A FIRST SIGHT TOWARDS PRIMITIVELY GENERATED CONNECTED BRAIDED BIALGEBRAS

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Abstract. The main aim of this paper is to investigate the structure of primitively generated connected braided bialgebras $A$ with respect to the braided vector space $P$ consisting of their primitive elements. When the Nichols algebra of $P$ is obtained dividing out the tensor algebra $T(P)$ by the two-sided ideal generated by its primitive elements of degree at least two, we show that $A$ can be recovered as a sort of universal enveloping algebra of $P$. One of the main applications of our construction is the description, in terms of universal enveloping algebras, of connected braided bialgebras whose associated graded coalgebra is a quadratic algebra.

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1. Introduction

Let $K$ be a fixed field, let $H$ be a pointed Hopf algebra over $K$ (this means that all its simple subcoalgebras are one dimensional). Denote by $G$ the set of grouplike elements in $H$. It is well known that the graded coalgebra $gr(H)$, associated to the coradical filtration of $H$, is a Hopf algebra itself and can be described as a Radford-Majid bosonization by $KG$ of a suitable graded connected braided bialgebra $R$, called diagram of $H$, in the braided monoidal category of Yetter-Drinfeld modules over $KG$. This is the starting point of the so called lifting method for the classification of finite dimensional pointed Hopf algebras, introduced by N. Andruskiewitsch and H.J. Schneider (see e.g. [ASc1]). Accordingly to this method, first one has to describe $R$ by generators and relations, then to lift the informations obtained to $H$. It is worth noticing that in the finite dimensional case there is a conjecture asserting that, in characteristic zero, $R$ is always primitively generated [ASc3, Conjecture 2.7] (see also [ASc2, Conjecture 1.4]).

Therefore, in many cases, for proving certain properties of Hopf algebras, it is enough to do it in the connected case. The price that one has to pay is to work with Hopf algebras in a braided category, and not with ordinary Hopf algebras. Actually nowadays people recognize that it is more convenient to work with braided bialgebras, that were introduced in [Ta].

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Motivated by these observations, in this paper we will investigate the structure of primitively generated connected braided bialgebras. Let \( (A,c_A) \) be such a bialgebra. Note that \( (A,c_A) \) is in particular a braided vector space i.e. \( c_A : A \otimes A \to A \otimes A \) is a map, called braiding, fulfilling the quantum Yang-Baxter equation \( [3] \). Furthermore \( c_A \) induces a braiding \( c_P : P \otimes P \to P \otimes P \) on the space \( P \) of primitive elements in \( A \). The tensor algebra \( T(P) \) carries a braided bialgebra structure depending on \( c_P \), that will be denoted by \( T(P,c_P) \), and there is a unique surjective braided bialgebra map \( \varphi : T(P,c_P) \to A \) that lifts the inclusion \( P \subseteq A \). Let \( E(P,c_P) \) be the space of primitive elements in \( T(P,c_P) \) of degree at least two. Since \( \varphi \) preserves primitive elements, we can define the map \( b_P : E(P,c_P) \to P, b_P(z) = \varphi(z) \) and consider the quotient

\[
U(P,c_P) := \frac{T(P,c_P)}{(\text{Id} - b_P)[E(P,c_P)]}.
\]

Now \( \varphi \) quotients to a surjective braided bialgebra map \( \overline{\varphi} : U(P,c_P,b_P) \to A \). It is remarkable here that the canonical \( K \)-linear map \( i_V : P \to U(P,c_P,b_P) \) is injective as \( \varphi \circ i_V \) is the inclusion \( P \subseteq A \). It is also notable that \( E(P,c_P) \) may contain homogeneous elements of arbitrary degree.

By a famous result due to Heyneman and Radford (see \([Mo, \text{Theorem 5.3.1}]\)), \( \overline{\varphi} \) is an isomorphism if the space of primitive elements in \( U(P,c_P,b_P) \) identifies with \( P \) via \( i_V \) (the converse is trivial).

We prove that this property holds whenever \( P \) belongs to a large class \( \mathcal{S} \) of braided vector spaces. It stems from our construction that \( \mathcal{S} \) can be taken to be the class of braided vector spaces \( (V,c) \) such that the Nichols algebra \( B(V,c) \), i.e. the image of the canonical graded braided bialgebra homomorphism from the tensor algebra \( T(V,c) \) into the quantum shuffle algebra \( T^c(V,c) \), is obtained dividing out \( T(V,c) \) by the two-sided ideal generated by \( E(V,c) \). We point out that the class \( \mathcal{S} \) is so large to enclose both the class of all braided vector spaces of diagonal type whose Nichols algebra is a domain of finite Gelfand-Kirillov dimension and also the class of all two dimensional braided vector spaces of abelian group type whose symmetric algebra has dimension at most 31. We note that in either one of these classes there are braided vector spaces \( (V,c) \) whose Nichols algebra is not quadratic and whose braiding has minimal polynomial of degree greater than two.

Meaningful examples of elements in \( \mathcal{S} \) are given. For instance, any braided vector space with braiding of Hecke type with regular mark (e.g. the braiding is a symmetry and the characteristic of \( K \) is zero) is in \( \mathcal{S} \).

An application of our construction is that \( A \cong U(P,c_P,b_P) \) whenever \( A \) is a connected braided bialgebra such that \( \text{gr}(A) \) is a quadratic algebra with respect to its natural braided bialgebra structure.

We point out that the structure and properties of \( U(P,c_P,b_P) \) are encoded in the datum \( (P,c_P,b_P) \). Indeed this leads to the introduction of what will be called a braided Lie algebra \( (V,c,b) \) for any braided vector space \( (V,c) \) and of the related universal enveloping algebra \( U(V,c,b) \). When \( c \) is a symmetry, i.e. \( c^2 = \text{Id}_{V \otimes V} \), and the characteristic of \( K \) is zero, our enveloping algebra reduces to the one introduced in \([Gu]\). In this case, the crucial property, mentioned above, that the space of primitive elements in \( U(V,c,b) \) identifies with \( V \) via \( i_V \) was proved in \([Kh6, \text{Lemma 6.2}]\) and it was applied to obtain a Cartier-Kostant-Milnor-Moore type theorem for connected braided Hopf algebras with involutive braidings \([Kh6, \text{Theorem 6.1}]\). Other notions of Lie algebra and enveloping algebra, extending the ones in \([Gu]\) to the non symmetric case, appeared in the literature. Let us mention some of them.

- Lie algebras for braided vector spaces \( (V,c) \) where \( c \) is a braiding of Hecke type i.e. \((c + \text{Id}_{V \otimes V})(c - q \text{Id}_{V \otimes V}) = 0 \) for some \( q \in K \setminus \{0\} \) which is called the mark of \( c \) \([Wz, \text{Definition 7.1}]\). See \([AMS]\, \text{Theorem 5.5}]\) for a strong version of Cartier-Kostant-Milnor-Moore theorem in this setting.
- Lie algebras for braided vector spaces \( (V,c) \) where \( c \) is not of Hecke-type but constructed by means of braidings of Hecke type \([Gu2, \text{Definition 1}]\). There a suitable quadratic algebra is requiring to be Koszul.
• Lie algebras for objects in the braided monoidal category of Yetter-Drinfeld modules over a Hopf algebra with bijective antipode [Pa, Definition 4.1] (here called Pareigis-Lie algebras - see Definition [0.4]). In the present paper we also stress the relation between our notion of Lie algebra and the one by Pareigis, which involves partially defined n-ary bracket multiplications as well.

• Lie algebras defined by considering quantum operations (see [Kh1, Definition 2.2]) as primitive polynomials in the tensor algebra. When the underline braided vector space is an object in the category of Yetter-Drinfeld modules over some group algebra, to any Lie algebra of this kind, the author associates a universal enveloping algebra which is not connected (see [Kh3]). Note that the universal enveloping algebra we introduce in the present paper is connected as the scope here is to investigate the structure of primitively generated connected braided bialgebras over $K$.

Finally, many authors have tried to characterize quantized enveloping algebras $U_q(\mathfrak{g})$ of Drinfeld and Jimbo or other quantum groups as bialgebras generated by a finite-dimensional Lie algebra like object via some kind of enveloping algebra construction (see the introduction of [Ma]). One idea is to attempt to build this on $\mathfrak{g}$ itself but with some kind of deformed bracket obeying suitable new axioms. Alternatively one can consider some subspace of $U_q(\mathfrak{g})$ endowed with some kind of ‘quantum Lie bracket’ based on the quantum adjoint action. Contributions to these problems can be found e.g. in [Wol, LS, DGG, GM, GS] and in the references therein.

Now, since $U_q(\mathfrak{g})$ is not connected but just pointed, we have no hope to describe it directly as an enveloping algebra of our kind (see [Kh1, Section 2] for a different approach). Furthermore, it is remarkable that, by [Mas], Theorem 7.8], $U_q(\mathfrak{g})$ can be viewed as a cocycle deformation of $\text{gr}(U_q(\mathfrak{g}))$. By the foregoing, $\text{gr}(U_q(\mathfrak{g}))$ is the Radford-Majid bosonization of the diagram $Q$ of $U_q(\mathfrak{g})$ by $KG$, where $G$ denotes the set of grouplike elements in $U_q(\mathfrak{g})$. One could believe that $Q$ is an enveloping algebra of our kind as it is a primitively generated connected braided bialgebra. The point is that $Q$ is much more. It is a Nichols algebra whence defined by quotienting the tensor algebra $T(P(Q))$ by homogeneous relations. Thus no enveloping algebra is required to describe $Q$.

**Methodology.** We proceed as follows. Let $(V,c)$ be a braided vector space and denote by $E(V,c)$ the space of primitive elements in $T(V,c)$ of degree at least two. A bracket on $(V,c)$ is a $K$-linear map $b : E(V,c) \rightarrow V$ that commutes with the braiding of $T(V,c)$ in the sense that (36) is satisfied. If $b$ is a bracket on $(V,c)$, then the universal enveloping algebra of $(V,c)$ is defined to be

$$U(V,c,b) := \frac{T(V,c)}{((\text{Id} - b)[E(V,c)])}.$$

Thus $U(V,c,b)$ carries a unique braided bialgebra structure such that the canonical projection $\pi_U : T(V,c) \rightarrow U(V,c,b)$ is a braided bialgebra homomorphism (Theorem 3.9). We say that $(V,c,b)$ is a braided Lie algebra whenever $(V,c)$ is a braided vector space, $b : E(V,c) \rightarrow V$ is a bracket on $(V,c)$ and the canonical $K$-linear map $i_U : V \rightarrow U(V,c,b)$ is injective (Definition 5.6). Natural examples of braided Lie algebras arise as follows:

1) If $(V,c)$ is a braided vector space, then $(V,c,0)$ is a braided Lie algebra (Proposition 5.7). The corresponding universal enveloping algebra is the symmetric algebra $S(V,c)$.

2) If $(A,c_A)$ is a braided algebra, then $(A,c_A,b_A)$ is a braided Lie algebra, where $b_A$ acts on the $t$-th graded component of $E(A,c_A)$ as the restriction of the iterated multiplication of $A$ (Proposition 5.8).

3) If $(A,c_A)$ is a connected braided bialgebra and $P = P(A)$ is the space of primitive elements of $A$, then $P$ forms a braided Lie algebra $(P,c_P,b_P)$ with structures induced by $(A,c_A,b_A)$ (Theorem 5.10). $(P,c_P,b_P)$ will be called the infinitesimal braided Lie algebra of $A$.

Theorem 5.3 contains the main characterization of the above notion of braided Lie algebra. Let $(V,c) \in \mathcal{S}$ and let $b : E(V,c) \rightarrow V$ be a bracket. Then, the following assertions are equivalent:

- $(V,c,b)$ is a braided Lie algebra.
- $U(V,c,b)$ is of PBW type in the sense of Definition 5.13.
\begin{itemize}
  \item $i_U : V \to U(V,c,b)$ induces an isomorphism between $V$ and $P(U(V,c,b))$.
\end{itemize}

Therefore, if $(V,c,b)$ is a braided Lie algebra whose underlying braided vector space lies in $\mathcal{S}$, then $P(U(V,c,b))$ identifies with $V$ via $i_U$. The main consequence of this property, which really justifies our construction, is Theorem $\ref{thm:main}$ concerning primitively generated connected braided bialgebras $(A,c_A)$. If $(P,cp,b) \in \mathcal{S}$, then $A$ is isomorphic to $U(P,cp,b)$ as a braided bialgebra, where $(P,cp,b)$ is the infinitesimal braided Lie algebra of $A$. The proof of this fact uses the universal property of the universal enveloping algebra (Theorem $\ref{thm:uni}$) to obtain a bialgebra projection of $U(P,cp,b)$ onto $A$; this projection comes out to be injective as $P(U(V,c,b))$ identifies with $V$ via $i_U$.

In Section $\ref{sec:example}$, the first meaningful examples of elements in $\mathcal{S}$ are given. Between them, braided vector spaces of diagonal type whose Nichols algebra is a domain of finite Gelfand-Kirillov dimension (Theorem $\ref{thm:finite}$) and two dimensional braided vector spaces of abelian group type whose symmetric algebra has dimension at most 31 (Proposition $\ref{prop:dim}$). In Examples $\ref{ex:2}$, $\ref{ex:3}$ and $\ref{ex:4}$, we provide instances of braided vector spaces which are not in $\mathcal{S}$ (the last two examples are due to V. K. Kharchenko).

The main applications and examples we are interested in are given in the last Sections $\ref{sec:appl}$ and $\ref{sec:exa}$. Explicitly, in Section $\ref{sec:appl}$ we deal with the class of braided vector spaces $(V,c)$ such that $c$ is a braiding of Hecke type i.e. it satisfies the equation $(c + \text{Id}_{V \otimes V})(c - q \text{Id}_{V \otimes V}) = 0$ for some regular $q \in K \setminus \{0\}$ (see Definition $\ref{def:hecke}$) called mark of $c$. In Theorem $\ref{thm:main}$ we prove that this class is contained in $\mathcal{S}$. Let $(V,c,b)$ be a braided Lie algebra such that $c$ is of Hecke type with regular mark $q$. Then, in Theorem $\ref{thm:main}$, using the results in $\cite{AMS1, AMS2}$, we show that the universal enveloping algebra of $(V,c,b)$ can be rewritten as follows

$$U(V,c,b) = \frac{T(V,c)}{(c(z) - qz - [z]_b | z \in V \otimes V)}$$

where $[-]_b : V \otimes V \to V$ is given by $[z]_b = b(c(z) - qz)$. Moreover if $q \neq 1$ and $\text{char}(K) \neq 2$, then $[-]_b = 0$.

Section $\ref{sec:example}$ is devoted to study the class of braided vector spaces $(V,c)$ such that the Nichols algebra $B(V,c)$ is a quadratic algebra (see Definition $\ref{def:quad}$). This also comes out to be a subclass of $\mathcal{S}$ (Theorem $\ref{thm:quad}$). In the literature there is plenty of examples of braided vector spaces of this kind (see Remark $\ref{rem:example}$). The main result of the section is Theorem $\ref{thm:quad}$ concerning connected braided bialgebras $(A,c_A)$ such that the graded coalgebra $gr(A)$ associated to the coradical filtration of $A$ is a quadratic algebra with respect to its natural braided bialgebra structure. Such an $A$ is proved to be isomorphic as a braided bialgebra to the enveloping algebra of its infinitesimal braided Lie algebra. The proof relies on the fact that the Nichols algebra associated to this braided Lie algebra results to be quadratic. Braided Lie algebras with this property will be further investigated in $\cite{AS}$.

In Section $\ref{sec:strong}$, we stress the relation between our notions of Lie algebra and universal enveloping algebra, and those introduced by Pareigis in $\cite{Pa}$. The main result of the section is Theorem $\ref{thm:pareigis}$ establishing that a Pareigis-Lie algebra $(V,c,[-])$ is associated to any braided Lie algebra $(V,c,b)$. Moreover there is a canonical braided bialgebra projection $p : U_P(V,c,[-]) \to U(V,c,b)$ where $U_P(V,c,[-])$ denotes the universal enveloping algebra of $(V,c,[-])$ (see Definition $\ref{def:uni}$). We prove this projection to be an isomorphism if condition (5) holds.

Note that, if $(V,c)$ lies in $\mathcal{S}$, then $p$ is an isomorphism if and only if $P(U_P(V,c,[-]))$ identifies with $V$ through the canonical map $i_{U_P} : V \to U_P(V,c,[-])$ (see Remark $\ref{rem:iso}$).

**Strongness degree.** The class $\mathcal{S}$ can be described as the class of braided vector spaces of strongness degree at most one. This terminology requires some explanation. First, we point out that many of the constructions and results we mentioned above concerning the tensor algebra $T(V,c)$ are indeed stated in the paper for a generic graded braided bialgebra $B$. In particular, in Definition $\ref{def:strong}$, we associate a symmetric algebra $S(B)$ to any graded braided bialgebra $B$. If $B$ is also endowed with a bracket $b$, then we can define the universal enveloping algebra $U(B,b)$ (see Definition $\ref{def:uni}$). Recall that a graded coalgebra $(C = \oplus_{n \in \mathbb{N}} C^n, \Delta_C, \varepsilon_C)$ is strongly $N$-graded whenever the $(i,j)$-homogeneous component $\Delta_C^{i,j} : C^{i+j} \to C^i \otimes C^j$ of $\Delta_C$ is a monomorphism.
for every $i, j \in \mathbb{N}$ (dually the notion of strongly $\mathbb{N}$-graded algebra can be introduced). Let $B$ be a graded braided bialgebra and consider $B, S(B), S(S(B))$ and so on. This yields a direct system whose direct limit is a graded braided bialgebra which is strongly $\mathbb{N}$-graded as a coalgebra (Theorem A.3) and that coincides with $B^0 [B^1]$, the braided bialgebra of Type one associated to $B^0$ and $B^1$ (see A.4), whenever $B$ is also strongly $\mathbb{N}$-graded as an algebra (Corollary A.4). We say that $B$ has strongness degree $n$ (sdeg $(B) = n$) if the direct system above is stationary exactly after $n$ steps (see Definition 4.4). V. K. Kharchenko pointed out to our attention that the notion of combinatorial rank in [Kh3, Definition 5.4] is essentially the same notion as the strongness degree. Nevertheless we decided here to keep our terminology as strongness degree is a measure of how far $B$ is to be strongly $\mathbb{N}$-graded as a coalgebra. In Section 4 we also provide criteria to compute sdeg $(B)$ in some cases. In appendix we collect some further results on strongness degree of graded braided bialgebras. In particular we see that under suitable assumptions the strongness degree of a graded braided bialgebra has an upper bound.

When we focus our attention on the case $B := T(V, c)$, the tensor algebra of a braided vector space $(V, c)$, we get $B(V, c) = B^0 [B^1], S(V, c) = S(B), U(V, c, b) = U(B, b)$. The strongness degree of a braided vector space is defined as the strongness degree of the corresponding tensor algebra. Hence a braided vector space $(V, c)$ is in $S$ if and only if it has strongness degree at most one.

Elsewhere, in the spirit of [Kh3, Section 6], we will investigate a notion of Lie algebra and enveloping algebra for braided vector spaces of arbitrary strongness degree.

2. Preliminaries

Throughout this paper $K$ will denote a field. All vector spaces will be defined over $K$ and the tensor product over $K$ will be denoted by $\otimes$.

In this section we define the main notions that we will deal with in the paper.

Definition 2.1. Let $V$ be a vector space over a field $K$. A $K$-linear map $c : V \otimes V \rightarrow V \otimes V$ is called a (quantum) Yang-Baxter operator (or $YB$-operator) if it satisfies the quantum Yang-Baxter equation

\begin{equation}
 c_1 c_2 c_1 = c_2 c_1 c_2
\end{equation}

on $V \otimes V \otimes V$, where we set $c_1 := c \otimes V$ and $c_2 := V \otimes c$. The pair $(V, c)$ will be called a braided vector space (or $YB$-space). A morphism of braided vector spaces $(V, c_V)$ and $(W, c_W)$ is a $K$-linear map $f : V \rightarrow W$ such that $c_W(f \otimes f) = (f \otimes f)c_V$.

Note that, for every braided vector space $(V, c)$ and every $\lambda \in K$, the pair $(V, \lambda c)$ is a braided vector space too. A general method for producing braided vector spaces is to take an arbitrary braided category $(\mathcal{M}, \otimes, K, a, l, r, c)$, which is a monoidal subcategory of the category of $K$-vector spaces. Hence any object $V \in \mathcal{M}$ can be regarded as a braided vector space with respect to $c := c_{V,V}$. Here, $c_{X,Y} : X \otimes Y \rightarrow Y \otimes X$ denotes the braiding in $\mathcal{M}$. The category of comodules over a coquasitriangular Hopf algebra and the category of Yetter-Drinfeld modules are examples of such categories. More particularly, every bicharacter of a group $G$ induces a braiding on the category of $G$-graded vector spaces.

Definition 2.2 (Baex, [Ba]). The quadruple $(A, m_A, u_A, c_A)$ is called a braided algebra if

- $(A, m_A, u_A)$ is an associative unital algebra;
- $(A, c_A)$ is a braided vector space;
- $m_A$ and $u_A$ commute with $c_A$, that is the following conditions hold:

\begin{align}
 c_A(m_A \otimes A) &= (A \otimes m_A)(c_A \otimes A)(A \otimes c_A), \\
 c_A(A \otimes m_A) &= (m_A \otimes A)(A \otimes c_A)(c_A \otimes A), \\
 c_A(u_A \otimes A) &= A \otimes u_A, & c_A(A \otimes u_A) &= u_A \otimes A.
\end{align}
A morphism of braided algebras is, by definition, a morphism of ordinary algebras which, in addition, is a morphism of braided vector spaces.

The quadruple \((C, \Delta_C, \varepsilon_C, c_C)\) is called a **braided coalgebra** if

- \((C, \Delta_C, \varepsilon_C)\) is a coassociative counital coalgebra;
- \((C, c_C)\) is a braided vector space;
- \(\Delta_C\) and \(\varepsilon_C\) commute with \(c_C\), that is the following relations hold:

\[
\begin{align*}
(5) \quad & (\Delta_C \otimes C)cc = (C \otimes cc)(cc \otimes C)(C \otimes \Delta_C), \\
(6) \quad & (C \otimes \Delta_C)c_c = (cc \otimes C)(C \otimes cc)(\Delta_C \otimes C), \\
(7) \quad & (\varepsilon_C \otimes C)c_c = C \otimes \varepsilon_C, \quad (C \otimes \varepsilon_C)c_c = \varepsilon_C \otimes C.
\end{align*}
\]

A morphism of braided coalgebras is, by definition, a morphism of ordinary coalgebras which, in addition, is a morphism of braided vector spaces.

**Definition 5.1** A sextuple \((B, m_B, u_B, \Delta_B, \varepsilon_B, c_B)\) is a called a **braided bialgebra** if

- \((B, m_B, u_B, c_B)\) is a braided algebra
- \((B, \Delta_B, \varepsilon_B, c_B)\) is a braided coalgebra
- the following relations hold:

\[
\Delta_B m_B = (m_B \otimes m_B)(B \otimes c_B \otimes B)(\Delta_B \otimes \Delta_B).
\]

Examples of the notions above are algebras, coalgebras and bialgebras in any braided category \((\mathcal{M}, \otimes, K, a, l, r, c)\), which is a monoidal subcategory of the category of \(K\)-vector spaces.

**Definition 2.3.** We will need graded versions of braided algebras, coalgebras and bialgebras. A **graded braided algebra** is a braided algebra \((A, m_A, u_A, c_A)\) such that \(A = \bigoplus_{n \in \mathbb{N}} A^n\) and \(m_A(A^n \otimes A^n) \subseteq A^{n+m},\) for every \(m, n \in \mathbb{N}\). The braiding \(c_A\) is assumed to satisfy \(c_A(A^n \otimes A^m) \subseteq A^m \otimes A^n\). It is easy to see that \(1_A = u_A(1_k) \in A^0\). Therefore a graded braided algebra is defined by maps \(m_{A, a, b}^n : A^a \otimes A^b \rightarrow A^{a+b}\) and \(c_{A, a, b}^n : A^a \otimes A^b \rightarrow A^b \otimes A^a\), and by an element \(1 \in A^0\) such that:

\[
\begin{align*}
(9) \quad & m_{A}^{n+m,p}(m_{A}^{n,m} \otimes A^p) = m_{A}^{n,m+p}(A^n \otimes m_{A}^{m,p}), \quad n, m, p \in \mathbb{N}, \\
(10) \quad & m_{A}^{0,n}(1 \otimes a) = m_{A}^{0,a}(1 \otimes 1), \quad \forall a \in A^n, \quad n \in \mathbb{N}. \\
(11) \quad & c_{A}^{n+m,p}(m_{A}^{n,m} \otimes A^p) = (A^p \otimes m_{A}^{n,m})(m_{A}^{n,m} \otimes A^p)(A^n \otimes c_{A}^{m,p}), \\
(12) \quad & c_{A}^{n,m+p}(A^n \otimes m_{A}^{m,p}) = (m_{A}^{m,p} \otimes A^n)(A^m \otimes c_{A}^{n,m+p}), \\
(13) \quad & c_{A}^{0,n}(1 \otimes 1) = 1 \otimes a \quad \text{and} \quad c_{A}^{0,a}(1 \otimes 1) = a \otimes 1, \quad \forall a \in A^n.
\end{align*}
\]

The multiplication \(m_A\) can be recovered from \((m_{A}^{n,m})_{n,m \in \mathbb{N}}\) as the unique \(K\)-linear map such that \(m_A(x \otimes y) = m_{A}^{p,q}(x \otimes y), \forall p, q \in \mathbb{N}, \forall x \in A^p, \forall y \in A^q\). Analogously, the braiding \(c_A\) is uniquely defined by \(c_A(x \otimes y) = c_{A}^{p,q}(x \otimes y), \forall p, q \in \mathbb{N}, \forall x \in A^p, \forall y \in A^q\). We will say that \(m_{A, a, b}^n\) and \(c_{A, a, b}^n\) are the \((n, m)\)-homogeneous components of \(\nabla\) and \(c_A\), respectively.

**Graded braided coalgebras** can be described in a similar way. By definition a graded coalgebra \((C, \Delta_C, \varepsilon_C, c_C)\) is graded if \(C = \bigoplus_{n \in \mathbb{N}} C^n\), \(\Delta_C(C^n) \subseteq \sum_{p+q=n} C^p \otimes C^q, c_C(C^m \otimes C^n) \subseteq C^{m+n}\) and \(\varepsilon_C|_{C^n} = 0, \forall n > 0\). If \(\pi\) denotes the projection onto \(C^p\) then the comultiplication \(\Delta_C\) is uniquely defined by maps \(\Delta_{C, p,q}^{n,m} : C^{p+q} \rightarrow C^p \otimes C^q\), where \(\Delta_{C, p,q}^{n,m} := (\pi^p \otimes \pi^q)\Delta_C|_{C^{p+q}}\). The counit is given by a map \(\varepsilon_{C}^0 : C^0 \rightarrow K\), while the braiding \(c_C\) is uniquely determined by a family \((c_{C}^{n,m})_{n,m \in \mathbb{N}}\), as for braided algebras. The families \((\Delta_{C, p,q}^{n,m})_{n,m \in \mathbb{N}}, (c_{C}^{n,m})_{n,m \in \mathbb{N}}\) and \(\varepsilon_{C}^0\) has to satisfy the relations that are dual to \((9) - (13)\):
We will say that $\Delta^{n,m}$ is the $(n,m)$-homogeneous component of $\Delta_C$.

A **graded braided bialgebra** is a braided bialgebra which is graded both as an algebra and as a coalgebra.

2.4. Recall that a coalgebra $C$ is called **connected** if $C_0$, the coradical of $C$ (i.e., the sum of simple subcoalgebras of $C$), is one dimensional. In this case there is a unique group-like element $1_C \in C$ such that $C_0 = K1_C$. A morphism of connected coalgebras is a coalgebra homomorphisms (clearly it preserves the group-like elements).

By definition, a braided coalgebra $(C, C_c)$ is **connected** if $C_0 = K1_C$ and, for any $x \in C$,

$$c_C(x \otimes 1_C) = 1_C \otimes x, \quad c_C(1_C \otimes x) = x \otimes g.$$  

**Definition 2.5.** Let $C = \bigoplus_{n \in \mathbb{N}} C^n$ be a graded braided coalgebra. We say that $C$ is 0-**connected**, whenever $C_0$ is a one dimensional vector space.

**Remark 2.6.** Let $C = \bigoplus_{n \in \mathbb{N}} C^n$ be a graded braided coalgebra. By [Ni, Proposition 11.1.1], if $(C_n)_{n \in \mathbb{N}}$ is the coradical filtration, then $C_n \subseteq \bigoplus_{m \leq n} C^m$. Therefore, $C$ is connected if it is 0-connected.

**Lemma 2.7.** [AMS, Lemma 1.1] Let $(C,c)$ be a connected braided coalgebra. Then $c$ induces a canonical braiding $c_{\text{gr}(C)}$ on the graded coalgebra $\text{gr}(C)$ associated to the coradical filtration on $C$ such that $(\text{gr}(C),c_{\text{gr}(C)})$ is a 0-connected graded braided coalgebra.

**Lemma 2.8.** Let $(B, m_B, u_B, \Delta_B, \varepsilon_B, c_B)$ be a graded bialgebra. Then

$$\Delta^{a,b} m^{c,d} = \delta_{b+a,c+d} \sum_{0 \leq u \leq c} \sum_{0 \leq u^\prime \leq d} \delta_{u,u^\prime} \left( m_B^{u,u^\prime} \otimes m_B^{c-u,d-u^\prime} \right) \left( B^u \otimes c_B^{c-u,u^\prime} \otimes B^{d-u^\prime} \right) \left( \Delta_B^{u,v} \otimes \Delta_B^{v,d-u^\prime} \right).$$

**Proof.** Since $\Delta_B^{a,b} m_B^{c,d} = (p_B^a \otimes p_B^b) \Delta_B m_B \left( i_B^a \otimes i_B^b \right)$, using (8) and the fact that $m_B, c_B, \Delta_B$ are graded maps one gets

$$\Delta_B^{a,b} m_B^{c,d} = \sum_{u+u^\prime=v} \delta_{b+a,c+d} \delta_{u,u^\prime} \left( m_B^{u,u^\prime} \otimes m_B^{c-u,d-u^\prime} \right) \left( B^u \otimes c_B^{v,d-u^\prime} \otimes B^{d-u^\prime} \right) \left( \Delta_B^{u,v} \otimes \Delta_B^{v,d-u^\prime} \right).$$

It is easy to check that the last term of this equality coincide with the second term of (20). \hfill \Box

### 3. Symmetric and enveloping algebras of graded braided bialgebras

In this section we introduce and investigate the notion of symmetric algebra of a graded braided bialgebra. We also introduce and study the notion of universal enveloping algebra of a graded braided bialgebra endowed with a bracket.

**Definitions 3.1.** Let $(C, \Delta_C, \varepsilon_C)$ be a graded coalgebra. Let

$$E(C) := \bigoplus_{n \in \mathbb{N}} E_n(C)$$

where $E_n(C) := \left\{ \begin{array}{ll} 0 & \text{if } n = 0, 1 \\ \bigcap_{a+b=n} \ker \left( \Delta^{a,b}_C \right) & \text{if } n \geq 2. \end{array} \right.$

Denote by $i^n_C : C^n \to C$ and $p^n_C : C \to C^n$ the canonical injection and projection respectively. Following [Ni, 1.2], we set

$$P_1(C) := \ker \left[ \Delta_C - (i^0_C p^0_C \otimes C) \Delta_C - (C \otimes i^0_C p^0_C) \Delta_C \right] \subseteq C.$$

**Remark 3.2.** Note that $P_1(C)$ is just the space $P(C)$ of primitive elements in $C$ whenever $C$ is 0-connected.

The following result shows that the elements of $E(C)$ are those of $P_1(C)$ of degree greater then 1.

**Lemma 3.3.** Let $(C, \Delta_C, \varepsilon_C)$ be a graded braided bialgebra. Then $P_1(C) = C^1 \oplus \bigoplus_{n \geq 2} E_n(C)$. Moreover $C^0 \wedge C^0 = C^0 \otimes C^1 \oplus \bigoplus_{n \geq 2} E_n(C)$. 


Definition 3.4. Let \((B, m_B, u_B, \Delta_B, \varepsilon_B, c_B)\) be a graded braided bialgebra. The **symmetric algebra** of \(B\) is defined to be

\[(21) \quad S(B) = \frac{B}{(E(B))}.\]

Denote by \(\pi_S := \pi_S^B : B \to S(B)\) the canonical projection.

**Theorem 3.5.** Let \((B, m_B, u_B, \Delta_B, \varepsilon_B, c_B)\) be a graded braided bialgebra. Then \(S(B)\) has a unique graded braided bialgebra structure such that the canonical projection \(\pi_S : B \to S(B)\) is a graded braided bialgebra homomorphism.

**Proof.** Since \(I := (E(B))\) is generated by homogeneous elements, it is a graded ideal of \(B\) with graded component \(I_n = (E(B)) \cap B^n\). Hence \(S := S(B)\) has a unique graded algebra structure such that \(\pi_S : B \to S(B)\) is a graded algebra homomorphism. Let us check it is also a graded coalgebra.

Since \(B\) is a graded coalgebra and \(E_0(B) = 0\) if \(n = 0, 1\), we have \(\varepsilon_B(I) = 0\) so that, in order to get that \(I\) is a coideal, it remains to prove that \(\Delta(I) \subseteq I \otimes B + B \otimes I\).
Firstly, let us check that
\[(22) \ c_B(B^a \otimes E_t(B)) \subseteq E_t(B) \otimes B^a \quad \text{and} \quad c_B(E_t(B) \otimes B^u) \subseteq B^u \otimes E_t(B) .\]
We focus on the second inclusion the first one having a similar proof. Since $B$ is a $c$-bracket, for every $z \in E_t(B)$, $y \in B^u$ and $a, b \geq 1, a + b = t$, we have
\[
\left( B^u \otimes \Delta_B^{a,b} \right) t^{u} \left( z \otimes y \right) \bigotimes \left( c_B^{a,u} \otimes B^b \right) \left( B^a \otimes c_B^{b,u} \right) \left( \Delta_B^{a,b} \otimes B^u \right) (z \otimes y) = 0
\]
Then $c_B^{t,u}(z \otimes y) \in \ker \left( B^u \otimes \Delta_B^{a,b} \right) = B^u \otimes \ker \left( \Delta_B^{a,b} \right)$, for every $a, b \geq 1, a + b = t$ so that
\[
c_B^{t,u}(z \otimes y) \subseteq \bigcap_{a,b \geq 1, a + b = t} B^u \otimes \ker \left( \Delta_B^{a,b} \right) = B^u \otimes E_t(B). 
\]
Thus \((22)\) holds.

Now, for every $z \in E_t(B), x \in B^u, y \in B^v, s \in B^w$, using \((11)\) and \((22)\) it is straightforward to check that $c_B^{u+v+w}(x \otimes y \otimes s) \in (B^v \otimes I_{a+t+v})$ so that
\[(23) \ c_B(B^u \otimes I_s) \subseteq I_t \otimes B^u \quad \text{and} \quad c_B(I_s \otimes B^u) \subseteq B^u \otimes I_t.
\]
Let $z \in E_t(B)$. Thus, for every $a, b \geq 1$ such that $a + b = t$, and we have
\[
\left( B^0 \otimes \Delta_B^{a,b} \right) \Delta_B^{0,t}(z) = \left( \Delta_B^{0,a} \otimes B^b \right) \Delta_B^{a,b}(z) = 0
\]
so that
\[
\Delta_B^{0,t}(z) \subseteq \bigcap_{a,b \geq 1, a + b = t} B^0 \otimes \ker \left( \Delta_B^{a,b} \right) = B^0 \otimes E_t(B).
\]
Therefore we have
\[(24) \ \Delta_B^{0,t}(E_t(B)) \subseteq B^0 \otimes E_t(B) \quad \text{and} \quad \Delta_B^{t,0}(E_t(B)) \subseteq E_t(B) \otimes B^0
\]
where the second inclusion can be checked similarly.

Let $x \in B^a$ and $z \in E_t(B)$. Using \((24)\), that $z \in E_t(B)$ and \((24)\), we get $\Delta_B^{a,b}(xz) \in B^a \otimes I_b + I_a \otimes B^b$ so that $\Delta_B^{a,b}(I_{a+b}) \subseteq B^a \otimes I_b + I_a \otimes B^b$, for every $a, b \in \mathbb{N}$.

Therefore $I$ is a graded coideal and hence $S := S(B)$ carries a unique graded coalgebra structure such that $\pi_S$ is a coalgebra homomorphism. In fact there is a unique morphism $\Delta_S^{a,b} : B^{a+b}/I_{a+b} \rightarrow B^a/I_a \otimes B^b/I_b$ such that $\Delta_S^{a,b} \pi_S^{a+b} = (\pi_S^a \otimes \pi_S^b) \Delta_B^{a,b}$, where $\pi_S^a : B^a \rightarrow S^n := B^n/I_n$ denotes the canonical projection. Using that $\pi_S^n$ is an epimorphism for every $n \in \mathbb{N}$, one gets that $\left( \Delta_S^{a,b} \otimes S^c \right) \Delta_S^{a+b,c} = \left( S^a \otimes \Delta_S^{b,c} \right) \Delta_S^{a+b+c}$. On the other hand $\pi_S^n$ is bijective so that we can set $\epsilon_S^n := \epsilon_S^{0,a} (\pi_S^n)^{-1}$. One easily checks that $\left( \epsilon_S^n \otimes S^c \right) \Delta_S^{0,c} = \Delta_S^{0,c} = \left( S^c \otimes \epsilon_S^n \right) \Delta_S^{0,c}$. By \([\text{AM}],\) Proposition 2.5, there is a unique map $\Delta_S : S \rightarrow S \otimes S$ such that $\Delta_S^n = \sum_{a+b=n} \left( \epsilon_S^{a} \otimes \epsilon_S^{b} \right) \Delta_S^{a+b}$ where $\epsilon_S^n : S^n \rightarrow S$ denotes the canonical injection. Furthermore $(S, \Delta_S, \epsilon_S) = (S, \epsilon_S)_{S_B}$ is a graded coalgebra where $p_B^n : S \rightarrow S^n$ denotes the canonical projection.

Let us prove that $c_B$ factors through a braiding $c_S$ of $S = S(B)$ that makes $S$ a braided bialgebra.

By the foregoing we get
\[
c_B^{a,b}(\ker (\pi_S^a \otimes \pi_S^b)) = c_B^{a,b}(B^a \otimes I_b + I_a \otimes B^b) \subseteq I_b \otimes B^a + B^b \otimes I_a = \ker (\pi_S^b \otimes \pi_S^b)
\]
so that, since $\pi_S \otimes \pi_S$ is surjective, there exists a unique $K$-linear map $c_S^{a,b} : S^a \otimes S^b \rightarrow S^b \otimes S^a$ such that
\[(25) \ c_S^{a,b}(\pi_S^b \otimes \pi_S^b) = (\pi_S^a \otimes \pi_S^b) c_B^{a,b}.
\]
This relation can be used to prove that $(S, c_S)$ is a graded braided bialgebra with structure induced by $\pi_S$. \qed
Proposition 3.6. Let \( f : (B, m_B, u_B, \Delta_B, \varepsilon_B, c_B) \to (B', m_{B'}, u_{B'}, \Delta_{B'}, \varepsilon_{B'}, c_{B'}) \) be a graded braided bialgebra homomorphism. Then there is a unique algebra homomorphism

\[
S(f) : S(B) \to S(B')
\]

such that \( S(f) \circ \pi_S = \pi_{S'} \circ f \). Moreover \( S(f) \) is a graded braided bialgebra homomorphism.

Proof. Let \( n \geq 2 \). Then, for every \( a, b \geq 1, a + b = n \), we have

\[
\Delta_B^{a,b} f_n (E_n (B)) = (f_a \otimes f_b) \Delta_B^{a,b} (E_n (B)) = 0
\]

so that \( f_n (E_n (B)) \subseteq \bigcap_{a,b \geq 1, a+b=n} \ker \left( \Delta_B^{a,b} \right) = E_n (B') \). Set \( I := E_n (B) \) and \( I' := E_n (B') \). Denote by \( I_n \) and \( I'_n \) the graded components of these ideals. Since \( f \) is a graded algebra homomorphism, we get \( f(I_n) \subseteq I'_n \), so that \( f \) is a morphism of graded ideals. Hence there is a unique algebra homomorphism \( S(f) : S(B) \to S(B') \) such that \( S(f) \circ \pi_S = \pi_{S'} \circ f \). Furthermore \( S(f) \) is a morphism of graded algebras with graded component \( S(f^n) = S^n \circ (S')^n \) uniquely defined by \( S(f^n) \circ \pi_S = \pi_{S'} \circ f^n \). Since \( \pi_S \) is an epimorphism, one gets that \( S(f) \) is a coalgebra homomorphism (and hence a graded coalgebra homomorphism).

Next aim is to introduce a notion universal enveloping algebra of a graded braided bialgebra endowed with a bracket and to see that it carries a braided bialgebra structure.

3.7. Let \( (B, m_B, u_B, \Delta_B, \varepsilon_B, c_B) \) be a graded braided bialgebra. By (22), we have the the inclusions \( c_B (B^u \otimes E_t (B)) \subseteq E_t (B) \otimes B^u \) and \( c_B (E_t (B) \otimes B^u) \subseteq B^u \otimes E_t (B) \) hold. Hence there exists a unique morphism \( c_{E_t(B), B^u} : E_t (B) \otimes B^u \to B^u \otimes E_t (B) \) such that

\[
(26) \quad (B^u \otimes j^{t}_{B}) \circ c_{E_t(B), B^u} = c_{B, t} \circ (j^{t}_{B} \otimes B^u),
\]

where \( j^{t}_{B} : E_t (B) \to B^t \) denotes the canonical injection. Similarly one gets \( c_{B^u, E_t(B)} \). Define now

\[
\begin{align*}
&c_{E_t(B), B^u}^1 : = \oplus_{t \in \mathbb{N}} c_{E_t(B), B^u} : E(B) \otimes B^u \to B^u \otimes E(B), \\
&c_{B^u, E_t(B)}^1 : = \oplus_{t \in \mathbb{N}} c_{B^u, E_t(B)} : B^u \otimes E(B) \to E(B) \otimes B^u.
\end{align*}
\]

Definition 3.8. A bracket for a graded braided bialgebra \( (B, m_B, u_B, \Delta_B, \varepsilon_B, c_B) \) is a \( K \)-linear map \( b = b_B : E (B) \to B^1 \) such that, for every \( t \in \mathbb{N} \)

\[
(27) \quad c_B^{1,t} (b \otimes B^t) = (B^t \otimes b) c_{E_t(B), B^u} \quad c_B^{1,t} (B^t \otimes b) = (b \otimes B^t) c_{B^u, E_t(B)}
\]

and

\[
(28) \quad \Delta_B^{1,0} b = (b \otimes B^0) \Delta_B^{0,1} b = (B^0 \otimes b) \Delta_B^{0,t}.
\]

The restriction of \( b \) to \( E_t (B) \) will be denoted by \( b^t : E_t (B) \to B^1 \). If \( b \) is a bracket for \( B \), then we define the universal enveloping algebra of \( (B, b) \) to be

\[
U (B, b) := \frac{B}{(\text{Id} - b) [E (B)]}.
\]

We will denote by \( i_U : B^1 \to U (B, b) \) the restriction to \( B^1 \) of the canonical projection \( \pi_U : B \to U (B, b) \). Note that \( 0 \) is always a bracket for \( B \) so that it makes sense to consider

\[
U (B, 0) = \frac{B}{[E (B)]} S (B).
\]

Let \( f : B \to B' \) be a graded braided bialgebra homomorphism. Then one has \( \Delta_B^{a,b} \circ f_{a+b} = (f_a \otimes f_b) \circ \Delta_B^{a,b} \), for every \( a, b \in \mathbb{N} \). This entails that \( f_{E_t(B)} \subseteq E_t (B') \) so that there exists a unique map

\[
E_t (f) : E_t (B) \to E_t (B')
\]

such that \( j^{t}_{B'} \circ E_t (f) = f_t \circ j^{t}_{B} \), for every \( t \geq 2 \), where \( j^{t}_{B} : E_t (B) \to B^t \) and \( j^{t}_{B'} : E_t (B') \to (B')^t \) are the canonical inclusions. We set

\[
E (f) := \bigoplus_{t \in \mathbb{N}} E_t (f).
\]
A morphism of brackets $f : (B, b_B) \to (B', b_{B'})$ is a graded braided bialgebra homomorphism $f : B \to B'$ such that $f_1 \circ b^t_B = b^t_{B'} \circ E_t(f)$, for every $t \in \mathbb{N}$.

**Theorem 3.9.** Let $(B, m_B, u_B, \Delta_B, \varepsilon_B, c_B)$ be a graded braided bialgebra endowed with a bracket $b : E(B) \to B^1$. Then $U(B, b)$ has a unique braided bialgebra structure such that the canonical projection $\pi_U : B \to U(B, b)$ is a braided bialgebra homomorphism.

**Proof.** Set $I = ((1d - b)(E(B)))$. Let us check that $I$ is a coideal i.e. that $\Delta(I) \subseteq I \otimes B + B \otimes I$ and $\varepsilon_B(I) = 0$. First, for every $z \in E_i(B)$, $x \in B^w$, $y \in B^v$, $s \in B^w$, using (11), (27) and (22) one can prove that $c_B(x(z - b(z)) y \otimes s) \in B^w \otimes I$ so that

$$c_B(I \otimes B^w) \subseteq B^w \otimes I \quad \text{and} \quad c_B(B^w \otimes I) \subseteq I \otimes B^w$$

where the second equality can be obtained similarly. Let $z \in E_i(B)$. By Lemma 3.3, one has

$$u = z - b(z) \in E_i(B) - B^1 \subseteq B^1 \oplus [\oplus_{n \geq 2} E_n(B)] = P_1(C)$$

so that $\Delta_B(u) = (i_B^0p_B^0 \otimes B) \Delta_B(u) + (B \otimes i_B^0p_B^0) \Delta_B(u)$. We have $\varepsilon_B(I) \subseteq \varepsilon_B(P_1(C)) = 0$. Moreover, using (28) and (24) one easily gets that $(p_B^0 \otimes B) \Delta_B(u) \in B^0 \otimes I$ and similarly $(B \otimes i_B^0) \Delta_B(u) \in I \otimes B^0$ so that $\Delta_B(u) \in B \otimes I + I \otimes B$. Let $x \in B^c$. Using [5], the last relation and (24) we get that $\Delta_B(xu) \in B \otimes I + I \otimes B$. Using [5], the last relation and (24), for every $y \in B^w$ we obtain $\Delta_B(xy) \subseteq I \otimes B + B \otimes I$.

In conclusion $\Delta(I) \subseteq I \otimes B + B \otimes I$. Therefore $I$ is coideal and hence $U = U(B, b)$ carries a unique coalgebra structure such that the canonical projection $\pi_U : T \to U$ is a coalgebra homomorphism. In fact there are unique morphisms $\Delta_U : B/I \to B/I \otimes B/I$ and $\varepsilon_U : B/I \to K$ such that $\Delta_U\pi_U = (\pi_U \otimes \pi_U) \Delta_B$ and $\varepsilon_U\pi_U = \varepsilon_B$. Using that $\pi_U$ is an epimorphism and that $B$ is a coalgebra one gets that $(U, \Delta_U, \varepsilon_U)$ is a coalgebra too. Let us prove that $c_B$ factors trough a braiding $c_U$ of $U$ that makes $U$ a braided bialgebra. We get

$$c_B(\ker (\pi_U \otimes \pi_U)) = c_B(I \otimes T + T \otimes I) \subseteq I \otimes T + T \otimes I = \ker (\pi_U \otimes \pi_U)$$

so that, since $\pi_U \otimes \pi_U$ is surjective, there exists a unique $K$-linear map $c_U : U \otimes U \to U \otimes U$ such that $c_U(\pi_U \otimes \pi_U) = (\pi_U \otimes \pi_U) c_U$. This relation can be used to prove that $(U, m_U, u_U, \Delta_U, \varepsilon_U, c_U)$ is a braided bialgebra with structure induced by $\pi_U$. Here $m_U : U \otimes U \to U$ and $u_U : K \to U$ denote the multiplication and the unit of $U$ respectively. \qed

4. Strongness degree

In this section we introduce and study the notion of strongness degree of a graded braided bialgebra $B$ which is somehow a measure of how far $B$ is to be strongly $\mathbb{N}$-graded as a coalgebra. Later on we will focus on braided bialgebras of strongness degree at most one. Further results on strongness degree of graded braided bialgebras are included in appendix.

**Definition 4.1.** [AM] Definition 3.5] Let $(A = \oplus_{n \in \mathbb{N}} A^n, m, u)$ be a graded algebra in $\mathcal{M}$. In analogy with the group graded case, we say that $A$ is a strongly $\mathbb{N}$-graded algebra whenever $m^{i,j}_A : A^i \otimes A^j \to A^{i+j}$ is an epimorphism for every $i, j \in \mathbb{N}$.

[AM] Definition 2.9] (see also [MS, Lemma 2.3]) Let $(C = \oplus_{n \in \mathbb{N}} C^n, \Delta, \varepsilon)$ be a graded coalgebra in $\mathcal{M}$. We say that $C$ is a strongly $\mathbb{N}$-graded coalgebra whenever $\Delta^i_C : C^{i+j} \to C^i \otimes C^j$ is a monomorphism for every $i, j \in \mathbb{N}$.

4.2. Recall that, given a coalgebra $C$ and a $C$-bicomodule $M$, one can consider the cotensor coalgebra $(T^c := T^c_C(M), \Delta^c, \varepsilon^c)$. It is defined as follows. As a vector space $T^c = C \oplus M \oplus M^{i+1} \oplus M^{i+2} \oplus \cdots$, where $\square_C$ denotes the cotensor product over $C$. Furthermore, $\Delta^c_{i,j}$ is given by the comodule structure map for $i = 0$ or $j = 0$; when $i, j > 0$ it is induced by the map $m_1 \otimes \cdots \otimes m_n \mapsto (m_1 \otimes \cdots \otimes m_i) \otimes (m_{i+1} \otimes \cdots \otimes m_n)$. The counit is zero on elements of positive degree. The cotensor coalgebra fulfill a suitable universal property that resemble the universal property of the tensor algebra (see [3, Proposition 1.4.2]). Note that $T^c$ is a strongly $\mathbb{N}$-graded coalgebra.
Theorem 4.3. Let \((B, m_B, u_B, \Delta_B, \varepsilon_B, c_B)\) be a graded braided bialgebra. The following assertions are equivalent.

1. The canonical projection \(\pi_S : B \to S(B)\) is an isomorphism.
2. \(E_n(B) = 0\) for every \(n \geq 2\).
3. \(B\) is strongly \(\mathbb{N}\)-graded as a coalgebra.
4. \(\Delta_B^n\) is injective for every \(n \in \mathbb{N}\).
5. The canonical map \(\psi : B \to T_{\varphi^0}(B)\) is injective.
6. \(B^0 \oplus B^1 \oplus \cdots \oplus B^n = B^0 \wedge_B \cdots \wedge_B B^0\), for every \(n \in \mathbb{N}\).
7. \(B^0 \oplus B^1 = B^0 \wedge_B B^0\).

Proof. By Theorem 3.3, \(S(B)\) is a graded coalgebra. Thus the equivalences (3) \(\Leftrightarrow\) (4) \(\Leftrightarrow\) (5) \(\Leftrightarrow\) (6) \(\Leftrightarrow\) (7) follows by applying [AM, Theorem 2.22] to the monoidal category of vector spaces. (1) \(\Leftrightarrow\) (2) It is trivial. (2) \(\Leftrightarrow\) (7) By Lemma 3.3, we have \(B^0 \wedge_B B^0 = B^0 \oplus B^1 \oplus [\oplus_{n \geq 2} E_n(B)]\). \(\square\)

Definition 4.4. Let \((B, m_B, u_B, \Delta_B, \varepsilon_B, c_B)\) be a graded braided bialgebra. Define iteratively \(S^{[i]}(B)\) by \(S^{[0]}(B) := B\) and \(S^{[n]}(B) := S(S^{[n-1]}(B))\), for \(n \geq 1\). This yields a direct system

\[\left(\left(S^{[i]}(B)\right)_{i \in \mathbb{N}}, \left(\pi^2_{i,j}\right)_{i,j \in \mathbb{N}}\right)\]

where \(\pi^2_j\) is defined in an obvious way for every \(j \geq i\). Denote by \((S^{[\infty]}(B), \pi^{\infty}) = \lim (S^{[i]}(B))\) the direct limit of this direct system. We say that \(B\) has strongness degree \(n\) if the system above is stationary and \(n\) is the least \(t \in \mathbb{N}\) such that \(\pi^{t+1}_t : S^{[t]}(B) \to S^{[t+1]}(B)\) is an isomorphism. In this case we will write \(\text{sdeg}(B) = n\). If the system above is not stationary we will write \(\text{sdeg}(B) = \infty\).

Remark 4.5. V. K. Kharchenko pointed out to our attention that the notion of combinatorial rank in [Kli, Definition 5.4] is essentially the same notion as the strongness degree. Nevertheless we decided here to keep our terminology as, in view of Theorem 4.3, strongness degree is a measure of how far \(B\) is to be strongly \(\mathbb{N}\)-graded as a coalgebra.

Definition 4.6. Let \((B, m_B, u_B, \Delta_B, \varepsilon_B, c_B)\) be a graded braided bialgebra. For every \(a, b, n \in \mathbb{N}\), we set

\[\Gamma_B^{a,b} := \Gamma_B^{a,b} \Delta_B^{a,b} \text{ and } \Gamma_B^n := \{ \text{Id}_{B^0} \} \oplus_{m_B} (\Gamma_B^{a,b} \Delta_B^{a,b}) \Delta_B^{a,b} \text{ if } n = 0 \]

Remark 4.7. Let \((V, c)\) be a braided vector space and let \(T = T(V, c)\) be the associated tensor algebra. Then \(m_T^{a,b}\) is just the juxtaposition. This entails that \(\Gamma_T^{a,b} = S_{a,b}\) and \(\Gamma_T^n = S_n\) where \(S_{a,b}\) and \(S_n\) are the morphisms defined in [Sch, page 2815].

Lemma 4.8. Let \(\pi : (B, m_B, u_B, \Delta_B, \varepsilon_B, c_B) \to (E, m_E, u_E, \Delta_E, \varepsilon_E, c_E)\) be a graded braided bialgebra homomorphism. Then

\[\Gamma_E^{a,b} \circ \pi^{a+b} = \pi^{a+b} \circ \Gamma_B^{a,b}, \text{ for every } a, b \in \mathbb{N}, a + b \geq 1.\]

where, for every \(n \in \mathbb{N}\), \(\pi^n : B^n \to E^n\) denotes the \(n\)-th graded component of \(\pi\).

Proof. We have \(\Gamma_E^{a,b} \pi^{a+b} = m_E^{a,b} \Delta_E^{a,b} \pi^{a+b} = m_E^{a,b} (\pi^a \otimes \pi^b) \Delta_B^{a,b} = \pi^{a+b} m_B^{a,b} \Delta_B^{a,b} = \pi^{a+b} \Gamma_B^{a,b}.\) \(\square\)

Lemma 4.9. Let \((B, m_B, u_B, \Delta_B, \varepsilon_B, c_B)\) be a graded braided bialgebra which is strongly \(\mathbb{N}\)-graded as an algebra. Then

\[\Gamma_B^{a,b} = m_B^{a,b} (\Gamma_B^a \otimes \Gamma_B^b) \Delta_B^{a,b}, \text{ for every } a, b \in \mathbb{N}.\]
Theorem 4.13\n
An isomorphism so that $\text{sdeg}(\Delta) = (34) \text{Γ}$ \n\nInductively, using (34) and the surjectivity of Theorem 4.12, we find (31) that restricted to $\text{sdeg}(\Delta)$ gives the canonical inclusions. Furthermore each graded component of $\text{Γ}B$ is epimorphism as $B$ is strongly $\mathbb{N}$-graded as an algebra. We have
\n$$m_{a,b}^n (\Gamma_a \otimes \Gamma_b) \Delta_{a,b}^n = m_{a,b}^n (\Gamma_a \otimes \Gamma_b) \Delta_{a,b}^n \quad \text{(*)}$$
\nwhere in (*) we applied [Sch (1)]. Since $\varphi_{a+b}$ is an epimorphism, we obtain (32). \n
We now restrict our attention to connected graded braided bialgebras.

Lemma 4.10. Let $(B, m_B, u_B, \Delta_B, \varepsilon_B, c_B)$ be a 0-connected graded braided bialgebra. Then $\Delta_{B}^0(z) = z \otimes 1$ and $\Delta_{B}^0 = 1 \otimes z$, for every $z \in B^n$. Moreover
\n$$\Delta_{B}^1 m_B^1 = \text{Id}_{B^n \otimes B^1} + \left( m_B^{n-1,1} \otimes B^1 \right) \left( B^{n-1} \otimes c_B^{1,1} \right) \left( \Delta_{B}^{n-1,1} \otimes B^1 \right) .$$
\nProof. The first assertion follows by [AMS1, Remark 1.16].

Since $B$ is a graded bialgebra we have
\n$$\Delta_{B}^1 m_B^1 = \text{Id}_{B^n \otimes B^1} + \left( m_B^{n-1,1} \otimes B^1 \right) \left( B^{n-1} \otimes c_B^{1,1} \right) \left( \Delta_{B}^{n-1,1} \otimes B^1 \right) .$$
\n\nLet $(B, m_B, u_B, \Delta_B, \varepsilon_B, c_B)$ be a 0-connected graded braided bialgebra. Next aim is to provide an upper bound of $\text{sdeg}(B)$ under suitable assumptions on $c_B^{1,1}$. These results will be needed in the last sections of the paper.

Definition 4.11. For every $n \geq 1$, we set
\n$$(n)_X := \frac{X^n - 1}{X - 1} = 1 + X + \cdots + X^{n-1}, \quad (n)_X! := (1)_X \cdot (2)_X \cdot \cdots \cdot (n)_X .$$
\nAn element $q \in K$ is called regular whenever $(n)_q \neq 0$, for all $n \geq 2$.

Theorem 4.12. Let $(B, m_B, u_B, \Delta_B, \varepsilon_B, c_B)$ be a 0-connected graded braided bialgebra. Assume that
\n- $B$ is strongly $\mathbb{N}$-graded as an algebra;
- $m_B^{1,1} \left( c_B^{1,1} - q \text{Id}_B \right) = 0$ for some $q \in K$ (e.g. $c_B^{1,1}$ has minimal polynomial of degree 1);
- $q$ is regular.

Then the canonical projection $\pi_S : B \to S(B)$ is an isomorphism so that $\text{sdeg}(B) = 0$.

Proof. By using the definition of $\Gamma_{n,1}^B$, (33), and the condition $m_B^{1,1} c_B^{1,1} = q m_B^{1,1}$, we arrive at
\n$$\Gamma_{n,1}^B m_B^{n,1} = m_B^{n,1} + q m_B^{n,1} \left( \Gamma_{n-1,1}^B \otimes B^1 \right) .$$
\nInductively, using (34) and the surjectivity of $m_B^{n,1}$ we obtain
\n$$\Gamma_{n,1}^B = (n+1) q \text{Id}_{B^{n+1}},$$
\nfor every $n \geq 1$.

Since $q$ is regular i.e. $(n)_q \neq 0, \forall n \geq 2$, we get that $\Gamma_{n,1}^B$ is bijective for every $n \geq 1$. Since, by definition, $\Gamma_{n,1}^B := m_B^{n,1} \Delta_{B}^n$, we find that $\Delta_{B}^n$ is injective for every $n \geq 1$ and hence for every $n \in \mathbb{N}$ $(\varepsilon_B^0 \otimes B^1) \Delta_{B}^0 = \text{Id}_{B^1}$ is bijective. By Theorem 1.3, the canonical projection $\pi_S : B \to S(B)$ is an isomorphism so that $\text{sdeg}(B) = 0$.

Theorem 4.13. Let $(B, m_B, u_B, \Delta_B, \varepsilon_B, c_B)$ be a connected graded braided bialgebra. Assume that
• \(m_{B_1}^{1.1}\) is an isomorphism.
• \(c_B^{1.1}\) has minimal polynomial \(f(X) \in K[X]\) where

\[f(-1) = 0\] i.e. \(f(X) = (X + 1)h(X)\) for some \(h \in K[X]\).

Then \(sdeg(B) = sdeg(S(B)) + 1\) and \(m_{S(B)}^{1.1}h(c_{S(B)}^{1.1}) = 0\).

**Proof.** Note that, if \(sdeg(B) = 0\), then \(\Delta_B^{1.1}\) is injective and hence \(c_B^{1.1} + \text{Id}_{B^2} = \Delta_B^{1.1}m_B^{1.1}\) is injective too. In this case \(f(c_B^{1.1}) = 0\) imply \(h(c_B^{1.1}) = 0\) so that \(f\) is not the minimal polynomial for \(c_B^{1.1}\), a contradiction. We have so proved that \(sdeg(B) \neq 0\). Thus \(sdeg(B) = sdeg(S(B)) - 1\).

By (23), we have \(c_S^{1.1} (\pi_S \otimes \pi_S) = (\pi_S \otimes \pi_S) c_B^{1.1}\). Since \(\pi_S\) is bijective, we infer that \(c_S^{1.1}\) has minimal polynomial \(f\). Thus \(0 = f(c_S^{1.1}) = (c_S^{1.1} + \text{Id}_{S^2}) h(c_S^{1.1}) = \Delta_S^{1.1}m_S^{1.1}h(c_S^{1.1})\). By Proposition A.2, \(\Delta_S^{1.1}\) is injective whence \(m_S^{1.1}h(c_S^{1.1}) = 0\). \(\square\)

5. Braided Lie algebras

Although many result in this section could be obtained for a general graded braided bialgebra \(B\), we focus here our attention on the case \(B = T(V,c)\), the tensor algebra of a braided vector space \((V,c)\).

5.1. Let \((V,c)\) be a braided vector space and let \(T = T(V,c)\). By the universal property of the tensor algebra there is a unique algebra homomorphism

\[\Gamma^T : T(V,c) \to T^c(V,c)\]

such that \(\Gamma^T_0 = \text{Id}_V\), where \(T^c(V,c)\) denotes the quantum shuffle algebra (see A.4). This is a morphism of graded braided bialgebras. By [Schl page 2815] (see also [AG1 page 25-26]), it is clear that the \(n\)-th graded component of \(\Gamma^T\) is the map \(\Gamma_n^T : V^\otimes n \to V^\otimes n\) in the sense of Definition 1.6 (see Remark 1.7). The bialgebra of Type one generated by \(V\) over \(K\) (or Nichols algebra) is by definition

\[B(V,c) = \text{Im}(\Gamma^T) \simeq \frac{T(V,c)}{\ker(\Gamma^T)}\]

We set

\[E_n(V,c) := E_n(T(V,c))\quad\text{and}\quad E(V,c) := E(T(V,c))\]

where we used the notations of Definition 5.3.

**Definition 5.2.** A *(braided)* bracket on a braided vector space \((V,c)\) is a \(K\)-linear map \(b : E(V,c) \to V\) such that

\[c(b \otimes V) = (V \otimes b)c_{E(V,c),V}\quad c(V \otimes b) = (b \otimes V)c_{V,E(V,c)}\]

The restriction of \(b\) to \(E_t(V,c)\) will be denoted by \(b_t : E_t(V,c) \to V\). A morphism of (braided) brackets \(f : (V,c_{V,V},b_V) \to (W,c_{W,W},b_W)\) is a morphism of braided vector spaces \(f : (V,c_{V,V}) \to (W,c_{W,W})\) such that

\[f \circ b_V = b_W \circ E_t(f)\]

**Lemma 5.3.** Let \((V,c)\) be a braided vector space. The following assertions are equivalent for a \(K\)-linear map \(b : E(V,c) \to V\):

(i) \(b\) is a bracket for the graded braided bialgebra \(T(V,c)\);

(ii) \(b\) is a bracket on \((V,c)\).

**Proof.** (i) \(\Rightarrow\) (ii) It is trivial.

(ii) \(\Rightarrow\) (i) Set \(T := T(V,c)\). Let us prove, by induction on \(t \geq 1\), that \(b\) fulfills (24) i.e. that

\[c_T^{1,t}(b \otimes V^\otimes t) = (V^\otimes t \otimes b)c_{E(V,c),V^\otimes t}\]

\[c_T^{1,t}(V^\otimes t \otimes b) = (b \otimes V^\otimes t)c_{V^\otimes t,E(V,c)}\]
For \( t = 1 \) there is nothing to prove as \( c_{T}^{1} = c \). Assume that the formulas are true for \( t - 1 \). Then, by construction of \( c_{T} \), we have

\[
c_{T}^{1} (b \otimes V^{\otimes t}) = \left( V^{\otimes t-1} \otimes c_{T}^{1} \right) \left( c_{T}^{1} \otimes V^{\otimes t-1} \otimes V \right) (b \otimes V^{\otimes t-1} \otimes V)
\]

\[
= \left( V^{\otimes t-1} \otimes c_{T}^{1} \right) \left( V^{\otimes t-1} \otimes b \otimes V \right) \left( c_{E(V,c),V^{\otimes t-1}} \otimes V \right)
\]

\[
= \left( V^{\otimes t-1} \otimes V \otimes b \right) \left( V^{\otimes t-1} \otimes c_{E(V,c),V} \right) \left( c_{E(V,c),V^{\otimes t-1}} \otimes V \right)
\]

and similarly we get the second equality. It remains to prove that \( b \) fulfills (28) i.e. that \( \Delta_{T}^{0} b = (b \otimes K) \Delta_{T}^{0,0} \) and \( \Delta_{T}^{b} b = (K \otimes b) \Delta_{T}^{0,1} \) but these equalities are trivially true by the definition of \( \Delta_{T} \).

**Example 5.4.** Let \( b \) be a bracket on a braided vector space \((V,c)\). In view of Lemma 5.3, it makes sense to define the **universal enveloping algebra** of \((V,c,b)\) to be

\[
U(V,c,b) := U(T(V,c),b) = \frac{T(V,c)}{(\text{Id} - b) [E(V,c)]}
\]

which, by Theorem 5.9, is a braided bialgebra quotient of the tensor algebra \( T(V,c) \). The **symmetric algebra** of \((V,c)\) is then

\[
S(V,c) := U(V,c,0) = U(T(V,c),0) = S(T(V,c))
\]

(see (28)).

We will denote by \( \pi_{U} : T(V,c) \to U(V,c,b) \) and \( \pi_{S} : T(V,c) \to S(V,c) \) the canonical projections.

**Remark 5.5.** Since \( \pi_{U} \) (resp. \( \pi_{S} \)) is an epimorphism, then \( U(V,c,b) \) (resp. \( S(V,c) \)) is a connected coalgebra (cf. [Mo, Corollary 5.3.5]).

**Definition 5.6.** We say that \((V,c,b)\) is a **braided Lie algebra** whenever

- \((V,c)\) is a braided vector space;
- \( b : E(V,c) \to V \) is a bracket on \((V,c)\);
- the canonical \( K \)-linear map \( i_{U} : V \to U(V,c,b) \) is injective i.e. \( V \cap \ker(\pi_{U}) = \ker(i_{U}) = 0 \).

Injectivity of \( i_{U} \) plays here the role of an implicit Jacobi identity. On the other hand, antisymmetry of the bracket is encoded in the choice of the domain of \( b \). This becomes clearer when \( c \) is of Hecke type as in Theorem 5.1. There it is shown that \( b \) can be substituted with the \( c \)-antisymmetric map \( [-]_{b} : V \otimes V \to V \) defined by setting \( [z]_{b} = b(c - q\text{Id}_{V \otimes V})(z) \), for every \( z \in V \otimes V \).

A first example of braided Lie algebra is given in the following Proposition. A large class of examples with not necessarily trivial brackets will be included in Proposition 5.8 or Theorem 5.10.

**Proposition 5.7.** Let \((V,c)\) is a braided vector space. Then \((V,c,0)\) braided Lie algebra.

**Proof.** We already observed that \( 0 \) is always a braided bracket for \((V,c)\) whence it remains to prove that \( i_{U} \) is injective. Set \( A := \frac{T(V,c)}{(V \otimes V)} \) where \( (V \otimes V) \) is the two sided ideal of \( T(V,c) \) generated by \( V \otimes V \). Denote by \( \pi : T(V,c) \to A \) and \( \pi_{S} : T(V,c) \to S(V,c) \) the canonical projections. Since \( E(V,c) \subseteq (V \otimes V) \), then \( \pi : T(V,c) \to A \) quotients to an algebra homomorphism \( \pi : S(V,c) = U(V,c,0) \to A \) such that \( \pi \circ \pi_{S} = \pi \). Hence \( \pi \circ i_{U} = \pi \circ \pi_{S} \circ i_{T} = \pi \circ i_{T} \) where \( i_{T} : V \to T(V,c) \) is the canonical injection. Note that as a vector space \( A \cong K \oplus V \) and through this isomorphism, \( \pi \circ i_{T} \) identifies with the canonical injection of \( V \) in \( K \oplus V \) whence \( i_{U} \) is injective too.

**Proposition 5.8.** Let \((A,c_{A})\) be a braided algebra. Then \((A,c_{A},b_{A})\) is a braided Lie algebra, where \( b_{A}(z) := m_{A}^{-1}(z) \), for every \( z \in E_{l}(A,c_{A}) \).
Proof. Let us check that $b_A$ is a $c_A$-bracket on the braided vector space $(A,c_A)$. For every $z \in E_t(A,c_A)$ and $x \in A$, we have
\[
c_A\left(b_A^t \otimes A\right)(z \otimes x) = c_A\left(m_A^{-1} \otimes A\right)(z \otimes x) = (A \otimes m_A^{-1})c_E^{1}\left(z \otimes x\right) = (A \otimes b_A^t)c_{E_t(A,c_A),A}(z \otimes x),
\]
so that the left-hand side of (36) is proved. Similarly one proves also the right-hand side. By the universal property of $T(A,c_A)$ (see [AMS1, Theorem 1.17]), $Id_A$ can be lifted to a unique braided algebra homomorphism $\varphi_A : T(A,c_A) \to A$.

For every $z \in E_t(A,c_A)$, we have $\varphi_A b_A(z) = \varphi_A m_A^{-1}(z) = m_A^{-1}(z) = \varphi_A (z)$ so that $\varphi_A$ quotients to a morphism $\varphi_A : U(A,c_A,b_A) \to A$. If we denote by $i_U : P \to U(A,c_A,b_A)$ the canonical map, we get that $\varphi_A \circ i_U = Id_A$. In particular $i_U$ is injective whence $(A,c_A,b_A)$ is a braided Lie algebra.

**Lemma 5.9.** Let $(V,c_V,b_V)$ a braided Lie algebra and let $h : (W,c_W) \to (V,c_V,b_V)$ be an injective morphism of braided vector spaces. If there exists a $K$-linear map $b_W : E(W,c_W) \to W$ such that $h \circ b_W = b_V \circ E(h)$, then $(W,c_W,b_W)$ is a braided Lie algebra too.

Proof. We have
\[
h \otimes h) c_W(b_W \otimes W) = c_V(h \otimes h)(b_W \otimes W) = c_V(b_W \otimes V)(E(h) \otimes h)
\]
\[
= (V \otimes b_V)c_{E(V,c_V),V}(E(h) \otimes h) = (V \otimes b_V)(h \otimes E(h))c_{E(W,c_W),W}
\]
\[
= (h \otimes h)(W \otimes b_W)c_{E(W,c_W),W}.
\]

Since $h$ is injective, we deduce that $c_W(b_W \otimes W) = (W \otimes b_W)c_{E(W,c_W),W}$ so that the left-hand side of (36) is proved for $(W,c_W)$. Similarly one proves also the right-hand side. If we denote by $i_U^W : W \to U(W,c_W,b_W)$ and $i_U^V : V \to U(V,c_V,b_V)$ the canonical maps, by the universal property of the universal enveloping algebra, there is a unique algebra homomorphism $U(h) : U(W,c_W) \to U(V,c_V)$ such that $U(h) \circ i_U^W = i_U^V \circ h$. Since both $i_U^W$ and $h$ are injective, we deduce that $i_U^W$ is injective too. Therefore $(W,c_W,b_W)$ is a braided Lie algebra.

**Theorem 5.10.** Let $(A,c_A)$ be a connected braided bialgebra. Let $P = P(A)$ be the space of primitive elements of $A$. Then
\[
c_A(P \otimes P) \subseteq P \otimes P.
\]
Let $c_P : P \otimes P \to P \otimes P$ be the restriction of $c_A$ to $P \otimes P$. Then
i) $m_A^{-1}(E_t(P,c_P)) \subseteq P$ for every $t \geq 2$, so we can define $b_P : E(P,c_P) \to P$ by $b_P(z) := m_A^{-1}(z)$, for every $z \in E_t(P,c_P)$.
ii) $(P,c_P,b_P)$ is a braided Lie algebra.
iii) Every morphism of braided brackets $f : (V,c_V,b_V) \to (P,c_P,b_P)$ can be lifted to a morphism of braided bialgebras $\overline{f} : U(V,c_V,b_V) \to A$.

Proof. The first assertion follows by [AMS1, Remark 1.12]. Set $T := T(P,c_P)$.

i) Since $T$ is connected, by Lemma 5.3 we have $P(T) = P_1(T) = T_1 \oplus [\oplus_{n \geq 2}E_n(P,c_P)]$. Thus $E_t(P,c_P) \subseteq P(T)$ for every $t \geq 2$. Let $\varphi_A : T \to A$ be the natural braided bialgebra homomorphism arising from the universal property of the tensor algebra. Then, for every $t \geq 2$, we have $m_A^{-1}(E_t(P,c_P)) = \varphi_A(E_t(P,c_P)) \subseteq \varphi_A(P(T)) \subseteq P(A) = P$.

ii) Let $b_P$ be defined as in the statement and denote by $i_A : P \to A$ the canonical inclusion. Then $i_A \circ b_P = b_A \circ E(i_A)$, where $b_A$ is the map defined in Proposition 5.8. By this proposition and Lemma 5.9 we have that $(P,c_P,b_P)$ is a braided Lie algebra.

iii) By the universal property of $T(V,c_V,b_V)$ (see [AMS1, Theorem 1.17]), $f$ can be lifted to a unique braided bialgebra homomorphism $f' : T(V,c_V,b_V) \to A$. We have
\[
f' \circ b_V(z) = f' b_V(z) = f' \circ b_P E_t(f)(z) = m_A^{-1}f \circ E_t(z) = f'(z)
\]
for every $z \in E_t(V,c_V,b_V)$. Therefore $f'$ quotients to a morphism $\overline{f} : U(V,c_V,b_V) \to A$. 

\[
\]
DEFINITION 5.11. With the same assumptions and notations of Theorem 5.10, \((P, cp, bp)\) will be called the infinitesimal braided Lie algebra of \(A\).

THEOREM 5.12. Let \((A, c_A)\) be a connected braided bialgebra and let \((P, cp, bp)\) be its infinitesimal braided Lie algebra. Then, every morphism of braided vector spaces \(f : (V, cv, V) \to (P, cp)\) such that \(m_A^{t-1} \varepsilon_t (f) = 0\), for every \(t \geq 2\), can be lifted to a morphism of braided bialgebras \(\mathcal{T} : S (V, cv, V) \to A\) which respects the gradings on \(S (V, cv, V)\) and \(A\) whenever \(A\) is graded.

Proof. By Proposition 5.7, we know that \((V, cv, V)\) is a braided Lie algebra for \(b_V = 0\). Let us prove that \(f : (V, cv, V) \to (P, cp, bp)\) is a morphism of braided Lie algebras. Since \(b_p\) acts like \(m_A^{-1}\), by hypothesis we get \(b_p \circ \varepsilon_t (f) = f \circ b_p\). Hence, we can apply Theorem 5.10. \(\square\)

5.13. Let \((V, c)\) be a braided vector space and let \(b : E (V, c) \to V\) be a \(c\)-bracket. Set \(T := T (V, c)\) and \(U := U (V, c, b)\).

Let \(\pi_U : T \to U\) be the canonical projection. By construction \(\pi_U\) is a morphism of braided bialgebras. Mimicking [AMS2, 4.11], set \(T(n) := \oplus_{0 \leq t \leq n} V^{\otimes t}\) and

\[
U'_n := \pi_U (T(n)).
\]

Then \((U'_n)_{n \in \mathbb{N}}\) is both an algebra and a coalgebra filtration on \(U\) which is called the standard filtration on \(U\). Note that this filtration is not the coradical filtration \((U_n)_{n \in \mathbb{N}}\) in general. Still one has \(U'_n = \pi_U (T(n)) \subseteq \pi_U (T_n) \subseteq U_n\) where \(T_n\) and \(U_n\) denote the \(n\)-th terms of the coradical filtration of \(T\) and \(U\) respectively. Denote by

\[
\text{gr}' (U) := \oplus_{n \in \mathbb{N}} \frac{U'_n}{U'_{n-1}}.
\]

the graded coalgebra associated to the standard filtration (see [SW, page 228]).

If \(b = 0\), then \(S (V, c) = U (V, c, 0)\) is a graded bialgebra \(S (V, c) = \oplus_{n \in \mathbb{N}} S^n (V, c)\). The standard filtration on \(S (V, c)\) is the filtration associated to this grading.

The following result is inspired to [AMS2, Proposition 4.19].

PROPOSITION 5.14. Let \((V, c)\) be a braided vector space and let \(b : E (V, c) \to V\) be a \(c\)-bracket. Then \(\text{gr}' (U (V, c, b))\) is a graded braided bialgebra and there is a canonical morphism of graded braided bialgebras \(\theta : S (V, c) \to \text{gr}' (U (V, c, b))\) which is surjective and lifts the map \(\theta_1 : V \to U_1/U_0 : v \mapsto \pi_U (v) + U_0\).

Proof. Set \(T := T (V, c), U := U (V, c, b), G := \text{gr}' (U (V, c, b)), G^n := U'_n/U'_{n-1}\) and let \(p_n : U'_n \to G^n\) be the canonical projection, for every \(n \in \mathbb{N}\). We have

\[
c_U (U'_a \otimes U'_b) = c_U (\pi_U \otimes \pi_U) (T(a) \otimes T(b)) = (\pi_U \otimes \pi_U) c_T (T(a) \otimes T(b))
\]

\[
\subseteq (\pi_U \otimes \pi_U) (T(b) \otimes T(a)) = U'_a \otimes U'_b.
\]

Hence \(c_U\) induces a braiding \(c_G : G \otimes G \to G \otimes G\). Now, we have already observed that \(G\) carries a coalgebra structure \((G, \Delta_G, \varepsilon_G)\). Since \((U'_n)_{n \in \mathbb{N}}\) is also an algebra filtration on \(U\), the multiplication of \(U\) induces, for every \(a, b \in \mathbb{N}\), maps \(m_G^{a,b} : G^a \otimes G^b \to G^{a+b}\) and the unit of \(U\) induces a map \(u_G^0 : K \to G^0\) such that (1) and (1) hold for "\(A\)" = \(G\). Thus there is a unique map \(m_G : G \otimes G \to G\) such that \(m_G (i_G^a \otimes i_G^b) = i_G^{a+b} m_G^{a,b}\) for every \(a, b \in \mathbb{N}\), where \(i_G^0 : G^0 \to G\) is the canonical map. Moreover \((G, m_G, u_G = i_G^0 u_G^0)\) is a graded algebra (see e.g. [AM, Proposition 3.4]).

It is straightforward to prove that \((G, m_G, u_G, \Delta_G, \varepsilon_G, c_G)\) is indeed a graded braided bialgebra.

Set \(P := P (G)\). Since \(G\) is a connected graded coalgebra, it is clear that \(\text{Im} (\theta_1) \subseteq G^1 \subseteq P\). Moreover \(\theta_1 : (V, c) \to (P, cp)\) is a morphism of braided vector spaces as, for every \(u, v \in V\), we have

\[
c_G (\theta_1 \otimes \theta_1) (u \otimes v) = c_G [(\pi_U (u) + U'_0) \otimes (\pi_U (v) + U'_0)]
\]

\[
= (p_1 \otimes p_1) c_U (\pi_U (u) \otimes \pi_U (v))
\]

\[
= (p_1 \otimes p_1) c_U (\pi_U \otimes \pi_U) (u \otimes v)
\]

\[
= (p_1 \otimes p_1) (\pi_U \otimes \pi_U) c_T (u \otimes v) = (\theta_1 \otimes \theta_1) c (u \otimes v).
\]
Let us prove that \( m_{G}^{t-1}E_t(\theta_1) = 0 \), for every \( t \geq 2 \). For every \( t \geq 2 \) and \( z \in E_t(V,c) \), we have
\[
m_{G}^{t-1}E_t(\theta_1)(z) = m_{G}^{t-1}\overline{\theta}^t(\pi_U(z)) = p_t\pi_Um_{G}^{t-1}(z) = p_t\pi_Um_{G}^{t-1}(z)
\]
so that \( m_{G}^{t-1} \circ E_t(\theta_1) = 0 \). By Theorem 5.13, there is a canonical morphism of graded braided bialgebras \( \theta : S(V,c) \to G \) lifting \( \theta_1 \). By construction \( \theta \) lifts the canonical map \( T(V,c_{V}) \to G \) which is surjective as \( \theta_1 \) is surjective and \( G \) is generated as an \( K \)-algebra by \( G^1 \). Thus \( \theta \) is surjective too.

**Definition 5.15.** Let \((V,c)\) be a braided vector space and let \( b : E(V,c) \to V \) be a \( c \)-bracket. Following [BC], Definition, page 316, we will say that \( U(V,c,b) \) is **Poincaré-Birkhoff-Witt (PBW) type** whenever the projection \( \theta : S(V,c) \to gr'(U(V,c,b)) \) of Proposition 5.14 is an isomorphism (compare with [Hu, page 92] for justifying this terminology).

### 6. Strongness degree of a braided vector space

This section mainly concerns the characterization of braided vector spaces of strongness degree not greater than 1 (see the following definition). As we will see in the sequel, braided vector spaces with braiding of Hecke type of regular mark \( q \) (Theorem 8.3) and braided vector spaces whose Nichols algebra is a quadratic algebra (Theorem 9.3) are examples of such a braided vector space.

**Definition 6.1.** Let \((V,c)\) be a braided vector space. The **strongness degree** of \((V,c)\) is defined to be
\[
sdeg(V,c) := sdeg(T(V,c)) .
\]
The following result shows how the investigation of braided vector spaces with finite strongness degree fits into the classification of finite dimensional Hopf algebras problem. In fact, in this direction, one of the steps of the celebrated lifting method by Andruskiewitsch and Schneider (see e.g. [ASC]) is the classification of finite dimensional Nichols algebras.

**Theorem 6.2.** Let \((V,c)\) be a braided vector space such that the Nichols algebra \( B(V,c) \) is finite dimensional. Then \( sdeg(V,c) \leq \deg(\mathfrak{h}(B(V,c))) - 1 \leq \dim_K B(V,c) - 1 \).

**Proof.** Observe that the tensor algebra \( T(V,c) \) is strongly \( \mathbb{N} \)-graded as an algebra. Since \( B(V,c) \) identifies with \( T^0(T^1) \) i.e. the braided bialgebra of type associated to \( T^0 \) and \( T^1 \), the conclusion follows by applying Corollary [A,11] to the case \( B = T \).

**Theorem 6.3.** Let \((V,c)\) be a braided vector space and let \( b : E(V,c) \to V \) be a braided bracket. Assume that \( sdeg(V,c) \leq 1 \). Then, the following assertions are equivalent.

(i) \((V,c,b)\) is a braided Lie algebra in the sense of Definition 5.6.

(ii) \( V \cap \ker(\pi_U) = 0 \).

(iii) The map \( \theta_1 : V \to U_1'/U_0' : v \mapsto \pi_U(v) + U_0' \) is injective.

(iv) \((V,c,b)\) is of PBW type in the sense of Definition 5.14.

(v) \( \iota_U : V \to U(V,c,b) \) induces an isomorphism between \( V \) and \( P(U(V,c,b)) \).

**Proof.** Set \( T := T(V,c) \), \( S := S(V,c) \) and \( U := U(V,c,b) \). Let \( p_t : U_t'/U_0' \to U_t'/U_{t-1}' \) be the canonical projection and let \( i : V \to U_1' \) be the corestriction of \( i_U \). Let \( \theta_1 : V \to U_1'/U_0' : v \mapsto \pi_U(v) + U_0' \). Then \( p_t \circ i = \theta_1 \). Let us prove that
\[
\ker(\theta_1) = V \cap \ker(\pi_U).
\]
Let \( x \in \ker(\theta_1) \). Then \( \pi_U(x) \in U_0' = \pi_U(K) \) so that there exists \( k \in K \) such that \( \pi_U(x) = \pi_U(k) \). From this we get \( k = e_Ux = e_U\pi_U(k) = e_U\pi_U(x) = x_U(x) = 0 \) where the last equality holds as \( V \subseteq P(T) \). Thus \( \pi_U(x) = 0 \) and hence \( x \in V \cap \ker(\pi_U) \). The other inclusion is trivial.

(i) \( \iff \) (ii) It follows by Definition 5.6.

(ii) \( \iff \) (iii) It follows trivially by (i).

(iii) \( \Rightarrow \) (iv) Since \( sdeg(V,c) \leq 1 \), we have that \( S(V,c) \) is strongly \( \mathbb{N} \)-graded as a coalgebra so that (cf. Theorem 1.3) \( P(S) = V \). Therefore, by [Mo, Lemma 5.3.3], we obtain that \( \theta \) is injective.
as it is injective its restriction \( \theta_1 \) to \( P(S(V,c)) \). In conclusion the surjective map \( \theta \) is indeed bijective.

\[ (iv) \Rightarrow (v) \] Let \( z \in P(U). \) Since \( z \in U \), there is \( n \in \mathbb{N} \) such that \( z \in U_n^{i} \setminus U_{n-1}^{i}. \) Let \( \mathfrak{z} := z + U_{n-1}^{i} \). By definition of the bialgebra structure of \( G := \text{gr}^i(U(V,c,b)) \) one has that \( \mathfrak{z} \in P(G). \) By \( (iv) \), we have \( P(G) = \theta_1(P(S)) = \theta_1(V) = U^i/P^i \). Thus \( \mathfrak{z} \in (U_n^i/U_{n-1}^i) \cap (U^i/P^i) \). Since \( \mathfrak{z} \neq 0 \), we get \( n = 1 \). Hence \( z \in U_n^i = U_1 = \pi_U(T(1)) = \pi_U(K \oplus V) \). Therefore, there are \( k \in K \) and \( v \in V \) such that \( z = \pi_U(k) + \pi_U(v) \) so that \( k = \varepsilon_U \pi_U(k) - \varepsilon_U \pi_U(v) = \varepsilon_U(z) - \varepsilon_U(v) = 0 \) where the last equality holds as \( z \in P(U) \) and \( v \in P(T) \). Then \( z = \pi_U(k) + \pi_U(v) = \pi_U(v) = \pi_U(v) \). We have so proved that \( P(U) \subseteq \text{Im}(i_U) \). The other inclusion is trivial so that \( \text{Im}(i_U) = P(U) \).

It remains to prove that \( i_U \) is injective or, equivalently, that \( i \) is injective. Recall that, by Proposition 5.14, the map \( \theta \) lifts the map \( \theta_1 \).

Since the canonical map \( V \to S \) is injective (cf. [5.4]) and since, by hypothesis, \( \theta \) is injective we infer that \( \theta_1 \) and hence \( i \) is injective too.

\[ (v) \Rightarrow (i) \] By hypothesis, \( i_U \) is injective. \( \square \)

**Definition 6.4.** A connected braided bialgebra \((A,c_A)\) is called **primitively generated** if it is generated as a \( K \)-algebra by its space \( P(A) \) of primitive elements.

The following is one of the main results of this section.

**Theorem 6.5.** Let \((A,c_A)\) be a primitively generated connected braided bialgebra and let \((P,c_P,b_P)\) be its infinitesimal braided Lie algebra. If \( \text{sdeg}(P,c_P) \leq 1 \), then \( A \) is isomorphic to \( U(P,c_P,b_P) \) as a primitively generated bialgebra.

**Proof.** Set \( U := U(P,c_P,b_P) \). By Theorem 5.10, the identity map of \( P \) can be lifted to a morphism of braided bialgebras \( \alpha : U \to A \). Since \( P \) generates \( A \) as a \( K \)-algebra, this morphism is surjective. On the other hand, by Theorem 5.3, \( i_U : P \to U \) induces an isomorphism between \( V \) and \( P(U) \). Hence the restriction of \( \alpha \) on \( P(U) \) is injective. By [Md, Lemma 5.3.3] \( \alpha \) is injective. \( \square \)

7. Some braided vector space of strongness degree not greater the one

In this section meaningful examples of braided vector space of strongness degree not greater the one are given. We also illustrate an example of a braided vector space of strongness degree two.

**Theorem 7.1.** Let \( L \subseteq E(V,c) \) and let \( R = T(V,c)/(L) \). If \( R \) inherits from \( T(V,c) \) a braided bialgebra structure such that the space \( P(R) \) of primitive elements of \( R \) identifies with the image of \( V \) in \( R \), then \( S(V,c) = R \) and \( \text{sdeg}(V) \leq 1 \).

**Proof.** Denote by \( \tau : R \to S := S(V,c), \pi_S : T(V,c) \to S \) and \( \pi_R : T(V,c) \to R \) the canonical projections. Since \( \tau \pi_R = \pi_S \) and both \( \pi_R \) and \( \pi_S \) are coalgebra homomorphism, then \( \tau \) is a coalgebra homomorphism too. By [Md, Lemma 5.3.3], we obtain that \( \tau \) is injective as its restriction to \( P(R) \) is injective (note that the map \( i_S : V \to S(V,c) \) is injective by Proposition 5.4). Hence \( \tau \) is an isomorphism so that \( S(V,c) = R \). Therefore \( P(S) \) identifies with \( V \), whence \( S \) is strongly \( N \)-graded as a coalgebra which means \( \text{sdeg}(V) \leq 1 \). \( \square \)

**Definition 7.2.** Recall that a braided vector space \((V,c)\) of diagonal type is there is a basis \( x_1, \ldots, x_n \) of \( V \) over \( K \) and a matrix \((q_{i,j}) \), \( q_{i,j} \in K \setminus \{0\} \), such that \( c(x_i \otimes x_j) = q_{i,j} x_j \otimes x_i, 1 \leq i,j \leq n \).

**Theorem 7.3.** Let \((V,c)\) be a braided vector space of diagonal type such that \( B(V,c) \) is a domain and its Gelfand-Kirillov dimension is finite. Then \( \text{sdeg}(V,c) \leq 1 \).

**Proof.** By hypothesis there is a basis \( x_1, \ldots, x_n \) of \( V \) over \( K \) and a matrix \((q_{i,j}) \), \( q_{i,j} \in K \setminus \{0\} \), such that \( c(x_i \otimes x_j) = q_{i,j} x_j \otimes x_i, 1 \leq i,j \leq n \). By [AA, Corollary 3.11] (which was obtained from results of Lusztig and Rosso), \( R := B(V,c) \) as a graded braided bialgebra divides out \( T(V,c) \) by the ideal generated by set \( L \) whose elements are \( z_{i,j} := (ad_{x_j})^{m_{i,j}+1}(x_j), 1 \leq i,j \leq n \), where \( i \neq j \) and \( m_{i,j} \) is a suitable non-negative integer. Since \( z_{i,j} \) is primitive in \( T(V,c) \) [see e.g. [Kh1, Theorem 6.1], where \( z_{i,j} \) is denoted by \( W(x_j,x_i) \) in formulae (18) and (19), or see [AS2, Lemma A.1]) and of degree at least two, we get \( L \subseteq E(V,c) \). By Theorem 7.3, \( \text{sdeg}(V,c) \leq 1 \). \( \square \)
Example 7.4. Assume $K$ is algebraically closed of characteristic 0. Let $V$ be a $n$-dimensional vector space with basis $x_1, \ldots, x_n$. Let $q_{i,j} \in K, i,j \in \{1, \ldots, n\}$, be such that $q_{i,i} \neq 1$ are $N_i$-roots of unity $(N_i > 1)$ and $q_{i,j}q_{j,i} = 1$ for $i \neq j$. Define $c : V \otimes V \to V \otimes V$ by setting $c(x_i \otimes x_j) := q_{i,j}x_j \otimes x_i$ (whence $c$ is of diagonal type). One checks that $c$ fulfills $\{\}$. By means of the quantum binomial formula (see [Asc3, Lemma 3.6]) one gets that $x_i \otimes x_j - q_{i,j}x_j \otimes x_i \in E_2(V,c)$ for $i \neq j$ and $x_i^{\otimes N_i} \in E_{N_i}(V,c)$. Consider the quantum linear space $R = K \times x_1, \ldots, x_n | x_i^{N_i} = 0; \ldots; x_i^{N_i} = 0; x_i \cdot x_j = q_{i,j}x_j \cdot x_i >$. By Theorem 7.1, $S(V,c) = R$ and sdeg $(V,c) \leq 1$. Since $S(V,c) \neq T(V,c)$ it is clear that sdeg $(V,c) = 1$.

Remark 7.5. Let $(V,c)$ be a braided vector space of diagonal type as in Definition 7.2. By [Kh2, Theorem 1], the space $E_n(V,c)$ contains an element of the form $\sum \lambda_\sigma x_\sigma(t_1) \otimes \cdots \otimes x_\sigma(t_m)$ with $0 \leq t_1 < \cdots < t_m \leq n, \lambda_\sigma \in K$, if and only if

$$\prod_{1 \leq i \neq j \leq n} q_{i,j}t_j = 1.$$ 

Furthermore, if this condition holds, then $(n - 2)! \leq \dim K (E_n(V,c))$.

For instance, in the previous example, condition $q_{i,j}q_{j,i} = 1$ is necessary to have that $x_i \otimes x_j - q_{i,j}x_j \otimes x_i \in E_2(V,c)$ for $i \neq j$.

Definition 7.6. [Asc3, Definition 1.6] Let $(V,c)$ be a braided vector space. Recall that $(V,c)$ is of group type if $V$ has a basis $x_1, \ldots, x_n$ such that $c(x_i \otimes x_j) = \gamma_i(x_j) \otimes x_i$ for some $\gamma_1, \ldots, \gamma_n \in GL(V)$. If in addition the subgroup $G(V,c)$ of $GL(V)$ generated by $\gamma_1, \ldots, \gamma_n$ is abelian we will say that $(V,c)$ is of abelian group type.

Proposition 7.7. Let $(V,c)$ be a two dimensional braided vector space of abelian group type. Suppose that $\dim K S(V,c) < 32$. Then sdeg $(V,c) \leq 1$.

Proof. We have two graded braided bialgebra homomorphisms $\pi_S : T(V,c) \to S(V,c)$ and $\pi^- : S(V,c) \to B(V,c)$. By a famous Takeuchi’s result (see [M], Lemma 5.2.10), these bialgebras are indeed Hopf algebras whence $\pi_S$ and $\pi^- S$ are fact graded braided Hopf algebra homomorphisms. It is well known that $V$ can be regarded as an object in the monoidal category $\mathcal{M}$ of Yetter-Drinfeld modules over $G(V,c)$. Furthermore $c$ coincides with the braiding of $V$ in the category. Clearly $T(V,c)$ and $B(V,c)$ are Hopf algebras in $\mathcal{M}$ and the same is true for $S(V,c)$ as $E(V,c)$ is an object in $\mathcal{M}$. Furthermore $\pi_S$ and $\pi^- S$ are morphism in $\mathcal{M}$. Since $B(V,c)$ is a quotient of $S(V,c)$, one has $\dim K B(V,c) \leq \dim K S(V,c) < 32$. By [G3, Lemma 6.2], we have that $\pi^- S$ is an isomorphism i.e. $\mathrm{sdeg}(V,c) \leq 1$.

Lemma 7.8. Let $(V,c)$ be a braided vector space. Then $I = (E(V,c))$ is a graded ideal of $T(V,c)$ with graded component $I_n := (E(V,c)) \cap V^{\otimes n}$. Moreover

$$I_n = I_{n-1} \otimes V + V \otimes I_{n-1} + E_n(V,c), \text{ for every } n \in \mathbb{N}.$$ 

Proof. For the first part see the proof of Theorem 3.3. The last part follows by induction on $n \in \mathbb{N}$. 

Theorem 7.9. Let $V \neq \{0\}$ be a $K$-vector space, let $q \in K \setminus \{0\}$ and set $c = q \mathrm{Id}_{V \otimes V}$. Then $(V,c)$ is a braided vector space and

i) $\mathrm{sdeg}(V,c) = 0$, if $q$ is regular.

ii) $\mathrm{sdeg}(V,c) = 1$, otherwise. In this case $q$ is a primitive $N$-th root of unity for some $N \geq 2$ and $S(V,c) = T(V,c) / (V^N)$.

Proof. Clearly $c : V \otimes V \to V \otimes V$ is a Yang-Baxter operator over $V$. 

i) It follows by Theorem 4.12.

ii) Set $T := T(V,c)$ and $S := S(V,c)$. By [B3], which holds for a not necessarily regular $q \in K$, we have $\Gamma^T_{1,1} = (n + 1) \mathrm{Id}_{B_{n+1}}$, for every $n \geq 1$. Using this equality and $\Gamma^T_n = (\Gamma^T_{n-1} \otimes V) \Gamma^T_{n-1,1}$, inductively one proves that

$$\Gamma^T_n = (n)_q \mathrm{Id}_{V^{\otimes n}}, \text{ for every } n \geq 2.$$
Now, since \( q \) is not regular, there is a minimal \( N \geq 2 \) such that \( \Gamma^T_N = 0 \). This entails that \((N)_q! = 0 \) but \((N - 1)_q! \neq 0 \) i.e. \( q \) is a primitive \( N \)-th root of unity.

Let \( I = (E(V,c)) \) and let \( I_n := (E(V,c)) \cap V^\otimes n \). Assume that
\[
\ker \left( \Gamma^T_n \right) \subseteq I_n \quad \text{for every } n \in \mathbb{N}.
\]
Since \( \Gamma^T : T(V,c) \to T^c(V,c) \) is a graded algebra homomorphism with \( n \)-th graded component \( \Gamma^T_n \) (see (1)), we get \( \Gamma^T(I_n) = \Gamma^T((E(V,c)) \cap V^\otimes n) \subseteq \Gamma^T((E(V,c)) \subseteq (\Gamma^T | E(V,c)) = 0 \) so that \( I_n \subseteq \ker (\Gamma^T_n) \) whence \( I_n = \ker (\Gamma^T_n) \). In this case
\[
S = \bigoplus_{n \in \mathbb{N}} \frac{V^\otimes n}{I_n} = \bigoplus_{n \in \mathbb{N}} \frac{V^\otimes n}{\ker (\Gamma^T_n)} = B(V,c).
\]
Moreover, in view of (13), we deduce
\[
\ker (\Gamma^T_n) = \begin{cases} 0, & \text{for every } 0 \leq n \leq N - 1, \\ V^n, & \text{for every } N \leq n \end{cases}
\]
whence \( B(V,c) = T(V,c) / (V^N) \). Since \( B(V,c) \) is strongly \( \mathbb{N} \)-graded as a coalgebra, we have that \( \text{sdeg} (V,c) = 1 \). It remains to prove that \( I_n = V^n \), for every \( N \leq n \). In view of (14), it suffices to prove that \( E_N(V,c) = V^N \). For every \( a, b \geq 1, a + b = N \), we have \( 0 = \Gamma^T_N = \Gamma^T_{a+b} \big( \Gamma^T_a \otimes \Gamma^T_b \big) \Gamma^T_{a,b} \). Since \( a, b \leq N - 1 \), then \( \Gamma^T_a \) and \( \Gamma^T_b \) are injective so that \( \Gamma^T_{a,b} = 0 \). Since \( n_{a,b}^T \) is an isomorphism, we get \( E_N(V,c) = V^N \).

Next we give some examples of braided vector spaces of strongness degree at least too. The most interesting of them are due to the work of V. K. Kharchenko on combinatorial rank (see Remark 4).

**Example 7.10.** From [AG1, Example 3.3.22], we quote the following example. Take \( H = K \mathbb{D}_4 \), where \( \mathbb{D}_4 = \{ \sigma, \rho \mid o(\sigma) = 2, o(\rho) = 4, \sigma \rho = \rho \sigma^{-1} \} \). Consider the module \( V_h \) with basis \( \{ z_0, z_1, z_2, z_3 \} \), with the structure given by
\[
\delta (z_i) = \rho^i \sigma \oplus z_i, \quad \rho^i \triangleright z_i = z_{i+2j}, \quad \sigma \triangleright z_i = -z_{-i}
\]
where we take subindexes of the \( z_i \) to be on \( \mathbb{Z} / 4 \mathbb{Z} \). The braiding is then given by \( c(z_i \otimes z_j) = -z_{2i-j} \otimes z_i \). Let \( T := T(V,c) \), let \( a := z_1 z_2 + z_0 z_1, b := z_1 z_0 + z_2 z_1 \) and let \( Z \) be the set consisting of the following elements:
\[
\begin{align*}
\{ z_0^2, z_1^2, z_2^2, z_3^2, z_0 z_2 + z_2 z_0, z_1 z_3 + z_3 z_1, \\
z_0 z_1 + z_2 z_2 + z_3 z_3 + z_3 z_0, z_0 z_3 + z_1 z_0 + z_2 z_1 + z_3 z_2
\}
\end{align*}
\]
(43)
(44)
(45)
Then the Nichols algebra \( B(V,c) \) comes out to be \( T / (Z) \). One can check that:
- \( E_2(V,c) \) is generated over \( K \) by the elements in (13) and (14);
- \( E_3(V,c) \) is generated over \( K \) by the elements \( z_1 z_2 + z_3 z_0, z_1 z_3 + z_3 z_1 \);
- \( E_4(V,c) = (V) \).

Thus, by (10) we obtain \( (E(V,c)) \cap V^\otimes 4 \subseteq E_2(V,c) \otimes V^\otimes 2 \otimes E_2(V,c) \otimes V^\otimes 2 \). Since \( a^2 = z_1 z_2 z_1 z_2 + z_1 z_2 z_0 z_1 + z_2 z_1 z_3 z_2 = z_2 z_3 z_2 z_3 + z_1 z_0 z_1 z_0 \) is not an element in the ideal generated by \( E_2(V,c) \), then \( \text{sdeg} (V,c) = \text{sdeg} (T) > 1 \). In fact the elements in (15) belongs to \( E_4(S(V,c)) \) so that we obtain \( \text{sdeg} (V,c) = 2 \).

**Example 7.11.** At the end of [K3], an example of a two dimensional braided vector space \( (V,c) \) of strongness degree 2 is given. The braiding \( c \) is of diagonal type of the form
\[
c(x_i \otimes x_j) = q_{i,j} x_j \otimes x_i, \quad 1 \leq i, j \leq 2,
\]
where \( x_1, x_2 \) is a basis of \( V \) over \( K \), \( q_{1,2} = 1 \neq -1 \) and \( q_{i,j} = -1 \) for all \( (i, j) \neq (1, 2) \).
Example 7.12. In [KA], one can find the strongness degree of a braided vector space with diagonal braiding of Cartan type $A_n$. That is, if $c(x_i \otimes x_j) = q_{i,j} x_j \otimes x_i$ with

$$q_{i,i} = q_{j,j} = q, \quad q_{i,i+1} = q^{-1}, \quad q_{i,j} = 1, i - j > 1$$

where $q$ is a primitive $t$-th root of unity, $t > 2$, then the strongness degree of the related braided vector space equals $\lceil 1 + \log_2(t) \rceil$. If $q$ is not a root of unity, then the strongness degree is one.

8. Braiding of Hecke type

Definition 8.1. (see e.g. [ASc3] Definition 3.3 or [AbA] Definition 3.1.1) Let $(V, c)$ be a braided vector space. We say that $c$ is a braiding of Hecke type (or a Hecke symmetry) with mark $q$ if $c$ satisfies the equation $(c + \text{Id}_V \otimes V)(c - q\text{Id}_V \otimes V) = 0$ for some $q \in K \setminus \{0\}$.

Theorem 8.2. Let $(B, m_B, u_B, \Delta_B, \varepsilon_B, c_B)$ be a 0-connected graded braided bialgebra. Assume that

- $B$ is strongly $\mathbb{N}$-graded as an algebra and $m_B^{1,1}$ is an isomorphism.
- $c_B^{1,1}$ has minimal polynomial $f(X) = (X + 1)(X - q)$, for some regular $q \in K$.

Then $\text{sdeg}(B) = 1$.

Proof. By Theorem 4.13, we have $\text{sdeg}(B) = \text{sdeg}(S(B)) + 1$ and $m_B^{1,1} \left(c_B^{1,1} - q\right) = 0$. Therefore, by Theorem 4.12, we get that $\text{sdeg}(S(B)) = 0$ whence $\text{sdeg}(B) = 1$.

The following result explains how braided vector spaces with braiding of Hecke type fits into our study of braided vector spaces of strongness degree at most 1.

Theorem 8.3. Let $(V, c)$ be braided vector space such that $c$ is a braiding of Hecke type with regular mark $q$. Then $\text{sdeg}(V, c) \leq 1$.

Proof. By hypothesis, $c$ satisfies the equation $(c + \text{Id}_V \otimes V)(c - q\text{Id}_V \otimes V) = 0$ so that $c$ has minimal polynomial a divisor $f$ of the polynomial $(X + 1)(X - q)$.

Now, the tensor algebra $T := T(V, c)$ is strongly $\mathbb{N}$-graded as an algebra and $m_T^{1,1}$ is an isomorphism. Moreover $c_T^{1,1} = c$. Thus, if $f = (X + 1)(X - q)$, by Theorem 8.2, we have $\text{sdeg}(V, c) := \text{sdeg}(T(V, c)) = 1$. On the other hand, by Theorem 7.9, if $f = X + 1$, we get $\text{sdeg}(V, c) = 1$ while, if $f = X - q$, we obtain $\text{sdeg}(V, c) = 0$.

Using the results in [AMS1] and [AMS2], we will now give a simple description of the universal enveloping algebra of a braided Lie algebra such that the braiding is of Hecke type with regular mark. We point out that this description does not depend on the bracket in the non-symmetric case whenever char$(K) \neq 2$.

Theorem 8.4. Let $(V, c, b)$ be a braided Lie algebra such that $c$ is of Hecke type with regular mark $q$. Then

$$U(V, c, b) = \frac{T(V, c)}{(c(z) - qz - [z]_b \mid z \in V \otimes V)}$$

where $[z]_b = b(c(z) - qz)$, for every $z \in V \otimes V$. Moreover if $q \neq 1$ and char$(K) \neq 2$, then $[-]_b = 0$.

Proof. We set $T := T(V, c)$. Let us prove that the following equality

$$(46) \quad \text{Im}(c - q\text{Id}_V \otimes V) = E_2(V, c)$$

holds. Since $(c + \text{Id}_V \otimes V)(c - q\text{Id}_V \otimes V) = 0$, it is clear that $\text{Im}(c - q\text{Id}_V \otimes V) \subseteq \ker(c + \text{Id}_V \otimes V)$. Let $z \in \ker(c + \text{Id}_V \otimes V)$. Then $(c - q\text{Id}_V \otimes V)(z) = c(z) - qz = -(1 + q)z = -(2)_q z$. Since $q$ is regular, we have $(2)_q \neq 0$, so that $z = -(2)_q^{-1}(c - q\text{Id}_V \otimes V)(z) \in \text{Im}(c - q\text{Id}_V \otimes V)$. Hence $\text{Im}(c - q\text{Id}_V \otimes V) = \ker(c + \text{Id}_V \otimes V) = \ker(\Delta_T^{1,1}) = E_2(V, c)$. We have so proved (46). By (46), we have that $(c - q\text{Id}_V \otimes V)(z)$ lies in the domain of $b$ for every $z \in V \otimes V$. Hence we can define a map $[-]_b : V \otimes V \to V$ by setting $[z]_b = b(c - q\text{Id}_V \otimes V)(z)$, for every $z \in V \otimes V$. 

Let us check that $[-]_b$ is a $c$-bracket in the sense of [AMS2, Definition 4.1] i.e. that $c([-]_b \otimes V) = (V \otimes [-]_b)_{c1}c2$ and $c(V \otimes [-]_b) = ([c]_b \otimes V)_{c2}c1$. By (46), $b$ is a bracket on $(V,c)$ if $c(b \otimes V) = (V \otimes b)_{c1}c2$ and $c(V \otimes b) = (b \otimes V)_{c1}c2$. By (46), the previous formulas rereads as follows:

\[
\begin{align*}
  & c(b \otimes V) [c - q_{\text{Id}}V] \otimes V(z) = (V \otimes b)_{c1}c2 \left([c - q_{\text{Id}}V] \otimes V\right)(z) \quad \text{and} \\
  & c(V \otimes b) [V \otimes (c - q_{\text{Id}}V)](z) = (b \otimes V)_{c1}c2 \left(V \otimes (c - q_{\text{Id}}V)\right)(z),
\end{align*}
\]

for every $z \in V^\otimes 3$. Thus $[-]_b$ is a $c$-bracket.

Set $U_H(V,c,[-]_b) := T(V,c)/\langle c(z) - qz - [z]_b \mid z \in V \otimes V \rangle$. This is the enveloping algebra as defined in [AMS1, Definition 2.1]. Let us prove that $U_H(V,c,[-]_b) = U(V,c,b)$.

By definition of $[-]_b$ and (46), we have

\[(c(z) - qz - [z]_b \mid z \in V \otimes V) = ((\text{Id}_{V \otimes V} - b)[E_2(V,c)]) .\]

Therefore there is a canonical projection $\gamma : U_H(V,c,[-]_b) \to U(V,c,b)$. Since $i_U : V \to U(V,c,b)$ is injective and $\gamma \circ i_U = i_U$, it is clear that the canonical map $\gamma : U_H(V,c,[-]_b) \to U(V,c,b)$ is injective too. By (AMS2, Theorem 4.20 and Corollary 4.23), the injectivity of $i_U$ imply that $U_H(V,c,[-]_b)$ is strictly graded as an algebra. Therefore $P(U_H(V,c,[-]_b))$ identifies with $V$ so that the restriction of the bialgebra homomorphism $\gamma$ to the space of primitive elements of $U_H(V,c,[-]_b)$ is injective. By [Mo, Lemma 5.3.3] this entails that $\gamma$ itself is injective and hence bijective. We have so proved that $U_H(V,c,[-]_b) = U(V,c,b)$.

Assume now $q \neq 1$ and char $(K) \neq 2$. Then, by [AMS1, Theorem 4.3], since $q$ is regular, the map $[-]_b$ is necessarily zero. □

Remark 8.5. Note that, when $q = 1$, then regularity of $q$ means char $(K) = 0$. Moreover $(V,c,[-]_b)$ becomes a Lie algebra in the sense of [Gu1] and $U_H(V,c,[-]_b)$ is the corresponding enveloping algebra. In this case $[-]_b$ is not necessarily trivial in general.

9. Quadratic algebras

Definition 9.1. Recall that a quadratic algebra [Man, page 19] is an associative graded $K$-algebra $A = \bigoplus_{n \in \mathbb{N}} A^n$ such that:

1) $A^0 = K$;

2) $A$ is generated as a $K$-algebra by $A^1$;

3) the ideal of relations among elements of $A^1$ is generated by the subspace of all quadratic relations $R(A) \subseteq A^1 \otimes A^1$.

Equivalently $A$ is a graded $K$-algebra such that the natural map $\pi : T_K(A^1) \to A$ from the tensor algebra generated by $A^1$ is surjective and ker $(\pi)$ is generated as a two sided ideal in $T_K(A^1)$ by ker $(\pi) \cap (A^1 \otimes A^1)$.

Corollary 9.2. Let $(B, m_B, u_B, \Delta_B, \varepsilon_B, c_B)$ be a graded braided bialgebra which is strongly $N$-graded as an algebra. Then $\text{sd}(B) \leq 1$ whenever $B^0 \left[ B^1 \right] \text{is a quadratic algebra}$.

Proof. By Definition 9.1, $B^0 = K$ and the ideal of relations among elements of $B^1$ is generated by the subspace of all quadratic relations $R(B) \subseteq B^1 \otimes B^1$. By Corollary A.11, we get $\text{sd}(B) \leq 1$. □

Theorem 9.3. Let $(V,c)$ be a graded vector space. Assume that $B(V,c)$ is a quadratic algebra. Then $\text{sd}(V,c) \leq 1$.

Proof. Apply Corollary 9.2 to the case $B := T(V,c)$ once observed that $B^0 \left[ B^1 \right] = B(V,c)$. □

Remark 9.4. Examples of braided vector spaces $(V,c)$ such that $B(V,c)$ is a quadratic algebra can be found e.g. in [MS] and in [AG2]. By [Asc3, Proposition 3.4], another example is given by braided vector spaces of Hecke-type with regular mark. This gives a different proof of Theorem 3.3.

Theorem 9.5. Let $(A,c_A)$ be a connected braided bialgebra such that the graded coalgebra associated to the coradial filtration is a quadratic algebra with respect to its natural braided bialgebra structure. Let $(P, c_P, b_P)$ be the infinitesimal braided Lie algebra of $A$. Then $\text{B}(P, c_P)$ is a quadratic algebra and $A$ is isomorphic to $U(P, c_P, b_P)$ as a braided bialgebra.
Proof. Since $A$ is connected, then graded coalgebra $B := \text{gr}(A)$ associated to the coradical filtration $(A_n)_{n \in \mathbb{N}}$ of $A$ is indeed a braided bialgebra. Since, by assumption, it is a quadratic algebra (see Definition 1.3), then $B$ is generated as a $K$-algebra by $B^1 = A_1/A_0$ so that $B$ is strongly $\mathbb{N}$-graded as an algebra. Since, by construction, it is also strongly $\mathbb{N}$-graded as a coalgebra, we get that $B \simeq B_0[B_1] \simeq B(P,cp)$. Hence $B(P,cp)$ is a quadratic algebra. By Theorem 9.3, $sdeg(P,cp) \leq 1$. On the other hand $B$ is generated as a $K$-algebra by $B^1 = A_1/A_0$ implies that $P$ generates $A$ as a $K$-algebra. Therefore, by Theorem 6.3, $A$ is isomorphic to $U(P,cp,bp)$ as a braided bialgebra. 

Example 9.6. We are going to apply the previous result to a braided algebra $A$ that appeared in [Gn2, page 325]. Let us recall its definition. Let $K$ be a field of characteristic 0 and let $q,\alpha,\beta \in K^*$ be such that $q \neq 1$ is regular and $(\alpha/\beta)^2 = q$. Set $m = -\alpha/\beta$ and consider the vector space $V$ with basis \{e_0, e_1, e_2\} and braiding given by $c(e_i \otimes e_j) = e_j \otimes e_i$ for $i = 0$ or $j = 0$, $c(e_i \otimes e_j) = q e_i \otimes e_j$ for $i = 1, 2$, $c(e_2 \otimes e_1) = me_1 \otimes e_2 + (q-1)e_2 \otimes e_1$ and $c(e_1 \otimes e_2) = q^{-1}e_2 \otimes e_1$. One can check that $E_2(V,c)$ is generated over $K$ by the elements $e_1 \otimes e_0 - e_0 \otimes e_1$, $e_2 \otimes e_0 - e_0 \otimes e_2$ and $e_2 \otimes e_1 - me_1 \otimes e_2$. Consider the $K$-linear map $b_2 : E_2(V,c) \rightarrow V$ defined, on generators, by

$$b_2(e_1 \otimes e_0 - e_0 \otimes e_1) = e_1, \quad b_2(e_2 \otimes e_0 - e_0 \otimes e_2) = e_2, \quad b_2(e_2 \otimes e_1 - me_1 \otimes e_2) = 0.$$ 

Then, by definition, $A$ is given by

$$A := \frac{T(V,c)}{([\text{Id} - b_2][E_2(V,c)])}.$$ 

Explicitly $A$ is generated over $K$ by $e_0, e_1, e_2$ with relations

$$e_1 \otimes e_0 - e_0 \otimes e_1 = e_1, \quad e_2 \otimes e_0 - e_0 \otimes e_2 = e_2, \quad e_2 \otimes e_1 - me_1 \otimes e_2 = 0.$$ 

Now, $A$ is a braided bialgebra quotient of $T(V,c)$ (to prove this one can adapt the proof of Theorem 3.9 by taking $E_2(V,c)$ instead of $E(V,c)$ and using the equalities $c(b_2 \otimes V) = (V \otimes b_2)c_{E_2(V,c),V}$ and $c(V \otimes b_2) = (b_2 \otimes V)c_{V,E_2(V,c)}$). Thus, by [ASc1, Corollary 5.3.5], $A$ is connected as $T(V,c)$ is connected. Let $(P,cp,bp)$ is the infinitesimal braided Lie algebra of $A$. Then $A$ has basis \{\epsilon_0^{n_0}e_1^{n_1}e_2^{n_2} | n_0, n_1, n_2 \in \mathbb{N}\} and, using regularity of $q$, one gets (by the same argument as in the proof of [ASc1, Lemma 3.3]) that $(P,cp)$ identifies with $(V,c)$. Thus, the graded coalgebra associated to the coradical filtration of $A$ is $B(V,c)$, which is isomorphic to $T(V,c)/E_2(V,c)$ which is quadratic. By Theorem 1.7, $A$ is isomorphic to $U(P,cp,bp)$ as a braided bialgebra. Note that, in this example, $U(P,cp,bp)$ is not trivial in the sense that it differs from $S(P,cp)$.

In this paper we are not interested in a classification of braided Lie algebras. Thus, for instance, we have not tried to list all possible brackets giving a braided Lie algebra structure on the braided vector space $(V,c)$ of Example 9.6. We just intended to provide an example having both not trivial bracket and not Hecke type braiding. On the other hand, an attempt of classification would require to rewrite the third condition in Definition 5.4 in a Jacobi identity type form. This will be concerned in [AS] using results in [BG]. Therein, other examples of braided Lie algebras whose associated Nichols algebra is quadratic will be investigated.

10. Pareigis-Lie algebras

In this section we will investigate the relation between our notions of Lie algebra and universal enveloping algebra and the ones introduced by Pareigis in [Pa2]. Although these constructions were performed for braided vector spaces which are in addition objects inside the category of Yetter-Drinfeld modules, we will not take care of this extra structure.

10.1. Let $B_n$ be the Artin braid group with generators $\tau_i, i \in \{1, \ldots, n-1\}$ and relations

$$\tau_i \tau_{i+1} \tau_i = \tau_{i+1} \tau_i \tau_{i+1} \quad \text{and} \quad \tau_i \tau_j = \tau_j \tau_i \quad \text{if} \quad |i-j| \geq 2.$$ 

Let $\nu : B_n \rightarrow S_n : \tau \mapsto \overline{\tau}$ be the canonical quotient homomorphism from the braid group onto the symmetric group and set $s_i := \overline{\tau_i}$ for every $i \in \{1, \ldots, n-1\}$.

Let $\sigma \in S_n$. The length $l(\sigma)$ of $\sigma$ is the smallest number $m$ such that there is a decomposition $\sigma = s_{i_1}s_{i_2} \cdots s_{i_m}$. Such a decomposition will be called reduced. A permutation $\sigma$ may have several
reduced decompositions, but if \( \sigma = s_{i_1} s_{i_2} \cdots s_{i_m} \) is one of them, then the element \( \tau_{i_1} \tau_{i_2} \cdots \tau_{i_m} \) is uniquely determined in \( B_n \).

It is well-known that the \( \nu_n \) has a canonical section \( \iota_n : S_n \rightarrow B_n \). By definition,

\[
\iota_n(\sigma) := \tau_{i_1} \tau_{i_2} \cdots \tau_{i_m},
\]

where \( \sigma = s_{i_1} s_{i_2} \cdots s_{i_m} \) is a reduced decomposition of \( \sigma \). Note that \( \iota_n \) is just a map, but

\[
\iota_n(\sigma \tau) = \iota_n(\sigma) \iota_n(\tau),
\]

for any \( \sigma, \tau \in S_n \) such that \( l(\sigma \tau) = l(\sigma) + l(\tau) \).

Recall that \( \sigma \in S_{p+q} \) is a \((p, q)\)-shuffle if \( \sigma(1) < \cdots < \sigma(p) \) and \( \sigma(p + 1) < \cdots < \sigma(p + q) \). The set of \((p, q)\)-shuffles will be denoted by \((p \mid q)\).

10.2. Let \((V, c)\) be a braided vector space. For \( n \in \mathbb{N}^* \), there is a canonical linear representation \( \rho_n : B_n \rightarrow \text{Aut}_K(V^\otimes n) \) such that \( \rho_n(\tau_i) = c_i \), where

\[
c_i = V^{\otimes (i-1)} \otimes c \otimes V^{\otimes (n-i-1)}.
\]

The induced action will be denoted by

\[
\triangleright : B_n \times V^\otimes n \rightarrow V^\otimes n.
\]

On generators we have \( \tau_i \triangleright x = c_i(x) \) for each \( x \in V^\otimes n \). Note that, since \( \iota_n \) is not a group homomorphism, this does not restrict in a natural way to an action of \( S_n \) on \( V^\otimes n \).

Let \( n \geq 2 \) and let \( \zeta \in K \setminus \{0\} \). Following [Pa, Definition 2.3], we set

\[
V^\otimes n(\zeta) := \{ x \in V^\otimes n \mid (\varphi^{-1} \tau_i^2 \varphi) \triangleright x = \zeta^2 x, \text{ for every } \varphi \in B_n, i \in \{1, \ldots, n-1\} \}.
\]

There is an action \( \triangleright : S_n \times V^\otimes n(\zeta) \rightarrow V^\otimes n(\zeta) \) defined on generators by

\[
\zeta s_i \triangleright x = \zeta^{-1} \tau_i \triangleright x.
\]

**Lemma 10.3.** Let \((V, c)\) be a braided vector space. For every \( n \geq 2 \) consider the map

\[
\Pi_n^\zeta : V^\otimes n(\zeta) \rightarrow V^\otimes n(\zeta), \Pi_n^\zeta(x) = \sum_{\sigma \in S_n} x
\]

for \( \zeta \in K \setminus \{0\} \). If \( \zeta \) is a primitive \( n \)-th root of unity then \( \Pi_n^\zeta(x) \in E_n(V, c) \) for each \( x \in V^\otimes n(\zeta) \).

**Proof.** We set \( T := T(V, c) \) by [AMS2, Corollary 1.22], for every \( 0 \leq i \leq n \) and for every \( x \in V^\otimes n \), we have \( \Delta_T^{i,n-i}(x) = \sum_{\sigma \in (i|n-i)} \iota_n(\sigma^{-1}) \triangleright x \). Note that, for every \( y \in V^\otimes n(\zeta) \) we have

\[
\sigma^{-1} \triangleright y = \zeta^{l(\sigma)} \iota_n(\sigma^{-1}) \triangleright y \]

as \( l(\sigma^{-1}) = l(\sigma) \) for every \( \sigma \in S_n \). Therefore, if \( x \in V^\otimes n(\zeta) \) we get

\[
\Delta_T^{i,n-i}[\Pi_n^\zeta(x)] = \sum_{\sigma \in (i|n-i)} \iota_n(\sigma^{-1}) \triangleright \Pi_n^\zeta(x) = \sum_{\sigma \in (i|n-i)} \sum_{\gamma \in S_n} \iota_n(\gamma^{-1}) \triangleright (\gamma \triangleright x) = \sum_{\sigma \in (i|n-i)} \sum_{\gamma \in S_n} \zeta^{l(\sigma)} (\sigma^{-1}\gamma) \triangleright x = \sum_{\sigma \in (i|n-i)} \sum_{\gamma \in S_n} \zeta^{l(\sigma)} \gamma \triangleright x = \left( \sum_{\sigma \in (i|n-i)} \zeta^{l(\sigma)} \right) \Pi_n^\zeta(x) = \binom{n}{i} \Pi_n^\zeta(x),
\]

where the last equality holds in view of [AMS2, 1.29]. If \( \zeta \) is a primitive \( n \)-th root of unity, one has that \( \binom{n}{i} = 0 \) unless \( i = 0 \) or \( i = n \). We have so proved that \( \Pi_n^\zeta(x) \in E_n(V, c) \) for each \( x \in V^\otimes n(\zeta) \).

We point out that the calculations above are inspired by part of the proof of [Pa, Theorem 5.3]. \( \square \)
where

\[ 10.6 \quad \text{Theorem projection } U(V,c,b) \]

every \( x \) so that (48) holds. In order to obtain (49) we adapt the proof of (3.4) as follows. For every \( n \in \mathbb{N}, \zeta \in \mathbb{P}_n \), we have

(48) \[ [\sigma \triangleright x]_\zeta^n = [x]_\zeta^n, \text{ for every } \sigma \in S_n, x \in V^{\otimes n}(\zeta), \]

(49) \[ \sum_{i=1}^{n+1} \{ [-]_{n+1}^i \circ \left( V \otimes [-]_\zeta^n \right) \} ((1, \ldots, i) \triangleright x) = 0, \text{ for every } x \in V^{\otimes n+1}(\zeta), \]

(50) \[ \{ [-]_{n+1}^i \circ \left( V \otimes [-]_\zeta^n \right) \} (x) = \sum_{i=1}^{n} \{ [-]_\zeta^n \circ \left( V^{\otimes i-1} \otimes [-]_{n+1-i} \circ V^{\otimes n-i} \right) \} ((\tau_{i-1} \cdots \tau_1) \triangleright x), \]

for every \( x \in V^{\otimes n+1}(-1, \zeta) \)

where

\[ V^{\otimes n+1}(-1, \zeta) := \{ x \in V \otimes V^{\otimes n}(\zeta) \mid (1 \otimes \varphi)^{-1} \triangleright \left( \tau_2^2 \triangleright ((1 \otimes \varphi) \triangleright x) \right) = x, \forall \varphi \in S_n \}. \]

Given a Pareigis-Lie algebra \((V,c,[-])\) one can define (see (Pa)) the universal enveloping algebra

\[ U_P(V,c,[-]) := T(V,c) \]

\[ \frac{\Pi^n(\zeta) - [z]_\zeta^n}{\Pi^n(\zeta)} \quad n \in \mathbb{N}, \zeta \in \mathbb{P}_n, z \in V^{\otimes n}(\zeta) \]

not to be mixed with (38).

Remark 10.5. V. K. Kharchenko drew our attention to (Kh5, Corollary 7.5) where it is proven there exist at least \((n-2)!\) possibilities to define symmetric Pareigis-Lie algebras in the case when the underline braided vector space is an object in the category of Yetter-Drinfeld modules over an abelian group algebra.

Next result associates a Pareigis-Lie algebra to any braided Lie algebra.

Theorem 10.6. Let \((V,c,b)\) be a braided Lie algebra. Then \((V,c,[-])\) is a Pareigis-Lie algebra where \([x]_\zeta^n := b_n \Pi^n_\zeta(x)\) for every \( x \in V^{\otimes n}(\zeta) \). Moreover there is a canonical braided bialgebra projection \(U_P(V,c,[-]) \to U(V,c,b)\) which is the identity map whenever

\[ \sum_{\zeta \in \mathbb{P}_n} \text{Im} \left( \Pi^n_\zeta \right) = E_n(V,c), \text{ for every } n \in \mathbb{N} \]

is fulfilled.

Proof. We keep many of the notations used in (Pa) and apply some properties proved therein. First, observe that, by Lemma 10.3, \( \Pi^n_\zeta(x) \) lies in the domain of \( b_n \), for every \( x \in V^{\otimes n}(\zeta) \) whence \([x]_\zeta^n\) is well defined. For every \( \sigma \in S_n, x \in V^{\otimes n}(\zeta) \), we have

\[ [\sigma \triangleright x]_\zeta^n = b_n \Pi^n_\zeta (\sigma \triangleright x) = b_n \left( \sum_{\gamma \in S_n} \gamma \triangleright (\sigma \triangleright x) \right) = b_n \left( \sum_{\gamma \in S_n} (\gamma \sigma) \triangleright x \right) \]

so that (48) holds. In order to obtain (49) we adapt the proof of (Pa, Theorem 3.4) as follows. For every \( x \in V^{\otimes n+1}(\zeta) \) and \( \zeta \in \mathbb{P}_n \), we have

\[ \sum_{i=1}^{n+1} \{ [-]_{n+1}^i \circ \left( V \otimes [-]_\zeta^n \right) \} ((1, \ldots, i) \triangleright x) = b_2 \sum_{i=1}^{n+1} (\Id_{V^{\otimes 2}} - c) \{ \left( V \otimes [-]_\zeta^n \right) ((1, \ldots, i) \triangleright x) \} \].
Let us focus on the argument of $b_2$ that we denote by $a$. Note that $a \in E_2(V, c)$ and observe that

$$c\left(V \otimes [-1]^n_{\zeta}\right)(z) \equiv \left[b_n \otimes V\right] c_{V,E_n(V,c)} \left[V \otimes \Pi^n_{\zeta}\right](z) = \left([-1]^n_{\zeta} \otimes V\right) \{(\tau_n \cdots \tau_1) \triangleright z\}$$

so that

$$\left(52\right) \quad c\left(V \otimes [-1]^n\right)(z) = \left([-1]^n \otimes V\right) \{(\tau_n \cdots \tau_1) \triangleright z\}, \text{ for every } z \in V \otimes V^{\otimes n}(\zeta).$$

Note also that $V^{\otimes n+1}(\zeta) \subseteq V \otimes V^{\otimes n}(\zeta)$ (see [Pa, Proof of Proposition 3.1]). We get

$$a = \sum_{i=1}^{n+1} \left( V \otimes [-1]^n_{\zeta} \right)((1, \ldots, i) \triangleright x) - \sum_{i=1}^{n+1} \left( [-1]^n_{\zeta} \otimes V \right)((n+1, \ldots, i) \triangleright x)$$

$$= \sum_{i=1}^{n+1} \left( V \otimes [-1]^n_{\zeta} \right)((1, \ldots, i) \triangleright x) - \sum_{i=1}^{n+1} \left( [-1]^n_{\zeta} \otimes V \right)((n+1, \ldots, i) \triangleright x)$$

$$= \sum_{i=1}^{n+1} \left( V \otimes [-1]^n_{\zeta} \right)((1, \ldots, i) \triangleright x) - \sum_{i=1}^{n+1} \left( b_n \Pi^n_{\zeta} \otimes V \right)((n+1, \ldots, i) \triangleright x)$$

$$= \sum_{i=1}^{n+1} \left( V \otimes b_n \Pi^n_{\zeta} \right)((1, \ldots, i) \triangleright x) - \sum_{i=1}^{n+1} \left( b_n \Pi^n_{\zeta} \otimes V \right)((n+1, \ldots, i) \triangleright x)$$

$$= \left( V \otimes b_n \right) \left( \sum_{\sigma \in S_{n+1}} \sigma \triangleright x \right) - \left( b_n \otimes V \right) \left( \sum_{\sigma \in S_{n+1}} \sigma \triangleright x \right)$$

Thus

$$\sum_{i=1}^{n+1} \left( [-1]^n_{\zeta} \otimes \left( V \otimes [-1]^n\right) \right)((1, \ldots, i) \triangleright x) = b_2 \left\{ \left( V \otimes b_n \right) \left( \Pi^{n+1}_{\zeta}(x) \right) - \left( b_n \otimes V \right) \left( \Pi^{n+1}_{\zeta}(x) \right) \right\}.$$
Let us check that \([\mathfrak{P}]\) is true. We will adapt the proof of \([\mathfrak{F}]\) Theorem 3.5. We get

\[
\sum_{i=1}^{n} \{-\zeta \circ \left(V^\otimes i-1 \otimes [-]_\zeta^2 \otimes V^\otimes n-i \right)\}((\tau_{i-1} \cdots \tau_1) \triangleright x)
\]

\[
= b_n \left( \sum_{i=1}^{n} \Pi_n \left(V^\otimes i-1 \otimes [-]_\zeta^2 \otimes V^\otimes n-i \right) \right) ((\tau_{i-1} \cdots \tau_1) \triangleright x)
\]

\[
= b_n \left( \sum_{i=1}^{n} \Pi_n \left(V^\otimes i-1 \otimes b_2 (\text{Id}_{V^\otimes 2} - c) \otimes V^\otimes n-i \right) \right) ((\tau_{i-1} \cdots \tau_1) \triangleright x)
\]

\[
= b_n \left( \sum_{i=1}^{n} \Pi_n \left(V^\otimes j-1 \otimes b_2 \otimes V^\otimes n-j \right) \right) ((\tau_{j-1} \cdots \tau_1) \triangleright x) + \sum_{j=1}^{n} \Pi_n \left(V^\otimes j-1 \otimes b_2 \otimes V^\otimes n-j \right) \left(V^\otimes j-1 \otimes c \otimes V^\otimes n-j \right) ((\tau_{j-1} \cdots \tau_1) \triangleright x)
\]

\[
\Rightarrow b_n \left( \sum_{i=1}^{n} \sum_{\sigma \in S_n} \sigma \left( (V^\otimes i-1 \otimes b_2 \otimes V^\otimes n-i) \right) \otimes \zeta^{-\ell(\tilde{\varphi})} \right) \left( (\tau_{i-1} \cdots \tau_1) \triangleright x \right) + \sum_{j=1}^{n} \sum_{\sigma \in S_n} \sigma \left( (V^\otimes j-1 \otimes b_2 \otimes V^\otimes n-j) \right) \otimes \zeta^{-\ell(\tilde{\varphi})} \left( (\tau_{j-1} \cdots \tau_1) \triangleright x \right)
\]

\[
\Rightarrow b_n \left( \sum_{i=1}^{n} \sum_{\varphi \in \text{Im}_n} \zeta^{-\ell(\tilde{\varphi})} \varphi \triangleright (V^\otimes i-1 \otimes b_2 \otimes V^\otimes n-i) \right) \otimes \left( (\tau_{i-1} \cdots \tau_1) \triangleright x \right) + \sum_{j=1}^{n} \sum_{\varphi \in \text{Im}_n} \zeta^{-\ell(\tilde{\varphi})} \left( \varphi \otimes \tilde{\varphi}(j) \right) \otimes \left( (\tau_{j-1} \cdots \tau_1) \triangleright x \right)
\]

where in (*) we used that, for every \(x' \in V^\otimes i-1 \otimes E_2 (V, c) \otimes V^\otimes n-i\),

\[
\varphi \triangleright \left( (V^\otimes i-1 \otimes b_2 \otimes V^\otimes n-i) \right) (x') = \left( V^\otimes i-1 \otimes b_2 \otimes V^\otimes n-i \right) \varphi(i)(x'),
\]

where \(\varphi(i) \in E_{n+1}\) is a suitable braid depending on \(\varphi\) and that was defined in \([\mathfrak{F}]\) Appendix; in (**) we applied \([\mathfrak{F}_4]\) Proposition 8.1.

On the other hand, we have

\[
[-]^2 \zeta \left( V \otimes [-]^n \zeta \right) (x)
\]

\[
= b_2 \left( \text{Id}_{V^\otimes 2} - c \right) \left( V \otimes [-]^n \zeta \right) (x)
\]

\[
= b_2 \left\{ \left( V \otimes [-]^n \zeta \right) (x) - c \left( V \otimes [-]^n \zeta \right) (x) \right\}
\]

\[
= b_2 \left\{ \left( V \otimes b_n \Pi_n \right) (x) - c \left( V \otimes b_n \Pi_n \right) (x) \right\}
\]

\[
\Rightarrow b_2 \left\{ \left( V \otimes b_n \right) \left( V \otimes \Pi_n \right) (x) - (b_n \otimes V) c_{V,E_n(V,c)} \left( V \otimes \Pi_n \right) (x) \right\}
\]

\[
= b_2 \left\{ \left( V \otimes b_n \right) - (b_n \otimes V) c_{V,E_n(V,c)} \right\} \left( V \otimes \Pi_n \right) (x).
\]

Let \(w := \left( V \otimes \Pi_n \right) (x)\). Hence it remains to check that

\(b_n \left( (b_2 \otimes V^\otimes n-1) - (V^\otimes n-1 \otimes b_2) c_{V,E_n(V,c)} \right) \left( V \otimes \Pi_n \right) (w) = b_2 \left\{ \left( V \otimes b_n \right) - (b_n \otimes V) c_{V,E_n(V,c)} \right\} \left( V \otimes \Pi_n \right) (w)\).
We have
\[
\begin{align*}
    s : &= \left( b_n \left( (b_2 \otimes V^{\otimes n-1}) - (V^{\otimes n-1} \otimes b_2) \right) c_{V,V^{\otimes n}} \right)(w) + \\
    &\quad - b_2 \left\{ (V \otimes b_n) - (b_n \otimes V) \right\} c_{V,E_n(V,c)}(w) \\
    &= \left( (b_2 - \text{Id}) \left( (b_2 \otimes V^{\otimes n-1}) - (V^{\otimes n-1} \otimes b_2) \right) c_{V,V^{\otimes n}} \right)(w) + \\
    &\quad + \left( (b_2 - \text{Id}) \otimes \left( b_2 - \text{Id} \right) \right) c_{V,V^{\otimes n}}(w) + \\
    &\quad - \left\{ (V \otimes (b_2 - \text{Id})) - ((b_2 - \text{Id}) \otimes V) \right\} c_{E_n(V,c)}(w).
\end{align*}
\]
so that \( s \in V \cap (F) = V \cap \ker(\pi_U) \) which is zero by Definition 5.6. Hence (60) is proved.

Thus \((V,c,[\cdot])\) is a Pareigis-Lie algebra. By the universal property of its universal enveloping algebra (see [Pa, Section 6]), there is a unique braided bialgebra homomorphism \( U_P(V,c,[\cdot]) \to U(V,c,b) \) that lifts the map \( \iota_U : V \to U(V,c,b) \). Assume that (51) is fulfilled. Then
\[
U(V,c,b) = \frac{T(V,c)}{(\text{Id} - b_n) \{ E(V,c) \}} = \frac{T(V,c)}{(\text{Id} - b_n) \left\{ \sum_{\xi \in \mathbb{F}_p} \text{Im}(\Pi^\xi) \right\} | n \in \mathbb{N}}
\]
\[
= \frac{T(V,c)}{(\text{Id} - b_n) \left\{ \Pi^\xi(z) \right\} | n \in \mathbb{N}, \xi \in \mathbb{F}_p, z \in V^{\otimes n}(\zeta)}
\]
\[
= \frac{T(V,c)}{\Pi^\xi(z) - [z]_\xi^n | n \in \mathbb{N}, \xi \in \mathbb{F}_p, z \in V^{\otimes n}(\zeta)} = U_P(V,c,[\cdot]).
\]

**Remark 10.7.** Note that, by [Ma, Lemma 5.3.3], the projection \( U_P(V,c,[\cdot]) \to U(V,c,b) \) in Theorem 10.4 becomes the identity whenever \( P(U_P(V,c,[\cdot])) \) identifies with \( V \) through the canonical map \( \iota_U : V \to U_P(V,c,[\cdot]) \) (in view of Theorem 5.3 the converse is true if \( \text{segl}(V,c) \leq 1 \)).

Theorem 10.4 leads to the following problem.

**Problem 10.8.** To determine under which conditions (51) is true for a given braided vector space \((V,c)\). Note that \( \text{Im}(\Pi^\xi) \) is a vector subspace of \( V^{\otimes n}(\zeta) \) so that a necessary condition is \( E_n(V,c) \subset \sum_{\xi \in \mathbb{F}_p} V^{\otimes n}(\zeta) \).

**Remark 10.9.** Assume \( \text{char}(K) \neq 2 \) and let \((V,c)\) be a braided vector space. Let us show (51) holds for \( n = 2 \) i.e. that \( \text{Im}(\Pi^2) = E_2(V,c) \). By definition, \( E_2(V,c) = \{ x \in V^{\otimes 2} | c(x) = -x \} \) and
\[
V^{\otimes 2}(-1) = \{ x \in V^{\otimes 2} | (\varphi^{-1}_1 \varphi) \triangleright x = x, \text{ for every } \varphi \in B_2 \}
\]
\[
= \{ x \in V^{\otimes 2} | \tau_1^2 \triangleright x = x \} = \{ x \in V^{\otimes 2} | c^2(x) = x \}.
\]
For each \( x \in V^{\otimes 2}(-1) \), we have \( \Pi^2(x) = \sum_{\sigma \in S_2} \sigma \triangleright x = x - \tau_1 \triangleright x = x - c(x) \). Let \( \gamma : E_2(V,c) \to V^{\otimes 2}(-1) \) be defined by \( \gamma(x) = x/2 \). Then, for every \( x \in E_2(V,c) \), we have \( \Pi^2_1 \gamma(x) = \frac{1}{2} (x - c(x)) = x \), where the last equality holds as \( x \in E_2(V,c) \). Hence \( \text{Im}(\Pi^2_1) = E_2(V,c) \).

Note that \( \Pi^2_1 \) is not injective in general. For, if \( x \in V^{\otimes 2}(-1) \) we have that \( \Pi^2_1(x) \in V^{\otimes 2}(-1) \) so that it makes sense to compute \( \Pi^2_1 \Pi^2_1(x) = x - c(x) - c(x) + c^2(x) = 2\Pi^2(x) \). Thus injectivity of \( \Pi^2_1 \) implies \( \Pi^2_1(x) = 2x \) i.e. \( c(x) = -x \) for every \( x \in V^{\otimes 2}(-1) \) which is not true in general (for example choose \( V \) to be a two dimensional vector space and \( c \) to be the canonical flip map defined by \( c(v \otimes w) = w \otimes v \) for every \( v, w \in V \)).

In view of Theorem 10.4, we can now recover two meaningful examples of Pareigis-Lie algebra.

**Corollary 10.10.** (cf. [Pa, Corollaries 4.2 and 5.4]) Let \((A,m_A,u_A,c_A)\) be a braided algebra. Then \((A,c_A,[-])\) is a Pareigis-Lie algebra where \( [x]_\xi^n := m_A^{n-1} \Pi^\xi(x) \), for every \( x \in V^{\otimes n}(\zeta) \).
Moreover, if \( A \) is a connected braided bialgebra, then the space \( P(A) \) of primitive elements of \( A \) forms a Pareigis-Lie algebra too.
Proof. By Proposition 5.8, \((A, c_A, b_A)\) is a braided Lie algebra, where \(b_A (z) := m^{-1}_A (z)\), for every \(z \in E_1 (A, c_A)\). By Theorem 5.10, also \(P (A)\) carries a Paeigis-Lie algebra structure which is induced by that of \(A\). We conclude by applying Theorem 10.4. 

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Appendix A. Further results on strongness degree

In this section we collect some further results on strongness degree of graded braided bialgebras. In particular we will see that under suitable assumptions the strongness degree of a graded braided bialgebra has an upper bound.

The following technical results are needed to prove Theorem A.5.

**Lemma A.1.** Let \((B, m_B, u_B, \Delta_B, \varepsilon_B, c_B)\) be a graded braided bialgebra. Assume there exists \(n \in \mathbb{N}\) such that \(\Delta_B^{1,n} : B^n \rightarrow B^1 \otimes B^{n-1}\) is injective for every \(0 \leq t \leq n\). Then

\[
\ker \left( \Delta_B^{n,1} \right) = \ker \left( \Delta_B^{n-1,2} \right) = \cdots = \ker \left( \Delta_B^1 \right).
\]

Furthermore, for every \(a, b \geq 1\) such that \(a + b = n + 1\), we have

\[
\ker \left( \Delta_B^{a,b} \right) = \ker \left[ \left( \Delta_B^{n-1,1} \otimes B^b \right) \Delta_B^{a,b} \right] = \ker \left[ \left( B^n \otimes \Delta_B^1 \right) \Delta_B^{a,b,1+1} \right] = \ker \left( \Delta_B^{a,b,1+1} \right).
\]

The last assertion follows by definition of \(E_{n+1} (B)\). 

**Proposition A.2.** Let \((B, m_B, u_B, \Delta_B, \varepsilon_B, c_B)\) be a graded braided bialgebra. Let \(n \in \mathbb{N}\) and let \(S_n := S^n (B)\). For every \(t \in \mathbb{N}\), set \(I_t^{S_n} := (E \left( S_n \right)) \cap (S_n)^t\). The following assertions hold.

i) \(I_t^{S_n} = \{0\}\) for every \(0 \leq t \leq n + 1\).

ii) \(I_{n+2}^{S_n} = \ker \left( \Delta_{S_n}^{a,b} \right)\), for every \(a, b \geq 1\) such that \(a + b = n + 2\).

iii) \(\Delta_{S_n}^{a,b}\) is injective for every \(a, b \geq 1\) such that \(0 \leq a + b \leq n + 1\).

**Proof.** Fix \(n \in \mathbb{N}\) and let us prove that i) and ii) follow by iii). Set \(A := S_n = S^n (B)\).

iii) \(\Rightarrow\) i) We have \(I_0^A := (E (A)) \cap A^0 = \{0\}\) and \(I_1^A := (E (A)) \cap A^1 = \{0\}\). From this and iii), since \(\Delta_A^{a,1} (I_{t+1}^A) \subseteq A^t \otimes I_{t+1}^A + I_{t+1}^A \otimes A^1\), for every \(t \in \mathbb{N}\), one easily proves that i) is satisfied by induction on \(t\).

iii) \(\Rightarrow\) ii) Let \(a, b \geq 1\) be such that \(a + b = n + 2\). Using i), we obtain \(\Delta_A^{a,b} (I_{n+2}^A) \subseteq A^a \otimes I_{n+2}^A + I_{n+2}^A \otimes A^b = \{0\}\) so that \(I_{n+2}^A \subseteq \ker \left( \Delta_A^{a,b} \right)\). By iii) and Lemma A.1, we have \(E_{n+2} (A) = \ker \left( \Delta_A^{a,b} \right)\) so that \(\ker \left( \Delta_A^{a,b} \right) \subseteq (E_{n+2} (A)) \cap A^{n+2} \subseteq (E (A)) \cap A^{n+2} = I_{n+2}^A\). Hence ii) holds.

Let us check that iii) holds, by induction on \(n \in \mathbb{N}\).

For \(n = 0\), by (iii), there is nothing to prove. Let \(n \geq 1\) and assume that iii) is true for \(n - 1\). Let us prove it for \(n\). Let \(R := S^{[n-1]} (B)\). Then, by inductive hypothesis and the first part, we have

i') \(I_t^B = \{0\}\) for every \(0 \leq t \leq n\).

ii') \(I_{n+1}^B = \ker \left( \Delta_B^{a,b} \right)\), for every \(a, b \geq 1\) such that \(a + b = n + 1\).

iii') \(\Delta_B^{a,b}\) is injective for every \(a, b \geq 1\) such that \(0 \leq a + b \leq n\).

For every \(i \in \mathbb{N}\), let \(\pi_i^A : R^i \rightarrow A^i := R^i / R_i^A\) be the canonical projection. Recall that \(\Delta_A^{a,b}\) is uniquely defined by \(\Delta_A^{a,b} \circ \pi_A^{a+b} = (\pi_A^a \otimes \pi_A^b) \circ \Delta_R^{a,b}\). By i'), \(\pi_A^i\) is an isomorphism for every \(0 \leq i \leq n\). Using this fact and hypothesis iii'), from the last displayed equality, we deduce that
\( \Delta^{a,b}_{A} \) is injective for every \( a, b \geq 1 \) such that \( 0 \leq a + b \leq n \). To conclude it remains to check that \( \Delta^{a,b}_{A} \) is injective for \( a, b \geq 1 \) such that \( a + b = n + 1 \). By \( iii \), we have that

\[
A^{n+1} = \frac{R^{n+1}}{ \ker \left( \Delta^{a,b}_{R} \right)}
\]

so that \( \Delta^{a,b}_{R} \) factors to a unique map \( \Delta^{a,b}_{R} : A^{n+1} \to R^{a} \otimes R^{b} \) such that \( \Delta^{a,b}_{R} \circ \pi^{n+1}_{A} = \Delta^{a,b}_{R} \). Furthermore \( \Delta^{a,b}_{R} \) is injective. We have \( \Delta^{a,b}_{A} \circ \pi^{n+1}_{A} = \left( \pi^{a}_{A} \otimes \pi^{b}_{A} \right) \circ \Delta^{a,b}_{R} = \left( \pi^{a}_{A} \otimes \pi^{b}_{A} \right) \circ \Delta^{a,b}_{R} \circ \pi^{n+1}_{A} \) and hence \( \Delta^{a,b}_{A} = \left( \pi^{a}_{A} \otimes \pi^{b}_{A} \right) \circ \Delta^{a,b}_{R} \). Since \( \pi^{a}_{A} \) and \( \pi^{b}_{A} \) are isomorphisms and \( \Delta^{a,b}_{R} \) is injective, it is clear that also \( \Delta^{a,b}_{A} \) is injective.

Lemma A.3. Let \( K \) be a field and let \( \left( B_{i} \right)_{i \in \mathbb{N}}, \left( \xi_{i}^{j} \right)_{i, j \in \mathbb{N}} \) be a direct system of \( K \)-vector spaces, where, for \( i \leq j, \xi_{i}^{j} : B_{i} \to B_{j} \). Assume that each \( B_{i} \) is endowed with a graded braided bialgebra structure \( (B_{i}, m_{B_{i}}, u_{B_{i}}, \Delta_{B_{i}}, \varepsilon_{B_{i}}, c_{B_{i}}) \) such that \( \xi_{i}^{j} \) is a graded braided bialgebra homomorphism, for every \( i, j \in \mathbb{N} \). Then \( \lim B_{i} \) carries a natural graded braided bialgebra structure that makes it the direct limit of \( \left( B_{i} \right)_{i \in \mathbb{N}}, \left( \xi_{i}^{j} \right)_{i, j \in \mathbb{N}} \) as a direct system of graded braided bialgebras. Furthermore the \( n \)-th graded component of \( \lim B_{i} \) is \( \lim B_{i}^{n} \), where \( B_{i}^{n} \) denotes the \( n \)-th graded component of \( B_{i} \).

A.4. Let \( (B, m_{B}, u_{B}, \Delta_{B}, \varepsilon_{B}, c_{B}) \) be a graded braided bialgebra. Then, by (4), (13), (9), (10) and (11) we have that \( \left( B^{0}, m_{B}^{0,0}, u_{B}^{0}, \Delta_{B}^{0,0}, \varepsilon_{B}^{0}, c_{B}^{0,0} \right) \) is a braided bialgebra. Denote by \( B^{0}M_{B}^{0} \) the category of Hopf bimodules over \( B^{0} \). By (12), \( \left( B^{1}, m_{B}^{0,1}, m_{B}^{1,0}, \Delta_{B}^{0,1}, \Delta_{B}^{1,0} \right) \) is an object in \( B^{0}M_{B}^{0} \). Both of them carry a graded braided bialgebra structure arising from their respective universal properties. Moreover by the universal property of the tensor algebra, there is a morphism

\[
F : T_{B^{0}}(B^{1}) \to T_{B^{0}}^{c}(B^{1})
\]

of graded algebras which is the identity on \( B^{0} \) and \( B^{1} \) respectively. \( F \) is in fact a graded braided algebra homomorphism. Thus the image of \( F \) is a graded braided bialgebra which is called the braided bialgebra of Type one associated to \( B^{0} \) and \( B^{1} \) and denoted by \( B^{0,1}(B^{1}) \). In case when \( c_{B} \) is the canonical flip map, this definition and notation goes back to \( \Delta\Lambda \). For further details, the reader is referred also to \( \Delta\Lambda \). Lemma 6.1 and Theorem 6.8]. The proofs there are performed inside braided monoidal categories, but can be easily adapted to a non categorical framework. This essentially works because of the definition of graded braided bialgebra which includes the compatibility of \( c_{B} \) with graded components of \( B \).

Theorem A.5. Let \( (B, m_{B}, u_{B}, \Delta_{B}, \varepsilon_{B}, c_{B}) \) be a graded braided bialgebra. Then

i) \( S^{[\infty]}(B) \) is a graded braided bialgebra which is strongly \( \mathbb{N} \)-graded as a coalgebra (i.e. \( \text{sdeg} \left( S^{[\infty]}(B) \right) = 0 \)).

ii) The \( n \)-th graded component of \( S^{[\infty]}(B) \) is given by

\[
S^{[\infty]}(B)^{n} = \left\{ \begin{array}{ll}
B^{0} & \text{for } n = 0, \\
S^{[n-1]}(B)^{n} & \text{for } n \geq 1,
\end{array} \right.
\]

so that \( S^{[\infty]}(B) = B^{0} \oplus S^{[0]}(B)^{1} \oplus S^{[1]}(B)^{2} \oplus \cdots \).

iii) There is a unique coalgebra homomorphism \( \psi_{B} : B \to T_{B^{0}}^{c}(B^{1}) = T_{c}^{c}(B^{1}) \) such that \( p_{0}^{c} \circ \psi_{B} = p_{B}^{0} \) and \( p_{1}^{c} \circ \psi_{B} = p_{B}^{1} \), where \( p_{i}^{c} : T_{c} \to (T_{c})^{i} \) and \( p_{i}^{B} : B \to B^{i} \) are the canonical projections. The map \( \psi_{B} \) is indeed a graded braided bialgebra homomorphism and as graded braided bialgebras we have \( (S^{[\infty]}(B), \pi_{B}^{c}) \simeq \text{Im} \psi_{B} \).

Proof. Set \( S := S^{[\infty]}(B) \) and set \( S_{i} := S^{[i]}(B) \) for each \( i \in \mathbb{N} \).
By Lemma A.3, $S$ carries a natural graded braided bialgebra structure that makes it the direct limit of $\left( (S_t)_{i \in \mathbb{N}}, (\pi_t^i)_{i,j \in \mathbb{N}} \right)$ as a direct system of graded braided bialgebras. Furthermore the $n$-th graded component of $S^{[\infty]}(B)$ is $S^n := \lim (S_t)^n$, where $(S_t)^n$ denotes the $n$-th graded component of $S_t$.

i) By construction, the family $\left( \Delta^{a,b}_{S_t} \right)_{a,b \in \mathbb{N}}$ of morphisms $\Delta^{a,b}_S : S^{a+b} \to S^a \otimes S^b$ is uniquely determined by $\Delta^{a,b}_S \circ (\pi^\infty)^{a+b} = \left( (\pi^\infty)^a \otimes (\pi^\infty)^b \right) \circ \Delta^{a,b}_{S_{a+b-1}}$, for every $i \in \mathbb{N}$, where $(\pi^\infty)^a : (S_t)^n \to S^n$ is the structural morphism of $S^n = \lim (S_t)^n$.

By Proposition A.2, $\Delta^{a,b}_{S_{a+b+1}}$ is injective for every $a, b \geq 1$ such that $0 \leq a + b \leq i + 1$. In particular for $i = a + b - 1$ we get the equality $\Delta^{a,b}_S \circ (\pi^\infty)^{a+b} = \left( (\pi^\infty)^a \otimes (\pi^\infty)^b \right) \circ \Delta^{a,b}_{S_{a+b-1}}$, where $\Delta^{a,b}_{S_{a+b+1}}$ is injective. Note also that the system

$$((S_0)^t \xrightarrow{(\pi^t_0)^f} (S_1)^t \xrightarrow{(\pi^t_1)^f} \cdots \xrightarrow{(S_{n-1})^t \xrightarrow{(\pi^t_{n-1})^f} (S_t)^t \xrightarrow{(\pi^t_t)^f} \cdots}$$

is stationary as, in view of Proposition A.2, the projection

$$(S_n)^t \xrightarrow{(\pi^t_{n+1})^f} (S_{n+1})^t = \frac{(S_n)^t}{I^n_t}$$

is an isomorphism for every $0 \leq t \leq n + 1$. This entails that $(\pi^\infty)^{t+1}_t : (S_{a+b+1})^t \to S^t$ is an isomorphism for $t \in \{a, b, a + b - 1\}$. Therefore $\Delta^{a,b}_S = \left( (\pi^\infty)^a \otimes (\pi^\infty)^b \right) \circ \Delta^{a,b}_{S_{a+b-1}}$ is injective for every $a, b \geq 1$ and hence for every $a, b \in \mathbb{N}$. Since $S$ is a graded braided bialgebra, by Theorem 1.3, it is strongly $\mathbb{N}$-graded as a coalgebra.

ii) For every $t \in \mathbb{N}$, the morphism $(\pi^t_{i+1})^0 : (S_t)^0 \to (S_{i+1})^0$ is an isomorphism so that we can identify $S^0$ with $(S_0)^0 = B^0$. The morphism $(\pi^\infty)^{t+1}_t : (S_t)^{t+1} \to S^{t+1}$ is an isomorphism so that we can identify $S^{t+1}$ with $(S_t)^{t+1}$ for every $t \in \mathbb{N}$. In particular we have $S^1 = (S_0)^1 = B^1$.

iii) The first part concerning $\psi_B$ follows by the universal property of the cotensor coalgebra. By Theorem 4, the canonical map $\psi_S : S \to T^g_{S^0} \left( S^1 \right) = T^g_{B^0} \left( B^1 \right)$ is injective. For $t = 1, 2$ we have

$$T^{g_0}_{S^0}(S^1) \circ \psi_S \circ \pi^\infty_0 = p^S_t \circ \pi^\infty_0 = (\pi^\infty_0)^t \circ p^B_t = (\pi^t_0)^0 \circ p^B_t \circ T^{g_0}_{B^0}(B^1) \circ \psi_B = p^t_{S^0} \circ T^{g_0}_{B^0}(B^1) \circ \psi_B.$$

Since $\pi^\infty_0$ is a graded braided bialgebra homomorphism, the universal property of cotensor coalgebra entails that $\psi_S \circ \pi^\infty_0 = \psi_B$. Since $\psi_S$ is injective and $\pi^\infty_0$ surjective we get that $S^{[\infty]}(B) = \text{Im}(\psi_B)$. \hfill $\square$

**Corollary A.6.** Let $(B, m_B, u_B, \Delta_B, \varepsilon_B, c_B)$ be a graded braided bialgebra which is strongly $\mathbb{N}$-graded as an algebra. Then as graded braided bialgebras we have $S^{[\infty]}(B) \simeq B^0 \left[ B^1 \right]$, the braided bialgebra of Type one associated to $B^0$ and $B^1$.

**Proof.** Set $S := S^{[\infty]}(B)$. By Theorem A.3, $S^{[\infty]}(B)$ is a graded braided bialgebra which is strongly $\mathbb{N}$-graded as a coalgebra so that, by Theorem 1.3, the canonical map $\psi_S : S \to T^g_{S^0} \left( S^1 \right) = T^g_{B^0} \left( B^1 \right)$ is injective. As a quotient of $B$, which is strongly $\mathbb{N}$-graded as an algebra, by Proposition 3.6, $S$ is strongly $\mathbb{N}$-graded as an algebra too. By Theorem 3.11, this means that the canonical map $\varphi_S : T^{g_0}_{S^0} \left( S^1 \right) \to S$ is surjective. Now, the composition $\psi_S \circ \varphi_S : T^{g_0}_{S^0} \left( S^1 \right) \to T^{g_0}_{S^0} \left( S^1 \right)$ is the unique graded braided bialgebra homomorphism that restricted to $S^0$ and $S^1$ gives the respective inclusion (compare with Theorem 6.8). Hence, its image is, by definition, $S^0 \left[ S^1 \right]$ that is the braided bialgebra of Type one associated to $S^0$ and $S^1$. We conclude by observing that $S^0 = B^0$ and $S^1 = B^1$. \hfill $\square$
Remark A.7. When $B$ is the tensor algebra $T(V,c)$ of a braided vector space $(V,c)$, the previous result should be compared with [Mas2, Proposition 3.2].

Next aim is to show that under suitable assumptions the strength degree of a graded braided bialgebra has an upper bound.

Theorem A.8. Let $(B,m_B,u_B,\Delta_B,\varepsilon_B,c_B)$ be a graded braided bialgebra. The following assertions are equivalent for $n \in \mathbb{N}$.

1. $\text{sdeg}(B) \leq n$.
2. $\pi^n_i : S^{[n]}(B) \to S^{[\infty]}(B)$ is an isomorphism.
3. $S^{[n]}(B)$ is strongly $\mathbb{N}$-graded as a coalgebra.

Proof. (1) $\Rightarrow$ (2) Let $N := \text{sdeg}(B)$. By definition $\pi^{i+1}_n$ is an isomorphism for every $i \geq N$. In particular $\pi^{i+1}_n$ is an isomorphism for every $i \geq n$ whence $\pi^{n}_n$ is an isomorphism too.

(2) $\Rightarrow$ (3) It follows by Theorem A.2.

(3) $\Rightarrow$ (1) By Theorem A.3 $\pi^{n+1}_n : S^{[n]}(B) \to S^{[n+1]}(B)$ is an isomorphism. Now, the uniqueness of $S(\pi_i^{i+1})$ entails $\pi_i^{i+2} = S(\pi_i^{i+1})$. Thus, for every $i \in \mathbb{N}$, inductively one proves that $\pi_i^{i+1}$ is an isomorphism for every $i \geq n$. Hence, the direct system

$$B = S^{[0]}(B) \to S^{[1]}(B) \to S^{[2]}(B) \to \cdots$$

is stationary and $\text{sdeg}(B) \leq n$. 

Theorem A.9. Let $(B,m_B,u_B,\Delta_B,\varepsilon_B,c_B)$ be a graded braided bialgebra which is strongly $\mathbb{N}$-graded as an algebra.

The following assertions are equivalent for $N \geq 1$.

1. $S^{[\infty]}(B)^N = 0$.
2. $S^{[\infty]}(B)^n = 0$, for every $n \geq N$.
3. $S^{[N-1]}(B)^N = 0$.
4. $S^{[N-1]}(B)^n = 0$, for every $n \geq N$.

If one of these conditions is fulfilled, then $\text{sdeg}(B) \leq N - 1$.

Proof. Since $B$ is strongly $\mathbb{N}$-graded as an algebra, by [AM, Proposition 3.6], each graded algebra quotient of $B$ is strongly $\mathbb{N}$-graded too. This entails that both $S^{[\infty]}(B)$ and $S^{[N]}(B)$ are strongly $\mathbb{N}$-graded as an algebras. Clearly, if $A$ is a graded algebra which is strongly $\mathbb{N}$-graded then $A^n = A^{n+N}$. Hence, we deduce that $(1) \Leftrightarrow (2)$ and $(3) \Leftrightarrow (4)$.

By Theorem A.2, $S^{[\infty]}(B)^N = S^{[N-1]}(B)^N$ so that $(1) \Leftrightarrow (3)$.

Now assume (4) and let us prove that $\text{sdeg}(B) \leq N$. By Theorem A.8, it is enough to prove that $S_{N-1} := S^{[N-1]}(B)$ is strongly $\mathbb{N}$-graded as a coalgebra. By Proposition A.2, $\Delta_{S_{N-1}}^{a,b}$ is injective for every $a,b \geq 1$ such that $0 \leq a + b \leq N$. On the other hand $(S_{N-1})^n = 0$, for every $n \geq N$, so that $\Delta_{S_{N-1}}^{a,b} : \{0\} \to (S_{N-1})^a \otimes (S_{N-1})^b$ is injective for every $a,b \geq 1$ such that $a + b \geq N$. In particular we have proved that $\Delta_{S_{N-1}}^{n,1}$ is injective for every $n \in \mathbb{N}$. By Theorem A.3, $S_{N-1}$ is strongly $\mathbb{N}$-graded as a coalgebra.

Definition A.10. Recall that the Hilbert-Poincaré series of a graded algebra $(B,m_B,u_B)$ is by definition

$$\mathfrak{h}(B) := \sum_{n \in \mathbb{N}} \dim_K (B^n) X^n \in K[[X]].$$

Corollary A.11. Let $(B,m_B,u_B,\Delta_B,\varepsilon_B,c_B)$ be a graded braided bialgebra which is strongly $\mathbb{N}$-graded as an algebra.

i) Assume that $B^0[B^1]$ as a graded braided bialgebra divides out $T_{B^0}(B^1)$ by relations in degree not greater then $N$. Then $\text{sdeg}(B) \leq N - 1$.

ii) If $B^0[B^1]$ is finite dimensional, then $\text{sdeg}(B) \leq \text{deg}(\mathfrak{h}(B^0[B^1])) - 1 \leq \dim_K B^0[B^1] - 1$. 

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Proof. Set $S := S^{[\infty]}(B)$ and set $S_i := S^{[i]}(B)$ for each $i \in \mathbb{N}$. Since $B$ is strongly $\mathbb{N}$-graded as an algebra, by Corollary 4.8, we have $B^0[B^1] = S$.

i) By Proposition 4.4, $\Delta_{S}^{a,b}$ is injective for every $a, b \geq 1$ such that $0 \leq a + b \leq n + 1$. In particular $\Delta_{S^{[n-1]}}^{a,b}$ is injective for every $a, b \geq 1$ such that $0 \leq a + b \leq N$. Now, by definition we have

$$S_N = S(S_{N-1}) = \frac{S_{N-1}}{(E(S_{N-1}))} = \left( \bigoplus_{n \in \mathbb{N}} E_n(S_{N-1}) \right) = \left( \bigoplus_{n \geq N+1} E_n(S_{N-1}) \right).$$

Since $S^m = \lim_{n \to \infty} S^n$ and $S$ is defined by relations in degree not greater then $N$, from $E_n(S_{N-1}) \subseteq (S_{N-1})^n$ we deduce that $\bigoplus_{n \geq N+1} E_n(S_{N-1}) = 0$. Hence $S_{N-1} \simeq S_N$, so that $S_{N-1}$ is strongly $\mathbb{N}$-graded as a coalgebra. By Theorem 4.3, $\text{sdeg}(B) \leq N$.

ii) Since $B^0[B^1]$ is finite dimensional, then $\text{deg}(\text{h}(B^0[B^1]))$ is finite and $S^{[\infty]}(B)^n = 0$, for every $n \geq \text{deg}(\text{h}(B^0[B^1]))$. Either by the first part or by Theorem 4.3, we conclude. \(\square\)

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