Transversal switching between generic stabilizer codes

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We propose a randomized variant of the stabilizer rewiring algorithm (SRA), a method for constructing a transversal circuit mapping between any pair of stabilizer codes. As gates along this circuit are applied, the initial code is deformed through a series of intermediate codes before reaching the final code. With this randomized variant, we show that there always exists a path of deformations which preserves the code distance throughout the circuit, while using at most linear overhead in the distance. Furthermore, we show that a random path will almost always suffice, and discuss prospects for implementing general fault-tolerant code switching circuits.

I. INTRODUCTION

It is an oft-cited fact that no quantum error-correcting code can implement a universal transversal logical gate set \([1,3]\). As a result, there have been several attempts to circumvent this no-go theorem to achieve universal fault-tolerant quantum computation. These candidates include magic state distillation \([4,5]\), gauge fixing \([6,7]\), and more recently pieceable fault-tolerance \([8,9]\). These last two candidates can be seen as a special case of the more general approach of code switching \([10–14]\).

Code switching is a natural idea: given two codes, map information encoded in one code to information encoded in the other. For this mapping to be fault-tolerant, we must often perform several intermediate error-correction steps to ensure that faults do not grow out of hand. Thus, it is essential that during a circuit switching between codes, the extremal error-correcting codes are deformed through a series of intermediate error-correcting codes from one to another. This notion of intermediate error-correction was used in \([11]\) to implement universal transversal computation by switching between the Steane and Reed-Muller codes, whose complementary transversal gate sets are universal when taken together. However, universal fault-tolerant computation is not the only consideration in choosing error-correcting codes, and different codes can be tailored to different tasks. For this reason, it would be nice to have a way of converting between different quantum codes fault-tolerantly.

Simply decoding and re-encoding information is undesirable, since the bare information becomes completely unprotected during this transformation. Past work has succeeded in constructing fault-tolerant circuits for switching between particular quantum error-correcting codes fault-tolerantly, while providing guarantees that these circuits are optimal within some framework \([10]\).

Recently, \([15]\) considered switching between generic stabilizer codes, and proposed the stabilizer rewiring algorithm (SRA) for constructing a transversal circuit mapping between any pair of stabilizer codes. The circuit complexity scales quadratically with the code length, and depends on a choice of presentation for the code generators. Different presentations will result in different circuits mapping between different sets of at most \(n\) intermediate codes. This circuit necessarily fails to be fault-tolerant when these intermediate codes have low distance. This leads to the central question: is there an efficient way of fault-tolerantly converting between generic stabilizer codes?

A. Results

Towards this goal, we propose a randomized variant of the SRA, the randomized SRA (rSRA). We show that for any pair of stabilizer codes, with at most linear overhead with respect to the distance of the codes, there always exists a transversal circuit that maps between intermediate codes of high distance. Furthermore, using slightly more overhead, such a path can be found with high probability. In particular, we show the following.

**Theorem 1** (Informal). For any two \([n, k, d]\) stabilizer codes \(S_1\) and \(S_2\), the rSRA scheme gives a transversal circuit mapping from \(S_1\) to \(S_2\) where each intermediate code has distance at least \(d\) with probability \(1 - \varepsilon\), using

\[
m = O\left(d \log \frac{n}{d} + \log \frac{1}{\varepsilon}\right)
\]

ancilla qubits.

This distance-preserving property is a necessary, but not sufficient condition to ensure a fault-tolerant mapping. While the algorithm does not necessarily yield a fault-tolerant conversion, it gives a universal upper bound on the number of ancilla qubits required for distance-preserving transversal code transformation. As was noted in \([15]\), the usefulness of this scheme is in its generality. While the upper bound may be of independent conceptual interest, we hope that with modification, the rSRA can be applied as a useful schema for searching...
for fault-tolerant paths between small codes. We provide
small examples of such transversal paths in Section IV,
including a path between the [[5, 1, 3]] and [[7, 1, 3]] codes
that without modification protects against erasure with no overhead.

B. Organization

In Section II we introduce some preliminaries and no-
tation that will be used throughout the paper. The main
rSRA schematic is presented in Section III. Some examples
illustrating different parts of the rSRA and demonstrating
small conversion paths are presented in Section IV. The proof of the main theorem is presented in
Section V. Some discussion on fault-tolerance and possi-
ble improvements for the rSRA can be found in Section VI. Readers interested in the technical details may
refer to the appendices for additional lemmas.

II. PRELIMINARIES

Let \( \mathcal{P}^n \) denote the \( n \)-qubit Pauli group. Then a stabilizer group \( S \subseteq \mathcal{P}^n \) is an abelian subgroup of the Pauli
group not containing \( -I \). To any such stabilizer group
\( S \), we can associate a subspace \( C_S \subseteq (\mathbb{C}^2)^\otimes n \) defined as
the simultaneous +1-eigenspace of all the operators in \( S \).
We call such a subspace \( C_S \) a stabilizer code.

A stabilizer code \( C_S \) has parameters \( [n, k, d] \). Here, \( n \)
is the number of physical qubits comprising the code, \( k \)
is the number of logical qubits of the code \( \log(\dim(C_S)) \),
and \( d \) is the distance of the code. More precisely, the
normalizer \( \mathcal{N}_{\mathcal{P}^n}(S) \) represents the set of logical Pauli op-
erators for \( C_S \), and so
\[
d := \min_{L \in \mathcal{N}(S)/S} (|L|)
\]
where \(|\cdot|\) denotes the weight of the Pauli operator. Note
that the number of stabilizer in the corresponding stabil-
izer subgroup is \( n - k \).

Given any stabilizer group \( S \), if we choose a generating
set \( G_S \) for \( S \), we can define a syndrome map
\[
\text{Syn}_G : \mathcal{P}^n \rightarrow \{0, 1\}^{n-k}
\]
\[
\text{Syn}_G(e)_i = \begin{cases} 
0 & \text{if } [e, g_i] = 0 \\
1 & \text{if } [e, g_i] = 0
\end{cases}
\]
for \( G = (g_1, \ldots, g_{n-k}) \). Then equivalently,
\[
d = \min_{L \in \ker(\text{Syn}_G)/S} (|L|)
\]
and is independent of the choice of \( G \).

Another convenient formalism for describing stabilizer
groups is as subspaces of symplectic vector spaces, and
we will use the two formulations interchangeably. For
any \( P \in \mathcal{P}^n \cup \mathcal{U}(1) \), if
\[
P = X^{a_1} Z^{b_1} \otimes X^{a_2} Z^{b_2} \cdots \otimes X^{a_n} Z^{b_n}
\]
then we can associate to \( P \) the vector \( \vec{P} := (\vec{a} \vec{b})^T \in \mathbb{F}_2^{2n} \).

EQUIP \( \mathbb{F}_2^2 \) with a symplectic bilinear form
\[
\langle \vec{v}, \vec{w} \rangle := \vec{v}^T B \vec{w}
\]
where \( B \) is the \( 2n \times 2n \) block matrix defined by
\[
B = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}.
\]

Then Paulis \( P, Q \) commute if any only if their associated
vectors \( \vec{P}, \vec{Q} \) are orthogonal in this vector space. Thus,
we can equivalently define a stabilizer group as a self-
\normalizer subgroup is
\( n \). Then equivalently, a logical state in the
\[ \mathcal{N}_{\mathcal{P}^n}(S) \]

Further note that for any \( A \in GL(\mathbb{F}_2, n - k) \), for any
generator matrix \( G \) for \( S \), \( G A^T \) is also a generator matrix
for \( S \). The syndrome map satisfies
\[
\text{Syn}_{G A^T}(\vec{P}) = (G A^T)^T B \vec{P} = A G^T B \vec{P} = A \cdot \text{Syn}_G(\vec{P}).
\]

So any action on the generator matrix induces a cor-
responding action on the syndrome vectors themselves.

Finally, we call a circuit \( C \) on a class of encoded inputs
t-fault-tolerant if it is t-fault-tolerant in the exRec
formalism \cite{10}. Formally, given error correction pro-
dure \( EC \), \( C \) is t-fault-tolerant if for any choice of \( t \) faulty
components in the combined circuit \( EC \cdot C \cdot EC \), a fault-
less version of \( EC \) applied to the output of the combined
circuit can successfully recover the data. If \( t \geq 1 \) we may
simply call the circuit fault-tolerant.

III. THE rSRA SCHEMATIC

The rSRA modifies the SRA presented in \cite{14}, whose
central insight is the following. Consider two stabilizer
groups \( S, S' \) with generating sets \( G, G' \) satisfying the fol-
lowing nice property:
\[
G = \{g, g_1, \ldots, g_l\}
\]
\[
G' = \{g', g_1, \ldots, g_l\}
\]
where \( \{g, g'\} = 0 \). We call two such codes for which one
can choose such generating sets adjacent. Then one can
readily check that the Clifford gate \( \frac{1}{\sqrt{2}}(1 + g' g) \) maps
information encoded in the stabilizer code defined by \( G \)
to the same information encoded in the stabilizer code
defined by \( G' \). Letting \( |\psi\rangle_G \) denote a logical state in the
code associated to \( G \), we see that
\[
\begin{align*}
g'_i \cdot \frac{1}{\sqrt{2}} (1 + g' g) |\psi\rangle_G &= \frac{1}{\sqrt{2}} (1 + g' g) |\psi\rangle_G, \\
g' \cdot \frac{1}{\sqrt{2}} (1 + g' g) |\psi\rangle_G &= \frac{1}{\sqrt{2}} (g' + 1) |\psi\rangle_G \\
&= \frac{1}{\sqrt{2}} (g' + 1) |\psi\rangle_G \\
&= \frac{1}{\sqrt{2}} (1 + g' g) |\psi\rangle_G.
\end{align*}
\]

The insight is that this mapping can be done transversally. While the Clifford transformation described need not be transversal, it can be simulated by a transversal Pauli measurement supplemented by a transversal Pauli gate controlled on classical information. This is similar to gauge-fixing, in which one measures a logical operator of the gauge and then applies a corresponding logical gauge operator conditioned on the outcome. To see this, consider the circuit described by:

1. Measure \( g' \).
2. Apply \( g \) conditioned on measurement outcome \(-1\).

Let \( P^\pm \) denote the projector onto the \(+1/−1\) eigenspace of \( g' \). Then, if the measurement outcome is \(+1\),
\[
\frac{1}{\sqrt{2}} (1 + g' g) |\psi\rangle_G = \frac{1}{\sqrt{2}} (1 + g' g) |\psi\rangle_G = \sqrt{2} P^+ |\psi\rangle_G.
\]

Furthermore, if the measurement outcome is \(-1\),
\[
\frac{1}{\sqrt{2}} (1 + g' g) |\psi\rangle_G = \frac{1}{\sqrt{2}} (g - g' g) |\psi\rangle_G = \frac{1}{\sqrt{2}} g (1 - g' g) |\psi\rangle_G = \frac{1}{\sqrt{2}} g P^- |\psi\rangle_G.
\]

Thus, we see that we can transversally perform the mapping \( |\psi\rangle_G \rightarrow |\psi\rangle_G \).

Now consider the more general case in which we have (non-adjacent) \( S, S' \) describing \([n, k]\) and \([n', k]\) codes respectively. We now describe a general randomized algorithm for outputting a circuit switching between these two codes, similar to [13], and will later show that this circuit is distance-preserving with high probability. The inputs are arbitrary generator matrices \( G, G' \) for stabilizer groups \( S, S' \), along with a choice of ancilla size \( m \in \mathbb{N} \).

### A. Preparing the generator matrices

1. Append \(|0\rangle\) ancilla to the smaller code so that the codes are of equal size. We now assume that both codes are \([n, k]\) codes.
2. Append \(|0\rangle^m\) to the first code, and \(|1\rangle^m\) to the second. Note that this is equivalent to defining a pair of new stabilizer codes
\[
\hat{S} = \langle S \otimes I^\otimes m, I^\otimes n \otimes Z \otimes I^\otimes m−1, \ldots, I^\otimes n+m−1 \otimes Z \rangle,
\]
\[
\hat{S}' = \langle S' \otimes I^\otimes m, I^\otimes n \otimes X \otimes I^\otimes m−1, \ldots, I^\otimes n+m−1 \otimes X \rangle.
\]
3. Choose \( G_A = G'_A \) to be a basis for the subspace defined by \( \hat{S} \cap \hat{S}' \).
4. Choose \( G_B \) to extend the basis of \( G_A \) to a basis for \( \mathcal{N}(\hat{S}') \cap S \) and choose \( G'_B \) to extend the basis of \( G_A \) to a basis for \( \mathcal{N}(\hat{S}) \cap S' \).
5. Choose \( G_C \) to extend the basis \( G_A \cup G_B \) to a basis for \( S \) and \( G'_C \) to extend the basis \( G'_A \cup G'_B \) to a basis for \( S' \).
6. Let \( H \) be the commutativity matrix for \( G_C, G'_C \), defined by \( H := G'_C \overline{T} B G_C \). By Lemma A.2, \( H \) is invertible with dimension \(|G_C| \times |G_C|\), where \(|G_C| \geq m\). So we can choose \( M, N \in GL(\mathbb{F}_2, |G_C|) \) : \( M^T H N = I_{|G_C|} \), and redefine
\[
G_C \leftarrow G_C \cdot M
\]
\[
G'_C \leftarrow G'_C \cdot N.
\]
7. Choose uniformly at random \( V, V' \in \mathbb{F}_2^{[G_C| \times |G_B]} \) and a \( U \in GL(\mathbb{F}_2, |G_C|) \).
8. Redefine
\[
G'_C \leftarrow U(V G_B^T + G'_C)
\]
\[
G'_C \leftarrow (U^{-1})^T (V' G_B^T + G'_C)^T
\]

Note that this does not change the commutativity matrix since
\[
U(V G_B^T + G'_C) B (G_C + G'_B V'^T) U^{-1} = I_{|G_C|}.
\]
9. Let \( G_B = \{g_1, \ldots, g_{|G_B|}\} \) and \( G'_B = \{g'_1, \ldots, g'_{|G_B|}\} \). For each \( g_i \in G_B \), choose \( \overline{g}_i \) satisfying
\[
[\overline{g}_i, G_A] = 0
\]
\[
[\overline{g}_i, G_C] = 0
\]
\[
[\overline{g}_i, G'_C] = 0
\]
\[
[\overline{g}_i, \{g_{i+1}, \ldots, g_{|G_B|}\}] = 0
\]
\[
[\overline{g}_i, \{g'_{i+1}, \ldots, g'_{|G_B|}\}] = 0
\]
\[
[\overline{g}_i, \{g_i, \ldots, g_{i-1}\}] = 0
\]
\[
\{\overline{g}_i, g_i\} = 0
\]
\[
\{\overline{g}_i, g'_i\} = 0.
\]
To see that such a choice of \( \overline{G} \) always exists, note that it must satisfy at most 2n affine linear equations, all of which are linearly independent, in a space of dimension 2n.

Now that we have prepared the generator matrices, we will step-by-step map between adjacent codes transversally.

**B. Applying the transformation**

10. For \( 1 \leq i \leq |G_B| \) indexing the elements of \( G_B \), perform the transformation \( g_i \mapsto \overline{g}_i \). Note that the resulting stabilizer codes are adjacent, and so the preceding discussion gives a transversal circuit for each mapping.

11. For \( 1 \leq i \leq |G_C| \) indexing the elements of \( G_C \), perform the transformation \( g_i \mapsto g'_i \). Again, since the codes are adjacent, the mapping can be done transversally.

12. For \( 1 \leq i \leq |G_B| \) indexing the elements of \( G_B' \), perform the transformation \( \overline{g}_i \mapsto g'_i \) starting from \( i = |G_B| \) and working backwards towards \( i = 1 \). Again, we have a transversal circuit for each mapping.

13. Discard the ancilla.

This randomized variant differs from the original SRA in several ways. First, there is the introduction of ancilla, which we will see are vital for preserving the distance. Next, the SRA fixes the generating sets \( G, G' \) subject to the same \( G_A \) and \( G_C \) conditions, but with different \( G_B \) conditions. Namely, the SRA fixes the \( \overline{G} \) to be the product of the complementary logical operators to those operators in \( G_B \) and \( G_B' \), which can be seen as nontrivial logical operators on the opposite code. This allows for a certain degree of freedom in choosing the order in which one converts between the two codes, but restricts the \( G_C, G_C' \) that are available to use. Also in the SRA, only the set of valid permutations among \( G_B \) and \( G_C \) are considered, which restricts the search for a distance-preserving mapping. In the rSRA, we consider the full set of invertible transformations on \( G_C \) for a better chance of success. Finally, the transformation described above is symmetric in the sense that switching from \( G \) to \( G' \) or \( G' \) to \( G \) after step 9 results in the same set of intermediate codes. We will see that this simplifies the set of errors we must consider.

**IV. DISTANCE-PRESERVATION FOR SMALL CODES**

We have now described a way of constructing a transversal circuit mapping information encoded in \( G \) to information encoded in \( G' \) through the use of Shor-style measurement (see Appendix E).

However, we have no a priori guarantee that these intermediate codes, resulting from the sequence of deformations, will themselves be error-correcting. In light of this, we offer several examples of small distance-preserving circuits generated from the rSRA. These illustrate the necessity of the aforementioned modifications, which are centered around choosing a path so that all of the intermediate codes have high distance. In these examples, the extremal codes all have distance 3, and so we call the circuit distance-preserving if the intermediate codes all have distance \( \geq 3 \).

**A. \([7,1,3]\) \(\leftrightarrow [5,1,3]\)**

With \( m = 0 \), one can generate a distance-preserving map from the \([7,1,3]\) Steane code to the perfect \([5,1,3]\) code using the rSRA with 17 multi-qubit gates. An optimal fault-tolerant (and so distance-preserving) transformation using \( CZ \) gates between these two codes was found via brute force search in [10] and involves 14 multi-qubit gates. The circuit output by the rSRA requires no overhead in data qubits compared to the three extra qubits required in [10]. However, because the \([5,1,3]\) code is perfect, any conversion without ancilla must only be able to protect against erasure, for reasons detailed in Section VI. Note also that there must be conversions with large separation between the circuit provided by the rSRA and the optimal fault-tolerant circuit, in particular when \( G \) and \( G' \) are locally unitarily equivalent.

| Type   | \([7,1,3]\) | \([5,1,3]\) |
|--------|-------------|-------------|
| \(G_A\) | \(YXXYZIZZ YXXYZIZZ\) | \(ZZZZIII IZZZZII\) |
| \(G_C\) | \(YXXYZIZZ XZZZZIII\) | \(IYXZYYZX XIYYZZII\) |
|        | \(XYYZYZZ XZXIXZII\) | \(ZYYZIXXX IIIZZI\) |

**TABLE I.** The generator matrices defining a distance-preserving conversion, proceeding from top to bottom. We follow steps 10-13 of the algorithm.

**B. \([34] \cdot [7,1,3] \leftrightarrow [9,1,3]\)**

With \( m = 0 \), one can convert from the \([34] \) permutation of the \([7,1,3]\) Steane code to Shor’s \([9,1,3]\) code while preserving the distance. However, for the standard choice of generator matrices, no permutation on the ordering of the deformations will suffice. Thus, we must choose \( U \in GL(\mathbb{F}_2, |G_C|) \) rather than restricting \( U \) to be a permutation matrix. A choice of generator matrices for which the circuit is distance-preserving is presented below.
TABLE II. The conversion proceeds from top to bottom. As the $G_B$ elements commute, we perform an intermediate conversion to the product of the complementary logical operators, which in this case are $XXXXXXX$ and $XXXXXXX$ respectively. This small modification is similar to the SRA [15], which we adopt here for ease of presentation.

C. $[7, 1, 3] \longleftrightarrow (34) \cdot [7, 1, 3]$  

For $m = 0$, it was observed in [15] that one cannot use the SRA to convert between the $[7, 1, 3]$ code, and the $(34)$ permutation of the $[7, 1, 3]$ code while preserving the distance. In fact, there does not exist a $U \in GL(\mathbb{F}_2, [G_C])$ that allows the intermediate codes to be error-correcting. In contrast, with $m = 2$, there does exist such a distance-preserving circuit, emphasizing the need for ancilla. Moreover, brute force search shows that this is the minimal number of ancilla required to produce a distance-preserving circuit within this framework. However, note that qubit permutations are themselves automatically fault-tolerant by simply relabeling the wires, rather than applying a fault-tolerant physical SWAP gate.

V. DISTANCE BOUNDS

We now show that, with low overhead and high probability, the described rSRA will yield a distance-preserving circuit. More specifically, we show that the intermediate codes preserve the distance of the extremal codes.

**Theorem 1.** Let $S, S'$ be any two stabilizer codes with parameters $[[n_1, k, d_1]]$ and $[[n_2, k, d_2]]$, respectively. Let $d = \min\{d_1, d_2\}$ and $n = \max\{n_1, n_2\}$. Then, the rSRA will output a distance-preserving circuit mapping information encoded in $S$ to information encoded in $S'$ with probability $1 - \epsilon$ using $m = O(d \log \frac{n}{d} + \log \frac{1}{\epsilon})$ ancilla qubits.

**Proof.** Consider a particular error $e : |e| < d$. There are four different types of errors to consider.

1. $e \in S \cap S'$: In this case, $e \in \text{Span}(G_A)$, and so remains passively corrected throughout the transformation.

2. $e \in S \setminus \mathcal{N}(S')$: In this case, we can decompose $e = g_A + g_B + g_C$ where $g_A \in \text{Span}(G_A), g_B \in \text{Span}(G_B)$, and $g_C \in \text{Span}(G_C)$. Furthermore, $g_C \neq 0$, or else $e$ would be a logical operator of weight $< d$ for $S'$. Thus, $e$ must be detected by $G_C$, and so it remains detectable after step $[11]$. In particular, before the end of step $[11]$, $e$ must fall out of the intermediate stabilizer group. Suppose this occurs for the first time when transforming between two adjacent codes whose stabilizer groups differ by $g, g'$. Then we can write $e = g + \sum_i a_i g_i$, and as $g'$ commutes with all other $g_i$, it must be that $\{e, g'\} = 0$. Since $g'$ remains in each intermediate code up through step $[11]$, $e$ must be detectable throughout.

3. $e \in S' \setminus \mathcal{N}(S)$: This error is just an error of type (2) when performing the opposite transformation from $S'$ to $S$. By symmetry of the scheme, the set of intermediate codes during this opposite transformation is the same, and so these errors remain detectable by the preceding argument.

4. $e \notin \mathcal{N}(S) \cup \mathcal{N}(S')$: Let $G_C^{(0)}, G_C^{(0)}$ be the bases $G_C$ and $G_C'$, we choose after step $[2]$ in the rSRA scheme, and let $G_C^{(1)}, G_C^{(1)}$ be the bases we choose after step $[3]$. Note that the syndrome map for $G_C^{(1)}$ can then be expressed as $\text{Syn}_{G_C^{(1)}}(e) = U(V \cdot \text{Syn}_{G_B}(e) + \text{Syn}_{G_C^{(0)}}(e))$.

In this case it must be that

$$(\text{Syn}_{G_A}(e) | \text{Syn}_{G_B}(e) | \text{Syn}_{G_C^{(0)}}(e))^T \neq 0,$$

$$(\text{Syn}_{G_A}(e) | \text{Syn}_{G_B}(e) | \text{Syn}_{G_C^{(0)}}(e))^T \neq 0.$$

Note that if $\text{Syn}_{G_A}(e) \neq 0$, then $e$ is always detectable since each intermediate code includes the check operators from $G_A$. Thus, we only need to consider the case where $\text{Syn}_{G_A}(e) = 0$, and so we can assume that $(\text{Syn}_{G_B}(e) | \text{Syn}_{G_C^{(0)}}(e))^T \neq 0$ and $(\text{Syn}_{G_B}(e) | \text{Syn}_{G_C^{(0)}}(e))^T \neq 0$.

Let $P_e$ denote the probability that the error $e$ is undetectable in some intermediate code over the random choices of $U, V,$ and $V'$. We divide $P_e$ into three parts. Let $A_e$ denote the event that $\text{Syn}_{G_C^{(1)}}(e) = 0$.
event that \( \text{Syn}_{G_C}^\epsilon (e) = 0 \), and let \( C_e \) denote the event that both \( \text{Syn}_{G_C}^{(1)} (e) \) and \( \text{Syn}_{G_C}^{(0)} (e) \) are nonzero, yet \( e \) becomes undetectable on some intermediate code during the transformation. Then \( P_c \leq \text{Pr}[A_e] + \text{Pr}[B_e] + \text{Pr}[C_e] \). We bound \( \text{Pr}[A_e] \), \( \text{Pr}[B_e] \), and \( \text{Pr}[C_e] \) separately. To bound \( \text{Pr}[A_e] \), note that
\[
\text{Syn}_{G_C}^\epsilon (e) = U(\lambda \cdot \text{Syn}_{G_{B}} (e) + \text{Syn}_{G_C}^{(0)} (e)).
\]
Since \( U \in GL(\mathbb{F}_2, n - k) \), \( A_e \) occurs if and only if \( \lambda \cdot \text{Syn}_{G_{B}} (e) + \text{Syn}_{G_C}^{(0)} (e) = 0 \). If \( \text{Syn}_{G_{B}} (e) = 0 \), it must be the case that \( \text{Syn}_{G_C}^{(1)} (e) \neq 0 \), and so \( \text{Syn}_{G_C}^{(1)} (e) \neq 0 \); otherwise \( \text{Syn}_{G_{B}} (e) \neq 0 \) and \( \lambda \cdot \text{Syn}_{G_{B}} (e) + \text{Syn}_{G_C}^{(0)} (e) \) is uniformly random over \( \{0, 1\}^{G_C} \). In either case, we have
\[
\text{Pr}[A_e] \leq 2^{-|G_C|}.
\]
Repeating the same argument shows that \( \text{Pr}[B_e] \leq 2^{-|G_C|} \) as well. To bound \( \text{Pr}[C_e] \), define
\[
v = \lambda \cdot \text{Syn}_{G_{B}} (e) + \text{Syn}_{G_C}^{(0)} (e), \quad w = \lambda' \cdot \text{Syn}_{G_{B}'} (e) + \text{Syn}_{G_C}^{(0)} (e).
\]
Since \( Uv, (U^{-1})^T w \neq 0 \), \( e \) will be detectable during steps 10 and 12 and so \( C_e \) occurs only if \( e \) becomes undetectable during step 11. Specifically, it must be that \( \text{Syn}_{G_{A}} (e) = 0, \text{Syn}_{G_{B}} (e) = 0 \), and the last 1 in the vector \( Uv \) occurs before the first 1 in the vector \( (U^{-1})^T w \). This is because we are sequentially replacing the check operators of \( G \) with the check operators of \( G' \), and so an error becomes undetectable for some intermediate code only if we produce some zero syndrome during this sequence of substitutions. By Lemma A.3, for two nonzero vectors \( v, w \in \{0, 1\}^{G_C} \), the probability that the last 1 in \( Uv \) comes before the first 1 in \( (U^{-1})^T w \) is bounded by \( (|G_C| - 1) \cdot 2^{-|G_C|} \).

Summing these three terms, we have \( P_c \leq (|G_C| + 1) \cdot 2^{-|G_C|} \). Taking a union bound, the probability \( P \) that any of the intermediate codes fail to detect any error of weight less than \( d \) is upper bounded by
\[
P \leq \sum_{e:|e| < d} P_c \leq |\{ e: |e| < d \}| \cdot (|G_C| + 1) \cdot 2^{-|G_C|}.
\]
Taking a Chernoff bound, we get that this is in turn upper bounded as
\[
P \leq 4^{n + m} \cdot e^{-D(\frac{d}{n+m})\frac{d}{2}(n+m)} \cdot (|G_C| + 1) \cdot 2^{-|G_C|}
\]
where \( D(|\cdot|) \) is the KL-divergence. By the quantum singleton bound, we can assume \( \frac{d}{n+m} < \frac{d}{n} < \frac{1}{3} \). Furthermore, by Lemma A.2 \( |G_C| \) is given by \( \text{rank}(G^T B G') \), which is at least \( m \). So the probability of failure can be further upper bounded by
\[
P \leq 4^{n + m} \cdot e^{-D(\frac{d}{n+m})\frac{d}{2}(n+m)} \cdot (m + 1) \cdot 2^{-m}.
\]
It suffices to choose \( m \) such that the above quantity is upper bounded by \( \epsilon \) in order to achieve a high probability of success. In particular, the case \( \epsilon = 1 \) upper bounds the minimum number of ancilla qubits required for a fault-tolerant transformation. By Lemma A.3, we observe that taking
\[
m = O(d \log \frac{n}{d} + \log \frac{1}{\epsilon})
\]
is sufficient for the rSRA scheme to succeed with probability \( 1 - \epsilon \).

\section{VI. Discussion}

Theorem 1 shows that with high probability, the rSRA will produce a transversal circuit with intermediate codes that have distances \textit{at least} the minimum of the distances of the extremal codes. It is important to note that this does \textit{not} necessarily imply fault-tolerance. The reason is because, when measuring \( g' \), the randomness in the outcome prevents us from using that syndrome bit during error-correction. More specifically, consider the following two scenarios.

1. We project onto the (+1)-eigenspace of \( g' \).
2. We project onto the (−1)-eigenspace of \( g' \) and simultaneously experience an error that anticommutes with only \( g' \).

Then we cannot distinguish these two scenarios using only our syndrome bits, and so cannot correct the resulting error. More generally, we can cast the property required for fault-tolerance in terms of subsystem codes. For every conversion between adjacent codes, we consider the subsystem code with a single gauge degree of freedom corresponding to gauge operators \( g' \) and \( g \). Then the resulting conversion will be \( t \)-fault-tolerant precisely when the resulting subsystem code has distance \( 2t + 1 \). This is because the redundant syndrome information can diagnoze errors without the syndrome bit associated to \( g' \), and so ensure that we project onto the correct eigenspace. For this reason, additional techniques may be required to achieve fault-tolerance using the rSRA, such as error-detection on the ancilla. We leave this to future work.

These techniques contrast with recent results from [3], where it was shown that pieceable fault-tolerance offers generic \textit{fault-tolerant} code switching between stabilizer codes subject to certain constraints. However, their techniques require that the codes are nondegenerate and have some set of native fault-tolerant Clifford gates, allowing a fault-tolerant SWAP gate \textit{between} different codes. One could also consider preparing a second code state and using logical teleportation to achieve a fault-tolerant mapping [12].

Practically, on small examples, one finds that often \textit{no} ancilla qubits are required to find a distance-preserving
circuit, which is desirable as the resulting circuit may then be fault-tolerant. In general, this can be attributed to a coarse accounting of $|G_C|$ in terms of the number $m$ of ancilla qubits. In most cases, $N(S) \cap N(S')$ will be small, and so the ancilla will be superfluous.

Moreover, the multi-qubit gate complexity of the algorithm is $\sum_{P \in \{P_U\} : U \in G_B \cup G_C} |P|$, so that choosing a low weight generating set is ideal for reducing the complexity of the code switching circuit. For this reason, LDPC codes might provide more efficient code switching circuits, although preserving the distance may depend on choosing a high weight set of generators.

This algorithm derives its usefulness from its generality. For specific code switching examples, it may be profitable to modify the circuit using the rSRA as a template, augmented with a larger class of fault-tolerant manipulations such as local Clifford gates, in order to search for a fault-tolerant mapping. For large code sizes, the use of high-weight Shor-style measurements is limiting as it requires large verified CAT states. Thus, this technique of high-weight Shor-style measurements is limiting as it requires large verified CAT states. Thus, this technique may be most useful as a step in a concatenated scheme, or simply as a search ansatz.

One subtlety about the rSRA is that, while it outputs a distance-preserving circuit switching between two codes, with high probability, this is difficult to check. This follows from the difficulty of computing the minimum distance of a generic error-correcting code, which is an NP-hard problem in general [17]. Indeed, even when restricting to a particular distance, this check remains extremely costly. This poses a barrier to derandomizing the algorithm, which would be one desirable avenue for future improvement.

Another such improvement would be to minimize overhead. One could imagine taking a random local Clifford transformation in order to increase the size of $G_C$, rather than introducing ancilla. Such a strategy would be interesting since locally equivalent codes have nearly identical properties. Of course, modifying the algorithm to ensure fault-tolerance is the most important improvement.

If it is true that one can always choose locally equivalent representatives for which the rSRA provides a distance-preserving conversion without ancilla, this would suggest that all error-protected information in stabilizer codes is, in some sense, “transversally equivalent”. This contrasts with the diverse set of equivalence classes of locally unitarily equivalent codes, which can be identified as distinct submanifolds of Grassmannians. Indeed, it may be of conceptual interest to interpret these upper-bounds in a broader framework of fault-tolerance, such as the one investigated in [18].

Similarly, the generality of the rSRA provides an aesthetically nice interpretation of error-protected information. It suggests that, with the addition of some minimal overhead, any stabilizer error-protected encoding of information is indeed “transversally equivalent” to any other.

VII. ACKNOWLEDGMENTS

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Appendix A: Technical lemmas

**Lemma A.1.** Let $v, w \in \{0,1\}^n \setminus \{0\}$ and $U \in \mathbb{F}_2 \cdot U \in \text{GL}(\mathbb{F}_2, n)$. Let $i_0 = \max\{i : (U \cdot v)_i = 1\}$ and $i_1 = \min\{i : ((U^{-1})^T \cdot w)_i = 1\}$. Then,

$$\text{Pr}[i_0 < i_1] \leq (n - 1) \cdot 2^{-n}.$$

**Proof.** Let $\langle \cdot, \cdot \rangle$ be the dot product over $\mathbb{F}_2$. Note that $\langle v, w \rangle = \langle U \cdot v, (U^{-1})^T \cdot w \rangle$. If $\langle v, w \rangle = 1$, then there must be at least one entry where both $U \cdot v$ and $(U^{-1})^T \cdot w$ are 1 for whichever $U$ we choose, and so $\text{Pr}[i_0 < i_1] = 0$.

Therefore we only need to consider the case in which $\langle v, w \rangle = 0$.

Consider the action of $\text{GL}(\mathbb{F}_2, n)$ on $A$ defined by $U \cdot (v, w) \rightarrow (U \cdot v, (U^{-1})^T \cdot w)$, where $A = \{(v, w) | (v, w) \in \{0,1\}^n \setminus \{0\}, \langle v, w \rangle = 0\}$. We show that the action is transitive by showing that for all such pairs $(v, w)$, there always exists a $U$ sending $(e_1, e_n)$ to $(v, w)$, where $e_1, e_n$ are $(1, 0, \ldots, 0)$ and $(0, 0, \ldots, 1)$, respectively. Given such a $(v, w)$, first extend $v$ to a basis for $w^T$, say $(u_1 = v, u_2, \ldots, u_{n-1})$, and then extend it to the whole space by adding in $u_n$. We claim that $U = (u_1, \ldots, u_n)$ is the desired matrix. It is sufficient to show that the last column $w'$ of $(U^{-1})^T$ is exactly $w$. We have $U^T w' = e_n$ given that $U^T (U^{-1})^T = I$, and that $U^T w = e_n$ by construction of $U$. Then, since $U$ is invertible, $w = w'$.

A uniformly random distribution over invertible $U$ then induces a uniformly random distribution over $A$. Then $\text{Pr}[i_0 < i_1]$ can then be bounded by counting the number of such pairs in $A$:

$$\text{Pr}[i_0 < i_1] = \sum_{i_0=1}^{n} 2^{n-i_0} \cdot \frac{(2^n - 1)}{(2^n - 1)} = \frac{(n-2)2^{n-1} + 1}{(2^n - 1)(2^n - 1)} \leq (n-1) \cdot 2^{-n}$$

when $n \geq 2$. Note that $|A| = 0$ when $n = 1$, so $\text{Pr}[i_0 < i_1] \leq (n-1) \cdot 2^{-n}$ holds for all $n \geq 0$. 

**Lemma A.2.** Let $G_A, G_B, G_C$ and $G_A', G_B', G_C'$ be the matrices defined up to step 8 in the rSRA scheme. The commutativity matrix $H = G_C' G_B G_C$ is invertible, and its dimension is $|G_C|$, with $|G_C| \geq m$.

**Proof.** For the two codes $\hat{S}$ and $\hat{S}'$, take arbitrary generator matrices $G, G'$ and define $H' = G^T B G'$. Note that
any two choices of generator matrices for the same code differ by an invertible row transformation, so the rank of $H'$ is invariant under different choices of the generator matrices. In particular, letting $G = (G_A|G_B|G_C), G' = (G'_A|G'_B|G'_C)$, we have

$$H' = \begin{bmatrix} 0 & 0 & H \end{bmatrix}.$$ 

Note that $\text{rank}(H) = |G_C|$, or else there would exist a combination of the rows of $G'_C^T$ that are orthogonal to all the columns of $G_C$. Since all the vectors in $G'_C$ are already orthogonal to $G_A$ and $G_B$ by definition, this cannot happen as no vector in $G'_C$ lies in $\mathcal{N}(S)$. The same argument applies to $G'_C$ as well. Therefore $H$ is invertible, with $\text{rank}(H') = \text{rank}(H) = |G_C|$, and is independent of the choice of $G_C$.

To show that $|G_C| > m$, take $G_C = (I^\otimes n \otimes Z \otimes I^\otimes m-1, \ldots, I^\otimes n+m-1 \otimes Z)$ and $G'_C = (I^\otimes n \otimes X \otimes I^\otimes m-1, \ldots, I^\otimes n+m-1 \otimes X)$, each of size $m$. By extending them to generator matrices $\tilde{G}$ and $\tilde{G}'$ for $\tilde{S}$ and $\tilde{S}'$ respectively, we get a commutativity matrix $\tilde{H}$ with an invertible submatrix of size $m \times m$, namely

$$\tilde{G}'_C B \tilde{G}'_C = I_m,$$

and so $\text{rank}(\tilde{H}) = |G_C| \geq m$.

**Lemma A.3.** For $D(\cdot||\cdot)$ the KL-divergence, let

$$P(n, m, d) = 4^{n+m} e^{-D\left(\frac{d}{n+1} \parallel \frac{3}{4}\right)(n+m) \cdot (m+1) \cdot 2^{-m}}.$$ 

Then $P < \varepsilon$ for some $m = O(d \log \frac{n}{d} + \log \frac{1}{\varepsilon})$.

**Proof.** Let $\alpha = m/n$. Then $P(n, m, d) < \varepsilon$ can be rewritten as

$$f(n, m, d) = \log P(n, m, d) = \log \frac{P(n, m, d)}{\varepsilon} = \log \frac{\varepsilon}{m+1} + n \left(2 + \alpha\right) \log 2 - (1 + \alpha) D\left(\frac{d}{n(1+\alpha)} \parallel \frac{3}{4}\right) < 0.$$ 

We first compute the dominant term, i.e. the $\alpha$ such that

$$(2 + \alpha) \log 2 - (1 + \alpha) D\left(\frac{d}{n(1+\alpha)} \parallel \frac{3}{4}\right) = 0.$$ 

Doing this we obtain

$$D\left(\frac{\alpha}{1+\alpha}\right) = D\left(\frac{d}{n(1+\alpha)} \parallel \frac{3}{4}\right)$$

$$\leq \frac{d}{\ln 2} \left(\ln \frac{3n(1+\alpha)}{d} + 1\right)$$

$$\alpha n \leq \frac{d}{\ln 2} (\ln \frac{d}{\ln 3})$$

$$m \leq \frac{1}{\ln 2} d (\ln \frac{n}{d} + (1 + \ln 3)),$$

where we have used convexity of $D(p||q) - p \ln p$ with respect to $p$. Letting $\alpha$ denote the solution to $(2 + \alpha) \log 2 - (1 + \alpha) D\left(\frac{d}{n(1+\alpha)} \parallel \frac{3}{4}\right) = 0$, we have that $m := \tilde{m} = \tilde{n} = O(d + d \log \frac{n}{d})$.

We now have that $f(n, m, d) = \log \frac{m+1}{\varepsilon}$. Taking the derivative of $f$ with respect to $m$, for all $\alpha > \tilde{\alpha}$ we have

$$\frac{\partial f(n, m, d)}{\partial m} = \frac{1}{m+1} - D\left(\frac{d}{n+m} \parallel \frac{3}{4}\right) - (m+n) \frac{\partial D\left(\frac{d}{n+m} \parallel \frac{3}{4}\right)}{\partial m}$$

$$\leq \frac{1}{m+1} - \frac{2 + \tilde{\alpha}}{1 + \tilde{\alpha}} \log 2 + \frac{d}{m+n} \left(\log \frac{d}{3(n+m-d)}\right)$$

$$\leq -\frac{1}{1 + \tilde{\alpha}} \log 2 + 0.1$$

for $m \geq 10$. For fixed $n$, $\tilde{\alpha}$ is monotonically increasing as a function of $d$. By the quantum singleton bound, $\frac{d-1}{m+1} < \frac{1}{2}$, and $\tilde{\alpha} < 3$ even in this case. Therefore $\frac{\partial f(n, m, d)}{\partial m} \leq -0.05$ when $m \geq 10$, so taking

$$m = \tilde{m} = 20 \log \frac{m+1}{\varepsilon} + O(1) = O(d \log \frac{n}{d} + \log \frac{1}{\varepsilon})$$

suffices to make $f(n, m, d) < 0$.

**Appendix B: Fault-tolerant measurement**

For completeness, we include a code switching circuit between adjacent codes, using Shor-style measurement [19]. We assume access to a collection of verified CAT states. Let $g = P_1 \otimes \ldots \otimes P_n$ and $g' = P'_1 \otimes \ldots \otimes P'_n$. The measurements are done on the supports of $g$ and $g'$. To make the diagram simpler, we suppose that the supports include qubits 1, 2, and $n$. Then the circuit obtained from the SRA to convert from the code with stabilizer $g'$ to the adjacent code with stabilizer $g$ is given by the following.

![FIG. 1. A generic circuit switching between adjacent codes using Shor-style measurement.](image-url)
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