Some Properties of Finitely Presented Groups with Topological Viewpoints

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Abstract

In this paper, using some properties of fundamental groups and covering spaces of connected polyhedra and CW-complexes, we present topological proof for some famous theorems about finitely presented groups.

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1 Introduction

There are some famous results about subgroups of free groups, free products and finitely presented groups with complicated group theoretical proofs. For example, a famous corollary of the Reidemeister-Schreier rewriting process[3] tells us that every subgroup of a finitely presented group with finite index is also finitely presented. In this paper, using some well-known relationship between covering spaces of connected polyhedra (simplicial complexes) and their fundamental groups, we intend to prove some results for finitely presented groups with a topological approach.

2 Notation and Preliminaries

We suppose that the reader is familiar with some well-known notion such as free groups, free products and presentation in group theory and simplicial complexes (polyhedra), covering spaces, and fundamental groups in algebraic topology.

Definition 2.1 Let $T$ be a connected simplicial complex, then $T$ is called a tree if $\dim T \leq 1$ and which contains no circuits. Let $K$ be a connected simplicial complex with a maximal tree $T$ in $K$. Define a group $G_{K,T}$ with the following presentation:

$$G_{K,T} = \langle (p, q) \in K \mid (p, q) \in T, (p, q)(q, r) = (p, r) \text{ if } \{p, q, r\} \text{ is a simplex in } K \rangle.$$ 

The following are some facts in algebraic topology which we need in the proof of main results.

Theorem 2.2 ([5]). Let $K$ be a connected polyhedron with a base point $p$. Then its fundamental group $\pi_1(K, p)$ is isomorphic to $G_{K,T}$, where $T$ is a maximal tree in $K$ (note that we identify the simplicial complex $K$ with its underlying set the polyhedron $|K|$).
Corollary 2.3. If $K$ is a graph i.e. a connected 1-complex, then $\pi_1(K, p)$ is a free group of rank $|\{(p, q) \in K \setminus T \mid T \text{ is a maximal tree in } K\}|$.

Theorem 2.4 ([5]). A group $G$ is finitely presented if and only if there exists a finite connected polyhedron $X$ with $G \cong \pi_1(X, p)$.

Theorem 2.5 ([5]). For any group $G$, there exists a CW-complex $K(G)$ with

$$\pi_1(K(G)) \cong G \text{ and } \pi_n(K(G)) = 1 \text{ for all } n \geq 2.$$ 

The space $K(G)$ is called Eilenberg-MacLane space of $G$.

Remark 2.6 ([5]). With respect to the way of constructing the Eilenberg-MacLane space, generators and relators of the group $G$ are in one to one corresponding to 1-cells and 2-cells in $K(G)$.

Corollary 2.7. A group $G$ is finitely presented if and only if the number of 1-cells and 2-cells in it’s Eilenberg-MacLane space $K(G)$ is finite.

Theorem 2.8 ([1]). For any group $G$ and its Eilenberg-MacLane space, $K$ say, we have

$$H_2(K) \cong M(G),$$

where $M(G)$ is the Schur multiplier of $G$.

Lemma 2.9 ([5]). Let $(\bar{X}, p)$ be a covering space of $X$, $x_0 \in X$, and $Y = p^{-1}(x_0)$ be the fiber over $x_0$. Then $|Y| = |\pi_1(X, x_0) : p_*(\pi_1(\bar{X}, x_0))|.$

Definition 2.10. A space $X$ is called semilocally 1-connected if for every $x \in X$ there exists an open neighborhood $U$ of $x$ so that every closed path at $x$ in $U$ is nullhomotopic in $X$.

Note that any CW-complex, particularly any Eilenberg-MacLane space, is semilocally 1-connected space.

Theorem 2.11 ([5]). If $X$ is connected, locally path connected, and semilocally 1-connected and $G \leq \pi_1(X, x_0)$, then there exists a constructed covering space of $X$, $(\bar{X}, p)$ such that

$$p_*(\pi_1(\bar{X}, \bar{x}_0)) = G.$$
Theorem 2.12 ([5]). If $X$ is a connected CW-complex and $\tilde{X}$ is a covering space of $X$, then $\tilde{X}$ is also a CW-complex with $\text{dim}\tilde{X} = \text{dim}X$. Moreover, if $X$ has $m$ $k$-cells, and $\tilde{X}$ is $n$-sheeted, then the number of $k$-cells in $\tilde{X}$ is exactly equal to $mn$.

Theorem 2.13 ([4]). For any two groups $G_1$ and $G_2$ with their Eilenberg-MacLane spaces $K_1$ and $K_2$, respectively, the topological wedge space $K_1 \vee K_2$ is an Eilenberg-MacLane space corresponding to the free product $G_1 \ast G_2$.

Theorem 2.14 ([4]). For any two groups $G_1$ and $G_2$ with their Eilenberg-MacLane spaces $K_1$ and $K_2$, respectively, the topological product space $K_1 \times K_2$ is an Eilenberg-MacLane space corresponding to the direct product $G_1 \times G_2$.

3 Main Results

The following theorem is a consequence of the Reidemeister-Schreier rewriting process [3, Prop. 4.2].

Theorem 3.1. Every subgroup of a finitely presented group with finite index is also finitely presented.

Proof. Let $G$ be a finitely presented group and $H \leq G$ with finite index. By Theorem 2.4, there exists a finite connected polyhedron $X$ with $G \cong \pi_1(X)$. Since $X$ is connected, locally path connected and semilocally 1-connected, there exists a covering space $\tilde{X}_H$ so that $\pi_1(\tilde{X}_H) \cong H$, by Theorem 2.11. Since $[G : H] \leq \infty$, $\tilde{X}_H$ is a finite sheeted covering space of $X$ and so by Theorem 2.12, $\tilde{X}_H$ is a finite polyhedron. Now, by Theorem 2.4, $\pi_1(\tilde{X}_H) \cong H$ is finitely presented. $\blacksquare$

Theorem 3.2. If $G$ is a finitely presented group, then its Schur multiplier $M(G)$ is finitely presented.

Proof. First, note that the Schur multiplier of any group $G$ is isomorphic to the second homology group of its corresponding Eilenberg-MacLane space [1], $K$ say. Now using the fact that the number of $i$-cells, for any $i \in \mathbb{N}$, in
the Eilenberg-MacLane space of any finitely presented group \( G \) is finite, any homology group of \( K \) and in particular, the Schur multiplier of \( G \) is finitely presented. □

**Corollary 3.3.** Any covering group of a finite group is also a finitely presented group.

**Proof.** Using the definition of covering group \( \tilde{G} \) considered as an extension of the Schur multiplier of \( G \) by the group \( G \) itself, this note is straightforward result of two recent theorems. □

**Theorem 3.4.** The number of finitely presented groups is countable.

**Proof.** First, recall that there exists a bijection between all finitely generated groups and special 2-simplicial complexes [6]. Hence to prove the result, it is sufficient to show the number of such spaces is countable. Note that each polyhedron corresponding to a finitely presented group \( G \), with a presentation \( G = \langle x_1, \ldots, x_n \mid r_1, \ldots, r_n \rangle \), is obtained by attaching \( r \) 2-cells to an \( n \)-rose via some particular maps.

Suppose \( K \) is an \( n \)-rose lying on the plane \( \mathbb{R}^2 \) and \( \{ K^n_\lambda \}_{\lambda \in \Lambda} \) be the family of all polyhedra obtained by attaching finitely many 2-cells to \( K \), in several ways.

Now by Whitney Theorem [7] which states that any \( n \)-simplicial complex can be embedded in \( \mathbb{R}^{2n+1} \), we can consider all the constructed complexes as above in the Euclidean space \( \mathbb{R}^5 \) and then using the axiom of choice and the denseness of \( \mathbb{Q}^5 \) in \( \mathbb{R}^5 \), we can consider the rational points \( x_\lambda \in \mathbb{Q}^5 \) belonging to one and only one \( K^n_\lambda \).

Finally, we conclude that all finitely presented groups with \( n \) generators in their presentations are in one to one corresponding to a subset of rational points in \( \mathbb{R}^5 \) and so we are done. □

**Theorem 3.5.** The free product of two finitely presented groups is finitely presented.

**Proof.** Suppose that \( G_1 \) and \( G_2 \) are finitely presented groups with Eilenberg-MacLane spaces \( K_1 \) and \( K_2 \), respectively. Using Theorem 2.13, \( K_1 \vee K_2 \) is an
Eilenberg-Maclane space corresponding to $G_1 \ast G_2$. Also, by the definition, clearly the number of $i$-cells in wedge space of two spaces having finitely many $i$-cells, is also finite and so by Theorem 2.7, the result satisfied. \qed

**Theorem 3.6.** The product of two finitely presented groups is finitely presented.

**Proof.** By the hypothesis of the previous proof, we only note Theorem 2.14, and the fact that the number of $i$-cells in product of two spaces having finitely many $i$-cells, is also finite. Hence similar to the above proof, we complete the proof. \qed

**Theorem 3.7.** The free amalgamated product of two finitely presented groups $G_1$ and $G_2$ over a finitely presented subgroup $H$ is also finitely presented.

**Proof.** First, we consider an Eilenberg-Maclane space corresponding to the presentation of $H$, $X$ say, and note that we can extend the algebraic presentation of $H$ to the presentations for $G_1$ and $G_2$.

Also, by joining some 1-cells and attaching 2-cells via the relations, similar to the method of [5, Theorem 7.45] and [6, Note 6.44], we extend the space $X$ to Eilenberg-Maclane spaces $X_1$ and $X_2$ corresponding to the presentations of $G_1$ and $G_2$, respectively. Note that the construction is considered so that $X$ is a deformation retract of the space $X_1 \cap X_2$.

Now using van-Kampen theorem, the fundamental group $\pi_1(X_1 \cup X_2)$ is the free amalgamated product of two groups $\pi_1(X_1) \cong G_1$ and $\pi_1(X_2) \cong G_2$ over the subgroup $\pi_1(Y) \cong \pi_1(X_1 \cap X_2) \cong H$ [5].

Hence by uniqueness of the free amalgamated product up to isomorphism, we conclude that

$$G \cong \pi_1(X_1 \cup X_2).$$

On the other hand, by the assumption of being finitely presented for the groups $H$, $G_1$ and $G_2$ we conclude the spaces $X$, $X_1$, $X_2$ and so the space $X_1 \cup X_2$ have finitely many cells, which implies the group $\pi_1(X_1 \cup X_2)$ to
be finitely presented. □

Finally, by the definition of two new concepts, the Schur multiplier of a pair and the Schur multiplier of a triple of groups [2], we conclude the following results. Note that for a pair of groups \((G, N)\), the natural epimorphism \(G \to G/N\) induces functorially the continuous map \(f : K(G) \to K(G/N)\). Suppose that \(M(f)\) is the mapping cylinder of \(f\) containing \(K(G)\) as a subspace and is also homotopically equivalent to the space \(K(G/N)\). We take \(K(G, N)\) to be the mapping cone of the cofibration \(K(G) \hookrightarrow M(f)\). The Schur multiplier of the pair \((G, N)\) is considered as the third homology group of the cofiber space \(K(G, N)\).

In addition, we can extend the above notes to a topological argument for the Schur multiplier of a triple of groups. If we consider the space \(X\) as the cofibration of the natural sequence \(K(G, N) \to K(G/M, MN/M)\), which is noted by \(K(G, M, N)\) [2, Sec. 6], then the Schur multiplier of the triple \((G, M, N)\) is defined to be the fourth homology group of the cofiber space \(K(G, M, N)\).

**Theorem 3.8.** The Schur multiplier of a pair of finitely presented groups is finitely presented.

**Proof.** We remark that a mapping cone space obtained from two spaces having finitely many cells, have also finitely many cells, which holds the result. □

Using a similar argument, we establish the following theorem:

**Theorem 3.9.** The Schur multiplier of a triple of finitely presented groups is finitely presented.

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