Quantum cosmology in the models of 2d and 4d dilatonic supergravity with WZ matter

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We consider $N = 1$ two-dimensional (2d) dilatonic supergravity (SG), 2d dilatonic SG obtained by dimensional reduction from $N = 1$ four-dimensional (4d) SG, $N = 2$ 2d dilatonic SG and string-inspired 4d dilatonic SG. For all the theories, the corresponding action on a bosonic background is constructed and the interaction with $N$ (dilatonic) Wess-Zumino (WZ) multiplets is presented. Working in the large-$N$ approximation, it is enough to consider the trace anomaly induced effective action due to dilaton-coupled conformal matter as a quantum correction (for 2d models s-waves approximation is additionally used). The equations of motion for all such models with quantum corrections are written in a form convenient for numerical analysis. Their solutions are numerically investigated for 2d and 4d Friedmann-Robertson-Walker (FRW) or 4d Kantowski-Sacks Universes with a time-dependent dilaton via exponential dilaton coupling. The evolution of the corresponding quantum cosmological models is given for different choices of initial conditions and theory parameters. In most cases we find quantum singular Universes. Nevertheless, there are examples of Universe non-singular at early times. Hence, it looks unlikely that quantum matter back reaction on dilatonic background (at least in large $N$ approximation) may really help to solve the singularity problem.

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1 Introduction

It is widely known that models of two-dimensional (2d) dilaton gravity may qualitatively describe the main properties of gravitational collapse and the evolution of the Universe taking into account quantum matter back reaction. This is due to the fact that reduction of 4d gravitational theories with matter often leads to effective two-dimensional (dilatonic) gravity with dilaton coupled matter. In a recent work [1], we studied the quantum cosmology of four-dimensional (4d) reduced Einstein gravity with \( N \) minimal scalars (working in large \( N \) and s-waves approximations). As a result of the reduction, one finds 2d dilatonic gravity with dilaton coupled scalars and quantum cosmology (of 4d Kantowski-Sacks form) where dilatonic coupling may be interpreted as a second scale factor of the singular, non-singular or big crunch type [1].

It is quite interesting to generalize such a study for supergravities (SGs) which will be the purpose of this paper. First, one can understand which different quantum SG may lead to which early Universe model in comparison with purely bosonic versions. Second, some SGs represent low-energy effective actions of superstring theory and so such cosmologies (limiting to our approach) may possess some characteristic stringy cosmological features. Third, such study may help to select the “best” theories in the cosmological sense from among the existing variety of SGs.

The present work is organized as follows. In the next section we review the construction of \( N = 1 \) 2d dilatonic SG [2] with \( N \) scalar supermultiplets. Section three is devoted to \( N = 1 \) Callan-Giddings-Harvey-Strominger (CGHS) SG with dilaton coupled scalar matter superfields. Working in the large \( N \) approximation, we take into account quantum matter effects and write the corresponding equations of motion (with non-zero matter fermions) on a bosonic background. The numerical study of such equations show 2d cosmologies where the Universe expands first and then shrinks, or where the Universe oscillates. In section four, we apply the same technique to study 2d dilatonic models with supermatter which is the reduced version from 4d \( N = 1 \) SG with matter. Then two-dimensional cosmology with a dilaton may be interpreted as a 4d Kantowski-Sacks Universe. Numerical estimations show that it is singular 4d Universe in the situations under discussion. In section five, we work with \( N = 2 \) 2d dilatonic SG [3] with dilaton-coupled matter. The numerical analysis of two-dimensional cosmolo-
gies with the back-reaction of quantum matter shows a qualitative difference from the $N = 1$ SG case due to the presence of vector field in the bosonic background. Section six is devoted to string-inspired 4d dilatonic SG (its bosonic version) with $N$ dilaton coupled matter multiplets. Working in the large $N$ approximation and using the 4d anomaly induced action, we analytically and numerically investigate equations of motion for conformally flat 4d cosmologies. In the cases under consideration we have found singular or non-singular at early times or trivial cosmological solutions.
2 General Action of $N = 1$ Dilatonic Supergravity

The supersymmetric extension of the CGHS model [4] was recently considered in [5]. This theory is a partial example of more general two-dimensional $N = 1$ dilatonic supergravity [2]. The quantum corrections in the large $N$ approximation for such theories with matter have been discussed in [2]. Using such quantum corrections one can also study Hawking radiation in 2d dilaton supergravity. In this section, we briefly review the general action using the component formulation of ref. [6].

In $N = 1$ dilatonic supergravity in this paper all the scalar fields are real and all the spinor fields are Majorana spinors.

The general $N = 1$ action of 2d dilatonic supergravity is given in terms

The conventions and notations are given as follows

- signature
  \[ \eta^{ab} = \delta^{ab} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \]

- gamma matrices
  \[ \gamma^a \gamma^b = \delta^{ab} + i \epsilon^{ab} \gamma^5 , \]
  \[ \sigma_{ab} = \frac{1}{4} [\gamma_a, \gamma_b] = \frac{i}{2} \epsilon_{ab} \gamma^5 . \]

- charge conjugation matrix $C$
  \[ C \gamma_a C^{-1} = -\gamma^T_a , \]
  \[ C = C^{-1} = -C^T , \]
  \[ \bar{\psi} = -\psi^T C . \]

Here the index $^T$ means transverse.

- Majorana spinor
  \[ \psi = \psi^c \equiv C \bar{\psi}^T . \]

- Levi-Civita tensor
  \[ \epsilon^{12} = \epsilon_{12} = 1 , \quad \epsilon^{ab} = -\epsilon^{ba} , \quad \epsilon_{ab} = -\epsilon_{ba} , \]
  \[ \epsilon^{\mu\nu} = e \epsilon_a^\mu e_b^\nu \epsilon^{ab} , \quad \epsilon_{\mu\nu} = e^{-1} \epsilon^a_\mu e^b_\nu \epsilon_{ab} . \]
of general functions of the dilaton $C(\phi)$, $Z(\phi)$, $f(\phi)$ and $V(\phi)$ as follows

$$\mathcal{L} = -\left[ C(\Phi) \otimes W\right]_{\text{inv}} + \frac{1}{2} \left[ \Phi \otimes \Phi \otimes T_P(Z(\Phi))\right]_{\text{inv}} - \left[ Z(\Phi) \otimes \Phi \otimes T_P(\Phi)\right]_{\text{inv}} + \sum_{i=1}^{N} \left\{ \frac{1}{2} \left[ \Sigma_i \otimes \Sigma_i \otimes T_P(f(\Phi))\right]_{\text{inv}} - \left[ f(\Phi) \otimes \Sigma_i \otimes T_P(\Sigma_i)\right]_{\text{inv}} \right\} + [V(\Phi)]_{\text{inv}}. \tag{1}$$

Here $\Phi = (\phi, \chi, F)$ is a dilaton multiplet and $\Sigma_i = (a_i, \xi_i, G_i)$ is a matter multiplet. These multiplets have the conformal weight $\lambda = 0$. $W$ is the curvature multiplet which is given by

$$W = \left( S, \eta, -S^2 - \frac{1}{2} R - \frac{1}{2} \bar{\psi}^\mu \gamma^\nu \psi_{\mu\nu} + \frac{1}{4} \bar{\psi}^\mu \psi_{\mu} \right) . \tag{2}$$

Here $R$ is the scalar curvature and

$$\eta = -\frac{1}{2} S \gamma^\mu \psi_{\mu} + \frac{i}{2} e^{-1} \epsilon^{\mu\nu} \gamma_5 \psi_{\mu\nu} ,$$
$$\psi_{\mu\nu} = D_{\mu} \psi_{\nu} - D_{\nu} \psi_{\mu} ,$$
$$D_{\mu} \psi_{\nu} = \left( \partial_{\mu} - \frac{1}{2} \omega_{\mu} \gamma_5 \right) \psi_{\nu} ,$$
$$\omega_{\mu} = -ie^{-1} e_{a\mu} \epsilon^{\lambda\nu} \partial_{\lambda} e_{a \nu} - \frac{1}{2} \bar{\psi}^\mu \gamma_5 \gamma^\lambda \psi_{\lambda} . \tag{3}$$

The curvature multiplet has the conformal weight $\lambda = 1$. $T_P(Z)$ is called the kinetic multiplet for the multiplet $Z = (\varphi, \zeta, H)$ (in terms of superfield operators this corresponds to $(\nabla^2 + \lambda W)Z$) and when the multiplet $Z$ has the conformal weight $\lambda = 0$, $T_P(Z)$ has the following form

$$T_P(Z) = (H, D\zeta, \Box \varphi) . \tag{5}$$

The kinetic multiplet $T_P(Z)$ has conformal weight $\lambda = 1$. The product of two multiplets $Z_k = (\varphi_k, \zeta_k, H_k)$ ($k = 1, 2$) with the conformal weight $\lambda_k$ is defined by

$$Z_1 \otimes Z_2 = (\varphi_1 \varphi_2, \varphi_1 \zeta_2 + \varphi_2 \zeta_1, \varphi_1 H_2 + \varphi_2 H_1 - \bar{\zeta}_1 \zeta_2) . \tag{6}$$

\(^6\)The multiplet containing $C(\phi)$, for example, is given by $(C(\phi), C'(\phi) \chi, C'(\phi) F - \frac{1}{2} C''(\phi) \bar{\chi} \chi)$. \[6]
This is familiar in the language of superfields as the components of $Z_1 Z_2$. The invariant Lagrangian $[Z]_{inv}$ for the multiplet $Z$ is defined by

$$[Z]_{inv} = e \left[ H + \frac{1}{2} \bar{\psi}_\mu \gamma^\mu \zeta + \frac{1}{2} \varphi \bar{\psi}_\mu \sigma^\mu\nu \psi_\nu + S \varphi \right],$$

(7)
corresponding to the superaction $S = \int d^2 \theta E^{-1} Z$.

We consider only bosonic background fields below since this will be sufficient for the study of the cosmological problems under consideration. On the bosonic background where the dilatino $\chi$ and the Rarita-Schwinger fields vanish, one can show that the gravity and dilaton part of the Lagrangian have the following forms:

$$[C(\Phi) \otimes W]_{inv} \sim e \left[ -C(\phi) \left( S^2 + \frac{1}{2} R \right) - C'(\phi) F S \right],$$

$$[\Phi \otimes \Phi \otimes T_P(Z(\Phi))]_{inv} \sim e \left[ \phi^2 \Box (Z(\phi)) + 2Z'(\phi) \phi F^2 \right],$$

$$[Z(\Phi) \otimes \Phi \otimes T_P(\Phi)]_{inv} \sim e \left[ Z(\phi) \phi \Box \phi + Z'(\phi) \phi F^2 + Z(\phi) F^2 \right],$$

$$[V(\Phi)]_{inv} \sim e \left[ V'(\phi) F + SV(\phi) \right].$$

(8)

For matter part we obtain

$$\sum_{i=1}^{N} \left\{ \frac{1}{2} \left[ \Sigma_i \otimes \Sigma_i \otimes T_P(f(\Phi)) \right]_{inv} - [f(\Phi) \otimes \Sigma_i \otimes T_P(\Sigma_i)]_{inv} \right\}$$

$$\sim e f(\phi) \sum_{i=1}^{N} \left( g^{\mu\nu} \partial_\mu a_i \partial_\nu a_i + \bar{\xi}_i \gamma^\mu \partial_\mu \xi_i - f(\phi) G_i^2 \right)$$

$$+ \text{total divergence terms}.$$

(9)

In eq.(8) and (9), "\sim" means that we neglect the terms containing fermionic fields except the kinetic term of $\xi_i$. Here we have used the fact that

$$\bar{\xi}_i \gamma_5 \xi = 0$$

(10)

for the Majorana spinors $\xi_i$. Using equations of motion with respect to the auxiliary fields $S$, $F$, $G_i$, on the bosonic background one can show that

$$S = \frac{C'(\phi) V'(\phi) - 2V(\phi) Z(\phi)}{C''(\phi) + 4C(\phi) Z(\phi)},$$

$$F = \frac{C'(\phi) V(\phi) + 2C(\phi) V'(\phi)}{C''(\phi) + 4C(\phi) Z(\phi)},$$

$$G_i = 0.$$

(11)
Especially for the supersymmetric extension \[5\] of the CGHS type model \[4\]

\[
C(\phi) = 2e^{-2\phi}, \quad Z(\phi) = 4e^{-2\phi}, \quad V(\phi) = 4e^{-2\phi},
\]

(12)
we find

\[
S = 0, \quad F = -\lambda, \quad G_i = 0.
\]

(13)

We should note \(\lambda^2 > 0\) in the supersymmetric theory or unitarity is broken.

### 3 N = 1 Supersymmetric CGHS Model with Dilaton Coupled Matter

After integrating out the auxilliary fields as we have shown in the previous section, the classical action of the CGHS type dilaton supergravity model with dilaton coupled supermatter in a purely bosonic background is given by

\[
S_c = \int d^2 x \left[ -\frac{e^{-2\phi}}{2\pi} \left\{ R + 4\partial_\mu \phi \partial^\mu \phi + 4\lambda^2 \right. \\
- \frac{1}{2} \sum_{i=1}^{N} \left( \partial_\mu a_i \partial^\mu a_i + \bar{\xi}_i \gamma^\mu D_\mu \xi_i \right) \right] \right)
\]

(14)

As we are going to work in large \(N\) approximation we may take into account only matter quantum effects. Then the bosonic part of the trace anomaly induced effective action \[3, 2\] (for a pure dilaton coupled scalar, see also \[4\]) together with classical action has the following form\[7\]

\[
S = S_c + W
\]
\[
= \int d^2 x \left[ -\frac{e^{-2\phi}}{2\pi} \left\{ R + 4\partial_\mu \phi \partial^\mu \phi - \frac{1}{2} \sum_{i=1}^{N} \left( \partial_\mu a_i \partial^\mu a_i + \bar{\xi}_i \gamma^\mu D_\mu \xi_i \right) + 4\lambda^2 \right. \\
- \frac{1}{2} \left\{ \frac{N}{32} R \frac{1}{\Box} R - \frac{N}{4} \partial_\mu \phi \partial^\mu \phi \frac{1}{\Box} R + \frac{N}{4} \phi \Box \right\} \right].
\]

(15)

\[7\] The inverse of \(\Box = -\frac{1}{\sqrt{-g}} \partial_\mu \sqrt{-g} g^\mu\nu \partial_\nu\) is defined by

\[\square_x \left( \frac{1}{\Box} (x, y) \right) = \sqrt{-g} \delta^4(x - y).\]

Here \(x\) and \(y\) are the coordinates of the space-time and the subscript \(x\) of \(\Box\) means the derivative with respect to \(x\). In the conformal gauge \[16\], the ambiguity of the boundary condition is absorbed into the function \(t^\pm(x^\pm)\) which appears in \[17\].
In the (super)conformal gauge,
\[ g_{\pm \mp} = -\frac{1}{2} e^{2\rho}, \quad g_{\pm \pm} = 0 \]  
the equations of motion given by the variations of \( g_{\pm \pm}, g_{\pm -}, \phi \) and \( a_i \) have the following form:

\[ 0 = T_{\pm \pm} \]
\[ = e^{-2\phi} \left( 4\partial_+ \rho \partial_+ \phi - 2 (\partial_+ \phi)^2 \right) + \frac{1}{2} e^{-2\phi} \sum_{i=1}^{N} (\partial_+ a_i)^2 + T_{\pm \pm}^f \]
\[ + \frac{N}{8} \left( \partial_+^2 \rho - \partial_+ \rho \partial_+ \rho \right) \]
\[ + \frac{N}{2} \left\{ (\partial_+ \phi \partial_+ \phi) \rho + \frac{1}{2} \partial_+^2 (\partial_+ \phi \partial_+ \phi) \right\} \]
\[ - \frac{N}{4} \left\{ -2 \partial_+ \rho \partial_+ \phi + \partial_+^2 \phi \right\} + t^\pm (x^\pm) \]  
(17)

\[ 0 = T_{\pm \mp} \]
\[ = e^{-2\phi} \left( 2\partial_+ \partial_- \phi - 4 \partial_+ \phi \partial_- \phi - \lambda^2 e^{2\rho} \right) + T_{\pm \mp}^f \]
\[ - \frac{N}{8} \partial_+ \partial_- \rho \]
\[ + \frac{N}{4} \partial_+ \phi \partial_- \phi \]
\[ + \frac{N}{4} \partial_- \partial_+ \phi \]  
(18)

\[ 0 = e^{-2\phi} \left( -4 \partial_+ \partial_- \phi + 4 \partial_+ \phi \partial_- \phi + 2 \partial_+ \partial_- \rho \right) \]
\[ - \frac{1}{2} \sum_{i=1}^{N} \partial_+ a_i \partial_- a_i + \lambda^2 e^{2\rho} \]  
\[ - T_{\pm \mp}^f \]
\[ + \left\{ - \frac{N}{4} \partial_+ (\rho \partial_- \phi) - \frac{N}{4} \partial_- (\rho \partial_+ \phi) - \frac{N}{4} \partial_+ \partial_- \rho \right\} \]  
(19)

Here \( t^\pm (x^\pm) \) is a function which is determined by the boundary condition and \( T_{\mu \nu}^f \) is the energy-momentum tensor of the matter fermion defined by

\[ T_{\mu \nu}^f = \frac{1}{2} e^{-2\phi} \sum_{i=1}^{N} \left( \xi_i \gamma_\mu \partial_\nu \xi_i + \xi_i \gamma_\nu \partial_\mu \xi_i - g_{\mu \nu} \xi_i \gamma_\lambda \partial_\lambda \xi_i \right) \]  
(21)

If we use the \( \xi \)-equation of motion, we find

\[ T_{\pm \pm}^f = T_{\pm \pm}^f (x^+) \], \[ T_{\pm \mp}^f = 0 \].  
(22)
Since we are presently considering the cosmological problem, we assume that all the fields depend only on time $t$ and replace $\partial_\pm \to \frac{1}{2} \partial_t$. Then Eq. (17) tells $t^\pm$ and $T^f_{\pm\pm}$ are constants: $t^\pm = \frac{N}{t} t_0$ and $T^f_{\pm\pm} = \frac{N}{t} T$. Then Eqs. (17), (18), (19) and (20) can be rewritten as

$$0 = e^{-2\phi} \left[ 4 \partial_t \rho \partial_t \phi - 2 (\partial_t \phi)^2 \right] + \frac{1}{2} e^{-2\phi} \sum_{i=1}^{N} (\partial_t a_i)^2 + NT$$

$$0 = e^{-2\phi} \left( \frac{N}{8} \left( \partial_t^2 \rho - (\partial_t \rho)^2 \right) \right)$$

$$- \frac{N}{2} \left( \partial_t \rho \partial_t \phi \right)$$

$$- \frac{N}{2} \left\{ -2 \partial_t \rho \partial_t \phi + \partial_t^2 \phi \right\} + Nt_0$$

$$0 = e^{-2\phi} \left( 2 \partial_t^2 \phi - 4 \partial_t \phi \partial_t \phi - 4 \lambda^2 e^{2\rho} \right)$$

$$- \frac{N}{8} \partial_t^2 \rho - \frac{N}{4} \partial_t \phi \partial_t \phi + \frac{N}{4} \partial_t^2 \phi$$

$$0 = e^{-2\phi} \left( -4 \partial_t^2 \phi + 4 (\partial_t \phi)^2 + 2 \partial_t^2 \rho + 4 \lambda^2 e^{2\rho} - \frac{1}{2} \sum_{i=1}^{N} (\partial_t a_i)^2 \right)$$

$$+ \left\{ - \frac{N}{2} \partial_t (\rho \partial_t \phi) - \frac{N}{4} \partial_t^2 \rho \right\}$$

$$0 = \partial_t (e^{-2\phi} \partial_t a_i) .$$

Eq. (26) can be integrated to be $\partial_t a_i = A_i e^{2\phi}$. Here $A_i$ is a constant of the integration. Note that $T$ can be absorbed into the redefinition of $t_0$

$$t_0 + T \to t_0 .$$

In the following we use the redefinition of (27).

Solving Eqs. (23) and (24), we obtain

$$\partial_t^2 \phi = \left( 3 - \frac{N}{4} e^{2\phi} \right) (\partial_t \phi)^2 - \left( 2 + \frac{N}{4} e^{2\phi} \right) \partial_t \rho \partial_t \phi + \frac{N}{16} e^{2\phi} (\partial_t \rho)^2$$

$$- \frac{N}{4} e^{4\phi} A^2 + 2 \lambda^2 e^{2\rho} - \frac{N}{2} e^{2\phi} t_0$$

$$\partial_t^2 \rho = \left( 1 + \frac{N}{8} e^{2\phi} \right) (\partial_t \rho)^2 - \frac{32}{N} e^{-2\phi} \left( 1 + \frac{N}{8} e^{2\phi} \right)^2 \partial_t \rho \partial_t \phi$$

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\[ + 4 \left\{ \left( 1 + \frac{4}{N} e^{-2\phi} \right) - \left( 1 + \frac{N}{8} e^{2\phi} \right) \right\} (\partial_t \phi)^2 \]
\[ - 4e^{2\phi} \left( 1 + \frac{N}{8} e^{2\phi} \right) A^2 - 8 \left( 1 + \frac{N}{8} e^{2\phi} \right) t_0 + 4\lambda^2 e^{2\rho} \cdot \]  

Here
\[ A^2 \equiv \frac{1}{N} \sum_{i=1}^{N} A_i^2 . \]  

Substituting (28) and (29) into (25) we obtain
\begin{align*}
t_0 &= - \left\{ -16e^{-2\phi} + 2N + \frac{N^2}{4} e^{2\phi} (\rho + 1) \right\}^{-1} \\
& \quad \times \left\{ \frac{32}{N} e^{-4\phi} - (8\rho + 4)e^{-2\phi} - N \left( 1 + \frac{\rho}{2} \right) + \frac{N^2}{8} e^{2\phi} (\rho + 1) \right\} (\partial_t \phi)^2 \\
& \quad + \left\{ \frac{64}{N} e^{-4\phi} + N \left( \rho + \frac{3}{2} \right) + \frac{N^2}{8} e^{2\phi} (\rho + 1) \right\} \partial_t \rho \partial_t \phi \\
& \quad + \left\{ 2e^{-2\phi} - \frac{N}{4} - \frac{N^2}{32} e^{2\phi} (\rho + 1) \right\} (\partial_t \rho)^2 \\
& \quad + \left\{ -8 + \frac{N}{2} e^{2\phi} + \frac{N^2}{8} e^{4\phi} (\rho + 1) \right\} A^2 \\
& \quad + \left\{ 4e^{-2\phi} - N(\rho + 1) \right\} \lambda^2 e^{2\rho} \right] \tag{31} \end{align*}

Eq. (31) determines \( t_0 \) from the initial conditions for \( \phi, \partial_t \phi, \rho \) and \( \partial_t \rho \). Note that attempt to study quantum cosmology in CGHS-like model (working in s-wave approximation and sigma-model description with ad hoc choice of dilatonic potential) has been done in ref.[7]. In our impression, above approach is better for treating the dynamical dilaton effects. Moreover, it can be applied to wider class of models.

Equations (25), (29) and (31) can be solved numerically for several parameters with the initial condition \( \phi = \rho = \frac{d\phi}{dt} = \frac{d\rho}{dt} = 0 \) at \( t = 0 \). Typical graphs are given in Figs.1–4. We also calculated the two-dimensional scalar curvature \( R \), which is given by
\[ R = 8e^{-2\rho} \partial_+ \partial_- \rho . \]  

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In Fig.1, both the conformal factor $\rho$ and dilaton field $\phi$ increase monotonically with time and there is a curvature singularity in a finite conformal time. We should note that there was not found a solution where $\rho$ monotonically increases with time in the bosonic model [1]. If $\rho$ diverges near the singularity, the singularity will appear in the infinite future of the cosmological time defined by

$$d\hat{t} = e^\rho dt.$$  \hfill (33)

Note $g_{tt} = 1$ when we use the cosmological time $\hat{t}$. It should be also noted that there will not appear any singularities in the two-dimensional scalar curvature in the solutions of Figs.2, 3 and 4. In Fig.2, $\phi$ increases monotonically in time but $\rho$ increases first and decreases later, which means that the universe expands first and then later shrinks. In Fig.3, $\phi$ oscillates and $\rho$ decreases with small oscillation which means that an oscillating shrinking universe is realized. The scalar oscillating curvature goes to zero. The behavior in Fig.3 is very similar to that found in case of the bosonic model reduced from 4 dimensional model with Einstein-Hilbert action [1] if we replace the two-dimensional curvature by the four-dimensional one.

In Fig.4, $\phi$ increases monotonically in time and $\rho$ increases first but decreases later, i.e., the universe expands first and then shrinks. The scalar curvature goes to vanishing. The behavior of solutions when $N = 100$ as in Fig.4 is not sensitive to the value of $\lambda^2$, which will imply that the system would be governed by the naive large-$N$ structure found previously. In the large $N$ limit, Eqs. (23), (24), (25) are rewritten as

\begin{align}
0 &= \frac{1}{2} e^{2\phi} A^2 + \frac{1}{8} \left( \partial_t^2 \rho - (\partial_t \rho)^2 \right) \\
&\quad + \frac{1}{2} \left( \rho + \frac{1}{2} \right) \partial_t \phi \partial_t \phi \\
&\quad - \frac{1}{4} \left\{ -2 \partial_t \rho \partial_t \phi + \partial_t^2 \phi \right\} + t_0 \hfill (34) \\
0 &= -\frac{1}{8} \partial_t^2 \rho - \frac{1}{4} \partial_t \phi \partial_t \phi + \frac{1}{4} \partial_t^2 \phi \hfill (35) \\
0 &= -\frac{1}{2} e^{2\phi} A^2 + \left\{ -\frac{1}{2} \partial_t (\rho \partial_t \phi) - \frac{1}{4} \partial_t^2 \rho \right\} \hfill (36).
\end{align}

Especially in case of $A^2 = 0$, we can delete $\phi$ from Eqs. (34), (33) and (36) and we find

$$(1 + \rho)(\partial_t \rho)^2 - 8t_0 \rho = c^2 \quad \text{(constant)}.$$  \hfill (37)
This tells us that there is a singularity when $\rho = -1$. This singularity corresponds also to a singularity in the scalar curvature $R$ in (32) since

$$R = 8e^{-2\rho} \partial_+ \partial_- \rho \sim \frac{2e^2 (-8t_0 + c^2)}{1 + \rho}$$

(38)

Using a new variable and the parameter

$$\hat{\rho} \equiv (1 + \rho)^{\frac{3}{2}}, \quad E \equiv \frac{9}{8} c^2,$$

(39)

we can rewrite (37) as follows

$$\frac{1}{2} (\partial_t \hat{\rho})^2 - 9t_0 (\hat{\rho}^{\frac{3}{2}} - 1) = E.$$  (40)

Therefore we can regard the system as that of a particle with unit mass in the potential $-9t_0 (\hat{\rho}^{\frac{3}{2}} - 1)$. When $t_0 > 0$, the system becomes unstable and $\hat{\rho}$ and $\rho$ increase monotonically. The $t_0 > 0$ case corresponds to an expanding universe. On the other hand, when $t_0 < 0$, the universe shrinks even if it expands for some initial period and then later the universe “encounters” the curvature singularity at $\rho = -1$.

Before closing this section, we make some remarks about the duality structure of $N = 1$ dilatonic supergravity on bosonic background. The duality might be useful when we consider the so-called “graceful exit” problem [10] for inflationary universe.

We now consider the following simplified classical action:

$$S = -\frac{1}{2\pi} \int d^2 x \sqrt{-g} \left[ e^{-2\phi} \left( R + 4(\nabla \phi)^2 \right) + \frac{1}{2} e^{2\alpha \phi} \sum_{i=1}^{N} (\nabla \chi_i)^2 \right].$$

(41)

Here we introduced a parameter $\alpha$ and we have put $\lambda^2 = 0$. Then the trace anomaly induced effective action [8] is given by

$$W = -\frac{1}{2} \int d^2 x \sqrt{-g} \left[ \frac{N}{32\pi} R \frac{1}{\Delta} R - \frac{N}{4\pi} \alpha^2 \nabla \phi \nabla \phi \frac{1}{\Delta} R + \frac{N}{4\pi} \alpha \phi R \right].$$

(42)

In the conformal gauge, the actions (H) and (H2) take the following forms:

$$S + W = -\frac{1}{2\pi} \int d^2 x \left[ e^{2\phi} \left\{ 4\partial_+ \partial_- \rho - 8\partial_+ \phi \partial_- \phi \right\} \right.$$

$$+ N \left\{ -\frac{1}{4} \rho \partial_+ \partial_- \rho - \alpha^2 \rho \partial_+ \phi \partial_- \phi + \alpha \phi \partial_+ \partial_- \rho \right\} \right].$$

(43)
Here we treat $S + W$ as a classical action and we solve the equations of motion. If we are interested in the case where all the fields only depend on time $t$, we can replace $\partial \mp$ by $\frac{1}{2} \partial_t$. Then the action (43) becomes

$$S + W = -\frac{1}{2\pi} \int d^2x \left[ e^{2\phi} \left\{ \partial_t^2 \rho - 2(\partial_t \phi)^2 \right\} 
+ N \left\{ -\frac{1}{16} \rho \partial_t^2 \rho - \frac{\alpha^2}{4} \rho (\partial_t \phi)^2 + \frac{\alpha}{4} \phi \partial_t^2 \rho \right\} \right].$$  

(44)

Furthermore, we replace the conformal time $t$ by the cosmological time $\hat{t}$ defined in Eq.(33). If we use a new variable $R$ instead of $\rho$

$$r = e^{-2\phi + 2\rho},$$  

(45)

we obtain

$$S + W = -\frac{1}{2\pi} \int d^2x \left[ \dot{r} \dot{\phi} + Ne^{2\phi} \left\{ \frac{1}{4} \dot{r}^2 + \left( \frac{1}{16} - \frac{\alpha}{8} \right) \dot{\phi} \dot{r} + \left( \frac{1}{16} - \frac{\alpha}{4} \right) r \dot{\phi}^2 
- \frac{\alpha^2}{4} \left( \frac{1}{2} \ln r + \phi \right) \dot{\phi}^2 \right\} \right].$$  

(46)

Here $\dot{\phi} = \frac{d\phi}{dt}$ and $\dot{r} = \frac{dr}{dt}$. The variables $\phi$ and $r$ express the classical duality $\phi \leftrightarrow r$. We now consider further redefinitions of the variables. When

$$\alpha^2 = 8$$  

(47)

defining the following new variables

$$U = r + Ne^{2\phi} \left( \frac{1}{4} \ln r + \frac{1}{32} \right)$$

$$V = \phi + Ne^{2\phi} r \left\{ \frac{1}{32} - \frac{\alpha}{8} - \frac{1}{3} \ln r + \frac{1}{3} (1 - 2\phi) \right\}$$

(48)

we obtain

$$S + W = -\frac{1}{2\pi} \int d^2x \dot{U} \dot{V} + \mathcal{O}(N^2).$$  

(49)
This tells that the system has a duality
\[ U \leftrightarrow V \] (50)

when \( N \) is large but \( Nh \) is small. Note that the Planck constant \( \hbar \) is implicitly contained in (49), i.e., \( \hbar \) was chosen to be a unity and \( \mathcal{O}(N^2) \) means \( \mathcal{O}(\hbar^2N^2) \). In case of the bosonic model, similar argument holds and we can find a “perturbative duality” in (50) if we change the coefficients and parameters a little bit, e.g. \( \alpha^2 = \frac{16}{3} \) instead of (17).

## 4 4d Reduced Supergravity Model

We now restrict our discussion to 4D action of Einstein gravity with a cosmological term and matter described by \( N \) minimal scalars \( a_i \) coupled to the metric with spherical symmetry:
\[ ds^2 = g_{\mu\nu}dx^\mu dx^\nu + e^{-2\phi}d\Omega \] (51)

Here the two-dimensional metric and dilaton depend only on time and radius. The spherically reduced action reads
\[ S_{\text{red}} = \int d^2x \sqrt{-g} e^{-2\phi} \left[ -\frac{1}{16\pi G} \left\{ R + 2(\nabla \phi)^2 - 2\Lambda + 2e^{2\phi} \right\} + \frac{1}{2} \sum_{i=1}^{N} (\nabla a_i)^2 \right] . \] (52)

In the 4-dimensional \( N = 1 \) supergravity model, the metric (51) can be realized by choosing the vierbein fields \( e^a_\mu \) as follows
\begin{align*}
  e^0_{\theta,\phi} &= e^3_{\theta,\phi} = e^1_t = e^2_r = 0, \\
  e^1_\theta &= e^2_\phi, \quad e^2_\phi = \sin \theta e^3_\phi, \quad e^1_\phi = e^2_\theta = 0 .
\end{align*}
(53)
The expression (53) is unique up to local Lorentz transformation. The local supersymmetry transformation for the vierbein field with the parameter \( \zeta \) and \( \bar{\zeta} \) is given by,
\[ \delta e^a_\mu = i \left( \psi_\mu \sigma^a \bar{\zeta} - \zeta \sigma^a \bar{\psi}_\mu \right) . \] (54)

Here \( \psi_\mu \) is the Rarita-Schwinger field (gravitino) and we follow the standard notations of ref. (11) (see also (12). If we require that the metric has the form
of (51) after the local supersymmetry transformation, i.e.

\[\delta g_{\theta \theta} = \delta g_{r \theta} = \delta g_{r \phi} = \delta g_{\theta \phi} = 0,\]

\[\delta g_{\varphi \varphi} = \sin \theta \delta g_{\theta \theta}.\]  

(55)

we obtain, up to local Lorentz transformation,

\[\zeta_1 = \bar{\zeta}^1, \quad \zeta_2 = \bar{\zeta}^2,\]  

(56)

\[\psi_{\varphi 1} = \sin \theta \psi_{\theta 1}, \quad \bar{\psi}_{\varphi 1} = \sin \theta \bar{\psi}_{\theta 1},\]

\[\psi_{\varphi 2} = -\sin \theta \psi_{\theta 2}, \quad \bar{\psi}_{\varphi 2} = -\sin \theta \bar{\psi}_{\theta 2},\]

\[\bar{\psi}_{\theta 1} = -\psi_{\theta 1}, \quad \bar{\psi}_{\theta 2} = -\psi_{\theta 2},\]

\[\psi_{r 1} - \bar{\psi}_{r 1} = -2e^{-\phi} e^3 \psi_{\theta 2}, \quad \psi_{t 1} - \bar{\psi}_{t 1} = -2e^{-\phi} e^3 \psi_{\theta 2},\]

\[\psi_{r 2} - \bar{\psi}_{r 2} = -2e^{-\phi} e^0 \psi_{\theta 1}, \quad \psi_{t 2} - \bar{\psi}_{t 2} = -2e^{-\phi} e^0 \psi_{\theta 1}.\]  

(57)

Eq. (56) shows that the local supersymmetry of the spherically reduced theory is \(N = 1\), which should be compared with the torus compactified case, where the supersymmetry becomes \(N = 2\). Let the independent degrees of freedom of the Rarita-Schwinger fields be denoted by

\[2\psi^1_r \equiv \psi_{r 1} + \bar{\psi}_{r 1}, \quad 2\psi^2_r \equiv \psi_{r 2} + \bar{\psi}_{r 2},\]

\[2\psi^1_t \equiv \psi_{t 1} + \bar{\psi}_{t 1}, \quad 2\psi^2_t \equiv \psi_{t 2} + \bar{\psi}_{t 2},\]

(58)

\[\chi_1 \equiv \psi_{\theta 1}, \quad \chi_2 \equiv \psi_{\theta 2},\]  

(59)

then we can regard \(\psi_{t, r}\) and \(\chi\) as the gravitino and the dilatino, respectively, in the spherically reduced theory.

We concentrate on the spherically symmetric metrics of Kantowski-Sacks form [13] in (52) where \(g_{\mu \nu} = a^2(t) \eta_{\mu \nu}\). Such a metric describes a Universe with a \(S^1 \times S^2\) spatial geometry (for some of its properties, see [14]). In the case of dilaton supergravity obtained by reducing of 4d supergravity with chiral scalar supermatter, the action in the bosonic background is given by

\[S = \int d^3 x e \left[ -\frac{e^{-2\phi}}{2\pi} \left\{ R + 2\partial_{\mu} \phi \partial^{\mu} \phi - 2\Lambda^2 + 2e^{2\phi} \right. \right.\]

\[+ \frac{1}{2} \sum_{i=1}^N \left( \partial_{\mu} a_i \partial^{\mu} a_i + \bar{\xi}_i \gamma^{\mu} D_{\mu} \xi_i \right) \left\} \right.\]

\[- \frac{1}{2\pi} \left\{ \frac{N}{32} R \frac{1}{\Box} R - \frac{N}{4} \partial_{\mu} \phi \partial^{\mu} \phi \frac{1}{\Box} R + \frac{N}{4} \frac{1}{\Box} R \right\}.\]  

(60)
Here, the last line term is the anomaly induced effective action due to matter. We obtain the following equations replacing (28), (29) and (31),

\[
\partial^2_t \phi = \left( \frac{5}{2} - \frac{N}{4} e^{2\phi} \right) (\partial_t \phi)^2 - \left( 2 + \frac{N}{4} e^{2\phi} \right) \partial_t \rho \partial_t \phi + \frac{N}{16} e^{2\phi} (\partial_t \rho)^2 - \frac{N}{4} e^{4\phi} A^2 + \lambda^2 e^{2\phi} + e^{2\phi+2\rho} - \frac{N}{2} e^{2\phi} t_0 \tag{61}
\]

\[
\partial^2_t \rho = \left( 1 + \frac{N}{8} e^{2\phi} \right) (\partial_t \rho)^2 - \frac{32}{N} e^{-2\phi} \left( 1 + \frac{N}{8} e^{2\phi} \right) (\partial_t \rho)^2 + 4 \left\{ \left( \frac{1}{4} + \frac{3}{N} e^{-2\phi} \right) - \left( 1 + \frac{N}{8} e^{2\phi} \right) \rho \right\} (\partial_t \phi)^2 - 4 e^{2\phi} \left( 1 + \frac{N}{8} e^{2\phi} \right) A^2 - 8 \left( 1 + \frac{N}{8} e^{2\phi} \right) t_0 + 2 \lambda^2 e^{2\rho}. \tag{62}
\]

\[
t_0 = - \left\{ -16 e^{-2\phi} + 2N + \frac{N^2}{4} e^{2\phi} (\rho + 1) \right\}^{-1} \times \left\{ \frac{24}{N} e^{-4\phi} - (8\rho + 7) e^{-2\phi} - \frac{N}{4} (\rho + 1) + \frac{N^2}{8} e^{2\phi} (\rho + 1) \right\} (\partial_t \phi)^2 + \left\{ -\frac{64}{N} e^{-4\phi} + N \left( \rho + \frac{3}{2} \right) + \frac{N^2}{8} e^{2\phi} (\rho + 1) \right\} \partial_t \rho \partial_t \phi + \left\{ 2 e^{-2\phi} - \frac{N}{4} - \frac{N^2}{32} e^{2\phi} (\rho + 1) \right\} (\partial_t \rho)^2 + \left\{ -8 + \frac{N}{2} e^{2\phi} + \frac{N^2}{8} e^{4\phi} (\rho + 1) \right\} A^2 + \left\{ 4 e^{-2\phi} - N(\rho + 1) \right\} \frac{\lambda^2}{2} e^{2\rho} - \left( 4 e^{-2\phi} + \frac{N}{2} \right) \right\} \right. \tag{63}
\]

Equations (61), (62) and (63) can be also solved numerically for several parameters with the initial condition \( \phi = \rho = \frac{d\phi}{dt} = \frac{d\rho}{dt} = 0 \) at \( t = 0 \). Typical graphs are given in Figs.5 and 6. We calculated the four-dimensional scalar curvature corresponding to the metric (51) which is given by

\[
R_{4d} = -e^{-2\rho} \left\{ 2 \partial^2_t \rho - 4 \partial^2_t \phi + 6 (\partial_t \phi)^2 \right\}. \tag{64}
\]

In the 4d reduced model, the curvature singularity always seems to appear, at least in cases under discussion. In Fig.5, both the dilaton field \( \phi \) and the
conformal factor $\rho$ increase monotonically in time as in Fig.1 in CGHS type model. If we regard the solution as describing a universe of the Kantowski-Sacks form with the topology $S^2 \times S^1$, the radius of $S^2$, which is given by $e^{-\phi}$, decreases to zero but the radius of $S^1$, which is given by $e^{2\rho}$, increases. Note that the radius of $S^1$ corresponds to the radius of the universe when we regard this model as a two dimensional one. The four-dimensional scalar curvature increases first and goes to minus infinity. In Fig.6, $\phi$ increases monotonically and $\rho$ decreases at first but increases infinitely after that. This means $S^2$ shrinks but the $S^1$ factor shrinks at first and later expands. The four-dimensional scalar curvature goes to zero at first but increases to infinity later. The dilaton field $\phi$ always increases and runs away to the singularity, which means $S_2$ in the universe of Kantowski-Sacks form shrinks to a point, which causes the curvature singularity. When we regard the model as a two-dimensional one, the radius of the universe goes to infinity at the final stage in the cases studied here. We see that at $t = 0$ there is no singularity in most of our cases. However, at late times there is as a rule singularity. Note that the behaviour of our cosmologies at late times is not so important as other effects should define late time structure of the Universe.

5 \textbf{N = 2 Dilaton Supergravity Model}

In this section, we discuss $N = 2$ dilaton supergravity. Notations and conventions are mainly following to [3]. However, the possibility of considering a purely twisted matter potential term point out in [3] is not treated. The general classical action of $U_A(1)$ $N = 2$ dilaton supergravity with matter is given by\footnote{We use $N = 1$ or $N = 2$ to denote type of SG but also we use $N$ as number of matter multiplets which should not cause confusion in the context used.}

$$S = \frac{1}{2\pi} \int d^2x d^2\theta e^{-1}C(\Phi)R + \frac{1}{2} \int d^2x d^4\theta e^{-1}Z(\Phi)Z(\bar{\Phi}) + \int d^2x d^2\theta e^{-1}V(\Phi) + \frac{1}{2} \int d^2x d^4\theta e^{-1}g(\Phi) \sum_{i=1}^{M} \Sigma_i \Sigma_i + \int d^2x d^4\theta e^{-1}f(\Phi) \sum_{i=1}^{N} \bar{\chi}_i \chi_i + h.c.$$
\[
\frac{1}{2\pi} \int d^2x e^{-1} \left[ C'(\phi)FB + C'(\bar{\phi})\bar{F}\bar{B} + 2(C(\phi) + C(\bar{\phi}))(\bar{B}\bar{B} - \mathcal{R}) + 2i(C(\phi) - C(\bar{\phi}))F \\
+ Z'(\phi)Z'(\bar{\phi})(-F\bar{F} - \partial_+ \phi \partial_- \bar{\phi}) + (V'(\phi)F + V'(\bar{\phi})\bar{F}) \\
- \sum_{i=1}^M \left\{ (g'(\phi)Fa_i + g(\phi)G_i)\bar{G}_i + G_i(g'(\bar{\phi})\bar{F}\bar{a}_i + g(\bar{\phi})\bar{G}_i) \\
+ \partial_+(g(\phi)a_i)\partial_-\bar{a}_i + \partial_+a_i\partial_-g(\bar{\phi})\bar{a}_i \right\} \\
-(f(\phi) + f(\bar{\phi})) \sum_{i=1}^N \left\{ \bar{H}_iH_i - \frac{1}{2} (\partial_-\bar{\chi}_i\partial_+\chi_i + \partial_+\bar{\chi}_i\partial_-\chi_i) \right\} \right] \\
+ \text{fermionic terms}
\]  

(65)

Here \( \Phi = (\phi, \chi, F) \), \( \Sigma_i = (a_i, \xi_i, G_i) \) and \( \mathcal{X} = (\chi_i, \zeta_i, H_i) \) are the dilaton, chiral matter and twisted chiral matter multiplets, respectively, \( \mathcal{R} \) is the scalar curvature and \( F = \partial_0 A_1 - \partial_1 A_0 \) is the field strength of the vector field \( A_\mu \). The action (65) shows that the imaginary part of \( C(\phi) \) can be identified with the axion field. Integrating out \( A_\mu \), we find

\[ C(\phi) - C(\bar{\phi}) = c \text{ (constant)} \]  

(66)

If we consider the solution where \( c = 0 \), the dilaton field becomes real

\[ \phi = \bar{\phi} . \]  

(67)

If we consider the case of \( N \gg M, 1 \), the bosonic part of the effective action would be given by

\[ W = -\frac{1}{2} \int d^2x \sqrt{g} \left[ \frac{N}{16\pi} \left( R \frac{1}{\Box} R - F \frac{1}{\Box} F \right) - \frac{N}{2\pi} \nabla^\lambda \varphi \nabla_\lambda \varphi \frac{1}{\Box} R + \frac{N}{2\pi} \varphi R \right] . \]  

(68)

Here

\[ \varphi \equiv -\frac{1}{2} \ln \left( \frac{f(\phi) + f(\bar{\phi})}{2} \right) \]  

(69)

and we used that the action of the fermion component of \( \mathcal{X}_i \) in the bosonic background (65) has the following form

\[ S_\zeta = i \int d^2x e^{-1}(f(\phi) + f(\bar{\phi})) \sum_{i=1}^N \left\{ (\partial_+\zeta_-)\zeta_- + (\partial_-\zeta_+)\zeta_+ \\
+ \frac{i}{2}(A_+\zeta_-\zeta_- - A_-\zeta_+\zeta_+) \right\} \]  

(70)
The action (68) is obtained by replacing $N$ by $2N$ in the $N = 1$ supergravity effective action and adding the second term coming from chiral anomaly. This is because the dilaton dependent function $(f(\phi) + f(\bar{\phi}))$ in front of the matter fermion kinetic term in Eq.(70) can be absorbed into the redefinition of $\zeta$ and the matter fermions do not give any contribution to the dilaton dependent terms of effective action. The contribution of the energy momentum tensor from the matter $\zeta_i$ fermions is, as in $N = 1$ case, absorbed into the redefinition of $t^0$. Therefore the behavior of the cosmological solution is almost identical with the $N = 1$ case in the bosonic background with $F = 0$.

We now consider the case where $F \neq 0$. For this case, we parametrize the vector field $A_\mu$ as follows:

$$A_\mu = \partial_\mu \tilde{\theta} + \frac{16}{N} g_{\mu\nu} \epsilon^{\nu\rho} \partial_\rho \theta .$$

(71)

Note that $\tilde{\theta}$ expresses the gauge degree of freedom. Then we can rewrite the chiral anomaly term in Eq.(68) in a local form,

$$\frac{N}{32\pi} \int d^2x \sqrt{g} F^1 \frac{1}{\Delta} F = \frac{8}{N\pi} \int d^2x \sqrt{\theta} \partial \theta .$$

(72)

The $\theta$-equation of motion obtained from $S + W$ has the following form

$$0 = \Box \left\{ 2i(C(\phi) - C(\bar{\phi})) + \theta \right\}$$

(73)

that is,

$$\theta = -2i(C(\phi) - C(\bar{\phi})) + \text{constant} .$$

(74)

Now we consider the case,

$$C(\phi) = -\frac{1}{4} e^{-2\phi}, \quad Z(\phi) = 4e^{-\phi}, \quad V(\phi) = \Lambda e^{-2\phi}, \quad f(\phi) = e^{-2\phi}$$

(75)

and the background where the chiral matter multiplet $\Sigma_i$ vanishes. Then integrating the auxiliary fields $B, \bar{B}, F$ and $\bar{F}$ and using (74), we can rewrite the bosonic part of $S + W$ as follows:

$$S + W = \frac{1}{2\pi} \int d^2 x e^{-1} \left[ -e^{-2\phi} R - \frac{1}{\sqrt{e^{-4\phi} + \theta^2}} \left\{ 4e^{-4\phi} \partial_\mu \varphi \partial^\mu \varphi + \partial_\mu \theta \partial^\mu \theta \right\} + \frac{4\Lambda^2(e^{-4\phi} + \theta^2)}{-\frac{1}{4}(e^{-4\phi} + \theta^2)e^{2\phi} + 16e^{-4\phi} + \theta^2} - \frac{16}{N} \theta \Box \theta + 2e^{-2\phi} \sum_{i=1}^{\infty} \partial_\mu \bar{\chi}_i \partial^\mu \chi_i \right]$$

$$- \frac{1}{2} \int d^2 x \sqrt{g} \left[ \frac{N}{16\pi} R \frac{1}{\Box} R - \frac{N}{2\pi} \nabla^\lambda \varphi \nabla_\lambda \varphi \frac{1}{\Box} R + \frac{N}{2\pi} \varphi R \right].$$

(76)
Here $\theta$ should be understood not to represent the degrees of freedom of the vector field but to be the imaginary part of $e^{-2\varphi}$ using (74), i.e. to be the axion field. Then the equations corresponding to (23) – (26) become \((T\text{ is absorbed in the redefinition of } t_0)\)

\[
0 = 4e^{-2\varphi}\partial_t \rho \partial_t \varphi - \frac{1}{2\sqrt{e^{-4\varphi} + \theta^2}} \left(4e^{-4\varphi}(\partial_t \varphi)^2 + (\partial_t \theta)^2\right) - \frac{4}{N}(\partial_t \theta)^2 \\
+ \frac{1}{2}e^{-2\varphi} \sum_{i=1}^{N} \partial_t \chi_i \partial_t \bar{\chi}_i + \frac{N}{4} \left(\partial_t^2 \rho - (\partial_t \rho)^2\right) + N \left(\rho + \frac{1}{2}\right)(\partial_t \varphi)^2 \\
- \frac{N}{2} \left\{ -2\partial_t \rho \partial_t \varphi + \partial_t^2 \varphi \right\} + 2N t_0
\]

\[
0 = e^{-2\varphi} \left(2\partial_t^2 \varphi - 4(\partial_t \varphi)^2\right) - \frac{4\Lambda^2(e^{-4\varphi} + \theta^2)e^{2\rho}}{4(e^{-4\varphi} + \theta^2)^{\frac{3}{2}}} + 16(e^{-4\varphi} + \theta^2)^{\frac{1}{2}} \\
- \frac{N}{4} \partial_t^2 \rho - \frac{N}{2}(\partial_t \varphi)^2 + \frac{N}{2} \partial_t^2 \varphi
\]

\[
0 = \frac{-\frac{4}{4(e^{-4\varphi} + \theta^2)^{\frac{3}{2}}}(\partial_t \varphi)^2 + \frac{4e^{-4\varphi}}{4(e^{-4\varphi} + \theta^2)^{\frac{3}{2}}} \partial_t^2 \varphi}{e^{-4\varphi} + \theta^2)^{\frac{4}{2}}} - \frac{4\theta e^{-4\varphi}}{(e^{-4\varphi} + \theta^2)^{\frac{3}{2}}} \partial_t \theta \partial_t \varphi + 2e^{-2\varphi} \partial_t^2 \rho \\
- 2\Lambda^2 e^{2\rho} \left\{ \frac{(e^{-4\varphi} + \theta^2)e^{2\varphi}}{4(e^{-4\varphi} + \theta^2)^{\frac{3}{2}}} + 16(e^{-4\varphi} + \theta^2)^{\frac{1}{2}} \right\}^{-2} \\
\times \left\{ \frac{1}{2}(e^{-4\varphi} + \theta^2)^{\frac{3}{2}}e^{2\varphi} - 32(e^{-4\varphi} + \theta^2)^{\frac{1}{2}}e^{-4\varphi} \right\} \\
- \frac{1}{2}e^{-2\varphi} \sum_{i=1}^{N} \partial_t \chi_i \partial_t \bar{\chi}_i + \left\{ -N \partial_t(\rho \partial_t \varphi) - \frac{N}{2}\partial_t^2 \rho \right\}
\]

\[
0 = \frac{\theta}{(e^{-4\varphi} + \theta^2)^{\frac{3}{2}}}(\partial_t \theta)^2 + \left\{ \frac{4}{(e^{-4\varphi} + \theta^2)^{\frac{3}{2}}} - \frac{8}{N}\right\} \partial_t^2 \theta \\
+ \frac{2\theta e^{-4\varphi}}{(e^{-4\varphi} + \theta^2)^{\frac{3}{2}}}(\partial_t \varphi)^2 + \frac{2e^{-4\varphi}}{(e^{-4\varphi} + \theta^2)^{\frac{3}{2}}} \partial_t \varphi \partial_t \theta \\
- 32\Lambda^2 e^{2\rho} \theta(e^{-4\varphi} + \theta^2)^{\frac{1}{2}} \left\{ \frac{(e^{-4\varphi} + \theta^2)e^{2\varphi}}{4(e^{-4\varphi} + \theta^2)^{\frac{3}{2}}} + 16(e^{-4\varphi} + \theta^2)^{\frac{1}{2}} \right\}^{-2}
\]

\[
0 = \partial_t(e^{-2\varphi} \partial_t \chi_i) = \partial_t(e^{-2\varphi} \partial_t \bar{\chi}_i).
\]

Eq. (81) can be also integrated to be $\partial_t \chi_i = A_i e^{2\varphi}$, $\partial_t \bar{\chi}_i = \bar{A}_i e^{-2\varphi}$. Defining
$A^2 = \frac{1}{2N} \sum_{i=1}^{N} A_i \bar{A}_i$, Eqs. (77), (78), (79) and (80) are rewritten as follows,

$$
\partial_t^2 \varphi = -2 \left( 1 + \frac{N}{4} e^{2\varphi} \right) \partial_t \rho \partial_t \varphi + \left( \frac{e^{-2\varphi}}{(e^{-4\varphi} + \theta^2)^{\frac{3}{2}}} + 2 - \frac{N}{2} \rho e^{2\varphi} \right) (\partial_t \varphi)^2
+ \left( \frac{1}{4(e^{-4\varphi} + \theta^2)^{\frac{3}{2}}} + \frac{2}{N} \right) e^{2\varphi} (\partial_t \varphi)^2 + \frac{N}{8} e^{2\varphi} (\partial_t \rho)^2
- \frac{N}{2} e^{4\varphi} A^2 + \frac{2\Lambda^2 (e^{-4\varphi} + \theta^2) e^{2\rho + 2\varphi}}{(e^{-4\varphi} + \theta^2)^{\frac{3}{2}}} - N t_0 e^{2\varphi} \quad (82)
$$

$$
\partial_t^2 \rho = -\frac{16}{N} \left( 1 + \frac{N}{4} e^{2\varphi} \right)^2 e^{-2\varphi} \partial_t \rho \partial_t \varphi
+ \left\{ 2 + \frac{8}{N} e^{-2\varphi} \left( 1 + \frac{N}{4} e^{2\varphi} \right) \left( \frac{e^{-2\varphi}}{(e^{-4\varphi} + \theta^2)^{\frac{3}{2}}} - \frac{N}{2} \rho e^{2\varphi} \right) \right\} (\partial_t \varphi)^2
+ \frac{8}{N} \left( 1 + \frac{N}{4} e^{2\varphi} \right) \left( \frac{1}{4(e^{-4\varphi} + \theta^2)^{\frac{3}{2}}} + \frac{2}{N} \right) (\partial_t \theta)^2
+ \left( 1 + \frac{N}{4} e^{2\varphi} \right) (\partial_t \rho)^2 - 4e^{2\varphi} \left( 1 + \frac{N}{4} e^{2\varphi} \right) A^2
+ \frac{4\Lambda^2 e^{2\rho + 2\varphi} (e^{-4\varphi} + \theta^2)}{(e^{-4\varphi} + \theta^2)^{\frac{3}{2}}} - 16(e^{-4\varphi} + \theta^2)^{\frac{3}{2}}
- 8 \left( 1 + \frac{N}{4} e^{2\varphi} \right) t_0 \quad (83)
$$

$$
\partial_t^2 \theta = \left\{ -\frac{4}{(e^{-4\varphi} + \theta^2)^{\frac{3}{2}}} + \frac{8}{N} \right\}^{-1} \left[ \frac{\theta}{(e^{-4\varphi} + \theta^2)^{\frac{3}{2}}} (\partial_t \theta)^2
+ \frac{2\theta e^{-4\varphi}}{(e^{-4\varphi} + \theta^2)^{\frac{3}{2}}} (\partial_t \varphi)^2 + \frac{2 e^{-4\varphi}}{(e^{-4\varphi} + \theta^2)^{\frac{3}{2}}} \partial_t \varphi \partial_t \theta \right]
- 32\Lambda^2 e^{2\rho} \theta e^{-4\varphi} + \theta^2 \left\{ -\frac{(e^{-4\varphi} + \theta^2) e^{2\varphi}}{4} + 16(e^{-4\varphi} + \theta^2)^{\frac{3}{2}} \right\}^{-2} \quad (84)
$$

$$
t_0 = \left\{ \frac{4Ne^{-2\varphi}}{(e^{-4\varphi} + \theta^2)^{\frac{3}{2}}} + 16e^{-2\varphi} \left( 1 - \frac{N^2}{16} e^{4\varphi} \right) - N^2 e^{2\varphi} \rho \right\}^{-1}
\times \left\{ -\frac{4(e^{-4\varphi} + 2\theta^2) e^{-4\varphi}}{(e^{-4\varphi} + \theta^2)^{\frac{3}{2}}} + \frac{4e^{-6\varphi}}{e^{-4\varphi} + \theta^2} + 4e^{-2\varphi} - N \right\} \quad (84)
$$
\begin{align*}
&+ \frac{8e^{-4\varphi} + \frac{16}{N} e^{-6\varphi} \left(1 - \frac{N^2}{16} e^{4\varphi}\right)}{(e^{-4\varphi} + \theta^2)^{\frac{3}{2}}} \\
&+ \left\{ - \frac{3N e^{-2\varphi}}{(e^{-4\varphi} + \theta^2)^{\frac{3}{2}}} - 2N - 8e^{-2\varphi} \left(1 - \frac{N^2}{16} e^{4\varphi}\right) \right\} \rho \\
&+ \frac{N^2}{2} e^{2\varphi} \rho^2 \right\} (\partial_t \varphi)^2 \\
&+ \left\{ - \frac{8e^{-4\varphi}}{(e^{-4\varphi} + \theta^2)^{\frac{3}{2}}} \left(1 + \frac{N}{4} e^{2\varphi}\right) - N \\
- \frac{32}{N} e^{-4\varphi} \left(1 - \frac{N^2}{16} e^{4\varphi}\right) \left(1 + \frac{N}{4} e^{2\varphi}\right)^2 \\
+ 2N \left(1 + \frac{N}{4} e^{2\varphi}\right) \rho \right\} \partial_t \rho \partial_t \varphi \\
&+ \left\{ \frac{N e^{-2\varphi}}{2(e^{-4\varphi} + \theta^2)^{\frac{3}{2}}} + 2e^{-2\varphi} \left(1 - \frac{N^2}{16} e^{4\varphi}\right) - \frac{N^2}{8} e^{2\varphi} \rho \right\} (\partial_t \rho)^2 \\
&+ \left\{ - \frac{e^{-4\varphi}}{(e^{-4\varphi} + \theta^2)^{\frac{3}{2}}} + \frac{e^{-2\varphi}}{(e^{-4\varphi} + \theta^2)^{\frac{3}{2}}} + \frac{12 N e^{-2\varphi} - \frac{N}{4} e^{2\varphi}}{(e^{-4\varphi} + \theta^2)^{\frac{3}{2}}} \\
+ \frac{32}{N^2} e^{-2\varphi} \left(1 - \frac{N^2}{16} e^{4\varphi}\right) - N e^{2\varphi} \left(\frac{1}{4(e^{-4\varphi} + \theta^2)^{\frac{3}{2}}} + \frac{2}{N} \rho\right) \right\} (\partial_t \theta)^2 \\
&- \frac{4\theta e^{-4\varphi}}{(e^{-4\varphi} + \theta^2)^{\frac{3}{2}}} \partial_t \theta \partial_t \varphi \\
&- 2\Lambda^2 e^{2\varphi} \left(\frac{e^{-4\varphi} + \theta^2)^{\frac{3}{2}}}{2} - \frac{32}{N^2} e^{4\varphi}\right) \\
&- \frac{2\Lambda^2 e^{2\varphi + 2\varphi} (e^{-4\varphi} + \theta^2)^{\frac{3}{2}}}{4} + 16(e^{-4\varphi} + \theta^2)^{\frac{3}{2}} \\
&+ 2\Lambda^2 e^{2\varphi + 2\varphi} (e^{-4\varphi} + \theta^2) \left(\frac{e^{-4\varphi}}{(e^{-4\varphi} + \theta^2)^{\frac{3}{2}}} - N \rho + 4e^{-2\varphi} - N\right) \\
&+ \left\{ - \frac{2N}{(e^{-4\varphi} + \theta^2)^{\frac{3}{2}}} + \frac{N^2}{2} e^{4\varphi} \rho - 8 \left(1 - \frac{N^2}{16} e^{4\varphi}\right) - Ne^{2\varphi} \right\} \right\} A^2 \\.
\end{align*}

Equations (82), (83), (84) and (85) can be also solved numerically for several parameters with the initial condition \( \varphi = \rho = \frac{d\varphi}{dt} = \frac{d\rho}{dt} = 0 \), \( \theta = 1 \) and \( \frac{d\theta}{dt} = 1.1 \) at \( t = 0 \). Typical graphs are given in Figs.7 – 10. In most of cases, \( \theta \) increases linearly with respect to the conformal time \( t \). As in the 4d reduced
model of Figs.5 and 6, the singularity always seems to appear, at least in the cases under discussion. As a specific feature in the $N = 2$ model, there is a case (Fig.10) where the dilaton field $\varphi$ goes to minus infinity. Such a case has not been found in the $N = 1$ model of Figs.1 - 6 but was found in the bosonic model [1]. When $\lambda^2 < 0$ in the bosonic model, the dilaton field $\varphi$ goes to minus infinity but note that $\lambda^2$ or $\Lambda^2$ is always positive or zero in the supersymmetric model. If we regard this $N = 2$ model as a four dimensional model with the topology $S^2 \times S^1$ of the Kantowski-Sacks form, the radius of $S^2$, which would be given by $e^{-\varphi}$, goes to infinity.

In Fig.7, the dilaton field $\varphi$ increases monotonically in time and goes to infinity, the conformal factor $\rho$ decreases at first but increases after that and goes to infinity and $\theta$ increases linearly in the conformal time $t$. In Fig.8, $\varphi$ increases monotonically in time and goes to infinity, $\rho$ increases at first but decreases after that and goes to minus infinity and $\theta$ increases linearly in the conformal time $t$. Fig.9 is the case that $\theta$ increases monotonically in time but not linearly with respect to the conformal time $t$. In Fig.9, both of $\varphi$ and $\rho$ decrease at first and increase after that and go to infinity. This case should also be compared with that of the $N = 1$ models, in most these the dilaton field $\varphi$ increases monotonically in time except in the case of Fig.3. Even in Fig.3, $\varphi$ cannot be negative. In Fig.10, the dilaton field $\varphi$ oscillates a little and finally goes to minus infinity, the conformal factor $\rho$ decreases monotonically and goes to minus infinity and $\theta$ increases linearly in the conformal time $t$. In the cases of Figs.7 and 9, the radius of the two dimensional universe, which is given by $e^{\rho}$, goes to infinity, that is, the universe expands. On the other hand, in the cases of Figs.8 and 10, the two dimensional universe shrinks to a point. It is, however, not clear what determines the fate of the two dimensional universe since it seems that there is no any clear correlation between the behavior of $\rho$ and the “cosmological constant” $\Lambda^2$ and/or the matter energy density $A^2$. We again see that at $t = 0$ there is no singularity in most of our cases. However, singularity appears in course of evolution. We again may argue that presented effects are getting negligible at late times where quantum corrections almost play no role.
6 String inspired model in 4 dimensions

The low-energy effective action of the dilaton-gravity sector in the superstring would be given by

$$S = \int d^4x \sqrt{-g} e^{-\phi - \phi^*} (R - 6 \nabla_\mu \phi \nabla^\mu \phi^*) + \cdots . \quad (86)$$

Here $\cdots$ denotes the terms containing moduli, gauge fields, which depend on the detail of the compactification, and fermionic fields. A special 4d dilatonic supergravity coming from string theories has been considered in [15] (recently it has been independently rediscovered in [16]). This model specifically (and is expected to be most directly related to superstring and heterotic strings) uses a real linear multiplet instead of the more familiar chiral multiplet description that follows. We now consider the system where the massless matter fields couple with the dilaton gravity system (86).

We first consider the case that the matter does not couple with the dilaton. In this case, the action of the matter field (WZ action) is given by

$$L = \int d^2\Theta^2 \mathcal{E} \left[ -\frac{1}{8} (\overline{D}D - 8R)\Phi \Phi^* + \text{h.c.} \right]. \quad (87)$$

Here we used the notations of [11]. Then the bosonic part of the action is given by

$$L^b = \frac{1}{6} eAA^* R - e\partial_m A\partial^m A$$

$$+ \frac{1}{9} (MA^* - 3F^*)(M^* A - 3F) - \frac{1}{9} eAA^* b^a b^a$$

$$- \frac{i}{3}(A^* \partial_m A - A\partial_m A^*) b^m . \quad (88)$$

Note that the coefficient $\frac{1}{6}$ in the first term, which assures the conformal invariance. The action (88) is, in fact, invariant under the following (local) conformal transfromation with the parameter $\sigma$:

$$g_{mn} \to g_{mn} e^{2\sigma}$$

$$(e_m^a \to e_m^a e^{\sigma})$$

$$M \to M e^{-\sigma}$$

$$b^m \to b^m e^{-\sigma}$$

$$A \to A e^{-\sigma}$$

$$F \to F e^{-2\sigma} . \quad (89)$$
The terms containing fermion fields are given by
\[ L_f = -\frac{i}{2} e \left[ \chi \sigma^m \partial_m \bar{\chi} + \bar{\chi} \bar{\sigma}^m \partial_m \chi \right] + \text{terms containing gravitino} \quad (90) \]

Here
\[
\begin{align*}
D_m \bar{\chi} &= \left( \partial_m - \bar{\sigma}^m \omega_{mnl} \right) \bar{\chi} \\
D_m \chi &= \left( \partial_m - \sigma^m \omega_{mnl} \right) \chi \\
\omega_{mnl} &= \frac{1}{2} \left\{ -e_{la} \left( \partial_n e^a_m - \partial_m e^a_n \right) - e_{ma} \left( \partial_l e^a_n - \partial_n e^a_l \right) \\
&\quad + e_{na} \left( \partial_m e^a_l - \partial_l e^a_m \right) \right\} \\
\bar{\sigma}^{mn} &= \frac{1}{4} \left( \bar{\sigma}^m \sigma^n - \bar{\sigma}^n \sigma^m \right) \\
\sigma^{mn} &= \frac{1}{4} \left( \sigma^m \bar{\sigma}^n - \bar{\sigma}^n \sigma^m \right) \quad (91)
\end{align*}
\]

Under the conformal transformation (89), \( \omega_{mnl} \) transform as
\[ \omega_{nml} \to e^{2\sigma} \left( \omega_{mnl} + g_{nt} \partial_n \sigma - g_{nm} \partial_l \sigma \right) \quad (92) \]

Then under the conformal transformation (89) and
\[ \chi \to \chi e^{-\frac{3}{2}\sigma} \quad (93) \]

the action (90) transform as
\[ L_f \to L_f + \delta L_f \]
\[ \delta L_f \propto (\chi \sigma^m \bar{\chi} + \bar{\chi} \bar{\sigma}^m \chi) \partial_m \sigma \quad (94) \]

Since \( \chi \sigma^n \bar{\psi} = -\bar{\psi} \bar{\sigma}^n \chi \), \( \delta L_f \) vanishes and \( L_f \) is invariant under the conformal transformation.

The conformal invariance holds if we introduce dilaton (chiral) multiplet \( \Psi \) whose first component \( \phi \) is invariant under the conformal transformation (89) and the last component \( G \) is transformed as
\[
\begin{align*}
\phi &\to \phi \\
G &\to Ge^{-\phi} \quad (95)
\end{align*}
\]
as follows

\[
\mathcal{L} = \int d^2 \Theta^2 2 \mathcal{E} \left[ \frac{1}{16} (\mathcal{D} \mathcal{D} - 8 R)(\Phi^+(f(\Psi)\Phi) + (f(\Psi^+)\Phi^+)) \right]
\]

\[
\begin{align*}
&= \frac{1}{12} e (f(\phi) + f(\phi^*)) AA^* \mathcal{R} \\
&- \frac{1}{2} e (\partial_m (f(\phi) A) \partial^m A^* + \partial_m A \partial^m (f(\phi^*) A^*)) \\
&- \frac{i}{4} e (f(\phi) + f(\phi^*)) [\chi \sigma^m \mathcal{D}_m \bar{\chi} + \bar{\chi} \bar{\sigma}^m \mathcal{D}_m \chi] \\
&+ \frac{1}{18} \{(MA^* - 3f(\phi^*) F^* - 3f'(\phi^*) G^* A^*)(M^* A - 3F) \\
&+ (MA^* - 3F^*)(M^* A - 3f(\phi) F - 3f'(\phi) GA)\} \\
&- \frac{1}{18} e (f(\phi) + f(\phi^*)) AA^* b_a b^a \\
&- \frac{i}{6} \{(A^* \partial_m (f(\phi) A) - f(\phi) A \partial_m A^*)b^m \}
\end{align*}
\]

(96)

Here we put the fermionic component of \( \Psi \) to vanish since we now regard \( \Psi \) as a background.

If we consider the background where the dilaton field \( \phi \) is real, the non-local effective action \( W \) induced by the conformal anomaly of the dilaton coupled matter fields has been calculated in ref.[17] and is given by:

\[
W = b \int d^4 x \sqrt{-g} F \sigma \\
+ b' \int d^4 x \sqrt{-g} \left\{ \sigma \left[ 2 \Box^2 + 4 R^\mu \nabla_\mu \nabla_\nu - \frac{4}{3} R \Box + \frac{2}{3} (\nabla^\mu R) \nabla_\mu \right] \sigma \right. \\
+ \left( G - \frac{2}{3} \Box R \right) \sigma \right} \\
- \frac{1}{12} \left[ \left( b'' + \frac{2}{3} (b + b') \right) \int d^4 x \sqrt{-g} \left[ R - 6 \Box \sigma - 6 (\nabla \sigma)(\nabla \sigma) \right]^2 \right. \\
+ \int d^4 x \sqrt{-g} \left\{ \frac{a_1}{8} \left[ (\nabla \varphi)(\nabla \varphi) \right]^2 \sigma + \frac{a_2}{4} (\nabla \varphi)(\nabla \varphi) \sigma \right. \\
+ \left. \frac{a_2}{4} (\nabla \varphi)(\nabla \varphi) \left[ (\nabla \sigma)(\nabla \sigma) \right] \right\}
\]

(97)

Here the \( \sigma \)-independent terms are dropped, the last terms represent the contribution from the dilaton dependent terms in trace anomaly. The dilaton
dependent function $\varphi$ is defined by

$$\varphi = -\frac{1}{2} \ln \left( \frac{f(\phi) + f(\phi^*)}{2} \right)$$

(98)

The coefficients are given by

$$b = \frac{1}{120(4\pi)^2} \times 4N, \quad b' = -\frac{1}{360(4\pi)^2} \times \frac{13N}{2},$$

$$a_1 = \frac{1}{32(4\pi)^2} \times 2N, \quad a_2 = \frac{1}{24(4\pi)^2} \times 2N.$$  

(99)

The coefficient $b''$ can be changed by a finite renormalization of local counterterm and we can put $b'' = 0$ (as is obtained also from direct calculation).

We consider below $N$ WZ multiplets and apply the large $N$ expansion which is why all the coefficients above have the multiplier $N$. In the following, we only consider the case where $f(\phi)$ is given by

$$f(\phi) = e^{-2\phi}$$

(100)

and we assume $\phi$ is real and that the conformally flat fiducial metric is $g_{\mu\nu} = e^{2\sigma} \eta_{\mu\nu}$. Then the total action has the following form:

$$S = W + S_{cl}$$

$$= \int d^4x \left\{ 2b' (\Box \sigma)^2 - [3b'' + 2(b + b')] \Box \sigma + (\partial_\mu \sigma)^2 \right\}$$

$$+ \frac{a_1}{8} [\partial_\mu \varphi \partial^\mu \varphi]^2 \sigma + \frac{a_2}{4} \Box (\partial_\mu \varphi \partial^\mu \varphi) \sigma$$

$$+ \frac{a_2}{4} \partial_\mu \varphi \partial^\mu \varphi \partial_\nu \sigma \partial^\nu \sigma$$

$$- 6e^{4\sigma/2} \partial_\mu \varphi \partial^\mu \varphi + e^{2\sigma/2} \left\{ -6 \Box \sigma - 6 (\partial_\mu \sigma)(\partial^\mu \sigma) \right\}.$$

(101)

The equations of the motion given by the variation of $\sigma$ and $\phi$ are given by

$$\Box^2 \sigma = 4b' \Box \sigma$$

$$-2[3b'' + 2(b + b')] \left[ \Box \left( \Box \sigma + (\partial_\mu \sigma)^2 \right) - 2\partial_\mu \left\{ \partial^\nu \sigma \left( \Box \sigma + (\partial_\mu \sigma)^2 \right) \right\} \right]$$

$$+ \frac{a_1}{8} [\partial_\mu \varphi \partial^\mu \varphi]^2 + \frac{a_2}{4} \Box (\partial_\mu \varphi \partial^\mu \varphi)$$

$$- \frac{a_2}{2} \partial_\nu (\partial^\nu \sigma \partial_\mu \varphi \partial^\mu \varphi)$$
\[-24e^{4\sigma-2\varphi}\partial_\mu\varphi\partial^\mu\varphi + 2e^{2\sigma-2\varphi}[-6\Box\sigma - 6(\partial_\mu\sigma)(\partial^\mu\sigma)] \]
\[-6\Box e^{2\sigma-2\varphi} + 12\partial_\mu(e^{2\sigma-2\varphi}\partial^\mu\sigma) . \] (102)

\[0 = \frac{a_1}{2} \partial_\nu\{\partial^\nu\varphi[\partial_\mu\varphi\partial^\mu\varphi]\sigma\} - \frac{a_2}{2} \partial_\nu(\partial^\nu\varphi\Box\sigma) - \frac{a_2}{2} \partial_\mu(\partial^\mu\varphi\nabla_\nu\sigma\nabla^\nu\sigma) + 12\partial_\mu(e^{4\sigma-2\varphi}\partial^\mu\varphi) + 12e^{4\sigma-2\varphi}\partial_\mu\varphi\partial^\mu\varphi - 2e^{2\sigma-2\varphi}[-6\Box\sigma - 6(\partial_\mu\sigma)(\partial^\mu\sigma)] . \] (103)

Since we are interested in cosmological problems, we only consider the uniform solution, where all the fields depend only on (conformal) time \(t\) and we obtain

\[0 = 4b'd^4\sigma dt^4 \]
\[-2[3b'' + 2(b + b')]\left[\frac{d^2\sigma}{dt^2} + \left(\frac{d\sigma}{dt}\right)^2\right] - 2\frac{d}{dt}\left(\frac{d\sigma}{dt} - \left(\frac{d\sigma}{dt}\right)^2\right) \]
\[+ \frac{a_1}{8} \left(\frac{d\varphi}{dt}\right)^4 + \frac{a_2}{4} \frac{d^2}{dt^2}\left(\left(\frac{d\varphi}{dt}\right)^2\right) - \frac{a_2}{2} \frac{d}{dt}\left(\frac{d\sigma}{dt}\left(\frac{d\varphi}{dt}\right)^2\right) \]
\[-24e^{4\sigma-2\varphi}\left(\frac{d\varphi}{dt}\right)^2 + 2e^{2\sigma-2\varphi}\left[-6\frac{d^2\sigma}{dt^2} - 6\left(\frac{d\sigma}{dt}\right)^2\right] \]
\[-6\frac{d^2e^{2\sigma-2\varphi}}{dt^2} + 12\frac{d}{dt}\left(e^{2\sigma-2\varphi}\frac{d\sigma}{dt}\right) . \] (104)

\[0 = \frac{a_1}{2} \frac{d}{dt}\left(\frac{d\varphi}{dt}\right)^3 - \frac{a_2}{2} \frac{d}{dt}\left(\frac{d\varphi}{dt}\right)\left(\frac{d^2\sigma}{dt^2}\right) - \frac{a_2}{2} \frac{d}{dt}\left(\frac{d\varphi}{dt}\left(\frac{d\sigma}{dt}\right)^2\right) \]
\[+ 12\frac{d}{dt}\left(e^{4\sigma-2\varphi}\frac{d\varphi}{dt}\right) + 12e^{4\sigma-2\varphi}\left(\frac{d\varphi}{dt}\right)^2 \]
\[-2e^{2\sigma-2\varphi}\left[-6\frac{d^2\sigma}{dt^2} - 6\left(\frac{d\sigma}{dt}\right)^2\right] . \] (105)

We can often drop the terms linear to \(\sigma\) in (101) [18]. (Actually, if one considers the correspondent theory as IR sector of quantum gravity [19] or IR sector of induced supergravity [20] the omitting of linear term on sigma
corresponds to trivial redefinition of the correspondent source.) In this case, (104) and (105) can be reduced to

\[
0 = 4b'' \frac{d^4 \sigma}{dt^4}
\]

\[
-2[3b'' + 2(b + b')] \left[ \frac{d^2 \sigma}{dt^2} \right]^2 - 2 \frac{d}{dt} \left\{ \frac{d \sigma}{dt} \left[ \frac{d^2 \sigma}{dt^2} + \left( \frac{d \sigma}{dt} \right)^2 \right] \right\}
\]

\[
- \frac{a_2}{2} \frac{d}{dt} \left( \frac{d \sigma}{dt} \left( \frac{d \varphi}{dt} \right)^2 \right)
\]

\[
-24e^{4\varphi - 2\varphi} \left( \frac{d \varphi}{dt} \right)^2 + 2e^{2\varphi - 2\varphi} \left[ -6 \frac{d^2 \sigma}{dt^2} - 6 \left( \frac{d \sigma}{dt} \right)^2 \right]
\]

\[
-6 \frac{d^2 e^{2\varphi - 2\varphi}}{dt^2} + 12 \frac{d}{dt} \left( e^{2\varphi - 2\varphi} \frac{d \sigma}{dt} \right).
\]

(106)

\[
0 = - \frac{a_2}{2} \frac{d}{dt} \left( \frac{d \sigma}{dt} \left( \frac{d \varphi}{dt} \right)^2 \right)
\]

\[
+12 \frac{d \varphi}{dt} \left( e^{4\varphi - 2\varphi} \frac{d \varphi}{dt} \right) + 12e^{4\varphi - 2\varphi} \left( \frac{d \varphi}{dt} \right)^2
\]

\[
-2e^{2\varphi - 2\varphi} \left[ -6 \frac{d^2 \sigma}{dt^2} - 6 \left( \frac{d \sigma}{dt} \right)^2 \right].
\]

(107)

In the large \(N\) limit, where the contribution from the classical part can be neglected in Eqs. (102) and (103), some analytical solutions can be found. When the terms linear to \(\sigma\) in (101) are dropped, such analytical solutions were found in [21]. On the other hand, when we include the \(\sigma\) linear terms, we can construct only a few closed form solutions. Some of them are given by

1. \(\varphi = \varphi_0 \) (constant), \(\sigma = 0\)
2. \(\varphi = \varphi_0 \) (constant), \(\sigma = \alpha \ln \left( \frac{t}{t_0} \right), \quad \alpha^2 \equiv 1 - \frac{2b'}{3b'' + 2(b + b')} > 1\)
3. \(\varphi = \beta \ln \left( \frac{t}{t_0} \right), \quad \sigma = 0, \quad \beta^2 \equiv 12 \frac{a_2}{a_1} > 0\)

(108)

In the case that the terms linear to \(\sigma\) in (101) are dropped [21] in the large
$N$ limit, (106) and (107) can be integrated to be

$$r \frac{d^3 \sigma}{dt^3} - 2q \left( \frac{d \sigma}{dt} \right)^3 - \frac{a_2}{4} \frac{d}{dt} \left( \frac{d \varphi}{dt} \right)^2 = c$$

$$\frac{d \varphi}{dt} \left( \frac{d \sigma}{dt} \right)^2 = e$$

(109)

where $c$ and $e$ are integration constants and $r = -(3b'' + 2b)$, $q = -[3b'' + 2b' + 2b]$. We can find some particular solutions of Eqs. (109).

1. For $r = 0$, we get

$$\sigma = \sigma_0 t + \text{const}$$

$$f = \text{const} \exp \left( \frac{et}{\sigma_0^2} \right)$$

(110)

where

$$\sigma_0 = \left( -\frac{a_2 e^2}{4q} \pm \sqrt{\frac{ae^2}{4q} - \frac{c}{2q}} \right)^{\frac{1}{3}}$$

(111)

That is a singular solution, it has an initial singularity at physical time equal to zero.

2. For $r \neq 0$, we find the general solution:

$$\sigma = \int \frac{yd y}{u(y)} , \quad t = \int \frac{d y}{u(y)} , \quad \varphi = -\frac{e}{2} \int \frac{d y}{y^2 u(y)}$$

(112)

where

$$u^2(y) = \frac{1}{r} \left( qy^4 - \frac{a_2 e^2}{y^2} + 2cy + e \right)$$

(113)

One particular solution in (113) is given by

$$\varphi = -\frac{e}{2} t , \quad \sigma = t , \quad c = -2q - a_2 e^2$$

(114)
That is the same solution as in (110). Another particular solution is given by $q = 0$, $c = 0$, $e = 1$, $r = 2b'$, and

$$\sigma = a_2 \sqrt{|2b'| \left[ \frac{1}{2} \arcsin \left( \frac{(|2b'|)^{-1/2}t + a_2}{a_2} \right) \right]} + \frac{1}{4} \sin \left\{ 2 \arcsin \left( \frac{(|2b'|)^{-1/2}t + a_2}{a_2} \right) \right\} ,$$

and similar solution for $\varphi$. That is again singular solution. In general, dilatonic coupling acts against such non-singular solutions. We did not find any non-singular solutions from the general solution (112).

We now consider solving (104) and (105) or (106) and (107) numerically. Some examples of the solution by numerical calculations are given in Figs.11 – 14. The results in case that we include the terms in (101) linear to $\sigma$ (104) and (103) are almost identical with the case that we drop the terms linear to $\sigma$ (106) and (107), as expected. For this reason, we only give the results in Figs.11 – 14 for the case that we drop the terms linear to $\sigma$. In Figs.11 – 14, we calculate $\varphi$ (solid line), $\sigma$ (broken line) and 4d scalar curvature (dotted line), which is given by

$$R = -6 \frac{\partial^2 \sigma}{\partial t^2} - 6 \left( \frac{\partial \sigma}{\partial t} \right)^2 ,$$

for several $N$ and for the several initial conditions. An interesting thing is that the 4d scalar curvature oscillates as a trigonometric function. In Fig.11, we choose $N = 1$ and the initial condition $\phi(0) = \sigma(0) = 0$, $\dot{\phi}(0) = \dot{\sigma}(0) = \ddot{\sigma}(0) = \sigma^{(3)}(0) = 1$. In Fig.11, both of the dilaton field $\phi$ and the conformal factor $\sigma$ slowly increase monotonically but the scalar curvature vibrates rather quickly like trigonometric functions but the amplitude slowly increases. In Fig.12, where we chose $N = 100$ and the initial condition $\phi(0) = \sigma(0) = \sigma^{(3)}(0) = 0$, $\dot{\phi}(0) = \dot{\sigma}(0) = \ddot{\sigma}(0) = 1$, the conformal factor $\rho$ behaves in a rather complicated way, i.e., $\rho$ increases at first but decreases after that with slow and small oscillations. On the other hand, $\phi$ increases monotonically and reaches a singularity in finite conformal time. Since the conformal factor $\rho$ does not become positive infinity, the finite conformal time corresponds to finite cosmological time. The 4d scalar curvature vibrates and
the amplitude increases quickly near the singularity. The 4d scalar curvature oscillates slowly in Fig.13, where $N = 100$ and the initial condition is $\phi(0) = \sigma(0) = \dot{\sigma}(0) = \ddot{\sigma}(0) = \sigma^{(3)}(0) = 0$, $\dot{\phi}(0) = 1$. The conformal factor $\rho$ increases at first but decreases after that with very slow oscillations and $\phi$ increases monotonically and reaches a singularity in finite conformal and cosmological times. In Fig.14, where $N = 100$ with the initial condition $\phi(0) = \dot{\phi}(0) = \sigma(0) = \dot{\sigma}(0) = \ddot{\sigma}(0) = \sigma^{(3)}(0) = 0$, $\dot{\sigma}(0) = 1$, the scalar curvature oscillates very quickly and the period becomes short with the passing of conformal time. In this solution, the dilaton field $\phi$ also oscillates but the amplitude becomes small and the conformal factor $\rho$ slowly and monotonically increase.

Hence, unlike the case of pure Einstein gravity with conformal matter back-reaction on gravitational background (see ref. [22]) it is more difficult to realise the inflationary Universe in the theory with dilaton under consideration.

### 7 Discussion

In the present work we studied quantum cosmology for some 2d and 4d dilatonic SGs with dilaton coupled matter. Working in the large $N$ approximation we demonstrated that quantum matter back reaction does not really help to solve the singularity problem unlike the case with no external dilaton. Of course, this does not mean that we claim that the singularity problem cannot be solved as result of quantum fluctuations. Rather our results indicate that the solution of this problem perhaps could be possible only in complete quantum SG models, i.e. taking into account quantum effects of all fields (graviton,dilaton,gravitino,...). It would be really interesting to study such a problem but it is not easy to do since we do not know the correspondent expression for effective action (at best, it is known only as an expansion in the curvature, see [13] for a review). Note, nevertheless, that we found few non-singular Universes which are non-singular at early times but become singular at late times (or vice-versa).

Another interesting aspect of our work is the following one. If we exchange the role of time and radius for many of our numerical solutions we find a static object which may be interpreted as a 2d black hole (BH). A few years ago there was a lot of activity in the study of quantum dilatonic 2d BHs (for
very incomplete list of references see \[23\] and refs. therein). Hence, using our approach one can study 2d dilatonic BHs for the models of dilaton SG with quantum matter. It is also interesting to note that recently discussed effect of BH anti-evaporation \[24\] also takes place in above BHs realized in SGs as it could be confirmed by easy calculation. (It is natural as only the coefficient of the Polyakov term in the anomaly induced effective action is changed due to the fermion's contribution).

As one more possible generalization of this work one can consider not only conformally flat but more complicated FRW cosmologies for 4d theory. However, we do not expect that the results will be qualitatively very different at least in the models of the same sort. The reason is that main effects come through time dependence which will be similar.

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References

[1] T. Kadoyoshi, S. Nojiri and S.D. Odintsov, preprint OCHA-PP-106, NDA-FP-40 (1997), \texttt{hep-th/9712015}, \textit{Phys. Lett. B}, to appear.

[2] S. Nojiri and S.D. Odintsov, \textit{Phys. Lett. B416} (1998) 85; \textit{Phys. Rev. D57} (1998) 4847.

[3] S.J. Gates, Jr., M.T. Grisaru and M.E. Wehlau, \textit{Nucl.Phys. B460} (1996) 579; S.J. Gates, Jr., L. Lu and R. Oerter, \textit{Phys.Lett. 218B} (1989) 33; S.J. Gates, Jr., \textit{Phys. Lett. 352B} (1995) 43.

[4] C.G. Callan, S.B. Giddings, J.A. Harvey and A. Strominger, \textit{Phys. Rev. D45} (1992) 1005.

[5] S. Nojiri and I. Oda, \textit{Mod.Phys.Lett. A8} (1993) 53.

[6] K. Higashijima, T. Uematsu and Y.Z. Yu, \textit{Phys.Lett. 139B} (1984) 161; T. Uematsu, \textit{Z.Phys. C29} (1985) 143; T. Uematsu, \textit{Z.Phys. C32} (1986) 33.
[7] K. Behrndt and T. Burwick, *Phys.Rev.* **D52** (1995) 1292.

[8] S. Nojiri and S.D. Odintsov, *Mod.Phys.Lett.* **A12** (1997) 2083.

[9] R. Bousso and S.W. Hawking, *Phys.Rev.* **D56** (1997) 7788.

[10] S. Bose and S. Kar, *Phys.Rev.* **D56** (1997) 4444.

[11] J. Wess and J. Bagger, “Supersymmetry and Supergravity”, Princeton University Press

[12] S.J. Gates, M.T. Grisaru, M. Rocek and W. Siegel, “Superspace or One Thousand and One Lessons in Supersymmetry”, Benjamin/cummings (1983) (Frontiers in Physics, 58).

[13] R. Kantowski and R. Sacks, *J.Math.Phys.* **7** (1967) 2315;

[14] R. Laflamme and E.P. Shellard, *Phys.Rev.* **D35** (1987); J. Louko and T. Vachaspati, *Phys.Lett.* **B223** (1989) 21.

[15] S.J. Gates, Jr., P. Majumdar, R. Oerter and A.E.M. van de Ven, *Phys.Lett.* **214B** (1988) 26.

[16] J. de Boer and K. Skenderis, *Nucl.Phys.* **B481** (1996) 129; P. Binetruy, M.K. Gaillard and Y-Y. Wu, *Nucl.Phys.* **B481** (1996) 109.

[17] S. Nojiri and S.D. Odintsov, *Phys.Rev.* **D57** (1998) 2363.

[18] I.L. Buchbinder, S.D. Odintsov and I.L. Shapiro, ‘Effective Action in Quantum Gravity”, IOP Publishing, Bristol and Philadelphia, 1992.

[19] I. Antoniadis and E. Mottola, *Phys.Rev.* **D45** (1992) 2013; S.D. Odintsov, *Z.Phys.* **C54** (1992) 531.

[20] I.L. Buchbinder and A. Petrov, *Mod.Phys.Lett.* **A11** (1996) 2159.

[21] V.V. Obukhov, S.D. Odintsov and L.N. Granda, preprint October 1997.

[22] A. Starobinsky, *Phys.Lett.* **B91** (1980) 99; S.G. Mamaev and V.M. Mostepanenko, *JETP* **78** (1980) 20.
[23] J.G. Russo, L. Susskind and L. Thorlacius, *Phys. Lett.* **B292** (1992) 13; S.P. de Alwis, *Phys. Lett.* **B289** (1992) 278; A. Bilal and C. Callan, *Nucl.Phys.* **B394** (1993) 73; S. Nojiri and I. Oda, *Phys. Lett.* **B294** (1992) 317; *Nucl.Phys.* **B406** (1993) 499; T. Banks, A. Dabholkar, M. Douglas and M. O’Loughlin, *Phys.Rev.* **D45** (1992) 3607; R.B. Mann, *Phys.Rev.* **D47** (1993) 4438; D. Louis-Martinez and G. Kunstatter, *Phys.Rev.* **D49** (1994) 5227; T. Klobsch and T. Strobl, *Class. Quant. Grav.* **13** (1996) 965; S. Bose, L. Parker and Y. Peleg, *Phys.Rev.* **D52** (1995) 3512; A. Strominger, Les Houches lectures on black holes, [hep-th/9501071](http://arxiv.org/abs/hep-th/9501071).

[24] R. Bousso and S.W. Hawking, *Phys.Rev.* **D57** (1998) 2436.
Figure Captions

Fig.1 $\phi$ (solid line), $\rho$ (broken line) and $R/100$ (dotted line) for $N = 1$, $A^2 = 1$ and $\lambda^2 = 1$.
Fig.2 $\phi/10$ (solid line), $\rho$ (broken line) and $R/100$ (dotted line) for $N = 10$, $A^2 = 0$ and $\lambda^2 = 1$.
Fig.3 $\phi$ (solid line), $\rho$ (broken line) and $R/30$ (dotted line) for $N = 10$, $A^2 = 1$ and $\lambda^2 = 0$.
Fig.4 $\phi$ (solid line), $\rho$ (broken line) and $R/500$ (dotted line) for $N = 100$, $A^2 = 1$ and $\lambda^2 = 1$.
Fig.5 $\phi$ (solid line), $\rho$ (broken line) and $R/100$ (dotted line) for $N = 1$, $A^2 = 0$ and $\lambda^2 = 1$.
Fig.6 $\phi$ (solid line), $\rho$ (broken line) and $R/100$ (dotted line) for $N = 10$, $A^2 = 1$ and $\lambda^2 = 0$.
Fig.7 $\phi$ (solid line), $\rho \times 10$ (broken line) and $\theta$ (dotted line) for $N = 1$, $A^2 = 0$ and $\Lambda^2 = 1$.
Fig.8 $\phi$ (solid line), $\rho \times 4$ (broken line) and $\theta$ (dotted line) for $N = 1$, $A^2 = 1$ and $\Lambda^2 = 0$.
Fig.9 $\phi \times 5$ (solid line), $\rho$ (broken line) and $\theta$ (dotted line) for $N = 10$, $A^2 = 1$ and $\Lambda^2 = 1$.
Fig.10 $\phi \times 100$ (solid line), $\rho \times 10$ (broken line) and $\theta$ (dotted line) for $N = 100$, $A^2 = 1$ and $\Lambda^2 = 0$.
Fig.11 $\phi$ (solid line), $\sigma$ (broken line) and 4d curvature divided by 100 (dotted line) for $N = 1$ with the initial condition $\phi(0) = \sigma(0) = 0$, $\dot{\phi}(0) = \dot{\sigma}(0) = \ddot{\sigma}(0) = \sigma^{(3)}(0) = 1$.
Fig.12 $\phi$ (solid line), $\sigma \times 5$ (broken line) and 4d curvature divided by 10 (dotted line) for $N = 100$ with the initial condition $\phi(0) = \sigma(0) = \sigma^{(3)}(0) = 0$, $\dot{\phi}(0) = \dot{\sigma}(0) = \ddot{\sigma}(0) = \sigma^{(3)}(0) = 1, \dot{\phi}(0) = 1$.
Fig.13 $\phi$ (solid line), $\sigma \times 10$ (broken line) and 4d curvature divided by 10 (dotted line) for $N = 100$ with the initial condition $\phi(0) = \sigma(0) = \dot{\sigma}(0) = \ddot{\sigma}(0) = \sigma^{(3)}(0) = 0, \dot{\phi}(0) = 1$.
Fig.14 $\phi \times 500$ (solid line), $\sigma$ (broken line) and 4d curvature divided by 5 (dotted line) for $N = 100$ with the initial condition $\phi(0) = \dot{\phi}(0) = \sigma(0) = \dot{\sigma}(0) = \sigma^{(3)}(0) = 0, \dot{\sigma}(0) = 1$.  

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