INFINITE-DIMENSIONAL FEATURES OF MATRICES AND PSEUDOSPECTRA

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Abstract. Given a Hilbert space operator $T$, the level sets of function $\Psi_T(z) = \| (T - z)^{-1} \|^{-1}$ determine the so-called pseudospectra of $T$. We set $\Psi_T$ to be zero on the spectrum of $T$. After giving some elementary properties of $\Psi_T$ (which, as it seems, were not noticed before), we apply them to the study of the approximation. We prove that for any operator $T$, there is a sequence $\{T_n\}$ of finite matrices such that $\Psi_{T_n}(z)$ tends to $\Psi_T(z)$ uniformly on $\mathbb{C}$. In this proof, quasitriangular operators play a special role. This is merely an existence result, we do not give a concrete construction of this sequence of matrices.

One of our main points is to show how to use infinite-dimensional operator models in order to produce examples and counterexamples in the set of finite matrices of large order. In particular, we get a result, which means, in a sense, that the pseudospectrum of a nilpotent matrix can be anything one can imagine. We also study the norms of the multipliers in the context of Cowen–Douglas class operators. We use these results to show that, to the opposite to the function $\Psi_S$, the function $\| \sqrt{S} - z \|$ for certain finite matrices $S$ may oscillate arbitrarily fast even far away from the spectrum.

1. Introduction

Let $\mathcal{H}$ be a complex separable Hilbert space and $\mathcal{B}(\mathcal{H})$ be the algebra of bounded operators on $\mathcal{H}$ with respect to supremum norm. Given an operator $T \in \mathcal{B}(\mathcal{H})$, put

$$\Psi_T(z) = \begin{cases} 0 & \text{if } z \in \sigma(T); \\ \| (T - z)^{-1} \|^{-1} & \text{if } z \notin \sigma(T). \end{cases}$$

This function is closely related with so-called $\varepsilon$-pseudospectra of $T$, defined by

$$\sigma_\varepsilon(T) = \{ z \in \mathbb{C} : \Psi_T(z) < \varepsilon \}$$

(here $\varepsilon > 0$). It is well known that

$$\sigma_\varepsilon(T) = \bigcup_{\|A\| < \varepsilon} \sigma(T + A),$$

see, for instance [13, 27, 44]. While the $\varepsilon$-pseudospectrum of a normal operator in a Hilbert space coincides with the $\varepsilon$-neighbourhood of the spectrum, the situation is more involved for non-normal operators. It is well-known that the spectral properties of a nonnormal operator (or matrix) not only depend on its spectrum, but are also influenced by the resolvent growth. The pseudospectra are a good language to describe this growth, and their importance has been widely recognized in the recent years. Their applications include the finite section method for Toeplitz matrices, growth bounds for semigroups, numerics for differential operators, matrix iterations, linear models for turbulence, etc. We refer to the book [50] by Trefethen and Embree and to the Trefethen’s review.
for comprehensive accounts. Much effort has been devoted to the calculation of pseudospectra of matrices [19].

By a filtration on $\mathcal{H}$, we mean a sequence $\{P_n\}$ of finite rank orthogonal projections such that $\text{Ran} \ P_n \subseteq \text{Ran} \ P_{n+1}$ and $\bigcup_n \text{Ran} \ P_n$ is dense in $\mathcal{H}$. The corresponding sequence of finite dimensional operators $T_n = P_n T |_{\text{Ran} \ P_n}$ will be referred to as finite sections of $T$.

Recall that an operator $T \in \mathcal{B}(\mathcal{H})$ is said to be quasitriangular if there is a filtration $\{P_n\}$ such that $\lim_{n \to \infty} \|(I - P_n)TP_n\| = 0$. If there is a filtration $\{P_n\}$ such that both $\lim_{n \to \infty} \|(I - P_n)TP_n\| = 0$ and $\lim_{n \to \infty} \|P_nT(I - P_n)\| = 0$, then $T$ is said to be quasidiagonal. In these cases, we will refer to $\{P_n\}$ as to a filtration, corresponding to a quasitriangular (quasidiagonal) operator $T$.

It is well known that spectra do not necessarily behave well under limiting procedures, even for a sequence of bounded operators on some Hilbert space $\mathcal{H}$ converging in operator norm. For example, consider the bilateral weighted shift on $\ell_2$ by $\{e_j : j \in \mathbb{Z}\}$.

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For the case of pseudospectra, the situation is better. It was noticed by many authors that, to the opposite to usual spectra, pseudospectra supply a vast quantitative information on the behavior of powers of operators, the semigroups they generate, etc. Our work also gives some results in this direction.

Our main results are as follows. In Section 2 we prove several elementary estimates and properties for the function $\Psi_T(z)$. In particular, we show that it is locally semiconvex (see the definition below). The list of these properties certainly can be extended. However, the question of describing all functions on $\mathbb{C}$ representable as $\Psi_T(z)$ for a Hilbert (or Banach) space operator $T$ seems to be open and might be interesting. We use the results of Section 2 in the next sections. We believe that these results may also be important for algorithms of numerical calculation of pseudospectra.

Section 3 is devoted to general convergence results for pseudospectra and for the function $\Psi_T(z)$. One of our starting points was the result by N. Brown, which says that if $T$ is quasidiagonal operator and $\{P_n\}$ is a corresponding filtration, then for any $\varepsilon$, the $\varepsilon$-pseudospectra of $T_n$ tend to the $\varepsilon$-pseudospectrum of $T$, see [13], Theorem 3.5 (1). We observe that a similar assertion holds also for quasitriangular operators. We prove that for any quasitriangular operator $T$ and the corresponding filtration $\{P_n\}$, the functions $\Psi_{T_n}$ tend uniformly to $\Psi_T$ on the whole complex plane. This permits us to show that for an arbitrary operator $T$, there is a sequence of matrices $S_n$ such that $\Psi_{S_n}$ tend uniformly to $\Psi_T$ on $\mathbb{C}$. Here we use the theorem by Apostol, Foiaş and Voiculescu, which characterizes quasitriangular operators in terms of semi-Fredholmness.

In Section 4 we use the above convergence results to prove that, in a sense, the function $\Psi_T(z)$, corresponding to a nilpotent matrix $T$, can have any imaginable shape. In this proof, we apply our approximation results to the adjoint to the operator of multiplication by the independent variable on the Hardy space $H^2$ of a domain in $\mathbb{C}$ and to direct sums of such operators.

The function $\Psi_T(z)$ only depends on the norms of the resolvent of $T$. One can ask about estimates of other functions of $T$. In Section 5 we prove an approximation result in this direction. We show that for a Cowen-Douglas class operator $T$, a function $f$, holomorphic at 0, an a filtration $\{P_n\}$, chosen in a special way (so that all finite sections $T_n$ are nilpotent), the norms of $f(T_n)$ are uniformly bounded if and only if $f$ belongs...
to a certain multiplier space. Notice that the Cowen-Douglas class is a particular (and well-understood) subclass of quasitriangular operators.

This result motivated Example [3, Example 5.8] where we show that, to the opposite to the function \( \Psi_S(z) \) (which is Lipschitz with constant 1), no uniform Lipschitz estimates for the function \( \|\sqrt{S - z}\| \) are possible in a neighbourhood of 1, even if \( S \) is assumed to be a finite nilpotent matrix.

2. Elementary estimates

Let \( T \) be an operator on a Hilbert space \( \mathcal{H} \). We denote the kernel of \( T \) and range of \( T \) by \( \ker T \) and \( \text{Ran} \, T \) respectively. If \( \mathcal{H}_0 \) is a closed subspace of \( \mathcal{H} \), then we shall write \( \mathcal{H}_0 \subseteq \mathcal{H} \). For \( T \in \mathcal{B}(\mathcal{H}) \), we shall denote the spectrum, point spectrum, left spectrum and right spectrum by \( \sigma(T), \sigma_p(T), \sigma_l(T) \) and \( \sigma_r(T) \) respectively.

Given a point \( z \in \mathbb{C} \), we recall that the injectivity radius \( j_T(z) \) and the surjectivity radius \( k_T(z) \) of \( T - z \) are defined by

\[
j_T(z) = \inf \{ \| (T - z)h \| : h \in \mathcal{H}, \| h \| = 1 \},
\]

\[
k_T(z) = \sup \{ r : (T - z)B_{\mathcal{H}} \supset rB_{\mathcal{H}} \};
\]

where \( B_{\mathcal{H}} = \{ h \in \mathcal{H} : \| h \| \leq 1 \} \). The following proposition gives a relation between these two characteristics.

**Proposition 2.1.** [3, Theorem 7, Theorem 8]:

(i) For any \( T \in \mathcal{B}(\mathcal{H}) \) and any \( z \in \mathbb{C} \), \( j_T(z) = k_T(z) \).

(ii) If \( T - z \) is invertible, then

\[
j_T(z) = k_T(z) = \Psi_T(z).
\]

As a consequence, we will prove the following lemma.

**Lemma 2.2.** The following assertions hold.

1. \( \Psi_T(z) = \min \{ j_T(z), k_T(z) \} \).
2. If \( j_T(z) > 0 \) and \( k_T(z) > 0 \), then \( j_T(z) = k_T(z) = \Psi_T(z) \).

**Proof.** Suppose first that both \( j_T(z) > 0 \) and \( k_T(z) > 0 \). Then using Proposition 2.1 we get \( j_T(z) = k_T(z) = \Psi_T(z) \), so that (1) holds in this case. This also gives (2).

Now suppose \( j_T(z) = 0 \). Then \( T - z \) is not invertible. Hence \( \Psi_T(z) = 0 = \min \{ j_T(z), k_T(z) \} \). Similarly, if \( k_T(z) = 0 \), then also \( \Psi_T(z) = 0 \). This completes the proof. \( \square \)

Let \( f : K \to \mathbb{C} \) be a function, defined on a subset \( K \) of the complex plane and let \( C > 0 \). In what follows, we will write \( f \in \text{Lip}_C(K) \) if \( f \) is a Lipschitz function with constant \( C \), that is, \( |f(z) - f(z')| \leq C|z - z'| \) for all \( z, z' \in K \).

**Lemma 2.3.** For any \( T \in \mathcal{B}(\mathcal{H}) \), \( j_T \in \text{Lip}_1(\mathbb{C}) \).

**Proof.** Take any \( z, z' \in \mathbb{C} \). Then we have \( \|(T - z')h\| \leq \|(T - z)h\| + |z - z'| \) for any \( h \in \mathcal{H} \) with \( \| h \| = 1 \). Therefore \( j_T(z') - j_T(z) \leq |z - z'| \). By symmetry, this implies the statement of Lemma. \( \square \)

Since \( \Psi_T(z) = \min \{ j_T(z), k_T(z) \} \) and the minimum of two \( \text{Lip}_1(\mathbb{C}) \) functions is again a \( \text{Lip}_1(\mathbb{C}) \) function, we get the following corollary.

**Corollary 2.4.** For any \( T \in \mathcal{B}(\mathcal{H}) \), \( \Psi_T \in \text{Lip}_1(\mathbb{C}) \).
This fact is known, see Theorem 9.2.15 from the E. Brian Davies’s book [21]. It holds, in fact, for any Banach space operator.

Put
\[ \rho_\theta(T) = \sup_{\|h\|=1} \Re \langle e^{-i\theta}Th, h \rangle, \quad \theta \in [0, 2\pi]. \]

The function \( \rho_\theta(T) \) has the following geometrical interpretation. Given a bounded convex subset \( A \) of \( \mathbb{C} \), its support function is defined as \( s_A(\theta) = \sup_{z \in A} \Re(e^{-i\theta}z) \) (so that \( A \) is contained in the half-plane \( \{ \Re(e^{-i\theta}z) \leq s_A(\theta) \} \), but is not contained in half-planes \( \{ \Re(e^{-i\theta}z) \leq \sigma \} \) for \( \sigma < s_A(\theta) \)). It is easy to see that
\[ \rho_\theta(T) = s_{W(T)}(\theta), \]
where \( W(T) = \{ \langle Th, h \rangle : \|h\| = 1 \} \) is the numerical range of \( T \) (it is always convex, by the Toeplitz-Hausdorff Theorem). Notice that \( \rho_\theta(T) \) is always a continuous function of \( \theta \).

By [50, Theorem 17.4],
\[ (2.1) \]
\[ \rho_\theta(T) = \lim_{r \to +\infty} r - \Psi_T(re^{i\theta}), \quad \theta \in [0, 2\pi]. \]

The following proposition is a slightly more precise version of this equality. It will be used in Section 3 below.

**Proposition 2.5.** (cf. [50, Theorem 17.4]) Let \( T \in \mathcal{B}(\mathcal{H}) \). Then for any \( z = re^{i\theta} \) with \( |z| = r > \rho_\theta(T), \)
\[ (2.2) \quad |z| - \rho_\theta(T) \leq \Psi_T(z) \leq \sqrt{|z|^2 - 2\rho_\theta(T)|z|} + \|T\|^2. \]

Notice that the inequality \( \sqrt{a^2 + b} \leq a + \frac{b}{2a} \) (valid for \( a > 0, a^2 + b > 0 \)) gives
\[ (2.3) \quad \sqrt{|z|^2 - 2\rho_\theta(T)|z|} + \|T\|^2 \leq \left( |z| - \rho_\theta(T) \right) + \frac{\|T\|^2 - \rho_\theta(T)^2}{2(|z| - \rho_\theta(T))}, \]
so that the difference between the upper and the lower estimates in (2.2) tends to 0 as \( |z| \to \infty \).

**Proof of Proposition 2.5.** Let \( z = re^{i\theta}, |z| > \rho_\theta(T) \). Then \( z \not\in \sigma(T), \) and
\[ \left( \Psi_T(z) \right)^2 = \inf_{\|h\|=1} \{ r^2 - 2r \Re \langle e^{-i\theta}Th, h \rangle + \|Th\|^2 \}. \]

Since \( \|Th\| \leq \|T\| \) for all \( h \) with \( \|h\| = 1 \), we get
\[ \left( \Psi_T(z) \right)^2 \leq r^2 - 2\rho_\theta(T)r + \|T\|^2. \]

On the other hand, since \( \|Th\| \geq \Re \langle e^{-i\theta}Th, h \rangle \), we see that
\[ \Psi_T(z)^2 \geq \inf_{\|h\|=1} \left( r - \Re \langle e^{-i\theta}Th, h \rangle \right)^2 = \left( r - \rho_\theta(T) \right)^2, \]
which gives the first inequality in (2.2). This completes the proof. \( \square \)

The following lemma estimates the ratio between the values of \( \Psi_T \) in two points of the plane.

**Lemma 2.6.** Suppose \( T \in \mathcal{B}(\mathcal{H}) \). Then
\[ \frac{\Psi_T(z_0)}{|z_0|} \leq \frac{\Psi_T(z)}{|z|} (1 + \varepsilon_{z, z_0}), \quad \text{where } \varepsilon_{z, z_0} = \frac{\|T\| |z - z_0|}{|z_0| \Psi_T(z)}. \]
Proof. For $T \in \mathcal{B}(\mathcal{H})$, we have

\[
(T - z)^{-1} = \frac{z_0}{z} (T - z_0)^{-1} S_{z,z_0},
\]
where $S_{z,z_0} = \frac{1}{z_0} (T - z_0)(T - z)^{-1}$. Also,

\[
\|S_{z,z_0} - I\| \leq \| (T - z)^{-1} \| \frac{\|z\|}{z_0} (T - z_0) - (T - z) \| = \frac{\|T\| \|z - z_0\|}{\|z_0\| \Psi_T(z)} = \varepsilon_{z,z_0}.
\]

Putting together (2.4) and (2.5), we get

\[
\Psi_T(z)^{-1} = \| (T - z)^{-1} \| \leq \| (T - z_0)^{-1} \| \| S_{z,z_0} \| \frac{z_0}{z} \leq \frac{1 + \varepsilon_{z,z_0}}{\Psi_T(z_0)} \frac{z_0}{z}.
\]

This completes the proof. \hfill \Box

Using the above Lemma, we will prove the following theorem.

**Theorem 2.7.** For any $c > \|T\|$, \(\frac{\Psi_T(z)}{|z|} \) is a Lip_{\eta(c)}(C) function, where \(\eta(c) = \frac{\|T\|}{c^2}\).

**Proof.** Take any $z, z_0 \in \mathbb{C}$ such that $|z|, |z_0| \geq c$. By applying twice Lemma 2.6, we get

\[
\frac{\Psi_T(z_0)}{|z_0|} - \frac{\Psi_T(z)}{|z|} \leq \varepsilon \frac{\Psi_T(z)}{|z|} \quad \text{and} \quad \frac{\Psi_T(z)}{|z|} - \frac{\Psi_T(z_0)}{|z_0|} \leq \delta \frac{\Psi_T(z_0)}{|z_0|},
\]

where

\[
\varepsilon = \frac{\|T\| |z - z_0|}{|z_0| \Psi_T(z)} \quad \text{and} \quad \delta = \frac{\|T\| |z - z_0|}{|z| \Psi_T(z_0)}.
\]

Therefore,

\[
\left| \frac{\Psi_T(z)}{|z|} - \frac{\Psi_T(z_0)}{|z_0|} \right| \leq \max \left( \frac{\Psi_T(z)}{|z|} \varepsilon, \frac{\Psi_T(z_0)}{|z_0|} \delta \right) = \max \left( \frac{\Psi_T(z)}{|z|} \frac{\|T\| |z - z_0|}{|z_0| \Psi_T(z_0)}, \frac{\Psi_T(z_0)}{|z_0|} \frac{\|T\| |z - z_0|}{|z| \Psi_T(z_0)} \right)
\]

\[
= \|T\| \max \left\{ \frac{1}{|z_0| |z|}, \frac{1}{|z_0| |z|} \right\} |z - z_0| \leq \frac{\|T\|}{c^2} |z - z_0| = \eta(c) |z - z_0|
\]

whenever $|z|, |z_0| \geq c$, and we are done. \hfill \Box

We recall the definition of semiconvex functions, see the book of P. Cannarsa and C. Sinestrari [14].

**Definition 2.8.** Let $A \subset \mathbb{R}^n$ be an open set and let $u : A \to \mathbb{R}$ be a continuous function.

1. We will say that $u$ is *semiconvex with a constant $C \geq 0$* if

\[
2u(\mu) - u(\mu + \eta) - u(\mu - \eta) \leq C|\eta|^2
\]

for all $\mu, \eta \in \mathbb{R}^n$ such that $[\mu - \eta, \mu + \eta] \subset A$.

2. Let $C : A \to \mathbb{R}$ be a positive continuous function. We will say that $u$ is *semiconvex with bound function $C(x)$* if for any compact convex subset $B$ of $A$, the restriction $u|_B$ is semiconvex with constant $C' = \max_{x \in B} C(x)$.

**Theorem 2.9.** The function $\Psi_T^{-1}$ is semiconvex on $\mathbb{C} \setminus \sigma(T)$ with bound function

\[
C(z) = 2\Psi_T(z)^{-3}.
\]
Proof. Let $B$ be a compact convex subset of $\mathbb{C} \setminus \sigma(T)$, and put $K = \max_{z \in B} \Psi^{-1}_T(z)$. Suppose that an interval $[\mu - \eta, \mu + \eta]$ is contained in $B$. Then
\[2(T - \mu)^{-1} - (T - \mu + \eta)^{-1} - (T - \mu - \eta)^{-1} = -2\eta^2 (T - \mu)^{-1} (T - \mu + \eta)^{-1} (T - \mu - \eta)^{-1},\]
which implies that
\[2\| (T - \mu)^{-1} \| \leq \| (T - \mu + \eta)^{-1} \| + \| (T - \mu - \eta)^{-1} \| + 2|\eta|^2 K^3.\]
This gives our statement. \(\square\)

Semiconvex functions admit some interesting characterizations and have good regularity properties. We can cite the following facts.

**Proposition 2.10.** (see [14].) Given a continuous function $u : B \to \mathbb{R}$ with $B \subset \mathbb{R}^n$ open and convex the following conditions are equivalent:

(a) $u$ is semiconvex in $B$ with a semiconvexity constant $C \geq 0$;

(b) $u$ satisfies
\[u(tx + (1 - t)y) - tu(x) - (1 - t)u(y) \leq C \frac{t(1 - t)}{2} |x - y|^2;\]
for all $x, y$ such that $[x, y] \subset B$ and for all $t \in [0, 1]$.

(c) The function $x \mapsto u(x) + \frac{C}{2} |x|^2$ is convex in $B$.

In particular, by applying the equivalence of (a) and (c), we get that for any $\lambda \notin \sigma(T)$ and any direction $\zeta \in \mathbb{C}$, $|\zeta| = 1$, $\Psi^{-1}_T$ possesses the one-sided directional derivative at $\lambda$
\[\lim_{s \to 0^+} \frac{\Psi^{-1}_T(\lambda + s\zeta) - \Psi^{-1}_T(\lambda)}{s}.\]
Hence, the same also holds for $\Psi_T$. It also follows that Alexandrov’s theorem applies to functions $\Psi^{-1}_T$ and $\Psi_T$, so that they are twice differentiable almost everywhere on $\mathbb{C} \setminus \sigma(T)$. We refer to [14, Theorem 2.3.1] for a precise statement.

It might also be worth recalling here that the function $-\log \Psi_T$ is subharmonic on $\mathbb{C} \setminus \sigma(T)$. Some of the above-stated properties that we state here are true for Banach space operators. However, there is a difference between the Hilbert space case and the Banach space case. For instance, the function $\Psi_T$ can be constant on an open set outside the spectrum for a Banach space operator, but this cannot happen in the Hilbert space case, see [33, 22] and references therein.

3. General theorems on convergence

Let $T$ be a bounded operator on a Hilbert space. The finite section method consists in approximating the spectrum of $T$ on a Hilbert space $H$ by spectra of the finite matrices $T_n = P_n TP_n$, where $\{P_n\}$ is a filtration on $H$. The possibility of doing it has been studied in several articles. In [35], it is shown that in general, there is no convergence of spectra and it is determined, for which subsets $K$ of $\mathbb{C}$ there exists a filtration $\{P_n\}$ such that $d_H(\sigma(T_n), K) \to 0$ as $n \to \infty$, where $d_H$ denotes the Hausdorff distance. On the other hand, there are also some positive results assuring the convergence of $\sigma(T_n)$ to $\sigma(T)$ under some restrictive hypotheses, see [24, 6] and references in [6]. Proposition 4.2 in [10] contains an abstract result on the partial limit set of $\varepsilon$-pseudospectra of $T_n$, under certain hypotheses. The approach related with $C^*$ algebras, originated in the works by Arveson [4, 5] turned out to be very useful, see the book [22]. In [13], this approach was applied to obtain positive results for the case of quasidiagonal operators. We also refer to Hansen [30, 31], Bögli [11] and Bögli and Siegl [12] and references therein for more
results on convergence of spectra for bounded and unbounded operators. In general, the convergence is only assured if either there is a kind of norm convergence of $T_n$ to $T$ or if $T$ belongs to a subclass of linear operators and the filtration $\{P_n\}$ is chosen in a special way.

In this section, we will prove that for a quasitriangular operator $T$ and the corresponding filtration $\{P_n\}$, the injectivity radius $j_{T_n}(z)$ converge uniformly to the injectivity radius $j_T(z)$ on $\mathbb{C}$. One of the main results of this section is Theorem AFV which asserts that for any $T \in \mathcal{B}(\mathcal{H})$, there exists a sequence of matrices $\{S_n\}$ such that the functions $\Psi_{S_n}$ converge uniformly to $\Psi_T$ on $\mathbb{C}$. This will be done with the use of the following powerful result.

**Theorem AFV** (the Apostol–Foiaș–Voiculescu theorem, see [1, 2, 3]). A Hilbert space operator $T$ is quasitriangular if and only if $\text{ind}(T - \lambda) \geq 0$ whenever $\lambda \in \mathbb{C}$ and $T - \lambda$ is Semi-Fredholm.

We recall that an operator $T \in \mathcal{B}(\mathcal{H})$ is said to be Semi-Fredholm if $\text{Ran} T$ is closed and at least one of $\ker T$ and $\ker T^*$ is finite dimensional. The index of a Semi-Fredholm operator $T \in \mathcal{B}(\mathcal{H})$ is defined by $\text{ind}(T) = \dim \ker T - \dim \ker T^*$.

The following lemma is an inequality between the injectivity radius of a quasitriangular operator $T$ and the injectivity radius of $T^*$.

**Lemma 3.1.** Suppose $T \in \mathcal{B}(\mathcal{H})$ is quasitriangular. Then $j_T(\lambda) \leq j_{T^*}(\overline{\lambda})$ for any $\lambda \in \mathbb{C}$.

**Proof.** Suppose that, to the contrary, $j_T(\lambda) > j_{T^*}(\overline{\lambda})$ for some $\lambda \in \mathbb{C}$. Then there exists an $\varepsilon > 0$, e.g. $\varepsilon = \frac{j_{T^*}(\overline{\lambda})}{2}$, such that $\|(T - \lambda)x\| \geq \varepsilon \|x\|$ for all $x \in \mathcal{H}$. Then $(T - \lambda)$ is one-to-one, $(T - \lambda)\mathcal{H}$ is closed and $(T^* - \lambda)\mathcal{H} = \mathcal{H}$. Hence $(T - \lambda)$ is semi-Fredholm. Using Theorem AFV, we conclude that $\text{ind}(T - \lambda) \geq 0$, which implies that $0 = \dim \ker (T - \lambda) \geq \dim \ker (T^* - \lambda)$. Hence, $(T^* - \lambda)$ is one-to-one, which implies that $(T - \lambda)\mathcal{H} = \mathcal{H}$. Hence $(T - \lambda)$ is invertible. Then by Proposition 2.1, $j_T(\lambda) = j_{T^*}(\overline{\lambda})$, a contradiction. □

**Lemma 3.2.** Suppose $T \in \mathcal{B}(\mathcal{H})$ is quasitriangular and $\{P_n\}$ is a corresponding filtration. Then $j_{T_n}(\lambda)$ converges pointwise to $j_T(\lambda)$, where $T_n = P_n T|_{P_n \mathcal{H}}$.

**Proof.** We have $\|(I - P_n)(T - \lambda)P_n\| \rightarrow 0$, for any $\lambda \in \mathbb{C}$. Take any $x \in P_n \mathcal{H}$ such that $\|x\| = 1$. Then

$$\|(T_n - \lambda)x\| = \|P_n(T - \lambda)x\| \geq \|(T - \lambda)x\| - \|(I - P_n)(T - \lambda)P_n x\| \geq j_T(\lambda) - \|(I - P_n)(T - \lambda)P_n\|.$$  

We get that $\lim \inf j_{T_n}(\lambda) \geq j_T(\lambda)$. If we can show that $\lim \sup j_{T_n}(\lambda) \leq j_T(\lambda)$, then we are done.

Take any $\varepsilon > 0$. Then from definition of $j_T(\lambda)$, we have $\|(T - \lambda)x\| \leq j_T(\lambda) + \varepsilon$, for some $x \in \mathcal{H}$ with $\|x\| = 1$. Since $P_n \rightarrow I$ strongly, given any $\varepsilon > 0$, there is a positive integer $N$ such that $\|x - P_n x\| < \varepsilon$ for all $n \geq N$. Now, for any $n \geq N$, we have

$$\|(T_n - \lambda)P_n x\| = \|P_n(T - \lambda)P_n x\| \leq \|P_n(T - \lambda)x\| + \|P_n(T - \lambda)(x - P_n x)\| \leq j_T(\lambda) + \varepsilon + \varepsilon\|T - \lambda\|.$$  

Since $\|P_n x\| \geq 1 - \varepsilon$, we get $j_{T_n}(\lambda) \leq \frac{j_T(\lambda) + \varepsilon(1 + \|T - \lambda\|)}{1 - \varepsilon}$ for all $n \geq N$. Hence we have $\lim \sup j_{T_n}(\lambda) \leq j_T(\lambda)$. This completes the proof. □
We will need a known analysis fact, which says that the pointwise convergence of functions implies the uniform convergence under some extra conditions.

**Proposition 3.3** (see [40], Theorem 7.13). Suppose $K$ is compact and $\{f_n\}$ is an increasing sequence of continuous functions on $K$ (so that $f_n \leq f_{n+1}$ for all $n$). If $f_n$ converge pointwise to a continuous function $f$ on $K$, then this convergence is uniform.

Using this proposition, we will prove the following lemma (which extends [13, Theorem 3.9]).

**Lemma 3.4.** Let $T \in \mathcal{B}(\mathcal{H})$ and put $T_n = P_n T |_{P_n \mathcal{H}}$, where $\{P_n\}$ is an arbitrary filtration on $\mathcal{H}$. Then $\rho_\theta(T_n)$ converges to $\rho_\theta(T)$ uniformly in $\theta \in [0,2\pi]$.

**Proof.** It is easy to see that $\rho_\theta(T_n) \leq \rho_\theta(T)$ for all $n$ and that the sequence $\{\rho_\theta(T_n)\}$ is increasing. Therefore for any $\theta$, there exists a finite limit $\lim_n \rho_\theta(T_n) \leq \rho_\theta(T)$. Now, fix some $\theta \in [0,2\pi]$ and some $\varepsilon > 0$. Find $h \in \mathcal{H}$, $\|h\| = 1$ such that $\Re \langle e^{-i\theta} Th, h \rangle > \rho_\theta(T) - \varepsilon$. Put $h_n = P_n h$. Since

$$\Re \langle e^{-i\theta} Th, h \rangle = \lim \Re \langle e^{-i\theta} T_n h, h_n \rangle = \lim \Re \langle e^{-i\theta} T_n h_n, h_n \rangle$$

and $\|h_n\| \to 1$, we get $\lim_n \rho_\theta(T_n) \geq \rho_\theta(T) - \varepsilon$. Hence $\rho_\theta(T_n)$ converge pointwise to $\rho_\theta(T)$ on $[0,2\pi]$. By Proposition 3.3 we conclude that $\rho_\theta(T_n)$ converge uniformly to $\rho_\theta(T)$ on $[0,2\pi]$.

As a consequence of the above lemmas, we will prove the following theorem.

**Theorem 3.5.** Suppose $T \in \mathcal{B}(\mathcal{H})$ is quasitriangular and $\{P_n\}$ is an associated filtration on $\mathcal{H}$. Then $\{j_{T_n}\}$ converges uniformly to $j_T$ on $\mathbb{C}$.

**Proof.** Fix any $R > 0$; first we check the uniform convergence on the closed ball $B_R(0) = \{z : |z| \leq R\}$. To this end, take some $\varepsilon > 0$. By compactness, $B_R(0)$ has a finite $\varepsilon$-net $\{\lambda_k : 1 \leq k \leq m\}$, so that $B_R(z) \subseteq \bigcup_{k=1}^m B_\varepsilon(\lambda_k)$.

By Lemma 3.2 there exists an integer $N$ such that $|j_{T_n}(\lambda_k) - j_T(\lambda_k)| < \varepsilon$ for all $k$ and all $n \geq N$. Since $\{j_{T_n}\}$ and $j_T$ are Lip$_1(\mathbb{C})$ functions, we can now apply a standard $3\varepsilon$ argument. Namely, let $\lambda$ be any point in $B_R(0)$. Then $\lambda \in B_\varepsilon(\lambda_k)$ for some $k$, and we get

$$|j_{T_n}(\lambda) - j_T(\lambda)| \leq |j_{T_n}(\lambda_k) - j_{T_n}(\lambda)| + |j_{T_n}(\lambda_k) - j_T(\lambda_k)| + |j_T(\lambda_k) - j_T(\lambda)| < 3\varepsilon$$

for all $n \geq N$. This implies the uniform convergence on $B_R(0)$.

Now we prove the uniform convergence on the whole complex plane. Once again, fix some $\varepsilon > 0$. Put $R = R(\varepsilon) = \|T\|^2/(2\varepsilon + \|T\|)$. Since $|\rho_\theta(T)| \leq \|T\|$, it follows from (2.2), (2.3) that for any $z = re^{i\theta}$ with $|z| > R$, one has

$$|j_T(z) - |z| + \rho_\theta(T)| \leq \frac{\|T\|^2}{2(R - \|T\|)} = \varepsilon.$$

We get in the same way that $|j_{T_n}(z) - |z| + \rho_\theta(T_n)| < \varepsilon$ for all $n$ and all $z$, $|z| > R$. Also, by Lemma 3.3, $\rho_\theta(T_n)$ converges uniformly to $\rho_\theta(T)$ on $[0,2\pi]$ as $n \to \infty$, that is, there exists a positive integer $N_0$ such that $|\rho_\theta(T_n) - \rho_\theta(T)| < \varepsilon$ for all $n \geq N_0$ and all $\theta$. This implies that for all $z = re^{i\theta}$ with $|z| > R$ and all $n \geq N_0$, we have

$$|j_{T_n}(z) - j_T(z)| \leq |j_T(z) - |z| + \rho_\theta(T)| + |j_{T_n}(z) - |z| + \rho_\theta(T_n)| + |\rho_\theta(T_n) - \rho_\theta(T)| < 3\varepsilon.$$

Now choose $N_1$ so that $|j_{T_n} - j_T| < 3\varepsilon$ on $B_R(e^{i\theta})(0)$ for all $n \geq N_1$. Then $|j_{T_n}(z) - j_T(z)| < 3\varepsilon$ for all $z \in \mathbb{C}$ whenever $n \geq \max(N_0, N_1)$. This proves that $j_{T_n}$ converges uniformly to $j_T$ on $\mathbb{C}$. □
Corollary 3.6. Let $T$ be an operator on $\mathcal{H}$ such that either $T$ or $T^*$ is quasitriangular. Let $\{P_n\}$ be a filtration on $\mathcal{H}$ that is associated to $T$ in the first case and is associated to $T^*$ in the second case. Then $\{\Psi_{T_n}\}$ converge uniformly to $\Psi_T$ on $\mathbb{C}$, where $T_n = P_n T P_n 5$.

Indeed, notice first that $\Psi_{T_n}(z) = j_{T_n}(z) = j_{T_2}(z)$ for all $n$ and all $z$. Next, $\Psi_T = j_T$ if $T$ is quasitriangular and $\Psi_T(z) = j_T^\ast(z)$ if $T^*$ is quasitriangular. So both cases follow from Theorem 3.5.

It follows that, under the above hypotheses on $T$, given any positive numbers $\varepsilon_1 < \varepsilon < \varepsilon_2$, one gets that

$$\sigma_{\varepsilon_1}(T_n) \subset \sigma_{\varepsilon}(T) \subset \sigma_{\varepsilon_2}(T_n)$$

for all sufficiently large $n$; in this sense, the pseudospectra $\sigma_{\varepsilon}(T)$ can be calculated with an arbitrary precision. Of course, it would be desirable to have estimates of the rate of the uniform convergence of $\Psi_{T_n}$ to $\Psi_T$ in some concrete terms.

The finite section method and convergence of pseudospectra for band-dominated operators is considered in [37, Chap. 6] for the $\ell^2$ case and in [32] for the case of $\ell^p$. We also refer to [8] for a discussion of spectral approximation for finite band selfadjoint operators. Herrero introduced several extensions of the notion of quasitriangularity (see [32]), and it would be interesting to know whether there is kind of extension of Theorem 3.5 for these classes.

The proof of the following proposition is very easy and we leave details to the reader.

**Proposition 3.7.** Let $T_1, T_2 \in \mathcal{B}(\mathcal{H})$. Then $j_{T_1 \oplus T_2}(z) = \min\{j_{T_1}(z), j_{T_2}(z)\}$.

The following is one of our main results.

**Theorem 3.8.** For any bounded linear operator $T$ on a Hilbert space $\mathcal{H}$, there exists a sequence $\{T_n\}$ of finite matrices such that $\Psi_{T_n}$ converges uniformly to $\Psi_T$ on $\mathbb{C}$.

**Proof.** Suppose $T \in \mathcal{B}(\mathcal{H})$ and let $J$ be the set of all isolated points of $\sigma(T)$. Set $K = \sigma(T) \setminus J$. Then $K$ is compact. Let $N$ be any normal operator on $\mathcal{H}$ with discrete spectrum, whose eigenvalues are contained in $K$ and are dense there.

Set $S = T \oplus N$. Then it is very easy to see that $\sigma(T) = \sigma(S)$. First we will show that $S$ is quasitriangular.

By Theorem AFV, we have to take an arbitrary point $\lambda \in \sigma(S)$ and to show that either $S - \lambda$ is not Semi-Fredholm or ind$(S - \lambda) \geq 0$. To do it, consider two cases.

**Case 1:** Suppose $\lambda \in K$. Then Ran$(N - \lambda)$ is not closed. This implies that Ran$(S - \lambda)$ is not closed. Hence $S - \lambda$ is not semi-Fredholm.

**Case 2:** Suppose $\lambda \in J$ and $S - \lambda$ is semi-Fredholm. There are points $\mu \notin \sigma(T)$ arbitrarily close to $\lambda$. By stability of the Fredholm index, we get ind$(S - \lambda) = \text{ind}(S - \mu) = 0$.

Hence we conclude that $S$ is quasitriangular.

Let $\{P_n\}$ be the corresponding filtration on $\mathcal{H} \oplus \mathcal{H}$, so that $\lim_{n \to \infty} \|(I - P_n)SP_n\| = 0$. Set $S_n = P_n S [\mathcal{H}_n]$, where $\mathcal{H}_n = P_n \mathcal{H}$. By applying Theorem 3.5 and Lemma 2.2, we get that $\Psi_{S_n}$ converges uniformly to $\Psi_S$ on $\mathbb{C}$. For any $\lambda \notin \sigma(T)$, $\|(N - \lambda)^{-1}\| = 1/\text{dist}(\lambda, K) \leq \|(T - \lambda)^{-1}\|$, and therefore

$$\Psi_S(\lambda)^{-1} = \max \{\|(T - \lambda)^{-1}\|, \|(N - \lambda)^{-1}\|\} = \|(T - \lambda)^{-1}\| = \Psi_T(\lambda)^{-1}.$$

Hence we conclude that $\Psi_S(\lambda) = \Psi_T(\lambda)$ for all $\lambda \in \mathbb{C}$, which completes the proof. □
Notice that for a concrete operator $T$, the above construction requires the knowledge of the spectrum of $T$, which is computationally difficult and requires, in general, three passages to limits (see the work \[9\] and its full version in arxiv, where the smallest number of limits necessary to solve a computational problem is studied in a systematic way). We refer to Part 3 of the book \[36\], to recent works \[17, 19, 34\] and references therein for diverse negative and positive results on computability of spectra and on the rate of convergence of approximations.

Even if the operator $N$ in the last proof is known, it does not seem so easy to construct the corresponding filtration $\{P_n\}$ on $\mathcal{H} \oplus \mathcal{H}$ explicitly. Therefore, to the opposite to Theorem \[3.5\], the above proof of the last theorem is not constructive, and a more explicit construction would be desirable.

**Remark 3.9.** It is well known that pseudospectra varies continuously with an operator $T$ in $\mathcal{B}(\mathcal{H})$. Hansen in his fundamental paper \[31\] introduced the notion of $(N, \varepsilon)$-pseudospectra defined by means of a modified function $\Psi_{T,N}$, which has all the nice continuity property that the function $\Psi_T$ has, but also allow one to approximate the spectrum arbitrarily well for large $N$. Later, M. Seidel \[41\] extended the concept of $(N, \varepsilon)$-pseudospectra of Hansen to the case of bounded linear operators on Banach spaces and proved several relations to the usual spectrum.

### 4. A THEOREM ABOUT SHAPES OF PSEUDOSPECTRA

Let $\Omega$ be a bounded domain in $\mathbb{C}$. We put $\overline{\Omega} = \{ \bar{w} \in \mathbb{C} : w \in \Omega \}$. Let $H^2(\overline{\Omega})$ stand for the Hardy space on $\overline{\Omega}$. We define the subnormal operator $M(\overline{\Omega})$ of multiplication by $z$ on $H^2(\overline{\Omega})$ by

$$M(\overline{\Omega})f(z) = zf(z), \quad f \in H^2(\overline{\Omega}).$$

If $G \supset \text{clos} \Omega$, then we write it as $G \ni \Omega$.

**Theorem 4.1.** Suppose $G_0 \ni G_1 \ni \ldots \ni G_m$ are bounded connected domains in $\mathbb{C}$, containing the origin, and $\varepsilon_1$ is any number such that $\varepsilon_1 < \text{dist}(\partial G_0, \partial G_1)$. Then there exist $\varepsilon_2, \varepsilon_3, \ldots, \varepsilon_m$ and a square nilpotent complex matrix $T$ such that $\varepsilon_1 > \varepsilon_2 > \ldots > \varepsilon_m > 0$ and

$$G_0 \ni \sigma_{\varepsilon_1}(T) \ni G_1 \ni \ldots \ni G_{m-1} \ni \sigma_{\varepsilon_m}(T) \ni G_m.$$  

**Proof.** Take any finitely connected domains $\Omega_j$, $j = 1, \ldots, m$ with smooth boundaries, such that $G_0 \ni \Omega_1 \ni G_1 \ni \ldots \ni \Omega_m \ni G_m$. We will assume that $\Omega_1$ is close to $G_1$, so that $\varepsilon_1 < \text{dist}(\partial G_0, \partial \Omega_1)$. Put

$$T_j(N) = M(\overline{\Omega_j})^* \ker M(\overline{\Omega_j})^N.$$

Operators $T_j(N)$ are nilpotent for any $N$. We will show that the pseudospectra of the operator

$$T = \bigoplus_{j=1}^m T_j(N_j)$$

(acting on a finite dimensional Hilbert space) satisfy the inclusions (4.1) if the numbers $N_j$ and $\varepsilon_j$ are properly chosen.

These numbers will be defined by an inductive construction. We set

$$\delta = \min_{j,k} \text{dist}(\partial \Omega_j, \partial G_k) > 0.$$

- First step: Notice that $M(\overline{\Omega_1})$ is subnormal and $\sigma(M(\overline{\Omega_1})^*) = \text{clos} \Omega_1$. Theorem \[3.5\] implies that $\Psi_{T_1(N)}(z) \to \text{dist}(z, \Omega_1)$ on $\mathbb{C} \setminus \Omega_1$ and $\Psi_{T_1(N)}(z) \to 0$ on $\Omega_1$ as $N \to \infty$. 


uniformly in both cases. Choose \( N_1 \) so that \( \Psi_{T_1(N_1)}(z) > \varepsilon_1 \) for \( z \) in \( \mathbb{C} \setminus G_0 \) and 
\( \Psi_{T_1(N_1)}(z) < \varepsilon_1/2 \) on \( \Omega \).

- \( k \)th step (\( 2 \leq k \leq m \)): Suppose \( N_1, \ldots, N_{k-1} \) and \( \varepsilon_j \) (\( 2 \leq j \leq k-1 \)) have been elected already. On this step, we choose \( \varepsilon_k \) and \( N_k \).

Choose any \( \varepsilon_k \) so that \( \varepsilon_k < \varepsilon_{k-1}, \varepsilon_k < \delta \), and 
\[
\max \left\{ \| (T_j(N_j) - z)^{-1} \| : 1 \leq j \leq k-1, z \in \mathbb{C} \setminus G_{k-1} \right\} < \varepsilon_k^{-1}.
\]
Notice that \( \Psi_{T_k(N)} \to 0 \) uniformly on \( \Omega_k \) as \( N \to \infty \). Choose \( N_k \) so that \( \Psi_{T_k(N_k)} \leq \varepsilon_k/2 \) on \( \Omega_k \).

After \( \varepsilon_2, \ldots, \varepsilon_m \) and \( N_1, \ldots, N_m \) have been chosen, define \( T \) by (4.2). It is a nilpotent operator on a finite dimensional Hilbert space.

If \( 1 \leq k \leq j \leq m \) and \( z \in \mathbb{C} \setminus G_{k-1} \), then 
\[
\| (T_j(N_j) - z)^{-1} \| \leq \| (M(N_j)^* - z)^{-1} \| \leq \frac{1}{\text{dist}(\partial G_{k-1}, \Omega_j)} \leq \frac{1}{\delta} \varepsilon_k^{-1}.
\]
It follows that 
\[
\max_{z \in \mathbb{C} \setminus G_{k-1}} \| (T_j(N_j) - z)^{-1} \| < \varepsilon_k^{-1} \text{ for all } j = 1, \ldots, m,
\]
so that 
\[
\max_{z \in \mathbb{C} \setminus G_{k-1}} \| (T - z)^{-1} \| < \varepsilon_k^{-1}.
\]
This implies that \( \sigma_{\varepsilon_k}(T) \subset G_{k-1} \). On the other hand, \( \Psi_{T_k(N_k)} \leq \varepsilon_k/2 \) on \( \Omega_k \) implies that \( \Psi_T \leq \varepsilon_k/2 \) on \( \Omega_k \), so that \( \sigma_{\varepsilon_k}(T) \supset \Omega_k \subset G_k \) for \( k = 1, \ldots, m \) (recall that \( \Psi_T \) is continuous on \( \mathbb{C} \)). It follows that \( T \) satisfies all inclusions in (4.1). \( \square \)

**Remark 4.2.** The inclusions given in (4.1) imply that for any \( \varepsilon \in [\varepsilon_m, \varepsilon_1] \), there exists an index \( j \), \( 0 \leq j \leq m-2 \) such that \( G_{j+2} \subset \sigma_j(T) \subset G_j \). So Theorem (4.1) shows that in some sense, the shape of pseudospectra of a finite matrix can be arbitrary. Certainly, we only are able to exhibit the example of this kind by taking the quotients \( \varepsilon_j/\varepsilon_{j+1} \) very large. As we mention in the Introduction, the problem of describing all possible functions \( \Psi_T(z) \) remains open.

5. Multipliers

### 5.1. Cowen-Douglas class and estimates of functions of nilpotent matrices.

First let us recall the well known class of operator from the fundamental paper of Cowen-Douglas [16].

**Definition 5.1.** For \( \Omega \) a connected open subset of \( \mathbb{C} \) and \( m \) a positive integer, let \( \mathcal{B}_m(\Omega) \) denote the set of operators \( T \) in \( \mathcal{B}(\mathcal{H}) \) which satisfy the following properties:

- \( \Omega \subset \sigma(T) \),
- \( \text{Ran}(T - w) = \mathcal{H} \) for all \( w \) in \( \Omega \),
- \( \bigvee_{w \in \Omega} \ker(T - w) = \mathcal{H} \),
- \( \text{dim} \ker(T - w) = m \) for \( w \in \Omega \).

Suppose \( T \) is in \( \mathcal{B}_m(\Omega) \) and \( 0 \in \Omega \). Put \( \mathcal{H}_n = \ker T^n \), \( T_n = T|_{\mathcal{H}_n} \) and \( P_n = P_{\mathcal{H}_n} \).

Then \( \{P_n\} \) is a filtration on \( \mathcal{H} \) and \( T \) is quasitriangular with respect to this filtration. As we will see a little bit later, \( \mathcal{H}_n \) is finite dimensional and \( T_n \) is nilpotent for any \( n \).

Let \( f \) be a function, defined and analytic in some (connected) neighborhood of \( 0 \). Then all operators \( f(T_n) \) are well defined. Notice also that \( \cup \mathcal{H}_n \) is dense in \( \mathcal{H} \) (we refer to [16] Section 1) for a background). Put \( f^*(z) = f(\overline{z}) \).

The main result of this section, Theorem 5.6, says that the norms \( \| f(T_n) \| \) are uniformly bounded if and only if \( f^* \) is a germ of a function in a certain multiplier space. This will motivate Example 5.8.

First we will need some preliminaries.
Put $N_\lambda = \ker(T - \lambda)$, where $\lambda \in \Omega$. As it follows from the Grauert theorem, the family of spaces $\{N_\lambda\}_{\lambda \in \Omega}$ possesses a global analytic frame: there exist analytic functions $\gamma_j : \Omega \to \mathcal{H}, 1 \leq j \leq m$ such that $\{\gamma_j(\lambda) : 1 \leq j \leq m\}$ is a basis of $N_\lambda$ for any $\lambda \in \Omega$ (see [16]). Let $\rho(\lambda) : \mathbb{C}^m \to N_\lambda$ be the isomorphism, defined by

$$\rho(\lambda)e_j = \gamma_j(\lambda), \quad j = 1, \ldots, m$$

(here $\{e_j\}$ is the standard basis of $\mathbb{C}^m$). Then $T\rho(\lambda) = \lambda \rho(\lambda), \lambda \in \Omega$.

The following proposition is rather standard. For reader’s convenience, we include a simple proof.

**Proposition 5.2.** There exists a Hilbert space $\tilde{\mathcal{H}}$ of holomorphic functions from $\overline{\Omega} = \{\tilde{w} : w \in \Omega\}$ to $\mathbb{C}^m$ and an isometric isomorphism $V : \mathcal{H} \to \tilde{\mathcal{H}}$ such that

$$T^* = V^{-1}M_zV,$$

where $M_z$ is the operator of multiplication by the co-ordinate function on $\tilde{\mathcal{H}}$.

**Proof.** This realization is provided by the injective map $V : \mathcal{H} \to \text{Hol}(\overline{\Omega}, \mathbb{C}^m)$, given by

$$Vx(\lambda) = (\rho(\lambda))^*x, \quad x \in \mathcal{H}, \lambda \in \overline{\Omega}$$

(here $\text{Hol}(\overline{\Omega}, \mathbb{C}^m)$ stands for the space of all analytic function from $\overline{\Omega}$ to $\mathbb{C}^m$). The identity $T\rho(\lambda) = \lambda \rho(\lambda)$ implies the intertwining property $VT^* = M_zV$, so that one has just to set $\tilde{\mathcal{H}} = V\mathcal{H}$.

Certainly, $\tilde{\mathcal{H}}$ can be seen as a vector-valued reproducing kernel Hilbert space. This is, in fact, an alternative point of view to the Cowen–Douglas class, which is discussed in the paper of Curto and Salinas [18]; in fact, $k$-tuples of operators were considered there. For one operator $T \in \mathcal{B}_1(\Omega)$, this fact is contained in [16] Subsection 1.15.

The commutant of $T$ is the weakly closed algebra of operators which commute with $T$. We denote it as $\{T\}'$. Notice that for any $S \in \{T\}'$, $SN_\lambda \subset N_\lambda$, $\lambda \in \Omega$. So there exist (uniquely defined) linear fibre maps $\Phi_S(\lambda) : N_\lambda \to N_\lambda$ such that

$$Sk = \Phi_S(\lambda)k \quad \text{for all } \lambda \in \Omega, k \in N_\lambda$$

(Cowen and Douglas in [16] use the notation $\Phi_S = \Gamma_TS$).

Given an operator $S \in \{T\}'$, put

$$\tilde{\Phi}_S(\lambda) = \rho(\lambda)\Phi_S(\lambda)\rho(\lambda)^{-1},$$

so that $\tilde{\Phi}_S(\lambda) \in \mathcal{B}(\mathbb{C}^m)$ and $\tilde{\Phi}_S(\lambda)$ is analytic in $\lambda$. It is easy to see that the matrix-valued function $\tilde{\Phi}_S$ is analytic in $\Omega$.

The multiplier algebra $\text{Mult}(\tilde{\mathcal{H}}) \subset \text{Hol}(\overline{\Omega})$ is defined as the set of (scalar) functions $\varphi$ on $\overline{\Omega}$ that multiply $\tilde{\mathcal{H}}$ into itself, i.e.

$$\{\varphi : \varphi f \in \tilde{\mathcal{H}}, \text{ for all } f \in \tilde{\mathcal{H}}\}.$$

It follows from the closed graph theorem that if $\varphi$ is a multiplier, then $M_\varphi$ is a bounded linear operator on $\tilde{\mathcal{H}}$.

The following fact follows immediately.

**Proposition 5.3.** Consider a subclass of the commutant, defined by

$$\mathcal{C}_T(\Omega) = \{S \in \{T\}' : \exists \text{ a scalar function } \varphi_S(\lambda) : \Phi_S(\lambda) = \varphi_S(\lambda)I_{N_\lambda}, \lambda \in \Omega\}.$$

Then

$$\{\varphi^*S : S \in \mathcal{C}_T(\Omega)\} = \text{Mult}(\tilde{\mathcal{H}}).$$
Lemma A. (see [16], Lemma 1.22). Suppose $T \in \mathcal{B}_m(\Omega)$. Then

1. $(T - \lambda)^\ell \gamma_n^{(\ell)}(\lambda) = \ell^\ell \gamma_n^{(\ell-1)}(\lambda)$ for all $\ell \geq 1$ and $k = 1, \ldots, m$;
2. For all $\lambda \in \Omega$,
   \[ \ker(T - \lambda)^n = \text{Span}\{ \gamma_k^{(\ell)}(\lambda) : 1 \leq k \leq m, 0 \leq \ell \leq n - 1 \} \]
   \[ = \text{Span}\{ \text{Ran} \rho^{(\ell)}(\lambda) : 0 \leq \ell \leq n - 1 \}. \]

It follows from this lemma that the vectors $\gamma_k^{(\ell)}(\lambda)$ $(1 \leq k \leq m, 0 \leq \ell \leq n - 1)$ form a basis of $\ker(T - \lambda)^n$ for any $\lambda \in \Omega$ and any $n$. In particular, $T_n$ is nilpotent, and its Jordan form has $m$ Jordan blocks of order $n$.

Lemma 5.4. Suppose $S \in \{ T \}'$ and $n \geq 1$. Then $S|_{\mathcal{F}_n} = 0$ if and only if $\Phi_S = z^n \Psi$, where $\Psi$ is analytic in the neighborhood of 0.

Proof. By Lemma A, $S|_{\mathcal{F}_n} = 0$ if and only if $S\rho^{(k)}(0) = 0$ for $k = 0, 1, \ldots, n - 1$.

By taking $k$th derivative in the identity $S\rho(\lambda) = \rho(\lambda)\tilde{\Phi}_S(\lambda)$, we get

\[ S\rho^{(k)}(\lambda) = \sum_{\ell=0}^{k-1} \binom{k}{\ell} k^{(k-\ell)}(\lambda)\tilde{\Phi}_S^{(\ell)}(\lambda) + \rho(\lambda)\tilde{\Phi}_S^{(k)}(\lambda). \]

By applying induction in $k$, we get that $S|_{\mathcal{F}_n} = 0$ if and only if $\tilde{\Phi}_S^{(k)}(0) = 0$ for all $k = 0, 1, \ldots, n - 1$ (notice that $\rho(0)$ is an isomorphism). This implies the statement of Lemma. \qed

Proposition 5.5. If $S \in \mathcal{C}_T(\Omega)$, then $S|_{\mathcal{F}_n} = \varphi_S(T_n)$ for all $n \geq 1$.

Proof. Fix some $n \geq 1$, and let $g$ be any polynomial such $\varphi_S = g = z^n \psi$, where $\psi$ is analytic at 0. Then $\Phi_{S-g(T)} = z^n \psi(z)I_{\mathcal{N}_n}$. By Lemma 5.4, $S-g(T)|_{\mathcal{F}_n} = 0$, and therefore $S|_{\mathcal{F}_n} = g(T)|_{\mathcal{F}_n} = \varphi_S(T_n)$. \qed

As consequence of the above lemma, we will prove the following theorem.

Theorem 5.6. Let $0 \in \Omega$ and let $T$ be an operator in $\mathcal{B}_n(\Omega)$. Put $\mathcal{H}_n = \ker T^n$, $T_n = T|_{\mathcal{F}_n}$. Let $f$ be a function, defined and analytic in the neighborhood of 0. Then the following properties are equivalent.

1. The norms $\|f(T_n)\|$ are uniformly bounded;
2. There exists an operator $S \in \mathcal{C}_T(\Omega)$ such that $\varphi_S = f$;
3. $f^* \in \text{Mult}(\tilde{\mathcal{H}})$ (or, more precisely, $f^*$ extends to a function in $\text{Mult}(\tilde{\mathcal{H}})$).

If these properties hold, then $S^*$ equals to the multiplication by $f^*$ on $\tilde{\mathcal{H}}$.

Proof. Since $T_n$ acts on a finite-dimensional $\mathcal{H}_n$ and is nilpotent, $f(T_n)$ is well-defined for all $n$. It is immediate that $f(T_n)|_{\mathcal{H}_m} = f(T_m)$ for all $n \geq m$, therefore the norms $\|f(T_n)\|$ increase as $n \to \infty$. First we show that (1) and (2) are equivalent.

(2) $\implies$ (1). Let $S \in \mathcal{C}_T(\Omega)$. Then, by Propositions 5.3 and 5.5 $\|\varphi_S(T_n)\| = \|S|_{\mathcal{F}_n}\| \leq \|S\|$ for all $n$.

(1) $\implies$ (2). Suppose that the norms $\|f(T_n)\|$ are uniformly bounded. Since $\mathcal{H}_n$ is dense in $\mathcal{H}$, the formula

\[ S|_{\mathcal{F}_n} = f(T_n), \quad n \geq 1 \]

defines correctly a bounded operator $S$ on $\mathcal{H}$. For any $h \in \mathcal{H}_n$, we have

\[ STh = f(T_n) T_n h = T_n f(T_n) h = T_n Sh = TSh. \]
Hence, \( S \in \{T\}' \). Now we can repeat the arguments used above in the proof of Proposition 5.5. Fix some \( n \geq 1 \), and let \( p_n \) be a polynomial such that \( f - p_n = z^n \psi \), where \( \psi \) is analytic at 0. We have

\[
S|_{\partial \Omega_n} = f(T_n) = p_n(T_n) = p_n(T)|_{\partial \Omega_n}
\]

(the equality \( f(T_n) = p_n(T_n) \) is due to the Jordan structure of \( T_n \)). By Lemma 5.4, this implies that \( \Phi_S - \Phi_{p_n(T)} = z^n \Psi \) for an analytic fibre map \( \Psi \). Since \( \Phi_{p_n(T)}(z) = p_n(z)I_{N_z} \), we get that for any \( n, \Phi_S(z) \) coincides with \( f(z)I_{N_z} \) at the origin, up to the \( n \)th order. Therefore \( \Phi_S(z) = f(z)I_{N_z} \) in a neighbourhood of 0, which gives (2).

The equivalence \( (2) \iff (3) \) and the last statement of the Theorem follow from Proposition 5.3. \( \square \)

5.2. An example. In what follows, we will denote by \( \sqrt{\text{the principal branch}} \) of the square root, defined for all \( z \neq 0 \) by \( \sqrt{z} = |z|^{1/2} \exp \left(i(\arg z)/2\right) \), where \( \arg z \in (-\pi, \pi] \) (so that the cut is along \( \mathbb{R}_- \)).

The next Lemma is auxiliary and will be used in Example 5.8 below.

**Lemma 5.7.** Define the function \( f_t(z) = \sqrt{z^2 - z + t} \), where \( t \in \mathbb{C} \), and let

\[
f_t(z) = \sum_{n=0}^{\infty} \hat{f}_n(t)z^n
\]

be its Taylor expansion at the origin. Let \( 0 < r < 1/4 \) and \( M > 0 \) be fixed. Then there exists some \( N = N(r, M) \) such that

\[
\max_{1 \leq n \leq N} |\hat{f}_n(t)| > M
\]

for all \( t \in \mathbb{C} \) such that \( |t - 1/4| = r \).

**Proof.** Fix some radius \( r \in (0, 1/4) \), and let \( |t - 1/4| = r \). The roots of \( z^2 - z + t \) are 

\[
z_{1,2} = z_{1,2}(t) = 1/2 \pm \sqrt{1/4 - t}. \]

Notice that \( |z_{1,2}| < 1 \). Set

\[
f_t(z) = \sqrt{z_1(t)} - z \sqrt{z_2(t)} - z.
\]

This coincides with the previous definition in a neighbourhood of 0, but now \( f_t(z) \) turns to be holomorphic in the disc \( |z| < \rho(t) \), where \( \rho(t) = \min(|z_1|, |z_2|) \). We have

\[
f_t(z) = g_t(z) + h_t(z),
\]

where

\[
\begin{align*}
g_t(z) &= (\sqrt{z_1} - \sqrt{z_1 - z_2}) (\sqrt{z_2} - z - \sqrt{z_2 - z_1}), \\
h_t(z) &= \sqrt{z_2 - z_1} \sqrt{z_1 - z} + \sqrt{z_1 - z_2} \sqrt{z_2 - z} - \sqrt{z_2 - z_1} \sqrt{z_1 - z_2}.
\end{align*}
\]

We denote by \( \hat{g}_n(t), \hat{h}_n(t) \) the Taylor coefficients of \( g_t(z) \) and \( h_t(z) \).

One has the formula

\[
\sqrt{1 - z} = 1 - \sum_{n=1}^{\infty} c_n z^n, \quad |z| < 1,
\]

where

\[
c_n = \frac{1}{2(n + \frac{1}{2})(n - \frac{1}{2})B\left(\frac{1}{2}, n + 1\right)} \sim \frac{1}{2\sqrt{\pi} n^{3/2}} \quad n \to \infty
\]

(\( B \) is the Beta function). By (5.2) and (5.3), we get

\[
\hat{h}_n(t) = -c_n (a(t)z_1(t)^{-n} + b(t)z_2(t)^{-n}), \quad n \geq 1,
\]
with
\[ a(t) = \sqrt{z_2(t) - z_1(t)} \sqrt{z_1(t)}, \quad b(t) = \sqrt{z_1(t) - z_2(t)} \sqrt{z_2(t)}. \]
Since \(|z_1(t) - z_2(t)| = 2 \sqrt{T}\), it is easy to see that there is a constant \(\varepsilon > 0\), independent of \(t\), such that for any \(n \geq 1\),
\[ |a(t)z_1(t)^{-n+1} + b(t)z_2(t)^{-n+1}| + |a(t)z_1(t)^{-n} + b(t)z_2(t)^{-n}| > \varepsilon \rho(t)^{-n}. \]
Therefore \(|\hat{h}_{n-1}(t)| + |\hat{h}_n(t)| > \varepsilon' n^{-3/2} \rho(t)^{-n}\), where \(\varepsilon' = \varepsilon'(r) > 0\).

We assert that a similar lower estimate holds for the Taylor coefficients of \(f_t\). To show it, consider the two-point set
\[ E(t) = \{ \rho(t) \frac{z_1(t)}{|z_1(t)|}, \rho(t) \frac{z_2(t)}{|z_2(t)|} \}. \]
Then for \(|z| < \rho(t)\), \( |g''_t(z)| \leq K \text{ dist} (z, E(t))^{-1/2} \), where \(K\) does not depend on \(t\) (here \(g''_t(z) = d^2 g_t(z)/dz^2\)). Hence \(\|g''_t\|_{H^1(B_{\rho(t)0})} \leq K_1 \) (\(H^1\) stands for the Hardy space). This gives
\[ |\hat{g}_n(t)| \leq \frac{K_2}{n^2} \rho(t)^{-n}. \]
The constants \(K_1\) and \(K_2\) only depend on \(r\). Fix any positive constant \(M\). Since \(\hat{f}_n(t) = \hat{g}_n(t) + \hat{h}_n(t)\), there exists a large \(N = N(r, M)\) such that
\[ |\hat{f}_{N-1}(t)| + |\hat{f}_N(t)| > \left( \frac{\varepsilon'}{N^{3/2}} - \frac{2K_2}{(N - 1)^2} \right) \rho(t)^{-N} > \frac{\varepsilon'}{2N^{3/2}} > 2M \]
for all \(t\), \(|t - 1/4| = r\). This implies the statement of Lemma. \(\square\)

**Example 5.8.** Given any real \(r\), \(0 < r < \frac{1}{2}\) and any (large) real number \(M\), there exists a nilpotent square matrix \(S\), whose size depends on \(r\) and \(M\), such that \(\|\sqrt{I - S}\| \leq 3\), whereas \(\|\sqrt{\tau - S}\| \geq M\) for any \(\tau\) on the circle \(|\tau - 1| = r\). Here \(\sqrt{\tau - S}\) is understood in the sense of the Riesz-Dunford calculus, applied to the function \(\sqrt{\tau - z}\), where the principal value of the square root is meant.

Indeed, consider the \(N \times N\) nilpotent lower triangular Toeplitz matrix
\[
S_N = \begin{pmatrix}
0 & 0 & 0 & \cdots & 0 & 0 \\
4 & 0 & 0 & \cdots & 0 & 0 \\
-4 & 4 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & -4 & 4 & 0
\end{pmatrix}
\]
(which has entries 4 on the first diagonal under the main one, entries -4 on the second diagonal and all other entries equal to 0). We assert that one can put \(S = S_N\), where \(N = N(r, M)\) is sufficiently large. To see this, notice first that \(S_N = 4J_N - 4J_N^2\), where \(J_N\) is the standard \(N \times N\) Jordan block with ones on the first diagonal under the main one. It is standard that for any function \(\varphi\), analytic in a neighbourhood of zero, \(\varphi(J_N)\) is well-defined and is a Toeplitz lower triangular matrix, whose entries in the first column are \(\varphi_0, \varphi_1, \ldots, \varphi_{N-1}\). Define \(f_t(z)\) as in Lemma 5.7. It follows that
\[ \sqrt{\tau - S_N} = 2f_{\tau/4}(J_N). \]
In particular, \(\sqrt{I - S_N} = I - 2J_N\). Therefore \(\|\sqrt{I - S_N}\| \leq 3\). Take any \(M > 0\). By Lemma 5.7, there is some \(N\) such that for any \(\tau\) on the circle \(|\tau - 1| = r\), the matrix \(\sqrt{\tau - S_N}\) has an entry, whose absolute value is greater than \(M\). This implies our assertion.
Notice that in fact, the above argument proves that for a fixed \( r \in (0, 1/2) \),
\[
\min_{|\tau - 1| = r} \| \sqrt{\tau - S_N} \|
\]
grows exponentially as a function of the size \( N \). The informal explanation of this example is that in the limit (as \( N \to \infty \)), the matrices \( S_N \) behave as the Toeplitz operator \( T_\psi \) with the analytic symbol \( \psi(z) = 4z - 4z^2 \) on \( H^2(B(0)) \). Then for \( \tau = 1 \), the square root \( \sqrt{T - T_\psi} \) exists as a bounded operator (and equals to \( T_{1 - z^2} \)), whereas a bounded operator square root \( \sqrt{T - T_\psi} \) does not exist if \( \tau \neq 1 \) is close to 1. We observe that the spectrum of the “limit operator” \( T_\psi \) is no longer one point, instead, it contains a neighbourhood of 1.

This example also implies that even for a nilpotent matrix \( S \), the values of \( \| \sqrt{\tau - S} \| \) can change very rapidly for \( \tau \) in a neighbourhood of 1. (Notice that for a fixed \( S \), \( \sqrt{T - S} \) is analytic in this neighbourhood.) In particular, to the contrary to Corollary 2.4, any estimate of the Lipschitz constants of the functions \( \tau \mapsto \| \sqrt{\tau - S} \| \), \( \tau \mapsto \| \sqrt{\tau - S} \|^{-1} \) should depend on the size of \( S \).

5.3. Final remarks on estimates of functions of operators and matrices. Here we discuss some relations between known results.

Suppose we have an operator \( T \) on a Hilbert space \( \mathcal{H} \) (which can be finite dimensional) and suppose that the function \( \Psi_T \) is known. One can ask, what can be said about the norms \( \| f(T) \| \), where \( f \) is analytic on \( \sigma(T) \). This question was raised in the work [28], which contains an example of two matrices \( T_1 \) and \( T_2 \) with simple eigenvalues and identical pseudospectra (that is, satisfying \( \Psi_{T_1}(z) = \Psi_{T_2}(z) \) for all \( z \)) and such that \( \| T_1^3 \| \neq \| T_2^3 \| \). The matrix norms here and in the definition of \( \Psi_T \) are induced by the Euclidean norm. This question was further investigated a series of papers by Ransford and his coauthors. The paper [39] by Ransford and Rostand gives another example of such type of matrices with simple eigenvalues. Moreover, the two matrices in this latter example have super-identical pseudospectra in the sense that all singular numbers of \( T_1 - z \) coincide with those of \( T_2 - z \), for any \( z \in \mathbb{C} \).

By a theorem in [33], given a domain \( \Omega \) and a function \( f \neq \text{const} \) in \( \text{Hol}(\Omega) \), which is not a Möbius transformation, for any \( N \geq 6 \) and any \( M > 1 \) one can find \( N \times N \) matrices \( T_1 \) and \( T_2 \) with identical pseudospectra such that \( \| f(T_1) \| \geq M \| f(T_2) \| \). On the other hand, it is known (see [26]) that, given matrices \( T_1 \) and \( T_2 \) of size \( N \times N \) with super-identical pseudospectra, one has
\[
N^{-1/2} \leq \frac{\| f(T_1) \|}{\| f(T_2) \|} \leq N^{1/2}
\]
for any function \( f \) holomorphic on \( \sigma(T_1) = \sigma(T_2) \). It is not known whether there is an estimate independent of \( N \).

There are also many other positive results on the estimation of functions of operators and matrices. For instance, the following assertion follows from the main result of [7].

**Theorem (7).** Let \( T \) be a Hilbert space operator and let \( z_1, \ldots, z_n \) be points outside its spectrum. Then for any bounded analytic function on the (unbounded) domain \( \Omega = \mathbb{C} \cup \bigcup_k \text{clos} B(z_j, \Psi_T(z_j)) \), one has \( \| f(T) \| \leq K \sup_{\Omega} | f | \), where \( K = n + n(n - 1)/\sqrt{3} \).

Notice that here \( K \) does not depend on the dimension of \( \mathcal{H} \).

Many other results have this form. For instance, suppose \( T \) is a Hilbert space operator, \( \sigma(T) \subset B(0) \) and \( \Psi_T(z) \geq r \) for any \( z \) on the circle \( |z| = 1 + r \). Then \( T \) is a \( \rho \)-contraction for \( \rho = 2 + 1/r \), which implies the estimate \( \| f(T) \| \leq \rho \sup_{B(0)} | f | \), for any function
holomorphic in $B_1(0)$ such that $f(0) = 0$ (see [45] Section I.11). It is easy to describe the numerical range of $T$ in terms of the behavior of the function $\Psi_T$, see (2.1). Therefore the variant of the von Neumann inequality given by B. Delyon and F. Delyon in [23] can also be seen as a positive result in this direction. We refer to [20] for a generalization of the result of [23] to certain non-convex sets associated with the operator.

As positive results on estimation of norms $\|f(T)\|$, one can mention the Kreiss matrix theorem (see, for instance, [51] Section 18) and the results by Szehr and Zarouf (see [46, 47] and references therein).

One can also relate the estimates of functions of an operator with the so-called weak resolvent sets. By definition (see [25]), an analytic function on $\mathbb{C} \setminus \sigma(T)$ is called a weak resolvent of a bounded operator $T$ on a Banach space $X$ if it has the form $z \mapsto G((T - z)^{-1}f)$ for some $f \in X$ and $G \in X^*$. The weak resolvent set $WR(T)$ of $T$ is the set of all its weak resolvents. This interesting notion was introduced in 1987 in a paper by Nordgren, Radjavi and Rosenthal and further studied by Fong and the named three authors in [25]. Since it makes no difference, let us consider the Banach space setting.

Let $T_j \in \mathcal{B}(H_j)$, $j = 1, 2$ be two Banach space operators. Following [25], we say that $WR(T_1) \subset WR(T_2)$ if $\sigma(T_1) \subset \sigma(T_2)$ and each function in $WR(T_1)$ is also in $WR(T_2)$. Let us cite the following result.

**Theorem** [25], Theorem 2.8. If $\sigma(T_1)$ has finitely many holes and $WR(T_1) \subset WR(T_2)$, then there is a constant $k$ such that $\|\varphi(T_2)\| \leq k\|\varphi(T_1)\|$ for any function $\varphi$, holomorphic on a neighbourhood of $\sigma(T_1)$.

In particular, it follows that $\Psi_{T_1} \leq k^{-1}\Psi_{T_2}$ on $\mathbb{C} \setminus \sigma(T_1)$. If $\sigma(T_1) = \sigma(T_2)$ and the weak resolvent sets of $T_1$ and $T_2$ coincide, then one has a two-sided estimate $\Psi_{T_1} \asymp \Psi_{T_2}$ on $\mathbb{C} \setminus \sigma(T_1)$.

One can observe that the statement from the above theorem is much stronger than just the relation $\Psi_{T_1} \asymp \Psi_{T_2}$. In fact, it is also proven in [25] that whenever the sets $WR(T_1)$ and $WR(T_2)$ coincide in a neighbourhood of $\infty$, operators $T_1$ and $T_2$ generate isomorphic uniformly closed algebras. If, moreover, both operators are strictly cyclic, then they are similar.

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**References**

[1] C. Apostol, C. Foiaş, L. Zsid, *Some results on non-quasitriangular operators*, Indiana Univ. Math. J. 22 (1972/73), 1151–1161.

[2] C. Apostol, C. Foiaş, D. Voiculescu, *Some results on non-quasitriangular operators. II, III, IV, V*, Rev. Roumaine Math. Pures Appl. 18 (1973), 159 - 181; ibid. 18 (1973), 309–324; ibid. 18 (1973), 487–514; ibid. 18 (1973), 1133–1149.

[3] C. Apostol, C. Foiaş, D. Voiculescu, *Some results on non-quasitriangular operators. VI. Hommage au Professeur Miron Nicolescu pour son 70eme anniversaire, I*, Rev. Roumaine Math. Pures Appl. 18 (1973), 1473–1494.

[4] W. Arveson, *The role of $C^*$-algebras in infinite-dimensional numerical linear algebra*, Contemp. Math., 167 (1994), 114–129.
[5] W. Arveson, $C^*$-algebras and numerical linear algebra, J. Funct. Anal. 122 (1994), no. 2, 333–360.
[6] O. F. Bandtlow, A. G"uven, Explicit upper bounds for the spectral distance of two trace class operators, Linear Algebra Appl. 466 (2015), 329–342.
[7] C. Badea, B. Beckermann, M. Crouzeix, Intersections of several disks of the Riemann sphere as $K$-spectral sets. Commun. Pure Appl. Anal. 8 (2009), no. 1, 37–54.
[8] A. Ben-Artzi, On approximation spectrum of bounded selfadjoint operators, Integral Equations Operator Theory 9 (1986), no. 2, 266–274.
[9] J. Ben-Artzi, A. C. Hansen, O. Nevanlinna, M. Seidel, New barriers in complexity theory: on the solvability complexity index and the towers of algorithms C. R. Math. Acad. Sci. Paris 353 (2015), no. 10, 931–936 (also see arXiv:1508.03289).
[10] A. B"ottcher, H. Wolf, Spectral approximation for Segal-Bargmann space Toeplitz operators, Linear operators (Warsaw, 1994), 25–48, Banach Center Publ., Polish Acad. Sci., Warsaw 38 (1997).
[11] S. B"ogli, Local convergence of spectra and pseudospectra, arXiv:1605.01041 to appear in J. Spectral Theory.
[12] S. B"ogli, P. Siegl, Remarks on the convergence of pseudospectra, Integr. Equ. Oper. Theory 80 (2014), no. 3, 303–321.
[13] N. P. Brown, Quasi-diagonality and the finite section method, Math. Comp. 76 (2007), no. 257, 339–360.
[14] P. Cannarsa, C. Sinestrari, Semiconcave Functions, Hamilton–Jacobi Equations, and Optimal Control, Birkh"auser, (2004).
[15] F. Chaitin-Chatelin and A. Harrabi, About definitions of pseudospectra of closed operators in Banach spaces, Tech. Rep. TR/PA/98/08, CERFACS.
[16] M. J. Cowen, R. G. Douglas, Complex geometry and operator theory, Acta Math. 141 (1978), no. 3–4, 187–261.
[17] T. Cubitt, D. Perez-Garcia, M. M. Wolf, Undecidability of the Spectral Gap (full version), arXiv preprint arXiv:1502.04573.
[18] R. E. Curto, N. Salinas, Generalized Bergman kernels and the Cowen-Douglas theory, Amer. J. Math. 106 (1984), no. 2, 447–488.
[19] M. Derevyagin, L. Perotti, M. Wojtylak, Truncations of a class of pseudo-Hermitian tridiagonal matrices. J. Math. Anal. Appl. 438 (2016), no. 2, 738–758.
[20] M. A. Dritschel, D. Estévez, D. Yakubovich, Tests for complete K-spectral sets, (2015) arXiv preprint arXiv:1510.08350.
[21] E. B. Davies, Linear operators and their spectra Cambridge University Press (2007).
[22] E. B. Davies, E. Shargorodsky, Level sets of the resolvent norm of a linear operator revisited, Mathematika 62 (2015), no. 1, 243–265.
[23] B. Delyon, F. Delyon, Generalization of von Neumann’s spectral sets and integral representation of operators, Bull. Soc. Math. France 127 (1999), no. 1, 25–41.
[24] L. Elsner, An optimal bound for the spectral variation of two matrices, Linear Algebra Appl. 71 (1985), 77–80.
[25] C. K. Fong, E. A. Nordgren, H. Radjavi, P. Rosenthal, Weak resolvents of linear operators. II, Indiana Univ. Math. J. 39 (1990), no. 1, 67–83.
[26] M. Fortier Bourque, T. Ransford, Super-identical pseudospectra, J. Lond. Math. Soc. 79 (2009), no. 2, 511–528.
[27] E. Gallestey, D. Hinrichsen, A. Pritchard, Spectral value sets of closed linear operators, Proc. R. Soc. London, 456 (2000), 930–937.
[28] A. Greenbaum, L.N. Trefethen, Do the pseudospectra of a matrix determine its behavior?, Technical Report TR 93-1371, Computer Science Department, Cornell University, (1993).
[29] R. Hagen, S. Roch, B. Silbermann, $C^*$-algebras and numerical analysis, Monographs and Textbooks in Pure and Applied Mathematics, Marcel Dekker, Inc., New York, 236 (2001).
[30] A. C. Hansen, On the approximation of spectra of linear operators on Hilbert spaces, J. Funct. Anal. 254 (2008), no. 8, 2092–2126.
[31] A. C. Hansen, On the solvability complexity index, the $n$-pseudospectrum and approximations of spectra of operators, J. Amer. Math. Soc. 24 (2011), no. 1, 81–124.
[32] D. A. Herrero, The diagonal entries in the formula “quasitriangular - compact = triangular” and restrictions of quasitriangularity, Trans. Amer. Math. Soc. 298 (1986), no. 1, 1–42.
[33] V. Müller, Spectral Theory of Linear Operators and Spectral Systems in Banach Algebras, Birkh"auser, (2000).
[34] M. Marletta, S. Naboko, *The finite section method for dissipative operators*, Mathematika 60 (2014), no. 2, 415–443.
[35] A. Pokrzywa, *Limits of spectra of strongly converging compressions*, J. Oper. Theory 12 (1984), no. 2, 199 - 212.
[36] M. B. Pour-El, J. I. Richards, *Computability in analysis and physics. Perspectives in Mathematical Logic*. Springer-Verlag, Berlin, 1989. xii+206 pp.
[37] V. Rabinovich, S. Roch, B. Silbermann, *Limit operators and their applications in operator theory*, Operator Theory: Advances and Applications, Birkhäuser Verlag, Basel, 150 (2004) xvi+392 pp.
[38] T. Ransford, S. Raoufi, *Pseudospectra and holomorphic functions of matrices*, Bull. Lond. Math. Soc. 45 (2013), no. 4, 693–699.
[39] T. Ransford, J. Rostand, *Pseudospectra do not determine norm behavior, even for matrices with only simple eigenvalues*, Linear Algebra and Its Applications 435 (2011), 3024–3028.
[40] W. Rudin, *Principles of mathematical analysis*, McGraw-Hill (1976).
[41] M. Seidel, *On \((N,\varepsilon)\)-pseudospectra of operators on Banach spaces*, J. Funct. Anal. 262 (2012), 4916–4927.
[42] M. Seidel, B. Silbermann, *Finite sections of band-dominated operators norms, condition numbers and pseudospectra*, Operator theory, pseudo-differential equations, and mathematical physics, Oper. Theory Adv. Appl., Birkhuser/Springer Basel AG, Basel, 228 (2013), 375-390.
[43] E. Shargorodsky, *On the level sets of the resolvent norm of a linear operator*, Bull. Lond. Math. Soc. 40 (2008), no. 3, 493–504.
[44] E. Shargorodsky, *On the definition of pseudospectra*, Bull. Lond. Math. Soc. 41 (2009), no. 3, 524–534.
[45] B. Sz.-Nagy, C. Foias, H. Bercovici, L. Kérchy, *Harmonic analysis of operators on Hilbert space*, Second edition, Revised and enlarged edition, Universitext, Springer, New York, (2010) xiv+474 pp.
[46] O. Szehr, *Eigenvalue estimates for the resolvent of a non-normal matrix*, J. Spectr. Theory, 4 (2014) no. 4, 783-813.
[47] O. Szehr, R. Zarouf, *Maximum of the resolvent over matrices with given spectrum*, arXiv preprint arXiv:1501.07007.
[48] L. N. Trefethen, *Pseudospectra of linear operators*, SIAM Rev. 39 (1997), no. 3, 383–406.
[49] L. N. Trefethen, *Computation of pseudospectra*, Acta Numer., 8, Cambridge Univ. Press, Cambridge,(1999), 247–295.
[50] L. N. Trefethen, M. Embree, *Spectra and pseudospectra*, Princeton University Press (2005).