Hasimoto Maps for Nonlinear Schrödinger Equations in Minkowski Space

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Abstract
In this paper, we study the vortex filament flow for timelike and spacelike curves in Minkowski 3-space. The vortex filament flow equations of the timelike and the spacelike curves are equivalent to the nonlinear Schrödinger equation and the heat equation, respectively. As a consequence, we prove that a soliton of the nonlinear Schrödinger equations of the timelike curve gives a solution of a traveling wave on a line at infinity. Also, we study a solution of a traveling wave of the nonlinear Schrödinger equations of the spacelike curve in terms of a new complex frame. Finally, we discuss the method to find the exact shape of the timelike and the spacelike curves from the vortex filament by solving the Frenet vectors of these curves and provide applications to illustrate the method.

Keywords Hasimoto map · Nonlinear Schrödinger equation · Travelling wave · Binormal flow · Evolution equation

Mathematics Subject Classification 35C07 · 53Z05
1 Introduction

Integrable nonlinear evolution equations occur in many branches of physics, applied mathematics and many fields [3, 4, 6, 14, 21, 24]. Such equations possess a number of interesting properties such as soliton solutions, infinite number of conservation laws, infinite number of symmetries, Bäcklund and Darboux transformations, bi-Hamiltonian structures and so on, see [5, 7, 8, 11, 13, 15]. The study of geometrical flows of curves has a deep connection with integrable nonlinear evolution equations. One of the simplest examples illustrating this connection arises from a vortex filament flow in two and three dimensional inviscid fluid dynamics. In three dimensional space, a vortex filament flow in an inviscid fluid can be described by the dynamical evolution of its vortex filament flow, which is given by the binormal flow $\gamma_t = \kappa \mathbf{b}$ for an arc-length parameterized curve $\gamma(s, t)$. In the work [12], Hasimoto discovered the relationship between integrable nonlinear evolution equations and vortex filament flows and he showed that the nonlinear Schrödinger (NLS) equation is equivalent to the binormal flow $\gamma_t = \kappa \mathbf{b}$ of the curve $\gamma(s, t)$ by using a transformation relating the wave function of the NLS equation to the curvature and the torsion of the curves (so-called Hasimoto transformation). Curve flows for the vortex filament have been studied by many experts and geometers [2, 10–11, 19], etc. Anco and Asadi [1] studied the general results on parallel frames and Hasimoto variables to extend the geometrical relationships among the NLS equation, the vortex filament equation and the Heisenberg spin model to the general setting of Hermitian symmetric spaces. Arroyo, Garay and Pámpano [2] studied curve motions by the binormal flow with the curvature and the torsion depending velocity and sweeping out immersed surfaces, and they obtained the vortex filaments evolving with constant torsion which arise from extremal curves of the curvature energy functionals. Also, Xu and Cao [23] gave three nonlinear partial differential equations which are associated with binormal flows of constant torsion curves in Minkowski 3-space, and the authors gave Bäcklund transformations for the equations, as well as for surfaces swept out by related moving curves. Mohamed [17] investigated the general description of the binormal flow of a spacelike and a timelike curve in a 3-dimensional de-Sitter space and gave some explicit examples of a binormal flow of the curves. In [22] Wang investigated the nonlinear stability of Hasimoto solitons, in energy space, for a fourth order NLS equation which arises in the context of the vortex filament. The traveling wave solutions play an important role in the long time dynamics of NLS equations at infinity. Ivey [13] discussed travelling wave solutions to the vortex filament flow generated by elastica produce surfaces in Euclidean 3-space that carry mutually orthogonal foliations by geodesics and by helices.

The outline of the paper is organized as follows: In Sect. 2, we give some geometric concepts of nonnull curves in Minkowski 3-space. In Sect. 3, we study nonlinear Schrödinger equations of timelike and spacelike curves and give Hasimoto travelling wave for these curves. In Sect. 4, we discuss the method to find the exact shape of the timelike and spacelike curve from the vortex filament flows by solving the tangent, principal normal and binormal vector in Minkowski
3-space. In the last section, we give some applications to find the position vector of the spacelike and timelike curve from the single soliton solution of the nonlinear Schrödinger equation.

2 Preliminaries

The Minkowski 3-space \( \mathbb{R}^3_1 \) is a real space \( \mathbb{R}^3 \) with the indefinite inner product \( \langle \cdot, \cdot \rangle \) defined on each tangent space by

\[
\langle x, y \rangle = -x_1 y_1 + x_2 y_2 + x_3 y_3,
\]

where \( x = (x_1, x_2, x_3) \) and \( y = (y_1, y_2, y_3) \) are vectors in \( \mathbb{R}^3 \).

A nonzero vector \( x \) in \( \mathbb{R}^3_1 \) is said to be spacelike or timelike if \( \langle x, x \rangle > 0 \) or \( \langle x, x \rangle < 0 \), respectively. Similarly, an arbitrary curve \( \gamma = \gamma(s) \) is spacelike or timelike if all of its tangent vector \( \frac{dy}{ds} = \gamma'(s) \) are spacelike or timelike, respectively.

For two vectors \( x = (x_1, x_2, x_3) \) and \( y = (y_1, y_2, y_3) \) in \( \mathbb{R}^3_1 \), a Lorentz cross product \( x \times y \) is defined by

\[
x \times y = (-x_2 y_3 + x_3 y_2, x_1 y_3 - x_3 y_1, x_2 y_1 - x_1 y_2).
\]

Let \( \gamma : I \rightarrow \mathbb{R}^3_1 \) be a spacelike or timelike curve parametrized by the arc-length \( s \) in Minkowski 3-space \( \mathbb{R}^3_1 \) and the vector \( \gamma_s(s) = t(s) \) be the unit tangent vector of \( \gamma \) with \( ||t(s)|| = \varepsilon_1 \). If \( \varepsilon_1 = 1 \), the curve \( \gamma \) is spacelike and if \( \varepsilon_1 = -1 \), the curve \( \gamma \) is timelike.

Since \( \gamma_{ss} \) is perpendicular to \( t \), we define the principal normal vector \( n \) as the normalized vector \( \gamma_{ss} \) and take the binormal vector \( b \) which is the unique vector perpendicular to the tangent plane generated by \( \{t(s), n(s)\} \) at every point \( \gamma(s) \) of \( \gamma \). In this case, \( \{t, n, b\} \) is the Frenet frame of the curve \( \gamma \) and the Frenet formulas are expressed in matrix notion as (cf. [16]):

\[
\begin{bmatrix}
  t \\
  n \\
  b
\end{bmatrix}_s =
\begin{bmatrix}
  0 & \varepsilon_2 \kappa & 0 \\
  -\varepsilon_1 \kappa & 0 & -\varepsilon_3 \tau \\
  0 & \varepsilon_2 \tau & 0
\end{bmatrix}
\begin{bmatrix}
  t \\
  n \\
  b
\end{bmatrix},
\]

where \( \kappa \) and \( \tau \) are the curvature and the torsion of the curve \( \gamma \). Here \( \varepsilon_2 \) and \( \varepsilon_3 \) are the signs of the vectors \( n \) and \( b \), respectively.

Suppose next that we have a fluid in Minkowski 3-space \( \mathbb{R}^3_1 \) that evolves according to a given one parameter family of diffeomorphism yielding the position of a fluid particle. The corresponding vortex filament flow is assumed to be parametrized by the arc length \( s \) and it is expressed as [12]

\[
\gamma_t = \gamma_s \times \gamma_{ss}.
\]

Now, we explain the geometric meaning of the evolution equation (2) of the spacelike or timelike curve in \( \mathbb{R}^3_1 \). We known that the flow is binormal, that is, \( \gamma_t = \kappa b \), the time evolution of the moving frames \( \{t, n, b\} \) of the timelike curve \( \gamma \) is expressed as (cf. [7])

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and the time evolution of the spacelike curve $\gamma$ is given by

$$
\begin{pmatrix}
  t \\
  n \\
  b 
\end{pmatrix}_t = \begin{pmatrix}
  0 & \kappa \tau & \kappa_s \\
  \kappa \tau & 0 & \kappa \frac{\kappa_s}{\kappa} - \tau^2 \\
  \frac{\kappa_s}{\kappa} - \kappa \frac{\kappa_s}{\kappa} + \tau^2 & 0 & 0 
\end{pmatrix} \begin{pmatrix}
  t \\
  n \\
  b 
\end{pmatrix},
$$

and the time evolution of the spacelike curve $\gamma$ is given by

$$
\begin{pmatrix}
  t \\
  n \\
  b 
\end{pmatrix}_t = \begin{pmatrix}
  0 & \varepsilon_2 \kappa \tau & -\varepsilon_2 \varepsilon_3 \kappa_s \\
  -\varepsilon_1 \kappa \tau & 0 & -\varepsilon_1 \varepsilon_3 \left( \frac{\kappa_s}{\kappa} - \tau^2 \right) \\
  -\varepsilon_1 \varepsilon_3 \kappa_s & \varepsilon_1 \varepsilon_2 \left( \frac{\kappa_s}{\kappa} + \tau^2 \right) & 0 
\end{pmatrix} \begin{pmatrix}
  t \\
  n \\
  b 
\end{pmatrix}.
$$

It is well-known that equation (2) is equivalent to the NLS equation for the timelike curve $\gamma$ as follows [10]:

$$
i \psi_t = -\psi_{ss} + \frac{1}{2} |\psi|^2 \psi
$$

via the Hasimoto transformations

$$
\psi(s,t) = \kappa(s,t) \exp \left( -i \int_s^t \tau(\rho,t) d\rho \right).
$$

The NLS equation describes a wide range of physical phenomena and it has many applications.

Next, if $\gamma$ is the spacelike curve with a timelike principal normal and a timelike binormal vector, equation (2) is equivalent to the nonlinear heat system [10]

$$
\phi_t = \phi_{ss} + \phi^2 \phi, \quad \phi_t = -\phi_{ss} - \phi^2 \phi
$$

in terms of the Hasimoto transformations

$$
\phi(s,t) = \kappa(s,t) \exp \left( -i \int_s^t \tau(\rho,t) d\rho \right), \quad \phi(s,t) = \kappa(s,t) \exp \left( i \int_s^t \tau(\rho,t) d\rho \right).
$$

### 3 Vortex Filament Equations

In this section, we construct the position vector of the spacelike or timelike curve from the single soliton solution of the nonlinear Schrödinger equation. It gives a geometrical tool to analyze localized evolution of dynamics in various dynamical systems. To obtain the main results we split it into two cases.
3.1 Timelike Vortex Filament

**Theorem 3.1** (cf. [7]) Let \( \gamma \) be a timelike curve parametrized by the arc length \( s \) with the curvature \( \kappa \) and the torsion \( \tau \) and satisfies the timelike binormal flow \( \gamma_t = \kappa \mathbf{b} \) in Minkowski 3-space. If we take the Hasimoto transformation

\[
\psi(s, t) = \kappa(s, t) \exp \left( -i \int_s^t \tau(s, t) d\tilde{s} \right),
\]  

then \( \psi \) satisfies the nonlinear Schrödinger equation

\[
i \psi_t = -\psi_{ss} + \frac{1}{2} \left( |\psi|^2 + R(t) \right) \psi
\]

for some smooth function \( R(t) \).

**Theorem 3.2** (Timelike Hasimoto travelling wave) If we consider a soliton solution of the NLS equation (9) such that \( \kappa \to 0 \) as \( s \to \infty \), then it gives a solution of a traveling wave with a kink that becomes a line at infinity.

**Proof** To prove the theorem, we take the new variable \( \rho = s + a - ct \) with a constant velocity \( c \) and a positive constant \( a \), then this variable will be used in order to obtain the our result with a soliton of the nonlinear Schrödinger equation. The variable implies

\[
\psi(\rho) = \kappa(\rho) \exp \left( -i \int_s^\rho \tau(\rho) d\rho \right).
\]  

Since (10) is the solution of (9), the real and imaginary parts of (9) lead to

\[
c \kappa(\rho)(\tau(\rho) - \tau(a - ct)) = \kappa''(\rho) - \kappa(\rho)\tau^2(\rho) - \frac{1}{2}(\kappa^2(\rho) + R(t))\kappa(\rho),
\]  

\[
c \kappa'(\rho) = -2\kappa'(\rho)\tau(\rho) - \kappa(\rho)\tau'(\rho).
\]

Then, we easily obtain

\[
(c + 2\tau)\kappa^2 = 0
\]

from this we have

\[
\tau = \tau_0 = -\frac{1}{2}c
\]

assuming that the curvature is not identically zero. So the torsion is constant along the vortex filament. It follows that a solution of the ODE (11) is given by

\[
\kappa(\rho) = 2b_1 \text{csch}(b_1 \rho),
\]
and in this case \( R = 2(b_1^2 - \tau_0^2) \) with a nonzero constant \( b_1 \). Thus the curvature and the torsion are determined, so we can construct the shape of the timelike binormal flow by using (13) and (14).

Now, we will construct the position vector of \( \gamma \) satisfying the timelike binormal flow. Since \( \gamma \) is the timelike curve, we take \( \epsilon_1 = -1, \epsilon_2 = 1 \) and \( \epsilon_3 = 1 \) in (1) and we obtain the following equation:

\[
\tau_0(t' - \kappa n) = \left( \frac{1}{\kappa}(b'' + \tau_0^2 b) \right)' - \kappa b' = 0.
\]

It is equivalently to

\[
\frac{d^3 b}{d\xi^3} + \coth(\xi) \frac{d^2 b}{d\xi^2} + (Q^2 - 4\csc^2(\xi)) \frac{db}{d\xi} + \coth(\xi) Q^2 b = 0,
\]

where \( \xi = b_1 \rho \) and \( Q = \frac{\tau_0}{b_1} \).

If we put

\[
c = \frac{db}{d\xi} + \coth(\xi)b,
\]

then equation (15) is rewritten as

\[
\frac{d^2 c}{d\xi^2} + (Q^2 - 2\csc^2(\xi))c = 0.
\]

It’s solution is given by

\[
c = (\coth(\xi) + iQ) \exp(-iQ\xi).
\]

By combining (16) and (18), we can solve the ODE and the solutions become

\[
b = \csch(\xi), \quad b = \left(1 - Q^2 + 2iQ \coth(\xi)\right) \exp(-iQ\xi).
\]

We substitute (19) into Frenet formula (1) with \( \epsilon_1 = -1, \epsilon_2 = 1 \) and \( \epsilon_3 = 1 \) and determine the coefficients so as to satisfy, without loss generality, the conditions for the vortex filament to be parallel to the \( x \) axis at infinity as:

\[
t_x \to 1 \quad \text{as} \quad \xi \to \infty
\]

\[
n_y + in_z = i(b_y + b_z) = \exp(-i(\tau_0 s + A(t))) \quad \text{as} \quad \xi \to \infty,
\]

where \( A(t) \) is a real function of \( t \) and the subscripts denote the \( x, y, \) and \( z \) components of the vector, respectively. The above conditions are suggested by asymptotic behaviour of the solution of Frenet formulas. First, we can find the unit spacelike binormal vector \( b \) as

\[
b_x = 2\mu Q\csch(\xi), \quad b_y + ib_z = i\mu(1 - Q^2 + 2iQ \coth(\xi)) \exp(-i(\tau_0 s + A(t)))
\]
and Frenet equation $\mathbf{b}_s = \tau_0 \mathbf{n}$ implies the unit spacelike principal normal vector $\mathbf{n}$ as follows:

$$\mathbf{n}_x = -2\mu \mathrm{csch}^2(\xi) \cosh(\xi),$$

$$\mathbf{n}_y + i\mathbf{n}_z = [-1 + 2\mu \coth(\xi)(iQ + \coth(\xi))] \exp(-i(\tau_0 s + A(t))).$$

So, we can compute the unit timelike tangent vector $\mathbf{t}$ by using the principal and binormal normal vectors and it leads to

$$\mathbf{t}_x = 2\mu \coth^2(\xi) + \mu \left( \frac{\tau_0^2 - b_1^2}{b_1^2} \right),$$

$$\mathbf{t}_y + i\mathbf{t}_z = -2\mu \mathrm{csch}(\xi)(iQ + \coth(\xi)) \exp(-i(\tau_0 s + A(t))),$$

where $\mu = \frac{b_1^2}{\tau_0^2 + b_1^2}$.

By integrating of the tangent vector, the timelike curve $\gamma$ is expressed as

$$\gamma_x = \mu \left( \frac{\tau_0^2 - b_1^2}{b_1^2} \right) s + 2\mu \frac{\tau_0}{b_1} (\xi - \coth(\xi)),
$$

$$\gamma_y + i\gamma_z = \frac{2\mu}{b_1} \mathrm{csch}(\xi) \exp(-i(\tau_0 s + (\tau_0^2 - b_1^2)t)),
$$

and the curve $\gamma$ satisfies the time evolution (3), in this case we can obtain $A(t) = (\tau_0^2 - b_1^2)t$. Thus the curve $\gamma$ determined by (20) satisfies the timelike binormal flow $\gamma_t = \kappa \mathbf{b}$ and gives a soliton solution of the traveling wave for the timelike curve.

### 3.2 Spacelike Vortex Filament

In Sect. 2, we explain briefly the vortex filament for a spacelike curve in Minkowski 3-space. In this case the corresponding vortex filament $\gamma_t = \gamma_s \times \gamma_{ss} = \kappa \mathbf{b}$ is equivalent to the nonlinear heat system (6) by using Hasimoto transformations (7). In this subsection, we want to give the parametrization of the spacelike curve from the single solitary wave solution of the nonlinear Schrödinger equation for a spacelike curve in Minkowski 3-space. In [18] and [15], authors considered the new complex frame in terms of the Frenet frame of a space curve in Euclidean 3-space and studied integrable system for a vortex filament flow.

Now, we consider a complex frame $\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_2^*\}$ for the spacelike curve $\gamma$ in Minkowski 3-space as follows:

$$\mathbf{p}_1 = \mathbf{b},$$
$$\mathbf{p}_2 = (\mathbf{n} + it)e^{-i/\kappa},$$
$$\mathbf{p}_2^* = (\mathbf{n} - it)e^{i/\kappa},\quad \text{(21)}$$
and the Hasimoto transformation

$$\phi(s, t) = \tau(s, t) \exp \left( -i \int^{s} \kappa(s') ds' \right). \quad (22)$$

In this case, if we take $\gamma_s = p_t$, the spacelike vortex filament equation is expressed by

$$\gamma'_t = \gamma_s \times \gamma_{ss} = \tau t. \quad (23)$$

On the other hand, we have the time evolution for Frenet frame of the spacelike curve as follows [10]:

$$\begin{pmatrix} t \\ n \\ b \end{pmatrix}_t = \begin{pmatrix} 0 & \frac{\nu}{\tau} - \kappa^2 & \tau_s \\ -\frac{\nu}{\tau} + \kappa^2 & 0 & \kappa \tau \\ \tau_s & \kappa \tau & 0 \end{pmatrix} \begin{pmatrix} t \\ n \\ b \end{pmatrix}. \quad (24)$$

**Theorem 3.3** ([10]) Let $\gamma$ be a spacelike curve parametrized by the arc length $s$ with the curvature $\kappa$ and the torsion $\tau$ and satisfies the spacelike vortex filament equation (23) in Minkowski 3-space. Then the Hasimoto transformation $\phi$ is a solution of the nonlinear Schrödinger equation

$$i\phi_t = -\phi_{ss} + \frac{1}{2} \left( |\phi|^2 + R(t) \right) \phi \quad (25)$$

for some smooth function $R(t)$.

**Theorem 3.4** (Spacelike Hasimoto traveling wave) If we consider a soliton solution of the nonlinear Schrödinger equation (25) such that $\tau \to 0$ as $s \to \infty$, then it gives a solution of a traveling wave with a kink that becomes a line at infinity.

**Proof** To prove the theorem, consider the new variable $\rho = s + a - ct$ with a constant velocity $c$ and a positive constant $a$, then the variable implies

$$\phi(\rho) = \tau(\rho) \exp \left( -i \int^{s} \kappa(\rho) ds \right). \quad (26)$$

Since (26) is the solution of (25), the real and imaginary parts of (25) lead to

$$c\tau(\rho)(\kappa(\rho) - \kappa(a - ct)) = \tau''(\rho) - \tau(\rho)\kappa^2(\rho) - \frac{1}{2}(\tau^2(\rho) + R(t))\tau(\rho), \quad (27)$$

$$c\tau'(\rho) = -2\tau'(\rho)\kappa(\rho) - \tau(\rho)\kappa'(\rho). \quad (28)$$

Look at (11), (12), (27) and (28), they are symmetric with respect to $\kappa$ and $\tau$, respectively. So, we have
\begin{equation}
\kappa = \kappa_0 = -\frac{1}{2} c,
\end{equation}

\begin{equation}
\tau(\rho) = 2b_1 \text{csch}(b_1 \rho),
\end{equation}

where \( R = 2(b_2^2 - \kappa_0^2) \) with a nonzero constant \( b_1 \).

Since \( \gamma \) is the spacelike curve, we take \( \varepsilon_1 = 1, \varepsilon_2 = 1 \) and \( \varepsilon_3 = -1 \) in (1) and we obtain the following equation:

\[ \kappa_0 (b' - \tau n) = \left( \frac{1}{\tau} (t'' + \kappa_0^2 t) \right)' - \tau t' = 0. \]

It is equivalently to

\begin{equation}
\frac{d^3 t}{d\xi^3} + \coth(\xi) \frac{d^2 t}{d\xi^2} + (Q^2 - 4 \text{csch}^2(\xi)) \frac{dt}{d\xi} + \coth(\xi) Q^2 t = 0,
\end{equation}

where \( \xi = b_1 \rho \) and \( Q = \frac{\kappa_0}{b_1} \).

Applying the similar method of the timelike vortex filament, we have

\[ t_x = 2\mu \text{csch}(\xi), \]

\[ t_y + t_z = i\mu(1 - Q^2 - 2iQ \coth(\xi)) e^{-i(\kappa_0 s + (\kappa_0^2 - b_1^2) t)}, \]

\[ n_x = -2\mu \text{csch}^2(\xi) \coth(\xi), \]

\[ n_y + n_z = [1 + 2\mu \coth(\xi) (iQ - \coth(\xi))] e^{-i(\kappa_0 s + (\kappa_0^2 - b_1^2) t)}, \]

\[ b_x = 2\mu \coth^2(\xi) + \mu \left( \frac{\kappa_0^2 - b_1^2}{b_1^2} \right), \]

\[ b_y + b_z = 2\mu \text{csch}(\xi) (-iQ + \coth(\xi)) e^{-i(\kappa_0 s + (\kappa_0^2 - b_1^2) t)}, \]

where \( \mu = \frac{b_1^2}{\kappa_0^2 + b_1^2} \). Thus, the spacelike curve \( \gamma_t = \tau t \) is determined by

\begin{equation}
\gamma_x = -\frac{2\mu}{b_1} (\xi - \coth(\xi)) - \mu \left( \frac{\kappa_0^2 - b_1^2}{b_1^2} \right) s,
\end{equation}

\[ \gamma_y + i\gamma_z = \frac{2\mu}{b_1} \text{csch}(\xi) \exp(i(\kappa_0 s + (\kappa_0^2 - b_1^2) t)). \]

This provides the traveling wave soliton solution of the spacelike curve.
4 Position Vectors of Vortex Filament

In this section, we give the method to find the exact shape of the timelike and spacelike curve from the vortex filament by solving the tangent, principal normal and binormal vector in Minkowski 3-space. The method is called an inverse Hashimoto transformation and we use the tool described by Shah [20].

First of all, let \( \gamma \) be a timelike curve with \( \varepsilon_1 = -1 , \varepsilon_2 = \varepsilon_3 = 1 \) in (1). Then these equations can be expressed as the first order ODEs:

\[
\frac{d}{ds} \begin{pmatrix} t \\ n \\ b \end{pmatrix} = \begin{pmatrix} 0 & \kappa(s,t) & 0 \\ \kappa(s,t) & 0 & -\tau(s,t) \\ 0 & \tau(s,t) & 0 \end{pmatrix} \begin{pmatrix} t \\ n \\ b \end{pmatrix},
\]

(34)

and this can be represented as follows

\[
\frac{dU}{ds} = A(s,t)U,
\]

(35)

where \( U = (t \ n \ b)^T \) and

\[
A(s,t) = \begin{pmatrix} 0 & \kappa(s,t) & 0 \\ \kappa(s,t) & 0 & -\tau(s,t) \\ 0 & \tau(s,t) & 0 \end{pmatrix}.
\]

To solve the ODE (35), we use \( \exp(- \int^s A(\sigma,t)d\sigma) \) as an ansatz for the integrating factor and multiples this ansatz both sides of (35), then we have

\[
\exp \left( - \int^s A(\sigma,t)d\sigma \right) \left( \frac{dU}{ds} - A(s,t)U \right) = 0,
\]

that is,

\[
\frac{d}{ds} \left( \exp \left( - \int^s A(\sigma,t)d\sigma \right) U(s,t) \right) = 0.
\]

It follows that we get

\[
\exp \left( - \int^s A(\sigma,t)d\sigma \right) U(s,t) = C(t),
\]

(36)

where \( C(t) \) is a matrix that is dependent on time \( t \).

If \( \exp(- \int^s A(\sigma,t)d\sigma) \) is non-singular, then equation (36) leads to

\[
U(s,t) = \exp(\mathcal{M}(s,t))C(t),
\]

(37)

where
\[ \mathcal{M}(s, t) = \int_{s}^{t} \mathcal{A}(\sigma, t) d\sigma = \begin{pmatrix} 0 & \int_{s}^{t} \kappa(\sigma, t) d\sigma & \int_{s}^{t} \tau(\sigma, t) d\sigma \\ \int_{s}^{t} \kappa(\sigma, t) d\sigma & 0 & 0 \\ \int_{s}^{t} \tau(\sigma, t) d\sigma & 0 & 0 \end{pmatrix}. \]

Now, we must show how to calculate \( U(s, t) \) from (37) to demonstrate the method. Applying matrix exponential \( \exp(\mathcal{M}) = \sum_{k=0}^{\infty} \frac{1}{k!} \mathcal{M}^k \) for the \( 3 \times 3 \) matrix \( \mathcal{M} \), \( \exp(\mathcal{M}) \) is given by

\[
\exp(\mathcal{M}) = \begin{pmatrix} \frac{1}{w^2}(q^2 - p^2 \cos w) & \frac{p}{w} \sin w & \frac{pq}{w^2}(\cos w - 1) \\
\frac{p}{w} \sin w & \cos w & -\frac{q}{w} \sin w \\
\frac{pq}{w^2}(1 - \cos w) & \frac{q}{w} \sin w & \frac{1}{w^2}(q^2 \cos w - p^2) \end{pmatrix},
\]

where

\[
p = \int_{s}^{t} \kappa(\sigma, t) d\sigma, \quad q = \int_{s}^{t} \tau(\sigma, t) d\sigma, \quad w^2 = q^2 - p^2. \quad (38)
\]

Let \( e_{ij} \) and \( c_{ij} \) be the \((i, j)\)-th components of the matrices \( \exp(\mathcal{M}) \) and \( C(t) \), respectively. Then equation (37) becomes

\[
U(s, t) = \begin{pmatrix} \sum_{k=1}^{3} e_{1k}(s, t)c_{k1}(t) & \sum_{k=1}^{3} e_{1k}(s, t)c_{k2}(t) & \sum_{k=1}^{3} e_{1k}(s, t)c_{k3}(t) \\
\sum_{k=1}^{3} e_{2k}(s, t)c_{k1}(t) & \sum_{k=1}^{3} e_{2k}(s, t)c_{k2}(t) & \sum_{k=1}^{3} e_{2k}(s, t)c_{k3}(t) \\
\sum_{k=1}^{3} e_{3k}(s, t)c_{k1}(t) & \sum_{k=1}^{3} e_{3k}(s, t)c_{k2}(t) & \sum_{k=1}^{3} e_{3k}(s, t)c_{k3}(t) \end{pmatrix}.
\]

Since \( U = (t \quad n \quad b)^T \), the tangent vector of the timelike curve \( \gamma \) to the vortex filament equation \( \gamma_t = \tau b \) is given by

\[
t(s, t) = \begin{pmatrix} \sum_{k=1}^{3} e_{1k}(s, t)c_{k1}(t) \\
\sum_{k=1}^{3} e_{1k}(s, t)c_{k2}(t) \\
\sum_{k=1}^{3} e_{1k}(s, t)c_{k3}(t) \end{pmatrix}.
\]

it follows that the position vector \( r(s, t) \) of the timelike curve \( \gamma \) can be expressed as

\[
r(s, t) = \begin{pmatrix} x_0(t) + \sum_{k=1}^{3} c_{k1}(t) \int_{s}^{t} e_{1k}(\sigma, t) d\sigma \\
y_0(t) + \sum_{k=1}^{3} c_{k2}(t) \int_{s}^{t} e_{1k}(\sigma, t) d\sigma \\
z_0(t) + \sum_{k=1}^{3} c_{k3}(t) \int_{s}^{t} e_{1k}(\sigma, t) d\sigma \end{pmatrix}. \quad (39)
\]

Thus, given the curvature \( \kappa(s, t) \) and the torsion \( \tau(s, t) \) of the timelike curve \( \gamma \) to the vortex filament we can construct the curve \( \gamma \) with the help of (39).

By the similar discussion as above, we give the position vector of the spacelike curve \( \gamma \) to the vortex filament equation \( \gamma_t = \tau b \). In this case,

\[
\mathcal{M}(s, t) = \begin{pmatrix} 0 & \int_{s}^{t} \kappa(\sigma, t) d\sigma & \int_{s}^{t} \tau(\sigma, t) d\sigma \\
\int_{s}^{t} \kappa(\sigma, t) d\sigma & 0 & 0 \\
\int_{s}^{t} \tau(\sigma, t) d\sigma & 0 & 0 \end{pmatrix}
\]

and \( \exp(\mathcal{M}) \) is given by
where \( r^2 = p^2 - q^2 \).

Thus the position vector \( r(s, t) \) of the spacelike curve \( \gamma \) to the vortex filament equation with the help of (37) is determined by

\[
\mathbf{r}(s, t) = \left( x_0(t) + \sum_{k=1}^{3} c_{k1}(t) \int s e^{3k(\sigma, t)} d\sigma, y_0(t) + \sum_{k=1}^{3} c_{k2}(t) \int s e^{3k(\sigma, t)} d\sigma, z_0(t) + \sum_{k=1}^{3} c_{k3}(t) \int s e^{3k(\sigma, t)} d\sigma \right).
\]

(40)

5 Applications

Consider the new variable

\[
\Phi = \varphi_\pm \exp \left( -\frac{1}{2} i \int R(t) dt \right),
\]

where \( \varphi_- \) and \( \varphi_+ \) are the Hasimoto transformations of the timelike and spacelike curves to the vortex filament, respectively, that is, \( \varphi_- = \psi \) and \( \varphi_+ = \phi \). Then, the NLS equations (9) and (25) of the timelike curve and the spacelike curve are written as

\[
i \Phi_t = -\Phi_{ss} + \frac{1}{2} | \Phi |^2 \Phi = 0.
\]

(41)

In order to obtain the solitary wave solution to the NLS equation (41), the starting ansatz is taken to be

\[
\Phi(s, t) = u(s, t) \exp(ivi(s, t)).
\]

This ansatz is used in the work of inverse Hasimoto transformation. Substituting the last equation into (41) and separating the real and imaginary parts, it is obtained as

\[
\begin{align*}
    u_t(s, t) &= -2v_s(s, t)u_s(s, t) - v_{ss}(s, t)u(s, t), \\
    v_t(s, t) &= u_{ss}(s, t) - v_s^2(s, t)u(s, t) - \frac{1}{2} u^3(s, t).
\end{align*}
\]

(42)

Example 5.1 If we take

\[
u(s, t) = 2 \tanh(s - 2ct), \quad v(s, t) = cs - (c^2 + 2)t,
\]

it is a solution of the PDEs (42) and from (38) we have...
\[ p(s, t) = 2 \ln \cosh(s - 2ct), \quad q(s, t) = -cs + (c^2 + 2)t \]

which are connected to the curvature and the torsion of the timelike and spacelike curve, respectively. Thus, the inverse Hasimoto parametrization can directly be constructed by

\[ \mathbf{r}(s, t) = (x(s, t), y(s, t), z(s, t)), \quad (43) \]

where

\[
\begin{aligned}
x(s, t) &= t \int_0^s \frac{2 \ln \cosh(s - 2t) \sin \left( \sqrt{(s+3t)^2 - (2 \ln \cosh(s-2t))^2} \right)}{\sqrt{(s+3t)^2 - (2 \ln \cosh(s-2t))^2}} \, ds, \\
y(s, t) &= t + \int_0^s \frac{(-s+3t)^2 - (2 \ln \cosh(s-2t))^2}{\sqrt{(s+3t)^2 - (2 \ln \cosh(s-2t))^2}} \cos \left( \sqrt{(s+3t)^2 - (2 \ln \cosh(s-2t))^2} \right) \, ds, \\
z(s, t) &= t \int_0^s \frac{2 \ln \cosh(s-2t) \cos \left( \sqrt{(s+3t)^2 - (2 \ln \cosh(s-2t))^2} \right) - 1}{\sqrt{(s+3t)^2 - (2 \ln \cosh(s-2t))^2}} \, ds,
\end{aligned}
\]

and

\[
\begin{aligned}
x(s, t) &= t \int_0^s \frac{(-s+3t) \sin \left( \sqrt{(2 \ln \cosh(s-2t))^2 - (s+3t)^2} \right)}{\sqrt{(2 \ln \cosh(s-2t))^2 - (s+3t)^2}} \, ds, \\
y(s, t) &= t + \int_0^s \frac{2 \ln \cosh(s-2t)(-s+3t)(\cos \left( \sqrt{(2 \ln \cosh(s-2t))^2 - (s+3t)^2} \right) - 1)}{(2 \ln \cosh(s-2t))^2 - (s+3t)^2} \, ds, \\
z(s, t) &= t \int_0^s \frac{(2 \ln \cosh(s-2t))^2 - (s+3t)^2 \cos \left( \sqrt{(2 \ln \cosh(s-2t))^2 - (s+3t)^2} \right)}{(2 \ln \cosh(s-2t))^2 - (s+3t)^2} \, ds,
\end{aligned}
\]

if \( \gamma(s) \) is a timelike curve,

in this case we choose a matrix \( C(t) \) given by

\[
C(t) = \begin{pmatrix} 0 & 1 & 0 \\ t & 0 & 0 \\ 0 & 0 & t \end{pmatrix},
\]

and \( x_0(t) = 0, y_0(t) = t, z_0(t) = 0, c = 1 \).

**Example 5.2** We consider that the torsion of the timelike curve \( \gamma \) vanishes, that is, \( \tau = 0 \). In this case, \( v = v(s, t) \) is constant in (42) and we have

\[ u_t = 0, \quad u_{ss} - \frac{1}{2} u^3 = 0. \]

Its solution is given by \( u(s, t) = -\frac{2}{s} \), which implies

\[ p = -2 \ln s, \quad q = q_0. \]

If we choose a matrix \( C(t) \) given by (46), then from (39) we have the inverse Hasimoto parametrization \( \mathbf{r}(s, t) \) as follows:
6 Conclusion

One of classical nonlinear differential equations by through inverse scattering transform is the vortex filament equation \( \gamma_t = \gamma_s \times \gamma_{ss} \) and this equation becomes the binormal flow \( \gamma_t = \mathbf{x} \times \mathbf{b} \) when the parameter \( s \) of the curve \( \gamma(s) \) is arc-length. We know that the binormal flow of the timelike curve or the spacelike curve is equivalent to the NLS equation or the heat equation, respectively. However, in [10] authors studied a spacelike curve with the new complex frame (21), in this case they showed that the vortex filament equation is equivalent to the nonlinear Schrödinger equation (25).

In this work, we construct the parametrizations of the timelike curve and the spacelike curve from the traveling wave soliton solution of the nonlinear Schrödinger equation. Also, we give the method to find the inverse Hasimoto transformation of the timelike and spacelike curve for the vortex filament by solving the Frenet vectors in Minkowski 3-space and provide applications to illustrate the inverse Hasimoto transformation.

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Declarations

Conflict of interest The authors report no declarations of interest.

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