osp(1|32) and Extensions of super-AdS$_5 \times S^5$ algebra

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Abstract

The super-AdS$_5 \times S^5$ and the four-dimensional $\mathcal{N}=4$ superconformal algebras play important roles in superstring theories. It is often discussed the roles of the osp(1|32) algebra as a maximal extension of the superalgebras in flat background. In this paper we show that the su(2,2|4), the super-AdS$_5 \times S^5$ algebra or the superconformal algebra, is not a restriction of the osp(1|32) though the bosonic part of the former is a subgroup of the latter. There exist only two types of u(1) extension of the super-AdS$_5 \times S^5$ algebra if the bosonic AdS$_5 \times S^5$ covariance is imposed. Possible significance of the results is also discussed briefly.

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1 Introduction

All possible extensions of the super-Poincare algebra correspond to branes that can exist on the flat background [1]. The maximal extension of the super-Poincare algebra in 10 dimensions contains 32 supercharges and 528 bosonic generators (in addition to Lorentz generators). It is the osp(1|32) algebra and the symmetries of IIA, IIB, M and F theories have been examined systematically [2].

A D-brane action consists of a Dirac-Born-Infeld (DBI) action and a Wess-Zumino (WZ) action. The WZ action is needed for the total action to possess $\kappa$-symmetry, which allows to project out half of the world-volume fermions and matches bosonic and fermionic degrees of freedom on the world-volume[4]. It was shown[5] that commutation relations among two of the Noether charges do not close but include topological brane charges, because the WZ action is quasi-superinvariant. In other words, the super-Poincare algebra is modified to include topological brane charges and becomes the extended super-Poincare algebra with brane charges\(^2\). The half of the 32 supercharges are those for Nambu-Goldstone (NG) fermions associated with supersymmetries broken by the brane. In fact, for a given brane configuration with the brane tension being equal to the brane charge, anti-commutation relation of two of supercharges turns out to be proportional to a projection operator which projects out half of the supersymmetries on the vacuum.

These brane backgrounds are constructed as solutions of supergravity theories. Constructing Killing vectors and spinors in a brane background, one can show that the super-isometry algebra contains a smaller number of supercharges than 32. In order to recover (super)symmetries broken by the brane, one examines small fluctuations around this solution. The broken symmetries are recovered in terms of NG fields and the resulting superalgebra contains 32 supercharges and brane charges.

For a curved background, D-brane actions are shown [6] to be $\kappa$-symmetric if the background satisfies the field equations of the target space supergravity. Since the AdS\(_5\) × S\(_5\) background is a solution of type-IIB supergravity, one can construct $\kappa$-symmetric D-brane action on it. Such D-brane actions have been examined in [7, 8, 9]. The amount of unbroken supersymmetries are determined by the balance between the $\kappa$-symmetry projection and Killing spinor equations.

One expects that the WZ action for such a brane produces brane charges in the superalgebra su(2,2|4), which is the super-isometry algebra of the AdS\(_5\) × S\(_5\) background, and the maximal extension is osp(1|32), referring the classification by Nahm [10]. The anti-commutator between two of supercharges are examined in [11] for a matrix theory on the eleven-dimensional pp-wave and in [12] for a superstring action on the AdS\(_5\) × S\(_5\) background, and are shown to include brane charges. However, the full supersymmetry

\(^1\)Superalgebras with a larger number of supercharges than 32 have been discussed in [3].

\(^2\)The brane charge is not a center of the extended super-Poincare algebra, because they have non-trivial commutation relations with Lorentz generators. These brane charges are ideal of the extended super-Poincare algebra, and the original algebra is represented as a coset, (the extended super-Poincare algebra)/ideal.
algebra has not been derived yet because of the complexity. To derive the full supersymmetry algebra, one may try to examine small fluctuations around a D-brane solution of a supergravity theory. But supergravity solutions for Dp-branes in the AdS$_5 \times S^5$ background have not been known well yet.

In order to achieve this, we take another route in this paper. We examine the osp(1|32) and relations to the super-AdS$_5 \times S^5$ algebra, su(2,2|4). The bosonic AdS$_5 \times S^5$ algebra, so(4,2)\times so(6), is a subalgebra of sp(32). However, su(2,2|4) is not a subalgebra of osp(1|32) because commutation relations, for example $\{Q, Q\}$, differ each other. Algebraically, the osp(1|32) is not an extension of su(2,2|4) because sp(32) is not an extension of so(4,2)\times so(6).

In this paper we find that the su(2,2|4) is not a restriction of the osp(1|32) in the following sense. Expressing the osp(1|32) algebra in an AdS$_5 \times S^5$ covariant basis and restricting the bosonic generators of sp(32) to those of so(4,2)\times so(6), the resulting commutation relations have forms of super-AdS$_5 \times S^5$ algebra apart from some sign differences. We find that they are neither those for the su(2,2|4) nor a consistent superalgebra. This is essentially because the Fierz identity for the one hand does not imply that for the other hand. The same argument is true for the four-dimensional $\mathcal{N} = 4$ superconformal algebra.

In the reverse, we examine possible generalizations of the super-AdS$_5 \times S^5$ algebra. When we consider some (brane) solutions in the super-AdS$_5 \times S^5$ background they break also manifest AdS$_5$ and/ or $S^5$ covariance. However it is a spontaneous symmetry breaking in the presence of a particular solution. The AdS$_5 \times S^5$ covariance of the theory would be still maintained. Thus, we impose the assumption that the bosonic AdS$_5 \times S^5$ algebra is a subalgebra of the bosonic part of the superalgebra with brane charges, and find that only two trivial types of u(1) extension of the super-AdS$_5 \times S^5$ algebra are allowed.

This paper is organized as follows. In the next section, we express the osp(1|32) algebra in an AdS$_5 \times S^5$ covariant basis. In section 3, we show that the super-AdS$_5 \times S^5$ algebra is not a restriction of the osp(1|32) algebra. In section 4, we find that only two types of u(1) extension of the super-AdS$_5 \times S^5$ algebra are allowed, under the assumption that bosonic AdS$_5 \times S^5$ algebra is a subalgebra of the bosonic part of the superalgebra with brane charges. The section 5 is devoted to a summary and discussions on possible significance. The uniqueness of the osp(1|32) algebra is presented in appendix A. In appendix B, complementary to the section 3, we show that the four-dimensional $\mathcal{N} = 4$ superconformal algebra is not a restriction of osp(1|32), by clarifying the relation between generators of the super-AdS$_5 \times S^5$ algebra and those of the four-dimensional $\mathcal{N} = 4$ superconformal algebra.

2 \quad \textbf{osp}(1|32)

The osp(1|32) is a maximally extended supersymmetry algebra whose generators are 32 supercharges $Q_A$ ($A = 1, 2, ..., 32$) and 528 bosonic $Z_{AB}$. The algebra is [2]

\[
\{Q_A, Q_B\} = Z_{AB},
\]  

(2.1)
\[ [Q_A, Z_{BC}] = Q_{(B\Omega A_C)}, \]  
\[ [Z_{AB}, Z_{CD}] = \Omega_{A(C} Z_{D)B} + \Omega_{B(C} Z_{D)A} \]  
\[ (2.2) \]
\[ (2.3) \]

where \( Z_{AB} \) is 32\times32 symmetric matrix and \( \Omega_{AB} \) is an anti-symmetric invertible symplectic metric. The Jacobi identities hold identically for any anti-symmetric \( \Omega \). It is shown in appendix A that any maximally extended supersymmetric algebra is expressed in the form of (2.1)-(2.3) by redefinitions.

The supercharge \( Q_A \)'s are 32 complex generators and are subject to a Majorana condition reducing the number of independent degrees of freedom to half,

\[ Q_A^\dagger = Q_{CA} B^{CA} = B^{tAC} Q_C \]  
\[ (2.4) \]

for some non-singular matrix \( B \) satisfying

\[ \Omega_{AB}^\dagger = (B^t\Omega B)^{AB}, \quad \text{and} \quad B^{AB} B_{BC}^* = \delta^A_C. \]  
\[ (2.5) \]

The \( \text{osp}(1|32) \) algebra (2.1)-(2.3) remains unchanged under the conjugation if \( Z \)'s satisfy

\[ Z_{AB}^\dagger = (B^tZB)^{AB}. \]  
\[ (2.6) \]
in addition to (2.4) and (2.5).

In this paper we are interested in relations among the \( \text{osp}(1|32) \), super-AdS_{5} \times S_{5} and superconformal algebras we represent the 32 component supercharge \( Q_A \) as a \( SO(4,1) \times SO(5) \times SO(2,1) \) spinor, [13]

\[ Q_A = Q_{\alpha\alpha'A}, \quad (\alpha = 1, 2, 3, 4, \quad \alpha' = 1, 2, 3, 4, \quad A = 1, 2), \]  
\[ (2.7) \]

where \( \mathcal{A} \) is a collective index of \((\alpha\alpha'A)\). The gamma matrices and charge conjugation for \( SO(4,1) \) satisfy,

\[ \{\gamma^a, \gamma^b\} = 2\eta^{ab} = 2 \text{diag}(-+++), \quad \gamma^0\gamma^a\gamma^0 = \gamma^a \]
\[ C^a = -C, \quad C^a C = 1, \quad C\gamma^a C^{-1} = (\gamma^a)^t, \]
\[ (C\gamma^a)^t = -(C\gamma^a) \quad (C\gamma^{ab})^t = (C\gamma^{ab}), \]  
\[ (2.8) \]

those for \( SO(5) \) are,

\[ \{\gamma^{a'}, \gamma^{b'}\} = 2\eta^{a'b'} = 2 \text{diag}(++++), \quad \gamma^{a'}\gamma^{a'} = \gamma^{a'} \]
\[ C^{a'} = -C', \quad C^{a'} C' = 1, \quad C'\gamma^{a'} C'^{-1} = (\gamma^{a'})^t, \]
\[ (C'\gamma^{a'})^t = -(C'\gamma^{a'}) \quad (C'\gamma^{a'b'})^t = (C'\gamma^{a'b'}). \]  
\[ (2.9) \]

For \( SO(2,1) \) they are, \((j = \hat{0}, \hat{1}, \hat{2})\),

\[ \{\rho^i, \rho^j\} = 2\eta^{ij} = 2 \text{diag}(-,+), \quad \rho^0\rho^i\rho^0 = \rho^i, \]
\[ c = -c^t, \quad c^t c = 1, \quad c\rho^i c^{-1} = -(\rho^i)^t \]
\[ (c\rho^i)^t = (c\rho^i). \]  
\[ (2.10) \]
The real forms of $c$ and $\rho$'s we use here are

$$
c = i\tau_2, \quad \rho^j = (i\tau_2, \tau_1, -\tau_3), \quad cp^j = (-1, \tau_3, \tau_1).
$$

(2.11)

Using these notations the symplectic metric $\Omega$ is taken to be proportional to a charge conjugation $C$,

$$
\Omega = s \, C, \quad s^* = s,
$$

(2.12)

where $C$ is a total charge conjugation defined as

$$
C \equiv c \, C \, C', \quad C^t = -C, \quad C^t C = 1.
$$

(2.13)

The Majorana condition (2.4) is imposed as

$$
Q^\dagger = Q \, B, \quad B = -e^{i\chi} (\rho^0 \gamma^0) C^{-1}.
$$

(2.14)

$e^{i\chi}$ is a phase ambiguity and taken to be 1 in the following. The explicit form of (2.14) in component form is one of [13]

$$
Q^i = Q_{\beta\alpha} \, \delta^B_{\beta\alpha} \, (\gamma^0 C^{-1})^{\beta\alpha} \, (C'^{-1})^{\beta'\alpha'}.
$$

(2.15)

The condition (2.5) is verified as

$$
(B^t \Omega B) = (\rho^0 \gamma^0 C^{-1})^t \, (s C) \, (\rho^0 \gamma^0 C^{-1}) = s \, C^{-1} = \Omega^\dagger.
$$

(2.16)

Using products of these gamma matrices we can construct a complete basis of symmetric matrices $\{C\Gamma^{(i)}\}_{AB}$, ($i = 1, 2, ..., 528 = \frac{32 \cdot 33}{2}$),

$$
C\Gamma^{(i)} = (C\Gamma^{(i)})^t, \quad i.e. \quad \Gamma^{(i)t} = -C\Gamma^{(i)}C^{-1}.
$$

(2.17)

They are

$$
\begin{align*}
\rho^j & \equiv \Gamma^j, \quad (3, 1, 1) \\
i \, \rho^j \gamma^a & \equiv \Gamma^{ja}, \quad (3, 5, 1) \\
\rho^j \gamma^{a'} & \equiv \Gamma^{ja'}, \quad (3, 1, 5) \\
\gamma^{ab} & \equiv \Gamma^{ab}, \quad (1, 10, 1) \\
\gamma^{a'b'} & \equiv \Gamma^{a'b'}, \quad (10, 1, 1) \\
\gamma^{ab} \gamma^{c'} & \equiv \Gamma^{abc'}, \quad (10, 5) \\
i \, \gamma^a \gamma^{b'c'} & \equiv \Gamma^{ad'b'}, \quad (10, 10) \\
i \, \rho^j \gamma^a \gamma^{a'} & \equiv \Gamma^{jaa'}, \quad (3, 5, 5) \\
\rho^j \gamma^{ab} \gamma^{a'b'} & \equiv \Gamma^{jaab'}, \quad (3, 10, 10)
\end{align*}
$$

(2.18)

where $(a, b, c)$ indicates the decomposition under $SO(2, 1) \times SO(4, 1) \times SO(5)$ representations. Factor “$i$” is introduced in (2.18) for the conjugation (2.15) so that they satisfy

$$
\rho^j \gamma^0 \Gamma^{(i)t} \rho^0 \gamma^0 = -\Gamma^{(i)}.
$$

(2.19)
\[ (C \Gamma^{(ab)}) = (C \gamma^{ab}) \quad \Rightarrow \quad (\Gamma_{(ab)} C^{-1}) = (\gamma_{ba} C^{-1}), \quad (2.20) \]
\[ (C \Gamma^{(ja)}) = (i C \rho^a \gamma^a) \quad \Rightarrow \quad (\Gamma_{(ja)} C^{-1}) = (-i \rho_j \gamma_a C^{-1}). \quad (2.21) \]

They satisfy orthonormal and completeness relations
\[
\frac{1}{32} (\Gamma_{(i)} C^{-1})^{BA}_{AB} (C \Gamma^{(j)})_{AB} = \delta_i^j. \quad (2.22)
\]
\[
\frac{1}{32} (C \Gamma^{(i)})_{AB} (\Gamma_{(i)} C^{-1})^{CD}_{CD} = \frac{1}{2} \delta_A^{(C} \delta_B^{D)}, \quad (2.23)
\]

or equivalently
\[
\frac{1}{32} (C \Gamma^{(i)})_{AB} (C \Gamma^{(j)})_{CD} = \frac{1}{2} C_{C(A} C_{B)D}. \quad (2.24)
\]

We expand the bosonic generators \( Z_{AB} \) using \( C \Gamma^{(i)} \) as
\[
Z_{AB} = a \sum_{i=1}^{528} (C \Gamma^{(i)})_{AB} Z_{(i)}, \quad Z_{(i)} = \frac{1}{32 a} (\Gamma_{(i)} C^{-1})^{BA}_{AB} Z_{AB}, \quad (2.25)
\]
where \( a \) is a constant. From \((2.6)\) \( Z_{(j)} \) 's are anti-hermitic for real choice of \( a \),
\[
Z_{(i)}^* = -Z_{(i)}, \quad a^* = a. \quad (2.26)
\]

In terms of \( Z_{(i)} \) the osp(1\(|32\)) algebra \((2.1)-(2.3)\) is expressed as
\[
\{ Q_A, Q_B \} = a (C \Gamma^{(i)})_{AB} Z_{(i)}, \quad (2.27)
\]
\[
[ Q_A, Z_{(i)} ] = \frac{1}{2} (Q \Gamma^{(i)})_A, \quad (2.28)
\]
\[
[ Z_{(i)}, Z_{(j)} ] = f_{ij}^{\hat{K}} Z_{(\hat{K})}, \quad (2.29)
\]

where \( f_{ij}^{\hat{K}} \) is the structure constants of \( sp(32) \),
\[
f_{ij}^{\hat{K}} = -\frac{1}{32} \text{tr}(\Gamma_{(i)}) \Gamma_{(j)}^{(\hat{K})} \Gamma_{(\hat{K})} \quad (2.30)
\]
\( s = -8a \) is a normalization convention. In this form the osp(1\(|32\)) algebra the Jacobi identities are guaranteed by \((2.24)\)
\[
\sum_{\text{cyclic } BCD} (C \Gamma^{(i)})_{AB} (C \Gamma^{(j)})_{CD} = 0, \quad (2.31)
\]

and the Jacobi identity of \( f_{ij}^{\hat{K}} \) as
\[
\sum_{\text{cyclic } i \hat{K} \hat{K}} f_{ij}^{\hat{L}} f_{\hat{L} \hat{K}}^{\hat{M}} = 0, \quad f_{ij}^{\hat{K}} = -f_{ji}^{\hat{K}}, \quad (2.32)
\]
\[
[\frac{1}{2} \Gamma_{(i)}, \frac{1}{2} \Gamma_{(j)}] = -f_{ij}^{\hat{K}} \frac{1}{2} \Gamma_{(\hat{K})}. \quad (2.33)
\]

\(^{3}\)The Killing metric \( g_{ij} = \frac{1}{32} \text{tr}(\Gamma_{(i)} \Gamma_{(j)}) \) is diagonal. The diagonal elements are either 1 or \(-1\) and \( \Gamma_{(i)} = g_{ij}^{\Gamma(j)} \) is inverse of \( \Gamma^{(i)} \). \( f_{ij}^{\hat{L}} g_{kL} f_{ij}^{\hat{L}} \) is totally anti-symmetric.
3 Super $\text{AdS}_5 \times S^5$

The bosonic part of the $\text{osp}(1|32)$ algebra is $sp(32)$ and contains a set of bosonic generators forming a subalgebra, $so(4,2) \times so(6) \subset sp(32)$. The $SO(4,2)$ is generated by

$$Z_{ab}, \quad Z_{0,a}, \quad (a, b = 0, 1, 2, 3, 4),$$

where $Z_{0,a}$ is $j = 0$ element of $Z_{ja}$. The algebra is

$$[Z_{0,a}, Z_{0,b}] = Z_{ab}, \quad [Z_{0,a}, Z_{cd}] = \eta_{a[c} Z_{0,d]},$$

$$[Z_{ab}, Z_{cd}] = \eta_{ad} Z_{bc} - \eta_{bd} Z_{ac} - \eta_{ac} Z_{bd} + \eta_{bc} Z_{ad}. \quad (3.2)$$

The $SO(6)$ is generated by

$$Z_{a'b'}, \quad Z_{0,a'}, \quad (a', b' = 1', 2', 3', 4', 5'),$$

as

$$[Z_{0,a'}, Z_{0,b'}] = -Z_{a'b'}, \quad [Z_{0,a'}, Z_{c'd'}] = \eta_{a'[c} Z_{0,d]},$$

$$[Z_{a'b'}, Z_{c'd'}] = \eta_{a'd'} Z_{b'c'} - \eta_{b'd'} Z_{a'c'} - \eta_{a'c'} Z_{b'd'} + \eta_{b'c'} Z_{a'd'}. \quad (3.4)$$

Note the sign difference of (3.2) and (3.4) so that their algebras are $so(4,2)$ and $so(6)$ respectively. They are identified with the $\text{AdS}_5 \times S^5$ generators and the algebra as

$$Z_{0,a} = P_a, \quad Z_{ab} = M_{ab}, \quad Z_{0,a'} = P_{a'}, \quad Z_{a'b'} = M_{a'b'}, \quad \eta_{c'} Z_{0,d'],}$$

$$[P_a, P_b] = M_{ab}, \quad [P_{a'}, P_{b'}] = -M_{a'b'},$$

$$[P_a, M_{bc}] = \eta_{ab} P_c - \eta_{ac} P_b, \quad [P_{a'}, M_{b'c'}] = \eta_{a'b'} P_{c'} - \eta_{a'c'} P_{b'},$$

$$[M_{ab}, M_{cd}] = \eta_{bc} M_{ad} + 3 \text{ terms}, \quad [M_{a'b'}, M_{c'd'}] = \eta_{b'c'} M_{a'd'} + 3 \text{ terms}, \quad (3.6)$$

where $P_a$ and $P_{a'}$ are $\text{AdS}_5$ and $S^5$ momenta respectively. The $\text{AdS}_5 \times S^5$ is a subalgebra of the $sp(32)$ thus of $\text{osp}(1|32)$.

The supersymmetric $\text{AdS}_5 \times S^5$ algebra is the $su(2,2|4)$ and has been extensively discussed in the superstring context. It has the same supercharge contents as those of $\text{osp}(1|32)$. As the $\text{osp}(1|32)$ is maximally generalized superalgebra the anti-commutator of two supercharges contains all bosonic charges as (2.27) while that of super-$\text{AdS}_5 \times S^5$ contains only bosonic $\text{AdS}_5 \times S^5$ charges. In this sense super-$\text{AdS}_5 \times S^5$ cannot be a subalgebra of $\text{osp}(1|32)$. In this section we examine if the super-$\text{AdS}_5 \times S^5$ algebra is obtained by a “restriction” of the $\text{osp}(1|32)$ algebra or not.\footnote{The “restriction” means 1) dropping the commutation relations for the extra generators $Z_{(I''}, \quad [*, Z_{(I'')} = ..., \text{ and 2) dropping the extra generators appearing in the r.h.s. of the (anti-)commutators.}}$ In other words if the $\text{osp}(1|32)$ is any generalized superalgebra associated with the super-$\text{AdS}_5 \times S^5$.

The $QZ$ commutators of the $\text{osp}(1|32)$ algebra are given in (2.28) and those for $\text{AdS}_5 \times S^5$ generators are read as

$$[Q_{\alpha \alpha'}, Z_{0,a}] = \frac{1}{2} (Q_{\gamma a} (-\iota \rho_0))_{\alpha \alpha'}, \quad [Q_{\alpha \alpha'}, Z_{0,a'}] = \frac{1}{2} (Q_{\gamma a'} \rho_0)_{\alpha \alpha'},$$

$$[Q_{\alpha \alpha'}, Z_{ab}] = \frac{1}{2} (Q_{\gamma ba})_{\alpha \alpha'}, \quad [Q_{\alpha \alpha'}, Z_{a'b'}] = \frac{1}{2} (Q_{\gamma b'a'})_{\alpha \alpha'}. \quad (3.7)$$
They show the covariance of $Q$ under the bosonic $\text{AdS}_5 \times S^5$ transformations,

\[ [Q_A, P_a] = \frac{i}{2} Q_B \gamma_a \epsilon_{BA}, \quad [Q_A, P_{a'}] = -\frac{1}{2} Q_B \gamma_{a'} \epsilon_{BA}, \]

\[ [Q_A, M_{ab}] = -\frac{1}{2} Q_A \gamma_{ab}, \quad [Q_A, M_{a'b'}] = -\frac{1}{2} Q_A \gamma_{a'b'}, \]

(3.8)

where $\epsilon = i \tau_2$. The explicit form of the osp$(1|32)$ $\{QQ\}$ algebra is

\[
\{Q_{\alpha\alpha'}, Q_{\beta\beta'}\} = \frac{a}{2} \sum_{C} [ (C_{\gamma}^{\alpha\beta})_{\alpha\beta'} (C^C_{\alpha'})_{\alpha'} Z_{ab} + (C_{\alpha\beta})_{\alpha'} (C^C_{\gamma}^{\alpha'\beta'})_{\alpha'} Z_{a'b'} + 
+ (C_{\gamma}^{\alpha})_{\alpha'} (C^C_{\gamma'}^{\alpha'})_{\alpha'} Z_{ab'} + (C_{\alpha\beta})_{\alpha'} (C^C_{\gamma'}^{\alpha'})_{\alpha'} Z_{a'b'} ] 
+ (C_{\alpha\beta})_{\alpha'} (C^C_{\alpha'})_{\alpha'} \left( Z_j + i (C_{j\alpha\beta})_{\alpha'} (C^C_{\gamma}^{\alpha'})_{\alpha'} Z_{ja'} + (C_{\alpha\beta})_{\alpha'} (C^C_{\gamma}^{\alpha'})_{\alpha'} Z_{ja''} + (C_{\alpha\beta})_{\alpha'} (C^C_{\gamma}^{\alpha'})_{\alpha'} Z_{ja'a''} + (C_{\alpha\beta})_{\alpha'} (C^C_{\gamma}^{\alpha'})_{\alpha'} Z_{ja'a''} \right) 
+ \sum_{\ldots} \ldots 

(3.9)

where $\ldots$ terms are the osp$(1|32)$ generators that are not included in the AdS$_5 \times S^5$. It is compared with that of the super-AdS$_5 \times S^5$ algebra

\[
\{Q_{\alpha\alpha'}, Q_{\beta\beta'}\} = 2\delta_{AB} \left[ -i (C_{\gamma}^{\alpha})_{\alpha'} (C^C_{\gamma}^{\alpha'})_{\alpha'} P_a + (C_{\alpha\beta})_{\alpha'} (C^C_{\gamma}^{\alpha'})_{\alpha'} P_{a'} \right] 
+ \epsilon_{AB} \left[ (C_{\alpha\beta})_{\alpha'} (C^C_{\gamma}^{\alpha'})_{\alpha'} M_{ab} + (C_{\alpha\beta})_{\alpha'} (C^C_{\gamma}^{\alpha'})_{\alpha'} M_{a'b'} \right]. 

(3.10)

The AdS$_5 \times S^5$ generators in the osp$(1|32)$ algebra (3.9) and the super-AdS$_5 \times S^5$ algebra in (3.10) have different signs. If we adjust the value $a = 2$ so that the coefficients of $P$ and $M$ in (3.9) and (3.10) coincide, those of $P'$ and $M'$ have opposite signs, and vice versa for $a = -2$.

The sign difference is required by the closure of the algebra. The (QQQ) Jacobi identity of (3.9) holds due to the presence of all osp$(1|32)$ generators in the r.h.s. of the (3.9) using with (2.31) obtained from the completeness relation (2.24)

\[
\sum_{\text{cyclic BC}D} \sum_{\text{all } i} (C^{(i)}_{AB})_{\text{CD}} (C^{(i)}_{CD})_{AB} = 0. 
\]

(3.11)

If we would drop $\ldots$ terms in the (3.9) the Jacobi identity no more holds

\[
\sum_{\text{cyclic BC}D} \sum_{1 \in \text{AdS}_5 \times S^5} (C^{(i)}_{AB})_{\text{CD}} (C^{(i)}_{CD})_{AB} \neq 0. 
\]

(3.12)

In (3.9) the relative signs of $P'$ and $M'$ terms and $P$ and $M$ terms are different from those of (3.10). It turns the Jacobi identity for (3.10) holds using an identity independent of (3.11):

\[
\sum_{\text{cyclic BC}D} \left( \frac{1}{2} (C^{\alpha\beta}_{AB})_{\text{CD}} (C^{\alpha\beta}_{CD})_{AB} + (C^{\hat{0}\alpha}_A)_{AB} (C^{\hat{0}\alpha}_A)_{CD} 
- \frac{1}{2} (C^{\alpha'b'}_{AB})_{\text{CD}} (C^{\alpha'b'}_{CD})_{AB} + (C^{\hat{0}\alpha'}_A)_{AB} (C^{\hat{0}\alpha'}_A)_{CD} \right) = 0. 
\]

(3.13)
It is proved by using the Fierz identities, which are valid both for $\gamma$, and $\gamma'$,

$$
\frac{1}{2} (C\gamma_{ab})_{\alpha\beta} (C\gamma_{ab})_{\gamma\delta} = 2 \ (C)_{(\alpha\gamma} (C)_{\beta)\delta},
$$

$$
(C\gamma_{a})_{\alpha\beta} (C\gamma_{a})_{\gamma\delta} = 2 \ (C)_{[\alpha\gamma} (C)_{\beta]\delta} - C_{\alpha\beta} C_{\gamma\delta}.
$$

(3.14)

In summary we have shown that the supersymmetric AdS$_5$×S$^5$ algebra is not a restriction of the osp(1|32) and maximal generalization of the super-AdS$_5$×S$^5$ algebra, if any, is not the osp(1|32) superalgebra. In appendix B we obtain the same result for the superconformal algebra in 4 dimensions as it is also isomorphic to the su(2,2|4).

4 Extensions of the super-AdS$_5$×S$^5$ algebra

In the last section we have shown the super-AdS$_5$×S$^5$ algebra is not a restriction of the osp(1|32) . In this section we will find possible generalization of the super-AdS$_5$×S$^5$ . “Generalization” here means that the AdS$_5$×S$^5$ algebra is obtained by a restriction of the generators to those of AdS$_5$×S$^5$ in the generalized algebras. In the generalized algebras we keep the covariance under AdS$_5$×S$^5$ . When we consider some (brane) solutions in the super-AdS$_5$×S$^5$ background they break also manifest AdS$_5$ and/or S$^5$ covariance. However it is a spontaneous symmetry breaking in the presence of a particular solution. The AdS$_5$×S$^5$ covariance of the theory would be still maintained. Thus we impose the assumption that the bosonic AdS$_5$×S$^5$ algebra is a subalgebra of the bosonic part of the superalgebra with brane charges.

Now we try to add bosonic generators on the super-AdS$_5$×S$^5$ algebra so that the commutators satisfy the Jacobi identities . We classify possible forms of the bosonic generators in the basis of (2.18) as

$$
\tilde{I} \rightarrow \begin{pmatrix}
I & : & Z_{(\hat{0},a)}, Z_{(ab)} \\
I' & : & Z_{(\hat{0},a')}, Z_{(a'b')} \\
I''_{(0)} & : & Z_{(0b')} \\
I''_{(1)} & : & Z_{(abc')}, Z_{(ab',c')}, Z_{(0,ab')} \\
I''_{(2)} & : & Z_{(j,a)}, Z_{(j,a')}, Z_{(j,ab')} \\
\end{pmatrix}
$$

(4.1)

$Z_{(I)}$, $(I = (\hat{0},a), (ab))$ are AdS$_5$ generators and $Z_{(I')}$, $(I' = (\hat{0},a'), (a'b'))$ are S$^5$ generators, $(\hat{0} \text{ means } j = 0)$. They have subalgebra structures

$$
(AdS_5 + S^5) \subset (AdS_5 + S^5 + u(1)) \subset (gl(16)) \subset (sp(32)).
$$

(4.2)

$(I + I' + I''_{0})$ can also be split into $(I + I' + I''_{1}) + (I''_{0})$, which is sl(16) + u(1).
The bosonic part of algebra is

\[ [Z(i), Z(j)] = s_Is_J f_{IJ}^K Z_{(K)} \equiv g_{IJ}^K Z_{(K)}, \quad (4.3) \]

where \( f_{IJ}^K \) is the structure constants of \( sp(32) \) (2.30). Values of \( s_I \)'s are listed below and are either “0” or “1” depending on which bosonic subalgebra is taken into account

\[
\begin{align*}
(AdS_5 + S^5) & \quad (AdS_5 + S^5 + U(1)) & \quad (gl(16)) & \quad (sp(32)) \\
s_I = 1 & \quad s_I = 1 & \quad s_I = 1 & \quad s_I = 1 \\
s_\nu = 1 & \quad s_\nu = 1 & \quad s_\nu = 1 & \quad s_\nu = 1 \\
s_{I_0''} = 0 & \quad s_{I_0''} = 1 & \quad s_{I_0''} = 1 & \quad s_{I_0''} = 1 \\
s_{I_1''} = 0 & \quad s_{I_1''} = 0 & \quad s_{I_1''} = 1 & \quad s_{I_1''} = 1 \\
s_{I_2''} = 0 & \quad s_{I_2''} = 0 & \quad s_{I_2''} = 0 & \quad s_{I_2''} = 1. \\
\end{align*}
\]

\( (ZZZ) \) Jacobi identity is verified since the structure constant \( g_{IJ}^K \) satisfies the Jacobi relation for each subalgebra of (4.4)

\[
\sum_{cyclic \, IJK} g_{IJ}^L g_{KL}^M = 0. \quad (4.5)
\]

\( QZ \) commutators are

\[
[Q_A, Z(i)] = \frac{s_I}{2} (Q \Gamma (i))_A, \quad (4.6)
\]

and \( (QZZ) \) Jacobi identity requires

\[
\begin{align*}
[Q_A, [Z(i), Z(j)]] & + [Z(j), [Q_A, Z(i)]] + [Z(i), [Z(j), Q_A]] \\
= \frac{s_K}{2} (Q \Gamma (K))_A g_{IJ}^K - \frac{s_I s_J}{4} (Q [\Gamma (i), \Gamma (j)])_A \\
= \frac{1}{2} (s_K - 1) s_I s_J \hat{f}_{IJ}^K (Q \Gamma (K))_A = 0. \quad (4.7)
\end{align*}
\]

The subgroup structure guarantees that it vanishes for each subalgebra of (4.4). For example for \( gl(16) \), \( s_{K_2''} = 0 \) but \( f_{IJ}^K \) for \( i, j \neq I_2'' \) which \( s_I = s_J = 1 \). Then the Jacobi identity is satisfied.

From the \( AdS_5 \times S^5 \) covariance the extension of the \( QQ \) anti-commutator of the super-\( AdS_5 \times S^5 \) will be

\[
\{Q_A, Q_B\} = a \left( (C \Gamma (i))_{AB} Z_{(i)} - (C \Gamma (i'))_{AB} Z_{(i')} + a_{i''} (C \Gamma (i''))_{AB} Z_{(i'')} \right) \\
= a \sum_{i=i', i', i''} a_I (C \Gamma (i))_{AB} Z_{(i)}. \quad (4.8)
\]

where the coefficients of \( AdS_5 \) and \( S^5 \)

\[
a_I = 1, \quad a_{i'} = -1 \quad (4.9)
\]
are fixed from the super-AdS$_5 \times $S$^5$ algebra (3.10). As will be clear from (4.11) $a_{I'_1}$ ($a_{I'_2}$) takes the same value for all generators of $Z(I'_1)$ ($Z(I'_2)$).

The (QQZ) Jacobi identity requires
\[
\{Q_A, [Q_B, Z(i_j)]\} + [Z(j), \{Q_A, Q_B\}] - \{Q_B, [Z(j), Q_A]\} = a \left( \frac{s_i a_{j} (C[\Gamma^{(j)}, \Gamma_{(j)}])_{AB}}{2} Z(j) + a_{j} (C[\Gamma^{(j)})_{AB} g_{Ij} K Z(K) \right) = a s_i (-a_K + a_{j} s_{j}) f_{Ij} K (C[\Gamma^{(j)})_{AB} Z(K) = 0,
\]
(4.10)
where we have used (2.17). The condition it vanishes is
\[
s_i (-a_K + a_{j} s_{j}) f_{Ij} K = 0 
\]
(4.11)
and satisfied for two cases,

Case 1, If $Z(i_j)$ (with non zero $s_{j}$) is mixed with $Z(K)$ by any of $Z(i_j)$ (with non zero
\[
s_{i}, \text{i.e. } f_{Ij} K \neq 0
\]
in the subalgebra under consideration then $a_K$ is
necessarily equal to $a_j$.

Case 2, If $Z(j)$ (with zero $s_{j}$) is mixed with $Z(K)$ by any of $Z(j)$ (with non zero $s_{i}$),
\[
i.e. f_{Ij} K \neq 0
\]
in the subalgebra under consideration then $a_K$ is necessarily
equal to 0.

In the following we will explain the values of $a_{i}'s$ listed below

\[
(AdS_5 + S^5) \quad (AdS_5 + S^5 + u(1)) \quad (gl(16)) \quad (sp(32))
\]
\[
s_{I} = 1 \quad s_{I} = 1 \quad s_{I} = 1 \quad s_{I} = 1
\]
\[
s_{I'} = 1 \quad s_{I'} = 1 \quad s_{I'} = 1 \quad s_{I'} = 1
\]
\[
s_{I''} = 0 \quad s_{I''} = 1 \quad s_{I''} = 1 \quad s_{I''} = 1
\]
\[
s_{I''} = 0 \quad s_{I''} = 0 \quad s_{I''} = 0 \quad s_{I''} = 1
\]
\[
a_{I} = 1 \quad a_{I} = 1 \quad a_{I} = 1 \quad a_{I} = 1
\]
\[
a_{I} = -1 \quad a_{I} = -1 \quad a_{I} = -1 \quad a_{I} = -1
\]
\[
a_{I''} = \text{any} \quad a_{I''} = \text{any} \quad a_{I''} = \text{any} \quad a_{I''} = \text{any}
\]
\[
a_{I'} = 0 \quad a_{I'} = 0 \quad a_{I'} = \times \quad a_{I'} = \times
\]
\[
a_{I'} = 0 \quad a_{I'} = 0 \quad a_{I'} = \times \quad a_{I'} = \times
\]

In the first AdS$_5 \times $S$^5$ algebra $\hat{I} = I, I'$ and $\hat{J} = J, J'$ are case 1 ($s_{j} = 1$). Since AdS$_5$ generators commute with the S$^5$ ones, $a_{I} \neq a_{I'}$ is not a contradiction. $\hat{I} = I, I'$ and $\hat{J} = J_0', J_1', J_2'$ are case 2 ($s_{j} = 0$). Since $Z_{0}$ commutes with AdS$_5 \times $S$^5$ there appears no condition on $a_{I''}$. $a_{I''} = 0$ comes, for example, from the following commutators,
\[
[Z_{0a}, Z_{bc}] = \eta_{a[0bc]} \epsilon_{a[0bc]}.
\]
(4.13)
Since $Z_{0a} \in I$, ($s_{i} = 1$) $Z_{bc}, Z_{0,cd} \in I_{1}', (s_{j} = 0)$ and $f_{Ij} K \neq 0$ then $a_K = a_{I''} = 0$. Similarly $a_{I''} = 0$ comes from the following commutators,
\[
[Z_{0a}, Z_{1,b}] = \eta_{ab} Z_{2}.
\]
(4.14)
Since \( Z_{0a} \in I, (s_j = 1) \) \( Z_{1b}, Z_2 \in I'_2, (s_j = 0) \) and \( f_{ij} \tilde{K} \neq 0 \) then \( a_{\tilde{K}} = a_i = 0 \).

In the second AdS5 \( \times S^5 + u(1) \) algebra \( \hat{I} = I, I', I_0' \) and \( \hat{J} = J, J', J_0' \) are case 1 \((s_j = 1)\). Since AdS5 and \( S^5 \) is commuting it is not necessary to be \( a_I = a_I' \). Since AdS5 \( \times S^5 \) and \( u(1) \) is commuting there appears no condition on \( a_{I_0}' \). \( \hat{I} = I, I', I_0' \) and \( \hat{J} = J_1'', J_2'' \) are case 2 \((s_j = 0)\). \( a_{I_0''} = a_{I_2''} = 0 \) comes from the same reasons as above.

For the \( gl(16) \) algebra there is no consistent solution. In order to see it, it is sufficient to observe

\[
[Z_{abc'}, Z_{0,cd}] = \eta_{[bd}Z_{0,a]c'},
\]

for which \( \hat{I} = (abc') \in I''_1, (s_j = 1) \), \( \hat{J} = (0d) \in I, (s_j = 1) \), \( \hat{K} = (0ac') \in I_1'' \) \((s_{\hat{K}} = 1)\) then

\[
a_{I''_1} = a_I = 1.
\]

On the other hand

\[
[Z_{ab'c'}, Z_{0,d'}] = \eta_{[c'd']Z_{0,ab']},
\]

for which \( \hat{I} = (ab'c') \in I''_1, (s_j = 1) \), \( \hat{J} = (0d') \in I', (s_j = 1) \), \( \hat{K} = (0ac') \in I_1'' \) \((s_{\hat{K}} = 1)\) then

\[
a_{I''_1} = a_{I'} = -1.
\]

(4.16) contradicts with (4.18) due to the opposite signs of \( a_I = 1 \) and \( a_{I'} = -1 \) (4.9).

The same argument can be applied to conclude that there is no consistent solution in \( sp(32) \) case. It is consistent with the result in the previous section that the super-AdS5 \( \times S^5 \) algebra is not a restriction of the osp(1\( | \)32). That is the generalization of AdS5 \( \times S^5 \) is failed when the \( QQ \) anti-commutator includes generators non-commuting both with AdS5 and \( S^5 \) generators. Values of \( a_I \) can differ only in different compact subalgebras. In the \( gl(16) \) and \( sp(32) \), \( a_I \) and \( a_{I'} \) are necessarily to have the same value in order to be consistent superalgebras and cannot be accommodated with (4.9).

There remains to check the \( (QQQ) \) Jacobi identity. It requires

\[
\frac{1}{2} \sum_{cyclic \ AB\ C} \ (QC^{-1})^D \ (C\Gamma(j))_{DA} \ a_{jsj} \ (C\Gamma(j'))_{BC} = 0
\]

then

\[
\sum_{AB\ C, cyclic} \ a_{jsj} \ (C\Gamma(j))_{DA} \ (C\Gamma(j'))_{BC} = 0.
\]

For AdS5 \( \times S^5 \) case \((s_j = 0)\), it is

\[
\sum_{AB\ C, cyclic} \ (C\Gamma(j))_{DA} \ (C\Gamma(j'))_{BC} - (C\Gamma(j))_{DA} \ (C\Gamma(j'))_{BC} = 0
\]
and is satisfied by the $\text{AdS}_5 \times S^5$ Fierz identity (3.13).

For $\text{AdS}_5 \times S^5 + u(1)$ case ($s_{\hat{\mu}} = 1$), there appears an additional term

$$\sum_{ABC, cyclic} \left( (\Gamma^{(I)})_{DA} (\Gamma^{(I')})_{BC} - (\Gamma^{(I')})_{DA} (\Gamma^{(I')})_{BC} + a_0 (\Gamma^{(0)})_{DA} (\Gamma^{(0)})_{BC} \right)$$

and vanishes only for $a_0 = 0$ case.

In summary under the present assumption that the $\text{AdS}_5 \times S^5$ is a bosonic subalgebra of bosonic part of the generalized algebra only two types of $u(1)$ extension are allowed. One is a central extension

$$[Q, Z_0] = 0,$$
$$\{Q_A, Q_B\} = a \left( (\Gamma^{(I)})_{AB} Z_{(I)} - (\Gamma^{(I')})_{AB} Z_{(I')} + a_0 (\Gamma^{(0)})_{AB} Z_{(0)} \right)$$

and the other is

$$[Q, Z_0] = \frac{1}{2} Q \Gamma_0,$$
$$\{Q_A, Q_B\} = a \left( (\Gamma^{(I)})_{AB} Z_{(I)} - (\Gamma^{(I')})_{AB} Z_{(I')} \right),$$

in which $Z_0$ is not an ideal of the algebra.

### 5 Summary and Discussions

We have shown that the $\text{su}(2,2\mid 4)$, the super-$\text{AdS}_5 \times S^5$ algebra or the four-dimensional $\mathcal{N}=4$ superconformal algebra, is not a restriction of the $\text{osp}(1\mid 32)$. The situation is the same for the super-pp-wave algebra as it is obtained by the Penrose limit of the super-$\text{AdS}_5 \times S^5$ algebra[14, 15]. In addition, we have shown that under the assumption that bosonic $\text{AdS}_5 \times S^5$ algebra is a subalgebra of the bosonic part of the superalgebra with brane charges, only two trivial types of $u(1)$ extension of the super-$\text{AdS}_5 \times S^5$ algebra are allowed.

Our results suggest that the generalized superalgebra associated with branes in the $\text{AdS}_5 \times S^5$ background, if any, cannot be any generalization of the super-$\text{AdS}_5 \times S^5$ algebra which contains the super-$\text{AdS}_5 \times S^5$ algebra as a restriction. This is completely different from the flat case, where extended superalgebra with brane charges contains the super-Poincare algebra as a restriction. The property that brane charge affects the algebra associated with the background may be related to a back reaction of the brane on the background. If this is the case, the brane probe analysis on the extended superalgebra must be modified appropriately.

There are possible ways to obtain generalized superalgebras with brane charges corresponding to branes in the $\text{AdS}_5 \times S^5$ background. First, one examines all possible
commutation relations among two of Noether charges. Secondly, one examines small fluctuations around a brane solution in the AdS$_5 \times S^5$ background of supergravity. Such a solution is not known well yet. Recently, D-brane supergravity solutions in the pp-wave background were constructed [16]. One may construct a generalized superalgebra with the brane charge recovering (super)symmetries broken by the brane. Thirdly, it is known that an intersection solution of a stack of D3-branes and a D$p$-brane in flat background is transformed to a D$p$-brane solution in the AdS$_5 \times S^5$ background under a near-horizon limit. One expects that the extended super-Poincare algebra with D3- and D$p$-brane charges is transformed to a generalized superalgebra with a D$p$-brane charge associated with a D$p$-brane solution in the AdS$_5 \times S^5$ background. In order to show this, at first, we must derive the AdS$_5 \times S^5$ algebra from the super Poincare algebra with D3-brane charges. We leave these issues for future investigations.

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A Uniqueness of the osp(1|32) algebra

We show that any maximally generalized supersymmetric algebra is expressed in the form of (2.1)-(2.3) by redefinitions explicitly. In a maximally generalized algebra $\frac{N(N+1)}{2}$ independent bosonic charges appear in the anti-commutator of the $N$-supercharges $Q_A$, (in the present case $N = 32$). Then the bosonic charges $Z_{AB}$ are defined by the first equation (2.1)

$$Z_{AB} \equiv \{Q_A, Q_B\} = Z_{BA}.$$  \hfill (A.1)

In the closed algebra $[Q_A, Z_{BC}]$ is odd and generally written as

$$[Q_A, Z_{BC}] = [Q_A, \{Q_B, Q_C\}] = Q_C \Omega^C_{A,BC}, \quad \Omega^C_{A,BC} = \Omega^C_{A,CB}.$$  \hfill (A.2)

The $QQQ$ Jacobi identity

$$\sum_{cyclic \ A,B,C} [\{Q_A, Q_B\}, Q_C] = 0,$$  \hfill (A.3)

requires

$$\sum_{cyclic \ A,B,C} \Omega^C_{A,BC} = 0.$$  \hfill (A.4)

The $QQZ$ Jacobi identity requires

$$[Z_{AB}, Z_{CD}] = Z_{(AB)} \Omega^E_{(CD),AB} = - Z_{(CE)} \Omega^E_{(D),AB}.$$  \hfill (A.5)

The last equality comes from anti-symmetry under $(AB \leftrightarrow CD)$. It follows

$$\delta^E_{(A-B)} \Omega^E_{(CD),AB} + \delta^E_{(C-D)} \Omega^E_{(AB),AB} + (\mathcal{F} \leftrightarrow \mathcal{G}) = 0.$$  \hfill (A.6)
By contraction with $\delta^A_F$
\begin{equation}
(N + 1) \Omega^F_{B,CD} + \delta^C_F \Omega^F_{F,CD} + \delta^F_{(C\Omega^F_D),FB} = 0.
\end{equation}
(A.7)

By further contraction with $\delta^B_F$ we find
\begin{equation}
\Omega^F_{B,CD} = 0
\end{equation}
and then
\begin{equation}
\Omega^F_{B,CD} = \delta^F_{(C\Omega^F_B),BD}, \quad \Omega_{BD} \equiv -\frac{1}{(N + 1)} \Omega^F_{D,FB}.
\end{equation}
(A.9)

The cyclic condition (A.4) means anti-symmetry of $\Omega_{AB}$ and (A.2) return to the form of (2.2)
\begin{equation}
[Q_A, Z_{BC}] = Q_{(B\Omega_{AC})}.
\end{equation}
(A.10)

The bosonic algebra (A.5) becomes symplectic form, $sp(N),$
\begin{equation}
[Z_{AB}, Z_{CD}] = Z_{(AE\Omega^E_{B}),CD} = Z_{A(C\Omega^E_B)D} + Z_{B(C\Omega^E_A)D}
\end{equation}
(A.11)
which coincide with (2.3). All the Jacobi identities hold identically.

In getting (A.6) we have assumed that all bosonic generators $Z_{AB}$ appear in the right hand of the $QQ$ anti-commutator (A.1). For a non-maximally generalized superalgebra $Z_{AB}$'s are not independent and (A.6) no more holds. The Jacobi identity is not trivially satisfied but is examined case by case.

### B. osp(1|32) and four-dimensional $\mathcal{N}=4$ superconformal algebra

In this appendix we show that the superconformal algebra $su(2,2|4)$ is also not a restriction of the osp(1|32) and there is no maximally generalized superalgebra. It is based on one to one correspondence between generators of the super-AdS$_5 \times S^5$ and superconformal algebras.

We explicitly write down relations among generators of the osp(1|32) , the super-AdS$_5 \times S^5$ and the superconformal algebras based on the isomorphism
\begin{equation}
AdS_5 \sim SO(4, 2) \sim 4D \text{ conformal}, \quad S^5 \sim SO(6) \sim SU(4)
\end{equation}
(B.1)
and their supercharges.

$AdS_5$ generators can be identified to the SO(4,2) generators by
\begin{equation}
P_a = Z_{0a} \equiv M_{a2}, \quad Z_{ab} \equiv M_{ab}.
\end{equation}
(B.2)
The so(4,2) algebra is

\[ [M_{AB}, M_{CD}] = \eta_{BC} M_{AD} + 3 \text{ terms}, \quad (B.3) \]

where \( A, B = 01234 \), and \( \eta_{AB} = (-; + + +; -) \). The sign of the metric in the direction comes from

\[ [M_{a\sharp}, M_{b\sharp}] = M_{ab} = -\eta_{\sharp\sharp} M_{ab}. \quad (B.4) \]

The conformal generators in 4 dimensions can be composed as

\[
\begin{align*}
\hat{P}_\mu &\equiv M_{\mu 4} + M_{\mu \sharp} = Z_{\mu 4} + Z_{\mu \sharp}, \\
K_\mu &\equiv M_{\mu 4} - M_{\mu \sharp} = Z_{\mu 4} - Z_{\mu \sharp}, \\
D &= M_{4\sharp} = Z_{\sharp 4},
\end{align*}
\]

where \( \mu = 0123 \), \( \eta_{\mu\nu} = (-; + + +; -) \). (B.5)

They satisfy

\[
\begin{align*}
\left[ \hat{P}_\mu, \hat{P}_\nu \right] &= [K_\mu, K_\nu] = 0, \\
\left[ \hat{P}_\mu, K_\nu \right] &= -2 M_{\mu\nu} + 2 \eta_{\mu\nu} D, \\
\left[ \hat{P}_\mu, M_{\rho\sigma} \right] &= \eta_{\mu[\rho} \hat{P}_{\sigma]}, \\
\left[ K_\mu, M_{\rho\sigma} \right] &= \eta_{\mu[\rho} K_{\sigma]}, \\
\left[ \hat{P}_\mu, D \right] &= \hat{P}_\mu, \\
\left[ K_\mu, D \right] &= -K_\mu.
\end{align*}
\] (B.6)

\( S^5 \) generators \( M_{a'b'} \) and \( P_{a'} \) become the SO(6) generators by

\[
\begin{align*}
P_{a'} &= Z_{0a'} \equiv M_{a'b'}, \\
Z_{a'b'} &\equiv M_{a'b'}, \\
[M_{A'B'}, M_{C'D'}] &= \eta_{B'C'} M_{A'D'} + 3 \text{ terms},
\end{align*}
\] (B.7, B.8)

where \( A', B' = 1'2'3'4'5'6 \), and \( \eta_{A'B'} = (+++++++;+) \). SU(4) generators are constructed from \( M_{A'B'} \) as

\[
U_{a'b'} = \frac{1}{4} M_{a'b'} (\gamma^{a'b'})_{\gamma'_{\delta'}} + \frac{i}{2} M_{a'b'} (\gamma^{a'}_{\delta'})_{\gamma'_{\delta'}}. \quad (B.9)
\]

It is traceless and hermitic and contains 15 independent components. Using the Fierz identities (3.14) it is inverted as

\[
M_{a'b'} = \frac{1}{2} \text{tr} (U_{a'b'}) \quad M_{a'b'} = -\frac{i}{2} \text{tr} (U_{a'b'}). \quad (B.10)
\]

The su(4) algebra follows from that of so(6) (B.8) as

\[
[U^{\alpha'_{\beta'}}_{\gamma'_{\delta'}}, U^{\gamma'_{\delta'}}_{\beta'_{\alpha'}}] = U^{\alpha'_{\beta'}}_{\gamma'_{\delta'}} \delta^{\gamma'_{\delta'}}_{\beta'_{\alpha'}} - U^{\gamma'_{\delta'}}_{\beta'_{\alpha'}} \delta^{\alpha'_{\beta'}}_{\gamma'_{\delta'}}. \quad (B.11)
\]

In the conformal group the generators are graded by their conformal dimensions. (B.6) shows \( \hat{P}_\mu \) has the conformal dimension +1, \( K_\mu \) has −1 and \( M_{\mu\nu} \) and \( D \) itself have 0. The
SU(4) generators $U^{\alpha'}{}_{\beta'}$ also have the conformal dimension 0. The 32 supercharges $Q_{\alpha'\alpha} A$ are recombined to be 16 $Q$ with the conformal dimension $\frac{1}{2}$ and $S$ with $-\frac{1}{2}$. They are

$$
Q_{\beta i} = (Q_{\alpha'\alpha 1} + iQ_{\alpha'\alpha 2}) (h_-)^\alpha_\beta \delta^{\alpha' i},
$$

$$
Q_{\beta i} = (Q_{\alpha'\alpha 1} - iQ_{\alpha'\alpha 2}) (h_+)^\alpha_\beta C\alpha'_{-1} \delta^{\alpha' i},
$$

$$
S_{\beta i} = (Q_{\alpha'\alpha 1} + iQ_{\alpha'\alpha 2}) (h_+)^\alpha_\beta \delta^{\alpha' i},
$$

$$
S_{\beta i} = (Q_{\alpha'\alpha 1} - iQ_{\alpha'\alpha 2}) (h_-)^\alpha_\beta C\alpha'_{-1} \delta^{\alpha' i}. \tag{B.12}
$$

Here $h_\pm$ are chiral projection operators in 4 dimensions ($\gamma^4$ is one usually denoted as $\gamma^5$)

$$
h_\pm = \frac{1 \pm \gamma^4}{2}, \quad h_\pm^\dagger = h_\pm. \tag{B.13}
$$

They are related by the hermitian conjugation defined by (2.15) as

$$
Q_{\beta i} \rightarrow (Q^\dagger)_{\beta i} = Q_{\gamma i} (-\gamma^0 C^{-1}) \gamma^\beta,
$$

$$
Q_{\beta i} \rightarrow (Q^\dagger)_{\beta i} = Q_{\gamma i} (\gamma^0 C^{-1}) \gamma^\beta,
$$

$$
S_{\beta i} \rightarrow (S^\dagger)_{\beta i} = S_{\gamma i} (-\gamma^0 C^{-1}) \gamma^\beta,
$$

$$
S_{\beta i} \rightarrow (S^\dagger)_{\beta i} = S_{\gamma i} (\gamma^0 C^{-1}) \gamma^\beta. \tag{B.14}
$$

The $Q Z$ commutators of the osp(1|32) are expressed as follows. The commutators with conformal generators are

$$
[Q_i, M_{\mu \nu}] = \frac{1}{2} Q_i \gamma_{\mu \nu}, \quad [Q_i, D] = \frac{1}{2} Q_i,
$$

$$
[Q_i, \tilde{P}_\mu] = 0, \quad [Q_i, K_\mu] = S_i \gamma_\mu,
$$

$$
[Q^i, M_{\mu \nu}] = \frac{1}{2} Q^i \gamma_{\mu \nu}, \quad [Q^i, D] = \frac{1}{2} Q^i,
$$

$$
[Q^i, \tilde{P}_\mu] = 0, \quad [Q^i, K_\mu] = -S^i \gamma_\mu, \tag{B.15}
$$

$$
[S_i, M_{\mu \nu}] = \frac{1}{2} S_i \gamma_{\mu \nu}, \quad [S_i, D] = -\frac{1}{2} S_i,
$$

$$
[S_i, \tilde{P}_\mu] = Q_i \gamma_\mu, \quad [S_i, K_\mu] = 0,
$$

$$
[S^i, M_{\mu \nu}] = \frac{1}{2} S^i \gamma_{\mu \nu}, \quad [S^i, D] = -\frac{1}{2} S^i,
$$

$$
[S^i, \tilde{P}_\mu] = -Q^i \gamma_\mu, \quad [S^i, K_\mu] = 0,
$$

and the commutators with SU(4) generators are obtained from (B.9)

$$
[Q_{\alpha i}, U^j k] = Q_{\alpha k} \delta^j_i - \frac{1}{4} Q_{\alpha i} \delta^j k,
$$

$$
[Q^i, U^j k] = -Q^i \delta^j k + \frac{1}{4} Q^i \delta^j k, \tag{B.16}
$$

$$
[S_{\alpha i}, U^j k] = S_{\alpha k} \delta^j_i - \frac{1}{4} S_{\alpha i} \delta^j k,
$$

$$
[S^i, U^j k] = -S^i \delta^j k + \frac{1}{4} S^i \delta^j k.
$$

There are also commutators of $Q$ with bosonic generators $Z_{(\mu)}$ which are not in AdS$_5 \times$S$^5$.

The osp(1|32) $QQ$ anti-commutators in terms of $Q$ and $S$ are

$$\{Q_{\alpha i}, Q_{\beta j}\} = 2a \left[ (Ch_-)_{\alpha \beta} C_{ij} \alpha' (Z_+ - iZ_+) + (Ch_-)_{\alpha \beta} (C' \gamma')_{ij} (Z_{+\alpha'} - iZ_{++4\alpha'}) \right].$$
\[
\{S_{\alpha i}, S_{\beta j}\} = 2a[i(C_\gamma h_+)_{\alpha\beta}C_{ij}(Z_+ + iZ_{+4}) + (C_\gamma h_+)_{\alpha\beta}(C_\gamma')_{ij}(Z_{+a'} + iZ_{+4a'}) \\
+ \frac{1}{4}(C_{\mu\nu}h_+)_{\alpha\beta}(C_{\alpha\beta}')_{ij}Z_{+\mu\nu a' a'}], 
\]
(B.18)

\[
\{Q_{\alpha i}, S_{\beta j}\} = 2a[i(C_\gamma h_+)_{\alpha\beta}C_{ij}(Z_+ + iZ_{+4}) + (C_\gamma h_+)_{\alpha\beta}(C_\gamma')_{ij}(Z_{+a'} + iZ_{+4a'}) \\
+ \frac{1}{2}(C_{\gamma h_+})_{\alpha\beta}(C_{\gamma'}_{ij})Z_{+\mu a' a'}], 
\]
(B.19)

\[
\{Q_{\alpha i}, Q_{\beta j}\} = 2ai[ (C_\gamma h_+)_{\alpha\beta}i_j \hat{P}_\mu \\
+ i(C_\gamma h_+)_{\alpha\beta} \left( \frac{1}{2}(\gamma^{\alpha\beta}')_{ij}(Z_{\mu a' a'} - Z_{\mu a' a'}) - i(\gamma^{\alpha'})_{ij}(Z_{\mu a' a'} + Z_{\mu a' a'}) \right)], 
\]
(B.23)

\[
\{Q_{\alpha i}, S_{\beta j}\} = 2ai[(C_\gamma h_+)_{\alpha\beta}i_j D - \frac{1}{2}(C_{\mu\nu}h_+)_{\alpha\beta}i_j M_{\mu\nu} - 2(C_\gamma h_+)_{\alpha\beta}U_{ij} \\
- i(C_\gamma h_+)_{\alpha\beta}i_j Z_{\mu a'} - (C_\gamma h_+)_{\alpha\beta}(\gamma^{\alpha'})_{ij}Z_{0a' a'} - i \frac{1}{4}(C_{\gamma h_+})_{\alpha\beta}(\gamma^{\alpha\beta}')_{ij}Z_{0\mu a' a'} \\
- \frac{1}{2}(C_{\gamma h_+})_{\alpha\beta}(\gamma^{\alpha'})_{ij}Z_{\mu a'} - i \frac{1}{2}(C_\gamma h_+)_{\alpha\beta}(\gamma^{\alpha\beta}')_{ij}(Z_{4a' a'})], 
\]
(B.24)

\[
\{S_{\alpha i}, Q_{\beta j}\} = 2ai[-(C_\gamma h_-)_{\alpha\beta}i_j D - \frac{1}{2}(C_{\mu\nu}h_-)_{\alpha\beta}i_j M_{\mu\nu} - 2(C_\gamma h_-)_{\alpha\beta}U_{ij} \\
- i(C_\gamma h_-)_{\alpha\beta}i_j Z_{\mu a'} - (C_\gamma h_-)_{\alpha\beta}(\gamma^{\alpha'})_{ij}Z_{0a' a'} - i \frac{1}{4}(C_{\gamma h_-})_{\alpha\beta}(\gamma^{\alpha\beta}')_{ij}Z_{0\mu a' a'} \\
- \frac{1}{2}(C_{\gamma h_-})_{\alpha\beta}(\gamma^{\alpha'})_{ij}Z_{\mu a'} + i \frac{1}{2}(C_\gamma h_-)_{\alpha\beta}(\gamma^{\alpha\beta}')_{ij}(Z_{4a' a'})], 
\]
(B.25)

\[
\{S_{\alpha i}, S_{\beta j}\} = 2ai[-(C_\gamma h_+)_{\alpha\beta}i_j K_{\mu} \\
- i(C_\gamma h_+)_{\alpha\beta} \left( \frac{1}{2}(\gamma^{\alpha\beta}')_{ij}(Z_{0a' a'} + Z_{a' a'}) + i(\gamma^{\alpha'})_{ij}(Z_{0a' a'} - Z_{a' a'}) \right)], 
\]
(B.26)

Here the bosonic generators other than the superconformal ones are \( Z \)'s and \( Z_{\pm} = Z_{1,\pm} \pm iZ_{2,\pm} \). These commutators are consistent under the conjugations (B.14). For example the conjugation of the (B.24) gives (B.25).

We have written the \( \text{osp}(1|32) \) algebra in terms of superconformal generators and other bosonic generators \( Z \)'s. The commutators (B.6), (B.11), (B.15)-(B.16) have the same forms as those of the superconformal algebra. For the anti-commutators of the
supercharges (B.17)-(B.26), if we would drop the bosonic generators $Z$’s other than $\hat{P}_\mu, K_\mu, M_{\mu\nu}, D$ and $U^i_{i\jmath}$, they appear those of the superconformal algebra except that the signs in front of the $U^i_{i\jmath}$ terms are reversed. The reason is the same as the $\text{AdS}_5 \times S^5$ case. The restriction of the extra generators $Z$’s from the $\text{osp}(1|32)$ is not sufficient to guarantee the Jacobi identities. In order to do it, it is necessary to reverse the signs of the $U^i_{i\jmath}$ terms in the anti-commutators of the supercharges since $\text{SU}(4)$ generator $U^i_{i\jmath}$ is related to the $S^5$ generators by (B.9) and (3.13) is applied.

The generalization of the superconformal algebra is discussed as was done in the case of the super-$\text{AdS}_5 \times S^5$ in section 4. It is not generalized up to maximal but only two types of $u(1)$ extension are allowed. The $U(1)$ generator $Z_0$ can be combined with the $\text{SU}(4)$ generator $U^i_{i\jmath}$ to form a $U(4)$ generator.

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