Limitations, insights and improvements to gyrokinetics

Peter J. Catto¹, Felix I. Parra¹, Grigory Kagan¹ and Andrei N. Simakov²

¹ MIT Plasma Science and Fusion Center, Cambridge, MA 02139, USA
² Theoretical Division, Los Alamos National Laboratory, Los Alamos, NM 87545, USA

E-mail: catto@psfc.mit.edu

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Abstract

We first consider gyrokinetic quasineutrality limitations when evaluating the axisymmetric radial electric field in a non-turbulent tokamak by an improved examination of intrinsic ambipolarity. We next prove that the background ions in a pedestal of poloidal ion gyroradius scale must be Maxwellian and nearly isothermal in Pfirsch–Schlüter and banana regime tokamak plasmas, and then consider zonal flow behaviour in a pedestal. Finally, we focus on a simplifying procedure for our transport time scale hybrid gyrokinetic-fluid treatment that removes the limitations of gyrokinetic quasineutrality and remains valid in the pedestal.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

Our recent work considers the limitations of gyrokinetic quasineutrality in evaluating the axisymmetric radial electric field in a tokamak [1]; derives a gyrokinetic entropy production banana regime restriction on the ion temperature pedestal and evaluates zonal flow behaviour in the pedestal [2]; and formulates a hybrid gyrokinetic-fluid treatment of electrostatic turbulence valid on slowly evolving transport time scales [3]. Here we extend and provide more insight into the detailed calculations published in [1–3] in a relatively equation free manner by further examining intrinsic ambipolarity, extending our ion temperature pedestal results to Pfirsch–Schlüter pedestals, and by considering a simplification of our hybrid gyrokinetic-fluid description enabling simulations on transport time scales that remain valid in the pedestal and remove the limitations of gyrokinetic quasineutrality.

Modern [4] and standard [3] gyrokinetics, which are equivalent [5], but typically only employed to first order in the gyroradius over major radius expansion, incorrectly determine the axisymmetric, long wavelength electrostatic potential. We have already illustrated this defect of gyrokinetics by considering a steady-state theta pinch with a distribution function correction to second order in the gyroradius expansion [1]. Here we extend the proof of the intrinsic ambipolarity [6, 7] to steady state, non-turbulent tokamaks to the same order with an improved and streamlined demonstration that gyrokinetic quasineutrality improperly determines the potential in the long wavelength, axisymmetric limit. The basic conceptual error is due to an inconsistent treatment of quasineutrality and must be corrected to recover agreement with the required intrinsic ambipolarity of tokamaks [6, 7] as we demonstrate here. We have recently shown that rather than evaluating quasineutrality to higher order it is more sensible to replace it by a toroidal angular momentum or vorticity conservation equation that does not require as accurate an ion distribution function [8].

Using canonical angular momentum as the radial variable allows strong radial gradients (as in the pedestal) to be treated gyrokinetically [2] while retaining all the other features of standard gyrokinetics. Entropy production is then found to require a physical lowest order banana regime ion distribution function to be nearly an isothermal Maxwellian with the ion temperature scale much greater than the poloidal ion gyroradius. Thus, absent very large sources or sinks in the pedestal, the background ion temperature profile in a tokamak cannot have a width as short as the poloidal ion gyroradius width of a banana regime density pedestal. Here, new insights are presented for a collisional plasma that lead to the conclusion that the ion temperature variation cannot be on the poloidal ion gyroradius scale for a pedestal in the Pfirsch–Schlüter regime. Weak ion temperature variation with subsonic pedestal flow requires electrostatically restrained ions and magnetically confined electrons. In the banana regime, we have shown that these features result in finite orbit modifications that increase the zonal flow residual [9]. The additional insights
we provide here into these results are valid in spite of the rather complicated ion trajectories we discuss in detail.

Simulating tokamaks on transport time scales require evolving drift wave turbulence with axisymmetric long wavelength and zonal flow radial electric field effects retained. However, full electric field effects are difficult to keep since they require evaluating the ion distribution function to higher order in the gyroradius expansion than in standard gyrokinetics and gyrokinetic quasi-neutrality. A hybrid gyrokinetic-fluid treatment of electrostatic turbulence that takes advantage of moments of the full Fokker–Planck equation removes the need to evaluate gyrokinetic quasi-neutrality to very high order in the gyroradius expansion [3] of the ion distribution function while remaining valid in the pedestal. This hybrid description self-consistently evolves potential as well as density and temperature profiles and flows, and models all electrostatic turbulence effects with wavelengths much longer than an electron gyroradius, while only requiring the ion distribution function to second order in gyroradius. Here we suggest that only a slight generalization of the standard gyrokinetic equation [1] is required if we further assume that the poloidal magnetic field is weak compared with the total magnetic field.

2. Limitations of the gyrokinetic determination of the radial electric field

A new recursive procedure is used to derive the electrostatic gyrokinetic equation for the full distribution function (a ‘full f’ description) accurate to first order in an expansion of gyroradius over the magnetic field characteristic length scale [1]. The procedure employs new, nonlinear gyrokinetic variables that are constructed to higher order than is typically the case by generalizing the linear procedure of [10]. The results of [1] are fully consistent with the constant magnetic field, homogeneous plasma limit of Dubin et al [4] as proven in [5], but also retain inhomogeneities to second order in \( \rho_i/L \), where \( \rho_i \) is the ion gyroradius and \( L \) is the perpendicular scale length of the background profiles. Of course, these results are completely consistent with other ‘modern’ gyrokinetic results [11]. The gyrokinetic procedure employed provides fresh insights into the limitations of the gyrokinetic quasi-neutrality equation that in the long wavelength limit must not determine the axisymmetric electrostatic potential because of intrinsic ambipolarity [6, 7] as proven in this section.

The axisymmetric radial electric field in a tokamak consists of two components that give rise to \( E \times \hat{B} \) drifts comparable to diamagnetic flows and magnetic drifts (this situation is normally referred to as the drift ordering). The relatively small amplitude, but rapidly radially varying zonal flow component of the electrostatic potential is generated by the turbulence associated with ion temperature gradient (ITG) modes, trapped electron modes (TEMs) and other tokamak instabilities. It is superimposed on a large amplitude component with a slow global (or long wavelength) structure on the scale of the background radial gradients. Gyrokinetic quasi-neutrality is expected to sensibly determine the temporally varying electrostatic potential \( \Phi \) in the short wavelength limit, but it would violate intrinsic ambipolarity if it also determined the steady state, axisymmetric, global long wavelength radial electric field component that impacts the transport time scale evolution of the turbulence. This error occurs because quasi-neutrality as used in standard gyrokinetics is only accurate to order \( \rho_i/L \), whereas corrections of higher order are required to evaluate the long wavelength potential.

In a steady state, axisymmetric, non-turbulent tokamak, intrinsic ambipolarity [6, 7] requires the heat and particle fluxes to be independent of electrostatic potential to second order in the expansion in ion gyroradius \( \rho_i \) divided by the local scale length \( L \). This property is most easily seen to order \( \rho_i/L \) by considering the axisymmetric drift kinetic equation for the leading correction \( f_{1i} \) to the lowest order ion Maxwellian \( f_{0i}(\Psi, E) \) found by solving

\[
v_{\parallel i} \hat{n} \cdot \nabla f_{1i} - C_{ii} f_{1i} = -v_{\parallel i} \cdot \nabla \Psi \frac{\partial f_{0i}}{\partial \Psi},
\]

where \( E = v^2/2 + e\Phi/M \) is the total energy (with \( e \) the magnitude of the electron charge and the ion mass), \( \Omega_i = eB/Mc \) is the parallel velocity, \( \bar{\nu}_a \) is the magnetic plus electric drifts, \( C_{ii} \) is the linearized ion–ion collision operator with \( C_{ii} v_{\parallel i} f_{0i} = 0 \) and \( B = f(\Psi)\nabla \zeta + \nabla \times \nabla \Psi = B_\parallel \) is the tokamak magnetic field with \( \zeta \) the toroidal angle and \( \Psi \) the poloidal flux function. Letting \( g_i = f_{0i} + (I/v_{\parallel i}/\Omega_i)(\partial f_{0i}/\partial \Psi) \) gives

\[
v_{\parallel i} \hat{n} \cdot \nabla g_i = C_{ii} \left[ g_i - \frac{1}{\Omega_i} \frac{\partial f_{0i}}{\partial \Psi} \right] = C_{ii} \left[ g_i - \frac{I f_{0i} v_{\parallel i}}{\Omega_i T} \left( \frac{Mv_i^2}{2T_i} - \frac{2}{5} \frac{\partial T_i}{\partial \Psi} \right) \right],
\]

showing that the only drive for \( g_i \) is the radial ion temperature gradient \( \partial T_i/\partial \Psi \), and giving a vanishing ion particle flux since \( \langle n V_i \cdot \nabla \Psi \rangle = \langle f d^3v f_{1i} \bar{v}_{\parallel i} \cdot \nabla \Psi \rangle = -\langle (I/\Omega_i) \rangle f d^3v v_{\parallel i} \hat{n} \cdot \nabla f_{1i} \rangle = 0 \), where \( \langle . . . \rangle \) denotes flux surface average and the \( \int v_{\parallel i}/\Omega_i \), moment of (1) is employed.

Note that \( \langle n V_i \cdot \nabla \Psi \rangle = 0 \) means that equation (1) of Wang et al [12], equation (3) of Satake et al [13] and equation (14) of Idoymura et al [14] cannot be used to determine and evolve the axisymmetric radial electric field. Satisfying the neoclassical relation for the parallel ion flow

\[
V_{\parallel i} = -\frac{IT_i}{M\Omega_i} \left( \frac{kB^2}{(B^2)_{\psi}} \frac{\partial}{\partial \psi} \ln T_i - \frac{\partial}{\partial \psi} \ln P_i - \frac{e}{T_i} \frac{\partial \Phi}{\partial \psi} \right)
\]

merely verifies that the ion gyrokinetic equation is being solved consistently to the order employed in codes, where the numerical coefficient \( k \) depends on the regime of collisionality. Equation (3) simply provides a relation between \( \partial \Phi/\partial \Psi \) and \( V_{\parallel i} \), and confirms that parallel ion momentum is being satisfied through leading order in the pressure anisotropy. It does not determine the radial electric field (unless it is mistakenly and arbitrarily assumed that the parallel ion flow can be set to zero or specified in some other ad hoc manner) and satisfying it is not a test that the correct rotation profile is obtained. Equation (3) places such a strong constraint on the relation between the parallel ion flow and the global radial electric field that it has recently been shown to be valid even at long (non-zonal flow) wavelengths in turbulent plasmas [8]. In brief, the ambipolarity constraint \( \langle n V_i \cdot \nabla \psi - n V_\psi \cdot \nabla \psi \rangle = 0 \) should be automatically satisfied to very high order for any long
wavelength radial electric field and should not determine the global axisymmetric radial electric field.

Gyrokinetic simulations using quasineutrality with polarization effects retained through second order in the ion poloidal gyroradius ($\rho_{pol}$) expansion and a guiding centre density valid only through first order in $\rho_{pol}$ determine an incorrect global axisymmetric radial electric field that is different from the one obtained by conservation of toroidal angular momentum. Effectively, such a gyrokinetic quasineutrality treatment adds charge sources and sinks that result in an incorrect global axisymmetric radial electric field. Indeed, in the pioneering banana regime evaluation of Rosenbluth et al [15] quasineutrality is not employed to determine the radial electric field since the second order in ($\rho_{pol}/L$)$^2$ corrections to quasineutrality exactly cancel. Instead, toroidal angular momentum conservation is used to determine the radial electric field at higher order in the combined gyroradius and collisionality expansions. More generally, a carefully constructed full $f$ turbulent gyrokinetic code should result in no net radial transport of toroidal momentum through order ($\rho_{pol}/L$)$^2$ in the absence of sources and sinks if quasineutrality is self-consistently treated to the same order [8].

In the remainder of this section we give an improved and streamlined proof that $\partial$/$\partial\psi$ cannot be determined in a turbulence-free tokamak without a moment approach if the ion distribution function is only known to second order in the gyroradius expansion.

A moment procedure for the electron particle flux using $C_{ec}(f_{0e}) = C_{ec}(f_{0e}) + C_{el}(f_{0e})$ with $C_{ec}$ the electron–electron operator and $C_{el}(f_{0e}) = L_{0e}(f_{0e} - (m/T_{e})V_{a}v_{0e}f_{0e})$ the unlike electron–ion Lorentz operator gives the electron radial particle flux as $n_{e}(\vec{V} \cdot \nabla(\rho_{0e}/\rho_{pol}/L)^2 \equiv \psi (m/T_{e})V_{a}v_{0e}f_{0e})(\vec{f}_{0e} - (m/T_{e})V_{a}v_{0e}f_{0e})_\psi$, with $f_{0e}$ the leading order correction to the electron Maxwellian $f_{0e}$ and $T_{e}$ and $m$ the electron temperature and mass. The electron drift kinetic equation can be written as $n_{e}(\vec{V} \cdot \nabla(\rho_{0e}/\rho_{pol}/L)^2 \equiv \psi (m/T_{e})V_{a}v_{0e}f_{0e})_\psi = C_{ec}(\vec{V})\nabla(\rho_{0e}/\rho_{pol}/L)^2 \equiv \psi (m/T_{e})V_{a}v_{0e}f_{0e})_\psi = C_{el}(\vec{V})\nabla(\rho_{0e}/\rho_{pol}/L)^2 \equiv \psi (m/T_{e})V_{a}v_{0e}f_{0e})_\psi = \partial \Phi/\partial \psi$, which drives terms in the collision operator cancel, making $g_{e}$ independent of the radial electric field so that $n_{e}(\vec{V} \cdot \nabla(\rho_{0e}/\rho_{pol}/L)^2 \equiv \psi (m/T_{e})V_{a}v_{0e}f_{0e})_\psi = C_{ec}(\vec{V})\nabla(\rho_{0e}/\rho_{pol}/L)^2 \equiv \psi (m/T_{e})V_{a}v_{0e}f_{0e})_\psi = C_{el}(\vec{V})\nabla(\rho_{0e}/\rho_{pol}/L)^2 \equiv \psi (m/T_{e})V_{a}v_{0e}f_{0e})_\psi = \partial \Phi/\partial \psi$ cannot depend on the radial electric field to order $(\rho_{pol}/L)^2$ since $C_{ec}/C_{el} \sim v_{a}/v_{ce} \sim (m/M)^{1/2} \sim \rho_{pol}/L$ is normally assumed, with $v_{a}$ and $v_{ce}$ the ion–ion and electron–electron collision frequencies. Alternately, a moment description can be used to further demonstrate that intrinsic ambipolarity must be satisfied to order $\rho_{pol}/L^2$ and demonstrate that it is the flux surface average of conservation of toroidal angular momentum that must give the radial electric field (pressure anisotropy does not enter this constraint). To order $\rho_{pol}/L^2$ the cross field viscosity is diamagnetic (and so collisionless to lowest order) and the radial flux of toroidal angular momentum may be written in terms of the ion gyroviscosity $\pi_{ig}$ within small up–down asymmetric contributions as $[16, 17] \langle R^2 \nabla \cdot \pi_{ig} \nabla \psi \rangle = \langle (M/I) B \times \vec{d} \cdot \vec{v}_{0e} \vec{v}_{0e} \nabla \psi \rangle$, with $R$ the major radius. Inserting $\vec{v}_{0e} = \vec{v}_{0e} - (I/v_{ce})\nabla(\rho_{0e}/\rho_{pol}/L)^2 \equiv \psi$, using $\langle R^2 \nabla \cdot \vec{d} \vec{V} f_{0e}(\vec{v}_{0e}/B) \nabla \psi \rangle = 0$, and recalling $g_{e}$ depends only on $\partial \Phi/\partial \psi$ gives a $\partial \Phi/\partial \psi$ independent result for $\langle R^2 \nabla \cdot \pi_{ig} \nabla \psi \rangle$. Hence, we have proven that the correct neoclassical radial electric field cannot be determined directly from toroidal angular momentum conservation knowing $f_{0e}$ to second order. As a result, a direct determination of $\partial \Phi/\partial \psi$ requires knowing the ion distribution function through third order in the gyroradius expansion, or a moment approach as outlined in section 4 must be used to save an order.

By considering a steady-state theta pinch using a model collision operator, we have explicitly shown that gyrokinetic quasineutrality cannot determine the axisymmetric, long radial wavelength electrostatic potential to order $\rho_{pol}/L^2$ [1]. Here we have proven the same situation occurs in axisymmetric tokamaks. In modern gyrokinetic treatments intrinsic ambipolarity is violated when the ion distribution function is retained to only order $\rho_{pol}/L$ in the guiding centre density, while being kept to order $\rho_{pol}/L^2$ in the finite orbit polarization term in gyrokinetic quasineutrality. However, when $f_{0e}$ is kept to order $\rho_{pol}/L^2$ in both places, the radial electric field does not enter and therefore cannot be determined, and no inconsistency arises. To determine this axisymmetric radial electric field higher order effects must be retained. The same conclusion holds in a turbulent tokamak but the proof is substantially more involved [8].

These results indicate that the gyrokinetic quasineutrality equation is not an effective tool for finding the electrostatic potential if the long wavelength components are to be properly retained in the analysis. In section 4 we discuss how second order accurate gyrokinetic variables can be employed [1, 2] in a hybrid gyrokinetic-fluid moment description to ensure the required accuracy in the gyrotorus expansion.

3. Gyrokinetics in the pedestal and internal barriers

A new gyrokinetic technique has been developed and applied to analysing pedestal and internal transport barrier (ITB) regions [2]. In contrast to typical gyrokinetic treatments [1, 4, 11], canonical angular momentum $\psi_{\ast} = \psi - (M/e)V \cdot \nabla(\rho_{0e}/\rho_{pol}/L)^2 \equiv \psi + \Omega_{1e}^{-1} \nabla \cdot \nabla \psi - (I/v_{ce})/\rho_{pol}/L)^2 \equiv \psi + \Omega_{1e}^{-1} \nabla \cdot \nabla \psi$. Such an approach allows strong radial plasma gradients to be treated, while retaining zonal flow, neoclassical behaviour and the effects of turbulence. The nonlinear gyrokinetic equation obtained is capable of handling such problems as large poloidal $E \times B$ drift and orbit squeezing effects on zonal flow, collisional zonal flow damping, as well as neoclassical transport in the pedestal or ITB. This choice of gyrokinetic variables allows the toroidally rotating Maxwellian solution of the isothermal tokamak limit to be exactly recovered [18].

More importantly, we prove in [2] that a physically acceptable solution for the lowest order ion distribution function in the banana regime in a tokamak pedestal must be nearly the same isothermal Maxwellian solution in the sense that the ion temperature variation radial scale must be much greater than poloidal ion gyroradius $\rho_{pol}$. Consequently, in the absence of a very strong heat sink or source in the pedestal, the background radial ion temperature profile there cannot be as narrow as that of plasma density or electron temperature if they vary on the scale of a poloidal ion gyroradius. We briefly summarize the banana regime proof of [2] before considering a Pfirsch–Schlüter pedestal.

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**References**

[1] Rosenbluth et al, Nucl. Fusion 49 (2009) 095026

[2] Catto et al.
Recall that the vanishing of the entropy production on a flux surface, \( \int d^3v \ln f_0 \nabla \ln f_0 = 0 \), requires the lowest order axisymmetric ion distribution function \( f_0 \) to be a local Maxwellian, with \( f_0 \) independent of poloidal angle in the banana regime. However, in the pedestal or an internal barrier (or on axis), drift departures from flux surfaces can become comparable to the local scale length \( (\rho_i \nabla \ln n \sim 1) \) with \( n \) the plasma density) and the entropy production argument has to be modified to account for the loss of locality due to finite poloidal ion gyroradius effects requiring an equilibrium to be established over the entire pedestal (or barrier). Using the new gyrokinetic variables, we find that entropy production must vanish in the pedestal [2]:

\[
\int_{\Delta V} d^3r \int d^3v \ln f_0 C_{1\mu}(f_0) = 0, \tag{4}
\]

where \( \Delta V \) is the volume of the pedestal (between the top of the pedestal where \( \rho_i \nabla \ln n \ll 1 \) and the separatrix) or the ITB (between inner and outer bounding flux surfaces having \( \rho_i \nabla \ln n \ll 1 \)) or even the on axis region from the magnetic axis to a neighbouring flux surface with \( \rho_i \nabla \ln n \ll 1 \) that bounds the potato orbit region. As a result, \( f_0 \) must be a drifting Maxwellian at most, giving \( f_0[f_{\text{eq}}] = 0 \). In the banana regime \( f_0 \) is independent of poloidal angle as well. Consequently, to make the Vlasov operator vanish \( f_0 = f_0(\Psi_s, E, \mu) \) must be Maxwellian, where \( E = v^2/2 + e \Phi/M \) is the total energy and \( \mu \) the magnetic moment. It is only possible to make a drifting Maxwellian out of these variables by ignoring the \( \mu \) dependence and assuming that the drift is nearly a rigid toroidal rotation of frequency \( \omega_i \) with the ion temperature variation slow compared with the poloidal ion gyroradius \( (\rho_i \nabla \ln T_i \ll 1, \rho_i \nabla \ln \omega_i \ll 1) \) as for an isothermal Maxwellian [2,18]:

\[
f_0(\psi, E, \mu) = n(M/2\pi T_i)^{3/2} \exp[-M(\vec{v} - \omega_i R^2 \nabla \chi)^2/2T_i] = \eta(M/2\pi T_i)^{3/2} \exp[-M/E - \omega_i \psi/eT_i],
\]

where \( \eta = n \exp[e(\psi/T_i) + (e\omega_i T_i)/(\nu_i + (M\omega_i^2 R^2/2T_i))] \) must also be nearly constant \( (\rho_i \nabla \ln \eta \ll 1) \). For a density pedestal having a scale length \( L \sim \rho_i \), the background ion temperature profile must have a much larger scale length than the pedestal—a restriction that will need to be satisfied in the absence of very strong pedestal sources or sinks.

When the pedestal is collisional the argument must be expanded to consider the behaviour of the leading correction to the lowest order ion Maxwellian to obtain the related arbitrary aspect ratio restriction in the Pfirsch–Schlüter regime. To do so we recall the short mean-free path result for gyrophase independent correction to the ion Maxwellian, \( f_{1i} \), from equations (4) and (15) of [19]

\[
\bar{f}_{1i} = \frac{2\nu_i f_0}{\nu_i T_i} \left[ 1 - \frac{M^2 v^2}{40T_i} - \frac{M^2 v^2}{40T_i^2} \right] \vec{n} \cdot \nabla T_i, \tag{5}
\]

and the Pfirsch–Schlüter relation between the parallel and radial derivatives of the ion temperature \( T_i \) [20] for which we use equation (10) of [17]

\[
\vec{n} \cdot \nabla T_i = \frac{16R B \nu_i}{25\nu_i} \left[ 1 - B^2/(B^2\nu_i) \right] \frac{\partial T_i}{\partial \psi}. \tag{6}
\]

The validity of the Pfirsch–Schlüter regime [20] requires \( \vec{n} \cdot \nabla T_i \sim \vec{n} \cdot \nabla B \sim (\nu_i/L_{\text{ia}})(qR/\lambda) \sim 1 \), with \( L_{\text{ia}} \) the width of the ion temperature pedestal, \( q \) the safety factor and the ion mean-free path \( \lambda = \nu_i/v_i \) with \( \nu_i = (2T_i/M)^{1/2} \). Consequently, in the Pfirsch–Schlüter regime \( f_{1i}/f_0 \sim \rho_i/L_{\text{ia}} \ll \lambda/qR \ll 1 \). Therefore, to maintain the required lowest order Maxwellian requires either large \( L_{\text{ia}} \) or a breakdown of the Pfirsch–Schlüter ordering. However, even if the Pfirsch–Schlüter ordering were violated (so that the collisional radial ion heat flux was at the ion diamagnetic heat flux level), we expect \( \vec{n} \cdot \nabla T_i \sim \vec{n} \cdot \nabla B \sim \rho_i/L_{\text{ia}} \sim \lambda/qR \ll 1 \) in a confined collisional plasma. As a result, \( \rho_i \sim L_{\text{ia}} \) should still hold in the pedestal.

In addition, for a density scale length of \( \rho_i \), lowest order perpendicular momentum balance gives \( \omega_i = -c(\Phi/d\psi + (en)^{-1}d(nT_i)/d\psi') \) with \( cR(en)^{-1}d(nT_i)/d\psi' \sim v_i = \text{ion thermal speed and } \Phi(\psi) \) the axisymmetric electrostatic potential. Consequently, in a subsonic pedestal it must be that to lowest order the ions are electrostatically confined [2] with \( en d\Phi/d\psi' \sim -d(nT_i)/d\psi' \sim -Tdn/d\psi' \). This behaviour is observed in the banana regime H mode pedestal of DIII-D [21] and in the slightly more collisional Alcator C-Mod H mode pedestal [22]. Using total pressure balance we then see that the electrons must be magnetically confined with a mean flow \( \vec{V}_e \) comparable to the ion thermal speed \( (\vec{V}_e \sim v_i) \). In this simple limit the electrons carry essentially all the pedestal current.

Next, we briefly address the issue of zonal flow in the pedestal to provide more insight into the result of [9]. Rosenbluth and Hinton [23] demonstrated that plasma polarization is the key factor affecting the linear stage of zonal flow dynamics in a tokamak. Namely, the tokamak plasma ‘shields’ the original zonal flow potential so that only some fraction of it, the residual, survives. Physically, this shielding is provided by the dipole moment induced in plasmas by the zonal flow. Accordingly, two components of polarization are distinguished in tokamak plasmas. The classical one originates from the dipole moment associated with ion gyrocentres. This plasma response is relevant on time scales greater than the cyclotron period. The other polarization is neoclassical in origin and due to shifting the centre of the banana or passing orbits of the ions in a tokamak. This mechanism takes effect on time scales greater than the bounce period. The residual is the ratio of the final perturbed potential \( \delta \Phi(t \to \infty) \) to its initial value \( \delta \Phi(t = 0) \) for a step function change in the perturbed density:

\[
\frac{\delta \Phi(t \to \infty)}{\epsilon_{\text{pol-cl}} + \epsilon_{\text{pol-nc}}} = \frac{1}{1 + 1.6(q^2/\nu_{\text{pol}}^2)Y}, \tag{7}
\]

where \( \epsilon_{\text{pol-cl}} \) and \( \epsilon_{\text{pol-nc}} \) are the classical and neoclassical plasma polarizations relating the perturbed polarization density \( \delta n_{\text{pol-cl}} \) and potential \( \delta \Phi \), assumed proportional to \( \exp(-i k \cdot \vec{d} \psi) \), in \( \epsilon_{\text{pol}} \delta \Phi \). Here \( q \) is the safety factor and \( \epsilon \) is the inverse aspect ratio, with \( Y \) unity for the familiar Rosenbluth and Hinton case.

The strong localized axisymmetric radial electric field that arises in the pedestal modifies the collisionless zonal flow residual of Rosenbluth and Hinton [23] due to the strong poloidal \( \vec{E} \times \vec{B} \) drift and its associated finite orbit effects (we ignore orbit squeezing [24]). For example, in the banana
regime the axisymmetric radial electric field of an ion poloidal gyroradius pedestal must approximately satisfy $\epsilon d\Phi_0/df = -(T_i/n)dn/d\psi$, with making the ion Maxwellian stationary. Then the residual associated with the small amplitude, shorter wavelength, axisymmetric zonal flow potential $\delta \Phi$ will differ substantially from that of (23) as shown in [9] and discussed in the following paragraphs.

Retaining the poloidal $\vec{E} \times \vec{B}$ drift as well as parallel streaming the ion poloidal drift frequency becomes

$$\dot{\theta} = \left[ v_i + c_i B^{-1} \Phi'(\psi) \right] \vec{n} \cdot \nabla \theta. \quad (8)$$

In the tokamak core, the second term on the right side of (8) is much less than the first one, whereas in the pedestal these terms are comparable, thereby modifying the poloidal motion of particles. We remark that the terms are comparable, thereby modifying the poloidal motion in the tokamak core, the second term on the right side of (8) becomes

Then the residual associated with the small amplitude, shorter particle to be trapped in the pedestal its poloidal motion projected onto the poloidal cross-section of a tokamak.

\[ \phi(\psi) = \phi_0 + \left( \psi - (\psi_0 - \Delta) \right) \phi'_0, \]

with $\phi_0 \equiv \phi(\psi_0 - \Delta)$, $\phi'_0 \equiv \phi'(\psi - \Delta)$ and $\delta \phi'_0 \equiv \delta \phi'(\psi_0 - \Delta)$. Assuming the radial variation of $B$ and $I$ are weak, equation (8) can be written as

$$qR \dot{\theta} = (v_1 + u). \quad (10)$$

where we continue to ignore orbit squeezing by taking $\phi'_0 = 0$ for simplicity. This result displays the $u$ shift to $v_1$ due to large $\vec{E} \times \vec{B}$.

The two main qualitative features are as follows. First, as the shift $u$ becomes comparable to the ion thermal speed, the trapped particle region moves towards the tail of the Maxwellian centred about $\nu = 0$ as shown in figure 3, dramatically reducing the number of trapped ions and the neoclassical polarization, and thereby enhancing the residual (7) to make it closer to unity.

The second feature is that in the pedestal the neoclassical polarization and therefore the residual become complex; a feature requiring a bit more discussion. Consider the polarization $\delta \phi$ caused by a radial dipole moment induced in a plasma that is linearly proportional to the dipole inducing electric field $\delta \vec{E} = -\vec{\nabla} \phi = -\vec{\nabla} \psi (\delta \Phi/\delta \psi)$. For a scalar susceptibility $\alpha = \alpha(\psi)$, $\delta \vec{E} = \alpha \delta \vec{E}$. Allowing for a strong gradient in the susceptibility and assuming $\nabla \psi$ is weakly varying, the polarization density $\delta n_{pol}$ given by

$$\delta n_{pol} = -4\pi \vec{\nabla} \cdot \delta \vec{E} \frac{\delta \Phi}{\delta \Phi} \left( -i k \vec{d} \psi \right)$$

Recalling that the polarization density and potential are related by $\epsilon_{pol} k^2 \phi = -4\pi e \delta n_{pol}$, we see that the neoclassical plasma polarization $\epsilon_{pol}$ in equation (7) becomes complex due to the strong radial dependence of the susceptibility that can vary on the scale of a poloidal ion gyroradius. The imaginary term corresponds to shifting the entire plasma pedestal as a whole in response to the applied electric field. The detailed result from [9] that connects smoothly to (23) is

\[ Y = \left( 1 + \frac{2iz}{k_r \rho_n} \right) \frac{4 \exp(-z^2)}{3\pi^{1/2}} \int_0^\infty dy (y + 2z^2)^{3/2} \exp(-y), \quad (12) \]

with $z = u/v_i$ and $k_r \approx \kappa R B_0$ the radial wavenumber. The result for $Y$ is sensitive to the electric field through $z$. As a result, the usual core paradigm that electric field shear controls turbulence must be modified in the pedestal. From (12) we can see that $Y$ becomes small and the residual of (7) approaches
Figure 2. Toroidal projections of trapped particle orbits for weak, subcritical $(v_\parallel + u < 0)$ and above critical $(v_\parallel + u > 0)$ $E \times B$ drift.

Figure 3. In the presence of a strong radial electric field the trapped region moves towards the tail of the Maxwellian distribution.

unity for $u/v_i > 1$, thereby causing the zonal flow to be undamped. Using $en\Phi' \sim T_i\nu'$ in $u \sim v_i$ gives a density pedestal width of order $\rho_{pi}$ as anticipated. Implicit in the derivation of (12) is that $\epsilon^{1/2}k_r\rho_{pi} \lesssim 1$ as in Rosenbluth and Hinton [23] who unlike us assume $u \ll v_i$.

A straightforward way to develop a density pedestal is to recall that particle transport must be ambipolar and therefore controlled by the electrons. If the anomalous electron particle diffusivity is reduced then a large plasma density gradient and a pedestal is allowed. Similarly, an electron temperature pedestal can form if the anomalous electron heat diffusivity becomes small since the electron poloidal gyroradius is much smaller than any width of interest. The reason these electron diffusivities are reduced is then the key issue for pedestal formation. However, the local maximum in $|Y|$ provides a threshold at $(k_r\rho_{pi})^2 = 4\zeta^2$ beyond which further steepening of the density profile (and, by pressure balance, a further decrease in the radial electric field) leads to a larger zonal flow residual, presumably causing a reduction in turbulence, and further enhancement of the density profile to complete the feedback loop. For a well-defined threshold to exist when the radial zonal flow wavelength $2\pi/k_r$ is comparable to the pedestal width $w$ requires $\pi\rho_{pi}/w \sim u/v_i \sim 1$. For weaker density gradients having $\pi\rho_{pi}/w \sim u/v_i \ll 1$ the zonal flow residual cannot be increased, no feedback occurs, and the usual Rosenbluth and Hinton result is recovered.

4. Gyrokinetic closure and radial electric field on transport time scales

Simulating electrostatic turbulence in tokamaks on transport time scales requires retaining and evolving a complete turbulence modified transport description, including all the axisymmetric long wavelength and zonal flow radial electric field effects, as well as the turbulent transport normally associated with drift instabilities. Gyrokinetic simulations normally use gyrokinetic quasineutrality to determine the electric field—a procedure that seems adequate for local or $\delta f$ gyrokinetic treatments. However, full $f$ gyrokinetic treatments also mistakenly attempt to determine the axisymmetric radial electric field from quasineutrality. The difficulty with gyrokinetic quasineutrality is that it only gives the proper electric field for $k_\perp\rho_i \sim 1$ or greater, begins to fail for $\rho_i/L_\perp \sim (k_\perp\rho_i)^2 \ll 1$, and is totally inadequate for the axisymmetric global radial electric field. To avoid this defect [8] derives equations that can be used in place of gyrokinetic quasineutrality. Full electric field effects and their evolution are more difficult to retain than density and temperature evolution effects since the need to satisfy intrinsic ambipolarity in the axisymmetric, long wavelength limit requires evaluating the ion distribution function to higher order in gyroradius over background scale length than standard gyrokinetic treatments as already noted earlier. To avoid having to derive and solve a gyrokinetic equation to order $(v_\parallel/\Omega_i)(\rho_i/L)^3$, an alternative hybrid gyrokinetic-fluid treatment is formulated that employs moments of the full
Fokker–Planck equation [3]. The description is an extension to gyrokinetics of drift kinetic treatments that yield expressions for the ion perpendicular viscosity as well as for the electron and ion parallel viscosities, gyrosviscosities and heat fluxes for arbitrary mean-free path plasmas, in which the lowest order distribution function is a Maxwellian [25]. Indeed, an implementable vorticity equation or, equivalently, toroidal angular momentum conservation equation, can be derived [8] to replace gyrokinetic quasi-neutrality in a way that removes the long wavelength defects of quasi-neutrality existing in gyrotropic simulations, but retains all the proper shorter wavelength features.

This hybrid description evolves electrostatic potential, plasma density, ion and electron temperatures and ion and electron flows using conservation of charge, number, ion and electron energy and total and electron momentum, respectively [3]. All electrostatic effects with wavelengths much longer than an electron gyroradius are retained so that ITG and TEM turbulence and the associated zonal flow as well as all neoclassical behaviour are treated. Closure for the electrons is obtained by solving the electron drift kinetic equation to find the leading order correction to the Maxwellian electron distribution function \( f_{0e} \) needed to evaluate the parallel electron viscosity (or pressure anisotropy) as well as the momentum and energy exchange terms with the ions. In addition, the \( \bar{v}^2/2 \) moment of the exact electron Fokker–Planck equation is used, along with this first order correction to \( f_{0e} \), to evaluate the electron heat flux (collisional plus diamagnetic), thereby achieving closure for the electrons. Ion closure is achieved similarly by solving the ion gyrokinetic equation to order (\( \rho_i/L \))^2. However, ion closure is somewhat more complicated because the \( \bar{v}^2 \) as well as the \( \bar{v}^2/2 \) moments of the ion Fokker–Planck equation must be used to evaluate the ion gyrosviscosity and perpendicular viscosity, along with the ion heat flux. Moreover, to recover the correct results in the axisymmetric, long wavelength limit and to treat turbulence, the gyrokinetic variables must be determined to one order higher than normal [1, 2]. This complication is a result of toroidal angular momentum attempting to flow in flux surfaces to lowest order, thereby reducing the size of the fast time and flux surface averages of the radial flux of toroidal angular momentum. Once this is done complete closure is obtained and a description valid on transport time scales is recovered that properly evolves the electrostatic potential and flows, as well as density and temperatures [3]. Instead of the distribution functions, the hybrid description employs the more accurate fluid conservation equations to evaluate and evolve the density, temperatures, electrostatic potential, ion flow and current density. In our hybrid description the distribution functions are only used to evaluate heat fluxes, viscosities and collisional exchanges in terms of the lower order moments determined from the conservation equations. Consequently, the gyrokinetic equation need only be solved intermittently using these evolved profiles. Specifically, the initial Maxwellian distributions and \( \vec{E} \times \vec{B} \) are updated to the new \( n, T_e, T_i, \Phi, \vec{V} \) and \( \vec{J} \) each time the gyrokinetic and drift equations are solved and evolved to a new steady turbulent state to determine the new distribution functions used for closure in the viscosities, heat flows and collisional exchange terms. This procedure ensures conservation of number, energy, charge and momentum are satisfied so the kinetic equations need not be solved in a conservative form or satisfy exact total energy or canonical angular momentum conservation.

To simplify the expression for the ion viscosity we may expand the ion distribution function in powers of \( \rho_i/L_{pe} = B_i/B \ll 1 \). In this case the ion gyrokinetic equation need only be derived to order \( \rho_i/L \), with \( \rho_i/L \) corrections retained as in drift kinetics [15, 20]. To order \( \rho_i/L^2 \) the Taylor expanded form of the gyrophase dependent portion of \( f \) from equation (80) of [1] is given by

\[
fi = (fi) = \Omega_i^{-1} \left[ \vec{v} \times \vec{n} \right] - 2(e/M)(\nabla \Phi) \delta / \partial \vec{v}^2 \right](f_{0i} + fi) - \left[ \vec{v} \cdot \vec{v}_d + (v_i/4\Omega_i)(\vec{v}_d \times \vec{n} + \vec{n} \times \vec{v}\vec{v}_d) : \nabla \vec{n} \right] \times B^{-1} \delta f_{0i} / \delta \mu_i, \tag{13}
\]

where order (\( \rho_i/L \)) corrections are ignored. Here \( f_{0i} \) is a stationary Maxwellian, \( f_{0i} \sim f_{0p0}/L \) is the leading order correction, \( (fi) \) is the gyrokinetic gyroaverage of \( f_i \), \( \vec{v}_d \) is the sum of the magnetic and \( \vec{E} \times \vec{B} \) drifts, \( \mu_i \) is the lowest order magnetic moment and \( \vec{V} \) is performed holding \( v \) and \( \mu_0 \) fixed. By using this \( f_{0i} \gg \rho_i \) form rather than the full result from equation (80) in [3] the evaluation of the ion viscosity can be considerably simplified. Indeed, we are in the process of using this approximation to obtain numerically implementable equations for properly determining the full radial electric field. The important point to make here is that a formulation is possible that requires only solving the standard full \( f \) gyrokinetic equation [1] if we assume that the poloidal magnetic field is weak compared with the total magnetic field.

In [3] explicit expressions were given using the gyrokinetic results of [1]. Alternatively, the gyrokinetic description of [2] that is specifically formulated to conveniently treat the pedestal can be employed. Rather than using the typical gyrokinetic variables and gyrokinetic equation of [1] in the pedestal, it might be advantageous to use the alternate gyrokinetic variables and gyrokinetic equation of [2] that exactly recover the isothermal Maxwellian as a lowest order solution.

In this hybrid description distribution functions are only used to evaluate moments needed for closure and collisional exchange [3]—they are not used to evaluate density, temperature and flows since these are found from the moment equations. The results are given in terms of a few velocity space integrals of the gyrokinetic distribution function and make possible a turbulent hybrid fluid-gyrokinetic description that includes the long wavelength or global radial electric field as well as arbitrary wavelength turbulence and zonal flow effects. Moment equations evolve all other quantities such as density, temperatures, flows and potential. As a result, either PIC or continuum, lowest order gyrokinetic and drift kinetic solvers may be employed, and the kinetic equations need not be solved in conservative form.

5. Discussion

In section 2 we have presented the most streamlined proof that we know of intrinsic ambipolarity through second order in the ion gyroradius expansion for a non-turbulent tokamak. We first show that in an axisymmetric, single ion species tokamak, intrinsic ambipolarity requires the distribution functions and
heat and particle fluxes to be independent of electrostatic potential to leading order in gyroradius. We then show that a moment description can be used to demonstrate that intrinsic ambipolarity must be satisfied to second order in the ion gyroradius expansion. Consequently, for global gyrokinetic simulations quasineutrality cannot determine the correct global axisymmetric radial electric field. It must be determined from toroidal angular momentum conservation, or equivalently, from the vorticity equation. These two equations are related by parallel momentum conservation.

Using canonical angular momentum as the radial variable allows strong gradients to be treated gyrokinetically [2] in spite of the rather complicated ion trajectories expected in the pedestal and illustrated in section 3. Entropy production requires a physical lowest order banana regime ion distribution function to be nearly an isothermal Maxwellian with the ion temperature scale much greater than the poloidal ion gyroradius [2]. Thus, the background ion temperature profile must have a pedestal with a width much larger than that of any density pedestal varying on an ion poloidal gyroradius scale. Here we prove for the first time that similar behaviour is to be expected in Pfirsch–Schluter regime plasmas. In addition, weak ion temperature variation with subsonic flow in such a pedestal requires electrostatically restrained ions and magnetically confined electrons thereby impacting zonal flow behaviour and altering the neoclassical polarization [9]. A heuristic discussion of the impact of these differences on zonal flow and pedestal formation is presented.

Simulating tokamaks on transport time scales requires evolving drift turbulence with axisymmetric long wavelength and zonal flow radial electric field effects retained. As discussed in section 4, full electric field effects are far more difficult to retain than density and temperature effects since they require evaluating the ion distribution function to higher order than standard gyrokinetics. An electrostatic hybrid gyrokinetic-fluid treatment using moments of the full Fokker–Planck equation removes the need to go to higher order while remaining valid in the pedestal and removing the limitations of gyrokinetic quasineutrality. This hybrid description evolves potential, density, temperatures and flows, and includes all electrostatic neoclassical and turbulent effects with wavelengths much longer than an electron gyroradius [3]. Here we illustrate a procedure whereby a hybrid description can be simplified by expanding in the smallness of the poloidal magnetic field compared with the total magnetic field. In this limit, only the standard full f gyrokinetic equation need be solved in conjunction with the fluid equations.

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