Critical dynamics of gauge systems: Spontaneous vortex formation in 2D superconductors

G. J. Stephens1, Luís M. A. Bettencourt2 and W. H. Zurek1
1Theoretical Division T-6 MS B288, Los Alamos National Laboratory, Los Alamos NM 87545
2Center for Theoretical Physics, Massachusetts Institute of Technology, Bldg. 6-308, Cambridge MA 02139

We examine the formation of vortices during the nonequilibrium relaxation of a high-temperature initial state of an Abelian-Higgs system. We equilibrate the scalar and gauge fields using gauge-invariant Langevin equations and relax the system by instantaneously removing thermal fluctuations. For couplings near critical, \( \kappa_c = \sqrt{\lambda/e} = 1 \), we observe the formation of large clusters of like-sign magnetic vortices. Their appearance has implications for the dynamics of the phase transition, for the distribution of topological defects and for late-time phase ordering kinetics. We offer explanations for both the observed vortex densities and vortex configurations.

PACS Numbers : 74.40.+k, 05.70.Fh, 11.27.+d, 98.80.Cq MIT-CTP-3109 LAUR-01-2579

Recently much progress has been made in the study of the dynamics of phase transitions. The emerging understanding may prove relevant for a variety of experiments, ranging from the collisions of heavy nuclei at RHIC [1] to the sudden cooling of condensed matter systems [2,7]. Nonequilibrium behavior is particularly important for understanding the population of topological defects which remains after a transition [8,9].

While most previous research focused on global symmetries, it is useful to explore systems with (local) gauge symmetry [10]. This includes critical phenomena in the early universe, heavy-ion collisions and superconductors [11]. With gauge symmetries, disorder is more subtle as local phase gradients can be removed by gauge transformations [12]. Moreover, gauge fields contribute to the (thermo)dynamics of the system, potentially leading to new phenomenology.

In this work we examine how critical dynamics is altered in the presence of gauge fields. We choose a simple and controlled physical setting, describing the behavior of long wavelength fields in the quench of 2D superconductors. Disorder in the superconducting state is associated with the presence of topological configurations (vortices). We analyze the formation and dynamics of these defects and trace their origins to thermal electromagnetic fluctuations. The most important result of this work is the detection of large clusters of like-sign vortices which form in the wake of a thermal quench.

Hindmarsh and Rajantie (HR) [13] recently argued that gauge fields alter the standard Kibble-Zurek predictions for the distribution of topological defects formed in a phase transition. They also observed that magnetic fluctuations can create vortex clusters. We provide evidence strongly correlating the density and distribution of defects to thermal magnetic fluctuations. In distinction to HR, we argue that an important role is played by the critical magnetic fields of a superconductor and we exhibit the importance of magnetic fluctuations on a variety of large length scales in seeding vortex clusters.

We investigate the dynamics of a complex scalar field minimally coupled to electromagnetism, the Abelian-Higgs model, in two dimensions. This provides, for example, a phenomenological description of a superconducting film. To accurately compare our results with experimental films the effects of a small third dimension need to be carefully considered. Nonetheless, this system provides a tractable and interesting model, allowing for a detailed examination of the importance of magnetic fluctuations on critical dynamics and defect formation. The Lagrangian density is

\[
\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} |D_\mu \phi|^2 - \frac{\lambda}{8} (|\phi|^2 - v^2)^2, \tag{1}
\]

where \( \phi \) is a complex scalar field, \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \) is the field strength tensor for electromagnetic gauge potential \( A_\mu \) and \( D_\mu = \partial_\mu - ie A_\mu \). Of the three parameters in the model \( (\lambda, e \text{ and } v) \) two may be rescaled away through coordinate and field redefinitions \( \phi \to \phi', A_\mu \to A_\mu v, x \to v\sqrt{\lambda} x, t \to v\sqrt{\lambda t} \) and \( \beta = v^{-2} \beta \), with \( \beta = (kbT)^{-1} \), where \( k_B \) is Boltzmann’s constant. The dimensionless parameter \( \kappa = \frac{\kappa^2}{v^2} \) (the ratio of the London penetration depth to the scalar correlation length) controls the relative strength of the gauge interactions. With these choices the crossover between type-I \( (\kappa < 1) \) and type-II \( (\kappa > 1) \) superconductivity occurs at critical coupling, \( \kappa_c = 1 \). When \( \kappa \to \infty \) the global phase invariance of the theory is recovered. In the following we work in temporal gauge, \( A_0 = 0 \), and therefore obtain Gauss’ law, \( \nabla \cdot E = 2e \text{ Im } |\phi|^2 \), as a constraint on the evolution.

To thermalize the system the scalar and gauge fields are coupled to a reservoir at a temperature \( T \). This is achieved by adding dissipative and stochastic Langevin sources to the equations of motion. These cannot break gauge invariance, i.e. the Gauss constraint must be preserved, resulting, generally, in multiplicative noise and fluctuations [8].

\[
D_\mu = \partial_\mu - ie A_\mu, \quad A_\mu = \frac{1}{2} F_{\mu\nu} \eta^{\nu}, \quad (\eta^{\mu} F_{\mu\nu} = 0), \quad \eta^{\mu} = (1, -e). \tag{2}
\]
dissipation terms \[14\]. There is more than one gauge invariant form for the equations of motion and we choose to couple our system to a reservoir through the observables \(\{\phi^2, \bar{E}\}\). The details of this choice are irrelevant to the state of canonical equilibrium reached by the system at long times.

After equilibrating the system, we instantaneously remove the noise and evolve with overdamped dynamics \[17\].

\[
\partial_t \phi_a = - \left[ \nabla^2 - e^2|A|^2 - \frac{1}{2}(|\phi|^2 - 1) \right] \phi_a + 2e\epsilon_{ab}A^i\partial_i\phi_b, \\
\partial_t A_i = -(\nabla \times B)_i + J_i, \\
J_i = -e^2|\phi|^2A_i - \epsilon\epsilon_{ab}\phi_a\partial_i\phi_b.
\]  

(2)

The system of equations (2), expressed in terms of lattice gauge invariant fields \[17\], was solved numerically. The Gauss constraint was satisfied to about 0.1\%. We used lattice spacing \(dx = 0.5\) and time step \(dt = 0.02\).

A snapshot taken after the quench reveals striking structure in the spatial distribution of vortices. In Fig. 1 we provide a contour plot of the magnetic flux across a section of the lattice with \(e = 0.5\) after a quench from \(T = 10\). The critical temperature is \(T_c \approx 0.35\). The snapshot is late enough in the quench that the magnetic flux is almost entirely localized into vortices, each of which carries the fundamental flux quantum \(\Phi_0 = 2\pi\). The large clusters of like-sign vortices are an obvious departure from the spatial vortex distribution observed after a quench of purely scalar fields. In the scalar case, defects and antidefects appear primarily as widely separated and anticorrelated pairs \[18\]. The appearance of closely-packed vortices is also surprising because (though they are not strongly coupled) like-sign vortices repel.

To explain the clustering of vortices we explore the dynamics of the electromagnetic fields. In equilibrium, at sufficiently high \(T\) above the transition, \(\langle B_k B_{-k} \rangle \sim T\). When we remove the external noise, all fields dissipate. However, not all length scales dissipate at the same rate. Long wavelength magnetic fluctuations suffer an analog of critical slowing down due to the absence of a magnetic thermal mass in an Abelian gauge theory \[16\]. Under these circumstances we expect

\[
B_k(t) \sim B_k(0) \exp \left[ -k^2t \right].
\]  

(3)

While strictly true only for free dynamics, we verified numerically that Eq. (3) adequately describes the behavior of \(B\) in the coupled system for the initial stages of cooling. This overdamped evolution results in the freezing of magnetic fluctuations on large spatial scales. Fluctuations in the scalar field are also damped during the cooling. When fluctuations of the scalar field have decayed sufficiently, long-wavelength modes feel the spinodal instability and the scalar field rolls to its nonzero minimum.

![Fig. 1. Contour plot of the magnetic flux across a section of the lattice after the quench. White, localized, regions denote vortices while black regions denote antivortices. The giant like-sign vortex clusters are surprising and are a result of the quench dynamics in a type-II superconductor, beginning with the formation of superconducting regions (in which the gauge fields are small \(B < B_{c1}\)) surrounding normal islands of coherent field. As the quench proceeds the normal islands fragment into clusters of individual vortices.](image)

The evolution of the system can be qualitatively understood with reference to the state of minimal free energy in an external magnetic field. There are three possibilities \[17\] delimited by

\[
B_{c1} = \Phi_0 \log \kappa/4\pi\xi_S^2, \quad B_{c2} = \Phi_0/2\pi\xi_S^2,
\]

(4)

where \(\xi_S\) and \(\xi_G\) are the scalar and gauge correlation lengths. Regions where \(B > B_{c2}\) are forced to remain in the normal phase. For \(B_{c1} < B < B_{c2}\) the system enters a vortex state, where \(B\) penetrates a superconducting state in an array of like-sign vortices. For \(B < B_{c1}\) the magnetic field is completely expelled and the system becomes a spatially homogeneous superconductor. In the presence of large thermal \(B\), the quench dynamics cause the system to enter the vortex state in certain regions of space. Once formed the vortex state is long-lived because topological charge is conserved and the evolution is overdamped. Where \(B < B_{c1}\), no vortices are formed by this mechanism; the flux is expelled by a Meissner effect and the picture of defect formation in terms of the dynamics of the scalar field alone is recovered. We arrive at a picture of spontaneous vortex formation with two primary qualifications:

1. The number of vortices formed in the transition is proportional to the amount of magnetic flux present when the system develops supercurrents.

2. The magnetic field splinters into vortex clusters when \(B > B_{c1}\). The clustered vortex distribution reflects the long-range correlations of the initial magnetic field, amplified by the expulsion of flux from regions where \(B < B_{c1}\).
Prediction 1 was first made by HR and our analysis offers quantitative evidence that this is correct. However, our second prediction is based on a qualitatively different picture of the dynamics of the gauge fields and order parameter during the phase transition.

To test the first prediction we examine the magnetic flux, \(|\Phi| = \int B_z dz x\). If we replace \(B^2\) by its average value then

\[ |\Phi| \sim \sqrt{\langle B^2 \rangle} A, \tag{5} \]

where \(A\) is the total area of the film. The determination of the time when supercurrents develop and therefore the point at which to evaluate \(\sqrt{\langle B^2 \rangle}\) depends on the coupled dynamics of the scalar and electromagnetic field and will be addressed elsewhere. However, we can make the following (parametric) prediction [13],

\[ \rho_{\text{def}} = C e \sqrt{T}, \tag{6} \]

where \(\rho_{\text{def}}\) is the (area) density of vortices and \(T\) is the initial temperature. \(C\) carries dimensions of inverse length and parameterizes the details of the cooling dynamics. In practice we found \(C \sim 0.01\) demonstrating that only a small fraction of the original thermal magnetic energy is available for vortex formation. We expect Eq. (5) to hold only when the magnetic field at the time of formation is greater than \(B_{c1}\).

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig2.png}
\caption{a) \(\rho_{\text{def}} \times 10^3\) vs. \(e\) from numerical evolutions. The solid line is the best linear fit. b) \(\rho_{\text{def}} \times 10^3\) vs. \(T\) from numerical simulations. The solid line is the best power-law fit \(\rho \sim T^\alpha\), where \(\alpha = 0.43 \pm 0.02\). In both plots error bars denote standard deviations among stochastic realizations.}
\end{figure}

Figs. 2a and 2b show \(\rho_{\text{def}}\) vs. \(e\) and the initial temperature \(T\) respectively. Both are in agreement with Eq. (6). The slightly smaller exponent in Fig. 2b is an indication that some fluctuations decay before vortices are formed.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig3.png}
\caption{a) \(N_c(l)\) for the cluster distribution of Fig. 1 (circles) compared against a random distribution with the same number of defects (triangles). The solid line shows the behavior expected from equilibrium fluctuations in the magnetic flux, \(\Phi_{\text{net}} = \sqrt{Nc}\). The close agreement between \(N_c(l)\) measured from the vortex distribution and \(N_c(l)\) calculated from equilibrium is strong evidence that vortex clusters in the gauge quench originate in equilibrium magnetic fluctuations. b) \(N_c(l)\) for the purely scalar theory, \(e = 0\), normalized to the same density as Fig. 3a. The solid line is the best fit to \(\sqrt{N}\).}
\end{figure}

While the number of vortices depends on the details of the cooling dynamics, the clustered distribution of vortices reflects initial magnetic thermal fluctuations on many scales. During the quench, magnetic fields are first damped and then redistributed by the Meissner effect: expelled where \(B < B_{c1}\) into initially normal magnetized islands, which eventually fragment into vortices. To quantify the properties of the vortex cluster distribution we examine the average excess topological charge \(N_c(l)\) within a square of side \(l\). \(N_c(l)\) can be measured in experiments using e.g. a SQUID, sensitive to flux over an area \(l^2\). To calculate \(N_c(l)\) we compute the absolute value of the net topological charge in a box of side length \(l\) centered on a lattice point. We then average over every lattice point. Fig. 3 shows the result for a typical run with \(e = 0.5\). \(N_c(l)\) observed for the gauge quench is indicative of clustering and is larger than that of a random distribution of the same number of defects. At small \(l\) the presence of clusters of like-sign vortices leads to faster than linear growth of \(N_c(l)\). At large \(l\) both distributions increase linearly. This is expected when defects are positioned at random; with average density \(\rho_{\text{def}}\) fluctuations are of order \(\sqrt{N(l)}\) where \(N(l)\) is the average number of defects in box of size \(l\). Therefore

\[ N_{c\text{random}}(l) = \sqrt{\rho_{\text{def}} l}. \tag{7} \]

The linear behavior and larger slope of \(N_c(l)\) measured from a gauge quench suggest that clusters of like-sign
defects are randomly distributed.

In scalar systems defect distributions are not random. Fluctuations of the net topological number over an area occur due to the fluctuations of the field phase along the boundary. After a quench, the phase is correlated on a scale $\xi$ with $l/\xi$ separate domains along a path of length $l$. Thus,

$$N_e(l) \sim (\rho_{\text{def}})^{1/4} \sqrt{l}. \quad (8)$$

Fig. 3a shows $N_e(l)$ measured from simulations of a purely scalar theory. Eq. (8) has also been verified experimentally in nematic liquid crystals [11] and is roughly consistent with measured fluctuations in the net flux trapped within a loop of Josephson junctions [12].

It is instructive to examine defect formation in the presence of an imposed magnetic field (see also [13]). We prepare an uncoupled initial state in which the scalar field is thermalized to a temperature close to $T_c$ and the magnetic field is

$$B(t=0) = B_0 \cos \left( \frac{2\pi x}{L} \right). \quad (9)$$

At the time of the quench we turn on the coupling (to $\epsilon = 0.5$) and evolve with the overdamped equations [14]. The results for two different $B_0$ are shown in Fig. 4.

For small amplitudes, the final defect distribution is not heavily influenced by the magnetic field and large vortex clusters are absent. In contrast for $B > B_{c1}$ the magnetic field has a profound influence on the final defect distribution. These simple considerations mimic the behavior occurring on many length scales in the thermal quench.

In conclusion, we have shown that the critical dynamics of 2D superconductors differs substantially from scalar field models. Specifically we observed that for quenches from high $T$ and for $\kappa \gtrsim \kappa_c$, remnants of long-wavelength magnetic fields seed the formation of topological defects in much larger numbers and in strikingly different configurations than scalar field theories. We established the dependence of the density of vortices on $\epsilon$ and the initial temperature $T$ and we connected the large-scale properties of the vortex distribution to the spectrum of initial thermal magnetic fluctuations. We also argued for the importance of $B_{c1}$ in determining the effect of magnetic fluctuations upon defect formation. With recent experimental progress in vortex imaging [15], clusters such as Fig. 1 might be discernible in superconductors with $\kappa \sim \kappa_c$ (e.g. Nb). A detailed analysis of the statistical properties of these clusters is under investigation.

We thank M. Hindmarsh, A. Rajantie and N. Goldenfeld for useful comments. This work was partly supported by D.O.E grant DF-FC02-94ER40818. Numerical work was performed on the T/CNLS Avalon cluster at LANL.