ALMOST BAND PRESERVERS

TIMUR OIKHBERG AND PEDRO TRADACETE

ABSTRACT. We study the stability of band preserving operators on Banach lattices. To this end the notion of ε-band preserving mapping is introduced. It is shown that, under quite general assumptions, a ε-band preserving operator is in fact a small perturbation of a band preserving one. However, a counterexample can be produced in some circumstances. Some results on automatic continuity of ε-band preserving maps are also obtained.

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1. Introduction

This paper is devoted to the stability of band preserving operators on a Banach lattice. Recall that a band $Y$ in a Banach lattice $X$ is an ideal (i.e. a subspace $Y$ such that if $y \in Y$ and $|x| \leq |y|$, then $x \in Y$) which is also closed under arbitrary suprema, i.e. for every collection $(y_\alpha)_{\alpha \in A}$ in $Y$ such that $\bigvee_{\alpha \in A} y_\alpha$ exists in $X$, this element must belong to $Y$. For instance, it is easy to see that on the spaces $L_p(\Omega, \Sigma, \mu)$, every band corresponds to the set of elements supported on some $A \in \Sigma$.

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A linear operator on a Banach lattice \( T : X \to X \) is band-preserving (BP for short) if \( T(Y) \subset Y \) for any band \( Y \subset X \). The study of band-preserving operators can be traced back to the work of H. Nakano [10]. A complete account on band-preserving operators can be found in [9, Section 3.1] and in [3, Sections 2.3 and 4.4], see also the survey paper [5]. Let us recall a useful characterization of BP operators on a Banach lattice due to Y. Abramovich, A. Veksler and A. Koldunov [1]: Given a Banach lattice \( X \), and an operator \( T \in B(X) \), the following are equivalent

1. \( T \) is band preserving.
2. \( T \) is an orthomorphism, i.e. \( T \) is order bounded and \( |x| \land |y| = 0 \) implies \( |x| \land |Ty| = 0 \).
3. \( T \) is in the center of \( X \), i.e. there is some scalar \( \lambda > 0 \) such that \( |Tx| \leq \lambda|x| \) for every \( x \in X \).

We say that a linear map \( T : X \to X \) is \( \varepsilon \)-band preserving (\( \varepsilon \)-BP in short) if, for any \( x \in X \),

\[
\sup\{\|Tx \land y\| : y \geq 0, y \perp x\} \leq \varepsilon\|x\|.
\]

Our main concern is to study when an \( \varepsilon \)-BP operator is a small perturbation of a band-preserving operator. That is, given a Banach lattice \( X \) and an \( \varepsilon \)-BP operator \( T \in B(X) \), when can one find a band-preserving \( S \in B(X) \) so that \( \|T - S\| \leq \phi(\varepsilon) \) for some function \( \phi \) satisfying \( \phi(\varepsilon) \to 0 \) as \( \varepsilon \to 0 \)?

We are also interested in quantitative versions of some well known facts concerning band-preserving operators. For instance, on a \( \sigma \)-Dedekind complete Banach lattice an operator is band-preserving if and only if it commutes with every band projection. A version of this result in terms of the size of the commutators \([T, P]\) where \( P \) is a band projection and \( T \) is \( \varepsilon \)-BP will be given in Proposition 2.4. As a consequence, we obtain a quantitative version of another stability property of band-preserving operators, due to C. B. Huijsmans and B. de Pagter, that the inverse of a bijective BP operator is also BP [6] (see Corollary 2.5).

The paper is organized as follows: A discussion of several properties of almost band preserving operators as well as some equivalent characterizations of this class can be found in Section 2. Recall that a band preserving map on a Banach lattice is always bounded [1]. Motivated by this fact, in Section 3, we will study the automatic continuity of \( \varepsilon \)-BP maps. Almost central operators and their connection with \( \varepsilon \)-BP operators will be studied in Section 4. In Section 5, we prove that any \( \varepsilon \)-BP map on a Banach lattice \( X \) is a small perturbation of a BP one, provided \( X \) is order continuous (Theorem 5.1) or has Fatou norm (Proposition 5.2).

Section 6 contains a similar result for \( \varepsilon \)-BP maps on \( C(K) \) spaces (Theorem 6.2). In Section 7, we present an example of a Banach lattice \( E \) with the property that, for every \( \varepsilon > 0 \), there exists an \( \varepsilon \)-BP contraction \( T \in B(E) \) whose distance from the set of BP maps is larger than 1/2 (Proposition 7.1).
Throughout, we use standard Banach lattice terminology and notation. For more information we refer the reader to the monographs [3] or [9]. The closed unit ball of a normed space $Z$ is denoted by $B(Z)$.

2. Basic properties of almost band-preserving operators

Definition 2.1. Given a Banach lattice $X$, we say that a linear mapping $T : X \to X$ is $\varepsilon$-band preserving ($\varepsilon$-BP) if, for any $x \in X$,
\[
\sup\{\|Tx\| \land y : y \geq 0, y \perp x\} \leq \varepsilon\|x\|.
\]
Observe that every bounded operator $T : X \to X$ is trivially $\|T\|$-BP. Thus, for a bounded operator $T$, $\varepsilon$-BP is meaningful only for $\varepsilon < \|T\|$. Note that if two operators $T_1, T_2 \in B(X)$ are such that $T_i$ is $\varepsilon_i$-BP for $i = 1, 2$, then $T_1 + T_2$ is $(\varepsilon_1 + \varepsilon_2)$-BP. Similarly, if $T \in B(X)$ is $\varepsilon$-BP, then, for any scalar $\lambda$, $\lambda T$ is $|\lambda|\varepsilon$-BP.

In order to reformulate Definition 2.1 in the language of bands, we need to recall the notion of band projection. A band $Y$ of a Banach lattice $X$ is called a projection band if $X = Y \oplus Y^\perp$, where
\[
Y^\perp = \{x \in X : |x| \land |y| = 0 \text{ for every } y \in Y\}.
\]
Several facts which arise for spaces with an unconditional basis can be generalized to more general Banach lattices by means of projection bands (see [8, 1.a]).

A characterization of projection bands can be found in [8, Proposition 1.a.10]. In particular, if $X$ is a $\sigma$-Dedekind complete Banach lattice (i.e. every bounded sequence has a supremum and an infimum), then for each $x \in X_+$ we can consider the principal band projection $P_x$ given by
\[
P_x(z) = \bigvee_{n=1}^\infty (nx \land z)
\]
for $z \in X_+$, and extended linearly as $P_x(z) = P_x(z_+) - P_x(z_-)$ for a general $z \in X$. This defines a projection onto the principal band generated by $x$. Recall that a Banach lattice $X$ is said to have the Principal Projection Property (PPP for short) if every principal band (a band generated by a single element) is a projection band. By [9, pp. 17-18], a Banach lattice has the PPP if and only if it is $\sigma$-Dedekind complete.

The study of projection bands was initiated in the classical work of S. Kakutani [7] concerning concrete representations of Banach lattices. For properties of band projections, see [9, Section 1.2]. Also recall that, by [9, Proposition 2.4.4], if $X$ is order continuous, then every closed ideal in $X$ is a projection band. The next proposition gives some equivalent reformulations of the definition of $\varepsilon$-BP operator by means of band projections.

Proposition 2.2. Given a Banach lattice $X$ and an operator $T : X \to X$, consider the following statements:

1. $T$ is $\varepsilon$-BP.
(2) For any band projection \( P \) and any \( x \in X \), \( \|PTx\| \leq \varepsilon \|x\| \) whenever \( Px = 0 \).

(3) For any principal band projection \( P \) and any \( x \in X \), \( \|PTx\| \leq \varepsilon \|x\| \) whenever \( Px = 0 \).

Then (1) \( \Rightarrow \) (2) \( \Rightarrow \) (3). Moreover, if \( X \) is \( \sigma \)-Dedekind complete, then (3) \( \Rightarrow \) (1).

We need a lemma, which may be known to experts (although we haven’t found it in the literature).

**Lemma 2.3.** Suppose \( X \) is a Banach lattice, \( x \) is a non-zero element of \( X \), and \( P \in B(X) \) is a band projection. Then the following are equivalent:

(1) \( Px = 0 \).

(2) \( P|x| = 0 \).

(3) \( x \perp P(X) \).

**Proof.** (1) \( \Rightarrow \) (2): write \( x = x_+ - x_- \). As \( 0 \leq Py \leq y \) for any \( y \in X_+ \), we conclude that \( 0 \leq Px_+ \leq x_+ \), and \( 0 \leq Px_- \leq x_- \). Consequently \( Px_+ \) and \( Px_- \) are disjoint, hence \( P|x| = Px_+ + Px_- = |Px| \). If \( Px = 0 \), then \( P|x| = |Px| = 0 \).

(2) \( \Rightarrow \) (3): It suffices to show that \( x \wedge z = 0 \) whenever \( x, z \in X_+ \) satisfy \( Px = 0 \) and \( Pz = z \). Let \( u = x \wedge z \). Then \( Pu \leq Px = 0 \). On the other hand, \( P \) is a band projection. As \( z \) belongs to the band \( P(X) \), the same must be true for \( u \), hence \( u = Pu = 0 \).

(3) \( \Rightarrow \) (1): By [9, Lemma 1.2.8], \( I - P \) is the band projection onto \( P(X) \). Then \( (I - P)x = x \), hence \( Px = 0 \). \( \square \)

**Proof of Proposition 2.2.** (1) \( \Rightarrow \) (2): Fix a norm one \( x \in X \), and suppose \( P \) is a band projection so that \( Px = 0 \). We have to show that \( \|PTx\| \leq \varepsilon \). If \( PTx = 0 \), we are done. Otherwise, let \( y = |PTx| \). By Lemma 2.3, \( P|x| = 0 \), and \( x \perp P(X) \). Moreover, by the proof of that lemma, \( y = P|Tx| \perp x \). Write \( |Tx| = P|Tx| + (I - P)|Tx| \). The ranges of \( P \) and \( I - P \) are mutually disjoint bands. Therefore,

\[
|Tx| \land y = (P|Tx| + (I - P)|Tx|) \land y = (P|Tx|) \land y = P(|Tx|).
\]

Indeed,

\[
(P|Tx| + (I - P)|Tx|) \land y \leq (P|Tx|) \land y + ((I - P)|Tx|) \land y = (P|Tx|) \land y,
\]

and on the other hand, by the positivity of \( I - P \), \( (P|Tx| + (I - P)|Tx|) \land y \geq (P|Tx|) \land y \). From (1), it follows that \( \|PTx\| \leq \varepsilon \|x\| \).

(2) \( \Rightarrow \) (3) is clear.

(3) \( \Rightarrow \) (1): Assume that \( X \) is \( \sigma \)-Dedekind complete. For every \( x \in X \) we can consider \( P_x \) the band projection onto the band generated by \( x \). Suppose \( y \) is positive, and disjoint from \( x \). Then \( Q = I - P_x \) is a band projection, with \( Qy = y \). As shown above,

\[
|Tx| \land y = (Q|Tx|) \land y,
\]
hence \(\|Tx \land y\| \leq \|QTx\|\). However, \(QTx = |QTx|\), hence, by (3),
\[
\|Tx \land y\| \leq \|QTx\| \leq \varepsilon \|x\|.
\]

Recall that if \(E\) has the PPP, then \(T \in B(E)\) is BP if and only if it commutes with any band projection [9, Proposition 3.1.3]. Given operators \(S, T \in B(E)\), we consider their commutator \([S, T] = ST - TS\).

**Proposition 2.4.** Let \(T\) be an operator on a \(\sigma\)-Dedekind complete Banach lattice.

1. If for every band projection \(P\), \(\|[P, T]\| \leq \varepsilon\), then \(T\) is \(\varepsilon\)-BP.
2. If \(T\) is \(\varepsilon\)-BP then for any band projection \(P\), \(\|[P, T]\| \leq 2\varepsilon\).

**Proof.** We will use the equivalence with (2) in Proposition 2.2. Suppose first that for every band projection \(P\), \(\|[P, T]\| \leq \varepsilon\). Let \(Q\) be a band projection and \(x\) be such that \(Qx = 0\), then we have
\[
\|QTx\| = \|(QT - TQ)x\| \leq \|[Q, T]\| \|x\| \leq \varepsilon \|x\|.
\]

For the second statement, given a band projection \(P\), let \(P^\perp\) denote its orthogonal band projection. For \(x \in X\) we have
\[
\|(PT - TP)x\| = \|(PT - TP)(Px + P^\perp x)\|
\]
\[
= \|PTPx - TPx + PTP^\perp x - TPP^\perp x\|
\]
\[
= \| - P^\perp TPx + PTP^\perp x\|
\]
\[
\leq \|P^\perp TPx\| + \|PTP^\perp x\|
\]

Now, since \(P^\perp Px = PP^\perp x = 0\) we get that \(\|(PT - TP)x\| \leq 2\varepsilon \|x\|\), i.e. \(\|[P, T]\| \leq 2\varepsilon\). \(\square\)

The following is a version of the result in [6] that the inverse of a bijective BP operator is also BP.

**Corollary 2.5.** Let \(X\) be a \(\sigma\)-Dedekind complete Banach lattice. If \(T \in B(X)\) is invertible and \(\varepsilon\)-BP, then \(T^{-1}\) is \((2\|T^{-1}\|^2\varepsilon)\)-BP.

**Proof.** Let \(P\) be any band projection. By Proposition 2.4, we have \(\|PT - TP\| \leq 2\varepsilon\). Therefore,
\[
\|T^{-1}P - PT^{-1}\| = \|T^{-1}(PT - TP)T^{-1}\| \leq 2\|T^{-1}\|^2\varepsilon,
\]
and the result follows by Proposition 2.4 again. \(\square\)

**Remark 2.6.** In general, the \(\|T^{-1}\|^2\) factor cannot be avoided in Corollary 2.5, even when \(T\) is positive. Indeed, consider
\[
T = \begin{pmatrix} 0 & \varepsilon \\ \varepsilon & 0 \end{pmatrix},
\]
acting on $\ell_p^2$ (with $1 \leq p \leq \infty$). Clearly $\|T\| = \varepsilon$, hence $T$ is $\varepsilon$-BP. However, $T^{-1}$ cannot be $c$-BP for $c < 1/\varepsilon$. Indeed, suppose $T^{-1}$ is $c$-BP, then we have

$$c\varepsilon \geq \left\| T^{-1} \left( \begin{array}{c} \varepsilon \\ 0 \\ 1 \end{array} \right) \right\| = 1.$$ 

Thus, Corollary 2.5 is sharp (up to a constant independent of $\|T^{-1}\|$).

Below we show that any band-preserving operator on a Köthe function space is a multiplication operator. Recall that a Köthe function space on a $\sigma$-finite measure space $(\Omega, \Sigma, \mu)$ is a Banach space $X$ consisting of equivalence classes, modulo equality almost everywhere, of locally integrable functions on $\Omega$ such that:

1. If $|f(\omega)| \leq |g(\omega)|$ holds a.e. on $\Omega$ with $f$ $\Sigma$-measurable and $g \in X$, then $f \in X$ and $\|f\| \leq \|g\|$.
2. If $S \in \Sigma$, and $\mu(S) \in (0, \infty)$, then $\chi_S \in X$.

**Proposition 2.7.** Suppose $E$ is a Köthe function space on a Borel measure space $(\Omega, \mu)$, and $T \in B(E)$ is band preserving. Then there exists $\phi \in L_\infty(\mu)$ so that $Tf = \phi f$ for any $f \in E$.

A similar result was established in [13], and our proof is similar to the one given there.

**Proof.** Suppose $T \in B(E(\Omega, \mu))$ is band-preserving. By [9, Theorem 3.1.5], $T$ is regular, hence (in the terminology of [9, Section 3.1]) an orthomorphism.

First suppose the measure $\mu$ is finite. Let $1 = 1_\Omega$, and set $\phi = T1$. Note that $\phi$ is essentially bounded, with $\|\phi\|_\infty \leq \|T\|$. Indeed, suppose otherwise, and find a set $S$ of positive measure so that $|\phi| > \|T\||1\| \mu$-a.e. on $S$. However,

$$\chi_S\phi = \chi_S[T(\chi_S + \chi_{\Omega \setminus S})] = \chi_S[T\chi_S] + \chi_S[T\chi_{\Omega \setminus S}].$$

As $T$ is band-preserving, the last term vanishes. Thus,

$$\|T\chi_S\| = \|\chi_S(T\chi_S)\| = \|\chi_S\phi\| > \|T\||\chi_S||,$$

which is a contradiction.

Define the operator $S \in B(E)$ via $Sf = \phi f$. Then $S1 = T1$. As 1 generates $E$ as a band, [9, Proposition 3.1.6] implies $T = S$.

Now suppose $\mu$ is $\sigma$-finite. Represent $\Omega$ as an increasing union of the sets $\Omega_i$, such that $\mu(\Omega_i)$ is finite. Let $\mu_i = \mu|_{\Omega_i}$, and define the operator $T_i \in B(E(\Omega_i, \mu_i))$ via $T_if = (Tf)1_{\Omega_i}$. Clearly $T_i$ is band-preserving, hence by the above, there exists $\phi_i \in L_\infty(\mu)$, supported on $\Omega_i$, so that $T_i f = \phi_i f$ for any $f$. It is clear that $\phi_i + 1_{\Omega_i} = \phi_i$. Due to the boundedness of $T$, $\phi \in L_\infty(\mu)$. 

**Remark 2.8.** By [9, Theorem 3.1.12], any band-preserving operator $T$ on a Banach lattice satisfies $-\|T\|I \leq T \leq \|T\|I$. 

\[\text{Remark 2.8.} \]
Recall that an ideal $U$ in a Banach lattice $X$ is a subspace with the property that $y \in U$ whenever $|y| \leq |x|$ and $x \in U$. One might consider ideal preserving operators $T : X \to X$, i.e. those satisfying that for every (closed) ideal $U \subset X$, $T(U) \subset U$. However, this notion is actually equivalent to that of band preserving operator: since a band is also an ideal, every ideal preserving operator is in particular band preserving; on the other hand, if $T$ is a band preserving operator, then $|Tx| \leq ||T|||x|$, and this in turn implies that $T$ is ideal preserving.

We show that the same holds for “almost” band preserving and ideal preserving maps.

**Definition 2.9.** Given a Banach lattice $X$, a linear map $T : X \to X$ is $\varepsilon$-ideal preserving ($\varepsilon$-IP, for short) if, for every ideal $U \subset X$ and $x \in B(X) \cap U$, there exist $y \in U$ and $z \in X$ with $\|z\| \leq \varepsilon$, such that $Tx = y + z$.

**Theorem 2.10.** Suppose $X$ is a Banach lattice, $T : X \to X$ is a linear map, and $\varepsilon > 0$. Consider the following statements:

1. $T$ is $\varepsilon$-BP.
2. If $x \in B(X)$ and $x^* \in B(X^*)$ satisfy $\langle |x^*|, |x| \rangle = 0$, then $\langle x^*, Tx \rangle \leq \varepsilon$.
3. For every $\varepsilon' > \varepsilon$, $T$ is $\varepsilon'$-IP.
4. For every $x \in B(X)$ and $\varepsilon > 0$, there is $\lambda > 0$ such that $|Tx| \leq \lambda|x| + z$ for some $\|z\| \leq \varepsilon'$.

Then (2) $\iff$ (3) $\iff$ (4) $\Rightarrow$ (1). If, in addition, $T$ is bounded, then (1) $\Rightarrow$ (2) (that is, all the four statements are equivalent).

**Proof.** (1) $\Rightarrow$ (2), for $T$ bounded: Suppose, for the sake of contradiction, that (1) holds, but (2) doesn’t. Then there exist $x \in B(X)$ and $x^* \in B(X^*)$ so that $\langle |x^*|, |x| \rangle = 0$ and

$$\langle |x^*|, |Tx| \rangle \geq \langle |x^*|, T x \rangle = c > \varepsilon.$$ 

Pick $\delta > 0$ so that $\varepsilon + \|T\|\|T\| + 1\delta < c$.

For brevity of notation, let $x' = |Tx|$. We find $y, y' \in X$ so that $|y| \leq |x|$, $\|x - y\| \leq (\|T\| + 1)\delta$, $y' \geq 0$, $y' \perp y$, and $\|x' \wedge y'\| \geq c$. Once this is achieved, the inequality

$$\|Tx' \wedge y'\| \geq \|Tx' \wedge y'\| - \|T\|\|x - y\| > \varepsilon$$

will give the desired contradiction.

Consider the (not necessarily closed) ideal $I \subset X$, generated by $x_0 = x' \vee |x|$. In a canonical fashion, we find a bijective lattice homomorphism $j : C(\Omega) \to I$, where $\Omega$ is a Hausdorff compact (so $j1 = x_0$). We have

$$K := \|j\|_{B(C(\Omega), X)} = \|j1\|_{X} \leq ||x|| + ||x'|| \leq \|T\| + 1.$$

Let $\phi$ and $\phi'$ in $C(\Omega)$ such that $x = j(\phi)$ and $x' = j(\phi')$. Set $\psi = (\phi_+ - \delta1)_+ - (\phi_- - \delta1)_+$, and $y = j(\psi)$. Then $|\psi| \leq |\phi|$, hence $|y| \leq |x|$. Furthermore, $\|\psi - \phi\|_{\infty} = \delta$, hence $\|x - y\| \leq K\delta$. 

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Now consider the closed sets $\Omega_1 = \{\omega \in \Omega : |\phi(\omega)| \leq \delta/2\}$ and $\Omega_0 = \{\omega \in \Omega : |\phi(\omega)| \geq \delta\}$. By Urysohn’s Lemma, there exists $h \in C(\Omega)$ so that $0 \leq h \leq 1$, $h|_{\Omega_1} = 1$, and $h|_{\Omega_0} = 0$.

Consider $\mu \in C(\Omega)^*$ given by $\langle \mu, f \rangle = \langle |x^*|, j(f) \rangle$ for $f \in C(\Omega)$. Clearly $\mu$ is a positive measure, and $\langle \mu, |\phi| \rangle = \langle |x^*|, |x| \rangle = 0$.

Now set $\psi' = \phi/h$. Note that $1 - h \leq 2\delta^{-1}|\phi|$, hence $\langle \mu, (1 - h)\eta \rangle = 0$ for any $\eta \in C(\Omega)$. Consequently, $\langle \mu, \psi' \rangle = \langle \mu, \phi' \rangle \geq c$. Set $y' = j(\psi')$. As $\psi \perp \psi'$, we also have $y \perp y'$. Further, $0 \leq \psi' \leq \phi'$, hence $0 \leq y' \leq x'$.

Consequently,

$$\|x' \wedge y'\| = \|y'\| \geq \langle |x^*|, y' \rangle = \langle \mu, \psi' \rangle = \langle \mu, \phi' \rangle \geq c,$$

which is the desired result.

(2) $\Rightarrow$ (3): Suppose that (3) does not hold. Then there exist $\varepsilon' > \varepsilon$, an ideal $U \subseteq X$ and $x \in B(U)$ such that for every $y \in U$, $\|Tx - y\| > \varepsilon'$. We can and do assume $U$ is closed. By Hahn-Banach Theorem, there is $x^* \in B(X^*)$ such that $\langle x^*, y \rangle = 0$ for every $y \in U$ and

$$\langle x^*, Tx \rangle = \text{dist}(Tx, U) \geq \varepsilon' > \varepsilon.$$

As $U$ is an ideal, we have

$$\langle |x^*|, |x| \rangle = \sup \{|\langle x^*, y \rangle| : |y| \leq |x|\} = 0.$$

This is impossible if (2) holds.

(3) $\Rightarrow$ (4) is immediate by considering the principal ideal generated by $x$. 

(4) $\Rightarrow$ (2): Suppose $x^*$ and $x$ are as in (2). Fix $\varepsilon' > \varepsilon$ and find $\lambda$ s.t. $|Tx| \leq \lambda|x| + z$, with $\|z\| \leq \varepsilon'$. Then

$$\langle x^*, Tx \rangle \leq \langle x^*, |Tx| \rangle \leq \langle x^*, z \rangle \leq \|z\| \leq \varepsilon'.$$

As $\varepsilon'$ can be arbitrarily close to $\varepsilon$, we obtain $\|\langle x^*, Tx \rangle\| \leq \varepsilon$.

(4) $\Rightarrow$ (1): Pick disjoint $x \in B(X)$ and $y \in X_+$. For every $\varepsilon' > \varepsilon > 0$, by (4), there exist $\lambda > 0$ and $z \in X$ with $\|z\| \leq \varepsilon'$ and $|Tx| \leq \lambda|x| + z$. Without loss of generality we can take $z \geq 0$. It follows that

$$\|\|Tx \wedge y\| \leq \|\lambda|x| + z \wedge y\| \leq \|z \wedge y\| \leq \varepsilon'.$$

Since this holds for arbitrary $\varepsilon' > \varepsilon$ we get that $T$ is $\varepsilon$-BP.  

Notice the following fact concerning the duality of band projections and almost band preserving operators. This will be useful in the stability results of Section 5.

**Proposition 2.11.** Let $X$ be a Banach lattice:

1. If $P$ is a band projection on $X$, then $P^*$ is a band projection on $X^*$.
2. If $X$ is order continuous and $P$ is a band projection on $X^*$, then $P^*|_X$ is a band projection on $X$.
3. If $T \in B(X)$ is such that $T^*$ is $\varepsilon$-BP, then $T$ is $\varepsilon$-BP.
4. If $X$ is order continuous and $T \in B(X)$ is $\varepsilon$-BP, then $T^*$ is $\varepsilon$-BP.
Proof. (1): This is a direct consequence of the fact that $P$ is a band projection if and only if $P^2 = P$ and $0 \leq P \leq I$.

(2): By part (1), we know that $P^*$ is a band projection on $X^{**}$. For $x \in X$, we have that $|P^*x| \leq |x|$, and since $X$ is an ideal in $X^{**}$ [9, Theorem 2.4.2], it follows that $P^*x \in X$. Thus, $P^*|_X$ is a band projection on $X$.

(3): Let $x \perp y$ in $X$ with $|x| \leq 1$ and $y \geq 0$, we have to show that $\|Tx \wedge y\| \leq \varepsilon$. First, since $x \perp ([Tx] \wedge y)$, by [9, Lemma 1.4.3], we can find $y^* \in B(X^*)$ so that $\langle|y^*|, |x|\rangle$ = 0, and $\langle|y^*|, [Tx] \wedge y\rangle = \|Tx \wedge y\|$. Passing to $|y^*|$, we can and do assume $y^* \geq 0$.

Thus, it suffices to prove that $\langle y^*, [Tx]\rangle \leq \varepsilon$. By [9, Lemma 1.4.4],

$$\langle y^*, [Tx]\rangle = \max \{|\langle z^*, Tx\rangle| : |z^*| \leq |y^*|\}.$$ For any such $z^*$, we have $\langle|z^*|, |x|\rangle = 0$. Therefore, $x$ annihilates on the principal ideal generated by $z^*$. Moreover, since the elements of $X$ acting on $X^*$ are order continuous functionals (cf. [3, p. 61]), it follows that $x$ also annihilates on the band generated by $z^*$.

Let $P$ denote the band projection onto the band generated by $z^*$. By the above, we have $P^*x = 0$. Therefore, by Proposition 2.2 we get

$$\langle T^*z^*, x\rangle = \langle P^1T^*z^*, x\rangle \leq \|P^1T^*z^*\| \leq \varepsilon.$$

(4): Suppose $T$ is $\varepsilon$-BP, and let $P$ be a band projection on $X^*$. Since $X^*$ is $\sigma$-Dedekind complete, by Proposition 2.2, it is enough to show that for every $x^* \in X^*$ such that $Px^* = 0$ we have $\|PT^*x^*\| \leq \varepsilon\|x^*\|$. According to (2), there is a band projection on $X$, given by $Q = P^*|_X$ such that $Q^* = P$. Let $Q^1 = I - Q$ be the band projection onto the complementary band. Since $T$ is $\varepsilon$-BP and $Q^1Qx = 0$, by Proposition 2.2, we have that $\|Q^1TQx\| \leq \varepsilon\|Qx\| \leq \varepsilon\|x\|$. Now, using the fact that $Px^* = 0$, we have

$$\|PT^*x^*\| = \sup\{\langle PT^*x^*, x\rangle : x \in X, \|x\| \leq 1\}$$

$$= \sup\{\langle x^*, TQx\rangle : x \in X, \|x\| \leq 1\}$$

$$= \sup\{\langle x^*, (I - Q)TQx\rangle : x \in X, \|x\| \leq 1\}$$

$$\leq \sup\{\|Q^1TQx\|||x^*| : x \in X, \|x\| \leq 1\} \leq \varepsilon\|x^*\|,$

as desired.

It is well known that any band-preserving operator is also disjointness preserving. For $\varepsilon$-BP maps, a similar result holds. Recall that an operator between Banach lattices $T : X \rightarrow Y$ is $\varepsilon$-disjointness preserving if $\|Tx \wedge Ty\| \leq \varepsilon$ whenever $x, y \in B(X)$ satisfy $x \perp y$. This class of operators has been the object of research in [11], where it was studied whether an $\varepsilon$-disjointness preserving operator can always be approximated by a disjointness preserving one.

**Proposition 2.12.** If $X$ is a Banach lattice, and $T \in B(X)$ is $\varepsilon$-BP, then $T$ is $2\varepsilon$-disjointness preserving.
The inequality (which can, in turn, be identified with $C(K)$), we see that $a \perp x'$ and $b \perp y'$. As $T$ is $\varepsilon$-BP, we have $||Tx'| \wedge a|| \leq \varepsilon$ and $||Ty' \wedge b|| \leq \varepsilon$. We have

$$|Tx'| \wedge |Ty'| \wedge (a + b) \leq |Tx'| \wedge a + |Ty'| \wedge b.$$ 

The inequality (2.1) now follows from $a + b \geq ne$. \hfill $\Box$

We do not know whether the continuity of $T$ is actually necessary in Proposition 2.12. However, for $\sigma$-Dedekind complete spaces we have:

**Proposition 2.13.** If $X$ is a $\sigma$-Dedekind complete Banach lattice, and $T$ is an $\varepsilon$-BP linear map, then $T$ is also $3\varepsilon$-disjointness preserving.

**Proof.** Suppose $x$ and $y$ are disjoint elements in the unit ball of $X$. Then

$$|Tx| \wedge |Ty| = (P_x|Tx| + P^+_x|Tx|) \wedge (P_y|Ty| + P^+_y|Ty|)$$

$$\leq (P_x|Tx|) \wedge (P_y|Ty|) + (P^+_x|Tx|) \wedge (P_y|Ty|)$$

$$+ (P_x|Tx|) \wedge (P^+_y|Ty|) + (P^+_x|Tx|) \wedge (P^+_y|Ty|).$$

By the triangle inequality, $||Tx| \wedge |Ty|| \leq 3\varepsilon$. \hfill $\Box$

Any disjointness preserving operator (hence also any band-preserving operator) is regular [9, Theorem 3.1.5]. Moreover, if $T \in B(X)$ is band-preserving, then so is $|T|$. One might wonder whether the modulus of a regular $\varepsilon$-BP operator is also $\varepsilon$-BP. This is the case for AM-spaces and AL-spaces. Recall that a Banach lattice is an AL-space if $||x + y|| = ||x|| + ||y||$.
whenever $x \land y = 0$; an AM-space if $\|x + y\| = \max\{\|x\|, \|y\|\}$ whenever $x \land y = 0$. AL-spaces are order isometric to spaces $L_1(\mu)$, while AM-spaces are order isometric to sublattices of spaces $C(K)$ [8, 1.b].

**Proposition 2.14.** Suppose $T \in B(X)$ is a $\varepsilon$-BP operator.

(1) If $X$ is an AM-space, and $T$ is regular, then $|T|$ is $\varepsilon$-BP.

(2) If $X$ is an AL-space, then $|T|$ is $\varepsilon$-BP.

**Proof.** (1) $X$ is an AM-space. Given $x, y \in X$ with $x \perp y$ we have

$$\|\|T||x| \land |y|\| \leq \|\|T||x| \land |y|\| = \left\| \left( \bigvee_{|z| \leq |x|} |Tz| \right) \land |y|\right\| = \left\| \bigvee_{|z| \leq |x|} |Tz| \land |y|\right\| \leq \bigvee_{|z| \leq |x|} \varepsilon \|z\| = \varepsilon \|x\|,$$

where the last inequality follows from the fact that $z \perp y$ for every $|z| \leq |x|$ and $T$ is $\varepsilon$-BP.

(2) $X$ is an AL-space, so in particular it is order continuous. By Proposition 2.11, we have that $T^*$ is $\varepsilon$-BP. Also, note that every operator on an AL-space is regular [3, Theorem 4.75]. By [9, Proposition 1.4.17], we have that $|T^*| = |T|^*$. Since $X^*$ is an AM-space, by part (1) we get that $|T|^*$ is $\varepsilon$-BP. Again, Proposition 2.11 yields that $|T|$ is $\varepsilon$-BP, as claimed. \(\square\)

**Remark 2.15.** Proposition 2.14 fails for general Banach lattices. For every $\varepsilon > 0$ there exists a regular $\varepsilon$-BP contraction $T \in B(\ell_2)$ so that $|T|$ is not $c$-BP whenever $c < 1/2$. An example can be found in [11, Proposition 9.4]. We briefly outline the construction.

For $i \in \mathbb{N}$ let $S_i$ be the $2^i \times 2^i$ Walsh unitary, and set $T_i = I_{\ell_2^{2^i}} + 2^{-i/2}S_i \in B(\ell_2^{2^i})$. For $\varepsilon > 0$, find $n \in \mathbb{N}$ so that $2^{-n/2} < \varepsilon$. Let $T = \oplus_{i \geq n} T_i$ be an operator on $E = (\bigoplus_{i=n}^{\infty} \ell_2^{2^i})_2$ (this space can be identified with $\ell_2$). Clearly $\|T - I_E\| < \varepsilon$, hence $T$ is $\varepsilon$-BP. However, as in [11, Proposition 9.4], $|T| = \oplus_i |T_i|$, where $|T_i| = I_{\ell_2^{2^i}} + \xi_i \otimes \xi_i$, with $\xi_i = 2^{-i/2}(1, \ldots, 1)$ is a unit vector in $\ell_2^{2^i}$. Taking $x = 2^{(1-i)/2}(1, \ldots, 1, 0, \ldots, 0)$ and $y = (0, \ldots, 0, 1, \ldots, 1)$ (both strings contain an equal number of 0’s and 1’s), we see that $\|\|T_i||x \land y\| = 1/2$.

One can use the same reasoning to construct, for $1 < p < \infty$ and $\varepsilon > 0$, a regular $\varepsilon$-BP operator $T \in B(\ell_p)$ so that $|T|$ is $c$-BP only when $c \geq c_p$, where $c_p > 0$ depends on $p$ only.

3. **Automatic continuity**

In certain situations, $\varepsilon$-BP linear maps are automatically continuous.
3.1. Köthe spaces. Recall that a Banach lattice $X$ has Fatou norm with constant $\hat{f}$ if, for any non-negative increasing net $(x_i) \subset X$, with sup$_{i} \|x_i\| < \infty$, and $\forall_i x_i \in X$, we have $\| \vee_i x_i \| \leq \hat{f} \sup_i \|x_i\|$. For Köthe function spaces this is equivalent to the following: if $f, f_1, f_2, \ldots \in X$ satisfy $f_n(\omega) \uparrow f(\omega)$ a.e., with $f_n(\omega) \geq 0$ a.e., then $\|f\| = \lim_n \|f_n\|$. Note that a Banach lattice which has a Fatou norm with constant $\hat{f}$ admits an equivalent lattice norm which is Fatou with constant $1$. Indeed, we can set

$$\|x\| = \inf \{\sup_i \|x_i\| : |x| = \vee_i x_i, \ \text{for } \ i \text{ increasing, sup } \|x_i\| < \infty\}.$$ 

If $(X, \| \cdot \|)$ is a Köthe function space, then the same is true for $(X, \| \cdot \|)$.

**Proposition 3.1.** Suppose $X$ is a Köthe function space on a $\sigma$-finite measure space $(\Omega, \mu)$, with Fatou norm. If $T : X \to X$ is a $\varepsilon$-BP linear map, then $T$ is continuous.

Let us first fix some notation. For a measurable $A \subset \Omega$, denote by $P_A$ the band projection onto the band generated by $A$ (i.e. $P_Ax = \chi_Ax$), and set

$$X_A = P_A(X) = \{ x \in X : x = \chi_Ax \}.$$ 

For any $x \in P_A(X)$, $\|P_ATx\| \leq \varepsilon \|x\|$. Indeed, it suffices to apply the definition of $\varepsilon$-BP to $y = |P_Ax|/\|P_Ax\|$.

For notational convenience, we assign infinite norm to any unbounded operator. By renorming if necessary, we can assume that the Fatou constant of $X$ equals 1.

**Lemma 3.2.** Suppose $X$ and $T$ are as in Proposition 3.1, and $(A_i)_{i \in I}$ is a family of disjoint subsets of $\Omega$, each having positive measure. Then there exists $C > 0$ so that $\|TP_{A_i}\| \leq C$ for all but finitely many indices $i \in I$.

**Proof.** Suppose otherwise. Then we can find a mutually disjoint sequence $(x_k)$ with $\text{supp} x_k \subset A_k$, so that, for each $k$, $\|x_k\| < 2^{-k}$, and $\|Tx_k\| > 2^k$. Let $x = \sum_{k=1}^{\infty} x_k$, and $\tilde{x}_k = x - x_k$. Then

$$\|Tx\| \geq \|PA_k(Tx_k + \tilde{x}_k)\| \geq \|PA_kTx_k\| - \|PA_kT\tilde{x}_k\|.$$ 

But $\|PA_kTx_k\| \geq \|Tx_k\| - \varepsilon \|x_k\| > 2^k - \varepsilon \|x_k\|$, while $\|PA_kT\tilde{x}_k\| \leq \varepsilon \|\tilde{x}_k\|$. Thus,

$$\|Tx\| \geq 2^k - \varepsilon(\|x_k\| + \|\tilde{x}_k\|) \geq 2^k - \varepsilon.$$ 

This inequality should hold for any $k$, which is impossible. \hfill $\Box$

**Lemma 3.3.** Suppose $X$ and $T$ are as in Proposition 3.1. If $(A_i)_{i \in \mathbb{N}}$ is an increasing sequence of measurable subsets of $\Omega$, so that for each $i \in \mathbb{N}$, $TP_{A_i}$ is bounded, then $\sup_n \|TP_{A_n}\| < \infty$.

**Proof.** Suppose $\sup_n \|TP_{A_n}\| = \infty$. Then there exist $1 \leq n_0 < n_1 < n_2 < \ldots$ so that $\|TP_{A_{n_0}}\| > 1$, and $\|TP_{A_{n_{k+1}}}\| > 3\|TP_{A_{n_k}}\|$ for every $k$. Consequently, $\|TP_{B_k}\| > 2^k$ for every $k$, where $B_0 = A_{n_0}$, and $B_k = A_{n_k} \setminus A_{n_{k-1}}$ for $k > 0$. This contradicts Lemma 3.2. \hfill $\Box$
Lemma 3.4. In the notation of Lemma 3.3, $TP_{\cup_i A_i}$ is bounded.

Proof. Let $A = \cup_i A_i$. Let $C = \sup_i \|TP_{A_i}\|$. Suppose, for the sake of contradiction, that there exists a norm one $x \in P_{A}(X)$, so that $\|Tx\| > C + 2\varepsilon$. Then $\|TP_{A_i}x\| > C + \varepsilon$. Since $X$ has the Fatou property we have that $\|TP_{A_n}x\| \rightarrow \|TP_{A}x\|$, thus $\|TP_{A_n}x\| > C + \varepsilon$ for $n$ large enough. Write $x = y + z$, where $y = P_{A_n}x$ and $z = P_{A \setminus A_n}x$. We have $\|TP_{A_n}y\| \leq C$ and $\|TP_{A_n}z\| \leq \varepsilon$, hence, by the triangle inequality, $\|TP_{A_n}x\| \leq C + \varepsilon$, yielding a contradiction. □

Proof of Proposition 3.1. Denote by $\Sigma$ the set of all equivalence classes of measurable subsets of $\Omega$, of positive measure (two sets are equivalent if the measure of their symmetric difference is 0). Abusing the notation slightly, we identify classes with their representatives. Denote by $\Sigma_b$ the set of all classes $S \in \Sigma$ so that, for any (equivalently, all) $S \in S$, $TP_{S}$ is bounded.

Note that $\Sigma_b$ is closed under finite or countably infinite unions. The finite case is clear. To handle the infinite case, consider $A_1, A_2, \ldots \in \Sigma_b$, and show that $A = \cup_k A_k \in \Sigma_b$ as well. Without loss of generality, we can assume $A_1, A_2, \ldots$ are disjoint. By Lemma 3.2, there exists $C > 0$ so that $\|TP_{A_i}\| < C$ for any $i$. Replacing now $A_k$ with $\cup_{i \leq k} A_i$, by Lemmas 3.3 and 3.4, it follows that $\|TP_{A}\| \leq C + 2\varepsilon$.

By Zorn’s Lemma (and taking the $\sigma$-finiteness of $\mu$ into account), we see that $\Sigma_b$ contains a maximal element $[A]$. We claim that $[A] = \Omega$. Indeed, otherwise $TP_{B}$ is unbounded for any $B \subset A^c$. If $A^c$ is a union of finitely many atoms, this is clearly impossible. Otherwise, write $A^c$ as a disjoint union of infinitely many sets $B_i$ of positive measure. By Lemma 3.2, $TP_{B_i}$ is bounded for some $i$ (in fact, for infinitely many $i$'s), hence $A \cup B_i \in \Sigma_b$, contradicting the maximality of $A$. □

Remark 3.5. In a similar fashion, one can prove the following: suppose $\varepsilon > 0$, and $X$ and $Y$ are Köthe function spaces on $(\Omega, \Sigma, \mu)$. Suppose a linear map $T : X \rightarrow Y$ has the property that, for any $S \in \Sigma$, and any $x \in X$ satisfying $x = \chi_S x$, we have $\|\chi_S [Tx]\| \leq \varepsilon \|x\|$. Then $T$ is continuous.

Corollary 3.6. For any $\varepsilon > 0$, any $\varepsilon$-BP linear map on an order continuous Banach lattice is continuous.

Proof. Suppose a Banach lattice $X$ is order continuous, and $T : X \rightarrow X$ is $\varepsilon$-BP. By the proof of [8, Proposition 1.a.9] (combined with [9, Proposition 2.4.4]), $X$ can be represented as an unconditional sum of mutually orthogonal projection bands $(X_\alpha)_{\alpha \in A}$, having a weak order unit. Denote the corresponding band projections by $P_\alpha$. For any $x \in X$, $\sum_\alpha P_\alpha x$ has at most countably many non-zero terms, and converges unconditionally. For $A \subset A$, $X_A = \oplus_{\alpha \in A} X_\alpha \subset X$ is the range of the band projection $P_A = \sum_{\alpha \in A} P_\alpha$ (indeed, $0 \leq P_A \leq I$). For each $\alpha$, $X_\alpha$ is order isometric to a Köthe function space [8, pp. 25-29].

Suppose, for the sake of contradiction, that $T : X \rightarrow X$ is an unbounded $\varepsilon$-BP map. As the unconditional decomposition of every $x \in X$ is at most
countable, there exists a countable set \( B \) so that \( P_B TP_B \) is unbounded. Write \( B = \{\beta_1, \beta_2, \ldots\} \).

By Proposition 3.1, \( P_\alpha TP_\alpha \) is bounded for any \( \alpha \), hence the same is true for \( TP_\alpha \). Note first that \( \sup_\alpha \|TP_\alpha\| < \infty \). Indeed, otherwise we can find distinct \( \alpha_i (i \in \mathbb{N}) \) and \( x_i \in X_{\alpha_i} \) so that \( \|x_i\| < 2^{-i} \), but \( \|P_{\alpha_i}Tx_i\| > 2^i + \varepsilon \).

Let \( x = \sum_i x_i \), and \( \tilde{x}_i = x - x_i \). Then for each \( i \),

\[
\|Tx\| \geq \|P_{\alpha_i}Tx\| \geq \|P_{\alpha_i}T\tilde{x}_i\| - \|P_{\alpha_i}T\tilde{x}_i\| > 2^i + \varepsilon - \varepsilon = 2^i,
\]

which is impossible.

Furthermore, let \( B_n = \{\beta_1, \ldots, \beta_n\} \). Then \( \sup_\alpha \|TP_{B_n}\| < \infty \). Indeed, otherwise we can find \( n_1 < n_2 < \ldots \) so that there exists \( x_k \in \bigoplus_{i \in B_{n_k} \setminus B_{n_{k-1}}} X_{\beta_i} \) with \( \|x_k\| < 2^{-k} \) and \( \|Tx_k\| > 2^k + \varepsilon \). Obtain a contradiction by considering \( x = \sum_k x_k \) (as in Lemma 3.3).

Finally set \( C = \sup_\alpha \|TB_n\| \). Pick a norm one \( x \in X_B \). By the order continuity of \( X \), \( P_{B_n} \to P_B \) point-norm, hence for every \( \delta > 0 \) there exists \( n \) so that \( \|P_{B_n}Tx\| > \|P_B Tx\| - \delta \). But (reasoning as in the proof of Lemma 3.4)

\[
\|P_{B_n}Tx\| \leq \|P_{B_n}TP_{B_n}x\| + \|P_{B_n}TP_{B_n \setminus B_n}x\| \leq C + \varepsilon,
\]

hence \( \|Tx\| \leq C + 2\varepsilon + \delta \). This contradicts our assumption that \( TP_B \) is unbounded. \( \square \)

3.2. \( C_0(K,X) \) spaces. If \( X \) is a Banach lattice, and \( K \) is a locally compact Hausdorff space, let \( C_0(K,X) \) denote the space of continuous functions \( f : K \to X \), having the property that, for any \( \varepsilon > 0 \), there exists a compact set \( \Omega \) so that \( \|f(t)\| < \varepsilon \) whenever \( t \notin \Omega \). We endow \( C_0(K,X) \) with the norm \( \|f\| = \sup_{t \in K} \|f(t)\|_X \), thus turning it into a Banach lattice with the pointwise order.

**Theorem 3.7.** Suppose \( X \) is a Köthe function space on a \( \sigma \)-finite measure space \( (\Omega, \mu) \) with the Fatou property, and \( K \) a locally compact Hausdorff space. Then any \( \varepsilon \)-BP linear map on \( C_0(K,X) \) is automatically continuous.

Applying this theorem with \( X = \mathbb{R} \), we conclude that any \( \varepsilon \)-BP linear map on \( C_0(K) \) is automatically continuous.

For the proof we need a topological result (cf. [12]).

**Lemma 3.8.** Suppose \((s_n)_{n \in \mathbb{N}}\) are distinct points in a locally compact Hausdorff space \( K \). Then there exist a family of disjoint open sets \((U_k)_{k \in \mathbb{N}}\) so that \( s_{n_k} \in U_k \) for any \( k \) \((n_1 < n_2 < \ldots)\).

**Proof.** We construct the sequence \((n_k)\), and the open sets \( U_k \), recursively.

Note first that for any sequence of distinct points \((t_i)_{i \in \mathbb{N}}\) in a Hausdorff space there is at most one natural number \( m \) so that any neighborhood of \( t_m \) contains all but finitely many members of the sequence \((t_i)\). Indeed, if there exist two numbers, say \( m \) and \( \ell \), with this property, then \( t_m \) and \( t_\ell \) cannot be separated, which cannot happen in a Hausdorff topology.
Consequently, if \((t_i)_{i \in \mathbb{N}}\) is a sequence of distinct points in a locally compact Hausdorff space, then for any \(i \in I\) (where \(I\) is either \(\mathbb{N}\) or \(\mathbb{N} \setminus \{m\}\), for the \(m\) corresponding to the sequence \((t_i)_{i \in \mathbb{N}}\)) there exists an open neighborhood \(V_i\) of \(t_i\) so that \(\{j \in I : t_j \notin V_i\}\) is infinite.

Let \(S_0 = \mathbb{N}\). Pick \(n_0 \in S_0\) in such a way that \(s_{n_0}\) has an open neighborhood \(U_0\) so that \(S_1 := \{n \in S_0 : s_n \notin U_0\}\) is infinite.

Now suppose we have already selected \(n_0 < \ldots < n_{k-1}\), and disjoint open sets \(U_0, \ldots, U_{k-1}\), so that \(s_{nj} \in U_j\) for \(0 \leq j \leq k - 1\), and

\[ S_k = \{n \in \mathbb{N} : s_n \notin \bigcup_{j=1}^{k-1} U_j\} \]

is infinite. Find \(n_k \in S_k\) with an open neighborhood \(V_k\) so that

\[ S_{k+1} := \{n \in S_k : s_n \notin V_k\} \]

is infinite. Note that the same property holds for \(U_k = V_k \setminus \bigcup_{j=0}^{k-1} U_j\).

Proceed further in the same manner to obtain a sequence with the desired properties. \(\Box\)

We now proceed to prove Theorem 3.7. For the rest of this subsection, \(K\) is locally compact Hausdorff, unless specified otherwise.

Suppose \(T : C_0(K, X) \to C_0(K, X)\) is an \(\varepsilon\)-BP linear map. For \(t \in K\), let

\[ \lambda_t = \| 4_t T \| = \sup \{ \| [T f](t) \|_X : \| f \| \leq 1 \} \in [0, \infty]. \]

We want to show that \(\sup_{t \in K} \lambda_t < \infty\).

**Lemma 3.9.** Suppose \(X\) is a Banach lattice, and \(T : C_0(K, X) \to C_0(K, X)\) is an \(\varepsilon\)-BP linear map. If \(f \in C_0(K, X)\) vanishes on an open set \(V \subset K\), then \(\| [T f](t) \|_X \leq \| f \|\) for any \(t \in V\).

**Proof.** By Urysohn’s Lemma, there is a continuous function \(h : K \to [0, 1]\) such that \(h(t) = 1\) and \(h(s) = 0\) for \(s \in V^c\). Let \(\phi = [T f](t) \cdot h \in C_0(K, X)\). We have that \(\phi \perp f\) and since \(T\) is \(\varepsilon\)-BP, it follows that

\[ \| [T f](t) \|_X = \| [T f](t) \|_X \cdot \| \phi(t) \|_X \leq \| [T f] \|_X \cdot \| \phi \| \leq \varepsilon \| f \|. \]

**Lemma 3.10.** For any \(t \in K\), any open neighborhood \(U\) with \(t \in U\), and any \(\sigma > 0\), there exists \(f \in B(C_0(K, X))\) so that \(f\) vanishes outside of \(U\), and \(\| [T f](t) \|_X > \lambda_t - \varepsilon - \sigma\).

**Proof.** Pick \(g \in B(C_0(K, X))\) so that \(\| [T g](t) \|_X > \lambda_t - \sigma\). Find an open set \(V\) so that \(\overline{V}\) is compact, and \(t \in V \subset \overline{V} \subset U\). Urysohn’s Lemma allows us to find a function \(h\) so that \(0 \leq h \leq 1\), \(h|_{\overline{V}} = 1\), and \(h|_{U^c} = 0\). Let \(f = hg\), and \(f' = (1 - h)g\). Since \(f'|_{V} = 0\), Lemma 3.9 gives \(\| [T f'](t) \|_X \leq \varepsilon \| f' \| \leq \varepsilon\).

By the triangle inequality,

\[ \| [T f](t) \|_X \geq \| [T g](t) \|_X - \| [T f'](t) \|_X > \lambda_t - \sigma - \varepsilon. \]

**Lemma 3.11.** If \((t_k)\) is a sequence of distinct points in \(K\), then \(\limsup_k \lambda_{t_k} < \infty\).
Proof. Suppose otherwise. Passing to a subsequence, we can assume that \( \lambda_{t_k} > 4^k + \varepsilon \) for any \( k \). Applying Lemma 3.8, and passing to a further subsequence if necessary, we can assume that there exist disjoint open sets \( U_k \) such that \( t_k \in U_k \) for every \( k \). By Lemma 3.10, we can find \( f_k \in B(C_0(K, X)) \), vanishing outside of \( U_k \), so that \( \|Tf_k(t_k)\|_X > 4^k \).

Now let \( f = \sum_k 2^{-k} f_k \). Clearly \( f \in C_0(K, X) \) (with \( \|f\| \leq 2 \)), and for every \( n \),

\[
\sum_{k \neq n} 2^{-k} f_k |U_n| = 0.
\]

Hence, by Lemma 3.9 we have

\[
\|Tf(t_n)\|_X \geq 2^{-n} \|Tf_n(t_n)\|_X - \varepsilon \| \sum_{k \neq n} 2^{-k} f_k \| > 2^n - 2\varepsilon,
\]

which contradicts the fact that \( Tf \in C_0(K, X) \). \( \square \)

**Lemma 3.12.** If \( t_n \to t \), then \( \lambda_t \leq \limsup \lambda_{t_n} \).

**Proof.** Suppose, for the sake of contradiction, that there exists a sequence \( (t_n) \) converging to \( t \), and \( \lambda_t > c > \sup_n \lambda_{t_n} \). Pick \( f \in B(C_0(K, X)) \) so that \( \|Tf(t)\|_X > c \). On the other hand,

\[
\|Tf(t)\|_X = \lim_n \|Tf(t_n)\|_X \leq c,
\]

a contradiction. \( \square \)

**Theorem 3.13.** Let \( K \) be a locally compact Hausdorff space without isolated points, and \( X \) a Banach lattice. If \( T : C_0(K, X) \to C_0(K, X) \) is a linear \( \varepsilon \)-BP mapping, then \( T \) is bounded.

**Proof.** As noted above, we need to show that \( \sup_{t \in K} \lambda_t < \infty \). Since \( K \) has no isolated points, for every \( t \in K \) there is a sequence \( (t_k) \) of distinct points in \( K \), such that \( t_k \to t \). By Lemmas 3.11 and 3.12, it follows that \( \lambda_t \) is finite for every \( t \in K \).

Suppose \( \sup_{t \in K} \lambda_t = \infty \), then it would be possible to find a sequence of distinct points \( (t_n) \), so that \( \lambda_{t_n} \) increases without a bound, which is impossible by Lemma 3.11. \( \square \)

**Proof of Theorem 3.7.** We will prove first that for every \( t \in K \), \( \lambda_t \) is finite. Suppose, for the sake of contradiction, that \( \lambda_t = \infty \) for some \( t \in K \). By Lemmas 3.11 and 3.12, \( t \) must be an isolated point in \( K \). Hence, we can consider the function \( \chi_{\{t\}} \in C_0(K) \) as well as the operators \( j_t : X \to C_0(K, X) \) and \( \delta_t : C_0(K, X) \to X \) given by \( j_t(x) = x \chi_{\{t\}} \) for \( x \in X \), and \( \delta_t(f) = f(t) \) for \( f \in C_0(K, X) \) respectively.

Let \( T_t = \delta_t j_t \). It is clear that \( T_t : X \to X \) is a linear mapping, and we claim it is \( \varepsilon \)-BP. Indeed, given \( x, y \in X \) with \( x \perp y \) and \( y \geq 0 \), we have that \( \chi_{\{t\}} x \perp \chi_{\{t\}} y \) in \( C_0(K, X) \), so as \( T \) is \( \varepsilon \)-BP it follows that

\[
\|T(\chi_{\{t\}} x) \land (\chi_{\{t\}} y)\| \leq \varepsilon \|\chi_{\{t\}} x\| = \varepsilon \|x\|.
\]
Therefore,
\[ ||T_t x \wedge y|| = ||\delta_t \left( |T(\chi_{\{t\}} x)\wedge (\chi_{\{t\}} y)| \right)|| \leq \varepsilon ||x||, \]
so \( T_t \) is \( \varepsilon \)-BP as claimed. Proposition 3.1 yields that \( T_t \) is bounded.

Since \( t \) is isolated, any \( f \in C_0(K, X) \) can be represented as \( f = f(t)\chi_{\{t\}} + f' \), where \( f' \) vanishes at \( t \) (equivalently, on a neighborhood of \( t \)), and moreover, \( ||f'|| \leq ||f|| \). If \( f \in B(C_0(K, X)) \), then by Lemma 3.9, we have
\[ \sup\{ ||T_f(t)||_{X} \leq ||(Tf(t)\chi_{\{t\}})(t)||_{X} + \varepsilon = ||T_{t}f|| + \varepsilon \} \]
Taking the supremum over all \( f \) as above, we obtain \( \lambda_t \leq ||T_t|| + \varepsilon < \infty \), a contradiction.

Suppose, for the sake of contradiction, that \( T \) is unbounded – that is, \( \sup_{t \in K} \lambda_t = \infty \). By the above, there must exist a sequence \( (t_k)_{k \in \mathbb{N}} \subseteq K \) so that \( \lim_{k} \lambda_{t_k} = \infty \). This, however, contradicts Lemma 3.11. \( \square \)

4. Some notions related to \( \varepsilon \)-band preservation

In this section, we consider some properties related to (and perhaps strengthening) band preservation.

**Definition 4.1.** An operator on a Banach lattice \( T : X \to X \) is \( \varepsilon \)-approximable by BP maps (in short \( T \in ABP(\varepsilon) \)) when there is a BP operator \( S \) such that \( ||T - S|| \leq \varepsilon \).

Clearly every \( T \in ABP(\varepsilon) \) is bounded, and \( \varepsilon \)-BP. In Section 5 we will study under which conditions every \( \varepsilon \)-BP operator is in \( ABP(\varepsilon) \).

Recall (Theorem 2.10) that \( T \in B(X) \) is \( \varepsilon \)-BP if and only if for every \( x \in B(X) \) and \( \varepsilon' > \varepsilon \) there exists \( \lambda = \lambda_x > 0 \) such that \( ||(|T x| - \lambda|x|)_{+}|| < \varepsilon' \). However, in principle, we have no control over \( \sup_{x \in B(X)} \lambda_x \). Strengthening this properties, we introduce:

**Definition 4.2.** An operator \( T : X \to X \) is in the \( \varepsilon \)-center (in short \( T \in \varepsilon - Z(X) \)) if there exists \( \lambda > 0 \) such that for every \( x \in B(X) \), there is \( z \in X \) with \( ||z|| \leq \varepsilon \) such that
\[ |Tx| \leq \lambda|x| + z. \]

Note that \( T \in 0 - Z(X) = Z(X) \) if and only if \( T \) is BP (\([1]\)). Moreover, if \( T \) is BP, and \( S \) is arbitrary, then \( T + S \in ||S|| - Z(X) \). In general, if \( T \in \varepsilon - Z(X) \), then \( T \) is \( \varepsilon \)-BP. We do not know whether the converse implication holds in general. However, if \( T \in ABP(\varepsilon) \), then \( T \in \varepsilon - Z(X) \). In Section 5 we will provide conditions for which every \( \varepsilon \)-BP operator is in \( ABP(4\varepsilon) \), hence it also belongs to \( 4\varepsilon - Z(X) \).

Note that \( T \in \varepsilon - Z(X) \) if and only if there is \( \lambda \geq 0 \) such that for every \( x \in B(X) \),
\[ ||(|T x| - \lambda|x|)_{+}|| \leq \varepsilon. \]

For \( T \in \varepsilon - Z(X) \), we define
\[ \rho_{\varepsilon}(T) = \inf\{ \lambda \geq 0 : \sup_{x \in B(X)} ||(|T x| - \lambda|x|)_{+}|| \leq \varepsilon \}. \]
Proposition 4.3. Let $X$ be a Banach lattice and $\varepsilon \geq 0$. Given $T \in B(X)$, we have $T \in \varepsilon - \mathcal{Z}(X)$ if and only if $T^* \in \varepsilon - \mathcal{Z}(X^*)$. Moreover, $\rho_\varepsilon(T) = \rho_\varepsilon(T^*)$.

Proof. Suppose $T \in \varepsilon - \mathcal{Z}(X)$ and take $\lambda > \rho_\varepsilon(T)$. By the Riesz-Kantorovich formulas (cf. [3, Theorem 1.18, and p. 58]), given $x \in B(X)_+$ and $x^* \in B(X^*)$ we have

$$\left< \left( |T^* x^*| - \lambda |x^*| \right)^+, x \right> = \sup_{0 \leq y \leq x} \left< |T^* x^*| - \lambda |x^*|, y \right>$$

$$= \sup_{0 \leq y \leq x} \left( \sup_{|z| \leq y} \left< T^* x^*, z \right> - \lambda \left| |x^*|, y \right> \right)$$

$$= \sup_{0 \leq y \leq x} \sup_{|z| \leq y} \left( \left| \left< x^*, Tz \right> - \lambda \left| |x^*|, y \right> \right) \right)$$

$$\leq \sup_{0 \leq y \leq x} \sup_{|z| \leq y} \|x^*\| \left( \left| Tz - \lambda |z| \right| \right)$$

$$\leq \sup_{0 \leq y \leq x} \sup_{|z| \leq y} \|x^*\| \left( \left| Tz - \lambda |z| \right| + \right) \leq \varepsilon.$$

Therefore, $T^* \in \varepsilon - \mathcal{Z}(X^*)$ and $\rho_\varepsilon(T^*) \leq \rho_\varepsilon(T)$.

Now, suppose $T^* \in \varepsilon - \mathcal{Z}(X^*)$. Applying the above argument to $T^*$ we obtain that $T^{**} \in \varepsilon - \mathcal{Z}(X^{**})$ with $\rho_\varepsilon(T^{**}) \leq \rho_\varepsilon(T^*)$. Since $T^{**}|_X = T$, this implies that $T \in \varepsilon - \mathcal{Z}(X)$ and

$$\rho_\varepsilon(T) \leq \rho_\varepsilon(T^{**}) \leq \rho_\varepsilon(T^*) \leq \rho_\varepsilon(T).$$

Definition 4.4. An operator on a Banach lattice $T : X \to X$ is locally $\varepsilon$-approximable by BP maps (in short $T \in \text{ABP}_{\text{loc}}(\varepsilon)$) provided for every $x \in X$, there is a BP operator $S_x$ such that

$$\|Tx - S_x x\| \leq \varepsilon \|x\|.$$

It is clear that every operator $T \in \text{ABP}_{\text{loc}}(\varepsilon)$ is $\varepsilon$-BP. Moreover, if the local approximants $S_x$ can be taken in such a way that $\sup_x \|S_x\| < \infty$, then $T \in \varepsilon - \mathcal{Z}(X)$, with $\rho_\varepsilon(T) \leq \sup_x \|S_x\|$. The following provides a converse:

Theorem 4.5. Suppose $E$ is a Banach lattice with a quasi-interior point, $\varepsilon > 0$, and $T \in B(E)$.

1. $T$ is $\varepsilon$-BP if and only if $T \in \text{ABP}_{\text{loc}}(\varepsilon')$ for every $\varepsilon' > \varepsilon$.
2. $T \in \varepsilon - \mathcal{Z}(E)$ with $\rho_\varepsilon(T) < C$ if and only if for every $x \in B(E)$ and every $\varepsilon' > \varepsilon$ there exists a BP map $T_x \in B(E)$ so that $\|T_x\| < C$, and $\|Tx - T_x x\| < \varepsilon'$.

Before the proof we need a decomposition result.

Lemma 4.6. Suppose $x$, $y$, and $z$ are elements of a Banach lattice $E$, so that $|y| \leq |x| + z$. Then there exists $u \in E$ so that $\|y - u\| \leq \|z\|$, and $|u| \leq |x|$.
Sketch of a proof. Without loss of generality, we may assume \( z \geq 0 \). We have
\[
||y| - |y| \wedge |x|| = ||y| \wedge (|x| + z) - |y| \wedge |x|| = ||y| \wedge (|(|x| + z) - |x||) || \leq ||z||.
\]
It remains to show that there exists \( u \in E \) so that \(|u| = a := |y| \wedge |x| \) and \( ||u - y|| = ||u|| - |y|| \). To this end, recall that the ideal \( I_y \) generated by \(|y|\) can be identified with \( C(K) \), for some \( K \) (with \(|y| \) corresponding to \( 1 \)). Further, \( y \) can be identified with the function \( y(t) = |y|(t)w(t) \), where \(|w| = 1\). We can set \( u(t) = a(t)w(t) \).

Proof of Theorem 4.5. (1) Suppose first that, for any \( x \in B(E) \), and any \( \varepsilon' > \varepsilon \), we can find a BP map \( T_x \) so that \( ||T_x - T_xx|| < \varepsilon' \). Then \( ||T_x|| \leq ||T_x x|| + ||T_x - T_xx|| \leq ||T_x|| |x| + ||T_x - T_xx||. \) If \( y \geq 0 \) is disjoint from \( x \), then \( ||T_x \wedge y|| \leq ||T_x|| |x| \wedge y + ||T_x - T_xx|| \wedge y \) has norm not exceeding \( ||T_x - T_xx|| \).

From the definition, \( T \) is \( \varepsilon \)-BP.

Suppose, conversely, that \( T \) is \( \varepsilon \)-BP. By Theorem 2.10, for any \( \varepsilon' > \varepsilon \), and any \( x \in B(E) \), there exists \( \lambda > 0 \) so that \( ||T_x|| \leq \lambda |x| + z \), where \( ||z|| < \frac{\varepsilon + \varepsilon'}{2} \).

By Lemma 4.6, there exists \( y \in E \) with \(|y| \leq \lambda |x| \), and \( ||y - T_x y|| < \varepsilon' \). Since \( E \) has a quasi-interior point, by [2, Lemma 4.17], there exists \( T_x \in B(E) \) band preserving such that \( ||T_xx - y|| \leq \frac{\varepsilon'}{2} ||z|| \), for every \( z \in E \) and \( ||T_xx - y|| \leq \frac{\varepsilon' - \varepsilon}{2} \). Hence,
\[
||T_x - T_xx|| \leq ||T_x - y|| + ||T_xx - y|| \leq \varepsilon'.
\]

Remark 4.7. For a Dedekind complete Banach lattice \( X \) and \( \varepsilon \geq 0 \), a similar argument using [3, Theorem 2.49] yields that if \( T \in (\varepsilon - Z)(X) \), then \( T \in ABP_{loc}(2\varepsilon) \) with local approximants satisfying \( \sup_x ||S_x|| \leq 2\rho_\varepsilon(T) \).

The following diagram illustrates the relation among the different notions introduced here, for bounded operators. The non-trivial implications are labeled with the reference of the corresponding result where they are proved. Note the values of \( \varepsilon \) may differ from one to another, and some of the implications are proved only for some classes of Banach lattices.

\[
\begin{array}{ccc}
ABP(\varepsilon) & 4.5 & ABP_{loc}(\varepsilon) \\
\varepsilon - Z & & \varepsilon - BP \\
\varepsilon - IP & 2.10 & \varepsilon - BP
\end{array}
\]

For unbounded operators the picture is different: we do not know whether \( \varepsilon \)-BP implies \( \varepsilon \)-IP.
In Section 7, we show that some of the arrows on the diagram cannot be reversed: for every $\varepsilon > 0$ there exists a contraction in $\varepsilon - Z$ (and in $\text{ABP}_{\text{loc}}(\varepsilon)$), but not in $\text{ABP}(\delta)$ for $\delta < 1/2$.

5. Stability of almost band preservers

We will show now that, under some mild hypothesis, an almost b and-preserving operator is close to a band-preserving one.

**Theorem 5.1.** If $E$ is an order continuous Banach lattice, and $T \in B(E)$ is $\varepsilon$-BP, then there exists a band-preserving $R \in B(E)$ so that $\|R\| \leq \|T\|$, and $\|T - R\| \leq 4\varepsilon$.

For positive $\varepsilon$-BP operators a similar result holds under weaker assumptions on $X$.

**Proposition 5.2.** Suppose $X$ is a Dedekind complete Banach lattice having a Fatou norm with constant $f$. Then for any positive $\varepsilon$-BP operator $T \in B(X)$ there exists $0 \leq S \leq T$ such that $S$ is band-preserving and $\|T - S\| \leq 4f\varepsilon$.

Note that order continuous Banach lattices, and dual Banach lattices, are Dedekind complete, and have Fatou norm with constant 1 (see e.g. [9, Theorem 2.4.2 and Proposition 2.4.19]).

For the proof of Proposition 5.2, we need to introduce an order in the family of finite sets of band projections. These can be considered as an abstract version of partitions of unity:

**Definition 5.3.** Given a Banach lattice $E$, let $\mathcal{P}$ be the family of finite sets of band projections $P = (P_1, \ldots, P_n)$ so that $P_iP_j = 0$ whenever $i \neq j$, and $\sum_{k=1}^n P_k = I_E$. We say that $P = (P_1, \ldots, P_n) \prec Q = (Q_1, \ldots, Q_m)$ if for $1 \leq i \leq n$ there exists a set $S_i \subset \{1, \ldots, m\}$ so that $\sum_{j \in S_i} Q_j = P_i$.

Note that the order $\prec$ makes $\mathcal{P}$ into a net: for $P = (P_1, \ldots, P_n), Q = (Q_1, \ldots, Q_m) \in \mathcal{P}$, we can define the family $R$ consisting of band projections $R_{ij} = P_iQ_j$, which satisfies $P, Q \prec R$.

As a preliminary step toward Theorem 5.1, we establish:

**Lemma 5.4.** Let $E$ be an order continuous Banach lattice, and $T \in B(E^*)$ is $\varepsilon$-BP, then there exists a band-preserving $U \in B(E^*)$ so that $\|U\| \leq \|T\|$, and $\|T - U\| \leq 4\varepsilon$.

**Proof.** For $P = (P_1, \ldots, P_n) \in \mathcal{P}$ on $E^*$, define $T_P = \sum_{k=1}^nP_kTP_k$. Since $T$ is $\varepsilon$-BP, by Proposition 2.2, for every $S \subset \{1, \ldots, n\}$ we have that

$$\left\| \left( \sum_{i \in S} P_i \right) T \left( \sum_{i \in S^c} P_i \right) \right\| \leq \varepsilon.$$ 

Note that

$$\sum_{S \subset \{1, \ldots, n\}} \sum_{i \in S} \sum_{j \in S^c} P_iTP_j = \sum_{i,j=1}^n \sum_{i \in S, j \notin S} P_iTP_j = 2^{n-2} \sum_{i,j=1}^n P_iTP_j.$$
Thus,
\[ T - T_P = \sum_{i \neq j} P_i T P_j = 4 \text{ Ave}_{S \subseteq \{1, \ldots, n\}} \left( \sum_{i \in S} P_i \right) T \left( \sum_{i \in S^c} P_i \right), \]
hence \( \|T - T_P\| \leq 4 \varepsilon \) for every \( P \in \mathcal{P}. \)

Recall that we have \( B(E^*) = (E^* \otimes E)^* \) via the trace duality: \( \langle A, e^* \otimes e \rangle = \langle Ae^*, e \rangle \), for \( e \in E, e^* \in E^* \), and \( A \in B(E^*) \) (see e.g. \cite[Section 1.1.3]{[reference]}).

Thus, the operators \( T_P \in B(E^*, E^*) \) have a subnet convergent weak* to \( U \in B(E^*, E^*) \), with \( \|T - U\| \leq 4 \varepsilon \).

Finally, we show that, for any band projection \( R \in B(E^*) \), we have \( RUR^\perp = 0 \) (as \( E^* \) is \( \sigma \)-Dedekind complete, the band-preserving property of \( U \) will follow). For “large enough” \( P \in \mathcal{P} \) (that is, when \( (R, R^\perp) < P \)), we have \( RT_P R^\perp = 0 \). From the definition of \( U \), \( T_P \to U \) in the point-weak* topology. By \cite[Corollary 2.4.7]{[reference]}, \( R \) and \( R^\perp \) are weak* to weak* continuous, hence \( T_P \to U \) in the point-weak* topology as well. Thus, \( RT_P R^\perp = 0 \). \( \square \)

**Proof of Theorem 5.1.** Since \( T \) is \( \varepsilon \)-BP and \( E \) is order continuous, by Proposition 2.11, we have that \( T^* \) is also \( \varepsilon \)-BP. By Lemma 5.4, there exists a band-preserving \( U \in B(E^*) \) so that \( \|U\| \leq \|T^*\| \), and \( \|T^* - U\| \leq 4 \varepsilon \).

Now, since \( U \) is band preserving we have that \( -\|U\|I \leq U \leq \|U\|I \), which means that for \( x \in E \),
\[ \|U^* x\| \leq \|U\| \|x\|. \]

Since \( E \) is order continuous, it is an ideal in \( E^{**} \), and the above inequality yields that \( U^*(E) \subseteq E \). In particular, \( R = U^*|E : E \to E \) is well defined and satisfies \( R^* = U \). By Proposition 2.11, it follows that \( R \) is band preserving in \( E \). Moreover, we have
\[ \|T - R\| = \|T^* - R^*\| = \|T^* - U\| \leq 4 \varepsilon. \] \( \square \)

The following easy lemma may well be known, but we haven’t seen it stated explicitly.

**Lemma 5.5.** Suppose \( A \) and \( B \) are bounded below sets in a Dedekind complete Banach lattice \( X \). Then
\[ \bigwedge_{a \in A} a + \bigwedge_{b \in B} b = \bigwedge_{a \in A, b \in B} (a + b). \]

**Proof.** Without loss of generality we can assume \( \bigwedge_{a \in A} a = 0 = \bigwedge_{b \in B} b \), then clearly \( \bigwedge_{a \in A, b \in B} (a + b) \geq 0 \). To prove the converse, note that, for any \( b_0 \in B \),
\[ \bigwedge_{a \in A, b \in B} (a + b) \leq \bigwedge_{a \in A} (a + b_0) = b_0. \]
Complete the proof by taking the infimum over \( b_0 \in B \). \( \square \)

**Proof of Proposition 5.2.** For \( P = (P_1, \ldots, P_n) \in \mathcal{P} \), we define
\[ T_P = \sum_{k=1}^n P_k T P_k. \]
As in the proof of Theorem 5.1, since $T$ is $\varepsilon$-BP, we have $\|T - T_P\| \leq 4\varepsilon$.

Since $T$ is positive, for every $x \in X_+$, the net $(T_P x)_{P \in \mathcal{P}}$ is decreasing. Indeed, let $P = (P_1, \ldots, P_n) \prec Q = (Q_1, \ldots, Q_m)$. Thus, for $1 \leq i \leq n$ there exists a set $S_i \subset \{1, \ldots, m\}$ so that $\sum_{j \in S_i} Q_j = P_i$. In particular, $P_i \geq Q_j$ for every $j \in S_i$, and we get

$$T_P x = \sum_{i=1}^n P_i T_P x = \sum_{i=1}^n \sum_{j \in S_i} Q_j T_P x \geq \sum_{i=1}^n \sum_{j \in S_i} Q_j T Q_j x = T_Q x.$$ 

Since $X$ is Dedekind complete, $\bigwedge_{P \in \mathcal{P}} T_P x$ exists. For each $x \in X_+$, let $S x = \bigwedge_{P \in \mathcal{P}} T_P x$. Then $S$ defines an additive positively homogeneous function on $X_+$. The homogeneity is easy to verify: for any $\lambda \geq 0$ and $x \in X_+$, we have

$$S(\lambda x) = \bigwedge_{P \in \mathcal{P}} (\lambda T_P x) = \lambda \bigwedge_{P \in \mathcal{P}} (T_P x) = \lambda S x.$$ 

The positive additivity follows directly from Lemma 5.5.

Clearly, $0 \leq S \leq T$. Also, for any $x \in X_+$, $(T - S)x = \bigvee_P (T - T_P)x$, hence, by the Fatou Property,

$$\|(T - S)x\| = \sup_{P \in \mathcal{P}} \|(T - T_P)x\| \leq 4\varepsilon\|x\|.$$ 

As $T - S$ is a positive operator, $\|T - S\| \leq 4\varepsilon$.

It remains to see that $S$ is band preserving. Given a band projection $R$ and $x \in X_+$, we have $RT_P R^\perp x = 0$ for $P$ “large enough” (that is, when $(R, R^\perp) \prec P$). Therefore, for $x \in X_+$,

$$0 \leq RSR^\perp x \leq \bigwedge_P RT_P R^\perp x = 0,$$

which implies $RSR^\perp = 0$. \hfill $\square$

It should be noted that the hypotheses of Theorem 5.1 and Propositions 5.2 are not always necessary: In Theorem 6.2 we will see that on $C(K)$ spaces every $\varepsilon$-BP operator is close to a BP one.

Suppose now that $E$ is a Banach lattice. Under what conditions on $E$ does there exist $c > 0$ so that, for every $\varepsilon > 0$, $E$ can be equipped with a new lattice norm $\|\cdot\|$ so that there exists a $\varepsilon$-BP operator on $(E, \|\cdot\|)$ with the property that $\|T - S\| \geq c$ for every BP operator $S$? By Theorem 5.1, this cannot happen when $E$ is order continuous (order continuity passes to renormings). A partial positive answer is given below.

**Proposition 5.6.** Suppose $E$ is a Banach lattice so that its dual has an atom $f$ with the property that $f^\perp = \{g \in E^*: f \perp g\}$ is not weak$^*$ closed. Then, for every $\varepsilon \in (0, 1)$, $E$ can be equipped with an equivalent lattice norm $\|\cdot\| = \|\cdot\|_\varepsilon$ so that there exists a positive rank one contraction $T \in \varepsilon - Z((E, \|\cdot\|_\varepsilon))$ so that $\|T - S\| \geq c$ whenever $S$ is a BP map ($c > 0$ is a constant depending on $E$).
As noted above, $E$ is order continuous if and only if any band in $E^*$ is weak* closed if and only if any band projection on $E^*$ is weak* continuous. Of course, there may be no rank one band projections on $E^*$ at all. We do not know whether Proposition 5.6 holds for general non-order continuous lattices.

Proposition 5.6 is applicable, for instance, when $E = C(K)$, where $K$ is an infinite compact Hausdorff space (cf. Remark 6.5).

**Proof.** Without loss of generality, we can assume that $f$ is positive and has norm one. Note that $f^\perp$ is a 1-codimensional sublattice of $E^*$. Indeed, let $P$ be the (one-dimensional) band projection corresponding to $f$, then $f^\perp$ is the range of $P^\perp = I - P$.

As $f^\perp$ is not weak* closed, by the Banach-Dieudonné Theorem, $f$ is a cluster point of $\{g \in f^\perp : \|g\| \leq C\}$, for some $C$.

Fix $\epsilon \in (0, 1)$, and equip $E$ with the new lattice norm

$$
\|x\| = \max \{\|x\|, \epsilon^{-1}\langle f, |x|\rangle\}.
$$

Note that $\|\cdot\| \leq \|\cdot\| \leq \epsilon^{-1}\|\cdot\|$. It is easy to check that the dual norm on $E^*$ is given by

$$
\|g\| = \inf_{\alpha \in \mathbb{K}} (\epsilon |\alpha| + \|g - \alpha f\|) = \inf_{g = \alpha f + h} (\epsilon |\alpha| + \|h\|).
$$

Fix $\delta > 0$, and find a positive norm one $e \in E$ so that $\langle f, e \rangle > 1 - \delta$. Define the rank one positive map $T : E \to E : x \mapsto \langle f, x e \rangle$. It is easy to check that $T$ acts contractively on $(E, \|\cdot\|)$. Indeed, if $\|x\| \leq 1$, then

$$
\|\langle f, x \rangle\| \leq \langle f, |x| \rangle \leq \epsilon,
$$

hence

$$
\|Tx\| = \|\langle f, x \rangle\| \max \{\|e\|, \epsilon^{-1}\langle f, e \rangle\} \leq 1.
$$

Further, $T^* g = (g, e) f$. We show that $T^* \in \mathcal{Z}(E^*, \|\cdot\|)$ (then, by Proposition 4.3, $T \in \mathcal{Z}(E, \|\cdot\|)$). Pick $g \in E^*$ with $\|g\| < 1$. Write $g = \alpha f + h$, with $\epsilon |\alpha| + \|h\| < 1$. We need to show that $|T^* g| \leq |g| + u$, with $\|u\| \leq \epsilon$. As $T$ is positive, we can and do restrict our attention to $g \geq 0$, and to the decompositions with $\alpha \geq 0$ and $h \geq 0$.

Then $T^* g = (\alpha \langle f, e \rangle + \langle h, e \rangle) f$, hence $T^* g \leq (\alpha + \|h\|) f \leq g + \|h\| f$. However, $\|f\| \leq \epsilon$, and we are done.

Next show that, if $S$ is a BP map on $E$, then $\|T - S\| \geq (1 - \delta)/(C + 1)$. Recall $P$ is the band projection associated with $f$, and then $P^\perp$ is the band projection onto $f^\perp$. We have

$$
\|T^* - S^*\| \geq \|P^\perp (T^* - S^*)|_{f^\perp}\| = \|S^*|_{f^\perp}\|
$$

(by the fact that $S^*$ maps $f^\perp$ into itself). Thus, for any $g \in f^\perp$, we have $|S^* g| \leq c |g|$, where $c = \|T - S\|$.

Using the band-preserving property of $S^*$ once more, we observe that $S^* f = \lambda f$, for some scalar $\lambda$. We claim that $|\lambda| \leq C$. Indeed, we know that $f$ is a weak* cluster point of $\{g \in f^\perp : \|g\| \leq C\}$. As $S^*$ is weak* to
Thus, start by presenting a criterion for a linear map on $C^*$, which, in turn, lies inside $\{g \in f^\perp : \|g\| \leq cC\}$.

On the other hand, note that

$$\|T^* f\| = |\langle f, e \rangle| \|f\| > (1 - \delta) \|f\|.$$ 

The triangle inequality implies

$$c = \|T^* - S^*\| \geq \frac{\|\langle x, \delta x \rangle\|}{\|f\|} = \frac{\|T^* f - \|S^* f\|}{\|f\|} > (1 - \delta - Cc).$$

Thus,

$$\|T - S\| \geq (1 - \delta)/(C + 1).$$

6. $\varepsilon$-BP OPERATORS ON $C(K)$ SPACES

In this section, we turn our attention to operators on $C(K)$ spaces. Let us start by presenting a criterion for a linear map on $C(K)$ to be $\varepsilon$-BP. Before the proof, recall that for $f \in C(K)$, we define its support as

$$\text{supp}(f) = \{t \in K : f(t) \neq 0\}.$$

**Lemma 6.1.** Suppose $K$ is a compact Hausdorff space. Then, for a linear map $T : C(K) \to C(K)$, the following statements are equivalent.

1. $T$ is $\varepsilon$-BP.
2. If $x \in C(K)$ and $t \in K$ satisfy $x(t) = 0$, then $\|Tx\| \leq \varepsilon \|x\|.$

**Proof.** (2) $\Rightarrow$ (1): Let $x \perp y$ in $C(K)$. We have that

$$\|[Tx] \perp y\| = \sup\{|[Tx](t) \perp y(t) : t \in K\}$$

$$= \sup\{|[Tx](t) \perp y(t) : t \in K, y(t) \neq 0\}$$

$$\leq \varepsilon \|x\|,$$

where the last inequality follows from the fact that if $y(t) \neq 0$, then $x(t) = 0$, together with the hypothesis.

(1) $\Rightarrow$ (2): Suppose first $t \notin \text{supp}(x)$. By Urysohn’s Lemma, there exists $y \in C(K)$ so that $0 \leq y \leq |[Tx](t)|$, $y|_{\text{supp}(x)} = 0$, and $y(t) = |[Tx](t)|$. Then $x \perp y$, and

$$|[Tx](t)| \leq \|[Tx] \perp y\| \leq \varepsilon \|x\|.$$

Now suppose $t \in \partial \text{supp}(x)$. For $\delta > 0$, let

$$x_\delta = (x_+ - \delta 1)_+ - (x_- - \delta 1)_+.$$ 

Note that $x_+ - \delta 1 \leq (x_+ - \delta 1)_+ \leq x_+$, hence $\|x_+ - (x_+ - \delta 1)_+\| \leq \delta$. Similarly, $\|x_- - (x_- - \delta 1)_+\| \leq \delta$. Thus, by the triangle inequality, $\|x - x_\delta\| \leq 2\delta$.

Moreover, we claim that $t \notin \text{supp}(x_\delta)$. Indeed, let us consider the open set $U = \{s \in K : |x(s)| < \delta\}$. Clearly, $t \in U$, and for every $s \in U$, $x_\delta(s) = 0$. Thus, $U \cap \text{supp}(x_\delta) = \emptyset$. By the preceding paragraph, $|[Tx_\delta](t)| \leq \varepsilon \|x_\delta\|$. As $T$ is continuous (Theorem 3.7), we are done.
Theorem 6.2. Suppose $K$ is a compact Hausdorff space. If $T \in B(C(K))$ is $\varepsilon$-BP, then there exists a BP operator $S \in B(C(K))$ so that $\|T - S\| \leq 2\varepsilon$. If $T$ is positive, then $S$ can be selected to be positive as well.

Proof. Let $\phi = T1$, and show that the multiplication operator $S$ defined via $Sf = \phi f$ has the desired properties. To this end, for $t \in K$, set $\mu_t = T^*\delta_t \in M(K) = C(K)^*$. Let $c_t = \mu_t(\{t\})$, and $\nu_t = \mu_t - c_t\delta_t$. Clearly $\nu_t(\{t\}) = 0$. We claim that $\|\nu_t\| \leq \varepsilon$—that is, for any $f \in B(C(K))$, $|\langle \nu_t, f \rangle| \leq \varepsilon$.

Suppose first $f$ satisfies an additional condition $f(t) = 0$. Then, since $T$ is $\varepsilon$-BP, by Lemma 6.1, we get

$$\varepsilon \geq |Tf(t)| = |\langle \delta_t, Tf \rangle| = |\langle T^*\delta_t, f \rangle| = |\langle \nu_t, f \rangle|.$$ 

For a generic $f \in B(C(K))$, fix $\sigma > 0$. By the regularity of $\nu_t$, there exists an open neighborhood $U \ni t$ so that $|\nu_t|(U) < \sigma$. Use Urysohn's Lemma to find $h \in C(K)$ so that $0 \leq h \leq 1$, $h(t) = 1$, and $h|_{K \setminus U} = 0$. By the above, $|\langle \nu_t, fh \rangle| \leq \sigma$, and $|\langle \nu_t, f(1 - h) \rangle| \leq \varepsilon$, which yields $|\langle \nu_t, f \rangle| \leq \varepsilon + \sigma$. As $\sigma$ can be arbitrarily small, $|\langle \nu_t, f \rangle| \leq \varepsilon$.

Next note that $\phi(t) = \langle T^*\delta_t, 1 \rangle = c_t + \langle \nu_t, 1 \rangle$, hence $|\phi(t) - c_t| \leq \varepsilon$. Finally, for $f \in B(C(K))$,

$$[Tf](t) = \langle \delta_t, Tf \rangle = (c_t\delta_t + \nu_t, f) = c_tf(t) + \langle \nu_t, f \rangle,$$

hence

$$|[Tf](t) - \phi(t)f(t)| \leq |c_t - \phi(t)| + \|\nu_t\| \leq 2\varepsilon.$$

As this holds for any $t$, we are done. \qed

Remark 6.3. As a consequence, on $(C(K), \| \cdot \|_{\infty})$, every $\varepsilon$-BP operator belongs to $2\varepsilon - Z(C(K))$.

Recall that an operator between Banach lattices $T : X \to Y$ is $\varepsilon$-disjointness preserving (in short, $\varepsilon$-DP) if for $x \perp y$ in $X$ with $\|x\|, \|y\| \leq 1$ we have $\|Tx \wedge Ty\| \leq \varepsilon \|x\| \|y\|$. 

Proposition 6.4. If $K$ is a compact Hausdorff space, and $T \in B(C(K))$ is $\varepsilon$-BP, then $T$ is $\varepsilon$-DP. Moreover, if $x, y \in C(K)$ are disjoint, then $\|T(x)(Ty)\| \leq \varepsilon \|x\| \|y\|$.

Proof. Consider disjoint $x, y \in C(K)$. By Lemma 6.1, $\|\|Tx\| \wedge |Ty\|\| \leq \varepsilon \max\{\|x\|, \|y\|\}$. Thus, $T$ is $\varepsilon$-DP.

Moreover, for $t \notin \text{supp}(x)$,

$$|[Tx](t)||Ty(t)| \leq \varepsilon \|x\| \cdot \|T\||\|y\| = \varepsilon \|T\||\|x\| \|y\|,$$

and the same inequality holds for $t \notin \text{supp}(y)$. \qed

Remark 6.5. Suppose $K$ is a Hausdorff compact. Recall that a space $C(K)$ is order continuous if and only if $K$ is a finite set. If $K$ is infinite, then Proposition 5.6 gives a renorming of $C(K)$ for which the conclusion of Theorem 6.2 no longer holds. We outline the construction from Proposition
for $C(K)$ spaces, as it may be instructive. In fact, for $\varepsilon > 0$ we equip $C(K)$ with an equivalent norm $\| \cdot \|$, and construct a positive contraction $T \in \varepsilon - Z(C(K), \| \cdot \|)$ so that $\| T - S \| \geq 1/2$ for any BP operator $S$.

Since $K$ is infinite, it has an accumulation point $k$. Consider the norm $\| x \| = \max\{ \| x \|_{\infty}, \varepsilon^{-1}|x(k)|\}$ ($\| \cdot \|_{\infty}$ stands for the canonical sup norm on $C(K)$). Clearly $\| \cdot \|$ is a lattice norm on $C(K)$, and $\| x \|_{\infty} \leq \| x \| \leq \varepsilon^{-1}\| x \|_{\infty}$ for $x \in C(K)$. We denote $(C(K), \| \cdot \|)$ by $E$.

Consider the rank one operator $T : E \to E : x \mapsto x(k)1$. Clearly $T \geq 0$, and for $x \in E$ we have $\| Tx \| = \| x(k)1 \| = \max\{ |x(k)|, \varepsilon^{-1}|x(k)|\} \leq \| x \|$.

Note also that $T1 = 1$, hence $\| T \| = 1$. Also, $T \in \varepsilon - Z(E)$. Indeed, for $x \in B(E)$, set $y = (|Tx| - |x|)$. Then $y(k) = 0$, while for $t \in K \setminus \{k\}$, $|y(t)| \leq \varepsilon$, hence $\| y \| \leq \varepsilon$.

Now suppose, for the sake of contradiction, that a BP map $S : E \to E$ satisfies $\| T - S \| < 1/2$. Note that $S$ is a multiplication operator: there exists $\phi \in C(K)$ so that $Sx = \phi x$. We claim that $|\phi(k)| < 1/2$. Indeed, take a net $(k_\alpha)_\alpha \subset K \setminus \{k\}$ such that $k_\alpha \to k$. By Urysohn’s Lemma, for every $\alpha$ there is $x_\alpha \in C(K)$ such that $0 \leq x_\alpha \leq x_\alpha(k_\alpha) = 1$, and $x_\alpha(k) = 0$. Then $Tx_\alpha = 0$, hence $|\phi(k_\alpha)| \leq \| (S - T)x_\alpha \| \leq \| T - S \| < 1/2$. By continuity, $|\phi(k)| \leq \| T - S \| < 1/2$ as well. On the other hand, since $\| 1 \| = \varepsilon^{-1}$, we have $\| T - S \| \geq \| (T - S)\varepsilon 1 \| \geq \varepsilon^{-1}|[(T - S)\varepsilon 1](k)| = |1 - \phi(k)| > 1/2$, which is the desired contradiction.

7. A COUNTEREXAMPLE

**Proposition 7.1.** There exists a Banach lattice $E$ so that, for every $\varepsilon > 0$, there exists an $\varepsilon$-BP contraction $T \in B(E)$ (actually, $T \in \varepsilon - Z(E)$) so that $\| T - S \| \geq 1/2$ whenever $S \in B(E)$ is BP.

**Lemma 7.2.** Suppose $K$ is a compact Hausdorff space. For $t_0 \in K$, $C(K; t_0) = \{ x \in C(K) : x(t_0) = 0 \}$ is a closed sublattice of $C(K)$. Then $T \in B(C(K; t_0))$ is BP if and only if there exists a uniformly bounded continuous function $\phi$ on $K \setminus \{t_0\}$ so that $Tx = \phi x$ for any $x$.

**Proof.** Clearly the operator given by $T(x) = \phi x$ is BP. Conversely, suppose $T$ is BP. Then $T$ is automatically bounded. Note that $C(K; t_0)^*$ is the quotient space of $C(K)^*$ by the set of linear functionals annihilating $\{t_0\}$. That is, we can identify $C(K; t_0)^*$ with the space of regular Radon measures $\mu$ on $K$ so that $\mu(\{t_0\}) = 0$.

Consider $T^* \in B(C(K; t_0)^*)$. We claim that there exists $\phi : K \setminus \{t_0\} \to \mathbb{K}$ so that $T^*\delta_t = \phi(t)\delta_t$ for any $t \in K \setminus \{t_0\}$. Indeed, fix $t$, and set $\mu_t = T^*\delta_t$. Suppose, for the sake of contradiction, that $|\mu_t|(K \setminus \{t\}) > 0$. Then there
exists an open set \( U \supset \{t, t_0\} \) so that \( |\mu_t|(K \setminus U) > 0 \). Then we can find \( x \in C(K) \), vanishing on \( U \), so that \( \langle \mu_t, x \rangle > 0 \).

Also, find \( y \in C(K; t_0) \), vanishing outside of \( U \), so that \( y(t) = 1 \). Then \( x \perp y \). However,

\[
[Tx](t) = \langle \delta_t, Tx \rangle = \langle T^* \delta_t, x \rangle = \langle \mu_t, x \rangle > 0,
\]

hence \( ||Tx| \wedge |y|| |(t) \neq 0 \), contradicting our assumption that \( T \) is BP.

Next show that \( \phi \) is continuous and uniformly bounded. For \( x \in C(K; t_0) \) and \( t \neq t_0 \), we have

\[
[Tx](t) = \langle \delta_t, Tx \rangle = \langle T^* \delta_t, x \rangle = \langle \phi(t) \delta_t, x \rangle = \phi(t)x(t).
\]

This shows the continuity of \( \phi \) away from \( t_0 \). If \( \phi \) is not uniformly bounded, then there exists a sequence \( (t_k) \), convergent to \( t_0 \), so that \( |\phi(t_k)| > 4^k \) for any \( k \). By Tietze Extension Theorem, we can find \( x \in C(K; t_0) \) so that \( x(t_k) = 2^{-k} \). Then \( Tx \) is unbounded, leading to a contradiction. \( \square \)

**Proof of Proposition 7.1.** The Banach lattice \( E \) consists of all continuous functions \( x \) on \( [0, 1] \), satisfying \( \lim_{n \to \infty} 2^n|x(2^{-n})| = 0 \) (consequently \( x(0) = 0 \)).

Set

\[
||x|| = \max \{||x||_\infty, \sup_{n \in \mathbb{N}} 2^n|x(2^{-n})|\}.
\]

For \( n \geq 2 \) let \( x_n \) be a continuous function such that \( 0 \leq x_n \leq 1 \), \( x_n(t) = 0 \) for \( t \leq 2^{-n-1} \) or \( t \geq 2^{1-n} \), and \( x_n(2^{-n}) = 1 \). Define \( T_n x = x(2^{-n})x_n \).

Clearly, \( T_n \) is a contraction. We next show that, for every \( x \in \mathcal{B}(E) \),

\[
||(T_n x) - |x||_+ || \leq 2^{-n} \text{ (hence, } T_n \in 2^{-n} - \mathcal{Z}(E) \text{, and in particular } T_n \text{ is } 2^{-n}\text{-BP}).
\]

If \( x(2^{-n}) = 0 \), then \( T_n x = 0 \). Otherwise \( x(2^{-n}) = [T_n x](2^{-n}) \), and \( [T_n x](2^{-m}) = 0 \) for \( m \neq n \). For \( t \notin \{2^{-m} : m \in \mathbb{N}\} \),

\[
||(T_n x) - |x||_+(t) \leq ||T_n x||_+(t) \leq |x(2^{-n})||x_n(t)| \leq 2^{-n}.
\]

Thus, we get

\[
||(T_n x) - |x||_+ || = \max \{||(T_n x) - |x||_+ ||_\infty, \sup_{m} 2^m||(T_n x) - |x||_+ ||(2^{-m})\} \leq 2^{-n}.
\]

Now suppose \( S : E \to E \) is band-preserving (hence bounded). We show that that there exists a uniformly bounded continuous function \( \phi : [0, 1] \to \mathbb{K} \) so that \( Sx = \phi x \) for any \( x \).

Indeed, for any \( n \in \mathbb{N} \), denote by \( E_n \) the sublattice of \( E \) consisting of functions vanishing on \( [0, 2^{-n}] \). Note that \( S \) takes \( E_n \) into itself. Clearly \( E_n \) is lattice isomorphic to \( C([2^{-n}, 1], 2^{-n}) \), hence, by Lemma 7.2, there exists a uniformly bounded continuous function \( \phi_n : [2^{-n}, 1] \to \mathbb{K} \) so that \( Sx = \phi_n x \) for any \( x \in E_n \). Clearly \( \phi_m|_{[2^{-n}, 1]} = \phi_n \) whenever \( m > n \). So there exists a function \( \phi \), continuous on \( (0, 1] \), so that \( Sx = \phi x \) for any \( x \in E_\infty \), where \( E_\infty = \bigcup_n E_n \) is the set of all elements of \( E \) vanishing on a neighborhood of \( 0 \).

Now set \( C = ||S|| \), and show that \( \sup_{t \in (0, 1]} |\phi(t)| \leq C \). Indeed, otherwise we can find \( t \in (0, 1] \setminus \{2^{-k} : k \in \mathbb{N}\} \) so that \( |\phi(t)| > C \). Find \( m \in \mathbb{N} \) so that
$2^{-m} < t < 2^{1-m}$, and consider $x \in C[0, 1]$ so that $0 \leq x \leq 1 = x(t)$, and $x = 0$ outside of $(2^{-m}, 2^{1-m})$. Then $\|x\| = 1$ and $\|Sx\| > C$, a contradiction.

It is easy to see that $E_\infty$ is dense in $E$, hence by continuity, $Sx = \phi x$ for any $x \in E$.

Now suppose, for the sake of contradiction, that there exist a BP map $S \in B(E)$ so that $\|T_n - S\| = c < 1/2$. We have shown that $S$ is implemented by multiplication by a function $\phi$, continuous on $(0, 1]$ and uniformly bounded. That is, for any $x \in B(E)$, we have $\|T_n x - \phi x\| \leq c$.

Show first that, for $t \notin \{2^{-k} : k \in \mathbb{N}\}$, $|\phi(t)| \leq c$. To this end, find $n \in \mathbb{N}$ so that $2^{-n-1} < t < 2^{-n}$. Pick $x \in E$ so that $0 \leq x \leq 1 = x(t)$, and $x = 0$ outside of $(2^{-n-1}, 2^{-n})$. Then $\|x\| = 1$, $T_n x = 0$, and $c \geq \|Sx\| \geq |\phi(t)|$. By continuity, $|\phi| \leq c$ everywhere.

Now consider $x \in E$ so that $x(2^{-n}) = 2^{-n}$, $0 \leq x \leq 1$, and $x = 0$ outside of $(2^{-n-1}, 2^{-n})$. Then $\|x\| = 1$, and

$$\|T_n x - Sx\| \geq 2^n \|T_n x(2^{-n}) - \phi(2^{-n}) x(2^{-n})\|$$

$$= 2^n \|2^{-n} - 2^{-n} \phi(2^{-n})\| \geq 2^n - 1 - |\phi(2^{-n})| \geq 1 - c > \frac{1}{2},$$

a contradiction. \( \square \)

**Remark 7.3.** The lattice $E$ from the proof of Proposition 7.1 is an AM-space. In fact, $j : E \to C[0, 1] \oplus_\infty c_0 : f \mapsto f \oplus (2^k f(2^{-k}))_{k \in \mathbb{N}}$ is a lattice isometry. Consequently, $E^*$ is an AL-space. As $T_n^* \in \mathcal{E}(E^*)$ for any $n$, Theorem 5.1 shows there exists a BP map $R_n \in B(E^*)$ so that $\|T_n^* - R_n\| \leq 2^{2-n}$ However, such an $R_n$ cannot be an adjoint operator, for $n > 3$.

**Remark 7.4.** Arguing as in Theorem 3.7 one can show that every $\varepsilon$-BP linear map on the lattice $E$ given in Proposition 7.1 is automatically continuous.

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DEPT. OF MATHEMATICS, UNIVERSITY OF ILLINOIS URBANA IL 61801, USA
E-mail address: oikhberg@illinois.edu

MATHEMATICS DEPARTMENT, UNIVERSIDAD CARLOS III DE MADRID, E-28911 LEGANÉS, MADRID, SPAIN.
E-mail address: ptradace@math.uc3m.es