UNIQUENESS AND STABILITY OF TRAVELING WAVES FOR A THREE-SPECIES COMPETITION SYSTEM WITH NONLOCAL DISPERAL

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Abstract. This paper is concerned with the traveling waves for a three-species competitive system with nonlocal dispersal. It has been shown by Dong, Li and Wang (DCDS 37 (2017) 6291-6318) that there exists a minimal wave speed such that a traveling wave exists if and only if the wave speed is above this minimal wave speed. In this paper, we first investigate the asymptotic behavior of traveling waves at negative infinity by a modified version of Ikehara’s Theorem. Then we prove the uniqueness of traveling waves by applying the stronger comparison principle and the sliding method. Finally, we establish the exponential stability of traveling waves with large speeds by the weighted-energy method and the comparison principle, when the initial perturbation around the traveling wavefront decays exponentially as $x \to -\infty$, but can be arbitrarily large in other locations.

1. Introduction. In this paper, we consider the following three species competition system with nonlocal dispersal

$$
\begin{align*}
\frac{\partial u}{\partial t} &= d_1(J_2 * u - u) + r_1 u [1 - u - b_{12} v - b_{13} w], \\
\frac{\partial v}{\partial t} &= d_2(J_1 * v - v) + r_2 v [1 - b_{21} u - v], \\
\frac{\partial w}{\partial t} &= d_3(J_3 * w - w) + r_3 w [1 - b_{31} u - w],
\end{align*}
$$

(1)

where $d_i > 0$, $r_i > 0$, $b_{ij} > 0$. Here $u$, $v$, $w$ are the population densities of species 1, 2, 3, respectively, $b_{ij}$ is the competition coefficient of species $j$ to species $i$, $r_i$ is the growth rate of species $i$, and $d_i$ is the diffusion coefficient of species $i$. Also, we have taken the scales of species so that the carrying capacity of each species is normalized to be 1. $J_i * \chi - \chi$ models nonlocal dispersal processes $[4, 10, 16]$, and

$$(J_i * \chi)(x, t) = \int_{\mathbb{R}} J_i(x - y) \chi(y, t) dy, \quad i = 1, 2, 3.$$

Throughout this paper, we always make the following assumptions on the kernels $J_i : \mathbb{R} \to \mathbb{R}^+$:

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\( \text{(J1):} \) \( J_i \in C(\mathbb{R}), J_i(x) = J_i(-x) \geq 0, x \in \mathbb{R}, \) and \( \int_{\mathbb{R}} J_i(x)dx = 1, i = 1, 2, 3; \)

\( \text{(J2):} \) For every \( \lambda > 0, \int_{\mathbb{R}} J_i(x)e^{-\lambda x}dx < \infty, i = 1, 2, 3. \)

The system (1) describes the relation that species \( u \) competes with both species \( v \) and \( w \), while species \( v \) and species \( w \) have no competition with each other. In this paper, we give a hypothesis on the competition coefficients \( b_{12}, b_{13}, b_{21} \) and \( b_{31} \):

\( \text{(H1):} \) \( b_{21}, b_{31} > 1, b_{12} + b_{13} < 1, \)

which means that the competition ability of the species \( u \) is stronger than the species \( v \) and \( w \).

In the case of diffusion free, (1) becomes an ODE system which has equilibria \((0, 0, 0), (1, 0, 0), (0, 1, 0), (0, 0, 1) \) and \((0, 1, 1)\). By (H1), it is easy to verify that \((0, 1, 1)\) is an unstable equilibrium and \((1, 0, 0)\) is a stable equilibrium.

In biology, it is very meaningful to study the population invasion between the residents \( v, w \) and the invader \( u \). Since the species \( v, w \) are weak competitors to the species \( u \), it is expected that the species \( u \) shall win the competition eventually. This dynamics of the invasion process can be mathematically characterized by a traveling wave solution connecting the equilibria \((0, 1, 1)\) and \((1, 0, 0)\).

Hereafter, a \textit{traveling wave solution} (in short, traveling wave) of (1) is a special translation invariant solution of the form

\[ (u(x,t), v(x,t), w(x,t)) = (U(\xi), V(\xi), W(\xi)), \quad \xi = x + ct, \]

where \((U, V, W)\) is the wave profile that propagates through the one-dimensional spatial domain at a constant velocity \( c \in \mathbb{R} \). If \( U(\xi), V(\xi) \) and \( W(\xi) \) are monotone in \( \xi \in \mathbb{R} \), then \((U, V, W)\) is called a traveling wavefront. Substituting \((U, V, W)\) into (1), we have the following wave profile system

\[ \begin{aligned}
&c U''(\xi) = d_1(J_1 * U(\xi) - U(\xi)) + r_1U(\xi)[1 - U(\xi) - b_{12}V(\xi) - b_{13}W(\xi)], \\
&c V''(\xi) = d_2(J_2 * V(\xi) - V(\xi)) + r_2V(\xi)[1 - b_{21}U(\xi) - V(\xi)], \\
&c W''(\xi) = d_3(J_3 * W(\xi) - W(\xi)) + r_3W(\xi)[1 - b_{31}U(\xi) - W(\xi)], \\
&0 \leq U, V, W \leq 1,
\end{aligned} \]  

with the asymptotic boundary conditions

\[ (U, V, W)(-\infty) = (0, 1, 1), \quad (U, V, W)(+\infty) = (1, 0, 0). \]  

As we all know, the traveling waves for nonlocal dispersal equations and systems has been extensively studied, see \([3, 9, 13–15, 20–22, 30, 31, 36, 39, 41, 43]\). In a recent paper, Dong, Li and Wang \([8]\) have established the existence and asymptotic behavior of traveling waves of (1), i.e., solutions of (2) with (3). Based on the asymptotic behavior of traveling waves, they further studied a new type of entire solutions which behave as two traveling waves coming from both sides of \( x \)-axis. It is very natural and interesting to investigate the uniqueness of the traveling waves (up to translation) and their asymptotic stabilities. This will be the main purpose of the present paper.

The uniqueness of traveling waves for various evolution equations has been established, for example, see \([1–3, 5–7, 11, 19]\) and the references therein. In the past few years, many researchers employed the strong comparison principle and the sliding method to investigate the uniqueness of traveling waves for two-component systems \([11, 13, 17–19]\). In particular, Guo and Wu \([11]\) considered a two-component non-delayed lattice dynamical system arising in competition models, and proved the
uniqueness of traveling wave solutions connecting two half-positive equilibria by the sliding method. Recently, Li et al. [19] further used the sliding method to prove the uniqueness of traveling wave fronts for a two-component nonlocal dispersal competitive system with time delay. Motivated by [11,13,17–19], in this paper, we apply the sliding method to prove the uniqueness of traveling waves for the three-component nonlocal dispersal competitive system (1). We should point out that a key step in proving uniqueness of traveling waves by the sliding method is to establish rather precise exponential decay rate of wave profiles when $x \to -\infty$. Although Dong, Li and Wang [8] have obtained some results on the asymptotic behavior of traveling waves of (1) at $\pm \infty$, it is not precise enough for the uniqueness of wave profiles. Inspired by Carr and Chmaj [3] for integro-differential equation without delay, we use a modified version of Ikehara’s Theorem to derive more precise asymptotically exponential tails of wave profiles, see also [11,19].

In addition to the uniqueness of traveling waves, the stability of traveling waves is an extremely important subject. In the past decades, there have been extensively investigations on the stability of traveling waves, see [5,14,15,20,23–29,32–35,40,42]. The main methods are the (technical) weighted energy method [15, 20, 23, 27, 28, 40], the sub- and supersolutions method and squeezing technique [5, 32], and the combination of the comparison principle and the weighted energy method [25, 26, 29, 42]. In 2014, Lv and Wang [24] used the third method to prove the stability of traveling waves for a delayed two-component Lotka-Volterra cooperative system with nonlocal dispersal. More recently, Yu et al. [35] took the same method to study the stability of traveling waves for a two-component Lotka-Volterra competitive system with nonlocal dispersal. Encouraged by the work of [24, 33, 35], it is very natural to expect that we can still apply the third method to prove the stability of traveling wave for our three-component Lotka-Volterra competition system (1). It is well known that when the component of the Lotka-Volterra competitive system is greater than or equal to three, it is hard to transform the competitive system into a cooperative system any more. Fortunately, since the nonlinearity of our system (1) has a special structure, and we consider the traveling waves connecting the equilibria $(0, 1, 1)$ and $(1, 0, 0)$, by a simple transformation $u_1 = u$, $u_2 = 1 - v$ and $u_3 = 1 - w$, the system (1) can be reduced into a cooperative system, see (5). Thus, the comparison principle works for the transformed system of (1). Therefore, in this paper, we take the weighted energy method together with the comparison principle to establish the stability of traveling waves of (1). We should point out that by this method, our stability result only holds for waves with larger speed due to the presence of the nonlocal dispersal term.

The rest of this paper is organized as follows. In Section 2, we give the main results of this paper. In Section 3, we are devoted to proving the monotonicity and uniqueness of traveling waves. In Section 4, the stability of traveling waves with large speeds is obtained.

2. Main results. Throughout this paper, we assume that (1) satisfies the following initial condition

$$u(x, 0) = u_0(x, 0), \quad v(x, 0) = v_0(x, 0), \quad w(x, 0) = w_0(x, 0), \quad x \in \mathbb{R}. \quad (4)$$

Let $u_1 = u$, $u_2 = 1 - v$ and $u_3 = 1 - w$. Then (1) with the initial condition (4) is transformed into a cooperative system as follows:
\[
\begin{align*}
\frac{\partial u_1}{\partial t} &= d_1 (J_1 * u_1 - u_1) + r_1 u_1 [1 - b_{12} - b_{13} - u_1 + b_{12} u_2 + b_{13} u_3], \\
\frac{\partial u_2}{\partial t} &= d_2 (J_2 * u_2 - u_2) + r_2 (1 - u_2) [b_{21} u_1 - u_2], \\
\frac{\partial u_3}{\partial t} &= d_3 (J_3 * u_3 - u_3) + r_3 (1 - u_3) [b_{31} u_1 - u_3],
\end{align*}
\]
\[
(5)
\]
with the initial condition
\[
\begin{align*}
u_1(x, 0) &= u_{01}(x, 0), \\
u_2(x, 0) &= u_{02}(x, 0), \\
u_3(x, 0) &= u_{03}(x, 0).
\end{align*}
\]
(6)

Correspondingly, by setting \((\varphi_1(\xi), \varphi_2(\xi), \varphi_3(\xi)) := (U(\xi), 1 - V(\xi), 1 - W(\xi))\), (2) and (3) are equivalent to
\[
\begin{align*}
\varphi_1'(\xi) &= d_1 (J_1 * \varphi_1(\xi) - \varphi_1(\xi)) + r_1 \varphi_1(\xi) [1 - b_{12} - b_{13} - \varphi_1(\xi) + b_{12} \varphi_2(\xi) + b_{13} \varphi_3(\xi)], \\
\varphi_2'(\xi) &= d_2 (J_2 * \varphi_2(\xi) - \varphi_2(\xi)) + r_2 (1 - \varphi_2(\xi)) [b_{21} \varphi_1(\xi) - \varphi_2(\xi)], \\
\varphi_3'(\xi) &= d_3 (J_3 * \varphi_3(\xi) - \varphi_3(\xi)) + r_3 (1 - \varphi_3(\xi)) [b_{31} \varphi_1(\xi) - \varphi_3(\xi)],
\end{align*}
\]
\[
(7)
\]

For any fixed \(c \neq 0\), define three functions \(\Delta_1(\lambda, c)\), \(\Delta_2(\lambda, c)\) and \(\Delta_3(\lambda, c)\) by
\[
\begin{align*}
\Delta_1(\lambda, c) &= c\lambda - d_1 \left( \int_{\mathbb{R}} J_1(y) e^{-\lambda y} dy - 1 \right) - r_1 (1 - b_{12} - b_{13}), \\
\Delta_j(\lambda, c) &= c\lambda - d_j \left( \int_{\mathbb{R}} J_j(y) e^{-\lambda y} dy - 1 \right) + r_j, \quad j = 2, 3,
\end{align*}
\]
where \(\lambda \in \mathbb{C}\). Define
\[
c_* := \inf_{\lambda > 0} \frac{d_1 \left( \int_{\mathbb{R}} J_1(y) e^{-\lambda y} dy - 1 \right) + r_1 (1 - b_{12} - b_{13})}{\lambda}.
\]
Then it can be seen from [8] that for \(c \geq c_*\), \(\Delta_1(\lambda, c) = 0\) admits two real roots \(0 < \lambda^- (c) \leq \lambda^+ (c)\), and \(\Delta_j(\lambda, c) = 0\) has a real root \(\nu_j(c) > 0\) with \(j = 2, 3\).

Clearly, if \(d_2 \leq d_1\), \(d_3 \leq d_1\), and \(J_1 = J_2 = J_3\) or \(J_1(x) \geq \max\{J_2(x), J_3(x)\}\) for all \(x \in \mathbb{R}\), then \(\nu_j(c) > \Lambda(c)\) for \(j = 2, 3\). In fact, when \(d_2 \geq d_1\), \(d_3 \leq d_1\), and \(J_1 = J_2 = J_3\) or \(J_1(x) \geq \max\{J_2(x), J_3(x)\}\) for all \(x \in \mathbb{R}\), we can get that
\[
d_1 \left( \int_{\mathbb{R}} J_1(y) e^{-\lambda y} dy - 1 \right) + r_1 (1 - b_{12} - b_{13}) > d_j \left( \int_{\mathbb{R}} J_j(y) e^{-\lambda y} dy - 1 \right) - r_j
\]
for any \(\lambda > 0\) and \(j = 2, 3\), which implies that \(\nu_j(c) > \Lambda(c)\), \(j = 2, 3\).

In [8], Dong et al. have established the existence of traveling waves of (5).

**Proposition 1.** (see [8, Theorem 1.4]) *Assume that (H1) holds, and \(J_i\) satisfies (J1) and is compactly supported, \(i = 1, 2, 3\). Then there exists a positive constant \(c_{\min} \) such that (7) admits a solution \((\varphi_1, \varphi_2, \varphi_3)\) satisfying \(\varphi_1 > 0, \varphi_2' > 0\) and \(\varphi_3' > 0\) in \(\mathbb{R}\) if and only if \(c \geq c_{\min}\).*

**Remark 1.** From Proposition 1, we can see that \(c_{\min}\) is the minimal wave speed. By Remark 2 in [8], \(c_{\min} \geq c_*\). Note that \(c_{\min} = c_*\) means that the minimal speed is linearly determined. From the random diffusion system and lattice dynamical system [12], we conjecture \(c_{\min} = c_*\) holds under some conditions on the coefficients and Kernel functions \(J_i, i = 1, 2, 3\), of (1).
Lemma 2.1. (see [8, Corollary 1]) Assume that (H1) holds, and \( J_i \) satisfies (J1) and is compactly supported, \( i = 1, 2, 3 \). Let \((\varphi_1, \varphi_2, \varphi_3)\) be a solution of (7). Then
\[
\lim_{\xi \to -\infty} \frac{\varphi'_1(\xi)}{\varphi_1(\xi)} = \Lambda(c),
\]
\[
\lim_{\xi \to -\infty} \frac{\varphi'_2(\xi)}{\varphi_2(\xi)} = \begin{cases} \nu_2(c), & \text{for } \nu_2(c) \leq \Lambda(c), \\ \Lambda(c), & \text{for } \nu_2(c) > \Lambda(c), \end{cases}
\]
\[
\lim_{\xi \to -\infty} \frac{\varphi'_3(\xi)}{\varphi_3(\xi)} = \begin{cases} \nu_3(c), & \text{for } \nu_3(c) \leq \Lambda(c), \\ \Lambda(c), & \text{for } \nu_3(c) > \Lambda(c). \end{cases}
\]
Moreover, if \( \nu_2(c) > \Lambda(c) \) (resp., \( \nu_3(c) > \Lambda(c) \)), then
\[
\lim_{\xi \to -\infty} \frac{\varphi'_1(\xi)}{\varphi_2(\xi)} = L_1 > 0 \quad \left( \text{resp., } \lim_{\xi \to -\infty} \frac{\varphi'_1(\xi)}{\varphi_3(\xi)} = L_2 > 0 \right)
\]
for some \( L_1 \) and \( L_2 \).

For any fixed \( c \neq 0 \), define \( \kappa_1(\lambda, c), \kappa_2(\lambda, c) \) and \( \kappa_3(\lambda, c) \) by
\[
\kappa_1(\lambda, c) = c\lambda - d_1 \left( \int_{\mathbb{R}} J_1(y)e^{-\lambda y}dy - 1 \right) + r_1,
\]
\[
\kappa_j(\lambda, c) = c\lambda - d_j \left( \int_{\mathbb{R}} J_j(y)e^{-\lambda y}dy - 1 \right) + r_j(b_{j1} - 1), \quad j = 2, 3,
\]
where \( \lambda \in \mathbb{C} \). We can easily verify that \( \kappa_1(\lambda, c) = 0 \) admits a real root \( \lambda_1(c) > 0 \), and \( \kappa_j(\lambda, c) \) has a real root \( \lambda_j(c) > 0 \) with \( j = 2, 3 \).

Lemma 2.2. (see [8, Corollary 2]) Assume that (H1) holds, and \( J_i \) satisfies (J1) and is compactly supported, \( i = 1, 2, 3 \). Let \((\varphi_1, \varphi_2, \varphi_3)\) be a solution of (7). Then
\[
\lim_{\xi \to +\infty} \frac{\varphi'_2(\xi)}{1 - \varphi_2(\xi)} = \lambda_2(c), \quad \lim_{\xi \to +\infty} \frac{\varphi'_3(\xi)}{1 - \varphi_3(\xi)} = \lambda_3(c)
\]
and
\[
\lim_{\xi \to +\infty} \frac{\varphi'_1(\xi)}{1 - \varphi_1(\xi)} = \begin{cases} \lambda_1(c), & \text{if } \lambda_1(c) \leq \min\{\lambda_2(c), \lambda_3(c)\}, \\ \min\{\lambda_2(c), \lambda_3(c)\}, & \text{if } \lambda_1(c) > \min\{\lambda_2(c), \lambda_3(c)\}. \end{cases}
\]
Moreover, there exists \( L_3 > 0 \) such that for any \( \xi \in \mathbb{R} \),
\[
\frac{(1 - \varphi_2(\xi)) + (1 - \varphi_3(\xi))}{1 - \varphi_1(\xi)} \leq L_3.
\]

Based on the asymptotic behavior of traveling waves of (5) in Lemmas 2.1 and 2.2, we can prove the monotonicity of wave profiles as follows.

Theorem 2.3 (Monotonicity of traveling waves). Assume that (H1) holds, and \( J_i \) satisfies (J1) and is compactly supported, \( i = 1, 2, 3 \). Then any traveling wave \((\varphi_1(\xi), \varphi_2(\xi), \varphi_3(\xi))\) \( \in C(\mathbb{R}, \mathbb{R}^3) \) of (5) with \( c \geq c_{\min} \) is strictly monotone.

In order to study the uniqueness, we shall need more precise information on the wave tails. Inspired by the approach developed by Carr and Chmaj [3] for integro-differential equation, we can establish the exact asymptotic behavior of traveling waves at negative infinity. Then the uniqueness of traveling waves of (5) with speed \( c \geq c_{\min} \) can be obtained by applying the sliding method.
Theorem 2.4 (Uniqueness of traveling waves). Assume that (H1) holds, and $J_i$ satisfies (31) and is compactly supported, $i = 1, 2, 3$, and $d_2 \leq d_1$, $d_3 \leq d_1$, $J_1 = J_2 = J_3$ or $J_1(x) \geq \max\{J_2(x), J_3(x)\}$ for all $x \in \mathbb{R}$. The wave profile of (5) is unique up to translation for a given wave speed $c \geq c_{\min}$.

Remark 2. The assumption in Theorems 2.3 and 2.4 that $c_i$ is compactly supported, $i = 1, 2, 3$, is given, due to the application of the asymptotic behavior in Lemmas 2.1 and 2.2. We should point out that the conclusions in Theorems 2.3 and 2.4 also hold, if the assumption that $J_i$ is compactly supported, $i = 1, 2, 3$, is replaced by (J2). This is because that we can directly use the Ikehara’s Theorem to show the asymptotic behavior of traveling waves, and then prove the monotonicity and uniqueness of traveling waves, see [19,37,38].

To obtain the stability of traveling wavefronts of (5), we need the following technical assumption:

(H2): $4r_1b_{12} + 4r_1b_{13} < 2r_1 - \frac{d_1}{2}, r_2b_{21} > 4r_2 + r_1b_{12} + \frac{d_2}{2}, r_3b_{31} > 4r_3 + r_1b_{13} + \frac{d_3}{2}$.

Define three functions on $\eta$ as follows:

$$M_1(\eta) = 2r_1 - 4r_1b_{12} - 4r_1b_{13} - d_1 \int_{-\infty}^{0} J_1(y)e^{-\eta y}dy,$$

$$M_2(\eta) = -4r_2 + r_2b_{21} - r_1b_{12} - d_2 \int_{-\infty}^{0} J_2(y)e^{-\eta y}dy,$$

$$M_3(\eta) = -4r_3 + r_3b_{31} - r_1b_{13} - d_3 \int_{-\infty}^{0} J_3(y)e^{-\eta y}dy.$$

By the assumption (H2), we can see that

$$M_1(0) = 2r_1 - 4r_1b_{12} - 4r_1b_{13} - \frac{d_1}{2} > 0,$$

$$M_2(0) = -4r_2 + r_2b_{21} - r_1b_{12} - \frac{d_2}{2} > 0,$$

$$M_3(0) = -4r_3 + r_3b_{31} - r_1b_{13} - \frac{d_3}{2} > 0.$$

Then by the continuity of $M_i(\eta)$ with respect to $\eta$, there exists $\eta_0 > 0$ such that $M_i(\eta_0) > 0$, $i = 1, 2, 3$.

Furthermore, define

$$N_1(\xi) = -2r_1 - 4r_1b_{12} - 4r_1b_{13} + 4r_1\varphi_1(\xi) - r_2b_{21} - r_3b_{31} + r_2b_{21}\varphi_2(\xi) + r_3b_{31}\varphi_3(\xi) - d_1 \int_{-\infty}^{0} J_1(y)e^{-\eta y}dy,$$

$$N_2(\xi) = -4r_2 + 2r_2b_{21}\varphi_1(\xi) - 2r_2b_{21} + r_2b_{21}\varphi_2(\xi) - r_1b_{12} - d_2 \int_{-\infty}^{0} J_2(y)e^{-\eta y}dy$$

and

$$N_3(\xi) = -4r_3 + 2r_3b_{31}\varphi_1(\xi) - 2r_3b_{31} + r_3b_{31}\varphi_3(\xi) - r_1b_{13} - d_3 \int_{-\infty}^{0} J_3(y)e^{-\eta y}dy,$$

where $(\varphi_1(\xi), \varphi_2(\xi), \varphi_3(\xi))$ is a traveling wavefront of (5). It is easy to see that

$$\lim_{\xi \to +\infty} N_i(\xi) = M_i(\eta_0) > 0, \quad i = 1, 2, 3,$$
which imply that there exists a number $\xi_0 > 0$ large enough such that
\[
\begin{cases}
N_1(\xi_0) = -2r_1 - 4r_1b_{12} - 4r_1b_{13} + 4r_1\varphi_1(\xi_0) - r_2b_{21} - r_3b_{31} + r_2b_{21}\varphi_2(\xi_0) \\
+ r_3b_{31}\varphi_3(\xi_0) - d_1 \int_{-\infty}^{0} J_1(y)e^{-\eta_0y}dy > 0,
\end{cases}
\]
\[
\begin{cases}
N_2(\xi_0) = -4r_2 + 2r_2b_{21}\varphi_1(\xi_0) - 2r_2b_{21} + r_2b_{21}\varphi_2(\xi_0) - r_1b_{12} \\
- d_2 \int_{-\infty}^{0} J_2(y)e^{-\eta_0y}dy > 0,
\end{cases}
\]
\[
\begin{cases}
N_3(\xi_0) = -4r_3 + 2r_3b_{31}\varphi_1(\xi_0) - 2r_3b_{31} + r_3b_{31}\varphi_3(\xi_0) - r_1b_{13} \\
- d_3 \int_{-\infty}^{0} J_3(y)e^{-\eta_0y}dy > 0.
\end{cases}
\]

\textbf{Notation.} Throughout the paper, $C > 0$ denotes a generic constant, while $C_i > 0 (i = 0, 1, 2 \cdots)$ represents a special constant. Letting $I$ be an interval, especially $I = \mathbb{R}$, $L^2(I)$ is the space of the square integrable function on $I$, and $H^k(I)(k \geq 0)$ is the Sobolev space of the $k$-function $f(x)$ defined on $I$ whose derivatives $\frac{d^i}{dx^i}f, i = 1, \cdots, k$, also belong to $L^2(I)$. $L^2_w(I)$ represents the weighted $L^2$-space with the weight $w(x) > 0$ and its norm is defined by
\[
||f(x)||_{L^2_w} = \left( \int_I w(x)f^2(x)dx \right)^{\frac{1}{2}}.
\]

$H^k_w(I)$ is the weighted Sobolev space with the norm
\[
||f(x)||_{H^k_w} = \left( \sum_{i=0}^{k} \int_I w(x) \left| \frac{d^i}{dx^i}f(x) \right|^2 dx \right)^{\frac{1}{2}}.
\]

Letting $T > 0$ and $\mathcal{B}$ space, we denote by $C^0([0, T]; \mathcal{B})$ the space of the $\mathcal{B}$-valued continuous functions on $[0, T]$, and $L^2([0, T]; \mathcal{B})$ as the space of $\mathcal{B}$-valued $L^2$-function on $[0, T]$. The corresponding spaces of the $\mathcal{B}$-valued function on $[0, \infty)$ are defined similarly.

For above $\eta_0$ and $\xi_0$, we define a weight function $w(\xi)$ by
\[
w(\xi) = \begin{cases}
  e^{-\eta_0(\xi-\xi_0)}, & \xi \leq \xi_0, \\
  1, & \xi > \xi_0.
\end{cases}
\]

Let
\[
c_1 = 2r_1 - \frac{d_1}{2} + 4r_1b_{12} + 4r_1b_{13} + r_2b_{21} + r_3b_{31} + d_1 \int_{\mathbb{R}} J_1(y)e^{-\eta_0y}dy,
\]
\[
c_2 = 4r_2 - \frac{d_2}{2} + 2r_2b_{21} + r_1b_{12} + d_2 \int_{\mathbb{R}} J_2(y)e^{-\eta_0y}dy,
\]
\[
c_3 = 4r_3 - \frac{d_3}{2} + 2r_3b_{31} + r_1b_{13} + d_3 \int_{\mathbb{R}} J_3(y)e^{-\eta_0y}dy.
\]

Now, we state our stability theorem.

\textbf{Theorem 2.5 (Stability of traveling waves).} Assume that (J1)-(J2) and (H1)-(H2) hold. For any given traveling wave $(\varphi_1(x + ct), \varphi_2(x + ct), \varphi_3(x + ct))$ of (5) with the wave speed $c > \max\{c_{\min}, \tilde{c}\}$, where
\[
\tilde{c} = \max\{c_1, c_2, c_3\},
\]

if the initial data satisfy
\[ (0, 0, 0) \leq (u_{10}(x, 0), u_{20}(x, 0), u_{30}(x, 0)) \leq (1, 1, 1), \quad x \in \mathbb{R}, \]
and the initial perturbations satisfy
\[ u_{i0}(x, 0) - \varphi_i(x) \in H_w^1(\mathbb{R}), \quad i = 1, 2, 3, \]
then the nonnegative solution of the Cauchy problem (5) and (6) uniquely exists and satisfies
\[ (0, 0, 0) \leq (u_1(x, t), u_2(x, t), u_3(x, t)) \leq (1, 1, 1), \quad x \in \mathbb{R}, t > 0, \]
and
\[ u_i(x, t) - \varphi_i(x + ct) \in C([0, +\infty); H^1_w(\mathbb{R})) \cap L^2([0, +\infty); H^1_w(\mathbb{R})), \quad i = 1, 2, 3, \]
where \( w(x) \) was defined by (9). Moreover, \((u_1(x, t), u_2(x, t), u_3(x, t))\) converges to the traveling wave \((\varphi_1(x + ct), \varphi_2(x + ct), \varphi_3(x + ct))\) exponentially in time \( t \), i.e.,
\[ \sup_{x \in \mathbb{R}} |u_i(x, t) - \varphi_i(x + ct)| \leq C e^{-\mu t}, \quad t > 0, \quad i = 1, 2, 3, \]
where \( C \) and \( \mu \) are some positive constants.

3. Monotonicity and uniqueness of traveling waves. In this section, we adopt the strong comparison principle and the sliding method to prove the monotonicity and uniqueness (up to a translation) of traveling waves of (1). We first give the strong comparison principle.

**Lemma 3.1** (Strong comparison principle). Let \((\varphi_1(\xi), \varphi_2(\xi), \varphi_3(\xi))\) and \((\psi_1(\xi), \psi_2(\xi), \psi_3(\xi))\) be two solutions of (7) with the wave speed \( c \geq c_{\text{min}} \) satisfying \( \varphi_1 \leq \psi_1, \varphi_2 \leq \psi_2 \) and \( \varphi_3 \leq \psi_3 \) in \( \mathbb{R} \). Then either \( \varphi_1 < \psi_1, \varphi_2 < \psi_2 \) and \( \varphi_3 < \psi_3 \) or \( \varphi_1 \equiv \psi_1, \varphi_2 \equiv \psi_2 \) and \( \varphi_3 \equiv \psi_3 \).

**Proof.** We only prove either \( \varphi_1 < \psi_1 \) in \( \mathbb{R} \) or \( \varphi_1 \equiv \psi_1 \) in \( \mathbb{R} \), the other case can be proved similarly. Assume \( \varphi_1(\xi_0) = \psi_1(\xi_0) \) for some \( \xi_0 \in \mathbb{R} \). According to (7) and taking \( \beta_1 > d_1 + 2r_1 \), we can see that \((\varphi_1, \varphi_2, \varphi_3)\) and \((\psi_1, \psi_2, \psi_3)\) satisfy
\[ \phi_1(\xi) = \frac{1}{c} e^{-\frac{\xi}{c}} \int_{-\infty}^{\xi} e^{\frac{s}{c}} d_1 (J_1 * \phi_1)(s) + F_1(\phi_1, \phi_2, \phi_3)(s) + (\beta_1 - d_1) \phi_1(s) \] ds,
where
\[ F_1(\phi_1, \phi_2, \phi_3)(s) = r_1 \phi_1(\xi)[1 - b_{12} - b_{13} - \phi_1(\xi) + b_{12} \phi_2(\xi) + b_{13} \phi_3(\xi)]. \]
Then we have
\[ 0 = \phi_1(\xi_0) - \psi_1(\xi_0) \]
\[ = \frac{1}{c} e^{-\frac{\xi_0}{c}} \int_{-\infty}^{\xi_0} e^{\frac{s}{c}} d_1 (J_1 * (\phi_1 - \psi_1))(s) + F_1(\phi_1, \phi_2, \phi_3)(s) \]
\[ - F_1(\psi_1, \psi_2, \psi_3)(s) + (\beta_1 - d_1)(\phi_1(s) - \psi_1(s)) \] ds.
Since \( \varphi_i(\cdot) \leq \psi_i(\cdot) \) in \( \mathbb{R}, i = 1, 2, 3 \), and \( \beta_1 > d_1 + 2r_1 \), we get
\[ F_1(\varphi_1, \varphi_2, \varphi_3)(s) - F_1(\psi_1, \psi_2, \psi_3)(s) + (\beta_1 - d_1)(\varphi_1(s) - \psi_1(s)) \leq 0 \]
for all \( s \). Hence,
\[ d_1 (J_1 * (\varphi_1 - \psi_1))(s) + F_1(\varphi_1, \varphi_2, \varphi_3)(s) - F_1(\psi_1, \psi_2, \psi_3)(s) \]
\[ + (\beta_1 - d_1)(\varphi_1(s) - \psi_1(s)) = 0 \]
Lemma 3.2

Theorem. If \( J_1 \ast (\varphi_1 - \psi_1)(s) = 0 \) and
\[
\mathcal{F}(\varphi_1, \varphi_2, \varphi_3)(s) - \mathcal{F}((\psi_1, \psi_2, \psi_3)(s) + (\beta_1 - d_1)(\varphi_1(s) - \psi_1(s)) = 0
\]
for all \( s \). Then from \( J_1 \ast (\varphi_1 - \psi_1)(s) = 0 \) for all \( s \), we can obtain that \( \varphi_1(\xi) \equiv \psi_1(\xi), \xi \in \mathbb{R} \). The proof is complete.

Proof of Theorem 2.3. By Lemmas 2.1 and 2.2, we can obtain that there exists \( N > 1 \) such that \( \varphi_i'(\xi) > 0, i = 1, 2, 3 \), for \( \xi \in \mathbb{R} \setminus [-N, N] \). Since \( \varphi_i(-\infty) = 0 \) and \( \varphi_i(\infty) = 1, i = 1, 2, 3 \), the set
\[ A := \{ \eta > 0 | \varphi_i(\xi + \eta) \geq \varphi_i(\xi), \forall \xi \in \mathbb{R} \} \]
is not empty. Hence, \( \eta_* := \inf A \) is well defined. By continuity, we have
\[ \varphi_i(\xi + \eta_*) \geq \varphi_i(\xi), \varphi_2(\xi + \eta_*) \geq \varphi_2(\xi), \varphi_3(\xi + \eta_*) \geq \varphi_3(\xi), \forall \xi \in \mathbb{R} \] We shall prove that \( \eta_* = 0 \). For a contradiction, we suppose that \( \eta_* > 0 \). By Lemma 3.1, we obtain
\[ \varphi_i(\xi + \eta_*) > \varphi_i(\xi), \varphi_2(\xi + \eta_*) > \varphi_2(\xi), \varphi_3(\xi + \eta_*) > \varphi_3(\xi), \forall \xi \in \mathbb{R} \]
Also, by the continuity of \( \varphi_i, i = 1, 2, 3 \), there exists \( \eta_0 \in (0, \eta_*) \) such that
\[ \varphi_1(\xi + s) > \varphi_1(\xi), \varphi_2(\xi + s) > \varphi_2(\xi), \varphi_3(\xi + s) > \varphi_3(\xi) \]
for \( \forall \xi \in [-N - \eta_0, N], s \in [\eta_0, \eta_*] \). Since \( \varphi_i'(\xi) > 0, i = 1, 2, 3 \), for \( \xi \in \mathbb{R} \setminus [-N, N] \), we have
\[ \varphi_1(\xi + s) \geq \varphi_1(\xi), \varphi_2(\xi + s) \geq \varphi_2(\xi), \varphi_3(\xi + s) \geq \varphi_3(\xi) \]
for \( \forall \xi \in \mathbb{R} \setminus [-N - \eta_0, N], s \in [\eta_0, \eta_*] \). Thus, \( \varphi_1(\xi + \eta) \geq \varphi_1(\xi), \varphi_2(\xi + \eta) \geq \varphi_2(\xi), \varphi_3(\xi + \eta) \geq \varphi_3(\xi) \) for all \( \xi \in \mathbb{R} \) and \( \eta > \eta_0 \). This contradicts the definition of \( \eta_* \)
Hence, \( \eta_* = 0 \) and it follows that \( \varphi_i'(\xi) \geq 0 \) for \( \xi \in \mathbb{R}, \ i = 1, 2, 3 \).

Now we prove \( \varphi_i'(\xi) > 0 \) for \( \xi \in [-N, N], i = 1, 2, 3 \). If there exists \( \xi_0 \in [-N, N] \) such that \( \varphi_i'(\xi_0) = 0 \), without loss of generality, we may assume \( \xi_0 \) is the left-most point such that \( \varphi_i'(\xi_0) = 0 \), then \( \varphi_i'(\xi_0) \) is the minimum of \( \varphi_i'(\xi) \), so \( \varphi_i''(\xi_0) = 0 \). By differentiating the first equation of (7), it yields
\[ 0 = d_1(J_1 \ast \varphi_i'(\xi_0) + r_1 \varphi_1(\xi_0)(b_1 \varphi_2(\xi_0) + b_3 \varphi_3(\xi_0)) \geq 0, \]
which implies that \( \varphi_i'(\xi_0 - y) = 0 \) for \( y \in \mathbb{R} \). It contradicts with the definition of \( \xi_0 \). Hence, we obtain that \( \varphi_i'(\xi) > 0 \) for \( \xi \in \mathbb{R} \). Similarly, we can prove \( \varphi_2'(\xi) > 0 \) and \( \varphi_3'(\xi) > 0 \) for \( \xi \in \mathbb{R} \). The proof is complete.

In order to study the uniqueness, we further investigate the asymptotic behavior of traveling waves of (5) at negative infinity by using a modified version of Ikehara’s Theorem.

Lemma 3.2 (Ikehara’s Theorem). For a positive non-decreasing function \( u(\xi) \), we define
\[ \mathcal{F}(\lambda) := \int_{-\infty}^{0} e^{-\lambda \xi} u(\xi) d\xi. \]
If \( \mathcal{F} \) has the representation \( \mathcal{F}(\lambda) = \frac{\mathcal{H}(\lambda)}{(\partial_0 - \lambda)^{\gamma}} \), where \( k > -1, \partial_0 > 0 \), and \( \mathcal{H}(\lambda) \) is analytic in the strip \( 0 < \text{Re} \lambda \leq \partial_0 \), then
\[ \lim_{\xi \to -\infty} \frac{u(\xi)}{\xi^{|k|e^{\partial_0 \xi}}} = \frac{\mathcal{H}(\partial_0)}{\Gamma(\partial_0 + 1)}. \]
In the sequel, we let
\[ I_1(\lambda) := \int_{\mathbb{R}} \varphi_1(\xi) \left[ -\varphi_1(\xi) + b_{12} \varphi_2(\xi) + b_{13} \varphi_3(\xi) \right] e^{-\lambda \xi} d\xi, \]
\[ I_j(\lambda) := \int_{\mathbb{R}} [\varphi_j^2(\xi) - b_j \varphi_1(\xi) \varphi_j(\xi)] e^{-\lambda \xi} d\xi, \quad j = 2, 3. \]

**Theorem 3.3.** Assume that \((\varphi_1, \varphi_2, \varphi_3)\) is any solution of (7) with \(c \geq c_{\min}\). Then there exists a constant \(k \in \{0, 1\}\) with \(k = 1\), if \(I_1(\Lambda) \neq 0\), and \(k = 0\), if \(I_1(\Lambda) = 0\), such that the following statements hold.

(i) There exist \(\theta_1, \theta_2 \in \mathbb{R}\) depending on \(\varphi_j\), \(j = 1, 2, 3\) such that
\[ \lim_{\xi \to -\infty} \frac{\varphi_1(\xi + \theta_1)}{e^{\Lambda(c)\xi}} = 1, \quad \text{if} \quad c > c_*, \quad \lim_{\xi \to -\infty} \frac{\varphi_1(\xi + \theta_2)}{|\xi|^{k+1} e^{\Lambda(c)\xi}} = 1, \quad \text{if} \quad c = c_. \]

(ii) For \(c > c_*\), there exist \(\theta_3, \tilde{\theta}_3, \theta_4, \tilde{\theta}_4, \theta_5, \tilde{\theta}_5 \in \mathbb{R}\), depending on \(\varphi_j\), \(j = 1, 2, 3\), such that
\[ \lim_{\xi \to -\infty} \frac{\varphi_2(\xi + \theta_3)}{e^{\Lambda(c)\xi}} = 1, \quad \text{if} \quad \nu_2(c) > \Lambda(c), \]
\[ \lim_{\xi \to -\infty} \frac{\varphi_2(\xi + \tilde{\theta}_3)}{e^{\nu_2(c)\xi}} = 1, \quad \text{if} \quad \nu_2(c) = \Lambda(c), \]
\[ \lim_{\xi \to -\infty} \frac{\varphi_3(\xi + \tilde{\theta}_3)}{|\xi|^{k+1} e^{\Lambda(c)\xi}} = 1, \quad \text{if} \quad \nu_3(c) > \Lambda(c), \]
\[ \lim_{\xi \to -\infty} \frac{\varphi_3(\xi + \tilde{\theta}_3)}{e^{\nu_3(c)\xi}} = 1, \quad \text{if} \quad \nu_3(c) < \Lambda(c). \]

(iii) For \(c = c_*\), there exist \(\theta_6, \tilde{\theta}_6, \theta_7, \tilde{\theta}_7, \theta_8, \tilde{\theta}_8 \in \mathbb{R}\), depending on \(\varphi_j\), \(j = 1, 2, 3\), such that
\[ \lim_{\xi \to -\infty} \frac{\varphi_2(\xi + \theta_6)}{|\xi|^{k+1} e^{\Lambda(c)\xi}} = 1, \quad \text{if} \quad \nu_2(c) > \Lambda(c), \]
\[ \lim_{\xi \to -\infty} \frac{\varphi_2(\xi + \tilde{\theta}_7)}{|\xi|^{k+1} e^{\Lambda(c)\xi}} = 1, \quad \text{if} \quad \nu_2(c) = \Lambda(c), \]
\[ \lim_{\xi \to -\infty} \frac{\varphi_3(\xi + \tilde{\theta}_8)}{|\xi|^{k+1} e^{\Lambda(c)\xi}} = 1, \quad \text{if} \quad \nu_3(c) > \Lambda(c), \]
\[ \lim_{\xi \to -\infty} \frac{\varphi_3(\xi + \tilde{\theta}_8)}{e^{\nu_3(c)\xi}} = 1, \quad \text{if} \quad \nu_3(c) < \Lambda(c). \]
Proof. By Lemma 2.1, we can define

$$L(\lambda, \varphi_1) := \int_{\mathbb{R}} \varphi_1(\xi)e^{-\lambda \xi}d\xi \quad \text{for} \ \lambda \in \mathbb{C} \ \text{with} \ 0 < \text{Re}\lambda < \Lambda,$$

$$L(\lambda, \varphi_j) := \int_{\mathbb{R}} \varphi_j(\xi)e^{-\lambda \xi}d\xi \quad \text{for} \ \lambda \in \mathbb{C} \ \text{with} \ 0 < \text{Re}\lambda < \sigma_j := \min\{\nu_j, \Lambda\}, \ j = 2, 3.$$

It follows from (7) that

$$\Delta_1(\lambda, c)L(\lambda, \varphi_1) = r_1I_1(\lambda) \quad (14)$$

for $\lambda \in \mathbb{C}$ with $0 < \text{Re}\lambda < \Lambda$ and

$$\Delta_j(\lambda, c)L(\lambda, \varphi_j) = \frac{r_j r_{j+1}I_1(\lambda)}{\Delta_1(\lambda, c)} + r_jI_j(\lambda) \quad (15)$$

for $\lambda \in \mathbb{C}$ with $0 < \text{Re}\lambda < \sigma_j, \ j = 2, 3$.

The following conclusions are obvious:

(a) $\lambda = \Lambda$ is a unique root with $\text{Re}\lambda = \Lambda$ of $\Delta_1(\lambda, c) = 0$. Indeed, let $\Lambda + i\beta$ be any root of $\Delta_1(\lambda, c) = 0$. Then we get

$$d_1 \int_{\mathbb{R}} J_1(y)e^{-\Lambda y} (\cos \beta y - 1)dy = 0$$

and

$$c\beta = -d_1 \int_{\mathbb{R}} J_1(y)e^{-\Lambda y} \sin \beta ydy.$$ 

It is easy to see that $\beta = 0$. Similarly, the only root of $\Delta_j(\lambda, c) = 0$ on $\{\text{Re}\lambda = \nu_j\}$ is $\lambda = \nu_j, \ j = 2, 3$.

(b) The functions $I_1(\lambda)$ is analytic in the strip $0 < \text{Re}\lambda < \Lambda + \min\{\sigma_2, \sigma_3\}$, and $I_j(\lambda)$ is analytic in the strip $0 < \text{Re}\lambda < 2\sigma_j, \ j = 2, 3$. Indeed,

$$\varphi_1(\xi)[-\varphi_1(\xi) + b_{12}\varphi_2(\xi) + b_{13}\varphi_3(\xi)] = O(e^{\gamma \xi}),$$

as $\xi \to -\infty, \ \forall \gamma \in (0, \Lambda + \min\{\sigma_2, \sigma_3\})$, 

$$\varphi_j^2(\xi) - b_{j1}\varphi_1(\xi)\varphi_j(\xi) = O(e^{\gamma \xi}), \quad \text{as} \ \xi \to -\infty, \ \forall \gamma \in (0, 2\sigma_j), \quad j = 2, 3.$$ 

In order to show that (i) holds, we rewrite (14) as

$$G_1(\lambda) := \int_{-\infty}^{0} \varphi_1(\xi)e^{-\lambda \xi}d\xi = \frac{r_1 I_1(\lambda)}{\Delta_1(\lambda, c)} - \int_{0}^{+\infty} \varphi_1(\xi)e^{-\lambda \xi}d\xi.$$ 

Define

$$H_1(\lambda) := (\Lambda - \lambda)^{q+1}G_1(\lambda)$$

$$= \frac{r_1 I_1(\lambda)}{\Delta_1(\lambda, c)/(\Lambda - \lambda)^{q+1}} - (\Lambda - \lambda)^{q+1} \int_{0}^{+\infty} \varphi_1(\xi)e^{-\lambda \xi}d\xi, \quad (16)$$

where $q = 0$ for $c > c_\star$, and $q = k$ for $c = c_\star$. By the relation between $G_1(\lambda)$ and $H_1(\lambda)$, we can obtain that $H_1(\lambda)$ is analytic in the strip $S := \{\lambda \in \mathbb{C} | 0 < \text{Re}\lambda < \Lambda\}$. By the facts (a), (b) and (16), we also get the analyticity of $H_1$ on $\{\text{Re}\lambda = \Lambda\}$. Hence, $H_1$ is analytic on the strip $0 < \text{Re}\lambda \leq \Lambda$.

By Theorem 2.3, $\varphi_1(\xi)$ is nondecreasing in $\mathbb{R}$. Then from the above discussions, we can apply Ikehara’s Theorem to obtain

$$\lim_{\xi \to -\infty} \frac{\varphi_1(\xi)}{|\xi|^q e^{\Lambda(\xi)}} = \frac{H_1(\Lambda)}{\Gamma(\Lambda + 1)}.$$
where \( q = 0 \) for \( c > c_* \), and \( q = k \) for \( c = c_* \). It is easy to see that if \( H_1(\lambda) \neq 0 \), then (i) holds. Hence, we only need to prove that \( H_1(\lambda) \neq 0 \).

For \( c > c_* \), since \( \Lambda \) is a simple root of \( \Delta_1(\lambda, c) \) and \( k = 0 \), from (16), we have

\[
H_1(\lambda) = r_1 I_1(\lambda)/g(\lambda, c),
\]

where \( g \) satisfies \( g(\lambda, c) \neq 0 \) and \( g(\lambda, c)(\Lambda - \lambda) = \Delta_1(\lambda, c) \) on \( 0 < \text{Re}\lambda < \Lambda(c) + \{\sigma_2, \sigma_3\} \). If \( I_1(\lambda) = 0 \), then \( I_1(\lambda) = 0 \). Hence, the singularity \( \lambda = \Lambda \) of \( \mathcal{P}(\lambda, c) := r_1 I_1(\lambda)/\Delta_1(\lambda, c) \) is removed. Thus, \( \mathcal{P}(\lambda, c) \) is analytic on \( 0 < \text{Re}\lambda < \Lambda + \varepsilon \) for some sufficiently small \( \varepsilon > 0 \). From (14), we conclude that \( \mathcal{L}(\lambda, \varphi_1) \) is well defined on \( 0 < \text{Re}\lambda < \Lambda + \varepsilon \). On the other hand, \( \lim_{\xi \to -\infty} \xi \varphi_j(\xi) \varphi_{j1}(\xi) = \Lambda \) implies \( \mathcal{L}(\lambda, \varphi_1) \) must diverge for \( \text{Re}\lambda > \Lambda \), which leads to a contradiction.

When \( c = c_* \), \( \Lambda = \Lambda(c_*) \) is a double root of \( \Delta_1(\lambda, c) = 0 \). If \( I_1(\lambda) \neq 0 \), then we can take \( k = 1 \) such that \( H_1(\lambda) \neq 0 \) by (16). On the other hand, if \( I_1(\lambda) = 0 \), then \( \lambda = \Lambda \) must be a simple root of \( I_1(\lambda) = 0 \). Otherwise, \( \mathcal{L}(\lambda, \varphi_1) \) will be well-defined on \( \text{Re}\lambda = \Lambda \) which leads a contradiction as above. Then we can take \( k = 0 \) such that \( H_1(\lambda) \neq 0 \) by (16). Thus (i) holds for \( c = c_* \).

Next, we shall prove (ii). By (15), we let

\[
G_j(\lambda) := \int_{-\infty}^{0} \varphi_j(\xi)e^{-\lambda\xi}d\xi = \frac{r_j r_1 b_1 I_1(\lambda)}{\Delta_j(\lambda, c)\Delta_1(\lambda, c)} + \frac{r_j I_1(\lambda)}{\Delta_1(\lambda, c)} - \int_{0}^{+\infty} \varphi_j(\xi)e^{-\lambda\xi}d\xi, \quad j = 2, 3.
\]

Define

\[
H_j(\lambda) := (\sigma_j - \lambda)^{q+1}G_j(\lambda) = \frac{r_1 r_j b_1 I_1(\lambda)}{\Delta_1(\lambda, c)\Delta_1(\lambda, c)/(\sigma_j - \lambda)^{q+1}} + \frac{(\sigma_j - \lambda)^{q+1}r_j I_1(\lambda)}{\Delta_1(\lambda, c)} - (\sigma_j - \lambda)^{q+1}\int_{0}^{+\infty} \varphi_j(\xi)e^{-\lambda\xi}d\xi
\]

in the strip \( 0 < \text{Re}\lambda \leq \sigma_j, \; j = 2, 3 \), where \( q = 0 \) when \( \nu_j \neq \Lambda, \; q = 1 \) when \( \nu_j = \Lambda \). By using a similar argument as above, it is easy to see that \( H_j(\lambda) \) is analytic in the strip \( 0 < \text{Re}\lambda < \sigma_j, \; j = 2, 3 \). From (a) and (b), we conclude that \( H_j(\lambda) \) is also analytic on \( \{\text{Re}\lambda = \sigma_j\} \), \( j = 2, 3 \). Then by Ikehara’s Theorem, we derive

\[
\lim_{\xi \to -\infty} \frac{\varphi_j(\xi)}{|\xi|^{q}(e^{\sigma_j(\xi)}\xi)} = \frac{H_j(\sigma_j)}{\Gamma(\sigma_j + 1)}, \quad j = 2, 3,
\]

where \( q = 0 \) when \( \nu_j \neq \Lambda, \; q = 1 \) when \( \nu_j = \Lambda \).

Now we prove that \( H_j(\sigma_j) \neq 0 \). If \( \nu_j \geq \Lambda \), then \( \sigma_j = \Lambda \). By (17), \( H_j(\Lambda) = 0 \) implies that \( I_1(\Lambda) = 0 \), since the second term and the third term of right-hand side of (17) become zero when \( \lambda = \Lambda \). But, this contradicts the fact \( I_1(\Lambda) \neq 0 \) and so \( H_j(\sigma_j) \neq 0 \) for \( \nu_j \geq \Lambda \). If \( \nu_j < \Lambda \), then \( \sigma_j = \nu_j \). From

\[
H_j(\lambda) = \frac{r_j \int_{\mathbb{R}} e^{-\lambda\xi} \left(b_1 \varphi_1(\xi)(1 - \varphi_j(\xi)) + \varphi_j^2(\xi)\right) d\xi}{\Delta_1(\lambda, c)/(\sigma_j - \lambda)} - (\nu_j - \lambda) \int_{0}^{+\infty} \varphi_j(\xi)e^{-\lambda\xi}d\xi,
\]

we can see that \( H_j(\nu_j) \neq 0 \). Indeed, if \( H_j(\nu_j) = 0 \), then

\[
\int_{\mathbb{R}} e^{-\lambda\xi} \left(b_1 \varphi_1(\xi)(1 - \varphi_j(\xi)) + \varphi_j^2(\xi)\right) d\xi = 0,
\]

which contradicts the fact that \( H_j(\nu_j) \neq 0 \).
which means $\varphi_1(\xi) = \varphi_3(\xi) = 0$ for $\xi \in \mathbb{R}$, $j = 2, 3$. It leads to a contradiction. The same argument can be used to show (iii), we omit the detail here. The proof is complete. \hfill \Box

**Remark 3.** We should point out that when $c_{\min} = c_*$, the statements (i), (ii) and (iii) hold. When $c_{\min} > c_*$, the statements (i) with $c > c_*$ and (ii) only hold.

**Lemma 3.4.** Let $(\varphi_1, \varphi_2, \varphi_3)$ and $(\psi_1, \psi_2, \psi_3)$ be two solutions of (7). If there exists $q > 0$ such that $(1 + q)\varphi_1(\cdot - kq) \geq \psi_1(\cdot)$, $(1 + q)\varphi_2(\cdot - kq) \geq \psi_2(\cdot)$ and $(1 + q)\varphi_3(\cdot - kq) \geq \psi_3(\cdot)$ in $\mathbb{R}$, then $\varphi_1(\cdot) \geq \psi_1(\cdot)$, $\varphi_2(\cdot) \geq \psi_2(\cdot)$ and $\varphi_3(\cdot) \geq \psi_3(\cdot)$ in $\mathbb{R}$, where $k := k(\varphi_1, \varphi_2, \varphi_3)$ is defined by

$$k := \max_{(\cdot - kq) > 0} \left\{ \frac{\varphi_1(\cdot)}{\varphi_2(\cdot)}, \frac{\varphi_2(\cdot)}{\varphi_3(\cdot)}, \frac{\varphi_3(\cdot)}{\varphi_1(\cdot)} \right\},$$

with $N_0 \gg 1$ such that $\frac{\varphi_1(\xi)}{\varphi_2(\xi) + \varphi_3(\xi)} > \max\{b_{12}, b_{13}, \frac{1}{b_{21}}, \frac{1}{b_{31}}\}$ for all $\xi > N_0$.

**Proof.** For $\xi \in \mathbb{R}$, we define

$$F_1(q, \xi) := (1 + q)\varphi_1(\xi - kq) - \psi_1(\xi),$$

$$F_2(q, \xi) := (1 + q)\varphi_2(\xi - kq) - \psi_2(\xi),$$

$$F_3(q, \xi) := (1 + q)\varphi_3(\xi - kq) - \psi_3(\xi).$$

Let

$$B := \{ q > 0 \mid F_1(q, \xi) \geq 0, F_2(q, \xi) \geq 0 \text{ and } F_3(q, \xi) \geq 0, \xi \in \mathbb{R} \}.$$

By assumption, we have $q_* := \inf B$ is well defined. We claim that $q_* = 0$. For a contradiction, we suppose that $q_* > 0$. By continuity, we get $F_1(q_*, \xi) \geq 0$, $F_2(q_*, \xi) \geq 0$ and $F_3(q_*, \xi) \geq 0$ for all $\xi \in \mathbb{R}$. It is easy to verify that

$$\frac{d}{dq} F_1(q, \xi) = \varphi_1(\xi - kq) \left( 1 - (1 + q)k \frac{\varphi_1'(\xi - kq)}{\varphi_1(\xi - kq)} \right) < 0$$

for all $\xi \leq N_0 + kq$. Similarly, $\frac{d}{dq} F_2(q, \xi) < 0$ and $\frac{d}{dq} F_3(q, \xi) < 0$ for all $\xi \leq N_0 + kq$. In addition, we can get

$$F_1(q_*, +\infty) = F_2(q_*, +\infty) = F_3(q_*, +\infty) = q_* > 0.$$

Hence, we obtain that there exists $\xi_0 > N_0 + kq_*$ such that one of the followings must happen:

$$\frac{d}{d\xi} F_1(q_*, \xi_0) = 0 = F_1(q_*, \xi_0), \; F_i(q_*, \xi) \geq 0, \; \xi \in \mathbb{R}, \; i = 1, 2, 3, \tag{18}$$

$$\frac{d}{d\xi} F_2(q_*, \xi_0) = 0 = F_2(q_*, \xi_0), \; F_i(q_*, \xi) \geq 0, \; \xi \in \mathbb{R}, \; i = 1, 2, 3, \tag{19}$$

$$\frac{d}{d\xi} F_3(q_*, \xi_0) = 0 = F_3(q_*, \xi_0), \; F_i(q_*, \xi) \geq 0, \; \xi \in \mathbb{R}, \; i = 1, 2, 3. \tag{20}$$

If (18) occurs, then

$$(1 + q_*)\varphi_1(\xi_0) = \psi_1(\xi_0), \; (1 + q_*)\varphi_1'(\xi_0) = \psi_1'(\xi_0), \tag{21}$$

$$(1 + q_*)\varphi_2(\xi) \geq \psi_2(\xi), \; (1 + q_*)\varphi_2'(\xi) \geq \psi_2'(\xi), \; (1 + q_*)\varphi_3(\xi) \geq \psi_3(\xi), \; \xi \in \mathbb{R}. \tag{22}$$
where $\xi_0 := \xi_0 - kq_*$ and $\bar{\xi} := \xi - kq_*$. By the first equation of (7), we have
\[
d_1(1 + q_*)(J_1 \ast \varphi_1)(\xi_0) + r_1(1 + q_*)\varphi_1(\xi_0)[1 - b_{12} - b_{13} - \varphi_1(\xi_0) + b_{12}\varphi_2(\xi_0) + b_{13}\varphi_3(\xi_0)]
\]
\[= \begin{array}{c}
d_1(J_1 \ast \psi_1)(\xi_0) \\
+ r_1\psi_1(\xi_0)[1 - b_{12} - b_{13} - \psi_1(\xi_0) + b_{12}\psi_2(\xi_0) + b_{13}\psi_3(\xi_0)].
\end{array} \tag{23}
\]
By the first inequality in (22), we can easily see that
\[
d_1(1 + q_*)(J_1 \ast \varphi_1)(\xi_0) - d_1(J_1 \ast \psi_1)(\xi_0)
= d_1 \int_{\mathbb{R}} J_1(y)[(1 + q_*)\varphi_1(\xi_0 - y) - \psi_1(\xi_0 - y)]dy \geq 0.
\]
Then from (23), we get
\[
\varphi_1(\xi_0) + b_{12}\varphi_2(\xi_0) + b_{13}\varphi_3(\xi_0) \geq \psi_1(\xi_0) + b_{12}\psi_2(\xi_0) + b_{13}\psi_3(\xi_0).
\]
Furthermore, by (21) and (22), we obtain
\[
\varphi_1(\xi_0) - b_{12}\varphi_2(\xi_0) - b_{13}\varphi_3(\xi_0) \leq 0.
\]
It then follows that
\[
\frac{\varphi_1(\xi_0)}{\varphi_2(\xi_0) + \varphi_3(\xi_0)} \leq \max\{b_{12}, b_{13}\},
\]
which contradicts the fact that $\frac{\varphi_1(\xi)}{\varphi_2(\xi) + \varphi_3(\xi)} > \max\{b_{12}, b_{13}\}$ for all $\xi > N_0$.

If (19) occurs, then
\[
(1 + q_*)\varphi_2(\xi_0) = \psi_2(\xi_0), \quad (1 + q_*)\varphi_3(\xi_0) = \psi_3(\xi_0),
\]
\[
(1 + q_*)\varphi_1(\xi) \geq \psi_1(\xi), \quad (1 + q_*)\varphi_2(\xi) \geq \psi_2(\xi), \quad (1 + q_*)\varphi_3(\xi) \geq \psi_3(\xi)
\]
for all $\xi \in \mathbb{R}$. By the second equation of (7), we have
\[
d_2(1 + q_*)(J_2 \ast \varphi_2 - \varphi_2)(\xi_0) + r_2(1 + q_*)(1 - \varphi_2(\xi_0))[b_{21}\varphi_1(\xi_0) - \varphi_2(\xi_0)]
= \begin{array}{c}
d_2(J_2 \ast \psi_2 - \psi_2)(\xi_0) + r_2(1 - \psi_2(\xi_0))[b_{21}\psi_1(\xi_0) - \psi_2(\xi_0)],
\end{array}
\]
which implies
\[
\psi_2(\xi_0) - \varphi_2(\xi_0) - b_{21}\psi_1(\xi_0)
\geq \frac{b_{21}}{\psi_2(\xi_0)} \{(1 + q_*)\varphi_1(\xi_0) - \psi_1(\xi_0)\} - b_{21}\varphi_1(\xi_0)
\geq b_{21}\{(1 + q_*)\varphi_1(\xi_0) - \psi_1(\xi_0) - \varphi_1(\xi_0)\}.
\]
Thus, one has
\[
\frac{\varphi_1(\xi_0)}{\varphi_2(\xi_0)} \leq \frac{1}{b_{21}}.
\]
It contradicts to the fact that $\frac{\varphi_1(\xi)}{\varphi_2(\xi)} > \frac{1}{b_{21}}$ for all $\xi > N_0$.

Similar to (19), if (20) occurs, then we can get
\[
\frac{\varphi_1(\xi_0)}{\varphi_3(\xi_0)} \leq \frac{1}{b_{31}},
\]
which contradicts to $\frac{\varphi_1(\xi)}{\varphi_3(\xi)} > \frac{1}{b_{31}}$ for all $\xi > N_0$. Therefore, we have $q_* = 0$, which implies $\varphi_1(\cdot) \geq \psi_1(\cdot)$, $\varphi_2(\cdot) \geq \psi_2(\cdot)$ and $\varphi_3(\cdot) \geq \psi_3(\cdot)$ in $\mathbb{R}$. The proof is complete. \hfill \square
Proof of Theorem 2.4. For any two solutions \((\varphi_1, \varphi_2, \varphi_3)\) and \((\psi_1, \psi_2, \psi_3)\), we may assume that \(\varphi_1(0) = \psi_1(0) = 1/2\) by a suitable translation. By Theorem 3.3 and exchanging \(\varphi_1\) and \(\psi_1\) (if it is necessary), we may assume

\[
\lim_{\xi \to -\infty} \frac{\varphi_1(\xi)}{\psi_1(\xi)} \geq 1. \tag{24}
\]

Since \(d_2 \leq d_1, d_3 \leq d_1\) and \(J_1 = J_2 = J_3\) or \(J_1(x) \geq \max\{J_2(x), J_3(x)\}\) for all \(x \in \mathbb{R}, \nu_j(c) > \Lambda(c), j = 2, 3\). Then by Theorem 3.3, we have

\[
\lim_{\xi \to -\infty} \frac{\varphi_2(\xi)}{\psi_2(\xi)} \geq 1 \text{ and } \lim_{\xi \to -\infty} \frac{\varphi_3(\xi)}{\psi_3(\xi)} \geq 1. \tag{25}
\]

Thus, for any \(n_0 > 0\), we obtain that \(\varphi_1(\cdot) > \psi_1(\cdot - n_0)\), \(\varphi_2(\cdot) > \psi_2(\cdot - n_0)\) and \(\varphi_3(\cdot) > \psi_3(\cdot - n_0)\) on \((-\infty, -\xi_0]\) for some \(\xi_0 \gg 1\). Since \(\varphi_i(+\infty) = \psi_i(+\infty) = 1, i = 1, 2, 3\), there exists \(\xi_1 \gg 1\) such that \(2\varphi_1(\cdot - k) \geq \psi_1(\cdot - \xi_1)\), \(2\varphi_2(\cdot - k) \geq \psi_2(\cdot - \xi_1)\) and \(2\varphi_3(\cdot - k) \geq \psi_3(\cdot - \xi_1)\) in \(\mathbb{R}\). It the follows from Lemma 3.4 that \(\varphi_1(\cdot) \geq \psi_1(\cdot - \xi_1)\), \(\varphi_2(\cdot) \geq \psi_2(\cdot - \xi_1)\) and \(\varphi_3(\cdot) \geq \psi_3(\cdot - \xi_1)\) in \(\mathbb{R}\).

Define

\[
\eta_* = \inf \{\eta > 0 | \varphi_1(\xi) \geq \psi_1(\xi - \eta), \varphi_2(\xi) \geq \psi_2(\xi - \eta), \varphi_3(\xi) \geq \psi_3(\xi - \eta), \xi \in \mathbb{R}\}.
\]

we claim that \(\eta_* = 0\). Otherwise, \(\eta_* > 0\). By (24) and (25), we get that there exists \(\xi_2 > 0\) such that

\[
\varphi_1(\xi - \eta_*/2) \geq \psi_1(\xi - \eta_*), \varphi_2(\xi - \eta_*/2) \geq \psi_2(\xi - \eta_*), \varphi_3(\xi - \eta_*/2) \geq \psi_3(\xi - \eta_*). \tag{26}
\]

for \(\xi \in (-\infty, -\xi_2]\).

Since \(\varphi_1(+\infty) = \varphi_2(+\infty) = \varphi_3(+\infty) = 1\) and \(\varphi'_1(+\infty) = \varphi'_2(+\infty) = \varphi'_3(+\infty) = 0\), there exists \(\xi_3 \gg 1\) such that

\[
\begin{aligned}
\left\{ \begin{array}{l}
\frac{d}{dq}(1 + q)\varphi_1(\xi - 2kq) = \varphi_1(\xi - 2kq) - 2k(1 + q)\varphi'_1(\xi - 2kq) > 0, \\
\frac{d}{dq}(1 + q)\varphi_2(\xi - 2kq) = \varphi_2(\xi - 2kq) - 2k(1 + q)\varphi'_2(\xi - 2kq) > 0, \\
\frac{d}{dq}(1 + q)\varphi_3(\xi - 2kq) = \varphi_3(\xi - 2kq) - 2k(1 + q)\varphi'_3(\xi - 2kq) > 0
\end{array} \right.
\end{aligned}
\]

for \(\xi \in [\xi_3, +\infty)\) and \(q \in [0, 1]\). Hence, one has

\[
\begin{aligned}
\begin{cases}
(1 + q)\varphi_1(\xi - 2kq) \geq \varphi_1(\xi) \geq \psi_1(\xi - \eta_*), \\
(1 + q)\varphi_2(\xi - 2kq) \geq \varphi_2(\xi) \geq \psi_2(\xi - \eta_*), \\
(1 + q)\varphi_3(\xi - 2kq) \geq \varphi_3(\xi) \geq \psi_3(\xi - \eta_*)
\end{cases}
\end{aligned} \tag{27}
\]

for all \(\xi \in [\xi_3, +\infty)\) and \(q \in [0, 1]\).

Note that \(\varphi_1(\xi) \geq \psi_1(\xi - \eta_*)\), \(\varphi_2(\xi) \geq \psi_2(\xi - \eta_*)\), \(\varphi_3(\xi) \geq \psi_3(\xi - \eta_*)\) for \(\xi \in \mathbb{R}\). Then by Lemma 3.1, we obtain that \(\varphi_1(\xi) > \psi_1(\xi - \eta_*)\), \(\varphi_2(\xi) > \psi_2(\xi - \eta_*)\), \(\varphi_3(\xi) > \psi_3(\xi - \eta_*)\) for \(\xi \in \mathbb{R}\). By continuity, there exists \(\varepsilon \in (0, \min\{1, \eta_*/4k\})\) such that

\[
\begin{aligned}
\varphi_1(\xi - 2k\varepsilon) \geq \psi_1(\xi - \eta_*), \varphi_2(\xi - 2k\varepsilon) \geq \psi_2(\xi - \eta_*), \varphi_3(\xi - 2k\varepsilon) \geq \psi_3(\xi - \eta_*)
\end{aligned} \tag{28}
\]

for \(\xi \in [-\xi_2, \xi_3]\).

Combining (26), (27) and (28), we have

\[
\begin{aligned}
\begin{cases}
(1 + \varepsilon)\varphi_1(\xi - 2k\varepsilon) \geq \psi_1(\xi - \eta_*), \xi \in \mathbb{R}, \\
(1 + \varepsilon)\varphi_2(\xi - 2k\varepsilon) \geq \psi_2(\xi - \eta_*), \xi \in \mathbb{R}, \\
(1 + \varepsilon)\varphi_3(\xi - 2k\varepsilon) \geq \psi_3(\xi - \eta_*), \xi \in \mathbb{R}
\end{cases}
\end{aligned}
\]
Then by Lemma 3.4, \( \varphi_1(\xi - k\varepsilon) \geq \psi_1(\xi - \eta_*), \varphi_2(\xi - k\varepsilon) \geq \psi_2(\xi - \eta_*), \varphi_3(\xi - k\varepsilon) \geq \psi_3(\xi - \eta_*) \) for \( \xi \in \mathbb{R} \). That is, \( \varphi_1(\xi) \geq \psi_1(\xi - (\eta_* - k\varepsilon)), \varphi_2(\xi) \geq \psi_2(\xi - (\eta_* - k\varepsilon)), \varphi_3(\xi) \geq \psi_3(\xi - (\eta_* - k\varepsilon)) \) for \( \xi \in \mathbb{R} \). This contradicts the definition of \( \eta_* \).

Hence, \( \eta_* = 0 \). So, we obtain that \( \varphi_1(\xi) \geq \psi_1(\xi), \varphi_2(\xi) \geq \psi_2(\xi), \varphi_3(\xi) \geq \psi_3(\xi) \) for \( \xi \in \mathbb{R} \). Since \( \varphi_1(0) = \psi_1(0) = 1/2 \), by Lemma 3.1, we have \( \varphi_1(\xi) \equiv \psi_1(\xi), \varphi_2(\xi) \equiv \psi_2(\xi), \varphi_3(\xi) \equiv \psi_3(\xi) \) for \( \xi \in \mathbb{R} \). The proof is complete.

\[ \square \]

4. Stability of traveling waves. In this section, we are going to prove the stability of traveling wavefronts by the weighted energy method with the comparison principle together. The global existence and uniqueness of the solution, and the comparison principle to the initial value problem (5) and (6) can be proved by an argument similar to [22, Lemma 3.2].

**Lemma 4.1** (Boundedness). Assume that (J1)-(J2) and (H1) hold, the initial data satisfy
\[
(0, 0, 0) \leq (u_{10}(x, 0), u_{20}(x, 0), u_{30}(x, 0)) \leq (1, 1, 1)
\]
for \( x \in \mathbb{R} \). Then the nonnegative solution \((u_1(x, t), u_2(x, t), u_3(x, t))\) of (5) and (6) satisfies
\[
(0, 0, 0) \leq (u_1(x, t), u_2(x, t), u_3(x, t)) \leq (1, 1, 1)
\]
for \( x \in \mathbb{R}, t > 0 \).

**Lemma 4.2** (Comparison principle). Assume that (J1)-(J2) and (H1) hold. Let \((u^-_1(x, t), u^-_2(x, t), u^-_3(x, t))\) and \((u^+_1(x, t), u^+_2(x, t), u^+_3(x, t))\) be the solution of (5) with the initial data \((u^-_{10}(x, 0), u^-_{20}(x, 0), u^-_{30}(x, 0))\) and \((u^+_{10}(x, 0), u^+_{20}(x, 0), u^+_{30}(x, 0))\), respectively. If
\[
(0, 0, 0) \leq (u^-_{10}(x, 0), u^-_{20}(x, 0), u^-_{30}(x, 0)) \leq (u^+_{10}(x, 0), u^+_{20}(x, 0), u^+_{30}(x, 0)) \leq (1, 1, 1)
\]
for \( x \in \mathbb{R} \), then
\[
(0, 0, 0) \leq (u^-_1(x, t), u^-_2(x, t), u^-_3(x, t)) \leq (u^+_1(x, t), u^+_2(x, t), u^+_3(x, t)) \leq (1, 1, 1)
\]
for \( x \in \mathbb{R}, t > 0 \).

Next, we mainly discuss the stability of the solution to (5) and (6). Define
\[
\begin{align*}
&u^-_{i0}(x, 0) = \min\{u_{i0}(x, 0), \varphi_i(x)\}, \\
&u^+_{i0}(x, 0) = \max\{u_{i0}(x, 0), \varphi_i(x)\},
\end{align*}
\]
for \( x \in \mathbb{R}, i = 1, 2, 3 \). It is easy to see that
\[
\begin{align*}
0 \leq u^-_{i0}(x, 0) &\leq u_{i0}(x, 0) \leq u^+_{i0}(x, 0) \leq 1, \\
0 \leq u^-_{i0}(x, 0) &\leq \varphi_i(x) \leq u^+_{i0}(x, 0) \leq 1,
\end{align*}
\]
for \( x \in \mathbb{R}, i = 1, 2, 3 \). Then by the comparison principle, we have
\[
\begin{align*}
0 \leq u^-_i(x, t) &\leq u_i(x, t) \leq u^+_i(x, t) \leq 1, \\
0 \leq u^-_i(x, t) &\leq \varphi_i(x + ct) \leq u^+_i(x, t) \leq 1,
\end{align*}
\]
for \( x \in \mathbb{R}, t > 0, i = 1, 2, 3 \).

In order to prove the stability of the traveling waves of (5), we need the following three steps:

**Step 1.** The convergence of \( u^+_i(x, t) \) to \( \varphi_i(x + ct) \), \( i = 1, 2, 3 \).

Let \( \xi := x + ct \) and
\[
U_i(\xi, t) = u^+_i(x, t) - \varphi_i(x + ct), \quad U_{i0}(\xi, 0) = u^+_{i0}(x, 0) - \varphi_i(x)
\]
for $i = 1, 2, 3$. Then by (29) and (30), we have

$$(0, 0, 0) \leq (U_{10}(\xi, 0), U_{20}(\xi, 0), U_{30}(\xi, 0)) \leq (1, 1, 1),$$

$$(0, 0, 0) \leq (U_1(\xi, t), U_2(\xi, t), U_3(\xi, t)) \leq (1, 1, 1).$$

By (5) and (7), it can be verify that $(U_1(\xi, t), U_2(\xi, t), U_3(\xi, t))$ satisfies

$$
\begin{align*}
U_{1t} + cU_{1\xi} &= d_1(J_1 * U_1 - U_1) \\
&\quad + U_1 \left\{ r_1 - r_1 b_{12} - r_1 b_{13} - 2 r_1 \varphi_1 + r_1 b_{12} U_2 + r_1 b_{13} U_3 \right. \\
&\quad \left. + r_1 b_{13} \varphi_3 \right\} - r_1 U_1^2 + r_1 b_{12} \varphi_1 U_2 + r_1 b_{13} \varphi_3 U_3, \\
U_{2t} + cU_{2\xi} &= d_2(J_2 * U_2 - U_2) \\
&\quad + U_2 \left\{ 2 r_2 \varphi_2 - r_2 b_{21} U_1 - r_2 b_{21} \varphi_1 \right\} + r_2 U_2^2 + r_2 b_{21} (1 - \varphi_2) U_1, \\
U_{3t} + cU_{3\xi} &= d_3(J_3 * U_3 - U_3) \\
&\quad + U_3 \left\{ 2 r_3 \varphi_3 - r_3 b_{31} U_1 - r_3 b_{31} \varphi_1 \right\} + r_3 U_3^2 + r_3 b_{31} (1 - \varphi_3) U_1,
\end{align*}
$$

(31)

with the initial data $(U_1(\xi, 0), U_2(\xi, 0), U_3(\xi, 0)) = (U_{10}(\xi, 0), U_{20}(\xi, 0), U_{30}(\xi, 0)), \xi \in \mathbb{R}.

It is easy to see that $U_{i0}(\xi, 0) \in H^1_w(\mathbb{R}), \ i = 1, 2, 3$. Then we can obtain that $U_i(\xi, t) \in C([0, +\infty), H^1_w(\mathbb{R})), \ i = 1, 2, 3$. In order to establish the energy estimates, sufficient regularity of the solution to (31) is required. We thus mollify the initial data as follows

$$U_{i0e}(\xi, 0) = (\Gamma_{\epsilon} * U_{i0})(\xi, 0) = \int_{\mathbb{R}} \Gamma_{\epsilon}(\xi - y) U_{i0}(y, 0) \, dy \in H^2_w(\mathbb{R}), \ i = 1, 2, 3,$$

where $\Gamma_{\epsilon}(\xi)$ is the mollifier. Let $U_{ie}(\xi, t), \ i = 1, 2,$ be the solution to (31) with this mollified initial data. Then we have

$$U_{ie}(\xi, t) \in C([0, +\infty), H^2_w(\mathbb{R})), \ i = 1, 2, 3.$$

By taking the limit $\epsilon \to 0$, we can obtain the corresponding energy estimate for the original solution $U_i(\xi, t)$. For the sake of simplicity, below we formally use $U_i(\xi, t)$ to establish the desired energy estimates.

Multiplying the three equations of (31) by $e^{2\mu t} w(\xi) U_1(\xi, t), e^{2\mu t} w(\xi) U_2(\xi, t)$ and $e^{2\mu t} w(\xi) U_3(\xi, t)$ respectively, where $\mu > 0$ will be specified later in Lemma 4.4, we obtain

$$
\begin{align*}
&\left( \frac{1}{2} e^{2\mu t} w U_1^2 \right)_t + \left( \frac{\xi}{2} e^{2\mu t} w U_1^2 \right)_\xi - d_1 e^{2\mu t} w U_1 \int_{\mathbb{R}} J_1(y) U_1(\xi - y, t) \, dy \\
&\quad + \left( d_1 - \frac{\epsilon w'}{2} - \mu - r_1 + r_1 b_{12} + r_1 b_{13} + 2 r_1 \varphi_1 \\
&\quad - r_1 b_{12} U_2 - r_1 b_{12} \varphi_2 - r_1 b_{13} U_3 - r_1 b_{13} \varphi_3 \right) e^{2\mu t} w U_1^2 \\
&= - r_1 e^{2\mu t} w U_1^3 + r_1 b_{12} \varphi_1 e^{2\mu t} w U_1 U_2 + r_1 b_{13} \varphi_1 e^{2\mu t} w U_1 U_3,
\end{align*}
$$

(32)
\[
\frac{1}{2} e^{2\mu t} U_2^2 + \left( \frac{c}{2} e^{2\mu t} w U_2^2 \right) d_2 e^{2\mu t} w U_2 \int_{\mathbb{R}} J_2(y) U_2(\xi - y, t) dy \\
+ \left( d_2 - \frac{c w'}{2} \right) - \mu - 2r_2 \varphi_2 + r_2 + r_2 b_21 U_1 + r_2 b_21 \varphi_1 \right) e^{2\mu t} w U_2^2 \\
= r_2 e^{2\mu t} w U_2^3 + r_2 b_21 (1 - \varphi_2) e^{2\mu t} w U_1 U_2
\]

(33)

and

\[
\frac{1}{2} e^{2\mu t} U_3^2 + \left( \frac{c}{2} e^{2\mu t} w U_3^2 \right) d_3 e^{2\mu t} w U_3 \int_{\mathbb{R}} J_3(y) U_3(\xi - y, t) dy \\
+ \left( d_3 - \frac{c w'}{2} \right) - \mu - 2r_3 \varphi_3 + r_3 + r_3 b_31 U_1 + r_3 b_31 \varphi_1 \right) e^{2\mu t} w U_3^2 \\
= r_3 e^{2\mu t} w U_3^3 + r_3 b_31 (1 - \varphi_3) e^{2\mu t} w U_1 U_3.
\]

(34)

Since \( U_i \in H^1_{\omega} \), \( i = 1, 2, 3 \), we obtain

\[
\left\{ \frac{c}{2} e^{2\mu t} w U_i^2 \right\} \bigg|_{\xi = -\infty}^{\infty} = 0, \quad i = 1, 2, 3.
\]

Integrating (32)-(34) over \( \mathbb{R} \times [0, t] \) with respect to \( \xi \) and \( t \), and noting the vanishing term at far fields, we get

\[
e^{2\mu t} ||U_1(t)||^2_{L^2} - 2d_1 \int_0^t \int_{\mathbb{R}} e^{2\mu s} w(\xi) U_1(\xi, s) \int_{\mathbb{R}} J_1(y) U_1(\xi - y, s) dy d\xi ds \\
+ 2 \int_0^t \int_{\mathbb{R}} \left( d_1 - \frac{c w'}{2} \right) - \mu - r_1 b_12 + r_1 b_13 + 2r_1 \varphi_1(\xi) - r_1 b_12 U_2(\xi, s) \\
- r_1 b_12 \varphi_2(\xi) - r_1 b_13 U_3(\xi, s) - r_1 b_13 \varphi_3(\xi) \right) e^{2\mu t} w(\xi) U_1^2(\xi, s) d\xi ds \\
\leq ||U_{10}(0)||^2_{L^2} + 2r_1 b_12 \int_0^t \int_{\mathbb{R}} \varphi_1(\xi) e^{2\mu s} w(\xi) U_1(\xi, s) U_2(\xi, s) d\xi ds \\
+ 2r_1 b_13 \int_0^t \int_{\mathbb{R}} \varphi_1(\xi) e^{2\mu s} w(\xi) U_1(\xi, s) U_3(\xi, s) d\xi ds,
\]

(35)

\[
e^{2\mu t} ||U_2(t)||^2_{L^2} - 2d_2 \int_0^t \int_{\mathbb{R}} e^{2\mu s} w(\xi) U_2(\xi, s) \int_{\mathbb{R}} J_2(y) U_2(\xi - y, s) dy d\xi ds \\
+ 2 \int_0^t \int_{\mathbb{R}} \left( d_2 - \frac{c w'}{2} \right) - \mu - 2r_2 \varphi_2(\xi) + r_2 + r_2 b_21 U_1(\xi, s) \\
+ r_2 b_21 \varphi_1(\xi) \right) e^{2\mu s} w(\xi) U_2^2(\xi, s) d\xi ds \\
\leq ||U_{20}(0)||^2_{L^2} + 2r_2 \int_0^t \int_{\mathbb{R}} e^{2\mu s} w(\xi) U_2^2(\xi, s) d\xi ds \\
+ 2r_2 b_21 \int_0^t \int_{\mathbb{R}} \left( 1 - \varphi_2(\xi) \right) e^{2\mu s} w(\xi) U_1(\xi, s) U_2(\xi, s) d\xi ds
\]

(36)

and

\[
e^{2\mu t} ||U_3(t)||^2_{L^2} - 2d_3 \int_0^t \int_{\mathbb{R}} e^{2\mu s} w(\xi) U_3(\xi, s) \int_{\mathbb{R}} J_3(y) U_3(\xi - y, s) dy d\xi ds
\]
By the Cauchy-Schwarz inequality $2xy \leq x^2 + y^2$, we have

$$2\tau \int_0^t \left( \int_R e^{2\mu w} w(\xi) U_i(\xi, s) J_i(y) U_i(\xi - y, s) dy d\xi ds \right) \leq d \int_0^t \left( \int_R e^{2\mu w} w(\xi) U_i^2(\xi, s) d\xi ds \right) + d \int_0^t \left( \int_R e^{2\mu w} w(\xi) \int_R J_i(y) U_i^2(\xi - y, s) dy d\xi ds \right)$$

for $i = 1, 2, 3$. Similarly, we obtain

$$2 \int_0^t \left( \int_R \varphi_1(\xi) e^{2\mu w} w(\xi) U_i(\xi, s) U_j(\xi, s) d\xi ds \right) \leq \int_0^t \left( \int_R \varphi_1(\xi) e^{2\mu w} w(\xi) U_i^2(\xi, s) d\xi ds \right) + \int_0^t \left( \int_R \varphi_1(\xi) e^{2\mu w} w(\xi) U_j^2(\xi, s) d\xi ds \right)$$

and

$$2 \int_0^t \left( \int_R (1 - \varphi_j(\xi)) e^{2\mu w} w(\xi) U_i(\xi, s) U_j(\xi, s) d\xi ds \right) \leq \int_0^t \left( \int_R (1 - \varphi_j(\xi)) e^{2\mu w} w(\xi) U_i^2(\xi, s) d\xi ds \right) + \int_0^t \left( \int_R (1 - \varphi_j(\xi)) e^{2\mu w} w(\xi) U_j^2(\xi, s) d\xi ds \right)$$

for $j = 2, 3$. Substituting (38)-(40) into (35)-(37), we have

$$e^{2\mu t} \| U_1(t) \|_{L^2_R}^2 + \int_0^t \int_R \left[ d_1 - \frac{c w' (\xi)}{w(\xi)} - 2\mu - 2r_1 b_1 - 2r_1 b_2 + 2r_1 b_3 + 4r_1 \varphi_1(\xi) - 2r_1 b_2 U_2(\xi, s) - 2r_1 b_2 \varphi_2(\xi) - 2r_1 b_3 U_3(\xi, s) - 2r_1 b_3 \varphi_3(\xi) - r_1 (b_1 + b_3) \varphi_1(\xi) - d_1 \int_R J_1(y) \frac{w(y) + \varphi_1(y)}{w(\xi)} dy \right] e^{2\mu w} w(\xi) U_2^2(\xi, s) d\xi ds \leq \| U_0(0) \|_{L^2_R}^2 + r_1 b_2 \int_0^t \int_R \varphi_1(\xi) e^{2\mu w} w(\xi) U_2^2(\xi, s) d\xi ds + r_1 b_3 \int_0^t \int_R \varphi_1(\xi) e^{2\mu w} w(\xi) U_3^2(\xi, s) d\xi ds$$

for $j = 2, 3$.
\[ e^{2\mu t} ||U_2(t)||_{L_x^2}^2 \]
\[ + \int_0^t \int_{\mathbb{R}} \left[ d_2 - c \frac{w'(\xi)}{w(\xi)} - 2\mu - 4r_2\varphi_2(\xi) + 2r_2(1 - U_2(\xi, s) + 2r_2b_{21}U_1(\xi, s) + 2r_2b_{21}\varphi_1(\xi) - r_2b_{21}(1 - \varphi_2(\xi)) - d_2 \int_{\mathbb{R}} J_2(y) \frac{w(\xi + y)}{w(\xi)}dy \right] e^{2\mu s} w(\xi) U_2^2(\xi, s)d\xi ds \]
\[ \leq ||U_{20}(0)||_{L_x^2}^2 + r_2b_{21} \int_0^t \int_{\mathbb{R}} (1 - \varphi_2(\xi)) e^{2\mu_s} w(\xi) U_2^2(\xi, s)d\xi ds \quad \text{(42)} \]

and
\[ e^{2\mu t} ||U_3(t)||_{L_x^2}^2 \]
\[ + \int_0^t \int_{\mathbb{R}} \left[ d_3 - c \frac{w'(\xi)}{w(\xi)} - 2\mu - 4r_3\varphi_3(\xi) + 2r_3(1 - U_3(\xi, s)) + 2r_3b_{31}U_1(\xi, s) + 2r_3b_{31}\varphi_1(\xi) - r_3b_{31}(1 - \varphi_3(\xi)) - d_3 \int_{\mathbb{R}} J_3(y) \frac{w(\xi + y)}{w(\xi)}dy \right] e^{2\mu_s} w(\xi) U_3^2(\xi, s)d\xi ds \]
\[ \leq ||U_{30}(0)||_{L_x^2}^2 + r_3b_{31} \int_0^t \int_{\mathbb{R}} (1 - \varphi_3(\xi)) e^{2\mu_s} w(\xi) U_1^2(\xi, s)d\xi ds. \quad \text{(43)} \]

Combining (41)-(43), one has
\[ e^{2\mu t} \sum_{i=1}^{3} ||U_i(t)||_{L_x^2}^2 + \sum_{i=1}^{3} \int_0^t \int_{\mathbb{R}} e^{2\mu_s} B_{\mu, w}^i(\xi, s) w(\xi) U_i^2(\xi, s)d\xi ds \]
\[ \leq \sum_{i=1}^{3} ||U_{i0}(0)||_{L_x^2}^2, \]

where
\[ B_{\mu, w}^i(\xi, t) = A_{w}^i(\xi, t) - 2\mu, \quad i = 1, 2, 3, \]

with
\[ A_{w}^1(\xi, t) = d_1 - c \frac{w'(\xi)}{w(\xi)} - 2r_1 + 2r_1b_{12} + 2r_1b_{13} + 4r_1\varphi_1(\xi) - 2r_1b_{12}U_2(\xi, t) - 2r_1b_{12}\varphi_2(\xi) - 2r_1b_{13}U_3(\xi, t) - 2r_1b_{13}\varphi_3(\xi) - r_1(b_{12} + b_{13})\varphi_1(\xi) - d_1 \int_{\mathbb{R}} J_1(y) \frac{w(\xi + y)}{w(\xi)}dy - r_2b_{21}(1 - \varphi_2(\xi)) - r_3b_{31}(1 - \varphi_3(\xi)), \]
\[ A_{w}^2(\xi, t) = d_2 - c \frac{w'(\xi)}{w(\xi)} - 4r_2\varphi_2(\xi) + 2r_2(1 - U_2(\xi, t)) + 2r_2b_{21}U_1(\xi, t) + 2r_2b_{21}\varphi_1(\xi) - r_2b_{21}(1 - \varphi_2(\xi)) - r_1b_{12}\varphi_1(\xi) - d_2 \int_{\mathbb{R}} J_2(y) \frac{w(\xi + y)}{w(\xi)}dy \]

and
\[ A_{w}^3(\xi, t) = d_3 - c \frac{w'(\xi)}{w(\xi)} - 4r_3\varphi_3(\xi) + 2r_3(1 - U_3(\xi, t)) + 2r_3b_{31}U_1(\xi, t) + 2r_3b_{31}\varphi_1(\xi) - r_3b_{31}(1 - \varphi_3(\xi)) - r_1b_{13}\varphi_1(\xi) - d_3 \int_{\mathbb{R}} J_3(y) \frac{w(\xi + y)}{w(\xi)}dy. \]

We shall prove \( B_{\mu, w}^i(\xi, t) > 0, \quad i = 1, 2, 3, \) which is one of the crucial steps in the proof of the stability of traveling wavefronts of (6). For this we need the following key lemma.
Lemma 4.3. Assume that (J1)-(J2) and (H1)-(H2) hold. For any \( c > \max\{c_{\min}, \hat{c}\} \), there exist some positive constants \( C_i \) such that
\[
A_w'(\xi, t) \geq C_i, \quad i = 1, 2, 3,
\]
for all \( \xi \in \mathbb{R} \), \( t \geq 0 \).

Proof. Since \( c > \max\{c_{\min}, \hat{c}\} \), we obtain \( c \eta_0 > c_i, i = 1, 2, 3 \), see (13). By (10)-(12), we get
\[
\begin{align*}
&c \eta_0 + \frac{d_1}{2} - 2r_1 - 4r_1 b_{12} - 4r_1 b_{13} - r_2 b_{21} - r_3 b_{31} - d_1 \int_\mathbb{R} J_1(y) e^{-\eta_0 y} dy > 0, \\
&c \eta_0 + \frac{d_2}{2} - 4r_2 - 2r_2 b_{21} - r_1 b_{12} - d_2 \int_\mathbb{R} J_2(y) e^{-\eta_0 y} dy > 0, \\
&c \eta_0 + \frac{d_3}{2} - 4r_3 - 2r_3 b_{31} - r_1 b_{13} - d_3 \int_\mathbb{R} J_3(y) e^{-\eta_0 y} dy > 0.
\end{align*}
\]
We first show that \( A_w'(\xi, t) \geq C_1 \) for some positive constant \( C_1 \).

Case 1. \( \xi \leq \xi_0 \). In this case, \( w(\xi) = e^{-\eta_0 (\xi - \xi_0)} \), \( \frac{w'(\xi)}{w(\xi)} = -\eta_0 \). Note that \((0, 0, 0) \leq (\varphi_1, \varphi_2, \varphi_3) \leq (1, 1, 1) \) and \((0, 0, 0) \leq (U_1, U_2, U_3) \leq (1, 1, 1) \). Then it can be verified that
\[
\begin{align*}
A_w'(\xi, t) = &d_1 - \frac{w'(\xi)}{w(\xi)} - 2r_1 + 2r_1 b_{12} + 2r_1 b_{13} + 4r_1 \varphi_1(\xi) - 2r_1 b_{12} U_2(\xi, t) \\
&- 2r_1 b_{12} \varphi_2(\xi) - 2r_1 b_{13} U_3(\xi, t) - 2r_1 b_{13} \varphi_3(\xi) - r_1 (b_{12} + b_{13}) \varphi_1(\xi) \\
&- d_1 \int_\mathbb{R} J_1(y) \frac{w(\xi + y)}{w(\xi)} dy - r_2 b_{21} (1 - \varphi_2(\xi)) - r_3 b_{31} (1 - \varphi_3(\xi)) \\
\geq &c \eta_0 + d_1 - 2r_1 - 3r_1 b_{12} - 3r_1 b_{13} - r_2 b_{21} - r_3 b_{31} \\
&- d_1 \int_{-\infty}^{\xi_0 - \xi} J_1(y) e^{-\eta_0 y} dy - d_1 \int_{\xi_0 - \xi}^{+\infty} J_1(y) e^{\eta_0 (\xi - \xi_0)} dy \\
\geq &c \eta_0 + d_1 - 2r_1 - 3r_1 b_{12} - 3r_1 b_{13} - r_2 b_{21} - r_3 b_{31} \\
&- d_1 \int_\mathbb{R} J_1(y) e^{-\eta_0 y} dy - \frac{d_1}{2} \\
\geq &c \eta_0 + d_1 - 2r_1 - 3r_1 b_{12} - 3r_1 b_{13} - r_2 b_{21} - r_3 b_{31} - d_1 \int_\mathbb{R} J_1(y) e^{-\eta_0 y} dy \\
> &r_1 b_{12} + r_1 b_{13} > 0.
\end{align*}
\]

Case 2. \( \xi > \xi_0 \). In this case, \( w(\xi) = 1, \frac{w'(\xi)}{w(\xi)} = 0 \). Then by (8), we obtain
\[
\begin{align*}
A_w'(\xi, t) = &d_1 - \frac{w'(\xi)}{w(\xi)} - 2r_1 + 2r_1 b_{12} + 2r_1 b_{13} + 4r_1 \varphi_1(\xi) - 2r_1 b_{12} U_2(\xi, t) \\
&- 2r_1 b_{12} \varphi_2(\xi) - 2r_1 b_{13} U_3(\xi, t) - 2r_1 b_{13} \varphi_3(\xi) - r_1 (b_{12} + b_{13}) \varphi_1(\xi) \\
&- d_1 \int_\mathbb{R} J_1(y) \frac{w(\xi + y)}{w(\xi)} dy - r_2 b_{21} (1 - \varphi_2(\xi)) - r_3 b_{31} (1 - \varphi_3(\xi)) \\
\geq &d_1 - 2r_1 - 3r_1 b_{12} - 3r_1 b_{13} + 4r_1 \varphi_1(\xi) - r_2 b_{21} - r_3 b_{31} + r_2 b_{21} \varphi_2(\xi) \\
&+ r_3 b_{31} \varphi_3(\xi) - d_1 \int_{-\infty}^{\xi_0 - \xi} J_1(y) e^{-\eta_0 (\xi + y - \xi_0)} dy - d_1 \int_{\xi_0 - \xi}^{+\infty} J_1(y) dy \\
\geq &- 2r_1 - 3r_1 b_{12} - 3r_1 b_{13} + 4r_1 \varphi_1(\xi_0) - r_2 b_{21} - r_3 b_{31} + r_2 b_{21} \varphi_2(\xi_0) \\
&+ r_3 b_{31} \varphi_3(\xi_0) - d_1 \int_{-\infty}^{\xi_0 - \xi} J_1(y) e^{-\eta_0 (\xi + y - \xi_0)} dy
\end{align*}
\]
Thus, we get
\[
\xi > \xi_0
\]
and
\[
C_\omega(t, \xi) - d_2 \int_\xi^{\xi_0-\xi} J_2(y) e^{-\gamma y} dy
\]
\[
= N_2(\xi_0) + r_2 b_{21} > r_2 b_{21} > 0.
\]
Let \( C_1 := r_1 b_{12} + r_1 b_{13} \). Then we obtain \( A_w^1(t, \xi) \geq C_1 \).

Next, we prove \( A_w^2(t, \xi) \geq C_2 \) for some positive constant \( C_2 \).

**Case 1.** \( \xi \leq \xi_0 \). In this case, \( w(\xi) = e^{-\eta_0(\xi-\xi_0)} \), \( \frac{w'(\xi)}{w(\xi)} = -\eta_0 \). Then we obtain
\[
A_w^2(\xi, t) = d_2 - \frac{w'(\xi)}{w(\xi)} - 4 r_2 \varphi_2(\xi) + 2 r_2 (1 - U_2(\xi, t)) + 2 r_2 b_{21} U_1(\xi, t)
\]
\[
+ 2 r_2 b_{21} \varphi_1(\xi) - r_2 b_{21} (1 - \varphi_2(\xi)) - r_1 b_{12} \varphi_1(\xi) - d_2 \int_\xi^{\xi_0-\xi} J_2(y) \frac{w(\xi + y)}{w(\xi)} dy,
\]
\[
\geq d_2 - 4 r_2 - 2 r_2 b_{21} - r_1 b_{12} - d_2 \int_\xi^{\xi_0-\xi} J_2(y) e^{-\gamma y} dy > d_2 \frac{d_2}{2}
\]
\[
= c_\eta_0 + d_2 - 4 r_2 - r_2 b_{21} - r_1 b_{12} - d_2 \int_\xi^{\xi_0-\xi} J_2(y) e^{-\gamma y} dy > r_2 b_{21} > 0.
\]

**Case 2.** \( \xi > \xi_0 \). In this case, \( w(\xi) = 1 \), \( \frac{w'(\xi)}{w(\xi)} = 0 \). Then we get
\[
A_w^2(\xi, t) = d_2 - \frac{w'(\xi)}{w(\xi)} - 4 r_2 \varphi_2(\xi) + 2 r_2 (1 - U_2(\xi, t)) + 2 r_2 b_{21} U_1(\xi, t)
\]
\[
+ 2 r_2 b_{21} \varphi_1(\xi) - r_2 b_{21} (1 - \varphi_2(\xi)) - r_1 b_{12} \varphi_1(\xi) - d_2 \int_\xi^{\xi_0-\xi} J_2(y) \frac{w(\xi + y)}{w(\xi)} dy,
\]
\[
\geq d_2 - 4 r_2 + 2 r_2 b_{21} \varphi_1(\xi_0) - r_2 b_{21} + r_2 b_{21} \varphi_2(\xi_0) - r_1 b_{12}
\]
\[
- d_2 \int_{-\infty}^{\xi_0-\xi} J_2(y) e^{-\gamma(y-\xi)} dy - d_2 \int_{\xi_0-\xi}^{+\infty} J_2(y) dy
\]
\[
> - 4 r_2 + 2 r_2 b_{21} \varphi_1(\xi_0) - r_2 b_{21} + r_2 b_{21} \varphi_2(\xi_0) - r_1 b_{12}
\]
\[
- d_2 \int_{-\infty}^{\xi_0-\xi} J_2(y) e^{-\gamma y} dy
\]
\[
= N_2(\xi_0) + r_2 b_{21} > r_2 b_{21} > 0.
\]

Thus, we get \( A_w^2(\xi, t) \geq C_2 \), where \( C_2 := r_2 b_{21} > 0 \).

Finally, by a similar argument as for \( A_w^2(\xi, t) \), we can show \( A_w^3(\xi, t) \geq C_3 := r_3 b_{31} \).

**Case 1.** \( \xi \leq \xi_0 \). In this case, \( w(\xi) = e^{-\eta_0(\xi-\xi_0)} \), \( \frac{w'(\xi)}{w(\xi)} = -\eta_0 \). Then we obtain
\[
A_w^3(\xi, t) \geq c_\eta_0 + \frac{d_3}{2} - 4 r_3 - r_3 b_{31} - r_1 b_{13} - d_3 \int_\xi^{\xi_0-\xi} J_3(y) e^{-\gamma y} dy > r_3 b_{31} > 0.
\]

**Case 2.** \( \xi > \xi_0 \). In this case, \( w(\xi) = 1 \), \( \frac{w'(\xi)}{w(\xi)} = 0 \). Then we get
\[
A_w^3(\xi, t) \geq -4 r_3 + 2 r_3 b_{31} \varphi_1(\xi_0) - r_3 b_{31} + r_3 b_{31} \varphi_2(\xi_0) - r_1 b_{13}
\]
\[
- d_3 \int_{-\infty}^{\xi_0-\xi} J_3(y) e^{-\gamma y} dy
\]
\[
= N_3(\xi_0) + r_3 b_{31} > r_3 b_{31} > 0.
\]
This completes the proof. \[ \Box \]

**Lemma 4.4.** Assume that (J1)-(J2) and (H1)-(H2) hold. For any \( c > \max \{ c_{\text{min}}, \tilde{c} \} \), there exist some positive constants \( C \) such that

\[
B^i_{\mu, w}(t, \xi) \geq C, \quad i = 1, 2, 3,
\]

for all \( \xi \in \mathbb{R} \) and \( 0 < \mu < \frac{\min_{i=1,2,3}(C_i)}{2} \).

**Lemma 4.5.** Assume that (J1)-(J2) and (H1)-(H2) hold. For any \( c > \max \{ c_{\text{min}}, \tilde{c} \} \), it holds

\[
\sum_{i=1}^{3} \left\| U_i(t) \right\|_{L^2_{\xi}}^2 + \int_0^t e^{-2\mu(t-s)} \left\| U_i(s) \right\|_{L^2_{\xi}}^2 \, ds \\
\leq Ce^{-2\mu t} \sum_{i=1}^{3} \left\| U_{i0}(0) \right\|_{L^2_{\xi}}^2
\]

for some positive constant \( C \).

Similarly, differentiating the three equations in (31) with respect to \( \xi \), we can obtain

\[
\begin{aligned}
U_{1\xi} + cU_{1\xi\xi} &= d_1(J_1 * U_{1\xi} - U_{1\xi}) \\
&\quad + U_1^2 \left\{ r_1 - r_1 b_{12} - r_1 b_{13} - 2r_1 \varphi_1 + r_1 b_{12} U_2 + r_1 b_{13} U_3 + r_1 b_{13} \varphi_3 \right\} \\
&\quad + U_1 \left\{ -2r_1 \varphi_1 + r_1 b_{12} U_{2\xi} + r_1 b_{13} \varphi_2 + r_1 b_{13} U_{3\xi} + r_1 b_{13} \varphi_3 \right\} \\
&\quad - 2r_1 U_1 U_{1\xi} + r_1 b_{12} \varphi_1 U_2 + r_1 b_{12} \varphi_1 U_{2\xi} + r_1 b_{13} \varphi_1 U_3 + r_1 b_{13} \varphi_1 U_{3\xi}, \\
U_{2\xi} + cU_{2\xi\xi} &= d_2(J_2 * U_{2\xi} - U_{2\xi}) \\
&\quad + U_{2\xi} \left\{ 2r_2 \varphi_2 - r_2 r_2 b_{21} U_1 - r_2 b_{21} \varphi_1 + 2r_2 U_2 U_{2\xi} - r_2 b_{21} \varphi_2 U_1 \right\} \\
&\quad + r_2 b_{21} (1 - \varphi_2) U_{1\xi} + U_2 \left\{ 2r_2 \varphi_2 - r_2 b_{21} U_{1\xi} - r_2 b_{21} \varphi_1 \right\}, \\
U_{3\xi} + cU_{3\xi\xi} &= d_3(J_3 * U_{3\xi} - U_{3\xi}) \\
&\quad + U_{3\xi} \left\{ 2r_3 \varphi_3 - r_3 r_3 b_{31} U_1 - r_3 b_{31} \varphi_1 + 2r_3 U_3 U_{3\xi} - r_3 b_{31} \varphi_2 U_1 \right\} \\
&\quad + r_3 b_{31} (1 - \varphi_3) U_{1\xi} + U_3 \left\{ 2r_3 \varphi_3 - r_3 b_{31} U_{1\xi} - r_3 b_{31} \varphi_1 \right\}.
\end{aligned}
\tag{45}
\]

Multiplying the three equations of (45) by \( e^{2\mu t} w(\xi) U_{1\xi}(\xi, t) \), \( e^{2\mu t} w(\xi) U_{2\xi}(\xi, t) \) and \( e^{2\mu t} w(\xi) U_{3\xi}(\xi, t) \), respectively, it holds

\[
\begin{aligned}
\left( \frac{1}{2} e^{2\mu t} w U_{1\xi}^2 \right)_t + \left( \frac{c}{2} e^{2\mu t} w U_{1\xi}^2 \right)_{\xi} - d_1 e^{2\mu t} w U_{1\xi} \int_{\mathbb{R}} J_1(y) U_{1\xi}(\xi - y, t) \, dy \\
+ \left( d_1 - \frac{c w_0}{2} \right) - \mu - r_1 - r_1 b_{12} + r_1 b_{13} + 2r_1 \varphi_1 - r_1 b_{12} U_2 - r_1 b_{12} \varphi_2 - r_1 b_{13} U_3 - r_1 b_{13} \varphi_3 \right\} e^{2\mu t} w U_{1\xi}^2 \\
= \left( -2r_1 \varphi_1 + r_1 b_{12} U_{2\xi} + r_1 b_{12} \varphi_2 + r_1 b_{13} U_{3\xi} + r_1 b_{13} \varphi_3 \right) e^{2\mu t} w U_{1\xi} \\
- 2r_1 e^{2\mu t} w U_{1\xi} U_{1\xi}^2 \\
+ (r_1 b_{12} \varphi_1 U_2 + r_1 b_{12} \varphi_1 U_{2\xi} + r_1 b_{13} \varphi_1 U_3 + r_1 b_{13} \varphi_1 U_{3\xi}) e^{2\mu t} w U_{1\xi},
\end{aligned}
\tag{46}
\]
Integrating (46)-(48) over $[0, t]$ with respect to $\xi$ and $t$, and noting the vanishing term at far fields, since $U_i \in H^2_{\text{loc}}, i = 1, 2, 3$, we can obtain

\[
\left. \left\{ \frac{c}{2} e^{2\mu t} w(\xi) U^2_{1\xi} \right\} \right|_{\xi = -\infty}^{\infty} = 0, \quad i = 1, 2, 3,
\]

\[
eq \frac{c}{2} e^{2\mu t} w(\xi) U^2_{1\xi} + 2 \int_0^t \int_R \left( d_1 - \frac{c w'(\xi)}{2 w(\xi)} - \mu - r_1 b_{12} + r_1 b_{13} + 2 r_1 \varphi_1(\xi) - r_1 b_{12} U_{2\xi}(\xi, s) \\
- r_1 b_{12} \varphi_2(\xi) - r_1 b_{13} U_{3\xi}(\xi, s) - r_1 b_{13} \varphi_3(\xi) \right) e^{2\mu s} w(\xi) U^2_{1\xi}(\xi, s) d\xi \]ds

\[
\leq ||U_{1\xi}(0)||_{L^2_{\infty}} + 2 r_1 b_{12} \int_0^t \int_R \varphi_1(\xi) e^{2\mu s} w(\xi) U_{1\xi}(\xi, s) U_{2\xi}(\xi, s) d\xi ds
\]

\[
+ 2 r_1 b_{13} \int_0^t \int_R \varphi_1(\xi) e^{2\mu s} w(\xi) U_{1\xi}(\xi, s) U_{3\xi}(\xi, s) d\xi ds
\]

\[
+ 2 \int_0^t \int_R \left( -2 r_1 \varphi'_1(\xi) + r_1 b_{12} U_{2\xi}(\xi, s) + r_1 b_{12} \varphi'_2(\xi) + r_1 b_{13} U_{3\xi}(\xi, s) \right) e^{2\mu s} w(\xi) U^2_{1\xi}(\xi, s) d\xi ds
\]

\[
+ 2 \int_0^t \int_R \varphi'_1(\xi) e^{2\mu s} w(\xi) U^2_{2\xi}(\xi, s) U_{1\xi}(\xi, s) d\xi ds
\]

\[
+ 2 \int_0^t \int_R \varphi'_1(\xi) e^{2\mu s} w(\xi) U^2_{3\xi}(\xi, s) U_{1\xi}(\xi, s) d\xi ds,
\]

\[
eq 2 \int_0^t \int_R \left( d_2 \frac{c w'(\xi)}{2 w(\xi)} - \mu - 2 r_2 \varphi_2(\xi) + r_2 + r_2 b_{21} U_{1\xi}(\xi, s) + r_2 b_{21} \varphi_1(\xi) \right) e^{2\mu s} w(\xi) U^2_{2\xi}(\xi, s) d\xi ds
\]

\[
+ 2 \int_0^t \int_R \left( d_2 \frac{c w'(\xi)}{2 w(\xi)} - \mu - 2 r_2 \varphi_2(\xi) + r_2 + r_2 b_{21} U_{1\xi}(\xi, s) + r_2 b_{21} \varphi_1(\xi) \right) e^{2\mu s} w(\xi) U^2_{2\xi}(\xi, s) d\xi ds.
\]
× e^{2μs} w(ξ)U_{2ξ}(ξ,s)dξds
≤||U_{2ξ,0}(0)||_{L^2_x}^2 + 4r_2 \int_0^t \int_\mathbb{R} e^{2μs} w(ξ)U_2(ξ,s)U_{2ξ}(ξ,s)dξds
+ 2r_2b_2 \int_0^t \int_\mathbb{R} (1 - ϕ_2(ξ))e^{2μs} w(ξ)U_{1ξ}(ξ,s)U_{2ξ}(ξ,s)dξds
- 2r_2b_2 \int_0^t \int_\mathbb{R} ϕ_2'(ξ)e^{2μs} w(ξ)U_1(ξ,s)U_{2ξ}(ξ,s)dξds
+ 2 \int_0^t \int_\mathbb{R} (2r_2ϕ_2(ξ) - r_2b_21 U_{1ξ}(ξ,s) - r_2b_21 ϕ_1'(ξ))w(ξ)e^{2μs} U_2(ξ,s)U_{2ξ}(ξ,s)dξds

and
\begin{align*}
e^{2μt}||U_{3ξ}(t)||_{L^2_x}^2 - 2d_3 \int_0^t \int_\mathbb{R} e^{2μs} w(ξ)U_{3ξ}(ξ,s) \int_\mathbb{R} J_1(y)U_{3ξ}(ξ - y, s)dxdξds
+ 2 \int_0^t \int_\mathbb{R} \left(d_3 - c \frac{w'(ξ)}{w(ξ)} - μ - 2r_3ϕ_3(ξ) + r_3 + r_3b_31 U_{1}(ξ,s) + r_3b_31 ϕ_1(ξ)\right)
× e^{2μs} w(ξ)U_{3ξ}^2(ξ,s)dξds
\leq||U_{3ξ,0}(0)||_{L^2_x}^2 + 4r_3 \int_0^t \int_\mathbb{R} e^{2μs} w(ξ)U_3(ξ,s)U_{3ξ}^2(ξ,s)dξds
+ 2r_3b_31 \int_0^t \int_\mathbb{R} (1 - ϕ_3(ξ))e^{2μs} w(ξ)U_{1ξ}(ξ,s)U_{3ξ}(ξ,s)dξds
- 2r_3b_31 \int_0^t \int_\mathbb{R} ϕ_2'(ξ)e^{2μs} w(ξ)U_1(ξ,s)U_{3ξ}(ξ,s)dξds
+ 2 \int_0^t \int_\mathbb{R} (2r_3ϕ_3(ξ) - r_3b_31 U_{1ξ}(ξ,s) - r_3b_31 ϕ_1'(ξ))w(ξ)e^{2μs} U_3(ξ,s)U_{3ξ}(ξ,s)dξds.
\end{align*}

By the Cauchy-Schwarz inequality, we further get
\begin{align*}
e^{2μt}||U_{1ξ}(t)||_{L^2_x}^2 + \int_0^t \int_\mathbb{R} \left[d_1 - c \frac{w'(ξ)}{w(ξ)} - 2μ - 2r_1 + 2r_1b_{12} + 2r_1b_{13} + 4r_1φ_1(ξ)\right]
- 2r_1b_{12} U_{2ξ}(ξ,s) - 2r_1b_{12}ϕ_2(ξ) - 2r_1b_{13} U_3(ξ,s) - 2r_1b_{13}φ_3(ξ)
- r_1(b_{12} + b_{13})φ_1(ξ) - r_1(b_{12} + b_{13})U_1(ξ,s)
- d_1 \int_\mathbb{R} J_1(y)\frac{w(ξ + y)}{w(ξ)} dy \int_\mathbb{R} e^{2μs} w(ξ)U_{1ξ}^2(ξ,s)dξds
\leq||U_{1ξ,0}(0)||_{L^2_x}^2 + r_1b_{12} \int_0^t \int_\mathbb{R} (φ_1(ξ) + U_1(ξ,s))e^{2μs} w(ξ)U_{2ξ}^2(ξ,s)dξds
+ r_1b_{13} \int_0^t \int_\mathbb{R} (φ_1(ξ) + U_1(ξ,s))e^{2μs} w(ξ)U_{3ξ}^2(ξ,s)dξds
+ 2 \int_0^t \int_\mathbb{R} (-2r_1φ_1(ξ) + r_1b_{12}ϕ_2(ξ) + r_1b_{13}φ_3(ξ))e^{2μs} w(ξ)U_{1ξ}(ξ,s)U_{ξ}(ξ,s)dξds
+ 2r_1b_{12} \int_0^t \int_\mathbb{R} φ_1'(ξ)e^{2μs} w(ξ)U_2(ξ,s)U_{1ξ}(ξ,s)dξds
+ 2r_1b_{13} \int_0^t \int_\mathbb{R} φ_1'(ξ)e^{2μs} w(ξ)U_3(ξ,s)U_{1ξ}(ξ,s)dξds,
\end{align*}
(49)
\[
e^{2\mu_t} \|U_{2\xi}(t)\|_{L_w^2}^2 + \int_0^t \int_{\mathbb{R}} \left[ d_2 - c \frac{w'(\xi)}{w(\xi)} - 2\mu - 4r_2\varphi_2(\xi) + 2r_2(1 - 2U_2(\xi, s)) + 2r_2b_{21}U_1(\xi, s) + 2r_2b_{21}\varphi_1(\xi) - r_2b_{21}(1 - \varphi_2(\xi)) - r_2b_{21}U_2(\xi, s) \right. \\
- d_2 \int_{\mathbb{R}} J_2(y) \frac{w(\xi + y)}{w(\xi)} dy \Bigg] e^{2\mu_s} w(\xi) U_{2\xi}^2(\xi, s) d\xi ds \\
\leq \|U_{2\xi,0}(0)\|_{L_w^2}^2 + r_2b_{21} \int_0^t \int_{\mathbb{R}} (1 - \varphi_2(\xi) + U_2(\xi, s)) e^{2\mu_s} w(\xi) U_{2\xi}^2(\xi, s) d\xi ds \\
- 2r_2b_{21} \int_0^t \int_{\mathbb{R}} \varphi_2'(\xi) e^{2\mu_s} w(\xi) U_1(\xi, s) U_{2\xi}(\xi, s) d\xi ds \\
+ 2 \int_0^t \int_{\mathbb{R}} (2r_2\varphi_2'(\xi) - r_2b_{21}\varphi_1'(\xi)) w(\xi) e^{2\mu_s} U_2(\xi, s) U_{2\xi}(\xi, s) d\xi ds \\
\tag{50}
\]

and
\[
e^{2\mu_t} \|U_{3\xi}(t)\|_{L_w^2}^2 + \int_0^t \int_{\mathbb{R}} \left[ d_3 - c \frac{w'(\xi)}{w(\xi)} - 2\mu - 4r_3\varphi_3(\xi) + 2r_3(1 - 2U_3(\xi, s)) + 2r_3b_{31}U_1(\xi, s) + 2r_3b_{31}\varphi_1(\xi) - r_3b_{31}(1 - \varphi_3(\xi)) - r_3b_{31}U_3(\xi, s) \right. \\
- d_3 \int_{\mathbb{R}} J_3(y) \frac{w(\xi + y)}{w(\xi)} dy \Bigg] e^{2\mu_s} w(\xi) U_{3\xi}^2(\xi, s) d\xi ds \\
\leq \|U_{3\xi,0}(0)\|_{L_w^2}^2 + r_3b_{31} \int_0^t \int_{\mathbb{R}} (1 - \varphi_3(\xi) + U_3(\xi, s)) e^{2\mu_s} w(\xi) U_{3\xi}^2(\xi, s) d\xi ds \\
- 2r_3b_{31} \int_0^t \int_{\mathbb{R}} \varphi_3'(\xi) e^{2\mu_s} w(\xi) U_1(\xi, s) U_{3\xi}(\xi, s) d\xi ds \\
+ 2 \int_0^t \int_{\mathbb{R}} (2r_3\varphi_3'(\xi) - r_3b_{31}\varphi_1'(\xi)) w(\xi) e^{2\mu_s} U_3(\xi, s) U_{3\xi}(\xi, s) d\xi ds. \\
\tag{51}
\]

Adding the (49)-(51) together, we get
\[
e^{2\mu_t} \sum_{i=1}^3 \|U_{i\xi}(t)\|_{L_w^2}^2 + \sum_{i=1}^3 \int_0^t \int_{\mathbb{R}} e^{2\mu_s} \tilde{B}_{\mu, w}^i(\xi, s) U_{i\xi}^2(\xi, s) w(\xi) d\xi ds \\
\leq \sum_{i=1}^3 \|U_{i\xi,0}(0)\|_{L_w^2}^2 + 2 \int_0^t \int_{\mathbb{R}} Q(s, \xi) w(\xi) e^{2\mu_s} d\xi ds, \\
\tag{52}
\]

where
\[
Q(t, \xi) = (-2r_1\varphi_1' + r_1b_{12}\varphi_2' + r_1b_{13}\varphi_3')U_1(\xi, t)U_{1\xi}(\xi, t) + r_1b_{12}\varphi_1'(U_2(\xi, t)U_{1\xi}(\xi, t)) + r_1b_{13}\varphi_1'(U_3(\xi, t)U_{2\xi}(\xi, t)) + (2r_2\varphi_2' - r_2b_{21}\varphi_1')U_2(\xi, t)U_{2\xi}(\xi, t) + (2r_3\varphi_3' - r_3b_{31}\varphi_1')U_3(\xi, t)U_{3\xi}(\xi, t) - r_2b_{21}\varphi_2'(U_1(\xi, t)U_{2\xi}(\xi, t)) - r_3b_{31}\varphi_3'(U_1(\xi, t)U_{3\xi}(\xi, t)) \\
\]

and
\[
\tilde{B}_{\mu, w}^i(t, \xi) = \tilde{A}_{w}^i(t, \xi) - 2\mu, \quad i = 1, 2, 3,
\]

with
\[
\tilde{A}_{w}^i(\xi, t) = d_1 - c \frac{w'(\xi)}{w(\xi)} - 2r_1 + 2r_1b_{12} + 2r_1b_{13} + 4r_1\varphi_1(\xi) - 2r_1b_{12}U_2(\xi, t) - 2r_1b_{12}\varphi_2(\xi) - 2r_1b_{13}U_3(\xi, t) - 2r_1b_{13}\varphi_3(\xi) - r_1(b_{12} + b_{13})\varphi_1(\xi)
\]
According to Lemma 4.5, it is easy to get the following inequality of traveling wave. Let

\[ \begin{align*}
- r_1(b_{12} + b_{13}) U_1(\xi, t) - r_2 b_{21}(1 - \varphi_2(\xi) + U_2(\xi, t)) \\
- r_3 b_{31}(1 - \varphi_3(\xi) + U_3(\xi, t)) - d_1 \int_{\mathbb{R}} J_1(y) \frac{w(\xi + y)}{w(\xi)} dy,
\end{align*} \]

\[ \begin{align*}
\tilde{A}_w^i(\xi, t) &= d_2 - c(\frac{w'(\xi)}{w(\xi)}) - 4 r_2 \varphi_2(\xi) + 2 r_2 b_{21} U_1(\xi, t) + 2 r_2(1 - 2 U_2(\xi, t)) \\
+ 2 r_2 b_{21} \varphi_1(\xi) - r_2 b_{21}(1 - \varphi_2(\xi)) - r_1 b_{12}(\varphi_1(\xi) + U_1(\xi, t)) \\
- r_2 b_{21} U_2(\xi, t) - d_2 \int_{\mathbb{R}} J_2(y) \frac{w(\xi + y)}{w(\xi)} dy
\end{align*} \]

and

\[ \begin{align*}
\tilde{A}_w^2(\xi, t) &= d_3 - c(\frac{w'(\xi)}{w(\xi)}) - 4 r_3 \varphi_3(\xi) + r_3 b_{31} U_1(\xi, t) + 2 r_3(1 - 2 U_3(\xi, t)) \\
+ 2 r_3 b_{31} \varphi_1(\xi) - r_3 b_{31}(1 - \varphi_3(\xi)) - r_1 b_{13}(\varphi_1(\xi) + U_1(\xi, t)) \\
- r_3 b_{31} U_3(\xi, t) - d_3 \int_{\mathbb{R}} J_3(y) \frac{w(\xi + y)}{w(\xi)} dy.
\end{align*} \]

Lemma 4.6. Assume that (J1)-(J2) and (H1)-(H2) hold. For any \( c > \max\{c_{\min}, \tilde{c}\} \), there exist some positive constants \( C_i, i = 1, 2, 3 \), such that

\[ \tilde{A}_w^i(t, \xi) \geq \tilde{C}_i, \quad i = 1, 2, 3, \]

for all \( \xi \in \mathbb{R}, t \geq 0 \).

The proof is similar to that in Lemma 4.4, so we omit the proof here.

Lemma 4.7. Assume that (J1)-(J2) and (H1)-(H2) hold. For any \( c > \max\{c_{\min}, \tilde{c}\} \), there exist some positive constants \( C \) such that

\[ \tilde{B}_{i,w}(t, \xi) \geq C \]

for all \( \xi \in \mathbb{R} \) and \( 0 < \mu < \frac{\min_{i=1,2,3} C_i}{2} \).

Now we estimate the last term on the right-hand side of (52). By the properties of traveling wave \((\varphi_1(\xi), \varphi_2(\xi), \varphi_3(\xi))\), we can obtain that \((\varphi_1(\xi), \varphi_2(\xi), \varphi_3(\xi))\) is bounded for all \( \xi \in \mathbb{R} \). Thus, there exists a positive constant \( C_5 \) such that

\[ \begin{align*}
| - r_1 \varphi_1' + r_1 b_{12} \varphi_2 + r_1 b_{13} \varphi_3' | &\leq C_4, \\
|r_1 b_{12} \varphi_1'_{12}| &\leq C_4, |r_1 b_{13} \varphi_1_{13}'| \leq C_4, \\
|2 r_2 \varphi_2' - r_2 b_{21} \varphi_1'| &\leq C_4, |2 r_3 \varphi_3' - r_3 b_{31} \varphi_1'| \leq C_4, \\
|r_2 b_{21} \varphi_2'_{12}| &\leq C_4, |r_3 b_{31} \varphi_3'_{13}| \leq C_4.
\end{align*} \]

According to Lemma 4.5, it is easy to get the following inequality

\[ \sum_{i=1}^{3} \int_{0}^{t} e^{2\mu s} ||U_i(s)||_{L^2_w}^2 ds \leq C \sum_{i=1}^{3} ||U_{i, 0}(0)||_{L^2_w}^2. \]

By using the Young-inequality \( 2xy \leq \eta x^2 + \frac{1}{\eta} y^2 \) and (44), we have

\[ \begin{align*}
2 \int_{0}^{t} \int_{\mathbb{R}} Q(s, \xi) w(\xi) e^{2\mu s} d\xi ds \\
\leq C_4 \int_{0}^{t} \int_{\mathbb{R}} e^{2\mu s} w(\xi) \left[ \frac{3}{\eta} \sum_{i=1}^{3} U_i^2(\xi, s) + \eta \sum_{i=1}^{3} U_i^2(\xi, s) \right] d\xi ds
\end{align*} \]
It holds that

\[ \eta > C \]

Choosing \( C > \eta \) such that \( C \rightarrow \eta > 0 \), we obtain the following results.

\[ \|U_{i0}(0)\|_{L^2}^2 + \int_0^t e^{2\mu s} \|U_{i\xi}(s)\|_{L^2}^2 \, ds \leq \sum_{i=1}^3 \|U_{i}(t)\|_{L^2}^2 \]

for some positive constant \( C \).

Lemma 4.8. Assume that (J1)-(J2) and (H1)-(H2) hold. For any \( c > \max\{c_{\min}, \tilde{c}\} \), it holds

\[ \|U_i(t)\|_{H^1_w}^2 \leq Ce^{-\mu t} \left( \sum_{j=1}^3 \|U_{j0}(0)\|_{H^1_w}^2 \right)^\frac{1}{2}, \quad i = 1, 2, 3, \]

for some positive constant \( C \), \( 0 < \mu < \min_{i=1,2,3} \frac{C_i}{\min(C_i, \tilde{C}_i)} \) and all \( t > 0 \).

According to the standard Sobolev embedding inequality \( H^1_w(\mathbb{R}) \hookrightarrow H^1(\mathbb{R}) \hookrightarrow C(\mathbb{R}) \) since \( w(\xi) \geq 1 \), one has

\[ \sup_{\xi \in \mathbb{R}} |U_i(\xi, t)| \leq C \|U_i(\xi, t)\|_{H^1} \leq C \|U_i(\xi, t)\|_{H^1_w}, \quad i = 1, 2, 3. \]

By Lemma 4.9, we obtain the following results.

Lemma 4.10. It holds that

\[ \sup_{x \in \mathbb{R}} |u_i^+(t, x) - \varphi_i(x + ct)| = \sup_{\xi \in \mathbb{R}} |U_i(\xi, t)| \leq Ce^{-\mu t}, \quad i = 1, 2, 3, \]

for some positive constant \( C \) and \( t > 0 \).

Step 2. The convergence of \( u_i^-(t, x) \) to \( \varphi_i(x + ct) \), \( i = 1, 2, 3 \).

Let \( \xi := x + ct \) and

\[ U_i(\xi, t) = \varphi_i(\xi) - u_i^-(x, t), \quad U_{i0}(\xi, 0) = \varphi_i(x) - u_{i0}(x, 0), \quad i = 1, 2, 3. \]

As shown in the process of Step 1, we can similarly prove the convergence of \( u_i^-(t, x) \) to \( \varphi_i(x + ct) \), \( i = 1, 2, 3 \), i.e.,
Lemma 4.11. It holds that
\[
\sup_{x \in \mathbb{R}} |u_i(x, t) - \varphi_i(x + ct)| = \sup_{\xi \in \mathbb{R}} |U_i(\xi, t)| \leq Ce^{-\mu t}, \quad i = 1, 2, 3,
\]
for some positive constant $C$ and $t > 0$.

Step 3. The convergence of $u_i(x, t)$ to $\varphi_i(x + ct)$, $i = 1, 2, 3$.

Lemma 4.12. It holds that
\[
\sup_{x \in \mathbb{R}} |u_i(x, t) - \varphi_i(x + ct)| \leq Ce^{-\mu t}, \quad i = 1, 2, 3,
\]
for some positive constant $C$ and $t > 0$.

Proof. From (30), we can see that
\[
(u_1^-(x, t), u_2^-(x, t), u_3^-(x, t)) \leq (u_1(x, t), u_2(x, t), u_3(x, t)) \leq (u_1^+(x, t), u_2^+(x, t), u_3^+(x, t)).
\]
Then we get
\[
|u_i(x, t) - \varphi_i(x + ct)| \\
\leq \max\{ |u_i^-(x, t) - \varphi_i(x + ct)|, |u_i^+(x, t) - \varphi_i(x + ct)| \}, \quad i = 1, 2, 3.
\]
In view of the convergence results in Lemmas 4.10 and 4.11, we have
\[
\sup_{x \in \mathbb{R}} |u_i(t, x) - \varphi_i(x + ct)| \leq Ce^{-\mu t}, \quad t > 0, \quad i = 1, 2, 3.
\]
The proof is complete.

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