The 1999 Heineman Prize Address

Integrable models in statistical mechanics:
The hidden field with unsolved problems

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Abstract

In the past 30 years there have been extensive discoveries in the theory of integrable statistical mechanical models including the discovery of non-linear differential equations for Ising model correlation functions, the theory of random impurities, level crossing transitions in the chiral Potts model and the use of Rogers-Ramanujan identities to generalize our concepts of Bose/Fermi statistics. Each of these advances has led to the further discovery of major unsolved problems of great mathematical and physical interest. I will here discuss the mathematical advances, the physical insights and extraordinary lack of visibility of this field of physics.

1 Introduction

It is a great honor to have been awarded the Dannie Heineman Prize in Mathematical Physics and it is a great pleasure to be in the company today of my fellow prize winners, C.N. Yang, this year’s winner of the Lars Onsager prize, A.B. Zamolodchikov, the inventor of conformal field theory, and especially my collaborator and thesis adviser T.T. Wu.

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2 Why integrable models is the invisible field of physics

The citation of this year’s Heineman Prize is for “groundbreaking and penetrating work on classical statistical mechanics, integrable models and conformal field theory” and in this talk I plan to discuss the work in statistical mechanics and integrable models for which the award was given. But since the Heineman prize is explicitly for publications in mathematical physics I want to begin by examining what may be meant by the phrase “mathematical physics.”

The first important aspect of the term “mathematical physics” is that it means something very different from most other kinds of physics. This is seen very vividly by looking at the list of the divisions of the APS. Here you will find 1) astrophysics, 2) atomic physics, 3) condensed matter physics, 4) nuclear physics, and 5) particles and fields but you will not find any division for mathematical physics. This lack of existence of mathematical physics as a field is also reflected in the index of Physical Review Letters where mathematical physics is nowhere to be found. Therefore we see that the Heineman prize in mathematical physics is an extremely curious award because it honors achievements in a field of physics which the APS does not recognize as a field of physics.

This lack of existence of mathematical physics as a division in the APS reflects, in my opinion, a deep uneasiness about the relation of physics to mathematics. An uneasiness I have heard echoed hundreds of times in my career in the phrase “It is very nice work but it is not physics. It is mathematics”. A phrase which is usually used before the phrase “therefore we cannot hire your candidate in the physics department.”

So the first lesson to be learned is that mathematical physics is an invisible field. If you want to survive in a physics department you must call yourself something else.

So what can we call the winners of the Heineman prize in mathematical physics if we cannot call them mathematical physicists? The first winner was Murray Gell–Mann in 1959. He is surely belongs in particles and fields; the 1960 winner Aage Bohr is surely a nuclear physicist; the 1976 winner Stephen Hawking is an astrophysicist. And in fact almost all winners of the prize in mathematical physics can be classed in one of the divisions of the
APS without much confusion.

But there are at least two past winners who do not neatly fit into the established categories, Elliott Lieb, the 1978 recipient, and Rodney Baxter, the 1987 recipient, both of whom have made outstanding contributions to the study of integrable models in classical statistical mechanics—the same exact area for which Wu and I are being honored here today. This field of integrable models in statistical physics is the one field of mathematical physics which does not fit into some one of the existing divisions of the APS.

It is for this reason that I have described integrable models as a hidden field. Indeed it is so hidden that it is sometimes not even considered to be statistical mechanics as defined by the IUPAP.

The obscurity of the field of integrable statistical mechanics models explains why there are less than a dozen physics departments in the United States where it is done. This makes the job prospects of a physicist working in this field very slim. But on the other hand it means that we in the field get to keep all the good problems to ourselves. So it is with mixed feelings that I will now proceed to discuss some of the progress made in the last 33 years and the some of the directions for future research.

3 The Ising Model

The award of the 1999 Heineman prize, even though the citation says that it is for work on integrable models, is in fact for work done from 1966–1981 on a very specific system: the two dimensional Ising model. This work includes boundary critical phenomena, randomly layered systems, the Painlevé representation of the two point function, and the first explicit results on the Ising model in a magnetic field. All of these pieces of work had results which were unexpected at the time and all have lead to significant extensions of our knowledge of both statistical mechanics and mathematics.

3.1 The boundary Ising model and wetting

The Ising model is a two dimensional collection of classical “spins” $\sigma_{j,k}$ which take on the two values $+1$ and $-1$ and are located at the $j$ row and $k$ column of a square lattice. For a translationally invariant system the interaction
energy of this system is

\[ E = - \sum_{j,k} \left( E^v\sigma_{j,k}\sigma_{j+1,k} + E^h\sigma_{j,k}\sigma_{j,k+1} + H\sigma_{j,k} \right). \] (1)

This is certainly one of the two most famous and important models in statistical mechanics (the Heisenberg-Ising chain being the other) and has been studied by some of the most distinguished physicists of this century including Onsager [1] who computed the free energy for \( H = 0 \) in 1944 and Yang [2] who computed the spontaneous magnetization in 1952.

In 1966 I began my long involvement with this model when, for part of my thesis, Prof. Wu suggested that I compute for an Ising model on a half plane the same quantities which Onsager and Yang computed for the bulk. At the time we both thought that because the presence of the boundary breaks translational invariance the boundary computations would be at least as difficult as the bulk computations. It was therefore quite surprising when it turned out that the computations were drastically simpler [3]. In the first place the model could be solved in the presence of a field on the boundary which meant that the computation of the magnetization came along for free once we could do the free energy but more importantly the correlation functions, which in the bulk were given by large determinants whose size increased with the separation of the spins, were here given by nothing worse than the product of two one-dimensional integrals for all separations. The key to this great simplification is the fact that the extra complication of the boundary magnetic field actually makes the problem simpler to solve (a realization I had in a dream at 3:00 AM after a New Years eve party).

This model is the first case where boundary critical exponents were explicitly computed. Indeed it remained almost the only solved problem of a boundary critical phenomena until it was generalized to integrable massive boundary field theory in 1993 [4].

This boundary field had the added virtue that we could analytically continue the boundary magnetization into the metastable region and explicitly compute a boundary hysteresis effect [5]. This leads to the lovely effect that near the value of the boundary magnetic field where the hysteresis curve ends that the spins a very long distance from the boundary turn over from pointing in the direction of the bulk magnetization to pointing in the direction of the metastable surface spin. At the value where the metastability ends this surface effect penetrates all the way to infinity and “flips” the spin in
the bulk. In later years this phenomena has been interpreted at a “wetting transition” \[6\] and the Ising intuition has been extended to many models where exact solutions do not exist.

### 3.2 Random layered Ising model and Griffiths-McCoy singularities

Our next major project was to generalize the translationally invariant interaction (1) to a non translationally invariant problem where not just a half plane boundary was present but to a case where

1. The interaction energies \( E^v \) were allowed to vary from row to row but translational invariance in the horizontal direction was preserved

2. The interaction energies \( E^v(j) \) between the rows were chosen as independent random variables with a probability distribution \( P(E^v) \).

This was the first time that such a random impurity problem had ever been studied for a system with a phase transition and the entire computation was a new invention \[7\]. In particular we made the first use in physics of Furstenburg’s theory \[8\] of strong limit theorems for matrices. We felt that the computation was a startling success because we found that for any probability distribution, no matter how small the variance, there was a temperature scale, depending on the variance, where there was new physics that is not present in the translationally invariant model. For example the divergence in the specific heat at \( T_c \) decreases from the logarithm of Onsager to an infinitely differentiable essential singularity. Moreover the average over the distribution \( P(E^v) \) of the correlation functions of the boundary could be computed \[9\] and it was seen that there was an entire temperature range surrounding \( T_c \) where the boundary susceptibility was infinite. Thus the entire picture of critical exponents which had been invented several years before to describe critical phenomena in pure systems was not sufficient to describe these random systems \[10\]. We were very excited.

But then something happened which I found very strange. Instead of attempting to further explore the physics we had found, arguments were given as to why our effect could not possibly be relevant to real systems. This has lead to arguments which continue to this day.
We were, and are, of the opinion that the effects seen in the layered Ising model are caused by the fact that the zeroes of the partition function do not pinch the real temperature axis at just one point but rather pinch in an entire line segment. A closely related effect was simultaneously discovered by Griffiths [11] and it was, and is, our contention that in this line segment there is a new phase of the system. But this line segment is not revealed by approximate computations and for decades it was claimed that our new phase was limited to layered systems; a claim which some continue to make to this day.

However, fortunately for us, there is an alternative interpretation of the Ising model in terms of a one-dimensional quantum spin chain in which our layered classical two-dimensional system becomes a randomly impure quantum chain [12]. In this interpretation there is no way to argue away the existence of our new phase and finally in 1995, a quarter century after we first found the effect, D. Fisher [13], in an astounding paper, was able to craft a theory of the physics of rare events based on an exact renormalization group computation which not only reproduced our the results of our layered model on the boundary but extended the computations to bulk quantities which in 1969 we had been unable to compute. With this computation I think the existence of what are now called Griffiths-McCoy singularities is accepted, but it has taken a quarter century for this to happen.

3.3 Painlevé Functions and difference equations

But perhaps the most dramatic discoveries were published from 1973 to 1981 on the spin correlation functions of the Ising model and most particularly the results that in the scaling limit where $T \to T_c$ and the separation of the spins $N \to \infty$ such that $|T - T_c|N = r$ is fixed that the correlation function \[ < \sigma_{0,0} \sigma_{0,N} > \] divided by $|T - T_c|^{1/4}$ is

\[
G_{\pm}(2r) = (1 \mp \eta)^{-1/2} \exp \int_r^\infty dx x^{-2} \eta^{-2} [(1 - \eta^2)^2 - \eta^2] \tag{2}
\]

where $\eta(x)$ satisfies the third equation of Painlevé [15]

\[
\frac{d^2 \eta}{dx^2} = \frac{1}{\eta} \left( \frac{d\eta}{dx} \right)^2 - \frac{1}{x} \frac{d\eta}{dx} + \frac{1}{x} (\alpha \eta^2 + \beta) + \gamma \eta^3 + \delta \tag{3}
\]
with $\alpha = \beta = 0$ and $\gamma = -\delta = 1$ and

\[ \eta(x) \sim 1 - \frac{2}{\pi} K_0(2x) \quad \text{as} \quad x \to \infty \quad (4) \]

where $K_0(2x)$ is the modified Bessel function of order zero. Furthermore on the lattice all the correlation functions satisfy quadratic difference equations [14].

This discovery of Painlevé equations in the Ising model was the beginning of a host of developments in mathematical physics which continues in an ever expanding volume to this day. It led Sato, Miwa, and Jimbo [17] to their celebrated series of work on isomonodromic deformation and to the solution of the distribution of eigenvalue of the GUE random matrix problem [18] in terms of a Painlevé V function. This has subsequently been extended by many people, including one of our original collaborators, Craig Tracy [19], to so many branches of physics and mathematics including random matrix theory matrix models in quantum gravity and random permutations that entire semester long workshops are now devoted to the subject. Indeed a recent book on special functions [20] characterized Painlevé functions as ”the special functions of the 21st century.” Rarely has the solution to one small problem in physics had so many ramifications in so many different fields.

### 3.4 Ising model in a field

The final piece of work in the Ising model to be mentioned is what happens to the two point function when a small magnetic field is put the system for $T < T_c$. At $H = 0$ for $T < T_c$ the two point function has the important property that it couples only to states with an even number of particles and thus, in particular the leading singularity in the Fourier transform is not a single particle pole but rather a two particle cut. In 1978 [21], as an application of our explicit formulas for the $n$ spin correlation functions [22] we did a (singular) perturbation computation to see what happens when in terms of the scaled variable $h = H/|T - T_c|^{15/8}$ a small value of $h$ is applied to the system. We found that the two particle cut breaks up into an infinite number of poles which are given by the zeroes of Airy functions. These poles are exactly at the positions of the bound states of a linear potential and are immediately interpretable as a weak confinement of the particles which are free at $H = 0$. This is perhaps the earliest explicit computation where
confinement is seen. From this result it was natural to conjecture that as we take the Ising model from $T > T_c$, $H = 0$ to $T < T_c$, $H = 0$ that as $h$ increases from 0 to $\infty$ ($T = T_c$, $H > 0$) that bound states emerge from the two particle cut and that as we further proceed from $T = T_c$, $H > 0$ down to $T < T_c$, $H = 0$ that bound states continue to emerge until at $H = 0$ an infinite number of bound states have emerged and formed a two particle cut. What this picture does not indicate is the remarkable result found 10 years later by A. Zamolodchikov [23] that at $T = T_c$, $H = 0$ the problem can again be studied exactly. This totally unexpected result will be discussed by Zamolodchikov in the next presentation.

4 From Ising to integrable

Even at the time when this Ising model work was initiated there were other models known such as the Heisenberg chain [24], [25] the delta function gases [24]-[28], the 6–vertex model [29], [30], the Hubbard model [31], and the 8–vertex model [32] for which the free energy (or ground state energy), and the excitation spectrum could be computed exactly. Since then it has been realized that a fundamental equation first seen by Onsager [1], [33] in the Ising model and used in a profound way by Yang [28] in the delta function gases and by Baxter [32] in the 8 vertex model could be used extend these computations to find many large classes of models for which free energies could be computed. These models which come from the Yang-Baxter (or star triangle equation) are what are now called the integrable models.

The Ising model itself is the simplest case of such an integrable model. It thus seems to be a very natural conjecture which was made by Wu, myself and our collaborators the instant we made the discovery of the Painlevé representation of the Ising correlation function that there must be a similar representation for the correlation functions of all integrable models. To be more precise I mean the following

Conjecture

The correlation functions of integrable statistical mechanical models are characterized as the solutions of classically integrable equations (be they differential, integral or difference).

One major step in the advancement of this program was made by our next speaker, Alexander Zamolodchikov, who showed, with the invention of
conformal field theory \cite{34}, that this conjecture is realized for models at the critical point. One of the major unsolved problems of integrable models today is to extend the linear equations which characterize correlation functions in conformal field theory to nonlinear equations for massive models. This will realize the goal of generalizing to all integrable models what we have learned for the correlation functions of the Ising model. This is an immense field of undertaking in which many people have and are making major contributions. It is surely not possible to come close to surveying this work in the few minutes left to me. I will therefore confine myself to a few remarks about things I have been personally involved with since completing the work with Wu in 1981 on the Ising model.

4.1 The chiral Potts model

In 1987 my coauthors H. Au-Yang, J.H.H. Perk, C.H. Sah, S. Tang, and M.L. Yan and I discovered \cite{35} the first example of an integrable model where, in technical language, the spectral variable lies on a curve of genus higher than one. This model has \( N \geq 3 \) states per site and is known as the integrable chiral Potts model. It is a particular case of a phenomenological model introduced for case \( N = 3 \) in 1983 by Howes, Kadanoff and den Nijs \cite{36} in their famous study of level crossing transitions and is a generalization of the \( N \) state model introduced by Von Gehlen and Rittenberg in 1985 \cite{37} which generalizes Onsager’s original solution of the Ising model \cite{1},\cite{38}.

The Boltzmann weights were subsequently shown by Baxter, Perk and Au-Yang \cite{39} to have the following elegant form for \( 0 \leq n \leq N - 1 \)

\[
\frac{W_{p,q}^{h}(n)}{W_{p,q}^{h}(0)} = \prod_{j=1}^{n} \left( \frac{d_{p}b_{q} - a_{p}c_{q}\omega^{j}}{b_{p}d_{q} - c_{p}a_{q}\omega^{j}} \right), \quad \frac{W_{p,q}^{v}(n)}{W_{p,q}^{v}(0)} = \prod_{j=1}^{n} \left( \frac{\omega a_{p}d_{q} - d_{p}a_{q}\omega^{j}}{c_{p}b_{q} - b_{p}c_{q}\omega^{j}} \right)
\]

\( (5) \)

where \( \omega = e^{2\pi i/N} \). The variables \( a_{p}, b_{p}, c_{p}, d_{p} \) and \( a_{q}, b_{q}, c_{q}, d_{q} \) satisfy the equations

\[
a^{N} + kb^{N} = k'd'^{N}, \quad ka^{N} + b^{N} = k'c'^{N}
\]

\( (6) \)

with \( k^{2} + k'^{2} = 1 \) and this specifies a curve of genus \( N^{3} - 2N^{2} + 1 \). When \( N = 2 \) the Boltzmann weights reduce to those of the Ising model \( (4) \) with \( H = 0 \) and the curve \( (3) \) reduces to the elliptic curve of genus 1. However when \( N \geq 3 \) the curve has genus higher than one. This is the first time that
such higher genus curves has arisen in the Boltzmann weights of integrable models.

This model is out of the class of all previously known models and raises a host of unsolved questions which are related to some of the most intractable problems of algebraic geometry which have been with us for 150 years. As an example of these new occurrences of ancient problems we can consider the spectrum of the transfer matrix

$$ T_{(l,l')} = \prod_{j=1}^{N} W_{p,q}^v(l_j - l'_j) W_{p,q}^h(l_j - l'_{j+1}). \quad (7) $$

This transfer matrix satisfies the commutation relation

$$ [T(p,q), T(p', q')] = 0 \quad (8) $$

and also satisfies functional equations on the Riemann surface at points connected by the automorphism $R(a_q, b_q, c_q, d_q) = (b_q, \omega_q, d_q, c_q)$. For the Ising case $N = 2$ this functional equation reduces to an equation which can be solved using elliptic theta functions. Most unhappily, however, for the higher genus case the analogous solution requires machinery from algebraic geometry which does not exist. For the problem of the free energy Baxter [40] has devised an ingenious method of solution which bypasses algebraic geometry completely but even here some problems remain in extending the method to the complete eigenvalue spectrum [41].

The problem is even more acute for the order parameter of the model. For the $N$ state models there are several order parameters parameterized by an integer index $n$ where $1 \leq n \leq N - 1$. For these order parameters $M_n$ we conjectured [42] 10 years ago from perturbation theory computations that

$$ M_n = (1 - k^2)^{n(N-1)/2N^2}. \quad (9) $$

When $N = 2$ this is exactly the result announced by Onsager [13] in 1948 and proven by Yang [4] for the Ising model in 1952. For the Ising model it took only three years to go from conjecture to proof. But for the chiral Potts model a decade has passed and even though Baxter [44]–[45] has produced several elegant formulations of the problem which all lead to the correct answer for the Ising case none of them contains enough information to solve the problem for $N \geq 3$. In one approach [44] the problem is reduced the the evaluation of
a path ordered exponential of noncommuting variables on a Riemann surface. This sounds exactly like problems encountered in non Abelian gauge theory but, unfortunately, there is nothing in the field theory literature that helps. In another approach [45] a major step in the solution involves the explicit reconstruction of a meromorphic function from a knowledge of its zeros and poles. This is a classic problem in algebraic geometry for which in fact no explicit answer is known either. Indeed the unsolved problems arising from the chiral Potts model are so resistant to all known mathematics that I have reduced my frustration to the following epigram;

_The nineteenth century saw many brilliant creations of the human mind. Among them are algebraic geometry and Marxism. In the late twentieth century Marxism has been shown to be incapable of solving any practical problem but we still do not know about algebraic geometry._

It must be stressed again that the chiral Potts model was not invented because it was integrable but was found to be integrable after it was introduced to explain experimental data. In a very profound way physics is here far ahead of mathematics.

### 4.2 Exclusion statistics and Rogers-Ramanujan identities

One particularly important property of integrable systems is seen in the spectrum of excitations above the ground state. In all known cases these spectra are of the quasiparticle form in which the energies of multiparticle states are additively composed of single particle energies $e_\alpha(P)$

$$E_{ex} - E_0 = \sum_{\alpha=1}^{n} \sum_{j=1}^{m_\alpha} e_\alpha(P_j^\alpha)$$

with the total momentum

$$P = \sum_{\alpha} \sum_{j=1}^{m_\alpha} P_j^\alpha \pmod{2\pi}.$$

Here $n$ is the number of types of quasi-particles and there are $m_\alpha$ quasiparticles of type $\alpha$. The momenta in the allowed states are quantized in units of $2\pi/M$ and are chosen from the sets

$$P_j^\alpha \in \{P_{\min}^\alpha(m), P_{\max}^\alpha(m) + \frac{2\pi}{M}, P_{\min}^\alpha(m) + \frac{4\pi}{M}, \ldots, P_{\max}^\alpha(m)\}$$
with the Fermi exclusion rule
\[ P_j^\alpha \neq P_k^\alpha \text{ for } j \neq k \text{ and all } \alpha \] (13)

and
\[ P_{\text{min}}^\alpha (m) = \frac{\pi}{M} [ (m(B-1))_\alpha - A_\alpha + 1 ] \text{ and } P_{\text{max}}^\alpha = -P_{\text{min}}^\alpha + \frac{2\pi}{M} \left( \frac{u}{2} - A_\alpha \right) \] (14)

where if some \( u_\alpha = \infty \) the corresponding \( P_{\text{max}}^\alpha = \infty \).

If some \( e_\alpha(P) \) vanishes at some momentum (say 0) the system is massless and for \( P \sim 0 \) a typical behavior is \( e_\alpha = v|P| \) where \( v \) is variously called the speed of light or sound or the spin wave velocity.

The important feature of the momentum selection rules (12) is that in addition to the fermionic exclusion rule (13) is the exclusion of a certain number of momenta at the edge of the momentum zones which is proportional to the number of quasiparticles in the state. For the Ising model at zero field there is only one quasiparticle and \( P_{\text{min}} = 0 \) so the quasiparticle is exactly the same as a free fermion. However, for all other cases the \( P_{\text{min}} \) is not zero and exclusion does indeed take place. This is a very explicit characterization of the generalization which general integrable models make over the Ising model.

The exclusion rules (14) lead to what have been called fractional (or exclusion) statistics by Haldane [46]. On the other hand they make a remarkable and beautiful connection with the mathematical theory of Rogers-Ramanujan identities and conformal field theory. We have found that these exclusion rules allow a reinterpretation of all conformal field theories (which are usually discussed in terms of a bosonic Fock space using a Feigin and Fuchs construction [47]) in terms of a set of fermionic quasiparticles [48]. What is most surprising is that there is not just one fermionic representation for each conformal field theory but there are at least as many distinct fermionic representations as there are integrable perturbations. The search for the complete set of fermionic representations is ongoing and I will only mention here that we have extensive results for the integrable perturbations of the minimal models \( M(p,p') \) for the \( \phi_{1,3} \) perturbation [49] and the \( \phi_{2,1} \) and \( \phi_{1,5} \) perturbations [50].
5 Beyond Integrability

There is one final problem of the hidden field of integrable models which I want to discuss. Namely the question of what is the relation of an integrable model to a generic physical system which does not satisfy a Yang-Baxter equation. For much of my career I have been told by many that these models are just mathematical curiosities which because they are integrable can, by that very fact, have nothing to do with real physics. But on the other hand the fact remains that all of the phenomenological insight we have into real physics phase transitions as embodied in the notions of critical exponents, scaling theory and universality which have served us well for 35 years either all come from integrable models or are all confirmed by the the solutions of integrable models. So if integrable models leave something out we have a very poor idea of what it is.

Therefore it is greatly interesting that several months ago Bernie Nickel [51] sent around a preprint in which he made the most serious advance in the study of the Ising model susceptibility since our 1976 paper [14]. In that paper in addition to the Painlevé representation of the two point function we derive an infinite series for the Ising model susceptibility where the \( n \)th term in the series involves an \( n \)th order integral.

When the integrals in this expansion are scaled to the critical point each term contributes to the leading singularity of the susceptibility \( |T - T_c|^{-7/4} \) However Nickel goes far beyond this scaling and for the isotropic case where \( E^v = E^h = E \) in term of the variable \( v = \sinh 2E/kT \) he shows that successive terms in the series contribute singularities that eventually become dense on the unit circle in the complex plane \( |v| = 1 \). From this he concludes that unless unexpected cancelations happen that there will be natural boundaries in the susceptibility on \( |v| = 1 \). This would indeed be a new effect which could make integrable models different from generic models, Such natural boundaries have been suggested by several authors in the past including Guttmann [52], and Orrick and myself [53] on the basis of perturbation studies of nonintegrable models which show ever increasingly complicated singularity structures as the order of perturbation increases; a complexity which magically disappears when an integrability condition is imposed. This connection between integrability and analyticity was first emphasized by Baxter [54] long ago in 1980 when he emphasized that the Ising model in a magnetic field satisfies a functional equation very analogous to the zero field Ising model.
but that the Ising model in a field lacks the analyticity properties need for a solution. The proof of Nickel’s conjecture will, if correct, open up a new view on what it means to be integrable.

6 Conclusion

I hope that I have conveyed to you some of the excitement and challenges of the field of integrable models in statistical mechanics. The problems are physically important, experimentally accessible, and mathematically challenging. The field has been making constant progress since the first work of Bethe in 1931 and Onsager in 1944. So it might be thought that, even though the problems are hard, it would command the attention of some of the most powerful researchers in a large number of institutions. But as I indicated in the beginning of this talk this is in fact not the case.

Most physics departments are more or less divided into the same divisions as is the APS. Thus it is quite typical to find departments with a condensed matter group, a nuclear physics group, and high energy group, an astrophysics group and an atomic and molecular group. But as I mentioned at the beginning, none of the work I have discussed in this talk fits naturally into these categories and thus if departments hire people in the mainstream of the existing divisions of the APS no one doing research in integrable models in statistical mechanics will ever be hired.

So while I am deeply honored and grateful for the award of the 1999 Heineman prize for mathematical physics there is still another honor I am looking for. It is to receive a letter from the chairman of a physics department which reads as follows:

Dear Prof. McCoy,

Thank you for the recommendation you recently made to us concerning the hiring of a new faculty member. We had not considered hiring anyone in the area of physics represented by your candidate, but after reading the resume and publications we decided that you were completely correct that the candidate is doing outstanding work which will bring an entirely new area of research to our department. We are very pleased to let you know that the university has made an offer of a faculty appointment to your candidate which has been accepted today. Thank you very much for your help and advice.

I have actually received one such letter in my life. If I am fortunate I
hope to receive a few more before the end of my career. The 21st century is long and anything is still possible.

Acknowledgments

This work is supported in part by the National Science Foundation under grant DMR 97-03543.

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