Heat and work are two fundamental quantities in thermodynamics. While these variables are deterministic in macroscopic systems [1], they become stochastic at the microscopic scale owing to the presence of thermal [2,3] or quantum [4,5] fluctuations. A central issue is then to determine their probability distributions. The nonequilibrium work statistics of classical driven systems has been extensively studied both theoretically and experimentally [6–8]. On the other hand, the investigation of heat fluctuations is more involved, even for simple systems at equilibrium [9–14]. The main reason is that heat depends nonlinearly on position even for a linear system like the harmonic oscillator. The heat distribution has been theoretically and experimentally analyzed for a classical harmonic oscillator in the overdamped limit in Ref. [15] and in the underdamped regime in Ref. [16].

At the quantum level, attention has so far mostly focused on nonequilibrium work. The work distributions of driven quantum oscillators have for instance been theoretically obtained in Refs. [17–19] and experimentally studied using a trapped ion [20]. At the same time, the quantum work statistics of a driven two-level system has been computed in Refs. [21,22] and determined experimentally in NMR [23] and cold-atom [24] setups. Recently, the quantum heat exchange statistics has been examined theoretically for exactly solvable two-level models [25,26] and the experimental reconstruction of such a heat distribution has been reported [27]. However, to our knowledge, the heat distribution of a quantum harmonic oscillator has neither been calculated nor measured, despite its essential role in many applications [28].

The aim of this paper is to analytically compute and analyze the properties of the heat distribution of a thermal quantum harmonic oscillator weakly coupled to a reservoir at a different temperature. To that end, we employ master equation methods of quantum optics [29]. We first determine the exact characteristic function of the heat statistics and demonstrate that it obeys the fluctuation theorem of Jarzynski and Wójcik [30]. We additionally derive closed form expressions for the heat distribution in the limit of long interaction times, both in the high and low temperature regimes. We finally study the time evolution of the heat probability density by analytically evaluating its first two cumulants.

Let us begin by considering a quantum harmonic oscillator with frequency $\omega$ and inverse temperature $\beta_1$ weakly coupled to a heat bath at a different inverse temperature $\beta_2$. We model the reservoir as an infinite set of quantum harmonic oscillators, as commonly done in condensed matter physics [28] and quantum optics [29]. The Hamiltonian of the combined system is $H = H_1 + H_2 + H_{12}$, where $H_1 = \hbar \omega (a^\dagger a + 1/2)$ and $H_2 = \sum_j \hbar \omega_j b_j^\dagger b_j$ are the respective Hamiltonians of system and bath, and $H_{12} = \hbar \sum_j \kappa_j (a^\dagger b_j + a b_j^\dagger)$ describes the interaction with coupling parameters $\kappa_j$ [29]. Here $a$ and $b_j$ denote the usual ladder operators. System and reservoir are brought into thermal contact at $t = 0$ and let to interact for a duration $t$. Since the oscillator-bath coupling is weak, heat may be identified with the energy exchanged between the two. The heat distribution at time $t$ is accordingly [30],

$$P(Q,t) = \sum_{n,m} \delta [Q - (E_m - E_n)] P_{n,m}^t P_n^0,$$

where $P_n^0 = \exp(-\beta_1 E_n)/Z$ is the initial thermal occupation probability of the oscillator with partition function $Z$, and $P_{n,m}^t$ are the transition probabilities between initial and final states $n$ and $m$ with corresponding energy eigenvalues $E_l = \hbar \omega (l + 1/2)$, $l = (n,m)$. The transition probabilities can be explicitly written in terms of the time evolution operator as $P_{n,m}^t = |\langle m | U(t) | n \rangle|^2 = \langle m | \rho(t) | m \rangle$, with the density operator $\rho(t) = U(t) | n \rangle \langle n | U(t)^\dagger$. We therefore need to determine the diagonal matrix element of the density operator in order to evaluate the heat statistics via Eq. (1).

The time evolution of the density operator $\rho$ of a damped harmonic oscillator in the weak-coupling limit is governed by the quantum-optical master equation [29],

$$-i\hbar \frac{d\rho(t)}{dt} = \omega [a^\dagger a, \rho] + \frac{\gamma}{2} \tilde{n}_2 (aa^\dagger \rho + \rho a a^\dagger - 2a^\dagger a \rho) + \frac{\gamma}{2} \tilde{n}_2 + 1 (a^\dagger a \rho + \rho a^\dagger a - 2a a^\dagger \rho),$$

where $\tilde{n}_2 = [\exp(\beta_2 \hbar \omega) - 1]^{-1}$ is the thermal occupation number at inverse temperature $\beta_2$ and $\gamma$ the damping constant. The quantum master equation (2) may be solved exactly using generating function techniques [31]. Writing concretely the diagonal matrix elements in the form $\langle m | \rho(\tau) | m \rangle = \sum_n X_{m,n}(\tau)p_n(0)$ with $\tau = \gamma t$ and
Heat distribution $P(Q)$, Eq. (1), for a harmonic oscillator at inverse temperature $\beta_1$, weakly coupled to a bath at inverse temperature $\beta_2$ (red squares), compared with the symmetric isothermal heat distribution $P_{iso}(Q)$, Eq. (11), obtained for $\beta = \beta_1 = \beta_2$ (blue dots). The respective blue dotted-dashed and red dashed lines represent the corresponding classical heat distribution given by Eq. (12). Parameters are $\beta_1 = 1, \beta_2 = 2.5$ and $\beta = 2.5$.

arbitrary initial condition $p_n(0)$, one finds \cite{31},

$$X_{m,n} = \frac{u^n}{(1+u)^{m+1}} \left(\frac{1+v}{1+u}\right)^n \sum_{j} \frac{(m+n-j)!}{(n-j)!j!(m-j)!} \left[\frac{-v(1+u)}{u(1+v)}\right]^j,$$

(3)

with the two parameters $u$ and $v$ defined as,

$$u = \bar{n}_2(1-e^{-\gamma}) \quad \text{and} \quad v = \bar{n}_2 - (\bar{n}_2 + 1)e^{-\gamma}.$$  

(4)

The heat distribution $P(Q)$ follows with $p_n(0) = 1$ as,

$$P(Q,\tau) = \frac{1}{Z} \sum_{m,n} \delta(Q - (E_m - E_n)) X_{m,n}(\tau) e^{-\beta_1 \hbar \omega m}.$$  

(5)

In order to analyze Eq. (5), we introduce the characteristic function $G(\mu,\tau) = \int dQ \exp(i\mu Q) P(Q,\tau)$ and obtain,

$$G(\mu,\tau) = \frac{1}{Z} \sum_{m,n} X_{m,n}(\tau) e^{-\beta_1 \hbar \omega m} e^{i\mu\hbar \omega (m-n)}.$$  

(6)

The three sums appearing in Eq. (6) can be performed explicitly, see details below, leading to,

$$G(\mu,\tau) = \frac{\exp(-\Delta \beta Q)}{e^{\beta_1 \hbar \omega}[(u + 1)e^{\beta_1 \hbar \omega} - (1+\beta_1)e^{\beta_1 \hbar \omega} + v] - v - 1}.$$  

(7)

The above expressions are exact and fully characterize the quantum heat fluctuations of a damped harmonic oscillator coupled to a reservoir at a different temperature. The characteristic function (7) satisfies the symmetry relation $G(i\Delta \beta - \mu,\tau) = G(\mu,\tau)$. We thus recover the fluctuation theorem for heat exchange, $P(Q,\tau)/P(-Q,\tau) = \exp(-\Delta \beta Q)$, derived by Jarzynski and Wójcik \cite{30}. In order to gain additional physical insight about the quantum heat statistics, we will now study different limits where closed form formulas can be derived.

We start by examining the long-time behavior of the heat statistics. In the limit $\tau \to \infty$, Eq. (7) reduces to,

$$G(\mu) = \frac{1}{Z} \sum_{m,n} e^{\beta_1 \hbar \omega e^{-i\mu \hbar \omega}} \left(\frac{n}{1+n}\right)^m \frac{1}{1+n}$$  

(8)

$$= \frac{1}{1 - e^{-\hbar \omega(\beta_2 - \mu)}} - e^{-\hbar \omega(\beta_1 + \beta_2)} + e^{-\hbar \omega(\beta_1 + \beta_2)}.$$  

Taking the inverse Fourier transform, we arrive at the asymptotic quantum heat distribution,

$$P(Q) = \frac{1 - e^{-\hbar \omega \beta_1} - e^{-\hbar \omega \beta_2} + e^{-\hbar \omega(\beta_1 + \beta_2)}}{1 - e^{-\hbar \omega(\beta_1 + \beta_2)}} \times \sum_n \delta(Q - n\hbar \omega) + \delta(Q + n\hbar \omega) \left\{ \begin{array}{ll} e^{-\beta_1 Q}, & Q \geq 0 \\ e^\beta Q, & Q < 0 \end{array} \right.$$  

(9)

In the isothermal case, $\beta = \beta_1 = \beta_2$, the characteristic function (8) further simplifies to,

$$G_{iso}(\mu) = \frac{\cosh([\hbar \omega]\beta) - 1}{\cosh([\hbar \omega]\beta) - \cos(\hbar \omega \mu)}.$$  

(10)

The corresponding probability distribution then reads,

$$P_{iso}(Q) = \frac{\cosh(\hbar \omega \beta) - 1}{\sinh(\hbar \omega \beta)} e^{-\beta Q} \times \sum_n \delta(Q - n\hbar \omega) + \delta(Q + n\hbar \omega).$$  

(11)

Equations (9) and (11) are shown in Fig. 1. We observe that the two heat distributions are discrete with spacing $\hbar \omega$, as expected for a quantized harmonic oscillator. We further note that they both decay exponentially for positive and negative arguments. In addition, the heat probability density is in general asymmetric, implying a non-zero mean heat current between oscillator and bath, except in the isothermal case since no average energy flows between two objects at the same temperature.

In the high-temperature limit, $\hbar \omega \beta_1, \beta_2 \ll 1$, the discrete heat distribution (9) becomes continuous and we recover the known classical expression (10) by Taylor expanding the exponential functions to lowest order,

$$P_{high}(Q) = \frac{\beta_1 \beta_2}{\beta_1 + \beta_2} \left\{ \begin{array}{ll} e^{-\beta_1 Q}, & Q \geq 0 \\ e^{\beta_1 Q}, & Q < 0 \end{array} \right.$$  

(12)

As seen in Fig. 1, the envelopes of the classical and quantum heat distributions are similar in shape, in contrast to the work distribution (17). The notable difference is that the quantum density is always narrower than the corresponding classical density, owing to the bosonic nature of the harmonic oscillator. In the opposite low-temperature regime, $\hbar \omega \beta_{1,2} \gg 1$, only the first three delta peaks at
FIG. 2. Evolution of the quantum heat distribution \( P(Q, \tau) \) computed as the inverse Fourier transform of the characteristic function \( G(\mu, \tau) \), Eq. \([7]\), for two thermalization times \( \tau_1 = 0.1 \) and \( \tau_2 = 2 \). Parameters are \( \beta_1 = 1 \), \( \beta_2 = 3 \).

\[
Q = 0, \pm \hbar \omega \text{ contribute significantly to the heat distribution.}
\]

\[
\sigma^2_Q(\tau) = \langle Q^2 \rangle(\tau) - \langle Q \rangle^2(\tau) = \frac{\hbar^2 \omega^2 \left[u(u + 1)e^{2\beta_1 \hbar \omega} + (1 - u(2v + 3) + v)e^{\beta_1 \hbar \omega} + v^2 + v\right]}{(e^{\beta_1 \hbar \omega} - 1)^2}.
\]

Equation \([16]\) is simply the difference between the mean energies at temperatures \( T_2 \) and \( T_1 \) and can be rewritten in terms of the thermal occupation probabilities as \( \langle Q \rangle = \hbar \omega (n_2 - n_1) \). We additionally notice that the heat fluctuations, as characterized by the variance, are left invariant when the temperatures of the harmonic oscillator and of the heat reservoir are switched. This is not the case for the average value of the heat which changes its sign, indicating a reversal of the energy current.

**Conclusions.** We have analytically computed the characteristic function of the quantum heat statistics of a harmonic oscillator weakly coupled to a heat reservoir at a different temperature. We have first shown that it satisfies the fluctuation theorem of Jarzynski and Wójcik \([30]\). We have additionally obtained closed form expressions for the quantum heat distribution in the asymptotic long-time limit, both in the low and high temperature regimes. The classical and quantum heat probability densities have the same exponential, and generally asymmetric, dependence on \( Q \). The quantum distribution is discrete with spacing corresponding to the level interval of the harmonic oscillator. It is moreover narrower than the classical distribution. We have finally investigated the time evolution of the quantum heat distribution by computing its first two cumulants using the formula \( \langle Q^n \rangle(\tau) = i^{-n}d^nG(\mu, \tau)/d\mu^n|\mu = 0 \) \([32]\). We obtain the average heat,

\[
\langle Q \rangle(\tau) = \frac{\hbar \omega (ue^{\beta_1 \hbar \omega} - v - 1)}{e^{\beta_1 \hbar \omega} - 1},
\]

and the variance,

\[
\sigma^2_Q = \frac{\hbar^2 \omega^2 \left[-4e^{\hbar \omega (\beta_1 + \beta_2)} + e^{\hbar \omega (2\beta_1 + \beta_2)} + e^{\hbar \omega (\beta_1 + 2\beta_2)} + e^{\beta_1 \hbar \omega} + e^{\beta_2 \hbar \omega}\right]}{(e^{\beta_1 \hbar \omega} - 1)^2(e^{\beta_2 \hbar \omega} - 1)^2}.
\]

Expression \([13]\) shows that quantum heat is strictly negative when the harmonic oscillator is initially in its ground state. This corresponds to the limiting situation where the quantum oscillator can only absorb energy.

It does not seem possible to analytically determine the quantum heat distribution \( P(Q, \tau) \) for arbitrary thermalization times \( \tau \) (see Fig. 2). In order to study its time evolution, we next compute its first two cumulants using the formula \( \langle Q^n \rangle(\tau) = i^{-n}d^nG(\mu, \tau)/d\mu^n|\mu = 0 \) \([32]\). We obtain the average heat,

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\]

As a result, we obtain the heat probability density,

\[
P_{\text{low}}(Q) = \frac{\delta(Q) + \delta(Q - \hbar \omega) + \delta(Q + \hbar \omega)}{1 + e^{-\hbar \omega \beta_1} + e^{-\hbar \omega \beta_2}}
\]

\[
\times \begin{cases} e^{-\beta_2 Q}, & Q \geq 0 \\ e^{\beta_1 Q}, & Q < 0 \end{cases}
\]

Expression \([13]\) shows that quantum heat is strictly negative when the harmonic oscillator is initially in its ground state. This corresponds to the limiting situation where the quantum oscillator can only absorb energy.

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\sigma^2_Q = \frac{\hbar^2 \omega^2 \left[-4e^{\hbar \omega (\beta_1 + \beta_2)} + e^{\hbar \omega (2\beta_1 + \beta_2)} + e^{\hbar \omega (\beta_1 + 2\beta_2)} + e^{\beta_1 \hbar \omega} + e^{\beta_2 \hbar \omega}\right]}{(e^{\beta_1 \hbar \omega} - 1)^2(e^{\beta_2 \hbar \omega} - 1)^2}.
\]
where we have defined the variable $y$. We show that the stationary limit is reached exponentially in time.

Appendix. Let us sketch the derivation of the characteristic function [7]. We first write Eq. (6) in terms of the ordinary hypergeometric function $\sum_{k,n} u^n y^k (1 + v)/(1 + u) e^{-\omega \beta_i}$

$G(\mu, \tau) = \frac{1}{Z(1 + u)} \sum_{k,n} u^n e^{-\omega \beta_i} \sum_{j=0}^\infty \frac{1}{1 + u} e^{i \mu \beta_i (m-n)} F[-n,-m,1;y]$,

where we have defined the variable $y = (u-v)/u(1+v)$. We next use the identity,

$F[-n,-m,1;y] = (1-y)^{1+m+n} F[1+n,1+m,1;1]$

and the explicit series representation of the ordinary hypergeometric function,

$F[1+n,1+m,1;1] = \sum_{k=0}^\infty y^k \binom{n+k}{k} \binom{m+k}{k}$.

We then obtain the characteristic function,

$G(\mu, \tau) = \frac{1 - y}{Z(1 + u)} \sum_{k} y^k \frac{(1+v)(1-y)}{1 + u} e^{-\omega (\beta_i + i \mu)} \binom{n+k}{k}$

$\times \sum_{m} \frac{u(1-y)}{1 + u} e^{i \mu \omega} \binom{m+k}{k}$.

The two sums over $m$ and $n$ are of the form,

$\sum_{j=0}^\infty a^j (j+k) = (1 - a)^{-k-1}, \quad |a| \leq 1$.

As a consequence, we find,

$G(\mu, \tau) = \frac{1 - y}{Z(1 + u)} \sum_{k} y^k (1-B)^{-k-1}(1-C)^{-k-1}$.

where we used introduced the two parameters,

$B = \frac{u(1-y)}{1 + u} e^{i \mu \omega}, \quad C = \frac{(1+v)(1-y)}{1 + u} e^{-\omega (\beta_i + i \mu)}$.

The final sum is a geometric series. We thus arrive at,

$G(\mu, \tau) = \frac{1 - y}{Z(1 + u)} \sum_{k} \frac{1}{(1-C)(1-B)-y}$. "Quantum coins and nano sensors".

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