kappa > 0$ for arbitrary $\delta > 0$. Hence, the continuity.

Next, we prove the time regularity. By using Lemma 3.13 we can write
\[
\|u(t) - u(s)\|_{L^2}^2 \leq \|H^{1/2}(u(t) - u(s))\|_{L^2}^2 \leq \left|\int_s^t H^{-1/2} \partial_t u(\tau) d\tau\right|_{L^2} \lesssim |t - s|.
\]
So, we can conclude that
\[
u \in C^{1/2}([0, T], L^2) \cap C([0, T]; \mathcal{D}(\sqrt{-H})).
\]

In the following corollary, we show that a solution can be obtained by solving the approximating PDEs.

**Corollary 3.15.** Consider the following PDE
\[
i \partial_t u_\epsilon = H\partial u_\epsilon - u_\epsilon |u_\epsilon|^2
\]
with initial data $u_\epsilon^0 = H^{-1}H(1 - \epsilon \sqrt{-H})^{-1}u_0$, where $u_0 \in \mathcal{D}(\sqrt{-H})$ and $0 < \epsilon < 1$. There exists a subsequence $\epsilon_n$ such that $u_{\epsilon_n} \to u$ and $\sqrt{-H}u_{\epsilon_n} \to \sqrt{-H}u$ in $L^2$. In addition, $u$ solves.

**Proof.** Consider the initial data $u_\epsilon^{0,\delta} = H^{-1}H(1 - \delta \sqrt{-H})^{-1}u_0$. Then, by Theorem 3.9 taking $\epsilon \to 0$ we obtain $u_\delta \in \mathcal{D}(H)$ which solves the equation
\[
i \partial_t u_\delta = Hu_\delta - u_\delta |u_\delta|^2
\]
with initial data $u_\delta^0 = (1 - \delta \sqrt{-H})^{-1}u_0 \in \mathcal{D}(H)$. For this solution, we also have $\sqrt{-H}u_{\epsilon_n,\delta} \to \sqrt{-H}u_\delta$ in $L^2$ and in particular $u_{\epsilon_n,\delta} \to u_\delta$. Now, as in Theorem 3.11 we take $\delta \to 0$ and obtain an energy solution to [50]. Taking a diagonal sequence yields the stated result.

In the following remarks, we compare those results with the ones in domain case.

**Remark 3.16.** Note that the solution we obtain is not necessarily unique, as opposed to the solution with initial data in $\mathcal{D}(H)$.

**Remark 3.17.** Observe that this result holds for the NLS treated here, with any power nonlinearity as in the domain case.
3.3 Two and three dimensional cubic wave equations

In this section, we consider the cubic wave equations

$$\partial_t^2 u = Hu - u^3 \text{ on } \mathbb{T}^d \quad (60)$$

$$(u, \partial_t u)|_{t=0} = (u_0, u_1),$$

in two and three dimensions simultaneously.

We are interested in the case $$(u_0, u_1) \in \mathcal{D}(H) \times \mathcal{D}(\sqrt{-H}).$$

However as we shall see, in a similar way, we can also consider the case $$(u_0, u_1) \in \mathcal{D}(\sqrt{-H}) \times L^2.$$

We refer to [21] and [11] for classical results about well-posedness of semilinear wave equations. We obtain global strong well-posedness for a range of exponents including the standard case $p = 3$, which we will consider in detail for simplicity. In also 3d, the range of exponents which are covered by our methods is as good as what one can achieve in the classical case with similar methods.

We fix an approximating sequence $$(u_0^\varepsilon, u_1^\varepsilon) \in \mathcal{H}^2 \times \mathcal{H}^1$$ such that

$$H_\varepsilon u_0^\varepsilon \rightarrow Hu_0 \text{ in } L^2,$$

$$(u_1^\varepsilon, H_\varepsilon u_1^\varepsilon) \rightarrow (u_1, Hu_1).$$

To be precise, we choose

$$u_0^\varepsilon := (-H_\varepsilon)^{-1}(-H)u_0$$

$$u_1^\varepsilon := (-H_\varepsilon)^{-1/2}(-H)^{1/2}u_1.$$

We will, as in the NLS case, prove that the solution to (60) is the limit of the solutions of the regularized equations (for $d = 2, 3$

$$\partial_t^2 u_\varepsilon = H_\varepsilon u_\varepsilon - u_\varepsilon^3 \text{ on } \mathbb{T}^d \quad (61)$$

$$(u_\varepsilon, \partial_t u_\varepsilon)|_{t=0} = (u_0^\varepsilon, u_1^\varepsilon),$$

in an appropriate sense.

We begin by proving global strong wellposedness of (60) and (61) by a fixed point argument as in section 3.2.1.

**Theorem 3.18.** For $$(u_0, u_1) \in \mathcal{D}(H) \times \mathcal{D}(\sqrt{-H})$$ and $$(u_0^\varepsilon, u_1^\varepsilon) \in \mathcal{H}^2 \times \mathcal{H}^1,$$ there exist unique global in time solutions $u \in C([0, T]; \mathcal{D}(H)) \cap C^1([0, T]; \mathcal{D}(\sqrt{-H}))$ and $u_\varepsilon \in C([0, T]; \mathcal{H}^2) \cap C^1([0, T]; \mathcal{H}^1) \cap C^2([0, T]; L^2)$ satisfying

$$u(t) = \cos \left( t\sqrt{-H} \right) u_0 + \frac{\sin \left( t\sqrt{-H} \right)}{\sqrt{-H}} u_1 + \int_0^t \frac{\sin \left( (t-s)\sqrt{-H} \right)}{\sqrt{-H}} u^3(s) \, ds$$

and

$$u_\varepsilon(t) = \cos \left( t\sqrt{-H_\varepsilon} \right) u_0^\varepsilon + \frac{\sin \left( t\sqrt{-H_\varepsilon} \right)}{\sqrt{-H_\varepsilon}} u_1^\varepsilon + \int_0^t \frac{\sin \left( (t-s)\sqrt{-H_\varepsilon} \right)}{\sqrt{-H_\varepsilon}} u_\varepsilon^3(s) \, ds$$

respectively.
Before we come to the proof, we prove some auxiliary lemmas. We define the conserved energies for (60) and (61) respectively as

\[ E(u) := \frac{1}{2} \langle \partial_t u, \partial_t u \rangle - \frac{1}{2} \langle u, Hu \rangle + \frac{1}{4} \int |u|^4, \]

and

\[ E(u_\varepsilon) := \frac{1}{2} \langle \partial_t u_\varepsilon, \partial_t u_\varepsilon \rangle - \frac{1}{2} \langle u_\varepsilon, H_\varepsilon u_\varepsilon \rangle + \frac{1}{4} \int |u_\varepsilon|^4. \]

Also, we introduce the *almost conserved* energies for the time derivatives

\[ \tilde{E}(\partial_t u) = \frac{1}{2} \langle \partial_t^2 u, \partial_t^2 u \rangle - \frac{1}{2} \langle \partial_t u, H \partial_t u \rangle + \frac{3}{2} \int |u|^2 |\partial_t u|^2, \]

and

\[ \tilde{E}(\partial_t u_\varepsilon) = \frac{1}{2} \langle \partial_t^2 u_\varepsilon, \partial_t^2 u_\varepsilon \rangle - \frac{1}{2} \langle \partial_t u_\varepsilon, H_\varepsilon \partial_t u_\varepsilon \rangle + \frac{3}{2} \int |u_\varepsilon|^2 |\partial_t u_\varepsilon|^2. \]

We clarify what we mean by almost conserved in the following lemma.

**Lemma 3.19.** Let \( u \in C([0, T]; \mathcal{D}(H)) \cap C^1([0, T]; \mathcal{D}(\sqrt{-H})) \cap C^2([0, T]; L^2) \) and \( u_\varepsilon \in C([0, T]; \mathcal{H}^2) \cap C^1([0, T]; \mathcal{H}^1) \cap C^2([0, T]; L^2) \) be solutions of (60) and (61) respectively. Then the energies \( \tilde{E}(\partial_t u) \) and \( \tilde{E}(\partial_t u_\varepsilon) \) satisfy the following bounds

\[ \tilde{E}(\partial_t u)(t) \lesssim \exp(t C \tilde{E}(u_1)) E(u_0), \]

\[ \tilde{E}(\partial_t u_\varepsilon)(t) \lesssim \exp(t C \tilde{E}(u_\varepsilon^1)) E(u_\varepsilon^0), \]

for some universal constant \( C > 0 \).

**Proof.** We give the proof only for the regularized case. The other case can be done analogously by replacing \( \mathcal{H}^2 \) by \( \mathcal{D}(H) \) and \( \mathcal{H}^1 \) by \( \sqrt{-H} \).

First note that \( \partial_t u_\varepsilon \) solves the equation

\[ \partial_t^2 u_\varepsilon = H_\varepsilon \partial_t u_\varepsilon - 3 \partial_t u_\varepsilon u_\varepsilon^2 \text{ in } C([0, T]; \mathcal{H}^{-1}). \]

Then one can formally compute

\[
\frac{d}{dt} \tilde{E}(\partial_t u_\varepsilon)(t) = \langle \partial_t^2 u_\varepsilon, \partial_t^2 u_\varepsilon - H_\varepsilon \partial_t u_\varepsilon + 3 \partial_t u_\varepsilon u_\varepsilon^2 \rangle + 3 \int u_\varepsilon \partial_t u_\varepsilon |\partial_t u_\varepsilon|^2 dt
= 3 \int u_\varepsilon \partial_t u_\varepsilon |\partial_t u_\varepsilon|^2 dt.
\]

and conclude by Gronwall. However, since \( \tilde{E}(\partial_t u_\varepsilon) \) is not \( C^1 \) in time, this is not justified. But one can argue that this computation is true in the integrated version. We claim that we get the following weak differentiability, for any \( \phi \in C_c([0, \infty)) \)

\[
\int_{\mathbb{R}} \phi'(t) \tilde{E}(\partial_t u_\varepsilon)(t) dt = -3 \int_{\mathbb{R}} \phi(t) \int_{\mathbb{R}} u_\varepsilon \partial_t u_\varepsilon |\partial_t u_\varepsilon|^2(t) dt + \tilde{E}(\partial_t u_\varepsilon(0)) \phi(0). \tag{62}
\]

Moreover, this also holds in the integrated form

\[
\tilde{E}(\partial_t u_\varepsilon)(t) = \tilde{E}(\partial_t u_\varepsilon)(0) + 3 \int_0^t u_\varepsilon \partial_t u_\varepsilon |\partial_t u_\varepsilon|^2(s) ds, \tag{63}
\]

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for any \( t \in [0, T] \). We prove this by a spectral approximation. For, consider \((e_n)_{n \in \mathbb{Z}}^3 \in \mathcal{H}^2\) an orthonormal eigenbasis of \( H_\varepsilon \) with eigenvalues \( \{\lambda_n\} \) and set
\[
u^{N}_\varepsilon(t,x) := \sum_{|n| \leq N} (u_\varepsilon(t,\cdot),e_n)e_n(x).
\]
Then one has
\[
\partial^k_t \nu^{N}_\varepsilon \to \partial^k_t u_\varepsilon \text{ in } C([0,T];\mathcal{H}^{2-k})
\]
for \(0 \leq k \leq 3\), which in turn implies that
\[
E(u^{N}_\varepsilon) \to E(u_\varepsilon) \text{ and } \tilde{E}(\partial_t u^{N}_\varepsilon) \to \tilde{E}(\partial_t u_\varepsilon).
\]
One also directly deduces
\[
\partial_t^2 u^{N}_\varepsilon = H_\varepsilon u^{N}_\varepsilon - 3 \sum_{|n| \leq N} (\partial_t u_\varepsilon^2(t),e_n)e_n(x).
\]
Thus, we have
\[
\int_{\mathbb{R}} \phi'(t) \tilde{E}(\partial_t u^{N}_\varepsilon)(t) \, dt = -\int_{\mathbb{R}} \phi(t) \frac{d}{dt} \tilde{E}(\partial_t u^{N}_\varepsilon)(t) \, dt + \tilde{E}(\partial_t u_\varepsilon(0))\phi(0)
\]
\[
= -\int_{\mathbb{R}} \phi(t) \left( (\partial^2_t u^{N}_\varepsilon, \partial^3_t u^{N}_\varepsilon)(t) - (\partial^2_t u^{N}_\varepsilon, H_\varepsilon \partial_t u^{N}_\varepsilon)(t) 
+ 3(\partial^2_t u^{N}_\varepsilon, \partial_t u^{N}_\varepsilon(u^{N}_\varepsilon)^2)(t) + 3(\partial_t u^{N}_\varepsilon, (\partial_t u^{N}_\varepsilon)(u^{N}_\varepsilon))(t) \right) \, dt
\]
\[
+ \tilde{E}(\partial_t u_\varepsilon(0))\phi(0)
\]
\[
= -\int_{\mathbb{R}} \phi(t) [3(\partial^2_t u^{N}_\varepsilon, \partial_t u^{N}_\varepsilon(u^{N}_\varepsilon)^2) - \sum_{|n| \leq N} (\partial_t u_\varepsilon^2, e_n)e_n(x)(t)
+ 3(\partial_t u^{N}_\varepsilon, (\partial_t u^{N}_\varepsilon^2)(u^{N}_\varepsilon))(t)] \, dt + \tilde{E}(\partial_t u_\varepsilon(0))\phi(0). \tag{64}
\]
Now, we can write
\[
\partial_t u^{N}_\varepsilon(u^{N}_\varepsilon)^2 \to \partial_t u_\varepsilon(u_\varepsilon)^2 \text{ in } L^2
\]
and
\[
\sum_{|n| \leq N} (\partial_t u_\varepsilon^2, e_n)e_n(x) \to \partial_t u_\varepsilon(u_\varepsilon)^2 \text{ in } L^2.
\]
Therefore, we see that for \( N \to \infty \) \[64\] converges to \(62\). To prove \[63\], it suffices to take a sequence \( \phi_n \) in \([62]\) that converges to the characteristic function \( \chi_{[0,t]} \) monotonically. We can thus compute
\[
\tilde{E}(\partial_t u_\varepsilon)(t) \leq \tilde{E}(u_\varepsilon^1) + 3 \int_{0}^{t} \int |u_\varepsilon(s)||\partial_t u_\varepsilon(s)||\partial_t u_\varepsilon|^2(s) \, ds
\]
\[
\leq \tilde{E}(u_\varepsilon^1) + 3 \int_{0}^{t} \|\partial_t u_\varepsilon\|_{L^2} \|u_\varepsilon\|_{L^6} \|\partial_t u_\varepsilon\|^2_{L^6}(s) \, ds
\]
\[
\leq \tilde{E}(u_\varepsilon^1) + 3CE(u_\varepsilon^0) \int_{0}^{t} \tilde{E}(\partial_t u_\varepsilon)(s) \, ds,
\]

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where we have used the bounds
\[ \| \partial_t u \|_{L^2}^2 \leq E(u), \quad \| u \|_{L^6}^2 \leq E(u), \quad \| \partial_t u \|_{L^6}^2 \leq \dot{E}(\partial_t u). \]
From this, we conclude by using Gronwall. \( \square \)

**Proof of Theorem 3.18.** This is similar to the NLS case (Section 3.2.1), except that the time \( T \) is going to depend on the conserved energy \( E(u) \) and the almost conserved energy \( \dot{E}(\partial_t u) \). We again give the proof only for the \( \mathcal{D}(H) \) case, as the \( \mathcal{H}^2 \) case can be proved in a similar way.

We claim that for \( (u_0, u_1) \in \mathcal{D}(H) \times \mathcal{D}(\sqrt{-H}) \) there exists a unique fixed point of
\[ \Phi(u)(t) = \cos \left( t\sqrt{-H} \right) u_0 + \frac{\sin \left( t\sqrt{-H} \right)}{\sqrt{-H}} u_1 + \int_0^t \frac{\sin \left( (t-s)\sqrt{-H} \right)}{\sqrt{-H}} u^3(s) \, ds \]
in \( X = C([0, T]; \mathcal{D}(H)) \cap C^1 ([0, T]; \mathcal{D}(\sqrt{-H})) \cap C^2([0, T]; L^2) \).

For the contraction property, we compute the following, for \( \|u\|_X \leq M \) with \( M > 0 \) fixed later,
\[
\| H \Phi(u)(t) - H \Phi(v)(t) \|_{L^2} \\
= \left\| \int_0^t \sqrt{-H} \sin((t-s)\sqrt{-H})(u^3(s) - v^3(s)) \, ds \right\|_{L^2} \\
= \left\| \int_0^t \partial_s(\cos((t-s)\sqrt{-H}))(u^3(s) - v^3(s)) \, ds \right\|_{L^2} \\
= \left\| \int_0^t \cos((t-s)\sqrt{-H})\partial_s(u^3(s) - v^3(s)) \, ds + v^3(t) - u^3(t) \right\|_{L^2} \\
\leq 2 \int_0^t \| \partial_s(u^3 - v^3(s)) \|_{L^2} \\
\leq 6 \int_0^t \| \partial_t u - \partial_t v \|_{L^\infty L^6} \| u \|_{L^6}^2(s) + \| \partial_t v \|_{L^\infty L^4} \| u - v \|_{L^\infty L^4}(\| u \|_{L^4}(s) + \| v \|_{L^4}(s)) \, ds \\
\leq C \| u - v \|_X \int_0^t (E(u)(s) + ME^{1/2}(u)(s) + ME^{1/2}(v)(s)) \, ds \\
\leq \frac{1}{3} \| u - v \|_X
\]
for small enough time depending on the energy and \( M \). Here we have used the bounds \( \| \partial_t u \|_{L^6} \lesssim \| \sqrt{-H} \partial_t u \|_{L^2} \) and \( \| u \|_{L^4} \lesssim E^{1/2}(u) \). For the other terms, we similarly compute
\[
\left\| \sqrt{-H} \partial_t \Phi(u)(t) - \sqrt{-H} \partial_t \Phi(v)(t) \right\|_{L^2} \leq 2 \int_0^t \| \partial_t(u^3 - v^3)(s) \|_{L^2} \\
\leq \frac{1}{3} \| u - v \|_X
\]
and
\[
\| \partial^2_t \Phi(u)(t) - \partial^2_t \Phi(v)(t) \|_{L^2} \leq \frac{1}{3} \| u - v \|_X.
\]
Lastly, we argue that $\Phi$ maps a ball to itself. Let $\|u\|_X \leq M$ for $M$ specified below, then we have
\[
\|H\Phi(u)(t)\|_{L^2} \lesssim \|Hu_0\|_{L^2} + \|(-H)^{1/2}u_1\|_{L^2} + \int_0^t \|\partial_t u\|_{L^6}(s)\|u\|_{L^6}^2(s) \, ds \\
\lesssim \|Hu_0\|_{L^2} + \|(-H)^{1/2}u_1\|_{L^2} + \int_0^t \tilde{E}^{1/2}(\partial_t u)(s)E(u)(s) \, ds \\
\leq \frac{M}{3}
\]
for large $M$ depending on the data and $t \leq T_E$, small depending on $E(u)$ and $\tilde{E}(\partial_t u)$. Analogously, we also have
\[
\|\partial_t^2\Phi(u)\|_{L^\infty L^2} \leq \frac{M}{3} \quad \text{and} \quad \left\|\sqrt{-H}\partial_t\Phi(u)\right\|_{L^\infty L^2} \leq \frac{M}{3}.
\]
Moreover, the time regularity is again a consequence of Stone’s Theorem. Thus there exists a unique strong solution up to the time $T$ that depends on (almost) conserved quantities and we can conclude that this yields a strong solution up to any time. More precisely, we get a priori estimates that allow us to choose a globally valid $T > 0$ and then iterate the solution map to obtain a solution up to any given time $T > 0$.

Assuming we have a solution on the interval $[0, T_E]$, then we can estimate similarly to above as,
\[
\|Hu(T_E)\|_{L^2} \lesssim \|Hu_0\|_{L^2} + \|(-H)^{1/2}u_1\|_{L^2} + \int_0^{T_E} \tilde{E}^{1/2}(\partial_t u)(s)E(u)(s) \, ds \\
\lesssim \|Hu_0\|_{L^2} + \|(-H)^{1/2}u_1\|_{L^2} + T \exp(CT\tilde{E}(u_1))E^{3/2}(u_0)
\]
and also similarly for $\left\|\sqrt{-H}\partial_t u(T_E)\right\|_{L^2}$. Thus we can choose $M$ globally and solve on the interval $[T_E, 2T_E]$ and so on.

\[\square\]

From the above considerations, we obtain a priori bounds for the quantities
\[
\|u_\varepsilon\|_{L^\infty L^2}, \|H_\varepsilon u_\varepsilon\|_{L^\infty L^2} \quad \text{and} \quad \sup_{t \in [0,T]} (\partial_t u_\varepsilon, H_\varepsilon \partial_t u_\varepsilon),
\]
independently of $\varepsilon$. By the same arguments, as in the previous sections, we can also prove convergence of the approximate solutions.

**Theorem 3.20.** Assume we are in the above setting, i.e., we have unique global strong solutions to (60) and (61) and the initial data are given by $(u_0, u_1) \in \mathcal{D}(H) \times \mathcal{D}(\sqrt{-H})$ and

\[
\begin{align*}
&u_0^\varepsilon := (-H_\varepsilon)^{-1}(-H)u_0 \\
&u_1^\varepsilon := (-H_\varepsilon)^{-1/2}(-H)^{1/2}u_1.
\end{align*}
\]

Then the solutions $u_\varepsilon$ converge to $u$ in the following way
\[
\begin{align*}
u_\varepsilon(t) &\to u(t) \quad \text{in} \ L^2 \\
H_\varepsilon u_\varepsilon(t) &\to Hu(t) \quad \text{in} \ L^2 \\
(-H_\varepsilon)^{1/2}\partial_t u_\varepsilon(t) &\to (-H)^{1/2}\partial_t u(t) \quad \text{in} \ L^2 \\
\partial^2_t u_\varepsilon(t) &\to \partial^2_t u(t) \quad \text{in} \ L^2
\end{align*}
\]

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Hence, the result.

Proof. The proof is similar to that of Theorem 3.9. Since we have strong convergence for the initial data, together with the fact that \( \sin(t \sqrt{-H}) \to \sin(t \sqrt{-H}) \) strongly, we can bound for any fixed time \( t \in [0, T] \) using the mild formulation for \( u \) and \( u_\varepsilon \). We have

\[
\| Hu(t) - H_\varepsilon u_\varepsilon(t) \|_{L^2} 
\leq O(\varepsilon) + \int_0^t \| \partial_t (u^\varepsilon) - \partial_t (u_\varepsilon^\varepsilon) \|_{L^2} \, ds
\]

Here, we have used the a priori bounds obtained in Theorem 3.18 and the estimate

\[
\| \partial_t u(s) - \partial_t u_\varepsilon(s) \|_{L^6} \lesssim \| \partial_t u^\varepsilon(s) - \partial_t u_\varepsilon^\varepsilon(s) \|_{L^6}
\]

which the first estimate follows by Sobolev embedding and Proposition 2.21 and 2.46. The second one can be proved analogously to Proposition 2.21 and 2.46 for 2 and 3d respectively. In a similar manner, we have the bound

\[
\| u(s) - u_\varepsilon(s) \|_{L^6} \lesssim \| Hu(s) - H_\varepsilon u_\varepsilon(s) \|_{L^2} + C(u_0, u_1, T) \| \Xi - \Xi_\varepsilon \|_{X^{\alpha}}.
\]

Analogously we can also write

\[
\| \sqrt{-H} \partial_t u(t) - \sqrt{-H_\varepsilon} \partial_t u_\varepsilon(t) \|_{L^2} \lesssim O(\varepsilon) + C(T, u_0, u_1) \int_0^t \| \sqrt{-H} \partial_t u(s) - \sqrt{-H_\varepsilon} \partial_t u_\varepsilon(s) \|_{L^2} \, ds,
\]

\[
\| \partial_t^2 u(t) - \partial_t^2 u_\varepsilon(t) \|_{L^2} \lesssim O(\varepsilon) + C(T, u_0, u_1) \int_0^t \| \sqrt{-H} \partial_t u(s) - \sqrt{-H_\varepsilon} \partial_t u_\varepsilon(s) \|_{L^2} \, ds
\]

Thus, by defining

\[
\phi_\varepsilon(t) := \| Hu(t) - H_\varepsilon u_\varepsilon(t) \|_{L^2} + \| \sqrt{-H} \partial_t u(t) - \sqrt{-H_\varepsilon} \partial_t u_\varepsilon(t) \|_{L^2}
\]

we can rewrite the above estimates as

\[
\phi_\varepsilon(t) \leq O(\varepsilon) + C(T, u_0, u_1) \int_0^t \phi_\varepsilon(s) \, ds
\]

and conclude by Gronwall that \( \phi_\varepsilon(t) \to 0 \) as \( \varepsilon \to 0 \) for all \( t \in [0, T] \).

Hence, the result.
Lastly, we state the analogous result for the energy space, i.e. with data \((u_0, u_1) \in \mathcal{D}(\sqrt{-H}) \times L^2\). In a nutshell, one can repeat the above arguments. For global well-posedness, one can use a fixed point argument in the space \(C([0, T]; \mathcal{D}(\sqrt{-H})) \cap C^1([0, T]; L^2) \cap C^2([0, T]; \mathcal{D}(\sqrt{-H})^*)\) together with energy conservation and convergence can also be proved as above. We omit the proofs.

**Theorem 3.21.** Let \((u_0, u_1) \in \mathcal{D}(\sqrt{-H}) \times L^2\) and \(T > 0\), then \((60)\) has a unique solution \(u \in C([0, T]; \mathcal{D}(\sqrt{-H})) \cap C^1([0, T]; L^2) \cap C^2([0, T]; \mathcal{D}(\sqrt{-H})^*)\). Moreover, \((61)\) has a unique solution \(u_\varepsilon \in C([0, T]; \mathcal{H}^1) \cap C^1([0, T]; L^2) \cap C^2([0, T]; \mathcal{H}^{-1})\) with initial data \((u_0^\varepsilon, u_1) \in \mathcal{H}^1 \times L^2\), where \(u_0^\varepsilon := (-H\varepsilon)^{-1/2}(-H)^{1/2}u_0\) and the following convergence holds

\[
\begin{align*}
  u_\varepsilon(t) & \to u(t) \text{ in } L^2 \\
  \sqrt{-H}u_\varepsilon(t) & \to \sqrt{-H}u(t) \text{ in } L^2 \\
  \partial_t u_\varepsilon(t) & \to \partial_t u(t) \text{ in } L^2
\end{align*}
\]

for all \(t \in [0, T]\).

**Remark 3.22.** The same result is also true in 2d for any power \(p \in (1, \infty)\) both for the domain and energy space case. In 3d, our proof for global wellposedness also works for powers up to 5 in the domain case using an analogue of Agmon’s inequality, which we included for completeness as Lemma 2.61.

## A Paracontrolled distributions and function spaces

We recall the definitions of Bony paraproducts, Besov and Sobolev spaces and collect some results about products of distributions. We work on the d-dimensional torus \(\mathbb{T}^d := \mathbb{R}^d / \mathbb{Z}^d\) for \(d = 2, 3\). For any \(f\) in the space \(\mathcal{S}'(\mathbb{T}^d, \mathbb{R})\) of tempered distributions on \(\mathbb{T}^d\), the Fourier transform of \(f\) will be denoted by \(\hat{f} : \mathbb{Z}^d \to \mathbb{C}\) (or sometimes \(\mathcal{F}f\)) and is defined for \(k \in \mathbb{Z}^d\) by

\[
\hat{f}(k) := (f, \exp(2\pi i \langle k, \cdot \rangle)) = \int_{\mathbb{T}^d} f(x) \exp(-2\pi i \langle k, x \rangle) dx.
\]

Recall that for any \(f \in L^2(\mathbb{T}^d, \mathbb{R})\) and a.e. \(x \in \mathbb{T}^d\), we have

\[
f(x) = \sum_{k \in \mathbb{Z}^d} \hat{f}(k) \exp(2\pi i \langle k, x \rangle).
\]  

The Sobolev space \(\mathcal{H}^\alpha(\mathbb{T}^d)\) with index \(\alpha \in \mathbb{R}\) is defined as

\[
\mathcal{H}^\alpha(\mathbb{T}^d) := \{ f \in \mathcal{S}'(\mathbb{T}^d, \mathbb{R}) : \sum_{k \in \mathbb{Z}^d} (1 + |k|^2)^\alpha |\hat{f}(k)|^2 < +\infty \}.
\]

Before introducing the Besov spaces, we recall the definition of Littlewood-Paley blocks. We denote by \(\chi\) and \(\rho\) two nonnegative smooth and compactly supported radial functions \(\mathbb{R}^d \to \mathbb{R}\) such that

1. The support of \(\chi\) is contained in a ball \(\{ x \in \mathbb{R}^d : |x| \leq R \}\) and the support of \(\rho\) is contained in an annulus \(\{ x \in \mathbb{R}^d : a \leq |x| \leq b \}\);
2. For all $\xi \in \mathbb{R}^d$, $\chi(\xi) + \sum_{j \geq 0} \rho(2^{-j}\xi) = 1$;

3. For $j \geq 1$, $\chi \rho(2^{-j} \cdot) \equiv 0$ and $\rho(2^{-i} \cdot) \rho(2^{-j} \cdot) \equiv 0$ for $|i - j| \geq 1$.

The Littlewood-Paley blocks $(\Delta_j)_{j \geq -1}$ acting on $f \in \mathcal{S}'(\mathbb{T}^d)$ are defined by

$$\mathcal{F}(\Delta_{-1} f) = \hat{\chi} \hat{f} \quad \text{and for } j \geq 0, \quad \mathcal{F}(\Delta_j f) = \rho(2^{-j} \cdot) \hat{f}.$$ 

Note that, for $f \in \mathcal{S}'(\mathbb{T}^d)$, the Littlewood-Paley blocks $(\Delta_j f)_{j \geq -1}$ define smooth functions, as their Fourier transforms have compact supports. We also set, for $f \in \mathcal{S}'(\mathbb{T}^d)$ and $j \geq 0$,

$$S_j f := \sum_{i = -1}^{j-1} \Delta_i f$$

and note that $S_j f$ converges in the sense of distributions to $f$ as $j \to \infty$.

The Besov space with parameters $p, q \in [1, \infty)$, $\alpha \in \mathbb{R}$ can now be defined as

$$B_{p,q}^\alpha(\mathbb{T}^d, \mathbb{R}) := \left\{ u \in \mathcal{S}'(\mathbb{T}^d); \quad \|u\|_{B_{p,q}^\alpha} = \left( \sum_{j \geq -1} 2^{jq\alpha} \|\Delta_j u\|_{L^p}^q \right)^{1/q} < +\infty \right\}. \quad (67)$$

We also define the Besov-Hölder spaces

$$B_{p,\infty}^\alpha = \mathcal{C}^\alpha,$$

which are naturally equipped with the norm $\|f\|_{B_{p,\infty}^\alpha} := \|f\|_{B_{p,\infty}^\alpha} = \sup_{j \geq -1} 2^{j\alpha} \|\Delta_j f\|_{L^\infty}$. For $\alpha \in (0, 1)$ these spaces coincide with the classical Hölder spaces.

We can formally decompose the product $fg$ of two distributions $f$ and $g$ as

$$fg = f \prec g + f \circ g + f \succ g$$

where

$$f \prec g := \sum_{j \geq -1} \sum_{i = -1}^{j-2} \Delta_i f \Delta_j g \quad \text{and} \quad f \succ g := \sum_{j \geq -1} \sum_{i = -1}^{j-2} \Delta_i g \Delta_j f$$

are usually referred to as the paraproducts whereas

$$f \circ g := \sum_{j \geq -1} \sum_{|i - j| \leq 1} \Delta_i f \Delta_j g$$

is called the resonant product.

Moreover, we define the notations $f \preceq g := f \prec g + f \circ g$ and $f \succeq g := f \succ g + f \circ g$.

The paraproduct terms are always well defined irrespective of regularities. The resonant product is a priori only well defined if the sum of regularities is strictly greater than zero. This is reminiscent of the well known fact that one can not multiply distributions in general. The following result makes those comments precise and gives simple but extremely vital estimates for paraproducts.

**Proposition A.1** (Bony estimates, [2]). Let $\alpha, \beta \in \mathbb{R}$. We have the following bounds:
1. If \( f \in L^2 \) and \( g \in \mathcal{C}^\beta \), then 
\[
||f \lesssim g||_{\mathcal{F}_\beta} \leq C_{\delta,\beta} ||f||_{L^2} ||g||_{\mathcal{C}^\beta} \text{ for all } \delta > 0.
\]
2. if \( f \in \mathcal{H}^\alpha \) and \( g \in L^\infty \) then 
\[
||f \succ g||_{\mathcal{H}^\alpha} \leq C_{\alpha,\beta} ||f||_{\mathcal{H}^\alpha} ||g||_{\mathcal{C}^\beta}.
\]
3. If \( \alpha < 0 \), \( f \in \mathcal{H}^\alpha \) and \( g \in \mathcal{C}^\beta \), then 
\[
||f \lesssim g||_{\mathcal{F}_\alpha} \leq C_{\alpha,\beta} ||f||_{\mathcal{H}^\alpha} ||g||_{\mathcal{C}^\beta}.
\]
4. If \( g \in \mathcal{C}^\beta \) and \( f \in \mathcal{H}^\alpha \) for \( \beta < 0 \) then 
\[
||f \succ g||_{\mathcal{H}^\alpha} \leq C_{\alpha,\beta} ||f||_{\mathcal{H}^\alpha} ||g||_{\mathcal{C}^\beta}.
\]
5. If \( \alpha + \beta > 0 \) and \( f \in \mathcal{H}^\alpha \) and \( g \in \mathcal{C}^\beta \), then 
\[
||f \odot g||_{\mathcal{F}_{\alpha+\beta}} \leq C_{\alpha,\beta} ||f||_{\mathcal{H}^\alpha} ||g||_{\mathcal{C}^\beta}.
\]
where \( C_{\alpha,\beta} \) is a finite positive constant.

**Proposition A.2.** Given \( \alpha \in (0, 1), \beta, \gamma \in \mathbb{R} \) such that \( \beta + \gamma < 0 \) and \( \alpha + \beta + \gamma > 0 \), there exists a trilinear operator \( C \) with the following bound 
\[
||C(f, g, h)||_{\mathcal{F}_{\alpha+\beta+\gamma}} \lesssim ||f||_{\mathcal{H}^\alpha} ||g||_{\mathcal{C}^\beta} ||h||_{\mathcal{C}^\gamma}
\]
for all \( f \in \mathcal{H}^\alpha, g \in \mathcal{C}^\beta \) and \( h \in \mathcal{C}^\gamma \).

The restriction of \( C \) to the smooth functions satisfies 
\[
C(f, g, h) = (f \lesssim g) \circ h - f(g \circ h).
\]

**Proof.** This is a restatement of the result (commutator Lemma) in [2], and the proof follows the same lines, with slight modifications. \( \square \)

We also prove the following modified version of the above Proposition, which suits our framework.

**Proposition A.3.** Let \( \alpha \in (0, 1), \beta, \gamma \in \mathbb{R} \) such that \( \beta + \gamma < 0 \) and \( \alpha + \beta + \gamma > 0 \). Then, there exists a trilinear operator \( C_N \) with the following bound 
\[
||C_N(f, g, h)||_{\mathcal{F}_{\alpha+\beta+\gamma}} \lesssim ||f||_{\mathcal{H}^\alpha} ||g||_{\mathcal{C}^\beta} ||h||_{\mathcal{C}^\gamma}
\]
for all \( f \in \mathcal{H}^\alpha, g \in \mathcal{C}^\beta \) and \( h \in \mathcal{C}^\gamma \).

The restriction of \( C_N \) to the smooth functions satisfies 
\[
C_N(f, g, h) := (\Delta_{\leq N}(f \lesssim g)) \circ h - f(g \circ h)
\]

**Proof.** Observe that we have 
\[
C(f, g, h) - C_N(f, g, h) = (\Delta_{> N}(f \lesssim g)) \circ h.
\]
So, we only need to show 
\[
|| (\Delta_{\leq N}(f \lesssim g)) \circ h ||_{\mathcal{F}_{\alpha+\beta+\gamma}} \lesssim ||f||_{\mathcal{H}^\alpha} ||g||_{\mathcal{C}^\beta} ||h||_{\mathcal{C}^\gamma}.
\]

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By product estimates, we obtain right away
\[
\| (\Delta \leq N(f \prec g)) \circ h \|_{\mathcal{F}^{\alpha+\beta}} \lesssim \| (\Delta \leq N(f \prec g)) \|_{\mathcal{F}^{\alpha+\beta}} \| h \|_{\mathcal{F}^{\gamma}}.
\]
We need to show
\[
\| (\Delta \leq N(f \prec g)) \|_{\mathcal{F}^{\alpha+\beta}} \lesssim \| f \|_{\mathcal{F}^{\alpha}} \| g \|_{\mathcal{F}^{\beta}}.
\]
We can write
\[
\| (\Delta \leq N(f \prec g)) \|_{\mathcal{F}^{\alpha+\beta}}^2 = \sum_{k=-1}^{\infty} 2^{2k(\alpha+\beta)} \| \Delta_k (\Delta \leq N(f \prec g)) \|_{L^2}^2.
\]
By the support of Fourier transforms we have that \( \Delta_k (\Delta \leq N(f \prec g)) = 0 \) for \( k > N + 1 \) so we obtain
\[
\| (\Delta \leq N(f \prec g)) \|_{\mathcal{F}^{\alpha+\beta}}^2 = \sum_{k=-1}^{N+1} 2^{2k(\alpha+\beta)} \| \Delta_k (\Delta \leq N(f \prec g)) \|_{L^2}^2.
\]
By using the convention \( \Delta_{<k} f := \sum_{i=-1}^{k-2} \Delta_i f \) we rewrite
\[
\| (\Delta \leq N(f \prec g)) \|_{\mathcal{F}^{\alpha+\beta}}^2 = \sum_{k=-1}^{N+1} 2^{2k(\alpha+\beta)} \| \Delta_k \left( \Delta \leq N(\sum_{i=-1}^{N+1} \Delta_{<i} f \Delta_i g) \right) \|_{L^2}^2.
\]
Again by support arguments this boils down to
\[
\| (\Delta \leq N(f \prec g)) \|_{\mathcal{F}^{\alpha+\beta}}^2 = \sum_{k=-1}^{N+1} 2^{2k(\alpha+\beta)} \| \Delta_k \left( \Delta \leq N(\sum_{i=-1}^{N+1} \Delta_{<i} f \Delta_i g) \right) \|_{L^2}^2.
\]
Applying two successive Young’s we obtain
\[
\| (\Delta \leq N(f \prec g)) \|_{\mathcal{F}^{\alpha+\beta}}^2 \leq \sum_{k=-1}^{N+1} 2^{2k(\alpha+\beta)} \| \phi_k \|_{L^1} \| \phi_{\leq N} \|_{L^1} \sum_{i=-1}^{N+1} \| \Delta_{<i} f \Delta_i g \|_{L^2}^2
\]
where on the right hand side we can write
\[
\| (\Delta \leq N(f \prec g)) \|_{\mathcal{F}^{\alpha+\beta}}^2 \leq \sum_{k=-1}^{N+1} 2^{2k(\alpha+\beta)} \| \phi_k \|_{L^1} \| \phi_{\leq N} \|_{L^1} \sum_{i=-1}^{N+1} \| \Delta_{<i} f \Delta_i g \|_{L^2}^2
\]
\[
\leq \sum_{k=-1}^{N+1} 2^{2k(\alpha+\beta)} \| \phi_k \|_{L^1} \| \phi_{\leq N} \|_{L^1} \left( \sup_{1 \leq i \leq N+1} 2^{2i(\alpha+\beta)} \| \Delta_i g \|_{L^2}^2 \right) \sum_{i=-1}^{N+1} 2^{2i(\alpha+\beta)} \| \Delta_{<i} f \|_{L^2}^2.
\]
At this point, it is clear that for a constant depending on $N$ we readily have

$$||| (\Delta_{\leq N} (f \prec g)) |||_{H^{\alpha, \beta}} \lesssim ||f||_{H^{\alpha}} ||g||_{H^{\beta}}^2$$

and the result follows. 

\[ \square \]

**Lemma A.4** (Bernstein’s inequality, [14]). Let $\mathcal{A}$ be an annulus and $\mathcal{B}$ be a ball. For any $k \in \mathbb{N}, \lambda > 0$, and $1 \leq p \leq q \leq \infty$ we have

1. if $u \in L^p(\mathbb{R}^d)$ is such that $\text{supp}(\mathcal{F}u) \subset \lambda \mathcal{B}$ then
   $$\max_{\mu \in \mathbb{N}^d : |\mu| = k} ||| \partial^\mu u |||_{L^q} \lesssim_k \lambda^{k+d(\frac{1}{p}-\frac{1}{q})} ||u||_{L^p}$$

2. if $u \in L^p(\mathbb{R}^d)$ is such that $\text{supp}(\mathcal{F}u) \subset \lambda \mathcal{A}$ then
   $$\lambda^k ||u||_{L^p} \lesssim_k \max_{\mu \in \mathbb{N}^d : |\mu| = k} ||\partial^\mu u||_{L^p}.$$

**Proposition A.5** (Paralinearisation, [15]). Let $\alpha \in (0, 1)$ and $F \in C^2$. Then there exists a locally bounded map $R_F : \mathcal{C}^\alpha \to \mathcal{C}^{2\alpha}$ such that

$$F(f) = F'(f) \prec f + R_F(f) \text{ for all } f \in \mathcal{C}^\alpha.$$

**Lemma A.6.** Let $\alpha, \beta, \gamma \in \mathbb{R}$ with $\alpha + \beta < 0$, $\alpha + \beta + \gamma \geq 0$, and $f \in \mathcal{H}^\alpha, g \in \mathcal{C}^\beta, h \in \mathcal{H}^\gamma$, then there exists a map $D(f, g, h)$ with the following bound

$$|D(f, g, h)| \lesssim ||g||_{\mathcal{C}^\beta} ||f||_{\mathcal{H}^\alpha} ||h||_{\mathcal{H}^\gamma}. \quad (69)$$

Moreover the restriction of $D(f, g, h)$ to the smooth functions $f, g, h$ is as follows:

$$D(f, g, h) = \langle f, h \circ g \rangle - \langle f \prec g, h \rangle.$$

**Proof.** We define

$$D(f, g, h) := \left( \sum_{i \geq k-1, |j-k| \leq L} - \sum_{i \sim k, 1 < |j-k| \leq L} \right) \langle \Delta_i f, \Delta_j h \Delta_k g \rangle.$$ 

So we get, for some $\delta > 0$,

$$|D(f, g, h)| \lesssim \sum_{i \geq k, j \sim k} |\langle \Delta_i f, \Delta_j h \Delta_k g \rangle|$$

$$\leq \sum_{i \geq k, j \sim k} ||| \Delta_i f |||_{L^2} ||| \Delta_j h \Delta_k g \|| L^\infty$$

$$\times \|g\|_{\mathcal{C}^{-1-\delta}} \sum_{i \geq k, j \sim k} 2^{k(1+\delta)} ||| \Delta_i f \|||_{L^2} ||| \Delta_j h \|| L^2$$

$$\leq \|g\|_{\mathcal{C}^{-1-\delta}} \|f\|_{\mathcal{H}^{(1+\delta)/2}} \|h\|_{\mathcal{H}^{(1+\delta)/2}}$$
and this argument can be adapted to show \((69)\) by simply observing \(1 \leq 2^{k(\beta + \alpha + \gamma)} = 2^{k\beta}2^{k(\alpha + \gamma)}\), since \(\beta + \alpha + \gamma \geq 0\). Moreover, for smooth functions \(f, g, h\); we can compute

\[
\langle f, h \circ g \rangle - (f \prec g, h) = \sum_{i, |j - k| \leq 1} \langle \Delta_i f, \Delta_j h \Delta_k g \rangle - \sum_{i < k - 1} \langle \Delta_i f \Delta_k g, \Delta_j h \rangle
\]

\[
= \left( \sum_{i, |j - k| \leq 1} - \sum_{i < k - 1, |j - k| \leq 1} \right) \langle \Delta_i f \Delta_k g, \Delta_j h \rangle
\]

\[
= \left( \sum_{i, |j - k| \leq L} - \sum_{i < k - 1, |j - k| \leq L} - \sum_{i, 1 < |j - k| \leq L} \right) \langle \Delta_i f, \Delta_j h \Delta_k g \rangle
\]

\[
= \left( \sum_{i \geq k - 1, |j - k| \leq L} - \sum_{i, 1 < |j - k| \leq L} \right) \langle \Delta_i f, \Delta_j h \Delta_k g \rangle
\]

\[
= \left( \sum_{i \geq k - 1, |j - k| \leq L} - \sum_{i \sim k, 1 < |j - k| \leq L} \right) \langle \Delta_i f, \Delta_j h \Delta_k g \rangle = D(f, g, h).
\]

Hence the result. \(\square\)

**Remark A.7.** Proposition\(\text{[A.6]}\) says that the paraproduct is *almost* the adjoint of the resonant product, meaning up to a more regular remainder term as is often the case in paradifferential calculus.

**Lemma A.8.** Let \(f \in H^{\alpha}, g \in C^{\beta}\), with \(\alpha \in (0, 1), \beta \in \mathbb{R}\), there exists a bilinear map \(R(f, g)\) that satisfies the following bound

\[
\|R(f, g)\|_{H^{\alpha + \beta + 2}} \lesssim \|f\|_{H^{\alpha}} \|g\|_{C^{\beta}},
\]

and restricts to smooth functions as

\[
R(f, g) = (1 - \Delta)^{-1}(f \prec g) - f \prec (1 - \Delta)^{-1}g.
\]

**Proof.** The proof is basically a straightforward modification of the proof of \([2]\) Proposition A.2]. which has almost the same statement. \(\square\)

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