Polarized 3-folds in a codimension 10 weighted homogeneous $F_4$ variety

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Abstract
We describe the construction of a codimension 10 weighted homogeneous variety $w\Sigma F_4(\mu, u)$ corresponding to the exceptional Lie group $F_4$ by explicit computation of its graded ring structure. We give a formula for the Hilbert series of the generic weighted $w\Sigma F_4(\mu, u)$ in terms of representation theoretic data of $F_4$. We also construct some families of polarized 3-folds in codimension 10 whose general member is a weighted complete intersection of some $w\Sigma F_4(\mu, u)$.

1 Introduction
We are interested in the study of projective algebraic varieties in terms of graded rings, which are usually Gorenstein in interesting cases. In lower codimension, one can describe the structure of varieties by using the standard structure theory of Gorenstein rings: they are defined by a single equation in codimension 1, in codimension 2 they are complete intersections [Ser], and in codimension 3 they are defined by the $2m \times 2m$ pfaffians of a $(2m + 1) \times (2m + 1)$ skew symmetric matrix [BE77]. The general structure theory for codimension 4 Gorenstein rings was developed by Reid in [Rei15] but, in the words of the author himself, is still some way from any tractable application. Techniques like unprojection have been successfully used in [BKR12] to construct Gorenstein rings in codimension 4 from rings in lower codimension, but usually there are obstructions to using unprojection.

To construct projective varieties having graded rings in codimension $> 4$, weighted homogeneous varieties $w\Sigma$ has been used as ambient (key) varieties to construct projective varieties as weighted complete intersections. The notion of weighted homogeneous variety was first introduced by Grojnowski and Corti–Reid in [CR02], a weighted projective analogue of the classical homogeneous variety $\Sigma = G/P$; $G$ a reductive Lie group and $P$ is a parabolic subgroup of $G$. They constructed some families of polarized 3-folds which are weighted complete intersections of the weighted orthogonal Grassmannian $OGr(5,10)$, by computing the corresponding Hilbert series and the graded ring structure of $wOGr(5,10)$. The orthogonal Grassmannian $OGr(5,10)$ is the quotient of the even orthogonal group $O(10,\mathbb{C})$ by one of its maximal parabolic subgroups, .

A formula for the Hilbert series of any weighted homogeneous variety and an algorithmic approach to compute their defining ideals has been given by Qureshi and Szendrői in [QST1]. This

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lead to an efficient approach of calculating graded rings and required free resolution information for any weighted homogeneous variety, allowing for the construction of projective varieties in higher codimension. In \[QS11, QS12\], a weighted $G_2$ variety was used to construct families of polarized 3-folds in codimension eight and the weighted Grassmannian $w\text{Gr}(2, 6)$ in codimension 6; to construct the polarized varieties in relatively higher codimension. The weighted Lagrangian Grassmannian $wL\text{Gr}(3, 6)$ and the weighted partial $A_3$ flag variety $w\text{FL}_{1,3}$ have been used to construct families of polarized 3-folds in codimension 7 and 9 respectively in \[Qur15\].

In this article, we construct some families of polarized 3 dimensional orbifolds, i.e. 3-folds with worst terminal quotient singularities, in codimension 10. We explicitly describe the construction of a new weighted homogeneous variety $(w\Sigma F_4, O_{w\Sigma F_4}(1))$: a homogeneous variety for the simple Lie group $F_4$. Our $w\Sigma F_4$ is a 15 dimensional variety and has an embedding in the weighted projective space $w\mathbb{P}^{25}$: a codimension ten embedding. We also give a compact formula for the Hilbert series of a generic weighted homogeneous $w\Sigma F_4(\mu, u)$. The explicit graded ring construction of $w\Sigma F_4$ will be given in terms of generators and relations, by using an algorithmic approach \[QS11\] Appendix A. Then we construct some families of polarized 3-folds as weighted completed intersections of $w\Sigma F_4$ by using the graded rings and Hilbert series of $w\Sigma F_4$. The graded rings of these polarized 3-folds by using the defining equations of $w\Sigma F_4$.

We construct 3-folds $(X, D)$ polarized by $\mathbb{Q}$-ample Weil divisor $D$ having a finitely generated graded ring

$$R(X, D) = \bigoplus_{m \geq 0} H^0(X, mD).$$

The surjective morphism from a free graded polynomial ring generated in degree $w_i$ by the variables $x_i$

$$\mathbb{C}[x_0, \ldots, x_n] \twoheadrightarrow R(X, D)$$

gives an embedding

$$i: X = \text{Proj} R(X, D) \hookrightarrow \mathbb{P}[w_0, \ldots, w_n].$$

The divisorial sheaf $O_X(D)$, a rank one reflexive sheaf, of $X$ is isomorphic to $O_X(1) = i^*O_{\mathbb{P}}(1)$.

We use the natural Plücker-type embeddings of the corresponding weighted homogeneous variety $(w\Sigma F_4, O_{w\Sigma F_4}(1))$ to construct examples in codimension 10 by taking quasilinear sections. We calculate the Hilbert series of a given weighted homogeneous variety to compute the canonical divisor class of $w\Sigma F_4$, corresponding to the choice of parameters $\mu$ and $u$. Then to construct a variety with a required canonical class we take the complete intersection of $w\Sigma F_4$ or of projective cone(s) over it, with hypersurfaces in weighted projective space of appropriate degree. We need the defining ideals of $w\Sigma F_4$ varieties to understand the type of singularities of their complete intersections. The defining ideals of homogeneous varieties appear in \[GKR07, \text{Sec 1}\]. We compute the equations by using the algorithmic approach of DeGraaf \[dG01\].

The Section 2 gives the required definitions and conventions used in the rest of the article which includes a recall of the weighted homogeneous varieties, a formula for their Hilbert series $P_{w\Sigma}(t)$ and computations of their defining equations. In Section 3 we study the structure of weighted homogeneous variety $w\Sigma F_4(\mu, u)$ and its quasilinear sections, leading to codimension ten polarized varieties.
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2 Definitions and notations

We work over a field complex numbers $\mathbb{C}$. A polarized variety is a pair $(X, D)$; $X$ is a normal projective algebraic variety and $D$ is a $\mathbb{Q}$-ample Weil divisor on $X$. All our varieties are well-formed and quasi-smooth, embedded in some weighted projective space. We use the standard notations $\mathbb{P}[w_0, w_1, \ldots, w_n]$ or $\mathbb{P}^n[w_i]$ to denote the weighted projective space.

A polarized variety $X \subset \mathbb{P}^n[w_i]$ of codimension $l$ is called well-formed, if no $n$ of $w_0, \ldots, w_n$ have a common factor and singularities of $X$ only appear in codimension greater than $l + 1$. If the affine cone $\widetilde{X} \subset \mathbb{A}^{n+1}$ over $X$ is smooth outside the origin then $X$ is called quasi-smooth. If $X$ is quasi-smooth, then it contain no other singularities than those coming from the weights of the embedding $\mathbb{P}^n[w_i]$, called quotient singularities. We assume that restriction of the tautological ample divisor $\mathcal{O}_{\mathbb{P}^n}(1)$ provides the polarization.

The Hilbert series of $(X, D)$ is given by

$$P_{(X,D)}(t) = \sum_{m \geq 0} \dim H^0(X, \mathcal{O}_X(mD)) t^m.$$  

Appropriate vanishing theorems and Riemann–Roch formulas can be used together to compute $h^0(X, mD) = \dim H^0(X, mD)$ in most of the cases. We will write $P_X(t)$ for the Hilbert series if no confusion can arise. A polarized 3-fold $(X, D)$ is a three dimensional Gorenstein, normal, projective algebraic variety with at worst terminal cyclic quotient singularities, consisting of isolated orbifold points of $X$. A projective variety $X$ is called Gorenstein if

1. $H^i(X, \mathcal{O}_X(mD)) = 0$ for all $m \geq 0$ and $0 < i < \dim(X)$.
2. $K_X \sim kD$ for some some integer $k$.

A cyclic quotient singularity $Q$ of type $\frac{1}{r}(a_1, \ldots, a_n)$ is the quotient $\mathbb{A}^n/\mathbb{Z}_r$ given by

$$\zeta : (x_1, \ldots, x_n) \mapsto (\zeta^{a_1}x_1, \ldots, \zeta^{a_n}x_n),$$

where $\zeta$ is a primitive $r$-th root of unity. It is called isolated if $\gcd(r, a_i) = 1$ for all $1 \leq i \leq n$ and terminal if

$$\frac{1}{r} \sum_{j=1}^{n} da_j > 1$$

for $k = 1, \ldots, r - 1$,

where $da_j$ is the smallest residue modulo $r$, see [Rei87].

Now we recall some background from representation theory which is primarily from [FH91]. Let $G$ be a linear algebraic group, $P$ be a parabolic subgroup of $G$ and $T$ be the maximal torus inside $G$ then $T \subset P \subset G$. Then $t \subset p \subset g$ are the inclusions of the corresponding Lie algebras. The quotient $\Sigma = G/P$ is called a homogeneous variety or generalized flag variety.
Let \( \Lambda_W = \text{Hom}(T, \mathbb{C}^*) \) denote the lattice of weights. Let \( V \) be an irreducible \( G \)-representation then \( V \) has a decomposition
\[
V = \bigoplus_{\alpha \in \Lambda_W} V_\alpha
\]
into eigenspaces of \( T \)-action; the \( \alpha \)'s with non-trivial \( V_\alpha \) are called the weights of the representation \( V \). If \( V = g \) then \( V \) is called the adjoint representation and the non-zero weights are called the roots \( \nabla \) of \( g \). Each element of \( \nabla \) can be written as a strictly positive or negative linear combination of the subset \( \nabla_0 \), called the simple roots of \( g \). The set of roots \( \nabla \) has a decomposition into the set of positive and negative roots \( \nabla = \nabla_+ \sqcup \nabla_- \).

Let \( V_\chi \) be the irreducible representation of \( G \) with highest (dominant) weight \( \chi \) with respect to certain partial order on the set of weights and \( \nabla(V) \) represents the set of weights of \( V_\chi \). Then the set of parabolic subgroups (up to conjugacy) of \( G \) are in 1–1 correspondence with the irreducible highest weight representations of \( G \). The character \( \chi : T \to \mathbb{C}^* \) give rise to a very ample line bundle \( L_\chi \) on the homogeneous variety \( \Sigma \). Then by Borel–Bott–Weil theorem \( H^0(\Sigma, L_\chi) = V_\chi \) and we get the embedding
\[
\Sigma = G/P_\chi \hookrightarrow \mathbb{P}V_\chi,
\]
where \( P = P_\chi \) is the parabolic corresponding to the subset of simple roots which are orthogonal to \( \chi \) in the character lattice \( \Lambda_W \). The dimension of \( V_\chi \) can be calculated by using the Weyl’s dimension formula:
\[
\dim(V_\chi) = \prod_{\alpha \in \nabla_+} \frac{(\chi + \rho, \alpha)}{(\rho, \alpha)},
\]
where the Weyl vector \( \rho \) is half the sum of the positive roots and \( (, \) is the Killing form on the Lie algebra \( g \). The full character of \( V_\chi \) can be computed by using the Weyl character formula:
\[
\text{Char}(V_\chi) = \frac{\sum_{\sigma \in W} (-1)^{\sigma} \rho^{\sigma(\chi + \rho)}}{\sum_{\sigma \in W} (-1)^{\sigma} \rho^{\sigma(\rho)}} = \sum_{\chi_i \in \nabla(V)} \dim(V_{\chi_i}) t^{\chi_i},
\]
where \( W \) is the Weyl group of the root system of \( g \) and \( \nabla(V) \) is the set of weights of the representation \( V \). If \( \sigma \) is a product of an even number of simple reflections in \( W \) then \((-1)^{\sigma} = 1 \) and -1 otherwise. Let \( \Lambda_W^* = \text{Hom}(\mathbb{C}^*, T) \) denote the lattice of 1-parameter subgroups of \( G \), then we have get a perfect pairing \( <, > : \Lambda_W \times \Lambda_W^* \to \mathbb{Z} \). We take an element \( \mu \) in \( \Lambda_W^* \) and a positive integer \( u \) such that
\[
<\sigma \chi, \mu > + u > 0,
\]
for all elements \( \sigma \) of the Weyl group \( W \). The inequality \( 3.3 \) makes sure that all the weights on the weighted projective space containing \( w\Sigma(\mu, u) \) are positive.

**Definition 2.1** [CR02] Let \( \Sigma \) be a homogeneous variety. Take the affine cone \( \tilde{\Sigma} \subset V_\chi \) of the embedding \( \Sigma \hookrightarrow \mathbb{P}V_\chi \) then the invariant part of the following \( \mathbb{C}^* \)-action on \( V_\chi \setminus \{0\} \)
\[
(\varepsilon \in \mathbb{C}^*) \mapsto (v \mapsto \varepsilon^u (\mu(\varepsilon) \circ v)),
\]
is called the weighted homogeneous variety embedded in \( \mathbb{P} \langle \chi_i, \mu > + u \rangle \), where \( \chi_i s \) are the weights of the presentation \( V_\chi \). We denote this variety by \( w\Sigma(\mu, u) \) as the weights depends on the parameter \( \mu \) and \( u \) or simply \( w\Sigma \), if no confusion can arise.
The following formula from [QS11, Thm. 3.1] can be used to figure out the Hilbert series of any weighted homogeneous variety.

\[ P_{w\Sigma}(t) = \frac{\sum_{\rho \in W} (-1)^{\langle \rho, \mu \rangle} t^{\langle \rho, \mu \rangle}}{\sum_{\sigma \in W} (-1)^{\langle \sigma, \mu \rangle} t^{\langle \sigma, \mu \rangle}}, \quad (2.4) \]

**Remark 2.1** The Hilbert series (2.4) simplifies to the expression

\[ P_{w\Sigma}(t) = \frac{N(t)}{\prod_{\chi_i \in \nabla(V_\chi)} (1 - t^{c_i, n} + u)}, \quad (2.5) \]

by using the standard Hilbert–Serre theorem [AM69, Theorem 11.1], where \( \chi_i \) are the weights of the representation \( V_\chi \).

The defining ideal \( I = \langle Q \rangle \) of a homogeneous variety \( \Sigma = G/P \hookrightarrow \mathbb{P}V_\chi \) is always generated by quadrics [GKR07, 2.1]. The second symmetric power of \( V_\chi^* \) has a decomposition

\[ S^2(V_\chi^*) = V_{2\nu} \oplus V_1 \oplus \cdots \oplus V_n \]

into irreducible \( G \)-representations, where \( \nu \) is the highest weight of \( V_\chi^* \). The generators of the subspace \( Q \subset S^2V_\chi^* \) consisting of all the summands except \( V_{2\nu} \), gives the defining equations of \( \Sigma \).

### 3 Weighted homogeneous \( F_4 \) varieties

In this section, we recall the representation theory of the Lie group \( F_4 \) and how we can use it to construct the weighted homogeneous variety \( w\Sigma F_4 \). We also compute a formula for the Hilbert series of \( w\Sigma F_4 \). At the end, we construct some families of 3-folds of general type as weighted complete intersection of some \( w\Sigma F_4 \).

#### 3.1 Generalities

In this section, we recall the basic algebraic structure of the Lie group \( F_4 \) and its corresponding Lie Algebra \( f_4 \) which can mostly be found in [Bou92]. Let \( G \) be the simple and simply connected Lie group of type \( F_4 \) with Lie algebra \( g = f_4 \). This is one of the five exceptional Lie groups. The rank of the Lie algebra \( f_4 \) is 4, which is the dimension of the Cartan subalgebra \( t \) of \( f_4 \). The weight lattice of \( f_4 \) is a rank four lattice

\[ \Lambda_W = \langle e_1, e_2, e_3, e_4 \rangle. \]

The fundamental weights of \( G \), which form a set of generators of the dominant Weyl chamber, are given by

\[ \omega_1 = e_1 + e_2, \quad \omega_2 = e_1 + e_2 + e_3, \quad \omega_3 = \frac{1}{2}(3e_1 + e_2 + e_3 + e_4), \quad \text{and} \quad \omega_4 = e_1. \]
The simple roots of the root system of $f_4$ are

$$\alpha_1 = 2\omega_1 - \omega_2 = e_2 - e_3, \quad \alpha_2 = -\omega_1 - 2\omega_3 + 2\omega_2 = e_3 - e_4$$

$$\alpha_3 = -\omega_4 + 2\omega_3 - \omega_2 = e_4, \quad \text{and} \quad \alpha_4 = 2\omega_4 - \omega_3 = \frac{1}{2}(e_1 - e_2 - e_3 - e_4).$$

The Weyl group $W$ of the root system $F_4$ is generated by the simple reflections $s_{\alpha_i}, \ 1 \leq i \leq 4$, and is the symmetry group of the convex regular 4-polytope, known as the 24-cell in convex geometry. The order of $W$ is 1152 and has a finite presentation given as follows.

$$W = \langle s_{\alpha_1}, s_{\alpha_2}, (s_{\alpha_1}s_{\alpha_2})^2 = (s_{\alpha_1}s_{\alpha_3})^2 = (s_{\alpha_2}s_{\alpha_3})^2 = (s_{\alpha_3}s_{\alpha_4})^3 = 1 \rangle$$

The Weyl vector $\rho$ is:

$$\rho = 8\alpha_1 + 15\alpha_2 + 21\alpha_3 + 11\alpha_4 = \frac{1}{2}(11e_1 + 5e_2 + 3e_3 + e_4).$$

Consider the irreducible fundamental representation $V_{\chi}$ of the Lie group $F_4$ with highest weight $\chi = \omega_4 = e_1$. Then the dimension of the irreducible representation $V_{\chi}$ is 26 by using the Weyl dimension formula [2,1]. Twenty-four of the weights appear with multiplicity one and the zero weight space appears with multiplicity two. This can be easily figured out by using the implementation of Weyl character formula [2,2] in some computer algebra system like SAGE [Sac] or LiE [MAAxLL92]. Sixteen of the 24 non-zero weights appear with multiplicity one are given by

$$\left\{ \frac{1}{2} \left( (-1)^i e_1 + (-1)^j e_2 + (-1)^k e_3 + (-1)^l e_4 \right) : 0 \leq i, j, k, l \leq 1 \right\}$$

and eight of them are

$$\left\{ (-1)^i e_j : 0 \leq i, 1 \leq j \leq 4 \right\}. \quad (3.1)$$

The Lie algebra $f_4$ is the adjoint representation of the Lie group $F_4$. In fact it is the irreducible representation with highest weight $\omega_1 = e_1 + e_2$. One can use the formula [2,1] to show that it is 52 dimensional. The simple roots $\alpha_2, \alpha_3$ and $\alpha_4$ are orthogonal to the dominant weight $\chi = \omega_4$ with respect to the the Killing form. Therefore, the parabolic subalgebra is

$$p_{\chi} = \bigoplus_{\alpha \in \nabla_+} t g_\alpha \bigoplus_{\alpha \in \nabla_p} g_{-\alpha},$$

where $\nabla_p$ is the subset of $\nabla_+$ consisting of the terms involving only $\alpha_2, \alpha_3$ and $\alpha_4$. The parabolic subalgebra $p_{\chi}$ is 37 dimensional. Consider the quotient of the Lie group $G$ by the parabolic subgroup $P_{\chi}$, corresponding to the subalgebra $p_{\chi}$, then $\Sigma F_4 = G/P_{\chi}$ is a homogeneous variety embedded in the projectivization of the $G$-representation $V_{\chi}$. The dimension of the homogeneous variety $\Sigma F_4$ is $52 - 37 = 15$ and therefore we get a codimension 10 embedding

$$\Sigma F_4 \hookrightarrow \mathbb{P}^{25}[V_{\chi}].$$

We denote the given homogeneous variety by $w\Sigma F_4$. Let

$$\Lambda_W = \{ f_1, f_2, f_3, f_4 \}$$
The weighted version of the homogenous variety $\Sigma$ is obtained by choosing

$$\mu = \sum_{i=1}^{4} a_i f_i \in \Lambda_W^*$$

and a positive integer $u$ such that $\langle s_\alpha \cdot \chi, \mu \rangle + u > 0$ for all the elements $s_\alpha$ of $W$. Then for each such choice of $\mu$ and $u$ we have the embedding of weighted homogeneous variety

$$w\Sigma^{15}(\mu, u) \hookrightarrow w\mathbb{P}^{25}_{\chi\mu},$$

where $\chi_i$'s are the weights of the representation $V_\chi$, $i = 1, \cdots, 26$. The element $\mu$ of the dual lattice is usually represented as a vector $\mu = (a_i)$.

**3.2 Hilbert series of weighted $w\Sigma F_4(\mu, u)$**

We use computer algebra systems SAGE and Mathematica together to compute the following Hilbert series formula of $w\Sigma F_4$. The SAGE has built in representation theoretic data which has been used to compute the numerator and denominator of the formula (3.3). A SAGE code is given in the Appendix B to compute the Hilbert series of $w\Sigma F_4$ which in principal can be modified to compute the Hilbert series of any weighted homogeneous variety. The further simplification has been performed by using Mathematica.

**Theorem 3.1** Consider the symmetric group $S_2$ with $\sigma(a_i) = a_i$ if $\sigma$ is an even and $\sigma(a_i) = -a_i$ if $\sigma$ is an odd permutation, where $a_i \in \mu$. Then the Hilbert series of the $w\Sigma F_4$ has the following compact form.

$$P_{w\Sigma F_4}(t) = \frac{1 - \sum_{k=2}^{3} (-1)^k P_{k-1}(t) \left( t^{ku} + t^{(15-k)u} \right) + \sum_{k=5}^{7} (-1)^k P_{k-2}(t) \left( t^{ku} + t^{(15-k)u} \right) - t^{15u}}{\prod_{\chi_i \in \Delta(V_\chi)} (1 - t^{\langle \chi_i, \mu \rangle + u})},$$

where

$$P_1(t) = \sum_{\sigma \in S_2} \sum_{i=1}^{4} t^{\sigma(a_i)} + \sum_{0 \leq (i,j,k,l) \leq 1} t^{\frac{1}{2} \left( (-1)^ia_1 + (-1)^ja_2 + (-1)^ka_3 + (-1)^la_4 \right) + 3},$$

$$P_2(t) = \sum_{\sigma \in S_2} \sum_{i=1}^{4} 2t^{\sigma(a_i)} + \sum_{0 \leq (i,j,k,l) \leq 1} 2t^{\frac{1}{2} \left( (-1)^ia_1 + (-1)^ja_2 + (-1)^ka_3 + (-1)^la_4 \right)}$$

$$+ \sum_{0 \leq (i,j) \leq 1} \left( \sum_{1 \leq m < n \leq 4} t^{(-1)^ia_m + (-1)^ja_n} \right) + 6.$$
and

\[ P_3(t) = \sum_{0 \leq (i,j,k,l) \leq 1} \left( 7t^4 \left( (-1)^i a_1 + (-1)^j a_2 + (-1)^k a_3 + (-1)^l a_4 \right) + 4 t^{a_m} \frac{1}{t^2} \left( (-1)^i a_1 + (-1)^j a_2 + (-1)^k a_3 + (-1)^l a_4 \right) \right) + \sum_{0 \leq (i,j) \leq 1} \left( \sum_{1 \leq m < n \leq 4} 3t (-1)^i a_m + (-1)^j a_n \right) + \sum_{0 \leq (i,j,k) \leq 1} \left( \sum_{1 \leq l < m < n \leq 4} t (-1)^i a_l + (-1)^j a_m + (-1)^k a_n \right) + \sum_{\sigma \in S_2} \sum_{i=1}^4 7t^{\sigma(a_i)} + 15 \]

Moreover, if \( w \Sigma \) is well-formed then the canonical line bundle \( K_{w \Sigma} = \mathcal{O}_{w \Sigma}(-11u) \).

**Proof**  We first calculate the orbit of the weight \( \chi \) under the action of the Weyl group \( W \). From the representation theory since the image of the non-zero weight is a non-zero weight under he action of Weyl group, only non-zero weight appear in the orbit of \( \chi \). By using SAGE we compute that all of the 24 non-zero weight appear in the orbit \( W \chi \). We evaluate the formula (2.3) for \( W, \rho, \mu \) and \( \chi \), as given in Section 3.1 to compute \( P_{w \Sigma F_4}(t) \); by using SAGE code in Appendix B. Then we perform simplification in Mathematica to obtain the below form of the Hilbert series of \( w \Sigma F_4 \).

\[ P_{w \Sigma F_4}(t) = \frac{1 + 2t - \sum_{i=1}^{24} t^{\langle \chi_i, \mu \rangle} + 2u + \cdots + 2t^{12u} + t^{13u}}{\prod_{i=1}^{24} (1 - t^{\langle \chi_i, \mu \rangle} + u)} \]  

(3.5)

where \( \chi_i \) are the collection of weights of \( V_\chi \) given by (3.1) and (3.2). The zero weight spaces do not appear in the orbit under the action of the Weyl group of \( F_4 \). Therefore the full expression
of type (2.5) for the Hilbert Series of \( w \Sigma F_4 \) is obtained obtained by multiplying and dividing the equation (3.5) by \((1 - t^u)^2\), which represents the zero weight spaces in the representation \( V_\chi \) as \( \chi = (0) \) in (3.3). This gives us the compact form (3.4) of the Hilbert series. The sum of the weights on \( P(\chi_i, \mu + u) \) is 26\( u \) and the adjunction number is 15\( u \). Therefore, if \( w \Sigma F_4 \) is well-formed (normal), then the canonical divisor class is \( K_{w \Sigma F_4} = \mathcal{O}(15u - 26u) = \mathcal{O}(-11u) \).

### 3.3 Families of 3-folds in codimension 10

In this section, we construct some families of 3-folds of general type with at worst terminal quotient singularities with the canonical divisor class \( K_X = kD, k \geq 5 \). We also tried to construct some families of Calabi–Yau and \( \mathbb{Q} \)-Fano threefolds in \( w \Sigma F_4 \) by using the algorithmic approach of [Qur17] but the computer search was not successful. We show that a threefold linear section of the straight homogeneous variety \( \Sigma \) is a smooth canonical 3-fold. The search for canonical 3-folds with non-trivial weights was also unsuccessful.

**Example 3.4** We consider the Hilbert series straight \( F_4 \) variety \( \Sigma F_4 \), which corresponds to choice of parameters \( \mu = (0, 0, 0, 0) \) and \( u = 1 \). The Hilbert series of \( \Sigma F_4 \) is given by

\[
P_{\Sigma F_4}(t) = \frac{1 - 27t^2 + 78t^4 - 351t^5 + \cdots - 351t^{10} + 78t^{12} - 27t^{13} + t^{15}}{(1 - t)^{26}}.
\]

The canonical divisor is \( K_{\Sigma F_4} = \mathcal{O}(-11) \), since \( \Sigma F_4 \) is a smooth projective variety. Consider the complete intersection

\[X = \Sigma F_4 \cap \bigcap_{i=1}^{12} H_i,
\]

where \( H_i \) represents a general hyperplane of \( \mathbb{P}^{25} \). Then \( X \) is a smooth 3-fold with

\[K_X = \mathcal{O}(-11 + 12) = \mathcal{O}(1).
\]

By using the Hilbert series we can compute the degree of \( X \) to be 78. Thus \( (X, K_X) \) is a smooth canonical threefold of general type with \( (K_X)^3 = 78 \).

**Remark 3.5** In fact we have a ladder of polarized varieties coming from Example 3.4. We can easily see that a 5-fold section \( V \) of \( \Sigma F_4 \) is a Gorenstein Fano 5-fold polarised by \(-K_V\). A general member \( Y \in |-(K_Y)| \) is a Calabi–Yau 4-fold polarized by \( D = -(K_Y)|_Y \), and \( X \) is a general member of \(|D|\) polarized by \( K_X = D|_X \). Thus we have a ladder of varieties

\[X \subset Y \subset V \subset \Sigma.
\]

**Example 3.6** We consider the case where ambient weighted homogeneous variety is not well-formed.

- Input parameters: \( \mu = (0, 0, 0, 0) \), \( u = 2 \)
- Embedding: \( w \Sigma F_4 \subset \mathbb{P}^{25}|_2 \), with each variable \( x_1, \ldots, x_{26} \) has weight 2.
- Hilbert numerator: \( 1 - 27t^4 - 78t^6 - 351t^{10} + \cdots - 351t^{20} + 78t^{24} - 27t^{28} + t^{30} \)
To construct a 3-fold of general type we take a triple projective cone over $w\Sigma$ which makes the given variety well-formed and we have an embedding of an 18-dimensional variety

$$C^3 w\Sigma F_4 \hookrightarrow \mathbb{P}^{28}[1^3, 2^{26}].$$

with $K_{C^3 w\Sigma F_4} = \mathcal{O}(-25)$; taking each projective cone adds 1 to the dimension and -1 to the canonical weight or index of $w\Sigma F_4$. Then we take the following weighted complete intersection

$$C^3 w\Sigma F_4 \cap (\bigcap_{i=1}^{15} Q_i) \hookrightarrow \mathbb{P}^{13}[1^3, 2^{11}]$$

with $K_X = \mathcal{O}_X(-25 + 2(15)) = \mathcal{O}_X(5)$ by the adjunction formula. The three new variables $y_1, y_2, y_3$, of degree 1 appear in the 15 degree two quadratics form. Each quadric $Q_i$ replaces a variable $x_i$ of weight 2 with a form of degree two in the equations of $X$. Without any loss of generality we assume that

$$x_i := Q_i(y_j s, x_k s) \text{ for } 12 \leq i \leq 26, 1 \leq j \leq 3, 1 \leq k \leq 11; \quad (3.6)$$

so we get the embedding

$$X \hookrightarrow \mathbb{P}[y_1, y_2, y_3, x_1, \ldots, x_{11}].$$

The base locus of the linear system $|\mathcal{O}(2)|$ of quadrics is empty, due to the weights of the embedding of $X$. Thus the only singularities of $X$ may occur due to the non-trivial weights of the embedding.

The locus of weights 2 variables in $w\Sigma F_4$ basically defines the whole 15-dimensional ambient variety. Then the complete intersection with 15 generic hypersurfaces of degree 2, linear equations in weight 2 variables, given by

$$X_0 := X \cap \{y_1 = y_2 = y_3 = 0\} \hookrightarrow \mathbb{P}^{10},$$

is 0-dimensional. Now from the Example 3.4 we conclude that the degree of $X_0$ is 78. Since $X_0$ is irreducible and reduced, so it consists of 78 distinct points. This can also be established by using the computer algebra on a specific example. Now to determine the local type of each singular point, we show that on each affine piece of the $\mathbb{P}^{10}[x_1]$, the weight 1 variables $y_1, y_2$ and $y_3$ are three local variables, so each point is locally of type $\frac{1}{2}(1, 1, 1)$. Now for affine piece $x_1 \neq 0$, we have

$$Q_{25} = x_4 + \cdots, \quad Q_{12} = x_5 + \cdots, \quad Q_{16} = x_6 + \cdots, \quad Q_{18} = x_7 + \cdots,$$

$$Q_{22} = x_8 + \cdots, \quad Q_{23} = x_9 + \cdots, \quad Q_{24} = x_{10} + \cdots, \quad Q_{20} = x_{11} + \cdots.$$

By using the implicit function theorem; from the equations A.1, A.2, A.3, A.4, A.5, A.6, A.7, A.8, A.9 and A.10 from Appendix A we can remove the variables $x_4, x_5, x_6, x_7, x_8, x_9, x_{10}, x_{11}, x_2$ and $x_3$ respectively. Thus $y_1, y_2, y_3$ are local variables near $x_1 \neq 0$. So the cyclic group $\mathbb{Z}_2$ acts by

$$\zeta : (y_1, y_2, y_3) \mapsto (\zeta y_1, \zeta y_2, \zeta y_3)$$

to give the quotient singularity of type $\frac{1}{2}(1, 1, 1)$. We can perform the similar calculation on each affine piece of $X$ to show that each point is a singular point of type $\frac{1}{2}(1, 1, 1)$. Thus $X$ is a family
of well-formed and quasi-smooth polarized 3-folds in the codimension 10 $w\Sigma F_4$. Moreover, the
degree of the polarizing divisor $D^3$ can easily be computed to be 39 by using the Hilbert series
of $X$. Thus the degree of the canonical divisor

$$(K_X)^3 = (5 \cdot D)^3 = 125 \cdot 39 = 4875.$$ 

**Remark 3.7** In fact, we can construct a finite number of families of smooth 3-folds of general
type by taking further projective cones over $w\Sigma(0, 2)$ and taking 3-fold linear sections $X$ of
relatively higher indices $k$ of $K_X = \mathcal{O}_X(k)$.

**Theorem 3.8** Let $w\Sigma(0, 2)$ be the weighted $F_4$-variety corresponding to the choice of parame-
ters $\mu = (0, 0, 0, 0)$ and $u = 2$. Then for each $k = 6, \cdots, 16$, we have a family of smooth polarized
3-fold $(X_k, D_k)$ of general type of index $k$, i.e. $K_{X_k} = \mathcal{O}_{X_k}(kD_k)$ which is the intersection of
$C^{k-2}w\Sigma(0, 2)$ with $(k+10)$ general quadric hypersurfaces of $\mathbb{P}^{k+23}$, with the following properties.

- Degree of the embedding: $D_k^3 = 39 \cdot (2)^{k-5}$
- Degree of canonical divisor: $(K_{X_k})^3 = (k \cdot D_k)^3$
- Weights of the embedding: $X_k \hookrightarrow \mathbb{P}^{13}[1^{k-2}, 2^{16-k}]$

**Proof** We start with the construction of a 3-fold with $k = 6$. Following the Example 3.3 if we
take a further projective cone over $w\Sigma(0, 2)$ then we add $-1$ to the canonical class of $w\Sigma(0, 2)$
and increase its dimension by one as well. Thus by taking four cones over $w\Sigma(0, 2)$ and taking
a weighted complete intersection with 16 generic forms of degree 2, we get a 3-fold

$$X = C^4w\Sigma \cap (\cap_{i=1}^{16}Q_i) \hookrightarrow \mathbb{P}^{13}[1^4, 2^{10}]$$

such that $K_X = \mathcal{O}(6)$. An extra quadric section will also change the degree $D^3$ of the new
threefold to be $D^3 = 39 \cdot 2 = 78$. The degree of the canonical class will be

$$(K_X)^3 = (6 \cdot D)^3 = 216 \cdot 78 = 16848.$$ 

Since we are taking one more quadric section, the singularities of $X$; thus

$$X \cap \{ \text{locus of degree 2 variables} \} = \emptyset.$$ 

So we get a smooth 3-fold of general type of index 6 in codimension 10. The rest of the cases
follow, by recursively using the same line of reasoning for $k = 7, \cdots, 16$.

**Remark 3.9** Recently, an algorithmic approach has been developed to find lists of isolated
orbifolds in a given weighted homogeneous variety, see [BKZ, Qur17]. A more complete list of
varieties in codimension 10 $w\Sigma F_4$ varieties will appear elsewhere [BKQ].

### A Equations of $F_4$ homogeneous variety

We compute the equations of the homogeneous variety $\Sigma F_4$ by the GAP4 code given in Ap-
pendix of [QS11]. We compute the decomposition of the 2nd symmetric power $S^2(V_\lambda^*)$ of the
representation dual to $V_\lambda$, into its direct summands as free modules over $f_4$. Now $S^2V_\lambda^*$ is a 351-dimensional vector space and we obtain the following decomposition into the subrepresentations of $f_4$:
\[
Z = S^2V_\lambda^* = V_1 \oplus V_{e_1} \oplus V_{e_2},
\]
having dimensions 324, 26, and 1 respectively. The set of defining equations of $\Sigma F_4$ is the union of the basis of the linear subspaces of $V_{e_1}$ and $V_{e_2}$.
\[
I = \langle Q \rangle = \langle V_{e_1} \cup V_{e_2} \rangle \subset S^2V_\lambda^*.
\]
Therefore the homogeneous ideal of $\Sigma$ is defined by 27 quadratic equations given below.

\[
x_{1}x_{25} - \frac{1}{7}x_{1}x_{26} - \frac{3}{7}x_{4}x_{5} - \frac{3}{7}x_{6}x_{7} - \frac{3}{7}x_{8}x_{9} + \frac{3}{7}x_{10}x_{11} \quad (A.1)
\]
\[
x_{1}x_{12} - \frac{1}{7}x_{4}x_{25} - \frac{1}{7}x_{4}x_{26} - \frac{1}{7}x_{6}x_{13} - \frac{1}{7}x_{8}x_{14} + \frac{1}{7}x_{10}x_{15} \quad (A.2)
\]
\[
x_{1}x_{16} - x_{4}x_{17} + \frac{1}{7}x_{6}x_{25} - \frac{2}{7}x_{6}x_{26} - x_{11}x_{14} + x_{9}x_{15} \quad (A.3)
\]
\[
x_{1}x_{18} - x_{4}x_{19} + \frac{1}{7}x_{8}x_{25} - \frac{2}{7}x_{8}x_{26} + x_{11}x_{13} - x_{7}x_{15} \quad (A.4)
\]
\[
x_{1}x_{20} - x_{4}x_{21} - \frac{1}{7}x_{10}x_{25} + \frac{2}{7}x_{10}x_{26} - x_{9}x_{7} + x_{7}x_{14} \quad (A.5)
\]
\[
x_{1}x_{22} - x_{6}x_{19} + x_{8}x_{17} - \frac{1}{7}x_{11}x_{25} - \frac{2}{7}x_{11}x_{26} + x_{5}x_{15} \quad (A.6)
\]
\[
x_{1}x_{23} - x_{6}x_{21} - \frac{2}{7}x_{9}x_{25} + \frac{1}{7}x_{9}x_{26} - x_{5}x_{14} \quad (A.7)
\]
\[
x_{1}x_{24} - x_{8}x_{21} - x_{10}x_{19} - \frac{1}{7}x_{7}x_{25} - \frac{2}{7}x_{7}x_{26} + x_{5}x_{13} \quad (A.8)
\]
\[
x_{1}x_{2} - x_{11}x_{21} - x_{9}x_{19} - x_{7}x_{17} - \frac{2}{7}x_{5}x_{25} + \frac{2}{7}x_{5}x_{26} \quad (A.9)
\]
\[
x_{1}x_{3} + x_{6}x_{24} - x_{8}x_{23} - x_{10}x_{22} - x_{15}x_{21} - x_{14}x_{19} + \]
\[
x_{5}x_{12} - x_{13}x_{17} - x_{13}x_{17} - \frac{1}{3}x_{25}^2 + \frac{1}{3}x_{26}^2 \quad (A.10)
\]
\[
x_{4}x_{22} - x_{6}x_{18} + x_{8}x_{16} - x_{11}x_{12} + \frac{2}{3}x_{15}x_{25} - \frac{2}{3}x_{15}x_{26} \quad (A.11)
\]
\[
x_{4}x_{23} - x_{6}x_{20} - x_{10}x_{16} + x_{9}x_{12} - \frac{2}{3}x_{14}x_{25} + \frac{2}{3}x_{14}x_{26} \quad (A.12)
\]
\[
x_{4}x_{24} - x_{8}x_{20} - x_{10}x_{18} - x_{7}x_{12} + \frac{2}{3}x_{13}x_{25} - \frac{2}{3}x_{13}x_{26} \quad (A.13)
\]
\[
x_{2}x_{4} - x_{6}x_{24} + x_{8}x_{23} + x_{10}x_{22} - x_{11}x_{20} - x_{9}x_{18} + \]
\[
x_{7}x_{16} - x_{5}x_{12} + \frac{2}{3}x_{25}x_{26} - \frac{1}{3}x_{26}^2 \quad (A.14)
\]
\[
x_{3}x_{4} - x_{15}x_{20} - x_{14}x_{18} - x_{13}x_{16} - \frac{1}{3}x_{12}x_{25} + \frac{2}{3}x_{12}x_{26} \quad (A.15)
\]
\[
x_{6}x_{2} - x_{11}x_{23} - x_{9}x_{22} + x_{16}x_{16} - \frac{2}{3}x_{17}x_{25} + \frac{2}{3}x_{17}x_{26} \quad (A.16)
\]
\[
x_{3}x_{6} - x_{15}x_{23} - x_{14}x_{22} + \frac{1}{3}x_{16}x_{25} + \frac{1}{3}x_{16}x_{26} - x_{12}x_{17} \quad (A.17)
\]
\[
x_{2}x_{8} - x_{11}x_{24} + x_{7}x_{22} + x_{5}x_{18} - \frac{2}{3}x_{19}x_{25} + \frac{2}{3}x_{19}x_{26} \quad (A.18)
\]
\[
x_{3}x_{8} - x_{15}x_{24} + x_{13}x_{22} + \frac{1}{3}x_{18}x_{25} + \frac{1}{3}x_{18}x_{26} - x_{12}x_{19} \quad (A.19)
\]
\[
x_{2}x_{10} - x_{9}x_{24} - x_{7}x_{23} - x_{5}x_{20} + \frac{2}{3}x_{21}x_{25} - \frac{2}{3}x_{21}x_{26} \quad (A.20)
\]
\[
x_{3}x_{10} - x_{14}x_{24} - x_{13}x_{23} - \frac{1}{3}x_{20}x_{25} - \frac{1}{3}x_{20}x_{26} + x_{12}x_{21} \quad (A.21)
\]
\[
x_{3}x_{11} - x_{2}x_{15} - \frac{1}{3}x_{22}x_{25} + \frac{2}{3}x_{22}x_{26} + x_{17}x_{18} - x_{16}x_{19} \quad (A.22)
\]
\[
x_{3}x_{9} - x_{2}x_{14} + \frac{1}{3}x_{23}x_{25} - \frac{2}{3}x_{23}x_{26} - x_{17}x_{20} + x_{16}x_{21} \quad (A.23)
\]
\[
x_{3}x_{7} - x_{2}x_{13} - \frac{1}{3}x_{24}x_{25} + \frac{2}{3}x_{24}x_{26} + x_{19}x_{20} - x_{18}x_{21} \quad (A.24)
\]
\[
x_{3}x_{5} - \frac{1}{3}x_{2}x_{25} - \frac{1}{3}x_{2}x_{26} + x_{17}x_{24} - x_{19}x_{23} + x_{21}x_{22} \quad (A.25)
\]
\[ x_3x_{25} - \frac{1}{2}x_3x_{26} - \frac{3}{2}x_{22}x_{12} + \frac{3}{2}x_{16}x_{24} - \frac{3}{2}x_{18}x_{23} + \frac{3}{2}x_{20}x_{22} \]  
(A.26)

\[ x_1x_3 - x_{22}x_4 + x_{6}x_{24} - x_{8}x_{23} + x_{10}x_{22} + x_{11}x_{20} + x_{9}x_{18} - x_{15}x_{21} + x_{7}x_{16} - x_{14}x_{19} - x_{5}x_{12} - x_{13}x_{17} + \frac{1}{3}x_{25}^2 - \frac{1}{3}x_{25}x_{26} + \frac{1}{3}x_{26}^2 \]  
(A.27)

B SAGE code for the Hilbert series of \( w \Sigma F_4 \)

```sage
F4=WeylCharacterRing(['F',4])
W=WeylGroup(['F',4])
L=W.domain()
lam=vector([1,0,0,0])
a,b,c,d,t,u= var('a,b,c,d,t,u')
rho=vector([11/2,5/2,3/2,1/2])
mu=vector([a,b,c,d])
N=[(-1)**(w.length())*(t**(((w*rho)*mu))/ (1-t**(((w*lam)*mu)+u))) for w in W]
num=sum(N)
D=[(-1)**(w.length())*t**(((w*rho)*mu)) for w in W]
den=sum(D)
HilbertSeries=num/den
```

References

[AM69] M. F. Atiyah and I. G. Macdonald, *Introduction to commutative algebra*, Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont., 1969.

[BE77] D. A. Buchsbaum and D. Eisenbud, *Algebra structures for finite free resolutions, and some structure theorems for ideals of codimension 3*, Amer. J. Math 99 (1977), 447–485.

[BKQ] G.D. Brown, A. M. Kasprzyk, and M. I. Qureshi, *Fano manifold in Gorenstein formats*, In preparation.

[BKR12] G.D. Brown, M. Kerber, and M.A. Reid, *Fano 3-folds in codimension 4, Tom and Jerry. I.*, Compos. Math. 148 (2012), no. 4, 1171–1194.

[BKZ] G. Brown, A. M. Kasprzyk, and L. Zhu, *Gorenstein formats, canonical and Calabi-Yau threefolds*, arXiv:1409.4644.

[Bou02] Nicolas Bourbaki, *Elements of Mathematics, Lie Groups and Lie Algebras, Chapters 4–6*, Springer-Verlag, 2002.

[CR02] A. Corti and M. Reid, *Weighted Grassmannians*, Algebraic geometry (M. C. Beltrametti, F. Catanese, C. Ciliberto, A. Lauter, and C. Pedrini, eds.), de Gruyter, Berlin, 2002, pp. 141–163.

[dG01] W. A. de Graaf, *Constructing representations of split semisimple Lie algebras*, J. Pure Appl. Algebra 164 (2001), no. 1-2, 87–107, Effective methods in algebraic geometry (Bath, 2000).
[FH91] W. Fulton and J. Harris, *Representation theory, a first course*, Graduate Text in Mathematics, 129, Springer-Verlag, 1991.

[GKR07] A. L. Gorodentsev, A. S. Khoroshkin, and A. N. Rudakov, *On syzygies of highest weight orbits*, Moscow Seminar on Mathematical Physics. II (V. I. Arnold, D.G. Gindikin, and V. P. Maslov, eds.), Amer. Math. Soc. Transl. Ser. 2, vol. 221, AMS, Providence, RI, 2007, pp. 79–120.

[MAAvLL92] A. M. Cohen M. A. A. van Leeuwen and B. Lisser, *"LiE, a Package for Lie Group Computations"*, 1992, Computer Algebra Nederland, Amsterdam, ISBN 90-74116-02-7.

[QS11] M. I. Qureshi and B. Szendrői, *Constructing projective varieties in weighted flag varieties*, Bull. Lond. Math Soc. 43 (2011), no. 2, 786–798.

[QS12] M. I. Qureshi and Balázs Szendrői, *Calabi-Yau threefolds in weighted flag varieties*, Adv. High Energy Phys. (2012), Art. ID 547317, 14 pp.

[Qur15] M. I. Qureshi, *Constructing projective varieties in weighted flag varieties II*, Math. Proc. Camb. Phil. Soc. 158 (2015), 193–209.

[Qur17] Muhammad Imran Qureshi, *Computing isolated orbifolds in weighted flag varieties*, Journal of Symbolic Computation 79, Part 2 (2017), 457 – 474.

[Rei87] M. Reid, *Young person’s guide to canonical singularities*, Algebraic geometry, Bowdoin, 1985 (Brunswick, Maine, 1985), Proc. Sympos. Pure Math., vol. 46, Amer. Math. Soc., Providence, RI, 1987, pp. 345–414.

[Rei15] ______, *Gorenstein in codimension 4 - the general structure theory*, Algebraic Geometry in East Asia (Nov 2011), Advanced Studies in Pure Mathematics, vol. 65, Taipei, 2015, pp. 201–227.

[Sag] *Sagemath, the Sage Mathematics Software System (Version 7.5.1)*, [http://www.sagemath.org](http://www.sagemath.org).

[Ser] J. P. Serre, *Sur les modules projectifs*, Séminaire Dubreil, 1960/61.