Linear optics based entanglement concentration protocols for Cluster-type entangled coherent state

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Abstract

We proposed two linear optics based entanglement concentration protocols (ECPs) to obtain maximally entangled 4-mode Cluster-type entangled coherent state (ECS) from less (partially) entangled Cluster-type ECS. The first ECP is designed using a superposition of single-mode coherent state with two unknown parameters, whereas the second ECP is obtained using a superposition of single-mode coherent state and a superposition of two-mode coherent state with four unknown parameters. The success probabilities have been calculated for both the ECPs. Necessary quantum circuits enabling future experimental realizations of the proposed ECPs are provided using linear optical elements. Further, the benefit of the proposed schemes is established in the context of long distance quantum communication where photon loss is an obstruction.

Keywords: Entanglement concentration, Coherent state, Maximally entangled state

1 Introduction

Quantum entanglement has already been established as an important resource that can accomplish various quantum information processing tasks. Almost all the existing applications of entangled states, such as quantum teleportation [1], dense coding [2], quantum key distribution [3–6], quantum key agreement [7,8], quantum secure direct communication [10–17], and many more [18–20] are achieved successfully by applying maximally entangled states (MES). However, in the practical scenarios, noise affects an MES during storing, transmitting and processing steps. Consequently, entanglement degradation happens unavoidably. This often leads to reduced communication efficiency and insecure quantum communication channel. In brief, noise is the biggest obstacle in maintaining a shared entangled state which is essential for quantum communication, and it (noise) often transforms an MES to a non-MES. Hence, it’s extremely important to design schemes for recovering MES from non-MESs. For a pure state such a scheme of recovering MES is referred to as ECP, whereas for a mixed state, such a scheme is called entanglement purification protocol (EPP) [21–27]. In this paper, we will restrict ourselves to the study of ECP for a particularly important continuous variable (CV) MES- 4-mode Cluster-type entangled coherent state (ECS). Before, we describe the specific reasons for selecting this state, it may be apt to note that in this work, we have proposed two ECPs, and ECPs as they are extremely important for quantum information processing in the noisy environment. Specifically, In 1996, Bennett et al. first introduced an ECP based on Schmidt decomposition [28]. Since then, many ECPs have been proposed to achieve MES for different quantum states [29–42]. Multi-partite Cluster state plays a significant role in several quantum communication [43–45] and computation [46,47] tasks. Specifically, it has a great importance and a unique characteristic due to its robust entanglement in noisy channels as compared to 2-qubit Bell and 3-qubit Greenberger-Horne-Zeilinger (GHZ) quantum states. In other words, Cluster state shows high level persistency of connectedness and is hard to be destroyed by a single-bit measurement, i.e., less susceptible to decoherence. Unfortunately, the Cluster states still interact with the noise just like other multi-partite states and consequently becomes less entangled. To avoid such noise effect, several ECPs

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have been reported for Cluster states in different forms [48–56]. However, we choose to propose two ECPs for 4-mode Cluster-type ECS. The motivation behind proposing ECPs for 4-mode Cluster-type ECS would become clear in the following text. Initially, the quantum communication protocols were proposed using discrete variable (DV). Nowadays, interest in design and development of quantum communication protocols, has been shifted from DV to continuous variable (CV) quantum cryptography [57]. Actually, there exists a kind of ECS, with the property of entanglement being encoded in CV, it is extensively attracting a lot of attention [58, 59] in performing various quantum information processing tasks. Recently, a few of the quantum cryptographic tasks have appeared using CV. For example, CV QKD [57, 60, 61] analogue of DV QKD [3]. Subsequently, an ECP has been reported for 4-qubit Cluster-type ECS using a superposition of single-mode coherent state [62]. In practical applications of ECS, the maximally ECSs are usually the necessary resources. It is to be noted that Sheng et al., [37] have shed light on the work of Nguyen et al.’s [63] study of an optimal QIP using W-type ECS that exhibits the existence of a quantum information protocol called remote symmetric entanglement, which could be done only using W-type ECS. Similarly, we expect that the maximally entangled Cluster-type ECS would be highly demanding as the necessary resources in certain practical applications of ECS. Interestingly, CV Cluster state has been utilized for quantum computation [64, 65], in fact the experimental realization has also been proposed for CV Cluster state [66, 67]. Hence, Cluster state is extremely important in both DV and CV quantum information tasks. The 4-mode Cluster-type ECS, which we have used to propose our ECP is a CV Cluster state, that has potential applications in quantum information. It is, therefore essential to design an ECP for Cluster-type ECS for the smooth operation of quantum cryptography protocols. To the best of our knowledge, no ECP has been proposed for Cluster-type ECS. Therefore, we have proposed two ECPs to convert partially entangled 4-mode Cluster-type ECS into the maximally entangled 4-mode Cluster-type ECS.

The paper is organized as follows: In Sec. 2 and Sec. 3, we have described our two ECPs to obtain maximally entangled 4-mode Cluster-type ECS from less (partially) entangled Cluster-type ECS. The first ECP is designed using a single-mode coherent state with two unknown parameters, whereas the second ECP is obtained using a two-mode coherent state with four unknown parameters. Further, in Sec. 4, the success probability has been calculated and plotted in Fig. 3 (a) and Fig. 3 (b) for two ECPs, respectively. Finally, the conclusion has been drawn in Sec. 5.

2 ECP for partially entangled 4-mode Cluster-type ECS assisted with a superposition of single-mode coherent state

First, we propose an ECP for 4-mode Cluster-type ECS using a superposition of single-mode coherent state having two terms (two unknown coefficients). Specifically, at the end of this protocol, we should obtain a maximally entangled CV 4-mode Cluster-type ECS expressed as

\[
|\psi\rangle_{abcd} = \frac{1}{2} \left[ (|\alpha\rangle_a|\alpha\rangle_b|\alpha\rangle_c|\alpha\rangle_d + | - \alpha\rangle_a| - \alpha\rangle_b|\alpha\rangle_c|\alpha\rangle_d \right.
\]

\[+ \left. (|\alpha\rangle_a|\alpha\rangle_b| - \alpha\rangle_c| - \alpha\rangle_d + | - \alpha\rangle_a| - \alpha\rangle_b| - \alpha\rangle_c|- \alpha\rangle_d) \right] . \tag{1}
\]

To perform entanglement concentration as shown in Fig. 1, we assume that Alice, Bob, Charlie and David used to share \(|\psi\rangle_{abcd}\), but due to noise this MES is transformed into a partially entangled 4-mode Cluster-type ECS of the form

\[
|\Psi\rangle_{abcd} = N_1 \left[ \beta (|\alpha\rangle_a|\alpha\rangle_b|\alpha\rangle_c|\alpha\rangle_d + | - \alpha\rangle_a| - \alpha\rangle_b|\alpha\rangle_c|\alpha\rangle_d \right.$
\]

\[+ \left. \gamma (|\alpha\rangle_a|\alpha\rangle_b| - \alpha\rangle_c| - \alpha\rangle_d + | - \alpha\rangle_a| - \alpha\rangle_b| - \alpha\rangle_c|- \alpha\rangle_d) \right] , \tag{2}
\]

where \(N_1\) is the normalization coefficient, which can be written as \(N_1 = \left[ 2\beta^2 + 2\gamma^2 + 2e^{-4|\alpha|^2} (\beta^2 - \gamma^2) \right]^{-\frac{1}{2}}\). For simplicity, we assume \(\beta\) and \(\gamma\) are real numbers. The subscripts \(a, b, c\) and \(d\) belong to Alice, Bob, Charlie and David, respectively. Subsequently, David prepares an ancilla in the superposition of single-mode coherent state in spatial mode \(e\) of the form

\[
|\Phi\rangle_e = N_2 (\beta|\alpha\rangle_e + \gamma| - \alpha\rangle_e) , \tag{3}
\]

where \(N_2\) is the normalization coefficient expressed as \(N_2 = \left[ \beta^2 + \gamma^2 + 2\beta\gamma e^{-2|\alpha|^2} \right]^{-\frac{1}{2}}\). Therefore, the combined state of the system can be expressed as
\[ |\xi\rangle_{abcdef} = |\Psi\rangle_{abcd} \otimes |\Phi\rangle_e \]

\[ = N_1 N_2 \left[ \beta \alpha_a |\alpha_b\rangle |\alpha_c\rangle |\alpha_d\rangle + | - \alpha_a\rangle | - \alpha_b\rangle |\alpha_c\rangle |\alpha_d\rangle \right] + \gamma (|\alpha_a\rangle |\alpha_b\rangle - |\alpha_b\rangle |\alpha_a\rangle) - \alpha_d | - \alpha_e\rangle + | - \alpha_e\rangle | - \alpha_d\rangle \bigg] \otimes |\beta|_e + \gamma | - \alpha\rangle_e \bigg], \]

(4)

\[ |\xi\rangle_{abcdef} = N_1 N_2 \left[ \beta^2 \alpha_a |\alpha_b\rangle |\alpha_c\rangle |\alpha_d\rangle |\alpha_e\rangle + | - \alpha_a\rangle | - \alpha_b\rangle |\alpha_c\rangle |\alpha_d\rangle | - \alpha_e\rangle \right] \otimes |\beta|_e + \gamma | - \alpha\rangle_e \bigg] \]

(5)

Further, they allow the spatial modes \(d\) and \(e\) to pass through the 50:50 beam splitter (BS) \((i.e.,\ BS1)\). A BS function is to transform the two coherent states \(|\alpha\rangle\) and \(|\beta\rangle\) as shown below

\[ B_{12}(\alpha|\beta) \rightarrow \frac{|\alpha + \beta\rangle}{\sqrt{2}} \frac{|\alpha - \beta\rangle}{\sqrt{2}}. \]

(6)

When the photons in the spatial mode \(d\) and \(e\) incident on a BS, there exists four different possibilities which can be expressed as

\[ |\alpha\rangle_d |\alpha_e\rangle \rightarrow |\sqrt{2} \alpha\rangle_{d_1} |0\rangle_{e_1}, \]

\[ |\alpha\rangle_d | - \alpha\rangle_e \rightarrow |0\rangle_{d_1} |\sqrt{2} \alpha\rangle_{e_1}, \]

\[ | - \alpha\rangle_d |\alpha_e\rangle \rightarrow |0\rangle_{d_1} | - \sqrt{2} \alpha\rangle_{e_1}, \]

\[ | - \alpha\rangle_d | - \alpha\rangle_e \rightarrow | - \sqrt{2} \alpha\rangle_{d_1} |0\rangle_{e_1}. \]

After the coherent states in the spatial modes \(d\) and \(e\) passing through the BS1, Eq. \([4]\) can be expressed as

\[ |\xi\rangle_{abcd,e_1} = N_1 N_2 \left[ \beta^2 \alpha_a |\alpha_b\rangle |\alpha_c\rangle |\sqrt{2} \alpha\rangle_{d_1} |0\rangle_{e_1} + | - \alpha_a\rangle | - \alpha_b\rangle |\sqrt{2} \alpha\rangle_{d_1} |0\rangle_{e_1} \right] + \gamma (|\alpha_a\rangle |\alpha_b\rangle - |\alpha_b\rangle |\alpha_a\rangle) - \alpha_d | - \alpha_e\rangle + | - \alpha_e\rangle | - \alpha_d\rangle \bigg] \otimes |\beta|_e + \gamma | - \alpha\rangle_e \bigg]. \]

(7)

It is clear from Eq. \([3]\) that components \(|\alpha\rangle_a |\alpha_b\rangle |\alpha_c\rangle |\sqrt{2} \alpha\rangle_{d_1} |0\rangle_{e_1}, | - \alpha_a\rangle | - \alpha_b\rangle |\sqrt{2} \alpha\rangle_{d_1} |0\rangle_{e_1}, |\alpha_a\rangle | - \alpha_b\rangle |\sqrt{2} \alpha\rangle_{d_1} |0\rangle_{e_1} \) do not contain photon in the spatial mode \(e_1\). However, the rest of the four components do not contain photon in the spatial mode \(d_1\). Therefore, they can adopt the post selection method to select only those four terms which the spatial mode \(d_1\) has no photon which can be written as

\[ |\xi\rangle_{abce_1} = N_1 N_2 \beta \gamma \left[ |\alpha\rangle_a |\alpha_b\rangle |\alpha_c\rangle |\sqrt{2} \alpha\rangle_{e_1} + | - \alpha\rangle_a | - \alpha_b\rangle |\sqrt{2} \alpha\rangle_{e_1} \right] \]

(8)
It is to be noted that the Eq. (8) has the same form with Eq. (1) with the only difference that the amplitude of the coherent state in spatial mode $e_1$ of Eq. (8) is $\sqrt{2}$ times higher than the coherent state in spatial mode $d$ of Eq. (1). However, this $\sqrt{2}$ times increased amplitude in the above equation, could be taken as advantage in long distance quantum communication, where photon loss is an obstruction. Subsequently, in order to obtain the maximally entangled 4-mode Cluster-type ECS as shown in Eq. (1), the coherent state in spatial mode $e_1$ passes through the $BS2$ and we obtain
\[
|\xi\rangle_{abe2e3} = N_1 N_2 \beta \gamma \left[ |\alpha\rangle_a |\alpha\rangle_b |\alpha\rangle_c |\alpha\rangle_d + | -\alpha\rangle_a | -\alpha\rangle_b |\alpha\rangle_c |\alpha\rangle_d + | -\alpha\rangle_c | -\alpha\rangle_a |\alpha\rangle_b |\alpha\rangle_d + | -\alpha\rangle_d | -\alpha\rangle_a | -\alpha\rangle_b |\alpha\rangle_c \right].
\]

Then, they perform the parity check measurement and can detect the coherent state in the spatial mode $e_2$ (without making any distinction between $|\alpha\rangle_{e_2}$ and $| -\alpha\rangle_{e_2}$) and obtain the maximally entangled 4-mode Cluster-type ECS of the same form as shown in Eq. (1).
\[
|\xi\rangle_{abe2} = N \left[ |\alpha\rangle_a |\alpha\rangle_b |\alpha\rangle_c |\alpha\rangle_d + | -\alpha\rangle_a | -\alpha\rangle_b |\alpha\rangle_c |\alpha\rangle_d + | -\alpha\rangle_c | -\alpha\rangle_a |\alpha\rangle_b |\alpha\rangle_d + | -\alpha\rangle_d | -\alpha\rangle_a | -\alpha\rangle_b |\alpha\rangle_c \right],
\]

where, $N = N_1 N_2 \beta \gamma$. Finally, they get the maximally entangled 4-mode Cluster-type ECS as shown in Eq. (1) with success probability $P = 4|N_1 N_2 \beta \gamma|^2$ and the same has been plotted in Fig. 3 (a) in Sec. 4.

3 ECP for partially entangled 4-mode ECS assisted with a superposition of single-mode coherent state and a superposition of a two-mode coherent state

In this section, we propose an ECP for 4-mode Cluster-type ECS using a superposition of single-mode coherent state and superposition of a two-mode coherent state having four terms (four unknown coefficients). As shown in Fig. 2, Alice, Bob, Charlie and David share partially entangled 4-mode Cluster-type ECS of the form
\[
|\Psi\rangle_{abcd} = N_3 \left[ \beta |\alpha\rangle_a |\alpha\rangle_b |\alpha\rangle_c |\alpha\rangle_d + | -\alpha\rangle_a | -\alpha\rangle_b |\alpha\rangle_c |\alpha\rangle_d + | -\alpha\rangle_c | -\alpha\rangle_a |\alpha\rangle_b |\alpha\rangle_d + | -\alpha\rangle_d | -\alpha\rangle_a | -\alpha\rangle_b |\alpha\rangle_c \right],
\]

where $N_3 = \left[ \beta^2 + \gamma^2 + \delta^2 + \eta^2 + 2(\beta \gamma + \beta \delta - \gamma \eta - \delta \eta) e^{-4|\alpha|^2} + 2(\delta \gamma - \eta \beta) e^{-8|\alpha|^2} \right]^{-\frac{1}{2}}$ is the normalization coefficient. Here, we assume the coefficients $\beta, \gamma, \delta$ and $\eta$ are real numbers. The subscripts $a, b, c$ and $d$ belong to Alice, Bob, Charlie and David, respectively. Further, David prepares the superposition of two-mode coherent state in spatial mode $e$ and $f$ of the form
\[
|\Phi\rangle_{ef} = N_4 \left[ |\alpha\rangle_e |\alpha\rangle_f + | -\alpha\rangle_e | -\alpha\rangle_f \right],
\]

where $N_4 = \left[ \beta^2 + \gamma^2 + \delta^2 + \eta^2 + 2(\beta \gamma + \beta \delta + \gamma \eta + \delta \eta) e^{-2|\alpha|^2} + 2(\delta \gamma + \eta \beta) e^{-4|\alpha|^2} \right]^{-\frac{1}{2}}$ is the normalization coefficient. Therefore, the combined state of the system can be expressed as
\[
|\xi\rangle_{abcdef} = |\Psi\rangle_{abcd} \otimes |\Phi\rangle_{ef}
\]

\[
|\xi\rangle_{abcdef} = N_3 N_4 \left[ \beta |\alpha\rangle_a |\alpha\rangle_b |\alpha\rangle_c |\alpha\rangle_d |\alpha\rangle_e + \gamma | -\alpha\rangle_a | -\alpha\rangle_b |\alpha\rangle_c |\alpha\rangle_d | -\alpha\rangle_e + \delta |\alpha\rangle_a |\alpha\rangle_b | -\alpha\rangle_c | -\alpha\rangle_d |\alpha\rangle_e + \eta | -\alpha\rangle_a | -\alpha\rangle_b | -\alpha\rangle_c | -\alpha\rangle_d | -\alpha\rangle_e \right],
\]

\[
|\xi\rangle_{abcdef} = N_3 N_4 \left[ \beta^2 |\alpha\rangle_a |\alpha\rangle_b |\alpha\rangle_c |\alpha\rangle_d |\alpha\rangle_e + \gamma | -\alpha\rangle_a | -\alpha\rangle_b |\alpha\rangle_c |\alpha\rangle_d | -\alpha\rangle_e + \delta |\alpha\rangle_a |\alpha\rangle_b | -\alpha\rangle_c | -\alpha\rangle_d |\alpha\rangle_e + \eta | -\alpha\rangle_a | -\alpha\rangle_b | -\alpha\rangle_c | -\alpha\rangle_d | -\alpha\rangle_e \right].
\]
Subsequently, David also prepares one more ancilla in the superposition of single-mode coherent state in spatial mode $g$ of the form

$$|\Phi\rangle_g = N_5 [\beta\gamma|\alpha\rangle_g + \delta\eta|\alpha\rangle_g],$$

(14)

where $N_5$ is the normalization coefficient and can be written as $N_5 = \left[\beta^2\gamma^2 + \delta^2\eta^2 + 2\beta\gamma\delta\eta e^{-2|\alpha|^2}\right]^{-\frac{1}{2}}$. The whole state of the system can be expressed as

$$|\zeta\rangle_{abcdefg} = |\zeta\rangle_{abcdef} \otimes |\Phi\rangle_g,$$

and same can be expanded as

\[
|\zeta\rangle_{abcdefg} = N_3 N_4 N_5 \left[ \beta^2 \gamma |\alpha\rangle_a |\alpha\rangle_b |\alpha\rangle_c |\alpha\rangle_d |\alpha\rangle_e |\alpha\rangle_f |\alpha\rangle_g + \beta^2 \delta \eta |\alpha\rangle_a |\alpha\rangle_b |\alpha\rangle_c |\alpha\rangle_d |\alpha\rangle_e |\alpha\rangle_f - |\alpha\rangle_g \right]_N \]

(15)
Figure 2: (Color online) The schematic diagram of the proposed 4-mode Cluster-type ECS has been shown where Alice, Bob, Charlie and David initially share partially entangled 4-mode Cluster-type ECS. The subscripts $a, b, c$ and $d$ belong to Alice, Bob, Charlie and David, respectively. Then, David prepares an ancilla in the superposition of single-mode coherent state in spatial mode $g$ and a superposition of two-mode coherent state in spatial mode $e$ and $f$. The parity check measurement is performed using the 50:50 beam splitters $BS1$, $BS2$ and $BS3$. Further, to get the maximally entangled 4-mode Cluster-type ECS, $BS5$, $BS4$ and $BS6$, respectively, are used that is helpful in reducing the increased amplitude obtained in Eq. (18).

They allow the photons in spatial mode $a$ and $f$, $c$ and $e$, $d$ and $g$ pass through the beam splitters $BS1$, $BS2$, $BS3$, respectively. Then the state becomes

\[
|\psi\rangle_{bc1d1f1g1} = N_3 N_4 N_5 \left[ (\beta^3 \gamma |\sqrt{2}\alpha\rangle_{a1}|\alpha\rangle_b|\sqrt{2}\alpha\rangle_{c1}|\sqrt{2}\alpha\rangle_{d1}|0\rangle_{e1}|0\rangle_{f1}|0\rangle_{g1} + \beta^2 \delta \eta |\sqrt{2}\alpha\rangle_{a1}|\alpha\rangle_b|\sqrt{2}\alpha\rangle_{c1}|0\rangle_{d1}|0\rangle_{e1}|0\rangle_{f1}|\sqrt{2}\alpha\rangle_{g1} + \beta \gamma^2 |\sqrt{2}\alpha\rangle_{a1}|\alpha\rangle_b|\sqrt{2}\alpha\rangle_{c1}|\sqrt{2}\alpha\rangle_{d1}|0\rangle_{e1}|0\rangle_{f1}|0\rangle_{g1} + \gamma^2 \delta \eta |\sqrt{2}\alpha\rangle_{a1}|\alpha\rangle_b|\sqrt{2}\alpha\rangle_{c1}|\sqrt{2}\alpha\rangle_{d1}|0\rangle_{e1}|0\rangle_{f1}|0\rangle_{g1} \right] + \delta^2 \gamma^2 |\sqrt{2}\alpha\rangle_{a1}|\alpha\rangle_b|\sqrt{2}\alpha\rangle_{c1}|\sqrt{2}\alpha\rangle_{d1}|0\rangle_{e1}|0\rangle_{f1}|0\rangle_{g1}.
\]

Further, they can adopt the post selection method to select the cases, when the spatial mode $a_1, d_1, e_1$ have no photons, then the state can be written as

\[
|\psi\rangle_{bc1f1g1} = N_3 N_4 N_5 \left[ (\beta \gamma |\sqrt{2}\alpha\rangle_{a1}|\alpha\rangle_b|\sqrt{2}\alpha\rangle_{c1}|\sqrt{2}\alpha\rangle_{d1}|0\rangle_{e1}|\sqrt{2}\alpha\rangle_{f1}|0\rangle_{g1} + |\alpha\rangle_b|\sqrt{2}\alpha\rangle_{c1}|\sqrt{2}\alpha\rangle_{d1}|0\rangle_{e1}|\sqrt{2}\alpha\rangle_{f1}|0\rangle_{g1} \right] + \delta \gamma^2 |\sqrt{2}\alpha\rangle_{a1}|\alpha\rangle_b|\sqrt{2}\alpha\rangle_{c1}|\sqrt{2}\alpha\rangle_{d1}|0\rangle_{e1}|0\rangle_{f1}|0\rangle_{g1}.
\]
for fixed value of \( \beta \), choose
\[
\langle \zeta \rangle_{b_1c_1g_1} = N_3N_4N_5\beta\gamma\delta\eta \left[ |\alpha_b\rangle|\sqrt{2}\alpha\rangle_{f_1} |\sqrt{2}\alpha\rangle_{c_1} |\sqrt{2}\alpha\rangle_{g_1} + | - \alpha_b\rangle - \sqrt{2}\alpha\rangle_{f_1} |\sqrt{2}\alpha\rangle_{c_1} |\sqrt{2}\alpha\rangle_{g_1} \right] 
+ |\alpha_b\rangle|\sqrt{2}\alpha\rangle_{f_1} |\sqrt{2}\alpha\rangle_{c_1} |\sqrt{2}\alpha\rangle_{g_1} + | - \alpha_b\rangle - \sqrt{2}\alpha\rangle_{f_1} |\sqrt{2}\alpha\rangle_{c_1} |\sqrt{2}\alpha\rangle_{g_1} \right].
\] (18)

Now, we can see that the Eq. (18) has the same form with Eq. (1) with the only difference that the amplitude of the coherent states in spatial modes \( f_1, c_1 \) and \( g_1 \) in Eq. (18) are \( \sqrt{2} \) times higher than the coherent states in spatial modes \( b, c, d \), respectively in Eq. (1). However, in order to obtain the maximally entangled 4-mode Cluster-type ECS as shown in Eq. (1), the photons in spatial modes \( c_2, f_2, g_2 \) and \( g_3 \), respectively (without making any distinction between \( |\alpha\rangle_{f_2} \) and \( | - \alpha\rangle_{f_2} \), \( |\alpha\rangle_{c_2}, \) and \( | - \alpha\rangle_{c_2} \), \( |\alpha\rangle_{g_3}, \) and \( | - \alpha\rangle_{g_3} \)) and the state becomes
\[
\langle \zeta \rangle_{b_2c_2g_2g_3} = N_3N_4N_5\beta\gamma\delta\eta \left[ |\alpha_b\rangle|\alpha_f\rangle_{f_2} |\alpha_c\rangle_{c_2} |\alpha_g\rangle_{g_2} |\alpha_g\rangle_{g_3} + | - \alpha_b\rangle - \alpha_f\rangle_{f_2} |\alpha_c\rangle_{c_2} |\alpha_g\rangle_{g_2} |\alpha_g\rangle_{g_3} \right] 
+ |\alpha_b\rangle|\alpha_f\rangle_{f_2} |\alpha_c\rangle_{c_2} |\alpha_g\rangle_{g_2} |\alpha_g\rangle_{g_3} + | - \alpha_b\rangle - \alpha_f\rangle_{f_2} |\alpha_c\rangle_{c_2} |\alpha_g\rangle_{g_2} |\alpha_g\rangle_{g_3} \right].
\] (19)

After performing the parity check measurement, they can detect the coherent state in the spatial modes \( f_3, c_3 \) and \( g_3 \), respectively (i.e., \( N_3N_4N_5\beta\gamma\delta\eta \) in Eq. (18) to (20) in Sec. 4). Finally, they obtain the maximally entangled 4-mode Cluster-type ECS of the same form as shown in Eq. (1) with success probability, \( P = 4|N_3N_4N_5\beta\gamma\delta\eta|^2 \) and the same has been plotted in Fig. 3(b) in Sec. 4.

4 Success probability

In this section, we have plotted the success probability \( P \) calculated for both the ECPs in Sec. 2 and Sec. 3, respectively. In Fig. 3(a), we have plotted the variation of success probability \( P \) with \( \beta \) for our first ECP, and it is shown that success probability \( P \) can be controlled by controlling coherent state parameters \( \alpha \) and \( \beta \). Specifically, in Fig. 3(a), variation of \( P \) with \( \beta \) is illustrated for \( \alpha = 0.5, 1, \) and \( 2 \), to reveal that the peak (maximum possible value) of the success probability \( P \) increases with the increase in \( \alpha \) with a slight increase in the corresponding value of \( \beta \). Further, in our second ECP, as \( \beta^2 + \gamma^2 + \delta^2 + \eta^2 = 1 \) and \( \beta, \gamma, \delta \) and \( \eta \) are real, we can parameterize these parameters (i.e., \( \beta, \gamma, \delta \) and \( \eta \)) in terms of new variables \( \theta_1, \theta_2 \) and \( \theta_3 \) as \( \beta = \cos[\theta_3], \gamma = \sin[\theta_3] \cos[\theta_2], \delta = \sin[\theta_3] \sin[\theta_2] \sin[\theta_1], \eta = \sin[\theta_3] \sin[\theta_2] \sin[\theta_1]. \) Subsequently, we plotted the variation of \( P \) with \( \theta_1 \) and \( \theta_2 \) both varying from \( 0 \rightarrow \frac{\pi}{2} \) for \( \theta_3 = \frac{\pi}{4} \) and \( \alpha = 2 \) as shown in Fig. 3(b).

![Figure 3: (Color online) (a) 2D Plot shows the variation of success probability \( P \) with \( \beta \) for the first ECP. Here, we choose \( \alpha = 0.5, 1, \) and \( 2 \). (b) 3D Plot shows the variation of success probability \( P \) with \( \theta_1 \{0 \rightarrow \frac{\pi}{2}\} \) and \( \theta_2 \{0 \rightarrow \frac{\pi}{2}\} \), for fixed value of \( \theta_3 = \frac{\pi}{4} \). Here, we assume \( \alpha = 2 \).](image-url)
5 Conclusion

Entanglement is used in several quantum information processing applications. In Sec. 1, we have already discussed the importance of MES and how it is an ultimate resource for quantum information applications. Unfortunately, it is a fact that these MES interact with the environment over the time while processing, which leads to the degradation in entanglement that lowers the efficiency of the quantum communication schemes. Our goal is to avoid such degradation in entanglement, and achieve the quantum communication applications with high fidelity. To do so, many ECPs have been proposed and all those existing ECPs have been designed for many different quantum states [28–42] as well as for Cluster state in different forms [48–56]. However, to the best of our knowledge, no ECP has been reported for 4-mode Cluster-type ECS. This is the motivation for us to propose ECPs for 4-mode Cluster-type ECS and it is the first ever ECP for such state proposed by us. Another inspiring point is that, a Cluster-type ECS [66,67] has extremely important and interesting applications in the recent past [64,65]. Keeping this in mind, we have proposed two ECPs for 4-mode Cluster-type ECS. A superposition of single-mode coherent state is used in the first ECP while a superposition of single-mode coherent state and a superposition of two-mode coherent state are used together in the second ECP. To achieve the coherent state, we have used the parity check measurement method, using a 50 : 50 BS, i.e., balanced beam splitter. We have also calculated the success probability $P$ of each ECP in Sec. 4 and shown its variation with the corresponding parameters in the Fig. 3 (a) and (b), respectively. Our ECPs have obtained the increased amplitude $\sqrt{2}$ times higher in Eq. 8 and 18, which could be advantageous for long distance quantum communication. We conclude the paper with the anticipation that our ECPs would be of practical interest and experimentally realizable with the present linear optical technology.

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References

[1] Bennett, C.H., Brassard, G., Crépeau,C., et al.: Teleporting an unknown quantum state via dual classical and Einstein–Podolsky–Rosen channels. Phys. Rev. Lett. 70, 1895–1899 (1993)

[2] Bennett, C.H., Wiesner, S.J.: Communication via one- and two-particle operators on Einstein– Podolsky–Rosen states. Phys. Rev. Lett. 69, 2881–2884 (1992)

[3] Bennett, C.H., Brassard, G.: Quantum cryptography: public key distribution and coin tossing. In Proceedings of the IEEE International Conference on Computers, Systems, and Signal Processing, Bangalore, India, pp. 175-179 (1984)

[4] Ekert, A.K.: Quantum cryptography based on Bell’s Theorem. Phys. Rev. Lett. 67, 661 (1991)

[5] Bennett, C.H.: Quantum cryptography using any two nonorthogonal states, Phys. Rev. Lett. 68, 3121 (1992)

[6] Goldenberg, L., Vaidman, L.: Quantum cryptography based on orthogonal states. Phys. Rev. Lett. 75, 1239 (1995)

[7] Shukla, C., Alam, N., Pathak, A.: Protocols of quantum key agreement solely using Bell states and Bell measurement. Quantum Inf. Process. 13, 2391 (2014)

[8] Shukla, C., Thapliyal, K., Pathak, A.: Semi-quantum communication: protocols for key agreement, controlled secure direct communication and dialogue. Quantum Inf. Process. 16, 295 (2017)

[9] Hillery, M., Buzek, V., Berthiaume, A.: Quantum secret sharing. Phys. Rev. A 59, 1829–1834 (1999)

[10] Long, G.L., Liu, X.S.: Theoretically efficient high-capacity quantum-key-distribution scheme. Phys. Rev. A 65, 032302 (2002)

[11] Deng, F. G., Long, G. L., Liu, X.S.: Two-step quantum direct communication protocol using the Einstein-Podolsky-Rosen pair block. Phys. Rev. A 68, 042317 (2003)
[12] Long, G.-L., Deng, F.-G., Wang, C., Li, X.-h., Wen, K., Wang, W.-y.: Quantum secure direct communication and deterministic secure quantum communication. Front. Phys. China 2, 251 (2007)

[13] Bostrom, K. Felbinger, T.: Deterministic secure direct communication using entanglement. Phys. Rev. Lett. 89, 187902 (2002)

[14] Shukla, C., Pathak, Srikanth, A.R.: Beyond the Goldenberg-Vaidman protocol: secure and efficient quantum communication using arbitrary, orthogonal, multi-particle quantum states. Int. J. Quantum Inf. 10, 1241009 (2012)

[15] Shukla, C., Banerjee, A., Pathak, A.: Improved protocols of secure quantum communication using W states. Int. J. Theor. Phys. 52, 1914 (2013)

[16] Shukla, C., Pathak, A.: Orthogonal-state-based deterministic secure quantum communication without actual transmission of the message qubits, Quantum Inf. Process. 13, 2099-2113 (2014)

[17] Shukla, C., Banerjee, A., Pathak, A. Srikanth, R.: Secure quantum communication with orthogonal states. Int. J. Quantum Inf. 14, 1640021 (2016)

[18] Shukla, C., Kothari, V., Banerjee, A., Pathak, A.: On the group-theoretic structure of a class of quantum dialogue protocols. Phys. Lett. A 377, 518 (2013)

[19] Banerjee, A., Shukla, C., Thapliyal, K., Pathak, A., Panigrahi, P.K.: Asymmetric quantum dialogue in noisy environment. Quantum Inf. Process. 16, 49 (2017)

[20] Banerjee, A., Thapliyal, K. Shukla, C., Pathak, A.: Quantum conference. Quantum Inf. Process. 17, 161 (2018)

[21] Bennett, C.H., Brassard, G., Popescu, S., Schumacher, B., et al.: Purification of noisy entanglement and faithful teleportation via noisy channel. Phys. Rev. Lett. 76, 722–725 (1996)

[22] J. W. Pan, C. Simon, Č. Brukner, A. Zeilinger.: Entanglement purification for quantum communication. Nature 410, 1067 (2001)

[23] Yamamoto, T., Koashi, M., Imoto, N.: Concentration and purification scheme for two partially entangled photon pairs. Phys. Rev. A 64, 012304 (2001)

[24] D. Gonză and P. van Loock.: High-fidelity entanglement purification using chains of atoms and optical cavities. Phys. Rev. A 86, 052312 (2012)

[25] M. Zwerger, H. J. Briegel, and W. Diöger.: Universal and optimal error thresholds for measurement-based entanglement purification, Phys. Rev. Lett. 110, 260503 (2013)

[26] M. Zwerger, H. J. Briegel, and W. Diöger.: Robustness of hashing protocols for entanglement purification, Phys. Rev. A 90, 012314 (2014)

[27] J. Z. Berničquez, J. M. Torres, L. Kunz, and G. Alber.: Multiphoton-state-assisted entanglement purification of material qubits, Phys. Rev. A 93, 032317 (2016)

[28] Bennett, C.H., Bernstein, H.J., Popescu, S., Schumacher, B.: Concentrating partial entanglement by local operations. Phys. Rev. A 53, 2046–2052 (1996)

[29] Bose, S., Vedral, V., Knight, P.L.: Purification via entanglement swapping and conserved entanglement. Phys. Rev. A 60, 194–197 (1999)

[30] Shi, B.S., Jiang, Y.K., Guo, G.C.: Optimal entanglement purification via entanglement swapping. Phys. Rev. A 62, 054301 (2000)

[31] Zhao, Z., Pan, J.W., Zhan, M.S.: Practical scheme for entanglement concentration. Phys. Rev. A 64, 014301 (2001)

[32] Sheng, Y.B., Deng, F.G., Zhou, H.Y.: Nonlocal entanglement concentration scheme for partially entangled multipartite systems with nonlinear optics. Phys. Rev. A 77, 062325 (2008)
[33] Sheng, Y.B., Zhou, L., Zhao, S.M., Zheng, B.Y.: Efficient single-photon-assisted entanglement concentration for partially entangled photon pairs. Phys. Rev. A 85, 012307 (2012)

[34] Sheng, Y.B., Zhou, L, Zhao, S.M.: Efficient two-step entanglement concentration for arbitrary W states. Phys. Rev. A 85, 042302 (2012)

[35] Deng, F.G.: Optimal nonlocal multipartite entanglement concentration based on projection measurements. Phys. Rev. A 85, 022311 (2012)

[36] Choudhury, B.S., Dhara, A.: An entanglement concentration protocol for cluster states. Quantum Inf. Process. 12, 2577–2585 (2013)

[37] Sheng, Y.B., Liu, J., Zhao, S.Y., Wang, L., Zhou, L.: Entanglement concentration for W-type entangled coherent state. Chin. Phys. B 23, 080305 (2014)

[38] Shukla, C., Banerjee, A., Pathak, A.: Protocols and quantum circuits for implementing entanglement concentration in cat state, GHZ-like state and 9 families of 4-qubit entangled states. Quantum Inf. Process. 14, 2077 (2015)

[39] Banerjee, A., Shukla, C., Pathak, A.: Maximal entanglement concentration for a set of (n + 1)-qubit states. Quantum Inf. Process. 14, 4523–4536 (2015)

[40] Wang, C., Shen, W.W., Mi, S.C., Zhang, Y., Wang, T.J.: Concentration and distribution of entanglement based on valley qubits system in graphene. Sci. Bull. 60, 2016–2021 (2016)

[41] Ding, S.P., Zhou, L., Gu, S.-P., Wang, X.F., Sheng, Y.B.: Electronic entanglement concentration for the concatenated Greenberger-Horne-Zeilinger state. Int. J. Theor. Phys. 56, 1912–1928 (2017)

[42] Liu, J., Zhou, L., Zhong, W., Sheng, Y.B.: Logic Bell state concentration with parity check measurement. Front. Phys. 14, 21601 (2019)

[43] Shukla, C., Pathak, A.: Hierarchical quantum communication. Phys. Lett. A 377, 1337–1344 (2013)

[44] Shukla, C., Thapliyal, K., Pathak, A.: Hierarchical joint remote state preparation in noisy environment. Quantum Inf Process. 16, 205 (2017)

[45] Shukla, C., Banerjee, A., Pathak, A.: Bidirectional controlled teleportation by using 5-qubit states: a generalized view. Int. J. Theor. Phys. 52, 3790–3796 (2013)

[46] Dür, W., Briegel, H.G.: Stability of macroscopic entanglement under decoherence. Phys. Rev. Lett. 92, 180403 (2005)

[47] Walther, P., Aspelmeyer, M., Resch, K.J., Zeilinger, A.: Experimental violation of a cluster state Bell inequality. Phys. Rev. Lett. 95, 020403 (2005)

[48] Lan, Z.: Consequent entanglement concentration of a less-entangled electronic cluster state with controlled-NOT gates. Chin. Phys. B 23, 050308 (2014)

[49] Zhao, S. Y., Liu, J., Zhou, L., Sheng, Y.B.: Two-step entanglement concentration for arbitrary electronic cluster state. Quantum Inf. Process. 12, 3633 (2013)

[50] Bin, S., Shi-Lei, S., Li-Li, S., Liu-Yong, C., Hong-Fu, W., Shou, Z.: Efficient three-step entanglement concentration for an arbitrary four-photon cluster state. Chin. Phys. B 22, 030305 (2013)

[51] Jiong, L., Sheng-Yang, Z., Lan, Z., Yu-Bo, S.: Electronic cluster state entanglement concentration based on charge detection. Chin. Phys. B 23, 020313 (2014)

[52] Sheng, Y.B., Zhao, S.Y., Liu, J., Wang, X.F., Zhou, L.: Arbitrary four-photon cluster state concentration with cross-Kerr nonlinearity. Int. J. Theor. Phys. 54, 1292–1303 (2015)

[53] Liu, H.J., Fan, L.L., Xia, Y., Song, J.: Efficient entanglement concentration for partially entangled cluster states with weak cross-Kerr nonlinearity. Quantum Inf. Process. 14, 2909–2928 (2015)
[54] Zhao, S.Y., Cai, C., Liu, J., Zhou, L., Sheng, Y.B.: Entanglement concentration for arbitrary four-photon cluster state assisted with single photons. Int. J. Theor. Phys. 55, 1128–1144 (2016)

[55] Du, F.F., Long, G.L.: Refined entanglement concentration for electron-spin entangled cluster states with quantum-dot spins in optical microcavities. Quantum Inf. Process. 16, 26 (2017)

[56] Song, T.T., Tan, X., Wang, T.: Entanglement concentration for arbitrary four-particle linear cluster states. Scientific Reports. 7, 1982 (2017)

[57] Luiz, F.S., Rigolin, G.: Teleportation-based continuous variable quantum cryptography. Quantum Inf. Process. 16, 58 (2017)

[58] Sanders, B.C.: Review of entangled coherent states. J. Phys. A :Math.Theor. 45, 244002 (2012)

[59] Sanders, B.C.: Entangled coherent states. Phys. Rev. A 45, 6811 (1992)

[60] Braunstein, S.L., Van Loock, P.: Quantum information with continuous variables. Rev. Mod. Phys. 77, 513 (2005)

[61] Weedbrook, C., Pirandola, S., García-Patrón, R., Cerf, N.J., et.al.: Gaussian quantum information. Rev. Mod. Phys. 84, 621 (2012)

[62] Guo, R., Zhou, L., Gu, S.P., Wang, X.F., Sheng, Y.B.: Hybrid entanglement concentration assisted with single coherent state. Chin. Phys. B 25, 030302 (2016)

[63] An, N.B.: Optimal processing of quantum information via W-type entangled coherent states. Phys. Rev. A 69, 022315 (2004)

[64] Menicucci, N.C., Loock, P.v., Gu, M., Weedbrook, C., Ralph, T.C., Nielsen, M.A.: Universal quantum computation with continuous-variable cluster states. Phys. Rev. Lett. 97, 110501 (2006)

[65] Gu, M., Weedbrook, C., Menicucci, N.C., Ralph, T.C., Loock, P.V.: Quantum computing with continuous-variable clusters. Phys. Rev. A 79, 062318 (2009)

[66] Su, X., Tan, A., Jia, X., Zhang, J., Xie, C., Peng, K.: Experimental preparation of quadripartite cluster and Greenberger-Horne-Zeilinger entangled states for continuous variables. Phys. Rev. Lett. 98, 070502 (2007)

[67] Yukawa, M., Ukai, R., van Loock, P., Furusawa, A.: Experimental generation of four-mode continuous-variable cluster states. Phys. Rev. A 78, 012301 (2008)