NON-VANISHING OF $L$-FUNCTIONS ASSOCIATED TO CUSP FORMS OF HALF-INTEGRAL WEIGHT

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Abstract. In this article, we prove non-vanishing results for $L$-functions associated to holomorphic cusp forms of half-integral weight on average (over an orthogonal basis of Hecke eigenforms). This extends a result of W. Kohnen [4] to forms of half-integral weight.

1. Introduction

In [4], W. Kohnen proved that, given a real number $t_0$ and a positive real number $\epsilon$, for all $k$ large enough the sum of the functions $L^*(f,s)$ with $f$ running over a basis of (properly normalized) Hecke eigenforms of weight $k$ does not vanish on the line segment $\text{Im } s = t_0, (k - 1)/2 < \text{Re}(s) < k/2 - \epsilon, k/2 + \epsilon < \text{Re}(s) < (k + 1)/2$. As a consequence, he proved that for any such point $s$, for $k$ large enough there exists a Hecke eigenform of weight $k$ on $SL_2(\mathbb{Z})$ such that the corresponding $L$-function value at $s$ is non-zero. Using similar methods, in [8], A. Raghuram generalised Kohnen’s result for the average of $L$-functions over a basis of newforms (of integral weight) of level $N$ with primitive character modulo $N$. In this article, we extend Kohnen’s method to forms of half-integral weight. As a consequence, we show that for any given point $s$ inside the critical strip $k/2 - 1/4 < \text{Re}(s) < k/2 + 3/4$, there exists a Hecke eigen cusp form $f$ of half-integral weight $k + 1/2$ on $\Gamma_0(4N)$ with character $\psi$ such that the corresponding $L$-function value at $s$ is non-zero, and the first Fourier coefficient of $f$ is non-zero. It should be noted that the normalisation of Fourier coefficients of forms of half-integral weight is still an open question. Also, contrary to the result of Kohnen in the case of integral weight modular forms on $SL_2(\mathbb{Z})$, we get the non-vanishing result inside the critical strip including the central line (see Remark 4.1). Our results are obtained for $N$ sufficiently large if $k$ is fixed and vice versa. In particular when $N = 1$, for sufficiently large $k$, either $f$ is a newform in the full space or $f$ is a Hecke eigenform in the Kohnen plus space.

2. Notations and Main Theorems

Let $N \geq 1, k \geq 3$ be integers and $\psi$ be an even Dirichlet character modulo $4N$. Let $S_{k+1/2}(4N, \psi)$ be the space of cusp forms of weight $k + 1/2$, level $4N$ with character $\psi$ [2, 10]. Let $L(f,s)$ be the $L$-function associated to the cusp form $f \in S_{k+1/2}(4N, \psi)$.

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defined by $L(f, s) = \sum_{n \geq 1} a_f(n)n^{-s}$, where $a_f(n)$ denotes the $n$-th Fourier coefficient of $f$. Then by [6] Proposition 1], the completed $L$-function defined by $L^*(f, s) := (2\pi)^{-s}(\sqrt{4N})^s\Gamma(s)L(f, s)$ has the following functional equation

$$L^*(f|H_{4N}, k + 1/2 - s) = L^*(f, s),$$

where $H_{4N}$ is the Fricke involution on $S_{k+1/2}(4N, \psi)$ defined by

$$f|H_{4N}(z) = i^{k+1/2}(4N)^{-k/2-1/4}z^{-k-1/2}f(-1/4Nz).$$

For $f, g \in S_{k+1/2}(4N, \psi)$, let $\langle f, g \rangle$ denote the Petersson scalar product of $f$ and $g$. It is known that the space $S_{k+1/2}(4N, \psi)$ has an orthogonal basis of Hecke eigenforms with respect to all Hecke operators $T(p^2)$, $p \not\mid 2N$. Let $\{f_1, f_2, \ldots, f_d\}$ be such an orthogonal basis of Hecke eigenforms, where $d$ is the dimension of the space $S_{k+1/2}(4N, \psi)$ (see for example [10]). Let $K$ be the operator defined by $f|K(z) = \overline{f(-z)}$. Since $KH_{4N} = H_{4N}K$ on $S_{k+1/2}(4N, \psi)$, we have $f|(KH_{4N})^2 = f$. Also, the operators $K$ and $H_{4N}$ commute with the Hecke operators $T(p^2)$, $p \not\mid 2N$. Therefore, for the basis elements $f_j$, $1 \leq j \leq d$, we may assume that $f_j|KH_{4N} = \lambda_{f_j}f_j$, where $\lambda_{f_j} = \pm 1$.

We now state the main results of this article.

**Theorem 2.1.** Let $N \geq 1$ be a fixed integer. Let $\{f_1, f_2, \ldots, f_d\}$ be an orthogonal basis as above. Let $r_0 \in \mathbb{R}$. Then there exists a constant $C = C(r_0)$ depending only on $r_0$ such that for $k > C$, the function

$$\sum_{j=1}^{d} \frac{L^*(f_j, s)}{\langle f_j, f_j \rangle} \lambda_{f_j} a_{f_j}(1)$$

doesn’t vanish for any point $s = \sigma + ir_0$ with $k/2 - 1/4 < \sigma < k/2 + 3/4$.

**Theorem 2.2.** Let $k \geq 3$ be a fixed integer. Let $\{f_1, f_2, \ldots, f_d\}$ be an orthogonal basis as above. Let $r_0 \in \mathbb{R}$. Then there exists a constant $C' = C'(r_0)$ depending only on $r_0$ such that for $N > C'$, the function

$$\sum_{j=1}^{d} \frac{L^*(f_j, s)}{\langle f_j, f_j \rangle} \lambda_{f_j} a_{f_j}(1)$$

doesn’t vanish for any point $s = \sigma + ir_0$ with $k/2 - 1/4 < \sigma < k/2 + 3/4$.

The following corollary is an easy consequence of the above two theorems.

**Corollary 2.3.** Let $s_0$ be a point inside the critical strip $k/2 - 1/4 < \Re(s_0) < k/2 + 3/4$. If either $k$ or $N$ is suitably large then there exists a Hecke eigenform $f$ belonging to $S_{k+1/2}(4N, \psi)$ such that $L(f, s_0) \neq 0$ and $a_f(1) \neq 0$.

**Remark 2.4.** Though we consider Hecke eigenforms of half-integral weight, the $L$-function corresponding to such a Hecke eigenform does not have an Euler product.
3. Proof

The proof is on the same lines as that of Kohnen [4] and so we give only a sketch. First, let us recall the Poincaré series in $S_{k+1/2}(4N, \psi)$. We define the $n$-th Poincaré series in $S_{k+1/2}(4N, \psi)$ as follows.

(2) $P_{n,k+1/2,4N,\psi}(z) = \frac{1}{2} \sum_{(c,d) \in \mathbb{Z}^2} \overline{\psi}(d) \left( \frac{c}{d} \right) \left( \frac{-4}{d} \right)^{k+1/2} (cz+d)^{-k+1/2} e\left( \frac{n}{cz+d} \right)$,

where in the summation above, for each coprime pair $(c, d)$ and $4N|c$, we make a fixed choice of $(a_0, b_0) \in \mathbb{Z}^2$ with $a_0d-b_0c = 1$ and $e(x)$ stands for $e^{2\pi ix}$. We have the following characterization of the Poincaré series.

(3) $\langle f, P_{n,k+1/2,4N,\psi} \rangle = \frac{\Gamma(k - 1/2)}{i_{4N}(4\pi N)^{k-1/2}} a_f(n), \quad f \in S_{k+1/2}(4N, \psi)$,

where $i_{4N}$ is the index of $\Gamma_0(4N)$ in $SL_2(\mathbb{Z})$.

Next, let us define the kernel function for the special values of the $L$-function associated to a cusp form of half-integral weight. A similar function for forms of integral weight was considered by Kohnen [3]. Let $z \in \mathcal{H}$ and $s \in \mathbb{C}$ with $1 < \sigma < k - 1/2$, $\sigma = \Re(s)$. Define

(4) $R_{s,k,N,\psi}(z) = \gamma_k(s) \sum' \overline{\psi}(d) \left( \frac{c}{d} \right) \left( \frac{-4}{d} \right)^{k+1/2} (cz+d)^{-k+1/2} \left( \frac{az+b}{cz+d} \right)^{-s}$,

where

(5) $\gamma_k(s) = \frac{1}{2} e^{\pi i s/2} \Gamma(s) \Gamma(k + 1/2 - s)$,

and the sum $\sum'$ varies over all matrices $\left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \Gamma_0(4N)$. The condition $1 < \sigma < k - 1/2$ ensures that the above series converges absolutely and uniformly on compact subsets of $\mathcal{H}$ and hence it represents an analytic function on $\mathcal{H}$. The function $R_{s,\psi}(z) := R_{s,k,N,\psi}(z)$ is a cusp form in $S_{k+1/2}(4N, \psi)$.

For a given $c, d \in \mathbb{Z}$ with $\gcd(c, d) = 1$ and $4N|c$, we choose $a_0, b_0$ such that $a_0d - b_0c = 1$. Then any other solution $a, b$ of $ad - bc = 1$ is given by $a = a_0 + nc$ and $b = b_0 + nd$ for some $n \in \mathbb{Z}$. Hence,

(6) $R_{s,\psi}(z) = \gamma_k(s) \sum_{(c,d)\in\mathbb{Z}\setminus\{0\}} \sum_{d\mid\gcd(4N,c)} \overline{\psi}(d) \left( \frac{c}{d} \right) \left( \frac{-4}{d} \right)^{k+1/2} (cz+d)^{-k+1/2} \left( \frac{a_0z+b_0}{cz+d} + n \right)^{-s}$.

Using Lipschitz’s formula

(7) $\sum_{n=-\infty}^{\infty} (z+n)^{-s} = \frac{e^{-\pi i s/2} (2\pi)^s}{\Gamma(s)} \sum_{n \geq 1} n^{s-1} e(nz) \quad (z \in \mathcal{H}, \sigma > 1)$,
we get

\[ R_{s,\psi}(z) = \sum_{(c,d)=1 \atop 4N|c} \overline{\psi}(d) \left( \frac{c}{d} \right) \left( -4 \right)^{k+1/2} \left( cz + d \right)^{-k-1/2} \frac{e^{-\pi is/(2\pi)s}}{\Gamma(s)} \]

\[ \times \sum_{n \geq 1} n^{s-1} e \left( n \frac{a_0 z + b_0}{cz + d} \right) \]

\[ = (2\pi)\Gamma(k + 1/2 - s) \sum_{n \geq 1} n^{s-1} \]

\[ \times \frac{1}{2} \sum_{(c,d)=1 \atop 4N|c} \overline{\psi}(d) \left( \frac{c}{d} \right) \left( -4 \right)^{k+1/2} \left( cz + d \right)^{-k-1/2} e \left( \frac{n}{cz + d} \right). \]

Here, we have used the absolute convergence of the above sum in the region \( 1 < \sigma < k - \beta - 1/2 \) and so the interchange of summations is allowed, where \( \beta = k/2 - 1/28 \) is the exponent of the estimate for the Fourier coefficients of a cusp form of weight \( k + 1/2 \) on \( \Gamma_0(4N) \) obtained by H. Iwaniec [1]. Thus, for \( 1 < \sigma < k - \beta - 1/2 \),

\[ R_{s,\psi}(z) = (2\pi)^s \Gamma(k + 1/2 - s) \sum_{n \geq 1} n^{s-1} P_{n,k+1/2,4N,\psi}(z). \]

Using equation (3) with the last equation, we get for \( 1 < \sigma < k - \beta - 1/2 \),

\[ \langle f, R_{s,\psi} \rangle = \frac{\pi \Gamma(k - 1/2)}{i4N^{2k-3/2}(4N)^{k+1/4-s/2}} L^*(f, k + 1/2 - s), \]

for all \( f \in S_{k+1/2}(4N, \psi) \). Using this, we have

\[ R_{s,\psi} = \frac{2^{-2k+1+s} \pi \Gamma(k - 1/2)}{i4N^{k+1/4-s/2}} \sum_{j=1}^{d} L^*(f_j|K, k + 1/2 - s) \frac{\lambda_{f_j}}{\langle f_j, f_j \rangle}, \]

where the sum varies over the orthogonal basis \( \{f_j\} \). Using \( f_j|KH_{4N} = \lambda_{f_j} f_j \) together with the functional equation (1), we get for \( 1 < \sigma < k - \beta - 1/2 \),

\[ R_{s,\psi} = \frac{2^{-2k+1+s} \pi \Gamma(k - 1/2)}{i4N^{k+1/4-s/2}} \sum_{j=1}^{d} L^*(f_j, s) \frac{\lambda_{f_j}}{\langle f_j, f_j \rangle} \]

This equality has been established for \( 1 < \sigma < k - \beta - 1/2 \). Since the right-hand side is an entire function of \( s \), this gives an analytic continuation of the kernel function \( R_{s,\psi} \), for all \( s \in \mathbb{C} \).

Next, we need the Fourier expansion of the function \( R_{s,\psi} \). In an earlier version of [6], the Fourier expansion of the function \( R_{s,\psi}(z) \) was derived, which we present here. Let

\[ R_{s,\psi}(z) = \sum_{n \geq 1} a_{s,\psi}(n)e^{2\piinz} \]
be the Fourier expansion of $R_{s, \psi}$, where the Fourier coefficients $a_{s, \psi}(n)$ are given by
\begin{equation}
  a_{s, \psi}(n) = (2\pi)^s\Gamma(k + 1/2 - s)n^{s-1} + e^{\pi is/2}(-2\pi i)^{k+1/2}n^{k-1/2}\nonumber
\end{equation}
\begin{equation}
  \times \sum_{(a, c) \in \mathbb{Z}, ac \neq 0, \gcd(a, c) = 1, 4N|c} \psi(a) \left(\frac{c}{a}\right) \left(\frac{-4}{a}\right)^{k+1/2} e^{s-k-1/2}a^{s}e^{2\pi in'/c} \lambda_{1}(s, k + 1/2; -2\pi in/ac),
\end{equation}
where $a'$ is an integer which is the inverse of $a$ modulo $c$ and
\begin{equation}
  \lambda_{1}(\alpha, \beta; z) = \frac{\Gamma(\alpha)\Gamma(\beta - \alpha)}{\Gamma(\beta)}_{1}F_{1}(\alpha, \beta; z).
\end{equation}
Here $_{1}F_{1}(\alpha, \beta; z)$ is the Kummer’s degenerate hypergeometric function.

We now give the proof of our theorems. Assume that $\sum_{j \in \mathbb{Z}} \frac{K_{s}(f_{j}, a)}{f_{j}(f_{j}, f_{j})} a_{f_{j}}(1) = 0$, for $s$ as in the theorem. This implies that (for the values of $s$) the first Fourier coefficient of $R_{s, \psi}$ is zero. Dividing by $(2\pi)^{s}\Gamma(k + 1/2 - s)$, we obtain,
\begin{equation}
  1 + e^{\pi is/2} \frac{(2\pi)^{k+1/2-s}}{i^{k+1/2}\Gamma(k + 1/2 - s)} \sum_{(a, c) \in \mathbb{Z}, ac \neq 0, \gcd(a, c) = 1, 4N|c} \psi(a) \left(\frac{c}{a}\right) \left(\frac{-4}{a}\right)^{k+1/2} e^{s-k-1/2}a^{s}e^{2\pi in'/c} \lambda_{1}(s, k + 1/2; -2\pi in/ac) = 0.
\end{equation}
In particular, let $s = k/2 + 1/4 - \delta + ir_{0}$, where $0 \leq \delta < 1/2$. Now, one has
\begin{equation}
  |_{1}f_{1}(s, k + 1/2; -2\pi in/ac)| \leq 1
\end{equation}
(see [4]). Taking the absolute value in (14) and using the above estimate, we get
\begin{equation}
  1 \leq A(r_{0}) \frac{\pi^{k/2+1/4+\delta}}{|\Gamma(k/2 + 1/4 + \delta) - ir_{0})|} \frac{1}{(2\pi)^{k/2+1/4+\delta}} \left(\sum_{a, c \in \mathbb{Z}, ac \neq 0, \gcd(a, c) = 1} \frac{1}{a^{k/2+1/4+\delta} e^{k/2+1/4+\delta}}
\end{equation}
where $A(r_{0})$ is a constant depending only on $r_{0}$ and $B > 0$ is an absolute constant. To prove Theorem 2.1, we fix $N$ and allow $k$ tend to infinity and for the proof of Theorem 2.2, we fix $k$ and allow $N$ tend to infinity. In either case, the right-hand side goes to zero (for fixed $N$ one should use the Stirling’s approximation), a contradiction. This completes the proof.

4. Remarks

Remark 4.1. In [4], the non-vanishing result was obtained for $s$ inside the critical strip with the condition that $\text{Re}(s) \neq k/2$, the center of the critical strip. However, since the level of the modular forms considered in this paper is greater than 1, we need not assume this condition and our results are valid for all $s$ inside the critical
strip $k/2 - 1/4 < \text{Re}(s) < k/2 + 3/4$. We also remark that the same is true in \cite{8}, since the level $M$ is greater than 1. The reason is as follows. When $M = 1$ (i.e., when one considers the case of forms of integral weight on $SL_2(\mathbb{Z})$), while deriving the Fourier expansion of the function $R_s$, the term corresponding to $ac = 0$ has two contributions ($c = 0$ and $a = 0$). Therefore, in the estimation of the first Fourier coefficient, there is an extra term on the right-hand side (see [4, p. 189, Eq.(10)]). Due to the appearance of this extra term, in order to get a contradiction, the central values have to be omitted. Since the case $a = 0$ doesn’t arise for the levels $M > 1$, we do not get the extra term on the right-hand side in the estimation. Therefore, this gives the advantage of considering all the values of $s$ inside the critical strip. In particular, one obtains non-vanishing results for forms at the center of the critical strip when the level is greater than 1.

**Remark 4.2.** The average sum in Theorem 2.1 (and also in Theorem 2.2) contains an extra factor $\lambda_f$ (and of course the first Fourier coefficient), which does not appear in Kohnen’s result. In the case of level 1, we have the functional equation $L^*(f,k-s) = (-1)^{k/2}L^*(f,s)$, whereas when the level $M$ is greater than 1, we have a different functional equation in the sense that on the one side we have $L^*(f,s)$ and on the other side we have $L^*(f|H_M,k-s)$ and therefore, in the final form of the functional equation, the root number will depend on the function, especially the eigenvalue of $f$ under the Fricke involution $H_M$. (In the case of half-integral weight, the Fricke involution is $H_M = H_{4N}$ in our notation.) Therefore, we will have an extra factor, which we call $\lambda_f$. Note that the extra factor corresponding to the eigenvalues under $H_M$ also appears in Raghuram’s results (see \cite{8}). Since normalization of Fourier coefficients is not known in the case of half-integral weight, we also have the first Fourier coefficients appearing in the average sum.

**Remark 4.3.** Let us consider the space $S_{k+1/2}(4N,\psi)$, where $\psi$ is an even primitive Dirichlet character modulo $4N$. Then it is known from the work of Serre and Stark \cite{9} that the space $S_{k+1/2}(4N,\psi)$ is the space of newforms. Hence, the orthogonal basis consists of newforms. In this case, the Hecke eigenform $f$ in Corollary 2.3 will be a newform of level $4N$.

**Remark 4.4.** Let us consider the case $N = 1$. That is, we consider the space $S_{k+1/2}(4)$. Let $\{f_1, f_2, \ldots, f_{d_1}\}$ be an orthogonal basis of $S_{k+1/2}^{\text{new}}(4)$ which are newforms (see \cite{7}). Let $\{g_1, g_2, \ldots, g_{d_3}\}$ be an orthogonal basis of $S_{k+1/2}^+(4)$ (see \cite{3}), which are Hecke eigenforms such that the set $\{g_1 \pm g_1|W(4), \ldots, g_{d_2} \pm g_{d_2}|W(4)\}$ forms an orthogonal basis of $S_{k+1/2}^{\text{old}}(4)$. Here, $d_1 + 2d_2 = d$ is the dimension of the space $S_{k+1/2}(4)$ and $W(4)$ is the Atkin-Lehner $W$-operator for the prime $p = 2$ on $S_{k+1/2}(4)$. Thus, an orthogonal basis of Hecke eigenforms for the space $S_{k+1/2}(4)$ is given as follows:

$$\{f_1, f_2, \ldots, f_{d_1}, g_1 \pm g_1|W(4), g_2 \pm g_2|W(4), \ldots, g_{d_2} \pm g_{d_2}|W(4)\}.$$  

In this case, we get the following result as a consequence of Theorem 2.1. Let $s_0$ be a point inside the critical strip $k/2 - 1/4 < \text{Re}(s_0) < k/2 + 3/4$. If $k$ is suitably large, then
there exists a \( j \), with \( 1 \leq j \leq d_1 \) or \( 1 \leq j \leq d_2 \) such that
\[
L(f_j, s_0) \neq 0, a_{f_j}(1) \neq 0 \quad \text{or} \quad L(g_j \pm g_j|W(4), s_0) \neq 0, a_{g_j}(1) \pm 2^{-k} a_{g_j}(4) \neq 0.
\]
For the last assertion in the above equation \([15]\), we use the fact that \( W(4) = 2^{-k}U(4) \) on \( S^+_{k+1/2}(4) \), where \( U(4) \) is the Hecke operator for \( p = 2 \) on \( S^+_{k+1/2}(4) \). Note that \( W(4) = H_4 \) on \( S^+_{k+1/2}(4) \) and therefore, in the second case of \([15]\), for a \( j \) with \( 1 \leq j \leq d_2 \), it follows from the functional equation that either \( L(g_j, s_0) \neq 0 \) or \( L(g_j, k+1/2-s_0) \neq 0 \). Hence, for any given point \( s \) inside the critical strip, our theorem gives (for sufficiently large \( k \)) the existence of a newform \( f \) in \( S^+_{k+1/2}(4) \) such that \( L(f, s) \neq 0 \) or a Hecke eigenform \( g \) in the plus space \( S^+_{k+1/2}(4) \) such that \( L(g, s) \neq 0 \) or \( L(g, k+1/2-s) \neq 0 \). Correspondingly, we also get the non-vanishing of the first Fourier coefficient (if it is a newform in \( S^+_{k+1/2}(4) \)) or the first or the 4-th Fourier coefficient (if it is a Hecke eigenform in \( S^+_{k+1/2}(4) \)).

4.1. Additional remarks: In view of a recent result of N. Kumar \([5]\), we make the following additional remarks.

Let \( N \) be any positive integer and let \( E(k+1/2, 4N) \) denote the set of all Hecke eigenforms \( h \in S_{k+1/2}(4N) \) (the vector space of cusp forms of weight \( k+1/2 \) on \( \Gamma_0(4N) \) with trivial character) such that the \( L \)-value \( L(h, k/2+1/4) \neq 0 \). In \([5, \S 4]\), the following results are obtained.

**Theorem 4.5.** For \( h \in E(k+1/2, 4N) \) one has
\[
\frac{L'(h, k/2+1/4)}{L(h, k/2+1/4)} = -\Psi(k/2 + 1/4) + \log(\pi),
\]
where \( \Psi \) is the logarithmic derivative of the gamma function \( \Gamma \). Further, for such an \( h \), \( L'(h, k/2+1/4) \neq 0 \) and the real number
\[
\exp \left( \frac{L'(h, k/2+1/4)}{L(h, k/2+1/4)} + \Psi(k/2 + 1/4) \right)
\]
is transcendental. Moreover, considering the quotient \( \frac{L'(h, k/2+1/4)}{L(h, k/2+1/4)} \) as a function of \( k \) one deduce that the function \( \frac{L'(h, k/2+1/4)}{L(h, k/2+1/4)} + \Psi(k/2 + 1/4) \) is independent of \( k \) and the function \( \frac{L'(h, k/2+1/4)}{L(h, k/2+1/4)} \) \( \to -\infty \) as \( k \to \infty \).

**Proposition 4.6.** If \( \frac{L'(h_0, k_0/2+1/4)}{L(h_0, k_0/2+1/4)} \) is algebraic (resp. transcendental) for some \( h_0 \in E(k_0 + 1/2, 4N) \) then \( \frac{L'(h, k/2+1/4)}{L(h, k/2+1/4)} \) is algebraic (resp. transcendental) for all \( h \in E(k+1/2, 4N) \) and for all \( k \in \mathbb{N} \) with \( k \equiv k_0 \pmod{2} \).

**Note:** In \([5, \S 4]\), the above results are proved for the case \( N = 1 \) and was remarked that a similar method will lead to the results for \( N \) square-free. In fact following the same arguments one can get the above results for any positive integer \( N \).

The above results (Theorem 4.5 and Proposition 4.6) are about the properties of the functions in the set \( E(k+1/2, 4N) \) under the assumption that it is a non-empty set.
It is to be noted that the main results of our present work guarantees the existence of an element in the set $E(k + 1/2, 4N)$. In fact, by taking $r_0 = 0$ and $\sigma = k/2 + 1/4$ in Theorem 2.1 and Theorem 2.2 we get the following two corollaries.

**Corollary 4.7.** There exists a constant $C_1$ such that for any $k > C_1$ and any $N$, the set $E(k + 1/2, 4N)$ is non-empty.

**Corollary 4.8.** There exists a constant $C_2 > 0$ such that for any $N > C_2$ and any $k \geq 3$, the set $E(k + 1/2, 4N)$ is non-empty.

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