Spontaneous Vortex Production in Driven Condensates with Narrow Feshbach Resonances

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(Dated: August 12, 2010)

We explore the possibility that, at zero temperature, vortices can be created spontaneously in a condensate of cold Fermi atoms, whose scattering is controlled by a narrow Feshbach resonance, by rapid magnetic tuning from the BEC to BCS regime. This could be achievable with current experimental techniques.

Causality imposes strong bounds on a system whose environment is changing rapidly. If external influences, such as quenches, try to make correlation lengths change faster than the relevant causal speed (e.g. the speed of sound) then the system will freeze, to unfreeze later when the effect has diminished. As proposed by Kibble and Zurek [1, 3], this then leads to frustration. In particular, vortices will arise spontaneously if they can be trapped by domain boundaries.

Spontaneous vorticity has already been observed in cold bose gases, on condensing them by rapid cooling [4]. In this letter we suggest, on similar causal grounds, that vortex sepa-
res will appear to accommodate the frustration of the field. In the KZ scenario it is suggested that vortex separation at their time of spontaneous production is \( \xi \approx \xi(\bar{t}) \). If \( \xi_0 = k_F^{-1} \), the inverse Fermi momentum which sets the atomic separation scale, and \( \bar{t}_0 = \hbar/\epsilon_F \), the inverse Fermi energy (in units of \( \hbar \)), we shall show that

\[
\bar{\xi} \approx \xi_0(\tau_Q/\tau_0)^{1/2},
\]

provided \( \tau_Q \gg \tau_0 \). The timescale \( \tau_Q \) is the quench time for the change in the inverse scattering length induced by the changing magnetic field, and is proportional to the quench time \( \tau_H \) for the field change. Experimentally, with current techniques, \( \tau_Q \) can be made comparable to \( \tau_0 \) itself, suggesting that spontaneous vortex production should be observable in realistic systems. In this regard there are many similarities with the analysis of [5] for thermal quenching of condensates (although in that case causality is determined by second sound). We stress that the mechanism that we are invoking here differs from that of defect formation in quantum phase transitions in \( T = 0 \) condensates [3].

For the sake of analytic simplicity, we restrict ourselves to narrow Feshbach resonances. Our starting point is the exemplary 'two-channel' microscopic action (in units in which \( \hbar = 1 \))

\[
S = \int dt d^3x \left\{ \sum_{\uparrow, \downarrow} \psi^*_{\sigma}(x) \left[ i \partial_t + \frac{\nabla^2}{2M} + \mu \right] \psi_{\sigma}(x) + \phi^*(x) \left[ i \partial_t + \frac{\nabla^2}{2M} + 2\mu - \nu \right] \phi(x) - g \left[ \phi^*(x) \psi_{\uparrow}(x) \psi_{\downarrow}(x) + \phi(x)\psi^*_{\uparrow}(x) \psi^*_{\downarrow}(x) \right] \right\} (3)
\]

for cold \( (T = 0) \) fermi fields \( \psi_{\sigma} \) with spin label \( \sigma = (\uparrow, \downarrow) \), which possess a narrow bound-state (Feshbach) resonance with tunable binding energy \( \nu \), represented by a diatomic field \( \phi \) with mass \( M = 2m \). This model has been discussed on great detail by Gurarie and Radzihovsky [9] and we borrow several results from their paper.

\( S \) is quadratic in the fermi fields. Integrating them out [10] enables us to write \( S \) in the non-local form

\[
S_{NL} = -i Tr \ln G^{-1} + \int dt d^3x \phi^*(x) \left[ i \partial_t + \frac{\nabla^2}{2M} + 2\mu - \nu \right] \phi(x)
\]

in which \( G^{-1} \) is the inverse Nambu Green function,

\[
G^{-1} = \begin{pmatrix}
i\partial_t - \varepsilon & -g\phi(x) \\
-g\phi^*(x) & i\partial_t + \varepsilon
\end{pmatrix} (5)
\]
where \(-g\, \phi(x) = g\, |\phi(x)| \, e^{i\theta(x)}\) represents the condensate (and \(\varepsilon = -\nabla^2 / (2m - \mu)\).

The s-wave scattering length \(a_S\) is determined from the binding energy as
\[
2\mu - \nu = \frac{g^2 m}{4\pi a_S}.
\]  
(6)

To see the effect of applying an external magnetic field \(H\), we adopt the parametrisation \([9]\)
\[
a_S = a_{bg} \left(1 - \frac{H}{H - H_0}\right),
\]
whence
\[
2\mu - \nu \approx -\frac{g^2 m}{4\pi a_{bg} H_0} (H - H_0),
\]
(8)

where \(a_{bg}\) is the background (off-resonance) scattering length and \(H_0\) the so-called ‘resonance width’ \([9]\). \(H_0\) is the field required to achieve infinite scattering length (the unitary limit). As \(H\) increases through \(H_0\) we pass from the BEC to the BCS regimes (with \(a_S\) going from +ve to -ve).

In this paper we restrict ourselves to the mean-field approximation, the general solution to \(S_{NL} = 0\), valid if \(\phi\) is a sufficiently narrow resonance \([9, 11]\). The action possesses a \(U(1)\) invariance under \(\theta \rightarrow \theta + \text{const.}\), which is spontaneously broken. \(S_{NL} = 0\) permits the spacetime constant gap solution \(|\phi(x)| = |\phi_0| \neq 0\). We perturb in the derivatives of \(\theta\) and the small fluctuation in the condensate density \(\delta |\phi| = |\phi| - |\phi_0|\) and its derivatives. \(\theta(x)\) is not small. Using the results of our earlier papers \([10, 12]\), for which \([9]\) is a limiting case, we can extract from \(S_{NL}\) a local effective Lagrangian density \(L_{eff}\),
\[
L_{eff} = -\frac{1}{2} \rho_0 G(\theta, \varepsilon) + \frac{N_0}{4} G^2(\theta, \varepsilon) - \alpha a G(\theta, \varepsilon) + \frac{1}{4} \eta \varepsilon^2 (\varepsilon, \theta) \frac{1}{4} \nabla^2 \epsilon^2,
\]
valid for long wavelength, low-frequency phenomena.

\(L_{eff}\) is given in terms of the Galilean scalar combinations \(G(\theta, \varepsilon) = \dot{\theta} + (\nabla \theta)^2 / 4m + (\nabla \varepsilon)^2 / 4m\), \(X(\theta, \varepsilon) = \dot{\epsilon} + \nabla \theta \cdot \nabla \varepsilon / 2m\), where the dimensionless \(\epsilon \propto \delta |\phi|\) is itself a scalar. We postpone a discussion of the coefficients of \([9]\), except that \(\rho_0 = \rho_0^F + \rho_0^B\) is the total (fixed) fermion number density where \(\rho_0^F\) is the explicit fermion density, and \(\rho_0^B = 2 |\phi_0|^2\) is due to molecules (two fermions per molecule). For the evolving system the molecular density is \(\rho^B = 2 |\phi|^2 = \rho_0^B + 4\delta |\phi| |\phi_0|\), showing that \(\epsilon \propto \delta \rho^B\), the molecular density fluctuation. Although the details are immaterial, we have scaled \(\epsilon\) so that it has the same coefficients as \(\theta\) in its spatial derivatives.

For small fluctuations, the linear approximation to the Euler-Lagrange equations for \(\theta\) and \(\varepsilon\), sufficient to determine the speed of sound, is
\[
\frac{N_0}{2} \ddot{\theta} - \rho_0 \frac{\nabla^2 \theta}{4m} - \alpha \dot{\varepsilon} = 0
\]
\[
\frac{\eta}{2} \ddot{\varepsilon} - \rho_0 \frac{\nabla^2 \varepsilon}{4m} + \frac{1}{2} \dot{\theta}^2 + \alpha \dot{\varepsilon} = 0.
\]
(10)

On diagonalising, we see that for long wavelengths the gapless phonon has dispersion relation \(\omega^2 = v_s^2 k^2\),
\[
v_s^2 = \frac{\rho_0 / 2m}{N_0 + 4\alpha^2 / M^2},
\]
(11)

independent of \(\eta\). With \(H\)-dependent coefficients, in the deep BCS regime (small \(\alpha\) \(v_s \rightarrow v_{BCS} = v_F / \sqrt{3}\) and in the deep BEC regime (large \(\alpha\)) \(v_s \rightarrow 0\).

Eqs. \([13, 14]\) represent a two-component system of molecules and atom pairs. Consider a spatially homogeneous condensate. Then these linearised EL equations, which ignore damping, display the oscillatory solution
\[
\delta \rho^B = \delta \rho_0^B \cos \Omega t,
\]
(12)

describing the repeated dissociation of molecules into atom pairs and their reconversion into molecules. The frequency \(\Omega\) of density fluctuations can be shown to increase monotonically from the exponentially dumped \(g|\phi_0| = O(\mu \exp(-\pi/2k_F |a_S|))\) in the BCS regime to \(\Omega = 2\sqrt{|g_0|^2 |\phi_0|^2 + \mu^2}\) as it crosses into the BEC regime \([13, 14]\). In the deep BEC regime \(\Omega \approx 2\mu\).

A two-component density in which fermions oscillate from one to the other is not the same as a two-fluid picture. In general, density fluctuations act as sources and sinks in the continuity equation and the condensate behaves as a fluid only when these are ignorable i.e. we can neglect the spatial and temporal variation in \(\dot{\epsilon}\), in comparison to \(\epsilon\) itself. As a result, \(\epsilon \approx -2\alpha G(\theta)/M^2\), a slave to the phase. We shall see that, for our quenches, this is the case, whence the Euler-Lagrange equation for \(\theta\) is, indeed, the continuity equation of a single fluid,
\[
\frac{\partial}{\partial t} \rho + \nabla \cdot (\rho v) = 0,
\]
(13)

with \(\rho = \rho_0 + 2\alpha \epsilon - \rho_0 G(\theta)\) and \(v = \nabla \theta / 2m\). To complete the fluid picture we observe that this definition of \(\rho\) is no more than the Bernoulli equation
\[
m \dot{\varepsilon} + \nabla \left[\delta \theta + \frac{1}{2} m \dot{\varepsilon}^2\right] = 0,
\]
(14)

where the enthalpy \(\delta h = m \dot{\varepsilon}^2 \delta \rho / \rho\).

Since the hydrodynamic equations can be derived from a Gross-Pitaevskii (GP) equation, it is more transparent to reconstitute this equation, with its natural vortex solutions. Consider the Lagrangian describing the wave-function \(\psi\) of a particle of mass \(2m\), interacting non-linearly with itself,
\[
L(\psi) = i\hbar \psi^* \dot{\psi} - \frac{\hbar^2}{4m} \nabla \psi^* \cdot \nabla \psi - \frac{m \dot{\varepsilon}^2}{\rho_0} (|\psi|^2 - \rho_0)^2
\]
(15)

where we have restored factors of \(\hbar\). The Gross-Pitaevskii equation following from \([15]\) is
\[
\frac{\hbar^2}{2m} \nabla^2 \psi^* + 2m v_s^2 \psi - \frac{2m v_s^2}{\rho_0} |\psi|^2 = 0.
\]
(16)
FIG. 1: A condensate with $\bar{g} = 1.5$, quenched by a magnetic field with quench rate $\tau_Q/\tau_0 = 16$. For the solid line the ordinate is $v_F^2/v_F^2$ and for the dashed line the ordinate is $\sqrt{\hbar/2M(dE/dt)}v_z^*$. In driving the system from the BEC regime (on the right) towards the BCS regime (on the left) the system unfreezes at $1/k_F a_S \approx 0.75$, when the two curves diverge, from [18]. The inset shows $\rho_0^*/\rho_0$ (dashed curve) and $\rho_0^*/\rho_0$ (solid curve).

with coefficients varying smoothly as we cross the unitary limit. If we set $\psi = \sqrt{\rho} \exp(i\theta)$ and solve [16] at the relevant order in derivatives, we recover [13] and [14] on ignoring $\epsilon$ density fluctuations (see [15] for a comparable analysis). Remarkably, the dependence of the GP equation on the coefficients of [9] is implicit, through $v_F^2$. The length scale of the system is $\xi = \hbar/2mv_s = \hbar/Mv_s$ and the time scale is $\tau = \hbar/Mv_s^2$.

The single-fluid model requires that density fluctuations can be averaged to zero on the timescale $\tau$ i.e. $\tau \Omega \gg 1$ which, for the moment, we assume. We stress the importance of the narrowness of the resonance. At best, broad resonances require a two-fluid description [10] from which we have no simple way to draw conclusions about defect formation in a field quench.

Suppose that $H$ increases uniformly in time with $H/H|_{t=0} = \tau_H^{-1}$, where $t = 0$ is the time at which the system is at the unitary limit. We write $1/k_F a_S = -t/\tau_Q$ for small $t$ where

$$\tau_Q = \tau_H \left( \frac{k_F a_S H_0}{H_0} \right).$$

(17)

The time $t$ of [11] at which the system unfreezes satisfies

$$(v_s^2(t))^2 \approx \frac{\hbar}{2M} \frac{d}{dt} v_z^2(t).$$

(18)

Empirically, we find the approximate universal behaviour

$$v_s^2(t) \approx v_F^2 (\tau_0/4\tau_Q)$$

(19)

for a very wide range of parameters. This universality is not surprising since, to a fair approximation, $v_s^2/v_F^2$ can be approximated as

$$v_s^2 \approx (v_F^2/6)(1 + \tanh(c_0 - b_0/k_F a_S)),$$

(20)

(of which we see only the tail in Fig. 1) where $b_0(g)$ is approximately constant for a wide range of $g$, which gives $v_s^2/v_F^2 \propto \tau_0/\tau_Q$ automatically. We only need the coefficients of $L_{eff}$ of [9] to improve upon [20] and determine the coefficient of proportionality. These can be obtained from [10] and [12] (in the limit of no contact interactions). The correlation length at the time of unfreezing then satisfies the scaling law [2]

$$\xi \approx \xi_0(\tau_Q/\tau_0)^{1/2}$$

(21)

provided $\tau_Q \gg \tau_0$.

Eq. [10] permits vortex solutions within which $\xi$ determines vortex width (in fermion number density). We can ignore the order parameter fluctuations $\delta \langle \phi \rangle$, since they are shorter-ranged and do not affect how vortices pack. The KZ picture then gives an estimated vortex separation at time of production as [21], as claimed in the introduction.

The quench parameters are related to the width of the resonance $\Gamma_0$ by [9] $\Gamma_0 \approx 4m\mu_B^2 a_S^2 H_0^2/\hbar^2$, where $\mu_B$ is the Bohr magneton. In practice, it is more convenient to work with the dimensionless width $\gamma_0 \approx \sqrt{\Gamma_0/\epsilon_F}$, whereby

$$\frac{\tau_Q}{\tau_0} = \frac{\tau_Q \epsilon_F}{\hbar} \approx \frac{\pi}{\mu_B \hbar} \frac{\epsilon_F}{\gamma_0}.\quad (22)$$

To be concrete, consider the resonance in $^6\text{Li}$ at $H_0 = 543.25 G$, discussed in some detail in [10]. As our benchmark we take the achievable number density $\rho_0 \approx 3 \times 10^{12} \text{cm}^{-3}$, whence $\epsilon_F \approx 7 \times 10^{-11} eV$ and $\gamma_0 \approx 0.2$. In terms of the dimensionless coupling $\bar{g}$, where $g^2 = (64k_F^2/3\hbar^2)\bar{g}^2$, $^6\text{Li}$ at the density above corresponds to $\bar{g}^2 \lesssim 1$. For a condensate of density $\rho$ it follows that $g^2 \approx (\rho_0/\rho)^{1/3}$ and

$$\frac{\tau_Q}{\tau_0} \approx \frac{1}{\hbar} \left( \frac{\rho}{\rho_0} \right),$$

(23)

where $\hat{H}$ is measured in units of Gauss (ms)$^{-1}$. Experimentally, it is possible to achieve quench rates as fast as $\dot{H} \approx 0.1 G/\text{ms}$ [10].

The condition $\tau \Omega \gg 1$ throughout the quench now becomes

$$\tau \Omega = \frac{\Omega v_F^2}{\epsilon_F 4v_F^2} \approx \frac{\Omega \tau_Q}{\epsilon_F \tau_0} \gg 1.$$

(24)

With $\Omega \approx 2|\mu|$, large in the BEC regime, the inequality is guaranteed there for all $\tau_Q/\tau_0 \gg 1$. However, in the deep
BEC regime $\Omega$ is exponentially damped, not sustaining the single fluid picture. Thus the single fluid approximation, on which our causal analysis depends, requires that the quench does not trespass beyond the unitarity limit. The question is then whether such a quench can be implemented sufficiently fast for the condensate to freeze.

We believe it possible for realistic quenches. It is simplest to adopt a density $\rho < \rho_0$, permitting smaller $\dot{\cal H}$ to maintain $\tau_Q > \tau_0$. As an example (see Fig.1), we take $\dot{\gamma} = 1.5$, corresponding to $\rho \approx 0.1\rho_0$, and $\tau_Q/\tau_0 = 16$. Then, the domain structure is formed when $v_s \approx v_{\text{BCS}}/4$, with $1/k_F a_s \approx 0.75$ and $\rho_0^B \approx 0.35\rho_0$. $\rho_0^B \approx 0.62\rho_0$. $\tau\Omega$ decreases throughout the quench, but its final value of $16\Omega/\xi_F \approx 27$ remains sufficiently large to justify the approximation. Adjacent parameter values are equally successful.

This suggests that, with $\xi \approx 4\xi_0$, spontaneous vortex creation should be possible, since the length scale $\xi_e$ for a condensate of $N = 10^5$ atoms at this density would give $\xi_e \approx 100\xi_0$. We should beware that the extent to which the KZ bound [4] is saturated depends on the system. To cite extremes for spontaneous vortex formation at thermal quenches, it is saturated for vortex production on quenching $^3\text{He} - B$ [17], but underestimates vortex separation strongly for high-$T_c$ superconductors [5]. The spontaneous creation of vortices in thermal quenches on low-$T_c$ superconductors [19] and fluxons in Josephson junctions [20] give results in between. Finally, we see that the approximation from Eq. (7) to Eq. (8) is justified, with a fractional error less than $k_F a_{\text{bg}} \ll 2.10^{-2}$.

Our conclusions that spontaneous vortex creation could be achievable with current experimental techniques needs further qualification. We stress that, since the vortices form early in the ramp we do not have to continue it far into the BCS regime, nor begin it from the deep BEC regime, where our idealised narrow-resonance approximation fails. This is rather like the situation in thermal quenches in which defects form so close to the critical temperature that there is no need to cool much below it. This has the further advantage in that, although our idealised calculations were for temperature $T = 0$, in reality temperature is finite. By stopping soon enough, we would hope to remain clear of critical thermal behaviour.

Narrow resonances are difficult to work with because of the required field stability, but we expect them to give most defects after a ramp. Increasing resonance width in (22) increases $\tau_Q$ and hence $\xi$ at fixed density. However, with $\xi \propto \gamma_0^{1/2}$ for moderately narrow resonances, the effect of broadening the resonance is, initially, weak and we can still anticipate observable spontaneous phase change for large condensates. [For very broad resonances we have no reliable analytic causal constraint for $\xi$.]

As a final caveat we do not have the homogeneous condensates assumed above and should take the details of their trapping into account. The causal length $\xi \propto (\rho_0 \sqrt{T_0})^{1/2}$ depends upon density and will vary across the trap, but vortices should still form in its centre if the condensate is sufficiently large, albeit with profile-dependent allometric scaling behaviour. In this regard there are many similarities with the analysis of [4] for thermal condensates and we would have to modify our analysis appropriately. This letter is rather aiming for a proof of principle, that causality could lead to observable changes of phase accessible by current experiments.

RR would like to thank the National Dong Hwa University, Hua-Lien, for support and hospitality, where much of this work was performed and Profs. Randy Hulet and Matt Davis for helpful comments. The work of DSL and CYL was supported in part by the National Science Council and the National Center for Theoretical Sciences, Taiwan.

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