RA TIONAL POINTS ON CERTAIN QUINTIC HYPERSURFACES

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Abstract. Let \( f(x) = x^5 + ax^3 + bx^2 + cx \in \mathbb{Z}[x] \) and consider the hypersurface of degree five given by the equation

\[ V_f : f(p) + f(q) = f(r) + f(s). \]

Under the assumption \( b \neq 0 \) we show that there exists \( \mathbb{Q} \)-unirational elliptic surface contained in \( V_f \). If \( b = 0 \), \( a < 0 \) and \(-a \not\equiv 2, 18, 34 \pmod{48}\) then there exists \( \mathbb{Q} \)-rational surface contained in \( V_f \). Moreover, we prove that for each \( f \) of degree five there exists \( \mathbb{Q}(i) \)-rational surface contained in \( V_f \).

1. Introduction

In this paper we are interested in the problem of the existence of integer and rational points on the hypersurface given by the equation

\[ V_f : f(p) + f(q) = f(r) + f(s), \]

where \( f \in \mathbb{Q}[X] \) and \( \deg f = 5 \). Moreover, we assume that for each pair \( a, b \in \mathbb{Q} \setminus \{0\} \) we have \( f(ax + b) \neq cx^5 + d \) for any \( c, d \in \mathbb{Q} \). This assumption guarantees that \( V_f \) is an affine algebraic variety of dimension three. The set of rational points on \( V_f \) we denote by \( V_f(\mathbb{Q}) \). In other words

\[ V_f(\mathbb{Q}) = \{(p, q, r, s) \in \mathbb{Q}^4 : f(p) + f(q) = f(r) + f(s)\}. \]

Similarly, by \( V_f(\mathbb{Z}) \) we denote the set of integer points on \( V_f \), so \( V_f(\mathbb{Z}) = V_f(\mathbb{Q}) \cap \mathbb{Z}^4 \).

We will say that the point \( P = (p, q, r, s) \in V_f \) is non-trivial if \( \{p, q\} \cap \{r, s\} = \emptyset \) and \( \{f(p), f(q)\} \cap \{f(r), f(s)\} = \emptyset \). By \( T_f \) we will be denoted the set of trivial rational points on \( V_f \). Let us note that each singular point is trivial and that the number of all singular points (rational or not) is finite. In the sequel by rational point we will understand non-trivial rational point.

The problem of the existence of integer points on the hypersurface \( V_f \) was investigated in the interesting work of Browning [1]. In this work it was shown that

\[ M(f; B) \ll_{\epsilon} f B^{1+\epsilon}(B^{1/3} + B^{2/\sqrt{5}+1/4}), \]

for each \( \epsilon > 0 \). Here \( M(f; B) \) is the number of solutions \( (p, q, r, s) \) of the equation which define \( V_f \) and with such a property that \( 0 < p, q, r, s \leq B \) and \( \{p, q\} \cap \{r, s\} = \emptyset \).

From the above estimation we can see that the set of positive integer points on \( V_f \) is rather "thin". According to the best Author knowledge we do not know any example of a polynomial \( f \) of degree five with such a property that the set \( V_f(\mathbb{Z}) \setminus T_f \) is infinite. Moreover, we are unable to find in the existing literature of subject any example of a polynomial \( f \) of degree five which gives a positive answer to the following:

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**Question 1.1.** Let $N > 1$ be given. Is it possible to construct a polynomial $f$ of degree five, such that $\sharp(\mathcal{V}_f(\mathbb{Z}) \setminus T_f) > N$?

It is clear that the question concerning the existence of a polynomial $f$ of degree five with such a property that the set $\mathcal{V}_f(\mathbb{Q})$ is infinite should be easier. So, it is natural to state the following:

**Question 1.2.** For which polynomials $f$ of degree five the set $\mathcal{V}_f(\mathbb{Q})$ is infinite?

It seems that these questions have not been considered earlier. It is also clear that in the case of Question 1.2 we can consider polynomials of the form $f(X) = X^5 + aX^3 + bX^2 + cX$, where $a, b, c \in \mathbb{Z}$ and at least one among the numbers $a, b, c$ is nonzero. We will see that if $b \neq 0$ then the diophantine equation $f(p) + f(q) = f(r) + f(s)$ has rational two parametric solution (Theorem 2.1). In geometrical terms this means that the there is a rational surface contained in $\mathcal{V}_f$. From this result we can deduce easily that the answer for the Question 1.1 is positive. Moreover, we will prove that for any polynomial $f$ of degree five there exists $\mathbb{Q}(i)$-rational surface contained in $\mathcal{V}_f$ (Theorem 3.1).

2. Construction of rational points on the $\mathcal{V}_f$

Let $f \in \mathbb{Q}[X]$ and suppose that $\deg f = 5$. In this section we will show how we can construct parametric solutions of the equation which define the hypersurface

$$\mathcal{V}_f : f(p) + f(q) = f(r) + f(s).$$

Because we are interested in rational solutions, so without loss of generality we can assume that $f(X) = X^5 + aX^3 + bX^2 + cX$, $a, b, c \in \mathbb{Z}$ and at least one among the numbers $a, b, c$ is nonzero.

Our aim is the proof of the following theorem.

**Theorem 2.1.** Let $f(X) = X^5 + aX^3 + bX^2 + cX \in \mathbb{Z}[X]$, where $b \neq 0$ and consider the hypersurface $\mathcal{V}_f$. Then, there exists $\mathbb{Q}$-unirational elliptic surface $\mathcal{E}_f$ such that $\mathcal{E}_f(\mathbb{Q}) \subset \mathcal{V}_f(\mathbb{Q})$. In particular, the set $\mathcal{V}_f(\mathbb{Q})$ is infinite.

**Proof.** In the equation which define $\mathcal{V}_f$ we make (non-invertible) change of variables given by

$$p = x, \quad q = y - x, \quad r = z, \quad s = y - z. \quad (1)$$

The result of this substitution is the following

$$f(x) + f(y - x) - f(z) - f(y - z) = (x - z)(x - y + z)G(x, y, z),$$

where $G(x, y, z) = 2b + 3ay + 5x^2y - 5xy^2 + 5y^3 - 5y^2z + 5yz^2$. From the geometric point of view the substitution we have used can be understood as the cross section of the hypersurface $\mathcal{V}_f$ with the hyperplane $L$ given by the equation $L : p + q = r + s$ (the system of equations (1) give the parametrization of $L$).

Let us note that the equation $G(x, y, z) = 0$ has a solution in rational numbers if and only if the discriminant of the polynomial $G$ in respect to $z$ is a square of a rational number, say $v$. So, we are interested in the construction of rational points on the surface

$$S : v^2 = -5y(15y^3 + 20xy(x - y) + 12ay + 8b) =: \Delta(x, y).$$
If we make a change of variables
\[(x, y, w) = \left( -\frac{5b(t + 1)}{X + 5a}, \frac{10b}{X + 5a}, \frac{20bY}{(X + 5a)^2} \right),\]
with the inverse
\[(X, t, Y) = \left( -\frac{5(2b + ay)}{y}, \frac{2x - y}{y}, \frac{5bw}{y^2} \right)\]
the surface \(S\) is transformed to the form
\[\mathcal{E} : Y^2 = X^3 - 75a^2X - 125(5bt^2 + 10b^2 + 2a^3).\]

Let us note that the surface \(\mathcal{E}\) is of degree three and contains rational curve at infinity \([X : Y : t : Z] = [0 : 1 : t : 0]\). So we can invoke Segre theorem which says that the surface of degree three with a rational point is unirational. In other words there is a rational function
\[\Phi : \mathbb{Q} \times \mathbb{Q} \ni (u, v) \mapsto \Phi(u, v) \in \mathcal{E}_f,\]
such that the set \(\Phi(\mathbb{Q} \times \mathbb{Q})\) is dense (in Zariski topology) in the set \(\mathcal{E}_f(\mathbb{C})\).

For convenience of the reader we will show how the function \(\Phi\) can be constructed.
Let us put \(F(X, Y, t) = Y^2 - (X^3 - 75a^2X - 125(5bt^2 + 10b^2 + 2a^3))\). We use the method of indetermined coefficients in order to find two-parametric solution of \(F(X, Y, t) = 0\). Let \(u, v\) be parameters and let us put
\[(2)\quad X = T^2 + 10uT + p, \quad Y = T^3 + qT^2 + rT + t = (v/5b)T^2 + s.\]
We want to find \(p, q, r, s, T \in \mathbb{Q}(u, v)\) with such a property that the equation \(F(X, Y, t) = 0\) is satisfied identically. For the quantities given by \((2)\) we have
\[F(X, Y, t) = a_0 + a_1T + a_2T^2 + a_3T^3 + a_4T^4 + a_5T^5,\]
where
\[
\begin{align*}
a_0 &= 250a^3 + 1250b^2 + 75a^2p - p^3 + 625b^2s^2, \quad a_1 = 30(5a - p)(5a + p)u, \\
a_2 &= 75a^2 - 3p^2 + r^2 - 300pu^2 + 250bsv, \quad a_3 = 2(qr - 30pu - 500u^2), \\
a_4 &= -3p + q^2 + 2r - 300u^2 + 25s^2, \quad a_5 = 2(q - 15u).
\end{align*}
\]
Let us notice that the system of equations \(a_2 = a_3 = a_4 = a_5 = 0\) has exactly one solution in \(\mathbb{Q}(u, v)\) given by
\[(3)\quad p = 25(u^2 - 3v^2)/3, \quad q = 15u, \\
r = 50(u^2 - v^2), \quad s = (25u^4 - 450u^2v^2 - 75v^4 - 9a^2)/30bv.\]

We can see that if \(p, q, r, s\) are given by \((3)\) then \(F(T^2 + 10uT + p, T^3 + qT^2 + rT, (v/5b)T^2 + s) \in \mathbb{Q}(u, v)[T]\) and \(\text{deg}_TF = 1\). So this polynomial has a root in the field \(\mathbb{Q}(u, v)\) of the form
\[T = -\frac{250a^3 + 1250b^2 + 75a^2p - p^3 + 625b^2s^2}{30(5a - p)(5a + p)u},\]
where \(p, q, r, s\) are given by \((3)\). Putting the calculated values \(p, q, r, s, T\) to the equations \((2)\) we get the solutions dependent on two parameters \(u, v\) we are looking for.

\[\square\]

**Remark 2.2.** It should be noted that the same method was used by Whitehead \([6]\) in order to give the proof of unirationality of the surface \(z^2 = h(x, y)\), where \(h \in \mathbb{Q}(x, y)\) is polynomial of degree three. This proof can be also found in \([4]\) p. 85.
From the above theorem we can prove easily that the answer for the Question 1.1 is positive.

**Corollary 2.3.** For any \( N \in \mathbb{N}_+ \), then there are infinitely many polynomials \( f \in \mathbb{Z}[X] \) of degree five with such a property that on the hypersurface \( V_f : f(p) + f(q) = f(r) + f(s) \) there are at least \( N \) nontrivial integer points.

**Proof.** Let \( b \neq 0 \) and consider the polynomial \( f(X) = X^5 + aX^3 + bX^2 + cX \). From the previous theorem we know that in this case the diophantine equation \( f(p) + f(q) = f(r) + f(s) \) has infinitely many solutions in rational numbers. Let us take \( N \) various rational solutions of our equation, say \((p_i/p'_i, q_i/q'_i, r_i/r'_i, s_i/s'_i)\) for \( i = 1, 2, \ldots, N \), and define

\[
d = \text{LCM}(p'_1, q'_1, r'_1, s'_1, \ldots, p'_N, q'_N, r'_N, s'_N).
\]

If we now define \( F(X) = X^5 + ad^2 X^3 + bd^3 X^2 + cd^4 X \), then on the hypersurface \( V_f : F(p) + F(q) = F(r) + F(s) \) we have the points \((dp_i/p'_i, dq_i/q'_i, dr_i/r'_i, ds_i/s'_i)\) for \( i = 1, 2, \ldots, N \), which are triplets of integers.

The above corollary give us a positive answer for the Question 1.1, but we see that if \( N \) is growing then the coefficients of the polynomial \( F \) are growing too. So, we can state the following:

**Question 2.4.** Let \( N > 1 \) be given. It is possible to construct a polynomial \( f(X) = X^5 + aX^3 + bX^2 + cX \) with such a property that at least one non-zero coefficient \( a, b, c \) is independent of \( N \) and for the set of integer points on the hypersurface \( V_f \) we have \( \#(V_f(\mathbb{Z}) \setminus T_f) \geq N \)?

As we will see, the answer on this question is positive too. Before we show how we can do that let’s go back to the Question 1.1 in the case when \( f \) is of the form \( f(X) = X^5 + aX^3 + cX \). Unfortunately, we are unable to prove theorem similar to the Theorem 2.1 in this case. However we can prove the following:

**Theorem 2.5.** Let \( f(X) = X^5 + aX^3 + cX \in \mathbb{Z}[X] \). If \( a < 0 \) and \( a \neq 2, 18, 34 \; (\text{mod} \; 48) \) then the diophantine equation \( f(p) + f(q) = f(r) + f(s) \) has two-parametric rational solution.

**Proof.** For the proof we put

\[
p = \frac{-x + y + 3z}{5}, \quad q = \frac{2x + y}{5}, \quad r = \frac{3y}{5}, \quad s = \frac{x - y + 3z}{5}.
\]

For \( p, q, r, s \) defined in this way we have

\[
f(p) + f(q) - f(r) - f(s) = \frac{6(x - y)(x + 2y - 3z)(x + 2y + 3z)(x^2 + 2y^2 + 3z^2 + 5a)}{625}.
\]

First three factors in the above identities lead to the trivial solutions of our equation. So, we can see that the quantities given by \( f \) lead to the nontrivial solution of the equation \( f(p) + f(q) = f(r) + f(s) \), if and only if \( x^2 + 2y^2 + 3z^2 + 5a = 0 \). In particular it must be \( a < 0 \). As we know, local to global principle of Hasse is true for diophantine equations of degree two. It means that the diophantine equation \( x^2 + 2y^2 + 3z^2 + 5a = 0 \) has solution in rational numbers if and only if it has solutions in the field of \( p \)-adic numbers \( \mathbb{Q}_p \) for any given \( p \in \mathbb{P} \cup \{\infty\} \), where as usual \( \mathbb{Q}_\infty = \mathbb{R} \).
In the theorem below we find the well known algorithm of the solvability of the diophantine equation of the form \(a_1X_1^2 + a_2X_2^2 + a_3X_3^2 + a_4X_4^2 = 0\).

**Theorem 2.6.** If \(f(x_1, x_2, x_3, x_4) = a_1X_1^2 + a_2X_2^2 + a_3X_3^2 + a_4X_4^2\), where \(a_i \in \mathbb{Z}\setminus\{0\}\) are square-free and no three have a factor in common, then \(f\) represents zero if and only if the following three conditions hold:

1. Not all coefficient have the same sign.
2. If \(p\) is an odd prime dividing two coefficients and for which \((d/p^2)|p = 1\), then \((-a_ia_j)p = 1\), where \(GCD(a_ia_j, p) = 1\) and \(d = a_1a_2a_3a_4\) is a discriminant of the form \(f\).
3. If \(d \equiv 1 \pmod{8}\) or \(d/4 \equiv 1 \pmod{8}\) then we have \((-a_1a_2, -a_2a_3)2 = 1\).

Proof of this theorem can be found in [3].

Let \(\alpha, \beta\) then we have that \((\alpha, \beta)_2 = (2|\alpha_1, \beta_2 = 2|\gamma_3)\) and \(GCD(2, \alpha_1, \beta_1) = 1\), then we have that \((\alpha, \beta)_2 = (2|\alpha_1\gamma_2, 2|\gamma_3)^u(-1)^{(\alpha_1-1)(\beta_1-1)/4}\). Here \((\cdot|\cdot)\) is an usual symbol of Legendre.

In order to finish the proof of our theorem we apply the above procedure to the quadratic form \(X_1^2 + 2X_2^2 + 3X_3^2 + 5aX_4^2\). We must consider four cases dependent on the values of \(GCD(a, 6)\). Because this reasoning is very simple we leave it to the reader.

**Example 2.7.** Let \(f(X) = X^5 - X^3 + cX\) and consider the equation \(f(p) + f(q) = f(r) + f(s)\). We will show how we can use the previous theorem in practice.

We consider the equation (*) \(x^2 + 2y^2 + 3z^2 - 5 = 0\). This equation has a rational solution \((x, y, z) = (0, 1, 1)\). Let us put \(x = uT, y = vT + 1, z = T + 1\) and next solve the equation \((uT)^2 + 2(vT + 1)^2 + 3(T + 1)^2 - 5 = 0\) in respect to \(T\). After some necessary simplifications we get parametrization of rational solutions of the equation (*) in the form

\[
x = \frac{-2u(2v + 3)}{u^2 + 2v^2 + 3}, \quad y = \frac{u^2 - 2v^2 - 6v + 3}{u^2 + 2v^2 + 3}, \quad z = \frac{u^2 + 2v^2 - 4v - 3}{u^2 + 2v^2 + 3}.
\]

Using the parametrization we have obtained we get the solution of the equation \(f(p) + f(q) = f(r) + f(s)\), where \(f(X) = X^5 - X^3 + cX\), in the form

\[
p = \frac{2(2u^2 + (2v + 3)u + 2v^2 - 9v - 3)}{5(u^2 + 2v^2 + 3)},
\]
\[
q = \frac{u^2 - 4(2v + 3)u - 2v^2 - 6v + 3}{5(u^2 + 2v^2 + 3)},
\]
\[
r = \frac{3(u^2 - 2v^2 - 6v + 3)}{5(u^2 + 2v^2 + 3)},
\]
\[
s = \frac{2(u^2 - (2v + 3)u + 4v^2 - 3(v + 2))}{5(u^2 + 2v^2 + 3)}.
\]

Using the method of proof of the Theorem 3.1 we will show the following:

**Corollary 2.8.** Answer to the Question 2.4 is positive.

Proof. This is a simple consequence of the fact that for any number \(N\) we can find a negative number \(a_N\) such that the equation \(x^2 + 2y^2 + 3z^2 = -5a_N\) has at least
N solutions in positive integers $x, y, z$ all divisible by 5. In order to prove this let us put $g_N = \prod_{k=1}^{N}(k^2 + 2)$ and $a_N = -(5g_N)^2$. Next, we define

$$x_k = \frac{5g_N}{k^2 + 2}(2k + 3), \quad y_k = \frac{5g_N}{k^2 + 2}(k^2 + 3k - 2), \quad z_k = \frac{5g_N}{k^2 + 2}(k^2 - 2k - 1),$$

for $k = 1, 2, \ldots, N$. Note that the numbers $x_k, y_k, z_k$ are integers and are divisible by 5.

Due to the fact that

$$\left(\frac{2k + 3}{k^2 + 2}\right)^2 + 2\left(\frac{k^2 + 3k - 2}{k^2 + 2}\right)^2 + 3\left(\frac{k^2 - 2k - 1}{k^2 + 2}\right)^2 = 5,$$

we see that

$$x_k^2 + 2y_k^2 + 3z_k^2 = -5a_N \quad \text{for} \quad k = 1, 2, \ldots, N.$$

Now define $f_N(x) = x^5 + a_N x^3 + cx$, where $c$ is an integer number. From your reasoning we can see that on the hypersurface $V_{f_N}$ there is at least $N$ integer points given by

$$p_k = \frac{-x_k + y_k + 3z_k}{5}, \quad q_k = \frac{2x_k + y_k}{5}, \quad r_k = \frac{3y_k}{5}, \quad s_k = \frac{x_k - y_k + 3z_k}{5},$$

for $k = 1, 2, \ldots, N$ and number $c$ is independent of $N$.

Results of this section suggest the following:

**Conjecture 2.9.** Let $f(x) = x^5 + ax^3 + cx$, where $a, c \in \mathbb{Z}$ not both zero and consider the hypersurface $V_f$. Then the set $V_f(\mathbb{Q}) \setminus T_f$ is infinite.

3. Construction of $\mathbb{Q}(i)$-rational points on $V_f$

In this section we will consider the problem of the construction of $\mathbb{Q}(i)$-rational points on the hypersurface $V_f$.

Let us go back to the equation of the surface $S$ from the proof of the Theorem 2.1 and let us note that for the degree of the polynomial $\Delta$ we have $\deg \Delta = 2$. Now we look on $S$ as on a curve defined over field $\mathbb{Q}(i)(y)$, where $i^2 + 1 = 0$. It is easy to see that $S$ is a rational curve. Indeed, on the curve $S$ there is a $\mathbb{Q}(i)(y)$-rational point $[x : v : u] = [i : 10y : 0]$ (it is a point at infinity). Putting $x = ip, w = 10yp + u$ and solving the obtained equation according to $p$ we get the parametrization of our curve given by

$$x = -\frac{u^2 + 75y^4 + 60ay^2 + 40by}{20y(u - 5iy^2)}, \quad w = \frac{u^2 - 10iuy^2 - 75y^4 - 60ay^2 - 40by}{2(u - 5iy^2)}.$$

Using the above parametrization we can find two-parametric solution of the equation defining $V_f$ in the form

$$p = -\frac{u^2 + 75y^4 + 60ay^2 + 40by}{20y(u - 5iy^2)},$$

$$q = \frac{u^2 - 10iuy^2 - 25y^4 + 60ay^2 + 40by}{20y(u - 5iy^2)},$$

$$r = \frac{u^2 + 10(1 - i)y^2u - 25(3 + 2i)y^4 - 60ay^2 - 40by}{20y(u - 5iy^2)},$$

$$s = -\frac{u^2 - 10(1 + i)y^2u - 25(3 - 2i)y^4 - 60ay^2 - 40by}{20y(u - 5iy^2)}.$$
We sum up the above discussion concerning the existence of \( \mathbb{Q}(i) \)-rational points on \( \mathcal{V}_f \) in the following:

**Theorem 3.1.** Let \( f(X) = X^5 + aX^3 + bX^2 + cX \in \mathbb{Z}[X] \) and consider the hypersurface \( \mathcal{V}_f \). If \( a = b = 0 \) then there exist \( \mathbb{Q}(i) \)-rational curve contained in \( \mathcal{V}_f \). If \( a \neq 0 \) or \( b \neq 0 \) then there exist \( \mathbb{Q}(i) \)-rational surface contained in \( \mathcal{V}_f \).

Let us note that in the above quantities for \( p, q, r, s \) the number \( c \) does not appear explicitly and that obtained solution is non-trivial for any choice of \( a, b, c \in \mathbb{Z} \). If we put \( a = b = c = 0 \), then we get a parametric solution (defined over \( \mathbb{Q}(i) \)) of the diophantine equation \( p^5 + q^5 = r^5 + s^5 \). After necessary simplifications our solution in this case is of the form (in homogenous form)

\[
\begin{align*}
p &= u^2 + 75v^2, \\
q &= -u^2 + 20iuv + 25v^2, \\
r &= iu^2 + 10(1 + i)uv + 25(2 - 3i)v^2, \\
s &= -iu^2 - 10(1 - i)uv + 25(2 + 3i)v^2.
\end{align*}
\]

Probably this solution is well known but we cannot find it in the literature of subject. We should note that this solution can be used in the construction of a parametric solution (defined over \( \mathbb{Q}(i) \)) of the diophantine equation

\[
p^{5n} + q^{5n} = r^5 + s^5,
\]

where \( n \) is a given positive integer. Indeed, it is easy to see that the diophantine equation \( u^2 + 75v^2 = X^n \) has a parametric solution given by the solution of the system of equations

\[
\begin{align*}u + \sqrt{-75}v &= (t_1 + \sqrt{-75}t_2)^n, \\
u - \sqrt{-75}v &= (t_1 - \sqrt{-75}t_2)^n, \\
X &= t_1^2 + 75t_2^2.
\end{align*}
\]

It is clear that the solutions \( u, v, X \) lead to the polynomial solution of the equation

\[
p^{5n} + q^{5n} = r^5 + s^5.
\]

4. **Possible generalizations of the results**

In this section we consider natural generalizations of the equation defining the hypersurface \( \mathcal{V}_f \) which has been considered in the previous paragraphs.

First natural generalization which came to mind is considering the following hypersurface

\[
\mathcal{V}_{F, G} : F(p) + G(q) = F(r) + G(s),
\]

where \( F(x) = x^5 + ax^3 + bx^2 + cx \), \( G(x) = x^5 + dx^3 + ex^2 + fx \) and \( F(x) - F(0) \neq G(x) - G(0) \). It is clear that in order to find rational points on the \( \mathcal{V}_{F, G} \) we can assume that \( a, b, \ldots, e \in \mathbb{Z} \).

As we will see it is possible to show that for given \( F, G \) satisfied the above conditions the hypersurface \( \mathcal{V}_{F, G} \) contain elliptic surface defined over \( \mathbb{Q} \). In order to show this let us define

\[
(5) \quad p = t - \frac{U}{V}, \quad q = \frac{U}{V}, \quad r = \frac{1}{V}, \quad s = t - \frac{1}{V}.
\]

For \( p, q, r, s \) defined in this way we get

\[
F(p) + G(q) - F(r) - G(s) = -\frac{tV - U - 1}{v^4}H(U, V, t),
\]

where \( H(U, V, t) = \sum_{i+j \leq 3} a_{i,j} U^i V^j \) and
If we make a change of variables

\[ a_{3, 0} = -a_{2, 0} = a_{1, 0} = -a_{0, 0} = 5t, \]
\[ a_{2, 1} = -a + d - 5t^2, \quad a_{1, 1} = a - d, \]
\[ a_{0, 1} = -a + d + 5t^2, \quad a_{1, 2} = b + e + (2a + d)t + 5t^3, \]
\[ a_{0, 2} = -b - c - (a + 2d)t - 5t^3, \quad a_{0, 3} = f - c + (e - b)t + (d - a)t^2. \]

Let us note that we can look on the surface \( S_{F, G} : H(U, V, t) = 0 \) as on the cubic curve defined over the field \( \mathbb{Q}(t) \). This has a \( \mathbb{Q}(t) \)-rational point \( P = (U, V) = (1, 0) \).

So, we can look on \( P \) as on the point at infinity and transform birationally \( S_{F, G} \) onto the elliptic surface \( E_{F, G} \) with the Weierstrass equation of the form

\[ E_{F, G} : Y^2 + a_1 XY + a_3 Y = X^3 + a_2 X^2 + a_4 X + a_6, \]

where \( a_i \) are certain polynomials in \( \mathbb{Z}[t] \) depending on the coefficients of polynomials \( F, G \).

Although it is possible to give exact values of polynomials \( a_i \) we do not give them here due to the fact that they are rather huge polynomials. Instead we give some numerical results concerning the existence of rational points on the hypersurface \( V_{F, G} \), where \( F(x) = x^5 + cx \), \( G(x) = x^5 + fx \) and \( c \neq f \).

In this case the surface \( S_{F, G} \) takes the form

\[ S_{F, G} : 5tU^3 - 5t^2 U^2 V + 5t^3 U V^2 - (c - f)V^3 - 5tU^2 - 5t^3 V^2 + 5tU + 5t^2 V - 5t = 0. \]

If we make a change of variables

\[ (u, v, t) = \left( \frac{50t^4 X^2 + 100t^2 Y(c - f) - X^3}{50t^4(100(c - f)^2 - 10X(c - f) + X^2) - X^3}, \right. \]
\[ \left. \frac{10t X(Y - 5t^2(10(c - f) - X))}{50t^4(100(c - f)^2 - 10X(c - f) + X^2) - x^3}, t \right), \]

with the inverse

\[ (X, Y, t) = \left( \frac{10t U V}{(U - 1)^2}, \right. \]
\[ \left. - \frac{10(c - f)t((c - f)V^3 - 5t^2(U - 1)V(tV - U) - 5t(U - 1)^2 U)}{(U - 1)^2}, t \right) \]

the surface \( S_{F, G} \) is transformed to the form

\[ E_{F, G} : Y^2 = X^3 - 25t^4 X^2 - 2500(c - f)^2 t^4. \]

Unfortunately, we are unable to show that for any pair of integers \( c, f \) it is possible to find a rational number \( t = t(c, f) \) with such a property that the elliptic curve \( E_t : Y^2 = X^3 - 25t^4 X^2 - 2500(c - f)^2 t^4 \) (which is a specialization of surface \( E_{F, G} \) in \( t \)) has a positive rank. However, we check that if \( D = |c - f| \leq 10^5 \) then there exists specialization of \( E_{F, G} \) with positive rank. In the Table below we give values for \( t \) in the case \( D \leq 100 \).
### Rational Points on Certain Quintic Hypersurfaces

| $D$ | $t$ | Nontorsion point on $\mathcal{E}_t$ | $D$ | $t$ | Nontorsion point on $\mathcal{E}_t$ |
|-----|-----|----------------------------------|-----|-----|----------------------------------|
| 1   | 2/45 | (1, 403/405)                     | 51  | 2/35 | (17, 17051/245)                  |
| 2   | 2/45 | (1, 397/405)                     | 52  | 1/15 | (26/5, 598/225)                  |
| 3   | 2/7  | (25, 6075/49)                    | 53  | 1/39 | (25/16, 85025/97344)             |
| 4   | 2/45 | (4/5, 1208/2025)                 | 54  | 4/25 | (17, 1429/125)                   |
| 5   | 1/15 | (5/4, 61/72)                     | 55  | 1/95 | (5/4, 3939/2888)                 |
| 6   | 1/5  | (6, 42/5)                        | 56  | 1/15 | (34/5, 2842/225)                 |
| 7   | 1/3  | (14, 308/9)                      | 57  | 3/35 | (38/5, 836/1225)                 |
| 8   | 2/15 | (25, 1123/9)                     | 58  | 1/5  | (29, 522/5)                      |
| 9   | 1/5  | (13, 216/5)                      | 59  | 2/15 | (25, 1021/9)                     |
| 10  | 1/3  | (25, 1000/9)                     | 60  | 1/17 | (25, 36000/289)                  |
| 11  | 1/15 | (2, 64/45)                       | 61  | 2/15 | (20, 640/9)                      |
| 12  | 2/11 | (25, 14925/121)                  | 62  | 4/5  | (164, 2232/5)                    |
| 13  | 7/11 | (50, 25800/121)                  | 63  | 14/43 | (49, 136857/1849)               |
| 14  | 1/9  | (25, 10100/81)                   | 64  | 4/45 | (64/5, 77312/2025)              |
| 15  | 1/9  | (6, 308/27)                      | 65  | 2/5  | (65, 39)                        |
| 16  | 4/45 | (16, 25792/405)                  | 66  | 1/15 | (6, 14/15)                      |
| 17  | 1/5  | (17, 306/5)                      | 67  | 8/35 | (32, 11136/245)                  |
| 18  | 2/5  | (36, 792/5)                      | 68  | 1/15 | (10, 250/9)                      |
| 19  | 1/5  | (17, 294/5)                      | 69  | 1/35 | (2, 64/245)                     |
| 20  | 1/3  | (25, 500/9)                      | 70  | 1/9  | (25, 9500/81)                    |
| 21  | 1/5  | (42, 134/45)                     | 71  | 4/141| (25, 2484475/19881)             |
| 22  | 1/3  | (25, 200/9)                      | 72  | 5/31 | (25, 79500/961)                  |
| 23  | 4/21 | (25, 51925/441)                  | 73  | 1/25 | (73/20, 19053/5000)             |
| 24  | 3/25 | (9, 2592/125)                    | 74  | 1/90 | (37/40, 49469/64800)            |
| 25  | 10/51| (25, 29987/2601)                 | 75  | 2/11 | (25, 1875/121)                  |
| 26  | 11/15| (169, 91468/45)                  | 76  | 3/29 | (25, 99400/841)                  |
| 27  | 8/25 | (32, 14464/125)                  | 77  | 1/15 | (34, 8888/45)                    |
| 28  | 1/9  | (29/4, 5873/648)                 | 78  | 1/5  | (29, 22/5)                      |
| 29  | 1/9  | (50, 28600/81)                   | 79  | 2/1  | (2084, 84048)                    |
| 30  | 2/7  | (25, 1125/49)                    | 80  | 2/15 | (20, 488/9)                      |
| 31  | 6/35 | (25, 5701/49)                    | 81  | 2/15 | (81, 3627/5)                     |
| 32  | 4/45 | (16, 25408/405)                  | 82  | 2/171| (1, 24209/29241)                |
| 33  | 1/5  | (22, 396/5)                      | 83  | 1/9  | (34, 15512/81)                   |
| 34  | 4/21 | (16, 7424/441)                   | 84  | 1/35 | (42/5, 4218/175)                 |
| 35  | 1/5  | (17, 6/5)                        | 85  | 2/165| (1, 4253/5445)                  |
| 36  | 1/5  | (18, 126/5)                      | 86  | 2/3  | (344, 54352/9)                   |
| 37  | 1/65 | (82/65, 74016/54925)             | 87  | 1/15 | (5597/324, 2016283/29160)        |
| 38  | 6/35 | (25, 5477/49)                    | 88  | 1/61 | (61/36, 1496489/803736)          |
| 39  | 1/45 | (6/5, 604/675)                   | 89  | 1    | (8381, 766104)                   |
| 40  | 3/13 | (25, 11000/169)                  | 90  | 5/31 | (25, 42000/961)                  |
| 41  | 1/5  | (125/4, 1233/8)                  | 91  | 1/85 | (26, 191568/1445)               |
| 42  | 1/13 | (6, 102/13)                      | 92  | 2    | (2516, 114264)                   |
| 43  | 1/45 | (10, 2560/81)                    | 93  | 1/22 | (93/8, 148893/3872)              |
| 44  | 5/21 | (25, 2000/441)                   | 94  | 2/5  | (89, 1833/5)                     |
| 45  | 1/34 | (25/16, 375/2312)                | 95  | 1/5  | (38, 684/5)                      |
| 46  | 1/15 | (46/5, 5842/225)                 | 96  | 5/31 | (25, 4500/961)                   |
| 47  | 12/53| (25, 91925/2809)                 | 97  | 4/15 | (97, 40061/45)                   |
| 48  | 1/34 | (17/8, 21261/9248)               | 98  | 1/3  | (6125/81, 267050/729)            |
| 49  | 1/3  | (49, 1862/9)                     | 99  | 1/7  | (50, 16600/49)                   |
| 50  | 1/45 | (13/4, 18557/3240)               | 100 | 2/15 | (20, 88/9)                      |
Our computations suggest the following

Conjecture 4.1. Let $c, f \in \mathbb{Z}$ and $c \neq f$ and let us consider the elliptic surface

\[ \mathcal{E} : Y^2 = X^3 - 25t^4X^2 - 2500(c - f)^2t^4. \]

Then the set

\[ S = \{ t \in \mathbb{Q} : \text{curve } \mathcal{E}_t \text{ is an elliptic curve and has a positive rank} \} \]

is nonempty.

We firmly believe that the following conjecture is also true.

Conjecture 4.2. Let $a, b, c, d, e, f \in \mathbb{Z}$ and let us consider the elliptic surface

\[ \mathcal{E} : H(U, V, t) = \sum_{i+j \leq 3} a_{i,j}U^iV^j = 0 \]

where $a_{i,j}$ are given by \([\text{I}].\) Then the set

\[ S = \{ t \in \mathbb{Q} : \text{curve } \mathcal{E}_t \text{ is an elliptic curve and has a positive rank} \} \]

is nonempty.

Second generalization which came to mind is considering the following hypersurface

\[ \mathcal{V}^f : f(p, q) = f(r, s), \]

where $f$ is a symmetric quintic polynomial (i.e. $f(x, y) = f(y, x)$), so a polynomial of the form

\[ f(x, y) = \sum_{i=1}^5 a_i (x^i + y^i) + xy \sum_{i=1}^3 b_i (x^i + y^i) + x^2y^2(c_0(x+y) + c_1). \]

As we will see it is quite easy task to show that on the $\mathcal{V}^f$ there is in general infinitely many $\mathbb{Q}(i)$-rational points. In order to prove this we use the substitution given by \([\text{II}].\) and get a quadratic curve, say $C$, defined over $\mathbb{Q}(t)$ with $\mathbb{Q}(i)$-rational points and thus $\mathbb{Q}(i)$-rational curve.

Using now substitution given by \([\text{III}].\) we get

\[ f(p, q) - f(r, s) = -\frac{(U - 1)(V - 1)}{V^4} G(U, V, t), \]

where $G(U, V, t) = \sum_{i+j \leq 2} b_{i,j}u^iv^j$ and

\[ b_{2,0} = 2a_4 - 2b_2 + c_1 + (5a_5 - 3b_3 + c_0), \quad b_{1,0} = 0, \quad b_{0,0} = b_{2,0}, \]
\[ b_{0,2} = 2a_2 + t(3a_3 - b_1) + t^2(4a_4 - b_2) + t^3(5a_5 - b_3), \quad b_{0,1} = b_{1,1}, \quad b_{1,1} = -tb_{2,0}. \]

In order to construct $\mathbb{Q}(i)$-rational points on $\mathcal{V}^f$ we must consider the quadratic $C : G(U, V, t) = 0$. Note that $G(i, 0, t) = 0$, so we can use standard method to parametrize $\mathbb{Q}(i)$-rational points on $C$ and in general we get two parametric solution of the equation $G(U, V, t) = 0$. This implies the existence of two parametric solution of the equation defining the hypersurface $\mathcal{V}^f$.

It is clear that this method cannot be used always. Indeed, if $b_{2,0} \equiv 0 \in \mathbb{Z}[t]$ then the equation $G(U, V, t) = 0$ is reduced to the equation $b_{0,2} = 0$ which has at most three solutions in $\mathbb{Q}(i)$. However, if $b_{2,0} \neq 0$ for some $t$ then the curve $C$ is nontrivial and we can apply our method in order to construct $\mathbb{Q}(i)$-rational points on $\mathcal{V}^f$. This suggests the following
Example 4.4. Consider the following polynomial

\[ f(x, y) = x^5 + y^5 - 5xy(x^2 + y^2) + 5xy(x + y) + 5(x^2 + y^2) - 5(x + y). \]

It should be noted that hypersurface \( V^f \) for this polynomial was considered by Consani and Scholten in [2].

In this case we have that \( C : U^2 - tUV + (1 - t + t^2)V^2 - tV + 1 = 0 \). Using the \( \mathbb{Q}(i) \)-rational point \( P = (i, 0) \) we get parametrization of \( C \) in the form

\[ U = \frac{-iu^2 - tu + i(t^2 - t + 1)}{u^2 - tu + t^2 - t + 1}, \quad V = \frac{(1 + i)(t - (1 + i)u)}{u^2 - tu + t^2 - t + 1}. \]

Using obtained parametrization of \( C \) we get two parametric solution of the equation defining \( V^f \) in the form

\[ p = \frac{(1 + i)(u^2 - (2 - i)tu - it^2 + t - 1)}{2(t - (1 + i)u)}, \quad q = \frac{(1 + i)(-u^2 - itu + t^2 - t + 1)}{2(t - (1 + i)u)}, \]

\[ r = \frac{(1 + i)(iu^2 - (2 + i)tu + t^2 - it + i)}{2(t - (1 + i)u)}, \quad s = \frac{(1 - i)(u^2 - tu + t^2 - t + 1)}{2(t - (1 + i)u)}. \]

We should note that for specific choice of numbers \( a_i, b_i, c_k \) it is very likely that there is a rational number \( t_0 \) with such a property that the quadric \( C_{t_0} : G(U, V, t_0) = 0 \) has a rational point. Then we can use standard method of parametrization of quadrics in order to get rational solutions of the equation defining the hypersurface \( V^f \). We illustrated this procedure with an example.

Example 4.5. Consider the polynomial \( f \) given by (7) and suppose that \( c_1 = 29 \) and all remaining coefficients in the polynomial \( f \) are equal to one. Let us put \( t = 2 \). Then on the curve \( C_2 : 32U^2 - 32UV + 11V^2 - 32V + 32 = 0 \) we have rational point \( P = (3, 8) \) and parametrization of the curve \( C_2 \) is of the form

\[ U = \frac{160w^2 - 144w + 33}{32w^2 - 32w + 11}, \quad V = \frac{8(32w^2 - 24w + 5)}{32w^2 - 32w + 11}. \]

Finally, we get rational curve on the hypersurface \( V^f : f(p, q) = f(r, s) \) of the form

\[ p = \frac{96w^2 - 48w + 7}{8(32w^2 - 24w + 5)}, \quad q = \frac{160w^2 - 144w + 33}{8(32w^2 - 24w + 5)}, \]

\[ r = \frac{224w^2 - 160w + 29}{8(32w^2 - 24w + 5)}, \quad s = \frac{32w^2 - 32w + 11}{8(32w^2 - 24w + 5)}. \]
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