A TROPICAL NULLSTELLENSATZ

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Abstract. We suggest a version of Nullstellensatz over the tropical semiring, the real numbers equipped with operations of maximum and summation.

1. Introduction

The tropical mathematics is a mathematics over the tropical semiring, the real numbers equipped with the operations of maximum and summation, corresponding to the classical operations of addition and multiplication, respectively (see in [17] the representation of the tropical semiring as a limit of the semiring of non-negative real numbers \((\mathbb{R}_+, +, \cdot)\)). Sometimes the tropical semiring is extended by \(-\infty\), the neutral element for the maximum operation, but we shall not need it. We write the tropical operations in quotes, i.e.,

\[
"a + b" = \max\{a, b\}, \quad "ab" = a + b.
\]

A rapid development of the tropical mathematics over the last years, especially, of the tropical algebraic geometry, has lead to spectacular applications in the classical algebraic geometry (see, for example, [2, 3, 4, 5, 8, 9, 10, 11, 13, 14, 15, 16, 17]). The tropical objects reveal unexpectedly much similarity with classical objects, which is based on the theory of large complex limits, logarithmic limits, non-Archimedean valuations, toric geometry etc. Here we suggest a tropical analogue of Nullstellensatz (in a bit different context this problem was stated in [1], Question 16).

Similarly to the classical case, we understand Nullstellensatz as a criterion for a polynomial to belong to the radical of an ideal. Let us give necessary definitions (cf. [3, 11, 13, 14, 15, 17]). A tropical polynomial in \(n\) variables is a function \(F : \mathbb{R}^n \to \mathbb{R}\) given by

\[
F(x) = \sum_{\omega \in \Omega} c_\omega x^\omega = \max_{\omega \in \Omega} \langle x, \omega \rangle + c_\omega,
\]

where \(\Omega\) is a non-empty finite set of points in \(\mathbb{Z}^n\) with non-negative coordinates, \(\langle *, * \rangle\) denotes the scalar product, and \(c_\omega \in \mathbb{R}, \omega \in \Omega\). This is a convex piece-wise linear function. Its Newton polytope \(\Delta(F)\) is the convex hull of the set \(\Omega\).

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The radical $F$ generated by $G$ (1.2) "For $F$, tropical polynomials belong to the locus of a tropical polynomial. whose complement consists of open convex polyhedra. It plays the role of the zero locus of the graph of $F$. Thus, the function $\partial F/\partial x_j |_D > 0$ as far as $\partial F_i/\partial x_j |_D > 0$.

Proof. 1. Necessity. The function $mF |_D$ is linear. Hence it must coincide with one of the terms $(h_i + F_i) |_D$ in the expression max$(h_i + F_i) |_D$, since otherwise the graph of the latter function would have a break inside $D$. Next, if $mF |_D = (h_i + F_i) |_D$, then both $h_i |_D$ and $F_i |_D$ must be linear in view of $A(h_i + F_i) = A(h_i) \cup A(F_i)$. Thus, $D \cap A(F_i) = \emptyset$. Observing that

$$m \frac{\partial F}{\partial x_j} |_D = \frac{\partial h_i}{\partial x_j} |_D + \frac{\partial F_1}{\partial x_j} |_D \geq \frac{\partial F_i}{\partial x_j} |_D,$$

we obtain (2.1).

2. Sufficiency. Distribute the connected components of $\mathbb{R}^n \setminus A(F)$ into disjoint subsets $\Pi_i$, $i \in J$, where $J \subset \{1, ..., k\}$, such that, for any $i \in J$ and $D \in \Pi_i$, we have $A(F_i) \cap D = \emptyset$ and relation (2.1).

Fix some $i \in J$. Condition (2.1) yields that there is $m_1$ such that, for any $m \geq m_1$, one has

$$m \frac{\partial F}{\partial x_j} |_D \geq \frac{\partial F_i}{\partial x_j} |_D, \quad D \in \Pi_i, \quad j = 1, ..., n,$$

and hence the gradients of the linear functions

(2.2) $L_{D,m} : \mathbb{R}^n \to \mathbb{R}, \quad L_{D,m}|_D = mF |_D - F_i |_D, \quad D \in \Pi_i, \quad m \geq m_1,$

1. The quotation marks mean that we supply this space by the tropical operations, maximum and summation.
have non-negative integral coordinates.

We claim that there is $m_2$ such that, for any $D \in \Pi_i$, in the complement of the closure of $D$, we have

$$mF > L_{D,m} + F_1$$

as far as $m \geq m_1$.

Indeed, write $F = L + \Phi$, $F_1 = L' + \Phi'$, where $L$, $L'$ are linear functions, and $\Phi$, $\Phi'$ are convex piece-wise linear functions, vanishing along $D$. Then $L_{D,m} = mL - L'$, and thus, $mF - L_{D,m} - F_1 = m\Phi - \Phi'$. Since $\Phi > 0$ and $|\partial \Phi / \partial x_j| \geq 1$, $j = 1, \ldots, n$, outside $D$, we obtain (2.3) when $m_2$ exceeds all the absolute values of the partial derivatives of $\Phi'$.

Define $h_i = \max_{D \in \Pi_i} L_{D,m}$. This is a tropical polynomial as $m \geq m_1$, and due to (2.2), (2.3) it satisfies

$$(h_i + F_1)|_D = mF|_D, \quad (h_i + F_1)|_{\mathbb{R}^n \setminus \mathcal{M}} < F|_{\mathbb{R}^n \setminus \mathcal{M}}, \quad D \in \Pi_i, \quad m \geq m_2.$$

That is $G = "F_m" = mF$ satisfies (1.2) for all sufficiently large $m$.

**Remark 2.2.** From the above proof, one can extract an explicit upper bound to the minimal value of $m$ such that "$F_m" \in I(F_1, \ldots, F_k)$.

**Example 2.3.** In the case $k = 1$, the criterion of Theorem 2.1 for $F \in \sqrt{I(F_1)}$ can be written as

- $A(F) \supset A(F_1)$, and
- for each $j = 1, \ldots, n$, $\partial F / \partial x_j(x) > 0$ as far as $\partial F_1 / \partial x_j(x) > 0$, $x \not\in A(F)$.

We shall comment on the first condition. One can assign integer positive weights to the $(n - 1)$-cells of an amoeba so that it will satisfy an equilibrium condition (see [12], section 2.1). Taking $m$ greater than the maximal ratio of weights in $A(F)$ and in $A(F_1)$, we then subtract the weight of an $(n - 1)$-cell $D$ of $A(F_1)$ from the multiplied by $m$ weights of those $(n - 1)$-cells of $A(F)$, whose interior intersects with $D$. The equilibrium condition persists after the subtraction due to its linearity, and we again obtain a balanced complex supported at $A(F)$. By [12], Proposition 2.4, it defines a convex piece-wise linear function $h_1$, which provides the relation $mF = h_1 + F_1$.

3. Modifications

**3.1. A tropical Laurent polynomial Nullstellensatz.** A tropical Laurent polynomial in $n$ variables is a function given by $\mathbf{131}$, where $\Omega$ is any non-empty finite subset of $\mathbb{Z}^n$. Denote the space of tropical Laurent polynomials by "$L[x_1, \ldots, x_n]$". Correspondingly we define the tropical Laurent polynomial ideal $I_{\text{tla}}(F_1, \ldots, F_k) \subset "L[x_1, \ldots, x_n]"$, generated by Laurent polynomials $F_1, \ldots, F_k$, as the set of tropical Laurent polynomials representable in the form (1.2) with any non-empty finite $J \subset \{1, \ldots, k\}$, and $h_i \in "L[x_1, \ldots, x_n]"$, $i \in J$.

**Theorem 3.1.** Let $F, F_1, \ldots, F_k$, $k \geq 1$, be tropical Laurent polynomials in $n$ variables. Then $F \in \sqrt{I_{\text{tla}}(F_1, \ldots, F_k)}$ if and only if, for any connected component $D$ of $\mathbb{R}^n \setminus A(F)$, there is $1 \leq i \leq k$ such that $D \cap A(F_i) = \emptyset$.

**Proof.** The necessity is established in the same way as in the proof of Theorem 2.1 if one "multiplies" $F, F_1, \ldots, F_k$ by suitable tropical monomials, turning $F, F_1, \ldots, F_k$ into tropical polynomials, and making all the partial derivatives of $F$ positive. □
3.2. **Restricted ideals and restricted Nullstellensatz.** In the space \( \mathbb{R}[x_1, \ldots, x_n] \) introduce the restricted ideal, generated by tropical polynomials \( F_1, \ldots, F_k \), as

\[
I'(F_1, \ldots, F_k) = \{ G \in \mathbb{R}[x_1, \ldots, x_n] : \}
\]

\[
G = \sum_{i=1}^{k} h_i F_i = \max_{1 \leq i \leq k} (h_i + F_i), \quad h_1, \ldots, h_k \in \mathbb{R}[x_1, \ldots, x_n].
\]

Notice that \( I'(F_1, \ldots, F_k) \subset I(F_1, \ldots, F_k) \), but they may differ, for example, \( x \not\in I'(x, x+1) \).

**Theorem 3.2.** Let \( F, F_1, \ldots, F_k, \ k \geq 1 \), be tropical polynomials in \( n \) variables. Then \( F \in \sqrt{I'(F_1, \ldots, F_k)} \) if and only if the following conditions hold:

1. for any connected component \( D \) of \( \mathbb{R}^n \setminus \mathcal{A}(F) \), there is \( 1 \leq i \leq k \) such that \( D \cap \mathcal{A}(F_i) = \emptyset \) and \( \mathcal{H}(i) \) is fulfilled;
2. there is \( m_0 \) such that \( m \Delta(F) \) contains a translate of each Newton polytope \( \Delta(F_i), \ldots, \Delta(F_k) \) as far as \( m \geq m_0 \).

**Remark 3.3.** Notice that condition (ii) always holds when \( \dim \Delta(F) = n \).

**Proof.** 1. **Auxiliary statement.** Let \( G, H \in \mathbb{R}[x_1, \ldots, x_n] \). We claim that

- if \( G \geq H, \) then \( \Delta(G) \supset \Delta(H) \);
- if \( \Delta(G) \supset \Delta(H) \), then there is \( c \in \mathbb{R} \) such that \( G \geq H + c \).

Represent \( \Delta(G) \) as intersection of finitely many halfspaces. Pick one of these halfspaces and apply an integral-affine automorphism \( Q \) of \( \mathbb{R}^n \), taking the halfspace to \( x_n \geq 0 \). Since \( G \circ Q^{-1} \geq H \circ Q^{-1} \), the Newton polytope \( \Delta(H \circ Q^{-1}) = Q(\Delta(H)) \) cannot contain points with negative \( n \)-th coordinate. Indeed, otherwise we would have that, for \( x_1, \ldots, x_{n-1} = \text{const}, \ x_n \to -\infty \), the function \( G(Q^{-1}(x)) \) does not increase, whereas \( H(Q^{-1}(x)) \) tends to \( +\infty \). Running over all halfspaces forming \( \Delta(G) \), we conclude that \( \Delta(G) \supset \Delta(H) \).

For the second statement, write

\[
G(x) = \max_{\omega \in \Delta(G) \cap \mathbb{Z}^n} \langle x, \omega \rangle + a_\omega, \quad H(x) = \max_{\omega \in \Delta(H) \cap \mathbb{Z}^n} \langle x, \omega \rangle + b_\omega.
\]

Then one can take

\[
c = \min_{\omega \in \Delta(H) \cap \mathbb{Z}^n} (a_\omega - b_\omega).
\]

2. **Necessity.** We have to prove only (ii). Observing that \( \Delta(h_i + F_i) \) is the convex hull of few translates of \( \Delta(F_i) \), we derive (ii) from the above auxiliary statement.

3. **Sufficiency.** A translate of \( \Delta(F_i) \) is the Newton polytope of a tropical polynomial \( "h_i F_i" \), where \( h_i \) is a tropical monomial (i.e., a linear function). Then, according to the auxiliary statement, \( m F_i \geq (h_i + c_i) + F_i \) for sufficiently large \( m \) and certain constant \( c_i \). Then, using Theorem 2.1 we obtain

\[
mF = \max_{i \in J} (h_i + F_i) = \max \left\{ \max_{i \in J} (h_i + F_i), \max_{i \notin J} (h_i + c_i + F_i) \right\}.
\]

\( \square \)

3.3. **Nullstellensatz for convex piece-wise linear functions of finite type.** A function given by \( \{ \Omega \} \), where \( \Omega \) is a finite subset of \( \mathbb{R}^n \) we call a convex piece-wise linear function of finite type in \( n \) variables. Denote the space of such functions by \( "PL(x_1, \ldots, x_n)" \). As in the polynomial case, we define amoebas and finitely generated ideals \( I_{\text{fun}}(F_1, \ldots, F_k) \) in \( "PL(x_1, \ldots, x_n)" \), and obtain a corresponding Nullstellensatz:
Theorem 3.4. Let $F, F_1, \ldots, F_k \in "P L(x_1, \ldots, x_n)^{\nu}\)$. Then $F \in \sqrt{I^{\text{fun}}(F_1, \ldots, F_k)}$ if and only if, for any connected component $D$ of $\mathbb{R}^n \setminus A(F)$, there is $1 \leq i \leq k$ such that $D \cap A(F_i) = \emptyset$.

Proof coincides with that of Theorem 3.1.

3.4. Nullstellensatz for polynomials over an extended tropical semiring. In [1], the second author introduced an extension $\mathbb{T}$ of the tropical semiring $\mathbb{R}$, obtained by adding one more copy of $\mathbb{R}$, which we denote by $\mathbb{R}^\nu$ and its elements by $a^\nu \in \mathbb{R}^\nu$ as $a \in \mathbb{R}$, and equipped with the tropical operations

$$a + b^\nu = \begin{cases} \max\{a, b\}, & \text{if } a \neq b, \\ a^\nu, & \text{if } a = b, \end{cases} \quad a^\nu + b^\nu = (\max\{a, b\})^\nu,$$

$$a + b = \begin{cases} a, & \text{if } a > b, \\ b, & \text{if } a \leq b, \end{cases} \quad ab^\nu = (a + b)^\nu,$$

$$ab = a + b, \quad a^\nu b = (a + b)^\nu, \quad a^\nu b^\nu = (a + b)^\nu$$

for all $a, b \in \mathbb{R}$. There is a natural epimorphism of tropical semirings:

$$\pi : \mathbb{T} \to \mathbb{R}, \quad \pi(a) = a, \quad \pi(a^\nu) = a, \quad \text{for all } a \in \mathbb{R},$$

which induces epimorphisms $\pi_\ast$ of the tropical polynomial and tropical Laurent polynomial rings. The above variants of Nullstellensatz can easily be translated to the case of base semiring $\mathbb{T}$. We present here such a translation of Theorem 2.1.

For a polynomial $F = \sum_{\omega \in \Omega} c_\omega \omega^\nu \in "\mathbb{T}[x_1, \ldots, x_n]^{\nu}\)”, where $\Omega$ is a non-empty finite set of points with non-negative integral coordinates, we introduce an amoeba $A(F) := A(\pi_\ast F) \subseteq \mathbb{R}^n$. The restriction of $\pi_\ast F$ to a connected component $D$ of $\mathbb{R}^n \setminus A(F)$ is a linear function $\langle \omega, x \rangle + \pi(c_\omega)$ for some $\omega = \omega(D) \in \Omega$. So we divide the set of the connected components of $\mathbb{R}^n \setminus A(F)$ into two disjoint subsets $\Pi(F)$ and $\Pi^\nu(F)$, letting $D \in \Pi(F)$ or $D \in \Pi^\nu(F)$ according as $c_\omega(D) \in \mathbb{R}$ or $c_\omega(D) \in \mathbb{R}^\nu$.

Theorem 3.5. Let $F, F_1, \ldots, F_k \in "\mathbb{T}[x_1, \ldots, x_n]^{\nu}\) \), $k \geq 1$. Then $F \in \sqrt{I(F_1, \ldots, F_k)}$ if and only if

(i) for any $D \in \Pi(F)$, there is $1 \leq i \leq k$ such that $D$ is contained in some component $D_i \in \Pi(F_i)$, and, for each $j = 1, \ldots, n$, relation (2.4) holds true;

(ii) for any $D \in \Pi^\nu(F)$, there is $1 \leq i \leq k$ such that $D \cap A(F_i) = \emptyset$, and, for each $j = 1, \ldots, n$, relation (2.4) holds true.

Proof. The necessity part is immediate in view of the proof of Theorem 2.1 and the fact that "$\mathbb{R}^\nu \subseteq \mathbb{R}^\nu\)". To prove the sufficiency, we construct coefficients $h_i$ of the expansion (1.2) to be in "$\mathbb{R}[x_1, \ldots, x_n]^\nu \subseteq \mathbb{T}[x_1, \ldots, x_n]^\nu\)" by the following modification of the procedure from the proof of Theorem 2.1:

- if $D \in \Pi(F)$, and $1 \leq i \leq k$ satisfies condition (i), then we put $h_i|_D = mF|_D - F_i|_D$;
- if $D \in \Pi^\nu(F)$, and $1 \leq i \leq k$ satisfies condition (ii), then we put $h_i|_D = m \cdot \pi_\ast F_i|_D - \pi_\ast F|_D$.

Choosing a sufficiently large $m$, we complete the proof as that of Theorem 2.1.

Corollary 3.6. Let $F, F_1, \ldots, F_k \in "\mathbb{T}[x_1, \ldots, x_n]^{\nu}\) \), $k \geq 1$, and $I(F_1, \ldots, F_k) \supset "\mathbb{R}^\nu[x_1, \ldots, x_n]^{\nu}\)\). Then $F \in \sqrt{I(F_1, \ldots, F_k)}$ if and only if condition (i) of Theorem 3.5 holds true.
Indeed, the relation \( I(F_1, ..., F_k) \supset \mathbb{R}^n[x_1, ..., x_n] \) means that the set of generators \( \{F_1, ..., F_k\} \) contains a constant polynomial \( F_i = a^\nu, a \in \mathbb{R} \), and thus, condition (ii) always holds.

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