The single-particle unit for alpha decay

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A salient feature of quantum mechanics is the inherent property of collective quantum motion, when apparent independent quasiparticles move in highly correlated trajectories, resulting in strongly enhanced transition probabilities. To assess the extend of a collective quantity requires an appropriate definition of the uncorrelated average motion, often expressed by single particle units. A well known example in nuclear physics is the Weisskopf unit for electromagnetic transitions which reveals different aspects of collective motion. In this paper we define the corresponding single particle unit for alpha decay following Weisskopf’s derivations. We evaluate the alpha decay amplitude as induced by four uncorrelated/non-interacting protons and neutrons and compare it with the one extracted from observed decay rates. Our definition elucidates the collectivity in alpha decay and facilitates an unified description of all alpha decay processes along the nuclear chart. Our formalism is also applicable to other particle decay processes.

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Alpha decay has been one of the most rewarding subjects in physics since Gamow was the first to apply the probabilistic interpretation of quantum mechanics to describe the penetrability of the Coulomb barrier by the α-particle [1]. The subsequent developments upon radioactive particle decay in nuclear physics has been outstanding [2, 3]. At present, α-decay is crucial for the identification of unstable nuclei far from stability, particularly super heavy and proton rich nuclei [4]. Yet there are unsolved fundamental problems even today: A microscopic description of the clustering of the two nucleons which eventually constitute the α-particle as provided by the nuclear configuration interaction shell model has not been able to account for the decay widths, even when high-lying configurations as well as the proper treatment of the nuclear continuum is incorporated into the calculations.

The understanding and quantification of collective motion in atomic nuclei have a long history. Enhanced decay probabilities in electromagnetic transitions are used to classify different excitation modes such as vibrations and rotations. These classifications of collectivity can be made through a reliable basic quantity, namely the single-particle Weisskopf unit (W.u.) [5]. Such a common reference enables one to differentiate between decays that are non collective and those that involve the coherent motion of many nucleons. Although called “unit”, the W.u. has not an universal value, since it depends upon the mass of the nucleus in question as well as upon the character of the transition (Eλ or MA).

Analogous to the W.u. for electromagnetic decay, we define in this letter an equivalent unit for α decay, the particle decay unit, p.d.u. This will relate the measured probability of α decay to an averaged single configuration in the description of the mother nucleus. We hope that this common reference will clarify the role played by nuclear collectivity in α decay. Our definition enables the appropriate comparison of all hitherto observed L = 0 α decay on the same footing, avoiding the multitudes of effective quantities found at present in the literature [6–8]. In addition, the formalism presented in this paper will enable one to quantify the role played by α clustering in heavy nuclei.

Below we present in detail the formalism. We start with the Thomas expression for the α decay width [9],

$$\Gamma_c(R) = \frac{\hbar^2 k R^2}{\mu} \left| F_c(R) \right|^2 \left| H_i^+(\chi, \rho) \right|^2$$

(1)

which often is written as

$$\Gamma_c(R) = \frac{\hbar^2 R}{\mu} \left| P_c(R) \right|^2 P_c(R)$$

(2)

where $P_c(R) = k R / |H_i^+(\chi, \rho)|^2$ is the penetrability of the already formed α particle through the Coulomb and centrifugal fields starting at the point R, which is the distance between the mass centres of the daughter and α cluster. In these equations c labels the decay channel, k is the linear momentum carried by the α-particle, μ is the reduced mass, $H_i^+$ is the Coulomb-Hankel function describing the two-body system in the outgoing channel. Its arguments are $\rho = \mu \nu R / \hbar$ and $\chi = 2Z_cZ_d e^2 / \hbar \nu$. $Z_c$ and $Z_d$ are the charge numbers of the cluster and daughter nucleus, respectively. The function $F_c(R)$ is the α formation amplitude, i.e., the mother wave function describing the motion of the α cluster in the field induced by the daughter nucleus at the point R. It is important to stress the difference between this exact treatment and the effective treatments mostly used in the literature. In Eq. (1) the evaluation of the formation amplitude is assumed to be performed within a microscopic framework [9, 10]. At the point R in Eq. (1) the α-particle is already formed and only the Coulomb and centrifugal interactions are relevant.
The greatest challenge facing microscopic treatments is nucleon-nucleon interactions inside the mother nucleus. In order to perform the radial part of this integral it is canceled in the angular and spin integrals in Eq. (3). To achieve the microscopic description of the α clusterization and the subsequent motion of the cluster at the surface \( R \) is a difficult undertaking \[2\] [10–12]. This explains why effective treatments are common in the literature, where the α effective particle is assumed to exist inside the mother nucleus. The decay is described as the penetration of a preformed α particle through the Coulomb barrier.

In our formalism the formation amplitude is determined following the microscopic treatment \[3\], i.e.,

\[
F_c(R) = \int d\hat{R}d\xi_d d\xi_\alpha [\Psi_d(\xi_d)\phi(\hat{R})]\Psi_m, \tag{3}
\]

where \( \xi_d \) and \( \xi_\alpha \) are the internal degrees of freedom determining the dynamics of the daughter nucleus and the α-particle. The wave functions \( \Psi_d(\xi_d) \) and \( \Psi_m(\xi_d, \xi_\alpha, \hat{R}) \) correspond to the daughter and mother nuclei respectively. The α-particle wave function has the form of a \( n = l = 0 \) harmonic oscillator eigenstate in the neutron-proton relative distances \( r_{nn} \), as well as in the proton-proton distance \( r_{pp} \) and in the distance \( r_{np} \) between the mass centres of the \( nn \) and \( pp \) pairs \[3\],

\[
\phi_\alpha(\xi_\alpha) = \left( \frac{1}{\pi} \frac{\nu_\alpha}{2} \right)^{3/4} e^{\nu_\alpha(r_{nn}^2 + r_{pp}^2 + 2r_{np}^2)/4} S_\alpha \tag{4}
\]

where \( S_\alpha \) is the α-spinor corresponding to the lowest harmonic oscillator wave function. The total angular momenta are \( L = S = 0 \). The quantity \( \nu_\alpha = 0.574 \) \( fm^{-2} \) is the α-particle harmonic oscillator parameter \[13\].

We consider decays involving uncorrelated states of even-even nuclei. We will focus our treatment on ground-state to ground-state transitions, implying that \( l = 0 \) and \( Y_{l = 0}(\hat{R}) = 1/\sqrt{4\pi} \). Uncorrelated decay means that the mother nucleus consists of the daughter nuclei times a pure configuration of a pair coupled to zero angular momentum times a similar proton pair, i.e.

\[
\Psi_m(\xi_d, \xi_\alpha, \hat{R}) = (\varphi(\mathbf{r}_1)\varphi(\mathbf{r}_2))_{00}(\varphi(\mathbf{r}_3)\varphi(\mathbf{r}_4))_{00} \tag{5}
\]

Writing the single-particle wave functions \( \varphi(\mathbf{r}) \) in their radial, angular and spin components, these last two are canceled in the angular and spin integrals in Eq. (5). In order to perform the radial part of this integral it is convenient to write the mother wave function in terms of the relative coordinates \( \mathbf{r}_{nn}, \mathbf{r}_{pp}, \mathbf{r}_{pn} \) and the centre of mass coordinate \( \hat{R} \). Since the Jacobian corresponding to the transformation from absolute to relative coordinates in the integral \[3\] is unity one can write

\[
\Psi_m(\xi_d, \xi_\alpha, \hat{R}) = \phi(\mathbf{r}_{nn})\phi(\mathbf{r}_{pp})\phi(\mathbf{r}_{pn})\phi(\hat{R})\Psi_d(\xi_d) \tag{6}
\]

where \( \phi \) are the wave functions in relative coordinates. These functions may diverge at \( r = 0 \) and therefore we use the standard function \( u(r) = r\phi(r) \). Following the method employed by Weisskopf, we assume that the radial single-particle wave function \( u(r) \) in Eq. (6) is constant inside the mother nucleus, with radius \( R \). As a result, the relative and centre of mass radial wave functions inside the mother nucleus are constants. Notice that according to our prescription the \( nn, pp \) and \( pn \) wave functions vanish outside the nuclear surface, while \( \phi(\hat{R}) \), the wave function corresponding to the motion of the α particle centre of mass, is constant inside the nucleus, but outside corresponds to an outgoing α particle, as seen below.

The normalization condition provides

\[
\int_0^R (u(r)/r)^2 r^2 dr = RC^2 = 1 \tag{7}
\]

where the constant \( C \) is the same for the \( pp, nn, pn \) and the centre of mass wave functions inside the mother nucleus resulting in \( C = 1/\sqrt{R} \).

The formation amplitude in Eq. (6) acquires the form,

\[
F_c(R) = \int d\hat{R} \int \nu_\alpha^2 r_{nn}^2 r_{pp}^2 r_{pn}^2 dr_{nn} dr_{pp} dr_{pn} \left[ \frac{1}{8} \left( \frac{\nu_\alpha}{\pi} \right)^{9/4} \right. \\
\times e^{\nu_\alpha(r_{nn}^2 + r_{pp}^2 + 2r_{pn}^2)/4} \frac{1}{\sqrt{4\pi}} \frac{C^4}{R_{nn} R_{pp} R_{pn}} \\
\left. \times e^{-\nu_\alpha(r_{nn}^2 + r_{pp}^2 + 2r_{pn}^2)/4} \frac{1}{\sqrt{4\pi}} \frac{C^4}{R_{nn} R_{pp} R_{pn}} \right] \tag{8}
\]

It is straightforward to perform the radial integrals. Thus for \( r_{nn} \) one obtains,

\[
\int r_{nn} dr_{nn} e^{\nu_\alpha r_{nn}^2/4} = \frac{2}{\nu_\alpha} \tag{9}
\]

The remaining integrals can be calculated in the same fashion. We are interested in the formation amplitude at the radius \( R \) and therefore integrate over the angle \( \hat{R} \) (which provides a factor \( 4\pi \)). The formation amplitude at the nuclear surface becomes,

\[
F_{\alpha pne}(R) = \frac{1}{\sqrt{8 \pi \nu_\alpha}} \left( \frac{\nu_\alpha}{\pi} \right)^{9/4} \frac{\sqrt{C^4 4}}{R^{3/2}} \frac{\nu_\alpha^4}{\pi^{7/4}} \frac{1}{R^3} \tag{10}
\]

which defines the particle decay unit (p.d.u.). It measures the α decay formation amplitude for decays from four uncorrelated single particle states. With \( R = 1.2(A^{1/3} + 4^{1/3}) \) \( fm \) one obtains

\[
F_{\alpha pne} = 0.335/(A^{1/3} + 4^{1/3})^3 \text{ fm}^{-3/2}. \tag{11}
\]
In order to clarify the procedure that we are following here it is important to remember that the neutrons and protons form the $\alpha$ particle at the nuclear surface because of the interactions among them inside the mother nucleus. At and inside the nuclear surface the $\alpha$ particle wave function has the constant value $u(r) = C$. Outside the nuclear surface, i.e. at $r > R$ (where only the Coulomb and centrifugal interactions are relevant), the wave function of the outgoing $\alpha$ particle becomes

$$u(r) = r\phi(r) = N[H^+_l(\chi, \rho)]$$  \hspace{1cm} (12)$$

where $N$ is the matching constant. The independence of the Thomas expression upon the distance $R$ (as pointed out above, $R$ should be beyond the nuclear surface) has often been used in microscopic calculations of $\alpha$ decay to probe whether the results are reliable [14].

Following Eq. (11), we extract the $\alpha$ decay formation amplitude measured in p.d.u. from the ratio between experiment and the corresponding p.d.u. value. Similar to the W.u. in electromagnetic decay, values that exceed the p.d.u. by an order of magnitude reflect the enhancement in $\alpha$ decay, pointing towards the collective nature of the process.

The value of the $\alpha$ formation amplitudes in p.d.u. of known $\alpha$ emitters in the mass $A = 180$ region and beyond are depicted in Fig. 1. The experimental half-lives are taken from Ref. [15] and references therein. The figure reveals distinctive features characterizing $\alpha$ decay. Thus and most conspicuous, the decay rates all exceed by far the value of a single particle unit. In other words, the $\alpha$ decay process in its nature reveals strong collectivity. Hence, it is not surprising that cluster components are needed in the shell model wave function to account for the experimental decay width [2, 16]. Other important feature revealed by the figure is the shell closure at $N = 126$. For heavier isotopes, i.e. above the magic number 126, the p.d.u. approach a constant value, somewhat above 20 p.d.u. For Po-, Pb- and Hg-isotopes below $N = 126$, we observe lower values of p.d.u. somewhat above 10 p.d.u. For the case of the Po isotopes, there are two branches below $N = 126$, where one branch is hindered in the decay due to configuration changes, as shown in Ref. [17, 18]. The reduced width of Pb and Hg isotopes reflects the restricted configuration space for cluster formation, particular due to the protons being at or just below shell closure. For $N > 126$, and $Z > 82$, neutrons and protons are above the shell gap opening a wide configuration space for cluster formation.

We have evaluated the $\alpha$ decay formation amplitude for the doubly magic nuclei $^{208}$Pb, which is stable due to low $Q$ value, following the microscopic treatment as described in Ref. [14]. What is striking is that the calculated $\alpha$ decay formation amplitude is nearly unity in p.d.u. This result is quite reasonable since one expects minimal collectivity in the nucleus $^{208}$Pb. It further validates the approximation we applied in deriving Eq. (10).

A strong reduction can also be expected for the decays from non-collective high-spin isomeric states [19].

In Fig. 2 we compare the p.d.u. of $\alpha$ formation amplitudes of nuclei above $^{100}$Sn. The decay widths of those nuclei have attracted significant attention in recent studies in relation to the search for the so-called superallowed $\alpha$ decay. This is expected due to the enhanced neutron-proton pairing when approaching the $N = Z$ line and hence an enhanced clustering effect [21, 22, 23, 24]. One can see from Fig. 2, that the formation amplitude of those nuclei follows the general average trend of $\alpha$ formation amplitude systematics even though it shows rather large fluctuations and uncertainties. Further experimental investigations are essential to clarify this issue. It may be useful to mention here that the systematics of formation probabilities for available $\alpha$ decays shows an increasing trend with decreasing mass number. Apparently, as our formula for p.d.u. shows, the formation of $\alpha$ scales with

\[ F(pdu) = \sqrt{\frac{1}{2}} \left( \frac{N - 2}{N - 20} \right) \]

for neutron-deficient Te (circle) and Xe (square) above $^{100}$Sn. Open symbols correspond to the decays of $\alpha$ particles carrying orbital angular momentum $l = 2$. The experimental data are extracted from Ref. [22–24].
the nuclear volume, 1/A. This important feature revealed by our results, needs to be taken into account in studies of \( \alpha \) decays of trans-tin nuclei, in particular when comparing to heavy nuclei including in particular \(^{212}\text{Po}\).

One can employ the \( \alpha \) decay formation amplitude in Eq. (10) to go one step further and evaluate even the \( \alpha \) decay width in p.d.u. This is an extension of the Weisskopf prescription. In electromagnetic transitions one evaluates the \( B(E\lambda/M\lambda) \) values (which is the equivalent of the alpha formation amplitude in our case) from the decay half-life/width by excluding the effect of the decay energy. In our case of particle decay, this can be easily performed by using Eq. [3]. Thus, the decay width in p.d.u. is,

\[
\Gamma_{\alpha\text{-p.d.u.}}(R) = \frac{\hbar^2 k}{\mu} R^2 F_{\alpha\text{-p.d.u.}}^2(R) \left| H_{\lambda}^f(\chi, \rho) \right|^2 \approx \frac{\hbar^2 k}{\mu} R^2 F_{\alpha\text{-p.d.u.}}^2(R) e^{-\frac{\mu}{2} \left[ \pi^2 / 2 - 2(\rho/\chi)^{1/2} + 1/3(\rho/\chi)^{3/2} \right] \cos^2 \beta}, \quad (13)
\]

where \( \cos^2 \beta = \rho/\chi \). The derived width above depends now upon the decay \( Q \)-value. For details of the approximate form of the Coulomb function used in this equation, where \( l = 0 \) was assumed, see Ref. 20.

Our derivation can also be extended to other decay processes including heavier cluster decays and decays that involve change of angular momentum. Similarly, we can evaluate proton decay, where the formation amplitude becomes much simpler than for \( \alpha \) decay in Eq. (3) since it involves no intrinsic structure (for details, see Ref. 3). With the same assumption as above for the single-particle wave function, the proton decay formation amplitude results to have the simple form of

\[
F_{\text{p-d.u.}}(R) = \frac{1}{R^{5/2}}. \quad (14)
\]

Using this value, we depict proton emitters in Fig. 3. As expected, since the proton already is formed inside the nucleus, the p.d.u. corresponding to unity sets the limit for the decay. Values smaller then one indicate a partial occupation of the particular proton-emitting \( l \)-state in the daughter nuclei and/or a change in structure/deformation between mother and daughter nuclei. Most decays in the figure show p.d.u. values between 0.1 and 0.8. The largest two values correspond to the proton decays from the odd-odd nuclei \(^{144,146}\text{Tm}\) which is enhanced due to the coupling of the decaying proton with the odd neutron \([28]\).

In conclusion, we have deduced a simple averaged single particle limit for the \( \alpha \) decay formation amplitude and decay width, which we call particle decay unit (p.d.u.). This definition enables a unified description and comparison of \( \alpha \) decay along the nuclear chart. The magnitude of the p.d.u. reveals the collectivity of \( \alpha \) decay. The decay pattern nicely reveals that a truly microscopic description requires the explicit presence of \( \alpha \) clustering elements at the nuclear surface. An important feature revealed by our formalism is that the \( \alpha \) formation amplitude in p.d.u. scales with the nuclear volume. Competing decay mechanisms within the same mother nucleus can be understood as changes of \( \alpha \) collectivity at the surface. One may expect a similar effect as induced by the competition between pairing and deformation in two-nucleon transfer reactions (see, e.g., Ref. 30). Our derivation can be extended to other decay processes including proton decay. We also hope the definition presented in this paper can be useful for quantifying the role played by \( \alpha \) clustering in heavy nuclei, which may be expected to exhibit a strong correlation to the slope of the nuclear symmetry energy and the underlying nuclear equation of state \([31]\). We presume that the present definition will greatly enhance the understanding of \( \alpha \) correlations as a probe to nuclear interaction.

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