Dynamics of a hybrid optomechanical system in the framework of the generalized linear response theory

B Askari\textsuperscript{1} and A Dalafi\textsuperscript{2,}\textsuperscript{*}

\textsuperscript{1} Department of Physics, Shahid Beheshti University, Tehran 19839, Iran
\textsuperscript{2} Laser and Plasma Research Institute, Shahid Beheshti University, Tehran 19839-69411, Iran

E-mail: a_dalafi@sbu.ac.ir

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Abstract

In this article, the linear response of a driven-dissipative hybrid optomechanical system consisting of an interacting one-dimensional Bose–Einstein condensate to an external time-dependent perturbation is studied in the framework of the generalized linear response theory (GLRT). It is shown that the Stokes and anti-Stokes amplitudes of the optical and atomic modes of the system can be obtained through the solutions to the equations of motion of the open quantum system Green’s function predicted by the GLRT. In this way, interesting phenomena like anti-resonance and Fano resonance are described and it is shown how the atom–atom interaction affects them. Furthermore, an interpretation of the anti-resonance phenomenon is presented based on the optical spectral function and self-energy.

Keywords: Green’s function, linear response theory, open quantum systems, quantum optomechanics, Bose–Einstein condensate, anti-resonance and Fano resonance

(Some figures may appear in colour only in the online journal)

1. Introduction

During the recent two decades, the science of quantum optomechanics, which deals with the radiation pressure coupling of a cavity optical field to the vibrational mode of a mechanical oscillator, has been developing very quickly both in the theoretical and experimental aspects [1, 2]. Optomechanical systems (OMSs) have had a significant contribution in many applica-
tions like: displacement and force sensing [3–9], ground-state cooling of the vibrational modes of a mechanical oscillator [10], synchronization of the mechanical oscillators [11], and generation of entanglement [12]. Another kind of OMSs can be formed by an optical cavity consisting of a Bose–Einstein condensate (BEC) or an ensemble of ultra-cold atoms where the fluctuation of the collective excitation of the atomic field behaves like an effective mechanical mode [13–15]. In such hybrid OMSs the radiation pressure of the cavity field can be coupled with the collective atomic mode as well as the vibrational mode of a mechanical oscillator [16–19].

In many interesting quantum optical phenomena like normal mode splitting [20–22], electromagnetically induced transparency (EIT) [23], optomechanically induced transparency (OMIT) [24–27], and Fano resonance [28–31] it is important to study the response of the system to an external time-dependent perturbation. In such cases the response of the system to the external source can be obtained by the standard linear response theory (SLRT) which is based on a closed model of the quantum system being initially in contact with a thermal bath at a finite temperature [32, 33]. The deficiency of the SLRT is that the effect of dissipation has to be entered into the theory phenomenologically since the environment which is responsible for fluctuation and dissipation is not modeled within the SLRT [34].

Recently, several researches have been conducted to generalize the SLRT to the theory of open quantum systems both in the Schrödinger picture through the master equation approach [35–37] and in the Heisenberg picture through the quantum Langevin equations (QLEs) [38, 39]. In the so-called generalized linear response theory (GLRT) the linear response of an open quantum system to an external time-dependent perturbation is investigated while the effect of the environment is taken into account in the mathematical modeling as a multi-mode quantum field with an infinite number of degrees of freedom which interacts with the open quantum system.

The GLRT predicts a set of equations of motion for the open quantum system Green’s functions which are derived in the Heisenberg picture through the QLEs [39]. The linear response of the open quantum system to the external source can be calculated through the solutions of Green’s functions equations of motion. It is worth mentioning again that the effect of dissipation is taken into consideration in the GLRT without necessity of any phenomenological manipulation and it is the superiority of GLRT over the SLRT. As a practical application, in reference [39] the linear response of a standard bare OMS to an external time-dependent perturbation has been studied in the framework of GLRT.

In the present article, we are going to investigate the linear responses of the optical and atomic modes of a hybrid OMS consisting of an interacting cigar-shaped BEC trapped inside an optical cavity to a time-dependent perturbation while both the optical and the atomic modes are investigated as open quantum systems. It is shown that the system behaves as a standard OMS with a difference that the hybrid system has an extra interaction term (in addition to the optomechanical interaction) which is due to inter-atomic collisions of the BEC atoms. Using the equations of motions of the open quantum system Green’s functions predicted by the GLRT, we obtain the linear responses of the system to a weak probe laser which acts as a time-dependent perturbation and show how the atom–atom interaction affects the optical and atomic responses of the system.

One of the most interesting features of the present hybrid OMS is that it behaves as a system consisting of two coupled quantum oscillators which one of them (the Bogoliubov mode of the BEC) functions as an atomic parametric amplifier through the atom–atom interaction of the BEC atoms which is the atomic analog of the optical parametric amplifier [40–43]. The study of the linear responses of such hybrid OMS shows that it has two resonance frequencies corresponding to the two normal modes of the system and an anti-resonance frequency [44, 45]
which occurs for the optical mode since it is the oscillator which is driven directly by the time-dependent perturbation. It is demonstrated how the position of the anti-resonance frequency can be manipulated by the s-wave scattering frequency of BEC atoms which itself is controllable through the transverse trapping frequency.

It is also shown through the anti-Stokes amplitude of the optical field that the system can switch from the OMIT to the normal mode splitting by controlling the pumping rate of the coupling laser. Furthermore, the appearance of the Fano resonance profiles in the optical field can be also explained when the effective cavity detuning deviates from the effective frequency of the Bogoliubov mode. Finally, an interpretation of the optical response behavior especially the manifestation of the anti-resonance phenomenon is presented based on the optical spectral function and self-energy. Our main goal has been to show that all of the above-mentioned phenomena can be exactly described (without necessity of any phenomenological manipulation) by the open system Green’s functions which are obtained by the GLRT.

The paper is structured as follows: in section 2 the system and its Hamiltonian are introduced. In section 3 the dynamics of the system is described using the theory of open quantum systems, and in section 4 the responses of the optical and atomic modes to the time-dependent perturbation are studied in the framework of GLRT. In section 5, the manifestation of the anti-resonance is explained by the optical spectral function and self-energy. Finally, our conclusions are summarized in section 6.

2. System Hamiltonian

Consider a hybrid OMS formed by an optical cavity with length $L$ and resonance frequency $\omega_0$ consisting of a BEC of $N$ two-level atoms with mass $m_a$ and transition frequency $\omega_a$ confined in a cylindrically symmetric trap with a transverse trapping frequency $\omega_\perp$ and negligible longitudinal confinement along the $x$ direction. The cavity is driven by a coupling laser with frequency $\omega_c$ and wave number $k = \omega_c/c$ at the rate of $\eta_c = \sqrt{2P/\kappa/\hbar \omega_c}$ through one of its mirrors where $P$ is the laser power and $\kappa$ is the cavity decay rate.

In the dispersive regime, where the coupling laser frequency is far detuned from the atomic resonance so that $\Delta_a = \omega_c - \omega_a \gg \Gamma_a$ with $\Gamma_a$ being the atomic linewidth, transition to the upper level is negligible and spontaneous emission is unlikely to happen and can be neglected. Consequently the dynamics of atoms can be described within an effective 1D model by quantizing the atomic motional degree of freedom along the cavity axis, $x$ [46–48]. Therefore, the system Hamiltonian in the frame rotating at the coupling laser frequency is given by

$$ H = \int_{-L/2}^{L/2} dx \left[ \Psi^\dagger(x) \left( -\frac{\hbar^2}{2m_a} \frac{d^2}{dx^2} + \hbar U_0 \cos^2(kx) a^\dagger a + \frac{1}{2} U_s \Psi^\dagger(x) \Psi(x) \right) \Psi(x) - \hbar \Delta_c a^\dagger a + i\hbar \eta_c (a - a^\dagger) \right], $$

where $\Psi(x)$ is the quantum field operator of the atoms in the framework of the second quantization formalism, and $a$ is the annihilation operator of the single optical mode of the cavity. Besides, $\Delta_c = \omega_c - \omega_0$ is the detuning of the coupling laser from the cavity resonance, $U_0 = g_0^2/\Delta_c$ is the optical lattice barrier height per photon with $g_0$ being the vacuum Rabi frequency, $U_s = 4\pi \hbar^2 a_s/m_a$ and $a_s$ is the two-body s-wave scattering length [46, 47]. The last terms in the Hamiltonian of equation (1) denotes the effect of the coupling laser which drives the optical mode of the cavity.
On the other hand, if the average number of the cavity photons is low enough so that the condition $U_0(a^\dagger a) \leq 10\omega_R$ is satisfied with $\omega_R = \frac{\hbar^2 L^2}{2m}$ being the recoil frequency of the condensate atoms, and under the Bogoliubov approximation [48], the atomic field operator can be approximated by the following single-mode quantum field [40]

$$\Psi(x) = \sqrt{\frac{N}{L}} + \frac{\sqrt{2}}{L} \cos(2kx)c. \quad (2)$$

Here, the first term, corresponding to the so-called condensate mode, has been considered as a $c$-number in the Bogoliubov approximation and the operator $c$ in the second term (the so-called Bogoliubov mode) corresponds to the quantum fluctuations of the atomic field about the classical condensate mode. By substituting the atomic field operator of equation (2) into the Hamiltonian of equation (1), the system Hamiltonian is simplified as

$$H = \hbar \delta_0 a^\dagger a + i\hbar \eta_k (a - a^\dagger) + \hbar \Omega c^\dagger c + \hbar \xi a^\dagger a (c + c^\dagger) + \frac{1}{4} \hbar \omega_{sw} (c^2 + c^\dagger^2), \quad (3)$$

where $\delta_0 = -\Delta_0 + \frac{1}{2}NU_0$ is the Stark-shifted cavity frequency due to the presence of the BEC, and $\Omega = 4\omega_R + \omega_{sw}$ is the frequency of the Bogoliubov mode. The important point that should be emphasized is that the resonance frequency of the cavity has been shifted from $\omega_0$ to $\tilde{\omega}_0 = \omega_0 + \frac{1}{2}NU_0$ due to the presence of the BEC. In this way, the hybrid system behaves effectively as a bare optomechanical cavity with the shifted optical frequency $\tilde{\omega}_0$ which interacts with an effective mechanical oscillator whose role is played by the Bogoliubov mode of the BEC through a radiation pressure interaction with the effective optomechanical coupling $\zeta = \sqrt{2\hbar}U_0$ (the fourth term in equation (3)).

The last term of equation (3) which behaves as an atomic parametric amplifier [40–43] corresponds to the atom–atom interaction with $\omega_{sw} = \frac{8\pi \hbar a_s N}{m s^2 w^2}$ being the $s$-wave scattering frequency of the atomic collisions ($w$ is the waist radius of the optical mode). The important point is that the strength of the atom–atom interaction which is determined by the $s$-wave scattering frequency can be controlled experimentally by manipulating the transverse trapping frequency $\omega_\perp$ which can change the waist radius of the optical mode $w$ [49].

3. Dynamics of the system

The dynamics of the hybrid OMS described by the Hamiltonian of equation (3) is fully characterized by the following set of nonlinear QLEs in the framework of open quantum systems [2, 50–52]

$$\dot{a} = -(i\delta_0 + \kappa/2)a - i\zeta (a^\dagger c + c^\dagger) - \eta_k c + \sqrt{\kappa} \delta a_{in}, \quad (4a)$$

$$\dot{c} = -(i\Omega c - \gamma/2)c - i\zeta a^\dagger a - \sqrt{\gamma} \delta c_{in}, \quad (4b)$$

where both the optical mode of the cavity and the Bogoliubov mode of the BEC are affected by their corresponding reservoirs. As is seen from QLEs, i.e. equations (4a) and (4b), the system dynamics is affected by two uncorrelated quantum noise sources.

(i) The optical input vacuum noise $\delta a_{in}$ arising from all the optical modes outside the cavity satisfying the Markovian correlation functions, i.e. $\langle \delta a_{in}(t)\delta a_{in}^\dagger(t') \rangle = (\eta_{th} + 1)\delta(t - t')$, $\langle \delta a_{in}(t)\delta a_{in}(t') \rangle = \eta_{ph}\delta(t - t')$ with the average thermal photon number $\eta_{th}$ which is nearly zero at optical frequencies [2, 50–52]. (ii) The atomic quantum noise operator $\delta c_{in}$ arising from the harmonic trapping potential in which the BEC has been confined and also from the extra modes
of the BEC which have been neglected in the single-mode approximation of equation (2) as has been discussed in references [43, 53–55]. All of these extra atomic modes behave as an atomic reservoir which not only injects the quantum noise $\delta c_{in}$ into the atomic system but also make the Bogoliubov mode of the BEC dissipate at the damping rate of $\gamma$. Besides, the atomic quantum noise also satisfies the same Markovian correlation functions as those of the optical noise [53].

The nonlinear equations (4a) and (4b) can be linearized by expanding the quantum operators around their respective classical mean values as $a = \alpha + \delta a$ and $c = \beta + \delta c$ where $\delta a$ and $\delta c$ are small quantum fluctuations around the mean fields $\alpha$ and $\beta$. In this way, the equations of motion of the mean fields as well as the linearized QLEs for the quantum fluctuations can be obtained through the nonlinear QLEs. In the steady state, where $\dot{\alpha}$ and $\dot{\beta}$ are set equal to zero, the mean field equations of motion lead to the following steady-state values

$$\alpha = -\frac{\eta_c}{i\Delta + \kappa/2}, \quad (5a)$$

$$\beta = -\zeta |\alpha|^2 \frac{\Omega^{(-)} + i\gamma/2}{\Omega^{(-)} + i\gamma/2 + \gamma^*/4}, \quad (5b)$$

where $\Delta = \delta_c + 2\beta_R\zeta$ is the effective frequency of the optical mode with $\beta_R$ being the real part of the complex mean field $\beta$, and $\Omega^{(\pm)} = \Omega_c \pm \frac{1}{2}\omega_{sw}$ while the quantum fluctuations together with their Hermitian conjugates satisfy the following linearized set of ordinary differential equations

$$\dot{u}(t) = \chi_0 u(t) + u_{in}(t), \quad (6)$$

where $u(t) = [\delta a(t), \delta a^\dagger(t), \delta c(t), \delta c^\dagger(t)]^\dagger$ is the vector of continuous variable fluctuation operators, $u_{in}(t) = [\sqrt{\kappa}\delta a_{in}, \sqrt{\kappa}\delta a_{in}^\dagger, \sqrt{\gamma}\delta c_{in}, \sqrt{\gamma}\delta c_{in}^\dagger]^\dagger$ is the corresponding vector of quantum noises, and also

$$\chi_0 = \begin{pmatrix}
-\frac{\kappa}{2} - i\Delta & 0 & -i\alpha\zeta & -i\alpha^*\zeta \\
0 & -\frac{\kappa}{2} + i\Delta & i\alpha^*\zeta & i\alpha\zeta \\
-i\alpha^*\zeta & -i\alpha\zeta & \frac{-\gamma}{2} - i\Omega_c & -\frac{i}{2}\omega_{sw} \\
i\alpha^*\zeta & i\alpha\zeta & \frac{i}{2}\omega_{sw} & \frac{-\gamma}{2} + i\Omega_c
\end{pmatrix}, \quad (7)$$

is the drift matrix. It is worth reminding that the solutions to equation (6) are stable only if all the eigenvalues of the matrix $\chi_0$ have negative real parts. The stability conditions can be obtained, for example, by using the Routh–Hurwitz criteria [56].

Now, the linearized QLEs of the system can be solved by taking the Fourier transform of equation (6) as

$$u(\omega) = \chi(\omega)u_{in}(\omega), \quad (8)$$

where the susceptibility matrix $\chi(\omega)$ is obtained as

$$\chi(\omega) = (-i\omega I - \chi_0)^{-1}, \quad (9)$$

where $I$ is the $4 \times 4$ identity matrix. It is obvious that the Fourier components of both the optical and the atomic quantum fluctuations can be obtained from equation (8) in terms of the Fourier components of the quantum noises of the system.
4. Linear response of hybrid OMS in the framework of GLRT

In this section we investigate the linear response of the hybrid OMS described in sections 2 and 3 to a weak external time-dependent perturbation which is executed by a weak probe laser with frequency $\omega_p$ which drives the cavity with the rate $\eta_p$ whose absolute value is much lower than that of the coupling laser. Therefore, the dynamics of the system is affected by the following time-dependent perturbation in the frame rotating at the coupling laser frequency

$$V(t) = \hbar \eta_p \delta a e^{i \omega_p t} + \hbar \eta_p \delta a^\dagger e^{-i \omega_p t},$$

(10)

where $\omega_{pc} = \omega_p - \omega_c$ is the detuning between the probe and coupling lasers frequencies.

Our main goal is to show how the dynamical behavior of the system and phenomena like the anti-resonance [44, 45], OMIT [24–27] and Fano resonance [28–31] can be described through the open system Green’s function which can be obtained by the GLRT. In reference [39], the linear response of a standard OMS has been investigated in the framework of the GLRT. On the other hand, in sections 2 and 3, it was shown that a hybrid OMS consisting of an atomic BEC fluctuations of the hybrid OMS to the external time-dependent perturbation can be obtained based on the GLRT as follows

$$\langle \delta a(t) \rangle = \langle \delta a \rangle_0 + \eta_p \int_{-\infty}^{+\infty} dt' G^R_{aa}(t-t') e^{i \omega_p t'} + \eta_p^* \int_{-\infty}^{+\infty} dt' G^R_{aa}(t-t') e^{-i \omega_p t'},$$

(11a)

$$\langle \delta c(t) \rangle = \langle \delta c \rangle_0 + \eta_p \int_{-\infty}^{+\infty} dt' G^R_{ca}(t-t') e^{i \omega_p t'} + \eta_p^* \int_{-\infty}^{+\infty} dt' G^R_{ca}(t-t') e^{-i \omega_p t'},$$

(11b)

where $\langle \delta a \rangle_0 = 0$ and $\langle \delta b \rangle_0 = 0$ are the steady-state mean values of the optical and atomic field fluctuations in the absence of the time-dependent perturbation and the system retarded Green’s functions have been defined as

$$G^R_{aa}(t) = -i \theta(t)[\delta a(t), \delta a(0)]_0, \quad \text{(12a)}$$

$$G^R_{ca}(t) = -i \theta(t)[\delta a(t), \delta a(0)]_0, \quad \text{(12b)}$$

$$G^R_{ca}(t) = -i \theta(t)[\delta c(t), \delta a(0)]_0, \quad \text{(12c)}$$

$$G^R_{ca}(t) = -i \theta(t)[\delta c(t), \delta a(0)]_0. \quad \text{(12d)}$$

The first two Green’s functions of equations (12a) and (12b) are related to the optical field while equations (12c) and (12d) represent the atomic field Green’s functions. It should be noted that the time evolutions of all the operators in equations (12a)–(12d) are obtained from the QLEs given by equation (6) derived in section 3 and the subscript 0 means that all the expectation values should be calculated in the steady-state of the system in the absence of the perturbation. On the other hand, equations (11a) and (11b) can be rewritten as

$$\langle \delta a(t) \rangle = \eta_p G^R_{aa}^{(0)}(\omega_{pc}) e^{-i \omega_p t} + \eta_p G^R_{aa}(\omega_{pc}) e^{i \omega_p t}, \quad \text{(13a)}$$

$$\langle \delta c(t) \rangle = \eta_p G^R_{ca}^{(0)}(\omega_{pc}) e^{-i \omega_p t} + \eta_p G^R_{ca}(\omega_{pc}) e^{i \omega_p t}, \quad \text{(13b)}$$

in terms of the definition of the Fourier transform of the Green’s function, i.e. $\tilde{G}(\omega) = \int_{-\infty}^{+\infty} d\tau \ G(\tau) e^{-i \omega \tau}$. 

\[ \text{[Image 96x761 to 468x761]} \]
Since \( \langle a(t) \rangle = \alpha + \langle \delta a(t) \rangle \), and \( \langle c(t) \rangle = \beta + \langle \delta c(t) \rangle \) are the expectation values of the optical and atomic fields in the rotating frame, the responses of the optical and atomic fields to the time-dependent perturbation in the laboratory frame are obtained as

\[
\langle a(t) \rangle = \alpha e^{i\omega_c t} + \eta p \tilde{G}^R_{aa}(\omega_{pc}) e^{-i\delta(\omega + \omega_{pc})t} + \eta p \tilde{G}^R_{mm}(\omega_{pc}) e^{i\delta(\omega - \omega_{pc})t},
\]

\[
\langle c(t) \rangle = \beta + \eta p \tilde{G}^R_{ca}(\omega_{pc}) e^{-i\delta(\omega + \omega_{pc})t} + \eta p \tilde{G}^R_{ca}(\omega_{pc}) e^{i\delta(\omega - \omega_{pc})t}.
\]

As is seen from equation (14a), the optical mode has a central band oscillating with \( \omega_c \) and two sidebands, the so-called Stokes and anti-Stokes sidebands, oscillating with \( \omega_c \pm \omega_{pc} \). For the atomic mode the situation is the same with the difference that the central mode, i.e. the mean field \( \beta \) in equation (14b), has no oscillation in laboratory frame.

Based on the GLRT, the Green’s functions of an open quantum system satisfy a system of ordinary differential equations which are derived through the linearized QLEs, i.e. equation (6) and are called the Green’s functions equations of motion which are given by the following compact equations

\[
\frac{d}{dt} G^R_{ai}(t) = -i\delta(t) V_{ai} + \chi a G^R_{ai}(t), \quad (15a)
\]

\[
\frac{d}{dt} G^R_{ai}(t) = +i\delta(t) V_{ai} + \chi a G^R_{ai}(t), \quad (15b)
\]

where \( G^R_{ai}(t) = (G^R_{ai}(t), G^R_{ai}(t), G^R_{ai}(t), G^R_{ai}(t))^T, \) \( G^R_{ai}(t) = (G^R_{ai}(t), G^R_{ai}(t), G^R_{ai}(t), G^R_{ai}(t))^T, \) \( V_{ai} := (1, 0, 0, 0)^T \) and \( V_a := (0, 1, 0, 0)^T \) are fixed four-dimensional vectors. Now, by taking the Fourier transforms of equations (15a) and (15b) the Fourier components of the Green’s function vectors are obtained as

\[
\tilde{G}^R_{ai}(\omega) = -i\chi a(\omega) V_{ai}, \quad (16a)
\]

\[
\tilde{G}^R_{ai}(\omega) = +i\chi a(\omega) V_{ai}, \quad (16b)
\]

where \( \chi a(\omega) \) is the susceptibility matrix defined by equation (9). As is seen from equations (16a) and (16b), the system Green’s functions in the frequency space, i.e. the Fourier transforms of equations (12a)–(12d) are obtained as

\[
\tilde{G}^R_{ai}(\omega) = +i\chi a(\omega), \quad (17a)
\]

\[
\tilde{G}^R_{ai}(\omega) = -i\chi a(\omega), \quad (17b)
\]

\[
\tilde{G}^R_{ai}(\omega) = +i\chi a(\omega), \quad (17c)
\]

\[
\tilde{G}^R_{ai}(\omega) = -i\chi a(\omega). \quad (17d)
\]

### 4.1. The effects of atomic collisions and coupling laser on the anti-resonance and resonances of the system

In this subsection, we investigate how the atom–atom interaction as well as the pumping rate of the coupling laser affect the linear responses of the optical and atomic modes of the hybrid OMS to the time dependent perturbation. Based on the linearized QLEs given by equation (6), it is obvious that the optical mode oscillates with frequency \( \Delta \) while it can be easily shown...
that the atomic mode oscillates effectively with following frequency

\[ \omega_m = \frac{1}{2} \left( 4 \omega_R + \frac{1}{2} \omega_{sw} \right) \left( 4 \omega_R + \frac{3}{2} \omega_{sw} \right), \tag{18} \]

which is called the effective mechanical frequency of the Bogoliubov mode of the BEC [57].

In the following we will study the dynamics of the hybrid system in the red detuned regime of \( \Delta = \omega_m \) which is possible by fixing the coupling laser frequency at \( \omega_c = \omega_0 - \omega_m \). Obviously, under this condition the effective frequencies of the optical and atomic modes are equal.

Furthermore, we present our results based on the experimentally feasible parameters given in [14, 15]. For this purpose, we consider a cavity with length \( L = 178 \mu m \), damping rate of \( \kappa = 10^3 \) Hz, and bare frequency \( \omega_0 = 2.41494 \times 10^{15} \) Hz corresponding to a wavelength of \( \lambda = 780 \) nm which contains \( N = 5 \times 10^6 \) Rb atoms. The atomic \( D_2 \) transition corresponding to the atomic transition frequency \( \omega_a = 2.41419 \times 10^{15} \) Hz couples to the mentioned mode of the cavity. The atom-field coupling strength is \( g_0 = 2\pi \times 14.1 \) MHz and the recoil frequency of the atoms is \( \omega_R = 23.7 \) KHz. Furthermore, we assume that the equilibrium temperature of the BEC is \( T = 0.1 \mu K \), and the damping rate of the Bogoliubov mode of the BEC is \( \gamma = 10^{-4} \kappa \).

In order to study the effect of atomic collisions on the responses of the optical and atomic modes of the system to the external time-dependent perturbation, in figure 1 we have plotted the normalized amplitudes of the anti-Stokes \( A_a = \omega_R |\tilde{G}_a| (\omega_{pc}/\omega_R) \) (figure 1(a)) and the Stokes \( S_c = \omega_R |\tilde{G}_c| (\omega_{pc}/\omega_R) \) (figure 1(b)) sidebands of the optical field as well as the anti-Stokes \( A_c = \omega_R |\tilde{G}_c| (\omega_{pc}/\omega_R) \) (figure 1(c)) and the Stokes \( S_c = \omega_R |\tilde{G}_c| (\omega_{pc}/\omega_R) \) (figure 1(d)) sidebands of the Bogoliubov mode of the BEC for three different values of the s-wave scattering frequency: \( \omega_{sw} = 40 \omega_R \) (red solid curve), \( \omega_{sw} = 45 \omega_R \) (black dashed curve), and \( \omega_{sw} = 50 \omega_R \) (blue dotted curve) while the cavity is driven at the rate of \( \eta_c = 0.5 \kappa \).

All the results of figure 1 have been obtained in the red detuned regime of \( \Delta = \omega_m \) where the coupling laser frequency is smaller than the effective frequency of the cavity as \( \omega_c = \omega_0 - \omega_m \). The condition \( \Delta = \omega_m \) leads to an algebraic equation of order three for \( \omega_c \) which can be solved for any fixed value of \( \omega_{sw} \). Based on our numerical calculations, for at most one of the solutions of \( \omega_c \), the system is stable based on the Routh–Hurwitz criteria [56]. As is seen from figure 1, the Stokes amplitudes of the optical and atomic modes are much smaller than the anti-Stokes ones. It is because of the fact that as the hybrid OMS is driven in the red detuned regime of \( \Delta = \omega_m \), the input photon absorbs a phonon from the mechanical oscillator (the Bogoliubov mode) and is reflected blue-shifted by \( \omega_m \) which leads to the attenuation of the Stokes amplitude and simultaneous amplification of the anti-Stokes sideband. It means that in the red detuned regime both the optical and the atomic modes oscillate effectively with the anti-Stokes amplitude.

Furthermore, figure 1 shows that for each value of the s-wave scattering frequency the anti-Stokes and Stokes amplitudes of the optical and atomic fields exhibit two peaks corresponding to the two resonances at the normal frequencies of the hybrid OMS due to the coupling between the optical and atomic modes. Since for \( \eta_c = 0.5 \kappa \) the enhanced optomechanical coupling, i.e. \( \zeta |c| \), is larger than the coupling strength threshold \( \kappa/4 \) [30], the system is in the normal mode splitting regime. The other important point is that for each value of the s-wave scattering frequency there is an anti-resonance between the two resonances of the normal modes which occurs at \( \omega_{pc} = \omega_m \) (equivalent to \( \omega_p \approx \omega_0 \)), where the anti-Stokes amplitude of the optical mode becomes zero (figure 1(a)) while the anti-Stokes amplitude of the atomic field reduces to a nonzero minimum (figure 1(c)).

It is well-known that for a driven dissipative system consisting of two coupled oscillators there are two resonance frequencies corresponding to the two normal modes where the
Figure 1. (a) and (b) The normalized amplitudes of the anti-Stokes $A_a = \omega_R |\tilde{G}_{aa}(\omega_{pc}/\omega_R)|$ and the Stokes $S_a = \omega_R |\tilde{G}_{aa}^\dagger(-\omega_{pc}/\omega_R)|$ sidebands of the optical field, and (c) and (d) the anti-Stokes $A_c = \omega_R |\tilde{G}_{ca}(\omega_{pc}/\omega_R)|$ and the Stokes $S_c = \omega_R |\tilde{G}_{ca}^\dagger(-\omega_{pc}/\omega_R)|$ sidebands of the Bogoliubov mode of the BEC for three different values of the s-wave scattering frequency: $\omega_{sw} = 40\omega_R$ (red solid curve), $\omega_{sw} = 45\omega_R$ (black dashed curve), and $\omega_{sw} = 50\omega_R$ (blue dotted curve). It has been assumed that the system is in the red detuned regime of $\Delta = \omega_m$ while the cavity is driven at the rate of $\eta_c = 0.5\kappa$ by the coupling laser.

The oscillation amplitudes of the two oscillators becomes very large. As has been investigated in references [44, 45], there is an anti-resonance frequency between the two resonance frequencies where the oscillation amplitude of the oscillator that is directly driven by the external source becomes zero in the limit where its damping rate is much larger than that of the other oscillator. The anti-resonance occurs when the frequency of the driving source reaches the resonance frequency of the second oscillator where the phase of the first oscillator suffers a sudden change and becomes out of phase with the second one so that the motion of the first oscillator is quenched effectively by the second one [44,45].

The present hybrid OMS investigated in this article, is a system consisting of two coupled quantum oscillators which is similar to the above-mentioned classical system of coupled oscillators. The single-mode optical field of the cavity that is driven by the probe laser plays the role of the first oscillator which has been coupled to the Bogoliubov mode of the BEC (the second oscillator) through a radiation pressure interaction which is triggered by the coupling laser. That is why the phenomenon of anti-resonance occurs just for the optical mode of the cavity since it is the oscillator that is directly driven by an external source. Furthermore, in the present hybrid OMS the condition $\kappa \gg \gamma$ is satisfied, which means that the damping rate of the first oscillator is much larger than the second one’s.

The most important result that has been demonstrated in figure 1 is the fact that the anti-resonance frequency is shifted to higher values as the s-wave scattering frequency increases.
Since the mechanical frequency of the Bogoliubov mode of the BEC, i.e. $\omega_m$, given by equation (18), increases by increasing the s-wave scattering frequency and as the anti-resonance frequency occurs at $\omega_{pc} = \omega_m$, the position of anti-resonance is shifted to higher values by increasing $\omega_{pc}$. The great advantage of hybrid OMSs consisting of BEC in comparison to the standard OMSs having fixed mechanical frequency, is the controllability of the effective mechanical frequency of the Bogoliubov mode of the BEC. Since the s-wave scattering frequency is itself controllable experimentally through the transverse trapping frequency [49], the anti-resonance frequency of the present hybrid system can be manipulated easily. On the other hand, any observed shift in the anti-resonance frequency can be considered as a signature of a change in the strength of atomic collisions of the BEC.

However, for an experimental observation of a pattern like that predicted theoretically in figure 1(a), one needs to have an estimation of the value of $\omega_{sw}$ which can be measured via a Feshbach resonance by the application of an appropriate magnetic field [58, 59]. Therefore, by having the value of $\omega_{sw}$, the appropriate value of $\omega_c$ satisfying the condition $\Delta = \omega_m$ can be determined. As has been explained previously, for any fixed value of $\omega_{sw}$ the condition $\Delta = \omega_m$ leads to a third order algebraic equation which can give us a value for $\omega_c$ for which the system is stable. Therefore by fixing the coupling laser frequency at this specified value and by scanning the probe laser frequency around the effective cavity frequency, i.e. $\tilde{\omega}_0$, an experimental observation of a pattern like figure 1 will be possible.

On the other hand, in order to show how the pumping rate of the coupling laser affects the optical and atomic linear responses of the hybrid OMS to the external time-dependent perturbation, in figure 2 we have plotted the normalized amplitudes of the anti-Stokes sideband $A_a = \omega_R \hat{G}^a_{rad} (\omega_{pc}/\omega_R)$ of the optical field (figure 2(a)), as well as the anti-Stokes sideband $A_c = \omega_R \hat{G}^c_{rad} (\omega_{pc}/\omega_R)$ of the Bogoliubov mode of the BEC (figure 2(b)), for three different values of the coupling laser pumping rate: $\eta_c = 0.04\kappa$ (red solid curve), $\eta_c = 0.15\kappa$ (blue dotted curve) while the s-wave scattering frequency has been fixed at $\omega_{sw} = 40\omega_R$ and the system is in the red detuned regime of $\Delta = \omega_m$. Based on the results demonstrated in figure 1, in the red detuned regime the Stokes sidebands are much weaker than the anti-Stokes ones and consequently the optical and atomic modes oscillate effectively with anti-Stokes amplitudes. Therefore, we no longer show the Stokes amplitudes in figure 2. Besides, as is seen from figure 2(a) the position of the anti-resonance frequency of the optical field is invariant for different values of the coupling laser pumping rate since for the specified value of the s-wave scattering frequency, i.e. $\omega_{sw} = 40\omega_R$, the effective mechanical frequency of the Bogoliubov mode of the BEC is fixed at $\omega_m \approx 39.2\omega_R$.

It should be noted that the behavior of the anti-Stokes amplitude of the optical field presented in figure 2(a) shows that the system can switch from the OMIT to the normal mode splitting by controlling the pumping rate of the coupling laser. For $\eta_c = 0.04\kappa$ (red solid curve) the enhanced optomechanical coupling, which is of order $\zeta[\alpha] \approx 0.16$, is smaller than $\kappa/4$ and the system is in the weak-driving regime [30] where both of the two normal frequencies of the system occur at $\omega_{pc} = \Delta = \omega_m$ but with different linewidths. On the other hand, the existence of the anti-resonance at $\omega_m$ generates a narrow transparency window which leads to the manifestation of the OMIT phenomenon.

However, by increasing the pumping rate of the coupling laser to $\eta_c = 0.15\kappa$, where the enhanced optomechanical coupling gets near to $\kappa/4$ the two normal modes start to split (black dotted curve) while for a larger value of $\eta_c = 0.5\kappa$ where the enhanced optomechanical coupling is pretty larger than $\kappa/4$ the system is in the strong-driving regime [30] and the two normal modes are completely separated (blue dotted curve). In fact, in the strong-driving regime if the
effective resonance frequencies of the two oscillators degenerate, i.e. for $\Delta = \omega_m$, the normal frequencies occur at $\omega_{pc} \approx \omega_m \pm \zeta |\alpha|$ [30]. Therefore, the amount of splitting between the normal modes becomes larger as the pumping rate of the coupling laser increases because the optical mean-field, i.e. $|\alpha|$, increases by increasing $\eta_c$.

On the other hand, a phenomenon like OMIT does not happen for the Bogoliubov mode of the BEC in the weak-driving regime of $\eta_c = 0.04\kappa$ (red solid curve in figure 2(b)). Since there is no anti-resonance frequency for the BEC mode, which plays the role of the second oscillator, no transparency window appears in the amplitude of the anti-Stokes of the atomic mode as is seen from the red solid curve in figure 2(b). Nevertheless, by increasing the pumping rate of the coupling laser when the enhanced optomechanical coupling exceeds the coupling strength threshold and the system reaches to the strong-driving regime, the normal mode splitting also occurs in the atomic mode as is seen from the blue dotted curve in figure 2(b).
4.2. Fano resonance

In the subsection 4.1, the dynamics of the hybrid OMS was studied in the red detuned regime of $\Delta = \omega_m$ where the effective resonance frequencies of the optical and mechanical modes degenerate. It was shown that under this condition an anti-resonance appears just in the middle of the two resonances of the anti-Stokes amplitude of the optical field which is the origin of the OMIT phenomenon in the weak-driving regime. It can also explain why in the case of $\Delta = \omega_m$ the two resonance peaks of the optical and mechanical anti-Stokes amplitudes are nearly symmetric in the strong-driving regime as was demonstrated in figure 1(a).

On the other hand, there is another interesting phenomenon named the Fano resonance which can be also observed in the red detuned regime of optomechanics [29] but for $\Delta = r\omega_m$ with $r$ being a positive real number. As is evident in this case, the two resonance frequencies of the normal modes are no longer equal. The Fano resonance was observed for the first time in some of the Rydberg spectral atomic lines exhibiting sharp asymmetric profiles in the absorption spectrum [28]. Ugo Fano [28] was the first one who could describe this interesting phenomenon theoretically based on the coupling of a discrete excited state of an atom with a continuum through an electromagnetic radiation which results in interference phenomena.

The Fano resonance like the OMIT phenomenon occurs in the weak-driving regime and can be also explained by the simple model of two coupled oscillators where one of them is driven by an external source with the difference that in the case of the Fano resonance the resonance frequencies of the two oscillators are no longer equal. In this case, there are again two resonance peaks with the difference that one of them is symmetric while the other one is asymmetric. The main reason for the asymmetry of one of the peaks is the fact that because of the inequality of the resonance frequencies of the two oscillators, the anti-resonance which occurs at the second oscillator frequency gets nearer to one of the peaks and leads to an asymmetrical shape.

In order to see how the Fano resonance appears in the present hybrid OMS, in figure 3 we have demonstrated the normalized amplitudes of the anti-Stokes sideband $A_a = \omega_R|\tilde{G}_{aa}^R(\omega_{pc}/\omega_R)|$ of the optical field (red solid curves) as well as the anti-Stokes sideband $A_c = \omega_R|\tilde{G}_{ca}^R(\omega_{pc}/\omega_R)|$ of the Bogoliubov mode of the BEC (blue dotted curves) for three different values of the effective cavity detuning: $\Delta = 0.7\omega_m$ (figure 3(a)), $\Delta = \omega_m$ (figure 3(b)), and $\Delta = 1.3\omega_m$ (figure 3(c)). The results have been obtained for a fixed value of the s-wave scattering frequency of $\omega_{sw} = 40\omega_R$ when the cavity is pumped by the coupling laser at the rate of $\eta_c = 0.04\kappa$ so that the system is in the weak-driving regime.

Figure 3 shows that in the hybrid OMS one can switch from OMIT with $\Delta = \omega_m$ (figure 3(b)) to Fano resonance with $\Delta = 0.7\omega_m$ (figure 3(a)) and $\Delta = 1.3\omega_m$ (figure 3(c)) by deviation of the effective cavity detuning ($\Delta$) from the effective frequency of the Bogoliubov mode ($\omega_m$). For this purpose, for each $\Delta = r\omega_m$ with $r = 0.7, 1, 1.3$ a third order algebraic equation is formed which can be solved for $\omega_{pc}$ for the fixed value of $\omega_{sw} = 40\omega_R$. In this way, for each $\Delta$ one can obtain a specified value for the coupling laser frequency for which the system is stable so that if the cavity is pumped by the coupling laser at that frequency at the rate of $\eta_c = 0.04\kappa$, the optical and atomic modes oscillate with amplitudes demonstrated in figure 3.

As is seen from the red curves in figure 3 which demonstrate the optical anti-Stokes amplitude, for $\Delta = 0.7\omega_m$ (figure 3(a)) the first resonance peak which is away from the anti-resonance point is symmetric and wide while the second one which is near to the anti-resonance is a small and narrow asymmetric peak. For $\Delta = \omega_m$ (figure 3(b)) the anti-resonance appears just in the middle of the wide peak and therefore a narrow transparency window appears just in the middle of the wide peak which leads to the manifestation of the OMIT. Finally, for $\Delta = 1.3\omega_m$ (figure 3(c)) the first peak gets nearer to the anti-resonance point and becomes
Figure 3. The normalized amplitudes of the anti-Stokes sideband $A_a = \omega_R |\tilde{G}^{\text{m}}(\omega_{\text{pc}}/\omega_R)|$ of the optical field (red solid curves) as well as the anti-Stokes sideband $A_c = \omega_R |\tilde{G}^{\text{p}}(\omega_{\text{pc}}/\omega_R)|$ of the Bogoliubov mode of the BEC (blue dotted curves) for three different values of the effective cavity detuning: (a) $\Delta = 0.7\omega_m$, (b) $\Delta = \omega_m$, and (c) $\Delta = 1.3\omega_m$. It has been assumed that $\omega_{\text{sw}} = 40\omega_R$ and the pumping rate of the coupling laser gas been fixed at $\eta_c = 0.04\kappa$ while the other parameters are like those of figure 1.
asymmetric and narrow while the second one gets away from the anti-resonance and becomes symmetric and wide.

On the other hand, the blue dotted curves in figure 3 represents the anti-Stokes amplitudes of the Bogoliubov mode of the BEC. In spite of the Fano profiles of the optical mode, the atomic anti-Stokes amplitude has a sharp peak near the anti-resonance point and a small one away from it (figures 3(a) and (c)). The sharpness of the mentioned peaks in the Fano profiles of the atomic modes is because of the low value of the damping rate of the Bogoliubov mode in comparison to that of the optical mode. Furthermore, since there is no anti-resonance phenomenon for the atomic mode no transparency window appears in the sharp peak of the anti-Stokes of the atomic mode where the OMIT occurs for the optical mode (figure 3(b)).

5. Spectral function and effective damping rate of the cavity

The response behavior of the optical mode to the external time-dependent perturbation can be explained in another way in terms of the self-energy and the spectral function of the optical mode. The spectral function [32] which is usually interpreted as an effective density of single-particle states is defined as

\[ A(\omega) = -\frac{2}{\pi} \text{Im}[\tilde{G}_{aa}^R(\omega)], \]  

(19)

for the optical field of the cavity. On the other hand, in order to obtain the optical self-energy [60], it is enough to algebraically eliminate the Bogoliubov mode operators in the Fourier transform of the linearized QLEs of equation (6) so that the optical field \( a(\omega) \) can be written as

\[ -i\omega \delta a(\omega) = \left( i\Delta + \frac{\kappa}{2} + i\Sigma_a(\omega) \right) \delta a(\omega) + \lambda_a(\omega) \delta a^\dagger(\omega) + \sqrt{\kappa} A_\text{in}(\omega), \]  

(20)

where the optical self-energy is obtained as

\[ \Sigma_a(\omega) = \frac{4|\alpha|^2 \zeta^2 (\omega_{sw} - 2\Omega_c)}{\gamma^2 - 4\gamma_0^2 - 4\omega^2 - \omega_{sw}^2}, \]  

(21)

and \( \lambda_a(\omega) = io\Sigma_a(\omega)/\alpha^* \). Furthermore, the last term in equation (20) is the effective quantum noise operator which is a complex combination of the optical and atomic quantum noises. Since it has no role in our discussion we do not represent here its explicit expression.

As is seen from equations (20) and (21), the self-energy is a complex function of frequency whose real part modifies the frequency of the optical mode while its imaginary part modifies the damping rate of the cavity. In fact due to the interaction of the atomic mode with the optical field the damping rate of the cavity is modified through the imaginary part of the optical self-energy so that the effective damping rate of the cavity can be defined as

\[ \kappa_{\text{eff}}(\omega) = \frac{\kappa}{2} - \text{Im}[\Sigma_a(\omega)]. \]  

(22)

Now, using the optical spectral function of equation (19) and the effective damping rate of the cavity, i.e. equation (22), one can find an interpretation of the optical response behavior. For this purpose, in figure 4 we have plotted the normalized optical spectral function \( \omega_{pc} A(\omega_{pc}/\omega_R) \) (figure 4(a)) and the normalized effective damping rate of the cavity \( \kappa_{\text{eff}}(\omega_{pc}/\omega_R)/\kappa \) (figure 4(b)) for three different values of the s-wave scattering frequency: \( \omega_{sw} = 40\omega_R \) (red solid curve), \( \omega_{sw} = 45\omega_R \) (black dashed curve), and
Figure 4. (a) The normalized optical spectral function $\omega R A(\omega_{pc}/\omega_R)$ (b) the normalized effective damping rate of the cavity $\kappa_{eff}(\omega_{pc}/\omega_R)/\kappa$ for three different values of the s-wave scattering frequency: $\omega_{sw} = 40\omega_R$ (red solid curve), $\omega_{sw} = 45\omega_R$ (black dashed curve), and $\omega_{sw} = 50\omega_R$ (blue dotted curve). It has been assumed that $\Delta = \omega_m$ and $\eta_c = 0.5\kappa$, and the other parameters are like those of figure 1.

$\omega_{sw} = 50\omega_R$ (blue dotted curve). It has been assumed that the system is in the red detuned regime of $\Delta = \omega_m$ while the cavity is driven at the rate of $\eta_c = 0.5\kappa$ by the coupling laser.

A comparison between figures 4 and 1(a) shows that at the two normal frequencies, where the system is at resonance and the optical anti-Stokes amplitude reaches to its peaks, i.e. at $\omega_{pc} \approx \omega_m \pm \zeta|\alpha|$, the optical spectral function also maximizes for each value of $\omega_{sw}$ (figure 4(a)). On the other hand, the anti-resonance occurs at the frequency $\omega_{pc} = \omega_m$, where the optical spectral function becomes zero and the effective damping rate of the cavity reaches to a very sharp peak which is of the order of $10^3\kappa$, for each value of $\omega_{sw}$ (figure 4(b)). In this way, an interesting physical interpretation for the manifestation of two resonances and one anti-resonance arises. It fact, at $\omega_{pc} = \omega_m$ where the effective damping rate of the cavity becomes very large and the optical spectral function is zero, it is expected that the oscillation amplitude
of the optical mode goes to zero and the phenomenon of anti-resonance occurs while at the normal frequencies where the effective damping rate of the cavity becomes minimum but the optical spectral function maximizes, the optical mode oscillates with the maximum amplitude and the resonances occur.

6. Conclusions

In conclusion, we have studied a hybrid OMS consisting of a cigar-shaped BEC which is affected by an external time-dependent perturbation. In the regime where the cavity photon number is low enough and under the Bogoliubov approximation, the BEC can be considered as a single mode quantum field which interacts with the cavity radiation pressure through an optomechanical coupling. In this way, the hybrid system behaves effectively as a standard ordinary OMS with the difference that there is an extra interaction in the system Hamiltonian which is due to the atomic collisions of the BEC atoms.

Using the GLRT, which deals with the linear response of an open quantum system to an external time-dependent perturbation, we investigate the linear responses of the optical and atomic modes of the hybrid OMS while both the optical and the atomic modes are investigated as open quantum systems. The linear responses of the hybrid OMS are obtained through the solutions of the Green’s functions equations of motion predicted by the GLRT. The great superiority of the GLRT over the SLRT is the fact that the dissipation is taken into consideration in the GLRT without necessity of any phenomenological manipulation. The main purpose of the present paper has been to show how some important phenomena like manifestation of the resonances and the anti-resonance in a hybrid OMS can be described based on a sophisticated theory which needs no phenomenological manipulations.

One of the most interesting features of the present hybrid OMS is that it behaves as a system consisting of two coupled quantum oscillators which one of them (the Bogoliubov mode of the BEC) functions as an atomic parametric amplifier through the atom–atom interaction of the BEC. The study of the linear responses of such hybrid OMS shows that it has two resonance frequencies corresponding to the two normal modes of the system and an anti-resonance frequency which occurs for the optical mode of the cavity which is driven directly by the time-dependent perturbation. It is demonstrated how the position of the anti-resonance frequency can be manipulated by the s-wave scattering frequency of BEC atoms which itself is controllable through the transverse trapping frequency.

It has also been shown through the anti-Stokes amplitude of the optical field that the system can switch from the OMIT to the normal mode splitting by controlling the pumping rate of the coupling laser. Furthermore, the appearance of the Fano resonance profiles in the optical field can be also explained when the effective cavity detuning deviates from the effective frequency of the Bogoliubov mode. Finally, an interpretation of the optical response behavior especially the manifestation of the anti-resonance phenomenon is presented based on the optical spectral density and self-energy.

Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).
ORCID iDs

A Dalafi https://orcid.org/0000-0002-5913-2425

References

[1] Aspelmeyer M, Kippenberg T J and Marquardt F 2014 Rev. Mod. Phys. 86 1391
[2] Bowen W P and Milburn G J 2016 Quantum Optomechanics (Boca Raton, FL: CRC Press)
[3] Kippenberg T J and Vahala K J 2007 Opt. Express 15 17172
[4] Tsang M and Caves C M 2012 Phys. Rev. X 2 031016
[5] Wimmer M H, Steinmeyer D, Hammerer K and Heurs M 2014 Phys. Rev. A 89 053836
[6] Motazedifard A, Bemani F, Naderi M H, Roknizadeh R and Vitali D 2016 New J. Phys. 18 073040
[7] Dalafi A, Naderi M H and Vitali D 2016 New J. Phys. 18 073040
[8] Fani M and Dalafi A 2020 J. Opt. Soc. Am. B 37 1263
[9] Dalafi A, Naderi M H 2017 Phys. Rev. A 95 043601
[10] Dalafi A, Naderi M H 2017 Phys. Rev. A 96 033631
[11] Dalafi A, Naderi M H and Motazedifard A 2018 Phys. Rev. A 97 043619
[12] Dobrindt J M, Wilson-Rae I and Kippenberg T J 2008 Phys. Rev. Lett. 101 263602
[13] Gröblacher S, Hammerer K, Vanner M R and Aspelmeyer M 2009 Nature 460 724
[14] Huang S and Agarwal G S 2010 Phys. Rev. A 81 033830
[15] Fleischhauer M, Imamoglu A and Marangos J P 2005 Rev. Mod. Phys. 77 633
[16] Weis S, Rivière R, Deleglise S, Gavartin E, Arcizet O, Schliesser A and Kippenberg T J 2010 Science 330 1520
[17] Xiong H and Wu Y 2018 Appl. Phys. Rev. 5 031305
[18] Wang T, Zheng M-H, Bai C-H, Wang D-Y, Zhu A-D, Wang H-F and Zhang S 2018 Ann. Phys. 530 1800228
[19] Miroshnichenko A E, Flach S and Kivshar Y S 2010 Rev. Mod. Phys. 82 2257
[20] Qu K and Agarwal G S 2013 Phys. Rev. A 87 063813
[21] Peng B, Ozdemir Ş K, Chen W, Nori F and Yang L 2014 Nat. Commun. 5 5082
[22] Abbas M, Ullah R, Chuang Y-L and Ziauddin 2019 J. Phys. B: At. Mol. Opt. Phys. 52 185502
[23] Stanescu C and van Leeuwen R 2013 Nonequilibrium Many-Body Theory of Quantum Systems: A Modern Introduction (Cambridge: Cambridge University Press)
[24] Stefanucci G and van Leeuwen R 2013 Nonequilibrium Many-Body Theory of Quantum Systems: A Modern Introduction (Cambridge: Cambridge University Press)
[25] Scarlatella O, Clerk A A and Schiro M 2019 New J. Phys. 21 043040
[26] Ban M, Kitajima S, Arimoto T and Shibata F 2017 Phys. Rev. A 95 022126
[27] Liu H Z, Li D X and Li X 2017 Phys. Rev. E 95 012156
[28] Shen H Z, Xu X, Li H, Wu S L and Yi X X 2018 Opt. Lett. 43 2852
[29] Motazedifard A, Dalafi A and Naderi M H 2021 J. Phys. A: Math. Theor. 54 215301
[30] Dalafi A, Naderi M H, Soltanolkotabi M and Barzanjeh S 2013 Phys. Rev. A 87 013417
[31] Dalafi A, Naderi M H, Soltanolkotabi M and Barzanjeh S 2013 J. Phys. B: At. Mol. Opt. Phys. 46 235502
[32] Dalafi A, Naderi M H and Soltanolkotabi M 2014 J. Mod. Opt. 61 1387
[33] Dalafi A, Naderi M H and Soltanolkotabi M 2015 J. Phys. B: At. Mol. Opt. Phys. 48 115507
[34] Belbasi S, Foulaadvand M E and Joe Y S 2014 Am. J. Phys. 82 32
[45] Joe Y S, Satanin A M and Kim C S 2006 Phys. Scr. 74 259
[46] Maschler C and Ritsch H 2004 Opt. Commun. 243 145
[47] Domokos P, Horak P and Ritsch H 2001 J. Phys. B: At. Mol. Opt. Phys. 34 187
[48] Nagy D, Domokos P, Vukics A and Ritsch H 2009 Eur. Phys. J. D 55 659
[49] Morsch O and Oberthaler M 2006 Rev. Mod. Phys. 78 179
[50] Scully M O and Zubairy M S 1997 Quantum Optics (Cambridge: Cambridge University Press)
[51] Gardiner C W and Zoller P 2000 Quantum Noise (Berlin: Springer)
[52] Carmichael H 1999 Statistical Methods in Quantum Optics 1: Master Equations and Fokker–Planck Equations (Berlin: Springer)
[53] Zhang K, Chen W, Bhattacharya M and Meystre P 2010 Phys. Rev. A 81 013802
[54] Kónya G, Szirmai G and Domokos P 2011 Eur. Phys. J. D 65 33
[55] Szirmai G, Nagy D and Domokos P 2010 Phys. Rev. A 81 043639
[56] Gradshteyn I S and Ryzhik I M 1980 Table of Integrals, Series and Products (New York: Academic)
  Hurwitz A 1964 Selected Papers on Mathematical Trends in Control Theory ed R Bellman and R Kabala (New York: Dover)
[57] Dalafi A and Naderi M H 2016 J. Phys. B: At. Mol. Opt. Phys. 49 145501
[58] Marte A, Volz T, Schuster J, Dürr S, Rempe G, Van Kempen E G M and Verhaar B J 2002 Phys. Rev. Lett. 89 283202
[59] Donley E A, Claussen N R, Cornish S L, Roberts J L, Cornell E A and Wieman C E 2001 Nature 412 295
[60] Levitan B A, Metelmann A and Clerk A A 2016 New J. Phys. 18 093014