Is the Universe logotropic?

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We consider the possibility that the universe is made of a single dark fluid described by a logotropic equation of state \( P = A \ln(\rho/\rho_0) \), where \( \rho \) is the rest-mass density, \( \rho_0 \) is a reference density, and \( A \) is the logotropic temperature. The energy density \( \epsilon \) is the sum of two terms: a rest-mass energy term \( \rho c^2 \) that mimics dark matter and an internal energy term \( u(\rho) = -P(\rho) - A \) that mimics dark energy. This decomposition leads to a natural, and physical, unification of dark matter and dark energy, and elucidates their mysterious nature. In the early universe, the dark fluid behaves as pressureless dark matter \( (P \approx 0, \epsilon \propto a^{-3}) \) and, in the late universe, it behaves as dark energy \( (P \approx -\epsilon, \epsilon \propto a^{-1}) \). The logotropic model depends on a single parameter \( B = A/\rho_0 c^2 \) (dimensionless logotropic temperature) where \( \rho_0 = 6.72 \times 10^{-24} \text{ g m}^{-3} \) is the cosmological density. For \( B = 0 \), we recover the \( \Lambda \)CDM model with a different justification. For \( B > 0 \), we can describe deviations from the \( \Lambda \)CDM model. Using cosmological constraints, we find that \( 0 \leq B \leq 0.09425 \). We consider the possibility that dark matter halos are described by the same logotropic equation of state. When \( B > 0 \), pressure gradients prevent gravitational collapse and provide halo density cores instead of cuspy density profiles, in agreement with the observations. The universal rotation curve of logotropic dark matter halos is consistent with the observational Burkert profile up to the halo radius. It decreases as \( r^{-3} \) at large distances, similarly to the profile of dark matter halos close to the core radius [Burkert, arXiv:1501.06604]. Interestingly, if we assume that all the dark matter halos have the same logotropic temperature, we find that their surface density \( \Sigma = \rho c^2 \) is constant. This result is in agreement with the observations [Donato et al., MNRAS 397, 1169 (2009)] where it is found that \( \Sigma_0 = 141 M_\odot/\text{pc}^2 \) for dark matter halos differing by several orders of magnitude in size. Using this observational result, we obtain \( B \approx 3.53 \times 10^{-3} \). Then, we show that the mass enclosed within a sphere of fixed radius \( r_a = 300 \text{ pc} \) has the same value \( M_{300} = 1.93 \times 10^7 M_\odot \) for all the dwarf halos, in agreement with the observations [Strigari et al., Nature 454, 1096 (2008)]. Finally, assuming that \( \rho = \rho_P \), where \( \rho_P = 5.16 \times 10^{39} \text{ g m}^{-3} \) is the Planck density, we predict \( B = 3.53 \times 10^{-3} \), in perfect agreement with the value obtained from the observations. We approximately have \( B \approx \frac{1}{\ln(\rho_P/\rho_0)} \) which is consistent with the minimum size of dark matter halos observed in the universe. Therefore, a logotropic equation of state is a good candidate to account for both galactic and cosmological observations. This may be a hint that dark matter and dark energy are the manifestation of a single dark fluid. If we assume that the dark fluid is made of a self-interacting scalar field, representing for example Bose-Einstein condensates, we find that the logotropic equation of state arises from the Gross-Pitaevskii equation with an inverted quadratic potential, or from the Klein-Gordon equation with a logarithmic potential. We also relate the logotropic equation of state to Tsallis generalized thermodynamics and to the Cardassian model (motivated by the existence of extra-dimensions).

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I. INTRODUCTION

The cosmological constant has an interesting history. Its pre-history dates back to 1896, when von Seeliger [1] and Neumann [2] introduced an attenuation factor in the Newtonian potential, \( \Phi = -G \rho e^{-\lambda r}/r \), in order to resolve the non-convergence of the gravitational force in Newtonian cosmology for an infinite homogeneous distribution of matter [3]. In 1917, Einstein [4] introduced a cosmological constant \( \Lambda \) in the equations of general relativity in order to obtain an infinite homogeneous static universe.\(^1\) After the discovery of the expansion of the universe, predicted theoretically by Friedmann [10] [11]

\(^1\) As a preamble of his paper, in order to motivate the cosmological constant, Einstein [4] considered the Newtonian approximation and replaced the Poisson equation by an equation of the form \( \Delta \Phi - (\Lambda/c^2) \Phi = 4\pi G \rho \), so as to obtain a homogeneous static solution \( \Phi = -4\pi G \rho /\Lambda \). In Einstein's belief, the Newtonian effect of the cosmological constant was to shield the gravitational interaction on a distance \( \sqrt{\Lambda} \), similarly to the proposal of Seeliger and Neumann. This is actually incorrect. Lemaître [5] was the first to understand that the cosmological constant can be interpreted as a force of “cosmic repulsion” (see also [6]–[9]). In the Newtonian context, the modified Poisson equation including the cosmological constant is \( \Delta \Phi = 4\pi G \rho - \Lambda \), leading to the homogeneous static solution \( \rho = \Lambda/4\pi G \).
and Lemaitre [12, 13], and ascertained by the observations of Hubble [14]. Einstein considered the cosmological constant as “the biggest blunder of his life” [15], and banished it. In 1932, Einstein and de Sitter [16] published a short paper describing the expansion of a pressureless universe without cosmological constant. This is the so-called Einstein-de Sitter (EdS) model. In this model, the universe undergoes a decelerating expansion. The discovery of the acceleration of the expansion of the universe in recent years [17] revived the interest in the cosmological constant since a positive cosmological constant can precisely account for such an acceleration. When present, the standard model of cosmology is based on a pressureless dark matter fluid ($\rho = 0$) was first considered by de Sitter [18, 19] in the context of an empty universe ($p = 0$) involving a negative pressure. Some authors [20, 21] have proposed to interpret the cosmological constant in terms of the energy of the vacuum. However, when one tries to explain the cosmological constant in relation to the vacuum energy, one is confronted to the so-called “cosmological constant problem” [22, 23]. Because quantum field theory predicts that the vacuum energy density should be of the order of the Planck density $\rho_P = c^5/\hbar G^2 = 5.16 \times 10^{99}$ g m$^{-3}$ which is 123 orders of magnitude larger than the value of the cosmological density $\rho_\Lambda = \epsilon_\Lambda/c^2 = 6.72 \times 10^{-24}$ g m$^{-3}$ deduced from the observations. To circumvent this problem, some authors have proposed to abandon the cosmological constant $\Lambda$ and to explain the acceleration of the universe in terms of a dark energy with a time-varying density associated to a scalar field called “quintessence” [24].

This mysterious concept of “dark energy” adds to the other mysterious concept of “dark matter” necessary to account for the missing mass of the galaxies inferred from the virial theorem [27] and to explain their flat rotation curves [28, 29]. As an alternative to quintessence, Kamenshchik et al. [30] proposed a heuristic unification of dark matter and dark energy in terms of an exotic fluid with an equation of state $P = -\epsilon/\epsilon_\Lambda$, called the Chaplygin gas. This equation of state provides a model of universe that behaves as a pressureless fluid (dark matter) at early times, and as a fluid with a constant energy density (dark energy) at late times, yielding an exponential acceleration similar to the effect of the cosmological constant. However, in the intermediate regime of interest, this model does not give a good agreement with the observations [31, 32]. Therefore, generalizations of the Chaplygin gas have been considered [33]. We note that an equation of state corresponding to a constant negative pressure $P = -\epsilon_\Lambda$ also provides a heuristic unification of dark matter and dark energy, and is simpler [34, 35]. Furthermore, this equation of state is consistent with the observations since it gives the same results as the $\Lambda$CDM model although its physical justification is different. Therefore, we can realize a simple unification of dark matter and dark energy by assuming that the universe is filled with a single dark fluid with a constant pressure. However, this model has two drawbacks. Since it has no free parameter, it cannot account for (small) deviations from the $\Lambda$CDM model.

On the other hand, a constant equation of state cannot describe dark matter halos. Indeed, since there is no pressure gradient ($\nabla P = 0$), this model behaves similarly to the $\Lambda$CDM model where $P = 0$, and generates cuspy density profiles at the center of the halos [36]. This is problematic because observations of dark matter halos favor density cores instead of cusps [39, 40]. Cored can be naturally explained by assuming that dark matter halos are collisional and that they are described by an equation of state $P(\rho)$ that strictly increases with the density so that pressure gradients can counterbalance the gravitational attraction. Therefore, if we want to realize a successful unification of dark matter and dark energy, the idea is to consider a dark fluid with an equation of state $P(\rho)$ increasing slowly with the density.

A natural idea is to consider a polytropic equation of state where the pressure is a power of the density with a small index $\gamma \simeq 0$. There are two types of polytropic equations of state in general relativity, as first realized by Tooper [41, 42] in the context of general relativistic stars. Model I corresponds to an equation of state of the form $P = K \rho^n$ with $\gamma = 1 + 1/n$, where the pressure is a power of the energy density. This model has been studied in detail in [43, 44] in a cosmological context. It can be viewed as a generalization of the Chaplygin gas. In this model, when $K < 0$ and $\gamma < 1$ (i.e. $n > 0$) [45], the dark fluid behaves as a pressureless fluid (dark matter) in the early universe and as a fluid with a constant energy density (dark energy) in the late universe. However, this distinction is only asymptotic and, in the

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2 The effect of the cosmological constant in an empty universe ($p = 0$) was first considered by de Sitter [18, 19] in the context of a static universe. It was understood later by Lemaître [20] that de Sitter’s solution actually describes a universe expanding exponentially with time.

3 The cosmological implications of these two types of polytropic models have been recently discussed by the author in [33, 43]. Similar arguments have been developed independently in [44] without reference to the work of Tooper.

4 When $K < 0$ and $\gamma > 1$ (i.e. $n < 0$) [31], this polytropic model describes a phase of early inflation followed by a decelerating expansion (modeling, e.g., radiation). In this model, the temperature is initially very low, increases exponentially rapidly during the inflation up to the Planck temperature (GUT scale), then decreases algebraically. By contrast, in the usual inflationary scenario [45–49], the temperature decreases during the inflation and one must advocate a phase of re-heating (not very well understood) to account for the observations. On the other hand, the polytropic model generates a value of the entropy as large as $S/k_B = 5.04 \times 10^{87}$ [42]. It is also possible to introduce a
intermediate regime, it is not possible to separate these two components unambiguously because they do not appear additively in the energy density unless $\gamma = 0$, which is equivalent to the $\Lambda$CDM model. Model II corresponds to an equation of state of the form $P = K \rho^n$ with $\gamma = 1 + 1/n$, where the pressure is a power of the rest-mass density. In this model, the energy density $\epsilon$ is the sum of two terms: a rest-mass energy $p c^2$ and an internal energy $u(\rho) = P(\rho)/(\gamma - 1)$. Since $\rho \propto a^{-3}$, the rest mass term mimics dark matter. When $K < 0$ and $\gamma < 1$ (i.e. $n < 0$), the internal energy term mimics dark energy. This decomposition leads to a natural, and physical, unification of dark matter and dark energy, and elucidates their mysterious nature. In order to account for the observations, these polytropic models must be relatively close to the $\Lambda$CDM model, equivalent to a constant negative pressure. This implies that the polytropic index $\gamma$ must be close to zero. For $\gamma = 0$ (i.e. $n = -1$), we recover the $\Lambda$CDM model exactly. For $\gamma$ different from zero, we can study deviations from the $\Lambda$CDM model. The polytropic models have been confronted to the observations in order to constrain the values of the admissible index $\gamma$ or $n$. For Model I, it is found that $n \simeq -1.05_{-0.16}^{+0.15}$ \cite{51}. For Model II, it is found that $-0.089 < \gamma < 0$ (corresponding to $-1 \leq n < -0.92$) \cite{15}. Although these polytropic models are interesting in themselves, we anticipate a problem in order to justify them from a fundamental theory. Actually, it is possible to justify a polytropic equation of state by assuming that the dark fluid is made of a self-interacting scalar field representing, for example, Bose-Einstein condensates (BECs). A polytropic equation of state with $\gamma \neq 0$ arises from the Gross-Pitaevskii (GP) equation with a power-law potential $|\psi|^{2(\gamma-1)}$ or from the Klein-Gordon (KG) equation with a power-law potential $|\phi|^{2\gamma}$. However, it is hard to imagine how one can justify from fundamental arguments why the universe is described by a polytropic equation of state with a “curious” index equal, for example, to $\gamma = -0.089$. The most natural value of the polytropic index is $\gamma = 0$ corresponding to a constant pressure but, in that case, the GP and KG equations degenerate. Furthermore, a polytropic equation of state with $\gamma = 0$ has no pressure gradient, which is problematic to account for the flat core of dark matter halos as previously mentioned.

In this paper, we develop another idea that was sketched in Appendix B of \cite{14}. We propose to describe the universe by a single dark fluid with a logotropic equation of state of the form $P = A \ln(\rho/\rho_*)$, where $\rho$ is the rest-mass density, $\rho_*$ is a reference density, and $A$ is the logotropic temperature. This equation of state was introduced phenomenologically in astrophysics by McLaughlin and Pudritz \cite{52} to describe the internal structure and the average properties of molecular clouds and clumps. It was also studied by Chavanis and Sire \cite{53} in the context of Tsallis generalized thermodynamics \cite{54} where it was shown to correspond to a polytropic equation of state with $\gamma \to 0$ and $K \to \infty$ in such a way that $A = \gamma K$ is finite.\(^5\) This limit is very relevant to the present cosmological context because we precisely require an equation of state with a polytropic exponent close to $\gamma = 0$. Therefore, the logotropic equation of state is a natural, and serious, candidate for the unification of dark matter and dark energy. Since the pressure must increase with the density, the logotropic temperature must be strictly positive ($A > 0$) which is satisfying from a thermodynamical point of view. In the early universe, the dark fluid behaves as pressureless dark matter ($P \simeq 0$, $\epsilon \propto a^{-3}$) and, in the late universe, it behaves as dark energy ($P \sim -\epsilon$, $\epsilon \propto \ln a$). The logotropic model depends on a single parameter $B = A/\rho_\Lambda c^2$ (dimensionless logotropic temperature), where $\rho_\Lambda$ is the cosmological density. For $B = 0$, we recover the $\Lambda$CDM model with a different justification. For $B > 0$, we can describe deviations from the $\Lambda$CDM model. Using cosmological constraints, we find that $0 \leq B \leq 0.09425$. For $B = 0.09425$, the universe is normal for $t < 20.7$ Gyrs (the energy density decreases as the scale factor increases) and becomes phantom afterwards (the energy density increases with the scale factor). Actually, the logotropic model breaks down before entering in the phantom regime because the velocity of sound exceeds the velocity of light when $t > 17.3$ Gyrs. However, this moment is quite remote in the future.

Since the logotropic equation of state is expected to provide a unification of dark energy and dark matter, it should also describe the structure of dark matter halos. In this respect, the logotropic equation of state has several nice properties. First of all, since the pressure increases with the density, pressure gradients prevent gravitational collapse and provide halo density cores instead of cuspy density profiles, in agreement with the observations \cite{39, 40}. On the other hand, the universal rotation curve of logotropic dark matter halos is consistent with the observational Burkert profile \cite{39} up to the halo radius.\(^6\)

\(^5\) In the framework of generalized thermodynamics, one can really interpret $A$ as a generalized temperature associated with a generalized entropy called Log-entropy \cite{38} (see Appendix C).\(^6\) The logotropic rotation curve continues to increase at larger distances while the Burkert rotation curve decreases. This feature comes from the fact that the logotropic density decreases as $r^{-1}$ while the Burkert density decreases as $r^{-3}$. Interestingly, we note that the profile of dark matter halos close to the core radius decreases as $r^{-1}$ \cite{55} in agreement with the logotropic profile. On the other hand, some galaxies such as low surface brightness (LSB) galaxies have rotation curves that strictly increase with the distance. These galaxies are particularly well isolated and little affected by tidal effects. In other cases, tidal effects or complex physical processes (e.g. incomplete relaxation) have to be taken into account. This can affect the behavior of the density profile and of the rotation curve at large distances and explain how we pass from a $r^{-1}$ behavior to a $r^{-3}$ behavior.
Finally, and very interestingly, if we assume that all the dark matter halos have the same logotropic temperature $B$ (a consequence of our model), we find that their surface density $\Sigma = \rho_0 r_h$ is constant. This result is in agreement with the observations, where it is found that $\Sigma_0$ has the same value $\Sigma_0 = 141 M_\odot/pc^2$ for dark matter halos differing by several orders of magnitude in size \cite{56,58}, a result unexplained so far. Using this observational result, we obtain $B = 3.53 \times 10^{-3}$. Then, we show that the mass enclosed within a sphere of fixed radius $r_u = 300$ pc has the same value $M_{300} = 1.93 \times 10^7 M_\odot$ for all the dwarf halos, in agreement with the observations of Strigari et al. \cite{59}. This so far unexplained result is a direct consequence of the logotropic equation of state. Finally, assuming that $\rho_\ast = \rho_P$, where $\rho_P$ is the Planck density, we predict $B = 3.53 \times 10^{-3}$, in perfect agreement with the value obtained from the observations.

We approximately apply the logotropic equation of state to the structure of dark matter halos and obtain $B = 3.53 \times 10^{-3}$ from observations. In Sec. \[VI\] we predict $B = 3.53 \times 10^{-3}$ from theoretical considerations. In Appendix \[A\] we provide general results concerning the thermodynamics of a relativistic fluid. In Appendix \[B\] we provide a simple argument to justify the logotropic equation of state and mention its relation to a fifth force. In Appendix \[C\] we develop a field theory for the dark fluid and determine the form of the GP and KG equations that generate a logotropic equation of state, as well as the corresponding generalized entropy.

Remark: Secs. \[II\] and \[III\] show the close connections between different cosmological models (many of these connections having not been discussed before) and explain the physical reasons why we consider a logotropic equation of state. These sections are important to motivate our study. They also introduce our general formalism. However, the reader just interested by the results may skip these sections and go directly to Sec. \[IV\] for the application of the logotropic equation of state to cosmology and to Sec. \[V\] for the application of the logotropic equation of state to dark matter halos.

II. COSMOLOGICAL MODELS

A. The Friedmann equations

We assume that the universe is homogeneous and isotropic, and contains a uniform perfect fluid of energy density $\epsilon(t)$ and isotropic pressure $P(t)$. The radius of curvature of the 3-dimensional space, or scale factor, is noted $a(t)$ and the curvature of space is noted $k$. The universe is closed if $k > 0$, flat if $k = 0$, and open if $k < 0$. We assume that the universe is flat ($k = 0$) in agreement with the observations of the cosmic microwave background (CMB) \cite{60}. In that case, the Einstein equations can be written as \[\[1\]:

$$\frac{de}{dt} + 3 \frac{a_\ast}{a} (\epsilon + P) = 0, \tag{1}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3c^2} (\epsilon + 3P) + \frac{\Lambda}{3}, \tag{2}$$

$$H^2 = \left(\frac{\ddot{a}}{a}\right)^2 = \frac{8\pi G}{3c^2} \epsilon + \frac{\Lambda}{3}, \tag{3}$$

where we have introduced the Hubble parameter $H = \dot{a}/a$ and accounted for a possible non-zero cosmological constant $\Lambda$.

Eqs. \[I\]-\[III\] are the well-known Friedmann equations describing a non-static universe. Among these three equations, only two are independent. The first equation can be viewed as an equation of continuity. For a given barotropic equation of state $P = P(\epsilon)$, it determines the relation between the energy density $\epsilon$ and the scale factor.
a. Then, the evolution of the scale factor $a(t)$ is given by Eq. (3).

Introducing the equation of state parameter $w = P/\epsilon$, and assuming $\Lambda = 0$, we see from Eq. (2) that the universe is decelerating if $w > -1/3$ (strong energy condition) and accelerating if $w < -1/3$. On the other hand, according to Eq. (1), the energy density decreases with the scale factor if $w > -1$ (null dominant energy condition) and increases with the scale factor if $w < -1$. The latter case corresponds to a “phantom” universe [62].

B. Einstein-de Sitter model

We consider a universe made of pressureless matter (dust) with an equation of state

$$P = 0,$$

and we assume that $\Lambda = 0$. This is the Einstein-de Sitter model [10]. The equation of continuity (1) can be integrated into

$$\epsilon = \epsilon_0 \left(\frac{a_0}{a}\right)^3,$$

where the subscript 0 refers to present-day values. The Friedmann equation (3) writes

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G\epsilon_0}{3c^2} \left(\frac{a_0}{a}\right)^3,$$

and it can be integrated into

$$\frac{a}{a_0} = \left(\frac{3}{2}H_0 t\right)^{2/3}, \quad \epsilon = \frac{\epsilon_0^2}{6\pi G t^2},$$

where $H_0 = (8\pi G\epsilon_0/3c^2)^{1/2}$ is the present value of the Hubble constant. From observations $H_0 = 70.2 \text{ km s}^{-1} \text{Mpc}^{-1} = 2.275 \times 10^{-18} \text{s}^{-1}$ yielding $\epsilon_0 = 8.32 \times 10^{-7} \text{g m}^{-3} \text{s}^{-2}$. In this model, the universe is always decelerating and its age is

$$t_{\text{EdS}}^{\text{EdS}} = \frac{2}{3H_0}.$$  

Numerically, $t_{\text{EdS}}^{\text{EdS}} = 9.29 \text{ Gyr}$.  

C. \( \Lambda \)CDM model

We still consider a universe made of pressureless matter (dust) with $P = 0$ but, in order to account for the acceleration of the expansion of the universe, we now assume that $\Lambda > 0$. This is the $\Lambda$CDM model. The Friedmann equation (3) now writes

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G\epsilon_0}{3c^2} \left(\frac{a_0}{a}\right)^3 + \frac{\Lambda}{3}.$$  

As discussed in the next section, this equation can be interpreted in terms of dark matter and dark energy, and it can be solved analytically.

D. Dark matter and dark energy

As is clear from Eq. (3), the cosmological constant is equivalent to a fluid with a constant energy density

$$\epsilon_\Lambda = \rho_\Lambda c^2 = \frac{\Lambda c^2}{8\pi G},$$

(10)

According to Eq. (1), the equation of state leading to a constant energy density is $P = -\epsilon$. Instead of introducing a cosmological constant $\Lambda$, we can consider that the universe is made of two fluids: a pressureless dark matter fluid with an equation of state $P = 0$ leading to a dark matter density $\epsilon_m = \epsilon_{m,0}(a_0/a)^3$, and a dark energy fluid with an equation of state

$$P = -\epsilon$$

leading to a constant density identified with the dark energy density, or the cosmological density. The total energy density of the universe ($\epsilon = \epsilon_m + \epsilon_\Lambda$) is therefore

$$\epsilon = \epsilon_{m,0} \left(\frac{a_0}{a}\right)^3 + \epsilon_\Lambda.$$  

As a result, the Friedmann equation (3) can be written as

$$\frac{H}{H_0} = \sqrt{\frac{\Omega_{m,0}}{(a_0/a)^3} + \Omega_{\Lambda,0}},$$

(13)

where we have introduced the present fractions of dark matter and dark energy $\Omega_{m,0} = \epsilon_{m,0}/\epsilon_0$ and $\Omega_{\Lambda,0} = \epsilon_\Lambda/\epsilon_0$. By construction, $\Omega_{m,0} + \Omega_{\Lambda,0} = 1$. From observations $\Omega_{m,0} = 0.274$ and $\Omega_{\Lambda,0} = 0.726$. Eq. (13) has a well-known analytical solution

$$\frac{a}{a_0} = \left(\frac{\Omega_{m,0}}{\Omega_{\Lambda,0}}\right)^{1/3} \sinh^{2/3} \left(\frac{3}{2} \sqrt{\Omega_{\Lambda,0}} H_0 t\right),$$

(14)

$$\frac{\epsilon}{\epsilon_0} = \frac{\Omega_{\Lambda,0}}{\tanh^2 \left(\frac{3}{2} \sqrt{\Omega_{\Lambda,0}} H_0 t\right)}.$$  

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7 This energy was initially interpreted as the vacuum energy [21, 22]. However, from particle physics and quantum field theory, the vacuum energy is expected to be of the order of the Planck density $\rho_P = 5.16 \times 10^{99} \text{ g m}^{-3}$ while cosmological observations give $\rho_\Lambda = 6.72 \times 10^{-24} \text{ g m}^{-3}$. These densities differ by the huge ratio $\rho_P/\rho_\Lambda \sim 10^{21}$. This is the so-called cosmological constant problem [23, 25].

8 In this paper, we consider a sufficiently old universe so that radiation can be neglected. On the other hand, baryonic matter can be treated as a pressureless fluid ($P_b = 0$) whose contribution adds to the contribution of dark matter. We can take it into account in the previous expressions by considering that $\epsilon_m$ refers to dark matter and baryonic matter.
For early times, we recover the EdS solution (with modified coefficients):

$$\frac{a}{a_0} \sim \left(\frac{3}{2} \sqrt{\Omega_{m,0}} H_0 t\right)^{2/3}, \quad \frac{\epsilon}{\epsilon_0} \sim \frac{4}{9H_0^2 t^2},$$

(16)

and for late times, we recover the de Sitter solution

$$\frac{a}{a_0} \sim \left(\frac{\Omega_{m,0}}{4\Omega_{\Lambda,0}}\right)^{1/3} e^{\sqrt{\Omega_{m,0}} H_0 t}, \quad \epsilon \simeq \epsilon_\Lambda.$$

(17)

In the ΛCDM model, the age of the universe is

$$t_0 = \frac{2}{3H_0 \sqrt{\Omega_{\Lambda,0}}} \sinh^{-1} \left[\left(\frac{\Omega_{\Lambda,0}}{\Omega_{m,0}}\right)^{1/2}\right].$$

(18)

Numerically, $t^-_{\Lambda \text{CDM}} = 13.8 \, \text{Gyrs}$.

In this model, the universe is decelerating at early times and accelerating at late times. The moment at which the universe starts accelerating ($\ddot{a} \geq 0$) takes place when $\epsilon_c = 3\epsilon_\Lambda$, i.e. $a_c/a_0 = (\Omega_{m,0}/2\Omega_{\Lambda,0})^{1/3} = 0.574$. This corresponds to a time

$$t_c = \frac{2}{3H_0 \sqrt{\Omega_{\Lambda,0}}} \sinh^{-1} \left(\frac{1}{\sqrt{2}}\right).$$

(19)

Numerically, $t^-_{\Lambda \text{CDM}} = 7.18 \, \text{Gyrs}$. The transition between the dark matter era and the dark energy era takes place when $\epsilon_m = \epsilon_\Lambda$, hence $\epsilon_2 = 2\epsilon_\Lambda$, i.e. $a_2/a_0 = (\Omega_{m,0}/\Omega_{\Lambda,0})^{1/3} = 0.723$. This corresponds to a time

$$t_2 = \frac{2}{3H_0 \sqrt{\Omega_{\Lambda,0}}} \sinh^{-1}(1).$$

(20)

Numerically, $t^-_{\Lambda \text{CDM}} = 9.61 \, \text{Gyrs}$.

### E. The Chaplygin gas

Kamenshchik et al. [30] have proposed to unify dark matter and dark energy by a single dark fluid described by the equation of state

$$P = -\frac{A}{\epsilon}.$$

(21)

This is called the Chaplygin gas because this equation of state was introduced by Chaplygin [33] as a convenient soluble model to study the lifting force on a plane wing in aerodynamics. This equation of state has also a connection with string theory and admits a supersymmetric generalization (see [30] for details). For the equation of state [21], the equation of continuity [1] can be integrated into

$$\epsilon = \sqrt{\frac{a_*}{a}} \left(\frac{a_*}{a}\right)^6 + 1)^{1/2},$$

(22)

where $a_*$ is a constant of integration. Eq. (22) can be viewed as a combination of dark matter and dark energy. For $a \to 0$, we have $\epsilon \propto a^{-3}$ which behaves as pressureless dark matter. For $a \to +\infty$, we have $\epsilon \propto \sqrt{A}$ which behaves as dark energy with a constant energy density $\epsilon_\Lambda = \sqrt{A}$, equivalent to a cosmological constant (see Secs. II C and II D). Although this model has an important historical interest (and can be motivated by string theory), it does not give a good agreement with the observations [31, 32]. Therefore, generalizations of the Chaplygin gas model have been considered.

### F. Constant pressure

As a first step, we consider a single dark fluid described by an equation of state corresponding to a constant negative pressure

$$P = -\epsilon_\Lambda.$$

(23)

The equation of continuity [1] can be integrated into

$$\epsilon = \epsilon_\Lambda \left[\left(\frac{a_*}{a}\right)^3 + 1\right],$$

(24)

where $a_*$ is a constant of integration. This model also provides a unification of dark matter and dark energy, and is simpler than the Chaplygin gas [34, 37]. Actually, this model gives the same result as the ΛCDM model of Secs. II C and II D although its physical justification is different. As a result, this model agrees with the cosmological observations, unlike the Chaplygin gas. In order to describe (small) deviations to the ΛCDM model, and in order to describe simultaneously the large scale structure of the universe and dark matter halos (as discussed in the Introduction), we should consider equations of state that are close to, but different from, a constant pressure. This is the aim of the polytropic and logotropic equations of state considered in the following sections.

### G. Polytropic model of type I

We assume that the universe is made of a single dark fluid. We can specify its equation of state in the form $P = P(\epsilon)$, where $\epsilon$ is the energy density. We consider a polytropic equation of state

$$P = K \left(\frac{\epsilon}{\epsilon_0}\right)\gamma, \quad \gamma = 1 + \frac{1}{n},$$

(25)

in which the pressure is a power of the energy density. This corresponds to the polytropic equation of state considered in the first paper of Tooper [41] in the context of
general relativistic stars. Therefore, we shall call it polytropic model of type I or, simply, Model I. This polytropic model has been studied in a cosmological context in [34-37]. It can be viewed as a generalization of the Chaplygin gas model.

For the equation of state (25), the equation of continuity (1) can be integrated analytically. Assuming \( w \equiv P/\epsilon \geq -1 \) (non-phantom), \( K < 0 \), and \( n < 0 \) (i.e. \( \gamma < 1 \)), the relation between the energy density and the scale factor can be written as [35]:

\[
\epsilon = \epsilon_* \left( \frac{a_*}{a} \right)^{3/[n]} + 1 \right)^{[n]},
\]

where \( \epsilon_* = (|K|/c^2)^{|n|}c^2 \) and \( a_* \) is a constant of integration. For \( n = -1/2, \gamma = 1 \), and \( K = -A/c^2 \), we recover the Chaplygin gas model (21) and (22). For \( n = -1, \gamma = 0, \) and \( K = -\epsilon_* \), we recover the constant pressure model (23) and (24), equivalent to the ΛCDM model. More generally, Eq. (26) can be viewed as a combination of dark matter and dark energy. Indeed, for \( a \to 0 \), we have \( \epsilon \to \epsilon_* (a_*/a)^3 \) which behaves as pressureless dark matter. For \( a \to +\infty \), we have \( \epsilon \to \epsilon_* \epsilon \) which behaves as dark energy. This allows us to make the identification \( \epsilon_* = \epsilon_\Lambda \). In this analogy, using Eq. (10) and the relation following Eq. (26), we find that the polytropic constant \( K \) is equivalent to a cosmological constant

\[
\Lambda = 8\pi G \left( \frac{|K|}{c^2} \right)^{|n|}.
\]

Introducing present-day variables, we can rewrite Eq. (26) as

\[
\epsilon = \epsilon_\Lambda \left\{ \left( \frac{\epsilon_0}{\epsilon_\Lambda} \right)^{1/[n]} - 1 \right\} \left( \frac{a_0}{a} \right)^{3/[n]} + 1 \right)^{[n]}. \tag{28}
\]

For \( a \to 0 \),

\[
\epsilon \simeq \epsilon_\Lambda \left( \frac{\epsilon_0}{\epsilon_\Lambda} \right)^{1/[n]} - 1 \right\} \left( \frac{a_0}{a} \right)^{3}. \tag{29}
\]

For \( a \to +\infty \),

\[
\epsilon \to \epsilon_\Lambda. \tag{30}
\]

This model behaves as pressureless dark matter at early times and as dark energy at late times. However, this decomposition is only asymptotic and, in the intermediate regime, it is not possible to separate these two components unambiguously because they do not appear additively in the energy density unless \( n = -1 \) (i.e. \( \gamma = 0 \)), corresponding to the ΛCDM model (see Sec. 11). We also note that the contribution of baryonic matter, considered as a pressureless fluid (\( P_b = 0 \)), must be added in the total energy density as an independent species.

The polytropic model (25) with \( K < 0 \) and \( n < 0 \) can be used to discuss deviations from the ΛCDM model (\( n = -1 \)). It is found from observations that \( n = -1.05^{+0.15}_{-0.16} \) [51]. If we impose that the velocity of sound \( c_s \) is real, the condition \( c_s^2 = P(\epsilon) c^2 = (K\gamma/c^2)(\epsilon/c^2)^{\gamma-1} \geq 0 \) constrains the index \( n \) in the range \(-1 \leq n \leq 0 \) (i.e. \( \gamma \leq 0 \)). Combining this condition with the observations, we get \(-1 \leq n \leq -0.9 \) (i.e. \(-0.11 \leq \gamma \leq 0 \)).

In conclusion, the polytropic model of type I offers a unified picture of dark matter and dark energy [35-51]. In this model, the universe starts from the matter dominated epoch and approaches a de Sitter phase at late times. Contrary to the ΛCDM model, the cosmic coincidence problem (namely why the ratio of dark energy and dark matter densities is of order unity today) [64] is solved naturally in the polytropic gas scenario [35-51]. We finally note that a theoretical justification of a polytropic equation of state of the form of Eq. (25) for a perfect relativistic fluid in cosmology has been given recently by Kazinski [65] in relation to the quantum gravitational anomaly. He predicts a polytropic equation of state of the form \( P = K(\epsilon/c^2)^{1+1/n} \) with \( n \in \mathbb{N} \).

H. Polytropic model of type II

As in the previous section, we assume that the universe is made of a single dark fluid but, here, we specify its equation of state in the form \( P = P(\rho) \), where \( \rho \) is the rest-mass density. In order to solve the Friedmann equation (3), we need to determine the energy density \( \epsilon \). To that purpose, we assume that the dark fluid is at \( T = 0 \), or that the evolution is adiabatic (which is the case for a perfect fluid [61]). In that case, the first law of thermodynamics reduces to (see Appendix A):

\[
d\epsilon = \frac{P + \epsilon}{\rho} d\rho. \tag{31}
\]

For a given \( P(\rho) \) this equation can be solved to give the relation between the energy density and the rest-mass density. This relation can be written as (see Appendix A):

\[
\epsilon = \epsilon_0 \rho c^2 + \rho \int_0^\rho \frac{P(\rho')}{\rho'^2} d\rho' = \rho c^2 + u(\rho), \tag{32}
\]

where \( \rho c^2 \) is the rest mass energy of the dark fluid and \( u(\rho) \) is its internal energy. On the other hand, combining the first law of thermodynamics (31) with the continuity equation (1), we get

\[
\frac{d\rho}{dt} + 3\dot{a} \rho = 0. \tag{33}
\]

We note that this equation is exact for a fluid at \( T = 0 \), or for a perfect fluid, and that it does not depend on the explicit form of the equation of state \( P(\rho) \). It expresses the conservation of the rest-mass density \( \rho \) (or number density \( n = \rho/m \)). It can be integrated into

\[
\rho = \rho_0 \left( \frac{a_0}{a} \right)^3, \tag{34}
\]
where $\rho_0$ is the present value of the rest-mass density and $a_0$ is the present value of the scale factor. Substituting Eq. (34) in Eq. (32), we obtain

$$
\epsilon = \rho_0 c^2 \left( \frac{a_0}{a} \right)^3 + u \left[ \rho_0 \left( \frac{a_0}{a} \right)^3 \right].
$$

(35)

For any barotropic equation of state $P(\rho)$, our treatment shows that the energy density in Eqs. (32) and (35) is the sum of two terms. The first term corresponds to the rest-mass energy of the fluid and the second term corresponds to its internal energy. In the present approach, there is just one (relativistic) dark fluid. If we want to relate our approach to the traditional viewpoint where the universe is made of dark matter and dark energy, we note that the first term in Eqs. (32) and (35) is equivalent to the term of Eq. (5) that describes a pressureless fluid (dark matter). As a result, we can interpret the second term in Eqs. (32) and (35) as the term corresponding to the dark energy. The equation

$$
u(\rho) = \rho \int^\rho \frac{P(\rho')}{\rho'^2} \, d\rho'
$$

(36)

clearly shows that the dark energy term arises from pressure effects (i.e., collisions between particles). If $P = 0$, the internal energy vanishes and we recover the standard EdS model describing only pressureless dark matter (see Sec. [11]). Alternatively, if we assume that the pressure is constant, $P = -\epsilon_0$, we find that the internal energy is constant, $u = \epsilon_0$, and recover the model of Sec. [11] that is equivalent to the $\Lambda$CDM model. Therefore, the rest-mass energy $\rho c^2$ of the dark fluid mimics “dark matter” and its internal energy $u(\rho)$ mimics “dark energy” (see Appendix B of [41]). This decomposition leads to a natural, and physical, unification of dark matter and dark energy, and elucidates their mysterious nature. We note that, in general, the dark energy term $u(\rho)$ in Eq. (35) is not constant (it depends on the rest-mass density $\rho$, and it does not even necessarily tend to a constant for $\rho \to 0$), contrary to the standard dark energy term $\epsilon_\Lambda$ in Eq. (24) corresponding to a constant negative pressure. Therefore, our approach can be used to obtain generalized models of dark energy by considering different expressions of the equation of state $P(\rho)$. If $P(\rho)$ is sufficiently close to a constant, the resulting models will be relatively close to the $\Lambda$CDM model. Therefore, we can use this approach to study small deviations from the $\Lambda$CDM model and constrain the possible equations of state $P(\rho)$ of dark energy from observations.

To be specific, we consider a polytropic equation of state

$$
P = K \rho^\gamma,
$$

(37)
in which the pressure is a power of the rest-mass density. This corresponds to the polytropic equation of state considered in the second paper of Tooper [42] in the context of general relativistic stars. Therefore, we shall call it polytropic model of type II, or simply, Model II. This polytropic model has been studied in a cosmological context in [13, 44], and independently in [15].

For the equation of state (37), using Eq. (32), we find that the energy density is related to the rest-mass density by [42, 43]:

$$
\epsilon = \rho c^2 + K \rho \ln \left( \frac{\rho}{\rho_*} \right),
$$

(38)

$$
\epsilon = \rho c^2 + \frac{K}{\gamma - 1} \rho^\gamma = \rho c^2 + n P(\rho),
$$

(39)

for $\gamma = 1$.

Combining Eq. (34) with Eqs. (38) and (39), we obtain for $\gamma = 1$:

$$
\epsilon = \rho_0 c^2 \left( \frac{a_0}{a} \right)^3 + K \rho_0 \left( \frac{a_0}{a} \right)^3 \ln \left[ \rho_0 \left( \frac{a_0}{a} \right)^3 \right]
$$

(40)

and for $\gamma \neq 1$:

$$
\epsilon = \rho_0 c^2 \left( \frac{a_0}{a} \right)^3 + \frac{K}{\gamma - 1} \rho_0 \left( \frac{a_0}{a} \right)^{3\gamma}.
$$

(41)

Introducing relevant (see below) notations $\Omega_{m,0} = \rho_0 c^2/\epsilon_0$, $\Omega_{\gamma,0} = K\rho_0/\epsilon_0$, and $\Omega_{\gamma,0}/[(\gamma - 1)\epsilon_0]$, the Friedmann equation (3) with $\Lambda = 0$ can be written as

$$
H / H_0 = \sqrt{\Omega_{m,0} + \Omega_{\gamma,0} \ln[(\rho_0/\rho_*)/(a_0/a)^3]}
$$

(42)

for $\gamma = 1$, and as

$$
H / H_0 = \sqrt{\Omega_{m,0} + \Omega_{\gamma,0} / [(a_0/a)^3]}
$$

(43)

for $\gamma \neq 1$. By construction, $\Omega_{m,0} + \Omega_{\gamma,0} \ln[(\rho_0/\rho_*)] = 1$ and $\Omega_{m,0} + \Omega_{\gamma,0} = 1$. As discussed previously, the rest-mass energy of the dark fluid mimics dark matter so that $\Omega_{m,0}$ represents the present fraction of dark matter. The contribution of baryonic matter, considered as a pressureless fluid ($P_b = 0$), must be added in the total energy density, as an independent species. Since it simply adds to the rest-mass energy of the dark fluid, we can take it into account by considering that $\Omega_{m,0}$ actually represents the present total fraction of matter (baryonic and dark).

For the polytropic equation of state (37), the energy density in Eqs. (38)-(41) is the sum of two terms. An ordinary term $\epsilon_m = \rho c^2 \propto a^{3(\gamma - 1)}$ (rest-mass energy) equivalent to pressureless dark matter and a new term $\epsilon_\gamma = u(\rho)/(\gamma - 1) a^{3\gamma}$ (internal energy) depending on the polytropic index $\gamma$. The equation of state of the new term is (see also Appendix B) $P = (\gamma - 1) u = u/n$. When $\gamma > 1$ (i.e., $n > 0$), we find that $P \sim \epsilon/n$ and $\epsilon \sim nK \rho^\gamma \propto a^{-3\gamma}$ in the “early” universe (small $a$, large $\rho$) and that $P \sim K(\epsilon/c^2)^\gamma$ and $\epsilon \sim \rho c^2 \propto a^{-3}$ in the “late” universe (large $a$, small $\rho$). When $\gamma < 1$ (i.e.,
Since \( n < 0 \), we find that \( P \sim K(\epsilon/c^2)\gamma \) and \( \epsilon \sim \rho c^2 \propto a^{-3} \) in the “early” universe (small \( a \), large \( \rho \)) and \( P \sim \epsilon/n \) and \( \epsilon \sim nK\rho^\gamma \propto a^{-3\gamma} \) in the “late” universe (large \( a \), small \( \rho \)).\(^{10}\) Therefore, the new term (i.e. the internal energy of the dark fluid) dominates in the late universe when \( \gamma < 1 \) (i.e. \( n < 0 \)), while it dominates in the early universe when \( \gamma > 1 \) (i.e. \( n > 0 \)). This remark allows us to refine the discussion given previously. The new term can mimic dark energy only for \( \gamma < 1 \) (i.e. \( n < 0 \)). For \( \gamma = 0 \) (i.e. \( n = -1 \)), the new term corresponds to a constant dark energy density \( (P = -u, \ u = \epsilon_\Lambda) \), equivalent to the \( \Lambda \)CDM model. By contrast, when \( \gamma > 1 \) (i.e. \( n > 0 \)), the new term can mimic the effect of an exotic constituent appearing in the early universe. For \( \gamma = 2 \) (i.e. \( n = 1 \)), the new term is equivalent to stiff matter \( (P = u, \ u = \epsilon_\Lambda \propto a^{-6}) \) as discussed in \([44]\) in relation to self-interacting BECs at \( T = 0 \) and in relation to the cosmological model of Zel’dovich \([66, 67]\). For \( \gamma = 4/3 \) (i.e. \( n = 3 \)), the new term is equivalent to the radiation of an ultra-relativistic gas \( (P = u/3, \ u = \epsilon_{\text{rad}} \propto a^{-4}) \).

We now restrict ourselves to the case \( \gamma < 1 \) in order to describe dark energy. If we assume that the energy density of the new term (internal energy) is positive in order to account for the observations (some energy is missing with respect of the EdS model where there is only dark matter), we conclude from the relation \( \epsilon_\gamma = P/(\gamma - 1) \) that the pressure must be negative \( (K < 0, \ P < 0) \). This is the case, in particular, for the constant equation of state \( P = -\epsilon_\Lambda < 0 \), corresponding to \( \gamma = 0 \), equivalent to the usual \( \Lambda \)CDM model. If we assume that the dark fluid is made of a self-interacting scalar field (e.g. a BEC), it is shown in Appendix \([C]\) that a negative pressure can arise from an attractive potential of interaction. Therefore, an equation of state with a negative pressure (such as the equation of state of dark energy) can be justified from a field theory.

The polytropic model \((41)\) with an index \( \gamma < 1 \) can be used to discuss deviations from the \( \Lambda \)CDM model \( (\gamma = 0) \). A detailed study has been done recently in \([35]\). It is found from cosmological constraints that \(-0.089 < \gamma \leq 0\) (corresponding to \(-1 \leq n < -0.92)\).\(^{11}\) We note that this range is very close to the one obtained from Model I (see Sec. \([II.G]\) but this may be a coincidence.

Before closing this discussion, we briefly recall the analogies and the differences between the two polytropic models: (i) Models I and II describe dark energy provided that \( \gamma < 1 \) (i.e. \( n < 0 \)) and \( K < 0 \) (negative pressure). They are equivalent to the \( \Lambda \)CDM model when \( \gamma = 0 \) (i.e. \( n = -1 \)). (ii) In Model II, the energy density \((41)\)

is the sum of two terms (rest-mass energy and internal energy) that can be associated with what has been called dark matter and dark energy, respectively. In Model I, this distinction is less clear because the expression \((28)\) of the energy density is non-additive, so the distinction between dark matter and dark energy is only asymptotic. (iii) In model I, the energy density tends to a constant \( \epsilon_A \) at late times (leading to a de Sitter era), which is equivalent to the cosmological constant or to the ordinary form of dark energy. In model II, the dark energy term is not constant (i.e. it depends on time), even asymptotically.

### I. Cardassian model

There is an interesting consequence of the ideas developed in the previous section. Using Eq. \((32)\), we can write the Friedmann equation \((3)\) with \( \Lambda = 0 \) in terms of the rest-mass density \( \rho \) as

\[
H^2 = \frac{8\pi G}{3}(\rho + \nu(\rho)),
\]

where \( \rho = \rho_0(a_0/a)^3 \) and \( \nu(\rho) = (8\pi G/3c^2)u(\rho) \). This equation is similar to the modified Friedmann equation introduced by Freese and Lewis \([68]\) in the so-called Cardassian model.\(^{12}\) For the polytropic equation of state \((37)\), it exactly coincides with the power law Cardassian model. Eq. \((44)\) was originally \([65]\) justified by the fact that the Friedmann equation is modified as a consequence of embedding our universe as a three-dimensional surface (3-brane) in higher dimensions. Hence, the modified Friedmann equation \((44)\) may result from the existence of extra-dimensions. Our approach provides another, simpler, justification of this equation from the ordinary four dimensional Einstein equations. In particular, starting from the usual Friedmann equation \((3)\) and considering a dark fluid at \( T = 0 \), or an adiabatic fluid, the “new” term \( \nu(\rho) \) in the “modified” Friedmann equation \((44)\) can be interpreted as the internal energy \((56)\) of the dark fluid while the “ordinary” term \((8\pi G/3)\rho \) corresponds to its rest-mass density.

### III. LOGOTROPIC EQUATION OF STATE

In Sec. \([III.\)] we have argued that dark matter and dark energy may be the manifestation of a unique dark fluid characterized by an equation of state \( P(\rho) \). If this claim is correct, the equation of state \( P(\rho) \) should describe both the cosmological evolution of the universe and the structure of the dark matter halos. Let us therefore consider the possibility that dark matter halos are described by an equation of state \( P(\rho) \), the same as the one governing the

\(^{10}\) Since \( P \sim (\gamma - 1)\epsilon \) in the late universe, we find that \( w = P/\epsilon \to \gamma - 1 \) when \( a \to +\infty \). Therefore, the universe is normal when \( \gamma > 0 \) (the energy density decreases as the scale factor increases) and phantom when \( \gamma < 0 \) (the energy density increases as the scale factor increases).

\(^{11}\) For \( \gamma < 0 \), the universe becomes phantom at late times (see footnote 10).

\(^{12}\) The name Cardassian refers to a humanoid race in Star Trek whose goal is to take over the universe \([68]\).
cosmological evolution of the universe. In that case, they satisfy the classical condition of hydrostatic equilibrium between pressure and gravity\textsuperscript{13}

\[
\nabla P + \rho \nabla \Phi = 0.
\]

(45)

We have seen that a dark fluid with a constant pressure \((P = -\epsilon A)\) generates a cosmological model equivalent to the \(\Lambda\)CDM model. Therefore, this equation of state seems to be suitable at the cosmological scale. However, when this equation of state is applied to dark matter halos, we find that \(\nabla P = 0\), so there is no pressure gradient to balance the gravitational force. Therefore, as in the classical CDM model where \(P = 0\), the dark matter halos described by a constant equation of state display cuspy density profiles while observations tend to favor a constant density core (cusp/core problem). This observational result is a strong argument against a constant density core (cusp/core problem). This observation is consistent with the cosmological bound \(K \leq 0\) obtained in \textsuperscript{15}.

Therefore, we turn next to a polytropic equation of state of the form of Eq. (37). The condition of hydrostatic equilibrium (45) becomes

\[
K \gamma \rho^{\gamma-1} \nabla \rho + \rho \nabla \Phi = 0.
\]

(46)

Because of pressure gradients when \(K \neq 0\) and \(\gamma \neq 0\), the dark matter halos described by a polytropic equation of state display a central core instead of a cusp.\textsuperscript{14} If we want this equation of state to describe both dark matter halos and the cosmological evolution of the universe, \(\gamma\) must be close to zero (i.e. the equation of state must be close to a constant pressure, equivalent to the \(\Lambda\)CDM model). But, in that case, we encounter the problem of the justification. Indeed, it seems difficult to theoretically justify why \(\gamma\) should take a value like \(-0.089\) for example \textsuperscript{45}. Therefore, we consider another possibility. We assume that \(\gamma \to 0\) and \(K \to \infty\) in such a way that \(A = K \gamma\) is finite \textsuperscript{53}. In that limit, Eq. (46) becomes

\[
A \frac{\nabla \rho}{\rho} + \rho \nabla \Phi = 0.
\]

(47)

Comparing Eq. (47) with Eq. (15), we see that the pressure term corresponds to the logotropic equation of state \textsuperscript{52,53}:

\[
P = A \ln \rho + C,
\]

(48)

where \(A\) and \(C\) are constants. As shown in Appendix C5\textsuperscript{c} the parameter \(A\) can be interpreted as a logotropic temperature. We note that \(A\) must be positive if we want that the pressure forces balance the gravitational attraction. This is satisfying from a thermodynamical point of view. In the following sections, we successively apply the logotropic equation of state to the evolution of the universe and to the structure of dark matter halos.

### IV. LOGOTROPIC COSMOLOGY

#### A. The equation of state

We assume that the universe is made of a single dark fluid described by a logotropic equation of state

\[
P = A \ln \left( \frac{\rho}{\rho_*} \right),
\]

(49)

where \(\rho\) is the rest-mass density, \(A\) is the logotropic temperature (we assume \(A \geq 0\), see Sec. \textsuperscript{III}, and \(\rho_*\) is a reference density.\textsuperscript{15} It will be called the logotropic dark fluid (LDF). According to Eq. (32), the relation between the energy density and the rest-mass density is

\[
\epsilon = \rho c^2 - A \ln \left( \frac{\rho}{\rho_*} \right) - A = \rho c^2 + u(\rho).
\]

(50)

The pressure is related to the internal energy by \(P = -u - A\). Using Eq. (34), we obtain

\[
\epsilon = \rho_0 c^2 \left( \frac{a_0}{a} \right)^3 - A \ln \left[ \frac{\rho}{\rho_*} \left( \frac{a_0}{a} \right)^3 \right] - A.
\]

(51)

As explained in Sec. \textsuperscript{III}, the first term (rest-mass energy) in Eqs. (50) and (51) mimics dark matter and the second term (internal energy) mimics dark energy. The evolution of the energy density with the scale factor is plotted in Fig. 1 and the relation between the energy density and the rest-mass density is plotted in Fig. 2. The universe starts at \(a = 0\) with an infinite rest-mass density \((\rho \to +\infty)\) and an infinite energy density \((\epsilon \to +\infty)\). The rest-mass density decreases as \(a\) increases, see Eq. (34). The energy density decreases as \(a\) increases (i.e. \(\rho\) decreases), reaches a minimum \(\epsilon_M = -A \ln(A/\rho_* c^2)\) at \(a_M = a_0 (\rho_0 c^2 / A)^{1/3}\) (i.e. \(\rho_M = A^{1/3}\)), increases as \(a\) increases (i.e. \(\rho\) decreases) further, and tends to \(\epsilon \to +\infty\) as \(a \to +\infty\) (i.e. \(\rho \to 0\)). The branch \(a \leq a_M\) (i.e. \(\rho \geq \rho_M\)) corresponds to a normal behavior in which the energy density decreases as the scale factor increases. The branch \(a \geq a_M\) (i.e. \(\rho \leq \rho_M\)) corresponds to a phantom behavior in which the energy density increases as the scale factor increases. We note that \(A\) is equal to the rest-mass energy at the point where the universe becomes phantom.

\textsuperscript{13} For dark matter halos, we can use Newtonian gravity.

\textsuperscript{14} For the pressure gradient to counterbalance the gravitational attraction, we need \(K \gamma > 0\). Since we have shown from cosmological considerations in Sec. \textsuperscript{III} that \(K < 0\), we must have \(\gamma < 0\). This is consistent with the cosmological bound \(\gamma \leq 0\) obtained in \textsuperscript{53}.

\textsuperscript{15} The meaning of this density will be unraveled in Sec. \textsuperscript{VI} but we leave it unspecified for the moment.
Combining Eqs. (49) and (50), we obtain the following relation between the pressure and the energy density

$$
\epsilon = \rho_0 c^2 e^{P/A} - P - A.
$$

(52)

This determines the equation of state $P = P(\epsilon)$ under the form $\epsilon = \epsilon(P)$. The pressure decreases as $\rho$ increases (i.e. $\rho \rightarrow +\infty$ at $a = 0$ (i.e. $\rho \rightarrow +\infty, \epsilon \rightarrow +\infty$), achieves the value $P_M = -\epsilon_M$ at $a_M$ (i.e. $\rho_M, \epsilon_M$) and tends to $-\infty$ when $a \rightarrow +\infty$ (i.e. $\rho \rightarrow 0, \epsilon \rightarrow +\infty$), see Fig. 4. The equation of state $P(\epsilon)$ is defined for $\epsilon \geq \epsilon_M$ and has two branches corresponding to a normal universe ($P \geq P_M$) and a phantom universe ($P \leq P_M$), as shown in Fig. 5. Therefore, the equation of state $P(\epsilon)$ is multi-valued.

In the early universe ($a \rightarrow 0, \rho \rightarrow +\infty$), the rest-mass energy (dark matter) dominates and we have

$$
\epsilon \sim \rho c^2 \sim \rho_0 c^2 \left( \frac{a_0}{a} \right)^3, \quad P \sim A \ln \left( \frac{\epsilon}{\rho_0 c^2} \right).
$$

(53)

For very small values of the scale factor, we recover the results of the CDM model ($P(0) = 0$) since $\epsilon \sim a^{-3}$. In the late universe ($a \rightarrow +\infty, \rho \rightarrow 0$), the internal energy (dark energy) dominates, and we have

$$
\epsilon \sim -A \ln \left( \frac{\rho}{\rho_0} \right) \sim 3A \ln a, \quad P \sim -\epsilon.
$$

(54)

We note that the equation of state $P(\epsilon)$ reduces to $P \sim -\epsilon$ for $\epsilon \rightarrow +\infty$, which is similar to the usual equation of state $P = \epsilon$ of the dark energy. However, this is only an asymptotic result, different from the usual equation of state of the dark energy where $P = -\epsilon$ exactly, implying a constant energy density $\epsilon = \epsilon_\Lambda$ (see Sec. III). As a result, in our model, the energy density does not tend to a constant $\epsilon_\Lambda$ for $a \rightarrow +\infty$, but slowly increases as $\ln a$. This corresponds to a phantom regime where $P/\epsilon < -1$ at high energy densities (we have $P + \epsilon \rightarrow -A < 0$ when $\epsilon \rightarrow +\infty$ on the phantom branch). As we shall see, this leads to a super de Sitter behavior. We also emphasize that a polytropic equation of state with an index $\gamma = 0$ (or $n = -1$), corresponding to a constant pressure $P = -\epsilon_\Lambda$, leads to different results since the energy density tends to a constant value $\epsilon_\Lambda$ for $a \rightarrow +\infty$ (see Sec. III), which is different from Eq. (54).

**B. The energy density**

Since, in our model, the rest-mass energy of the dark fluid in Eq. (51) mimics dark matter, we identify $\rho_0 c^2$ with the present energy density of dark matter. We set

$$
\rho_0 c^2 = \Omega_{m,0} \epsilon_0.
$$

(55)

where $\epsilon_0$ is the present energy density of the universe and $\Omega_{m,0}$ is the present fraction of dark matter.\(^{17}\) As a result, the present internal energy of the dark fluid $\epsilon_0 = \epsilon_0 - \rho_0 c^2$ is identified with the present dark energy density $\epsilon_\Lambda = (1 - \Omega_{m,0}) \epsilon_0$ where $\Omega_{m,0} = 1 - \Omega_{m,0}$ is the present fraction of dark energy. From the observations, one has $H_0 = 70.2 \text{ km s}^{-1} \text{ Mpc}^{-1} = 2.275 	imes 10^{-18} \text{ s}^{-1}, \Omega_{m,0} = 0.274$ and $\Omega_{\Lambda,0} = 0.726$, yielding $\epsilon_0 = 3H_0^2 c^2 / 8\pi G = 8.32 \times 10^{-7} \text{ g m}^{-3} \text{ s}^{-2}$ and $\epsilon_0 / c^2 = 9.26 \times 10^{-24} \text{ g m}^{-3}$. Therefore, $\rho_0 = 2.54 \times 10^{-24} \text{ g m}^{-3}$ and $\rho_\Lambda = \epsilon_\Lambda / c^2 = 6.72 \times 10^{-24} \text{ g m}^{-3}$.

Applying Eq. (51) with Eq. (55) at $a = a_0$, we get

$$
\epsilon_0 = \Omega_{m,0} \epsilon_0 - A \ln \left( \frac{\Omega_{m,0} \epsilon_0}{\rho_0 c^2} \right) - A,
$$

(56)

which determines $\rho_0$ as a function of $A$, for known (measured) values of $\epsilon_0$ and $\Omega_{m,0}$. Using Eqs. (55) and (56), the relation (51) between the energy density and the scale factor can be rewritten as

$$
\frac{\epsilon}{\epsilon_0} = \frac{\Omega_{m,0}}{(a/a_0)^3} + (1 - \Omega_{m,0}) \left[ 1 + \frac{3A}{\epsilon_0 (1 - \Omega_{m,0})} \ln \left( \frac{a}{a_0} \right) \right].
$$

(57)

It is convenient to introduce the dimensionless logotropic temperature

$$
B = \frac{A}{\epsilon_\Lambda} = \frac{A}{\epsilon_0 (1 - \Omega_{m,0})}
$$

(58)

and the normalized scale factor

$$
R = \frac{a}{a_0}.
$$

(59)

In terms of these variables, the characteristic density $\rho_\star$ is given from Eq. (56) by

$$
\rho_\star c^2 = \epsilon_0 (1 - \Omega_{m,0}) B, \quad \frac{\rho_\star c^2}{\epsilon_0 \Omega_{m,0}} = e^{1 + 1/B},
$$

(60)

and Eq. (57) becomes

$$
\frac{\epsilon}{\epsilon_0} = \frac{\Omega_{m,0}}{R^3} + (1 - \Omega_{m,0}) (1 + 3B \ln R).
$$

(61)

The dimensionless logotropic temperature $B$ is the only free parameter of the model.

For $B = 0$, Eq. (61) reduces to

$$
\frac{\epsilon}{\epsilon_0} = \frac{\Omega_{m,0}}{R^3} + 1 - \Omega_{m,0}.
$$

(62)

This returns the LCDM model with a different justification. This is because when $B \rightarrow 0$, using Eqs. (58) and

\[^{16}\text{It is interesting to note that the equation of state } P = -\epsilon \text{ can be obtained as an asymptotic limit of the logotropic equation of state.}\]

\[^{17}\text{As explained in Sec. III, we also include baryonic matter in } \rho_0 \text{ so that } \Omega_{m,0} \text{ actually represents the present total fraction of mass (baryonic and dark).}\]
the equation of state \( P = -\frac{A}{B} = -\epsilon_0 (1 - \Omega_{m,0}) = -\epsilon_\Lambda \) equivalent to Eq. \( \frac{23}{18} \).

For \( B \neq 0 \), our model differs from the \( \Lambda \)CDM model. The relation between the energy density and the scale factor is represented in Fig. 1 for different values of \( B \).

For \( R \to 0 \),

\[
\frac{\epsilon}{\epsilon_0} \sim \frac{\Omega_{m,0}}{R^3} \quad (64)
\]

and, for \( R \to +\infty \),

\[
\frac{\epsilon}{\epsilon_0} \sim 3B (1 - \Omega_{m,0}) \ln R, \quad (B \neq 0). \quad (65)
\]

When \( B > 0 \), the curve \( \epsilon(R) \) presents a minimum at \( (R_M, \epsilon_M) \), as detailed in the next section. When \( R < R_M \), the energy density decreases with the scale factor, which corresponds to a “normal” universe. When \( R > R_M \), the energy density increases with the scale factor, which corresponds to a “phantom” universe.

It is useful to write the relation between the energy density and the rest-mass density using dimensionless variables. According to Eqs. \( \frac{34}{}, \frac{55}{}, \) and \( \frac{59}{}, \) we have

\[
\rho = \frac{\Omega_{m,0} \epsilon_0}{\epsilon^2} \frac{1}{R^3}. \quad (66)
\]

Eqs. \( \frac{61}{}, \) and \( \frac{66}{}, \) determine the relation \( \epsilon(\rho) \) in parametric form. Eliminating \( R \) between these two relations,

\[
\frac{\epsilon}{\epsilon_0} = \frac{\rho^2}{\epsilon_0^2} + (1 - \Omega_{m,0}) \left[ 1 + B \ln \left( \frac{\Omega_{m,0} \epsilon_0}{\rho^2} \right) \right], \quad (67)
\]

as represented in Fig. 2. For \( B = 0 \), this equation reduces to \( \epsilon = \rho \epsilon_0^2 + 3 \epsilon_\Lambda \) equivalent to the \( \Lambda \)CDM model.

C. The point of minimum energy density

The minimum of the curve \( \epsilon(R) \) defined by Eq. \( \frac{61}{}, \) is given by

\[
R_M = \left[ \frac{\Omega_{m,0}}{B(1 - \Omega_{m,0})} \right]^{1/3}, \quad (68)
\]

\[
\left( \frac{\epsilon}{\epsilon_0} \right)_M (0) = (1 - \Omega_{m,0}) \left[ B + 1 + B \ln \left( \frac{\Omega_{m,0}}{1 - \Omega_{m,0}} \right) - B \ln B \right]. \quad (69)
\]

The function \( \epsilon_M(B) \) is represented in Fig. 3. For the \( \Lambda \)CDM model, corresponding to \( B = 0 \), we have

\[
\left( \frac{\epsilon}{\epsilon_0} \right)_M (0) = 1 - \Omega_{m,0} = 0.726. \quad (70)
\]

In that case, \( R_M \to +\infty \), so the minimum \( \epsilon_M = \epsilon_\Lambda \) is rejected at infinity. For \( B \to 0 \),

\[
\left( \frac{\epsilon}{\epsilon_0} \right)_M = (1 - \Omega_{m,0})(1 - B \ln B), \quad (71)
\]

and for \( B \to +\infty \),

\[
\left( \frac{\epsilon}{\epsilon_0} \right)_M = -(1 - \Omega_{m,0}) B \ln B \to -\infty. \quad (72)
\]

We note that the function \( \epsilon_M(B) \) first increases with \( B \) before decreasing. Its maximum is located at

\[
B_1 = \frac{\Omega_{m,0}}{1 - \Omega_{m,0}} = 0.377, \quad \left( \frac{\epsilon}{\epsilon_0} \right)_M (B_1) = 1. \quad (73)
\]

\[\]
The function $\epsilon_M(B)$ recovers its “initial” value $\epsilon_M(0)$ at the point

$$B_2 = \frac{\Omega_{m,0}}{1 - \Omega_{m,0}} e = 1.03,$$  \hspace{1cm} (74)

$$\left(\frac{\epsilon}{\epsilon_0}\right)_M(B_2) = 1 - \Omega_{m,0} = 0.726. \hspace{1cm} (75)$$

Finally, the minimum energy density $\epsilon_M(B)$ vanishes at $B_{\text{max}}$ given by the implicit equation

$$\ln(B_{\text{max}}) - \frac{1}{B_{\text{max}}} = 1 + \ln \left(\frac{\Omega_{m,0}}{1 - \Omega_{m,0}}\right). \hspace{1cm} (76)$$

Solving this equation numerically, we find $B_{\text{max}} = 1.79$. In the following, we assume $0 \leq B < B_{\text{max}} = 1.79$ so the energy density $\epsilon(R)$ remains always positive (which is of course a necessary condition). This puts a first constraint on the allowable values of $B$. If we demand that the present-day universe is not phantom, corresponding to the condition $R_M > 1$, using Eqs. (68) and (74), we obtain the more stringent constraint $0 \leq B < B_1 = 0.377$. This gives a physical interpretation to $B_1$.

![FIG. 3: Minimum energy density $\epsilon_M(B)$ as a function of the logotropic temperature $B$.](image)

**D. The pressure**

Using Eqs. (74), (55), (58), (59) and (60), the pressure can be expressed in terms of the scale factor as

$$P = -\epsilon_0(1 - \Omega_{m,0})(B + 1 + 3B \ln R). \hspace{1cm} (77)$$

For $B = 0$, we obtain a constant pressure $P = -\epsilon_0(1 - \Omega_{m,0}) = -\epsilon_A$, equivalent to the $\Lambda$CDM model. For $B \neq 0$, our model differs from the $\Lambda$CDM model. For $R \to 0$ and for $R \to +\infty$,

$$P \sim -3B\epsilon_0(1 - \Omega_{m,0}) \ln R. \hspace{1cm} (78)$$

The pressure vanishes at

$$R_w = e^{-(B+1)/3B}. \hspace{1cm} (79)$$

At that point, according to Eq. (61), the energy density is

$$\left(\frac{\epsilon}{\epsilon_0}\right)_w = \Omega_{m,0}e^{(B+1)/B} - (1 - \Omega_{m,0})B. \hspace{1cm} (80)$$

The pressure is positive for $R < R_w$ and negative for $R > R_w$. Since $R_w < 1$, the pressure is always negative in the present-day universe, for any $B \geq 0$. Its value is $P_w = -(B+1)\epsilon_A$. The relation between the pressure and the scale factor is plotted in Fig. 4 for different values of $B$. We note the “exceptional” point $R_w = e^{-1/3} = 0.7165$ at which the pressure takes the same value $P_w = -\epsilon_A = -6.04 \times 10^{-7} \text{g m}^{-1} \text{s}^{-2}$, independently of $B$.

![FIG. 4: Evolution of the pressure as a function of the scale factor.](image)

The equation of state $P(\epsilon)$ is given in parametric form by Eqs. (61) and (77). Using Eqs. (75), (58) and (60),

![FIG. 5: Equation of state $P(\epsilon)$ giving the pressure as a function of the energy density (for illustration we have taken $B = 0.2$).](image)
we also have
\[
\frac{\epsilon}{\epsilon_0} = \Omega_{m,0} e^{(B+1)/B} e^{P/[\epsilon_0(1-\Omega_{m,0})B]}
\]
\[-(1 - \Omega_{m,0}) \left( \frac{P}{\epsilon_0(1-\Omega_{m,0})} + B \right)
\]
which determines \(\epsilon(P)\) in dimensionless form. The equation of state \(P(\epsilon)\) presents two branches corresponding to a normal universe and a phantom universe (see Fig. [6]). For \(\epsilon \to +\infty\), we get
\[
P \sim B\epsilon_0(1-\Omega_{m,0}) \ln \epsilon
\]
on the normal branch \((R \to 0)\) and
\[
P \sim -\epsilon
\]
on the phantom branch \((R \to +\infty)\). The pressure at the point of minimum energy \(\epsilon_M\) is
\[
P_M = -\epsilon_0(1-\Omega_{m,0}) \left[ B + 1 + B \ln \left( \frac{\Omega_{m,0}}{1-\Omega_{m,0}} \right) - B \ln B \right].
\]
Comparing Eqs. (69) and (84), we find that \(P_M = -\epsilon_M\) (see also Sec. [IV A].)

E. The equation of state parameter \(w\)

The equation of state parameter \(w\) is defined by
\[
P = w\epsilon.
\]
According to Eqs. (61) and (77), it can be expressed in terms of the scale factor as
\[
w = -(1 - \Omega_{m,0}) \left( \frac{B + 1 + 3B \ln R}{\Omega_{m,0}} + (1 - \Omega_{m,0})(1 + 3B \ln R) \right).
\]
For \(R \to 0\),
\[
w \sim -3b \frac{1 - \Omega_{m,0}}{\Omega_{m,0}} R^3 \ln R.
\]
For \(R \to +\infty\),
\[
w \to -1.
\]
More precisely, \(w + 1 \sim -1/(3 \ln R)\) for \(R \to +\infty\). We note that \(w > 0\) for \(R < R_w\) and \(w < 0\) for \(R > R_w\). Therefore, the pressure is positive for \(R < R_w\) and negative for \(R > R_w\), in agreement with the results of Sec. [IV B]. We also note that \(w > -1\) for \(R < R_M\) and \(w < -1\) for \(R > R_M\). Therefore, the universe is normal for \(R < R_M\) (the energy density decreases as the scale factor increases) and phantom for \(R > R_M\) (the energy increases as the scale factor increases), in agreement with the results of Sec. [IV B].

The curve \(w(R)\) is plotted in Fig. [6]. It starts from \(w = 0\) at \(R = 0\), increases, reaches a maximum (see Fig. [7]), decreases, becomes negative after \(R = R_w\), passes below \(-1\) after \(R = R_M\), reaches a minimum, increases, and tends to \(-1\) as \(R \to +\infty\).

For \(B = 0\), corresponding to the ΛCDM model, the function \(w(R)\) is given by
\[
w = \frac{-(1 - \Omega_{m,0})}{\Omega_{m,0} - 1 - \Omega_{m,0}}.
\]
It starts from \(w = 0\) at \(R = 0\), decreases monotonically, and tends to \(-1\) as \(R \to +\infty\).

The present value of the equation of state parameter is
\[
w_0 = -(1 - \Omega_{m,0})(B + 1).
\]
For \(B = 0\), corresponding to the ΛCDM model, \(w_0 = -(1 - \Omega_{m,0}) = -0.726\).
F. The velocity of sound

The velocity of sound $c_s$ is defined by $c_s^2 = P'(\epsilon)c^2$. Taking the derivative of Eq. (52) with respect to $\epsilon$, we obtain

$$\frac{c_s^2}{c^2} = \frac{1}{\frac{c_s^2}{c^2} - 1}. \quad (91)$$

This relation requires that $A \geq 0$, hence $B \geq 0$, otherwise the velocity of sound would always be imaginary ($c_s^2 \leq 0$). The constraint $A \geq 0$ is consistent with the results of Sec. [11]. It also reinforces the interpretation of $A$ as a (logotropic) temperature.\footnote{For a classical isothermal gas with $P = \rho k_B T/m$ and $c_s^2 = P'(\rho) = k_B T/m$, the velocity of sound is real at positive temperatures and imaginary at negative temperatures, implying that negative temperatures are (usually) forbidden.} Substituting Eqs. (34), (58) and (68) in Eq. (91), we get

$$\frac{c_s^2}{c^2} = \frac{1}{(\frac{R_M}{R})^3 - 1}. \quad (92)$$

For $B = 0$, corresponding to the $\Lambda$CDM model, $c_s^2 = 0$ since the pressure is constant ($P = -\epsilon_A$). For $B > 0$, the velocity of sound is real for $R < R_M$ (i.e. when the universe is normal) and imaginary for $R > R_M$ (i.e. when the universe is phantom). The relation between the velocity of sound and the scale factor is plotted in Fig. 8.

![FIG. 8: Evolution of the square of the velocity of sound as a function of the scale factor for illustration we have taken $B = 0.2$.](image)

The velocity of sound in the present-day universe is

$$\left(\frac{c_s^2}{c^2}\right)_0 = \frac{1}{R_M^3 - 1}. \quad (93)$$

If we require that $(c_s^2)_0 \geq 0$, we must have $R_M > 1$, hence $B < B_1 = 0.377$. The condition that the present-day velocity of sound is real coincides with the condition that the present-day universe is non-phantom.

More stringent constraints on $B$ can be obtained from the following arguments. In order to have $(c_s^2/c^2)_0 \leq 1$, the dimensionless logotropic temperature must satisfy

$$B \leq \frac{\Omega_{m,0}}{2(1 - \Omega_{m,0})} = \frac{B_1}{2} = 0.1885. \quad (94)$$

This ensures that the velocity of sound in the present universe is less than the speed of light. In order to have $(c_s^2/c^2)_0 \leq 1/3$, the dimensionless logotropic temperature must satisfy

$$B \leq \frac{\Omega_{m,0}}{4(1 - \Omega_{m,0})} = \frac{B_1}{4} = 0.09425. \quad (95)$$

This ensures that the present universe is non-relativistic (i.e. the velocity of sound in the present universe is less than the velocity of sound in a radiation-dominated universe for which $P = c/3$ and $c_s^2/c^2 = 1/3$). We shall take the value $B_1/4 = 0.09425$ as an upper bound for the allowable values of $B$.

For a given $B$, the velocity of sound becomes larger than the speed of light ($c_s > c$) when $R > R_s$ with

$$R_s = \left[\frac{\Omega_{m,0}}{B(1 - \Omega_{m,0})}\right]^{1/3} \approx \frac{R_M}{2^{1/3}}. \quad (96)$$

Since $R_s < R_M$, the logotropic model may break down before the universe reaches the phantom regime at $R = R_M$.

Remark: If we compute the velocity of sound from the relation $c_s^2 = P'(\rho)$, we obtain

$$c_s^2 = A \rho = \left(\frac{R}{R_M}\right)^3 c^2. \quad (97)$$

This approximation is valid when $\rho \gg A$ or $R \ll R_M$.

G. The deceleration parameter

In a flat universe without cosmological constant ($k = \Lambda = 0$), the deceleration parameter $q = -\ddot{a}a/\dot{a}^2$ is given by (see Eqs. (2), (3) and (87)):

$$q = \frac{1 + 3w}{2}. \quad (98)$$

Using the expression of $w$ given by Eq. (86), we obtain

$$q = \frac{\Omega_{m,0}}{R^3} - \frac{(1 - \Omega_{m,0})(3B + 2 + 6B \ln R)}{2 \left[\Omega_{m,0} + (1 - \Omega_{m,0})(1 + 3B \ln R)\right]}. \quad (99)$$

For $R \to 0$,

$$q \sim \frac{1}{2} \left[1 - 9B \frac{1 - \Omega_{m,0}}{\Omega_{m,0}} R^3 \ln R\right]. \quad (100)$$
For $R \to +\infty$, 

$$q \to -1.$$  \hfill (101)

The deceleration parameter $q(R)$ vanishes at $R = R_c$ determined in the next section. When $R < R_c$, the universe is decelerating ($q > 0$) and when $R > R_c$, the universe is accelerating ($q < 0$).

The curve $q(R)$ is plotted in Fig. 9. Its behavior follows the one of $w(R)$ described in Sec. IV.E.

For $B = 0$, corresponding to the ΛCDM model, the function $q(R)$ is given by

$$q = \frac{\Omega_{m,0}}{R^2} - 2(1 - \Omega_{m,0}) \frac{1}{2} + \frac{1}{1 - \Omega_{m,0}}.$$  \hfill (102)

FIG. 9: Evolution of the deceleration parameter as a function of the scale factor for $B = 0$, $B = 0.1$, and $B = 0.2$ (top to bottom).

H. The point at which the universe accelerates

According to Eq. [99], the point $R_c(B)$ at which the universe starts accelerating ($q = 0$) is determined by the equation

$$B = \frac{\Omega_{m,0}}{1 - \Omega_{m,0}} - \frac{1}{3(1 + 2 \ln R_c)}.$$  \hfill (103)

For $B = 0$, corresponding to the ΛCDM model,

$$R_c = \left[ \frac{\Omega_{m,0}}{2(1 - \Omega_{m,0})} \right]^{1/3} = 0.574.$$  \hfill (104)

For $B \to +\infty$,

$$R_c \to e^{-1/2} = 0.6065.$$  \hfill (105)

The curve $R_c(B)$ is plotted in Fig. 10. Since $R_c < 1$, the present-day universe is always accelerating, whatever the value of $B$. We also note that, in the framework of the logotropic model, $R_c$ lies in the small interval $[0.574, 0.6065]$. Since $R_c(B)$ increases, we note that when $B > 0$ the acceleration of the universe occurs later than in the ΛCDM model.

FIG. 10: The scale factor $R_c$ at which the universe accelerates as a function of $B$.

The present value of the deceleration parameter is

$$q_0 = \frac{\Omega_{m,0}}{2} - \frac{(1 - \Omega_{m,0})(3B + 2)}{2}. \hfill (106)$$

For $B = 0$, corresponding to the ΛCDM model,

$$q_0 = \frac{3\Omega_{m,0}}{2} - 2 = -0.589. \hfill (107)$$

The condition for the present-day universe to be accelerating is that $\Omega_{m,0} < 2/3$. This inequality is indeed realized by the observational value $\Omega_{m,0} = 0.274$.

I. The transition between matter and dark energy

If we interpret in Eq. (61) the rest-mass density as “dark matter” and the internal energy as “dark energy”, their ratio evolves with the scale factor as

$$\frac{\rho_{DE}}{\rho_{DM}} = \frac{u(\rho)}{\rho c^2} = \frac{1 - \Omega_{m,0}}{\Omega_{m,0}} R^3 (1 + 3B \ln R). \hfill (108)$$

The transition between dark matter and dark energy takes place at a scale factor $R_2$ determined by

$$\frac{\Omega_{m,0}}{R_2^2} = (1 - \Omega_{m,0})(1 + 3B \ln R_2). \hfill (109)$$

For $B = 0$, corresponding to the ΛCDM model, we get $R_2(0) = 0.723$. For $B \to +\infty$, we get $R_2 \sim \exp \left\{ (2\Omega_{m,0} - 1)/(1 - \Omega_{m,0})B \right\} \to 1$. More generally, the function $R_2(B)$ is plotted in Fig. 11.
The scale factor $R^2$ corresponding to the transition between dark matter and dark energy as a function of $B$.

Using Eq. (61), the Friedmann equation (3) with $\Lambda = 0$ takes the form

$$H = \frac{\dot{R}}{R} = H_0 \sqrt{\frac{\Omega_{m,0}}{R^3} + (1 - \Omega_{m,0})(1 + 3B \ln R)}. \quad (110)$$

The temporal evolution of the scale factor $R(t)$ is given by

$$\int_0^R \frac{dx}{x^{3/2} + (1 - \Omega_{m,0})(1 + 3B \ln x)} = H_0 t. \quad (111)$$

For $t \to 0$,

$$R \sim \left(\frac{3}{2} \sqrt{\Omega_{m,0} H_0 t}\right)^{2/3}, \quad \epsilon \sim \frac{4}{9H_0^2 t^2}, \quad (112)$$

like in the EdS universe (see Sec. II B). For $t \to +\infty$,

$$R \propto e^{3B(1-\Omega_{m,0})H_0^2 t^2}, \quad \epsilon \sim \left[\frac{3B}{2} (1 - \Omega_{m,0}) H_0^2 t\right]^2. \quad (113)$$

This solution, which is valid in the regime where the universe is phantom, has a super-de Sitter behavior. However, as indicated in Sec. IV F, the logotropic model may break down before the universe enters in this regime because the velocity of sound exceeds the speed of light when $R > R_s$ with $R_s < R_M$.

The curve $R(t)$ is plotted in Fig. 12. For $B = 0$, corresponding to the $\Lambda$CDM model, we recover the analytical solution of Eq. (114).

The age of the universe

Taking $R = 1$ in Eq. (111), the age of the universe is given as a function of the logotropic temperature by

$$t_0(B) = \frac{1}{H_0} \int_0^{R_0 = 1} \frac{dx}{x^{3/2} + (1 - \Omega_{m,0})(1 + 3B \ln x)}. \quad (114)$$

The characteristic times of the logotropic model as a function of $B$ (see the text for details).

For $B = 0$, corresponding to the $\Lambda$CDM model, we recover Eq. (18) leading to the standard value $t_0(0) = 13.8$ Gyrs. More generally, the function $t_0(B)$ is plotted in Fig. 13. We also plot the functions $t_w(B)$, $t_s(B)$, $t_2(B)$, $t_3(B)$ and $t_M(B)$ corresponding to the time at which (i) the pressure becomes negative, (ii) the universe accelerates, (iii) the transition from dark matter to dark energy occurs, (iv) the velocity of sound becomes higher than the speed of light, (v) the universe enters in the
phantom regime. They are obtained by replacing \( R_0 = 1 \) in Eq. \( (111) \) by \( R_e, R_c, R_2, R_s \) and \( R_M \) given by Eqs. \( (79), (103), (109), (96), \) and \( (68). \) For \( B = 0, \) we recover the standard values \( t_1(0) = 7.18 \text{ Gyrs} \) and \( t_2(0) = 9.61 \text{ Gyrs} \) of the \( \Lambda \text{CDM} \) model. For \( B \to 0, \) we have the asymptotic behavior

\[
t_w(B) \sim \frac{2}{3H_0 \sqrt{\Omega_{m,0}}} e^{-\frac{1}{3}} \to 0. \tag{115}
\]

### L. The Hubble diagram

In terms of the redshift parameter

\[
z + 1 = \frac{a_0}{a}, \tag{116}
\]

the Hubble function \( H(z) \) can be written as

\[
H(z) = H_0 \sqrt{\Omega_{m,0}(z + 1)^3 + (1 - \Omega_{m,0})[1 - 3B \ln(z + 1)]}. \tag{117}
\]

The Hubble function \( H(z) \) is plotted in Fig. 14 for different values of \( B \) and compared with the observational data of Refs. \( [69, 70]. \)

The history of the universe expansion is revealed by the Hubble diagram giving the distance modulus \( \mu(z) = m - M \) as a function of the redshift \( z, \) where \( m \) is the apparent luminosity and \( M \) the absolute luminosity of a light-emitting source (standard candle) such as supernovae of Type Ia. The procedure to obtain this curve is explained in Ref. \( [45]. \) In a spatially flat universe, the luminosity distance of a source with redshift \( z \) is \( [71]: \)

\[
\frac{d_L(z)}{\text{Mpc}} = c(1 + z) \int_0^{z} \frac{dz'}{H(z')} . \tag{118}
\]

The \( K \)-corrected distance modulus of a light-emitting source is then given by \( [72]: \)

\[
\mu(z) = 5 \log \left( \frac{d_L(z)}{\text{Mpc}} \right) + 25. \tag{119}
\]

The function \( \mu(z) \) corresponding to the logotropic model is plotted in Fig. 14 for different values of \( B, \) and compared with observational data of the Union 2.1 Compilation \( [73]. \) Even for values of \( B \) as large as 0.2, the curves are very close to the \( \Lambda \text{CDM} \) model and can hardly be distinguished. They are all consistent with the observational results.

### M. The shift parameter from CMB

The shift parameter \( R, \) which is related to the position of the first acoustic peak in the power spectrum of the temperature anisotropies \( [74], \) provides an information from the CMB \( [45]. \) It is defined by

\[
R = \sqrt{\Omega_{m,0}} \int_0^{z_{\ast}} \frac{dz'}{H(z')/H_0}, \tag{120}
\]

where \( z_{\ast} \) is the value of the cosmological redshift at photon decoupling. We adopt the nine-year WMAP survey final result \( z_{\ast} = 1091.64 \pm 0.47 \) \( [75]. \) The value of the shift parameter obtained by these authors is \( R_{\text{obs}} = 1.7329 \pm 0.0058. \) The \( \Lambda \text{CDM} \) model gives \( R_{\Lambda \text{CDM}} = 1.7342. \) On the other hand, using a combination of the Planck first-data release with those of the WMAP survey \( [76], \) the value \( R_{\text{obs}} = 1.7407 \pm 0.0091 \) is obtained. The curve \( R(B) \) obtained from the logotropic model is plotted in Fig. \( [16]. \) From the observational value \( R_{\text{obs}} \) we find the constraint \( B \leq 0.0262, \) and from the observational value \( R_{\text{obs}} \) we find the constraint \( B \leq 0.0379. \) These values are about 3 times smaller than the bound \( B_1/4 = 0.09425 \) obtained in Sec. \( [IVF]. \)
Combining the condition of hydrostatic equilibrium with the Poisson equation
\[ \Delta \Phi = 4\pi G \rho. \] (122)
we obtain the differential equation
\[ \nabla \cdot \left( \frac{1}{\rho} \nabla P \right) = -4\pi G \rho. \] (123)

For the logotropic equation of state \[ 49 \], using Eq. (121), we find that the density is related to the gravitational potential by the Lorentzian-type distribution
\[ \rho(r) = \frac{1}{\alpha + \frac{1}{A} \Phi(r)}. \] (124)

where \( \alpha \) is a constant.\(^{20}\) On the other hand, Eq. (123) takes the form
\[ \Delta \left( \frac{1}{\rho} \right) = 4\pi G \frac{A}{\rho}. \] (125)

Assuming spherical symmetry, and introducing the notations
\[ \theta = \frac{\rho_0}{\rho}, \quad \xi = \frac{r}{r_0}, \] (126)
where \( \rho_0 \) is the central density and
\[ r_0 = \left( \frac{A}{4\pi G \rho_0} \right)^{1/2}, \] (127)
is a typical core radius, we obtain
\[ 1 \frac{d}{\xi^2} \frac{d^2}{d\xi^2} \left( \xi^2 \frac{d\theta}{d\xi} \right) = \frac{1}{\theta}, \] (128)
with
\[ \theta(0) = 1, \quad \theta'(0) = 0. \] (129)

This equation coincides with the Lane-Emden equation of index \( n = -1 \) \[ 27 \]. We note that the Lane-Emden equation of index \( n = -1 \) cannot be obtained from a polytropic equation of state of index \( n = -1 \) (which corresponds to a constant pressure \( P = K \)) because, when \( P \) is a constant, there is no pressure gradient in Eq. (121) to counterbalance gravity, and the system collapses. This is the reason for the occurrence of density cusps in the CDM model for which \( P = 0 \). By contrast, a logotropic equation of state can sustain gravity and develops flat density cores instead of cusps. The fact that the logotropic equation of state \[ 49 \] yields a Lane-Emden equation of index \( n = -1 \) clearly demonstrates that it can be interpreted as the limit of a polytropic equation of state with \( \gamma \to 0 \) (i.e. \( n \to -1 \)) and \( K \to \infty \) such that \( K^2 = A \) is finite \[ 53 \]. In a sense, the logotropic equation of state is a “regularization” of the polytropic equation of state with index \( n = -1 \).

\( ^{20} \) A self-gravitating system described by a distribution function of the form \( f(r, v) = f(\epsilon) \), where \( \epsilon = v^2 / 2 + \Phi \) is the individual energy of the particles, has a barotropic equation of state \( P(\rho) \) obtained by eliminating \( \Phi(r) \) from the relations \( \rho = \int f \, dv = \rho(\Phi) \) and \( P = (1/3) \int f v^2 \, dv = P(\Phi) \) \[ 60 \]. For example, a system described by a polytropic distribution function has a polytropic equation of state \[ 60 \]. In a sense, the logotropic equation of state \[ 49 \] is associated with a Lorentzian distribution function \[ 53 \]. However, the connection is not direct because the Lorentzian is not normalizable in \( d = 3 \), except if we introduce a maximum bound on the velocity. Therefore, rigorously speaking, the logotropic equation of state cannot be derived from a distribution function and, consequently, \( P/\rho \) does not represent a velocity dispersion \( \sigma^2 \). It is more likely that the logotropic equation of state comes from a field theory at \( T = 0 \) as discussed in Appendix \[ 58 \].
The King model at the limit of microcanonical stability taking tidal effects into account is the King model \([78]\). A famous example of distribution functions generating a universal density profile is the isotermal sphere \([77]\). This can be compared to the \(r^{-2}\) behavior of the isothermal sphere \([77]\). Because of this slow decay, the total mass of a logotropic sphere is infinite. This implies that the logotropic equation of state cannot hold at large distances. We note that the empirical Burkert density profile, that provides a good fit of many dark matter halos, decreases at large distances as \(r^{-3}\) \([39]\). This is substantially steeper than the \(r^{-1}\) behavior of a logotropic halo. In practice, the finite mass and finite radius of dark matter halos result from tidal effects or from incomplete violent relaxation. Tidal effects and incomplete violent relaxation alter the density profile at large distances and steepen it. A famous example of distribution functions taking tidal effects into account is the King model \([78]\). The King model at the limit of microcanonical stability can be approximated by the modified Hubble profile that has a core and that decreases at large distances as \(r^{-3}\), similarly to the Burkert profile \([79]\). Since tidal effects, or other complicated effects such as incomplete relaxation, are not taken into account in the logotropic model, we should not give too much credit on the asymptotic behavior of its density profile at very large distances.\(^{22}\) However, it is relevant to mention that, in a recent paper, Burkert \([55]\) observes that the slope of the density profile of dark matter halos close to the core radius is approximately equal to \(-1\) (see the upper right panel of his Fig. 1) which is precisely the density exponent of a logotropic. This is a first hint that a logotropic equation of state may be relevant in the case of dark matter halos.

Using the Lane-Emden equation \([128]\), the mass profile
\[
M(r) = \int_0^r \rho(r') 4\pi r'^2 \, dr' \quad \text{is given by}
\]
\[
M(r) = 4\pi \rho_0 \theta_0^3 \xi^2 \theta'(\xi). \tag{130}
\]
The circular velocity defined by
\[
v_c^2(r) = GM(r)/r \quad \text{can be expressed as}
\]
\[
v_c^2(r) = 4\pi G \rho_0 r_0^3 \xi \theta'(\xi). \tag{131}
\]
We define the halo radius \(r_h\) as the radius at which \(\rho/\rho_0 = 1/4\). The dimensionless halo radius is the solution of the equation \(\theta(\xi_h) = 4\). We numerically find \(\xi_h = 5.8458\) and \(\theta'(\xi_h) = 0.69343\). Then, \(r_h = \xi_h r_0\). The normalized halo mass at the halo radius is given by
\[
\frac{M_h}{\rho_0 r_h^3} = \frac{v_c^2(r_h)}{G \rho_0 r_h^3} = 4\pi \frac{\theta'(\xi_h)}{\xi_h} = 1.49. \tag{132}
\]
This value is relatively close to the value \(M_h/\rho_0 r_h^3 = 1.60\) \([79, 80]\) obtained with the Burkert profile defined by
\[
\frac{\rho(r)}{\rho_0} = \frac{1}{(1 + x)(1 + x^2)}, \quad x = \frac{r}{r_h}, \tag{133}
\]
\[
\frac{v_c(r)}{v_c(r_h)} = 1.98 \sqrt{x} \left[ \ln(1 + x) - \arctan x + \frac{1}{2} \ln(1 + x^2) \right]^{1/2}. \tag{134}
\]

![FIG. 17: Normalized density profile of a logotropic sphere. It decays as \(\xi^{-1}\) when \(\xi \to +\infty\).](image)

\(\xi = \frac{\rho}{\rho_0}\)

21 The singular logotropic sphere is \(\rho_s = (A/8\pi G)1/2 r^{-1}\) \([53]\) and the singular isothermal sphere is \(\rho_s = (1/2\pi G 3m) r^{-2}\) \([77]\).

22 In the context of hierarchical clustering (small gravitationally bound clumps of dark matter halos merge to form progressively larger objects), we suggest that the core of the objects after merging remains logotropic while their halo develops a density profile decreasing as \(r^{-3}\) due to tidal effects and incomplete relaxation.
The relation \( \xi_h = 4\pi \rho_0 r_0 u' \frac{r_u}{r_0} \) allows us to determine the logotropic temperature \( A \) from the measurement of the surface density \( \Sigma_0 \). We find

\[
A = 4\pi G \left( \frac{\Sigma_0}{\xi_h} \right)^2 = 2.13 \times 10^{-9} \text{ g m}^{-1} \text{s}^{-2}.
\]

The dimensionless logotropic temperature is then given by

\[
B = \frac{4\pi G}{c_0} \frac{1}{1 - \Omega_{m,0}} \left( \frac{\Sigma_0}{\xi_h} \right)^2 = 3.53 \times 10^{-3}.
\]

We note that this value satisfies the cosmological bound \( 0 \leq B \leq 0.09425 \) obtained in Sec. IVF as well as the bounds \( 0 \leq B \leq 0.0262 \) and \( 0 \leq B \leq 0.0379 \) obtained from the measurement of the shift parameter from CMB in Sec. IVM. This agreement is particularly rewarding because the value of \( B \) obtained in Eq. (137) is based on galactic observations that are completely different from the cosmological observations of Sec. IVF and IVM.

Therefore, a logotropic equation of state can simultaneously describe dark matter halos and account for cosmological constraints. This may be a hint that dark matter and dark energy are the manifestations of a single dark fluid.

There are interesting consequences of this result. According to Eq. (132), the mass of the halos calculated at the halo radius \( r_h \) is given by \( M_h = 1.49 \Sigma_0 r_h^2 \). Since the surface density of the dark matter halos is constant, Eq. (132) implies that \( M_h/M_\odot = 210/(r_h/\text{pc})^2 \propto r_h^2 \). This scaling is consistent with the observations. On the other hand, Strigari et al. have proposed that all dwarf spheroidal galaxies (dSphs) of the Milky Way have the same total dark matter mass contained within a radius of \( r_u = 300 \text{ pc} \). From the observations, they obtain \( \log(M_{300}/M_\odot) = 7.6 \pm 0.3 \). This result is still unexplained.

We now show how the logotropic equation of state immediately leads to this result. According to Eq. (130), using \( \xi_h = r_u/r_0 \), we have

\[
M_{300} = 4\pi \rho_0 r_0^2 u' \frac{r_u}{r_0}.
\]

Introducing the halo radius \( r_h = \xi_h r_0 \), we obtain

\[
M_{300} = 4\pi \rho_0 r_0^2 \frac{r_u^2}{\xi_h} \frac{r_u}{r_h} \left( \frac{r_u}{r_h} \frac{r_h}{r_0} \right).
\]

As we have already indicated, it is an observational evidence that the surface density \( \rho_0 r_h \) of the halos is a constant. In principle, the last term in Eq. (139) is not a constant since it depends on \( r_h \) which substantially changes from halo to halo. However, for the logotropic
distribution, we have the asymptotic result \( \theta(\xi) \sim \xi/\sqrt{2} \)
for \( \xi \to +\infty \) \[^{53}\] . More precisely, for \( \xi \geq 8 \) (typically), \( \theta'(\xi) \) undergoes damped oscillations about \( 1/\sqrt{2} \). Therefore, \( \theta'(\xi) \) quickly reaches a constant value \( 1/\sqrt{2} \). For the dSphs considered in \[^{59}\] , \( \xi_{h_d}/r_h \gg 1 \) (see, e.g., Table 2 of \[^{81}\] ), so Eq. (130) may be replaced by

\[
M_{300} = \frac{4\pi \Sigma_0 \rho_c^2}{\xi_0 \sqrt{2}}, \quad (140)
\]

which is a constant in agreement with the claim of Strigari et al. \[^{59}\] . Furthermore, using \( \Sigma_0 = 141 M_\odot/p_c^2 \), the numerical application gives

\[
M_{300} = 1.93 \times 10^7 M_\odot, \quad (141)
\]

leading to \( \log(M_{300}/M_\odot) = 7.28 \) in very good agreement with the observational value. It is also relevant to express this result directly in terms of the logotropic temperature \( \Lambda \) under the form

\[
M_{300} = r_u^2 \left( \frac{2\pi A}{G} \right)^{1/2}, \quad (142)
\]

which shows that the constancy of \( M_{300} \) is due to the universality of \( A \).

VI. A PREDICTION OF THE LOGOTROPIC TEMPERATURE

In the previous section, the values of \( A \) and \( B \) have been deduced from the observations. In this section, we show that they can actually be predicted from general considerations.

Introducing the cosmological density \( \rho_\Lambda = \rho_\Lambda c^2 = (1 - \Omega_{m,0})\epsilon_0 \), which corresponds to the present value of the dark energy term (internal energy) in Eq. (57), we can rewrite Eq. (60) as

\[
\frac{\rho \epsilon c^2}{\epsilon_\Lambda} = \frac{\Omega_{m,0}}{1 - \Omega_{m,0}} e^{1 + 1/B}. \quad (143)
\]

Since \( B \) is expected to be very small, this ratio is huge. This makes us think of the huge ratio \( \rho_P/\rho_\Lambda \sim 10^{423} \) between the vacuum energy, associated to the Planck density \( \rho_P = 5.16 \times 10^{99} \text{ g m}^{-3} \), and the dark energy (or cosmological constant \( \Lambda \)), associated to the cosmological density \( \rho_\Lambda = 6.72 \times 10^{-24} \text{ g m}^{-3} \). In the framework of the \( \Lambda \)CDM model, this huge ratio is at the origin of the cosmological constant problem \[^{24}\] . In the present approach, in which there is no cosmological constant nor dark energy, it has a completely different significance. We propose that it determines the dimensionless logotropic temperature \( B \). Therefore, we propose to identify the density \( \rho_\Lambda \) in the logotropic equation of state (49) with the Planck density \( \rho_P \). In that case, Eq. (143) becomes

\[
\frac{\rho_P}{\rho_\Lambda} = \frac{\Omega_{m,0}}{1 - \Omega_{m,0}} e^{1 + 1/B}. \quad (144)
\]

This yields approximately \( B \approx 1/\ln(\rho_P/\rho_\Lambda) \approx 1/(123 \ln(10)) \), where 123 is the famous number occurring in the ratio \( \rho_P/\rho_\Lambda \approx 10^{123} \). More precisely,

\[
B = \frac{1}{\ln \left( \frac{1 - \Omega_{m,0} \rho_P}{1 - \Omega_{m,0} \rho_\Lambda} \right) - 1} = 3.53 \times 10^{-3}. \quad (145)
\]

This value turns out to be in perfect agreement with the value obtained from the measurement of the surface density of dark matter halos in Sec. \[^{7}\] . If we accept that this agreement is not fortuitous, or coincidental, this is a strong argument in favor of the logotropic model.

Therefore, by taking \( \rho_\star = \rho_P \) in the logotropic equation of state (49), we obtain a prediction of the parameter \( B \) which is in perfect agreement with the observations. With this value, there is no free parameter in our model, so we can predict all the measurable quantities that occur in cosmology and in the study of dark matter halos (see Table I in Sec. \[^{7}\] ). We emphasize that our estimate of \( B \) only depends on the values of \( \rho_P, \rho_\Lambda \) and \( \Omega_{m,0} \) that are known accurately from the observations.

In conclusion, the logotropic equation of state (49) can be written as

\[
P = B \epsilon_\Lambda \ln \left( \frac{\rho}{\rho_P} \right), \quad (146)
\]

with \( \rho_P = 5.16 \times 10^{99} \text{ g m}^{-3} \), \( \epsilon_\Lambda = (1 - \Omega_{m,0})\epsilon_0 = 6.04 \times 10^{-7} \text{ g m}^{-1} \text{ s}^{-2} \), and \( B = 3.53 \times 10^{-3} \). The logotropic temperature is

\[
A = B \epsilon_\Lambda = 2.13 \times 10^{-9} \text{ g m}^{-1} \text{ s}^{-2}. \quad (147)
\]

It is of the order of the cosmological density \( \epsilon_\Lambda \) divided by \( \ln(\rho_P/\rho_\Lambda) \approx |123 \ln(10)| \).

The \( \Lambda \)CDM model is recovered for \( B = 0 \). This corresponds to \( \rho_P \to +\infty \), hence \( h \to 0 \). Therefore, the fact that \( B \) is small but nonzero shows that quantum mechanics plays a role in the late universe in relation to dark energy.

The logotropic equation of state (146) has several nice properties.

(i) If our approach is correct, there is no dark matter nor dark energy. There is just one dark fluid described by the equation of state (146). In that case, it may be relevant to interpret the logotropic temperature \( A = 2.13 \times 10^{-9} \text{ g m}^{-1} \text{ s}^{-2} \) as a fundamental constant which supersedes the cosmological constant. We note that it depends on all the fundamental constants of physics \( h, G, c, \) and \( \Lambda \) [see Eqs. (145) and (147)]. Then, Eq. (60) with \( \rho_\star = \rho_P \) may be used to predict \( \Omega_{m,0} \) for a given value of the present energy density \( \epsilon_0 \) determined from the observations by the Hubble constant. In the present interpretation \( \Omega_{m,0} \) does not represent the proportion of dark matter. It simply gives the proportion of the rest-mass energy of the dark fluid as compared to its total energy (the remaining energy being internal energy).
(ii) Since dark matter and dark energy are the manifestation of a single dark fluid, there is no cosmic coincidence problem. On the other hand, the cosmological constant problem $\rho_p/\rho_A \sim 10^{123}$ is translated into an equation [Eq. (144)] that determines the logotropic temperature $B$.

(iii) In the polytropic model (see Sec. II), the pressure is negative because $K < 0$, i.e. because the polytropic temperature is negative. In the logotropic model, the logotropic temperature $A$ is positive, and the pressure is negative because $\rho < \rho_p$. Positive temperatures are more satisfying on a physical point of view than negative temperatures. On the other hand, while the pressure $P(\rho)$ is negative, which is required at the cosmological scale to produce an acceleration of the expansion of the universe, its derivative $P'(\rho)$ is positive, which is required at the scale of dark matter halos to counteract self-gravity and avoid density cusps.

(iv) It is interesting to see that both the cosmological density $\rho_A$ and the Planck density $\rho_P$ appear in the logotropic equation of state (146). The presence of the Planck density is rather unexpected, and intriguing, because the logotropic equation of state only describes a relatively old universe, dominated by dark matter and dark energy ($\rho_A$) well after the phase of early inflation dominated by quantum mechanics ($\rho_P$). The fact that the Planck density appears as a density scale in the logarithm of the ratio of the cosmological density over the Planck density has a physical meaning since these quantities differ by 123 orders of magnitude. The occurrence of the Planck density suggests that the logotropic equation of state (146) may be the limit of a more general equation of state connecting the very early universe ($\rho_P$) to the very late universe ($\rho_A$). In relation to this observation, we note that the pressure in the logotropic model vanishes precisely at the Planck scale, so our treatment breaks down at that scale.

Now that $B$ and $A$ have been derived from theoretical considerations [see Eqs. (145) and (147)], we can reverse the arguments of Sec. IV and predict the values of $\Sigma_0$ and $M_{300}$ [see Eqs. (135) and (142)]. We obtain $\Sigma_0 = 141 M_\odot/pc^2$ and $M_{300} = 1.93 \times 10^7 M_\odot$ in perfect agreement with the observations, and without any free parameter. We can also obtain an estimate of the Jeans length $\lambda_J$ at the beginning of the matter era where perturbations start to grow. We assume that the matter era starts at $R_i = 10^{-4}$, corresponding to the epoch of matter-radiation equality. In this era, we can make the approximation $\epsilon = \rho c^2$, so the Jeans wavenumber is given by [61]:

$$k_j^2 = \frac{4\pi G \rho R^2}{c_s^2}. \quad (148)$$

From Eqs. (66) and (97), we find $\rho_i = 2.54 \times 10^{-12} g/m^3$ and $(c_s^2/c^2)_0 = 9.33 \times 10^{-15}$. This leads to a Jeans length $\lambda_J = 2\pi/k_j = 1.25 \times 10^{18} m = 40.4 pc$ which is of the order of magnitude of the smallest known dark matter halos such as Willman I ($r_h = 33 pc$), see, e.g., Table 2 of [81]. We predict that there should not exist halos of smaller size since the perturbations are stable for $\lambda < \lambda_J$. This is in agreement with the observations. By contrast, in the CDM model, since $P = 0$, the velocity of sound $c_s = 0$. As a result, the Jeans length is zero ($\lambda_J = 0$), implying that the homogeneous background is unstable at all scales so that halos of any size should be observed in principle, which is not the case. Therefore, a small but nonzero value of $B$, yielding a nonzero velocity of sound and a nonzero Jeans length, is able to account for the minimum observed size of dark matter halos. It also puts a cut-off in the density power spectrum of the perturbations and sharply suppresses small-scale linear power. Finally, it solves the cuspy problem and the missing satellite problem. On the other hand, if we make the numerical application for the present-day universe ($R_0 = 1$), we find $\rho_0 = 2.54 \times 10^{-24} g/m^3$, $(c_s^2/c^2)_0 = 9.42 \times 10^{-3}$ and $\lambda_J = 1.25 \times 10^{26} m = 4.06 10^9 pc$. The Jeans length is so large (of the order of the horizon $\lambda_H = c/H_0 = 1.32 \times 10^{26} m$ since $\lambda_J/\sqrt{G\rho_0} \sim c/H_0$) that there is no Jeans instability in the present universe. A detailed study of the perturbations in the logotropic model is beyond the scope of this paper and will be considered in a future work.

VII. NUMERICAL APPLICATIONS

In this section, we provide the values of the principal quantities that occur in cosmology and in the study of DM halos (see Table I). We make the numerical application for: (i) $B = 0$, corresponding to the $\Lambda$CDM model, (ii) $B = 0.09425$, corresponding to the cosmological constraint of Sec. IV and (iii) $B = 3.53 \times 10^{-3}$, corresponding to the prediction of Sec. VI.

We recall that ($R_{su}$, $\epsilon_{su}$, $t_0$) refer to the values at which the pressure of the universe becomes negative, ($R_c$, $\epsilon_c$, $t_c$) refer to the values at which the universe accelerates, ($R_2$, $\epsilon_2$, $t_2$) refer to the transition between dark matter and dark energy (i.e. when the internal energy of the dark fluid dominates its rest-mass energy), $t_0$ is the age of the universe, $q_0$ is the present value of the deceleration parameter, $w_0$ is the present value of the equation of state parameter, $c_s,0$ is the present velocity of sound, ($R_s$, $\epsilon_s$, $t_s$) refer to the values at which the velocity of sound exceeds the speed of light, ($R_M$, $\epsilon_M$, $t_M$) refer to the values at which the universe becomes phantom, $R$ is the shift parameter from CMB, $A$ is the logotropic temperature, $\Sigma_0$ is the surface density of DM halos, and $M_{300}$ is the mass of dwarf halos enclosed within a sphere of radius $r_u = 300 pc$.

We note that for the predicted value of $B = 3.53 \times 10^{-3}$, the cosmological parameters are almost the same as in the $\Lambda$CDM model. This is satisfactory since the $\Lambda$CDM model works well at the cosmological scale. In this sense, there is no important difference, from the observational viewpoint, between the logotropic model ($B = 3.53 \times 10^{-3}$) and the $\Lambda$CDM model ($B = 0$) at
| $B = 0$ (LCDM) | $B = 0.09425$ | $B = 3.53 \times 10^{-3}$ |
|----------------|----------------|------------------|
| $R_w$          | $2.09 \times 10^{-2}$ | $7.00 \times 10^{-42}$ |
| $(\epsilon/\epsilon_0)_w$ | $3.02 \times 10^4$ | $7.97 \times 10^{122}$ |
| $t_w$ (Gyrs)   | 0.0535          | $3.29 \times 10^{-51}$ |
| $R_c$          | 0.574           | 0.576             |
| $(\epsilon/\epsilon_0)_c$ | 2.17         | 2.05              |
| $t_c$ (Gyrs)   | 7.18            | 7.33              |
| $R_2$          | 0.723           | 0.744             |
| $(\epsilon/\epsilon_0)_2$ | 1.45         | 1.33              |
| $t_2$ (Gyrs)   | 9.61            | 10.1              |
| $t_0$ (Gyrs)   | 13.8            | 14.0              |
| $q_0$          | −0.589          | −0.692            |
| $w_0$          | −0.726          | −0.794            |
| $(c_s^2/c^2)_0$ | 0.331          | $9.42 \times 10^{-3}$ |
| $R_s$          | 1.26            | 3.77              |
| $(\epsilon/\epsilon_0)_s$ | 0.910        | 0.741             |
| $t_s$ (Gyrs)   | 17.3            | 34.5              |
| $R_M$          | 1.59            | 4.75              |
| $(\epsilon/\epsilon_0)_M$ | 0.889       | 0.7405            |
| $t_M$ (Gyrs)   | 20.7            | 38.3              |
| $A$ (g m$^{-1}$ s$^{-2}$) | $5.69 \times 10^{-8}$ | $2.13 \times 10^{-9}$ |
| $\Sigma_0$ (M$_\odot$/pc$^2$) | 1523          | 141               |
| $M_{300}$ (M$_\odot$) | $2.08 \times 10^8$ | $1.93 \times 10^7$ |

**TABLE I**: Quantities that occur in cosmology and in the study of dark matter halos for different values of $B$ (see the text for details). Some observational values are $R_c = 0.571 \pm 0.013$ [73], $q_0 = -0.53^{+0.17}_{-0.13}$ [82], and $R_2 = 0.719 \pm 0.017$ [73].

the cosmological scale. Differences will appear in the remote future, in about 20 Gyrs, when the velocity of sound approaches the speed of light and the universe tends to become phantom (the logotropic model may break down before). However, on a theoretical point of view, the logotropic model solves the cosmic coincidence problem (it gives an interpretation to the Planck density $\rho_P$ and to the cosmological density $\rho_A$ [see Eqs. (135), (142), (145), and (147)]. Such a dependence was unexpected and remains relatively mysterious. Our model first suggests that dark matter and dark energy are the manifestation of a unique entity (dark fluid) since the properties of dark matter halos such as $\Sigma_0$ and $M_{300}$ are related, through the logotropic temperature $A$, to cosmological observables such as $\rho_A$ (dark energy). On the other hand, the fact that the Planck density enters in the logotropic equation of state initially designed to model dark matter and dark energy is intriguing. It suggests that quantum mechanics manifests itself at the cosmological scale in relation to dark energy. This may be a hint for a fundamental theory of quantum gravity. This also suggests that the logotropic equation of state may be the limit of a more general equation of state providing a possible unification of dark energy ($\rho_A$) in the late universe and inflation (vacuum energy $\rho_P$) in the primordial universe. This may be a clue for the elaboration of a more general theory incorporating inflation.

**VIII. CONCLUSION**

We have proposed that the universe is made of a single dark fluid described by a logotropic equation of state.
In our model, there is no cosmological constant, no dark matter, and no dark energy. As a result, the cosmological constant problem and the cosmic coincidence problem do not arise. What we traditionally call dark matter, dark energy, and cosmological constant can be related to the intrinsic properties of the dark fluid. Dark matter corresponds to the rest-mass energy of the dark fluid and dark energy corresponds to its internal energy. The logotropic temperature $A$ of the dark fluid can be interpreted as a fundamental constant of physics superseding the cosmological constant.

We have first determined bounds on the dimensionless logotropic temperature $B$ by using cosmological constraints:

(i) The condition that the velocity of sound is not always imaginary implies $B \geq 0$.

(ii) The condition that the present velocity of sound is real implies $B \leq B_1 = 0.377$. This condition coincides with the condition that the present universe is non-phantom.

(iii) The condition that the present velocity of sound is less than the speed of light implies $B \leq B_1/2 = 0.1885$.

(iv) The condition that the present universe is non-relativistic implies $B \geq B_1/4 = 0.09425$.

(v) The measurement of the shift parameter from CMB implies $B \leq 0.0379$ or $B \leq 0.0262$ depending on the estimates.

We have then proposed that dark matter halos are described by the same logotropic equation of state. For $B > 0$, this equation of state leads to flat cores instead of density cusps, so it solves the cusp-core problem. Furthermore, it generates a universal density profile and a universal rotation curve that agree with the observational Burkert profiles up to the halo radius $r_h$. More specifically, and more strikingly, a logotropic equation of state explains naturally (i) the recent observation of Burkert, (ii) that the density of dark matter halos decreases as $r^{-3}$ close to the core radius, (ii) the observation that the surface density of dark matter halos is the same for all the halos, and (iii) the observation that the mass of dark matter halos contained in a fixed radius (e.g. 300 pc) is the same for all the dwarf halos. Using the measured value of the surface density $\Sigma_0 = 141 \ M_\odot /pc^2$, we obtain a value $B = 3.53 \times 10^{-3}$ for the logotropic temperature. From this value, we find that the mass of dark matter halos contained in a radius of 300 pc is $M_{300} = 1.93 \times 10^7 \ M_\odot$, in agreement with the value obtained by Strigari et al.

Finally, we have argued that the reference density $\rho_*$ appearing in the logotropic equation of state should be identified with the Planck density $\rho_p$. This predicts the value of the logotropic temperature to be $B = 3.53 \times 10^{-3}$ in perfect agreement with the value deduced from the observations. The corresponding value of the dimensional logotropic temperature is $A = B\epsilon_\Lambda = 2.13 \times 10^{-9} \ g \ m^{-1} \ s^{-2}$. Such a small value of $B$ yields sensibly the same results as the $\Lambda$CDM model ($B = 0$) at the cosmological scale. However, having $B = 3.53 \times 10^{-3}$ instead of $B = 0$ is crucial at the scale of dark matter halos to explain their properties, as shown above. Furthermore, it implies a nonzero velocity of sound and a nonzero Jeans length that is, at the beginning of the matter era, of the order of the minimum size ($\sim 10$ pc) of the observed dark matter halos. This also solves the missing satellite problem.

The next step is to justify the logotropic equation of state from first principles. We note that our approach is already very economical. Instead of having dark matter and dark energy with two different equations of state, we just have one dark fluid with one equation of state. The logotropic equation of state involves two parameters, the reference density $\rho_*$ and the logotropic temperature $A$. They are related to each other by Eq. which depends on the known (observed) values of $\epsilon_0$ and $\Omega_{m,0}$. Assuming that $\rho_\star = \rho_p$, this equation determines $A$, so there is no free parameter in our model. The polytropic equation of state (Sec. ) also involves two parameters, the polytropic index $\gamma$ and the polytropic temperature $K$. They are related to each other by an equation similar to Eq. However, it seems difficult to justify why the polytropic index should have a particular value such as $\gamma = -0.089$. On the other hand, in the logotropic model, the logotropic temperature $A$ is positive so it can be related to an energy scale $A = 3.53 \times 10^{-3}\epsilon_\Lambda$. In the polytropic model, the polytropic temperature $K$ is negative so its physical interpretation is not direct.

We have proposed different possibilities to justify the logotropic equation of state from first principles:

(i) The logotropic equation of state is related to Tsallis generalized thermodynamics. It is associated with the Log-entropy (see Appendix) which is a sort of regularized Tsallis entropy with index $q = 0$ (i.e. $\gamma = 0$ in the polytropic terminology). This marginal index, where power laws degenerate into a logarithm, may be viewed as a fixed point, or as a critical point, in the framework of Tsallis generalized thermodynamics. Therefore, if our cosmological model is correct, it would be a nice confirmation of the interest of generalized thermodynamics in physics and astrophysics.

(ii) We have proposed that the dark fluid could be a relativistic self-interacting scalar field representing, for example, a self-interacting BEC at $T = 0$. In that context, the logotropic equation of state arises from the GP equation with an inverted quadratic potential [Eq. ], or from the KG equation with a logarithmic potential [Eq. ]. These potentials may be simpler to justify from fundamental physics than power-law potentials with ex-

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24 The rotation curve of logotropic dark matter halos continues to increase (instead of decreasing) at larger distances. This is consistent with the rotation curves of certain galaxies, such as low surface brightness (LSB) galaxies, that are particularly well isolated. For other galaxies, tidal effects or complex physical processes (e.g. incomplete relaxation) have to be taken into account at large distances to confine the system.
ponents $2(\gamma - 1) = -2.18$ [Eq. (C42)] and $2\gamma = -0.178$ [Eq. (C44)] associated with a polytropic equation of state with $\gamma = -0.089$ [45]

(iii) In Sec. [41] we have mentioned that our approach provides a new justification of the Cardassian model [68]. Inversely, the original justification of the Cardassian model, namely that the term $\nu(\rho)$ arising in the modified Friedmann equation [44] may result from the existence of extra-dimensions, could also be a way to justify the logotropic model corresponding to

$$\nu(\rho) = -7\pi G A/3c^4 \ln(\rho/\rho_P) + 1.$$ 

In Appendix [3] we present a simple argument to justify the logotropic equation of state and we relate this equation of state to a long-range confining force that could be a fifth force or an effective description of higher dimensional physics.

In conclusion, the logotropic equation of state may be a good candidate for the unification of dark matter and dark energy. Indeed, it provides a good description of the cosmological evolution of the universe as a whole and, at the same time, accounts for many properties of dark matter halos, some of them being until now unexplained. The logotropic equation of state is rather unique in accounting for all these observational results in a unified manner. It is related to generalized thermodynamics and corresponds to the Log-entropy which is a sort of degenerate Tsallis entropy with index $q = 0$. It is also related to the Cardassian model. Finally, it can be given a field theory interpretation as it arises from a nice simplified form of GP and KG equations corresponding to an inverted quadratic potential or to a logarithmic potential, respectively. All these elements suggest that dark matter and dark energy may be the manifestation of a unique dark fluid with a logotropic equation of state. Some developments of this model will be given in future works.

Appendix A: Relativistic thermodynamics

The local form of the first law of thermodynamics can be expressed as

$$d \left( \frac{\epsilon}{\rho} \right) = -P d \left( \frac{1}{\rho} \right) + T d \left( \frac{s}{\rho} \right),$$

where $\rho = nm$ is the mass density, $n$ is the number density, and $s$ is the entropy density in the rest frame. For a system at $T = 0$, or for an adiabatic evolution such that $d(s/\rho) = 0$, the first law of thermodynamics reduces to

$$d\epsilon = \frac{P + \epsilon}{\rho} d\rho.$$  \hspace{1cm} (A2)

For a given equation of state, Eq. (A2) can be integrated to obtain the relation between the energy density $\epsilon$ and the rest-mass density.

If the equation of state is prescribed under the form $P = P(\epsilon)$, Eq. (A2) can be immediately integrated into

$$\ln \rho = \int \frac{d\epsilon}{P(\epsilon) + \epsilon}. \hspace{1cm} (A3)$$

If, as an example, we consider the "gamma law" equation of state [83, 84]:

$$P = (\gamma - 1)\epsilon,$$  \hspace{1cm} (A4)

we get

$$P = K \rho^\gamma, \hspace{0.5cm} \epsilon = \frac{K}{\gamma - 1} \rho^{\gamma - 1}, \hspace{1cm} (A5)$$

where $K$ is a constant of integration.

We now assume that the equation of state is prescribed under the form $P = P(\rho)$. In that case, Eq. (A2) reduces to the first order linear differential equation

$$\frac{d\epsilon}{d\rho} - \frac{1}{\rho} \epsilon = \frac{P(\rho)}{\rho}. \hspace{1cm} (A6)$$

Using the method of the variation of the constant, we obtain

$$\epsilon = A \rho c^2 + \rho \int_\rho^\infty \frac{P(\rho')}{\rho'^{\gamma - 1}} d\rho', \hspace{1cm} (A7)$$

where $A$ is a constant of integration.

As an example, we consider the polytropic equation of state

$$P = K \rho^\gamma, \hspace{1cm} \gamma = 1 + \frac{1}{n}. \hspace{1cm} (A8)$$

For $\gamma = 1$, we get

$$\epsilon = A \rho c^2 + K \rho \ln \rho. \hspace{1cm} (A9)$$

For $\gamma \neq 1$, we obtain

$$\epsilon = A \rho c^2 + K \rho^\gamma = A \rho c^2 + nP. \hspace{1cm} (A10)$$

Taking $A = 0$, we recover Eqs. (A4) and (A5). Taking $A = 1$, a choice that we shall make in the following, we obtain Eqs. (38) and (39).

We first assume $n > 0$ (i.e. $\gamma > 1$). For $\rho \to 0$,

$$\epsilon \sim \rho c^2, \hspace{1cm} P \sim K(\epsilon/c^2)^\gamma. \hspace{1cm} (A11)$$

and for $\rho \to +\infty$,

$$\epsilon \sim nK \rho^\gamma, \hspace{1cm} P \sim \epsilon/n \sim (\gamma - 1)\epsilon. \hspace{1cm} (A12)$$

We now assume $n < 0$ (i.e. $\gamma < 1$). For $\rho \to 0$,

$$\epsilon \sim nK \rho^\gamma, \hspace{1cm} P \sim \epsilon/n \sim (\gamma - 1)\epsilon, \hspace{1cm} (A13)$$

and for $\rho \to +\infty$,

$$\epsilon \sim \rho c^2, \hspace{1cm} P \sim K(\epsilon/c^2)^\gamma. \hspace{1cm} (A14)$$
For a general equation of state $P(\rho)$, we obtain
\[ \epsilon = \rho c^2 + \rho \int^\rho \frac{P(\rho')}{\rho'^2} \, d\rho' = \rho c^2 + u(\rho), \tag{A15} \]
where the primitive is determined such that $u(\rho)$ does not contain terms in $\rho c^2$. We note that $u(\rho)$ may be interpreted as an internal energy density (see \[85\] and Appendix C). Therefore, the energy density $\epsilon$ is the sum of the rest-mass energy $\rho c^2$ and the internal energy $u(\rho)$. The rest-mass energy is positive while the internal energy can be positive or negative. Of course, the total energy $\epsilon = \rho c^2 + u(\rho)$ is always positive.

Remark: according to Eq. (A2), the velocity of sound $c_s$ defined by $c_s^2 = \frac{\partial P}{\partial \rho} = \frac{\partial^2 \epsilon}{\partial \rho^2}$ satisfies the identity
\[ \frac{c_s^2}{c^2} = \frac{\rho c_s^2}{\rho c^2} = \frac{d\epsilon}{d\rho}. \tag{A16} \]

Appendix B: A simple argument to justify the logotropic equation of state and its relation to a fifth force

We have seen in Sec. II H that the energy density $\epsilon$ of a relativistic dark fluid at $T = 0$ (or a perfect fluid) is the sum of two terms: a rest-mass term $\rho c^2$ mimicking dark matter and an internal energy term
\[ u(\rho) = \rho \int^\rho \frac{P(\rho')}{\rho'^2} \, d\rho' \tag{B1} \]
mimicking a “new fluid”. If we want the new fluid to mimic dark energy, we should have $P = -u$. Substituting the relation $u(\rho) = -P(\rho)$ in Eq. (B1) and solving the resulting differential equation, we find $P = \text{const}$, which returns the model of Sec. II H equivalent to the $\Lambda$CDM model. However, we have explained that a constant pressure does not account for the structure of dark matter halos. Therefore, a next guess is $P = -u - A$ where $A$ is a constant. Substituting the relation $u(\rho) = -P(\rho) - A$ in Eq. (B1) and solving the resulting differential equation, we find the logotropic equation of state $P = A \ln(\rho/\rho_0)$. We note that $P \sim -u$ as $\rho \to 0$, similarly to the equation of state of dark energy. Another possible guess is $P = u/n$ with $n \to -1$. Substituting the relation $u(\rho) = nP(\rho)$ in Eq. (B1) and solving the resulting differential equation, we find the polytropic equation of state $P = K \rho^{1+1/n}$. For $n = -1$, we obtain $P = K$. The polytropic equation of state with an arbitrary index $n$ is interesting at a general level (beyond dark energy). For example, the index $n = 1$, leading to a quadratic equation of state $P = K \rho^2$, corresponds to a stiff fluid ($P = u$) possibly existing in the very early universe [44].

The pressure $P$ can be associated with a self-interaction potential between particles. If we consider a power-law interaction of the form
\[ U(r_{ij}) = U_0 r_{ij}^{-\alpha}, \tag{B2} \]
corresponding to a force $F(r_{ij}) = -\alpha U_0 r_{ij}^{\alpha-1}$, the virial theorem writes (see, e.g., Appendix I of \[59\]):
\[ 2K - \alpha W = 3PV, \tag{B3} \]
where $K$ is the kinetic energy, $W$ is the potential energy, $P$ is the pressure, and $V$ is the volume of the system. Taking $K = 0$ for a dark fluid at $T = 0$ and identifying $W/V$ as the internal energy $u$, we obtain (see an alternative derivation of this result in the Appendix of \[68\]):
\[ P = -\frac{1}{3} \alpha u. \tag{B4} \]

The equation of state $P = -u$ of dark energy corresponds to $\alpha = 3$ yielding $U = U_0 r_{ij}^3$ and $F = -3U_0 r_{ij}^2$. This is a long-range confining force that could be a fifth force or an effective description of higher dimensional physics \[68\]. The equation of state $P = u/n$ associated with a polytrope of index $n$ corresponds to $\alpha = -3/n$ yielding $U = U_0 r_{ij}^{-3/n}$ and $F = 3U_0 r_{ij}^{-2}$. In particular, the equation of state of radiation $P = u/3$ associated with a polytrope of index $n = 3$ corresponds to $\alpha = -1$ yielding $U = U_0 r_{ij}^{-1}$ and $F = U_0 r_{ij}^{-2}$ (Coulombian force). The stiff equation of state $P = u$ associated with a polytrope of index $n = 1$ corresponds to $\alpha = -3$ yielding $U = U_0 r_{ij}^{-3}$ and $F = 3U_0 r_{ij}^{-4}$ (attractive force).\[26\] We note that the dark energy equation of state and the stiff equation of state correspond to a polytropic index $n = -1$ and $n = +1$, respectively, and to a potential with an exponent $+3$ and $-3$. This symmetry may be related to the comment following Eq. (C56).

For a logarithmic potential of interaction of the form
\[ U(r_{ij}) = \tilde{U}_0 \ln r_{ij}, \tag{B5} \]
the virial theorem writes (see, e.g., Appendix I of \[59\]):
\[ 2K - \tilde{U}_0 N^2 = 3PV, \tag{B6} \]
where $N$ is the number of particles in the volume $V$. Therefore, it is tempting to associate the constant $A$ (logotropic temperature) in the equation of state $P = -u - A$ as arising from a logarithmic potential of interaction of the form of Eq. (B5), such that
\[ A = \frac{\tilde{U}_0 N^2}{6V}. \tag{B7} \]

The total potential of interaction is therefore $U = U_0 r_{ij}^3 + \tilde{U}_0 \ln r_{ij}$ and the total force is $F = -3U_0 r_{ij}^2 - \tilde{U}_0 r_{ij}$. The meaning of Eq. (B7) is not clear, but it is interesting to note that $A > 0$ for an attractive potential $\tilde{U}_0 > 0$ (expected for dark energy), which is the correct sign of the logotropic temperature.

\[26\] Actually, in Zel’dovich model [77], the stiff equation of state arises from a screened Coulombian potential (see Sec. 4.2. of [77]).
Appendix C: Field theory

1. The Klein-Gordon equation

The Klein-Gordon (KG) equation for a complex scalar field $\phi$ writes

$$\frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2} - \Delta \phi + \frac{m^2 c^2}{\hbar^2} \left(1 + \frac{2 \Phi}{c^2}\right) \phi + 2 \frac{dV}{d|\phi|^2} \phi = 0,$$

(C1)

where $V = V(|\phi|^2)$ is the self-interaction potential and $\Phi$ is an external potential that, in a simplified model, can be identified with the gravitational potential (see [88] for a fully general relativistic treatment). The energy density and the pressure of the scalar field are given by

$$\epsilon = \frac{1}{2c^2} \left| \frac{\partial \phi}{\partial t} \right|^2 + \frac{1}{2} |\nabla \phi|^2 + V(|\phi|^2) + \frac{m^2 c^2}{2\hbar^2} |\phi|^2,$$

(C2)

$$P = \frac{1}{2c^2} \left| \frac{\partial \phi}{\partial t} \right|^2 + \frac{1}{2} |\nabla \phi|^2 - V(|\phi|^2) - \frac{m^2 c^2}{2\hbar^2} |\phi|^2. \quad \text{(C3)}$$

The KG equation without self-interaction can be viewed as the relativistic generalization of the Schrödinger equation. Similarly, the KG equation with a self-interaction can be viewed as the relativistic generalization of the GP equation. In order to recover the Schrödinger and GP equations in the nonrelativistic limit $c \to +\infty$, we make the transformation

$$\phi(r, t) = Ae^{-imc^2 t/\hbar} \psi(r, t), \quad \text{(C4)}$$

where $A$ is a constant. This constant can be determined by the following argument. Substituting Eq. (C4) in the expression of the energy density (C2), we get

$$\epsilon = \frac{A^2}{2c^2} \left| \frac{\partial \psi}{\partial t} \right|^2 + \frac{A^2 m^2 c^2}{\hbar^2} |\psi|^2 + \frac{A^2 m}{\hbar} \text{Im} \left( \psi \left( \frac{\partial \psi^*}{\partial t} \right) \right)$$

$$+ \frac{A^2}{2} |\nabla \psi|^2 + V(|\psi|^2). \quad \text{(C5)}$$

In the nonrelativistic limit $c \to +\infty$, we have $\epsilon \sim \rho c^2$ where $\rho$ is the rest-mass density. On the other hand, according to Eq. (C5),

$$\frac{\epsilon}{c^2} \sim \frac{A^2 m^2}{\hbar^2} |\psi|^2. \quad \text{(C6)}$$

If we interpret $\psi$ as the wavefunction normalized such that $|\psi|^2 = \rho$, we find by identification that

$$A = \frac{\hbar}{m}. \quad \text{(C7)}$$

Therefore, Eq. (C4) can be rewritten as

$$\phi(r, t) = \frac{\hbar}{m} e^{-imc^2 t/\hbar} \psi(r, t). \quad \text{(C8)}$$

Mathematically, we can always make this change of variables. However, we emphasize that it is only in the nonrelativistic limit $c \to +\infty$ that $\psi$ has the interpretation of a wave function, and that $|\psi|^2 = \rho$ has the interpretation of a rest-mass density. In the relativistic regime, $\psi$ and $\rho = |\psi|^2$ do not have a clear physical interpretation. We will call them “pseudo wave function” and “pseudo rest-mass density”. Nevertheless, it is perfectly legitimate to work with these variables [88].

Substituting Eq. (C8) in the KG equation (C1), we obtain

$$\frac{\hbar^2}{2mc^2} \frac{\partial^2 \psi}{\partial t^2} - i\hbar \frac{\partial \psi}{\partial t} - \frac{\hbar^2}{2m} \Delta \psi + m \Phi \psi + m \frac{dV}{d|\psi|^2} \psi = 0. \quad \text{(C9)}$$

On the other hand, the energy density and the pressure can be written in terms of $\psi$ as

$$\epsilon = \frac{\hbar^2}{2m^2 c^2} \left| \frac{\partial \psi}{\partial t} \right|^2 + \frac{\hbar}{m} \text{Im} \left( \psi \left( \frac{\partial \psi^*}{\partial t} \right) \right)$$

$$+ \frac{\hbar^2}{2m^2} |\nabla \psi|^2 + V(|\psi|^2). \quad \text{(C10)}$$

$$P = \frac{\hbar^2}{2m^2 c^2} \left| \frac{\partial \psi}{\partial t} \right|^2 + \frac{\hbar}{m} \text{Im} \left( \psi \left( \frac{\partial \psi^*}{\partial t} \right) \right)$$

$$+ \frac{\hbar^2}{2m^2} |\nabla \psi|^2 - V(|\psi|^2). \quad \text{(C11)}$$

2. The Gross-Pitaevskii equation

Taking the nonrelativistic limit $c \to +\infty$ of the KG equation (C9), we obtain the nonlinear Schrödinger equation

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \Delta \psi + m \Phi \psi + m \frac{dV}{d|\psi|^2} \psi. \quad \text{(C12)}$$

It can be written as a GP equation of the form

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \Delta \psi + m \Phi \psi + m h(|\psi|^2) \psi \quad \text{(C13)}$$

with a potential

$$h(|\psi|^2) = \frac{dV}{d|\psi|^2} \quad \text{i.e.} \quad h(\rho) = V'(\rho). \quad \text{(C14)}$$

The GP equation describes a BEC at $T = 0$. The GP equation with a cubic nonlinearity (corresponding to $h(\rho) \propto \rho$) can be derived from the mean field Schrödinger equation with a pair contact potential (see Sec. II.A. of [89]). The present approach shows that the GP equation with an arbitrary potential $h(\rho)$ can be derived from the KG equation with a self-interaction potential $V(\rho)$. More precisely, Eq. (C14) shows that the potential $h(\rho)$ in the GP equation is equal to the derivative of the potential $V(\rho)$ in the KG equation. Reciprocally, $V(\rho)$ is a primitive of $h(\rho)$. The primitive of $h(\rho)$ played some role in
our previous studies [89], and was noted $H(\rho)$. When the GP equation is derived from the KG equation, we have

$$H(\rho) = V(\rho).$$

(C15)

The KG equation can be written in terms of $h$ as

$$\frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2} - \Delta \phi + \frac{m^2 c^2}{\hbar^2} \left(1 + \frac{2\Phi}{c^2}\right) \phi + 2 \frac{m^2}{\hbar^2} \hbar \left( \frac{m^2}{\hbar^2} |\phi|^2 \right) \phi = 0.$$  

We also have

$$V(|\phi|^2) = H \left( \frac{m^2}{\hbar^2} |\phi|^2 \right).$$

(C17)

3. The Madelung transformation

Using the Madelung [90] transformation

$$\psi = \sqrt{\rho} e^{iS/h}, \quad u = \frac{1}{m} \nabla S,$$

(C18)

where $\rho(r, t)$ is the density, $S(r, t)$ is an action, and $u(r, t)$ is interpreted as an irrotational velocity field, we can rewrite the GP equation (C13) in the form of hydrodynamic equations (see, e.g., [89]):

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho u) = 0,$$

(C19)

$$\frac{\partial u}{\partial t} + (u \cdot \nabla) u = -\frac{1}{m} \nabla P - \nabla \Phi - \frac{1}{m} \nabla Q,$$

(C20)

where

$$Q = -\frac{\hbar^2}{2m} \frac{\Delta \sqrt{\rho}}{\sqrt{\rho}}$$

(C21)

is the Bohm quantum potential and $P$ is the pressure. The pressure is given by a barotropic equation of state $P = P(\rho)$ determined by the potential $h(\rho)$ according to

$$h'(\rho) = \frac{P'(\rho)}{\rho}.$$  

(C22)

Since $\nabla h = (1/\rho)\nabla P$, the potential $h$ in the GP equation has the interpretation of an enthalpy in the quantum Euler equation (C20). We note that the pressure is explicitly given by

$$P(\rho) = \rho h(\rho) - H(\rho) = \rho V'(\rho) - V(\rho).$$

(C23)

4. Energy functional

The barotropic quantum Euler equations (C19) and (C20) conserve the mass and the free energy functional$^{27}$

$$F = E + U,$$

(C24)

where $E = \Theta_c + \Theta_Q + W$ is the sum of the classical kinetic energy $\Theta_c$, the quantum kinetic energy $\Theta_Q$ and the potential energy $W$, and $U$ is the internal energy. They are defined by (see, e.g., [89]):

$$\Theta_c = \int \rho u^2 \frac{1}{2} \, d\mathbf{r}, \quad \Theta_Q = \frac{1}{m} \int \rho Q \, d\mathbf{r},$$

(C25)

$$W = \int \rho \Phi \, d\mathbf{r}, \quad U = \int \rho \int \frac{P(\rho')}{\rho'^2} \, d\rho' \, d\mathbf{r}.$$  

(C26)

The internal energy can be written as

$$U = \int u(\rho) \, d\mathbf{r},$$

(C27)

where

$$u(\rho) = \rho \int \rho P(\rho') - \rho d\rho'.$$

(C28)

is the internal energy density.

The internal energy density $u(\rho)$ defined by Eq. (C28) corresponds precisely to the term that appears in the energy density of Eq. (A15) in addition to the rest-mass density $\rho c^2$. As we have indicated in Sec. II H, depending on the pressure law $P(\rho)$, this term mimics a “new fluid” that adds to “dark matter”. This new fluid may be an exotic constituent (e.g. a stiff fluid) appearing in the early universe [44]. It may also represent the “dark energy” in the late universe (see Sec. II H). Using the present formalism, we can make a connection between this new fluid and the potential that appears in the GP and KG equations. Integrating Eq. (C28) by parts and using Eq. (C22), the internal energy density may be rewritten as

$$u(\rho) = \rho h(\rho) - P(\rho).$$

(C29)

According to Eqs. (C15), (C23) and (C29), we obtain

$$u(\rho) = H(\rho) = V(\rho),$$

(C30)

or, equivalently,

$$u'(\rho) = H'(\rho) = V'(\rho) = h(\rho).$$

(C31)

Therefore, the internal energy density $u(\rho)$, which mimics a “new fluid” in Eq. (A15), is equal to the potential $V(\rho)$ appearing in the KG equation. $^{28}$

Remark: We note that $H(\rho)$ and $V(\rho)$ are defined up to a term of the form $a\rho + b$. Therefore, we can adapt the coefficients $a$ and $b$ in order to have the simplest expressions of $H$ and $V$. We will use this prescription in the following section.

$^{27}$ As a result, a minimum of free energy $F$ at fixed mass $M$ is a stable stationary state of the GP and quantum Euler equations.

$^{28}$ Actually, this equivalence is valid only in the nonrelativistic limit $c \rightarrow +\infty$. The relativistic regime is more complicated to investigate (since $\rho$ is not the rest-mass energy) and will be considered specifically in a future communication.
5. Particular examples

The internal energy density \( u(\rho) \) is determined by the equation of state \( P(\rho) \). As we have seen, this equation of state can be obtained from a field theory based on the KG or GP equation. In this section, we consider particular examples of equations of state.

a. Isothermal equation of state

We consider the isothermal equation of state \(^{77}\):

\[
P = \frac{k_B T}{\rho},
\]

(C32)

In the present context, this equation of state is expected to arise from a self-interaction potential, not from thermal motion. As a result, the temperature has to be regarded as an effective temperature which can be either positive or negative. The internal energy is

\[
u = \frac{k_B T}{\rho} \rho \ln \rho.
\]

(C33)

The potential in the GP equation is

\[
h = \frac{k_B T}{m} \rho \ln \rho, \quad H = \frac{k_B T}{m} \rho \rho \ln \rho.
\]

(C34)

The GP equation takes the form

\[
i \hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \Delta \psi + m \Phi \psi + 2k_B T \ln |\psi| |\psi|.
\]

(C35)

The free energy can be written as \( F = E - TS \) where \( E \) is the energy, \( T \) is the temperature and \( S \) is the Boltzmann entropy

\[
S = -k_B \int \frac{\rho}{m} \ln \frac{\rho}{m} d\mathbf{r}.
\]

(C36)

The potential in the KG equation \(^{1}\) is

\[
V(|\phi|) = 2m k_B T / h^2 |\phi|^2 \ln |\phi|,
\]

(C37)

so the KG equation writes

\[
\frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2} - \Delta \phi + \frac{m^2 c^2}{h^2} \left(1 + \frac{2\Phi}{c^2}\right) \phi + \frac{4mk_B T}{h^2} \ln \left(\frac{m|\phi|}{h}\right) \phi = 0.
\]

(C38)

b. Polytropic equation of state

We consider the polytropic equation of state \(^{77}\):

\[
P = K \rho^\gamma, \quad \gamma = 1 + \frac{1}{n},
\]

(C39)

where the polytropic constant \( K \) may be positive or negative (for the same reason as before), and the polytropic index \( \gamma \) is arbitrary. The internal energy is

\[
u = \frac{K}{\gamma - 1} \rho^\gamma = \frac{P}{\gamma - 1}.
\]

(C40)

Since \( P = (\gamma - 1)u \), this component in Eq. \(^{82}\) can mimic dark energy when \( \gamma \to 0 \) (see Sec. \(^{110}\)). For \( \gamma = 0 \), the pressure is constant \( (P = K) \) and we recover the equation of state \( P = -u \) of dark energy. The potential in the GP equation is

\[
h = \frac{K \gamma}{\gamma - 1} \rho^{\gamma - 1}, \quad H = \frac{K}{\gamma - 1} \rho^\gamma.
\]

(C41)

The GP equation takes the form

\[
i \hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \Delta \psi + m \Phi \psi + K(n + 1)m|\psi|^{2/n} |\psi|.
\]

(C42)

The free energy can be written as \( F = E - KS \), where \( E \) is the energy, \( K \) is the polytropic temperature and \( S \gamma \) is the Tsallis entropy \(^{51} \):

\[
S_\gamma = -\frac{1}{\gamma - 1} \int (\rho^\gamma - \rho) d\mathbf{r}.
\]

(C43)

The potential in the KG equation \(^{1} \) is

\[
V(|\phi|) = \frac{K}{\gamma - 1} \left(\frac{m}{h}\right)^{2\gamma} |\phi|^{2\gamma},
\]

(C44)

so the KG equation writes

\[
\frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2} - \Delta \phi + \frac{m^2 c^2}{h^2} \left(1 + \frac{2\Phi}{c^2}\right) \phi + \frac{2m^2 K \gamma}{(\gamma - 1)h^2} \left(\frac{m|\phi|}{h}\right)^{2(\gamma - 1)} \phi = 0.
\]

(C45)

The usual GP equation writes \(^{91} \):

\[
i \hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \Delta \psi + m \Phi \psi + \frac{4\pi \alpha_s \hbar^2}{m^2} |\psi|^2 |\psi|.
\]

(C46)

It describes a BEC at \( T = 0 \) with a quadratic self-interaction \( (\hbar \propto |\psi|^2) \). The GP equation \(^{46} \) can be derived from the mean field Schrödinger equation with a pair contact potential (see Sec. \(^{II.A.} \) of \(^{89} \)). In that case, \( \alpha_s \) represents the scattering length of the bosons. It is positive when the self-interaction is repulsive and negative when the self-interaction is attractive. The potential in the GP equation \(^{46} \) is

\[
h = \frac{4\pi \alpha_s \hbar^2}{m^3} \rho, \quad H = \frac{2\pi \alpha_s \hbar^2}{m^3} \rho^2.
\]

(C47)

The pressure is

\[
P = \frac{2\pi \alpha_s \hbar^2}{m^3} \rho^2.
\]

(C48)
It corresponds to a polytrope of index \( n = 1 \) (i.e. \( \gamma = 2 \)) and polytropic constant \( K = 2\pi a_s h^2/m^3 \). The internal energy is

\[ u = K \rho^2 = P. \quad (C49) \]

Since \( P = u \), this component in Eq. (32) can mimic stiff matter \([14]\). Historically, the stiff equation of state was introduced by Zel’dovich \([66, 67]\) in the context of baryon stars in which the baryons interact through a vector meson field. The free energy can be written as \( F = E - KS_2 \) where \( E \) is the energy, \( K \) is the polytropic temperature and \( S_2 \) is the Tsallis entropy of index \( \gamma = 2 \):

\[ S_2 = -\int \rho^2 \, dr. \quad (C50) \]

The potential in the KG equation (C1) is

\[ V(\phi) = \frac{2\pi a_s m}{h^2} |\phi|^4, \quad (C51) \]

so the KG equation writes

\[ \frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2} - \Delta \phi + \frac{m^2 c^2}{h^2} \left( 1 + \frac{2\Phi}{c^2} \right) \phi + \frac{8\pi a_s m}{h^2} |\phi|^2 \phi = 0. \quad (C52) \]

c. Logotropic equation of state

We consider the logotropic equation of state \([52, 53]\):

\[ P = A \ln(\rho/\rho_s), \quad (C53) \]

where the logotropic constant \( A \) may be positive or negative. The logotropic equation of state \([C53]\) can be viewed as the limiting form of the polytropic equation of state \([C39]\) when \( \gamma \to 0 \) (\( n \to -1 \)) and \( K \to \infty \) with \( A = K \gamma \) finite. However, it cannot be obtained from a polytropic equation of state with \( \gamma = 0 \) (which corresponds to a constant pressure \( P = K \)) because a constant pressure yields \( h = 0 \) (or \( h = \text{const.} \)) in the GP equation, which is different from Eq. (C56). This shows the “regularizing” property of the logotropic equation of state when \( \gamma \to 0 \), as in the case of the Lane-Emden equation (see Sec. V). The free energy can be written as \( F_L = E - AS_L \) where \( E \) is the energy, \( A \) is the logotropic temperature and \( S_L \) is the log-entropy \([53]\):

\[ S_L = \int \ln \rho \, dr. \quad (C57) \]

The potential in the KG equation (C1) is

\[ V(|\phi|) = -2A \ln |\phi|, \quad (C58) \]

so the KG equation writes

\[ \frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2} - \Delta \phi + \frac{m^2 c^2}{h^2} \left( 1 + \frac{2\Phi}{c^2} \right) \phi - \frac{2A}{|\phi|^2} \phi = 0. \quad (C59) \]

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