Inverse scattering and the symplectic form for sine-Gordon solitons

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Abstract

We consider the canonical symplectic form for sine-Gordon evaluated explicitly on the solitons of the model. The integral over space in the form, which arises because the canonical argument uses the Lagrangian density, is done explicitly in terms of functions arising in the group doublecrossproduct formulation of the inverse scattering procedure, and we are left with a simple expression given by two boundary terms. The expression is then evaluated explicitly in terms of the changes in the positions and momenta of the solitons, and we find agreement with a result of Babelon and Bernard who have evaluated the form using a different argument, where it is diagonal in terms of ‘in’ or ‘out’ co-ordinates. Using the result, we also investigate the higher conserved charges within the inverse scattering framework, check that they Poisson commute and evaluate them on the soliton solutions.
1 Introduction

The purpose of this paper is to calculate the symplectic form restricted to the phase space describing the dynamics of solitons in the sine-Gordon model, in terms of the loop groups of the inverse scattering procedure.

The aim is to be able to do Hamiltonian mechanics with solitons, while retaining all their positions and phases. The reader may remember that any method proceeding only via the monodromy matrix, such as the quantum inverse scattering method [4], looses information on the positions and phases. Some work has already been done on the soliton symplectic form restricted to the solitons of sine-Gordon by Babelon and Bernard [1], where a result was obtained in terms of the positions and momenta of the solitons by using an intuitive argument based on ‘in’ and ‘out’ co-ordinates; the form becoming diagonal in these co-ordinates. This argument solved for the form without evaluating it explicitly (the integrals over space were not performed).

The approach adopted in this paper relies on an interesting simplification which occurs when the symplectic form is expressed in the language of inverse scattering and group factorisations. The space integral in the form can be done explicitly by an ‘integration by parts trick’, and the form is then a difference of two boundary terms. This general expression for the form (5.13) is of interest by itself, and could be useful for applications such as quantisation – in fact quantisation is the real motivating force behind our construction. However, we go further and calculate it explicitly in terms of the changes in position and momenta, where we indeed find agreement with the results of Babelon and Bernard [1].

The second purpose of the paper is to relate this to an abstract integrable system consisting of a group factorisation and a classical vacuum map. From this abstract point of view we do Hamiltonian mechanics by studying Hamiltonian functions arising as ‘higher momenta’ in integrable theories, and we calculate the momenta for the sine-Gordon case. We explicitly verify a result of Olive and Turok [8], restricted to sine-Gordon, that the values of the higher charges with odd Lorentz spins are zero.

2 Preliminaries

We firstly define light cone coordinates on $\mathbb{R}^{1,1}$ by $x_\pm = t \pm x$, and let $\partial_\pm$ denote differentiation with respect to $x_\pm$ respectively. In these coordinates the sine-Gordon equation is written as

$$\partial_+ \partial_- u = -\frac{m^2}{\beta} \sin(\beta u),$$

where $u(x_+, x_-)$ is a real valued function on $\mathbb{R}^{1,1}$.

Now consider the following linear system for a function

$$\Psi : \mathbb{R}^{1,1} \times \mathbb{C}^* \rightarrow GL_2(\mathbb{C})$$

$$\partial_+ \Psi(x_+, x_-, \lambda) = \Psi A(x_+, x_-, \lambda), \quad \partial_- \Psi(x_+, x_-, \lambda) = \Psi B(x_+, x_-, \lambda).$$

The linear system is overdetermined, and the compatibility condition derived from

$$\partial_- \partial_+ \Psi = \partial_- \partial_+ \Psi$$

is precisely

$$\partial_- A - \partial_+ B = [A, B].$$

We shall take $A$ and $B$ of the form

$$A = -\beta(\partial_+ u)s_3 + 2m\lambda(\cos(\frac{\beta u}{2})s_1 - \sin(\frac{\beta u}{2})s_2)$$

$$B = \beta(\partial_- u)s_3 + 2m\lambda^{-1}(\cos(\frac{\beta u}{2})s_1 + \sin(\frac{\beta u}{2})s_2)$$
\[ \beta (\partial_- u) s_3 + 2m\lambda^{-1}e^{\beta u s_3} s_1 e^{-\beta u s_3}, \] (2.4)

where \( \lambda \) can be an arbitrary complex number and is called the spectral parameter, and \( s_1, s_2, \) and \( s_3 \) are the standard anti-hermitian basis elements of the Lie algebra \( \mathfrak{sl}(2) \)

\[
\begin{align*}
[s_1, s_2] &= \frac{1}{2} s_3, & [s_2, s_3] &= \frac{1}{2} s_1, & [s_3, s_1] &= \frac{1}{2} s_2.
\end{align*}
\] (2.5)

Observe that the spectral parameter \( \lambda \) is in the principal gradation of the underlying loop algebra. The compatibility conditions (2.3) then immediately yield

\[ (\partial_+ \partial_- u + \frac{m^2}{\beta} \sin(\beta u)) s_3 = 0, \]

in other words \( u \) is a solution to the sine-Gordon equation (2.1).

For convenience we shall want to have \( s_3 \) diagonal, so we choose the explicit basis

\[
\begin{align*}
s_1 &= \frac{1}{4} \left( \begin{array}{cc} 0 & i \\ i & 0 \end{array} \right), & s_2 &= \frac{1}{4} \left( \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right), & s_3 &= \frac{1}{4} \left( \begin{array}{cc} i & 0 \\ 0 & -i \end{array} \right).
\end{align*}
\] (2.6)

We have a simple vacuum solution to these equations given by \( u \) constant with \( \cos(u^2) = 1 \). In this case we define \( J \) and \( K \) by

\[
\begin{align*}
A &= J = 2m\lambda s_1, & B &= K = 2m\lambda^{-1} s_1.
\end{align*}
\] (2.7)

The equations

\[
\begin{align*}
\partial_+ \Psi_0 &= \Psi_0 J, & \partial_- \Psi_0 &= \Psi_0 K
\end{align*}
\]

have a simple exponential solution, and if we then subtract off this vacuum solution from the general solution \( \Psi \) by defining

\[
\phi = \Psi_0^{-1} \Psi,
\]

we see that \( \phi \) obeys the equations

\[
\begin{align*}
\partial_+ \phi &= \phi A - J \phi, & \partial_- \phi &= \phi B - K \phi.
\end{align*}
\] (2.8)

We demand that \( \lambda = 0 \) and \( \lambda = \infty \) are regular points of the function \( \phi(\lambda) \), by requiring that

\[
\begin{align*}
\phi^{-1} \partial_+ \phi &= A - \phi^{-1} J \phi, & \phi^{-1} \partial_- \phi &= B - \phi^{-1} K \phi
\end{align*}
\]

are free from singularities at 0 and \( \infty \), and from this we deduce that

\[
e^{-\beta u s_3} s_1 e^{\beta u s_3} = \phi(\infty)^{-1} s_1 \phi(\infty),
\] (2.10)

and

\[
e^{\beta u s_3} s_1 e^{-\beta u s_3} = \phi(0)^{-1} s_1 \phi(0).
\] (2.11)

We can solve (2.10) by setting the normalisation of \( \phi(\lambda) \), that is \( \phi(\infty) = e^{\beta u s_3} \). We can then compute the explicit soliton solutions from \( \phi(0) \) and (2.11), which must satisfy \( \phi(0) = N e^{-\beta u s_3} \), for \( N \) a constant matrix such that \( N^{-1} s_1 N = s_1 \).

To be consistent with the linear system we impose the condition, as shown in [3], that

\[
U \phi(\lambda) U^\dagger = f(\lambda) \phi(-\lambda),
\] (2.12)

where \( f(\lambda) \) is a scalar function and \( U = 4s_3 \).

It is well known, for example see [3, 4], or [2, 3], that the meromorphic solutions \( \phi(\lambda) \), which are unitary on the real axis, can be written as products of

\[
\left( P_i^\perp + \frac{\lambda - \bar{\alpha}_i}{\lambda - \alpha_i} P_i \right),
\] (2.13)
where \( i \) runs from one to the number of solitons in the system. Here \( P_i \) is a Hermitian projection, i.e. \( P_i^2 = P_i \) and \( P_i^\dagger = P_i \). The product can be taken in any order, but then different \( P_i \)'s will apply. The positions of the solitons are encoded into the \( P_i \)'s, and the momenta of the \( i \)th soliton is related in a simple way to \( |\alpha_i| \), that is the rapidity, \( \theta_i \), is given by \( |\alpha_i| = e^{-\theta_i} \).

The condition (2.12) restricts the positions of the poles \( \alpha_i \) to either the imaginary axis, which corresponds to solitons and anti-solitons (corresponding to a single factor of the form (2.13)), or if not on the imaginary axis, the poles come in pairs, one at \( \alpha_i \) and the other at \(-\alpha_i \). This latter situation corresponds to a breather – a bound state of a soliton and anti-soliton.

3 The doublecross product method.

At this point we must beg the reader's indulgence as we consider a very abstract picture. The justifications for our digression are:

(1) We will derive concrete formulae which we will apply to our present case, the sine-Gordon model.

(2) We shall find general formulae for the ‘higher conserved momenta’ typical of 1+1 dimensional integrable field theories.

(3) The expression found for the symplectic form will be based on functions defined in this section.

Consider a group factorisation \( \mathcal{X} = \mathcal{G}\mathcal{M} = \mathcal{M}\mathcal{G} \), where \( \mathcal{G} \) and \( \mathcal{M} \) are subgroups of \( \mathcal{X} \), and \( \mathcal{G} \cap \mathcal{M} \) consists only of the identity element. We require that any element of \( \mathcal{X} \) can be uniquely factorised as \( c\psi \), where \( c \in \mathcal{M} \) and \( \psi \in \mathcal{G} \), and can be uniquely factorised as \( \phi f \), where \( f \in \mathcal{M} \) and \( \phi \in \mathcal{G} \). This is the definition of a group doublecross product, and appears in the construction of Hopf algebra bicrossproducts [6], however we shall not consider Hopf algebras in this paper.

To construct an example of an abstract integrable field theory, take a spacetime \( S \), and a function \( a : S \to \mathcal{M} \), which we shall call the classical ‘vacuum’ map. The group \( \mathcal{G} \) may be called the classical phase space. For a given classical solution \( \phi_0 \) in the phase space, we perform a factorisation for all space-time positions \( s \in S \):

\[
a(s) \phi_0 = \phi(s) b(s) , \quad \phi(s) \in \mathcal{G} , \ b(s) \in \mathcal{M} .
\]

The solution to the field theory, the fields at the point \( s \in S \), is encoded into \( b(s) \) and \( \phi(s) \). If \( S \) is a differential manifold we can recover a linear system from the factorisation (3.1) by differentiating it along a vector \( g \) in \( S \) to give \( a_g \phi_0 = \phi_g b + \phi b_g \) (we use subscripts for differentiation). This can be rearranged to give

\[
\phi_g = a_g \phi^{-1} - \phi b_g b^{-1} .
\]

For the sine-Gordon model, we take \( S \) to be the ordinary 1+1 dimensional flat Lorentzian spacetime. The group \( \mathcal{M} \) consists of functions from \( \mathbb{C}^* \) to \( GL_2(\mathbb{C}) \) which are complex analytic (so that essential singularities are likely to be present in elements of \( \mathcal{M} \) at 0 and \( \infty \)), and are unitary on \( \mathbb{R}^+ \). The group \( \mathcal{G} \) consists of meromorphic functions from \( \mathbb{C}_\infty \) to \( GL_2(\mathbb{C}) \) which are unitary on \( \mathbb{R}_\infty \), and which satisfy the symmetry condition (2.12). The solution \( \phi(\lambda) \) to the linear system (2.12) is valued in \( \mathcal{G} \), and is the same \( \phi(\lambda) \) which appears in the factorisation (3.1). Equation (3.2) has the same form as (2.9), provided that \( a_+a^{-1} = -J, a_-a^{-1} = -K, b_+b^{-1} = -A, \) and \( b_-b^{-1} = -B \). Therefore the ‘vacuum’ map \( a(\lambda, x, t) \), in this case, is defined as

\[
a(\lambda, x, t) = e^{-Jx + Kx -}.\]

The proof of the corresponding statement for the \( A_{n-1} \) affine Toda field theories can be found in [3]. Sine-Gordon can be recovered from the analysis in [3], if we restrict ourselves to the case \( n = 2 \). In the more general affine Toda case the unitarity condition has to be dropped, and a generalisation of (2.12) applies. The loss of unitarity means that the residue \( P_i \) of the poles in each of the simple pole factors (2.13) is no longer Hermitian. For sine-Gordon the specification of the kernel of \( P_i \) is fixed in terms of the image, therefore the problems pointed out in [3] due to the kernel do not apply.
4 Explicit calculation of some soliton solutions

The explicit form of the soliton solutions are extremely familiar, and have been known for quite some time, however we shall need to know precisely how the positions of the solitons are encoded into the residues \( P_i \) and for this we need to calculate some solutions using the method. We will also use the group factorisation (3.1) to our advantage, to simplify the calculations.

We write \( \phi(\lambda) \) as a product of meromorphic loops, of the form \( (2.13) \), which we call \( \psi(\lambda) \), times the non-trivial normalization matrix

\[
g = \phi(x, t, \infty) = e^{\beta u(x, t)s_3},
\]

which we place on the right:

\[
\phi(\lambda) = \psi(\lambda)g.
\] (4.1)

The condition \( (2.10) \) is then automatically satisfied. We can now calculate \( u(x_+, x_-) \) from the projections \( P(x_+, x_-) \) by using the regularity condition at \( \lambda = 0 \) \( (2.11) \)

\[
e^{2\beta u s_3 s_1}e^{-2\beta u s_3} = \psi(0)^{-1}s_1\psi(0).
\] (4.2)

It is simple to check that the unique Hermitian projection \( P^\perp \) that annihilates the vector \( \begin{pmatrix} 1 \\ \mu \end{pmatrix} \) is

\[
P^\perp = \begin{pmatrix}
\frac{|\mu|^2}{1+|\mu|^2} & -\bar{\mu} \\
\bar{\mu} & \frac{1+|\mu|^2}{1+|\mu|^2}
\end{pmatrix}.
\]

The one-soliton solution

We take one simple pole on the imaginary axis

\[
\psi(x_+, x_-, \lambda) = (P^\perp(x_+, x_-) + \frac{\lambda + i\kappa}{\lambda - i\kappa}P(x_+, x_-)) \quad (4.3)
\]

where \( \kappa \) is real and positive. The symmetry condition \( (2.12) \) implies that

\[
U(P^\perp + \frac{\lambda + i\kappa}{\lambda - i\kappa}P)U^\dagger = f(\lambda)(P^\perp + \frac{\lambda - i\kappa}{\lambda + i\kappa}P)
\]

\[
\frac{\lambda + i\kappa}{\lambda - i\kappa}(UPU^\dagger + \frac{\lambda - i\kappa}{\lambda + i\kappa}UP^\dagger U^\dagger) = f(\lambda)(P^\perp + \frac{\lambda - i\kappa}{\lambda + i\kappa}P).
\]

We see that

\[
f(\lambda) = \frac{\lambda - i\kappa}{\lambda + i\kappa},
\]

and

\[
UPU^\dagger = P^\perp = 1 - P.
\] (4.4)

The group factorisation (3.1) implies that the image of \( P \) projects onto the space

\[
V(x_+, x_-) = e^{2im(\kappa x_+ - \kappa^{-1}x_-)s_1}V_0,
\]

where \( V_0 \) is an arbitrary initial space. \( V_0 \) must be one dimensional for there to be non-trivial solutions. Alternatively \( P^\perp \) is the unique projection which annihilates this space.

We take \( V_0 \) to be spanned by the vector

\[
\begin{pmatrix} 1 \\ 1 \end{pmatrix}(iQ)^{-1/2} + \begin{pmatrix} 1 \\ -1 \end{pmatrix}(iQ)^{1/2},
\] (4.5)

where \( Q \) is real. Then \( V(x_+, x_-) \) is spanned by the vector

\[
\begin{pmatrix} 1 \\ 1 \end{pmatrix}(iW)^{-1/2} + \begin{pmatrix} 1 \\ -1 \end{pmatrix}(iW)^{1/2} = \begin{pmatrix} (iW)^{-1/2} + (iW)^{1/2} \\ (iW)^{-1/2} - (iW)^{1/2} \end{pmatrix},
\] (4.6)
where \[ W = Q e^{m(\kappa x + \kappa^{-1} x)} , \]
and thus the hermitian projection \( P^\perp(x_+, x_-) \) which annihilates \( V(x_+, x_-) \) is
\[
P^\perp(x_+, x_-) = \frac{1}{2} \begin{pmatrix} 1 & -\frac{1-iW}{1+iW} \\ -\frac{1+iW}{1-iW} & 0 \end{pmatrix}
\] (4.7)
This projection is also consistent with the symmetry condition (2.12) which has now been translated into the form (4.4). Any other choice of initial space \( V \) would not have satisfied this condition, and indeed it is this choice which enforces the reality of the solution.
We have
\[
\psi(x_+, x_-, 0) = (1 - 2P) = \begin{pmatrix} 0 & -(1+iW) \\ -\frac{1+iW}{1-iW} & 0 \end{pmatrix}
\]
Now
\[
i \frac{1}{4} \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-i\beta u} = (1 - 2P) s_1 (1 - 2P) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = -\frac{1-iW}{1+iW} (1 - 2P) s_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \left( \frac{1-iW}{1+iW} \right)^2 i \frac{1}{4} \begin{pmatrix} 0 \\ 1 \end{pmatrix},
\] (4.8)
so the one-soliton solution is
\[
e^{-i\frac{\beta u}{4}} = \frac{1-iW}{1+iW}.
\] (4.9)
We note that \( \log |Q| \) is essentially the position \( x_0 \) of this soliton (after suitable normalisation) at some fixed time, since \( x_0 \) is given through \( Q \) by
\[
\text{sign}(Q) W = e^{m((\kappa - \kappa^{-1}) t + (\kappa + \kappa^{-1})(x - x_0))},
\] (4.10)
so
\[
\log(|Q|) = -m(\kappa - \kappa^{-1})x_0 .
\] (4.11)
Now \( Q \in \mathbb{R} \), and the choice \( Q < 0, Q > 0 \) corresponds to taking solitons or anti-solitons. Note that we can also change the sign of \( \kappa \) to move from a soliton to an anti-soliton.

The position of the pole at \( i\kappa \) is related to the rapidity \( \theta \) of the soliton by \( |\kappa| = e^{-\theta} \).

**The two-soliton solution**

We take
\[
\psi(x_+, x_-, \lambda) = \left( P^\perp_1 + \frac{\lambda + i\kappa_1}{\lambda - i\kappa_1} P_1 \right) \left( P^\perp_2 + \frac{\lambda + i\kappa_2}{\lambda - i\kappa_2} P_2 \right),
\]
for \( \kappa_1 \) and \( \kappa_2 \) real and positive. We can choose to write the unitary meromorphic loop corresponding to the pole at \( i\kappa_2 \) to the left in the product:
\[
\left( P^\perp_1 + \frac{\lambda + i\kappa_1}{\lambda - i\kappa_1} P_1 \right) \left( P^\perp_2 + \frac{\lambda + i\kappa_2}{\lambda - i\kappa_2} P_2 \right) = \left( P^\perp_3 + \frac{\lambda + i\kappa_2}{\lambda - i\kappa_2} P_3 \right) \left( P^\perp_4 + \frac{\lambda + i\kappa_1}{\lambda - i\kappa_1} P_4 \right). \] (4.12)
The projections \( P_3 \) and \( P_4 \) are uniquely determined. We do this because we cannot immediately solve for \( P_2(x_+, x_-) \) since it is not ordered to the left, but on the other hand we can of course solve for \( P_3(x_+, x_-) \).

Imposing the symmetry condition (2.12) implies
\[
U(P^\perp_1 + \frac{\lambda + i\kappa_1}{\lambda - i\kappa_1} P_1) U^\dagger U(P^\perp_2 + \frac{\lambda + i\kappa_2}{\lambda - i\kappa_2} P_2) U^\dagger
\]
\[
= f(\lambda)(P^\perp_1 + \frac{\lambda - i\kappa_1}{\lambda + i\kappa_1} P_1)(P^\perp_2 + \frac{\lambda - i\kappa_2}{\lambda + i\kappa_2} P_2),
\]
while
and hence,
\[
\frac{\lambda + \imath \kappa_1}{\lambda - \imath \kappa_1}(\frac{\lambda + \imath \kappa_2}{\lambda - \imath \kappa_2})(UP_1U^\dagger + \frac{\lambda - \imath \kappa_1}{\lambda + \imath \kappa_1}UP_1U^\dagger)(UP_2U^\dagger + \frac{\lambda - \imath \kappa_2}{\lambda + \imath \kappa_2}UP_2U^\dagger)
\]
\[= f(\lambda)(P_1^\perp + \frac{\lambda - \imath \kappa_1}{\lambda + \imath \kappa_1}P_1)(P_2^\perp + \frac{\lambda - \imath \kappa_2}{\lambda + \imath \kappa_2}P_2).\]

It follows that
\[UP_1U^\dagger = P_1^\perp \quad UP_2U^\dagger = P_2^\perp,
\]
repeating the argument for the simple pole factors reversed implies
\[UP_3U^\dagger = P_3^\perp \quad UP_4U^\dagger = P_4^\perp.
\]

Only the conditions for \(P_1\) and \(P_3\) are strictly needed for the calculation of \(P_2\). Indeed we can take the one-soliton solution (4.7) for \(P_1\) and \(P_3\) separately.

\[P_1^\perp = \frac{1}{2} \begin{pmatrix}
1 & -(1+iW_1) \\
-(1-iW_1) & 1
\end{pmatrix}
\]

and
\[V_3(x_+, x_-) = \begin{pmatrix}
(iW_2)^{-1/2} + (iW_2)^{1/2} \\
(iW_2)^{-1/2} - (iW_2)^{1/2}
\end{pmatrix}
\]

where
\[W_1 = Q_1 e^{\imath(m(\kappa_1 x_+-\kappa_1^{-1}x_-)} \quad \text{and} \quad W_2 = Q_2 e^{\imath(m(\kappa_2 x_+-\kappa_2^{-1}x_-)}.
\]

We rewrite equation (4.12) as
\[U_1(\lambda)((P_2^\perp + \frac{\lambda + \imath \kappa_2}{\lambda - \imath \kappa_2}P_2) = (P_3^\perp + \frac{\lambda + \imath \kappa_2}{\lambda - \imath \kappa_2}P_3)U_4(\lambda)
\]
evaluating the residue at \(\lambda = \imath \kappa_2\)
\[U_1(\imath \kappa_2)P_2 = P_3U_4(\imath \kappa_2)
\]
\[P_2 = U_1(\imath \kappa_2)^{-1}P_3U_4(\imath \kappa_2).
\]

\(P_2\) projects onto the image of \(U_1(\imath \kappa_2)^{-1}P_3U_4(\imath \kappa_2)\) and this is the image of \(U_1(\imath \kappa_2)^{-1}P_3\) since \(U_4(\imath \kappa_2)\) is invertible. This space is the space annihilated by \(P_2^\perp\). \(P_2\) can now be computed explicitly. So we have
\[V_2(x_+, x_-) = U_1(\imath \kappa_2)^{-1}V_3(x_+, x_-),
\]
and it follows that
\[V_2 = \begin{pmatrix}
\frac{1+W_1W_2-iY^{-1}W_2+iY^{-1}W_2}{1+W_1W_2+iY^{-1}W_2-iY^{-1}W_2} \\
\frac{1-iW_1(1+iW_1)W_2+iY^{-1}W_2-iY^{-1}W_2}{1+iW_1(1-iW_1)W_2+iY^{-1}W_2-iY^{-1}W_2}
\end{pmatrix}
\]

and
\[P_2 = \begin{pmatrix}
\frac{\frac{1}{2}}{1-iW_1(1+iW_1)(1+iW_1)W_2+iY^{-1}W_2-iY^{-1}W_2} & \frac{1}{2} \\
\frac{1}{2} & \frac{1}{2}
\end{pmatrix}
\]

where
\[Y = \frac{\kappa_2 - \kappa_1}{\kappa_2 + \kappa_1}.
\]

We then find that
\[\psi(x_+, x_-, 0) = (1 - 2P_1)(1 - 2P_2) = \begin{pmatrix}
\frac{1+iY^{-1}W_2-iY^{-1}W_2+iY^{-1}W_2}{1+iY^{-1}W_2-iY^{-1}W_2+iY^{-1}W_2} & 0 \\
0 & \frac{1+iY^{-1}W_2-iY^{-1}W_2+iY^{-1}W_2}{1+iY^{-1}W_2-iY^{-1}W_2+iY^{-1}W_2}
\end{pmatrix}.
\]
Now
\[ \frac{i}{4} \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-i\beta u} = \left( (1 - 2P_1)(1 - 2P_2) \right)^{-1} s_1(1 - 2P_1)(1 - 2P_2) \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \]
so we get
\[ e^{-i\beta u} = \left( \frac{(1 + iY^{-1}W_1 - iY^{-1}W_2 + W_1W_2)}{(1 - iY^{-1}W_1 + iY^{-1}W_2 + W_1W_2)} \right)^2. \] (4.16)

If we adjust the initial subspaces \( V_1(0,0) \) and \( V_2(0,0) \) by
\[ W_1 \rightarrow -YW_1, \quad W_2 \rightarrow YW_2, \] (4.17)
and let
\[ X = Y^2 = \left( \frac{\kappa_2 - \kappa_1}{\kappa_2 + \kappa_1} \right)^2, \] (4.18)
then we get the familiar two-soliton solution:
\[ e^{-i\beta u} = \frac{(1 - iW_1 - iW_2 - XW_1W_2)}{(1 + iW_1 + iW_2 - XW_1W_2)}. \] (4.19)

The solution (4.16) is written in ‘left-most’ ordered co-ordinates, that is coordinates given by taking the \( Q \)’s defined by the projection for each pole in the left-most ordered position in the factorisation.

We note that the adjustment (4.17) to these co-ordinates is a subtle re-normalisation of the standard \( Q \)’s, which enter the standard form of the solution (4.19). This adjustment will be crucial for us in obtaining the final version of the symplectic form written in terms of the positions and momenta of the solitons.

**The breather solution**

We choose the poles to be at \( \alpha \) and \(-\alpha\), where \( \alpha \) is not on the imaginary axis.
\[ \psi(x_+,x_-,\lambda) = (P_1^\perp + \frac{\lambda - \bar{\alpha}}{\lambda - \alpha} P_1)(P_2^\perp + \frac{\lambda + \bar{\alpha}}{\lambda + \alpha} P_2) \]
\[ = (P_3^\perp + \frac{\lambda + \bar{\alpha}}{\lambda + \alpha} P_3)(P_4^\perp + \frac{\lambda - \bar{\alpha}}{\lambda - \alpha} P_4). \] (4.20)

The symmetry condition (2.12) implies that
\[ UP_1U^\dagger = P_3, \quad UP_2U^\dagger = P_4. \] (4.21)
This time we take for \( Q \in \mathbb{C} \),
\[ V_1 = Q^{-1/2} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + Q^{1/2} \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \] (4.22)
and then, writing \( W(\alpha) = e^{im(\alpha x_+ + \alpha^{-1} x_-)} \),
\[ V_1(x_+,x_-) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} (QW(\alpha))^{-1/2} + \begin{pmatrix} -1 \\ 0 \end{pmatrix} (QW(\alpha))^{1/2}, \]
splitting \( QW(\alpha) \) into a pure phase and modulus, \( QW(\alpha) = UW \), we have
\[ V_1(x_+,x_-) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} (UW)^{-1/2} + \begin{pmatrix} -1 \\ 0 \end{pmatrix} (UW)^{1/2} \]
\[ = \begin{pmatrix} 1 \\ 0 \end{pmatrix} (UW)^{-1/2} - (UW)^{1/2}, \] (4.23)
where, setting \( \alpha = v + ik \),
\[ U = e^{im(vx_+ + \frac{\alpha x_+}{v + \alpha})} = e^{im((v + \frac{\alpha x_+}{v + \alpha})(t-t_0)+(-v + \frac{-\alpha x_+}{v + \alpha})(x-x_0))} \]
and
\[
W = e^{m(-kx + \frac{kx}{\lambda})} = e^{m((k + \frac{k}{\lambda x})x - (k + \frac{k}{\lambda x})(x - x_0))}.
\]

Here \(x_0\) and \(t_0\) are fixed through the complex valued \(Q\). We can write any \(Q \in \mathbb{C} - \{0\}\) as \(Q = e^{m(\alpha q_1 + \alpha^{-1} q_2)}\), since \(\alpha\) has a non-zero imaginary part, so that \(\alpha\) and \(\alpha^{-1}\) span the complex plane, and where \(q_1, q_2 \in \mathbb{R}\). Then, on remembering that \(x = \pm t + x_0\) and \(t_0\) are related to \(q_1\) and \(q_2\) by
\[
q_1 = -(x_0 + t_0), \quad q_2 = -t_0 + x_0.
\]

Then
\[
P_1 = \frac{1}{2} \begin{pmatrix}
\frac{(U-W)(1-UW)}{U(1+W)} & -\frac{(-1+UW)(W+U)}{U(1+W)} \\
-\frac{(-U+W)(UW+1)}{U(1+W)} & \frac{(U+W)(1+UW)}{U(1+W)}
\end{pmatrix},
\]
and to be consistent with Eqn. (4.21), we must start with the vector \(V_3(0,0)\):
\[
V_3(0,0) = Q^{1/2} \begin{pmatrix} 1 \\ i \\ t \\ -1 \\ 1 \end{pmatrix},
\]
and we get the evolution (recall that the pole is at \(\lambda = -\alpha\))
\[
V_3(x, x_0, x_0, t_0) = \begin{pmatrix}
-((UW)^{-1/2} - (UW)^{1/2}) \\
(UW)^{-1/2} + (UW)^{1/2}
\end{pmatrix}.
\]

Using the same arguments as for the two soliton case we find that, putting \(s = k/v\),
\[
V_2 = \begin{pmatrix}
\frac{-i(s^2W + iW^2W - U - CW^2)}{UW + iW(U + CW + iW^2)} \\
\frac{-1}{UW + iW(U + CW + iW^2)}
\end{pmatrix},
\]
and
\[
P_2 = \begin{pmatrix}
\frac{-1}{UW + iW(U + CW + iW^2)} & -\frac{1}{UW + iW(U + CW + iW^2)} \\
\frac{1}{UW + iW(U + CW + iW^2)} & \frac{1}{UW + iW(U + CW + iW^2)}
\end{pmatrix},
\]
so
\[
\psi(0) = (1 - \frac{2is}{UW + iW(U + CW + iW^2)})P_1(1 - \frac{2is}{UW + iW(U + CW + iW^2)})P_2
\]
\[
= \begin{pmatrix}
\frac{-1}{(s^2 + i)(sW + iW^2W - U - CW^2)(1+iW)} & 0 \\
\frac{-1}{(s^2 + i)(sW + iW^2W - U - CW^2)(1+iW)}
\end{pmatrix}.
\]

The result follows:
\[
e^{-i\beta s} = \begin{pmatrix}
isU^{-1} + isU + W^{-1} + W \\
isU^{-1} + isU - W^{-1} - W
\end{pmatrix}.
\]

5 The symplectic form

We have seen that the loop \(\phi_0\) specifies a solution to the sine Gordon equation for all space-time. We can consider the set of such \(\phi_0\) to be the phase space of the sine Gordon system. Then a change \(v_0\) in \(\phi_0\) would represent a change in \(u\) at all values in space-time. Given two such changes \(v_0\) and \(w_0\) we should be able to calculate the canonical symplectic form (5.3), and this is what we shall do in this section. Here we denote the time derivative of \(u\) by \(\dot{u}\).

Lemma 5.1

With \(b\) defined by the factorisation (3.1)
\[
(b^{-1}v_e) = \beta(\delta_e \dot{u}) s_3 + 2m\beta(\delta_e u)(\lambda^{-1} e^{\beta u s_3} [s_3, s_1] e^{-\beta u s_3} + \lambda e^{-\beta u s_3} [s_3, s_1] e^{\beta u s_3})
\]
Proof Using the results for \( b \) just prior to Eqn. (3.3) we can write
\[
b_x b^{-1} = b_+ b^{-1} - b_- b^{-1} = B - A
\]
\[
= \beta(\partial_- u + \partial_+ u) s_3 + 2m(\lambda^{-1} e^{\beta u s_3} s_1 e^{-\beta u s_3} - \lambda e^{-\beta u s_3} s_1 e^{\beta u s_3})
\]
\[
= \beta \dot{s}_3 + 2m(\lambda^{-1} e^{\beta u s_3} s_1 e^{-\beta u s_3} - \lambda e^{-\beta u s_3} s_1 e^{\beta u s_3}). \tag{5.2}
\]
Now vary \( u \) by a parameter we refer to as \( v \). We obtain
\[
(b_x b^{-1})_v = \beta(\delta_v \dot{u}) s_3 + 2m \beta(\delta_v u)(\lambda^{-1} e^{\beta u s_3}[s_3, s_1] e^{-\beta u s_3} + \lambda e^{-\beta u s_3}[s_3, s_1] e^{\beta u s_3}). \tag{5.3}
\]

The canonical symplectic form \([1]\), derived from the Lagrangian density formulation of the sine-Gordon model, is
\[
\omega = \frac{\beta^2}{4} \int_{-\infty}^{\infty} \delta u \wedge \dot{\delta} \dot{u} dx.
\]
This can be written as
\[
\omega(v, w) = \frac{\beta^2}{4} \int_{-\infty}^{\infty} (\delta_v u \delta_w \dot{u} - \delta_w u \delta_v \dot{u}) dx \tag{5.3}
\]
in terms of variations \( v = \delta_v \phi \) and \( w = \delta_w \phi \).

**Proposition 5.2**
The form (5.3) is given by the formula
\[
\omega(v, w) = \frac{1}{2\pi i} \int_{\gamma} d\lambda \frac{\beta}{\lambda} \operatorname{Trace} \left[ \int_{-\infty}^{\infty} (b_x b^{-1})_v \phi^{-1} w \right] dx \tag{5.4}
\]
where the contour \( \gamma \) is a small clockwise circle and a large anti-clockwise circle around the origin. Here 'small' means that all the poles of the meromorphic function \( \phi \) lie outside the contour, and 'large' means that all the poles lie inside it.

**Proof**
Using Lemma 5.1,
\[
\omega(v, w) = \frac{1}{2\pi i} \int_{\gamma} d\lambda \frac{\beta}{\lambda} \operatorname{Trace} \left[ \int_{-\infty}^{\infty} (b_x b^{-1})_v \phi^{-1} w \right] dx \tag{5.5}
\]
We calculate this integral in two parts, for the first part
\[
\frac{1}{2\pi i} \beta \int_{\gamma} d\lambda \frac{\beta}{\lambda} \operatorname{Trace} \left[ \int_{-\infty}^{\infty} (\delta_v \dot{u}) s_3 \phi^{-1} w \right] dx.
\]
The \( \frac{1}{2\pi i} \int_{\gamma} d\lambda \frac{\beta}{\lambda} \) is equivalent to evaluating at \( \lambda \to \infty \) minus evaluating at \( \lambda = 0 \), that is
\[
\beta \int_{-\infty}^{\infty} (\delta_v \dot{u}) \operatorname{Trace} \left( s_3 \phi^{-1} w(\infty) - s_3 \phi^{-1} w(0) \right) dx,
\]
recall that \( \phi(\infty) = e^{\beta u s_3} \) and \( \phi(0) = N e^{-\beta u s_3} \), where \( N \) is some constant normalisation matrix. Then, we get
\[
\frac{\beta^2}{4} \int_{-\infty}^{\infty} (\delta_v \dot{u})(\delta_w u) dx = - \frac{\beta^2}{4} \int_{-\infty}^{\infty} (\delta_v \dot{u})(\delta_w u) dx. \tag{5.6}
\]
The second term is
\[ \frac{2m\beta}{2\pi i} \int \frac{d\lambda}{\lambda} \lambda^0 \text{Trace} \int_{-\infty}^{\infty} (\delta_v u) \left( \lambda^{-1} e^{\beta u s_3} [s_3, s_1] e^{-\beta u s_3} + \lambda e^{-\beta u s_3} [s_3, s_1] e^{\beta u s_3} \right) \phi^{-1} w \, dx, \]
and integrating around the small circle in $\gamma$ gives the $\lambda^0$ coefficient of the expansion about $\lambda = 0$ of
\[ -2m\beta \int_{-\infty}^{\infty} (\delta_v u) \lambda^{-1} \text{Trace} \ s_3 [e^{\beta u s_3} s_1 e^{-\beta u s_3}, \phi^{-1} w] \, dx \]
\[ = -2m\beta \int_{-\infty}^{\infty} (\delta_v u) \text{Trace} \ s_3 [e^{\beta u s_3} s_1 e^{-\beta u s_3}, (\phi^{-1} w)'(0)] \, dx. \]
To calculate this, consider the derivative of $a\phi = \phi b$ in the direction $x_-$;
\[ \lambda \phi^{-1} \phi_+ = \lambda \beta (\partial_- u) s_3 + 2m e^{\beta u s_3} s_1 e^{-\beta u s_3} - 2m \phi^{-1} s_1 \phi, \]
and now apply $\frac{d}{dx}$ at $\lambda = 0$ to get
\[ \phi^{-1} \phi_-(0) = \beta (\partial_- u) s_3 - 2m (\phi^{-1} s_1 \phi'(0) - \phi^{-1} \phi' \phi^{-1} s_1 \phi(0)), \]
(we use prime to denote $\frac{d}{dx}$) and apply $\phi^{-1} \phi_-(0) = -\beta (\partial_- u) s_3$,\[ -\beta (\partial_- u) s_3 = -m (\phi^{-1} s_1 \phi'(0) - \phi^{-1} \phi' \phi^{-1} s_1 \phi(0)). \]
Conjugating this with $\phi(0)$, and noting that $\phi(0) s_3 \phi^{-1}(0) = N s_3 N^{-1}$, we have\[ -\beta (\partial_- u) N s_3 N^{-1} = m (\phi' \phi^{-1}(0) s_1 - s_1 \phi' \phi^{-1}(0)) \]
\[ = m [\phi' \phi^{-1}(0), s_1]. \quad (5.7) \]
Now vary this in the direction $w$
\[ -\beta (\partial_- u w) N s_3 N^{-1} = m [w' \phi^{-1}(0) - \phi' \phi^{-1} w \phi^{-1}(0), s_1], \]
and conjugate by $\phi(0)$ again get
\[ -\beta (\partial_- u w) s_3 = m [\phi^{-1} w'(0) - \phi^{-1} \phi' \phi^{-1} w(0) - \phi^{-1}(0) s_1 \phi(0)] \]
\[ = m [(\phi^{-1} w)'(0), e^{\beta u s_3} s_1 e^{-\beta u s_3}]. \quad (5.8) \]
This gives the integral of the second term around the small circle in $\gamma$ as
\[ \frac{\beta^2}{4} \int_{-\infty}^{\infty} (\delta_v u) (\delta_w (\partial_- u)) \, dx. \]
Now we turn to the integral around the large circle in $\gamma$. This is the $\lambda^0$ part of the Laurent expansion of
\[ 2m\beta \int_{-\infty}^{\infty} (\partial_v u) \lambda \text{Trace} \ s_3 [e^{-\beta u s_3} s_1 e^{\beta u s_3}, \phi^{-1} w] \, dx \]
about $\lambda = \infty$. We shall calculate this from the $x_+$ equation of the linear system:
\[ \phi^{-1} \phi_+ = -\beta (\partial_+ u) s_3 + 2m \lambda e^{-\beta u s_3} s_1 e^{\beta u s_3} - 2m \lambda \phi^{-1} s_1 \phi, \]
taking the limit as $\lambda \to \infty$, and using $\phi^{-1} \phi_+ (\infty) = \beta (\partial_+ u) s_3$,\[ \beta (\partial_+ u) s_3 = m \lim_{\lambda \to \infty} \lambda (e^{-\beta u s_3} s_1 e^{\beta u s_3} - \phi^{-1} s_1 \phi), \]

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if we set $\phi'(\infty) = \lim_{\lambda \to \infty} \lambda (\phi(\lambda) - \phi(\infty))$, then

$$\beta(\partial_u) s_3 = m(\phi^{-1}\phi' s_1 \phi(\infty) - \phi^{-1}s_1 \phi'(\infty)),$$

since $\phi(\infty)$ commutes with $s_3$, we can conjugate to get

$$\beta(\partial_u) s_3 = m[\phi' \phi^{-1}(\infty), s_1],$$

and differentiating along the direction $w$ to get

$$\beta(\partial_u w) s_3 = m[w' \phi^{-1}(\infty) - \phi' \phi^{-1}w \phi^{-1}(\infty), s_1],$$

conjugating again

$$\beta(\partial_u w) s_3 = m[\phi' \phi^{-1}(\infty) - \phi' \phi^{-1}w \phi^{-1}(\infty), s_1]$$

so the total contribution from the second term is

$$\frac{\beta^2}{4} \int_{-\infty}^{\infty} (\delta_x u)(\delta_w(\partial_u u)) dx ,$$

where

$$\phi^{-1}w = \lim_{\lambda \to \infty} \lambda e^{-\beta u s_3} e^{\beta u s_1}.$$

Then the contribution to the second term for the large circle in $\gamma$ is

$$\frac{\beta^2}{4} \int_{-\infty}^{\infty} (\delta_x u)(\delta_w(\partial_u u)) dx ,$$

and the total is the required formula

$$\omega(v, w) = \frac{\beta^2}{4} \int_{-\infty}^{\infty} (\delta_x u \delta_w \dot{u} - \delta_w u \delta_x \dot{u}) dx . \Box$$

Now we are left with the problem of how to perform the $x$ integral in

$$\omega(v, w) = \frac{1}{2\pi i} \int_{\gamma} d \lambda \text{ Trace } \int_{-\infty}^{\infty} \left( b_x b^{-1} \right) \phi^{-1} w dx ,$$

and for this we shall need the following Lemma:

**Lemma 5.3**

$$-(\phi^{-1}w)_x + [b_x b^{-1}, \phi^{-1}w] = (b_x b^{-1})_w$$

**Proof**

Begin by varying the equation $a\phi_0 = \phi b$ by the parameter $w_0$, a change in $\phi_0$, and find

$$aw_0 = wb + \phi b w,$$

where $w$ is the corresponding change in $\phi$. On differentiating with respect to $x$ we get

$$a_x w_0 = w_x b + w b_x + \phi_x b w + \phi b_{wx}.$$
Multiplying by $\phi^{-1}$ on the left and rearranging gives the required result

$$-(\phi^{-1}w)_x + [b_xb^{-1}, \phi^{-1}w] = (b_xb^{-1})_w.$$  \hfill \Box

**Proposition 5.4**

$$\omega(v, w) = \frac{1}{2\pi i} \int \frac{d\lambda}{\lambda} \text{Trace} \left[ b_xb^{-1}\phi^{-1}w \right]_{x=\infty}^{x=-\infty}$$

(5.13)

**Proof**

We can write

$$\omega(v, w) = \frac{1}{2\pi i} \int \frac{d\lambda}{\lambda} \text{Trace} \left\{ (b_xb^{-1})_x\phi^{-1}w + [b_xb^{-1}, b_xb^{-1}]\phi^{-1}w \right\} dx.$$  \hfill (5.14)

Now integrate by parts and reorder the commutation relation using the trace property:

$$\omega(v, w) = \frac{1}{2\pi i} \int \frac{d\lambda}{\lambda} \text{Trace} \left\{ b_xb^{-1}\phi^{-1}w \right\}_{x=\infty}^{x=-\infty} + \int_{-\infty}^{\infty} dx b_xb^{-1}(- (\phi^{-1}w)_x + [b_xb^{-1}, \phi^{-1}w]) \right\}$$

Using Lemma 5.3, we can write

$$\omega(v, w) = \frac{1}{2\pi i} \int \frac{d\lambda}{\lambda} \text{Trace} \left\{ b_xb^{-1}\phi^{-1}w \right\}_{x=\infty}^{x=-\infty} + \int_{-\infty}^{\infty} dx b_xb^{-1}(b_xb^{-1})_w,$$

and we observe that the integral around $\gamma$ of $b_xb^{-1}(b_xb^{-1})_w$ vanishes since the function is analytic between the circles comprising $\gamma$.  \hfill \Box

### 6 The abstract symplectic form.

It might be thought that the form of the symplectic form we derived in the last section would be highly dependent on the structure of the sine-Gordon equation. However the formula (5.13) actually gives a closed 2-form associated to a group doublecross product under very general conditions, as we now show.

Let $\mathcal{X} = \mathcal{G}M = \mathcal{MG}$ be a group doublecross product with an adjoint invariant inner product $\langle , \rangle$ on its Lie algebra. From the factorisation $a\phi_0 = \phi b$ and a change $(\phi_0; v_0)$ in $\phi_0$ we define derivatives $v = \phi_v = D(\phi_0; v_0)\phi$ and $b_v = D(\phi_0; v_0)b$ (in these derivatives $a$ is kept constant).

Our aim is to define a closed 2-form over the phase space $\mathcal{G}$ with coordinate $\phi_0$. To do this we first define another 2-form $\tau_a$ on $\mathcal{G}$, for a given $a \in \mathcal{M}$, as

$$\tau_a(\phi_0; v_0, w_0) = \langle b_vb^{-1}, \phi^{-1}w \rangle - \langle b_wb^{-1}, \phi^{-1}v \rangle.$$

**Proposition 6.1**

$$d\tau_a(\phi_0; y_0, v_0, w_0) = \langle \left[ w_0\phi_0^{-1}, y_0\phi_0^{-1} \right], v_0\phi_0^{-1} \rangle - \langle \left[ \phi^{-1}w, \phi^{-1}y \right], \phi^{-1}v \rangle$$

- $$- \langle \left[ b_wb^{-1}, b_yb^{-1} \right], b_vb^{-1} \rangle.$$  

**Proof**

First calculate the derivative

$$\tau_a(\phi_0; v_0, w_0; y_0) = \langle b_vb^{-1} - b_vb^{-1}b_yb^{-1}, \phi^{-1}w \rangle + \langle b_xb^{-1}, \phi^{-1}\phi_wy - \phi^{-1}y\phi^{-1}w \rangle$$

- $$- \langle b_wy^{-1} - b_wb^{-1}b_yb^{-1}, \phi^{-1}v \rangle - \langle b_wb^{-1}, \phi^{-1}\phi_yv - \phi^{-1}y\phi^{-1}v \rangle.$$

By adding the other two terms and using symmetry of the double derivative, we find

$$d\tau_a(\phi_0; y_0, v_0, w_0) = \langle \left[ b_yb^{-1}, b_vb^{-1} \right], \phi^{-1}w \rangle + \langle \left[ b_wb^{-1}, b_yb^{-1} \right], \phi^{-1}v \rangle$$

- $$+ \langle \left[ b_xb^{-1}, b_vb^{-1} \right], \phi^{-1}y \rangle + \langle b_vb^{-1}, \left[ \phi^{-1}w, \phi^{-1}y \right] \rangle$$

- $$+ \langle b_yb^{-1}, \left[ \phi^{-1}v, \phi^{-1}w \right] \rangle + \langle b_wb^{-1}, \left[ \phi^{-1}y, \phi^{-1}v \right] \rangle.$$
Now we try to simplify this expression, starting with the factorisation \(a\phi_0 = \phi b\), and differentiating in the direction \((\phi_0; v_0)\) to get \(a v_0 = \phi b + \phi b v_o\), or in a more useful form
\[ \phi^{-1} a v_0 b^{-1} = b v + \phi^{-1} v. \]

Now using this result successively,
\[
\langle [b_w b^{-1}, b_y b^{-1}], \phi^{-1} v \rangle = \langle [b_w b^{-1}, b_y b^{-1}], \phi^{-1} a v_0 b^{-1} \rangle - \langle [b_w b^{-1}, b_y b^{-1}], b_v b^{-1} \rangle \\
= \langle [b_w b^{-1}, \phi^{-1} a b y b^{-1}], \phi^{-1} a v_0 b^{-1} \rangle - \langle [b_w b^{-1}, \phi^{-1} y], \phi^{-1} a v_0 b^{-1} \rangle \\
- \langle [b_w b^{-1}, \phi^{-1} y], \phi^{-1} a v_0 b^{-1} \rangle - \langle [b_w b^{-1}, b_y b^{-1}], b_v b^{-1} \rangle \\
= \langle [\phi^{-1} a b b^{-1}, \phi^{-1} y b^{-1}], \phi^{-1} a v_0 b^{-1} \rangle \\
- \langle [\phi^{-1} w, \phi^{-1} a b y b^{-1}], \phi^{-1} a v_0 b^{-1} \rangle \\
- \langle [\phi^{-1} w, \phi^{-1} y b^{-1}], \phi^{-1} a v_0 b^{-1} \rangle - \langle [b_w b^{-1}, b_y b^{-1}], b_v b^{-1} \rangle \\
= \langle [w_0 \phi_0^{-1}, y_0 \phi_0^{-1}], v_0 \phi_0^{-1} \rangle - \langle [\phi^{-1} w, b_y b^{-1}], b_v b^{-1} \rangle \\
- \langle [\phi^{-1} w, b_y b^{-1}], \phi^{-1} v \rangle - \langle [\phi^{-1} w, \phi^{-1} y b^{-1}], b_v b^{-1} \rangle \\
- \langle [\phi^{-1} w, \phi^{-1} y b^{-1}], \phi^{-1} v \rangle - \langle [b_w b^{-1}, b_y b^{-1}], b_v b^{-1} \rangle.
\]

Substitution of this into the formula above for \(d\tau_a\), and using adjoint invariance of the inner product, gives the result. \[ \square \]

Now we define a 2-form \(\omega\) on \(G\) by the formula
\[ \omega(\phi_0; v_0, w_0) = \lim_{R \to \infty} \left[ \left\langle b_v b^{-1}, \phi^{-1} w \right\rangle - \left\langle b_w b^{-1}, \phi^{-1} v \right\rangle \right]_x^{-R}, \]
where we remember that \(a\) is a function on space-time. This is the difference in the values of \(\tau_a\) between two points in space-time lying either side of the ‘interesting’ region, that is where the fields are substantially different from the vacuum. If the fields are not compactly supported, but merely decreasing, we take a limit \(x \to \pm\infty\), as the points tend to regions where the fields take more vacuum-like values. But what does ‘vacuum’ mean in our group picture? A brief look at the asymptotic form of the soliton solutions will give a clear answer. For the sine-Gordon or principal chiral model the function \(\phi(t, x) : \mathbb{C} \to GL_n\) tends to a limit matrix valued function commuting with \(J\) and \(K\) as \(x \to \pm\infty\), c.f. Lemma 10.1, below. The subgroup consisting of such functions is abelian. In general, we assume that \(\phi(s)\) tends to a limiting value in an abelian subgroup, which we call \(\mathcal{G}_\infty \subset G\).

**Proposition 6.2**

Suppose that \(\phi(s)\) in the factorisation \(a(s)\phi_0 = \phi(s)b(s)\) tends to a limiting value in the abelian subgroup \(\mathcal{G}_\infty \subset G\) as \(x \to \infty\) and as \(x \to -\infty\). Also suppose that the inner product \(\langle , \rangle\) vanishes on the Lie algebra of the group \(M\). Then \(d\omega = 0\).

**Proof**

This is more or less direct from the previous proposition. The term \(\left\langle [w_0 \phi_0^{-1}, y_0 \phi_0^{-1}], v_0 \phi_0^{-1} \right\rangle\) in \(d\tau_a\) cancels on taking differences. The term \(\left\langle [b_w b^{-1}, b_y b^{-1}], b_v b^{-1} \right\rangle\) is zero since in the inner product vanishes on \(m\). The term \(\left\langle [\phi^{-1} w, \phi^{-1} y], \phi^{-1} v \right\rangle\) tends to zero as \(x \to \pm\infty\) since the commutator tends to zero by the abelian subgroup condition. \[ \square \]

## 7 The abstract higher momenta.

Having decided that \(d\omega = 0\) under certain realistic conditions, we can try to calculate values of \(\omega\) on certain vectors \((\phi_0; w_0)\). But which vectors to use? There is a certain easy choice of vector, which will lead to the higher conserved momenta for sine-Gordon.
Proposition 7.1
Consider \( c \) in the Lie algebra of \( \mathcal{M} \) which commutes with \( J \) and \( K \), that is it commutes with \( s_1 \). Then \( c \) brings about a change in \( \phi_0 \) through the factorisation \( c^{tr} \phi_0 = \phi_0(r)m \), for some \( m \in \mathcal{M} \). The corresponding infinitesimal change \( w_0 \) of \( \phi_0 \) is given by the formula \( c\phi_0 = w_0 + \phi_0d \), where \( d \) is also in the Lie algebra of \( \mathcal{M} \). Then the vector field \( (\phi_0; w_0) \) is a Hamiltonian flow and is generated by the Hamiltonian

\[
f_c(\phi_0) = 2 \lim_{R \rightarrow \infty} \left[ \langle \vartheta(\phi)\phi^{-1}, h \rangle \right]_{x=-R}^R,
\]

where \( \vartheta \) is a derivation operator acting on the Lie algebra of \( \mathcal{M} \), defined by Eqn. (7.8), and \( h \) is an element in the Lie algebra of \( \mathcal{M} \) such that \( \vartheta h = c \). This means that

\[
\omega(\phi_0; v_0, w_0) = D_{(\phi_0; v_0)} f_c(\phi_0).
\]

Proof
If we insert the formula for \( w_0 \) into \( aw_0 = wb + \phi b_w \), we find

\[
\phi^{-1}aca^{-1} \phi - dbb^{-1} = \phi^{-1}w + b_wb^{-1}.
\]

Using our assumption that the inner product is zero on \( m \), we obtain

\[
\begin{align*}
\langle b_wb^{-1}, \phi^{-1}w \rangle &= \langle b_wb^{-1}, \phi^{-1}aca^{-1} \phi \rangle \\
&= \langle \phi b_wb^{-1} \phi^{-1}, aca^{-1} \rangle \\
&= \langle av_0\phi_0^{-1}a^{-1} - \phi^{-1}, aca^{-1} \rangle \\
&= \langle v_0\phi_0^{-1}, a \rangle - \langle \psi \phi^{-1}, aca^{-1} \rangle \\
\langle b_wb^{-1}, \phi^{-1}v \rangle &= \langle \phi^{-1}aca^{-1} \phi - dbb^{-1} - \phi^{-1}w, \phi^{-1}v \rangle \\
&= \langle \phi^{-1}aca^{-1} \phi, \phi^{-1}v \rangle - \langle dbb^{-1}, \phi^{-1}v \rangle - \langle \phi^{-1}w, \phi^{-1}v \rangle \\
&= \langle aca^{-1}, \phi^{-1}v \rangle - \langle dbb^{-1}, \phi^{-1}v \rangle - \langle \phi^{-1}w, \phi^{-1}v \rangle.
\end{align*}
\]

We now subtract Eqs. (7.4) and (7.3) and calculate \( \omega(\phi_0; v_0, w_0) \), for some arbitrary change \( v_0 \). On taking differences between \( x = \pm \infty \), we see that the \( \langle d, \phi_0^{-1}v \rangle \) and \( \langle v_0\phi_0^{-1}, c \rangle \) terms vanish as \( c \) and \( d \) are independent of \( a \), so we are left with

\[
\omega(\phi_0; v_0, w_0) = \lim_{R \rightarrow \infty} \left[ \langle \phi^{-1}w, \phi^{-1}v \rangle - 2 \langle \phi^{-1}w, \phi^{-1}v \rangle \right]_{x=-R}^R.
\]

In what follows, we restrict ourselves to the case where \( c \) commutes with every \( a \), and with the limit subgroup \( \mathcal{G}_\infty \subset \mathcal{G} \). Then \( \phi^{-1}c\phi - dbb^{-1} = \phi^{-1}w + b_wb^{-1} \), where \( \phi^{-1}c\phi \rightarrow c \) as \( x \rightarrow \pm \infty \). This means that \( \phi^{-1}w = \pi_g(\phi^{-1}c\phi - dbb^{-1}) \rightarrow 0 \) as \( x \rightarrow \pm \infty \) (\( \pi_g \) is the projection to the Lie algebra of \( \mathcal{G} \)). Then the first term of (7.6) vanishes and we can rewrite Eqn. (7.6) as

\[
\omega(\phi_0; v_0, w_0) = -2 \lim_{R \rightarrow \infty} \left[ \langle v\phi^{-1}, c \rangle \right]_{x=-R}^R.
\]

This might be a sufficiently simple formula to calculate \( \omega(\phi_0; v_0, w_0) \) for various vectors \( (\phi_0; v_0) \). However it is our purpose to do Hamiltonian mechanics, so we would like to answer the question "is there a Hamiltonian function giving rise to the vector field \( (\phi_0; v_0) \) generated by \( c \in m \)?". To do this we would have to show that \( \omega(\phi_0; v_0, w_0) \) was the derivative in the direction \( (\phi_0; v_0) \) of a function of \( \phi_0 \) and \( c \).

Suppose that there is a 1-parameter automorphism \( \Theta : \mathbb{R} \times X \rightarrow X \) which preserves the subgroups \( \mathcal{G} \) and \( \mathcal{M} \), and the inner product on the Lie algebra of \( X \). Then there is a derivation \( \vartheta \) on the Lie algebra defined by

\[
\vartheta(y) = \Theta''(0, c; 0, y; 1, 0)
\]

\( (e \) is the group identity), and this preserves the inner product, that is \( \langle \vartheta y, z \rangle + \langle y, \vartheta z \rangle = 0 \).

\[\text{1Loosely speaking, in the applications, } \vartheta \text{ can be thought of as } \vartheta = \lambda \frac{d}{dx}\]
If there is an $h$ in the Lie algebra of $\mathcal{M}$ so that $\partial h = c$, then

$$\omega(\phi_0; v_0, w_0) = -2 \lim_{R \to \infty} \left( \frac{\partial \ln}{\partial h} \right)_{x=-R}^R = 2 \lim_{R \to \infty} \left( \frac{\partial (v \phi^{-1})}{\partial h} \right)_{x=-R}^R . (7.9)$$

For our purposes we wish to swap the order of $\partial$ and $D_{(\phi_0; v_0)}$ in this expression. To do this we proceed cautiously and apply $\partial$ to $v \phi^{-1}$ to get $\partial(v) \phi^{-1} - v \phi^{-1} \phi \phi^{-1}$, where $\partial(v) = \Theta''(0, \phi; 0; v; 1, 0)$ and $\phi = \Theta'(0, \phi; 1, 0)$. From the last formula and the symmetry of double derivatives we see that $D_{(\phi_0; v_0)} \phi = \partial(v)$, so

$$\partial (v \phi^{-1}) = D_{(\phi_0; v_0)} \left( \phi \phi^{-1} \right) - \left[ v \phi^{-1}, \phi \phi^{-1} \right],$$

$$\left\langle \partial (v \phi^{-1}), h \right\rangle = D_{(\phi_0; v_0)} \left( \phi \phi^{-1}, h \right) - \left\langle \left[ v \phi^{-1}, \phi \phi^{-1} \right], h \right\rangle .$$

If $\partial$ preserves the abelian subgroup $\mathcal{G}_\infty$ then the Lie bracket $\left[ \phi \phi^{-1}, v \phi^{-1} \right]$ tends to zero as $x \to \pm \infty$, and we can write Eqn. (7.9)

$$\omega(\phi_0; v_0, w_0) = D_{(\phi_0; v_0)} \left( 2 \lim_{R \to \infty} \left[ \phi \phi^{-1}, h \right]_{x=-R}^R \right) ,$$

or in other words the vector field $(\phi_0; v_0)$ generated by an element $c \in m$ is a Hamiltonian flow, with Hamiltonian

$$f_c(\phi) = 2 \lim_{R \to \infty} \left[ \partial (\phi) \phi^{-1}, h \right]_{x=-R}^R .$$

Now that we have expressions for Hamiltonian functions $f_c$ we should show that these functions Poisson commute.

**Proposition 7.2**

The Hamiltonians $\{ f_c \}$ Poisson commute.

**Proof**

Suppose that the vector $(\phi_0; v_0)$ is generated by another element $c$ in the Lie algebra of $\mathcal{M}$ which commutes with $s_1$. Then $\dot{c} \phi_0 = v_0 + \phi_0 \dot{d}$, where $\dot{d}$ is also in the Lie algebra of $\mathcal{M}$. Eqn. (7.5) applied for an arbitrary $v_0$, we can insert this particular $v_0$ to find,

$$\omega(\phi_0; v_0, w_0) = 2 \lim_{R \to \infty} \left[ \phi \phi^{-1}, c \right]_{x=-R}^R = -2 \lim_{R \to \infty} \left[ \phi^{-1} \phi \phi^{-1}, c \right]_{x=-R}^R = 0 .$$

This means that the function $f_c$ Poisson commutes with $f_{\dot{c}}$, and therefore the $\{ f_c \}$ are in involution.

As we shall see, in the case of the sine-Gordon equation, we can take $c = J$ or $c = K$ to obtain the Hamiltonians representing the total energy and momentum in light-cone co-ordinates. The higher momenta are obtained by taking the infinite number of other possible $c$'s. Since the energy is the Hamiltonian generator of time translations, and the momentum is the generator of space translations, all the higher momenta are conserved by these translations. The 1-parameter flow corresponding to the Hamiltonian $f_c$ on the phase space is given by the factorisation $(r, \phi_0) \mapsto \phi_0(r)$, where $\phi_0(r) \in \mathcal{G}$ is the solution to the factorisation problem $e^{rc} \phi_0 = \phi_0(r)d$ for some $d \in \mathcal{M}$.

**8 The higher momenta for sine-Gordon.**

Here we shall specialise the results of the last section to the solitons in the sine-Gordon model. The 1-parameter automorphism of the loop group $X$ is given by $\Theta(s, \rho)(\lambda) = \rho(s \lambda)$ for $\lambda \in \mathbb{C}^*$, which is actually the Lorentz boost for the system, giving $\vartheta = \lambda \frac{d}{d\lambda}$. The inner product is

$$\left( y, z \right) = \frac{1}{2\pi i} \text{Tr} \int \frac{d\lambda}{\lambda} y(\lambda)z(\lambda) ,$$
which is \( \Theta \) invariant. We can choose \( h(\lambda) = s_1 \lambda^n \), giving \( c(\lambda) = n s_1 \lambda^n \). The meromorphic loops for sine-Gordon split into two cases, solitons and breathers, and we shall calculate the higher momenta for both cases.

The loop \( \phi \) for a single soliton is \( \phi = N \psi e^{iux_3} \), where \( N \) commutes with \( s_1 \), and

\[
\psi = \left( P_\perp + \frac{\lambda + i\kappa}{\lambda - i\kappa} P \right),
\]

for \( \kappa \) real. Then

\[
\langle \artial(\phi) \phi^{-1}, h \rangle = \frac{1}{2\pi i} \operatorname{Trace} \int \frac{-2i\kappa P s_1 \lambda^n}{(\lambda - i\kappa)(\lambda + i\kappa)} d\lambda = -(i\kappa)^n (1 + (-1)^{n-1}) \operatorname{Trace}(Ps_1),
\]

which is zero if \( n \) is even, and if \( n \) is odd we calculate

\[
f_c = -4\kappa^n i (-1)^{(n-1)/2} \left[ \operatorname{Trace}(Ps_1) \right]_{x = -\infty}^\infty = -4\kappa^n i (-1)^{(n-1)/2} \left[ \frac{i}{4} \frac{\bar{\mu} + \mu}{1 + \mu \bar{\mu}} \right] x = -\infty,
\]

and inserting the \( x \) dependence of \( \mu \) shows that

\[
f_c = -2 |\kappa|^n (-1)^{(n-1)/2}.
\]

The calculation for the breather is more complicated. In this case \( \phi = \psi e^{iux_3} \), where \( \psi \) is given by (4.20). We write \( \psi = \psi_1 \psi_2 \), where

\[
\psi_1 = \left( P_{1\perp} + P_1 \frac{\lambda - \alpha}{\lambda - \alpha} \right) \quad \text{and} \quad \psi_2 = \left( P_{2\perp} + P_2 \frac{\lambda + \alpha}{\lambda + \alpha} \right).
\]

Now \( \artial(\phi) \phi^{-1} = \artial(\psi_1) \psi_1^{-1} + \partial(\psi_2) \psi_2^{-1} \psi_1^{-1} \), and we can calculate

\[
\langle \artial(\psi_1) \psi_1^{-1}, h \rangle = \frac{1}{2\pi i} \operatorname{Trace} \int \frac{-\alpha - \bar{\alpha})P s_1 \lambda^n}{(\lambda - \alpha)(\lambda - \bar{\alpha})} d\lambda = -(\alpha^n - \bar{\alpha}^n) \operatorname{Trace}(Ps_1).
\]

The term \( \langle \psi_1 \partial(\psi_2) \psi_2^{-1} \psi_1^{-1}, h \rangle \) has the same limit as \( x \to \pm \infty \) as the simpler term \( \langle \partial(\psi_2) \psi_2^{-1}, h \rangle \), because \( \psi_1 \) tends to a limiting loop which commutes with \( s_1 \). Now we can write

\[
f_c = 2 \left[ \langle \partial(\phi) \phi^{-1}, h \rangle \right]_{x = -\infty}^\infty = -2(\alpha^n - \bar{\alpha}^n) \left[ \operatorname{Trace}(Ps_1) + (-1)^n \operatorname{Trace}(Ps_2) \right]_{x = -\infty}^\infty.
\]

For the breather case we can assume that \( \alpha \) lies in the upper half plane, in which case the equation above gives zero for \( n \) even, and for \( n \) odd,

\[
f_c = 2i(\alpha^n - \bar{\alpha}^n).
\]

The integer \( n \) of the \( n \)-th conserved charge is the Lorentz spin. The fact that these charges were zero for \( n \) even was observed beforehand in [8], in the more general context of the affine Toda field theories, since the elements of the principal Heisenberg subalgebra of the Lie algebra of \( \mathcal{M} \), only has odd principal grades. In [8] it was shown that elements of the principal Heisenberg subalgebra generated the higher conserved charges, with the derivation operator in the principle grade acting as the Lorentz boost, indeed the principal Heisenberg subalgebra is nothing more than the subalgebra of the Lie algebra of \( \mathcal{M} \) which commutes with \( J \) and \( K \). It is then an immediate consequence that the Lorentz spins of the charges, which are measured by the values of the principal grade, are only non-zero for the exponents of the affine algebra \( m \). For \( m = su(2) \), these are the odd integers. The argument presented here is a concrete verification of this.

We would also like to bring to the reader’s attention the expressions given by the central parts of \( \partial^{-1} J \partial \) and \( \partial^{-1} K \partial \),

\[
< J, \partial(\phi) \phi^{-1} > \quad \text{and} \quad < K, \partial(\phi) \phi^{-1} >,
\]

respectively, discussed in our previous paper [8], which were identified as the energy and momentum densities in light-cone co-ordinates integrated up to a point \( x \). It is evident that these are the same as the Hamiltonian expressions [7,2], for \( c = J \) and \( c = K \), provided we evaluate the differences as \( x \to \pm \infty \) of the integrated expressions. This is because the Hamiltonian expressions only give the total quantities.
9 The splitting of the symplectic form.

We must now address the practical concerns of calculating the symplectic form. First we deal with the normalisation, that is \( \phi = N \psi e^{\beta u s} \), where \( \psi = 1 \) at \( \lambda = \infty \), and where the constant matrix \( N \) (which commutes with \( s_1 \)) may be needed to satisfy the symmetry condition. We wish to rewrite the expression for the symplectic form in terms of \( \psi \), which is easier to deal with. After doing this we see that the form splits into two parts, one due to ‘self’ interactions and the other due to ‘mutual’ interactions.

It is not too difficult to see that \( N \) cancels immediately from the expression for the symplectic form, as it cancels from \( \phi^{-1} \phi_w \), and does not affect \( b \) at all. We shall therefore continue assuming that \( N = 1 \).

**Proposition 9.1**

Consider the pole at \( \alpha \), we write \( \psi = \zeta \chi \), where \( \zeta \) is of the form \( z_1 \) having a pole at \( \alpha \), and \( \chi \) is regular at \( \alpha \). Perform the usual factorisation (3.1) on \( \chi \) only, that is \( a \zeta_0 = \zeta d \), this defines \( d \in M \). Then the \( \alpha \) contribution to the symplectic form (5.13) is

\[
\omega(v, w)_{\alpha} = \left[ \langle d \zeta_0^{-1} \zeta_0 d^{-1} - \zeta^{-1} \chi, \zeta^{-1} \psi \rangle - \langle \chi \psi^{-1}, \zeta^{-1} \psi \rangle_{\alpha, \bar{\alpha}} \right]_{x = -\infty} .
\] (9.1)

The notation \( <,>_{\alpha, \bar{\alpha}} \) means take the contributions to the integral in the inner product only from the poles at \( \alpha \) and \( \bar{\alpha} \).

We can interpret the first term here as a self interaction term of the \( \alpha \) pole with itself, and the second term as the mutual interaction of the \( \alpha \) pole with the other poles. The total form is evidently the sum of such terms for each of the poles \( \alpha \).

**Proof**

First note that

\[
\phi^{-1} \phi_w = e^{-\beta u s} \psi^{-1} \psi_w e^{\beta u s} + \beta u w s_3 ,
\]

and

\[
b^{-1} = \beta u w b s_3 b^{-1} + e^{-\beta u s} \left( \psi^{-1} a \psi_0 \psi_0^{-1} a^{-1} \psi - \psi^{-1} \psi \right) e^{\beta u s} - \beta u w s_3 ,
\]

Now look at the contribution to \( \langle b v b^{-1}, \phi^{-1} \phi_w \rangle \), which makes up the form (5.13), from the \( \beta u w b s_3 b^{-1} \) term in (9.2), which is

\[
\langle \beta u w b s_3 b^{-1}, \phi^{-1} \phi_w \rangle = \beta u w \langle s_3, b^{-1} \phi^{-1} \phi_w b \rangle = \beta u w \langle s_3, \phi^{-1} \phi_w \rangle - \beta u w \langle s_3, b^{-1} b_w \rangle .
\]

Here the second term is zero as the inner product vanishes on analytic functions, and the first term cancels on taking the difference of its values between \( x = R \) and \( x = -R \). The contribution to the inner product \( \langle b v b^{-1}, \phi^{-1} \phi_w \rangle \) from the \( \beta u w s_3 \) term of (9.2) tends to zero as \( x \to \pm \infty \), and we are left with the contribution

\[
\langle e^{-\beta u s} \left( \psi^{-1} a \psi_0 \psi_0^{-1} a^{-1} \psi - \psi^{-1} \psi \right) e^{\beta u s}, e^{-\beta u s} \psi^{-1} \psi_w e^{\beta u s} + \beta u w s_3 \rangle ,
\]

which simplifies to

\[
\langle \psi^{-1} a \psi_0 \psi_0^{-1} a^{-1} \psi - \psi^{-1} \psi, \psi^{-1} \psi_w + \beta u w s_3 \rangle ,
\]

and the part containing \( \beta u w s_3 \) vanishes as it is the inner product of two analytic functions, giving the simpler result

\[
\langle \psi^{-1} a \psi_0 \psi_0^{-1} a^{-1} \psi - \psi^{-1} \psi, \psi^{-1} \psi_w \rangle = \langle c \psi_0^{-1} \psi_0 c^{-1} - \psi^{-1} \psi, \psi^{-1} \psi_w \rangle .
\]

Here we have used the factorisation \( a v_0 = v c \). As \( c \psi_0^{-1} \psi_0 c^{-1} - \psi^{-1} \psi \) is analytic, we only have to sum over the residues of \( \psi^{-1} \psi_w \) to calculate the inner product.

Let us look at the contribution from the poles at \( \alpha \) (and \( \bar{\alpha} \)) to the inner product. We suppose that \( \psi = \zeta \chi \), where \( \zeta \) has a pole at \( \alpha \), and \( \chi \) is regular at \( \alpha \). Then we can write

\[
\psi^{-1} \psi_w = \chi^{-1} \chi_w + \chi^{-1} \zeta w \chi .
\]
If we also write $\alpha \zeta_0 = \zeta d$, then $c = \chi^{-1} \chi_0$, and the $\alpha$ and $\bar{\alpha}$ contribution to the inner product is

$$
\langle c \psi_0^{-1} \psi_0 c^{-1} - \psi^{-1} \psi_0 \psi^{-1} \psi_0 \rangle_{\alpha, \bar{\alpha}} = \langle c \psi_0^{-1} \psi_0 c^{-1} - \psi^{-1} \psi_0 \chi^{-1} \zeta \chi_0 \rangle_{\alpha, \bar{\alpha}}
$$

$$
= \langle d \chi_0 \chi_0^{-1} d^{-1}, \zeta^{-1} \zeta \chi_0 \rangle_{\alpha, \bar{\alpha}} - \langle \chi \chi^{-1}, \zeta^{-1} \chi_w \rangle_{\alpha, \bar{\alpha}}
$$

$$
+ \langle d \chi_0^{-1} \zeta \chi_0 d^{-1} - \zeta^{-1} \zeta \chi_0 \rangle_{\alpha, \bar{\alpha}}
$$

(9.3)

The first term in (9.3) is

$$
\langle \chi_0 \chi_0^{-1}, d^{-1} \zeta^{-1} \zeta d \rangle_{\alpha, \bar{\alpha}} = \langle \chi_0 \chi_0^{-1}, \zeta_0^{-1} \chi_0 \rangle_{\alpha, \bar{\alpha}} - \langle \chi_0 \chi_0^{-1}, d^{-1} d \rangle_{\alpha, \bar{\alpha}},
$$

the last term of this vanishes as there is no pole at $\alpha$ or $\bar{\alpha}$, and the first term vanishes on taking the difference between the two asymptotic values of $x$. We can now write the $\alpha$-contribution to the inner product as

$$
\langle d \chi_0^{-1} \zeta \chi_0 d^{-1} - \zeta^{-1} \zeta \chi_0 \rangle_{\alpha, \bar{\alpha}} - \langle \chi \chi^{-1}, \zeta^{-1} \chi_w \rangle_{\alpha, \bar{\alpha}},
$$

which can be rewritten as

$$
\langle d \chi_0^{-1} \zeta \chi_0 d^{-1} - \zeta^{-1} \zeta \chi_0 \rangle_{\alpha, \bar{\alpha}} - \langle \chi \chi^{-1}, \zeta^{-1} \chi_w \rangle_{\alpha, \bar{\alpha}}.
$$

as $\zeta^{-1} \chi_w$ only has poles at $\alpha$ and $\bar{\alpha}$.

10 The mutual interaction term.

The limiting behaviour of the meromorphic loops.

Lemma 10.1

The projections $P_i$, making up the product of meromorphic loops of the form (2.13) for $\phi(\lambda)$, commute with $s_1$ in the limits $x \to \pm \infty$.

Proof

From the equations of motion of the linear system

$$
\phi^{-1} P_+ \phi = A - \phi^{-1} J \phi, \quad \text{and} \quad \phi^{-1} P_- \phi = B - \phi^{-1} K \phi.
$$

in the limits $x \to \pm \infty$, (for finite $t$), the left-hand sides of these equations are zero, and $A \to J$, $B \to K$, so we conclude that $\phi(\lambda, \pm \infty)$ commutes with $s_1$. We add a soliton to the system by multiplying a previous $\phi$ by a meromorphic unitary loop on the right of the form (2.13), and adjusting the normalisation. The space-time behaviour of the previous $\phi$ is unchanged. We also find that the limit of the resultant $\phi$ commutes with $s_1$, so we see that the limit of the meromorphic loop we have added also commutes with $s_1$. By taking residues therefore any of the $P_i$ in the products of the form (2.13) making up $\phi$ also commute with $s_1$. □

For the following, recall that all matrices commuting with $s_1$ commute amongst themselves.

Lemma 10.2

We factor $\psi$ into the form $\psi = \zeta \chi \beta$, where $\zeta$ has only a pole at $\beta$, and $\chi \beta$ is regular at $\beta$. $\zeta$ is of the form (2.13), where we denote the relevant projection by $P_\beta(x)$. Then for any two poles $\beta_1$ and $\beta_2$ in $\psi$

$$
\text{Trace}(P_{\beta_1}(\infty)P_{\beta_2}(\infty) - P_{\beta_1}(-\infty)P_{\beta_2}(-\infty)) = 0.
$$

Proof

This result is almost immediate from the form (1.7) of the projection for the one-soliton solution. In the limit $(x \to -\infty, \text{Im}(\beta) > 0)$ or $(x \to \infty, \text{Im}(\beta) < 0)$,

$$
P_\beta \to \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix},
$$

as $\zeta^{-1} \chi$ only has poles at $\alpha$ and $\bar{\alpha}$. □
and for \((x \to \infty, \text{Im}(\beta) > 0)\) or \((x \to -\infty, \text{Im}(\beta) < 0)\),

\[ P_\beta \to \frac{1}{2} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}. \]

A simple calculation establishes the validity of the statement. □

**Proposition 10.3**

The mutual interaction term, the second term of (10.1), vanishes, i.e.

\[ \bigg[ \langle \chi_\alpha \chi_\alpha^{-1}, \zeta_\alpha^{-1} \zeta_\alpha \rangle_{\alpha, \bar{\alpha}} \bigg]_{x=-\infty}^{\infty} = 0. \]

**Proof**

Given a unitary meromorphic loop \(\psi\) with \(\psi(\infty) = 1\), we can factor it into

\[ \psi = \zeta_\alpha \chi_\beta, \quad (10.1) \]

where \(\zeta_\beta\) has only a pole at \(\beta\), and \(\chi_\beta\) is regular at \(\beta\), but will contain a pole at \(\alpha\), c.f. equation (10.1), unless we choose \(\beta = \alpha\). In the generic case where poles are simple,

\[ \zeta_\beta = P_\beta^\perp + \frac{\lambda - \beta}{\lambda - \beta} P_\beta, \quad \zeta_\beta^{-1} = P_\beta^\perp + \frac{\lambda - \beta}{\lambda - \beta} P_\beta, \]

\[ \zeta_\beta^{-1} \zeta_\beta v = \frac{\beta - \bar{\beta}}{\lambda - \beta} P_{\beta v} P_\beta + \frac{\beta - \bar{\beta}}{\lambda - \beta} P_\beta P_{\beta v} + \frac{\chi(\beta_v - \bar{\beta}_v) + \bar{\beta}_v - \beta_\beta \bar{\beta}_v}{(\lambda - \beta)^2} P_\beta, \quad (10.2) \]

where \(\beta_v\) and \(P_{\beta v}\) are the changes in \(\beta\) and \(P_\beta\), respectively, due to the vector \(v_0\). In deriving this formula we must remember that \(P^\perp P_v = P_\alpha P\), which comes from differentiating the equation \(P^2 = P\). From (10.1), we compute

\[ \psi^{-1}\psi_v = \chi^{-1}_{\beta} \zeta^{-1}_{\beta} \chi_\beta v + \chi^{-1}_{\beta} \chi_\beta v, \quad (10.3) \]

and noting that \(\psi^{-1}\psi_v\) is a meromorphic function, zero at infinity, and has simple poles at \(\beta\) and \(\bar{\beta}\), we can write \(\psi^{-1}\psi_v\) in terms of its simple poles as

\[ \psi^{-1}\psi_v = \sum_\beta \left( \frac{\text{res}_\beta(\psi^{-1}\psi_v)}{\lambda - \beta} + \frac{\text{res}_{\bar{\beta}}(\psi^{-1}\psi_v)}{\lambda - \bar{\beta}} \right). \]

From (10.3) this is equal to

\[ \psi^{-1}\psi_v = \sum_\beta \left( \frac{\chi_\beta(\beta)^{-1} \text{res}_\beta(\zeta^{-1}_{\beta} \zeta_{\beta v}) \chi_\beta(\beta)}{\lambda - \beta} + \frac{\chi_\beta(\bar{\beta})^{-1} \text{res}_{\bar{\beta}}(\zeta^{-1}_{\beta} \zeta_{\beta v}) \chi_\beta(\bar{\beta})}{\lambda - \bar{\beta}} \right). \quad (10.4) \]

We re-arrange formula (10.3):

\[ \chi_\alpha \chi_\alpha^{-1} = \chi_\alpha \psi^{-1}\psi_v \chi_\alpha^{-1} - \zeta^{-1}_{\alpha} \zeta_\alpha, \quad (10.5) \]

and calculate \(\chi_\alpha \chi_\alpha^{-1}(\alpha)\), which we will eventually insert into \(\chi_\alpha \chi_\alpha^{-1}, \zeta_\alpha^{-1} \zeta_\alpha >_{\alpha, \bar{\alpha}}\). Insert the form (10.4) into (10.5), and consider the contribution to the sum from the \(\beta \neq \alpha\) poles.

From Eqn. (10.2), and noting that \(P_{\beta v} \to 0\), as \(x \to \pm \infty\), \(\text{res}_\beta(\zeta^{-1}_\beta \zeta_{\beta v}) \to \beta_\beta^\perp P_\beta(\pm \infty)\), and \(\text{res}_{\bar{\beta}}(\zeta^{-1}_\beta \zeta_{\beta v}) \to -\beta_\beta^\perp P_\beta(\pm \infty)\), as \(x \to \pm \infty\), and since \(\chi_\alpha(\alpha)\chi_\beta(\beta)^{-1}\), and \(\chi_\alpha(\alpha)\chi_\beta(\bar{\beta})^{-1}\) tend to matrices which commute with \(s_1\), c.f. Lemma 10.1, and therefore commute respectively with \(\text{res}_\beta(\zeta^{-1}_\beta \zeta_{\beta v})\), and \(\text{res}_{\bar{\beta}}(\zeta^{-1}_\beta \zeta_{\beta v})\) after the limits \(x \to \pm \infty\), we get the contribution to \(\chi_\alpha \chi_\alpha^{-1}(\alpha)\) of

\[ \sum_{\beta \neq \alpha} \left( \frac{\beta_\beta}{\alpha - \beta} - \frac{\bar{\beta}_\beta}{\alpha - \beta} \right) P_\beta(\pm \infty). \quad (10.6) \]
The contribution to $\chi_\alpha \chi_\alpha^{-1}(\alpha)$ from the $\bar{\alpha}$ pole is zero, and for the $\alpha$ pole we get the limit as $\lambda \to \alpha$ of

$$\chi_\alpha(\lambda) \chi_\alpha(\alpha)^{-1} \text{res}_\alpha(\zeta_\alpha^{-1} \zeta_{\omega\nu}) \chi_\alpha(\alpha) \chi_\alpha(\lambda)^{-1} - \text{res}_\alpha(\zeta_\alpha^{-1} \zeta_{\omega\nu}).$$

This is

$$[\chi_\alpha'(\lambda) \chi_\alpha(\alpha)^{-1}, \text{res}_\alpha(\zeta_\alpha^{-1} \zeta_{\omega\nu})] + O(\lambda - \alpha),$$

which tends to zero on putting $\lambda = \alpha$ and taking limits as $x \to \pm \infty$, since after these limits, both terms in the commutator separately commute with $s_1$, c.f. Lemma 10.1, and so commute with each other. Thus $\chi_\alpha \chi_\alpha^{-1}(\alpha)$ is equal to formula (10.6), at the limits when $x \to \pm \infty$. In the formula (10.2) for $\zeta_\alpha^{-1} \zeta_{\omega\nu}(\lambda)$ we note that $P_{\beta\nu} \to 0$, as $x \to \pm \infty$, so that only the last term of (10.2) contributes. Since $\chi_\alpha \chi_\alpha^{-1}(\lambda)$ is regular at $\alpha$, we see that the $\alpha$ pole contribution to the inner product $\langle \chi_\alpha \chi_\alpha^{-1}, \zeta_\alpha^{-1} \zeta_{\omega\nu}\rangle_{\alpha, \bar{\alpha}}$ is made up of sums of terms proportional to

$$\text{trace}(P_\beta(\infty) P_\alpha(\infty) - P_\beta(-\infty) P_\alpha(-\infty))$$

for $\beta \neq \alpha$. From Lemma 10.2, these terms are all zero. Likewise the contribution from the $\bar{\alpha}$ pole gives zero, so we conclude that the mutual interaction term $\langle \chi_\alpha \chi_\alpha^{-1}, \zeta_\alpha^{-1} \zeta_{\omega\nu}\rangle_{\alpha, \bar{\alpha}}$ contributes nothing to the symplectic form. □

11 The self interaction term.

Proposition 11.1

Consider the self-interaction term, $\langle d\zeta^{-1}_0 \zeta_{\nu} d^{-1} - \zeta^{-1}_v, \zeta^{-1}_w \rangle$, the first term of (11.1), then we have

$$\langle d\zeta^{-1}_0 \zeta_{\nu} d^{-1} - \zeta^{-1}_v, \zeta^{-1}_w \rangle = \pm 2 \text{Re} \frac{\alpha_\nu Q_w - \alpha_w Q_\nu}{\alpha Q},$$

(11.1)

(11.2)

Here $Q$ is the coefficient specifying the initial projection of the left-most ordered meromorphic loop in the product of loops. For the two-soliton case, comparing with Eqn. (11.4), this $Q$ is either $Q_1$ or $Q_2$ depending on whether the pole at $i\kappa_1$ or $i\kappa_2$ is considered. The total form is the sum of the contributions of this type.

Proof

We take

$$\zeta = P^\perp + P \frac{\lambda - \bar{\alpha}}{\lambda - \alpha},$$

which gives, as before,

$$\zeta^{-1} \zeta_v = \frac{\alpha - \bar{\alpha}}{\lambda - \alpha} P_\nu P + \frac{\alpha - \bar{\alpha}}{\lambda - \alpha} P P_\nu + \frac{\lambda (\alpha_v - \bar{\alpha}_v) + \bar{\alpha}_v \alpha - \alpha_v \bar{\alpha}}{(\lambda - \alpha)(\lambda - \bar{\alpha})} P,$$

(11.3)

and for convenience we shall write $n = \lambda (\alpha_v - \bar{\alpha}_v) + \bar{\alpha}_v \alpha - \alpha_v \bar{\alpha}$. We also need to look at the value of $d = \zeta^{-1} a_\zeta$ at $\alpha$ and $\bar{\alpha}$. We recall that $\zeta^{-1} = (P^\perp + \frac{\lambda - \alpha}{\lambda - \bar{\alpha}} P)$, and then

$$d(\lambda) = (P^\perp + \frac{\lambda - \alpha}{\lambda - \bar{\alpha}} P) a(\lambda) (P^\perp + \frac{\lambda - \bar{\alpha}}{\lambda - \alpha} P_0).$$

(11.4)

Now, from (11.3), we find that the first entry in the inner product to be calculated is

$$H(\lambda) = d\zeta_v^{-1} a_\zeta d^{-1} - \zeta^{-1}_v$$

$$= \frac{\alpha - \bar{\alpha}}{\lambda - \alpha} (dP_\nu P_0d^{-1} - P_\nu P) + \frac{\alpha - \bar{\alpha}}{\lambda - \alpha} (dP_\nu P_0d^{-1} - P P_\nu) + \frac{n(dP_0d^{-1} - P)}{(\lambda - \alpha)(\lambda - \bar{\alpha})}.$$
The contribution to the total inner product from the $\alpha$ and $\bar{\alpha}$ poles, noting that $P^2 = P$, using cyclicity of the trace, and $P^\perp P_w = P_w P$, is:

\[
\text{Trace}\left(P\left(\frac{\alpha_w}{\alpha}H(\alpha) - \frac{\bar{\alpha}_w}{\bar{\alpha}}H(\bar{\alpha})\right)\right) + (\alpha - \bar{\alpha}) \text{Trace}\left(\frac{PH(\alpha)P^\perp}{\alpha} + \frac{P^\perp H(\bar{\alpha})P}{\bar{\alpha}}\right) P_w ,
\] (11.5)

and we calculate, by substituting (11.4) for $d(\lambda)$, and writing $a$ for $a(\lambda)$,

\[
\begin{align*}
PH(\lambda)P & = (\alpha - \bar{\alpha})\left(\frac{PaP_0P_0a^{-1}P}{\lambda - \bar{\alpha}} + \frac{PaP_0P_0\bar{a}^{-1}P}{\lambda - \bar{\alpha}}\right) + n \frac{PaP_0a^{-1}P - P}{(\lambda - \bar{\alpha})}\lambda - \bar{\alpha}\right)\left(\frac{PaP_0P_0a^{-1}P}{(\lambda - \bar{\alpha})^2} + \frac{PaP_0P_0\bar{a}^{-1}P}{(\lambda - \bar{\alpha})^2}ight) + n \frac{PaP_0a^{-1}P^\perp}{(\lambda - \bar{\alpha})^2}
\end{align*}
\]

\[
PH(\lambda)P^\perp = (\alpha - \bar{\alpha})\left(\frac{PaP_0P_0a^{-1}P}{(\lambda - \bar{\alpha})^2} + \frac{PaP_0P_0\bar{a}^{-1}P}{(\lambda - \bar{\alpha})^2}\right) + n \frac{PaP_0a^{-1}P^\perp}{(\lambda - \bar{\alpha})^2}
\]

\[
P^\perp H(\lambda)P = (\alpha - \bar{\alpha})\left(\frac{P^\perp aP_0P_0a^{-1}P - P_v P}{\lambda - \bar{\alpha}} + \frac{(\lambda - \bar{\alpha})P^\perp aP_0P_0a^{-1}P}{(\lambda - \bar{\alpha})^2}\right) + n \frac{P^\perp aP_0a^{-1}P}{(\lambda - \bar{\alpha})^2}.
\]

We know that $P$ is the orthogonal projection to $a(\alpha)\lim P_0$, which gives rise to the equations

\[
Pa(\alpha)P_0 = a(\alpha)P_0 \quad \text{and} \quad P_0a(\alpha)^{-1}P = a(\alpha)^{-1}P .
\] (11.6)

Using these equations (11.4), we find

\[
PH(\alpha)P = PaP_0P_0a^{-1}P - (\alpha - \bar{\alpha})PaP_0P_0a^{-1}a_\lambda a^{-1}P + a_v(Pa_\lambda P_0a^{-1}P - PaP_0a^{-1}a_\lambda a^{-1}P).
\]

If we take the equation $Pa(\alpha)P_0a(\alpha)^{-1}P = P$, derived from (11.6), and differentiate it, we find that

\[
P_v aP_0a^{-1}P + PaP_0a^{-1}P + PaP_0a^{-1}P_v + a_v(Pa_\lambda P_0a^{-1}P - PaP_0a^{-1}a_\lambda a^{-1}P) = P_v ,
\]

and using this we can write

\[
\text{Trace}(PH(\alpha)P) = \text{Trace}(aP_0a^{-1}P_v + (\alpha - \bar{\alpha})PaP_0P_0a^{-1}a_\lambda a^{-1}P).
\]

We also find that

\[
PH(\alpha)P^\perp = aP_0P_0a^{-1}P^\perp - PP_v + \frac{\alpha_v}{\alpha - \bar{\alpha}}aP_0a^{-1}P^\perp
\]

\[
\text{Trace}(PH(\alpha)P^\perp P_w) = \text{Trace}(aP_0P_0P_0P_0a^{-1}P_w - PP_v P_w + \frac{\alpha_v}{\alpha - \bar{\alpha}}aP_0a^{-1}P_w P).\]

The combination from the $\lambda = \alpha$ terms of (11.5) is, writing $a$ for $a(\alpha)$,

\[
\text{Trace}(\alpha_w PH(\alpha)P + (\alpha - \bar{\alpha})PH(\alpha)P^\perp P_w) / \alpha = \alpha^{-1} \times
\]

\[
\text{Trace}(\alpha_w aP_0a^{-1}P_w P_0P_0a^{-1}P_v P + (\alpha - \bar{\alpha})(PaP_0P_0P_0a^{-1}P_w + \alpha_w PaP_0P_0P_0(a^{-1}P - PP_v P_w))).
\]

By differentiating the equation $P_0^\perp a^{-1}(\alpha) = 0$, derived from (11.4), we obtain

\[
-PP_0a^{-1}P + \alpha_w P_0^\perp (a^{-1})_P + P_0^\perp a^{-1}P = 0 ,
\]

so

\[
\text{Trace}(\alpha_w PH(\alpha)P + (\alpha - \bar{\alpha})PH(\alpha)P^\perp P_w) / \alpha =
\]

\[
\text{Trace}(\alpha_w aP_0a^{-1}P_w P_0P_0a^{-1}P_v P + (\alpha - \bar{\alpha})(P_0P_0P_0P_0 - PP_v P_w)) / \alpha .
\]

The term $PP_v P_w$ tends to zero as $x \to \pm \infty$, and $P_0P_0P_0P_0$ vanishes on taking the difference between two values of $x$, leaving the $\lambda = \alpha$ terms from (11.5) as

\[
\left[\frac{1}{\alpha} \text{Trace}(\alpha_w PH(\alpha)P + (\alpha - \bar{\alpha})PH(\alpha)P^\perp P_w)\right]_{x=\pm \infty} =
\]

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Let $P$ be the orthogonal projection to the complex vector $(1, \mu)$. Then

$$P = \frac{1}{1 + \mu \bar{\mu}} \begin{pmatrix} 1 & \mu \\ \mu & \bar{\mu} \end{pmatrix} \quad \text{and} \quad P \bar{P} = \frac{\mu_\nu}{(1 + \mu \bar{\mu})^2} \begin{pmatrix} -\bar{\mu} & -\mu^2 \\ 1 & \bar{\mu} \end{pmatrix}. $$

Since $a(\lambda) = e^{-jx_+ - \kappa x_-}$, it can be written in the form

$$a(\alpha) = \frac{1}{2} \begin{pmatrix} r + 1/r & r - 1/r \\ r - 1/r & r + 1/r \end{pmatrix},$$

where $r = \exp(-imx(\alpha - \alpha^{-1})/2)$. Here we have set $t = 0$, although the limits derived below are equally valid for any finite $t$. Then it is possible to calculate

$$\frac{1}{\alpha} \left[ \text{Trace}(a_\nu a_{0\mu}^{-1} P_w P - a_{\mu\nu} a_0^{-1} P_v P) \right]_{x = -\infty}^\infty = \frac{2(\alpha_\nu \mu_{\nu\mu} - \alpha_{\nu\mu} \mu_{\nu\nu})}{\alpha(\mu_\nu^2 - 1)}. $$

The overall sign of (11.8) is reversed if $\text{Im}(\alpha) < 0$. If we use the coordinate $Q$, c.f. Eqn (4.13), where

$$\mu_\nu = \frac{1 - iQ}{1 + iQ},$$

we can rewrite Eqn. (11.8) as

$$\frac{1}{\alpha} \left[ \text{Trace}(a_\nu a_{0\mu}^{-1} P_w P - a_{\mu\nu} a_0^{-1} P_v P) \right]_{x = 0}^\infty = \frac{\alpha_\nu Q_{\nu\mu} - \alpha_{\nu\mu} Q_{\nu\nu}}{\alpha Q}. $$

We also calculate the $\lambda = \bar{\alpha}$ parts of (11.3), and find the total contribution to the symplectic form

$$\langle d\zeta_0^{-1} \zeta_{\nu\mu} a^{-1} - \zeta^{-1} \zeta_{\nu\mu} \rangle = \pm 2\text{Re} \frac{\alpha_\nu Q_{\nu\mu} - \alpha_{\nu\mu} Q_{\nu\nu}}{\alpha Q},$$

where $+$ corresponds to $\text{Im}(\alpha) > 0$ and $-$ to $\text{Im}(\alpha) < 0$.

## 12 Final statement of result.

The two-soliton case. By combining Propositions 9.1, 10.3 and 11.1, and by considering the contributions from the poles at $i\kappa_1$ and $i\kappa_2$, we have derived the symplectic form $\omega$ for the two-soliton solution

$$\omega = \frac{d\kappa_1 \wedge dQ^\prime_1}{\kappa_1 Q_1^\prime} + \frac{d\kappa_2 \wedge dQ^\prime_2}{\kappa_2 Q_2^\prime}.$$ 

Throughout this section, we must take the real parts of the expressions giving the forms. Also, for the solitons, without loss of generality we can take $\kappa > 0$, and therefore take the plus sign of (11.2). If $\kappa < 0$, the minus sign in (11.2) is to be expected because of the relation of the position $x_0$ of the soliton to $Q$, see Eqn. (4.11). If we wished to write the form in terms of the rapidity $\theta$ and position
of the soliton, we must substitute $|\kappa| = e^{-\theta}$, and therefore pick up a minus sign if $\kappa < 0$, this cancels with the minus appearing in (11.2), so in both cases we would have an overall plus sign.

Here the explicit two-soliton solution in these co-ordinates is written as

$$e^{-i\frac{2\pi}{2}} = \frac{(1 + iY^{-1}Q_1'W_1 - iY^{-1}Q_2'W_2 + Q_1'Q_2'W_1W_2)}{(1 - iY^{-1}Q_1'W_1 + iY^{-1}Q_2'W_2 + Q_1'Q_2'W_1W_2)},$$

(12.1)

with

$$Y = \frac{\kappa_2 - \kappa_1}{\kappa_2 + \kappa_1},$$

compare with Eqn. (4.16), and following. Also, $W_i = e^{m(\kappa_i,x_i - \kappa_i'x_i')}$, and $\kappa_i$ is related to the rapidity $\theta_i$ of the soliton by $e^{-\kappa_i} = |\kappa_i|$.

If we write (12.1) in terms of the co-ordinates $Q_1, Q_2$ which are more familiar to us, so that the two-soliton solution is

$$e^{-i\frac{2\pi}{2}} = \frac{(1 + iQ_1W_1 - iQ_2W_2 + Y^2Q_1Q_2W_1W_2)}{(1 - iQ_1W_1 + iQ_2W_2 + Y^2Q_1Q_2W_1W_2)},$$

by defining

$$Q_1 = Y^{-1}Q_1', \quad Q_2 = Y^{-1}Q_2'.$$

Then the symplectic form is

$$\frac{\omega}{2} = \frac{d\kappa_1 \wedge d(YQ_1)}{\kappa_1 YQ_1} + \frac{d\kappa_2 \wedge d(YQ_2)}{\kappa_2 YQ_2}$$

$$= \frac{d\kappa_1 \wedge dQ_1}{\kappa_1 Q_1} + \frac{d\kappa_2 \wedge dQ_2}{\kappa_2 Q_2} + \frac{d\kappa_1 \wedge dY}{\kappa_1 Y} + \frac{d\kappa_2 \wedge dY}{\kappa_2 Y}.$$  \hspace{1cm} (12.2)

However, it is easy to show that

$$\frac{d\kappa_1 \wedge dY}{\kappa_1} = \frac{d\kappa_2 \wedge dY}{\kappa_2} = \frac{2}{(\kappa_1 + \kappa_2)^2}d\kappa_1 \wedge d\kappa_2$$

so we can rewrite the form as

$$\frac{\omega}{2} = \frac{d\kappa_1 \wedge d(Y^2Q_1)}{\kappa_1 Y^2Q_1} + \frac{d\kappa_2 \wedge dQ_2}{\kappa_2 Q_2}$$

or

$$\frac{\omega}{2} = \frac{d\kappa_1 \wedge dQ_1}{\kappa_1 Q_1} + \frac{d\kappa_2 \wedge d(Y^2Q_2)}{\kappa_2 Y^2Q_2}.$$  \hspace{1cm} (12.3)

these last two expressions are ones familiar from Babelon and Bernard [1], where the form is represented in terms of “in” and “out” co-ordinates, respectively. The multiplication of one of the $Q$’s by $Y^2$, is precisely the shift in one of the co-ordinates of the solitons by the standard time delay, proportional to log ($Y^2$), for sine-Gordon.

The $n$-soliton case. The form is evidently the sum of single soliton contributions given by the leftmost ordered $Q$’s, denoted $Q_i'$. However, it must be remembered that these $Q_i'$ are not the co-ordinates which appear in the standard $n$-soliton solution, that is the $n$-soliton version of (4.13), or the $n$-soliton solutions given in [1]. (in [1], they are denoted by $a_i$). We denote these co-ordinates $Q_i$. The two sets of co-ordinates are related by

$$Q_i' = Q_i \prod_{j \neq i} \frac{\kappa_i - \kappa_i}{\kappa_i + \kappa_i}.$$  \hspace{1cm} (12.4)

Exactly as we have seen for the two-soliton case above, we can then make contact with the diagonal result of [1], where the form is given in terms of ‘in’ or ‘out’ co-ordinates.
The breather case. Begin with the factorisation (4.20), where the left ordered projections are $P_1$ and $P_3$. Without loss of generality we may assume that the imaginary part of $\alpha$ is positive. Then the contribution to the symplectic form from the $\alpha$ pole is

$$\frac{\omega}{2} \bigg|_{\alpha} = \frac{d\alpha \wedge dQ_1}{\alpha Q_1}.$$ 

To find the contribution from the $-\alpha$ pole we use equation (4.21) and (4.24) to find $Q_3 = 1/Q_1$, and we can now give the total symplectic form for a breather, remembering to take the real part of this, as

$$\frac{\omega}{2} \bigg|_{\alpha, -\alpha} = 2 \frac{d\alpha \wedge dQ_1}{\alpha Q_1}.$$ 

The $Q$ used here is the one appearing in the left-most ordered factor, the factor with a pole at $\alpha$, in a product of loops representing many breathers and solitons. It is crucial to remember that this $Q$ will be shifted, as compared with the $Q$ in the single breather solution (4.26), by factors which arise when the other breathers and solitons to the right of the left-most ordered factors are taken into account. Compare with the $n$-soliton case.

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