A HIGHER ORDER BLOKH-ZYablov PROPAGATION RULE FOR HIGHER ORDER NETS

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Abstract

Higher order nets were introduced by Dick as a generalization of classical \((t, m, s)\)-

nets, which are point sets frequently used in quasi-Monte Carlo integration algo-

rithms. Essential tools in finding such point sets of high quality are propagation

rules, which make it possible to generate new higher order nets from existing higher

order nets and even classical \((t, m, s)\)-nets. Such propagation rules for higher order

nets were first considered by the authors in [9] and further developed in [2]. In

[4] Blokh and Zyablov established a very general propagation rule for linear codes.

This propagation rule has been extended to \((t, m, s)\)-nets by Schürer and Schmid

in [19]. In this paper we show that this propagation rule can also be extended to

higher order nets. Examples indicate that this propagation rule yields new higher

order nets with significantly higher quality.

Keywords: Higher order nets, linear codes, duality theory, propagation rules.

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1 Introduction

Quasi-Monte Carlo (QMC) rules are quadrature rules for approximating a high-dimensional

integral \(\int_{[0,1]^s} f(x) \, dx\) by the average of certain function values, \(\frac{1}{N} \sum_{i=0}^{N-1} f(x_i)\), where

\(x_0, x_1, \ldots, x_{N-1}\) are deterministically chosen integration nodes. The crucial question is

how to choose the points \(x_i\) in order to obtain low integration errors. For this reason,

several different classes of point sets and their properties in relation to integration prob-

lems have been studied over the past decades (see, e.g., the monographs [11, 13, 20] for

introductions to this topic). Note that by a point set we always mean a multiset, i.e.,

points are allowed to occur repeatedly. One prominent class of point sets commonly used

in QMC algorithms are \((t, m, s)\)-nets as introduced by Niederreiter (cf. [12, 13, 14]). It is

known that \((t, m, s)\)-nets can yield the optimal order of convergence (up to powers of the

logarithm of the total number of points) of the integration error in QMC algorithms for

functions of finite variation in the sense of Hardy and Krause (see again [12, 13, 14]).

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In [5, 6], Dick generalised Niederreiter’s \((t, m, s)\)-nets to so-called higher order nets which have the convenient property that the corresponding QMC integration algorithms yield higher order convergence of the error for smoother functions. To be more precise, in [6] digital higher order nets were introduced, the construction of which, as it is also the case for Niederreiter’s digital nets, is based on linear algebra over finite fields. In what follows, let \( \mathbb{F}_q \) be the finite field of prime-power order \( q \). Moreover, we denote by \( \mathbb{N} \) and \( \mathbb{N}_0 \) the set of positive integers and nonnegative integers, respectively.

**Definition 1.** Let \( q \) be a prime power and let \( n, m, s \in \mathbb{N} \). Let \( C_1, \ldots, C_s \) be \( n \times m \) matrices over the finite field \( \mathbb{F}_q \) of order \( q \). We construct \( q^m \) points in \([0,1]^s\) in the following way: For \( 0 \leq h < q^n \) let \( h = h_0 + h_1 q + \cdots + h_{m-1} q^{m-1} \) be the base \( q \) representation of \( h \).

Consider an arbitrary but fixed bijection \( \eta : \{0,1,\ldots,q-1\} \to \mathbb{F}_q \) where \( \eta(0) \) is the zero element in \( \mathbb{F}_q \). Identify \( h \) with the vector \( h := (\eta(h_0), \ldots, \eta(h_{m-1})) \in \mathbb{F}_q^m \). For \( 1 \leq j \leq s \), we multiply the matrix \( C_j \) by \( h \),

\[
C_j \cdot h^\top := (y_{j,1}(h), \ldots, y_{j,n}(h)) \in \mathbb{F}_q,
\]

and set

\[
x_h^{(j)} := \frac{\eta^{-1}(y_{j,1}(h))}{q} + \cdots + \frac{\eta^{-1}(y_{j,n}(h))}{q^n}.
\]

Finally, set \( x_h := (x_h^{(1)}, \ldots, x_h^{(s)}) \). The point set consisting of the points \( x_0, \ldots, x_{q^n-1} \) is called a digital net over \( \mathbb{F}_q \). The matrices \( C_1, \ldots, C_s \) are called the generating matrices of the digital net.

As it can be seen from Definition 1, the properties of the points of a digital net (such as, e.g., their distribution in the unit cube) are determined by properties of the generating matrices \( C_1, \ldots, C_s \). These properties are, in the currently most general form of digital nets as introduced in [6], described by additional parameters \( t, \alpha, \beta \), which is why those nets are referred to as \((t, \alpha, \beta, n \times m, s)\)-nets. The exact roles of the parameters \( t, \alpha \), and \( \beta \) are outlined in the following definition.

**Definition 2.** Let \( n, m, \alpha \in \mathbb{N} \), let \( 0 < \beta \leq \min(1, \alpha m/n) \) be a real number. Let \( \mathbb{F}_q \) be the finite field of prime power order \( q \) and let \( C_1, \ldots, C_s \in \mathbb{F}_q^{n \times m} \) with \( C_j := (\vec{c}_{j,1}, \ldots, \vec{c}_{j,n})^\top \). The digital net with generating matrices \( C_1, \ldots, C_s \) is called a digital \((t, \alpha, \beta, n \times m, s)\)-net for a natural number \( t \), \( 0 \leq t \leq \beta n \), if the following condition is satisfied. For each choice of \( 1 \leq i_{\nu_j} < \cdots < i_{j,1} \leq n \), where \( \nu_j \geq 0 \) for \( j = 1, \ldots, s \), with

\[
i_{1,1} + \cdots + i_{1,\min(\nu_1, \alpha)} + \cdots + i_{s,1} + \cdots + i_{s,\min(\nu_s, \alpha)} \leq \beta n - t
\]

the vectors

\[
\vec{c}_{1,i_{1,\nu_1}}, \ldots, \vec{c}_{1,i_{1,1}}, \ldots, \vec{c}_{s,i_{s,\nu_s}}, \ldots, \vec{c}_{s,i_{s,1}}
\]

are linearly independent over \( \mathbb{F}_q \).

If \( t \) is the smallest non-negative integer such that the digital net generated by \( C_1, \ldots, C_s \) is a digital \((t, \alpha, \beta, n \times m, s)\)-net, then we call the digital net a strict digital \((t, \alpha, \beta, n \times m, s)\)-net.

Note that Definition 2 implies that \( t \) must be chosen such that \( \nu_1 + \cdots + \nu_s \leq m \) holds whenever (1) is satisfied. (Note that \( \nu_j \leq i_{j,1} \)).
The definition of classical digital \((t, m, s)\)-nets is obtained by choosing \(\alpha = \beta = 1\) and \(m = n\) in Definition 2.

Digital higher order nets are a subclass of general higher order nets which were introduced in \([8]\). While we denote digital higher order nets as \((t, \alpha, \beta, n \times m, s)\)-nets, by which we emphasise the role of the generating matrices in the construction, general higher order nets are denoted as \((t, \alpha, \beta, n, m, s)\)-nets.

We give the definitions and some properties of \((t, \alpha, \beta, n, m, s)\)-nets in base \(b\). We follow \([2]\) in our presentation.

Let \(n, s \geq 1, b \geq 2\) be integers. For \(\nu = (\nu_1, \ldots, \nu_s) \in \{0, \ldots, n\}^s\) let \(|\nu|_1 = \sum_{j=1}^{s} \nu_j\) and define \(i_{\nu} = (i_{1,1}, \ldots, i_{1,\nu_1}, \ldots, i_{s,1}, \ldots, i_{s,\nu_s})\) with integers \(1 \leq i_{j,\nu_j} < \cdots < i_{j,1} \leq n\) in case \(\nu_j > 0\) and \(\{i_{j,1}, \ldots, i_{j,\nu_j}\} = \emptyset\) in case \(\nu_j = 0\), for \(j = 1, \ldots, s\). For given \(\nu\) and \(i_{\nu}\) let \(a_{\nu} \in \{0, \ldots, b-1\}^{|\nu|_1}\), which we write as \(a_{\nu} = (a_{1,1}, \ldots, a_{1,\nu_1}, \ldots, a_{s,1}, \ldots, a_{s,\nu_s})\).

A generalised elementary interval in base \(b\) is a subset of \([0, 1)^s\) of the form

\[
J(i_{\nu}, a_{\nu}) = \prod_{j=1}^{s} \bigcup_{l \in \{1, \ldots, n\} \setminus \{i_{j,1}, \ldots, i_{j,\nu_j}\}} \left[ \frac{a_{j,1}}{b} + \cdots + \frac{a_{j,n}}{b^n}, \frac{a_{j,1}}{b} + \cdots + \frac{a_{j,n}}{b^n} + \frac{1}{b^n} \right].
\]

From \([3]\) Lemmas 1 and 2 it is known that for \(\nu \in \{0, \ldots, n\}^s\) and \(i_{\nu}\) defined as above and fixed, the generalised elementary intervals \(J(i_{\nu}, a_{\nu})\) where \(a_{\nu}\) ranges over all elements from the set \(\{0, \ldots, b-1\}^{\nu_1}\) form a partition of \([0, 1)^s\) and the volume of \(J(i_{\nu}, a_{\nu})\) is \(b^{-|\nu|_1}\).

We can now give the definition of \((t, \alpha, \beta, n, m, s)\)-nets based on \([3]\) Definition 4.

**Definition 3.** Let \(n, m, s, \alpha \geq 1\) be natural numbers, let \(0 < \beta \leq \min(1, \alpha m/n)\) be a real number, and let \(0 \leq t \leq \beta n\) be an integer. Let \(b \geq 2\) be an integer and \(P = \{x_0, \ldots, x_{b^m-1}\}\) be a multiset in \([0, 1)^s\). We say that \(P\) is a \((t, \alpha, \beta, n, m, s)\)-net in base \(b\), if for all integers \(1 \leq i_{j,\nu_j} < \cdots < i_{j,1}\), where \(0 \leq \nu_j \leq n\), with

\[
\sum_{j=1}^{s} \sum_{i=1}^{\min(\nu_j, \alpha)} i_{j,i} \leq \beta n - t,
\]

where for \(\nu_j = 0\) we set the empty sum \(\sum_{i=1}^{0} i_{j,i} = 0\), the generalised elementary interval \(J(i_{\nu}, a_{\nu})\) contains exactly \(b^{|\nu|_1}\) points of \(P\) for each \(a_{\nu} \in \{0, \ldots, b-1\}^{\nu_1}\).

A \((t, \alpha, \beta, n, m, s)\)-net in base \(b\) is called a strict \((t, \alpha, \beta, n, m, s)\)-net in base \(b\), if it is not a \((u, \alpha, \beta, n, m, s)\)-net in base \(b\) with \(u < t\).

Informally we refer to \((t, \alpha, \beta, n, m, s)\)-nets as higher order nets.

Note that in the definition above the specific order of elements of a multiset is not important. The parameter \(t\) is often referred to as the quality parameter of the net. By the strength of the net one means the quantity \(\beta n - t\).

The advantage of the more general concept due to \([8]\) (in comparison to classical \((t, m, s)\)-nets) is that \((t, \alpha, \beta, n, m, s)\)-nets in base \(b\) can exploit the smoothness \(\alpha\) of a function \(f\) (which is not the case for the classical concepts of \((t, m, s)\)-nets and \((t, s)\)-sequences). More precisely, we have the following result from \([1]\) (and \([5, 6]\) for the digital case).
Proposition 1. Let \( \{x_0, \ldots, x_{b^m-1}\} \) be a \((t, \alpha, \beta, n, m, s)\)-net in base \( b \). Let \( f : [0, 1]^s \to \mathbb{R} \) have mixed partial derivatives up to order \( \alpha \geq 2 \) in each variable which are square integrable. Then
\[
\left| \int_{[0,1]^s} f(x) \, dx - \frac{1}{b^m} \sum_{h=0}^{b^m-1} f(x_h) \right| = O \left( b^{-(1-1/\alpha)(\beta n - t)}(\beta n - t)^{\alpha s} \right).
\]
If \( \{x_0, \ldots, x_{b^m-1}\} \) is a digital \((t, \alpha, \beta, n, m, s)\)-net in base \( b \), then
\[
\left| \int_{[0,1]^s} f(x) \, dx - \frac{1}{b^m} \sum_{h=0}^{b^m-1} f(x_h) \right| = O \left( b^{-(\beta n - t)}(\beta n - t)^{\alpha s} \right).
\]
Additionally, the following results are known. If \( \alpha = \beta = 1 \) and \( n = m \), then the integration error is of order \( O(b^{-m+1+m^s}) \), see [13].

Proposition 1 underlines the importance of knowing explicit constructions of higher order nets with a large value of \( \beta n - t \).

In a series of papers, see for example [8, 15, 17, 18], so-called propagation rules for classical \((t, m, s)\)-nets were introduced, which allow one to construct new digital nets from known ones and thereby improve on the parameters, in particular on the strength, of those nets. That such constructions are very useful can be seen in [19], where the best known parameters of classical \((t, m, s)\)-nets are listed. A first step in establishing and using propagation rules for digital higher order nets (i.e., \((t, \alpha, \beta, n \times m, s)\)-nets) was made in the paper [9], where a series of propagation rules for such point sets were discussed. In the later paper [2], these propagation rules were extended to the case where the underlying nets need no more be digital, i.e., the rules apply to the more general class of \((t, \alpha, \beta, n, m, s)\)-nets.

In this paper, it is our aim to show a further propagation rule for higher order nets, which is a modification of a propagation rule for linear codes by Blokh and Zyablov [4]. Our main result, which will be prepared in Section 3 and stated in Section 4, is an extension of a Blokh-Zyablov propagation rule for classical \((t, m, s)\)-nets by Schürer and Schmid in [18]. Before (Section 2), we are going to review duality theory of higher order nets, as it was introduced in [9] and [2]. Duality theory will be the main tool in proving our new propagation rule for higher order nets. Finally, we outline exemplary numerical results in Section 5 which are going to highlight the strength of our new propagation rule.

2 Duality theory

In this section we review the basics of duality theory for higher order nets, as this is going to be the key ingredient for showing our main results. Duality theory, as introduced for classical \((t, m, s)\)-nets by Niederreiter and Pirsic [16], is a helpful tool in the analysis and construction of digital nets. In [9] it was extended to digital higher order nets, and it was further extended to the non-digital case in [2]. We keep close to [2] and [9] in the following two sections.
2.1 Duality theory for digital higher order nets

In [9], the dual of the row space of a digital \((t, \alpha, \beta, n \times m, s)\)-net was introduced. Given the generating matrices \(C_1, \ldots, C_s\) of a digital higher order net, let

\[
C = (C_1^\top | \cdots | C_s^\top) \in \mathbb{F}_q^{m \times sn}.
\]

The points of a digital net are obtained from the linear subspace \(C\) of \(\mathbb{F}_q^{ns}\) of dimension at most \(m\), given as the row space of \(C\) by Definition 1. The dual space of \(C\) is given by

\[
N = \{x \in \mathbb{F}_q^{ns} : x \cdot y = 0 \in \mathbb{F}_q\ \text{for all} \ y \in \mathcal{P}\},
\]

i.e., \(N\) is the null space of \(C\).

Let \(A = (a_1, \ldots, a_s) \in \mathbb{F}_q^{ns}\), where \(a_j = (a_{j,1}, \ldots, a_{j,n}) \in \mathbb{F}_q^n\). For \(1 \leq j \leq s\) with \(a_j \neq 0\) let \(1 \leq \nu_j \leq n\) denote the number of nonzero elements of \(a_j\) and let \(1 \leq i_{j,\nu_j} < \cdots < i_{j,1} \leq n\) denote the indices of the nonzero elements \(a_{j,i_{j,\nu_j}}, \ldots, a_{j,i_{j,1}} \in \mathbb{F}_q^* = \mathbb{F}_q \setminus \{0\}\) of \(a_j\) (thus, \(a_{j,l} = 0\) for \(l \in \{1, \ldots, n\} \setminus \{i_{j,1}, \ldots, i_{j,\nu_j}\}\)). Let \(\alpha \geq 1\) be an integer. Then we define

\[
\mu_{\alpha,n}(a_j) = \begin{cases} 0 & \text{for } a_j = 0, \\ \sum_{k=1}^{\min(\alpha, \nu_j)} i_{j,k} & \text{otherwise,} \end{cases}
\]

and

\[
\mu_{\alpha,n}(A) = \sum_{j=1}^s \mu_{\alpha,n}(a_j).
\]

Let \(N \subseteq \mathbb{F}_q^{ns}\) and let \(|N|\) denote the number of elements of \(N\). For \(A, B \in \mathcal{N}\) we define the distance

\[
d_{\alpha,n}(A, B) = \mu_{\alpha,n}(A - B).
\]

Further we define

\[
\delta_{\alpha,n}(N) = \min_{A, B \in \mathcal{N}} d_{\alpha,n}(A, B).
\]

We always have \(\delta_{\alpha,n}(N) \geq 1\) and \(\delta_{\alpha,n}(N) \geq \delta_{\alpha',n}(N)\) for \(\alpha \geq \alpha' \geq 1\).

The following definition and result was first shown in [9] and is a generalisation of [16].

**Proposition 2.** The matrices \(C_1, \ldots, C_s \in \mathbb{F}_q^{m \times m}\) generate a digital \((t, \alpha, \beta, n \times m, s)\)-net over \(\mathbb{F}_q\) if and only if

\[
\delta_{\alpha,n}(N) \geq \beta n - t + 1,
\]

where \(N\) is the dual space of the row space \(C\). If \(C_1, \ldots, C_s\) generate a strict digital \((t_0, \alpha, \beta, n \times m, s)\)-net over \(\mathbb{F}_q\), where we assume that \(\beta n\) is an integer, then

\[
\delta_{\alpha,n}(N) = \beta n - t_0 + 1.
\]

2.2 Duality theory for general higher order nets

Here we review duality theory for higher order nets which also applies to point sets not obtained by the digital construction scheme. The basic tool are Walsh functions in integer base \(b \geq 2\) whose definition and basic properties are recalled in the following. We repeat some background from [2], where also the proofs of the results mentioned in this section can be found.
Definition 4. Let $b \geq 2$ be an integer and represent $k \in \mathbb{N}_0$ in base $b$, $k = \kappa_{a-1} b^{a-1} + \cdots + \kappa_0$, with $\kappa_i \in \{0, \ldots, b - 1\}$. Further let $\omega_b = e^{2\pi i / b}$ be the $b$th root of unity. Then the $k$th $b$-adic Walsh function \( \text{wal}_k(x) : [0, 1) \rightarrow \{1, \omega_b, \ldots, \omega_b^{b-1}\} \) is given by
\[
\text{wal}_k(x) = \omega_b^{\xi_1 \kappa_0 + \cdots + \xi_a \kappa_{a-1}},
\]
for $x \in [0, 1)$ with base $b$ representation $x = \xi_1 b^{-1} + \xi_2 b^{-2} + \cdots$ (unique in the sense that infinitely many of the $\xi_i$ are different from $b - 1$).

For dimension $s \geq 2$, $\mathbf{x} = (x_1, \ldots, x_s) \in [0, 1)^s$, and $\mathbf{k} = (k_1, \ldots, k_s) \in \mathbb{N}_0^s$, we define $\text{wal}_\mathbf{k} : [0, 1)^s \rightarrow \{1, \omega_b, \ldots, \omega_b^{b-1}\}$ by
\[
\text{wal}_\mathbf{k}(\mathbf{x}) = \prod_{j=1}^s \text{wal}_{k_j}(x_j).
\]

The following notation will be used throughout the paper: By $\oplus$ we denote digitwise addition modulo $b$, i.e., for $x, y \in [0, 1)$ with base $b$ expansions $x = \sum_{i=1}^\infty \xi_i b^{-i}$ and $y = \sum_{i=1}^\infty \eta_i b^{-i}$, we define
\[
x \oplus y = \sum_{i=1}^\infty \zeta_i b^{-i},
\]
where $\zeta_i \in \{0, \ldots, b - 1\}$ is given by $\zeta_i \equiv \xi_i + \eta_i \pmod{b}$. Let $\ominus$ denote digitwise subtraction modulo $b$ (for short we use $\ominus x := 0 \ominus x$). In the same fashion we also define digitwise addition and digitwise subtraction of nonnegative integers based on the $b$-adic expansion. For vectors in $[0, 1)^s$ or $\mathbb{N}_0^s$, the operations $\oplus$ and $\ominus$ are carried out componentwise. Throughout the paper, we always use the same base $b$ for the operations $\oplus$ and $\ominus$ as is used for Walsh functions. Further, we call $x \in [0, 1)$ a $b$-adic rational if it can be written in a finite base $b$ expansion. The following simple properties of Walsh functions will be used several times.

For all $k, l \in \mathbb{N}_0$ and all $x, y \in [0, 1)$, with the restriction that if $x, y$ are not $b$-adic rationals, then $x \oplus y$ is not allowed to be a $b$-adic rational, we have $\text{wal}_k(x) \cdot \text{wal}_l(x) = \text{wal}_{kl}(x)$ and $\text{wal}_k(x) \cdot \text{wal}_l(y) = \text{wal}_k(x \oplus y)$. Furthermore, $\text{wal}_k(x) = \text{wal}_{c_k}(x)$.

Now we turn to duality theory for nets. Let $K^*_{b,r} = \{0, \ldots, b^r - 1\}^s$. We also assume there is an ordering of the elements in $K^*_{b,r}$ which can be arbitrary but needs to be the same in each instance, i.e., let $K^*_{b,r} = \{k_0, \ldots, k_{b^r - 1}\}$. (Note that $|K^*_{b,r}| = b^{sr}$.) By this we mean that in expressions like $\sum_{k \in K^*_{b,r}} (a_{k,l})_{k,l \in K^*_{b,r}}$, and $(c_k)_{k \in K^*_{b,r}}$ the elements $\mathbf{k}$ and $\mathbf{l}$ run through the set $K^*_{b,r}$ always in the same order.

The following $b^{sr} \times b^{sr}$ matrix plays a central role in the duality theory for higher order nets,
\[
W_r := \left(\text{wal}_k(b^{-r}\mathbf{l})\right)_{k,l \in K^*_{b,r}}.
\]
We call $W_r$ a Walsh matrix.

In the following we denote by $A^*$ the conjugate transpose of a matrix $A$ over the complex numbers $\mathbb{C}$, i.e., $A^* = \overline{A^T}$. The following lemma was shown in [2].

Lemma 1. The Walsh matrix $W_r$ is invertible and its inverse is given by $W_r^{-1} = b^{-sr}W_r^*$.

Let now $b \geq 2$ and $r, N \geq 1$ be integers. For a multiset $P = \{\mathbf{x}_0, \ldots, \mathbf{x}_{N-1}\}$ in $[0, 1)^s$ and $\mathbf{k} \in K^*_{b,r}$ we define
\[
c_k = c_k(P) := \sum_{h=0}^{N-1} \text{wal}_k(\mathbf{x}_h).
\]
(note that $|c_k| \leq N$ and $c_0 = N$) and the vector
\[ \vec{C} = \vec{C}(P) := (c_k)_{k \in \mathcal{K}_{b,r}}. \]  \hfill (2)

For $a = (a_1, \ldots, a_s) \in \mathcal{K}_{b,r}^s$ define the elementary $b$-adic interval
\[ E_a := \prod_{j=1}^{s} \left[ \frac{a_j}{b^r}, \frac{a_j + 1}{b^r} \right). \]

We also have the following lemma.

**Lemma 2.** We have
\[ \sum_{k \in \mathcal{K}_{b,r}^s} \text{wal}_k(x \ominus y) = \begin{cases} |\mathcal{K}_{b,r}^s| & \text{if } x, y \in E_a \text{ for some } a \in \mathcal{K}_{b,r}^s, \\ 0 & \text{otherwise.} \end{cases} \]

Let $x \in E_a$ for some $a \in \mathcal{K}_{b,r}^s$. Then, using Lemma 2 we have
\[
\frac{1}{|\mathcal{K}_{b,r}^s|} \sum_{k \in \mathcal{K}_{b,r}^s} c_k \text{wal}_k(x) = \frac{1}{|\mathcal{K}_{b,r}^s|} \sum_{k \in \mathcal{K}_{b,r}^s} \sum_{h=0}^{N-1} \text{wal}_k(x_h \ominus x) \\
= \sum_{h=0}^{N-1} \frac{1}{|\mathcal{K}_{b,r}^s|} \sum_{k \in \mathcal{K}_{b,r}^s} \text{wal}_k(x_h \ominus x) \\
= |\{ h : x_h \in E_a \}| = m_a.
\]

**Definition 5.** Let $b \geq 2$ and $r, N \geq 1$ be integers. Let $P = \{x_0, \ldots, x_{N-1}\}$ be a multiset in $[0, 1)^s$ and let $\mathcal{K}_{b,r}^s = \{0, \ldots, b^r - 1\}^s$.

1. For $a \in \mathcal{K}_{b,r}^s$ let
\[ m_a = m_a(P) := |\{ h : x_h \in E_a \}| \]
and
\[ \vec{M} = \vec{M}(P) := (m_a)_{a \in \mathcal{K}_{b,r}^s}. \]

Then we call the vector $\vec{M}$ the **point set vector** (with resolution $r$).

2. The vector $\vec{C} = \vec{C}(P)$ from (2) is called the **dual vector** (with respect to the Walsh matrix $W_r$).

3. The set
\[ \mathcal{D}_r = \mathcal{D}_r(P) := \{ k \in \mathcal{K}_{b,r}^s : c_k \neq 0 \} \]
is called the **dual set** (with respect to the Walsh matrix $W_r$).

The relationship between a point set vector and its dual vector is stated in the following theorem.

**Theorem 1.** Let $P = \{x_0, \ldots, x_{N-1}\}$ be a multiset in $[0, 1)^s$ and let $r \in \mathbb{N}$. Let $\vec{M}$ be the point set vector with resolution $r$ and $\vec{C}$ be the dual vector with respect to $W_r$ defined as above. Then
\[
\frac{1}{|\mathcal{K}_{b,r}^s|} W_r \vec{C} = \vec{M} \quad \text{and} \quad \vec{C} = W_r^* \vec{M}. \] \hfill (3)
The vector $\vec{C}$ carries sufficient information to construct a point set in the following way: Given $\vec{C}$, we can use Theorem 1 to determine how many points are to be placed in the interval $E_{\alpha, a} \in K_{b,r}^s$. We remark that at the functional level, the vector $\vec{C}$ is comparable to the generator matrices of a digital net, which completely determine the point set.

Note that for the $(t, \alpha, \beta, n, m, s)$-net property it is of no importance where exactly within an interval $E_{\alpha, a} \in K_{b,n}^s$, the points are placed. Hence we can reconstruct a net from a dual vector with respect to $W_r$ provided that $r \geq |\beta n| - t$. In words, if one knows the dual vector of a net, then one can use this dual vector to obtain the net via Theorem 1 provided that the resolution is greater than or equal to the strength of the net.

In analogy, the dual space of a digital net also allows us to reconstruct the original point set, see [16]. Although $\vec{C}$ is different from the dual space for digital nets, it contains the same information and can be used in a manner similar to the dual space. In case $P$ is a digital $(t, \alpha, \beta, n \times m, s)$-net, the dual set $D_n$ defined in Definition 5 coincides with the dual space defined in [9] intersected with $K_{b,n}^s$, and if $P$ is a digital $(t, m, s)$-net, it coincides with the dual space in [16] intersected with $K_{b,m}^s$.

Although the above results hold for arbitrary point sets, in the following we consider point sets which are nets and show how to relate the quality of a $(t, \alpha, \beta, n, m, s)$-net to its dual set. To this end we need to introduce a function which was first introduced in [6] in the context of applying digital nets to quasi-Monte Carlo integration of smooth functions and which is related to the quality of suitable digital nets. For $k \in \mathbb{N}_0$ and $\alpha \geq 1$ let

$$
\mu_\alpha(k) = \begin{cases} 
a_1 + \cdots + a_{\min(\nu, \alpha)} & \text{for } k > 0, \\
0 & \text{for } k = 0,
\end{cases}
$$

where for $k > 0$ we assume that $k = \nu_1 b^{a_1} - 1 + \cdots + \nu_\alpha b^{a_\nu} - 1$ with $0 < \nu_1, \ldots, \nu_\alpha < b$ and $1 \leq a_\nu < \cdots < a_1$. Note that for $\alpha = 1$ we obtain the well-known Niederreiter-Rosenbloom-Tsfasman (NRT) weight (see, for example, [11] Section 7.1).

For a vector $k = (k_1, \ldots, k_s) \in \mathbb{N}_0^s$ we define $\mu_\alpha(k) = \mu_\alpha(k_1) + \cdots + \mu_\alpha(k_s)$ and for a subset $Q$ of $K_{b,r}^s$ with $Q \setminus \{0\} \neq \emptyset$ and $\alpha \geq 1$ define

$$
\rho_\alpha(Q) := \min_{k \in Q \setminus \{0\}} \mu_\alpha(k).
$$

For $Q \subseteq \{0\}$ we set $\rho_\alpha(Q) = r + 1$.

Let $P = \{x_0, \ldots, x_{N-1}\} \subset [0, 1)^s$. In the following we consider for which cases we have $D_r(P) = \{0\}$ (note that $0 \in D_r(P)$ for any point set $P$ with at least one point). If $D_r(P) = \{0\}$, then we have $c_0 \neq 0$ and $c_k = 0$ for all $k \in K_{b,r}^s \setminus \{0\}$. By Theorem 4 we have $\hat{M}(P) = c_0 b^{-r} (1, 1, \ldots, 1)^T$, that is, each box $E_{\alpha, a}$ contains exactly $c_0 b^{-r}$ points for all $a \in K_{b,r}^s$ and $P$ consists of $N = c_0$ points altogether. This is the only case for which $D_r(P) = \{0\}$.

Conversely, since the number of points in $E_{\alpha, a}$ must be an integer, it follows that $c_0 b^{-r} \in \mathbb{N}$, i.e., $b^r$ divides $c_0$ and therefore $b^s$ divides $N$. From this we conclude that if we choose a resolution $r \in \mathbb{N}$ such that $b^s > N$, i.e., $r > \frac{1}{s} \log_b N$, then $D_r(P) \neq \{0\}$. For a net with $N = b^m$ points this means that we require $r > m/s$.

The following theorem establishes a relationship between $\rho_\alpha(Q)$ and the quality of a $(t, \alpha, \beta, n, m, s)$-net.
Theorem 2. Let \( P = \{\mathbf{x}_0, \ldots, \mathbf{x}_{2^n-1}\} \subset [0,1)^s \) be a multiset. Then \( P \) is a \((t, \alpha, \beta, n, m, s)\)-net in base \( b \) and only if \( \rho_a(\mathcal{D}_{[\beta n]^{-1}}) \geq \lfloor \beta n \rfloor - t + 1 \). If \( P \) is a strict \((t_0, \alpha, \beta, n, m, s)\)-net in base \( b \), then \( \rho_a(\mathcal{D}_{[\beta n]-t_0}) = \lfloor \beta n \rfloor - t_0 + 1 \).

Let now \( P \) be a strict \((t_0, \alpha, \beta, n, m, s)\)-net in base \( b \). Let \( r \geq \lfloor \beta n \rfloor - t_0 \). Then \( \mathcal{D}_r \supseteq \mathcal{D}_{[\beta n]-t_0} \) and \( \mathcal{D}_r \setminus \mathcal{D}_{[\beta n]-t_0} \subseteq \mathcal{K}_k^s \setminus \mathcal{K}_k^s ([\beta n]-t_0) \). For any \( k \in \mathcal{K}_k^s \setminus \mathcal{K}_k^s ([\beta n]-t_0) \) we have \( \mu_k(\mathcal{D}_r) \geq \lfloor \beta n \rfloor - t_0 + 1 \). Theorem 2 implies that \( \rho_{\alpha}(\mathcal{D}_{[\beta n]-t_0}) = \lfloor \beta n \rfloor - t_0 + 1 \) and hence \( \rho_{\alpha}(\mathcal{D}_r) = \rho_{\alpha}(\mathcal{D}_{[\beta n]-t_0}) = \lfloor \beta n \rfloor - t_0 + 1 \). In particular, for all \( r, r' \geq \lfloor \beta n \rfloor - t_0 \) we have

\[
\rho_{\alpha}(\mathcal{D}_r) = \rho_{\alpha}(\mathcal{D}_{r'}) = \rho_{\alpha}(\mathcal{D}_n) = \lfloor \beta n \rfloor - t_0 + 1, \tag{4}
\]

since \( n \geq \lfloor \beta n \rfloor - t_0 \).

3 A generalized Blokh-Zyablov construction

In this section we generalize the Blokh-Zyablov construction for codes from [4] and for classical nets from [18]. The proofs closely follow those in [18].

We call \( \mathcal{N} \subseteq \mathbb{F}_q^{ns} \) an \((s, n), \alpha, K, \delta\)\(q\)-space if \( K = [\mathcal{N}] \) and \( \delta = \delta_{\alpha n}(\mathcal{N}) \). If \( \mathcal{N} \) is a linear subspace of \( \mathbb{F}_q^{ns} \) then we call \( \mathcal{N} \) a linear \([s, n, \alpha, k, \delta]\)\(q\)-space if \( [\mathcal{N}] = q^k \) and \( \delta = \delta_{\alpha n}(\mathcal{N}) \).

Note that an \((s, n), 1, K, \delta\)\(q\)-space is a generalized \((s, n), K, \delta\)\(q\)-code and a linear \([s, n), 1, k, \delta\)\(q\)-space is a generalized linear \([s, n), k, \delta]\)\(q\)-code in the sense of Schürer and Schmid [18, 19]. Note, furthermore, that in the special case \( n = 1 \) an \((s, 1), k, \delta\)\(q\)-code is an ordinary linear \([s, k, \delta]\)\(q\)-code (see [18] or [19] for details).

Lemma 3. Let \( \alpha, \alpha' \geq 1 \) and let

\[
\{0\} = \mathcal{N}_0' \subset \mathcal{N}_1' \subset \cdots \subset \mathcal{N}_r' = \mathbb{F}_q^{ns'}
\]

be a chain of linear \([s', n', \alpha', k'_u, \delta'_u]\)\(q\)-spaces and let \( \mathbf{v}_1, \ldots, \mathbf{v}_{n's'} \in \mathbb{F}_q^{n's'} \) denote vectors such that \( \mathcal{N}_u' \) is generated by \( \mathbf{v}_1, \ldots, \mathbf{v}_{k'_u} \) for \( u = 1, \ldots, r \). We call the spaces \( \mathcal{N}_u' \) the inner spaces.

Let \( \mathcal{N}_u \) denote (not necessarily linear) \((s, n), \alpha, K_u, \delta_u\)\(q\)-spaces with \( q_u = q^{e_u} \) and \( e_u = k'_u - k'_{u-1} \) for \( u = 1, \ldots, r \). We call the spaces \( \mathcal{N}_u \) the outer spaces.

Let \( \mathcal{N} \) be given by

\[
\mathcal{N} = \left\{ \sum_{u=1}^{r} \phi_u(\mathbf{x}_u) : \mathbf{x}_u \in \mathcal{N}_u \text{ for } u = 1, \ldots, r \right\}, \tag{5}
\]

where \( \phi_u : \mathbb{F}_q^{ns} \to \mathbb{F}_q^{n's's'} \) replaces each symbol (regarded as a vector of length \( e_u \) over \( \mathbb{F}_q \)) of an element from \( \mathcal{N}_u \) by the corresponding linear combination of the \( e_u \) vectors \( \mathbf{v}_{k'_{u-1}+1}, \ldots, \mathbf{v}_{k'_u} \). The elements in the resulting vector are grouped such that column \((a', \tau')\) (with \( 1 \leq a' \leq s' \) and \( 1 \leq \tau' \leq n' \)) from \( \mathcal{N}_u' \) and column \((a, \tau)\) (with \( 1 \leq a \leq s \) and \( 1 \leq \tau \leq n \)) from \( \mathcal{N}_u \) determine column \(((a-1)s' + a' + \tau - 1)n' + \tau') \) in the resulting element in \( \mathcal{N} \).

Then \( \mathcal{N} \) is an

\[
((ss', mn'), \alpha a', K_1 \cdots K_r, \min_{1 \leq a \leq r, |\mathcal{N}_u| > 1} \delta_u \delta'_u)_q \text{-space}.
\]
Proof. The proof follows along the same lines as the proof of [19, Theorem 3]. We have \( N \subseteq \mathbb{F}^m \) by the definition of the mappings \( \phi_u \). The number of elements in \( N \) is given by \( K_1 \cdots K_r \) which follows from [5] and the fact that all elements given by \( \sum_{u=1}^r \phi_u(x_u) \) for \( x_u \in N_u \) and \( u = 1, \ldots, r \) are distinct, which we show in the following.

Let \( x, y \) be two elements \( N \) and let \( x_u, y_u \in N_u, 1 \leq u \leq r \), denote the elements defining \( x \) and \( y \) using the sum in [5]. If \( x_u = y_u \) for \( 1 \leq u \leq r \), then it follows that \( x = y \). Assume now that \( x \neq y \), then there exists a largest integer \( 1 \leq u \leq r \) such that \( x_u \neq y_u \). This implies that \( |N_u| > 1 \).

Since \( \phi_u \) is linear, we have

\[
x - y = \sum_{u=1}^{u^* - 1} \phi_u(x_u - y_u) + \phi_{u^*}(x_{u^*} - y_{u^*}) + \sum_{u=u^*+1}^r \phi_u(x_u - y_u),
\]

where the first sum is a concatenation of elements from \( N_{u^* - 1} \) and the second sum is 0 (since \( x_u = y_u \) and the linearity of \( \phi_u \) for \( u < u^* \leq r \)). Further, \( \phi_{u^*}(x_{u^*} - y_{u^*}) \) is a concatenation of elements from \( N_{u^*} \setminus N_{u^* - 1} \). We denote the elements in this concatenation by \( w(j,k) \) for \( 1 \leq j \leq s \) and \( 1 \leq k \leq n \).

Thus, \( x - y \) is a concatenation of elements from \( N_{u^*} \setminus N_{u^* - 1} \), since the sum of an element from \( N_{u^* - 1} \) and an element from \( N_{u^*} \setminus N_{u^* - 1} \) is again an element of \( N_{u^*} \setminus N_{u^* - 1} \).

Let

\[
\mu_{\alpha,n}(x_u, y_u) = \sum_{j=1}^s \sum_{k=1}^{\min(\alpha, \nu_j)} i_{j,k},
\]

where \( \nu_j \) denotes the number of nonzero elements and \( i_{j,k} \) are the positions of the nonzero elements in \( x_u - y_u \). Furthermore, let

\[
\mu_{\alpha', n'}(w(j,k)) = \sum_{j'=1}^{s'} \sum_{k'=1}^{\min(\alpha', \nu'_{j,k})} i'_{j', k', j,k},
\]

where \( \nu'_{j,k} \) denotes the number of nonzero elements and \( i'_{j', k', j,k} \) denotes the positions of the nonzero elements in \( w(j,k) \). Then we have

\[
d_{\alpha, n'}(x, y) = \mu_{\alpha', n'}(x - y)
\]

\[
\geq \sum_{j=1}^s \sum_{j'=1}^{s'} \sum_{k=1}^{\min(\alpha, \nu_j)} \sum_{k'=1}^{\min(\alpha', \nu'_{j,k})} \left\{ \begin{array}{ll}
(i_{j,k} - 1)\nu_j + i'_{j', k', j,k} & \text{if } \nu_j, \nu'_{j,k} > 0 \\
0 & \text{otherwise}
\end{array} \right.
\]

\[
\geq \sum_{j=1}^s \sum_{j'=1}^{s'} \sum_{k=1}^{\min(\alpha, \nu_j)} \sum_{k'=1}^{\min(\alpha', \nu'_{j,k})} i_{j,k} i'_{j', k', j,k}
\]

\[
= \sum_{j=1}^s \sum_{k=1}^{\min(\alpha, \nu_j)} \sum_{j'=1}^{s'} \sum_{k'=1}^{\min(\alpha', \nu'_{j,k})} i'_{j', k', j,k}
\]

\[
\geq \delta_{\alpha,n}(N_{u^*}) \mu_{\alpha', n'}(w(j,k))
\]

\[
\geq \delta_{\alpha,n}(N_{u^*}) \delta_{\alpha', n'}(N_{u^*}).
\]

The result thus follows. \( \square \)
Lemma 4. Let \( \{0\} = \mathcal{N}_0 \subset \mathcal{N}_1 \subset \cdots \subset \mathcal{N}_s = \mathbb{F}_q^s \) be as in Lemma 3 with \( \alpha' = n' = 1 \). Let \( \mathbf{v}_1, \ldots, \mathbf{v}_s' \in \mathbb{F}_q^s \) be defined as in Lemma 3, where we assume without loss of generality that the first \( i-1 \) elements of \( \mathbf{v}_i \) are 0. This means that \( (\mathbf{v}_1, \ldots, \mathbf{v}_s')^\top \) forms an \( s \times s' \) upper triangular matrix.

Moreover, let \( q_u \) and \( e_u \) be defined as in Lemma 3. For positive integers \( s_1 \leq s_2 \leq \cdots \leq s_r \) let \( \mathcal{N}_u, u = 1, \ldots, r \), be a (not necessarily linear) \( ((s_u, n), \alpha, K_u, \delta_u)_{q_u} \)-space.

Then we can construct an

\[
((s_1e_1 + \cdots + s_re_r, n), \alpha, K_1 \cdots K_r, \min_{1 \leq u \leq r} \delta_u \delta_u') - \text{space}.
\]

Proof. The construction is analogous to the construction in the proof of [19, Theorem 4] and follows in three steps.

- Let \( s = s_r \) and construct new spaces \( \mathcal{M}_u \) by embedding each space \( \mathcal{N}_u \) in \( \mathbb{F}_q^{(s_u, n)} \). This is achieved by prepending \( \mathbf{0} \in \mathbb{F}_q^{(s-s_u, n)} \) to each element of \( \mathcal{N}_u \).
- Use Lemma 3 with \( \mathcal{N}_0, \ldots, \mathcal{N}_r \) as inner spaces and \( \mathcal{M}_1, \ldots, \mathcal{M}_r \) as outer spaces. This yields a new space \( \mathcal{M} \).
- Finally, for \( i = 1, \ldots, s' \), choose \( u \) minimal such that the condition \( e_1 + \cdots + e_u \geq i \) holds and construct a space \( \mathcal{N} \) from \( \mathcal{M} \). This can be done by deleting all columns \((k, r) \) in \( \mathcal{M} \) with \( k = 0s' + i, 1s' + i, \ldots, (s - s_u - 1)s' + i \).

The total number of deleted blocks is \( e_1(s - s_1) + \cdots + e_r(s - s_r) \), so the length of \( \mathcal{N} \) is \( s_1e_1 + \cdots + s_re_r \). The deleted positions are \( \mathbf{0} \in \mathbb{F}_q^{n} \) for each \( \phi_u(x_u) \), either due to a 0 in \( \mathbf{v}_i \) or due to a \( \mathbf{0} \) appended to \( x_u \) in \( \mathcal{N}_u \). Note that this procedure neither influences the dimension nor the weight of \( \mathcal{M} \).

\[ \Box \]

4 A Blokh-Zyablov propagation rule

4.1 A Blokh-Zyablov propagation rule for digital higher order nets

Lemma 4 can be applied to digital higher order nets which yields a new propagation rule. This propagation rule generalises the Matrix-product construction in [9]. As there were Propagation Rules I–XIV in [2], we call our new rule Propagation Rule XV.

Theorem 3 (Propagation Rule XV). Let \( \mathbf{v}_1, \ldots, \mathbf{v}_s' \) and \( \mathcal{N}_u, q_u \) and \( e_u \) for \( u = 1, \ldots, r \) be defined as in Lemma 4. Let \( \alpha \geq 1 \). For positive integers \( s_1 \leq s_2 \leq \cdots \leq s_r \) let \( \mathcal{P}_u, 1 \leq u \leq r \), denote digital \((t_u, \alpha, \beta_u, m_u \times m_u, s_u)\)-nets over \( \mathbb{F}_{q_u} \).

Then a digital \((t, \alpha, \beta, n \times m, s)\)-net over \( \mathbb{F}_q \) can be constructed, where

\[
s = \sum_{u=1}^{r} e_u s_u,
\]

\[
m = \sum_{u=1}^{r} e_u m_u,
\]
\[ n = \sum_{u=1}^{r} e_u n_u, \]
\[ \beta = \min(1, \alpha m/n), \]
\[ t \leq \beta n + 1 - \min_{1 \leq u \leq r} (\beta u n_u - t_u + 1) \delta'_u. \]

**Proof.** Let \( N_u \) denote the dual space of \( P_u \) for \( u = 1, \ldots, r \). Then \( N_u \) is a linear \([s, n), \alpha, m_u, \beta u n_u - t_u + 1]_{q_u}\)-space. The result then follows by applying Lemma 4 using \( N'_0, \ldots, N'_r \) as inner spaces and \( N_1, \ldots, N_r \) as outer spaces to obtain a space \( N \) and applying Proposition 2.

**4.2 A Blokh-Zyablov propagation rule for higher order nets**

In this subsection we consider higher order nets which are not necessarily digital. Again we have the same theorem as in the previous case by considering the dual set instead of the dual net. Applying Lemma 4 and Theorem 2 yields the following result, which is Propagation Rule 17 for general higher order nets (as there were Propagation Rules 1–16 for general higher order nets in [2]).

**Theorem 4** (Propagation Rule 17). Let \( v_1, \ldots, v_s' \) and \( N'_u, q_u \) and \( e_u \) for \( u = 1, \ldots, r \) be defined as in Lemma 4. Let \( \alpha \geq 1 \). For positive integers \( s_1 \leq s_2 \leq \cdots \leq s_r \), let \( P_u \) denote \((t_u, \beta u, \alpha, n_u \times m_u, s_u)\)-nets in base \( b_u = b^{e_u} \) for \( 1 \leq u \leq r \).

Then a \((t, \alpha, \beta, n \times m, s)\)-net in base \( b \) can be constructed, where

\[ s = \sum_{u=1}^{r} e_u s_u, \]
\[ m = \sum_{u=1}^{r} e_u m_u, \]
\[ n = \sum_{u=1}^{r} e_u n_u, \]
\[ \beta = \min(1, \alpha m/n), \]
\[ t \leq \beta n + 1 - \min_{1 \leq u \leq r} (\beta u n_u - t_u + 1) \delta'_u. \]

**Remark 1.** Note that in Lemma 3, Lemma 4, Theorem 3 and Theorem 4 it is sufficient to require \( N'_r \subseteq \mathbb{F}_q' \) instead of \( N'_r = \mathbb{F}_q' \). Indeed, this slight modification is possible as one can use a trivial space \( N_u = \{0\} \) as an outer space in Lemma 4, which does not influence the minimal distance of the newly obtained space (cf. Remark 4 in [18]).

**5 Numerical results**

Theorems 3 and 4 provide previously unknown propagation rules for higher order nets. To show how powerful these are, we state exemplary numerical results for the digital case, but it should be noted that the results are also true for the more general case, as the propagation rules yield the same parameters and we always start from existing digital nets, which are by definition a subclass of general higher order nets. In Tables 1–3 we present,
for selected values of \(m\) and \(s\), the results obtained when we use different propagation rules for the case \(q = 5\), \(\alpha = 2\), and \(\beta = 1\). As we restrict ourselves to considering only digital nets, we only refer to propagation rules with roman numbers within this section.

Table 1 covers the case where \(s = 5\), Table 2 the case \(s = 15\) and Table 3 the case \(s = 25\). In all tables we consider \(m\) between 15 and 30. Since \(\alpha\) and \(\beta\) are fixed, and different propagation rules might yield different ratios of \(n\) and \(m\), it is most useful to compare the strengths of the nets obtained. As outlined above, the strength of a digital net rules, one can make many different choices of smaller nets that might yield a bigger net with the same parameters. We only give the best values of the strength \(\sigma\) we can obtain by going through a number of possible choices of the smaller nets involved.

Our tables below can be seen as an extension of Table 3 in [9, Section 4]. To be more precise, we consider the following quantities in Tables 1 and 2.

- \(\sigma_{\text{dir}}\): The strength of a digital \((t, 2, 1, 2m \times m, s)\)-net over \(F_5\) using the generating matrices of an existing classical digital \((t', m, 2s)\)-net over \(F_5\), where we then obtain (cf. [6])

\[
t \leq 2 \min \left\{ m, t' + \left\lfloor \frac{s}{2} \right\rfloor \right\}.
\]

(6)

We call this construction method the direct construction method.

- \(\sigma_{\text{VII}}\): The strength of a digital net constructed from a digital \((t_1, 2, 1, 2m_1 \times m_1, s_1)\)-net \(P_1\) and a digital \((t_2, 2, 1, 2m_2 \times m_2, s_2)\)-net \(P_2\) over \(F_5\) using Propagation Rule VII in [9], where \(P_1\) and \(P_2\) are obtained by the direct construction method from classical nets. Here, \(n = 2m\).

- \(\sigma_{\text{VIII}}\): The strength of a digital net constructed from a digital \((t_1, 2, 1, 2m_1 \times m_1, s_1)\)-net \(P_1\) and a digital \((t_2, 2, 1, 2m_2 \times m_2, s_2)\)-net \(P_2\) \((s_1 \leq s_2)\) over \(F_5\) using Propagation Rule VIII in [9], where \(P_1\) and \(P_2\) are obtained by the direct construction method. Here, \(n = 2m\).

- \(\sigma_{\text{IX}}\): The strength of a digital net constructed from a digital \((t_1, 2, 1, 2m_1 \times m_1, s_1)\)-net \(P_1\), a digital \((t_2, 2, 1, 2m_2 \times m_2, s_2)\)-net \(P_2\), and a digital \((t_3, 2, 1, 2m_3 \times m_3, s_3)\)-net \(P_3\) \((s_1 \leq s_2 \leq s_3)\) over \(F_5\) using Propagation Rule IX in [9], where \(P_1\), \(P_2\), and \(P_3\) are obtained by the direct construction method. Again, \(n = 2m\).

- \(\sigma_{\text{XI}}\): The strength of a digital net obtained by using Propagation Rule XI from [9] with \(r = 2\), where the higher order nets plugged into Rule XI are obtained by the direct construction method from classical nets. Note that this rule can, since \(r = 2\), only be applied for the cases where \(m\) is even, and we then have \(n = m\). As Propagation Rule XI with \(r = 2\) only yields higher order nets with an even value of \(s\), we obtain the values in the Tables [1-3] by projection from 6-dimensional, 16-dimensional, and 26-dimensional nets, respectively (this is allowed due to Propagation Rule V in [9]).

- \(\sigma_{\text{XI}+\text{VIII}}\): The strength of a digital net obtained by first applying the direct construction method to classical nets, then applying Propagation Rule XI with \(r = 2\) in [9], and then using pairs of the newly obtained nets to apply Propagation Rule VIII in [9]. Note again that we need to restrict ourselves to even cases of \(m\) here, as we first apply Rule XI with \(r = 2\). For \(\sigma_{\text{XI}+\text{VIII}}\), we again have \(n = m\).
Table 1: $\sigma$-values depending on $m$ (15 ≤ $m$ ≤ 30) for $\alpha = 2$, $\beta = 1$, $q = 5$, and $s = 5$.

| $m$ | $\sigma_{\text{dir}}$ | $\sigma_{\text{VIII}}$ | $\sigma_{\text{VH}}$ | $\sigma_{\text{IX}}$ | $\sigma_{\text{XI}}$ | $\sigma_{\text{XI+VIII}}$ | $\sigma_{\text{XV}}$ |
|-----|----------------------|-------------------------|----------------------|----------------------|----------------------|-------------------------|----------------------|
| 15  | 24  | 12  | 17  | 14  |                      |                      |                      |
| 16  | 26  | 14  | 18  | 16  | 14  | 9  | 19  |
| 17  | 28  | 14  | 20  | 17  |                      |                      |                      |
| 18  | 30  | 16  | 21  | 18  | 16  | 10 | 21  |
| 19  | 32  | 16  | 22  | 20  |                      |                      |                      |
| 20  | 34  | 18  | 24  | 20  | 18  | 12 | 25  |
| 21  | 36  | 18  | 25  | 21  |                      |                      |                      |
| 22  | 38  | 20  | 26  | 22  | 20  | 13 | 27  |
| 23  | 40  | 20  | 28  | 24  |                      |                      |                      |
| 24  | 42  | 22  | 29  | 25  | 22  | 14 | 29  |
| 25  | 44  | 22  | 30  | 26  |                      |                      |                      |
| 26  | 46  | 24  | 32  | 26  | 24  | 16 | 33  |
| 27  | 48  | 24  | 33  | 28  |                      |                      |                      |
| 28  | 50  | 26  | 34  | 29  | 26  | 17 | 35  |
| 29  | 52  | 26  | 36  | 30  |                      |                      |                      |
| 30  | 54  | 28  | 37  | 32  | 28  | 18 | 37  |

- $\sigma_{\text{XV}}$: The strength of a digital net by first applying the direct construction method to classical nets, and then Propagation Rule XV with $r = 2$. (Theorem 3). As the inner spaces (see the proof of Theorem 3 and Lemma 4) we use (extended) Reed Solomon Codes over $\mathbb{F}_5$. These codes are linear $[5, j, 6-j]$-codes over $\mathbb{F}_5$ (see [19]). In view of Remark 1 we can use Reed Solomon Codes with parameters $[5, 0, 6]$, $[5, 2, 4]$, and $[5, 4, 2]$ as the inner codes, hence $r$ indeed equals 2 and $e_1 = e_2 = 2$ in Theorem 3 which is why we again need to restrict ourselves to even cases of $m$ here (in general one can of course also obtain odd values for $m$ by different choices of $e_i$ and $m_i$). In the language of Theorem 3 we have $\delta_1 = 4$ and $\delta_2 = 2$ in this case. For $\sigma_{\text{XV}}$ we have $n = 2m$, and, as for $\sigma_{\text{XI}}$, we obtain the values in the Tables 1–3 by projection from 6-, 16-, and 26-dimensional nets, respectively (again by Propagation Rule V in [9]).

The results in Tables 1–3 show that our new Propagation Rule XV is superior to the other propagation rules tested for dimensions 15 and 25, and that also the combination of other rules, such as that of Rules XI and VIII, does not yield better results. In dimension 5, though still competitive, Propagation Rule XV is outperformed by the direct construction method. It is likely that the direct construction performs better for lower dimensions as the $t$-values of classical digital nets are very small in this case. Furthermore, the bound (6) depends critically on the value of $s$, so it is natural that lower dimensional results for the direct construction method are stronger. The fact that the direct construction method works well for low dimensions is in line with numerical results in [10], where a similar phenomenon was observed.

We emphasise that our examples are just illustrations and can by no means systematically cover all cases one might theoretically consider; to be more precise, we have the following restrictions in Tables 1–3:

- We only show particular choices of the parameters involved. We restrict ourselves to some illustrative cases.
Table 2: $\sigma$-values depending on $m$ ($15 \leq m \leq 30$) for $\alpha = 2$, $\beta = 1$, $q = 5$ and $s = 15$.

| $m$ | $\sigma_{\text{dir}}$ | $\sigma_{\text{VII}}$ | $\sigma_{\text{VIII}}$ | $\sigma_{\text{IX}}$ | $\sigma_{\text{XI}}$ | $\sigma_{\text{XI+VIII}}$ | $\sigma_{\text{XV}}$ |
|-----|------------------------|------------------------|-------------------------|------------------------|------------------------|-----------------------------|------------------------|
| 15  | 0                      | 0                      | 1                       | 2                      |                        |                             |                        |
| 16  | 0                      | 0                      | 1                       | 2                      | 4                      | 2                           | 5                      |
| 17  | 0                      | 0                      | 1                       | 2                      |                        |                             |                        |
| 18  | 0                      | 0                      | 1                       | 2                      | 6                      | 4                           | 9                      |
| 19  | 0                      | 0                      | 1                       | 2                      |                        |                             |                        |
| 20  | 0                      | 0                      | 1                       | 2                      | 8                      | 5                           | 11                     |
| 21  | 0                      | 0                      | 1                       | 4                      |                        |                             |                        |
| 22  | 0                      | 0                      | 1                       | 4                      | 10                     | 6                           | 13                     |
| 23  | 0                      | 2                      | 2                       | 5                      |                        |                             |                        |
| 24  | 0                      | 2                      | 2                       | 5                      | 12                     | 8                           | 17                     |
| 25  | 0                      | 2                      | 4                       | 6                      |                        |                             |                        |
| 26  | 0                      | 2                      | 4                       | 6                      | 14                     | 9                           | 19                     |
| 27  | 2                      | 4                      | 5                       | 8                      |                        |                             |                        |
| 28  | 2                      | 4                      | 5                       | 8                      | 16                     | 10                          | 21                     |
| 29  | 4                      | 4                      | 5                       | 9                      |                        |                             |                        |
| 30  | 6                      | 6                      | 6                       | 10                     | 18                     | 12                          | 25                     |

Table 3: $\sigma$-values depending on $m$ ($15 \leq m \leq 30$) for $\alpha = 2$, $\beta = 1$, $q = 5$, and $s = 25$.
• We do not consider all possible combinations of different propagation rules, as this would lead to a too high number of parameters.

• Not all propagation rules are applicable for all sets of parameters. This is indicated by void cells in the tables in cases where a certain propagation rule was not applicable.

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