Phase Structure of Higher Spin Black Holes

Abhishek Chowdhury 1, Arunabha Saha 2

Harish-Chandra Research Institute
Chhatnag Road, Jhusi, Allahabad - 211 019, India

Abstract

We revisit the study of the phase structure of higher spin black holes carried out in arXiv: 1210.0284 using the “canonical formalism”. In particular we study the low as well as high temperature regimes. We show that the Hawking-Page transition takes place in the low temperature regime. The thermodynamically favoured phase changes from conical surplus to black holes and then again to conical surplus as we increase temperature. We then show that in the high temperature regime the diagonal embedding gives the appropriate description. We also give a map between the parameters of the theory near the IR and UV fixed points. This makes the “good” solutions near one end map to the “bad” solutions near the other end and vice versa.

1E-mail: abhishek@hri.res.in
2E-mail: arunabha@hri.res.in
1 Introduction

Higher spin theories in various dimensions have been the object of interest for quite sometime now [1, 2, 3, 4, 5, 6]. They have become a very useful arena for studying the nature of AdS/CFT dualities. In [7] and [8, 9] the first example of a higher spin AdS/CFT duality was given. There it was conjectured that a theory of O(N) vector models in $2 + 1$ dimensions is dual to the higher spin theories in $\text{AdS}_4$. In general these theories have a spectrum of spins ranging from 2 to $\infty$ [6].

In 3 dimensions the complexity reduces quite a bit due to the fact that there are no bulk propagating degrees of freedom and due to the related fact that the spectrum can be truncated to any finite maximal spin $N$. In spite of all these simplifications the theories in 3 dimensions serve as good toy models to understand various aspects of both the higher spin theories and also the AdS/CFT dualities. Higher spin theory in 3 space-time dimensions was studied in [10, 11]. In the latter the $\text{SL}(2, R) \times \text{SL}(2, R)$ Chern-Simons formulation for gravity in AdS space was extended to $\text{SL}(N, R) \times \text{SL}(N, R)$ theory and it was shown that the spectrum is that of fields of spin ranging from 2 to $N$. The classical asymptotic symmetry algebra of higher spin theories in $\text{AdS}_3$ has been shown to match with W-symmetry algebra in [12, 13]. See also [14, 15] for further analysis of the asymptotic symmetry algebra. The first indication that the symmetry is present in the quantum regime was provided in [16], where the one loop partition function in the bulk was calculated and shown to be equal to the vacuum character of the W-symmetry enhanced CFTs. Based on this, a duality between higher spin theories in $\text{AdS}_3$ and CFT with W-symmetry was proposed in [17]. Further elaborations of the proposal were done [18, 19, 20, 21] and for a review see [22]. The topologically massive versions of these higher spin theories and their possible dualities to logarithmic CFTs was shown in [23, 24].

1.1 Higher Spin Black Holes in 3 dimensions

In 3 dimensions, the topology of a space-time with asymptotic AdS geometry is a solid torus. The contractible cycle is either spatial or temporal depending on whether we are in a thermal AdS background or a black hole background. In black holes the non-contractible cycle being spatial points towards the existence of a “horizon”. For Euclidean black holes the temperature is defined by assuming that the time cycle is periodic. The periodicity is such that at the horizon there are no conical singularities i.e. the horizon is smooth. This periodicity in time cycle is related to the inverse temperature of the black hole.

In higher spin theories the concept of a metric is blurred by the fact that there are higher spin gauge transformations under which the metric is not invariant. Hence, the normal procedure of identifying black hole geometries to metrics with horizons doesn’t work here. In [25], a procedure to identify the higher spin black hole geometry in $\text{AdS}_3$, in the Chern-Simons formulation was given. There the black hole geometry was identified with those configurations where the connection is smooth in the interior of the torus geometry with a temporal contractible cycle. This is equivalent to demanding a trivial holonomy for the connection along the temporal cycle (i.e. falls in the centre of the gauge group). This ensures that when the contractible cycle is shrunk to zero, the connection comes back to itself after moving around the cycle once. But this does not ensure that the corresponding metric will look like that of an ordinary
black hole. In [26] a gauge transformation was found in which the metric obtained resembled that of a conventional black hole. It was also shown that the RG flow by an irrelevant deformation triggered by a chemical potential corresponding to a spin 3 operator takes us from the principle embedding of $SL(2, R)$ to the diagonal embedding of $SL(2, R)$ in $SL(3, R)$. Also to get a a higher spin black black hole a chemical potential corresponding to the independent charges had to be added so that the system is stable thermodynamically. So, in higher spin theories a black hole solution also causes the system to flow from one fixed point to another. In [27] the partition function for the black hole solution was obtained as a series expansion in spin 3 chemical potential with $hs[\lambda] \times hs[\lambda]$ algebra (this gives the higher spin symmetry algebra when the spin is not truncated to any finite value) and matched with the known CFT results for free bosonic ($\lambda = 1$) and free fermionic case ($\lambda = 0$). This answer also matches the one for general $\lambda$ obtained from CFT calculations in [33]. A review of these aspects of black holes in higher spin theories can be found in [28].

A different approach to study the thermodynamics of these black holes was carried out in [29, 30, 32, 31]. A good variational principle was obtained by adding proper boundary terms to Chern Simons theories on manifolds with boundaries. The free energy was obtained from the on-shell action and an expression for entropy was obtained from that. This expression for entropy was different from that obtained in [25]. It was also shown that the stress energy tensor obtained from the variational principle mixes the holomorphic and antiholomorphic components of the connection. This formalism for obtaining the thermodynamics variables is referred to as the “canonical formalism” in the literature. The CFT calculations done in [33] seem to match with the “holomorphic formalism” given in [25], but the canonical approach seems to be much more physically plausible. In [34] a possible solution to this discrepancy was suggested, where they changed the bulk to boundary dictionary in a way suited to the addition of chemical potential which deforms the theory.

In [36], the process of adding chemical potential was unified for the full family of solutions obtained by modular transformation from the conical defect solution. The black holes that we talked about is only one member of the family. It was shown that the same boundary terms need to be added to the action to get a good variational principle for all members of the family. The definitions for all thermodynamic quantities for any arbitrary member of the family were obtained there.

1.2 Phase structure of higher spin black holes in $AdS_3$

The phase structure of spin 3 black holes in $AdS_3$ was studied in [34] using the holomorphic variables. In the principle embedding (with spectrum consisting of fields with spin 2 and 3) of $SL(2, R)$ in $SL(3, R)$, they found 4 solutions to the equations corresponding to a trivial holonomy along the time circle. They relaxed the condition that the spin 3 charge has to approach 0 as the corresponding chemical potential is taken to 0. It was shown there that of the 4 branches one is unphysical as its entropy is negative. Of the remaining three branches one is the BTZ branch (here spin 3 charge goes to zero as chemical potential goes to 0), one is the extremal branch (having a non-zero charge configuration at zero temperature) and a third branch. The negative specific heat of the extremal branch makes it an unstable branch.

The phase diagram given there shows that the BTZ and extremal branch exist only in the low temperature regime, after which the thermodynamic quantities for this two
branches do not remain real. The third branch that is present has real thermodynamic variables at all temperatures. It is shown there that in the low temperature regime only the BTZ branch has the expected scaling behavior while the third branch does not have the correct scaling behavior at high temperature. By scaling behavior they meant that charges and other thermodynamic quantities have the right power law behavior with temperature, e.g. spin 3 charge $\propto T^3$ etc. From the free energy perspective the BTZ branch dominates over all other branches. In the region of existence of the BTZ and extremal branch, the extremal branch dominates over the third branch and at high enough temperature only the third branch survives. The free energy of the unphysical 4th branch is greatest among all the branches.

They then argued that the correct thermodynamics at high temperature is given by the diagonal embedding. The diagonal embedding is obtained as the end point of the RG flow initiated by the addition of chemical potential corresponding to the spin 3 charge in the principle embedding. The spectrum in the diagonal embedding has a pair of fields with spin $\frac{3}{2}$, a pair with spin 1 and a spin 2 field. It also has a chemical potential corresponding to the spin $\frac{3}{2}$. The scaling behavior of the thermodynamic quantities was found to be correct at high temperature for this embedding. Near the zero chemical potential (for spin $\frac{3}{2}$) limit, i.e. near the end point of the RG flow, the holonomy equations have 2 real solutions. Of them the one with the lower free energy was conjectured to be the “third branch” at the end of the RG flow. Since that branch has the correct scaling behaviour it was argued that after the point where BTZ and extremal branches of the principle embedding cease to exist the third branch takes over and it is actually the black hole solution in the diagonal embedding.

In summary they showed that the principle embedding is the correct IR picture valid at low temperature regime and diagonal embedding is the correct UV picture valid in the high temperature regime.

1.3 Our Work

In this work we study the phase structure of $Sl(3, R) \times Sl(3, R)$ higher spin system in the canonical formalism. We will first work in the principle embedding. We will use the definition of thermodynamic quantities for conical surplus solution (which go to the thermal AdS branch when chemical potential and spin 3 charges are taken to zero) given in [36]. The conical surplus has a contractible spatial cycle. So, we demand that the holonomy of connection along this cycle be trivial. Using this condition we are able to get the parameters of the theory (The undeformed spin 2 and 3 charge) in terms of temperature and chemical potential for spin 3 charge. We use this to study the phase structure of the conical surplus. From the phase diagram that we plot we see that there are 2 branches of thermodynamic parameters for a given temperature and chemical potential. One of the branch reduces to the thermal AdS branch when the chemical potential ($\mu$) is taken to zero. The other one is a new branch which like the extremal black holes has a non-trivial charge configuration even when $\mu \rightarrow 0$. This we call the “extremal thermal AdS” branch. This extremal branch has lower free energy for all values of $\mu$ and $T$.

We then study the phase structure of black hole in this embedding. Here the contractible cycle is the temporal cycle and using this condition, we again solve for the parameters of the theory in terms of $\mu$ and temperature ($T$). Here we get 4 branches of solutions. We see that two of these branches are unphysical (we call them the bad
branches) because they have negative entropy. One among the other two branches is the BTZ branch (which reduces to BTZ black hole when chemical $\mu \to 0$) and the other branch is the extremal branch (having a non-trivial charge configuration at $T = 0$). The extremal branch has negative specific heat and hence is unstable and might decay. The BTZ branch is stable. The BTZ branch has lower specific heat and is the dominant of the two good solutions. For a given chemical potential both the black hole and the thermal AdS solutions exist till a certain value in temperature (different for the black hole and thermal AdS). Crossing the temperature leads to imaginary thermodynamic quantities.

We then undertake a study of the phase structure for the conical surplus and black hole together. Between the 3 solutions- the BTZ black hole, the extremal branch and the thermal AdS branch we study which branch has the minimum free energy for a given chemical potential and temperature. We notice that for a particular chemical potential, at very low temperature the thermal AdS has the lowest free energy and then as we gradually increase the temperature the BTZ branch starts dominating over the thermal AdS. After a particular temperature the extremal black hole also dominates over the thermal AdS though it is sub-dominant to the BTZ branch. This is the analogue of Hawking-page transition. Still increasing the temperature further the black holes cease to exist and the thermal AdS is the only solution. In all this we have not considered the extremal thermal AdS branch which has the lowest free energy of all the branches. If this branch is not absent (due to some physical reasons that we are unaware of) it will be the dominant branch all through the low temperature regime. We will comment further about this branch later in the paper.

Next part of our study involves studying black holes in the diagonal embedding of $SL(2, R)$ in $SL(3, R)$. There is a consistent truncation where the spin $\frac{3}{2}$ fields are put to zero [40]. But here we don’t want to do this. The reason being that we want to use the fact that this diagonal embedding is actually the UV limit of the flow initiated in principal embedding by the spin 3 chemical potential. We want to study the full theory obtained from this procedure and there all the above mentioned fields are present. First of all we will be able to give a map between the parameters of the theory at UV and IR fixed points. Also, here we obtain 4 solutions to the holonomy equations and by similar arguments as above two of them are unphysical. Of the other two branches the one with the lower free energy is throughout stable. We also showed that the good solution near one end point actually maps to the bad solution in the other end and vice versa. We give a plausible reasoning for this mapping between the good and bad branches.

### 1.4 Organization of the paper

In section 2 we give a brief review of the geometry of higher spin theories and their thermodynamics. In the section 3 we give the analysis for the thermodynamics of conical surplus, black hole and Hawking-Page transition for principle embedding. In section 4 we give the description for black hole in the diagonal embedding. Lastly we give a summary of our results in section 5 and some possible directions for future studies in 6.
2 Review of higher spin geometry in $AdS_3$ and thermodynamics

Let us briefly elaborate on the ‘Canonical formalism’ for BTZ Black Holes in higher spin scenarios. We will mostly follow the conventions given in [30, 36]. In 2 + 1 dimensions Einstein gravity with negative cosmological constant can be written as a Chern-Simons theory with gauge group $G \simeq SL(N, \mathbb{R}) \times SL(N, \mathbb{R})$ [11]. For $N = 2$ it reduces to ordinary gravity but for $N \geq 3$ depending on possible embeddings of the $SL(2, \mathbb{R})$ subgroup into $SL(N, \mathbb{R})$ it generates a spectrum of higher spin fields. We are mostly interested in an Euclidean Chern-Simons theory on a three-dimensional manifold $M$ with the topology $S^1 \times D$ where the $S^1$ factor is associated with the compactified time direction and $\partial D \simeq S^1$. It is customary to introduce coordinates $(\rho, z, \bar{z})$ on $M$, where $\rho$ is the radial coordinate and $\rho \to \infty$ is the boundary with the topology of a torus where the $z, \bar{z}$ coordinates are identified as $z(\bar{z}) \simeq z(\bar{z}) + 2\pi \simeq z + 2\pi \tau(\bar{\tau})$. For Chern-Simons theory the field strength is zero, so the connection is pure gauge. We will be working in a gauge where the connections have a radial dependence given by

$$A = b^{-1}db + b^{-1}ab \quad \bar{A} = bdb^{-1} + b\bar{a}b^{-1}$$

with $b = b(\rho) = e^{\rho L_0}$ and $a, \bar{a}$ being functions of boundary $z, \bar{z}$ coordinates only.

The holonomies associated with the identification along the temporal direction are

$$Hol_{\tau, \bar{\tau}}(A) = b^{-1}e^{\bar{h}b} \quad Hol_{\tau, \bar{\tau}}(\bar{A}) = be^{\bar{h}b^{-1}}$$

(1)

where the matrices $h$ and $\bar{h}$ are

$$h = 2\pi(\tau a_z + \bar{\tau a}_z) \quad \bar{h} = 2\pi(\bar{\tau a}_z + \tau a_z)$$

(2)

Triviality of the holonomy forces it to be an element of the center of the gauge group and a particularly interesting choice which corresponds to the choice for uncharged BTZ black hole gives

$$Tr[h \cdot h] = -8\pi^2 \quad Tr[h \cdot h \cdot h] = 0$$

(3)

A different choice of the center element is synonymous to a scaling of $\tau$ and hence is not very important in the context of studies that we undertake in this work.

With this setup in mind the Euclidean action is

$$I^{(E)} = I^{(E)}_{CS} + I^{(E)}_{Bdy}$$

where

$$I^{(E)}_{CS} = CS[A] - CS[\bar{A}], \quad CS[A] = \frac{ik_{cs}}{4\pi} \int_M Tr[A \wedge dA + \frac{2}{3} A \wedge A \wedge A]$$

For a good variational principal on the manifold we need to add some boundary terms to the above action. To get a variation of action of the form $\delta I \sim Q_i \delta \mu_i$ we need to add a boundary term of the form

$$I^{(E)}_{Bdy} = -\frac{k_{cs}}{2\pi} \int_{\partial M} d^2z \ Tr[(a_z - 2L_1)a_{\bar{z}}] - \frac{k_{cs}}{2\pi} \int_{\partial M} d^2z \ Tr[(\bar{a}_{\bar{z}} - 2L_{-1})a_z]$$
We will be interested in an asymptotically AdS boundary which will give rise to the \( W_N \) algebra as the asymptotic symmetry algebra in the absence of any chemical potential. This is satisfied by the connections written in the Drinfeld-Sokolov form
\[
a = (L_1 + Q) \, dz - (M + \ldots) \, d\bar{z} \\
\bar{a} = (L_{-1} - \bar{Q}) \, d\bar{z} + (\bar{M} + \ldots) \, dz
\]
with \( [L_{-1}, Q] = [L_1, M] = 0 \) (and similarly for \( \bar{Q}, \bar{M} \)). We adopt a convention that the highest (lowest) weights in \( a_z (\bar{a}_z) \) are linear in the charges, and the highest (lowest) weights in \( \bar{a}_z (a_z) \) are linear in the chemical potentials corresponding to spin 3 and higher charges with the definition (in principal embedding). The convention for definition of chemical potential that we use is given by
\[
\text{Tr} \left[ (a_z - L_1) (\bar{\tau} - \tau) a_z \right] = \sum_{i=3}^{N} \mu_i Q_i \\
\text{Tr} \left[ (-\bar{a}_z + L_{-1}) (\bar{\tau} - \tau) \bar{a}_z \right] = \sum_{i=3}^{N} \bar{\mu}_i \bar{Q}_i
\]
Varying \( I^{(E)} \) on-shell we arrive at
\[
\delta I_{os}^{(E)} = - \ln Z = -2\pi i k_{cs} \int_{\partial M} \frac{d^2 z}{4\pi^2 Im(\tau)} \left[ \text{Tr} \left[ (a_z - L_1) \delta (\bar{\tau} - \tau) a_z \right] + \left( \frac{a_z^2}{2} + a_z a_{\bar{z}} - \frac{\bar{a}_z^2}{2} \right) \delta \tau \\
- \left( -\bar{a}_z + L_{-1} \right) \delta (\bar{\tau} - \tau) \bar{a}_z \right] + \sum_{i=3}^{N} (Q_i \delta \mu_i - \bar{Q}_i \delta \bar{\mu}_i)
\]
where \( T \) and \( Q_i \)’s are the zero modes of the stress tensor and Spin \( \geq 3 \) currents respectively. So, the added boundary terms are the correct one as we get the desired variation of the action on-shell.

The black hole geometry that we discussed can be obtained by a \( SL(2, Z) \) modular transformation acting on a conical surplus geometry and vice versa. This property was used to show in [36] that the variational principle for either geometry (or for that matter any geometry obtained by a \( SL(2, Z) \) transformation on the conical surplus geometry) goes through correctly if we use the boundary terms given above. This in principle means that we have the same definition of stress tensor for all the members of the ’\( SL(2, Z) \)’ family and is given by
\[
\mathcal{T} = \text{Tr} \left[ \frac{a_z^2}{2} + a_z a_{\bar{z}} - \frac{\bar{a}_z^2}{2} \right], \quad \bar{\mathcal{T}} = \text{Tr} \left[ \frac{\bar{a}_z^2}{2} + \bar{a}_z \bar{a}_z - \frac{a_z^2}{2} \right]
\]
The on-shell action evaluated for a particular member gives the free energy for that member. The information about the family for which this is evaluated comes about in this way. The identification \( z \simeq z + 2\pi \tau \) is problematic when you want to evaluate the on-shell action as \( \tau \) is the chemical potential corresponding to the stress tensor.
So, you need to go to coordinates whose periodicity is independent of the parameters of the theory and you push back the information regarding the chemical potential in periodicities to some explicit dependence of metric on these. This procedure of making periodicities of coordinates constant is dependent upon which member of the \( SL(2, \mathbb{Z}) \) family one is working with. The free energy for any arbitrary member was evaluated in [36] and is given by

\[
-\beta F = -I_{\text{on-shell}} + 2\pi i k_{cs} \text{Tr} \left[ (h_A h_B - \bar{h}_A \bar{h}_B) - 2i(a_z - 2L_1)a_{\bar{z}} - 2i(\bar{a}_z - 2L_{-1})\bar{a}_\bar{z} \right] \tag{10}
\]

where \( h_A \) and \( h_B \) are respectively the holonomy along the contractible and non-contractible cycles.

Performing a Legendre transform of the free energy (i.e. from a function of chemical potentials/sources to function of charges) we arrive at an expression for the entropy. The expression for entropy of the black hole solution turns out to be

\[
S = -2\pi i k_{cs} \text{Tr} \left[ (a_z + a_{\bar{z}})(\tau a_z + \bar{\tau} a_{\bar{z}}) - (\bar{a}_z + \bar{a}_\bar{z})(\tau \bar{a}_z + \bar{\tau} \bar{a}_\bar{z}) \right] \tag{11}
\]

The entropy for a conical surplus turns out to be zero which is along expected lines as conical surplus is a geometry without any horizons.

All the things stated above can very easily be generalized to non-principal embedding. The things that will be different will be the value of \( k \) which is the label associated with the \( SL(2, \mathbb{R}) \) embedding in \( SL(3, \mathbb{R}) \) for different embeddings and their definition of charges and chemical potentials. The value of \( k \) is related to \( k_{cs} \) by

\[
k_{cs} = \frac{k}{2 \text{Tr} [\Lambda^0 \Lambda^0]}, \tag{12}
\]

where \( k_{cs} \) is the label associated with the \( SL(3, \mathbb{R}) \) CS theory. The central charge of the theory for a particular embedding is given by \( c = 6k \). \([\Lambda_{-1}, \Lambda_0, \Lambda_1]\) are the generators giving rise to the \( Sl(2, \mathbb{R}) \) sub-algebra in the particular embedding.

3 The principal embedding for \( Sl(3, \mathbb{R}) \)

We give the conventions for connections and the thermodynamic quantities that we use in our paper here. The connection that we use here is based on [25, 36, 30]. We will confine our connections in the radial gauge and use the conventions for generators of \( SL(3, \mathbb{R}) \) given in [28].

\[
a = (L_1 - 2\pi \mathcal{L}L_{-1} - \frac{\pi}{2} \mathcal{W}W_{-2})dz + \frac{mT}{2}(W_2 + 4\pi \mathcal{W}L_{-1} - 4\pi \mathcal{L}W_0 + 4\pi^2 \mathcal{L}^2 W_{-2})d\bar{z},
\]

\[
\bar{a} = (L_{-1} - 2\pi \bar{\mathcal{L}}L_1 - \frac{\pi}{2} \bar{\mathcal{W}}\bar{W}_2)d\bar{z} + \frac{\bar{m}T}{2}(W_{-2} + 4\pi \bar{\mathcal{W}}L_{-1} - 4\pi \bar{\mathcal{L}}W_0 + 4\pi^2 \bar{\mathcal{L}}^2 W_2)dz.
\]

We are interested in studying only non-rotating solutions, hence we require \( g_{zz} = g_{\bar{z}\bar{z}} \) for the metric which when converted to the language of connections in radial gauge becomes

\[
\text{Tr}[a_z a_{\bar{z}} - 2a_z \bar{a}_z + \bar{a}_z a_{\bar{z}}] = \text{Tr}[a_z a_{\bar{z}} - 2a_z \bar{a}_z + \bar{a}_z a_{\bar{z}}]. \tag{14}
\]

\(^{3}\)Here we redefine our variables to absorb the \( k \) appearing in the connections given in [30].
With our convention this is satisfied if $\bar{m} = -m$, $\bar{W} = -W$, $\bar{L} = L$.

These connections automatically satisfy the equations of motion $[a_z, a_{\bar{z}}] = 0$. With our conventions the equation (6) becomes

$$Tr[(a_z - L_1) a_{\bar{z}} (\overline{\tau} - \tau)] = 4im\pi W.$$  

So, demanding that $W$, which is the measure of spin 3 charge in our conventions be real, the chemical potential $\mu_3$ is imaginary, whose measure is given by ‘$im$’.

### 3.1 The Conical Surplus Solution

Here as stated above the contractible cycle is spatial and hence we demand that the holonomy of connection defined as $h = 2\pi(a_z + a_{\bar{z}})$ to be trivial along the contractible cycle. It follows the same holonomy equation as that given in (3). This choice of center is the same as that for thermal AdS.

The boundary terms that we use (given by the equation above the Drienfeld-Sokolov connection in (4)) is suited for a study in grand canonical ensemble where, the chemical potentials are the parameters of the theory.

The first of the 2 holonomy equations in (3) can be used to get $W$ in terms of $L$

$$W_{CS} = \frac{1}{12m\pi T} + \frac{2\mathcal{L}}{3mT} + \frac{16}{9}m\pi\mathcal{L}^2T$$

and then put in the second equation to get an equation for $L$ in terms of $m$ and $T$ given by

$$-\frac{1}{mT} - \frac{8\pi\mathcal{L}}{mT} + \frac{8m\pi\mathcal{L}T + 64m\pi^2\mathcal{L}^2T^2}{3} + \frac{128}{9}m^3\pi^2\mathcal{L}^2T^3 - \frac{512}{3}m^3\pi^3\mathcal{L}^3T^3 + \frac{4096}{27}m^5\pi^4\mathcal{L}^4T^5 = 0.$$  

The stress energy tensor of equation (9) in this case is given by

$$\mathcal{T}_{CS} = 8\pi\mathcal{L} - 12m\pi WT - \frac{64}{3}m^2\pi^2\mathcal{L}^2T^2,$$  

and the spin 3 charge $W$ is left unchanged from its definition in absence of chemical potential. The free energy given in equation (10) in this case becomes

$$F_{CS} = 16\pi\mathcal{L} - 8m\pi WT - \frac{128}{3}m^2\pi^2\mathcal{L}^2T^2.$$  

Now there are 4 solutions to (17) out of which only 2 turn out to be real solutions. Using this we can get the solutions for $\mathcal{T}_{CS}$ and $F_{CS}$ in terms of $m$ and $T$. From equation (17) we see that all relevant quantities are functions of $\mu_c = mT$ and hence we plot them in terms $\mu_c$ in figure (1).

From the figure (1) we see that the blue branch is the branch that goes to thermal AdS (without charges) when $\mu_c \rightarrow 0$. The other branch which is red is a special branch where as $\mu_c \rightarrow 0$ we have $\frac{\sqrt{W}}{L} = -\frac{1}{6(2\pi)^\frac{1}{2}}$. This special branch starts from an “extremal
point” analogous to black holes discussed in [25] and [37]. Let us call it, the “extremal branch”. This is a bit of a misnomer as for thermal AdS in any gauge there is no concept of horizon. We will discuss a little bit more about it in section (5). The two branches merge at the value of the parameter \( \mu_c = \frac{4}{3} \sqrt{3} + 2\sqrt{3} \), and after that the conical surplus solution ceases to exist.

### 3.2 The black hole solution

The black hole solution is obtained by demanding that the time circle is contractible and holonomy defined in equation (11) satisfy the equations in (3). The holonomy equations in this case are

\[
2 - \frac{32m^2\mathcal{L}^2}{3} - \frac{4\mathcal{L}}{\pi T^2} - \frac{6m\mathcal{W}}{\pi T} = 0,
- \frac{128}{9}m^3\mathcal{L}^3 + \frac{6m^3\mathcal{W}^2}{\pi} + \frac{3\mathcal{W}}{2\pi^2 T^3} + \frac{16m\mathcal{L}^2}{\pi T^2} + \frac{12m^2\mathcal{L}\mathcal{W}}{\pi T} = 0. \tag{20}
\]

Using the same procedure as after (16) we get the final holonomy equation for the black hole as

\[
\frac{4m\mathcal{L}}{3} - \frac{64}{3}m^3\mathcal{L}^3 - \frac{\mathcal{L}}{m\pi^2 T^4} + \frac{1}{2m\pi T^2} + \frac{8m\mathcal{L}^2}{\pi T^2} + \frac{2}{3}m\pi T^2 - \frac{64}{9}m^3\pi^2 T^2 + \frac{512}{27}m^5\pi^4 T^2 = 0. \tag{21}
\]

The free energy of equation (11) in this case is given by

\[
F_{BH} = -16\pi\mathcal{L} - 8m\pi\mathcal{W}T + \frac{128}{3}m^2\pi^2 \mathcal{L}^2 T^2. \tag{22}
\]

The entropy defined in equation (11) in our case becomes

\[
S = \frac{32\pi\mathcal{L}}{T} - \frac{256}{3}m^2\pi^2 \mathcal{L}^2 T. \tag{23}
\]
We see that equation (21) is an equation for \( l = \frac{c}{T^2} \) in terms of \( \mu_b = mT^2 \). So, \( \mu_b \) is a good variable to study the phase structure for the black hole. The phase diagram for spin 3 black hole is given in figure (2). We denote the 4 branches of solution here with the following color code- branch-1-Blue, branch-2-Red, branch-3-Orange and branch-4-greeen.

Figure 2: This figure gives the phase structure for spin 3 black hole. The horizontal axis is \( \mu_b \) and the vertical axis on the upper panel are respectively the spin 3 charge \( W_{BH} \) and stress tensor \( T_{BH} \) and in the lower panel are free energy \( F_{BH} \) and entropy \( S \) respectively.

From the plots we see that branches 3 and 4 are unphysical with negative entropy. Branches 1 and 2 merge at the point \( \mu_b = \frac{3\sqrt{-3+2\sqrt{3}}}{8\pi} \). Beyond this point the black hole solutions cease to exist. For branch 2 the stress tensor decreases with \( \mu_d = mT^2 \), i.e. it decreases with \( T^2 \) if we keep chemical potential \( m \) fixed, so this branch has negative specific heat and hence is unstable. So, the branches 1 and 2 correspond respectively to the large (stable) and small (unstable) black hole solutions in AdS space [35] [37]. In

\[ t = \frac{T}{T^2}, \quad w = \frac{W}{T^2}, \quad f = \frac{F}{T^2}, \quad s = \frac{S}{T} \]

(24)
Figure 3: Comparison between free energy of black hole and conical surplus at m=1. The blue branch is the BTZ branch of black hole and red branch is extremal branch. The brown branch is the conical surplus branch which goes to pure AdS in absence of chemical potential and the green branch is the new “extremal branch” of conical surplus.

3.3 The “Hawking-Page” Transition

We will now study a phase transition first studied for Einstein theory on $AdS_4$ in [38]. There it was shown that in asymptotically AdS space, out of the two phases 1) a gas of gravitons and 2) a black hole, the former dominates at low temperature and after a particular temperature the black hole solution becomes more dominant. The dominant phase was obtained by identifying which solution had the lowest free energy for a particular temperature. Both pure AdS (gas of gravitons) and black hole were put at the same temperature by keeping the identification of the time circle at the same value. The free energy was calculated by calculating the on shell action in Euclidean signature with proper boundary terms added. For the case of $AdS_3$ the thermal AdS and BTZ black hole configurations are related by a modular transformation $\tau_{BTZ} = -\frac{1}{\tau_{AdS}}$. At the point of Hawking-Page transition i.e. $\tau_{BTZ} = \tau_{AdS}$, $T = \frac{1}{2\pi}$ (putting the AdS radius to unity). We are studying the phase structure in a grand canonical ensemble and we will try to find out the regions in parameter space where this phase transition takes place.

At $m = 0$ the temperature at which transition takes place is $T = \frac{1}{2\pi}$. Let us introduce a chemical potential for spin 3 and see how the temperature deviates from this point. For the moment we will study the branches which go to BTZ black holes and thermal AdS in the limit $m \rightarrow 0$ i.e. the branch 1 in both cases. We will assume the following form for the transition temperature after the introduction of a non zero chemical potential.
chemical potential $m$.

$$T = \frac{1}{2\pi} + \#_1 m + \#_2 m^2 + \#_3 m^3 + \#_4 m^4 + \ldots \ .$$ \hspace{1cm} (25)

We will find the difference between the free energy of black hole given in equation (22) and that of conical surplus (the thermal AdS upon introduction of chemical potential) given in equation (19), both the configuration being put at the same temperature and chemical potential. We then find where this difference is zero which will give the various coefficients in (25) order by order. Upon doing this we arrive at the following temperature where the transition takes place to $O(m^6)$

$$T_{HP} = \frac{1}{2\pi} - \frac{1}{12\pi^3} m^2 + \frac{7}{144\pi^5} m^4 - \frac{71}{1728\pi^7} m^6 + \ldots \ .$$ \hspace{1cm} (26)

For a chemical potential given by $m = 1 \frac{1}{5}$ we plot the free energies of both the black hole and the conical surplus in figure (3). The color coding is explained in the caption there. We see that the ”thermal AdS branch” conical surplus dominates over the black hole for low temperature and the BTZ branch black hole solutions take over at higher temperatures. The unstable (extremal) black hole always is the sub-dominant contribution to the free energy compared to BTZ branch. The extremal black hole branch also starts dominating over the thermal AdS as we increase the temperature further. Beyond the temperature of existence of the black hole the ”thermal AdS branch” conical surplus is the only solution available. This phase transition can be explained by a physical argument based on the fact that all even spin fields are self-attractive and all odd spin fields are self-repulsive. So, at very low temperature when there are very few excitations the thermal AdS is the dominating solution. As we increase the temperature the number of excitation of both the spin 2 and 3 fields increase but the attractive nature of spin 2 field dominates and formation of a black hole is more favourable. Further increasing the temperature causes the number of excitations to increase further and the repulsive nature of spin 3 dominating over the attractive nature of spin 2 and makes it unfavourable to form a black hole. The only issue of concern is that the “extremal” conical surplus solution seems to dominate over all solutions in the low temperature regime. We will try to shed some more light on this issue later.

We numerically give the region where the Conical Surplus solution dominates, where Black hole solution dominates as well as where both this solution exist together in the figure (4). We see that the temperature where the ”Hawking-Page” transition takes place is lower for higher values of chemical potential.

From the figure (4) we see that for high enough chemical potential at any temperature the black hole solution ceases to exist and only the conical surplus is the dominant solution. To put this in perspective we plot the region of existence of the black hole and conical surplus in the parameter space on the right hand side of figure (4). The

---

5This is for the purpose of illustration only as this helps in bringing out all the features nicely in a single diagram. Since introduction of chemical potential violates the boundary falloff conditions we want $m << 1$ if we want the theory to be studied for high enough temperature as the deformation is by a term of the form $mTW_2$

6This was brought to our notice by Arnab Rudra and the physical argument rose from a discussion with him.
Figure 4: In these figures the x axis represents temperature (T) and y axis the chemical potential (m). For the upper figure pink region is where the conical surplus dominates and the blue region indicates where black hole dominates. The boundary between this two regions represents the temperature where the “Hawking-Page” transition takes place for a particular chemical potential. The figure on right represents the region of existence of conical surplus and black holes solutions. The black hole solutions exist in the region bound by the axes and the blue line boundary and the conical surplus solution exists in the full coloured region.
region of existence of the conical surplus is much larger (the full coloured region) than the black hole (region bounded by the axes and the blue line boundary). Beyond the region of existence of the black hole solution the conical surplus is the only solution valid in the low temperature regime of this theory.

In all this we have to be careful of the fact that since introducing a chemical potential corresponds to breaking the asymptotic AdS boundary conditions. The asymptotic AdS falloff conditions which gives rise to the Virasoro symmetry algebra is 
\[ A - A_{\text{AdS}} = O(1), \]
but by introducing a chemical potential this gets broken down to
\[ A - A_{\text{AdS}} = mT e^{2\mu}. \]
So, the definition of charges that we are using are not valid if we move too far away from the fixed point. Since, we want to study the property of the system for high enough temperatures we have to confine ourselves to very small values of chemical potential. Also introduction of this deformation induces a RG flow which takes us to another non trivial fixed point in the UV with a completely different spectrum, to be studied next. Hence for large values of \( m \) the parameters of the UV fixed point may be the correct parameters to use.

Across the point of transition we see that not only does the stress energy tensor changes sign which was expected, but the spin 3 charge also changes sign. This can be inferred from the fact that in the allowed regime for conical surplus the spin 3 charge is always positive which can be seen from figure 1, and that for black hole it is always negative as can be seen in figure 2.

4 The diagonal embedding for \( SL(3) \)

The definition of \( SL(2,R) \) sub-algebra generators in diagonal embedding in terms of generators of principal embedding is given in [37] [26] by
\[ \varphi, \vartheta, \varsigma, \varepsilon, \lambda, \mu, \nu. \] The spectrum here consists of fields of spin 2, spin \( \frac{3}{2} \) and spin 1. The generators for spin \( \frac{3}{2} \) multiplet in the bulk are given by \( (W_1, L_{-1}) \) and \( (W_{-1}, L_1) \) and those for spin 1 are \( W_0 \). The on-shell connection for this theory is given by
\[
\begin{align*}
a &= \left( \frac{1}{4} W_2 + \mathcal{G} L_{-1} + \mathcal{J} W_0 + \mathcal{J}^2 W_{-2} \right) dz + \frac{\lambda T_d}{2} \left( L_1 + 2 \mathcal{J} L_{-1} - \frac{\mathcal{G}}{2} W_{-2} \right) d\bar{z} \\
\bar{a} &= -\frac{\bar{\lambda} T_d}{2} \left( L_{-1} + 2 \bar{\mathcal{J}} L_1 - \frac{1}{2} \bar{\mathcal{G}} W_2 \right) dz - \left( \frac{1}{4} W_{-2} + \mathcal{G} L_1 + \mathcal{J} W_0 + \mathcal{J}^2 W_2 \right) d\bar{z}
\end{align*}
\]

The non rotating condition [14] applied here gives
\[
\mathcal{G} = -\mathcal{G}, \quad \mathcal{J} = \mathcal{J}, \quad \bar{\lambda} = -\lambda
\] (28)

Though this embedding looks like an independent theory by itself. But in [26] it was shown that after adding a deformation with chemical potential corresponding to spin \( \frac{3}{2} \) (\( \lambda \) above), this theory becomes the correct UV behaviour of a theory whose behaviour near IR fixed point is given by the principal embedding studied earlier. If we reintroduce the radial dependence in (27) the leading term comes from \( \frac{1}{4} W_2 \). So, the way to go to the UV theory from the IR side is to change the coefficient of \( W_2 \) in \( \bar{z} \).

\footnote{We have reintroduced the radial dependence by \( A = b^{-1} db + b^{-1} ab \)}
component of connection in equation (13) from \( \frac{mT}{2} \) to \( \frac{1}{4} \) by a similarity transformation (also found out in [37]).

\[
\begin{align*}
\tilde{a}_{UV}^z &= e^{xL_0} a_z^R e^{-xL_0}, \\
\tilde{a}_{UV}^R &= e^{-xL_0} a_z^L e^{xL_0},
\end{align*}
\]
where \( x = \ln(\sqrt{2mT}) \), (29)

where \( a^{UV} \) is the connection given in equation (27) and \( a^{IR} \) is the one given in equation (13). We see from the map given in (29) that the holomorphic and anti holomorphic components change into each other in going from the IR to UV picture. Demanding that equation (29) holds we get a relation between parameters of the undeformed theories in the UV and IR like in [37] given by

\[
G = 2\sqrt{2\pi}W(mT)\frac{3}{2}, \quad J = -2\pi\mathcal{L}mT, \quad \lambda T_d = \frac{\sqrt{2}}{mT}
\]

The holonomy equation calculated here as in the case of principle embedding is given by

\[
-\frac{8J^3}{3\pi^2T_d^2} + \frac{2J}{3\pi T_d} - \frac{64J^4}{27\pi^3T_d^5\lambda^2} + \frac{32J^2}{9\pi T_d^3\lambda^2} - \frac{4\pi}{3T_d\lambda^2} - \frac{J^2\lambda^2}{4\pi - \frac{J^2T_d\lambda^4}{8\pi^3}} = 0
\]

The value of spin \( \frac{3}{2} \) charge is obtained in terms of the spin 1 field using the holonomy condition and is given by

\[
G_{\text{diag}} = -\frac{16J^2}{9T_d\lambda} + \frac{4\pi^2T_d}{3\lambda} + \frac{2JT_d\lambda}{3}.
\]

The holonomy equation should also evolve along the RG flow from IR to UV, i.e. the holonomy equation (31) with the variable transformation given in (34) should reduce to (21). This happens if over and above the above transformation we assume that the definition of temperature on both limits is the same i.e, \( T_d = T \) and the chemical potentials are related by \( \lambda = \frac{\sqrt{T}}{T_d^{\sqrt{mT}}} \).

The definition of the thermodynamic quantities in terms of connection are the same as they were for principal embedding given in [30, 36]. Here their definition in terms of the parameters of diagonal embedding are given by

\[
\begin{align*}
T_{\text{diag}} &= \frac{16J^2}{3} - 3\lambda GT_d^2 \lambda^2, \\
F_{\text{diag}} &= \frac{-32J}{3} + 4\lambda GT_d^2 \lambda^2, \\
S_{\text{diag}} &= \frac{64J^2}{3T_d} + 8\lambda GT_d^2 
\end{align*}
\]

The equation (31) is an equation for \( \frac{J}{T_d} \) in \( l = \lambda\sqrt{T_d} \). So, The correct parameter for drawing phase diagram is \( l \) and the quantities which are a function of \( l \) only, are

\[
\begin{align*}
g &= \frac{G}{T_d^2}, \quad j = \frac{J}{T_d}, \quad t = \frac{T}{T_d^2}, \quad s = \frac{S_{\text{CS}}}{T_d}, \quad f = \frac{F_{\text{CS}}}{T_d^2}
\end{align*}
\]

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Figure 5: Phase structure for spin 3 black hole in diagonal embedding. The horizontal axis is $l$ parameter that we used.
In the phase diagram for black holes given in figure (5) the 4 branches of solution are color coded as branch 1-Blue, branch 2-Red, branch 3-Orange and branch 4-Green. From the phase diagram we see that branches 3 and 4 are unphysical because they have negative entropy. If we assume that the chemical potential is fixed at some value then these are plots with respect to square root of temperature. So, if somewhere the gradient of stress tensor is negative then in those region it decreases with temperature and hence the system has negative specific heat. The 2nd branch has a region of negative specific heat for lower temperature but at higher temperatures it is stable for a given chemical potential. The branch 1 is the dominant solution when we look at the free energy plot. Another interesting thing that we notice is that for branch 1 and 2 as $\lambda \to 0$ we have

$$G \to -\frac{2\sqrt{2\pi} T_d^3}{3^\frac{3}{2}}, \quad J \to -\frac{1}{2}\sqrt{3\pi} T_d.$$  \hspace{1cm} (35)$$

So, both the spin $\frac{3}{2}$ and spin 1 charges are non-zero even when the chemical potential corresponding to that charge is zero, i.e. even when the theory is undeformed. This was also obtained in [37]. This is different than the principle embedding case where spin 3 charge goes to zero when the chemical potential goes to zero for the dominant branch. This stems from the fact that $\lambda \to 0$ limit corresponds to $m \to \infty$ limit and hence it is not exactly an undeformed theory that we are studying but a theory which has been deformed in IR.

We see that this embedding has a valid high temperature behaviour i.e. as $T_d \to \infty$ we have

$$G \to -\frac{\lambda^3 T_d^3}{2 T_d^3}, \quad J \to -\frac{2\lambda^2 T_d^3}{8} \to -\frac{2\lambda^2 T_d}{8}.$$  \hspace{1cm} (36)$$

We see that in the high temperature limit the charges have the correct scaling behaviour in terms of the only dimensionful parameter ($T$) and they are real. So, to study the high temperature behaviour the diagonal embedding is the correct theory to use.

In equation (30) we have the mapping between the parameters of the UV and IR theory. Now if we substitute solutions of branch 1, 2, 3, 4 of the IR theory in this map, it respectively matches with branches 4, 3, 2, 1 of the UV theory. So, this suggests that along the flow the good solutions in one end go to the bad solutions in the other and vice versa. This is easy to see if we plot branches of $-2\pi L m T$ and the corresponding branches of $J$ with its parameters $\lambda$ and $T_d$ replaced by $m$ and $T$ using equation (30), the two plots merge with the identifications of the branches at two ends that we mentioned above. This is expected since from the expression of entropy given in [11] we see that the sign of the expressions changes if we replace the $z$-component of connection by $\bar{z}$ component. From the data at the end points of RG flow we see that the mapping between the $z$ and $\bar{z}$ components changes as can be seen from equation (29). Also, since we expect the RG flow from IR to UV to happen when $m$ goes from 0 to $\infty$, we see that initially the $\bar{z}$ component of the connection acts like a perturbation near IR fixed point as can be seen from equation (13), but near UV fixed point due to $m \to \infty$ $\bar{z}$ component is dominant part. So, the sign of entropy of the branches changes between the UV and IR fixed points and hence the good and the bad solutions

---

\*1 is dimensionless as in terms of $l$ all thermodynamic parameters have correct scaling behaviour with temperature as seen from equation (34).
get swapped. This is the reason for the bad branches in IR being able to explain the high temperature behaviour of the theory in the UV.

5 Summary and Discussions

At first glance our analysis may look very similar to [37]. But we differ on many aspects of formulation, convention and results. We differ from [37] in the following respect

- We use a convention for chemical which is more suited for a partition function of the type

\[ Z = e^{\tau L + \mu_i Q_i} \]  

(37)

where \( \mu_i \) are chemical potential corresponding to the charges \( Q_i \). Whereas the convention of [37] is much suited to a definition of the form

\[ Z = e^{\tau (L + \mu_i Q_i)} \]  

(38)

The conventions differ in the definition of the deformation added to the holomorphic component of connection ‘a’ in [13]. We used a form which suits the variational principle of [30, 36] in the canonical formalism and the integrability condition given in [25] for the holomorphic formulation. The disadvantage we faced while using these parameters is that quantities in the extremal branches \( \rightarrow \infty \) as \( T \rightarrow 0 \). As can be seen from the plots used in [37], all quantities are finite as \( T \rightarrow 0 \) for all branches. But, to use this parameter in our case we have to reformulate the variational principle analysis. It looks plausible that the change in the definition is not going to affect the conclusions considerably. Also, in the holomorphic formalism to use the definition (38), the definition of charges have to be altered to satisfy the integrability condition. This was accounted for in section (2.2) in [37] by adding to the action a suitable boundary term and thus making it more suitable for the \( \mu \) and \( T \) variables. The new action supports the correct variational principle and doesn’t rely on the integrability conditions for consistency.

- In the study of black hole phase structure in principle embedding in [37] they get one unphysical branch and 3 physical branch. Of the remaining 3 branches they have one unstable branch with negative specific heat. Of the remaining 2 stable branches they have one branch with the expected scaling behavior and one with wrong scaling behaviour with temperature for quantities having a CFT description. We get 2 unphysical and 2 physical branch. Our BTZ branch has the correct scaling behavior (e.g. Stress Tensor \( \propto T^2 \)) whereas our extremal branch apparently does not have one (\( \propto \frac{1}{mT^2} \)). But using the correct dimensionless thermodynamical variable that we use to plot our phase plots, the scaling of the extremal branch goes like \( \propto \frac{T^2}{\mu^2} \) and hence, it too has the correct scaling behaviour. So, both our branches can have a consistent CFT description.

- In [37] they argued that the third physical branch which was real for high temperature but which had the wrong scaling with temperature at high temperature for the principal embedding is actually the branch with correct scaling with temperature at high temperatures from the diagonal embedding perspective. They argued this by stating that in the diagonal embedding, of the two real branches
near the fixed point, the one with the lower free energy must map to the lower free energy branch among the two surviving branches from the principal embedding. Whereas in our case we explicitly show that the good branches in the principal embedding map to the bad branches in diagonal embedding and vice-versa by giving an explicit one-to-one mapping between the branches at the two ends. We also show that in terms of temperature and the dimensionless parameter used the thermodynamic quantities have the correct scaling behaviour at high temperature in the diagonal embedding.

- In addition to this we studied the thermodynamics of the conical surplus also in the principal embedding and in the process we were able to show that a “Hawking-Page” like transition takes place in the low temperature regime. This study also brought out the fact that after the temperature when all the black hole solutions in the principal embedding ceases to exist the conical surplus solution again takes over as its regime of existence extends to a higher temperature than that of the black hole for a particular chemical potential. As we have stated earlier this was due to the self repulsive nature of spin 3 fields due to which at high enough temperatures the repulsive nature of spin 3 field prevents formation of black holes.

In all this we have turned a blind eye towards the “extremal thermal AdS” branch which dominates all the other 3 physical branch in principal embedding at low temperature regime. The physics of this branch needs to be studied better as it is a consistent solution at least mathematically, to our holonomy equations with the desired boundary falloff conditions. If this were not to be excluded by some physical argument (which we are currently unaware of) then at low temperature regime we will not see a black hole phase and this ”extremal thermal AdS“ branch will be the only observable solution. Only at high temperature will there be a stable black hole solution but with the spectrum of the diagonal embedding. This is a picture where after adding deformation the RG flow takes us to the UV fixed point and only there a black hole can form. This aspect needs to be studied better and we defer further discussions to future works. In the diagonal embedding a similar analysis shows that the free energy of a conical surplus there has higher free energy than the black hole solution in the high temperature regime.

Also, the behaviour near the zero temperature shows that when \( m = 0 \) both the extremal black hole and extremal thermal AdS solutions are absent. By introducing a non zero \( m \) either of the two extremal branches becomes dominant depending on whether the solution acquires a positive or negative spin 3 charge. This looks like that we are breaking some sort of discrete symmetry which does not distinguish between positive or negative spin 3 charge, by introduction of a chemical potential.

### 6 Further Directions

We did our calculations in the canonical formalism which is different from the holomorphic formalism used in [31]. In our methods the criteria for discarding solutions look much more physical (at least to us). But this holomorphic formalism is in apparent disagreement with the CFT calculations of [32]. But in the canonical formalism the
thermodynamic quantities are obtained much more naturally so we think that there has to be a way to see if it matches with the CFT calculation in the highest spin going to $\infty$ limit, where the CFT calculations have been done. A possible solution for this was suggested in [34].

In the the recent paper [35] where it has been proposed that for higher spin theories the correct way to add chemical potential preserving the Brown-Henneaux fall off conditions necessary for definition of charges that we are using is to add them along the time component of the connection rather than the antiholomorphic component. In the light of this our analysis should be redone to see if some extra features emerge other than that we already have here. The ideal situation would be to derive the the asymptotic charges in the presence of a chemical potential exactly.

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