Regularity of intrinsically convex $H^2$ surfaces and the derivation of homogenized bending shell models

Peter Hornung  
FB Mathematik, TU Dresden  
01062 Dresden (Germany)

Igor Velčić  
University of Zagreb,  
Faculty of Electrical Engineering and Computing  
Unska 3, Zagreb (Croatia)

Abstract

We prove smoothness of $H^2$ isometric immersions of surfaces endowed with a smooth Riemannian metric of positive Gauss curvature. We use this regularity result to rigorously derive homogenized bending models of convex shells from three-dimensional nonlinear elasticity.

Keywords: isometric immersions, positive Gauss curvature, regularity, elasticity, dimension reduction, homogenization, shell theory, two-scale convergence, Gamma convergence.

1 Introduction

For a local $C^2$ isometric immersion $u$ of a two-dimensional Riemannian manifold with positive Gauss curvature into $\mathbb{R}^3$, there is a link between the regularity of the metric and the regularity of $u$; in particular, if the metric is smooth then so is $u$. Without a priori assumptions on the regularity of $u$ this link is broken; as shown by classical results of Nash and Kuiper even $C^1$ regularity is not enough.
In the present paper, we show that square integrability of the second fundamental form of \( u \) is sufficient for the link to persist. In particular, if the metric is smooth, then \( u \) is smooth in the interior, provided that initially it belongs to the Sobolev space \( H^2 \).

We note in passing that replacing the assumption of positive Gauss curvature by zero Gauss curvature also breaks the regularity link. Indeed, local isometric immersions of class \( H^2 \) of the standard flat metric in \( \mathbb{R}^2 \) into \( \mathbb{R}^3 \) may fail to be \( C^2 \), see [Hor11b] and [Hor11a] for examples in this direction.

Our regularity results for metrics with positive Gauss curvature rely upon earlier work by Sverák on the Monge-Ampère equation. Due to the low regularity, the passage from the scalar problem to the vectorial problem addressed here is not trivial.

Relaxing \( C^2 \) regularity to regularity on the Sobolev scale is important for variational problems: the \( H^2 \) isometric immersions studied here arise naturally in thin film elasticity. In the present paper, we use this regularity result to derive homogenized bending models for convex shells from three dimensional nonlinear elasticity.

Much work has been done on shell theories in elasticity. An overview of the derivation of models for linear and nonlinear shells by the method of formal asymptotic expansions can be found in [Cia00]. In the case of linearly elastic shells, the models thus obtained can also be justified by a rigorous convergence result, starting from three dimensional linearized elasticity.

In the last two decades, rigorous justifications of nonlinear models for rods, curved rods, plates and shells were obtained by means of \( \Gamma \)-convergence, starting from three dimensional nonlinear elasticity. The first papers in that direction are [ABP91], [LDR95], [LDR96] for strings and membranes (plates and shells, respectively). For plates, the nonlinear bending theory the Föppl-von Kármán theory were derived in [FJM02] and [FJM06]. For shells, the nonlinear bending theory was derived in [FJMM03] and the von Kármán theory in [LMP10]; see also [LMP11] for limiting models in an intermediate energy scaling regime between bending and von Kármán theories, for the particular case of convex surfaces.

Here we are interested in the derivation of the homogenized bending theory of shells by means of simultaneous homogenization and dimension reduction. Our starting point is the energy functional of three dimensional nonlinear elasticity: the elastic energy stored in a deformation \( u \in H^1(S^h, \mathbb{R}^3) \) of a reference shell \( S^h \subset \mathbb{R}^3 \) of thickness \( h > 0 \) around a surface \( S \subset \mathbb{R}^3 \) is given by

\[
\frac{1}{h^2 |S^h|} \int_{S^h} W_\varepsilon(x, \nabla u(x)) \, dx.
\]
Here, $W_\varepsilon$ is a non-degenerate stored energy function that oscillates periodically in $x$, with some period $\varepsilon \ll 1$. We are interested in the effective behavior when both the thickness $h$ and the period $\varepsilon$ are small.

The separate limits $h \to 0$ and $\varepsilon \to 0$ are reasonably well understood: in \cite{FJMM03} it is shown that, when $W_\varepsilon$ does not depend on $\varepsilon$, then the functionals (1) $\Gamma$-converge, as $h \to 0$, to a two-dimensional bending shell theory.

Regarding the limit $\varepsilon \to 0$, which is related to homogenization, the first rigorous results relevant in nonlinear elasticity were obtained by Braides \cite{Bra85} and independently by Müller \cite{Mull87}. They proved that, under suitable growth assumptions on $W_\varepsilon$, the energy (1) $\Gamma$-converges as $\varepsilon \to 0$ (for fixed $h$) to the functional obtained by replacing $W_\varepsilon$ in (1) with the homogenized energy density given by an infinite-cell homogenization formula.

In this paper we study the asymptotic behavior when both the thickness $h$ and the period $\varepsilon$ tend to zero simultaneously. Such a combination of dimensional reduction and homogenization has been the subject of numerous papers: in the paper \cite{BFF00} the authors study the same effects for nonlinear systems (membrane plate) by means of $\Gamma$-convergence, also without periodicity assumptions. In their general approach they also consider the possibility of oscillating boundary. The 1d case of rods was addressed in \cite{Neu12}; more recently the plate model in the von Kármán regime (see \cite{NV13}) and in the bending regime (see \cite{HNV14, Vel15}) were analyzed. In these cases one does not obtain an infinite-cell homogenization formula as in the membrane case (see \cite{BFF00}). This is because the energy is essentially convex for small strain. Therefore one can use two-scale convergence techniques in all these cases.

However, each case has its own peculiarities. In the von Kármán theory of plates, one obtains a limiting quadratic energy density which is continuous in the asymptotic ratio $\gamma$ between $h$ and $\varepsilon$, for all $\gamma \in [0, \infty]$. Moreover, the case $\gamma = 0$ corresponds to the situation when the dimensional reduction dominates and the resulting model is just the homogenized plate model in von Kármán regime. The situation $\gamma = \infty$ corresponds to the case when homogenization dominates and the resulting model is the plate model in the von Kármán regime corresponding to the homogenized functional. The same phenomenology is observed in bending regime for rods in \cite{Neu12}. The limiting model of the homogenized plate in von Kármán regime without periodicity assumption is obtained in \cite{Vel} and the model of rod in bending regime in \cite{MV}. The nonlinear bending theory of plates is more involved in the periodic case. In \cite{HNV14} the authors obtained asymptotic models in the case $\gamma \in (0, \infty)$. In \cite{Vel15} the author obtained the asymptotic model corresponding to the regime $\gamma = 0$ under the additional assumption that $\varepsilon^2 \ll h \ll \varepsilon.$
In the critical case $h \sim \varepsilon^2$ one may expect that the oscillations needed for the relaxation would have to include some other frequencies in addition to the one on which the material is oscillating. So two-scale analysis might not be sufficient in this case. In [NO15] the authors derived the homogenized bending plate model by performing homogenization of 2D bending plate. The homogenization of shells in the von Kármán regime (cf. [HV14]) also displayed a separation of the models for the case $\gamma = 0$. For generic shells, we were able to identify the model for the situations $\varepsilon^2 \ll h \ll \varepsilon$ and, surprisingly, also for the situation $h \sim \varepsilon^2$. Moreover, for convex shells we were able to completely resolve the case $\gamma = 0$. The geometric framework developed in [HV14] will be used in the present paper as well. As it was already emphasized in [HV14] the bending theory for shells seems likely to be more involved. In this paper we address that case, but only for convex shells. To answer the question of homogenization of bending shell we need to answer on the question of additional regularity of isometries on convex surfaces, for which we apriori know (by the compactness result) that are $H^2$. The main results of this paper are Theorem 2.1 and Theorem 3.4. We emphasize the fact that this regularity result is also needed for the simpler case of studying the shells that are layered in the thickness direction (this model also incorporated in the analysis here). For the plate model of layered materials in the bending regime, see [Sch07]. Let us explain the technical relevance of the regularity result Theorem 2.1 for the homogenization. In the compactness result (Proposition 3.7) we are unable to identify — i.e., to obtain any additional information on — the $L^2$ matrix field $B$ which arises as (part of) the weak limit of approximate strains. In the construction of the recovery sequence it is natural to try to recover this field by adding to the standard recovery sequence an additional term of order $h$; this term is called $w$ in our proof of the upper bound. Under the convexity assumption, we can prove that with this term we can indeed recover any $L^2$ matrix field $B$. This completes the proof. In order to carry out this plan, one must analyze a system of equations whose coefficients depend on the limiting deformation, which is an $H^2$ isometry. The analysis of this system requires regularity of its coefficients, see Lemma 3.10. No such regularity result was needed in [FJMM03]. Instead, the gradient truncation method used in [FJM02] was employed. However, such a truncation (Lusin type approximation) cannot be used in the present paper. Note that in contrast to the situation encountered here, in the case of plates (see [Sch07] [HNV14]) not every $L^2$ matrix field $B$ can be recovered by adding an additional term. But one can recover any matrix field that is zero on all planar parts of the deformation. And this class of fields contains the ones
relevant for the relaxation.

Notation

$C^{k,\alpha}$ denotes the space of functions that are $k$-times differentiable and their $k$-th derivatives are Hölder continuous with exponent $\alpha$. $W^{k,p}$, $H^k$ denote the standard Sobolev spaces. For $A \subset \mathbb{R}^n$ we denote by $C_0^k(A)$ the space of $k$-differentiable functions with compact support in $A$. We set $Y = [0,1)^2$ and we denote by $\mathcal{Y}$ the Euclidean space $\mathbb{R}^2$ equipped with the torus topology, that is for all $z \in \mathbb{Z}^2$ the points $y+z$ and $y$ are identified in $\mathcal{Y}$. We write $C_0^k(Y)$ to denote the space of $k$-differentiable functions with continuous derivatives that have compact support in $A$. We denote $Y = \mathbb{R}^2$ and we denote by $\mathcal{Y}$ the Euclidean space $\mathbb{R}^2$ equipped with the torus topology, that is for all $z \in \mathbb{Z}^2$ the points $y+z$ and $y$ are identified in $\mathcal{Y}$. We write $C_0^k(Y)$ to denote the space of continuous functions $f : \mathbb{R}^2 \to \mathbb{R}$ satisfying $f(y+z) = f(y)$ for all $z \in \mathbb{Z}^2$. We denote by $C^k(\mathcal{Y})$ those functions in $C^k(\mathbb{R}^2) \cap C^0(\mathcal{Y})$ whose derivatives up to the $k$-th order belong to $C^0(\mathcal{Y})$. We denote by $L^2(\mathcal{Y})$, $H^1(\mathcal{Y})$ and $H^1(S \times \mathcal{Y})$ the Banach spaces obtained as the closure of $C^\infty(\mathcal{Y})$ and $C^\infty(S, C^\infty(\mathcal{Y}))$ with respect to the norm in $L^2(\mathcal{Y})$, $H^1(\mathcal{Y})$ and $H^1(S \times \mathcal{Y})$, respectively. By $L^2(\mathcal{Y}), H^k(\mathcal{Y})$ etc. we denote the subspaces of $L^2(\mathcal{Y}), H^k(\mathcal{Y})$ etc. whose mid-value over $\mathcal{Y}$ is zero. For $A \subset \mathbb{R}^d$ measurable and $X$ a Banach space, $L^2(A, X)$ is understood in the sense of Bochner. We identify the spaces $L^2(A, L^2(B))$ and $L^2(A \times B)$ in usual way. Standard basis vectors in $\mathbb{R}^2$ are denoted by $e_i$. By $\text{SO}(3)$ we denote the set of rotational matrices in $\mathbb{R}^{3 \times 3}$, by $\text{so}(3)$ the space of skew symmetric matrices in $\mathbb{R}^{3 \times 3}$, while by $\mathbb{R}^{3 \times 3}_{\text{sym}}$ the space of symmetric matrices in $\mathbb{R}^{3 \times 3}$. For a matrix $M$ by $\text{sym} M$ we denote the symmetric part of $M$, $\text{sym} M = \frac{1}{2}(M + M^t)$. For $A, B \subset \mathbb{R}^n$, with $A \Subset B$ we mean that $A$ is compactly contained in $B$.

2 Regularity of intrinsically convex $H^2$ surfaces

The purpose of this chapter is to prove the following result:

**Theorem 2.1.** Let $U \subset \mathbb{R}^2$ be a bounded domain and let $g \in C^\infty(U, \mathbb{R}^{2 \times 2}_{\text{sym}})$ be a smooth Riemannian metric on $U$. Assume that the Gauss curvature $K$ of $g$ is positive and let $u \in H^2_g(U)$, i.e., $u \in H^2(U, \mathbb{R}^3)$ solves

$$(\nabla u)^T(\nabla u) = g \text{ almost everywhere on } U.$$  

Then $u \in C^\infty(U)$.

We use ideas and a key result from the unpublished (but widely circulated) manuscript [Sve]. For our purposes, its main result is to deduce convexity of
$H^2$ solutions $f$ of the Monge-Ampère inequality $\det \nabla^2 f \geq c > 0$, cf. Lemma 2.9 below. In [Sve], this result is combined with a local graphical representation to prove smoothness of $C^{1,1}$ isometric immersions of subdomains of the sphere, endowed with the standard metric.

Our proof of Theorem 2.1 also uses this idea of representing $u$ locally as a graph of a function $f$. However, a priori $u$ is not $C^1$. Instead, we show that the normal $n_u$ to $u$ is continuous. This is enough for $u$ to be locally a graph. Finally, a bootstrap argument, using classical facts about two-dimensional Monge-Ampère equations on one hand and exploiting the link between $u$ and its graphical representation on the other hand, implies that $u$ is smooth.

2.1 Continuity of the normal

The purpose of this section is to provide a fairly self-contained proof of Proposition 2.3 below. In doing so, we combine ideas from [BN95, BN96] and others, and we introduce a suitable notion of topological degree. For the reader’s convenience, we include proofs of its relevant properties.

In what follows, we use the notation $|g| = \det g$ and $\sqrt{g} = (\det g)^{1/2}$. The Christoffel symbols of $g$ are denoted by $\Gamma^\gamma_{\alpha\beta}$. Define $h : B \to \mathbb{R}^{2 \times 2}$ by $h = n_u \cdot \nabla^2 u$. The Gauss equation is easily seen to remain true for $u \in H^2$. It reads:

$$\partial_\alpha \partial_\beta u = h_{\alpha\beta} n_u + \Gamma^\gamma_{\alpha\beta} \partial_\gamma u. \tag{2}$$

Since we are dealing with $H^2$ maps, we should verify the validity of Gauss’ Theorema Egregium.

Lemma 2.2. If $u \in H^2_g(U)$ and $h = n_u \cdot \nabla^2 u$, then $\det h = K|g|$ almost everywhere on $U$.

Proof. As shown in [FJM06, Proof of Proposition 6], by approximation it is easy to see that the map $u$ satisfies

$$|\partial_1 \partial_2 u|^2 - \partial_1 \partial_1 u \cdot \partial_2 \partial_2 u = \frac{1}{2} (\partial_2 \partial_2 g_{11} + \partial_1 \partial_1 g_{22} - 2 \partial_1 \partial_2 g_{12}) \tag{3}$$

almost everywhere on $U$. Denote by $P(x)$ the orthogonal projection from $\mathbb{R}^3$ onto the subspace spanned by $\partial_1 u$ and $\partial_2 u$. Recall that $h = n_u \cdot \nabla^2 u$. Then we deduce from (3) that

$$\det h = -|P(\partial_1 \partial_2 u)|^2 + P(\partial_1 \partial_1 u) \cdot P(\partial_2 \partial_2 u) + \frac{1}{2} (\partial_2 \partial_2 g_{11} + \partial_1 \partial_1 g_{22} - 2 \partial_1 \partial_2 g_{12}). \tag{4}$$
But in view of (2) we have

\[ P(\partial_\alpha \partial_\beta u) \cdot P(\partial_\gamma \partial_\delta u) = \Gamma^\rho_{\alpha \beta} \Gamma^\sigma_{\gamma \delta} g_{\rho \sigma}. \]

We conclude that the right-hand side of (4) can be computed from \( g \) and its derivatives. Since \( g \) is smooth, the right-hand side of (4) agrees with \( K|g| \).

**Proposition 2.3.** Let \( U \subset \mathbb{R}^2 \) be a bounded domain and let \( g \in C^\infty(U) \) be a smooth Riemannian metric on \( U \). Assume that the Gauss curvature \( K \) of \( g \) is positive and let \( u \in H^2_g(U) \). Then the normal

\[ n_u = \frac{\partial_1 u \times \partial_2 u}{|\partial_1 u \times \partial_2 u|} \]

to \( u \) is continuous on \( U \).

To prove Proposition 2.3 let \( R > 0 \) and let \( \varphi \in H^1(B_{2R}, \mathbb{S}^2) \). Then, for \( L^1 \) almost every \( r \in (0,R) \) we have \( \varphi|_{\partial B_r} \in H^1(\partial B_r) \), hence by Sobolev embedding

\[ \varphi|_{\partial B_r} \in C^0(\partial B_r). \]  

By a classical result of Schoen and Uhlenbeck, there exist \( \varphi_k \in C^\infty(B_{2R}, \mathbb{S}^2) \) converging strongly in \( H^1 \) to \( \varphi \). After possibly passing to a subsequence, we may assume that \( \varphi_k \to \varphi \) in \( H^1(\partial B_r) \) for almost every \( r \in (0,R) \), hence

\[ \varphi_k \to \varphi \text{ uniformly on } \partial B_r. \]  

In fact, setting \( f_k(r) = \int_{\partial B_r} |\nabla \varphi_k - \nabla \varphi|^2 \), by the coarea formula we have

\[ \|f_k\|_{L^1(B_R)} = \int_0^R dr \int_{\partial B_r} |\nabla \varphi_k - \nabla \varphi|^2 = \int_{B_R} |\nabla \varphi_k - \nabla \varphi|^2 \to 0. \]

Hence there is a subsequence such that \( f_k(r) \to 0 \) for \( L^1 \) almost every \( r \in (0,R) \).

For the rest of this section we fix \( r \in (0,R) \) such that (5) and (6) are satisfied simultaneously. Note in particular that \( \varphi(\partial B_r) \) is compact, due to (5). Define the degree \( Q : \mathbb{S}^2 \to \mathbb{R} \) of \( \varphi \) with respect to \( B_r \) by setting

\[ Q(y) = \int_{\partial B_r} \varphi^* \eta, \]

where \( \eta \) is any smooth 2-form on \( \mathbb{S}^2 \) with \( \int_{\mathbb{S}^2} \eta = 1 \) which is supported in the connected component \( \Lambda_y \) of \( \mathbb{S}^2 \setminus \varphi(\partial B_r) \) that contains \( y \).

We claim that \( d \) is well-defined, i.e., that it is independent of the choice of \( \eta \). We use the following lemma.
Lemma 2.4. Let $\Lambda \subset S^2$ be connected and let $\tilde{\eta}$ be a smooth 2-form on $S^2$ whose support is contained in $\Lambda$ and which is such that $\int_{S^2} \tilde{\eta} = 0$. Then there exists a smooth 1-form $w$ on $S^2$ with support in $\Lambda$ and such that $\eta = dw$.

Proof. The proof can be found in [Sch69].

In view of the lemma it remains to show that if $w$ is a smooth 1-form supported in $\Lambda_y$ then $\int_{B_r} \varphi^*(dw) = 0$.

Since $dw$ is a 2-form and since $\varphi_k \to \varphi$ strongly in $H^1(B_r)$, we see that $\varphi^*_k(dw) \to \varphi^*(dw)$ strongly in $L^1(B_r)$.

Hence
\[ \int_{B_r} d(\varphi^*_k w) = \int_{B_r} \varphi^*_k(dw) \to \int_{B_r} \varphi^*(dw). \tag{8} \]

Due to (8), the compact set $\varphi_k(\partial B_r)$ does not intersect the support of $w$ for $k$ large enough, because the latter has positive distance from the compact set $\varphi(\partial B_r)$. Therefore, $\varphi^*_k w$ has compact support in $B_r$. Hence, by Stokes' theorem, the left-hand side of (8) is zero. This concludes the proof showing that $Q$ is well-defined by (7).

Recall that the essential range of $\varphi|_{B_r}$ is the smallest closed set $F$ such that $\varphi(x) \in F$ for almost every $x \in B_r$; as shown in [BN95] it is well-defined. More or less directly from the definition of $Q$, we see the following:

Lemma 2.5. Let $\varphi \in H^1(U, S^2)$ and define $Q$ as in (7). Then the following are true:

(i) $Q$ is constant on every connected component of $S^2 \setminus \varphi(\partial B_r)$;

(ii) If $Q(y) \neq 0$ then $\Lambda_y$ is contained in the essential range $F$ of $\varphi|_{B_r}$.

(iii) $Q$ takes integer values.

Proof. To prove (iii), assume that $\Lambda_y$ is not contained in $F$. Then there exists an open set $\Lambda \subset \Lambda_y \setminus F$ and a normalized smooth 2-form $\eta$ supported on $\Lambda$. So $\varphi^*\eta = 0$ almost everywhere on $B_r$. Hence we would have $Q(y) = 0$.

To prove (iii) just note that the last convergence in (8) is also true for any other 2-form; in particular for the form $\eta$ in (7). But for smooth $\varphi$, the right-hand side of (7), which is known to attain only integer values.

Lemma 2.6. If $\varphi = n_u$, then we have $Q \geq 0$ on $S^2 \setminus n_u(\partial B_r)$. Moreover, if $Q(y) = 0$ then $\Lambda_y$ does not intersect the essential range of $n|_{\partial B_r}$.
Proof. Denote by $\eta_{S^2}$ the standard area form on $S^2$. Then $n_u^* \eta_{S^2} = K d \text{vol}_g$, due to Lemma 2.2. Applying (7) with $\eta = \rho \eta_{S^2}$, we see that

$$\left( \int_{B_r} \rho \eta_{S^2} \right) Q(y) = \int_{B_r} (\rho \circ n_u) K d \text{vol}_g.$$ 

for every $\rho \in C_0^\infty(\Lambda_y)$. Since $K > 0$ on $B_r$, we conclude that if $Q(y) = 0$ then $\rho \circ n = 0$ almost everywhere on $B_r$. Since $\rho$ was arbitrary, this implies that $n(x) \in S^2 \setminus \Lambda_y$ for almost every $x \in B_r$. Since $S^2 \setminus \Lambda_y$ is closed, by minimality of the essential range we conclude that it must be contained in $S^2 \setminus \Lambda_y$.

Lemma 2.7. If $n_u = \varphi$ and if $R > 0$ is small enough, then $Q$ is zero at some point in $S^2 \setminus n_u(\partial B_r)$.

Proof. There exists a constant $C$ depending only on $g$ such that

$$\int_{B_r} K d \text{vol}_g \leq CR^2.$$ 

We choose $R$ so small that the right-hand side is bounded by $1/4$ times the area of $S^2$.

Assume for contradiction that $Q \neq 0$ on $S^2 \setminus n_u(\partial B_r)$. Then by Lemma 2.6 we know that $Q$ is positive and so by Lemma 2.5 we have $Q \geq 1$ on $S^2 \setminus n_u(\partial B_r)$. Since $n_u \in H^1(\partial B_r)$, it maps $\partial B_r$ into a set of zero area, cf. [Res89]. So the area of $S^2 \setminus n_u(\partial B_r)$ is that of $S^2$.

Hence there exist finitely many pairwise disjoint connected components $\Lambda_1, \ldots, \Lambda_M$ of $S^2 \setminus n_u(\partial B_r)$ and $\psi_i \in C_0^\infty(\Lambda_i)$ satisfying $\psi_i \in [0, 1]$, such that

$$\sum_{i=1}^M H^2(\{\psi_i = 1\}) \geq \frac{1}{2} H^2(S^2). \tag{9}$$

Let $y_i \in \Lambda_i$ and note that $\sum_i \psi_i \leq 1$ pointwise on $S^2$. Hence, recalling that
\[ Q(y_i) \geq 1, \]
\[
\int_{B_r} K \, d\text{vol}_g \geq \sum_{i=1}^M \int_{B_r} (\psi_i \circ n_u) \, K \, d\text{vol}_g
\]
\[
= \sum_{i=1}^M \int_{B_r} n_u^*(\psi_i \eta_{S^2})
\]
\[
= \sum_{i=1}^M Q(y_i) \int_{S^2} \psi_i \eta_{S^2}
\]
\[
\geq \sum_{i=1}^M \int_{S^2} \psi_i \eta_{S^2}
\]
\[
\geq \sum_{i=1}^M H^2 (\{\psi_i = 1\}).
\]

In view of (9) this contradicts our choice of \( R \).

Since \( Q \) vanishes at some point, it is in fact zero on a whole connected component \( \Lambda \) of the open set \( S^2 \setminus n_u(\partial B_r) \). But this means that (after possibly redefining \( n_u \) on a set of measure zero) \( n_u \) does not take values in \( \Lambda \).

We assume without loss of generality that \( e_3 \in \Lambda \) and we denote by \( \Psi : S^2 \setminus \{e_3\} \to \mathbb{R}^2 \) the stereographic projection. Since \( \Lambda \subset S^2 \setminus n_u(\partial B_r) \) is open, there exists \( \rho > 0 \) such that \( S^2 \cap B_{2\rho}(e_3) \) does not intersect \( n_u(\overline{B_r}) \).

And \( \Psi \in C^\infty \left( \mathbb{R}^3 \setminus \overline{B_\rho(e_3)} \right) \). Hence \( \Psi \circ n \in (H^1 \cap L^\infty)(U, \mathbb{R}^2) \). Since \( \Psi \) is conformal, we deduce from \( K > 0 \) that the Jacobian of \( \Psi \circ n_u \) is positive (or negative) and bounded away from zero. Hence \( \Psi \circ n_u \) is continuous, cf. [Res89]. (In fact, our arguments essentially even show that \( \Psi \circ n_u \) is monotone in the Lebesgue sense, which quite directly implies continuity.) Hence \( n_u \) is continuous as well.

### 2.2 Proof of Theorem 2.1

Fix a point \( x_0 \in U \) and let us prove that \( u \) is smooth in a neighbourhood of this point. We assume without loss of generality that \( x_0 = 0 \) and that \( n_u(0) = e_3 \). We write \( u = \begin{pmatrix} \Psi \\ V \end{pmatrix} \) where \( V = e_3 \cdot u \) and \( \Psi : U \to \mathbb{R}^2 \) is the in-plane component.

**Lemma 2.8.** There exists a positive constant \( c \) such that \( \det \nabla \Psi \geq c \) near 0.
Proof. We compute
\[
\det \nabla \Psi = e_3 \cdot \begin{pmatrix} \frac{\partial_1 \Psi}{0} \\ \frac{\partial_2 \Psi}{0} \end{pmatrix} \times \begin{pmatrix} \frac{\partial_1 \Psi}{0} \\ \frac{\partial_2 \Psi}{0} \end{pmatrix} = e_3 \cdot \frac{\partial_1 u \times \partial_2 u}{\sqrt{g(n_u \cdot e_3)}}.
\]
By continuity of \(n_u\) and since \(\sqrt{g} \geq c\), there is a neighbourhood of 0 on which the right-hand side is bounded from below by a positive constant.

Since \(u\) is Lipschitz by hypothesis, we see that \(\Psi\) is Lipschitz and, by Lemma 2.8 that its Jacobian is bounded from below by a positive constant. Hence Clarke’s inverse function theorem [Cla76] shows that \(\Psi\) is locally bi-Lipschitz on some ball \(B\) centered at 0.

Define \(\Phi = \Psi^{-1}\) and define \(f = V \circ \Phi\), which is a map \(\Psi(B) \to \mathbb{R}\). Then \(u(B) = \text{graph}\ f|_{\Psi(B)}\). For \(z \in \Psi(B)\) we define
\[
G(z) = (z, f(z)),
\]
so that \(u = G(\Psi)\). We assume without loss of generality that \(\Psi(0) = 0\).

Clearly, \(f \in W^{1,\infty}(B)\). Moreover, \(f \in H^2\) by the chain rule for the composition of Sobolev functions with a bi-Lipschitz map \(\Phi\). The usual chain rule applies. We claim that \(f\) satisfies the prescribed Gauss curvature equation
\[
\det \nabla^2 f = K(\Phi) \cdot (1 + |\nabla f|^2)^2 \quad (10)
\]
almost everywhere on \(\Psi(B)\). In fact, denoting the metric induced by \(G\) by
\[
\tilde{g} = (\nabla G)^T(\nabla G) = I + \nabla f \otimes \nabla f,
\]
and the normal to \(G\) by \(\tilde{n}_u = \frac{\partial_1 G \times \partial_2 G}{|\partial_1 G \times \partial_2 G|}\), we have
\[
\tilde{n}_u = \frac{(-\nabla f, 1)^T}{(1 + |\nabla f|^2)^{1/2}}, \quad (11)
\]
because \(\det \tilde{g} = 1 + |\nabla f|^2\). Moreover, \(\partial_{\alpha} \partial_{\beta} G = (0, 0, \partial_{\alpha} \partial_{\beta} f)^T\). Therefore, \(\tilde{h} = \tilde{n}_u \cdot \nabla^2 G\) satisfies
\[
\tilde{h} = \frac{\nabla^2 f}{(1 + |\nabla f|^2)^{1/2}}. \quad (12)
\]
Taking determinants in (12), we see that
\[
\det \nabla^2 f = (1 + |\nabla f|^2) \det \tilde{h}. \quad (13)
\]
Using the chain rule, it is easy to verify that

\[ h = (\nabla \Psi)^T \tilde{h}(\Psi) (\nabla \Psi) \text{ almost everywhere.} \]  

(14)

A similar relation applies to \( g \) and \( \tilde{g} \). Therefore, using Lemma 2.2, we see that (13) implies (10).

Observe that the right-hand side of (10) is positive and bounded away from zero and infinity. Hence the following lemma implies that \( f \) is a (locally) convex function.

**Lemma 2.9.** Suppose \( f \in H^2(B) \) satisfies \( \det \nabla^2 f \geq c > 0 \). Then \( f \) is either locally convex on \( B \) or locally concave on \( B \).

**Proof.** This is proven in [ˇSve]. As observed in [ˇSve], the results in [IS93] (which were conjectured in [ˇSve]) indeed allow to relax the \( W^{2,\infty} \) hypothesis in [ˇSve] to the \( H^2 \) hypothesis used here. We refer to [Horon, LMP15] and the references therein for more on this.

We may assume that \( f \) is convex, and then, e.g. by a result of [Hei60] or by [Cal90b, Cal91], we may assume that \( f \) is strictly convex on \( B \). One can then deduce from (10) that \( f \in C^1 \), but we give a direct proof of this:

**Lemma 2.10.** We have \( f \in C^1(\Psi(B)) \).

**Proof.** We know that \( n_u : B \to S^2 \) is continuous. And so is \( \Phi \). Since \( \tilde{n}_u = n_u(\Phi) \), we see that \( \tilde{n}_u \) is continuous. Upon scalar multiplication of (11) with \( e_3 \), we have

\[ (1 + |\nabla f|^2)^{-1/2} = n_u(\Phi) \cdot e_3. \]

Since \( n_u(\Phi) \cdot e_3 \) is continuous and strictly positive, we conclude that \( (1 + |\nabla f|^2)^{-1/2} \) is continuous, hence so is \( \nabla f \), by (11).

The following lemma is [NS85, Theorem 1’]; cf. also [Cal90a].

**Lemma 2.11.** Let 0 < \( m < M < \infty \) and let \( \tilde{F} : B_R \to [m, M] \). If \( \varphi \in C^0(\overline{B}_R) \) is a convex Aleksandrov solution of

\[ \det \nabla^2 \varphi = \tilde{F}, \]

then there exists \( p \geq 1 \) such that \( \varphi \in W^{2,p}_{\text{loc}}(B_R) \). Moreover, \( p \to \infty \) as \( M/m \to 1 \).
Since $F$ is continuous, its oscillation is as small as we please on $B_R$, provided $R$ is small enough. Hence Lemma 2.11 implies that there exists $R > 0$ and $p > 2$ such that $f \in W^{2,p}(B_R)$.

Since $f \in W^{2,p}(B_R)$, also $G \in W^{2,p}(B_R)$. And (12) implies that $\tilde{h} \in L^p(B_R)$. But $\nabla \Psi \in L^\infty$ because $u \in W^{1,\infty}$. Therefore, (14) shows that $h \in L^p(\Phi(B_R))$.

Since $\Gamma$ is smooth and $\nabla u$ is bounded, we deduce from (2) that $u \in W^{2,p}(\Phi(B_R))$. In particular, by Morrey-Sobolev embedding, there exists a constant $\delta > 0$ such that $\Psi \in C^{1,\delta}(\Phi(B_R))$ and $f \in C^{1,\delta}(B_R)$. Since $K$ is, in particular, Lipschitz on $B$, we have $F \in C^{0,\delta}$. E.g. by the results in [Sch82], we therefore deduce from det $\nabla^2 f = F$ that $f \in C^{2,\delta}$. Hence (12) shows that $\tilde{h} \in C^{0,\delta}$. Hence $h \in C^{0,\delta}$ by (14). Thus (2) implies $u \in C^{2,\delta}$ for some positive $\delta$ in some neighborhood of the origin.

Since on some neighborhood of the origin $u \in C^{2,\delta}$, for every constant unit vector $e$ the function $v = e \cdot u$ is a $C^{2,\delta}$ solution of the Darboux equation

$$\det \left( \nabla^2 v + \Gamma \cdot \nabla v \right) = K |g| \left( 1 - g^{-1} : (\nabla v \otimes \nabla v) \right),$$

where $\Gamma \cdot \nabla v$ denotes a suitable contraction of indices. This equation is elliptic with respect to $v$. Since $g^{-1}$, $\Gamma$ and $K$ are smooth, and since $v \in C^{2,\delta}$, we conclude by standard theory that $v \in C^\infty$, hence $u$ is smooth in a neighborhood of the origin.

**Remark 2.12.** Notice that if we only assume that $g \in C^{2,\alpha}(U, \mathbb{R}^{2\times 2})$ then we can conclude that $u \in C^{2,\delta}(U; \mathbb{R}^3)$, for some $\delta > 0$. If $g \in C^{k,\alpha}(U, \mathbb{R}^{2\times 2})$, for $k \geq 3$ then we can conclude that $u \in C^{k+2,\alpha}(U, \mathbb{R}^3)$, by standard elliptic regularity (see, e.g., [GT01, Lemma 17.16]).

## 3 Homogenization for shells

### 3.1 Preliminaries

Unless additional properties are stated, $\omega \subset \mathbb{R}^2$ is assumed to be a bounded domain with boundary of class $C^3$. We set $I = (-\frac{1}{2}, \frac{1}{2})$ and $\Omega^h = \omega \times (hI)$, and $\Omega = \omega \times I$. From now on, $S$ denotes a compact connected oriented surface with boundary which is embedded in $\mathbb{R}^3$. For convenience we assume that $S$ is parametrized by a single chart: From now on, $\xi \in C^4(\overline{\omega}; \mathbb{R}^3)$ denotes an embedding with $\xi(\omega) = S$. The inverse of $\xi$ is denoted by $r : S \to \omega$, and we assume it to be $C^4$ up to the boundary.

As before by $g = (\nabla \xi)^T(\nabla \xi)$ we denote the Riemannian metric on $\omega$ induced by $\xi$. Its Christoffel symbols are denoted by $\Gamma^\gamma_{\alpha\beta}$. In what follows we recall
some standard notions and set the notation. All notions are discussed in
detail in most basic textbooks on Riemannian geometry. By $H^2_{\text{iso}}(S)$
we denote the $H^2$ isometries of the surface $S$. Notice that
\[ u \in H^2_{\text{iso}}(S) \iff \tilde{u} \in H^2_g(U), \]
where $\tilde{u} = u \circ \xi$. We repeat the geometrical framework we used in [HV14].

- The volume element on $S$ is denoted by $d\text{vol}_S$.
- The scalar product on a vector space $V$ is denoted by $\langle x, y \rangle_V$, and we define $x \cdot y = \langle x, y \rangle_{\mathbb{R}^k}$, for $x, y \in \mathbb{R}^k$.
- We denote by $T_S$ the tangent bundle over $S$, i.e., the collection of tangent spaces $T_xS$ with $x \in S$. A basis of $T_xS$ is given by $\tau_1(x), \tau_2(x)$, defined by $\tau_\alpha(x) = (\partial_\alpha \xi)(r(x))$ for all $x \in S$.

\[ \nabla r = \begin{pmatrix} \tau^1 \\ \tau^2 \end{pmatrix}. \]  \hspace{1cm} (15)

- By $n : S \to \mathbb{S}^2$ we denote the unit normal to $S$, i.e.,
\[ n(x) = \frac{\tau_1(x) \times \tau_2(x)}{|\tau_1(x) \times \tau_2(x)|} \text{ for all } x \in S. \]

We define $\tau^3 = \tau_3 = n$. The normal bundle of $S$ is denoted by $NS$ and
by definition has fibers $N_xS$ given by the span of $n(x)$. We denote by $T_xS = I - n(x) \otimes n(x)$
the orthogonal projection from $\mathbb{R}^3$ onto $T_xS$. We will frequently deal
with vector fields $V : S \to \mathbb{R}^3$ on the surface.
• The tensor product bundles $TS \otimes TS$ etc. are defined fiberwise. If $T_xS$ is regarded as a subspace of $\mathbb{R}^3$, then $T_x^*S \otimes T_x^*S$ can be regarded as a subspace of $\mathbb{R}^{3 \times 3}$.

The symmetric product $E \odot F$ of two vector spaces (or bundles) $E$ and $F$ by definition consists of elements of the form

$$a \odot b := \frac{1}{2} (a \otimes b + b \otimes a)$$

with $a \in E$ and $b \in F$.

Sections of the bundle $T^*S \otimes T^*S$ are called quadratic forms on $S$.

Sections $B$ of $T^*S \otimes T^*S$ can be regarded as maps from $S$ into $\mathbb{R}^{3 \times 3}$ via the embedding $\iota$ defined by $\iota(B) = B(T_S, T_S)$. (On the right-hand side and elsewhere we identify $(\mathbb{R}^3)^* \otimes (\mathbb{R}^3)^*$ with $\mathbb{R}^{3 \times 3}$). By definition, $B(T_S, T_S) : S \to \mathbb{R}^{3 \times 3}$ takes the vector fields $v, w : S \to \mathbb{R}^3$ into the function $x \mapsto B(x)(T_S(x)v(x), T_S(x)w(x))$.

• For any vector bundle $E$ over $S$ we denote by $L^2(S; E)$ the space of all $L^2$-sections of $E$. The spaces $H^1(S; E)$ etc. are defined similarly. Explicitly, e.g. for $E = TS$ we have

$$L^2(S, TS) = \{ f \in L^2(S, \mathbb{R}^3) : f(x) \in T_xS \text{ for a.e. } x \in S \}.$$

• For any vector bundle $E$ over $S$ with fibers $E_x$, we denote by $L^2(\mathcal{Y}, E)$ the vector bundle over $S$ with fibers $L^2(\mathcal{Y}, E_x)$. The bundles $H^1(\mathcal{Y}, E)$ etc. are defined similarly. For example, $L^2$-sections of the bundle $H^1(\mathcal{Y}, TS)$ are given by

$$L^2(S, H^1(\mathcal{Y}, TS)) = \{ Z \in L^2(S, H^1(\mathcal{Y}, \mathbb{R}^3)) : Z(x) \in H^1(\mathcal{Y}, T_xS) \text{ for a.e. } x \in S \}.$$

• For a scalar function $f : S \to \mathbb{R}$ its gradient field along $S$ will be denoted by $df$, which is also the notation for the corresponding 1-form. In other words,

$$df(x)(\tau) = \nabla_\tau f(x) \text{ for all } \tau \in T_xS.$$

Here and elsewhere $\nabla$ denotes the usual gradient on $\mathbb{R}^3$ (or on $\mathbb{R}^2$) of the extension of $f$, and $\nabla_\tau f = \tau \cdot \nabla f = \sum_i \tau_i \partial_i f$. We extend these definitions componentwise to maps into $\mathbb{R}^3$.

• For functions $f \in L^2(S, H^2(\mathcal{Y}))$ we define $\text{Hess}_\mathcal{Y} f$ to be the section of the bundle $L^2(\mathcal{Y}; T^*S \otimes T^*S)$ over $S$ given by

$$(\text{Hess}_\mathcal{Y} f)(x, y) = (\nabla^2_y f)_{\alpha\beta}(x, y) \tau^\alpha(x) \otimes \tau^\beta(x),$$

where $(\nabla^2_y f)_{\alpha\beta} = \partial_{y\alpha} \partial_y \beta f$.
For $$v \in L^2(S, H^1(\mathcal{Y}; \mathbb{R}^2))$$ we define the section $$\text{Def}_Y v$$ of the bundle $$L^2(\mathcal{Y}, T^*S \otimes T^*S)$$ by

$$(\text{Def}_Y v)(x, y) = (\text{sym} \nabla_y v(x, y))_{\alpha\beta}\tau^\alpha(x) \otimes \tau^\beta(x).$$

Here and elsewhere $$\nabla_y$$ is gradient in $$Y$$ with respect to the variable $$y$$.

The Weingarten map $$S$$ of $$S$$ is given by $$S = d\pi$$, i.e.,

$$S(x)\tau = (\nabla_\tau n)(x)$$ for all $$x \in S, \tau \in T_xM.$$  

We extend $$S$$ to a linear map on $$TS \oplus NS \cong \mathbb{R}^3$$ by setting $$S = S \circ T_s$$, i.e., we define $$S(x)n(x) = 0$$.

### 3.1.1 The nearest point retraction

The nearest point retraction of a tubular neighbourhood of $$S$$ onto $$S$$ will be denoted by $$\pi$$. Hence

$$\pi(x + tn(x)) = x$$ whenever $$|t|$$ is small enough.

After rescaling the ambient space, we may assume that the curvature of $$S$$ is as small as we please. In particular, we may assume without loss of generality that $$\pi$$ is well-defined on a domain containing the closure of $$\{x + tn(x); x \in S, -1 < t < 1\}$$, and that $$|Id + tS(x)| \in (1/2, 3/2)$$ for all $$t \in (-\frac{1}{2}, \frac{1}{2})$$ and all $$x \in S$$.

For a subset $$A \subset S$$ and $$h \in (0, 1]$$ we define $$A^h = \{x + tn(x); x \in S, -h/2 < t < h/2\}$$. In particular, the shell is given by

$$S^h = \{x + tn(x); x \in S, t \in (-h/2, h/2)\}.$$ 

We introduce the function $$t : S^1 \to \mathbb{R}$$ by

$$t(x) = (x - \pi(x)) \cdot n(x)$$ for all $$x \in S^1.$$ 

We extend all maps $$f : S \to \mathbb{R}^k$$ trivially from $$S$$ to $$S^1$$, simply by defining

$$f(x) = f(\pi(x))$$ for all $$x \in S^1.$$ 

In particular, we extend $$r, TS$$ and $$S$$ trivially to $$S^1$$, i.e., we have $$S(x) = S(\pi(x))$$ and $$T_S(x) = T_S(\pi(x))$$ and $$r(x) = r(\pi(x))$$ for all $$x \in S^1$$. The following identity is easy to verify and is given in {HV14}

$$\nabla \pi)(x) = T_S(x) (I + t(x)S(x))^{-1}. $$

By using the chain rule we have for $$f : S \to \mathbb{R}^k$$ the following identity

$$\nabla (f \circ \pi) = df \circ \pi T_S (I + tS)^{-1}.$$
### 3.1.2 Thin films

To deal with thin films, we introduce the map \( \Xi : \omega \times \mathbb{R} \to \mathbb{R}^3 \) by setting

\[
\Xi(z', z_3) = \xi(z') + z_3 n(\xi(z')) \quad \text{for all} \quad z' \in \omega \quad \text{and} \quad z_3 \in \mathbb{R}.
\]

As in [FJM02] we will use the diffeomorphism \( \tilde{\Theta}^h : \Omega_h \to \Omega \) given by

\[
\tilde{\Theta}^h(z_1, z_2, z_3) = (z_1, z_2, z_3/h),
\]

and for a map \( \tilde{y} : \Omega \to \mathbb{R}^3 \) we introduce the scaled gradient \( \tilde{\nabla}_h \tilde{y} = (\partial_1 \tilde{y}, \partial_2 \tilde{y}, \frac{1}{h} \partial_3 \tilde{y}) \).

The counterpart of \( \tilde{\Theta}^h \) on the shell is the diffeomorphism \( \Theta^h : S^h \to S^1 \) given by

\[
\Theta^h(x) = \pi(x) + \frac{t(x)}{h} n(x).
\]  

It is easy to see that

\[
\Theta^h \circ \Xi = \Xi \circ \tilde{\Theta}^h \quad \text{on} \quad \Omega_h. \tag{21}
\]

For given \( u : S^h \to \mathbb{R}^3 \) we define its pulled back version \( \tilde{u} : \Omega_h \to \mathbb{R}^3 \) by \( \tilde{u} = u \circ \Xi \). We also define its rescaled version \( y : S^1 \to \mathbb{R}^3 \) by \( y(\Theta^h) = u \) on \( S^h \) and we define the pulled back version \( \tilde{y} \) of this map by \( \tilde{y} = y \circ \Xi \). Then it is easy to see that

\[
(\tilde{\nabla}_h \tilde{y}) \circ \tilde{\Theta}^h = \nabla \tilde{u} \quad \text{on} \quad \Omega_h. \tag{22}
\]

We define the rescaled gradient \( \nabla_h y \) of \( y \) by the condition

\[
(\nabla_h y) \circ \Theta^h = \nabla u \quad \text{on} \quad S^h. \tag{23}
\]

This corresponds to the fact that the rescaled gradient on the canonical domain is just the physical gradient (on the physical domain) translated to the canonical domain. Using (21) and (22) it is easy to see that

\[
\tilde{\nabla}_h \tilde{y} = \nabla u(\Xi) \left( (\nabla \Xi) \circ (\tilde{\Theta}^h)^{-1} \right). \tag{24}
\]

Since \( \nabla t = n \), formula (23) and formula (19) show (recall that \( n \) is extended trivially to \( S^1 \)):

\[
\nabla \Theta^h = \nabla \pi + \frac{t}{h} \nabla n + \frac{1}{h} n \otimes n = T_S(I + tS)^{-1} + \frac{t}{h} S T_S(I + tS)^{-1} + \frac{1}{h} n \otimes n.
\]

Since \( T_S \) clearly commutes with \( S \), we see that \( T_S \) commutes with \( (I + tS)^{-1} \) as well. Hence

\[
\nabla \Theta^h = (I_h + \frac{t}{h} S)(I + tS)^{-1} \quad \text{on} \quad S^h, \tag{25}
\]

where \( I_h = T_S + \frac{1}{h} n \otimes n \). To express \( \nabla_h y \) in terms of \( \nabla y \), insert the definition of \( y \) into (23) and use (25) to find

\[
\nabla_h y = \nabla y (I_h + tS)(I + thS)^{-1} \quad \text{on} \quad S^1. \tag{26}
\]
3.1.3 Two-scale convergence

Recall that we extend the chart \( r \) trivially from \( S \) to \( S^1 \). We make the following definitions:

- A sequence \((f_h) \subset L^2(S^1)\) is said to converge weakly two-scale on \( S^1 \) to the function \( f \in L^2(S^1, L^2(Y))\) as \( h \to 0 \), provided that the sequence \((f_h)\) is bounded in \( L^2(S^1)\) and
  \[
  \lim_{h \to 0} \int_{S^1} f_h(x) \rho(x, r(x)/\varepsilon) \, dx = \int_{S^1} \int_Y f(x, y) \rho(x, y) \, dy \, dx.
  \] (27)
  for all \( \rho \in C_0^\infty(S^1, C_0^\infty(Y)) \).

- We say that \( f_h \) strongly two-scale converges to \( f \) if, in addition,
  \[
  \lim_{h \to 0} \|f_h\|_{L^2(S^1)} = \|f\|_{L^2(S^1 \times Y)}.
  \]

We write \( f_h \overset{\text{wss}}{\rightharpoonup} f \) to denote weak two-scale convergence and \( f_h \overset{\text{sps}}{\rightharpoonup} f \) to denote strong two-scale convergence. If \( f_h \overset{\text{sps}}{\rightharpoonup} f \) then \( f_h \rightharpoonup \int_Y f(\cdot, y) \, dy \) weakly in \( L^2 \).

If \( f_h \) is bounded in \( L^2(S^1) \) then it has a subsequence which converges weakly two-scale to some \( f \in L^2(S^1; L^2(Y)) \). These and other facts can be deduced from the corresponding results on planar domains (cf. [All92, Vis06]) by means of the following simple observations.

Defining \( \tilde{f}_h = f_h \circ \Xi \) and \( \tilde{f}(z, y) = f(\Xi(z), y) \), and taking
\[
\tilde{\rho}(z, y) = \rho(\Xi(z), y) \left( \det \nabla \Xi^T(z) \nabla \Xi(z) \right)^{1/2},
\]
a change of variables shows that (27) is equivalent to
\[
\int_{\Omega} \tilde{f}_h(z) \tilde{\rho}(z, z'/\varepsilon) \, dz \to \int_{\Omega} \int_Y \tilde{f}(z, y) \tilde{\rho}(z, y) \, dy \, dz,
\] (28)
where \( z' \) is the projection of \( z \) onto \( \mathbb{R}^2 \). Hence \( f_h \overset{\text{sps}}{\rightharpoonup} f \) on \( S^1 \) if and only if \( \tilde{f}_h \overset{\text{sps}}{\rightharpoonup} \tilde{f} \) on \( \Omega \) in the usual sense.

When \( f_h : S \to \mathbb{R} \), then \( f_h \overset{\text{sps}}{\rightharpoonup} f \) on \( S \) means, by definition, that the trivial extensions converge weakly two-scale on \( S^1 \). In particular, \( f_h \overset{\text{sps}}{\rightharpoonup} f \) on \( S \) if and only if \( \tilde{f}_h \overset{\text{sps}}{\rightharpoonup} \tilde{f} \) on \( \omega \). All these definitions are extended componentwise to vector-valued maps. For quadratic forms \( q, q_h \) on \( S \) we say \( q_h \overset{\text{sps}}{\rightharpoonup} q \) if \( q_h(\tau, \sigma) \overset{\text{sps}}{\rightharpoonup} q(\tau, \sigma) \) for all \( \tau, \sigma \in C^1(S, TS) \). A similar definition applies to other bundles. In the end we give the compactness lemma which gives the information about two-scale limit of scaled gradients. For the proof see [HV14, Lemma 4.3].
Lemma 3.1. Let $w^h \in H^1(S^1, \mathbb{R}^3)$ be such that

$$\limsup_{h \to 0} \left( \|w^h\|_{L^2(S^1)} + \|\nabla_h w^h\|_{L^2(S^1)} \right) < \infty.$$  

Then there exists a map $w_0 \in H^1(S, \mathbb{R}^3)$ and a field $H_\gamma \in L^2(S \times I \times \mathcal{Y}, \mathbb{R}^{3 \times 3})$ of the form

$$H_\gamma = \begin{cases} 
(\nabla y w_1, w_2) & \text{for some } \begin{cases} w_1 \in L^2(S; \dot{H}^1(\mathcal{Y}, \mathbb{R}^3)) \\
2 \in L^2(S \times \mathcal{Y} \times I, \mathbb{R}^3) \end{cases} \\
(\nabla y w_1, \frac{1}{\gamma} \partial_3 w_1) & \text{for some } w_1 \in L^2(S; \dot{H}^1(I \times \mathcal{Y}, \mathbb{R}^3)) \\
(\nabla y w_1, w_2) & \text{for some } \begin{cases} w_1 \in L^2(S \times I; \dot{H}^1(\mathcal{Y}, \mathbb{R}^3)) \\
w_2 \in L^2(S \times I, \mathbb{R}^3) \end{cases} \\
(\nabla y w_1, w_2) & \text{for some } \begin{cases} w_1 \in L^2(S \times I, \dot{H}^1(\mathcal{Y}, \mathbb{R}^3)) \\
w_2 \in L^2(S \times I, \mathbb{R}^3) \end{cases} \\
\end{cases} \quad (29)$$

such that, up to a subsequence, $w^h \to w_0$ in $L^2$ and

$$\nabla_h w^h \rightharpoonup dw_0 \circ T_S + \sum_{i,j=1}^3 (\dot{H}_\gamma)_{ij} \tau^i \otimes \tau^j \quad \text{weakly two-scale on } S^1.$$  

Here, $w_0$ is the weak limit in $H^1(S)$ of $\int_I w^h(x + tn(x))dt$ and $\dot{H}_\gamma \in L^2(S^1 \times \mathcal{Y}; \mathbb{R}^{3 \times 3})$ is defined by $\dot{H}_\gamma(x, y) = H_\gamma(\pi(x), t(x), y)$.

3.1.4 Elasticity framework

Throughout this paper we assume that the limit

$$\gamma := \lim_{h \to 0} \frac{h}{\varepsilon(h)}$$

exists in $[0, \infty]$. We will frequently write $\varepsilon$ instead of $\varepsilon(h)$, but always with the understanding that $\varepsilon$ depends on $h$. In the case $\gamma = 0$ we will additionally assume that $\varepsilon(h)^2 \ll h$.

From now on we fix a Borel measurable energy density

$$W : S^1 \times \mathbb{R}^2 \times \mathbb{R}^{3 \times 3} \to [0, \infty]$$

with the following properties:
(W1) (continuity) $W(\cdot, y, F)$ is continuous for almost every $y \in \mathbb{R}^2$ and $F \in \mathbb{R}^{3 \times 3}$.

(W2) (periodicity) $W(x, \cdot, F)$ is $Y$-periodic for all $x \in S^1$ and almost every $F \in \mathbb{R}^{3 \times 3}$.

(W3) (objectivity) For all $(x, y) \in S^1 \times Y$ we have $W(x, y, I) = 0$ and $W(x, y, RF) = W(x, y, F)$ for all $F \in \mathbb{R}^{3 \times 3}$, $R \in \text{SO}(3)$.

(W4) (uniform quadratic growth near identity) There exist constants $0 < \alpha \leq \beta$ and $\rho > 0$ such that for all $(x, y) \in S^1 \times Y$ we have

\[
W(x, y, F) \geq \alpha \text{dist}^2(F, \text{SO}(3)) \quad \text{for all } F \in \mathbb{R}^{3 \times 3}
\]

\[
W(x, y, F) \leq \beta \text{dist}^2(F, \text{SO}(3)) \quad \text{for all } F \in \mathbb{R}^{3 \times 3} \text{ with dist}^2(F, \text{SO}(3)) \leq \rho.
\]

(W5) (uniform expansion) For each $(x, y) \in S^1 \times Y$ there exists a quadratic form $Q(x, y, \cdot) : \mathbb{R}^{3 \times 3} \to \mathbb{R}$ such that

\[
\text{ess sup}_{(x,y) \in S^1 \times Y} \frac{|W(x, y, I + G) - Q(x, y, G)|}{|G|^2} \to 0 \text{ as } G \to 0. \quad (30)
\]

In the following lemma we collect the properties of $Q(\cdot, \cdot, \cdot)$. They are direct consequence of the properties of $W$. For a proof see [Neu12, Lemma 2.7].

**Lemma 3.2.** Let $W$ be as above and let $Q$ be the quadratic form associated to $W$ through the expansion (30). Then

(Q1) $Q(\cdot, y, \cdot)$ is continuous for almost every $y \in \mathbb{R}^2$,

(Q2) $Q(x, \cdot, G)$ is $Y$-periodic and measurable for all $x \in \Omega$ and all $G \in \mathbb{R}^{3 \times 3}$,

(Q3) for all $x \in \Omega$ and almost every $y \in \mathbb{R}^2$ the map $Q(x, y, \cdot)$ is quadratic and satisfies

\[
\alpha|\text{sym} G|^2 \leq Q(x, y, G) = Q(x, y, \text{sym} G) \leq \beta|\text{sym} G|^2 \quad \text{for all } G \in \mathbb{R}^{3 \times 3}.
\]

The elastic energy per unit thickness of a deformation $u^h \in H^1(S^h; \mathbb{R}^3)$ of the shell $S^h$ is given by

\[
J^h(u^h) = \frac{1}{h} \int_{S^h} W(\Theta^h(x), r(x)/\varepsilon, \nabla u^h(x)) \, dx.
\]
In order to express the elastic energy in terms of the new variables, we associate with $y : S^1 \to \mathbb{R}^3$ the energy
\[ I^h(y) = \int_{S^1} W(x, r(x)/\varepsilon, \nabla_h y(x)) \det(I + t(x)S(x))^{-1} \, dx \]
\[ = \int_S \int_I W(x + tn(x), r(x)/\varepsilon, \nabla_h y(x + tn(x))) \, dt \, d\text{vol}_S(x). \]

By a change of variables we have
\[ J^h(u^h) = \frac{1}{h} \int_{S^1} W(x, r(x)/\varepsilon, \nabla_h y^h(x)) \left| \det(\Theta^h)^{-1}(x) \right| \, dx, \]
where $y^h(\Xi^h) = u^h$. Using (25) it is easy to see that
\[ J^h(u^h) = I^h(y^h) (1 + O(h)) \text{ as } h \to 0, \quad (31) \]
where $|O(h)| \leq Ch$.

### 3.2 Asymptotic energy functionals and main result

Next we will introduce the asymptotic energy functionals. In order to do so, we need the definition of the relaxation fields and the cell formulas. Recall that $a \circ b = \frac{1}{2}(a \otimes b + b \otimes a)$. We now define relaxation operators with range in the space of $L^2$-sections of the vector bundle over $S$ with fibers given for each $x \in S$ by
\[ L^2(I \times Y, (T_x^* S \otimes T_x^* S) \oplus (T_x^* S \otimes N_x^* S) \oplus (N_x^* S \otimes N_x^* S)). \quad (32) \]

Of course each of these fibers is isomorphic to $L^2(I \times Y; \mathbb{R}^{3 \times 3})$. We now make the following definitions: Set $D(U_0) = \hat{H}^1(Y, \mathbb{R}^2) \times \hat{H}^2(Y) \times L^2(I \times Y, \mathbb{R}^3)$ and define
\[ U_0(\zeta, \varphi, g) = \text{Def}_Y \zeta + 2g_\alpha \tau^\alpha \otimes n + g_3 n \otimes n - t \text{Hess}_Y \varphi \text{ for all } (\zeta, \varphi, g) \in L^2(S, D(U_0)). \]

Set $D(U_\infty) = L^2(I, \hat{H}^1(Y, \mathbb{R}^2)) \times L^2(I, \hat{H}^1(Y)) \times L^2(I, \mathbb{R}^3)$ and define
\[ U_\infty(\zeta, \rho, c) = \text{Def}_Y \zeta + 2(\partial_{y_\alpha} \rho + c_\alpha) \tau^\alpha \otimes n + c_3 n \otimes n \text{ for all } (\zeta, \rho, c) \in L^2(S, D(U_\infty)). \]

Set $D(U_\gamma) = \hat{H}^1(I \times Y, \mathbb{R}^2) \times H^1(I \times Y)$ and define
\[ U_\gamma(\zeta, \rho) = \text{Def}_Y \zeta + (\partial_{y_\alpha} \rho + \frac{1}{\gamma} \partial_3 \zeta_\alpha) \tau^\alpha \otimes n + (\frac{1}{\gamma} \partial_3 \rho) n \otimes n \]
for all \((\zeta, \rho) \in L^2(S, D(U_0))\).

By trivially embedding \(D(U_0)\) as constant maps into \(L^2(S, D(U_0))\), we can regard \(U_0\) also as a map from \(D(U_0)\) itself into \((32)\). For each \(x \in S\) the fiberwise action \(U_0^{(x)}\) of \(U_0\) is

\[
U_0^{(x)}(\zeta, g) = (\text{Def}_Y \zeta)(x) + 2g_{\alpha\alpha}(x)\circ n(x) + g_{33} n(x)\circ n(x) \text{ for all } (\zeta, g) \in D(U_0).
\]

For each \(x \in S\) we define \(L_0^{(x)}(I \times Y) = U_0^{(x)}(D(U_0))\), i.e.,

\[
L_0^{(x)}(I \times Y) = \left\{(\zeta, g) : (\zeta, g) \in D(U_0)\right\}.
\]

This is a subspace of \((32)\), i.e., of \(L^2(I \times Y; \mathbb{R}^{3\times 3})\). We denote by \(L_0(I \times Y)\) the vector bundle over \(S\) with fibers \(L_0^{(x)}(I \times Y)\); in what follows we will frequently omit the index \((x)\) for the fibers. The bundles \(L_\gamma(I \times Y)\), for \(\gamma \in (0, \infty)\) are defined analogously. The elements of these spaces are the relaxation fields.

For \(\gamma \in [0, \infty]\) and \(x \in S\) we define \(Q_\gamma(x) : T_x^* S \otimes T_x^* S \to \mathbb{R}\), by setting

\[
Q_\gamma(x, q) = \inf_{U \in L_0^{(x)}(I \times Y)} \int_I \int_Y Q(x + t \eta(x), y, p + tq + U(t, y)) \, dy \, dt, \quad (33)
\]

for each \(x \in S\) and \(q \in T_x^* S \otimes T_x^* S\). For \(x \in S\) and \(q \in T_x^* S \otimes T_x^* S\) define the homogeneous relaxation (cf. [LMP10]):

\[
Q_2(x, t, q) = \min_{M \in \mathbb{R}^{3\times 3}_{\text{sym}}} \{Q(x + t \eta(x), M) : M(T_S, T_S) = q(T_S, T_S)\}. \quad (34)
\]

Then it is easy to see that

\[
Q_0(x, q) = \inf_{I \times Y} Q_2(x + t \eta(x), y, p + tq + \text{Def}_Y \zeta - t \text{Hess}_Y \varphi) \, dt \, dy, \quad (35)
\]

where both infima are taken over all \(\zeta \in H^1(Y; \mathbb{R}^2)\), all \(\varphi \in H^2(Y)\) and all \(p \in T^* S \otimes T^* S\). In the case when \(Q\) does not depend on \(t\), i.e., the material is homogeneous in the thickness direction we find:

\[
Q_0(x, q) = \frac{1}{12} \inf_{\varphi \in H^2(Y)} \int_Y Q_2(x, y, q + \text{Hess}_Y \varphi) \, dy.
\]

**Remark 3.3.** The formulae \((34)\) and \((34)\) correspond to the fact that in the regime when dimensional reduction dominates, one can, at least on the level of energies, first do the dimensional reduction \((34)\) and then the homogenization \((35)\). However, doing homogenization of the bending shell model is a different problem, since one has to deal with the restriction of being an isometry (see [NO15] for the planar case).
As in [NV13], one can prove pointwise continuity of the energies in the parameter $\gamma$, i.e., that for all $x \in S$, $q \in T_x^* S \otimes T_x^* S$ we have

$$\lim_{\gamma \to \infty} Q_\gamma(x, q) = Q_\infty(x, q) \quad \text{and} \quad \lim_{\gamma \to 0} Q_\gamma(x, q) = Q_0(x, q).$$

For $u \in H^2_{iso}$, by $S^*_u$ we denote the relative change of Weingarten maps

$$S^*_u = (R \circ \Xi^{-1})^\dagger \nabla (R \circ \Xi^{-1}) T_S \cdot n \quad (36)$$

$$= (R \circ \Xi^{-1})^\dagger \nabla ((R \circ \Xi^{-1}) n) T_S - S \quad (37)$$

$$= (R \circ \Xi^{-1})^\dagger S_{u(S)} (R \circ \Xi^{-1}) T_S - S, \quad (38)$$

where $R = \nabla \tilde{u} \in H^1(\omega; \text{SO}(3))$ and we have denoted by $\nabla (\tilde{R}^h \circ \Xi^{-1}) T_S \cdot n$ the operator on $\mathbb{R}^3$ given by

$$\nabla (\tilde{R}^h \circ \Xi^{-1}) T_S \cdot n = \nabla (\tilde{R}^h \circ \Xi^{-1}) T_S v \cdot n, \quad \text{for all} \quad v \in \mathbb{R}^3.$$

Notice that $S^*_u \in L^2(S; T^* S \otimes T^* S)$.

For $\gamma \in [0, \infty] \quad \text{we define the functionals} \quad I_\gamma : H^1(S; \mathbb{R}^3) \to \mathbb{R}$ by setting

$$I_\gamma(u) = \begin{cases} \int_S Q_\gamma(x, S^*_u(x)) \ dv_S(x), & \text{if} \ u \in H^2_{iso}, \\ +\infty, & \text{otherwise.} \end{cases}$$

We are now ready to state the main result of this part.

**Theorem 3.4.** Let $W$ be as above and assume that $u^h \in H^1(S^h; \mathbb{R}^3)$ satisfy

$$\lim_{h \to 0} \sup h^{-2} J^h(u^h) < \infty. \quad (39)$$

Define $y^h : S^1 \to \mathbb{R}^3$ by $y^h(\Theta^h) = u^h$. Put $\gamma = \lim_{h \to 0} \frac{h}{\epsilon(h)^2}$ and in the case $\gamma = 0$ assume additionally $\epsilon(h)^2 \ll h$. Then the following are true:

(i) (lower bound). If $y^h$ is such that $y^h - \int_{S^1} y^h \to u$, strongly in $L^2$, then we have

$$\lim_{h \to 0} \inf h^{-2} J^h(u^h) \geq I_\gamma(u).$$

(ii) (recovery sequence). Assume additionally that $\omega$ is star shaped with respect to some ball $B \subset \omega$ and that $S$ is convex, i.e., there exists $C > 0$ such that for every $x \in S$ we have

$$\frac{1}{C} |\tau|^2 \leq S(x) \tau \cdot \tau \leq C |\tau|^2, \quad \forall \tau \in T_x S.$$

Then we have that for any $u \in H^2_{iso}$ there exist $u^h \in H^1(S^h; \mathbb{R}^3)$ satisfying (40), and such that $y^h \to u$, strongly in $H^1$. Moreover, we have

$$\lim_{h \to 0} h^{-2} J^h(u^h) = I_\gamma(u).$$
This is complemented by the compactness result, given in [FJMM03, Theorem 1].

**Theorem 3.5.** Suppose a sequence $u^h \in H^1(S^h, \mathbb{R}^3)$ has finite bending energy, that is (40) is satisfied. Define $y^h : S^1 \to \mathbb{R}^3$ as above. Then there exists $u \in H_{iso}^2(S)$ such that on a subsequence

$$y^h - \int_{S^1} y^h \, dx \to u, \quad \text{strongly in } H^1(S^1, \mathbb{R}^3),$$

$$\nabla_h y^h \to R, \quad \text{strongly in } L^2(S^1, \mathbb{R}^{3 \times 3}),$$

as $h \to 0$ after passing to subsequences and extending $u$ and $n$ trivially to $S^1$. Here $R \in H^1(S, SO(3))$ is defined by $RT_S = \nabla u T_S$ and $Rn = R\tau_1 \times R\tau_2$.

### 3.3 Two scale compactness and lower bound

We impose that the sequence $(u^h) \subset H^1(S^h, \mathbb{R}^3)$ satisfies

$$\limsup_{h \to 0} h^{-2} J^h(u^h) < \infty, \quad (40)$$

and for a given sequence $(u^h) \subset H^1(S^h, \mathbb{R}^3)$ we define the sequence $(y^h) \subset H^1(S^1, \mathbb{R}^3)$ by the expression $y^h(\Theta^h) = u^h$. The following lemma gives us the compactness result we need. It is a direct consequence of [FJM02, Theorem 3.1] and of the arguments in [FJMM]. We refer also to [FJMM03]. The result is obtained by dividing the set $\omega$ into the small cuboids $(C^h_i)_{i=1}^n$ of size $\delta(h)$ in the in-plane direction and height $h$ and then applying the theorem on geometric rigidity on the domains $\Xi(C^h_i)$ for every $i$ and $h > 0$ and by smoothening.

**Lemma 3.6.** There exist constants $C, c > 0$, depending only on $\omega$, such that the following is true for every $h > 0$ sufficiently small: if $u \in H^1(S^h, \mathbb{R}^3)$ then there exists a map $R : \omega \to SO(3)$ which is piecewise constant on each cube $x + \delta(h)Y$ with $x \in \delta(h)\mathbb{Z}^2$ and there exists $\bar{R} \in H^1(\omega, \mathbb{R}^{3 \times 3})$ such that for each $\xi \in \mathbb{R}^2$ which satisfy $|\xi|_\infty = \max\{|\xi \cdot e_1|, |\xi \cdot e_2|\} \leq h$ and for each $\bar{\omega} \subset \omega$ which satisfy $\text{dist}(\bar{\omega}, \partial \omega) > ch$ we have:

$$\|\nabla u \circ (\Theta^h)^{-1} \circ \Xi - R\|^2_{L^2(\bar{\omega} \times I)} + \|R - \bar{R}\|^2_{L^2(\bar{\omega})} + h^2\|R - \bar{R}\|^2_{L^\infty(\bar{\omega})},$$

$$+ h^2\|\nabla^\prime \bar{R}\|^2_{L^2(\bar{\omega})} + \|\tau \bar{R} - R\|^2_{L^2(\bar{\omega})} \leq C \|\text{dist} (\nabla u \circ (\Theta^h)^{-1} \circ \Xi, SO(3))\|^2_{L^2(\Omega)},$$

where $\tau \bar{R} = \bar{R} \circ \tau$.
where \((\tau_\xi R)(z') := R(z' + \xi)\). Here

\[
\delta(h) := \begin{cases} 
\varepsilon(h), & \text{if } \lim_{h \to 0} \frac{h}{\varepsilon(h)} > 0 \\
\left\lfloor \frac{h}{\varepsilon(h)} \right\rfloor \varepsilon(h), & \text{if } \lim_{h \to 0} \frac{h}{\varepsilon(h)} = \infty, \\
h, & \text{otherwise.}
\end{cases}
\]

For a sequence \((u^h)\) that satisfies (40), we define the sequence \((G^h) \subset L^2(S^1; \mathbb{R}^{3 \times 3})\) of approximate strains with

\[
G^h = \frac{(R^h \circ \Xi^{-1})^* \nabla u^h \circ (\Theta^h)^{-1} - I}{h},
\]

where \(R^h : \omega \to SO(3)\) is a sequence of piecewise constant maps given by Lemma 3.6. Notice that \(\nabla u^h \circ (\Theta^h)^{-1} = \nabla_h y^h\), where the operator \(\nabla_h\) is given by (23).

For \(\tilde{\omega} \in \omega\) we denote by \(\tilde{S} = \Xi(\omega)\) and by \(\tilde{S}^1 = \tilde{\omega} \times I\). We are interested in identifying two-scale limits of the sequence \((G^h)\). We will not give the complete proof, only reduce the case to the planar situation, when the rest of the proof goes by not so difficult adaptation to the proofs given in \cite{HNV14, Vel15}.

**Proposition 3.7.** Let \((u^h)\) be a sequence of deformations that satisfies (40), let \(G^h\) be defined with (41) and let \(u \in H^{2}_{\text{iso}}(S)\) be determined by Theorem 3.5. Then for every \(\tilde{\omega} \in \omega\) such that \(\partial \tilde{\omega}\) is \(C^{1,1}\) we have the following convergence on a subsequence:

- **case \(\gamma \in (0, \infty)\):** there exist \(B \in L^2(\tilde{S}, T^*S \odot T^*S)\) and \((\zeta, \rho) \in L^2(\tilde{S}, D(U_\gamma))\) such that
  \[
  \text{sym } G^h \overset{2}{\rightharpoonup} B + tS^r_u + U_\gamma(\zeta, \rho);
  \]

- **case \(\gamma = \infty\):** there exist \(B \in L^2(S, T^*S \odot T^*S)\) and \((\zeta, \rho, c) \in L^2(\tilde{S}; D(U_\infty))\) such that
  \[
  \text{sym } G^h \overset{2}{\rightharpoonup} B + tS^r_u + U_\infty(\zeta, \rho, c);
  \]

- **case \(\gamma = 0, \varepsilon^2 \ll h \ll \varepsilon\):** there exist \(B \in L^2(\tilde{S}; T^*S \odot T^*S)\) and \((\zeta, \varphi, g) \in L^2(\tilde{S}, D(U_0))\) such that
  \[
  \text{sym } G^h \overset{2}{\rightharpoonup} B + tS^r_u + U_0(\zeta, \varphi, g).
  \]
Proof. Take \( \tilde{\omega} \in \omega \) such that \( \tilde{\omega} \) has \( C^{1,1} \) boundary. We decompose the deformation \( y^h \) on the set \( \tilde{S}^1 = \tilde{\omega} \times I \), where \( \tilde{\omega} \in \omega \) is such that \( \partial \tilde{\omega} \) is \( C^{1,1} \).

\[
y^h(x + tn) = \tilde{y}^h(x) + th(\tilde{R}^h \circ \Xi^{-1})n + hz^h(x + tn),
\]

where \( \tilde{R}^h \) is given by Lemma 3.6 and \( \tilde{y}^h(x) = \int_0^1 y^h(x + tn(x))dt \). Notice that

\[
\int_I z^h(x + tn)dt = 0.
\]

We compute using (19) and (26)

\[
\nabla_h y^h = \nabla \tilde{y}^h T_S(I + thS)^{-1} + (\tilde{R}^h \circ \Xi^{-1})n \otimes n
\]

\[
+ th\nabla (\tilde{R}^h \circ \Xi^{-1})T_S(I + thS)^{-1} \cdot n + th(\tilde{R}^h \circ \Xi^{-1})ST_S(I + thS)^{-1}
\]

\[
+ h\nabla_h z^h.
\]

Notice that

\[
\nabla \tilde{y}^h = \int_I \nabla y^h T_S(I + tS)dt.
\]

From Lemma 40 and (3.6) we conclude that

\[
\| \nabla_h y^h - \tilde{R}^h \circ \Xi^{-1} \|_{L^2(\tilde{S}^1)} \leq Ch.
\]

From the identity (26) and (45) we also conclude that

\[
\| \nabla \tilde{y}^h - (R^h \circ \Xi^{-1})T_S \|_{L^2(\tilde{S}^1)} \leq Ch.
\]

From the last two expressions and (44) we can conclude that

\[
\limsup_{h \to 0} \| \nabla_h z^h \|_{L^2(\tilde{S}^1)} < \infty.
\]

By using Poincare inequality and (43) we also conclude that

\[
\limsup_{h \to 0} \| z^h \|_{L^2(\tilde{S}^1)} < \infty.
\]

In order to identify the two scale limit of the approximate strain we decompose it in the following form

\[
C^h = (R^h \circ \Xi^{-1})^t \frac{\nabla \tilde{y}^h T_S(I + thS)^{-1} - (R^h \circ \Xi^{-1})T_S}{h}
\]

\[
+ (R^h \circ \Xi^{-1})^t \frac{(\tilde{R}^h \circ \Xi^{-1}) - (R^h \circ \Xi^{-1})}{h} n \otimes n
\]

\[
+ t(R^h \circ \Xi^{-1})^t \nabla (\tilde{R}^h \circ \Xi^{-1})T_S(I + thS)^{-1} \cdot n
\]

\[
+ t(R^h \circ \Xi^{-1})^t (\tilde{R}^h \circ \Xi^{-1})ST_S(I + thS)^{-1}
\]

\[
+ (R^h \circ \Xi^{-1})^t \nabla_h z^h.
\]
where, similarly as before, by $\nabla(\tilde{R}^h \circ \Xi^{-1})T_S(I + thS)^{-1} \cdot n$ we have denoted the operator on $\mathbb{R}^3$ given by

$$(\nabla(\tilde{R}^h \circ \Xi^{-1})T_S(I + thS)^{-1} \cdot n)v = \nabla(\tilde{R}^h \circ \Xi^{-1})T_S(I + thS)^{-1}v, \text{ for all } v \in \mathbb{R}^3.$$

Notice that

$$\nabla \tilde{y}^h T_S(I + thS)^{-1} - (R^h \circ \Xi^{-1})T_S = \frac{\nabla \tilde{y}^h T_S - (R^h \circ \Xi^{-1})T_S}{h} - t\nabla \tilde{y}^h ST_S + O(h^2),$$

where $\|O(h^2)\|_{L^2(S_1)} \leq Ch^2$. Notice that $\nabla \tilde{y}^h T = \partial_n \tilde{y}^h$, for $\alpha = 1, 2$. Since now we can, by pulling back, easily translate the problem to the planar one by using the techniques developed in [HNV14] (see the proof of Proposition 3.2) and Lemma 3.1 we can conclude the following

- case $\gamma \in (0, \infty)$

There exist $w^0 \in L^2(\tilde{S}, \mathbb{R}^3)$, $w, v \in L^2(\tilde{S}, \tilde{H}^1(Y, \mathbb{R}^3))$, $\tilde{B} \in L^2(\tilde{S}, \mathbb{R}^{3 \times 2})$, $z \in L^2(\tilde{S}, \tilde{H}^1(I \times Y, \mathbb{R}^3))$ such that

$$\nabla \tilde{y}^h T_S - (R^h \circ \Xi^{-1})T_S \overset{h}{\geq} \sum_{i \leq 3; j \leq 2} \tilde{B}_{ij}(x) \tau^i \otimes \tau^j + \frac{1}{\gamma}(y \cdot \nabla')R \circ \Xi^{-1}(x)T_S + \sum_{i \leq 3; j \leq 2} \partial_{y_j} v_i(x, y) \tau^i \otimes \tau^j;$$

$$\frac{(\tilde{R}^h \circ \Xi^{-1}) - (R^h \circ \Xi^{-1})}{h} \overset{h}{\geq} \frac{1}{\gamma} \sum_{i=1}^3 (w^0_i(x) + w_i(x, y)) \tau^i + \frac{1}{\gamma} (y \cdot \nabla')R \circ \Xi^{-1}(x) n;$$

$$\nabla (\tilde{R}^h \circ \Xi^{-1})T_S(I + thS)^{-1} \cdot n \overset{h}{\geq} \nabla (R \circ \Xi^{-1}(x))T_S \cdot n + \sum_{i \leq 3; j \leq 2} \partial_{y_j} w_i(x, y) \tau^i \otimes \tau^j;$$

$$\nabla_{h,z} \overset{h}{\geq} \sum_{i,j=1}^3 \left( \nabla y z(x, t, y), \frac{1}{\gamma} \partial_z z(x, t, y) \right)_{ij} \tau^i \otimes \tau^j.$$

Here $R$ is $L^2$ limit of the sequence $\tilde{R}^h$, i.e., $R^h$ and we denoted by $(y \cdot \nabla')R = y_1 \partial_1 R + y_2 \partial_2 R$. Using the fact that $R^h \partial_n R$ for $\alpha = 1, 2$ is skew symmetric matrix we conclude that there exist $B \in L^2(\tilde{S}, T^*S \otimes T^*S)$ and $(\zeta, \rho) \in L^2(\tilde{S}, D(U_t))$ such that

$$\text{sym } G^h \overset{h}{\geq} B + tS^*_u + U_t(\zeta, \rho),$$

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where
\[ B = \sum_{\alpha, \beta = 1, 2} \left( (M\tilde{B})_{\alpha \beta} + (M\tilde{B})_{\beta \alpha} \right) \tau^\alpha \otimes \tau^\beta, \]

\[ \left( \begin{array}{c} \zeta \\ \rho \end{array} \right) = M(z + v + tw^0 + tw) + \gamma t \left( \begin{array}{c} (M\tilde{B})_{31} \\ (M\tilde{B})_{32} \end{array} \right), \]

and where \( M \in L^\infty(S; \mathbb{R}^{3 \times 3}); \) \( M_{ij} = (R \circ \Xi^{-1})^T \tau^i \cdot \tau^j. \)

- case \( \gamma = \infty. \) In the same way as before we can conclude that there exist \( w^0 \in L^2(\tilde{S}, \mathbb{R}^3), w, v \in L^2(\tilde{S}, H^1(\mathcal{Y}, \mathbb{R}^3)), \tilde{B} \in L^2(\tilde{S}, \mathbb{R}^{3 \times 3}), z \in L^2(\tilde{S} \times I, H^1(\mathcal{Y}, \mathbb{R}^3)), d \in L^2(\tilde{S} \times I, \mathbb{R}^3) \) such that

\[ \nabla (R^h \circ \Xi^{-1}(x))TS_T S - \frac{(R^h \circ \Xi^{-1}(x))TS_T}{h} \geq 2 \sum_{i \leq 3, j \leq 2} B_{ij}(x) \tau^i \otimes \tau^j \]

\[ + \sum_{i \leq 3, j \leq 2} \partial_y v_i(x, y) \tau^i \otimes \tau^j; \]

\[ \frac{(\tilde{R}^h \circ \Xi^{-1}) - (R^h \circ \Xi^{-1})}{h} \n  \geq 2 \sum_{i=1}^3 w^0_i(x) \tau^i; \]

\[ \nabla (\tilde{R}^h \circ \Xi^{-1})TS(I + thS)^{-1} \cdot n \geq 2 \nabla (R \circ \Xi^{-1}(x))TS_T \cdot n \]

\[ + \sum_{i \leq 3, j \leq 2} \partial_y w_i(x, y) \tau^i \otimes \tau^j; \]

\[ \nabla_h z^h \geq 2 \sum_{i,j=1}^3 (\nabla_y z(x, t, y), d(x, t))_{ij} \tau^i \otimes \tau^j. \]

From this we conclude that there exist \( B \in L^2(S; T^*S \otimes T^*S) \) and \( (\zeta, \rho, c) \in L^2(\tilde{S}; \mathcal{D}(U_{\infty})) \) such that

\[ \text{sym} G^h \geq B + tS_T^\infty + U_{\infty}(\zeta, \rho, c), \]

where

\[ B = \left( (M\tilde{B})_{\alpha \beta} + (M\tilde{B})_{\beta \alpha} \right) \tau^\alpha \otimes \tau^\beta, \]

\[ \left( \begin{array}{c} \zeta \\ \rho \end{array} \right) = M \left( \begin{array}{c} \frac{1}{2}(w_1 + v_1) \\ \frac{1}{2}(w_2 + v_2) \\ w_3 + v_3 \end{array} \right) + Mz, \]

\[ c = M \left( \begin{array}{c} \frac{1}{2}(d_1 + w^0_1) \\ \frac{1}{2}(d_2 + w^0_2) \\ d_3 \end{array} \right) + \left( \begin{array}{c} \frac{1}{2}(M\tilde{B})_{31} \\ \frac{1}{2}(M\tilde{B})_{32} \end{array} \right). \]
and $M$ is defined above.

The case $\gamma = 0$, $\varepsilon(h)^2 \ll h \ll \varepsilon(h)$ requires different approach. We will explain how to adapt the approach from [Vel15]. Firstly by using (46) we conclude that

$$\|\nabla'\tilde{y}^h - (\tilde{R}^h(\tau_1 \circ \xi), \tilde{R}^h(\tau_2 \circ \xi))\|_{L^2(\tilde{\omega}; \mathbb{R}^{3 \times 2})} \leq C h.$$ 

Denote by $p^h = (\tilde{R}^h(\tau_1 \circ \xi), \tilde{R}^h(\tau_2 \circ \xi))$. We look the following minimization problem

$$\min_{\tilde{v}^h \in H^1(\tilde{\omega}; \mathbb{R}^3)} \int_{\tilde{\omega}} |\nabla'\tilde{v}^h - p^h|^2 \, dx^\prime.$$ 

The associated Euler-Lagrange equation reads

$$\begin{cases} 
-\Delta'\tilde{v}^h = -\nabla \cdot p^h & \text{in } \tilde{\omega} \\
\partial_\nu \tilde{v}^h = p^h \cdot \nu & \text{on } \partial \tilde{\omega}, 
\end{cases}$$

subject to $\int_{\tilde{\omega}} \tilde{v}^h \, dx = 0$. Above, $\nu$ denotes the normal on $\partial \tilde{\omega}$. We obtain by standard regularity estimates that $\tilde{v}^h \in H^2(\tilde{\omega}; \mathbb{R}^3)$ under the assumption that $\partial \tilde{\omega}$ is $C^{1,1}$. In this case it holds

$$\|\tilde{v}^h\|_{H^2(\tilde{\omega}; \mathbb{R}^3)} \leq C \left( \|\nabla \cdot p^h\|_{L^2(\tilde{\omega}; \mathbb{R}^3)} + \|p^h\|_{L^2(\tilde{\omega}; \mathbb{R}^3)} \right) \leq C h,$$

$$\|\nabla'\tilde{v}^h - \nabla\tilde{y}^h\|_{L^2(\tilde{\omega}; \mathbb{R}^3)} \leq C h,$$

$$\|\nabla'\tilde{v}^h - (\tilde{R}^h(\tau_1 \circ \xi), \tilde{R}^h(\tau_2 \circ \xi))\|_{L^2(\tilde{\omega}; \mathbb{R}^{3 \times 2})} \leq C h,$$

for some $C > 0$. We now decompose the sequence of the deformations $(y^h)$ in the following way

$$y^h(x + tn) = v^h(x) + (\tilde{y}^h(x) - v^h(x)) + th(\tilde{R}^h \circ \Xi^{-1}) n + hz^h(x + tn), \quad (47)$$

where $v^h = \tilde{v}^h \circ \xi$. We also decompose the sequence of strains $(G^h)$ in the
following way

\[ G^h = \frac{1}{h} \sum_{\alpha,\beta=1,2} \left( (R^h \circ \Xi^{-1}) \tau_\alpha \cdot \nabla v^h T_S (I + thS)^{-1} \tau_\beta \right) \tau_\alpha \otimes \tau_\beta \]

\[ + \frac{1}{h} \sum_{\alpha=1,2} \left( (R^h \circ \Xi^{-1}) n \cdot \nabla v^h T_S (I + thS)^{-1} \tau_\alpha \right) n \otimes \tau_\alpha \]

\[ + \frac{1}{h} \left( R^h \circ \Xi^{-1} \right) ((\nabla \bar{g}^h T_S (I + thS)^{-1} - \nabla v^h T_S (I + thS)^{-1}) \right) \]

\[ + \frac{1}{h} \left( (R^h \circ \Xi^{-1}) \tilde{\nabla} \right) \n \otimes \n \]

\[ + t (R^h \circ \Xi^{-1} \right) \nabla \left( \tilde{R}^h \circ \Xi^{-1} \right) T_S (I + thS)^{-1} \cdot n \]

\[ + t (R^h \circ \Xi^{-1} \right) \left( \tilde{R}^h \circ \Xi^{-1} \right) S T_S (I + thS)^{-1} \]

\[ + (R^h \circ \Xi^{-1}) \nabla h z^h. \]

By using the tools developed in [Vel15] for the planar case, adapted to this framework, we can conclude that there exist \( B^1 \in L^2(\tilde{S}, T^* S \otimes T^* S) \), \( B^2 \in L^2(\tilde{S}, [\mathbb{R}^3]^{2 \times 2}) \), \( w \in L^2(\tilde{S}, H^1(\mathcal{Y}, [\mathbb{R}^2])) \), \( p \in L^2(\tilde{S} \times [\mathcal{Y}, [\mathbb{R}^2])) \), \( k \in L^2(\tilde{S}, H^1(\mathcal{Y}, [\mathbb{R}^3])) \), \( r \in L^2(\tilde{S}, H^1(\mathcal{Y}, [\mathbb{R}^3])) \), \( \varphi \in H^2(\mathcal{Y}) \), \( \phi \in L^2(\tilde{S} \times I, H^1(\mathcal{Y}, [\mathbb{R}^3])) \), \( d \in L^2(\tilde{S} \times I \times \mathcal{Y}, [\mathbb{R}^3]) \) such that

\[ \frac{1}{h} \sum_{\alpha,\beta=1,2} \left( (R^h \circ \Xi^{-1}) \tau_\alpha \cdot \partial_{\tau_\beta} v^h \right) \tau_\alpha \otimes \tau_\beta \overset{2}{=} B^1(x) + \text{Def}_y w, \]

\[ \frac{1}{h} \sum_{\alpha=1,2} \left( (R^h \circ \Xi^{-1}) n \cdot \partial_{\tau_\alpha} v^h \right) n \otimes \tau_\alpha \overset{2}{=} p_\alpha(x, y) n \otimes \tau_\alpha, \quad \text{for} \ \alpha = 1, 2, \]

\[ \frac{1}{h} \left( R^h \circ \Xi^{-1} \right) ((\nabla \bar{g}^h T_S - \nabla v^h T_S) \right) \overset{2}{=} \sum_{i \leq j \leq 2} B_{ij}^2(x) \tau^i \otimes \tau^j \]

\[ + \sum_{i \leq j \leq 2} \partial_{y_j} k_i(x, y) \tau^i \otimes \tau^j, \]

\[ \left( R^h \circ \Xi^{-1} \right) ((\tilde{R}^h \circ \Xi^{-1}) - I) n \overset{2}{=} \sum_{i=1}^3 l_i(x, y) \tau^i, \]

\[ \nabla (\tilde{R}^h \circ \Xi^{-1}) T_S \cdot n \overset{2}{=} \text{R Hess} \varphi, \]

\[ \nabla h z^h \overset{2}{=} \sum_{i, j=1}^3 \left( \nabla g \phi(x, y), d(x, y, t) \right)_{ij} \tau^i \otimes \tau^j. \]

From this we conclude that there exist \( B \in L^2(\tilde{S}, T^* S \otimes T^* S) \) and \((\zeta, \varphi, g) \in \)
\[ L^2(\tilde{S}, D(\mathcal{U}_0)) \text{ such that} \]
\[
\text{sym } G^h \rightharpoonup B + tS_u^* + \mathcal{U}_0(\zeta, \varphi, g),
\]

where
\[
B = B^1 + ((MB^2)_{\alpha\beta} + (MB^2)_{\beta\alpha}) \tau^\alpha \otimes \tau^\beta
\]
\[
\zeta = w + ((Mk)_1, (Mk)_2)^t + (\phi_1, \phi_2)^t
\]
\[
g = \left( \frac{1}{2}(p_1 + l_1 + d_1) \right) + \left( \frac{1}{2}(MB^2)_{31} \right) + \frac{1}{2} \left( \frac{1}{2}(MB^2)_{32} \right).
\]

Remark 3.8. We were able to completely identify oscillatory part of two-scale limit of strain without using the convexity of the shell. The weak limit contained in the field \( B \) remains unknown. We will use the convexity in the construction of the recovery sequence by showing that in that case an arbitrary matrix field \( B \) can be recovered. In the plate case this is not completely true, but it is true that some appropriate class of the fields \( B \) can be recovered. However, it is commented that the class contains the minimizers that we need for the relaxation.

We will just give the sketch of the proof of the lower bound since it follows standard arguments (truncation, Taylor expansion and lower semicontinuity of integral functional with respect to two-scale convergence).

Proof of Theorem 3.4 - lower bound

Without loss of generality we may assume that \( \int_{\tilde{S}} y^h \, dx = 0 \) and \( \limsup_{h \to 0} h^{-2} J^h(u^h) < \infty \). We only consider the case \( \gamma \in (0, \infty) \). The proof in other cases is similar. By using Theorem 3.5 we conclude that \( u \in H^2_{iso}(S) \) and that stronger convergence, given in that theorem are satisfied. Take an arbitrary \( \tilde{\omega} \in \omega \) with \( C^{1,1} \) boundary. By using Proposition 3.7 we have that there exists \( B \in L^2(\tilde{S}; T^* S \otimes T^* S) \) and \( (\zeta, \rho) \in L^2(\tilde{S}, D(\mathcal{U}_\gamma)) \) such that
\[
\text{sym } G^h \rightharpoonup B + tS_u^* + \mathcal{U}_\gamma(\zeta, \rho);
\]
As explained in [Neu12] (cf. [FJM02] for the corresponding argument in the homogeneous case), a careful Taylor expansion of \( W(x + tn(x), r(x)/\varepsilon, I + \)
\( hG^h(x) \) combined with the lower semi-continuity of convex integral functionals with respect to weak two-scale convergence (see e.g. \cite[Proposition 1.3]{Vis07}) and using relations (30), (31) yields the lower bound

\[
\lim_{h \to 0} h^{-2} J^h(u^h) = \lim_{h \to 0} h^{-2} I^h(y^h)
\]

\[
= \int_{\tilde{S}^1} \int_{I \times Y} Q(x, t, y, B + tS^h_u(x) + U_\gamma(\zeta, \rho)) \, dt \, dy \, dvol_S.
\]

Minimization over \( B \in L^2(\tilde{S}, \mathbb{R}^{2 \times 2}) \) and \( (\zeta, \rho) \in L^2(\tilde{S}, D(U_\gamma)) \) yields

\[
\lim_{h \to 0} h^{-2} J^h(u^h) \geq \int_{\tilde{S}^1} Q_\gamma(x, S^h_u(x)) dvol_S.
\]

By using the properties of \( Q_\gamma \) (see Lemma 3.9) and exhausting \( \omega \) with the sequence \((\tilde{\omega}_n)\) such that for each \( n \in \mathbb{N} \) we have \( \tilde{\omega}_n \subseteq \omega \) and \( \tilde{\omega}_n \) has \( C^{1,1} \) boundary we have the claim.

### 3.4 Upper bound

The proof of upper bound has the analogy with the planar cases. The crucial ingredient here is, as already commented in Remark 3.8 that every matrix field \( B \) can be recovered. It is easy to see, by using Korn’s inequality, that each fiber of \( L^2(\tilde{S}, \mathbb{R}^{2 \times 2}) \) for \( \gamma \in [0, \infty] \) is a closed subspace of \( L^2(I \times Y, \mathbb{R}^{3 \times 3}_{sym}) \).

Also by Korn’s inequality it is easy to see (see also \cite{Neu10, Neu12}) that the following coercivity bound is satisfied:

\[
\|U_0(\zeta, g)\|_{L^2}^2 \geq \frac{C}{\|\zeta\|_{H^1}^2 + \|g\|_{L^2}^2} \text{ for all } (\zeta, g) \in D(U_0),
\]

where the constant \( C \) depends on the embedding \( \xi \). Analogous bounds are satisfied by \( U_\gamma \), for \( \gamma \in (0, \infty] \) with the obvious norms on their respective domains of definition.

The following lemma is analogous to \cite[Lemma 2.10, 2.11]{NV13}.

**Lemma 3.9.** For \( \gamma \in [0, \infty] \) we have the following

(a) the function \( Q_\gamma : S \times TS \otimes TS \to [0, \infty) \) is continuous, and there exists a constant \( C > 0 \) depending only on the energy density \( W \) and the surface \( S \) such that, for all \( x \in S \) and all \( q \in T_x^* S \otimes T_x^* S \),

\[
C^{-1}|q|^2 \leq Q_\gamma(x, q) \leq C|q|^2.
\]

For fixed \( x \in S \), the function \( Q_\gamma(x) : T_x^* S \otimes T_x^* S \to [0, \infty) \) is a quadratic form.
(b) there exists a bounded linear operator

$$\Pi_\gamma : L^2(S, T^*S \otimes T^*S) \to L^2(S, T^*S \otimes T^*S) \times L^2(S, L_\gamma(I \times \mathcal{Y})),$$

such that for almost every \( x \in S \) we have

$$Q_\gamma(x, q) = \int_I \int_Y Q(x + tn(x), y, \Pi_\gamma[q](x) + tQ_\gamma(x, t, y))\,dy\,dt,$$

where \( \Pi_\gamma[q], \Pi_\gamma^2[q] \) are projections of \( \Pi_\gamma[q] \) on \( L^2(S, TS \otimes TS) \) and \( L^2(S, L_\gamma(I \times \mathcal{Y})) \) respectively. Moreover, if \( q \in C^0(S, TS \otimes TS) \) then \( \Pi_\gamma[q] \in C^0(S, T^*S \otimes T^*S) \times C^0(S, L_\gamma(I \times \mathcal{Y})) \).

Proof. We will just give the sketch of the proof for \( \gamma \in (0, \infty) \). We introduce the space \( \mathcal{V}_x = T^*S \otimes T^*S + L_\gamma(I \times \mathcal{Y}) \) (the sum is orthogonal in the \( L^2 \)-norm and thus the projections on the components are well defined and linear). Notice that for fixed \( x \in S \) and \( q \in T^*_xS \otimes T^*_xS \) we can look the minimization equation (33) as projecting the function \( t\text{sym}q \) on the linear space \( \mathcal{V}_x \) in the norm determined by the quadratic form \( Q \) (which is equivalent to the \( L^2 \)-norm). From this we obtain the uniqueness and the linearity of the minimizer for \( q \in T^*_xS \otimes T^*_xS \). (a) can be now easily proved (see also [NV13]). Also the part (b) is easily proved, firstly for \( q \in C^0(S, TS \otimes TS) \) and then for \( q \in L^2(S, TS \otimes TS) \) by extension, using the inequalities in (48). The continuity of \( \Pi_\gamma[q] \) for \( q \in C^0(S, TS \otimes TS) \) follows from the property \((Q1)\) stated in Lemma 3.2. \( \square \)

The proof of the following lemma is given in [LMP10, Lemma 5.6]. It uses the results in [Nir53]. We just augment it with the regularity statement that can be easily seen by the arguments given there and standard elliptic regularity.

**Lemma 3.10.** Let \( \omega \subset \mathbb{R}^2 \) be simply connected with \( C^2 \) boundary. Let \( s : \overline{\omega} \to \mathbb{R}^3 \) be of class \( C^{2,1}(\overline{\omega}, \mathbb{R}^3) \). Assume additionally that \( s(\omega) \) defines the local surface such the Weingarten map \( S_s \) is strictly positive (or negative) definite up to the boundary, i.e., there exists \( C > 0 \) such that

$$\forall z' \in \omega, \forall v \in \mathbb{R}^2, \quad \frac{1}{C} |v|^2 \leq S_s(z')v \cdot v \leq C |v|^2,$$

where \( (S_s)_{\alpha\beta} = \frac{\partial_\alpha n_s \partial_\beta s}{|\partial_\alpha n_s \partial_\beta s|} \). Then for every \( B \in L^2(\omega, \mathbb{R}^{2\times 2}) \) the system

$$q_w^s = B, \quad (q_w^s)_{\alpha\beta} = \frac{1}{2}(\partial_\alpha s \partial_\beta w + \partial_\alpha w \partial_\beta s), \quad \text{for } \alpha, \beta = 1, 2,$$

has a solution \( w \in H^1(\omega, \mathbb{R}^3) \). If we assume that \( s \in C^{3,\alpha}(\omega, \mathbb{R}^3), B \in C^{2,\alpha}(\omega, \mathbb{R}^{2\times 2}) \), then we have that \( w \in C^{3,\alpha}(\omega, \mathbb{R}^3) \).
Remark 3.11. The system $q_w^* = B$ arises naturally in the geometrical question of the problem of continuation of the infinitesimal bendings (see, e.g., [HLPT9, LMPT11]). In [Nir53] this system arises as one step linearization of solving the Weyl’s problem (finding a convex surface with prescribed metric) by a continuation method.

Proof of Theorem 3.4 - upper bound

- case $\gamma \in (0, \infty)$: Without loss of generality we may assume that $\omega$ is star-shaped with respect to some ball $B(0, r)$ centered at the origin. Notice that, as a consequence of this, we have that for $\omega^\lambda := \lambda \omega$, $(0 < \lambda < 1)$, it holds $\omega^\lambda \in \Omega$. Take an arbitrary $u \in H^2_{\text{iso}}(S)$. By Remark 3.12 we have that $u \in C^{3,0}(\omega, \mathbb{R}^3)$. Define $\tilde{w} \in C^{3,0}(\omega, \mathbb{R}^3)$ by $\tilde{w} = u \circ r$. Let $R \in C^{1,0}(S, SO(3))$ be such that $RT_S = \nabla u$. Take arbitrary $B \in C^0_0(S, T^*S \otimes T^*S)$, $(\zeta, \rho) \in C^0_0(S, C^1(I \times \mathcal{Y}, \mathbb{R}^2)) \times C^0_0(S, C^1(I \times \mathcal{Y})).$

Define $\tilde{u} \in C^{3,0}(\omega, \mathbb{R}^3)$, $\tilde{B} \in C^0_0(\omega, \mathbb{R}^{2 \times 2})$, $\tilde{R} \in C^{1,0}(\omega, SO(3))$, $(\tilde{\zeta}, \tilde{\rho}) \in C^0_0(S, \tilde{C}^1(I \times \mathcal{Y}, \mathbb{R}^2)) \times C^0_0(S, \tilde{C}^1(I \times \mathcal{Y}))$ by

$$\tilde{u} = u \circ r, \quad \tilde{B}_{ij} = (B \circ \xi)_{ij}, \quad \tilde{R} = R \circ r.$$ 

For fixed $0 < \lambda_0 < 1$, sufficiently close to 1, define $\tilde{w}^0 \in C^{3,0}(\lambda_0 \omega, \mathbb{R}^3)$ as the solution of the system $q_{\tilde{w}}^0 = \tilde{B}|_{\lambda_0 \omega}$. For $0 < \lambda < \lambda_0$, define $\tilde{w}^\lambda$, $\tilde{w}^\lambda \in C^{3,0}(\lambda \omega, \mathbb{R}^3)$, $\tilde{B}^\lambda \in C^0_0(\omega, \mathbb{R}^{2 \times 2})$, $\tilde{R}^\lambda \in C^{1,0}(\omega, SO(3))$ by

$$\tilde{w}^\lambda(\zeta') = \frac{1}{\lambda} \tilde{w}(\lambda \zeta'), \quad \tilde{w}^\lambda(\zeta') = \frac{1}{\lambda} \tilde{w}(\lambda \zeta'), \quad \tilde{B}^\lambda(\zeta') = B(\lambda \zeta'), \quad \tilde{R}^\lambda(\zeta') = R(\lambda \zeta').$$

In an obvious way, by pushing forward, define $u^\lambda, w^\lambda \in C^{3,0}(S, \mathbb{R}^3)$, $B^\lambda \in C^0_0(S, T^*S \otimes T^*S)$, $R^\lambda \in C^{1,0}(S, SO(3))$. Notice that

$$\lim_{\lambda \to \lambda_0} \|\tilde{w}^\lambda - \tilde{w}\|_{H^2(\omega, \mathbb{R}^3)} = 0, \quad \lim_{\lambda \to \lambda_0} \|\tilde{B}^\lambda - \tilde{B}\|_{L^2(\omega, \mathbb{R}^{2 \times 2})} = 0,$$

$$\lim_{\lambda \to \lambda_0} \|\tilde{R}^\lambda - \tilde{R}\|_{H^1(\omega, SO(3))} = 0,$$  \hspace{1cm} (49)

i.e.,

$$\lim_{\lambda \to \lambda_0} \|u^\lambda - u\|_{H^2(S, \mathbb{R}^3)} = 0, \quad \lim_{\lambda \to \lambda_0} \|B^\lambda - B\|_{L^2(S, T^*S \otimes T^*S)} = 0,$$

$$\lim_{\lambda \to \lambda_0} \|R^\lambda - R\|_{H^1(S, SO(3))} = 0.$$ \hspace{1cm} (50)
We define the recovery sequence

$$y^h(x + tn(x)) = u^\lambda(x) + thR^\lambda(x)n(x) + hw^\lambda(x)$$

$$-th^2 (\partial_{\tau_\alpha} w^\lambda(x) \cdot R^\lambda(x)n(x)) \tau^\alpha(x)$$

$$+h\varepsilon R^\lambda(x) (\zeta_\alpha(x, t, r(x)/\varepsilon) \tau^\alpha(x) + \rho(x, t, r(x)/\varepsilon)n(x)).$$

Notice that

$$\lim_{h \to 0} \|y^h - u^\lambda\|_{H^1(S, \mathbb{R}^3)} = 0.$$ (51)

After a short computation we obtain

$$(R^\lambda)^t \nabla_h y^h = I + thS^\lambda_{\tau_\alpha}(x) + h(R^\lambda(x))^t \nabla w^\lambda(x)T_S$$

$$-h (\partial_{\tau_\alpha} w^\lambda(x) \cdot R^\lambda(x)n(x)) \tau^\alpha(x) \otimes n(x)$$

$$+ h\mathcal{U}_\gamma(\zeta, \rho)(x, t, r(x)/\varepsilon) + o(h),$$

where \(\lim_{h \to 0} \|o(h)/h\|_{L^\infty} = 0\). Here we used (15), (19), (26), (36).

Notice that

$$\operatorname{sym} ((R^\lambda)^t \nabla w^\lambda T_S - (\partial_{\tau_\alpha} w^\lambda(x) \cdot R^\lambda(x)n(x)) \tau^\alpha(x) \otimes n(x)) = B^\lambda(x).$$

By using the objectivity property (W3), (30) we easily obtain from (52)

$$\lim_{h \to 0} \left| \frac{1}{h^2} W(x + tn(x), r(x)/\varepsilon, \nabla_h y(x + tn(x)))$$

$$- \mathcal{Q}(x + tn(x), r(x)/\varepsilon, S^\lambda_{\tau_\alpha}(x) + B^\lambda(x) + \mathcal{U}_\gamma(\zeta, \rho)(x, t, r(x)/\varepsilon)) \right| = 0.$$ From this we easily obtain

$$\lim_{h \to 0} h^{-2} I^h(y^h) =$$

$$\int_S \int_{I \times \mathcal{Y}} \mathcal{Q}(x + tn(x), y, S^\lambda_{\tau_\alpha}(x) + B^\lambda(x) + \mathcal{U}_\gamma(\zeta, \rho)(x, t, y)) dy dtd \operatorname{vol}_S(x).$$

By using diagonalization argument and letting \(\lambda, \lambda_0 \to 1\), we can construct a sequence \((y^h)\) such that \(\lim_{h \to 0} \|y^h - u\|_{H^1(S, \mathbb{R}^3)} = 0\) and

$$\lim_{h \to 0} h^{-2} I^h(y^h) =$$

$$\int_S \int_{I \times \mathcal{Y}} \mathcal{Q}(x + tn(x), y, S^\lambda_{\tau_\alpha}(x) + B(x) + \mathcal{U}_\gamma(\zeta, \rho)(x, t, y)) dy dtd \operatorname{vol}_S(x).$$

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Next we take $B_l \in L^2(S, T^*S \otimes T^*S)$, $(\zeta_l, \rho_l) \in L^2(S, \mathcal{D}(U_\gamma))$ by defining

$$B_l = \Pi^1_{\gamma} \left[ S_{u^\lambda}^\gamma \right], \quad U_\gamma (\zeta_l, \rho_l) = \Pi^2_{\gamma} \left[ S_{u^\lambda}^\gamma \right].$$

By using approximation with the sequence $(B^n) \subset C^0_0(S, T^*S \otimes T^*S)$, $(\zeta^n, \rho^n) \subset C^1_0(S, \hat{C}^1(I \times \mathcal{Y}, \mathbb{R}^3)) \times C^2_0(S, \hat{C}^2(I \times \mathcal{Y}))$ such that

$$\lim_{n \to \infty} ||B^n - B||_{L^2(S, T^*S \otimes T^*S)} = 0, \quad \lim_{n \to \infty} \|(\zeta^n, \rho^n) - (\zeta_l, \rho_l)\|_{L^2(S, D(U_\gamma))} = 0,$$

and the diagonalization argument we easily obtain the recovery sequence.

- case $\gamma = \infty$ the only difference is in the relaxation part of the recovery sequence. We will just give the recovery sequence. For $(\zeta, \rho, c) \in C^1_0(S, C^1_0(I, \hat{C}^1(\mathcal{Y}, \mathbb{R}^3))) \times C^1_0(S, C^2_0(I, \hat{C}^2(\mathcal{Y}))) \times C^1_0(S, C^1_0(I, \mathbb{R}^3))$ we define

$$y^h(x + tn(x)) = u^\lambda(x) + thR^\lambda(x)n(x) + hw^\lambda(x)$$

$$-th^2 \left( \partial_{\tau^\alpha} R^\lambda(x) \right) \tau^\alpha(x) + h\varepsilon R^\lambda(x) \left( \zeta(x, t, r(x)/\varepsilon) \tau^\alpha(x) + \rho(x, t, r(x)/\varepsilon) \right) + h^2 R^\lambda(x) \left( 2 \int_0^t c_a(x, s) \, ds \tau^\alpha + \int_0^t c_3(x, s) \, ds \right).$$

The results from the previous case can be easily repeated.

- case $\gamma = 0, \varepsilon^2 \ll h \ll \varepsilon$. For $(\zeta, \varphi, g) \in C^1_0(S, \hat{C}^1(\mathcal{Y}, \mathbb{R}^3)) \times C^2_0(S, \hat{C}^2(\mathcal{Y})) \times C^1_0(S, C^1_0(I \times \mathcal{Y}, \mathbb{R}^3))$ we define

$$y^h(x + tn(x)) = u^\lambda(x) + thR^\lambda(x)n(x) + hw^\lambda(x)$$

$$+ \varepsilon^2 R^\lambda(x) \varphi(x, r(x)/\varepsilon) - th\varepsilon R^\lambda(x) \left( \partial_{\varphi^\alpha} \varphi(x, r(x)/\varepsilon) \tau^\alpha \right) - th^2 \left( \partial_{\tau^\alpha} R^\lambda(x) \right) \tau^\alpha(x) + h\varepsilon R^\lambda(x) \left( \zeta(x, r(x)/\varepsilon) \tau^\alpha(x) \right)$$

$$+ h^2 R^\lambda(x) \left( 2 \int_0^t g_a(x, s, r(x)/\varepsilon) \, ds \tau^\alpha(x) + \int_0^t g_3(x, s, r(x)/\varepsilon) \, ds \right).$$

In this case we have

$$(R^\lambda)^T \nabla_h y^h = I + thS^\lambda_{u^\lambda}(x) + h(R^\lambda(x))^T \nabla w^\lambda(x) T_S$$

$$- h \left( \partial_{\tau^\alpha} R^\lambda(x) \right) \tau^\alpha(x) \otimes n(x) + \varepsilon \partial_{\varphi^\alpha} \varphi(x, r(x)/\varepsilon) \otimes \tau^\alpha(x) + \varepsilon \partial_{\varphi^\alpha} \varphi(x, r(x)/\varepsilon) \tau^\alpha(x) \otimes n(x)$$

$$+ h\varepsilon \lambda_0 (\zeta, \varphi, g)(x, t, r(x)/\varepsilon) + o(h).$$

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Although in the expansion we have the dangerous term of order $\varepsilon \gg h$, the symmetric part of this term vanishes (see [Vel15] for the planar case). This means that

$$
(R^\varepsilon)^t(R^{\lambda})^t \nabla h y^h = I + th S_{u,\lambda}(x) + h(R^{\lambda}(x))^t \nabla w(x) T_S \\
-h \left( \partial_{\alpha_\lambda} w^\lambda(x) \cdot R^{\lambda}(x)n(x) \right) \tau^\alpha(x) \otimes n(x) \\
+ h U_0(\zeta, \varphi, g)(x, t, r(x)/\varepsilon) + o(h),
$$

where $R^\varepsilon = \exp(A^\varepsilon)$ and

$$A^\varepsilon = \varepsilon \partial_{\alpha} \varphi(x, r(x)/\varepsilon)n(x) \otimes \tau^\alpha(x) - \varepsilon \partial_{\alpha} \varphi(x, r(x)/\varepsilon)\tau^\alpha(x) \otimes n(x).$$

Again we can repeat the results from the first case (by using the objectivity property (W3) and (30)).

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