The second Chern class in Spinning System

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Abstract

Topological property in a spinning system should be directly associated with its wavefunction. A complete decomposition formula of SU(2) gauge potential in terms of spinning wavefunction is established rigorously. Based on the φ-mapping theory and this formula, one proves that the second Chern class is inherent in the spinning system. It is showed that this topological invariant is only determined by the Hopf index and Brouwer degree of the spinning wavefunction.

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I. INTRODUCTION

Topology now becomes absolutely necessary in physics [1-4]. The $\phi$-mapping theory and the gauge potential decomposition theory [5,6] are found to be significant in exhibiting the topological structure of physics systems [7-10]. Topological properties of a quantum system should be directly associated with its wavefunction. In our previous paper hep-th/9908168, based on $\phi$-mapping theory and gauge potential decomposition theory, we reveal the inner relation between the topological property of Schrödinger system and the intrinsic properties of its wavefunction. We point out that a topological invariant, the first Chern class, is inherent in the Schrödinger system, which is only associated with the wave function. This intrinsic relation is the topological source of the inner structure of London equations in superconductor [10].

In this paper, we extend our work to spinning system. Based on the gauge potential decomposition theory, we investigate the $SU(2)$ gauge potential in terms of the spinning wavefunction. The complete decomposition formula of the $SU(2)$ gauge potential is obtained rigorously. By making use of this result, the second Chern class is studied. One proves that the second Chern class is exclusively labeled by the topological index of the spinning wavefunction without using any particular models or hypotheses. In other words, one shows that the second Chern class is inherent in spinning system. This relation between Chern class and wavefunction is intrinsic property of quantum system, and gives a basic concept in topological quantum mechanics.

II. $SU(2)$ GAUGE POTENTIAL IN TERMS OF SPINNING WAVEFUNCTION

Considering a spinning wavefunction $\psi = \left( \begin{array}{c} \psi^1 \\ \psi^2 \end{array} \right)$, we have the convariant derivative denoted as

$$D\psi = d\psi - A\psi,$$  \hspace{1cm} (1)
where \( A = \frac{i}{2} A^a \sigma^a \) \((a = 1, 2, 3)\) is a \( SU(2) \) gauge potential, and \( \sigma^a \) is the Pauli Matrices. Its complex conjugate is given by

\[
D\psi^+ = d\psi^+ + \psi^+ A,
\]

in which \( \psi^+ = \begin{pmatrix} \psi^1* & \psi^2* \end{pmatrix} \). By making use of the equation

\[
\sigma^a \sigma^b + \sigma^b \sigma^a = 2I \delta^{ab},
\]

one can prove that

\[
A^a = -\frac{i}{\psi^+\psi} [(\psi^+ \sigma^a d\psi - d\psi^+ \sigma^a \psi) - (\psi^+ \sigma^a D\psi - D\psi^+ \sigma^a \psi)].
\]

From this formula, it can be proved that

\[
A^a = -\frac{i}{\psi^+\psi} Tr(d\psi^+ \sigma^a - \psi d\psi^+ \sigma^a) + \frac{i}{\psi^+\psi} Tr(D\psi^+ \sigma^a - \psi D\psi^+ \sigma^a).
\]

Using the properties of Pauli matrices and considering \( Tr(A) = 0 \), we can obtain that

\[
A = \frac{1}{\psi^+\psi} [(d\psi^+ - \psi d\psi^+) - Tr(d\psi^+ - \psi d\psi^+)I]
- \frac{1}{\psi^+\psi} [(D\psi^+ - \psi D\psi^+) - Tr(D\psi^+ - \psi D\psi^+)I].
\]

From (3), it is easy to prove that \( A \) satisfies the gauge transformation

\[
A^* = SAS^{-1} + dSS^{-1},
\]

under the \( SU(2) \) transformation

\[
\psi^* = S\psi, \quad \psi^* + = \psi^+ S^+.
\]

Here, we know that \( S^+ = S^{-1} \).

The main feature of the decomposition theory of the gauge potential is that the gauge potential \( A \) can be generally decomposed as [5,6,11]

\[
A = a + b,
\]
where $a$ and $b$ are required to satisfy the following respective transformation

$$a' = SaS^{-1} + dSS^{-1}, \quad (6)$$

$$b = SbS^{-1}. \quad (7)$$

Investigating Eq. (5), one finds that the $SU(2)$ gauge potential can be decomposed as

$$a = \frac{1}{\psi^+\psi}[(d\psi\psi^+ - \psi d\psi^+) - Tr(d\psi\psi^+ - \psi d\psi^+)I], \quad (8)$$

and

$$b = -\frac{1}{\psi^+\psi}[(D\psi\psi^+ - \psi D\psi^+) - Tr(D\psi\psi^+ - \psi D\psi^+)I], \quad (9)$$

which satisfy the Eqs. (8) and (7) respectively. We know that the topological property is independent with the choice of the gauge potential [4]. So, we can take $A$ as

$$A = \frac{1}{\psi^+\psi}[(d\psi\psi^+ - \psi d\psi^+) - Tr(d\psi\psi^+ - \psi d\psi^+)I]. \quad (10)$$

One can regard it as a special gauge. In fact, this choice is equal to the common condition, $D\psi = 0$, which is often used in studying spinning problem.

On the other hand, the spinning wavefunction can be denoted as

$$\psi = \begin{pmatrix} \phi^1 + i\phi^2 \\ \phi^3 + i\phi^4 \end{pmatrix},$$

where $\phi^a(a = 1, 2, 3, 4)$ are real function. One can regard the spinning wavefunction as the complex representation of a vector field $\vec{\phi} = (\phi^1, \phi^2, \phi^3, \phi^4)$. Let us define a unit vector $\vec{n}$ as

$$n^a = \frac{\phi^a}{||\phi||}, \quad ||\phi|| = (\phi^a\phi^a)^{\frac{1}{2}} = (\psi^+\psi)^{\frac{1}{2}}, \quad (11)$$

and

$$n^a n^a = 1. \quad (12)$$
If we denote that

\[ N = \begin{pmatrix} n^1 + i n^2 \\ n^3 + i n^4 \end{pmatrix}, \quad \quad N^+ = \begin{pmatrix} n^1 - i n^2 \\ n^3 - i n^4 \end{pmatrix}, \]

then one has

\[ N^+ N = n^a n^a = 1 \quad (13) \]

The expression \( (10) \) can be rewrote as

\[ A = (NdN^+ - dNN^+) - Tr(NdN^+ - dNN^+)I. \quad (14) \]

**III. THE SECOND CHERN CLASS INHERITS IN SPINNING SYSTEM**

We know that the unit vector \( \vec{n} \) can be expressed in terms of Clifford algebra \([12, 13]\) as

\[ n = n^A s_A, \quad A = 0, 1, 2, 3. \quad (15) \]

with \( n^0 = n^4 \), and where

\[ s = (I, i\vec{\sigma}), \quad s^\dagger = (I, -i\vec{\sigma}). \quad (16) \]

Then one can rewrite \( (12) \) as

\[ nn^\dagger = 1. \quad (17) \]

If we choose the spinor as \( \psi = \begin{pmatrix} \phi^1 - i \phi^2 \\ -\phi^3 - i \phi^4 \end{pmatrix} \), it can be proved that the gauge potential \( A \) given in \( (14) \) has the expression as

\[ A = dnn^\dagger. \quad (18) \]

It is interesting that this result is the same as the decomposition formula of the \( SU(2) \) gauge potential in terms of a four-dimensional vector field which we have obtained in \([11]\).

One knows that the second Chern class can be wrote in terms of Chern-Simon \([14]\)
\[ C_2(P) = \frac{1}{8\pi^2} d\Omega, \]  

(19)

and

\[ \Omega = \frac{1}{8\pi^2} Tr \left( A \wedge dA - \frac{2}{3} A \wedge A \wedge A \right) \]  

(20)

which is known as Chern-Simon form \( \text{[15]} \).

Substituting (18) into (19) and considering (17), we obtain

\[ C_2(P) = \frac{1}{24\pi^2} Tr (dn \wedge dn^\dagger \wedge dn \wedge dn^\dagger). \]  

(21)

From (15), one gets

\[ C_2(P) = \frac{1}{12\pi^2} \varepsilon_{\mu\nu\lambda\rho} \partial_\mu n^A \partial_\nu n^B \partial_\lambda n^C \partial_\rho n^D \]  

(22)

By substituting (11) into (22), and considering

\[ dn^A = d\phi^A \frac{1}{||\phi||} + \phi^A d \left( \frac{1}{||\phi||} \right), \]  

(23)

we have

\[ C_2(P) = -\frac{1}{4\pi^2} \frac{\partial^2}{\partial \phi^A \partial \phi^A} \left( \frac{1}{||\phi||^2} \right) D(\phi/x) d^4x, \]  

(24)

where \( D(\phi/x) \) is the Jacobian defined as

\[ \varepsilon^{ABCD} D(\phi/x) = \varepsilon^{\mu\nu\lambda\rho} \partial_\mu \phi^A \partial_\nu \phi^B \partial_\lambda \phi^C \partial_\rho \phi^D. \]  

(25)

By means of the general Green function formula

\[ \frac{\partial^2}{\partial \phi^A \partial \phi^A} \left( \frac{1}{||\phi||^2} \right) = -4\pi \delta^4 \left( \vec{\phi} \right), \]  

(26)

we get

\[ C_2(P) = \delta^4 \left( \vec{\phi} \right) D(\phi/x) d^4x. \]  

(27)
Suppose $\phi^A(x)$ $(A = 0, 1, 2, 3)$ have $m$ isolated zeros at $x_\mu = z_\mu^i$ $(i = 1, 2, \cdots, m)$, according to the $\delta$-Function theory [2], $\delta(\vec{\phi})$ can be expressed by

$$\delta(\vec{\phi}) = \sum_{i=1}^{m} \frac{\beta_i \delta(\vec{x} - \vec{z}_i)}{|D(\phi/\mu)|_{\vec{x} = \vec{z}_i}}, \tag{28}$$

and one then obtains

$$C_2(P) = \sum_{i=1}^{m} \eta_i \beta_i \delta^4(x - z_i) d^4x, \tag{29}$$

where $\beta_i$ is a positive integer (the Hopf index of the $i$th zeros) and $\eta_i$ is the Brouwer degree [16]:

$$\eta_i = \frac{D(\phi/\mu)}{|D(\phi/\mu)|} = sgn[D(\phi/\mu)]|_{x = z_i} = \pm 1. \tag{30}$$

The meaning of the Hopf index $\beta_i$ is that the vector field function $\vec{\phi}$ covers the corresponding region $\beta_i$ times while $\vec{x}$ covers the region neighborhood of zero $z_i$ once. From above discussion, the Chern density $\rho(x)$ is defined as:

$$\rho(x) = \sum_{i=1}^{m} \eta_i \beta_i \delta^4(x - z_i), \tag{31}$$

which shows that the topological structure of Chern density $\rho$ is labeled by the Brouwer degrees and the Hopf index. The integration of $\rho(x)$

$$C_2 = \int \rho(x) d^4x = \sum_{i=1}^{m} \eta_i \beta_i \tag{32}$$

is integer called Chern number which is a topological invariant.

The result (32) suggests that the topological invariant is determined only by the zero points of the vector field $\vec{\phi}$, i.e. the zeroes of the spinning wavefunction $\psi$. The topological indexes of zero points are intrinsic properties of the wavefunction, so the topological invariant given in (32) is naturally inherent in the spinning system.

IV. DISCUSSION

Based on the gauge potential decomposition method, we established the gauge potential decomposition theory in terms of the spinning wavefunction. One finds that the $SU(2)$ gauge
potential can be completely decomposed by the wavefunction of spinning system. By making use of the $\phi$-mapping theory, we reveal that the second Chern class is naturally inherent in spinning system, which is only associated with the intrinsic properties of the spinning wavefunction, and labeled by the topological indexes of the wavefunction. We believe that this intrinsic topological property is a fundamental property of quantum system and is the source of many topological effects in quantum system. This gives a new concept in topological quantum mechanics.

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