Research Article

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Characterizations for the potential operators on Carleson curves in local generalized Morrey spaces

Abstract: In this paper, we give a boundedness criterion for the potential operator $I^a$ in the local generalized Morrey space $L^{1,p}(t;\nu)$ and the generalized Morrey space $M^{1,p}(t;\nu)$ defined on Carleson curves $\Gamma$, respectively. For the operator $I^a$, we establish necessary and sufficient conditions for the strong and weak Spanne-type boundedness on $L^{1,p}(t;\nu)$ and the strong and weak Adams-type boundedness on $M^{1,p}(t;\nu)$.

Keywords: Carleson curve, local generalized Morrey space, potential operator, Adams-type inequalities

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1 Introduction

Let $\Gamma = \{ t \in \mathbb{C} : t = t(s), \ 0 \leq s \leq l \leq \infty \}$ be a rectifiable Jordan curve in the complex plane $\mathbb{C}$ with arc-length measure $\nu(t) = s$, where $l = \nu(\Gamma)$ = lengths of $\Gamma$. We denote

$\Gamma(t, r) = \Gamma \cap B(t, r), \quad t \in \Gamma, \quad r > 0,$

where $B(t, r) = \{ z \in \mathbb{C} : |z - t| < r \}$. We also denote for brevity $\nu\Gamma(t, r) = |\Gamma(t, r)|$.

A rectifiable Jordan curve $\Gamma$ is called a Carleson curve if the condition

$\nu\Gamma(t, r) \leq c_0 r$

holds for all $t \in \Gamma$ and $r > 0$, where the constant $c_0 > 0$ does not depend on $t$ and $r$.

Let $f \in L^{1,\infty}(\Gamma)$. The maximal operator $M$ and the potential operator $I^a$ on $\Gamma$ are defined by

$Mf(t) = \sup_{t \in \Gamma} (\nu\Gamma(t, r))^{-1} \int_{\Gamma(t, r)} |f(\tau)| d\nu(\tau)$

and

$I^a f(t) = \int_{\Gamma} \frac{f(\tau) d\nu(\tau)}{|t - \tau|^{1-a}}, \quad 0 < a < 1,$

respectively.

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Maximal operator and potential operator in various spaces, in particular, defined on Carleson curves have been widely studied by many authors (see, for example, \([1–14]\)).

The main purpose of this paper is to establish the boundedness of potential operator \(I^a\), \(0 < a < 1\) in local generalized Morrey spaces \(LM^{[q,\alpha]}_{p,\phi}(\Gamma)\) defined on Carleson curves \(\Gamma\). We shall give characterizations for the strong and weak Spanne-type boundedness of the operator \(I^a\) from \(LM^{[q,\alpha]}_{p,\phi}(\Gamma)\) to \(LM^{[q,\alpha]}_{p,\phi_1}(\Gamma)\), \(1 < p < q < \infty\), \(1/p − 1/q = \alpha\) and from the space \(LM^{[q,\alpha]}_{p,\phi}(\Gamma)\) to the weak space \(WLM^{[q,\alpha]}_{p,\phi_1}(\Gamma)\), \(1 < q < \infty\), \(1 − 1/q = \alpha\). Also, we study Adams-type boundedness of the operator \(I^a\) from generalized Morrey spaces \(M^{[q,\alpha]}_{p,\phi}(\Gamma)\) to \(M^{[q,\alpha]}_{p,\phi_1}(\Gamma)\), \(1 < p < q < \infty\), and from the space \(M^{[q,\alpha]}_{p,\phi}(\Gamma)\) to the weak space \(WM^{[q,\alpha]}_{p,\phi_1}(\Gamma)\), \(1 < q < \infty\). We shall give characterizations for the Adams-type boundedness of the operator \(I^a\) in generalized Morrey spaces, including weak versions.

By \(A \leq B\) we mean that \(A \leq CB\) with some positive constant \(C\) independent of appropriate quantities. If \(A \leq B\) and \(B \leq A\), we write \(A \approx B\) and say that \(A\) and \(B\) are equivalent.

2 Preliminaries

Morrey spaces were introduced by C. B. Morrey [15] in 1938 in connection with certain problems in elliptic partial differential equations and calculus of variations. Later, Morrey spaces found important applications to Navier-Stokes and Schrödinger equations, elliptic problems with discontinuous coefficients, and potential theory.

Let \(L_p(\Gamma)\), \(1 \leq p < \infty\) be the space of measurable functions on \(\Gamma\) with finite norm

\[
\|f\|_{L_p(\Gamma)} = \left(\int_{\Gamma}|f(t)|^p\,dv(t)\right)^{1/p}.
\]

**Definition 2.1.** Let \(1 \leq p < \infty\), \(0 \leq \lambda \leq 1\), \([r]_1 = \min\{1, r\}\). We denote by \(L_{p,\lambda}(\Gamma)\) the Morrey space, and by \(\tilde{L}_{p,\lambda}(\Gamma)\) the modified Morrey space, the set of locally integrable functions \(f\) on \(\Gamma\) with the finite norms

\[
\|f\|_{L_{p,\lambda}(\Gamma)} = \sup_{t \in \Gamma, r > 0} r^{-\lambda} \|f\|_{L_p(\Gamma(t, r))}, \quad \|f\|_{\tilde{L}_{p,\lambda}(\Gamma)} = \sup_{t \in \Gamma, r > 0} \|f\|_{L_p(\Gamma(t, r))},
\]

respectively.

Note that (see \([16,17]\)) \(L_{p,0}(\Gamma) = \tilde{L}_{p,0}(\Gamma) = L_p(\Gamma)\), \(\tilde{L}_{p,\lambda}(\Gamma) = L_{p,\lambda}(\Gamma) \cap L_p(\Gamma)\) and \(\|f\|_{\tilde{L}_{p,\lambda}(\Gamma)} = \max\{\|f\|_{L_{p,\lambda}(\Gamma)}, \|f\|_{L_p(\Gamma)}\}\) if \(\lambda < 0\) or \(\lambda > 1\), then \(L_{p,\lambda}(\Gamma) = \tilde{L}_{p,\lambda}(\Gamma) = \Theta\), where \(\Theta\) is the set of all functions equivalent to 0 on \(\Gamma\).

We denote by \(WL_{p,\lambda}(\Gamma)\) the weak Morrey space, and by \(\tilde{W}L_{p,\lambda}(\Gamma)\) the modified Morrey space, as the set of locally integrable functions \(f\) on \(\Gamma\) with finite norms

\[
\|f\|_{WL_{p,\lambda}(\Gamma)} = \sup_{t \in \Gamma, r > 0} \|f\|_{WL_p(\Gamma(t, r))},
\]

\[
\|f\|_{\tilde{W}L_{p,\lambda}(\Gamma)} = \sup_{t \in \Gamma, r > 0} \|f\|_{\tilde{W}L_p(\Gamma(t, r))}.
\]
Samko [14] studied the boundedness of the maximal operator $M$ defined on quasimetric measure spaces, in particular on Carleson curves in Morrey spaces $L_{p,\lambda}(\Gamma)$:

**Theorem A.** Let $\Gamma$ be a Carleson curve, $1 < p < \infty$, $0 < \alpha < 1$ and $0 \leq \lambda < 1$. Then $M$ is bounded from $L_{p,\lambda}(\Gamma)$ to $L_{p,\lambda}(\Gamma)$.

Kokilashvili and Meskhi [18] studied the boundedness of the operator $I^a$ defined on quasimetric measure spaces, in particular on Carleson curves in Morrey spaces and proved the following:

**Theorem B.** Let $\Gamma$ be a Carleson curve, $1 < p < q < \infty$, $0 < \alpha < 1$, $0 < \lambda < 1$, $0 < \lambda_i < \frac{p}{q}$, $\frac{\lambda_i}{p} \leq \frac{\lambda}{q}$ and $\frac{1}{p} - \frac{1}{q} = \alpha$. Then the operator $I^a$ is bounded from the spaces $L_{p,\lambda}(\Gamma)$ to $L_{q,\lambda_i}(\Gamma)$.

The following Adams boundedness (see [19]) of the operator $I^a$ in Morrey space defined on Carleson curves was proved in [20].

**Theorem C.** Let $\Gamma$ be a Carleson curve, $0 < \alpha < 1$, $0 \leq \lambda < 1 - \alpha$ and $1 \leq p < \frac{1-\lambda}{\alpha}$.

1. If $1 < p < \frac{1-\lambda}{\alpha}$, then the condition $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{1-\lambda}$ is sufficient and in the case of infinite curve also necessary for the boundedness of the operator $I^a$ from $L_{p,\lambda}(\Gamma)$ to $L_{q,\lambda}(\Gamma)$.

2. If $p = 1$, then the condition $1 - \frac{1}{q} = \frac{\alpha}{1-\lambda}$ is sufficient and in the case of infinite curve also necessary for the boundedness of the operator $I^a$ from $L_{1,\lambda}(\Gamma)$ to $WL_{q,\lambda}(\Gamma)$.

The following Adams boundedness of the operator $I^a$ in modified Morrey space $\tilde{L}_{p,\lambda}(\Gamma)$ defined on Carleson curves was proved in [16], see also [17].

**Theorem D.** Let $\Gamma$ be a Carleson curve, $0 < \alpha < 1$, $0 \leq \lambda < 1 - \alpha$ and $1 \leq p < \frac{1-\lambda}{\alpha}$.

1. If $1 < p < \frac{1-\lambda}{\alpha}$, then the condition $\alpha \leq \frac{1}{p} - \frac{1}{q} \leq \frac{\alpha}{1-\lambda}$ is sufficient and in the case of infinite curve also necessary for the boundedness of the operator $I^a$ from $\tilde{L}_{p,\lambda}(\Gamma)$ to $\tilde{L}_{q,\lambda}(\Gamma)$.

2. If $p = 1$, then the condition $1 - \frac{1}{q} \leq \frac{\alpha}{1-\lambda}$ is sufficient and in the case of infinite curve also necessary for the boundedness of $I^a$ from $L_{1,\lambda}(\Gamma)$ to $WL_{q,\lambda}(\Gamma)$.

We use the following statement on the boundedness of the weighted Hardy operator:

$$H_w g(t) := \int_{t}^{\infty} g(s) w(s) \, ds, \quad 0 < t < \infty,$$

where $w$ is a weight.

The following theorem was proved in [21].

**Theorem 2.1.** Let $v_1$, $v_2$ and $w$ be weights on $(0,\infty)$ and $v_i(t)$ be bounded outside a neighborhood of the origin. The inequality

$$\text{ess sup}_{t \to 0} v_2(t) H_w g(t) \leq C \text{ ess sup}_{t \to 0} v_1(t) g(t)$$

holds for some $C > 0$ for all non-negative and non-decreasing $g$ on $(0,\infty)$ if and only if

$$B = \sup_{t \to 0} v_2(t) \int_{t}^{\infty} \frac{w(s) \, ds}{\text{ess sup}_{s \to 0} v_1(s)} < \infty.$$

Moreover, the value $C = B$ is the best constant for (2.1).
3 Local generalized Morrey spaces

We find it convenient to define the local generalized Morrey spaces in the form as follows, see [21,22].

**Definition 3.2.** Let \( 1 \leq p < \infty \) and \( \varphi(t, \tau) \) be a positive measurable function on \( \Gamma \times (0, \infty) \). Fixed \( t_0 \in \Gamma \), we denote by \( L_{p, \varphi}(\Gamma) \) (\( WLM_{p, \varphi}(\Gamma) \)) the local generalized Morrey space (the weak local generalized Morrey space), the space of all functions \( f \in L_{p, \varphi}(\Gamma) \) by the finite quasinorm

\[
\|f\|_{L_{p, \varphi}(\Gamma)} = \sup_{r > 0} \frac{1}{\varphi(t_0, r)} \left( \frac{1}{(\varphi \Theta(t_0, r))^{-\frac{1}{p}}} \|f\|_{L_{p, \varphi}(\Gamma(t_0, r))} \right).
\]

\[
\|f\|_{WLM_{p, \varphi}(\Gamma)} = \sup_{r > 0} \frac{1}{\varphi(t_0, r)} \left( \frac{1}{(\varphi \Theta(t_0, r))^{-\frac{1}{p}}} \|f\|_{WLM_{p, \varphi}(\Gamma(t_0, r))} \right).
\]

**Definition 3.3.** Let \( 1 \leq p < \infty \) and \( \varphi(t, \tau) \) be a positive measurable function on \( \Gamma \times (0, \infty) \). The generalized Morrey space \( M_{p, \varphi}(\Gamma) \) is defined as the set of all functions \( f \in L_{p, \varphi}(\Gamma) \) by the finite norm

\[
\|f\|_{M_{p, \varphi}(\Gamma)} = \sup_{t \in \Gamma, r > 0} \frac{1}{\varphi(t_0, r)} \left( \frac{1}{(\varphi \Theta(t_0, r))^{-\frac{1}{p}}} \|f\|_{L_{p, \varphi}(\Gamma(t_0, r))} \right).
\]

Also, the weak generalized Morrey space \( WLM_{p, \varphi}(\Gamma) \) is defined as the set of all functions \( f \in L_{p, \varphi}(\Gamma) \) by the finite norm

\[
\|f\|_{WM_{p, \varphi}(\Gamma)} = \sup_{t \in \Gamma, r > 0} \frac{1}{\varphi(t_0, r)} \left( \frac{1}{(\varphi \Theta(t_0, r))^{-\frac{1}{p}}} \|f\|_{WLM_{p, \varphi}(\Gamma(t_0, r))} \right).
\]

It is natural, first the set of all, to find conditions ensuring that the spaces \( L_{p, \varphi}(\Gamma) \) and \( M_{p, \varphi}(\Gamma) \) are non-trivial, that is, consist not only of functions equivalent to 0 on \( \Gamma \).

**Lemma 3.1.** [23] Let \( t_0 \in \Gamma \) and \( \varphi(t, \tau) \) be a positive measurable function on \( \Gamma \times (0, \infty) \). If

\[
\sup_{r < \tau < \infty} \frac{1}{\varphi(t_0, r)} \left( \frac{1}{(\varphi \Theta(t_0, r))^{-\frac{1}{p}}} \|f\|_{L_{p, \varphi}(\Gamma(t_0, r))} \right) = \infty \quad \text{for some } r > 0,
\]

then \( L_{p, \varphi}(\Gamma) = \Theta \).

**Remark 3.1.** We denote by \( \Omega_{p, \varphi} \) the set of all positive measurable functions \( \varphi \) on \( \Gamma \times (0, \infty) \) such that for all \( r > 0 \),

\[
\left\| \frac{1}{\varphi(t_0, \tau)} \left( \frac{1}{(\varphi \Theta(t_0, \tau))^{-\frac{1}{p}}} \right) \right\|_{L_{p, \varphi}(\Gamma)} < \infty.
\]

In what follows, keeping in mind Lemma 1, for the non-triviality of the space \( L_{p, \varphi}(\Gamma) \) we always assume that \( \varphi \in \Omega_{p, \varphi} \).

**Lemma 3.2.** [23] Let \( \varphi(t, \tau) \) be a positive measurable function on \( \Gamma \times (0, \infty) \).

(i) If

\[
\sup_{r < \tau < \infty} \frac{1}{\varphi(t, \tau)} \left( \frac{1}{(\varphi \Theta(t, \tau))^{-\frac{1}{p}}} \right) = \infty \quad \text{for some } r > 0 \text{ and for all } t \in \Gamma,
\]

then \( M_{p, \varphi}(\Gamma) = \Theta \).

(ii) If

\[
\sup_{0 < \tau < r} \varphi(t, \tau)^{-1} = \infty \quad \text{for some } r > 0 \text{ and for all } t \in \Gamma,
\]

then \( M_{p, \varphi}(\Gamma) = \Theta \).
Remark 3.2. We denote by $\Omega_p$ the sets of all positive measurable functions $\varphi$ on $\Gamma \times (0, \infty)$ such that for all $r > 0$,
\[
\sup_{t \epsilon \Gamma} \left\| \frac{1}{\varphi(t, \tau)} \right\|_{L^p(\varphi(t, \tau))} < \infty \quad \text{and} \quad \sup_{t \epsilon \Gamma} \|\varphi(t, \tau)^{-1}\|_{L_0(\varphi(t, \tau))} < \infty,
\]
respectively. In what follows, keeping in mind Lemma 2, we always assume that $\varphi \in \Omega_p$.

A function $\varphi : (0, \infty) \to (0, \infty)$ is said to be almost increasing (resp. almost decreasing) if there exists a constant $C > 0$ such that
\[
\varphi(r) \leq C \varphi(s) \quad \text{(resp. } \varphi(r) \geq C \varphi(s)) \text{ for } r \leq s.
\]
Let $1 \leq p < \infty$. Denote by $\mathcal{G}_p$ the set of all almost decreasing functions $\varphi : (0, \infty) \to (0, \infty)$ such that $t \in (0, \infty) \mapsto t^p \varphi(t) \in (0, \infty)$ is almost increasing.

Seemingly, the requirement $\varphi \in \mathcal{G}_p$ is superfluous but it turns out that this condition is natural. Indeed, Nakai established that there exists a function $\rho$ such that $\rho(t) t^p \leq \rho(T) T^p$ for all $0 < t \leq T < \infty$ and that $LM_{p, \varphi}(\Gamma) = LM_{p, \varphi}^{[\rho]}(\Gamma)$.

By elementary calculations we have the following, which shows particularly that the spaces $LM_{p, \varphi}$, $\mathcal{W}_{p, \varphi}$ and $\mathcal{M}_{p, \varphi}(\Gamma)$ are not trivial, see, for example, [23–25].

Lemma 3.3. [23] Let $\varphi \in \mathcal{G}_p$, $1 \leq p < \infty$, $\Gamma_0 = \Gamma(t_0, r_0)$ and $\chi_{\Gamma_0}$ be the characteristic function of the ball $\Gamma_0$, then $\chi_{\Gamma_0} \in LM_{p, \varphi}^{[\rho]}(\Gamma) \cap \mathcal{M}_{p, \varphi}(\Gamma)$. Moreover, there exists $C > 0$ such that
\[
\frac{1}{\varphi(r_0)} \leq \|\chi_{\Gamma_0}\|_{W_{LM_{p, \varphi}^{[\rho]}}} \leq \|\chi_{\Gamma_0}\|_{LM_{p, \varphi}^{[\rho]}} \leq \frac{C}{\varphi(r_0)}
\]
and
\[
\frac{1}{\varphi(r_0)} \leq \|\chi_{\Gamma_0}\|_{W_{LM_{p, \varphi}}} \leq \|\chi_{\Gamma_0}\|_{LM_{p, \varphi}} \leq \frac{C}{\varphi(r_0)}.
\]

4 Maximal operator in the spaces $LM_{p, \varphi}^{[\rho]}(\Gamma)$ and $\mathcal{M}_{p, \varphi}(\Gamma)$

We denote by $L_{\infty, \varphi}(0, \infty)$ the set of all functions $g(t)$, $t > 0$ with finite norm
\[
\|g\|_{L_{\infty, \varphi}(0, \infty)} = \text{ess sup}_{t>0} \nu(t) g(t)
\]
and $L_{\infty, \varphi}(0, \infty) \equiv L_{\infty, \varphi}(0, \infty)$. Let $\mathcal{M}(0, \infty)$ be the set of all Lebesgue-measurable functions on $(0, \infty)$ and $\mathcal{M}^+(0, \infty)$ its subset consisting of all non-negative functions on $(0, \infty)$. We denote by $\mathcal{M}^+(0, \infty ; \uparrow)$ the cone of all functions in $\mathcal{M}^+(0, \infty)$ which are non-decreasing on $(0, \infty)$ and
\[
A = \{ \varphi \in \mathcal{M}^+(0, \infty ; \uparrow) : \lim_{t \to 0^+} \varphi(t) = 0 \}.
\]

Let $u$ be a continuous and non-negative function on $(0, \infty)$. We define the supremal operator $S_u$ on $g \in \mathcal{M}(0, \infty)$ by
\[
(S_u g)(t) = \|ug\|_{L_{\infty, \varphi}(0, \infty)}, \quad t \in (0, \infty).
\]

The following theorem was proved in [26].
Theorem 4.2. Let \( v_1, v_2 \) be non-negative measurable functions satisfying \( 0 < \| v_1 \|_{L_0(\mathbb{R}, \infty)} < \infty \) for any \( t > 0 \) and let \( u \) be a non-negative continuous function on \( (0, \infty) \).

Then the operator \( S_u \) is bounded from \( L_{0,v_2}(0, \infty) \) to \( L_{0,v_1}(0, \infty) \) on the cone \( A \) if and only if

\[
\| v_2 \|_{L_0(\mathbb{R}, \infty)}^{-1} \| v_1 \|_{L_0(\mathbb{R}, \infty)} < \infty. \tag{4.5}
\]

The following Guliyev-type local estimate for the maximal operator \( M \) is true, see for example, [27,28].

Lemma 4.4. Let \( \Gamma \) be a Carleson curve, \( 1 \leq p < \infty \) and \( t_0 \in \Gamma \). Then for \( p > 1 \) and any \( r > 0 \) the inequality

\[
\| Mf \|_{L_p(\Gamma, t_0, r)} \leq \| f \|_{L_p(\Gamma, t_0, 2r)} + r \sup_{r > \frac{t}{2}} \| f \|_{L_p(\Gamma, t_0, r)} \tag{4.6}
\]

holds for all \( f \in L_p^\text{loc}(\Gamma) \).

Moreover, for \( p = 1 \) the inequality

\[
\| Mf \|_{W^{\frac{1}{2},p}(\Gamma, t_0, r)} \leq \| f \|_{L_1(\Gamma, t_0, 2r)} + r \sup_{r > \frac{t}{2}} \| f \|_{L_1(\Gamma, t_0, r)} \tag{4.7}
\]

holds for all \( f \in L_1^\text{loc}(\Gamma) \).

Proof. Let \( 1 < p < \infty \). For arbitrary ball \( \Gamma(t_0, r) \) let \( f = f_1 + f_2 \), where \( f_1 = f \chi_{\Gamma(t_0, 2r)} \) and \( f_2 = f \chi_{\Gamma(t_0, 2r)} \cap \chi_{\Gamma(t_0, 2r)} \).

By the continuity of the operator \( M : L_p(\Gamma) \rightarrow L_p(\Gamma) \) from Theorem A we have

\[
\| Mf \|_{L_p(\Gamma, t_0, 2r)} \leq \| f \|_{L_p(\Gamma, t_0, 2r)} + \| Mf \|_{L_p(\Gamma, t_0, r)} \tag{4.6}
\]

Let \( y \) be an arbitrary point from \( \Gamma(t_0, r) \). If \( \Gamma(y, r) \cap \Gamma(t_0, 2r) \neq \emptyset \), then \( r > r \). Indeed, if \( z \in \Gamma(y, r) \cap \Gamma(t_0, 2r) \), then \( r > |y - z| \geq |t - z| - |t - y| > 2r - r = r \).

On the other hand, \( \Gamma(y, r) \cap \Gamma(t_0, 2r) \subset \Gamma(t_0, 2r) \). Indeed, \( z \in \Gamma(y, r) \cap \Gamma(t_0, 2r) \), then \( |z| \leq |y - z| + |t - y| < r + r < 2r \).

Hence,

\[
Mf_2(y) \leq 2 \sup_{r > \frac{t}{2}} \frac{1}{\nu(\Gamma(t_0, 2r) \cap \Gamma(t_0, r))} \int_{\Gamma(t_0, 2r)} |f(z)| \, dv(z) = 2 \sup_{r > \frac{t}{2}} \frac{1}{\nu(\Gamma(t_0, t_0) \cap \Gamma(t_0, r))} \int_{\Gamma(t_0, t_0) \cap \Gamma(t_0, r)} |f(z)| \, dv(z) \leq 2 \sup_{r > \frac{t}{2}} \int_{\Gamma(t_0, t_0) \cap \Gamma(t_0, r)} |f(z)| \, dv(z).
\]

Therefore, for all \( y \in \Gamma(t_0, t) \) we have

\[
Mf_2(y) \leq 2 \sup_{r > \frac{t}{2}} \int_{\Gamma(t_0, t_0) \cap \Gamma(t_0, r)} |f(z)| \, dv(z). \tag{4.8}
\]

Thus,

\[
\| Mf \|_{L_p(\Gamma, t_0, r)} \leq \| f \|_{L_p(\Gamma, t_0, 2r)} + r \sup_{r > \frac{t}{2}} \int_{\Gamma(t_0, r)} |f(z)| \, dv(z).
\]

Let \( p = 1 \). It is obvious that for any ball \( \Gamma(t_0, r) \)

\[
\| Mf \|_{W^{\frac{1}{2},p}(\Gamma, t_0, r)} \leq \| Mf_1 \|_{W^{\frac{1}{2},p}(\Gamma, t_0, r)} + \| Mf_2 \|_{W^{\frac{1}{2},p}(\Gamma, t_0, r)} \tag{4.6}
\]

By the continuity of the operator \( M : L_1(\Gamma) \rightarrow W^{\frac{1}{2},p}(\Gamma) \) from Theorem A we have

\[
\| Mf_1 \|_{W^{\frac{1}{2},p}(\Gamma)} \leq \| f \|_{L_1(\Gamma, t_0, 2r)} \tag{4.7}
\]

Then by (4.8) we get inequality (4.7). \( \Box \)
Lemma 4.5. Let $\Gamma$ be a Carleson curve, $1 \leq p < \infty$ and $t_0 \in \Gamma$. Then for $p > 1$ and any $r > 0$ in $\Gamma$, the inequality

$$\|Mf\|_{L^p(\Gamma(t_0, r))} \leq r^{\frac{1}{p}} \sup_{r > 2r} \left( \int_{\Gamma(t_0, r)} |f(z)|^p \, dv(z) \right)^{\frac{1}{p}}$$

(4.9)

holds for all $f \in L^1_{\text{loc}}(\Gamma)$.

Moreover, for $p = 1$ the inequality

$$\|Mf\|_{W(L^1(\Gamma(t_0, r)))} \leq r \sup_{r > 2r} \left( \int_{\Gamma(t_0, r)} |f(z)|^p \, dv(z) \right)^{\frac{1}{p}}$$

(4.10)

holds for all $f \in L^1_{\text{loc}}(\Gamma)$.

**Proof.** Let $1 < p < \infty$. Denote

$$M_1 = r^{\frac{1}{p}} \sup_{r > 2r} \left( \int_{\Gamma(t_0, r)} |f(z)|^p \, dv(z) \right)^{\frac{1}{p}}, \quad M_2 = \|f\|_{L^p(\Gamma(t_0, 2r))}.$$ 

Applying Hölder’s inequality, we get

$$M_1 \leq r^{\frac{1}{p}} \sup_{r > 2r} \left( \int_{\Gamma(t_0, r)} |f(z)|^p \, dv(z) \right)^{\frac{1}{p}} \geq r^{\frac{1}{p}} \left( \sup_{r > 2r} \|f\|_{L^p(\Gamma(t_0, 2r))} \right) = M_2.$$ 

On the other hand,

$$r^{\frac{1}{p}} \sup_{r > 2r} \left( \int_{\Gamma(t_0, r)} |f(z)|^p \, dv(z) \right)^{\frac{1}{p}} \geq r^{\frac{1}{p}} \left( \sup_{r > 2r} \|f\|_{L^p(\Gamma(t_0, 2r))} \right) \approx M_2.$$

Since by Lemma 4.4

$$\|Mf\|_{L^p(\Gamma(t_0, r))} \leq M_1 + M_2,$$ 

we arrive at (4.9).

Let $p = 1$. The inequality (4.10) directly follows from (4.7). □

The following theorem is valid.

**Theorem 4.3.** Let $\Gamma$ be a Carleson curve, $1 \leq p < \infty$, $t_0 \in \Gamma$ and $(\varphi_1, \varphi_2)$ satisfies the condition

$$\sup_{r < \infty} \sup_{r < \infty} \frac{1}{r} \varphi_1(t_0, s) \frac{1}{s} \varphi_2(t_0, r) \leq C \varphi_2(t_0, r),$$

(4.11)

where $C$ does not depend on $r$. Then for $p > 1$ the operator $M$ is bounded from $L^1_{\text{loc}}(\Gamma)$ to $L^{\frac{1}{p}}_{\varphi_2}(\Gamma)$ and for $p = 1$ the operator $M$ is bounded from $L^{\frac{1}{p}}_{\varphi_1}(\Gamma)$ to $W^1_{\varphi_2}(\Gamma)$.

**Proof.** By Theorem 4.2 and Lemma 4.5, we get

$$\|Mf\|_{L^{\frac{1}{p}}_{\varphi_1}(\Gamma)} \leq \sup_{r \in (0, \infty)} \sup_{r \in (0, \infty)} \varphi_1(t_0, r) r^\frac{1}{p} \|f\|_{L^p(\Gamma(t_0, r))} \leq \sup_{r \in (0, \infty)} \varphi_1(t_0, r) r^\frac{1}{p} \|f\|_{L^{\frac{1}{p}}_{\varphi_2}(\Gamma)}$$

if $p \in (1, \infty)$ and

$$\|Mf\|_{W^1_{\varphi_2}(\Gamma)} \leq \sup_{r \in (0, \infty)} \sup_{r \in (0, \infty)} \varphi_2(t_0, r) r^{-\frac{1}{p}} \|f\|_{L^p(\Gamma(t_0, r))} \leq \sup_{r \in (0, \infty)} \varphi_2(t_0, r) r^{-\frac{1}{p}} \|f\|_{W^1_{\varphi_2}(\Gamma)}$$

if $p = 1$. □

From Theorem 4.3, we get the following.
Corollary 4.1. Let $\Gamma$ be a Carleson curve, $1 \leq p < \infty$ and $\varphi_1, \varphi_2 \in \Omega_p$ satisfies the condition
\begin{equation}
\sup_{r < t < \infty} \tau^{-1} \mathrm{osc} \inf_{t \leq s \leq r} \varphi_1(t, s) \leq C \varphi_2(t, r),
\end{equation}
where $C$ does not depend on $t$ and $r$. Then for $p > 1$ the operator $M$ is bounded from $M_{p, \varphi_1}(\Gamma)$ to $M_{p, \varphi_2}(\Gamma)$ and for $p = 1$ the operator $M$ is bounded from $M_{1, \varphi_1}(\Gamma)$ to $WM_{1, \varphi_2}(\Gamma)$.

Corollary 4.2. Let $\Gamma$ be a Carleson curve, $1 \leq p < \infty$ and $\varphi \in \mathcal{G}$. Then for $p > 1$ the operator $M$ is bounded on $M_{p, \varphi}(\Gamma)$ and for $p = 1$ the operator $M$ is bounded from $M_{1, \varphi}(\Gamma)$ to $WM_{1, \varphi}(\Gamma)$.

5 Fractional integral operator in the spaces $LM_{p, \varphi}^{\{t_0\}}(\Gamma)$ and $M_{p, \varphi}(\Gamma)$

5.1 Spanne-type results

The following local estimate is true, see for example, [28].

Theorem 5.4. Let $\Gamma$ be a Carleson curve, $1 \leq p < \infty$, $t_0 \in \Gamma$, $0 < \alpha < \frac{1}{p'}$, $\frac{1}{q} = \frac{1}{p} - \alpha$ and $f \in L^p_{\text{loc}}(\Gamma)$. Then for $p > 1$
\begin{equation}
\|I^a f\|_{L^q(\Gamma(t_0, r))} \leq C r^\frac{\alpha}{2} \int_{\frac{r}{2}}^r t^{\frac{1}{q} - 1} \|f\|_{L^p(\Gamma(t_0, t))} \, \mathrm{d}t,
\end{equation}
and for $p = 1$
\begin{equation}
\|I^a f\|_{W^{1,q}(\Gamma(t_0, r))} \leq C r^\frac{\alpha}{2} \int_{\frac{r}{2}}^r t^{\frac{1}{q} - 1} \|f\|_{L^p(\Gamma(t_0, t))} \, \mathrm{d}t,
\end{equation}
where $C$ does not depend on $f$, $t_0 \in \Gamma$ and $r > 0$.

Proof. For a given ball $\Gamma(t_0, r)$, we split the function $f$ as $f = f_1 + f_2$, where $f_1 = f\chi_{\Gamma(t_0, 2r)}$, $f_2 = f\chi_{(\Gamma(t_0, 2r)}$, and then
\[ I^a f(t) = I^a f_1(t) + I^a f_2(t). \]

Let $1 < p < \infty$, $0 < \alpha < \frac{1}{p'}$, $\frac{1}{q} = \frac{1}{p} - \alpha$. Since $f_i \in L^p(\Gamma)$, by the boundedness of the operator $I^a$ from $L^p(\Gamma)$ to $L^q(\Gamma)$ (see Theorem B) it follows that
\begin{equation}
\|I^a f_i\|_{L^q(\Gamma)} \leq C \|f_i\|_{L^p(\Gamma)} = C \|f\|_{L^p(\Gamma(t_0, 2r))} \leq C r^\frac{\alpha}{2} \int_{\frac{r}{2}}^r t^{\frac{1}{q} - 1} \|f\|_{L^p(\Gamma(t_0, t))} \, \mathrm{d}t,
\end{equation}
where the constant $C$ is independent of $f$.

Observe that the conditions $z \in \Gamma(t_0, r)$, $y \in \mathcal{C}(\Gamma(t_0, 2r))$ imply
\[ \frac{1}{2}|z - y| \leq |t - y| \leq \frac{3}{2}|t - z|. \]

Then for all $z \in \Gamma(t_0, r)$ we get
\[ |I^a f_2(z)| \leq \left(\frac{3}{2}\right)^{1-a} \int_{\mathcal{C}(\Gamma(t_0, 2r))} |t - y|^{1-a} |f(y)| \, \mathrm{d}v(y). \]
By Fubini’s theorem, we have
\[
\int |t - y|^{n-1}|f(y)|\,dv(y) = \int \int t^{n-2}dr = \int \int |f(y)|\,dv(y) r^{n-2}dr.
\]

Applying Hölder’s inequality, we get
\[
\int |t - y|^{n-1}|f(y)|\,dv(y) \leq \int \|f\|_{L_p(\Gamma_0, t, r)} r^{-\frac{1}{q}}dr.
\]

Moreover, for all \(p \in [1, \infty)\) the inequality
\[
\|I^z f\|_{L_q(\Gamma_0, t, r)} \leq C \int \frac{1}{r} \frac{1}{r} r^{-\frac{1}{q}}dr
\]
is valid. Thus, from (5.15) and (5.17) we get inequality (5.13).

Finally, in the case \(p = 1\) by the weak (1, q)-boundedness of the operator \(I^z\) (see Theorem B) it follows that
\[
\|I^z f\|_{WLM_q(\Gamma_0, t, r)} \leq C \int \frac{1}{r} \frac{1}{r} r^{-\frac{1}{q}}dr,
\]
where \(C\) does not depend on \(t_0\) and \(r\). Then from (5.17) and (5.18) we get inequality (5.14).

**Theorem 5.5.** Let \(\Gamma\) be a Carleson curve, \(1 \leq p < \infty\), \(t_0 \in \Gamma\), \(0 < \alpha < \frac{1}{p} - \frac{1}{q} = \frac{1}{p} - \alpha\), \(\varphi_1 \in \Omega_{p,1}, \varphi_2 \in \Omega_{q,1}\) and the pair \((\varphi_1, \varphi_2)\) satisfy the condition
\[
\int \frac{\text{ess inf}}{r} \frac{s^h}{r} \,dr \leq C \varphi_2(t_0, r),
\]
where \(C\) does not depend on \(t_0\) and \(r\). Then for \(p > 1\) the operator \(I^z\) is bounded from \(LM^1_{\varphi_1}(\Gamma)\) to \(LM^1_{\varphi_2}(\Gamma)\) and for \(p = 1\) the operator \(I^z\) is bounded from \(LM^1_{\varphi_1}(\Gamma)\) to \(WLM^1_{\varphi_2}(\Gamma)\).

**Proof.** By Theorems 2.1 and 5.4 with \(v_2(r) = \varphi_2(t_0, r)^{-1}\), \(v_1(r) = \varphi_1(t_0, r)^{-1}r^{-\frac{1}{q}}\) and \(w(r) = r^{-\frac{1}{q}}\) we have for \(p > 1\)
\[
\|I^z f\|_{LM^1_{\varphi_2}(\Gamma)} \leq \sup_{r > 0} \varphi_2(t_0, r)^{-1} \int r^{-\frac{1}{q}} \frac{1}{r} \,dr \leq \sup_{r > 0} \varphi_1(t_0, r)^{-1} r^{-\frac{1}{q}} \|f\|_{L_1(\Gamma_0, t_0, r)} = \|f\|_{LM^1_{\varphi_1}(\Gamma)}
\]
and for \(p = 1\)
\[
\|I^z f\|_{WLM^1_{\varphi_2}(\Gamma)} \leq \sup_{r > 0} \varphi_2(t_0, r)^{-1} \int r^{-\frac{1}{q}} \frac{1}{r} \,dr \leq \sup_{r > 0} \varphi_1(t_0, r)^{-1} r^{-\frac{1}{q}} \|f\|_{L_1(\Gamma_0, t_0, r)} = \|f\|_{LM^1_{\varphi_1}(\Gamma)}.
\]

From Theorem 4.3 we get the following.
Corollary 5.3. Let $\Gamma$ be a Carleson curve, $1 \leq p < \infty$, $0 < \alpha < \frac{1}{p}$, $\frac{1}{q} = \frac{1}{p} - \alpha$, $\varphi_1 \in \Omega_p$, $\varphi_2 \in \Omega_q$ and the pair $(\varphi_1, \varphi_2)$ satisfy the condition

$$\int_0^\infty \inf_{r \leq |s|} \frac{\varphi_1(t, s)}{s^\alpha} \frac{ds}{r} \leq C \varphi_2(t, r),$$

(5.20)

where $C$ does not depend on $t$ and $r$. Then for $p > 1$ the operator $I^a$ is bounded from $M_{p, \varphi_1}(\Gamma)$ to $M_{q, \varphi_2}(\Gamma)$ and for $p = 1$ the operator $I^a$ is bounded from $M_{q, \varphi_1}(\Gamma)$ to $WM_{q, \varphi_2}(\Gamma)$.

For proving our main results, we need the following estimate.

Lemma 5.6. Let $\Gamma$ be a Carleson curve and $\Gamma_0 = \Gamma(t_0, r_0)$, then $r_0^a \leq I^a_{\oo}(t)$ for every $t \in \Gamma_0$.

Proof. If $t, y \in \Gamma_0$, then $t - y \leq |t - t_0| + |t_0 - y| < 2r_0$. Since $0 < \alpha < 1$, we get $r_0^{a-1} \leq 2^{1-a}|t - y|^{a-1}$. Therefore,

$$I^a_{\oo}(t) = \int_{\Gamma_0} |t - y|^{a-1}d\nu(y) = \int_{t_0} |t - y|^{a-1}d\nu(y) \geq c_0 2^{1-a}r_0^a. \quad \Box$$

The following theorem is one of our main results.

Theorem 5.6. Let $\Gamma$ be a Carleson curve, $0 < \alpha < 1$, $t_0 \in \Gamma$ and $p, q \in [1, \infty)$.

1. If $1 \leq p < \frac{1}{\alpha}$ and $\frac{1}{q} = \frac{1}{p} - \alpha$, then condition (5.20) is sufficient for the boundedness of the operator $I^a$ from $LM_{p, \varphi_1}^{\alpha}(\Gamma)$ to $WLM_{q, \varphi_2}^{\alpha}(\Gamma)$. Moreover, if $1 < p < \frac{1}{\alpha}$, condition (5.20) is sufficient for the boundedness of the operator $I^a$ from $LM_{p, \varphi_1}^{\alpha}(\Gamma)$ to $LM_{q, \varphi_2}^{\alpha}(\Gamma)$.

2. If the function $\varphi_1 \in G_p$, then the condition

$$r_0^a \varphi_1(r) \leq C \varphi_2(r),$$

(5.21)

for all $r > 0$, where $C > 0$ does not depend on $r$, is necessary for the boundedness of the operator $I^a$ from $LM_{p, \varphi_1}^{\alpha}(\Gamma)$ to $WLM_{q, \varphi_2}^{\alpha}(\Gamma)$ and $LM_{p, \varphi_1}^{\alpha}(\Gamma)$ to $LM_{q, \varphi_2}^{\alpha}(\Gamma)$.

3. Let $1 \leq p < \frac{1}{\alpha}$ and $\frac{1}{q} = \frac{1}{p} - \alpha$. If $\varphi_1 \in G_p$ satisfies the regularity condition

$$\int_0^\infty s^{\alpha-1} \varphi_1(s) ds \leq C r^a \varphi_1(r),$$

(5.22)

for all $r > 0$, where $C > 0$ does not depend on $r$, then condition (5.21) is necessary and sufficient for the boundedness of the operator $I^a$ from $LM_{p, \varphi_1}^{\alpha}(\Gamma)$ to $WLM_{q, \varphi_2}^{\alpha}(\Gamma)$. Moreover, if $1 < p < \frac{2}{\alpha}$, then condition (5.21) is necessary and sufficient for the boundedness of the operator $I^a$ from $LM_{p, \varphi_1}^{\alpha}(\Gamma)$ to $LM_{q, \varphi_2}^{\alpha}(\Gamma)$.

Proof. The first part of the theorem is proved in Theorem 5.3.

We shall now prove the second part. Let $\Gamma_0 = \Gamma(t_0, r_0)$ and $t \in \Gamma_0$. By Lemma 5.6, we have $r_0^a \leq CI\oo_{\oo}(t)$. Therefore, by Lemmas 3.3 and 5.6

$$r_0^a \leq (s(\Gamma_0))^{-\frac{1}{2}} \|I^a_{\oo}\|_{L_{p, \varphi_1}} \leq \varphi_2(r_0) \|I^a_{\oo}\|_{M_{q, \varphi_2}} \leq \varphi_2(r_0) \|\oo\|_{M_{p, \varphi_1}} \leq \frac{\varphi_2(r_0)}{\varphi_1(r_0)}$$

or

$$r_0^a \leq \frac{\varphi_2(r_0)}{\varphi_1(r_0)} \quad \text{for all } r_0 > 0 \Rightarrow r_0^a \varphi_1(r_0) \leq \varphi_2(r_0) \quad \text{for all } r_0 > 0.$$
Remark 5.3. If we take \( \phi_1(r) = r^{\frac{1}{p}} \) and \( \phi_2(r) = r^{\frac{1}{q}} \) at Theorem 5.6, then conditions (5.22) and (5.21) are equivalent to \( 0 < \lambda < 1 - ap \) and \( \frac{a}{p} = \frac{b}{q} \), respectively. Therefore, we get Theorem C from Theorem 5.6.

5.2 Adams-type results

The following pointwise estimate plays a key role where we prove our main results.

Theorem 5.7. Let \( \Gamma \) be a Carleson curve, \( 1 \leq p < \infty \), \( 0 < \alpha < 1 \) and \( f \in L^p_{\infty}(\Gamma) \). Then

\[
|I^a f(t)| \leq C r^a Mf(t) + \frac{C}{r} \int s^{\alpha - \frac{1}{p} - 1} \|f\|_{L_p(t,s)} ds,
\]

where \( C \) does not depend on \( f, t \in \Gamma \) and \( r > 0 \).

Proof. Write \( f = f_1 + f_2 \), where \( f_1 = f_{\chi_{(t,2r)}} \) and \( f_2 = f_{\chi_{(t,\infty)}} \). Then

\[
I^a f(t) = I^a f_1(t) + I^a f_2(t).
\]

For \( I^a f_1(t) \), following Hedberg’s trick (see for instance [29, p. 354]), for all \( z \in \Gamma \) we obtain \( I^a f_1(z) \leq C r^a Mf(z) \). For \( I^a f_2(z) \) with \( z \in D \) from (5.16) we have

\[
I^a f_2(z) \leq \int_{D \cup \partial D(z)} |t - y|^a |f(y)| dy \leq C \int_{2r}^\infty s^{\alpha - \frac{1}{p} - 1} \|f\|_{L_p(t,s)} ds,
\]

which proves (5.23).

The following is a result of Adams type for the fractional integral on Carleson curves (see [28]).

Theorem 5.8. (Adams-type result) Let \( \Gamma \) be a Carleson curve, \( 1 \leq p < q < \infty \), \( 0 < \alpha < \frac{1}{p} \) and let \( \varphi \in \Omega_p \) satisfy condition

\[
\sup_{r \in \Gamma} \tau^{-\frac{1}{p}} \inf_{s \in \Gamma} \varphi(t, s) s \leq C \varphi(t, r),
\]

and

\[
\int_{r}^{\infty} \tau^{a - \frac{1}{p}} \varphi(t, r) \tau ds \leq C \tau^{1 - \frac{a}{p}},
\]

where \( C \) does not depend on \( t \in \Gamma \) and \( r > 0 \). Then for \( p > 1 \) the operator \( I^a \) is bounded from \( M_{p,\varphi}(\Gamma) \) to \( M_{q,\varphi}(\Gamma) \) and for \( p = 1 \) the operator \( I^a \) is bounded from \( M_1(\Gamma) \) to \( WM_{q,\varphi}(\Gamma) \).

Proof. Let \( 1 \leq p < \infty \) and \( f \in M_{p,\varphi}(\Gamma) \). By Theorem 5.7, inequality (5.23) is valid. Then from condition (5.26) and inequality (5.23) we get

\[
|I^a f(t)| \leq r^a Mf(t) + \int_{r}^{\infty} s^{\alpha - \frac{1}{p} - 1} \|f\|_{L_p(t,s)} ds
\]

\[
\leq r^a Mf(t) + \|f\|_{M_{p,\varphi}(\Gamma)} \int_{r}^{\infty} s^{\alpha - \frac{1}{p}} \varphi(t, s) ds
\]

\[
\leq r^a Mf(t) + r^{1 - \frac{a}{p}} \|f\|_{M_{p,\varphi}(\Gamma)}.
\]
Hence, choosing \( r = \left(\frac{M_p\varphi(t)\Gamma}{M_f(t)}\right) \) for every \( t \in \Gamma \), we have
\[
|I^\alpha f(t)| \leq (M_f(t))^{1-\frac{1}{p}} \left\| f \right\|_{M_p\varphi,\Gamma}^{1-\frac{1}{p}}.
\]

Hence, the statement of the theorem follows in view of the boundedness of the maximal operator \( M \) in \( M_{p,\varphi}(\Gamma) \) provided by Theorem 3, by virtue of condition (5.25).

\[
\left\| I^\alpha f \right\|_{M_{\varphi,\Gamma}} \leq \left\| f \right\|_{M_{p,\varphi,\Gamma}}^{1-\frac{1}{p}} \sup_{(t,r) \in \Gamma \times \mathbb{R}_+} |\varphi(t,r)| r^{-\frac{1}{p}} |M_f(r)|^{\frac{1}{p}} \leq \left\| f \right\|_{M_{p,\varphi,\Gamma}}^{1-\frac{1}{p}} \left\| M f \right\|_{p,\varphi,\Gamma}^{\frac{1}{p}} \leq \left\| f \right\|_{M_{p,\varphi,\Gamma}}
\]

if \( 1 < p < q < \infty \) and
\[
\left\| I^\alpha f \right\|_{WM_{\varphi,\Gamma}} \leq \left\| f \right\|_{M_{\varphi,\Gamma}}^{1-\frac{1}{q}} \sup_{(t,r) \in \Gamma \times \mathbb{R}_+} |\varphi(t,r)| r^{-\frac{1}{q}} |M_f(r)|^{\frac{1}{q}} \leq \left\| f \right\|_{M_{\varphi,\Gamma}}^{1-\frac{1}{q}} \left\| M f \right\|_{\varphi,\Gamma}^{\frac{1}{q}} \leq \left\| f \right\|_{M_{\varphi,\Gamma}}
\]

if \( p = 1 < q < \infty \). \( \Box \)

The following theorem is another of our main results.

**Theorem 5.9.** Let \( \Gamma \) be a Carleson curve, \( 0 < a < 1 \), \( 1 \leq p < q < \infty \) and \( \varphi \in \varphi_p \).

1. If \( \varphi(t,r) \) satisfies condition (5.25), then condition (5.26) is sufficient for the boundedness of the operator \( I^\alpha \) from \( M_{p,\varphi}(\Gamma) \) to \( WM_{q,\varphi}(\Gamma) \). Moreover, if \( 1 < p < q < \infty \), then condition (5.26) is sufficient for the boundedness of the operator \( I^\alpha \) from \( M_{p,\varphi}(\Gamma) \) to \( M_{q,\varphi}(\Gamma) \).

2. If \( \varphi \in \mathcal{G}_p \), then the condition
\[
r^a \varphi(r) = Cr^{1-a}, \tag{5.28}
\]

for all \( r > 0 \), where \( C > 0 \) does not depend on \( r \), is necessary for the boundedness of the operator \( I^\alpha \) from \( M_{p,\varphi}(\Gamma) \) to \( WM_{q,\varphi}(\Gamma) \) and from \( M_{p,\varphi}(\Gamma) \) to \( M_{q,\varphi}(\Gamma) \).

3. If \( \varphi \in \mathcal{G}_p \) satisfies the regularity condition
\[
\int_r^\infty s^{-1-a} \varphi(s) \, ds \leq Cr^a \varphi(r)^{1-a}, \tag{5.29}
\]

for all \( r > 0 \), where \( C > 0 \) does not depend on \( r \), then condition (5.28) is necessary and sufficient for the boundedness of the operator \( I^\alpha \) from \( M_{p,\varphi}(\Gamma) \) to \( WM_{q,\varphi}(\Gamma) \). Moreover, if \( 1 < p < q < \infty \), then condition (5.28) is necessary and sufficient for the boundedness of the operator \( I^\alpha \) from \( M_{p,\varphi}(\Gamma) \) to \( M_{q,\varphi}(\Gamma) \).

**Proof.** The first part of the theorem is a corollary of Theorem 5.8.

We shall now prove the second part. Let \( \Gamma_0 = \Gamma(t_0, r_0) \) and \( t \in \Gamma_0 \). By Lemma 5.6, we have \( r_0^a \leq I^\alpha \varphi_{r_0}(t) \).

Therefore, by Lemmas 3.3 and 5.6 we have
\[
r_0^a \leq (\varphi(\Gamma_0))^{1-a} \left\| I^\alpha \chi_{\Gamma_0} \right\|_{L^1(\Gamma_0)} \leq \varphi(t_0)^{1-a} \left\| I^\alpha \chi_{\Gamma_0} \right\|_{M_{p,\varphi,\Gamma}} \leq \varphi(t_0)^{1-a} \left\| \chi_{\Gamma_0} \right\|_{M_{p,\varphi,\Gamma}} \leq \varphi(t_0)^{1-a} \varphi(r_0)^{1-a}
\]
or
\[
r_0^a \varphi(t_0)^{1-a} \leq 1 \quad \text{for all} \quad r_0 > 0 \Leftrightarrow r_0^a \varphi(t_0)^{1-a} \leq r_0^{-a} \varphi(r_0)^{1-a}.
\]

Since this is true for every \( t \in \Gamma \) and \( r_0 > 0 \), we are done.

The third statement of the theorem follows from first and second parts of the theorem. \( \Box \)

The following is a result of Adams type for the fractional integral on Carleson curves.
Theorem 10. (Adams-type result). Let $\Gamma$ be a Carleson curve, $0 < \alpha < 1$, $1 \leq p < q < \infty$ and $\varphi \in \Omega_p$ satisfy condition (5.25) and

$$r^\alpha \varphi(t, r) + \int_{r}^{\infty} s^{\alpha-1} \varphi(t, s) \, ds \leq C \varphi(t, r)^{\frac{\alpha}{\gamma}},$$  \hspace{1cm} (5.30)

where $C$ does not depend on $t \in \Gamma$ and $r > 0$. Then for $p > 1$ the operator $I^a$ is bounded from $M_{p, \varphi^p}(\Gamma)$ to $M_{q, \varphi^q}(\Gamma)$ and for $p = 1$ the operator $I^a$ is bounded from $M_1(\varphi)(\Gamma)$ to $WM_{q, \varphi^q}(\Gamma)$.

Proof. Let $1 \leq p < \infty$ and $f \in M_{p, \varphi}(\Gamma)$. By Theorem 5.7, inequality (5.23) is valid. Then from condition (5.26) and inequality (5.23), we get

$$|I^a f(t)| \leq r^\alpha M_{p, \varphi}(f(t) + \int_{r}^{\infty} s^{\alpha-1} \|f\|_{L_\varphi(t, s)} \, ds \leq r^\alpha M_{p, \varphi}(f(t) + \|f\|_{M_{p, \varphi}(\Gamma)} \int_{r}^{\infty} s^{\alpha-1} \varphi(t, s) \, ds. \hspace{1cm} (5.31)$$

Thus, by (5.30) and (5.31) we obtain

$$|I^a f(t)| \leq \min \left\{ \varphi(t, r)^{\alpha-1} M_{p, \varphi}(f(t), \varphi(t, r)^{\beta} \|f\|_{M_{p, \varphi}(\Gamma)} \right\} \leq \sup_{r > 0} \|r^{\alpha-1} M_{p, \varphi}(f(t), r\|f\|_{M_{p, \varphi}(\Gamma)} \right\} \hspace{1cm} (5.32)$$

where we have used that the supremum is achieved when the minimum parts are balanced. From Theorem 4.3 and (5.32), we get

$$\|I^a f\|_{M_{q, \varphi^q}(\Gamma)} \leq \|f\|_{M_{p, \varphi^p}(\Gamma)} \|M_{p, \varphi}(f(t), \|f\|_{M_{p, \varphi}(\Gamma)} \|f\|_{M_{p, \varphi}(\Gamma)} \leq \|f\|_{M_{p, \varphi^p}(\Gamma)},$$

if $1 < p < q < \infty$ and

$$\|I^a f\|_{WM_{q, \varphi^q}(\Gamma)} \leq \|f\|_{M_{p, \varphi^p}(\Gamma)} \|M_{p, \varphi}(f(t), \|f\|_{M_{p, \varphi}(\Gamma)} \|f\|_{M_{p, \varphi}(\Gamma)} \leq \|f\|_{M_{p, \varphi^p}(\Gamma)},$$

if $p = 1 < q < \infty$, \hspace{1cm} $\square$

The following theorem is another of our main results.

Theorem 5.11. Let $\Gamma$ be a Carleson curve, $0 < \alpha < 1$, $1 \leq p < q < \infty$ and $\varphi \in \Omega_p$.

1. If $\varphi(t, r)$ satisfies condition (5.25), then condition (5.30) is sufficient for the boundedness of the operator $I^a$ from $M_{p, \varphi^p}(\Gamma)$ to $M_{q, \varphi^q}(\Gamma)$. Moreover, if $1 < p < q < \infty$, then condition (5.30) is sufficient for the boundedness of the operator $I^a$ from $M_{p, \varphi^p}(\Gamma)$ to $M_{q, \varphi^q}(\Gamma)$.

2. If $\varphi \in \mathcal{G}_p$, then the condition

$$r^\alpha \varphi(r) \leq C \varphi(r)^{\frac{\alpha}{\gamma}},$$ \hspace{1cm} (5.33)

for all $r > 0$, where $C > 0$ does not depend on $r$, is necessary for the boundedness of the operator $I^a$ from $M_{p, \varphi^p}(\Gamma)$ to $WM_{q, \varphi^q}(\Gamma)$ and from $M_{p, \varphi^p}(\Gamma)$ to $M_{q, \varphi^q}(\Gamma)$.

3. If $\varphi \in \mathcal{G}_p$ satisfies the regularity condition (5.25), then condition (5.33) is necessary and sufficient for the boundedness of the operator $I^a$ from $M_{p, \varphi^p}(\Gamma)$ to $WM_{q, \varphi^q}(\Gamma)$. Moreover, if $1 < p < q < \infty$, then condition (5.33) is necessary and sufficient for the boundedness of the operator $I^a$ from $M_{p, \varphi^p}(\Gamma)$ to $M_{q, \varphi^q}(\Gamma)$.

Proof. The first part of the theorem is a corollary of Theorem 5.10.

We shall now prove the second part. Let $\Gamma_0 = \Gamma(0, r_0)$ and $t \in \Gamma_0$. By Lemma 5.6 we have $r_{0}^{\alpha} \leq C I^a \chi_{\Gamma_0}(t)$. Therefore, by Lemmas 3.3 and 5.6 we have

$$r_0^\alpha \leq (\nu(\Gamma_0))^{\frac{\gamma}{\alpha}} \|I^a \chi_{\Gamma_0}\|_{L_\varphi(\Gamma_0)} \leq \varphi(r_0)^{\frac{\alpha}{\gamma}} \|I^a \chi_{\Gamma_0}\|_{M_{q, \varphi^q}(\Gamma_0)} \leq \varphi(r_0)^{\frac{\alpha}{\gamma}} \|\chi_{\Gamma_0}\|_{M_{p, \varphi^p}(\Gamma_0)} \leq \varphi(r_0)^{\frac{\alpha}{\gamma}}.$$
or

\[ r_0^\alpha \varphi(r_0)^{\frac{1}{\theta}} \leq 1 \text{ for all } r_0 > 0 \Leftrightarrow r_0^\alpha \varphi(r_0)^{\frac{1}{\theta}} \leq \varphi(r_0)^{\frac{1}{\theta}}. \]

Since this is true for every \( t \in \Gamma \) and \( r_0 > 0 \), we are done.

The third statement of the theorem follows from first and second parts of the theorem. \( \square \)

**Remark 5.4.** If we take \( \varphi(r) = r^{\lambda - 1} \) in Theorem 5.9, then condition (5.29) is equivalent to \( 0 < \lambda < 1 - \alpha p \) and condition (5.28) is equivalent to \( \frac{1}{p} - \frac{1}{q} = \frac{\alpha}{1 - \lambda} \). Therefore, from Theorem 5.9 we get Theorem C.

**Remark 5.5.** If we take \( \varphi(r) = [r_1]_{\lambda}^{\beta - 1} \) in Theorem 5.9, then condition (5.29) is equivalent to \( 0 < \lambda < 1 - \alpha \) and condition (5.28) is equivalent to \( \alpha \leq \frac{1}{p} - \frac{1}{q} \leq \frac{\alpha}{1 - \lambda} \). Therefore, from Theorem 5.9 we get Theorem D.

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**References**

[1] R. P. Agarwal, S. Gala, and M. A. Ragusa, *A regularity criterion in weak spaces to Boussinesq equations*, Mathematics 8 (2020), no. 6, 920.

[2] A. Böttcher and Yu. I. Karlovich, *Carleson Curves, Muckenhoupt Weights, and Toeplitz Operators*, Basel, Boston, Berlin: Birkhäuser Verlag, 1997.

[3] A. Böttcher and Yu. I. Karlovich, *Toeplitz operators with PC symbols on general Carleson Jordan curves with arbitrary Muckenhoupt weights*, Trans. Amer. Math. Soc. 351 (1999), 3143–3196.

[4] A. Eridani, V. Kokilashvili, and A. Meskhi, *Morrey spaces and fractional integral operators*, Expo. Math. 27 (2009), no. 3, 227–239.

[5] A. Eroglu and J. V. Azizov, *A note on the fractional integral operators in generalized Morrey spaces on the Heisenberg group*, Trans. Natl. Acad. Sci. Azerb. Ser. Phys. Tech. Math. Sci. 37 (2017), no. 1, 86–91.

[6] V. S. Guliyev, R. V. Guliyev, and M. N. Omarova, *Riesz transforms associated with Schrödinger operator on vanishing generalized Morrey spaces*, Appl. Comput. Math. 17 (2018), no. 1, 56–71.

[7] A. Yu. Karlovich, *Maximal operators on variable Lebesgue spaces with weights related to oscillations of Carleson curves*, Math. Nachr. 283 (2010), no. 1, 85–93.

[8] V. Kokilashvili, *Fractional integrals on curves*, (Russian) Trudy Tbiliss. Mat. Inst. Razmadze Akad. Nauk Gruzin. SSR 95 (1990), 56–70.

[9] V. Kokilashvili and A. Meskhi, *Fractional integrals on measure spaces*, Frac. Calc. Appl. Anal. 4 (2001), no. 1, 1–24.

[10] V. Kokilashvili and S. Samko, *Boundedness of maximal operators and potential operators on Carleson curves in Lebesgue spaces with variable exponent*, Acta Math. Sin. (Engl. Ser.) 24 (2008), no. 11, 1775–1800.

[11] M. A. Ragusa, *Operators in Morrey type spaces and applications*, Eurasian Math. J. 3 (2012), no. 3, 94–109.

[12] M. A. Ragusa and A. Scapellato, *Mixed Morrey spaces and their applications to partial differential equations*, Nonlinear Anal. 151 (2017), 51–65.

[13] M. A. Ragusa and A. Tachikawa, *Regularity for minimizers for functionals of double phase with variable exponents*, Adv. Nonlinear Anal. 9 (2020), no. 1, 710–728.

[14] N. Samko, *Weighted Hardy and singular operators in Morrey spaces*, J. Math. Anal. Appl. 350 (2009), no. 1, 56–72.

[15] C. B. Morrey, *On the solutions of quasi-linear elliptic partial differential equations*, Trans. Amer. Math. Soc. 43 (1938), 126–166.

[16] I. B. Dadashova, C. Aykol, Z. Cakir, and A. Serbetci, *Potential operators in modified Morrey spaces defined on Carleson curves*, Trans. A. Razmadze Math. Inst. 172 (2018), no. 1, 15–29.

[17] V. S. Guliyev, I. J. Hasanov, and Y. Zeren, *Necessary and sufficient conditions for the boundedness of the Riesz potential in modified Morrey spaces*, J. Math. Inequal. 5 (2011), no. 4, 491–506.

[18] V. Kokilashvili and A. Meskhi, *Boundedness of maximal and singular operators in Morrey spaces with variable exponent*, Arm. J. Math. (Electronic) 1 (2008), no. 1, 18–28.
[19] D. R. Adams, *A note on Riesz potentials*, Duke Math. 42 (1975), 765–778.
[20] A. Eroglu and I. B. Dadashova, *Potential operators on Carleson curves in Morrey spaces*, Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat. 66 (2018), no. 2, 188–194.
[21] V. S. Guliyev, *Generalized local Morrey spaces and fractional integral operators with rough kernel*, J. Math. Sci. (N. Y.) 193 (2013), no. 2, 211–227.
[22] V. S. Guliyev, M. N. Omarova, M. A. Ragusa, and A. Scapellato, *Commutators and generalized local Morrey spaces*, J. Math. Anal. Appl. 457 (2018), no. 2, 1388–1402.
[23] H. Armutcu, A. Eroglu, and F. Isayev, *Characterizations for the fractional maximal operators on Carleson curves in local generalized Morrey spaces*, Tbil. Math. J. 13 (2020), no. 1, 23–38.
[24] A. Eridani, M. I. Utoyo, and H. Gunawan, *A characterization for fractional integrals on generalized Morrey spaces*, Anal. Theory Appl. 28 (2012), no. 3, 263–268.
[25] A. Eroglu, V. S. Guliyev, and C. V. Azizov, *Characterizations for the fractional integral operators in generalized Morrey spaces on Carnot groups*, Math. Notes 102 (2017), no. 5, 127–139.
[26] V. Burenkov, A. Gogatishvili, V. S. Guliyev, and R. Mustafayev, *Boundedness of the fractional maximal operator in local Morrey-type spaces*, Complex Var. Elliptic Equ. 55 (2010), no. 8–10, 739–758.
[27] A. Akbulut, V. S. Guliyev, and R. Mustafayev, *On the boundedness of the maximal operator and singular integral operators in generalized Morrey spaces*, Math. Bohem. 137 (2012), no. 1, 27–43.
[28] V. S. Guliyev, *Boundedness of the maximal, potential and singular operators in the generalized Morrey spaces*, J. Inequal. Appl. 2009 (2009), 503948.
[29] E. M. Stein, *Harmonic Analysis: Real-Variable Methods, Orthogonality and Oscillatory Integrals*, Princeton Univ. Press, Princeton, NJ, 1993.