Variable Importance Based Interaction Modeling with an Application on Initial Spread of COVID-19 in China

Jianqiang Zhang
Center for Applied Statistics and School of Statistics, Renmin University of China, Beijing, China.
E-mail: zhangjqs@163.com
Ze Chen
Center for Applied Statistics and School of Statistics, Renmin University of China, Beijing, China.
E-mail: chze96@163.com
Yuhong Yang†
School of Statistics, University of Minnesota, Minneapolis, USA.
E-mail: yangx374@umn.edu
Wangli Xu‡
Center for Applied Statistics and School of Statistics, Renmin University of China, Beijing, China.
E-mail: wlxu@ruc.edu.cn

†Correspondence Yuhong Yang, School of Statistics, University of Minnesota, Minneapolis, USA. E-mail: yangx374@umn.edu
‡Correspondence Wangli Xu, Center for Applied Statistics and School of Statistics, Renmin University of China, Beijing, China. E-mail: wlxu@ruc.edu.cn
Summary. Interaction selection for linear regression models with both continuous and categorical predictors is useful in many fields of modern science, yet very challenging when the number of predictors is relatively large. Existing interaction selection methods focus on finding one optimal model. While attractive properties such as consistency and oracle property have been well established for such methods, they actually may perform poorly in terms of stability for high-dimensional data, and they do not typically deal with categorical predictors. In this paper, we introduce a variable importance based interaction modeling (VIBIM) procedure for learning interactions in a linear regression model with both continuous and categorical predictors. It delivers multiple strong candidate models with high stability and interpretability. Simulation studies demonstrate its good finite sample performance. We apply the VIBIM procedure to a Corona Virus Disease 2019 (COVID-19) data used in Tian et al. (2020) and measure the effects of relevant factors, including transmission control measures on the spread of COVID-19. We show that the VIBIM approach leads to better models in terms of interpretability, stability, reliability and prediction.

1. Introduction

Interaction terms play an important role in many linear regression applications, e.g., applying user-item interactions to identify higher level patterns in the recommender systems (Koren, 2009) and identifying interactions of single nucleotide polymorphisms (SNPs) for complex diseases classification (Schwender, 2008). Even if the number of predictors is moderate, in practice of linear regression, interactions are often barely considered or even completely ignored. For instance, Tian et al. (2020) did not consider interaction terms at all in studying the impact of transmission control measures on the initial spread of COVID-19. For a linear regression model with $p$ main effects, the total number of all pairwise interaction terms is $\binom{p}{2} = O(p^2)$, which increases drastically with $p$. Hence, identifying important interaction effects when $p$ is relatively large is very challenging.

In the existing literature, there are mainly two types of procedures for interaction selection, namely, joint analysis (Yuan et al., 2009; Choi et al., 2010) and stage-wise analysis (Jiang and Liu, 2014; Hao and Zhang, 2014; Hao et al., 2018). The joint analysis approach selects the main and interaction effects simultaneously by making a global
search over all possible models with interactions. The stage-wise analysis procedure selects significant main effects only at the first stage, then identifies main effects and important interactions among the reduced list of main effects. Although such procedures enjoy appealing asymptotic properties, they may suffer from three weaknesses, as will be elaborated later: (a) they are often unstable; (b) the procedures developed so far do not deal with categorical predictors; (c) they only select a single model among possibly many almost equally good models. As a consequence, a novel approach is needed to overcome these shortcomings.

Regardless of which approach is used to select a final model, there is inherent model selection uncertainty. In fact, the problem of instability associated with variable selection has been well recognized in literature (see, e.g., Draper (1995); Chatfield (1995); Yuan and Yang (2003); Yu (2017)). Due to rapid increase in dimensionality incurred by interactions, the instability of interaction selection tends to be even higher. To measure the instability, there are several methods proposed in the existing literature, such as bootstrap resampling based measures (Diaconis and Efron, 1983; Breiman, 1996; Buckland et al., 1997), sequential instability measure (Chen et al., 2007) and variable selection deviation measure (Nan and Yang, 2014).

To address high variable selection uncertainty as revealed by the above measures, one may stabilize a method (e.g., Meinshausen and Bühlmann (2010), Lim and Yu (2016)), or rely on a method via minimizing a variable selection uncertainty measure (e.g., Yang and Yang (2017)). Such methods have been shown to perform better. However, there is still an inherent drawback of the standard model selection approach, regardless of how it is done: conclusions based on the luckily selected model may not be reliable, since it presents only one story from the angle of the single model. When the data are not fully informative, as is often the case in moderate or high-dimensional data, it is more likely that multiple models are equally well supported by data at hand. In this scenario, as noted in Nan and Yang (2014), any selection procedure will not yield a single model that significantly dominates all possible competitors, so that any selection method chooses a lucky winner almost randomly among many more or less equally promising models.
A well established approach to address the issue of “winner takes all” is to use a model confidence set, which contains the true model with a given level of confidence; see Hansen et al. (2011), Ferrari and Yang (2015), Li et al. (2019), Zheng et al. (2019) and references therein. From a practical point of view, however, the size of model confidence set is often too large to be directly useful. In addition, models contained in a model confidence set may sometimes be drastically different in their compositions with few common variables. Thus, the interpretability and applicability of the models in the model confidence set may not be necessarily satisfactory in our context.

In this paper, we propose a methodology to deal with all of the three weaknesses at the same time. To address the model selection instability issue, we take advantage of model averaging that mitigates model selection uncertainty by weighting estimators across some models. There are various model averaging approaches proposed, e.g., Bayesian model averaging (Hoeting et al., 1999), adaptive regression by mixing (ARM) (Yang, 2001), model averaging based on a local asymptotic framework (Claeskens, 2003), model averaging with estimator’s risk characteristics (Leung and Barron, 2006), Mallows model averaging (Hansen, 2007) and parsimonious model averaging (Zhang et al., 2020).

Note that model averaging does not perform variable selection. However, model averaging weights can be used in this regard. For instance, Yang and Yang (2017) showed that such weighting based methods for variable selection become more stable. Thus, combining the advantages of model averaging and variable importance, we develop a variable importance based interaction modeling (VIBIM) procedure using the sparsity oriented importance learning (SOIL) proposed by Ye et al. (2018). The detailed introduction of the VIBIM procedure is presented in Section 2.

Moreover, SOIL can be applied to models with categorical variables and its theoretical properties are established in Chen et al. (2022). Thus, VIBIM overcomes the second weakness of the current two types of interaction selection procedures mentioned earlier. Further, instead of glorifying a single selected model, VIBIM delivers multiple candidate models with high stability and interpretability.

We evaluate the performance of VIBIM by simulation studies. The results indicate the superiority of VIBIM compared with existing approaches. We apply VIBIM to a data
set on the initial spread of COVID-19 in China, pioneered in Tian et al. (2020) with the goal to measure the effects of relevant factors, including transmission control measures on the spread of the disease. We identify three major weaknesses of the published work in Science, and show that the VIBIM approach leads to better models in terms of interpretability, stability, reliability and prediction. In addition, we correct a puzzling sign of an important covariate reported in Tian et al. (2020).

The remainder of the paper is organized as follows. In Section 2, the proposed VIBIM methodology is described in details. Simulation analysis is carried out in Section 3 to illustrate the finite sample performance of the proposed procedure. In Section 4, the COVID-19 data and methods—linear and Poisson regression models are introduced. Section 5 applies the proposed procedure to investigate the effects of the relevant factors, including transmission control measures on the spread of COVID-19 with interpretation and discussions. Concluding remarks are offered in Section 6. Additional analysis results are provided in the Supplementary Material.

2. Methodology

2.1. Preliminaries

For a linear regression model with both categorical and continuous predictors, without loss of generality, we assume that, among the \( p \) predictors \( \{X_1, \ldots, X_p\} \), the predictors \( \{X_1, \ldots, X_q\} \) are categorical, while \( \{X_{q+1}, \ldots, X_p\} \) are continuous. The categorical levels of \( \{X_1, \ldots, X_q\} \) are denoted by \( \{J_1, \ldots, J_q\} \) respectively. For each categorical variable \( X_i \), we define dummy variables \( X_{i,j} \) pertaining to the \( j \)-th categorical level for \( j = 1, \ldots, J_i - 1 \), and put \( X_{I_i} = (X_{i,1}, \ldots, X_{i,J_i-1})^T \) with \( I_i \equiv \{(i,1), \ldots, (i,J_i-1)\} \) in the regression. In a similar fashion, put \( X_{I_i} = X_i \) with \( I_i \equiv \{i\} \) for each continuous predictor \( X_i \). A linear regression model with categorical and continuous predictors is written as

\[
Y = \beta_0 + \sum_{i=1}^{p} X_{I_i}^T \beta_{I_i} + \epsilon. \tag{1}
\]

Here, \( \beta_{I_i} = (\beta_{i,1}, \ldots, \beta_{i,J_i-1})^T \) for \( i = 1, \ldots, q \) and \( \beta_{I_i} = \beta_i \) for \( i = q+1, \ldots, p \). The error \( \epsilon \) is assumed to be normally distributed with mean zero and variance \( \sigma^2 \). The total
number of predictors in model (1) is $p^* = \sum_{i=1}^q J_i + p - 2q$. For $N > 1$, let $\mathcal{M} \stackrel{\text{def}}{=} \{ M_i, i = 1, \ldots, N \}$ be a candidate model set, where $M_i = \bigcup_{j \in A_i} I_j$, $A_i \subseteq \{1, \ldots, p\}$.

2.2. A new interaction selection procedure

To address the high instability of model selection approaches associated with a single selection criterion, one constructive technique is to take advantage of a sound variable importance (VI) measure. As an effective tool, VI can be used to arrive at more stable results. To be specific, we consider interactions among variables with reasonably large VI values. We propose to rank all the main effects and interactions according to VI values in a descending order and sort out top nested models. In this way, VI offers most plausible models. Various methods are available for evaluating variable importance. For instance, RFI1 and RFI2 investigated by Breiman (2001) are two measures of variable importance in random forest based on out-of-bag assessment and node impurities, respectively.

The aforementioned SOIL can be generalized to models with categorical variables. For each element $M$ in $\mathcal{M}$, we calculate the corresponding weight $w_M$ below according to the BIC-p weighting (Nan and Yang, 2014), which is defined as

$$w_M \stackrel{\text{def}}{=} \exp \left( -\frac{I_M}{2} - \psi C_M \right) / \sum_{M' \in M} \exp \left( -\frac{I_{M'}}{2} - \psi C_{M'} \right), \text{ for } M \in \mathcal{M},$$

(2)

where $I_M$ is the BIC information criterion for model $M$, $C_M = |M| \log (e \cdot p^*/|M|) + 2 \log (|M| + 2)$, $| \cdot |$ denotes cardinality, and $\psi$ is a positive constant. According to the model set $\mathcal{M} = \{ M_i : i = 1, \ldots, N \}$ and weighting vector $w = (w_1, \ldots, w_N)^T$ defined in (2), the importance of the $j$th variable $X_j$, $j \in \{1, \ldots, p\}$, is defined as

$$S_j \stackrel{\text{def}}{=} S(j; w, \mathcal{M}) = \sum_{i=1}^N w_i I (I_j \subseteq M_i),$$

(3)

where $I( \cdot )$ denotes the indicator function. Thus, the importance of the $j$th variable $X_j$ is the sum of weights of the candidate models including the variable $X_j$. The choice of $\psi$ can be specified by the users. Choosing too small a $\psi$ may result in significant SOIL importances of unimportant variables and vice versa. As suggested in Ye et al. (2018), SOIL under $\psi = 0.5$ or $\psi = 1$ generally performs well. In practice, it is computationally impractical to consider all subsets as candidate model set $\mathcal{M}$ for $p \gg n$. Our
implementation strategy is to make use of three group variable selection methods (e.g.,
group LASSO (Yuan and Lin, 2006), group SCAD (Wang et al., 2007), and group MCP
(Huang et al., 2012)) first, and put together the resultant models with various levels of
sparsity to obtain the candidate models.

SOIL has a desirable theoretical property, that is, it can well separate the true predic-
tors from the others asymptotically (Ye et al., 2018) under the condition of consistency
of the weighting. Chen et al. (2022) has demonstrated the rationality of this consistency
condition based on BIC-p weighting. Thus, SOIL is a recommendable method in both
theory and practical application.

In this work, with SOIL, we propose a variable importance based interaction modeling
(VIBIM) procedure for linear models with categorical predictors. The procedure consists
of the following four steps:

Step 1. Using \( S^1, \ldots, S^p \) to compute the SOIL importances based on the linear
candidate models without any interaction term.

Step 2. Set up a threshold value \( c \in (0, 1) \), and define a submodel \( M_c = \{ i : S_i > c, \text{for} \ 1 \leq i \leq p \} \). Then add the interactions \( \{ X_{I_i}X_{I_j} : i, j \in M_c \} \) to \( \{ X_{I_1}, \ldots, X_{I_p} \} \),
where \( X_{I_i}X_{I_j} \) denotes the vector consisting of all pairwise products of elements of \( X_{I_i} \)
and \( X_{I_j} \) with length \( |I_i| \cdot |I_j| \). For example, \( X_{I_i}X_{I_j} = (X_{i,1}X_{j,1}, \ldots, X_{i,1}X_{j,J-1}, \ldots, X_{i,J-1}X_{j,1}, \ldots, X_{i,J-1}X_{j,J-1})^T \) for two categorical variables.

Step 3. Recompute the SOIL importance for the augmented dataset to sort the \( p \)
main effects together with \( \frac{1}{2}|M_c| \cdot (|M_c| - 1) \) interactions \( \{ X_{I_i}X_{I_j} : i, j \in M_c, i \neq j \} \) in
a descending order \( S'_{j_1} \geq \cdots \geq S'_{j_{p'}} \), where \( p' = p + \frac{1}{2}|M_c| \cdot (|M_c| - 1) \).

Step 4. Obtain a sequence of nested models \( M_{(1)} \subseteq \cdots \subseteq M_{(K)} \), the \( s \)th one of which
includes the first \( s \) predictors according to the SOIL importance calculated in Step 3.

Step 5. Narrow down the above nested models. One can choose a small model \( M_{(L)} \)
by minimizing BIC and a large model \( M_{(U)} \) by minimizing AIC, and the models between
\( M_{(L)} \) and \( M_{(U)} \) are considered as most plausible. In high-dimensional situations, we can
use AIC-p (see, e.g., Yang (1999)) and BIC-p information criteria to avoid overfitting.

Unlike the existing model selection methods that rely on a single final model, and
model confidence set approaches that typically give out too many models, VIBIM delivers
a relatively small number of nested models. We point out that the most appropriate threshold value \( c \) and the number of nested models \( K \) are not specified, since they may depend on the goal of the analysis and application. Model selection methods may suffer from the disadvantage of excluding predictors that are highly correlated with ones that are already in a model, whether or not they should be included. Our method alleviates this problem, as we will see in Section 5, where we identify a significant interaction that is highly correlated with its main effect. By taking advantage of model averaging and variable importance, VIBIM can yield good models in terms of interpretability, stability, reliability and prediction.

3. Simulation Study

We demonstrate the performance of interaction selection of the VIBIM method in various high-dimensional scenarios. For all the numerical analysis in this paper, we obtain the candidate models using the R package \texttt{grpreg} with the default settings, which determines a grid of 100 tuning parameters that are equally spaced on the log scale over a range \([\lambda_{\text{min}}, \lambda_{\text{max}}]\) (see, e.g., Breheny and Huang (2011) and Huang et al. (2012) for choices of \( \lambda_{\text{min}} \) and \( \lambda_{\text{max}} \)). We choose \( \psi = 1 \) in (2) for the BIC-p weighting and the threshold value \( c = 1e-4 \) in \textit{Step} 2 for VIBIM procedure. For comparison, we also consider three procedures: group LASSO, group SCAD, and group MCP (abbreviated as gLASSO, gSCAD, gMCP, respectively). The gLASSO first selects main effects by 10-fold cross-validation, then identifies main effects and important interactions among the reduced list of main effects. The gSCAD and gMCP are conducted similarly.

We generate the simulation data by the linear model (1) with simple size \( n \), where the error \( \epsilon \sim N(0, \sigma^2) \). Let \((Z_1, \ldots, Z_p) \sim N_p(0, \Sigma)\) with \( \Sigma = (\rho^{i-j})_{p \times p} \). Notice that \( X_{i,j} = (X_{i,1}, \ldots, X_{i,J_i-1})^T \) for \( i = 1, \ldots, q \), let \( X_{i,j} = I(\Phi_{(j-1)/J_i} < Z_i \leq \Phi_{j/J_i}) \) for \( j = 1, \ldots, J_i - 1 \), where \( \Phi_{(j-1)/J_i} \) is the \((j - 1)/J_i\)-th quantile of the standard normal distribution. Let \( X_i = Z_i \) for \( i \in \{q + 1, \ldots, p\} \). In all the examples, we set \( \beta_0 = 1 \), \((\rho, \sigma) = (0.5, 1)\). The number of categorical variables is \( q = 6 \) and \( J_1 = J_2 = J_3 = J_4 = 2, J_5 = J_6 = 6 \). We choose the threshold value \( c = 1e-4 \) in \textit{Step} 2 of VIBIM procedure. Three different settings of \((n, p)\) are considered: \((100, 1000), (200, 1000), (3000, 1000)\), the
results of \((n, p) = (100, 1000)\) and \((n, p) = (300, 1000)\) are provided in the supplementary materials due to the space limitation. All simulation examples are repeated 100 times and the corresponding values are averaged.

Two examples are considered in the simulations. First, we consider interaction models that obey strong heredity, i.e., the interactions are only among pairs of nonzero main effects. Second, we simulate interaction models that obey weak heredity, i.e., each interaction has only one of its main effects present.

**Example 1** (Strong heredity). We first consider the following six interaction models:

1. **Model I:** \(Y = \beta_1^T X_1 + \beta_2^T X_3 + \beta_5^T X_3 + X_7 \beta_7 + X_8 \beta_8 + X_9 \beta_9 + \beta_{7,9} X_7 X_9 + \epsilon,\)
2. **Model II:** \(Y = \beta_1^T X_1 + \beta_3^T X_3 + \beta_5^T X_3 + X_7 \beta_7 + X_8 \beta_8 + X_9 \beta_9 + \beta_{1,8} X_1 X_8 + \epsilon,\)
3. **Model III:** \(Y = \beta_1^T X_1 + \beta_3^T X_3 + \beta_5^T X_3 + X_7 \beta_7 + X_8 \beta_8 + X_9 \beta_9 + \beta_{1,3} X_1 X_3 + \epsilon.\)
4. **Model IV:** \(Y = \beta_1^T X_1 + \beta_3^T X_3 + \beta_5^T X_3 + X_7 \beta_7 + X_8 \beta_8 + X_9 \beta_9 + \beta_{1,3} X_1 X_3 + \epsilon.\)
5. **Model V:** \(Y = \beta_3^T X_3 + \beta_5^T X_3 + X_7 \beta_7 + X_8 \beta_8 + X_9 \beta_9 + \beta_{1,3} X_1 X_3 + \epsilon.\)
6. **Model VI:** \(Y = \beta_3^T X_3 + \beta_5^T X_3 + X_7 \beta_7 + X_8 \beta_8 + X_9 \beta_9 + \beta_{1,3} X_1 X_3 + \epsilon.\)

The true \((\beta_1, \beta_3, \beta_7, \beta_8, \beta_9) = (2, 3, 2, 3, -2), \beta_{1,8} = (-2, -3, -4, -5, 0)^T.\) The coefficients of the interactions are \(\beta_{7,9} = 1.5, \beta_{1,8} = 1.5, \beta_{1,3} = 2.\) The first three models Model I–III include an interaction between two continuous variables, a continuous variable and a categorical variable and two categorical variables, respectively. Models IV and V include two interactions and Model VI contains three interactions.

**Example 2** (Weak heredity).

1. **Model I:** \(Y = \beta_1^T X_1 + \beta_3^T X_3 + \beta_5^T X_3 + X_7 \beta_7 + X_8 \beta_8 + \beta_{7,9} X_7 X_9 + \epsilon,\)
2. **Model II:** \(Y = \beta_1^T X_1 + \beta_3^T X_3 + \beta_5^T X_3 + X_7 \beta_7 + X_8 \beta_8 + X_9 \beta_9 + \beta_{1,8} X_1 X_8 + \epsilon.\)
3. **Model III:** \(Y = \beta_1^T X_1 + \beta_3^T X_3 + \beta_5^T X_3 + X_7 \beta_7 + X_8 \beta_8 + X_9 \beta_9 + \beta_{1,3} X_1 X_3 + \epsilon.\)
4. **Model IV:** \(Y = \beta_1^T X_1 + \beta_3^T X_3 + \beta_5^T X_3 + X_7 \beta_7 + X_8 \beta_8 + X_9 \beta_9 + \beta_{1,3} X_1 X_3 + \epsilon.\)

We adopt the same setup of coefficients as in Example 1. In Model I, we consider an interaction among two continuous variables with only one parent effect. Model II contains an interaction between a continuous variable and a categorical variable which
has only its categorical main effect present, whereas Model III includes this interaction and its continuous main effect. In the last model, an interaction between two categorical variables with only one parent effect are considered.

To evaluate the variable selection performance of each method, we employ two performance measures, $F$- and $G$-measures. The $F$-measure for a given model $M_0$ is defined as the harmonic mean of the precision and recall, and the $G$-measure is defined as the geometric mean of the two. Since we focus on the task of interaction selection, the precision and recall are calculated with respect to interaction terms. Specifically, $F(M_0) = 2|\tilde{M}_0 \cap M^*|/(|\tilde{M}_0| + |M^*|)$ and $G(M_0) = |\tilde{M}_0 \cap M^*|/\sqrt{|\tilde{M}_0| \cdot |M^*|}$, where $\tilde{M}_0$ denotes interaction terms in $M_0$ and $M^*$ represents interactions in the true model. To ensure a fair comparison, we present simulation results for multiple models $M_{(i)}, i = 5, \ldots, 10$ consisting of $i$ predictors obtained by VIBIM, gLASSO, gSCAD, gMCP, respectively. To be specific, for VIBIM, $M_{(i)}, i = 5, \ldots, 10$ stand for the nested models in Step 4, and for three variable selection methods, $M_{(i)}, i = 5, \ldots, 10$ are obtained from solution path which have the model size $i = 5, \ldots, 10$, respectively.

The means and standard errors (in parentheses) of $F$- and $G$-measures for Examples 1–2 are summarized in Tables 1–2. It is seen that when the strong heredity assumption holds, the VIBIM performs best, across all of the settings, with the highest $F$- and $G$-measures. The $F$- and $G$-measures are generally very close to 1 when the model size of VIBIM is around the true model size. In Example 2, when the weak heredity condition holds, the VIBIM outperforms the other methods in Model II and Model IV. In Model I and Model III, however, no method could identify correctly the important interactions. This is perhaps because the low correlation between the interaction and its parent effects. For instance, the Pearson correlation coefficient between $X_1, X_8$ and $X_8$ is around 0.709, and thus $X_8$ is typically identified as an important main effect in the first stage, while that between $X_1, X_8$ and $X_8$ is only around $-0.003$. Therefore, in Example 2, the four methods tend to select $X_1, X_8$ in Model II, but fail in Model III. The gMCP performs mostly well from the perspective of $G$-measure, but less so in $F$-measure. Overall, the VIBIM is the best among all the methods in terms of $F$- and $G$-measures.

We also conduct another version of the procedures based on the three variable se-
lection methods, i.e., the selected main effects are obtained from the solution path of each of the selection methods at the first stage, and the number of the main effects are all equal to the cardinality of $M_e$ in Step 2 of VIBIM. These results can be found in Section S.1 of the Supplementary Material, which shows that our proposed method has a clear advantage over the other methods. In addition, in Supplementary Material Section S.1, we present simulation results of importance measures SOIL, RFI1 and RFI2 for Examples 1–2 at the first stage of VIBIM procedure.

4. COVID-19 Data and Methods

4.1. Data Source and Modification

Based on a dataset about COVID-19 with 296 cities across China, Tian et al. (2020) explored the associations between the number of confirmed cases ($\text{Sevendays.Cucase}$) during the first week of the outbreak (i.e. reported the first case) in a city and three transmission control measures: closing entertainment venues, banning intra-city public transport and prohibition of railway from and to other cities. For the sake of description, we denote the three control measures as $\text{Enter}$, $\text{Bus}$ and $\text{Railway}$, respectively. Specifically, using Poisson regression model, Tian et al. (2020) investigated the associations between $\text{Sevendays.Cucase}$ and 10 explanatory variables: a binary variable indicating whether control measure $K$ ($K = \text{Enter, Bus, Railway}$) was implemented in city $i$ before or during its first week of outbreak ($K.\text{Resp}^*$), the timing of implementing transmission control measure $K$ in the city, the date of the first confirmed case in the city ($\text{Arr.Time}$), the distance from Wuhan City ($\text{Dis.WH}$), the population of the city in 2018 ($\text{Pop.2018}$), and the passenger volume from Wuhan in 2018 ($\text{Total.Flow}$). A more detailed description of these variables can be found in Table 3.

Upon a careful examination, we feel there may be several issues in their analysis. First, the causal relationships between variables $K.\text{Resp}^*$ and $\text{Sevendays.Cucase}$, and between variables $K.\text{Date}^*$ and $\text{Sevendays.Cucase}$ are not clear. More precisely, the percent of cities that implemented at least one measure after the outbreak is $208/296 = 70.27\%$. Such measures may have been implemented as a response to, rather than a prevention of, the arrival of COVID-19 in most cities. In other words, the causality
Table 1. Results for the Models I–VI that are considered in Example 1.

| Method | F-measure | G-measure |
|--------|-----------|-----------|
|        | M(7) | M(8) | M(9) | M(10) | M(7) | M(8) | M(9) | M(10) | M(11) | M(12) | M(13) | M(14) | M(15) | M(16) |
| Model I | VIBIM | 1.000 | 0.967 | 0.932 | 0.840 | 1.000 | 0.971 | 0.940 | 0.860 | 0.889 | 0.972 | 0.945 | 0.943 | 0.900 | 0.912 |
|        | gLASSO | 0.552 | 0.546 | 0.441 | 0.402 | 0.677 | 0.623 | 0.576 | 0.532 |
|        | gSCAD | 0.703 | 0.688 | 0.534 | 0.464 | 0.740 | 0.718 | 0.743 | 0.705 |
|        | gMCP | 0.873 | 0.823 | 0.693 | 0.617 | 0.889 | 0.972 | 0.945 | 0.943 |
| Model II | VIBIM | 1.000 | 0.920 | 0.840 | 0.755 | 1.000 | 0.930 | 0.860 | 0.788 |
|        | gLASSO | 0.583 | 0.577 | 0.478 | 0.388 | 0.679 | 0.660 | 0.587 | 0.548 |
|        | gSCAD | 0.265 | 0.597 | 0.557 | 0.408 | 0.272 | 0.676 | 0.754 | 0.743 |
|        | gMCP | 0.887 | 0.747 | 0.620 | 0.485 | 0.898 | 0.940 | 0.935 | 0.898 |
| Model III | VIBIM | 0.917 | 0.852 | 0.800 | 0.761 | 0.927 | 0.870 | 0.826 | 0.793 |
|        | gLASSO | 0.706 | 0.538 | 0.462 | 0.357 | 0.762 | 0.655 | 0.593 | 0.528 |
|        | gSCAD | 0.643 | 0.463 | 0.415 | 0.440 | 0.840 | 0.747 | 0.632 | 0.687 |
|        | gMCP | 0.530 | 0.523 | 0.505 | 0.507 | 0.885 | 0.866 | 0.880 | 0.900 |
| Model IV | VIBIM | 1.000 | 0.966 | 0.930 | 0.879 | 1.000 | 0.969 | 0.936 | 0.891 |
|        | gLASSO | 0.645 | 0.646 | 0.605 | 0.494 | 0.759 | 0.734 | 0.698 | 0.656 |
|        | gSCAD | 0.596 | 0.676 | 0.610 | 0.470 | 0.605 | 0.740 | 0.776 | 0.764 |
|        | gMCP | 0.933 | 0.700 | 0.614 | 0.421 | 0.937 | 0.950 | 0.949 | 0.923 |
| Model V | VIBIM | 0.974 | 0.955 | 0.927 | 0.901 | 0.976 | 0.959 | 0.933 | 0.910 |
|        | gLASSO | 0.743 | 0.581 | 0.537 | 0.438 | 0.811 | 0.743 | 0.675 | 0.630 |
|        | gSCAD | 0.725 | 0.600 | 0.549 | 0.422 | 0.824 | 0.766 | 0.742 | 0.707 |
|        | gMCP | 0.717 | 0.706 | 0.585 | 0.539 | 0.864 | 0.862 | 0.884 | 0.867 |
| Model VI | VIBIM | 0.998 | 0.977 | 0.960 | 0.947 | 0.998 | 0.978 | 0.962 | 0.951 |
|        | gLASSO | 0.899 | 0.772 | 0.692 | 0.616 | 0.888 | 0.843 | 0.792 | 0.741 |
|        | gSCAD | 0.823 | 0.760 | 0.560 | 0.554 | 0.835 | 0.815 | 0.811 | 0.775 |
|        | gMCP | 0.896 | 0.694 | 0.571 | 0.474 | 0.947 | 0.907 | 0.898 | 0.901 |
|        |        |        |        |        |        | (0.002) | (0.005) | (0.007) | (0.008) |
|        |        |        |        |        |        | (0.002) | (0.005) | (0.007) | (0.008) |
|        |        |        |        |        |        | (0.027) | (0.024) | (0.025) | (0.012) |
|        |        |        |        |        |        | (0.020) | (0.024) | (0.039) | (0.036) |
|        |        |        |        |        |        | (0.023) | (0.039) | (0.044) | (0.046) |

Note: Standard errors are in parentheses.
Table 2. Results for the Models I–IV that are considered in Example 2.

| Method | \( F \)-measure | \( G \)-measure |
|--------|-----------------|-----------------|
|        | \( M_6 \) | \( M_7 \) | \( M_8 \) | \( M_9 \) | \( M_6 \) | \( M_7 \) | \( M_8 \) | \( M_9 \) |
| VIBIM  | 0.030 | 0.030 | 0.030 | 0.027 | 0.031 | 0.030 | 0.030 | 0.027 |
|        | (0.017) | (0.017) | (0.017) | (0.015) | (0.017) | (0.017) | (0.017) | (0.016) |
| gLASSO | 0.100 | 0.082 | 0.047 | 0.047 | 0.111 | 0.098 | 0.061 | 0.063 |
|        | (0.027) | (0.023) | (0.015) | (0.014) | (0.023) | (0.025) | (0.018) | (0.018) |
| gSCAD  | 0.033 | 0.027 | 0.015 | 0.005 | 0.036 | 0.032 | 0.018 | 0.008 |
|        | (0.017) | (0.013) | (0.011) | (0.005) | (0.018) | (0.015) | (0.012) | (0.007) |
| gMCP   | 0.010 | 0.010 | 0.000 | 0.010 | 0.010 | 0.011 | 0.000 | 0.013 |
|        | (0.010) | (0.010) | (0.000) | (0.010) | (0.010) | (0.011) | (0.000) | (0.011) |
| VIBIM  | 0.980 | 0.913 | 0.847 | 0.774 | 0.980 | 0.921 | 0.864 | 0.802 |
|        | (0.014) | (0.019) | (0.021) | (0.023) | (0.014) | (0.018) | (0.020) | (0.020) |
| gLASSO | 0.233 | 0.355 | 0.420 | 0.342 | 0.280 | 0.396 | 0.505 | 0.489 |
|        | (0.033) | (0.031) | (0.022) | (0.019) | (0.036) | (0.033) | (0.022) | (0.014) |
| gSCAD  | 0.407 | 0.665 | 0.647 | 0.470 | 0.425 | 0.722 | 0.767 | 0.725 |
|        | (0.041) | (0.022) | (0.032) | (0.037) | (0.042) | (0.018) | (0.020) | (0.020) |
| gMCP   | 0.830 | 0.697 | 0.692 | 0.585 | 0.854 | 0.903 | 0.878 | 0.833 |
|        | (0.023) | (0.041) | (0.040) | (0.043) | (0.021) | (0.019) | (0.020) | (0.024) |
| VIBIM  | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
|        | (0.000) | (0.000) | (0.000) | (0.000) | (0.000) | (0.000) | (0.000) | (0.000) |
| gLASSO | 0.043 | 0.038 | 0.037 | 0.022 | 0.068 | 0.053 | 0.047 | 0.033 |
|        | (0.019) | (0.017) | (0.017) | (0.011) | (0.024) | (0.020) | (0.019) | (0.014) |
| gSCAD  | 0.010 | 0.000 | 0.007 | 0.007 | 0.032 | 0.000 | 0.014 | 0.014 |
|        | (0.010) | (0.000) | (0.007) | (0.007) | (0.018) | (0.000) | (0.010) | (0.010) |
| gMCP   | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
|        | (0.000) | (0.000) | (0.000) | (0.000) | (0.000) | (0.000) | (0.000) | (0.000) |
| VIBIM  | 1.000 | 0.923 | 0.863 | 0.811 | 1.000 | 0.933 | 0.881 | 0.836 |
|        | (0.000) | (0.014) | (0.018) | (0.020) | (0.000) | (0.012) | (0.016) | (0.017) |
| gLASSO | 0.823 | 0.658 | 0.515 | 0.430 | 0.884 | 0.743 | 0.651 | 0.588 |
|        | (0.032) | (0.023) | (0.024) | (0.023) | (0.024) | (0.014) | (0.011) | (0.011) |
| gSCAD  | 0.923 | 0.757 | 0.625 | 0.518 | 0.947 | 0.866 | 0.802 | 0.768 |
|        | (0.023) | (0.035) | (0.038) | (0.017) | (0.019) | (0.016) | (0.019) | (0.019) |
| gMCP   | 0.940 | 0.770 | 0.752 | 0.633 | 1.000 | 0.946 | 0.927 | 0.895 |
|        | (0.024) | (0.038) | (0.038) | (0.042) | (0.000) | (0.011) | (0.013) | (0.015) |
confusion arises because of a possible opposite causal connection: the response variable may be a causal factor in explanatory variables related to control measures. Second, the definition of $K.\text{Date}_i^*$ is questionable. Recall the definition of $K.\text{Date}_i^*$, which is the timing of implementing control measure $K$ in city $i$, where 31 December 2019 is day 0, if $K.\text{Resp}_i^* = 1$; $K.\text{Date}_i^* = 0$ if $K.\text{Resp}_i^* = 0$. We argue that the setting that $K.\text{Date}_i^* = 0$ when $K.\text{Resp}_i^* = 0$ has the potential to mislead. Since $K.\text{Date}_i^* = 0$ suggests that city $i$ did not implement $K$ in practice before or during its first week of outbreak, but measure $K$ may be implemented after first week of outbreak in city $i$, and in this case, $K.\text{Date}_i^*$ is in fact greater than 0 by the definition of $K.\text{Date}_i^*$, which seems contradictory. Third, quite surprisingly, Tian et al. (2020) drew a conclusion that $\text{Dis.WH}$ had a significant positive effect on the response variable, which indicates that cities farther away from Wuhan had more confirmed cases when the other covariates were controlled.

In view of these issues, we modify variable $K.\text{Resp}^*$ as the variable $K.\text{Resp}$ indicating whether measure $K$ has been implemented before the first case was reported. In the meanwhile, variable $K.\text{Date}$ represents the number of days that measure $K$ was implemented before the outbreak of COVID-19. In addition, the puzzling sign issue of $\text{Dis.WH}$ might be due to improper modeling, or might be caused by omitting some important variables in light of the weak correlations between $\text{Dis.WH}$ and the other explanatory variables. Therefore, we supplement 4 continuous explanatory variables: average travel intensity ($\text{Travel.Intensity}$), per capita gross regional product ($\text{PGRP}$), number of grade-III level-A hospitals ($\text{3A.Hospital}$), daily average temperature ($\text{Temperature}$) and 2 categorical variables: tier of the city ($\text{City.Tier}$) and region of the city ($\text{Region}$), which might be potentially important. A detailed description of the above variables is summarized in Table 3.

As for the source of data about the explanatory variables in Table 3 the travel intensity in each city ($\text{Travel.Intensity}$) was obtained from the Baidu Migration Map (http://qianxi.baidu.com). We calculated the average of daily temperatures in each city ($\text{Temperature}$) over the first 14 days before the arrival of the first confirmed case from the China Weather website (http://www.weather.com.cn). From the Baidu Baike (https://baike.baidu.com), we acquired the China’s city-tier classification ($\text{City.Tier}$).
Table 3. Description of variables used in the study.

| Variable               | Description                                                                 |
|------------------------|-----------------------------------------------------------------------------|
| 7edays.Cucase\textsubscript{i} | Number of confirmed cases during the first seven days of the outbreak in city \textit{i} |
| City.Tier\textsubscript{i} | Tier of city \textit{i} which has six levels, namely 1, \ldots, 6          |
| Travel.Intensity\textsubscript{i} | Average travel intensity of 14 days prior to the date of the first confirmed case in city \textit{i} |
| Pop.2018\textsubscript{i} | Population size (in millions) in log scale of city \textit{i} in 2018       |
| Region\textsubscript{i} | Region of city \textit{i} which has seven levels, namely North, East, South, Central, Northwest, Southwest and Northeast |
| Dis.WH\textsubscript{i} | Distance in log10 scale between Wuhan and city \textit{i}                 |
| Total.Flow\textsubscript{i} | Passenger volume (in millions) in log scale from Wuhan to city \textit{i} by various means of transportation during the whole of 2018 |
| PGRP\textsubscript{i} | Per capita gross regional product in 2018 of city \textit{i}               |
| 3A.Hospital\textsubscript{i} | Number of grade-III level-A hospitals in city \textit{i}                  |
| Arr.Time\textsubscript{i} | Date of the first confirmed case in city \textit{i}, where 31 December 2019 is day 0 |
| Temperature\textsubscript{i} | Average daily temperature of fourteen days prior to the date of the first confirmed case in city \textit{i} |
| K.Resp\textsubscript{i} | A binary variable indicating whether or not control measure \textit{K} was implemented in city \textit{i}, \textit{K.Resp}_\textsubscript{i} = 1 if city \textit{i} has implemented \textit{K} prior to the outbreak |
| K.Resp\textsubscript{*} | A binary variable indicating whether or not control measure \textit{K} was implemented in city \textit{i}, \textit{K.Resp}_\textsubscript{*} = 1 if city \textit{i} has implemented \textit{K} before or during its first week of outbreak |
| K.Date\textsubscript{i} | Number of days that control measure \textit{K} has been implemented \textit{K} prior to the outbreak in city \textit{i}, if \textit{K.Resp}_\textsubscript{i} = 0, then \textit{K.Date}_\textsubscript{i} = 0 For example, if the date of the first confirmed case in city \textit{i} is on 28 January 2020, and \textit{K} was implemented in city \textit{i} on 25 January 2020, then \textit{K.Date}_\textsubscript{i} = 3  |
| K.Date\textsubscript{*} | Timing of implementing control measure \textit{K} in city \textit{i}, where 31 December 2019 is day 0, if \textit{K.Resp}_\textsubscript{*} = 0, then \textit{K.Date}_\textsubscript{*} = 0 For example, if the date of the first confirmed case in city \textit{i} is on 28 January 2020, then \textit{K.Date}_\textsubscript{*} = 28 |
and the region (Region) in which each city is located. The latest data of the per capital gross regional product (PGRP) was from the China City Statistical Yearbook (http://www.stats.gov.cn). The number of grade-III level-A hospitals (3A.Hospital) in each city was collected from the China Kang website (http://www.cnkang.com). The sources of the data of the remaining 10 explanatory variables and the response variable in Table 3 are described in Tian et al. (2020).

4.2. Modeling the Effects of COVID-19 Control Measures

4.2.1. Linear Regression Model With Categorical and Continuous Predictors

We apply linear regression model (1) to investigate the effects of the relevant factors on the spread of COVID-19. Here, $Y$ is the response variable $\log(\text{Sevendays.Cucase})$, $X_1 = (X_{i,1}, \ldots, X_{i,5})^T$ and $X_2 = (X_{i,1}, \ldots, X_{i,6})^T$ are used to represent the categorical variable City.Tier and Region, respectively. The remaining explanatory variables are 3 binary variables: Enter.Resp, Bus.Resp, Railway.Resp and 11 continuous explanatory variables: Travel.Intensity, Pop.2018, Dis.WH, Total.Flow, PGRP, 3A.Hospital, Arr.Time, Temperature, Enter.Date, Bus.Date, Railway.Date. We will apply VIBIM to analyze the data.

4.2.2. Poisson Regression Model

Tian et al. (2020) constructed a Poisson regression model

$$\log(\mathbb{E}[Y|\tilde{X}_1, \ldots, \tilde{X}_{10}]) = \tilde{\beta}_0 + \tilde{\beta}_1 \tilde{X}_1 + \tilde{\beta}_2 \tilde{X}_2 + \ldots + \tilde{\beta}_p \tilde{X}_{10} \tag{4}$$

of the response variable Sevendays.Cucase on 10 explanatory variables: Dis.WH, Arr.Time, Total.Flow, Pop.2018, Enter.Date*, Enter.Resp*, Bus.Date*, Bus.Resp*, Railway.Date*, Railway.Resp*, where Total.Flow and Pop.2018 are offset variables. In (4), $Y$ is assumed to be a Poisson random variable given $\tilde{X}_1, \ldots, \tilde{X}_p$. Note that we use $\tilde{X}$ and $\tilde{\beta}$ to avoid confusion with the notation in (1). For comparison purposes, we also consider Tian et al. (2020)’s model and denote it by “Poisson” in the following.
5. Analysis of the COVID-19 Data

In this section, we continue the analysis of linear model (1) with the response and the 16 explanatory variables described in Table 3. In subsection 5.1, three variable selection methods: group LASSO, group SCAD, and group MCP together with four importance measures SOIL (Ye et al., 2018), LMG (Lindeman et al., 1980), RFI1 and RFI2 (Breiman, 2001) are used to select predictors related to S7. Subsection 5.2 presents the estimates for the resultant models from VIBIM as well as the three selection methods. Comparisons of different models in terms of information criteria and prediction accuracy are provided in subsection 5.3. We assess the stabilities of the selected models and perform diagnostic analyses in subsection 5.4. A discussion of an interesting collinearity issue is provided in subsection 5.5.

5.1. A Preliminary Analysis of Variable Selection and Variable Importance

We first apply the three penalized regression methods with the tuning parameters selected by the standard BIC. The selected sets of important explanatory variables from group MCP, group SCAD and group LASSO are abbreviated as follows:

\[
\begin{align*}
M_{\text{gMCP}}^\dagger &= \{ \text{Dis.WH}, \text{3A.Hospital}, \text{Total.Flow}, \text{Pop.2018}, \text{Enter.Date} \} \\
M_{\text{gSCAD}}^\dagger &= \{ \text{Dis.WH}, \text{3A.Hospital}, \text{Total.Flow}, \text{Pop.2018}, \text{Enter.Date}, \text{Region}, \text{Temperature}, \text{Railway.Resp} \} \\
M_{\text{gLASSO}}^\dagger &= \{ \text{Dis.WH}, \text{3A.Hospital}, \text{Total.Flow}, \text{Pop.2018}, \text{Enter.Date}, \text{PGRP}, \text{Temperature}, \text{Region} \}.
\end{align*}
\]

These models are labeled as “gMCP\dagger”, “gSCAD\dagger”, “gLASSO\dagger” respectively.

As can be seen, the different methods produce different results, making it difficult to identify which variables are really important. In addition, as mentioned already, the stability of such results is a big issue. To get a good sense of instability of the three methods, we calculate the size of the symmetric difference between the originally selected model and that based on the perturbed data and the reduced data, denoted by PIVS and SIVS, respectively; see Nan and Yang (2014) for the implementation details. In our procedure, we denote by \( \tau \) the perturbation size in PIVS and let \( \rho \) be the proportion of observations removed from the original dataset in SIVS. The results based on 100
replications are gathered in Table S14 in Section S.2 of the Supplementary Material to save the space. We can see that the instabilities of group LASSO and group SCAD are unacceptably high, especially group SCAD. For example, when only 5% of the data are removed, the SIVS value of group SCAD is 5.11, that is, group SCAD would choose a model with more than 5 variables different on average.

We now apply the VIBIM procedure in Section 2.2 to address the observed high instabilities of model selection methods. Besides SOIL, we also consider several existing methods, namely LMG, RFI1 and RFI2. For SOIL, we choose $\psi = 1$ in (2) for the BIC-p weighting. The importances of the top 10 variables measured by SOIL, LMG, RFI1 and RFI2 are shown in Table 4.

It can be seen that the SOIL importance values of Dis.WH, 3A.Hospital, Total.Flow and Pop.2018 are close to 1, and they are also in the top 10 for the LMG, RFI1 and RFI2. The importances of Dis.WH, Total.Flow and Pop.2018 are in line with our expectations. Given the fact that hospitals play an important part in the early detection of coronavirus disease, it is appropriate that 3A.Hospital is considered as an important variable. Moreover, Enter.Date is regarded as a slightly important variable by SOIL, which indicates that the timing of closing entertainment venues affected the number of confirmed cases to a certain degree.

The importance value of Temperature assigned by SOIL is only 0.005, which is coincident with Yao et al. (2020)’s conclusion that there is no significant correlation between

| Rank | SOIL  | LMG    | RFI1   | RFI2   |
|------|-------|--------|--------|--------|
| 1    | Dis.WH | 1.000  | Dis.WH | 0.281  | Total.Flow | 1.000 | Total.Flow | 1.000 |
| 2    | 3A.Hospital | 1.000  | Region | 0.245  | Pop.2018  | 0.753  | Dis.WH    | 0.590 |
| 3    | Total.Flow | 0.990  | City.Tier | 0.101 | Region    | 0.666  | Region    | 0.516 |
| 4    | Pop.2018  | 0.883  | Total.Flow | 0.090 | Dis.WH    | 0.610  | Pop.2018  | 0.443 |
| 5    | Enter.Date | 0.048  | Pop.2018  | 0.090 | City.Tier | 0.541  | Temperature | 0.319 |
| 6    | PGRP   | 0.017  | 3A.Hospital | 0.055 | Temperature | 0.487  | City.Tier | 0.281 |
| 7    | Temperature | 0.005  | Arr.Time | 0.033 | 3A.Hospital | 0.382  | PGRP      | 0.171 |
| 8    | Railway.Resp | 9e-4   | PGRP   | 0.024  | PGRP     | 0.280  | 3A.Hospital | 0.164 |
| 9    | Region  | 5e-9   | Temperature | 0.023 | Arr.Time | 0.164  | Travel.Intensity | 0.154 |
| 10   | Bus.Resp | 4e-10  | Travel.Intensity | 0.017 | Travel.Intensity | 0.078  | Arr.Time | 0.127 |
the cumulative number of confirmed cases and temperature, whereas RFI1 and RFI2 presented a different view. Also, the top 5 variables for SOIL are all included in gMCP†, gSCAD†, and gLASSO†. In addition, except for those produced by SOIL, the importance values of the other methods are not very discriminative, making it difficult to compare the importance of variables. Overall, SOIL seems to be more informative. To save space, the analysis results without interactions are provided in Section S.2 of the Supplementary Material, including coefficient estimations, information criterion values, prediction accuracies, instability measures and regression diagnostics.

The results of diagnostic analyses for no interaction models (see Table S19 in the Supplementary Material) show that the residuals exhibit autocorrelation, and there may be interactions between some predictors. Thus, in addition to the 16 explanatory variables in Table 3, we add the interactions between top 8 variables according to SOIL importances in Table 4 whose importance values are greater than the threshold value \(c =1e^{-4}\). From now on, the set of variables for constructing candidate models is given by the 16 main effects together with all possible interactions between the above 8 variables identified by SOIL, hence a total of \(16 + \binom{8}{2} = 44\) variables.

For comparison, we also apply penalized regression methods to the data with the added interactions. The selected sets of important explanatory variables with interaction from group MCP, group SCAD and group LASSO are as follows:

\[
M_{gMCP} = \{ \text{Pop.2018, Dis.WH, Total.Flow, Enter.Date, 3A.Hospital,} \\
\quad 3A.Hospital*Railway.Resp, \text{PGRP*Temperature}\}
\]

\[
M_{gSCAD} = \{ \text{Dis.WH, Total.Flow, 3A.Hospital, Region, Dis.WH*Enter.Date,} \\
\quad \text{Total.Flow*Pop.2018, Total.Flow*PGRP, 3A.Hospital*Temperature,} \\
\quad 3A.Hospital*Railway.Resp, \text{Enter.Date*Railway.Resp, PGRP*Temperature}\}
\]

\[
M_{gLASSO} = \{ \text{Pop.2018, Dis.WH, Total.Flow, 3A.Hospital, PGRP, City.Level,} \\
\quad \text{Region, Dis.WH*Enter.Date, 3A.Hospital*Temperature} \\
\quad 3A.Hospital*Railway.Resp, \text{Pop.2018*PGRP, PGRP*Temperature}\}
\]

These models are labeled as “gMCP”, “gSCAD” and “gLASSO”, respectively. It is seen that the models are quite different. Considering that the interaction terms and their main effects are possibly highly correlated, group MCP, group SCAD and group LASSO may be unstable and fail to identify some important interaction terms.
Table 5. Top 12 variables for SOIL (after including interactions between top 8 variables).

| Rank | Variable           | Importance | Rank | Variable            | Importance |
|------|--------------------|------------|------|---------------------|------------|
| 1    | Dis.WH             | 1.000      | 2    | 3A.Hospital         | 0.960      |
| 3    | Total.Flow         | 0.914      | 4    | Pop.2018            | 0.731      |
| 5    | 3A.Hospital*Railway.Resp | 0.055 | 6    | Pop.2018*PGRP       | 0.051      |
| 7    | Dis.WH*Total.Flow  | 0.048      | 8    | PGRP*Temperature    | 0.003      |
| 9    | Dis.WH*Enter.Date  | 0.001      | 10   | Enter.Date          | 0.001      |
| 11   | Bus.Resp           | 5e-5       | 12   | 3A.Hospital*Temperature | 7e-08    |

Table 6. AIC and BIC values of various regression models.

| SOIL5 | SOIL6 | SOIL7 | SOIL8 | SOIL9 | gLASSO | gSCAD | gMCP |
|-------|-------|-------|-------|-------|--------|-------|------|
| AIC   | 684.278 | 685.849 | 674.953 | 671.371 | 672.426 | 655.116 | 674.940 |
| BIC   | 710.110 | 715.372 | 717.714 | 711.856 | 711.965 | 757.305 | 725.232 | 708.153 |

With VIBIM, the top 12 variables in SOIL importance are presented in Table 5. It can be seen that the top 4 variables have the same rank and their importance values are close to those in Table 4. The main difference between SOIL importance values in Table 4 and Table 5 is the addition of 6 interactions: 3A.Hospital*Railway.Resp, Pop.2018*PGRP, Dis.WH*Total.Flow, PGRP*Temperature, Dis.WH*Enter.Date, 3A.Hospital*Temperature. It is also of interest to note that 3A.Hospital*Railway.Resp is the most important interaction although its main effect Railway.Resp is less important.

We narrow down the sequence of nested models by information criteria, standard AIC and BIC. Denote by SOIL\(_i\) (i = 5, \ldots, 9) the resultant linear models based on the first \(i\) variables of \{Dis.WH, 3A.Hospital, Total.Flow, Pop.2018, 3A.Hospital*Railway.Resp, Pop.2018*PGRP, Dis.WH*Total.Flow, PGRP*Temperature, Dis.WH*Enter.Date\}, then the minimizers of BIC and AIC are SOIL5 and SOIL9, respectively. Thus, for VIBIM, the most plausible models are \{SOIL5, \ldots, SOIL9\}. Table 6 summarizes AIC and BIC values of various regression models. We can see that no model can be optimal in aspects of both AIC and BIC. The overall performances of SOIL8, SOIL9 and gMCP seem comparable in terms of the two information criteria, although significant differences will be seen later.
5.2. Model Estimation and Results

The corresponding coefficient estimates of above models are reported in Table 7. As can be seen, Dis.WH, 3A.Hospital, Pop.2018 and Total.Flow are highly significant and the signs of the estimated coefficients in accordance with our prior expectations. In addition, the coefficients of 5 interactions in SOIL9 are significant at the significance level 0.05 except Pop.2018*PGRP. We know from the fact that the coefficient on Dis.WH*Enter.Date is negative, i.e., Enter.Date has a negative effect (note that Dis.WH takes only positive values), which provides supporting evidence that the longer timing of closing entertainment venues before the date of the first confirmed case, the fewer number of confirmed cases. It is notable that the marginal effect of 3A.Hospital on the response variable varies with the presence or absence of Railway.Resp. For instance, in SOIL9, when Railway.Resp = 0, the effect of 3A.Hospital is 0.032; when Railway.Resp = 1, the effect of 3A.Hospital is 0.032 + 0.141 = 0.173. Perhaps this is because grade-III level-A hospitals in cities where the railway was banned had higher daily nucleic acid testing capability.

For SOILi (i = 5, . . . , 9), any two nested models of them coincide, that is, the signs of estimated coefficients of the smaller model are exactly the same as those in the larger model, with only one exception of Pop.2018*PGRP. This is reasonable because Pop.2018*PGRP is insignificant in SOILi (i = 6, . . . , 9) and therefore its sign may be unstable in different nested models. Therefore, our procedure offers reliable insights of the relationship between Sevendays.Cucase and important predictors based on multiple models. In contrast, group LASSO and group SCAD select too many variables that are not significant. Also, group MCP possibly excludes at least one of the relevant variables, such as Dis.WH*Total.Flow. To verify the usefulness of this term, we fit the model by adding Dis.WH*Total.Flow to gMCP. The result shows that Dis.WH*Total.Flow is highly significant (p-value is 0.008), suggesting that group MCP may be overly parsimonious.
Table 7. Coefficients ($p$-values) of the variables of various regression models, and “–” means that the variable does not exist in models.

|                         | SOIL5       | SOIL6       | SOIL7       | SOIL8       | SOIL9       | gLASSO      | gSCAD       | gMCP       |
|-------------------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|------------|
| Dis.WH                  | -2.259(2e-16) | -2.245(2e-16) | -2.592(2e-16) | -2.573(2e-16) | -2.575(2e-16) | -1.569(8e-9) | 1.535(7e-9) | -2.213(2e-16) |
| 3A.Hospital             | 0.032(2e-6)  | 0.028(0.002) | 0.029(0.002) | 0.032(3e-4)  | 0.032(4e-4)  | 0.038(0.001) | 0.034(8e-8) | 0.029(1e-5)  |
| Total.Flow              | 0.066(5e-6)  | 0.066(5e-6)  | 0.299(0.022) | 0.428(0.001) | 0.411(0.002) | 0.388(0.024) | 0.143(3e-7) | 0.058(5e-5)  |
| Pop.2018                | 0.251(8e-5)  | 0.219(0.007) | 0.232(0.004) | 0.329(1e-4)  | 0.319(2e-4)  | 0.314(0.031) | –           | 0.270(2e-5)  |
| 3A.Hospital*Railway.Resp| 0.135(0.014) | 0.138(0.013) | 0.117(0.037) | 0.1244(0.025)| 0.141(0.011) | 0.134(0.018) | 0.189(0.001)| 0.170(0.002)|
| Pop.2018*PGRP           | –           | 0.008(0.518) | 0.009(0.442) | -0.003(0.835) | -0.002(0.864) | 0.025(0.299) | –           | –          |
| Dis.WH*Total.Flow       | –           | –           | -0.086(0.071)| -0.137(0.006)| -0.131(0.008) | –           | –           | –          |
| PGRP*Temperature        | –           | –           | –           | 0.003(0.001) | 0.003(7e-4)  | 0.001(0.275) | 0.001(0.001)| 0.002(0.006)|
| Dis.WH*Enter.Date       | –           | –           | –           | –           | -0.039(0.020)| -0.046(0.009)| -0.046(0.016)| –          |
| Enter.Date              | –           | –           | –           | –           | –           | –           | –           | -0.119(0.016)|
| PGRP                    | –           | –           | –           | –           | –           | 0.032(0.309) | –           | –          |
| City.Tier1              | –           | –           | –           | –           | –           | -0.001(0.914) | –           | –          |
| City.Tier2              | –           | –           | –           | –           | –           | 0.346(0.392) | –           | –          |
| City.Tier3              | –           | –           | –           | –           | –           | 0.233(0.406) | –           | –          |
| City.Tier4              | –           | –           | –           | –           | –           | 0.267(0.113) | –           | –          |
| City.Tier5              | –           | –           | –           | –           | –           | 0.251(0.055) | –           | –          |
| Region.North            | –           | –           | –           | –           | –           | -0.167(0.424) | 0.138(0.493) | –          |
| Region.East             | –           | –           | –           | –           | –           | 0.097(0.702) | 0.110(0.653) | –          |
| Region.South            | –           | –           | –           | –           | –           | -0.030(0.909) | 0.032(0.897) | –          |
| Region.Central          | –           | –           | –           | –           | –           | 0.723(0.008) | 0.724(0.006) | –          |
| Region.Northwest        | –           | –           | –           | –           | –           | -0.048(0.818) | 0.027(0.895) | –          |
| Region.Southwest        | –           | –           | –           | –           | –           | 0.012(0.956) | 0.053(0.796) | –          |
| 3A.Hospital*Temperature | –           | –           | –           | –           | –           | 9e-4(0.252)  | 9e-4(0.193)  | –          |
| Bus.Resp                | –           | –           | –           | –           | –           | 0.382(0.085) | –           | –          |
| Total.Flow*Pop.2018     | –           | –           | –           | –           | –           | -0.056(4e-7) | –           | –          |
| Total.Flow*PGRP         | –           | –           | –           | –           | –           | -0.004(0.152) | –           | –          |
| Enter.Date*Railway.Resp | –           | –           | –           | –           | –           | -0.231(0.044) | –           | –          |
Table 8. Prediction errors of various regression models.

|           | Poisson | SOIL₅ | SOIL₆ | SOIL₇ | SOIL₈ |
|-----------|---------|-------|-------|-------|-------|
| MSE       | 6600.871| 885.373| 825.305| 754.367| 741.382|
|           | (356.499)| (24.047)| (24.664)| (19.081)| (18.896)|
|           | SOIL₇  | gLASSO| gSCAD | gMCP  |
| MSE       | 736.289 | 960.775| 900.339| 890.761†|
|           | (18.422)| (24.344)| (26.537)| (24.172†)|

Note: The standard errors are given in parenthesis.
† The results for gMCP are obtained by removing overly large outliers (greater than 10⁵) over 1000 runs.

5.3. Model Comparison

We compare the performances of various models in terms of prediction accuracy (i.e., mean square error (MSE)) using cross-validation. The COVID-19 dataset is randomly split into a training set with proportion of 4/5 and a test set with proportion of 1/5. On the training set, we select important variables and estimate the coefficients of the model by applying the least squares method. The test set is used to evaluate the prediction performance of the models built on the training set. Table 5.3 reports the average results over 1000 runs of MSE of prediction on the test set, as measured in the original scale of the response variable (i.e., for the linear regression models, the predictions are exponentiated and compared to the true non-transformed response values).

The results in Table 5.3 indicate that the proposed procedure yields more accurate predictions than group LASSO, group SCAD and group MCP. In addition, with the same number of variables, SOIL₇ yields more accurate predictions than group MCP. In contrast, the prediction accuracies of Poisson are quite poor, that is, the MSE of Poisson is one magnitude bigger than that of model SOIL₇ (i = 7, 8, 9). It is also worth pointing out that, compared to the results of no interaction models (see Table S18 in Section S.2 of the Supplementary Material), taking interaction into account significantly improves the prediction accuracy.
Table 9. PIVS and SIVS of various methods.

| Method        | PIVS $\tau = 0.1$ | PIVS $\tau = 0.2$ | SIVS $\rho = 0.05$ | SIVS $\rho = 0.1$ |
|---------------|-------------------|-------------------|-------------------|-------------------|
| SOIL$_5$      | 1.66              | 1.86              | 1.78              | 2.04              |
| SOIL$_6$      | 1.76              | 2                 | 1.92              | 2.06              |
| SOIL$_7$      | 1.58              | 1.82              | 1.8               | 1.72              |
| SOIL$_8$      | 1.2               | 1.56              | 1.86              | 2.06              |
| SOIL$_9$      | 0.64              | 1.18              | 1.52              | 1.86              |
| group LASSO   | 3.96              | 4.03              | 5.03              | 6.58              |
| group SCAD    | 8.4               | 8.8               | 9.82              | 9.93              |
| group MCP     | 3.49              | 3.11              | 3.23              | 2.69              |

Note: We use the averaged size of the symmetric difference between the originally top $i$ variables and that based on the modified version of the data to measure the instability of our procedure.

5.4. Model Stability and Diagnostic

In this subsection, we examine the stabilities of the selected models. Also, diagnostic analyses are carried out to check whether model assumptions have been violated.

We use the PIVS and SIVS measures introduced in Section 5.1 to evaluate the instability of various methods. The PIVS and SIVS results in Table 9 show that SOIL outperforms group MCP, group SCAD and group LASSO across all settings. More specifically, both PIVS and SIVS values of group MCP, group LASSO and group SCAD are typically greater than 3, 4, 8, respectively. In contrast, in nearly all of the settings we consider, our method would choose a model with no more than 2 variables different on average.

In order to verify the assumptions of the models, we perform a series of regression diagnostics for plausible candidate models: SOIL$_i$ ($i = 5, \ldots, 9$), in regard to model significance, heteroscedasticity, normality, multicollinearity and error independence. For testing these five problems, we use $F$ test, Breusch-Pagan test, Shapiro-Wilk test, Variance Inflation Factor (VIF) and Durbin-Watson test, respectively. If the VIF of a variable
Table 10. Results of diagnostic analyses for SOIL$_i$, $i = 4, \ldots, 7$.

| Property                | SOIL$_5$ | SOIL$_6$ | SOIL$_7$ | SOIL$_8$ | SOIL$_9$ |
|-------------------------|----------|----------|----------|----------|----------|
| Model significance      | ✓ (✓)    | ✓ (✓)    | ✓ (✓)    | ✓ (✓)    | ✓ (✓)    |
| Homoscedasticity        | ✓ (✓)    | ✓ (✓)    | ✓ (✓)    | ✓ (✓)    | ✓ (✓)    |
| Normality               | ✓ (✓)    | ✓ (✓)    | ✓ (✓)    | ✓ (✓)    | ✓ (✓)    |
| Error independence      | ✓ (✗)    | ✓ (✓)    | ✓ (✓)    | ✓ (✓)    | ✓ (✓)    |
| No multicollinearity    | ✓        | ✓        | ✓        | ✓        | ✗        |

Note: The significance level $\alpha = 0.05$ and the results of the case $\alpha = 0.01$ are given in parenthesis.

is greater than 4, we consider that the variable has a collinearity problem with other variables. The detailed results are reported in Table 5.4 and “✓” indicates that a specified model has the given property. It can be seen that compared to no interaction models (see Table S19 in Section S.2 of the Supplementary Material), SOIL$_i$ ($i = 5, \ldots, 9$) all have desirable properties: model significance, homoscedasticity and normality. Moreover, the results show no evident correlation between the residuals of SOIL$_8$ and SOIL$_9$ at the significance level 0.05.

Through the comparison among models from variable selection methods and the proposed procedure in different aspects, including coefficient estimations, information criterion values, prediction accuracies and instability measures, we proclaim that our procedure outperforms the group variable selection methods.

5.5. Discussion of Collinearity

As we can see from Table 5.4, there exists collinearity among some variables in SOIL$_i$ ($i = 6, \ldots, 9$). Further examination shows that the collinearity issue lies with Total.Flow and Dis.WH*Total.Flow (the Pearson correlation coefficient between them is 0.990). Nonetheless, from Table 7 we can see that the above two variables are significant for models SOIL$_8$ and SOIL$_9$ at the significance level 0.05 (the $p$-value of Dis.WH*Total.Flow in SOIL$_9$ is 0.007 after netting out the effect of others explanatory variables, see Section S.4 of the Supplementary Material for more details). In this section, we consider SOIL$_9$ as an example to show the reliability of the inclusion of Dis.WH*Total.Flow.
Fig. 1. Relationship between Total.Flow and the fitted values given different values of Dis.WH.

The marginal effect of Total.Flow on log(SevenDays.CuCase) across a substantively meaningful range of Dis.WH can be depicted graphically in Figure 1. We can clearly see that the larger value of Dis.WH, the smaller slope of the line. In other words, an increased Total.Flow yields a higher increase in the number of confirmed cases for smaller Dis.WH. In fact, cities far away from Wuhan typically have a lower population density, which may have led to a less within-population contact, thereby exhibiting a slower speed of growth in cases. Therefore, the sign of the interaction term seems to be a sensible representation of reality to some extent.

When faced with severe collinearity, the “standard” approach is to drop one of the collinear variables. However, omitting an important variable generally results in biased (and inconsistent) estimators of the coefficients (Wooldridge, 2015). For instance, when we drop Dis.WH*Total.Flow from SOIL7, we obtain SOIL6, whose estimated coefficient of Dis.WH and Total.Flow are dramatically different from those of SOIL7. In cases of high collinearity, one worries about possibly much inflated standard errors of the corresponding coefficient estimates and it is also likely to encounter that the estimators and their significances can be unstable to small changes in the data. Hence, we conduct further analysis of the credibility and stability of the results in Table 7 and significance of Total.Flow and Dis.WH*Total.Flow.
Variable importance based interaction modeling

We perform a guided simulation study to check credibility of the inclusion of $Dis.WH*Total.Flow$. To be specific, we generate the simulation data by model $\text{SOIL}_8$, i.e.,

$$
\tilde{Y} = \hat{\beta}_0 + \hat{\beta}_1 Dis.WH + \hat{\beta}_2 3A.Hospital + \hat{\beta}_3 Total.Flow + \hat{\beta}_4 Dis.WH * Total.Flow
+ \hat{\beta}_5 Pop.2018 + \hat{\beta}_6 3A.Hospital * Railway.Resp + \hat{\beta}_7 Pop.2018 * PGRP
+ \hat{\beta}_8 PGRP * Temperature + \epsilon,
$$

where $\hat{\beta}_i$ is the estimated coefficients of model $\text{SOIL}_8$, $\epsilon$ is drawn from normal distribution $N(0, \hat{\sigma}^2)$, where $\hat{\sigma}^2$ is the estimated error variance of $\text{SOIL}_8$. Then we apply the VIBIM procedure to the simulated dataset and obtain top 8 predictors $\text{SOIL}_8^*$, the threshold value $c$ in Step 2 of VIBIM procedure is set as 1e-4. Also, we generate the simulation data by model $\text{SOIL}_{8-1}$, that is, variables in $\text{SOIL}_8$ with $Dis.WH*Total.Flow$ excluded and obtain top the 7 predictors $\text{SOIL}_8^{*\_1}$ by VIBIM on the simulated dataset for comparison. We record the number of times that $Dis.WH$, $Total.Flow$ and $Dis.WH*Total.Flow$ are all included in $\text{SOIL}_8^*$ ($\text{SOIL}_{8-1}$) and all significant at the significance level 0.05 over 1000 replications. We also apply a similar procedure to $\text{SOIL}_9$. The results are summarized in Table 11. It can be seen that when $Dis.WH*Total.Flow$ is included in the true data generating model, the VIBIM procedure can correctly identify this interaction and its parent effects around 30% of times. In contrast, if $Dis.WH*Total.Flow$ is excluded from the true data generating model, VIBIM rarely produces models such that $Dis.WH$, $Total.Flow$ and $Dis.WH*Total.Flow$ are all included and all significant. The percentage difference provides a support to the VIBIM finding on the interaction term.

We have also performed an instability analysis on the issue (see Section S.3 of the Supplementary Material). The results show that the instabilities of the significance of $Dis.WH*Total.Flow$ and $Total.Flow$ are really low. Also, it is instructive to point out that the partial correlation between $Dis.WH*Total.Flow$ and $Total.Flow$ in $\text{SOIL}_9$ is $-0.638$. Thus, there is a strong reason to proclaim that the model $\text{SOIL}_9$ with interaction term is a stable and accurate description of the relationship between
log(Sevendays.Cucase) and relevant predictors.

6. Conclusion

In this article, we consider the problem of learning pairwise interactions in a linear regression model with both continuous and categorical predictors. To overcome the weaknesses of existing approaches, we proposed a new interaction selection procedure (VIBIM) that delivers multiple strong candidate models with high stability and interpretability. Our numerical results suggest that VIBIM gives superior performance for high-dimensional data. The VIBIM procedure is applied to analyze the COVID-19 data. In light that some variables related to the control measures in Tian et al. (2020) have possible causality flaws, we properly modified the questionable variables, and also added some potentially important variables to the COVID-19 dataset. We found a newly added important explanatory variable $3A.Hospital$ and 5 relatively important interaction terms $3A.Hospital*Railway.Resp, Pop.2018*PGRP, Dis.WH*Total.Flow, PGRP*Temperature, Dis.WH*Enter$. Unlike the “standard” approach of dealing with collinearity, we actually include some collinear variables, and this is supported by strong evidence. Moreover, in contrast to the coefficient estimates reported in Tian et al. (2020), the distance variable in models $SOIL_i$ ($i = 5, \ldots, 10$) are all of the expected sign. A series of analysis results demonstrate that our proposed procedure can lead to better models in terms of interpretability, stability, reliability and accurate prediction than commonly used alternative methods.

SUPPLEMENTARY MATERIAL

Text document: Supplementary Material for “Variable Importance Based Interaction Modeling with an Application on Initial Spread of COVID-19 in China”. (.pdf file)

R source code for VIBIM: R source code to perform analysis described in the article. (.zip file)

COVID-19 dataset: Dataset used in the illustration of the VIBIM method in Sections 4-5. (.csv file)
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