Trigonometric Solutions of the WDVV Equations from Root Systems

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Abstract

By introduction of an additional variable and addition of a Weyl invariant correction term to the perturbative prepotential in five-dimensional Seiberg-Witten theory we construct solutions of the WDVV equations of trigonometric type for all crystallographic root systems.

1 Introduction

In two-dimensional topological conformal field theory the following remarkable system of third order nonlinear partial differential equations for a function $F$ of $N$ variables emerged

$$F_i F_j^{-1} F_j^{-1} = F_j F_i^{-1} F_i^{-1} \quad i, j = 1, \ldots, N$$  \hspace{1cm} (1)

Here $F_i$ is the matrix

$$(F_i)_{kl} = \frac{\partial^3 F}{\partial a_i \partial a_k \partial a_l}.$$  

Moreover, it is required that $F_1$ is a constant and invertible matrix. Usually this system is called the WDVV equations. Generalizations, not requiring $F_1$ to be constant, have been introduced and studied in the context of four- and five-dimensional $N = 2$ supersymmetric gauge theory.

Although extremely difficult to solve in general, this overdetermined system (1) of nonlinear partial differential equations admits exact solutions. For instance, within the theory of Frobenius manifolds, a substantial class of polynomial solutions has been constructed by Dubrovin [1] for any Coxeter group. Furthermore, for any gauge group, perturbative approximations to exact prepotentials in four-dimensional Seiberg-Witten theory satisfy the (generalized) WDVV equations. These solutions are of rational type and may be constructed for any root system (see [2] and [3]).

In this note we shall construct solutions of system (1) of trigonometric type for any crystallographic root system. This construction is achieved by the introduction of an additional variable and addition of a Weyl invariant correction term to the perturbative prepotential in five-dimensional Seiberg-Witten theory.

2 Main result

For our convenience by renumbering the variables we may suppose that $F_N$ is constant and invertible instead of $F_1$. More precisely we have the following result
2 MAIN RESULT

2.1 Theorem

Let $R$ be a crystallographic root system in $\mathbb{R}^n$ and $W$ the corresponding Weyl group. Then the function $F$ of $n+1$ variables $a_1, \ldots, a_n, a_{n+1}$ given by

$$F(a_1, \ldots, a_n, a_{n+1}) = \frac{1}{2} \sum_{\alpha \in R} f((\alpha, a)) + \gamma \left( \frac{1}{6} a_{n+1}^3 + \frac{1}{2} a_{n+1}(a, a) \right)$$

satisfies the system 1 of WDVV equations. Here $(\alpha, a) = \alpha_1 a_1 + \cdots + \alpha_n a_n$ is the standard Euclidean inner product in $\mathbb{R}^n$ and $f$ is the function given by

$$f(x) = \frac{1}{6} x^3 - \frac{1}{4} Li_3(e^{-2x}) = \frac{1}{6} x^3 - \frac{1}{4} \sum_{k=1}^{\infty} \frac{e^{-2kx}}{k^3}$$

so that

$$f'''(x) = \coth(x).$$

$\gamma$ is some fixed constant satisfying $-\gamma^2 = c$ for some fixed number $c$ depending on the root system $R$. For each type of crystallographic root system the value of $c$ is given in table 1.

**Proof**

Obviously the matrix $F_{n+1}$ equals $\gamma I$, where $I$ is the identity matrix. So in this case the WDVV conditions reduce to

$$F_i F_j = F_j F_i$$

and are automatically satisfied if $i = n+1$ or $j = n+1$. Therefore we may restrict ourselves to $i, j \leq n$ and in order that the expression $F$ satisfies the WDVV equations the expression

$$\sum_{k=1}^{n} F_{il} F_{km} + F_{i,n+1} F_{n+1,jm} = \sum_{k=1}^{n} F_{il} F_{km} + \gamma^2 \delta_{il} \delta_{jm}$$

should be symmetric in $i$ and $j \leq n$. In other words we should have

$$\sum_{k=1}^{n} (F_{il} F_{km} - F_{jik} F_{kim}) + \gamma^2 (\delta_{il} \delta_{jm} - \delta_{jl} \delta_{im}) = 0$$

(5)

The left hand side of the equation (5) above is anti symmetric in $l$ and $m$, thus this condition is equivalent to

$$\sum_{k=1}^{n} F_{il} F_{km} - F_{jik} F_{kim} - F_{ilm} F_{kl} + F_{jlm} F_{kli} + 2\gamma^2 (\delta_{il} \delta_{jm} - \delta_{jl} \delta_{im}) = 0$$

(6)

Since

$$F_{il} = \frac{1}{2} \sum_{\alpha \in R} f'''((\alpha, x))\alpha_i \alpha_l \alpha_k$$

$(i, l, k \leq n)$

where $f'''(u) = \coth(u)$, a simple calculation shows that this condition becomes

$$\frac{1}{4} \sum_{\alpha \in R, \beta \in R} f'''((\alpha, x)) f'''((\beta, x)) (\alpha, \beta)(\alpha_i \beta_j - \alpha_j \beta_i)(\alpha_l \beta_m - \alpha_m \beta_l) + 2\gamma^2 (\delta_{il} \delta_{jm} - \delta_{jl} \delta_{im}) = 0$$

(7)

Considering only positive roots we finally should have

$$\sum_{\alpha > 0, \beta > 0} f'''((\alpha, x)) f'''((\beta, x)) (\alpha, \beta)(\alpha_i \beta_j - \alpha_j \beta_i)(\alpha_l \beta_m - \alpha_m \beta_l) = 2\gamma^2 (\delta_{il} \delta_{jm} - \delta_{jl} \delta_{im})$$

(8)
For a moment we restrict our attention to the left hand side of the last equality (8). We split this expression into a partition of pairs of positive roots \( \alpha > 0, \beta > 0 \) such that the product \( s_\alpha s_\beta \) of the corresponding Weyl reflections \( s_\alpha, s_\beta \) equals a \( w \) in the Weyl group \( W \). So we split the sum into

\[
\sum_{w \in W} \sum_{\alpha > 0, \beta > 0} f'''((\alpha, x)) f'''((\beta, x)) s_\alpha s_\beta = w
\]

(9)

Applying the Dunkl identity (see Matsuo [5], proof of proposition 3.3.1) we see that this sum (9) equals

\[
\sum_{w \in W} \sum_{\alpha > 0, \beta > 0} (\alpha, \beta)(\alpha_i \beta_j - \alpha_j \beta_i)(\alpha_l \beta_m - \alpha_m \beta_l)
\]

(10)

and simplifying this sum again, it equals

\[
\frac{1}{4} \sum_{\alpha \in R, \beta \in R} (\alpha, \beta)(\alpha_i \beta_j - \alpha_j \beta_i)(\alpha_l \beta_m - \alpha_m \beta_l)
\]

(11)

We want to evaluate this last expression. To this end we introduce the homogeneous 4-form \( A \) by

\[
\frac{1}{4} \sum_{\alpha \in R, \beta \in R} (\alpha, \beta)((\alpha, x)(\beta, y) - (\alpha, y)(\beta, x))((\alpha, u)(\beta, v) - (\alpha, v)(\beta, u)) = A(x, y; u, v)
\]

Obviously the form \( A \) is antisymmetric in \( x, y \) and in \( u, v \). Moreover it is invariant under the Weyl group \( W \) and under permutation of \( x, y \) by \( u, v \). Consequently a small calculation shows that we necessarily have

\[
A(x, y; u, v) = c((x, u)(y, v) - (x, v)(y, u))
\]

(12)

for some fixed constant \( c \). With respect to the Euclidean coordinates \( e_1, \ldots, e_n \) this means that the expression (11) equals

\[
c(\delta_{ij}\delta_{jm} - \delta_{il}\delta_{lm})
\]

Hence the WDVV condition (8) reduces to \( c = -2\gamma^2 \). This completes the proof of the theorem.

The precise value of \( c \) is evaluated with the help of appendix of Bourbaki [6] and is listed in table 1. The numbers listed in this table are in agreement with the results in the paper [4].

### Table 1: The numbers \( c \) for each Lie algebra

| \( A_N \) | \( B_N \) | \( C_N \) | \( D_N \) | \( E_6 \) | \( E_7 \) | \( E_8 \) | \( F_4 \) |
|---|---|---|---|---|---|---|---|
| \( 2(N + 2) \) | \( 4(2N - 3) \) | \( 8(N + 2) \) | \( 8(N - 2) \) | \( 6 \) | \( 96 \) | \( 320 \) | \( 30 \) |

2.2 Remark

We may insert a \( W \)-invariant set of complex numbers, i.e. \( k_w \alpha = k_\alpha(w \in W) \) into the expression \( F \) in (2) in the following way

\[
F(a_1, \ldots, a_n, a_{n+1}) = \frac{1}{2} \sum_{\alpha \in R} k_\alpha f((\alpha, a)) + \gamma(\frac{1}{6}a_{n+1}^3 + \frac{1}{2}a_{n+1}(a, a))
\]

(13)
The proof of the theorem has to be modified in a rather obvious way. Note that the Dunkl identity remains in force in this case. Of course the value of $\gamma$ has to be modified correspondingly.

References

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