ON THE CHARACTERISATIONS OF WAVE FRONT SETS VIA THE SHORT-TIME FOURIER TRANSFORM

STEVAN PILIPOVIĆ AND BOJAN PRANGOSKI

Abstract. It is well known that the classical and Sobolev wave fronts were extended into non-equivalent global versions by the use of the short-time Fourier transform. In this very short paper we give complete characterisations of initial wave front sets via the short-time Fourier transform.

The goal of this note is to give descriptions of the classical (local) wave front set as well as the Sobolev wave front set, both in the sense of Hörmander [6]–[8], of a distribution $f$, by its short-time Fourier transform, from now on abbreviated as STFT. The STFT is also known as the wave-packet transform and it was introduced by Córdoba and Fefferman [2] (see also [5]). The STFT of $f \in \mathcal{S}'(\mathbb{R}^d)$ with a window (also known as wave packet) $\chi \neq 0 \in \mathcal{S}(\mathbb{R}^d)$ is defined by $V_\chi f(x, \xi) = \mathcal{F}_{t \to \xi} f(t) \chi(t-x)$, where $\mathcal{F}$ is the Fourier transform given by $\mathcal{F}g(\xi) = \int_{\mathbb{R}^d} e^{-ix\xi} g(x) dx$, $g \in L^1(\mathbb{R}^d)$. In fact, $V_\chi f$ is a smooth function in $(x, \xi)$.

When the window $\chi$ is compactly supported, we can naturally extend the definition of the STFT even when $f \in \mathcal{D}(\mathbb{R}^d)$; namely $V_\chi f(x, \xi) = \langle e^{-i\xi \cdot } f, \chi(\cdot - x) \rangle$. Clearly, even in this case, $V_\chi f \in C^\infty(\mathbb{R}^{2d})$.

Folland [4] gave a characterisation of the wave front set of $f \in \mathcal{S}'(\mathbb{R}^d)$ via its STFT under some restrictions on the window $\chi \in \mathcal{S}(\mathbb{R}^d)$. Later Ókaji [15] relaxed the restrictions on the window and only recently Kato, Kobayashi and Ito [9] managed to give a characterisation without any restriction on $\chi$. Our paper is motivated by [9] where the authors have given a nice application to the Schrödinger equation.

In several recent papers it was shown that the homogeneous wave front (cf. [10], [11], [12], [13], [14], [17]) and the equivalent to it, Gabor wave front set (cf. [15], [1], [3], [19], [20]) are equivalent to the global one of Hörmander. Up to our knowledge, the local (classical) $C^\infty$ and Sobolev-type Hörmander’s definitions of wave fronts based on the STFT are not given in the literature. We do this in Theorem 1.1. (iv) and Theorem 1.2 (iv). Moreover, Theorem 1.1 (iii) as well as Theorem 1.2 (ii)–(iii) give new characterisations of both wave front sets. The purpose of this article is to fill in these gaps. Note that the $L^q$, $q \in [1, \infty)$, versions, so called $\mathcal{F}L^q_\omega$ weighted wave fronts [16] (equal to the Sobolev one for $q = 2$) can be treated in the same way as in the case $q = 2$.

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1. The main results

We begin by recalling the definition of Hörmander [6] (cf. [7, Definition 8.1.2, p. 254]) for the wave front set \(WF(f) \subseteq \mathbb{R}^d \times (\mathbb{R}^d \setminus \{0\})\) of \(f \in \mathcal{D}'(\mathbb{R}^d)\):

\[(x_0, \xi_0) \in \mathbb{R}^d \times (\mathbb{R}^d \setminus \{0\})\] does not belong to \(WF(f)\) if there exists \(\chi \in \mathcal{D}(\mathbb{R}^d)\) with \(\chi(x_0) \neq 0\) and a cone neighbourhood \(\Gamma\) of \(\xi_0\) such that for every \(n \in \mathbb{N}\) there exists \(C_{n,\chi} > 0\) such that

\[
|\mathcal{F}(\chi f)(\xi)| \leq C_{n,\chi}(1 + |\xi|)^{-n}, \quad \forall \xi \in \Gamma.
\]

**Theorem 1.1.** Let \(f \in \mathcal{D}'(\mathbb{R}^d)\) and \((x_0, \xi_0) \in \mathbb{R}^d \times (\mathbb{R}^d \setminus \{0\})\). The following conditions are equivalent.

1. \((x_0, \xi_0) \notin WF(f)\).
2. There exist a compact neighbourhood \(K\) of \(x_0\) and a cone neighbourhood \(\Gamma\) of \(\xi_0\) such that for every \(n \in \mathbb{N}\) and \(\chi \in \mathcal{D}_K\) there exists \(C_{n,\chi} > 0\) such that (1) is valid.
3. There exist a compact neighbourhood \(K\) of \(x_0\) and a cone neighbourhood \(\Gamma\) of \(\xi_0\) such that for each \(n \in \mathbb{N}\) there exist \(C_n > 0\) and \(k_n \in \mathbb{N}\) such that

\[
|V_{\chi}(f)(x, \xi)| \leq C_n \sup_{|\alpha| \leq k_n} \|D^\alpha \chi\|_{L^\infty(\mathbb{R}^d)} (1 + |\xi|)^{-n}, \quad \forall x \in K, \forall \xi \in \Gamma, \forall \chi \in \mathcal{D}_{K - \{x_0\}},
\]

where \(K - \{x_0\} = \{y \in \mathbb{R}^d | y + x_0 \in K\}\).
4. There exist a compact neighbourhood \(K\) of \(x_0\), a cone neighbourhood \(\Gamma\) of \(\xi_0\) and \(\chi \in \mathcal{D}(\mathbb{R}^d)\), with \(\chi(0) \neq 0\) such that for each \(n \in \mathbb{N}\) there exists \(C_{n,\chi} > 0\) such that

\[
|V_{\chi}(f)(x, \xi)| \leq C_{n,\chi}(1 + |\xi|)^{-n}, \quad \forall x \in K, \forall \xi \in \Gamma.
\]

**Proof.** (i) \(\Rightarrow\) (ii) The fact that \((x_0, \xi_0) \notin WF(f)\) implies the existence of \(\chi \in \mathcal{D}(\mathbb{R}^d)\) with \(\chi(x_0) \neq 0\) and a cone neighbourhood \(\Gamma\) of \(\xi_0\) for which (1) is valid for \(\xi \in \Gamma\). There exists a compact neighbourhood \(K\) of \(x_0\) where \(\chi\) never vanishes. Fix a cone neighbourhood \(\Gamma\) of \(\xi_0\) such that \(\Gamma \subseteq \Gamma' \cup \{0\}\). By employing exactly the same technique as in the proof of [7, Lemma 8.1.1, p. 253] we obtain that for each \(n \in \mathbb{N}\) and \(\psi \in \mathcal{D}_K\) we have

\[
|\mathcal{F}(\psi \chi f)(\xi)| \leq C_{n,\psi,\chi}(1 + |\xi|)^{-n}, \quad \forall \xi \in \Gamma.
\]

Then (ii) immediately follows from this since \(\psi f = (\psi/\chi)f\) where \(\psi/\chi \in \mathcal{D}_K\).

(ii) \(\Rightarrow\) (iii) By (ii), there exists a compact neighbourhood \(K_1\) of \(x_0\) and a cone neighbourhood \(\Gamma\) of \(\xi_0\) such that for every \(n \in \mathbb{N}\) and \(\chi \in \mathcal{D}_{K_1}\), there exists \(C_{n,\chi} > 0\) such that (1) is valid. Of course, we can assume \(K_1 = B_r(x_0)\), for some \(r > 0\), where \(B_r(x_0)\) stands for the open ball with centre \(x_0\) and radius \(r\). Notice that (i) implies that for each \(n \in \mathbb{N}\), the set \(H_n = \{(1 + |\xi|)^n e^{-i\xi \cdot x_0} \cdot f| \xi \in \Gamma\}\) is weakly bounded in \(\mathcal{D}'_{K_1}\) and hence equicontinuous as \(\mathcal{D}_{K_1}\) is barreled. Put \(K = \overline{B_{r/2}(x_0)}\). For each \(\chi \in \mathcal{D}_{K - \{x_0\}}\) and \(x \in K\) the function \(t \mapsto \chi(t - x)\) is in \(\mathcal{D}_{K_1}\) and the equicontinuity of \(H_n\) implies the existence of \(C_n > 0\) and \(k_n \in \mathbb{N}\) such that

\[
|\langle e^{-i\xi \cdot f}, \chi(t - x) \rangle| \leq C_n(1 + |\xi|)^{-n} \sup_{|\alpha| \leq k_n} \sup_{t \in K_1} |D^\alpha \chi(t - x)| = C_n \sup_{|\alpha| \leq k_n} \|D^\alpha \chi\|_{L^\infty(\mathbb{R}^d)} (1 + |\xi|)^{-n}, \quad \forall \xi \in \Gamma, \forall x \in K,
\]

which implies the validity of (iii). Notice that (iii) \(\Rightarrow\) (iv) is trivial and (iv) \(\Rightarrow\) (i) follows immediately by specialising the estimate in (iv) for \(x = x_0\). \(\square\)

1. As usual, \(\mathcal{D}_K\) stands for the Fréchet space of all smooth functions supported by \(K\).
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We can also give similar characterisation of the Sobolev wave front. Before we give the result we recall its definition (see [3] Definition 8.2.5, p. 188; Proposition 8.2.6, p. 189):

Let \( f \in \mathcal{D}'(\mathbb{R}^d) \) and \( s \in \mathbb{R} \). We say that \( (x_0, \xi_0) \in \mathbb{R}^d \times (\mathbb{R}^d \setminus \{0\}) \) does not belong to \( WF_{H^s}(f) \) if there exist \( \chi \in \mathcal{D}(\mathbb{R}^d) \) with \( \chi(x_0) \neq 0 \) and a cone neighbourhood \( \Gamma \) of \( \xi_0 \) such that \( \langle \cdot \rangle^s \mathcal{F}(\chi f) \|_{L^2(\Gamma)} \leq \infty \).

**Theorem 1.2.** Let \( f \in \mathcal{D}'(\mathbb{R}^d) \), \( (x_0, \xi_0) \in \mathbb{R}^d \times (\mathbb{R}^d \setminus \{0\}) \) and \( s \in \mathbb{R} \). The following conditions are equivalent.

(i) \( (x_0, \xi_0) \notin WF_{H^s}(f) \).

(ii) There exist a compact neighbourhood \( K \) of \( x_0 \) and a cone neighbourhood \( \Gamma \) of \( \xi_0 \) such that the mapping \( \chi \mapsto \langle \cdot \rangle^s \mathcal{F}(\chi f) \), \( \mathcal{D}_K \rightarrow L^2(\Gamma) \), is well-defined and continuous.

(iii) There exist a compact neighbourhood \( K \) of \( x_0 \), a cone neighbourhood \( \Gamma \) of \( \xi_0 \) and \( C > 0 \) and \( k \in \mathbb{N} \) such that

\[
\sup_{x \in K} \| \langle \cdot \rangle^s \mathcal{F}(\chi f)(x, \cdot) \|_{L^2(\Gamma)} \leq C \sup_{|\alpha| \leq k} \| D^\alpha \chi \|_{L^\infty(\mathbb{R}^d)}, \quad \forall \chi \in \mathcal{D}_K - \{x_0\}.
\]

(iv) There exist a compact neighbourhood \( K \) of \( x_0 \), a cone neighbourhood \( \Gamma \) of \( \xi_0 \) and \( \chi \in \mathcal{D}(\mathbb{R}^d) \) with \( \chi(0) \neq 0 \) such that \( \sup_{x \in K} \| \langle \cdot \rangle^s \mathcal{F}(\chi f)(x, \cdot) \|_{L^2(\Gamma)} < \infty \).

**Proof.** (i) \( \Rightarrow \) (ii) There exist a cone neighbourhood \( \Gamma' \) of \( \xi_0 \) and \( \chi \in \mathcal{D}(\mathbb{R}^d) \) with \( \chi(0) \neq 0 \) such that \( C_\chi = \| \langle \cdot \rangle^s \mathcal{F}(\chi f) \|_{L^2(\Gamma')} < \infty \). There exists a compact neighbourhood \( K \) of \( x_0 \) where \( \chi \) never vanishes and there are \( C_1, m \geq 1 \) such that \( |\mathcal{F}(\chi f)(\xi)| \leq C_1|\xi|^m \), \( \forall \xi \in \mathbb{R}^d \). Let \( \Gamma \) be a cone neighbourhood of \( \xi_0 \) such that \( \overline{\Gamma} \subseteq \Gamma' \cup \{0\} \). One can find \( 0 < c < 1 \) such that

\[
\{ \eta \in \mathbb{R}^d | \exists \xi \in \Gamma \text{ such that } |\xi - \eta| \leq c|\xi| \} \subseteq \Gamma'.
\]

For \( \psi \in \mathcal{D}_K \), we have \( \mathcal{F}(\psi \chi f) = (2\pi)^{-d} \mathcal{F}(\psi) * \mathcal{F}(\chi f) \) and the Minkowski integral inequality yields

\[
\| \langle \cdot \rangle^s \mathcal{F}(\psi \chi f) \|_{L^2(\Gamma)} \leq \frac{1}{(2\pi)^d} \int_{\mathbb{R}_n^d} \left( \int_{\mathbb{R}_n^d} \langle \xi, \eta \rangle^{2s} |\mathcal{F}(\psi)(\eta)|^2 |\mathcal{F}(\chi f)(\xi - \eta)|^2 d\eta \right)^{1/2} d\xi
\]

\[
\leq I_1/(2\pi)^d + I_2/(2\pi)^d,
\]

where

\[
I_1 = \int_{\mathbb{R}_n^d} |\mathcal{F}(\psi)(\eta)| \left( \int_{|\xi| \geq |\eta|} \langle \xi, \eta \rangle^{2s} |\mathcal{F}(\chi f)(\xi - \eta)|^2 d\xi \right)^{1/2} d\eta,
\]

\[
I_2 = \int_{\mathbb{R}_n^d} |\mathcal{F}(\psi)(\eta)| \left( \int_{|\xi| < |\eta|} \langle \xi, \eta \rangle^{2s} |\mathcal{F}(\chi f)(\xi - \eta)|^2 d\xi \right)^{1/2} d\eta.
\]

First we assume \( s \geq 0 \). A change of variables in the inner integral in \( I_1 \) gives

\[
I_1 = \int_{\mathbb{R}_n^d} |\mathcal{F}(\psi)(\eta)| \left( \int_{|\xi + \eta| \geq |\eta|} \langle \xi + \eta, \eta \rangle^{2s} |\mathcal{F}(\chi f)(\xi)|^2 d\xi \right)^{1/2} d\eta
\]

\[\text{as usual, } \langle \xi \rangle \text{ stands for } (1 + |\xi|^2)^{1/2}\]
\[
\leq (1 - c)^{-s} \int_{\mathbb{R}^d_+} |\mathcal{F}\psi(\eta)| \left( \int_{\Gamma'} |\xi|^2 |\mathcal{F}(\chi f)(\xi)|^2 d\xi \right)^{1/2} d\eta = \frac{C_\chi \|\mathcal{F}\psi\|_{L^1(\mathbb{R}^d)}}{(1 - c)^s},
\]
where, in the inequality we have used \( \{\xi \in \Gamma - \{\eta\} \mid |\xi + \eta| \geq |\eta|/c\} \subseteq \Gamma' \) which easily follows from [2]. For \( I_2 \) we have
\[
I_2 \leq C_1 \int_{\mathbb{R}^d_+} |\mathcal{F}\psi(\eta)| \left( \int_{|\xi| < |\eta|/c} |\xi|^2 |\psi/\chi-f|^{2m} d\xi \right)^{1/2} d\eta \leq C_1 (1 + c^{-1}) m c^{-s-d-1} \|\langle \cdot \rangle^{-d-1} \|_{L^2(\mathbb{R}^d)} \|\langle \cdot \rangle^{m+s+d+1} \mathcal{F}\psi\|_{L^1(\mathbb{R}^d)}^2.
\]
Combining these estimates together we conclude that there exists \( C = C(\chi) > 0 \) such that \( \|\langle \cdot \rangle^s \mathcal{F}(\chi f)\|_{L^2(\Gamma)} \leq C\|\langle \cdot \rangle^{s+m+d+1} \mathcal{F}\psi\|_{L^1(\mathbb{R}^d)}, \forall \psi \in \mathcal{D}_K \). Since for \( \psi \in \mathcal{D}_K \), we have \( \psi f = (\psi/\chi)\chi f \) with \( \psi/\chi \in \mathcal{D}_K \), the claim in (ii) easily follows from this. The case when \( s = 0 \) is similar and we omit it.

(ii) \( \Rightarrow \) (iii) Let \( K_1 \) be a compact neighbourhood of \( x_0 \) and \( \Gamma \) a cone neighbourhood of \( \xi_0 \) such that the mapping \( \chi \mapsto \langle \cdot \rangle^s \mathcal{F}(\chi f), \mathcal{D}_K \rightarrow L^2(\Gamma) \), is well-defined and continuous; of course, without losing generality, we can assume \( K_1 = \overline{B}_r(x_0) \), for some \( r > 0 \). There exist \( C > 0 \) and \( k \in \mathbb{N} \) such that
\[
\|\langle \cdot \rangle^s \mathcal{F}(\chi f)\|_{L^2(\Gamma)} \leq C \sup_{|\alpha| \leq k} \|D^\alpha \chi\|_{L^\infty(K_1)}, \forall \chi \in \mathcal{D}_K.
\]
Put \( K = \overline{B}_{\sqrt{2}}(x_0) \). For \( \chi \in \mathcal{D}_{K-K_1} \) and \( x \in K \), the function \( \chi_x : t \mapsto \chi(t-x) \) belongs to \( \mathcal{D}_{K_1} \) and, as \( \mathcal{F}(\chi_x f)(\xi) = V_x f(x,\xi) \), we have
\[
\sup_{x \in K} \|\langle \cdot \rangle^s V_x f(x,\cdot)\|_{L^2(\Gamma)} \leq C \sup_{x \in K} \sup_{|\alpha| \leq k} \|D^\alpha \chi_x\|_{L^\infty(K_1)} = C \sup_{|\alpha| \leq k} \|D^\alpha \chi\|_{L^\infty(\mathbb{R}^d)}.
\]
The implications (iii) \( \Rightarrow \) (iv) is trivial and (iv) \( \Rightarrow \) (i) follows immediately by specialising \( x = x_0 \).

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