On $q$-deformed Stirling numbers

Yilmaz Simsek
University of Akdeniz, Faculty of Arts and Science, Department of Mathematics, 07058 Antalya, Turkey
E-Mail: ysimsek@akdeniz.edu.tr,

Abstract

The purpose of this article is to introduce $q$-deformed Stirling numbers of the first and second kinds. Relations between these numbers, Riemann zeta function and $q$-Bernoulli numbers of higher order are given. Some relations related to the classical Stirling numbers and Bernoulli numbers of higher order are found. By using derivative operator to the generating function of the $q$-deformed Stirling numbers of the second kinds, a new function is defined which interpolates the $q$-deformed Stirling numbers of the second kinds at negative integers. The recurrence relations of the Stirling numbers of the first and second kind are given. In addition, relation between $q$-deformed Stirling numbers and $q$-Bell numbers is obtained.

2000 Mathematics Subject Classification. 11B39, 11B68, 11B73.

Key Words and Phrases. $q$-Bernoulli numbers and polynomials, $q$-Stirling numbers first and second kind, fermionic Stirling numbers first and second kind. $q$-Bell Numbers.

1. Introduction, Definitions and Notations

The $q$-deformed Stirling numbers of the first and second kind are denoted by $s(n,k,q)$, $S(n,k,q)$, respectively. The fermionic Stirling numbers of the first and second kind are denoted by $s_f(n,k)$, $S_f(n,k)$, respectively. In this paper, we use notation in the work of Kim[9] and Schork[20]. $q$-Stirling numbers were first defined in the work of Carlitz[1]. A lot of combinatorial work has centered around the $q$-analogue, the earliest by Milne[17], also see ([20, 3, 6, 9, 13, 16, 22]). In [8], Kim constructed $q$-Bernoulli numbers of higher order associated with the $p$-adic $q$-integers. $q$-Volkenborn integral is origianally constructed by Kim[9]. By using the $q$-Volkenborn integral, Kim[9] evaluated complete sum for $q$-Bernoulli polynomials. He also obtained relations between $q$-Bernoulli numbers and $q$-analogs of the Stirling numbers. The fermionic and bosonic Stirling numbers were given in detail by Schork[20, 21]. In [22], Wagner studied three partition statistics and the $q$-Stirling and $q$-Bell numbers that serve as their generating functions, evaluating these numbers when $q = -1$.

In this paper, we give $q$-deformed Stirling numbers of the second kinds. Relations between these numbers, Riemann zeta functions and $q$-Bernoulli numbers higher order are given. We also give some relations related to the classical Stirling numbers and Bernoulli numbers higher order. By using derivative operator to the generating function of the $q$-deformed Stirling numbers of the second kinds, we define new function, which interpolates the $q$-deformed Stirling numbers of the second kinds at negative integers. The recurrence relations of the Stirling numbers of the first and second kind are given. We also give relation between $q$-deformed Stirling numbers and $q$-Bell numbers.

Let $q \in (-1,1]$. The fundamental properties of the $q$-integers and $q$-deformed numbers are given by

$$[n,q] = [n] = \frac{1 - q^n}{1 - q}. \quad (1.1)$$
\[
[n]! = [n][n-1]...[2][1], [0]! = 1 \text{ and } \binom{n}{k} = \frac{[n]!}{[n-k]![k]!}
\]

Note that \(\lim_{q \to 1}[n] = n\), cf. \(\text{(6), (7), (10), (11), (12), (20), (13), (14), (15)}\).

The generating function of the \(q\)-Stirling numbers of the second kind is given by defined by \(\text{Kim}\):

\[
F_{S,q}(t) = \sum_{k=0}^{\infty} \frac{t^k}{k!} \Delta_q^k 0^n = \sum_{n=0}^{\infty} S(n, k, q) t^n.
\]

Let \((Eh)(x) = h(x + 1)\) be the shift operator. Let \(\Delta_q^n = \prod_{j=0}^{n-1}(E - q^j I)\) be the \(q\)-difference operator cf. \(\text{Kim}\).

\[
[n] = \sum_{k=0}^{n} (-1)^k \binom{n}{k} q^k \Delta_q^n 0^n
\]

Kim\(\text{[9]}\) defined \(q\)-analog of the Newton-Gregory expansion as follows:

\[
S(n, k, q) = \frac{q^{\binom{n+k}{2}}}{[k]!} \sum_{j=0}^{k} \binom{k-j}{j} q^j \Delta_q^n 0^n
\]

2. \(q\)-deformed Stirling Numbers

When \(q < 0\), we write \(q \equiv -q^*\) with \(q^* > 0\). By \(\text{(1.1)}\), we have

\[
[n] = \frac{1 - (q^-)^n}{1 + q^-} = [n, q^*]^F \text{ cf. \(\text{(20)}\)}
\]

\([n, q^*]^F\) is called \(q^*\)-fermionic basic numbers. These numbers are appearing in recent studies of the \(q^*\)-deformed fermionic oscillator (see \(\text{[16], (20), (18), (21)}\)). Observe that \(q \to -1\), i.e., \(q^* \to 1\) yields \(\text{(20)}\).

\[
[n, q = -1] = [n, q^* = 1]^F = \frac{1 - (-1)^n}{2} = \epsilon_n = \begin{cases} 0, \text{ if } n \text{ is even integers} \\ 1, \text{ if } n \text{ is odd integers.} \end{cases}
\]

Hence, for \(n \geq 2\)

\[
[n, q = -1]!=0.
\]

The \(q^*\)-fermionic basic numbers were given in detail by Schork\(\text{[20], (21)}\).

By using \(\text{(1.2)}\), the \(q\)-deformed Stirling numbers of the second kind, \(S(n, k, q)\) are given by (in the version of Kim\(\text{[9]}\)):

\[
S(n, k, q) = \sum_{j=0}^{k} (-1)^j \binom{k-j}{j} q^{\binom{k-j}{2} + j(j-1)/2} \frac{[j]^{n-1}}{[j-1]![k-j]!}.
\]

The recurrence relations of \(S(n, k, q)\), with \(S(1, 0, q) = 0\) and \(S(1, 1, q) = 1\), is given by

\[
S(n + 1, k, q) = q^{-k} S(n, k - 1, q) + [k] S(n, k, q), \text{ cf. \(\text{(2), (4), (9), (17), (20)}\)}.
\]

By applying the derivative operator \(\frac{d^n}{dt^n} F_{S,q}(t) \big|_{t=0}\) to \(\text{(1.2)}\), we arrive at the following theorem:

\[
S(n, k, q) = \frac{d^n}{dt^n} F_{S,q}(t) \big|_{t=0} = \sum_{j=0}^{k} (-1)^j \binom{k-j}{j} q^{\binom{k-j}{2} + j(j-1)/2} [j]^n.
\]

By using \(\text{(2.4)}\), we define \(Y_S(z, k, q)\) function as follows:
Definition 1. Let $z \in \mathbb{C}$. We define
\[ Y_S(z, k, q) = \frac{q^{k(1-k)}}{[k]!} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} q^{\frac{(k-j)(k-j-1)}{2}+2jd} j!^{-z} . \] (2.5)

Observe that if $z \in \mathbb{C}$, then $Y_S(z, k, q)$ is an analytic function. The function $Y_S(z, k, q)$ interpolates the $q$-deformed Stirling numbers of the second kind $S(n, k, q)$ at negative integers, which is given in Theorem 1 below.

By (1.1), (2.4), we have the following Corollary:

Corollary 1.
\[ S(n, k, q) = \frac{q^{k(1-k)}}{(1-q)^n[k]!} \sum_{j=0}^{n} (-1)^{k-j-d} \binom{k}{j} \sum_{d=0}^{n} q^{\frac{(k-j)(k-j-1)+2jd}{2}} j!^{-z}. \]

By substituting $z = -n$, with $n$ is a positive integer, into (2.5), and using (2.4), we arrive at the following theorem:

Theorem 1. Let $n$ be a positive integer. Then we have
\[ Y_S(-n, k, q) = S(n, k, q). \] (2.6)

The $q$-deformed Bell numbers are defined by [20]
\[ B(n, q) = \sum_{k=0}^{n} S(n, k; q). \]

By using (2.4) and (2.6), we give relation between $Y_S(z, k, q)$ and $B(n, q)$ as follows:

Theorem 2. Let $n$ be a positive integer. Then we have
\[ B(n, q) = \sum_{k=0}^{n} \frac{q^{k(1-k)}}{[k]!} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} q^{\frac{(k-j)(k-j-1)+2jd}{2}} j!^{-z}. \]

Remark 1. Observe that when $\lim_{q \to 1} B(n, q) = B(n) = \sum_{k=0}^{n} S(n, k)$, where $B(n)$ denotes the classical Bell numbers cf. [20, 17, 22]. In [5], Gessel gave relation between the classical Stirling numbers of first kind, $S(n, k)$ and the classical Bernoulli numbers of higher order $B_k^{(n)}$ as follows:
\[ S(n+k, n) = \binom{n+k}{k} B_k^{(n-k)}, \] (2.7)
where
\[ \sum_{j=0}^{\infty} \frac{B_j^{(n)}}{j!} t^j = \left( \frac{t}{e^t-1} \right)^n. \]

In [9], Kim gave relation between $S(n, k; q)$ numbers and $q$-Bernoulli numbers of higher-order as follows: Let $m \geq 0$ and $h, k$ are natural numbers.
\[ \begin{align*}
 m^k \sum_{j=0}^{m} \binom{m}{j} (q-1)^j \beta_j(0, -k, q) &= \sum_{j=0}^{k} \frac{q^{k(k-1)}}{[j]!} S(k, j; q) \binom{m}{j}, \\
 \beta_m(h, k, q) &= (1-q)^{-m} \sum_{j=0}^{m} \binom{m}{j} (-1)^j \left( \frac{h+j}{[h+j]} \right)^k.
\end{align*} \]
In [19], Rassias and Srivastava gave relation between Riemann zeta functions and the classical Stirling numbers of first kind, \( s(n, k) \) as follows:

\[
\zeta(k + 1) = \sum_{n=k}^{\infty} \frac{(-1)^{n-k}}{n!} s(n, k). \tag{2.8}
\]

For each \( k = 0, 1, ..., n - 1, \ (n \geq 1) \), the Eulerian numbers \( E(n, k) \) are given by

\[
E(n, k) = \sum_{j=0}^{k} (-1)^j \binom{n + 1}{j} (k + 1 - j)^n. \tag{2.11}
\]

Relation between \( E(n, k) \) and the classical Stirling numbers of the second kind, \( S(n, k) \) is given by

\[
S(n, m) = \frac{1}{m!} \sum_{j=0}^{n-1} E(n, j) \binom{j}{n - m}, \ n \geq m, n \geq 1. \tag{2.9}
\]

By (2.7) and (2.9), after some elementary calculations, we arrive at the following corollary:

**Corollary 2.** Let \( k = 0, 1, ..., n - 1, \ (n \geq 1) \). Then we have

\[
B^{(-n)}_k = \binom{n + k}{k} \sum_{j=0}^{n+k-1} E(n + k, j) \binom{j}{k}. \tag{2.10}
\]

By (2.6) with \( q \to 1 \) and (2.9), we have the following Corollary:

**Corollary 3.** Let \( n \) be a positive integer and \( n \geq k, n \geq 1 \). Then we have

\[
Y_s(-n, k) = \frac{1}{k!} \sum_{j=0}^{n-1} E(n, j) \binom{j}{n - k}. \tag{2.12}
\]

The \( q \)-deformed Stirling number of the first kind \( s(n, k, q) \), with \( s(1, 0, q) = 0 \) and \( s(1, 1, q) = 1 \), satisfy the following recurrence relations ( see [20], [2]):

\[
s(n + 1, k, q) = q^{-n} \left( s(n, k - 1, q) - [n]s(n, k, q) \right). \tag{2.11}
\]

If \( q \to 1 \) in the above, we have

\[
s(n + 1, k) = s(n, k - 1) - ns(n, k) \]

For \( k \geq 1 \), we set

\[
[x]^k = [x][x - 1]...[x - k + 1], \text{ cf. } (1), (6), (20), (21), (18). \]

By using the above relation, the \( q \)-deformed Stirling numbers is defined as

\[
[x]^n = \sum_{j=0}^{n} S(n, j, q)[x]^j \text{ and } [x]^n = \sum_{j=0}^{n} s(n, j, q)[x]^j. \tag{2.12}
\]

By using (2.12), the \( q \)-deformed Stirling numbers of the first and the second kind satisfies for \( n \geq m \) the inversion relations, which are given by the following theorem:

**Theorem 3.** Let \( n \) and \( m \) be non-negative integers. Then we have

\[
\sum_{k=m}^{n} s(n, k, q)S(k, m, q) = \delta_{n,m},
\]

\[
\sum_{k=m}^{n} S(n, k, q)s(k, m, q) = \delta_{n,m}. \]

Proofs of this theorem were given by Schork [20] and Charalambides [2]. From the above theorem, \( q \)-Stirling numbers of the first and the second kind satisfy the orthogonality relations.
3. Further Remarks and Observations

The fermionic Stirling numbers of the first and the second kind studied by [17], [20], [21], [11], [4], [22], [13], [10], [13], [9]. In this section, we can use some notations which are due to Schork [20], and Kim [9]. In [20], Schork gave the recurrence relations of the fermionic Stirling numbers of first kind and second kind as follows:

\[ s_f(n + 1, k) = (-1)^n s_f(n, k - 1) + (-1)^{n+1} \epsilon_n s_f(n, k), \]  
(3.1)

with \( s_f(1, 0) = 0 \) and \( s_f(1, 1) = 1 \). For the convention, here we take \( s_f(n, 0) = 0 \) and 

\[ S_f(n + 1, k) = (-1)^{k-1} S_f(n, k - 1) + \epsilon_k S_f(n, k), \]  
(3.2)

with \( S_f(1, 0) = 0 \) and \( S_f(1, 1) = 1 \), where \( \epsilon_k \) is defined in (2.1). Schork gave the values \( S_f(n, k) \) for maximal and small \( k \) and found \( S_f(n, n) = (-1)^{\frac{n(n-1)}{2}} \) as well as \( S_f(n, 1) = 1, S_f(n, 2) = -1, S_f(n, 3) = 2 - n, S_f(n, 4) = n - 3 \) and by (3.1), we easily see that \( s_f(n, n) = (-1)^{\frac{n(n-1)}{2}} \).

By (3.1) and (2.1), many Stirling numbers of the first kind vanish. By induction over \( k \), Schork [20] prove \( s_f(n, k) = 0 \) for \( n > 2k \). By the same method, \( s_f(n, k^\sim) = 0 \) for \( 1 \leq k^\sim \leq k \) and \( n > 2k^\sim \) and \( s_f(n, k+1) = 0 \) for \( n \geq 2k + 3 \) (for detail see [20]). From (3.1), we arrive at the following theorem [20]:

**Theorem 4.** Let \( k \) be non-negative integer. Then we have 

\[ s_f(2k + 3, k + 1) = s_f(2k + 2, k) - \epsilon_{2k+2} s_f(2k + 2, k + 1). \]

Note that by the induction hypothesis, the first summand vanishes, whereas by the \( \epsilon_{2k+2} = 0 \), the second summand vanish. Hence, for a given \( n \), the first \( n \) Stirling numbers \( s_f(n, k) \) vanish. Let 

\[ [n]_f = [n, q = -1] = \epsilon_n \] cf. [20]

for the fermionic basic numbers, by (2.1). The fermionic Stirling numbers are connection coefficients for the fermionic basic numbers [20]:

\[ [x]_f^n = \sum_{j=0}^{n} S_f(n, j)[x]_f^j \] and 
\[ [x]_f^m = \sum_{j=0}^{n} s_f(n, j)[x]_f^j. \]  
(3.3)

By induction over \( n \), Schork [20] proved the first equation. We give sketch of the proof as follows: Let the assertion obtain for \( n \). Thus, the induction hypothesis implies that 

\[ [x]_f^{n+1} = \left( \sum_{j=0}^{n} S_f(n, j)[x]_f^j \right) [x]_f. \]

By using \( f \)-arithmetic relations, \( [x]_f = [j]_f + (-1)^j [x-j]_f \), \( [n+m]_f = [m]_f + (-1)^m [n]_f \) and \( [n+1]_f = 1 - [n]_f \), we easily find 

\[ [x]_f^{n+1} = \sum_{j=0}^{n} S_f(n, j)[x]_f^j [x]_f^j = \sum_{j=0}^{n} \left( S_f(n, j)[j]_f + (-1)^j [x-k]_f S_f(n, j) \right) [x]_f^j \]

\[ = \sum_{j=0}^{n} S_f(n, j)[j]_f [x]_f^j + \sum_{j=0}^{n} (-1)^j [x-k]_f S_f(n, j) [x]_f^j \]

\[ = \sum_{j=0}^{n} S_f(n, j)[j]_f [x]_f^j + \sum_{j=0}^{n} (-1)^j S_f(n, j) [x]_f^{j+1} \]

\[ = \sum_{j=0}^{n+1} (S_f(n, j)[j]_f + (-1)^{j-1} S_f(n, j - 1)) [x]_f^j. \]  
(3.4)
Since $S_f(n+1,0) = 0$, we have

$$[x]^{n+1}_f = \sum_{j=0}^{n+1} S_f(n+1,j)[x]_f^j = \sum_{j=1}^{n+1} S_f(n+1,j)[x]_f^j. \quad (3.5)$$

By (3.4) and (3.5) we arrive at (3.2).

Since $[x-n]_f = (-1)^n[x]_f + (-1)^{n+1}\epsilon_n$, we have

$$[x]^{n+1}_f = [x-n]_f \sum_{j=0}^{n} s_f(n,j)[x]_f^j = \sum_{j=0}^{n} s_f(n,j)\left((-1)^n[x]_f + (-1)^{n+1}\epsilon_n\right)[x]_f^j$$

$$= \sum_{j=0}^{n} (-1)^n s_f(n,j)[x]_f^{j+1} + \sum_{j=0}^{n} (-1)^{n+1} s_f(n,j)\epsilon_n[x]_f^j$$

$$= \sum_{j=1}^{n+1} \left((-1)^n s_f(n,j-1) + (-1)^{n+1} s_f(n,j)\epsilon_n\right)[x]_f^j,$$

by the above equation, we obtain (3.1).

Observe that by (3.1) and (3.2), Schork[20] defined the fermionic Stirling numbers of first and second kind, respectively.

It is well-known that if $j \geq 2$, then $[x]_f^j$ vanishes, so by using the first equation of (3.3)

$$[x]_f^n = S_f(n,1)[x]_f = [x]_f.$$

It is easy to prove the following theorem.

**Theorem 5.** Let $n$ and $m$ be non-negative integers. Then we have

$$\sum_{j=0}^{n} s_f(n,j)s_f(j,m) = \delta_{n,m}, \text{ and } \sum_{j=m}^{n} S_f(n,j)s_f(j,m) = \delta_{n,m}. \quad (3.6)$$

By (3.6), the fermionic Stirling numbers satisfy the inversion relations. The proof of (3.6), was given by Schork[20]. He also proved the following the recurrence relations of the fermionic Stirling numbers of second kind: For odd $j > 3$ one can find that $S_f(n+1,j) = S_f(n,j) + S_f(n,j-1)$. Let $n$ and $j$ be non-negative integers.

$$S_f(n+1,j) = S_f(n,j) - S_f(n-1,j-2).$$

**Acknowledgement 1.** We would like to thank to Professor Matthias Schork for his many valuable suggestions and comments on this paper. We also would like to thank to Professor Taekyun Kim for his many valuable comments on this paper.

*This work was supported by Akdeniz University Scientific Research Projects Unit.*

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