On the distance in some bipartite graphs $L_{k,n}$

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Abstract

The paper presents some bipartite graph $L_{k,n}$, so called $(k,n)$-level graph, that arise by taking $k$-th and $(n-k)$-th levels of $n$-dimensional Boolean algebra. Two results are established: (1) precise description of a distance (a shortest path) between arbitrary vertices and (2) solution of the problem how many vertices may be reached in $i$ steps starting from some initial point.

Keywords: bipartite graph, path, distance, level of Boolean algebra.

1. Preliminaries. For the standard notions such as simple graph, connected graph, path and so see for example [1]. Let us consider a finite and connected simple graph $G = (V,E)$. For arbitrary vertices $u$ and $v$, let $||uv||$ denote the distance from $u$ to $v$ (the length of a shortest path from $u$ to $v$), and assume that $||uv|| = 0$ iff $u = v$.

Definition 1 A map $d: V \times V \rightarrow \mathbb{N}$ is called a metric in $G$, iff for any vertices $u, v, w$ hold the following conditions:

\[
\begin{align*}
    d(u,v) &= 0 \iff u = v, \quad (1) \\
    d(u,v) &= d(v,u), \quad (2) \\
    d(u,v) + d(v,w) &\geq d(u,w). \quad (3)
\end{align*}
\]

Then a couple $(G,d)$ is called a metric graph.

Lemma 1 Let $(G,d)$ be a metric graph, and moreover:

\[
    d(u,v) = 1 \iff uv \in E,
\]

for any $u,v \in V$. Then $d(u,v) \leq ||uv||$.

Proof goes by induction on $n = ||uv||$.

(1) For $n = 0$ or $n = 1$ the thesis follows from (1) and (2), respectively.

(2) For $n > 1$ assume induction hypothesis:

\[
\forall u,v \in V (||uv|| < n \Rightarrow d(u,v) \leq ||uv||).
\]
(3) Fix \( u, v \in V \) such that \( \|uv\| = n \) and take arbitrary path from \( u \) to \( v \): \[
  u = x_0 \to x_1 \to \ldots \to x_{n-1} \to x_n = v.
\]

Since \( n > 1 \), there is some intermediate vertex \( x_i \) (i.e. \( 0 < i < n \)), so \( \|ux_i\| + \|x_i v\| = \|uv\| \), because there is no shorter path from \( u \) to \( v \). Applying the induction hypothesis and (3) we compute:

\[
d(u, v) \leq d(u, x_i) + d(x_i, v) \leq \|ux_i\| + \|x_i v\| = \|uv\|.
\]

**Remark 1** Condition (4) is important.

**Proof.** Consider a graph \( G = ([1, 2, 3], \{12, 13, 23\}) \) and a map \( d \) such that: \( d(1, 2) = d(1, 3) = d(2, 3) = 2 \). It easy to verify that \( (G, d) \) is a metric graph.

2. **The distance in a graph** \( L_{k,n} \). Let us consider a finite set \( X = \{1, \ldots, n\} \) and fix a natural \( k \) such that \( k < n - k \). Let \( [X]^k \) stands for the sets of all \( k \)-element subsets of \( X \), and similarly \( [X]^{n-k} \) stands for the \((n-k)\)-element subsets of \( X \). Let us define a bipartite graph \( L_{k,n} = (V, E) \) in the following way:

\[
  V = [X]^k \cup [X]^{n-k}, \quad E = \{AB : A \in [X]^k \land B \in [X]^{n-k} \land A \subseteq B\}.
\]

This graph can be regarded as built-up from two levels of \( n \)-dimensional Boolean algebra, namely the \( k \)-th and \((n-k)\)-th levels. Hence one can call it a \((k, n)\)-level graph.

**Theorem 1** If \( A, B \in V \) and \( |A \cap B| = i \), then:

1. \( \|AB\| \leq 2 \left[ \frac{k+i}{n-2k} \right] + 1 \), if \( |A| \neq |B| \),
2. \( \|AB\| \leq 2 \left[ \frac{|A|-i}{n-2k} \right] \), if \( |A| = |B| \).

**Proof.** For short, put \( t = n - 2k \). Moreover, \( [a, b] \) denotes the set \( \{x \in \mathbb{N} : a \leq x \leq b\} \).

Add (11). Without loss of the generality assume that \( A \in [X]^k \), \( B \in [X]^{n-k} \), and moreover:

\[
  A = [1,i] \cup [i + 1, k], \quad B = [1,i] \cup [k + 1, n - i].
\]

Define a sequence of sets \( C_{0}^{i}, C_{1}^{i}, \ldots, C_{j}^{i}, C_{j+1}^{i}, C_{j+1}^{i+1}, \ldots, C_{s-1}^{i}, C_{s}^{i}, C_{s+1}^{i} \), in the following way:

\[
  C_{0}^{i} = [1,i] \cup [i + 1 + 0t, k + 0t],
  C_{1}^{i} = [1,i] \cup [i + 1 + 0t, k + 1t],
  \vdots
  C_{j}^{i} = [1,i] \cup [i + 1 + jt, k + jt],
  C_{j+1}^{i} = [1,i] \cup [i + 1 + jt, k + (j + 1)t],
  \vdots
  C_{s}^{i} = [1,i] \cup [i + 1 + st, k + st],
\]

where \( s \) is the smallest natural such that \( C_{s}^{i} \subseteq B \), i.e. \( s \) satisfies inequalities:

\[
  k + 1 \leq i + 1 + st, \quad i + 1 + (s-1)t < k + 1,
\]
hence we achieve a path from $A$ of the length $2$ so $C_t = A, C_s \subseteq B$. Henceforth we just constructed a path in $G$:

$$C_0 \rightarrow C_1 \rightarrow C_1 \rightarrow \ldots \rightarrow C_s \rightarrow B,$$

of the length $2 \left\lceil \frac{n-k-1}{t} \right\rceil + 1$, which ends the proof of (1).

Ad (2). Assume that $A, B \in [X]^{n-k}$ and

$$A = [1, i] \cup [i + 1, n - k], \quad B = [1, i] \cup [n - k + 1, 2n - 2k - i],$$

and simultaneously $2n - 2k - i \leq n$ i.e. $t \leq i$. Just like before we define sets $C_0^s, C_1^s, \ldots, C_{s-2}^s, C_{s-1}^s, C_s^s$ such that:

$$C_0^s = [1, i] \cup [i + 1 + 0t, n - k + 0t],$$

$$C_1^s = [1, i] \cup [i + 1 + 1t, n - k + 0t],$$

$$\vdots$$

$$C_j^s = [1, i] \cup [i + 1 + jt, n - k + jt],$$

$$C_{j+1}^s = [1, i] \cup [i + 1 + (j + 1)t, n - k + jt],$$

$$\vdots$$

$$C_{s-1}^s = [1, i] \cup [i + 1 + st, n - k + (s - 1)t],$$

where $s$ is the smallest natural such that $C_{s-1}^s \subseteq B$, i.e. $s$ satisfies inequalities:

$$n - k + 1 \leq i + 1 + st, \quad i + 1+ (s-1)t < n-k+1,$$

so $s = \left\lceil \frac{n-k-1}{t} \right\rceil$. Then we have: $C_j^s \in [X]^{n-k}$, $C_{j+1}^s \in [X]^k$, $C_j^s \supseteq C_{j+1}^s \subseteq C_{j+1}^s$ and $C_0^s = A, C_{s-1}^s \subseteq B$. Finally, exists a path in $G$ from $A$ to $B$ of the length $2 \left\lceil \frac{n-k-1}{t} \right\rceil = 2 \left\lceil \frac{|A|-i}{t} \right\rceil$:

$$C_0^s \rightarrow C_0^s \rightarrow C_1^s \rightarrow \ldots \rightarrow C_{s-1}^s \rightarrow \ldots$$

Ad (2). Assume that $A, B \in [X]^{k}$ and

$$A = [1, i] \cup [i + 1, k], \quad B = [1, i] \cup [k + 1, 2k - i].$$

We will reduce this case to (1). Consider two possibilities: first, if $k - i \geq t$ then putting $C = [1, i+t] \cup [k+1, 2k-i]$, we get $C \subseteq [X]^{n-k}$ and $B \subseteq C$, and moreover $|A \cap C| = i + t$. By part (1) we have a path from $A$ to $C$ of the length $2 \left\lceil \frac{k-i+t}{t} \right\rceil + 1$. Observe also that:

$$2 \left\lceil \frac{k - (i + t)}{t} \right\rceil + 1 = 2 \left\lceil \frac{k - i}{t} - 1 \right\rceil + 1 = 2 \frac{k-i}{t} - 1;$$

hence we achieve a path from $A$ to $B$ of the length $2 \left\lceil \frac{k-i}{t} \right\rceil$ which ends the proof. The second possibility $k - i < t$ is trivial, because then we obtain $|A \cup B| < n - k$ so the path from $A$ to $B$ is of the length $2 = 2 \cdot 1 = 2 \left\lceil \frac{k-i}{t} \right\rceil$. 

The above proof give an algorithm of constructing a path from arbitrary vertex $A$ to arbitrary $B$. Let us define the function $d$ for $A, B \in V$ (just like before $|A \cap B| = i$):

$$d(A, B) = \begin{cases} 2 \left\lceil \frac{k-i}{n-2k} \right\rceil + 1, & \text{if } |A| \neq |B| \\ 2 \left\lceil \frac{|A|-i}{n-2k} \right\rceil, & \text{if } |A| = |B| \end{cases}.$$
Lemma 2  The map \( d \) is a metric and satisfies (4).

Proof. It is easy to see that \( d \) fulfils conditions (1), (2) and (4). The proof of (3) is also simple but quite arduous (eight cases). Assume abbreviations \(|(A \cap B) \setminus C| = p, |(B \cap C) \setminus A| = q, |(A \cap C) \setminus B| = r, |A \cap B \cap C| = x, \) and moreover \(|A \cap B| = i, |B \cap C| = j, |A \cap C| = l|.

(1) Consider the the case when \( A, B, C \in [X]^k \). To prove \( d(A, B) + d(B, C) \geq d(A, C) \) it is sufficient to show that:

\[
\frac{k - i}{n - 2k} + \frac{k - j}{n - 2k} \geq \frac{k - l}{n - 2k}.
\]

However it is clear that the above inequality is equal to:

\[
k + q \geq p + x + r,
\]

which is obviously true.

(2) If \( A, C \in [X]^{n-k}, B \in [X]^k \), it is sufficient to show that:

\[
\frac{k - i}{n - 2k} + \frac{k - j}{n - 2k} + 1 \geq \frac{n - k - l}{n - 2k},
\]

which is also equivalent to (5).

(3) Next six cases we easy check in similar way.  

The main result of this section is

Corollary 1  \( d(A, B) = \|AB\| \), for any \( A, B \in V \).

Proof. Inequality \( \leq \) follows forom lemmas 1 and 2, inequality \( \geq \) is obvious, since number \( d(A, B) \) is length of concrete path from \( A \) to \( B \).

3. The number of vertices reachable in \( i \) steps. The set \( P = \{1, \ldots, k\} \) is called an initial vertex of a graph \( L_{k,n} \). Our aim is to find a pattern of the function \( f \) that describe a cardinality of the set of vertices, that we reach in consecutive steps, starting from the initial vertex.

To illustrate the problem let us consider the graph \( L_{2,5} \). For simplify notation, vertex \( \{a, b\} \) will be denoted \( ab \) and similarly \( abc \) stands for the vertex \( \{a, b, c\} \).

The initial vertex is the only one that is reached in 0 steps, so \( \Gamma(0) = \{12\} \). In one step we reach three vertices: 123, 124, 125, so we write \( \Delta(0) = \{123, 124, 125\} \). In two steps we reach six vertices: \( \Gamma(1) = \{13, 14, 15, 23, 24, 25\} \) and so on. So the function that we are looking for, in the case of graph \( L_{2,5} \) is: \( f(0) = 1, f(1) = 3, f(2) = 6, f(3) = 6, f(4) = 3, f(5) = 1 \) (see figure below).
We assume that the Newton’s symbol $\binom{n}{k}$ have a sense for any integers $n$ and $k$:

$$\binom{n}{k} = \begin{cases} \frac{n!}{k!(n-k)!}, & \text{if } k \geq 0 \land k \leq n \\ 0, & \text{if } k < 0 \lor k > n \end{cases}$$

Fix $n$ and $k$ such that $k < n - k$ and assume $t = n - 2k$, $s = \left\lceil \frac{t}{2} \right\rceil$. For $i = 0, \ldots, s$ put:

$$\Gamma(i) = \{A \in [X]^k : d(P, A) = 2i\},$$
$$\Delta(i) = \{B \in [X]^{n-k} : d(P, B) = 2i + 1\}.$$  

The set $\Gamma(i)$ is just a set of vertices that may be reached from $P$, in precisely $2i$ steps. Similarly, $\Delta(i)$ is a set of vertices that may be reached from $P$, in precisely $2i + 1$ steps. By Corollary 1 it easy follows that:

$$\Gamma(i) \cap \Gamma(j) = \emptyset, \quad \Delta(i) \cap \Delta(j) = \emptyset, \quad \text{for } i \neq j,$$

and

$$[X]^k = \bigcup_{i=0}^{s} \Gamma(i), \quad [X]^{n-k} = \bigcup_{i=0}^{s} \Delta(i).$$

Let us compute the cardinality of $\Gamma(i)$; first observe that:

$$d(P, A) = 2i \iff 2 \left\lceil \frac{k - \mid P \cap A \mid}{t} \right\rceil = 2i \iff \left\lceil \frac{\mid P \setminus A \mid}{t} \right\rceil = i \iff$$

$$i - 1 < \frac{\mid P \setminus A \mid}{t} \leq i \iff (i - 1)t < \mid P \setminus A \mid \leq it \iff \bigvee_{j=1}^{t} \mid P \setminus A \mid = (i - 1)t + j,$$

so:

$$\Gamma(i) = \bigcup_{j=1}^{t} \{A \in [X]^k : \mid P \setminus A \mid = (i - 1)t + j\}.$$  

The set $\Gamma(i)$ has been presented as a union of disjoint sets. It is also clear that:

$$\mid \{A \in [X]^k : \mid P \setminus A \mid = l\} \mid = \binom{k}{k-l} \binom{n-k}{l},$$

so we achieve:

$$\gamma(i) = \mid \Gamma(i) \mid = \sum_{j=1}^{t} \binom{k}{k - ((i - 1)t + j)} \binom{n-k}{(i - 1)t + j}.$$  

Now let us compute the cardinality of $\Delta(i)$. Similarly like previously we show that:

$$\Delta(i) = \bigcup_{j=1}^{t} \{B \in [X]^{n-k} : \mid P \setminus B \mid = (i - 1)t + j\},$$

and since

$$\mid \{B \in [X]^{n-k} : \mid P \setminus B \mid = l\} \mid = \binom{k}{k-l} \binom{n-k}{t+l},$$

we obtain:

$$\delta(i) = \mid \Delta(i) \mid = \sum_{j=1}^{t} \binom{k}{k - ((i - 1)t + j)} \binom{n-k}{it + j}.$$  

The main result of this section is
Corollary 2 The function $f$ that gives the cardinality of the set of all vertices that is reached in precisely $x \in \{0, 1, \ldots, 2 \left\lceil \frac{k}{t} \right\rceil + 1\}$ steps is

$$f(x) = \begin{cases} 
\gamma(i), & \text{if } x = 2i \\
\delta(i), & \text{if } x = 2i + 1 
\end{cases}.$$ 

From the above calculation we obtain also a pure combinatorial corollary:

Corollary 3 For $n$ and $k$ such that $2k < n$ hold:

1. \[ \binom{n}{k} = \sum_{i=0}^{\left\lfloor \frac{k}{t} \right\rfloor} \sum_{j=1}^{n-2k} \left( k \right. \left. \begin{array}{c} k \\ i(n-2k) + j \end{array} \right) \left. \begin{array}{c} n-k \\ (i-1)(n-2k) + j \end{array} \right) \]

2. \[ \binom{n}{k} = \sum_{i=0}^{\left\lfloor \frac{k}{t} \right\rfloor} \sum_{j=1}^{n-2k} \left( k \right. \left. \begin{array}{c} k \\ i(n-2k) + j \end{array} \right) \left. \begin{array}{c} n-k \\ (i-1)(n-2k) + j \end{array} \right) \]

References

[1] R. Diestel, Graph Theory, Springer-Verlag New York 2000.