Influence of anharmonic convex interaction and Shapiro steps in the opposite direction of driving force

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The response function and largest Lyapunov exponent analysis were applied to the driven overdamped Frenkel-Kontorova model with two types of anharmonic convex interparticle interactions. In both cases model reduces to a single particle model for integer values of winding number. It is shown that the mirror image of the amplitude dependence of critical depinning force and largest Lyapunov exponent observed recently in the standard Frenkel-Kontorova model [Commun. Nonlinear Sci. Numer. Simul. 47, 100 (2017)] is not retained generally. Behaviour of systems with relatively strong anharmonic interaction force was examined and evidence for the appearance of mode-locking phenomenon in both directions of particles’ motion is presented.

I. INTRODUCTION

In the past few decades, synchronization phenomenon has become a subject of particular interest. Although it can be detected in many systems, the focus is on examination of dynamical mode-locking, which emerges when systems with a characteristic internal frequency are subjected to an external periodic drive [1, 2]. For instance, Shapiro steps (i.e., step-like macroscopic response) are experimentally observed in various colloidal systems [3–7], charge-density wave systems [8–10], vortex lattices [11–13], Josephson junction arrays [14–18] and others. Regardless of the response function type (average velocity as a function of $F_{dc}$ force, $IV$ characteristics), both pinning-sliding transition and Shapiro steps can be recognized. In addition, special attention is given to the critical force and Shapiro steps width dependence on value of the ac force amplitude in these systems, which has proven to be Bessel-like [9, 10, 19–21].

A model which is used to successfully capture such behaviour in many condensed matter systems, micro- and nanotechnologies in the past few decades is the Frenkel-Kontorova model (FK model) driven by periodic forces [22–24]. The FK model is commonly used to describe the dynamics of Josephson junction arrays [25–28], DNA chains [29], charge-density waves [19, 30, 31], incommensurate phases in dielectrics, dynamics of domain walls in ferromagnetic and antiferromagnetic chains [23], etc. The one-dimensional Frenkel-Kontorova model describes chain of identical particles which are coupled to their nearest neighbours and subjected to a substrate potential. In a special case of standard FK model, interaction between the particles is harmonic and substrate potential is sinusoidal [30]. A special attention is given to theoretical research of the dissipative FK model driven by dc and ac forces in both overdamped [32–35] and underdamped [36–38] regimes nowadays.

From the theoretical point of view, several interesting results were found for the standard dissipative FK model driven by external periodic forces [30]. For example, width of the first harmonic Shapiro step and the critical force exhibit Bessel-like behaviour and the maxima of one function correspond to the minima of the other [32]. Also, the standard FK model cannot be used for modeling phenomena related to subharmonic steps since subharmonic steps do not exist in the commensurate structures with integer values of winding number [39, 40], while for the non-integer values their size is too small [30]. On the other hand, dissipative ac+dc driven FK model with various forms of deformable substrate potentials was very successful in describing most of the phenomena related to the Shapiro steps such as the origin of subharmonic steps, their amplitude and frequency dependence [33, 41]. Since detection of subharmonic steps can be particularly difficult, the largest Lyapunov exponent (largest LE) analysis is often used instead of the response function as it represents a more sensitive way to detect subharmonic steps [34].

Most of the research done so far regarded the harmonic type of interparticle interactions and different forms of periodic substrate potentials with a periodic driving force (see [32] for example), but a more general type of interaction has not been widely examined yet. However, some properties of the FK model with non-convex interparticle interactions [42, 43], dynamics of the FK systems with Morse type of interaction [44] and Toda interaction [45] have been investigated in the last few decades. An anharmonic interaction form is present in the model description of Josephson junctions [46] and many other systems [47]. Consequently, it is of significant importance to investigate the effect of anharmonic interactions on dynamics in the FK systems driven by external periodic force.

The purpose of this paper is to demonstrate how a change of harmonic interaction affects the model dynamics using both response function and largest LE analysis and compare it to the previously well-investigated standard FK model case. Since in the one-dimensional overdamped FK model with convex interparticle interaction the particles move following Middleton’s no passing rule (i.e., they preserve their order during motion), the
system possesses the property of asymptotic uniqueness. This means that, in the limit of long times, the average velocities of the particular commensurate structure approach a unique solution [48, 49]. Therefore, the overdamped FK models with sinusoidal substrate potential and convex anharmonic interactions that depend only on distance between the nearest neighbours are studied in this paper. Two types of interparticle interactions are taken into account and their influence on the depinning force, size of the steps and their amplitude dependence is examined.

The paper is organized as follows. A brief description of the model is given in Section II, while the results are presented in Section III. The paper ends with discussion of the results in Section IV.

II. MODEL

Total potential energy of the standard FK model is given by [30]:

\[ H = \sum_j (V(u_j) + W(u_{j+1} - u_j)), \]  

(1)

where \( V(u_j) \) is sinusoidal substrate potential of the form:

\[ V(u_j) = \frac{K}{(2\pi)^2} (1 - \cos(2\pi u_j)), \]  

(2)

with strength of the periodic substrate potential \( K \), and \( W(u_{j+1} - u_j) \) is pairwise harmonic interparticle interaction, which is a function of distance between the nearest neighbours:

\[ W(u_{j+1} - u_j) = \frac{1}{2}(u_{j+1} - u_j)^2, \]  

(3)

where \( j = 1, 2, ..., N \). In this paper the overdamped FK model subjected to the influence of both ac and dc external force is considered. The corresponding equations of motion are:

\[ \dot{u}_j = \nabla^2 u_j + \frac{K}{2\pi} \sin(2\pi u_j) + F(t), \]  

(4)

with \( \nabla^2 \) being the lattice Laplacian:

\[ \nabla^2 u_j = u_{j+1} + u_{j-1} - 2u_j, \]  

(5)

and external force \( F \) has the following form:

\[ F(t) = F_{dc} + F_{ac} \cos(2\pi \nu_0 t). \]  

(6)

\( F_{dc} \) is the dc force, \( F_{ac} \) is amplitude of the ac force and \( \nu_0 = \frac{1}{T} \) is frequency of the ac force with \( T \) being its period.

Due to the competition between two frequency scales (one of the external periodic force and the other of the motion associated with the periodic substrate potential) the dynamics of this model is characterized by the appearance of Shapiro steps. These steps correspond to the resonant solutions of (4). If \( \{u_j(t)\} \) is the solution of (4) with the initial condition \( \{u_j(t_0)\} \), then:

\[ \sigma_{r,m,s} \{u_j(t)\} = \{u_{j+r}(t - \frac{s}{\nu_0}) + m\} \]  

(7)

is also a solution of the same equations corresponding to the initial condition \( \sigma_{r,m,s} \{u_j(t_0)\} \), where \( r, m, s \) are arbitrary integers. This can easily be verified by a simple substitution:

\[ \dot{u}_{j+r}(t - \frac{s}{\nu_0}) = u_{j+r+1}(t - \frac{s}{\nu_0}) + u_{j+r-1}(t - \frac{s}{\nu_0}) - 2u_{j+r}(t - \frac{s}{\nu_0}) - \frac{K}{2\pi} \sin \left( 2\pi \left( u_{j+r}(t - \frac{s}{\nu_0}) + m \right) \right) + F(t). \]  

(8)

The corresponding average velocity is then easily evaluated:

\[ \bar{v} = \frac{1}{sT} \sum_{j=M}^{N-1} (u_j(t + sT) - u_j(t)) \]

\[ = \frac{\nu_0}{s} \frac{1}{N - M} \sum_{j=M}^{N-1} (u_{j+r}(t) + m - u_j(t)) \]  

(9)

and the final form is well-known:

\[ \frac{\bar{v}}{\nu_0} = \frac{r\omega + m}{s}, \]  

(10)

where \( \omega \) is winding number.
Although equivalence of some non-convex and convex models was shown previously [50], this sort of difficulty is evaded since only convex interactions are considered. First off, it is of particular interest to establish how anharmonic terms affect the model dynamics. Thus, quartic polynomial interaction was used:

\[ W(u_{j+1} - u_j) = \frac{g}{2}(u_{j+1} - u_j)^2 + \frac{h}{3}(u_{j+1} - u_j)^3 + \frac{f}{4}(u_{j+1} - u_j)^4 \]  

where constants \( g, h \) and \( f \) are chosen conveniently in order to obtain convex interaction. The corresponding equations of motion are:

\[ \dot{u}_j = g \nabla^2 u_j - h(\nabla^2 u_j)^2 + f(\nabla^2 u_j)^3 + \frac{K}{2\pi} \sin(2\pi u_j) + F(t), \]  

where \( F(t) \) is given by (6) and \( j = 1, 2, ..., N \). Parameters \( g, h \) and \( f \) will be varied and their influence on the model dynamics examined in the following section.

The second type of interaction examined in this paper is given by exponential form:

\[ W(u_{j+1} - u_j) = e^{-(u_{j+1} - u_j)} \]  

with the corresponding equations of motion:

\[ \dot{u}_j = e^{-(u_j - u_{j-1})} - e^{-(u_{j+1} - u_j)} + \frac{K}{2\pi} \sin(2\pi u_j) + F(t). \]  

From the form of convex interactions (11) and (13) one can see that they are once again a function of distance between the nearest neighbours and, since they are independent of time, the symmetry of the solution is the same as in (7) and the resonant velocities are given by (10).

The response functions were obtained from the equations of motion (12) and (14), which had been integrated using the fourth order Runge-Kutta method with periodic boundary conditions. The dc force was varied adiabatically with the step \( \Delta F_{dc} = 10^{-5} \). The largest LE \( \lambda \) was calculated according to the algorithm given in details in Reference [51].

### III. RESULTS

In this section, the study of FK model with two types of anharmonic convex interactions is presented. Both response function and largest LE analysis were used.

#### A. Subharmonic steps and the largest Lyapunov exponent analysis

The response functions \( \bar{v} = \bar{v}(F_{dc}) \) corresponding to the equations of motion (12) and (14) are presented in Figure 1. Since the same symmetry (7) accounts for two chosen types of interparticle interactions, the same resonant velocities (10) are expected to be observed in the \( \bar{v} = \bar{v}(F_{dc}) \) plot. Namely, for \( \omega = \frac{1}{2} \), one can always choose \( m = 0 \) (see [30]), which simplifies the expression (10):

\[ \bar{v} = \frac{r\omega}{s} \nu_0. \]  

Given the equation (15), all the Shapiro steps observed in Figure 1 are harmonic. The response functions are somewhat similar to the ones of the standard FK model [52]. In addition, high resolution analysis reveals the subharmonic steps as can be seen in Figure 2.

**FIG. 1.** Average velocity as a function of driving force for two anharmonic interactions: quartic polynomial interaction (11) with \( g, h, f = 1 \) (a) and exponential interaction (13) (b) and three values of the ac force amplitude. The chosen set of parameters is: \( \omega = \frac{1}{2}, K = 4.0 \) and \( \nu_0 = 0.2 \). The numbers mark harmonic steps.
Further, we have verified that for \( \omega = 1 \) subharmonic steps are not detected and that the response functions corresponding to equations (12) and (14) coincide with the standard FK model case for the same set of parameters. It was shown previously that for the standard harmonic interaction and integer values of winding number, the model reduces to a single particle model [30]. The examination of response functions for multiple sets of parameters and various integer values of winding number has shown that subharmonic steps do not exist even in the case of anharmonic interactions (11) and (13) and thus new degrees of freedom are not added to the system.

In spite of analogous behaviour, there are some deviations from the standard model in two cases considered in this paper. A simple comparison between the Figures 1 a) and b) allows one to deduce that the values of both critical force \( F_c \) and size of Shapiro steps vary with the change of interaction form. We notice that the values of critical force are a bit larger than the one obtained in purely harmonic case \([32, 52]\). This may lead one to the conclusion that the anharmonic form of interparticle interaction affects particles’ tendency to leave their positions pinned by the periodic substrate potential and, for the given set of parameters, they need to be stimulated by a larger value of dc drive.

One can expect that, since the quartic polynomial interaction (11) reduces to the standard harmonic form for \( g \to 1, \ h, f \to 0 \), plot in Figure 1 a) turns into its standard FK form from Reference [52] (the same applies for other ac force frequencies). This situation is presented in Figure 3. The critical depinning force tends towards its value in case of the standard FK model (approximately \( F_c = 1.6 \) for given set of parameters [32, 52]). One can also notice that varying the values of this set of parameters does not influence widening of the subharmonic steps, as only harmonic steps are detected in Figure 3, and high resolution analysis is necessary in order to observe them.

The largest LE analysis is useful since it allows one to examine the appearance of chaos in a system [35, 38, 53]. It is well-known that chaotic behaviour has not been observed in the overdamped FK model with various types of substrate potentials [54]. However, the question remains whether taking anharmonic interactions (11) and (13) into account would result in the appearance of chaos in the system. The results of comparison between the response function and largest LE analysis are given in Figure 4.

For both interactions ((11) and (13)) it is obvious that the largest LE is always non-positive and thus the corresponding dynamics is non-chaotic. One may draw the conclusion that this is yet another property that is common for the family of overdamped FK models with different convex interactions. Also, just like in the Reference [34], the largest LE is negative in the pinned regime, reaching zero at the value \( F_c \) for the first time. Afterwards the behaviour is consistent and the largest LE is negative for the entire interval of dc force where the corresponding Shapiro step is observed. In Figure 5, which represents a zoomed segment of Figure 4 a), one can see that this tool is appropriate for detecting some subharmonic steps that could not be seen in the corresponding response function plot.

### B. Bessel-like behaviour and Shapiro steps in the opposite direction of driving force

Reference [34] provides evidence that, in the case of standard FK model, the largest LE can be used to investigate some characteristic properties of the model since the dependence of LE on \( F_{ac} \) for \( F_{dc} = 0 \) gives reverse image of the critical force amplitude dependence. Furthermore, amplitude dependencies of the critical force and first har-
FIG. 4. Average velocity and largest LE as functions of driving force in case of two interactions: a) quartic polynomial interaction (11) and b) exponential interaction (13). The chosen set of parameters is: $\omega = \frac{1}{2}$, $F_{ac} = 0.2$ $K = 4.0$, $\nu_0 = 0.2$ and $g = h = f = 1$. The numbers mark harmonic steps.

FIG. 5. Zoomed segment of Figure 4 a). The numbers mark subharmonic steps.

monic step width exhibit Bessel-like behaviour and the maxima of one function correspond to the minima of the other [32, 33]. However, different choice of the substrate potential in FK system leads to the violation of this rule [34] and thus it would be interesting to examine whether the same would happen if some anharmonic interactions are taken into account. Amplitude dependencies of the critical force $F_c$, first harmonic Shapiro step width $\Delta F_1$ and largest LE $\lambda$ for $F_{dc} = 0$ for the FK model with interactions (11) and (13) are shown in Figure 6.

It is obvious that amplitude dependence of the critical force and width of the first harmonic step preserves the previously described behaviour in the standard case. This serves as yet another proof of the correspondence between the FK models with different convex interactions. However, the reverse image of largest LE and critical force is retained only for the exponential interaction (13). Since there are some deviations in the case of quartic polynomial interaction (11), this mirror image cannot be outlined as a general rule.

Yet another deviation from the standard model is observed in Figure 6 a). While in the standard case it was shown that the value of critical depinning force is clearly equal to zero, or at least very close to it, for several values of the ac force amplitude that are in the same range as the one given in Figure 6, that is not the case when quartic polynomial interaction (11) is taken into account. It becomes apparent that, in the case of interaction (11), system has to overcome some additional force that tends to push the particles in the direction opposite to the di-
direction of dc force. Simple analysis of the corresponding
equations of motion (12) provides a credible explanation.
Indeed, when one plots the interaction force between the
nearest neighbours \( F_{\text{int}} = g\nabla^2 u_j - h(\nabla^2 u_j)^2 + f(\nabla^2 u_j)^3 \) as a function of discrete Laplacian term \( \nabla^2 u_j \), as it is
done in Figure 7, it becomes straightforward.

In Figure 7 one can observe that, in the purely harmonic case (\( g = 1, h = f = 0 \)), interaction force \( F_{\text{int}} \) is obviously an odd function of discrete Laplacian term \( \nabla^2 u_j \). In this case, critical depinning force has the smallest possible value (close to \( F_{\text{dc}} = 1.6 \)). However, in the case of non-zero values of parameters \( h \) and \( f \), one talks about anharmonic interactions and it is clearly seen that the influence of this term is biased towards the negative values of interaction force.

To check whether this deduction is correct and get a better insight into the particles’ motion, Poincaré sections for two neighbouring particles with coordinates \( u_1 \) and \( u_2 \) are shown in Figures 8 and 9. It is important to outline that, in order to be able to obtain such results, periodic boundary conditions were used and \( N = 8 \) particles considered. Due to the fact that the period of substrate potential is 1 and the winding number is \( \omega = \frac{1}{2} \), the behaviour presented in these figures corresponds to the relative motion of two neighbouring particles in four potential wells. The results are presented for two values of ac force amplitude \( F_{\text{ac}} = 0.2 \) and \( F_{\text{ac}} = 0.4 \) and four dc force values that are close to the corresponding critical depinning forces in both pinning and sliding regime. While the positions of the particles are bound to a really tight band in the pinning regime, one can see that this is not the case with the sliding regime and the particles’ collective motion is quite noticeable.

Although discrete Laplacian term \( \nabla^2 u_j \) is both positive and negative at different moments in time, the un
 doubted favouring of the interaction force with direction opposite to the direction of driving dc force eventually gives rise to the critical depinning force \( F_c \). This is rather an interesting observation and implies that the convenient choice of parameters \( h \) and \( f \) could cause reduction of the depinning force to zero value. This situation would occur in the case when the interaction force term is strong enough to push the particles out of their pinned positions in the direction opposite to the direction of dc force. In Figure 10 one can see evidence that this assumption is accurate.

In Figure 10 a) interaction force is still insufficient to push the particles out of their positions and the critical depinning force is a bit larger than it was the case for the set of parameters considered previously (\( g = h = f = 1 \)). This is purely the consequence of the fact that the biased direction of the interaction force is the one opposite to the direction of dc force. However, with the increase in values of parameters \( h \) and \( f \), particles are pushed out of their positions in the direction opposite to the dc force and average velocity has a non-zero value even for \( F_{\text{dc}} = 0 \).

Of course, the negative sign of the average velocity is merely a consequence of the fact that its direction is opposite to the direction of dc force and is thus purely a
FIG. 8. Poincaré sections for two neighboring particles with coordinates \( u_1 \) and \( u_2 \) in the case of model with quartic polynomial interaction (11) and chosen set of parameters: \( \omega = \frac{1}{2}, \ K = 4.0, \ \nu_0 = 0.2 \) and \( g = h = f = 1 \). The values of dc force are: a) \( F_{dc} = 0.25 \), b) \( F_{dc} = 0.265 \), c) \( F_{dc} = 0.1 \) and d) \( F_{dc} = 0.12 \).

FIG. 9. Poincaré sections for two neighboring particles with coordinates \( u_1 \) and \( u_2 \) in the case of model with exponential interaction (13) and chosen set of parameters: \( \omega = \frac{1}{2}, \ K = 4.0 \) and \( \nu_0 = 0.2 \). The values of dc force are: a) \( F_{dc} = 0.23 \), b) \( F_{dc} = 0.25 \), c) \( F_{dc} = 0.09 \) and d) \( F_{dc} = 0.1 \).
FIG. 10. Average velocity as a function of driving force in case of quartic polynomial interaction (11) for $\omega = \frac{1}{2}$, $F_{dc} = 0.2$, $K = 4.0$, $\nu_0 = 0.2$ $g = 1$ and several values of parameters $h$ and $f$: a) $h = f = 1.25$, b) $h = f = 2$, c) $h = 4$, $f = 10$ and d) $h = 20$, $f = 140$. The numbers mark harmonic steps.

statement of the convenient convention. Mode-locking is observed even in this region and thus the appearance of Shapiro steps is detected at the already expected values of average velocity: $\bar{v} = 0.1$ (Figure 10 b), c) and d)) and $\bar{v} = 0.2$ (Figure 10 d)) but now with a negative sign due to the opposite direction. When dc force becomes strong enough, average velocity decreases to zero. Afterwards, the behaviour is analogous to the one given in Figure 1 and the dc force prevails over the interaction force, leading the particles in the opposite direction.

IV. CONCLUSION

In this paper, generalized dissipatively driven FK models with anharmonic convex interactions are considered. The research was conducted for two types of interactions that resemble the ones that are already discussed in uniform atomic chains [55] and one-dimensional Hamiltonian systems perturbed by a conservative noise [56].

The response functions were obtained in the case of quartic polynomial interaction (11) and exponential interaction (13). Clear signs of correspondence between the FK models with different types of convex interactions were observed, but with two major differences: value of critical depinning force $F_c$ and size of Shapiro steps. The response functions for the model with anharmonic interactions (11) and (13) match completely the results obtained in the standard case for integer values of winding number and the model reduces to a single particle model. However, the amplitude dependencies of the critical force $F_c$, size of the first harmonic step $\Delta F_1$ and largest LE $\lambda$ for $F_{dc} = 0$ have shown that there are some deviations from the standard FK model if the considered interaction is anharmonic. Although the mirror image of the amplitude dependence of critical depinning force and largest LE observed in the standard case holds out in the case of exponential interaction (13), this conclusion cannot be drawn generally since in the case of quartic polynomial interaction (11) this image is not retained. We have shown that the reason for critical depinning force non-zero minima in the case of quartic polynomial interaction (11) lies in the fact that anharmonic interactions tend to push the particles in the direction opposite to the direction of the dc drive. However, when the dc force becomes large enough, it overcomes the influence of the interaction force and the particles are pushed in the direction of the dc force. Another interesting result is that mode-locking phenomenon is observed in both directions of particles’ motion. Namely, if one chooses anharmonic interaction forces that are strong enough to solely push the particles out of their pinned positions and applies external dc drive
in the opposite direction, Shapiro steps are detected in the response function plots at both positive and negative values of average velocities. These values are determined by the equation (15) and the sign of negative average velocities is merely a consequence of the reverse direction of particles’ motion.

The FK models with anharmonic interactions have not been widely studied so far [44–47]. In the present paper, not only is it shown that it is possible to change the direction of motion by conveniently choosing an anharmonic convex interaction but also to vary the critical depinning force value and the size of Shapiro steps. Although it was already known that the critical depinning force decreases to zero for certain values of ac force amplitude in the standard FK model, this paper has provided evidence that this can also be done by choosing a convenient anharmonic form. Furthermore, critical depinning force non-zero minima have already been observed in experiments [9]. Therefore, our results could be significant for such systems as we have shown that anharmonicity might be the cause of it. The presented results could be important for the study of overdamped systems like irradiated Josephson junction arrays, where capacitance of junctions is small enough [35], colloidal systems [3–7] and charge-density wave systems [8–10] since pure harmonicity is not so common in nature.

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