Order estimates of best orthogonal trigonometric approximations of classes of infinitely differentiable functions

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Abstract In this paper we establish exact order estimates for the best uniform orthogonal trigonometric approximations of the classes of $2\pi$-periodic functions, whose $(\psi, \beta)$-derivatives belong to unit balls of spaces $L_p$, $1 \leq p < \infty$, in the case, when the sequence $\psi(k)$ tends to zero faster, than any power function, but slower than geometric progression. Similar estimates are also established in the $L_s$-metric, $1 < s \leq \infty$ for the classes of differentiable functions, which $(\psi, \beta)$-derivatives belong to unit ball of space $L_1$.

1 Introduction

Let $L_p$, $1 \leq p < \infty$, be the space of $2\pi$-periodic functions $f$ summable to the power $p$ on $[0, 2\pi)$, with the norm $\|f\|_p = \left( \frac{2\pi}{0} |f(t)|^p dt \right)^{\frac{1}{p}}$; $L_\infty$ be the space of $2\pi$- periodic functions $f$, which are Lebesque measurable and essentially bounded with the norm $\|f\|_\infty = \text{ess sup}_t |f(t)|$.

Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be the function from $L_1$, whose Fourier series is given by

$$\sum_{k=-\infty}^{\infty} \hat{f}(k)e^{ikx},$$

where $\hat{f}(k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)e^{-ikt} dt$ are the Fourier coefficients of the function $f$. $\psi(k)$ is an arbitrary fixed sequence of real numbers and $\beta$ is a fixed real number. Then, if

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the series

\[ \sum_{k \in \mathbb{Z} \setminus \{0\}} \frac{\hat{f}(k)}{\psi(|k|)} e^{i(kx + \frac{\beta \pi}{2} \text{sign} k)} \]

is the Fourier series of some function \( \varphi \) from \( L_1 \), then this function is called the \((\psi, \beta)\)-derivative of the function \( f \) and is denoted by \( f^{\psi}_{\beta} \). A set of functions \( f \), whose \((\psi, \beta)\)-derivatives exist, is denoted by \( L^{\psi}_{\beta} \) (see [13]).

Let

\[ B^p_\beta := \{ \varphi \in L_p : ||\varphi||_p \leq 1, \varphi \perp 1 \}, \quad 1 \leq p \leq \infty. \]

If \( f \in L^{\psi}_{\beta} \), and, at the same time \( f^{\psi}_{\beta} \in B^p_\beta \), then we say that the function \( f \) belongs to the class \( L^{\psi}_{\beta, p} \).

By \( M \) we denote the set of all convex (downward) continuous functions \( \psi(t) \), \( t \geq 1 \), such that \( \lim_{t \to \infty} \psi(t) = 0 \). Assume that the sequence \( \psi(k), \, k \in \mathbb{N} \), specifying the class \( L^{\psi}_{\beta, p}, \, 1 \leq p \leq \infty \), is the restriction of the functions \( \psi(t) \) from \( \mathcal{M} \) to the set of natural numbers.

Following Stepanets (see, e.g., [13]), by using the characteristic \( \mu(\psi; t) \) of functions \( \psi \) from \( \mathcal{M} \) of the form

\[ \mu(t) = \frac{t}{\eta(t) - t}; \quad (1) \]

where \( \eta(t) = \eta(\psi; t) := \psi^{-1}(\psi(t)/2) \), \( \psi^{-1} \) is the function inverse to \( \psi \), we select the following subsets of the set \( \mathcal{M} \):

\[ \mathcal{M}^+_{\infty} = \{ \psi \in \mathcal{M} : \mu(\psi; t) \uparrow \infty \}. \]

\[ \mathcal{M}''_{\infty} = \{ \psi \in \mathcal{M}^+_{\infty} : \exists K > 0 \, \eta(\psi; t) - t \geq K \quad t \geq 1 \}. \]

The functions \( \psi_{r, \alpha}(t) = \exp(-\alpha t^r) \) are typical representatives of the set \( \mathcal{M}^+_{\infty} \). Moreover, if \( r \in (0, 1) \), then \( \psi_{r, \alpha} \in \mathcal{M}''_{\infty}. \) The classes \( L^{\psi}_{\beta, p} \), generated by the functions \( \psi = \psi_{r, \alpha} \) are denoted by \( L^{\psi}_{\beta, r, p} \).

For functions \( f \) from classes \( L^{\psi}_{\beta, p} \), we consider: \( L_{s} \)-norms of deviations of the functions \( f \) from their partial Fourier sums of order \( n - 1 \), i.e., the quantities

\[ \| \rho_n(f; \cdot) \|_s = \| f(\cdot) - S_{n-1}(f; \cdot) \|_s, \quad 1 \leq s \leq \infty, \quad (2) \]

where

\[ S_{n-1}(f; x) = \sum_{k=-n+1}^{n-1} \hat{f}(k) e^{ikx}; \]

and the best orthogonal trigonometric approximations of the functions \( f \) in metric of space \( L_s \), i.e., the quantities of the form

\[ \| f - \hat{f} \|_s = \| f(\cdot) - \hat{f}(\cdot) \|_s. \]
Order estimates of best orthogonal trigonometric approximations

\[ e_n^k(f)_s = \inf_{\gamma_n} \| f(\cdot) - S_{\gamma_n}(f; \cdot) \|_s, \ 1 \leq s \leq \infty; \]  \hspace{1cm} (3)

where \( \gamma_n, m \in \mathbb{N} \), is an arbitrary collection of \( m \) integer numbers, and

\[ S_{\gamma_n}(f; x) = \sum_{k \in \gamma_n} \hat{f}(k)e^{ikx}. \]

We set

\[ \varepsilon_n(L_{\beta,p})_s = \sup_{f \in L_{\beta,p}} \| p_n(f; \cdot) \|_s, \ 1 \leq p, s \leq \infty; \]  \hspace{1cm} (4)

\[ e_n^\perp(L_{\beta,p})_s = \sup_{f \in L_{\beta,p}} e_n^\perp(f)_s, \ 1 \leq p, s \leq \infty. \]  \hspace{1cm} (5)

The following inequalities follow from given above definitions [4] and [5]

\[ e_{2n}\perp(L_{\beta,p})_s \leq e_{2n-1}\perp(L_{\beta,p})_s \leq \varepsilon_n(L_{\beta,p})_s, \ 1 \leq p, s \leq \infty. \]  \hspace{1cm} (6)

In the case when \( \psi(k) = k^{-r}, r > 0 \), the classes \( L_{\beta,p}^\psi, 1 \leq p \leq \infty, \ \beta \in \mathbb{R} \) are well-known Weyl–Nagy classes \( W_{\beta,p}^\psi \). For these classes, the order estimates of quantities \( e_n(L_{\beta,p}^\psi)_s \) are known for \( 1 < p, s < \infty \) (see [4], [5]), for \( 1 \leq p < \infty, s = \infty, r > \frac{1}{p} \) and also for \( p = 1, 1 < s < \infty, r > \frac{1}{2}, \frac{1}{2} + \frac{1}{p} = 1 \) (see [5], [6]).

In the case, when \( \psi(k) \) tends to zero not faster than some power function, order estimates for quantities (6) were established in [1], [2], [11] and [12]. In the case, when \( \psi(k) \) tends to zero not slower than geometric progression, exact order estimates for \( e_n^\perp(L_{\beta,p})_s \) were found in [10] for all \( 1 \leq p, s \leq \infty \).

Our aim is to establish the exact-order estimates of \( e_n^\perp(L_{\beta,p})_\infty, 1 \leq p < \infty \), and \( e_n^\perp(L_{\beta,1})_s, 1 < s < \infty \), in the case, when \( \psi \) decreases faster than any power function, but slower than geometric progression (\( \psi \in \mathcal{M}^\psi_\infty \)).

2 Best orthogonal trigonometric approximations of the classes \( L_{\beta,p}^\psi, 1 < p < \infty \), in the uniform metric

We write \( a_n \asymp b_n \) to mean that there exist positive constants \( C_1 \) and \( C_2 \) independent of \( n \) such that \( C_1 a_n \leq b_n \leq C_2 a_n \) for all \( n \).

**Theorem 1.** Let \( 1 < p < \infty, \ \psi \in \mathcal{M}^\psi_\infty \) and the function \( \frac{\psi(t)}{\psi(t)} \uparrow \infty \) as \( t \to \infty \). Then, for all \( \beta \in \mathbb{R} \) the following order estimates hold

\[ e_{2n-1}\perp(L_{\beta,p})_\infty \asymp e_{2n}\perp(L_{\beta,p})_\infty \asymp \psi(n)(\eta(n) - n)^{\frac{1}{p}}. \]  \hspace{1cm} (7)

**Proof.** According to Theorem 1 from [8] under conditions \( \psi \in \mathcal{M}^\psi_\infty, \ \beta \in \mathbb{R}, \ 1 \leq p < \infty, \) for \( n \in \mathbb{N} \), such that \( \eta(n) - n \geq a > 2, \ \mu(n) \geq b > 2 \) the following esti-
mate is true
\[ \varepsilon_n(L^\varphi_{\beta,p}) = K_{a,b} (2p)^{1 - \frac{1}{p}} \psi(n)(\eta(n) - n)^{\frac{1}{p}}, \]  
(8)

where
\[ K_{a,b} = \frac{1}{\pi} \max \left\{ \frac{2b}{b - 2} + \frac{1}{a}, \ 2\pi \right\}. \]

Using inequalities (6) and (8), we obtain
\[ e_\perp^2_n(L^\varphi_{\beta,p}) \leq e_\perp^2_n(L^\varphi_{\beta,p}) \leq K_{a,b} (2p)^{1 - \frac{1}{p}} \psi(n)(\eta(n) - n)^{\frac{1}{p}}. \]  
(9)

Let us find the lower estimate for the quantity \( e_\perp^2_n(L^\varphi_{\beta,p}) \). With this purpose we construct the function
\[ f_{p,n}(t) = f_{p,n}(\psi; t) := \frac{\lambda_p}{\psi(n)(\eta(n) - n)^{\frac{1}{p}}} \left( \frac{1}{2} \psi(1) \psi(2n) + \sum_{k=1}^{n-1} \psi(k) \psi(2n - k) \cos kt + \sum_{k=n}^{2n} \psi^2(k) \cos kt \right), \quad \frac{1}{p} + \frac{1}{p'} = 1. \]  
(10)

Let us show that \( f_{p,n} \in L^\varphi_{\beta,p} \). The definition of \((\psi, \beta)\)-derivative yields
\[ (f_{p,n}(t))^{\psi}_{\beta} = \frac{\lambda_p}{\psi(n)(\eta(n) - n)^{\frac{1}{p}}} \left( \sum_{k=1}^{n-1} \psi(2n - k) \cos \left( kt + \frac{\beta \pi}{2} \right) \right. \]
\[ \left. + \sum_{k=n}^{2n} \psi(k) \cos \left( kt + \frac{\beta \pi}{2} \right) \right). \]  
(11)

Obviously
\[ |(f_{p,n}(t))^{\psi}_{\beta}| \leq \frac{\lambda_p}{\psi(n)(\eta(n) - n)^{\frac{1}{p}}} \left( \sum_{k=1}^{n-1} \psi(2n - k) + \sum_{k=n}^{2n} \psi(k) \right) < \]
\[ \frac{2\lambda_p}{\psi(n)(\eta(n) - n)^{\frac{1}{p}}} \left( \sum_{k=n}^{2n} \psi(k) \right) \leq \frac{2\lambda_p}{\psi(n)(\eta(n) - n)^{\frac{1}{p}}} \left( \psi(n) + \int_{n}^{m} \psi(u) du \right). \]  
(12)

To estimate the integral from the right part of formula (12), we use the following statement [7, p. 500].

**Proposition 1.** If \( \psi \in \mathfrak{M}_{\varphi}^\infty \), then for arbitrary \( m \in \mathbb{N} \), such that \( \mu(\psi, m) > 2 \) the following condition holds
\[ \int_{m}^{\infty} \psi(u) du \leq \frac{2}{1 - \frac{2}{\mu(m)}} \psi(m)(\eta(m) - m). \]  
(13)
Formulas (12) and (13) imply that
\[ |(f_{p,n}(t))_\beta^\gamma| \leq \frac{2\lambda_p}{\psi(n) \psi(\eta(n) - n)} \left( \psi(n) + \frac{2b}{b-2} \psi(n)(\eta(n) - n) \right) < \frac{5\lambda_p b}{b-2} (\eta(n) - n)^\gamma. \] (14)

We denote
\[ D_{k,\beta}(t) := \frac{1}{2} \cos \frac{\beta \pi}{2} + \sum_{j=1}^{k} \cos \left( j t + \frac{\beta \pi}{2} \right). \] (15)

Applying Abel transform, we have
\[ \sum_{k=1}^{n-1} \psi(2n - k) \cos \left( \frac{kt + \beta \pi}{2} \right) = \sum_{k=1}^{n-2} (\psi(2n - k + 1) - \psi(2n - k))D_{k,\beta}(t) + \psi(n + 1) D_{n-1,\beta}(t) - \psi(2n - 1) \frac{1}{2} \cos \frac{\beta \pi}{2} \] (16)

and
\[ \sum_{k=n}^{2n} \psi(k) \cos \left( \frac{kt + \beta \pi}{2} \right) = \sum_{k=n}^{2n-1} (\psi(k) - \psi(k+1))D_{k,\beta}(t) + \psi(2n) D_{2n,\beta}(t) - \psi(n) D_{n-1,\beta}(t). \] (17)

Since
\[ \sum_{k=0}^{N-1} \sin(\gamma + kt) = \sin \left( \gamma + \frac{N-1}{2} \right) \sin \frac{N\gamma}{2} \sin \frac{t}{2} \] (18)

(see, e.g., [2, p.43]), for \( N = k + 1 \), \( \gamma = (\beta - 1)\frac{\pi}{2} \), the following inequality holds
\[ |D_{k,\beta}(t)| = \left| \frac{\cos \left( \frac{k\pi + \beta \pi}{2} \right) \sin \frac{k+1}{2} t}{\sin \frac{\pi}{2}} - \frac{1}{2} \cos \frac{\beta \pi}{2} \right| \]
\[ = \left| \sin \left( (k + \frac{1}{2}) t + \frac{\beta \pi}{2} \right) - \cos \frac{\beta \pi}{2} \sin \frac{\beta \pi}{2} \frac{\pi}{2} \frac{\pi}{2} \right| \leq \frac{\pi}{t}, \quad 0 < |t| \leq \pi. \] (19)

According to (11), (16), (17) and (19), we obtain
\[ |(f_{p,n}^*)(t))_\beta | \leq \frac{\lambda_p}{\psi(n)(\eta(n) - n)^{\frac{1}{p'}}} \frac{\pi}{|t|} \sum_{k=1}^{n-2} |\psi(2n - k) - \psi(2n - k - 1) + \psi(n + 1) + \psi(2n - 1) + \sum_{k=n}^{2n-1} |\psi(k) - \psi(k + 1)| + \psi(2n) + \psi(n)| \]
\[ = \frac{\lambda_p}{\psi(n)(\eta(n) - n)^{\frac{1}{p'}}} \frac{2\pi}{|t|} (\psi(n + 1) + \psi(n)) \leq \frac{4\pi \lambda_p}{(\eta(n) - n)^{\frac{1}{p'}}} \frac{1}{|t|}. \] (20)

So, (14) and (20) imply
\[ \| (f_{p,n}^*)(t))_\beta \|_p \leq \lambda_p \max\left\{ \frac{5b}{b - 2}, 4\pi \right\} \left( \int_{|t| \leq \eta(n) - n} (\eta(n) - n) dt + \frac{1}{(\eta(n) - n)^{\frac{1}{p'}}} \int_{\eta(n) - n \leq |t| \leq \pi} dt \right)^{\frac{1}{p'}} \]
\[ \leq 2\lambda_p \max\left\{ \frac{5b}{b - 2}, 4\pi \right\} \left(1 + \frac{1}{p - 1}\right)^{\frac{1}{p'}} = 2\lambda_p \max\left\{ \frac{5b}{b - 2}, 4\pi \right\} (p')^{\frac{1}{p'}}. \]

Hence, for
\[ \lambda_p = \frac{1}{2(p')^{\frac{1}{p'}} \max\left\{ \frac{5b}{b - 2}, 4\pi \right\}} \]
the embedding \( f_{p,n}^* \in L_{\beta,p}^\psi \) is true.

Let us consider the quantity
\[ I_1 := \inf_{\gamma_{2n}} \left| \int_{-\pi}^{\pi} (f_{p,n}^*(t) - S_{\gamma_{2n}}(f_{p,n}^*(t))V_{2n}(t)) dt \right|, \] (21)
where \( V_{2n} \) are de la Vallée-Poisson kernels of the form
\[ V_m(t) := \frac{1}{2} + \sum_{k=1}^{m} \cos kt + \sum_{k=m+1}^{2m-1} \left(1 - \frac{k}{2m}\right) \cos kt, m \in \mathbb{N}. \] (22)

Proposition A1.1 from [3] implies
\[ I_1 \leq \inf_{\gamma_{2n}} \| f_{p,n}^*(t) - S_{\gamma_{2n}}(f_{p,n}^*(t)) \|_\infty \| V_{2n} \|_1 = e^{\frac{1}{2n}}(f_{p,n}^*)_\infty \| V_{2n} \|_1. \] (23)

Since (see, e.g., [14, p.247])
\[ \| V_m \|_1 \leq 3\pi, \ m \in \mathbb{N}, \] (24)
from (23) and (24) we can write down the estimate.
Order estimates of best orthogonal trigonometric approximations

\[ e_{2n}^1(f_{p,n}) \geq \frac{1}{3\pi} I_1. \]  

(25)

Notice, that

\[ f_{p,n}(t) - S_{2n}(f_{p,n}^{*}t) \]

\[ = \frac{\lambda_p}{2\psi(n)(\eta(n) - n)} \int \left( \sum_{|k| \leq n-1, \ k \neq 0} \psi(|k|) \psi(2n - |k|)e^{ikt} + \sum_{n \leq |k| \leq 2n, \ k \neq 0} \psi^2(|k|)e^{ikt} \right) \]

(26)

where \( \psi(0) := \psi(1) \)

Whereas

\[ \int_{-\pi}^{\pi} e^{ikt} e^{imt} dt = \begin{cases} 0, & k + m \neq 0, \\ 2\pi, & k + m = 0, \ k, m \in \mathbb{Z}, \end{cases} \]

(27)

and taking into account [22], we obtain

\[ \int_{-\pi}^{\pi} (f_{p,n}^{*}(t) - S_{2n}(f_{p,n}^{*}t))V_{2n}(t)dt \]

(28)

\[ = \frac{\lambda_p}{4\psi(n)(\eta(n) - n)^2} \int_{-\pi}^{\pi} \left( \sum_{0 \leq k \leq n-1, \ k \neq 0} \psi(k) \psi(2n - k)e^{ikt} + \sum_{-n+1 \leq k \leq -1, \ k \neq 0} \psi(|k|) \psi(2n - |k|)e^{ikt} \right. \]

\[ + \sum_{n \leq k \leq 2n, \ k \neq 0} \psi^2(k)e^{ikt} + \sum_{-2n \leq k \leq -n, \ k \neq 0} \psi^2(|k|)e^{ikt} \]

\[ \times \left( \sum_{0 \leq k \leq 2n} e^{ikt} + \sum_{2n+1 \leq |k| \leq 4n-1} 2 \left( 1 - \frac{|k|}{4n} \right) e^{ikt} \right) dt \]

(29)

\[ = \frac{\lambda_p \pi}{2\psi(n)(\eta(n) - n)^2} \left( \sum_{|k| \leq n-1, \ k \neq 0} \psi(|k|) \psi(2n - |k|) + \sum_{n \leq |k| \leq 2n, \ k \neq 0} \psi^2(|k|) \right). \]

(30)

The function \( \phi_n(t) := \psi(t)\psi(2n - t) \) decreases for \( t \in [1, n] \). Indeed

\[ \phi_n'(t) = |\psi'(t)| |\psi'(2n - t)| \left( \frac{\psi(t)}{\psi'(t)} - \frac{\psi(2n - t)}{\psi'(2n - t)} \right) \leq 0, \]

because \( \frac{\psi(t)}{\psi'(t)} \uparrow \infty \) for large \( n \).

Thus, the monotonicity of function \( \phi_n(t) \) and [50] imply
\[ I_1 = \frac{\pi \lambda_p}{2 \psi(n)(\eta(n) - n)^p} \left( \psi^2(n) + \sum_{n+1 \leq |k| \leq 2n} \psi^2(|k|) \right) \]
\[ > \frac{\pi \lambda_p}{2 \psi(n)(\eta(n) - n)^p} \sum_{k=n}^{2n} \psi^2(k) \geq \frac{\pi \lambda_p}{2 \psi(n)(\eta(n) - n)^p} \int_n^{\eta(n)} \psi^2(t) dt \]
\[ > \frac{\pi \lambda_p}{2 \psi(n)(\eta(n) - n)^p} \psi^2(\eta(n)) (\eta(n) - n) = \frac{\pi \lambda_p}{8} \psi(n)(\eta(n) - n)^{\frac{1}{p}}. \quad (31) \]

By considering (25) and (31) we can write
\[ e_\perp^2 n(L^\psi_{\beta, p})_{\infty} \geq e_\perp^2 n(f_{p,n})_{\infty} \geq \frac{1}{3\pi} I_1 \geq \frac{\lambda_p}{24} \psi(n)(\eta(n) - n)^{\frac{1}{p}}. \quad (32) \]

Theorem 1 is proved.

**Remark 1.** Let \( \psi \in \mathcal{M}^+_\infty \), \( \beta \in \mathbb{R} \), \( 1 < p < \infty \), \( \frac{1}{p} + \frac{1}{p'} = 1 \), and the function \( \frac{\psi(t)}{|\psi'(t)|} \uparrow \infty \) for \( t \to \infty \). Then for \( n \in \mathbb{N} \) the following estimates hold
\[ K_{b,p} \psi(n)(\eta(n) - n)^{\frac{1}{p}} \leq e_\perp^2 n(L^\psi_{\beta, p})_{\infty} \leq \frac{1}{3\pi} I_1 \geq \frac{\lambda_p}{24} \psi(n)(\eta(n) - n)^{\frac{1}{p}}. \quad (33) \]

where
\[ K_{a,b,p} = \frac{1}{\pi} \max \left\{ \frac{2b}{b-2} + \frac{1}{a}, \frac{2\pi}{2p} \right\} (2p)^{\frac{1}{p'}}. \quad (34) \]
\[ K_{b,p} = \frac{1}{48} \max \left\{ \frac{2b}{b-2}, \frac{4\pi}{4\pi} \right\} (p')^{\frac{1}{p'}}. \quad (35) \]

### 3 Best orthogonal trigonometric approximations of the classes \( L^\psi_{\beta, 1} \) in the uniform metric

**Theorem 2.** Let \( \psi \in \mathcal{M}^+_\infty \). Then for all \( \beta \in \mathbb{R} \) order estimates are true
\[ e_\perp^2 n(L^\psi_{\beta, 1})_{\infty} \asymp e_\perp^2 n(f_{1,n})_{\infty} \asymp \psi(n)(\eta(n) - n). \quad (36) \]

**Proof.** According to formula (48) from [8] under conditions \( \psi \in \mathcal{M} \), \( \sum_{k=1}^{\infty} \psi(k) < \infty \), \( \beta \in \mathbb{R} \), for all \( n \in \mathbb{N} \) the following estimate holds
\[ e_n(L^\psi_{\beta, 1})_{\infty} \leq \frac{1}{\pi} \sum_{k=n}^{\infty} \psi(k). \quad (37) \]

Using Proposition 1, we have
In [14, p. 263–265] it was shown that
\[ \|e_{2n}^\psi(L_{\beta,1})\|_\infty \leq e_{2n-1}^\psi(L_{\beta,1}) \leq e_n^\psi(L_{\beta,1}) \leq \frac{1}{\pi} \sum_{k=n}^m \psi(k) \]
\[ \leq \frac{1}{\pi} \left( \psi(n) + \int_n^\infty \psi(u) du \right) \leq \frac{\psi(n)}{\pi} \left( 1 + \frac{b}{b-2}(\eta(n) - n) \right). \quad (38) \]

Let us find the lower estimate for the quantity \( e_{2n}^\psi(L_{\beta,1}) \).

We consider the quantity
\[ I_2 := \inf_{\gamma_{2n}} \left| \int_0^\pi (f_{2n}^\psi(t) - S_{\gamma_{2n}}(f_{2n}^\psi; t)) V_{2n}(t) dt \right|, \quad (39) \]

where \( V_m \) are de la Vallée-Poisson kernels of the form (22), and
\[ f_{2n}^\psi(t) = f_{2n}^\psi(t) := \frac{1}{\pi m} \left( \psi(1) + \sum_{k=1}^m k \psi(k) \cos kt + \sum_{k=m+1}^{2m} (2m+1-k) \psi(k) \cos kt \right). \quad (40) \]

In [14, p. 263–265] it was shown that \( \|f_{2n}^\psi\|_1 \leq 1 \), i.e., \( f_{2n}^\psi \) belongs to the class \( L_{\beta,1}^\psi \) for all \( m \in \mathbb{N} \).

Using Proposition A1.1 from [13] and inequality (24), we have
\[ I_2 \leq \inf_{\gamma_{2n}} \|f_{2n}^\psi(t) - S_{\gamma_{2n}}(f_{2n}^\psi; t)\|_1 \|V_{2n}\|_1 \leq 3\pi e_{2n}^\psi(f_{2n}^\psi) \|V_{2n}\|_1. \quad (41) \]

Assuming again \( \psi(0) := \psi(1), \) from (22) and (40), we derive
\[ I_2 = \frac{1}{20\pi n} \inf_{\gamma_{2n}} \left| \int_0^\pi \left( \sum_{|k| \leq 2n} |k| \psi(|k|) e^{ikt} + \sum_{2n+1 \leq |k| \leq 4n} (4n+1-|k|) \psi(|k|) e^{ikt} \right) \times \right. \]
\[ \left. \times \left( \sum_{|k| \leq 2n} e^{ikt} + 2 \sum_{2n+1 \leq |k| \leq 4n-1} \left( 1 - \frac{|k|}{4n} \right) e^{ikt} \right) dt \right| \]
\[ = \frac{1}{10n} \inf_{\gamma_{2n}} \left( \sum_{|k| \leq 2n} |k| \psi(|k|) + \sum_{2n+1 \leq |k| \leq 4n} \left( 1 - \frac{|k|}{4n} \right) \psi(|k|) \right) \]
\[ > \frac{1}{10n} \inf_{\gamma_{2n}} \sum_{|k| \leq 2n} |k| \psi(|k|) = \frac{1}{10n} \left( n \psi(n) + 2 \sum_{k=n+1}^{2n} k \psi(k) \right) \]
\[ > \frac{1}{10} \sum_{k=n}^{2n} \psi(k) > \frac{1}{10} \int_n^\infty \psi(t) dt > \frac{1}{20} \psi(n)(\eta(n) - n). \quad (42) \]

Formulas (41) and (42) imply
Theorem 3. Let
\[ e_{2n}(L_{β,1}^γ) ≥ e_{2n}(f_{2n}^*=) ≥ \frac{1}{3π}l_2 > \frac{1}{60π}ψ(n)(η(n) - n). \]

Theorem 2 is proved.

Remark 2. Let \( ψ \in \mathcal{M}_{k∞}^∞ \) and \( β \in \mathbb{R} \). Then for \( n \in \mathbb{N} \), such that \( μ(n) ≥ b > 2 \) the following estimate holds

\[ \frac{1}{60π}ψ(n)(η(n) - n) ≤ e_{2n}(L_{β,1}^γ) ≤ e_{2n−1}(L_{β,1}^γ) ≤ \frac{1}{π} \left( \frac{1}{b} + \frac{b}{b-2} \right)ψ(n)(η(n) - n). \] (43)

Corollary 1. Let \( r \in (0, 1) \), \( α > 0 \), \( 1 ≤ p < ∞ \) and \( β \in \mathbb{R} \). Then for all \( n \in \mathbb{N} \) the following estimates are true

\[ e_n^1(L_{β,p}^α) ≥ \exp(-αn^p) n^{-p}. \] (44)

4 Best orthogonal trigonometric approximations of the classes \( L_{β,1}^γ \) in the metric of spaces \( L_s \), \( 1 < s < ∞ \)

Theorem 3. Let \( 1 < s < ∞ \), \( ψ \in \mathcal{M}_{k∞}^∞ \) and function \( \frac{ψ(t)}{ψ(0)} \uparrow ∞ \) as \( t \to ∞ \). Then for all \( β \in \mathbb{R} \) order estimates hold

\[ e_{2n−1}(L_{β,1}^γ) ≥ e_{2n}(L_{β,1}^γ) ≥ ψ(n)(η(n) - n)^{\frac{s}{s'}} ≥ \frac{1}{s} + \frac{1}{s'} = 1. \] (45)

Proof. According to Theorem 2 from [8] under conditions \( ψ \in \mathcal{M}_{k∞}^∞ \), \( β \in \mathbb{R} \), \( 1 < s ≤ ∞ \) for \( n \in \mathbb{N} \), such that \( η(n) − n ≥ a > 2 \), \( μ(n) ≥ b > 2 \) the following estimate holds

\[ e_n(L_{β,1}^γ) ≤ K_{a,b} (2s')^{\frac{1}{s}}ψ(n)(η(n) - n)^{\frac{1}{2}}. \] (46)

Using inequalities (6) and (46), we get

\[ e_{2n}(L_{β,1}^γ) ≤ e_{2n−1}(L_{β,1}^γ) ≤ K_{a,b,s'} (2s')^{\frac{1}{s'}}ψ(n)(η(n) - n)^{\frac{1}{2}}. \] (47)

Let us find the lower estimate of the quantity \( e_{2n}(L_{β,1}^γ) \).

We consider the quantity

\[ I_3 := \inf_{\gamma_{2n}} \left| \frac{π}{2n} \int_{-π}^{π} (f_{2n}^{*}\gamma(t) - S_{2n}(f_{2n}^{*}\gamma(t)))f_{s,p}^{*}\gamma(t)dt \right|, \] (48)

where

\[ f_{m}^{*}\gamma(t) = \frac{1}{3π}V_m(t), \]
and $f^*_{s,n}$ is defined by formula (10).

On the basis of Proposition A1.1 from [3] we derive

$$I_3 = \inf_{\gamma} \| f^*_{2n} - S_{\gamma}(f^*_{2n}) \|, \| f^*_{s,n} \| \leq e_{2n}(f^*_{2n}),$$  \hspace{1cm} (49)

On other hand, using formulas (27), we write

$$I_3 = \frac{\lambda_s}{6\psi(n)(\eta(n) - n)^{\frac{1}{2}}} \inf_{\gamma} \left( \sum_{|k| \leq n, k \neq \gamma} \psi(|k|)\psi(2n - |k|) + \sum_{n \leq |k| \leq 2n} \psi^2(|k|) \right)$$

$$= \frac{\lambda_s}{6\psi(n)(\eta(n) - n)^{\frac{1}{2}}} \left( \psi^2(n) + 2 \sum_{k=n+1}^{2n} \psi^2(k) \right) \geq \frac{\lambda}{6\pi \psi(n)(\eta(n) - n)^{\frac{1}{2}}} \sum_{k=n}^{2n} \psi^2(k)$$

$$> \frac{\lambda_s}{6\psi(n)(\eta(n) - n)^{\frac{1}{2}}} \left( \psi^2(n) + 2 \sum_{k=n+1}^{2n} \psi^2(k) \right) \geq \frac{\lambda}{6\pi \psi(n)(\eta(n) - n)^{\frac{1}{2}}} \sum_{k=n}^{2n} \psi^2(k) \geq \frac{\lambda_s}{24} \psi(n)(\eta(n) - n)^{\frac{1}{2}}. \hspace{1cm} (50)$$

Hence, formulas (49) and (50) imply

$$e_{2n}(L^\psi f_{s,n}) \geq e_{2n}(f^*_{s,n}) \geq I_3 \geq \frac{\lambda_s}{24} \psi(n)(\eta(n) - n)^{\frac{1}{2}}. \hspace{1cm} (51)$$

Theorem 3 is proved.

Note, that functions

1) $e^{-\alpha t^2}$, $\alpha > 0$, $r \in (0, 1)$, $\gamma \in \mathbb{R}$;

2) $e^{-\alpha t \ln(t + K)}$, $\alpha > 0$, $r \in (0, 1)$, $K > e - 1$, etc., can be regarded as examples of functions $\psi$, which satisfy the conditions of Theorem 1 and Theorem 3.

Remark 3. Let $\psi \in M^+_n$, $\beta \in \mathbb{R}$, $1 \leq p < \infty$ and function $\frac{\psi(t)}{|\psi(t)|} \uparrow \infty$ as $t \to \infty$. Then for all $n \in \mathbb{N}$, such the following estimates are true

$$K_{p,b,\alpha} \psi(n)(\eta(n) - n)^{\frac{1}{p}} \leq e_{2n}^{\\psi}(L^\psi f_{s}) \leq e_{2n}^{\\psi}(L^\psi f_{s,n}) \leq K_{p,b,\alpha} \psi(n)(\eta(n) - n)^{\frac{1}{p}}, \hspace{1cm} (52)$$

where $K_{p,b,\alpha}$ and $K_{p,b,\alpha}$ are defined by formulas (54) and (55) respectively.
Corollary 2. Let $r \in (0,1)$, $\alpha > 0$, $1 < s < \infty$ and $\beta \in \mathbb{R}$. Then for all $n \in \mathbb{N}$ the following estimates are true
\[
e_n^\perp (L^{\alpha,r}_{\beta,1})_r \asymp \exp(-\alpha n^r) n^{\frac{1}{r}}, \quad \frac{1}{s} + \frac{1}{s'} = 1. \tag{53}
\]

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