ON THE EXISTENCE OF FOUR OR MORE CURVED FOLDINGS WITH COMMON CREASES AND CREASE PATTERNS

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Abstract. Consider a curve $\Gamma$ in a domain $D$ in the plane $\mathbb{R}^2$. Thinking of $D$ as a piece of paper, one can make a curved folding in the Euclidean space $\mathbb{R}^3$. This can be expressed as the image of an ‘origami map’ $\varphi : D \to \mathbb{R}^3$ such that $\Gamma$ is the singular set of $\varphi$, the word ‘origami’ coming from the Japanese term for paper folding. We call the singular set image $C := \varphi(\Gamma)$ the crease of $\varphi$ and the singular set $\Gamma$ the crease pattern of $\varphi$. We are interested in the number of origami maps whose creases and crease patterns are $C$ and $\Gamma$ respectively. Two such possibilities have been known. In the previous authors’ work, two other new possibilities and an explicit example with four such non-congruent distinct curved foldings were established. In this paper, we show that the case of four mutually non-congruent curved foldings with the same crease and crease pattern occurs if and only if $\Gamma$ and $C$ do not admit any symmetries. Moreover, when $C$ is a closed curve, we show that there are infinitely many distinct possibilities for curved foldings with the same crease and crease pattern, in general.

Figure 1. A given crease pattern $\Gamma$ and its realization (right) as a curved folding along the crease $C$.

Introduction

The geometry of curved foldings is, nowadays, an important subject not only from the viewpoint of mathematics but also from the viewpoint of engineering. Work on this topic have been published by numerous authors; for example, [1, 7, 9, 12] and [2, Lecture 15] are fundamental references.

Drawing a curve $\Gamma$ in $\mathbb{R}^2$, we think of a tubular neighborhood of $\Gamma$ as a piece of paper. Then we can fold this along $\Gamma$, and obtain a paper folding in the Euclidean space $\mathbb{R}^3$, which is a developable surface whose singular set image is a space curve $C (\subset \mathbb{R}^3)$. We call this surface a curved folding along $C$. The initially given curve $\Gamma$ in the plane is called the crease pattern and $C$ is called the crease of the given
curved folding (cf. Figure 1). In this paper, we focus on curved foldings which are produced from a single curve $\Gamma$ in $\mathbb{R}^2$ satisfying the following properties:

(i) The length of $\Gamma$ is finite and is equal to that of $\mathcal{C}$ ([1, Page 29 (5)]).
(ii) The curvature function of $\mathcal{C}$ is positive everywhere and $\Gamma$ has no inflection points ([1, Page 29 (3)]).
(iii) The maximum of the absolute value of the curvature function of $\Gamma$ is less than the minimum of the curvature function of $\mathcal{C}$ ([1, Page 28 (1)]).
(iv) The curve $\mathcal{C}$ (resp. $\Gamma$) is embedded in $\mathbb{R}^3$ (resp. $\mathbb{R}^2$).

If such a pair $(\mathcal{C}, \Gamma)$ is given, there are two possibilities for corresponding curved foldings (see [1]). Moreover, in the authors’ previous work [6], two additional possibilities were found, and an explicit example of four non-congruent curved foldings with the same crease and crease pattern was given. The purpose of this paper is to further develop the discussions in [6]: We let $\mathcal{P}(\mathcal{C}, \Gamma)$ be the set of curved foldings whose creases and crease patterns are $\mathcal{C}$ and $\Gamma$ satisfying (i)–(iv), respectively.

**Definition 0.1.** A subset $A$ of $\mathbb{R}^n$ ($n = 2, 3$) is said to have a *symmetry* if $T(A) = A$ holds for an isometry $T$ of $\mathbb{R}^n$ which is not the identity map, and $T$ is called a *symmetry* of $A$. Moreover, if there is a point $x \in A$ such that $T(x) \neq x$, then $A$ is said to have a *non-trivial symmetry* $T$. On the other hand, a symmetry $T$ of $\mathcal{A}$ is called positive (resp. negative) if it is an orientation preserving (resp. reversing) isometry of $\mathbb{R}^n$.

A curved folding $Q \in \mathcal{P}(\mathcal{C}, \Gamma)$ is called a C-*isomer* of a given curved folding $P \in \mathcal{P}(\mathcal{C}, \Gamma)$ if $P$ does not coincide with $Q$ as a subset of $\mathbb{R}^3$. Using these terminologies, one of our main results is stated as follows, which is a refinement of [6] Theorems A and B:

**Theorem A.** Let $\mathcal{C}$ be a non-closed space curve. Then the following assertions hold:

(a) If $\Gamma$ admits no symmetries (resp. a symmetry), then $P \in \mathcal{P}(\mathcal{C}, \Gamma)$ has three C-*isomers* (resp. one C-*isomer*), that is, $\mathcal{P}(\mathcal{C}, \Gamma)$ consists of four (resp. two) elements.

(b) Denoting by $\langle \mathcal{P}(\mathcal{C}, \Gamma) \rangle$ the set of congruence classes of the four elements in $\mathcal{P}(\mathcal{C}, \Gamma)$, the cardinality $\# \langle \mathcal{P}(\mathcal{C}, \Gamma) \rangle$ of the set $\langle \mathcal{P}(\mathcal{C}, \Gamma) \rangle$ is four only when both $\mathcal{C}$ and $\Gamma$ have no symmetries. Otherwise, $\# \langle \mathcal{P}(\mathcal{C}, \Gamma) \rangle \leq 2$ holds.

(c) $\# \langle \mathcal{P}(\mathcal{C}, \Gamma) \rangle = 1$ holds if and only if either

- $\mathcal{C}$ is planar and has a non-trivial symmetry, or
- $\mathcal{C}$ admits a non-trivial positive symmetry (cf. Definition 0.2) and $\Gamma$ has a symmetry.

In particular, if $\mathcal{C}$ is planar, then $\# \langle \mathcal{P}(\mathcal{C}, \Gamma) \rangle \leq 2$. Moreover, $\# \langle \mathcal{P}(\mathcal{C}, \Gamma) \rangle = 1$ holds if and only if this planar $\mathcal{C}$ has a non-trivial symmetry.

As mentioned in above, the authors [6] found an example of $(\mathcal{C}, \Gamma)$ satisfying $\# \langle \mathcal{P}(\mathcal{C}, \Gamma) \rangle = 4$. This fact was proved by computing the mean curvature functions. However, to prove Theorem A, the computation of mean curvature functions is not sufficient, since there is an example satisfying $\# \langle \mathcal{P}(\mathcal{C}, \Gamma) \rangle = 4$ but the corresponding four mean curvature functions are not mutually distinct (see Example 2.14). So to prove the theorem, we need a different approach (see Section 2).

We next consider the case that $\mathcal{C}$ is a closed curve of length $a$ in $\mathbb{R}^3$ that is a crease of a curved folding, and $\Gamma$ is a curve of length $a$ in $\mathbb{R}^2$ as a crease pattern. Since $\mathcal{C}$ is closed (even when $\mathcal{C}$ is closed, $\Gamma$ may not be a closed curve in general), there exists a smooth arc-length parametrization $\gamma : \mathbb{R} \to \mathbb{R}^2$ of $\Gamma$ such that

- $\gamma(t + a) = \Phi \circ \gamma(t)$ for $t \in \mathbb{R}$ for an isometry $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$,
- $\Gamma = \gamma([-a/2, a/2])$ has no self-intersections.
In this situation, $\Gamma$ is said to have a hidden symmetry if there exists another isometry $\Psi$ on $\mathbb{R}^2$ such that $\Psi \circ \gamma(R) = \gamma(R)$ and $\Psi \neq \Phi^n$ for any integer $n$. If $\Gamma$ is closed, that is, $\gamma(t + a) = \gamma(t)$ ($t \in R$), then each hidden symmetry is an actual symmetry of $\Gamma$.

![Figure 2. The shape of $\Gamma_1$ (left) and $\Gamma_2$ (right)](image)

For example, we consider a trochoid given by

$$\gamma(t) := \left(2t/3 - \sin t, 1 - \cos t\right) (t \in \mathbb{R}).$$

Two subarcs $\Gamma_1 = \gamma([0, 2\pi])$ and $\Gamma_2 = \gamma([\pi, 3\pi])$ can be considered as fundamental pieces of the periodic curve $\gamma$ (cf. Figure 2). The arc $\Gamma_1$ admits a symmetry with respect to the dotted vertical line as in the left of Figure 2, which is one of the hidden symmetries of $\Gamma_1$. Moreover, the reflectional symmetry of $\Gamma_2$ with respect to the dotted vertical line as in the right of Figure 2 also gives another hidden symmetry of $\Gamma_1$. The following is our second main result.

**Theorem B.** If $C$ is a closed curve embedded in $\mathbb{R}^3$, then there exist four continuous families $\{P_{x}^i\}_{x \in C}$ ($i = 1, 2, 3, 4$) of curved foldings satisfying the following properties:

(a) The set $\mathcal{P}(C, \Gamma)$ coincides with $\bigcup_{x \in C} \{P_{x}^1, P_{x}^2, P_{x}^3, P_{x}^4\}$.

(b) Suppose that $C$ is not a circle and $\Gamma$ is not a part of a circle. Then, for each $P_{x}^i$ ($i \in \{1, 2, 3, 4\}$, $x \in C$), the set

$$\Lambda_{x}^i := \{Q \in \mathcal{P}(C, \Gamma); Q \text{ is congruent to } P_{x}^i\}$$

is finite. In particular, each $Q \in \mathcal{P}(C, \Gamma)$ has uncountably many $C$-isomers, which are not congruent to each other.

(c) If $C$ has no symmetries, and $\Gamma$ has no hidden symmetry, then for each $i \in \{1, 2, 3, 4\}$ and $x \in C$, the set $\Lambda_{x}^i$ consists of a single element, that is, any two curved foldings in $\mathcal{P}(C, \Gamma)$ are mutually non-congruent.

The most interesting case is that $C$ and $\Gamma$ are both closed: At the end of Section 3, we concretely give such an example of a $\mathcal{P}(C, \Gamma)$ containing uncountably many congruence classes. The existence of $C$-isomers of curved foldings is analogous to that for cuspidal edge singularities in $\mathbb{R}^3$, as mentioned in the introduction of [11]. Theorems A and B can be considered as the analogues of [11, Theorem IV] and [5, Theorem 1.4], respectively. In the final section (Section 4) of this paper, we explain how the two subjects are related to each other.

1. Developable strips and associated origami maps

In this section, we briefly review several fundamental facts on developable surfaces and curved foldings.
Developable strips. We fix a positive number $a$ and set $I_a := [-a/2, a/2]$. We let $c(t)$ $(t \in I_a)$ be a $C^\infty$-differentiable curve embedded in the Euclidean space $\mathbb{R}^3$ parametrized by arc-length, and denote by $\kappa(t) := |c''(t)|$ its curvature function, where $c'' := d^2c/dt^2$. We denote by $C$ the image $c(I_a)$. We assume that $C$ is a non-closed space curve (the case that $C$ is closed is discussed in Section 3). Take a $C^\infty$-vector field $\xi(t)$ along $c(t)$ satisfying

\begin{equation}
\xi(t) \cdot c'(t) \neq 0 \quad (0 := (0, 0, 0)),
\end{equation}

where $\times$ is the vector product of $\mathbb{R}^3$. Consider a ruled surface (cf. Appendix A)

\begin{equation}
f(t, v) := c(t) + v\xi(t) \quad (t \in I_a, |v| < \varepsilon)
\end{equation}

along $C$ with a unit ruling vector field $\xi$ satisfying (1.1), which is an embedding when $\varepsilon(> 0)$ is sufficiently small. Throughout this paper, we fix such a number $\varepsilon$. Then there exist smooth functions $\alpha$ and $\beta$ defined on $I_a$ such that

\begin{equation}
\xi(t) := \cos \beta(t)c(t) + \sin \beta(t)(\cos \alpha(t)n(t) + \sin \alpha(t)b(t)),
\end{equation}

where $c(t)$, $n(t)$ and $b(t)$ are the unit velocity vector, the unit principal normal vector, and the unit bi-normal vector of $c(t)$, respectively. We call $\xi$, written as in the form (1.3), a unit ruling vector field of $f$ (cf. Appendix A).

Remark 1.1. In [6], instead of the unit ruling vector field, we used the normalized vector field

\[ \tilde{\xi}(t) := \cot \beta(t)c(t) + \cos \alpha(t)n(t) + \sin \alpha(t)b(t). \]

The images of two ruled strips $c(t) + v\xi(t)$ and $c(t) + v\tilde{\xi}(t)$ determine the same map germs along $C$ in the sense of Definition [A.1] in Appendix A. In this paper, we prefer $\xi(t)$ rather than $\tilde{\xi}(t)$, since we can then apply Proposition [A.2].

We assume

\begin{align}
0 &< |\alpha(t)| < \pi \quad (t \in I_a), \\
0 &< \beta(t) < \pi \quad (t \in I_a),
\end{align}

and $f$ is a developable strip, that is, the Gaussian curvature of $f$ vanishes identically, which is equivalent to the condition (cf. [H])

\begin{equation}
cot \beta(t) = \frac{\alpha'(t) + \tau(t)}{\kappa(t) \sin \alpha(t)},
\end{equation}

where $\alpha' = dc/dt$ and $\tau(t) := n'(t) \cdot b(t)$ is the torsion function of $c(t)$. It should be remarked that the sign of the torsion function in [H] is opposite to that in this paper. In this setting, if the function $\alpha$ satisfies

\begin{equation}
0 < |\alpha(t)| < \frac{\pi}{2} \quad (t \in I_a),
\end{equation}

then $f$ is called an admissible developable strip. Moreover, we call $\alpha(t)$ $(t \in I_a)$ the first angular function and $\beta(t)$ the second angular function. The second angular function $\beta$ plays an important role since it gives a ruling direction along a given crease pattern of a curved folding associated with $f$ (see Figure [I]). The following lemma is a consequence of the expression (1.6).

Lemma 1.2. If $f$ is admissible (i.e. $\alpha$ satisfies (1.6)), then

\begin{equation}
\xi(t) \cdot n(t) > 0 \quad (t \in I_a).
\end{equation}

We denote by $\mathcal{D}(C)$ (resp. $\tilde{\mathcal{A}}(C)$) the set of germs of developable strips (resp. admissible developable strips) written in the form (1.2) such that $\xi(t)$ is expressed as in the form (1.3) and the first angular function satisfies (1.6) (resp. (1.6')). By definition, $\tilde{\mathcal{A}}(C)$ is a proper subset of $\mathcal{D}(C)$. We remark that the developable strip $g(t, v) := f(t, -v)$ associated to $f \in \tilde{\mathcal{A}}(C)$ does not belong to $\tilde{\mathcal{A}}(C)$, but is an
element of $\mathcal{D}(C)$. For $f \in \mathcal{D}(C)$, one can observe that the restriction $H(t)$ of the mean curvature function of $f$ to the curve $c(t)$ satisfies (cf. [3])

$$
\pm H = \frac{\kappa \sin \alpha \cosec^2 \beta}{2} = \frac{\kappa^2 \sin^2 \alpha + (\alpha' + \tau)^2}{2 \kappa \sin \alpha}.
$$

By (1.3), the right-hand side never vanishes. In particular, $f$ has no umbilics and the ruling direction $\xi(t) := f_c(t,0)$ along $C$ points in the (uniquely determined) asymptotic direction of $f$. We let

$$
(1.10) \quad C_{\pi/2}(I_a) \quad (\text{resp. } C_{\pi}^\infty(I_a))
$$

be the set of $C_{\infty}$-functions which map $I_a$ to

$$
(-\pi/2,0) \cup (0,\pi/2), \quad (\text{resp. } (-\pi,0) \cup (0,\pi)).
$$

By the above discussions, each function $\alpha \in C_{\pi/2}^\infty(I_a)$ induces a germ of an admissible developable strip, which we shall denote by $f^\alpha$. The next assertion follows immediately from (1.3), (1.4) and (1.5):

**Proposition 1.3.** The map defined by $S: C_{\pi/2}^\infty(I_a) \ni \alpha \mapsto f^\alpha \in \mathcal{D}(C)$ is bijective.

In this situation, we define the first involution $\mathcal{I}_1: \mathcal{D}(C) \ni f \mapsto \hat{f} \in \mathcal{D}(C)$ by

$$
(1.11) \quad \hat{f} = \mathcal{I}_1(f) := f^{-\alpha}.
$$

We call $\hat{f}$ the dual of $f$ (cf. [3]). By (1.6), the second angular function $\hat{\beta}(t)$ of $\hat{f}$ satisfies

$$
(1.12) \quad \cot \hat{\beta}(t) = \frac{\alpha'(t) - \tau(t)}{\kappa(t) \sin \alpha(t)}.
$$

Note that $C$ lies in a plane if and only if $\tau(t)$ vanishes identically. This is equivalent to the condition $\beta = \hat{\beta}$.

We set

$$
(1.13) \quad \Omega_+ := I_a \times (0,\varepsilon), \quad \Omega_- := I_a \times (-\varepsilon,0), \quad \Omega := I_a \times (-\varepsilon,\varepsilon),
$$

and show the following:

**Lemma 1.4.** For $f \in \mathcal{D}(C)$, the following assertions hold:

1. $f(\Omega_+) \cap f(\Omega_-)$ is empty.
2. letting $T$ be an isometry of $\mathbb{R}^3$ satisfying $T(\mathcal{C}) = C$, then for each $\sigma_i \in \{+, -\}$ $(i = 1, 2)$, the surface $h(t,v) := T \circ f(\sigma_1 t, \sigma_2 v)$ also belongs to $\mathcal{D}(C)$, and
3. $f(\Omega_+) = g(\Omega_\sigma)$ for some $g \in \mathcal{D}(C)$ and $\sigma \in \{+, -\}$ implies $f(\Omega) = g(\Omega)$.

**Proof.** Since $\varepsilon$ was chosen so that $f$ is embedded on $\Omega$, (1) is obvious. Since $t$ (resp. $v$) is an arc-length parameter of $C$ (resp. a parameter of each asymptotic line of $f$), it also is for $h(t,v)$. Thus (2) holds. Since $f$ and $g$ are ruled strips with unit ruling vector fields, (3) follows (cf. Proposition 1.2). \qed

**Lemma 1.5.** Let $f,g \in \tilde{\mathcal{A}}(C)$ be two admissible developable strips along $C$ (here the case $f = g$ is included as a special case). Then $f(\Omega_+)$ (resp. $f(\Omega_-)$) is disjoint from $g(\Omega_-)$ (resp. $g(\Omega_+)$).

**Proof.** We denote by $\xi_f$ and $\xi_g$ the unit ruling vector fields of $f$ and $g$, respectively. Since $f,g \in \tilde{\mathcal{A}}(C)$, $\xi_f \cdot \mathbf{n}$ and $\xi_g \cdot \mathbf{n}$ are positive, by [1,3]. So $\xi_f(t) \neq -\xi_g(t)$ for each $t \in I_a$. Thus, we obtain the conclusion. \qed

**Proposition 1.6.** $f(\Omega_+) \cap f(\Omega)$ is empty for $f \in \tilde{\mathcal{A}}(C)$. 

Proof. By Lemma 1.5, \( f(\Omega_+) \cap \tilde{f}(\Omega_-) \) is empty. On the other hand, the unit ruling vector field \( \xi \) associated with \( f \) satisfies
\[
\frac{-\xi \cdot b}{\sin \beta} = \frac{\xi \cdot b}{\sin \beta} = \sin \alpha (\neq 0).
\]
So we can conclude \( f(\Omega_+) \cap \tilde{f}(\Omega_+) \) is also empty. \( \square \)

**Lemma 1.7.** Let \( f, g \in D(C) \) be two developable strips along \( C \). Suppose that the image of \( f \) coincides with that of \( g \) and \( f(t, 0) = g(t, 0) \) for \( t \in I_a \). Then there exists \( \sigma \in \{+, -\} \) such that \( f(t, v) \) coincides with \( g(t, \sigma v) \) as a map germ. Moreover, if \( f, g \in A(C) \), then \( f \) coincides with \( g \).

Proof. Since \( f(t, 0) = g(t, 0) \) for \( t \in I_a \), the fact that the image of \( f \) coincides with that of \( g \) implies the unit ruling vector fields of \( f \) and \( g \) coincide along \( C \) up to a \( \pm \)-multiplication. So \( f(t, v) = g(t, \sigma v) \) holds for some \( \sigma \in \{+, -\} \) (cf. Proposition 1.3). If \( f, g \in \tilde{A}(C) \), then \( \sigma = + \) (cf. Lemma 1.5). \( \square \)

**Proposition 1.8.** Let \( f, g \in D(C) \) be two developable strips along \( C \). Suppose that \( f(\Omega_+) \) is congruent to \( g(\Omega_+) \) for some \( \sigma \in \{+, -\} \). Then there exist an isometry \( T \) of \( R^3 \) and a sign \( \sigma' \in \{+, -\} \) such that \( T(C) = C \) and \( T \circ f(t, v) = g(\sigma' t, \sigma v) \) hold for \( t \in I_a \) and \( |v| < \varepsilon \).

Proof. Since \( f(\Omega_+) \) is congruent to \( g(\Omega_+) \), there exists an isometry \( T \) of \( R^3 \) such that \( T \circ f(\Omega_+) = g(\Omega_+) \). By Lemma 1.3, \( T \circ f(\Omega) = g(\Omega) \) holds. Proposition 1.2 yields \( T \circ f(I_a \times (-\delta, \delta)) = g(I_a \times (-\delta, \delta)) \) for sufficiently small \( \delta \in (0, \varepsilon] \). In particular, \( T \) fixes \( C \), and so we may assume that \( T \circ f(t, 0) = g(\sigma' t, 0) \) holds for a sign \( \sigma' \in \{+, -\} \). Then, Lemma 1.7 implies \( T \circ f(\sigma' t, v) = g(t, \sigma v) \) for a sign \( \sigma_2 \in \{+, -\} \). Thus \( g(\Omega_+) = T \circ f(\Omega_+) = g(\Omega_{\sigma_2}) \), and so \( \sigma = \sigma_2 \) is obtained. \( \square \)

**Admissible developable strips and origami maps.** According to [1], we define origami maps induced by admissible developable strips as follows:

**Definition 1.9.** For each \( f \in \tilde{A}(C) \), we define a pair of folded developable surfaces \( \varphi_f \) and \( \psi_f \) by
\[
\varphi_f(t, v) := \begin{cases} f(t, v) & (v \geq 0), \\ \tilde{f}(t, v) & (v < 0), \end{cases} \quad \psi_f(t, v) := \begin{cases} f(t, v) & (v \geq 0), \\ \tilde{f}(t, v) & (v < 0), \end{cases}
\]
where \( \tilde{f} \) is the dual of \( f \). We call \( \varphi_f \) and \( \psi_f \) origami maps associated with \( f \). Here, \( \psi_f \) (resp. \( \varphi_f \)) is called the adjacent origami map with respect to \( \varphi_f \) (resp. \( \psi_f \)). Moreover, \( C \) is called the crease of \( \varphi_f \) (resp. \( \psi_f \)).

Figure 4 indicates the second angular functions \( \beta \) and \( \tilde{\beta} \) of \( f \) and \( \tilde{f} \) along a crease and the associated crease pattern. By definition, it holds that \( \varphi_f = \psi_f \) (resp. \( \psi_f = \varphi_f \)). We set \( \tilde{O}(C) := \{ \varphi_f \; f \in \tilde{A}(C) \} \), which gives the set of origami maps associated with admissible developable strips. Obviously, there exists a canonical involution
\[
\tilde{O}(C) \ni \varphi_f \mapsto \psi_f \in \tilde{O}(C).
\]
As in [1], the intersection of the images of \( \varphi_f \) and \( \psi_f \) is \( C \), and the union of the two images coincides with the union of the images of \( f \) and \( \tilde{f} \). We now fix \( f \in \tilde{A}(C) \) arbitrarily, and write \( f = f^\alpha (\alpha \in C_{3/2}^\infty(I_a)) \). Since
\[
N(t) := \cos \alpha(t)n(t) + \sin \alpha(t)b(t)
\]
gives a unit co-normal vector of \( f \) (cf. [4]), the geodesic curvature function \( \mu_f(t) := f_{tt}(t, 0) \cdot N(t) \) is computed as
\[
\mu_f(t) := \kappa(t) \cos \alpha(t)(> 0),
\]
(1.14)
where \( f_{tt} := \frac{\partial^2 f}{\partial t^2} \). By definition, we have

\begin{equation}
\mu_f(t) = \mu_f(t).
\end{equation}

Since \( f \) is developable, the image of \( f \) can be developed to a plane, and \( C \) corresponds to a plane curve \( \gamma_f(t) \ (t \in I_a) \) whose curvature function is \( \mu_f(t) \). To fix such a curve \( \gamma_f(t) \) uniquely, we set

\begin{equation}
\gamma_f(t) := \int_0^t (\cos \theta(u), \sin \theta(u))du, \quad \theta(t) := \int_0^t \mu_f(u)du.
\end{equation}

Then \( t \) gives an arc-length parameter of \( \gamma_f(t) \). We denote by \( \Gamma_f \) its image and call it the generator of \( f \) or the crease pattern of the origami map \( \varphi_f \). The two images of \( \varphi_f \) and \( \psi_f \) are the two curved foldings with the same crease and crease pattern (see Fact 1.13 and [6, Theorem 3.1] for the reason why \( \varphi_f \) and \( \psi_f \) can be constructed from a piece of paper). We let \( \langle \Gamma_f \rangle \) denote the congruence class of \( \Gamma_f \). Then, the map

\[ J_C : \tilde{A}(C) \ni f \mapsto \langle \Gamma_f \rangle \in E(I_a, \mathbb{R}^2) \]

is induced, where \( E(I_a, \mathbb{R}^2) \) is the set of the congruence classes of the image of regular curves \( \gamma : I_a \to \mathbb{R}^2 \) parametrized by arc-length without inflections. To make a curved folding from a piece of paper, we need to assume that the crease pattern \( \Gamma_f \) has no self-intersections. So it is natural to consider the following subclass \( A(C) \):

**Remark 1.10.** We set

\begin{equation}
\beta_L := \beta, \quad \beta_R := \pi - \beta,
\end{equation}

where \( \beta \) and \( \dot{\beta} \) are the second angular functions of \( f \) and \( \dot{f} \), respectively. Then \( \beta_L \) (resp. \( \beta_R \)) gives the left-ward (resp. right-ward) angular function of the ruling direction from the tangential direction \( \gamma'(t) \) in the plane containing the crease pattern of \( \varphi_f \), see Figure 3.

**Figure 3.** Angular functions \( \beta_L \) and \( \beta_R \) on the crease pattern of \( \varphi_f \).

**Definition 1.11.** An admissible developable strip \( f \in \tilde{A}(C) \) belongs to the subclass \( A(C) \) if the generator \( \Gamma_f \) has no self-intersections.

Since \( \dot{f}(= \dot{I}_1(f)) \) has the same generator as \( f \), we have

\begin{equation}
\dot{I}_1(A(C)) \subset A(C).
\end{equation}

The following example shows that \( A(C) \) is generally a proper subset of \( \tilde{A}(C) \):
Example 1.12. We fix a positive number $a > 0$. Let $C$ be a helix with $\kappa = \tau = 1/2$, parametrized as 

$$c(t) = \left( \cos \left( \frac{t}{\sqrt{2}} \right), \sin \left( \frac{t}{\sqrt{2}} \right), \frac{t}{\sqrt{2}} \right), \quad \left( |t| \leq \frac{a}{2} \right),$$

and set $\alpha(t) = \pi/4$. The second angular function $\beta$ is given by $\beta = \cot^{-1}(\sqrt{2})$. Figure 4 shows the images of $f = f^\alpha$ and $\tilde{f}$ for $t \in [-\pi, \pi]$ (i.e. $a := 2\pi$). Since the curvature function of the corresponding plane curve $\Gamma$ is constant, it is a part of a circle of radius $2\sqrt{2}$. If $a > 4\sqrt{2}\pi$, then the curve $\gamma_f(t)$ is not injective on $I_\alpha$, although $t \mapsto f(t, 0)$ is injective. So this shows that $f \in \mathring{\mathcal{A}}(C) \setminus \mathcal{A}(C)$ for $a > 4\sqrt{2}\pi$.

![Figure 4](image_url)
Proposition 1.16. and together with (1.18), we have
\( (1.26) \)

By (1.14) and (1.20), we have that
\( \Gamma \)

\( \mu_f(-t) = \kappa(t) \cos \alpha_*(t), \)
\( \alpha(t)\alpha_*(t) > 0. \)

Thus, we obtain a new admissible developable strip
\( f_* := f^{\alpha_*} \in \mathcal{A}_*(C). \)

We call it the second involution on \( \mathcal{A}_*(C) \). By definition,
\( \mathcal{I}_2 : \mathcal{A}_*(C) \ni f \mapsto f_* \in \mathcal{A}_*(C). \)

We call it the second involution on \( \mathcal{A}_*(C) \). By definition,
\( \mathcal{I}_i \circ \mathcal{I}_2 = \mathcal{I}_2 \circ \mathcal{I}_i, \)
\( \mu_{f_*}(t) = \mu_f(-t), \)
\( \mu_{\tilde{f}_*}(t) = \mu_f(-t), \)
and together with (1.14), the first and second involutions \( \mathcal{I}_i (i = 1, 2) \) satisfy
\( \mathcal{J}_C \circ \mathcal{I}_i = \mathcal{J}_C \) \( (i = 1, 2). \)

Proposition 1.16. The map \( f \in \mathcal{A}_*(C) \) coincides with its inverse \( f_* \) if and only if \( \Gamma_f \) has a symmetry (cf. Definition 0.1).

Proof. By (1.14) and (1.20), we have that
\( (1.26) \)

\( \frac{\mu_f(t)}{\cos \alpha(t)} = \kappa(t) = \frac{\mu_f(-t)}{\cos \alpha_*(t)}. \)

In particular, \( \cos \alpha(t) = \cos \alpha_*(t) \) holds if and only if
\( (1.27) \)

\( \mu_f(t) = \mu_f(-t). \)

Moreover, (1.21) yields \( \cos \alpha(t) = \cos \alpha_*(t) \) holds if and only if \( \alpha = \alpha_* \). In particular, (1.27) holds if and only if \( f = f_* \), by Proposition 1.15. So we obtain the assertion by applying Proposition 1.11. \( \square \)

Proposition 1.17 (6 Theorem A). \( \mathcal{J}_C^{-1}(\mathcal{J}_C(f)) = \{ f, \tilde{f}, f_*, \tilde{f}_* \} \) holds for each \( f \in \mathcal{A}_*(C) \).

Proof. We can write \( f := f^\alpha \) for \( \alpha \in C^\infty_{\pi/2}(I_0) \). Obviously, \( f, \tilde{f}, f_* \) and \( \tilde{f}_* \) belong to \( \mathcal{J}_C^{-1}(\mathcal{J}_C(f)) \). Since \( \mu_f(t) = \kappa(t) \cos \alpha(t) \), the geodesic curvature \( \mu_g(t) \) of \( \gamma_g(t) \) for \( g \in \mathcal{J}_C^{-1}(\mathcal{J}_C(f)) \) must coincide with either \( \mu_f(t) \) or \( \mu_f(-t) \). Then \( g \) coincides with \( f \) or \( \tilde{f} \) if \( \mu_g(t) = \mu_f(t) \), and \( g \) coincides with \( f_* \) or \( \tilde{f}_* \) if \( \mu_g(t) = \mu_f(-t) \) (cf. Proposition 1.13). \( \square \)

Proposition 1.18. For \( f \in \mathcal{A}_*(C) \), the following assertions hold:

(1) \( f(\Omega_+) \cap f_*(\Omega) \) is empty,
(2) \( f(\Omega_+) = f_*(\Omega) \) holds for some \( \sigma \in \{ +, - \} \) if and only if \( \sigma = + \) and \( \Gamma_f \) has a symmetry.
Proof. We may assume that \( f = f^\alpha \) for \( \alpha \in C_{x/2}(I_a) \). We prove (1): Since the first angular function of \( f^\alpha \) has the opposite sign of that of \( f^\alpha \), the ruling vector \( \xi^\alpha \) has the property that the sign of \( \xi \cdot b \) is opposite of that of \( \xi \cdot b \). Thus, we have \( \xi^\alpha \neq \xi \), and \( f(\Omega_+) \) is disjoint from \( f^\alpha(\Omega_+) \). On the other hand, by Lemma 1.4, \( f(\Omega_+) \) is disjoint from \( f^\alpha(\Omega_-) \). Consequently, \( f(\Omega_+) \cap f^\alpha(\Omega_-) \) is empty. We next prove (2). If \( \Gamma_f \) has a symmetry, then \( f = f^\alpha \) by Proposition 1.10 which implies \( f(\Omega_+) = f^\alpha_*(\Omega_+) \). Conversely, we suppose \( f(\Omega_+) = f^\alpha_*(\Omega_+) \). Then \( f(\Omega) = f^\alpha_*(\Omega) \), by Lemma 1.4. Since \( f, f^\alpha \in \mathcal{A}(C) \), \( f = f^\alpha \) holds by Lemma 1.7 and \( \Gamma_f \) has a symmetry (cf. Proposition 1.16).

\[ \Box \]

**Theorem 1.19.** Suppose that \( f \in \mathcal{A}_r(C) \). If \( \Gamma_f \) has no symmetries, then the set \( \mathcal{J}_{\mathcal{C}}^{-1}(\mathcal{J}_C(f)) \) consists of four elements, otherwise \( \mathcal{J}_{\mathcal{C}}^{-1}(\mathcal{J}_C(f)) \) consists of two elements.

Proof. We may assume that \( f = f^\alpha \) for \( \alpha \in C_{x/2}(I_a) \). If \( \Gamma_f \) has a symmetry, then \( f = f^\alpha \) by Proposition 1.10. So the images of the four maps consist of two subsets in \( \mathbb{R}^3 \). On the other hand, suppose that the number of the images is less than four. If necessary, replacing \( f \) by one of \( \{ f, f^\alpha, \tilde{f}, \tilde{f}^\alpha \} \), we may assume that the image \( f(\Omega) \) coincides with one of \( \{ f(\Omega), f^\alpha(\Omega), \tilde{f}(\Omega), \tilde{f}^\alpha(\Omega) \} \). By Propositions 1.16 and 1.18, \( f(\Omega) \) must coincide with \( f^\alpha(\Omega) \). Then, by the same argument as in the last part of the proof of Proposition 1.18, \( \Gamma_f \) has a symmetry. \( \Box \)

Since \( \mathcal{P}(C, \Gamma) \) is represented as \( \{ \varphi_f(\Omega), \varphi_f^\alpha(\Omega), \varphi_f^\alpha(\Omega), \varphi_f^\alpha(\Omega) \} \), it has the same cardinality as \( \{ f(\Omega), f(\Omega), f^\alpha(\Omega), f^\alpha(\Omega) \} \), by Proposition 1.16. So we have obtained the following:

**Corollary 1.20.** If \( \Gamma \) has no symmetries, then \( \mathcal{P}(C, \Gamma) \) consists of four elements, and otherwise \( \mathcal{P}(C, \Gamma) \) consists of two elements.

Although the tool for the proof of Corollary 1.20 is different, a similar assertion for generic real analytic cuspidal edge singularities is proved in [4] Corollary III.

2. The congruence classes of \( \mathcal{P}(C, \Gamma) \).

We give the following definition to recognize the shape of subsets of \( \mathbb{R}^n \).

**Definition 2.1.** Let \( \Sigma := \{ S_1, ..., S_r \} \) be a finite family of subsets in \( \mathbb{R}^n \), \( n = 2, 3 \). We denote by \#\( \Sigma \) the number of the elements of \( \Sigma \). We let \( \langle S_i \rangle \) be the congruence class of \( S_i \) in \( \mathbb{R}^n \). In particular, \( \langle S_i \rangle = \langle S_j \rangle \) (\( i \neq j \)) means that \( T(S_i) = S_j \) holds for a certain isometry \( T \) of \( \mathbb{R}^n \).

By definition, \( \#\langle \Sigma \rangle \leq \#\Sigma \leq r \) holds for \( \Sigma := \{ S_1, ..., S_r \} \), where \( \langle \Sigma \rangle = \{ \langle S_1 \rangle, ..., \langle S_r \rangle \} \).

We first consider the case that \( C \) has a symmetry: Let \( T \) be a symmetry (cf. Definition 0.1) of \( C \) which reverses the orientation of \( C \). We let the origin \( 0 \) be the midpoint of \( C \), by a suitable translation of \( C \), and identify \( T \) with an orthogonal matrix. Then either \( T \circ c(t) = c(t) \) or \( T \circ c(t) = c(-t) \) holds. The former case happens if and only if \( T \) is a trivial symmetry on \( C \).

**Proposition 2.2.** The space curve \( C \) admits a trivial symmetry (i.e., a symmetry fixing all points of \( C \)) if and only if \( C \) lies in a plane. In this case \( T \) is the reflection with respect to the plane.

Proof. The "if" part is obvious. So we assume the existence of a symmetry \( T \) fixing all points of \( C \). Since \( C \) is a fixed point set of \( T \), we have \( T e(t) = e(t) \) and \( T e'(t) = e'(t) \). In particular, we have \( T n(t) = n(t) \). Since \( T \) is not the identity,
Theorem 2.7. Let \( T \) be a non-trivial symmetry of \( C \) that is, cos \( \alpha \) (resp. positive) symmetry of \( C \). Suppose that \( C \) admits a symmetry \( T \) of \( C \).

Corollary 2.3. If the space curve \( C \) admits a symmetry \( T \), then \( T \) is an involution.

Proof. By the last assertion of Proposition 2.2, we may assume that \( T \) is non-trivial, namely \( T \circ c(-t) = c(t) \) holds. By differentiating this identity, the following formulas are obtained:

\[
T \mathbf{e}(-t) = \mathbf{e}(t), \quad \kappa(-t) = \kappa(t), \quad T \mathbf{n}(-t) = \mathbf{n}(t),
\]

\[-T \mathbf{b}(-t) = \sigma \mathbf{b}(t), \quad \tau(-t) = \sigma \tau(t),\]

where \( \sigma \in \{+,-\} \) is the sign of \( \det(T) \). In particular,

\[
T(\mathbf{e}(0), \mathbf{n}(0), \mathbf{b}(0)) = (-\mathbf{e}(0), \mathbf{n}(0), -\sigma \mathbf{b}(0))
\]

holds, which implies \( T \) is an involution.

Proposition 2.4. The space curve \( C \) admits a positive symmetry and a negative symmetry (cf. Definition 0.1) at the same time if and only if \( C \) lies in a plane and \( C \) has a non-trivial symmetry.

Proof. Suppose that \( C \) admits a positive symmetry \( T_1 \) and a negative symmetry \( T_2 \) at the same time. If \( T_1 \) or \( T_2 \) is trivial, \( C \) is a plane curve and \( T_2 \) must give the trivial symmetry, by Proposition 2.2. Then \( T_1 \) must be a non-trivial symmetry as a consequence. So we may assume that \( T_1 \) is non-trivial. Since \( T_1 \circ c(t) = c(-t) \) (\( i = 1, 2 \)), the torsion function \( \tau(t) \) of \( c(t) \) satisfies \((-1)^{i+1} \tau(-t) = \tau(t) \). So \( \tau \) must vanish identically.

Conversely, suppose that \( C \) lies in a plane. Then each point of \( C \) is fixed by the reflection \( T_0 \) with respect to the plane. We let \( T_1 \) be a non-trivial symmetry of \( C \). If \( T_1 \) is a positive (resp. negative) symmetry of \( C \), then \( T_2 := T_0 \circ T_1 \) is a negative (resp. positive) symmetry of \( C \). So we obtain the conclusion.

We next prepare the following:

Lemma 2.5. Suppose that \( C \) lies in a plane \( \Pi \), and let \( T_0 \) be the reflection with respect to \( \Pi \). Then \( T_0 \circ f(t, v) = f(t, v) \) holds for each \( f \in D(C) \).

Proof. We may assume that \( C \) lies in the \( xy \)-plane in \( \mathbb{R}^3 \). \( T_0 \) maps \((x, y, z) \in \mathbb{R}^3 \) to \((x, y, -z) \in \mathbb{R}^3 \). Since \( \mathbf{b} = (0, 0, 1) \) and the second angular function of \( f \) coincides with that of \( f \), the formula follows by a direct calculation.

Lemma 2.6. Suppose that \( C \) has a non-trivial symmetry \( T \). Let \( \alpha(t) \) be the first angular function of \( f \in \tilde{A}_s(C) \). Then the first angular function \( \alpha_s(t) \) of \( f_s(t) \) coincides with \( \alpha(-t) \).

Proof. We can write \( f = f^\alpha \) for \( \alpha \in C_{\pi/2}^\infty(I_n) \). Since \( C \) admits a non-trivial symmetry, we have \( \kappa(-t) = \kappa(t) \). By the definition of \( \alpha_s(t) \), we have

\[
\kappa(t) \cos \alpha(t) = \mu_f(-t) = \kappa(-t) \cos \alpha(-t) = \kappa(t) \cos \alpha(-t),
\]

that is, \( \cos \alpha_s(t) = \cos \alpha(-t) \). By (1.21), we can conclude \( \alpha_s(t) = \alpha(-t) \).

The following assertion plays an important role in the latter discussions:

Theorem 2.7. Let \( f \in \tilde{A}_s(C) \) be an admissible developable strip along \( C \). Let \( T \) be a non-trivial symmetry of \( C \). Then, the following two assertions hold:
Corollary 2.8. If \( T \) is a positive symmetry, then
\[
T \circ f(-t, v) = \tilde{f}_s(t, v).
\]
Moreover, if \( \Gamma_f \) has a symmetry, then \( T \circ f(-t, v) = \tilde{f}(t, v) \).

(2) If \( T \) is a negative symmetry, then
\[
T \circ f(-t, v) = f_s(t, v).
\]
Moreover, if \( \Gamma_f \) has a symmetry, then \( T \circ f(-t, v) = f(t, v) \).

We remark that the developable surface \( f \) given in Example 1.12 satisfies (1).

Proof. We may assume \( T \) is an orthogonal matrix, and \( f = f^\alpha \) for \( \alpha \in C_{\pi/2}^\infty(I_a) \). By (2.2), we have
\[
T \xi(t) = \cos \beta(t)Te(t) + \sin \beta(t)\left( \cos \alpha(t)Tn(t) + \sin \alpha(t)Tb(t) \right).
\]
We let \( \sigma \in \{+, -\} \) be the sign of \( \det(T) \). Since \( \alpha_+(t) = \alpha(-t) \) (cf. Lemma 2.10), we have (cf. (2.2))
\[
T \xi(-t) = \cos(\pi - \beta(-t))e(t) + \sin(\pi - \beta(-t))\left( \cos(-\sigma \alpha_+(t))n(t) + \sin(-\sigma \alpha_+(t))b(t) \right),
\]
which gives the unit ruling vector of \( T \circ f(-t, v) \) satisfying (1.8). Thus \( T \circ f(-t, v) \) belongs to \( \mathcal{A}(C) \), and its first and second angular functions are given by \( -\sigma \alpha_+(t) \) and \( \pi - \beta(-t) \), respectively. By Proposition 1.15, we have
\[
T \circ f(-t, v) = \tilde{f}_s(t, v),
\]
if \( \sigma = + \), and
\[
T \circ f(-t, v) = f_s(t, v),
\]
if \( \sigma = - \). Then we suppose \( \Gamma_f \) has a symmetry. By Proposition 1.16, \( \mu_f(t) = \mu_f(-t) \) holds. Then by Proposition 1.16, we have \( f = f_s \). Thus,
\[
T \circ f(-t, v) = \tilde{f}(t, v) \quad \text{(resp. } T \circ f(-t, v) = f(t, v))
\]
holds if \( \sigma = + \) (resp. \( \sigma = - \)).

Corollary 2.8. If \( C \) has a non-trivial symmetry, then \( \{\langle f(\Omega), \langle \tilde{f}(\Omega) \rangle \} \) coincides with \( \{\langle f_s(\Omega), \langle \tilde{f}_s(\Omega) \rangle \} \} \).

Proof. If \( T \) is positive (resp. negative), then \( T \circ f(-t, v) = \tilde{f}_s(t, v) \) (resp. \( T \circ f(-t, v) = f_s(t, v) \)) and \( T \circ f(-t, v) = f_s(t, v) \) (resp. \( T \circ f(-t, v) = \tilde{f}_s(t, v) \)).

Proposition 2.9. If \( f \in \mathcal{A}_s(C) \), then \( f(\Omega_+) \) is not congruent to \( f(\Omega_-) \).

Proof. We may write \( f = f^\alpha \) for \( \alpha \in C_{\pi/2}^\infty(I_a) \). Suppose that \( f(\Omega_+) \) is congruent to \( f(\Omega_-) \). There exists a symmetry \( T \) of \( C \) such that \( T \circ f(\Omega_+) = f(\Omega_-) \). By Proposition 1.15, \( T \circ f(t, v) = f(\sigma t, -v) \) holds for a certain \( \sigma \in \{+, -\} \). If \( \sigma = + \), then \( C \) lies in a plane, by Proposition 1.16. In this case, Lemma 2.3 yields that
\[
f(t, -v) = T \circ f(t, v) = \tilde{f}(t, v),
\]
contradicting Proposition 1.15. So we may set \( \sigma = - \). If \( T \) is positive (resp. negative), then
\[
f(t, -v) = T \circ f(-t, v) = \tilde{f}_s(t, v) \quad \text{(resp. } f(t, -v) = T \circ f(-t, v) = f_s(t, v))
\]
holds by Theorem 2.4, which contradicts 1.8. □
Remark 2.10. We can give an alternative proof of the above proposition as follows: Since $f$ is a developable surface, it can be developed to a plane $\mathbb{R}^2$. Since the geodesic curvature of $\gamma_f(\subset \mathbb{R}^3)$ is positive, $f(\Omega_+)$ (resp. $f(\Omega_-)$) corresponds to the left-hand side (resp. right-hand side) of $\gamma_f$. So it is a convex (concave) domain and so $f(\Omega_+)$ is not congruent to $f(\Omega_-)$.

The following assertion is a refinement of [3, Lemma 3.2]:

**Proposition 2.11.** If $f \in A_\ast(C)$, then $f(\Omega_\sigma)$ is congruent to $\tilde{f}(\Omega_\sigma)$ for $\sigma \in \{+,-\}$ if and only if $\sigma = +$ and

1. $C$ lies in a plane, or
2. $C$ has a positive non-trivial symmetry and $\Gamma_f$ also has a symmetry.

**Proof.** Applying Theorem 2.7, it can be easily checked that (1) or (2) implies $f(\Omega_+)$ is congruent to $\tilde{f}(\Omega_\sigma)$. So it is sufficient to show the ‘only if’ part. Suppose that $f(\Omega_\sigma)$ is congruent to $\tilde{f}(\Omega_\sigma)$, that is, there exists an isometry $\tilde{T}$ on $\mathbb{R}^3$ such that $\tilde{T} \circ f(\Omega_\sigma) = f(\Omega_\sigma)$. To get the conclusion, we may assume that $C$ is not planar. In this case $T$ is non-trivial, that is, it reverses the orientation of $C$ (cf. Proposition 2.2). By Proposition 1.8, we have

\begin{equation}
T \circ \tilde{f}(t,v) = f(-t,\sigma v).
\end{equation}

By (1.13), it holds that $\mu_f(t) = \mu_f(-t)$, which implies $\Gamma_f$ has a symmetry (cf. Lemma 1.4). We first consider the case that $T$ is a negative symmetry. Then (2) of Theorem 2.7 and (2.3) yield

\begin{equation}
\tilde{f}(t,v) = T \circ f(-t,\sigma v) = f(t,\sigma v),
\end{equation}

which is impossible, by Proposition 1.4. Thus $T$ must be a positive symmetry. In this case, (1) of Theorem 2.7 and (2.3) yield

\begin{equation}
\tilde{f}(t,v) = T \circ f(-t,\sigma v) = \tilde{f}(t,\sigma v).
\end{equation}

So we can conclude $\sigma = +$. \qed

**Corollary 2.12.** Let $f,g \in A_\ast(C)$. Then $\varphi_f(\Omega)$ is congruent to $\varphi_g(\Omega)$ if and only if $f(\Omega)$ is congruent to $g(\Omega)$.

**Proof.** By Proposition 2.11, $f(\Omega_\sigma)$ cannot be congruent to $\tilde{f}(\Omega_-)$. So $\varphi_f(\Omega)$ is congruent to $\varphi_g(\Omega)$ if and only if $f(\Omega_\sigma)$ is congruent to $g(\Omega_\sigma)$. By Proposition 2.9, $f(\Omega_\sigma)$ is congruent to $g(\Omega_\sigma)$ if and only if $f(\Omega)$ is congruent to $g(\Omega)$. So we obtain the conclusion. \qed

**Proposition 2.13.** If $f \in A_\ast(C)$, then $f(\Omega_\sigma)$ is congruent to $f_\ast(\Omega_\sigma)$ for a certain $\sigma \in \{+,-\}$ if and only if $\sigma = +$ and

1. $C$ has a non-trivial negative symmetry, or
2. $\Gamma_f$ has a symmetry.

**Proof.** It is easy to check that (1) or (2) implies $f(\Omega_\sigma)$ is congruent to $f_\ast(\Omega_\sigma)$ (cf. Theorem 2.7 and Proposition 1.10). So it is sufficient to show the ‘only if’ part. We suppose that $f(\Omega_\sigma)$ is congruent to $f_\ast(\Omega_\sigma)$ and $\Gamma_f$ has no symmetries. Then $f(\Omega_\sigma) \neq f_\ast(\Omega_\sigma)$ by Proposition 1.13. So $C$ has a non-trivial symmetry $T$ such that $T \circ f(\Omega_\sigma) = f_\ast(\Omega_\sigma)$. Since $T$ is non-trivial, it reverses the orientation of $C$ (cf. Proposition 2.2). By Proposition 1.8

\begin{equation}
T \circ f_\ast(t,v) = f(-t,\sigma v)
\end{equation}

holds. Suppose that $T$ is a positive symmetry. By (1) of Theorem 2.7 and (2.3), we have

\begin{equation}
f_\ast(t,v) = T \circ f(-t,\sigma v) = \tilde{f}_\ast(t,\sigma v),
\end{equation}

which implies $\sigma = +$.
but this never occurs, by Proposition 1.16. So we obtain (1) and $T$ is a negative symmetry. By (2) of Theorem 2.7 and (2.4), we have

$$f_\ast(t,v) = T \circ f(-t,\sigma v) = f_\ast(t,\sigma v).$$

So we obtain $\sigma = +$. \qed

Similarly, the following assertion holds:

**Proposition 2.14.** Let $f \in A_\ast(C)$. Then the image of $f(\Omega_+)$ is congruent to $\hat{f}_\ast(\Omega_\ast)$ for $\sigma \in \{+,-\}$ if and only if $\sigma = +$ and $C$ has a non-trivial positive symmetry.

**Proof.** If $C$ has a non-trivial positive symmetry, then (2) of Theorem 2.7 implies $f(\Omega_+)$ is congruent to $\hat{f}_\ast(\Omega_\ast)$. So it is sufficient to prove the ‘only if’ part. We assume that there exists an isometry $T$ such that $T \circ f(\Omega_+) = \hat{f}_\ast(\Omega_\ast)$. Since $f(\Omega_+) \neq \hat{f}_\ast(\Omega_\ast)$ (cf. Proposition 1.18), $T$ is a non-trivial symmetry of $C$. By Proposition 1.8, we have

$$(2.5) \quad T \circ \hat{f}_\ast(t,v) = f(-t,\sigma v).$$

If $T$ is a negative symmetry, then (2) of Theorem 2.7 and (2.5) yield

$$f_\ast(t,v) = T \circ f(-t,\sigma v) = f_\ast(t,\sigma v),$$

which never happens, by Proposition 2.11. So $T$ must be a positive symmetry. Then (1) of Theorem 2.7 and (2.5) yield $f_\ast(t,v) = T \circ f(-t,\sigma v) = f_\ast(t,\sigma v)$. So $\sigma = +$ holds. \qed

We set (cf. (2.1))

$$\langle J_C^{-1}(J_C(f)) \rangle := \{\text{Image of $f$, Image of $\hat{f}$, Image of $f_\ast$, Image of $\hat{f}_\ast$}\}.$$ Using the above three propositions, we can prove the following:

**Theorem 2.15.** For $f \in A_\ast(C)$, then the following assertions hold:

1. If $C$ and $\Gamma$ have no symmetries, then $\# \langle J_C^{-1}(J_C(f)) \rangle = 4$.
2. Otherwise $\# \langle J_C^{-1}(J_C(f)) \rangle \leq 2$ holds.
3. Moreover, $\# \langle J_C^{-1}(J_C(f)) \rangle = 1$ holds if and only if either
   - (3-a) $C$ is planar and $C$ has a non-trivial symmetry, or
   - (3-b) $C$ admits a positive symmetry and $\Gamma$ has a symmetry.

The corresponding results for real analytic cuspidal edge singular points are given in [1, Section 5], which are somewhat different from the case of curved folding: For example, the case that the images of three isomers of a given cuspidal edge consist of only one congruence class happens only when $C$ is planar. On the other hand, for developable surfaces, such a case occurs not only in the case that $C$ lies in a plane, but also the case that $C$ is non-planar, as stated above.

**Proof.** The first assertion (1) immediately follows from Propositions 2.9, 2.11 and 2.13. If $\Gamma$ has a symmetry, then $f = f_\ast$ and $\hat{f} = \hat{f}_\ast$, and so $\# \langle J_C^{-1}(J_C(f)) \rangle \leq 2$. On the other hand, if $C$ has a symmetry, then $\langle J_C^{-1}(J_C(f)) \rangle \leq 2$ (cf. Corollary 2.5). So (2) is obtained. We prove (3). Since the ‘if’ part follows by Propositions 2.9, 2.11 and 2.13, it is sufficient to prove the ‘only if’ part. We first observe that $\# \langle J_C^{-1}(J_C(f)) \rangle = 1$ implies $\langle \hat{f}(\Omega) \rangle = \langle f(\Omega) \rangle$. So, by Proposition 2.11 we may assume either

1. $C$ is planar, or
2. $C$ admits a positive symmetry and $\Gamma$ has a symmetry.
In the case of (ii), \( \#(J_C^{-1}(J_C(f))) = 1 \) actually holds, by Corollary 2.8. We next consider the case that \( \mathcal{C} \) is planar. The condition \( \#(J_C^{-1}(J_C(f))) = 1 \) also implies \( \langle f, (\Omega) \rangle = \langle f, (\Omega) \rangle \). Since \( f \) and \( f_* \) belong to \( \mathcal{A}_x(\mathcal{C}) \), we have \( \langle f, (\Omega_+ \rangle = \langle f, (\Omega_+ \rangle \). By Proposition 2.11, \( \Gamma \) must have a symmetry. So the assertion is obtained. \( \square \)

**Proof of Theorem A.** We let \( f \in \mathcal{A}_x(\mathcal{C}) \) be a developable strip satisfying \( \Gamma_f = \Gamma \). The first assertion (a) follows from Corollary 1.20. On the other hand, by Corollary 2.12, \( \#(P(\mathcal{C}, \Gamma)) \) is equal to \( \#(J_C^{-1}(J_C(f))) \). So (b) and (c) follow from Theorem 2.10. \( \square \)

Examples for the case (3-a) (resp. (3-b)) have been discussed at the end of Section 6 (resp. Example 1.12). Also in [6], the authors showed that the image of four \( \mathcal{C} \) surfaces \( \mathcal{C} \) and \( \hat{\mathcal{C}} \) are non-congruent in general by showing an example so that the absolute values of the surfaces' mean curvature functions are distinct. As mentioned in the introduction, this criterion using mean curvature might fail as in the following example:

**Example 2.16.** Let \( \tau(t) \) be a fixed smooth function defined on a neighborhood of \( t = 0 \). We set \( \mu(t) := 1 + t \) and \( \kappa(t) := 2 + t \). We let \( c(t) \) be a space curve with arc-length parameter whose curvature function and torsion function are \( \kappa \) and \( \tau \), respectively. We consider the developable strip \( f \) along \( c \) whose first angular function is \( \arccos(\mu/\kappa) \). Then, the geodesic curvature of \( c \) as a curve on the surface \( f \) is equal to \( \mu \). By using (1.39), the mean curvatures \( H(t) \) and \( H_*(t) \) of \( f \) and \( f_* \) are given by

\[
H = \frac{37 + t(7 + 2t)(2t^2 + 7t + 12) - 2x(2 + t)\varphi(t) + (2 + t)^2\varphi(t)^2}{2(2 + t)^2\varphi(t)^3},
\]

\[
H_* = \frac{15 + 3(t(5 + 2t)(2t^2 + 5t + 4) + 4x(2 + t)\psi(t) + \sqrt{3}(2 + t)^2\psi(t)^2}{2\sqrt{3}(2 + t)^3\psi(t)^3},
\]

where \( \varphi(t) := 3 + 2t \) and \( \psi(t) := 1 + 2t \). If we think the torsion function \( \tau(t) \) of the curve \( c(t) \) is an unknown variable and denote it by \( s \), then we have the following expression

\[
H(t) - H_*(t) = a(t)x^2 + b(t)x + c(t)
\]

such that

- if \( t = 0 \), then \( a(0)x^2 + b(0)x + c(0) = 0 \) implies \( a(0) = 0 \) and \( x = -1/(2\sqrt{3}) \),
- \( a(t)/t \) gives a smooth function which does not vanish at \( t = 0 \),
- \( D(t) := b(t)^2 - 4a(t)c(t) \) satisfies \( D(0) = b(0)^2 > 0 \) at \( t = 0 \).

So there exists a unique smooth function \( \tau(t) \) such that \( \tau(0) = -1/(2\sqrt{3}) \) and \( H - H_* \) vanishes identically. Since \( \kappa(t) \neq \kappa(-t) \) and \( \mu(t) \neq \mu(-t) \), the two curves \( c \) and \( \gamma \) have no symmetries. Thus, the congruence classes of the images of \( f, \hat{f}, f_* \) and \( \hat{f}_* \) are distinct, by Theorem A in the introduction. We denote by \( H \) and \( H_* \) the mean curvature functions of \( f \) and \( f_* \), along \( c(t) \). Then the graphs of the four functions \( H_+, |H|, H_* \) and \( |H_*| \) are indicated in Figure 5. One can observe that \( |H| < H = H_* < |H_*| \).

**Remark 2.17.** We remark that the images of \( f \) and \( \hat{f} \) (resp. \( f_* \)) meet only along the space curve \( \mathcal{C} \) (cf. Proposition 1.6 and (1) of Proposition 1.18). On the other hand, the images of \( f \) and \( f_* \) or \( f_* \) and \( f \) might have an intersection which does not lie on \( \mathcal{C} \). In fact, we consider a helix \( c(t) := (\cos t, \sin t, t) \) whose curvature and torsion are \( 1/2 \). We set \( \alpha := \pi/4 + t \) and consider the developable strip \( \hat{f} = f^\alpha \) associated to the angular function \( \alpha \) on \( I_\alpha \times (-\varepsilon, \varepsilon) \). The image of the helix \( c \) has
a positive symmetry associated with the special orthogonal matrix
\[ T := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \]

By (1) of Theorem 2.7, the image of \( T \circ f \) coincides with the image of \( \hat{f}^* \). Using this fact, one can easily observe that the image of \( f \) meets the image of \( \hat{f}^* \) at points not on the curve \( C \).

3. The case that \( C \) is closed

**Definition of origami maps.** We denote by \( S^1_a \) the 1-dimensional torus \( \mathbb{R}/a \mathbb{Z} \).

We let \( C^\infty_{\pi/2}(S^1_a) \) be the set of \( C^\infty \)-functions which map \( S^1_a \) to \((-\pi/2,0) \cup (0,\pi/2)\). In this section, we consider the case that \( C \) is the image of an embedded curve \( c : S^1_a \to \mathbb{R}^3 \), that is, \( c(t+a) = c(t) \) \( (t \in \mathbb{R}) \). Like as in the case of \( I_a \), we can define the class \( \tilde{A}(C) \) consisting of admissible developable strips along \( C \); that is, \( f \in \tilde{A}(C) \) if and only if there exists \( \alpha \in C^\infty_{\pi/2}(S^1_a) \) such that
\[ f(t,v) := c(t) + v\xi(t) \quad (t \in I_a, |v| < \varepsilon), \]
\[ \xi(t) := \cos \beta(t)e(t) + \sin \beta(t)\left( \cos \alpha(t)n(t) + \sin \alpha(t)b(t) \right), \]
where \( e \), \( n \) and \( b \) denote the unit velocity vector, the unit principal normal vector, and the unit bi-normal vector of \( c \), respectively, and \( \beta : S^1_a \to (0,\pi) \) is the smooth function satisfying (1.6). For \( f \in \tilde{A}(C) \), we call \( \alpha \) the *first angular function* and \( \beta \) the *second angular function* of \( f \). Since, the first angular function \( \alpha \) is uniquely determined by \( f \), we also use the notations \( f^\alpha \) and \( \alpha_f \) instead of \( f \) and \( \alpha \), respectively.

We then set
\[ \mu_f(t) := \kappa(t)\cos \alpha_f(t) \]
where \( \kappa \) is the curvature function of \( c \). In particular, the map
\[ J_C : \tilde{A}(C) \ni f \mapsto \mu_f \in C^\infty(S^1_a) \]
is defined, where \( C^\infty(S^1_a) \) is a smooth function on \( S^1_a \). There exists a regular curve \( \gamma_f(t) \) \( (t \in \mathbb{R}) \) in the plane \( \mathbb{R}^2 \) whose curvature function is \( \mu_f(t) \). To assign \( \gamma_f \) from \( f \) uniquely, we define \( \gamma_f \) by (1.10). Since \( \mu_f \) is periodic, there exists an isometry \( \Phi \) in \( \mathbb{R}^2 \) such that
\[ \gamma_f(t) = \Phi \circ \gamma_f(t+a). \]
We set \( f \in \mathcal{R}(S^1_a, \mathbb{R}^2) \) the set of such regular curves in \( \mathbb{R}^2 \) defined on \( S^1_a \) with arc-length parameter.

Let \( \gamma_1, \gamma_2 \in \mathcal{R}(S^1_a, \mathbb{R}^2) \). Then we say that \( \gamma_2 \) is equivalent to \( \gamma_1 \) if there exist an isometry \( \Phi \) of \( \mathbb{R}^2 \), \( b \in [0, a) \) and a sign \( \sigma \in \{+, -\} \) such that
\[
\gamma_2(t) = \Phi \circ \gamma_1(\sigma t + b) \quad (t \in \mathbb{R})
\]
holds. In this situation, we also say that \( \gamma_2 \) is \((\Phi, \sigma, b)\)-related to \( \gamma_1 \). Moreover, if \( \gamma_1 \) itself is \((\Phi, \sigma, b)\)-related to \( \gamma_1 \) and \( (\sigma, b) \neq (+, 0) \), then we say that \( \Phi \) is a non-trivial symmetry of \( \gamma_1 \). We denote by \( \mathcal{E}(S^1_a, \mathbb{R}^2) \) the quotient space of \( \mathcal{R}(S^1_a, \mathbb{R}^2) \) by this equivalence relation. We denote by \([\gamma_1]\) the equivalence class representing \( \gamma_1 \). It can be easily checked that if \([\gamma_1] = [\gamma_2]\), then \( \gamma_1 \) has a non-trivial symmetry, if and only if so does \( \gamma_2 \). We can define a map
\[
\mathcal{J}_C : \tilde{\mathcal{A}}(C) \ni f \mapsto [\gamma_f] \in \mathcal{E}(S^1_a, \mathbb{R}^2).
\]
By definition, we have \( \mu_f = \mu_f \). So the involution
\[
\mathcal{I}_1 : \tilde{\mathcal{A}}(C) \ni f \mapsto \tilde{f} \in \tilde{\mathcal{A}}(C)
\]
satisfying \( \mathcal{I}_1 \circ \mathcal{J}_C = \mathcal{I}_1 \) is defined. We call \( \tilde{f} \) the dual of \( f \).

**Definition 3.2.** We denote by \( \mathcal{A}(C) \) the subset of \( \tilde{\mathcal{A}}(C) \) consisting of developable strips such that
\[
\Gamma_f(b) := \gamma_f([b, a + b])
\]
has no self-intersections for a certain choice of \( 0 \leq b < a \). Moreover, such a \( \Gamma_f(b) \) is called a crease pattern of \( f \).

By definition, each hidden symmetry of \( \Gamma_f(b) \) corresponds to a non-trivial symmetry of \( \gamma_f \).

**Remark 3.3.** The property that \( \Gamma_f(b) \) has no self-intersections depends on the choice of \( b \). In fact, the trochoid \( \gamma \) defined by \( (L, L) \) has two fundamental pieces \( \Gamma_1 := \gamma([0, 2\pi]) \) and \( \Gamma_2 := \gamma([\pi, 3\pi]) \). As pointed out in the introduction, \( \Gamma_1 \) has no self-intersections, but \( \Gamma_2 \) does.

We next define a subclass \( \tilde{\mathcal{A}}_*(C) \) of \( \tilde{\mathcal{A}}(C) \) so that \( f \in \tilde{\mathcal{A}}_*(C) \) if and only if max \( |\mu_f(t)| \) is less than min \( \kappa(t) \). Let \( \alpha \) be the first angular function of \( f \in \tilde{\mathcal{A}}_*(C) \). Then there exists a unique \( f_* \in \tilde{\mathcal{A}}_*(C) \) (called the inverse of \( f \)) whose angular function satisfies
\[
(3.6) \quad \mu_f(-t) = \kappa(t) \cos \alpha_*(t),
(3.7) \quad \alpha(t)\alpha_*(t) > 0,
\]
which induces a map \( \mathcal{I}_2 : \tilde{\mathcal{A}}_*(C) \ni f \mapsto f_* \in \tilde{\mathcal{A}}_*(C) \), called the second involution. We set
\[
\mathcal{A}_*(C) := \tilde{\mathcal{A}}_*(C) \cap \mathcal{A}(C).
\]
By definition, \( \mathcal{I}_2(\mathcal{A}_*(C)) \subset \mathcal{A}_*(C) \) holds. Moreover, \( f, \tilde{f}, f_* \) and \( \tilde{f}_* \) also belong to \( \mathcal{A}_*(C) \) (resp. \( \mathcal{A}_*(C) \)). Moreover, we have
\[
\mathcal{J}_C(f) = \mathcal{J}_C(\tilde{f}) = \mathcal{J}_C(f_*) = \mathcal{J}_C(\tilde{f}_*).
\]
We set \( \mathcal{O}_*(C) := \{ \varphi_f ; f \in \mathcal{A}_*(C) \} \). Then it can be easily checked that
\[
\mathcal{P}(C, \Gamma) := \{ \varphi_f(\Omega) ; f \in \mathcal{A}_*(C) \text{ and } \Gamma = \Gamma_f(b) \text{ for some } b \in S^1_a \}.
\]
coincides with the set of curved foldings given in the introduction in the case of $C$ is closed.

**Proof of Theorem B.** Since the congruence class plane curve is determined by its curvature function, the following assertion can be easily proved using the fundamental theorem of curves in the Euclidean plane.

**Lemma 3.4.** Let $\gamma_1, \gamma_2 \in \mathcal{R}(S^1_a, \mathbb{R}^2)$. Then $[\gamma_1] = [\gamma_2]$ holds in $\mathcal{E}(S^1_a, \mathbb{R}^2)$ if and only if there exist $b \in S^1_a$ and signs $\sigma \in \{+,-\}$ ($i = 1, 2$) such that $\mu(t) = \mu(\sigma t + b)$ for $t \in S^1_a$.

We set
\begin{equation}
\Omega_k := S^1_a \times (0,\varepsilon), \quad \Omega := S^1_a \times (-\varepsilon,0), \quad \Omega := S^1_a \times (-\varepsilon,\varepsilon).
\end{equation}

Similarly, the following lemma can be also proved easily:

**Lemma 3.5.** Let $c_i(t)$ ($t \in S^1_a$, $i = 1, 2$) be closed curves embedded in $\mathbb{R}^3$ with arc-length parameter. If the images of the two curves are congruent, then there exists an isometry $T$ of $\mathbb{R}^3$, a sign $\sigma \in \{+,-\}$ and $b \in [0,a)$ such that
\begin{equation}
c_2(t) = T \circ c_1(\sigma t + b).
\end{equation}

**Proposition 3.6.** We let $f,g \in \tilde{A}(C)$. If $f(\Omega)$ is congruent to $g(\Omega)$, then there exist an isometry $T$ of $\mathbb{R}^3$, two signs $\sigma_1, \sigma_2 \in \{+,-\}$ and $b \in [0,a)$ such that
\begin{equation}
f(t,0) = T \circ g(\sigma t + b,0).
\end{equation}

Here, $f(t,v)$ and $g(t,v)$ are both ruled surfaces, and their normalized ruling vector fields must coincide up to a $\pm$-multiplication. So we obtain the assertion. \hfill \Box

We fix $f \in A_4(C)$. We let $\kappa(t)$ be the curvature function of $c(t)$, and $\mu_f(t)$ is the curvature function of $\gamma_f(t)$. For each $b \in [0,a)$, four functions $\alpha^i_b \in C^\infty_K(S^1_a)$ ($i = 1, 2, 3, 4$) of period $a$ are defined by
\begin{align*}
\kappa(t) \cos \alpha^1_b(t) &= \mu_f(t + b), & \kappa(t) \cos \alpha^2_b(t) &= -\mu_f(t + b), \\
\kappa(t) \sin \alpha^3_b(t) &= \mu_f(-t + b), & \kappa(t) \sin \alpha^4_b(t) &= -\mu_f(-t + b).
\end{align*}

Using this, we set
\begin{align*}
f^1_b &:= f^{2\alpha^1_b}, & f^2_b &:= f^{2\alpha^2_b} := (f^1_b)\ast, & f^3_b &:= f^{2\alpha^3_b} := (f^1_b)\ast, & f^4_b &:= f^{2\alpha^4_b} := (f^1_b)\ast.
\end{align*}

These four developable strips belong to $A_4(C)$ and $\mathcal{J}_C(f^i_b) = \mathcal{J}_C(f) \ast (\Gamma)$ holds for $i = 1, 2, 3, 4$ and $b \in [0,a)$.

**Theorem 3.7.** Let $C$ be a closed curve embedded in $\mathbb{R}^3$, and $f \in \tilde{A}_4(C)$. Then the four continuous families $\{f^i_b\}_{b \in [0,a)}$ ($i = 1, 2, 3, 4$) satisfy the following properties:

(a) The set $\mathcal{J}_C^{-1}(\mathcal{J}_C(f))$ coincides with $\bigcup_{b \in [0,a)} \{f^1_b, f^2_b, f^3_b, f^4_b\}$.

(b) Suppose that $C$ is not a circle and $\Gamma_f$ is a curve with non-constant curvature. Then, for each $f^i_b$ ($i \in \{1, 2, 3, 4\}$, $b \in [0,a)$), the set $\Lambda^i_b := \{g \in \mathcal{J}_C^{-1}(\mathcal{J}_C(f)) ; \text{ the image of } g \text{ is congruent to that of } f^i_b\}$ is finite.

(c) If $C$ has no symmetries and $\Gamma_f$ has no hidden symmetry, then the set $\Lambda^i_b$ consists of a single point, that is, any two surfaces in $\mathcal{J}_C^{-1}(\mathcal{J}_C(f))$ are non-congruent.
A similar assertion for real analytic cuspidal edge singularity is proved in [5].

We show this assertion and Theorem B at the same time. For the proof of Theorem B, we fix $P \in \mathcal{P}(C, \Gamma)$. We may assume that $P = \varphi_f(\Omega)$ for $f \in \mathcal{A}_t(C)$. 

**Proof of Theorems B and 3.7** The method to prove the several assertions for the local geometry of curved foldings given in the previous section is different from that for corresponding assertions for cuspidal edges given in [4]. However, when $C$ is a closed curve, the proof of the corresponding assertion for closed cuspidal edges as in [5] can be modified for our present situation as follows:

If there exists $g \in \mathcal{A}_v(C)$ such that $\mathcal{J}_C(g) = \mathcal{J}_C(f)$, then by definition, there exist $\sigma \in \{+,-\}$ and $b \in [0,a)$ such that the curvature function of $\gamma_g$ coincides with $\mu_f(\sigma t + b)$. So $g$ coincides with one of $\{f_b^n\}_{i=1,2,3,4}$. Thus, (a) of Theorem 3.7 is proved.

Fix $Q \in \mathcal{P}(C, \Gamma)$. There exists $g \in \mathcal{A}_v(C)$ such that $Q = \varphi_g(\Omega)$. Then, $g(\Omega_{\sigma_0})$ must coincide with one of two sheets of $Q$. In particular, we have $\mathcal{J}_C(g) = \mathcal{J}_C(f)$. Thus, by (a) of Theorem 3.7 there exist $b \in [0,a)$ and $i \in \{1,2,3,4\}$ such that $g = f_b^i$. Then $\tilde{g} = \tilde{f}_b^i$ holds and we obtain (a) of Theorem B.

We next prove (b) of Theorems 3.7 and B. If Theorem 3.7 (resp. Theorem B) fails, then there exist $f_0 := f_b^i (n \geq 0)$, a sequence $\{b_n\}_{n=1}^{\infty}$ and $\sigma(n) \in \{+,-\}$ such that $f_{\sigma(n)}(\Omega_+)$ $(n \geq 1)$ are all congruent to $f_0(\Omega_{\sigma(n)})$ (resp. $f_0(\Omega_{\sigma(n)})$ or $f_0(\Omega_{\sigma(n)})$).

Replacing $\{b_n\}_{n=1}^{\infty}$ by a suitable subsequence and replacing $f_0$ by $f_0$, we may assume that $f_{\sigma(n)}(\Omega_+)$ $(n \geq 1)$ are all congruent to $f_0(\Omega_{\sigma(n)})$ in both of the two cases. Since $f$ is chosen arbitrarily, we may also assume $f_0 = f$ without loss of generality. Since the number of possibilities for the $f_n$ are finite, by replacing $\{b_n\}_{n=1}^{\infty}$ by a suitable subsequence, we may assume that $j := j_n$ and $\sigma_0 := \sigma(n)$ do not depend on $n$. Then $f_{\sigma_0}(\Omega_+)$ $(n \geq 1)$ is congruent to $f_0(\Omega_{\sigma_0})$. In particular, $f_{\sigma_0}(\Omega)$ $(n \geq 1)$ is congruent to $f_0(\Omega)$. By Proposition 3.6 there exist an isometry $T_n$ of $R^3$, $d_n \in [0,a)$ and signs $\sigma_n, \sigma'_n \in \{+,-\}$ such that

$$f_n(t,v) = T_n \circ f_0(\sigma_n t + d_n, \sigma'_n v).$$

Since the number of possibilities for the $\sigma_n, \sigma'_n$ are finite, replacing $\{b_n\}_{n=1}^{\infty}$ by a suitable subsequence, we may assume that $\sigma = \sigma_n$ and $\sigma' = \sigma'_n$ do not depend on $n$. So we have $f_n(t,v) = T_n \circ f_0(\sigma t + d_n, \sigma' v)$ for $n \geq 1$. Substituting $v = 0$,

$$f_n(t,0) = T_n \circ f_0(\sigma t + d_n, 0)$$

holds. In particular, there exists a sign $\sigma''_n \in \{+,-\}$ such that

$$\kappa(t) = \kappa(\sigma t + d_n), \quad \tau(t) = \sigma''_n \tau(\sigma t + d_n),$$

where $\tau(t)$ is the torsion function of $c(t)$ and $\sigma''_n = +$ (resp. $\sigma''_n = -$) if det$(T_n)$ is positive (resp. negative). Since $C$ is not a circle, the number of symmetries of $C$ is finite (see Proposition B.2 in the appendix). So not only for $\sigma''_n$, the number of possibilities for $d_n$ and $T_n$ are also finite. Thus, by replacing $\{b_n\}_{n=1}^{\infty}$ with a suitable subsequence, we may assume $\sigma'' := \sigma''_n$, $T := T_n$ and $(d :=) d_n$ are all not depending on $n$. Since $f = f_0$, we have

$$f_b^i(t,v) = f_{\sigma''}(t,v) = T \circ f_0(\sigma t + d_n, v) = T \circ f(\sigma t + d_n, \sigma' v),$$

which implies $\mu_f(\sigma_0 t + b_n) = \mu_f(\sigma_0 t + b_1)$. Without loss of generality, we may assume that $\mu(0)$ is a regular value of the function $\mu_f : R/aZ \to R$, since $\mu_f$ is not constant. Substituting $t = -b_1/\sigma_0$, we have $\mu_f(-b_1 + b_n) = \mu_f(0)$. Since $S^1_a$ is compact and $\{b_n\}_{n=1}^{\infty}$ consists of distinct points, this contradicts that $\mu(0)$ is a regular value. So we obtain (b).

Finally, we prove (c) of Theorems 3.7 and B. If Theorem 3.7 (resp. Theorem B) fails, then there exists $f_0 := f_b^i (n = 0,1, b_0,b_1 \in [0,a))$ and $\sigma_0 \in \{+,-\}$ such that $f_1(\Omega_+)$ is congruent to $f_0(\Omega_{\sigma_0})$ (resp. congruent to $f_0(\Omega_{\sigma_0})$ or $f_0(\Omega_{\sigma_0})$). We
may replace \( f_0 \) by \( \hat{f}_0 \) if necessary. So we may assume that \( f_2(\Omega_\ast) \) is congruent to \( f_0(\Omega_{\ast\alpha}) \) in both of the two cases. In particular, \( f_1(\Omega) \) \((n \geq 1)\) is congruent to \( f_0(\Omega) \).

Without loss of generality, we may assume \( f = f_0 \). By Proposition \ref{prop:congruence} there exist an isometry \( T \) of \( \mathbb{R}^3 \), \( d \in [0,a] \) and signs \( \sigma, \sigma' \in \{+,-\} \) such that

\[
f_1(t,v) = T \circ f_0(\sigma t + d, \sigma' v).
\]

So we have \( f_1(t,v) = T \circ f_0(\sigma t + d, \sigma' v) \). Substituting \( v = 0 \), \( f_1(t,0) = T \circ f_0(\sigma t + d, 0) \) holds. Since \( C \) has no symmetries, \( T \) is the identity map and \((\sigma, d) = (+,0)\). Since \( f_0 = f \), we have \( f_1(t,v) = f(t, \sigma' v) \) and \( \mu_f(\sigma t + b_1) = \mu_f(t) \). Since \( \Gamma_f \) has no hidden symmetries, Corollary \ref{cor:isometry} yields that \((\sigma, b_1) = (+,0)\), a contradiction. \( \blacksquare \)

We fix \( b_0 \in [0,a] \) arbitrarily. Since the crease patterns of the map \( f_j^1(t + (b_0 - b), \nu) \) is congruent to \( \Gamma_f(b_0) \), for each \( b \in [0,a] \) and \( j = 1, 2, 3, 4 \), we have the following assertion:

**Corollary 3.8.** \( \mathcal{P}(C, \Gamma_f(b)) \) does not depend on a choice of \( b \in [0,a] \), that is, the choice of crease pattern does not affect the definition of \( \mathcal{P}(C, \Gamma_f(b)) \).

It is interesting to consider the class that \( \Gamma_f \) is a closed curve without self-intersections. Since \( \Gamma_f \) has no inflection point, it must be convex. So we set

\[
\mathcal{A}_\ast(C) := \{ f \in \mathcal{A}_\ast(C) ; \ \Gamma_f \ is \ a \ closed \ convex \ curve \}.
\]

If \( f \in \mathcal{A}_\ast(C) \), then \( \varphi_f \) and \( \psi_f \) can be actually realized as curved foldings. Moreover, if \( f \in \mathcal{A}_\ast(C) \), then the common crease pattern of \( \varphi_f \) and \( \psi_f \) is a closed convex curve. By definition,

\[
\mathcal{A}_\ast(C) \subset \mathcal{A}_\ast(C) \subset \mathcal{A}(C) \subset \mathcal{A}(C)
\]

holds, and we have the following:

**Proposition 3.9.** If \( \min_{t \in (0,2\pi)} \kappa(t) > 2\pi/a \), then the set \( \mathcal{A}_\ast(C) \) is non-empty. Moreover, if \( f \in \mathcal{A}_\ast(C) \), then \( f, f_\ast \) and \( f_\ast \) also belong to \( \mathcal{A}_\ast(C) \).

**Proof.** The latter assertion is obvious, so it is sufficient to show that \( \mathcal{A}_\ast(C) \) is non-empty. Let \( \Gamma \) be a circle whose length is \( a \). Then its curvature function \( \mu \) is identically \( 2\pi/a \). There exists a unique smooth function \( \alpha : S^1 \rightarrow (0,\pi/2) \) such that \( \kappa(t) \cos \alpha(t) = 2\pi/a \) \((t \in S^1)\). Then \( f = f^\alpha \) belongs to \( \mathcal{A}_\ast(C) \). \( \blacksquare \)

**Example 3.10.** Consider a curve

\[
c_m(t) := ((2 + \cos mt) \cos t, (2 + \cos mt) \sin t, \sin mt)
\]

lying on a torus. We denote by \( L_m \) the length of \( c_m \). Then for each \( m \geq 2 \), the inequality \( \min_{t \in (0,2\pi)} \kappa_m(t) > 2\pi/L_m \), holds, where \( \kappa_m(t) \) is the curvature of \( c_m(t) \). So \( \mathcal{A}_\ast(C_m) \) is non-empty.

**Remark 3.11.** If we slightly perturb \( \gamma_m \) so that it is not a circle, while keeping its length, then the absolute value of the curvature of the resulting plane curve satisfies the property \((*)\), and we can find an example of \( \mathcal{P}(C, \Gamma) \) having uncountably many congruence classes, where \( \Gamma \) is a closed plane curve. Moreover, one can also perturb \( C_m \) so that it has no symmetries, while keeping its length, and then an example of a curved folding satisfying \((b)\) of Theorem B is obtained.

**Example 3.12.** Consider an ellipse \( \gamma(t) := (\cos t, a \sin t) \) for \( a := 6/5 \), whose curvature function is given by \( \mu(t) = a(\sin^2 t + a^2 \cos^2 t)^{-3/2} \). On the other hand, let \( c_\ast(s) := c_m(t(s)) \) be the arc-length parametrization of the above closed space curve \( c_m(t) \) for \( m = 3 \). If we set

\[
\hat{\gamma}(t) := k \gamma(t), \quad k := L_\beta/L_\gamma \approx 3.31,
\]
then the two curves \( \tilde{\gamma} \) and \( c_3 \) have common length. We reparametrize \( \tilde{\gamma} \) by the arc-length parameter \( s \) \((0 \leq s \leq L_3)\) so that \( \tilde{\gamma}(s) = (k, 0) \) at \( s = 0 \). Let \( f \in \tilde{\mathcal{A}}(C_3) \) be the developable strip along \( C_3 \) such that \( \mu_f(t) = \tilde{\mu}(t) \), where \( \tilde{\mu}(s) \) is the curvature function of \( \tilde{\gamma}(s) = \tilde{\gamma}(t(s)) \). Figure 6 indicates the crease patterns of \( f \) (left), \( f^b \) (center) and \( f^{2b} \) (right) for \( b = 1/8 \). These three crease patterns are mutually non-congruent.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{crease_patterns.png}
\caption{The ruling directions on the ellipse of three non-congruent curved folding along \( C_3 \).}
\end{figure}

We remark that the possible topologies of closed developable strips along an embedded closed curve \( C \) without assuming (i)–(iv) are discussed in [10].

4. A CONNECTION WITH CUSPIDAL EDGE SINGULARITIES

In this section, we discuss a relationship between curved foldings and the authors’ recent works on cuspidal edges with a common first fundamental form and singular set image: By ‘\( C^r \)-differentiable’ we mean \( C^\infty \)-differentiability if \( r = \infty \) and real analyticity if \( r = \omega \). We set 
\[
J := I_a \quad \mathrm{or} \quad S_1^a,
\]
and fix a closed \( C^r \)-embedded curve \( c : J \to \mathbb{R}^3 \) with positive curvature function, and denote by \( C \) the image of \( c \). Let \( \mathcal{F}^r(C) \) be the set of \( C^r \)-cuspidal edge germs along \( C \), that is
\[
\mathcal{F}^r(C) := \left\{ F : J \times (-\varepsilon, \varepsilon) \to \mathbb{R}^3 : F(t, 0) \text{ is a cuspidal edge along } C \text{ for each } t \in J \right\}.
\]
We let \( \mathbf{n}(t) \) (resp. \( \mathbf{b}(t) \)) be the unit principal normal (resp. bi-normal) vector of \( c(t) \). For each \( F \in \mathcal{F}^r(C) \), there exists a unique parametrization \((t, v)\) of \( F(C) \) such that, for sufficiently small \( \varepsilon > 0 \),
\[
F(t, v) := c(t) + (v^2, v^3 \beta(t, v)) \begin{pmatrix} \cos \theta(t) & -\sin \theta(t) \\ \sin \theta(t) & \cos \theta(t) \end{pmatrix} \begin{pmatrix} \mathbf{n}(t) \\ \mathbf{b}(t) \end{pmatrix} \quad (t \in J, \ |v| < \varepsilon),
\]
where
\begin{itemize}
\item \( t \) is an arc-length parameter of \( c \),
\item \( \beta(t, v) \) is a \( C^r \)-function, and
\item for each \( t \in J \), the map \((-\varepsilon, \varepsilon) \ni v \mapsto (v^2, v^3 \beta(t, v)) \in \mathbb{R}^2 \) is a cusp at \( v = 0 \) with the normalized half-arc-length parameter (see [4, Appendix A]). In particular \( \beta(t, 0) \neq 0 \) for each \( t \in J \).
\end{itemize}
This expression (4.1) is called \textit{Fukui’s representation formula} (cf. Fukui [3] and also [1]). Here, the \( C^r \)-function \( \theta(t) \) in (4.1) is called the \textit{cuspidal angle} at \( c(t) \). We denote by \( \kappa(t) \) the curvature function of \( c(t) \). In this situation, the \textit{singular}
curvature $\kappa_s(t)$ and the limiting normal curvature $\kappa_n(t)$ along the singular set of $F \in F^*(C)$ are given by (cf. [1])
\begin{equation}
(4.2) \quad \kappa_s(t) = \kappa(t) \cos \theta(t), \quad \kappa_n(t) = \kappa(t) \sin \theta(t).
\end{equation}

**Definition 4.1** ([1, 5]). A cuspidal edge $G \in F^*(C)$ is called a $C$-isomer of $F \in F^r(C)$ if

- there exist $\varepsilon > 0$ and a $C^\infty$-diffeomorphism $\varphi$ on $\Omega := J \times (-\varepsilon, \varepsilon)$ such that the pull-back of the first fundamental form of $G$ by $\varphi$ coincides with that of $F$ on $\Omega$ and,
- the image of $F$ does not coincide with that of $G$.

In the authors’ previous work, the following assertion was proved by applying the Cauchy-Kowalevski theorem:

**Fact 4.2** ([11, 4, 5]). For each $F \in F^\omega(C)$, there exists a unique $C$-isomer $\tilde{K} \in F^\omega(C)$ (called the dual of $F$) whose singular curvature function is the same as that of $F$.

For $F \in F^r(C)$, the singular curvature function $\kappa_s : J \to \mathbb{R}$ is defined along its singular set. By (4.2), $|\kappa_s(t)| \leq |\kappa(t)|$ holds, and $\kappa_s(t)$ depends only on the first fundamental form of $F$. We then consider the condition (which corresponds to the condition $(\ast)$ in Section 1)
\begin{equation}
(4.3) \quad 0 < |\kappa_s(t)| < \min_{t \in J} |\kappa(t)|,
\end{equation}
and consider the subclass
\[
F_s^r(C) := \{ F \in F^r(C) ; F \text{ satisfies (4.3)} \}
\]
of $F^r(C)$. We denote by $\theta_F$ the cuspidal angle of each $F \in F_s^r(C)$. By (4.3), we may assume
\begin{equation}
(4.4) \quad 0 < |\theta_F| < \frac{\pi}{2}.
\end{equation}
If we replace $v^2$ with $v$ and set $\beta = 0$ in (4.1), a regular surface is obtained whose unit normal vector $\nu(t)$ along $C$ is the same as that of $F$ and is given by
\[
\nu(t) = \cos \theta(t) \mathbf{n} + \sin \theta(t) \mathbf{b}.
\]
We set $(\alpha := |\alpha_F| := -\theta_F$. Since the unit normal vector field of the developable surface $f^\alpha$ for $\alpha \in C_{\pi/2}(J)$ along $c(t)$ coincides with $\nu(t)$ up to a sign, the map
\[
\Xi : F_s^r(C) \ni F \mapsto \mathcal{S}(\alpha_F) \in \mathcal{A}_s(C)
\]
is well-defined, where $\mathcal{S}$ is the map defined in Proposition 1.3. The developable surface $\Xi(F) (= \mathcal{S}(\alpha_F))$ is defined in Izumiya, Saji and Takeuchi [8] and is called the osculating developable surface associated with $F$. It can be easily checked that the geodesic curvature $\mu(t)$ of $f := \Xi(F)$ coincides with the singular curvature $\kappa_s(t)$ of $F$. This is the reason why the condition (4.3) implies $(\ast)$ in Section 1. The following fact was also shown:

**Fact 4.3** ([11, 4, 5]). For each $F \in F_s^\omega(C)$, there exist two involutions
\[
I_1 : F_s^\omega(C) \ni F \mapsto \check{F} \in F_s^\omega(C), \quad I_2 : F_s^\omega(C) \ni F \mapsto F_* \in F_s^\omega(C)
\]
such that for each $F \in F_s^\omega(C)$,
\begin{itemize}
  \item $\theta_{I_1(F)} = -\theta_F$ and the singular curvature of $I_1(F)$ coincides with that of $F$,
  \item the sign of $\theta_{I_2(F)}$ coincides with that of $\theta_F$ and the singular curvature of $I_2(F)$ coincides with $\kappa_s(-t)$, where $\kappa_s(t)$ is the singular curvature of $F$,
  \item $I_1 \circ I_2 = I_2 \circ I_1$.
\end{itemize}
Moreover, $\check{F}, F_*, \check{F}$ have common first fundamental form.
Although Ξ⁻¹(f) (f ∈ A*(C)) has a freedom of choice of function β, it can be easily checked that the following holds:

**Proposition 4.4.** If we set f := Ξ(F) for F ∈ F*(C), then f bisects each sectional cusp v → F(t, v) of F along C and
\[ \Xi(f) = \hat{f}, \quad \Xi(F_+) = f_+, \quad \Xi(F_-) = \hat{f}. \]

**Appendix A. Images of Ruled Strips**

Let \( c : I_a \to R^3 \) (\( I_a := [-a/2, a/2] \)) be a regular curve with arc-length parametrization, and set C := c(I_a). A ruled strip along C is a continuous map \( f : I_a \times (-\varepsilon, \varepsilon) \to R^3 \) satisfying the following properties:
- \( f(t, 0) = c(t) \) for \( t \in I_a \),
- For each \( t \in I_a \), the two curves \( (0, \varepsilon) \ni v \mapsto f(t, v) \in R^3 \) and \( (-\varepsilon, 0) \ni v \mapsto f(t, v) \in R^3 \) parametrize line segments.

**Definition A.1.** Two ruled strips \( f \) and \( g \) along \( C \) are said to have the *same image* (as map germs along \( C \)) if there exists \( \delta \in (0, \varepsilon] \) such that \( f(I_a \times (-\delta, \delta)) \) is contained in \( g(I_a \times (-\varepsilon, \varepsilon)) \).

It is obvious that an arbitrarily given ruled strip along \( C \) has an expression
\[
f(t, v) = \begin{cases} 
    c(t) + v\xi_1(t) & (v \in (0, \varepsilon)), \\
    c(t) + v\xi_2(t) & (v \in (-\varepsilon, 0))
\end{cases}
\]
such that \( \xi_i(t) (i = 1, 2) \) are (continuous) unit vector fields along \( C \), which we call the *normal form* of the given ruled strip \( f \). The origami maps defined in Section 1 are ruled strips written in the normal form. Since \( \xi_i(t) \) is a unit vector and \( t \) is the arc-length parameter of \( C \), the following assertion holds:

**Proposition A.2.** Let \( f \) and \( g \) be two ruled strips along \( C \) written in the normal form. Let \( T \) be an isometry of \( R^3 \). Then the following assertions are equivalent:
1. \( T \circ f \) and \( g \) have the same images (as map germs along \( C \)),
2. for sufficiently small \( \delta \in (0, \varepsilon] \), \( T \circ f(I_a \times (-\delta, \delta)) = g(I_a \times (-\delta, \delta)) \).

**Proof.** If \( f \) is written in the normal form, and so is \( T \circ f \). Thus, we obtain the assertion. □

**Appendix B. Symmetries of Curves**

In this appendix, we prove two assertions on symmetries of curves in \( R^2 \) or \( R^3 \):

**Proposition B.1.** Let \( \gamma : I_a \to R^2 \) be a regular curve without self-intersections, which is parametrized by arc-length. Suppose that \( \gamma \) has no inflection points. Then the image \( \Gamma := \gamma(I_a) \) has a symmetry \( T \) if and only if \( \mu(t) = \mu(-t) \) holds for \( t \in I_a \), where \( \mu(t) \) is the curvature function of \( \gamma \). In this case, \( T \) is an orientation reversing isometry in \( R^2 \).

**Proof.** Suppose that \( \Gamma \) has a symmetry \( T \). Since \( \Gamma \) has no self-intersections, we have
\[
\gamma(t) = T \circ \gamma(\sigma t)
\]
for some sign \( \sigma \in \{+, -\} \). Suppose that \( \sigma = + \). Since the curvature function of \( T \circ \gamma(t) \) is \( \det(T)\mu(t) \), \( \text{(B.1)} \) implies that \( \det(T) = 1 \), that is, \( T \) must be an orientation preserving isometry. Since \( x_0 := \gamma(0) \) is the midpoint of \( \Gamma \), \( T \) preserves the tangential direction of \( \Gamma \) at \( x_0 \), and must be 180°-rotation at \( x_0 \). However,
since ζ has no inflection point, this is a contradiction. Thus we may assume σ = −. Since the curvature function of T ∘ γ(−t) is −det(T)µ(−t), (B.1) yields the identity µ(t) = −det(T)µ(−t). In particular, det(T) = −1 and µ(t) = µ(−t).

Conversely, we suppose µ(t) = µ(−t). Then two curves γ(t) and T₀ ∘ γ(−t) have the same curvature function µ(t) if T₀ is an orientation reversing isometry of R².

By the fundamental theorem of curves in the Euclidean plane (cf. [13, Chapter 2]), γ(t) is congruent to T₀ ∘ γ(−t) by a certain orientation preserving isometry T₁ of R². So, ζ has a symmetry T₁ ∘ T₀, which is orientation reversing. □

Proposition B.2. Let c : S¹ → R³ be a C∞-regular curve with curvature κ(> 0). Then the number of non-trivial symmetries of C := c(S¹) is finite, unless C is a circle.

Proof. We assume that C is not a circle. Then κ(t) is non-constant. By Sard’s theorem, the set of critical values of κ is of measure zero. We choose a regular value r of κ such that κ⁻¹(p) consists of finitely many points t₁,...,tₙ ∈ S¹ and the torsion function at t₁ is non-zero when C is not a plane curve. Then we set pᵢ = c(tᵢ) (i = 1,...,n). Suppose that there are infinitely many distinct non-trivial symmetries (Tⱼ) 있게 of C. Then Tⱼ(p₁) ∈ {p₁,...,pₙ}. So, there exists j₀ ∈ {1,...,n} such that Tⱼ(p₁) = p{j₀} for infinitely many j. So without loss of generality, we may assume T₀⁻¹ ∘ Tⱼ(p₁) = p₁ for j ≥ 1. Since T₀ ≠ Tⱼ, and there exists at most one non-trivial symmetry R_p of C fixing p, we have T₀⁻¹ ∘ Tⱼ = R_p. In particular, T₁ = T₀ ∘ R_p = T_j for j ≥ 2, a contradiction. □

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