QUARK MASS EFFECTS IN FERMIONIC DECAYS
OF THE HIGGS BOSON IN $O(\alpha_s^2)$ PERTURBATIVE QCD

Levan R. Surguladze
Institute of Theoretical Science, University of Oregon
Eugene, OR 97403, USA

Abstract

The results of analytical evaluation of $O(\alpha_s^2)$ QCD contributions due to the nonvanishing quark masses to $\Gamma_{H\rightarrow q\bar{q}J}$ are presented. The “triangle anomaly” type contributions are included. As a byproduct the $O(\alpha_s^3)$ logarithmic contributions are evaluated. The results are presented both in terms of running and pole quark masses. The partial decay modes $H\rightarrow b\bar{b}$ and $H\rightarrow c\bar{c}$ are considered. The calculated corrections decrease the absolute value of large and negative $O(\alpha_s^2)$ massless limit coefficient by $\leq 1\%$ in the intermediate mass region and by $1\%$–$20\%$ in the low mass region which, however, is experimentally ruled out. The results are relevant for $H\rightarrow t\bar{t}$ decay mode for the higher Higgs mass region where the mass effects are large and important. The high order corrections remove a very large discrepancy between the results for $\Gamma_{H\rightarrow q\bar{q}J}$ in terms of running and pole quark masses almost completely and reduce the scale dependence from about $40\%$ to nearly $5\%$. The remaining theoretical uncertainties are discussed.

1. Introduction

The discovery of the Higgs boson, which would be a crucial confirmation of the Standard Model (SM), is one of the primary goals of modern high energy physics [1]. The unsuccessful search of the Higgs particle ($H$) at LEP 100 has established the present lower bound on its mass: $M_H > 63.5$ GeV (95\% c.l.). On the other hand, the theory is able to provide only a broad range of the allowed Higgs mass with the upper limit about $9M_W - (8\pi\sqrt{2}/3G_F)^{1/2} \approx 1$ TeV (see, e.g., [2]). However, if the standard spontaneous symmetry breaking mechanism is correct, then the SM Higgs particle is expected to show up at LEP 200 or/and LHC (for the review of Higgs search at LEP see, e.g., [3]) and possibly at the upgraded Fermilab Tevatron [4].

Prior to the experimental discovery, important constraints on the Higgs properties and on its mass can be obtained when the loop radiative corrections are taken into account. Moreover, in the case of discovery of the Higgs boson, the precise evaluation of the characteristics of its
production and decay processes from the first principles of the theory would be important in order to distinguish the SM Higgs from its extended theoretical versions. (For the analysis of the loop radiative effects in the SM Higgs phenomenology see [2].)

The explicit calculations show that the high order radiative effects are large and important. For example, the calculated QCD corrections at $M_H \sim 120$ GeV reduce the Born result for $H \to c\bar{c}$ by about 20% and for the $b\bar{b}$ mode by about 42% (see, e.g., [2] and references therein). There are similar size effects for $H \rightarrow q\bar{q}$ and even larger for $pp \rightarrow H + X$.

The leading order QCD correction to the decay width $H \rightarrow q\bar{q}$, where $q$ is the quark with flavor $f$ and mass $m_f$, has been calculated by several groups [5]-[7]. In [8] this correction has been evaluated in the zero quark mass limit. The $O(\alpha_s^2)$ QCD correction to the $\Gamma_{H \rightarrow q\bar{q}}$ has been computed in [9] (see also [10]) in the zero quark mass limit and the results were presented in terms of running quark mass. As was mentioned, although the QCD corrections depend on $M_H$, their impact on the decay width is very large for the wide range of the Higgs mass. Indeed, the massless limit $(m_f/M_H \to 0)$ three-loop correction [14, 15] is about 10% of the Born result and 30% of the leading order QCD correction (with \(\alpha_s(M_H) \approx 0.15\)). On the other hand, presently there are no analytical methods to carry out the complete three-loop calculations involving massive particles (as an exceptions see [11, 12]). However, the contributions due to the nonvanishing quark masses might be important for modeling the behavior near the flavor thresholds including the $t\bar{t}$ threshold. Moreover, the possibility exists that the large $O(\alpha_s^2)$ QCD correction [9, 10] is an artifact of the limit $m_f \to 0$.

In the present work the $O(\alpha_s^2)$ corrections $\sim m_f^2/M_H^2$ ($f = u, d, s, c, b, t$) to the decay width $\Gamma_{H \rightarrow q\bar{q}}$ are calculated. The results are obtained both in terms of running and pole quark masses. The calculated $\sim m_f^2/M_H^2$ corrections are not entirely negligible in the intermediate and low Higgs mass regions. For instance, at $M_H = (100, 40, 20)$ GeV the $O(\alpha_s^2)$ mass corrections increase the previous result for $\Gamma_{H \rightarrow b\bar{b}}$ by about (1%, 5%, 23%). However, the low Higgs mass region is experimentally ruled out [13]. On the other hand, the mass corrections for the $t\bar{t}$ decay mode in the high Higgs mass region are very large.

2. Preliminary Relations. The standard $SU(2) \times U(1)$ Lagrangian density of fermion-Higgs interaction is:

$$L = -g_Y q_f H = -\sqrt{2} G_F \frac{1}{2} m_f q_f H = -\sqrt{2} G_F \frac{1}{2} j_f H$$

The two-point correlation function of the scalar currents $j_f = m_f \bar{q}_f q_f$ has the following form:

$$\Pi(Q^2 = -s, m_f) = i \int e^{i q x} T j_f(x) j_f(0) >_0 d^4 x$$

The decay width can be expressed as the imaginary part of $\Pi(s + i0, m_f)$ in the standard way:

$$\Gamma_{H \rightarrow q\bar{q}} = \frac{\sqrt{2} G_F}{M_H} m \Pi(s + i0, m_f) \bigg|_{s = M_H^2}$$

2
The total hadronic decay width will be the sum over all participating (depending on $M_H$) quark flavors:

$$\Gamma_{H \rightarrow \text{hadrons}}^{\text{tot}} = \sum_{f = u,d,s,...} \Gamma_{H \rightarrow q_f \bar{q}_f}$$

For the experimentally accessible quantity - the branching fraction of the particular decay mode one has:

$$BR(H \rightarrow q_f \bar{q}_f) = \frac{\Gamma_{H \rightarrow q_f \bar{q}_f}}{\Gamma_{H \rightarrow \text{hadrons}}^{\text{tot}} + \Gamma_{H \rightarrow \tau^+ \tau^-}}$$

The full $O(\alpha_s)$ analytical result for the decay rate of $H \rightarrow q_f \bar{q}_f$ in terms of pole quark masses looks like [5]-[7]:

$$\Gamma_{H \rightarrow q_f \bar{q}_f} = \frac{3\sqrt{2}G_F M_H}{8\pi} m_f^2 \left(1 - \frac{4m_f^2}{M_H^2}\right)^{\frac{3}{2}} \left[1 + \frac{\alpha_s(M_H)}{\pi} \delta^{(1)} \left(\frac{m_f^2}{M_H^2}\right) + O(\alpha_s^2)\right]$$

where:

$$\delta^{(1)} = \frac{4}{3} \left[\frac{a(\beta)}{\beta} + \frac{3 + 34\beta^2 - 13\beta^4}{16\beta^3} \log \gamma + \frac{21\beta^2 - 3}{8\beta^2}\right]$$

$$a(\beta) = (1 + \beta^2) \left[4Li_2(\gamma^{-1}) + 2Li_2(-\gamma^{-1}) - \log \gamma \log \frac{8\beta^2}{(1 + \beta)^2}\right] - \beta \log \frac{64\beta^4}{(1 - \beta^2)^3}$$

$$\gamma = (1 + \beta)/(1 - \beta), \quad \beta = (1 - 4m_f^2/M_H^2)^{1/2}$$

and the Spence function is defined as usual:

$$Li_2(x) = -\int_0^x dx \frac{\log(1 - x)}{x} = \sum_{n=1}^{\infty} \frac{x^n}{n^2}$$

The expansion of the r.h.s of eq.(3) in a power series in terms of small $m_f^2/M_H^2$ has the following form:

$$\Gamma_{H \rightarrow q_f \bar{q}_f} = \frac{3\sqrt{2}G_F M_H}{8\pi} m_f^2 \left\{\left(1 - \frac{6m_f^2}{M_H^2} + \ldots\right) + \frac{\alpha_s(M_H)}{\pi} \left[3 - 2\log \frac{M_H^2}{m_f^2} - \frac{m_f^2}{M_H^2} \left(8 - 24\log \frac{M_H^2}{m_f^2}\right) + \ldots\right] + O(\alpha_s^2)\right\}$$

where the period covers high order terms $\sim (m_f/M_H)^{2k}, k = 2, 3,...$.

3. O($\alpha_s^2$) QCD contribution to $\Gamma_{H \rightarrow q_f \bar{q}_f}$. The expansion of the full two-point correlation function (3) in powers of $m_f^2/Q^2$ in the “deep” Euclidean region ($Q^2 \gg m_f^2$) has the following form:

$$\frac{1}{m_f^2 Q^2} \Pi(Q^2, m_f, m_v) = \Pi_1(Q^2) + \frac{m_f^2}{Q^2} \Pi_{m_f^2}(Q^2) + \sum_{v=u,d,s,c,b} \frac{m_v^2}{Q^2} \Pi_{m_v^2}(Q^2) + \ldots$$

The last term in the above expansion is due to the certain topological types of three-loop diagrams containing virtual fermionic loop. In fact, in these loops the virtual top quark can also appear. This issue will be discussed later.

3
In order to evaluate the coefficient functions on the r.h.s of eq.(8), it is sufficient to write the diagrammatic representation for the $\Pi(Q^2, m_f^B, m_v^B)$ up to the desired level of perturbation theory and apply the appropriate projector. To $O(\alpha_s^2)$ one has:

$$\Pi_{m_f^2 m_v^2}(Q^2) = \frac{1}{(2n)!(2k)!} \left(\frac{d}{dm_f^B}\right)^{2n} \left(\frac{d}{dm_v^B}\right)^{2k} \left\{\Pi(Q^2, m_f^B, m_v^B)\right\}_{m_f^B=m_v^B=0}$$

(9)

where $n, k = 0, 1$; $n + k \leq 1$ and superscript “$B$” denotes the bare quantities.

The one-, two- and some of the typical three-loop diagrams contributing to the $\Pi_i$ are pictured on the fig.1.

The obtained expressions for $\Pi_i$ at each order of $\alpha_s$ are the polynomials with respect to $1/\varepsilon$ and $\log \mu_{\overline{MS}}^2/Q^2$. However, there are no terms like $(1/\varepsilon^n)(\log \mu_{\overline{MS}}^2/Q^2)^k$. They will appear only at higher orders $\sim m_f^2 m_v^4/Q^4$ and represent the infrared mass logarithms. The poles can be removed by an additive renormalization. The imaginary part of each of the $\Pi_i$ is finite after the
regularization is removed. Here the usual way \[9, 10\] of the introduction of Adler $D$-function \[18\] (the $\Pi$-function differentiated with respect of $Q^2$) which is finite and renormalization group (RG) invariant, can be avoided. Instead, one can operate directly with the correlation function $\Pi_i$. Namely, one should analytically continue it from Euclidean to Minkowski space and take the imaginary part at $s = M_H^2$ (eq.(3)). One obtains the following analytical result for the standard QCD with $SU_c(3)$ gauge group:

$$
\Gamma_{H \rightarrow q_f q_f} = \frac{3\sqrt{2}G_F M_H}{8\pi} m_f^2 \left\{ 1 + \frac{\alpha_s}{\pi} \left( \frac{17}{3} + 2 \log \frac{\mu^2}{M_H^2} \right) 
+ \left( \frac{\alpha_s}{\pi} \right)^2 \left[ \frac{10801}{144} - \frac{19}{2} \zeta(2) - \frac{39}{2} \zeta(3) + \frac{106}{3} \log \frac{\mu^2}{M_H^2} + \frac{19}{4} \log^2 \frac{\mu^2}{M_H^2} 
- N \left( \frac{65}{24} - \frac{1}{3} \zeta(2) - \frac{2}{3} \zeta(3) + \frac{11}{9} \log \frac{\mu^2}{M_H^2} + \frac{1}{12} \log^2 \frac{\mu^2}{M_H^2} \right) \right]
- \frac{m_f^2}{M_H^2} \left( 6 + \frac{\alpha_s}{\pi} \left( 40 + 24 \log \frac{\mu^2}{M_H^2} \right) \right)
+ \left( \frac{\alpha_s}{\pi} \right)^2 \left[ \frac{2383}{4} - 162 \zeta(2) - 166 \zeta(3) + 371 \log \frac{\mu^2}{M_H^2} + 81 \log^2 \frac{\mu^2}{M_H^2} 
- N \left( \frac{313}{18} - 4 \zeta(2) - 4 \zeta(3) + 10 \log \frac{\mu^2}{M_H^2} + 2 \log^2 \frac{\mu^2}{M_H^2} \right) \right] 
+ \left( \frac{\alpha_s}{\pi} \right)^2 \sum_{v=u,d,s,c,b} \frac{m_v^2}{M_H^2} \right\} \right \}

(12)

where the Riemann function $\zeta(2) = \pi^2/6$ arose from the analytical continuation of $\log^3 \mu^2/\mu^2_{\overline{MS}}$ terms and $\zeta(3) = 1.202056903$. The last term in eq.(12) represents the contributions from the three-loop diagrams in fig.1 containing the virtual quark loop and the “triangle anomaly” type corrections from the graphs in fig.2. Their contribution vanishes in the massless quark limit.

\[fig.2\] The “triangle anomaly” type diagrams.

The leading three-loop term $\sim m_f^2$ (the massless approximation) coincides with the one obtained in \[9,10\], while the three-loop results $\sim m_f^4, m_f^4 m_v^2$ are new.

4. **Renormalization group analysis.** An observable quantity, in particular the calculated decay width is invariant under the RG transformations and obeys the homogeneous
The solution of the RG eq. (13) can conveniently be written as follows:

\[ \Gamma_{H \rightarrow q_n \bar{q}_f} = \Gamma_0 m_f^2(\mu) \sum_{0 \leq i \leq j \leq i} \left( \frac{\alpha_s(\mu)}{\pi} \right)^i \log^j \frac{\mu}{M_H^2} \left( a_{ij} \right) + \sum_v \frac{m_v^2(\mu)}{M_H^2} c_{ij} \]

where the coefficients \( a_{ij}, b_{ij} \) and \( c_{ij} \) are the same as the ones in eq. (14). Applying the differential operator \( \mu^2 d/d\mu^2 \) to the both sides of the eq. (18) and taking into account the RG-invariance of \( \Gamma_{H \rightarrow q_n \bar{q}_f} \) and the eqs. (14), one obtains at the \( O(\alpha_s) \):

\[ a_{11} = 2\gamma_0 a_{00}, \quad b_{11} = 4\gamma_0 b_{00}, \]

at the \( O(\alpha_s^2) \):

\[ a_{21} = 2\gamma_1 a_{00} + (\beta_0 + 2\gamma_0) a_{10}, \]
\[ a_{22} = (\beta_0 + 2\gamma_0) a_{11}/2 = (\beta_0 + 2\gamma_0) \gamma_0 a_{00}, \]
\[ b_{21} = 4\gamma_1 b_{00} + (\beta_0 + 4\gamma_0) b_{10}, \]
\[ b_{22} = (\beta_0 + 4\gamma_0) b_{11}/2 = 2(\beta_0 + 4\gamma_0) \gamma_0 b_{00}, \]

and at the \( O(\alpha_s^3) \):

\[ a_{31} = 2(\beta_0 + \gamma_0) a_{20} + (\beta_1 + 2\gamma_1) a_{10} + 2\gamma_2 a_{00}, \]
\[ a_{32} = (\beta_0 + \gamma_0) a_{21} + (\beta_1 + 2\gamma_1) a_{11}/2 = (\beta_0 + \gamma_0) [2\gamma_1 a_{00} + (\beta_0 + 2\gamma_0) a_{10}] + (\beta_1 + 2\gamma_1) \gamma_0 a_{00}, \]
\[ a_{33} = 2(\beta_0 + \gamma_0) a_{22}/3 = 2\gamma_0 (\beta_0 + \gamma_0) (\beta_0 + 2\gamma_0) a_{00}/3, \]

The known \( \beta \)-function coefficients are [19]:

\[ \beta_0 = \frac{1}{4} \left( 11 - \frac{2}{3} N \right), \quad \beta_1 = \frac{1}{16} \left( 102 - \frac{38}{3} N \right), \quad \beta_2 = \frac{1}{64} \left( \frac{2857}{2} - \frac{5033}{18} N + \frac{325}{54} N^2 \right) \]

and for the \( \gamma_m(\alpha_s) \) one has [20]:

\[ \gamma_0 = 1, \quad \gamma_1 = \frac{1}{16} \left( \frac{202}{3} - \frac{20}{9} N \right), \quad \gamma_2 = \frac{1}{64} \left[ 1249 - \left( \frac{2216}{27} + \frac{160}{3} \zeta(3) \right) N - \frac{140}{81} N^2 \right]. \]
\[ b_{31} = 2(\beta_0 + 2\gamma_0)b_{20} + (\beta_1 + 4\gamma_1)b_{10} + 4\gamma_2b_{00}. \]
\[ b_{32} = (\beta_0 + 2\gamma_0)b_{21} + (\beta_1 + 4\gamma_1)\frac{b_{11}}{2} = (\beta_0 + 2\gamma_0)[4\gamma_1b_{00} + (\beta_0 + 4\gamma_0)b_{10}] + (\beta_1 + 4\gamma_1)2\gamma_0b_{00}, \]
\[ b_{33} = 2(\beta_0 + 2\gamma_0)b_{22}/3 = 4\gamma_0(\beta_0 + 2\gamma_0)(\beta_0 + 4\gamma_0)b_{00}/3. \]
\[ c_{31} = 2(\beta_0 + 2\gamma_0)c_{20} \]

Since the diagrams with the virtual fermionic loop first appear at \(O(\alpha_s^2)\), \(c_{00} = c_{1j} = c_{21} = c_{22} = 0\). The relations (19,20) provide a powerful check of the \(O(\alpha_s^2)\) calculations, while the relations (21) allow one to evaluate the log-terms at \(O(\alpha_s^2)\), without explicit calculations of the corresponding four-loop diagrams. With those relations, the information available at present, namely the QCD \(\beta\)-function, mass anomalous dimension and the 2-point correlation function up to the corresponding order, is fully exploited. In fact, the similar relations can be derived for the correlation function \(\Pi\). However, the RG-equation for \(\Pi\) is not a homogeneous one and the anomalous dimension function up to the three-loop level is needed. In fact, the similar relations can be derived for the correlation function \(\Pi\). However, the RG-equation for \(\Pi\) is not a homogeneous one and the anomalous dimension function up to the corresponding order of \(\alpha_s\) is needed.

The solution of the RG-equation (13) at \(\mu_{\overline{MS}}^2 = M_H^2\) has the following form:

\[ \Gamma_{H \rightarrow q_j\pi_f} = \frac{3\sqrt{2}G_F M_H}{8\pi} m_f^2(M_H) \left( 1 - 6 \frac{m_f^2(M_H)}{M_H^2} + \frac{\alpha_s(M_H)}{\pi} (5.66667 - 40 \frac{m_f^2(M_H)}{M_H^2}) \right) \]
\[ + \left( \frac{\alpha_s(M_H)}{\pi} \right)^2 \left[ 35.93996 - 1.35865N - \frac{m_f^2(M_H)}{M_H^2}(129.72924 - 6.00093N) \right] + 12 \sum_{v=u,d,s,c,b} m_v^2(M_H)/M_H^2 \]  \( (22) \)

The running coupling is parametrized as follows:

\[ \frac{\alpha_s(M_H^2)}{\pi} = \frac{1}{\beta_0 L} - \frac{1}{\beta_0^3 L^2} L \left( \beta_1^2 \log^2 L - \beta_1^2 \log L + \beta_2 - \beta_2^2 \right) + O(L^{-4}), \]  \( (23) \)

where \(L = \log(M_H^2/\Lambda_{\overline{MS}}^2)\). For the running mass one has:

\[ m_f(\mu_1)/m_f(\mu_2) = \phi(\alpha_s(\mu_1))/\phi(\alpha_s(\mu_2)), \]  \( (24) \)

where,

\[ \phi(\alpha_s(\mu)) = \left( 2\beta_0 \frac{\alpha_s(\mu)}{\pi} \right)^{\gamma_0} \left( 1 + \frac{\gamma_1}{\beta_0} - \frac{\beta_1}{\beta_0^2} \right) \frac{\alpha_s(\mu)}{\pi} \]
\[ + \frac{1}{2} \left( \frac{\gamma_1}{\beta_0} - \frac{\beta_1}{\beta_0^2} \right)^2 + \frac{\gamma_2}{\beta_0} - \frac{\beta_1 \gamma_1}{\beta_0^2} - \frac{\beta_2 \gamma_0}{\beta_0^2} + \frac{\beta_1^2 \gamma_0}{\beta_0^2} \]  \( (25) \)

In the eqs. (22)-(25) all appropriate quantities are evaluated for the \(N\) active quark flavors. \(N\) can be determined according to the scale of \(M_H\). At present one usually considers \(N = 5\).

5. \( \Gamma_{H \rightarrow q_j\pi_f} \) in terms of pole quark mass. For the heavy flavor decay mode of the Higgs, it is relevant to parametrize the decay rate in terms of pole quark mass (see, e.g., [2]). Below
the result (23) will be rewritten in terms of pole quark mass, assuming that heavy quark is not exactly on-shell.

First, one derives the following general evolution equation for the running coupling to \( O(\alpha_s^3) \):

\[
\frac{\alpha_s^{(n)}(\mu)}{\pi} = \frac{\alpha_s^{(N)}(M)}{\pi} + \left( \frac{\alpha_s^{(N)}(M)}{\pi} \right)^2 \left( \beta_0^{(N)} \log \frac{M^2}{\mu^2} + \frac{1}{6} \sum_l \log \frac{m_l^2}{\mu^2} \right) + \left( \frac{\alpha_s^{(N)}(M)}{\pi} \right)^3 \left[ \beta_1^{(N)} \log \frac{M^2}{\mu^2} + \frac{19}{24} \sum_l \log \frac{m_l^2}{\mu^2} + \left( \beta_0^{(N)} \log \frac{M^2}{\mu^2} + \frac{1}{6} \sum_l \log \frac{m_l^2}{\mu^2} \right)^2 - \frac{25}{72} (N - n) \right]
\] (26)

where the superscript \( n \) (\( N \)) indicates that the corresponding quantity is evaluated for \( n \) (\( N \)) numbers of participating quark flavors. Conventionally (see, e.g., [21]), \( n \) (\( N \)) is specified to be the number of quark flavors with mass \( \leq \mu \) (\( \leq M \)). However, the eq.(26) is relevant for any \( n \leq N \) and arbitrary \( \mu \) and \( M \), regardless the conventional specification of the number of quark flavors. The log \( m_l/\mu \) terms are due to the “quark threshold” crossing effects and the constant coefficients \( 1/6 = \beta_0^{(k-1)} - \beta_0^{(k)} \), \( 19/24 = \beta_1^{(k-1)} - \beta_1^{(k)} \) represent the contributions of the quark loop in the \( \beta \)-function. The sum runs over \( N - n \) quark flavors (e.g., \( l = 6 \) if \( n = 4 \) and \( N = 5 \)). Note that \( m_l \) is the pole mass of the quark with flavor \( l \). For the on-shell definition of the quark masses the eq.(23) changes - the constant \(-25/72\) should be substituted by \(+7/72\). The above equation is derived based on the eq.(23), the QCD matching conditions for \( \alpha_s \) at “quark thresholds” [22, 23] and the one-loop relation between on-shell and pole quark masses. The eq.(26) is consistent with QCD matching relation at \( m_f(m_f) \) [28] (see also [24]):

\[
\alpha_s^{(N_f-1)}(m_f(m_f)) = \alpha_s^{(N_f)}(m_f(m_f)) + (\alpha_s^{(N_f)}(m_f(m_f)))^2 (C_A/9 - 17C_F/96)/\pi^2
\] (27)

Here and below \( N_f \) is the number of quark flavors \( u,d,...,f \). Note that the nonlogarithmic constant at \( O(\alpha_s^2) \) in eq.(26) will not contribute in further analysis.

Next, using the scaling properties of the \( MS \) running mass and the eq.(26), one obtains the following matching condition:

\[
m_f^{(N-1)}(\mu) = m_f^{(N)}(\mu) \left\{ 1 + \left( \frac{\alpha_s^{(N)}(\mu)}{\pi} \right)^2 \left[ \delta(m_f,m_f') - \frac{5}{36} \log \frac{\mu^2}{m_f^2} - \frac{1}{12} \log^2 \frac{\mu^2}{m_f^2} \right] + \frac{1}{6} \log \frac{\mu^2}{m_f^2} \log \frac{\mu^2}{m_f'} - \frac{2}{9} \log \frac{m_f^2}{m_f'} \right\}
\] (28)

where the constant terms are: \( 1/12 = \gamma_0(\beta_0^{(k-1)} - \beta_0^{(k)})/2 \), \( 5/36 = \gamma_1^{(k-1)} - \gamma_1^{(k)} \) and \( 2/9 = C_F(\beta_0^{(k-1)} - \beta_0^{(k)}) \). In general the \( \delta(m_f,m_f') \) is the finite contribution of the single virtual heavier quark with mass \( m_f' \), entering when one increases the number of flavors from \( N - 1 \) to \( N \) (one can also consider the particular case \( m_f' = m_f \)). From the two-loop on-shell quark mass renormalization one has [23]:

\[
\delta(m_f,m_f') = -\zeta(2)/3 - 71/144 + (4/3) \Delta(m_f/m_f)
\] (29)

where

\[
\Delta(r) = \frac{1}{4} \left[ \log^2 r + \zeta(2) - \left( \log r + \frac{3}{2} \right)^2 - (1+r)(1+r^3)L_+(r) - (1-r)(1-r^3)L_-(r) \right]
\] (30)
\[ L_{\pm}(r) = \int_{0}^{1/r} dx \frac{\log x}{x \pm 1}. \]

\( L_{\pm}(r) \) can be evaluated for different quark mass ratios \( r \) numerically (table 1).

One can relate the \( \overline{MS} \) quark mass \( m_f(m_f) \) to the pole mass \( m_f \) using the \( O(\alpha_s^2) \) on-shell results of [25]:

\[ m_f^{(N)}(m_f) = m_f[1 - 4 \alpha_s^{(N)}(m_f)/3 + (16/9 - K_f)(\alpha_s^{(N)}(m_f)/\pi)^2], \]  

where

\[ K_f = \frac{3817}{288} + \frac{2}{3}(2 + \log 2)\zeta(2) - \frac{1}{6} \zeta(3) - \frac{N_f}{3} \left( \zeta(2) + \frac{71}{48} \right) + \frac{4}{3} \sum_{m_i \leq m_f} \Delta \left( \frac{m_i}{m_f} \right). \]  

The first four terms in \( K_f \) represent the QCD contribution with \( N_f \) massless quarks, while the sum is the correction due to the \( N_f \) nonvanishing quark masses.

Combining the eqs. (24,25) and the eqs. (26)-(31), one obtains the relation between the pole quark mass \( m_f^{(N)}(M_H) \) renormalized at \( M_H \) and evaluated for the \( N \)-flavor theory and the pole quark mass \( m_f^{(N)}(M_H) = m_f \left\{ 1 - \frac{\alpha_s^{(N)}(M_H)}{\pi} \left[ \frac{4}{3} + \gamma_0 \log \frac{M_H^2}{m_f^2} \right] - \frac{\alpha_s^{(N)}(M_H)}{\pi} \right\} \left[ K_f + \sum_{m_f < m_f' < M_H} \delta(m_f, m_f') \right. \]  

\[ \left. - \frac{16}{9} + \left( \gamma_1^{(N)} - \frac{4}{3} \gamma_0 + \frac{4}{3} \beta_0^{(N)} \right) \log \frac{M_H^2}{m_f^2} + \frac{\gamma_0}{2} (\beta_0^{(N)} - \gamma_0) \log \frac{M_H^2}{m_f^2} \right\}. \]  

Note that \( N \) is specified according to the size of \( M_H \) and has no correlation with the quark mass \( m_f \). Thus, for instance, one can apply the eq.(33) to the charm mass \( m_c^{(5)}(M_H) \) evaluated for five-flavor theory.

Substituting eqs. (33,32,17,16) into the eq.(22), one obtains the general form for the decay rate \( \Gamma_{H\to q\bar{q}} \) in terms of the pole quark masses:

\[ \Gamma_{H\to q\bar{q}} = \frac{3\sqrt{2}G_F M_H m_f^2}{8\pi} \left\{ 1 - 6 \frac{m_f^2}{M_H^2} + \frac{\alpha_s^{(N)}(M_H)}{\pi} \left[ 3 - 2 \log \frac{M_H^2}{m_f^2} - \frac{m_f^2}{M_H^2} \left( 8 - 24 \log \frac{M_H^2}{m_f^2} \right) \right] \right. \]  

\[ + \left( \frac{\alpha_s^{(N)}(M_H)}{\pi} \right)^2 \left[ \frac{697}{6} + \frac{3}{2} \log 2 \right] \left( \zeta(2) - \frac{115}{6} \zeta(3) \right) - \frac{m_f^2}{M_H^2} \left[ 171 - (194 + 16 \log 2) \zeta(2) - 162 \zeta(3) \right] - \frac{50}{9} (12 \zeta(2) - 4 \zeta(3)) \right) \]  

\[ \left. - \frac{87}{6} - \frac{13}{18} N - \frac{m_f^2}{M_H^2} \left[ 221 - \frac{26}{3} N \right] \log \frac{M_H^2}{m_f^2} - \left[ 3 \frac{1}{4} - \frac{1}{6} N + \frac{m_f^2}{M_H^2} (15 + 2N) \right] \log M_H^2 \right\} \]  

\[ - \frac{8}{3} - 32 \frac{m_f^2}{M_H^2} \sum_{m_i < M_H} \Delta \left( \frac{m_i}{m_f} \right) + 12 \sum_{m_i < M_H} \frac{m_f^2}{M_H^2} \right\}. \]  

The above result confirms the asymptotic form (3) of the two-loop exact result (3), while the \( O(\alpha_s^2) \) expression is new.
The numerical values of $\Delta(\frac{m_l}{m_f})$ defined in the eq.(30) are given in the table 1. Quark masses can be estimated from QCD sum rules: $m_b = 4.72\text{GeV}$, $m_c = 1.46\text{GeV}$, $m_s = 0.27\text{GeV}$, $m_u + m_d \approx m_s / 13 \approx 0.02\text{GeV}$. For the upper bound of the top mass the latest CDF result $m_t \leq 174 + 10 + 13 \text{GeV}$ is used. On the other hand, for the lower bound more conservative value $m_t > 131 \text{GeV}$ given by D0 is used. The values of $\Delta(\frac{m_l}{m_t})$ are given in the table 1 for both the lower and the upper bounds of the top mass.

Table 1. The numerical values of $\Delta(\frac{m_l}{m_f})$.

| $f$-flavor | $l$-flavor | $\Delta(\frac{m_l}{m_f})$ |
|------------|------------|---------------------------|
| $t$        | $u$ or $d$ | $\leq 0.0001$             |
| $t$        | $s$        | 0.0025-0.0017              |
| $t$        | $c$        | 0.0137-0.0091              |
| $t$        | $b$        | 0.0435-0.0291              |
| $t$        | $t$        | 0.8587                     |
| $b$        | $u$ or $d$ | 0.0027                     |
| $b$        | $s$        | 0.0673                     |
| $b$        | $c$        | 0.3290                     |
| $b$        | $b$        | 0.8587                     |
| $c$        | $u$ or $d$ | 0.0084                     |
| $c$        | $s$        | 0.2045                     |
| $c$        | $c$        | 0.8587                     |
| $c$        | $b$        | 1.9015                     |
| $s$        | $u$ or $d$ | 0.0454                     |
| $s$        | $s$        | 0.8587                     |
| $s$        | $c$        | 2.5767                     |
| $s$        | $b$        | 4.5605                     |

In order to achieve better numerical precision within the wider range of $M_H$ it is more appropriate to use full one- and two-loop results (eq.(30)) and the mass corrected three-loop result given by the eq.(34). For the dominant decay mode $H \to b\bar{b}$ with five participating quark flavors one obtains:

$$
\Gamma_{H \to b\bar{b}} = \frac{3\sqrt{2}G_FM_H}{8\pi}m_b^2\left\{ \left(1 - \frac{4m_b^2}{M_H^2}\right)^{\frac{3}{2}} + \frac{\alpha_5^{(5)}(M_H)}{\pi}\delta^{(1)} \left(1 - \frac{4m_b^2}{M_H^2}\right)^{\frac{3}{2}} 
\right\}
$$

$$
+ \left(\frac{\alpha_5^{(5)}(M_H)}{\pi}\right)^2\left[ -2.23039 + 266.13393\frac{m_b^2}{M_H^2} 
- \left(18.13889 - 177.66667\frac{m_b^2}{M_H^2}\right)\log\frac{M_H^2}{m_b^2} + \left(0.08333 - 25\frac{m_b^2}{M_H^2}\right)\log^2\frac{M_H^2}{m_b^2} 
- \left(2.66667 - 32\frac{m_b^2}{M_H^2}\right)\sum_{m_l \leq m_b}\Delta\left(\frac{m_l}{m_b}\right) + 12\sum_{m_v \leq m_b}\frac{m_v^2}{M_H^2}\right] 
$$

(35)

where $\delta^{(1)}$ is defined in eq.(3).
At the present LEP lower bound on Higgs mass $M_H \approx 60\text{GeV}$ one has:

$$
\Gamma_{H \rightarrow b\bar{b}} = \frac{3\sqrt{2}G_F M_H}{8\pi} m_b^2 \left\{ (1 - 0.037) + \frac{\alpha_s^{(5)}(M_H)}{\pi} (-7.17 + 0.71) + \left( \frac{\alpha_s^{(5)}(M_H)}{\pi} \right)^2 (-95.67 + 3.57) \right\}
$$

where the second number in each bracket corresponds to the leading order quark mass correction, while the first one is the massless contribution. One can see that the mass corrections always decrease the large massless contribution. For the lower values of $M_H$ the relative mass corrections increase rapidly. For example, at $M_H = 40\text{GeV}$ the $O(\alpha_s^2)$ mass correction decrease the large and negative massless coefficient by more than 10% and correspondingly increase the decay rate by about 5%.

Finally, one also needs to include weak and electromagnetic contributions [32, 2]. However, they will not affect further analysis here.

The dependence of the partial decay rate $\Gamma_{H \rightarrow b\bar{b}}$ on the mass of the Higgs particle $M_H$ is plotted in fig.3.

---

**fig.3** The partial decay rate $\Gamma_{H \rightarrow b\bar{b}}$ vs. Higgs mass

In the region of $M_H$ shown in the fig.3 the three-loop mass corrections are small and does not affect the overall picture. However, in the low mass region their impact is significant (fig.4).

To obtain the decay rate $\Gamma_{H \rightarrow c\bar{c}}$ one should substitute $b \rightarrow c$ everywhere in eq.(36) except the summation bounds, which remain the same. The $b$-quark “threshold” crossing effect will be represented by the term $\sim \Delta(m_b/m_c)$.
fig. 4 The partial decay rate $\Gamma_{H \to b\bar{b}}$ vs. Higgs mass (low mass region)

Since the derivation of the eqs. (22,34) has not been restricted by the particular quark flavor, one could apply these results to the $H \to t\bar{t}$ decay channel. The only restriction that apply is that the Higgs mass $M_H$ must be sufficiently larger then $2m_t$ in order the process to be allowed and the expansion in terms of $m_t^2/M_H^2$ to be a legitimate. Note, that in those equations for the $t\bar{t}$ mode one should consider six flavor theory and the summation index $v$ will run over all six quark flavors. Thus, from the eq. (34) at $m_f = m_t$ and $N_f = 6$ one obtains:

$$\Gamma_{H \to t\bar{t}} = \frac{3\sqrt{2}G_F M_H}{8\pi} m_t^2 \left\{ \left(1 - \frac{4m_t^2}{M_H^2}\right)^{\frac{3}{2}} + \frac{\alpha_s^{(6)}(M_H)}{\pi} \delta^{(1)} \left( \frac{m_t^2}{M_H^2} \right) \left( 1 - \frac{4m_t^2}{M_H^2}\right)^{\frac{3}{2}} \right. $$

$$+ \left( \frac{\alpha_s^{(6)}(M_H)}{\pi} \right)^2 \left[ -1.50631 + 247.14204 \frac{m_t^2}{M_H^2} \right.$$  

$$- \left( 17.41667 - 169 \frac{m_t^2}{M_H^2} \right) \log \frac{M_H^2}{m_t^2} + \left( 0.25 - 27 \frac{m_t^2}{M_H^2} \right) \log^2 \frac{M_H^2}{m_t^2} \right.$$  

$$- \left( 2.66667 - 32 \frac{m_t^2}{M_H^2} \right) \sum_{m_i \leq m_t} \Delta \left( \frac{m_i}{m_t} \right) + 12 \sum_{m_i \leq m_t} \frac{m_i^2}{M_H^2} \right\}$$

(37)

The numerical values of $\Delta(m_i/m_t)$ defined in the eq.(31) are given in the table 1. In the above equation the known exact dependence on the top mass at one and two-loop levels is used (see eq.(3)). It is straightforward to express the $\alpha_s^{(6)}$ in terms of $\alpha_s^{(5)}$ using the relation (27). For the conservative lower bound on the top mass $m_t = 131$ GeV and for the Higgs mass at, for instance, $M_H = 400$ GeV one obtains:

1The author is grateful to the referee for Physics Letters B for his suggestion.
\[
\Gamma_{H \rightarrow t \bar{t}} = \frac{3\sqrt{2} G_F M_H}{8\pi} m_t^2 \left\{ (1 - 0.644 + 0.069 + 0.005 + 0.0008 + ...) + \frac{\alpha_s^{(6)}(M_H)}{\pi} (-1.465 + 4.889 - 1.739 - 0.005 - 0.0003 + ...) + \left( \frac{\alpha_s^{(6)}(M_H)}{\pi} \right)^2 (-41.59 + 57.02 + ...) \right\}
\]

(38)

In the above equation at each order of \( \alpha_s^{(6)} \) the expansion in \( m_t^2/M_H^2 \) (at \( m_t = 131 \) GeV and \( M_H = 400 \) GeV) is given. One can see that for the case \( H \rightarrow t \bar{t} \) the mass corrections are important. In fact, the massless limit coefficients (e.g., at \( O(\alpha_s) \)) does not reproduce even a correct sign.

6. Scheme-scale dependence and the theoretical uncertainties. The results for the decay rates are scheme-scale dependent. The present analysis is restricted by the one parametric family of the \( MS \)-type schemes [14]. The ambiguity is due to the dependence on the parameter \( \mu \), which enters via dimensional regularization. (For the similar analysis of the \( \Gamma_{Z \rightarrow \text{hadrons}} \) see [33].) In the solution of the RG eq.(13) one replaces \( \mu^2_{MS} = e^t M_H^2 \) and \( \alpha_s \rightarrow \alpha_s^t \), where the parametrization of \( \alpha_s^t \) is given by the eq.(23) with \( \Lambda_t = e^{-t/2} \Lambda_{MS} \).

fig.5 The approximants of the \( \Gamma_{H \rightarrow b \bar{b}} \) vs the scale parameter \( t \).

The one-, two- and three-loop approximants for the \( \Gamma_{H \rightarrow b \bar{b}} \) in terms of running quark mass (eq.(22), with \( N = 5 \) and \( m_f = m_b \)) vs. the scale parameter \( t \) are plotted in the fig.5. One can see that the higher order corrections diminish the scale dependence from 40% to nearly 5%. The solid curve (corresponding to the three-loop result) became flat in the wide range of the logarithmic scale parameter \( t \). Moreover, the choice \( t = 0 \) (\( MS \)-scheme) satisfies Stevenson’s
“minimal sensitivity” principle [34]. The mass corrections does not change the overall picture. It should be stressed once again that the present consideration is restricted by the MS-type schemes. On the other hand one could carry out the scheme invariant analysis along the lines of [34, 35] (see also [11]).

It is known that the dependence on the unphysical scale parameter is due to the truncation of perturbation series at particular order. If one sums up to “all orders”, the scale dependence will disappear. Thus, one may try to relate the remainder dependence on the scale to the sum of all uncalculated terms and estimate the size of the theoretical uncertainty in the reasonably wide neighborhood of the initial choice of scale (in our case \( t = 0 \)). The deviation of the approximant from the constant is used as a measure of theoretical error. From the fig.5 one estimates the theoretical uncertainty at 5%.

The comparison of fig.3 and fig.5 shows that the high order corrections resolve the very large discrepancy (more then factor 2!) between the results for \( \Gamma_{H \rightarrow b\bar{b}} \) in terms of running and pole quark masses. The remaining discrepancy is about 5%, which is the same as the estimated 5% uncertainty due to the uncalculated higher order terms.

Besides the above estimated theoretical error, there are two more contributions, with the potentially important effects comparable with the present uncertainty. The first most likely non negligible contribution comes from the diagrams in fig.2, with the virtual top quark in the triangle fermionic loop. (The similar diagrams in the case of \( Z \)-boson decay gave sizable correction [11].) The other contribution may come from the diagrams in fig.1 with the virtual top quark. The decoupling theorem may not apply in this case because of possible comparable size of the top and the Higgs masses. The contributions from these diagrams (and those in fig.2) with virtual bottom and lighter quarks are already included in the result. However, in the case of top, one needs to evaluate these diagrams explicitly. The similar diagrams for the \( Z \)-boson decay gave a somewhat moderate but not entirely negligible correction [12]. The evaluation of the higher order mass corrections (\( \sim m_t^6 \) and higher) requires an accurate treatment of infrared divergences. On the other hand those corrections are heavily suppressed by the powers of \( m_f^2/M_H^2 \) and numerically will be negligible. The calculation of the \( O(\alpha_s^3) \) corrections requires the evaluation of relevant four-loop diagrams up to the finite terms in their expansion in Laurent series in \( \varepsilon \) and the method used in [30] to simplify the \( O(\alpha_s^2) \) calculations of the decay rate of \( Z \rightarrow \text{hadrons} \) does not apply. The above discussion of the theoretical uncertainties is mainly for the \( b\bar{b} \) and \( cc \) modes. For the case of \( H \rightarrow t\bar{t} \) the uncertainty should be somewhat higher because of large mass corrections, unless the Higgs mass is in the TeV region.

7. Summary. The quark mass corrections to the decay rate of the SM Higgs particle into the quark-antiquark pair to \( O(\alpha_s^2) \) perturbative QCD are calculated. The previously known \( O(\alpha_s) \) results with explicit mass dependence and the three-loop massless limit result with running quark mass parametrization are independently confirmed. The expression for the decay rate is obtained both with running and pole quark mass parametrizations. It was found that the quark mass corrections are not entirely negligible and they decrease the large and negative massless coefficients. For the decay mode \( H \rightarrow t\bar{t} \) the mass corrections are large
and important. On the other hand, the higher order corrections resolve the large discrepancy (more than factor 2) between the results for the decay width in terms of running and pole quark masses. It was also found that the three-loop QCD correction reduces the scale dependence significantly. The theoretical error of evaluation of the QCD contribution was estimated at 5% for the decay mode $H \rightarrow b\bar{b}$ plus, the possible effects of the virtual heavy top quark are emphasized.

Finally one should mention the recent work [37], where the similar problem without calculating three-loop mass corrections has been discussed. However, the relation between running and pole quark masses used in [37], seems to be incorrect.

**Acknowledgements** The author is grateful to D.E.Soper for discussions and the advice concerning the relation between running and pole quark masses. It is a pleasure to thank D.Broadhurst, B.Kniehl and A. Sirlin for helpful communications and D.Strom for the discussion of the experimental status of the problem at LEP. This work was supported by the U.S. Department of Energy under grant No. DE-FG06-85ER-40224.

**References**

[1] J.F.Gunion, H.E. Haber, G.Kane and S.Dawson, *The Higgs Hunter’s Guide* (Addison-Wesley, Redwood City) 1990;
V. Barger and R.J.N. Phillips, *Collider Physics* (Addison-Wesley, Redwood City) 1987.

[2] B.A.Kniehl, preprint DESY 93-069, 1993; Phys. Rep. (to be published)

[3] E.Gross and P.Yepes, Intern.J.Mod.Phys. A8 (1993) 407.

[4] A.Stange, W.Marciano and S.Willenbrock, preprint ILL-(TH)-94-8, 1994.

[5] E.Braaten and J.P.Leveille, Phys. Rev. D22 (1980) 715.

[6] T.Inami and T.Kubota, Nucl. Phys. B179 (1981) 171.

[7] M.Drees and K.Hikasa, Phys. Rev. D41 (1990) 1547.

[8] N.Sakai, Phys. Rev. D22 (1980) 2220.

[9] S.G.Gorishny, A.L.Kataev, S.A.Larin and L.R.Surguladze, Mod. Phys. Lett. A5 (1990) 2703.

[10] S.G.Gorishny, A.L.Kataev, S.A.Larin and L.R.Surguladze, Phys. Rev. D43 (1991) 1633.

[11] B.A.Kniehl and J.H.Kuhn, Nucl.Phys. B329 (1990) 547.

[12] D.E.Soper and L.R.Surguladze, University of Oregon preprint OITS-545, 1994; Phys. Rev. Lett. (in print).
[13] G.t’Hooft and M.Veltman, Nucl.Phys. B44 (1972) 189.
[14] G.t’Hooft, Nucl.Phys. B61 (1973) 455.
[15] W.Bardeen, A.Buras, D.Duke and T.Muta, Phys.Rev. D18 (1978) 3998.
[16] L.R.Surguladze, FERMILAB-PUB-92/191-T, 1992.
[17] J.A.M.Vermaseren, FORM Manual, (NIKHEP, Amsterdam), 1989.
[18] S.L.Adler, Phys.Rev. D10 (1974) 3714.
[19] O.V.Tarasov, A.A.Vladimirov and A.Yu.Zharkov, Phys. Lett. B93 (1980) 429.
[20] O.V.Tarasov, Dubna preprint JINR P2-82-900, 1982 (unpublished).
[21] W.J.Marciano, Phys.Rev. D29 (1984) 580.
[22] M.R.Barnett, H.E.Haber and D.E.Soper, Nucl.Phys. B306 (1988) 697.
[23] W.Bernreuter and W.Wentzel, Nucl.Phys. B197 (1982) 228.
[24] G.Rodrigo and A.Santamaria, Phys.Lett. B313 (1993) 441.
[25] D.J.Broadhurst, N.Gray and K.Schilcher, Z.Phys. C52 (1991) 111.
[26] C.A.Dominguez and N.Paver, Phys.Lett. 293B (1992) 197.
[27] J.Gasser and H.Leutwyler, Phys.Rep. 87 (1982) 77; see also S. Narison, Phys. Lett. B197 (1987) 405.
[28] C.A.Dominguez, C. van Gend and N.Paver, Phys.Lett. 253B (1991) 241.
[29] J.Gasser and H.Leutwyler, Nucl.Phys. B250 (1985) 465.
[30] F. Abe et al., CDF Collaboration, Fermilab preprint FERMILAB-PUB-94/097-E.
[31] T. Ferbel et al., D0 Collaboration, Fermilab UR-1347, March 1994.
[32] B.A.Kniehl, Nucl. Phys. B376 (1992) 3.
[33] L.R.Surguladze and M.A.Samuel, Phys.Lett. B309 (1993) 157.
[34] P.M.Stevenson, Phys.Lett. B100 (1981) 61; Phys.Rev. D23 (1981) 2916.
[35] A.C.Mattingly and P.M.Stevenson, Phys.Rev. D49 (1994) 437.
[36] L.R.Surguladze and M.A.Samuel, Phys.Rev.Lett. 66 (1991) 560.
[37] A.L.Kataev and V.T.Kim, preprint ENSLAPP-A-407/92, 1992.
This figure "fig1-1.png" is available in "png" format from:

http://arxiv.org/ps/hep-ph/9405325v3
This figure "fig2-1.png" is available in "png" format from:

http://arxiv.org/ps/hep-ph/9405325v3
This figure "fig1-2.png" is available in "png" format from:

http://arxiv.org/ps/hep-ph/9405325v3
This figure "fig2-2.png" is available in "png" format from:

http://arxiv.org/ps/hep-ph/9405325v3
This figure "fig1-3.png" is available in "png" format from:

http://arxiv.org/ps/hep-ph/9405325v3