Asymptotic regularity of powers of ideals of points in a weighted projective plane

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To the memory of Professor Masayoshi Nagata

Abstract In this article we study the asymptotic behavior of the regularity of symbolic powers of ideals of points in a weighted projective plane. By a result of Cutkosky, Ein, and Lazarsfeld, regularity of such powers behaves asymptotically like a linear function, which is deeply related to the Seshadri constant of a blowup. We study the difference between regularity of such powers and this linear function. Under some conditions, we prove that this difference is bounded or eventually periodic.

As a corollary, we show that if there exists a negative curve, then the regularity of symbolic powers of a monomial space curve is eventually a periodic linear function. We give a criterion for the validity of Nagata’s conjecture in terms of the lack of existence of negative curves.

1. Introduction

Suppose that $H$ is an ample $\mathbb{Q}$-Cartier divisor on a normal projective variety $V$ and $\mathcal{I}$ is an ideal sheaf on $V$. Let $\nu: W \to V$ be the blowup of $\mathcal{I}$. Let $E$ be the effective Cartier divisor on $W$ defined by $\mathcal{O}_W(-E) = \mathcal{I}\mathcal{O}_W$. The $s$-invariant, $s_{\mathcal{O}_V(H)}(\mathcal{I})$, is defined by

$$s_{\mathcal{O}_V(H)}(\mathcal{I}) = \inf \{ s \in \mathbb{R} \mid \nu^*(sH) - E \text{ is an ample } \mathbb{R}\text{-divisor on } W \}.$$ 

The reciprocal, $1/s_H(\mathcal{I})$, is the Seshadri constant of $\mathcal{I}$.

Examples in [3] and [4] show that $s_H(\mathcal{I})$ can be irrational, even when $\mathcal{O}_V(H) \cong \mathcal{O}_{\mathbb{P}}(1)$ on ordinary projective space.

Suppose that $K$ is a field, and suppose that $a_0, \ldots, a_\pi$ are positive integers. Let $S = K[x_0, \ldots, x_\pi]$ be a polynomial ring graded by the weighting $\text{wt}(x_i) = a_i$ for $0 \leq i \leq \pi$. Let $\mathfrak{m}$ be the graded maximal ideal of $S$. Let $\mathbb{P} = \mathbb{P}(a_0, a_1, \ldots, a_\pi) = \text{proj}(S)$ be the associated weighted projective space. $\mathbb{P}$ is a normal projective variety. $\mathbb{P}$ is isomorphic to a weighted projective space in which $\text{gcd}(a_0, a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_\pi) = 1$ for $0 \leq i \leq \pi$ (see [9], [10]).
We suppose through most of this article that $a_0, \ldots, a_n \in \mathbb{Z}_+$ satisfy the condition that $\gcd(a_0, a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n) = 1$ for $0 \leq i \leq n$. With this assumption on the $a_i$, there exists a Weil divisor $H$ on $\mathbb{P}$ such that $O_\mathbb{P}(r) \cong O_\mathbb{P}(rH)$ is a divisorial sheaf of $O_\mathbb{P}$ modules (reflexive of rank 1) for all $r \in \mathbb{Z}$ (see [24]).

Let $M$ be a finitely generated, graded $S$-module. The local cohomology modules $H^i_s(M)$ are naturally graded. The regularity of $M$ is defined (see [12]) by

$$\text{reg}(M) = \max\{i + j \mid H^i_s(M)_j \neq 0\}.$$ 

Suppose that $I \subset S$ is a homogeneous ideal. Let $I^{\text{sat}}$ be the saturation of $I$ with respect to the graded maximal ideal of $I$. Let $\mathcal{I}$ be the sheaf associated to $I$ on $\mathbb{P}$. Let $X = X(I) = \text{proj}(\bigoplus_{m \geq 0} I^m)$ be the blowup of $\mathcal{I}$, with natural projection $f : X \to \mathbb{P}$.

In Section 2, we develop the basic properties of regularity on weighted projective space and show that the theory of asymptotic regularity on ordinary projective space extends naturally to weighted projective space. For instance, the statement on ordinary projective space, proven in [5, Theorem 1.1] or in [20], extends to show that $\text{reg}(I^m)$ is a linear function for $m \gg 0$. We also establish the following basic result, which generalizes the statement for ample line bundles proven in [4, Theorem B] to the $\mathbb{Q}$-Cartier Weil divisor $O_\mathbb{P}(1)$ on weighted projective space (with the $a_i$ pairwise relatively prime).

**Theorem 1.1**

We have

$$\lim_{m \to \infty} \frac{\text{reg}((I^m)^{\text{sat}})}{m} = s_{O_\mathbb{P}(1)}(\mathcal{I}).$$

In general, as commented above, this limit is irrational, so $\text{reg}((I^m)^{\text{sat}})$ is, in general, far from being a linear function.

We write $[x]$ for the greatest integer in a real number $x$. We may define a function $\sigma_I : \mathbb{N} \to \mathbb{Z}$ by

$$\text{reg}((I^m)^{\text{sat}}) = [ms_{O_\mathbb{P}(1)}(\mathcal{I})] + \sigma_I(m).$$

By Theorem 1.1, we have

$$\lim_{m \to \infty} \frac{\sigma_I(m)}{m} = 0.$$

An interesting question is to determine when $\sigma_I(m)$ is bounded. We do not know of an example where $\sigma_I(m)$ is not bounded.

In this article we study the case where $I \subset S = K[x, y, z]$ is the ideal of a set of nonsingular points with multiplicity (a fat point) in a weighted two-dimensional projective space $\mathbb{P} = \mathbb{P}(a, b, c)$ with $\text{wt}(x) = a$, $\text{wt}(y) = b$, $\text{wt}(z) = c$ and $a, b, c$ pairwise relatively prime. We also assume that $K$ is algebraically closed. Suppose that $P_1, \ldots, P_r$ are distinct nonsingular closed points in $\mathbb{P}(a, b, c)$ and that $e_i$ are positive integers. Let $I_{P_i} \subset S = K[x, y, z]$ be the weighted homogeneous ideal of
the point $P_i$, and let $I = \bigcap_{i=0}^{r} I_{P_i}$. Let $\mathcal{I} = \hat{I}$ be the sheafication of $I$ on $\mathbb{P}$, and define

$$s(I) = s_{\mathcal{O}_{\mathbb{P}}(1)}(\mathcal{I}).$$

Let $u = \sum_{i=1}^{r} e_i^2$. We have $s(I) \geq \sqrt{abcu}$. If $s(I) > \sqrt{abcu}$, then $s(I)$ is a rational number.

Nagata’s conjecture states that $s(I) = \sqrt{r}$ if $r \geq 9$, $e_i = 1$ for $1 \leq i \leq r$, and $P_1, \ldots, P_r$ are independent generic points in ordinary projective space $\mathbb{P}^2$ (see [26, p. 772]). Nagata proved this conjecture in [26, p. 772] in the case when $r$ is a perfect square as a critical ingredient in his counterexample to Hilbert’s fourteenth problem. A proof of Nagata’s conjecture in the case of an $r$ which is not a perfect square would give a set of points in $\mathbb{P}^2$ for which $s(I)$ is not rational. Some recent articles on regularity and $s$-invariants of points in $\mathbb{P}^2$ are [1], [8], [14], [17], [19], and [22].

Let $I^{(m)}$ be the $m$th symbolic power of $I$, which is also, in our situation, the saturation $(I^{(m)})^{\text{sat}}$ of $I^m$ with respect to the graded maximal ideal $\mathfrak{m}$ of $S$.

We prove the following asymptotic statements about regularity in Section 3.

A function $\sigma : \mathbb{N} \rightarrow \mathbb{Z}$ is bounded if there exists $c \in \mathbb{N}$ such that $|\sigma(m)| < c$ for all $m \in \mathbb{N}$. In Theorem 4.6, we prove the following.

Let $I$ be the ideal of a set of fat points in a weighted projective plane. Then $\sigma_1(m)$ is a bounded function.

If the graded $K$-algebra $\bigoplus_{m \geq 0} I^{(m)}$ is a finitely generated $K$-algebra, then $\text{reg}(I^{(m)})$ must be a quasi-polynomial for large $m$. A quasi-polynomial is a polynomial in $m$ with coefficients which are periodic functions in $m$. In general, $\bigoplus_{m \geq 0} I^{(m)}$ is not a finitely generated $K$-algebra. Some examples where this algebra is not finitely generated are given by Nagata’s theorem (see [26]), showing that it is not finitely generated when $r \geq 9$ is a perfect square and $r$ generic points in $\mathbb{P}^2$ are blown up. Goto, Nishida, and Watanabe [15] give examples of monomial primes $P(a, b, c)$ such that the symbolic algebra is not finitely generated.

A function $\sigma : \mathbb{N} \rightarrow \mathbb{Z}$ is eventually periodic if $\sigma(m)$ is periodic for $m \gg 0$. In Theorem 4.7, we prove the following.

Suppose that $s(I) > \sqrt{abcu}$ and that $K$ has characteristic zero or is the algebraic closure of a finite field. Then the function $\sigma_1(m)$ is eventually periodic.

An example, defined over a field $K$ which is of positive characteristic and is transcendental over the prime field, where $s(I) > \sqrt{abcu}$ but $\sigma(m)$ is not eventually periodic, is given in [5, Example 4.4]. In this example, constructed from 17 special points $P_i$ in ordinary projective space $\mathbb{P}^2$, $I = I_{P_1} \cap \cdots \cap I_{P_{13}} \cap I_{P_{14}} \cap \cdots \cap I_{P_{27}}$. We have $s(I) = 29/5 > \sqrt{abcu} = \sqrt{29}$ in this example.

Let $H$ be a Weil divisor on $\mathbb{P}$ such that $\mathcal{O}_{\mathbb{P}}(1) \cong \mathcal{O}_{\mathbb{P}}(H)$, and let $A = f^*(H)$. An effective divisor $D$ on $X$ such that $(D \cdot D) < 0$ is called a negative curve.
effective divisor $D$ such that $D \sim aA - mE$ for some positive integers $a$ and $m$ is called an $E$-uniform curve.

We establish the following in Corollary 4.8.

**Suppose that there exists an $E$-uniform negative curve on $X(I)$, and suppose that $K$ has characteristic zero or is a finite field. Then $s(I)$ is a rational number, and the function $\sigma_I(m)$ of Theorem 4.6 is eventually periodic.**

An important case of this construction is when $i = 1$ and $I = P(a, b, c)$ is the prime ideal of a monomial space curve. The ideal $P(a, b, c)$ is defined to be the kernel of the $K$-algebra homomorphism $K[x, y, z] \to K[t]$ given by $x \mapsto t^a$, $y \mapsto t^b$, $z \mapsto t^c$. As a corollary to Theorems 4.6 and 4.7, we have the following application, Corollary 4.9, to monomial space curves.

**Suppose that $I = P(a, b, c)$ is the prime ideal of a monomial space curve, and suppose that there exists a negative curve on $X(I)$. Then $s(I)$ is a rational number, and the function $\sigma_I(m)$ is eventually periodic.**

We do not know of an example of a monomial prime $I = P(a, b, c)$ where there does not exist a negative curve. This interesting problem is discussed in [21].

In Section 5, we give a criterion for the validity of Nagata’s conjecture in terms of the lack of existence of uniform negative curves on certain weighted projective planes.

2. Regularity on weighted projective space

In this section we define the regularity of a finitely generated graded module over a nonstandard graded polynomial ring.

Let $K$ be a field, and let $B = K[x_1, \ldots, x_s]$ be a graded polynomial ring with $\text{wt}(x_1) = d_1, \ldots, \text{wt}(x_s) = d_s$, where $d_1, \ldots, d_s$ are positive integers. Set $m = (x_1, \ldots, x_s)B$.

**Definition 2.1**

For a finitely generated $B$-module $M \neq 0$, we define $a_i(M)$, $\text{reg}(M)$, $\text{reg}_i(M)$, and $\text{reg}'(M)$ as follows:

$$a_i(M) = \begin{cases} \max \{j \in \mathbb{Z} \mid H^i_m(M)_j \neq 0\} & \text{if } H^i_m(M) \neq 0, \\ -\infty & \text{otherwise}, \end{cases}$$

$$\text{reg}(M) = \max \{i + j \mid H^i_m(M)_j \neq 0\} = \max \{a_i(M) + i \mid 0 \leq i \leq \dim M\},$$

$$\text{reg}_i(M) = \begin{cases} \max \{j \in \mathbb{Z} \mid \text{Tor}_i^B(M, B/m)_j \neq 0\} - i & \text{if } \text{Tor}_i^B(M, B/m) \neq 0, \\ -\infty & \text{otherwise}, \end{cases}$$

$$\text{reg}'(M) = \max \{\text{reg}_i(M) \mid i \geq 0\}.$$
It is not difficult to prove the following theorem (cf. [7, Theorem 3.5]). We omit a proof.

**THEOREM 2.2**

*With notation as above,*

\[
\text{reg}(M) = \text{reg}'(M) + s - \sum_{i=1}^{s} d_i.
\]

**REMARK 2.3**

Assume \(d_1 = \cdots = d_s = 1\). Let \(I\) be a homogeneous ideal of \(B\). Put \(\mathbb{P} = \text{proj}(B)\).

The regularity of a coherent \(\mathcal{O}_{\mathbb{P}}\)-module \(\mathcal{F}\) is

\[
\text{reg}(\mathcal{F}) = \min \{ l \mid H^i(\mathbb{P}, \mathcal{F}(j-i)) = 0 \text{ for all } j \geq l \text{ and } i > 0 \}.
\]

Let \(\mathcal{I}\) be the ideal sheaf on \(\mathbb{P}\) associated to \(I\). We have

\[
\text{reg}((I^m)_{\text{sat}}) = \text{reg}(I^m)
\]

for all \(m \geq 0\), as follows from [11, Theorem A4.1].

For each \(i \geq 0\), \(\text{reg}_i(I^m)\) is eventually linear on \(m\) by [5, Theorem 3.1].

Hence, \(\text{reg}(I^m)\) is eventually linear on \(m\) as in [5, Theorem 1.1(ii)].

On the other hand, there exists an example that \(a_2(I^m)\) is not eventually linear on \(m\) as follows. Let \(I\) be the ideal in [5, Example 4.4] (which was cited in the introduction). Then

\[
\text{reg}((I^m)_{\text{sat}}) = \max \{ a_2((I^m)_{\text{sat}}) + 2, a_3((I^m)_{\text{sat}}) + 3 \}
\]

\[
= \max \{ a_2(I^m) + 2, 0 \} = a_2(I^m) + 2
\]

since \(\text{reg}((I^m)_{\text{sat}}) = \text{reg}'((I^m)_{\text{sat}}) > 0\), and \(\text{reg}((I^m)_{\text{sat}})\) is not eventually linear in \(m\). Therefore, \(a_2(I^m)\) is not eventually linear in \(m\) in this case.

In Section 3, we consider the case when \(S = K[x, y, z]\) with \(\text{wt}(x) = a, \text{wt}(y) = b, \text{and wt}(z) = c\) for pairwise relatively prime positive integers \(a, b, c\), and \(I = I_{P_1}^{x_1} \cap \cdots \cap I_{P_r}^{x_r}\) with \(P_i\) distinct nonsingular points of \(\mathbb{P}(a, b, c)\). We then have

\[
\text{reg}((I^m)_{\text{sat}}) = \max \{ a_2((I^m)_{\text{sat}}) + 2, a_3((I^m)_{\text{sat}}) + 3 \}
\]

\[
= \max \{ a_2(I^m) + 2, 3 - a - b - c \} = a_2(I^m) + 2
\]

since \(\text{reg}((I^m)_{\text{sat}}) = \text{reg}'((I^m)_{\text{sat}}) + 3 - a - b - c > 3 - a - b - c\), and (with the notation defined in the introduction)

\[
a_2(I^m) = \max \{ n \in \mathbb{Z} \mid H^1(X, \mathcal{O}_X(nA - mE)) \neq 0 \}.
\]

**REMARK 2.4**

Let \(B_1 = K[x_1, \ldots, x_s]\) and \(B_2 = K[y_1, \ldots, y_s]\) be graded polynomial rings with \(\text{wt}(x_i) = d_i\) \((i = 1, \ldots, s)\) and \(\text{wt}(y_j) = d_j'\) \((j = 1, \ldots, s)\), where the \(d_i\)'s and \(d_j'\)'s are positive integers.
Let $\delta : B_1 \rightarrow B_2$ be a flat $K$-algebra graded homomorphism. Assume that $B_2/(x_1, \ldots, x_s)B_2$ is of finite length.

Set $m_1 = (x_1, \ldots, x_s)B_1$ and $m_2 = (y_1, \ldots, y_s)B_2$.

Let $M$ be a finitely generated graded $B_1$-module. Then

$$H^i_{m_1}(M) \otimes_{B_1} B_2 = H^i_{m_1B_2}(M \otimes_{B_1} B_2) = H^i_{m_2}(M \otimes_{B_1} B_2).$$

Here, set

$$\xi = \max\{n \in \mathbb{Z} \mid [B_2/m_1B_2]_n \neq 0\}.$$

By (4), we obtain

$$a_i(M \otimes_{B_1} B_2) = a_i(M) + \xi$$

for $i = 0, \ldots, \dim M$. Therefore,

$$\text{reg}(M \otimes_{B_1} B_2) = \text{reg}(M) + \xi.$$

From the above remark, we see that the statement on ordinary projective space, proven in [5, Theorem 1.1] or in [20], extends to show that $\text{reg}(I^m)$ is a linear function for $m \gg 0$ when $I$ is a homogeneous ideal with respect to our weighting.

**Remark 2.5**

Let $B_1 = K[x_1, \ldots, x_s]$ and $B_2 = K[y_1, \ldots, y_s]$ be graded polynomial rings with \(\text{wt}(x_i) = d_i\) \((i = 1, \ldots, s)\) and \(\text{wt}(y_j) = 1\) \((j = 1, \ldots, s)\), where the $d_i$'s are positive integers satisfying the condition $\gcd(d_1, \ldots, d_{i-1}, d_{i+1}, \ldots, d_s) = 1$ for $1 \leq i \leq s$.

Let $\delta : B_1 \rightarrow B_2$ be the $K$-algebra graded homomorphism satisfying $\delta(x_i) = y_i^{d_i}$ for $i = 1, \ldots, s$. Note that $\delta$ is flat, $B_2/(x_1, \ldots, x_s)B_2$ is of finite length, and

$$\xi = \max\{n \in \mathbb{Z} \mid B_2/(x_1, \ldots, x_s)B_2]_n \neq 0\} = \sum_{i=1}^{s} d_i - s.$$

Let $I$ be a homogeneous ideal of $B_1$. Then

$$(I^m)^{\text{sat}} \otimes_{B_1} B_2 = (I^m)^{\text{sat}}B_2 = (I^m B_2)^{\text{sat}}$$

for any $m > 0$. Thus, using (5), we have

$$\text{reg}((I^m B_2)^{\text{sat}}) = \text{reg}((I^m)^{\text{sat}}) + \sum_{i=1}^{s} d_i - s$$

for any $m > 0$. Therefore,

$$\lim_{m \to \infty} \frac{\text{reg}((I^m B_2)^{\text{sat}})}{m} = \lim_{m \to \infty} \frac{\text{reg}((I^m)^{\text{sat}})}{m}.$$

Here, set $X = \text{proj}(B_1)$, $Z = \text{proj}(B_2)$. By our condition on the $d_i$, there exists a Weil divisor $H$ on $X$ such that $\mathcal{O}_X(r) \cong \mathcal{O}_X(rH)$ is a reflexive, rank 1 sheaf of $\mathcal{O}_X$-modules for all $r \in \mathbb{Z}$ (see [24]). Let $Y$ (resp., $W$) be the blowups of $X$ (resp., $Z$) along the ideal sheaf $\mathcal{I} = \mathcal{I}$ (resp., $\mathcal{I}_Z = \mathcal{I}_B$). Then we have the
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following Cartesian diagram:

\[
\begin{array}{ccc}
W & \longrightarrow & Z \\
f \downarrow & & \downarrow \\
Y & \xrightarrow{\pi} & X
\end{array}
\]

Set \( E = \pi^{-1}(\text{proj}(B_1/I)) \). Let \( \ell \) be a positive integer such that \( \mathcal{O}_X(\ell) \) is invertible. (We can take \( \ell = \text{lcm}(d_1, \ldots, d_s) \)). By the projection formula, for any positive integers \( \alpha \) and \( \beta \), \( \mathcal{O}_Y(-\ell\beta E) \otimes \pi^*\mathcal{O}_X(\ell\alpha) \) is nef if and only if \( f^*(\mathcal{O}_Y(-\ell\beta E) \otimes \pi^*\mathcal{O}_X(\ell\alpha)) \) is also. Hence, \( \alpha/\beta \geq s_{\mathcal{O}_Z(1)(I)} \) if and only if \( \alpha/\beta \geq s_{\mathcal{O}_X(1)(I)} \).

Therefore, we obtain

(7) \( s_{\mathcal{O}_Z(1)(I)} = s_{\mathcal{O}_X(1)(I)} \).

On the other hand,

(8) \( \lim_{m \to \infty} \frac{\text{reg}((I^mB_2)^{\text{sat}})}{m} = s_{\mathcal{O}_Z(1)(I)} \mathcal{O}_Z \) by [4, Theorem B]. By (6), (7), and (8), we obtain

(9) \( \lim_{m \to \infty} \frac{\text{reg}((I^m)^{\text{sat}})}{m} = s_{\mathcal{O}_X(1)(I)} \),

which is the statement of Theorem 1.1.

3. Blowups of a weighted projective plane

Suppose that \( G \) is a subgroup of \( \mathbb{R} \). Then \( G_+ \) denotes the semigroup of positive elements of \( G \), and \( G_{\geq 0} \) denotes the semigroup of nonnegative elements of \( G \).

In this section, we suppose that \( K \) is an algebraically closed field and that \( a, b, c \in \mathbb{Z}_+ \) are pairwise relatively prime. Let \( \mathbb{P} = \mathbb{P}(a, b, c) \) be the corresponding weighted projective space. Suppose that \( P_1, \ldots, P_r \) are distinct nonsingular closed points in \( \mathbb{P}(a, b, c) \), and suppose that \( e_1, \ldots, e_r \in \mathbb{Z}_+ \).

The coordinate ring of \( \mathbb{P}(a, b, c) \) is the graded polynomial ring

\[
S = K[x, y, z] = \bigoplus_{n \geq 0} H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(n)),
\]

which is graded by \( \text{wt}(x) = a, \text{wt}(y) = b, \text{wt}(z) = c \). Let \( m = (x, y, z) \) be the graded maximal ideal of \( S \).

Some references on the geometry of weighted projective spaces are [10] and [24]. We have that \( \mathbb{P}(a, b, c) \) is a normal surface which is nonsingular, except possibly at the three points \( Q_1 = V(x, y), Q_2 = V(x, z), \) and \( Q_3 = V(y, z) \). Since \( a, b, c \) are pairwise relatively prime, there exists a Weil divisor \( H \) on \( \mathbb{P} \) such that \( \mathcal{O}_{\mathbb{P}}(r) \cong \mathcal{O}_{\mathbb{P}}(rH) \) is a reflexive, rank 1 sheaf of \( \mathcal{O}_{\mathbb{P}} \)-modules for all \( r \in \mathbb{Z} \) (see [24]). The canonical divisor on \( \mathbb{P} \) is \( \mathcal{O}_{\mathbb{P}}(-a - b - c) \). We have that \( \mathcal{O}_{\mathbb{P}}(\ell) \) is an ample invertible sheaf if \( \ell = \text{lcm}(a, b, c) \).

Suppose that \( L \) is a finitely generated graded \( S \)-module. Recall (see Section 2) that the regularity, \( \text{reg} L \), of \( L \) is the largest integer \( t \) such that there exists an index \( j \) such that \( H^t_m(L)_{t-j} \neq 0 \). We denote the sheaf associated to \( L \) on \( \mathbb{P} \) by \( \tilde{L} \).
For all \( j \in \mathbb{Z} \), we have a natural morphism of sheaves of \( \mathcal{O}_P \)-modules

\[
\Lambda : \hat{L}(j) := \hat{L} \otimes \mathcal{O}_P(j) \to \hat{L}(j)
\]

which is an isomorphism whenever \( \mathcal{O}_P(j) \) is Cartier.

Let \( I_{P_i} \subset K[x, y, z] \) be the weighted homogeneous ideal of the point \( P_i \), and let \( I = \bigcap_{i=0}^{r} I_{P_i} \). An important case is when \( i = 1 \), and \( I = P(a, b, c) \) is the prime ideal of a monomial space curve.

Let \( \mathcal{I} = \hat{I} \) be the sheaf associated to \( I \) on \( \mathbb{P} \). For \( m \in \mathbb{N} \) and \( n \in \mathbb{Z} \),

\[
\mathcal{I}^m(n) \cong \hat{I}^m(n) \cong \mathcal{I}^m \otimes \mathcal{O}_{\mathbb{P}}(n)
\]

since \( \mathcal{O}_P(1) \) is locally free at the support of \( \mathcal{O}/\mathcal{I} \).

Let \( I^{(m)} \) be the \( m \)th symbolic power of \( I \), which is also in our situation, the saturation \( (I^m)^{\text{sat}} \) of \( I^m \) with respect to the graded maximal ideal \( m \) of \( S \). We have, as follows from [11, Theorem A4.1], that the graded local cohomology satisfies

\[
H^0_m(I(m)) = H^1_m(I(m)) = 0
\]

for all \( m \in \mathbb{N} \), and for \( i \geq 1 \), we have graded isomorphisms

\[
H^i_m(I(m)) \cong \bigoplus_{n \in \mathbb{Z}} H^i(\mathbb{P}, \mathcal{I}^m(n)) \cong \bigoplus_{n \in \mathbb{Z}} H^i(\mathbb{P}, \mathcal{I}^m \otimes \mathcal{O}_{\mathbb{P}}(n)).
\]

Let \( f : X = X(I) \to \mathbb{P}(a, b, c) \) be the blowup of these points. Let \( E_i \) be the exceptional curves mapping to \( P_i \) for \( 1 \leq i \leq r \), and let \( E = e_1E_1 + \cdots + e_rE_r \).

Let \( A \) be a Weil divisor on \( X \) such that \( \mathcal{O}_X(A) \cong f^*\mathcal{O}_P(1) \). (Recall that \( \mathcal{O}_P(1) \) is locally free at points where \( f \) is not an isomorphism.)

Since \( f \) is the blowup of the nonsingular points \( P_1, \ldots, P_r \) for \( m \geq 0 \), \( f_*\mathcal{O}_X(-mE) \cong \mathcal{I}^m \) and \( R^i f_*\mathcal{O}_X(-mE) = 0 \) for \( i > 0 \) and \( m \geq 0 \) (for instance, by [23, Proposition 10.2]). Since \( \mathcal{O}_P(1) \) is locally free above all points on \( \mathbb{P} \) where \( f \) is not an isomorphism, by the projection formula,

\[
f_*\mathcal{O}_X(nA - mE) \cong \mathcal{I}^m \otimes \mathcal{O}_P(n) \cong \mathcal{I}^m(n)
\]

for all \( m \in \mathbb{N} \) and \( n \in \mathbb{Z} \). By the Leray spectral sequence, we have

\[
H^i(X, \mathcal{O}_X(nA - mE)) \cong H^i(\mathbb{P}, \mathcal{I}^m(n))
\]

for all \( m \in \mathbb{N} \), \( n \in \mathbb{Z} \), and \( i \geq 0 \).

The \( m \)th symbolic power of \( I \) can be computed as

\[
I^{(m)} = \bigoplus_{n \geq 0} H^0(X, \mathcal{O}_X(nA - mE)).
\]

Let \( u = \sum_{i=1}^{r} e_i^2 \).

We have

\[
(A \cdot A) = \frac{1}{abc}, \quad (E_i \cdot E_i) = -1, \quad (A \cdot E_i) = 0,
\]

\[
(E \cdot E) = -u, \quad \text{and} \quad (A \cdot E) = 0.
\]
Let $\text{Div}(X)$ be the group of Weil divisors on $X$. There is an intersection theory on $\text{Div}(X)$, developed in [25], which associates to Weil divisors $D_1$ and $D_2$ on $X$ a rational number $(D_1 \cdot D_2)$. Divisors $D_1$ and $D_2$ are numerically equivalent, written $D_1 \equiv D_2$ if $(D_1 \cdot C) = (D_2 \cdot C)$ for every Weil divisor $C$ on $X$. A $\mathbb{Q}$-divisor $D$ on $X$ is called numerically ample if $(D \cdot C) > 0$ for all curves $C$ on $X$ and $(D \cdot D) > 0$. Let $N_1(X) = (\text{Div}(X)/\equiv) \otimes \mathbb{R}$. We write $\mathcal{D}$ to denote the class in $N_1(X)$ of a Weil divisor $D$ on $X$.

Let $L$ be the real vector subspace of $N_1(X)$ spanned by the (classes of) $E$ and $A$. Let $NL = NE(X) \cap L$, and let $AL = \overline{\text{AMP}}(X) \cap L$, where $NE(X)$ is the closure of the cone of curves on $X$, and $\overline{\text{AMP}}(X)$ is the closure of the ample cone on $X$.

We now make a sketch of the cones $NL$ and $AL$. $NL$ is a cone with boundary rays $E \mathbb{R}_{\geq 0}$, and $R = (\tau A - E)\mathbb{R}_{\geq 0}$ for some $\tau = \tau(I) \in \mathbb{R}$. $AL$ is a cone with boundary rays $A\mathbb{R}_{\geq 0}$, and $T = (sA - E)\mathbb{R}_{\geq 0}$, where $s = s(I) = s_{O_P(1)}(I) \in \mathbb{R}$ is the $s$-invariant of $I$.

Let $g : Y \to X$ be the minimal resolution of singularities. Let $\mathcal{L}$ be a line bundle on $X$. The fact that $X$ has rational singularities implies $H^i(Y, \mathcal{M}) = H^i(X, \mathcal{L})$, where $\mathcal{M} = g^*(\mathcal{L})$. By the Riemann-Roch theorem on $Y$, $(\chi(\mathcal{O}_Y) = 1$ since $Y$ is rational) \[
\chi(\mathcal{L}) = \chi(\mathcal{M}) = \frac{1}{2}(\mathcal{M} \cdot \mathcal{M} \otimes \omega_Y^{-1}) + 1.
\]
By the projection formula,
\[\chi(\mathcal{L}) = \frac{1}{2}(\mathcal{L} \cdot \mathcal{L} \otimes \omega_X^{-1}) + 1.\]

Since $X$ has rational singularities, $g_*\omega_Y = \omega_X = \mathcal{O}_X(-(a + b + c)A + E_1 + \cdots + E_r)$.

**PROPOSITION 3.1**

We have vanishing of cohomology $H^2(X, \mathcal{O}_X(\alpha A - \beta E)) = 0$ if $\beta \geq 0$ and $\alpha > -(a + b + c)$.

**Proof**

We have $H^2(X, \mathcal{O}_X(\alpha A - \beta E)) \cong H^2(\mathbb{P}, \mathcal{I}^\beta(\alpha))$. From the exact sequence
\[0 \to \mathcal{I}^\beta(\alpha) \to \mathcal{O}_\mathbb{P}(\alpha) \to (\mathcal{O}_\mathbb{P}/\mathcal{I}^\beta)(\alpha) \to 0\]
and the fact that $(\mathcal{O}_\mathbb{P}/\mathcal{I}^\beta)(\alpha)$ has zero-dimensional support, we have
\[H^2(\mathbb{P}, \mathcal{I}^\beta(\alpha)) \cong H^2(\mathbb{P}, \mathcal{O}_\mathbb{P}(\alpha)) = 0\]
for $\alpha > -(a + b + c)$. \qed

From (13), (14), and Proposition 3.1, we see that $\sqrt{abcu} A - E \in NL$ since
\[((\sqrt{abcu} A - E) \cdot (\sqrt{abcu} A - E)) = 0.\]

Thus,
\[0 < \tau(I) \leq \sqrt{abcu} \leq s(I).\]
Suppose that \( D \) is a Weil divisor on \( X \); \( g^*(D) \) is defined in [25] as the \( \mathbb{Q} \)-divisor on \( Y \) which agrees with the strict transform of \( D \) away from the exceptional locus of \( g \) and has intersection number zero with all exceptional curves. If \( F = \sum \alpha_i E_i \) is a \( \mathbb{Q} \)-divisor on \( Y \) (with \( \alpha_i \in \mathbb{Q} \)), then we define a \( \mathbb{Z} \)-divisor by
\[
\lfloor F \rfloor = \sum \lfloor \alpha_i \rfloor E_i.
\]
If \( L \) is a line bundle on \( X \), then as follows, for instance, from the projection formula of [27, Theorem 2.1],
\[
H^0(X, \mathcal{O}_X(D) \otimes L) = H^0(Y, \mathcal{O}_Y(\lfloor g^*(D) \rfloor) \otimes g^*L).
\]

**Lemma 3.2**
Suppose that \( \mathcal{F} \) is a coherent sheaf on \( X \), and suppose that \( \mathcal{B} \) is a line bundle on \( X \).

1. The Euler characteristic \( \chi(\mathcal{F} \otimes \mathcal{B}^n) \) is a polynomial in \( n \) for \( n \in \mathbb{N} \).
2. If \( (\mathcal{B} \cdot \mathcal{O}_X(A)) > 0 \), then \( H^2(X, \mathcal{F} \otimes \mathcal{B}^n) = 0 \) for \( n \gg 0 \).

**Proof**
We first prove (1). Let \( \mathcal{M} \) be an ample line bundle on the projective surface \( X \). Thus, there is a composition series of \( \mathcal{F} \) by \( \mathcal{O}_X \)-modules \( \mathcal{O}_{Z_i} \otimes \mathcal{M}^{e_i} \) with \( 1 \leq i \leq m \), where \( m \) is a positive integer, \( Z_i \) are (integral) subvarieties of \( X \), and \( e_i \in \mathbb{Z} \) (cf. [18, Chapter 1, Section 7]). Thus,
\[
\chi(\mathcal{F} \otimes \mathcal{B}^n) = \sum_{i=1}^{m} \chi(\mathcal{O}_{Z_i} \otimes \mathcal{M}^{e_i} \otimes \mathcal{B}^n).
\]
If \( Z_i = X \), we have that
\[
\chi(\mathcal{O}_{Z_i} \otimes \mathcal{M}^{e_i} \otimes \mathcal{B}^n)
\]
is a polynomial in \( n \) by the Riemann-Roch formula (14). If \( Z_i \) is an (integral) curve, then we have an exact sequence
\[
0 \to \mathcal{O}_{Z_i} \to \mathcal{O}_{\overline{Z}_i} \to \mathcal{G}_i \to 0
\]
of \( \mathcal{O}_{Z_i} \)-modules, where \( \overline{Z}_i \) is the normalization of \( Z_i \). Since \( \mathcal{G}_i \) has finite support, \( \chi(\mathcal{O}_{Z_i} \otimes \mathcal{M}^{e_i} \otimes \mathcal{B}^n) \) is a polynomial in \( n \) by the Riemann-Roch theorem on the nonsingular projective curve \( \overline{Z}_i \). In the case when \( Z_i \) is a point, \( \chi(\mathcal{O}_{Z_i} \otimes \mathcal{M}^{e_i} \otimes \mathcal{B}^n) = \chi(\mathcal{O}_{Z_i}) = 1 \) for all \( n \).

Now we prove (2). By the consideration of the composition sequence constructed in the first part of the proof, we are reduced to showing that for \( 1 \leq i \leq m \), \( H^2(Z_i, \mathcal{O}_{Z_i} \otimes \mathcal{M}^{e_i} \otimes \mathcal{B}^n) = 0 \) for \( n \gg 0 \). If \( Z_i = X \), this follows from Proposition 3.1. Otherwise, \( Z_i \) has dimension smaller than 2, so the vanishing must hold.

**4. Regularity of symbolic powers**
We continue with the assumptions of Section 3.
PROPOSITION 4.1
There exist positive integers $b_0$ and $t_0$ such that if $D$ is a Weil divisor on $X$ such that $\overline{D}$ is in the translation of $AL$ by $t_0(b_0abcA - E) + abcA$, then $H^i(X, D) = 0$ for $i > 0$.

Proof
Let $F_m = O_X(mA)$ for $0 \leq m < abc$. There exists a positive integer $b_0$ such that $b_0abcA - E$ is ample on $X$. Let $M = O_X(b_0abcA - E)$.

We use the following vanishing theorem, proven in [13, Theorem 5.1]. Let $F$ be coherent on a projective scheme $Y$, and let $M$ be an ample line bundle. Then there exists an integer $t$ such that $H^i(Y, F \otimes L \otimes M^t) = 0$ for all nef line bundles $L$ on $Y$ and for all $i > 0$.

Choose $t_0$ so that $t_0$ satisfies the condition for $t$ in the above vanishing theorem for $F_m$ with $0 \leq m < abc$ and for all $i > 0$.

Suppose that $\overline{D}$ is in the translation of $AL$ by $t_0(b_0abcA - E) + abcA$. Then $D \sim \alpha A - \beta E + t_0(b_0abcA - E) + abcA$ with $\alpha \geq s\beta$. Expand $\alpha = nabc + m$ with $0 \leq m < abc$. Then we have

$$O_X(D) \cong F_m \otimes L \otimes M^{t_0},$$

where $L = O_X((n + 1)abcA - \beta E)$ is nef. Thus, the conclusions of the proposition hold for $D$. \qed

Suppose that $D$ is a divisor on $X$. Define

$$\overline{D}^\perp = \{ \varphi \in N_1(X) \mid (D \cdot \varphi) = 0 \}.$$  

An effective divisor $D$ such that $(D \cdot D) < 0$ is called a negative curve. An effective divisor $D$ such that $D \sim aA - mE$ for some positive integers $a$ and $m$ is called an $E$-uniform curve.

LEMMA 4.2
Either $T = (\sqrt{ab cu}A - E)_{\mathbb{R} > 0}$ or $T = C^\perp \cap AL$, where $C$ is an irreducible negative curve.

Proof
Recall from (15) that $T = (s(I)A - E)_{\mathbb{R} > 0}$ with $s = s(I) \geq \sqrt{ab cu}$. Suppose that $s > \sqrt{ab cu}$. There exists $\alpha \in \mathbb{Q}$ such that $s > \alpha > \sqrt{ab cu}$. Write $\alpha = c/d$, where $c, d \in \mathbb{Z}_+$. By (13), (14), and Proposition 3.1, we have $h^0(X, O_X(m(cA - dE))) > 0$ for $m \gg 0$. Thus, there exist only a finite number of irreducible curves $C_1, \ldots, C_t$ on $X$ such that $(C_i \cdot (\alpha A - E)) < 0$.

Suppose that $(C \cdot (sA - E)) > 0$ for all irreducible curves $C$ on $X$. In particular, $(C_i \cdot (sA - E)) > 0$ for all $1 \leq i \leq t$. This implies that there exists a real number $\beta$ with $\alpha < \beta < s$ such that $(C_i \cdot (\beta A - E)) > 0$ for $1 \leq i \leq t$. If $C$ is an irreducible curve on $X$ other than one of the $C_i$, then we have $(C \cdot (\alpha A - E)) \geq 0$ and $(C \cdot A) \geq 0$, so that $(C \cdot (\beta A - E)) \geq 0$. Thus, $\beta A - E \in AL$, a contradiction.
Thus, there exists an irreducible curve $C$ on $X$ such that $(C \cdot (sA - E)) = 0$. (The only irreducible curves $C$ on $X$ with $(C \cdot A) = 0$ are the $E_i$.)

**THEOREM 4.3**

Let $u = \sum_{i=1}^{r} e_i^2$. We have $s(I) \geq \sqrt{abc} u$. If $s(I) > \sqrt{abc} u$, then $s(I)$ is a rational number.

**Proof**

The proof is immediate from (15) and Lemma 4.2.

Negative curves and $E$-uniform curves are defined before Lemma 4.2.

**LEMMA 4.4**

Suppose that there exists an $E$-uniform negative curve $F$. Then $s(I) > \sqrt{abc} u$, and $T = \overline{C} \cap AL$, where $C$ is an irreducible negative curve in the support of $F$.

**Proof**

Let $F$ be an $E$-uniform negative curve. $F \sim mA - nE$ for some $m, n \in \mathbb{Z}_+$. Since $(F^2) < 0$ and $F$ is effective, there exists an irreducible curve $C$ in the support of $F$ such that $(C \cdot F) < 0$. Since $((sA - E) \cdot (\tau A - E)) \geq 0$ and $\tau < \sqrt{abc} u$, we have $s > \sqrt{abc} u$. We have $T = (\alpha A + F)^{\mathbb{R}_{\geq 0}}$ for some $\alpha > 0$. Since $T$ is a boundary ray of $AL$ for all $\varepsilon > 0$, there exists an irreducible curve $C_\varepsilon$ on $X$ such that $(C_\varepsilon \cdot ((\alpha - \varepsilon)A + F)) < 0$. Since $(C_\varepsilon \cdot A) \geq 0$, we must have $(C_\varepsilon \cdot F) < 0$, so that $C_\varepsilon$ is in the support of $F$. Since $F$ has only a finite number of irreducible components, we have $T = \overline{C} \cap AL$ for some irreducible component $C$ of $F$.

**PROPOSITION 4.5**

There exist $t_1 > 0$ and $m_0 > 0$ such that $m \geq m_0$ implies that there exists a Cartier divisor $D = \alpha A - mE$ such that $D$ lies between the rays $T$ and the translation of $T$ by $-t_1A$ such that $h^1(X, O_X(D)) \neq 0$.

**Proof**

Let $\gamma = \sqrt{abc} u - E$. Let

$$d = (e_1 + \cdots + e_r) \sqrt{abc} u.$$ 

Observe that $\sqrt{abc} u \in \mathbb{Q}$ if and only if $d \in \mathbb{Q}$. Suppose that $n \in \mathbb{Z}_+$ and $\alpha(n) \in \mathbb{R}_{\geq 0}$ are such that $n\gamma - \alpha(n)A$ is a Cartier divisor. Then by (13) and (14),

$$\chi(O_X(n\gamma - \alpha(n)A))$$

$$= n - \sqrt{n} \left( (a + b + c) - d - 2\alpha(n) \right) + \frac{1}{2abc} (\alpha(n)^2 - \alpha(n)(a + b + c)) + 1. $$

By (15), we always have $s \geq \tau$, so that we reduce to establishing the Proposition in the two cases $s = \tau$ and $s > \tau$. 

Case 1. Assume that \( s = \tau \), so that by (15), \( T = R = \gamma \mathbb{R}_{\geq 0} \). For \( n \in \mathbb{Z}_+ \), choose \( \alpha(n) \) in (18) such that \( 2abc \leq \alpha(n) < 3abc \). Then \( (a + b + c) - d - 2\alpha(n) < 0 \), so that
\[
-h^1(X, \mathcal{O}_X(n\gamma - \alpha(n)A)) \leq \chi(\mathcal{O}_X(n\gamma - \alpha(n)A)) < 0
\]
for \( n \gg 0 \).

Case 2. Assume that the boundary ray \( R = (\tau A - E)\mathbb{R}_{\geq 0} \) of \( NL \) and the boundary ray \( T = (sA - E)\mathbb{R}_{\geq 0} \) of \( NA \) satisfy \( s > \tau \).

We must have \( s > \sqrt{abcu} \) with these assumptions for, otherwise, \( \gamma \in AL \), and there would be an effective divisor \( F \equiv \alpha A - \beta E \) such that \( \alpha/\beta < \sqrt{abcu} \), so that \( F \) is an \( E \)-uniform negative curve. We would then have \( (F \cdot \gamma) < 0 \), a contradiction.

By Lemma 4.2, we have \( T = C^\perp \), where \( C \) is an irreducible negative curve. Let \( p_a(C) \) be the arithmetic genus of \( C \). Let \( \delta = sA - E \). Let
\[
\beta = \max\{0, 1-p_a(C)\/(C \cdot A)\}
\]
For \( n \in \mathbb{Z}_+ \), let \( \alpha(n) \) be such that
\[
\beta < \alpha(n) \leq \beta + abc
\]
and \( n\delta - \alpha(n)A \) is a Cartier divisor.

We have an exact sequence of \( \mathcal{O}_X \)-modules
\[
0 \to \mathcal{O}_X(n\delta - \alpha(n)A - C) \to \mathcal{O}_X(n\delta - \alpha(n)A) \to \mathcal{O}_X(n\delta - \alpha(n)A) \otimes \mathcal{O}_C \to 0.
\]
(20)

Since \( s > \sqrt{abcu} \), we have \( h^0(X, \mathcal{O}_X(n\delta - \alpha(n)A)) > 0 \) for \( n \gg 0 \). We further have \( ((n\delta - \alpha(n)A) \cdot C) < 0 \), so since \( C \) is an integral curve for \( n \gg 0 \),
\[
h^0(X, \mathcal{O}_X(n\delta - \alpha(n)A - C)) = h^0(X, \mathcal{O}_X(n\delta - \alpha(n)A)) > 0.
\]

We have
\[
h^2(X, \mathcal{O}_X(n\delta - \alpha(n)A - C)) = h^2(X, \mathcal{O}_X(n\delta - \alpha(n)A)) = 0
\]
by Proposition 3.1.

From (20) we now have
\[
h^1(X, \mathcal{O}_X(n\delta - \alpha(n)A - C)) - h^1(X, \mathcal{O}_X(n\delta - \alpha(n)A))
\]
\[
= \chi(\mathcal{O}_X(n\delta - \alpha(n)A)) - \chi(\mathcal{O}_X(n\delta - \alpha(n)A - C))
\]
\[
= \chi(\mathcal{O}_X(n\delta - \alpha(n)A) \otimes \mathcal{O}_C)
\]
\[
= (C \cdot (n\delta - \alpha(n)A)) + 1-p_a(C) < 0,
\]
where the last equality is by the Riemann-Roch theorem for the curve \( C \), and (19). Thus, \( h^1(X, \mathcal{O}_X(n\delta - \alpha(n)A)) > 0 \) for \( n \gg 0 \). \qed
Recall that \( \lfloor x \rfloor \) is the greatest integer in a real number \( x \).

**THEOREM 4.6**

There exists a bounded function \( \sigma_I : \mathbb{N} \to \mathbb{Z} \) such that

\[
\text{reg}(I^m) = \lfloor s(I)m \rfloor + \sigma_I(m)
\]

for all \( m \in \mathbb{N} \).

**Proof**

For \( i \geq 2 \),

\[
H^i_m(I^m) = H^{i-1}(X, \mathcal{O}_X(nA - mE))
\]

by equations (11) and (12). The theorem now follows from Propositions 4.1, 3.1, and 4.5 for large \( m \), and thus the theorem is true for all \( m \).

\( \square \)

A function \( \sigma : \mathbb{N} \to \mathbb{Z} \) is *eventually periodic* if \( \sigma(m) \) is periodic for \( m \gg 0 \).

Recall (see Theorem 4.3) that \( s(I) \geq \sqrt{abcu} \); and if \( s(I) > \sqrt{abcu} \), then \( s(I) \) is a rational number.

**THEOREM 4.7**

Assume that \( s(I) > \sqrt{abcu} \), and suppose that \( K \) has characteristic zero or is an algebraic closure of a finite field. Then the function \( \sigma_I(m) \) of Theorem 4.6 is eventually periodic.

**Proof**

By Propositions 4.1, 3.1, and 4.5, we need only compute \( h^1(X, \mathcal{O}_X(nA - mE)) \) for \( nA - mE \) between the rays \( T \) translated up by \( (t_0b_0 + 1)abcA \) and \( T \) translated down by \( -t_1A \). Call this region \( \Delta \).

By Theorem 4.3, there exists a numerically effective Cartier divisor \( G \) such that \( T = \mathcal{O}^\mathbb{R} \geq 0 \). We have \( (G^2) > 0 \). Since \( \mathcal{O} \) is rational, there exist a finite number of Weil divisors \( D_i \) with \( D_i \in \Delta \) such that every divisor \( D \) with \( D \in \Delta \) can be written as \( D \sim D_i + nG \) for some \( i \) and some \( n \in \mathbb{N} \). Let \( \mathcal{L} = \mathcal{O}_X(G) \).

Since \( (g^*\mathcal{O}_X(abc) \cdot g^*\mathcal{L}) > 0 \), Serre duality on \( Y \) implies

\[
(21) \quad h^2(Y, \mathcal{O}_Y(\lfloor g^*(D_i) \rfloor) \otimes g^*\mathcal{L}^n) = 0
\]

for all \( i \) and for \( n \gg 0 \).

By Lemma 3.2(2), (16), and (21), for all \( i \) and for \( n \gg 0 \) we have

\[
\begin{align*}
    h^1(X, \mathcal{O}_X(D_i) \otimes \mathcal{L}^n) &= h^0(X, \mathcal{O}_X(D_i) \otimes \mathcal{L}^n) - \chi(\mathcal{O}_X(D_i) \otimes \mathcal{L}^n) \\
    &= h^0(Y, \mathcal{O}_Y(\lfloor g^*(D_i) \rfloor) \otimes g^*(\mathcal{L}^n)) - \chi(\mathcal{O}_X(D_i) \otimes \mathcal{L}^n) \\
    &= h^1(Y, \mathcal{O}_Y(\lfloor g^*(D_i) \rfloor) \otimes g^*(\mathcal{L}^n)) \\
    &\quad + \chi(\mathcal{O}_Y(\lfloor g^*(D_i) \rfloor) \otimes g^*(\mathcal{L}^n)) - \chi(\mathcal{O}_X(D_i) \otimes \mathcal{L}^n).
\end{align*}
\]
By the Riemann-Roch theorem on \( Y \) and Lemma 3.2(1), we have that
\[
\chi(O_Y([g^*(D_i)]) \otimes g^*(L^n)) - \chi(O_X(D_i) \otimes L^n)
\]
is a polynomial in \( n \) for all \( i \).

By [6, Proposition 13], there exists an effective divisor \( C \) on \( Y \) such that
\[
g^*(L) \otimes O_C
\]
is numerically trivial, and the restriction maps
\[
H^1(Y, O_Y([g^*(D_i)]) \otimes g^*(L^n)) \rightarrow H^1(C, O_C \otimes O_Y([g^*(D_i)]) \otimes g^*(L^n))
\]
are isomorphisms for \( n \gg 0 \) and all \( i \).

In the case when \( K \) has characteristic zero, [6, Theorem 8] shows that for all \( i \), \( h^1(C, O_C \otimes O_Y([g^*(D_i)]) \otimes g^*(L^n)) \) is eventually periodic in \( n \) for all \( i \). In the case when \( K \) is an algebraic closure of a finite field, then the numerically trivial invertible sheaf \( O_C \otimes g^*(L) \) must be torsion, so some power is isomorphic to \( O_C \). Thus we trivially have that \( h^1(C, O_C \otimes O_Y([g^*(D_i)]) \otimes g^*(L^n)) \) is eventually periodic in \( n \) for all \( i \).

In either case of \( K \), \( h^1(Y, O_Y([g^*(D_i)]) \otimes g^*(L^n)) \) is eventually periodic as a function of \( n \). Thus \( \sigma(m) \) is eventually periodic. \( \square \)

When \( K \) is a field of positive characteristic which has positive transcendence degree over the prime field, the conclusions of Theorem 4.7 may fail. An example of a set of points in ordinary projective space \( \mathbb{P}^2 \) where \( \sigma(m) \) is not eventually periodic is given in [5, Example 4.4].

**Corollary 4.8**

Suppose that there exists an \( E \)-uniform negative curve, and suppose that \( K \) has characteristic zero or is a finite field. Then the function \( \sigma_I(m) \) of Theorem 4.6 is eventually periodic.

**Proof**

This follows from Lemma 4.4 and Theorem 4.7. \( \square \)

An important case of this construction is when \( i = 1 \), and \( I = P(a,b,c) \) is the prime ideal of a monomial space curve. As a corollary to Theorems 4.6 and 4.7, we have the following application to monomial space curves.

**Corollary 4.9**

Suppose that \( I = P(a,b,c) \) is the prime ideal of a monomial space curve, and suppose that there exists a negative curve on \( X(I) \). Then \( s(I) \) is a rational number, and the function \( \sigma_I(m) \) of Theorem 4.6 is eventually periodic.

### 5. Uniform negative curves and Nagata’s conjecture

Let \( S \) be a polynomial ring as in Section 3. Let \( C = K[u,v,w] \) be a polynomial ring with \( \text{wt}(u) = \text{wt}(v) = \text{wt}(w) = 1 \).
Consider the \( K \)-algebra homomorphism 
\[ \delta : S \rightarrow C \]
defined by \( \delta(x) = u^a, \delta(y) = v^b, \delta(z) = w^c \), where \( a, b, c \) are pairwise relatively prime positive integers.

Let 
\[ I = I_{P_1}^{e_1} \cap \cdots \cap I_{P_r}^{e_r} \]
be an ideal of \( S \) as in Section 3.

Consider the following two conditions.

(A1) \( K \) is an algebraically closed field such that \( \text{ch}(K) \) is 0 or \( \text{ch}(K) \) does not divide \( abc \).

(A2) \( I_{P_i} \) is not contained in \( xyz \) for \( i = 1, \ldots, r \).

**Lemma 5.1**
Assume conditions (A1) and (A2) as above. There are distinct prime ideals \( Q_{i_1}, Q_{i_2}, \ldots, Q_{i_{abc}} \) of \( C \) such that
\[ I_{P_i}^{(m)} C = \bigcap_{j=1}^{abc} Q_{ij}^{(m)} \]
for \( m > 0 \) and \( i = 1, \ldots, r \).

**Proof**
Let \( Q \) be a prime ideal of \( C \) lying over \( I_{P_i} \). Then there exists a point \((\alpha : \beta : \gamma) \in \mathbb{P}_K^2\) such that
\[ Q = I_2 \left( \begin{array}{ccc} u & v & w \\ \alpha & \beta & \gamma \end{array} \right), \]
where \( I_2(\ ) \) is the ideal generated by all the \((2 \times 2)\)-minors of the given matrix.

We remark that \( Q \) is the kernel of the \( K \)-algebra homomorphism
\[ \varphi_{(\alpha;\beta;\gamma)} : C \rightarrow K[t] \]
defined by \( \varphi_{(\alpha;\beta;\gamma)}(u) = \alpha t, \varphi_{(\alpha;\beta;\gamma)}(v) = \beta t, \varphi_{(\alpha;\beta;\gamma)}(w) = \gamma t \). Then \( I_{P_i} \) is the kernel of the \( K \)-algebra homomorphism
\[ \varphi = \varphi_{(\alpha;\beta;\gamma)} \delta : S \rightarrow K[t] \]
defined by \( \varphi(x) = \alpha^a t^a, \varphi(y) = \beta^b t^b, \varphi(z) = \gamma^c t^c \). Let \( \zeta_q \) be a primitive \( q \)th root of 1 for a positive integer \( q \).

Set
\[ Q_{n_1,n_2,n_3} = I_2 \left( \begin{array}{ccc} u & v & w \\ \zeta_{n_1} & \zeta_{n_2} & \zeta_{n_3} \end{array} \right), \]
It is the kernel of the \( K \)-algebra homomorphism \( \varphi_{(\zeta_{n_1}^a;\zeta_{n_2}^b;\zeta_{n_3}^c)} \). For any \( n_1, n_2, \) and \( n_3, Q_{n_1,n_2,n_3} \) is a prime ideal of \( C \) lying over \( I_{P_i} \) since \( \varphi_{(\zeta_{n_1}^a;\zeta_{n_2}^b;\zeta_{n_3}^c)} \delta = \varphi. \)
By our assumption (A2), all of $\alpha$, $\beta$, and $\gamma$ are not zero. By (A1),
\[
\{(\zeta_n^1 : \zeta_n^2 : \zeta_n^3) \in \mathbb{P}_K^2 \mid n_1 = 0, \ldots, a - 1; n_2 = 0, \ldots, b - 1; n_3 = 0, \ldots, c - 1\}
\]
are distinct $abc$ points in $\mathbb{P}_K^2$. Therefore,
\[
\{Q_{n_1,n_2,n_3} \mid n_1 = 0, \ldots, a - 1; n_2 = 0, \ldots, b - 1; n_3 = 0, \ldots, c - 1\}
\]
are distinct prime ideals of $C$ lying over $I_{P_i}$. Here we have
\[
\text{Ass}_C(C/I_{P_i}C) = \text{Min}_C(C/I_{P_i}C) = \{Q \in \text{Spec}(C) \mid Q \cap S = I_{P_i}\} \supseteq \{Q_{n_1,n_2,n_3} \mid n_1 = 0, \ldots, a - 1; n_2 = 0, \ldots, b - 1; n_3 = 0, \ldots, c - 1\}.
\]
Since $C$ is an $S$-free module of rank $abc$, $C/I_{P_i}C$ is an $S/I_{P_i}$-free module of rank $abc$. Then it is easy to see
\[
abc = \text{rank}_{S/I_{P_i}}(C/I_{P_i}C) = \sum_{Q \in \text{Spec}(C)} \ell_C(C_Q/I_{P_i}C_Q) \cdot \text{rank}_{S/I_{P_i}}(C/Q).
\]
Therefore,
\[
\{Q \in \text{Spec}(C) \mid Q \cap S = I_{P_i}\} = \{Q_{n_1,n_2,n_3} \mid n_1 = 0, \ldots, a - 1; n_2 = 0, \ldots, b - 1; n_3 = 0, \ldots, c - 1\}
\]
and
\[
\ell_C(C_{Q_{n_1,n_2,n_3}}/I_{P_i}C_{Q_{n_1,n_2,n_3}}) = 1
\]
for each $n_1$, $n_2$, $n_3$. It follows from the equation as above that
\[
I_{P_i}C_{Q_{n_1,n_2,n_3}} = Q_{n_1,n_2,n_3}C_{Q_{n_1,n_2,n_3}}
\]
for each $n_1$, $n_2$, $n_3$. Since
\[
\text{Ass}_C(C/I_{P_i}^{(m)}C) = \text{Min}_C(C/I_{P_i}^{(m)}C) = \text{Min}_C(C/I_{P_i}C) = \{Q_{n_1,n_2,n_3} \mid n_1 = 0, \ldots, a - 1; n_2 = 0, \ldots, b - 1; n_3 = 0, \ldots, c - 1\},
\]
we have
\[
I_{P_i}^{(m)}C = \bigcap_{n_1,n_2,n_3} (I_{P_i}^{(m)}C_{Q_{n_1,n_2,n_3}} \cap C) = \bigcap_{n_1,n_2,n_3} (Q_{n_1,n_2,n_3}^m C_{Q_{n_1,n_2,n_3}} \cap C) = \bigcap_{n_1,n_2,n_3} Q_{n_1,n_2,n_3}^{(m)}.
\]
By Lemma 5.1,
\[ I^{(m)}C = \bigcap_{i=1}^{r} I^{(e_{i}m)}_{P_i} C = \bigcap_{i=1}^{r} \left( \bigcap_{j=1}^{abc} Q_{ij}^{(e_{i}m)} \right). \]

In this case,
\[ \max\{ n \in \mathbb{Z} \mid [C/(u^{a}, v^{b}, w^{c})C]_n \neq 0 \} = a + b + c - 3. \]

Then, by (5),
\[ \text{reg}(I^{(m)}C) = \text{reg}(I^{(m)}) + a + b + c - 3. \]

By (22), we may assume \( a = b = c = 1 \) in Theorems 4.6 and 4.7 if (A1) and (A2) are satisfied.

Let \( q_1, \ldots, q_n \) be independent generic points in \( \mathbb{P}^2 \). Suppose that \( n \geq 10 \). Nagata conjectured that
\[ [I_{q_1}^m \cap \cdots \cap I_{q_n}^m]_d = 0 \]
if \( d \leq \sqrt{nm} \). Nagata [26] solved it affirmatively when \( n \) is a square.

Consider the following two conditions:
\begin{enumerate}
    \item[(A0)] \( K = \mathbb{C} \), the field of complex numbers, and
    \item[(A3)] \( I = \sqrt{I} \); that is, \( e_1 = e_2 = \cdots = e_r = 1 \) and \( E = E_1 + \cdots + E_r \).
\end{enumerate}

**Proposition 5.2**

Suppose that \( n \) is a positive integer which has a factorization \( n = abcr \) by positive integers with \( a, b, c \) pairwise relatively prime. If there exist distinct points \( P_1, \ldots, P_r \) on the weighted projective space \( \mathbb{P}_{\mathbb{C}}(a, b, c) \) satisfying (A2), such that there does not exist an \( E \)-uniform negative curve on the blowup of \( \mathbb{P}_{\mathbb{C}}(a, b, c) \) defined by (A3), then Nagata’s conjecture is true for \( abcr \) general points in \( \mathbb{P}^2 \).

**Proof**

Assume that Nagata’s conjecture is not true for \( abcr \) general points in \( \mathbb{P}^2 \). If \( abcr \) is a square, Nagata solved the conjecture affirmatively. Therefore, we may assume that \( \sqrt{abcr} \) is not a rational number.

Let \( q_1, \ldots, q_{abcr} \) be independent generic points in \( \mathbb{P}_{\mathbb{C}}^2 \). \( I_{q_i} \) is the defining ideal of \( q_i \). By our assumption, there exist positive integers \( m_0 \) and \( d_0 \) such that
\[ d_0 \leq \sqrt{abcrm_0} \quad \text{and} \quad [I_{q_1}^{m_0} \cap \cdots \cap I_{q_{abcr}}^{m_0}]_{d_0} \neq 0. \]

Since \( \sqrt{abcr} \) is not a rational number, we have
\[ d_0 < \sqrt{abcrm_0}. \]

Assume that there does not exist an \( E \)-uniform negative curve for some \( P_i \)'s satisfying (A0), (A2), and (A3). Then we have
\[ [I^{(m)}]_d = [I_{P_1}^{(m)} \cap \cdots \cap I_{P_r}^{(m)}]_d = 0 \]
if \( d < \sqrt{abcrm} \). Considering the \( \mathbb{C} \)-algebra homomorphism
\[
\delta : S = \mathbb{C}[x, y, z] \to \mathbb{C}[u, v, w],
\]
we obtain
\[
0 = [I(m)C]_d = \left[ \bigcap_{i=1}^r I_{P_i}^{(m)} C \right]_d = \left[ \bigcap_{i=1}^r \left( \bigcap_{j=1}^{abc} Q_{ij}^{(m)} \right) \right]_d
\]
for \( d < \sqrt{abcrm} \). This contradicts (23) since we can specialize \( \{q_1, \ldots, q_{abcr} \} \) to
\[
\{Q_{ij} \mid i = 1, \ldots, r; j = 1, \ldots, abc\}.
\]
\[\square\]

**REMARK 5.3**

Let \( K \) be a field, and let \( a, b, c \) be pairwise relatively prime integers. Let \( P_K(a, b, c) \) be the kernel of the \( K \)-algebra homomorphism
\[
\delta : S = K[x, y, z] \to K[t]
\]
defined by \( \delta(x) = t^a, \delta(y) = t^b, \delta(z) = t^c \).

Let \( X_K(a, b, c) \) be the blowup of the weighted projective space \( \mathbb{P}(a, b, c) \) at the point corresponding to \( P_K(a, b, c) \).

Assume that \( K \) is of positive characteristic. If there exists a negative curve on \( X_K(a, b, c) \), then the symbolic Rees ring
\[
S \oplus P_K(a, b, c) \oplus P_K(a, b, c)^{(2)} \oplus P_K(a, b, c)^{(3)} \oplus \cdots
\]
is Noetherian by [2].

Here assume that the symbolic Rees ring (24) is not Noetherian for some \( a_0, b_0, c_0 \) over some field \( K_0 \) of positive characteristic. Since \( P_{K_0}(a, b, c) \) is not a complete intersection, we may assume \( 3 \leq a_0 < b_0 < c_0 \). In particular, \( a_0b_0c_0 \geq 60 > 10 \). Then by [2], there is no negative curve on \( X_{K_0}(a_0, b_0, c_0) \). By a standard method of mod \( p \) reduction, there is no negative curve on \( X_{\mathbb{C}}(a_0, b_0, c_0) \). Then, by Proposition 5.2, Nagata’s conjecture is true for \( a_0b_0c_0 \).

**References**

[1] M. V. Catalisano, N. V. Trung, and G. Valla, *A sharp bound for the regularity index of fat points in general position*, Proc. Amer. Math. Soc. **118** (1993), 717–724.

[2] S. D. Cutkosky, *Symbolic algebras of monomial primes*, J. Reine Angew. Math. **416** (1991), 71–89.

[3] _____, *Irrational asymptotic behaviour of Castelnuovo-Mumford regularity*, J. Reine Angew. Math. **522** (2000), 93–103.

[4] S. D. Cutkosky, L. Ein, and R. Lazarsfeld, *Positivity and complexity of ideal sheaves*, Math. Ann. **321** (2001), 213–234.
[5] S. D. Cutkosky, J. Herzog, and N. V. Trung, *Asymptotic behaviour of the Castelnuovo-Mumford regularity*, Compositio Math. **118** (1999), 243–261.

[6] S. D. Cutkosky and V. Srinivas, *On a problem of Zariski on dimensions of linear systems*, Ann. of Math. (2) **137** (1993), 531–559.

[7] G. Dalzotto and E. Sbarra, *On non-standard graded algebras*, Toyama Math. J. **31** (2008), 33–57.

[8] E. D. Davis and A. V. Geramita, “The Hilbert function of a special class of 1-dimensional Cohen-Macaulay graded algebras” in *The Curves Seminar at Queen’s, Vol. III (Kingston, Ont., 1983)*, Queen’s Papers in Pure and Appl. Math. **667**, Queen’s Univ., Kingston, Ont., Canada, 1984, exp. no. H.

[9] C. Delorme, *Espaces projectifs anisotropes*, Bull. Soc. Math. France **103** (1975), 203–223; Erratum, Bull. Soc. Math. France **103** (1975), 510.

[10] I. Dolgachev, “Weighted projective varieties” in *Group Actions and Vector Fields (Vancouver, B.C., 1981)*, Lecture Notes in Math. **956**, Springer, Berlin, 1982, 34–71.

[11] D. Eisenbud, *Commutative Algebra: With a View Toward Algebraic Geometry*, Grad. Texts in Math. **150**, Springer, New York, 1995.

[12] D. Eisenbud and S. Goto, *Linear free resolutions and minimal multiplicity*, J. Algebra **88** (1984), 89–133.

[13] T. Fujita, *Semipositive line bundles*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **30** (1983), 353–378.

[14] A. Gimigliano, *Regularity of linear systems of plane curves*, J. Algebra **124** (1989), 447–460.

[15] S. Goto, K. Nishida, and K. Watanabe, *Non-Cohen-Macaulay symbolic blow-ups for space monomial curves and counterculmeples to Cowsik’s question*, Proc. Amer. Math. Soc. **120** (1994), 383–392.

[16] A. Grothendieck and J. Dieudonné, *Éléments de géométrie algébrique, II: Étude globale élémentaire de quelques classes de morphismes*, Inst. Hautes Études Sci. Publ. Math. **8** (1961).

[17] B. Harbourne and J. Roé, *Linear systems with multiple base points in $\mathbb{P}^2$*, Adv. Geom. **4** (2004), 41–59.

[18] R. Hartshorne, *Algebraic Geometry*, Grad. Texts in Math. **52**, Springer, New York, 1977.

[19] A. Hirschowitz, *Une conjecture pour la cohomologie des diviseurs sur les surfaces rationnelles générées*, J. Reine Angew. Math. **397** (1989), 208–213.

[20] V. Kodiyalam, *Asymptotic behaviour of Castelnuovo-Mumford regularity*, Proc. Amer. Math. Soc. **128** (2000), 407–411.

[21] K. Kurano and N. Matsuzaka, *On finite generation of symbolic Rees rings of space monomial curves and existence of negative curves*, J. Algebra **322** (2009), 3268–3290.

[22] R. Lazarsfeld, *Positivity in Algebraic Geometry*, Vols I and II, Springer, Berlin, 2004.
[23] H. Matsumura, *Geometric structure of the cohomology rings in abstract algebraic geometry*, Mem. Coll. Sci. Univ. Kyoto Ser. A Math. **32** (1959), 33–84.

[24] S. Mori, *On a generalization of complete intersections*, J. Math. Kyoto Univ. **15** (1975), 619–646.

[25] D. Mumford, *The topology of normal singularities of an algebraic surface and a criterion for simplicity*, Inst. Hautes Études Sci. Publ. Math. **9** (1961), 5–22.

[26] M. Nagata, *On the 14-th problem of Hilbert*, Amer. J. Math. **81** (1959), 766–772.

[27] F. Sakai, *Weil divisors on normal surfaces*, Duke Math. J. **51** (1984), 877–887.

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