Singular Density Dependent Stochastic Differential Equations *

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Abstract

The (strong and weak) well-posedness is proved for singular SDEs depending on the distribution density point-wisely and globally, where the drift satisfies a local integrability condition in time-spatial variables, and is Lipschitz continuous in the distribution density with respect to a local $L^k$-norm. Density dependent reflecting SDEs are also studied.

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1 Introduction

The study of distribution dependent SDEs goes back to McKean’s pioneering work [9] where an expectation dependent SDE is proposed to characterize Maxwellian gas. Comparing with the dependence on the global distribution, the point-wise dependence on the density function is more singular for SDEs. The point-wisely density dependent SDE is called Nemytskii-type McKean-Vlasov SDE, see [1] [2] for the correspondence of this type SDEs and nonlinear PDEs.

In recent years, distribution dependent SDEs have been intensively investigated and a plenty of results have been derived. However, much less is known for density dependent SDEs, see Remark 1.1 below for existing results.

Let $\ell_\xi : \mathbb{R}^d \to [0, \infty)$ be the distribution density function of an absolutely continuous random variable $\xi$ on $\mathbb{R}^d$. We investigate the following SDE depending on the distribution

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density point-wisely and globally:

\[ (1.1) \quad dX_t = b_t(X_t, \ell_{X_t}(X_t), \ell_{X_t})dt + \sigma_t(X_t, \ell_{X_t})dW_t, \quad t \in [0, T], \]

where \( T > 0 \) is fixed, \( \{W_t\}_{t \in [0, T]} \) is an \( m \)-dimensional Brownian motion on a complete filtration probability space \((\Omega, \{\mathcal{F}_t\}_{t \in [0, T]}, \mathbb{P})\), and

\[ b : [0, T] \times \mathbb{R}^d \times [0, \infty) \times \mathcal{D}_+^1 \to \mathbb{R}^d, \quad \sigma : [0, T] \times \mathbb{R}^d \times \mathcal{D}_+^1 \to \mathbb{R}^d \otimes \mathbb{R}^m \]

are measurable, where

\[ \mathcal{D}_+^1 := \left\{ f \in L^1(\mathbb{R}^d) : f \geq 0, \int_{\mathbb{R}^d} f(x)dx \leq 1 \right\} \]

is a closed subspace of \( L^1(\mathbb{R}^d) \).

**Definition 1.1.** A continuous adapted process \((X_t)_{t \in [0, T]}\) on \( \mathbb{R}^d \) is called a (strong) solution of \((1.1)\), if

\[ \int_0^T \mathbb{E}\left[ |b_s(X_s, \ell_{X_s}(X_s), \ell_{X_s})| + \|\sigma_s(X_s, \ell_{X_s})\|^2 \right] ds < \infty \]

and \( \mathbb{P}\)-a.s.

\[ X_t = X_0 + \int_0^t b_s(X_s, \ell_{X_s}(X_s), \ell_{X_s})ds + \int_0^t \sigma_s(X_s, \ell_{X_s})dW_s, \quad t \in [0, T]. \]

A pair \((X_t, W_t)_{t \in [0, T]}\) is called a weak solution of \((1.1)\), if \((W_t)_{t \in [0, T]}\) is an \( m \)-dimensional Brownian under a complete filtration probability space \((\Omega, \{\mathcal{F}_t\}_{t \in [0, T]}, \mathbb{P})\) such that \((X_t)_{t \in [0, T]}\) solves \((1.1)\). We identify any two weak solutions \((X_t, W_t)_{t \in [0, T]}\) and \((\tilde{X}_t, \tilde{W}_t)_{t \in [0, T]}\) if \((X_t)_{t \in [0, T]}\) and \((\tilde{X}_t)_{t \in [0, T]}\) have the same distribution under the corresponding probability spaces.

**Remark 1.1.** We introduce below some existing results concerning the well-posedness of \((1.1)\) in two special situations.

1. When \( m = d, \sigma = I_d \) (the \( d \times d \) identity matrix), and \( b_t(x, r, \rho) = b_t(x, r) \) does not depend on \( \rho \), the weak solutions are studied in [5, 6]. In [5], the weak existence is proved for \( b_t(x, r) \) bounded and continuous in \((t, r)\) locally uniformly in \( x \), and the weak and strong uniqueness holds when \( b_t(x, r) \) is furthermore Lipschitz continuous in \( r \) uniformly in \((t, x)\). In [6] the initial density is in \( C^{\beta^+} := \cup_{\beta > 0} C^\beta \) for some \( \beta \in (0, \frac{1}{2}) \), the weak well-posedness is proved for \( b_t(x, r) := F(r)b_t(x) \), where \( b \in C([0, T]; C^{-\beta}) \) and \( F \) is bounded and Lipschitz continuous such that \( rF(r) \) is Lipschitz continuous in \( r \geq 0 \). See [7] and references within for the case with better drift.

2. In \((1.1)\) the noise does not point-wisely depend on the density. It seems that to solve SDEs with point-wisely density dependent noise, one needs stronger regularity for the initial density and the coefficients. For instance, [8] proved the well-posedness and studied the propagation of chaos for the following SDE with point-wisely density dependent noise:

\[ dX_t = b(\ell_{X_t}(X_t))dt + \sigma(\ell_{X_t}(X_t))dW_t, \]
where the initial distribution density is $C^{2+}$-smooth, $b$ is $C^2$-smooth, and $\sigma$ is uniformly elliptic and $C^3$-smooth.

In this paper, we only consider density dependent SDEs with singular (non-continuous in spatial) drifts, and leave to a forthcoming paper for the study of regular SDEs with point-wisely density dependent noise.

In Section 2, we state two main results of the paper which provide the well-posedness of (1.1) for density free noise and density dependent noise respectively, and explain the main idea of the proof. To realize the idea, in Section 3 we recall some heat kernel estimates based on [11] and present new estimates. With these estimates we prove the main results in Sections 4 and 5 respectively, and finally make an extension to the reflecting setting in Section 6.

## 2 Main results and idea of proof

To characterize the time-spatial singularity, we recall some spaces of locally integrable functions introduced in [13].

For $p \in [1, \infty]$ and $f \in B(\mathbb{R}^d)$, the space of measurable functions on $\mathbb{R}^d$, let

$$\|f\|_{L^p} := \left(\int_{\mathbb{R}^d} |f|^p(x) dx\right)^{\frac{1}{p}}, \quad \|f\|_{\tilde{L}^p} := \sup_{x \in \mathbb{R}^d} \|1_{B(x,1)}f\|_{L^p} < \infty,$$

where $B(z, r) := \{x \in \mathbb{R}^d : |x - z| \leq r\}, r > 0, z \in \mathbb{R}^d$. We write $f \in L^p (f \in \tilde{L}^p)$ if $\|f\|_{L^p} < \infty (\|f\|_{\tilde{L}^p} < \infty)$.

For any $p, q \in [1, \infty]$ and $f \in B([0, T] \times \mathbb{R}^d)$, the space of measurable functions on $[0, T] \times \mathbb{R}^d$, let

$$\|f\|_{L^p_q} := \left(\int_0^T \|f_t\|_{L^p}^q dt\right)^{\frac{1}{q}}, \quad \|f\|_{\tilde{L}^p_q} := \sup_{z \in \mathbb{R}^d} \left(\int_0^T \|1_{B(z,1)}f_t\|_{L^p}^q dt\right)^{\frac{1}{q}}.$$

We denote $f \in L^p_q (f \in \tilde{L}^p_q)$ if $\|f\|_{L^p_q} < \infty (\|f\|_{\tilde{L}^p_q} < \infty)$.

In the following the parameter $(p, q)$ will be taken from the class

$$\mathcal{K} := \{(p, q) \in (2, \infty] : \frac{d}{p} + \frac{2}{q} < 1\}.$$

For simplicity, we identify $L^\infty = \tilde{L}^\infty$ with $B_b(\mathbb{R}^d)$, the space of bounded measurable functions on $\mathbb{R}^d$, equipped with the uniform norm

$$\|f\|_\infty = \|f\|_{\tilde{L}^\infty} = \|f\|_\infty := \sup_{\mathbb{R}^d} |f|.$$

This uniform norm is defined for real functions on an abstract space. Similarly, $\tilde{L}^\infty = L^\infty = B_b([0, T] \times \mathbb{R}^d)$ is the space of bounded measurable functions on $[0, T] \times \mathbb{R}^d$ equipped with the uniform norm, and $L^\infty_k (\tilde{L}^\infty_k)$ is the space of functions $f \in B([0, T] \times \mathbb{R}^d)$ such that

$$\|f\|_{L^\infty_k} := \sup_{t \in [0, T]} \|f_t\|_{L^k} < \infty (\|f\|_{\tilde{L}^\infty_k} := \sup_{t \in [0, T]} \|f_t\|_{\tilde{L}^k} < \infty).$$
Finally, let $\nabla$ be the gradient in $\mathbb{R}^d$, and let $\|\nabla f\|_\infty$ denote the Lipschitz constant of a real function $f$ on $\mathbb{R}^d$.

In the following, we state our main results for density free noise and density dependent noise respectively, and briefly explain the main idea of proof.

### 2.1 Density free noise

In this part, we let $\sigma_t(x, \rho) = \sigma_t(x)$ do not depend on $\rho$. For $k > 1$ and a signed measure $\mu$ with density function $\ell_\mu(x) := \frac{\mu(dx)}{dx}$, let

$$\|\mu\|_{L^k} := \|\ell_\mu\|_{L^k}, \quad \|\mu\|_{\tilde{L}^k} := \|\ell_\mu\|_{\tilde{L}^k}.$$  

When $k = 1$, we define

$$\|\mu\|_{L^1} := \sup_{\|f\|_\infty \leq 1} |\mu(f)|, \quad \|\mu\|_{\tilde{L}^1} := \sup_{z \in \mathbb{R}^d} \sup_{\|f\|_\infty \leq 1} |\mu(1_{B(z,1)}f)|,$$

where $\mu(f) := \int_{\mathbb{R}^d} f d\mu$. Note that $\|\cdot\|_{L^1}$ is the total variation norm.

Let $\mathcal{P}$ be the set of all probability measures on $\mathbb{R}^d$. We will solve (1.1) with initial distributions in the classes

$$\mathcal{P}^k := \{\nu \in \mathcal{P} : \|\nu\|_{L^k} < \infty\}, \quad \tilde{\mathcal{P}}^k := \{\nu \in \mathcal{P} : \|\nu\|_{\tilde{L}^k} < \infty\}, \quad k \in [1, \infty],$$

which are complete metric spaces under distances $\|\nu_1 - \nu_2\|_{L^k}$ and $\|\nu_1 - \nu_2\|_{\tilde{L}^k}$ respectively.

**A** $a_t(x) := (\sigma_t, \sigma_t^\gamma)(x)$ and $b_t(x, r, \rho) = b_t^{(1)}(x) + b_t^{(0)}(x, r, \rho)$ satisfy the following conditions for some $k \in [1, \infty]$.

(A1) $a_t(x)$ is invertible with $\|a\|_{\infty} + \|a^{-1}\|_{\infty} < \infty$, and there exist constants $\alpha \in (0, 1)$ and $C > 0$ such that

$$\sup_{t \in [0,T]} \|a_t(x) - a_t(y)\| \leq C|x - y|^\alpha, \quad x, y \in \mathbb{R}^d.$$  

(A2) There exist $(p_0, q_0) \in \mathcal{K}$, $\theta > \frac{2}{q_0} + \frac{d}{p_0} - 1$, and $1 \leq f_0 \in \tilde{L}^{p_0}_{q_0}$ such that

$$|b_t^{(0)}(x, r, \rho) - b_t^{(0)}(x, \bar{r}, \bar{\rho})| \leq f_0(t, x) t^\theta (|r - \bar{r}| + |\rho - \bar{\rho}|)_{L_k},$$

$$|b_t^{(0)}(x, r, \rho)| \leq f_0(t, x), \quad (t, x) \in (0, T) \times \mathbb{R}^d, \quad r, \rho \in [0, \infty), \rho, \bar{\rho} \in \tilde{L}^k \cap \mathcal{D}^+_1.$$  

(A3) $b_t^{(1)}(0)$ is bounded in $t \in [0, T]$ and

$$\|\nabla b_t^{(1)}\|_{\infty} := \sup_{t \in [0,T]} \sup_{x \neq y} \frac{|b_t^{(1)}(x) - b_t^{(1)}(y)|}{|x - y|} < \infty.$$  

To ensure $\ell_X \in L^k$ for $\ell_X \in L^k$, we replace (A2) by the following (A$'_2$).
There exist $C \in (0, \infty), (p_0, q_0) \in \mathcal{X}, \theta > \frac{2}{q_0} + \frac{d}{p_0} - 1,$ and $0 \leq f_0 \in L_{q_0}^{p_0}$ such that
\[
|b^{(0)}_t(x, r, \rho) - b^{(0)}_s(x, s, \bar{\rho})| \leq \theta^{|f_0(t, x)}(|r - s| + \|\rho - \bar{\rho}\|_{L^k}),
\]
\[
|b^{(0)}_t(x, r, \rho)| \leq f_0(t, x), \quad (t, x) \in (0, T] \times \mathbb{R}^d, r, s \in [0, \infty), \rho, \bar{\rho} \in L^k \cap \mathcal{D}_+.
\]

Under the above assumptions, the following result ensures the well-posedness of (1.1) for initial distributions in $\mathcal{P}^k$ or $\mathcal{P}_\infty$ for
\[
k \in \left[\frac{p_0}{p_0 - 1}, \infty \right) \cap (k_0, \infty], \quad k_0 := \frac{d}{2\theta + 1 - 2q_0 - 1 - dp_0 - 1}.
\]
This explains the role played by the quantity $\theta$ in $(A_2)$ and $(A'_2)$: for bigger $\theta$, $(A_2)$ and $(A'_2)$ provide stronger upper bound conditions on $|b^{(0)}_t(x, r, \rho) - b^{(0)}_t(x, \tilde{\rho})|$ for small $t$, so that the SDE is solvable for initial distributions in larger classes $\mathcal{P}^k$ and $\mathcal{P}_\infty$. In particular, when $p_0 = \infty$ and $\theta$ is large enough such that $k_0 < 1$, we may take $k = 1$ so that the SDE is well-posed for any initial distribution $\nu \in \mathcal{P}$.

**Theorem 2.1.** Let $k \in \left[\frac{p_0}{p_0 - 1}, \infty \right)$ with $k > k_0 := \frac{d}{2\theta + 1 - 2q_0 - 1 - dp_0 - 1}$.

1. Under (A), for any $\nu \in \mathcal{P}^k$, (1.1) has a unique weak solution with $\mathcal{L}_{X_0} = \nu$ satisfying $\ell_X \in \tilde{L}_{\infty}^k$, and there exist an increasing function $\Lambda : [0, \infty) \to (0, \infty)$ such that for any two weak solutions $\{X^i\}_{i=1,2}$ of (1.1) with $\ell_{X^i} \in \tilde{L}_{\infty}^k$,
\[
\sup_{t \in [0, T]} \|\ell_{X^1_t} - \ell_{X^2_t}\|_{L^k} \leq \Lambda(\|\mathcal{L}_{X^1_0}\|_{L^k} \wedge \|\mathcal{L}_{X^2_0}\|_{L^k})\|\mathcal{L}_{X^1_0} - \mathcal{L}_{X^2_0}\|_{L^k}.
\]

If moreover $\sigma_t$ is weakly differentiable with
\[
\|\nabla \sigma\| \leq \sum_{i=1}^l f_i, \text{ for some } l \in \mathbb{N}, 0 \leq f_i \in \tilde{L}_{p_i}^{q_i}, (p_i, q_i) \in \mathcal{X}, 1 \leq i \leq l,
\]
then for any $X_0$ with $\mathcal{L}_{X_0} \in L^k$, (1.1) has a unique strong solution with $\ell_X \in \tilde{L}_{\infty}^k$.

2. Under (A) with $(A'_2)$ replacing $(A_2)$, assertions in (1) hold for $(\mathcal{P}^k, L^k, L^k)$ replacing $(\mathcal{P}^k, \tilde{L}_{\infty}, \tilde{L}_{\infty})$.

### 2.2 Density dependent noise

In this part, we allow $\sigma$ to be density dependent but make stronger assumptions on the initial density and the coefficients in the spatial variable.

For any $n \in \mathbb{Z}^+$, let $C^n_0(\mathbb{R}^d)$ be the class of real functions $f$ on $\mathbb{R}^d$ with continuous derivatives $\{\nabla^i f\}_{0 \leq i \leq n}$ such that
\[
\|f\|_{C^n_0} := \sum_{i=0}^n \|\nabla^i f\|_{\infty} < \infty.
\]

For any $n \in \mathbb{Z}^+$ and $\alpha \in (0, 1)$, $C^{n+\alpha}_b(\mathbb{R}^d)$ is the space of functions $f \in C^n_0(\mathbb{R}^d)$ such that
\[
\|f\|_{C^{n+\alpha}_b} := \|f\|_{C^n_0} + \sup_{x \neq y} \frac{|\nabla^n f(x) - \nabla^n f(y)|}{|x - y|^{\alpha}} < \infty.
\]
(B) There exist $1 \leq f_0 \in \tilde{L}^p_{q_0}$, $C \in (0, \infty)$ and $\alpha \in (0, 1)$, such that the following conditions hold for all $t \in (0, T]$, $x, y \in \mathbb{R}^d$, $r, \tilde{r} \in [0, \infty)$ and $\rho, \tilde{\rho} \in L^\infty \cap \mathcal{D}_+$:

$$
|b_t(x, r, \rho)| \leq f_0(t, x),
$$

$$
|b_t(x, r, \rho) - b_t(x, \tilde{r}, \tilde{\rho})| \leq C(|r - \tilde{r}| + \|\rho - \tilde{\rho}\|_\infty),
$$

$$
\|\sigma\|_\infty + \|\nabla \sigma\|_\infty + \|(\sigma \sigma^*)^{-1}\|_\infty \leq C,
$$

$$
\|\nabla \sigma_t(\cdot, \rho)(x) - \nabla \sigma_t(\cdot, \rho)(y)\| \leq C|x - y|^\alpha,
$$

$$
\|\sigma_t(\cdot, \rho) - \sigma_t(\cdot, \tilde{\rho})\|_{C_b} \leq C\|\rho - \tilde{\rho}\|_\infty.
$$

Theorem 2.2. Assume (B) and let $\beta \in (0, 1 - \frac{d}{p_0} - \frac{2}{q_0})$. For any initial value (initial density) with $\ell_{X_0} \in C_b^2(\mathbb{R}^d)$, (1.1) has a unique strong (weak) solution satisfying $\ell_X \in L^\infty$, and there exists a constant $c > 0$ such that

$$
\sup_{t \in [0, T]} \|\ell_{X_t}\|_{C_b^\beta} \leq c\|\ell_{X_0}\|_{C_b^\beta}.
$$

Moreover, there exists an increasing function $\Lambda : (0, \infty) \to (0, \infty)$ such that for any two solutions $\{X^i_t\}_{i=1,2}$ with $\ell_{X_0^i} \in C_b^2(\mathbb{R}^d)$ and $\ell_{X_t^i} \in L^\infty$,

$$
\sup_{t \in [0, T]} \|\ell_{X_t^1} - \ell_{X_t^2}\|_\infty \leq \Lambda(\|\ell_{X_0^1}\|_{C_b^\beta} \wedge \|\ell_{X_0^2}\|_{C_b^\beta})\|\ell_{X_0^1} - \ell_{X_0^2}\|_\infty.
$$

2.3 Idea of proof

For fixed $k \geq 1$ and $\nu \in \tilde{\mathcal{P}}^k$, let $\tilde{\mathcal{P}}^k_{\nu, T}$ be the set of all bounded measurable maps $\gamma : (0, T] \to \tilde{L}^k \cap \mathcal{D}_+$, $\gamma_0 = \nu$.

When $k = 1$, the initial value $\gamma_0$ may be singular, and if it is absolutely continuous we regard it as its density function.

Then $\tilde{\mathcal{P}}^k_{\nu, T}$ is complete under the metric

$$
\tilde{d}_{k, \lambda}(\gamma^1, \gamma^2) := \sup_{t \in [0, T]} e^{-\lambda t}\|\gamma^1_t - \gamma^2_t\|_{L^k}, \quad \gamma^1, \gamma^2 \in \tilde{\mathcal{P}}^k_{\nu, T}
$$

for $\lambda > 0$. We define $(\mathcal{P}_{\nu, T}^k, d_{k, \lambda})$ in the same way with $(L^k, \mathcal{P}^k)$ replacing $(\tilde{L}^k, \tilde{\mathcal{P}}^k)$.

For any $\gamma \in \tilde{\mathcal{P}}^k_{\nu, T}$, let

$$
b^1_t(x) := b_t(x, \gamma_t(x), \gamma_t), \quad \sigma^1_t(x) := \sigma_t(x, \gamma_t), \quad t \in (0, T], x \in \mathbb{R}^d.
$$

Then for $\nu := \mathcal{L}_{X_0} \in \tilde{L}^k$, (1.1) has a unique (weak or strong) solution with $\ell_X \in \tilde{L}_\infty^k$ if we could verify the following two things:
1) For any $\gamma \in \tilde{\mathcal{P}}^{k,\nu,T}$, the SDE

\begin{equation}
\text{d}X^\gamma_t = b^\gamma_t(X^\gamma_t)\text{d}t + \sigma^\gamma_t(X^\gamma_t)\text{d}W_t, \quad t \in [0,T], \quad X^\gamma_0 = X_0
\end{equation}

is (weakly or strongly) well-posed, and

$$\gamma \mapsto \Phi^{\nu}_t : \ell^X_{\gamma} t, t \in (0,T], \Phi^{\nu}_0 : \gamma_0 = \nu$$

provides a map $\Phi^{\nu} : \tilde{\mathcal{P}}^{k,\nu,T} \to \tilde{\mathcal{P}}^{k,\nu,T}$.

2) $\Phi^{\nu}$ has a unique fixed point $\bar{\gamma}$ in $\tilde{\mathcal{P}}^{k,\nu,T}$.

Indeed, from these we see that $X_t := X^\bar{\gamma}_t$ is the unique (weak or strong) solution of (1.1) with $L^X \in \tilde{L}^{k}$.

To verify 1) and 2), in Section 2 we recall some heat kernel upper bounds of [11], and estimate the $\tilde{L}^{p}_q$-$\tilde{L}^{p'}_q$ norm for time inhomogeneous semigroups.

### 3 Heat kernel estimates

We first recall a result of [11]. Let

$$a : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^d, \quad b : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d.$$

We consider heat kernel estimates for the time dependent second order differential operator

$$L^{a,b}_t := \frac{1}{2} \text{tr}\{a_t \nabla^2\} + \nabla b_t$$

satisfying the following conditions.

- $(H^{a,b})$ $a_t(x)$ is invertible and there exist constants $C > 0$ and $\alpha \in (0,1)$ such that

\begin{align*}
\|b_t(0)\|_\infty + \|a\|_\infty + \|a^{-1}\|_\infty & \leq C, \\
\sup_{t \in [0,T]} \|a_t(x) - a_t(y)\| & \leq C|x - y|^\alpha, \\
\sup_{t \in [0,T]} |b_t(x) - b_t(y)| & \leq C(|x - y| + |x - y|^\alpha), \quad x, y \in \mathbb{R}^d.
\end{align*}

- $(H^a)$ $a_t(x)$ is differentiable in $x$, and there exist constants $C \in (0,\infty)$ and $\alpha \in (0,1)$ such that

\begin{equation}
\|\nabla a\|_\infty \leq C, \quad \sup_{t \in [0,T]} \|\nabla a_t(x) - \nabla a_t(y)\| \leq C|x - y|^\alpha, \quad x, y \in \mathbb{R}^d.
\end{equation}

Under $(H^{a,b})$, for any $s \in [0,T)$, the SDE

$$\text{d}X^x_{s,t} = b_s(X^x_{s,t})\text{d}s + \sqrt{a_s(X^x_{s,t})}\text{d}W_s, \quad t \in [s,T], \quad X^x_{s,s} = x \in \mathbb{R}^d$$

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is weakly well-posed with semigroup \( \{P_{s,t}^{a,b}\}_{0 \leq s \leq t \leq T} \) and transition density \( \{p_{s,t}^{a,b}\}_{0 \leq s \leq t \leq T} \) given by

\[
P_{s,t}^{a,b} f(x) = \int_{\mathbb{R}^d} p_{s,t}^{a,b}(x,y)f(y)dy = \mathbb{E}[f(X_{s,t}^x)], \quad f \in \mathcal{B}(\mathbb{R}^d),
\]

and we have the following Kolmogorov backward equation (see Remark 2.2 in [11])

\[
\partial_s P_{s,t}^{a,b} f = -L_s P_{s,t}^{a,b} f, \quad f \in C_b^\infty(\mathbb{R}^d), s \in [0,t], t \in (0,T].
\]

Next, we denote \( \psi_{s,t} = \theta_{t,s}^{(1)} \) presented in [11]. Then \( \psi_{s,t} \) is a family of diffeomorphisms on \( \mathbb{R}^d \) satisfying

\[
\sup_{0 \leq s \leq t \leq T} \{ \|\nabla \psi_{s,t}\|_\infty + \|\nabla \psi_{s,t}^{-1}\|_\infty \} \leq \delta
\]

for some constant \( \delta > 0 \) depending on \( \alpha, C \). For any \( \kappa > 0 \), consider the Gaussian heat kernel

\[
p_t^\kappa(x) := (\kappa \pi t)^{-\frac{d}{2}} e^{-\frac{|x|^2}{\kappa t}}, \quad t > 0, \quad x \in \mathbb{R}^d.
\]

The following result is taken from [11] Theorem 1.2.

**Theorem 3.1** ([11]). **Assume** \((H^{a,b})\). **Then** there exist constants \( c, \kappa > 0 \) depending on \( C, \alpha \) such that

\[
|\nabla \psi_{s,t}^{a,b}(\cdot, y)(x)| \leq c(t-s)^{-\frac{1}{2}} p_{t-s}^\kappa(\psi_{s,t}(x) - y),
\]

\[
i = 0, 1, 2, \quad 0 \leq s < t \leq T, \quad x, y \in \mathbb{R}^d.
\]

If moreover \((H^a)\) holds, then

\[
|\nabla p_{s,t}^{a,b}(x, \cdot)(y)| \leq c(t-s)^{-\frac{1}{2}} p_{t-s}^\kappa(\psi_{s,t}(x) - y), \quad 0 \leq s < t \leq T, \quad x, y \in \mathbb{R}^d,
\]

and for any \( \beta \in (0,1) \) there exists a constant \( c' > 0 \) depending on \( C, \alpha, \beta \) such that

\[
|\nabla p_{s,t}^{a,b}(\cdot, y)(x) - \nabla p_{s,t}^{a,b}(\cdot, y')(x)| + |\nabla p_{s,t}^{a,b}(x, \cdot)(y) - \nabla p_{s,t}^{a,b}(x, \cdot)(y')| \leq c'|y - y'|^{1-\beta}(t-s)^{-\frac{\beta}{2}} \{(p_{t-s}^\kappa(\psi_{s,t}(x) - y) + p_{t-s}^\kappa(\psi_{s,t}(x) - y'))
\]

\[
0 \leq s < t \leq T, \quad x, x', y \in \mathbb{R}^d.
\]

For any \( f \in \mathcal{B}(\mathbb{R}^d) \cup \mathcal{B}^+(\mathbb{R}^d) \), let

\[
P_t^\kappa f(x) := \int_{\mathbb{R}^d} p_t^\kappa(x-y)f(y)dy,
\]

\[
\hat{P}_{s,t}^\kappa f(x) := \int_{\mathbb{R}^d} p_{t-s}^\kappa(\psi_{s,t}(x) - y)f(y)dy,
\]

\[
\tilde{P}_{s,t}^\kappa f(x) := \int_{\mathbb{R}^d} p_{t-s}^\kappa(\psi_{s,t}(x) - y)f(y)dy, \quad 0 \leq s < t \leq T, \quad x \in \mathbb{R}^d.
\]

It is well known that for some constant \( c > 0 \),

\[
\|P_t^\kappa\|_{L^p \rightarrow L^{p'}} := \sup_{\|f\|_p \leq 1} \|P_t^\kappa f\|_{L^{p'}} \leq ct^{-\frac{d(p'-p)}{2pp'}}, \quad t > 0, 0 \leq p \leq p' \leq \infty.
\]
Combining this with (3.2) we obtain

\[
\|\hat{P}^n_{s,t}\|_{L^p \to L^{p'}} + \|\tilde{P}^n_{s,t}\|_{L^p \to L^{p'}} \leq C(t - s) \frac{d(y-p)}{2^{pp'}} , \quad 0 \leq s < t \leq T, 1 \leq p \leq p' \leq \infty.
\]

for some different constant \( c > 0 \). Below we extend this estimate to the \( \tilde{L}_q^p \)-norm. For any \( t \in (0, T] \), let

\[
\|f\|_{\tilde{L}_q^p(t)} := \sup_{z \in \mathbb{R}^d} \left( \int_0^t \|1_{B(z,t)} f_s\|^q_{L^p} ds \right)^{\frac{1}{q}}, \quad p, q \in [1, \infty].
\]

**Lemma 3.2.** There exists a constant \( c > 0 \) such that for any \( 0 \leq s < t \leq T, \ 1 \leq p \leq p' \leq \infty \) and \( q \in [1, \infty] \),

\[
\|\hat{P}^n_{s,t} f\|_{\tilde{L}_q^p(t)} + \|\tilde{P}^n_{s,t} f\|_{\tilde{L}_q^p(t)} \leq C(t - s)^{-\frac{d(p'-p)}{2pp'}} \|f\|_{\tilde{L}_q^p(t)}, \quad f \in \mathcal{B}^+(\mathbb{R}^d),
\]

where and in the sequel, \( (t - \cdot)(s) := t - s \) is a function on \( [0, t] \), and

\[
\sup_{z \in \mathbb{R}^d} \|g \hat{P}^n_{s,t} (1_{B(z,t)} f)\|_{L^1} \leq C(t - s)^{-\frac{d(p'-p)}{2pp'}} \|g\|_{\tilde{L}_q^p(t)} \|f\|_{L^p}, \quad f, g \in \mathcal{B}^+(\mathbb{R}^d).
\]

**Proof.** Let \( B_n := \{v \in \mathbb{Z}^d : |v|_1 := \sum_{i=1}^d |v_i| = n\}, n \geq 0 \). By (3.2), we find a constant \( \varepsilon \in (0, 1) \) such that for any \( n \geq 0 \) and \( 0 \leq s < t \leq T \),

\[
|\psi_{s,t}(x) - y|^2 \geq \varepsilon n^2, \quad x \in B(\psi_{s,t}^{-1}(z), \varepsilon), \quad y \in \bigcup_{v \in B_n} B(z + v, d), \quad z \in \mathbb{R}^d.
\]

Combining this with (3.7), we find constants \( c_2, c_3, c_4 > 0 \) such that for any \( z \in \mathbb{R}^d, 0 \leq s < t \leq T, \) and \( f, g \in \mathcal{B}^+_b(\mathbb{R}^d), \)

\[
\|1_{B(\psi_{s,t}^{-1}(z),\varepsilon)} g \hat{P}^n_{s,t} f\|_{L^1} \leq \sum_{n=0}^{\infty} \sum_{v \in \mathbb{Z}^d} |v|_1 = n \|1_{B(\psi_{s,t}^{-1}(z),1)} g \hat{P}^n_{s,t} (1_{B(z+v,d)} f)\|_{L^1}
\]

\[
\leq \sum_{n=0}^{\infty} \sum_{v \in \mathbb{B}_n} \int_{\mathbb{R}^d \times \mathbb{R}^d} \|1_{B(\psi_{s,t}^{-1}(z),\varepsilon)} g(x) p^n_{s-t}(\psi_{s,t}(x) - y)\|_{1_{B(z+v,d)}} \|f\|_{L^d} dxdy
\]

\[
\leq c_2 \sum_{n=0}^{\infty} \sum_{v \in \mathbb{B}_n} e^{-c_3 n^2 (t-s)^2} \int_{\mathbb{R}^d \times \mathbb{R}^d} \|1_{B(\psi_{s,t}^{-1}(z),\varepsilon)} g(x) p^n_{s-t}(\psi_{s,t}(x) - y)\|_{1_{B(z+v,d)}} \|f\|_{L^d} dxdy
\]

\[
\leq c_3 \sum_{n=0}^{\infty} \sum_{v \in \mathbb{B}_n} e^{-c_3 n^2 (t-s)^2} \|P^n_{s-t} (1_{B(\psi_{s,t}^{-1}(z),\varepsilon)} g)\|_{1_{B(z+v,d)}} \|f\|_{L^d}
\]

\[
\leq c_3 \sum_{n=0}^{\infty} \sum_{v \in \mathbb{B}_n} e^{-c_3 n^2 (t-s)^2} \|P^n_{s-t} (1_{B(\psi_{s,t}^{-1}(z),\varepsilon)} g)\|_{L^{p'}} \|1_{B(z+v,d)} f\|_{L^p}
\]

\[
\leq c_4 (t - s)^{-\frac{d(p'-p)}{2pp'}} \|g\|_{L^{p'}} \sum_{n=0}^{\infty} \sum_{v \in \mathbb{B}_n} e^{-c_3 n^2 (t-s)^2} \|1_{B(z+v,d)} f\|_{L^p}.
\]
Since
\begin{equation}
(3.11) \sup_{z \in \mathbb{R}^d} \left( \int_0^t \|1_B(z,d)f\|_{L^p}\,ds \right)^{\frac{1}{q}} \leq c_5 \|f\|_{\tilde{L}^q_0(t)}
\end{equation}
holds for some constant $c_5 > 0$, we find a constant $c_6 > 0$ such that this and Hölder’s inequality imply
\begin{align*}
&\sup_{z \in \mathbb{R}^d} \left( \int_0^t \|1_B(z,d)\bar{f}_s\|_{L^p}\,ds \right)^{\frac{1}{q}} = \sup_{z \in \mathbb{R}^d} \left( \int_0^t \|1_B(\psi_{s,t}(z),\varepsilon)\bar{f}_{s,t}f_s\|_{L^p}\,ds \right)^{\frac{1}{q}} \\
&\leq \sup_{z \in \mathbb{R}^d} \left( \int_0^t \left\{ c_4 \|1_B(z,d)f_s\|_{L^p}(t-s)^{-\frac{d(p'-p)}{2pp'}} \right\}^{\frac{1}{q}} \,ds \right) \sup_r \sum_{n=0}^\infty \sum_{v \in \mathbb{B}_n} e^{-\frac{n^2}{c_3 T}} \\
&\leq c_6 (t-s)^{-\frac{d(p'-p)}{2pp'}} \|f\|_{\tilde{L}^q_0(t)} \sum_{n=0}^\infty \sum_{v \in \mathbb{B}_n} e^{-\frac{n^2}{c_3 T}}.
\end{align*}

This implies the upper bound for $\bar{f}^k_\nu$ in (3.9), by noting that for some constant $K > 0$
\begin{equation}
(3.12) \sum_{n=0}^\infty \sum_{v \in \mathbb{B}_n} e^{-\frac{n^2}{c_3 T}} \leq \sum_{n=0}^\infty K(1 + n^{d-1})e^{-\frac{n^2}{c_3 T}} < \infty.
\end{equation}

By (3.2) and integral transforms, the estimate on $\bar{f}_{s,t}$ follows from that of $\bar{f}^k_\nu$.

Similarly, we find a constant $K > 1$ such that
\begin{align*}
\|g\bar{f}^k_\nu(1_B(\psi_{s,t}(z),1)f)\|_{L^1} &\leq \sum_{n=0}^\infty \sum_{v \in \mathbb{B}_n} \|1_B(z+v,d)g\bar{f}^k_\nu(1_B(\psi_{s,t}(z),\varepsilon)f)\|_{L^1} \\
&\leq \sum_{n=0}^\infty \sum_{v \in \mathbb{B}_n} \int_{\mathbb{R}^d \times \mathbb{R}^d} |1_B(z+v,d)g|\bar{f}^k_\nu(\psi_{s,t}(z)-y)|1_B(\psi_{s,t}(z),\varepsilon)f|(y)\,dy \\
&\leq K(t-s)^{-\frac{d(p'-p)}{2pp'}} \sum_{n=0}^\infty \sum_{v \in \mathbb{B}_n} e^{-\frac{n^2}{c_3 (t-s)}} \|g\|_{L_p} \|1_B(z,d)f\|_{L^p}.
\end{align*}

This together with (3.11) and (3.12) implies (3.10) for some $c > 0$.

\section{Proof of Theorem \ref{thm2.1}}

We first prove assertion 1), i.e. the well-posedness of (2.5). For $\gamma \in \mathcal{H}^k_\nu_t$, we denote
\begin{align*}
\sigma^\gamma_t(x) &:= \sigma_t(x, \gamma_t), \quad b^{\gamma,0}_t(x) := b^{(0)}_t(x, \gamma_t(x), \gamma_t), \\
b^\gamma_t(x) &:= b_t(x, \gamma_t(x), \gamma_t) = b^{(1)}_t(x) + b^{\gamma,0}_t(x), \quad t \in [0,T], x \in \mathbb{R}^d.
\end{align*}
Lemma 4.1. Assume (A) with (A1) holding for $\sigma^\gamma$ replacing $\sigma$ uniformly in $\gamma \in \tilde{\mathcal{D}}_{\nu,T}$, where $k \in [\frac{p_0}{p_{01}} , \infty)$. Then (2.3) is weakly well-posed for any $L_{X_0} \equiv L^k$ and $\gamma \in \tilde{\mathcal{D}}_{\nu,T}$. If for any $\beta \in (0,1)$ there exists a constant $c > 1$ independent of $\nu$ and $\gamma$ such that $\Phi^\gamma_{\nu} := \ell_{X_{\nu}^\gamma}$ for $L_{X_0} = \nu$, satisfies

\begin{equation}
\|\Phi^\gamma_{\nu}\|_{L_{\nu}^0} \leq c\|\nu\|_{\tilde{L}^k}.
\end{equation}

Moreover, under the assumption with (A'$_2$) replacing (A$_2$), the assertion holds for $(\mathcal{D}_{\nu,T}^k, L^k)$ replacing $(\tilde{\mathcal{D}}_{\nu,T}^k, \tilde{L}^k)$.

**Proof.** (a) By (A$_2$), we have

\begin{equation}
\sup_{\gamma \in \tilde{\mathcal{D}}_{\nu,T}^k} \|b^{\gamma,0}\| \leq f_0, \quad \|f_0\|_{L_{p_0}^0} < \infty.
\end{equation}

According to [17], see also [13, Theorem 1.1(1)], this together with (A$_1$) and (A$_3$) imply the well-posedness of (2.5). Moreover, by Theorem 6.2.7(ii)-(iii) in [3], the distribution density \( \ell_{X_{\gamma}^\nu} \) exists.

(b) To estimate $\Phi^\gamma_{\nu}$ for $\gamma \in \tilde{\mathcal{D}}_{\nu,T}^k$, consider the SDE

\begin{equation}
d\tilde{X}_{\gamma}^\nu = b^{(1)}_{\gamma}(\tilde{X}_{\gamma}^\nu)ds + \sigma^{\gamma}(\tilde{X}_{\gamma}^\nu)dW_s, \; s \in [0,t], \; \tilde{X}_{\gamma}^\nu = X_{0^\gamma} = X_0 \text{ with } L_{X_0} = \nu.
\end{equation}

Let $a^{\gamma} := \sigma^\gamma(\sigma^\gamma)^*$, then

\[
\mathbb{E}[f(\tilde{X}_{\gamma}^\nu)] = \mathbb{E}[(P_{0,t}^{\gamma,a^{(1)}} f)(X_0)] = \int_{\mathbb{R}^d \times \mathbb{R}^d} P_{0,t}^{\gamma,a^{(1)}}(x,y)f(y)\nu(dx)dy, \; f \in \mathcal{B}^+(\mathbb{R}^d),
\]

and (3.3) holds for $P_{s,t}^{\gamma,a^{(1)}}$ with constants $c, \kappa > 0$ uniformly in $\gamma$. So, we find a constant $c_1 > 0$ such that

\begin{equation}
\mathbb{E}[f(\tilde{X}_{t}^\nu)] \leq c_1 \int_{\mathbb{R}^d} (P_{0,t}^{\kappa} f)(x)\nu(dx) = c_1 (P_{0,t}^{\kappa} \nu)(f), \; f \in \mathcal{B}^+(\mathbb{R}^d),
\end{equation}

where

\begin{equation}
(P_{0,t}^{\kappa} \nu)(dy) := \left( \int_{\mathbb{R}^d} P_{0,t}^{\kappa}(x,y)\nu(dx) \right)dy, \; t \in (0,T], \nu \in \mathcal{P}.
\end{equation}

On the other hand, let

\[
R_t := e^{\int_0^t (\xi_s dW_s) - \frac{1}{2} f_0^2 |\xi_s|^2 ds}, \; \xi_s := \{\sigma_s^\gamma(\sigma_s^\gamma)^{-1} b_s^{\gamma,0}\}(X_s).
\]

By (4.2), the uniform boundedness of $\|\sigma^\gamma(\sigma^\gamma)^{-1}\|_{\infty}$, and Khasminskii’s estimate implied by the Krylov’s estimate in [17, Theorem 3.1] (see the proof of [13, Lemma 4.1(ii)]), we find a map $K_\gamma : [1, \infty) \rightarrow (0, \infty)$ such that

\begin{equation}
K_\gamma(p) := (\mathbb{E}[R_T^p])^{\frac{1}{p}} < \infty, \; p \geq 1.
\end{equation}
By Girsanov’s theorem,
\[ \tilde{W}_s := W_s - \int_0^s \xi_r \, dr, \quad s \in [0, t] \]
is an \( m \)-dimensional Brownian motion under the probability measure \( Q_t := R_t \mathbb{P} \), with which the SDE \( \text{(4.3)} \) reduces to
\[ d\tilde{X}_s = b^\gamma_s(\tilde{X}_s) \, ds + \sigma^\gamma_s(\tilde{X}_s) \, d\tilde{W}_s, \quad s \in [0, t], \tilde{X}_0 = X^\gamma_0. \]

By the weak uniqueness, the law of \( X^\gamma_t \) under \( \mathbb{P} \) coincides with that of \( \tilde{X}_t \) under \( Q_t \). Combining this with \( \text{(4.4)}, \text{(4.6)} \) and \( \text{(3.10)} \), for any \( p > 1 \) and \( k' \geq k \) we find constants \( c_1(p), c_2(p) > 0 \) such that
\[\int_{\mathbb{R}^d} \left\{ \left( \Phi^\nu_1(1_B) \right) f \right\}(y) \, dy = \mathbb{E}\left[ (1_B(1,1)) (X^\gamma_t) \right] = \mathbb{E}\left[ R_t (1_B(1,1)) (X^\gamma_t) \right] \]
\[ \leq \left( \mathbb{E} \left[ R_t^{\frac{\nu^p}{p}} \right] \right)^{\frac{1}{p}} \left( \mathbb{E} \left[ (1_B(1,1))^p \right] \right)^{\frac{1}{p}} \leq c_1(p) \left( \int_{\mathbb{R}^d} \left\{ \left( \hat{P}^\nu_{0,t} (1_B(1,1)) \right) f \right\}(x) \nu(dx) \right)^{\frac{1}{p}} \]
\[ \leq c_2(p) \left\| \nu \right\|^{\frac{1}{p}} t^{\frac{k'(k'-k) - 1}{2k'}}, \quad t \in (0, T), f \in \mathcal{B}^+(\mathbb{R}^d). \]

Therefore, for any \( \nu \in \mathcal{B}^k \),
\[\| \Phi^\nu_1 \|_{L^{\frac{pk'}{pk'-k+1}}} \leq c_2(p) \left\| \nu \right\|^{\frac{1}{p}} t^{\frac{k'(k'-k) - 1}{2k'}}, \quad p > 1, k' \geq k, \gamma \in \mathcal{B}_T^k, t \in (0, T), \]
where for \( k' = k = \infty \) we set \( \frac{pk'}{pk'-k+1} := \frac{p}{p-1}, \frac{d(k'-k)}{2kk'} := 0 \). Using \( \text{(3.3)} \) replacing the estimate in Lemma \( \text{[3.2]} \) we find a \( c : (1, \infty) \rightarrow (0, \infty) \) such that
\[\| \Phi^\nu_1 \|_{L^{\frac{pk'}{pk'-k+1}}} \leq c(p) \left\| \nu \right\|^{\frac{1}{p}} t^{\frac{k'(k'-k) - 1}{2k'}}, \quad p > 1, k' \geq k, \gamma \in L^\infty, t \in (0, T). \]

(c) By the backward Kolmogorov equation \( \text{[3.1]} \) and Itô’s formula, for any \( f \in C^\infty_0(\mathbb{R}^d) \) we have
\[d\{ (P_{s,t}^\gamma \gamma b^{(1)} (X^\gamma_s)) \} = \{ (\partial_s + L_{s,t}^\gamma b^{(1)}) P_{s,t}^\gamma \gamma b^{(1)} (X^\gamma_s) \} ds + dM_s \]
\[= \{ \nabla b^{(1)}_{s,t} f \} (X^\gamma_s) ds + dM_s, \quad s \in [0, t] \]
for some martingale \( M_s \). Then
\[\mathbb{E}[f(X^\gamma_t)] = \mathbb{E}[P_{t,t}^{\gamma, b^{(1)}} f(X^\gamma_t)] \]
\[= \mathbb{E}[P_{0,t}^{\gamma, b^{(1)}} f(X^\gamma_0)] + \int_0^t \mathbb{E}[(\nabla b^{(1)}_{s,t} f) \{ X^\gamma_s \}] ds, \quad s \in [0, t]. \]
We explain that the last term in \( \text{(4.9)} \) exists. Indeed, by \( \text{[13]} \) Theorem 1.1(2)], there exists a constant \( c_2 > 0 \) such that
\[\| \nabla P_{s,t}^{\gamma, b^{(1)}} f \|_\infty \leq c_2 \| \nabla f \|_\infty, \quad 0 \leq s \leq t, f \in C^1_0(\mathbb{R}^d), \]
so that (4.2) and Krylov’s estimate (see Theorem 3.1 in [17]) yield
\[\mathbb{E}\left(\int_0^t |(\nabla b_{k,0} P_{s,t}^{0,1}) f(X_s)\,ds\right) \leq \mathbb{E}\left(\int_0^t c_2 \|f\|_\infty |b_{k,0}(X_s)|\,ds\right) < \infty, \quad n \geq 1.\]

Noting that \(\Phi_\nu \gamma := \ell_{X_\gamma} \) and \(P_{s,t}^{\alpha,1} f(x) = \int_{\mathbb{R}^d} p_{s,t}^{\alpha,1} (x, y) f(y)\,dy\), (4.9) is equivalent to
\[\int_{\mathbb{R}^d} \{\Phi_\nu f\}(y)\,dy = \int_{\mathbb{R}^d \times \mathbb{R}^d} \nu(x) p_{s,t}^{\alpha,1}(x, y) f(y)\,dx\,dy + \int_0^t ds \int_{\mathbb{R}^d \times \mathbb{R}^d} (\Phi_\nu \gamma)(x) \{\nabla b_{k,0} P_{s,t}^{\alpha,1}(\cdot, y)(x)\} f(y)\,dy,\quad f \in C_0^\infty(\mathbb{R}^d), s \in [0, t].\]

Thus,
\[(\Phi_\nu \gamma)(y) = \int_{\mathbb{R}^d} p_{0,t}^{\alpha,1}(x, y) \nu(dx) + \int_0^t ds \int_{\mathbb{R}^d} (\Phi_\nu \gamma)(x) \{\nabla b_{k,0} P_{s,t}^{\alpha,1}(\cdot, y)(x)\} f(y)\,dy, \quad t \in [0, T].\]

By (3.3) for \(p = p', \|\hat{P}_t^{\nu,\nu}\|_{L^1} \leq K\|\nu\|_{L^1}\) holds for some constant \(K > 0\). Combining this with (4.9, 4.11) and (4.10), we find a constant \(c_3 > 0\) such that
\[
\|\Phi_\nu \gamma\|_{L^1} \leq c_3 \|\nu\|_{L^1} + c_3 \sup_{z \in \mathbb{R}^d} \int_0^t (t - s)^{-\frac{3}{2}} \|\hat{P}_t^{\nu,\nu} \{\Phi_\nu \gamma\} f_0(s, \cdot)\|_{L^1} \,ds, \quad l \in [1, \infty].
\]

By \(k > k_0\) and \(k \geq \frac{p_0}{p_0 - 1}\), for any \(l \in (k_0, k] \cap [\frac{p_0}{p_0 - 1}, k]\) we have
\[
q_l := \frac{p_0 l}{p_0 + l} \in (1, l], \quad \frac{1}{q_l} = \frac{1}{p_0} + \frac{1}{l},
\]
and \((p_0, q_0) \in \mathcal{K}\) implies
\[
\frac{1}{2} + \frac{d(l - q_l)}{2lq_l} = \frac{1}{2} + \frac{d}{2p_0} = : \delta' < \frac{q_0 - 1}{q_0}.
\]

Combining these with (3.9) for \((p', p) = (l, q_l)\) and applying Hölder’s inequality, we find a constant \(c_4 > 0\) such that
\[
\int_0^t (t - s)^{-\frac{3}{2}} \|\hat{P}_t^{\nu,\nu} \{\Phi_\nu \gamma\} f_0(s, \cdot)\|_{L^1} \,ds \leq c_4\|f_0\|_{L^{q_0,1}(t)}\|\Phi_\nu \gamma\|_{L^{q_0,1}(t)} \leq c_4\|f_0\|_{L^{q_0,1}(t)}\|\Phi_\nu \gamma\|_{L^{q_0,1}(t)}, \quad l \in (k_0, k] \cap \left[\frac{p_0}{p_0 - 1}, k\right],
\]
where \{(t - \cdot)^{-\delta'} \Phi_\nu\}(s, x) := (t - s)^{-\delta'} \Phi_\nu(x). This together with (4.11) implies that for some constant \(c_5 > 0\),
\[
\|\Phi_\nu \gamma\|_{L^1} \leq c_5\|\nu\|_{L^1} + c_5\|f_0\|_{L^{q_0,1}(t)}\left(\int_0^t (t - s)^{-\delta'}\|\Phi_\nu \gamma\|_{L^{q_0,1}}\,ds\right)^{\frac{q_0}{q_0 - 1}}, \quad t \in [0, T], l \in (k_0, k) \cap \left[\frac{p_0}{p_0 - 1}, k\right].
\]
Similarly, using (3.8) replacing Lemma 3.2, we derive the same estimate for \( L^l \) replacing \( \hat{L}^l \) and \( \| f_0 \|_{L_{q_0}^{p_0}} \) replacing \( \| f_0 \|_{\hat{L}_{q_0}^{p_0}} \):

\[
\| \Phi_t^\nu \|_{L^l} \leq c_5 \| \nu \|_{L^l} + c_5 \| f_0 \|_{L_{q_0}^{p_0}} \left( \int_0^t \left\{ (t-s)^{-d'} \| \Phi_s^\nu \|_{L^l} \right\}^{q_0/q_0-1} ds \right)^{q_0/q_0-1},
\]

(4.15)

\( t \in [0, T], l \in (k_0, k] \cap \left[ \frac{p_0}{p_0 - 1}, k \right] \).

Below we prove (4.1) by considering two different situations.

\((c_1) k < \infty.\) For any \( k' \in (k, \infty) \) we have

\[
p_{k,k'} := \frac{k(k' - 1)}{k'(k - 1)} > 0, \quad \frac{p_{k,k'}k'}{p_{k,k'}k' - k' + 1} = k.
\]

Noting that

\[
\lim_{k' \downarrow k} \frac{d(k' - k)}{2kk'p_{k,k'}} = 0,
\]

by (4.13) we find \( k' > k \) such that

\[
\varepsilon_{k,k'} := \frac{d(k' - k)}{2kk'p_{k,k'}} \in \left( 0, 1 - \frac{\delta'q_0}{q_0 - 1} \right).
\]

Combining this with (4.7) and (4.14) for \( l = k \), we find a constant \( K > 0 \) such that

\[
\sup_{t \in [0, T]} \| \Phi_t^\nu \|_{\hat{L}^k} \leq K \| \nu \|_{\hat{L}^k} + K \sup_{t \in [0, T]} \left( \int_0^t (t-s)^{-\frac{q_0k'}{q_0-1}s^{-\varepsilon_{k,k'}L}p_{k,k'}^{q_0/q_0-1}} ds \right)^{q_0/q_0-1} < \infty.
\]

Therefore, by the generalized Gronwall inequality (see [16]), (4.13) and (4.14) implies (4.1).

When \( f_0 \in L_{q_0}^{p_0} \), by using (4.8) and (4.15) replacing (4.7) and (4.14), we obtain this estimate for \( L \) replacing \( \hat{L} \).

\((c_2) k = \infty.\) We take \( k' = k = \infty \), so that by (4.7), for any \( p > 1 \) we find a constant \( c(p) > 0 \) such that

\[
\| \Phi_t^\nu \|_{\hat{L}^k} \leq c(p) \| \nu \|_{\hat{L}^k}.
\]

Combining this with (4.14) for \( l \in \left( \frac{p_0}{p_0 - 1} \lor k_0, \infty \right) \) and \( p := \frac{l}{l-1} > 1 \), we obtain

\[
\sup_{t \in [0, T]} \| \Phi_t^\nu \|_{\hat{L}^l} < \infty,
\]

so that by the generalized Gronwall inequality, (4.14) implies (4.1) for \( l \in \left( \frac{p_0}{p_0 - 1} \lor k_0, \infty \right) \) replacing \( k = \infty \) with a uniform constant \( c > 0 \). By letting \( l \uparrow k = \infty \), we prove (4.1).

Noting that a probability density function \( \rho \in L^\infty \) implies \( \rho \in L^l \) for any \( l \geq 1 \), when \( f_0 \in L_{q_0}^{p_0} \) we prove (4.1) for \( L \) replacing \( \hat{L} \) by using (4.8) and (4.15) replacing (4.7) and (4.14).

\( \square \)
Proof of Theorem 2.1(1). By Lemma 3.2, (2.5) is weakly well-posed. By [17] or [13], it is also strongly well-posed provided (2.2) holds. Thus, as explained in the end of Section 1 that for the weak or strong well-posedness of (1.1), it suffices to prove that $\Phi^\nu$ has a unique fixed point in $\mathcal{D}^k_{\nu,T}$. In general, for any $\nu_1, \nu_2 \in \mathcal{D}^k_{\nu,T}$ and $\gamma^1, \gamma^2 \in \mathcal{D}^k_{\nu,T}$, we estimate

$$
\tilde{d}_{k,\lambda}(\Phi^{\nu_1,\gamma^1}, \Phi^{\nu_2,\gamma^2}) := \sup_{t \in [0,T]} e^{-\lambda t} \|\Phi^\nu_t \gamma^1 - \Phi^\nu_t \gamma^2\|_{\tilde{L}^k}, \ \lambda > 0.
$$

By (4.10), (A2) and (3.3), we find a constant $c_1 > 0$ such that

$$
\|\Phi^{\nu_1} \gamma^1 - \Phi^{\nu_2} \gamma^2\|_{\tilde{L}^k} 
\leq c_1 \int_0^t (t - s)^{-\frac{1}{2}} \left\|\hat{P}_{s,t} \left\{ f_0(s, \cdot) \left[ |\Phi^{\nu_1}_s \gamma^1 - \Phi^{\nu_2}_s \gamma^2| + s^\theta (|\Phi^{\nu_1}_s \gamma^1| + |\gamma^1_s - \gamma^2_s| + \|\gamma^1_s - \gamma^2_s\|_{\tilde{L}^k}) \right] \right\|_{\tilde{L}^k} ds.
$$

Letting

$$
F_l(s, x) := (t - s)^{-\frac{d(l-1)}{2k} - \frac{1}{2}} \left[ |\Phi^{\nu_1}_s \gamma^1 - \Phi^{\nu_2}_s \gamma^2| + s^\theta (|\Phi^{\nu_1}_s \gamma^1| + |\gamma^1_s - \gamma^2_s| + \|\gamma^1_s - \gamma^2_s\|_{\tilde{L}^k}) \right](x)
$$

for $l \in [1, \frac{k_p}{k + p_0}]$, by (3.9) for $q = 1$ and $(p', p) = (k, l)$, and applying Hölder’s inequality, we find a constant $c_2 > 0$ such that

$$
\int_0^t (t - s)^{-\frac{1}{2}} \left\|\hat{P}_{s,t} \left\{ f_0(s, \cdot) \left[ |\Phi^{\nu_1}_s \gamma^1 - \Phi^{\nu_2}_s \gamma^2| + s^\theta (|\Phi^{\nu_1}_s \gamma^1| + |\gamma^1_s - \gamma^2_s| + \|\gamma^1_s - \gamma^2_s\|_{\tilde{L}^k}) \right] \right\|_{\tilde{L}^k} ds
\leq c_2 \|f_0\|_{\tilde{L}^l} \leq c_2 \|f_0\|_{\tilde{L}^{p_0}_l} \left\| F_l \right\|_{\tilde{L}^{l_0}_{q_0}} \left\| f_0 \right\|_{\tilde{L}^{p_0}_{l_0}} \left\| f_0 \right\|_{\tilde{L}^{p_0}_{l_0}}
\leq c_2 \|f_0\|_{\tilde{L}^{p_0}_{l_0}} \left( \int_0^t \left\{ (t - s)^{-\frac{d(l-1)}{2k} - \frac{1}{2}} \left[ \|\Phi^{\nu_1}_s \gamma^1 - \Phi^{\nu_2}_s \gamma^2\|_{\tilde{L}^{l_0}_{q_0}} + s^\theta (|\Phi^{\nu_1}_s \gamma^1| + |\gamma^1_s - \gamma^2_s| + \|\gamma^1_s - \gamma^2_s\|_{\tilde{L}^k}) \right] \right\|_{\tilde{L}^{l_0}_{q_0}} ds \right)^{\frac{q_0}{q_0 - 1}} \right)^{\frac{q_0}{q_0 - 1}}.
$$

Since $l \in [1, \frac{k_p}{k + p_0}]$ implies $\frac{p_0}{q_0 - 1} \leq k$, combining this with (4.16) and applying Hölder’s inequality, we find a constant $c_3 > 0$ such that

$$
\|\Phi^{\nu_1}_t \gamma^1 - \Phi^{\nu_2}_t \gamma^2\|_{\tilde{L}^k} 
\leq c_3 \left( \int_0^t (t - s)^{-\frac{d(l-1)}{2k} - \frac{1}{2}} \left[ \|\Phi^{\nu_1}_s \gamma^1 - \Phi^{\nu_2}_s \gamma^2\|_{\tilde{L}^k} + s^\theta (|\Phi^{\nu_1}_s \gamma^1| + |\gamma^1_s - \gamma^2_s| + \|\gamma^1_s - \gamma^2_s\|_{\tilde{L}^k}) \right] \right)^{\frac{q_0}{q_0 - 1}} ds \right)^{\frac{q_0}{q_0 - 1}}, \ l \in [1, \frac{k_p}{k + p_0}].
$$

Letting

$$
\alpha_l := \frac{q_0}{q_0 - 1} \left( \frac{d(k - l)}{2k} + 1 \right), \ \beta_l := \frac{k_p q_0}{k(p_0 - l) - p_0},
$$

we find a constant $c_4 > 0$ such that

$$
\|\Phi^{\nu_1}_t \gamma^1 - \Phi^{\nu_2}_t \gamma^2\|_{\tilde{L}^k} 
\leq c_4 \left( \int_0^t (t - s)^{-\frac{d(l-1)}{2k} - \frac{1}{2}} \left[ \|\Phi^{\nu_1}_s \gamma^1 - \Phi^{\nu_2}_s \gamma^2\|_{\tilde{L}^k} + s^\theta (|\Phi^{\nu_1}_s \gamma^1| + |\gamma^1_s - \gamma^2_s| + \|\gamma^1_s - \gamma^2_s\|_{\tilde{L}^k}) \right] \right)^{\frac{q_0}{q_0 - 1}} ds \right)^{\frac{q_0}{q_0 - 1}}, \ l \in [1, \frac{k_p}{k + p_0}].
$$
by the definition of $\tilde{d}_{k,\lambda}$, this implies that for any $\lambda > 0$ and $l \in [1, \frac{kp_0}{k+p_0}]$,

$$
\tilde{d}_{k,\lambda}(\Phi^{\nu_1,1}, \Phi^{\nu_2,2}) \leq c_1 \|\nu_1 - \nu_2\|_{L^k} + c_3 \tilde{d}_{k,\lambda}(\Phi^{\nu_1,1}, \Phi^{\nu_2,2}) \sup_{t \in [0, T]} \left( \int_0^t (t - s)^{-\alpha_l} e^{-\frac{\lambda q_0}{q_0 - 1} (t-s)} ds \right) \sup_{t \in [0, T]} \left( \int_0^t (t - s)^{-\alpha_l} e^{-\frac{\lambda q_0}{q_0 - 1} (t-s)} ds \right) \sup_{t \in [0, T]} \left( \int_0^t (t - s)^{-\alpha_l} e^{-\frac{\lambda q_0}{q_0 - 1} (t-s)} ds \right).
$$

(4.19)

Below we complete the proof by considering two different situations respectively.

(a) Let $k < \infty$. By $(p_0, q_0) \in \mathcal{K}$ and $k > k_0 := \frac{d}{2(d+1-\theta_0 - 2q_0)}$, $\alpha_l$ in (4.18) satisfies

$$
\lim_{l \uparrow \frac{kp_0}{k+p_0}} \alpha_l + \frac{q_0}{q_0 - 1} \left( \frac{d}{2k} - \theta \right) < 1.
$$

(4.20)

By (4.7) for $k' = \infty$ and $p = \frac{\beta_1}{\beta_1 - 1}$, there exists a constant $c_4 > 0$ such that

$$
\|\Phi^{\nu_1,1}\|_{L^{p_0}} \leq c_4 \|\nu_1\|_{L^k} s^{-\frac{d}{2k}}.
$$

Combining this with (4.19) and (4.20), when $\lambda$ is large enough increasing in $\|\nu_1\|_{L^k} (\leq \|\nu_2\|_{L^k})$, we obtain

$$
\tilde{d}_{k,\lambda}(\Phi^{\nu_1,1}, \Phi^{\nu_2,2}) \leq c_1 \|\nu_1 - \nu_2\|_{L^k} + \frac{1}{4} \tilde{d}_{k,\lambda}(\Phi^{\nu_1,1}, \Phi^{\nu_2,2}) + \frac{1}{4} \tilde{d}_{k,\lambda}(\gamma^1, \gamma^2).
$$

Taking $\nu_1 = \nu_2 = \nu$ we prove the contraction of $\Phi^\nu$ on the complete metric space $(\tilde{L}^{k,T}, \tilde{d}_{k,\lambda})$, and hence $\Phi^\nu$ has a unique fixed point. This implies the weak (also strong under (2.2)) well-posedness of (1.1). Moreover, for two solutions $(X^i)_{i=1,2}$ of this SDE with initial distribution densities $(\nu_i)_{i=1,2}$, by taking $\gamma^i = \mathcal{L}_{X^i}$ we have $\gamma^i = \Phi^{\nu_i,1}$, so that this estimate implies (2.1) for some increasing function $\Lambda$.

(b) Let $k = \infty$. By taking $l = p_0$, we have $\beta_l = \infty$ and $\theta > 2q_0 + \frac{d}{p_0} - 1$ in (A2) implies

$$
\alpha_l + \frac{q_0}{q_0 - 1} \left( \frac{d}{2k} - \theta \right) < 1.
$$

Combining (4.19) with (4.1) for $k = \infty$, we derive that for a large enough $\lambda > 0$ increasing in $\|\nu_1\|_{L^\infty} (\leq \|\nu_2\|_{L^\infty})$, $\tilde{d}_{k,\lambda}(\Phi^{\nu_1,1}, \Phi^{\nu_2,2}) \leq c_1 \|\nu_1 - \nu_2\|_{L^\infty}$.
\[ + c_3 \tilde{d}_{k,\lambda}(\Phi^{\nu_1 \gamma_1}, \Phi^{\nu_2 \gamma_2}) \sup_{t \in (0, T]} \left( \int_0^t (t-s)^{-\alpha_1} e^{-\frac{\lambda_{\nu_1}}{\gamma_0}(t-s)} \, ds \right)^{\frac{q_{\nu_1}-1}{q_{\nu_1}}} \]
\[ + c_3 \tilde{d}_{k,\lambda}(\gamma_1^2, \gamma_2^2) \sup_{t \in (0, T]} \left( \int_0^t (t-s)^{-\alpha_1} e^{-\frac{\lambda_{\nu_1}}{\gamma_0}(t-s)} \left( s^\theta \| \nu_1 \|_{L^\infty} \right)^{\frac{q_{\nu_1}-1}{q_{\nu_1}}} \, ds \right)^{\frac{q_{\nu_1}-1}{q_{\nu_1}}} \]
\[ \leq c_1 \| \nu_1 - \nu_2 \|_{L^\infty} + \frac{1}{4} \tilde{d}_{k,\lambda}(\Phi^{\nu_1 \gamma_1}, \Phi^{\nu_2 \gamma_2}) + \frac{1}{4} \tilde{d}_{k,\lambda}(\gamma_1^2, \gamma_2^2). \]

Then we finish the proof as shown in step (a).

**Proof of Theorem 2.7 (2).** Let (A) hold for \((A'_2)\) replacing \((A_2)\). By \((3.8)\) and Hölder’s inequality, we find constants \(c_1, c_2 > 0\) such that for any \(0 \leq s < t \leq T\) and \(l \in [1, \frac{kp_0}{k+p_0}]\),

\[
\left\| \tilde{P}^\nu_{s,t}\left( \left( C + f_0(s, \cdot) \right) \left( |\Phi^{\nu_1 \gamma_1} - \Phi^{\nu_2 \gamma_2}| + s^\theta (\Phi^{\nu_1 \gamma_1}) |\gamma_1^2 - \gamma_2^2| \right) \right) \right\|_{L^k} \leq c_1 (t-s)^{-\frac{d(k-l)}{2kl}} \left\{ \left\| \Phi^{\nu_1 \gamma_1} - \Phi^{\nu_2 \gamma_2} \right\|_{L^l} + s^\theta \left\| (\Phi^{\nu_1 \gamma_1}) |\gamma_1^2 - \gamma_2^2| \right\|_{L^l} \right\}
\]

\[
+ \left\| f_0(s, \cdot) \left( \Phi^{\nu_1 \gamma_1} - \Phi^{\nu_2 \gamma_2} \right) \right\|_{L^l} + s^\theta \left\| f_0(s, \cdot) \left( \Phi^{\nu_1 \gamma_1} \right) |\gamma_1^2 - \gamma_2^2| \right\|_{L^l} \right\} \leq c_1 (t-s)^{-\frac{d(k-l)}{2kl}} \left\{ \left\| \Phi^{\nu_1 \gamma_1} - \Phi^{\nu_2 \gamma_2} \right\|_{L^l} + s^\theta \left\| \Phi^{\nu_1 \gamma_1} \right\|_{L^k} \right\} \left\| |\gamma_1^2 - \gamma_2^2| \right\|_{L^k} \right\}
\]

Noting that \(l \in [1, \frac{kp_0}{k+p_0}]\) implies \(l \leq k\) and \(\frac{kl}{k-l} \leq \frac{p_{k,l}}{p_{k,l}-kl-p_{l}}\) by combining this with \((A'_2)\), \((3.3)\), \((4.10)\) and Hölder’s inequality, we find constants \(c_3, c_4 > 0\) such that

\[
\left\| \Phi_t^{\nu_1 \gamma_1} - \Phi_t^{\nu_2 \gamma_2} \right\|_{L^k} - c_1 \| \nu_1 - \nu_2 \|_{L^k} \]
\[
\leq c_3 \int_0^t (t-s)^{-\frac{d(k-l)}{2kl}} \left\| \tilde{P}^\nu_{s,t}\left( \left( C + f_0(s, \cdot) \right) \left( |\Phi^{\nu_1 \gamma_1} - \Phi^{\nu_2 \gamma_2}| + s^\theta (\Phi^{\nu_1 \gamma_1}) |\gamma_1^2 - \gamma_2^2| \right) \right) \right\|_{L^k} \, ds \]
\[
\leq c_4 \left( 1 + \| f_0 \|_{L^p} \right) \left( \int_0^t \left\| \Phi^{\nu_1 \gamma_1} - \Phi^{\nu_2 \gamma_2} \right\|_{L^k} \, ds \right)^{\frac{q_{\nu_1}-1}{q_{\nu_1}}} \leq \frac{q_{\nu_1}-1}{q_{\nu_1}}, \quad l \in [1, \frac{kp_0}{k+p_0}].
\]

Then the remainder of the proof is similar to that of Theorem 2.1 (1) from \((4.17)\) with \(L\) replacing \(\tilde{L}\).

**5 Proof of Theorem 2.2**

Let \(\nu \in \mathcal{P}\) with \(\ell_\nu \in C_0^\gamma\). By Theorem 2.1 and (B), for any \(\gamma \in \mathcal{P}_{\nu,T}^{\infty}\), the following density dependent SDE has a unique (weak and strong) solution with \(\ell_{X_0,\gamma} \in L^\infty\):

\[
dX_t^{\gamma,\nu} = \beta_t(X_t^{\gamma,\nu}, \ell_{X_t^{\gamma,\nu}}, \ell_{X_t^{\gamma,\nu}}) \, dt + \sigma_t^{\gamma}(X_t^{\gamma,\nu}), \quad L_{X_0}^{\gamma,\nu} = \nu, \quad t \in [0, T],\]

\[
X_t^{\gamma,\nu} = \Phi^{\nu_2 \gamma_2}(X_0^{\gamma,\nu}, \int_0^t \beta_s^{\gamma}(X_s^{\gamma,\nu}, \ell_{X_t^{\gamma,\nu}}, \ell_{X_t^{\gamma,\nu}}) \, ds + \sigma_s^{\gamma}(X_s^{\gamma,\nu}) \, \tilde{d}_s), \quad t \in [0, T].\]
We aim to show that the map \( \gamma \mapsto \ell_{X,\gamma,\nu} \)
has a unique fixed point in \( L_\infty \cap \mathcal{D}_0^1 \), such that the (weak and strong) well-posedness of (5.11) implies that of (1.1). As shown in the proof of Theorem 2.1, we will need heat kernel estimates presented in Section 2 for the operator \( L_t^{\alpha,\beta} \), where

\[
\| \ell_{X,\gamma,\nu} \|_{C_0^1} \leq c \| \ell_{\nu} \|_{C_0^1}, \quad t \in (0, T].
\]

To this end, we first prove the Hölder continuity of \( b_i^{\gamma,\nu} \). By (B), this follows from the Hölder continuity of \( \ell_{X,\gamma,\nu} \).

**Lemma 5.1.** Assume (B) and let \( \beta \in (0, 1 - \frac{d}{p_\infty} - \frac{2}{q_\infty}) \). Then there exists a constant \( c > 0 \) such that for any \( \gamma \in L_\infty \cap \mathcal{D}_0^1 \) and \( \nu \in \mathcal{P}_0^\infty \) with \( \ell_{\nu} \in C_0^\beta \),

\[
\| \ell_{X,\gamma,\nu} \|_{C_0^\beta} \leq c \| \ell_{\nu} \|_{C_0^\beta}, \quad t \in (0, T].
\]

**Proof.** Simply denote \( \ell \equiv \ell_{X,\gamma,\nu} \). Let \( p_{s,t}^\gamma \) be the heat kernel for the operator

\[
\ell_{s,t}^\gamma := \frac{1}{2} \text{div} \{ a_i^\gamma \nabla \} = L_t^{\alpha,\beta},
\]

where

\[
a_i^\gamma := \frac{1}{2} \sigma_i^\gamma (\sigma_i^\gamma)^*, \quad (b_i^\gamma)_i := \frac{1}{2} \sum_{j=1}^d \partial_j (a_i^\gamma)_{ij}.
\]

Then \( p_{s,t}^\gamma(x, y) = p_{s,t}^\gamma(y, x) \), and by (B) and Theorem 3.1, there exist constants \( c, \kappa > 0 \) depending on \( C, \alpha, \beta \) such that for some diffeomorphisms \( \{ \psi_{s,t} \}_{0 \leq s \leq t \leq T} \) satisfying (3.2),

\[
|\nabla_i^\gamma p_{s,t}^\gamma(\cdot, y)(x)| \leq c_1 (t-s)^{-\frac{d}{2}} p_{t-s}^\kappa(\psi_{s,t}(x) - y), \quad i = 0, 1, 2,
\]

\[
|\nabla p_{s,t}^\gamma(x, \cdot)(y)| \leq c_1 (t-s)^{-\frac{d}{2}} p_{t-s}^\kappa(\psi_{s,t}(x) - y),
\]

\[
|\nabla p_{s,t}^\gamma(\cdot, y)(x) - \nabla p_{s,t}^\gamma(\cdot, y)(x')| 
\leq c_1 |y - y'|^{\beta} (t-s)^{1+\beta} \{ p_{t-s}^\kappa(\psi_{s,t}(x) - y) + p_{t-s}^\kappa(\psi_{s,t}(y) - y') \},
\]

\[
0 \leq s < t \leq T, \quad x, y, y' \in \mathbb{R}^d.
\]

By the argument leading to (4.10) for \( \bar{b}^\gamma \) replacing \( b^{(1)} \), we obtain

\[
\ell_{s}(y) = \int_{\mathbb{R}^d} p_{0,s}^\gamma(x, y) \ell_{\nu}(x) dx + \int_{0}^{s} ds \int_{\mathbb{R}^d} \ell_{s}(x) \{ \nabla_{b_i(x, \ell_{s}(x), \ell_{s}) - \bar{b}_i(x)} p_{s,t}^\gamma(\cdot, y) \}(x) dx.
\]

By the symmetry of \( p_{0,t}^\gamma(x, y) \) we have

\[
\int_{\mathbb{R}^d} p_{0,t}^\gamma(x, y) \ell_{\nu}(x) dx = \int_{\mathbb{R}^d} p_{0,t}^\gamma(y, x) \ell_{\nu}(x) dx =: (P_{0,t} \ell_{\nu})(y).
\]
Let $X_t^x$ solve the SDE
\[ dX_t^x = b_t^0(X_t^x)dt + \sigma_t^0(X_t)\,dW_t, \quad t \in [0, T], X_0 = x. \]

By [14] (4.8), we find a constant $c_1 > 0$ depending on $C, \alpha$ in (B) such that
\[ \mathbb{E} \left[ \sup_{t \in [0,T]} |X_t^x - X_t^y| \right] \leq c_1|y - x|, \quad x, y \in \mathbb{R}^d. \]

Then (5.6) implies
\[ \left| (P_{0,t}^\gamma \ell_\nu)(y) - (P_{0,t}^\gamma \ell_\nu)(y') \right| = \left| \mathbb{E} [\ell_\nu(X_t^y) - \ell_\nu(X_t^{y'})] \right| \leq \| \ell_\nu \|_{C_b^\alpha} \mathbb{E} \left[ |X_t^y - X_t^{y'}|^{\beta} \right] \leq \| \ell_\nu \|_{C_b^\alpha} (c_1|y - y'|)^\beta. \]

Since (B) implies $|b| + |\bar{b}^\gamma| \leq cf_0$ for some constant $c > 0$, by combining this with (5.2), the last inequality in (5.4), and (5.5), we find a constant $c_2 > 0$ independent of $\gamma, \nu$ such that
\[ |\ell_t(y) - \ell_t(y')| - c_2|y - y'|^{\beta} \leq c_2\| \ell_\nu \|_{C_b^{\alpha}}|y - y'|^{\beta} \int_0^t (t - s)^{-\frac{(1 + \beta)}{2}} \left\{ \tilde{P}_{s,t}^\kappa f_0(s, \cdot)(y) + \tilde{P}_{s,t}^\kappa f_0(s, \cdot)(y') \right\} ds, \]
where $\tilde{P}_{s,t}$ is in (3.6). By (3.9) for $(p, q) = (p_0, q_0)$ and $p' = \infty$, we find a constant $c_3 > 0$ such that this implies
\[ \frac{|\ell_t(y) - \ell_t(y')| - c_2|y - y'|^{\beta}}{c_2\| \ell_\nu \|_{C_b^{\alpha}}|y - y'|^{\beta}} \leq \int_0^t (t - s)^{-\frac{(1 + \beta)}{2} + \frac{d}{2p_0}} \left( \tilde{P}_{s,t}^\kappa ((t - s)^{\frac{d}{2p_0}} f_0(s, \cdot))(y) + \tilde{P}_{s,t}^\kappa ((t - s)^{\frac{d}{2p_0}} f_0(s, \cdot))(y') \right) ds \leq 2c_2 \left( \int_0^t (t - s)^{-\frac{(1 + \beta)}{2} + \frac{d}{2p_0}} \frac{q_0}{\gamma_0} ds \right)^{\frac{\gamma_0 - 1}{\gamma_0}} \left\| \ell_t^{\gamma_0} ((t - s)^{\frac{d}{2p_0}} f_0) \right\|_{L_{\gamma_0}(t)} \leq c_3\| f \|_{L_{\gamma_0}^{p_0}}, \quad y \neq y', t \in (0, T], \]
where we have used the fact that $\| \cdot \|_{L_{\gamma_0}^{p_0}} = \| \cdot \|_{L_{\gamma_0}^{q_0}}$ and $(\frac{1 + \beta}{2} + \frac{d}{2p_0})\frac{q_0}{\gamma_0} - 1 < 1$ due to $\beta \in (0, 1 - \frac{2}{q_0} - \frac{d}{p_0})$. Combining this with (5.2), we finish the proof.

The next lemma contains two classical estimates on the operator $1 - \Delta$ and the heat semigroup $P_t = e^{t\Delta}$.

**Lemma 5.2.** Let $P_t = e^{t\Delta}$.

1. For any $\beta > 0$, there exists a constant $c > 0$ such that
   \[ \|(1 - \Delta)^{\frac{\beta}{2}} f \|_{L_{\infty}} \leq c \| f \|_{C_b^\alpha}. \]

2. For any $\alpha, \beta, k \geq 0$, there exists a constant $c > 0$ such that
   \[ \|(1 - \Delta)^{-k} P_t f \|_{C_b^{\alpha + \beta}} \leq c \tau^{-\frac{\alpha}{2} + k} \| f \|_{C_b^\beta}, \quad t > 0. \]
Proof of Theorem 2.2. For \( \mathcal{L}_{X_0} = \nu_t \) with \( \ell \nu_t \in C^3_b(\mathbb{R}^d) \) and \( \gamma^i \in L^\infty \cap \mathcal{D}_r^1 \), simply denote

\[
\ell^i = \ell_{X^i \nu_t}, \quad b^i_t := b_t(\cdot, \ell^i(\cdot), \ell^i_t), \quad t \in [0, T], \quad i = 1, 2.
\]

Without loss of generality, let \( \| \ell \nu_2 \|_{C^3_b} \leq \| \ell \nu_1 \|_{C^3_b} \).

By (5.5) with \((\nu, \gamma) = (\nu_1, \gamma^1)\), we obtain

\[
\ell^1_t(y) = P_{0,t}^\gamma \ell_{\nu_1}(y) + \int_0^t \int_{\mathbb{R}^d} \ell^1_s(x) \left\{ \nabla b^1_t(x) - \bar{b}^1_t(x) \right\} p^\gamma_{s,t}(\cdot, y) \right\} dx.
\]

By the argument leading to (4.10) for \((p^\gamma_{s,t}, X^2_{\gamma^2, \nu_2})\) replacing \((p^\gamma_{s,t}, X^2_{\gamma^2, \nu_2})\), we derive

\[
\ell^2_t(y) = P_{0,t}^\gamma \ell_{\nu_2}(y) + \int_0^t \int_{\mathbb{R}^d} \ell^2_s(x) \left\{ \nabla b^2_t(x) - \bar{b}^2_t(x) \right\} p^\gamma_{s,t}(\cdot, y) \right\} dx
\]

\[+ \frac{1}{2} \sum_{i,j=1}^d \int_0^t \int_{\mathbb{R}^d} \left\{ \ell^2_s(a^2_s - a^1_s)_{ij} \partial_i \partial_j p^\gamma_{s,t}(\cdot, y) \right\} dx.
\]

Thus,

\[
\| \ell^1_t - \ell^2_t \|_{\infty} \leq I_1 + I_2 + \sum_{i,j=1}^d I_{ij},
\]

where

\[
I_1 := \| P_{0,t}^\gamma \ell_{\nu_1} - P_{0,t}^\gamma \ell_{\nu_2} \|_{\infty} \leq \| \ell \nu_1 - \ell \nu_2 \|_{\infty},
\]

and

\[
I_2 := \int_0^t \int_{\mathbb{R}^d} \left\{ \left[ \ell^2_s(b^2_s - \bar{b}^2_s) - \ell^1_s(b^1_s - \bar{b}^1_s) \right] \nabla p^\gamma_{s,t}(\cdot, y) \right\} dx,
\]

\[
I_{ij} := \frac{1}{2} \sup_{y \in \mathbb{R}^d} \left| \int_0^t \int_{\mathbb{R}^d} \left\{ \ell^2_s(a^2_s - a^1_s)_{ij} \partial_i \partial_j p^\gamma_{s,t}(\cdot, y) \right\} dx \right|.
\]

Below we estimate \( I_2 \) and \( I_{ij} \) respectively.

Firstly, by (B) and (5.2), we find a constant \( c_1 > 0 \) such that

\[
\left| \ell^2_s \{ b^2_s(x) - \bar{b}^2_s(x) \} - \ell^1_s \{ b^1_s(x) - \bar{b}^1_s(x) \} \right|
\leq \| \ell^1_s - \ell^2_s \|_{\infty} \| b^2_s(x) - \bar{b}^2_s(x) \| + \| \ell^2_s \|_{\infty} \| b^2_s(x) - \bar{b}^2_s(x) \|
\leq c_1 \| \ell \nu_2 \|_{\infty} \| \ell^1_s - \ell^2_s \|_{\infty} f_0(s, x), \quad s \in [0, T], \quad x \in \mathbb{R}^d.
\]

Combining this with (5.4) for \( i = 1 \), (3.9) for \((p, q) = (p_0, q_0)\) and \( p' = \infty \), and applying Hölder’s inequality, we find constant \( c_2, c_3 > 0 \) such that

\[
I_2 \leq c_2 \| \ell \nu_2 \|_{\infty} \int_0^t (t - s)^{-\frac{1}{2}} \| \ell^1_s - \ell^2_s \|_{\infty} P^\kappa_{s,t} f_0(s, \cdot)(y) ds
\]

\[
\leq c_2 \| \ell \nu_2 \|_{\infty} \left( \int_0^t \left( (t - s)^{-\frac{1}{2}} \int_0^t \| \ell^1_s - \ell^2_s \|_{\infty} ds \right)^{\frac{q_0}{q_0 - 1}} \right)^{\frac{q_0 - 1}{q_0}} \| (t - \cdot)^{-\frac{d}{2q_0}} f_0 \|_{L^q_{00}}
\]

\[
\leq c_3 \| \ell \nu_2 \|_{\infty} \| f_0 \|_{L^q_{00}} \left( \int_0^t (t - s)^{-\frac{q_0}{q_0 - 1} - \frac{d}{2q_0}} \| \ell^1_s - \ell^2_s \|_{\infty} ds \right)^{\frac{q_0 - 1}{q_0}}, \quad t \in [0, T].
\]
Next, by integration by parts formula, (B), (5.3), (5.4) for $i = 1$ and Lemma 5.2 for any $\delta := \alpha \wedge \beta$, we find constants $c_4, c_5 > 0$ such that

\[
\left| \int_{\mathbb{R}^d} \left\{ \ell_s^2(a_{s,i}^2 - a_{s,i}^{-1}) \partial_t \partial_j \partial_{j_s} \partial_{s,t} p_{s,t}^\gamma(\cdot, y) \right\}(x) \right| dx \leq \left\| (1 - \Delta)^{\frac{\delta}{2}} \{ \ell_s^2(a_{s,i}^2 - a_{s,i}^{-1}) \} \right\|_{\infty} \int_{\mathbb{R}^d} \left| \partial_t \partial_j (1 - \Delta)^{-\frac{\delta}{2}} p_{s,t}^\gamma(\cdot, y) \right| dx \leq c_4 \| \ell_s^2(a_{s,i}^2 - a_{s,i}^{-1}) \|_{C^0_{b \wedge \alpha}}(t - s)^{\frac{\delta}{2} - 1} \| \gamma_s - \gamma_s^2 \|_{\infty} \leq c_5 \| \ell_{s,t}^\gamma \|_{C^0_{b \wedge \alpha}}(t - s)^{\frac{\delta}{2} - 1} \| \gamma_s - \gamma_s^2 \|_{\infty}.
\]

By combining this with (5.7), (5.8) and (5.9), we arrive at

\[
\| \ell_{t}^1 - \ell_{t}^2 \|_{\infty} \leq \| \ell_{s,t}^\gamma - \ell_{s,t}^\gamma^2 \|_{\infty} + c_3 \| \ell_{s,t}^\gamma \|_{\infty} \left( \int_0^t (t - s)^{-\frac{\delta}{2} + \frac{\delta}{2\theta_0}} \left\| \ell_s^1 - \ell_s^2 \right\|_{\infty} ds \right)^{\frac{\delta}{2} - 1} \| \gamma_s - \gamma_s^2 \|_{\infty} \leq c_5 \| \ell_{s,t}^\gamma \|_{C^0_{b \wedge \alpha}}(t - s)^{\frac{\delta}{2} - 1} \| \gamma_s - \gamma_s^2 \|_{\infty} ds, \quad t \in [0, T].
\]

Consequently, for any $\lambda > 0$,

\[
d_{\infty,\lambda}(\mathcal{L}_{X,\gamma^1,\nu_1}, \mathcal{L}_{X,\gamma^2,\nu_2}) := \sup_{t \in [0, T]} e^{-\lambda t} \| \ell_{t}^1 - \ell_{t}^2 \|_{\infty} \leq \| \ell_{s,t}^\gamma - \ell_{s,t}^\gamma^2 \|_{\infty} + \varepsilon(\lambda) \{ d_{\infty,\lambda}(\mathcal{L}_{X,\gamma^1,\nu_1}, \mathcal{L}_{X,\gamma^2,\nu_2}) + d_{\infty,\lambda}(\gamma^1, \gamma^2) \}
\]

holds for

\[
\varepsilon(\lambda) := \sup_{t \in [0, T]} \left\{ c_3 \| \ell_{s,t}^\gamma \|_{\infty} \left( \int_0^t (t - s)^{-\frac{\delta}{2} + \frac{\delta}{2\theta_0}} e^{-\frac{\theta_0 \lambda}{2\theta_0 - 1} (t - s)} ds \right)^{\frac{\theta_0 - 1}{\theta_0}} + \frac{d^2 c_5}{2} \| \ell_{s,t}^\gamma \|_{C^0_{b \wedge \alpha}} \int_0^t (t - s)^{\frac{\delta}{2} - 1} e^{-\lambda (t - s)} ds \right\}.
\]

Since $(p_0, q_0) \in \mathcal{K}$ implies $\frac{\theta_0}{\theta_0 - 1} (\frac{1}{2} + \frac{d}{2p_0}) < 1$, and since $1 - \frac{\delta}{2} < 1$, by taking large enough $\lambda > 0$ increasing in $\| \ell_{s,t}^\gamma \|_{C^0_{b \wedge \alpha}}$, we obtain

\[
(5.10) \quad d_{\infty,\lambda}(\mathcal{L}_{X,\gamma^1,\nu_1}, \mathcal{L}_{X,\gamma^2,\nu_2}) \leq \| \ell_{s,t}^\gamma - \ell_{s,t}^\gamma^2 \|_{\infty} + \frac{1}{4} \{ d_{\infty,\lambda}(\mathcal{L}_{X,\gamma^1,\nu_1}, \mathcal{L}_{X,\gamma^2,\nu_2}) + d_{\infty,\lambda}(\gamma^1, \gamma^2) \}.
\]

Taking $\nu_1 = \nu_2 = \nu$, we see that the map $\gamma \mapsto \ell_{X,\gamma,\nu}$ is contractive on the complete metric space $(L^\infty_{\infty} \cap C^1_+, d_{\infty,\lambda})$, so that it has a unique fixed point. Therefore, (1.1) is well-posed. Estimate (2.3) follows from Lemma 5.1 for $\gamma_t = \ell_{X_t}$, for the solution to (1.1), while (2.4) follows from (5.10) for $\gamma^i_t := \ell_{X_t}^{\gamma^i}, \nu^i \in \mathcal{L}_{X_0^i}, i = 1, 2$.  

\[\square\]
6 Density dependent reflecting SDEs

In this section, we extend Theorem 2.1 to density dependent reflecting SDEs on a domain $D$. There exists additional difficulty to extend Theorem 2.2, for instance, in the proof of Theorem 2.2 we used

$$(1 - \Delta)^{-\frac{\delta}{2}} \partial_i \partial_j = \partial_t \partial_j (1 - \Delta)^{-\frac{\delta}{2}}$$

which is no longer true for the Neumann Laplacian in a domain.

Let $D \subset \mathbb{R}^d$ be a connected $C^2$-smooth open domain. Consider the following density dependent reflecting SDE on the closure $\overline{D}$ of $D$:

(6.1) \[ dX_t = b_t(X_t, \ell_X(X_t), \ell_X)dt + \sigma_t(X_t)dW_t + n(X_t)dl_t, \quad t \in [0, T], \]

where $n$ is the unit inward normal vector field on the boundary $\partial D$, $l_t$ is a continuous adapted increasing process with $dl_t$ supported on $\{ t : X_t \in \partial D \}$, and

$$b : [0, T] \times \overline{D} \times [0, \infty) \times \mathcal{D}_1 \to \mathbb{R}^d, \quad \sigma : [0, T] \times \overline{D} \to \mathbb{R}^d \otimes \mathbb{R}^m$$

are measurable. Here and in the following, $\mathcal{D}_1^1, \mathcal{L}_q, \mathcal{L}_p, \mathcal{L}_p, \mathcal{L}_p, \dot{\mathcal{P}}_p, \mathcal{P}_p$ and $\mathcal{P}$ are defined as before for $\overline{D}$ replacing $\mathbb{R}^d$.

We will assume $\partial D \in C^{2,L}_b$ in the following sense: there exists a constant $r_0 > 0$ such that the polar coordinate map

$$\Psi : \partial D \times [-r_0, r_0] \ni (\theta, r) \mapsto \theta + r n(\theta) \in \partial_{\pm r_0} D : = \{ x \in \mathbb{R}^d : \rho_0(x) := \text{dist}(x, \partial D) \leq r_0 \}$$

is a $C^2$-diffeomorphism, such that $\Psi^{-1}(x)$ have bounded and continuous first and second order derivatives in $x \in \partial_{\pm r_0} D$, and $\nabla^2 \rho_0$ is Lipschitz continuous on $\partial_{\pm r_0} D$.

Note that $\partial D \in C^{2,L}_b$ does not imply the boundedness of $D$ or $\partial D$, but any bounded $C^{2,L}$ domain satisfies $\partial D \in C^{2,L}_b$.

**Definition 6.1.** (1) A pair $(X_t, l_t)_{t \in [0, T]}$ is called a (strong) solution of (6.1), if $(X_t)_{t \in [0, T]}$ is a continuous adapted process on $\overline{D}$, $(l_t)_{t \in [0, T]}$ is a continuous adapted increasing process with $l_0 = 0$ and $dl_t$ supported on $\{ t \in [0, T] : X_t \in \partial D \}$, such that

$$\int_0^T \mathbb{E} \left[ |b_s(X_s, \ell_{X_s}(X_s), \ell_X)| + \|\sigma_s(X_s)\|^2 \right] ds < \infty$$

and $\mathbb{P}$-a.s.

$$X_t = X_0 + \int_0^t b_s(X_s, \ell_{X_s}(X_s), \ell_X)ds + \int_0^t \sigma_s(X_s)dW_s + \int_0^t n(X_s)dl_s, \quad t \in [0, T].$$

(2) A triple $(X_t, l_t, W_t)_{t \in [0, T]}$ is called a weak solution of (6.1), if $(W_t)_{t \in [0, T]}$ an $m$-dimensional Brownian under a complete filtration probability space $(\Omega, \{ \mathcal{F}_t \}_{t \in [0, T]}, \mathbb{P})$ such that $(X_t, l_t)_{t \in [0, T]}$ solves (6.1). We identify any two weak solutions $(X_t, l_t, W_t)$ and $(\tilde{X}_t, \tilde{l}_t, \tilde{W}_t)$ if $(X_t, l_t)_{t \in [0, T]}$ and $(\tilde{X}_t, \tilde{l}_t)_{t \in [0, T]}$ have the same distribution under the corresponding probability spaces.
To extend assumption (A) to the present setting, we introduce the Neumann semigroup 
\{P_{s,t}^{a,b}(1)\}_{0 \leq s \leq t \leq T}
generated by \(L_s^{a,b}(1)\) on \(\bar{D}\) for \(a_t := \sigma_t^0\), that is, for any \(\phi \in C^2_b(\bar{D})\), and any \(t \in (0, T]\), \((P_{s,t}^{a,b}(1) \phi)_{s \in [0, t]}\) is the unique solution of the PDE

\[
\partial_s u_s = -L_s^{a,b}(1) u_s, \quad \nabla_n u_s|_{\partial D} = 0 \quad \text{for } s \in [0, t), u_t = \phi.
\]

For any \(t > 0\), let \(C_b^{1,2}([0, t] \times \bar{D})\) be the set of functions \(f \in C_b([0, t] \times \bar{D})\) with bounded and continuous derivatives \(\partial_t f, \nabla f\) and \(\nabla^2 f\).

We now extend (A) to the domain setting as follows.

(C) Let \(k \in [1, \infty]\), \(\partial D \in C^{2L}_b\), \(\sigma_t(x, \rho) = \sigma_t(x)\) and \(b_t(x, r, \rho) = b_t^{(1)}(x) + b_t^{(0)}(x, r, \rho)\) satisfy the following conditions.

\((C_1)\) \(a_t(x) := (\sigma_t, \sigma_t^0)(x)\) is invertible for \((t, x) \in [0, T] \times \bar{D}\), \(\|a_t\|_\infty + \|a_t^{-1}\|_\infty < \infty\), and

\[
\lim_{\varepsilon \downarrow 0} \sup_{t \in [0, T]} \sup_{x, y \in \bar{D}, |x-y| \leq \varepsilon} \|a_t(x) - a_t(y)\| = 0.
\]

Moreover, \(\sigma_t\) is weakly differentiable with \(\|\nabla \sigma\| \leq \sum_{i=1}^l \|f_i\|_{\bar{D}}\) for some \(l \in \mathbb{N}\), \(0 \leq f_i \in \bar{L}^{p_i}_{q_i}\), \((p_i, q_i) \in \mathcal{K}, 1 \leq i \leq l\).

\((C_2)\) \(a_t\) holds for \(D\) replacing \(\mathbb{R}^d\).

\((C_3)\) For any \(\phi \in C_b^2(\bar{D})\) and \(t \in (0, T]\), the PDE \((6.2)\) has a unique solution \(P_{s,t}^{a,b}(1) \phi \in C_b^{1,2}([0, t] \times \bar{D})\), such that for some constants \(c, \kappa > 0\) and diffeomorphisms \(\{\psi_{s,t}\}_{0 \leq s \leq t \leq T}\) on \(\bar{D}\) satisfying \((3.2)\), the heat kernel \(p_{s,t}^{a,b}(1)\) of \(P_{s,t}^{a,b}(1)\) satisfies

\[
(6.3) \quad \|\nabla_i p_{s,t}^{a,b}(1) \cdot (\cdot, y)\|_1(x) \leq c(t-s)^{\frac{\beta}{2}} p_{s,t}^\kappa(\psi_{s,t}(x) - y), \quad 0 \leq s < t \leq T, x, y \in \bar{D}, i = 1, 2.
\]

By [4, Theorem VI.3.1], \((C_3)\) holds if \(D\) is bounded with \(\partial D \in C^{2+\alpha}\) for some \(\alpha \in (0, 1)\), and there exists \(c > 0\) such that

\[
\{ |b_t^{(1)}(x) - b_t^{(1)}(y)| + \|a_t(x) - a_t(y)\| \} \leq c(|t-s|^\alpha + |x-y|^\alpha), \quad s, t \in [0, T], x, y \in \bar{D}.
\]

If moreover \(\nabla a_t\) is Hölder continuous uniformly in \(t \in [0, T]\), then for any \(\beta \in (0, 1)\) there exists a constant \(c > 0\) such that \((3.5)\) holds for \(p_{s,t}^{a,b}(1)\):

\[
(6.4) \quad \frac{\beta}{2} \int_0^1 \{ p_{t-s}^\kappa(\psi_{s,t}(x) - y) + p_{t-s}^\kappa(\psi_{s,t}(x) - y) \},
\]

\[
0 \leq s < t \leq T, \ x, x', y \in \mathbb{R}^d.
\]

The following result extends Theorem 2.1 to the reflecting setting.

**Theorem 6.1.** Assume (C) for some \(k \in \left[\frac{p_0}{p_0 - 1}, \infty\right] \cap (k_0, \infty)\), where \(k_0 := \frac{d}{2q+1-dp_0 - 2d - 1}r\).
(1) For any $\nu \in \mathcal{P}^k$, (6.1) has a unique strong (respectively weak) solution with $L_{X_0} = \nu$ satisfying $\ell_x \in L^k_{\infty}(\bar{D})$. Moreover, there exists a constant $c > 0$ such that for any two solutions $X_t^1$ and $X_t^2$ of (6.1) with initial distributions $L_{X_1}^1, L_{X_2}^2 \in L^k$, 

$$\sup_{t \in [0,T]} \| \ell_{X_t^1} - \ell_{X_t^2} \|_{L^k} \leq c \| L_{X_1}^1 - L_{X_2}^2 \|_{L^k}.$$

(2) Assertions in (1) hold for $(\mathcal{P}^k, L^k_{\infty}, L^k)$ replacing $(\mathcal{P}^k, \bar{L}^k_{\infty}, \bar{L}^k)$, provided in $(C_2)$ the condition $(A_2)$ is replaced by $(A_2')$ for $\bar{D}$ replacing $\mathbb{R}^d$.

**Proof.** As explained in the proof of Theorem 2.1(2), we only prove the first assertion.

According to [12, Theorem 2.2(ii)], for any $\gamma \in \bar{L}^k_{\infty} \cap \mathcal{P}^1_+$, the reflecting SDE

$$dX_t^\gamma = b_t(X_t^\gamma)dt + \sigma_t(X_t^\gamma)dW_t + n(X_t^\gamma)dl_t^\gamma, \quad t \in [0,T]$$

is well-posed. Let $X_t^{\gamma,x}$ denote the solution with initial value $X_0^\gamma = x \in \bar{D}$, and simply denote $X_t^\gamma$ for the solution with $X_0^\gamma = X_0$ for $L_{X_0} = \nu$.

By Theorem 6.2.7(ii)-(iii) in [3], the distribution density function $\ell_{X_t^{\gamma,x}}$ exists for $t \in (0,T]$ and $x \in D$. Next, by [12, Theorem 4.1] for distribution independent drift, there exists $c > 0$ such that the following log-Harnack inequality holds for the associated semigroup:

$$P_t^\gamma \log f(x) \leq \log P_t^\gamma f(y) + c|y - x|^2/t, \quad t \in (0,T], x, y \in \mathbb{R}^d, f > 0.$$

This implies that $\{L_{X_t^{\gamma,x}}\}_{x \in \bar{D}}$ are mutually equivalent for $t \in (0,T]$. Thus, the existence of $\{\ell_{X_t^{\gamma,x}}\}_{t \in (0,T]}$ for $x \in D$ implies that for $x \in \bar{D}$. Consequently,

$$\Phi_t^\nu \gamma := \ell_{X_t^\gamma} = \int_D \ell_{X_t^{\gamma,x}} L_{X_0}(dx)$$

exists for any $t \in (0,T]$. Since $\{\psi_{s,t}\}_{0 \leq s \leq t \leq T}$ are diffeomorphisms on $\bar{D}$ satisfying (3.2),

$$\hat{P}_{s,t}^\nu f(y) := \int_D p_{t-s}^\nu(y - \psi_{s,t}(x))f(x)dx, \quad y \in \bar{D}$$

gives rise to a family of linear operators satisfying (3.8) and Lemma 3.2 for norms defined with $\bar{D}$ replacing $\mathbb{R}^d$. Then by repeating the proof of Lemma 4.1 using the present estimates, we conclude that $\Phi^\nu$ maps $\mathcal{P}^k_{\nu,T}$ into $\mathcal{P}^k_{\nu,T}$ such that (4.1) and (4.10) hold for $\bar{D}$ replacing $\mathbb{R}^d$. With this result and using $(C_3)$ replacing (3.3), we prove Theorem 6.1(1) by the means in used the proof of Theorem 2.1(1).

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