COMPUTATION OF TWIN-WIDTH OF GRAPHS

KAJAL DAS

Abstract: Twin-width is a recently introduced graph parameter. In this article, we compute twin-width of various finite graphs. In particular, we prove that the twin-widths of finite graphs with 4 and 5 vertices are less than equal to 1 and 2, respectively. We show that the constructions of dual graph and line graph do not preserve twin-width. Also, we give upper bounds for the twin-width of King’s graph and Rook’s graph.

Mathematics Subject Classification (2020): 05C30, 05C38, 05C76, 68R10.

Key terms: Graph, Twin-width, Complement graph, Dual graph, Line graph, Graphs with 4 and 5 vertices, King’s graph, Rook’s graph.

1. Introduction

Twin-width is an invariant of graphs introduced in [BKTW20]. It is used to study the parameterized complexity of graph algorithms. It has applications in logic, enumerative combinatorics etc. Recently, it has appeared in many articles ([BGKTW21], [BGKTW21′], [BGMSTT21], [BKRT22], [BGTT22], [BCKKLT22]). Moreover, it has been studied in the context of finitely generated groups [BGTT22]. The twin-width is defined for a finite simple graph, later it is extended for a simple infinite graph. The computation of twin-width of a finite graph is extremely difficult. It has been computed before for complete graphs, path graphs, cyclic graphs (or graphs with at most one cycle), Paley graphs, Caterpillar tree, planar graphs etc. In this article, we compute twin-width for all graphs with vertices 4 and 5 and prove the following theorems.

Theorem 1.1. The twin-width of a graph with 4 vertices is less than equal to 1.

Theorem 1.2. The twin-width of a graph with 5 vertices is less than equal to 2.

It is an open problem to determine whether there is an n-vertex graph having twin-width at least n/2 (see [AHKO22], page 3). Therefore, the above-mentioned theorems show that we should look for n ≥ 3. It is known that twin-width is invariant under taking complement graphs. It was not known how twin-width behaves under other graph operations. In this article, we prove that they are not invariant under taking dual graphs and line graphs.

Theorem 1.3. Let C be the collection of simple connected planar graphs whose dual are also a simple connected planar graphs. The construction of dual graphs does not preserve twin-width, i.e., there exists a graph G in C such that the twin-width of G and G* are not equal, where G* is the dual graph of G.
Theorem 1.4. The construction of line graph does not preserve twin-width.

However, there are some graphs, King’s graph and Rook’s graph, which are associated with Chess and are important in Graph Theory. In this article, we study twin-width of these graphs. We briefly recall the definitions of King’s graph and Rook’s graph. A King’s graph is a graph that represents all legal moves of the king chess piece on a chessboard where each vertex represents a square on a chessboard and each edge is a legal move. More specifically, an \((n \times m)\)-King’s graph is a King’s graph of an \((n \times m)\)-chessboard. On the other hand, a Rook’s graph is a graph that represents all legal moves of the rook chess piece on a chessboard. Each vertex of a rook’s graph represents a square on a chessboard, and each edge connects two squares on the same row or on the same column (each edge connects the squares that a rook can move between). In this article, we prove the following two theorems.

Theorem 1.5. The twin-width of a \((n \times m)\)-King’s graph is less than 7.

Theorem 1.6. The twin-width of a \((n \times m)\) Rook’s graph is less than equal to \(2(m - 1)\).

1.1. Organization. In Section 2, we introduce our necessary definitions, notations and abbreviations. In Section 3, we survey the known results of twin-width of finite graphs. We study the behaviour of twin-width under graph operations in Section 4. In Section 5, we compute the twin-width of finite graphs with 4 and 5 vertices. In Section 6, we provide upper bounds of twin-width of King’s graph and Rook’s graph.

2. Preliminaries: some definitions, notations and abbreviations:

A trigraph \(G\) is a graph with a vertex set \(V(G)\), a black edge set \(E(G)\), and a red edge set \(R(G)\) (the error edges), where \(E(G)\) and \(R(G)\) are disjoint. The set of neighbours of a vertex \(v\) in a trigraph \(G\), denoted by \(N_G(v)\), consists of all the vertices adjacent to \(v\) by a black or red edge. The degree of a vertex \(v\) is defined by the number \(|N_G(v)|\). A \(d\)-trigraph is a trigraph \(G\) such that the red graph \((V(G), R(G))\) has degree at most \(d\). In this situation, we also say that the trigraph has red degree at most \(d\).

A contraction or identification in a trigraph \(G\) consists of merging two (non-necessarily adjacent) vertices \(u\) and \(v\) into a single vertex \(w\), and defining the edges of \(G'\) (the new graph after contraction) in the following way: Every vertex of the symmetric difference \(N_G(u) \Delta N_G(v)\) is linked to \(w\) by a red edge. Every vertex \(x\) of the intersection \(N_G(u) \cap N_G(v)\) is linked to \(w\) by a black edge if both \(ux \in E(G)\) and \(vx \in E(G)\), and by a red edge otherwise. The rest of the edges (not incident to \(u\) or \(v\)) remain unchanged. Also, the vertices \(u\) and \(v\) (together with the edges incident to these vertices) are removed from the trigraph.

A sequence of \(d\)-contractions is a sequence of \(d\)-trigraphs \(G_n, G_{n-1}, \ldots, G_1\), where \(G_n = G\), \(G_1 = K_1\), where \(K_1\) is the graph on a single vertex, and \(G_{i-1}\)
is obtained from $G_i$ by performing a single contraction of two (non-necessarily adjacent) vertices. We observe that $G_i$ has precisely $i$ vertices, for every $i \in \{1, \cdots, n\}$. The twin-width of $G$, denoted by $tww(G)$, is the minimum integer $d$ such that $G$ admits a $d$-sequence.

Now, we provide an example of a sequence contractions of a finite graph. In the sequence of graphs depicted below, we start with the given finite graph in the extreme left end and we label the vertices by $a, b, c, d, e, f, g$. The next diagram is the result of the contraction of the vertices $e$ and $f$ and in the resulting graph we label the new vertex by $ef$. In this way, we obtain a sequence of graphs by gradually contacting other vertices. The graph in the extreme left end of the second line of this sequence is obtained by contracting the vertices $ad$ and $g$ in the graph depicted in the extreme right end of the first line of the sequence.

Moreover, we will draw every contraction sequence by this fashion in this article. We end this section by defining twin-width of an infinite graph. It is defined by the maximum of the twin-widths of its induced finite subgraphs.

3. A survey of the known results of twin-width of finite graphs

In this section, we survey the results regarding the twin-width of complete graphs, planar graphs, graphs with at most one cycle, Caterpillar tree and Paley graph.

**Theorem 3.1.** The complete graph with $n$-vertices, denoted by $K_n$, and the complete bipartite with $n, m$-vertices, denoted by $K_{n,m}$, have twin-width zero.

**Theorem 3.2.** [H22] The twin-width of any simple planar graph $G$ is at most $9$.

**Theorem 3.3.** [AHKO22] If every component of a graph $G$ has at most one cycle, then $tww(G) \leq 2$.

We obtain the following corollary from the above-mentioned theorem.
Corollary 3.4. The cyclic graph with \( n \) vertices (denoted by \( C_n \)) has twin-width less than equal to 2.

A caterpillar tree is a tree in which all the vertices are within distance 1 of a central path. We draw an example of a Caterpillar tree below.

\[
\begin{array}{c}
\text{a} \quad \text{b} \quad \text{c} \quad \text{e} \quad \text{k} \\
\text{d} \quad \text{i} \\
\text{f} \quad \text{g} \quad \text{h}
\end{array}
\]

Theorem 3.5. \[\text{AHKO22}\] For a tree \( T \), \( \text{tww}(T) \leq 1 \) if and only if \( T \) is a Caterpillar tree.

The above-mentioned theorem gives rise to the following corollary.

Corollary 3.6. The path graph with \( n \) vertices, denoted by \( P_n \), has twin-width less than equal to 1.

Let \( q \) be a prime power such that \( q \equiv 1 \pmod{4} \). The Paley graph of order \( q \), denoted by \( P(q) \), is defined as follows: The vertices of the graph are the elements of the field \( \mathbb{F}_q \) and the vertices \( i \) and \( j \) are adjacent if \( j - i \) is a quadratic residue in \( \mathbb{F}_q \).

Theorem 3.7. \[\text{AHKO22}\] For each prime \( q \) with \( q \equiv 1 \pmod{4} \), the Paley graph \( P(q) \) has twin-width exactly \( (q - 1)/2 \).

Remark 3.8. The twin-width of the class of Paley graphs is unbounded.

Finally, we end this section by mentioning results on groups. We say that a finitely generated group has bounded or unbounded twin-width if one of its Cayley graphs have bounded or unbounded twin-width.

Theorem 3.9. \[\text{BGTT22}\] The solvable, hyperbolic, ordered finitely generated groups have finite twin-width.

Theorem 3.10. \[\text{BGTT22}\] There is a finitely generated group with infinite twin-width.

4. Graph operations and their twin-width

In this section, we study the behaviour of twin-width under graph operations, like complement graph, dual graph and line graph.
4.1. **Complement of a graph.** The *complement of a graph* $G$ is a graph $H$ on the same vertices such that two distinct vertices of $H$ are adjacent if and only if they are not adjacent in $G$. We obtain the following theorem from [BKTW20] (see Subsection 4.1).

**Theorem 4.1.** Twin-width is invariant under complementation.

4.2. **Dual Graph.** The *dual graph* of a planar graph $G$ is a graph that has a vertex for each face of $G$. The dual graph has an edge for each pair of faces in $G$ that are separated from each other by an edge, and a self-loop when the same face appears on both sides of an edge. Now, we prove Theorem 1.3.

**Proof of Theorem 1.3:** Let $G$ be the following graph:

```
    a
   /\  \\
  d -- e
  |    |  \\
 b -- f -- c
```

If we contract any two vertices of $G$, it generates a red edge. Therefore, the twin-width of $G$ is greater than equal to 1. Now, we compute the dual graph of $G$. We denote the triangular region ‘def’ by $r_1$, the trapezium ‘bcfe’ by $r_2$, the trapezium ‘adfc’ by $r_3$, the trapezium ‘abed’ by $r_4$ and the region outside $\triangle abc$ by $r_5$. Therefore, the dual of $G$, denoted by $G^*$, will be the following graph:

```
    r1
   / \  \\
  r2 -- r3
  |    |  \\
  r4 -- r5
```

We apply the following 0-contraction sequence to $G^*$. 
Therefore, $G^*$ has twin-width zero. Hence, the twin-widths of $G$ and $G^*$ are different.

4.3. **Line graph of a graph.** The line graph of an undirected graph $G$ is another graph $L(G)$ that represents the adjacencies between edges of $G$. $L(G)$ is constructed in the following way: for each edge in $G$, make a vertex in $L(G)$; for every two edges in $G$ that have a vertex in common, make an edge between their corresponding vertices in $L(G)$. Now, we prove the Theorem 1.4.

**Proof of Theorem 1.4:** Let, $G$ be the following graph:

```
  a   e  
  |   /  |
  v b  d  |
  | /    
  v e    
```

Since $G$ is a complete bipartite graph, it has twin-width zero. However, the line graph of $G$, denoted by $L(G)$, is the following graph:

```
 b'   f'  
|   /    |
| / v d'  |
|v e'     |
```

We observe that if we contract any two vertices of $L(G)$, it generates a red edge. Therefore, the twin-width of $L(G)$ is greater than equal to 1, which implies that the twin-width is not preserved under taking line graph.

5. **Computation of twin-width of finite graphs with 4 and 5 vertices**

In this section, we prove Theorem 1.1 and Theorem 1.2.

**Proof of Theorem 1.1:** First, we make a list of graphs which are disconnected.
Since the twin-width of a (disconnected) graph is the maximum of the twin-widths of its components, it is easy to see that the twin-widths of the disconnected graphs with 4 vertices are zero. Now, we make a list of graphs which are Caterpillar tree.

By Theorem 3.5, these two graphs have twin-width less than equal to 1. Finally, we make a list of graphs whose complement graphs are disconnected.

Since the twin-width of a graph is same as the twin-width of the complement graph by Theorem 4.1 and the twin-width of a disconnected graph with 4 vertices is less than equal to 1, we obtain that the twin-widths of the graphs in the above list is zero. Hence, we have our theorem.

**Proof of Theorem 1.2** First, we make a list of disconnected simple graphs with 5 vertices:
Since the twin-width of a disconnected graph is the maximum of the twin-widths of its components and the twin-width of a simple graph with 4 vertices is less than equal to 1 by Theorem 1.1, the twin-width of the disconnected graphs with 5 vertices are less than equal to 1. Now, we make a list of graphs which are Caterpillar tree and have 5 vertices.

Since a Caterpillar tree has twin-width less than equal to 1 by 3.5, the graphs in the above list have twin-width less than equal to 1. Next, we make a list of graphs which have at most one cycle.

Since the graphs with at most one cycle have twin-width less than equal to 2, the graphs in the above-mentioned list have twin-width less than equal to 2. Now, we make a list of graphs which can be reduced to a graph with 4 vertices after applying a 1-step contraction (all edges will remain black).
Since the graphs with 4 vertices have twin-width less than equal to 1, the graphs in the above-mentioned list will also have twin-width less than equal to 1. Next, we have a graph with twin-width less than equal to 2.

Now, we make a list of graphs whose complement graphs are disconnected. Since a graph has same twin-width as its complement graphs by Theorem 4.1, the graphs in this list will have twin-width less than equal to 1.
(The vertex ‘c’ (or ‘d’) is connected by edges with every other vertices. Therefore, the complement graph is disconnected.)

(The vertex ‘c’ is connected by edges with every other vertices. Therefore, the complement graph is disconnected.)

(The vertex ‘c’ is connected by edges with every other vertices. Therefore, the complemented graph is disconnected.)

(The vertex ‘c’ (or ‘d’) is connected by edges with other vertices. Therefore, the complement graph is disconnected.)

(The vertex ‘c’ is connected by edges with every other vertices. Therefore, the complement graph is disconnected.)
(The vertex ‘c’ (or ‘d’ or ‘e’) is connected by edges with every other vertices. Therefore, the complement graph is disconnected.)

Finally, we are left with the following complete graph whose twin-width is zero.

Hence, we have our theorem □

6. Upper bound of twin-width of King’s graph and Rook’s graph

In this section, we prove Theorem 1.5 and Theorem 1.6. Before going into the proof, we introduce two types of graph products Strong Product and Cartesian Product which will be required in the proofs of the theorems. The strong product of graphs $G$ and $H$, denoted by $G \boxtimes H$, is a graph such that the vertex set of $G \boxtimes H$ is the Cartesian product $V(G) \times V(H)$, where $V(G)$ and $V(H)$ are the set of vertices of $G$ and $H$, respectively; and distinct vertices $(u, u')$ and $(v, v')$ are adjacent in $G \boxtimes H$ if and only if $u = v$ and $u'$ is adjacent to $v'$, or $u' = v'$ and $u$ is adjacent to $v$, or $u$ is adjacent to $v$ and $u'$ is adjacent to $v'$. On the other hand, the Cartesian product of graphs $G$ and $H$, denoted by $G \square H$, is a graph such that the vertex set of $G \square H$ is the Cartesian product $V(G) \times V(H)$; and two vertices $(u, u')$ and $(v, v')$ are adjacent in $G \square H$ if and only if either $u = v$ and $u'$ is adjacent to $v'$ in $H$, or $u' = v'$ and $u$ is adjacent to $v$ in $G$. However, we now prove Theorem 1.5.

**Proof of Theorem 1.5** The $(n \times m)$-King’s graph, is a strong product of two path graphs $P_n$ and $P_m$. We obtain from [BGKTW21] (Theorem 9) that

$$tww(G \boxtimes H) \leq \max\{tww(G)(\Delta(H) + 1) + 2\Delta(H), tww(H) + \Delta(H)\}.$$ 

Since the twin-width of a path graph is less than equal to 1 and $\Delta(P_m) = 2$, we obtain that the twin-width of a King’s graph with $n$-vertices is less than equal to 7.

Now, we prove Theorem 1.6.

**Proof of Theorem 1.6** The $(n \times m)$-Rook’s graph is a Cartesian product of two Complete graphs $K_n$ and $K_m$. We obtain from [PS22] (Theorem 3.1) that
for any graphs $G$ and $H$,
\[ \text{tww}(G \Box H) \leq \max\{\text{tww}(G) + \Delta(H), \text{tww}(H)\} + \Delta(H). \]
Since the twin-width of a complete graph is zero and $\Delta(K_n)$ is $(m - 1)$, we have the twin-width of $(n \times m)$-Rook’s graph is less than equal to $2(m - 1)$.

**References**

[AHKO22] Jungho Ahn, Kevin Hendrey, Donggyu Kim, and Sang-il Oum. Bounds for the Twin-width of Graphs. https://arxiv.org/pdf/2110.03957.pdf, 2022.

[BCKKLT22] Édouard Bonnet, Dibyayan Chakraborty, Eun Jung Kim, Noleen Köhler, Raul Lopes, and Stéphan Thomassé. Twin-width VIII: delineation and win-wins. CoRR, abs/2204.00722, 2022.

[BGKTW21] Édouard Bonnet, Colin Geniet, Eun Jung Kim, Stéphan Thomassé, and Rémi Watrigant. Twin-width II: small classes. In Proceedings of the 2021 ACM-SIAM Symposium on Discrete Algorithms (SODA 2021), pages 1977-1996, 2021.

[BGKTW21'] Édouard Bonnet, Colin Geniet, Eun Jung Kim, Stéphan Thomassé, and Rémi Watrigant. Twin-width III: max independent set, min dominating set, and coloring. In Nikhil Bansal, Emanuela Merelli, and James Worrell, editors, 48th International Colloquium on Automata, Languages, and Programming, ICALP 2021, July 12-16, 2021, Glasgow, Scotland (Virtual Conference), volume 198 of LIPIcs, pages 35:1-35:20. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2021.

[BGMSTT21] Édouard Bonnet, Ugo Giocanti, Patrice Ossona de Mendez, Pierre Simon, Stéphan Thomassé, and Szymon Toruńczyk. Twin-width IV: ordered graphs and matrices. CoRR, abs/2102.03117, 2021, to appear at STOC 2022.

[BGTT22] Édouard Bonnet, Colin Geniet, Romain Tessera, Stéphan Thomassé. Twin-width VII: groups. https://arxiv.org/abs/2204.12330, 2022.

[BKRT22] Édouard Bonnet, Eun Jung Kim, Anadeus Reinald, and Stéphan Thomassé. Twin-width VI: the lens of contraction sequences. CoRR, abs/2111.00282, 2021. To appear in the proceedings of SODA 2022.

[BKTW20] Édouard Bonnet, Eun Jung Kim, Stéphan Thomassé, and Rémi Watrigant. Twin-width I: tractable FO model checking. In Sandy Irani, editor, 61st IEEE Annual Symposium on Foundations of Computer Science, FOCS 2020, pages 601-612. IEEE, 2020. doi:10.1109/FOCS46700.2020.00062.

[H22] Petr Hliněný. Twin-width of Planar Graphs is at most 9. https://arxiv.org/pdf/2205.05378.pdf, 2022.

[PS22] William Pettersson, John Sylvester. Bounds on the Twin-Width of Product Graphs. https://arxiv.org/pdf/2202.11556.pdf, 2022.

Indian Statistical Institute, 203 Barrackpore Trunk Road, Kolkata 700 108
Email address: kdas.math@gmail.com