Edge-matching Problems with Rotations

Martin Ebbesen
Appstract Consulting
DK-1300 København K
martin@appstract.dk

Paul Fischer
DTU Compute
Technical University of Denmark
DK-2800 Lyngby
pafi@dtu.dk

Carsten Witt
DTU Compute
Technical University of Denmark
DK-2800 Lyngby
cawi@dtu.dk

March 29, 2017

Abstract

Edge-matching problems, also called edge matching puzzles, are abstractions of placement problems with neighborhood conditions. Pieces with colored edges have to be placed on a board such that adjacent edges have the same color. The problem has gained interest recently with the (now terminated) Eternity II puzzle, and new complexity results. In this paper we consider a number of settings which differ in size of the puzzles and the manipulations allowed on the pieces. We investigate the effect of allowing rotations of the pieces on the complexity of the problem, an aspect that is only marginally treated so far. We show that some problems have polynomial time algorithms while others are NP-complete. Especially we show that allowing rotations in one-row puzzles makes the problem NP-hard. The proofs of the hardness result uses a large number of colors. This is essential because we also show that this problem (and another related one) is fixed-parameter tractable\(^1\), where the relevant parameter is the number of colors.

Keywords: Edge-matching puzzles, complexity, fixed-parameter tractability

1 Introduction

We consider the following combinatorial optimization problem. Let \( N \) and \( M \) be positive integers. The puzzles considered in this paper consist of \( NM \) quadratic pieces whose edges are colored. Let \( c_0, c_1, \ldots, c_K \) denote the colors. The pieces are placed in the cells of a rectangular \( N \times M \) grid. The edges of a piece are denoted right, top, left, bottom in the obvious way. A

\(^*\)A preliminary version of this paper appeared in Proceedings of FCT 2011, [1].

\(^1\)An NP-complete problem is fixed-parameter tractable if there is algorithms which is exponential in only one parameter which specifies the problem size an polynomial in the size of the input. When this parameter is constant (or “small”), then the problem is efficiently solvable.
piece \( P \) can then be specified by its position \((i, j)\) on the grid and the colors \( c_{r,P}, c_{t,P}, c_{l,P}, c_{b,P} \) on its right, top, left, and bottom edge, in this order. The image below shows a piece specified by the color pattern \((c_1, c_2, c_3, c_4)\).

Two pieces are *neighbors* if they are placed in the same column and the difference of their row coordinates in the grid is 1 (or with the roles of rows and columns interchanged), i.e., they share an edge. Given two pieces \( P \) and \( P' \) on neighboring positions, say at \((i, j)\) and \((i, j+1)\) respectively, we say that they *match*, if the adjacent edges have the same color, that is \( c_{r,P} = c_{l,P'} \). In this case we also say that the edge they have in common *matches*. Similarly, if they are vertically adjacent, say at \((i, j)\) and \((i+1, j)\), they *match*, if \( c_{b,P} = c_{t,P'} \), see example below.

We say that a board is *solved* if the pieces are placed in such a way that all edges match. At this point one has to specify rules for the “border edges”, that is, the edges facing the border of the board. We consider three cases here

**Free** There is no restriction, any color matches the border of the board.

**Monochrome** There is a single color for all border edges.

**Cyclic** We assume that the board actually is a torus, that is, the top edges are aligned with the bottom ones and the left edges are aligned with the right ones.

An arrangement of the pieces which solves the board is called a *solution*.

The manipulations allowed in a board are:

**Swap** Two pieces \( P \) and \( P' \) interchange their position, without being rotated.

**Rotate** A piece \( P \) is rotated counter-clockwise in-place by 0 deg, 90 deg, 180 deg, or 270 deg.

**Edge permutation** The colors on the edges are permuted in one of the 24 possible ways. The position of the piece is unchanged. Note that this manipulation includes rotations.

**Flip** The colors of the left and right or up and down edges of piece \( P \) are interchanged (that is \( P = (c_1, c_2, c_3, c_4) \) becomes \( P_{\text{flipped}} = (c_3, c_2, c_1, c_4) \) or \( P = (c_1, c_2, c_3, c_4) \) becomes \( P_{\text{flipped}} = (c_1, c_4, c_3, c_2) \)). Flipping is an in-place operation.

Combinations of the manipulations are possible.

The *edge matching problem* is then formulated as follows.
Problem 1.1 Given is a $N \times M$ board, $K$ colors, $NM$ pieces, a border rule and a set of manipulations. The decision problem is given by the question “is it possible to solve the board?”

In the optimization version of the problem, a solution has to be produced.

Edge-matching has found applications in biology where it is used in a method for DNA fragment assembly [2]. The problem has also gained interest recently with the Eternity II puzzle, a $16 \times 16$-puzzle, with a $\$2$ million prize for a solution (which was not found, though). Another area where this kind of problems appears is chip-layout, where interfaces have to be placed on a rectangular chip but their order is arbitrary.

1.1 Previous Work

Edge-matching with one row, swaps and no rotation corresponds to domino tiling [3] and is, as we discuss in this paper (and as covered in [3]), equivalent to the problem of finding an Eulerian path in a multi-graph. The computational complexity of edge-matching and related problems has been studied for several decades. Early results show that the more general tiling problem, i.e., solving the edge-matching problem with swaps for a quadratic board using only a subset of a given set of un-rotatable pieces is NP-complete [4]. Goles and Rapaport showed in [5] that the edge-matching problem with only in-place rotations (disallowing swaps) and free border rule is NP-complete. More recently, Demaine and Demaine [6] proved that the edge-matching variant considered here is NP-complete for quadratic boards with swaps and rotations. In 2010, Antoniadis and Lingas showed that this problem is even APX-hard, i.e., hard to approximate, already for boards with at least two rows [7].

1.2 Overview of the Paper

Most of the above-mentioned analyses of the edge-matching problem consider swaps, but do not allow rotating pieces. The NP-hardness proof in [6], even though it formally allowed rotations, forces the pieces to be used in a fixed orientation. Only recently, the APX-hardness proof in [7] explicitly made use of rotation and swaps at the same time. In this paper we address the question of what changes in the complexity of the problem occur when rotations of pieces are allowed.

In Section 2 we consider puzzles with only one row ($1 \times M$). For these puzzles there is a known correspondence to Euler paths, which we describe along with some previous results. We then show that single-row puzzles where only in-place rotations are allowed can be efficiently solved. In contrast we show that solving single-row puzzles with rotations and swaps allowed is an NP-hard problem. The proofs implicitly use the Euler path formulation of the problem.

In Section 3 we strengthen a result of [6] by showing that already boards with two rows with swaps only are NP-hard to solve.

Having observed that the hardness proofs use a number of colors which is proportional to the number of pieces in the puzzle, we show in Section 4 that the problem of solving a single-row puzzle with rotations and swaps is efficiently solvable when the number of colors is constant or logarithmic in the number of pieces. Thus this problem is fixed-parameter tractable. Similarly, an efficient solution is also possible for the case of constant board size and number of colors.

2 Boards with One Row

Edge-matching with one row ($N = 1$) without rotation is equivalent to the problem of finding an Eulerian trail in a multi-graph allowing loops. An Eulerian trail is a trail that visits each
edge of the graph exactly once, and the concept is applicable to both directed and undirected graphs: An undirected graph is Eulerian (i.e. contains an Eulerian trail) iff it is connected and has no more than 2 vertices with an uneven number of edges. An Eulerian circuit (a trail starting and ending on the same vertex) requires that all vertices must have an even number of edges. A directed graph has an Eulerian circuit iff it is weakly connected and all vertices has equal in- and out-degree, while a path requires connectedness and 0 or 2 vertices with difference in in- and out-degree equal to 1. The above facts are shown in, e.g., [3].

Swaps and flipping

An edge-matching instance with flipping, with $M$ pieces and $K$ colors is transformed to an undirected multigraph containing $K$ vertices and $M$ edges. Each piece corresponds to an edge connecting the vertices corresponding to the colors on opposing edges of the piece. Now traversing a vertex using two different edges corresponds to matching two pieces having a common color. A trail in the graph corresponds to a matched chain of pieces, and an Eulerian trail corresponds to a solution where all pieces are fitted, and vice versa. Hence this variant of the problem is efficiently solvable. Figure 1 shows an example, see [8] for more details.

Figure 1: An example for a single-row puzzle with swaps and flips. The colors of the vertical edges are denoted 1, 2, 3, and 4; the irrelevant color at the top and bottom is 0. The multigraph has a one node for each relevant color and one edge for every piece. The edge connects the nodes corresponding to the colors of the vertical edges. The graph has an Euler trail, 2, 1, 3, 4, 3 and thus the puzzle is solvable. The corresponding solution is shown at right.

Swaps only

Edge-matching with swaps (no flipping) correspond to Eulerian cycles in directed multigraphs, and vice versa. Hence this variant of the problem is efficiently solvable.

Rotations only

With free border rule, if the board consists of only one piece it will always be solved. Obviously, for a board with two pieces there will be a solution if the two pieces can be rotated such that their touching edges match. This can be generalized:

**Theorem 2.1** Single-row edge matching puzzles with in-place rotations can be solved or determined to be unsolvable by an algorithm that has time complexity linear in the number of pieces.

**Proof** The proof is by induction. The pieces are numbered from left to right as $p_0, p_1, ..., p_{M-1}$. $L(p_i)$ is the set of colors that can be on the left edge of piece $p_i$ which is equivalent to the set
of unique colors on the 4 edges of \( p_i \). \( R(p_i) \) is the set of colors that can be on the right edge of \( p_i \) such that \( p_i \) fits \( p_{i-1} \), that is, such that \( L(p_i) \cap R(p_{i-1}) \neq \emptyset \). \( L(p_i) \) and \( R(p_i) \) can both have at most 4 members since a piece has 4 edges.

**Base case:** For \( p_0 \) there is no constraint, as one piece always represents a solution to a 1 \times 1 board: \( R(p_0) = L(p_0) \).

**Induction step:** Given that \( R(p_i) \) is known and the board is solvable up to \( p_i \), then piece \( p_{i+1} \) can be fitted if \( R(p_i) \cap L(p_{i+1}) \neq \emptyset \). If not, the board must be unsolvable. \( R(p_{i+1}) \) can be calculated by finding the color on the opposite edge of \( p_{i+1} \) for each member in \( R(p_i) \cap L(p_{i+1}) \). This operation takes constant time because of the bound on the sizes of the two sets.

The theorem holds because each step in the induction, i.e., each additional piece, only adds a constant time overhead.

**q.e.d.**

**Swaps and Rotations**

**Theorem 2.2** If both swaps and rotations are allowed, single-row edge-matching with free border rule is NP-complete.

**Proof** We use a polynomial-time reduction from the NP-complete Monotone 1-in-3-SAT problem [9]: given \( m \) monotone (i.e., disallowing negation) clauses \( c_1, \ldots, c_m \) over \( n \) variables \( x_1, \ldots, x_n \), the question is whether there exists an assignment which satisfies exactly one variable in each clause.

Let \( m_i, i = 1, \ldots, n \), denote the number of occurrences of variable \( i \), i.e., \( m_1 + \cdots + m_n = 3m \). The instance of Monotone 1-in-3-SAT is mapped to a single-row board with free border at the top and bottom and cyclic border to the left and right (which can be simulated by a free border using a polynomial number of extra pieces). There are \( M = 10m \) pieces and \( K = 13m - n + 1 \) colors. We denote colors by lower-case and pieces by upper-case letters. The set of pieces consists of all \( V_{j,q}, V'_{j,q}, S_j \), where \( j = 1, \ldots, m \) and \( q = 1, 2, 3, \) and \( A_{i,k} \), where \( i = 1, \ldots, n \) and \( k = 1, \ldots, m_i \). The colors are \( \ell, t_{j,q}, t'_{j,q}, f_{j,q}, s_j \), and \( a_{i,k} \), where \( k = 1, \ldots, m_i - 1 \). We define and name the different classes of pieces as follows:

**Value** \( V_{j,q} := (f_{j,q}, t_{j,q}, s_j, \ell); V'_{j,q} = (\ell, t'_{j,q}, f_{j,q}, \ell). \)

**Satisfying** \( S_j := (s_j, s_j, s_j, s_j) \).

**Accordance** If \( m_i = 1 \) then \( A_{i,1} := (\ell, \ell, \ell, \ell) \). Otherwise, let the \( k \)-th occurrence of \( x_i \) be at position \( q_k \in \{1, \ldots, 3\} \) in clause \( c_{jk}, k = 1, \ldots, m_i \). Then define \( A_{i,k} := (t'_{j_k,q_k}, a_{i,k-1}, t_{j_k,q_k}, a_{i,k}) \), where \( a_{i,0} := a_{i,m_i} := \ell \).

This completes the transformation, which is obviously polynomial-time computable.

Rotations can deactivate up to two colors from a piece, namely those that are facing the border and thus need not be matched. Active colors must occur an even number of times in a solved board. When placing a piece \( V_{j,q} \) either \( f_{j,q} \) and \( s_j \), or \( t_{j,q} \) and \( \ell \) will be deactivated. Actually, since \( f_{j,q} \) only appears in \( V_{j,q} \) and \( V'_{j,q} \), a solution requires that these pieces either both deactivate \( f_{j,q} \) or that both activate this color. If they both activate \( f_{j,q} \) (which also activates \( s_j \) once), they will be adjacent in a solution. Later, active \( f_{j,q} \) will model that the \( q \)-th variable in \( c_j \) is false, and inactive \( f_{j,q} \) will model a true setting. Since \( s_j \) appears only in \( V_{j,1}, V_{j,2}, V_{j,3} \) and \( S_j \), but cannot be deactivated in \( S_j \), where it will be used twice, it is necessary for a solution that an even, positive number of pieces from \( V_{j,1}, V_{j,2}, V_{j,3} \) use \( s_j \). This forces two of the variables in \( c_j \) to a “false” and one to a “true” setting.

Finally, the interplay of the \( t_{j,q} \) and \( t'_{j,q} \), within the \( V \)- and \( A \)-pieces will ensure the consistency of the truth assignment. We fix a variable \( x_i \) and assume \( m_i > 1 \) as there is nothing to show
otherwise. Since color $a_{i,k}$, $k = 1, \ldots, m_i - 1$, is only used in $A_{i,k}$ and $A_{i,k+1}$, these two pieces must either both be rotated to activate the $t$ and $t'$ colors on their edges, or the pieces must be adjacent, which inductively requires that the solution contains all $A_i$-pieces as a chain $A_{i,1}, A_{i,2}, \ldots, A_{i,m_i}$ in this or reverse order. In the case of a chain, the colors $t_{j,k,q}$ and $t'_{j,k,q}$ must be inactive for all $k = 1, \ldots, m_i$, which means that the pieces $V_{j,k,q}$ and $V'_{j,k,q}$ are adjacent by virtue of color $f_{j,k,q}$, hence a “false” setting of $x_i$. Otherwise, the colors are all active, which is (by the uniqueness of the colors) only possible if $A_{i,k}$ is fit between the pieces $V_{j,k,q}$ and $V'_{j,k,q}$, both of which are rotated to a “true” setting. Hence, all $m_i$ occurrences of $x_i$ must be consistent.

We now prove that there is a solution to Monotone 1-in-3-SAT if and only if there is one to the puzzle. Let us first prove the implication $\Rightarrow$, i.e., we consider an assignment to $x_1, \ldots, x_n$ which satisfies exactly one variable in each clause. We lay down the pieces as subsequences in clause-wise order. When considering a clause $c_j = (u_{j,1} \lor u_{j,2} \lor u_{j,3})$, $j = 1, \ldots, m$, let $q \in \{1, \ldots, 3\}$ be the index of its unique satisfied variable, say this variable is $x_i$ in its $k$-th occurrence. The subsequence for $c_j$ starts with $V_{j,q}$ rotated by $270^\circ$ (left-hand side has color $\ell$), followed by $A_{i,k}$ unrotated and $V'_{j,q}$ rotated by $90^\circ$. Note that the right-hand side is $\ell$. Let $q'$ and $q''$ be the indices of the other two unsatisfied variables in $c_j$. We proceed by placing $V'_{j,q'}$ rotated by $180^\circ$, then $V_{j,q''}$ rotated by $180^\circ$ and afterwards $S_j$. The construction for clause $j$ is continued by placing $V_{j,q''}$ and $V_{j,q'}$ unrotated, which again ends in color $\ell$. Note that we have placed all $V_{j,q}$-pieces and the $S_j$-piece as well as a single $A$-piece corresponding to the occurrence of the satisfied variable. Finally, if they have not been placed before, we use all $A$-pieces corresponding to the unsatisfied variables as follows: If $u_{j,q'} = x_r$, then $A_{r,1}, A_{r,2}, \ldots, A_{r,m_r}$ are appended in the chained way described above (again ending in $\ell$), completed by the chain for variable $u_{j,q''}$. The construction ends with $\ell$ on the right-hand side and is continued with the next clause, resulting in all pieces being used and the puzzle being solved.

For the implication $\Leftarrow$, assume now that the puzzle is solved. We have already argued that for any $j$, there must be exactly one $q$ such that the pair $(V_{j,q}, V'_{j,q})$ is rotated according a “true” setting and two other $q$ such that the pair is in a “false” setting. We set the variables in $c_j$ accordingly. If a variable contains in another clause, we already know that the corresponding pieces must be rotated in a consistent way, which proves that we have a solution to Monotone 1-in-3-SAT.

q.e.d.

3 Boards with at Least Two Rows

We consider boards with two rows and arbitrarily many columns. Of course, the roles of columns and rows can be interchanged.

In [6] edge-matching is shown to be NP-complete for quadratic boards. More recently, [7] showed in a much more involved proof that the problem is even APX-complete, already for rectangular boards with only two rows. We focus on Demaine’s and Demaine’s technique from [6], which does not use rotations, and show that it can be strengthened to also include boards with row-count two – by extension edge-matching with any width-height ratio is NP-complete (with the exception of single-row puzzles), since it is trivial to force a board to contain more rows by adding uniformly colored pieces. However, as stated in Section 2 edge-matching without rotation is efficiently solvable when the row-count is 1.

Theorem 3.1 Two-row edge matching puzzles with swaps and free or monochromatic border is NP-complete.

Proof The transformation is from 3-partition in which the task is to partition a set of $3m$ positive integers into $m$ sets, each consisting of 3 integers such that the integers in each set
sum to the same value $S$. This problem can be visualized as the task of, given a collection of bars of varying length, placing the bars in rows, such that each row has 3 bars and all rows have the width $S$. Importantly 3-partition is also NP-complete when all the integers are limited to values in the range $(S/4, S/2)$, meaning that any row with width $S$ must contain exactly 3 bars – this is the version of the problem used in this proof. The problem remains NP-complete when $S$ and $m$ are polynomially related.

Converting this problem into an edge-matching puzzle with height 2 proceeds as follows: A section of pieces (a ‘bar’) is defined for each of the $3m$ integers. The internal left-right edge-color (shown as ‘x’ in the figure) is unique for each bar, and every bar starts and ends on the specific color ‘$’.

A board with height $N = 2$ and width $M = mS + m – 1$ is constructed, where the upper row is forced to have a particular layout as illustrated in the figure by using unique colors for every edge pair, and the lower row is separated into areas of length $S$, each of which can contain 3 bars. All separators will fit any bar left and right (color ‘$’), and all bars will fit in any position below the fixed upper row (color ‘%’). If the 3-partition has a solution then it will be possible to place the $m$ bars into the board giving a solution to the edge matching puzzle.

If the edge matching puzzle can be solved it will be because all sections can be placed into the forced layout of the grid, meaning that there is a solution to the corresponding 3-partition problem.

q.e.d.

4 Dependence on the number of colors.

In the proof of Theorem 2.2 (and other hardness results concerning puzzles), the number of colors used is linear in the the size of the reference problem. It is natural to ask whether the problem remains hard, when the number of colors is limited to a constant. We now show that the problem of Theorem 2.2 becomes efficiently solvable if the number of colors is at most $c(\log(M))^{1/4}$, for $c$ constant.

\textbf{Theorem 4.1} If both swaps and rotations are allowed, single-row edge-matching with free border and two colors is solvable in time $O(M)$, where $M$ is the number of pieces. For $K$ colors the problem is solvable in time $O(M + 5^K)$.

The last part of the theorem leads to the following corollary.

\textbf{Corollary 4.2} If both swaps and rotations are allowed, the single-row edge-matching problem with free border is fixed-parameter tractable.

\textbf{Proof} (Of the theorem) We start by exemplifying the proof method by looking at the case of one and two colors, black ($b$) and white ($w$).

If there is only one color, the puzzle is solvable, by placing the pieces in a row.
With exactly two colors, the pieces can be classified as either *constants* or *switches*. Constants are pieces that have the same color on the left and right edge, regardless of the rotation. Switches can be rotated such that there are different colors on the left and right edge.

\[ \begin{array}{c|c}
\text{Constants} & \text{Switches} \\
\hline
\begin{array}{c}
\begin{tikzpicture}
\draw[fill=black] (0,0) -- (1,0) -- (1,1) -- (0,1) -- cycle;
\end{tikzpicture}
\end{array} & \\
\begin{array}{c}
\begin{tikzpicture}
\draw[fill=black] (0,0) -- (1,0) -- (1,1) -- (0,1) -- cycle;
\draw[fill=white] (0,0) -- (1,0) -- (1,1) -- (0,1) -- cycle;
\end{tikzpicture}
\end{array} & \\
\begin{array}{c}
\begin{tikzpicture}
\draw[fill=black] (0,0) -- (1,0) -- (1,1) -- (0,1) -- cycle;
\draw[fill=white] (0,0) -- (1,0) -- (1,1) -- (0,1) -- cycle;
\end{tikzpicture}
\end{array} & \\
\end{array} \]

If the puzzle consists of constants only it can be solved if and only if it does not contain both a completely black and a completely white piece.

Otherwise there is both a monochromatic black piece and a white one. We claim that the puzzle has a solution if and only if there is at least one switch. Consider the multi-graph modeling as defined in Section 2. Recall that an Euler path corresponds to a solution when swaps and flips are allowed, but not general rotations. We now show, how this method can be used also in the case of few colors when also rotations are allowed. For two colors the graph has two nodes \( w \) and \( b \), one for each color. Consider the constant pieces first. The monochromatic ones give rise to loops at the node corresponding to their color. The non-monochromatic pieces give rise to a loop on one of the nodes. Which node this is depends on the rotation. In absence of a switch piece there is no edge between the nodes, hence no Euler path and no solution.

If the puzzle contains at least one switch, then this can be rotated in such a way that it forms an edge \( \{b, w\} \) between the two nodes. Regardless how the other pieces are rotated, the resulting graph has an Euler path. One starts at, say, \( w \) and traverses all loops at that node, then one uses one edge \( \{w, b\} \) to get to node \( b \). Then one traverses all loops at \( b \). Finally one uses the remaining \( \{w, b\}\)-edges (if any) exactly once. The Euler path corresponds to a solution as described above.

Consider the case of a puzzle with \( M \) pieces and \( K \) colors. Let \( X = abcd \) be a *color scheme*, i.e., a counter-clockwise order of four (not necessarily distinct) colors. We assume that the cyclic order of the color scheme is such that it is lexicographically minimal. This is to avoid treating an \( abcd \) and \( bcda \) as different color schemes.

Every arrangement of the \( M \) pieces gives rise to a multi-graph. The graph has \( K \) nodes and one edge for every piece. A piece with color scheme \( abcd \) will correspond to an edge \( \{a, c\} \) if \( a \) and \( c \) are the colors on the vertical edges. Otherwise the piece gives rise to an edge \( \{b, d\} \). Let \( m_{a,b} \) denote the number of edges between nodes \( a \) and \( b \). Loops \( (a = b) \) are allowed, however they do not have any influence in the solvability of the puzzle, as discussed in the example for two colors.

Given an arrangement of the \( M \) pieces, let \( n_X \) be the number of pieces having color scheme \( X = abcd \). Let \( n_{X,ac} \) denote the number of pieces with color scheme \( X \) which are rotated to become an \( \{a, c\}\)-edge. Similar \( n_{X,bd} \). We make the convention that \( n_{X,ac} = 0 \) if \( X \) does not contain the colors \( a \) and \( c \) on opposing edges. Then

\[
n_X = n_{X,ac} + n_{X,bd} \tag{1}\]

and

\[
m_{ac} = \sum_X n_{X,ac} \tag{2}\]

Furthermore

\[
\deg(a) = \sum_{b: b > a} m_{ab} + \sum_{c: c < a} m_{ca} + m_{aa}
\]
For the graph to have an Euler cycle the $m_{xy}$ have to be chosen such that they obey the constraints (1) and (2) and all degrees are even. For the graph to have an Euler path the $m_{xy}$ have to be chosen such that they obey the constraints (1) and (2) and exactly two nodes have odd degrees.

As we are only interested in connectivity and parity of degrees, we replace the $n_X$ as follows (that is we remove pieces with color scheme $X$):

- If $n_X \leq 4$ it is unchanged.
- If $n_X > 4$ and even then set $n_X = 4$.
- If $n_X > 4$ and odd then set $n_X = 3$.

We claim that this does not change the fact whether the puzzle is solvable or not. Assume $n_X > 4$ and is even. The number of pieces rotated such that they produce an $\{a, c\}$-edge is $n_{X,ac}$, number of pieces rotated such that they produce an $\{b, d\}$-edge is $n_{X,bd}$ and $n_X = n_{X,ac} + n_{X,bd}$.

Assume that the puzzle is solvable. Then there is a way to choose $n_{X,ab}$ and $n_{X,bd}$ such that the resulting graph has an Euler cycle or path, especially it is connected. We only discuss the case of an Euler cycle, the arguments for a path are identical. Then the degrees of all nodes, especially $a, b, c, d$ are even. As $n_X$ is reduced by an even number but remains at least 3, we can reduce both $n_{X,ac}$ and $n_{X,bd}$ (that is remove the corresponding edges) by an even number each, without being forced to make one of them zero. If some $n_{X,ac}$ is zero in the beginning, it will remain unchanged, therefore we only consider $n_{X,ac} > 0$ in the following. The resulting graph is connected and all degrees are even. Hence it still has an Euler cycle.

Now assume that the puzzle is not solvable. Then there is no way to choose the numbers $n_{X,ab}$ and $n_{X,bd}$ such that the resulting graph has an Euler cycle. That is the graph is un-connected or has nodes with odd degree for any such choice. If there is no way to choose $n_{X,ab}$ and $n_{X,bd}$ to make the graph connected, then reducing the number of pieces maintains that property.

Assume therefore that for some choices of the numbers $n_{X,ab}$ and $n_{X,bd}$ the graph is connected, but always has nodes with odd degree.

Consider a given partition of the $n_X = n_{X,ac} + n_{X,bd}$. If $n_X$ is odd then one of $n_{X,ac}$ and $n_{X,bd}$ is odd the other is even. Replacing $n_X$ by 3 (and changing $n_{X,ac}$ and $n_{X,bd}$ to satisfy (1)) maintains this property. Thus it is not possible to introduce a new distribution of parities of the nodes (for example one with only even node degrees).

If $n_X$ is even then $n_{X,ac}$ and $n_{X,bd}$ are either both even of both odd. Replacing $n_X$ by 4 and changing $n_{X,ac}$ and $n_{X,bd}$ to satisfy (1)) maintains this property. If $n_{X,ac}$ and $n_{X,bd}$ were both even and positive, we set both to 2. If both were odd and positive, we set them to 3 and 1. Again it is not possible to introduce a new distribution of parities of the nodes (for example one with only even node degrees).

Hence reducing the $n_X$ as described still allows the graph to be connected and does not allow the graph to have only vertices of even degree.

Now, check all possibilities to partition the reduced $n_X$ into $n_{X,ac}, n_{X,bd}$ and compute the $m_{xy}$. For each partition check whether it leads to a degree sequence ensuring an Euler cycle.

Each $n_X$ can be partitioned into $n_{X,ac}$ and $n_{X,bd}$ in 4, or 5 ways, depending on it being 3 or 4; these are the possibilities for the latter case (0, 4), (1, 3), (2, 2), (3, 1), and (4, 0).

Hence the number of possibilities to try is at most $5^\ell$, where $\ell$ is the number of color schemes. The quantity $\ell$ can be upper-bounded by $K^4$, where $K$ is the number of colors. For each partition it has to be checked, whether the resulting degree sequence is Eulerian. Computing the $n_X$ is linear in the number of pieces.
Hence the total time for deciding a puzzle with $M$ pieces and $K$ colors is

$$O(M + 5^K)$$

which is linear for constant $K$, and polynomial for $K = c(\log(M))^{1/4}$, for $c$ constant. q.e.d.

The last result shows that the number of colors seems to play an essential role for complexity of puzzle problems. The following result supports this by showing that also puzzles with in-place rotations can be efficiently solved when the number of colors and rows is constant. As mentioned in section the unrestricted problem is NP-complete, see [5]. This resembles fixed-parameter tractability, but requires two parameters to be constant.

**Theorem 4.3** Puzzles with $N$ rows, $M$ columns, and $K$ colors and in-place rotations are solvable in polynomial time for constant $N$ and $K$.

**Proof** Consider a column in the puzzle. For each of the $N$ pieces in the column, there are 4 possible rotations, giving $4^N$ configurations for the column. An $N$-color pattern is a sequence $[c_1, \ldots, c_N]$ of $N$ colors. Given $K$ colors there are at most $K^N$ $N$-color patterns, which we number $1, \ldots, K^N$. For each column there are at most $4^N$ possible color patterns at the left and at the right edge of the column. Consider column $j$ and construct a bipartite graph $G_j$ as follows. The vertex set is $V = V_{j, \text{left}} \cup V_{j, \text{right}}$ where $V_{j, \text{left}}$ and $V_{j, \text{right}}$ each contain $K^N$ vertices, one for each color pattern. Let $V_{j, \text{left}} = \{v_{j,1}, \ldots, v_{j,K^N}\}$ and $V_{j, \text{right}} = \{w_{j,1}, \ldots, w_{j,K^N}\}$. We then try all $4^N$ configurations of the pieces in column $j$. If a configuration gives color pattern $a$ at left and pattern $b$ at right and the colors at the $N - 1$ horizontal edges match then we add a directed edge $(v_{j,a}, w_{j,b})$ to the graph. Let $G_j$ denote the resulting bipartite graph. The construction of $G_j$ can be performed in time $O((4^N + (K^N))$ given that the index of the color pattern can be computed in time $\Theta(K + N)$. Graph $G_j$ has $2K^N$ nodes and at most $4^N$ edges.

Next we connect the consecutive graphs $G_j, G_{j+1}, j = 1, \ldots, M - 1$ by adding the directed edges $(w_{j,a}, v_{j+1,a})$, $a = 1, \ldots, K^N$. Finally we add a source node $s$ and connect it to all left nodes of $G_1$ by directed edges $(s, v_{1,a})$. We also add a sink node $t$ and connect it to all right nodes of $G_M$ by directed edges $(w_{M,a}, t)$. Let $G = (V, E)$ be the resulting graph. Note that $|V| = 2MK^N + 2$ and $|E| \leq 4M^N + (M - 1)K^N + 2K^N$ and that $G$ can be constructed in time $O(M(K^M + 4^N))$.

Now, compute whether there is directed path from $s$ to $t$ in $G$. We claim, that if so, there is solution for the puzzle otherwise there is not. Assume there is a directed path from $s$ to $t$. Let $(s, v_{1,a_1})$ be the first edge of the path and $(w_{n,b_n}, t)$ be the last one. In between are alternating edges inside the $G_j$ and in between the $G_j$. This means that for every $j = 1, \ldots, n-1$ the path contains an edge $(w_{j,c_j}, v_{j+1,c_j})$ for some color pattern $c_j$. Hence there are matching color patterns between all rows. Within every $G_j$, the path uses an edge $(v_{j,a_j}, w_{j,b_j})$. That is, the color patterns at the left and at the right of column $j$ can be realized simultaneously by an appropriate configuration of the pieces in that column. Hence the puzzle has a solution. As finding the shortest path can be done in time linear in the graph size, the total running time is polynomial in for constant $N$ and $K$.

Conversely, any solution of the puzzle gives rise to at least one path from $s$ to $t$ by the definition of the graph.

q.e.d.

5 Summary and Conclusions

In this paper, we have focused on several problem aspects of edge-color puzzles whose impact on complexity was unknown or only marginally treated before, in particular rotating pieces,
in-place pieces, border rules, number of colors and shapes of pieces. With regard to single-row boards, it has been shown that introducing rotations makes the otherwise easy problem NP-hard. In basically all hardness results, a large number of colors is used, usually linear in the number of pieces. For two cases we have shown that the problem is fixed-parameter tractable, i.e., it becomes efficiently solvable when the number of colors is constant or logarithmic in the number of pieces.

The table below summarizes the known hardness results for boards with one row compared to boards with at least two rows.

| Rows: | 1 | ≥ 2 |
|-------|---|-----|
| Swap  | P | NP-complete |
| Rotation | P | NP-complete |
| Both  | NP-complete, fixed-parameter tractable | NP-complete |

This paper has raised further questions concerning the hardness of edge-matching problems. For example, are there more hard problems which are fixed-parameter tractable? Furthermore, it is unknown whether the single-row case with rotations is hard to approximate.

**Acknowledgement**

The second author gratefully acknowledges support by the DTU’s Corrit travel grant.

**References**

[1] M. Ebbesen, P. Fischer, C. Witt, Edge-matching problems with rotations, in: Fundamentals of Computation Theory - 18th International Symposium, FCT 2011, Oslo, Norway, August 22-25, 2011. Proceedings, 2011, pp. 114–125.

[2] P. Pevzner, H. Tang, M. Waterman, An Eulerian path approach to DNA fragment assembly, Proceedings of the National Academy of Sciences of the United States of America 98, no. 17 (2001) 9748–9753.

[3] O. R. Lesniak, Linda; Oellermann, An Eulerian exposition, Journal of Graph Theory 10 (1986) 277–297.

[4] M. Garey, D. S. Johnson, Computers and Intractability: A Guide to the Theory of NP-Completeness, W.H. Freeman and Company, 1979.

[5] E. Goles, I. Rapaport, Complexity of tile rotation problems, Theoretical Computer Science 188 (1997) 129–159.

[6] E. D. Demaine, M. L. Demaine, Jigsaw puzzles, edge matching, and polyomino packing: Connections and complexity, Graphs and Combinatorics 23 (2007) 195–208.

[7] A. Antoniadis, A. Lingas, Approximability of edge matching puzzles, in: Proc. of SOFSEM 2010, Vol. 5901 of LNCS, Springer, 2010, pp. 153–164.

[8] M. Ebbesen, Analysis of restricted edge-matching problems, Master’s thesis, Technical University of Denmark, master thesis, Technical University of Denmark, reference no. IMM-M.Sc.-2011-08, http://etd.dtu.dk/thesis/275156/ (2011).
[9] T. J. Schaefer, The complexity of satisfiability problems, in: Proceedings of the tenth annual ACM symposium on Theory of computing, STOC ’78, 1978, pp. 216–226.