Cylindrical surface plasmon mode coupling as a three-dimensional Kaluza-Klein theory

Igor I. Smolyaninov
Department of Electrical and Computer Engineering
University of Maryland, College Park,
MD 20742
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Cylindrical surface plasmon (CSP) mode coupling in a nanowire or a nanochannel is described in terms of three-dimensional Kaluza-Klein theory in which the compactified third dimension is considered to be the angular coordinate of the metal cylinder. Higher \((n \neq 0)\) CSP modes are shown to possess a quantized effective charge proportional to their angular momentum \(n\). In a nanowire these slow moving charges exhibit strong long-range interaction via fast massless CSPs with zero angular momentum \((n = 0)\). Such a mode-coupling theory may be used in description of nonlinear optics of cylindrical nanowires and nanochannels (for example, in single-photon tunneling effect), and may be further extended to describe interaction of electrons with nonzero angular momenta in thin gold wires and carbon nanotubes.

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Quantum communication and information processing require unusual sources of light with strong quantum correlations between single photons. This necessity drives strong current interest in exploration and understanding of nonlinear optical phenomena at single photon levels. One example of such phenomena is photon blockade in a nonlinear optical cavity \([1]\) introduced in close analogy with the well-known phenomenon of Coulomb blockade for quantum-well electrons. Another recent example is the proposed observation of sound waves and, possibly, superfluid state in a weakly interacting “photon gas” in a nonlinear Fabry-Perot cavity \([2]\). In the latter proposal photons in the nonlinear cavity are treated as massive particles weakly interacting with each other via some unspecified nonlinear optical mechanism.

Very recently strong evidences of nonlinear optical effects such as single-photon tunneling \([3]\) and light-controlled photon tunneling \([4]\) occurring at very low light intensities in natural nanometer-scale pinholes and artificial cylindrical nanochannels drilled using focused ion beam milling in thick gold films and membranes, and filled with nonlinear polymer have been observed. Transmission of nanoholes in gold and silver films is commonly accepted to happen via excitation of intermediary surface plasmons (SP) propagating along the metal film surfaces, and cylindrical surface plasmons (CSP) excited in the cylindrical nanochannels \([5]\). In the case of long cylindrical nanochannels filled with nonlinear material the theoretical description of nonlinear optical transmission of the nanochannels requires understanding of CSP behavior (such as mode propagation and interaction) at a level of single optical quanta in the presence of strong nonlinear optical interaction between them. So far to my knowledge, no such theory has been developed.

In this Letter I analyze CSP behavior in two closely related cylindrical geometries, such as a long cylindrical nanochannel drilled in a metal that supports surface plasmons (such as gold and silver) and filled with nonlinear material, and a long cylindrical metal nanowire embedded in a nonlinear material. In both geometries a specific case of chiral nonlinear media will be addressed (it is important to mention that metal itself becomes a chiral material in an external magnetic field \([5]\), and all metal interfaces exhibit pronounced nonlinear behavior, so that chirality and nonlinear optical effects must be considered in cylindrical surface plasmon propagation). Since CSPs may be considered as if they live in a “three-dimensional space-time” with two spatial dimensions being a very long \(z\)-dimension along the axis of the cylinder, and a small “compactified” \(\phi\)-dimension along the circumference of the cylinder, the theory of CSP mode propagation and interaction is formulated similar to the three-dimensional Kaluza-Klein theory \([6]\). In the most interesting cylindrical nanowire case, higher \((n > 0)\) CSP modes are shown to possess a quantized effective charge (not of electric origin) proportional to their angular momentum. These slow moving effective charges exhibit strong long-range interaction via fast uncharged massless CSPs with zero angular momentum \((n = 0)\). The character of CSP mode interaction in a nanochannel is somewhat similar, but in this case the zero angular momentum modes acquire mass, so that interaction of higher mode quanta becomes short-ranged. Such a mode-coupling theory may be further extended to describe mode propagation and interaction in other chiral optical waveguides, as well as interaction of nonzero angular momentum electron states in thin gold wires and carbon nanotubes.

Let us recall the basic features of Kaluza-Klein theories. In modern Kaluza-Klein theories \([6]\) the extra N-4 space-time dimensions are considered to be compact and small (with characteristic size on the order of the Planck length). The symmetries of this internal space are chosen to be the gauge symmetries of some gauge theory \([7]\), so a unified theory would contain gravity together with the other observed fields. In the original form of the the-
ory a five-dimensional space-time was introduced where the four dimensions \(x^1, ..., x^4\) were identified with the observed space-time. The associated \(10\) components of the metric tensor \(g_{\alpha\beta}\) were used to describe gravity. After a compactified fifth dimension \(x^5\) with a small circumference \(L\) was added, the extra four metric components \(g_{5\alpha}\) connecting \(x^5\) to \(x^1, ..., x^4\) gave four extra degrees of freedom, which were interpreted as the electromagnetic potential (here we use the following convention for greek and latin indices: \(\alpha = 1, ..., 4; \ i = 1, ..., 5\)). An additional scalar field \(g_{55}\) or dilaton may be either set to a constant, or allowed to vary.

When a quantum field \(\psi\) coupled to this metric via an equation

\[
\Box_5 \psi + a \psi = 0
\]

is considered, where \(\Box_5\) is the covariant five-dimensional d’Alembert operator, the solutions for the field \(\psi\) must be periodic in the \(x^5\) coordinate. This leads to the appearance of an infinite "tower" of solutions with quantized \(x^5\)-component of the momentum:

\[
g_{n5}^2 = 2\pi n / L
\]

where \(n\) is an integer. In our four-dimensional space-time on a large scale such solutions with \(n \neq 0\) interact with the electromagnetic potential \(g_{5\alpha}\) as charged particles with an electric charge \(e_n\) and mass \(m_n\):

\[
e_n = \hbar q_n (16\pi G)^{1/2} / c
\]

\[
m_n = \hbar (q_n^2 - a)^{1/2} / c
\]

where \(G\) is the gravitational constant (see for example the derivation in [9]). For the purposes of discussion below let us follow this derivation when an angular coordinate \(\phi^5\) varying within an interval from 0 to \(2\pi\) is introduced, so that

\[
\phi^5 = 2\pi x^5 / L
\]

Now the metric can be written as

\[
ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta + 2 g_{\alpha5} dx^\alpha d\phi^5 + g_{55} d\phi^5 d\phi^5,
\]

Here we are not interested in possible spatial dependence of \(g_{55}\) and consider all \(g_{5\alpha}\) components to be independent of \(\phi^5\). Equation (1) with \(a = 0\) for the quantum field \(\psi\) in this metric should be written as

\[
\frac{\partial}{\partial x^\alpha}(g^{\alpha\beta} \frac{\partial \psi}{\partial x^\beta}) + \frac{\partial}{\partial x^\alpha}(g^{\alpha5} \frac{\partial \psi}{\partial \phi^5}) + \frac{\partial}{\partial \phi^5}(g^{5\alpha} \frac{\partial \psi}{\partial x^\alpha}) + \frac{\partial}{\partial \phi^5}(g^{55} \frac{\partial \psi}{\partial \phi^5}) = 0
\]

Since we assume that the \(g_{5\alpha}\) do not depend explicitly on \(\phi^5\), we should not address the terms like \(\partial g^{55} / \partial \phi^5\). Thus, we may search for the solutions in the usual form as

\[
\psi = \Psi(x^\alpha) e^{i q \phi^5}, \text{ where periodicity in } \phi^5 \text{ requires } q_n = n.
\]

As a result, we obtain

\[
\Box_5 \psi - q_n^2 \frac{1 - g_{55} g^{55}}{g_{55}} \psi + 2i q_n g^{\alpha5} \frac{\partial \psi}{\partial x^\alpha} + i q_n \frac{\partial g^{\alpha5}}{\partial x^\alpha} \psi = 0
\]

(8)

This is the same as the Klein-Gordon equation in the presence of an electromagnetic field: in four-dimensional space-time it describes a particle of mass

\[
m = \frac{2\pi \hbar q_n}{c g_{55}^{1/2} (q_n)}
\]

(9)

which interacts with a vector field \(g^{5\alpha}\) through a quantized charge \(e_n \sim q_n\). In this theory the conservation of the electric charge is a simple consequence of the conservation of the \(x^5\)-component of the momentum. Similar Kaluza-Klein theory may be formulated in a three-dimensional space-time, which has one compactified spatial dimension. Again, quantized \(\phi\)-component of the momentum will play a role of an effective charge, which interacts with a two-component vector field \(g^{a3}\).

It is well-known that Maxwell equations in a general curved space-time background \(g_{ik}(x,t)\) are equivalent to the macroscopic Maxwell equations in the presence of matter background with some nontrivial electric and magnetic permeability tensors \(\epsilon_{ik}(x,t)\) and \(\mu_{ik}(x,t)\). Thus, strong similarity between the results of the three-dimensional Kaluza-Klein theory and the solutions of Maxwell equations in some quasi-three-dimensional cylindrical waveguide geometries may be expected: in both cases we consider a massless field in a geometry which has a compactified cyclic \(\phi\)-dimension in the presence of nontrivial space-time curvature or permeability tensors, respectively. This similarity is expected to be especially strong in the case of surface plasmon waveguides, since surface plasmons live in (almost) three-dimensional space-times on the metal interfaces.

Let us first consider general properties of Maxwell equations in a medium (media), which has cylindrically symmetric geometry. The insight gained from the previous discussion tells us that in order to look similar to the three-dimensional Kaluza-Klein theory, the medium should discriminate between left- and right-circular polarized waves (the waves which have opposite angular momenta, and thus expected to possess opposite effective Kaluza-Klein charges). This means that the medium should be chiral or optically active. There are different ways of introducing optical activity (gyration) tensor in the macroscopic Maxwell equations. It can be introduced in a symmetric form, which is sometimes called Condon relations [10]:

\[
\vec{B} = \epsilon \vec{E} + \gamma \frac{\partial \vec{B}}{\partial t}
\]

(10)

\[
\vec{H} = \mu^{-1} \vec{B} + \gamma \frac{\partial \vec{E}}{\partial t}
\]

(11)
Or it can be introduced only in an equation for \( \vec{D} \) (see \([\text{11}]\)). In our consideration I will follow Landau and Lifshitz \([\text{3}]\), and for simplicity use only the following equation valid in isotropic or cubic-symmetry materials:

\[
\vec{D} = \varepsilon \vec{E} + i \vec{E} \times \vec{g},
\]

where \( \vec{g} \) is called the gyration vector. If the medium exhibits magneto-optical effect and does not exhibit natural optical activity \( \vec{g} \) is proportional to the magnetic field \( \vec{H} \):

\[
\vec{g} = f \vec{H},
\]

where the constant \( f \) may be either positive or negative. For metals in the Drude model at \( \omega >> eH/me \)

\[
f(\omega) = \frac{4\pi N e^3}{cm^2\omega^3} = -\frac{e\omega_p^2}{mc\omega^3},
\]

where \( \omega_p \) is the plasma frequency and \( m \) is the electron mass \([\text{3}]\).

Initially, let us consider a medium with \( \vec{g} = \vec{g}(r, z, t) \) directed along the \( \phi \)-coordinate. Such a distribution may be produced, for example, in a medium exhibiting magneto-optical effect around a cylindrical nanowire if a current is passed through the wire. After simple calculations we obtain a wave equation in the form:

\[
\nabla \times \nabla \times \vec{B} = -\Delta \vec{B} = -\frac{\epsilon}{c^2} \frac{\partial^2 \vec{B}}{\partial t^2} + i \frac{\partial (\nabla \times [\vec{E} \times \vec{g}])}{\partial t}
\]

\[(15)\]

The \( z \)-component of this wave equation for a solution \( \sim e^{i\omega t} \) is:

\[
\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial B_z}{\partial r} \right) + \frac{\partial^2 B_z}{\partial z^2} - \frac{c^2 \partial^2 B_z}{\partial t^2} - \frac{n^2}{r^2} B_z + \frac{ng}{rc} \frac{\partial (IE_z)}{\partial t} + \frac{in}{rc} \frac{\partial g}{\partial t} (IE_z) = 0
\]

\[(16)\]

It can be re-written in the form similar to equations (1) and (8) as follows:

\[
\hat{a}_r B_z + \Box_2 B_z - \frac{n^2}{g_{\phi\phi}} B_z + i n \frac{g}{r} \left( \frac{\partial E_z}{\partial t} \right) + i \frac{n \partial (g/r)}{\partial t} E_z = 0,
\]

\[(17)\]

where \( \hat{a}_r \) plays the role of factor \( a \) in equation (1), and \( g_{\phi\phi} = r^2 \). Similarity of equations (8) and (17) becomes more evident if we identify the Kaluza-Klein vector field \( g_{\phi\phi} \) as \( g/r \), disregard terms higher than linear in \( g \), and recall that for cylindrical surface plasmon modes \( \imath E_z = \alpha B_z \), where the coefficient of proportionality \( \alpha \) is real and is determined by the boundary conditions \([\text{12}]\). The missing factor of 2 in front of the fourth term in equation (17) originates from the asymmetric way of introducing the gyration in the macroscopic Maxwell equations (see equations (10-12) and the relevant discussion).

Thus, from the macroscopic Maxwell equations we arrive to a picture of \( n > 0 \) waveguide modes interacting with the "gyration potential" \( g/r \) via quantized effective "chiral charges" proportional to the angular momenta of the modes.

The dispersion law and electromagnetic field distribution of cylindrical surface plasmon modes of a cylindrical metal wire may be found in \([\text{12}]\). The \( \omega(k) \) of the \( n = 0 \) CSP mode goes down to \( \omega \rightarrow 0 \) approaching the light line \( \omega = kc/e^1/2 \) from the right, as \( k \rightarrow 0 \). The field of this mode has only the following nonzero components:

\[
E_r, E_z, \text{ and } H_{\phi}, \text{ thus satisfying our initial requirements for the direction of } \vec{g}, \text{ and if we recall equation (13), we notice that in the media exhibiting magneto-optical effect, such as the cylindrical metal wire itself, the higher } n > 0 \text{ CSP modes interact with the } H_{\phi} \text{ field of the } n = 0 \text{ CSPs via their effective "chiral charges" proportional to } n.
\]

Thus, CSP quanta with \( n = 0 \) may be considered as massless (in the \( k \rightarrow 0 \) limit) quanta of the "gyration field". The dispersion laws of the CSP modes with \( n > 0 \) start at some nonzero frequencies and intersect the light line (become nonradiative and de-couple from free-space photons) at some finite \( \omega \). The modes of different \( n \) are well separated from each other, and there is no crossing. In the \( k \rightarrow \infty \) limit the \( \omega(k) \) of all the CSP modes saturates at \( \omega = \omega_p/2^{1/2} \). Thus, the group velocity \( \omega(k)/dk \) of the higher modes goes to 0 in this limit: the "charged" quanta are slow.

Let us now derive an analog of the Poisson equation for the "gyration potential" and "chiral charges" around the cylindrical metal wire. Let us search for the solutions of the wave equation (15) in the form \( \vec{B} = \vec{B}_0 + \vec{B}_n \) and \( \vec{E} = \vec{E}_0 + \vec{E}_n \), where \( \vec{B}_0 \) and \( \vec{B}_n \) are the CSP fields with zero and nonzero \( (n) \) angular momenta, respectively, and the "gyration field" in the magneto-active media is obtained in a self-consistent manner as \( \vec{g} = f(\vec{H} + \vec{B}_0 + \vec{B}_n) \), where \( \vec{H} \) is a constant external field. We are interested in the solution for the field \( \vec{B}_0 \) in the limit \( \omega_p \rightarrow 0 \) in the presence of the \( \vec{B}_n \) field. The resulting nonlinear Maxwell equation may be simplified assuming that the field \( \vec{B}_n \) is supposed to be the solution of linear Maxwell equation, and the terms proportional to \( f^2 \) and higher may be neglected. As a result, we obtain:

\[
\Delta \vec{B}_0 = -i \frac{\omega_n}{c} \frac{\partial (\nabla \times [\vec{E}_n \times \vec{B}_n])}{\partial t} - i \frac{n \omega_n}{c} \frac{\partial (\nabla \times [\vec{E}_0 \times \vec{B}_n])}{\partial t} - \frac{n^2 \omega_n}{c} \frac{\partial (\nabla \times [\vec{E}_0 \times \vec{B}_0])}{\partial t}
\]

\[(18)\]

Since the fields \( \vec{B}_0 \) and \( \vec{B}_n \) are not supposed to be coherent, their products disappear after time averaging, and we are left with the equivalent Poisson equation:

\[
\Delta \vec{B}_0 = \frac{f \omega_n}{c} \frac{\nabla \times [\vec{E}_n \times \vec{B}_n]}{c^2} = \frac{4\pi f \omega_n}{c^2} \nabla \times \vec{S}_n,
\]

\[(19)\]

where \( \vec{S}_n \) is the Pointing vector of the CSP field with the nonzero \( (n) \) angular momentum. As an interesting con-
sequence of this equation, let us note that in an isotropic medium

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{B}_0 - \frac{4\pi f\omega_n}{c^2} \vec{S}_n) = 0$$  \hspace{1cm} (20)$$

This observation lets us to conclude what is the real physical nature of the "chiral charge" in the normal language of four-dimensional physics. Due to magneto-optical effect, cylindrical surface plasmons with nonzero angular momenta behave as current loops bound to the metal wire. Unlike dipole-dipole interaction of the current loops in free space, the interaction of CSPs is mitigated by the presence of cylindrical surface plasmons with zero angular momentum. Thus, interaction of current loops becomes quasi one-dimensional and very long-ranged.

Using the same approximations as before, we can also derive an analog of the Gauss theorem for the "chiral charges". Let us consider a cylindrical volume $V$ around a cylindrical metal wire (see Fig.1), such that the side wall of the volume $V$ is located very far from the wire and the CSP fields are zero at this wall.

Using the vector calculus theorem we can write

$$\int_V \vec{\nabla} \times \vec{S} d^3 x = \int_S \vec{n} \times \vec{S} da = \int_{S_2} \vec{N} \times \vec{S} da - \int_{S_1} \vec{N} \times \vec{S} da,$$

where $S$ is the closed two-dimensional cylindrical surface bounding $V$, with area element $da$ and unit outward normal $\vec{n}$ at $da$, $S_1$ and $S_2$ are the front and the back surfaces of $V$, and $\vec{N}$ is the chosen direction of the wire. Using equation (20) we obtain

$$\int_V \frac{4\pi f\omega_n}{c^2} \vec{\nabla} \times \vec{S} d^3 x = \int_{S_2} \vec{N} \times [\vec{\nabla} \times \vec{B}_0] da - \int_{S_1} \vec{N} \times [\vec{\nabla} \times \vec{B}_0] da$$  \hspace{1cm} (21)

Since $\vec{N} \times [\vec{\nabla} \times \vec{B}_0] = \partial B_{n0}/\partial z$, we see that a "chiral charge" produces a local step in the "gyration field". If this effective Gauss theorem would be precise at any distance, we would come up with an unphysical result that a localized "chiral charge" would produce an infinite field $B_0$ at infinity. In reality, one has to take into account all the effects that we neglected while deriving the effective Poisson equation and Gauss theorem, such as higher than quadratic terms in the nonlinear Maxwell equations, and also take into account final lifetime of the cylindrical surface plasmons. Nevertheless, the underlying Kaluza-Klein mechanism of the "chiral charge" interaction lends credence to the conclusion that CSPs of a cylindrical metal nanowire exhibit very strong long-range interaction. The case of cylindrical nanochannel is somewhat different, since in this case the zero angular momentum mode has nonzero frequency in the limit $k \to 0$ (in other words it acquires some effective mass), so that interaction of higher mode quanta via exchange of zero-angular momentum CSPs becomes more short-ranged. Similar mode-coupling theory may be further developed to describe mode propagation and interaction in other chiral optical waveguides.

Let us estimate the strength of chiral charges interaction. Using equations (17) and (22) within the approximations described above the potential energy of two equal chiral charges $n$ separated by distance $z$ can be written as

$$W = \frac{2\pi \hbar \omega_n f^2 n S_{n0}}{cr},$$  \hspace{1cm} (23)

where $S_{n0} = n\omega_n(\epsilon E_s^2 + B_s^2)/(4\pi\beta_n^2 r)$ is the $\phi$-component of the Pointing vector, $\beta_n = \epsilon\omega_n^2/c^2 - k_n^2$, is defined by the dispersion law of the $n$th mode, and $r$ is the radius of the cylindrical metal wire. Assuming $f(\omega)$ from (14) and $\beta_n \sim k_n$ the order of magnitude estimate of $W$ may be written as

$$W = zn^2(\frac{e^2}{\hbar c})\hbar \omega_n(\frac{\omega_n}{\omega_n^2})(\frac{\lambda}{r}) \frac{\hbar^2}{r^4 m^2 c^4},$$  \hspace{1cm} (24)

where $\lambda$ is the wavelength of $\omega_n$ light in vacuum. As may be expected, the nonlinear optical interaction of CSP modes grows inversely proportional to $r^4$, and achieves considerable strength in nanometer scale metallic nanowires and nanochannels even without addition of nonlinear dielectrics inside a nanochannel or around a nanowire. If we assume $\hbar\omega_n = 1eV$, $\hbar\omega_p = 7eV$, and $r = 1nm$ (disregarding nonzero skin depth) the potential energy of two unit chiral charges is of the order of $W \approx 2 \times 10^4 zeV/cm$, so even interaction of single CSP quanta separated by submicrometer distances is considerable.

It is also important to mention that electron states with nonzero angular momenta in metal nanowires (such as gold, silver, copper, aluminum and many other metals which support surface plasmons, as well as carbon nanotubes with metallic conductance) should also behave like chiral charges described above. Their long-range current-loop-like interaction via exchange of $n = 0$
CSPs must be taken into account while considering mesoscopic conductance of thin wires. The strength of such interaction should be much stronger than the interaction of CSPs with each other, since the potential energy of two electron current-loops is proportional to $f$ (and not to $f^2$ as in equation (24)). Long range attractive interaction between two electron current loops may have important consequences for superconductive properties of metal nanowires and carbon nanotubes, with considerable strength of such interaction indicative of potential high $T_c$ superconductivity.

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