Spectrum Generating Conformal and Quasiconformal 
U-Duality Groups, Supergravity and Spherical Vectors

Murat Günyaydın†∗
† Institute for Gravitation and the Cosmos
Physics Department
Penn State University
University Park, PA 16802, USA

Oleksandr Pavlyk‡†
‡Wolfram Research Inc.
100 Trade Center Dr.
Champaign, IL 61820, USA

Abstract: After reviewing the algebraic structures that underlie the geometries of \( N = 2 \) Maxwell-Einstein supergravity theories (MESGT) in five and four dimensions with symmetric scalar manifolds, we give a unified realization of their three dimensional U-duality groups as spectrum generating quasiconformal groups. They are \( F_4(4), E_6(2), E_7(-5), E_8(-24) \) and \( SO(n + 2, 4) \). Our formulation is covariant with respect to U-duality symmetry groups of corresponding five dimensional supergravity theories, which are \( SL(3, \mathbb{R}) \), \( SL(3, \mathbb{C}) \), \( SU^*(6) \), \( E_6(6) \) and \( SO(n - 1, 1) \times SO(1, 1) \), respectively. We determine the spherical vectors of quasiconformal realizations of all these groups twisted by a unitary character. We also give their quadratic Casimir operators and determine their values. Our work lays the algebraic groundwork for constructing the unitary representations of these groups induced by their geometric quasiconformal actions, which include the quaternionic discrete series. For rank 2 cases, \( SU(2, 1) \) and \( G_2(2) \), corresponding to simple \( N = 2 \) supergravity in four and five dimensions, this program was carried out in arXiv:0707.1669. We also discuss the corresponding algebraic structures underlying symmetries of \( N \geq 4 \) supergravity theories. They lead to quasiconformal realizations of split real forms of U-duality groups as a straightforward extension of the quaternionic real forms.

Keywords: Supergravity, Duality, Black Holes

∗murat@phys.psu.edu
†pavlyk@wolfram.com
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1. Introduction

Earliest studies of unitary representations of U-duality groups of extended supergravity theories were given in [1, 2, 3]. These studies were motivated mainly by the idea that in a quantum theory global symmetries must be realized unitarily on the spectrum as well as by the composite scenarios that attempted to connect maximal $N = 8$ supergravity with observation [4]. In the composite scenario of [4] $SU(8)$ local symmetry of $N = 8$ supergravity was conjectured to become dynamical and act as a family unifying grand unified theory (GUT) which contains $SU(5)$ GUT as well as a family group $SU(3)$. A similar scenario leads to $E_6$ GUT with a family group $U(1)$ in the exceptional supergravity theory [4]. With the discovery of counter terms at higher loops suggesting that divergences at higher orders would eventually spoil the finiteness properties of $N = 8$ supergravity and the discovery of Green-Schwarz anomaly cancellation in superstring theory [5], the attempts at composite scenarios were abandoned. However, recent discovery of unexpected cancellations of divergences in supergravity theories [6–10, 11, 12, 13, 14, 15, 16] has brought back the question of finiteness of $N = 8$ supergravity as well as of exceptional supergravity.

Over the last decade or so, work on unitary representations of U-duality groups of extended supergravity theories has accelerated, leading to considerable progress. Some of the renewed interest in unitary realizations of U-duality groups was due to the proposals that certain extensions of U-duality groups may act as spectrum generating symmetry groups. First, based on geometric considerations involving orbits of extremal black hole solutions in $N = 8$ supergravity and $N = 2$ Maxwell-Einstein supergravity theories with symmetric scalar manifolds, it was suggested that four dimensional U-duality groups act as spectrum generating conformal symmetry groups of the corresponding five dimensional supergravity theories [17, 18, 19, 20, 21]. This proposal was extended to the proposal that three dimensional U-duality groups act as spectrum generating quasiconformal groups of the corresponding four dimensional supergravity theories [18, 19, 20, 21]. Quasiconformal realization of the U-duality group $E_8(8)$ of maximal supergravity in three dimension is the first known geometric realization of $E_8(8)$ and its quantization leads to the minimal unitary representation of $E_8(8)$ [24]. Remarkably, quasiconformal realizations exist for different real forms of all noncompact groups and their quantizations yield directly the minimal unitary representations of the respective groups [22, 23, 21, 25]. In fact, the construction of minimal unitary representations of noncompact groups via the quasiconformal method gives a unified approach to these representations and extends also to supergroups [25]. For symplectic groups these minimal unitary representations are simply the singleton representations.

Five dimensional Maxwell-Einstein supergravity theories with symmetric scalar manifolds $\mathcal{M}_5 = G_5/K_5$ such that $G_5$ is a symmetry of the Lagrangian are in one-to-one correspondence with Euclidean Jordan algebras $J$ of degree three and their scalar manifolds are of the form

$$\mathcal{M}_5 = \frac{Str_0(J)}{Aut(J)}$$

\footnote{For further references on the subject, see [1]}
where \( \text{Str}_0(J) \) and \( \text{Aut}(J) \) are the reduced structure and automorphism groups of the Jordan algebra \( J \), respectively. The scalar manifolds of these theories in four dimensions are

\[
\mathcal{M}_4 = \frac{\text{Conf}(J)}{\text{Str}_0(J) \times U(1)}
\]

where \( \text{Conf}(J) \) is the conformal group of the Jordan algebra \( J \) and \( \widetilde{\text{Str}}_0(J) \) is the compact form of the reduced structure group. Upon further dimensional reduction to three dimensions they lead to scalar manifolds of the form

\[
\mathcal{M}_3 = \frac{\text{QConf}(J)}{\text{Conf}(J) \times SU(2)}
\]

where \( \text{QConf}(J) \) is the quasiconformal group associated with the Jordan algebra \( J \) and \( \widetilde{\text{Conf}}(J) \) is the compact form of the conformal group of \( J \).

Since they were first proposed many results have been obtained that support the proposals that four and three dimensional U-duality groups act as spectrum generating conformal and quasiconformal groups of five and four dimensional supergravity theories with symmetric scalar manifolds. First, the work relating black hole solutions in four and five dimensions \([26, 27, 28, 29]\) is in complete accord with the proposal that four dimensional U-duality groups act as spectrum generating conformal symmetry groups of five dimensional supergravity theories from which they descend. Furthermore, applying the two proposals in succession leads to the proposal that three dimensional U-duality groups should act as spectrum generating symmetry groups of five dimensional supergravity theories. The work of \([30, 31]\) on using solution generating techniques to relate the known black hole solutions of five dimensional ungauged supergravity theories to each other and generate new solutions using symmetry groups of the corresponding three dimensional supergravity theories and related work on gauged supergravity theories \([32]\) lend further support to these proposals.

A concrete framework for implementation of the proposal that three dimensional U-duality groups act as spectrum generating quasiconformal groups was given in \([33, 34, 35]\) for spherically symmetric stationary BPS black holes of four dimensional supergravity theories. The basic starting point is the observation that the attractor equations \([36, 37]\) for symmetric stationary black holes of four dimensional supergravity theories are equivalent to the geodesic motion of a fiducial particle on the moduli space \( \mathcal{M}_3^* \) of the three dimensional supergravity obtained by reduction on a time-like circle\(^2\). A related analysis on non-BPS extremal black holes in theories with symmetric target manifolds was carried out in \([41]\). For \( N = 2 \) MESGTs defined by Jordan algebras of degree three the manifolds \( \mathcal{M}_3^* \) are para-quaternionic symmetric spaces

\[
\mathcal{M}_3^* = \frac{\text{QConf}(J)}{\text{Conf}(J) \times SU(1, 1)}
\]

\(^2\)This was first observed in \([38]\) and used in \([39, 40]\) to construct static and rotating black holes in heterotic string theory.
Quantization of the motion of fiducial particle leads to quantum mechanical wave functions that provide the basis of a unitary representation of $Q\Conf(J)$. BPS black holes correspond to a special class of geodesics which lift holomorphically to the twistor space $Z_3$ of $\mathcal{M}_3$ and $Z_3$ can be identified as the BPS phase space. Spherically symmetric stationary BPS black holes of $N = 2$ MESGT’s are described by holomorphic curves in $Z_3$ [33, 34, 35, 42]. The quasiconformal group actions discovered in [18] extend to the complexified groups. As a consequence, one finds that the action of three dimensional U-duality group $Q\Conf(J)$ on the natural complex coordinates of corresponding twistor space is precisely of the quasiconformal type [35]. Therefore the unitary representations of $Q\Conf(J)$ relevant for BPS black holes are those induced by holomorphic quasiconformal actions of $Q\Conf(J)$ on the corresponding twistor spaces $Z_3$, which belong in general to quaternionic discrete series representations [35].

Further support for the proposal that three dimensional U-duality groups act as spectrum generating groups of the corresponding four dimensional theories comes from the connection established in [4] between the harmonic superspace (HSS) formulation of $N = 2$, $d = 4$ supersymmetric quaternionic Kähler sigma models that couple to $N = 2$ supergravity and the minimal unitary representations of their isometry groups. In particular, for $N = 2$ sigma models with quaternionic symmetric target spaces of the form $Q\Conf(J)/\tilde{\Conf}(J) \times SU(2)$ one finds that there exists a one-to-one mapping between the quartic Killing potentials that generate the isometry group $Q\Conf(J)$ under Poisson brackets in the HSS formulation, and the generators of minimal unitary representation of $Q\Conf(J)$. This suggests that in the corresponding quantum theory the “fundamental spectrum” fits into the minimal unitary representation of $Q\Conf(J)$ and the full spectrum is obtained by tensoring of the minimal unitary representation obtained from quantization of the quasiconformal action. We should perhaps stress that the results of [4] apply to all quaternionic Kähler sigma models and not only those that are in the C-map of four dimensional $N = 2$ MESGT’s.

In [35] unitary representations of two quaternionic groups of rank two, namely $SU(2,1)$ and $G_{2(2)}$, induced by their geometric quasiconformal actions were studied in great detail. These groups are the isometry groups of four and five dimensional simple $N = 2$ supergravity theories dimensionally reduced to three dimensions, respectively. Unitary representations induced by the quasiconformal action twisted by a unitary character include the quaternionic discrete series representations that were studied in mathematics literature using algebraic methods [14]. The starting point of the constructions of unitary representations of $SU(2,1)$ and $G_{2(2)}$ given in [35] are the spherical vectors of maximal compact subgroups under their quasiconformal actions. In this paper we lay the algebraic groundwork for extending the results of [35] to all quaternionic quasiconformal groups $Q\Conf(J)$ that arise as three dimensional U-duality groups of five dimensional $N = 2$ MESGT’s defined by Euclidean Jordan algebras $J$ of degree three. We also present some results for extending this program to quasiconformal groups associated with non-Euclidean Jordan algebras of degree three that correspond to three dimensional U-duality groups of supergravity theories with $N \geq 4$ supersymmetries in $d = 4$ or $d = 5$. Study of the unitary representations induced from the quasiconformal actions on the spherical vectors will be the subjects of separate studies.
In section 2 we review the five and four dimensional $N = 2$ MESGT’s with symmetric scalar manifolds and the algebraic structures that underlie their geometries, namely, Euclidean Jordan of degree three and Freudenthal triple systems defined over them, respectively. Section 3 reviews the symmetries of supergravity theories with $N \geq 4$ in five, four and three dimensions and discuss their relation to non-Euclidean Jordan algebras of degree three and associated Freudenthal triple systems. In section 4 we give a unified realization of all quaternionic quasiconformal groups $QConf(J)$ in a basis covariant with respect to their five dimensional U-duality groups, which are simply the reduced structures groups of Euclidean Jordan algebras $J$ and and identify the generators of their maximal compact subgroups. Again in section 4 we present the quadratic Casimir operators of all the quasiconformal algebras $QConf(J)$ and determine their values. Section 5 is devoted to the study of the quasiconformal group $SO(4,4)$ of the Jordan algebra $J = \mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R}$ corresponding to the direct sum of three irreducible idempotents. Every element of a Euclidean Jordan algebra of degree three can be brought to this form by the action of its automorphism group and hence study of $SO(4,4)$ as a quasiconformal group is important. The corresponding supergravity theory is the STU model. We give the commutation relations in a noncompact basis covariant with respect to the reduced structure group $SO(1,1) \times SO(1,1)$ as well as in a compact basis. We give the spherical vector of quasiconformal $SO(4,4)$ annihilated by all the generators of its maximal compact subgroup $SO(4) \times SO(4) = SU(2) \times SU(2) \times SU(2) \times SU(2)$. In section 6 we determine the spherical vectors of all quaternionic quasiconformal groups $QConf(J)$ annihilated by the generators of their maximal compact subgroups $Conf(J) \times SU(2)$. The quasiconformal groups associated with simple Euclidean Jordan algebras are $F_4(4), E_6(2), E_7(-5)$ and $E_8(-24)$ and those associated with nonsimple Euclidean Jordan algebras are $SO(n+2,4)$. In section 7 we briefly discuss how the results of section 4 extend to split exceptional groups $E_6(6), E_7(7)$ and $E_8(8)$ and to $SO(m+4,n+4)$ by simply replacing the Euclidean Jordan algebras by split Jordan algebras. The appendix A gives a review of relevant Jordan algebra theory. Appendix B reviews the construction of conformal algebras of Jordan algebras and Appendix C reviews the quasiconformal group actions over Freudenthal triple systems extended by an extra coordinate.

2. Conformal and Quasiconformal Groups of Jordan Algebras and Maxwell-Einstein Supergravity Theories

Five dimensional Maxwell-Einstein supergravity theories with symmetric scalar manifolds $G/H$ such that $G$ is a symmetry of the Lagrangian are uniquely defined and classified by Euclidean Jordan algebras of degree three. The corresponding four dimensional MESGTs obtained by dimensional reduction are similarly described by Freudenthal triple systems defined over these Jordan algebras. These triple systems were introduced by Freudenthal in his study of metaplectic geometries associated with the exceptional groups $F_4, E_6, E_7$ and $E_8$ [13, 14]. They correspond to the last row of some very remarkable geometries associated with the groups of the Magic Square. Referring to appendices A,B and C for further details.
on Jordan algebras, Freudenthal triple systems and their symmetry groups we shall in this section review these MESGTs and associated algebraic and geometric structures.

2.1 5D, \( N = 2 \) Maxwell-Einstein Supergravity Theories and Jordan Algebras

Five dimensional MESGTs that describe the coupling of an arbitrary number of \( N = 2 \) (Abelian) vector multiplets to \( N = 2 \) supergravity were constructed in [7, 47, 48, 49]. The bosonic part of five dimensional \( N = 2 \) MESGT Lagrangian is given by

\[
e^{-1} \mathcal{L}_{\text{bosonic}} = -\frac{1}{2} R - \frac{1}{4} a_{IJK} F_{\mu\nu}^{I} F_{\mu\nu}^{J} - \frac{1}{2} g_{xy}(\partial_\mu \varphi^x)(\partial^\mu \varphi^y) + \frac{e^{-1}}{6\sqrt{6}} C_{IJK} \varepsilon^{\mu\nu\rho\sigma\lambda} F_{\mu\nu}^{I} F_{\rho\sigma}^{J} A_{\lambda}^{K}, \tag{2.1.1}
\]

where

\[
I = 1, \ldots, n_V \\
a = 1, \ldots, (n_V - 1) \\
x = 1, \ldots, (n_V - 1) \\
\mu, \nu, \ldots = 0, 1, 2, 3, 4.
\]

\( e \) and \( R \), respectively, denote the fünfbein determinant and scalar curvature of spacetime. \( F_{\mu\nu}^{I} \) are field strengths of the vector fields \( A_{\mu}^{I} \). The metric, \( g_{xy} \), of the scalar manifold \( \mathcal{M}_5 \) and the metric \( a_{IJK} \) both depend on the scalar fields \( \varphi^x \). On the other hand, the completely symmetric tensor \( C_{IJK} \) is constant as required by local Abelian gauge symmetries of vector fields.

One remarkable feature of these theories is the fact that the entire \( N = 2 \) MESGT is uniquely determined by the constant tensor \( C_{IJK} \) [47]. In particular, geometry of the scalar manifold \( \mathcal{M}_5 \) is determined by \( C_{IJK} \), which can be used to define a cubic polynomial, \( \mathcal{V}(h) \), in \( n_V \) real variables \( h^I \) (\( I = 1, \ldots, n_V \)),

\[
\mathcal{V}(h) := C_{IJK} h^I h^J h^K. \tag{2.1.2}
\]

One defines a metric, \( a_{IJ} \), of an ambient space \( \mathcal{C}_{n_V} \) coordinatized by \( h^I \):

\[
a_{IJ}(h) := -\frac{1}{3} \frac{\partial}{\partial h^I} \frac{\partial}{\partial h^J} \ln \mathcal{V}(h). \tag{2.1.3}
\]

and \((n_V - 1)\)-dimensional manifold, \( \mathcal{M}_5 \), of scalar fields \( \varphi^x \) can then be represented as an hypersurface defined by the condition [47]

\[
\mathcal{V}(h) = C_{IJK} h^I h^J h^K = 1, \tag{2.1.4}
\]

\(^3\)We use the conventions of [47] in this section.
in this ambient space $\mathcal{C}_{nV}$. The ambient space $\mathcal{C}_{nV}$ is the 5 dimensional counterpart of the hyper Kähler cone of the twistor space of the corresponding three dimensional quaternionic geometry of the scalar manifold $\mathcal{M}_3$. The metric $g_{xy}$ is simply the pull-back of \((2.1.3)\) to $\mathcal{M}_5$: 

$$g_{xy}(\varphi) = h^I_x h^J_y a_{IJ}|_{\mathcal{V}=1}$$

\[(2.1.5)\]
where $h^I_x = \sqrt{\frac{3}{2}} \frac{\partial}{\partial \varphi^x}$ and the “metric” $\delta_{IJ}(\varphi)$ of kinetic energy term of the vector fields is given by the componentwise restriction of the metric $a_{IJ}$ of the ambient space $\mathcal{C}_{nV}$ to $\mathcal{M}_5$: 

$$\delta_{IJ}(\varphi) = a_{IJ}|_{\mathcal{V}=1} .$$

Riemann curvature tensor of the scalar manifold has a very simple form

$$K_{xyzu} = \frac{4}{3} (g_{x[u}g_{z]y} + T_{x[u}^w T_{z]yw})$$

\[(2.1.6)\]
where $T_{xyz}$ is the symmetric tensor 

$$T_{xyz} = h^I_x h^J_y h^K_z C_{IJK}$$

\[(2.1.7)\]
From the form of the Riemann curvature tensor $K_{xyzu}$ it follows that the covariant constancy of $T_{xyz}$ implies the covariant constancy of $K_{xyzu}$:

$$T_{xyzw} = 0 \longrightarrow K_{xyzu,w} = 0$$

Therefore scalar manifolds $\mathcal{M}_5$ with covariantly constant $T$ tensors are locally symmetric spaces.

If $\mathcal{M}_5$ is a homogeneous space the covariant constancy of $T_{xyz}$ was shown to be equivalent to the “adjoint identity” \[47\]:

$$C^{IJK} C_{J(MNC_PQ)_K} = \delta^I_M C_{NPQ}$$

\[(2.1.8)\]
where the indices are raised by the inverse $\delta^{IJ}$ of $\delta_{IJ}$. Furthermore, cubic forms defined by $C_{IJK}$ of $N = 2$ MESGT’s that satisfy the adjoint identity are in one-to-one correspondence with norm forms of Euclidean (formally real) Jordan algebras of degree 3 \[17\]. These theories exhaust the list of 5D MESGTs with symmetric target spaces $G/H$ such that $G$ is a symmetry of their Lagrangians \[50\]. Remarkably, the list of cubic forms that satisfy the adjoint identity coincides with the list of Legendre invariant cubic forms that were classified more recently \[51\].

Symmetric target spaces of $N = 2$ MESGTs defined by Euclidean Jordan algebras of degree three are of the form

$$\mathcal{M} = \frac{\text{Str}_0 (J)}{\text{Aut} (J)}$$

\[(2.1.9)\]
where $\text{Str}_0 (J)$ and $\text{Aut} (J)$ are the reduced structure (“Lorentz”) group and automorphism (“rotation”) group of the Jordan algebra $J$ respectively. Vector fields of these theories are in
one-to-one correspondence with elements of the underlying Jordan algebra \( J \) and transform linearly under the action of \( \text{Str}_0(J) \). Similarly, charges to which vector fields couple transform linearly under the action of \( \text{Str}_0(J) \), which can be interpreted as the “Lorentz” group of the Jordan algebra. Therefore, to a black hole solution of the corresponding MESGTs with charges \( q^I \) one can associate an element of the underlying Jordan algebra \( q = e_I q^I \), where \( e_I \) form a basis of \( J \). The entropy of a spherically symmetric stationary extremal black hole solution is then determined by the cubic norm \( \mathcal{V}(q) \) of \( q \). The orbits of spherically symmetric stationary extremal black holes of five dimensional \( N = 2 \) MESGT’s defined by Jordan algebras were first classified in \[17\]. It was found that the solutions with vanishing entropy have larger symmetries beyond those of the reduced structure group, namely they are invariant under “special conformal transformations” of the Jordan algebra \( J \) that lie outside the Lorentz group of \( J \). As reviewed in Appendix B one can define the action of a generalized conformal group \( \text{Conf}(J) \) acting on the Jordan algebra \( J \) \[52, 53, 18\], which leaves light-like separations \( \mathcal{V}(q - q') = 0 \) invariant. Conformal group \( \text{Conf}(J) \) changes the norm of a general element \( q \in J \) and hence the corresponding entropy of black hole solutions. Therefore conformal groups \( \text{Conf}(J) \) of Jordan algebras that define \( N = 2 \) MESGT’s were proposed as spectrum generating symmetry groups of the solutions of these theories in five dimensions \[17, 18, 21, 19, 21\]. Furthermore, \( \text{Conf}(J) \) is isomorphic to the U-duality symmetry group of the corresponding four dimensional MESGT obtained by dimensional reduction (R-map).

### 2.2 4D, \( N = 2 \) Maxwell-Einstein Supergravity Theories and Freudenthal Triple Systems

Let us now briefly review the dimensional reduction of the 5D, \( N = 2 \) MESGTs to four dimensions (R-map), restricting ourselves to the bosonic sector. The metric of the target space of the four-dimensional scalar fields of dimensionally reduced theories were first obtained in \[17\], where it was shown that these four dimensional target spaces are generalized upper half-spaces (tube domains) over the convex cones defined by the cubic norm. More specifically, the four dimensional scalar manifold are parametrized by complex coordinates \[17\],

\[
z^I := \frac{1}{\sqrt{3}} A^I + \frac{i}{\sqrt{2}} \tilde{h}^I
\]

(2.2.1)

where \( A^I \) denote the 4d scalars descending from the 5D vectors. Imaginary components are given by

\[
\tilde{h}^I := e^\sigma h^I.
\]

(2.2.2)

where \( \sigma \) is the scalar field coming from 5D graviton and they satisfy the positivity condition

\[
C_{IJK} \tilde{h}^I \tilde{h}^J \tilde{h}^K = e^{3\sigma} > 0
\]

Geometry of four dimensional \( N = 2 \) MESGTs obtained by dimensional reduction from five dimensions (R-map) was later called “very special geometry” and has been studied extensively \[14\]. Dimensional reduction of the full bosonic sector of gauged 5D \( N = 2 \) MESGTs with
tensor multiplets and its reformulation in the language of special Kähler geometry was given in \[55\], which we follow in our summary here, restricting ourselves to the ungauged theory.

The \( n_V \) complex coordinates \( z^I \) can be interpreted as inhomogeneous coordinates of a \((n_V + 1)\)-dimensional complex vector

\[
X^A = \begin{pmatrix} X^0 \\ X^I \end{pmatrix} = \begin{pmatrix} 1 \\ z^I \end{pmatrix}.
\] (2.2.3)

Taking as “prepotential” the cubic form

\[
F(X^A) = -\sqrt{2} \, C_{IJK} \frac{X^I X^J X^K}{X^0}
\] (2.2.4)
and using the symplectic section

\[
\begin{pmatrix} X^A \\ F_A \end{pmatrix} = \begin{pmatrix} X^A \\ \partial_A F \end{pmatrix} \equiv \begin{pmatrix} X^A \\ \frac{\partial F}{\partial X^A} \end{pmatrix}
\] (2.2.5)

one obtains the Kähler potential

\[
K(X, \bar{X}) := -\ln[\sqrt{\frac{2}{3}} C_{IJK}(z^I - \bar{z}^I)(z^J - \bar{z}^J)(z^K - \bar{z}^K)]
\] (2.2.6)

and the “period matrix”

\[
N_{AB} := \bar{F}_{AB} + 2i \frac{\text{Im}(F_{AC}) \text{Im}(F_{BD}) X^C X^D}{\text{Im}(F_{CD}) X^C X^D}
\] (2.2.8)

where \( F_{AB} \equiv \partial_A \partial_B F \) etc. Components of the resulting period matrix \( N_{AB} \) are:

\[
N_{00} = \frac{2\sqrt{2}}{9\sqrt{3}} C_{IJK} A^I A^J A^K - \frac{i}{3} \left( e^\sigma \tilde{a}_{IJ} A^I + \frac{1}{2} e^{3\sigma} \right)
\] (2.2.9)

\[
N_{0I} = \frac{\sqrt{2}}{3} C_{IJK} A^I A^K + \frac{i}{\sqrt{3}} e^\sigma \tilde{a}_{IJ} A^J
\] (2.2.10)

\[
N_{IJ} = -\frac{2\sqrt{2}}{\sqrt{3}} C_{IJK} A^K - i e^\sigma \tilde{a}_{IJ}
\] (2.2.11)

Prepotential (2.2.4) leads to the Kähler metric

\[
g_{IJ} \equiv \partial_I \partial_J K = \frac{3}{2} e^{-2\sigma} \tilde{a}_{IJ}
\] (2.2.12)

for the scalar manifold \( \mathcal{M}_{(4)} \) of four-dimensional theory. Denoting the field strength of vector field that comes from the graviton in five dimensions as \( F^0_{\mu\nu} \), bosonic sector of dimensionally reduced Lagrangian can be written as

\[
e^{-1} \mathcal{L}^{(4)} = -\frac{1}{2} R - g_{IJ} (\partial_\mu z^I)(\partial^\mu \bar{z}^J)
+ \frac{1}{4} \text{Im}(N_{AB}) F_{\mu\nu}^A F^{\mu\nu B} - \frac{1}{8} \text{Re}(N_{AB}) e^{\nu\rho\sigma} F_{\mu}^A F_{\rho\sigma}^B.
\] (2.2.13)
Since scalar fields $z^I$ are restricted to the domain $\mathcal{V}(\text{Im}(z)) > 0$, the scalar manifolds of 4D, $N = 2$ MESGT’s defined by Euclidean Jordan algebras $J$ of degree three are simply a Köcher “upper half spaces” of these Jordan algebras, which belong to the family of Siegel domains of the first kind \[56\]. The “upper half spaces” of Jordan algebras can be mapped into bounded symmetric domains, which can be realized as hermitian symmetric spaces of the form

$$\mathcal{M}_4 = \frac{\text{Conf}(J)}{\text{Str}J}$$

(2.2.14)

where maximal compact subgroup $\tilde{\text{Str}}J$ of the conformal group of $J$ is the compact real form of the structure group $\text{Str}(J)$ generated by dilatations and Lorentz transformations of $J$. We should also stress the important point that the Kähler potential \[2.2.7\] that one obtains directly under dimensional reduction from 5 dimensions is given by the “cubic light-cone”

$$\mathcal{V}(z - \bar{z}) = C_{IJK}(z^I - \bar{z}^I)(z^J - \bar{z}^J)(z^K - \bar{z}^K)$$

(2.2.15)

which is manifestly invariant under the 5 dimensional U-duality group $\text{Str}_0(J)$ and real translations

$$\text{Re}(z^I) \Rightarrow \text{Re}(z^I) + a^I$$

$$a^I \in \mathbb{R}$$

which follows from Abelian gauge invariances of vector fields of the five dimensional theory. Under dilatations it gets simply rescaled. Infinitesimal action of special conformal generators $K^I$ on the “cubic light-cone” gives \[18\]

$$K^I\mathcal{V}(z - \bar{z}) = (z^I + \bar{z}^I)\mathcal{V}(z - \bar{z})$$

(2.2.16)

which can be integrated to give the global transformation:

$$\mathcal{V}(z - \bar{z}) \Rightarrow f(z^I)\bar{f}(\bar{z}^I)\mathcal{V}(z - \bar{z})$$

(2.2.17)

Thus the cubic light-cone defined by $\mathcal{V}(z - \bar{z}) = 0$ is invariant under the full conformal group $\text{Conf}(J)$. Furthermore, the above global transformation leaves the metric $g_{IJ}$ invariant since it corresponds simply to a Kähler transformation of the Kähler potential $\ln \mathcal{V}(z - \bar{z})$.

In $N = 2$ MESGTs defined by Euclidean Jordan algebras $J$ of degree three, one-to-one correspondence between vector fields of five dimensional theories (and hence their charges) and elements of $J$ gets extended, in four dimensions, to a one-to-one correspondence between field strengths of vector fields plus their magnetic duals and Freudenthal triple systems defined over $J$ \[47, 18, 17, 21, 24\]. An element $X$ of Freudenthal triple system (FTS) $\mathcal{F}(J)$ \[43, 46\] over $J$ can be represented formally as a $2 \times 2$ “matrix”:

$$X = \begin{pmatrix} \alpha & x \\ y & \beta \end{pmatrix} \in \mathcal{F}(J)$$

(2.2.18)
where $\alpha, \beta \in \mathbb{R}$ and $x, y \in J$

Denoting the “bare” four dimensional graviphoton field strength and its magnetic dual as $F^0_{\mu\nu}$ and $\tilde{F}^0_{\mu\nu}$, respectively, we have the correspondence
\[
\begin{pmatrix}
F^0_{\mu\nu} & F^I_{\mu\nu} \\
\tilde{F}^I_{\mu\nu} & \tilde{F}^0_{\mu\nu}
\end{pmatrix} \iff \begin{pmatrix}
e_0 & e_I \\
\tilde{e}^I & \tilde{e}^0
\end{pmatrix} \in \mathcal{F}(J),
\]
where $e_I(\tilde{e}^I)$ are the basis elements of $J$ (its conjugate $\tilde{J}$). Consequently, one can associate with a black hole solution with electric and magnetic charges (fluxes) $(q^0, q_I, p^0, p^I)$ of the 4D MESGT defined by $J$ an element of FTS $\mathcal{F}(J)$
\[
\begin{pmatrix}
p^0 e_0 & p^I e_I \\
q_I \tilde{e}^I & q^0 \tilde{e}^0
\end{pmatrix} \in \mathcal{F}(J),
\] (2.2.19)

U-duality group $G_4$ of such a four dimensional MESGT acts as the automorphism group of FTS $\mathcal{F}(J)$, which is endowed with an invariant symmetric quartic form and a skew-symmetric bilinear form. The entropy of a spherically symmetric stationary extremal black with charges $(p^0, p^I, q^0, q_I)$ is determined by the quartic invariant $Q_4(q, p)$ of $\mathcal{F}(J)$. With this identification the orbits of extremal black holes of 4D, $N=2$ MESGT’s with symmetric scalar manifolds were classified in [17, 57].

As discussed above, four dimensional U-duality groups $G_4$ were proposed as spectrum generating conformal symmetry groups in five dimensions that leave a cubic light-cone invariant. On the other hand, three dimensional U-duality group $G_3$ of $N=2$ MESGTs defined by Jordan algebras of degree three do not, in general, have any conformal realizations on the $2n_V+2$ dimensional space of the FTS. It was shown in [18] that three dimensional U-duality groups $G_3$ have geometric realizations as quasi-conformal groups on the vector spaces of FTS’s extended by an extra coordinate and leave invariant a generalized light-cone with respect to a quartic distance function. This quasiconformal action of three dimensional U-duality group $G_3$ was proposed as a spectrum generating symmetry group of corresponding four dimensional supergravity theories [18, 19, 20, 21, 33, 34]. We shall denote the quasiconformal groups defined over FTS’s $\mathcal{F}$ extended by a singlet coordinate as $Q\text{Conf}(\mathcal{F})$. If the FTS is defined over a Jordan algebra $J$ of degree three we shall denote the corresponding quasiconformal groups either as $Q\text{Conf}(\mathcal{F}(J))$ or simply as $Q\text{Conf}(J)$. For $N=2$ MESGTs defined by Jordan algebras of degree three, quasiconformal group actions of their three dimensional U-duality groups $G_3$ were given explicitly in [24], in a basis covariant with respect to U-duality groups of corresponding six dimensional supergravity theories.

Upon further dimensional reduction to three dimensions (C-map) $N=2$ MESGTs lead to $N=4$, $d=3$ quaternionic Kähler $\sigma$ models coupled to supergravity [17, 58]. In Table we give the symmetry groups of $N=2$ MESGTs defined by Euclidean Jordan algebras in $d = 5, 4$ and $3$ dimensions and their scalar manifolds. We should note that five and
three dimensional U-duality symmetry groups $Str_0(J)$ and $QConf(J)$, respectively, act as symmetries of supergravity Lagrangians, while four dimensional U-duality groups $Conf(J)$ are on-shell symmetries.

### 3. Symmetries of Supergravity Theories with $N \geq 4$ and non-Euclidean Jordan Algebras of Degree Three

C-tensors and scalar manifolds of simple supergravity theories with $N > 4$ and $N = 4$ MESGTs in five spacetime dimensions can also be described by Jordan algebras of degree three. Referring to appendix A for further details we summarize the main results in this section. The C-tensor of five dimensional maximal $N = 8$ supergravity can be identified with the symmetric tensor defining the cubic norm of split exceptional Jordan algebra $J_3^{os}$ whose invariance group is $E_6(6)$ \cite{52,17,18,33}. Scalar manifold of 5$d$, $N = 8$ supergravity is

$$\mathcal{M}_5 = \frac{E_{6(6)}}{USp(8)}$$

and 27 vector fields of the theory correspond to elements of $J_3^{os}$, whose reduced structure group is $E_{6(6)}$. Under dimensional reduction to $d = 4$ this correspondence gets extended to a correspondence between 28 vector field strengths and their magnetic duals and the Freudenthal triple system $\mathcal{F}(J_3^{os})$ defined over $J_3^{os}$. U-duality group $E_7(7)$ of four dimensional $N = 8$ supergravity is then simply the automorphism group of $\mathcal{F}(J_3^{os})$, which is isomorphic

| $J$       | $\mathcal{M}_4 = \frac{Conf(J)}{\tilde{Str}(J)}$ | $\mathcal{M}_5 = \frac{Str_0(J)}{Aut(J)}$ | $\mathcal{M}_3 = \frac{QConf(\mathcal{F}(J))}{\tilde{Conf}(J) \times SU(2)}$ |
|-----------|-----------------------------------------------|------------------------------------------|-----------------------------------------------|
| $J_3^\mathbb{R}$ | $SL(3,\mathbb{R})/SO(3)$                        | $Sp(6,\mathbb{R})/U(3)$                    | $F_4(4)/USp(6) \times SU(2)$                  |
| $J_3^\mathbb{C}$ | $SL(3,\mathbb{C})/SU(3)$                        | $SU(3,3)/S(U(3) \times U(3))$             | $E_6(3)/SU(6) \times SU(2)$                  |
| $J_3^\mathbb{H}$ | $SU^*(6)/USp(6)$                                  | $SO^*(12)/U(6)$                           | $E_7(-5)/SO(12) \times SU(2)$                |
| $J_3^\mathbb{D}$ | $E_{6(-20)}/F_4$                                 | $E_{7(-25)}/E_6 \times U(1)$             | $E_8(-24)/E_7 \times SU(2)$                  |
| $\mathbb{R} \oplus \Gamma_n(\mathbb{Q})$ | $SO(n-1,1) \times SO(1,1)/SO(n-1)$             | $SO(n,2) \times SU(1,1)/SO(n) \times SO(2) \times U(1)$ | $SO(n+2,4)/SO(n+2) \times SO(4)$             |

Table 1: Above we list the scalar manifolds $\mathcal{M}_d$ of $N = 2$ MESGT’s defined by Euclidean Jordan algebras $J$ of degree 3 in $d = 3, 4, 5$ dimensions. $J_3^\mathbb{A}$ denotes the Jordan algebra of $3 \times 3$ Hermitian matrices over the division algebra $\mathbb{A} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$. The last row $\mathbb{R} \oplus \Gamma_n(\mathbb{Q})$ are the reducible Jordan algebras which are direct sums of Jordan algebras $\Gamma_n$ defined by a quadratic form $\mathbb{Q}$ and a one dimensional Jordan algebra $\mathbb{R}$. $Str(J)$ and $Conf(J)$ denote the compact real forms of the structure group $Str(J)$ and conformal group $Conf(J)$ of a Jordan algebra $J$. $QConf(\mathcal{F}(J))$ denotes the quasiconformal group defined by the FTS $\mathcal{F}(J)$ defined over $J$. 


to the conformal group of $J_3^{O_8}$

$$Aut(F(J_3^{O_8})) \simeq Conf(J_3^{O_8}) \quad (3.1)$$

U-duality group of 3 dimensional maximal supergravity is $E_{8(8)}$ which is simply the quasi-conformal group associated with the Freudenthal triple system $F(J_3^{O_8})$ [18]

$$QConf(F(J_3^{O_8})) = E_{8(8)} \quad (3.2)$$

Remarkably the bosonic sector of $N=6$ supergravity coincide with bosonic sector of $N=2$ MESGT defined by the Euclidean Jordan algebra $J_3^{E_6}$ and hence their U-duality groups coincide in $d=5,4$ and 3 dimensions [7]. Corresponding scalar manifolds are given in the third row of Table 1.

As for $N=4$ MESGTs describing the coupling of $n$, $N=4$ vector multiplets to $N=4$ supergravity, one finds that their C-tensors in $d=5$ can be identified with non-Euclidean Jordan algebras

$$(\mathbb{R} \oplus \Gamma(5,n)) \quad (3.3)$$

where $\Gamma(5,n)$ is the Jordan algebra of degree two associated with a quadratic form of signature $(5,n)$. Scalar manifolds of these theories in five dimensions are symmetric spaces

$$\mathcal{M}_5 = SO(5,n) \times SO(1,1)/SO(5) \times SO(n)$$

and their vector fields are in one-to-one correspondence with elements of $(\mathbb{R} \oplus \Gamma(5,n))$. In corresponding four dimensional theories U-duality groups are again automorphism groups of the underlying Freudenthal triple sytems which are isomorphic to conformal groups of $(\mathbb{R} \oplus \Gamma(5,n))$

$$Aut(F(\mathbb{R} \oplus \Gamma(5,n))) \simeq Conf(\mathbb{R} \oplus \Gamma(5,n)) \simeq SO(6,n) \times SU(1,1) \quad (3.4)$$

and scalar manifolds of four dimensional theories are

$$\mathcal{M}_4 = \frac{SO(6,n+1) \times SU(1,1)}{SO(6) \times SO(n+1) \times U(1)}$$

In three dimensions their isometry groups are given by quasiconformal groups of the corresponding Freudenthal triple systems

$$QConf(F(\mathbb{R} \oplus \Gamma(5,n))) = SO(8,n+3) \quad (3.5)$$

and their scalar manifolds are simply:

$$\mathcal{M}_3 = \frac{SO(8,n+3)}{SO(8) \times SO(n+3)}$$

In Table 2 we list the symmetry groups $Aut(J), Str_0(J), Conf(J)$ and $QConf(J)$ associated with non-Euclidean Jordan algebras of degree three.
The quartic norm $Q_4(X)$ of $X = (\alpha, \beta, x, y)$ is defined as

$$Q_4(X) \equiv \frac{1}{48} \langle (X, X, X), X \rangle$$

where $(X, Y, Z)$ denotes the Freudenthal triple product. For FTS's defined over Jordan algebras $J$ of degree three the quartic norm of an element $X = (\alpha, \beta, x, y)$ takes the following

Table 2: Above we give the automorphism $(Aut(J))$, reduced structure $Str_0(J)$, conformal $(Conf(J))$ and quasiconformal groups $(QConf(J))$ associated with non-Euclidean Jordan algebras of degree three.

4. Quasiconformal Groups associated with Euclidean Jordan Algebras of Degree Three

4.1 Quasiconformal Group Actions and Quartic Light-cones

General theory of novel quasiconformal realizations of Lie groups over Freudenthal triple systems extended by an extra singlet coordinate was given in [18] which we review in appendix C. Since the automorphism group of a Freudenthal triple system $F(J)$ defined over a Jordan algebra $J$ of degree three is isomorphic to the four dimensional U-duality group $G_4$ of MESGT defined by $J$ the original formulation of [18] is covariant with respect to $G_4$. In this section we shall study quasiconformal realizations of groups associated with FTS's defined by Jordan algebras of degree three in a basis covariant with respect to U-duality groups of five dimensional $N = 2$ MESGTs with symmetric target spaces. For convenience we shall label the elements of FTS $F(J)$ defined over $J$ as follows

$$\begin{pmatrix} \alpha \\ \beta \\ x \\ y \end{pmatrix} \equiv (\alpha, \beta, x, y) \quad (4.1.1)$$

where $\alpha, \beta \in \mathbb{R}$ and $x, y \in J$. The skew-symmetric bilinear form of two elements $X = (\alpha, \beta, x, y)$ and $Y = (\gamma, \delta, w, z)$ of $F(J)$ is given by

$$\langle X, Y \rangle \equiv \alpha \delta - \beta \gamma + (x, z) - (y, w) \quad (4.1.2)$$

where $(x, z)$ is the symmetric bilinear form over $J$ defined by the trace form $Tr$ of $J$:

$$\langle x, z \rangle \equiv Tr(x \circ z) \quad (4.1.3)$$

The quartic norm $Q_4(X)$ of $X = (\alpha, \beta, x, y)$ is defined as

$$Q_4(X) \equiv \frac{1}{48} \langle (X, X, X), X \rangle \quad (4.1.4)$$
form \[59\]
\[\mathcal{Q}_4(X) = \frac{1}{8} \left\{ (\alpha \beta - (x, y))^2 - 4 \left( x^2, y^2 \right) + 4\alpha N(y) + 4\beta N(x) \right\} \]  
(4.1.5)

where the adjoint map \( \sharp : J \to J \) is defined such that
\[ \left( x^2 \right) \sharp = N(x) x \]  
(4.1.6)

and \( N(x) \) is the cubic norm of the element \( x \in J \). Using the adjoint map one can define the symmetric Freudenthal product \( \sharp \) among two elements of \( J \):
\[ x \sharp y = (x + y)^2 - (x)^2 - (y)^2 \]  
(4.1.7)

For simple Jordan algebras \( J^3 \) of degree three the \( \sharp \) product can be expressed in terms of the Jordan product and trace form \( \text{Tr} \) as follows:
\[ x \sharp y = 2x \circ y - \text{Tr}(x) y - \text{Tr}(y) x + (\text{Tr}(x) \text{Tr}(y) - (x, y)) I_3 \]  
(4.1.8)

where \( I_3 \) is the identity element of \( J \). The adjoint \( x^2 \) of \( x \) is then
\[ x^2 = \frac{1}{2} (x \sharp x) \]  
(4.1.9)

Referring to Appendix C for a brief summary and to the original references \([18, 24]\) for details let us write down the action of Lie algebra of the quasiconformal group, associated with a FTS \( \mathcal{F} \), on the vector space \( T \) coordinatized by elements \( X \) of \( \mathcal{F} \) and an extra single variable \( x \) \([18, 24]\):
\[
\begin{align*}
K(X) &= 0 & U_A(X) &= A & S_{AB}(X) &= (A, B, X) \\
K(x) &= 2 & U_A(x) &= \langle A, X \rangle & S_{AB}(x) &= 2 \langle A, B \rangle x \\
\tilde{U}_A(X) &= \frac{1}{2} (X, A, X) - Ax \\
\tilde{U}_A(x) &= -\frac{1}{6} \langle (X, X, X), A \rangle + \langle X, A \rangle x \\
\tilde{K}(X) &= -\frac{1}{6} (X, X, X) + X x \\
\tilde{K}(x) &= \frac{1}{6} \langle (X, X, X), X \rangle + 2 x^2 
\end{align*}
\]  
(4.1.10)

where \( A, B \in \mathcal{F} \). The quartic norm over the space \( T \) is defined as
\[ \mathcal{N}_4(x) := \mathcal{Q}_4(x) - x^2 \]  
(4.1.11)

where \( \mathcal{Q}_4(x) \) is the quartic invariant of \( \mathcal{F} \). Quartic “symplectic distance” \( d(\mathcal{X}, \mathcal{Y}) \) between any two points \( \mathcal{X} = (X, x) \) and \( \mathcal{Y} = (Y, y) \) in \( T \) is defined as the quartic norm of “symplectic difference” \( \delta(\mathcal{X}, \mathcal{Y}) := (X - Y, x - y + (X, Y)) \) of two vectors
\[ d(\mathcal{X}, \mathcal{Y}) := \mathcal{N}_4(\delta(\mathcal{X}, \mathcal{Y})) = \mathcal{Q}_4(X - Y) - (x - y + (X, Y))^2 \]  
(4.1.12)
Invariance of this quartic “symplectic distance function” \(d(\mathcal{X}, \mathcal{Y})\) under the action of automorphism group of \(\mathcal{F}\) generated by \(S_{(AB)}\) and under “symplectic translations” generated by \(U_A\) and \(K\) is manifest. Generator \(\Delta\) simply rescales \(d(\mathcal{X}, \mathcal{Y})\) and under the action of generators \(\tilde{U}_A\) and \(\tilde{K}\) the distance function \(d(\mathcal{X}, \mathcal{Y})\) gets multiplied by terms linear \(X\) and \(Y\). Hence the quasiconformal group action preserves light-like separations

\[d(\mathcal{X}, \mathcal{Y}) = 0\]

with respect to quartic distance function \(13\). Hence we shall refer to the quartic distance function \(1.1.12\) as “quartic light-cone”. Quasiconformal realization of a simple Lie algebra \(g\) over a FTS \(\mathcal{F}\) extended by an extra coordinate \(x\) carries over, in a straightforward manner, to that of the complex Lie algebra \(g(\mathbb{C})\) by complexifying \(\mathcal{F}\) and \(x\).

As discussed above logarithm of cubic distance function \(V(z - \bar{z})\), written in complex coordinates, that we refer to as “cubic light cone”, is simply the Kähler potential of complex special geometry of four dimensional MESGT that descend from five dimensional MESGT uniquely determined by the C-tensor. This holds true for all four dimensional theories that descend from five dimensions and not only for those theories with symmetric scalar manifolds \(13\). Translational symmetries of the cubic light-cone (hence the Kähler potential) follows from Abelian gauge symmetries of the five dimensional supergravity theories. For theories with symmetric scalar manifolds defined by Jordan algebras there are additional symmetries, namely the special conformal transformations which lead to Kähler transformations under which the Kähler metric is invariant. There is a parallel picture in going from four to three dimensions. In four dimensional MESGT’s defined by Freudenthal triple systems \(\mathcal{F}\) the space coordinatized by real coordinates of \(\mathcal{X} = (X, x)\) on which the three dimensional U-duality group acts as a quasiconformal group correspond to the boundary coordinates of the twistor space \(35\)

\[Z_3 = \frac{QConf(J)}{Conf(J) \times SU(2)} \times SU(2_1) = \frac{QConf(J)}{U(1)} U(1)
\]

of the quaternionic symmetric scalar manifold

\[M_3 = \frac{QConf(J)}{Conf(J) \times SU(2)}
\]

Since the quasiconformal action of a group \(G\) extends to its complexification, the complex coordinates of the twistor space can be taken to be the complex extensions of the coordinates \(\mathcal{X} = (X, x)\) which we shall denote as \(Z = (Z, z)\). The Kähler potential of the twistor space \(4.1.13\) is then simply given by the logarithm of quartic distance function \(4.1.12\) written in complex “quasiconformal coordinates” \(35\):

\[K(Z, \bar{Z}) = \ln d(Z, \bar{Z}) = \ln \left[ Q_4(Z) - Z) + (z - \bar{z} + (Z, \bar{Z}))^2 \right]
\]

The quartic light-cone is manifestly invariant under the Heisenberg symmetry group corresponding to “symplectic translations” generated by \(U_A\) and \(K\) in \(4.1.10\). “Symplectic special
conformal generators” \( \tilde{U}_A \) and \( \tilde{K} \) also form an Heisenberg subalgebra. The global action of symplectic special conformal transformations on the quartic light-cone \( d(\mathcal{Z}, \bar{\mathcal{Z}}) \) results in overall multiplicative factors which are holomorphic or anti-holomorphic \( [35] \)

\[
d(\mathcal{Z}, \bar{\mathcal{Z}}) \Longrightarrow f(\mathcal{Z}, z) \bar{f}(\bar{\mathcal{Z}}, \bar{z}) d(\mathcal{Z}, \bar{\mathcal{Z}})
\]

(4.1.15)

This proves that the light-like separations are left invariant under the full quasiconformal group action. Furthermore the Kähler potential \( [4.1.14] \) of the twistor space undergoes Kähler transformations under the quasiconformal group action and hence leaves the Kähler metric invariant. These results obtained first for quaternionic symmetric spaces \( [35] \) extend to general quaternionic manifolds that are in the C-map \( [4.2] \) in complete parallel to the situation in going from five to four dimensions. In fact, the correspondence established between harmonic superspace formulation of \( N = 2 \) sigma models coupled to \( N = 2 \) supergravity and quasiconformal actions of their isometry groups \( [4.3] \) suggests that Kähler potentials of quartic light-cone type should exist for all quaternionic manifolds that couple to four dimensional \( N = 2 \) supergravity and not only those that are in the C-map.

Let us choose a basis \( f_\alpha \in \mathcal{F}(J) \) for the FTS and let \( d_{\alpha\beta\gamma\delta} \) denote the structure constants of \( \mathcal{F}(J) \) defined as follows

\[
(f_\alpha, f_\beta, f_\gamma) = d_{\alpha\beta\gamma\delta} f_\delta
\]

(4.1.16)

Under the action of its automorphism group elements of a FTS \( \mathcal{F} \) transform in a symplectic representation. Therefore one can raise and lower their indices with invariant symplectic metric \( \Omega_{\alpha\beta} \) and its inverse:

\[
V^\alpha = \Omega^{\alpha\beta} V_\beta \quad \quad \Omega_{\alpha\beta} \Omega^{\beta\gamma} = -\delta_\alpha^\gamma
\]

\[
V_\alpha = V^\beta \Omega_{\beta\alpha} \quad \quad V_\alpha W^\alpha = -V^\alpha W_\alpha
\]

(4.1.17)

Now

\[
\langle (f_\alpha, f_\beta, f_\gamma), f_\delta \rangle = d_{\alpha\beta\gamma\delta} \langle f_\epsilon, f_\delta \rangle = d_{\alpha\beta\gamma\delta} \Omega_{\epsilon\delta} = d_{\alpha\beta\gamma\delta}
\]

(4.1.18)

where we assumed the normalization \( \langle f_\alpha, f_\beta \rangle = \Omega_{\alpha\beta} \). Thus the quartic norm \( Q_4(X) \) of an element \( X = X^\alpha f_\alpha \in \mathcal{F} \) is given as

\[
Q_4 = \frac{1}{48} \langle (X, X, X, X) \rangle = \frac{1}{48} d_{\alpha\beta\gamma\delta} X^\alpha X^\beta X^\gamma X^\delta = \frac{1}{48} S_{\alpha\beta\gamma\delta} X^\alpha X^\beta X^\gamma X^\delta
\]

(4.1.19)

where \( S_{\alpha\beta\gamma\delta} \equiv d_{(\alpha\beta\gamma\delta)} \) is the completely symmetrized structure constants \( d_{\alpha\beta\gamma\delta} \) and can be written as

\[
S_{\alpha\beta\gamma\delta} = d_{\alpha\beta\gamma\delta} + \text{products of } \Omega \Omega
\]

(4.1.20)

4.2 Quasiconformal Lie Algebras twisted by a unitary character

As already stated above automorphism group \( \text{Aut}(\mathcal{F}(J)) \) of a Freudenthal triple system \( \mathcal{F}(J) \) defined over Jordan algebras \( J \) of degree three is isomorphic to the conformal group \( \text{Conf}(J) \) of \( J \):

\[
\text{Conf}(J) \cong \text{Aut}(\mathcal{F}(J))
\]
Table 3: Above we give the $7 \times 5$ grading of the quasiconformal Lie algebra associated with the Freudenthal triple system $F(J)$. The vertical 5-grading is determined by $D = -\Delta$ that commutes with the generators $R^I_J$ of the structure group of $J$ and the horizontal 7-grading is determined by $R \equiv \frac{1}{n_V} R^I_J$, which commutes with the generators of the reduced structure group $Str_0(J)$ of $J$. The generators $\tilde{R}^I_J, R^I_J$ and $R_I$ generate the automorphism group of the Freudenthal triple system $F(J)$ that is isomorphic to the conformal group $Conf(J)$ of the underlying Jordan algebra $J$, under which the grade +1 (-1) generators with respect to $D$, namely $U_0, U_I, V^I, V^0 (\tilde{U}_0, \tilde{U}_I, \tilde{V}^I, \tilde{V}^0)$ transform linearly in a symplectic representation.

The reduced structure group $Str_0(J)$ is a subgroup of $Conf(J)$ under which the elements $J$ and $\tilde{J}$ transform in conjugate representations. Hence we shall label the basis vectors of $F(J)$ as follows

$$\begin{pmatrix} \alpha & x \\ y & \beta \end{pmatrix} = \alpha \tilde{e}^0 + \beta e^I + x^I e_I + y_I \tilde{e}^I$$  \hspace{1cm} (4.2.1)$$

where $I = 1, \ldots, n_V$ and $n_V = dim(J)$.

The quasiconformal Lie algebra $QConf(F(J))$ can be given a 7 by 5 graded decomposition that is covariant with respect to the reduced structure group $Str_0(J)$ as shown in Table 3. With applications to supergravity theories in mind we shall label the elements $X, Y, \ldots$ of FTS $F(J)$ in terms of coordinates $(q_0, q_I)$ and momenta $(p^0, p^I)$ as follows

$$X = q_0 \tilde{e}^0 + q_I \tilde{e}^I + p^I e_I + p^0 e_0 \hspace{1cm} (4.2.2)$$

We shall normalize the basis elements and cubic norm (C-tensor) such that the quartic invariant is given by

$$I_4(X) = \left( p^0 q_0 - p^I q_I \right)^2 - \frac{4}{3} C_{IJKL} p^I p^K q^J q^K + \frac{4}{3\sqrt{3}} q_0 C_{IJKL} p^I p^K$$

$$= \left( p^0 q_0 - p^I q_I \right)^2 - \frac{4}{3} (p^*_I q^*_J)(q^*_I) +$$

$$\frac{4}{3\sqrt{3}} p^0 N(q) + \frac{4}{3\sqrt{3}} q_0 N(p)$$

$^4 p^*_I q^*_J = (e_I p^*_J, q^*_I \tilde{e}^J)$ and $(p^*_I q^*_J)^t = (\tilde{p}^*_I \tilde{e}^I, q^*_I e_I)$.

$^5$We should note that $C_{IJK} = C^{IJK}$ is an invariant tensor of $Str_0(J)$. 

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\[ N(q) \equiv C_{IJK} q_I q_J q_K \]
\[ (q^*)^I \equiv C_{IJK} q_I q_J q_K \]
\[ N(p) \equiv C_{IJK} p_I p_J p_K \]
\[ (p^*)^I \equiv C_{IJK} p_I p_J p_K \]

The quartic invariant \( I_4 \) agrees with the quartic invariant \[ \text{4.1.3} \] used in mathematics literature and in \[ \text{[18]} \] up to an overall factor. \( I_4(X) = 8Q_4(X) \) under the identifications

\[ N(x) = \frac{1}{3\sqrt{3}}N(x) \]
\[ x^* = \frac{1}{\sqrt{3}}x^* \]

The normalization of basis vectors \( e_I (\tilde{e}^I) \) of the Jordan algebra \( J \) (and its conjugate \( \tilde{J} \)) such that

\[ (\tilde{e}^I, \tilde{e}^J) = \text{Tr} \tilde{e}^I \circ \tilde{e}^J = \delta^{IJ} \quad (4.2.4) \]
\[ (e_I, e_J) = \text{Tr} e_I \circ e_J = \delta_{IJ} \quad (4.2.5) \]

The action of quasiconformal group \( QConf(J) \) on the space \( T = \mathcal{F}(J) \oplus \mathbb{R} \) with coordinates \( q_0, q_I, p^0, p^I \) of \( \mathcal{F}(J) \) plus an extra singlet coordinate \( x \in \mathbb{R} \), twisted by a unitary character \( \nu \), is given by the following differential operators:

\[ K = \partial_x \quad (4.2.6) \]
\[ U_0 = \partial_{p^0} + q_0 \partial_x \quad (4.2.7) \]
\[ U_I = -\partial_{p^I} + q_I \partial_x \quad (4.2.8) \]
\[ V^0 = \partial_{q_0} - p^0 \partial_x \quad (4.2.9) \]
\[ V^I = \partial_{q_I} + p^I \partial_x \quad (4.2.10) \]
\[ R_I = -\sqrt{2}C_{IJK} p_K \partial_{q_K} - \frac{3}{2} (p^0 \partial_{p^0} + q_I \partial_{q_I}) \quad (4.2.11) \]
\[ \hat{R}^I = \sqrt{2}C_{IJK} q_J \partial_{p^K} + \frac{3}{2} (q_0 \partial_{q_I} + p^I \partial_{p^0}) \quad (4.2.12) \]
\[ R^I_J = \frac{3}{2} \delta^I_J (p^0 \partial_{p^0} - q_0 \partial_{q_0}) + \frac{3}{2} \left( \delta^I_N \delta^K_J - \frac{4}{3} C_{IKL} C^{JNL} \right) (q_N \partial_{q_K} - p^K \partial_{p^N}) \quad (4.2.13) \]
\[ \mathcal{R} = \frac{1}{n\nu} R^I_J = \frac{3}{2} (p^0 \partial_{p^0} - q_0 \partial_{q_0}) + \frac{1}{2} (p^I \partial_{p^I} - q_I \partial_{q_I}) \quad (4.2.14) \]
\[ \Delta = = -\mathcal{D} = -(p^0 \partial_{p^0} + p^I \partial_{p^I} + q_0 \partial_{q_0} + q_I \partial_{q_I} - \nu) - 2x \partial_x \quad (4.2.15) \]
\[ \tilde{K} = x (p^0 \partial_{p^0} + p^I \partial_{p^I} + q_0 \partial_{q_0} + q_I \partial_{q_I} - \nu) + (x^2 + I_4) \partial_x \]
\[ + \frac{1}{2} \left( \frac{\partial I_4}{\partial p^0} \partial_{q_0} - \frac{\partial I_4}{\partial q_0} \partial_{p^0} + \frac{\partial I_4}{\partial q_I} \partial_{p^I} - \frac{\partial I_4}{\partial p^I} \partial_{q_I} \right) \quad (4.2.16) \]
The vertical five grading is given by the adjoint action of $\mathcal{D}$
\[
[\mathcal{D}, \begin{pmatrix} U_0 \\ U_I \\ V^I \\ V^0 \end{pmatrix}] = \begin{pmatrix} U_0 \\ U_I \\ V^I \\ V^0 \end{pmatrix}
\] (4.2.17)

The grade $-1$ generators (with respect to $\mathcal{D} = -\Delta$) are obtained from the above expressions by commutation with the grade $-2$ generator $\tilde{K}$
\[
\tilde{U}_0 = \begin{bmatrix} U_0, \tilde{K} \end{bmatrix}
\] (4.2.18)
\[
\tilde{V}^0 = \begin{bmatrix} V^0, \tilde{K} \end{bmatrix}
\] (4.2.19)
\[
\tilde{U}_I = \begin{bmatrix} U_I, \tilde{K} \end{bmatrix}
\] (4.2.20)
\[
\tilde{V}^I = \begin{bmatrix} V^I, \tilde{K} \end{bmatrix}
\] (4.2.21)

and satisfy
\[
[\mathcal{D}, \begin{pmatrix} \tilde{U}_0 \\ \tilde{U}_I \\ \tilde{V}^I \\ \tilde{V}^0 \end{pmatrix}] = - \begin{pmatrix} \tilde{U}_0 \\ \tilde{U}_I \\ \tilde{V}^I \\ \tilde{V}^0 \end{pmatrix}
\] (4.2.22)

The remaining non-vanishing commutation relations of Lie algebra of $QConf(J)$ are as follows:
\[
\begin{aligned}
[K, \tilde{K}] &= \Delta \\
[\Delta, K] &= -2\tilde{K} \\
[\Delta, \tilde{K}] &= 2\tilde{K} \\
[U_I, V^J] &= -2\delta^J_I \tilde{K} \\
[U_0, V^0] &= -2\tilde{K} \\
[K, \tilde{U}_0] &= U_0 \\
[K, \tilde{U}_I] &= U_I \\
[K, \tilde{V}^I] &= V^I \\
[K, \tilde{V}^0] &= V^0 \\
[\tilde{U}_I, \tilde{V}^J] &= -2\delta^J_I \tilde{K} \\
[\tilde{U}_0, \tilde{V}^0] &= -2\tilde{K} \\
[U_0, \tilde{V}^0] &= -2\mathcal{R} + \mathcal{D} \\
[V^0, \tilde{U}_0] &= -2\mathcal{R} - \mathcal{D}
\end{aligned}
\] (4.2.23–4.2.35)
\[
[R_I^l, R_K] = \frac{3}{2} \Lambda_{IK}^{JL} R_L
\]
\[
[R_I^l, \tilde{R}^L] = -\frac{3}{2} \Lambda_{IL}^{JK} \tilde{R}^L
\]
\[
[R_I^l, U_K] = \frac{3}{2} \Lambda_{IK}^{JL} U_L - \frac{3}{2} \delta_I^J U_K
\]
\[
[R_I^l, V^K] = -\frac{3}{2} \Lambda_{IK}^{JK} V^L + \frac{3}{2} \delta_I^J V^K
\]
\[
[R_I^l, \tilde{U}_K] = \frac{3}{2} \Lambda_{IK}^{JL} \tilde{U}_L - \frac{3}{2} \delta_I^J \tilde{U}_K
\]
\[
[R_I^l, \tilde{V}^K] = -\frac{3}{2} \Lambda_{IK}^{JK} \tilde{V}_L + \frac{3}{2} \delta_I^J \tilde{V}_K
\]
\[
[R_I, \tilde{R}^l] = -R_I^l
\]
\[
[U_0, \tilde{V}^l] = 2 \sqrt{\frac{2}{3}} \tilde{R}^l
\]
\[
[\tilde{U}_0, V^l] = -2 \sqrt{\frac{2}{3}} \tilde{R}^l
\]
\[
[V^0, \tilde{U}_I] = -2 \sqrt{\frac{2}{3}} R_I
\]
\[
[\tilde{V}^0, U_I] = -2 \sqrt{\frac{2}{3}} R_I
\]
\[
[U_I, \tilde{V}^J] = \frac{4}{3} R_I^J - \delta_I^J (\Delta + 2R)
\]
\[
[\tilde{U}_I, V^J] = -\frac{4}{3} R_I^J - \delta_I^J (\Delta - 2R)
\]
\[
[U_I, \tilde{U}_J] = -\frac{4}{3} \sqrt{2} C_{IJK} \tilde{R}_K
\]
\[
[V^I, \tilde{V}^J] = -\frac{4}{3} \sqrt{2} C^{IJK} R_K
\]
\[
[V^I, R_J] = -\sqrt{\frac{3}{2}} \delta^J_I V^0
\]
\[
[\tilde{V}^I, R_J] = -\sqrt{\frac{3}{2}} \delta^J_I \tilde{V}^0
\]
\[
[\tilde{U}_I, \tilde{R}^J] = -\sqrt{\frac{3}{2}} \delta^J_I \tilde{U}_0
\]
\[
[U_I, \tilde{R}^J] = -\sqrt{\frac{3}{2}} \delta^J_I U_0
\]

where
\[
\Lambda_{IK}^{JL} := \delta_K^I \delta_L^J + \delta_L^I \delta_K^J - \frac{4}{3} C^{IJM} C_{KLM}
\]

The generators \( K, \tilde{K}, U_0, \tilde{U}_0, V^0, \tilde{V}^0, R \) and \( \Delta \) form the Lie subalgebra \( SL(3, \mathbb{R}) \) whose max-
imal compact subalgebra $SO(3)$ is generated by

\[
T_1 := \frac{1}{\sqrt{2}} \left( U_0 - \tilde{V}^0 \right) \\
T_2 := \frac{1}{\sqrt{2}} \left( V^0 + \tilde{U}_0 \right) \\
T_3 := - \left( K + \tilde{K} \right)
\]

They satisfy the commutation relations

\[
[T_i, T_j] = \epsilon_{ijk} T_k
\]

where $i, j, k = 1, 2, 3$. The noncompact generators of $SL(3, \mathbb{R})$ are $R$, $(K - \tilde{K})$, $\Delta$, $(U_0 + \tilde{V}^0)$ and $(V^0 - \tilde{U}_0)$. They transform in the spin 2 representation under $SO(3)$ subgroup. Centralizer of this $SL(3, \mathbb{R})$ subgroup inside $QConf(J)$ is the reduced structure group $Str_0(J)$ of the underlying Jordan algebra $J$:

\[
QConf(J) \supset Str_0(J) \times SL(3, \mathbb{R})
\]

Maximal compact subgroups of quasiconformal groups $QConf(J)$ associated with Euclidean Jordan algebras of degree three are of the form

\[
\widetilde{Conf}(J) \times SU(2)_L \subset QConf(J)
\]

where the $\widetilde{Conf}(J)$ is the compact real form of the conformal group of the Jordan algebra $J$. Hence the quotient

\[
\frac{QConf(J)}{\widetilde{Conf}(J) \times SU(2)_L}
\]

is a quaternionic symmetric space whose coset generators transform in the $(2n_V + 2, 2)$ representation of $\widetilde{Conf}(J) \times SU(2)_L$. The generators of the maximal compact subgroup $\widetilde{Conf}(J) \times SU(2)_L$ are as follows:

\[
(U_I - \tilde{V}^I), \\
(V^I + \tilde{U}_I), \\
(R_I + \tilde{R}^I), \\
A_{IJ} = -A_{JI} = (R^I_I - R^I_J), \\
(\tilde{U}_0 + V^0), \\
(-U_0 + \tilde{V}^0), \\
(K + \tilde{K})
\]

Automorphism group of the underlying Jordan algebra is generated by $A_{IJ}$ and is compact for Euclidean Jordan algebras. The compact structure group $\widetilde{Str}(J)$ is a subgroup of $\widetilde{Conf}(J)$ and is generated by $A_{IJ}$ and $(R_I + \tilde{R}^I)$:

\[
\widetilde{Str}(J) \iff A_{IJ} \oplus (R_I + \tilde{R}^I) \quad ; I, J = 1, \ldots, n_V
\]
SU(2)$_L$ subalgebra is generated by the following linear combinations

\[
L_3 = \frac{1}{4} \left( K + \hat{K} + \sqrt{\frac{2}{3}} \left( \sum_{I=1}^{n_V} (R_I + \hat{R}_I) \right) \right)
\]

\[
L_1 = \frac{1}{4\sqrt{2}} \left( -U_0 + \hat{V}^0 + \sum_{I=1}^{n_V} (\hat{U}_I + V^I) \right)
\]

\[
L_2 = \frac{1}{4\sqrt{2}} \left( \hat{U}_0 + V^0 + \sum_{I=1}^{n_V} (U_I - \hat{V}^I) \right)
\]

and satisfy the commutation relations

\[
[L_i, L_j] = \epsilon_{ijk} L_k \quad ; i, j, \ldots = 1, 2, 3
\]

We will label the basis elements of the Jordan algebra $J$ such that $e_1, e_2$ and $e_3$ are the three irreducible idempotents of $J$ and the identity element $I$ is simply

\[
I = e_1 + e_2 + e_3
\]

With this labelling the operators

\[
\sum_{i=1}^{3} (U_i - \hat{V}^i) \ , \ \sum_{i=1}^{3} (V^i + \hat{U}_i) \ \ , \ \sum_{i=1}^{3} (R_i + \hat{R}_i)
\]

generate an SU(2)$_S$ subgroup of $\widetilde{\text{Conf}}(J)$ whose centralizer inside $\widetilde{\text{Conf}}(J)$ is the automorphism group Aut($J$) of Jordan algebra $J$ generated by $A_{IJ} = R_I^J - R_J^I$. Thus we have the inclusions

\[
Q\text{Conf}(J) \supset \widetilde{\text{Conf}}(J) \times SU(2)_L \supset Aut(J) \times SU(2)_S \times SU(2)_L
\]

The centralizer of Aut($J$) inside the full QConf($J$) is the split exceptional group $G_{2(2)}$:

\[
Q\text{Conf}(J) \supset Aut(J) \times G_{2(2)}
\]

4.3 Quadratic Casimir Operators of Quasiconformal Lie algebras

The generators $S^I_J$ of the reduced structure (Lorentz) group of a Jordan algebra are given by the traceless components of $R_I^J$:

\[
S^I_J = R^I_J - \frac{1}{n_V} \delta^I_J (R^K_K) = R^I_J - \delta^I_J R
\]

The quadratic Casimir operator of the quasiconformal group QConf($J$) of a simple Jordan algebra $J$ of degree three can then be written in a general form involving a single parameter $\alpha$:
\[ C_2 = \alpha S_I S_J - \frac{4}{3} (R^I R_I + R_I R^I) + (U_0 \bar{V}^0 + U_I \bar{V}^I + \bar{V}^0 U_0 + \bar{V}^I U_I) + \frac{(n_V - 3) (R_2^2 + R_3^3)}{9} + \Delta^2 \]

(4.3.2)

\[-(\bar{U}_0 V^0 + \bar{U}_I V^I + V^0 \bar{U}_0 + V^I \bar{U}_I) - 2(K \bar{K} + \bar{K} K) + \Delta^2\]

where \( \alpha \) takes on the following values for different quasiconformal Lie groups \( \text{QCG}(J) \):

\[
\begin{align*}
\alpha(F(4)) &= \frac{16}{45} \\
\alpha(E_{6(2)}) &= \frac{8}{27} \\
\alpha(E_{7(-5)}) &= \frac{2}{9} \\
\alpha(E_{8(-24)}) &= \frac{4}{27}
\end{align*}
\] (4.3.3)

As for the quasiconformal groups associated with the generic family of reducible Jordan algebras \( J = \mathbb{R} \oplus \Gamma(1,n_{V-1}) \) the quadratic Casimir can be written in the form:

\[
C_2(SO(n_V + 1,4)) = \frac{4}{9} (R_I^I R_I^I - \frac{4}{3} (\bar{R}_I^I R_I^I + R_I \bar{R}_I^I)) - \frac{(n_V - 3)}{9} (R_2^2 + R_3^3)^2
\]

\[+(U_0 \bar{V}^0 + U_I \bar{V}^I + \bar{V}^0 U_0 + \bar{V}^I U_I) \]

\[-(\bar{U}_0 V^0 + \bar{U}_I V^I + V^0 \bar{U}_0 + V^I \bar{U}_I) - 2(K \bar{K} + \bar{K} K) + \Delta^2\]

(4.3.4)

The quadratic Casimir operators for the quasiconformal realizations given above reduce to a c-number whose value can be expressed universally as:

\[ C_2(\text{QCG}(J)) = \nu(\nu + 2n_V + 4) \]

(4.3.5)

where \( n_V \) is equal to the dimension \( \text{Dim}(J) \) of the underlying Jordan algebra. As a consequence the representations induced by the quasiconformal action of \( \text{QConf}(J) \) over the space of square integrable functions \( L^2(p^i,q^r,x) \) are unitary representations belonging to the principle series under the scalar product

\[ \langle f | g \rangle = \int \tilde{f}(p,q,x)g(p,q,x)d^dpd^dqdx \]

(4.3.6)

for

\[ \nu = -(n_V + 2) + i \rho \]

(4.3.7)

where \( \rho \in \mathbb{R} \). For special real discrete values of the twisting parameter \( \nu \) one can realize the unitary representations of \( \text{QConf}(J) \) belonging to the quaternionic discrete series and their continuations over the space of holomorphic functions of complexified quasiconformal coordinates. These holomorphic coordinates can be identified with the natural complex coordinates of the twistor spaces associated with the quaternionic symmetric spaces

\[ \text{QConf}(J) \times \text{Conf}(J) \times SU(2) \]
For rank two quaternionic groups these representations were studied in [35]. Detailed studies of discrete series representations induced by the quasiconformal actions for higher rank groups $QConf(J)$ will be subjects of separate studies.

5. Jordan Algebra of STU Model and its Quasiconformal Group $SO(4, 4)$

To illustrate the general structure of quasiconformal group actions formulated above we shall study in detail the quasiconformal group associated with the smallest nontrivial Euclidean Jordan algebra of degree three

$$J = \mathbb{R} \oplus \Gamma_{(1,1)} \quad (5.0.1)$$

which belongs to the generic Jordan family of reducible Jordan algebras. Its cubic norm form

$$\mathcal{N} = \alpha Q(\beta^0, \beta^1) = \alpha [(\beta^0)^2 - (\beta^1)^2] \quad (5.0.2)$$

can be factorized as

$$\mathcal{N} = \alpha \beta_+ \beta_- \quad (5.0.3)$$

where $\beta_\pm = \beta^0 \pm \beta^1$. We shall choose the normalization of the cubic norm such that the symmetric $C$-tensor is given by the absolute value of the Levi-Civita tensor:

$$C_{IJK} = \frac{\sqrt{3}}{2} |\epsilon_{IJK}| \quad (5.0.4)$$

so that

$$C_{IJK} x^I x^J x^K = 3\sqrt{3} x^1 x^2 x^3 \quad (5.0.5)$$

The structure group of the Jordan algebra is

$$Str(J = \mathbb{R} \oplus \Gamma_{(1,1)}) = SO(1, 1) \times SO(1, 1) \times SO(1, 1) \quad (5.0.6)$$

and the $d = 5, N = 2$ MESGT defined by $J = \mathbb{R} \oplus \Gamma_{(1,1)}$ describes the coupling of two vector multiplets to $N = 2$ supergravity. In four dimensions it gives the STU model whose U-duality group is isomorphic to the conformal group of $J = \mathbb{R} \oplus \Gamma_{(1,1)}$

$$Conf(J = \mathbb{R} \oplus \Gamma_{(1,1)}) = SL(2, \mathbb{R}) \times SL(2, \mathbb{R}) \times SL(2, \mathbb{R}) \quad (5.0.7)$$

U-duality group of the STU model in three dimensions is the quasiconformal group defined over the FTS associated with $J = \mathbb{R} \oplus \Gamma_{(1,1)}$

$$QConf(\mathbb{R} \oplus \Gamma_{(1,1)}) = SO(4, 4) \quad (5.0.8)$$
Table 4: Above we give the 7 by 5 grading of $SO(4,4)$ with respect to the generators $\mathcal{R}$ and $\mathcal{D}$ respectively.

5.1 Noncompact Basis of Quasiconformal $SO(4,4)$

In the standard 7 $\times$ 5-grading of the Lie algebra of $SO(4,4)$ as a quasiconformal Lie algebra given in Table 4 the indices $I, J, ..$ run over 1,2,3 and $R_I J = 0$ if $I \neq J$. Explicit expressions for the generators of grade zero subspace (with respect to $SO(1,1)$ generator $\mathcal{D} = -\Delta$ ) are as follows:

$$R_I = \frac{3}{2} \left( p^0 \frac{\partial}{\partial p^0} + p^K \frac{\partial}{\partial p^K} - q_0 \frac{\partial}{\partial q_0} - q_K \frac{\partial}{\partial q_K} - 2p^I \frac{\partial}{\partial p^I} + 2q_I \frac{\partial}{\partial q_I} \right)$$

\[(I = 1, 2, 3) \text{ , } I \text{ not summed over} \]

$$R_1 = -\sqrt{\frac{3}{2}} \left( p^0 \frac{\partial}{\partial p^0} + q_1 \frac{\partial}{\partial q_0} + p^2 \frac{\partial}{\partial q_3} + p^3 \frac{\partial}{\partial q_2} \right)$$

$$\tilde{R}_1 = \sqrt{\frac{3}{2}} \left( p^1 \frac{\partial}{\partial p^0} + q_0 \frac{\partial}{\partial q_1} + q_2 \frac{\partial}{\partial p^1} + q_3 \frac{\partial}{\partial p^2} \right)$$

$$R_2 = -\sqrt{\frac{3}{2}} \left( p^0 \frac{\partial}{\partial p^2} + q_2 \frac{\partial}{\partial q_0} + p^3 \frac{\partial}{\partial q_1} + p^1 \frac{\partial}{\partial q_3} \right)$$

$$\tilde{R}_2 = \sqrt{\frac{3}{2}} \left( p^2 \frac{\partial}{\partial p^0} + q_0 \frac{\partial}{\partial q_2} + q_3 \frac{\partial}{\partial p^1} + q_1 \frac{\partial}{\partial p^2} \right)$$

$$R_3 = -\sqrt{\frac{3}{2}} \left( p^0 \frac{\partial}{\partial p^3} + q_3 \frac{\partial}{\partial q_0} + p^1 \frac{\partial}{\partial q_2} + p^2 \frac{\partial}{\partial q_1} \right)$$

$$\tilde{R}_3 = \sqrt{\frac{3}{2}} \left( p^3 \frac{\partial}{\partial p^0} + q_0 \frac{\partial}{\partial q_3} + q_1 \frac{\partial}{\partial p^1} + q_2 \frac{\partial}{\partial p^2} \right)$$

$$\mathcal{D} = -\Delta = - \left( p^0 \frac{\partial}{\partial p^0} + p^I \frac{\partial}{\partial p^I} + q_1 \frac{\partial}{\partial q_1} + q_0 \frac{\partial}{\partial q_0} + 2x \frac{\partial}{\partial x} - \nu \right)$$

They generate the subgroup $Conf(J) \times SO(1,1)$. The generators of three $SL(2,\mathbb{R})$ factors of $Conf(J)$ are

$$SL(2,\mathbb{R})_I \iff (R_I \oplus R_I^J \oplus \tilde{R}_I^J) \text{ , } I = 1, 2, 3$$
They satisfy the commutation relations:

\[
\begin{align*}
[R_I, \hat{R}^I] &= R^I_i \\
[R^I_i, R^I_i] &= 3R^I_i \\
[R^I_i, \hat{R}^I] &= -3\hat{R}^I_i
\end{align*}
\] (5.1.18)

(5.1.19)

(5.1.20)

I not summed over

The linear combinations \((R_I + \hat{R}^I)\) for \(I = 1, 2, 3\) generate the compact \(U(1)_I\) subgroups of the three \(SL(2, \mathbb{R})_I\). The grade \(\pm 2\) generators \(K\) and \(\tilde{K}\) together with \(\Delta = -D\) generate another \(SL(2, \mathbb{R})\) subgroup of \(SO(4, 4)\):

\[
SL(2, \mathbb{R})_0 \rightarrow K \oplus \Delta \oplus \tilde{K}
\] (5.1.21)

whose compact \(U(1)\) subgroup is generated by \((K + \tilde{K})\). Explicitly we have

\[
K = \frac{\partial}{\partial x}
\] (5.1.22)

\[
\tilde{K} = x \left( p^0 \frac{\partial}{\partial p^0} + p^I \frac{\partial}{\partial p^I} + q_0 \frac{\partial}{\partial q_0} + q_I \frac{\partial}{\partial q_I} \right) + \left( x^2 + I_4 \right) \frac{\partial}{\partial x}
\]

where the quartic invariant \(I_4\) is

\[
I_4 = 4p^0 q_1 q_2 q_3 + 4q_0 p^1 p^2 p^3 + \left( p^0 q_0 - p^1 q_1 - p^2 q_2 - p^3 q_3 \right)^2
\]

\[
- 4 \left( p^2 q_2 p^3 q_3 + p^1 q_1 p^2 q_2 + p^1 q_1 p^3 q_3 \right) \] (5.1.23)

They satisfy the commutation relations;

\[
\begin{align*}
[K, \tilde{K}] &= \Delta \\
[\Delta, K] &= -2K \\
[\Delta, \tilde{K}] &= 2\tilde{K}
\end{align*}
\] (5.1.24)

(5.1.25)

(5.1.26)

Four mutually commuting \(SL(2, \mathbb{R})\) subgroups correspond to the decomposition

\[
SO(4, 4) \supset SO(2, 2) \times SO(2, 2) \cong SL(2, \mathbb{R}) \times SL(2, \mathbb{R}) \times SL(2, \mathbb{R}) \times SL(2, \mathbb{R})
\] (5.1.27)

Vertical grade +1 generators (with respect to \(D\) are

\[
U_0 = \frac{\partial}{\partial p^0} + q_0 \frac{\partial}{\partial x}
\] (5.1.28)

\[
V^0 = \frac{\partial}{\partial q_0} - p^0 \frac{\partial}{\partial x}
\] (5.1.29)

\[
U_I = -\frac{\partial}{\partial p^I} + q_I \frac{\partial}{\partial x}
\] (5.1.30)

\[
V^I = -\frac{\partial}{\partial q_I} + p^I \frac{\partial}{\partial x}
\] (5.1.31)
The generators belonging to vertical grade $-1$ subspace are given by the differential operators

\begin{align*}
\vec{U}_0 &= 2q_0^2 \left( \frac{\partial}{\partial q_0} \right) - \nu q_0 + 2q_1 q_0 \left( \frac{\partial}{\partial q_1} \right) + 2q_2 q_0 \left( \frac{\partial}{\partial q_2} \right) + 2q_3 q_0 \left( \frac{\partial}{\partial q_3} \right) \\
&\quad + (-p^0 q_0 + p^1 q_1 + p^2 q_2 + p^3 q_3 + x) \left( \frac{\partial}{\partial p^0} \right) \\
&\quad + 2q_2 q_3 \left( \frac{\partial}{\partial p^1} \right) + 2q_1 q_3 \left( \frac{\partial}{\partial p^2} \right) + 2q_1 q_2 \left( \frac{\partial}{\partial p^3} \right) \\
&\quad + (p^0 q_0 - p^1 q_1 - p^2 q_2 - p^3 q_3 + x q_0 + 2q_1 q_2 q_3) \left( \frac{\partial}{\partial x} \right) \tag{5.1.32}
\end{align*}

\begin{align*}
\vec{V}^0 &= -2(p^0)^2 \left( \frac{\partial}{\partial p^0} \right) + \nu p^0 - 2p^1 p^0 \left( \frac{\partial}{\partial p^1} \right) - 2p^2 p^0 \left( \frac{\partial}{\partial p^2} \right) - 2p^3 p^0 \left( \frac{\partial}{\partial p^3} \right) \\
&\quad + (p^0 q_0 - p^1 q_1 - p^2 q_2 - p^3 q_3 + x) \left( \frac{\partial}{\partial q_0} \right) \\
&\quad - 2p^2 p^3 \left( \frac{\partial}{\partial q_1} \right) - 2p^1 p^3 \left( \frac{\partial}{\partial q_2} \right) - 2p^1 p^2 \left( \frac{\partial}{\partial q_3} \right) \\
&\quad + (p^1 (2p^2 p^3 - p^0 q_1) + p^0 (p^0 q_0 - p^2 q_2 - p^3 q_3 - x)) \left( \frac{\partial}{\partial x} \right) \tag{5.1.33}
\end{align*}

\begin{align*}
\vec{U}_1 &= 2q_1^2 \left( \frac{\partial}{\partial q_1} \right) - \nu q_1 + 2p^2 q_1 \left( \frac{\partial}{\partial p^1} \right) + 2p^3 q_1 \left( \frac{\partial}{\partial p^2} \right) + 2q_0 q_1 \left( \frac{\partial}{\partial q_0} \right) + 2p^2 p^3 \left( \frac{\partial}{\partial p^3} \right) \\
&\quad + (p^0 q_0 - p^1 q_1 - p^2 q_2 + p^3 q_3 - x) \left( \frac{\partial}{\partial q_0} \right) + 2p^3 q_0 \left( \frac{\partial}{\partial q_2} \right) + 2p^2 q_0 \left( \frac{\partial}{\partial q_3} \right) \\
&\quad + (p^2 (q_1 q_2 - 2p^3 q_0) + q_1 (p^0 q_0 - p^1 q_1 + p^3 q_3 + x)) \left( \frac{\partial}{\partial x} \right) \tag{5.1.34}
\end{align*}

\begin{align*}
\vec{V}^1 &= 2(p^1)^2 \left( \frac{\partial}{\partial p^1} \right) - \nu p^1 + 2p^0 p^1 \left( \frac{\partial}{\partial p^0} \right) + 2q_2 p^1 \left( \frac{\partial}{\partial q_2} \right) + 2q_3 p^1 \left( \frac{\partial}{\partial q_3} \right) + 2p^0 q_1 \left( \frac{\partial}{\partial p^2} \right) \\
&\quad + 2p^0 q_2 \left( \frac{\partial}{\partial p^3} \right) + 2q_2 q_3 \left( \frac{\partial}{\partial q_0} \right) + (p^0 q_0 - p^1 q_1 + p^2 q_2 + p^3 q_3 + x) \left( \frac{\partial}{\partial q_1} \right) \\
&\quad + (p^1 (2q_2 q_3 - q^0) + p^1 (p^1 q_1 - p^2 q_2 - p^3 q_3 + x)) \left( \frac{\partial}{\partial x} \right) \tag{5.1.35}
\end{align*}

\begin{align*}
\vec{U}_2 &= 2q_2^2 \left( \frac{\partial}{\partial q_2} \right) - \nu q_2 + 2p^1 q_2 \left( \frac{\partial}{\partial p^1} \right) + 2p^3 q_2 \left( \frac{\partial}{\partial p^3} \right) + 2q_0 q_2 \left( \frac{\partial}{\partial q_0} \right) + 2p^1 p^3 \left( \frac{\partial}{\partial p^3} \right) \\
&\quad + (p^0 q_0 + p^1 q_1 - p^2 q_2 + p^3 q_3 - x) \left( \frac{\partial}{\partial q_0} \right) + 2p^3 q_0 \left( \frac{\partial}{\partial q_1} \right) + 2p^1 q_0 \left( \frac{\partial}{\partial q_3} \right) \\
&\quad + (p^1 (q_1 q_2 - 2p^3 q_0) + q_2 (p^0 q_0 - p^2 q_2 + p^3 q_3 + x)) \left( \frac{\partial}{\partial x} \right) \tag{5.1.36}
\end{align*}

\begin{align*}
\vec{V}^2 &= 2(p^2)^2 \left( \frac{\partial}{\partial p^2} \right) - \nu p^2 + 2p^0 p^2 \left( \frac{\partial}{\partial p^0} \right) + 2q_1 p^2 \left( \frac{\partial}{\partial q_1} \right) + 2q_3 p^2 \left( \frac{\partial}{\partial q_3} \right) + 2p^0 q_3 \left( \frac{\partial}{\partial p^1} \right) \\
&\quad + 2p^0 q_1 \left( \frac{\partial}{\partial p^3} \right) + 2q_1 q_3 \left( \frac{\partial}{\partial q_0} \right) + (p^0 q_0 + p^1 q_1 - p^2 q_2 + p^3 q_3 + x) \left( \frac{\partial}{\partial q_2} \right) \\
&\quad + (p^0 (2q_1 q_3 - p^2 q_0) + p^2 (-p^1 q_1 + p^2 q_2 - p^3 q_3 + x)) \left( \frac{\partial}{\partial x} \right) \tag{5.1.37}
\end{align*}
\[ \tilde{U}_3 = 2q_3^2 \left( \frac{\partial}{\partial q_3} \right) - \nu q_3 + 2p_1 q_3 \left( \frac{\partial}{\partial p_1} \right) + 2p_2 q_3 \left( \frac{\partial}{\partial p_2} \right) + 2q_0 q_3 \left( \frac{\partial}{\partial q_0} \right) + 2p_1 p_2 \left( \frac{\partial}{\partial p_1} \right) \\
+ (p^0 q_0 + p^1 q_1 + p^2 q_2 - p^3 q_3 - x) \left( \frac{\partial}{\partial p_1} \right) + 2p_2 q_0 \left( \frac{\partial}{\partial q_1} \right) + 2p_1 q_2 \left( \frac{\partial}{\partial q_2} \right) \\
+ (p^1 (q_1 q_3 - 2p^2 q_0) + q_3 (p^0 q_0 + p^2 q_2 - p^3 q_3 + x)) \left( \frac{\partial}{\partial x} \right) \] (5.1.38)

\[ \tilde{V}^3 = 2(p^3)^2 \left( \frac{\partial}{\partial p^3} \right) - \nu p^3 + 2p^0 p^3 \left( \frac{\partial}{\partial p^0} \right) + 2q_1 p^3 \left( \frac{\partial}{\partial q_1} \right) + 2q_2 p^3 \left( \frac{\partial}{\partial q_2} \right) + 2p_0 q_2 \left( \frac{\partial}{\partial p_1} \right) \\
+ 2p_0 q_1 \left( \frac{\partial}{\partial p^2} \right) + 2q_1 q_2 \left( \frac{\partial}{\partial q_0} \right) + (p^0 q_0 + p^1 q_1 + p^2 q_2 - p^3 q_3 + x) \left( \frac{\partial}{\partial q_3} \right) \\
+ (p^0 (2q_1 q_2 - p^3 q_0) + p^3 (-p^1 q_1 - p^2 q_2 + p^3 q_3 + x)) \left( \frac{\partial}{\partial x} \right) \] (5.1.39)

The horizontal 7-grading of the Lie algebra of $SO(4,4)$ is given by the adjoint action of $R = \frac{1}{3} R_I$:

\[ [\mathcal{R}, \left( V^0 \right)] = \frac{3}{2} \left( \tilde{V}^0 \right) \] (5.1.40)

\[ [\mathcal{R}, R_I] = R_I \] (5.1.41)

\[ [\mathcal{R}, \left( V^I \right)] = \frac{1}{2} \left( \tilde{V}^I \right) \] (5.1.42)

\[ [\mathcal{R}, R^I_I] = 0 \] (5.1.43)

\[ [\mathcal{R}, \left( U_I \right)] = -\frac{1}{2} \left( \tilde{U}_I \right) \] (5.1.44)

\[ [\mathcal{R}, \tilde{R}^I] = -\tilde{R}^I \] (5.1.45)

\[ [\mathcal{R}, \left( U_0 \right)] = -\frac{3}{2} \left( \tilde{U}_0 \right) \] (5.1.46)

The remaining nonvanishing commutation relations of $SO(4,4)$ are

\[ [U_0, \tilde{V}^0] = -2\mathcal{R} + \mathcal{D} \] (5.1.47)

\[ [V^0, \tilde{U}_0] = -2\mathcal{R} - \mathcal{D} \] (5.1.48)

\[ [U_0, V^0] = -K \] (5.1.49)
\[
\begin{aligned}
\left[ \tilde{U}_0, \tilde{V}^0 \right] &= -2\tilde{K} \quad (5.1.50) \\
\left[ U_I, \tilde{V}^J \right] &= \frac{4}{3} R_I^J - \delta_I^J (2\mathcal{R} - D) \quad (5.1.51) \\
\left[ V^I, \tilde{U}_J \right] &= \frac{4}{3} R_I^J - \delta_I^J (2\mathcal{R} + D) \quad (5.1.52) \\
\left[ U_I, \tilde{U}_J \right] &= -2\sqrt{\frac{2}{3}} |\epsilon_{IJK}| \tilde{R}^K \quad (5.1.53) \\
\left[ V^I, \tilde{V}^J \right] &= -2\sqrt{\frac{2}{3}} |\epsilon^{IJK}| R_K \quad (5.1.54) \\
\left[ U_0, \tilde{V}^I \right] &= 2\sqrt{\frac{2}{3}} \tilde{R}^I \quad (5.1.55) \\
\left[ \tilde{U}_0, V^I \right] &= -2\sqrt{\frac{2}{3}} \tilde{R}^I \quad (5.1.56) \\
\left[ V^0, \tilde{U}_I \right] &= -2\sqrt{\frac{2}{3}} R_I \quad (5.1.57) \\
\left[ \tilde{V}^0, U_I \right] &= 2\sqrt{\frac{2}{3}} R_I \quad (5.1.58) \\
\left[ U_I, R_J \right] &= \sqrt{\frac{2}{3}} |\epsilon_{IJK}| V^K \quad (5.1.59) \\
\left[ \tilde{U}_I, R_J \right] &= \sqrt{\frac{2}{3}} |\epsilon_{IJK}| \tilde{V}^K \quad (5.1.60) \\
\left[ V^I, \tilde{R}^J \right] &= -\sqrt{\frac{2}{3}} |\epsilon^{IJK}| U_K \quad (5.1.61) \\
\left[ \tilde{V}^I, \tilde{R}^J \right] &= -\sqrt{\frac{2}{3}} |\epsilon^{IJK}| \tilde{U}_K \quad (5.1.62) \\
\end{aligned}
\]

Recalling that
\[
|\epsilon_{IJK}| = \frac{2}{\sqrt{3}} C_{IJK}
\]
covariance of the commutation relations above with respect to the U-duality group \(SO(1, 1) \times SO(1, 1)\) of the five dimensional STU supergravity model \([47, 48]\) becomes manifest. The generators of the maximal compact subgroup \(SO(4) \times SO(4) = SU(2) \times SU(2) \times SU(2) \times SU(2)\) of \(SO(4, 4)\) are the following:

\[
(K + \tilde{K}), (U_0 - \tilde{V}^0), (\tilde{U}_0 + V^0) \\
(R_I + \tilde{R}^I), \\
(U_I - V^I), \\
(\tilde{U}_I + V^I), \\
(I = 1, 2, 3)
\]

(5.1.64)
Note that the maximal compact subgroup is of the form

$$\widehat{\text{Conf}}(J) \times SU(2)_L$$

where \(\widehat{\text{Conf}}(J)\) is the compact real form of the conformal group of the underlying Jordan algebra \(J = (\mathbb{R} \oplus \Gamma_{(1,1)})\)

$$\widehat{\text{Conf}}(\mathbb{R} \oplus \Gamma_{(1,1)}) = SU(2)_M \times SU(2)_N \times SU(2)_O$$  \hspace{1cm} (5.1.65)

The group \(SU(2)_S\) defined in the previous section is simply the diagonal subgroup of the three \(SU(2)\) subgroups and commutes with \(SU(2)_L\).

### 5.2 Compact Basis of \(SO(4,4)\)

The generators of the maximal compact subgroup \(\widehat{\text{Conf}} \times SU(2)_L = SU(2)_M \times SU(2)_N \times SU(2)_O \times SU(2)_L\) are given by the following linear combinations of the generators studied in the previous subsection

\[
\begin{align*}
M_3 & := 1/4 \left( K + \tilde{K} - \sqrt{2/3}(R_1 + \tilde{R}^1 + R_2 + \tilde{R}^2 - R_3 - \tilde{R}^3) \right) \hspace{1cm} (5.2.1) \\
M_1 & := \frac{1}{4\sqrt{2}} \left( -R_0 - \tilde{R}_1 + \tilde{R}_2 + \tilde{R}_3 - V^1 - V^2 + V^3 + \tilde{V}^0 \right) \hspace{1cm} (5.2.2) \\
M_2 & := \frac{1}{4\sqrt{2}} \left( -R_1 - \tilde{R}_2 + \tilde{R}_3 + U^1 + \tilde{U}^1 + V^2 - \tilde{V}^1 + V^3 + \tilde{V}^0 \right) \hspace{1cm} (5.2.3) \\
N_3 & := 1/4 \left( K + \tilde{K} - \sqrt{2/3}(R_1 + \tilde{R}^1 + R_2 + \tilde{R}^2 + R_3 + \tilde{R}^3) \right) \hspace{1cm} (5.2.4) \\
N_1 & := \frac{1}{4\sqrt{2}} \left( R_0 + \tilde{R}_1 - \tilde{R}_2 + \tilde{R}_3 + V^1 - V^2 + V^3 + \tilde{V}^0 \right) \hspace{1cm} (5.2.5) \\
N_2 & := \frac{1}{4\sqrt{2}} \left( R_1 - \tilde{R}_2 + \tilde{R}_3 + U^1 - \tilde{U}^1 + V^2 - \tilde{V}^1 + V^3 - \tilde{V}^0 \right) \hspace{1cm} (5.2.6) \\
O_3 & := 1/4 \left( K + \tilde{K} + \sqrt{2/3}(R_1 + \tilde{R}^1 - R_2 - \tilde{R}^2 + R_3 + \tilde{R}^3) \right) \hspace{1cm} (5.2.7) \\
O_1 & := \frac{1}{4\sqrt{2}} \left( R_0 - \tilde{R}_1 + \tilde{R}_2 + \tilde{R}_3 - V^1 + V^2 + V^3 - \tilde{V}^0 \right) \hspace{1cm} (5.2.8) \\
O_2 & := \frac{1}{4\sqrt{2}} \left( -R_1 + \tilde{R}_2 + \tilde{R}_3 + U^1 + \tilde{U}^1 - V^2 - \tilde{V}^1 + V^3 - \tilde{V}^0 \right) \hspace{1cm} (5.2.9) \\
L_3 & := 1/4 \left( K + \tilde{K} + \sqrt{2/3}(R_1 + \tilde{R}^1 + R_2 + \tilde{R}^2 + R_3 + \tilde{R}^3) \right) \hspace{1cm} (5.2.10) \\
L_1 & := \frac{1}{4\sqrt{2}} \left( \tilde{R}_0 + \tilde{R}_1 + \tilde{R}_2 + \tilde{R}_3 - \tilde{V}^1 + V^2 + V^3 + \tilde{V}^0 \right) \hspace{1cm} (5.2.11) \\
L_2 & := \frac{1}{4\sqrt{2}} \left( R_1 + \tilde{R}_2 + \tilde{R}_3 + U^1 - \tilde{U}^1 + V^2 - \tilde{V}^1 + V^3 + \tilde{V}^0 \right) \hspace{1cm} (5.2.12)
\end{align*}
\]
They satisfy the commutation relations

\[
\begin{align*}
[M_i, M_j] &= \epsilon_{ijk} M_k \quad (5.2.13) \\
[N_i, N_j] &= \epsilon_{ijk} N_k \quad (5.2.14) \\
[O_i, O_j] &= \epsilon_{ijk} O_k \quad (5.2.15) \\
[L_i, L_j] &= \epsilon_{ijk} L_k \quad (5.2.16)
\end{align*}
\]

where the indices \(i, j, k\) run from 1 to 3. The noncompact generators decompose as 8 doublets under each \(SU(2)\) subgroup and all together form the \((j_L = 1/2, j_M = 1/2, j_N = 1/2, j_O = 1/2)\) representation of \(SU(2)_L \times SU(2)_M \times SU(2)_M \times SU(2)_O\). We shall work in basis in which the noncompact generators are labelled by the eigenvalues of \(iL_3, iM_3, iN_3\) and \(iO_3\) and use the 5-grading with respect to the eigenvalues of \(iL_3\):

\[
\begin{align*}
[iL_3, K^\pm (m, n, o)] &= \pm \frac{1}{2} K^\pm (m, n, o) \quad (5.2.17) \\
[iM_3, K^\pm (m, n, o)] &= m K^\pm (m, n, o) \quad (5.2.18) \\
[iN_3, K^\pm (m, n, o)] &= n K^\pm (m, n, o) \quad (5.2.19) \\
[iO_3, K^\pm (m, n, o)] &= o K^\pm (m, n, o) \quad (5.2.20)
\end{align*}
\]

where \(m, n\) and \(o\) take on values \(\pm 1/2\). They are given by the following linear combinations of the generators defined above

\[
\begin{align*}
K^-(-1/2, -1/2, -1/2) &= (-iK + i\tilde{K} + \Delta) \quad (5.2.21) \\
K^+(1/2, 1/2, 1/2) &= (iK - i\tilde{K} + \Delta) \quad (5.2.22) \\
K^-(-1/2, -1/2, 1/2) &= -\frac{1}{2\sqrt{2}} (-iU_0 - U_1 + U_2 + U_3 + \tilde{U}_0 - i\tilde{U}_1 + i\tilde{U}_2 + \tilde{U}_3 \\
&\quad - V^0 + iV^1 - iV^2 - i\tilde{V}^0 - \tilde{V}^1 + \tilde{V}^2 + \tilde{V}^3) \quad (5.2.23) \\
K^+(1/2, 1/2, -1/2) &= -\frac{1}{2\sqrt{2}} (iU_0 - U_1 + U_2 + U_3 + \tilde{U}_0 + i\tilde{U}_1 - i\tilde{U}_2 + \tilde{U}_3 \\
&\quad - V^0 - iV^1 + iV^2 + i\tilde{V}^0 + \tilde{V}^1 + \tilde{V}^2 + \tilde{V}^3) \quad (5.2.24) \\
K^-(-1/2, 1/2, 1/2) &= \frac{1}{3} (i\sqrt{6} R_3 - i\sqrt{6} \tilde{R}_3 + 2R_3^3) \quad (5.2.25) \\
K^+(1/2, -1/2, -1/2) &= \frac{1}{3} (-i\sqrt{6} R_3 + i\sqrt{6} \tilde{R}_3 + 2R_3^3) \quad (5.2.26)
\end{align*}
\]
\[
K^{-}(1/2, 1/2, 1/2) = \frac{1}{2\sqrt{2}}(-iU_0 + U_1 + U_2 + U_3 - \bar{U}_0 - i\bar{U}_1 - i\bar{U}_2 - i\bar{U}_3
+ V^0 + iV^1 + iV^2 + iV^3 - i\bar{V}^0 + \bar{V}^1 + \bar{V}^2 + \bar{V}^3)
\] (5.2.27)

\[
K^{+}(-1/2, -1/2, -1/2) = \frac{1}{2\sqrt{2}}(iU_0 + U_1 + U_2 + U_3 - \bar{U}_0 + i\bar{U}_1 + i\bar{U}_2 + i\bar{U}_3
+ V^0 - iV^1 - iV^2 - iV^3 + i\bar{V}^0 + \bar{V}^1 + \bar{V}^2 + \bar{V}^3)
\] (5.2.28)

\[
K^{-}(-1/2, 1/2, -1/2) = -\frac{1}{2\sqrt{2}}(-iU_0 + U_1 - U_2 + U_3 + \bar{U}_0 + i\bar{U}_1 - i\bar{U}_2 + i\bar{U}_3
- V^0 - iV^1 + iV^2 - iV^3 - i\bar{V}^0 + \bar{V}^1 - \bar{V}^2 + \bar{V}^3)
\] (5.2.29)

\[
K^{+}(1/2, -1/2, 1/2) = -\frac{1}{2\sqrt{2}}(iU_0 + U_1 - U_2 + U_3 + \bar{U}_0 - i\bar{U}_1 + i\bar{U}_2 - i\bar{U}_3
- V^0 + iV^1 - iV^2 + iV^3 + i\bar{V}^0 - \bar{V}^1 + \bar{V}^2 + \bar{V}^3)
\] (5.2.30)

\[
K^{-}(1/2, -1/2, -1/2) = -\frac{1}{2\sqrt{2}}(iU_0 - U_1 - U_2 + U_3 - \bar{U}_0 - i\bar{U}_1 - i\bar{U}_2 + i\bar{U}_3
+ V^0 + iV^1 + iV^2 - iV^3 - i\bar{V}^0 - \bar{V}^1 - \bar{V}^2 + \bar{V}^3)
\] (5.2.31)

\[
K^{+}(-1/2, 1/2, 1/2) = -\frac{1}{2\sqrt{2}}(-iU_0 - U_1 - U_2 + U_3 - \bar{U}_0 + i\bar{U}_1 + i\bar{U}_2 - i\bar{U}_3
+ V^0 - iV^1 - iV^2 + iV^3 - i\bar{V}^0 + \bar{V}^1 - \bar{V}^2 + \bar{V}^3)
\] (5.2.32)

\[
K^{-}(1/2, -1/2, 1/2) = \frac{1}{3}(i\sqrt{6}R_2 - i\sqrt{6}R_1^2 + 2R_2^2)
\] (5.2.33)

\[
K^{+}(-1/2, 1/2, -1/2) = \frac{1}{3}(-i\sqrt{6}R_2 + i\sqrt{6}R_1^2 + 2R_2^2)
\] (5.2.34)

\[
K^{-}(1/2, 1/2, -1/2) = \frac{1}{3}(i\sqrt{6}R_1 - i\sqrt{6}R_1^2 + 2R_1^2)
\] (5.2.35)

\[
K^{+}(-1/2, -1/2, 1/2) = \frac{1}{3}(-i\sqrt{6}R_1 + i\sqrt{6}R_1^2 + 2R_1^2)
\] (5.2.36)

They satisfy the commutation relations:

\[
[K^{-}(m_1, n_1, o_1), K^{-}(m_2, n_2, o_2)] = 4i(m_1 - m_2)(n_1 - n_2)(o_1 - o_2)L^{(-1)}
\] (5.2.37)

\[
[K^{+}(m_1, n_1, o_1), K^{+}(m_2, n_2, o_2)] = -4i(m_1 - m_2)(n_1 - n_2)(o_1 - o_2)L^{(+1)}
\] (5.2.38)

\[
[K^{+}(m_1, n_1, o_1), K^{-}(m_2, n_2, o_2)] = -4i((m_1 + m_2)(n_1 - n_2)(o_1 - o_2)M^{(m_1+m_2)}
+ (m_1 - m_2)(n_1 + n_2)(o_1 - o_2)N^{(n_1+n_2)}
+ (m_1 - m_2)(n_1 - n_2)(o_1 + o_2)O^{(o_1+o_2)}
- (m_1 - m_2)(n_1 - n_2)(o_1 - o_2)L_3
- (m_1 - m_2)(n_1 - n_2)(o_1 - o_2)|O_3
- (m_1 - m_2)(o_1 - o_2)|N_3
- (n_1 - n_2)(o_1 - o_2)|M_3)
\] (5.2.39)

where

\[
M^{(±1)} = M_1 ± iM_2
\]
\[ N^{(\pm 1)} = N_1 \pm iN_2 \]
\[ O^{(\pm 1)} = O_1 \pm iO_2 \]

5.3 Spherical Vector of Quasiconformal Action of \( SO(4, 4) \)

Unitary representations induced by the quasiconformal action with unitary character \( \nu \) include the quaternionic discrete series representations of Gross and Wallach [44] as was shown explicitly for rank two cases in [35]. Critical to the analysis of [35] is the explicit expression for the spherical vector of quasiconformal realizations of \( SU(2, 1) \) and \( G_2(2) \). The quaternionic discrete series representations and their continuations appear as submodules in the Verma modules generated by the action of noncompact generators on the spherical vectors for special values of the parameter \( \nu \). To carry out this program for \( SO(4, 4) \) and higher groups we need to determine the spherical vector of their quasiconformal realizations. The spherical vectors can, in principle, be obtained by solving the differential equations

\[ C_M \Phi_\nu(p, q, x) = 0 \quad (5.3.1) \]

where \( C_M \) denote the compact generators in the quasiconformal realization. This is however quite unwieldy considering the fact that the relevant differential operators are highly nonlinear and involve 9 variables for \( SO(4, 4) \). The situation gets much worse for the quasiconformal groups of higher Jordan algebras. However, we were able to deduce the spherical vector of \( SO(4, 4) \) from that of \( G_2(2) \) given in [35] simply by using properties of Jordan algebras of degree three and the fact that the quasiconformal realization of \( G_2(2) \) can be obtained from that of any Jordan algebra of degree three by restricting the Jordan algebra to its identity element.

We find that the spherical vector of \( SO(4, 4) \) is given simply by

\[ \Phi_\nu(p, q, x) = [(1 + x^2)^2 + (I_4)^2 + 8I_4 - 2(1 + x^2)(I_4 - I_2)
+ \frac{1}{2}J_6 + 8xJ_4 + \frac{4}{81}H_4]^{\frac{1}{2}} \quad (5.3.2) \]

where

\[ I_4 = 4p^0q_1q_2q_3 + 4q_0p^1p^2p^3 + (p^0q_0 - p^1q_1 - p^2q_2 - p^3q_3)^2 \quad (5.3.3) \]
\[ -4(p^2q_2p^3q_3 + p^1q_1p^2q_2 + p^1q_1p^3q_3) \]
\[ I_2 = (p^0)^2 + (q_0)^2 + p^Ip^I + q_1q_1 \quad (5.3.4) \]
\[ J_4 = \frac{1}{4} \left( p_0 \frac{\partial I_4}{\partial q_0} - q_0 \frac{\partial I_4}{\partial p_0} + q_1 \frac{\partial I_4}{\partial p^I} - p^I \frac{\partial I_4}{\partial q_1} \right) \quad (5.3.5) \]
\[ J_6 = \left( \frac{\partial I_4}{\partial p^0} \right)^2 + \left( \frac{\partial I_4}{\partial q_0} \right)^2 + \left( \frac{\partial I_4}{\partial p^I} \right)^2 + \left( \frac{\partial I_4}{\partial q_1} \right)^2 + \left( \frac{\partial I_1}{\partial q_1} \right) \left( \frac{\partial I_1}{\partial q_1} \right) \quad (5.3.6) \]
\[ H_4 = 27 \left( (p^0)_I - \sqrt{3}q_0p^I - \sqrt{3}p^0q_I + (q_#)^1 \right) \left( (p^0)_I - \sqrt{3}q_0p^I - \sqrt{3}p^0q_I + (q_#)^1 \right) \quad (5.3.7) \]
where \((p^#)_I = C_{IJK}p^Jp^K\) and \((q^#)_I = C_{IJK}q_Jq_K\) and \(I, J, K, \ldots = 1, 2, 3\).

The generators \(C_M\) of the maximal compact subgroup \(SO(4) \times SO(4)\) all annihilate the vector \(\Phi_\nu(p, q, x)\) for arbitrary values of \(\nu\). Unitary irreducible representations, including the quaternionic discrete series induced by the above quasi-conformal realization starting from the spherical vector will be studied elsewhere [60].

6. Spherical Vectors of Quasi-conformal Groups associated with general Euclidean Jordan algebras of Degree Three

Every Jordan algebra \(J\) of degree three admits three mutually orthogonal irreducible idempotents \(P_1, P_2, P_3:\)

\[
P_i \circ P_j = \delta_{ij}P_i \tag{6.1}
\]

\[
Tr(P_i) = 1 \quad i, j, .. = 1, 2, 3
\]

By the action \(U\) of the automorphism group \(Aut(J)\) one can bring a general element \(X\) of the Jordan algebra to the form

\[
Aut(J) : \quad X \longrightarrow UXU^{-1} = \lambda_1P_1 + \lambda_2P_2 + \lambda_3P_3 \tag{6.2}
\]

such that its norm is simply

\[
N(X) = \lambda_1\lambda_2\lambda_3 \tag{6.3}
\]

For simple Jordan algebras \(J^3_{\mathbb{R}}, J^3_{\mathbb{C}}\) and \(J^3_{\mathbb{H}}\) this corresponds simply to diagonalizing a \(3 \times 3\) symmetric, complex Hermitian and quaternionic Hermitian matrix by an \(SO(3), SU(3)\) and \(USp(6)\) transformation. For the exceptional Jordan algebra \(J^O_3\) the diagonalization procedure is more subtle and can be achieved by an \(F_4\) transformation [51]. For the generic Jordan family \(J = \mathbb{R} \oplus \Gamma(1, n-1)\), by the action of the automorphism group \(SO(n-1)\) one can rotate the cubic norm of a general element to that of an element of subalgebra \(\mathbb{R} \oplus \Gamma(1,1)\) whose norm can be written in the above form as we saw in the previous section. Thus the general quasi-conformal group actions on a Jordan algebra of degree three can be transformed into the action of \(SO(4, 4)\) of previous section.

To construct the spherical vector of a general quasi-conformal group starting from that of \(SO(4, 4)\) we will use the fact that closure of the Lie algebra \(SO(4, 4)\) and Lie algebra of automorphism group \(Aut(J)\) of the Euclidean Jordan algebra \(J\) is the full quasi-conformal Lie algebra \(QCon(\mathcal{F}(J))\). Similarly the closure of Lie algebra of maximal compact subgroup \(SO(4) \times SO(4)\) of \(SO(4, 4)\) and Lie algebra of \(Aut(J)\) is the Lie algebra of maximal compact subgroup \(\tilde{Conf}(J) \times SU(2)\) of the quasi-conformal group \(QConf(J)\). Written in terms of the \(C\)-tensor (or equivalently the cubic norm) the extension of the terms in the spherical vector

\footnote{As stated above we shall work in a basis \(e_i\) where the three irreducible idempotents are the first three basis elements \(P_i = e_i\) for \(i = 1, 2, 3\). The parameters \(\lambda_i\) are the analogs of the light-cone coordinates in Minkowskian spacetimes.}
of $SO(4,4)$ to those that occur in the spherical vector of general $QConf(J)$ is straightforward. However, one complication is the fact that there are terms in the spherical vector of general $QConf(J)$ that vanish when restricted to $SO(4,4)$ subalgebra. One writes down an Ansatz for such terms with arbitrary coefficients and then determines these coefficients by the requirement of invariance under the maximal compact subgroup $Con(J) \times SU(2)$.

We find that the spherical vector of a general quasiconformal group $QConf(J)$ associated with a Euclidean Jordan algebra $J$ is given by

$$\Phi_{\nu}(p,q,x) = [(1 + x^2 + I_2 - I_4)^2 - (I_2)^2 + 8I_4 + \frac{1}{2}I_6 + 8xJ_4 + \frac{4}{81}H_4]^{\frac{\nu}{2}}$$  \hspace{1cm} (6.4)

where

$$I_2 = (p^0)^2 + (q_0)^2 + p^Ip^I + qIqI$$ \hspace{1cm} (6.5)

$$I_4 = (p^0q_0 - p^Iq_I)^2 - \frac{4}{3}C_{IJK}p^Ip^KC_{LMN}q_Lq_M$$

$$+ \frac{4}{3\sqrt{3}}p^0C_{IJK}q_Iq_Jq_K + \frac{4}{3\sqrt{3}}q_0C_{IJK}p^Ip^K$$ \hspace{1cm} (6.6)

$$J_4 = \frac{1}{4} \left( p^0 \frac{\partial I_4}{\partial q_0} - q_0 \frac{\partial I_4}{\partial p^0} + qI \frac{\partial I_4}{\partial p^I} - p^I \frac{\partial I_4}{\partial qI} \right)$$ \hspace{1cm} (6.7)

$$I_6 = \left( \frac{\partial I_4}{\partial p^0} \right)^2 + \left( \frac{\partial I_4}{\partial q_0} \right)^2 + \left( \frac{\partial I_4}{\partial p^I} \right) \left( \frac{\partial I_4}{\partial q_I} \right) + \left( \frac{\partial I_4}{\partial pI} \right) \left( \frac{\partial I_4}{\partial qI} \right) + 4I_4I_2$$ \hspace{1cm} (6.8)

$$H_4 = 27 \left( (p^#)_I - \sqrt{3}q_0p^I - \sqrt{3}p^0q_I + (q^#)_I \right) \left( (p^#)_I - \sqrt{3}q_0p^I - \sqrt{3}p^0q_I + (q^#)_I \right) + C_4(J)$$ \hspace{1cm} (6.9)

where $(p^#)_I = C_{IJK}p^Ip^K$ and $(q^#)_I = C_{IJK}q_Iq_K$. $C_4(J)$ is the “correction” term that vanishes when restricted to the subalgebra $SO(4,4)$ and has a different form for simple Jordan algebras and non-simple ones. For simple Euclidean Jordan algebras of degree three the quartic correction term $C_4(J)$ is given by

$$C_4(J^\pm) = 81 (Tr [M_0(p) \circ M_0(q)])^2 + \frac{81}{2} Tr [M_0(p)^2] \cdot Tr [M_0(q)^2]$$

$$- 243 Tr \{[M_0(p), M_0(q), M_0(p)] \circ M_0(q)\}$$ \hspace{1cm} (6.10)

where

$$M_0(q) = M(q) - \frac{1}{3} Tr M(q)$$ \hspace{1cm} (6.11)

$$M(q) = e^I q_I \in J^\pm_3$$ \hspace{1cm} (6.12)

and similarly for $M_0(p)$. \{A, B, C\} denotes the Jordan triple product

$$\{A, B, C\} = A \circ (B \circ C) + C \circ (B \circ A) - (A \circ C) \circ B$$ \hspace{1cm} (6.13)
For special Jordan algebras with the Jordan product
\[ A \circ B = \frac{1}{2}(AB + BA) \]
we have
\[ \{A, B, A\} = ABA \]
(6.14)

Hence for Jordan algebras \( J^R_3 \), \( J^C_3 \) and \( J^H_3 \) the term \( C_4(J) \) can be written as
\[ C_4(J^\mathbb{A}_3) = 81 \left( \text{Tr} [M_0(p)M_0(q)] \right)^2 + \frac{81}{2} \text{Tr} \left[ M_0(p)^2 \right] \text{Tr} \left[ M_0(q)^2 \right] - 243 \text{Tr} \left[ M_0(p)M_0(q)M_0(p)M_0(q) \right] \]
(6.15)

We recall that the basis elements \( e_I \) of \( J^\mathbb{A}_3 \) are normalized such that
\[ \text{Tr}(e_I \circ e_J) = \delta_{IJ} \]
\[ \text{Tr}(\tilde{e}^I \circ \tilde{e}^J) = \delta^{IJ} \]
and
\[ \mathcal{N}(M) = 3\sqrt{3}\det M \]

For \( J^\mathbb{R}_3 \) we label our basis elements such that
\[ M(p) = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{2}p^1 & p^6 & p^5 \\ p^6 & \sqrt{2}p^2 & p^4 \\ p^5 & p^4 & \sqrt{2}p^3 \end{pmatrix} \]
(6.16)
and
\[ \mathcal{N}(M(p)) = 3\sqrt{3} \left\{ p^1p^2p^3 - \frac{1}{2} \left[ p^1(p^4)^2 + p^2(p^5)^2 + p^3(p^6)^2 \right] + \frac{1}{\sqrt{2}}p^4p^5p^6 \right\} \]
(6.17)

For the Jordan algebras \( J^\mathbb{A}_3 \), where \( \mathbb{A} = \mathbb{C}, \mathbb{H} \) or \( \mathbb{O} \) coordinates \( p^4, p^5 \) and \( p^6 \) become elements of \( \mathbb{A} \), which we will denote by capital letters \( P^4, P^5 \) and \( P^6 \). Thus for \( M(p) \in J^\mathbb{A}_3 \) we have
\[ M(p) = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{2}p^1 & P^6 & P^5 \\ P^6 & \sqrt{2}p^2 & P^4 \\ P^5 & P^4 & \sqrt{2}p^3 \end{pmatrix} \]
(6.18)
and the cubic norm of \( M(p) \) becomes
\[ \mathcal{N}(M(p)) = 3\sqrt{3} \left\{ p^1p^2p^3 - \frac{1}{2} \left( p^1|P^4|^2 + p^2|P^5|^2 + p^3|P^6|^2 \right) + \frac{1}{\sqrt{2}} \text{Re}(P^4P^5P^6) \right\} \]
(6.19)
where \( \text{Re}(X) \) denotes the real part of \( X \in \mathbb{A} \) and \( |X|^2 = X \bar{X} \). If we expand the elements \( P^4, P^5 \) and \( P^6 \) in terms of their real components

\[
\begin{align*}
P^4 &= p^4 + p^{4+3A} j_A \\
\bar{P}^4 &= p^4 - p^{4+3A} j_A \\
P^5 &= p^5 + p^{5+3A} j_A \\
\bar{P}^5 &= p^5 - p^{5+3A} j_A \\
P^6 &= p^6 + p^{6+3A} j_A \\
\bar{P}^6 &= p^6 - p^{6+3A} j_A
\end{align*}
\]

where the index \( A \) is summed over and using the fact that the imaginary units satisfy

\[
\delta_{AB} + \eta_{ABC} j_C = 0
\]

we can write the cubic norm as

\[
\mathcal{N}(M(p)) = 3\sqrt{3} \left\{ p^1 p^2 p^3 - \frac{1}{2} p^1 [(p^4)^2 + p^{4+3A} p^{4+3A}] - \frac{1}{2} p^2 [(p^5)^2 + p^{5+3A} p^{5+3A}] - \frac{1}{2} p^3 [(p^6)^2 + p^{6+3A} p^{6+3A}] + \frac{1}{2} [p^4 p^5 p^6 - p^4 p^{(5+3A)} p^{(6+3A)} - p^4 p^{(4+3A)} p^{(5+3A)} - p^4 p^{(4+3A)} p^{(6+3A)}] - \frac{1}{2} \eta_{ABC} p^{4+3A} p^{5+3A} p^{6+3B} \right\}
\]

The indices \( A, B, C \) take on the single value 1 for \( \mathbb{C} \), run from 1 to 3 for \( \mathbb{H} \) and from 1 to 7 for \( \mathbb{O} \). Note that \( \eta_{ABC} \) vanishes for \( \mathbb{C} \).

For the generic nonsimple Jordan algebras \( (\mathbb{R} \oplus \Gamma_{(1,n-1)}) \) of degree three the cubic form is

\[
\mathcal{N}(q) = C^{IJK} q_I q_J q_K = \frac{3\sqrt{3}}{2} q_1 [2q_2 q_3 - (q_4)^2 - (q_5)^2 - \cdots - (q_{n+1})^2]
\]

and the quartic correction term \( C_4 \) that appears in \( H_4 \) is given by

\[
C_4(\mathbb{R} \oplus \Gamma_{(1,n-1)}) = -\frac{81}{2} \left\{ (p^2 - p^3) (q_2 - q_3) + 2p^4 q_4 + \cdots + 2p^{n+1} q_{n+1} \right\}^2 + \frac{81}{2} \left( (p^2 - p^3)^2 + 2(p^4)^2 + \cdots + 2(p^{n+1})^2 \right) \left( (q_2 - q_3)^2 + 2(q_4)^2 + \cdots + 2(q_{n+1})^2 \right)
\]

7. Exceptional Groups \( E_6(6), E_7(7), E_8(8) \) and \( SO(m+4, n+4) \) as Quasiconformal Groups and Non-Euclidean Jordan Algebras

Jordan algebras of degree three that define \( N = 2 \) MESGTs in \( d = 5 \) with symmetric target spaces are all Euclidean and the quasiconformal groups associated with them are of the quaternionic real form that we studied in the previous section. The general formulas given for the quasiconformal Lie algebra in a basis covariant with respect to the reduced structure

\[
-38-
\]
group of the Jordan algebra are valid for all real forms of the underlying Jordan algebras of degree three. Split exceptional group $E_{8(8)}$ is the U-duality group of maximal supergravity in three dimensions and its quasiconformal realization was first given in [13], in a basis covariant with respect to the four dimensional U-duality group $E_{7(7)}$. By truncation the quasiconformal realizations of $E_{7(7)}$, $E_{6(6)}$ and $F_{4(4)}$ were obtained in a basis covariant with respect to $SO(6,6), SL(6,\mathbb{R})$ and $Sp(6,\mathbb{R})$, respectively [13]. The groups $E_{7(7)}, SO(6,6), SL(6,\mathbb{R})$ and $Sp(6,\mathbb{R})$ are the conformal groups of split Jordan algebras of $J^O_{3}, J^H_{3}, J^C_{3}$ and $J^R_{3}$, respectively.

To obtain the quasiconformal realizations of split exceptional groups in a basis covariant with respect to Lorentz (reduced structure) groups of underlying Jordan algebras we simply need to substitute in the formulas of section 4 cubic norm forms (or C-tensors) of simple split Jordan algebras in place of those of Euclidean Jordan algebras.

The quasiconformal group associated with $J^R_{3}$ is the split exceptional group $F_{4(4)}$ whose quotient with respect to its maximal compact subgroup $USp(6)\times USp(2)$ is a quaternionic symmetric space. In other words for $J^R_{3}$ the quasiconformal group is both split and quaternionic real. However if we replace the underlying division algebras of the other simple Jordan algebras of degree three to be of the split form $\mathbb{C}, \mathbb{H}$ and $\mathbb{O}$, then the resultant quasiconformal groups are no longer quaternionic real. They yield the quasiconformal realizations of split exceptional groups $E_{6(6)}, E_{7(7)}$ and $E_{8(8)}$, respectively, in a basis covariant with respect to their reduced structure groups $SL(3,\mathbb{R})\times SL(3,\mathbb{R}), SL(6,\mathbb{R})$ and $E_{6(6)}$, respectively.

Similarly, in the formulas of section four if we replace the norm forms (C-tensors) of non-simple Euclidean Jordan algebras $(\mathbb{R}\oplus \Gamma(1,m))$ with those of non-Euclidean Jordan algebras $(\mathbb{R}\oplus \Gamma(n,m))$ $(n \neq 1)$ we get the quasiconformal realizations of groups $SO(n+3, m+3)$ covariant with respect their reduced structure groups $SO(1,1)\times SO(n,m)$.

The formulas for spherical vectors of the quasiconformal groups associated with euclidean Jordan algebras do not however carry over directly to the spherical vectors of quasiconformal groups of non-Euclidean Jordan algebras and will be studied elsewhere.

Acknowledgement: We would like thank Andy Neitzke, Boris Pioline and Andrew Waldron for discussions in the early stages of this work. One of us (M.G) would like to acknowledge the hospitality extended to him during his sabbatical stay at the Institute for Advanced Study, Princeton, where part of this work was done and support of Monell Foundation during his stay. This work was supported in part by the National Science Foundation under grant number PHY-0555605. Any opinions, findings and conclusions or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of the National Science Foundation.

8. Appendices

A. Euclidean and Split Jordan algebras of Degree Three

Referring to the recent monograph [12] for references and details we shall give a brief review
of Jordan algebras in this Appendix. A Jordan algebra over a field \( F \) is an algebra, \( J \) with a symmetric product \( \circ \)
\[
X \circ Y = Y \circ X \in J, \quad \forall \ X, Y \in J,
\]
that satisfies the Jordan identity
\[
X \circ (Y \circ X^2) = (X \circ Y) \circ X^2,
\]
where \( X^2 \equiv (X \circ X) \). Hence a Jordan algebra is commutative and in general not associative.

Given a Jordan algebra \( J \), one can define a norm form, \( N : J \to \mathbb{R} \) over it that satisfies the composition property [13]
\[
N[2X \circ (Y \circ X) - (X \circ X) \circ Y] = N^2(X)N(Y).
\]
The degree, \( p \), of the norm form as well as of \( J \) is defined by \( N(\lambda X) = \lambda^p N(X) \), where \( \lambda \in \mathbb{R} \).

A Euclidean Jordan algebra is a Jordan algebra for which the condition \( X \circ X + Y \circ Y = 0 \) implies that \( X = Y = 0 \) for all \( X, Y \in J \). They are sometimes called compact Jordan algebras since the automorphism groups of Euclidean Jordan algebras are compact.

As was shown in [17], given a Euclidean Jordan algebra of degree three one can identify its norm form \( N \) with the cubic polynomial \( V \) defined by the C-tensor of a 5D, \( N = 2 \) MESGT with a symmetric scalar manifold. Euclidean Jordan algebras of degree three fall into an infinite family of non-simple Jordan algebras of the form
\[
J = \mathbb{R} \oplus \Gamma_{(1,n-1)}
\]
which is referred to as the generic Jordan family. The scalar manifolds of corresponding 5D, \( N = 2 \) MESGT’s are
\[
\mathcal{M} = \frac{SO(n-1,1)}{SO(n-1)} \times SO(1,1)
\]
\( \Gamma_{(1,n-1)} \) is an \( n \) dimensional Jordan algebra of degree two associated with a quadratic norm form in \( n \) dimensions that has a “Minkowskian signature” \((+,-,\ldots,-)\). A simple realization of \( \Gamma_{(1,n-1)} \) is provided by \((n-1)\) Dirac gamma matrices \( \gamma^i \) \((i,j,\ldots=1,\ldots,(n-1))\) of an \((n-1)\) dimensional Euclidean space together with the identity matrix \( \gamma^0 = 1 \) and the Jordan product \( \circ \) being one half the anticommutator:
\[
\gamma^i \circ \gamma^j = \frac{1}{2} \{\gamma^i, \gamma^j\} = \delta^{ij} \gamma^0
\]
\[
\gamma^0 \circ \gamma^j = \frac{1}{2} \{\gamma^0, \gamma^j\} = \gamma^0
\]
\[
\gamma^i \circ \gamma^0 = \frac{1}{2} \{\gamma^i, \gamma^0\} = \gamma^i.
\]
The quadratic norm of a general element \( X = X_0\gamma^0 + X_i\gamma^i \) of \( \Gamma_{(1,n-1)} \) is defined as
\[
Q(X) = \frac{1}{2^{[n/2]}} Tr X\bar{X} = X_0X_0 - X_iX_i,
\]

where
\[ \bar{X} \equiv X_0 \gamma^0 - X_i \gamma^i. \]

The norm of a general element \( y \oplus X \) of the non-simple Jordan algebra \( J = \mathbb{R} \oplus \Gamma_{(1,n-1)} \) is simply given by
\[ N(y \oplus X) = yQ(X) \quad (A.5) \]
where \( y \in \mathbb{R} \).

There exist four simple Euclidean Jordan algebras of degree three. They are generated by Hermitian (3×3)-matrices over the four division algebras \( \mathbb{A} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O} \)
\[ J = \begin{pmatrix} \alpha & Z & \bar{Y} \\ \bar{Z} & \beta & X \\ Y & \bar{X} & \gamma \end{pmatrix} \]
where \( \alpha, \beta, \gamma \in \mathbb{R} \) and \( X, Y, Z \in \mathbb{A} \) with the product being one half the anticommutator. They are denoted as \( J_3^\mathbb{R}, J_3^\mathbb{C}, J_3^\mathbb{H}, J_3^\mathbb{O} \), respectively, and the corresponding \( N = 2 \) MESGts are called “magical supergravity theories”. They have the 5D scalar manifolds:
\[ J_3^\mathbb{R} : \quad \mathcal{M} = SL(3, \mathbb{R})/SO(3) \]
\[ J_3^\mathbb{C} : \quad \mathcal{M} = SL(3, \mathbb{C})/SU(3) \]
\[ J_3^\mathbb{H} : \quad \mathcal{M} = SU^*(6)/USp(6) \]
\[ J_3^\mathbb{O} : \quad \mathcal{M} = E_6(-26)/F_4 \quad (A.6) \]
\[ J_3^\mathbb{O} \]
\[ J_3^\mathbb{O} \quad (A.7) \]

The cubic norm form, \( N \), of these Jordan algebras is given by the determinant of the corresponding Hermitian (3×3)-matrices (modulo an overall scaling factor).
\[ N(J) = \alpha \beta \gamma - \alpha X \bar{X} - \beta YY - \gamma Z \bar{Z} + 2Re(XYZ) \quad (A.8) \]
where \( Re(XYZ) \) denotes the real part of \( XYZ \) and bar denotes conjugation in the underlying division algebra.

For a real quaternion \( X \in \mathbb{H} \) we have
\[ X = X_0 + X_1 j_1 + X_2 j_2 + X_3 j_3 \]
\[ \bar{X} = X_0 - X_1 j_1 - X_2 j_2 - X_3 j_3 \]
\[ X \bar{X} = X_0^2 + X_1^2 + X_2^2 + X_3^2 \quad (A.9) \]
where the imaginary units \( j_i \) satisfy
\[ j_i j_j = -\delta_{ij} + \epsilon_{ijk} j_k \quad (A.10) \]
For a real octonion $X \in \mathbb{O}$ we have

$$X = X_0 + X_1 j_1 + X_2 j_2 + X_3 j_3 + X_4 j_4 + X_5 j_5 + X_6 j_6 + X_7 j_7$$

$$\bar{X} = X_0 - X_1 j_1 - X_2 j_2 - X_3 j_3 - X_4 j_4 - X_5 j_5 - X_6 j_6 - X_7 j_7$$

(A.11)

$$X \bar{X} = X_0^2 + \sum_{A=1}^{7} (X_A)^2$$

Seven imaginary units of real octonions satisfy

$$j_{AB} = -\delta_{AB} + \eta_{ABC} j_C$$

(A.12)

where $\eta_{ABC}$ is completely antisymmetric and in the conventions of [14] take on the values

$$\eta_{ABC} = 1 \Leftrightarrow (ABC) = (123), (471), (572), (624), (435), (516)$$

(A.13)

In the generic infinite family of non-simple Jordan algebras of degree three, $\mathbb{R} \oplus \Gamma$, one can take the quadratic form defining the Jordan algebra $\Gamma$ of degree two to be of arbitrary signature different from Minkowskian, which result in non-compact or non-Euclidean Jordan algebras. If the quadratic norm form has signature $(p, q)$ we shall denote the Jordan algebra as $\Gamma_{(n,m)}$. It is generated by $(n + m - 1)$ Dirac gamma matrices $\gamma_\mu$ satisfying

$$\{\gamma_\mu, \gamma_\nu\} = 2\eta_{\mu\nu}\gamma^0$$

(A.14)

where $\eta_{\mu\nu}$ has signature $(n - 1, m)$. Then the quadratic norm is invariant under $SO(n, m)$ and the reduced structure group of the Jordan algebra $\mathbb{R} \oplus \Gamma_{(n,m)}$ is

$$SO(n, m) \times SO(1, 1)$$

The invariant tensor $C_{IJK}$ defining the cubic norm of the Jordan algebra $(\mathbb{R} \oplus \Gamma_{(n,5)})$ can be identified with the invariant tensor in 5D $N = 4$ MESGT’s describing the $F \wedge F \wedge A$ coupling of $n$, $N = 4$ vector multiplets coupled to $N = 4$ supergravity. Scalar manifolds of these theories are symmetric spaces

$$SO(p, 5) \times SO(1,1)/SO(5) \times SO(p)$$

(A.15)

Scalar manifold of $N = 6$ supergravity is the same as that of $N = 2$ MESGT defined by the simple Jordan algebra $J_3^{\mathbb{H}}$, namely

$$SU^*(6)/USp(6)$$

Therefore its invariant C-tensor is simply the one given by the cubic norm of $J_3^{\mathbb{H}}$.

As for $N=8$ supergravity in five dimensions its C-tensor is simply the one given by the cubic norm of the split exceptional Jordan algebra $J_3^{\mathbb{O}_s}$ defined over split octonions $\mathbb{O}_s$. Four of the seven “imaginary units” of split octonions square to +1. If we denote the split imaginary
units as $j^s_\mu$ ($\mu = 4, 5, 6, 7$) and the imaginary units of the real quaternion subalgebra as $j_i, (i = 1, 2, 3)$ we have:

$$
\begin{align*}
\dot{j}_\mu \dot{j}_\nu &= \delta_{\mu\nu} - \eta_{\mu\nu} j_i \\
\dot{j}_i \dot{j}_j &= -\delta_{ij} + \epsilon_{ijk} \dot{j}_k \\
\dot{j}_i \dot{j}_i^s &= \eta_{\mu\nu} \dot{j}_\nu^s
\end{align*}
$$

where $\eta_{ABC} (A, B, C = 1, 2, 3)$ are the structure constant of the real octonion algebra $\mathbb{O}$ defined above. For a split octonion $O_s = o_0 + o_1 j_1 + o_2 j_2 + o_3 j_3 + o_4 j_4^s + o_5 j_5^s + o_6 j_6^s + o_7 j_7^s$

the norm is

$$
O_s \bar{O}_s = o_0^2 + o_1^2 + o_2^2 + o_3^2 - o_4^2 - o_5^2 - o_6^2 - o_7^2
$$

where $\bar{O}_s = o_0 + o_1 j_1 + o_2 j_2 + o_3 j_3 - o_4 j_4^s - o_5 j_5^s - o_6 j_6^s - o_7 j_7^s$. The norm has the invariance group $SO(4, 4)$. The automorphism group of the split exceptional Jordan algebra defined by $3 \times 3$ Hermitian matrices of the form

$$
J^s = \begin{pmatrix}
\alpha & Z^s & \bar{Y}^s \\
\bar{Z}^s & \beta & X^s \\
Y^s & \bar{X}^s & \gamma
\end{pmatrix}
$$

is the noncompact group $F_{4(4)}$ and its reduced structure group is $E_{6(6)}$ under which the C-tensor is invariant. $E_{6(6)}$ is the invariance group of maximal supergravity in five dimensions whose scalar manifold is $E_{6(6)} / USp(8)$

The split quaternion algebra $\mathbb{H}^s$ has two “imaginary units” $j_m^s$ ($m=2,3$) that square to +1:

$$
\begin{align*}
\dot{j}_m^s \dot{j}_n^s &= \delta_{mn} - \epsilon_{mnk} \dot{j}_k \\
(j_1)^2 &= -1 \\
\dot{j}_1 \dot{j}_1^s &= \epsilon_{1mn} \dot{j}_n^s
\end{align*}
$$

For a split quaternion $Q_s = q_0 + q_1 j_1 + q_2 j_2^s + q_3 j_3^s$

the norm is

$$
Q_s \bar{Q}_s = q_0^2 + q_1^2 - q_2^2 - q_3^2
$$

where $\bar{Q}_s = q_0 - q_1 j_1 - q_2 j_2^s - q_3 j_3^s$ and it is invariant under $SO(2, 2)$. The automorphism group of the split Jordan algebra $J^s_3$ is $Sp(6, \mathbb{R})$ with the maximal compact subgroup $SU(3) \times U(1)$. Its reduced structure group is $SL(6, \mathbb{R})$.

The split complex numbers have an “imaginary unit” that squares to +1 and its norm has $SO(1, 1)$ invariance. The automorphism group of split complex Jordan algebra $J^c_3$ is $SL(3, \mathbb{R})$ and its reduced structure group is

$$
SL(3, \mathbb{R}) \times SL(3, \mathbb{R})
$$
B. Conformal Groups of Jordan Algebras

Generalized conformal group $\text{Conf}(J)$ of a Jordan algebra $J$ is generated by translations $T_a$, special conformal generators $K_a$, Lorentz transformations and dilatation generators $M_{ab}$. Lorentz transformations and dilatations generate the structure algebra $\text{str}(J)$ of $J$. \cite{2, 53, 54}. Lie algebra $\text{conf}(J)$ of the conformal group $\text{Conf}(J)$ has a 3-grading with respect to the generator $D$ of dilatations:

$$\text{conf}(J) = K_a \oplus M_{ab} \oplus T_b$$ (B.1)

The conformal Lie algebra $\text{conf}(J)$ acts on the elements $x$ of a Jordan algebra $J$ as follows:

$$T_ax = a$$
$$M_{ab}x = \{abx\}$$
$$K_ax = -\frac{1}{2}\{xax\}$$ (B.2)

where

$$\{abx\} := a \circ (b \circ x) - b \circ (a \cdot x) + (a \circ b) \circ x$$

$a, b, x \in J$

and $\circ$ denotes the Jordan product. They satisfy the commutation relations

$$[T_a, K_b] = M_{ab}$$
$$[M_{ab}, T_c] = T_{\{abc\}}$$
$$[M_{ab}, K_c] = K_{\{bac\}}$$
$$[M_{ab}, M_{cd}] = M_{\{abc\}d} - M_{\{bad\}c}$$ (B.3)

corresponding to the well-known Tits-Kantor-Koecher construction of Lie algebras from Jordan triple systems \cite{53, 54}. \cite{53} We note that $M_{ab}$ can be written as

$$M_{ab} = D_{a,b} + L_{a \cdot b}$$ (B.4)

where $D_{a,b}$ generate the automorphism (rotation) group of $J$

$$D_{a,b}x = a \circ (b \circ x) - b \circ (a \circ x)$$

and $L_c$ denotes multiplication by the element $c \in J$. The generator $D$ is proportional to the multiplication operator by the identity element of $J$.

Choosing a basis $e_I$ for the Jordan algebra such that an element $x \in J$ can be written as

$$x = e_I q^I = \tilde{e}^I q_I$$

one can write the generators of $\text{conf}(J)$ as differential operators acting on the “coordinates” $q^I$. \footnote{We should note that there are, in general, two inequivalent actions of the reduced structure group on the Jordan algebra and its conjugate. The tilde refers to the conjugate basis such that $q_I p^I$ is invariant under the action of the reduced structure group. For details on this issue see \cite{53}.}

These generators can be twisted by a unitary character $\lambda$ and take
the simple form

\[ T_I = \frac{\partial}{\partial q^I} \]

\[ R_I^J = -\Lambda^I_{JK} q^L \frac{\partial}{\partial q^K} - \lambda^I J \]  \hspace{1cm} (B.5)

\[ K_I^J = \frac{1}{2} \Lambda^I_{JK} q^L \frac{\partial}{\partial q^K} + \lambda^I J \]  \hspace{1cm} (B.6)

where

\[ \Lambda^I_{KL} := \delta^I_K \delta^J_L + \delta^I_L \delta^J_K - \frac{4}{3} C^{IJK} C^{KL} \]

They satisfy the commutation relations

\[ [T_I, K^J] = -R_I^J \]  \hspace{1cm} (B.7)

\[ [R_I^J, T_K] = \Lambda^I_{JK} T_L \]  \hspace{1cm} (B.8)

\[ [R_I^J, K^K] = -\Lambda^I_{JK} K^L \]  \hspace{1cm} (B.9)

The generator of automorphism group are

\[ A_{IJ} = R_I^J - R_J^I \]  \hspace{1cm} (B.10)

C. Quasiconformal realizations of Lie groups and Freudenthal triple systems

In this appendix we shall review the general theory of quasiconformal realizations of noncompact groups over Freudenthal triple systems that was given in [18].

Every simple Lie algebra \( g \) of \( G \) can be given a 5-graded decomposition, determined by one of its generators \( \Delta \), such that grade \( \pm 2 \) subspaces are one dimensional:

\[ g = g^{-2} \oplus g^{-1} \oplus g^0 \oplus g^1 \oplus g^{2} . \]  \hspace{1cm} (C.1)

where

\[ g^0 = \mathfrak{h} \oplus \Delta \]  \hspace{1cm} (C.2)

and

\[ [\Delta, t] = mt \quad \forall t \in g^m , \quad m = 0, \pm 1, \pm 2 \]  \hspace{1cm} (C.3)

A simple Lie algebra with such a 5-graded decomposition can be constructed over a Freudenthal triple system \( \mathcal{F} \) \(^8\). Freudenthal introduced these triple systems in his study of the

\(^8\)For \( sl(2) \) the 5-grading degenerates into a 3-grading.
geometries associated with exceptional groups \[46\]. A Freudenthal triple system (FTS) is defined as a vector space \(F\) equipped with a triple product \((X,Y,Z)\)
\[
(X,Y,Z) \in F \quad \forall \ X,Y,Z \in F
\]
and a skew symmetric bilinear form \(\langle X,Y \rangle = -\langle Y,X \rangle\) such that the triple product satisfies the identities
\[
\begin{align*}
(X,Y,Z) &= (Y,X,Z) + 2 \langle X,Y \rangle Z , \\
(X,Y,Z) &= (Z,Y,X) - 2 \langle X,Z \rangle Y , \\
\langle (X,Y,Z), W \rangle &= \langle (X,W,Z), Y \rangle - 2 \langle X,Z \rangle \langle Y,W \rangle , \\
(X,Y,(V,W,Z)) &= (V,W,(X,Y,Z)+(X,Y,V),W,Z) \\
&\quad + (V,(Y,X,W),Z) .
\end{align*}
\]
(C.5)

The construction of the associated Lie algebra \(g(F)\) labels the generators belonging to subspace \(g^{+1}\) by the elements of \(F\)
\[
U_A \in g^{+1} \leftrightarrow A \in F
\]
(C.6)
and by using the involution \(\tilde{\phantom{\quad}}\), that reverses the grading, elements of \(g^{-1}\) can also be labeled by elements of \(F\)
\[
\tilde{U}_A \in g^{-1} \leftrightarrow A \in F
\]
(C.7)

Elements of \(g^{\pm 1}\) generate the Lie algebra \(g(F)\) by commutation, and hence the remaining generators can be labelled by a pair of elements of \(F\)
\[
\begin{align*}
[U_A,\tilde{U}_B] &= S_{AB} \in g^0 \\
[U_A,U_B] &= -K_{AB} \in g^2 \\
[\tilde{U}_A,\tilde{U}_B] &= -\tilde{K}_{AB} \in g^{-2}
\end{align*}
\]
(C.8)

Commutation relations are all determined by the Freudenthal triple product \((A,B,C)\)
\[
\begin{align*}
[S_{AB},U_C] &= -U_{(A,B,C)} \\
[S_{AB},\tilde{U}_C] &= -\tilde{U}_{(B,A,C)} \\
[K_{AB},U_C] &= U_{(A,C,B)} - U_{(B,C,A)} \\
[\tilde{K}_{AB},U_C] &= \tilde{U}_{(B,C,A)} - \tilde{U}_{(A,C,B)} \\
[S_{AB},S_{CD}] &= -S_{(A,B,C)D} - S_{C(B,A,D)} \\
[S_{AB},K_{CD}] &= K_{A(C,B,D)} - K_{A(D,B,C)} \\
[[S_{AB},\tilde{K}_{CD}]] &= \tilde{K}_{(D,A,C)B} - \tilde{K}_{(C,A,D)B} \\
[K_{AB},\tilde{K}_{CD}] &= S_{(B,C,A)D} - S_{(A,C,B)D} - S_{(B,D,A)C} + S_{(A,D,B)C}
\end{align*}
\]
(C.9)
Since the grade ±2 subspaces are one dimensional one can write

\[ K_{AB} := K_{(A,B)} := \langle A, B \rangle K \tag{C.10} \]

\[ \tilde{K}_{AB} := \tilde{K}_{(A,B)} := \langle A, B \rangle \tilde{K} \tag{C.11} \]

Furthermore, the defining identities of the FTS imply that

\[ S_{AB} - S_{BA} = -2 \langle A, B \rangle \Delta \tag{C.12} \]

where \( \Delta \) is the generator that determines the 5-grading

\[
\begin{align*}
[\Delta, U_A] &= U_A \\
[\Delta, \tilde{U}_A] &= -\tilde{U}_A \\
[\Delta, K] &= 2K \\
[\Delta, \tilde{K}] &= -2\tilde{K}
\end{align*}
\tag{C.13}
\]

and generates the distinguished \( sl(2) \) subalgebra together with \( K, \tilde{K} \)

\[ [K, \tilde{K}] = -2\Delta \tag{C.14} \]

The 5-grading of \( g \) is then given as

\[ g = \tilde{K} \oplus \tilde{U}_A \oplus [S_{(AB)} + \Delta] \oplus U_A \oplus K \]

where

\[ S_{(AB)} := \frac{1}{2} (S_{AB} + S_{BA}) \]

are the generators of \( Aut(F) \) and commute with \( \Delta \)

\[ [\Delta, S_{(AB)}] = 0 \tag{C.15} \]

The remaining non-zero commutators are

\[
\begin{align*}
[U_A, \tilde{U}_B] &= S_{(AB)} - \langle A, B \rangle \Delta \\
[K, \tilde{U}_A] &= -2\tilde{U}_A \\
[\tilde{K}, U_A] &= 2\tilde{U}_A \\
[S_{(AB)}, K] &= 0
\end{align*}
\tag{C.16}
\]

A quartic invariant \( Q_4 \) can be defined over the FTS \( F \) using the triple product and the bilinear form as

\[ Q_4(X) := \frac{1}{48} \langle (X, X, X, X) \rangle \tag{C.17} \]
which is invariant under the automorphism group $\text{Aut}(\mathcal{F})$ of $\mathcal{F}$ generated by $S_{(AB)}$.

As was shown in [18] one can realize the 5-graded Lie algebra $\mathfrak{g}$ geometrically as a quasi-conformal Lie algebra over a vector space $\mathcal{T}$ coordinatized by the elements $X$ of the FTS $\mathcal{F}$ and an extra single variable $x$ [18, 24]:

\[
\begin{align*}
K(X) &= 0 \quad U_A(X) = A \quad S_{AB}(X) = (A, B, X) \\
K(x) &= 2 \quad U_A(x) = \langle A, X \rangle \quad S_{AB}(x) = 2 \langle A, B \rangle x \\
\tilde{U}_A(X) &= \frac{1}{2} (X, A, A) - A x \\
\tilde{U}_A(x) &= -\frac{1}{6} (\langle (X, X, X), A \rangle + \langle X, A \rangle x \\
\tilde{K}(X) &= \frac{1}{6} (X, X, X) + X x \\
\tilde{K}(x) &= \frac{1}{6} (\langle (X, X, X), X \rangle + 2 x^2
\end{align*}
\]

(C.18)

The geometric nature of quasiconformal actions is made manifest by first defining a quartic norm over the space $\mathcal{T}$ as

\[
\mathcal{N}_4(\mathcal{X}) := \mathcal{Q}_4(X) - x^2
\]

where $\mathcal{Q}_4(X)$ is the quartic invariant of $\mathcal{F}$ and then a “distance” function between any two points $\mathcal{X} = (X, x)$ and $\mathcal{Y} = (Y, y)$ in $\mathcal{T}$ as

\[
d(\mathcal{X}, \mathcal{Y}) := \mathcal{N}_4(\delta(\mathcal{X}, \mathcal{Y}))
\]

(C.20)

where $\delta(\mathcal{X}, \mathcal{Y})$ is the “symplectic” difference of two vectors $\mathcal{X}$ and $\mathcal{Y}$:

\[
\delta(\mathcal{X}, \mathcal{Y}) := (X - Y, x - y + \langle X, Y \rangle)
\]

One can then show that the light-like separations with respect to this quartic distance function

\[
d(\mathcal{X}, \mathcal{Y}) = 0
\]

is left invariant under the action of quasiconformal group [18].

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