Balayage of measures on a locally compact space

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Dedicated to Professor Bent Fuglede on the occasion of his 95th birthday

Abstract. We develop a theory of inner balayage of a positive Radon measure $\mu$ of finite energy on a locally compact space $X$ to arbitrary $A \subset X$, generalizing Cartan’s theory of Newtonian inner balayage on $\mathbb{R}^n$, $n \geq 3$, to a suitable function kernel on $X$. As an application of the theory thereby established, we show that if the space $X$ is perfectly normal and of class $K_\sigma$, then a recent result by B. Fuglede (Anal. Math., 2016) on outer balayage of $\mu$ to quasiclosed $A$ remains valid for arbitrary Borel $A$. We give in particular various alternative definitions of inner (outer) balayage, provide a formula for evaluation of its total mass, and prove convergence theorems for inner (outer) swept measures and their potentials. The results obtained do hold (and are new in part) for most classical kernels on $\mathbb{R}^n$, $n \geq 2$, which is important in applications.

1. Introduction

The paper deals with balayage of a positive Radon measure $\mu$ of finite energy on a locally compact (Hausdorff) space $X$ to a set $A \subset X$ in the setting of potentials with respect to a symmetric, lower semicontinuous (l.s.c.) kernel $\kappa : X \times X \to [0, \infty]$. Throughout the present section as well as Sects. 3.2, the kernel $\kappa$ is assumed to satisfy the energy, consistency, and domination principles [15, 24].

It has recently been shown by Bent Fuglede [18, Theorem 4.12] that if a set $A$ is quasiclosed (that is, if it can be approximated in outer capacity by closed sets), then there is a unique positive Radon measure $\mu^*A$ of finite energy that is concentrated on $A$ (that is, $\mu^*A \in \mathcal{E}_A^+$) and has the property

$$\kappa \mu^*A = \kappa \mu \quad \text{q.e. on } A,$$

where $\kappa \nu(\cdot) := \int \kappa(\cdot, y) \, d\nu(y)$ is the potential of a Radon measure $\nu$ on $X$, and q.e. (quasi-everywhere) means that the equality holds everywhere on $A$ except for a subset of outer capacity zero. Such a $\mu^*A$ is said to be the outer balayage of $\mu$ onto $A$.

We shall show below that if the space $X$ is perfectly normal and of class $K_\sigma$, then the quoted Fuglede’s result remains valid for arbitrary Borel $A$ (Sect. 6). The outer balayage $\mu^*A$ now, however, is no longer concentrated on the set $A$ itself (as it was for $A$ quasiclosed), but on the closure of $A$ in $X$. It is still characterized uniquely by the above display, but now within $\mathcal{E}_A^+$, the closure of $\mathcal{E}_A^+$ in the topology determined by the energy norm $\|\nu\| := \sqrt{\int \kappa \nu \, d\nu}$. The outer balayage $\mu^*A$ is actually the (unique) limit of the net $(\mu^*K)$ in both the vague and the energy norm topologies when $K$ increases along the upper directed family of all compact subsets of $A$. Furthermore, it can alternatively be determined as a solution (which exists and is unique) to either of the following two extremal problems.

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Problem 1.1. In the class \( \Lambda^{A}_{\kappa,\mu} \) of positive Radon measures \( \nu \) on \( X \) of finite energy and such that \( \kappa \nu \geq \kappa \mu \) q.e. on \( A \), find \( \lambda^{*}_{A,\mu} \in \Lambda^{A}_{\kappa,\mu} \) of minimal potential:

\[
\kappa \lambda^{*}_{A,\mu} = \min_{\nu \in \Lambda^{A}_{\kappa,\mu}} \kappa \nu \quad \text{everywhere on } X.
\]

Problem 1.2. Find \( \mu_{\mathcal{E}} \in \mathcal{E}'_{A} \) such that

\[
\| \mu - \mu_{\mathcal{E}} \| = \min_{\nu \in \mathcal{E}_{A}} \| \mu - \nu \| = \inf_{\nu \in \mathcal{E}_{A}} \| \mu - \nu \|.
\]

If moreover \( \mu \) is bounded (i.e. \( \mu(X) < \infty \)) while \( \kappa \) satisfies Frostman’s maximum principle, then the outer balayage \( \mu^{*}_{A} \) can also be found as the (unique) solution to the problem obtained from Problem 1.1 resp. Problem 1.2 by requiring additionally that all the \( \nu \) involved have the total mass \( \nu(X) \leq q, \mu(X) \leq q < \infty \) being given.

These results are obtained by a direct application of the theory of inner balayage, developed in Sects. 4, 8 below and generalizing H. Cartan’s theory [8] of inner Newtonian balayage on \( \mathbb{R}^{n}, n \geq 3 \), to a locally compact (l.c.) space \( X \) endowed with a kernel described above. (In this part of the study, both \( X \) and \( A \) are arbitrary.)

We give in particular various alternative definitions of inner (outer) balayage, provide a formula for evaluation of its total mass, and prove convergence theorems for inner (outer) swept measures and their potentials (see Sects. 4, 7, 9).

The results obtained do hold (and are new in part) for the Newtonian kernel \( |x - y|^{2-n} \) and, more generally, the \( \alpha \)-Riesz kernel \( |x - y|^{\alpha-n} \) of order \( 0 < \alpha < 2 \) on \( \mathbb{R}^{n}, n \geq 3 \), as well as for the associated \( \alpha \)-Green kernel \( g_{\alpha}^{n} \) on an open set \( D \subset \mathbb{R}^{n} \). The same is true for the (2-)Green kernel on a planar Greenian set. This suggests that the present work can be useful, for instance, in connection with optimal point configurations and minimum energy problems (see e.g. [25] [10] [26] [27] [6] [11] [20] [12] [29] for some applications of balayage, the papers [25]–[27] dealing with minimum energy problems with respect to a general function kernel \( \kappa \) on a l.c. space \( X \).

To begin with, we review some basic facts of the theory of potentials on a l.c. space \( X \), using the fundamental study by Fuglede [15] as a guide.

2. Basic facts of potential theory on a locally compact space

Denote by \( \mathcal{M} = \mathcal{M}(X) \) the linear space of all (real-valued scalar Radon) measures \( \mu \) on a l.c. space \( X \), equipped with the vague (= weak) topology of pointwise convergence on the class \( C_{0} = C_{0}(X) \) of all continuous functions \( f : X \rightarrow \mathbb{R} \) with compact support, and by \( \mathcal{M}^{+} = \mathcal{M}^{+}(X) \) the cone of all positive \( \mu \in \mathcal{M} \).

Lemma 2.1 (see e.g. [15] Section 1.1]). For any l.s.c. function \( \psi : X \rightarrow [0, \infty] \), the mapping \( \mu \mapsto \int \psi \, d\mu \) is vaguely l.s.c. on \( \mathcal{M}^{+} \) (the integral here being understood as upper integral, see e.g. [5] Section IV.1.1)).

Given a (positive, symmetric, l.s.c.) kernel \( \kappa \) on \( X \) and given (signed) measures \( \mu, \nu \in \mathcal{M} \), define the potential and the mutual energy by

\[
\kappa \mu(x) := \int \kappa(x,y) \, d\mu(y), \quad x \in X;
\]

\[
\kappa(\mu, \nu) := \int \kappa(x,y) \, d(\mu \otimes \nu)(x, y),
\]

\[1^\text{For the theory of outer, resp. inner, Riesz balayage, we refer the reader to [2], resp. [28] [30], the investigation in [2] being carried out in the general framework of balayage spaces.}\]

\[2^\text{Regarding the logarithmic kernel on } \mathbb{R}^{2}, \text{ see Example 2.4 below and footnote 7 attached to it.}\]
respectively, provided the right-hand side is well defined as a finite number or \( \pm \infty \) (for more details, see e.g. [15, Section 2.1]). For \( \mu = \nu, \kappa(\mu, \nu) \) defines the energy \( \kappa(\mu, \mu) \) of \( \mu \). In particular, if the measures are positive, then \( \kappa(\mu, x) \), resp. \( \kappa(\mu, \nu) \), is well defined and represents a positive l.s.c. function of \( (x, \mu) \in X \times \mathcal{M}^+ \), resp. of \( (\mu, \nu) \in \mathcal{M}^+ \times \mathcal{M}^+ \) (the principle of descent [15, Lemma 2.2.1], cf. Lemma 2.1).

Throughout the remainder of the present paper, we shall tacitly assume a kernel in question to satisfy the energy and consistency principles. Recall that a kernel \( \kappa \) is said to satisfy the energy principle (or to be strictly positive definite) if for any (signed) \( \mu \in \mathcal{M} \), \( \kappa(\mu, \mu) \) is \( \geq 0 \) whenever defined, and it is zero only for \( \mu = 0 \). All the \( \mu \in \mathcal{M} \) with finite energy then form a pre-Hilbert space \( \mathcal{E} = \mathcal{E}(X) \) with the inner product \( (\mu, \nu) := \kappa(\mu, \nu) \) and the norm \( \| \mu \| := \sqrt{\kappa(\mu, \mu)} \), see [15, Section 3.1]. The (Hausdorff) topology on \( \mathcal{E} \) defined by the norm \( \| \cdot \| \) is said to be strong.

We shall use the following lemma from the geometry of the pre-Hilbert space \( \mathcal{E} \).

**Lemma 2.2** (see [15, Lemma 4.1.1]). Suppose that \( \Gamma \) is a convex subset of \( \mathcal{E} \), and there exists \( \lambda \in \Gamma \) of minimal norm:

\[
\| \lambda \| = \min_{\nu \in \Gamma} \| \nu \|. 
\]

For any \( \nu \in \Gamma \), then

\[
\| \nu - \lambda \|^2 \leq \| \nu \|^2 - \| \lambda \|^2.
\]

A (strictly positive definite) kernel \( \kappa \) is called consistent [15, Section 3.3] if every strong Cauchy net in the cone \( \mathcal{E}^+ = \mathcal{E}^+(X) := \mathcal{E} \cap \mathcal{M}^+ \) converges strongly to any of its vague limit points. The cone \( \mathcal{E}^+ \) then becomes strongly complete, a strongly bounded part of \( \mathcal{E}^+ \) being vaguely compact by [15, Lemma 2.5.1]. As the strong limit of a strong Cauchy net \( (\mu_k) \subset \mathcal{E}^+ \) is unique, all the vague limit points of such a \( (\mu_k) \) must be equal. The vague topology on \( \mathcal{M} \) being Hausdorff, we thus conclude by applying [3, Section I.9.1, Corollary] that any strong Cauchy net in \( \mathcal{E}^+ \) converges to some (unique) limit points both strongly and vaguely. In Fuglede’s terminology [15, Section 3.3], a strictly positive definite, consistent kernel is, therefore, perfect.

Given \( A \subset X \), denote by \( \mathcal{M}^+_A \) the class of all \( \mu \in \mathcal{M}^+ \) concentrated on \( A \), which means that \( A^c := X \setminus A \) is locally \( \mu \)-negligible, or equivalently, that \( A \) is \( \mu \)-measurable and \( \mu = \mu|_A, \mu|_A := 1_A \cdot \mu \) being the trace of \( \mu \) to \( A \) [14, Section IV.14.7]. (Note that for \( \mu \in \mathcal{M}^+_A \) is \( \mu(X) = \mu_+(A), \mu_-(A) \) and \( \mu^+(A) \) denoting the inner and outer measure of \( A \), respectively. If moreover \( A \) is closed, or if \( A^c \) is contained in a countable union of sets \( Q_k \) with \( \mu^+(Q_k) < \infty \), then for any \( \mu \in \mathcal{M}^+_A \), \( A^c \) is \( \mu \)-negligible, i.e. \( \mu^+(A^c) = 0 \).

\[3\] If the energy principle holds, then a (signed) measure \( \mu \in \mathcal{M} \) has finite energy if and only if so do both \( \mu^+, \mu^- \in \mathcal{M}^+ \), the positive and negative parts of \( \mu \) in the Hahn–Jordan decomposition.

\[4\] As the vague topology on \( \mathcal{M}^+ \) does not satisfy the first axiom of countability, the vague convergence cannot in general be described in terms of sequences. We follow Moore and Smith’s theory of convergence [22], based on the concept of nets. However, if the space \( X \) has a countable base of open sets, the use of nets in \( \mathcal{M}^+ \) may be avoided. Indeed, such an \( X \) is metrizable (with a metric \( g_X \)) and of class \( K_\sigma \) [4, Section IX.2, Corollary to Proposition 16], and hence it has a countable dense subset \( (x_k) \subset X \) [4, Section IX.2, Proposition 12]. Therefore,

\[
\mathcal{U}_k = \left\{ \nu \in \mathcal{M}^+: \int (1 - k g_X(x_k, x))^+ d|\nu - \nu_0|(x) < 1/k \right\}, \quad k \in \mathbb{N},
\]

form a countable base of vague neighborhoods of \( \nu_0 \in \mathcal{M}^+ \). Here \(|\mu| := \mu^+ + \mu^- \), \( \mu \in \mathcal{M} \).

\[5\] If the latter holds, \( A^c \) is said to be \( \mu \)-\( \sigma \)-finite [14, Section IV.7.3]. This occurs e.g. if the measure \( \mu \) is bounded, or if the space \( X \) is of class \( K_\sigma \).
In particular, if $A$ is closed, then $\mathfrak{M}^+_A$ consists of all $\mu \in \mathfrak{M}^+$ with $S(\mu) \subset A$, $S(\cdot)$ being the support of a measure.

Denoting now $\mathcal{E}^+_A := \mathfrak{M}^+_A \cap \mathcal{E}$, we define the inner capacity of $A$ by

$$\kappa_*(A) := \inf_{\mu \in \mathcal{E}^+_A : \mu(X) = 1} \kappa(\mu, \mu)^{-1}.$$  

(The infimum over the empty set is interpreted as $+\infty$.) Then \cite{15} p. 153, Eq. (2)]

\begin{equation}
\kappa_*(A) = \sup_{K \subset A} \kappa_*(K) \quad (K \subset A \text{ compact}).
\end{equation}

Also, by homogeneity reasons (cf. \cite{15} Lemma 2.3.1),

\begin{equation}
\kappa_*(A) = 0 \iff \mathcal{E}^+_A = \{\{0\} \iff \mathcal{E}^+_K = \{\{0\} \text{ for every compact } K \subset A.
\end{equation}

We are thus led to the following conclusion, to be often used in what follows.

**Lemma 2.3.** Given $\mu \in \mathcal{E}^+$ and a $\mu$-measurable set $A \subset X$, $A$ is locally $\mu$-negligible if $\kappa_*(A) = 0$, and it is $\mu$-negligible if moreover it is $\mu$-$\sigma$-finite.

Due to the perfectness of the kernel, for any $A$ with $\kappa_*(A) < \infty$ (thus in particular for any compact $A$), there is a unique $\gamma_A \in \mathcal{E}^+$, called the inner equilibrium measure for $A$, that minimizes the energy $\kappa(\nu, \nu)$ among all $\nu \in \mathcal{E}^+$ having the property $\kappa \nu \geq 1$ n.e. on $A$, where n.e. (nearly everywhere) means that the equality holds everywhere on $A$ except for a subset of inner capacity zero. Furthermore \cite{15} Theorem 4.1,

\begin{align*}
\kappa(\gamma_A, \gamma_A) &= \gamma_A(X) = \kappa_*(A), \\
\kappa \gamma_A &\geq 1 \text{ n.e. on } A, \\
\kappa \gamma_A &\leq 1 \text{ on } S(\gamma_A).
\end{align*}

Thus, if moreover Frostman’s maximum principle holds\(^6\) then also

\begin{equation}
\kappa \gamma_A = 1 \text{ n.e. on } A.
\end{equation}

Defining further the outer capacity of $A$ by

$$\kappa^*(A) := \inf D \sup \kappa_*(D),$$

where $D$ ranges over all open sets that contain $A$, we have $\kappa_*(A) \leq \kappa^*(A)$. A set $A$ is said to be capacitable if $\kappa_*(A) = \kappa^*(A)$, and then we shall simply write

$$\kappa(A) := \kappa_*(A) = \kappa^*(A);$$

this occurs e.g. whenever $A$ is open or compact. If $A$ is capacitable and $\kappa(A) < \infty$, then the inner equilibrium measure $\gamma_A$ serves simultaneously as the (unique) outer equilibrium measure $\gamma_A^*$, minimizing the energy $\kappa(\nu, \nu)$ among all $\nu \in \mathcal{E}^+$ with the property $\kappa \nu \geq 1$ q.e. on $A$ (see \cite{15} Theorem 4.3, Lemma 4.3.4).

Throughout Sects. \[11-15\] in addition to the (permanent) requirement of perfectness for the kernel $\kappa$, we shall tacitly assume that $\kappa$ satisfies the domination principle (= the second maximum principle), which means that for any $\mu, \nu \in \mathcal{E}^+$ such that $\kappa \nu \leq \kappa \nu$ $\mu$-a.e., the same inequality holds on all of $X$.

**Example 2.4.** The Newtonian kernel $|x - y|^{2-n}$ and, more generally, the $\alpha$-Riesz kernel $|x - y|^{\alpha-n}$ of order $\alpha \in (0, 2]$ on $\mathbb{R}^n$, $n \geq 3$, satisfy the energy and consistency principles as well as the first and second maximum principles \cite{21} Theorems 1.10, 1.15, 1.18, 1.27, 1.29]. So does the associated $\alpha$-Green kernel on an arbitrary open

\begin{footnote}
\(6\) A kernel $\kappa$ is said to satisfy Frostman’s maximum principle (= the first maximum principle) if for any $\mu \in \mathfrak{M}^+$ with $\kappa \mu \leq 1$ on $S(\mu)$, the same inequality holds on all of $X$. For the purposes of the present paper, it is enough, in fact, to suppose that this is true for any $\mu$ of finite energy.
\end{footnote}
subset of $\mathbb{R}^n$, $n \geq 3$ [19, Theorems 4.6, 4.9, 4.11]. The (2-)Green kernel on a planar Greenian set is likewise strictly positive definite [9, Section I.XIII.7] and consistent [13, Section I.V.10]). Finally, the logarithmic kernel $-\log |x-y|$ on a closed disc in $\mathbb{R}^2$ of radius $< 1$ is strictly positive definite and satisfies Frostman’s maximum principle [21, Theorems 1.6, 1.16], and hence it is perfect by [15, Theorem 3.4.2]. However, the domination principle then fails in general; it does hold only in a weaker sense where the measures $\mu, \nu$ involved in the above-quoted definition satisfy the additional requirement that $\nu(\mathbb{R}^2) \leq \mu(\mathbb{R}^2)$, cf. [23, Theorem 3.2].

3. Preliminaries

In all that follows, $\kappa$ denotes a positive, symmetric, l.s.c., perfect kernel on a l.c. space $X$. Given $\mu \in \mathcal{E}^+$ and $F \subset \mathcal{E}^+$, write

$$\rho(\mu, F) := \inf_{\nu \in F} ||\mu - \nu||.$$  

**Theorem 3.1.** Assume that $F$ is a strongly closed, convex subset of $\mathcal{E}^+$. For any $\mu \in \mathcal{E}^+$, there is a unique $\mu_F \in F$ having the property

$$||\mu - \mu_F|| = \min_{\nu \in F} ||\mu - \nu|| = \rho(\mu, F);$$  

it is called the orthogonal projection of $\mu$ in the pre-Hilbert space $\mathcal{E}$ onto $F$. If moreover $F$ is a cone in $\mathcal{E}^+$ that contains zero measure, then such a $\mu_F$ is characterized uniquely within $F$ by the two relations

$$\begin{align*}
(\mu - \mu_F, \nu) &\leq 0 \quad \text{for all } \nu \in F, \\
(\mu - \mu_F, \mu_F) &= 0.
\end{align*}$$  

**Proof.** We conclude this from [14] (Theorem 1.12.3 and Proposition 1.12.4(2)) by noting that $F$ is complete in the induced strong topology, $F$ being a strongly closed subset of the strongly complete cone $\mathcal{E}^+$. (The existence of $\mu_F$ is due to the consistency of the kernel, whereas the uniqueness is implied by the energy principle.) □

**Definition 3.2** (Fuglede [16]). A set $A \subset X$ is said to be quasiclosed if

$$\inf \left\{ c^*(A \triangle F) : F \text{ closed, } F \subset X \right\} = 0,$$

where $\triangle$ denotes the symmetric difference.

**Lemma 3.3.** For $A \subset X$ quasiclosed, the cone $\mathfrak{M}_A^+$ and its truncated subcone $\bar{\mathfrak{M}}_A^+ := \{ \nu \in \mathfrak{M}_A^+ : \nu_s(A) \leq 1 \}$ are both vaguely closed.

**Proof.** For $A$ quasiclosed, $\mathfrak{M}_A^+$ is vaguely closed according to [17, Corollary 6.2]. Given a vague limit point $\nu$ of $\mathfrak{M}_A^+$, choose a net $(\nu_k) \subset \mathfrak{M}_A^+$ that converges to $\nu$ vaguely. Then $\nu \in \mathfrak{M}_A^+$ (see above), and moreover

$$\nu_s(A) = \nu(X) \leq \liminf_k \nu_k(X) = \liminf_k (\nu_k)_s(A) \leq 1,$$

the former inequality being valid by Lemma 3.4 below. Thus indeed $\nu \in \mathfrak{M}_A^+$. □

**Lemma 3.4.** The mapping $\mu \mapsto \mu(X)$ is vaguely l.s.c. on $\mathfrak{M}^+$.

**Proof.** This is obvious from Lemma 2.1 with $\psi := 1_X$. □

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7Because of this obstacle, the theory of inner and outer balayage on a l.c. space $X$, developed in the present paper, does not cover the case of the logarithmic kernel on $\mathbb{R}^2$.

8For the notation and terminology used here, see Sects. [1] [2].
Lemma 3.5. If $A \subset X$ is quasiclosed, then the cone $E_A^+$ is strongly closed; hence

$$E_A^+ = E_A',$$

$E_A'$ denoting the strong closure of $E_A^+$. The same holds if $E_A^+$ is replaced by its truncated subcone $\hat{E}_A^+ := \mathfrak{N}_A^+ \cap \mathcal{E}$ while $E_A'$ by $\hat{E}_A'$, the strong closure of $\hat{E}_A^+$.

Proof. Given $\nu \in E_A'$, choose a net $(\nu_k) \subset E_A^+$ that converges to $\nu$ strongly. The kernel being perfect, $\nu_k \to \nu$ also vaguely (Sect. 2). As the cone $\mathfrak{N}_A^+$ is vaguely closed (Lemma 3.3), we have $\nu \in \mathfrak{N}_A^+$, and hence actually $\nu \in E_A^+$. Since the truncated cone $\hat{\mathfrak{N}}_A^+$ is likewise vaguely closed (Lemma 3.3), the rest of the proof runs as before. □

4. INNER BALAYAGE

From now on, a (perfect) kernel $\kappa$ on a l.c. space $X$ is supposed to satisfy the domination principle.

Fix arbitrary $\mu \in E^+$ and $A \subset X$.

Definition 4.1. $\mu^A = \mu^{A, \ast} \in E^+$ is said to be an inner balayage of $\mu$ to $A$ if

$$\kappa \mu^A = \inf_{\nu \in \Lambda_{A, \mu}} \kappa \nu$$
on $X$,

where

$$\Lambda_{A, \mu} := \{ \nu \in E^+ : \kappa \nu \geq \kappa \mu \text{ n.e. on } A \}.$$n

As seen from [8, Section 19, Theorem 1], this definition is in agreement with Cartan’s (classical) concept of inner Newtonian balayage on $\mathbb{R}^n$, $n \geq 3$ (cf. also [28, Theorem 4.3] providing a characteristic property of inner Riesz balayage).

Lemma 4.2. The inner balayage $\mu^A$ is unique (if it exists).

Proof. If both $\theta_1, \theta_2 \in E^+$ satisfy (4.1), then $\kappa \theta_1$ and $\kappa \theta_2$ take finite equal values q.e. on $X$, cf. [15, Corollary to Lemma 3.2.3]. Applying [15, Lemma 3.2.1(a)] therefore gives $\|\theta_1 - \theta_2\| = 0$, and hence $\theta_1 = \theta_2$, by the energy principle. □

4.1. Existence of the inner balayage. Alternative definitions. Given $\mu \in E^+$ and $A \subset X$, denote by $\mu_{E_A'}$ the orthogonal projection of $\mu$ in the pre-Hilbert space $\mathcal{E}$ onto $E_A'$, the strong closure of $E_A^+$. Such a projection $\mu_{E_A'}^\perp$ exists and is unique by Theorem 3.1 applied to the (strongly closed, convex) set $E_A'$.

Theorem 4.3. Given $\mu \in E^+$ and $A \subset X$, the inner balayage $\mu^A$, introduced by Definition 4.1, does exist. Actually,

$$\mu^A = \mu_{E_A'},$$

and hence the inner balayage $\mu^A$ can alternatively be determined by the two formulae

$$\mu^A \in E_A', \quad \|\mu - \mu^A\| = \min_{\nu \in E_A'^+} \|\mu - \nu\| = \inf_{\nu \in E_A'^+} \|\mu - \nu\|.$$n

Furthermore, $\mu^A$ has the properties

$$\kappa \mu^A = \kappa \mu \text{ n.e. on } A,$$

$$\kappa \mu^A = \kappa \mu \text{ a.e.,}$$

$$\kappa \mu^A \leq \kappa \mu \text{ on } X,$$n

and it can equivalently be defined as the only measure in $E_A'$ satisfying (1.3).

9That the inner balayage $\mu^A$ is uniquely determined by (13) within $E_A'$ (A being arbitrary) seems to be unknown before even for the (classical) Newtonian, $\alpha$-Riesz, and $\alpha$-Green kernels.
For a proof of Theorem 4.3 see Sect. 5 below. If $A = K$ is compact, the proof is based on the classical Gauss variational method [7, 8], generalized to a perfect kernel satisfying the domination principle. In order to extend the relation

$$\mu^K = \mu^{E_K^+} = \mu^{E_K^-},$$

thereby obtained to $A$ arbitrary, we shall establish convergence assertions for the nets $(\mu^{E_K^+})$ and $(\kappa \mu^{E_K^+})$ when $K$ increases along the upper directed family of all compact subsets of $A$ (see Eqs. (5.4) and (5.5) in Theorem 5.1).

**Corollary 4.4.** Variational problem (4.1) on minimizing the potential $\kappa \nu$ among the measures $\nu \in \Lambda_{A,\mu}$ has the unique solution $\mu^A$, determined by Theorem 4.3.

**Proof.** By Theorem 4.3 and Lemma 4.2 the infimum in (4.1) is achieved at the unique an actual minimum. $\square$

The inner balayage $\mu^A$ is not concentrated on the set $A$ itself, but on its closure, and this occurs even in the case of the Newtonian kernel on $\mathbb{R}^n$, $n \geq 3$, and an open ball $\mathbb{B}$. The inclusion $\mu^A \in E_A^+$, however, does already hold if $A$ is quasiclosed.

**Corollary 4.5.** If $A$ is quasiclosed (more generally, if $E_A^+$ is strongly closed), the inner balayage $\mu^A$ can be found as the orthogonal projection of $\mu$ onto $E_A^+$, i.e.

$$\mu^A = \mu^{E_A^+},$$

or alternatively, as the only measure in $E_A^+$ having property (4.3).

**Proof.** For quasiclosed $A$, we have $E_A^+ = E_A'$ (Lemma 3.5), hence $\mu^{E_A^+} = \mu^{E_A'}$, which substituted into Theorem 4.3 yields the corollary. $\square$

Assume now that $\mu \in E^+$ is bounded. Given $A \subset X$, denote by $\hat{E}_{A,\mu}$, resp. $\hat{\Lambda}_{A,\mu}$, the (convex) class of all $\nu \in E^+_A$, resp. $\nu \in \Lambda_{A,\mu}$, having the property

$$\nu(X) \leq \mu(X),$$

and by $\hat{E}_{A,\mu}'$ the strong closure of $\hat{E}_{A,\mu}$.

**Theorem 4.6.** Suppose that $\mu \in E^+$ is bounded while $\kappa$ satisfies Frostman’s maximum principle. Then Definition 4.1 and that obtained from it by replacing $\Lambda_{A,\mu}$ by $\hat{\Lambda}_{A,\mu}$ lead to the same concept of inner balayage. Furthermore, Theorem 4.3 and Corollaries 4.4 and 4.5 remain valid if $\Lambda_{A,\mu}$, $E_A^+$, and $E_A'$ are replaced throughout by $\hat{\Lambda}_{A,\mu}$, $\hat{E}_{A,\mu}$, and $\hat{E}_{A,\mu}'$, respectively. Thus, under the stated assumptions, the inner balayage $\mu^A$ can equivalently be defined by either of the two formulae

$$\kappa \mu^A = \inf_{\nu \in \hat{\Lambda}_{A,\mu}} \kappa \nu \text{ on } X, \quad (4.7)$$

$$\mu^A = \mu^{\hat{E}_{A,\mu}'}, \quad (4.8)$$

the infimum in (4.7) being an actual minimum. If $A$ is quasiclosed, then also

$$\mu^A = \mu^{\hat{E}_{A,\mu}}. \quad (4.9)$$

\footnote{It may even happen that $S(\mu^A) \cap A = \emptyset$. For instance, for any $\nu \in \mathfrak{M}^+(\mathbb{R}^n)$, $n \geq 3$, and any open set $D \subset \mathbb{R}^n$ with connected complement $D^c$ of nonzero Newtonian capacity, we have $S(\nu^D) = \partial \mathfrak{S}, D$, and hence $S(\nu^D) \cap D = \emptyset$, $\nu^D$ denoting the inner Newtonian balayage of $\nu$ to $D$. This follows e.g. from [30, Theorems 4.1, 5.1].}

\footnote{The orthogonal projections involved in (4.8) and (4.9) do exist (Theorem 3.1 and Lemma 3.3).}
Remark 4.7. Theorem 4.6 still holds if (4.6) is weakened to $\nu(X) \leq q \mu(X)$, where $q \in [1, \infty)$ (cf. Sect. 8.1).

4.2. Convergence theorems for inner balayage. Theorem 4.8 below shows that both $\mu^A$ and $\kappa \mu^A$ are continuous under the exhaustion of $A$ by $K$ compact, which justifies the term ‘inner’ balayage.

**Theorem 4.8.** For $\mu \in \mathcal{E}^+$ and $A \subset X$ arbitrary,

$$
\mu^K \to \mu^A \text{ strongly and vaguely in } \mathcal{E}^+ \text{ as } K \uparrow A,
$$

where the abbreviation $K \uparrow A$ means that $K$ increases along the upper directed family $\mathcal{C} = \mathcal{C}_A$ of all compact subsets of $A$. Furthermore,

$$
\kappa \mu^K \uparrow \kappa \mu^A \text{ pointwise on } X \text{ as } K \uparrow A,
$$

and hence

$$
\kappa \mu^A = \sup_{K \in \mathcal{C}_A} \kappa \mu^K \text{ on } X.
$$

**Theorem 4.9.** Suppose that $A$ is the union of an increasing sequence $(A_k)$ of universally measurable subsets of $X$. For any $\mu \in \mathcal{E}^+$, then

$$
\mu^{A_k} \to \mu^A \text{ strongly and vaguely in } \mathcal{E}^+ \text{ as } k \to \infty,
$$

$$
\kappa \mu^{A_k} \uparrow \kappa \mu^A \text{ pointwise q.e. on } X \text{ as } k \to \infty.
$$

**Theorem 4.10.** Let $A$ be the intersection of a decreasing net (resp. a decreasing sequence) $(A_t)_{t \in T}$ of closed (resp. quasiclosed) sets. Given $\mu \in \mathcal{E}^+$, the following two limit relations hold as $t$ increases along $T$:

$$
\mu^{A_t} \to \mu^A \text{ strongly and vaguely in } \mathcal{E}^+,
$$

$$
\kappa \mu^{A_t} \downarrow \kappa \mu^A \text{ pointwise q.e. on } X.
$$

In view of the equality $\mu^A = \mu_{\mathcal{E}_A}$ (see Eq. (4.2)), the proofs of Theorems 4.8–4.10 (Sects. 6, 8.2, 8.3) are mainly based on a careful analysis of the orthogonal projection $\mu_{\mathcal{E}_A}$, provided by Theorem 5.1 below, as well as on some further properties of inner swept measures and their potentials, presented in Sect. 7.

5. On the orthogonal projection of $\mu \in \mathcal{E}^+$ onto $\mathcal{E}_A$

**Theorem 5.1.** Given $\mu \in \mathcal{E}^+$ and $A \subset X$,

$$
\kappa \mu_{\mathcal{E}_A} = \kappa \mu \text{ n.e. on } A,
$$

$$
\kappa \mu_{\mathcal{E}_A} = \kappa \mu \text{ $\mu_{\mathcal{E}_A}$-a.e.,}
$$

$$
\kappa \mu_{\mathcal{E}_A} \leq \kappa \mu \text{ on } X.
$$

Furthermore,

$$
\mu_{\mathcal{E}_K^+} \to \mu_{\mathcal{E}_A^+} \text{ strongly and vaguely in } \mathcal{E}^+ \text{ as } K \uparrow A,
$$

$$
\kappa \mu_{\mathcal{E}_K^+} \uparrow \kappa \mu_{\mathcal{E}_A^+} \text{ pointwise on } X \text{ as } K \uparrow A.
$$

**Proof.** For any $K, K' \in \mathcal{C} = \mathcal{C}_A$ such that $K \subset K'$, $\mu_{\mathcal{E}_K^+} \in \mathcal{E}^+_K \subset \mathcal{E}^+_A$, and

$$
\min_{\nu \in \mathcal{E}_K^+} \|\mu - \nu\| = \|\mu - \mu_{\mathcal{E}_K^+}\| \leq \|\mu - \mu_{\mathcal{E}_K^+}\|,
$$

12These two inclusions follow from the fact that any subset of a locally negligible set is likewise locally negligible [5, Section IV.5.2], which will often be used in the sequel.
by the definition of orthogonal projection. Lemma 2.2 with \( \Gamma := \{ \mu - \nu : \nu \in E^+_K \} \) and \( \lambda := \mu - \mu_{E^+_K} \) therefore yields
\[
\| \mu_{E^+_K} - \mu_{E^+_K} \| = \| (\mu - \mu_{E^+_K}) - (\mu - \mu_{E^+_K}) \| \leq \| \mu - \mu_{E^+_K} \|^2 - \| \mu - \mu_{E^+_K} \|^2.
\]
Being decreasing and lower bounded, the net (\( \| \mu - \mu_{E^+_K} \| \))\( K \in \mathbb{E} \) is Cauchy in \( \mathbb{R} \), which together with the preceding display implies that \( (\mu_{E^+_K})_{K \in \mathbb{E}} \) is strong Cauchy in \( E^+ \).
In view of the perfectness of the kernel, there is a unique \( \mu_0 \in E^+ \) such that
\[
\mu_{E^+_K} \to \mu_0 \quad \text{strongly and vaguely as } K \uparrow A,
\]
and hence \( \mu_0 \in E'_{A} \). To prove (5.4), it is thus enough to verify the equality
\[
(5.6) \quad \| \mu - \mu_0 \| = \varrho(\mu, E_{A}^+) \quad (:= \inf_{\nu \in E_{A}^+} \| \mu - \nu \|).
\]
It follows from the above that
\[
(5.7) \quad \varrho(\mu, E_{A}^+) = \varrho(\mu, E_{A}^+) \leq \| \mu - \mu_0 \| = \lim_{K \uparrow A} \| \mu - \mu_{E^+_K} \| = \lim_{K \uparrow A} \varrho(\mu, E_{K}^+),
\]
the first equality being evident by definition. On the other hand, for every \( \nu \in E_{A}^+ \),
\( A \) is \( \nu \)-measurable and \( \nu = \nu|_A \). Applying [15, Lemma 1.2.2] to an arbitrary function \( f \) from \( C_0^+ := \{ g \in C_0 : \ g \geq 0 \} \) we therefore get
\[
\nu(f) = \nu|_A(f) = \int f \, d\nu|_A = \sup_{K \in E_{A}} \int f \, d\nu|_K = \lim_{K \uparrow A} \nu|_K(f).
\]
Thus, for every \( \nu \in E_{A}^+ \),
\[
\nu|_K \to \nu \quad \text{vaguely as } K \uparrow A,
\]
which gives, by the principle of descent,
\[
\kappa(\nu, \nu) \leq \lim_{K \uparrow A} \kappa(\nu|_K, \nu|_K), \quad \kappa(\mu, \nu) \leq \lim_{K \uparrow A} \kappa(\mu, \nu|_K).
\]
The opposite being obvious in view of the positivity of the kernel, equality in fact prevails in these two inequalities. So,
\[
\| \mu - \nu \| = \lim_{K \uparrow A} \| \mu - \nu|_K \| = \lim_{K \uparrow A} \varrho(\mu, E_{K}^+) \quad \text{for every } \nu \in E_{A}^+,
\]
and consequently
\[
\varrho(\mu, E_{A}^+) \geq \lim_{K \uparrow A} \varrho(\mu, E_{K}^+).
\]
Combining this with (5.7) establishes (5.6), and hence (5.4).

The remainder of the proof is based on the fact that the orthogonal projection \( \mu_{E^+_A} \) is characterized uniquely within \( E^+_A \) by (3.1) and (3.2) with \( \mathcal{F} := E^+_A, E^+_A \) being a convex subcone of \( E^+ \) which contains zero measure. If \( A = K \) is compact, this actually holds with \( E^+_K \) in place of \( E^+_A \), see (3.3).

Assuming first that \( A = K \) is compact, we begin by showing that \( c_\nu(E) = 0 \), where \( E := \{ x \in K : \kappa\mu(x) > \kappa\mu_{E^+_K}(x) \} \). For any \( \sigma \in E^+ \) and any compact \( Q \subset E \),
we have \( \sigma|_Q \in E^+_Q \subset E^+_K \), and hence, by (3.1) with \( \mathcal{F} := E^+_K \),
\[
(\mu - \mu_{E^+_K}, \sigma|_Q) = \int (\kappa\mu - \kappa\mu_{E^+_K}) \, d\sigma|_Q \leq 0.
\]
As seen from the definition of \( E \), equality must prevail here; or, equivalently,
\[
\kappa\mu - \kappa\mu_{E^+_K} = 0 \quad \sigma|_Q\text{-a.e.}
\]
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(cf. [5, Section IV.2, Theorem 1]), which is possible only if \( \sigma|_Q = 0 \). Applying (2.2), we therefore conclude that, indeed, \( c_*(E) = 0 \), and so

\[
(5.8) \quad \kappa \mu_{\xi^+_K} \geq \kappa \mu \text{ n.e. on } K.
\]

Since \( c_*(E) = 0 \) while \( \mu_{\xi^+_K} \in \mathcal{E}^+_K \) is bounded, the (Borel) set \( E \) is \( \mu_{\xi^+_K} \)-negligible (cf. Lemma 2.3). Thus \( \kappa \mu_{\xi^+_K} \geq \kappa \mu \mu_{\xi^+_K} \)-a.e., which in view of (3.2) implies that

\[
\kappa \mu_{\xi^+_K} = \kappa \mu \mu_{\xi^+_K} \text{ a.e. on } K.
\]

By the domination principle, \( \kappa \mu_{\xi^+_K} \leq \kappa \mu \) on \( X \), which combined with (5.8) gives

\[
(5.9) \quad \kappa \mu_{\xi^+_K} = \kappa \mu \text{ n.e. on } K.
\]

Dropping now the current requirement on \( A \) of being compact and applying (5.9) to each of \( K, K' \in \mathcal{C}_A \) such that \( K \subset K' \), we get

\[
\kappa \mu_{\xi^+_K} = \kappa \mu = \kappa \mu_{\xi^+_K'} \text{ n.e. on } K,
\]

the inner capacity being countably subadditive on universally measurable subsets of \( X \) [15, Lemma 2.3.5]. Similarly as in the preceding paragraph, this actually holds \( \mu_{\xi^+_K} \)-a.e., which yields, again by the domination principle,

\[
\kappa \mu_{\xi^+_K} \leq \kappa \mu_{\xi^+_K'} \leq \kappa \mu \text{ on } X.
\]

Hence, for some function \( \psi \) that is \( \leq \kappa \mu \) on \( X \),

\[
(5.10) \quad \kappa \mu_{\xi^+_K} \uparrow \psi \text{ pointwise on } X \text{ as } K \uparrow A.
\]

To establish both (5.3) and (5.5), it thus remains to show that

\[
(5.11) \quad \psi = \kappa \mu_{\xi^+_A} \text{ on } X.
\]

The strong topology on \( \mathcal{E} \) having a countable base of neighborhoods, it follows from (5.4) that there is a subsequence \((\mu_{\xi^+_K})_{j \in \mathbb{N}}\) of the net \((\mu_{\xi^+_K})_{K \in \mathcal{E}}\) which converges strongly (hence vaguely) to \( \mu_{\xi^+_A} \). Therefore, by [18, Theorem 3.6],

\[
\kappa \mu_{\xi^+_K_j} \to \kappa \mu_{\xi^+_A} \text{ pointwise q.e. on } X \text{ as } j \to \infty,
\]

which in view of (5.10) implies that \( \psi = \kappa \mu_{\xi^+_A} \) q.e. on \( X \). Thus, for every \( K \in \mathcal{E}_A \), the inequality

\[
\kappa \mu_{\xi^+_K} \leq \kappa \mu_{\xi^+_A}
\]

holds q.e. on \( X \), hence \( \mu_{\xi^+_A} \)-a.e., and so everywhere on \( X \), the last two conclusions being obtained in the same manner as above. Letting now \( K \uparrow A \) gives

\[
\psi = \limsup_{K \uparrow A} \kappa \mu_{\xi^+_K} \leq \kappa \mu_{\xi^+_A} \leq \liminf_{K \uparrow A} \kappa \mu_{\xi^+_K} \text{ on } X,
\]

the latter inequality being derived from (5.4) by use of the principle of descent. This establishes (5.11), and hence also both (5.3) and (5.5).

The proof is completed by verifying (5.1) and (5.2). The function \( \varphi := \kappa \mu - \kappa \mu_{\xi^+_A} \) is \( \nu \)-negligible for every \( \nu \in \mathcal{E}'_A \) (cf. [5] Section IV.2, Definition 1)], for

\[
\int \varphi \, d\nu = (\mu - \mu_{\xi^+_A}, \nu) \leq 0
\]

The set \( E \) is Borel because \( \kappa(\mu - \mu_{\xi^+_K}) \) is Borel measurable on \( X \), being the difference between two l.s.c. functions.
by (3.1) with $\mathcal{F} := \mathcal{E}'_A$, whereas $\varphi \geq 0$ by (3.3). Again by [5, Section IV.2, Theorem 1],

$$\kappa \mu_{\mathcal{E}'_A} = \kappa \mu \ ν \text{-a.e. for every } \nu \in \mathcal{E}'_A,$$

which leads to (5.1) in a manner similar to that in the proof of (5.9). Indeed, for every $\sigma \in \mathcal{E}^+$ and every compact subset $Q$ of the set $E_1$ of all $x \in A$ with $\kappa \mu_{\mathcal{E}'_A}(x) < \kappa \mu(x)$, we have $\sigma|_Q = 0$, hence $\mathcal{E}'_A \cap Q = 0$, and consequently $c_* (E_1) = 0$, by (2.2). Finally, (5.12) with $\nu := \mu_{\mathcal{E}'_A}$ yields (5.2).

**Remark 5.2.** For a compact set $A$, the orthogonal projection $\mu_{\mathcal{E}'_A} (= \mu_{\mathcal{E}^+_A})$ is characterized uniquely within $\mathcal{E}'_A (= \mathcal{E}^+_A)$ by (5.1). Indeed, if $\kappa \theta = \kappa \mu$ n.e. on $A$ for some $\theta \in \mathcal{E}^+_A$, then, by (5.1), the equality $\kappa \theta = \kappa \mu_{\mathcal{E}^+_A}$ holds true n.e. on $A$, and hence $(\theta + \mu_{\mathcal{E}^+_A})$-a.e. (Lemma 2.3), the set $\{x \in A : \kappa \theta(x) \neq \kappa \mu_{\mathcal{E}^+_A}(x)\}$ being Borel measurable while the measure $\theta + \mu_{\mathcal{E}^+_A}$ being bounded. This implies by integration $\|\theta - \mu_{\mathcal{E}^+_A}\| = 0$, and consequently $\theta = \mu_{\mathcal{E}^+_A}$, by the energy principle.

6. **Proofs of Theorems 4.3 and 4.8**

Due to Theorem 5.1, we are now able to establish Theorems 4.3 and 4.8.

Fix $\mu \in \mathcal{E}^+$ and $A \subset X$. Noting from (5.1) that $\mu_{\mathcal{E}'_A} \in \Lambda_{A,\mu}$ (see Definition 4.1), we shall verify (4.2) by proving that for any given $\nu \in \Lambda_{A,\mu}$,

$$\kappa \mu_{\mathcal{E}'_A} \leq \kappa \nu \text{ on } X. \quad (6.1)$$

Combining $\kappa \nu \geq \kappa \mu$ n.e. on $A$ with (5.9) applied to any $K \in \mathcal{C}_A$ shows that

$$\kappa \mu_{\mathcal{E}^+_K} \leq \kappa \nu \text{ holds n.e. on } K,$$

holds n.e. on $K$, hence $\mu_{\mathcal{E}^+_K}$-a.e. by Lemma 2.3 and consequently on all of $X$, by the domination principle. On account of (5.5), letting here $K \uparrow A$ results in (6.1). Also note that, because of (4.2),

$$\mu^A = \mu \text{ for every } \mu \in \mathcal{E}'_A. \quad (6.2)$$

By (4.2) and (3.3) with $A := K$ compact, $\mu^K = \mu_{\mathcal{E}^+_K}$. Combining this and (4.2) with (5.1) and (5.5) yields (4.10) and (4.11), thereby establishing Theorem 4.8.

To complete the proof of Theorem 4.3, it remains to verify that $\mu^A$ is characterized uniquely within $\mathcal{E}'_A$ by (4.3). Having fixed $\theta \in \mathcal{E}'_A$ with the property

$$\kappa \theta = \kappa \mu \text{ n.e. on } A, \quad (6.3)$$

we need to show that then necessarily

$$\theta = \mu^A. \quad (6.4)$$

Applying (5.9) to $K \in \mathcal{C}_A$ and each of $\theta$ and $\mu$, we conclude from (6.3) that

$$\kappa \theta_{\mathcal{E}^+_K} = \kappa \theta = \kappa \mu = \kappa \mu_{\mathcal{E}^+_K} \text{ n.e. on } K.$$

14The same actually holds if $A$ is closed or even quasiclosed, which is seen from Corollary 1.5. Moreover, this can be generalized to arbitrary $A$ as follows: the orthogonal projection $\mu_{\mathcal{E}'_A}$ is characterized uniquely within $\mathcal{E}'_A$ by (5.1). Indeed, on account of (4.2), such an extension is derived directly from the characteristic property of inner balayage given in Theorem 4.3.

15The application of Lemma 2.3 is justified in the same manner as above, see e.g. Remark 5.2.
In view of the characteristic property of orthogonal projection onto $E^+_K$ observed in Remark 5.2, this gives $\theta_{E^+_K} = \mu_{E^+_K}$, and hence, by (4.2),

$$\theta^K = \mu^K$$

for every $K \in \mathcal{E}_A$.

Letting now $K \uparrow A$ and using (4.10) we get

$$\theta^A = \mu^A,$$

the vague limit of any convergent net in the (Hausdorff) space $\mathcal{M}$ being unique. As $\theta^A = \theta$ by (6.2) applied to $\theta \in \mathcal{E}'_A$, (6.4) follows.

7. Further properties of inner balayage

**Proposition 7.1** (Monotonicity property). If $A \subset Q$, then for any $\mu \in \mathcal{E}^+$,

$$\kappa \mu^A \leq \kappa \mu^Q.$$

*Proof.* Since $\mathcal{E}_A \subset \mathcal{E}_Q$, this follows directly from (4.12). □

**Proposition 7.2** (Balayage with a rest). If $A \subset Q$, then for any $\mu \in \mathcal{E}^+$,

$$\mu^A = (\mu^Q)^A.$$

*Proof.* As seen from Theorem 4.3 both $\mu^A$ and $(\mu^Q)^A$ belong to $\mathcal{E}_A'$, and moreover

$$\kappa(\mu^Q)^A = \kappa \mu^Q = \kappa \mu = \kappa \mu^A$$

n.e. on $A$.

By the characteristic property of inner balayage given in Theorem 4.3 $\mu^A$ and $(\mu^Q)^A$ are indeed equal. □

**Proposition 7.3.** Given $A \subset X$,

$$\mu^A = \mu$$

for every $\mu \in \mathcal{E}_A'$ (thus in particular for every $\mu \in \mathcal{E}_A^+$).

*Proof.* Since $\mu_{\mathcal{E}'_A} = \mu$ for every $\mu \in \mathcal{E}_A'$, this is obvious from (12). □

In the remainder of this section, the kernel $\kappa$ is supposed to satisfy Frostman’s maximum principle (see footnote 6 for details).

**Proposition 7.4** (Principle of positivity of mass). Given $\mu \in \mathcal{E}^+$ and $A \subset X$,

$$\mu^A(X) \leq \mu(X).$$

*Proof.* For any $K \in \mathcal{E}_A$, consider the equilibrium measure $\gamma_K$ on $K$ (Sect. 2). Then, by Fubini’s theorem,

$$\mu^K(X) = \int 1 d\mu^K = \int \kappa \gamma_K d\mu^K$$

$$= \int \kappa \mu^K d\gamma_K = \int \kappa \mu d\gamma_K = \int \kappa \gamma_K d\mu,$$

**Footnote:** Eq. (7.1) is implied by the following strengthened version of countable subadditivity for inner capacity [15, p. 158, Remark]: For arbitrary $A \subset X$ and universally measurable $U_k \subset X$, $k \in \mathbb{N}$,

$$c_* \left( \bigcup_{k \in \mathbb{N}} A \cap U_k \right) \leq \sum_{k \in \mathbb{N}} c_*(A \cap U_k).$$
because $\kappa \gamma_K = 1$, resp. $\kappa \mu^K = \kappa \mu$, holds n.e. on $K$ (cf. (2.3) and (4.3)), and hence $\mu^K$-a.e., resp. $\gamma_K$-a.e. (Lemma 2.3). Since $\kappa \gamma_K \leq 1$ on $X$, this gives

$$\mu^K(X) \leq \mu(X).$$

Letting $K \uparrow A$ we get (7.2) by use of Lemma 3.4 (noting that $\mu^K \rightarrow \mu^A$ vaguely).

Before providing a formula for evaluation of the total mass $\mu^A(X)$ for the inner swept measure $\mu^A$ (Proposition 7.6), we shall first analyze the continuity of the inner equilibrium potential $\kappa \gamma_A$ under the exhaustion of $A$ by $K$ compact.

**Lemma 7.5.** Given $A \subset X$ with $c_*(A) < \infty$,

$$\kappa \gamma_K \uparrow \kappa \gamma_A \text{ pointwise on } X \text{ as } K \uparrow A,$$

$\gamma_A$ denoting the inner equilibrium measure for $A$.

**Proof.** Since $c(K) \uparrow c_*(A)$ as $K \uparrow A$ (cf. (2.1)), we conclude in much the same way as it was done in [15, Proof of Theorem 4.1] that

$$\kappa \gamma_K \rightarrow \kappa \gamma_A \text{ strongly (hence, vaguely) in } E^+ \text{ as } K \uparrow A.$$

But for any $K, K' \in C_A$ such that $K \subset K'$,

$$1 = \kappa \gamma_K = \kappa \gamma_{K'} = \kappa \gamma_A \text{ n.e. on } K,$$

hence $\gamma_K$-a.e. Therefore, by the domination principle, the net $(\kappa \gamma_K)_{K \in E_A}$ increases pointwise on $X$ to some function that does not exceed $\kappa \gamma_A$. To establish (7.2), we thus only need to verify the inequality

$$\kappa \gamma_A \leq \lim_{K \uparrow A} \kappa \gamma_K \text{ on } X,$$

which follows directly from (7.3) in view of the vague lower semicontinuity of the mapping $\nu \mapsto \kappa \nu(\cdot)$, $\nu \in \mathcal{M}^+$ (the principle of descent [15, Lemma 2.2.1(b)]).

**Proposition 7.6.** For any $\mu \in E^+$ and any $A \subset X$ with $c_*(A) < \infty$\footnote{For the $\alpha$-Riesz kernel of order $\alpha \in (0, 2]$ on $\mathbb{R}^n$, $n \geq 3$, Proposition 7.6 remains valid for any $A \subset \mathbb{R}^n$ that is inner $\alpha$-thin at infinity (even if $c_*(A) = \infty$), which is seen by combining Theorems 2.5 and 5.1 from [30]. (For the concept of inner $\alpha$-thinness at infinity, see [30, Definition 2.2].) Moreover, then the requirement on $\mu$ of having finite energy can be simply omitted.}

$$\mu^A(X) = \int \kappa \gamma_A \, d\mu.$$

**Proof.** Combining Propositions 7.2 and 7.3 yields

$$\mu^K(X) = (\mu^A)^K(X) \leq \mu^A(X) \text{ for every } K \in C_A,$$

hence

$$\limsup_{K \uparrow A} \mu^K(X) \leq \liminf_{K \uparrow A} \mu^K(X),$$

the latter inequality being obtained from (4.10) with the aid of Lemma 3.4. On account of (7.3), this gives

$$\mu^A(X) = \lim_{K \uparrow A} \mu^K(X) = \lim_{K \uparrow A} \int \kappa \gamma_K \, d\mu.$$

Since the net $(\kappa \gamma_K)_{K \in E_A}$ increases pointwise on $X$ to $\kappa \gamma_A$ (Lemma 7.5), we get (7.6) by applying [5, Section IV.1, Theorem 1] to the integral on the right.
8. Proofs of Theorems 4.6, 4.9 and 4.10

8.1. Proof of Theorem 4.6. Fix \( \mu \in \mathcal{E}^+ \) and \( A \subset X \). According to Theorems 4.3 and 4.8, the inner balayage \( \mu^A \), introduced by Definition 4.1, can be found as the orthogonal projection \( \mu_{\mathcal{E}'_A} \) of \( \mu \) onto the cone \( \mathcal{E}'_A \), and moreover

\[
\mu^K \to \mu^A \text{ strongly (hence vaguely) as } K \uparrow A.
\]

Assume now that \( \mu \) is bounded while \( \kappa \) satisfies Frostman’s maximum principle. By Proposition 7.4, \( \mu^K(X) \leq \mu(X) \) for all \( K \in \mathcal{E}_A \), hence \( \mu^K \in \mathcal{E}_{K,\mu}^+ \subset \mathcal{E}_{A,\mu}^+ \) (cf. footnote [12]), which in view of the preceding display implies that, actually\[13\]

\[
(8.1) \quad \mu^A \in \mathcal{E}'_{A,\mu}.
\]

Noting that \( \mathcal{E}'_{A,\mu} \subset \mathcal{E}'_A \), we therefore get

\[
\varrho(\mu, \mathcal{E}'_{A,\mu}) \leq \|\mu - \mu^A\| = \varrho(\mu, \mathcal{E}'_A) \leq \varrho(\mu, \mathcal{E}'_{A,\mu}),
\]

hence

\[
\varrho(\mu, \mathcal{E}'_A) = \varrho(\mu, \mathcal{E}'_{A,\mu}),
\]

and so \( \mu^A \) serves simultaneously as the orthogonal projection of \( \mu \) onto the (strongly closed, convex) truncated cone \( \mathcal{E}'_{A,\mu} \). This proves (4.8).

As \( \mu^A \in \Lambda_{A,\mu} \) (Corollary 4.4) and \( \mu^A(X) \leq \mu(X) \) (Proposition 7.4),

\[
(8.2) \quad \mu^A \in \bar{\Lambda}_{A,\mu}.
\]

Thus

\[
\inf_{\nu \in \Lambda_{A,\mu}} \kappa \nu \leq \kappa \mu^A = \min_{\nu \in \Lambda_{A,\mu}} \kappa \nu \leq \inf_{\nu \in \bar{\Lambda}_{A,\mu}} \kappa \nu,
\]

and so \( \mu^A \) gives indeed a solution to problem (4.7).

If \( A \) now is quasiclosed, then the (convex) set \( \mathcal{E}'_{A,\mu} \) is strongly closed, which is seen from the latter part of Lemma 3.5 by homogeneity reasons. Hence, the orthogonal projection \( \mu_{\mathcal{E}'_{A,\mu}^+} \) does exist (see the former part of Theorem 3.1)\[10\] and moreover it equals \( \mu_{\mathcal{E}'_{A,\mu}} \). Substituting this into (4.8) establishes (4.9).

8.2. Proof of Theorem 4.9. Fix \( \mu \in \mathcal{E}^+ \), and assume that \( A \) is the union of an increasing sequence \( (A_k) \). Then \( \mathcal{E}^+_A \subset \mathcal{E}^+_A \subset \mathcal{E}^+_A \) for any \( k, p \in \mathbb{N} \) (cf. footnote [12]), hence \( \mathcal{E}'_{A_k} \subset \mathcal{E}'_{A_{k+p}} \subset \mathcal{E}'_A \). As \( \mu^{A_k} = \mu_{\mathcal{E}'_{A_k}} \), we therefore have \( \mu^{A_k} \in \mathcal{E}'_{A_k} \subset \mathcal{E}'_{A_{k+p}} \) and

\[
\|\mu - \mu^{A_k}\| = \min_{\nu \in \mathcal{E}'_{A_k}} \|\mu - \nu\| \geq \min_{\nu \in \mathcal{E}'_{A_{k+p}}} \|\mu - \nu\| = \|\mu - \mu^{A_{k+p}}\|.
\]

Thus, by Lemma 2.2 with \( \Gamma := \{\mu - \nu : \nu \in \mathcal{E}'_{A_{k+p}}\} \) and \( \lambda := \mu - \mu^{A_{k+p}} \),

\[
\|\mu^{A_k} - \mu^{A_{k+p}}\|^2 = \|(\mu - \mu^{A_k}) - (\mu - \mu^{A_{k+p}})\|^2 \leq \|\mu - \mu^{A_k}\|^2 - \|\mu - \mu^{A_{k+p}}\|^2,
\]

which together with the preceding display implies that the sequence \( (\mu^{A_k}) \subset \mathcal{E}'_A \) is strong Cauchy. Since \( \mathcal{E}'_A \) is strongly complete, there is a unique \( \mu_0 \in \mathcal{E}'_A \) such that

\[
(8.3) \quad \mu^{A_k} \to \mu_0 \text{ strongly and vaguely.}
\]

\[10\] For the notation used here, see the paragraph followed by Theorem 4.6.

\[19\] Observe that the latter part of Theorem 3.1 is not applicable to \( \mu_{\mathcal{E}'_{A,\mu}^+} \), for \( \mathcal{E}'_{A,\mu}^+ \) is not a cone.
In turn, all this gives
\[ \varrho(\mu, \mathcal{E}_{A_k}^+) = \varrho(\mu, \mathcal{E}_A^+) \leq \|\mu - \mu_0\| = \lim_{k \to \infty} \|\mu - \mu_k\| = \lim_{k \to \infty} \varrho(\mu, \mathcal{E}_{A_k}^+) = \lim_{k \to \infty} \varrho(\mu, \mathcal{E}_A^+) = \lim_{k \to \infty} \varrho(\mu, \mathcal{E}_{A_k}) \]

Assuming now the sets \(A_k\) to be universally measurable, we shall show that then
\[ \lim_{k \to \infty} \varrho(\mu, \mathcal{E}_{A_k}^+) \leq \varrho(\mu, \mathcal{E}_A^+) \]
which combined with \((8.4)\) will imply \(\|\mu - \mu_0\| = \varrho(\mu, \mathcal{E}_A^+)\), hence \(\mu_0 = \mu_{\mathcal{E}_A^+} = \mu^A\), and consequently \((4.13)\), by \((8.3)\).

In fact, for every \(\nu \in \mathcal{E}_{A_k}^+\) and every \(f \in C_0^+\),
\[ \lim_{k \to \infty} \nu|_{A_k}(f) = \lim_{k \to \infty} \int 1_{A_k} f d\nu = \int 1_A f d\nu = \nu(f), \]
where the first and last equalities hold by the definition of the trace of a positive Radon measure to a measurable set (also noting that the first and last equalities hold by the definition of the trace of a positive function on \(A\)), while the second equality is obtained by applying \([5, \text{Section IV.1, Theorem 3}]\) to the positive functions \(1_{A_k} f, k \in \mathbb{N}\), with the upper envelope \(1_A f\). Thus
\[ \nu|_{A_k} \to \nu \text{ vaguely,} \]
which yields, by the principle of descent,
\[ \kappa(\nu, \nu) \leq \lim_{k \to \infty} \kappa(\nu|_{A_k}, \nu|_{A_k}), \quad \kappa(\mu, \nu) \leq \lim_{k \to \infty} \kappa(\mu, \nu|_{A_k}). \]
Since the kernel is positive, equality prevails in these two inequalities; therefore,
\[ \|\mu - \nu\| = \lim_{k \to \infty} \|\mu - \nu|_{A_k}\| \geq \lim_{k \to \infty} \varrho(\mu, \mathcal{E}_{A_k}^+) \text{ for every } \nu \in \mathcal{E}_{A_k}^+, \]
and \((8.5)\) follows. (The inequality in \((8.6)\) is valid because for any \(\sigma \in \mathcal{E}^+\) and any \(\sigma\)-measurable set \(Q \subset X\), we have \(\sigma|_Q \in \mathcal{E}_{Q}^+\).)

Having thus established \((4.13)\), we complete the proof by verifying \((4.14)\). By the monotonicity of inner balayage (Proposition \((7.4)\)) that for \(F \subset X\) quasiclosed (thus in particular closed),
\[ \mu^F = \mu_{\mathcal{E}_{F}^+}, \]
the convex cone \(\mathcal{E}_{F}^+\) being strongly closed (Lemma \((3.3)\)).

Suppose first that \(A\) is the intersection of a lower directed family \((A_t)_{t \in T}\) of closed sets. In view of the monotonicity of \((\mathcal{E}_{A_t}^+)_{t \in T}\), we see in a manner similar to that in Sect. \((8.2)\) that \((\mu^A_t)_{t \in T}\) is a strong Cauchy net in \(\mathcal{E}^+\), and hence there is a unique \(\mu_0 \in \mathcal{E}^+\) with the property
\[ \mu^A_t \to \mu_0 \text{ strongly and vaguely as } t \text{ increases along } T. \]
Such a limit \(\mu_0\) belongs to the class \(\mathcal{E}_{A_t}^+\), for every \(t \in T\), \(\mathcal{E}_{A_t}^+\) being strongly closed. Since for a closed set \(F \subset X\), \(\mathcal{E}_{F}^+\) consists of all \(\nu \in \mathcal{E}^+\) supported by \(F\) (see Sect. \((2)\)), \(\mu_0\) is supported by every \(A_t\), and hence by the intersection of \(A_t\) over all \(t\). Thus
\[ \mu_0 \in \mathcal{E}_{A_t}^+, \]
and consequently
\[ \varrho(\mu, \mathcal{E}_A^+) \leq \| \mu - \mu_0 \| = \lim_{t \to 0} \| \mu - \mu_{\mathcal{E}_A^+} \| = \lim_{t \to 0} \varrho(\mu, \mathcal{E}_A^+) \leq \varrho(\mu, \mathcal{E}_A^+), \]
the latter inequality being valid because \( \mathcal{E}_A^+ \subset \mathcal{E}_{A_t}^+ \) for each \( t \in T \). This implies that, actually, \( \mu_0 = \mu_{\mathcal{E}_A^+} \), and (4.13) follows.

By Proposition 7.1 the net \( (\kappa \mu^A_t)_{t \in T} \) decreases pointwise on \( X \), and moreover
\[ \kappa \mu^A(x) \leq \lim_{t \to 0} \kappa \mu^A_t(x) \quad \text{for all} \quad x \in X. \]

The strong topology on \( \mathcal{E} \) having a countable base of neighborhoods, it follows from (4.13) that there is a subsequence \( (\mu^A_{i_j})_{j \in \mathbb{N}} \) of the net \( (\mu^A_t)_{t \in T} \) that converges strongly (hence vaguely) to \( \mu^A \). Applying [18, Theorem 3.6] we therefore conclude that equality, in fact, prevails in (8.9) for quasi all \( x \in X \), which establishes (4.14).

Let \( A \) now be the intersection of a decreasing sequence \( (A_i)_{i \in T} \) of quasiclosed sets. In view of the fact that a countable intersection of quasiclosed sets is likewise quasiclosed [10, Lemma 2.3], the proof of (4.15) and (4.16) is essentially the same as above, the only difference being in that of (8.9). As \( \mu_0 \in \mathcal{E}_A^+ \) (see above), each \( (A_i)^c \) is locally \( \mu_0 \)-negligible. Being thus a countable union of locally \( \mu_0 \)-negligible sets, \( A^c \) is likewise \( \mu_0 \)-negligible [3, Section IV.5.2], and hence (8.9) indeed holds.

9. Outer balayage

The approach to balayage problems, utilized in the present paper, is mainly based on an analysis of convergence of inner swept measures and their potentials under the exhaustion of \( A \subset X \) by \( K \) compact. We shall now show that, under suitable topological assumptions on \( X \) and \( A \), this (typically inner) approach is still efficient when dealing with outer balayage problems.

**Definition 9.1.** \( \mu^A \in \mathcal{E}^+ \) is said to be an outer balayage of \( \mu \in \mathcal{E}^+ \) to \( A \) if
\[ \kappa \mu^A = \inf_{\nu \in \mathcal{E}_A^+} \kappa \nu \quad \text{on} \quad X, \quad (9.1) \]
where
\[ \Lambda_{A, \mu} := \{ \nu \in \mathcal{E}^+ : \kappa \nu \geq \kappa \mu \ \text{q.e. on} \ A \}. \]

By the same proof as in Lemma 4.2, the outer balayage \( \mu^A \) is unique (if it exists). Observe also that this definition is in agreement with Cartan’s (classical) concept of outer Newtonian balayage on \( \mathbb{R}^n \), \( n \geq 3 \), cf. [3, Section 19, Theorem 1].

**Definition 9.2** (Fuglede [16]). A set \( B \subset X \) is said to quasicontain a set \( A \subset X \) if \( c^*(A \setminus B) = 0 \). Two sets \( A, B \subset X \) are said to be \( c^* \)-equivalent if \( c^*(A \Delta B) = 0 \).

**Remark 9.3.** If the outer balayage \( \mu^A \) exists, then so does \( \mu^{c^* A} \) for any \( A \subset X \) that is \( c^* \)-equivalent to \( A \), and moreover \( \mu^A = \mu^{c^* A} \). Indeed, then \( \Lambda_{A, \mu} = \Lambda_{B, \mu} \), the outer capacity being countably subadditive on any subsets of \( X \) [15, Lemma 2.3.5].

Unless explicitly stated otherwise, throughout Sect. 9 we shall tacitly assume that a l.c. space \( X \) is perfectly normal\(^{20}\) and of class \( K_\sigma \), and that \( A \subset X \) is Borel. Suppose as before that the kernel is perfect and satisfies the domination principle.

\(^{20}\)By Urysohn’s theorem [3, Section IX.1, Theorem 1], a topological Hausdorff space \( Y \) is said to be normal if for any two disjoint closed subsets \( F_1, F_2 \) of \( Y \), there exist disjoint open sets \( D_1, D_2 \) such that \( F_i \subset D_j \) (\( i = 1, 2 \)). Further, a normal space \( Y \) is said to be perfectly normal [21, Section IX.4, Exercise 7] if each closed subset of \( Y \) is a countable intersection of open sets (or, equivalently, if each open subset of \( Y \) is a countable union of closed sets).
9.1. Existence of the outer balayage. Alternative definitions. Fix $\mu \in \mathcal{E}^+$. 

**Theorem 9.4.** The outer balayage $\mu^* A$, introduced by Definition 9.1, does exist, and moreover it coincides with the inner balayage $\mu^*_A$, introduced by Definition 4.1. Thus, by Theorem 4.3, 

$$\mu^* A = \mu^*_A = \mu_{\mathcal{E}_A^+},$$

and hence the outer balayage $\mu^* A$ can equivalently be determined by the two formulae 

$$\begin{align*}
\mu^* A \in \mathcal{E}_A^+, & \quad \|\mu - \mu^* A\| = \min_{\nu \in \mathcal{E}_A^+} \|\mu - \nu\| = \inf_{\nu \in \mathcal{E}_A^+} \|\mu - \nu\|.
\end{align*}$$

Furthermore, $\mu^* A$ has the properties 

$$\begin{align*}
\kappa \mu^* A &= \kappa \mu \text{ q.e. on } A, \tag{9.3} \\
\kappa \mu^* A &= \kappa \mu \text{ $\mu^*$-a.e.,} \tag{9.4} \\
\kappa \mu^* A &\leqslant \kappa \mu \text{ on } X, \tag{9.5}
\end{align*}$$

and it can equivalently be defined as the only measure in $\mathcal{E}_A^+$ satisfying (9.3).

**Corollary 9.5.** Variational problem (9.1) on minimizing the potential $\kappa \nu$ among the measures $\nu \in \Lambda^*_A, \mu$ has the unique solution $\mu^* A$, given by Theorem 9.4.

**Proof.** According to Theorem 9.4, the infimum in (9.1) is achieved at the (unique) measure $\mu^* A$, determined for instance by (9.2), and moreover $\mu^* A \in \Lambda^*_A, \mu$, by (9.3). Hence the infimum in (9.1) is indeed an actual minimum. \qed

**Corollary 9.6.** Assume that a (Borel) set $A$ is quasiclosed (or, more generally, that $E_+^A$ is strongly closed). Then the outer balayage $\mu^* A$ is actually the orthogonal projection of $\mu$ onto the cone $\mathcal{E}_A^+$, i.e. 

$$\mu^* A = \mu_{\mathcal{E}_A^+}.$$ 

Alternatively, $\mu^* A$ can be found as the only measure in $\mathcal{E}_A^+$ having property (9.3). 

**Remark 9.7.** If $A$ is quasiclosed while $X$ arbitrary, the existence of $\mu^* A \in \mathcal{E}_A^+$, determined uniquely within $\mathcal{E}_A^+$ by $\kappa \mu^* A = \kappa \mu$ q.e. on $A$, has been established before by Fuglede [18, Theorem 4.12]. Theorem 9.4 shows that this result, suitably modified, remains valid for any Borel subset $A$ of a perfectly normal, l.c. space $X$ of class $K_\sigma$, thereby presenting a further development of Fuglede’s theory on outer balayage. See also Theorems 9.8, 9.10, and 9.11 below providing some additional properties of $\mu^* A$, which seem to be new in part even for quasiclosed $A$.

Assume now that $\mu \in \mathcal{E}^+$ is bounded. For a given (Borel) set $A \subset X$, denote by $\hat{\Lambda}^*_A, \mu$ the convex, truncated cone of all $\nu \in \Lambda^*_A, \mu$ having property (4.6).

**Theorem 9.8.** Suppose that $\mu \in \mathcal{E}^+$ is bounded while $\kappa$ satisfies Frostman’s maximum principle. Then Definition 9.1 and that obtained from it by replacing $\Lambda^*_A, \mu$ by $\hat{\Lambda}^*_A, \mu$ lead to the same concept of outer balayage. Furthermore, Theorem 9.4 and Corollaries 9.5 and 9.6 remain valid if $\Lambda^*_A, \mu$, $\mathcal{E}_A^+$, and $\mathcal{E}_A$ are replaced throughout by $\hat{\Lambda}^*_A, \mu$, $\hat{\mathcal{E}}_A^+$, and $\hat{\mathcal{E}}_A$, respectively. 

---

21For the notations $\hat{\mathcal{E}}_A^+$ and $\hat{\mathcal{E}}_A$, see the paragraph followed by Theorem 4.6. Also note that Theorem 9.8 still holds if (4.6) is weakened to $\nu(X) \leqslant q \mu(X)$, where $q \in [1, \infty)$ (cf. Remark 4.7).
These results, as well as those in Sect. 9.3, can easily be derived from the theory of inner balayage, developed in Sects. 4–8 above. For the sake of completeness, we shall nevertheless sketch their proofs.

9.2. Proofs of Theorems 9.4 and 9.8 and Corollary 9.6. The analysis given below is based substantially on the following theorem on capacitability, obtained by a direct application of [15, Theorem 4.5].

Theorem 9.9. Any Borel subset of a perfectly normal, l.c. space $X$ of class $K_\sigma$, endowed with a perfect kernel $\kappa$, is capacitable.

This enables us to show that for any $\mu \in \mathcal{E}^+$ and any Borel $A \subset X$, the classes $\Lambda^*_{A,\mu}$ appearing in Definitions 4.1 and 9.1 of inner and outer balayage, respectively, coincide:

$$\Lambda_{A,\mu} = \Lambda^*_{A,\mu}. \quad (9.6)$$

To this end, it is enough to verify that for any $\nu \in \mathcal{E}^+$ with $\kappa \nu \geq \kappa \mu$ n.e. on $A$, the same inequality holds q.e. on $A$. Being the potential of a (signed) measure of finite energy, $\kappa (\nu - \mu)$ is well defined and finite q.e. on $X$ [15, Corollary to Lemma 3.2.3], and it is Borel measurable. Applying Theorem 9.9 to $E := A \cap \{x \in X : \kappa \nu(x) < \kappa \mu(x)\}$, we therefore get $c^*(E) = c_*(E) = 0$, and (9.6) follows.

It follows directly from (9.6) that the inner balayage $\mu^A$ (whose existence was justified by Theorem 4.3) gives actually a (unique) solution to the problem on the existence of outer balayage of $\mu$ to $A$:

$$\mu^A = \mu^*A. \quad (9.7)$$

When substituted into (4.2), (4.4), and (4.5), this results in (9.2), (9.4), and (9.5), respectively.

Applying now Theorem 9.9 to the (Borel) set $A \cap \{x \in X : \kappa \mu^A(x) < \kappa \mu(x)\}$, we infer from (9.7) and (4.3) that

$$\kappa \mu^A = \kappa \mu^*A = \kappa \mu \text{ n.e. (hence q.e.) on } A, \quad (9.8)$$

which proves (9.3). Moreover, $\mu^*A$ is the only measure in $\mathcal{E}'_A$ satisfying (9.3), for (4.3) characterizes $\mu^A$ uniquely within $\mathcal{E}'_A$ (see Theorem 4.3).

Having thus verified Theorem 9.4, assume now that the (Borel) set $A$ is quasi-closed. According to Lemma 3.5, the convex cone $\mathcal{E}^+_A$ then coincides with its strong closure $\mathcal{E}'_A$; hence, the orthogonal projection $\mu^E_A$ exists (Theorem 3.1), and moreover $\mu^E_A = \mu^*E_A$. Substituting this into Theorem 9.4 yields Corollary 9.6.

Returning again to arbitrary Borel $A$, suppose finally that $\mu$ is bounded while the kernel satisfies Frostman’s maximum principle. Then the inner balayage does not increase the total mass of a measure (Proposition 7.4), and we have thus been led to (8.1) and (8.2). Combining these two with (9.6) and (9.7) shows that the outer balayage $\mu^*A$ belongs, in fact, to both $\mathcal{E}'_{A,\mu}$ and $\hat{\Lambda}^*_{A,\mu}$, which establishes Theorem 9.8 in the same manner as it did in Sect. 8.1.

9.3. Further properties of outer balayage. Convergence assertions. Recall that we require the space $X$ to be perfectly normal and of class $K_\sigma$.

Theorem 9.10. For any $\mu \in \mathcal{E}^+$, the following assertions (a)–(c) on convergence of outer swept measures and their potentials hold true.
(a) For a Borel, the following two limit relations hold when $K \uparrow A$:
\[
\mu^K \to \mu^A \text{ strongly and vaguely in } \mathcal{E}^+,
\kappa\mu^K \uparrow \kappa\mu^A \text{ pointwise on } X.
\]

(b) If $A$ is the union of an increasing sequence $(A_k)$ of Borel sets, then
\[
\mu^A_k \to \mu^A \text{ strongly and vaguely in } \mathcal{E}^+,
\kappa\mu^A_k \uparrow \kappa\mu^A \text{ pointwise on } X.
\]

(c) If $A$ is the intersection of a lower directed family (resp. a decreasing sequence) $(A_t)$ of closed (resp. quasiclosed and Borel) sets, then
\[
\mu^A_t \to \mu^A \text{ strongly and vaguely in } \mathcal{E}^+,
\kappa\mu^A_t \downarrow \kappa\mu^A \text{ pointwise q.e. on } X.
\]

Proof. This follows by substituting (9.2) into Theorems 4.8–4.10.

Theorem 9.11. Given Borel $A,Q \subset X$ and $\mu \in \mathcal{E}^+$, the following (d)–(g) hold.

(d) (Monotonicity property) If $A \subset Q$, then
\[
\kappa\mu^A \leq \kappa\mu^Q.
\]

(e) (Balayage with a rest) If $A \subset Q$, then
\[
\mu^A = (\mu^Q)^A.
\]

(f) If $\mu \in \mathcal{E}'_A$ (thus in particular if $\mu \in \mathcal{E}^+_A$), then
\[
\mu^A = \mu.
\]

(g) Assume Frostman’s maximum principle holds. Then

\[
\mu^A(X) \leq \mu(X).
\]

If moreover $c^*(A) < \infty$, then actually
\[
\mu^A(X) = \int \kappa\gamma^*_A \, d\mu,
\]
\[
\gamma^*_A \text{ being the outer equilibrium measure for } A.
\]

Proof. After applying (9.2) to either of $A$ and $Q$, we deduce (d)–(f) from Propositions 7.1–7.3. If Frostman’s maximum principle holds, then combining (9.2) with Proposition 7.4 leads to (9.8). Assume moreover that $c^*(A) < \infty$. The set $A$ being capacitable (Theorem 9.9), we conclude from [15] (Theorems 4.1, 4.3 and Lemma 4.3.4) that the outer and inner equilibrium measures for $A$ (exist and) coincide:
\[
\gamma^*_A = \gamma_A.
\]
Substituting this equality and (9.2) into (7.6) results in (9.9).

Remark 9.12. Both (d) and (e) still hold if $Q$ quasicontains $A$, cf. Remark 9.3.
10. Comments

1. The concepts of inner and outer balayage of $\mu \in \mathcal{E}^+$ to $A \subset X$, introduced by Definitions 4.1 and 9.1 respectively, and further clarified by a number of subsequent assertions, are in agreement with Cartan’s concepts of inner and outer Newtonian balayage on $\mathbb{R}^n$, $n \geq 3$ (cf. [8, Section 19, Theorems 1, 1']).

2. If the space $X$ has a countable base of open sets, then another approach to outer balayage of $\mu \in \mathcal{E}^+$ to arbitrary $A \subset X$ was suggested by Fuglede [18, Theorem 4.15]. The outer balayage to a set $A$ was defined there as that to its quasiclosure (for the concept of quasiclosure, see [16, Section 2.8]). Our approach to outer balayage is relevant to a wider class of l.c. spaces $X$ though being limited only to Borel sets $A$. But when these two approaches can be applied simultaneously, they turned out to be equivalent. (A concept of inner balayage, basic to the present study, was not considered in [18].)

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Indeed, according to [4] Section IX.2, Corollary to Proposition 16], a l.c. space $X$ has a countable base of open sets if and only if it is metrizable and of class $K_\sigma$. Being therefore metrizable, a second-countable, l.c. space $X$ must be perfectly normal [4 Section IX.1, Proposition 2], whereas the converse is false in general [4 Section IX.2, Exercise 13(b)].
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