Data-driven Variable Speed Limit Design for Highways via Distributionally Robust Optimization

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Abstract—This paper introduces an optimization problem (P) and a solution strategy to design variable-speed-limit controls for a highway that is subject to traffic congestion and uncertain vehicle arrival and departure. By employing a finite data-set of samples of the uncertain variables, we aim to find a data-driven solution that has a guaranteed out-of-sample performance. In principle, such formulation leads to an intractable problem (P) as the distribution of the uncertainty variable is unknown. By adopting a distributionally robust optimization approach, this work presents a tractable reformulation of (P) and an efficient algorithm that provides a suboptimal solution that retains the out-of-sample performance guarantee. A simulation illustrates the effectiveness of this method.

I. INTRODUCTION

Transportation networks constitute one of the most critical infrastructure sectors today, with a major impact on the economics, security, public health, and safety of a community. In these networks, the accessibility of routes between increasingly-larger geographical locations is highly dependent on the network connectivity as well as on the traffic congestion on the available roads. New advances on smart infrastructure, computation, and communication make possible the collection of real-time traffic data as well as the implementation of novel control policies that can alleviate traffic problems. Motivated by this, we consider a problem of traffic congestion reduction via variable speed-limits and the assimilation of traffic data.

Literature Review: Several congestion control schemes have been proposed in the literature with the goal of mitigating congestion, such as ramp metering control [1], [2], lane assignment [3], [4], optimal control [5], [6], logic-based control [7] and many other innovative control strategies [8]–[10]. More recently, variable speed limits have been proposed as an effective congestion control mechanism in transportation [11]–[14]. Such works exploit the Cell Transmission Model to capture the deterministic distribution of traffic densities along a road [15], [16]. In practice, these approaches may be limited, due to the uncertainty on traffic density subject to unknown actions by various drivers as well as vehicle arrival and departure. However, the wide availability of data in real time [17], [18] can help reduce this uncertainty and opens the way to the application of novel data-driven optimization methods for control. In this way, we consider here a distributionally robust optimization (DRO) framework [19]–[22] for data assimilation. DRO uses finite data to make decisions with desirable out-of-sample performance guarantees, and as such, it paves the way for real-time decisions to dynamical transportation systems. Here, we aim to answer two questions; that is, 1) what role variable speed limits play in congestion, and 2) find an efficient approach for the computation of data-driven variable speed limit controls with performance guarantees.

Statement of Contributions: In this work, we consider a highway divided into equal-size segments. Vehicle arrival and departure into each segment represent inflow and outflow disturbances to traffic, and we model these disturbances as unknown stochastic processes. Further, we assume that finite realizations of such random variables can be acquired in real time and that a transportation network operator can prescribe variable speed limits to control congestion on each of the segments. In this setting, we propose a novel data-driven variable-speed-limit control to limit congestion and maximize the throughput of the road. To do this, we first leverage the effect of variable speed limits to limit traffic congestion. This is achieved by exploiting approximations of the well-known Fundamental Diagram for various speed limits. To ensure the performance of a data-driven solution with a given confidence, we generalize the DRO framework in the literature to handle the dynamical system constraints of our control problem. Specifically, we define ambiguity sets, or the sets of system trajectory distributions, to contain the distribution of the true system trajectory with high probability. The proposed DRO approach then allows us to obtain a set of speed limits with an out-of-sample performance bound defined as the optimal objective value of a worst-case optimization problem over the ambiguity set. As the resulting problem is infinite-dimensional and intractable, we further obtain an equivalent reformulation that reduces it into a finite-dimensional problem. Still the resulting problem is nonconvex, and our third contribution provides an integer-solution search algorithm to find feasible data-driven variable speed limits. This algorithm is based on the decomposition of the nonconvex problem into mix-integer linear programs and, as such, has certain convergence properties guarantees. We establish that this solution procedure guarantees a feasible solution with the out-of-sample performance guarantee with high probability. We finally illustrate the performance of the proposed algorithm in simulation.

II. PRELIMINARIES

Let \( \mathbb{R}^{m \times n} \) denote the \( m \times n \)-dimensional real vector space, and let the shorthand notations \( 1_m \) and \( 0_m \) denote the column vector \( (1, \cdots, 1)^\top \in \mathbb{R}^m \) and \( (0, \cdots, 0)^\top \in \mathbb{R}^m \), respectively. Any letter \( x \) may have appended the following indices and arguments: it may have the subscript \( x_e \), with \( e \in \mathbb{N} \), the argument \( x_{e}(t) \), \( t \in \mathbb{R} \), and further a superscript \( l \in \mathbb{N} \) as in \( x_{e}^{(l)}(t) \). We assume that the dimension of the letter with the most indexes belongs to \( \mathbb{R} \), while their removal
increases its dimension. In this way, given \( x^{(i)}(t) \in \mathbb{R} \), for several \( e, t, \) and \( l \), we denote \( x^{(i)}(t) := (x^{(i)}_1(t), x^{(i)}_2(t), \ldots) \), then further \( x^{(i)} := (x^{(i)}(1), x^{(i)}(2), \ldots) \), and finally \( x := (x^{(1)}, x^{(2)}, \ldots) \). The inner and component-wise products of any two vectors \( x, y \in \mathbb{R}^m \) are denoted by \( \langle x, y \rangle \) and \( x \odot y \), respectively. In addition, the Kronecker product of any two vectors \( x, y \) with arbitrary dimension is denoted by \( x \otimes y \).

The 1-norm of the vector \( x \in \mathbb{R}^m \) is denoted by \( \|x\|_1 := \sum_{i=1}^{m} |x_i| \) and its dual norm is denoted by \( \|x\|_* := \sup_{\|z\| \leq 1} \langle x, z \rangle \). We have that \( \|x\|_* = \|x\|_1 \).

Let \( X \subseteq \mathbb{R}^n \) be a subspace and let \( X^* \) denote the dual space of \( X \). For each \( x, y \in X \), the dual \( x^* \in X^* \) is defined as \( x^*(y) = \langle x, y \rangle \), for any \( y \in X \). Let \( f : X \to \mathbb{R} \) be a function on \( X \) and we define its domain of interest by \( \text{dom } f := \{ x \in X \mid -\infty < f(x) < +\infty \} \). We say \( f \) is convex on \( X \), if \( f(x + (1 - \lambda) y) \leq \lambda f(x) + (1 - \lambda) f(y) \) for all \( x, y \in X \) and \( 0 \leq \lambda \leq 1 \). We call \( f \) lower semi-continuous on \( X \), if \( f(x) \leq \liminf_{y \to x} f(y) \) for all \( x \in X \). A function \( f \) is lower semi-continuous on \( X \) if and only if its sublevel sets \( \{ x \in X \mid f(x) \leq \gamma \} \) are closed for each \( \gamma \in \mathbb{R} \). We denote the convex conjugate of \( f \) by \( f^* : \mathbb{R} \cup \{ +\infty \} \to \mathbb{R} \cup \{ -\infty \} \), which is defined as \( f^*(x) := \sup_{y \in X} \{ f(y) - y x \} \). Let \( f \) and \( g \) denote two functions on \( X \). The infimal convolution of \( f \) and \( g \) is defined as \( (f \square g)(x) := \inf_{y \in X} f(x - y) + g(y) \).

Let \( A \) be a set in \( X \). We use the notion \( \chi_A : X \to \mathbb{R} \cup \{ +\infty \} \) to denote the characteristic function of \( A \), i.e., \( \chi_A(x) = 1 \) if \( x \in A \) and \( +\infty \) otherwise. The support function of \( A \) is defined as \( \sigma_A : X \to \mathbb{R} \), \( \sigma_A(x) := \sup_{y \in X} \{ y \chi_A(x) \} \). It can be verified that \( \sigma_A(x) = |\chi_A|^*(x) \) for all \( x \in X \), and \( \chi_A \) is lower semi-continuous if and only if \( A \) is closed. Let \( f \) and \( g \) denote two convex and lower semi-continuous functions on \( X \) with \( \text{dom } f \cap \text{dom } g \neq \emptyset \). The conjugate of \( f + g \) has the following property: \( (f + g)^* = (f^* \square g^*) \). For more information and details of these properties we refer readers to [23, Theorem 11.23(a), Dual operations] and references therein.

### III. Problem Statement

In this section, we first introduce the traffic model that we consider in this paper, and then we propose a stochastic optimal control framework to solve the proposed variable-speed limit design problem.

#### A. Transportation System Model

Consider a one-way road of length \( L \), and divide the road into \( n \) segments of equal size \( L/n \). The road can be modeled as a chain directed graph \( G = (V, E) \) with set of nodes \( V := \{0, 1, \ldots, n\} \) and set of edges \( E := \{(0, 1), \ldots, (n - 1, n)\} \). Each edge \( e \in E \) corresponds to a road segment and each node \( v \in V \) corresponds to a link between two road segments. We call node 0 the source node, and call node \( n \) the sink node. Further, a node \( v \in V \) is called an arrival node if there exists a non-zero inflow at node \( v \). Similarly, a node \( v \) is called a departure node if there exists a non-zero outflow at node \( v \). Let \( V_A \) and \( V_D \) denote the set of arrival nodes and departure nodes, respectively. By convention, we have node 0 \( \in V_A \) and node \( n \) \( \in V_D \).

Let us consider a time horizon of length \( Q \), and assume that time is divided into time slots of size \( \delta \). Let \( T := \{0, 1, \ldots, T\} \) denote the set of time slots, where \( T = Q/\delta \). We denote by \( \pi_e, \pi_n \) and \( \bar{f}_e \) the maximal free flow speed, the jam density, and the capacity of edge \( e \), respectively. Let \( u_e \in [0, \pi_e] \) denote the speed limit of the vehicles on edge \( e \) and we consider \( u_e \) to be constant over the set of time slots \( T \). At each time slot \( t \), we denote by \( \rho_e(t) \in [0, \bar{\pi}_e] \) the density of the vehicles on edge \( e \), and the allowable flow rate on each edge \( e \) by \( f_e(t) \in [0, \bar{f}_e] \), depending on the density \( \rho_e(t) \) and speed limit \( u_e \) of the segment. Given a (constant) speed limit \( u_e \), the relationship between \( f_e(t) \) and \( \rho_e(t) \) can be characterized by the fundamental diagram of edge \( e \) \( \in \mathcal{E} \), (e.g., see [11, 24]). This diagram determines the nonlinear relationship \( f_e(t) := f_e(\rho_e(t), u_e) \) between the allowable flow rate \( f_e(t) \) and the density \( \rho_e(t) \) for a given \( u_e \), as shown in Fig. 1. In this study, we define the fundamental diagrams for various speed limits as the curves shown in Fig. 1. Note that for each edge \( e \in \mathcal{E} \), the critical density \( \rho^*_e(u_e) \) is defined as the density at which the maximum allowable flow is achievable for a given speed limit \( u_e \). More precisely, given \( u_e \), the function \( f_e(\rho, u_e) \) is increasing if \( \rho \leq \rho^*_e(u_e) \) and decreasing if \( \rho > \rho^*_e(u_e) \).

Let \( s_e(t) \in [0, \pi_e] \) denote the average speed of the vehicles on edge \( e \in \mathcal{E} \) at time \( t \in \mathcal{T} \), and let us assume that the majority of drivers have no incentive to exceed the speed limit \( u_e \), i.e., \( s_e(t) \leq u_e \). Further, let \( q_e(t) \) denote the demand on edge \( e \in \mathcal{E} \) at time slot \( t \in \mathcal{T} \). The flow \( q_e(t) \) is equal to \( \rho_e(t)s_e(t) \), for all \( e \in \mathcal{E} \) and \( t \in \mathcal{T} \). For each speed limit \( u_e \), edge \( e \in \mathcal{E} \) will be congested at a time \( t \) if the flow rate \( q_e(t) \) is greater than the allowable flow rate \( f_e(\rho_e(t), u_e) \) and the density \( \rho^*_e(u_e) \) satisfies \( q_e(t) > f_e(\rho^*_e(u_e), u_e) \) and, thus, \( \rho_e(t) > \rho^*_e(u_e) \). To prevent congestion at each time slot \( t \in \mathcal{T} \), we must have:

\[
\rho_e(t) \leq \rho^*_e(u_e), \quad \forall e \in \mathcal{E}. 
\]  

The above constraints are sufficient to guarantee that the road
is not congested, regardless of the flow rates.

The computation of critical density $\rho_c(u_e)$ and allowable flow $f_e$ of each segment $e \in E$ for different values of speed limits $u_e$ is highly dependent on the fundamental diagram of the segment. In this study, we approximate the fundamental diagram of each segment with a finite set of piecewise linear functions as shown in Fig. 1. Each of these functions corresponds to a speed limit. Let $\Gamma := \{\gamma^{(1)}, \ldots, \gamma^{(m)}\}$ denote the set of fixed non-zero speed limits. For each edge $e \in E$ and $u_e \in \Gamma$, we approximate the fundamental diagram of segment $e$ by

$$f_e(\rho_e(t), u_e) = \begin{cases} u_e \rho_e(t), & \text{if } \rho_e(t) \leq \rho^*_e(u_e), \\ \tau_e \pi_e (\pi_e - \rho_e(t)), & \text{o.w.}, \end{cases}$$

with

$$\rho^*_e(u_e) := (\tau_e \pi_e) / (\tau_e \pi_e + u_e),$$

where the time $\tau_e := \tau_e \pi_e / (\pi_e \pi_e - \tau_e)$.

For each time $t \in T$, let $\omega^\mu(t)$ denote the random inflow of the starting node $\mu$ of the edge $e \in E$. Let $\omega^\nu(t)$ denote the random outflow of the ending node $\nu$ of the same edge, and let $\omega_e(t)$ denote the difference between the inflow and outflow of edge $e$. In this setting, each random variable $\omega^\mu(t)$ has nontrivial support $Z_{\omega^\mu(t)} \subseteq R_{\geq 0}$ for $\mu \in V_A$. Similarly, each random variable $\omega^\nu(t)$ has nontrivial support $Z_{\omega^\nu(t)} \subseteq R_{\geq 0}$ for $\nu \in V_D$. Without loss of generality, we assume that the random inflows and outflows are independent from the speed limits $u$. Let $\rho(0) = (\rho_1(0), \ldots, \rho_n(0))$ denote the random initial density of the road $G$ with nontrivial support $Z_{\rho(0)} \subseteq R_{\geq 0}$. The dynamics of the density on each edge $e \in E$ can be represented by [15]

$$\rho_e(t) = \rho_e(t) + h(f_e(t) - f_e(t) + \omega_e(t)), \quad \forall \, t,$$

where $h := n \delta / L$ is determined by $n$, $\delta$ and $L$ such that $h \leq 1 / \max_{e \in E}(\pi_e)$, the subscript $s \in E \cup \emptyset$ denotes the preceding edge of edge $e$, and $f_e(t) := u_e \rho_e(t)$ for each $e \in E$, $t \in T$.

Random events, such as accidents on different segments of the road and temporary lane closure, can affect the capacity and jam density of each segment. In this step, we use $f^U_e$ and $\rho^U_e$ to denote the temporary conditions on the capacity and jam density. We assume that the system is in a specific condition, and hence the values of the parameters $f^U_e \leq \tau_e$ and $\rho^U_e \leq \pi_e$ are fixed and known to the operator. Since the values of $f^U_e$ and $\rho^U_e$ are known, for all $e \in E$, we can compute the maximum and minimum speed limits of each segment under the certain event. For given $f^U_e$ and $\rho^U_e$ of each segment $e$, we have $\rho^*_e(u_e) \leq f^U_e$ and $\rho^*_e(u_e) \leq \rho^U_e - \pi$ with a small but positive threshold $\pi$ to ensure non-zero flows on edge $e$. For each edge $e \in E$, we need to ensure that the variable speed limit of the segment satisfies the following constraint:

$$u_e \leq u_e \leq u^*_e,$$

$$u_e \in \Gamma := \{\gamma^{(1)}, \ldots, \gamma^{(m)}\}.$$  

B. Problem Formulation

We aim to maximize the average flow passing through the highway. To achieve this goal, we select our objective function to be $E \rho_\pi \{ \frac{1}{T} \sum_{e \in E, t \in T} f_e(t) \}$, where $f_e(t) := \rho_e(t) u_e$, for each $e \in E$, $t \in T$, and the notion $E \rho_\pi$ is the distribution of the concatenated random variable $\pi := (\rho(0), \omega)$. Given the parameters $\{f_e(t), \pi_e \}, \{\rho_e(t)\} \in E$, $\{\pi_e\} \in E$ and $\Gamma$, the problem of computing variable speed limits which are robust to the uncertainty $\pi$ can be formulated as follows:

$$\max_{u_\rho} \quad E \rho_\pi \left( \frac{1}{T} \sum_{e \in E, t \in T} \rho_e(t) u_e \right),$$

s.t. \quad (1), (2), (3),

where $\rho := (\rho_1(1), \rho_2(1), \ldots, \rho_n(1), \rho_1(2), \ldots, \rho_n(T))$ is the concatenated variable of $\{\rho_e(t)\} \in E, t \in T \setminus \{0\}$ and $u := (u_1, \ldots, u_n)$ is that of $\{u_e \} \in E$.

The probability distribution $P_\pi$ is needed in order to compute a set of speed limits which are robust to the uncertainty $\pi$ and solve problem (P). However, this distribution $P_\pi$ is unknown, and we assume that we have access to $N$ samples of the random variable $\pi$. Thus, we investigate the computation of a set of feasible variable speed limits that possess certain out-of-sample guarantees within a distributionally robust optimization framework [19], [22]. In this way, we seek to find a set of feasible $u$ with certificates $J(u)$, such that the out-of-sample performance of $u$, $E \rho_\pi \{ \frac{1}{T} \sum_{e \in E, t \in T} \rho_e(t) u_e \}$, has the following performance guarantee with a given confidence level $\beta \in (0, 1)$:

$$P^N \left( \mathbb{E}_{\rho_\pi} \left( \frac{1}{T} \sum_{e \in E, t \in T} \rho_e(t) u_e \right) \geq J(u) \right) \geq 1 - \beta,$$

where $P^N$ denotes the probability that the event $E \rho_\pi \{ \frac{1}{T} \sum_{e \in E, t \in T} \rho_e(t) u_e \} \geq J(u)$ happens on the $N$ product of the sample space that defines $\pi$.

IV. PERFORMANCE GUARANTEED REFORMULATION

Problem (P) is intractable mainly due to the uncertainty $\pi$. We aim to obtain a tractable formulation of (P) that enables us to compute the variable speed limits with performance guarantees, as shown in (4). To achieve this goal, we follow a four-step procedure. First, we treat the density trajectories as random variables and formulate Problem (P) into an equivalent problem, Problem (P1). Second, we propagate the sample trajectories via the measurements of $\pi$. This step enables the the distributionally robust optimization framework for dynamical systems with performance guarantees be equivalent to (4). Third, we adapt the distributionally robust optimization approach to Problem (P1) for certificates. Finally, we obtain a tractable problem reformulation for data-driven solutions and certificates.

Step 1: (Equivalent Formulation of (P)) The random inflows and outflows along the highway result in random density dynamics characterized by (2). Therefore, the densities $\rho_e(t)$, for all $e \in E$ and $t \in T \setminus \{0\}$, will be random variables whose distributions are determined by speed limits $u$, inflows and outflows $\omega$, and the initial density $\rho(0)$. In this step, we...
take the decision variable $\rho$ in Problem (P) as the random variable. Using Probability Theory, we derive an equivalent Problem (P1) via a reformulation of the constraints in (P).

Let us take the variable $\rho$ considered in (P) as the random variable. For each speed limit $u$ characterized by (3), let $\mathcal{Z}(u)$ and $\mathcal{P}(u)$ denote the support of $\rho$ and the probability distribution of $\rho$, respectively. Recall that the support of $\rho$ is the smallest closed set such that the probability $P(\rho \in \mathcal{Z}(u)) = 1$. Note that in Problem (P), constraints (1) on $\rho$ and $u$ ensure no congestion. Therefore, to obtain an equivalent problem, we need to select $\mathcal{Z}(u)$ such that $\mathcal{Z}(u) = \rho \in \mathbb{R}^{n_T}$ | (1)). Without loss of generality, we select $\mathcal{Z}(u) := \rho \in \mathbb{R}^{n_T}$ | (1)). To fully characterize the random variable $\rho$, we need to determine the distribution $\mathcal{P}(u)$. Using the density dynamics (2), we can represent $\mathcal{P}(u)$ as a convolution of the distribution $\mathcal{P}(u)$. Given that $\mathcal{P}(u)$ is unknown, the characterization of $\mathcal{P}(u)$ is done in later steps.

We denote by $\mathcal{M}(\mathcal{Z}(u))$ the space of all probability distributions supported on $\mathcal{Z}(u)$, and equivalently write the unsolvable Problem (P) as

(P1) $\max_{u} \mathbb{E}_{\mathcal{P}(u)} \{H(u; \rho) := \frac{1}{T} \sum_{t \in \mathcal{T}} \rho_{t}(u)\},$

subject to $\mathcal{P}(u)$ characterized by (2) and $\mathcal{P}(u), \mathcal{P}(u) \in \mathcal{M}(\mathcal{Z}(u))$, (3).

We can now obtain the performance guarantee of (P1) by considering the induced out-of-sample performance on $\mathcal{P}(u)$, written as $\mathbb{E}_{\mathcal{P}(u)} \{\frac{1}{T} \sum_{t \in \mathcal{T}} \rho_{t}(u)\}$. For all Problems derived later, we will use the performance guarantees equivalent to (4), as follows:

$$P_N \left( \mathbb{E}_{\mathcal{P}(u)} \left\{ \frac{1}{T} \sum_{t \in \mathcal{T}} \rho_{t}(u) \right\} \geq J(u) \right) \geq 1 - \beta, \quad (5)$$

where notions $P_N$, $J(u)$ and $\beta$ in (5) are those as in (4).

**Step 2: (Sample Trajectory Propagators)** In this step, we obtain samples of $\rho$ and use them to deal with $\mathcal{P}(u)$. Consequently, these samples enable the distributionally robust optimization framework for (P1).

Given the speed limit $u \in \Gamma$, the density dynamics represented by (2) reduce to a linear system. As the result of the Uniqueness Solutions of Linear Systems, we can use (2) to obtain a unique density trajectory $\rho$ for each measurement of $\varpi$. As mentioned earlier, we assume that a set of data comprising $N$ samples of random variable $\varpi$ is available. Let $\mathcal{L} = \{1, \ldots, N\}$ denote the index set for realizations of the random variable $\varpi$, and let us denote the set of independent and identically distributed (iid) realizations of $\varpi$ by $\{\varpi^{(i)} := (\rho^{(i)}(0), \omega^{(i)})\}_{i \in \mathcal{L}}$. Given these realizations $\{\varpi^{(i)}\}_{i \in \mathcal{L}}$, the sample trajectories $\{\rho^{(i)}\}_{i \in \mathcal{L}}$ of the random traffic flow dynamics for each edge $e \in \mathcal{E}$ with its precedent edge $s \in \mathcal{E} \cup \emptyset$, are given by

$$\rho^{(i)}(t+1) = \rho^{(i)}(t) + h(u_{s} \rho^{(i)}(t) - u_{e} \rho^{(i)}(t) + \omega^{(i)}(t)), \quad (2a)$$

for all $t \in \mathcal{T} \setminus \{T\}$ and sample index $i \in \mathcal{L}$. The following lemma establishes that $\{\rho^{(i)}\}_{i \in \mathcal{L}}$ are iid samples from $\mathcal{P}(u)$.

**Lemma IV.1 (Iid sample generators of $\rho$)** Given $u \in \Gamma$ and iid realizations $\{\varpi^{(i)}\}_{i \in \mathcal{L}}$ of $\varpi$, the system dynamics (2a) generates iid sample trajectories $\{\rho^{(i)}\}_{i \in \mathcal{L}}$ of $\mathcal{P}(u)$.

**Proof:** We know that continuous functions of iid random variables are iid, therefore the sample trajectories $\{\rho^{(i)}\}_{i \in \mathcal{L}}$ generated by (2a) are iid realizations of $\mathcal{P}(u)$. Consider the random variable $\varpi$ with unknown distribution $\mathcal{P}(\varpi)$. Let $\mathcal{M}_{u}(\mathcal{Z}(u)) \subset \mathcal{M}(\mathcal{Z}(u))$ denote the space of all light-tailed probability distributions supported on $\mathcal{Z}(u)$. We make the following assumption on $\mathcal{P}(\varpi)$:

**Assumption IV.1 (Light tailed unknown distributions)** It holds that $\mathcal{P}(\varpi) \in \mathcal{M}_{u}(\mathcal{Z}(u))$, i.e., there exists an exponent $a > 1$ such that: $b := \mathbb{E}_{\mathcal{P}(\varpi)}[\exp(\|\varpi\|^{a})] < \infty$.

The above assumption invokes the following lemma:

**Lemma IV.2 (Light-tailed distribution of $\rho$)** If Assumption IV.1 holds, then $\mathcal{P}(u) \in \mathcal{M}_{u}(\mathcal{Z}(u))$.

**Proof:** To show the random variable $\rho$ has a light-tailed distribution, we bound its norm by $\|\varpi\|$ via an norm equivalence and dynamics on $\rho$. In this way, by norm equivalence, there exists $M_{1} > 0$ such that $\|\rho\| \leq M_{1}||\rho||_{\infty}$. Let $t^{*} \in \arg\max_{t \in \mathcal{T} \setminus \{0\}} \|\rho(t)\|_{\infty}$, we have:

$$\|\rho\| \leq M_{1}||\rho||_{\infty},$$

$$= M_{1} \max_{t \in \mathcal{T} \setminus \{0\}} \|\rho(t)\|_{\infty} = M_{1}||\rho(t^{*})||_{\infty}.$$
Again using norm equivalence, there exists $M_3 > 0$ such that $\|\varpi\|_{\infty} \leq M_3 \|\varpi\|$. This results in

$$\|\rho\| \leq M_2 (t^* + 1) \|\varpi\|_{\infty} \leq M_2 M_3 (t^* + 1) \|\varpi\|.$$ 

Let $M_4 = (M_2 M_3 (t^* + 1))^a < \infty$. Then for each $u$ and any $a > 1$ such that $E_{\mathbb{P}_u}[\exp(\|\varpi\|^a)] < \infty$, we have $E_{\mathbb{P}_u}[\exp(\|\rho\|^a)] = E_{\mathbb{P}_u}[\exp(\|\rho(u, \varpi)\|^a)] \leq E_{\mathbb{P}_w}[\exp(\|M_4 \varpi\|^a)] < \infty$, that is, $\mathbb{P}(u)$ is light tailed. □

The above lemma is the last ingredient to enable the distributionally robust optimization framework for (P1) in the next step.

**Step 3: (Certificates)** We now design a certificate to satisfy the performance guarantee (5) using the distributionally robust optimization approach. To design a certificate $J(u)$ for a given set of speed limits $u$, we need to estimate the probability distribution $\mathbb{P}(u)$ empirically. To do so, we use the sample trajectories $\{\rho^{(i)}\}_{i \in \mathcal{L}}$ obtained from sample generators (2a).

Let $\tilde{\mathbb{P}}(u) := (1/N) \sum_{i \in \mathcal{L}} \delta(\rho^{(i)})$ denote the estimated probability distribution. In this way, by application of the point mass operator $\delta$, we have $\mathbb{E}_{\tilde{\mathbb{P}}(u)}[H(u; \rho)] = (1/N) \sum_{i \in \mathcal{L}} H(u; \rho^{(i)})$, which is taken to be the candidate certificate for the performance guarantee (5).

Note that such certificates only result in an approximation of the out-of-sample performance if $\mathbb{P}$ is unknown, and (5) cannot be guaranteed in probability. To achieve the out-of-sample performance, we follow the procedure proposed in [19, 20]. More precisely, we determine an ambiguity set $\mathcal{P}(u)$ containing all the possible probability distributions supported on $\mathcal{Z}(u) \subseteq \mathbb{R}^{nT}$ that can generate the sample trajectories $\{\rho^{(i)}\}_{i \in \mathcal{L}}$ with high confidence. Then, with the given data-driven solution $u$, it is plausible to consider the worst-case expectation of the out-of-sample performance for all distributions contained in $\mathcal{P}(u)$. Such worst-case distribution offers a lower bound for the out-of-sample performance with high probability.

Lemma IV.2 on the light-tailed distribution of $\rho$ validates the modern measure concentration result [25, Theorem 2] on $\mathcal{M}_a(\mathbb{L}(u))$, which provides an intuition for considering the Wasserstein ball $\mathcal{B}_\varepsilon(\tilde{\mathbb{P}}(u))$ of center $\tilde{\mathbb{P}}(u)$ and radius $\varepsilon$ as the ambiguity set $\mathcal{P}(u)$. The ambiguity sets $\mathcal{P}(u) := \mathcal{B}_\varepsilon(\tilde{\mathbb{P}}(u))$ allow us to provide the certificate that ensures the performance guarantee (5) for any data-driven solution $u$, by taking $J(u) := \inf_{\mathcal{Q} \in \mathcal{P}(u)} \mathbb{E}_{\mathcal{Q}}[H(u; \rho)]$.

**Step 4: (Tractable Reformulation of (P1))** To obtain the certificate $J(u)$, we need to solve an infinite-dimensional optimization problem, which is generally hard. With an extended version of the strong duality results for moment problem [26, Lemma 3.4], we can reformulate the optimization problem for $J(u)$ into a finite-dimensional convex programming problem as the following:

$$J(u) = \sup_{\lambda \geq 0} - \lambda \epsilon(\beta) + \frac{1}{N} \sum_{i \in \mathcal{L}} \inf_{\rho \in \mathcal{Z}(u)} \left\{ \lambda \|\rho - \rho^{(i)}\| + H(u; \rho) \right\},$$

s.t. $\{\rho^{(i)}\}_{i \in \mathcal{L}}$ is obtained from (2a),

where the parameter $\beta$ is the confidence level in (5) and the value $\epsilon(\beta)$ is the radius of $\mathcal{B}_\varepsilon(\beta)$ as calculated in [20].

To obtain a data-driven speed limits $u$ with a good out-of-sample performance of (P1), we need to obtain $u$ with a high certificate $J(u)$. Finally, we can obtain a data-driven solution $u$ with a high certificate $J(u)$, by solving the problem:

$$\max_{u, \text{s.t.} \; (3)} J(u).$$

Problem (P2) consists of many inner optimization problems. To propose a solution method, we consider an equivalent optimization problem given as follows:

$$\max_{u, \rho, \lambda, \mu, \nu, \eta} - \lambda \epsilon(\beta) - \frac{1}{N} \sum_{e, t, l} \mathcal{J}_e \mathbb{P}_e \mathcal{Q}_l^{(e)}(t) \quad + \frac{1}{N} \sum_l \left\{ \nu(l, \rho(l)) \right\},$$

s.t. $[\mathcal{J} + u(\mathcal{J} - \rho(\mathcal{J}))] \otimes 1_T \otimes \eta(l)$

$- \mu(l) \geq 0_{nT}, \forall l \in \mathcal{L},$

$\|\nu(l)\|_\ast \leq \lambda, \forall l \in \mathcal{L},$

$\eta(l) \geq 0_{nT}, \forall l \in \mathcal{L},$ (3), (2a),

where decision variables $(u, \rho, \lambda, \mu, \nu, \eta)$ are concatenated versions of $(u_e, \rho_e^{(i)}(t), \lambda, \mu_e^{(i)}(t), \nu_e^{(i)}(t), \eta_e^{(i)}(t)) \in \mathbb{R}$, for all $l \in \mathcal{L}, t \in \mathcal{T} \setminus \{0\}$, and $e \in \mathcal{E}$. The parameter $\epsilon(\beta)$ is the radius of $\mathcal{B}_\varepsilon(\beta)$, the value $\rho(\mathcal{J}) := \hat{f}/\mathcal{H}$ is the critical density under the free flow and $\mathcal{J}$ is the jam density.

The following lemma shows that problems (P2) and (P3) are equivalent for $(u, J)$.

**Lemma IV.3 (Tractable reformulation of (P2))** Consider the DRO setting as in (P2). Then Problem (P2) is equivalent to (P3) in the sense that their optimal objective value are the same and the set of optimizers of (P2) is the projection of that of (P3). Further, for any feasible point $(u, \rho, \lambda, \mu, \nu, \eta)$ of (P3), let $\hat{J}(u)$ denote the value of its objective function. Then the pair $(u, \hat{J}(u))$ gives a data-driven solution $u$ with an estimate of its certificate $J(u)$ by $\hat{J}(u)$, such that the performance guarantee (5) holds for $(u, \hat{J}(u))$.

**Proof:** Following [19, 20], and references therein, we write Problem (P2) as follows:

$$\sup_{u, \lambda \geq 0} - \lambda \epsilon(\beta) + \frac{1}{N} \sum_{i \in \mathcal{L}} \inf_{\rho \in \mathcal{Z}(u)} \left\{ \lambda \|\rho - \rho^{(i)}\| + H(u; \rho) \right\},$$

s.t. (3), (2a).

Using the definition of the dual norm and moving its sup operator we can write the above problem as:

$$\sup_{u, \lambda \geq 0} - \lambda \epsilon(\beta) + \frac{1}{N} \sum_{i \in \mathcal{L}} \inf_{\rho \in \mathcal{Z}(u)} \sup_{\|\nu(l)\|_\ast \leq \lambda} \left\{ \left(\nu(l, \rho - \rho^{(i)}) + H(u; \rho) \right), \right\},$$

s.t. (3), (2a).
Given $\lambda \geq 0$, the sets $\{ \nu(l) \in \mathbb{R}^{n_T} \mid \|\nu(l)\|_* \leq \lambda \}$ are compact for all $l \in \mathcal{L}$. We then apply the minimax theorem between inf and the second sup operators. This results in the switch of the operators, and by combining the two sup operators we have:

$$\sup_{u,\lambda,\nu} -\lambda \epsilon(\beta) + \frac{1}{N} \sum_{l \in \mathcal{L}} \inf_{\rho \in \mathcal{Z}(u)} \left( \left\langle \nu(l), \rho - \rho^l \right\rangle + H(u; \rho) \right),$$

s.t. (3), (2a), $\lambda \geq 0$,

$$\|\nu(l)\|_* \leq \lambda, \forall l \in \mathcal{L}.$$  

The objective function can be simplified as follows:

$$-\lambda \epsilon(\beta) + \frac{1}{N} \sum_{l \in \mathcal{L}} \left\langle -\nu(l), \rho \right\rangle + \frac{1}{N} \sum_{l \in \mathcal{L}} h^{(l)}(u),$$

where

$$h^{(l)}(u) := \inf_{\rho \in \mathcal{Z}(u)} \left( \left\langle \nu(l), \rho \right\rangle + H(u; \rho) \right), \forall l \in \mathcal{L}.$$  

For each $l \in \mathcal{L}$, we rewrite $h^{(l)}(u)$ by firstly taking a minus sign out of the inf operator, then exploiting the equivalent representation of sup operator, and finally using the definition of conjugate functions. The function $h^{(l)}(u)$ results in the following form:

$$h^{(l)}(u) = -\sup_{\rho \in \mathcal{Z}(u)} \left( \left\langle -\nu(l), \rho \right\rangle - H(u; \rho) \right),$$

$$= -\sup_{\rho \in \mathcal{Z}(u)} \left( \left\langle -\nu(l), \rho \right\rangle - H(u; \rho) - \chi_{\mathcal{Z}(u)}(\rho) \right),$$

Further, we apply the property of the inf-convolution operation and push the minus sign back into the inf operator, for each $h^{(l)}(u), l \in \mathcal{L}$. The representation of $h^{(l)}(u)$ results in the following relation:

$$h^{(l)}(u) = -\inf_\mu \left[ \left\langle H(u; \cdot) \right\rangle^* \left( -\mu(l) - \nu(l) \right) + \left[ \chi_{\mathcal{Z}(u)}(\cdot) \right]^* \left( \mu(l) \right) \right],$$

$$= \sup_\mu \left[ -\left\langle H(u; \cdot) \right\rangle^* \left( -\mu(l) - \nu(l) \right) - \left[ \chi_{\mathcal{Z}(u)}(\cdot) \right]^* \left( \mu(l) \right) \right].$$

By substituting $-\nu(l)$ by $\nu(l)$, the resulting optimization problem has the following form:

$$\sup_{u,\lambda,\mu,\nu} -\lambda \epsilon(\beta) - \frac{1}{N} \sum_{l \in \mathcal{L}} \left[ \left\langle H(u; \cdot) \right\rangle^* \left( -\mu(l) + \nu(l) \right) + \left[ \chi_{\mathcal{Z}(u)}(\cdot) \right]^* \left( \mu(l) \right) - \left\langle \nu(l), \rho \right\rangle \right],$$

s.t. (3), (2a), $\lambda \geq 0$,

$$\|\nu(l)\|_* \leq \lambda, \forall l \in \mathcal{L}.$$  

Given $u$, the strong duality of linear programs are applicable for the conjugate of the function $H(u; \cdot)$ and the support function $\sigma_{\mathcal{Z}(u)}(\cdot)$. Using the strong duality and the definition of the support function, we compute

$$\left\langle H(u; \cdot) \right\rangle^* \left( \nu(l) - \mu(l) \right) := \begin{cases} 0, & \nu(l) = \mu(l) + \frac{1}{T} u \otimes 1_T, \forall l \in \mathcal{L}, \\ \infty, & \text{otherwise}. \end{cases}$$

and

$$[\chi_{\mathcal{Z}(u)}(\cdot)]^* (\mu(l)) = \sigma_{\mathcal{Z}(u)}(\mu(l))$$

$$= \sup_{\xi} \left\langle \mu(l), \xi \right\rangle,$$

s.t. $0 \leq \xi(t) = \rho^l(e_u), \forall e \in \mathcal{E},$

$$\inf_{\eta} \left\langle \eta(t), \sum_{e \in \mathcal{T}, t \in \mathcal{L}} \mathcal{T} \mathcal{E} \eta^l(t) \rightangle,$$

s.t. $[\mathcal{J} + u \otimes (\mathcal{F} - \rho^l(\mathcal{E}))] \otimes 1_T \circ \eta^l,$

$$\eta^l \geq 0, \forall l \in \mathcal{L},$$

By substituting these functions into the objective function and take a minus sign out of the resulting inf operator above, we obtain the form of Problem (P3). Given that the relations hold with equalities, we therefore claim that (P2) is equivalent to (P3).

Further, given any feasible point $(\nu, \rho, \lambda, \mu, \nu, \eta)$ of (P3), we denote its objective value by $\tilde{J}(u)$. The value $\tilde{J}(u)$ is a lower bound of (P3) and therefore a lower bound for (P2), i.e., $\tilde{J}(u) \leq J(u)$. Then the value $J(u)$ is an estimate of the certificate for the performance guarantee (5). Therefore, $(u, \tilde{J}(u))$ is a data-driven solution and certificate pair for (P1).

Problem (P3) is inherently difficult to solve due to the discrete decision variables $u$, bi-linear terms in the first group of constraints $u \otimes 1_T \circ \eta^l$, and the nonlinear sample trajectories $(\rho(l))_{l \in \mathcal{L}}$, which motivates our next section.

V. Solution Techniques for Nonconvex Problem (P3)

To compute high-quality solutions, we follow a two-step procedure. In the first step, we transform Problem (P3) into a mixed-integer bi-linear program with a linear constrained set. We call it Problem (P4). Finally, we propose an integer-solution search algorithm to compute high-quality solutions to Problem (P4).

Step 1: In this step, we represent the speed limits $u$ with a set of binary variables, and then represent each bi-linear term that is comprised of a continuous variable and a binary variable, with a set of linear constraints.

**Binary Representation of Speed Limit $u$:** For each edge $e \in \mathcal{E}$ and speed limit value $\gamma^i \in \Gamma$ with $i \in \mathcal{O} := \{1, \ldots, m\}$, let us define the binary variable $x_{e,i}$ to be equal to one if $u_e = \gamma^i$; otherwise $x_{e,i} = 0$. We will then have $u_e = \sum_{i \in \mathcal{O}} \gamma^i x_{e,i}$ for each $e \in \mathcal{E}$. Using this representation, we reformulate the speed limit constraints (3) into the following:

$$u_e \leq \sum_{i \in \mathcal{O}} \gamma^i x_{e,i} \leq u_e^U, \sum_{i \in \mathcal{O}} x_{e,i} = 1, \forall e \in \mathcal{E},$$

and we update the sample trajectories formula (2a) as follows:

$$\rho^l(t + 1) = \rho^l(t) + h \omega^l(t) + \sum_{i \in \mathcal{O}} \gamma^i (x_{e,i} \rho^l(t) - x_{e,i} \rho^l(t)), \forall e \in \mathcal{E}, t \in \mathcal{T} \setminus \{T\}, l \in \mathcal{L},$$

(7)
Reformulation of Bi-linear Terms: In Problem (P3), there are three groups of bi-linear terms: 1) the bi-linear terms $\nu^{(i)}_c(t)\rho^{(i)}_c(t)$ in the objective function written as $\nu^{(i)}_c(t)\rho^{(i)}_c(t)$, 2) the bi-linear terms $\sum_{i\in O} \gamma_i x_{e,i} \eta_i^{(i)}(t)$ which appear in the first set of constraints (e.g., $u \otimes 1_T \otimes \eta_i^{(i)}$), and 3) the bi-linear terms $x_{e,i} \rho_c^{(i)}(t)$ in the sample trajectories formula (7). In the group 2) and 3), each bi-linear term is comprised of a continuous variable and a binary variable. In this regard, we simplify these bi-linear terms by using the linearization technique under the following assumption:

Assumption V.1 (Bounded dual variable $\eta$) There exist large enough scalar $\overline{\eta}$ such that $\eta_i^{(i)}(t) \leq \overline{\eta}$ for all $e \in \mathcal{E}$, $t \in T \setminus \{0\}$ and $l \in \mathcal{L}$.

Proposition V.1 (Equivalence reformulation for bi-linear terms in group 2) and 3) [27, Section 2] Let $\mathcal{Y} \subset \mathbb{R}$ be a compact set. Given a binary variable $x$ and a linear function $g(y)$ in a continuous variable $y \in \mathcal{Y}$, $z$ equals the quadratic function $x g(y)$ if and only if

$$
\begin{align*}
&g x \leq z \leq \overline{g} x, \\
g(y) - \overline{g} \cdot (1 - x) \leq z \leq g(y) - g \cdot (1 - x),
\end{align*}
$$

where $\underline{g} = \min_{y \in \mathcal{Y}} \{g(y)\}$ and $\overline{g} = \max_{y \in \mathcal{Y}} \{g(y)\}$. \hfill \square

Applying Proposition V.1, we introduce variables $z_{e,i}^{(i)}(t)$ to represent the bi-linear terms $x_{e,i} \rho_c^{(i)}(t)$ in the group 2), via the following constraints:

$$
\begin{align*}
&\sum_{i\in O} z_{e,i}^{(i)}(t) = \eta_i^{(i)}(t), &\forall e \in \mathcal{E}, t \in T \setminus \{0\}, l \in \mathcal{L}, \\
&0 \leq z_{e,i}^{(i)}(t) \leq \overline{\gamma}_{e,i}, &\forall e \in \mathcal{E}, i \in \mathcal{O}, t \in T \setminus \{0\}, l \in \mathcal{L}, \\
&\eta_i^{(i)}(t) - \overline{\gamma}(1 - x_{e,i}) \leq z_{e,i}^{(i)}(t) \leq \eta_i^{(i)}(t), &\forall e \in \mathcal{E}, i \in \mathcal{O}, t \in T \setminus \{0\}, l \in \mathcal{L}.
\end{align*}
$$

Similarly, we introduce variables $y_{e,i}^{(i)}(t)$ to represent the bi-linear terms $x_{e,i} \rho_c^{(i)}(t)$ in the group 3), via the following constraints:

$$
\begin{align*}
&y_{e,i}^{(i)}(0) = x_{e,i} \rho_c^{(i)}(0), &\forall e \in \mathcal{E}, l \in \mathcal{L}, i \in \mathcal{O}, \\
&\rho_c^{(i)}(t) - \overline{\gamma}_{e,i} \leq y_{e,i}^{(i)}(t) \leq \rho_c^{(i)}(t), &\forall e \in \mathcal{E}, i \in \mathcal{O}, t \in T \setminus \{0\}, l \in \mathcal{L}, \\
&\sum_{i\in O} y_{e,i}^{(i)}(t) = \rho_c^{(i)}(t), &\forall e \in \mathcal{E}, i \in \mathcal{O}, t \in T \setminus \{0\}, l \in \mathcal{L}, \\
&0 \leq y_{e,i}^{(i)}(t) \leq \overline{\rho}_c^{(i)}(t), &\forall e \in \mathcal{E}, i \in \mathcal{O}, t \in T \setminus \{0\}, l \in \mathcal{L}.
\end{align*}
$$

Using variables $y_{e,i}^{(i)}(t)$ to reformulate the sample trajectories formula (7), we have the following constraints:

$$
\begin{align*}
\rho_c^{(i)}(t + 1) &= \rho_c^{(i)}(t) + h \omega_c^{(i)}(t), \\
&+ h \sum_{i\in O} \gamma_i(y_{e,i}^{(i)}(t) - y_{e,i}^{(i)}(t)), &\forall e \in \mathcal{E}, t \in T \setminus \{T\}, l \in \mathcal{L},
\end{align*}
$$

By the above reformulation, the bi-linear terms in group 2) and 3) will be linear, and Problem (P3) can be equivalently written as the following optimization problem:

$$
\begin{align*}
\max_{x,y,z,\nu,\mu,\nu,\eta} & -\lambda \epsilon(\beta) - \frac{1}{N} \sum_{e,t,l} \mathcal{F}_e \mathcal{P}_e \eta_i^{(i)}(t) + \nu_i^{(i)}(t) \rho_c^{(i)}(t), \\
\text{s.t.} & \sum_{i\in O} \gamma_i x_{e,i} \otimes 1_T \otimes \eta_i^{(i)}(t) \geq 0, \forall l \in \mathcal{L}, \\
& \nu_i^{(i)}(t) = \mu_i^{(i)} + \frac{1}{T} \sum_{i\in O} \gamma_i x_{e,i} \otimes 1_T, \forall l \in \mathcal{L}, \\
& \|\nu_i^{(i)}\|_\infty \leq \lambda, \forall l \in \mathcal{L}, \\
& 0 \leq \eta_i^{(i)}(t) \leq \overline{\eta}, \forall l \in \mathcal{L}, \\
& \text{speed limits (6), dual variable (8),} \\
& \text{sample trajectories \{(9), (10), (11)\}.}
\end{align*}
$$

Further, let $\tilde{J}(u)$ denote the value of the objective function of (P4) at a computed feasible solution $(x, y, z, \rho, \lambda, \mu, \nu, \eta)$. Then, the resulting speed limits $u := \sum_{i\in O} \gamma_i(x_{1,1}, \ldots, x_{n,i})$ provide a data-driven solution such that $(u, \tilde{J}(u))$ satisfies the performance guarantee (5).

Step 2: Problem (P4) is computationally intractable since its objective function is still nonlinear in its arguments due to the bi-linear terms $\nu_i^{(i)}(t) \rho_c^{(i)}(t)$. To compute high-quality feasible solutions to (P4), we propose an integer-solution search algorithm. The proposed algorithm is a prototype of the decomposition-based methods in the literature [28], [29]. These methods can handle specialized mix-integer nonlinear programs and achieve suboptimal solutions efficiently.

We propose an integer-solution search algorithm as shown in Algorithm 1. The idea of the algorithm is to iteratively solve 1) upper-bounding problems to (P4), and 2) lower-bounding problems to (P4), until a stopping criteria is met. In each iteration $k$ of this process, we construct an upper-bounding problem (UBP$_k$) through McCormick relaxations of the bi-linear terms $\nu_i^{(i)}(t) \rho_c^{(i)}(t)$. This upper bounding problem is a mixed-integer linear program and its solution gives the upper bound of (P4) and candidate variable speed limits $x^{(k)}$. These $x^{(k)}$ can be used to construct sample trajectories $\{(\rho^{(i,k)}, t)\}_{i\in E}$ and a linear lower-bounding problem (LBPP$_k$) for potential feasible solutions of (P4).

Upper-bounding Problems: At each iteration $k$, the upper-bounding problem (UBP$_k$) is constructed using two extra ingredients: 1) a McCormick relaxation of the bi-linear terms $\nu_i^{(i)}(t) \rho_c^{(i)}(t)$ in the objective function of (P4), and 2) canonical integer cuts that exclude the previous visited candidate variable speed limits $\{x^{(p)}_{(k)}\}_{p=1}^{k-1}$.

1) The McCormick envelope [30] provides relaxations of bi-linear terms, which is stated in the following remark:

Remark V.1 (McCormick envelope) Consider two variables $x, y \in \mathbb{R}$ with upper and lower bounds, $\underline{x} \leq x \leq \overline{x}$, $\underline{y} \leq y \leq \overline{y}$. The McCormick envelope of the variable...
Algorithm 1 Integer solution search algorithm

1: Initialize $k = 0$
2: repeat
3: $k ← k + 1$
4: Solve Problem (UBP$_k$), return $x^{(k)}$ and UBP
5: Generate sample trajectories \{$\rho^{(k)}_t$\}$_{t \in \mathcal{L}}$
6: Solve Problem (LBP$_k$), return obj$_k$ and LBP
7: until $\text{UBP}_k - \text{LB}_k \leq \epsilon$, or (UBP$_k$) is infeasible, or a satisfactorily suboptimal solution is found after certain running time $T_{\text{run}}$
8: return data driven solution $u_{\text{best}} := u^{(q)}$ with certificate $J(u^{(q)})$ such that $q \in \text{argmax}_{p=1,\ldots,k} \{\text{obj}_p\}$

$s := xy \in \mathbb{R}$ is characterized by the following constraints:

$$ s \geq xy + x\overline{y} - x\overline{y}, \quad s \geq xy + y\overline{x} - y\overline{x}, $$

$$ s \leq xy + x\overline{y} - x\overline{y}, \quad s \leq xy + y\overline{x} - y\overline{x}. $$

To construct a McCormick envelope for (UBP$_k$), let us denote $\overline{\tau}_e := \tau_e (T - 1 + \overline{\nu}_e)$ with $e \in \mathcal{E}$. For each $e \in \mathcal{E}$, $t \in \mathcal{T}$ and $l \in \mathcal{L}$, we have $0 \leq \nu_e(t) \leq \nu_e(t) \leq \overline{\nu}_e$, and the McCormick envelope of $s_e(t)$ is given by

$$ s_e(t) \geq \sum_{i \in \mathcal{I}} \nu_e(t), t \geq 0,$$  

$$ s_e(t) \leq \sum_{i \in \mathcal{I}} \nu_e(t), t \leq \nu_e(t),$$

(16)

2) The canonical integer cuts prevent (UBP$_k$) from choosing examined candidate variable speed limits \{$x^{(p)}_e, 1 \leq e \leq n, p = 1, \ldots, k$\}. Let $\Omega^p := \{(e, i) \in \mathcal{E} \times \mathcal{O} | x^{(p)}_{e,i} = 1\}$ denote the index set of $x$ for which the value $x^{(p)}_{e,i}$ is 1 at the previous iteration $p$. Let $c^p := |\Omega^p|$ denote the cardinality of the set $\Omega^p$ and let $\Omega^p := (\mathcal{E} \times \mathcal{O}) \setminus \Omega^p$ denote the complement of $\Omega^p$. The canonical integer cuts of Problem (UBP$_k$) at iteration $k$ are given by:

$$ \sum_{(e, i) \in \Omega^p} x^{(p)}_{e,i} - \sum_{(e, i) \in \Omega^p} x^{(p)}_{e,i} \leq c^p - 1, \quad \forall p \in \{1, \ldots, k - 1\}. $$

(17)

At each iteration $k$, the upper-bounding problem (UBP$_k$) has the following form:

$$ \text{max} \quad x, y, z, s, \rho \quad \lambda, \mu, \nu, \eta $$

$$ \text{s.t.} \quad \text{speed limits (6)}, \text{ sample trajectories } \{9, 10, 11\}, \text{no congestion } \{8, 12, 13, 14, 15\}, \text{McCormick envelope (16), integer cuts (17)}.$$

We denote by UBP$_k$ the optimal objective value of (UBP$_k$) and UB$_k$ is an upper bound of the original nonconvex problem (P4). We denote by $x^{(k)}$ the integer part of the optimizers of (UBP$_k$) and use it as a candidate speed limit in the lower-bounding problem LBP$_k$ of (P4).

Lower-bound Problems: To exploit the structure of (P4) and find lower-bounding problems, let us define the set $\Psi(x) := \{(x, \lambda, \mu, \nu, \eta) | \text{no congestion}\},$ $\Psi(x) := \{(y, \rho) | \text{sample trajectories}\}$ and $X := \{x | \text{speed limits}\}$. Problem (P4) can be equivalently written as:

$$ \text{max} \quad \sum_{e, i, t} \overline{\tau}_e \nu_e(t) - \lambda \overline{\epsilon} \beta - \frac{1}{N} \sum_{e, i, t} \overline{\tau}_e \nu_e(t) \rho^{(k)}_e(t), $$

s.t. $(z, \lambda, \mu, \nu, \eta) \in \Psi(x), (y, \rho) \in \Psi(x), x \in X$.

Given $x^{(k)} \in X$ solved by (UBP$_k$) at iteration $k$, we have a candidate speed limit $u^{(k)} := \sum_{i \in \mathcal{O}} \gamma(i)(x^{(k)}_{1,i}, \ldots, x^{(k)}_{n,i})$. For each $l \in \mathcal{L}$ with given $u^{(k)}$, the sample trajectory $\nu^{(l)}(t)$ is uniquely determined by $(\rho^{(l)}(0), \omega^{(l)}(t))$, via the uniqueness solution of the linear time-invariant systems. Therefore, the element $(y, \rho) \in \Psi(x^{(k)})$ is unique. Using the constraints set $\Psi(x^{(k)})$, we then construct the unique sample trajectories \{$\nu^{(l)}(t)$\}$_{l \in \mathcal{L}}$. The unique sample trajectories enable us to define the linear lower bounding problem at iteration $k$, as follows:

$$ \text{min} \quad z, \lambda, \mu, \nu, \eta $$

$$ \text{s.t.} \quad (z, \lambda, \mu, \nu, \eta) \in \Psi(x^{(k)}).$$

Let obj$_k$ denote the optimal objective value of (LBP$_k$). If Problem (LBP$_k$) is solved to optimality with a finite obj$_k$, we then obtain a feasible solution of (P4) with speed limit $u^{(k)} := \sum_{i \in \mathcal{O}} \gamma(i)(x^{(k)}_{1,i}, \ldots, x^{(k)}_{n,i})$ and certificate $J(u^{(k)}) := \text{obj}_k$. Otherwise, Problem (LBP$_k$) is either infeasible or unbounded and we let obj$_k = -\infty$. The lower bound of (P4) is then calculated by $\text{LB}_k = \max_{p=1,\ldots,k} \{\text{obj}_p\}$. The stopping criteria of the algorithm can be determined by 1) $\text{UB}_k - \text{LB}_k \leq \epsilon$, or 2) (UBP$_k$) is infeasible, or 3) a satisfactory suboptimal solution is found after certain running time $T_{\text{run}}$. We refer to [28] for the finite convergence of Algorithm 1 to a global $\epsilon$-optimal solution using both the first and second stopping criteria. To find a potentially good feasible solution within certain running time $T_{\text{run}}$, we further propose the third criteria. A satisfactory suboptimal solution after running time $T_{\text{run}}$ is then a feasible solution that achieves the lower bound of the algorithm. If no feasible solution is found within time $T_{\text{run}}$, we wait until a feasible solution is obtained.

VI. SIMULATIONS

In this section, we demonstrate in an example how to find a solution to (P4) that results in a data-driven variable-speed limit $u \in \mathbb{R}^5$ with performance guarantee (5). We consider a highway with length $L = 10km$ and we divide it into $n = 5$ segments. Let the unit size of each time slot $\delta = 30sec$ and consider $T = 20$ time slots for a 10min planning horizon. For each edge $e \in \mathcal{E}$, we assume a jam density of $\overline{\tau}_e = 1050vec/km^2$, a capacity of $\overline{\tau}_e =$...
3.1 \times 10^4 \text{vec/h} and a maximal free flow of } \overline{\mu} = 140 \text{km/h.}

Let us consider } m = 5 \text{ different candidate speed limits } \Gamma = \{40 \text{km/h}, 60 \text{km/h}, 80 \text{km/h}, 100 \text{km/h}, 120 \text{km/h}\}. \text{ On the } 4^{th} \text{ edge } e \in \mathcal{E}, \text{ we assume an accident happens during } \mathcal{T} \text{ with parameters } f_{\text{acc}} = 2.7 \times 10^4 \text{vec/h} \text{ and } \rho_{\text{acc}} = \overline{\mu}. \text{ To evaluate the effect of the proposed algorithm, samples of the random variables } w \text{ and } \rho(0) \text{ are needed. In real-case studies, samples } \{\rho^{(i)}(0)\}_{i \in \mathcal{E}} \text{ can be obtained from road sensors (loop detectors), while samples of the uncertain flows } \{w^{(i)}(t)\}_{i \in \mathcal{E}} \text{ can be constructed either from a database of flow data on the road, or from the current measurements of ramp flows with the assumption that the stochastic process } \{\omega(t)\}_{t \in \mathcal{T}} \text{ is stationary.}

In this simulation example, the index set of accessible samples is given by } \mathcal{L} = \{1, 2, 3\}. \text{ For each } l \in \mathcal{L}, \text{ let us assume that each segment } e \in \mathcal{E} \text{ initially operates under a free flow condition with an initial density } \rho^{(l)}(0) = 260 \text{vec/km}. \text{ For each edge } e \in \mathcal{E} \setminus \{1\} \text{ and time } t \in \mathcal{T}, \text{ we will assume that samples } \{\omega^{(l)}(t)\}_{i \in \mathcal{E}} \text{ are generated from a uniform distribution within interval } [-1500, 2500] \text{vec/h}. \text{ To ensure significant inflows of the system, we further let the samples } \{\omega^{(l)}(t)\}_{i \in \mathcal{E}} \text{ of the first segment to be chosen from the uniform distribution within interval } [2 \times 10^4, 2.4 \times 10^4] \text{vec/h. We also let the confidence level be } \beta = 0.95 \text{ and the radius of the Wasserstein Ball } \epsilon(\beta) = 0.985 \text{ as calculated in [20].}

To generate feasible solutions that can be carried out for a real time transportation system, we allocate } T_{\text{run}} = 5 \text{ min execution time to the proposed Algorithm } 1, \text{ and run it on a machine with 3.4GHz CPU and 4G RAM. In 5 minutes, the algorithm computed 5 feasible candidate speed limits and discarded 13 infeasible speed limits. The feasible solutions were obtained after 120 sec, 138 sec, 174 sec, 189 sec and 270 sec, respectively. We verified that } \hat{J}(u^{(3)}) = 2.435 \times 10^4 \text{vec/h} \text{ is the highest certificate obtained, i.e., } \hat{J}(u^{(3)}) \in \text{argmax}_{l=1,..,5} \{\hat{J}(u^{(l)}) | u^{(l)} \text{ is feasible}\}, \text{ and the desired speed limits are } u^{(3)} = \{100, 120, 100, 80, 120\} \text{km/h}. \text{ The algorithm terminated at iteration } k = 18, \text{ with bounds LB}_k = \hat{J}(u^{(3)}) \text{ and UB}_k = 9.0 \times 10^4 \text{vec/h}. \text{ It can be seen that the upper bound of the algorithm is loose, but the implementable solutions can be obtained in reasonable computational time. With knowledge of the underlying distribution, we see that the value of the certificate averaged on segments, given by } \hat{J}(u^{(3)})/5, \text{ is higher than the upper bound of the random flows injected in the first segment of the highway. This indicates that, with } 95\% \text{ confidence, the speed limit } u^{(3)} \text{ guarantees no congestion flows along the highway although initially the highway is congested.}

To evaluate the out-of-sample performance of the speed limits } u^{(3)}, \text{ we generated } N_{\text{val}} = 10^3 \text{ validation samples of } (\omega, \rho(0)) \text{ and simulated the cell transmission model [15] with the same parameter settings but a } 30\text{min time horizon. Fig. 2 shows the average of the sample trajectories over time, i.e., the function } \frac{1}{N_{\text{val}}} \sum_{t=1}^{N_{\text{val}}} \rho^{(i)}(t) \text{ for each segment } e, \text{ with and without speed limits. For the density evolution with speed limits } u^{(3)}, \text{ we verified that the density trajectory of accident edge (4) did not exceed its critical density } \rho^{(3)}(80 \text{km/h}) = 335 \text{vec/km and thus the road } G \text{ kept free of congestion in this planning horizon } \mathcal{T}. \text{ However, for the density evolution without speed limits, vehicles were accumulated on edge (4) and the congestion was propagated along edges of the road } G.

VII. CONCLUSIONS

In this paper, we proposed a traffic model that considered uncertain inflow, outflow, and random events along a highway. We then formulate a control problem in form of (P), where realizations of the unknown inflows and outflows are employed to derive data-driven variable speed limits that have guaranteed out-of-sample performance. We achieved this by adopting DRO theory to the equivalent Problem (P1), which further results into the mix-integer bilinear Problem (P4). Problem (P4) is solved by means of a proposed integer-solution search algorithm that is derived from decomposition-based method. The focus of our current work is on considering more complex traffic networks and the use of moving horizons to derive data-driven variable speed limits by leveraging real-time dynamic data.

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