On Invariant Operations on a Manifold with a Linear Connection and an Orientation

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Abstract: We prove a theorem that describes all possible tensor-valued natural operations in the presence of a linear connection and an orientation in terms of certain linear representations of the special linear group. As an application of this result, we prove a characterization of the torsion and curvature operators as the only natural operators that satisfy the Bianchi identities.

Keywords: natural tensors; linear connections; torsion tensor; curvature operator

1. Introduction

Since the very early days of differential geometry, the idea of natural operation played a major role in the development theory. As an example, let us point out the applications of this notion of naturalness in the inception of general relativity (cf. [1]). In the course of the years, there also appeared some striking mathematical results, such as Gilkey’s characterization of Pontryagin forms on Riemannian manifolds [2,3] or his proof of the uniqueness of the Chern–Gauss–Bonnet formula [4]. By the end of the last century, the modern development of this theory was summarized in the monograph by Kolůr-Michor-Slovák [5]. That book contained all of the main results and techniques that were known so far, and thus became the standard reference in the subject since then.

On the other hand, the notion of covariance or naturalness is, in some sense, ubiquitous in physics and mathematics. For that reason, the renewed interest in this theory of natural operations that has been raised in recent years is not surprising, with the appearance of new results and applications in contact geometry [6], homotopy theory [7,8], Riemannian and Kähler geometry [9–12], general relativity [13], or quantum field theory [14,15].

In this paper, we focus our attention on the vector space of tensor-valued natural operations that can be performed in the presence of a linear connection and an orientation. Our main result, Theorem 8, establishes that such a vector space is isomorphic to the space of invariant maps between certain linear representations of the special linear group. Thus, the description of these spaces can, in certain cases, be completely achieved using classical invariant theory. As an example of this philosophy, in the final section, we characterize the torsion and the curvature as the only natural tensors satisfying the Bianchi identities (Corollary 13 and Theorem 15).

These results generalize analogous statements that were recently proven in [16], where we studied natural tensors associated with a linear connection. This was also the situation considered in a landmark paper by Slovák [17], whose results were included—and expanded—in [5]. Nevertheless, the non-specialist may find it difficult to understand the precise meaning of some statements of this book due to the functorial language and the generality of its setting.

For this reason, we outlined in [16] the foundations of an alternative approach, which we hope will be accessible to a wider audience. The present paper lays out complete proofs...
of the main results of this approach, whose novelties are a systematic use of the language
of sheaves, ringed spaces, and a more elementary—yet equivalent (cf. [18])—notion of the
natural bundle. In our opinion, the heart of the matter in this theory is the existence of an
analogue of a Galois theorem (cf. [18], Thm. 1.6), which allows the use of group theory to
infer theorems in many areas of differential geometry, in many of which (such as Fedosov,
contact, or Finsler geometry) this idea is still to be exploited.

2. The Category of Ringed Spaces

In this section, we firstly introduce the category of ringed spaces, which is a framework
adequate for our purposes: It will allow us to treat certain “infinite dimensional” spaces—
such as the $\infty$-jet space or a countable product of vector spaces—and quotients of smooth
manifolds by the actions of groups on equal footing as usual smooth, finite-dimensional
manifolds.

Secondly, we state Theorem 4, which is an important characterization of differential
operators as the morphisms of sheaves that transform smooth families of sections into
smooth families of sections.

**Definition 1.** A ringed space is a pair $(X, \mathcal{O}_X)$, where $X$ is a topological space and \( \mathcal{O}_X \) is a
sub-algebra of the sheaf of real-valued continuous functions on $X$.

A morphism of ringed spaces $\varphi: (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a continuous map $\varphi: X \to Y$ such
that composition with $\varphi$ induces a morphism of sheaves $\varphi^*: \mathcal{O}_Y \to \varphi_* \mathcal{O}_X$, that is, for any open set
$V \subset Y$ and any function $f \in \mathcal{O}_Y(V)$, the composition $f \circ \varphi$ lies in $\mathcal{O}_X(\varphi^{-1}V)$.

Any smooth manifold $X$ is a ringed space, where $\mathcal{O}_X = C^\infty_X$ is the sheaf of smooth
real-valued functions. If $X$ and $Y$ are smooth manifolds, a morphism of ringed spaces
$X \to Y$ is just a smooth map.

By analogy with this example, on any ringed space $(X, \mathcal{O}_X)$, the sheaf $\mathcal{O}_X$ will be called the sheaf of smooth
functions, and morphisms of ringed spaces $X \to Y$ will be often referred to as smooth morphisms.

2.1. Limits of Ringed Spaces

This category possesses limits; nevertheless, in what follows, only the following
particular case appear.

**Definition 2.** The inverse limit of a sequence of smooth manifolds and smooth maps between them

$$
\ldots \to X_{k+1} \xrightarrow{\varphi_{k+1}} X_k \xrightarrow{\varphi_k} X_{k-1} \to \ldots
$$

is the ringed space $(X_\infty, \mathcal{O}_\infty)$, which is defined as follows:

- The underlying topological space is the inverse limit of the topological spaces $X_k$, i.e., the set
  
  $X_\infty := \lim_k X_k$

  is endowed with the minimum topology for which the canonical projections $\pi_k: X_\infty \to X_k$
  are continuous.

- Its sheaf of smooth functions is the direct limit $\mathcal{O}_\infty := \lim_k \pi_k^\ast \mathcal{O}_X$.
That is to say, for any open set \( U \subseteq X_{\infty} \), a continuous map \( f : U \to \mathbb{R} \) lies in \( O_{\infty}(U) \) if and only if, for any point \( x \in U \), there exist \( k \in \mathbb{N} \), an open neighborhood \( \pi_k(x) \in V_k \subseteq X_k \), and a smooth map \( f_k : V_k \to \mathbb{R} \) such that the following triangle commutes:

\[
\begin{array}{ccc}
\pi_k^{-1}(V_k) & \xrightarrow{\pi_k} & V_k \\
\downarrow f & & \downarrow f_k \\
\mathbb{R} & & \end{array}
\]

Later, we will need the following two properties regarding the smooth structure of this inverse limit:

**Universal property of the inverse limit:** For any smooth manifold \( Y \), the projections \( \pi_k : X_{\infty} \to X_k \) induce a bijection that is functorial on \( Y \),

\[
C^\infty(Y, X_{\infty}) = \lim_{\kappa} C^\infty(Y, X_k), \quad \varphi \mapsto (\pi_k \circ \varphi)
\]

where \( C^\infty(\_\_\_) \) denotes the set of morphisms of ringed spaces.

**Proof.** The projections \( \pi_k \) are smooth maps, so one inclusion is trivial. As for the other, let \( \varphi : Y \to X_{\infty} \) be a continuous map such that \( \pi_k \circ \varphi \) is smooth for any \( k \in \mathbb{N} \).

Let \( f \in O_{\infty}(U) \) be a smooth function and let \( y \in \varphi^{-1}(U) \). On a neighborhood \( V \) of \( \varphi(y) \), there exists an smooth map \( f_k : X_k \to \mathbb{R} \) such that \( f = f_k \circ \pi_k \), and therefore:

\[
f \circ \varphi = (f_k \circ \pi_k) \circ \varphi = f_k \circ (\pi_k \circ \varphi)
\]

which is smooth because \( \pi_k \circ \varphi \) is a smooth map. \( \square \)

**Proposition 1.** Let \( Z \) be a smooth manifold. A continuous map \( \varphi : X_{\infty} \to Z \) is smooth if and only if it locally factors through a smooth map defined on some \( X_k \).

**Proof.** Let \( \varphi : X_{\infty} \to Z \) be a smooth map; let \( x \in X_{\infty} \) be a point and let \( (U, z_1, \ldots, z_n) \) be a coordinate chart around \( \varphi(x) \) in \( Z \). Each of the functions \( z_1 \circ \varphi, \ldots, z_n \circ \varphi \in O_{\infty}(\varphi^{-1}(U)) \) locally factors through some \( X_j \); as they are a finite number, there exists \( k \in \mathbb{N} \) and an open neighborhood \( V \) of \( x \) such that all of them, when restricted to \( V \), factor through \( X_k \). Hence, \( \varphi|_V = (z_1 \circ \varphi, \ldots, z_n \circ \varphi) \).

The converse is obvious because the composition of morphisms of ringed spaces is a morphism of ringed spaces. \( \square \)

As examples, the space \( J^\infty F \) of \( \infty \)-jets of sections of a fiber bundle \( F \) is defined as the inverse limit of the sequence of \( k \)-jets fiber bundles:

\[
\ldots \to J^k F \to J^{k-1} F \to \ldots \to F \to X.
\]

In addition, if \( N_0, N_1, N_2, \ldots \) is a countable family of finite-dimensional \( \mathbb{R} \)-vector spaces, the vector space \( \prod_{i=1}^\infty N_i \) is, by definition, the inverse limit of the projections:

\[
\ldots \to \bigoplus_{i=1}^{k+1} N_i \to \bigoplus_{i=1}^k N_i \to \ldots \to N_2 \times N_1 \to N_1.
\]

2.2. Quotients by the Action of Groups

Let \( G \) be a group acting on a ringed space \( X \). Let us denote by \( X/G \) the quotient topological space and by \( \pi : X \to X/G \) the quotient map.
Definition 3. The quotient ringed space \((X/G, \mathcal{O}_{X/G})\) is the ringed space whose underlying topological space is the quotient topological space \(X/G\) and whose sheaf of smooth functions is defined, on any open set \(U \subseteq X/G\) as:

\[
\mathcal{O}_{X/G}(U) := \{ f \in \mathcal{C}(U, \mathbb{R}) : f \circ \pi \in \mathcal{O}_X(\pi^{-1}(U)) \} = \mathcal{O}_X(\pi^{-1}(U))^G,
\]
where \(\mathcal{O}_X(\pi^{-1}(U))^G\) stands for the set of maps \(f \in \mathcal{O}_\infty(\pi^{-1}(U))\) such that \(f(g \cdot p) = f(p)\) for any \(g \in G\) and \(p \in \pi^{-1}(U)\).

It is then routine to check that the quotient map \(\pi : X \to X/G\) is a morphism of ringed spaces that satisfies the following property:

Universal property of the quotient: For any ringed space \(Y\), the quotient map \(\pi : X \to X/G\) induces a functorial bijection:

\[
\left\{ \text{Morphisms of ringed spaces } X \to Y \text{ constant along the orbits of } G \right\} \leftrightarrow \left\{ \text{Morphisms of ringed spaces } X/G \to Y \right\}.
\]

Corollary 2 (Orbit reduction). Let \(G\) be a group acting on a ringed space \(X\), and let \(f : X \to Y\) be a surjective morphism of ringed spaces that, locally on \(Y\), admits smooth sections passing through any point of \(X\).

If the orbits of \(G\) coincide with the fibers of \(f\), then the corresponding map \(\bar{f} : X/G \to Y\) is an isomorphism of ringed spaces.

Proof. The hypothesis on the fibers assures that the induced morphism \(\bar{f} : X/G \to Y\) is bijective. The inverse map \(\bar{f}^{-1}\) is also a morphism of ringed spaces because it locally coincides with the projection into the quotient of any smooth section of \(f\). \(\Box\)

There is also the following corollary, whose proof is routine:

Corollary 3. Let \(G\) be a group acting on two ringed spaces \(X\) and \(Y\), and let \(H \subseteq G\) be a subgroup that acts trivially on \(Y\).

Then, the universal property of the quotient restricts to a bijection:

\[
\left\{ \text{G-equivariant morphisms of ringed spaces } X \to Y \right\} \leftrightarrow \left\{ \text{G/H-equivariant morphisms of ringed spaces } X/H \to Y \right\}.
\]

2.3. Differential Operators

Let \(F \to X\) and \(F' \to X\) be fiber bundles over a smooth manifold \(X\).

Definition 4. A differential operator is a morphism of ringed spaces \(P : J^n F \to F'\) such that the following triangle commutes:

\[
\begin{array}{ccc}
J^n F & \xrightarrow{P} & F' \\
\downarrow{j^n \pi} & & \downarrow{\pi'} \\
X & \xrightarrow{\pi} & \\
\end{array}
\]

Let us denote by \(\mathcal{F}\) and \(\mathcal{F}'\) the sheaves of smooth sections of \(F\) and \(F'\), respectively.

Definition 5. A family of sections \(\{ s_t : U_t \to F \}_{t \in T}\) is smooth if \(T\) is a smooth manifold and the following conditions are satisfied:

1. \(U = \bigcup_{t \in T} U_t\) is an open set of \(X \times T\).
2. The map \(s : U \to F\), defined as \(s(t, x) := s_t(x)\), is smooth.
A morphism of sheaves $\phi : F \to F'$ is regular if, for any smooth family of sections $\{s_t : U_t \to F\}_{t \in T}$, the family $\{\phi(s_t) : U_t \to F'\}_{t \in T}$ is also smooth.

Any differential operator $P : \mathcal{I} \to F' \to F$ defines a morphism of sheaves $\phi_P : F \to F'$, $\phi_P(s)(x) := P(s(x))$, and the chain rule proves that it is a regular morphism of sheaves.

The following statement is a particular case of a deep result due to J. Slovák (see [5], Sect. 19.7, or [19] for a proof of the specific statement below):

**Theorem 4** (Peetre-Slovák). If $F \to X$ and $F' \to X$ are fiber bundles over a smooth manifold $X$, then the assignment $P \to \phi_P$ explained above establishes a bijection:

\[
\begin{array}{c}
\{ \text{Differential operators} \} \\
\{ \mathcal{I} \to F' \to F \}
\end{array}
\longleftrightarrow
\begin{array}{c}
\{ \text{Regular morphisms of sheaves} \} \\
\{ F \to F' \}
\end{array}
\]

3. Natural Operations in the Presence of an Orientation

The purpose of this section is twofold: On the one hand, we present the notion of natural operation (Definition 7); our definition strongly differs from the standard one (cf. [5]), although it is equivalent to it ([18]). On the other hand, we prove a general result—Theorem 6—that relates these natural operations with certain smooth equivariant morphisms.

3.1. Natural Bundles

Let $\text{Diff}(X)$ denote the set of diffeomorphisms $\tau : U \to V$ between open sets of a smooth manifold $X$.

If $\pi : F \to X$ is a bundle over $X$, a lifting of diffeomorphisms is a map:

$$\tau \mapsto \tau_*$$

such that if $\tau : U \to V$ is a diffeomorphism between open sets in $X$, then $\tau_* : F_U \to F_V$ is a diffeomorphism covering $\tau$; that is to say, making the following square commutative

\[
\begin{array}{ccc}
F_U & \xrightarrow{\tau_*} & F_V \\
\pi \downarrow & & \downarrow \pi \\
U & \xrightarrow{\sim} & V
\end{array}
\]

where $F_U := \pi^{-1}(U)$ and $F_V := \pi^{-1}(V)$.

**Definition 6.** A natural bundle over a smooth manifold $X$ is a bundle $F \to X$ together with a lifting of diffeomorphisms satisfying the following properties:

1. Functorial character: $\text{Id}_x = \text{Id}$ and $(\tau \circ \tau')_* = (\tau)_* \circ (\tau')_*$.
2. Local character: For any diffeomorphism $\tau : U \to V$ and any open subset $U' \subset U$,

\[
(\tau|_{U'})_* = (\tau_*)|_{F_{U'}}
\]

3. Regularity: If $\{\tau_t : U_t \to V_t\}_{t \in T}$ is a smooth family of diffeomorphisms between open sets on $X$, then the family $\{(\tau_t)_* : F_{U_t} \to F_{V_t}\}_{t \in T}$ is also smooth.

A sub-bundle $E$ of a natural bundle $F$ is said to be natural if it is a natural bundle and its lifting of diffeomorphisms is the restriction of the lifting of diffeomorphisms of $F$. 
A morphism of natural bundles is a morphism of bundles \( \varphi : F \to F' \) that commutes with the lifting of diffeomorphisms; that is, such that for any diffeomorphism \( \tau : U \to V \), the following square commutes:

\[
\begin{array}{ccc}
F_U & \xrightarrow{\varphi} & F'_U \\
\downarrow{\tau_*} & & \downarrow{\tau_*} \\
F_V & \xrightarrow{\varphi} & F'_V \\
\end{array}
\]

The tangent and cotangent bundles, or, more generally, the bundles of \((r, s)\)-tensors \( T^r_s \), are examples of natural bundles. The sub-bundle of \( k \)-forms \( \Omega^k \subset T^0_k \) is a natural sub-bundle of the bundle of \( k \)-covariant tensors \( T^0_k \).

If \( F \to X \) is a natural bundle, its \( k \)-jet prolongation \( J^k F \) is also a natural bundle for all \( k \in \mathbb{N} \). Thus, if \( \tau : U \to V \) is a diffeomorphism, its liftings to these jet spaces \( J^k F \) allow the definition of a lifting to the \( \infty \)-jet space—in other words, a morphism of ringed spaces

\[ \tau_* : J^\infty F_U \longrightarrow J^\infty F_V \]

covering the diffeomorphism \( \tau \).

Let \( \pi : F \to X \) and \( \pi' : F' \to X \) be natural bundles over \( X \), and let \( \mathcal{F} \) and \( \mathcal{F}' \) be their sheaves of smooth sections, respectively.

**Definition 7.** A differential operator \( P : J^\infty F \longrightarrow F' \) is natural if it is a morphism of ringed spaces that commutes with the lifting of diffeomorphisms.

A morphism of sheaves \( \phi : \mathcal{F} \to \mathcal{F}' \) is natural if it is a regular morphism of sheaves that commutes with the action of diffeomorphisms on sections; that is to say, if for any diffeomorphism \( \tau : U \to V \) between open sets of \( X \), the following square commutes:

\[
\begin{array}{ccc}
\mathcal{F}(U) & \xrightarrow{\phi} & \mathcal{F}'(U) \\
\downarrow{\tau_*} & & \downarrow{\tau_*} \\
\mathcal{F}(V) & \xrightarrow{\phi} & \mathcal{F}'(V) \\
\end{array}
\]

where \( \tau_* : \mathcal{F}(U) \to \mathcal{F}(V) \) is defined as \( \tau_* (s) := \tau_* \circ s \circ \tau^{-1} \) for any \( s \in \mathcal{F}(U) \).

**Theorem 5.** The choice of a point \( p \in X \) allows the definition of a bijection:

\[
\begin{align*}
\begin{Bmatrix}
\text{Natural morphisms of sheaves} \\
\phi : \mathcal{F} \longrightarrow \mathcal{F}'
\end{Bmatrix}
\quad & \quad
\begin{Bmatrix}
\text{Diff}_p\text{-equivariant smooth maps} \\
J^\infty_p F \longrightarrow F'_p
\end{Bmatrix}
\end{align*}
\]

where \( \text{Diff}_p \) stands for the group of germs of diffeomorphisms \( \tau \) between open sets of \( X \) such that \( \tau(p) = p \).

**Proof.** In this context, where both \( F \) and \( F' \) are natural bundles, the bijection of Theorem 4 specializes to a bijection:

\[
\begin{align*}
\begin{Bmatrix}
\text{Natural morphisms of sheaves} \\
\phi : \mathcal{F} \longrightarrow \mathcal{F}'
\end{Bmatrix}
\quad & \quad
\begin{Bmatrix}
\text{Natural differential operators} \\
P : J^\infty F \longrightarrow F'
\end{Bmatrix}
\end{align*}
\]
Then, a standard argument—using that the pseudogroup $\text{Diff}(X)$ acts transitively on $X$—allows one to prove that restriction to the fiber of the point $p$ establishes a bijection:

\[
\begin{align*}
\begin{array}{c}
\text{Natural differential operators} \\
\quad P: \mathcal{F}^\infty \rightarrow \mathcal{F}' \\
\end{array}
\quad \longleftrightarrow \quad \begin{array}{c}
\text{Diff}_p\text{-equivariant smooth maps} \\
\quad f_p: \mathcal{F}^\infty_p \rightarrow \mathcal{F}'_p \\
\end{array}
\end{align*}
\]

To be precise, if $f_p: \mathcal{F}^\infty_p \rightarrow \mathcal{F}'_p$ is a $\text{Diff}_p$-equivariant map, the corresponding differential operator

\[
P: \mathcal{F}^\infty \rightarrow \mathcal{F}'
\]

is defined, over the fiber of any other point $q \in X$, as the composition $\tau^{-1}_q \circ f_p \circ \tau_q$, where $\tau: U_q \rightarrow V_p$ is any diffeomorphism such that $\tau(q) = p$. The choice of a different $\tau'$ produces the same $P$ due to the $\text{Diff}_p$-equivariance of $f_p$, whereas the smoothness of $P$ is a consequence of the smoothness assumptions on the liftings on $\mathcal{F}$ and $\mathcal{F}'$.

### 3.2. Natural Operations in the Presence of an Orientation

Let us now explain how to generalize Theorem 5 to the case of natural operations that depend on an orientation.

First of all, we observe that the orientation bundle $\text{Or}_X \rightarrow X$ is a natural bundle: The lifting of a diffeomorphism $\tau$ at a point $p$ is the identity in the case that $\det \tau^* p$ is positive, and the other map otherwise.

On the other hand, let us also observe that the direct product $\mathcal{F} \times \mathcal{F}'$ of natural bundles is also a natural bundle with the obvious lifting of diffeomorphisms.

**Theorem 6.** Let $\mathcal{F}$ and $\mathcal{F}'$ be natural bundles over $X$, and let $\mathcal{F}$ and $\mathcal{F}'$ be their sheaves of smooth sections, respectively.

The choice of a point $p \in X$ and an orientation $\text{Or}_X$ at $p$ produces a bijection:

\[
\begin{align*}
\begin{array}{c}
\text{Natural morphisms of sheaves} \\
\quad \mathcal{F} \times \text{Or}_X \rightarrow \mathcal{F}' \\
\end{array}
\quad \longleftrightarrow \quad \begin{array}{c}
\text{SDiff}_p\text{-equivariant smooth maps} \\
\quad f_p: \mathcal{F}^\infty_p \times \mathcal{F}^\infty_{\text{Or}_X} \rightarrow \mathcal{F}'_p \\
\end{array}
\end{align*}
\]

where $\text{Or}_X$ denotes the sheaf of orientations on $X$, and $\text{SDiff}_p$ stands for the group of germs at $p$ of diffeomorphisms $\tau$ such that $\tau(p) = p$ and $\det \tau^* p > 0$.

**Proof.** Due to Theorem 5, the choice of a point $p$ allows the definition of a bijection:

\[
\begin{align*}
\begin{array}{c}
\text{Natural morphisms of sheaves} \\
\quad \mathcal{F} \times \text{Or}_X \rightarrow \mathcal{F}' \\
\end{array}
\quad \longleftrightarrow \quad \begin{array}{c}
\text{Diff}_p\text{-equivariant smooth maps} \\
\quad f: \mathcal{F}^\infty \times \mathcal{F}^\infty_{\text{Or}_X} \rightarrow \mathcal{F}' \\
\end{array}
\end{align*}
\]

As the action of the group $\text{Diff}_p$ on the ringed space $\mathcal{F}^\infty_{\text{Or}_X}$ is transitive, a general statement about ringed spaces—Proposition 7 below—permits us to conclude.

**Proposition 7.** Let $G$ be a group acting on three ringed spaces $X$, $Y$, and $Z$.

If the action on $Y$ is transitive, then the choice of a point $\delta \in Y$ allows the definition of a bijection:

\[
\begin{align*}
\begin{array}{c}
\text{G-equivariant smooth maps} \\
\quad f: X \times Y \rightarrow Z \\
\end{array}
\quad \longleftrightarrow \quad \begin{array}{c}
\text{l}_\delta\text{-equivariant smooth maps} \\
\quad \bar{f}: X \rightarrow Z \\
\end{array}
\end{align*}
\]

where $l_\delta \subseteq G$ denotes the isotropy group of $\delta$.

**Proof.** For any smooth map $f: X \times Y \rightarrow Z$, the restriction to the subspace $X \times \{\delta\}$ defines a smooth $l_\delta$-equivariant map $\bar{f}: X \times \{\delta\} = X \rightarrow Z$. 
Conversely, any smooth $I_\delta$-equivariant map $\bar{f} : X \rightarrow Z$ can be extended to a smooth $G$-equivariant map as follows:

$$f : X \times Y \rightarrow Z, \quad f(x, y) := g \cdot (\bar{f}(g^{-1} \cdot y)),$$

where $g \in G$ is any element such that $x = g \cdot \delta$.

Finally, it is not difficult to check that this extension is well defined, as well as that both assignments are mutually inverse. \(\square\)

4. Invariants of Linear Connections and an Orientation

This section is devoted to proving Theorem 8, which is a description of the space of natural tensors associated to a linear connection and an orientation.

Let $\nabla$ be the germ of a linear connection at a point $p \in X$, and let $\bar{\nabla}$ be the germ of the flat connection at $p \in X$ corresponding, via the exponential map, to the flat connection of $T_p X$.

Let $T_p : \nabla - \bar{\nabla}$ be the $(2, 1)$-tensor:

$$T(\omega, D_1, D_2) := \omega(D_{\nabla} D_1 D_2 - D_{\bar{\nabla}} D_1 D_2).$$

**Definition 8.** For any integer $m \geq 0$, the $m$-th normal tensor of $\nabla$ at $p$ is $\bar{\nabla}^m p T$.

In a system of normal coordinates $(x_1, \ldots, x_n)$ for $\nabla$ at $p$:

$$\bar{\nabla}^m p T = \sum_{i,j,k} \frac{\partial^m r_{ij}}{\partial x_{\alpha_1} \ldots \partial x_{\alpha_m}}(p) \left( \frac{\partial}{\partial x_k} \right)_p \otimes dx_i \otimes dx_j \otimes dx_{\alpha_1} \otimes \ldots \otimes dx_{\alpha_m}. $$

**Definition 9.** The space $N^m$ of normal tensors of order $m$ at $p$ is the vector subspace of $(m + 2, 1)$-tensors $T$ at $p$ satisfying the following symmetries:

1. They are symmetric in the last $m$ covariant indices:

$$T_{ij \ldots k} = T_{ij \sigma(1) \ldots \sigma(m)}, \quad \forall \sigma \in S_m; $$

2. The symmetrization of the $m + 2$ covariant indices is zero:

$$\sum_{\sigma \in S_{m+2}} T_{i \sigma(j) \sigma(k) \sigma(\alpha_1) \ldots \sigma(\alpha_m)} = 0.$$ 

Normal tensors $\bar{\nabla}^m p T$ lie in $N^m$ [20] (Prop. 3.4). Thus, it makes sense to consider the following maps for any $m \geq 0$:

$$\phi_m : J^m p \text{Conn} \rightarrow N_0 \times \ldots \times N_m$$

$$j^m p \nabla \mapsto (T_p, \ldots, \bar{\nabla}^m p T),$$

where $\text{Conn} \rightarrow X$ denotes the bundle of linear connections on $X$ (not necessarily symmetric).

These maps $\phi_m$ are compatible in the sense that the following diagrams commute:

$$\begin{array}{ccc}
J^{m+1} p \text{Conn} & \xrightarrow{\phi_{m+1}} & N_0 \times \ldots \times N_{m+1} \\
\downarrow & & \downarrow \\
J^m p \text{Conn} & \xrightarrow{\phi_m} & N_0 \times \ldots \times N_m,
\end{array}$$
and hence, they define a morphism of ringed spaces between the corresponding inverse limits:

\[ \phi_\infty : f_p^\infty \text{Conn} \longrightarrow \prod_{i=0}^\infty N_i \]

\[ j_p^\infty \nabla \longmapsto (T_p, \nabla^1_p, \ldots). \]

For any \( m \geq 1 \), let us consider the Lie groups \( \text{Diff}_p^m := \{ j^m_p \tau : \tau \in \text{Diff}_p \} \) as well as their subgroups \( \text{NDiff}_p^m := \{ j^m_p \tau \in \text{Diff}_p^m : j^1_p \tau = j^1_p \text{Id} \} \).

Their inverse limits define groups

\[ \text{Diff}_p^\infty := \lim_{\leftarrow} \text{Diff}_p^m \quad \text{and} \quad \text{NDiff}_p^\infty := \lim_{\leftarrow} \text{NDiff}_p^m, \]

that can be related via a short, exact sequence of groups:

\[ 1 \longrightarrow \text{NDiff}_p^\infty \longrightarrow \text{Diff}_p^\infty \longrightarrow \text{GL} \longrightarrow 1, \tag{3} \]

where \( \text{GL} := \text{Diff}_p^1 = \{ d_p \tau : \tau \in \text{Diff}_p \} \).

**Reduction Theorem.** The \( \text{Diff}_p^\infty \)-equivariant morphism of ringed spaces

\[ \phi_\infty : f_p^\infty \text{Conn} \longrightarrow \prod_{i=0}^\infty N_i \]

\[ j_p^\infty \nabla \longmapsto (T_p, \nabla^1_p, \ldots), \]

is surjective, its fibers are the orbits of \( \text{NDiff}_p^\infty \), and it admits smooth sections passing through any point of \( f_p^\infty \text{Conn} \).

As a consequence, \( \phi_\infty \) induces a \( \text{GL} \)-equivariant isomorphism of ringed spaces:

\[ (f_p^\infty \text{Conn})/\text{NDiff}_p^\infty \cong \prod_{i=0}^\infty N_i. \]

**Proof.** By [20] (Thm. 3.6), the \( \text{Diff}_p^{m+2} \)-equivariant maps

\[ \phi_m : f_p^m \text{Conn} \longrightarrow N_0 \times \ldots \times N_m \]

\[ j_p^m \nabla \longmapsto (\nabla^0_p, \ldots, \nabla^m_p) \]

are surjective, regular projections whose fibers are the orbits of \( \text{NDiff}_p^{m+2} \) for any \( m \geq 0 \).

Let us explain how these facts imply the statement above that deals with formal developments of connections. Firstly, as \( \phi_m \) is \( \text{Diff}_p^{m+2} \)-equivariant and surjective for all \( m \), it follows that \( \phi_\infty \) is \( \text{Diff}_p^\infty \)-equivariant and surjective.

Next, let us check that the fibers of \( \phi_\infty \) are the orbits of \( \text{NDiff}_p^\infty \). On the one hand, if \( j^m_p \nabla = \tau_\infty \cdot j^m_p \nabla' \) for some \( \tau_\infty \in \text{NDiff}_p^\infty \), the condition of \( \phi_\infty \) being \( \text{Diff}_p^\infty \)-equivariant implies

\[ \phi_\infty (j^m_p \nabla) = \phi_\infty (\tau_\infty \cdot j^m_p \nabla') = \tau_\infty \cdot \phi_\infty (j^m_p \nabla') = \phi_\infty (j^m_p \nabla'). \]

Conversely, if \( \phi_\infty (j^m_p \nabla) = \phi_\infty (j^m_p \nabla') \), then \( \phi_m (j^m_p \nabla) = \phi_m (j^m_p \nabla') \) for all \( m \). Therefore, there exists \( \tau_m \in \text{NDiff}_p^{m+2} \) such that \( j^m_p \nabla = \tau_m \cdot j^m_p \nabla \). The sequence \( (\tau_m)_{m \in \mathbb{N}} \) defines an element \( \tau_\infty \in \text{NDiff}_p^\infty \) that verifies

\[ j^\infty_p \nabla = \tau_\infty \cdot j^\infty_p \nabla', \]

so that both formal developments are in the same orbit of \( \text{NDiff}_p^\infty \).
As for the existence of smooth sections, let us choose a local coordinate system centered at \( p \). For any given formal development \( f_p^\infty \nabla \), the proof of [20] (Thm. 3.6) shows how these coordinates define a global section \( \sigma_m \) that passes through \( f_p^\infty \nabla \). These sections are easily checked to be compatible with the projections \( f_p^{m+1} \text{Conn} \to f_p^m \text{Conn} \) and \( \prod_{i=0}^{m+1} N_i \to \prod_{i=0}^m N_i \) for all \( m \), so that, in turn, define a morphism of ringed spaces that is a section of \( \phi_m \) and passes through \( f_p^\infty \nabla \).

Finally, the last assertion of the statement is a consequence of Corollary 2. \( \Box \)

**Theorem 8.** Let \( X \) be a smooth manifold and let \( C \) and \( \text{Or}_X \) denote the sheaves of connections and orientations on \( X \), respectively.

Let \( F \) be a natural sub-bundle of the bundle of \((r,s)\)-tensors \( T^r_p \) and let \( \mathcal{F} \) be its sheaf of smooth sections.

If we fix a point \( p \in X \) and an orientation \( \text{or}_p \) at \( p \), there exists an \( \mathbb{R} \)-linear isomorphism

\[
\left\{ \begin{array}{c}
\text{Natural morphisms of sheaves} \\
C \times \text{Or}_X \to \mathcal{T}
\end{array} \right\} \cong \bigoplus_{d_i} \text{Hom}_{\mathbb{R}}(S_{d_0} N_0 \otimes \ldots \otimes S_{d_k} N_k, T_p),
\]

where \( d_0, \ldots, d_k \) run over the non-negative integer solutions of the equation

\[
d_0 + \ldots + (k+1)d_k = r - s,
\]

and where \( \text{Gl} := \{ d_p \tau : \tau \in \text{Diff}_p \} \) and \( \text{Sl} := \{ d_p \tau : \tau \in \text{SDiff}_p \} \).

**Proof.** Theorem 6 yields the isomorphism:

\[
\left\{ \begin{array}{c}
\text{Natural morphisms of sheaves} \\
C \times \text{Or}_X \to \mathcal{T}
\end{array} \right\} \cong \left\{ \begin{array}{c}
\text{SDiff}_p\text{-equivariant smooth maps} \\
f_p^\infty \text{Conn} \to T_p
\end{array} \right\}.
\]

Observe that the action of \( \text{SDiff}_p \) over \( f_p^\infty \text{Conn} \) and \( F_p \) coincides with that of \( \text{SDiff}_p^\infty \), so that, in the formula above, we may consider \( \text{SDiff}_p^\infty \)-equivariant maps instead.

In addition, notice that the following sequence of groups is exact:

\[
1 \to \text{NDiff}_p^\infty \to \text{SDiff}_p^\infty \to \text{Sl} \to 1
\]

The subgroup \( \text{NDiff}_p^\infty \) acts by the identity over \( T_p \) so that Corollary 3, in conjunction with the exact sequence above, assures the existence of an isomorphism:

\[
\left\{ \begin{array}{c}
\text{SDiff}_p^\infty\text{-equivariant smooth maps} \\
f_p^\infty \text{Conn} \to T_p
\end{array} \right\} \cong \left\{ \begin{array}{c}
\text{Sl-equivariant smooth maps} \\
f_p^\infty \text{Conn} / \text{NDiff}_p^\infty \to T_p
\end{array} \right\}.
\]

Now, the Reduction Theorem above allows us to replace this quotient ringed space with an infinite product of vector spaces via the isomorphism:

\[
\left\{ \begin{array}{c}
\text{Sl-equivariant smooth maps} \\
f_p^\infty \text{Conn} / \text{NDiff}_p^\infty \to T_p
\end{array} \right\} \cong \left\{ \begin{array}{c}
\text{Sl-equivariant smooth maps} \\
t : \prod_{i=0}^\infty N_i \to T_p
\end{array} \right\}.
\]

Finally, in the last step, we make use of the equivariance by homotheties \( h_\lambda : T_p X \to T_p X \) of ratio \( \lambda > 0 \). As \( h_{\lambda^{-1}} \in \text{Sl} \), the equivariance of these maps \( t \) implies

\[
t(\ldots, \lambda^{m+1} \Gamma_p^m, \ldots) = t(h_{\lambda^{-1}}(\ldots, \Gamma_p^m, \ldots)) = h_{\lambda^{-1}} \cdot t(\ldots, \Gamma_p^m, \ldots) = \lambda^{-s} t(\ldots, \Gamma_p^m, \ldots)
\]

for all \( \lambda > 0, (\ldots, \Gamma_p^m, \ldots) \in \prod_{i=0}^\infty N_i. \)
In view of this property of the smooth maps \( t \), the Homogeneous Function Theorem stated below (to be precise, Formula (7)) allows us to conclude with the isomorphism:

\[
\begin{bmatrix}
\text{Sl-equivariant smooth maps} \\
t: \prod_{i=0}^{\infty} N_i \to T_p \\
\end{bmatrix} \cong \bigoplus_{d_i} \text{Hom}_{\text{Sl}}(S^{d_0}N_0 \otimes \ldots \otimes S^{d_k}N_k, T_p),
\]

where \( d_0, \ldots, d_k \) are non-negative integers running over the solutions of the equation

\[d_0 + \ldots + (k + 1)d_k = r - s.\]

\( \square \)

**Homogeneous Function Theorem.** Let \( \{ E_i \}_{i \in \mathbb{N}} \) be finite-dimensional vector spaces.

Let \( f : \prod_{i=1}^{\infty} E_i \to \mathbb{R} \) be a smooth function such that there exist positive real numbers \( a_i > 0 \) and \( w \in \mathbb{R} \) satisfying:

\[f(\lambda^a_1 e_1, \ldots, \lambda^a_n e_n, \ldots) = \lambda^w f(e_1, \ldots, e_r)\]  

(4)

for any positive real number \( \lambda > 0 \) and any \( (e_1, \ldots, e_r, \ldots) \in \prod_{i=1}^{\infty} E_i \).

Then, \( f \) depends on a finite number of variables \( e_1, \ldots, e_k \), and it is a sum of monomials of degree \( d_i \) in \( e_i \) satisfying the relation

\[a_1d_1 + \cdots + a_kd_k = w.\]  

(5)

If there are no natural numbers \( d_1, \ldots, d_r \in \mathbb{N} \cup \{ 0 \} \) satisfying this equation, then \( f \) is the zero map.

**Proof.** Firstly, if \( f \) is not the zero map, then we observe \( w \geq 0 \) because, otherwise, (4) is contradictory when \( \lambda \to 0 \).

As \( f \) is smooth, there exists a neighbourhood \( U = \{ |e_1| < e_1, \ldots, |e_k| < e_k \} \subset \prod_1^{\infty} E_i \) of the origin and a smooth map \( \tilde{f} : \pi_k(U) \to \mathbb{R} \) such that \( f|_U = (\tilde{f} \circ \pi_k)|_U \).

As the \( a_1, \ldots, a_k \) are positive, there exist a neighborhood of zeros, \( \mathbb{V}_0 \subset \mathbb{R} \), and a neighborhood of the origin \( V \subset \pi_k(U) \) such that, for any \( (e_1, \ldots, e_k) \in V \) and any \( \lambda \in \mathbb{V}_0 \) that are positive, the vector \( (\lambda^{a_1} e_1, \ldots, \lambda^{a_k} e_k) \) lies in \( V \).

On that neighborhood \( V \), the function \( \tilde{f} \) satisfies the homogeneity condition:

\[\tilde{f}(\lambda^{a_1} e_1, \ldots, \lambda^{a_k} e_k) = \lambda^w \tilde{f}(e_1, \ldots, e_k)\]  

(6)

for any positive real number \( \lambda \in \mathbb{V}_0 \).

Differentiating this equation, we obtain analogous conditions for the partial derivatives of \( \tilde{f} \); v.g.r.:

\[\frac{\partial \tilde{f}}{\partial x_1}(\lambda^{a_1} e_1, \ldots, \lambda^{a_k} e_k) = \lambda^{w-a_1} \frac{\partial \tilde{f}}{\partial x_1}(e_1, \ldots, e_k).\]

If the order of derivation is big enough, the corresponding partial derivative is homogeneously of negative weight and, hence, zero. This implies that \( \tilde{f} \) is a polynomial; the homogeneity condition (6) is then satisfied for any positive \( \lambda \in \mathbb{V}_0 \) if and only if its monomials satisfy (5).

Finally, given any \( e = (e_1, \ldots, e_n, \ldots) \in \prod_{i=1}^{\infty} E_i \), we take \( \lambda \in \mathbb{R}^+ \) such that the vector \( (\lambda^{a_1} e_1, \ldots, \lambda^{a_k} e_k, \ldots) \) lies in \( U \). Then:

\[f(e) = \lambda^{-w} f(\lambda^{a_1} e_1, \ldots, \lambda^{a_n} e_n, \ldots) = \lambda^{-w} \tilde{f}(\lambda^{a_1} e_1, \ldots, \lambda^{a_k} e_k) = \tilde{f}(e_1, \ldots, e_k)\]

and \( f \) only depends on the first \( k \) variables. \( \square \)
This statement readily generalizes to say that, for any finite-dimensional vector space \( W \), there exists an \( \mathbb{R} \)-linear isomorphism:

\[
\left. \text{Smooth maps } f: \prod_{i=1}^{\infty} E_i \to W \text{ satisfying (4)} \right\} \cong \bigoplus_{d_1, \ldots, d_k} \text{Hom}_{\mathbb{R}}(S^{d_1}E_1 \otimes \ldots \otimes S^{d_k}E_k, W)
\]

(7)

where \( d_1, \ldots, d_k \) run over the non-negative integer solutions of (5).

5. An Application

Finally, as an application of Theorem 8, in this section, we compute some spaces of vector-valued and endomorphism-valued natural forms associated to linear connections and orientations, thus obtaining characterizations of the torsion and curvature operators (Corollary 13 and Theorem 15).

5.1. Invariant Theory of the Special Linear Group

Let \( V \) be an oriented \( \mathbb{R} \)-vector space of finite dimension \( n \), and let \( \text{Sl}(V) \) be the real Lie group of its orientation-preserving \( \mathbb{R} \)-linear automorphisms. Our aim is to describe the vector space of \( \text{Sl}(V) \)-invariant linear maps:

\[
V^* \otimes \ldots \otimes V^* \otimes V \otimes \ldots \otimes V \to \mathbb{R}.
\]

For any permutation \( \sigma \in S_p \), there exist the so-called total contraction maps, which are defined as follows:

\[
C_\sigma(\omega_1 \otimes \ldots \otimes \omega_p \otimes e_1 \otimes \ldots \otimes e_p) := \omega_1(e_{\sigma(1)}) \ldots \omega_p(e_{\sigma(p)}).
\]

Moreover, let \( \Omega \in \Lambda^n V^* \) be a representative of the orientation, and let \( \epsilon \) be the dual \( n \)-vector; that is to say, the only element in \( \Lambda^n V \) such that \( \Omega(\epsilon) = 1 \). For any permutation \( \sigma \in S_{p+k} \), the following linear maps are also \( \text{Sl}(V) \)-invariant:

\[
(\omega_1, \ldots, \omega_p, e_1, \ldots, e_p) \mapsto C_\sigma(\Omega \otimes \ldots \otimes \omega_1 \otimes \ldots \otimes \omega_p \otimes \epsilon \otimes \ldots \otimes \epsilon \otimes e_1 \otimes \ldots \otimes e_p).
\]

Classical invariant theory proves that these maps suffice to generate the vector space under consideration.

**Theorem 9.** The real vector space \( \text{Hom}_{\text{Sl}(V)}(V^* \otimes \ldots \otimes V^* \otimes V \otimes \ldots \otimes V, \mathbb{R}) \) of invariant linear forms on \( V^* \otimes \ldots \otimes V \) is spanned by

\[
(\omega_1, \ldots, \omega_p, e_1, \ldots, e_p) \mapsto C_\sigma(\Omega \otimes \ldots \otimes \omega_1 \otimes \ldots \otimes \omega_p \otimes \epsilon \otimes \ldots \otimes \epsilon \otimes e_1 \otimes \ldots \otimes e_p),
\]

where \( k \) is a non-negative integer such that \( 0 \leq k \leq p/n \).

In particular, for \( p < n \), the vector space of \( \text{Sl}(V) \)-invariant linear maps coincides with the vector space of \( \text{Gl}(V) \)-invariant linear maps.

In the applications, we will also require the following facts:

**Proposition 10.** Let \( E \) and \( F \) be (algebraic) linear representations of \( \text{Sl}(V) \).

- There exists a linear isomorphism \( \text{Hom}_{\text{Sl}(V)}(E, F) = \text{Hom}_{\text{Sl}(V)}(E \otimes F^*, \mathbb{R}) \).
- If \( W \subset E \) is a sub-representation, then any equivariant linear map \( W \to F \) is the restriction of an equivariant linear map \( E \to F \).
5.2. Uniqueness of the Torsion and Curvature Operators

**Definition 10.** Let $E \to X$ be a natural vector bundle. An $E$-valued natural $k$-form (associated to linear connections and orientations) is a regular and natural morphism of sheaves

$$\omega : \mathcal{C} \times \text{Or}_X \to \Omega^k \otimes E,$$

where $\Omega^k$ denotes the sheaf of differential $k$-forms on $X$ and $E$ stands for the sheaf of smooth sections of $E$.

Theorem 8 implies, in particular, that the space of $E$-valued natural forms associated to linear connections and orientations is a finite-dimensional real vector space. Moreover, as the exterior differential commutes with diffeomorphisms, it induces $\mathbb{R}$-linear maps:

$$\begin{bmatrix}
\text{E-valued natural}
\downarrow k\text{-forms}
\end{bmatrix}
\xrightarrow{d}
\begin{bmatrix}
\text{E-valued natural}
\downarrow (k+1)\text{-forms}
\end{bmatrix},$$

where it should be understood that, if $\omega$ is an $E$-valued natural $k$-form, the differential $d\omega : \mathcal{C} \times \text{Or}_X \to \Omega^{k+1} \otimes E$ is defined, on each section $(\nabla, \text{or})$, with respect to the linear connection on $E$ induced by $\nabla$.

**Definition 11.** A closed $E$-valued natural $k$-form (associated to linear connections and orientations) is an element in the kernel of the map above.

5.3. Vector-Valued Natural Forms

The torsion tensor of a linear connection can be understood as a vector-valued natural $2$-form; that is to say, as a regular and natural morphism of sheaves

$$\text{Tor} : \mathcal{C} \times \text{Or}_X \to \Omega^2 \otimes \mathcal{D},$$

where $\mathcal{D}$ stands for the sheaf of vector fields on $X$.

To be precise, the value of that tensor on a linear connection $\nabla$ and an orientation $\text{or}$ on an open set $U \subseteq X$ is

$$\text{Tor}_\nabla(D_1, D_2) := \nabla_{D_1} D_2 - \nabla_{D_2} D_1 - [D_1, D_2],$$

so that, in particular, it is independent of the orientation.

On the other hand, if $I : \mathcal{D} \to \mathcal{D}$ denotes the identity map and $c^1_1$ stands for the trace of the first covariant and contravariant indices, the tensor $H := c^1_1(\text{Tor}) \wedge I$ defines another vector-valued natural $2$-form:

$$H : \mathcal{C} \times \text{Or}_X \to \Omega^2 \otimes \mathcal{D}.$$

**Lemma 11.** If $\dim X \geq 3$, then $\text{Tor}$ and $H$ are a basis of the $\mathbb{R}$-vector space of vector-valued natural $2$-forms.

**Proof.** Looking at Theorem 8, we first compute the non-negative integer solutions of

$$d_0 + 2d_1 + \ldots + (k+1)d_k = 2 - 1 = 1.$$

There is only one solution, namely $d_0 = 1, d_i = 0$, for $i > 0$, so Theorem 8 assures that the vector space under consideration is isomorphic to the space of $\text{SL}$-equivariant linear maps:

$$N_0 = \Lambda^2 T_p^* X \otimes T_p X \to \Lambda^2 T_p^* X \otimes T_p X.$$

Thus, the problem is reduced to a question of invariants for the special linear group, and we can invoke Theorem 9 and Proposition 10 to obtain generators for this vector space.
According to those results, if \( \dim X > 3 \), then the space of \( \text{SL}\)-equivariant linear maps that we are considering coincides with the space of \( \text{GL}\)-equivariant linear maps, which, in turn, are proved in [16] (Lemma 3.5) to be spanned by \( H \) and \( \text{Tor} \).

If \( \dim X = 3 \), there may exist another generator; namely, the map \( \varphi: \Lambda^2 T_p^*X \otimes T_p X \to \Lambda^2 T_p^*X \otimes T_p X \), which, in coordinates around \( p \), reads:

\[
(d x_i \wedge d x_j) \otimes \partial_{x_k} \mapsto \Omega(\partial_{x_i}, \partial_{x_j}, \partial_{x_k}) \cdot e(d x_i, d x_j, d x_k),
\]

where \( \Omega = d x_1 \wedge d x_2 \wedge d x_3 \) and \( e \) is its dual 3-vector.

If \( \Gamma^k_{ij} \) denote the Christoffel symbols, then a trivial computation allows us to express

\[
\begin{align*}
\varphi &= d x_1 \wedge d x_2 \otimes \left( \Gamma^3_{23} \cdot \partial_{x_1} + \Gamma^3_{31} \cdot \partial_{x_2} + \Gamma^3_{12} \cdot \partial_{x_3} \right) \\
&+ d x_2 \wedge d x_3 \otimes \left( \Gamma^1_{23} \cdot \partial_{x_1} + \Gamma^1_{31} \cdot \partial_{x_2} + \Gamma^1_{12} \cdot \partial_{x_3} \right) \\
&+ d x_3 \wedge d x_1 \otimes \left( \Gamma^2_{23} \cdot \partial_{x_1} + \Gamma^2_{31} \cdot \partial_{x_2} + \Gamma^2_{12} \cdot \partial_{x_3} \right),
\end{align*}
\]

as well as the linear relation \( \varphi = \text{Tor} + H \).

**Theorem 12.** If \( \dim X \geq 3 \), then the exterior differential is an injective \( \mathbb{R} \)-linear map:

\[
\begin{bmatrix}
\text{Vector-valued natural} \\
2 \text{-forms}
\end{bmatrix}
\xrightarrow{d}
\begin{bmatrix}
\text{Vector-valued natural} \\
3 \text{-forms}
\end{bmatrix}.
\]

**Proof.** It is a consequence of both Lemma 11 and the fact that \( d H \) and \( d \text{Tor} \) are \( \mathbb{R} \)-linearly independent [16] (Theorem 3.6).

The so-called first Bianchi identity for the torsion tensor describes its differential in terms of the curvature, \( R \), and the identity map, \( I \): It is the following equality of vector-valued natural 3-forms:

\[
d \text{Tor} = R \wedge I.
\]

Therefore, an immediate corollary of Theorem 12 is:

**Corollary 13.** The torsion tensor is characterized as the only vector-valued natural 2-form \( \omega \) that satisfies the first Bianchi identity, i.e., such that \( d \omega = R \wedge I \).

### 5.4. Endomorphism-Valued Natural Forms

In this section, we restrict our attention to symmetric linear connections.

As in the case of the torsion tensor, the curvature tensor can also be thought of as an endomorphism-valued natural 2-form; that is to say, as a (regular and natural) morphism of sheaves

\[
R: \mathcal{C}^2 \times \text{Or}_X \to \Omega^2 \otimes \text{End}(\mathcal{D}),
\]

whose value on a symmetric linear connection \( \nabla \) and an orientation or defined on an open set \( U \subset X \) are the following endomorphism-valued 2-form \( R_\nabla \) on \( U \):

\[
R_\nabla(D_1, D_2)D_3 := \nabla_{D_1} \nabla_{D_2} D_3 - \nabla_{D_2} \nabla_{D_1} D_3 - \nabla_{[D_1, D_2]} D_3.
\]

**Definition 12.** An endomorphism-valued natural 2-form \( \omega \) satisfies the first Bianchi identity if, for any symmetric linear connection \( \nabla \), any orientation or and any vector fields \( D_1, D_2, \) and \( D_3 \):

\[
\omega_{(\nabla, \omega)}(D_1, D_2)D_3 + \omega_{(\nabla, \omega)}(D_2, D_3)D_1 + \omega_{(\nabla, \omega)}(D_3, D_1)D_2 = 0.
\]
The curvature tensor satisfies the first Bianchi identity. Moreover, if \( \text{Ric}^a \) and \( \text{Ric}^h \) denote the symmetric and skew-symmetric parts of the Ricci tensor \( \text{Ric} \), then the following tensors also satisfy the first Bianchi identity:

\[
\begin{align*}
C_1(D_1, D_2, D_3, \omega) & := \text{Ric}^a(D_1, D_3)\omega(D_2) - \text{Ric}^a(D_2, D_3)\omega(D_1), \\
C_2(D_1, D_2, D_3, \omega) & := \text{Ric}^h(D_1, D_3)\omega(D_2) - \text{Ric}^h(D_2, D_3)\omega(D_1) + 2\text{Ric}^h(D_1, D_2)\omega(D_3).
\end{align*}
\]

**Lemma 14.** If \( \dim X > 3 \), then the tensors \( C_1, C_2, \) and \( R \) are a basis of the \( \mathbb{R} \)-vector space of endomorphism-valued natural 2-forms that satisfy the first Bianchi identity.

If \( \dim X = 3 \), then that vector space has dimension four.

**Proof.** Let \( \mathcal{R} \) be the vector space of endomorphism-valued 2-forms at a point that satisfies the first Bianchi identity. Theorem 8 describes the space of the natural 2-forms under consideration as the vector space:

\[
\bigoplus_{d_1, \ldots, d_k} \text{Hom}_{\text{Sl}(T_x X)}(S^{d_1}N_1^3 \otimes \ldots \otimes S^{d_k}N_k^3, \mathcal{R}),
\]

where \( d_1, \ldots, d_k \) are non-negative integers verifying the equation:

\[
2d_1 + \ldots + (k+1)d_k = 3 - 1 = 2.
\]

The only solution to this equation is \( d_1 = 1, d_2 = \ldots = d_k \), so that the vector space to analyze is the space of Sl-equivariant linear maps:

\[
N_1^3 \rightarrow \mathcal{R}.
\]

First of all, recall that the maps induced by the tensors \( R, C_1 \), and \( C_2 \) are a basis of the space of Gl-equivariant linear maps \( N_1^3 \rightarrow \mathcal{R} \); see [16] (Lemma 3.11).

A systematic application of Theorem 9 now allows us to find generators for the space of Sl-equivariant maps.

If \( \dim X > 5 \), then the vector space of Sl-equivariant maps coincides with the space of Gl-equivariant maps and, hence, is generated by these three elements.

In case \( \dim X = 4 \), there is another possible generator: the map \( N_1^3 \rightarrow \mathcal{R} \) defined as

\[
dx_i \otimes dx_j \otimes dx_k \otimes dx_l \rightarrow \Omega(\partial_{x_i, \sigma}, \sigma, \sigma) \cdot \epsilon(dx_j, dx_l, dx_k, \sigma).
\]

However, as any tensor in \( N_1^3 \) is symmetric in the first two indices, it readily follows that this map is identically zero.

If \( \dim X = 3 \), let us first describe the Sl-equivariant endomorphisms \( T_3^1 \rightarrow T_3^1 \).

To this end, let \( x_1, x_2 \) and \( x_3 \) be coordinates centered at \( p \) such that \( \Omega = dx_1 \wedge dx_2 \wedge dx_3 \) is positively oriented, and let \( \epsilon \) be its dual 3-vector.

Using \( \Omega \) and \( \epsilon \), we can construct 16 generators, and they can all be expressed as a permutation of the factors of \( T_3^1 \) followed by one of these four maps:

(a) \( dx_i \otimes dx_j \otimes dx_k \otimes \partial_{x_l} \rightarrow \epsilon(dx_l, dx_j, dx_k) \cdot \Omega \otimes \partial_{x_i} \),
(b) \( dx_i \otimes dx_j \otimes dx_k \otimes \partial_{x_l} \rightarrow \epsilon(dx_l, dx_i, dx_k) \cdot \Omega(\partial_{x_j, \sigma}, \sigma) \otimes I \),
(c) \( dx_i \otimes dx_j \otimes dx_k \otimes \partial_{x_l} \rightarrow \delta_{ij}^l \cdot \Omega \otimes \epsilon(dx_{\sigma(i)}, dx_{\sigma(k)}, \sigma), \quad \sigma \in S_3 \),
(d) \( dx_i \otimes dx_j \otimes dx_k \otimes \partial_{x_l} \rightarrow \Omega(\partial_{x_j, \sigma}, \sigma) \otimes dx_{\sigma(i)} \otimes \epsilon(dx_{\sigma(j)}, dx_{\sigma(k)}), \quad \sigma \in S_3 \).

As the first two covariant indices of \( N_1^3 \) are symmetric, the following maps are identically zero: (a), (b), and raising the first two indices at (c) and (d).

That leaves eight non-zero generators. However, this symmetry also makes raising indices 1, 3 and 2, 3 indistinguishable, hence reducing to just four generators.
The last step is to check which of these maps take values in $\mathcal{R}$. Out of these four generators, only the following two produce tensors that are skew-symmetric in the first two covariant indices:

$$
\phi_1(dx_i \otimes dx_j \otimes dx_k \otimes \partial_{\eta}) = \Omega(\partial_{\xi/\eta - \chi}) \otimes dx_i \otimes \epsilon(dx_j, dx_k, \delta^i_j)
$$

and the skew-symmetrization of the remaining two is a linear combination of these.

None of these two tensors satisfy the first Bianchi identity, but the linear combination $\phi := 3\phi_1 - \phi_2$ does.

Finally, all that is left to prove is that $\phi$ is $\mathbb{R}$-linearly independent of $R, C_1,$ and $C_2$. In order to do that, it is enough to find a symmetric linear connection and an orientation on a 3-manifold $X$ such that the aforementioned tensors on $X$ are $\mathbb{R}$-linearly independent.

The following example works: Let $\nabla$ be the linear connection on $\mathbb{R}^3$ whose only non-zero Christoffel symbols in cartesian coordinates are

$$
\Gamma^1_{11} = x_2 x_3 \quad \text{and} \quad \Gamma^2_{23} = \Gamma^3_{32} = x_1 x_2 .
$$

Assume that $dx_1 \wedge dx_2 \wedge dx_3$ is positively oriented, and denote $T_{ij} := dx_i \otimes \partial_{x_j}$.

Direct computation gives the following linearly independent tensors, thus finishing the proof:

$$
R = dx_1 \wedge dx_2 \otimes (-x_3 T_{11} + x_2 T_{32}) + dx_1 \wedge dx_3 \otimes (-x_2 T_{11} + x_2 T_{22})
$$

$$
+ dx_2 \wedge dx_3 \otimes (x_1 T_{22} - x_1 T_{32}^2 T_{32}),
$$

$$
C_1 = dx_1 \wedge dx_2 \otimes \left(\frac{1}{2} x_3 T_{11} - \frac{1}{2} x_3 T_{22} - \frac{1}{2} x_1 T_{31} - x_2 T_{32}\right) +
$$

$$
+ dx_1 \wedge dx_3 \otimes \left(x_2 T_{11} - \frac{1}{2} x_1 T_{21} - \frac{1}{2} x_3 T_{23} + x_1^2 T_{31} - x_2 T_{33}\right) +
$$

$$
+ dx_2 \wedge dx_3 \otimes \left(x_3 T_{12} - \frac{1}{2} x_1 T_{22} - x_1^2 x_2 T_{32} + \frac{1}{2} x_1 T_{33}\right),
$$

$$
C_2 = dx_1 \wedge dx_2 \otimes \left(\frac{3}{2} x_3 T_{11} + \frac{3}{2} x_3 T_{22} + \frac{1}{2} x_1 T_{31} + x_3 T_{33}\right) +
$$

$$
+ dx_1 \wedge dx_3 \otimes \left(-\frac{1}{2} x_1 T_{21} + \frac{1}{2} x_3 T_{23}\right) +
$$

$$
+ dx_2 \wedge dx_3 \otimes \left(-x_1 T_{11} - \frac{1}{2} x_3 T_{13} - \frac{3}{2} x_1 T_{22} - \frac{3}{2} x_1 T_{33}\right),
$$

$$
\phi = dx_1 \wedge dx_2 \otimes (x_1 T_{31} - x_3 T_{33}) +
$$

$$
+ dx_1 \wedge dx_3 \otimes (2x_1 T_{21} - 3x_2 T_{22} + x_3 T_{23} + 3x_2 T_{33}) +
$$

$$
+ dx_2 \wedge dx_3 \otimes (x_1 T_{11} - 3x_2 T_{12} + 2x_3 T_{13}).
$$

\[\square\]

**Definition 13.** An endomorphism-valued natural 2-form $\omega$ is said to satisfy the second Bianchi identity if it is closed in the sense of Definition 11.

**Theorem 15.** The constant multiples of the curvature are the only endomorphism-valued natural 2-forms that satisfy both the first and second Bianchi identities.
Proof. The curvature tensor $R$ is always a closed natural 2-form, so, by the previous Lemma, it is enough to analyze the $\mathbb{R}$-linear span of the differentials of $C_1, C_2$, and, in dimension 3, of $\varphi$.

If $\dim X > 3$, then $dC_1$ and $dC_2$ are linearly independent by [16] (Thm. 3.13), and the statement follows.

If $\dim X = 3$, a direct computation, using the same example as in the previous Lemma, proves that $dC_1$, $dC_2$, and $d\varphi$ are $\mathbb{R}$-linearly independent tensors:

$$d_C C_1 = dx_1 \wedge dx_2 \wedge dx_3 \otimes \left( -\frac{1}{2} T_{11} - x_2^2 x_3 T_{12} + \frac{1}{2} x_2 x_3^2 T_{13} + \frac{1}{2} (x_1 x_2 x_3 - 2) T_{22} + \right.$$

$$- \frac{5}{2} x_1^2 x_2 T_{31} + 2 x_1 x_2^2 T_{32} - \frac{1}{2} (x_1 x_2 x_3 - 3) T_{33} \bigg),$$

$$d_C C_2 = dx_1 \wedge dx_2 \wedge dx_3 \otimes \left( \frac{1}{2} T_{11} + \frac{1}{2} x_2 x_3^2 T_{13} - \frac{1}{2} x_1 x_2 x_3 T_{22} + \right.$$

$$- \frac{1}{2} x_1^2 x_2 T_{31} + \frac{1}{2} (x_1 x_2 x_3 - 1) T_{33} \bigg),$$

$$d_C \varphi = dx_1 \wedge dx_2 \wedge dx_3 \otimes \left( T_{11} + 3 x_2 x_3 T_{12} - 2 x_2 x_3^2 T_{13} - (x_1 x_2 x_3 - 3) T_{22} + \right.$$

$$+ 2 x_1^2 x_2 T_{31} - 6 x_1 x_2^2 T_{32} + (x_1 x_2 x_3 - 4) T_{33} \bigg).$$

$\square$

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