Is there a map between Galilean relativity and special relativity?

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(Dated: January 28, 2014)

Mandanici in [1] has provided a map which he claims to be a two way map between Galilean relativity and special relativity. We argue that this map is simply a curvilinear coordinate system on a subset of the two-dimensional Minkowski space-time, and is not a two way map between 1+1 dimensional Galilean relativity and 1+1 dimensional special relativity.

PACS numbers: 03.30.+p, 02.20.-a

Keywords: Special Relativity, Doubly Special Relativity, Lorentz Group

I. INTRODUCTION

In concise mathematical terms, special relativity means covariance under the action of the Poincaré group while Galilean relativity means covariance under the action of the full Galilean group. The space-time of special relativity is the 3 + 1-dimensional Minkowski spacetime, \(\mathbb{M}^4 \sim \mathbb{R}^4\), which is a manifold on which a semi-Riemannian metric \(ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2\) is defined [see for example 2, p. 118]. The space-time of the Galilean relativity is the 3 + 1-dimensional Galilean space-time, \(\mathbb{G}^4 \sim \mathbb{R}^4\), which is an affine space, with an absolute time function [see 3, pp. 3-6].

The symmetry group of the \((n+1)\)-dimensional Minkowski space-time, \(\mathbb{M}^{n+1}\), is the Poincaré group \(\text{Poi}(n+1)\), consisting of translations, rotations, and Lorentz transformations. Rotations and Lorentz transformations form a group, \(\text{O}(n,1)\). The subgroup \(\text{SO}^+(n,1)\), consisting of special, orthochronous transformations, is denoted by \(\text{Lor}(n+1)\). The symmetry group of the \((n+1)\)-dimensional Galilean space-time, \(\mathbb{G}^{n+1}\), is the ‘Newtonian’ group \(\text{Newt}(n+1)\), consisting of translations, rotations, and Galilean transformations. The special orthochronous sub-group of the rotations and the Galilean transformations is the Galilean group, which we denote by \(\text{Gal}(n+1)\).

As groups, \(\text{Lor}(2)\) and \(\text{Gal}(2)\) are both isomorphic to \((\mathbb{R},+)\). In higher dimensions the situation is different. For \(n > 1\) the groups \(\text{Lor}(n+1)\) and \(\text{Gal}(n+1)\) are both \(\frac{1}{2} n(n+1)\)-dimensional, but they are not homomorphic, because their Lie algebras are not homomorphic. The groups \(\text{Poi}(n+1)\) and \(\text{Lor}(n+1)\) depend on a parameter, \(c\). It can be shown that the \(c \to \infty\) limit of these groups are, respectively \(\text{Newt}(n+1)\) and \(\text{Gal}(n+1)\)—the Inönü-Wigner contraction [4]. It is also important to notice that the symmetry algebra of a non-relativistic quantum system, is not the Lie algebra of \(\text{Newt}(n+1)\), but a larger central extension of that—for a textbook introduction to Inönü-Wigner contraction and the emergence of this central extension see [5, pp. 61-62].

II. THE MANDANICI’S CLAIM

In a recent article, G. Mandanici introduced a map, which he claims to be a two way map between the special and the Galilean relativities [1].

The Mandanici map. Consider the two-dimensional Minkowski space-time \(\mathbb{M}^2\) with Cartesian coordinates \((T, X)\) and metric \(ds^2 = c^2 T^2 - dX^2\). Now consider curvilinear coordinates \((t, x)\) defined by the following formulas:

\[
X = \alpha c t \sinh \left( \frac{x}{c t} \right) + \beta c t \cosh \left( \frac{x}{c t} \right),
\]

\[
T = \alpha t \cosh \left( \frac{x}{c t} \right) + \beta t \sinh \left( \frac{x}{c t} \right),
\]

where \(\alpha\) and \(\beta\) are dimensionless parameters. Calculating the Jacobian is straightforward:

\[
J = \frac{\partial T}{\partial t} \cdot \frac{\partial X}{\partial x} - \frac{\partial T}{\partial x} \cdot \frac{\partial X}{\partial t} = \alpha^2 - \beta^2,
\]

from which it follows that this transformation is invertible, only if \(\alpha \neq \beta\). Consider the special case of \(\alpha = 1, \beta = 0\), which means

\[
X = c t \sinh \left( \frac{x}{c t} \right), \quad T = t \cosh \left( \frac{x}{c t} \right).
\]

Obviously \(c^2 T^2 - X^2 = c^2 t^2\) and \(X/c T = \tanh(x/c t)\). From these two equations, it is obvious that this transformation \((\alpha = 1, \beta = 0)\) is defined only for the region \(c^2 T^2 \geq X^2\) of the 2D Minkowski space-time.

On \(\mathbb{M}^2\), the vector field \(K = c^{-1} X \partial T + c T \partial X\) is a Killing vector field, which generates Lorentz boosts. (A Killing vector field is a vector field which generates an isometry of the space-time, that is, a symmetry of the space-time.) Now let’s see what is the vector field \(\partial_x\). It is easy to see that

\[
\frac{\partial}{\partial x} = \frac{\partial T}{\partial x} \frac{\partial}{\partial T} + \frac{\partial X}{\partial x} \frac{\partial}{\partial X}
\]

\[
= \frac{1}{c} \sinh \left( \frac{x}{c t} \right) \frac{\partial}{\partial T} + \cosh \left( \frac{x}{c t} \right) \frac{\partial}{\partial X}
\]

\[
\partial_x = \frac{1}{c t} \left( c^{-1} X \partial T + c T \partial X \right)
\]
from which we see that
\[
ct \partial_x = c^{-1} X \partial_T + cT \partial_X = K. \tag{6}
\]

In words: The vector field \( K \), in terms of the curvilinear coordinates \((t, x)\) reads \(ct \partial_x\). The transformation this vector field generates, in terms of the curvilinear coordinates \((t, x)\), is \((t, x) \mapsto (t, x + \theta c t)\), where \(\theta\) is the dimensionless parameter of the flow. Defining \(v := c \theta\), this is the famous Galilean transformation. This does not mean that transformation (4) transforms the 2D Minkowski space-time into the 2D Galilean space-time! What is given by the map (4), is a change of coordinates in a subset of \(\mathbb{M}^2\). On this subset—the interior of the light-cone—the Killing vector field \( K \), in terms of the curvilinear coordinates \((t, x)\) has the form \(K = ct \partial_x\), which is the generating vector field of the Galilean transformations. However, this does not mean that we have found a map from the Newtonian group \(\text{Newt}(2)\) to the Poincaré group \(\text{Poi}(2)\)?

Remembering that both \(\text{Gal}(2)\) and \(\text{Lor}(2)\) are isomorphic to \((\mathbb{R}, +)\), this is not strange. In fact, according to the straightening-out theorem, any smooth vector field, in the neighborhood of a point at which the vector field is non-zero, could be written as \(\partial_t\), for a suitably chosen coordinate system \((x^1, x^2)\) [see for example 6, p. 177], which means that if we wish, we could find a map such that the Lorentz boosts in \(\mathbb{M}^2\) have the form of translations—locally, of course. However, besides \(K\), the Minkowski space-time \(\mathbb{M}^2\) has two other Killing vector fields, \(\partial_T\), and \(\partial_X\), which are generators of translations in time and space. Transformation (4) does not map these two vector fields to \(\partial_t\) and \(\partial_x\).

**Minor comments.** The notation of Mandanici’s article is misleading. The infinitesimal action of a vector field on the manifold is \(x^\mu \mapsto x^\mu + \delta x^\mu\). Writing it as \(\Delta x^\mu - \Delta x^\nu + \delta \Delta x^\nu\) caused the author to consider it, not as a flow on the manifold, but as a map of the tangent space (the velocity space), which is not correct—see equations (1) and (2) of [1]. Due to this confusion, the author confused \(X/cT\) and \(x/ct\), with velocities (in his notation with \(v_E\) and \(v_G\)). But these ratios are not velocities—velocities of what object should they be? The relation \(X/cT = \tanh (x/ct)\), which the author writes as \(v_E = \tanh v_G\), simply shows that the map (4) is defined on the closure of the interior of the light cone, \(i.e., c^2 T^2 - X^2 \geq 0\).

\(\text{Newt}(n + 1)\) is not isomorphic to \(\text{Poi}(n + 1)\), because, as Lie groups, the corresponding Lie algebras are not homomorphic. Therefore, from the onset, we know that one cannot find a two way map sending a subset of \(\mathbb{M}^{n+1}\) to a subset of \(\mathbb{G}^{n+1}\), mapping the corresponding structures.
2. Because the Lie algebra of the full Galilean group and the Poincaré group are not homomorphic, one cannot find a coordinate system on $\mathbb{M}^4$ such that the Killing vector fields are transformed to the vector fields generating the full Galilean group of transformations.

ACKNOWLEDGMENTS

This work was supported by the research council of Alzahra University.

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