Regge-like spectra of excited singly heavy mesons

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In this work, we study the Regge-like spectra of excited singly heavy mesons by proposing a general Regge-like mass relations in which the slope ratio $\alpha'/\beta'$ between the radial and angular-momentum Regge trajectories is $\pi/2$ and the hadron mass undergoes a shift including the heavy quark mass and an extra binding energy between heavy quark and strange anti-quark. The relation is successfully tested against the observed spin-averaged data of the singly heavy mesons in their radially and angularly excited states. Some new predictions are made for more excited excitations and the discussion is given associated with the QCD string (flux tube) picture.

I. INTRODUCTION

During the last two decades, heavy meson spectroscopy has been greatly enlarged due to the discovery of numerous excited charm and charm-strange states (charm mesons hereafter) by the B-Factory experiments BaBar, Belle, CLEO and recently by the LHCB experiment [1]. One of recent examples is the observations of the heavy mesons $B(5970)$ by the CDF Collaboration in the $B^+\pi^-$ and $B^0\pi^0$ mass distributions [2] and the $B_J(5960)$ by the LHCB Collaboration in the $B\pi\pi$ mass distributions [3]. The first observation determined the mass of the $B(5970)$ resonances to be $5978 \pm 5 \pm 12$ MeV for the neutral state and to be $5961 \pm 5 \pm 12$ MeV for the charged state. The second observation, giving the mass of $5969.2 \pm 2.9 \pm 5.3$ MeV for the neutral state and $82.3 \pm 7.7 \pm 9:4$ MeV for the charged state for $B_J(5960)$, seems to be consistent with the $B(5970)$ in their properties, and they may be the same state. Another example is the observation of the charmed states of $D_J(3000)$ by the LHCB Collaboration in the $D\pi$ mass distribution data from pp collision [4, 5]. These experimental findings arouse theoretical interests to explore and accommodate the newly-discovered states with various approaches, e.g., with the potential quark models [6–18], Lattice QCD [19, 20] and other approaches (see Ref. [21] for a recent review). For the $B(5970)$, the two main interpretations exist, with the assignments $2^3S_1$ [22] and the candidate of $1^3D_1$ state [23]. For the $D_J(3000)$ and $D^{*}(3000)$, there are the different interpretations, the $D(3^3S_0)$ and the $D(3^3S_1)$ states, respectively, or the candidates of $D(1F)$ or $D(2P)$. For the $D^*_J(3000)$ the prediction favoring $D(3^3P_2)$ was given [24]. In Ref. [25], a comprehensive analysis was made for the whole singly heavy mesons and baryons using the Regge-like mass relation,

$$(M-M_Q)^2 = \alpha L + c,$$

with $M_Q$ the heavy quark mass, supporting that all heavy-light hadrons fall on straight lines in a “shifted” orbital trajectory plane: $M \rightarrow M - M_Q$, with the slope $(\alpha)$ nearly half of that for the light hadrons. Here, $L$ is the orbital angular momentum of the hadron system. This reflects that the notion of Regge trajectory, which is widely-used in light-flavor sector and stems from the well-known Regge theory of the scattering matrix [26], remains to be useful in the spectra of the heavy flavor hadrons if one shifts the hadron mass $M$ by a constant. In addition, the knowledge of these trajectories is also valuable in the modeling of the recombination and fragmentation for hadrons transition in the scattering region $(t < 0)$ [27]. For the light mesons, an updated picture emerges in Ref. [28] and in Refs. [29–36] where a more general linear Regge trajectories,

$$M^2 = \alpha J + \beta n + a_0,$$

were proposed, with $n$ the principle quantum number and $a_0$ a constant. This raises a question as to if the relation (2) applies for the heavy mesons and how much is the slope ratio $\beta/\alpha$ is if it applies. Differing from an existing pattern of approximate degeneracy $(\alpha \approx \beta)$ in the excited light hadron spectra [30, 31, 33, 37], in the case of the heavy hadrons with limited number of observed states, this type of degeneracy remains to be explored, and study of possible pattern in them will be of interest for accommodating or searching for some of the higher excitations of the heavy mesons that have surfaced after the $B(5970)^{+}/^{0}$ observation.

Purpose of this work is to propose and support that the relation (2) applies for the heavy-light (HL) mesons if the slope ratio $\beta/\alpha$ between the radial and angular-momentum Regge trajectories is chosen to be $\pi/2$ and the mass $M$ of meson is shifted by a mount $M_{Q} - \mu_{Q}$ with $M_{Q}$ the heavy quark mass and $\mu_{Q}$ the effective reduced mass of the heavy quark and light (anti)quark. We use the mass relation to plot the orbital and angular-momentum Regge trajectories combined for the observed spectroscopy of heavy-light mesons [38] and infer thereby that $D(3000)^0$, $B_J(5840)$ and $B_J(5970)$ are most likely, the $D(3P)$ and $B(2S)$ respectively. A semiclassical illustration for the mass relation including the slope ratio $\pi/2$ is presented within the picture of QCD string or flux tube. The flavor-dependence of the trajectory parameters in the mass relation is noted.

It is well known that the parent linear Regge trajectories (2) with $n = 0$ can be explained in the rotating string picture [39–41], where the quark and antiquark in meson are assumed to be massless and tied by the gluon flux tube (QCD string). In the massive case, the corrections to linear Regge trajectories were explored in Refs. [42, 43] and recently in Refs. [44, 45]. When the radially excitations involved, the issue of how well the Regge trajectory applies is of interest for understanding the excited heavy mesons. For more discussions of the Regge-
like relations for the mesons, see [29–32, 34, 46] and [28, 47] for instance.

In Section II, we present the Regge-like mass relation for the heavy-light mesons and outline the existing proposals that is directly related to the our mass relation. In Section III, we confront the mass relation proposed with the observed masses of the excited HL mesons and thereby some of predictions are given. A simple semiclassical picture that supports Regge-like mass relation is outlined in the QCD string model in Section IV and the conclusions are summarized in Section V.

II. REGGE-LIKE MASS RELATION FOR ORBITALLY AND RADIALLY EXCITATIONS

Our proposal for heavy-light (HL) mesons spectrum is to extend the Regge-like relation (2) to the general case that applies for the orbital and angularly excited states, by choosing the slope ratio between the radial and angular-momentum Regge trajectories to be \( \pi : 2 \), that is, \( \beta/\alpha = \pi/2 \), and the mass \( M \) of mesons to be shifted by \( M_Q - \mu_Q \) with \( \mu_Q = km_l/(km_q + M_Q) \) the effective reduced mass of the heavy quark and light (anti)quark:

\[
M \rightarrow M - M_Q + \mu_Q = M - M_Q/(1 + km_q/M_Q). \tag{3}
\]

Here, the prefactor \( k \) is introduced to describe effectively the scale-dependent mass running of the light quarks that may be relevant in the higher excitations. The emergence of \( \pi/2 \) seems to be quite unusual in the light of pure phenomenology. From the viewpoint of this work, however, it stems from the fact that the energy cost for orbital motion of a string linked to quarks differs itself from that for radial motion (see Section IV for details). The main proposal is then

\[
\left( M - \frac{M_Q}{1 + km_l/M_Q} \right)^2 = \pi b \left( L + \frac{\pi}{2} n \right) + \left( m_l + \frac{P^2}{M_Q} \right)^2, \tag{4}
\]

where \( L \) is the quantum numbers of the orbital angular momentum, \( n \) the radial quantum number, \( b \) the slope parameter, and \( M, M_Q \) and \( m_l \) are the masses of the HL meson, heavy quark and light quark, respectively. In the intercept part of the relation (4),

\[
P_Q = M_Q v_Q = M_Q \left( 1 - \frac{m_{bareQ}}{M_Q^2} \right)^{1/2}, \tag{5}
\]

that is,

\[
\text{Intercept} = \left[ m_l + M_Q \left( 1 - \frac{m_{bareQ}^2}{M_Q^2} \right) \right]^2 ,
\]

where \( m_{bareQ} = 1.275 GeV \) and 4.18GeV are the bare masses of the heavy quark \( Q = c, b \), respectively.

It is useful to view Eq. (4) as an extension of the mass relation suggested by Selem and Wilczek [43] for the singly heavy baryons,

\[
M - M_Q = \sqrt{\frac{\alpha}{2}} + 2^{1/4} k \frac{\beta}{L^{1/4}}, \tag{6}
\]

which implies \( (M - M_Q)^2 \sim (\alpha/2) L \) when the orbital angular momentum of hadrons \( L \rightarrow \infty \) or the mass of light quark \( \mu \rightarrow 0 \), as examined in Ref. [25]. It agrees with the relation (2) with \( n = 0 \) up to a mass shift \( M \rightarrow M - M_Q \) or \( M \rightarrow M - O(M_Q) \).

In the light of Ref. [43], the established states of mesons and baryons can be accommodated with appropriate quantum numbers and approximate mass using the hypotheses of the loaded flux-tube (string) and the emergent diquark. In the case of the HL systems, the hypotheses predict (6) for the hadron mass relation with \( L \) (the orbital angular momentum) and \( \mu \) (the effective mass of the light quark or diquark). The relation (6) works well for the excited D/B mesons, as examined in Ref. [25], but evidently, suffers from the singularity \( (L = 0, \text{S-wave states}) \) occurred in the second term in its right hand side (RHS). One way to avoid the singularity may be to view the \( \kappa \)-term in Eq. (6) as a subleading correction in the large \( L \) expansion. This can be done by going back to the picture of loaded string (LS) and rederiving the mass relation using expansion analysis of the LS model as in [43], but with a different small parameter. The result is [45] (Appendix A)

\[
(E - M_Q)^2 = \pi TL + a_0, \tag{7}
\]

with \( a_0 = (m_l + M_Q v_Q^2) \) depending upon \( v_Q \) and two effective masses of quarks, defined by

\[
M_Q = \frac{m_l}{\sqrt{1 - v_Q^2}}, m_l = \frac{m_q}{\sqrt{1 - v_q^2}}, \tag{8}
\]

(\( m_q \) is the bare mass of the light quark \( u \) or \( d, s \)).

The relation (7) was derived in Refs. [45, 48] for the singly heavy baryons. A phenomenological study [25] of the relation (7) was made more recently for both heavy-light mesons and singly heavy baryons, favoring (7) to be a mass relation mapping experimental data for all heavy-light hadron systems. When \( L \) is very large, Eq. (7) becomes \( E - M_Q \approx \sqrt{\alpha L}/2 + a_0/\sqrt{2\alpha L} \), quite similar to the Selem-Wilczek relation (6), whereas it avoids the singularity when \( L \rightarrow 0 \). For similar discussions for the singly heavy baryons see Ref. [45]. Considering the loaded string model is classical and valid only in the quasi-classical region (with large quantum number), it is expected that (7) needs quantum correction far from the quasi-classical region (e.g., \( L \rightarrow 0 \)). However, the full quantum treatment is nontrivial for the low-lying hadrons.

While the quasi-classical model can not give the intercept \( \alpha(0) \) or \( a_0 \) of the trajectory, which was rooted in a quantum effect of the HL system, the fact that Eqs. (2) and (1) with constant intercept \( a_0(c) \) maps the excited mass of light mesons quite well is very impressing [42, 44, 47]. Therefore, it is hoped that a modified version of the quasi-classical prediction may fit the mass spectrum in full quantum theory, if appropriately using the Regge phenomenology of hadrons [42, 44]. In fact, in the string picture, the full quantum treatment provides merely the “quantum defect” \( \alpha(0) \) in \( L [49] \).
Our approach for meson excitations is to find a Regge-like mass relation in the quasi-classical region (at large-\(L\) or \(n\)) at first, and then extrapolate it to the lower-\(L\) region by taking into account the enhancement of the binding effect between the heavy-quark and strange quark appropriately. This extrapolation relies on the empirical fact that the hadronic trajectories are nearly linear even for lower excitations, though the trajectory parameters may be flavor-dependent weakly.

To account for the measured mass data of the heavy-light mesons comprehensively, it is necessary to consider the radial excitations. The simplest way is to assume the Regge-like excitations. The simplest way is to assume the Regge-like nonstrange \(\bar{c}n\) and \(\bar{b}n\) mesons for which \(m_l/M_Q\) is small. This is not the case for the strange \(\bar{c}s\) and \(\bar{b}s\) mesons. Qualitatively, it accounts for phenomenological energies of heavy quark's binding with the strange quarks [56, 57], approximately linear in the effective reduced mass \(\mu_Q\) of the heavy quark and antiquark [58].

(i) The flavor-dependence of the mass relation, which is embodied in the effective mass of quarks, is incorporated to the mass-shifted Regge-like relation. The correction to the mass relation (7) due to the replacement (3) is tiny for the nonstrange \(\bar{c}n\) and \(\bar{b}n\) mesons for which \(m_l/M_Q\) is small. This is not the case for the strange \(\bar{c}s\) and \(\bar{b}s\) mesons. Qualitatively, it accounts for phenomenological energies of heavy quark's binding with the strange quarks [56, 57], approximately linear in the effective reduced mass \(\mu_Q\) of the heavy quark and antiquark [58].

(ii) The slope ponderation (\(\pi : 2\)) between radial and angular trajectories is examined successfully (Section III) and enables us to accommodate the newly-observed \(D_J(3000)\) and \(D^*_J(3000)\).

(iii) A prefactor \(k\) for the light quark mass \(m_l\) is added phenomenologically when the excited states of the HL mesons are involved. It employs to describe the dressing effect reduction of the light quark when it is highly excited.

### III. NUMERICAL PLOTS AND TEST OF THE MASS RELATIONS

To confront the mass relation (4) with the experimental spectra of the HL mesons, we list, in Table II and III, the all observed charmed and charmed strange mesons and their masses, and the corresponding predictions by the relativistic quasipotential model [47]. In this work, some mesons, though they are listed, do not enter during computing spin-averaged masses as their mass may be shifted due to the near-threshold effects [21]. For example, the mesons \(D^*_s(2317)\) and \(D_s(2460)\) are not used when spin-averaging of the \(1P\) \(D_s\) meson masses as they are light abnormally in the light of the quark models, due probably to the near-threshold effects or the exotic natures of these mesons (see [21], for recent review).

With the data in Table II and Table III, one can obtain the spin-averaged (spin-AV) masses by

\[
\overline{M}_{nl} = \frac{1}{N_{nl}} \sum (2J + 1)\overline{M}_{nl}^{\text{Exp}}(J)
\]

and then map the relation (4) of them by using the \(\chi^2\) fitting,

\[
\chi^2 = \frac{1}{N} \sum (\overline{M}_{nl} - \overline{M}_{nl})^2,
\]

in which the estimated mass, from the proposal (4), is given explicitly by

\[
M_{nl} = \frac{M_Q}{1 + k_{13} m_l/M_Q} \left[ \sqrt{\pi b \left( L + \frac{m_l}{2} \right)} + \left( m_l + M_Q \left( 1 - \frac{m_l^2}{M_Q^2} \right) \right) \right]
\]

Here, \(N = 13\) corresponds to the available number of the spin-averaged data chosen from the Table I and II.

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**TABLE I**: The predictions or estimates of the ratio factor \(\beta/\alpha\) in the existing literatures quoted.

| Refs. | System          | \(\beta/\alpha\) | Method (framework)         |
|-------|-----------------|-----------------|---------------------------|
| [50]  | Light-light      | 1               | Old dual amplitudes       |
| [51]  | Light-light      | 1               | Old dual amplitudes       |
| [52]  | Light-light      | 1               | Old dual amplitudes       |
| [29]  | Light-light      | 2               | Semiclassical quantization (String) |
| [30, 31] | Light-light | 1               | Phenomenological analysis (Exp.) |
| [32]  | Light-light      | \(\pi/2\)       | Semiclassical Analysis (Potential QM) |
| [46]  | Heavy-light      | \(\sqrt{2} - \sqrt{6}\) | Semiclassical Analysis (Potential QM) |
| [33]  | Light-light      | \(\approx 1\)   | Linear fit (Exp.)         |
| [36]  | Light-light      | \(\approx 1.23\) | Linear fit (Exp.)         |
| [34, 53] | Light-light       | 1               | Holographic QCD           |
| [54]  | Light-meson      | 1.3             | Relativistic quasipotential (QM) |
| [47]  | Heavy-light      | 1.4             | Relativistic quasipotential (QM). |
| [55]  | Heavy baryon     | 1.5             | Relativistic quasipotential (quark-diquark) |
| [45]  | Heavy baryon     | 1.57            | The \(v_0\)-expansion (Relativistic String) |

The predictions of the ratio \(\beta/\alpha\) are about 1 mostly in the light-light mesons while it is close to 1.5 in the heavy-light mesons in the literatures cited. In the case of heavy mesons, the actual value of the ratio relies on how much the hadron mass is shifted, as the mass squared \(M^2\), instead of the shifted mass squared \((M - m_0)^2\), is nonlinear in \(L\), with slightly larger slope in the Regge plot near the low-\(L\) excitations. Given the limited number of observed states, it of interest to explore the flavor-dependence of the trajectory parameters in the heavy meson case. The feature of our proposal (4) lies in:
TABLE II: The observed masses (in MeV) of the charmed and charmed strange mesons [1]. The some of quantum numbers indicated by question marks are quark model predictions, which has not been established experimentally. The errors less than 5 MeV are not indicated.

| State $J^P$ | Meson | Mass | EFG [47] | Meson | Mass | EFG [47] |
|------------|-------|------|------|-------|------|------|
| $1^3S_0$ 0$^-$ | $D^*$ | 1869.7 | 1871 | $D_1$ | 1968.3 | 1969 |
| $1^3S_1$ 1$^-$ | $D^*(2010)^+$ | 2010.3 | 2010 | $D^*_1(J^P?)$ | 2112.2 | 2111 |
| $1^3P_0$ 0$^+$ | $D^*_1(2400)^+$ | 2351(7) | 2406 | $D^*_0(2317)$ | 2317.7 | 2509 |
| $1^3P_1$ 1$^+$ | $D_1(2430)^+$ | 2427(40) | 2469 | $D_1(2460)$ | 2459.5 | 2574 |
| $1^3P_2$ 2$^+$ | $D^*_1(2460)^+$ | 2465.4 | 2460 | $D^*_2(2573)$ | 2569.1 | 2571 |
| $2^1S_0$ 0$^-$ | $D(2550)^0 [J^P?]$ | 2564(20) | 2581 | | | 2688 |
| $2^1S_1$ 1$^-$ | $D^*(2640)^+[J^P?]$ | 2637(6) | 2632 | $D^*_1(2700)$ | 2708.3 | 2731 |
| $1^3D_0$ 1$^-$ | | 2788 | | $D^*_1(2860)$ | 2859(27) | 2913 |
| $1D_2$ 2$^-$ | | 2850 | | | | 2961 |
| $1D_2$ 2$^-$ | $D(2740)^0 [J^P?]$ | 2737(12) | 2806 | | | 2931 |
| $1^3D_1$ 3$^-$ | $D^*_1(2750)$ | 2763.5 | 2863 | $D^*_1(2860)$ | 2860(7) | 2971 |
| $2^3P_0$ 0$^+$ | | 2919 | | | | 3054 |
| $2P_1$ 1$^+$ | | 3021 | | | | 3154 |
| $2P_2$ 2$^+$ | | 2932 | $D_{1J}(3040)[J^P?]$ | 3044$^{+31}_{-29}$ | 3067 |
| $3^3P_0$ 0$^+$ | | 3346 | | | | 3513 |
| $3P_1$ 1$^+$ | | 3461 | | | | 3618 |
| $3P_2$ 1$^+$ | | 3365 | | | | 3519 |
| $3^3P_2$ 2$^+$ | $D(3000)^0 [J^P?]$ | 3214(60) | 3407 | | | 3580 |

TABLE III: The observed masses of the bottomed and bottomed strange mesons [1]. The some of quantum numbers as shown are quark model predictions. The mass is in MeV. The errors less than 5 MeV are not indicated.

| State $J^P$ | Meson | Mass | EFG [47] | Meson | Mass | EFG [47] |
|------------|-------|------|------|-------|------|------|
| $1^3S_0$ 0$^-$ | $B^0$ | 5279.6 | 5280 | $B_s$ | 5366.9 | 5372 |
| $1^3S_1$ 1$^-$ | $B^*$ | 5324.7 | 5326 | $B^*_1$ | 5415.4 | 5414 |
| $1^3P_0$ 0$^+$ | $B^*_1(5732)[J^P?]$ | 5698(8) | 5749 | | | 5833 |
| $1P_1$ 1$^+$ | | 5774 | | $B^*_s(5850)[J^P?]$ | 5853(15) | 5865 |
| $1P_2$ 1$^+$ | $B^*_1(5721)^0$ | 5726.0 | 5723 | $B^*_1(5830)$ | 5828.6 | 5831 |
| $1^3P_2$ 2$^+$ | $B^*_2(5747)^0$ | 5739.5 | 5741 | $B^*_2(5840)$ | 5839.9 | 5842 |
| $2^3S_0$ 0$^-$ | $B_s(5840)^0 [J^P?]$ | 5863(9) | 5890 | | | 5976 |
| $2^3S_1$ 1$^-$ | $B^*_s(5970)^0 [J^P?]$ | 5971(5) | 5906 | | | 5992 |
TABLE IV: The effective masses of quarks determined via mapping the Table I and II. The mass is in GeV and $b$ is in GeV$^2$; $k = 0.551$.

| Parameters      | $M_f$ | $M_b$ | $m_n$ | $m_s$ | $b(c\bar{n})$ | $b(c\bar{s})$ | $b(b\bar{n})$ | $b(b\bar{s})$ | $\chi_{SM}$ |
|-----------------|-------|-------|-------|-------|---------------|---------------|---------------|---------------|--------------|
| This work       | 1.46  | 4.52  | 0.31  | 0.49  | 0.264         | 0.314         | 0.306         | 0.372         | 0.0116       |
| EFG [47]        | 1.55  | 4.88  | 0.33  | 0.5   | 0.64/0.58     | 0.68/0.64     | 1.25/1.21     | 1.28/1.23     |              |

TABLE V: The trajectory parameters ($\alpha', \alpha_0$) in (4) and predicted by Ref. [47]. The unit of the $\alpha'$ is in GeV$^{-2}$.

| Traj. Parameters | $c\bar{n}$(natural $J^P$) | $c\bar{s}$(natural $J^P$) | $b\bar{n}$(natural $J^P$) | $b\bar{s}$(natural $J^P$) |
|------------------|---------------------------|---------------------------|---------------------------|---------------------------|
| This work($\alpha', \alpha_0$) | $(1.21, -0.52)$          | $(1.01, -0.72)$          | $(1.04, -0.97)$          | $(0.86, -1.13)$          |
| EFG ($\alpha', \alpha_0$) [47]    | $(0.494, -1.00(4))$      | $(0.469, -1.10(4))$      | $(0.254, -6.30(36))$     | $(0.249, -6.43(51))$     |
|                  | $(0.548, -3.21(12))$     | $(0.497, -3.16(12))$     | $(0.263, -8.77(47))$     | $(0.259, -8.87(58))$     |

TABLE VI: The masses of the charmed and charmed strange mesons [1]. The mass is in MeV.

| $J^P$     | Meson   | Mass   | CDLLM [25] | Exp. (Spin-Av.) | Meson   | Mass   | CDLLM [25] | Exp. (Spin-Av.) |
|-----------|---------|--------|------------|-----------------|---------|--------|------------|-----------------|
| (0, 1)$^-$| $D(1S)$ | 1964   | 1910.6     | 1975.1          | $D_s(1S)$| 2071   | 1991.6     | 2076.2          |
| (0, 1, 2)$^+$| $D(1P)$| 2430   | 2441.6     | 2436(12)        | $D_s(1P)$| 2532   | 2516.4     | 2556.4          |
| (0, 1)$^-$| $D(2S)$ | 2624   | 2619(10)   | 2733            | $D_s(2S)$| 2733   | 2708       |                 |
| (1, 2, 3)$^-$| $D(1D)$| 2752   | 2758.3     | 2752(7)         | $D_s(1D)$| 2868   | 2843.5     | 2860(13)        |
| (0, 1, 2)$^+$| $D(2P)$| 2908   | 3007.7     | 3214(60)        | $D_s(2P)$| 3288   |            |                 |

TABLE VII: The masses of the bottomed and bottomed strange mesons [1]. The mass is in MeV.

| $J^P$     | Meson   | Mass   | CDLLM [25] | Exp. (Spin-Av.) | Meson   | Mass   | CDLLM [25] | Exp. (Spin-Av.) |
|-----------|---------|--------|------------|-----------------|---------|--------|------------|-----------------|
| (0, 1)$^-$| $B(1S)$ | 5320   | 5303.1     | 5313.4          | $B_s(1S)$| 5412   | 5390.9     | 5403.3          |
| (0, 1, 2)$^+$| $B(1P)$| 5730   | 5732.6     | 5730            | $B_s(1P)$| 5840   | 5834.5     | 5840            |
| (0, 1)$^-$| $B(2S)$ | 5917   | 5944(6)    | 6039            | $B_s(2S)$| 6175   | 6129.4     |                 |
| (1, 2, 3)$^-$| $B(1D)$| 6044   | 6009.2     |                 | $B_s(1D)$| 6459   | 6366.9     |                 |
| (0, 1, 2)$^+$| $B(2P)$| 6199   | 6230.2     |                 | $B_s(2P)$| 6606   |            |                 |
| (0, 1)$^-$| $B(3S)$ | 6342   |            |                 | $B_s(3S)$| 6744   |            |                 |
| (2, 3, 4)$^+$| $B(1F)$| 6308   |            |                 | $B_s(1F)$| 6606   |            |                 |
| (1, 2, 3)$^-$| $B(2D)$| 6443   |            |                 | $B_s(2D)$| 6744   |            |                 |
| (0, 1, 2)$^+$| $B(3P)$| 6571   |            |                 | $B_s(3P)$| 6744   |            |                 |
The mapped values of the parameters are listed in Table IV and that of the ensuing trajectory parameters determined by the RHS of the relation (4) are listed in Table V. The trajectory parameters $(\alpha', a_0)$ in Table V are related to the parameters in Table IV through

$$\alpha' = \frac{1}{\pi b}, \quad a_0 = \frac{1}{\pi b} \left( m_q + M_Q - m_{bareQ}/M_Q \right)^2.$$  

(12)

As is shown in Table IV, the mapped effective masses of the quarks in this work agree remarkably with that in the quark model [47]. According to the plots [25] for the HL mesons, the Regge-like relation fitting of the observed masses of the HL mesons by Eq. (7) yields a close but different trajectories (CDLLM) in contrast with that of Ref. [47]. We show in Table VI the masses of the $D/D_s$ mesons determined by Eq. (4) (denoted as this work) and by Eq. (7) (denoted as CDLLM) in Ref. [25], and the experimental masses of the spin-averaged, and in Table VII the corresponding masses of the $B/B_s$ mesons by two equations and the experimental masses of the spin-averaged. The comparisons are plotted in Figs. 1-2 for the Table VI and in Figs. 3-4 for the Table VII, correspondingly.

We end this section by giving the following remarks:

(1) As far as the spin-averaged spectra concern, the observed excited states of the $D/D_s$ and $B/B_s$ mesons can be reasonably described by the mass relation (11), with the slope parameters depending weakly upon the flavors.

(2) Once determined through the known data, the numerical values of the parameters (the effective mass $m_i(i = n, s, c, b)$ of quarks and the slope $b$) can be used to predict the spin-averaged masses of the higher excitations of the HL mesons, as shown in Table VI and VII. For instance, the mass of the $D(2P)$ is about 2908MeV and that of the $D(2D)$ about
3148MeV.

(3) The relation (4) gives a straightforward way to determine the effective mass of quarks, which is nontrivial in the potential quark models. The last term $a_0$ in Eq. (7), when divided by $\pi T$, may play the role of intercept of the trajectory considered, which embodies the short-distance QCD correction to the QCD string model [60].

IV. THE QCD STRING (FLUX TUBE) PICTURE

We show in this section how the general mass relation (4) is related to the QCD string (flux tube) picture. For this, we rewrite the relativistic Hamiltonian [42, 43] of the quark-antiquark bound system as

$$H = \sqrt{\mathbf{p}^2 + m_Q^2} + \sqrt{\mathbf{p}^2 + m_q^2} + V_{\text{string}},$$

(13)

$$V_{\text{string}} = \frac{T}{\omega} [\arcsin(\omega r_q) + \arcsin(\omega r_Q)]$$

(14)

where $V_{\text{string}}$ is the relativistic energy of string with constant tension $T$, tied to the heavy quark with mass $m_Q$ and light antiquark with mass $m_q$. Here, $v_i = \omega r_i (i = Q, q)$ denote the velocity of the string end $i$ to which the quark $i$ is tied, and $\omega$ stands for the angular velocity of the rotating system.

In the light of this work, we attribute the success of the LS picture in [43] to the fact that it appropriately encodes the nonperturbative behavior of QCD [30, 31, 40, 52, 61, 62] (See [25, 44, 45] for recent study). Since states with the large quantum numbers are semiclassical, in which the antiquark is mostly at large distances from the quark, their energy levels obey the WKB quantization condition or semiclassical mass relation both in the large limit of the quantum number and in the massless limit of the light quark, one can incorporate the short-distance correction to string-like interquark interactions. For similar discussion see Ref. [63] for the hydrogen-like atoms. We will supply the LS picture with the short-distance binding effect between heavy (anti)quark and strange quarks to improve the Regge-like mass relation (7) in the low-lying region.

When choosing $v_Q$, the velocity of the string end $\bar{Q}$, as the expanding parameter, which is conserved in the heavy quark limit, one can show in the LS picture (see Appendix A)

$$(E - M_Q)^2 = \pi \sigma L + \left[ m_l + M_Q \left( 1 - \frac{m_l^2}{M_Q^2} \right) \right]^2,$$

(15)

This gives the relation (7). Considering that this is a mass relation both in the large limit of the quantum number and in the massless limit of the light quark, one can incorporate the short-distance effect due to the two-body binding interaction using the replacement (3), so that

$$M_{nl} = M_Q - \tilde{\mu} Q + \sqrt{\pi b L} + a_0.$$

(16)

For the angularly excited HL mesons, (16) gives the Regge-like relation (4) (with $n = 0$). For the radially excited HL mesons ($n > 0$), we use the quasi-classical WKB approach for the string model Hamiltonian (13).

Since $\arcsin(\omega r_q)/\omega$ behaves linear in $r_q$ when the velocity $v_q = \omega r_q$ of the light quark approaches the speed of light: $v_q \approx 1$, the potential (14) leads to an asymptotically linear confining potential for large $r_q$. This is, however, not true for the heavy quark part of potential since since $v_Q < 1$ notably. Up to the leading order of Eq. (A5), one has for the heavy quark, $T/\omega = M_Q \omega r_Q$, which yields

$$\omega = \sqrt{\frac{T}{M_Q r_Q}}, v_Q = \sqrt{\frac{T r_Q}{M_Q}},$$

(17)

$$\frac{Tr_Q}{v_Q} \arcsin(v_Q) = T r_Q + \frac{(T r_Q)^2}{6M_Q}.$$ (18)

For the string energy part of the light quark, one has

$$\frac{Tr_q}{v_q} \arcsin(v_q) = \frac{\pi}{2} T r_q.$$ When $v_q \approx 1$. (18)

It follows from (17) and (18) that the string energy for high-$L$ states becomes

$$V_{\text{string}}(r_q, r_Q) = \frac{\pi}{2} T r_q + M_Q + \frac{(T r_Q)^2}{6M_Q}.$$ (19)

Comparing (19) with the string energy for the light-light mesons,

$$V_{\text{string}}(r_q, r_q) \approx \frac{\pi}{2} T r_q + \frac{\pi}{2} T r_q = \frac{\pi}{2} T r,$$

one sees that the tensions for light quark $q$ and for the heavy antiquark $\bar{Q}$ differ by a factor of $\pi/2$. Using $r_q = r - r_Q$, the total Hamiltonian (13) of the bound system becomes, up to a $1/M_Q$ correction,

$$H = M_Q + \sqrt{\mathbf{p}^2 + m_q^2} + \frac{\pi}{2} T r + T r_Q \left[ 1 - \frac{\pi}{2} \right].$$ (20)

Given that the recoil effect has been taken into account in (3), one can choose $r_Q \to 0$ in the heavy quark limit, so that the problem becomes that of the light quark with Hamiltonian

$$H = M_Q + \sqrt{\mathbf{p}^2 + \omega^2 m_q^2} + \frac{\pi}{2} T r,$$ (21)

where the chiral limit($m_q \to 0$) has been taken. Assuming the orbital angular momentum $L$ of a meson is dominated by that of the string, which is used in the native rotating string picture [39–41],

$$l_q = \frac{m_q v_q^2/\omega}{\sqrt{1 - v_q^2}} = \omega m_q r_q^2 \ll L_{\text{string}},$$

one can take $l_q \to 0$ so that $|\mathbf{p}|^2 = p_r^2 + (l_q (l_q + 1))/r^2 \approx p_r^2$. This reduces the radial version of WKB approximation to that of one-dimensional effectively, in which the CM of meson is located at $r_Q$ and the light quark $q$ moving in the force field of
\[ \pi r/2, \text{ with } r \text{ ranging from 0 to } \infty. \] The WKB quantization condition for (21) gives [60, 64]

\[ 2 \int_{0}^{(E-M_Q)/a} (E - M_Q - a|x|) dx = \pi(n + b), \quad (22) \]

with \( a = \pi T/2 \), that is,

\[ (E - M_Q)^2 = \pi a(n + b) = \pi T \left( \frac{\pi}{2} n + \frac{\pi}{2} b \right), \quad (23) \]

where \( r = (E - M_Q)/a \) corresponds to the physically possible turning point. This confirms the slope ratio \( \pi/2 \) in the relation (4) by simply comparing (23) with the relation (7) for the angular excitations.

V. SUMMARY AND DISCUSSIONS

We study the Regge-like spectra of singly heavy mesons, that is, the \( D/D_s \) and \( B/B_s \) mesons, by proposing a general Regge-like mass relations in which the slope ratio between the radial and angular-momentum Regge trajectories is \( \pi/2 \) and the binding effect of heavy quark and flavored light quarks has been taken into account. We test the proposed mass relation against the spin-averaged observed data of the singly heavy mesons in their radially and angularly excited states and find that the agreement is remarkable for the reasonable values of effective quark masses. An argument is outlined for the mass relations using semiclassical WKB analysis of the relativistic interquark dynamics in the QCD string (flux tube) picture.

Some new predictions are made for more excited excitations. For instance, the \( D(3000)^0 \) is more likely to be \( 3P \) state, and the \( B_j(5840) \) and \( B_j(5970) \) can be the candidates of \( 2S \). It is expected that the forthcoming Belle II and LHCb experiments can test our predictions.

We note that the limitation of our mass relation (4) may stem from that it simply assumes the short-distance binding between quarks to be linear in the reduced quark mass, which is confirmed only for the ground S-wave states [56, 57]. To embody the possible deviations from the linearity for the highly-excited HL mesons, a simple prefactor \( k \) for the light quark mass is employed which may vary with \( L \) or \( n \). We have checked that this dependence is weak in the case of the relation (4). Moreover, our predictions can not distinguish the spin multiplets due to neglecting of the spin-dependent interactions, as shown in Table VI and Table VII. The predictions by our relation (4), however, is generic in that they exempt the subtle influences due to the near-threshold effects hidden in the strange HL meson families\( (D_s^*(2317) \) and \( D_s(2460) \) are avoided to use when when spin-averaging). We await the further explorations in the future study.

ACKNOWLEDGMENTS

D. J thanks Bing Chen, X Liu and Atsushi Hosaka for many useful discussions and valuable comments. D. J is supported by the National Natural Science Foundation of China under the no. 11565023 and the Feitian Distinguished Professor Program of Gansu (2014-2016). W. D is supported by Undergraduate Innovative Ability Program 2018 (Grants No. CX2018B338).

APPENDIX A

For the orbital excitations, the classical energy (13) and orbital angular momentum for the loaded string can be written as [42, 43, 65]

\[ E = \frac{m_Q}{\sqrt{1 - v_Q^2}} + \frac{m_l}{\sqrt{1 - v_l^2}} + \frac{T}{\omega} [\arcsin(v_q) + \arcsin(v_Q)], \quad (A1) \]

\[ L = \sum_{i=Q,q} \frac{m_i v_i^2}{\omega} + \sum_{i=Q,q} \int_0^{v_i} \frac{u^2 du}{\sqrt{1 - u^2}}, \quad (A2) \]

where \( m_{Q,q} \) are the bare masses of the heavy and light quarks and \( v_i = \omega r_i (i = Q, q) \). The Semen-Wilczek relation (6) follows from [43] upon the \( (m_i/\omega)/T \)-expansion of (A1) and (A2). Applying (8), one has

\[ E = M_Q + m_l + \frac{T}{\omega} [\arcsin(v_q) + \arcsin(v_Q)], \quad (A3) \]

The orbital angular momentum \( L \) of the system is [43, 65]

\[ L = \frac{1}{\omega} (M_Q v_Q^2 + m_l v_l^2) + \frac{T}{\omega} \sum_{i=Q,q} [\arcsin(v_i) - v_i \sqrt{1 - v_i^2}], \quad (A4) \]

in which the last term is the orbital angular momentum due to the string rotating.

The boundary condition of string at ends with heavy quark gives

\[ \frac{T}{\omega} = \frac{m_Q v_Q}{1 - v_Q^2}, \quad (A5) \]

which implies,

\[ \frac{T}{\omega} = \frac{M_Q v_Q}{\sqrt{1 - v_Q^2}} \approx M_Q v_Q + \frac{1}{2} M_Q v_Q^3. \quad (A6) \]

Expanding Eqs. (A3) and (A4) up to \( v_i^4 \),

\[ E = M_Q + m_l + \frac{\pi \sigma}{2 \omega} \left( v_Q - \frac{m_l}{m_l} \right) + \frac{1}{6} M_Q v_Q^3 + O[v_Q^5]. \quad (A7) \]

\[ \omega L = m_l + M_Q v_Q^2 + \frac{T}{\omega} \left( \frac{\pi}{4} - \frac{m_l}{m_l} \right) + \frac{T}{3 \omega} M_Q v_Q^3 + O[v_Q^5]. \quad (A8) \]

With the help of Eq. (A6), Eqs. (A7) and (A8) become
which, upon eliminating $\omega$ and ignoring the small term $m_q/m_t$, leads to

$$(E - M_Q)^2 = \pi \sigma L + \left( m_t + \frac{P_Q^2}{M_Q} \right)^2 - 2m_Q P_Q, \quad (A9)$$

with $P_Q = M_Q v_Q = m_q v_q$ the conserved momentum of the heavy quark relativistically. Rewriting the velocity $\frac{v}{Q} = 1 - x_Q^2$ in terms of the mass ratio $x_Q = m_q/M_Q$, Eq. (A9) yields (7) or (15), where the bare mass term $2m_q P_Q$ has been ignored.
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