CHARACTERISING ACYLINDRICAL HYPERBOLICITY VIA PERMUTATION ACTIONS

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Abstract. We characterise acylindrical hyperbolicity of a group in terms of properties of an action of the group on a set (without any extra structure). In particular, this applies to the action of the group on itself by left multiplication, as well as the action on a (full measure subset of the) Furstenberg-Poisson boundary.

1. Introduction

The goal of this note is to characterise acylindrical hyperbolicity of a group $G$ in terms of the action of $G$ on itself, or on some other set. We do so by defining the notion of axial element for an action (on a set) which is inspired by the notion of WPD element for an action on a hyperbolic space, as first considered in [5].

Consider an action of the group $G$ on the non-empty set $X$ and fix an element $g \in G$ and a fundamental domain $D$ for the $\langle g \rangle$-action on $X$. A subset of $X$ is called bounded if it is contained in a finite union of translates of $D$ by elements of $\langle g \rangle$ and it is called unbounded otherwise. An element of $G$ is called tame if it translates every bounded subset of $X$ into a bounded subset. The pair $(g, D)$ is said to be an axial pair for the action of $G$ on $X$ if the following are satisfied.

Axiom 1: The subset of tame elements $t \in G$ such that $tD \cap D \neq \emptyset$ is finite.

Axiom 2: For every $h \in G$ there exists a bounded subset $B \subset X$ such that for every element $w \in G$ for which $hwD$ is unbounded, $X = wB \cup g^nB$ for some $n \in \mathbb{Z}$.

The element $g \in G$ is said to be an axial element for the action of $G$ on $X$ if there exists a fundamental domain $D \subset X$ such that the pair $(g, D)$ is an axial pair and it is said to be an axial element in $G$ if it is axial for the left regular action of $G$.

Recall that $g \in G$ is said to be a generalised loxodromic element if there is an acylindrical action of $G$ on a hyperbolic space where $g$ acts loxodromically. Our main theorem gives equivalent characterisations of generalised loxodromicity. We refer to a probability measure as admissible if the semigroup generated by its support is the whole group.

Theorem 1.1. Let $G$ be a finitely generated, non-virtually cyclic group, and let $g \in G$. Then the following are equivalent.

1. $g$ is generalised loxodromic.
2. $g$ is axial for some action of $G$ on a set.
3. $g$ is an axial element in $G$.
4. $g$ is axial for the action of $G$ on some full-measure subset of its Furstenberg-Poisson boundary for some (any) admissible measure, with respect to a measurable fundamental domain.
By definition, a non-virtually cyclic group is acylindrically hyperbolic if and only if it contains a generalised loxodromic element (see [8] for various characterisations of acylindrical hyperbolicity). We thus immediately get:

**Corollary 1.2.** Let $G$ be a finitely generated, non-virtually cyclic group. Then the following are equivalent:

1. $G$ is acylindrically hyperbolic.
2. $G$ has an action on some set that admits an axial element.
3. $G$ contains an axial element.
4. $G$ contains an axial element for its action on some full-measure subset of its Furstenberg-Poisson boundary for some (any) admissible measure, with respect to a measurable fundamental domain.

**Outline.** In Section 2 we study axial elements, proving various properties reminiscent of those of loxodromic WPD elements. These allow us to use the Bestvina-Bromberg-Fujiwara construction [3] to construct hyperbolic spaces on which axial elements are loxodromic, see Theorem 2.12. This shows that any of items (2),(3),(4) of Theorem 1.1 implies item (1).

In Section 3 we show Proposition 3.2 which says that generalised loxodromic elements are axial for various actions, proving that item (1) of Theorem 1.1 implies items (2),(3),(4). We note that Proposition 3.2 also says that the loxodromic elements for a specific acylindrical action on a hyperbolic space $Z$ are each axial for the action of $G$ on $Z$ (regarded as a set), as well as for the action on a natural subset of the boundary.

2. **Axial elements are generalised loxodromics**

2.A. **Heuristics.** In this subsection we briefly explain some of the heuristics behind the definition of axial element. One setup that one might think of is a group $G$ acting on a hyperbolic space $X$, which we want to only regard as a set, with $g$ being a loxodromic WPD. The fundamental domain $D$ that we would like to consider is, roughly, the set of points that closest-point project to an axis of $g$ to a fixed fundamental domain there. Now, it is known that $g$ is contained in a unique maximal virtually cyclic subgroup $E(g)$, and the definition of tame element is designed to capture the elements of said subgroup. Axiom (1) is the interpretation of $E(g)$ being virtually cyclic. Axiom (2) has a more involved relationship with properties of $g$ (and indeed, it will take some work to prove it in Proposition 3.2 below). As it will become clearer below, it simultaneously capture two facts. First, it captures the fact that translates of an axis of $g$ by elements outside of $E(g)$ have uniformly bounded projection to the axis. Secondly, it captures the fact that, given two translates of an axis of $g$, there are only finitely many translates "in between" the two.

2.B. **Properties of axial elements.** In this section we fix an action of the group $G$ on a non-empty set $X$. We note that if $G$ is finite then any of its elements is axial in a trivial way. We thus assume that $G$ is infinite. We fix once and for all an axial pair $(g,D)$.

It follows from Axiom (1) that $g$ is of infinite order (indeed, if not then every element of $G$ would be tame, and every tame element $t$ can be multiplied on the left by a power of $g$ to obtain an element as in Axiom (1)). We denote $D_t = g^t D_0$, in particular $D_0 = D$, and get a partition of $X$, $X = \bigsqcup_{t \in \mathbb{Z}} D_t$. We let an interval
be a subset of $X$ of the form $D_{[a, b]} = \bigcup_{i=a}^{b} D_i$, for $a \leq b$, and note that a subset of $X$ is bounded if and only if it is contained in some interval. We rewrite axiom (2) as follows:

(2') For every $h \in G$ there exists $m \in \mathbb{Z}$ such that for every element $w \in G$ for which $hwD$ is unbounded we have $X = wD_{[-m, m]} \cup g^n D_{[-m, m]}$ for some $n \in \mathbb{Z}$.

Given $h \in G$ we let $m(h)$ be the minimal $m \in \mathbb{N}$ satisfying (2'). We note that for a tame element $t \in G$ we have $m(th) = m(h)$, since then $thwD$ is unbounded if and only if $thwD$ is unbounded. We observe that $m : G \to \mathbb{N}$ is subadditive.

**Lemma 2.1.** For $h_1, h_2 \in G$, $m(h_1h_2) \leq m(h_1) + m(h_2)$.

**Proof.** We denote $m_i = m(h_i)$. Fix $w \in G$ for which $h_1h_2wD$ is unbounded. Then (using (2') with $h_1$ and $h_2w$ replacing $h$ and $w$) we have that $h_2wD_{[-m_1, m_1]}$ contains $X - g^{n_1} D_{[-m_1, m_1]}$ for some $n_1 \in \mathbb{Z}$ and in particular it is unbounded. It follows that there exists $j \in [-m_1, m_1]$ such that $h_2wD_j = h_2wgjD$ is unbounded. Then $wgjD_{[-m_2, m_2]}$ contains $X - g^{n_2} D_{[-m_2, m_2]}$ for some $n_2 \in \mathbb{Z}$. Writing $m = m_1 + m_2$, we deduce that

$$wD_{[-m, m]} \supseteq wgjD_{[-m_2, m_2]} \supseteq X - g^{n_2} D_{[-m_2, m_2]} \supseteq X - g^{n_2} D_{[-m, m]}.$$ 

thus indeed $m(h_1h_2) \leq m(h_1) + m(h_2)$.

We denote by $T \subseteq G$ the subset of tame elements (we will see soon that this is a subgroup) and by $F \subset T$ its finite subset $\{t \in T \mid tD \cap D \neq \emptyset\}$. We denote $W = G - T$ and call the elements of $W$ wild elements. We denote $M = m(e)$.

The following lemma, in the setup of the heuristic discussion above, is best interpreted as saying that the $w^{-1}$-translate of the axis of $g$ has bounded projection to the axis of $g$.

**Lemma 2.2.** For $w \in W$ there exists $i \in \mathbb{Z}$ such that $wD_i$ is unbounded. For every such $i$, $wD_{[-M, i+M]}$ contains $X - D_{[a, b]}$ for some $a, b \in \mathbb{Z}$ with $b - a = 2M$.

**Proof.** As in the previous proof, if the $w$-image of an interval is unbounded, then the $w$-image of one of the fundamental domains that it contains is unbounded as well. The second part of the statement is Axiom (2') for the case $h = e$, applied for $wg^i$, where $a = k - M$ and $b = k + M$.

A subset of $X$ is called *cobounded* if its complement is bounded.

**Lemma 2.3.** An element $w \in G$ is wild if and only if there exists a bounded subset $B \subset X$ such that $wB$ is cobounded.

**Proof.** The "if" part is clear and the "only if" part follows from Lemma 2.2 by setting $B = D_{[-M, i+M]}$.

The following lemma describes the same virtually cyclic subgroup $E(g)$ that was mentioned in the heuristic discussion above.

**Lemma 2.4.** The subset $T \subseteq G$ is a virtually cyclic subgroup and it coincides with the commensurating subgroup of the subgroup $\langle g \rangle$ in $G$.

**Proof.** It is clear that $T$ is closed under taking products. By Lemma 2.3, $W$ is closed under taking inverses: for $w \in W$ and bounded $B \subset X$ such that $wB$ is cobounded, $w^{-1}(X - wB)$ is unbounded, thus $w^{-1} \in W$. It follows that $T$ is also closed under taking inverses, thus it is a subgroup of $G$. 

From Axiom (1) and the fact that \( g \in T \) we get that for every \( t \in T \) there exists \( n \in \mathbb{Z} \) such that \( g^n t \) belongs to the finite subset \( F \). It follows that \( T \) is virtually cyclic. Since \( T \) is virtually cyclic and \( g \) is an element of infinite order, we get that \( T \) normalizes a finite index subgroup of \( \langle g \rangle \). In particular, \( T \) commensurates \( \langle g \rangle \).

We are left to show that the commensurating subgroup of \( \langle g \rangle \) is contained in \( T \). To see this, we assume by contradiction that \( w \) is a wild element which commensurates \( \langle g \rangle \). We thus have, by Lemma 2.3 a bounded subset \( B \subset X \) such that \( wB \) is cobounded and \( a, b \in \mathbb{N} \) such that \( wg^aw^{-1} = g^b \). Then the set \( wB \) contains the sets \( D_{nb} \) for infinitely many values of \( n \in \mathbb{Z} \), but for every such \( n \), \( B \supset w^{-1}D_{nb} = w^{-1}g^{bn}D = g^{an}w^{-1}D \) and the union of the sets \( g^{an}w^{-1}D \) is unbounded. This contradiction finishes the proof.

We conclude that \( G \) is virtually cyclic if and only if \( G = T \). For the rest of the section we assume that \( G \) is not virtually cyclic, thus \( W \neq \emptyset \). We note that \( W \) is stable under taking inverses and multiplying by tame elements from either side, as \( T \) is a group.

For every \( w \in W \) there exists \( i \in \mathbb{Z} \) such that \( wD_1 = wg^i D \) is unbounded. We denote the set of such \( i \)'s by \( I(w) \). This is where \( w \) gets really wild. It turns out that these sets are uniformly bounded. Heuristically, \( I(w) \) determines the projection of the \( w^{-1} \)-translate of the axis of \( g \).

**Lemma 2.5.** For every \( w \in W \), the diameter of \( I(w) \subset \mathbb{Z} \) is bounded from above by \( 2M \).

**Proof.** Fix \( w \in W \) and \( i, j \in I(w) \). By Lemma 2.2 the sets \( wD_{[i-M,i+M]} \) and \( wD_{[j-M,j+M]} \) are cobounded, thus intersect non-trivially. It follows that the intervals \([i-M,i+M]\) and \([j-M,j+M]\) intersect non-trivially, thus \(|i-j| \leq 2M \). □

We define the function \( i : W \to \mathbb{Z} \) by

\[
i(w) = \lfloor \min(I(w)) + \max(I(w)) \rfloor / 2.
\]

Note that \( I(w) \subseteq [i(w) - M, i(w) + M] \) (though \( i(w) \) might not lie in \( I(w) \)). We think of \( i(w) \) as the wilderness center of \( w \).

The following lemmas build up to Lemma 2.9 which is the interpretation in our setup of the Behrstock inequality (Axiom (P1) of [3]) in our context. Once one rewrites said inequality as we did in Lemma 2.9 it is natural to try to understand the quantities \(|i(u) - i(v)|\):

**Lemma 2.6.** For every \( u, v \in W \) we have \(|i(u) - i(v)| \leq m(uw^{-1}) + 2M \).

**Proof.** Fix \( i \in I(u) \). Then \(|i - i(u)| \leq M \). Set \( m = m(uw^{-1}) \). Since \( ug^iD = \langle uw^{-1} \rangle vg^iD \) is unbounded, we get that \( vg^iD_{[-m,m]} \) is cobounded. It follows that for some \( j \in [-m,m] \), \( vg^jD_j = vg^{i+j}D \) is unbounded, thus \( i + j \in I(v) \) and therefore \(|i + j - i(v)| \leq M \). We conclude that \(|i(u) - i(v) + j| \leq 2M \), thus indeed, \(|i(u) - i(v)| \leq m + 2M \). □

**Lemma 2.7.** For every \( w \in W \), \( wD_{[i(w) - 2M,i(w) + 2M]} \) contains \( X - D_{[a,b]} \) for some \( a, b \in \mathbb{Z} \) with \( b - a = 2M \).

**Proof.** Follows immediately from Lemma 2.2 as for \( i \in I(w) \), \( D_{[i-M,i+M]} \) is contained in \( D_{[i(w) - 2M,i(w) + 2M]} \). □
Lemma 2.8. For every \( w \in W \),
\[
wD_{[i(w) - 4M, i(w) + 4M]} \text{ contains } X - D_{[i(w-1) - 2M, i(w-1) + 2M]}.
\]

Proof. By Lemma 2.7 and the fact that the inverse of a wild element is also wild, the sets \( wD_{[i(w) - 2M, i(w) + 2M]} \) and \( w^{-1}D_{[i(w-1) - 2M, i(w-1) + 2M]} \) are cobounded. In fact, \( w^{-1}D_{[i(w-1) - 2M, i(w-1) + 2M]} \) contains \( X - D_{[a,b]} \) for some \( a, b \in \mathbb{Z} \) with \( b - a = 2M \). Hence \( wD_{[a,b]} \) contains \( X - D_{[i(w-1) - 2M, i(w-1) + 2M]} \). In particular, \( wD_{[a,b]} \) is cobounded, thus it intersects non-trivially the cobounded set \( wD_{[i(w) - 2M, i(w) + 2M]} \).

It follows that
\[
[a, b] \cap [i(w) - 2M, i(w) + 2M] \neq \emptyset,
\]
hence
\[
[a, b] \subset [i(w) - 4M, i(w) + 4M].
\]

We conclude that \( wD_{[a,b]} \) is contained in \( wD_{[i(w) - 4M, i(w) + 4M]} \), thus
\[
X - D_{[i(w-1) - 2M, i(w-1) + 2M]} \subseteq wD_{[a,b]} \subseteq wD_{[i(w) - 4M, i(w) + 4M]}.
\]

As mentioned above, the following lemma will give us one of the axioms from [3].

Lemma 2.9. Assume \( u, v \in W \) are such that also \( u^{-1}v \in W \). Then
\[
\min\{|i(u) - i(v^{-1}u)|, |i(v) - i(u^{-1}v)|\} \leq 5M.
\]

Proof. Assume \( |i(u) - i(v^{-1}u)| > 4M \). Then
\[
[i(u) - 2M, i(u) + 2M] \cap [i(v^{-1}u) - 2M, i(v^{-1}u) + 2M] = \emptyset
\]
and we get by Lemma 2.8 (applied with \( w = u^{-1} \) and \( w = u^{-1}v \)) that
\[
X = u^{-1}D_{[i(u-1) - 4M, i(u-1) + 4M]} \cup u^{-1}vD_{[i(u-1)v - 4M, i(u-1)v + 4M]}.
\]

Therefore,
\[
X = D_{[i(u-1) - 4M, i(u-1) + 4M]} \cup vD_{[i(u-1)v - 4M, i(u-1)v + 4M]}.
\]

It follows that \( vD_{[i(u-1)v - 4M, i(u-1)v + 4M]} \) is unbounded, thus
\[
[i(u-1)v - 4M, i(u-1)v + 4M] \cap [i(v) - M, i(v) + M] \neq \emptyset.
\]

Therefore \( |i(u) - i(u^{-1}v)| \leq 5M \).

At this point, we would be able to show all axioms from [3] except one. This remaining axiom says, roughly, that given two translates of an axis of \( g \) there are only finitely many other translates onto which the two fixed translates project far away from each other.

It is convenient to consider \( i : G \to \mathbb{Z} \cup \{\infty\} \) by setting \( i(t) = \infty \) for \( t \in T \). Note that for every \( h \in G \), \( i(g^nh) = i(h) \) and \( i(hg^n) = i(h) - n \), with the usual conventions regarding \( \infty \). Given \( h \in G \), we consider the function \( w \mapsto i(hw) - i(w) \in \mathbb{Z} \cup \{\infty\} \), which is well defined for \( w \in W \). Note also that \( i(hwg^n) - i(wg^n) = i(hw) - i(w) \), thus we get a well defined function
\[
f_h : W/\langle g \rangle \to \mathbb{Z} \cup \{\infty\}, \quad f_h(wg) = i(hw) - i(w).
\]

The next lemma will give the remaining axiom from [3]. The proof uses the Behrstock inequality (Lemma 2.9) combined with the consequence of Axiom (2) that, roughly, the number of large projections under consideration is a Lipschitz function on the group \( G \).
**Lemma 2.10.** Assume $G$ is finitely generated. Then there exists $N \geq 0$ such that for every $h \in G$, $|f_h(w(g))| \leq N$ for all but finitely many cosets in $W/\langle g \rangle$.

**Proof.** Fix a finite generating set $S$ of $G$ and set $L = \max\{m(s) \mid s \in S\} + 2M$. Set $N = 20M + L$. We fix $h \in G$, consider the set of exceptional cosets,

$$E = \{w(g) \mid |f_h(w(g))| > N\} \subseteq W/\langle g \rangle$$

and prove the lemma by showing that $|E| \leq 2|h| \cdot |T : \langle g \rangle|$, which is finite by Lemma 2.4. Here $|h|$ denotes the word length of $h$ with respect to $S$.

We write $h = s_1 \ldots s_{|h|}$ for $s_k \in S$ and denote $h_k = s_k \ldots s_{|h|}$. The size of the set

$$E_0 = \{h_k^{-1}t(g) \mid k = 1, \ldots , |h|, \ t \in T\} \subseteq G/\langle g \rangle$$

is bounded by $|h| \cdot |T : \langle g \rangle|$. Defining $E_1 = E - E_0$, it is thus enough to show that $|E_1| \leq |h| \cdot |T : \langle g \rangle|$. Letting $E_1 \subset W$ be a set of representatives of left $T$-cosets of elements of $E_0$ in $W$, we are left to show that $|E_1| \leq |h|$.

Note that for every pair of distinct elements $u, v \in E_1$, we have $u^{-1}v \in W$ and, by the fact that $E_1 \cap E_0 = \emptyset$, for every $k = 1, \ldots , |h|$ we have $h_ku \in W$, and thus by Lemma 2.6

$$|i(h_ku) - i(h_{k-1}u)| \leq m(h_kh_{k-1}^{-1}) = m(s_k) \leq L.$$ 

We conclude that for every $u \in E_1$ there exists $k(u)$ such that both

$$|i(h_ku) - i(u)|, |i(h_ku) - i(hu)| > 10M.$$ 

Indeed, the first integer $k$ such that $|i(h_ku) - i(u)| > 10M$ will do, as we have $|i(hu) - i(u)| > N = 20M + L$. We claim that the resulting map

$$k : E_1 \to \{1, \ldots , |h|\}, \ u \mapsto k(u)$$

is injective. Proving this claim will guarantee $|E_1| \leq |h|$, thus concluding the proof.

We now fix a pair of distinct elements $u, v \in E_1$ and argue to show that indeed $k(u) \neq k(v)$. Without loss of the generality, using Lemma 2.4, we assume that $|i(u) - i(v^{-1}u)| \leq 5M$. Since $|i(h_k(u)) - i(u)| > 10M$, we conclude that $|i(h_k(u)) - i(v^{-1}u)| > 5M$. Applying again Lemma 2.4 for $h_k(u)$ and $h_k(v)$, we get $|i(h_k(u)) - i(u^{-1}v)| \leq 5M$. Similarly, since $|i(hu) - i(u)| > N > 10M$, we get $|i(hu) - i(v^{-1}u)| > 5M$, thus $|i(hu) - i(u^{-1}v)| \leq 5M$. Therefore we have $|i(h_k(u)) - hv| \leq 10M$. Since $|i(h_k(v)) - hv| > 10M$ we must have $k(u) \neq k(v)$, as required. This finishes the proof of the claim. 

Recall that $I(w)$ represents the wilderness zone of $w \in W$. For precise estimations we replaced it by $i(w)$. We think of $I(w^{-1})$ as the rough image in $Z$ of $\infty$ under $w$. Identifying $Z$ with $(g)$ via $n \mapsto g^n$, we observe that $I : W \to \langle g \rangle$ is equivariant in the sense that $I((wg)^{-1}) = I(w^{-1})$ and $I((gw)^{-1}) = g^{-1}I(w^{-1})$. Next we turn it into a map which takes values in bounded subsets of $T$, rather than of $\langle g \rangle$, maintaining the equivariance properties. We set

$$\pi : W \to 2^T, \ \pi(w) = \bigcup_{s,t \in T} tI(sw^{-1}t),$$

thinking of $\pi(w)$ as the rough image in $T$ of $\infty$ under $w$. The following lemma summarises all properties of $\pi$, essentially saying that all axioms from $\mathbb{K}$ are satisfied.

**Lemma 2.11.** We fix a word distance $d$ on $T$. The map $\pi : W \to 2^T$ satisfies the following properties:
• π is bi-$T$-invariant, that is for $w \in W, s,t \in T$, $\pi(sw) = \pi(w)$.

• There is a uniform bound on the diameters of the subsets $\pi(w) \subseteq T, w \in W$.

• There is a uniform bound on the set of numbers

$$\min\{d(\pi(u), \pi(vu^{-1}), d(\pi(v), \pi(uv^{-1}))\}$$

which are obtained by running over all pairs $u,v \in W$ such that $uv^{-1} \in W$.

• There is a uniform constant $C > 0$ such that for every $h \in G$,

$$|\{w \in W \cap Wh^{-1} \mid d(\pi(w), \pi(wh)) > C\}| < \infty.$$

**Proof.** That $\pi$ is bi-$T$-invariant follows at once from the equivariance properties of $I$. It follows that $\pi(w) = \cup_{i \in F} I(w_i^{-1}t) \subseteq F \cdot \cup_{i \in F} I(w_i^{-1}t)$, where $F \subseteq T$ is a set of cosets representatives for $\langle g \rangle$. For every $w \in W$ we have $I(w) \subseteq [i(w) - M, i(w) + M]$, and hence it follows from Lemma 2.6 that $\cup_{i \in F} I(w_i^{-1}t) \subseteq [i(w) - 2M, i(w) + 2M]$ (since $m(t) = m(e) = M$). We conclude that the diameters of the sets $\pi(w)$ are uniformly bounded. Similarly, the third item follows from Lemma 2.9 and the fourth item follows from Lemma 2.10. \qed

Finally, we combine all results in this section with [3] to construct actions on hyperbolic spaces starting from axial elements.

**Theorem 2.12.** Let the finitely generated group $G$ act on the set $X$, and let $g \in G$ be axial. Assume $G$ is not virtually cyclic. Then $G$ acts acylindrically on some quasi-tree $Z$ such that $g$ is a loxodromic WPD element for the action on $Z$.

**Proof.** We will apply the Bestvina-Bromberg-Fujiwara construction [3] to the collection of all cosets of $T$ in $G$ (each endowed with a word metric). To do so, we need to define projections $\pi_{h1}(h_2T) \subseteq h_1T$ for all $h_1T \neq h_2T$ satisfying suitable properties. The output of the construction is a quasi-tree on which $G$ acts and which contains isometrically embedded copies of all cosets of $T$, so that in particular $g$ is loxodromic, see [3, Theorem B].

Given $h_1, h_2 \in T$ with $h_1T \neq h_2T$ we define projections equivariantly by

$$\pi_{h_1}(h_2T) = h_1 \pi(h_1^{-1}h_2T).$$

Boundedness of projections is axiom (P0) of Bestvina-Bromberg-Fujiwara [3, Section 1], and the remaining 2 axioms (P1),(P2) require that there exists a constant $B$ such that:

• for all $h_1, h_2, h_3 \in G$ such that the cosets $h_iT$ are pairwise distinct we have

$$\min\{d_{h_1T}(\pi_{h_1}(h_2T), \pi_{h_1}(h_3T)), d_{h_2T}(\pi_{h_2}(h_1T), \pi_{h_2}(h_3T)) \leq B,$$

and

• for all $h_1, h_2 \in G$ such that the cosets $h_iT$ are distinct we have

$$|\{hT \mid d_{hT}(\pi_{hT}(h_1T), \pi_{hT}(h_2T)) \geq B\}| < +\infty.$$

These axioms are satisfied by Lemma 2.11. We are now in a position to apply [3, Theorem B] and [3, Theorem 6.4] (for the acylindricity conclusion) to obtain the required space. \qed
3. Generalised loxodromics are axial

The aim of this section is to show that a generalised loxodromic is axial for various actions of the ambient group. We start with a lemma that says, roughly, that axiality can be "pulled-back".

Lemma 3.1. Let $G$ be a group acting on the sets $X, Y$, and let $f : X \to Y$ be a $G$-equivariant map. If the pair $(g, D)$ is axial for the action of $G$ on $Y$, then $(g, f^{-1}D)$ is axial for the action of $G$ on $X$.

Proof. Consider an axial pair $(g, D)$ for the action of $G$ on $Y$. We will use the terminology, e.g., "$X$-bounded" or "$Y$-bounded" to clarify which action we are considering.

First of all, note that if $w \in G$ is $Y$-wild then it is also $X$-wild for. In fact, by Lemma 2.8 there exists a $Y$-bounded set $B$ such that $wB$ is $Y$-cobounded, and taking preimages we see that $f^{-1}B$ is $X$-bounded but $wf^{-1}(B) = f^{-1}(wB)$ is $X$-cobounded.

In particular, we see that the set of $X$-tame elements $t$ with $tf^{-1}D \cap f^{-1}D \neq \emptyset$ is contained in the set of $Y$-tame elements with $tD \cap D \neq \emptyset$, yielding Axiom (1).

Regarding Axiom (2), note that, given $h \in G$, the set of elements $w$ such that $hwD$ is $Y$-unbounded is contained in the set of elements $w$ such that $hwf^{-1}D$ is $X$-unbounded. Hence, given a $B$ as in Axiom (2) for the action on $Y$, the subset $f^{-1}(B)$ satisfies Axiom (2) for the action on $X$. □

We are now ready to show that generalised loxodromics are axial, for various actions.

Proposition 3.2. Let $G$ be a non-virtually cyclic group and let $g \in G$ be a loxodromic WPD for the action of $G$ on the hyperbolic space $Z$. Then $g$ is axial for:

(1) The action of $G$ on $Z$ (regarded as a set).
(2) The action of $G$ on itself by left multiplication.
(3) The action of $G$ on $\partial Z - (G \cdot g^+ \cup G \cdot g^-)$, where $\partial Z$ is the Gromov boundary and $g^\pm$ are the fixed points of $g$ in $\partial Z$.
(4) The action of $G$ on a full-measure set of the Furstenberg-Poisson boundary $B$ associated with any given admissible measure $\mu$.

In the third and fourth case, the fundamental domain for $g$ can be taken to be a Borel set.

Proof. We will show that $g$ is axial for the action on $X = Z \cup \partial Z - (G \cdot g^+ \cup G \cdot g^-)$, with fundamental domain $D$ such that $D \cap (\partial Z - (G \cdot g^+ \cup G \cdot g^-))$ is a Borel set. Once we do this, items (1) and (3) follow from Lemma 3.1 applied to the obvious inclusions and item (2) will follow by considering the orbit map $G \to X$ associated with an arbitrary element $x \in X$. Item (4) follows as well from Lemma 3.1 combined with the fact that there exists an equivariant measurable map $B \to \partial Z - G \cdot g^+ \cup G \cdot g^-$. The existence of an equivariant map $B \to \partial Z$ is a consequence of \textit{[7] Theorem 1.1}, but the statement of \textit{[1] Theorem 2.3} (in view of \textit{[2] Theorem 2.7}, and \textit{[6] Remark 4} to drop the separability assumption) gives this more directly. To conclude, we only have to argue that the preimage of $G \cdot g^+ \cup G \cdot g^-$ has measure 0. This holds because otherwise that preimage would have full measure by ergodicity of $B$, and hence there would be an equivariant map $B \to G \cdot g^+ \cup G \cdot g^-$ endowed with the discrete metric. Since $G$ is not virtually cyclic and $g$ is loxodromic WPD,
the stabilisers of $g^\pm$ are not the whole $G$ (for example because the stabiliser of $\{g^\pm\}$ is the subgroup $E(g)$ used below). Hence, the action is fixed-point free, and this violates metric ergodicity, see [2] Page 2; Theorem 2.7.

We thus proceed to show that $g$ is axial for the above mentioned space $X$. To avoid confusion between the metric notion of boundedness and being contained in finitely many fundamental domains, we will use the terminology “bounded in $X$” for the latter. We will use the terminology “unbounded in $X$” similarly.

Denote $\gamma = \langle g \rangle x_0$, for some fixed basepoint $x_0 \in Z$ (we think of $\gamma$ as the axis of $g$). We consider a $g$-equivariant (closest point projection) map $\pi : Z \to \gamma$ that associates to $z \in Z$ a closest point in $\gamma$. In fact, we can extend $\pi$ to $\tilde{Z} = Z \cup (\partial Z - \{g^\pm\})$ to a coarsely locally constant map, meaning that there exists a constant $C_0$ such that any point of $\tilde{Z}$ has a neighborhood whose image under $\pi$ has diameter at most $C_0$. We can find a fundamental domain $D$ such that $\pi(D)$ is a bounded subset of $\gamma$ and such that any bounded subset of $\gamma$ is contained in finitely many fundamental domains, that is, it is bounded in $X$. Moreover, we can assume that the intersection of $D$ with $\partial Z - (G \cdot g^+ \cup G \cdot g^-)$ is Borel.

The WPD element $g$ is contained in a virtually cyclic subgroup $E(g)$ which has the property that there exists a constant $C_1$ such that for all $h \notin E(g)$ we have that $\pi(h\gamma)$ has diameter at most $C_1$, see e.g. [9] Corollary 4.4. Up to enlarging $C_1$, for all $h \notin E(g)$ there are neighborhoods $N^{\pm}$ in $Z \cup \partial Z$ of $g^\pm$ such that $hN^{\pm} \cap \{g^\pm\} = \emptyset$ and $\pi(hN^{\pm})$ has diameter at most $C_1$. This means that, given $h \notin E(g)$, all but finitely many fundamental domains are mapped into a certain interval $D_{[-n,n]}$ for some $n$ (depending on $h$). Those finitely many fundamental domains are then necessarily mapped to an unbounded subset of $X$. In summary, if $h \notin E(g)$, then $h$ is not tame, and therefore tame elements all lie in $E(g)$. We can now show Axiom (1), by showing that any coset $\langle g \rangle t$ in $E(g)$ can only contain finitely many elements $t'$ with $t'D \cap D \neq \emptyset$. Indeed, since $t$ is tame we have that $tD$ is bounded in $X$, so that for any sufficiently large $n$ we have that $g^{\pm n}tD \cap D = \emptyset$, concluding the proof of Axiom (1).

To show Axiom (2), we will consider projections to translates of $\gamma$, which we denote $\pi_{k\gamma} : Z \cup (\partial Z - \{kg^\pm\}) \to k\gamma$ (with $\pi_\gamma = \pi$). Fix $h \in G$. We claim that there exists $C_3 \geq 0$ (which is allowed to depend on $h$) such that if $hwD$ is unbounded in $X$ then $\pi_{w\gamma}(h^{-1}\gamma)$ lies within $C_3$ of $wx_0$. To prove the claim, note that it suffices to show that $\pi(w^{-1}h^{-1}g^+)$ or $\pi(w^{-1}h^{-1}g^-)$ project close to $x_0$, since $\pi$ is coarsely locally constant and $w^{-1}h^{-1}\gamma$ has bounded projection to $\gamma$ (note that $hw$ is wild, hence it does not lie in $E(g)$, and therefore the same is true of $w^{-1}h^{-1}$). Now, $hwD$ being unbounded in $X$ means that there are points $z$ in $Z \cup \partial Z$ arbitrarily close (in the topology of $Z \cup \partial Z$) to either $g^+$ or $g^-$ and such that $w^{-1}h^{-1}z \in D$, and in particular $w^{-1}h^{-1}z$ projects within bounded distance of $x_0$ in $\gamma$ (since $\pi(D)$ is a bounded subset of $\gamma$). Since $\pi$ is coarsely locally constant, $w^{-1}h^{-1}g^+$ or $w^{-1}h^{-1}g^-$ also project uniformly close to $x_0$ in $\gamma$.

We claim that there exists a bounded subset $B_0$ of $X$ and a constant $C_4$ such that for all wild elements $w$ there exists some $n$ such that $\pi(w(X - g^nB_0))$ lies within Hausdorff distance $C_4$ of $\pi(w\gamma)$. This is equivalent to finding $C_4$ and $B_0$ such that for all wild $w$ we have that $\pi_{w\gamma}(X - g^nB_0)$ lies within Hausdorff distance $C_4$ of $\pi_{w\gamma}(\gamma)$ for some $n$. We will use the fact that there exists a constant $C_5$ such that for all wild elements $w$ and $x \in X$ we have

$$\min\{d_Z(\pi(w\gamma), \pi(x)), d_Z(\pi_{w\gamma}(\gamma), \pi_{w\gamma}(x))\} \leq C_5$$
this is essentially the fact that axes of WPD elements satisfy Axiom (P1) of [3], see e.g. the arguments of [9 Lemma 2.5]. Since \( \pi(w^\gamma) \) is bounded uniformly over all wild \( w \), we can take \( B_0 \) such that for all wild \( w \) there exists \( n \) with the property that if \( x \in X - g^n B_0 \) then \( d_Z(\pi(x), \pi(w^\gamma)) \) is larger than \( C_5 \). We then have that \( \pi_w^\gamma(x) \) lies within distance \( C_5 \) of \( \pi_w^\gamma(\gamma) \), as required.

On the other hand, for all \( C \geq 0 \) there is a bounded subset of \( X \) that contain all points \( x \) with \( \pi(x) \) within \( C \) of \( x_0 \). Hence, for any constant \( C \geq 0 \) there exists a bounded subset \( B \supseteq B_0 \) of \( X \) with the property that if \( wB \) does not contain any \( X - g^n B \) for any \( n \), then \( \pi(w^{-1}(g^\pm)) \) is \( C \)-far from \( x_0 \), and hence \( \pi_w^\gamma(\gamma) \) is \( C \)-far from \( wx_0 \). Hence, if for all bounded subsets \( B \) of \( X \) we had some \( w \) with \( hwD \) unbounded but \( wB \) not containing any \( X - g^n B \), then we would have elements \( w \) such that the distances \( d_Z(\pi_w^\gamma(\gamma), \pi_w^\gamma(h^{-1}\gamma)) \) are arbitrarily large. But any such distance is bounded in terms of \( d_Z(\gamma, h^{-1}\gamma) \) since closest point projections in a hyperbolic space are coarsely Lipschitz, a contradiction. \( \square \)

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