Abstract

Qualitative numerical planning is classical planning extended with non-negative real variables that can be increased or decreased “qualitatively”, i.e., by positive random amounts. While deterministic planning with numerical variables is undecidable in general, qualitative numerical planning is decidable and provides a convenient abstract model for generalized planning. Qualitative numerical planning, introduced by Srivastava, Zilberstein, Immerman, and Geffner (2011), showed that solutions to qualitative numerical problems (QNPs) correspond to the strong cyclic solutions of an associated fully observable non-deterministic (FOND) problem that terminate. The approach leads to a generate-and-test algorithm for solving QNPs where solutions to a FOND problem are generated one by one and tested for termination. The computational shortcomings of this approach, however, are that it is not simple to amend FOND planners to generate all solutions, and that the number of solutions to check can be doubly exponential in the number of variables. In this work we address these limitations, while providing additional insights on QNPs. More precisely, we introduce two reductions, one from QNPs to FOND problems and the other from FOND problems to QNPs both of which do not involve termination tests. A result of these reductions is that QNPs are shown to have the same expressive power and the same complexity as FOND problems.

1. Introduction

Qualitative numerical problems (QNPs) are classical planning problems extended with non-negative numerical variables $X$ that can be decreased of increased “qualitatively”, i.e., by random amounts. Since such numerical variables cannot be used for counting, QNP planning, unlike most other general forms of planning with numbers (Helmert, 2002), turns out to be decidable. QNPs were introduced by Srivastava et al. (2011) as a useful model for generalized planning, namely, the synthesis of plans that solve multiple classical planning instances (Levesque, 2005; Bonet, Palacios, & Geffner, 2009; Srivastava, Immerman, & Zilberstein, 2011; Hu & De Giacomo, 2011; Bonet & Geffner, 2015; Belle & Levesque, 2016; Jiménez, Segovia-Aguas, & Jonsson, 2019). Basically, collections $Q$ of planning instances $P$ that share the same set of actions and state features may be often expressed as a single QNP problem $Q$ whose solutions, that map state features into actions, then solve all problems $P$ in $Q$ (Bonet & Geffner, 2018).

QNPs can be solved in two steps (Srivastava et al., 2011). First, the QNP $Q$ is converted into a standard fully observable non-deterministic (FOND) problem $P$ (Cimatti, Pistore, Roveri, & Traverso, 2003). Then, solutions of $P$ obtained by an off-the-shelf FOND planner are tested for termination. This last step is required because the non-determinism in the FOND problem $P$ is not fair but conditionally fair: infinite qualitative decrements of a numerical variable $X$ make the expression $X = 0$ true eventually, provided that $X$ is increased not more than a finite number of

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times (Bonet, De Giacomo, Geffner, & Rubin, 2017). The policies that solve \( P \) and terminate are then exactly the policies that solve the QNP \( Q \) (Srivastava et al., 2011).

The computational shortcomings of solving QNPs following this generate-and-test approach, however, are two. First, it is not simple to amend FOND planners to generate all the solutions of a FOND problem because FOND plans are not action sequences but closed-loop policies. Second, the number of policies that need to be tested for termination may be huge: exponential in the number of FOND states, and hence, doubly exponential in the number of variables.

In this work we address these limitations while providing additional insights on QNPs. We introduce two polynomial-time reductions, one from QNPs to FOND problems and the other from FOND problems to QNPs. As every (formal) reduction, both reduction are sound and complete, and hence do not require termination tests. A result of these reductions is that QNPs are shown to have the same expressive power and in particular, the plan-existence decision problem for both have the same complexity EXP-Complete (Littman, Goldsmith, & Mundhenk, 1998; Rintanen, 2004). The new QNP to FOND translator is implemented and available. In combination with sound and complete FOND planners, the translation yields the only sound and complete QNP planner available.

The structure of the paper is as follows. We review first classical planning, FOND planning, QNPs, the direct translation of QNPs into FOND problems, and the termination test. We follow ideas from Srivastava et al. (2011) but in a slightly different and more expressive formulation. We then introduce the two new reductions: from FOND problems into QNPs, and from QNPs into FOND problems. This last reduction is very different from the one sketched in (Bonet et al., 2017) which is buggy (we illustrate this with an example). We then consider variations and extensions of QNPs and FOND problems, look at the relation between the two in terms of the fairness assumptions that each model makes, discuss related work, and report experimental results.

2. Classical and FOND planning

A classical planning problem is a sequential decision problem where a goal is to be reached by performing actions with deterministic effects from a given initial state. These problems are usually expressed in compact form in planning languages such as STRIPS (Fikes & Nilsson, 1971; Russell & Norvig, 2002). A (grounded) STRIPS planning problem (with negation) is a tuple \( P = \langle F, I, O, G \rangle \) where \( F \) denotes a set of propositional variables, \( I \) and \( G \) are sets of \( F \)-literals representing the initial and goal situation, and \( O \) is a set of actions \( a \) with preconditions and effects \( \text{Pre}(a) \) and \( \text{Eff}(a) \) given by sets of \( F \)-literals.

The state model \( S(P) \) for the problem \( P = \langle F, I, O, G \rangle \) is a tuple \( S(P) = \langle S, s_0, \text{Act}, A, f, S_G \rangle \) where \( S \) is the set of possible truth-valuations over the \( F \) literals, called the states, \( s_0 \) is the initial state, \( \text{Act} = O, A(s) \) represents the actions \( a \) in \( \text{Act} \) whose preconditions are true in \( s \), \( f(a, s) \) represents the state \( s' \) that follows action \( a \) in \( s \) for \( a \in A(s) \), and \( S_G \) is the set of goal states. It is assumed that the problem \( P \) is consistent in the sense that \( s_0 \) and \( f \) are well-defined and \( S_G \) is not empty. A solution to a classical problem \( P \) is an action sequence \( a_0, \ldots, a_n \) that generates a state sequence \( s_0, \ldots, s_{n+1} \) over the model \( S(P) \) that reaches the goal. In this sequence, \( a_i \in A(s_i) \) and \( s_{i+1} = f(a_i, s_i) \) for \( i = 0, \ldots, n \), and \( s_{n+1} \in S_G \).

A fully-observable non-deterministic (FOND) problem \( P \) is like a classical planning problem except that actions \( a \) may have non-deterministic effects expressed as \( \text{Eff}_1(a) | \cdots | \text{Eff}_n(a) \) where \( \text{Eff}_i(a) \) is a set of \( F \)-literals as above (Cimatti et al., 2003; Geffner & Bonet, 2013; Ghallab, Nau, & Traverso, 2016). The state model \( S(P) \) determined by a FOND problem \( P = \langle F, I, O, G \rangle \) is a tuple \( S(P) = \langle S, s_0, \text{Act}, A, F, S_G \rangle \) as above with the difference that the state transition function \( F \) is non-deterministic, and maps an action \( a \) and state \( s \) into a non-empty set \( F(a, s) \) of possible
successor states. As usual, the non-deterministic transition function $F$ is given in factored form. That is, for action $a$ made of multiple effects $\text{Eff}_1 \cdots \text{Eff}_n$ (possibly deterministic when $n = 1$), each outcome $s'$ in $F(a, s)$ results of the choice of one $\text{Eff}_i$ for each non-deterministic effect of $a$.\footnote{As it is standard, any choice of effects is assumed to be consistent (i.e., any pair of choices for two different non-deterministic effects of the same action contain no complementary literals). However, with some (polynomially bounded) extra work, our methods, algorithms and results still apply if the model is extended with constraints that every outcome $s'$ must satisfy, when such constraints are given in suitable form; e.g. DNF formulas over $F$.}

The solutions of FOND problems ensure that the goal is reached with certainty under certain fairness assumptions. Policies or plans in the FOND setting are partial functions $\pi$ mapping states $s$ into actions $\pi(s)$. A state trajectory $s_0, s_1, \ldots, s_n$ finite or infinite is induced by $\pi$ over the model $S(P)$ if the action $a_i = \pi(s_i)$ is defined, it is applicable in the state $s_i$, i.e., $a_i \in A(s_i)$, and $s_{i+1}$ is in $F(a_i, s_i)$, $i = 1, \ldots, n-1$. The trajectory is said to be a $\pi$-trajectory. The trajectory is maximal if A) it is infinite, i.e., $n = \infty$, and does not include a goal state, B) if $s_n$ is the first goal state in the sequence, or C) the action $\pi(s_n)$ is not defined or not applicable in $s_n$.

A policy $\pi$ is a solution of the FOND problem $P$ if all the fair maximal trajectories induced by $\pi$ over the model $S(P)$ are goal reaching (Cimatti, Roveri, & Traverso, 1998; Cimatti et al., 2003). The so-called strong solutions assume that all state trajectories are fair. Strong-cyclic solutions, on the other hand, assume that all trajectories are fair except the infinite trajectories where a state $s$ occurs infinitely often but a state transition $(s, s')$ for some $s' \in F(a, s)$ for $a = \pi(s)$, occurs finitely often. The latter trajectories are deemed to be unfair.

Other equivalent characterizations of strong and strong cyclic solutions are common. For example, a strong cyclic solution $\pi$ for a FOND problem $P$ is also a policy $\pi$ such that for each $\pi$-trajectory connecting an initial state to a state $s$, there is a $\pi$-trajectory connecting $s$ to a goal state. Similarly, a strong solution is a strong cyclic solution $\pi$ with no cycles; i.e., one where no $\pi$-trajectory visits the same state twice.

Strong solutions can also be thought as winning strategies against an adversary, while strong cyclic solutions as winning strategies against nature. Indeed, there is a well known relation between (proper) policies that achieve the goal with probability 1 in goal-based MDPs (Markov Decision Processes) and the strong cyclic policies that solve the FOND problem associated with the MDP, where the transition function is such that $F(a, s)$ collects the states $s'$ that are possible after action $a$ in $s$, i.e., for which $P_a(s'|s) > 0$ (Geffner & Bonet, 2013).

From now, by solution of a FOND problem we mean a strong cyclic solution, and by a FOND planner, we mean a strong cyclic planner. There are some good FOND planners available, including PRP (Muise, McIlraith, & Beck, 2012), based on classical planners, MyND (Bercher & Mattmüller, 2009), based on heuristic AND/OR search, and FOND-SAT (Geffner & Geffner, 2018), based on a reduction to SAT.

3. Qualitative Numerical Problems

Qualitative numerical problems (QNPs) are classical planning problems extended with numerical variables that can be decremented or incremented “qualitatively”. We make this formal below.

3.1 QNPs: Syntax

The syntax of QNPs is defined as an extension of the STRIPS language with negation. A QNP is a tuple $Q = \langle F, V, I, O, G \rangle$ where the new component is a set $V$ of non-negative numerical variables $X \in V$. These variables introduce the non-propositional atoms $X = 0$ and their negations, denoted as $X > 0$. These literals can appear in the initial situation, action preconditions, and goals of $Q$.\footnote{The syntax of QNPs is defined as an extension of the STRIPS language with negation. A QNP is a tuple $Q = \langle F, V, I, O, G \rangle$ where the new component is a set $V$ of non-negative numerical variables $X \in V$. These variables introduce the non-propositional atoms $X = 0$ and their negations, denoted as $X > 0$. These literals can appear in the initial situation, action preconditions, and goals of $Q$.}
The effects of actions \( a \) on a numerical variable \( X \) can be just qualitative increments or qualitative decrements denoted by the expressions \( \text{Inc}(X) \) and \( \text{Dec}(X) \). We refer to \( X = 0 \) and \( X > 0 \) as the \( V \)-literals for \( X \in V \), and to \( p \) and \( \neg p \) for \( p \in F \), as the \( F \)-literals in \( Q \).

**Definition 1.** A QNP is a tuple \( Q = \langle F, V, I, O, G \rangle \) where \( F \) and \( V \) are sets of propositional and numerical variables respectively, \( I \) and \( G \) denote the initial and goal situations, and \( O \) is a set of actions \( a \) with preconditions, and propositional and numerical effects that are denoted as \( \text{Pre}(a) \), \( \text{Eff}(a) \), and \( \text{N}(a) \) respectively. The \( F \)-literals can appear in \( I \), \( G \), \( \text{Pre}(a) \), and \( \text{Eff}(a) \), while \( V \)-literals can appear in \( I \), \( G \), and \( \text{Pre}(a) \). The numerical effects \( \text{N}(a) \) only contains special atoms of the form \( \text{Inc}(X) \) or \( \text{Dec}(X) \) for the variables \( X \) in \( V \). Actions with the \( \text{Dec}(X) \) effect must feature the precondition \( X > 0 \) for any variable \( X \) in \( V \).

QNPs are assumed to be syntactically consistent by requiring that no pair of complementary literals or qualitative effects appears in the initial situation, action effects, or goals. A pair of complementary literals or qualitative effects has the form \( \{p, \neg p\} \) for some \( p \) in \( F \), or \( \{X = 0, X > 0\} \) or \( \{X \downarrow, X \uparrow\} \) for some \( X \) in \( V \).

The preconditions, propositional and numerical effects for an action \( a \) are denoted by pairs \( \langle \text{Pre}(a); \text{Eff}(a), \text{N}(a) \rangle \) where the \( \text{Inc}(X) \) and \( \text{Dec}(X) \) expressions in \( \text{N}(a) \) are abbreviated by \( X \uparrow \) and \( X \downarrow \) respectively. Before defining the semantics of QNPs, let us consider an example.

**Example.** An abstraction that is suitable for expressing the generalized problem of achieving the goal \( \text{clear}(x) \) over an arbitrary Blocksworld instance (Bonet & Geffner, 2018) is given in terms of the QNP \( Q_{\text{clear}} = \langle F, V, I, O, G \rangle \) where \( F = \{H\} \) contains a boolean variable \( H \) that represents if the gripper is holding a block, \( V = \{n(x)\} \) contains a numerical variable \( n(x) \) that represents the number of blocks above \( x \), and \( I = \{\neg H, n(x) > 0\} \) and \( G = \{n(x) = 0\} \) represent the initial and goal situations. The actions \( O = \{a, b\} \) are

\[
a = \langle \neg H, n(x) > 0; H, n(x) \downarrow \rangle \tag{1}
\]

and

\[
b = \langle H; \neg H \rangle. \tag{2}
\]

It is easy to see that the first action \( a \) picks up blocks that are above \( x \); its first precondition \( \neg H \) expresses that the gripper is holding no block while the second \( n(x) > 0 \) that there is at least one block above \( x \). The effects, on the other hand, make \( H \) true (expressing that some block is being held) and decrease the number \( n(x) \) of blocks above \( x \). The other action \( b \) puts blocks being held away from block \( x \) as expressed by its precondition \( H \) and effect \( \neg H \). The fact that \( b \) puts blocks away from block \( x \) is reflected in that it does not affect the variable \( n(x) \).

The QNP \( Q_{\text{clear}} \) captures the relevant part of the infinite collection of Blocksworld instances where the goal if to achieve the atom \( \text{clear}(x) \) for some block \( x \). The solution to \( Q_{\text{clear}} \) provides the general strategy for solving all such instances. For ways of learning such abstractions automatically; see (Bonet, Francès, & Geffner, 2019).

### 3.2 QNPs: Semantics

A state \( s \) for QNP \( Q = \langle F, V, I, O, G \rangle \) is a valuation that assigns a truth value \( s[p] \) to each boolean variable \( p \in F \), and a non-negative real value \( s[X] \) to each numerical variable \( X \in V \). Since the initial situation \( I \) can only feature atoms of the form \( X = 0 \) or \( X > 0 \), there is a set \( S_0 \) of possible initial states \( s_0 \) that correspond to the valuations that satisfy the literals in \( I \). For example, in
$Q_{\text{clear}(x)}$, $I$ is given by the literals $I = \{ \neg H, n(x) > 0 \}$, meaning that $S_0$ contains all and only the valuations that make $H$ false and $n(x) = r$ for some positive real number $r$. The use of variables that can take real values for representing integer counters illustrates that the semantics of QNPs is coarse-grained, and for this reason, decidable. Indeed, QNPs use just one qualitative property of numbers; namely, that a non-negative variable that is decreased infinitely often and increased finitely often, eventually must reach the value of zero. This property is true for integers, and it also true for reals, as long as the magnitude of the decrements is bounded from below by some positive $\varepsilon$-parameter. More about this below. The state model $S(Q)$ represented by a QNP can be characterized as follows:

**Definition 2.** A QNP $Q = \langle F, V, I, O, G \rangle$ determines a non-deterministic state model $S(Q) = \langle S, S_0, Act, A, F, S_G \rangle$ where

- the states $s$ in $S$ are the valuations that assign a truth value to the boolean variables in $F$ and a non-negative real value to the numerical variables in $V$,
- the initial states $s_0$ in $S_0$ are those that satisfy the literals in $I$ under a closed-world assumption ($s_0$ makes $p$ and $X = 0$ false if the literals $p$ and $X = 0$ are not in $I$),
- the actions in $Act$ are those in $O$; i.e., $Act = O$,
- the actions $A(s)$ applicable in $s$ are those in $Act$ such that $Pre(a)$ is true in $s$,
- the goal states in $S_G$ are those that satisfy $G$,
- the transition function $F$ is such that $s' \in F(a, s)$ for $a \in A(s)$ if
  - (a) $s'[p]$ is true (resp. false) if $p$ (resp. $\neg p$) is in $Eff(a)$,
  - (b) $s[X] < s'[X]$ if $Inc(X)$ is in $N(a)$,
  - (c) $s'[X] < s[X]$ if $Dec(X)$ is in $N(a)$,
  - (d) $s'[p] = s[p]$ if neither $p$ nor $\neg p$ in $Eff(a)$,
  - (e) $s'[X] = s[X]$ if neither $X$ nor $\neg X$ in $N(a)$.

A trajectory $s_0, a_0, s_1, a_1, \ldots, s_n$ is compatible with the model $S(Q) = \langle S, S_0, Act, A, F, S_G \rangle$ if $s_0 \in S_0$, and $a_i \in A(s_i)$ and $s_{i+1} \in F(a_i, s_i)$ for each $a_i$ in the sequence. The trajectory is an $\varepsilon$-bounded trajectory or $\varepsilon$-trajectory if the numerical changes are bounded from below by a parameter $\varepsilon > 0$, except when this would make the variable negative:

**Definition 3.** A trajectory $s_0, a_0, s_1, a_1, \ldots, s_n$ is an $\varepsilon$-trajectory iff for any variable $X$ and time point $i$, with $i < n$, $s_{i+1}[X] \neq s_i[X]$ implies $|s_{i+1}[X] - s_i[X]| \geq \varepsilon$ or $0 = s_{i+1}[X] < s_i[X] < \varepsilon$.

Trajectories bounded by $\varepsilon > 0$ cannot decrease the value of a variable asymptotically without ever reaching the value of zero. This is in agreement with the key assumption in QNPs by which variables that are decreased infinitely often and increased finitely often must eventually reach the value zero. From now, trajectories over QNPs will refer to $\varepsilon$-trajectories for some $\varepsilon > 0$.

### 3.3 QNPs: Solutions

Solutions to QNPs take the form of partial functions or policies $\pi$ that map states into actions. The choice of the action $\pi(s)$ to be done in a state $s$, however, can only depend on the truth values $s[p]$ associated with the boolean variables $p$ in $F$ and the truth values of the expressions $s[X] = 0$
associated with the numerical variables \( X \) in \( V \). If we use the notation \( s[X = 0] \) to refer to \( s[X] = 0 \), then \( \pi(s) \) must depend solely on the truth-valuation over the \( F \)-literals \( p \) and the \( V \)-literals \( X = 0 \) that are determined by the state \( s \). There is indeed a finite number of such truth valuations but an infinite number of states. We refer to such truth valuations as the boolean states of the QNP and denote the boolean state associated with a state \( s \) as \( \bar{s} \).

**Definition 4 (Policy).** A policy \( \pi \) for a QNP \( Q = \langle F, V, I, O, G \rangle \) is a partial mapping of states into actions such that \( \pi(s) = \pi(s') \) if \( \bar{s} = \bar{s'} \).

A trajectory \( s_0, a_0, s_1, a_1, \ldots, s_n \) compatible with the model \( S(Q) \) is said to be \( \pi \)-trajectory for \( Q \) if \( \pi \) if \( a_i = \pi(s_i) \). The \( \pi \)-trajectory is also said to be a trajectory induced by \( \pi \) or compatible with \( \pi \). As before, a \( \pi \)-trajectory is maximal if A) the trajectory is infinite and does not include a goal state, B) \( s_n \) is the first goal state in the trajectory, or C) \( \pi(s_n) \) is undefined or denotes an action that is not applicable in \( s_n \). The solutions to QNPs are defined then as follows:

**Definition 5 (Solution).** Let \( Q \) be a QNP and let \( \pi \) be a policy for \( Q \). The policy \( \pi \) solves \( Q \) iff for every \( \epsilon > 0 \), all the maximal \( \epsilon \)-trajectories induced by \( \pi \) reach a goal state.

**Example.** Consider the QNP \( Q_{clear(x)} = \langle F, V, I, O, G \rangle \) from above with \( F = \{H\}, V = \{n(x)\}, I = \{-H, n(x) > 0\}, G = \{n(x) = 0\}, \) and \( O = \{a, b\} \) where \( a = \langle -H, n(x) > 0; H, n(x) \downarrow \rangle \) and \( b = \langle H; \neg H \rangle \). Let \( \pi \) be the policy defined by the rules:

\[
\begin{align*}
&\text{if } \neg H \text{ and } n(x) > 0, \text{ then do } a, \\
&\text{if } H \text{ and } n(x) > 0, \text{ then do } b.
\end{align*}
\]

All the maximal \( \epsilon \)-bounded trajectories that are induced by the policy \( \pi \) on \( Q_{clear(x)} \) have the form

\[
s_0, a, s_1, b, s_2, a, s_3, b, \ldots, s_{2m}, a, s_{2m+1}
\]

where \( s_m \), for positive integer \( m \), is the first state where \( n(x) = 0 \) is true. The actions \( a \) and \( b \) alternate because the first makes \( H \) false and the second makes it true. In each transition \( (s_i, s_{i+1}) \) for a non-negative even integer \( i \), the numerical variable \( n(x) \) decreases by \( \epsilon \) or more, or \( 0 = s_{i+1}[n(x)] < s_i[n(x)] < \epsilon \). The former case cannot happen more than \( s_0[n(x)]/2\epsilon \) times, as the numerical variable \( n(x) \) is decreased every two steps and is never increased. Thus, in all cases and for any \( \epsilon > 0 \), any \( \epsilon \)-trajectory induced by the policy \( \pi \) reaches a goal state in a finite number of steps, regardless of the initial value \( s_0[n(x)] \) of \( n(x) \), and regardless of the actual magnitude of the changes \( |s_{i+1}[n(x)] - s_i[n(x)]| \).

**Example.** A more interesting QNP that requires “nested loops” is \( Q_{nest} = \langle F, V, I, O, G \rangle \) with \( F = \emptyset, V = \{X, Y\}, I = \{X > 0, Y > 0\}, G = \{X = 0\}, \) and \( O = \{a, b\} \) where

\[
\begin{align*}
a &= \langle X > 0, Y = 0; X \downarrow, Y \uparrow \rangle, \\
b &= \langle Y > 0; Y \downarrow \rangle.
\end{align*}
\]

The policy \( \pi \) is given by the rules:

\[
\begin{align*}
&\text{if } X > 0 \text{ and } Y = 0, \text{ then do } a, \\
&\text{if } X > 0 \text{ and } Y > 0, \text{ then do } b.
\end{align*}
\]
The policy decrements $Y$ using action $b$ until the action $a$ that decreases $X$ and increases $Y$ can be applied, and the process is repeated until $X = 0$. The $\epsilon$-trajectories induced by $\pi$ have the form
\[
s_0, \ a, s_1^1, b, \ldots, b, s_{k_1}, \ a, s_1^2, b, \ldots, b, s_{k_2}, \ldots, \ a, s_1^m, b, \ldots, b, s_{k_m}, \ a, s_G.
\]
where there is an outer loop that is executed a number of times $m$ bounded by $s_0[X]/\epsilon$, as $X$ is decreased by $\epsilon$ or more, but is not increased. In the iteration $i$ of such a loop, the action $b$ is executed a number of times $k_i$ bounded by $s_i[Y]/\epsilon$ as in such inner loop $Y$ begins with value $s_i[Y]$ and it is decreased and not increased. The result is that all the $\epsilon$-trajectories induced by $\pi$ reach a goal state in a finite number of steps that cannot be bounded a priori because the increments of $Y$ produced by the action $a$ are finite but not bounded. The policy $\pi$ thus solves $Q_{nest}$. 

4. Direct Translation and Termination Test

The problem of deciding the existence of a policy that solves a given QNPs is decidable as noted by Srivastava et al. (2011). They hint a generate-and-test procedure to find such a policy where the QNP is first translated into a FOND problem, and then all the possible strong cyclic policies for the FOND problem are enumerated and tested for termination. The translation runs in polynomial (lineal) time in the number of boolean states for the QNP while the termination test for a given strong cyclic solution is polynomial in the number of FOND states. However, the number of strong cyclic solutions that need to be tested is exponential in the number of FOND states in the worst case. The generate-and-test approach is not efficient but it is complete and runs in finite time. In contrast, the plan-existence problem for numerical planning is undecidable even in the classical setting where there is a single initial state and the action effects are deterministic; e.g., Post’s correspondence problem can be reduced to a numerical planning problem (Helmert, 2002). The decidability of plan existence for QNPs is due to the “qualitative” behaviour of the numerical variables that cannot keep track of counts; in particular, the variables cannot be incremented or decremented by specific amounts nor queried about specific values. We review the translation and the termination test for QNPs before considering a novel polynomial translation which does not require such tests and which thus is a true reduction of QNPs into FOND problems.

The translation $T_D$ from a QNP $Q$ to a FOND $P = T_D(Q)$ is simple and direct, and it involves three steps: 1) the literals $X = 0$ and $X > 0$ are made propositional with the numerical variables $X$ eliminated, 2) $Inc(X)$ effects are converted into deterministic boolean effects $X > 0$, and 3) $Dec(X)$ effects are converted into non-deterministic boolean effects $X > 0 \mid X = 0$.

**Definition 6** (Direct Translation $T_D$). For QNP $Q = (F, V, I, O, G)$, the FOND problem $P = T_D(Q)$ is $P = (F', I', O', G')$ with

1. $F' = F \cup \{X = 0 : X \in V\}$, where $X = 0$ stands for a new propositional symbol $p_{X = 0}$ and $X > 0$ stands for $\neg p_{X = 0}$.
2. $I' = I$ but with $X = 0$ and $X > 0$ denoting $p_{X = 0}$ and $\neg p_{X = 0}$.
3. $O' = O$ but with $Inc(X)$ effects replaced by the deterministic propositional effects $X > 0$, and $Dec(X)$ effects replaced by non-deterministic propositional effects $X > 0 \mid X = 0$.
4. $G' = G$ but with $X = 0$ and $X > 0$ denoting $p_{X = 0}$ and $\neg p_{X = 0}$.

The problem $P = T_D(Q)$ is a special type of FOND problem. For example, from its definition, there is no action in $P$ that can achieve a proposition $X = 0$ deterministically. We refer to actions in the FOND $P$ with effects $X > 0$ and $X > 0 \mid X = 0$ as $Inc(X)$ and $Dec(X)$ actions, as such effects
in $P$ may only come from $Inc(X)$ and $Dec(X)$ effects in $Q$. Also, observe that the FOND $P$ has a unique initial state even though there QNP $Q$ may have an infinite number of initial states.

The states of the FOND problem $P = T_D(Q)$ are related to the boolean states over $Q$, i.e., the truth-assignments over the atoms $p$ and $X = 0$, the latter of which stand for (abbreviation of) symbols in $P$. A policy $\pi$ for the QNP $Q$ thus induces a policy over the FOND problem $P$ and vice versa. Moreover, the FOND problem $P = T_D(Q)$ captures the possible boolean state transitions in $Q$ exactly. More precisely, $(s, a, s')$ is a possible transition in $Q$ iff $(\bar{s}, a, \bar{s}')$ is a possible transition in $P$. Indeed, if we extend the notion of strong cyclic policies to QNPs:

**Definition 7.** Let $Q$ be a QNP and let $\pi$ be a policy for $Q$. $\pi$ is strong cyclic for $Q$ iff for every $\pi$-trajectory connecting $s_0$ with a state $s$, there is a $\pi$-trajectory connecting $s$ with a goal state.

The following correspondence between boolean states in $Q$ and the states of the boolean FOND problem $T_D(Q)$ results:

**Theorem 8.** Let $Q$ be a QNP and let $\pi$ be a policy for $Q$. $\pi$ is strong cyclic solution for $Q$ iff $\pi$ is strong cyclic policy for the FOND problem $T_D(Q)$.

**Proof.** Let $S(Q) = \langle S, S_0, Act, F, S_G \rangle$ and $S(P) = \langle S', S'_0, Act', F', S'_G \rangle$ be the state models for the QNP $Q$ and the FOND problem $P = T_D(Q)$. From the definition of the $T_D$ translation, the state $s$ is in $S_0$ (resp. in $S_G$) iff $\bar{s} = s'_0$ (resp. in $S'_G$), and the state $s' \in F(a, s)$ for $a \in A(s)$ iff $\bar{s}' \in F'(a, \bar{s})$ for $a \in A'(\bar{s})$. This means that there is a $\pi$-trajectory connecting an initial state $s_0$ in $S_0$ with a state $s$ in $S$ iff there is a corresponding $\pi$-trajectory connecting $s'_0$ with $\bar{s}$ in $S'$, and similarly, there is a $\pi$-trajectory connecting $s$ with a goal state $s'$ iff there is a corresponding $\pi$-trajectory connecting $\bar{s}$ with $s'$ in $S(Q)$. \hfill \square

The correspondence between the $\pi$-trajectories connecting states $s$ in $Q$ and the $\pi$-trajectories connecting the states $\bar{s}$ in $P = T_D(Q)$ does not imply however that the solutions of $P$ and $Q$ are the same. Indeed, the $Dec(x)$ effects of an action $a$ in $Q$ are mapped into the non-deterministic propositional effects $X > 0 \mid X = 0$ in $P = T_D(Q)$ which implies that $X = 0$ will be true if the action $a$ is repeated infinitely often. On the other hand, a $Dec(X)$ effect in $Q$ ensures that $X = 0$ will be true if $a$ is repeated infinitely often as long as no $Inc(X)$ action is performed infinitely often as well.

In other words, the correspondence between the state transitions $(s, a, s')$ in $Q$ and the state transitions $(\bar{s}, a, \bar{s}')$ in $P = T_D(Q)$ does not extend to infinite trajectories (Bonet et al., 2017). Recall that trajectories in $Q$ refer to $\epsilon$-trajectories for some $\epsilon > 0$ that exclude “infinitesimal” changes. As a result:

**Theorem 9.** Let $Q$ be a QNP and let $\pi$ be a policy for $Q$. If $\tau = s_0, s_1 \ldots$ is an infinite $\pi$-trajectory in $Q$, then $\bar{\tau} = \bar{s}_0, \bar{s}_1, \ldots$ is an infinite $\pi$-trajectory in $P = T_D(Q)$. Yet if $\bar{\tau} = \bar{s}_0, \bar{s}_1, \ldots$ is an infinite $\pi$-trajectory in $P = T_D(Q)$, there may not be an infinite trajectory $\tau = s_0, s_1 \ldots$ in $Q$.

**Proof.** For the first part, if $\tau$ is an infinite $\pi$-trajectory over $Q$, then $s_{i+1} \in F(a_i, s_i)$ for $a_i = \pi(s_i)$; therefore $\bar{s}_{i+1} \in F(\bar{a}_i, \bar{s}_i)$ for $\bar{a}_i = \pi(\bar{s}_i)$, and hence $\bar{\tau} = \bar{s}_0, \bar{s}_1, \ldots$ is a infinite $\pi$-trajectory over $P$. For the second part, one example suffices. Let $Q$ be a QNP with a single variable $X$ that is numerical, a single action $a$ with precondition $X > 0$ and effect $Dec(X)$, initial condition $X > 0$, and goal $X = 0$. In the state model $S(P)$ associated with the FOND problem $P = T_D(Q)$, there are

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2. The policy $\pi$ over states $s$ of $Q$ determines the policy $\pi'$ over the FOND $P$ where $\pi'(t) = \pi(s)$ if $t = \bar{s}$, and vice versa, a policy $\pi'$ for $P$ determines a policy $\pi$ for $Q$ where $\pi(s) = \pi'(t)$ if $\bar{s} = t$. For simplicity, we use the same notation $\pi$ to refer to the policy $\pi$ over $Q$ and the policy $\pi'$ that it induces over $P = T_D(Q)$. 

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two states $t$ and $t'$, the first where $X = 0$ is true and the second where $X > 0$ is true, and there is an infinite trajectory $\bar{\tau} = \bar{s}_0, \bar{s}_1, \ldots$ where all $\bar{s}_i = t$ and $\pi(\bar{s}_i) = a$, but there is no infinite trajectory $\tau = s_0, s_1, \ldots$ with $\pi(s_i) = a$ and where $X > 0$ stays true forever while being decremented. Indeed, for any $\epsilon > 0$ and any initial value of $X$, $s_0[X] > 0$, it is the case that $s_n[X] = 0$ for $n > s_0[X]/\epsilon$. \hfill \Box

The notion of termination is aimed at capturing the infinite $\pi$-trajectories over the FOND problem $P = T_D(Q)$ that do not map into infinite $\pi$-trajectories over $Q$. Let

$$\bar{s}_0, \bar{s}_1, \ldots, [\bar{s}_i, \ldots, \bar{s}_m]^*$$

(11)

denote any infinite $\pi$-trajectory on the FOND $P$ where the states $\bar{s}_i, \ldots, \bar{s}_m$ in brackets make the non-empty set of recurring states; namely those that occur infinitely often in the trajectory (not necessarily in that order). We refer to such set of recurrent states as the loop of the trajectory. Termination imposes the following condition on loops:

**Definition 10** (Terminating Trajectories). Let $Q$ be a QNP and let $\pi$ be a policy for $Q$. An infinite $\pi$-trajectory $s_0, \ldots, [s_i, \ldots, s_m]^*$ is terminating in $P = T_D(Q)$ if there is a variable $X$ in $Q$ that is decremented but not incremented in the loop; i.e., if $\pi(s)$ is a $\text{Dec}(X)$ action for some $k \in [i, m]$, and $\pi(s_j)$ is not an $\text{Inc}(X)$ action for any $k \in [i, m]$.

The notion of termination is a notion of fairness that is different from the one underlying strong cyclic planning that says that infinite but terminating trajectories in $P$ are not “fair” and hence can be ignored. Indeed, this notion of termination closes the gap in Theorem 9:

**Theorem 11.** Let $Q$ be a QNP and let $\pi$ be a policy for $Q$. $\bar{\tau} = \bar{s}_0, \bar{s}_1, \ldots$ is an infinite non-terminating $\pi$-trajectory in $P = T_D(Q)$ iff there is an infinite $\pi$-trajectory $\tau = s_0, s_1, \ldots$ in $Q$.

**Proof.** Let $\tau = s_0, s_1, \ldots$ be an infinite $\pi$-trajectory in $Q$, and let us assume that the infinite trajectory $\bar{\tau} = \bar{s}_0, \bar{s}_1, \ldots$ is terminating. Then there must be a variable $X$ that is decremented by $\pi(s)$ in some recurring state $s$ in $\tau$ and which is not incremented by $\pi(s')$ on any recurrent state $s'$ in $\tau$. Let $s(t)$ denote the state at time point $t$ in $\tau$, let $t$ be the last time point where variable $X$ is increased in $\tau$ ($t = -1$ if $X$ is not increased in $\tau$), and let $X(t+1)$ be the value of variable $X$ at the next time point. The maximum number of times that $X$ can be decreased after $t + 1$ is bounded by $X(t+1)/\epsilon$, and after this, $X$ must have zero value. But in $\tau$, $X$ is decreased an infinite number of times, in contradiction with the assumption that any action that decrements $X$ features $X > 0$ as precondition.

For the converse, we show that one such trajectory $\tau$ in $Q$ can be constructed for any $\epsilon > 0$, given that the trajectory $\bar{\tau}$ in $P$ is non-terminating. We do so by adjusting the non-deterministic increments and decrements of the actions, all of which have to be greater than or equal to $\epsilon$, except when this would result in negative values that are increased back to zero. We construct $\tau = s_0, s_1, \ldots$ from $\bar{\tau} = \bar{s}_0, \bar{s}_1, \ldots$ as follows. The value of the boolean variables is the same in $s_1$ as in $\bar{s}_i$, and in addition, $s_i[X] = 0$ iff $X = 0$ is true in $\bar{s}_i$ for all $i$. We just have to find exact values for the numerical variables $X$ in each of the states $s_i$ in $\tau$, and this is a function of the their initial values $s_0[X]$ when $s_0[X] > 0$, and the positive decrements or increments $\Delta(X, s_i)$ when $\pi(s_i)$ is a $\text{Dec}(X)$ or $\text{Inc}(X)$ action, and $\Delta(X, s_i) \geq \epsilon$. For simplicity and without loss of generality, let us assume that $\epsilon < 1$.

All the positive initial values of numerical variables, increments, and decrements are set to **positive integers** by considering the sequence of actions $\pi(s_i)$, $i = 1, \ldots$. The initial values $s_0[X]$ are set to $\max(k(X, 0), 1)$ where $k(X, i)$ stands for the number of $\text{Dec}(X)$ actions that occur between the state $s_i$ and the first state $s_j$ after $s_i$ where an $\text{Inc}(X)$ action occurs (if no $\text{Inc}(X)$
action occurs after state $s_i$, $k(X, i)$ is the number of $Dec(X)$ actions after $s_i$. That is, $k(X, i)$ is the cardinality of the set

$$\{ j : i \leq j < ind(X, i) \text{ and } \pi(s_j) \text{ is } Dec(X) \text{ action} \}$$

(12)

where $ind(X, i)$ is the minimum index $j > i$ such that $\pi(s_j)$ is an $Inc(X)$ action, or $\infty$ if there is no such action after $s_i$. Observe that $k(X, i)$ is bounded. The only way it could be infinite is when no $Inc(X)$ action occurs after $s_i$ while at the same time an infinite number of $Dec(X)$ actions occur; yet, this is impossible since then $X$ eventually becomes zero after which no $Dec(X)$ action may occur as such actions feature the precondition $X > 0$. Likewise, the increments $\Delta(X, s_i)$ are set to $\max(k(X, i), 1)$, and the decrements $\Delta(X, s_i)$ are set to $s_i[X]$ if $X = 0$ is true in $\bar{s}_{i+1}$ and to 1 is $X > 0$ if true in $\bar{s}_{i+1}$. It is not difficult to verify that these choices define a trajectory $\tau$ in $Q$ that corresponds to the assumed trajectory $\bar{\tau}$ in $P$. \hfill \square

The full correspondence between infinite $\pi$-trajectories $\tau$ in $Q$ and infinite non-terminating $\pi$-trajectories $\tau$ in $P = T_D(Q)$ suggests the following definition of termination in QNPs $Q$ and FOND problems $T_D(Q)$:

**Definition 12** (Termination in $Q$). A policy $\pi$ for the QNP $Q$ is terminating iff all the $\pi$-trajectories on $Q$ are of finite length. In such a case, we say that $\pi$ is $Q$-terminating.

**Definition 13** (Termination in $P$). Let $Q$ be a QNP. A policy $\pi$ for the FOND problem $P = T_D(Q)$ is terminating iff all the infinite $\pi$-trajectories on $P$ are terminating. In such a case, we say that $\pi$ is $P$-terminating.

The correspondence between policies can then be expressed as:

**Theorem 14.** Let $Q$ be a QNP, let $P = T_D(Q)$ be its direct translation, and let $\pi$ be a policy for $Q$ (and thus also for $P$). Then, $\pi$ is $Q$-terminating iff $\pi$ is $P$-terminating.

**Proof.** Direct from Theorem 11. For on direction, assume that $\pi$ is $Q$-terminating and let $\bar{\tau}$ be a $\pi$-trajectory in $P$. If $\bar{\tau}$ is not terminating, by Theorem 11, $\tau$ is infinite and thus $\pi$ would not be $Q$-terminating. Therefore, every $\pi$-trajectory $\bar{\tau}$ in $P$ is terminating and thus $\pi$ is $P$-terminating. The other direction is established similarly. \hfill \square

The soundness and completeness of the direct translation extended with termination can be expressed as following:

**Theorem 15** (Soundness and Completeness $T_D$). Let $Q$ be a QNP, let $P = T_D(Q)$ be its direct translation, and let $\pi$ be a policy for $Q$ (and thus also for $P$). The following are equivalent:

1. $\pi$ solves $Q$,

2. $\pi$ is a strong cyclic solution of $Q$ and $\pi$ is $Q$-terminating,

3. $\pi$ is a strong cyclic solution of $P$ and $\pi$ is $P$-terminating.

**Proof.** ($1 \Leftrightarrow 2$) Assume that $\pi$ solves $Q$. If there is $\pi$-trajectory connecting an initial state with state $s$, there must be $\pi$-trajectory connecting $s$ with a goal state. Otherwise, $\pi$ would not be a solution for $Q$. Likewise, if $\tau$ is an infinite $\pi$-trajectory in $Q$, then $\tau$ does not reach a goal state and thus $\pi$ would not solve $Q$. For the converse direction, assume that $\pi$ is a strong cyclic solution for $Q$ and that $\pi$ is $Q$-terminating, and suppose that $\pi$ does not solve $Q$. Then, there is a maximal $\pi$-trajectory $\tau$ in $Q$ that does not reach a goal state. It cannot be the case that $\tau$ ends in a state.
s where $\pi(s)$ is undefined or non-applicable as $\pi$ then would not be a strong cyclic solution for $Q$. Hence, $\tau$ must be infinite but this contradicts the assumption that $\pi$ is $Q$-terminating.

(2 $\leftrightarrow$ 3) By Theorem 8, $\pi$ is a strong cyclic solution for $Q$ iff it is strong cyclic solution for $P$. By Theorem 14, $\pi$ is $Q$-terminating iff it is $P$-terminating. 

5. Checking Termination with SIEVE

SIEVE is the procedure introduced by Srivastava et al. (2011) to test whether a policy terminates. It runs in time that is polynomial in the number of states of the FOND problem reached by the policy.\(^3\) For this, the algorithm takes as input a policy graph $G(P, \pi) = \langle V, E \rangle$ constructed from the FOND problem $P = T_D(Q)$ and a strong cyclic policy $\pi$ for $P$. The nodes in the policy graph are the states $\bar{s}$ in the state model $S(P)$ that are reachable from the initial state and the policy $\pi$, and the directed edges in $E$ are the pairs $(\bar{s}, \bar{s}')$ for $\bar{s}' \in F(a, \bar{s})$ and $\pi(\bar{s}) = a$. These edges are labeled with the action $a$. The algorithm iteratively removes edges from the graph $G(P, \pi)$ until the graph becomes acyclic or no additional edge can be removed.

For incrementally removing edges from the graph, SIEVE identifies first its strongly connected components by a single depth-first search traversal, following Tarjan’s algorithm (Tarjan, 1972). A strongly connected component (SCC) is a partition of the nodes of the graph such that if a node $\bar{s}$ belongs to a partition, any node $\bar{s}'$ that can be reached from $\bar{s}$ and that can reach $\bar{s}$ back in the graph, is placed in the same partition as $\bar{s}$.

The algorithm then picks a variable $X$ and a SCC such that the variable $X$ is decremented but not incremented in the SCC. That is, there must be a state $\bar{s}$ in the SCC such that $\pi(\bar{s})$ is a Dec($X$) action and no $\pi(\bar{s}')$ is an Inc($X$) action for any $\bar{s}'$ in the SCC. The algorithm then removes all the edges $(\bar{s}, \bar{s}')$ in the SCC such that $\pi(\bar{s})$ is a Dec($X$) action. We abbreviate this by saying that variable $X$ is removed from the SCC, which means that the edges associated with Dec($X$) actions are removed. Following the edge removals, the SCCs must be recomputed and the process is repeated until the graph becomes acyclic or no more edges can be removed in this manner. The result is that:

**Theorem 16.** Let $Q$ be a QNP. A policy $\pi$ for the FOND problem $P = T_D(Q)$ is $P$-terminating iff SIEVE reduces the policy graph $G(P, \pi)$ to an acyclic graph.

**Proof.** For the first direction, let us assume that $\pi$ is $P$-terminating and suppose that SIEVE terminates with a cyclic graph $G$. Then, there must there must be a $\pi$-trajectory $\bar{\tau}$ in $P$ of the form $\bar{s}_0, \ldots, [\bar{s}_i, \ldots, \bar{s}_m]^*$ where the recurrent states $\bar{s}_0, \ldots, \bar{s}_m$ in $\bar{\tau}$ are exactly all the states in a SCC $C$ of $G$. By the assumption, there must be a variable $X$ such that $\pi(\bar{s})$ is a Dec($X$) action for some state $\bar{s}$ in the loop, and $\pi(\bar{s}')$ is not an Inc($X$) action for any of the states $\bar{s}'$ in the loop. But then SIEVE should have removed the variable $X$ from $C$, or any other similar variable, breaking the SCC $C$ into smaller components.

For the converse, let us assume that $\pi$ is not $P$-terminating. Then, there must be an infinite $\pi$-trajectory $\bar{\tau}$ in $P$ of the form $\bar{s}_0, \ldots, [\bar{s}_i, \ldots, \bar{s}_m]^*$ and some variable $X$ such that $X$ is decremented by some action $\pi(\bar{s}_j)$, $j \in [i, m]$, and not decremented by any action $\pi(\bar{s}_j)$, $j \in [i, m]$. We want to show that SIEVE terminates with a graph that has one SCC that includes all the states in the loop. Indeed, initially, all the states in the loop must be in one component as they are all reachable from each other. Then, notice that SIEVE is incapable of removing the variable $X$ from the loop as the

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3. In (Srivastava et al., 2011), the notion of termination is not articulated independently of the algorithm, so our account departs from the one in that paper. A second difference is that our version of the algorithm is developed and applied to QNPs that involve both numerical and boolean variables.
Sieve (Graph $\mathcal{G} = \mathcal{G}(P, \pi)$):

repeat
  Compute the strongly connected components (SCC) of $\mathcal{G}$.
  Choose an SCC $C$ and a variable $X$ that is decreased in $C$ but is not increased in $C$;
  i.e., for some $\bar{s}$ in $C$, $\pi(\bar{s})$ is a $\text{Dec}(X)$ action, and for no $\bar{s}$ in $C$, $\pi(\bar{s})$ is an $\text{Inc}(X)$ action.
  Remove the edges $(\bar{s}, \bar{s}')$ such that $\bar{s}$ and $\bar{s}'$ are in $C$, and $\pi(\bar{s})$ is a $\text{Dec}(X)$ action.
until $\mathcal{G}$ is acyclic (terminating) or there is no SCC $C$ and variable $X$ to choose (non-terminating)

Figure 1: Sieve procedure for testing whether policy $\pi$ for FOND problem $P = T_D(Q)$ terminates (Srivastava et al., 2011).

Example. The policy $\pi$ for $Q_{nest}$ above is given by the rules:

\begin{align*}
\text{if } X > 0 \text{ and } Y = 0, \text{ then do } a, \quad (13) \\
\text{if } X > 0 \text{ and } Y > 0, \text{ then do } b \quad (14)
\end{align*}

where recall that $Q_{nest} = \langle F, V, I, O, G \rangle$ with $F = \emptyset$, $V = \{X, Y\}$, $I = \{X > 0, Y = 0\}$, $G = \{X = 0\}$, and $O = \{a, b\}$ where $a = \langle X > 0, Y > 0; X\downarrow, Y\uparrow \rangle$ and $b = \langle Y > 0; Y\downarrow \rangle$. The policy decrements $Y$ using the action $b$ until the action $a$ that decreases $X$ and increases $Y$ can be applied. The process is repeated until $X = 0$. The nested loops in the policy graph $\mathcal{G}(P, \pi)$ are shown in Figure 2. The policy graph contains three states: the leftmost one is the initial state, and the rightmost one is a goal state. The two states on the left are reachable from each other, and hence, define a strongly connected component (SCC). In this SCC, the $\text{Dec}(X)$ edges are removed by Sieve because $X$ is not increased anywhere. Once this is done, the $\text{Dec}(Y)$ edges are removed by Sieve because the edges associated with $\text{Inc}(Y)$ effects are gone. The resulting graph is acyclic establishing thus that the policy $\pi$ terminates in $P = T_D(Q_{nest})$. □

Using Sieve it is easy to see that the problem of checking the existence of plans for QNPs can be decided in exponential space:

**Theorem 17** (Srivastava et al. (2011)). Deciding plan existence for QNPs is in EXPSPACE.

**Proof.** Let $Q = \langle F, V, I, O, G \rangle$ be a QNP. The number of boolean states for $Q$ is exponential in the number of fluents and variables; i.e., $|F| + |V|$. A policy $\pi$ for $Q$ can be described in exponential space as a mapping from boolean states into actions. A brute-force algorithm enumerates all policies one by one using exponential space. Each one is tested for strong cyclicity and termination. The former is a straightforward test in a graph while the latter is done with Sieve. If the policy is strong cyclic and terminating, $Q$ is accepted. Otherwise, if no policy is found to be strong cyclic and terminating, $Q$ is rejected. Since testing strong cyclicity and running Sieve both require polynomial time in the size of the input policy, the whole algorithm can be implemented in space that is exponential in the size of $Q$. □
Below we improve this bound and show exponential time (EXP) solvability through a more complex translation of QNPs into FOND problems that is also polynomial. The novelty of the new translation is that the strong-cyclic policies of the resulting FOND problems do not need to be checked for termination. QNPs are thus fully reduced to FOND problems. Since FOND problems can be reduced to QNPs as well, we will show indeed that FOND problems and QNPs have the same expressive power, and the complexity of plan existence for FOND problems is known to be EXP-Complete (Littman et al., 1998; Rintanen, 2004), then these reductions show the EXP-Completeness of the plan existence decision problem for QNPs. In addition to the establishment of novel theoretical results, these reductions are also of practical importance as they permit the computation of solutions; once a QNP is reduced to FOND, a solution for the QNP can be recovered in linear time from a solution to the FOND problem, and the same for the reduction from FOND problems into QNPs.

The distinction between the classes EXP and EXPSPACE is important. EXPSPACE contains the (decision) problems that can be solved with Turing machines (TMs) that operate within exponential space (as a function of the input size), yet such TMs may in fact run in doubly exponential time as the running time is bounded, in the worst case, by an exponential function of the bound in space (Sipser, 2006). On the other hand, EXP comprises the problems that can be solved with TMs that run within exponential time. The difference is analogous to the difference between the classes P (polynomial time) and PSPACE (polynomial space).

6. First Reduction: FOND problems into QNPs

The first reduction that we introduce is from FOND problems into QNPs. It is a non-trivial reduction yet simpler than the inverse reduction from QNPs into FOND problems. The main obstacle to overcome is that the non-deterministic effects in FOND problems are over boolean variables, while those in QNPs are only on numerical variables through the decrement effects. Another important obstacle is that strong cyclic solutions in QNPs are not QNP solutions unless they are terminating.

Let $P = \langle F, I, O, G \rangle$ be a FOND problem and let us denote the non-deterministic effects of action $a$ as $E_1^a \cup E_2^a \cup \cdots \cup E_{k_a}^a$, where each $E_i^a$ is a set (conjunction) of $F$-literals, and $k_a$ denotes the number of non-deterministic effects of the action $a$. If $k_a = 1$, the action $a$ is deterministic,
else it is non-deterministic. For simplicity, we assume that the set of effects \( \{E^a_i\}_i \) for action \( a \) aggregates all the multiple non-deterministic effects in the description of \( a \) in \( P \), and the reduction below is presented under this assumption. Afterwards, we discuss how to remove this requirement for handling in polynomial time FOND problems whose transitions are factorized.

We map \( P \) into a QNP \( Q = \langle F', V', I', O', G' \rangle \) that extends \( P \) with numerical variables \( V' = \{X\} \cup \{Y_{a,i} : a \in O, 1 \leq i \leq k_a\} \), extra boolean variables, and extra actions; i.e., \( F \subseteq F' \), \( I \subseteq I' \), \( O' \subseteq O \), and \( G' = G \).

The heart of the reduction lies in the way in which the non-deterministic effects of each action are captured in \( Q \). For this, the collection of non-deterministic effects of the action \( a \) are replaced by an \( \text{Inc}(X) \) action, for the unique numerical variable \( X \), followed by a fixed loop where the variable \( X \) is decremented until becoming zero. The alternative effects \( E^a_i \) are then triggered when \( X = 0 \) becomes true in the corresponding part of the loop. The increments and decrements of the extra variables \( Y_{a,i} \) ensure that only strong cyclic policies \( \pi \) for \( P \) induce policies \( \pi' \) for \( Q \) that are terminating. The fixed loop sequence associated with action \( a \) performs the following steps, that are implemented with the help of new auxiliary actions and propositional symbols:

1. \( \text{Inc}(X) \) (implemented by modified action \( a \)),
2. \( \text{Dec}(X) \) (implemented by new action \( \text{Start} \)),
3. If \( X = 0 \), apply the effects in \( E^a_1 \), increment \( Y_{a,1} \), decrement \( Y_{a,j} \), \( j \neq 1 \), and break loop (implemented by new action \( \text{Exit}(a,i) \) for \( i = 1 \)),
4. \( \text{Dec}(X) \) (implemented by new action \( \text{Cont}(a,i) \) for \( i = 1 \))
5. If \( X = 0 \), apply the effects in \( E^a_2 \), increment \( Y_{a,2} \), decrement \( Y_{a,j} \), \( j \neq 2 \), and break loop (implemented by new action \( \text{Exit}(a,i) \) for \( i = 2 \)),
6. \( \text{Dec}(X) \) (implemented by the new action \( \text{Cont}(a,i) \) for \( i = 2 \))
7. If \( X = 0 \), apply the effects in \( E^a_{k_a} \), increment \( Y_{a,k_a} \), decrement \( Y_{a,j} \), \( j \neq k_a \), and break loop (implemented by new action \( \text{Exit}(a,i) \) for \( i = k_a \)),
8. Go back to 3 (implemented by new action \( \text{Loop}(a) \))

To show that the resulting mapping is indeed a reduction (i.e., the mapping is sound and complete), two things need to be established: that a policy \( \pi \) that solves \( Q \) induces a policy \( \pi' \) that solves \( P \) (soundness), and vice versa, that a policy \( \pi' \) that solves the FOND problem \( P \) induces a policy \( \pi \) that solves \( Q \) (completeness).

The fixed sequence of effects for each action \( a \) in \( Q \) is implemented using the following additional boolean variables:

- A boolean \text{normal} that is false when the sequence is entered for some action \( a \) and made true when the loop is exit.
- A boolean \text{ex}(a) to express that the sequence for the action \( a \) is being executed. The reduction is such that the atoms in \( \{\text{normal}\} \cup \{\text{ex}(a) : a \in O\} \) are pairwise mutex.
- A counter from 0 to \( K \) encoded using mutex atoms \( \text{cnt}(\ell), \ell = 0, 1, \ldots, K + 1 \), that is set to 1 when the loop is entered and re-entered (step 8 above) and where \( K \) is the maximum number of non-deterministic outcomes of any action in \( P \).
The actions that implement the fixed loop sequence are the following:

- **Start** = \(\langle \neg normal, cnt(0), X > 0; \neg cnt(0), cnt(1), X\downarrow \rangle\).
- **Exit(a, i)** = \(\langle ex(a), cnt(i), X = 0, Y_{a,i} > 0; \neg ex(a), \neg cnt(i), normal, cnt(0), E_i^a, Y_{a,i}, Y_{a,j}\downarrow \rangle\) where the increment is for every variable \(Y_{a,j}\) with \(j \neq i\).
- **ExitG(a, i)** = \(\langle ex(a), cnt(i), X = 0, Y_{a,i} = 0; \neg ex(a), \neg cnt(i), normal, cnt(0), G \rangle\) that may be used to achieve the goal \(G\) when \(X = 0\) and \(Y_{a,i} = 0\).
- **Cont(a, i)** = \(\langle ex(a), cnt(i), X > 0; \neg cnt(i), cnt(1+i), X\downarrow \rangle\) to advance along the fixed loop sequence by decrementing variable \(X\).
- **Loop(a)** = \(\langle ex(a), cnt(k_a), X > 0; \neg cnt(k_a), cnt(1) \rangle\) to start a new iteration of the loop.

The initial state of the QNP includes the atoms normal and cnt(0). Deterministic actions \(a\) in \(P\) “pass directly” into the QNP \(Q\) with these two atoms added as extra preconditions. Non-deterministic actions, however, are handled differently by replacing their effects \(E_1^a | E_2^a | \cdots | E_{k_a}^a\) by deterministic effects \(\{\neg normal, ex(a), X^i\}\) after which the only applicable action would be the Start action from above. The idea of the construction is illustrated in Figure 3.
Definition 18. Let $P = \langle F, I, O, G \rangle$ be a FOND problem. The reduction $R$ maps $P$ into the QNP $Q = \langle F', V', I', O', G' \rangle$ given by

1. $F' = F \cup \{\text{normal}\} \cup \{\text{ex}(a) : a \in O\} \cup \{\text{cnt}(\ell) : \ell \in [0, K]\}$ where $K = \max_{a \in O} k_a$.
2. $V' = \{X\} \cup \{Y_{a,i} : a \in O, i \in [1, k_a]\}$,
3. $I' = I \cup \{\text{normal, cnt}(0)\} \cup \{X = 0\} \cup \{Y_{a,i} > 0 : a \in O, i \in [1, k_a]\}$,
4. $O' = O \cup \{\text{Start}\} \cup \{A(a,i) : A \in \{\text{Exit, ExitG, Cont}\}, a \in O, i \in [1, k_a]\} \cup \{\text{Loop}(a) : a \in O\}$,
5. $G' = G$.

For stating the formal properties of the reduction, we need to specify how a policy $\pi$ for $Q$ induces a policy $\pi'$ for $P$, and vice versa, as $Q$ involves extra variables and actions.

Let us split the non-goal states in $Q$ into two sets: the normal states, where the booleans normal and cnt$(0)$ are true, and the rest of non-goal states where normal or cnt$(0)$ is false. (Observe that there cannot be a normal state where some variable $Y_{a,i}$ is zero as when that happens, the loop must exit with the ExitG$(a,i)$ that reaches a goal state.) In the normal states, the (qualitative) value of all the extra variables is the same: all the extra boolean variables are false except for normal and cnt$(0)$ that are true, $X = 0$, and $Y_{a,i} > 0$ for $a \in O$ and $i = 1, 2, \ldots, k_a$. Moreover, for every state $s$ over the FOND $P$, there is a unique normal state $\tilde{s}$ over the QNP $Q$ that extends $s$ with this fixed value of the extra variables in $Q$, and vice versa, for any normal state $\tilde{s}$ over $Q$ there is a unique state $s$ over $P$. Taking advantage of this relation and leaving the part of the normal states over $Q$ that is fixed implicit, we obtain that a policy $\pi$ for $P$ represents a unique policy $\pi'$ over the normal states in $Q = R(P)$, and vice versa, a policy $\pi$ over the normal states represents a unique policy for $P$. The policy over the non-normal states $t$ in $Q$ is determined as for each such state there is only one (new) action applicable; i.e., non-normal states $t$ are mapped to one and only one of the actions in $O' \setminus O$ according to the action precondition that is true in $t$; observe that the preconditions of these actions are mutually exclusive. The main properties of the reduction can be then expressed as follows:

Theorem 19 (Reduction FOND to QNP). The mapping $R$ is a polynomial-time reduction from FOND problems into QNPs. That is, a FOND problem $P$ has solution iff the QNP $Q = R(P)$ has solution. Moreover, a solution for $P$ (resp. $Q$) can be recovered in polynomial time from a solution for $Q$ (resp. $P$).

Proof. It is straightforward to see that $R$ is computable in time that is polynomial in the size of the input $P$ and the maximum value $k_a$ for any action $a$. (See below for a discussion on how to deal with FOND problems that may have multiple non-deterministic effects per action.) In the following we show that $R$ is indeed a reduction.

We establish a correspondence between the FOND problem $P$ and the FOND problem $P' = T_D(Q)$ associated with $Q$, by focusing on the normal states over $Q$ and $P'$, leaving the values of the extra variables out, so that such reduced states over $P'$ are states over $P$ and vice versa.

The problem $P'$ has indeed an associated state model $S(P')$ with a transition function where $s' \in F'(a,s)$ if $s'$ is a first normal state that may follow the normal state $s$ after an action $a$ common to $P$ and $P'$ (i.e., $a$ is not an extra action from the translation). There must be such first normal states as for non-deterministic actions $a$, the loop through non-normal states, as shown in Fig. 3, must eventually terminate with a non-normal state or in a non-normal goal state of $P'$.

If $a$ is a deterministic action in $P$, $F'(a,s)$ is given, but for non-deterministic actions $a$, $F'(a,s)$ follows from the definition of $R(Q)$. It is not difficult to show that 1) $a \in A(s)$ iff $a \in A'(s)$ where
A and \(A'\) denote the applicable actions in the models associated with \(P\) and \(P'\) in normal states \(s\) that are common to both \(P\) and \(P'\), and 2) \(s' \in F(a, s)\) iff \(s' \in F'(a, s)\), where \(F\) and \(F'\) are the transition functions associated with \(P\) and \(P'\) respectively, and \(a\) is an action common to \(P\) and \(P'\). Since the initial states of \(P\) and \(P'\) coincide, this means that infinite trajectories over \(P\) induce infinite trajectories over \(P'\) (and hence \(Q\)), and that infinite trajectories over \(P'\) (and \(Q\)) induce infinite trajectories over \(P\).

For proving the two implications of the theorem, we need to show that 1) infinite fair trajectories over \(P\) yield non-terminating trajectories over \(P'\) (and hence \(Q\)), and that 2) non-terminating trajectories over \(P'\) yield infinite fair trajectories over \(P\). Then, if \(\pi\) solves \(P\) but the policy \(\pi'\) induced by \(\pi\) over \(P'\) does not solve \(P'\), it would mean that there must be a non-terminating \(\pi'\)-trajectory over \(P'\), and hence, due to 2), that there must be an infinite fair \(\pi\)-trajectory over \(P\), in contradiction with \(\pi\) solving \(P\). Similarly, if \(\pi'\) solves \(P'\) but the induced policy \(\pi\) does not solve \(P\), then by 1), there must be an infinite \(\pi\)-trajectory over \(P\) that is fair, and hence a non-terminating trajectory over \(P'\), in contradiction with \(\pi'\) solving \(P'\). We are thus left to show 1) and 2).

For proving 1), if \(\tau\) is an infinite fair \(\pi\)-trajectory over \(P\), then the policy graph determined by \(\pi\) over the states of \(P\), has a strongly connected component \(C\) such that for each non-deterministic action \(a\) applied in a state \(s \in C\), all its successor states \(s' \in F(a, s)\) are also in \(C\). This implies that in the policy graph determined by the policy \(\pi'\) over the (normal) states \(s'\) of \(P'\) has a corresponding strongly connected component \(C'\) that includes \(s\) and all the (normal) successor states \(s' \in F(a, s)\). The component \(C'\) is non-terminating as all the variables \(X\) and \(Y_{a,i}\) that are used to capture the non-determinism of \(a\), are all decremented and incremented in \(C'\). Then, there is a non-terminating trajectory over \(P'\), one that enters the component \(C'\) infinitely often.

The converse, required for proving 2), is also direct. If \(C'\) is a non-terminating loop that includes a state \(s\) where \(a = \pi'(s)\) is a non-deterministic action, then \(C'\) must include all the (normal) successor states \(s' \in F(a, s)\), as otherwise, variable \(Y_{a,i}\) would be decremented in \(C'\) but not incremented, and hence the loop would be terminating. But then the loop \(C\) corresponding to \(C'\) in \(P\) is also closed in this sense, and represents an infinite \(\pi\)-trajectory in \(P\) that is thus fair.

Theorem 19 states that the strong cyclic policies for a FOND problem \(P\) correspond exactly with the policies of the QNP \(Q = R(P)\) that can be obtained from \(P\) in polynomial time. There is an analogous translation for computing the strong policies of \(P\); namely, the strong cyclic policies that are actually acyclic (no state visited twice along any trajectory). For this, let \(R'\) be the translation that is like \(R\) above but with the numerical variables \(Y_{a,i}\) removed; i.e., initial and goal conditions on \(Y_{a,i}\) are removed from \(R'(P)\) as well as the conditions and effects on \(Y_{a,i}\) in the actions \(Exit(a, i)\), and also the actions \(ExitG(a, i)\) are removed.

**Theorem 20** (Reduction Strong FOND to QNP). The mapping \(R'\) is a polynomial time reduction from FOND problems with strong solutions into QNPs. That is, a FOND problem \(P\) that has a strong solution iff the QNP \(Q = R'(P)\) has solution. Moreover, a strong solution for \(P\) (resp. solution for \(Q\)) can be recovered in polynomial time from a solution for \(Q\) (resp. strong solution for \(P\)).

**Proof.** In the absence of the variables \(Y_{a,i}\), the only solutions to \(Q = R'(P)\) must be acyclic, as any cycle would involve increments and decrements of the single numerical variable \(X\), and thus, would not be terminating. The rest of the proof follows from the correspondence layed out in the previous proof between the trajectories over the normal states in \(Q\) and the trajectories in \(P\).

Let us now consider the case when the FOND problem \(P\) contains actions with multiple non-deterministic effects. Let \(a\) be one such action, and let \(n\) be the number of non-deterministic effects in \(a\), each one denoted by \(E_{1}^{i} | \cdots | E_{k_{j}}^{j}\) where \(j \in [1, n]\) and \(k_{j}\) is the number of outcomes for the
\(j\)-th non-deterministic effect of \(a\). By assumption, every choice of outcomes for the effects yields a consistent (aggregated) outcome for \(a\) (cf. footnote 1). The easiest way to accomodate such actions is to “preprocess” \(P\) by replacing the action \(a\) by the sequence \(\langle a(1), a(2), \ldots, a(n) \rangle\) of \(n\) non-deterministic actions, each one featuring exactly one effect; i.e., the effect of \(a(j)\) is \(E_j^1 \mid \cdots \mid E_j^n\). For this, the precondition of \(a(1)\) is set to the precondition of \(a\), that of \(a(j)\) to \(\{seq(a), next(j)\}\), \(j \in [2, n]\), and the precondition of every other action is extended with \(¬seq(a)\). Likewise, the effect of \(a(1)\) is extended with \(\{seq(a), next(2)\}\), that of \(a(j)\) with \(\{¬next(j), next(j + 1)\}\), \(j \in [2, n − 1]\), and that of \(a(n)\) with \(\{¬next(n), ¬seq(a)\}\). The new atoms \(seq(a)\) and \(next(j)\) denote that the sequence for \(a\) is being applied and the next action to apply in the sequence is \(a(j)\) respectively. In this way, the FOND problem \(P\) is converted in linear time into an equivalent FOND problem where each action has exactly one non-deterministic effect. In other words, we may assume without loss of generality that the FOND problem \(P\) is such that each action has at most one non-deterministic effect since if this is not the case, \(P\) can be converted into an equivalent FOND problem \(P′\) in linear time. By equivalent, we mean that any solution \(\pi\) for \(P\) can be converted in polynomial time into a solution \(\pi′\) for \(P′\), and vice versa.

We have shown how strong cyclic and strong planning over a FOND problem \(P\) translates into QNP planning over a QNP \(Q\) obtained from \(P\): in one case, with all the variables \(X\) and \(\{Y_{a,i}\}_{a,i}\) in place, in the second, with no such variables. The translation with these variables offers the possibility of capturing more subtle forms of fairness. For example, if we just remove from the translation a variable \(Y_{a,i}\) along with the effects on it, the resulting QNP would assume that all the outcomes \(E_j\), \(j \neq i\), of action \(a\) are fair (i.e., they cannot be skipped forever in a fair trajectory) but that the outcome \(E_i\) is not. In other words, while in strong cyclic planning, all the non-deterministic actions are assumed to be fair, and in strong planning, all of them to be unfair, in QNP planning, it is possible to handle a combination of fair and unfair actions (as in dual FOND planning (Geffner & Geffner, 2018)), as well as a combination of fair and unfair outcomes of the same action.

7. Second Reduction: QNPs into FOND problems

We have shown that FOND problems can be reduced in polynomial time to QNPs. We show now the other direction: QNPs can be reduced in polynomial time to FOND problems. The two results imply a new complexity result; namely, that QNPs have the same expressive power as FOND problem and that the plan existence decision problem for both models have the same complexity. This second translation \(T\) is more subtle than the first and unlike the direct translation \(T_D\) above, it is a full reduction which does not require termination tests.

The first attempt at such a translation was sketched in (Bonet et al., 2017) but the reduction is buggy as it is not always sound. The intuition, however, is useful and we build on it. Basically, that reduction introduces boolean variables \(q_X\) that when set to true preclude increments of variable \(X\), hence making the decrements of \(X\) “fair”. The variable \(q_X\) can be reset to false when the loop “finishes”, i.e., when \(X = 0\) is true. This idea, however, does not fully avoid non-terminating loops and hence, by itself, does not produce a sound reduction.4 The new translation replaces

\begin{footnotesize}
4. Consider a QNP \(Q = \{F, V, I, O, G\}\) with a single numerical variable \(X\) and four actions \(a, b, c, \) and \(d\) that result in a loop where \(a = (p_1, X > 0; ¬p_1, p_2, X↓)\), \(b = (p_2; p_3, ¬p_2)\), \(c = (p_3, X > 0; X↓)\), \(d = (p_3, X = 0; ¬p_3, p_1, X↓)\). Let us assume that \(I = \{p_1, X > 0\}\) and \(G = \{X = 0, p_2\}\). There is a single policy \(\pi\) for \(Q\), as in all the (non-goal) states that can be reached from \(I\), there is a single applicable action. This policy \(\pi\) is strongly cyclic but is not terminating. The reason is that one of the trajectories determined by the policy is a non-terminating loop \(s_0, a, s_1, b, s_2, c, s_3, d, s_0, \ldots\) where the single variable that is decremented (\(X\)) is also incremented, and where \(s_0 = (p_1, X > 0)\), \(s_1 = (p_3, X > 0)\), \(s_2 = (p_3, X > 0)\), and \(s_3 = (p_3, X = 0)\). The FOND problem that results from the translation sketched in (Bonet et al., 2017) accepts this policy \(\pi\) as a solution, which is incorrect. This
\end{footnotesize}
the $q_X$ variables by a **bounded stack** that keeps the variables $X$ that are being decremented *in order*, and suitable *counters*. The new variables and actions enforce that solutions of the FOND problem $P = T(Q)$, unlike the solutions of the direct translation $T_D(Q)$, are all terminating. For this, the new translation introduces conditions that mirror those captured by the *SIEVE* procedure. In particular, for capturing policies that terminate, the variables are to be placed on the stack following the order by which *SIEVE* removes them.

### 7.1 Extra Variables and Actions

The reduction $T(Q)$ introduces a bounded stack $\alpha$ where numerical variables from $V$ can be pushed and popped, and bounded counters $c(d)$, for $d = 0, \ldots, |V|$ that are associated with the possible levels (depths) $d = |\alpha|$ of the stack. There is also a top counter $c_T$ that may only increase. The stack starts empty and may grow to contain all the variables in $V$, but no variable can appear in the stack more than once. The stack is represented as growing from left to right; e.g., $\alpha X$ is the stack that results of pushing the variable $X$ in the stack $\alpha$. The $c$ counters start at 0 and may grow up to a Max number, that for completeness must be set to $1 + 2^n$ where $n$ is the total number of boolean and numerical variables in $Q$. In practice, Max can be set to a much small number.\(^5\)

In any case, the counters and the stack are captured in terms of a polynomial number of boolean variables and the whole reduction $T(Q)$ is computed in polynomial time. The state of the stack $\alpha$ is captured by the atoms $in(X)$, $depth(d)$, and $index(X,d)$ that represent whether $X$ is in the stack, the depth of the stack, and the depth at which $X$ is in the stack, respectively. $X$ is the top element in the stack when $index(X,d)$ and $depth(d)$ are true (i.e., the stack is $\alpha X$ and $|\alpha| = d - 1$), and it is bottom element when $index(X,1)$ is true (i.e., the stack is $X$). The stack is empty when $depth(0)$ holds.

The **extra actions** in $P = T(Q)$ are those for pushing and popping variables to and from the stack, and for advancing the top counter $c_T$.

1. **Actions** $Push(X,d)$ for variable $X$ and depth $d \in [0, |V| - 1]$ have preconditions $\neg in(X)$, $depth(d)$ and $c(d) < Max$, and effects:

   a) $in(X)$, $index(X,d + 1)$, $depth(d + 1)$ and $\neg depth(d)$ to push $X$ and increase stack depth,

   b) $c(d) := c(d) + 1$ to increment counter for old level,

   c) $c(d + 1) := 0$ to initialize counter for new level.

2. **Actions** $Pop(X,d)$ for variable $X$ and depth $d \in [1, |V|]$ have preconditions $in(X)$, $index(X,d)$ and $depth(d)$, and effects:

   a) $\neg in(X)$, $\neg index(X,d)$, $\neg depth(d)$, $depth(d - 1)$ to pop $X$ and decrease stack depth.

3. **Action** $Move$ advances the top counter $c_T$ by 1 when the stack is empty; i.e., it has preconditions $depth(0)$ and $c_T < Max$, and effect $c_T := c_T + 1$.

   For simplicity, we assume that the language of our FOND problems $P$ makes room for the integer counters $c(d)$, $d = 0, \ldots, |V|$, that may be increased by 1, from 0 up to a fixed number Max

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\(^5\) For structured policies that result in loops that can be entered and left through single entry and exit points, Max is the bound on the number of consecutive loops (blocks), possibly with other loops nested, that the policy can generate at the same level.
and that may be reset back to 0. In the implementation, these counters are represented in terms of a linear number of boolean variables.\footnote{In the actual encoding, the counters $c(d)$ are represented with $1 + n$ atoms (bits), $b_i(d)$, $i = 0, \ldots, n$, where $n$ is total number of variables in $Q$, i.e., $n = |F| + |V|$. A preconditions such as $c(d) < Max$ then translates into the precondition $\neg b_0(d)$; the least and most significant bits for $c(d)$ are $b_0(d)$ and $b_n(d)$ respectively. Increments of $c(d)$ by 1 may be translated in two different ways, either by using conditional effects, or by increasing the number of actions. For the former, conditional effects of the form $b_0(d), \ldots, b_{i-1}(d), \neg b_i(d) \to \neg b_0(d), \ldots, \neg b_{i-1}(d), b_i(d)$, for $i \in [0, n]$, are used. For the latter, each action act that increases $c(d)$ is replaced by $n$ actions act($i$), $i \in [0, n]$, that are like act but have the extra precondition $b_0(d), \ldots, b_{i-1}(d), \neg b_i(d)$ and the extra effects $b_0(d), \ldots, \neg b_{i-1}(d), b_i(d)$. The first translation, however, when compiled into STRIPS introduces additional actions as well (Nebel, 2000). Finally, a reset effect $c(d) := 0$ is obtained by setting all atoms $b_0(d), i \in [0, n]$, to false.}

The actions $a$ that belong to $Q$ are split into two classes. Actions $a$ that do not decrement any variable, keep their names in $P = T(Q)$ but replace their $Inc(X)$ effects by propositional effects $X > 0$, and add the precondition $\neg in(X)$ that disables $a$ when $X$ is in the stack:

4. **Actions $a$ in $Q$ that decrement no variable**, keep the same names in $T(Q)$, the same preconditions and same effects, except that the effects $Inc(X)$ are replaced by propositional effects $X > 0$, if any, and in such a case, the precondition $\neg in(X)$ is added.

Actions $a$ from $Q$ that decrement variables (and hence introduce non-determinism) map into actions in $P$ of type $a(X, d)$ where $X$ is a variable decremented by $X$ that is in the stack at depth $d$: more of the variables that are decremented are in the stack when the action is applied:

5. **Actions $a(X, d)$ for $a$ in $Q$ that decrement a variable $X$ in the stack at level $d$ (and possibly others)** inherit propositional preconditions and effects of $a$, and for each variable $Y$ that is increased by $a$, they include the precondition $\neg in(Y)$ and the effect $Y > 0$. The parameter $d \in [0, |V|]$ stands for the current stack depth. The action $a(X, d)$ also has:

   a) extra precondition $index(X, d)$ (i.e., $d$ is level at which $X$ appears in stack),
   b) extra non-deterministic effects $X_i > 0 \mid X_i > 0$ for each $Dec(X_i)$ effect, and
   c) extra effects $c(d') := 0$ for each $d'$ such that $d \leq d' \leq |V|$ to reset the counters for the levels above or equal to $d$.

In words, actions $a$ from $Q$ that do not decrement any variable map into a single action of the form $a$ in $P = T(Q)$, while actions $a$ from $Q$ that decrement variables map into actions $a(X, d)$ applicable only when a variable $X$ decremented by $a$ is in the stack at level $d$. The actions of the form $a$ in $P = T(Q)$ are deterministic, and only actions $a(X, d)$ can generate cycles in a strong cyclic policy for $P$. The reduction $P = T(Q)$ can be summarized as follows:

**Definition 21** (Reduction QNP to FOND). Let $Q(F, V, I, O, G)$ be a QNP. The FOND problem $P = T(Q)$ is $P = \langle F', I', O', G' \rangle$ with:

1. $F' = F \cup \{c_T\} \cup \{\text{depth}(d), c(d)\} \cup \{\text{in}(X)\} \cup \{\text{idx}(X, d')\}$,
2. $I' = I \cup \{\text{depth}(0), c_T = 0, c(0) = 0\}$,
3. $G' = G$,
4. $O' = \{a : a \in O^+\} \cup \{a(Y, d') : a \in O^-\} \cup \{\text{Push}(X, d' - 1), \text{Pop}(X, d')\} \cup \{\text{Move}\}$

where $X$ ranges over $V$, $d$ and $d'$ range over $[0, |V|]$ and $[1, |V|]$ respectively, and $O^-$ and $O^+$ stand for the sets of actions in $O$ that decrement and do not decrement a variable respectively, and the variable $Y$ in $a(Y, d')$ ranges among the variables decremented by the action $a$ in $Q$. Preconditions and effects of the actions in $O'$ are described above in the text.
8. Properties

Clearly, the reduction $P = T(Q)$ can be computed in polynomial time:

**Theorem 22.** Let $Q = \langle F, V, I, O, G \rangle$ be a QNP. The reduction $P = T(Q)$ can be computed in time that is polynomial in the size of $Q$.

*Proof.* Let $n = |F| + |V|$ be the number of variables, propositional or numerical, in $P = T(Q)$. $P$ has $1 + n$ counters of capacity $1 + 2^n$, each one requiring $1 + n$ bits: the counter $c(d)$ is encoded in binary with bits $c(d, i)$, $i \in [0, n]$. $P$ also has $n$ atoms of form $\text{depth}(d)$, $|V| = O(n)$ atoms of form $in(X)$, and $n|V| = O(n^2)$ atoms of form $\text{index}(X, d)$. Therefore, $P$ has $O(|F| + n^2) = O(n^2)$ propositional variables.

$P$ has $|V|^2 = O(n^2)$ push actions. Since $\text{Push}(X, d)$ has precondition $c(d) < \text{Max}$ and effect $c(d) := c(d) + 1$, it gets compiled into $n$ actions of the form $\text{Push}(X, d, i)$, $i \in [0, n - 1]$, where precondition $c(d) < \text{Max}$ is expressed as $\{\neg c(d, i)\} \cup \{c(d, j) : j \in [0, i - 1]\}$, and effect $c(d) := c(d) + 1$ is expressed as $\{c(d, i)\} \cup \{\neg c(d, j) : j \in [0, i - 1]\}$. The pop actions do not modify counters, so there are $O(n^2)$ of them. The $\text{Move}$ action increments the counter $c_T$ and then, like for $\text{Push}(X, d)$, it gets compiled into $n$ different actions. Actions $a$ in $Q$ that do not decrement variables are translated into actions $a$ in $P$. Actions $a$ that decrement a variable get translated into actions $a(X, d)$; there are $O(n^2)$ such actions $a(X, d)$ in $P$ for each such action $a$ in $Q$.

In total, $P$ has $O(n^2)$ propositional variables and $O(|O|^2 + n^3)$ actions, where the cubic term accounts for the $\text{Push}(X, d, i)$ actions. These numbers (polynomially) bound the size of $P$. It is clear that producing each action in $P$ is straightforward and can be done in polynomial time. □

The second direct property of the translation is that due to the use of the counters, all strong cyclic policies $\pi$ for $P$ must terminate:

**Theorem 23.** Let $Q$ be a QNP. Any strong cyclic policy $\pi$ for $P = T(Q)$ is $P$-terminating.

*Proof.* Let $\pi$ be a strong cyclic policy for $P$ and let $\tau = \bar{s}_0, \ldots, [\bar{s}_i, \ldots, \bar{s}_m]^*$ be an infinite $\pi$-trajectory. We need to show that there is some variable $X$ that is decreased in one of these states and increased in none. Clearly, $\pi(\bar{s})$ for some $\bar{s}$ in the recurrent set must be a non-deterministic action, and this means it is an action of form $a(X, d)$. The actions $a(X, d)$ require $X$ to be in the stack and then resets all counters $c(d')$ for $d' \geq d$ back to 0.

Let us pick an action $a(X, d)$ in the recurrent states of $\tau$ to be one with smallest stack depth $d$, and let $\bar{s}$ be one of such states where $\pi(\bar{s}) = a(X, d)$. In the state $\bar{s}$, the variable $X$ is in the stack at level $d$; i.e., $\text{index}(X, d)$ is true. We show next that this same atom must be true in all the other recurrent states in $\tau$. Indeed, if there is a recurrent state where $\text{index}(X, d)$ is false, it means that there are recurrent states where $X$ is popped from the stack, and others where it is pushed back at level $d$, as $\bar{s}$ is a recurrent state where $\text{index}(X, d)$ holds. Yet, each occurrence of the action $\text{Push}(X, d - 1)$ needed to make $\text{index}(X, d)$ true increases the counter $c(d - 1)$ that no action $a(Y, d')$ can reset with $d' < d$, due our choice of the action $a(X, d)$ as one with minimum $d$. As a result, it has to be the case that $X$ is in the stack at level $d$ in all the recurrent states of $\tau$, and hence no action that increases $X$ is applied while in a recurrent state (since increments of $X$ are disabled when $X$ is in the stack). Then, since there is a recurrent state where $X$ is decremented, the infinite $\pi$-trajectory $\tau$ is terminating. Therefore, the policy $\pi$ is $P$-terminating. □

In order to prove soundness and completeness, we establish a correspondence between the strong cyclic policies of $Q$ and the strong cyclic policies of $P = T(Q)$. The policies cannot be the same, however, as the reduction $T$, unlike the direct translation $T_D$, adds extra variables and actions.
Indeed, $T$ preserves the atoms $p$ and $X = 0$ from $Q$, the latter being propositional, but adds boolean variables and actions that ensure that the policies over of $T(Q)$, unlike the policies over $T_D(Q)$, terminate.

Let $Q_M$ be the QNP obtained from the FOND problem $P = T(Q)$ by 1) adding the numerical variables $X$ from $Q$, 2) replacing the effects $X > 0$ by $Inc(X)$, and the non-deterministic effects $X > 0 | X = 0$ by $Dec(X)$, and 3) interpreting the preconditions and goal of the form $X = 0$ and $X > 0$ in terms of such variables (i.e., non-propositionally).

**Theorem 24.** If $\pi$ solves the FOND problem $P = T(Q)$, $\pi$ solves the QNP $Q_M$.

*Proof.* $P$ is the direct translation of $Q_M$, i.e. $P = T_D(Q_M)$. $\pi$ is $P$-terminating by Theorem 23 and strong cyclic for $P$ as it solves it. Therefore, by Theorem 15, $\pi$ solves $Q_M$. □

The QNP $Q_M$ can be thought of as the composition of the original QNP $Q$ and with a deterministic model that encodes the state of the stack and counters. From this perspective, the policy $\pi$ that solves $Q_M$ stands for a controller $\pi_M$ for $Q$ with that has an internal memory $M$ comprised of the atoms that encode the stack and counters: actions like $Push(X, d)$, $Pop(X, d)$ and $Move$ only affect the internal memory $M$ of the controller $\pi_M$, actions $a$ that do not decrement any variable in $Q$, only affect the state of $Q$, while actions $a(X, d)$ affect both the state of $Q$ and the internal memory $M$. Due to the correspondence between the application of policy $\pi$ to the QNP $Q_M$ and the application of the controller with memory $\pi_M$ to the QNP $Q$, it is then direct that:

**Theorem 25** (Soundness). If policy $\pi$ solves the FOND problem $P = T(Q)$, the controller $\pi_M$ solves $Q$.

*Proof.* Each execution of $\pi$ in $Q_M$ generates a trajectory over $M$ and one over $Q$. Since $\pi$ solves $Q_M$, the latter must be terminating and goal reaching, but then they must be terminating and goal reaching in $Q$ that shares the same goal as $Q_M$ and the same $Dec(X)$ and $Inc(X)$ actions. □

The inverse direction of this theorem is also true, but it does not give us a completeness result. For that, we need to show that a policy $\pi$ that solves $Q$ determines a policy $\pi'$ that solves $P = T(Q)$.

### 8.1 Completeness

We now assume that there is a policy $\pi$ for $Q$ that solves it, and want to show that then there is a policy $\bar{\pi}$ for $P = T(Q)$ that solves it. Since $\pi$ solves $Q$, $\pi$ is $Q$-terminating and also $P'$-terminating where $P' = T_D(Q)$ is the direct translation of $Q$ (cf. Theorem 15).

Let $G$ be the policy graph associated with $\pi$ in $P'$. By Theorem 16, SIEVE reduces $G$ to an acyclic graph. For the rest of this section, we assume that SIEVE is run until all edges that are associated with actions that decrement variables are eliminated rather than stopping as soon as the graph becomes acyclic.\(^7\) Each edge removed by SIEVE can be identified with a variable, and edges are removed in batches by SIEVE, each such batch($C$) associated with a component $C$ and a variable $X$ chosen by SIEVE; i.e., in a given iteration, SIEVE chooses a component $C$ and a variable $X$, and removes all edges $(\bar{s}, \bar{s}')$ from $C$ such that $\pi(\bar{s})$ is a $Dec(X)$ action (cf. Figure 1).

Let us index the top SCCs processed by SIEVE in topological order (i.e., if $C_i$ reaches $C_j$ for $j \neq i$, then $i < j$), and let $scc(\bar{s})$ be the index of the (top) component that includes $\bar{s}$ (i.e., the index of the component that includes $\bar{s}$ in the graph $G$). SIEVE decomposes each component $C$ into

\(^7\) Clearly, the modification on the stopping condition for SIEVE does not affect its correctness since an acyclic graph remains acyclic when one or more edges are removed, and, on the other hand, if the original SIEVE cannot reduce a component, the modified algorithm is not able to reduce it either.
a collection of nested SCCs that result of recursively removing edges from $C$. For each state $\bar{s}$, let $C_{\bar{s}} = \{C^j_{\bar{s}}\}_{j \geq 1}$ be the collection of nested SCCs that contain $\bar{s}$; i.e.,

- $C^1_{\bar{s}} = C_{k_1}$ where $k_1 = \text{scc}(\bar{s})$ is the index of the (top) component that contains $\bar{s}$, and

- for $j \geq 1$, $C^j_{\bar{s}} = C_{k_{j+1}}$, where $\bar{s}$ is in the component $C_{k_{j+1}}$ of the graph that results when all edges in $\cup \{\text{batch}(C_{k_i}) : i \in [1, j]\}$ have been removed by SIEVE.

For each state $\bar{s}$, let $\text{stack}(\bar{s})$ be the sequence of variables chosen by SIEVE for each component in $C_{\bar{s}}$. Observe that such sequence contains no repetitions since once SIEVE chooses variable $X$ for a component $C$, the same $X$ cannot be chosen later for another component $C'$ contained in $C$. Also, if the action $\pi(\bar{s})$ is a $\mathit{Dec}(X)$ action for variable $X$, then $\text{stack}(\bar{s})$ contains $X$ by the assumption that SIEVE is run until all all edges associated with decrements of variables are eliminated.

We compare the stack $\alpha$ and $\text{stack}(\bar{s})$, and say that $\alpha = X_1 \cdots X_n$ is a prefix of $\text{stack}(\bar{s}) = Z_1 \cdots Z_m$ if the latter can be obtained from the former by pushing variables only; i.e., if $n \leq m$ and $X_i = Z_i$ for $i \in [1, n]$. A property of the SIEVE algorithm that we exploit in the completeness proof is the following:

**Theorem 26.** Let $Q$ be a QNP and let $P' = T_D(Q)$ be its direct translation. If $\pi$ solves $Q$ and $\bar{\tau} = s_0, \ldots, [s_i, \ldots, s_m]^*$ is an infinite $\pi$-trajectory in $P'$, there is a variable $X$ and a recurrent state $\bar{s}$ such that $\pi(\bar{s})$ is a $\mathit{Dec}(X)$ action, and $X$ is in $\text{stack}(\bar{s})'$ for every recurrent state $\bar{s}'$ in $\bar{\tau}$.

**Proof.** If $\pi$ solves $Q$, $\pi$ must be $P'$-terminating. Thus, there must be a variable that is decreased by $\pi$ in some recurrent state, and increased by $\pi$ in no recurrent state. At the same time, SIEVE is complete and must remove variables (edges) in the policy graph until it becomes acyclic (cf. Theorem 16). Initially, all the recurrent states in $\bar{\tau}$ are in the same component but at one point SIEVE removes a variable $X$ and splits the set of recurrent states into different and smaller components. From its definition, this means that $X$ appears in $\text{stack}(\bar{s})$ for each recurrent state $\bar{s}$ in $\bar{\tau}$. Moreover, since the removal of $X$ leaves these states into two or more components, $\pi(\bar{s})$ for one such state must be a $\mathit{Dec}(X)$ action. \hfill $\Box$

We now define the policy $\pi^*$ for $P = T(Q)$ that is determined by the policy $\pi$ that solves $Q$. In the definition, we use the two functions $\text{scc}(\bar{s})$ and $\text{stack}(\bar{s})$ defined above in terms of the execution of SIEVE on the policy graph for $\pi$ on $T_D(Q)$. The states over $P$ are denoted by triplets $\langle \bar{s}, c, \alpha \rangle$ where $\bar{s}$ is the state in $T_D(Q)$, $c$ stands for the state of the counters $c(d)$, $d \in [0, |V|]$, and $c_T$, and $\alpha$ stands for the state of the stack (given by the atoms $\text{depth}(d)$, $\text{in}(X)$, $\text{index}(X, d)$). The policy $\pi^*$ for $P$ is defined at triplet $\langle \bar{s}, c, \alpha \rangle$ by

\begin{align}
\text{Pop}(X, d) &\quad \text{if } c_T < \text{scc}(\bar{s}), X \text{ is top variable in } \alpha, \text{ and } d = |\alpha|, \text{ else} \\
\text{Move} &\quad \text{if } c_T < \text{scc}(\bar{s}) \text{ and empty stack, else} \\
\text{Pop}(X, d) &\quad \text{if } X \text{ is top variable in } \alpha, d = |\alpha|, \text{ and } \alpha \text{ is not a prefix of } \text{stack}(\bar{s}), \text{ else} \\
\text{Push}(X, d) &\quad \text{if } \alpha X \text{ is a prefix of } \text{stack}(\bar{s}) \text{ and } d = |\alpha|, \text{ else} \\
\text{a} &\quad \text{if } \pi(\bar{s}) = a \text{ decrements no variable, else} \\
a(X, d) &\quad \text{if } \pi(\bar{s}) = a \text{ decrements } X \text{ at depth } d \text{ in } \alpha \text{ but no other var at depth } d' < d. 
\end{align}

A first observation is that the policy $\pi^*$ is defined on every triplet $\langle \bar{s}, c, \alpha \rangle$ such that $\pi(\bar{s})$ is defined. The policy $\pi^*$ for the FOND problem $P = T(Q)$ on a triplet $\langle \bar{s}, c, \alpha \rangle$ advances the $c_T$ counter until it becomes equal to the index $\text{scc}(\bar{s})$ of the SCC in $G$ that contains the node $\bar{s}$. It then performs pops and pushes until $\alpha$ becomes equal to $\text{stack}(\bar{s})$, and finally applies the action $a$
selected by the policy \( \pi \) on \( Q \) using the action names \( a \) or \( a(X,d) \) according to whether \( a \) decrements no variable or decrements a variable, which must be in \( \text{stack}(\bar{s}) \) as discussed above. In the latter case, the variable \( X \) for the action \( a(X,d) \) is the variable \( X \) decremented by \( a \) that is deepest in the stack, at depth \( d \). The completeness result can then be expressed as follows:

**Theorem 27** (Completeness). Let \( Q \) be a QNP and let \( \pi \) be a policy for \( Q \). If \( \pi \) solves \( Q \), then the policy \( \pi^* \) defined by (15) solves \( P = T(Q) \).

**Proof.** From Theorem 15, \( \pi \) solves \( Q \) iff \( \pi \) solves and terminates in \( P' = T_D(Q) \). We will show that if \( \pi \) solves and terminates in \( P' \), then \( \pi^* \) must solve \( P = T(Q) \). Therefore, by forward reasoning, given that \( \pi \) solves \( Q \), then \( \pi \) solves and terminates in \( P' \) from which we obtain that \( \pi^* \) solves \( P \).

We need to show that the policy \( \pi^* \) is executable in \( P \), and more precisely that 1) \( \pi^* \) cannot generate non-goal states \( \langle \bar{s}, c, \alpha \rangle \) where \( \pi^* \) is not defined or defined but non applicable, and 2) \( \pi^* \) cannot get trapped in a loop that only involves the extra actions \( \text{Move}, \text{Pop}(X,d) \) and \( \text{Push}(X,d) \).

These two properties ensure that in any \( \pi^* \)-trajectory over \( P \), if a non-goal state \( \langle \bar{s}, c, \alpha \rangle \) is reached, an action \( a \) or \( a(X,d) \) will be the one changing the component \( \bar{s} \) of the state when \( \pi(\bar{s}) = a \), and that this will happen after a bounded number of applications of the extra actions \( \text{Move}, \text{Pop}(X,d) \) and \( \text{Push}(X,d) \) that do not change \( \bar{s} \). Since the effect of the actions \( a \) or \( a(X,d) \) on \( \bar{s} \) in \( P \) is the same as the effect of \( a \) on \( s \) in \( P' \), it follows that \( \pi^* \) will be strong cyclic for \( P \) if \( \pi \) is strong cyclic for \( P' \). Alternatively, 1) and 2) ensure that if \( \pi^* \) is executable in \( P \), it generate trajectories over the \( \bar{s} \) components that are the same as those obtained by the policy \( \pi \) over \( P' \) except for a bounded number of steps where the \( \bar{s} \) component in the states \( \langle \bar{s}, c, \alpha \rangle \) does not change.

Point 2) is direct. \( \text{Move} \) increases the counter \( c_T \) that no other action decreases. Pushes and pops are applied in order, either to flush out the stack when \( c_T < \text{scs}(\bar{s}) \), or to make \( \alpha = \text{stack}(\bar{s}) \). In the latter case, \( \alpha \) is popped until it becomes a prefix of \( \text{stack}(\bar{s}) \) (flushed out in the extreme case), and then pushes take place to make \( \alpha \) equal to \( \text{stack}(\bar{s}) \). Hence, no loops that only involve \( \text{Move}, \text{Pop}(X,d) \) and \( \text{Push}(X,d) \) actions are possible.

Point 1) is more subtle. The policy \( \pi^* \) is defined on all triplets \( \langle \bar{s}, c, \alpha \rangle \) for which \( \bar{s} \) is reachable by \( \pi \). We first argue, that except for the preconditions on the counters \( c(d) \) and \( c_T \), the rest of the preconditions are true for the actions selected by \( \pi^* \). Observe that every triplet \( \langle \bar{s}, c, \alpha \rangle \) reached by \( \pi^* \) is such that \( \bar{s} \) is reachable by \( \pi \), and thus \( \pi(\bar{s}) \) is defined and applicable in \( \bar{s} \). Second, for an action selected by \( \pi^* \), its easy to see, except for \( \neg \text{in}(Y) \) when the action is \( a \) or \( a(X,d) \) and it increments \( Y \), that its preconditions hold. To see that \( \neg \text{in}(Y) \) also holds, observe that if the actions increments \( Y \), then \( \text{stack}(\bar{s}) \) cannot contain \( Y \); if so, the collection \( C_s \) of nested components for \( \bar{s} \) has a component \( C \) that contains a state where \( Y \) is decremented while being incremented in \( \bar{s} \), thus making \( Y \) ineligible by \( \text{Sieve} \).

The actions that have preconditions on counters are of type \( \text{Push}(\square, d) \) with precondition \( c(d) < \text{Max} \), and \( \text{Move} \) with precondition \( c_T < \text{Max} \). Here, \( \square \) is a placeholder that denotes any variable \( X \) in \( V \). For the top counter, \( c_T < \text{Max} \) always hold since \( c_T \) starts at 0, it is only increased to make it equal to \( \text{scs}(\bar{s}) \), and the number of components in \( G \) is less than or equal the number of subsets of states which is less than \( \text{Max} \). We are thus left to show \( c(d) < \text{Max} \) by considering the only type of actions that increase \( c(d) \): \( \text{Push}(\square, d) \).

For this, we show that \( \pi^* \) cannot generate a trajectory \( \bar{\tau} \) in \( P \) that contains a fragment \( \bar{\tau}' \) with \( 1 + 2^n \) (i.e. \( \text{Max} \)) actions of the form \( \text{Push}(\square, d) \) while no action of the form \( a(\square, d') \), \( d' \leq d \), or \( \text{Push}(\square, d - 1) \) as this would be the only way in which \( c(d) \) may grow up to \( 1 + 2^n \): actions of the form \( \text{Push}(\square, d) \) increase \( c(d) \) by 1, and the only actions that decreases \( c(d) \), back to 0, have the form \( a(\square, d') \) for \( d' \leq d \), or \( \text{Push}(\square, d - 1) \).
Indeed, let \( \tilde{\tau}' = (\tilde{s}_1, c_1, \alpha_1), (\tilde{s}_2, c_2, \alpha_2), \ldots \) be such a fragment, and let \( 1 = i_1 < i_2 < \cdots < i_m \), for \( m = 1 + 2^n \), be the indices for the triplets in \( \tilde{\tau}' \) on which the policy \( \pi^* \) selects an action of type \( \text{Push}(\square, d) \). Observe that between each pair of such indices, there must be one triplet where an action of type \( a \) or \( a(\square, d') \) is applied: two pushes at the same stack depth must be mediated by at least one such action.

Let \( i_1^* < i_2^* < \cdots \) be the indices such that \( i_k^* \) is the first index after \( i_k \) where the action selected by \( \pi^* \) is of type \( a \) or \( a(\square, d') \), \( k \in [1, m] \). Since the total number of states is less than or equal to \( m \), there is some \( \tilde{s} \) that repeats. Without loss of generality, let us assume that \( \tilde{s}_{i_1} = \tilde{s}_{i_m} \). The policy \( \pi \) loops in \( P \) on a set \( R \) of recurrent states that includes \( \{ \tilde{s}_{i_k^*} : k \in [1, m] \} \). By Theorem 26, there is a variable \( X \) that is decremented by \( \pi \) while looping in \( R \) such that \( X \) belongs to each \( \text{stack}(\tilde{s}_{i_k^*}), k \in [1, m] \). We choose \( X \) to be such variable appearing deepest in the stacks. Therefore, there is index \( k \geq 1 \) such that \( \pi^*((\tilde{s}_{i_k}, c_k, \alpha_k)) = a(X, d') \) where \( d' \) is the depth of \( X \) in \( \alpha_k \). Since \( X \) also belongs to \( \alpha_1 \), it must be the case \( d' \leq |\alpha_1| = d \), the latter inequality since \( \pi^*((\tilde{s}_{i_1}, c_{i_1}, \alpha_{i_1})) \) is of type \( \text{Push}(\square, d) \). This is a contradiction with the assumption that \( \tilde{\tau}' \) contains no action of type \( a(\square, d') \) for \( d' \leq d \). \( \square \)

The second reduction from QNPs into FOND problems may be used to compute policies for a given QNP from policies of the resulting FOND. The reduction is a sound and complete mapping. As mentioned above, the resulting QNP policies correspond to controllers that map states \( \tilde{s} \) and controller states, pairs \( (c, \alpha) \) that encode the state of the (bounded) counters and stack, into actions. Hence, there is still the question of whether a QNP solvable by such a controller is solvable by a flat policy (as given in Definition 5). This question remains open and it is beyond the scope of this paper. We have spent time trying to answer it without success. Indeed, there are arguments that support both an affirmative and a negative answer. For the former, QNPs contain no hidden information and thus it seems that they satisfy the Markov property: the current state decouples the past (history) from the future. For the latter, it seems that in order to select an appropriate action, an agent must keep in memory what variables it is currently trying to get to zero to avoid any increment of them (such information is in the stack). On the other hand, the other general approach for solving QNPs, based on LTL synthesis and discussed below, is also incapable of resolving this issue as well as the inefficient algorithm of brute-force enumeration and termination test of all strong-cyclic policies.

9. Extensions and Variations of QNPs

For simplicity, QNPs have been defined with certain syntactic restrictions that do not limit their expressive power. In particular, there are no actions with non-deterministic effects on boolean variables, and there are no effects that can map a literal \( X = 0 \) non-deterministically into the literals \( X > 0 \) and \( X = 0 \), as decrements require \( X > 0 \) as a precondition, and increments yield the outcome \( X > 0 \). Yet, these two restrictions are not essential and can be bypassed. As we have seen before, non-deterministic effect on boolean variables as found in strong and strong cyclic FOND planning can be obtained in QNPs by using additional numerical variables (Section 6). Likewise, a sequence of two consecutive effects \( \text{Inc}(X) \) and \( \text{Dec}(X) \) can be used to emulate the effect of an action that leave the value of \( X \) completely uncertain; i.e., that it may increase \( X \), decrease \( X \), or leave the value of \( X \) unchanged.

There are also syntactic conditions that when satisfied, make QNPs simpler. For example, if the numerical variables \( X_i \) in a QNP \( Q \) can be linearly ordered so that the actions that increment a variable \( X_i \) also decrease a variable \( X_j \) that appears later in the ordering, then every policy \( \pi \) that solves the direct translation \( P = T_D(Q) \) of \( Q \) will necessarily solve \( Q \) as well, as any such
policy will terminate in \( Q \). Indeed, if \( X_i \) is the latest variable in the ordering that is decreased in a cycle induced by \( \pi \), the cycle cannot include a different state where the variable \( X_i \) is increased, as otherwise, the condition implies that another variable \( X_j \) appearing later in the ordering must be decreased in the cycle. For such **well-ordered** QNPs, the simpler, direct translation \( T_D \) is thus both sound and complete.

### 10. Implementation

The reduction from QNPs to FOND problems has been implemented and it can be found in the GitHub repository [https://github.com/bonetblai/qnp2fond](https://github.com/bonetblai/qnp2fond). The reduction produces a FOND problem without conditional effects. The reduction may be parametrized in terms of the maximum capacity for counters and the maximum stack depth whose default values are the ones used in the proofs. In some cases, the reduction can be made simpler.

Variables \( X \) that are decremented but which are not incremented by any action can be translated as in the direct translation \( T_D \) bypassing the more complex translation \( T \); i.e., the decrements of \( X \) are translated as non-deterministic boolean outcomes \( X = 0 \mid X > 0 \). Likewise, if there is a subset of numerical variables that can be ordered as \( X_1, \ldots, X_n \), such that actions that increment a variable \( X_i \) decremented a variable \( X_j \) in the subset with \( i < j \), then the effects of all the variables \( X_i \) in this subset can be translated as in the direct translation \( T_D \) as well. This is because, loops that involve changes in these variables only, must necessarily be terminating: the variable \( X_i \) with the largest index in the loop that is decremented in the loop, cannot be incremented as well, as else, there will be another variable \( X_j \) in the set, \( j > i \) such that \( X_j \) is decremented.

Our implementation supports the optimization described above where variables \( X \) that are incremented by no action are translated as in the direct translations. There is also the option to disable this optimization and an option to force the direct translation independently of the increments in the actions. The default mode, used in the examples, is to use the optimization whenever possible, to use the direct translation when no variable is incremented by any action, and to use the maximum capacity for counters and maximum stack depth. In the examples, the resulting FOND problems are solved with FOND-SAT (Geffner & Geffner, 2018), a general SAT-based solver that is available at [https://github.com/tomsons22/FOND-SAT](https://github.com/tomsons22/FOND-SAT). This planner calls a SAT solver multiple times. The SAT solver used by FOND-SAT is Minisat (Een & Sorensson, 2004).

### 11. Examples

We consider some examples of QNPs and their translations. There is no useful baseline for evaluating the use of the translator in combination with FOND planners. The only other complete QNP planner would result from translating QNP problems into LTL synthesis tasks but the comparison would be unfair because, as mentioned earlier, the LTL synthesis is computationally harder than QNP planning. There is also no complete generate-and-test QNP planner reported, which would have to generate the strong cyclic policies of the direct FOND translation, one by one, while checking them for termination.

The QNPs below represent abstractions of particular generalized planning problems; i.e, families of concrete planning problems, and they have been learned automatically (except for the action names) from PDDL representations of the planning domains and sampled plans (Bonet et al., 2019). These QNPs involve a small number of boolean and numerical variables, something which is common in QNPs that represent abstractions where the number of variables is bounded and does not grow with the size of the concrete instances (Bonet & Geffner, 2015). The QNP is a **sound abstraction** of the concrete problems when the boolean and numerical variables \( p \) and \( n \) in the QNP...
accurately represent and track the value changes of certain boolean and numerical state features $\phi_p$ and $\phi_n$. In such a case, the sequences of boolean valuations over the literals $p$ and $n = 0$ in the abstraction can be emulated by sequences of states in the concrete problems with the corresponding values over the features. All QNPs below are sound abstractions in this sense. Soundness implies that the abstract policy obtained by solving the QNP can be applied successfully on the concrete problems by making use of the state feature functions $\phi_p$ and $\phi_n$ (Bonet & Geffner, 2018).

Let us recall that a policy $\pi$ that solves the FOND $P$ obtained from the translation $P = T(Q)$ of a QNP $Q$, defines a policy that can be understood as a memory-extended controller that solves $Q$ using extra boolean variables and actions. Often, however, the policy $\pi(s, m)$ obtained from $P$, where $s$ is the state over $Q$ and $m$ is the memory state, can be projected onto a memoryless policy $\pi'$ for $Q$ which does not use either the extra variables or actions. This happens when there is no state $s$ where the actions $\pi(s, m)$ and $\pi(s, m')$ selected by the policy over two different memory states are associated with different $Q$ actions. In such a case, all states $s$ are associated with a single action $a$ from $Q$ (there must be one such action as otherwise $\pi$ could not solve $P$), and the memoryless policy $\pi'$ for $Q$ can be defined so that $\pi'(s) = a$.

### 11.1 Clearing a Block

A general plan for clearing a given block $x$ in any Blocksworld instance featuring the block $x$ can be obtained by solving the following abstraction, expressed as a QNP problem $Q = \langle F, V, I, O, G \rangle$ where $F = \{H\}$ contains a boolean variable $H$ that represents if a block is being held, $V = \{n\}$ contains a numerical variable that counts the number of blocks above $x$, the initial situation $I = \{\neg H, n > 0\}$ assumes that block $x$ is not clear and that gripper is empty, and the goal situation $G = \{n = 0\}$ expresses that there are no blocks on top of $x$. The two actions in $O$ are

- $\text{Putaway} = \langle H; \neg H \rangle$ to put on the table or on a block not above $x$ the block being held, and

- $\text{Pick} = \langle \neg H, n > 0; H, n \downarrow \rangle$ to pick the topmost block above $x$.

A policy $\pi$ that solves $Q$ is $\pi(s) = \text{Putaway}$ when $H$ is true in $s$, and $\pi(s) = \text{Pick}$ when $H$ and $n = 0$ are both false in $s$. The QNP $Q$ translates into a FOND problem $P$ with 2 atoms and 2 actions since the direct translation is triggered as no action in $Q$ increments any variable, and it runs in less than 0.01 seconds. FOND-SAT solves $P$ and outputs the solution shown in Figure 4 in 0.06 seconds; the planner makes 2 calls to the SAT solver requiring less than 0.01 seconds in total.
11.2 Placing a Block on Top of Another

A general plan for placing block $x$ on top of block $y$ (both fixed) may be obtained by solving a suitable QNP. For simplicity, we only consider the case when the blocks $x$ and $y$ are initially in different towers. In this case, the QNP $Q = \langle F, V, I, O, G \rangle$ has boolean and numerical variables $F = \{E, X, D\}$ and $V = \{n, m\}$ respectively. The general case when $x$ and $y$ may be in the same tower requires a QNP that involves more features and more actions. The boolean variables in $Q$ represent whether the gripper is empty ($E$), holding block $x$ ($X$), or the goal has been achieved ($D$), while the numerical variables count the number $n$ and $m$ of blocks above $x$ and $y$ respectively. The initial $I = \{E, \neg X, \neg D, n > 0, m > 0\}$ describes a configuration where no block is being held, there are blocks above $x$ and above $y$, but $x$ and $y$ are in different towers; the goal is simply $G = \{D\}$. The QNP has 7 different actions:

- $\text{Pick-above-}x = \langle E, \neg X, \neg D, n > 0, m > 0; \neg E, n \downarrow \rangle$ to pick the topmost block that is above $x$,
- $\text{Pick-above-}y = \langle E, \neg X, \neg D, n = 0, m > 0; \neg E, m \downarrow \rangle$ to pick the topmost block that is above $x$,
- $\text{Put-aside-1} = \langle \neg E, \neg X, \neg D, n = 0; E \rangle$ to put aside (not above $x$ or $y$) the block being held,
- $\text{Put-aside-2} = \langle \neg E, \neg X, \neg D, n > 0, m > 0; E \rangle$ to put aside (not above $x$ or $y$) the block being held,
- $\text{Pick-}x = \langle E, \neg X, \neg D, n = 0, m = 0; \neg E, X \rangle$ to pick block $x$,
- $\text{Put-x-aside} = \langle \neg E, \neg X, \neg D, n = 0, m > 0; E, \neg X \rangle$ to put block $x$ aside (not above $y$), and
- $\text{Put-x-on-}y = \langle \neg E, \neg X, \neg D, n = 0, m = 0; E, \neg X, D, m \uparrow \rangle$ to put $x$ on $y$.

Observe that some actions have preconditions that are not strictly necessary, and that there are two different actions to put a block aside. The reason is that this abstraction is automatically learned from a sample of plans and thus there is no guarantee on that it will be the simplest possible one.

Since the QNP features increments, the translator switches to the complete translation, but it notices that there are no increments for $n$ (the number of blocks above $x$), and thus the optimization for $n$ triggers. The translator runs in less than 0.01 seconds and generates a FOND problem $P$ with 74 atoms and 60 actions (69 atoms for encoding the counters and the stack, and 48 actions to manipulate the stack and move the top counter). FOND-SAT finds the solution shown in Figure 5 in 5.14 seconds; it makes 8 calls to the SAT solver that require 0.49 seconds in total.

11.3 Gripper

The task involves a robot with grippers whose goal is to move a number of balls from one room into a target room. Each gripper may carry one ball at a time. The STRIPS predicates are $\text{at-robby}(l)$, $\text{at-ball}(b, l)$, $\text{free}(g)$, $\text{carry}(b, g)$ that denote, respectively, whether the robot (the ball) is at location $l$, whether gripper $g$ is free, and whether gripper $g$ carries ball $b$, plus unary type predicates $\text{room}$, $\text{ball}$, and $\text{gripper}$.

From two plans for this task, one for an instance with 4 balls and 2 grippers and the other with 5 balls and 2 grippers, the approach of Bonet et al. (2019) learns the QNP $Q = \langle F, V, I, O, G \rangle$ that involves one boolean feature $T$ that indicates whether the robot is in the target room, and 3 numerical features that count the number of balls still to be moved ($b$), the number of balls being carried ($c$), and the number of empty grippers ($g$). The set of (abstract) actions in $Q$ is:

- $\text{Drop} = \langle T, c > 0; c \downarrow, g \downarrow \rangle$ to drop balls in the target room,
- $\text{Pick} = \langle \neg T, b > 0, g > 0; b \downarrow, c \downarrow, g \downarrow \rangle$ to pick balls in the other room,
Figure 5: Resulting policy and policy graph for the problem of placing a block on top of another. Nodes represent states in the QNP abstraction and edges represent actions with the effects on numerical variables as shown. The initial state is the top left state and the goal state is the bottom left state (only one goal state shown; goal is $D$). The policy is strong cyclic and terminating, as it involves two non-trivial (non-singleton) components where a numerical variable is decremented but not incremented.

- $Move$-half-loaded $= \langle \neg T, b = 0, c > 0, g > 0; T \rangle$ to move to the target room when there is still capacity to carry balls,

- $Move$-fully-loaded $= \langle \neg T, c > 0, g = 0; T \rangle$ to move to the target room where there is no more capacity to carry balls, and

- $Leave$-target $= \langle T, c = 0, g > 0; \neg T \rangle$ to move to the other room.

It is easy to see that this abstraction correctly captures any instance involving any number of balls and, remarkably, also works for instances where the robot has any positive number of grippers.

The translator switches to the complete translation as there are actions that increment variables but, as in the previous case, the variable $b$ is optimized because no action increments it. The complete translation runs in less than 0.01 seconds and produces a FOND problem $P$ that has 63 atoms and 40 actions (59 atoms for encoding the counters and the stack, and 35 actions to manipulate the stack and move the top counter). FOND-SAT finds the solution shown in Figure 6 in 2.53 seconds; it makes 7 calls to the SAT solver that require 0.30 seconds in total.

12. Related Work

QNPs have been introduced as a decidable planning model able to account for plans with loops (Srivastava et al., 2011; Srivastava, Zilberstein, Gupta, Abbeel, & Russell, 2015). In addition, by defining the boolean and numerical variables of QNPs as suitable general boolean and numerical features over a given domain, it has been shown that QNPs can be used to express abstract models for generalized planning, in particular when the ground actions change from instance to instance (Bonet & Geffner, 2018). More recently, it has been shown that these QNP abstractions can be learned automatically from a given planning domain and sampled plans (Bonet et al., 2019). QNPs thus provide a convenient language for a model-based approach for the computation of general
plans where such plans are derived from a (QNP) planning model. If the model is sound, the general plans are guaranteed to be correct (Bonet & Geffner, 2018; Bonet et al., 2019). This is contrast with the more common inductive or learning-based approaches where plans computed to solve a few sampled instances are assumed to generalize to other instances by virtue of the compact form of the plans (Khardon, 1999; Martin & Geffner, 2004; Fern, Yoon, & Givan, 2004). These learning approaches do not construct or solve a suitable abstraction of the problems as expressed by QNPs. Inductive approaches have been used recently to learn general plans in the form of finite-state controllers (Bonet, Palacios, & Geffner, 2009; Hu & De Giacomo, 2013), finite programs (Aguas, Celorrio, , & Jonsson, 2016), and deep neural nets learned in a supervised manner (Toyer, Trevizan, Thiébaux, & Xie, 2018; Bueno, de Barros, Mauá, & Sanner, 2019; Issakkimuthu, Fern, & Tadepalli, 2018; Bajpai, Garg, et al., 2018). A key difference between learning-based and model-based approaches is that the correctness of the latter follows from the soundness of the model. Deep reinforcement learning methods have also been used recently for computing generalized plans with no supervision (Groshev, Goldstein, Tamar, Srivastava, & Abbeel, 2018; Sukhbaatar, Szlam, Synnaeve, Chintala, & Fergus, 2015), yet by not using first-order symbolic representations, they have difficulties in dealing with relational domains that involve objects and relations (Garnelo &

Figure 6: Resulting policy and policy graph for Gripper. Nodes represent states in the QNP abstraction and edges represent actions with the effects on numerical variables as shown. The initial state is the top left state and the goal state is the bottom right state (only one goal state shown; goal is $G = \{b = 0, c = 0\}$). The policy is strong cyclic and terminating. The main component is $\{\tilde{s}_0, \tilde{s}_1, \tilde{s}_2, \tilde{s}_5, \tilde{s}_8, \tilde{s}_9\}$ where the variable $b$ is decreased but not increased. SIEVE removes the edges associated with such decrements, then allowing the elimination of the edges associated with the decrements of $g$, and then the edges for the decrements of $c$, leaving the graph acyclic. The policy has been derived by solving the FOND translation $T$ of the QNP that ensures termination and strong cyclicity.
Shanahan, 2019). Forms of generalized planning have also been formulated using first-order logic (Srivastava et al., 2011; Illanes & McIlraith, 2019), and general plans over finite horizons have been derived using first-order regression as well (Boutilier, Reiter, & Price, 2001; Wang, Joshi, & Khardon, 2008; Van Otterlo, 2012; Sanner & Boutilier, 2009). The use of QNPs for expressing (or learning) abstractions for generalized planning problems, combined with the compilation of QNPs into FOND problems, allows us to benefit from the performance of propositional off-the-shelf FOND planners like PRP (Muise et al., 2012), MyND (Bercher & Mattmüller, 2009), or FOND-SAT (Geffner & Geffner, 2018) in order to find policies for generalized planning.

QNPs can be easily translated into LTL planning problems with FOND domains, reachability goals, and a particular type of trajectory constraints that can be expressed as compact LTL formula (Bonet et al., 2017). The trajectory constraints use a fragment of LTL (Pnueli, 1977) to express the QNP fairness constraints; namely, that for each numerical variable $X$ in a QNP, it is always the case that infinite decrements of $X$ combined with finite increments of $X$ must eventually drive the variable $X$ to 0. As a result, QNP planning can be translated quite efficiently (linear time) into LTL synthesis. The translation, however, is not particularly useful computationally, as QNP planning, like FOND planning, is EXP-Complete, while LTL synthesis is 2EXP-Complete (doubly exponential in time) (Pnueli & Rosner, 1989). In LTL planning, i.e., FOND planning with LTL goals and trajectory constraints, the double exponential growth is in the number of variables that appear in such formulas (Camacho, Bienvenu, & McIlraith, 2019; Aminof, Giacomo, Murano, & Rubin, 2019).

For the specific type of LTL trajectory constraints that QNPs convey, the general method of Bonet et al. (2017) results in an EXPSPACE algorithm for the synthesis of a tree automaton that solves the given QNP (or to determine that such automaton does not exist). Indeed, the method first computes a deterministic parity word (DPW) automaton that accepts the models of an LTL formula that captures the QNP; this automaton may be of doubly exponential size and with an exponential number of priorities for general types of LTL trajectory constraints, but it is “only” of exponential size and with a bounded number of priorities for QNPs. Then, a deterministic parity tree automaton $A_t$, that accepts the policies for the QNP and is built from the DPW automaton, must be tested for non-emptiness. The tree automaton $A_t$ has size that is polynomial in the size of the DPW automaton and with the same number of priorities. The non-emptiness test requires time that is polynomial in the size of $A_t$ but exponential in the number of priorities. For QNPs, the number of priorities is bounded and thus this method can be implemented in exponential space since the DPW automaton must be explicitly built. Like the reduction from QNPs into FOND problems, this method does not solve the question posed above about the solvability of QNPs by memoryless policies since the automaton $A_t$ captures all history-based policies for the input QNP, not only memoryless policies.

13. Conclusions

QNPs are convenient abstract models for generalized planning. In this work we have studied QNPs and placed them on firmer ground by revising their theoretical foundations, and have shown that FOND problems can be reduced into QNPs, and vice versa, that QNPs can reduced into FOND problems. Both translations are new and polynomial-time computable, hence establishing that the two models have the same expressive power and the same complexity. The previous, direct translation $T_D(Q)$ for QNPs $Q$ also yields a FOND problem but with fairness assumptions that do not match those underlying strong cyclic FOND planning, and this is why solutions to this translation need to be checked for termination. QNPs can be reduced to LTL synthesis and planning but these are harder computational tasks. In the future, it would be interesting to
study more general types of fairness assumptions and the fragments of LTL that can be handled efficiently with methods similar to the ones developed for QNPs that are based on polynomial-time translations and off-the-shelf FOND planners.

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