Abstract

In many problems of quantum chaos the calculation of sums of products of periodic orbit contributions is required. A general method of computation of these sums is proposed for generic integrable models where the summation over periodic orbits is reduced to the summation over integer vectors uniquely associated with periodic orbits. It is demonstrated that in multiple sums over such integer vectors there exist hidden saddle points which permit explicit evaluation of these sums. Saddle point manifolds consist of periodic orbits vectors which are almost mutually parallel. Different problems has been treated by this saddle point method, e.g. Berry’s bootstrap relations, mean values of Green function products etc. In particular, it is obtained that suitably defined 2-point correlation form-factor for periodic orbit actions in generic integrable models is proportional to quantum density of states and has peaks at quantum eigenenergies.
1 Introduction

The trace formulas are the main instrument in the investigation of relations between quantum and classical properties of a given system (see e.g. [1]). These formulas connect the quantum density of states, \( d(E) \), in the semi-classical limit with a sum over classical periodic orbits \( (p_0) \) [1]-[4]

\[
d(E) = \bar{d}(E) + \sum_{p_0} A_p \exp(\frac{i S_p(E)}{\hbar}),
\]

where \( \bar{d}(E) \) is the mean density, \( S_p(E) \) is the classical action along a periodic orbit, and \( A_p \) is a pre-factor build from classical quantities.

There is two main applications of these formulas. First, one tries to use them to compute smoothed density of states (or even approximate positions of energy levels [5]). Second, one uses trace formulas to understand statistical properties of energy levels [6]. In the latter approach one starts with a formal expression of \( n \)-point correlation function, \( R_n(\epsilon_1, \epsilon_2, \ldots, \epsilon_n) \), as a mean value of product of \( n \) density of states, \( d(E) \), at different points

\[
R_n(\epsilon_1, \epsilon_2, \ldots, \epsilon_n) = \langle d(E + \epsilon_1)d(E + \epsilon_2) \cdots d(E + \epsilon_n) \rangle,
\]

where the brackets denote an average over an interval of energy \( \Delta E \) which is classically small but includes a large number of quantum levels. Substituting in this formula the semi-classical expression (1) for the density of states one obtains a semi-classical approximation for correlation functions.

The main difficulty of such method is the calculation of mean values of products of contributions of different periodic orbits

\[
\langle \sum_{p_1,p_2,\ldots} \exp(i \frac{1}{\hbar} \sum_j e_j S_{p_j}(E)) \rangle,
\]

where \( e_j = \pm 1 \). When all \( e_j \) are of the same sign this average equals zero but if signs of \( e_j \) are different the calculations are far from being clear.

The simplest approximation (called the diagonal approximation) consists of taking into account only terms with exactly the same action

\[
\langle \exp(i \frac{1}{\hbar}(S_{p_1} - S_{p_2})) \rangle = \begin{cases} 1, & \text{if } S_{p_1} = S_{p_2} \\ 0, & \text{if } S_{p_1} \neq S_{p_2} \end{cases}.
\]
Berry [6] showed that for integrable systems this approximation gives the correct 2-point correlation form-factor but for chaotic systems the validity of diagonal approximation is limited to short-time behavior of the form-factor. For more complicated quantities (even for integrable systems) this approach is insufficient and more refined methods are needed [7]-[9].

The purpose of this paper is to develop a method which permits explicit computations of the expressions (3) for generic integrable systems. We shall show that in these expressions there exist hidden saddle points which give dominant contribution in the semi-classical limit.

The usual trace formulas can be applied only to quantum mechanical quantities which (like in Eq. (1)) are expressed through one Green function. The method discussed in this paper permits to construct a new type of trace formulas which express a product of such quantities for integrable systems through a sum over classical periodic orbits.

The plan of the paper is the following. In Section 2 for completeness the derivation of the trace formula for a rectangular billiard is presented. This is a simple prototype of integrable models where all formulas are transparent without unnecessary complications. In Section 3 on the example of Berry’s bootstrap we demonstrate the existence and importance of quantities which include sums like in Eq. (3) and whose semi-classical calculation is the main topic of the rest of the paper. The saddle point method for oscillatory and smooth terms is discussed in Sections 4 and 5 respectively. In is shown that the dominant saddle point configurations are build from periodic orbits whose integer vectors are almost mutually parallel. In Section 6 the calculation of mean values of products of different powers of retarded and advanced Green functions is presented. In this Section we also discuss the calculation of next-to-leading terms by the saddle point method. The correlation functions for classical actions are derived in Section 7. An interesting consequence of the discussed formalism is the fact that (properly defined) 2-point correlation form-factor for actions of classical periodic orbits is proportional to quantum density of states. The generalization of the proposed method for generic integrable models is performed in Section 8.
2 Trace formula for rectangular billiard

Let us consider a plane rectangular billiard with sides $a$ and $b$ with (for simplicity) periodic boundary conditions. The functions

$$
\Psi_{m,n}(x, y) = \frac{1}{\sqrt{ab}} \exp\left(\frac{2\pi i}{a}nx + \frac{2\pi i}{b}my\right), \quad (5)
$$

with integers $m, n = 0, \pm 1, \pm 2, \ldots$ are periodic solutions of the (Schrödinger) equation

$$(E_{mn} + \Delta)\Psi_{m,n}(x, y) = 0 \quad (6)$$

with energy

$$E_{mn} = \left(\frac{2\pi}{a}\right)^2 n^2 + \left(\frac{2\pi}{b}\right)^2 m^2. \quad (7)$$

The quantum density of these states

$$d(E) = \sum_{m,n=-\infty}^{\infty} \delta(E - E_{mn}) \quad (8)$$

may be rewritten by the Poisson summation formula in the following way

$$d(E) = \sum_{M,N=-\infty}^{\infty} \int e^{2\pi i(Mm+Nn)} \delta(E - \left(\frac{2\pi}{a}\right)^2 n^2 - \left(\frac{2\pi}{b}\right)^2 m^2) dndm. \quad (9)$$

The substitution $n = a r \cos \theta / (2\pi)$ and $m = b r \sin \theta / (2\pi)$ after simple algebra gives

$$d(E) = \frac{ab}{4\pi} \sum_{M,N=-\infty}^{\infty} J_0(kL_{MN}), \quad (10)$$

where $E = k^2$,

$$L_{MN} = \sqrt{(Na)^2 + (Mb)^2}, \quad (11)$$

and

$$J_0(x) = \frac{1}{2\pi} \int_0^{2\pi} e^{ix \cos \theta} d\theta \quad (12)$$

is the Bessel function of zero order. It is evident that $L_{MN}$ is the length of a classical periodic orbit or, more strictly, the length of a classical trajectory on a resonant (= periodic) torus. In integrable models almost all periodic
orbits belong to resonant tori and we shall use these two notions on the equal footing.

As most of the terms in both parts of Eq. (10) are 4 times degenerated it is convenient to define the density of non-degenerate states by dividing both parts of this equation by 4

\[ d(E) = \bar{d}(E) + d^{osc}(E). \]  \hspace{1cm} (13)

Here

\[ \bar{d}(E) = \frac{A}{16\pi}, \]  \hspace{1cm} (14)

where \( A = ab \) is the area of the rectangle and

\[ d^{osc}(E) = \frac{A}{4\pi} \sum_{L_p} e_p J_0(kL_p), \]  \hspace{1cm} (15)

where the sum is taken over all periodic orbits defined by two integers \( M \) and \( N \) whose lengths \( L_p \) are given by Eq. (11). \( e_p = 1 \) if both \( M \) and \( N \) are nonzero and \( e_p = 1/2 \) otherwise (the term with \( M = N = 0 \) gives \( \bar{d}(E) \)).

In semi-classical limit \( k \to \infty \) and

\[ d^{osc}(E) = \frac{A}{2\pi} \sum_{L_p} \frac{e_p}{\sqrt{8\pi kL_p}} e^{ikL_p-i\pi/4} + c.c. \]  \hspace{1cm} (16)

Eqs. (13)-(16) define the trace formulas for a rectangular billiard (with periodic boundary conditions). Trace formula for general integrable systems have similar form and will be discussed in Section 8.

### 3. Berry’s bootstrap

The above trace formula for rectangular billiard is the simplest example of trace formulas for integrable systems. We know explicitly everything. The periodic orbit lengths are given by simple expression (11) and all semi-classical corrections to Eq. (11) (coming from asymptotic expansion of the Bessel function) are easy to take into account.

But even in this case quantities like those in Eq. (3) is difficult to compute. The necessity of such calculation is clearly seen e.g. from Berry’s remark 6
that for any quantum system with non-degenerated discrete spectrum the square of the density of states should be proportional to the density itself.

This fact can be demonstrated as follows. The density of states at real values of energy is usually defined as the limit $\epsilon \to 0$ of the sum over all energy eigenvalues $e_n$

$$d_\epsilon(E) = \frac{1}{2\pi i} \sum_n \left( \frac{1}{E - e_n - i\epsilon} - \frac{1}{E - e_n + i\epsilon} \right) = \frac{\epsilon}{\pi} \sum_n \frac{1}{(E - e_n)^2 + \epsilon^2}. \quad (17)$$

In the semi-classical trace formula a finite value of $\epsilon$ corresponds to a regularization of divergent sums by adding (e.g. in Eq. (16)) a small imaginary part to the energy $E \to E + i\epsilon$.

As it is evident from Eq. (17) at finite (but small) values of $\epsilon$ the function $d_\epsilon(E)$ has peaks near each energy eigenvalues, the peak width being proportional to $\epsilon$. Now let us compute $d_\epsilon^2(E)$ assuming that all eigenvalues are non-degenerated. It is clear that in the limit $\epsilon \to 0$

$$d_\epsilon^2(E) = \frac{\epsilon^2}{\pi^2} \sum_n \frac{1}{((E - e_n)^2 + \epsilon^2)^2}, \quad (18)$$

and cross terms with different $E_n$ will give negligible contributions at small $\epsilon$.

Because

$$\int_{-\infty}^{\infty} \frac{dx}{(x^2 + \epsilon^2)^2} = \frac{\pi}{2\epsilon^3}, \quad (19)$$

one gets that

$$\lim_{\epsilon \to 0} \frac{\epsilon^3}{(x^2 + \epsilon^2)^2} = \frac{\pi}{2} \delta(x), \quad (20)$$

and therefore

$$\lim_{\epsilon \to 0} 2\pi \epsilon d_\epsilon^2(E) = d(E). \quad (21)$$

Similarly by computing higher powers of $d_\epsilon(E)$ one obtains that

$$\lim_{\epsilon \to 0} \pi^{n-1} \frac{(2n - 2)!!}{(2n - 3)!!} \epsilon^{n-1} d_\epsilon^n(E) = d(E). \quad (22)$$

These bootstrap relations reflect the analytical structure of the density of states, namely that it should have $\delta$-function singularities at real values of
energy with unit residues. Therefore these type of relations is very general
and our first purpose is to find a method which will permit to prove these
relations for integrable systems starting from the semi-classical trace formula
(16).

4 Saddle points for oscillatory terms

Let us begin with the case $n = 2$. The density of states may be written as a
sum of two terms $d_{\epsilon}(E) = d_{+\epsilon}(E) + d_{-\epsilon}(E)$ where $\pm$ corresponds to the sign
of the exponent in two terms in Eq. (16)

\[
d_{+\epsilon}(E) = \frac{A}{2\pi} \sum_{L_p} \frac{e^{i\epsilon L_p}}{\sqrt{8\pi k L_p}} e^{i k L_p - i\pi/4 - \epsilon L_p/2k},
\]

and $d_{-\epsilon}(E) = d_{+\epsilon}^*(E)$. Of course, the contribution of $\bar{d}(E)$ to the left-
hand side of bootstrap relations is negligible at small $\epsilon$. Therefore $d_{\epsilon}^2(E) = d_{+\epsilon}^2(E) + 2d_{+\epsilon}(E)d_{-\epsilon}(E) + d_{-\epsilon}^2(E)$. The periodic orbit expansion of $d_{\pm\epsilon}^2(E)$ contains only terms with the same sign of the actions in the exponent and for
any fixed total action it includes only a finite number of terms, and in the
limit $\epsilon \to 0$ their contributions are negligible. On the contrary the periodic
orbit expansion of $d_{+\epsilon}(E)d_{-\epsilon}(E)$ for any finite total action contains infinite
number of terms and it is this term which can give contribution in the limit
of small $\epsilon$. Therefore the following relation should be true

\[
\lim_{\epsilon \to 0} 2\pi \epsilon A^2 \frac{1}{2\pi^2} \sum_{p_1, p_2} \frac{e_{p_1} e_{p_2}}{8\pi k \sqrt{l_1 l_2}} e^{i k (l_1 - l_2) - (l_1 + l_2)\epsilon/2k} = \bar{d}(E) + \frac{A}{2\pi} \sum_p \left( \frac{e_{p}}{\sqrt{8\pi k l_p}} e^{i k l_p - i\pi/4} + c.c. \right).
\]

(24)

Here in the left-hand side $l_{1,2}$ are the lengths of periodic orbits $p_1$ and $p_2$ and
the sum is taken over all such orbits.

The smooth part (=mean value) of this expression is easy to compute. It was shown in [2] that diagonal approximation is applicable and the mean
value of the level density is expressed through the sum over periodic orbits as follows

\[
\lim_{\epsilon \to 0} 2\pi \epsilon A^2 \frac{1}{2\pi^2} \sum_p e^{-\epsilon l / k} \frac{1}{8\pi k l} = \bar{d}(E).
\]

(25)
The density of periodic orbits can easily be calculated from Eq. (11). If \( N(l) \) is the number of periodic orbits with the lengths less than \( l \) then for large \( l \)

\[
N(l) = \frac{\pi l^2}{4A},
\]

(26)

where \( A = ab \) is the area of the rectangle and the factor \( 1/4 \) comes because we consider only orbits with positive integers \((M, N)\).

Changing the summation over periodic orbits to the integration over periodic orbit length with the above density one concludes that the left hand side of Eq. (25) in the limit of small \( \varepsilon \) equals \[6\]

\[
2\pi\varepsilon A^2 \int_0^\infty \frac{e^{-\varepsilon l/k}}{16Akl} l dl \equiv \frac{A}{16\pi}.
\]

(27)

But this is exactly equal to \( \tilde{d}(E) \) as it was predicted by the bootstrap equation (21).

Though in this case the computation of the smooth part is simple the bootstrap formula (21) should be valid for oscillatory terms as well for which the diagonal approximation cannot be applied. It is the calculation of oscillatory terms (proportional to \( \exp(ikl_p) \)) which is the main subject of this Section.

The left-hand side of (21) contains a double sum over pairs of periodic orbits. Let us try to find saddle points in this double sum.

A periodic orbit for a rectangular billiard (and for any integrable models in 2 dimensions (see Section 8)) is defined by two integers \( M \) and \( N \) (or integer vector \((M, N)\)) and its length equals the modulus of vector with components \( Ma \) and \( Nb \), \( L(M, N) = \sqrt{(Ma)^2 + (Nb)^2} \). The length of a periodic orbit defined by integers \( M + \delta M \) and \( N + \delta N \) has the following expansion up to the second order in \( \delta M \) and \( \delta N \)

\[
L(M + \delta M, N + \delta N) \equiv \sqrt{a^2(M + \delta M)^2 + b^2(N + \delta N)^2} = L(M, N) + \frac{a^2M\delta M + b^2N\delta N}{2L^3(M, N)} + \frac{A^2}{2L^5(M, N)}(N\delta M - M\delta N)^2.
\]

(28)

The double sum in Eq. (21) has the form \( \sum_{p_1,p_2} \exp ik(l_1 - l_2) \). Each periodic orbit in this sum is defined by two integers and the sum is really the sum over 4 integers. Let us assume that saddle point manifolds consist of
periodic orbits with integers \((M_1 + \delta M_1, N_1 + \delta N_1)\) for the first sum and \((M_2 + \delta M_2, N_2 + \delta N_2)\) for the second one with unknown integers \((M_i, N_i), (\delta M_i, \delta N_i)\). The necessary condition for the existence of a saddle point in the double sum is the cancellation of linear terms in the exponent. From Eq. (28) one concludes that the saddle point condition has the form

\[
\frac{a^2 M_1 \delta M_1 + b^2 N_1 \delta N_1}{L(M_1, N_1)} - \frac{a^2 M_2 \delta M_2 + b^2 N_2 \delta N_2}{L(M_2, N_2)} = 0. \tag{29}
\]

The important restriction is that we are looking at the integer solutions of this equation. We shall also assume that the ratio \(a^2/b^2\) is a ‘good’ irrational number (which is a necessary condition that the spectral statistics of such billiard will be close to the Poisson statistics \([10]\)). In this case the only possibility to find non-trivial solutions of the above saddle point equation is to require that the lengths \(L(M_1, N_1)\) and \(L(M_2, N_2)\) are commensurable

\[
\frac{L(M_1, N_1)}{L(M_2, N_2)} = \frac{r_1}{r_2}, \tag{30}
\]

with certain integers \(r_1\) and \(r_2\). This condition means that saddle point values of integers \(M_i\) and \(N_i\) are the following

\[
M_1 = r_1 m, \quad M_2 = r_2 m, \quad N_1 = r_1 n, \quad N_2 = r_2 n, \tag{31}
\]

where the pair of integers \(m\) and \(n\) has no common factors (i.e. they are co-prime integers).

Now the saddle point condition (29) takes the form

\[
a^2 m \delta M_1 + b^2 n \delta N_1 - a^2 m \delta M_2 - b^2 n \delta N_2 = 0. \tag{32}
\]

As we have assumed that the ratio \(a^2/b^2\) is an irrational number the only way to fulfill this equation is to cancel terms in front of \(a^2\) and \(b^2\) separately

\[
\delta M_1 - \delta M_2 = 0, \quad \delta N_1 - \delta N_2 = 0. \tag{33}
\]

This argumentation shows that in the double sum (21) there exists saddle point manifold consisted from periodic orbits defined by the following pairs of integers

\[
M_1 = r_1 m + \delta M, \quad N_1 = r_1 n + \delta N, \quad M_2 = r_2 m + \delta M, \quad N_2 = r_2 n + \delta N, \tag{34}
\]
for arbitrary co-prime integers \( m \) and \( n \) and arbitrary \( \delta M \) and \( \delta N \).

Note that the knowledge of pairs \( (M_1, N_1) \) and \( (M_2, N_2) \) uniquely defines the pair \( (m, n) \) and the difference \( r_1 - r_2 \) provided that it is non-zero. Namely, \( r_1 - r_2 \) is the greatest common factor of the pair \( (M_1 - M_2, N_1 - N_2) \) and \( (m, n) \) is the ratio of the division of \( (M_1 - M_2, N_1 - N_2) \) on \( r_1 - r_2 \).

To find the value of the double sum (21) in the saddle point approximation it is necessary to perform the summation over all saddle point manifold defined in Eq. (34) in the Gaussian approximation (i.e. expanding all actions up to the quadratic terms in \( \delta M \) and \( \delta N \) as in Eq. (28)).

Let us denote the left-hand part of Eq. (21) by \( D(E) \). Using (28) and (34) one obtains

\[
D(E) = \frac{A^2\epsilon}{8\pi^2k} \sum_{l_0} \sum_{r_1, r_2} \frac{1}{l_0 \sqrt{r_1 r_2}} \exp(ik(r_1 - r_2)l_0 - (r_1 + r_2)\frac{\epsilon l_0}{2k} + ik\frac{A^2}{2l_0^3}(\frac{1}{r_1} - \frac{1}{r_2})t^2) + c.c.,
\]

(35)

where \( t = n\delta M - m\delta N \), and \( l_0 = \sqrt{a^2m^2 + b^2n^2} \).

Note that the quadratic form in the exponent is not positive definite. But it is easy to check that if the quantity \( t \) has two identical values

\[ n\delta M_1 - m\delta N_1 = n\delta M_2 - m\delta N_2, \]

and integers \( m \) and \( n \) have no common factor (as it was assumed above) then \( \delta M_2 = \delta M_1 + lm \) and \( \delta N_2 = \delta N_1 + ln \) for a certain integer \( l \). But from Eq. (34) it follows that this analog of zero modes corresponds to a change of the repetition number: \( r_2 = r_2 + l \). Therefore the restriction of the summation to orbits with fixed repetition numbers is equivalent to the summation over all values of \( t \) only once.

It means

\[
D(E) = \frac{A^2\epsilon}{8\pi^2k} \sum_{l_0} \sum_{r_1r_2=1}^{\infty} \frac{1}{l_0 \sqrt{r_1 r_2}} \exp(ik(r_1 - r_2)l_0 - (r_1 + r_2)\frac{\epsilon l_0}{2k})
\]

\[
\times \sum_{t=-\infty}^{\infty} \exp(ikr_1 - r_2)\frac{A^2}{2l_0^3}(\frac{1}{r_1} - \frac{1}{r_2})t^2) + c.c.,
\]

(36)

Let \( r_1 = r_2 + r \). Changing the summation over \( t \) to the integration one finds

\[
D(E) = \frac{A\epsilon}{8\pi^2k\sqrt{k}} \sum_{l_0} \sum_{r=1}^{\infty} \frac{\sqrt{2\pi l_0^3}}{l_0 \sqrt{r}} \exp(ikr l_0 - r \frac{\epsilon l_0}{2k} - i\frac{\pi}{4}) \sum_{r_1=0}^{\infty} e^{-r_2\epsilon l_0/k} + c.c.
\]

(37)
At small $\epsilon$ the last sum tends to $k/(\varepsilon l_0)$ and

$$D(E) = \frac{A}{2\pi} \sum_{l=0}^{\infty} \frac{1}{\sqrt{8\pi kl_0 r}} \exp(ikrl_0 - r_0 \frac{\epsilon}{2k} - i\frac{\pi}{4}) + c.c. \quad (38)$$

which is exactly the right hand side of Eq. (21).

Note that the changing the summation over $t$ to the integration in the sum $\sum_{t=-\infty}^{\infty} \exp(i\pi xt^2)$ is equivalent to ignoring terms of the form $e^{-i\pi m^2/x}/\sqrt{x}$ with integer $m \neq 0$. This is correct when $x \to 0$ and only in a weak sense (i.e. after the multiplication of both sides of the equality

$$\sum_{t=-\infty}^{\infty} e^{i\pi xt^2} - \frac{1}{\sqrt{x}} = \frac{1}{\sqrt{x}} \sum_{m=-\infty, m \neq 0}^{\infty} e^{-i\pi m^2/x} \quad (39)$$

by a suitable chosen test function). It is in such weak sense that one should treat equalities similar to Eq. (38). This is not a restriction of our method as all semi-classical trace formulas have mathematical meaning only in a weak sense.

The above calculations clearly demonstrate that the proposed saddle point method is sufficient to obtain the bootstrap condition (21) for $n = 2$. Below we show that exactly the same considerations permit to verify the bootstrap conditions (22) for all $n$. Because $d_\epsilon(E) = d_{+\epsilon}(E) + d_{-\epsilon}(E)$, $d^n_\epsilon(E)$ is given by the sum

$$d^n_\epsilon(E) = \sum_{\nu_1, \nu_2} C^n_{\nu_1} d^\nu_1_{+\epsilon}(E)d^\nu_2_{-\epsilon}(E), \quad (40)$$

where $\nu_1 \geq 0$ and $\nu_1 + \nu_2 = n$. Substituting instead of $d_{\pm\epsilon}(E)$ its semi-classical expression (16) one gets

$$d^\nu_1_{+\epsilon}(E)d^\nu_2_{-\epsilon}(E) = \left(\frac{A}{2\pi \sqrt{8\pi k}}\right)^{\nu_1 + \nu_2} \sum_{l,l'} e^{ik(L-L')/2k - i\pi(\nu_1 - \nu_2)/4} \sqrt{(l_1 \cdots l_{\nu_1})(l'_1 \cdots l'_{\nu_2})}, \quad (41)$$

and $L = l_1 + \ldots + l_{\nu_1}$, $L' = l'_1 + \ldots + l'_{\nu_2}$. The summation here is performed over all periodic orbits with lengths $l_j$, $j = 1, \ldots, \nu_1$ corresponding to terms with positive exponent and over orbits with lengths $l'_k$, $k = 1, \ldots, \nu_2$ from terms with negative exponent.

Let each ‘positive’ periodic orbit is defined by a pair of integers $(M_j, N_j)$ and respectively the ‘negative’ orbit is associated with a pair $(M'_k, N'_k)$. The
same arguments as above prove that the saddle point manifold (i.e. the set of integer vectors for which linear terms in the difference $L - L'$ cancel) has the following form

$$M_j = r_j m + \delta M_j, \quad N_j = r_j n + \delta N_j, \quad M'_k = r'_k m + \delta M'_k, \quad N'_k = r'_k n + \delta N'_k,$$

(42)

with 2 restrictions

$$\sum_{j=1}^{\nu_1} \delta M_j - \sum_{k=1}^{\nu_2} \delta M'_k = 0, \quad \sum_{j=1}^{\nu_1} \delta N_j - \sum_{k=1}^{\nu_2} \delta N'_k = 0.$$

(43)

It is convenient to rewrite these conditions in terms of integer vectors

$$\vec{N}_j = (M_j, N_j),$$

(44)

associated with each periodic trajectory. Now Eqs. (42)-(43) take the form

$$\sum_{j=1}^{\nu_1} \vec{N}_j - \sum_{k=1}^{\nu_2} \vec{N}'_k = r \vec{n},$$

(45)

where

$$r = \sum_{j=1}^{\nu_1} r_j - \sum_{k=1}^{\nu_2} r'_k,$$

(46)

and

$$\vec{n} = (m, n).$$

(47)

Because we assume that $m$ and $n$ are co-prime integers they are defined uniquely from the knowledge of $\vec{N}_j$ and $\vec{N}'_k$ and each term in the multiple sum in Eq. (41) can be uniquely attributed to the new summation of the form of Eqs. (42)-(43).

The total contribution of these saddle points to the $n$-fold sum in Eq. (41) in the Gaussian approximation is

$$d_{x+}(E) d_{x-}(E) = \sum_{l_0} P^{\nu_1+\nu_2} \sum_{R,R'} \frac{1}{\sqrt{(r_1 \cdots r_{\nu_1})(r'_1 \cdots r'_{\nu_2})}} \times \exp(ik(R - R')l_0 - \frac{\epsilon(R + R')l_0}{2k} - \frac{i\pi}{4}(\nu_1 - \nu_2))S,$$

(48)

where

$$P = \frac{A}{2\pi \sqrt{8\pi kl_0}},$$

(49)
\[ S = \sum_{t,t'} \exp(iQ(t_1^2/r_1 + \ldots + t_{\nu_1}^2/r_{\nu_1} - t_1^2/r_1' - \ldots - t_{\nu_2}^2/r_{\nu_2}')) \delta(t_1 + \ldots + t_{\nu_1} - t_1' - \ldots - t_{\nu_2}). \]  

(50)

Here

\[ Q = k \frac{A^2}{2l_0^2}, \]

(51)

\[ R = r_1 + \ldots + r_{\nu_1}, \quad R' = r_1' + \ldots + r_{\nu_2}', \quad t_j = n\delta M_j - m\delta N_j, \quad t_k' = n\delta M_k' - m\delta N_k', \]

and the \( \delta \)-function is a consequence of the saddle point conditions (43) in variables \( t \) and \( t' \).

Changing the summation over \( t \) and \( t' \) into the integration and using the usual representation of the \( \delta \)-function

\[ \delta(x) = \frac{1}{2\pi} \int d\alpha e^{i\alpha x}, \]

(52)

one can easily perform all integrals and one obtains

\[ S = e^{i\frac{\pi}{4}(\nu_1 - \nu_2 - \text{sgn}(r))} \frac{(r_1 \cdot \ldots \cdot r_{\nu_1})(r_1' \cdot \ldots \cdot r_{\nu_2}')}{|r|} \left( \frac{\pi}{Q} \right)^{\frac{\nu_1 + \nu_2 - 1}{2}}, \]

(53)

where \( r = R - R' \) (note that for oscillating terms \( r \neq 0 \)).

This results leads to the following expression for \( d^n_\epsilon(E) \)

\[ d^n_\epsilon(E) = \sum_{l_0,r} P^n \left( \frac{\pi}{Q} \right)^{n-1} \sum_{R_1-R_2=r} N(R_1, R_2) e^{iklo_r - is\text{sgn}(r)/4 - e_{l_0}R_2/k} \frac{e^{i\delta_M - is\text{sgn}(r)R_1/4}}{|r|}, \]

(54)

where

\[ N(R_1, R_2) = \sum_{\nu_1=1}^{n} C_n^{\nu_1} n(R_1, R_2; \nu_1), \]

(55)

and \( n(R_1, R_2; \nu_1) \) is the number of terms with fixed number of positive, \( R_1 \), and negative, \( R_2 \), repetitions.

To find this number let us first compute the number of representation of a given number \( M \) into a sum of \( k \) integers: \( M = n_1 + \ldots + n_k \) and all \( n_i \geq 1 \). It is easy to see that this number is the coefficient of \( x^M \) in the expansion
\(x^k/(1 - x)^k\), therefore there exists \(C_{M-1}^{k-1}\) ways of representing \(M\) as a sum of \(k\) non-zero integers and
\[
N(R_1, R_2) = \sum_{\nu_1=1}^{n-1} C_n^{\nu_1} C_{R_1-1}^{\nu_1-1} C_{R_2-1}^{m-\nu_1-1}. \tag{56}
\]

Because we are interested in the region \(R_1, R_2 \to \infty\) with fixed difference \(R_1 - R_2 = r\) we can use the following asymptotics of the binomial coefficients
\[
C_{R_1-1}^{\nu_1-1} \to \frac{R_1^{\nu_1-1}}{(\nu_1 - 1)!}.
\]

Finally
\[
N(R_1, R_2) = \sum_{\nu_1=1}^{n-1} C_n^{\nu_1} C_{n-2}^{\nu_1-1} \frac{R_2^{n-2}}{(n-2)!}. \tag{57}
\]

The remaining sum
\[
\sum_{\nu_1=1}^{n-1} C_n^{\nu_1} C_{n-2}^{\nu_1-1} = \sum_{\nu_1=1}^{n-1} C_n^{\nu_1} C_{n-2}^{n-\nu_1-1}
\]
is the coefficient of \(x^{n-1}\) in the expansion \((1 + x)^n(1 + x)^{n-2}\) therefore
\[
\sum_{\nu_1=1}^{n-1} C_n^{\nu_1} C_{n-2}^{\nu_1-1} = C_{2n-2}^{m-1}. \tag{58}
\]

The total contribution is
\[
d^n_\epsilon(E) = C_{2n-2}^{m-1} \sum_{l_0,r} P^n \left(\frac{\pi}{Q}\right)^{(n-1)/2} \frac{e^{ikl_0 r - i\pi \text{sgn}(r)/4}}{\sqrt{|r|}} \sum_{R_2=1}^{\infty} \frac{R_2^{n-2}}{(n-2)!} e^{-\epsilon l_0 R_2/k}. \tag{59}
\]

When \(\epsilon \to 0\) the last sum tends to \((k/\epsilon l_0)^{n-1}\) and because
\[
P \sqrt{\frac{\pi}{Q}} \frac{2k}{l_0} = \frac{1}{2\pi}, \tag{60}
\]
one obtains the following relation
\[
\lim_{\epsilon \to 0} \frac{(4\pi\epsilon)^{n-1}}{C_{2n-2}^{m-1}} d^n_\epsilon(E) = \frac{A}{2\pi} \sum_{l_p} \frac{e^{ikl_p - i\pi/4}}{\sqrt{8\pi kl_p}} + c.c. = d^{\text{osc}}(E). \tag{61}
\]
As
\[ \frac{4^{n-1}}{C_{2n-2}^{n-1}} = \frac{(2n-2)!!}{(2n-3)!!} \] (62)
this result coincides with oscillating part of the general bootstrap condition (22) for all \( n \).

These calculations demonstrate that our saddle point method reproduces correctly the oscillating part of bootstrap condition but it remains the question how to apply this method for the smooth part of the bootstrap condition with \( n > 2 \).

## 5 Saddle points for smooth terms

Immediate difficulty of generalization of the above method to the computation of the smooth part of the bootstrap condition (22) in the fact that now one cannot attribute uniquely values of \((m, n)\) to each term in the multiple sum over periodic orbits (41). Even for \( n = 2 \) the calculation of the smooth term is done by the diagonal approximation and requires the knowledge of the number of periodic orbits in contrast to the saddle point method discussed in the above Section.

Each periodic orbit of rectangular billiard is defined by an unique vector\( \vec{N} = (M, N) \),
\[ \vec{N} = (M, N), \] (63)
with positive integers \( M, N \) and the length of this periodic orbit is given by the following expression
\[ L(\vec{N}) = \sqrt{(aM)^2 + (bN)^2}. \] (64)

In the polar coordinates with \( 0 \leq \phi \leq \pi/2 \)
\[ M = R \cos \phi, \quad N = R \sin \phi, \] (65)
the local density of periodic orbits \( \rho(l, \phi) \) may be computed from the relation
\[ \int \rho(l, \phi)dl d\phi \approx \int_0^\infty dMdN \delta(L(\vec{N}) - l)dl, \] (66)
\footnote{For the purpose of this Section it will be natural to consider vectors with component \((Ma, Nb)\) but to be consistent with more general case discussed in Section 8 this definition is more convenient.}
which leads to
\[ \rho(l, \phi) = \frac{l}{L_0^2(\phi)}, \]  
(67)
and
\[ L_0(\phi) = \sqrt{a^2 \cos^2 \phi + b^2 \sin^2 \phi}. \]  
(68)
As
\[ \int_0^{\pi/2} \frac{d\phi}{L_0^2(\phi)} = \frac{\pi}{2ab}, \]  
(69)
the integrated density of periodic orbit length
\[ \rho(l) = \int_0^{\pi/2} \rho(l, \phi) d\phi = \frac{\pi l}{2ab}, \]  
(70)
in agreement with Eq. (26).

To apply the saddle point method it is necessary to construct a configurations of vectors (saddle point manifold) such that any variations of them will decrease (or increase) the total length (i.e. the expansion of the total length will not contain terms linear on variations). From geometrical considerations it is clear that the only possibility for such a configuration is the case when all vectors are parallel. To build the saddle point manifold we shall proceed as follows.

Let us consider \( n \) vectors \( \vec{N}_i \), \( i = 1, \ldots, n \) in polar coordinates
\[ \vec{N}_i = R_i (\cos \phi_i, \sin \phi_i). \]  
(71)
The condition that all these vectors are almost parallel is equivalent to the statement that all polar angles are close to each other
\[ \phi_i = \phi + \delta \phi_i, \]  
(72)
and \( \delta \phi_i \ll \phi \). Under these conditions \( L(\vec{N}_i) \) can be calculated from Eq. (28) up to the second order of \( \delta \phi_i \)
\[ L(\vec{N}_i) = R_i L_0 + \frac{(b^2 - a^2) \sin 2\phi}{2L_0} R_i \delta \phi_i + \frac{A^2}{2L_0^3} R_i (\delta \phi_i)^2. \]  
(73)
Let us now compute the difference of actions
\[ S_n = k \left( \sum_{i=1}^{n} L(\vec{N}_i) - L(\sum_{i=1}^{n} \vec{N}_i) \right). \]  
(74)
Note that this operation is possible because the sum of integer vectors (63) with positive components is also an integer vector with positive components and therefore it defines a periodic orbit. Direct application of Eq. (73) gives

\[
S_n = Q \left( \sum_{i=1}^n R_i (\delta \phi_i)^2 \right) = \frac{Q}{R_1 + \ldots + R_n} \left( \sum_{i=1}^n (R_i R_j (\phi_i - \phi_j)^2) \right),
\]

(75)

where

\[
Q = \frac{k A^2}{2 L_0^3(\phi)}.
\]

(76)

As in the previous Section

\[
d_{\nu_1 + \nu_2 = n}^\alpha (E) = \sum_{\nu_1 + \nu_2 = n} C_n^{\nu_1} d_{+\epsilon}^{\nu_1} (E) d_{-\epsilon}^{\nu_2} (E),
\]

and under the same notation as above

\[
d_{+\epsilon}^{\nu_1} (E) d_{-\epsilon}^{\nu_2} (E) = \left( \frac{A}{2\pi \sqrt{8\pi k}} \right)^{\nu_1 + \nu_2} \sum_{l, l'} \frac{1}{\sqrt{(l_1 \ldots l_{\nu_1})(l'_1 \ldots l'_{\nu_2})}} \times
\]

\[
\exp(i k (L - L') - \frac{\varepsilon (L + L')}{2k} - i \frac{\pi}{4}(\nu_1 - \nu_2)),
\]

(77)

with \( L = l_1 + \ldots + l_{\nu_1} \) and \( L' = l'_1 + \ldots + l'_{\nu_2} \).

Each periodic orbit is defined by an integer vector \( \vec{N}_i \) and according to the above discussion the saddle point manifold for the smooth part of this expression is constructed from vectors \( \vec{N}_i \) and \( \vec{N}'_j \) which are almost parallel each to other and obey the following relation (because it should be the smooth part)

\[
\sum_{j=1}^{\nu_1} \vec{N}_j = \sum_{k=1}^{\nu_2} \vec{N}'_k.
\]

(78)

One may easily check that the regrouping terms in Eq. (77) according to the sum of vectors \( \vec{N}_j \) and that of \( \vec{N}'_k \) avoids the double counting and permits the correct arrangement of different contributions.

In polar coordinates the saddle point condition (78) means that

\[
R_1 + \ldots + R_{\nu_1} = R'_1 + \ldots + R'_{\nu_2},
\]

(79)
and

\[
R_1 \delta \phi_1 + \ldots + R_{\nu_1} \delta \phi_{\nu_1} = R'_1 \delta \phi'_1 + \ldots + R'_{\nu_2} \delta \phi'_{\nu_2}.
\]

(80)

From general considerations (and from Eq. (75)) it follows that the difference of lengths \((L - L')\) depends only on differences of angles between different vectors and therefore one of these angles (say \(\phi_1 = \phi\)) can be chosen arbitrary and under the condition of smallness of these differences the sum over periodic orbits in Eq. (77) can be substitute by the integration over components of \(\vec{N}_i\) and Eq. (77) can be transformed as follows

\[
d^\mu_{\nu_1}(E)d^\nu_{\nu_1}(E) = \int_0^{\pi/2} d\phi P^{(\nu_1+\nu_2)} \int \prod_{j=1}^{\nu_1} dR_j \prod_{k=1}^{\nu_2} dR'_k \prod_{j=2}^{\nu_1} d\phi_j \prod_{k=1}^{\nu_2} d\phi'_k \times
\]

\[
\delta(R - R') \delta(R\phi - R'\phi') \sqrt{(R_1 \ldots R_{\nu_1})(R'_1 \ldots R'_{\nu_2})} \times
\]

\[
\exp(iQ(\sum_{j=2}^{\nu_1} R_j(\phi_j)^2 - \sum_{k=1}^{\nu_2} R'_k(\phi'_k)^2) - \frac{(R + R')}{2k} - i\frac{\pi}{4}(\nu_1 - \nu_2))
\]

where \(R = R_1 + \ldots + R_{\nu_1}, R' = R'_1 + \ldots + R'_{\nu_2}, R\phi = R_2 \phi_2 + \ldots + R_{\nu_1} \phi_{\nu_1}, R'\phi' = R'_1 \phi'_1 + \ldots + R'_{\nu_2} \phi'_{\nu_2},\) and

\[
P = \frac{A}{2\pi \sqrt{8\pi k L_0(\phi)}},
\]

(82)

Note the absence of \(\phi_1\) in these expressions.

Using the standard representation of the \(\delta\)-function

\[
\delta(R\phi - R'\phi') = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{iQ(R\phi - R'\phi')},
\]

(83)

one can easily perform the angular integration over all independent angles \(\phi_j\) and \(\phi'_k\)

\[
<\sum_{l,l'} e^{i(L-L')} \delta(R\phi - R'\phi') >= \frac{\pi}{Q}^{(\nu_1+\nu_2)/2-1} \frac{e^{i\pi(\nu_1-\nu_2)/4}}{\sqrt{(R_1 \ldots R_{\nu_1})(R'_1 \ldots R'_{\nu_2})}}.
\]

(84)

Combining this result together with Eq. (77) one obtains

\[
< d^\mu_{\nu_1}(E)d^\nu_{\nu_1}(E) >= \int_0^{\pi/2} d\phi P^{(\nu_1+\nu_2)} \left(\frac{\pi}{Q}\right)^{(\nu_1+\nu_2)/2-1} \times
\]

\[
\int_0^{\infty} \prod_{j=1}^{\nu_1} dR_j \prod_{k=1}^{\nu_2} dR'_k \delta(R - R')e^{-(R + R')L_0/2k}.
\]

(85)
The integrals over \( R \) and \( R' \) may be computed either by using the representation \((83)\) of the \( \delta \)-function or by the integration first by one variable e.g. \( R'_{\nu_2} \). Performing the change of variables \( R_j = k\tau_j/L_0 \), \( R'_k = k\tau'_k/L_0 \) and taking into account that \( R'_{\nu} \geq 0 \) one obtains

\[
\int_0^\infty \prod_{j=1}^{\nu_1} dR_j \prod_{k=1}^{\nu_2} dR'_k \delta(R - R') e^{-(R + R')L_0/2k} = \left( \frac{k}{L_0} \right)^{(\nu_1 + \nu_2 - 1)} \int_0^\infty \mathcal{D}\tau \mathcal{D}\tau' \theta(\tau - \tau') e^{-\epsilon\tau}, \tag{86}
\]

where \( \tau = \tau_1 + \ldots + \tau_{\nu_1}, \tau' = \tau'_1 + \ldots + \tau'_{\nu_2 - 1}, \mathcal{D}\tau = d\tau_1 \cdots d\tau_{\nu_1}, \) and \( \mathcal{D}\tau' = d\tau'_1 \cdots d\tau'_{\nu_2 - 1} \).

The integral over \( \mathcal{D}\tau' \) has the form

\[
g(\tau) = \int_0^\infty d\tau'_1 \cdots \tau'_{\nu_2 - 1} \theta(\tau - \tau'_1 + \ldots + \tau'_{\nu_2 - 1}), \tag{87}
\]

and may be computed by the Laplace transform over variable \( \tau \). The result is

\[
g(\tau) = \frac{\tau^{\nu_2 - 1}}{(\nu_2 - 1)!}. \tag{88}
\]

The same method applied to the integral over \( \mathcal{D}\tau \) gives

\[
\int \mathcal{D}\tau \mathcal{D}\tau' \theta(\tau - \tau') e^{-\epsilon\tau} = \int d\tau g(\tau) e^{-\epsilon\tau} \int d\tau_1 \cdots \tau_{\nu_1} \delta(\tau - \tau_1 - \ldots - \tau_{\nu_1})
\]

\[
= \int_0^\infty d\tau \frac{\tau^{\nu_1 + \nu_2 - 2}}{(\nu_1 - 1)!(\nu_2 - 1)!} e^{-\epsilon\tau} = \frac{(\nu_1 + \nu_2 - 2)!}{(\nu_1 - 1)!(\nu_2 - 1)!} \frac{1}{(4\epsilon)\nu_1^{\nu_1 - 1}}. \tag{89}
\]

Taking into account Eq. \((83)\) and that (compare with Eq. \((60)\))

\[
P \sqrt{\frac{\pi}{Q}} 2k = \frac{1}{2\pi}, \tag{90}
\]

and

\[
\int_0^{\pi/2} P \sqrt{\frac{Q}{\pi}} d\phi = \frac{A}{16\pi} = \bar{d}, \tag{91}
\]

one obtains

\[
<d_{\nu_1}^{\nu_1}(E)d_{\nu_2}^{\nu_2}(E) > = C_{\nu_1 + \nu_2 - 2}^{\nu_1 - 1} \frac{A}{(4\pi\epsilon)^{\nu_1 - 1} 16\pi}, \tag{92}
\]

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and
\[ < d^n_r > = \frac{A}{(4\pi \epsilon)^{n-1} 16\pi} \sum_{\nu_1=1}^{n-1} C_{n-2}^{\nu_1} C_{n-2}^{\nu_1-1}. \] (93)

But this sum equals \( C_{2n-2}^{n-1} \) (see Eq. (58)) and
\[ \lim_{\epsilon \to 0} \frac{(4\pi \epsilon)^{n-1}}{C_{2n-2}^{n-1}} < d^n_r (E) > = \frac{A}{16\pi} = \bar{d}, \] (94)
which agrees with the mean part of the bootstrap condition (22).

6 Mean value of Green function products

Problems similar to the ones described above appear also in the computation of mean value of products of the advanced and retarded Green functions of the type
\[ < G^m_+ (E_1) G^n_+ (E_2) >, \] (95)
where \( G_+ (E) = G(E \pm i\epsilon) \) and \( G(E) \) is the Green function of an integrable model. A typical example is the semi-classical computation of spectral correlation functions in the Seba billiard [11, 12] or, more generally, in integrable models with diffraction centers.

For clarity as in the previous Sections let us consider a rectangular billiard with periodic boundary conditions. The Green function for this problem has the form
\[ G_\pm (\vec{x}, E) = \frac{1}{A} \sum_n \frac{e^{i\vec{k}_n \vec{x}}}{E - e_n \pm i\epsilon}, \] (96)
where \( e_n = k_n^2 \) are eigenvalues of the billiard problem, \( \vec{k}_n \) is the vector with components \( 2\pi (m/a, n/b) \) with integers \( m, n \), and \( A \) is the area of the rectangle. This Green function permits also an exact representation as a sum over all classical orbits connecting points 0 and \( \vec{x} = (x, y) \)
\[ G_+ (\vec{x}, E) = -\frac{i}{4} \sum_{m,n=-\infty}^{\infty} H^{(1)}_0 (k \sqrt{(x+ma)^2 + (y+nb)^2}), \] (97)
and \( H^{(1)}_0 (x) \) is the Hankel function, \( G_- (\vec{x}, E) = G^*_+ (\vec{x}, E) \).

In diffraction problems one is interested in the Green function when \( x \to 0 \). In this limit the Green function diverges and requires a cut-off
at small $x$ (i.e. a regularization). The divergent part comes only from the contribution of the shortest trajectory with $m = n = 0$. Taking into account the asymptotic behavior of the Hankel function at small $x$ one concludes that the ‘renormalized’ Green function (97) (when $|x| \to \mu$ and $\mu$ is small) is a sum of 2 terms

$$G_+(E) = \bar{g}_+(E) + g_+^{\text{osc}}(E), \quad (98)$$

where (in 2 dimensions)

$$\bar{g}_+(E) = \frac{1}{2\pi} \ln(k\mu) - \frac{i}{4} \quad (99)$$

and

$$g_+^{\text{osc}}(E) = -\frac{i}{4} \sum_p H^{(1)}_0(kL_p), \quad (100)$$

where $L_p$ denotes the length of a periodic orbit defined by 2 integers $m$ and $n$, $L_p = \sqrt{(ma)^2 + (nb)^2}$, the summation is done over all $m, n$ except $m = n = 0$, and parameter $\mu$ is a renormalization parameter (a cut-off at small distances). In the quantization of the Seba billiard the necessity of the renormalization is connected with the fact that the $\delta$-function potential is too singular in dimensions $\geq 2$ and Eq. (100) defines one parameter self-adjoint extension (defined by $\mu$) of a singular Hamiltonian [11]. Note that $\langle G_\pm \rangle = \bar{g}_\pm(E)$ and $\langle g_\pm^{\text{osc}}(E) \rangle = 0$.

The semi-classical approximation corresponds to $k \to \infty$ and the oscillating part of the Green function takes the form

$$g_+^{\text{osc}}(E) = \sum_p \frac{e^{ikL_p - 3\pi i/4 - \epsilon L_p/2k}}{\sqrt{8\pi kL_p}}. \quad (101)$$

Let

$$g_{mn}(E_1, E_2) = \frac{1}{A^{m+n}} \sum_n \frac{1}{(E_1 - e_n + ie)^m (E_2 - e_n - ie)^n}. \quad (102)$$

Assuming that all energy levels have the Poisson distribution (i.e. they are independent random variables with mean density $\bar{d}$) it follows that

$$g_{mn}(E_1, E_2) = \frac{1}{A^{m+n}} \int \frac{d\epsilon e^{i\epsilon L_p - 3\pi i/4 - \epsilon L_p/2k}}{(E_1 - e + i\epsilon)^m (E_2 - e - i\epsilon)^n} = \frac{2\pi i d(-1)^{n-1}}{A((E_1 - E_2)A)^{m+n-1} C_{n+m-2}}. \quad (103)$$

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For rectangular billiard $d = A/(16\pi)$ and
\[ g_{mn}(E_1, E_2) = \frac{i(-1)^{n-1}}{8((E_1 - E_2)A)^{m+n-1}} C_{n+m-2}^{m-1}. \] (104)

It is easy to check (see below) that the most divergent part of the product (95) when $E_2 \to E_1$ equals $g_{mn}(E_1, E_2)$
\[ < G^m_+(E_1)G^n_-(E_2) > \to g_{mn}(E_1, E_2), \text{ when } E_2 \to E_1. \] (105)

The purpose of this section is to check that exactly the same answer can be obtained by using the semi-classical expression (101) for the Green function and applying the saddle point method discussed in the previous Sections. The differences between the computation of $< d + \epsilon_1(E) \nu_1 d - \epsilon_2(E) \nu_2 >$ performed in the last Section and that of $< G^m_+(E_1)G^n_+(E_2) >$ are (i) the absence of factor $A/2\pi$ and (ii) the change of the definition of $\epsilon \epsilon \to -i(E_1 - E_2)/2$ (and, of course, it is necessary to substitute $\nu_1 \to m$ and $\nu_2 \to n$). From Eq. (92) one concludes that the saddle point method gives
\[ < G^m_+(E_1)G^n_-(E_2) > = C_{n+m-2}^{m-1} \frac{2^{m+n-1}A}{16\pi(i(E_2 - E_1))^{m+n-1}} \left(\frac{2\pi}{A}\right)^{m+n} \]
\[ = i(-1)^{n-1} C_{n+m-2}^{m-1} \frac{1}{8((E_1 - E_2)A)^{m+n-1}}, \] (106)
which agree with the value (105) obtained by direct calculations.

Till now we have considered only the dominant term of the mean value of Green functions product (95) when $E_2 \to E_1$ (i.e. we look for contributions with the highest negative power of the difference of energies). But this restriction is not necessary and other terms can also be computed by the same method.

The $m$-th power of the Green function (10) may be written as follows
\[ G^m_+(E) = \frac{1}{A^m} \sum_{m_i} \frac{m!}{m_1!m_2! \cdots m_p!} \]
\[ \times \sum_{e_j} \frac{1}{(E - e_1 + i\epsilon)^{m_1}(E - e_2 + i\epsilon)^{m_2} \cdots (E - e_p + i\epsilon)^{m_p}}, \]
where the first sum is taken over all positive integers $m_i$ whose sum equals $m$, $m_1 + \ldots + m_p = m$, and the second summation is performed over different energy eigenvalues $e_j$ such that $e_1 \leq e_2 \leq \ldots$. Representing $G^n_-(E)$
in the similar way and taking into account that the connected part of the
Green function products should not include mean values of individual Green
functions one concludes that
\[
< G^m_+ (E_1) G^n_- (E_2) >^c = \min(m,n) \sum_{p=1}^{\min(m,n)} \frac{1}{p!} \times \sum_{m_i, n_i} \left( \frac{m!}{m_1! m_2! \cdots m_p!} \right) \left( \frac{n!}{n_1! n_2! \cdots n_p!} \right) g_{m_1 n_1} g_{m_2 n_2} \cdots g_{m_p n_p},
\]
where \( g_{mn} = g_{mn}(E_1, E_2) \) is given by Eq. \( \text{(104)} \) and the summation is performed over all positive integers \( m_i \geq 1, n_i \geq 1 \) such that \( \sum_{i=1}^{p} m_i = m \) and \( \sum_{i=1}^{p} n_i = n \). The factor \( 1/p! \) appears from the assuming ordering of variables \( e_i \). Because of the symmetry with respect to permutations this factor can be removed by assuming the ordering of (say) \( m_i \) variables and the above expression may be rewritten as follows
\[
< G^m_+ (E_1) G^n_- (E_2) >^c = \min(m,n) \sum_{p=1}^{\min(m,n)} \sum_{\tilde{m}_i, \tilde{n}_i} \left( \frac{m!}{m_1! m_2! \cdots m_p!} \right) \left( \frac{n!}{n_1! n_2! \cdots n_p!} \right) g_{m_1 n_1} g_{m_2 n_2} \cdots g_{m_p n_p},
\]
where \( m_1 \leq m_2 \cdots \leq m_p \).

Each \( g_{m_i n_i} \) is proportional to \( ((E_1 - E_2) A)^{1-m_i-n_i} \), therefore terms with fixed \( p \) in these sums will be proportional to \( ((E_1 - E_2) A)^{p-m-n} \) and terms with higher values of \( p \) correspond to the expansion of \( < G^m_+ (E_1) G^n_- (E_2) >^c \) \( (E_1 - E_2)^{m+n} \) on positive powers of \( (E_1 - E_2) A \).

The above expression represents the value of the connected part of the
mean value of Green function product calculated from the assumption of the
Poisson distribution of energy eigenvalues. The total answer includes also
terms coming from the mean value of the Green function itself (see Eq. \( \text{(103)} \))
\[
< G^m_+ (E_1) G^n_- (E_2) >^c = \sum_{k,p} C_m^k C_n^p g^k_+ g^-_p < G^k_+ (E_1) G^p_- (E_2) >^c,
\]
and due to Eq. \( \text{(108)} \) this expression can also be organized as a series on
decreasing powers of \( (E_1 - E_2) A \).

Let us check that this answer may also be obtained by the saddle point
method. As was demonstrated in Section \( 4 \) the saddle point manifold for
smooth terms consists of integer vectors $\vec{N}_i$ and $\vec{N}'_j$ such that

$$\sum_{i=1}^{m} \vec{N}_i = \sum_{j=1}^{n} \vec{N}'_j. \quad (111)$$

The dominant contribution discussed in the previous Sections corresponds to the integration over all possible deviations on this manifold. But by fixing certain deviations it is possible to find lower dimensional manifolds and the integration over them will give corrections to the dominant result.

Let as above $n_i \geq 1$ and $m_i \geq 1$ be partitions of $n$ and $m$ into positive integers: $n = n_1 + \ldots + n_p$ and $m = m_1 + \ldots + m_p$. Each such partition gives rise to a possible regrouping of integer vectors

$$\sum_{i=1}^{m_1} \vec{N}_i = \sum_{j=1}^{n_1} \vec{N}'_j \quad \sum_{i=1}^{m_2} \vec{N}'_{m_1+i} = \sum_{j=1}^{n_2} \vec{N}'_{n_1+j} \quad \ldots \quad \ldots \quad \sum_{i=1}^{m_p} \vec{N}'_{m_1+\ldots+m_{p-1}+i} = \sum_{j=1}^{n_p} \vec{N}'_{n_1+\ldots+n_{p-1}+j}. \quad (112)$$

Such manifold is a part of co-dimension $(p - 1)$ of the full saddle point manifold (111) and the summation over all possible deviations on it will evidently give $g_{m_1,n_1}g_{m_2,n_2}\ldots g_{m_p,n_p}$. As there exists $m!/(m_1!\ldots m_p!)$ possible partitions with fixed $m_i$ and $n!/(n_1!\ldots n_p!)$ possible partitions with fixed $n_i$ the total contribution will be equal exactly to Eq. (109) which demonstrates that all correction terms can be computed correctly by the saddle point method.

Let us note that in problems considered there exist 4 parameters with dimensions of energy. One is the energy itself or more precisely the center of energy window which defines mean values, the second one is the width of energy window, the third is the difference of 2 energies, $\epsilon = E_1 - E_2$, and the forth is the mean distance between quantum levels equal $1/\bar{d}$ where $\bar{d}$ is the mean density of levels. In the end of this Section we discussed the corrections corresponding to powers of the most important dimensionless parameter $\tilde{d}\epsilon$. All other dimensionless parameters are assumed to be large and we consider only the first terms of the expansion on them. The computation of
higher order terms of these parameters, coming e.g. from corrections to the Gutzwiller trace formulas, are also possible but is beyond the scope of this paper.

7 Correlation functions of classical actions

It is well known [13], [14] that there exists a duality between quantum spectral statistics and statistics of classical actions. If one is known the other can be computed by the Fourier transformation.

In this Section we demonstrate that the results of the previous Sections can be rewritten in the form of correlation functions of classical actions. Let us consider for simplicity the 2-point correlation function of periodic orbit length for rectangular billiard (calculation of other correlation functions will be discussed elsewhere [16])

\[ R_2(s) = \sum_{l,l'} \delta(s - l + l')a(l,l'), \quad (113) \]

where \( a(l,l') \) is a certain weighted function which should dominates by large values of \( l \) and \( l' \). A typical example is

\[ a(l,l') = \delta(L - \frac{l + l'}{2}) \quad (114) \]

and \( L \) is assumed to be a large quantity. The correlation function with this weight will be denoted by \( R_2(s, L) \)

\[ R_2(s, L) = \sum_{l,l'} \delta(s - l + l') \delta(L - \frac{l + l'}{2}). \quad (115) \]

It is this function which appears naturally in many applications (see below).

The summation in (113) is done over pairs of periodic orbit lengths. We implicitly assume that all values of \( l \) and \( l' \) are permitted. There are few other possibilities e.g. one can choose \( l' \geq l \), or only \( l' > l \) which lead to slight modifications of the formulas below.

One has

\[ R_2(s) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\tau f(\tau)e^{is\tau}, \quad (116) \]
where
\[ f(\tau) = \sum_{l, l'} e^{i\tau(l'-l)} a(l, l'). \] (117)

This sum splits into 2 parts
\[ f(\tau) = f^{(\text{diag})}(\tau) + f^{(\text{osc})}(\tau), \] (118)

where
\[ f^{(\text{diag})}(\tau) = \sum_{l = l'} a(l, l') \approx \bar{\rho} \int l a(l, l) dl, \] (119)

where we have used the mean density of periodic orbits (26) with \( \bar{\rho} = \pi/2A \) and
\[ f^{(\text{osc})}(\tau) = \sum_{l \neq l'} e^{i\tau(l'-l)} a(l, l'). \] (120)

To compute this function we shall use the saddle point method developed in the previous Sections. As above the dominant contribution will come from the saddle point manifold (34) and
\[ f^{(\text{osc})}(\tau) = \sum_{l_0, r_1, r_2} \int dt \exp(i\tau l_0 (r_2 - r_1) + iQt^2(1/r_2 - 1/r_1)) a(r_1 l_0, r_2 l_0), \] (121)

where \( Q = A^2\tau/(2l_0^3) \). Performing the integration and taking into account that \( r_1, r_2 \to \infty \) but \( r_2 - r_1 = r \) is fixed one obtains
\[ f^{(\text{osc})}(\tau) = \sum_{l_0} D e^{i\tau l_0 r - i\pi \text{sgn}(\tau) r/4} \sqrt{\frac{\pi}{Q|r|}}, \] (122)

where
\[ D = \int dr_1 r_1 a(r_1 l_0, r_1 l_0) = \frac{1}{l_0^2} \int l a(l, l). \] (123)

Because
\[ \sqrt{\frac{\pi}{Q|l| l_0^2}} = \frac{A\bar{\rho}}{2\pi d \sqrt{8\pi\tau l_0^4}}, \] (124)

we finally get
\[ f(\tau) = \frac{\bar{a}}{d} (d + \frac{A}{2\pi} \sum_{l_p} \frac{e^{i\pi l_p - i\pi/4}}{\sqrt{8\pi\tau l_p} + \text{c.c.}}), \] (125)
where
\[ \bar{a} = \sum_l a(l, l) = \bar{\rho} \int l a(l, l) dl. \] (126)

But the sum in Eq. (125) is exactly the semi-classical expansion of oscillating part of the trace formula for rectangular billiard (see Eq. (11)) with energy equals \( \tau^2 \) which means that 2-point correlation form-factor for periodic orbit lengths is proportional to the semi-classical trace formula

\[ f(\tau) = \frac{\bar{a}}{d} d(\tau^2). \] (127)

In particular for rectangular billiard with weighted function (114)

\[ R_2(s, L) = 4\pi A^2 \int d\tau d(\tau^2) e^{i\tau s} \] (128)

These results can be considered as a generalization of results of Refs. [13], [14]. In these papers only the smooth part of this relation has been considered \( f(\tau) = \bar{d}(\tau^2) \). But relation (127) contains more information. In particular it states that the Fourier transform of classical 2-point correlation function of periodic orbit lengths for rectangular billiard should have peaks exactly at eigenvalues of quantum problem with Newmann boundary conditions. The generalization of this relation to general integrable models is performed in the next Section. Numerical calculations for different integrable billiards confirm well this prediction [13].

The knowledge of the 2-point correlation function of classical actions permits to compute easily any quantities depending on a product of 2 densities. For example,

\[ d^2(\epsilon) = 2\left(\frac{A}{2\pi}\right)^2 \sum_{l,l'} e^{\frac{ik(l-l')-\epsilon(l+l')}{2k}} \int_{-\infty}^{\infty} dse^{iks-\epsilon L/k} R_2(s, L) dL \]

\[ = \frac{1}{2\pi k} d(E) \int_0^\infty e^{-\epsilon L/k} dL = \frac{1}{2\pi \epsilon} d(E) \] (129)

which coincides with the bootstrap condition (21).

It is also instructive to compute the 2-point correlation function of energy levels

\[ R_2(\epsilon) = <d(E + \frac{\epsilon}{2})d(E - \frac{\epsilon}{2})> \]

\[ = <d^2(E)> + \left(\frac{A}{2\pi}\right)^2 \sum_{l,l'} e^{\frac{ik(l-l')-\epsilon(l+l')}{4k}} \frac{1}{8\pi k\sqrt{l'l'}} + c.c. > . \] (130)
The last sum equals
\[
\left( \frac{A}{2\pi} \right)^2 \int_{-\infty}^{\infty} \frac{dL}{8\pi k L} e^{i \frac{eL}{2k}} \int_{-\infty}^{\infty} dse^{iks} R_2(s, L) = \frac{1}{4\pi k} d(E) \int_{-\infty}^{\infty} e^{i \frac{eL}{2k}} dL = \delta(e) d(E),
\]
and the 2-point correlation function for energy levels of rectangular billiard (and for general integrable systems as well) has the following form
\[
R_2(e) = \langle d^2(E) \rangle + \delta(e) \langle d(E) \rangle.
\]
Here the brackets denote a local average over a certain energy window. When this window is very large \( \langle d(E) \rangle = \bar{d} \) but in general this smoothing defines a new local scale and \( \langle d(E) \rangle \) can deviates from \( \bar{d} \). This result is exactly as was expected. The energy levels of typical integrable systems (in particular for rectangular billiard) are distributed as independent random variables but at the local scale (or equivalently only after unfolding). The correlation function of classical lengths (127) does the required transformation from global scale to the local one.

8 General integrable systems

In the previous Sections for clarity we discussed the case of rectangular billiard but the method used is not restricted only to this example and can be generalized for general integrable systems as well.

The starting point of the derivation of the trace formula for integrable systems is a formal expression of a Hamiltonian as a function of action variables
\[
E = H(I_1, I_2).
\]
Semi-classical quantization conditions consist in fixing the values of these action variables
\[
I_i = \hbar (n_i + \frac{1}{4} \mu_i),
\]
where \( n_i \) are integers and \( \mu_i \) are certain phases connected with the type of classical motion (the Maslov indices). Therefore for 2-dimensional integrable systems (multi-dimensional case will be discussed elsewhere) the quantum energies are
\[
E_{n_1n_2} = H(n_1 + \frac{1}{4} \mu_1, n_2 + \frac{1}{4} \mu_2),
\]
and the quantum density of states is

\[ d(E) = \sum_{n_1,n_2} \delta(E - E_{n_1n_2}). \] (136)

Using the Poisson summation formula one obtains

\[ d(E) = \sum_{N_1,N_2} \int e^{2\pi i(N_1n_1+N_2n_2)}\delta(E - E_{n_1n_2})dn_1dn_2. \] (137)

Separating the term with \( N_1 = N_2 = 0 \) and setting \( n_i = I_i/h - \mu_i/4 \) leads to

\[ d(E) = \bar{d}(E) + d^{osc}(E), \] (138)

where

\[ \bar{d}(E) = \frac{1}{\hbar^2} \int \delta(E - H(I_1, I_2))dI_1dI_2 = \frac{1}{(2\pi\hbar)^2} \int \delta(E - H(\vec{p}, \vec{q}))d\vec{p}d\vec{q}, \] (139)

and

\[ d^{osc}(E) = \frac{1}{\hbar^2} \sum_{N_1,N_2} e^{-i(N_1\mu_1+N_2\mu_2)/2} \int e^{2\pi i(N_1I_1+N_2I_2)/\hbar}\delta(E - H(I_1, I_2))dI_1dI_2. \] (140)

It is convenient instead of variables \( \vec{I} \) to use variables \( e \) and \( t \) defined by

\[ \frac{\partial \vec{I}}{\partial e} = \vec{\nu}, \quad \frac{\partial \vec{I}}{\partial t} = \vec{\tau}, \] (141)

where the unit vectors \( \vec{\nu} \) and \( \vec{\tau} \) have the following components

\[ \nu_i = \frac{1}{\sqrt{\omega_1^2 + \omega_2^2}} \omega_i, \quad \tau_i = \frac{1}{\sqrt{\omega_1^2 + \omega_2^2}} \sum_{j=1}^{2} e_{ij}\omega_j, \] (142)

and

\[ \omega_i = \frac{\partial H(I_1, I_2)}{\partial I_i} \] (143)

are frequencies of the classical motion, and \( e_{ij} \) is \( 2 \times 2 \) antisymmetric tensor with components \( e_{11} = e_{22} = 0 \) and \( e_{21} = -e_{12} = 1 \).

Vector \( \vec{\nu} \) is the unit vector perpendicular to the line of fixed energy, vector \( \vec{\tau} \) is the unit vector tangent to this line, and \( \vec{\nu}^2 = \vec{\tau}^2 = 1, \vec{\nu} \vec{\tau} = 0 \).
It is easy to check that \(dI_1dI_2 = dedt\) and
\[
\int \delta(E - H(e))de = \frac{1}{\sqrt{\omega_1^2 + \omega_2^2}}. \tag{144}
\]
The integral over the surface of constant energy
\[
\int e^{2\pi i(N_1I_1 + N_2I_2)/\hbar} dt \tag{145}
\]
can be taken by saddle point method. The saddle points are points where
\[
\sum_i N_i \frac{\partial I_i}{\partial t} = 0. \tag{146}
\]
This condition means that in saddle points the classical frequencies should be commensurable
\[
\frac{\omega_1}{\omega_2} = \frac{N_1}{N_2}. \tag{147}
\]
Straightforward calculation gives
\[
\frac{\partial \vec{\tau}}{\partial t} = -\vec{\nu} K \frac{1}{\sqrt{\omega_1^2 + \omega_2^2}} \tag{148}
\]
where the curvature \(K\) is
\[
K = \frac{1}{\omega_1^2 + \omega_2^2}(\omega_1^2 \frac{\partial^2 H}{\partial I_2^2} + \omega_2^2 \frac{\partial^2 H}{\partial I_1^2} - 2\omega_1\omega_2 \frac{\partial^2 H}{\partial I_1 \partial I_2}). \tag{149}
\]
As
\[
\frac{\partial^2 (N_1I_1 + N_2I_2)}{\partial t^2} = \sum_{i=1}^2 N_i \frac{\partial \tau_i}{\partial t} = -\frac{K}{\sqrt{\omega_1^2 + \omega_2^2}} \sum_{i=1}^2 N_i \nu_i, \tag{150}
\]
the oscillating part of the level density takes the form
\[
d^{\text{osc}}(E) = \frac{1}{\hbar^{3/2}} \sum_{N_1, N_2 \geq 0} P \exp(iS_{cl}(E) - \frac{i\pi}{2}(N_1\mu_1 + N_2\mu_2) - i\frac{\pi}{4}\text{sgn}K) + \text{c.c.}, \tag{151}
\]
where the pre-factor \(P\) is
\[
P = \frac{1}{\sqrt{(N_1\omega_1 + N_2\omega_2)|K|}} \tag{152}
\]
The classical action of resonant periodic tori (= periodic orbit)

\[ S_{cl}(E) = 2\pi(N_1 I_1 + N_2 I_2), \quad (153) \]

is defined by special values of action variables \( I_i = I_i(N_1, N_2, E) \) for which the following two equations are valid

\[ H(I_1, I_2) = E \quad (154) \]

and

\[ N_2 \frac{\partial H(I_1, I_2)}{\partial I_1} = N_1 \frac{\partial H(I_1, I_2)}{\partial I_2}. \quad (155) \]

Introducing the classical period of motion

\[ T(E) = \frac{\partial S_{cl}(E)}{\partial E} = \frac{2\pi N_1}{\omega_1}, \quad (156) \]

one can rewrite the pre-factor in the following form

\[ P = \sqrt{\frac{T(E)}{2\pi|\kappa|}}, \quad (157) \]

where

\[ \kappa = N_1^2 \frac{\partial^2 H}{\partial I_2^2} + N_2^2 \frac{\partial^2 H}{\partial I_1^2} - 2N_1N_2 \frac{\partial^2 H}{\partial I_1 \partial I_2}. \quad (158) \]

Eq. (151) is the semi-classical trace formula for general integrable systems and is a generalization of the trace formula for rectangular billiard given by Eq. (14). The important difference between these two formulas is that the periodic torus action is given by a simple formula (11) for the latter but only by implicit formulas (153)-(155) for the former.

The first step to apply the saddle point method similar to the one discussed in previous Sections to the general trace formula (151) is to find the expansion of the classical action \( S_{cl}(N_1 + \delta N_1, N_2 + \delta N_2, E) \) into series of \( \delta N_i \).

By differentiating Eqs. (153)-(155) one obtains that this expansion (which is a generalization of Eq. (28)) has the following form

\[ S_{cl}(N_1 + \delta N_1, N_2 + \delta N_2, E) = S_{cl}(N_1, N_2, E) + 2\pi(I_1\delta N_1 + I_2\delta N_2) + Q(N_2\delta N_1 - N_1\delta N_2)^2, \quad (159) \]
where
\[ Q = \frac{2\pi^2}{T(E)\kappa}, \]  
and all other notations are the same as above.

Taking into account that \( I_1, I_2 \) do not depend on the common factor of integers \( N_1, N_2 \) and assuming that values of action variables \( I_i \) for different primitive periodic tori are non-commensurable one concludes that the only possibility of existence of non-trivial saddle points (i.e. the cancellation of linear terms from different periodic tori in the exponent) is exactly the same conditions as in previous Sections. For example, for two periodic tori the saddle point manifold has the form exactly (up to notations) the same as in Eq. (34)

\[ N_1^{(1)} = r^{(1)}n_1 + \delta N_1, \quad N_2^{(1)} = r^{(1)}n_2 + \delta N_2, \]
\[ N_1^{(2)} = r^{(2)}n_1 + \delta N_1, \quad N_2^{(2)} = r^{(2)}n_2 + \delta N_2, \]  
where superscript denotes values for different periodic tori. As before, we assume that integers \( n_1 \) and \( n_2 \) have no common factor.

All further calculations can be performed exactly as it has been done for rectangular billiard in Section 4. The only difference with the latter case is different expressions of the pre-factor in the trace formula (157), the coefficient of the quadratic form (160), and the period of the motion. But it is easy to check that
\[ P \sqrt{\frac{\pi}{|Q/T|}} = \frac{1}{2\pi}, \]  
which agree with Eq. (90) and all formulas derived for the oscillating parts of products of periodic orbit contributions for rectangular billiard remain valid for generic integrable systems as well. Note that from Eq. (45) it follows that the total Maslov phase will have the correct value.

To apply the saddle point method to smooth terms one need to know the mean density of periodic orbits for general integrable systems. Though it is possible to do it by applying the Hannay-Ozorio de Almeida method to integrable systems [4] it is instructive to compute it directly from Eqs. (153)-(155).

Let us define the density of periodic orbits with fixed period (156) as
\[ \rho(\tau) = \int_0^\infty \delta(T(N_1, N_2, E) - \tau) dN_1 dN_2. \]  

Introducing polar coordinates $N_1 = R \cos \phi$, $N_2 = R \sin \phi$ one gets

$$
\rho(\tau) = \int \delta \left( \frac{2\pi R \cos \phi}{\omega_1} - \tau \right) R dR d\phi = \frac{\tau}{4\pi^2} \int (\omega_1^2 + \omega_2^2) d\phi,
$$

(164)

This integral can be transformed into an integral over classical actions $I_i$ by noting that they are functions of $\phi$ and $E$ defined by Eqs. (154) and (155). Straightforward calculations give

$$
\det \left( \begin{array}{cc} \partial I_1 / \partial E & \partial I_1 / \partial \phi \\ \partial I_2 / \partial E & \partial I_2 / \partial \phi \end{array} \right) = \frac{1}{K},
$$

(165)

where $K$ is the curvature defined in Eq. (149). As $\omega_2/\omega_1 = \tan \phi$

$$
K = \cos^2 \phi \frac{\partial^2 H}{\partial I_2^2} + \sin^2 \phi \frac{\partial^2 H}{\partial I_1^2} - 2 \cos \phi \sin \phi \frac{\partial^2 H}{\partial I_1 \partial I_2}.
$$

(166)

Therefore

$$
d\phi = K \delta (E - H(I_1, I_2)) dI_1 dI_2,
$$

(167)

and in particular

$$
\rho(\tau) = \frac{\tau}{4\pi^2} \int (\omega_1^2 + \omega_2^2) K \delta (E - H(I_1, I_2)) dI_1 dI_2.
$$

(168)

The computation of smooth terms can be performed exactly as above but with different values of pre-factor, $P$ and the coefficient, $Q$, of the quadratic form in the exponent. We have

$$
P = \sqrt{\frac{\cos \phi}{\omega_1 K}},
$$

(169)

and

$$
Q = \frac{\pi \omega_1}{\cos \phi K}.
$$

(170)

Because the reduced period

$$
T = 2\pi \frac{\cos \phi}{\omega_1},
$$

(171)
the combination of these terms in Eq. (162) has the same value as before and all integrals except the integral over \( \phi \) rest exactly as in Section 5. The last integral takes the form
\[
\int P \sqrt{\frac{Q}{\pi}} d\phi = \int \frac{d\phi}{K} = \int \delta(E - H(I_1, I_2))dI_1dI_2 = \bar{d},
\] (172)
as it should be to reproduce the smooth part of bootstrap conditions (22).

Though one can compute any correlation functions of classical actions with arbitrary definition it is convenient instead of Eq. (113) to define the 2-point correlation function of differences of actions for general integrable systems as follows
\[
R_2(s) = \sum_{p,p'} \delta(s - S_p + S_{p'})P_pP_{p'}e^{-i\pi(\mu_{p'} - \mu_p)/2}a(T_p, T_{p'}),
\] (173)
where \( T_p \) are periodic orbit periods, \( \mu_p \) are the Maslov indices for the problem considered, \( P_p \) are pre-factors in the trace formula given by Eq. (157) (without powers of the Planck constant), and \( a(T_p, T_{p'}) \) is a certain weighted function.

Separating the diagonal and oscillatory contributions as in Section 7
\[
R_2(s) = \frac{1}{2\pi} \int d\tau e^{i\tau s} (f^{(diag)}(\tau) + f^{(osc)}(\tau)),
\] (174)
and using Eqs. (169)-(172) one gets
\[
f^{(diag)}(\tau) = \sum_p P_p^2 a(T_p, T_p) = \frac{1}{2\pi} \int dTa(T, T) \int \frac{d\phi}{K} = \bar{a}\bar{d},
\] (175)
where
\[
\bar{a} = \frac{1}{2\pi} \int dTa(T, T).
\] (176)
The computation of oscillatory terms can be performed exactly as in Section 7. Taking into account the relation (162) it is easy to check that
\[
f^{(osc)}(\tau) = \frac{\bar{a}}{\sqrt{\tau}} \sum_p P_p e^{i\tau S_p - i\pi\mu_p/2 - i\pi sgnK/4}.
\] (177)
Therefore
\[
f(\tau) = \hbar^2d(E, \hbar)|_{\hbar=1/\tau},
\] (178)
where \( d(E, \hbar) \) is the level density of the quantum problem considered which generalizes the results in Refs. [13], [14] about duality in integrable systems.
9 Conclusion

The main goal of this paper was to develop a new method which permits to calculate sums of product of periodic orbit contributions for generic integrable systems beyond the diagonal approximation. In the previous Sections it was demonstrated how different quantities like bootstrap conditions (22), mean value of Green function products (23), 2-point correlation function of classical actions (113), etc., can be computed for integrable systems directly from semi-classical trace formulas using this method. More applications of this method will be discussed elsewhere [16].

The main ingredient of the method is the existence of hidden saddle points in the summation over periodic orbits. The saddle point configurations consist of periodic orbits whose integer vectors are mutually parallel. The saddle point manifold includes orbits which (i) are almost mutually parallel and (ii) are chosen in such a way that the total action is quadratic on deviations from the saddle point configuration. The integration over these quadratic forms gives the dominant contribution to the summation over periodic orbits. By considering low-dimensional sub-manifolds it is also possible to compute corrections to this term.

In the calculations it was assumed that for generic integrable systems classical actions of (almost) all resonant tori are non-commensurable but the method can be generalized for systems with geometrical symmetries and (or) accidental degeneracy. In Ref. [17] the method has been applied for integrable quantum maps.

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