Numerical study of spin quantum Hall transitions in superconductors with broken time-reversal symmetry

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We present results of numerical studies of spin quantum Hall transitions in disordered superconductors, in which the pairing order parameter breaks time-reversal symmetry. We focus mainly on \(p\)-wave superconductors in which one of the spin components is conserved. The transport properties of the system are studied by numerically diagonalizing pairing Hamiltonians on a lattice, and by calculating the Chern and Thouless numbers of the quasiparticle states. We find that in the presence of disorder, (spin-)current carrying states exist only at discrete critical energies in the thermodynamic limit, and the spin-quantum Hall transition driven by an external Zeeman field has the same critical behavior as the usual integer quantum Hall transition of non-interacting electrons. These critical energies merge and disappear as disorder strength increases, in a manner similar to those in lattice models for integer quantum Hall transition.

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I. INTRODUCTION

Transport properties of quasiparticles in unconventional superconductors have been of strong interest to condensed matter physicists since the discovery of high \(T_c\) cuprate superconductors. The cuprates, which are \(d\)-wave superconductors, support gapless nodal quasiparticle excitations; these nodal quasiparticles dominate heat and spin transport at low temperatures. Some heavy fermion superconductors are also known to have \(d\)-wave pairing. Another class of unconventional superconductors that have been receiving increasing attention are \(p\)-wave superconductors. While the \(p\)-wave pairing was originally found in superfluid \(^3\)He\(^1\) interest in it has been renewed more recently due to advances in two distinct systems. Firstly, fractional quantum Hall effects at fillings factor \(\nu = 5/2\) are believed to be the manifestation of the \(p\)-wave pairing of composite fermions.\(^{1,3,6,7}\) Secondly, a growing number of results on the unconventional superconductivity of Sr\(_2\)RuO\(_4\) emerge in favor of a triplet-pairing order parameter.\(^8\)

In addition to the experimental relevance, unconventional, disordered superconductors are also of great interest for theoretical reasons, as they represent new symmetry classes in disordered non-interacting fermion problems that are not realized in metals. A classification of these symmetry classes have been advanced recently.\(^{9,10}\) Depending on the existence (or the lack) of time-reversal and spin-rotation symmetries, dirty superconductors can be classified into four symmetry classes, CI, DIII, C, and D in Cartan’s classification scheme. These classes are believed to complete the possible universality classes\(^{11}\) in disordered single-particle systems.\(^{9,10}\)

Systems of class C with broken time-reversal invariance but preserved spin-rotation symmetry exhibit universal critical behavior which has been under extensive investigation recently.\(^{12,13,14,15,16}\) Such systems, which can be realized in two-dimensional superconductors with \(d_{xy}/d_{x^2−y^2}\) symmetry, have a distinct signature of a critical density of states at criticality.\(^{13,14}\) Analogous to the conventional quantum Hall effect where the Hall conductance is quantized, the Hall conductance of the spin current of such systems is quantized. This effect is, therefore, called spin quantum Hall effect, because spin, rather than charge, is conserved in such a superconductor.

Studies of the critical behavior of class D with both time-reversal and spin-rotation symmetries broken have a longer history, starting from the two-dimensional random-bond Ising model (RBIM).\(^{17,18,19,20,21,22,23,24}\) The RBIM can be mapped onto the Cho-Fisher network model\(^{25}\) which resembles the original Chalker-Coddington network model\(^{26}\) for the integer quantum Hall plateau transition, but with a distinct symmetry. The class D models may have an even richer phase diagram; in addition to the spin (or thermal, if the spin-rotation symmetry is completely broken) quantum Hall phase mentioned above and an insulating phase (which is always possible), they may also support a metallic phase.\(^{26}\)

The possibility of spin (or thermal) quantum Hall states in unconventional superconductors allows one to draw a close analogy between the quantum Hall effect and superconductivity, as well as to transfer theoretical or numerical methods developed in the study of one system to another. One of the well-developed numerical methods in the study of the quantum Hall effect is the calculation of Chern numbers of either single- or many-electron states, which allows one to distinguish between current carrying and insulating states unambiguously, even in a finite-size system. The Chern number method has been very successful in the studies of quantum Hall transitions, for both integer\(^{27,28,29,30}\) and fractional effects\(^{31}\) as well as in bilayer systems\(^{32}\) and also in other contexts.\(^{33,34,35}\) For example, by finite-size scaling, the localization length
exponent $\nu \approx 2.3$ has been obtained, consistent with other estimates. More recent application of this method to fractional quantum Hall states (where one inevitably has to deal with interacting electrons) allows one to determine the transport gap numerically, which is not possible from other known numerical methods.

In this paper, we report results of numerical studies on a lattice model of disordered $p$-wave superconductors with $p_x+ip_y$ pairing, which conserves the $z$-component of electron spin. As we will show later, this is an example of the class D model, and in certain sense the simplest model that supports a spin quantum Hall phase. We study the localization properties of the quasiparticle states by calculating the Chern and Thouless numbers in the present context, respectively, which are also related to the Hall and longitudinal thermal conductivities. We note that while it has been pointed out earlier that quasiparticle bands or individual quasiparticle states can be labeled by their topological Chern numbers, the present work represents the first attempt to calculate them numerically and use them to study the localization properties of the quasiparticle states in the context of unconventional superconductors.

Our main findings are summarized as the following. We find that the $p$-wave model we study supports an insulating phase and a spin quantum Hall phase with spin Hall conductance one in appropriate unit. For relatively weak disorder, there exist two critical energies at which current-carrying states exist, carrying a total Chern number; they are responsible for the spin quantum Hall phase. Phase transitions between these two phases may be induced either by changing the disorder strength, or by applying and sweeping a Zeeman field. The field-driven transition is found to have the same critical behavior as the integer quantum Hall transition of non-interacting electrons. As disorder strength increases the two critical energies both move toward $E = 0$, and annihilate at certain critical disorder strength, resulting in an insulating phase in which all quasiparticle states are localized. No metallic phase is found in our model.

The remainder of the paper is organized as the following. In section II we introduce the model Hamiltonian of our numerical study, and discuss its symmetry. We also describe the application of Chern and Thouless number methods to the present problem in some detail. We present our numerical results in section III, including results of the finite-size scaling analysis of the numerical data. Section IV is reserved for a summary and the discussion of our results.

II. MODEL AND NUMERICAL METHODS

We consider electrons moving on a two-dimensional square lattice with linear size $L$, in the presence of pairing and random potentials, described by the Hamiltonian

$$\mathcal{H} = -t \sum_{<i,j>} (c_{i\uparrow}^\dagger c_{j\uparrow} + c_{i\downarrow}^\dagger c_{j\downarrow}) + \sum_{<i,j>} (\Delta_{ij} c_{i\uparrow}^\dagger c_{j\downarrow} + \Delta_{ij}^* c_{j\uparrow} c_{i\downarrow}) + \sum_i (u_i - \mu)(c_{i\uparrow}^\dagger c_{i\uparrow} + c_{i\downarrow}^\dagger c_{i\downarrow}),$$

where $u_i$ is the random potential on site $i$ which are independent random variables distributed uniformly from $-W/2$ to $W/2$, and $\mu$ is the chemical potential of the electrons. $t$ is the nearest neighbor hopping integral and we choose $t = 1$ as the unit of energy from now on. We define, for a $p$-wave superconductor, the pairing potential $\Delta_{ij,\pm e_{\sigma}} = \pm \Delta$ and $\Delta_{ij,\mp e_{\sigma}} = \mp i\Delta$, where $e_x$ and $e_y$ are unit vectors along $x$- and $y$-axis, respectively. For a finite-size system, we introduce the generalized periodic boundary condition $c_{i+L\sigma} = e^{i\theta} c_{i\sigma}$, $c_{i+L\downarrow} = e^{-i\theta} c_{i\downarrow}$ ($i = x, y$). We note that while the total spin of the system is not conserved for the $p$-wave pairing, the $z$-component of the spin is conserved due to our choice that pairing only occurs between electrons with opposite spin. This becomes especially clear if we rewrite the Hamiltonian in terms of particle-hole transformed operators for the electrons with down spins:

$$d_i^\dagger = c_{i\uparrow}, \quad d_i = c_{i\downarrow}^\dagger,$$

so that

$$\mathcal{H} = (d_i^\dagger d_i) \left( \frac{\hbar}{\Delta^+} \right) \left( \begin{array}{c} d_i \\ d_i^\dagger \end{array} \right) = d^\dagger H d,$$

where $H$ is the Hamiltonian without pairing potentials for each spin component, and $\Delta = (\Delta_{\sigma})$. Clearly the number of $d$ particles is conserved, reflecting the conservation of the $z$-component of the total electron spin. Thus the corresponding transport properties of the $z$-component spin are well-defined; in the following we simply use the word spin to refer to its $z$-component, and spin conductances refer to the ratios between the $z$-component of the spin current and the gradient of the $z$-component of the Zeeman field.

The $p$-wave pairing symmetry $\Delta_{\sigma\sigma'} = -\Delta_{\sigma'\sigma}$ leads to a special symmetry of the Hamiltonian:

$$\sigma_x H \sigma_x = -H^T, \quad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

which makes the current problem distinct from the usual problem of electrons moving in a random potential. In fact, it is an example of the symmetry class D in the classification of Altland and Zirnbauer.
We note that the model we study here, Eq. (1), is not of the most general form of symmetry class D, for spin-1/2 fermions, which, by classification, has no spin-rotation symmetry along any direction. Consider a generic Hamiltonian for quasiparticles in a superconductor:

$$\mathcal{H} = \sum_{\alpha\beta} \left( \frac{\hbar}{2} \delta_{\alpha\beta} \epsilon^\dagger \epsilon + \frac{1}{2} \delta_{\alpha\beta} \epsilon^\dagger \epsilon + \frac{1}{2} \delta_{\alpha\beta} \epsilon^\dagger \epsilon \right),$$  \hspace{1cm} (5)

where $\alpha$ and $\beta$ are indices that label both lattice site and spin of the electron, running from 1 to $2N$ (if $N$ is the number of lattice sites). The Hamiltonian can be solved by the Bogoliubov transformation, or, explicitly, by the diagonalization of the $4N \times 4N$ matrix

$$\hat{H} = \left( \begin{array}{cc} -\delta^* & -\hbar T \\ \delta & -\hbar T \end{array} \right).$$  \hspace{1cm} (6)

In the generic case of class D (without time-reversal and spin-rotation symmetries), the only constraint on $\hat{H}$ is

$$\hat{H}^\dagger = \hat{H} = -\Sigma_x \hat{H}^\dagger \Sigma_x,$$  \hspace{1cm} (7)

where

$$\Sigma_x = \left( \begin{array}{cc} 0 & 1_{2N} \\ 1_{2N} & 0 \end{array} \right) = \sigma_x \otimes 1_{2N}. \hspace{1cm} (8)$$

This constraint, which comes from both hermiticity and Fermi statistics, is the same as Eq. (4). Note that this Hamiltonian is twice as large as the $p$-wave pairing Hamiltonian we study [Eq. (6)], although they belong to the same symmetry class for the following reasons. Interestingly, the partial spin-rotation symmetry along $z$-axis leads to a decomposition of the $4N \times 4N$ matrix $\hat{H}$ into two homomorphic subblocks. One subblock corresponds to spin-up particles and spin-down holes, and the other to spin-up holes and spin-down particles. Without additional symmetry, each subblock belongs to the conventional unitary ensemble $\mathbb{U}$. On the other hand, if spin-rotation symmetry in other directions is present (as in $d$-wave or other singlet pairing), the spin-up particles and spin-down holes are equivalent, and the coupling between them is symmetric; this is, in fact, the case of class C. In the present case, however, the coupling between spin-up particles and spin-down holes is antisymmetric, required by the special $p$-wave pairing we introduce. Therefore, the model we study is equivalent to pairing between spinless or spin-polarized fermions, also described by a Hamiltonian of the form [Eq. (6)], with $\alpha$ and $\beta$ label lattice sites only. It is in this sense that we can study well-defined spin transport in a class D model.

It is also useful for us to consider the presence of a uniform Zeeman field:

$$\mathcal{H}_B = \mu_B B \sum_m \epsilon_{m\uparrow}^\dagger \epsilon_{m\uparrow} - \epsilon_{m\downarrow}^\dagger \epsilon_{m\downarrow}$$

$$= \mu_B B \sum_m (d_{m\uparrow}^\dagger d_{m\uparrow} + d_{m\downarrow}^\dagger d_{m\downarrow}) + \text{const}. \hspace{1cm} (9)$$

We note that the Zeeman field plays a role of the Fermi energy for the (conserved) $d$ particles. More importantly, its presence changes the symmetry property of the system, because $H_B$ does not obey Eq. (4).

The spin Hall conductance of an individual quasiparticle eigenstate $|m\rangle$ can be calculated by the Kubo formula

$$\sigma_{xy}(m) = \frac{i \hbar}{A} \sum_{n \neq m} \frac{\langle m | j^S | n \rangle \langle n | j^S | m \rangle - \langle m | j^S | n \rangle \langle n | j^S | m \rangle}{(E_n - E_m)^2},$$  \hspace{1cm} (10)

where $A = L^2$ is the area of the system. $|m\rangle$, $|n\rangle$ are quasiparticle eigenstates of the Hamiltonian [Eq. (1)] and $j^S_x$, $j^S_y$ the components of the spin current operator. Following Thouless and co-workers, we can show that the spin Hall conductance averaged over boundary conditions is related to a topological quantum number:

$$\langle \sigma_{xy}(m) \rangle = \frac{\hbar}{8\pi} \int \int d\theta_x d\theta_y \frac{1}{2\pi i} \left[ \frac{\partial m}{\partial \theta_y} \frac{\partial m}{\partial \theta_x} \right]$$

$$- \left[ \frac{\partial m}{\partial \theta_x} \frac{\partial m}{\partial \theta_y} \right],$$  \hspace{1cm} (11)

where $C_1(m)$ is an integer and known as the first Chern index. As is widely used in quantum Hall transitions and other contexts, $C_1(m)$ can be used to distinguish current carrying states from localized states unambiguously, even in finite-size systems, thus providing a powerful method to study the localization properties of the quasiparticle states.

An alternative way to study the localization properties of the states is to calculate the Thouless number (also known as the Thouless conductance) of the states at a given Fermi energy $E$, defined as

$$\gamma_T(E) = \frac{\langle |\delta E| \rangle}{\Delta E} \sim \frac{8\pi}{\hbar} \sigma_{xx}^S,$$  \hspace{1cm} (12)

where $\Delta E$ is the average energy level spacing at energy $E$, and $\langle |\delta E| \rangle$ is the average energy level shift caused by the change of the boundary condition from periodic to anti-periodic in one spatial direction. It was argued in the context of electron localization that $\gamma_T(E)$ is proportional to the longitudinal conductance of the system in the present context we expect it to provide a measure of the longitudinal spin conductance of the superconductor. Thouless numbers have also been numerically studied for the conventional integer quantum Hall effect, in both fully and projected lattice models.

In this work we carry out numerical calculations to diagonalize the Hamiltonian $H$ to obtain the exact quasiparticle eigen wave functions. We calculate their Chern and Thouless numbers to study their localization properties, and perform finite-size scaling analysis to extract critical behavior of the transitions driven by the change of the disorder strength $W$ or the Zeeman field.
III. NUMERICAL RESULTS

In the absence of the random potential we can diagonalize the Hamiltonian [Eq. (1)] in the momentum space, and the energy spectrum is

\[ E_k = \sqrt{\varepsilon_k^2 + |\Delta_k|^2}, \]

where \( \varepsilon_k = -2t(\cos k_x + \cos k_y) - \mu \) is the single-particle kinetic energy, and \( \Delta_k = 2i\Delta(\sin k_x + i\sin k_y) \) the p-wave pairing order parameter. From Eq. (13), we expect that there is an energy gap between two bands, which should be stable against weak disorder, while the gap will be closed when the disorder becomes strong enough.

In this work, we choose \( \Delta = 0.5 \) and the chemical potential \( \mu = 3.0 \) to avoid the van Hove singularity at zero energy in the single electron spectrum. To calculate the Chern number of each eigenstate, we evaluate the integral in Eq. (14) numerically over the boundary phase space \( 0 \leq \theta_x, \theta_y \leq 2\pi \). We divide the boundary phase space into \( M \times M \) square grids with \( M = 20-80 \), depending on the system size \( L = 10-40 \) to achieve desired precision. Figure 1 shows the density of states (DOS) (per lattice site and spin species) \( \rho(E) \) for a system with \( L = 10 \) and \( W = 4.0 \). For such a relatively weak disorder, the superconducting gap is still visible. Also shown is the spin Hall conductance \( \sigma_{xy}^S \) as a function of quasiparticle Fermi energy \( E = \mu_B B \), calculated by summing up Chern number of states below the Fermi energy. We find that \( \sigma_{xy}^S \) jumps from zero up by one unit near the (disorder-broadened) lower band edge, and jumps back to zero above the gap. Therefore, a plateau in \( \sigma_{xy}^S \) is well developed around \( E = 0 \), clearly indicating the existence of a spin quantum Hall phase. This phase with topological Chern number equal to one is the simplest possible spin quantum Hall phase for non-interacting quasiparticles; it is simpler, for example, than the corresponding phase of an \( d_{x^2-y^2} + id_{xy} \) superconductor, which carries a total Chern number two.

In the following discussion, we focus on cases with disorder strong enough to close the gap, and look for transitions from the spin quantum Hall phase to other possible phases, driven by either the disorder strength \( W \) or the quasiparticle Fermi energy. In Fig. 2 we plot the total DOS \( \rho(E) \) (which is roughly system size independent) and the density of current carrying states (defined as states with non-zero Chern number) \( \rho_c(E) \) for systems with \( L = 10-40 \). We find that \( \rho_c(E) \) has a weak double-peak structure near \( E = 0 \) for large \( L \), whose width shrinks as \( L \) increases. This behavior is reminiscent of those seen in the numerical study of current carrying states in the integer quantum Hall effect,28,29 where the current carrying states exist only at discrete critical energies in the thermodynamic limit and, thus, the width of \( \rho_c(E) \) peak(s) shrinks to zero as \( L \) increases toward infinity. In the present case the two peaks correspond to two such critical energies, carrying a total Chern number \( +1 \) and \( -1 \), respectively, which are responsible for the spin quantum Hall plateau when the Fermi energy is between them (so that only the lower critical energy is below the Fermi energy). According to the scaling theory of localization, \( \rho_c(E) \) depends on \( L \) only through a dimensionless ratio \( L/\xi(E) \) when the system size becomes sufficiently large; the localization length \( \xi \) diverges in the vicinity of a critical energy \( E_c \) as \( \xi(E) \sim |E - E_c|^{-\nu} \). Therefore, the number of current carrying states \( N_c(L) \) behaves as

\[ N_c(L) = 2L^2 \int_{-\infty}^{\infty} \rho_c(E) dE \sim L^{2-1/\nu}, \]

from which we can estimate \( \nu \). Assuming we have a simi-
Two methods are complementary to each other. Figure 4 shows the density of states \( \rho(E) \) and Thouless number \( g_T(E) \) for systems with \( L = 40-80 \) and \( W = 8.0 \). This situation here, we plot \( N_c(L) \), normalized by the total number of states \( N(L) = 2L^2 \), on a log-log scale in Fig. 3. Just as in the quantum Hall case\(^{28,29} \), we can fit the data to a power law (a straight line in the log-log plot) as in Eq. (14) reasonably well, and obtain
\[
\nu = 2.6 \pm 0.2.
\]

This is close to the corresponding exponent \( \nu = 2.3 \pm 0.1 \) for the integer quantum Hall transition. These results suggest that just as in the case of the integer quantum Hall effect, current carrying states exist at discrete critical energies in the thermodynamic limit, and the spin quantum Hall transition driven by the Zeeman field (or equivalently, the quasiparticle Fermi energy) has the same critical behavior as the integer quantum Hall transition. This is expected on the symmetry ground, because in this case the critical energies are away from \( E = 0 \), and thus can only be reached in the presence of the Zeeman field. As discussed earlier, the Zeeman field breaks the symmetry of Eq. (4) and reduces the symmetry of the present problem to that of electrons moving in a magnetic field and a random potential.

While the Chern numbers measure the ability of individual states to carry spin Hall current, we have also calculated the Thouless conductance \( g_T(E) \), which is a measure of the longitudinal spin conductance. Unlike the Chern number calculation which requires the diagonalization of the Hamiltonian for many different boundary conditions, the Thouless number calculation only needs the diagonalization at two different boundary conditions, thus allowing us to study larger systems. On the other hand, it is known in the numerical study of quantum Hall effect that Chern number calculation reaches the scaling behavior at smaller system sizes. Therefore, these two methods are complementary to each other. Figure 3 shows \( \rho(E) \) and \( g_T(E) \) for systems with \( L = 40-80 \), and with \( W = 8.0 \). We find that \( g_T(E) \) has a similar double-peak structure as \( \rho_e(E) \) with peaks locating at the same energies, and that the peaks become narrower as \( L \) increases. In the following we perform the similar scaling analysis based on the zeroth moment of \( g_T(E) \) as for the Chern numbers. Namely, we compute the area \( A(L) \) under \( g_T(E) \) and expect
\[
A(L) = \int_{-\infty}^{\infty} g_T(E) dE \sim L^{-1/\nu}.
\]

One slight complication is that unlike \( \rho_e(E) \), \( g_T(E) \) has long tails extending to the edges of \( \rho(E) \), which clearly has no connection to the critical behavior near the critical energies. To eliminate the influence of these artificial tails, we introduce a cutoff energy \( E_{\text{cut}} \), and exclude contributions from \( |E| > E_{\text{cut}} \). Based on the Chern number calculation above (Fig. 2), as well as the \( g_T(E) \) curves themselves, we can safely choose \( E_{\text{cut}} \) between 3.0 and 4.0, beyond which we find essentially no current carrying states for \( L \geq 40 \). In Fig. 4, we plot, on a log-log scale, the area \( A(L) \) normalized by the area under the DOS curve between \(-E_{\text{cut}} \) and \( E_{\text{cut}} \):
\[
N_{\text{cut}} = \int_{-E_{\text{cut}}}^{E_{\text{cut}}} \rho(E; L) dE
\]
for a series of different \( E_{\text{cut}} \), and list the corresponding \( \nu \) in Table I. We find that \( \nu \) has very weak dependence on the choice of the cutoff energy and its variation between 2.54 and 2.79 is consistent with the results obtained from the Chern number calculation.

We also studied other disorder strengths. In the case of the integer quantum Hall transition\(^{29} \), it is known that as the disorder strength increases, the critical energies that carry opposite Chern numbers move close together,
merge, and disappear at some critical disorder strength $W_c$. In the present case, we expect the same to happen and due to the symmetry of the Hamiltonian, the critical energies can only merge at $E = 0$. We present the results for $W = 9.0$ in Fig. 5. In this case we no longer see two split critical energies, suggesting that the two critical energies that were clearly distinguishable at $W = 8.0$ either (i) have moved too close to be distinguishable at the accessible system sizes, or (ii) have just merged. We believe scenario (i) is much more likely than (ii) based on the following observations. (a) We find that the peak value of $g_T(E)$ is independent of system size and takes the same value as that of $W = 8.0$. (b) We have performed the same scaling analysis of $g_T(E)$ as we did above for $W = 8.0$ and obtained a similar exponent $\nu \approx 2.3$ (see inset of Fig. 5), which is even closer to the known value of the integer quantum Hall transition. However, there is another possibility that instead of entering the insulating phase (in which all quasiparticles states are localized) immediately, the system is in a metallic phase, after the two critical energies merge so that the system is no longer in the spin quantum Hall phase. Senthil and Fisher suggested that in this phase both the DOS $\rho(E)$ and the conductance diverge logarithmically at the band center. Interestingly, we indeed find $\rho(E)$ to be enhanced at $E = 0$. We believe, however, this is not associated to the metallic phase for the following reasons. (i) No such enhancement is seen in the Thouless number, which is a measure of the longitudinal conductance. (ii) We find $\rho(E)$ to be essentially system size independent between $L = 40$ and $L = 80$, even at $E = 0$, while one expects $\rho(L) \sim \log L$ in the metallic phase. (iii) We find that (see below) the enhancement of $\rho(E)$ at $E = 0$ is also present at stronger disorder when the system is clearly insulating. Thus it appears unlikely that the metallic phase is responsible for the single peak in $g_T(E)$.

The situation is quite different as $W$ further increases. In Fig. 5 we present results for $W = 10.0$ and see a very different behavior. Here the peak value of $g_T(E)$ systematically decreases as the system size increases, exhibiting a characteristic insulating behavior. Combined with results of smaller $W$, we conclude that in the absence of the Zeeman field (or when the quasiparticle Fermi energy is at $E = 0$), the system is driven into the insulating phase from the spin quantum Hall phase as the disorder strength $W$ increases. The critical strength $W_c$ is slightly above 9.0 and clearly below 10.0. No evidence has been found for the existence of an intermediate metallic phase that separates these two phases for our choice of model parameters ($\mu = 3.0$, $\Delta = 0.5$, etc.).

The critical behavior of the transition driven by increasing $W$ is expected to be different from the one driven by changing the Zeeman field discussed above, due to the additional symmetry. In order to study the critical property one first needs to determine the critical disorder strength $W_c$ accurately, which we are unable to do within the accessible system size in our study. It would be of significant interest to study this transition with more powerful computers and/or other computational methods.

We give the results of $W = 15.0$ in Fig. 5 as an example of strong disorder, where all states are clearly localized. Here, the Thouless number drops rapidly as the system size increases as expected. Interestingly, the enhancement of the DOS at $E = 0$ remains to be quite pronounced, suggesting that it is not associated with possible metallic behavior discussed above. For comparison, we have also calculated the DOS for a $d_{x^2-y^2} + id_{xy}$ superconductor, by choosing the pairing order parameter to be $\Delta_{j,i+y+e} = -\Delta_{j,j+y+e} = \Delta_{x^2-y^2}$, $\Delta_{j,j+i+e+y} = -\Delta_{j,j+i+e-y} = i\Delta_{xy}$. In the $d$-wave superconductor the total spin of the system is conserved, due to the singlet nature of the pairing. As a consequence it belongs to the symmetry class C in the classification by Altland and Zirnbauer This model has been studied in considerable detail in Refs. 12–15. We plot the DOS for a $d_{x^2-y^2} + id_{xy}$ superconductor for different values of $W$ in Fig. 5. While the gap vanishes just like the $p$-wave case for sufficiently large $W$, the DOS exhibits a pseudogap behavior at $E = 0$ for large $W$, in the vicinity of which the DOS vanishes in an (apparently) sublinear power law as predicted. 13,14 This is a good example that the change of symmetry profoundly affects the critical behavior as well as other properties of the system.

### Table I: Critical exponent $\nu$ for different cutoff energy $E_c$ with $W = 8.0$

| $E_c$  | $\nu$  | $\delta\nu$ |
|-------|--------|-------------|
| 3.0   | 2.79   | 0.08        |
| 3.2   | 2.73   | 0.08        |
| 3.4   | 2.68   | 0.08        |
| 3.6   | 2.64   | 0.07        |
| 3.8   | 2.58   | 0.07        |
| 4.0   | 2.54   | 0.07        |

![FIG. 5: Area $A(L)$ of Thouless number $g_T(E)$ normalized by number of states $N_{cut}$ counted, versus system size $L$ on a log-log scale for different cutoff energy $E_c$ and $W = 8.0$.](image-url)
FIG. 6: Density of states $\rho(E)$ and Thouless number $g_T(E)$ for systems with $L = 40-80$ and $W = 9.0$. The inset shows the area of Thouless number $g_T(E)$ divided by $N_{\text{cut}}$ for $E_{\text{cut}} = 4.0$.

FIG. 7: Density of states $\rho(E)$ and Thouless number $g_T(E)$ for systems with $L = 40-80$ and $W = 10.0$. The inset is a blow-up of the Thouless number curves near $E = 0$, which shows that $g_T(E = 0)$ decreases with increasing $L$.

FIG. 8: Density of states $\rho(E)$ and Thouless number $g_T(E)$ for systems with $L = 20-40$ and $W = 15.0$.

FIG. 9: Density of states of $d_{x^2-y^2} + id_{xy}$ superconductor with $L = 60$, $\Delta_{x^2-y^2} = 1$, $\Delta_{xy} = 0.6$. We average over 80 samples of different random potential realizations.

IV. DISCUSSION AND SUMMARY

In this paper we have studied the localization properties of the quasiparticle states in superconductors with spontaneously broken time-reversal symmetry, which support spin quantum Hall phases. Our study is based on the exact diagonalization of microscopic lattice models and the consequent numerical calculation of the Chern and Thouless numbers of the quasiparticle states. Our microscopic study is complementary to previous numerical work on this subject, which have been based almost exclusively on effective network models with appropriate symmetries.

We have focused mostly on a $p$-wave pairing model in which the time-reversal symmetry is broken by the (complex) pairing order parameter, while the $z$-component of the total spin is conserved so that the transport properties of the $z$-component of the spin is well defined. We find the system supports a spin quantum Hall phase with spin Hall conductance one in appropriate unit, and an insulating phase. Transitions between these two phases may be induced either by changing the disorder strength, or by applying and sweeping a Zeeman field. The field-driven transition is found to have the same critical behavior as the integer quantum Hall transition of non-interacting electrons as expected on symmetry grounds.
The disorder-driven transition in the absence of the Zeeman field is expected to have different critical properties due to additional symmetry of the Hamiltonian. However, we have not been able to study the critical behavior of this transition.

The symmetry properties of the p-wave pairing model in the absence of the Zeeman field belongs to class D in the classification of general fermion pairing models of Altland and Zirnbauer. It has been suggested that in addition to the quantum Hall and the insulating phases, class D models may also support a metallic phase,

which has logarithmically divergent density of states and conductance. Such a system can have either a direct transition between the quantum Hall and the insulating phases, or a metallic phase separating these two phases. In our model we find a direct transition between the spin quantum Hall and insulating phases, but no definitive evidence for a metallic phase. This is not unusual as it is known that specific microscopic models may or may not support the metallic phase.

For comparison, we have also calculated the density of states of a d-wave superconductor with $d_{x^2-y^2} + id_{xy}$ pairing order parameter, which supports a spin quantum Hall phase with spin Hall conductance two in the same unit. This model has different symmetry properties and belongs to class C in the classification of Altland and Zirnbauer. We find that the density of states vanishes with sublinear power law near $E = 0$, in agreement with earlier studies.

This is in sharp contrast to the p-wave case in which we observe an enhanced density of states at $E = 0$ for sufficiently strong disorder, demonstrating the profound effect of symmetries on the low-energy properties of the system. While this enhancement is somewhat reminiscent of the divergent density of states of the possible metallic phase, further analysis suggests this is not the case. The origin of this enhancement is currently unclear.

Finally we note that recently there is interest in the spin Hall effect in semiconductors with spin-orbit coupling, which is driven by an electric field. Physically this effect is quite different from the spin Hall effect discussed here, in the following ways. (i) Our spin Hall effect is induced by the gradient of a Zeeman field that couples to spin, while the other effect is induced by an electric field that couples to charge. (ii) The existence of the spin Hall effect in our case relies on the broken time-reversal symmetry in the pairing Hamiltonian, while the time-reversal symmetry is intact in the Hamiltonians used in Refs. [13,16]. Instead the spin-Hall effect is present due to the presence of spin-orbit coupling. In Ref. [16] a universal spin Hall conductance was found in a clean system; it is not clear at present if this value has a topological origin as in our case, and how stable this result is in the presence of disorder.

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