An exact mobility edge in one dimension

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We investigate localization properties of a family of deterministic (i.e. no disorder) nearest neighbor tight binding models with quasiperiodic onsite modulation. We prove that this family is self-dual under a generalized duality transformation. The self-dual condition for this general model turns out to be a simple closed form function of the model parameters and energy. We numerically verify that this self-dual line indeed defines a mobility edge in energy separating localized and extended states. Our model is a first example of a nearest neighbor tight binding model manifesting a mobility edge protected by a duality symmetry. Although primarily of fundamental theoretical significance as a one of a kind example of a true one-dimensional mobility edge, we also propose a simple scheme to experimentally realize our results in atomic optical lattices and photonic waveguides.

Anderson localization [1] is a universal and extensively studied property of a quantum particle or a wave in a disordered medium. An interesting consequence of Anderson localization is a quantum phase transition between extended and localized states as a function of the disorder strength. In three dimensional systems with random (uncorrelated) disorder, a localization transition occurs as the strength of disorder crosses a critical value forming a sharp energy dependent mobility edge at the phase boundary separating localized/extended states below/above the mobility edge. Scaling theory [2] has shown the absence of this critical behavior in one and two dimensions where all states, at least in the absence of interaction, are known to be localized in a disordered system, pushing the mobility edge effectively to infinite energy. However, in one dimension this picture changes for a quasiperiodic system with two incommensurate (but deterministic) lattice potentials, which in a loose qualitative sense might be construed to be a highly correlated disorder, albeit perfectly well-defined with no randomness whatsoever. Aubry and Andre [3] showed that a 1D tight binding model with an onsite cosine modulation incommensurate with the underlying lattice has a self-dual symmetry and manifests an energy independent localization transition as a function of the modulation strength, i.e., all states are either localized or extended depending on the relative strength of the incommensurate modulation potential with respect to the lattice potential. The same model was earlier considered by Harper [4] and by Azbel [5] and Hofstadter [6] to study the self similar spectrum of conduction electrons in an external magnetic field (we abbreviate this model as AAH from hereon).

This result [3] has led to an extensive theoretical investigation of the AAH model in the context of localization [7–12] in an incommensurate potential. Recent experimental developments in photonic crystals [13–16] and ultracold atoms [17–19] have led to the implementation of the quasiperiodic AAH model where this localization transition has been observed. Recent works have used analytical [20] and numerical methods [21] to show the existence of a many body localization transition in the quasiperiodic AAH model in the presence of weak interactions. The duality-driven and energy independent localization transition in the AAH model does not manifest a mobility edge, which is a hallmark of the disorder tuned localization transition in 3D. The 1D localization transition in the AAH model, defined by the self-duality point, has thus no analog in the disorder-driven Anderson localization, and does not give any insight into the physics of 3D mobility edges.

In this letter we show that there exists a general family of quasiperiodic models with nearest neighbor hopping that are self-dual under a generalized transformation. We analytically show that this general family has a true 1D mobility edge, not allowed in the Anderson model and the AAH model, which can be expressed as a closed form expression involving energy and system parameters. This self-duality is robust in the sense that as long as some basic symmetry conditions are satisfied, we can deform this model without destroying self-duality. We organize this letter by first writing the self-duality condition for a specific onsite model. We then provide a physical intuition for this novel self-dual critical condition and present numerical verification for the existence of the mobility edge. We then present a different model satisfying exactly the same critical condition with totally different energy spectrum. After developing some physical intuition of our result, we explicitly prove that the mobility edge is precisely the self-duality condition for a broad class of nearest neighbor models. We conclude by presenting a concrete schematic to engineer our model in ultracold atoms and photonic waveguides. The experimental observation of a 1D mobility edge would be an exciting and surprising result, leading to a deeper understanding of quantum localization phenomena.

**Model:** We consider a family of 1D tight binding chain with an onsite modulation $V_n$ (Eq. (1)). The onsite potential, $V_n(\alpha, \phi)$, is characterized by the deformation parameter $\alpha$, onsite modulation strength $\lambda$, period $1/b$ and
the phase parameter \( \phi \). For a quasiperiodic modulation, we set \( b \) to be irrational (we choose \( 1/b = \pm \sqrt{2} \) for our numerical work although any other irrational choice for \( b \) is equally acceptable).

\[
t(u_{n-1} + u_{n+1}) + V_n(\alpha, \phi) u_n = E u_n. \tag{1}
\]

where, \( V_n(\alpha, \phi) = 2\lambda \cos(2\pi nb + \phi) / [1 - \alpha \cos(2\pi nb + \phi)] \) \tag{2}

This onsite potential is a smooth function of \( \alpha \) in the open interval \( \alpha \in (-1, 1) \). \( V_n(\alpha, \phi) \) has singularities at \( \alpha = \pm 1 \) which we approach only in a limiting sense. Each value of \( \alpha \) corresponds to a different tight binding model containing the AAH (\( \alpha = 0 \)) model as a limiting case and a general quasi periodic model with correlated singularities at \( \alpha = \pm 1 \).

The AAH limit (\( \alpha = 0 \)) with irrational \( b \) manifests a localization transition at the self-dual point \( \lambda = t \). A vast body of numerical work \([7, 9, 12]\) has been done to understand how this critical point changes once the duality symmetry of the AAH model is broken in some controlled fashion. Based on numerical results, a general consensus has prevailed that the duality symmetry of the AAH model is broken in some critical point. Note that for the AAH model, the duality symmetry is a simple Fourier transformation that modifies into a mobility edge. The critical mobility edge for the model defined in Eq. (2) is given by the following extremely simple closed form expression,

\[
\alpha E = 2(t - \lambda). \tag{3}
\]

This is our central result which we prove by identifying a generalized duality symmetry. Before deriving this result, we show analytical consistency and numerical verification of this condition representing as a mobility edge. Note that the critical condition must reduce to that of the AAH model for \( \alpha = 0 \). This is indeed the case for \( \alpha = 0 \), as the critical condition (Eq. (3)) becomes energy independent giving the familiar self-dual Auby-Andre \( \lambda = t \) critical point. Note that for the AAH model, the duality transformation is a simple Fourier transformation that maps extended (localized) states in the real space to the localized (extended) states in the Fourier space, leading to a very special singular continuous Cantor set spectrum at the self-dual point where the critical states can neither be extended nor localized. This mapping will not be obvious for the general case and we numerically confirm that this critical condition, defined by Eq. (3), is indeed a localization transition point. We numerically diagonalize the tight binding model defined in Eq. (1) for \( N = 500 \) sites with periodic boundary conditions. The localization properties of an eigenstate can be numerically quantified using the Inverse Participation Ratio (IPR).

IPR for an eigenstate \( \alpha \) is given as,

\[
\text{IPR}^\alpha = \frac{\sum_n |u_n^\alpha|^4}{(\sum_n |u_n^\alpha|^2)^2} \tag{4}
\]

For a localized eigenstate, the IPR approaches the maximum possible value \( \sim 1 \). For an extended state, the IPR is of the order \( 1/N \), which is vanishingly small.
in the large system size limit. We plot IPR values for each energy eigenstate as a function of the dimensionless deformation parameter \( \alpha \) for different values of \( \lambda/t \). The IPR value of an eigenstate is given by the coloring scheme defined in the legend of the plot. The color scheme is a blend of cyan and black where pure cyan corresponds to the maximally extended state with IPR=0 and pure black denotes the maximally localized state with IPR=1. The mobility edge conjectured in Eq. (3) is plotted in red. We superimpose the IPR result with the critical condition conjectured in Eq. (3). Fig. (a, b, c, d) is a plot of energy eigenvalue \( E/t \) as a function of \( \alpha \) for fixed values of \( \lambda/t \). In Fig. (a) we fix \( \lambda/t = 0.9 \). The \( \alpha = 0 \) slice of the plot is the AAH model for which all the states are extended as demonstrated by the cyan color corresponding to IPR=0. For \( \alpha \neq 0 \), the states remain extended till it encounters the mobility edge (red line). Across this mobility edge, the IPR value jumps to a finite value (by two orders of magnitude for 500 sites) corresponding to a localized state. The region \( \alpha E < 2(t - \lambda) \) encloses the localized states with IPR \( \sim 1 \). In Fig. (a, d), we fix \( \lambda/t = 1.0 \), which is the critical condition for the AAH model. We show that the critical line is formed by \( \alpha = 0 \) and \( E = 0 \) which divides the \((\alpha, E/t)\) space into four quadrants. The IPR indicates a localized (extended) to extended (localized) transition across the critical lines.

![Figure 2](image_url)

Figure 2. Numerical energy spectrum \( E/t \) \((t = -1.0)\) as a function of \( \alpha \) for \( N=500 \) sites tight binding model for fixed \( \lambda/t \). IPR value is encoded in a blend between cyan and black. Pure cyan denotes IPR=0 and pure black denotes IPR=1. Red line is a plot of the analytically obtained critical condition defined in Eq. (3). (a) \( \lambda/t = 0.8 \), (b) \( \lambda/t = 1.0 \), (c) \( \lambda/t = 2.0 \), (d) \( \lambda/t = 4.0 \).

in exact agreement with the analytical critical boundary defined by Eq. (3). Figs. (a, d) are IPR plots for \( \lambda/t = 1.1, 1.5 \) respectively. The AAH model slice for these values of \( \lambda/t \) is completely localized for all energy eigenstates. For \( \lambda/t > 1.0 \), the duality condition switches sign and the region \( \alpha E > 2(t - \lambda) \) encloses the extended states. Note that IPR values indicate localization transition at the analytical mobility edge with remarkable accuracy. Fig. (d) shows the energy spectrum as a function of \( \lambda/t \) for \( \alpha = 0, 0.99 \). The critical condition in Eq. (3) is a straight line in the \((\lambda/t, E/t)\) space with slope \( 2/\alpha \). The localization transition indicated by the IPR exactly coincides with the critical condition. Fig. (c) is a well known plot of the AAH model where the critical condition is a line parallel to the energy axis \((2/\alpha = \infty)\). For \( \alpha = 0.99 \) in Fig. (c), the critical line has slope 2 and introduces an energy dependence in the critical condition in excellent agreement with IPR results.

In the following, we consider a different family of nearest neighbor models that satisfies the same critical condition defined in Eq. (3). This family is defined by the following onsite potential,

\[
V_\alpha(\alpha, \phi) = 2\lambda \frac{1 - \cos(2\pi nb + \phi)}{1 + \cos(2\pi nb + \phi)}
\]

\[
= 2\lambda \begin{cases} 
1 & \text{as } \alpha \to -1 \\
1 - \cos(2\pi nb + \phi) & \text{as } \alpha \to 0 \\
\tan^2\left(\frac{2\pi nb + \phi}{2}\right) & \text{as } \alpha \to 1 
\end{cases}
\]

Note that this onsite modulation connects different tight binding models compared to Eq. (2). As shown in Eq. (3), the onsite potential for \( \alpha = -1 \) corresponds to a constant onsite energy \( 2\lambda \). \( \alpha = 0 \) corresponds to a rescaled AAH model. \( \alpha = 1 \) corresponds to the closed form singular potential given by \( \tan^2\left(\frac{2\pi nb + \phi}{2}\right) \). In Fig. (3), we plot the numerical spectrum as a function of \( \alpha \) with color coded IPR for \( \lambda/t = 0.8, 1.0, 2.0, 4.0 \). The critical condition (shown in red line) is in excellent agreement with the separation of the localized and extended states as indicated by the IPR. We now emphasize a special feature of this 1D model demonstrating the 3D Anderson localization behavior in its full glory. Note that \( \alpha = -1 \) gives a disorder free constant potential and all the states must be extended. Consequently, all the states lie just below the critical line in the extended regime for all values of \( \lambda/t \). Infinitesimal deviation from the \( \alpha = -1 \) point manifests localized states across this critical line. The number of available localized states depends on the value of \( \lambda/t \). This feature, showing up clearly in our 1D incommensurate model, is one of the striking manifestations of 3D Anderson localization phenomenon, where any infinitesimal disorder completely localizes some eigenstates forming a sharp mobility edge defined by Eq. (3). It is
surprising that our simple 1D model captures a key element of 3D Anderson localization transition.

**Self-duality:** Having presented compelling numerical evidence that the condition conjectured in Eq. (3) is a critical point of a localization transition, we now analytically derive this condition. We rewrite the model defined in Eq. (1) for onsite potentials defined in Eqs. (2)–(5) in the following form,

\[ t(u_{p-1} + u_{p+1}) + g \chi_p u_p = (E + 2\lambda \cosh \beta) u_p \] (6)

Where we have defined the onsite potential \( \chi_p(\beta) \)

\[ \chi_p(\beta) = \frac{\sinh \beta}{\cosh \beta - \cos(2\pi p\phi)} \]

\( \chi_p = \sum_r e^{-\beta |r|} e^{i r (2\pi p b)} \) (7)

We have defined \( 1/\alpha = \cosh \beta \) for \( \alpha > 0 \) (we can absorb the overall sign of \( \alpha \) as a phase shift in the cosine term). The parameter \( g \) is model dependent and is given by \( g = 2\lambda \cosh \beta / \tan \beta \) for the onsite potential defined in Eq. (2) and \( g = 2\lambda (1 + \cosh \beta) / \sinh \beta \) for the onsite potential in Eq. (6). The above parametrization will help us to identify the hidden duality symmetry of this model. Note that Eq. (6) can be deformed without breaking the duality symmetry by making an arbitrary choice for the parameter \( g \equiv g(\alpha, \lambda) \) and \( E \equiv E(\alpha, \lambda, t) \) to design several onsite potentials with exact mobility edges. In the following we can set the overall phase \( \phi = 0 \) without loss of generality. Now we define the following ansatz for the duality transformation under which the model in Eq. (6) is self-dual:

\[ f_k = \sum_{mnp} e^{i 2\pi b(mn + np)} \chi_{n-1} u_p \] (8)

The above transformation can be viewed as three independent transformations acting on \( u_p \). In the following, we show how these three transformations act on the tight binding model in Eq. (6) resulting in the final tight binding model for \( f_k \). We multiply Eq. (6) by \( e^{i 2\pi b np} \) and performing a summation over \( p \), we obtain

\[ 2t \cos(2\pi nb) v_n + g \sum_p e^{i 2\pi b np} \chi_p u_p = (E + 2\lambda \cosh \beta) v_n \] (9)

Here we have defined \( v_n = \sum_p e^{i 2\pi b np} u_p \). Using the identity defined in Eq. (7) we can rewrite the above equation as,

\[ \omega \chi_n^{-1}(\beta_0) v_n = g \sum_r e^{-\beta |n-r|} v_r \] (10)

Where we have defined,

\[ \cosh \beta_0 = \frac{E + 2\lambda \cosh \beta}{2t}, \omega = 2t \sinh \beta_0 \] (11)

Now we multiply Eq. (10) by \( e^{i 2\pi b mn} \) and carry out summation over \( n \). The resulting equation can be rewritten in terms of \( w_m = \sum_n e^{i m(2\pi bn)} \chi_n^{-1}(\beta_0) v_n \) as,

\[ \omega \chi_n^{-1}(\beta) w_m = g \sum_r e^{-\beta |m-r|} w_r \] (12)

Note that the above step is a generalized transformation encoded in the definition of \( w_m \). This is the final operation of our generalized transformation where we define \( f_k = \sum_m e^{i 2\pi b nk} w_m \). In the final step, we multiply Eq. (12) by \( e^{i 2\pi b nk} \) and sum over \( m \) to obtain the following tight binding model for \( f_k \),

\[ t(f_{k+1} + f_{k-1}) + g \frac{\sinh \beta}{\sinh \beta_0} \chi_k(\beta) f_k = (E + 2\lambda \cosh \beta) f_k \] (13)

Now we have derived the result we set out to prove. The above tight binding model in terms of \( f_k \) in Eq. (13) is explicitly self dual to the original tight binding model defined in Eq. (6) if \( \beta_0 = \beta \). From Eq. (11), this self duality condition can be expressed in terms of \( E, \alpha, t \) and \( \lambda \),

\[ \cosh \beta = \frac{E + 2\lambda \cosh \beta}{2t} \] (14)

\[ \alpha E = 2(t - \lambda) \] (15)

Thus we have proved the condition for self duality which we proposed and numerically verified as defining the mobility edge of a localization transition.

**Experimental design:** Having proved the existence of a mobility edge, we show that this localization physics can be realized in optical lattices and photonic waveguides within existing experimental technology. Quasiperiodic 1D lattices have been realized in ultra cold atoms (Bose-Einstein condensate (BEC) of R\(^{39}\) atoms) by a standing wave arrangement of two laser beam with mutually incommensurate wave vector \([18]\). The quasiperiodic potentials considered in this letter can be systematically engineered by a controlled application of series of standing wave laser beams. The experimental schematic becomes transparent by considering the Cosine Fourier series of the onsite potential defined in Eqs. (2) and (5),

\[ V_n(\beta, \phi) = \frac{a_0(\beta)}{2} + \sum_{r=1}^{\infty} a_r(\beta) \cos(r(2\pi nb + \phi)) \] (16)

where the coefficients are given by \( a_0 = -4\lambda \cosh \beta \), \( a_r = 2ge^{-\beta r} \). Note that \( g \) is an overall model dependent constant defined in Eq. (6). The \( r = 1 \) term is the AAH term which has been realized by a standing wave laser of wave vector \( k_1 \) superimposed on the underlying lattice potential generated by another standing wave laser beam of wave vector \( k_2 \) \([18, 19]\). The incommensuration parameter is defined by the wave vector ratio \( b = k_1/k_2 \), \( r > 1 \) terms can be realized by simply adding
a series of standing wave laser beams with a wave vector that is an integer multiple of $k_1$. The intensity of the $r^{th}$ harmonic laser beam is determined by the coefficient $a_r$. For small value of $\alpha$ (large $\beta$), the Fourier series can be truncated with few Fourier components. The mobility edge is extremely pronounced even for small values of $\alpha$ if we fine tune the model parameters to the critical AAH model ($\lambda = t$). As shown in Figs. (1b) and (2b), even an infinitesimal $\alpha$ manifests a sharp mobility edge. Experiments should trace both static (momentum distribution of a stationary state) and dynamical (diffusion dynamics) properties of the localized BEC condensate to demonstrate a clear localization transition. In addition to the diffusion dynamics, the momentum distribution of the stationary state should be a useful tool in mapping out the mobility edge physics.

In addition to ultracold atoms, the localization of the AAH model was also observed in quasiperiodic photonic waveguides. For this setup, the localization is quantified by directly monitoring the IPR of the injected wave packet [13]. Recent photon pumping experiments with complex quasiperiodic models [16] demonstrate the capability for design flexibility towards realizing onsite potentials considered in this letter.

**Conclusion:** In this work we have analytically demonstrated the surprising existence of a mobility edge in a wide class of 1D nearest neighbor tight binding models with quasiperiodic onsite potentials. We show that the analytical critical condition is in excellent agreement with the localization properties obtained from the numerical computation of IPR. We outlined a concrete design schematic and observation methodology of our results within existing experimental setups.

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