DE FACTORISATIONE NUMERORUM I:
IN PURSUIT OF THE ERYMANTHIAN BOAR

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Abstract. We introduce a novel glance at factoring. The technique broached here departs from any known (at least to the author) factoring method. In this paper, we show, given a product of two large primes \(N\) (a RSA modulus), how to select a multiplicative function \(\sigma_k\) (dependent on \(N\)) related to the sum of divisors function and produce a nontrivial small linear relation among \(\exp(\log^\epsilon N)\) values of \(\sigma_k(n)\) for \(|n - N| = O(\exp(\log^\epsilon N))\), (subject to a plausible conjecture). The tools to achieve this don’t go beyond classical analytic number theory, as known one hundred years ago.

Keywords. Riemann zeta function, RSA moduli, complex analysis.

1. Introduction

The problem of quickly factoring large integers is central in cryptography and computational number theory. It is in fact a pity that no more effort has been deployed to investigate the problem of integer factorization, when in comparison the number of cryptographic protocols relying on the difficulty of factoring integers has exploded in the last decades.

The current state of the art in factoring large integers \(N\), the Number Field Sieve algorithm [2,3], stems from the earlier Quadratic Sieve [8] and Continued Fraction [6]. We should also mention the Elliptic Curve Method (ECM) by H. Lenstra [4], which is particularly useful when \(N\) has a small prime factor \(p\). They are all probabilistic factoring algorithms.

These algorithms have heuristic running times respectively \(O(\exp(c(\log N)^{1/3}(\log \log N)^{2/3}))\), \(O(\exp(c(\log N)^{1/2}(\log \log N)^{1/2}))\) and \(O(\exp(c(\log p)^{1/2}(\log \log p)^{1/2}))\), for some constant \(c\) (not always the same). The first two strive to find nontrivial arithmetical relations of the form \(x^2 \equiv y^2 \pmod{N}\) (which lead to a nontrivial factor by computing \(\gcd(N, x + y)\)), whereas the third is a generalisation of Pollard’s \(p-1\) method [7], involving computations in some elliptic curve group instead of \(\mathbb{Z}/N\). We should note, however, that there exist probabilistic algorithms with proved running time \(O((1 + o(1))(\log N)^{1/2}(\log \log N)^{1/2})) [5]\). As far as the author is aware, no such rigorous bound exists in the form \(O(\exp((\log N)^c))\) for \(c < 1/2\). Similarly, no deterministic subexponential algorithm is currently known, the best one being Shank’s square form factorization SQUFOF which runs in \(O(N^{1/4+\epsilon})\), or in \(O(N^{1/5+\epsilon})\) on the Extended Riemann Hypothesis.

In this work, given an arbitrary \(\epsilon > 0\) we look for nontrivial relations (with \(c_0 = 1\), say)

\[
\sum_{-\exp(\log^\epsilon N) \leq j \leq \exp(\log^\epsilon N)} c_j \sigma_k(N + j)
\]

which are negligibly small, where \(\sigma_k\) is a multiplicative function related to \(\sigma(n) = \sum_{d|n} d\) and \(N\) is some number which we want to factor. If we then know the factorizations of all \(n + j\) for \(|j| \leq \exp(\log^\epsilon N)\) then we can recover \(\sigma_k(N)\) and from there the factorization of \(N\) in polynomial time (if \(N = pq\) is the product of two primes). In other terms we produce a polynomial reduction from the problem of factoring \(N\) to the problem of factoring \(O(\exp(\log^\epsilon N))\) integers around \(N\).
For instance, recall that for $N = pq$ with $p \neq q$ primes, we have

$$\sigma(N) = 1 + N + p + q$$

and hence the computation (or even a fair approximation) of $\sigma(N)$ yields the sum $p + q$, which together with the product $N = pq$ is sufficient to recover the factors. However, an absolute arithmetic function is too “coarse” and erratic to be computed by analytic means. We discovered that to find anything useful, we have to select an arithmetic function depending on $N$. This will be made explicit in Section 3.

2. Notations

This work borrows heavily from standard notations in analytic number theory and indeed a classical reference on the subject is the treatise of Davenport [1]. In particular, we will make liberal use of the $O$ notation in Landau’s as well as Vinogradov’s form ($\ll$). Hence, for instance

$$f(u) = O(g(u)) \iff f(u) \ll g(u)$$

means that $g(u) > 0$ and $|f(u)|/g(u)$ is bounded above (usually as $u \to \infty$ or $u \to 0^+$, depending clearly on the context). Similarly, $f(u) = o(g(u))$ (resp. $f(u) = \Omega(g(u))$) means $g(u) > 0$ and $|f(u)|/g(u)$ goes to zero (resp. $|f(u)|/g(u)$ is bounded below). Unless specified, the implied constants are absolute, as well as undisclosed positive quantities that we label $c_1, c_2, \ldots$. Any sum such as

$$\sum_{abc=n} a^2bc$$

is to be understood as taken over all positive integers $a, b, c$ such that $abc = n$. We also define

$$\sum_{a|n} f(a) = \sum_{ab=n} f(a)$$

so that for instance the number of divisors of $n$ is $\sum_{d|n} 1$ and its sum of divisors $\sum_{d|n} d$.

3. Choice of the Multiplicative Function $\sigma_k$

Let $k \in \mathbb{N}_{>1}$ be an integer. Define the multiplicative function

$$\sigma_k(n) = \sum_{d_1d_2 \cdots d_k = n} d_1^{1/k} d_2^{2/k} \cdots d_k^{(k-1)/k}$$

Let $\chi$ be a primitive Dirichlet character to a modulus $\ell$. We have the Dirichlet series expansion

$$L(s, \chi)L\left(s - \frac{1}{k}, \chi \right) \cdots L\left(s - \frac{k-1}{k}, \chi \right) = \sum_{n=1}^{\infty} \frac{\sigma_k(n) \chi(n)}{n^s}$$

whenever $\Re s > 2 - 1/k$. Here $L(s, \chi)$ is the usual Dirichlet L-function

$$L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}$$

convergent (resp. absolutely convergent) for $\Re s > 0$ (resp. $\Re s > 1$).

Now let $\nu \in \mathbb{N}$ ($\nu \geq 2$) to be defined later as a function of $k$ alone, and

$$f_\nu(t) = \begin{cases} 
(1 - t)^{\nu-1} & 0 \leq t \leq 1, \\
0 & t \geq 1.
\end{cases}$$

The Mellin transform of $f_\nu$ is by definition the beta function
In order to fix notations let us suppose $p > \sqrt{N}$.

Let $\bar{y} = \sigma_k(N) + O(N^{-2}) = g(p^{1/k}) + O(N^{-2})$. Define $\bar{x} \geq \sqrt{N^{1/k}}$ so that $g(\bar{x}) = \bar{y}$ (note that without loss of generality, we can always suppose that $\bar{y} \geq g(\sqrt{N^{1/k}})$ and therefore such a value $\bar{x}$ exists and is unique).

\[ B(\nu, s) = \frac{\Gamma(\nu)\Gamma(s)}{\Gamma(\nu + s)} = \int_0^\infty f_{\nu}(t)t^{s-1}dt \]

hence by the inverse Mellin transform.

\[ \frac{1}{2\pi i} \int_{2-\infty}^{2+i\infty} L(s, \chi)L\left(s - \frac{1}{k}, \chi\right)\cdots L\left(s - \frac{k-1}{k}, \chi\right) \frac{\Gamma(\nu)\Gamma(s)}{\Gamma(\nu + s)} x^s ds = \sum_{n \leq x} \sigma_k(n)\chi(n)f_{\nu}\left(\frac{n}{x}\right). \]

But this is the convolution of $f_{\nu}$ with the function $\sum_{n \leq x} \sigma_k(n)\chi(n)x^{-\nu}$. Hence by the inverse Mellin transform

\[ F(x) = \sum_{n \leq x} \sigma_k(n)\chi(n)f_{\nu}\left(\frac{n}{x}\right) = \sum_{n \leq x} \sigma_k(n)\chi(n)\left(1 - \frac{n}{x}\right)^{-\nu} \]

and note that

\[ P_{\nu}(x) = x^{\nu-1}F(x) = \sum_{n \leq x} \sigma_k(n)\chi(n)(x - n)^{\nu-1} \]

is a piecewise polynomial (given by a different expression between consecutive integers) of degree $\nu - 1$.

4. Relation to Factoring

We will see later in Section 7 how to relate calculations on the polynomial $P_{\nu}$ to values of $\sigma_k(n)$.

In this section we suppose that $N = pq$ is a product of two different primes (e.g. as in a RSA modulus). We derive how the computation of $\sigma_k(N)$ to within $O(N^{-2})$ can be used to factor $N$.

Remark that

\[ \sigma_k(p) = 1 + p^{1/k} + p^{2/k} + \cdots + p^{(k-1)/k} \]

and by multiplicativity $\sigma_k(N) = \sigma_k(p)\sigma_k(q) = \sigma_k(p)\sigma_k(N/p)$. Define

\[ g(X) = (1 + X + X^2 + \cdots + X^{k-1})(1 + N^{1/k}X^{-1} + N^{2/k}X^{-2} + \cdots + N^{(k-1)/k}X^{-(k-1)}) \]

so that $\sigma_k(N) = g(p^{1/k})$. Observe that

\[ g(X) = \sum_{m=-\infty}^{k-1} a_m X^m \]

where $a_m > 0 \, (|m| \leq k-1)$ and given by polynomials in $N^{1/k}$. Hence $g$ is convex in $(0, \infty)$ and $g''(X) > N^{1/k} > 1$ (if $k > 3$). Additionally, since $g(X) = g(N^{1/k}/X)$, we have $g'\left(\sqrt{N^{1/k}}\right) = 0$. It follows that

\[ |g'(X)| > N^{-1} \quad \text{if} \quad X \notin \left(\sqrt{N^{1/k}} - N^{-1}, \sqrt{N^{1/k}} + N^{-1}\right). \]

We derive from this estimation that if $|g(X_1) - g(X_2)| < N^{-2}$ and $X_1, X_2 > \sqrt{N^{1/k}} + N^{-1}$, say then

\[ |X_1 - X_2| < N^{-2}N = N^{-1} . \]

In order to fix notations let us suppose $p$ is the greatest of the prime factors of $N$, so that $p > \sqrt{N}$.

Let $\bar{y} = \sigma_k(N) + O(N^{-2}) = g(p^{1/k}) + O(N^{-2})$. Define $\bar{x} \geq \sqrt{N^{1/k}}$ so that $g(\bar{x}) = \bar{y}$ (note that without loss of generality, we can always suppose that $\bar{y} \geq g(\sqrt{N^{1/k}})$ and therefore such a value $\bar{x}$ exists and is unique).

\[ ^{1}\text{We will also use the notation } \int_{(c)} \text{ instead of } \int_{c-\infty}^{c+\infty}. \]
(1) If $\sqrt{N^{1/k}} < p^{1/k} < \sqrt{N^{1/k} + 1/N}$, define $r = \sqrt{N^{1/k}}$. This case is highly improbable and can be excluded a priori by a quick computation (see below).

(2) If $p^{1/k} > \sqrt{N^{1/k} + 1/N}$ we can suppose that $\bar{y} \geq g(\sqrt{N^{1/k}} + 1/N)$ since otherwise again $g(\sqrt{N^{1/k}} + 1/N)$ is a better approximation to $\sigma_k(N)$. Therefore $\bar{x} \geq \sqrt{N^{1/k} + 1/N}$. Hence, by virtue of (3) and (4), $\bar{x} = p^{1/k} + O(N^{-1})$. Note that in this case, a dichotomy argument can produce in $O(\log N)$ steps a value $r = \bar{x} + O(N^{-1})$.

In all cases, we end up with

$$p^{1/k} = r + O(N^{-1})$$

We will assume henceforth in the paper, that $k = o(\log N/\log \log N)$. We have

$$N^{1/k} - p^{1/k} > \frac{N - p}{kN^{1-1/k}}$$

and hence

$$r = p^{1/k} + O\left(\frac{1}{N}\right) < N^{1/k}.$$

Therefore,

$$p = (r + O(N^{-1}))^k = r^k + O\left(\frac{k r^{k-1}}{N}\right) + \sum_{n=2}^{k} \binom{k}{n} O\left(\frac{r^{k-n}}{N^{n}}\right) = r^k + O(k N^{1/(2k)}) + O(3^k N^{-1})$$

Note that since these last error terms are $o(1)$ we get $p = \lfloor r^k \rfloor$ (the nearest integer to $r^k$). Remark also that in the improbable first case that $p^{1/k}$ be exceptionally close to $\sqrt{N^{1/k}}$, this can be revealed immediately by checking if $p$ is the integer nearest to $\sqrt{N}$. The next task is write $P_\nu(x)$ in a different way.

5. **Functional Equation of the Dirichlet L-Functions**

The Dirichlet L-function of a primitive character $\chi$ of modulus $\ell > 1$ is an entire function satisfying the functional equation (given here in asymmetric form)

$$L(s, \chi) = \frac{1}{2\pi i} \left(\frac{2\pi}{\ell}\right)^s \tau(\chi) \Gamma(1 - s) L(1 - s, \bar{\chi}) \left(e^{i\pi s/2} - \chi(-1)e^{-i\pi s/2}\right),$$

where $\tau(\chi)$ is the Gauss sum

$$\tau(\chi) = \sum_{m=1}^{\ell} \chi(m) e^{2\pi im/\ell}.$$ 

6. **A New Expression for $P_\nu(x)$**

A standard “integration line moving” (to $\Re s = -1/2$) argument in the integral of (2) will get us to the following.

$$F(x) = L(0, \chi) \cdots L(-(k-1)/k, \chi) + \frac{1}{2\pi i} \int_{(-1/2)}^{k-1} \prod_{m=0}^{k-1} L\left(s - m/k, \chi\right) \frac{\Gamma(\nu) \Gamma(s)}{\Gamma(\nu + s)} x^s ds$$

We should note here that the moving the line of integration to the line $\Re s = -1/2$ is by no means automatic here, and can be rigorously justified provided $\nu$ is sufficiently large. For if $s = \sigma + it$ with $\sigma, t \in \mathbb{R}$ it can be recovered from the functional equation and Stirling’s formula for $\Gamma$ that

$$|L(s, \chi)| \ll |\ell t|^{1/2-\sigma} \quad \text{as} \quad |t| \to \infty,$$
for $\sigma \leq 1/2$. In particular, $L(s - m/k, \chi) \ll |t|^{1 + m/k} < |t|^2$ for $m = 0, \ldots, k - 1$ and $\sigma \geq -1/2$, from which we deduce that the line moving argument to $\Re s = -1/2$ is justified provided, say,

$$\nu \geq 6k$$

which we will suppose from now on.

The integral in the previous expression is the focus of our investigation. It is very natural at this point to use the functional equation. We get quite straightforwardly

$$\frac{1}{2\pi i} \int_{(-1/2)} \prod_{m=0}^{k-1} L\left(s - \frac{m}{k}, \chi\right) \frac{\Gamma(\nu)\Gamma(s)}{\Gamma(\nu + s)} x^s ds = \frac{1}{2\pi i} \sum_{r=0}^{k} C_r \Gamma(\nu) \left(\frac{\tau(\chi)}{2\pi i}\right)^k \left(\frac{\ell}{2\pi}\right)^{(k-1)/2}$$

$$\times \int_{(-1/2)} \prod_{m=0}^{k-1} L\left(1 - s + \frac{m}{k}, \bar{\chi}\right) \Gamma\left(1 - s + \frac{m}{k}\right) \frac{\Gamma(s)}{\Gamma(\nu + s)} \left(\frac{2\pi}{\ell} e^{i\pi\omega_r/2}\right)^k x^s ds$$

$$= \frac{1}{2\pi i} \sum_{r=0}^{k} C_r \Gamma(\nu) \left(\frac{\tau(\chi)}{2\pi i}\right)^k \left(\frac{\ell}{2\pi}\right)^{(k-1)/2} \left(\frac{2\pi}{\ell} e^{i\pi\omega_r/2}\right)^k x$$

$$\times \int_{(3/2)} \prod_{m=0}^{k-1} L\left(s + \frac{m}{k}, \bar{\chi}\right) \Gamma\left(s + \frac{m}{k}\right) \frac{\Gamma(1 - s)}{\Gamma(\nu + 1 - s)} \left(\frac{2\pi}{\ell} e^{i\pi\omega_r/2}\right)^k x^{-s} ds ,$$

where we have last performed the change of variables $s \to 1 - s$ and

$$C_0 = \exp(-i\pi(k - 1)/4), \quad C_k = -\chi(-1)^k \exp(i\pi(k - 1)/4), \quad \omega_0 = 1, \quad \omega_k = -1.$$ 

Otherwise $|C_r| \leq 2^k$ and

$$|\omega_r| \leq 1 - \frac{2}{k}, \quad 0 < r < k.$$ 

We have singled out the extremal terms with $r = 1, 2^k$ since they will lead to the crucial (and most difficult) series to evaluate. The other terms do not actually constitute a problem. Let us now turn our attention to a typical integral in the last sum, of the form

$$\frac{1}{2\pi i} \int_{(3/2)} \prod_{m=0}^{k-1} L\left(s + \frac{m}{k}, \bar{\chi}\right) \Gamma\left(s + \frac{m}{k}\right) \frac{\Gamma(1 - s)}{\Gamma(\nu + 1 - s)} \left(\frac{2\pi}{\ell} e^{i\pi\omega_r/2}\right)^k x^{-s} ds .$$

Notice that

$$\frac{\Gamma(s)\Gamma(1 - s)}{\Gamma(\nu + 1 - s)} = \Gamma(s - \nu) (\cos \pi\nu - \cot \pi s \sin \pi\nu) = \Gamma(s - \nu) ,$$

by (5). Therefore we get

\[ \text{Lindelöf's conjecture asserts that the same should be true (up to } |t|^{\nu} \text{) for } \sigma \leq 1/2. \text{ This conjecture is still open at the moment. It is know to be a consequence of the Generalized Riemann Hypothesis for } L(s, \chi) \text{ which asserts that } L(s, \chi) = 0 \text{ and } 0 < \sigma < 1 \text{ implies } \sigma = 1/2. \]
\[
\frac{1}{2\pi i} \int_{(3/2)} \left\{ \prod_{m=0}^{k-1} L \left( s + \frac{m}{k} \xi \right) \Gamma \left( s + \frac{m}{k} \right) \right\} \frac{\Gamma(1-s)}{\Gamma(s+1)} \left( \frac{2\pi i e^{i\pi \nu/2}}{\ell} \right)^k x^{-s} ds
\]

\[
= \frac{1}{2\pi i} \int_{(3/2)} L(s, \chi) \left\{ \prod_{m=1}^{k-1} L \left( s + \frac{m}{k} \xi \right) \Gamma \left( s + \frac{m}{k} \right) \right\} \frac{\Gamma(s-\nu)}{\Gamma(s-\nu)} \left( \frac{2\pi i e^{i\pi \nu/2}}{\ell} \right)^k x^{-s} ds .
\]

By our choice of \( \nu \), this integral is absolutely convergent. Furthermore, since the L-functions on \( \Re s = 3/2 \) are all absolutely convergent, we can replace the product of L-functions in the integrand by their Dirichlet series expansion, interchange summation and integration, so that we are reduced to considering terms, for \( y = (2\pi i e^{i\pi \nu/2})^k x^1 \) \( n \in \mathbb{N} \), of the form

\[
\frac{1}{2\pi i} \int_{(3/2)} \Gamma(s-\nu) \prod_{m=1}^{k-1} \Gamma \left( s + \frac{m}{k} \right) y^{-s} ds = \frac{1}{2\pi i} \int_{(3/2)} \prod_{m=0}^{k-1} \Gamma \left( s + \frac{m}{k} \right) \frac{y^{-s}}{(s-\nu) \cdots (s-1)} ds
\]

\[
= (2\pi)^{\frac{k+1}{2} - k} I ,
\]

where we have used repeatedly the functional equation \( \Gamma(s+1)/s = \Gamma(s) \), as well as Gauss’ multiplication formula for the Gamma function

\[
\Gamma(s) \Gamma \left( s + \frac{1}{k} \right) \cdots \Gamma \left( s + \frac{k-1}{k} \right) = (2\pi)^{\frac{k-1}{2} + ks} \Gamma(k) .
\]

Let us suppose at first that \( y > 0 \) is real. We can then move the line of integration in \( I \) from \( \Re s = 3/2 \) to \( \Re s = \nu + 1 \) and get

(6) \[
I = \sum_{m=2}^{\nu} \frac{\Gamma(km)}{\prod_{r=1}^{\nu} (m-r)} (k^r y)^{-m} + \frac{1}{2\pi i} \int_{(1+\nu)} \Gamma(k) \frac{(k^r y)^{-s}}{(s-\nu) \cdots (s-1)} ds ,
\]

where the “widehat” notation means that the term corresponding to \( r = m \) is skipped. Let us name \( \tilde{I} \) the second addend and

\[
\tilde{I} = (k^r y)^\nu \tilde{I} = \frac{1}{2\pi i} \int_{(1+\nu)} \Gamma(k) \frac{(k^r y)^{-(\nu-s)}}{(s-\nu) \cdots (s-1)} ds .
\]

The \( \nu \)-th derivative \( \frac{\partial \nu \tilde{I}}{\partial y^\nu} \) of the latter integral equals

(7) \[
\frac{(-1)^\nu k^{\nu \nu}}{2\pi i} \int_{(1+\nu)} \Gamma(k) (k^r y)^{-s} ds = \frac{(-1)^\nu k^{\nu \nu}}{k} e^{-ky^{1/k}} ,
\]

since there is no problem in differentiating under the integral sign if \( y > 0 \). To get back \( \tilde{I} \) we note that the right-hand side of (7) can be integrated back \( \nu \) times with elementary functions. We add a word of explanation about the constant of integration, which has to be set to zero at each step,
because
\[
\lim_{y \to \infty} \frac{\partial^m}{\partial y^m} \int_{(1+\nu)}^\infty \Gamma(ks) \frac{(k^ky)^-(s-\nu)}{(s-\nu) \cdots (s-1)} \, ds = 0 , \quad \forall 0 \leq m \leq \nu .
\]

We work out these steps in an example with \( \nu = 2 \) and \( k = 3 \) (at this point, it is irrelevant to use (5), as we are assuming \( y > 0 \) so the integrand is always decreasing exponentially along vertical lines). We then want to compute
\[
\frac{1}{2\pi i} \int_{(3/2)} \Gamma(3s) \frac{(27y)^-(s-2)}{(s-2)(s-1)} \, ds = 120 + \frac{1}{2\pi i} \int_{(3)} \Gamma(3s) \frac{(27y)^-(s-2)}{(s-2)(s-1)} \, ds .
\]

The second derivative of the right-hand term is
\[
\frac{729}{2\pi i} \int_{(3)} \Gamma(3s) (27y)^-s \, ds = 243 e^{-3y^{1/3}} .
\]

Therefore
\[
-\frac{27}{2\pi i} \int_{(3)} \Gamma(3s) \frac{(27y)^-(s-1)}{s-1} \, ds = 243 \int e^{-3y^{1/3}} \, dy + C = 243 \left( 3 \int e^{-3u^2} \, du \right) + C
\]
\[
= -243 \left( e^{-3u^2} + \frac{2}{3} e^{-3u} \right) + C
\]
with \( u = y^{1/3} \). Now taking the limit on both sides as \( y \to \infty \) we get that \( C = 0 \). Another round of integration will give
\[
\frac{1}{2\pi i} \int_{(3/2)} \Gamma(3s) \frac{(27y)^-(s-2)}{(s-2)(s-1)} \, ds = -243 \int e^{-3u^2} + \frac{2}{3} e^{-3u} \, dy + C
\]
\[
= -729 \int e^{-3u^4} + \frac{2}{3} e^{-3u^2} \, du + C
\]
\[
= 3e^{-3u}(81u^4 + 162u^3 + 180u^2 + 120u + 40) + C .
\]

By the same argument as above, here too \( C = 0 \), so that in the end
\[
\frac{1}{2\pi i} \int_{(3/2)} \Gamma(3s) \frac{(27y)^-(s-2)}{(s-2)(s-1)} \, ds = 3e^{-3y^{1/3}} (81y^{4/3} + 162y + 180y^{2/3} + 120y^{1/3} + 40) + 120 .
\]

In general, it is not hard to see that \( I \) can be always given in closed form as a function of \( y \), in the shape
\[
\int \frac{(k^ky)^-s}{(s-\nu) \cdots (s-1)} \, ds = I = \frac{1}{(k^ky)^\nu} \left( p_{k,\nu}(y^{1/k}) e^{-ky^{1/k}} + q_{k,\nu}(y) \right)
\]
where \( p_{k,\nu}(X) \) and \( q_{k,\nu}(X) \) are polynomials with leading terms respectively
\[
k^{k\nu} k^{k\nu} X^{(k-1)\nu} \quad \text{and} \quad (-1)^\nu \frac{\Gamma(2k)}{(\nu-2)!} k^{\nu(\nu-2)} X^{\nu-2} .
\]
It should be noted that the result was derived for a real valued $X = y^{1/k}$ and we need it for $X = y^{1/k} = 2\pi e^{i\pi \omega /2 (x n)}^{1/k}$ where $|\omega| \leq 1$. In other words, we have proved

\[ \frac{1}{2\pi i} \int \frac{\Gamma(k s)}{(s - \nu) \cdots (s - 1)} \, ds = I = \frac{1}{(kX)^{k \nu}} \left( p_{k,\nu}(X) e^{-kX} + q_{k,\nu}(X^k) \right) \]

for $X > 0$ and need to extend it to the positive half-plane $\Re X \geq 0$ (up to the boundary, possibly excluding $X = 0$). This can be achieved by analytic continuation, by noticing that each expression is analytic in $\Re X > 0$ and continuous up to the border $\Re X = 0$ (possibly excluding $X = 0$), and therefore we have equality throughout this region.

We will now bound the coefficients of $p_{k,\nu}$ more closely. We have

\[ \mathcal{I} = \frac{p_{k,\nu}(y^{1/k})}{(k y)^\nu} \cdot \]

Writing a partial sum decomposition

\[ \frac{1}{(s - \nu) \cdots (s - 1)} = \sum_{u=1}^{\nu} b_u \frac{1}{s - u} \]

we get

\[ \mathcal{I} = \sum_{u=1}^{\nu} b_u \frac{1}{(k y)^u} \frac{1}{2\pi i} \int \Gamma(k s) \frac{(k y)^-(s - u)}{(s - u)} \, ds \cdot \]

As before, the derivative $\partial/\partial y$ of the last integral (times $1/2\pi i$) will equal

\[ -\frac{(k y)^u}{k y} e^{-k y^{1/k}} \]

and after integrating this last expression (substituting $t = y^k$ and then by parts $k u - 1$ times) we arrive at the result

\[ (10) \quad \mathcal{I} = e^{-k y^{1/k}} \times \sum_{u=1}^{\nu} b_u \left( \frac{1}{k y^{1/k}} + \frac{k u - 1}{k^2} \frac{1}{y^{2/k}} + \cdots + \frac{(k u - 1) \cdots (k u - (\nu - 1))}{k^\nu} \frac{1}{y^{\nu/k}} + \cdots + \frac{(k u - 1) \cdots (k u - (m - 1))}{k^m} \frac{1}{y^{m/k}} \right) = e^{-k y^{1/k}} \sum_{u=1}^{\nu} b_u \sum_{m=1}^{k u} \frac{P_{m,u}}{y^{m/k}} \]

with

\[ P_{m,u} = \frac{(k u - 1) \cdots (k u - (m - 1))}{k^m} \cdot \]

Note that after summing over $u$, the first nonzero power of $y$ in the sum is $y^{-\nu/k}$. Let

\[ \prod_{m=0}^{k-1} L\left(s + \frac{m}{k}, \frac{\chi}{k}\right) = \sum_{n \geq 1} \frac{a_n \chi(n)}{n^s}, \quad \Re s > 1, \]

where

\[ a_n = \sum_{d_1 \cdots d_{k-1} | n} d_1^{-1/k} d_2^{-2/k} \cdots d_{k-1}^{-(k-1)/k}. \]

Note that

\[ (11) \quad |a_n| \ll 2^k n^3 \]
as can be inferred from the inverse Mellin expression

$$\Phi(x) = \sum_{m \leq x} a_n \tilde{\chi}(m)(x - m) = \frac{1}{2\pi i} \int \prod_{m=0}^{k-1} L(s + m\frac{k}{k}, \tilde{\chi}) \frac{x^{s+1}}{(s+1)(s+2)} ds \ll \zeta(k)^k x^3$$

and

$$a_n \tilde{\chi}(n) = 2(\Phi(n + 3/4) - \Phi(n + 1/4) - \Phi(n - 1/4) + \Phi(n - 3/4)) \quad .$$

Then we deduce the following expression for $P_\nu(x) = x^{\nu - 1} F(x)$:

$$P_\nu(x) = x^{\nu - 1} L(0, \chi) \cdots L(-(k - 1)/k, \chi) + \sum_{r=0}^{k} C_r \Gamma(\nu) \left( \frac{\tau(\chi)}{2\pi i} \right)^k \left( \frac{\ell}{2\pi} \right)^{(k-1)/2} \left( \frac{2\pi \ell e^{i\pi \nu/2}}{n^{m/k}} \right)^k x^\nu$$

(12)

$$P_\nu(x) = x^{\nu - 1} L(0, \chi) \cdots L(-(k - 1)/k, \chi) + \sum_{r=0}^{k} C_r \Gamma(\nu) \left( \frac{\tau(\chi)}{2\pi i} \right)^k \left( \frac{\ell}{2\pi} \right)^{(k-1)/2} \left( \frac{2\pi \ell e^{i\pi \nu/2}}{n^{m/k}} \right)^k x^\nu$$

(13)

$$+ \sum_{m=2}^{\nu} \frac{\Gamma(km)}{\prod_{r=1}^{\nu} (m - r)} \left( \frac{2\pi k}{\ell} e^{i\pi \nu / 2} x^{1/k} \right)^{-km} \sum_{n \geq 1} a_n \tilde{\chi}(n)$$

(14)

$$(2\pi)^{k+1} \sqrt{k} \left( \prod_{u=1}^{\nu} b_u \right) m_{\nu} \sum_{m \geq 1} a_n \tilde{\chi}(n) \exp\left( -2\pi ke^{i\pi \nu / 2} (xn)^{1/k} / \ell \right)$$

(15)

7. Choosing Small Linear Combinations of the $\sigma_k(N + j)$

It is actually quite a challenging task that of finding an approximation of all the inner series of (13). However, an easier task consists in relating the values of $\sigma_k(n)$ for $|n - N| \leq J$ for some small enough $J$ by finding a linear combination

$$\sum_{-J \leq j \leq J} C_j \sigma_k(N + j) \chi(N + j) \quad ,$$

which equals a negligible quantity, for suitable $C_j \in \mathbb{C}$, not all zero. This will relate the problem of factoring $N$ to the problem of factoring $N + j$ for $|j| \leq J$, which is often easier.

In the following, let $x = N$. The idea here is to consider linear combinations

$$\sum_{-J \leq j \leq J} c_j P_\nu(x + j)$$

and impose that the unknowns $c_j$'s satisfy (polynomially many) homogeneous linear equations. Since we can scale these solutions, we will also assume that $\max |c_j| = 1$. If we insure that $J$ is larger than this number of equations, there will exist a solution not identically zero and this will give what we want.

We can write

$$\sum_{-J \leq j \leq J} c_j P_\nu(x + j) = \sum_{0 \leq m \leq \nu - 1} \sum_{0 \leq h \leq \nu - 1 - m} (-1)^h \binom{\nu - 1}{h} \binom{\nu - 1 - h}{m} x^m \times \sum_{n \leq x + J} \left( \sum_{j \leq x - n} j^{\nu - 1 - h - m} c_j \right) \sigma_k(n) \chi(n) n^h \quad ,$$

and therefore imposing the $\nu$ linear conditions

$$\sum_{-J \leq j \leq J} j^h c_j = 0 \quad , \quad h = 0, \ldots, \nu - 1$$

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will insure that the sum on the n’s in the right-hand side will actually be supported on $x - J \leq n \leq x + J$. Put it otherwise, for those $c_j$’s satisfying (15), we have

\begin{equation}
\sum_{-J \leq j \leq J} c_j P_{\nu}(x + j) = \sum_{-J \leq j \leq J} C_j \sigma_k(x + j) \chi(x + j)
\end{equation}

where

\begin{equation}
C_j = \sum_{\ell \geq j} c_{\ell} (\ell - j)^{\nu - 1}
\end{equation}

being understood that $c_{\ell} = 0$ if $\ell \notin [-J, J]$. We will take, given any $\epsilon > 0$,

$$J = x^\delta, \quad \delta = \frac{1}{(\log x)^{1 - \epsilon}}, \quad k = 2, \quad \nu = \lfloor (\log x)^{1 - \epsilon/2} \rfloor, \quad \ell = \text{largest prime} \leq x^{1/k}$$

In particular, we will get a linear relation among $2J = O(\exp(\log^\epsilon N))$ terms close to $N$.

It is the inner series (which we call the singular series) in (13) which present the greatest challenge. We will see that taking appropriate linear combinations as above will allow us to discard the contribution of these series into a small error term.

We now make a conjecture on the size of $C_0$, for random coefficients $c_j$ satisfying (15), (19) and the following:

\begin{equation}
(18) \quad c_j = 0 \ , \ j < J/\log x .
\end{equation}

**Conjecture 1.** In (17), $C_0 = \Omega \left( J^{\nu/2} \right)$ for a random choice of $c_i$ satisfying all the necessary linear equations (15), (18) and (19), with max $|c_i| = 1$. (Note: We could replace it with the weaker statement that $C_0 = \Omega(J^{\theta\nu}$ for some $\theta > 0$.)

The reason to impose (18) comes from the fact that we believe the $c_j$ with smallest $|j|$ will be biggest and then decay as $|j|$ increases, as (15) seems to indicate. This is probably not necessary.

We then see from (16) that computing $\sum_{-J \leq j \leq J} c_j P_{\nu}(x + j)$ with an error term $O(x^{\delta\nu/3}) = O \left( J^{\nu/2} x^{-2} \right)$ will lead, if we know the values $\sigma_k(x + j)$ for $j \in [-J, J], j \neq 0$, to the determination of $\sigma_k(N)$ with error term $O(N^{-2})$ and hence the factorisation of $N$, as we have seen (in doing this, we may take $\chi$ to be the quadratic character mod $\ell$, so that $\chi(N) = \pm 1$ and can be easily computed, although not necessary here, since one can try the possible two values). We are left to compute $\sum_{-J \leq j \leq J} c_j P_{\nu}(x + j)$ with an error term $O(x^{\nu/2})$. Let us call $c = \nu/k$. Our present choice, as we have seen, is to let $c = \lfloor (\log x)^{1-\epsilon/2} \rfloor/2$, but we keep it open, for future different choices. Define also

$$X = x^{(1-\delta/4)\nu}$$
The first point is to approximate the singular series in (13) for $P_\nu(x+j)$ by dropping the contribution of all $n > X$. The error term so obtained is, for an appropriate (and explicit) absolute constant $A$,

$$\ll A^k \Gamma(\nu)(x+j)^\nu \sum_{u=1}^{\nu} |b_u| \sum_{m=\nu}^{\nu} \left( \frac{\ell}{2\pi(x+j)^{1/k}} \right)^m P_{m,u} \sum_{n>X} \frac{a_n}{n^{m/k}}$$

$$\leq (2A)^k 2^\nu \Gamma(\nu) x^\nu \sum_{m=\nu}^{2(2\nu)^m} \sum_{n>X} \frac{1}{n^{m/k-3}}$$

$$\leq (2A)^k 2^\nu \Gamma(\nu) x^\nu \sum_{m=\nu}^{2(2\nu)^m} \left( \frac{2\nu^m}{X^{m/k-4}} \right) \leq (2A)^k 2^\nu \Gamma(\nu) x^{\delta/3} \sum_{m=\nu}^{2(2\nu)^m} \frac{2\nu^m x^{4\nu/c}}{x^{\delta m/12}}$$

$$= (2A)^k 2^\nu \Gamma(\nu) x^{\delta/3} \sum_{m=\nu}^{2(2\nu)^m} \frac{2\nu^m x^{4\nu/c}}{x^{\delta m/12}}$$

$$\ll (2A)^k 2^\nu \Gamma(\nu) x^{\delta/3} k^2(2\nu)^\nu x^{9\nu-(\log x)^2/12} \ll x^{\delta/3} x^{-1}$$

Therefore, the contribution of terms with $n > X$ in the singular series of $\sum_{-j \leq j \leq J} c_j P_\nu(x+j)$ is

$$\ll J x^{\delta/3} / x \ll x^{\delta/3}$$

Secondly, we use a dichotomy argument and write $x_1$ for the smallest $j \geq -J$ and $x_{\alpha+1} - x_{\alpha} = [\log^3 x]$ if $\alpha \geq 1$. We thus get a partition of $[-J, J]$ as a union of intervals of the form $[x_\alpha, x_{\alpha+1})$. We adjust this partition so that the right-most interval has length at least $[\log^3 x]$ and at most $2[\log^3 x]$, by merging it with its adjacent interval (i.e., if $J \in [x_{\alpha+1}, x_{\alpha+2})$ we redefine $x_{\alpha+1} = [J]$). Note that the number of such $\alpha$’s is $O(J/\log^3 x)$. If $x_{\alpha} \leq x + j < x_{\alpha+1}$, we write $x + j = x_{\alpha} + h$ (hence $h \leq 2\log^3 x$). Express in (13) with $n \leq X$,

$$\exp \left( -\frac{2\pi ke^{i\omega r/2}(x+j)n^{1/k}}{\ell} \right) = \exp \left( -\frac{2\pi ke^{i\omega r/2}(x_{\alpha}n^{1/k})}{\ell} \right)$$

$$\times \exp \left( -\frac{2\pi ke^{i\omega r/2}(x_{\alpha}+h)^{1/k} - x_{\alpha}^{1/k}}{\ell} n^{1/k} \right)$$

and note that the argument of the second exponential is

$$\ll \frac{k \left( (x_{\alpha} + h)^{1/k} - x_{\alpha}^{1/k} \right) X^{1/k}}{\ell} \ll h x^{\frac{1}{k} + 1 - \frac{4}{k} - \frac{1}{\mu}} \ll h x^{-\frac{1}{4 \log x} + \varepsilon} = \frac{h}{\exp \left( \frac{\log^2 x}{4} \right)} = o(1)$$

This means that we can approximate this exponential to within $O(\exp(-\log^2 x))$ by replacing it with a truncated Maclaurin series

$$\sum_{\mu=0}^{[\log^3 x]} \frac{1}{\mu!} \left( -\frac{2\pi ke^{i\omega r/2}(x_{\alpha} + h)^{1/k} - x_{\alpha}^{1/k}}{\ell} n^{1/k} \right)^\mu.$$
that using (12), we see that if we impose the linear conditions

\[ \sum \] where

\[ \sum \]

we will obtain that

\[ \sum \] and therefore we can find solutions which are not identically zero.

\[ + \] which is what we want. We can remark that the number of variables \( \nu \) run in an interval of length between \( \log \) and 2 \( \log \) \( x \).

\[ \sum \]

\[ \sum \]

\[ \sum \]

\[ \sum \]

where \( \sum' \) means that corresponding to the largest value of \( x_\alpha < J \), the parameter \( h \) runs in an interval of length between \( \log^3 x \) and 2 \( \log^3 x \). Note that because of (15), we have

\[ \sum \]

and therefore

\[ \sum \]

\[ \sum \]

\[ \sum \]

\[ \sum \]

At this point we see that if we impose the linear conditions

\( \sum' \)

\[ (x_\alpha + h)^{1/k} - x^{1/k}_\alpha = 0, \quad (\mu = 0, \ldots, \lfloor \log^2 x \rfloor), (m = \nu, \ldots, k\nu), \]

we will obtain that

\[ \sum \]

which is what we want. We can remark that the number of variables \( c_{\alpha h} \) is \( \Omega(J) \), while the number of homogeneous linear equations that we impose on them (15), (18) and (19) is

\[ \ll \nu + \frac{J}{\log x} + (\#\alpha)k\nu \log^2 x \ll \log x + \frac{J}{\log x} + (\#\alpha) \log^{3-\epsilon/2} x \ll \frac{J}{(\log x)^{\epsilon/2}} \]

and therefore we can find solutions which are not identically zero.
Remark: One may wonder what the limit of this method is. We believe that actually one should be able to bring the linear relation down to $J = [\log^{3+\epsilon} N]$ terms. This will be made clear in future versions.

Remark: We changed the setting a little from the last version of this work, because we realized that Conjecture 1 does not hold if $c$ is too small, for the following reason: in (19), taking $\mu = 0$, we see that we have

$$\sum_{0 \leq h < [\log x]}' c_{\alpha,h}(x_{\alpha} + h)^m = 0, \quad m = 0, \ldots, \nu - c.$$ 

By using Leibniz’ rule and induction, this implies that

$$\sum_{0 \leq h < [\log x]}' c_{\alpha,h}h^m = 0, \quad m = 0, \ldots, \nu - c.$$ 

Writing as before $x + j = x_{\alpha} + h$ and calling $J_{\alpha} = x_{\alpha} - x$, we see that

$$C_0 = \sum_{\alpha: J_{\alpha} \geq 0} \sum_{0 \leq h < [\log x]}' c_{\alpha,h}(J_{\alpha} + h)^{\nu - 1}.$$ 

Then, from (20), we can deduce, by Leibniz’ rule again, that

$$\sum_{0 \leq h < [\log x]}' c_{\alpha,h}(J_{\alpha} + h)^{\nu - 1} = \sum_{m \leq c - 2} \left(\nu - 1\right) J_{\alpha}^m \sum_{0 \leq h < [\log x]}' c_{\alpha,h}h^{\nu - 1 - m} \leq c 4^\nu \log^3 x J^c (\log x)^{3\nu}.$$ 

Therefore, the choice of $c$ cannot be made too small if we want $C_0 = \Omega(J^\theta)$ for some $\theta > 0$. However, there is no apparent reason that would make $\sum_{0 \leq h < [\log x]}' c_{\alpha,h}h^m$ small for $m > \nu - c$ (when $\max |c_{\alpha,h}| = 1$), although at the moment we are unable to show this.

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