Optimal Renormalization-Group Improvement of $R(s)$ via the Method of Characteristics

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Abstract

We discuss the application of the method of characteristics to the renormalization-group equation for the perturbative QCD series within the electron-positron annihilation cross-section. We demonstrate how one such renormalization-group improvement of this series is equivalent to a closed-form summation of the first four towers of renormalization-group accessible logarithms to all orders of perturbation theory.

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The renormalization group (RG) has played a central role in our understanding of quantum field theory [1-10] especially since the discovery of asymptotic freedom [11-14]. The central idea of the renormalization group is the insensitivity of physical quantities to the mass scale $\mu^2$ introduced in the course of regularizing and eliminating infinities within perturbative calculations. Explicit dependence of a perturbative series on $\mu^2$ is compensated by $\mu^2$ dependence in masses and coupling constants characterising that series. Indeed, the replacement of such quantities by running quantities that are explicitly functions of $\mu^2$ constitutes what is generally denoted by “RG-improvement” of a perturbative expression [15]. The numerical value of a calculation to a given order of perturbation theory still depends upon the numerical value of $\mu^2$, entailing the introduction of either prescriptions (e.g., $m_b/2 \leq \mu \leq 2m_b$ for semileptonic $b$-decays) or procedures [16, 17] to obtain optimal values of $\mu^2$.

However, such substitutions do not in themselves take full advantage of all information accessible from the renormalization-group equation (RGE), which also determines portions of the perturbative series beyond the order of perturbation theory to which calculations have been explicitly performed. Application of the RGE to one-loop expressions has long been known to determine the leading logarithm contribution to each subsequent order of perturbation theory. The RGE can similarly be used in conjunction with two-loop calculations to determine next-to-leading logarithm contributions to all subsequent orders in perturbation theory – indeed the application of the RGE to an $n^{th}$ loop perturbative
expression is sufficient to determine the contribution of \( n \) successively-subleading logarithms to all orders in the perturbative expansion parameter.\(^1\) Such RGE methods for obtaining and summing “RG-accessible” logarithms to all orders of perturbation theory have been applied to effective potentials [19] and actions [20, 21], QCD correlation functions [22, 23], QCD contributions to decay rates [22], and even the high-energy behaviour of the \( WW \rightarrow ZZ \) cross-section [22], a process dominated by Higgs boson exchanges. A related RG-summation of dimensionality poles in the expansion of the bare coupling constant in terms of its renormalized analog has been developed in ref. [24] and (for thermal field theory) in ref. [25].

The point we wish to emphasize is that the summation of higher order logarithmic contributions is quite distinct (and a substantial improvement over) what is usually understood to be RG-improvement, the incorporation of running masses and coupling constants into perturbative expressions taken to a given order. Indeed, such inclusion of all RG-accessible logarithms within perturbative series is forcefully advocated in ref. [26]. Series which incorporate summation of RG-accessible logarithmic contributions to all orders of perturbation theory have been seen to exhibit much less dependence on \( \mu^2 \) than series which utilize running masses and coupling constants to a fixed calculational order [22, 23]. This latter approach, however, devolves from the method of characteristics [27], a standard approach to first-order partial differential equa-

\(^1\)The possibility that infrared effects might alter this is addressed in ref. [18]
tions such as the RGE [28]. We demonstrate below how this same method of characteristics can be extended to obtain summations of leading and three successively-subleading towers of logarithms to all orders of the perturbative QCD series for the electron-positron annihilation cross-section.

The total cross-section for $e^+e^-$-annihilation, $R(s) \equiv \sigma(e^+e^- \rightarrow \text{hadrons})/\sigma(e^+e^- \rightarrow \mu^+\mu^-)$ can be extracted from the imaginary part of the QCD vector-current correlation function [29], a perturbative expression that necessarily depends upon a renormalization mass scale $\mu$:

$$R(s) = 3 \sum_f Q_f^2 S \left[ x(\mu^2), \log(\mu^2/s) \right].$$

The expansion parameter $x(\mu^2) \equiv \alpha_s(\mu^2)/\pi$ is proportional to the running QCD coupling constant, and renormalization mass scale $\mu$ is a by-product of the regularization procedure for indentifying and excising infinities from the underlying correlation function, as discussed above. Since $R(s)$ cannot depend on this unphysical scale parameter, it follows that

$$\mu^2 \frac{dR(s)}{d\mu^2} = 0 = \left( \mu^2 \frac{\partial}{\partial \mu^2} + \beta(x) \frac{\partial}{\partial x} \right) R(s)$$

where

$$\beta(x(\mu^2)) \equiv \mu x \left( \frac{dx}{d\mu^2} \right)$$

with an appropriately chosen boundary condition for (3) [e.g. $x(\Lambda^2) = \infty$ or $x(M_z^2) = 0.118/\pi$]. Perturbative series expansions of $S[x(\mu^2), \log \left( \frac{\mu^2}{s} \right)]$ and $\beta(x)$ are seen to take the form
\[ S[x, L] = 1 + \sum_{n=1}^{\infty} \sum_{m=0}^{n-1} T_{n,m} x^n L^m \]  

(4a)

\[ \beta(x) = -x^2 \sum_{k=0}^{\infty} \beta_k x^k \]  

(4b)

where \( x = x(\mu^2) \) and \( L \equiv \log(\mu^2/s) \). Generally \( \beta(x) \) is determined by relating the bare coupling to the renormalized coupling \([7, 9, 24]\), although it can also be extracted directly from eq. (2) \([20, 21, 30]\). Indeed, explicit Feynman diagrammatic calculations to four-loop order have determined \( \beta_0, \beta_1, \beta_2, \beta_3 \) \([13, 14, 31]\) as well as \( T_{1,0}, T_{2,0}, T_{3,2}, T_{3,1} \) and \( T_{3,0} \) \([29]\), and these results (as tabulated in Table I of ref. [23]) are manifestly consistent with eq. (2).

However, it is possible to utilize eq. (2) to extract higher-order coefficients within \( S[x, L] \) than those tabulated in Table 1. It is easily seen \([23]\) that \( T_{1,0} \) and \( \beta_0 \) determine all leading logarithm coefficients \( T_{n,n-1} \) for \( n > 1 \); similarly additional knowledge of \( T_{2,0} \) and \( \beta_1 \) is sufficient to determine all next-to-leading logarithm coefficients \( T_{n,n-2} \) for \( n > 2 \); knowledge of \( T_{3,0} \) and \( \beta_2 \) permits determination of \( T_{n,n-3} \) for \( n > 3 \), and so forth. In ref. [23], the double summation in eq. (4a) is reorganised into the form

\[ S[x, L] = 1 + \sum_{n=1}^{\infty} x^n S_n(xL) \]  

(5)

where the functions

\[ S_n(u) = \sum_{k=0}^{\infty} T_{n+k,k} u^k \]  

(6)

are completely determined by knowledge of the “RG-accessible” coefficients.
One can show that eq. (2) gives rise to a nested set of first order
differential equations for the functions $S_n(u)$:

$$
\frac{dS_k}{du} - \frac{k\beta_0}{1 - \beta_0 u} S_k = \left(1 - \delta_{k,1}\right) \frac{1}{1 - \beta_0 u} \sum_{\ell=1}^{k-1} \beta_{\ell} \left(u \frac{d}{du} + k - \ell \right) S_{k-\ell}(u), \quad S_n(0) = T_n, 0.
$$

(7)

These equations are derived and sequentially solved in ref. [23]. When one
applies this “RG-summation” to the series (5) within $R(s)$, the dependence of
$R(s)$ on $\mu^2$ is substantially reduced [22, 23]. This is not surprising, as the exact
result for $R(s)$ is necessarily independent of $\mu^2$ [the RGE is just a statement
of this independence], and the inclusion of higher-order logarithm contributions
to $R(s)$ via (5) is expected to approximate the exact result more closely than
truncation of eq. (4) to a given order.

As discussed above, the method of characteristics [27,28] provides a comple-
mentary procedure for obtaining information from the renormalization group
equation (2). To illustrate this method, consider the first-order partial differen-
tial equation

$$
\left[ f(x, y) \frac{\partial}{\partial x} + g(x, y) \frac{\partial}{\partial y} \right] A(x, y) = 0
$$

(8)

where $f$ and $g$ are given functions, and where $A(x, y)$ may be indentified as some
field-theoretical amplitude characterised by quantities (e.g, coupling constants)
x and y. If $A_0(x, y)$ is a solution to eq. (8), then so is $A_0(\bar{x}(t), \bar{y}(t))$, provided
that

$$
\frac{d\bar{x}}{dt} = f(\bar{x}(t), \bar{y}(t)) \quad \frac{d\bar{y}}{dt} = g(\bar{x}(t), \bar{y}(t))
$$

(9)
with initial conditions $\bar{x}(0) = x$, $\bar{y}(0) = y$. One then sees from eqs. (9) that

$$0 = \left[ f(\bar{x}, \bar{y}) \frac{\partial}{\partial \bar{x}} + g(\bar{x}, \bar{y}) \frac{\partial}{\partial \bar{y}} \right] A_0(\bar{x}, \bar{y}) = \frac{d}{dt} A_0(\bar{x}(t), \bar{y}(t))$$  \hspace{1cm} (10)$$

The initial conditions ensure that $A_0(\bar{x}(t), \bar{y}(t))$ is a solution of eq. (8) when $t = 0$. Since eq. (10) implies that $A_0(\bar{x}(t), \bar{y}(t))$ is independent of $t$, $A_0(\bar{x}(t), \bar{y}(t))$ is necessarily a solution to eq. (8) for all values of $t$. Eqs. (9) and (10) provide the justification for replacing the variables $x$ and $y$ with their corresponding characteristic functions $\bar{x}, \bar{y}$ in the amplitude $A(x, y)$.

For the RGE (2), as applied to the field theoretical series $S[x, \log(\mu^2/s)]$, the role of $f$ and $g$ as dependent variables is assumed by $\mu^2$ and $\beta$, in which case correspondence to eqs. (9) requires running values for these functions

$$\frac{d\tilde{\mu}^2(t)}{dt} = \tilde{\mu}^2(t),$$  \hspace{1cm} (11)$$

$$\frac{d\tilde{x}(t)}{dt} = \beta(\tilde{x}(t)).$$  \hspace{1cm} (12)$$

The usual prescription for RG improvement is to identify $t$ with $\log(\mu^2)$ \cite[\text{i.e.}, \(\tilde{\mu}^2 = e^t = \mu^2\)]{eq:11}, in which case eq. (12) becomes eq. (3). Indeed, this construction provides the justification for having the coupling constant $x$ run with $\mu^2$ within the perturbative series (4a) \cite{27}.

However, it is entirely valid to let $t$ be arbitrarily chosen in eqs. (11) and (12), up to initial conditions $\tilde{\mu}^2(0) = \mu^2$, $\tilde{x}(0) = x(\mu^2)$ that establish contact with a known solution to eq. (2). Thus the “running coupling” $x(\mu^2)$ may be employed to serve as an initial condition for the characteristic function $\bar{x}(t)$.

\footnote{The dimensional regularization equation relating the bare ($g_B$) and renormalized ($g$) cou-}
In order to keep track of the order of perturbation theory to which we are working, we follow the approach of ref. [32] by introducing an expansion parameter $\bar{h}$ such that

\begin{align}
t & \rightarrow t/\bar{h} \\
\bar{x} & \rightarrow \bar{x}\bar{h}
\end{align}

so that the characteristic equations (9) become

\begin{align}
\bar{h} \frac{d\bar{\mu}^2(t)}{dt} & = \bar{\mu}^2(t) \\
\bar{h}^2 \frac{d\bar{x}(t)}{dt} & = -\bar{x}^2 \bar{h}^2 \sum_{n=0}^{\infty} \bar{x}^n \bar{h}^n \beta_n
\end{align}

and the series expansion (4a) becomes

\begin{equation}
S[\bar{x}, \bar{L}] = 1 + \sum_{n=1}^{\infty} \sum_{m=0}^{n-1} T_{n,m} \bar{h}^n \bar{x}^m \bar{L}^m
\end{equation}

[\bar{L} \equiv \log (\bar{\mu}^2(t)/s)]. From eq. (14) and the $\bar{\mu}(0) = \mu$ initial condition, we see that

\begin{equation}
\bar{\mu}^2(t) = \mu^2 e^{t/\bar{h}}.
\end{equation}

We now express $\bar{x}(t)$ as a perturbative expansion

\begin{equation}
\bar{x}(t) = \sum_{n=0}^{\infty} \bar{x}_n(t) \bar{h}^n
\end{equation}

pling constants is $g_B = \mu^s \sum_{k=0}^{\infty} \sum_{k=0}^{\infty} a_{k,\ell} g^{2k+1}/\epsilon^\ell$ [7, 9]. Since $g_B$ is a bare parameter independent of $\mu$, the renormalized coupling-constant $g$ is necessarily a $\mu$-dependent quantity, i.e., a function of $\mu$. 

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with $\bar{x}_0(0) = x(\mu^2)$ and $\bar{x}_n(0) = 0$ for $n > 0$. Upon substituting eq. (18) into eq. (15), we obtain a nested set of linear first-order differential equations for the variables $\bar{x}_n(t)$ when $n > 0$:

\[
\frac{d\bar{x}_0}{dt} = -\beta_0 \bar{x}_0^2, \quad \bar{x}_0(0) = x(\mu^2),
\]

\[
\frac{d\bar{x}_1}{dt} + (2\beta_0 \bar{x}_0(t)) \bar{x}_1 = -\beta_1 \bar{x}_0^3(t), \quad \bar{x}_1(0) = 0,
\]

\[
\frac{d\bar{x}_2}{dt} + (2\beta_0 \bar{x}_0(t)) \bar{x}_2 = -\beta_2 \bar{x}_0^4(t) - 3\beta_1 \bar{x}_0^2(t)\bar{x}_1(t) - \beta_0 \bar{x}_1^2(t), \quad \bar{x}_2(0) = 0,
\]

\[
\frac{d\bar{x}_3}{dt} + (2\beta_0 \bar{x}_0(t)) \bar{x}_3 = -\beta_3 \bar{x}_0^5(t) - 4\beta_2 \bar{x}_0^3(t)\bar{x}_1(t) - 3\beta_1 \bar{x}_0^2(t)\bar{x}_2(t) + \bar{x}_0(t)\bar{x}_1^2(t)
\]

\[
- 2\beta_0 \bar{x}_1(t)\bar{x}_2(t), \quad \bar{x}_3(0) = 0.
\]

The solution to eq. (19) is

\[
\bar{x}_0(t) = \frac{x(\mu^2)}{1 + \beta_0 x(\mu^2)t}.
\]

If we substitute eq. (23) into eq. (20), we find the solution to eq. (20) to be

\[
\bar{x}_1(t) = -\frac{\beta_1 x^2(\mu^2) \log(1 + \beta_0 x(\mu^2)t)}{\beta_0 (1 + \beta_0 x(\mu^2)t)^2}.
\]

Similarly, substitution of eq. (23) and (24) into eq. (21) leads to a solution for $\bar{x}_2(t)$

\[
\bar{x}_2(t) = \frac{x^3(\mu^2)}{(1 + \beta_0 x(\mu^2)t)^3} \left[ \left( \frac{\beta_1}{\beta_0} - \frac{\beta_2}{\beta_0} \right) \beta_0 x(\mu^2)t - \frac{\beta_2^2}{\beta_0} \log(1 + \beta_0 x(\mu^2)t) \right]
\]

\[
+ \frac{\beta_2^2}{\beta_0^2} \log^2(1 + \beta_0 x(\mu^2)t),
\]
and substitution of eqs. (23), (24) and (25) into eq. (22) leads to a solution of \( \bar{x}_3(t) \):

\[
\bar{x}_3(t) = \frac{x^4(\mu^2)}{(1 + \beta_0 x(\mu^2)t)^4} \left[ \left( -\frac{\beta_1}{\beta_0^2} + \frac{\beta_1 \beta_2}{\beta_0^3} - \frac{\beta_3}{\beta_0^2} \right) (1 + \beta_0 x(\mu^2)t)^2 
+ \left( \frac{\beta_1}{\beta_0} - \frac{\beta_1 \beta_2}{\beta_0^2} \right) (1 + \beta_0 x(\mu^2)t) \right] 
+ \left( -\frac{\beta_1}{2\beta_0} + \frac{\beta_1 \beta_2}{\beta_0^2} \right) + \left( \frac{2\beta_1 \beta_2}{\beta_0^3} - \frac{3\beta_1 \beta_2}{\beta_0^3} \right) \log (1 + \beta_0 x(\mu^2)t) 
+ \frac{5\beta_1^3}{2\beta_0^2} \log^2 (1 + \beta_0 x(\mu^2)t) - \frac{\beta_1^3}{\beta_0^3} \log^3 (1 + \beta_0 x(\mu^2)t) \right].
\]

(Eqs. (23), (24) and (25) also appear in refs. [21,32].) Eqs. (23) - (26) provide an expansion (18) of the solution to (15) that is distinct from the usual perturbative expansion.

If we substitute the series (18) into the expansion (16) for \( S[\bar{x}, \bar{L}] \) we find the following solution to the renormalization group equation (2):

\[
S[\bar{x}, \bar{L}] = 1 + \hbar[T_{1,0}\bar{x}_0(t)] 
+ \hbar^2 \left[ T_{1,0}\bar{x}_1(t) + (T_{2,0} + T_{2,1}\bar{L})\bar{x}_0^2(t) \right] 
+ \hbar^3 \left[ T_{1,0}\bar{x}_2(t) + (T_{2,0} + T_{2,1}\bar{L}) (2\bar{x}_0(t)\bar{x}_1(t)) 
+ (T_{3,0} + T_{3,1}\bar{L} + T_{3,2}\bar{L}^2) \bar{x}_0^3(t) \right] 
+ \hbar^4 \left[ T_{1,0}\bar{x}_3(t) + (T_{2,0} + T_{2,1}\bar{L}) (\bar{x}_1^2(t) + 2\bar{x}_0(t)\bar{x}_2(t)) 
+ (T_{3,0} + T_{3,1}\bar{L} + T_{3,2}\bar{L}^2) (3\bar{x}_0^2(t)\bar{x}_1(t)) 
+ (T_{4,0} + T_{4,1}\bar{L} + T_{4,2}\bar{L}^2 + T_{4,3}\bar{L}^3) \bar{x}_0^4(t) \right] 
+ \ldots
\]

(27)

Now, if \( t = 0 \), we find from eq. (17) that \( \tilde{\mu}^2 = \mu^2 \), \( \bar{L} = \log(\mu^2/s) = L \). Since
\( \bar{x}_0(0) = x(\mu^2) \) and \( \bar{x}_k(0) = 0 \) for \( k \geq 1 \), we see that eq. (27) recovers the original expansion (4a) when \( \bar{h} = 1 \).

It \( t \) is a non-zero constant, the solution (27) provides a means for obtaining all coefficients \( T_{n,m} \) with \( m \neq 0 \) in terms of coefficients \( T_{k,0} \). To see this, let \( t = \bar{h} \log k \), in which case we see from eq. (17) that \( \bar{\mu}^2 = k\mu^2 \), \( \bar{L} = \log(k\mu^2/s) = L + \log k \). If we substitute \( t = \bar{h} \log k \) into eqs. (23) - (26) and note that

\[
\log \left(1 + \bar{h} \beta_0 x(\mu^2) t\right) = \bar{h} \beta_0 x(\mu^2) \log k - \frac{\bar{h}^2}{2} \beta_0^2 x^2(\mu^2) \log^2 k + \frac{\bar{h}^3}{3} \beta_0^3 x^3(\mu^2) \log^3 k + \ldots, \tag{28}
\]

we find upon further substitution into eq. (27) that

\[
S[\bar{x}, \bar{L}]|_{t=\bar{h} \log k} = 1 + \bar{h} x(\mu^2) T_{1,0} + \bar{h}^2 x^2(\mu^2) \left[(T_{2,0} - \beta_0 T_{1,0} \log k) + T_{2,1} \log(k\mu^2/s)\right] + \bar{h}^3 x^3(\mu^2) \left\{[T_{3,0} - (2T_{2,0} \beta_0 + T_{1,0} \beta_1) \log k + T_{1,0} \beta_0^2 \log^2 k]ight. \\
+ [T_{3,1} - 2\beta_0 T_{2,1} \log k] \log(k\mu^2/s) + T_{3,2} \log^2(k\mu^2/s)\right\} + \bar{h}^4 x^4(\mu^2) \left\{[T_{4,0} - (3\beta_0 T_{3,0} + 2\beta_1 T_{2,0} + \beta_2 T_{1,0}) \log k \right. \\
+ \left. \left(3\beta_0^2 T_{2,0} + \frac{5}{2} \beta_0 \beta_1 T_{1,0}\right) \log^2 k - 3\beta_1^2 T_{1,0} \log^3 k\right]ight. \\
+ [T_{4,1} - (3\beta_0 T_{3,1} + 2\beta_1 T_{2,1}) \log k + 3\beta_1^2 T_{2,1} \log^2 k] \log(k\mu^2/s) \\
+ [T_{4,2} - 3\beta_0 T_{3,2} \log k] \log^2(k\mu^2/s) + T_{4,3} \log^3(k\mu^2/s)\right\} + \mathcal{O}(\bar{h}^5 x^5(\mu^2)). \tag{29}
\]

Now if we rewrite the original series expansion (4a) with \( L = \log(\mu^2/s) = \ldots \)
\( \log(k\mu^2/s) - \log k \), we obtain

\[
S[x, L] = 1 + x(\mu^2)T_{1,0}
\]

\[
+ x^2(\mu^2) \left[ (T_{2,0} - T_{2,1} \log k) + T_{2,1} \log(k\mu^2/s) \right]
\]

\[
+ x^3(\mu^2) \left[ (T_{3,0} - T_{3,1} \log k + T_{3,2} \log^2 k) \right.
\]

\[
+ (T_{3,1} - 2T_{3,2} \log k) \log(k\mu^2/s)
\]

\[
+ T_{3,2} \log^2(k\mu^2/s)]
\]

\[
+ x^4(\mu^2) \left[ (T_{4,0} - T_{4,1} \log k + T_{4,2} \log^2 k - T_{4,3} \log^3 k) \right.
\]

\[
+ (T_{4,1} - 2T_{4,2} \log k + 3T_{4,3} \log^2 k) \log(k\mu^2/s)
\]

\[
+ (T_{4,2} - 3T_{4,3} \log k) \log^2(k\mu^2/s) + T_{4,3} \log^3(k\mu^2/s)]
\]

\[
+ O(x^5(\mu^2)). \tag{30}
\]

When \( \bar{h} = 1 \), eqs. (29) and (30) must be equal, since the solution (27) to the renormalization-group equation (2) has been constructed...

1) ...to coincide with eq. (30) at \( t = 0 \),

2) ...to be independent of the choice for \( t \) via the method of characteristics [eq. (10)], and

3) ...to be given by eq. (29) when \( t = \bar{h} \log k \).

Direct comparison of eqs. (29) and (30) when \( \bar{h} = 1 \) shows that

\[
T_{2,1} = \beta_0 T_{1,0},
\]

\[
T_{3,1} = 2T_{2,0}\beta_0 + \beta_1 T_{1,0},
\]
\begin{align*}
T_{3,2} &= \beta_0^2 T_{1,0}, \\
T_{4,1} &= 3 \beta_0 T_{3,0} + 2 \beta_1 T_{2,0} + \beta_2 T_{1,0}, \\
T_{4,2} &= 3 \beta_0^2 T_{2,0} + \frac{5}{2} \beta_0 \beta_1 T_{1,0}, \\
T_{4,3} &= \beta_0^3 T_{1,0},
\end{align*}

relations that can also be obtained [22] by direct substitution of the series (4a) into the renormalization-group equation (2). Thus, the method of characteristics is seen to determine all logarithmic coefficients to the order of perturbation theory considered.

However, a more powerful application of the solution (27) occurs by setting \( t = \hbar \log(s/\mu^2) \), ensuring via eq. (17) that \( \bar{\mu}^2 = s \), that \( \bar{L} = 0 \), and that factors of \( 1 + \hbar \beta_0 x(\mu^2) t \) in eqs. (23) - (26) become \( 1 - \beta_0 x(\mu^2) \log(\mu^2/s) \equiv w \) in the \( \hbar \to 1 \) limit. In this limit, eq. (27) generates the following series:

\begin{align*}
S[\bar{x}, \bar{L}] &= 1 + x(\mu^2) \frac{T_{1,0}}{w} + x^2(\mu^2) \left[ T_{2,0} - \frac{\beta_1}{\beta_0} T_{1,0} \log w \right] w^{-2} \\
+& x^3(\mu^2) \left[ T_{3,0} - 2 T_{2,0} \frac{\beta_1}{\beta_0} \log w + \left( \frac{\beta_2}{\beta_0} - \frac{\beta_3}{\beta_0} \right) (w - 1) - \frac{\beta_2^2}{\beta_0^2} \log w + \frac{\beta_2^2}{\beta_0^2} \log^2 w \right] w^{-3} \\
+& x^4(\mu^2) \left[ \left( -\frac{\beta_1^3}{2 \beta_0^3} + \frac{\beta_1 \beta_2}{\beta_0^2} - \frac{\beta_3}{2 \beta_0} \right) w^2 + \left( \frac{\beta_3^2}{\beta_0^2} + \frac{\beta_1 \beta_2}{\beta_0^2} \right) + 2 T_{2,0} \left( \frac{\beta_1^2}{\beta_0^2} - \frac{\beta_2}{\beta_0} \right) + T_{4,0} \right] w \\
+& \left( \frac{2 \beta_3^3}{\beta_0^3} - \frac{3 \beta_1 \beta_2}{\beta_0^2} - 2 T_{2,0} \frac{\beta_2^2}{\beta_0^2} - 3 T_{3,0} \frac{\beta_1}{\beta_0} \right) \log w \\
+& \left( \frac{5 \beta_3^3}{2 \beta_0^3} + 3 T_{2,0} \frac{\beta_2^2}{\beta_0^2} \right) \log^2 w \\
-& \frac{\beta_1^3}{\beta_0^3} \log^3 w \right] w^{-4} \quad \text{(32)}
\end{align*}

This series explicitly reproduces the series (5) obtained via the successive solu-
tions to the differential equations (7). The coefficient functions $S_1(xL)$, $S_2(xL)$, $S_3(xL)$ and $S_4(xL)$, as calculated in ref. [23], are reproduced in eq. (32), demonstrating how the method-of-characteristics approach to the renormalization group equation can recover the results obtained via summation of leading, next-to-leading, and successively subleading logarithm factors to all orders of perturbation theory. Such results as eq. (32) represent the optimal possible RG-improvement of the original perturbative series (4a), insofar as they incorporate all RG-accessible coefficients of logarithms occurring within that series.

As a final note, the ambiguity in the choice of $k$ such that $t = \hbar \log k$ [thereby leading to the series of eq. (29)] is equivalent to the ambiguity noted in ref. [25]. In [25], this ambiguity was viewed as a consequence of shifting the initial condition of eq. (7) to the equation (6) defining $S_n(u)$; by replacing $\log(\mu^2/s)$ with $\log(k\mu^2/s) - \log(k)$ in the series (4a), that series becomes

$$S[x, L] = 1 + \sum_{n=1}^{\infty} \sum_{m=0}^{n-1} T'_{n,m} x^n (L')^m$$

where $L' \equiv \log(k\mu^2/s)$ and where $T'_{1,0} = T_{1,0} = 1$, $T'_{2,1} = T_{2,1}, T'_{2,0} = T_{2,0} - T_{2,1} \log(k)$, etc. The initial condition in eq. (7) is now replaced by $S_n(0) = T'_{n,0}$.

Of course, when one sums to all orders in perturbation theory, the dependence on $k$ within eq. (29) will drop out.

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References

[1] E. C. G. Stueckelberg and A. Peterman, Helv. Phys. Acta 26 (1953) 499.

[2] N. W. Bogoliubov and D. V. Shirkov, Introduction to the Theory of Quantized Fields (John Wiley, New York, 1980) Chapter 9.

[3] M. Gell-Mann and F. Low, Phys. Rev. 95 (1954) 1300.

[4] K. G. Wilson, Phys. Rev. D3 (1971) 1818.

[5] C. G. Callan, Phys. Rev. D2 (1970) 1541.

[6] K. Symanzik, Commun. Math. Phys. 18 (1970) 227.

[7] G. ’t Hooft, Nucl. Phys. B61 (1973) 455.

[8] S. Weinberg, Phys. Rev. D8 (1973) 3497.

[9] J. C. Collins and A. J. Macfarlane, Phys. Rev. D10 (1974) 1201.

[10] P. M. Stevenson, Ann. of Phys. 132 (1981) 383.

[11] V. S. Vanyashin and M. V. Terentev, JETP 21 (1965) 375.

[12] I. B. Khriplovich, Sov. J. Nucl. Phys. 10 (1970) 235.

[13] D. J. Gross and F. Wilczek, Phys. Rev. Lett. 30 (1973) 1343.

[14] H. D. Politzer, Phys. Rev. Lett. 30 (1973) 1346.
[15] S. Coleman and E. Weinberg, Phys. Rev. D 7 (1973) 1888.

[16] P. M. Stevenson, Phys. Rev. D 23 (1981) 2916.

[17] G. Grunberg, Phys. Lett. B 95 (1980) 70 and Phys. Rev. D 29 (1984) 2315.

[18] G. Dunne, H. Geis and C. Schubert, JHEP 0211 (2002) 032.

[19] B. Kastening, Phys. Lett. B 283 (1992) 287; M. Bando, T. Kugo, N. Maskawa and H. Nakano, Phys. Lett. B 301 (1993) 83; C. Ford, Phys. Rev. D 50 (1994) 7531.

[20] D. G. C. McKeon, Int. J. Mod. Phys. 37 (1998) 817.

[21] M. R. Ahmady, V. Elias, D. G. C. McKeon, A. Squires and T. G. Steele, Nucl. Phys. B (to appear) hep-ph/0211227.

[22] M. R. Ahmady et. al., Phys. Rev. D 66 (2002) 014010.

[23] M. R. Ahmady et. al., Phys. Rev. D (to appear) hep-ph/0208025.

[24] V. Elias and D. G. C. McKeon, Mod. Phys. Lett. A (to appear) hep-th/0209151.

[25] A. Rebhan and D. G. C. McKeon, Phys. Rev. D (to appear) hep-ph/0210163.

[26] C. J. Maxwell, Nucl. Phys. B (Proc. Suppl.) 86 (2000) 74.

[27] A. Peterman, Phys. Rep. 53 C (1979) 157.
[28] R. Courant and D. Hilbert, *Methods of Mathematical Physics Vol. II* (Interscience, NY, 1966) Ch. II.

[29] K. G. Chetyrkin, Phys. Lett. B 391 (1997) 402; see also S. G. Gorishny, A. L. Kataev and S. A. Larin, Phys. Lett. B 259 (1991) 144, and L. R. Surguladze and M. A. Samuel, Phys. Rev. Lett. 66 (1991) 560 and 2416 (E).

[30] L. Culumovic, M. Leblanc, R. B. Mann, D. G. C. McKeon and T. N. Sherry, Phys. Rev. D 41 (1990) 514; L. Culumovic, D. G. C. McKeon and T. N. Sherry, Ann. Phys. 197 (1990) 94.

[31] W. E. Caswell, Phys. Rev. Lett. 33 (1974) 244; D. R. T. Jones, Nucl. Phys. B 75 (1974) 531; E. S. Egorian, O. V. Tarasov, Theor. Mat. Fiz. 41 (1979) 26; O. V. Tarasov, A. A. Vladimirov and A. Yu. Zharkov, Phys. Lett. B 93 (1980) 429; S. A. Larin and J. A. M. Vermaseren, Phys. Lett. B 303 (1993) 334; T. van Ritbergen, J. A. M. Vermaseren and S. A. Larin, Phys. Lett. B 400 (1997) 379.

[32] J. M. Chung and B. K. Chung, Phys. Rev. D 60 (1999) 105001.