THICKNESS OF SKELETONS OF ARITHMETIC HYPERBOLIC ORBIFOLDS

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Abstract. We show that closed arithmetic hyperbolic \( n \)-dimensional orbifolds with larger and larger volumes give rise to triangulations of the underlying spaces whose 1-skeletons are harder and harder to embed nicely in Euclidean space. A related question: can 1-skeletons of triangulations of a sphere form an expander family?

1. Introduction

Our main theorem is about the complexity of 1-skeletons of triangulations of arithmetic hyperbolic orbifolds, as measured by how nicely those 1-skeletons can be embedded in Euclidean space. Following Gromov and Guth in [GG12], we say that an embedding of a graph \( G \) into \( \mathbb{R}^N \) has **combinatorial thickness** at least 1 if disjoint vertices and edges have disjoint 1-neighborhoods; that is, every two distinct vertices have distance at least 2, as do every two edges without a vertex in common and every edge and a vertex other than the edge’s two endpoints. Let \( V_{1,N}(G) \) denote the infimum, over all embeddings of \( G \) into \( \mathbb{R}^N \) with combinatorial thickness at least 1, of the \( N \)-dimensional volume of the 1-neighborhood of the image of the embedding. We define the **Cheeger constant** \( h(X) \) of an \( n \)-dimensional metric space \( X \) as the greatest number such that for all open subsets \( A \subseteq X \) with Hausdorff measurable boundary \( \partial A \), we have

\[
h(X) \leq \frac{\text{Area} \partial A}{\min\{\text{Vol} A, \text{Vol} X \setminus A\}}.
\]

Our main theorem is stated as follows.

**Theorem 1.** Let \( \{X_k\}_{k=1}^{\infty} \) be a sequence of closed arithmetic hyperbolic orbifolds of dimension \( n \geq 3 \). Suppose that the Cheeger constants \( h(X_k) \) are uniformly bounded by

\[
0 < \frac{1}{C} < h(X_k) < C,
\]

and suppose that the hyperbolic volumes satisfy

\[
\text{Vol} X_k \rightarrow \infty.
\]

Then for any fixed dimension \( N \geq 3 \), there exist triangulations of the underlying spaces of the orbifolds \( X_k \), such that the 1-skeletons \( X_k^1 \) have a uniform bound on the number of edges at each vertex and satisfy

\[
\frac{V_{1,N}(X_k^1)}{\# \text{vertices}(X_k^1)} \rightarrow \infty.
\]
The theorem is most relevant when the orbifolds $X_k$ all have the same underlying space, such as, for example, the sphere $S^n$. In that case it does not appear to be known whether there exist triangulations whose 1–skeletons satisfy the conclusion of the theorem. If no such family of triangulations exists, it would imply that the volumes $\text{Vol} X_k$ are uniformly bounded, which would then imply that certain families of orbifolds have only finitely many elements.

**Question 1.** Does there exist a sequence $\{T_k\}_{k=1}^{\infty}$ of triangulations of $S^n$, $n \geq 3$, such that the 1–skeletons $T^1_k$ have bounded degree and satisfy

$$\frac{V_{1,n}(T^1_k)}{\#\text{vertices}(T^1_k)} \to \infty?$$

In particular, the conclusion would hold if $T^1_k$ were an expander family; Gromov and Guth explain this statement in [GG12] and essentially attribute it to Kolmogorov and Barzdin in a paper from the 1960’s, reprinted as [KB93].

**Question 2.** Does there exist a sequence $\{T_k\}_{k=1}^{\infty}$ of triangulations of $S^n$, $n \geq 3$, such that the 1–skeletons $T^1_k$ have bounded degree and form an expander family?

A discussion related to Question 2 can be found in Kalai’s chapter 19 of the Handbook of Discrete and Computational Geometry [Kal18]. This part of the chapter has been changing from one edition to the other. It was first conjectured that graphs of polytopes cannot have very good expansion properties, namely that the graph of every simple $d$–polytope with $n$ vertices can be separated into two parts, each having at least $n/3$ vertices, by removing $O(n^{1-1/(d-1)})$ vertices. Recall that a $d$–polytope is called *simple* if each vertex is contained in exactly $d$ facets, so the dual of a simple polytope is simplicial. However, the conjecture was disproved by Loiskekoski and Ziegler [LZ17]. Supporting an earlier claim by Thurston, whose proof seems to be lost, they show that there exist simple 4–dimensional polytopes with $n$ vertices such that all separators of the graph have size at least $\Omega(n/(\log n)^{3/2})$. We note in passing that the dual triangulations of these polytopes do not have bounded degree. It is still an open problem if there are examples of simple $d$-polytopes with $n \to \infty$ vertices whose graphs are expanders.

Our proof of Theorem 1 is based on two other theorems which may have independent interest. The first of them finds a triangulation of a closed arithmetic hyperbolic orbifold, with bounded degree and with not too many simplices compared to the hyperbolic volume. We define a *good triangulation* of an orbifold to be any triangulation of the underlying space such that all simplices are geodesic and for every dimension $\ell$, the $\ell$–stratum of the singular set of the orbifold is contained in the $\ell$–skeleton of the triangulation. The *vertex degree* of a triangulation is the maximum number of 1–dimensional edges incident to a vertex.

**Theorem 2.** For any $\delta > 0$ and any dimension $n \geq 2$, there is a constant $V_0 = V_0(\delta, n)$ such that any closed arithmetic hyperbolic $n$–orbifold of volume $\text{Vol}(\mathcal{O}) \geq V_0$ has a good triangulation with at most $\text{Vol}(\mathcal{O})^{1+\delta}$ simplices and vertex degree bounded above by a constant $D = D(n)$.

The proof of the theorem uses some deep results about volumes of arithmetic orbifolds and their relation to Lehmer’s problem in number theory, to bound the injectivity radius. A good triangulation is then obtained as a barycentric subdivision of a certain equivariant Voronoi complex in the hyperbolic $n$–space.
The other theorem is an inequality about the triangulated orbifolds, relating the Cheeger constant and the hyperbolic volume to $V_{1,N}$, the infimal $N$–volume of the 1–neighborhood of an embedding of the 1–skeleton in $\mathbb{R}^N$ with combinatorial thickness 1. It is based on Theorem 3.2 from [GG12]. We define a closed piecewise hyperbolic pseudomanifold to be a finite simplicial complex in which the top-dimensional simplices form a fundamental cycle under mod 2 coefficients, and each simplex is isometric to a geodesic hyperbolic simplex. In particular, a closed hyperbolic orbifold endowed with a good triangulation is a piecewise hyperbolic pseudomanifold.

**Theorem 3.** Let $X$ be a closed piecewise hyperbolic pseudomanifold of dimension $n \geq 3$, triangulated with vertex degree at most $D$. Then for all $N \geq 3$, we have

$$V_{1,N}(X^1) \geq \text{const}(n, N, D) \cdot \left( \frac{h(X)}{h(X) + 1} \cdot \text{Vol } X \right)^{\frac{N-1}{N}},$$

where the constant is positive and $X^1$ denotes the 1–skeleton of $X$.

Assuming these two results, it is straightforward to finish the proof of Theorem 1.

**Proof of Theorem 1.** Given $N$, we choose $\delta < \frac{1}{N-1}$ and apply Theorem 2 to triangulate each sufficiently large $X_k$ with vertex degree at most $D$ and with

$$\#\text{simplices}(X_k) \leq (\text{Vol } X_k)^{1+\delta}.$$

We have

$$\#\text{vertices}(X_k) \leq (n+1) \cdot \#\text{simplices}(X_k),$$

and thus

$$1 \geq \frac{1}{n+1} \cdot \#\text{vertices}(X_k) \cdot (\text{Vol } X_k)^{-(1+\delta)}.$$

The result is a closed piecewise hyperbolic pseudomanifold, so we may apply Theorem 2 to get

$$V_{1,N}(X_k^1) \geq \text{const}(n, N, D) \cdot \left( \frac{h(X_k)}{h(X_k) + 1} \cdot \text{Vol } X_k \right)^{\frac{N-1}{N}} \geq \text{const}(n, N, D) \cdot \left( \frac{1}{C(C+1)} \right)^{\frac{N-1}{N}} \cdot \frac{1}{n+1} \cdot \#\text{vertices}(X_k) \cdot (\text{Vol } X_k)^{\frac{N}{N-2}-(1+\delta)},$$

and thus

$$\frac{V_{1,N}(X_k^1)}{\#\text{vertices}(X_k)} \to \infty.$$

In Section 2 we prove Theorem 2 and in Section 3 we prove Theorem 3.

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2. Triangulating arithmetic hyperbolic orbifolds

In this section we prove Theorem 2. A similar result for hyperbolic 3–orbifolds was presented in [Bel17, Section 5]. The three-dimensional case allows several simplifications; moreover, the construction of a triangulation as a barycentric subdivision of a Voronoi complex presented in Lemma 4 below is new.

Arithmeticity of the orbifolds is essential for Theorem 2. We begin with recalling the definition of arithmetic subgroups. Let $H$ be a linear semisimple Lie group defined over a number field $k$ such that $G(k \otimes \mathbb{Q})$ is isogenous to $H \times K$, where $K$ is a compact Lie group. Consider a natural projection $\phi: G(k \otimes \mathbb{Q}) \rightarrow H$. The image of the group of $k$–integral points $\phi(G(O_k))$ and all subgroups of $H$ which are commensurable with it are called arithmetic subgroups of $H$ defined over $k$. Arithmetic subgroups are lattices, i.e., they are discrete and have finite colume in $H$. Their quotient spaces are called arithmetic orbifolds. In our case, $H = \text{PO}(n,1)$ is the group of isometries of the hyperbolic space $\mathbb{H}^n$ and the quotient orbifolds are hyperbolic $n$–orbifolds. We refer to [Mor15] for a comprehensive introduction to the theory of arithmetic subgroups.

Let $\mathcal{O} = \mathbb{H}^n/\Gamma$ be a closed hyperbolic orbifold with singular set $\Sigma$, and let $\pi: \mathbb{H}^n \rightarrow \mathcal{O}$ be the covering map. The elements of the group $\Gamma$ fall into two types: elliptic are those which have fixed points in $\mathbb{H}^n$ and hyperbolic are those which act freely. For a hyperbolic isometry $\gamma \in \Gamma$ its displacement at $x \in \mathbb{H}^n$ is defined by $\ell(\gamma, x) = \text{dist}(x, \gamma x)$ and the displacement of $\gamma$ (also called its translation length) is

$$\ell(\gamma) = \inf_{x \in \mathbb{H}^n} \ell(\gamma, x).$$

It is equal to the displacement of $\gamma$ at the points of its axis. We will define the orbifold injectivity radius by $r_{\text{inj}}(\mathcal{O}) = \inf \{ \frac{1}{2} \ell(\gamma) \}$, where the infimum is taken over all hyperbolic elements $\gamma \in \Gamma$. It is equal to half of the smallest length of a closed geodesic in $\mathcal{O}$. When $\mathcal{O}$ is a manifold, this definition is equivalent to the usual definition of the injectivity radius as the supremum of $r$ such that any point $p \in \mathcal{O}$ admits an embedded ball $B(p, r) \subset \mathcal{O}$. This is not the case in general; the points in the singular set only admit embedded folded balls (see [Sam13] for the definition of folded balls).

We will first assume that $r_{\text{inj}}(\mathcal{O}) \geq r > 0$ and that any finite subgroup $F < \Gamma$ has order $|F| \leq q$. A similar problem was considered before by Gelander and Samet (see [BGLS10, Section 2] and [Sam13]). The difference in our case is that we require an explicit control over the constants and that we want to construct a good triangulation of $\mathcal{O}$, not just a simplicial complex homotopy equivalent to it.

By the Margulis lemma there exist constants $\mu_n > 0$ and $m_n \in \mathbb{N}$ depending only on the dimension $n$, such that any subgroup of $\Gamma$ generated by the elements whose displacements at some point $x$ are bounded above by $\mu_n$ contains a normal nilpotent subgroup of index at most $m_n$. We refer to [BGSS5] Theorem 8.3 for the general statement and the proof of the lemma. We will use this result to obtain certain constraints on the position of the singular set in $\mathcal{O}$.

**Lemma 4.** Let $\mathcal{O} = \mathbb{H}^n/\Gamma$ be a closed hyperbolic orbifold, and let $\varepsilon = \min\{ \frac{\mu_n}{2}, \frac{1}{2m_n}, \}$, where $\mu_n$ and $m_n$ are dimensional constants arising from the Margulis lemma, and $r \leq r_{\text{inj}}(\mathcal{O})$. Then there is a good triangulation $T$ of $\mathcal{O}$ such that the vertex degree is bounded by $D(n)$ and the number of simplices is bounded by $C(n)\frac{q\text{Vol}(\mathcal{O})}{v_\varepsilon}$, where $V_\varepsilon = \varepsilon^{d+1}$.
where $C(n)$ and $D(n)$ are dimensional constants, $q$ is the maximum size of a finite subgroup $F < \Gamma$, and $v_\varepsilon$ denotes the volume of a ball of radius $\varepsilon$ in $\mathbb{H}^n$.

Proof. Let $S$ be any maximal $2\varepsilon$-separated set of points in $O$ that are not in the singular set, let $\overline{S}$ be the set of lifts of those points in $\mathbb{H}^n$, and let $P$ be the Voronoi decomposition of $\mathbb{H}^n$ corresponding to $\overline{S}$. It is a cell decomposition with one top-dimensional cell for each point of $\overline{S}$, and this top-dimensional cell is equal to the convex hyperbolic polytope consisting of all points of $\mathbb{H}^n$ that are closer to our selected point than to any other point of $\overline{S}$.

We define a barycentric subdivision of $P$ as follows. For any convex hyperbolic polytope, there is a unique point that minimizes the sum of squared distances to the vertices of the polytope; this is because squared distance to a point is a strictly convex function on $\mathbb{H}^n$ [BO69 Theorem 4.1(2)]. We refer to this point as the barycenter of the polytope. The barycenter is in the relative interior of the polytope, because for every point, the negative gradient of the sum of squared distances to the vertices is a sum of vectors pointing toward the vertices. Thus, we can form a triangulation $\mathcal{T}$ of $\mathbb{H}^n$, in which the vertices are the barycenters of all the faces of $P$ of all dimensions, and the simplices (all equal to the convex hulls of their vertices) correspond to chains of faces of $P$, under the partial ordering by inclusion of closures. Because $P$ is $\Gamma$–invariant, so is $\mathcal{T}$, and so we can set $T$ to be the triangulation of $O$ corresponding to $\mathcal{T}$.

First we check that $T$ is a good triangulation, that is, that for every dimension $\ell$, the $\ell$–stratum of the singular set of $O$ is contained in the $\ell$–skeleton of $T$. Let $x \in O$ be any point, and let $d$ be the least dimension of any simplex of $T$ containing $x$. Consider the stabilizer in $\Gamma$ of any lift $\overline{x}$ of $x$. Any $g \in \Gamma$ that fixes $\overline{x}$ must send the whole $d$–simplex containing $\overline{x}$ to itself. But the $d + 1$ vertices of this simplex all come from different-dimensional faces of $P$, so $g$ cannot permute them in any way other than by the identity. Thus the whole $d$–simplex is in the fixed-point set of the stabilizer of $\overline{x}$, and so if $x$ is in the $\ell$–stratum, then $\ell \geq d$.

Next we check that there is a bound on the vertex degree that depends only on the dimension $n$. For $i = 0, 1, \ldots, n$, let $P_i$ be the set of vertices of $T$ that are the images in $O$ of barycenters of $i$–dimensional faces of $P$. First, for each vertex $v \in P_n$, let us bound the number of neighbors of $v$ in $P_{n-1}$. This is equivalent to counting top-dimensional cells in $P$ that neighbor the cell of a lift of $v$ and have distinct projections to $O$. Let $x$ be the point of $S$ that corresponds to $v$, and let $y_1, \ldots, y_k$ be the points of $S$ such that the cells of $y_1, \ldots, y_k$ share an $(n - 1)$–dimensional face with the cell of $x$. Then each $y_i$ is within $4\varepsilon$ of $x$, and the $\varepsilon$–balls around $x, y_1, \ldots, y_k$ are all disjoint. We can choose lifts $\overline{x}, \overline{y}_1, \ldots, \overline{y}_k$ such that the same is true for the lifts. The number of disjoint $\varepsilon$–balls that can fit within $4\varepsilon$ of a given point in $\mathbb{H}^n$ is monotonic in $\varepsilon$, so because we have assumed $\varepsilon \leq \frac{4\varepsilon}{n}$, we have a dimensional upper bound $d_i(n)$ on $k$, the number of such $P_{n-1}$–neighbors of each $v \in P_n$.

Then, we can use this bound to bound the total number of neighbors of each vertex $v \in P_n$. Consider the cell in $P$ of a lift $\overline{x}$ of $v$. The point $s$ in $S$ corresponding to this cell is not in the singular set of $O$, and we claim that this implies that the interior of this cell maps injectively to $O$. Suppose to the contrary that some nontrivial element $g$ of $\Gamma$ takes this cell to itself. Then it fixes the barycenter $\overline{x}$ of the cell but must move the lift $\overline{s}$ of $s$ because $s$ is not in the singular set, but this means that $\overline{x}$ is equidistant between $\overline{x}$ and $g\overline{x}$, contradicting the definition of the
Voronoi decomposition because we know that $\overline{v}$ is in the interior of the cell. Thus every top-dimensional cell in $P$ maps injectively to $O$.

This implies that each $(n - 1)$-dimensional face in $P$ has the same image in $O$ as at most one other face. Thus, the total number of $(n - 1)$-dimensional faces of the cell of $\overline{v}$ is at most $2d_1(n)$. Every subset of $(n - 1)$-dimensional faces intersects in at most one arbitrary-dimensional face of the cell of $\overline{v}$, so the total number of faces of the cell of $\overline{v}$ is at most $2d(n)$, and thus the total degree of $v$ is at most $2d(n)$.

Similarly, if instead we let $v$ be a vertex in any $P_t$, we can bound the number of vertices in $P_n$ that are neighbors of $v$. Equivalently, if $\overline{v}$ is a lift of $v$, then we count the number of top-dimensional cells in $P$ that have $\overline{v}$ as a boundary point and have distinct projections to $O$. Let $y_1, \ldots, y_k$ be the points of $S$ corresponding to those cells. Then some lifts $\overline{y}_1, \ldots, \overline{y}_k$ are the closest points in $\overline{S}$ to the point $\overline{v}$. They are within $2\varepsilon$ of $\overline{v}$, and the $\varepsilon$-balls around $\overline{y}_1, \ldots, \overline{y}_k$ are all disjoint. Thus, again taking the maximum number of balls that fit when $\varepsilon = \frac{\mu_n}{2}$, there is a dimensional upper bound $d_2(n)$ on $k$, the number of $P_n$-neighbors of each $v \in P_0$.

Putting these two bounds together, we can bound the vertex degree of $T$. Given any vertex $v \in P_t$, if $u$ is any neighbor of $v$, then $u$ and $v$ have a common neighbor $w \in P_n$. Thus, the total number of neighbors of $v$ is at most $d_2(n) \cdot 2d_1(n)$, and so we set $D(n) = d_2(n) \cdot 2d_1(n)$.

Finally, we prove the bound on the number of top-dimensional simplices in $T$. Each simplex has one vertex in $P_n$, and each vertex in $P_n$ is in at most $\binom{D(n)}{n}$ simplices, so the total number of simplices is at most $\binom{D(n)}{n}$ times the number of points in our original $2\varepsilon$-separated set $S$, and therefore it suffices to show that

$$|S| \leq \frac{q \text{Vol}(O)}{v_\varepsilon}.$$  

To show this, we claim that every $\varepsilon$-ball in $H^n$ maps to $O$ with multiplicity at most $q$ at each point. The constants $\mu_n$ and $m_n$ in the Margulis lemma have the following property. For any $x \in H^n$ and any $t \in \mathbb{R}$, let $\Gamma_t(x)$ denote the subgroup of $\Gamma$ generated by the elements that move $x$ by distance less than $t$. Then if $t \leq \mu_n$ and if $\Gamma_t(x)$ is infinite, there is an element in $\Gamma$ of infinite order that moves $x$ by distance less than $2m_n t$ [Sam13 Lemma 2.3]. We have chosen $\varepsilon$ such that $2\varepsilon \leq \mu_n$ and $2m_n(2\varepsilon) \leq r \leq \text{inj}(O)$. By definition, every element in $\Gamma$ of displacement less than $\text{inj}(O)$ has a fixed point and therefore has finite order. Thus, $\Gamma_{2\varepsilon}(x)$ must be finite for all $x \in H^n$. Let $x_1, \ldots, x_k$ be points in some $\varepsilon$-ball in $H^n$ that all map to the same point of $O$. Then they are all in the orbit of $x_1$ under $\Gamma_{2\varepsilon}(x_1)$, and $\Gamma_{2\varepsilon}(x_1)$ has order at most $q$ (we have assumed this for every finite subgroup of $\Gamma$), so $k \leq q$ and the multiplicity of the map is at most $q$ on the $\varepsilon$-ball.

Thus, the $\varepsilon$-balls around the points of $S$ are disjoint in $O$ and each has volume at least $\frac{v_\varepsilon}{q}$, where $v_\varepsilon$ denotes the volume of a ball of radius $\varepsilon$ in $H^n$. In total, the volume is at most $\text{Vol}(O)$, so we have $|S| \leq \frac{q \text{Vol}(O)}{v_\varepsilon}$, and thus

$$\#\text{simplices}(T) \leq \binom{D(n)}{n} \cdot \frac{q \text{Vol}(O)}{v_\varepsilon} = C(n) \cdot \frac{q \text{Vol}(O)}{v_\varepsilon}.
$$

We now bring in the arithmetic information for estimating the number of simplices in terms of volume.
Given an integral monic polynomial $P(x)$ of degree $d$, its **Mahler measure** is defined by

$$M(P) = \prod_{i=1}^{d} \max(1, |\theta_i|),$$

where $\theta_1, \ldots, \theta_d$ are the roots of $P(x)$.

Let $\gamma \in \Gamma$ be a hyperbolic transformation. By [Gre62, Proposition 1(1,4)], the eigenvalues of $\gamma$ considered as an element of $O(n, 1)$ are $e^{\pm \ell(\gamma)}$ and $n - 1$ eigenvalues whose absolute value is 1. We would like to relate $e^{\ell(\gamma)}$ to the Mahler measure of a certain polynomial naturally associated to $\gamma$. To this end we can adapt the argument of [Gel04, Section 10]. Let $H^0$ be the identity component of the group $H = \text{Isom}(\mathbb{H}^n)$. It is center-free and connected so we can identify it with its adjoint group $\text{Ad}(H^0) \leq \text{GL}(g)$, where $g$ denotes the Lie algebra of $H$. We have $\Gamma' = \Gamma \cap H^0$, a cocompact arithmetic lattice, and $\gamma^2 \in \Gamma'$. Since $\Gamma'$ is arithmetic, there is a compact extension $H^\circ \times K$ of $H^0$ and a $\mathbb{Q}$-rational structure on the Lie algebra $g \times \mathfrak{k}$ of $H^\circ \times K$, such that $\Gamma$ is the projection to $H^0$ of a lattice $\tilde{\Gamma}$, which is contained in $(H^\circ \times K)_\mathbb{Q}$ and commensurable to the group of integral points $(H^\circ \times K)_\mathbb{Z}$ with respect to some $\mathbb{Q}$-base of $(g \times \mathfrak{k})_\mathbb{Q}$. By changing this $\mathbb{Q}$-base we can assume that $\tilde{\Gamma}$ is contained in $(H^\circ \times K)_\mathbb{Z}$. This means that the characteristic polynomial $P_{\tilde{\gamma}}$ of any $\tilde{\gamma} \in \tilde{\Gamma}$ is a monic integral polynomial of degree $(n + 1)\deg(k)$, where $k$ is the field of definition of the arithmetic group. Since $K$ is compact, any eigenvalue of $\tilde{\gamma}$ with absolute value different from 1 is also an eigenvalue of its projection in $H^0$. Therefore,

$$e^{\ell(\gamma^2)} = M(P_{\gamma^2});$$

(1)

This implies that $r_{\text{min}}(\mathcal{O}) \geq \min\{\frac{1}{2} \log M(P_{\gamma^2})\}$, where the minimum is taken over all $\tilde{\gamma} \in \tilde{\Gamma}$ which project to hyperbolic elements in $\Gamma'$. Moreover, our argument shows that the degrees of the irreducible integral monic polynomials whose Mahler measures appear in this bound satisfy

$$d \leq (n + 1)\deg(k);$$

(2)

Let us mention in passing that more precise versions of inequalities (1) and (2) for arithmetic subgroups of the simplest type were obtained in [ERT16].

Now recall that the celebrated Lehmer’s problem says that the Mahler measures of non-cyclotomic polynomials are expected to be uniformly bounded away from 1. A special case of this conjecture also known as the Margulis conjecture implies a uniform lower bound for the lengths of closed geodesics of arithmetic locally symmetric $n$-dimensional manifolds (see [Gel04, Section 10]). These conjectures have attracted a lot of interest but still remain wide open. Nevertheless, there are some quantitative number-theoretic results towards Lehmer’s problem which we can use for our estimates.

In [Dob79], Dobrowolski proved the following lower bound for the Mahler measure:

$$\log M(P) \geq c_1 \left( \frac{\log \log d}{\log d} \right)^3,$$

(3)
where $d$ is the degree of the polynomial $P$ and $c_1 > 0$ is an explicit constant.

We can relate the degree $d$ to the volume by using an important inequality relating the volume of a closed arithmetic orbifold and the degree of its field of definition:

\[(4) \quad \deg(k) \leq c_2 \log \text{Vol}(O) + c_3.\]

For hyperbolic orbifolds of dimension $n \geq 4$ this inequality follows from [Bel07, Section 3.3] and Minkowski’s bound for discriminant. In dimensions 2 and 3 this inequality is a result of Chinburg and Friedman [CF86], and in the form stated here it can be found in [BGLST10, Section 3].

For sufficiently large $x$ the function $\frac{\log x}{x}$ is monotonically decreasing, hence for sufficiently large volume we obtain

\[(5) \quad r_{\text{inj}}(O) \geq \frac{c_1}{4} \left( \frac{\log \log \log \text{Vol}(O)^c}{\log \text{Vol}(O)^c} \right)^3.\]

We note that this is a very slowly decreasing function.

Next we need to bound the order $q$ of finite subgroups $F < \Gamma$ in terms of volume. This can be done using the Margulis lemma once again, this time applied to the discrete subgroups of $O(n)$. The details for arithmetic subgroups of the simplest type can be found in [ABSW08, Lemma 4.4 and Corollary 4.5]. A similar argument applies in general: Consider a $k$–embedding of $\Gamma$ into $\text{GL}(m, k)$ with $m = n + 1$ if $n$ is even, $m = 2(n + 1)$ if $n$ is odd and $\neq 7$ (cf. [Mor15, Proposition 6.4.8]), and $m = 24$ if $n = 7$. The last embedding comes from the fact that an adjoint simple group of type $D_4$ over $k$ is a connected component of the automorphism group of a trialetarian algebra [KMRT98, Chapter X]. Starting from this place we can repeat the proof of the lemma and the corollary cited above. The resulting inequality is

\[(6) \quad q \leq c_4 \deg(k)^{c_5},\]

with the constants $c_4, c_5 > 0$ depending only on $n$.

Together with (4) it implies

\[(7) \quad q \leq c_6 (\log \text{Vol}(O))^{c_5}.\]

It remains to apply inequalities (5) and (7) for estimating the number of simplices in a good triangulation provided by Lemma 4. For sufficiently large volume, we have

\[r_{\text{inj}}(O) \geq \frac{c_1}{4} \left( \frac{\log \log \log \text{Vol}(O)^c}{\log \text{Vol}(O)^c} \right)^3 \geq \frac{c_1}{4} \left( \frac{1}{\log \text{Vol}(O)^c} \right)^3 \geq \frac{c_1}{4} \left( \frac{1}{\log \text{Vol}(O)} \right)^3,\]

so for $\varepsilon = \min \left\{ \frac{\mu_n}{2}, \frac{r}{2m_n} \right\}$ we have

\[v_\varepsilon \geq C(n) \cdot \varepsilon^n \geq C(n) \cdot \left( \frac{1}{\log \text{Vol}(O)} \right)^{3n}.\]
Thus for sufficiently large volume we obtain
\[
\# \text{simplices}(T) \leq C(n) \cdot q \cdot \text{Vol}(O) \cdot \frac{1}{v_\varepsilon} \leq C(n) \cdot (\log \text{Vol}(O))^{c_5} \cdot \text{Vol}(O) \cdot (\log \text{Vol}(O))^{3n} \leq \text{Vol}(O)^{1+\delta}.
\]
This finishes the proof of the theorem. \qed

3. Slicing piecewise hyperbolic manifolds

In this section we prove Theorem 3 based on the proof of Theorem 3.2 of [GG12]. For convenience we restate it below.

**Theorem 3.** Let \(X\) be a closed piecewise hyperbolic pseudomanifold of dimension \(n \geq 3\), triangulated with vertex degree at most \(D\). Then for all \(N \geq 3\), we have
\[
V_{1,N}(X^1) \geq \text{const}(n, N, D) \cdot \left( \frac{h(X)}{h(X) + 1} \cdot \text{Vol} X \right)^{\frac{n}{N-1}},
\]
where the constant is positive and \(X^1\) denotes the 1–skeleton of \(X\).

**Proof.** Let \(i: X^1 \hookrightarrow \mathbb{R}^N\) be an embedding of the 1–skeleton into \(\mathbb{R}^N\). Let \(N_1(X^1)\) denote the 1–neighborhood of the image, and let \(V_1(X^1)\) denote its volume. The Falconer slicing inequality (from [Fal80], recalled in [GG12]) guarantees that we can rotate the coordinates of \(\mathbb{R}^N\) to get the \(x_N\) coordinate pointing in a good direction so that for every \(t \in \mathbb{R}\), the \((n-1)\)–dimensional volume of the slice \(N_1(X^1) \cap \{x_N = t\}\) is at most \(\text{const}(N) \cdot V_1(X^1)^{\frac{n-1}{n}}\).

We view \(\mathbb{R}^N\) as broken into slabs,
\[
\text{Slab}(j) = \{j \leq x_N \leq j + 1\}.
\]
For each Slab\((j)\), we let \(S_j\) be the subcomplex of \(X\) consisting of all simplices that have a 1–dimensional edge that intersects Slab\((j)\). We claim that the number of top-dimensional simplices in \(S_j\) is at most \(\text{const}(n, N, D) \cdot V_1(X^1)^{\frac{n-1}{n}}\). To show this, suppose that there are \(M\) top-dimensional simplices in \(S_j\). Select one edge of each of these simplices that intersects Slab\((j)\). Because each edge is in at most \(\binom{D-1}{n-1} = \text{const}(n, D)^{-1}\) top-dimensional simplices, after removing duplicates we have at least \(\text{const}(n, D)^{-1} \cdot M\) edges through Slab\((j)\). Because each edge is incident to at most \(2D - 2\) other edges, we may greedily choose a subset of disjoint edges containing at least \((2D - 1)^{-1} \cdot \text{const}(n, D)^{-1} \cdot M = \text{const}(n, D) \cdot M\) of the original edges. On each edge in this matching, we select a point in Slab\((j)\); the 1–balls around these points are disjoint and are contained in the union of Slab\((j - 1)\), Slab\((j)\), and Slab\((j + 1)\). Thus, using the Falconer slicing inequality, we have
\[
\text{Vol(balls)} \leq \text{const}(N) \cdot V_1(X^1)^{\frac{n-1}{n}} \cdot 3,
\]
and thus
\[
M \leq \text{const}(n, N, D) \cdot V_1(X^1)^{\frac{n-1}{n}}.
\]
Next, we extend the embedding \(i\) to a map
\[
i: X \to \mathbb{R}^N
\]
that is smooth on each simplex—it doesn’t matter whether it is an embedding—such that the image of a given simplex intersects Slab\((j)\) only if it is in \(S_j\). We also
assume that every integer $j$ is a regular value of the restriction of $x_N \circ i$ to every open simplex; by Sard’s theorem this can be achieved by slightly perturbing the slab boundaries for every $j$.

The remainder of the proof is very much like the proof of Theorem 3.2 in [GG12]. We let $X_j$ be the preimage $i^{-1} \text{Slab}(j)$ in $X$, and view it as a chain in homology with coefficients in $\mathbb{Z}_2$, so that

$$[X] = \left[ \sum_j X_j \right].$$

We let $Z_j$ be the preimage $i^{-1}\{x_N = j\}$ in $X$, so that

$$\partial X_j = Z_j + Z_{j+1}.$$ We homotope the identity map on $X$ to a map that sends each $Z_j$ to the $(n-1)$–skeleton of $X$, with the property that the image of each $X_j$ remains in $S_j$. We can find this homotopy by choosing in each top-dimensional simplex a small ball not in any $Z_j$, and stretching that ball to cover the simplex so that the rest of the simplex maps to the boundary of the simplex.

Let $X'_j$ be the image of each $X_j$ under this homotopy. Taking the degree mod 2 of $X'_j$ with respect to each top-dimensional simplex, we can replace $X'_j$ by a simplicial chain $\overline{X}_j$, so that the fundamental class $[X]$ is the sum

$$[X] = \left[ \sum_j \overline{X}_j \right].$$

Similarly, we define $Z'_j$ and $\overline{Z}_j$ so that

$$\partial \overline{X}_j = \overline{Z}_j + \overline{Z}_{j+1}.$$ Because each $S_j$ has at most $\text{const}(n, N, D) \cdot V_1(X^1)^{N-1}$ top-dimensional simplices, each $\overline{X}_j$ and each $\overline{Z}_j$ has at most $\text{const}(n, N, D) \cdot V_1(X^1)^{N-1}$ simplices also.

Notice that each $\overline{Z}_j$ is null-homologous because it is the boundary of $\sum_{i<j} \overline{X}_j$. Thus, the definition of the Cheeger constant $h(X)$ implies that for every $\overline{Z}_j$ we can find a chain $\overline{Y}_j$ with $\partial \overline{Y}_j = \overline{Z}_j$ that satisfies

$$\text{Vol} \overline{Y}_j \leq h(X)^{-1} \cdot \text{Area} \overline{Z}_j.$$ In the sum

$$\sum_j (\overline{X}_j + \overline{Y}_j + \overline{Y}_{j+1}),$$

each $\overline{Y}_j$ is counted twice and cancels, so we can write the fundamental class $[X]$ as the sum of cycles

$$[X] = \sum_j [\overline{X}_j + \overline{Y}_j + \overline{Y}_{j+1}].$$

Thus, not every $\overline{X}_j + \overline{Y}_j + \overline{Y}_{j+1}$ can be null-homologous, and so at least one of them must be homologous to $[X]$ and must have total volume at least $\text{Vol} X$. Thus,
using the fact that geodesic hyperbolic simplices have volume bounded above, for this $j$ we have
\[
\operatorname{Vol} X \leq \operatorname{Vol} X_{j+1} + \operatorname{Vol} Y_{j} + \operatorname{Vol} Y_{j+1} \leq \\
\leq \operatorname{const}(n, N, D) \cdot V_1(X) \frac{1}{n} + 2 \cdot h(X)^{-1} \cdot \operatorname{const}(n, N, D) \cdot V_1(X) \frac{1}{n} \leq \\
\leq \operatorname{const}(n, N, D) \cdot (1 + h(X)^{-1}) \cdot V_1(X) \frac{1}{n}.
\]
\[\square\]

When a pseudomanifold $X$ is a closed hyperbolic $n$–manifold both our Theorem 3 and the Gromov–Guth Theorem 3.2 can be applied to it, and it would be interesting to compare the results. This leads to a question about the relation between combinatorial thickness and the retraction thickness from [GG12]. We recall the definitions:

**Definition 1.** A manifold $X$ embedded in $\mathbb{R}^N$ is said to have **retraction thickness** at least $T$ if the $T$–neighbourhood of $X$ retracts to $X$.

**Definition 2.** A pseudomanifold $X$ whose 1–skeleton is embedded in $\mathbb{R}^N$ with combinatorial thickness $T$ is said to have **thickness** at least $T$.

Given a subset $Y \subset \mathbb{R}^N$, denote by $V_T(Y)$ the $N$–dimensional volume of its $T$–neighbourhood. Now assume that a closed hyperbolic manifold of dimension $n \geq 3$ has an embedding $i: X \hookrightarrow \mathbb{R}^N$ with retraction thickness $T$. One can then try to construct a triangulation of the image $i(X)$ whose 1–skeleton has a combinatorial thickness $T$ (or at least $T - \varepsilon$ for an arbitrary small $\varepsilon > 0$). If there is such a triangulation, then we can apply the simplex straightening to its simplices and obtain a piecewise hyperbolic pseudomanifold isometric to $X$ such that $V_{T-\varepsilon}(X) \leq V_T(i(X))$. It would then allow us to deduce Theorem 3.2 from [GG12] from our Theorem 3. So far we were not able to verify all the steps of this argument. We leave this as an open problem.

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