ON THE COHOMOLOGY OF LOOP SPACES FOR SOME THOM SPACES

ANDREW BAKER

Abstract. In this paper we identify conditions under which the cohomology $H^*(\Omega M\xi; k)$ for the loop space $\Omega M\xi$ of the Thom space $M\xi$ of a spherical fibration $\xi \to B$ can be a polynomial ring. We use the Eilenberg-Moore spectral sequence which has a particularly simple form when the Euler class $e(\xi) \in H^n(B; k)$ vanishes, or equivalently when an orientation class for the Thom space has trivial square. As a consequence of our homological calculations we are able to show that the suspension spectrum $\Sigma\Omega M\xi$ has a local splitting replacing the James splitting of $\Sigma\Omega M\xi$ when $M\xi$ is a suspension.

Introduction

In [1], topological methods were used to prove the algebraic Ditter’s conjecture on quasi-symmetric functions, which is equivalent to the assertion that $H^*(\Omega\Sigma CP^\infty; \mathbb{Z})$ is a polynomial ring (infinitely generated but of finite type). Most of the ingredients of the proof given there are essentially formal within algebraic topology, the exception being James’s splitting of $\Sigma\Omega\Sigma CP^\infty$.

The purpose of this paper is to identify circumstances in which the cohomology $H^*(\Omega M\xi; k)$ of the loop space $\Omega M\xi$ of the Thom space $M\xi$ of a spherical fibration $\xi \to B$ can be a polynomial ring. In place of the James splitting we use the Eilenberg-Moore spectral sequence which has a particularly simple form when the Euler class $e(\xi) \in H^n(B; k)$ vanishes, or equivalently when an orientation class for the Thom space has trivial square. As a consequence of our homological calculations we are able to show that the suspension spectrum $\Sigma\Omega M\xi$ has a local splitting generalizing that for $\Sigma\Omega M\xi$ when $M\xi$ is a suspension. Our results appear to be more general and essentially formal in that only generic properties of the Eilenberg-Moore spectral sequence are used; however, the above stable splitting is a weaker result than the James splitting.

Although our examples are all associated with vector bundles, our methods are valid for arbitrary spherical fibrations, and even more generally they apply to $p$-local or $p$-complete spherical fibrations. We hope to consider examples associated with $p$-compact groups in future work.

We were very influenced by the discussion of the cohomology of $\Omega\Sigma X$ in Smith’s article [15]. Massey’s paper [5] provides a useful background to our work. Although we do not make direct use of it, Ray’s paper [8] has ideas that might allow generalizations to other mapping cones. Although we do not make direct use of the results of these papers, we remark that Bott & Samelson [2] and Petrie [7] gave earlier versions of the arguments we use, however neither paper...
contains the full range of our results; in particular the latter does not deal with questions about multiplicative structure.

1. Thom complexes of spherical fibrations

Let $B$ be space and let $\xi : S^{n-1} \to S \to B$ be a spherical fibration with associated disc bundle $D^n \to D \to B$. The Thom space $M = M \xi$ is the cofibre of the inclusion $S \to D$, i.e., the quotient space $D/S$. In each fibre this corresponds to the inclusion $S^{n-1} \to D^n$ and there is a cofibre sequence of based spaces

$$S_+ \to D_+ \to M \delta \to \Sigma S_+.$$  

Here we implicitly allow for generalizations to include localized spheres as fibres and bundles with structure monoids obtained from the invertible components of $\text{Maps}(S^{n-1}, S^{n-1})$.

We are interested in the based loop space $\Omega M$. There is an obvious unbased map $S \to \Omega M$ which sends $v \in S_b$ (the fibre above $b \in B$) to the non-constant loop $[0, 1] \to M$ given by $t \mapsto [(2t - 1)v]$, running through $b$ parallel to $v$ and passing through the base point at times $t = 0, 1$. This extends to a based map $\theta : S_+ \to \Omega M$. We write $ev : \Sigma \Omega M \to M$ for the evaluation map. See [8] for a related construction.

Our next result is surely standard but we don’t know an explicit reference.

**Lemma 1.1.** The composition

$$M \delta \to \Sigma S_+ \xrightarrow{\Sigma \theta} \Sigma \Omega M \xrightarrow{ev} M$$

is a homotopy equivalence.

**Proof.** This follows by unravelling definitions. Depending on the sign conventions used for the coboundary map of a cofibration, it is homotopic to $\pm \text{Id.}$ \qed

**Corollary 1.2.** Let $h^*(-)$ be a reduced cohomology theory. Then the cohomology suspension map

$$h^*(M) \xrightarrow{\text{ev}^*} h^*(\Sigma \Omega M) \xrightarrow{\cong} h^{*-1}(\Omega M)$$

is a monomorphism.

These two results are analogues of results for a suspension $\Sigma X$ in [15, section 2] which depend on the fact that $\Sigma, \Omega$ is an adjoint pair.

The next result is standard, although it seems to be hard to find it stated in this form in the literature, see for example [7, section 1]. To clarify what is involved, we give details. First recall an algebraic notion.

Let $k$ be a commutative unital ring; tensor products will be taken over $k$ unless otherwise specified. Let $A$ be a commutative unital graded $k$-algebra with product $\varphi : A \otimes A \to A$.

**Definition 1.3.** A non-unital $A$-algebra is a left $A$-module $M$ with multiplication

$$A \otimes M \to M; \quad a \otimes m \mapsto a \cdot m$$
and a non-unital associative product $\mu: M \otimes_A M \to M$. Thus the following diagram commutes, where $T: M \otimes A \to A \otimes M$ is the switch map with appropriate signs based on gradings.

For homogeneous elements $a_1, a_2 \in A$, $m_1, m_2 \in M$ and $m_1m_2 = \mu(m_1 \otimes m_2)$,

$$(a_1a_2) \cdot (m_1m_2) = (-1)^{|a_2||m_1|}\mu(a_1 \cdot m_1) \otimes (a_2m_2).$$

There is a Thom diagonal map $\tilde{\Delta}: M \to B_+ \wedge M$ fitting into a strictly commutative diagram

\begin{equation}
\begin{array}{ccc}
D_+ & \xrightarrow{\Delta} & D_+ \wedge D_+ \\
\text{quot.} & & \text{quot.}
\end{array}
\end{equation}

whose vertical maps are the evident quotient maps. If $h^*(-)$ is a multiplicative cohomology theory, then $\tilde{\Delta}$ induces an external product

\begin{equation}
\cdot : h^*(B) \otimes \tilde{h}^*(M) \to \tilde{h}^*(B_+ \wedge M) \xrightarrow{\tilde{\Delta}^*} \tilde{h}^*(M); \quad b \otimes m \mapsto b \cdot m,
\end{equation}

where $\tilde{h}^*(-)$ denotes the reduced theory.

**Theorem 1.4.** Suppose that $h^*(-)$ is a commutative multiplicative cohomology theory. Then the external product induced from $\tilde{\Delta}$ makes $\tilde{h}^*(M)$ into a left $h^*(B)$-module enjoying the following properties.

(a) If $M$ has an orientation $u \in \tilde{h}^n(M)$ then the associated Thom isomorphism

$$h^*(B) \xrightarrow{\cong} \tilde{h}^*(M); \quad x \leftrightarrow x \cdot u$$

makes $\tilde{h}^*(M)$ into a free $h^*(B)$-module of rank 1.

(b) The cup product on $\tilde{h}^*(B)$ makes it a commutative non-unital $h^*(B)$-algebra.

(c) When $h^*(-) = H^*(-; \mathbb{F}_p)$ for a prime $p$, the mod $p$ Steenrod algebra acts compatibly so that the Cartan formula holds for products of the form $t \cdot w$ with $t \in \tilde{H}^*(B; \mathbb{F}_p)$ and $w \in \tilde{H}^*(M; \mathbb{F}_p)$.

**Proof.** The main point is to verify that the following diagram commutes, where $\Delta$ always denotes an internal based diagonal map $X \to X \wedge X$.

\begin{equation}
\begin{array}{ccc}
M \wedge M & \xrightarrow{\Delta} & M \\
\tilde{\Delta} \wedge \Delta & & \\
B_+ \wedge M & \xrightarrow{\text{switch}} & B_+ \wedge B_+ \wedge M \wedge M
\end{array}
\end{equation}
Making use of the commutative diagram (1.2), this follows from properties of the diagonal $\Delta: D_+ \to D_+ \wedge D_+$ which is (strictly) coassociative, cocommutative and counital (the counit is the projection $D_+ \to S^0$). The diagram

$$
\begin{array}{ccc}
D_+ \wedge D_+ & \xrightarrow{\Delta \wedge \Delta} & D_+ \wedge D_+ \wedge D_+ \\
\Delta & & D_+ \wedge D_+ \wedge D_+ \\
\Delta & & D_+ \wedge D_+ \wedge D_+ \wedge D_+ \\
\end{array}
$$

commutes, so by passing to the diagram of quotients we obtain commutativity of (1.3).

Applying $h^*(\cdot)$ and $\tilde{h}^*(\cdot)$ now give the algebraic properties asserted. Of course $h^*(M)$ is also a commutative unital $h^*$-algebra.

The statement about the Steenrod action follows from the Cartan formula for external smash products and naturality. □

**Corollary 1.5.** If the orientation $u$ satisfies $u^2 = 0$, then the product in $\tilde{h}^*(M)$ is trivial.

Notice that the condition $u^2 = 0$ for one orientation implies that the same is true for any orientation.

We end with another result involving the external diagonal.

**Lemma 1.6.** The following diagram commutes.

$$
\begin{array}{ccc}
\Sigma S_+ & \xrightarrow{\Sigma \theta} & \Sigma \Omega M \\
\Sigma S_+ \wedge B_+ & \xrightarrow{\cong} & B_+ \wedge \Sigma S_+ \\
\Sigma S_+ \wedge S_+ & \xrightarrow{\Sigma \Delta} & \Sigma \Omega S_+ \\
\end{array}
$$

Hence if $h^*(-)$ is a multiplicative cohomology theory, then $(\text{ev} \circ \Sigma \theta)^*: \tilde{h}^*(M) \to h^*(S)$ is a homomorphism of $h^*(B)$-modules.

## 2. Recollections on the Eilenberg-Moore spectral sequence

There is of course an extensive literature on Eilenberg-Moore spectral sequence, but for our purposes most of what we need can be found in Smith’s excellent survey article [15], together with Rector and Smith’s papers on Steenrod operations [9, 14]. For the homological algebra background and construction, see [11]. Other useful sources are [3, 10, 12, 13].

In the following we will assume that $k$ is a field, and $H^*(-) = H^*(-; k)$. We will also assume that our Thom space $M$ from Section 1 has an orientation in $H^*(\cdot)$, $M$ is simply connected, and $H^*(B)$ has finite type; these conditions are needed for convergence of the Eilenberg-Moore spectral sequence we will use.

**Theorem 2.1.** There is a second quadrant Eilenberg-Moore spectral sequence of $k$-Hopf algebras $(E_r^{*,*}, d_r)$ with differentials

$$
d_r: E_r^{s,t} \to E_r^{s+r,t-r+1}
$$
and
\[ E_2^{s,t} = \text{Tor}_{H^*(M)}^{s,t}(k,k) \Rightarrow H^{s+t}(\Omega M). \]

The grading conventions here give
\[ \text{Tor}_{H^*(M)}^{s,t} = \text{Tor}_{-s,-t}^{t,s} \]
in the standard homological grading.

When \( k = \mathbb{F}_p \) for a prime \( p \), this spectral sequence admits Steenrod operations; see [9,10,12,14]. We denote the mod \( p \) Steenrod algebra by \( A(p)^\ast \) or \( A^\ast \) when the prime \( p \) is clear.

**Theorem 2.2.** If \( H^\ast(-) = H^\ast(-;\mathbb{F}_p) \) for a prime \( p \), the Eilenberg-Moore spectral sequence is a spectral sequence of \( A^\ast \)-Hopf algebras.

We will need explicit formulae for the Steenrod action. The main result is the following.

**Proposition 2.3.** Suppose that \( X \) is a based space. Then in the Eilenberg-Moore spectral sequence
\[ E_2^{s,*} = \text{Tor}_{H^*(X;\mathbb{F}_p)}^{s,*}(\mathbb{F}_p,\mathbb{F}_p) \Rightarrow H^\ast(\Omega X;\mathbb{F}_p) \]
the action of the Steenrod operations on the \( E_2 \)-term is given in terms of the cobar construction by
\[
\begin{align*}
\text{Sq}^s[x_1|\cdots|x_n] &= \sum_{s_1+\cdots+s_n=s} [\text{Sq}^{s_1}x_1|\cdots|\text{Sq}^{s_n}x_n] \quad \text{if } p = 2, \\
\mathcal{P}^s[x_1|\cdots|x_n] &= \sum_{s_1+\cdots+s_n=s} [\mathcal{P}^{s_1}x_1|\cdots|\mathcal{P}^{s_n}x_n] \quad \text{if } p \text{ is odd}.
\end{align*}
\]

**Sketch of Proof.** There is a construction of the Eilenberg-Moore spectral sequence for the pull-back of a fibration \( q \) along a map \( f \).

\[
\begin{array}{ccc}
E' & \longrightarrow & E \\
\downarrow q' & & \downarrow q \\
B' & \longrightarrow & B
\end{array}
\]

For details see [3,14]. This approach involves the cosimplicial space \( C^\ast \) with
\[ C^s = E \times B^{\times s} \times B' \]
and structure maps \( h_t: C^s \rightarrow C^{s+1} \) (\( 0 \leq t \leq s + 1 \)),
\[
h_t(c,b_1,\ldots,b_s,b') = \begin{cases} (e,h(e),b_1,\ldots,b_s,b') & \text{if } t = 0, \\
(e,b_1,\ldots,b_{t-1},b_t,b_1,\ldots,b_{s+1},b') & \text{if } 1 \leq t \leq s, \\
(e,b_1,\ldots,b_s,q(b'),b') & \text{if } t = s + 1.
\end{cases}
\]
The geometric realisation \( |C^\ast| \) admits a map \( E' \rightarrow |C^\ast| \), and on applying \( H^\ast(-;\mathbb{F}_p) \) to the coskeletal filtration of \( |C^\ast| \) we obtain the Eilenberg-Moore spectral sequence for \( H^\ast(E';\mathbb{F}_p) \). Then the \( E_1 \)-term can be identified with bar construction on \( H^\ast(B;\mathbb{F}_p) \) and comes from the cohomology of the filtration quotients which are suspensions of the spaces \( E \wedge B^{(s)} \wedge B' \). The action of Steenrod operations on \( \tilde{H}^\ast(E \wedge B^{(s)} \wedge B';\mathbb{F}_p) \) is determined using the Cartan formula, and gives the claimed formulae in the \( E_2 \)-term. \( \square \)

Now we come to a special situation that is our main concern.
Theorem 2.4. Suppose that the orientation \( u \in H^n(M) = H^n(M; \mathbb{k}) \) satisfies \( u^2 = 0 \). Then there is an isomorphism of Hopf algebras

\[
\text{Tor}^*_H(M, k) = B^*(H^*(M)),
\]

where \( B^*(H^*(M)) \) denotes the bar construction with

\[
B^{-s}(H^*(M)) = (\tilde{H}^*(M))^\otimes s
\]

for \( s \geq 0 \). The coproduct

\[
\psi: B^{-s}(H^*(M)) \rightarrow \bigoplus_{i=0}^{s} B^{-i}(H^*(M)) \otimes B^{i-s}(H^*(M))
\]

is the usual one with

\[
\psi([u_1| \cdots |u_s]) = \sum_{i=0}^{s} [u_1| \cdots |u_i] \otimes [u_{i+1}| \cdots |u_s],
\]

where we use the traditional bar notation \([w_1| \cdots |w_r] = w_1 \otimes \cdots \otimes w_r\).

Proof. The proof is identical to that for the case of \( \Sigma X \) in [15, section 2, example 4], and uses the fact that \( \tilde{H}^*(N) \) has only trivial products by Corollary 1.5. \( \Box \)

Remark 2.5. The product in the \( E_2 \)-term is the shuffle product,

\[
[u_1| \cdots |u_r] \triangleright [v_1| \cdots |v_s] = \sum_{(r,s) \text{ shuffles } \sigma} (-1)^{\text{Sgn}(\sigma)} [w_{\sigma(1)}|w_{\sigma(2)}| \cdots |w_{\sigma(r+s)}],
\]

where \( \sigma \in \Sigma_{r+s} \) is an \((r,s)\)-shuffle if

\[
\sigma(1) < \sigma(2) < \cdots < \sigma(r), \quad \sigma(r+1) < \sigma(r+2) < \cdots < \sigma(r+s),
\]

and

\[
w_{\sigma(i)} =\begin{cases} u_{\sigma(i)} & \text{if } 1 \leq \sigma(i) \leq r, \\ v_{\sigma(i)-r} & \text{if } r+1 \leq \sigma(i) \leq r+s,\end{cases}
\]

and

\[
\text{Sgn}(\sigma) = \sum_{(i,j)} (\deg w_i + 1)(\deg w_{i+j} + 1)
\]

where the summation is over pairs \((i,j)\) for which \( \sigma(i) > \sigma(r+j) \).

In the situation of this Theorem we have

Corollary 2.6. The Eilenberg-Moore spectral sequence of Theorem 2.1 collapses at the \( E_2 \)-term.

The proof is similar to that of [15, section 2, example 4], and depends on two observations on this spectral sequence for \( H^*(\Omega M) \) under the conditions of Theorem 2.1.

Lemma 2.7. The edge homomorphism \( e: E_2^{-1,s+1} \rightarrow H^*(\Omega M) \) can be identified with the composition

\[
H^{s+1}(M) \xrightarrow{\text{ev}} H^{s+1}(\Sigma \Omega M) \xrightarrow{\cong} H^s(\Omega M)
\]

using the canonical isomorphism \( E_2^{-1,s+1} \cong H^{s+1}(M) \).

Corollary 2.8. The edge homomorphism \( e: E_2^{-1,s+1} \rightarrow H^*(\Omega M) \) is a monomorphism.

Proof. This follows from Lemma 1.4 since \( (\Sigma \theta \circ \delta)^* \) provides a left inverse for \( e \). \( \Box \)
In this section we recall some results of Massey \[5\] part II. We continue to use the notation and general set-up of Section 1.

We assume that our spherical fibration $\xi$ is orientable in $H^*(-) = H^*(-; k)$. Choosing an orientation class $u \in H^n(M)$, we also suppose that $u^2 = 0$. Then (1.1) induces an exact sequence

$$0 \rightarrow H^*(B) \rightarrow H^*(S) \xrightarrow{\delta^*} \tilde{H}^{*+1}(M) \rightarrow 0$$

in which $\delta^*$ is a an $H^*(B)$-module homomorphism with respect to the obvious module structure on $H^*(S)$ and the Thom module structure on $\tilde{H}^*(M)$. Since the left hand map is a monomorphism we regard $H^*(B)$ as a subring of $H^*(S)$.

Now choose $v \in H^{n-1}(S)$ so that $\delta^*(v) = u$. Then by [5, (8.1)] there is a relation of the form

$$v^2 = s + tv,$$

where $s \in H^{2n-2}(B)$ and $t \in H^{n-1}(B)$. If we make a different choice $v' \in H^{n-1}(S)$ with $\delta^*(v') = u$, then $w = v' - v \in H^{n-1}(B)$ and we find that

$$(v')^2 = s' + t'v',$$

where

$$s' = s - wt - w^2,$$

$$t' = \begin{cases} t & \text{if } n \text{ is even}, \\ t + 2w & \text{if } n \text{ is odd}. \end{cases}$$

Massey also shows that when $n$ is odd and $k = \mathbb{F}_2$,

$$t = w_{n-1}(\xi).$$

(3.2)

Here we define the Stiefel-Whitney class through the Wu formula in $H^*(M)$,

$$w_{n-1}(\xi) \cdot u = Sq^{n-1}u.$$

Of course this makes sense for any spherical fibration, not just those associated with vector bundles.

Here are two examples that we will discuss again later.

**Example 3.1.** Consider the universal Spin(2) and Spin(3) bundles $\zeta_2 \downarrow B\text{Spin}(2)$ and $\zeta_3 \downarrow B\text{Spin}(3)$ obtained from the canonical representations into $\text{SO}(2)$ and $\text{SO}(3)$. Of course the bases of these bundles can be taken to be

$$B\text{Spin}(2) = \mathbb{C}P^\infty, \quad B\text{Spin}(3) = \mathbb{H}P^\infty,$$

and $\zeta_2 = \eta^2$, the square of the universal complex line bundle $\eta \downarrow \mathbb{C}P^\infty$. Since there are Spin(3)-equivariant homeomorphisms

$$\text{Spin}(3)/\text{Spin}(2) \cong \text{SO}(3)/\text{SO}(2) \cong S^2,$$

the sphere bundle of $\zeta_3$

$$E\text{Spin}(3)/\text{Spin}(2) \xrightarrow{\pi} E\text{Spin}(3) \times_{\text{Spin}(3)} \text{Spin}(3)/\text{Spin}(2) \rightarrow E\text{Spin}(3)/\text{Spin}(3)$$
can be realised as the natural map $CP^\infty \to \mathbb{H}P^\infty$. In cohomology this induces a monomorphism

$$H^*(\mathbb{H}P^\infty; \mathbb{F}_2) = \mathbb{F}_2[y] \to H^*(CP^\infty; \mathbb{F}_2) = \mathbb{F}_2[x]; \quad y \mapsto x^2.$$ 

It is clear that in $H^*(-; \mathbb{F}_2)$, $w_2(\zeta_2) = 0 = w_2(\zeta_3)$ and also $w_3(\zeta_4) = 0$ since $H^3(\mathbb{H}P^\infty) = 0$.

So we can take $v = x$ and then (3.1) becomes

$$x^2 = y + 0x,$$

since $t = w_2(\zeta_3) = 0$. Similarly, if $p$ is an odd prime, we have $t = 0$ and the analogous relations hold in $H^*(CP^\infty; \mathbb{F}_p)$ and in $H^*(CP^\infty; \mathbb{Q})$.

4. Results on cohomology over $\mathbb{F}_2$

Now we can give some general results for the case $k = \mathbb{F}_2$. Here $H^*(-) = H^*(-; \mathbb{F}_2)$.

We recall Borel’s theorem on the structure of Hopf algebras over perfect fields, see [6, theorem 7.11 and proposition 7.8].

**Theorem 4.1.** Suppose that the orientation $u \in H^n(M)$ satisfies $u^2 = 0$, $H^*(B)$ has no nilpotents, and $Sq^{n-1}u \neq 0$. Then $H^*(\Omega M)$ is a polynomial algebra.

**Proof.** Let $0 \neq x \in H^k(B)$ and consider $[x \cdot u] \in E_2^{-1,k+n}$. Then the Steenrod operation $Sq^{n+k-1}$ satisfies

$$Sq^{n+k-1}[x \cdot u] = [Sq^{n+k-1}(x \cdot u)]$$

$$= [(Sq^k x) \cdot Sq^{n-1}u]$$

$$= [x^2 \cdot Sq^{n-1}u] \neq 0,$$

since all other terms in the sum $\sum_i Sq^ix \cdot Sq^{n+k-1-i}u$ are easily seen to be trivial. It follows that the element of $H^*(\Omega M)$ represented in the spectral sequence by $[x \cdot u]$ has non-trivial square since this is represented by $Sq^{n+k-1}[x \cdot u] = [x^2 \cdot Sq^{n-1}u] \neq 0$.

More generally, using the description of the $E_2$-term in Theorem 2.1, we can similarly see that an element $[x_1 \cdot u] \cdots [x_\ell \cdot u]$ with $x_i \in H^{k_i}(B)$ has

$$Sq^{k_1+\cdots+k_\ell+n-\ell}[x_1 \cdot u] \cdots [x_\ell \cdot u] = [x_1^2 \cdot Sq^{n-1}u] \cdots [x_\ell^2 \cdot Sq^{n-1}u] \neq 0.$$

Thus the algebra generators of $H^*(\Omega M)$ are not nilpotent, so by Borel’s theorem we see that $H^*(\Omega M)$ is a polynomial algebra. \hfill \Box

**Theorem 4.2.** Suppose that the orientation $u \in H^n(M) = H^n(M; \mathbb{F}_2)$ satisfies $u^2 = 0$ and $Sq^{n-1}u = 0$. Then $H^*(\Omega M)$ is an exterior algebra.

**Proof.** First consider an element of $w \in H^{n+k-1}(\Omega M)$ in filtration 1. We can assume that this is represented in the Eilenberg-Moore spectral sequence by $[x \cdot u]$ for some $x \in H^k(B)$. Then

$$w^2 = Sq^{n+k-1}w$$

is also in filtration 1. Since in positive degrees, filtration 0 is trivial, we have $w^2 = 0$.

Now we proceed by induction on the filtration $r$. Suppose that for every positive degree element $z \in H^*(\Omega M)$ of filtration $r \geq 1$, we have $z^2 = 0$. Suppose that $w \in H^*(\Omega M)$ has filtration $r + 1$. We can assume that $w$ is represented by $[x_1 \cdot u] \cdots [x_{r+1} \cdot u]$ where $x_j \in H^{k_j}(B)$. \hfill 8
Applying the Steenrod operation $\text{Sq}^{k_1+\cdots+k_{r+1}+(r+1)n-1}$ we see that $w^2$ is also in filtration $r+1$ and is represented by

$$\text{Sq}^{k_1+\cdots+k_{r+1}+(r+1)(n-1)}[x_1 \cdot u] \cdots [x_{r+1} \cdot u] = [(\text{Sq}^{k_1}x_1) \cdot \text{Sq}^{n-1}u] \cdots [(\text{Sq}^{k_{r+1}}x_{r+1}) \cdot \text{Sq}^{n-1}u] = 0.$$

On the other hand, the coproduct on $w$ is

$$\psi(w) = w \otimes 1 + 1 \otimes w + \sum_i w'_i \otimes w''_i$$

where the $w'_i, w''_i$ all have filtration in the range 1 to $r$. On squaring and using the inductive assumption we find that

$$\psi(w^2) = w^2 \otimes 1 + 1 \otimes w^2,$$

so $w^2$ is primitive and decomposable. By [6, proposition 4.21], the kernel of the natural homomorphism $PH^*(\Omega M) \rightarrow QH^*(\Omega M)$ consists of squares of primitives. Since the primitives must all have filtration 1, all such squares are trivial, hence $w^2 = 0$. This shows that all elements of filtration $r+1$ square to zero, giving the inductive step.

Borel’s theorem now implies that $H^*(\Omega M)$ is an exterior algebra. □

5. Results on cohomology over $\mathbb{F}_p$ with $p$ odd

In this we give analogous results for the case $k = \mathbb{F}_p$ where $p$ is an odd prime. Here $H^*(-) = H^*(-; \mathbb{F}_p)$. We assume that $n$ is odd, say $n = 2m+1$, and that $M$ has an orientation class $u \in H^{2m+1}(M)$. For degree reasons, $u^2 = 0$.

**Theorem 5.1.** Suppose that $H^*(B)$ has no nilpotents, and $\mathcal{P}^m u \neq 0$. Then $H^*(\Omega M)$ is a polynomial algebra.

Of course $\mathcal{P}^m u$ defines a Wu class $W_m(\xi)$ by the formula

$$W_m(\xi) \cdot u = \mathcal{P}^m u,$$

and the condition $\mathcal{P}^m u \neq 0$ amounts to its non-vanishing. The no nilpotents condition implies that $H^*(B)$ is concentrated in even degrees.

**Proof.** Let $0 \neq x \in H^{2k}(B)$ and consider $[x \cdot u] \in \mathbb{E}_2^{-1,2k+2m+1}$. Then the Steenrod operation $\mathcal{P}^{m+k}$ satisfies

$$\mathcal{P}^{m+k}[x \cdot u] = [\mathcal{P}^{m+k}(x \cdot u)] = (\mathcal{P}^k x) \cdot \mathcal{P}^m u = x^p \cdot \mathcal{P}^m u \neq 0,$$

since all other terms in the sum $\sum_i \mathcal{P}^i x \cdot \mathcal{P}^{m+k-i} u$ are easily seen to be trivial. It follows that the element of $H^*(\Omega M)$ represented in the spectral sequence by $[x \cdot u]$ has non-trivial $p$-th power since it is represented by

$$\mathcal{P}^{m+k}[x \cdot u] = [x^p \cdot \mathcal{P}^m u] \neq 0.$$

Similarly every element represented by $[x_1 \cdot u] \cdots [x_\ell \cdot u]$ with $x_i \in H^{2k_i}(B)$ has non-zero $p$-th power since

$$\mathcal{P}^{k_1+\cdots+k_\ell+m\ell}[x_1 \cdot u] \cdots [x_\ell \cdot u] \neq 0.$$

Thus the algebra generators of $H^*(\Omega M)$ are not nilpotent, so by Borel’s theorem we see that $H^*(\Omega M)$ is a polynomial algebra. □
We will say that a connective commutative graded $F_p$-algebra is $p$-truncated if every positive degree element $x$ satisfies $x^p = 0$. When $p = 2$, being 2-truncated is equivalent to being exterior.

**Theorem 5.2.** Suppose that $P^m u = 0$. Then $H^*(\Omega M)$ is a $p$-truncated algebra.

**Proof.** First consider an element of $w \in H^{2m+2k}(\Omega M)$ in filtration 1. We can assume this is represented in the Eilenberg-Moore spectral sequence by $[x \cdot u] \in E_2^{-1,2m+2k+1}$ for some $x \in H^{2k}(B)$. Then $w^p = P^m+k w$ is represented by $[((P^k x) \cdot P^m u)] = 0$,

and is also in filtration 1. Since filtration 0 is trivial in positive degrees, we have $w^p = 0$.

Now as in the proof of Theorem 4.2, we prove by induction on the filtration $r$ that for every positive degree element $z \in H^*(\Omega M)$ of filtration $r \geq 1$ has $z^p = 0$. Borel’s theorem now implies that every element of $H^*(\Omega M)$ has trivial $p$-th power. \qed

6. **Rational results**

In this section we take $k = \mathbb{Q}$. By Borel’s Theorem [6] theorem 7.11 and proposition 7.8], we have

**Theorem 6.1.** There is an isomorphism of algebras

$$H^*(\Omega M; \mathbb{Q}) \cong \bigotimes_i \mathbb{Q}[x_i] \otimes \bigotimes_j \mathbb{Q}[y_i]/(y_j^2),$$

where deg $x_i$ is even and deg $y_i$ is odd. In particular, if $H^*(M; \mathbb{Q})$ is concentrated in odd degrees then $H^*(\Omega M; \mathbb{Q})$ is a polynomial algebra on even degree generators.

7. **Local to global results**

Before giving some examples, we record a variant of the local-global result [1] proposition 2.4]. We follow the convention that a prime $p$ can be 0 or positive, and set $F_0 = \mathbb{Q}$.

Let $S \subseteq \mathbb{N}$ be the multiplicatively closed set generated by a set of non-zero primes (if this set is empty then $S = \{1\}$). Then

$$\mathbb{Z}[S^{-1}] = \{a/b : a \in \mathbb{Z}, \ b \in S\}.$$

In the following, whenever $p \notin S$, $F_p = \mathbb{Z}[S^{-1}]/(p)$.

**Proposition 7.1.** Let $H^*$ be a graded commutative connective $\mathbb{Z}[S^{-1}]$-algebra which is concentrated in even degrees and with each $H^{2n}$ a finitely generated free $\mathbb{Z}[S^{-1}]$-module. Suppose that for each prime $p \notin S$, $H(p)^* = H^* \otimes \mathbb{F}_p$ is a polynomial algebra, then $H^*$ is a polynomial algebra and for every prime $p$,

$$\text{rank}_{\mathbb{Z}[S^{-1}]} \mathcal{Q}H^{2n} = \dim_{\mathbb{F}_p} \mathcal{Q}H(p)^{2n}.$$

**Proof.** The proof of [1] proposition 2.4] can be modified by systematically replacing $\mathbb{Z}$ with the principal ideal domain $\mathbb{Z}[S^{-1}]$ and working only with primes not contained in $S$ (including 0). \qed
8. SOME EXAMPLES

Our first example is a recasting of the main result of \[1\].

**Example 8.1.** Consider the universal line bundle \(\eta \downarrow \mathbb{C}P^\infty\), viewed as a real 2-plane bundle. Then the 3-dimensional bundle \(\xi = \eta \oplus \mathbb{R}\) has Thom space \(M\xi = \Sigma MU(1) \sim \mathbb{C}P^\infty\). It is straightforward to verify that the conditions of Theorems 4.1 and 5.1 apply. Thus \(H^*(\Omega \Sigma \mathbb{C}P^\infty; \mathbb{Z})\) is polynomial.

**Example 8.2.** Recall Example 3.1.

Here \(w_2(\xi_3) = 0 = w_2(\xi_2)\), so \(H^*(\Omega M\text{Spin}(3); \mathbb{F}_2)\) and \(H^*(\Omega \Sigma M\text{Spin}(2); \mathbb{F}_2)\) are exterior algebras.

For an odd prime \(p\), the natural map \(\Sigma M\text{Spin}(2) \rightarrow M\text{Spin}(3)\) induces a monomorphism in \(H^*(-; \mathbb{F}_p)\) and in \(H^*(M\text{Spin}(2); \mathbb{F}_p) = H^*(\mathbb{C}P^\infty; \mathbb{F}_p)\) we see that for the generator \(x \in H^2(\mathbb{C}P^\infty; \mathbb{F}_p)\), \(P^1x = x^p \neq 0\). Therefore \(H^*(\Omega M\text{Spin}(3); \mathbb{F}_p)\) and \(H^*(\Omega \Sigma M\text{Spin}(2); \mathbb{F}_2)\) are polynomial algebras.

Combining these results we see that \(H^*(\Omega M\text{Spin}(3); \mathbb{Z}[1/2])\) and \(H^*(\Omega \Sigma M\text{Spin}(2); \mathbb{Z}[1/2])\) are polynomial algebras.

9. HOMOLOGY GENERATORS AND A STABLE SPLITTING

The map \(\theta: S_+ \rightarrow \Omega M\) introduced in Section 11 allows us to define a canonical choice of generator \(v \in H^{n-1}(S)\) in the sense of Massey’s paper \[5\], namely

\[v = (ev \circ \Sigma \theta)^* u.\]

This follows from Lemma 1.1. When \(n = 2m + 1\) is odd, in mod \(p\) cohomology \(H^*(-) = H^*(-; \mathbb{F}_p)\), from (3.1) we obtain

\[v^2 = s + tv,\]

where

\[t = \begin{cases} w_{2m}(\xi) & \text{if } p = 2, \\ W_m(\xi) & \text{if } p \text{ is odd}. \end{cases} \]

and we define these invariants by

\[w_{2m}(\xi) \cdot u = S^2u, \quad W_m(\xi) \cdot u = P^m u.\]

Notice that the multiplicativity given by Lemma 1.6 implies that for \(x \in H^*(B)\),

\[ (ev \circ \Sigma \theta)^*(x \cdot u) = xv. \]

Now let \(b_i \in H^*(B)\) form an \(\mathbb{F}_p\)-basis for \(H^*(B)\), where we suppose that \(b_0 = 1\). Then the elements \(b_i v, b_i \in H^*(S)\) form a basis for \(H^*(S)\), and the \(b_i \cdot u\) form a basis for \(\widetilde{H}^*(M)\). Since

\[\delta^*(b_i v) = b_i \cdot u, \quad \delta^*(b_i) = 0,\]

for the dual bases \((b_i \cdot v)^\circ, (b_i)^\circ\) of \(H^*(S)\) and \((b_i \cdot u)^\circ, (b_i)^\circ\) of \(\widetilde{H}^*(M)\) we have

\[\delta_*((b_i \cdot u)^\circ) = (b_i v)^\circ.\]

Furthermore, \((\Sigma \theta \circ \delta)_*((b_i \cdot u)^\circ)\) is dual to the class represented in the Eilenberg-Moore spectral sequence by the primitive \([b_i \cdot u]\), hence the \((\Sigma \theta \circ \delta)_*((b_i \cdot u)^\circ)\) form a basis for the indecomposables
QH\_*((\Omega M))$. Using the bar resolution description of the Eilenberg-Moore spectral sequence and the dual cobar resolution for the homology spectral sequence

\[ E^2_{*,*} = \text{Cotor}^{H_*(M)}(F_p,F_p) \implies H_*(\Omega M) \]

we obtain

**Proposition 9.1.** The homology algebra \( H_*(\Omega M;F_p) \) is the free non-commutative algebra on the elements \((\Sigma \theta \circ \delta)_*((b \cdot u)^o)\).

Now we can give an analogue of the James splitting. We need the free \( S \)-algebra functor \( T \) of [4, section II.4]. This is defined for an \( S \)-module \( X \) by

\[ TX = \bigvee_{k \geq 0} X^{(k)}, \]

where \((-)^{(k)}\) denotes the \( k \)-th smash power. The map \( \Sigma \theta \circ \delta \) gives rise to a map of spectra

\[ \Theta: \Sigma^{-1}\Sigma^\infty M \rightarrow \Sigma^\infty \Omega M \]

and by the freeness property of \( T \), there is an induced morphism of \( S \)-algebras

\[ \tilde{\Theta}: T(\Sigma^{-1}\Sigma^\infty M) \rightarrow \Sigma^\infty (\Omega M)_+, \]

where \( \Sigma^\infty (\Omega M)_+ \) becomes an \( S \)-algebra using the natural \( A_\infty \) structure on \( \Omega M \).

**Theorem 9.2.** Suppose that \( p \) is a prime for which Proposition 9.1 is true. Then \( \tilde{\Theta} \) is an \( HF_p \)-equivalence of \( S \)-algebras.

**Proof.** Under the map \( \tilde{\Theta}_* \), an exterior product of classes in \( H_*(\Sigma^{-k}\Sigma^\infty M^{(k)};F_p) \) goes to their internal product in \( H_*(\Omega M;F_p) \). Now Proposition 9.1 shows that \( \tilde{\Theta} \) is an \( F_p \)-equivalence for such a prime \( p \).

Combining our results and using an arithmetic square argument we obtain

**Theorem 9.3.** Let \( S \subseteq \mathbb{N} \) be the multiplicatively closed set generated by all the primes \( p \) for which Proposition 9.1 is false. Then \( \tilde{\Theta} \) is an \( H\mathbb{Z}[S^{-1}] \)-equivalence of \( S \)-algebras. Hence there is an \( H\mathbb{Z}[S^{-1}] \)-equivalence

\[ \bigvee_{k \geq 1} \Sigma^{-k}\Sigma^\infty M^{(k)} \rightarrow \Sigma^\infty \Omega M. \]

Of course, this stable splitting is very different from the James splitting for a connected based space \( X \),

\[ \Sigma \Omega \Sigma X \sim \bigvee_{k \geq 1} \Sigma X^{(k)}. \]

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School of Mathematics & Statistics, University of Glasgow, Glasgow G12 8QW, Scotland.

E-mail address: a.baker@maths.gla.ac.uk
URL: http://www.maths.gla.ac.uk/~ajb