FPT Algorithms to Compute the Elimination Distance to Bipartite Graphs and More*

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Abstract. For a hereditary graph class \( \mathcal{H} \), the \( \mathcal{H} \)-elimination distance of a graph \( G \) is the minimum number of rounds needed to reduce \( G \) to a member of \( \mathcal{H} \) by removing one vertex from each connected component in each round. The \( \mathcal{H} \)-treewidth of a graph \( G \) is the minimum, taken over all vertex sets \( X \) for which each connected component of \( G - X \) belongs to \( \mathcal{H} \), of the treewidth of the graph obtained from \( G \) by replacing the neighborhood of each component of \( G - X \) by a clique and then removing \( V(G) \setminus X \). These parameterizations recently attracted interest because they are simultaneously smaller than the graph-complexity measures treedepth and treewidth, respectively, and the vertex-deletion distance to \( \mathcal{H} \).

For the class \( \mathcal{H} \) of bipartite graphs, we present non-uniform fixed-parameter tractable algorithms for testing whether the \( \mathcal{H} \)-elimination distance or \( \mathcal{H} \)-treewidth of a graph is at most \( k \). Along the way, we also provide such algorithms for all graph classes \( \mathcal{H} \) defined by a finite set of forbidden induced subgraphs.

1 Introduction

Background Assuming some structure on the input of a computational problem can greatly decrease its difficulty. For instance, it is well known that many NP-hard graph problems can be computed efficiently on graphs of bounded treewidth using dynamic programming over so-called tree decompositions \cite{1}. The analysis of computational problems in terms of the input size and an additional parameter such as treewidth is the main objective in the field of parameterized complexity \cite{10,11}. A parameter similar to treewidth is treedepth \cite{26,§6.4}. It can be defined as the minimum number of rounds needed to get to the empty graph, where in each round we can delete one vertex from each connected component (formal definitions in the preliminaries). Some NP-hard graph problems become solvable in polynomial time if the input graph is restricted to be in a certain class. For instance the NP-hard VERTEX COVER can be solved in polynomial time in chordal graphs; those graphs without induced cycles of length at least four. A parameter that naturally follows from this observation is the minimum cardinality of a set of vertices whose deletion results in a graph contained in graph class \( \mathcal{H} \). Such a set is called an \( \mathcal{H} \)-deletion set. This parameter essentially indicates how far the problem is from being a trivial case (cf. \cite{18}). The size of a feedback vertex set \cite{22,20} or vertex cover number \cite{13,14} of the graph are often used examples of such parameters, where \( \mathcal{H} \) is the class of forests and edgeless graphs respectively.

Recently there has been a push \cite{12,16,17} in obtaining parameterized algorithm where the parameter is a hybrid of some overall structure of the graph, like treewidth and treedepth, and some distance to triviality. One such example introduced by Bulian and Dawar is \( \mathcal{H} \)-elimination distance (\( \text{ed}_{\mathcal{H}} \)) \cite{6,7}, which can be defined as the minimum number of deletion rounds needed to obtain a graph in \( \mathcal{H} \) by removing one vertex from each connected component in each round; recall that in the elimination-based definition of treedepth, the goal is to eliminate the entire graph. Hence \( \text{ed}_{\mathcal{H}} \) is never larger than the treedepth or the (vertex-)deletion distance to \( \mathcal{H} \). Bulian and Dawar showed that \( \text{ed}_{\mathcal{H}} \) can be computed in FPT time when \( \mathcal{H} \) is minor-closed \cite{7}.

A related hybrid variant of treewidth was introduced by Eiben et al. \cite{12}, namely \( \mathcal{H} \)-treewidth (\( \text{tw}_{\mathcal{H}} \)). The \( \mathcal{H} \)-treewidth of a graph can be defined as the minimum treewidth of the torso graph of a vertex set whose removal ensures each component belongs to \( \mathcal{H} \). This gives rise to tree decompositions in which each bag has

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size at most \( k + 1 \), apart for an arbitrarily large set of vertices that occurs in no other bags and induces a subgraph from \( \mathcal{H} \). Similarly as before, \( \text{tw}_H(G) \) is not larger than \( \text{tw}(G) \) or the deletion distance from \( G \) to \( \mathcal{H} \). For minor-closed graph classes \( \mathcal{H} \) it can be shown that graphs of \( \mathcal{H} \)-treewidth at most \( k \) are minor-closed and therefore characterized by a finite set of forbidden minors. This leads to non-uniform algorithms to recognize graphs of \( \mathcal{H} \)-treewidth at most \( k \) for minor-closed \( \mathcal{H} \) using the Graph Minor algorithm \cite{fomin2010graph}.

Apart from minor-closed families \( \mathcal{H} \), some isolated results are known about FPT algorithms to compute \( \text{ed}_H \) and \( \text{tw}_H \) exactly, parameterized by the parameter value. In recent work, Agrawal and Ramamurthy \cite{agrawal2018vertex} give an FPT algorithm to compute the elimination distance to a cluster graph, as part of a kernelization result using the corresponding structural parameterization. Eiben et al. \cite{eiben2017parameterizations} show that when \( \mathcal{H} \) is the class of graphs of rankwidth at most \( c \) for some constant \( c \), then \( \text{tw}_H \) is FPT. Bulian and Dawar \cite{bulian2017approximating} considered the elimination distance to graphs of bounded degree \( d \) and gave an FPT approximation algorithm. Lindermayr et al. \cite{lindermayr2018approximating} showed that the elimination distance of a planar graph to a bounded-degree graph can be computed in FPT time. Very recently, Agrawal et al. \cite{agrawal2018multi} obtained non-uniform FPT algorithms for computing the elimination distance to any family \( \mathcal{H} \) defined by a finite number of forbidden induced subgraphs, thereby settling the case of bounded-degree graphs as well.

**Results and techniques** We show that \( \text{tw}_H \) and \( \text{ed}_H \) are non-uniformly fixed parameter tractable parameterized by the solution value when \( \mathcal{H} \) is the class of bipartite graphs. As a side-product of our proof, we show that \( \text{tw}_H \) is non-uniformly FPT when \( \mathcal{H} \) is defined by a finite number of forbidden induced subgraphs, generalizing the results of Agrawal et al. \cite{agrawal2018multi} for \( \text{ed}_H \). The non-uniformity of our algorithms stems from the use of a meta-theorem by Lokshatanov et al. \cite{lokshatanov2011parameterized} which encapsulates the technique of recursive understanding. This theorem essentially states that for any problem expressible in Counting Monadic Second Order (CMSO) logic, the effort of classifying whether the problem is in FPT is reduced to inputs that are \((s, c)\)-unbreakable (formally defined later). The theorem allows us to use the technique of recursive understanding in a black box matter, leading to a streamlined proof at the expense of obtaining non-uniform algorithms. We believe that uniform algorithms can be obtained using the same approach by implementing the recursive understanding step from scratch and deriving an explicit bound on the sizes of representatives for the canonical congruence for \( \text{ed}_H \) and \( \text{tw}_H \) on \( t \)-boundaryed graphs. As the running times would not be practical in any case, we did not pursue this route.

Our proof is independent of that of Agrawal et al. \cite{agrawal2018multi}, but is based on an older approach inspired by the earlier work of Ganian et al. \cite{ganian2015comparative} that contains similar ideas. The key ingredient for our work is the insight that the approach based on recursive understanding used by Ganian et al. \cite{ganian2015comparative} to compute a hybrid parameterization for instances of constraint satisfaction problems, can be applied more generally to aid in the computation of \( \text{ed}_H \) and \( \text{tw}_H \). We can lift one of their main lemmas to a more general setting, where it roughly shows that given a \((s(k), 2k)\)-unbreakable graph \( G \) (definitions in Section 2) and a deletion set \( X \) from \( G \) to \( \mathcal{H} \) that is a subset of some (unknown) structure that witnesses the value of \( \text{tw}_H \) or \( \text{ed}_H \), we can determine in FPT time whether such a witness exists. This allows \( \text{ed}_H \) and \( \text{tw}_H \) to be computed in FPT time if we can efficiently find a deletion set with the stated property. For families \( \mathcal{H} \) defined by finitely many forbidden induced subgraphs, a simple bounded-depth branching algorithm suffices. Our main contribution is for bipartite graphs, where we show that the relation between odd cycle transversals and graph separators that lies at the heart of the iterative compression algorithm for OCT \cite{fomin2010graph}, can be combined with the fact that there are only few minimal \((u, v)\)-separators of size at most \( 2k \) in \((s(k), 2k)\)-unbreakable graphs, to obtain an \( \mathcal{H} \)-deletion set with the crucial property described above.

**Related work** Hols et al. \cite{hols2018vertex} used parameterizations based on elimination distance to obtain kernelization algorithms for Vertex Cover.

In recent work \cite{seppala2018approximate}, a superset of the authors gave FPT algorithms to approximate \( \text{ed}_H \) and \( \text{tw}_H \) for several classes \( \mathcal{H} \), including bipartite graphs and all classes defined by a finite set of forbidden induced subgraphs. That work employed completely different techniques than used here, and left open the question whether the parameters can be computed exactly in FPT time.

## 2 Preliminaries

We consider simple undirected graphs without self-loops. The vertex and edge set of a graph \( G \) are denoted by \( V(G) \) and \( E(G) \) respectively. When the graph is clear from context, we denote \(|V(G)|\) by \( n \) and \(|E(G)|\) by
m. For each \( X \subseteq V(G) \), the graph induced by \( X \) is denoted by \( G[X] \). We denote \( G[V(G) \setminus X] \) by \( G - X \), and write \( G - v \) instead of \( G - \{v\} \). The open and closed neighborhoods of \( v \in V(G) \) are denoted \( N_G(v) \) and \( N_G[v] \) respectively. For \( X \subseteq V(G) \), \( N_G[X] = \bigcup_{v \in X} N_G[v] \) and \( N_G(X) = N_G[X] \setminus X \). The subscript \( G \) is omitted if it is clear from context. The graph obtained from \( G \) by contracting an edge \( e = \{u, v\} \in E(G) \) is the graph obtained by deleting \( u \) and \( v \) and inserting a new vertex that is adjacent to all of \( (N_G(u) \cup N_G(v)) \setminus \{u, v\} \).

A graph \( H \) is a minor of \( G \), if it can be obtained from a subgraph of \( G \) by a number of edge contractions. A parameter is a function that assigns an integer to each graph. A parameter \( f \) is minor-closed if \( f(H) \leq f(G) \) for each minor \( H \) of \( G \). The connected components of \( G \) are denoted by \( cc(G) \). A set \( \Pi \subseteq E(G) \) is an \( \Pi \)-deletion set if \( \forall x, k \in \Pi \) such that for every \( \{u, v\} \in E(G) \) it holds that \( f(u) \neq f(v) \). A graph is bipartite if and only if it has a proper 2-coloring. For sets \( X, Y \subseteq V(G) \), we say that \( S \subseteq V(G) \) is an \( (X, Y) \)-separator if the graph \( G - S \) does not contain a vertex \( u \in X \setminus S \) and \( v \in Y \setminus S \) in the same connected component.

A parameterized problem \( H \) is a subset \( \Sigma^* \times \mathbb{N} \) for some finite alphabet \( \Sigma \). A parameterized problem is non-uniformly fixed-parameter tractable (FPT) if there exists a fixed \( d \) such that for every fixed \( k \in \mathbb{N} \), there exists an algorithm that determines whether \((x, k) \in H \) in \( O(|x|^d) \) time. (Hence there is a different algorithm for each value of \( k \).)

### 2.1 Treewidth

**Definition 1.** A tree decomposition of a graph \( G \) is a pair \((T, \{X_t\}_{t \in V(T)})\), where \( T \) is a tree and each \( t \in V(T) \) is assigned a vertex subset \( X_t \subseteq V(G) \), such that the following holds:

1. For every \( \{u, v\} \in E(G) \), there exists \( t \in V(T) \) with \( \{u, v\} \subseteq X_t \).
2. \( \bigcup_{t \in V(T)} X_t = V(G) \).
3. For every \( v \in V(G) \), the set \( T_v = \{ t \in V(T) \mid u \in X_t \} \) induces a connected subtree of \( T \).

The width of tree decomposition \((T, \{X_t\}_{t \in V(T)})\) equals \( \max_{t \in V(T)} |X_t| - 1 \). The treewidth of a graph, denoted \( tw(G) \), is the minimum possible width over all possible tree decompositions of \( G \).

### 2.2 \( \mathcal{H} \)-treewidth and \( \mathcal{H} \)-elimination distance

**Definition 2.** [17] Let \( G \) be a graph and \( X \subseteq V(G) \). The torso of \( X \), denoted by \( T_G(X) \), is the graph obtained by turning the neighborhood of every connected component of \( G - X \) into a clique, followed by deleting all of \( V(G) \setminus X \).

Eiben et al. [12] use the term of collapsing \( V(G) \setminus X \) instead of the torso of \( X \). Since our algorithms try to identify \( X \), the torso terminology is more natural.

**Definition 3.** [12] The \( \mathcal{H} \)-treewidth of a graph \( G \) is the smallest integer \( k \) such that there exists a set \( X \subseteq V(G) \) with \( tw(T_G(X)) \leq k \) and for each connected component \( C \in cc(G - X) \) we have \( C \in \mathcal{H} \). We call \( X \) an \( tw_\mathcal{H} \) witness of width \( k \).

**Definition 4.** [17] The \( \mathcal{H} \)-elimination distance of \( G \) for a hereditary graph class \( \mathcal{H} \), denoted by \( ed_\mathcal{H}(G) \), is defined as:

\[
ed_\mathcal{H}(G) = \begin{cases} \max_{C \in cc(G)} ed_\mathcal{H}(C) & \text{if } G \text{ not connected} \\ 0 & \text{if } G \text{ connected and } G \in \mathcal{H} \\ 1 + \min_{v \in V(G)} ed_\mathcal{H}(G - v) & \text{otherwise} \end{cases} \]

The treedepth of a graph, denoted \( td(G) \), is equivalent to \( ed_\mathcal{H}(G) \) where \( \mathcal{H} \) only contains the empty graph.

Note that the definition above is well defined when \( \mathcal{H} \) is hereditary, since each hereditary graph class contains the empty graph. We argue that \( \mathcal{H} \)-elimination distance has an equivalent definition similar to that of \( \mathcal{H} \)-treewidth.
Proposition 1. A graph has $ed_H(G) \leq k$ if and only if there exists $X \subseteq V(G)$ such that $td(T_G(X)) \leq k$ and $C \in cc(G - X)$.

Proof. For the first direction, we prove by induction on $k$. A graph has $ed_H(G) \leq k$ if and only if there exists $X \subseteq V(G)$ such that $G - X \in H$ and $td(T_G(X)) \leq k$, then $ed_H(G) \leq k$.

For the base case $k = 0$, note that $td(T_G(X)) = 0$ implies that $X = \emptyset$, so that $G \in H$. By Definition 4 we have $ed_H(G) = 0 \leq k$.

For the induction step we have $k > 0$. To show that $ed_H(G) \leq k$, by Definition 3 it suffices to prove that each $C \in cc(G)$ satisfies $ed_H(C) \leq k$. Let $X_C := C \cap X$. If $X_C = \emptyset$ then $C \in H$ (since $H$ is hereditary) so $ed_H(C) = 0 \leq k$. In the remainder assume that $X_C \neq \emptyset$. Observe that $T_G(X) = T_G(X_C)$ as a connected component, and that $T_G(X_C)$ is connected since $C$ is connected and $X_C \subseteq C$. By definition of $td$ there exists a vertex $x \in X_C$ such that $td(T_G(X_C) - x) = td(T_G(X)) - 1$. Let $X_C' := X_C \setminus \{x\}$ and let $C' := C - x$. Note that $T_G(X_C') = T_G(X_C) - x$: it makes no difference whether we first turn the neighborhood of each component of $C - X_C$ into a clique, remove $C \setminus X_C$, and then remove $x$, or whether we start from $C' = C - x$, turn the neighborhood of each component of $C' - X_C$ into a clique, and then remove $C' \setminus X_C'$. By induction on $C'$ and $X_C'$, with $k' := td(T_C(X_C')) \leq k$, it follows that $ed_H(C') \leq td(T_C(X_C')) = td(T_G(X)) - 1 \leq td(T_G(X)) - 1$, where the last inequality follows since $T_G(X_C)$ is a connected component of $T_G(X)$. By Definition 4, since $C$ is connected we have $ed_H(C) \leq 1 + \min_{v \in cc(G)} ed_H(G - v) \leq 1 + ed_H(C - x) \leq 1 + (td(T_G(X)) - 1) = td(T_G(X)) \leq k$, which completes this direction of the proof.

For the converse direction, we prove that if $ed_H(G) \leq k$ then $G$ has a vertex set $X$ such that $G - X \in H$ and $td(T_G(X)) \leq k$. We use an induction on $|V(G)|$. If $ed_H(G) = 0$ then by Definition 4 we have $G \in H$ so that $X = \emptyset$ suffices. For the induction step we have $ed_H(G) > 0$. We distinguish two cases, depending on the connectivity of $G$.

If $G$ is connected, then since $ed_H(G) > 0$ we have $G \notin H$. Hence by Definition 3 we have $ed_H(G) = 1 + \min_{v \in V(G)} ed_H(G - v)$. Let $x$ be a vertex for which equality is attained. Since $ed_H(G - x) = ed_H(G) - 1 < k$, by induction on $G' := G - x$ there exists a set $X' \subseteq V(G')$ such that $td(T_{G'}(X')) \leq k - 1$. Define $X := X' \cup \{x\}$ and note that $td(T_G(X)) \leq 1 + td(T_{G'}(X')) \leq 1 + (k - 1)$ since the graph $T_{G'}(X')$ can be obtained from $T_G(X)$ by removing the vertex $x$. Hence $td(T_G(X)) \leq k$, proving the claim.

Now suppose that $G$ is disconnected, so that $ed_H(G) = \max_{C \in cc(G)} ed_H(C)$. For each $C \in cc(G)$ we have that $|V(C)| < |V(G)|$ and $ed_H(C) \leq ed_H(G) = k$, so we may apply the induction hypothesis to $C$ to obtain a set $X_C \subseteq V(C)$ such that $td(T_G(X_C)) \leq ed_H(C) \leq k$. Let $X := \bigcup_{C \in cc(G)} X_C$. Observe that each connected component of the graph $T_G(X)$ is equal to $T_G(X_C)$ for some $C \in cc(G)$, so that each connected component $H$ of $T_G(X)$ satisfies $td(H) \leq td(T_G(X_C)) \leq k$ for some $C \in cc(G)$. By Definition 4 the fact that each component of $T_G(X)$ has treedepth at most $k$ ensures $td(T_G(X)) \leq k$, which concludes the proof.

Similar to $tw_H$ witnesses, we call $X$ an $ed_H$ witness of depth $k$. Since the torso operation on $X$ turns the neighborhood of each connected component of $G - X$ into a clique, the following note follows.

Note 1. If $X$ is a $tw_H$ witness of width $k - 1$ (respectively $ed_H$ witness of depth $k$), then $|N(C)| \leq k$ for every $C \in cc(G - X)$.

We are ready to introduce the main problem we try to solve.

\textbf{Parameter: $k$}

\textbf{Input:} A graph $G$, an integer $k$.

\textbf{Question:} Decide whether $tw_H(G) \leq k - 1$ / $ed_H(G) \leq k$.

Definition 5. \cite{23} Let $G$ be a graph and $s, c \in \mathbb{N}$. A partition $(X, C, Y)$ of $V(G)$ is an $(s, c)$-separation in $G$ if:

- $C$ is a separator, that is, no edge has one endpoint in $X$ and one in $Y$,
- $|X| \leq c$, $|X| \geq s$, and $|Y| \geq s$.

A graph $G$ is $(s, c)$-unbreakable if there is no $(s, c)$-separation in $G$.

The following proposition is similar to Lemma 21 of Ganian et al. \cite{17}.
Proposition 2. Let \( G \) be an \((s,c)\)-unbreakable graph for \( s, c \in \mathbb{N} \) and \( \mathcal{H} \) be a graph class such that \( \text{tw}_\mathcal{H}(G) \leq k - 1 \) (resp. \( \text{ed}_\mathcal{H}(G) \leq k \)) and \( c \geq k \). Then at least one of the following holds:

1. \( \text{tw}(G) \leq s + k - 1 \) (resp. \( \text{td}(G) \leq s + k - 1 \)),
2. each \( \text{tw}_\mathcal{H} \) (resp. \( \text{ed}_\mathcal{H} \)) witness \( X \) of \( G \) satisfies the following:
   - \( G - X \) has exactly one connected component \( C \) of size at most \( s \), and
   - \( |V(G) \setminus N[C]| \leq s \) and \( |X| \leq s + k - 1 \)

Proof. Consider an arbitrary witness \( X \). If all connected components of \( G - X \) have size at most \( s - 1 \), then \[\text{[1]}\] holds. Otherwise, let \( X \) be some component of \( G - X \) of size at least \( s \). First observe that \( |V(G) \setminus N[C]| \leq s \). Since for any connected component \( C' \) of \( G - X \) besides \( C \) it holds that \( |V(C') \setminus N[C]| < s \) too. Finally note that \( X \subseteq V(G) \setminus N[C] \) and hence \( |X| \leq s + k - 1 \) for any witness \( X \) and hence \[\text{[2]}\] holds.

The following lemma bounds the number of small connected vertex sets with a small neighborhood. It was originally stated for connected sets of exactly \( b \) vertices with an open neighborhood of exactly \( f \) vertices.

Lemma 1. [L3, cf. Lemma 3.1] Let \( G \) be a graph. For every \( v \in V(G) \) and \( b, f \geq 0 \), the number of connected vertex sets \( B \subseteq V(G) \) such that (a) \( v \in B \), (b) \( |B| \leq b + 1 \), and (c) \( |N(B)| \leq f \) is at most \( b \cdot f \cdot (b + f) \).

Furthermore they can be enumerated in \( O(n \cdot b^2 \cdot f \cdot (b + f) \cdot (b + f)) \) time using polynomial space.

2.3 CMSO

Our exposition of CMSO roughly follows that of Lokshtanov et al. [24]. Monadic second order logic (MSO) is a logic that can be used to express properties of graphs. The syntax includes logical connectives such as \( \lor, \land, \neg, \leftrightarrow, \Rightarrow \), and variables for single vertices, single edges, sets of vertices, and sets of edges, which can be quantified using \( \forall \) and \( \exists \). Furthermore there are binary relations for set membership \((\in)\), equality of variables \( (=)\), testing whether edge \( e \) incident to vertex \( v \) \((\text{inc}(v, e))\), and finally testing whether two vertices are adjacent \((\text{adj}(u, v))\). Counting monadic second order logic (CMSO) is an extension of MSO that includes a cardinality test \( \text{card}_{d,r}(S) \), which is true if and only if \( |S| \equiv q \mod r \). For a more complete introduction to CMSO we refer to the book of Courcelle and Engelfriet [9].

Let \( \mathcal{H} \) be a graph class. We say that containment in \( \mathcal{H} \) is expressible in CMSO if there exists a CMSO formula \( \varphi_\mathcal{H} \) such that for any graph \( G \) it holds that \( G \models \varphi_\mathcal{H} \) if and only if \( G \in \mathcal{H} \).

Lemma 2. There exist CMSO-formulas with the following properties:

1. For any graph \( H \), there exists a formula \( \varphi_{\mathcal{H} - \text{MINOR}}(X) \) such that for any graph \( G \) and any \( X \subseteq V(G) \) it holds that \( (G,X) \models \varphi_{\mathcal{H} - \text{MINOR}}(X) \) if and only if \( H \) is a minor of \( G[X] \).
2. For any graph class \( \mathcal{H} \) characterized by a finite set of forbidden induced subgraphs, there exists a formula \( \varphi_\mathcal{H} \) such that for any graph \( G \) it holds that \( G \models \varphi_\mathcal{H} \) if and only if graph \( G \in \mathcal{H} \).
3. There exists a formula \( \varphi_{\text{BIP}} \) such that for any graph \( G \) it holds that \( G \models \varphi_{\text{BIP}} \) if and only if graph \( G \) is bipartite.
4. For each \( k \in \mathbb{N} \), for each graph class \( \mathcal{H} \) such that containment in \( \mathcal{H} \) is CMSO expressible, and for each minor-closed parameter \( f \), there exists a formula \( \varphi_{(k, \mathcal{H}, f)}(X) \) such that for any graph \( G \) and any \( X \subseteq V(G) \) we have \( (G,X) \models \varphi_{(k,\mathcal{H},f)}(X) \) if and only if \( f(\mathcal{T}_G(X)) \leq k \) and \( C \in \mathcal{H} \) for each \( C \in \text{cc}(G - X) \).

Proof. For \[\text{[4]}\] see for instance Corollary 1.14 [9], we repeat it here as we adapt it for \[\text{[4]}\]

\[
\begin{align*}
\text{CONN}(X,V,E) &= \forall Y \subseteq V ((\exists u \in X : u \in Y \land \exists v \in X : v \notin Y) \Rightarrow \\
&((\exists e \in E \exists u,v \in X : \text{inc}(u,e) \land \text{inc}(v,e) \land u \in Y \land v \notin Y)) \\
\varphi_{\mathcal{H} - \text{MINOR}}(X) &= \exists Y_1, \ldots, Y_n \subseteq X : \\
& \bigwedge_{1 \leq i \leq n} \neg \exists y(y \in Y_i \land y \in Y_j) \\
& \bigwedge_{1 \leq i < j \leq n} \exists u, v(u \in Y_i \land v \in Y_j \land \text{adj}(u,v))
\end{align*}
\]
For \[^2\] let \( \mathcal{F}_H \) be the forbidden induced subgraph characterization of \( H \), where \( H \in \mathcal{F}_H \) is a graph on vertex set \(|V(H)|\). A formula \( \varphi_H \) is given below and is similar to that of checking for a minor.

\[
\varphi_H = \bigwedge_{H \in \mathcal{F}_H} \neg \exists v_1, \ldots, v_{|V(H)|} \in V(G) : \left( \bigwedge_{1 \leq i < j \leq |V(H)|} v_i \neq v_j \right) \\
\land \bigwedge_{(i,j) \in E(H)} \text{adj}(v_i, v_j) \land \bigwedge_{(i,j) \notin E(H)} \neg \text{adj}(v_i, v_j))
\]

Since a graph is bipartite if and only if it has a proper 2-coloring, the following formula shows item \[^3\].

\[
\text{PARTITION}(V, X_1, X_2) = \forall v \in V \left[ (v \in X_1 \land v \notin X_2) \lor (v \notin X_1 \land v \in X_2) \right] \\
\text{INDP}(X) = \forall u, v \in X \neg \text{adj}(u, v)
\]

Finally for \[^4\] note that since \( f \) is minor-closed, the set of graphs \( F \) with \( f(F) \leq k \) has a finite set of forbidden minors by the Graph Minor Theorem of Robertson and Seymour. Using formula \[^5\] we can check whether \( T_G(X) \) contains a forbidden minor. The only thing we need to change is that an edge \( \{u, v\} \) is in \( T_G(X) \) if either \( u \) and \( v \) are adjacent, or if there is a path whose internal vertices are not in \( X \).

\[
\text{TADJ}(u, v, X) = \text{adj}(u, v) \lor \exists P \subseteq V(G)(u, v \in P \land \text{conn}(P, V, E) \\
\land \forall w \in P(w = u \lor w = v \lor w \notin X))
\]

Finally we can check if each connected component \( C \) of \( cc(G - X) \) is in \( H \) by going over every vertex subset and verifying that if it is connected, disjoint from \( X \), and maximal, then it induces a graph in \( H \). □

Note 2. For each \( k \in \mathbb{N} \) and graph class \( \mathcal{H} \) such that containment in \( \mathcal{H} \) is CMSO-expressible, there exists a formula \( \varphi(k, \mathcal{H}, tw) \) (respectively \( \varphi(k, \mathcal{H}, td) \)) such that \( (G, k) \) is a YES-instance of \( \mathcal{H}\text{-TREewidth} \) (respectively \( \mathcal{H}\text{-Elimination Distance} \)) if and only if \( G \models \varphi(k, \mathcal{H}, tw) \) (respectively \( G \models \varphi(k, \mathcal{H}, td) \)).

CMSO formulas can have free variables. A graph together with an evaluation of free variables is called a structure. We denote the problem of evaluating a CMSO formula \( \varphi \) on a structure by CMSO[\( \varphi \)]. The following theorem is the main tool used to achieve our algorithms; we apply it only to formulas without free variables. The formulation is slightly different from its original form, see the appendix for details.

**Theorem 1.** \[^{[27]}\] **Theorem 23** Let \( \hat{\varphi} \) be a CMSO formula. For all \( \hat{c} : \mathbb{N}_0 \to \mathbb{N}_0 \), there exists \( \hat{s} : \mathbb{N}_0 \to \mathbb{N}_0 \) such that if CMSO[\( \hat{\varphi} \)] parameterized by \( k \) is FPT on \( (\hat{s}(k), \hat{c}(k)) \)-unbreakable structures, then CMSO[\( \varphi \)] parameterized by \( k \) is FPT on general structures.

### 3 Algorithms for computing \( ed_\mathcal{H} \) and \( tw_\mathcal{H} \)

In this section we present our algorithms. In Section 3.1 we present a key lemma. In Section 3.2 we use it to deal with \( \mathcal{H} \) characterized by a finite number of forbidden induced subgraphs, and in Section 3.3 we deal with bipartite graphs.

#### 3.1 Extracting witnesses from deletion sets contained in them

Our strategy for solving \( \mathcal{H}\text{-TREewidth} \) and \( \mathcal{H}\text{-Elimination Distance} \) is similar to that of lemmas 9 and 10 of Ganian et al. \[^{[17]}\] and is based on Proposition \[^2\]. Given an \( (s(k), c(k)) \)-unbreakable graph, either the treewidth of the graph is bounded \(^{[1]}\) and we can solve the problem directly using Courcelle’s Theorem, or each witness is of bounded size and introduces some structure \(^{[2]}\).

In the following lemma we assume we are in the latter case (hence the \( tw(G) > s(k) + k \) condition) and are given some \( \mathcal{H}\)-deletion set \( Y \). We show that given an \( (s(k), c(k)) \)-unbreakable graph, in FPT time we can find a witness \( X \) such that \( Y \subseteq X \) if such a witness exists.
Lemma 3. Consider some $k \in \mathbb{N}$ and $c: \mathbb{N} \to \mathbb{N}$ such that $c(k) \geq k$. Let $H$ be a graph class such that containment in $H$ is solvable in polynomial time. There is an algorithm that runs in FPT time that, given an $(s(k), c(k))$-unbreakable graph for any $s: \mathbb{N} \to \mathbb{N}$ with $tw(G) > s(k) + k$ and an $H$-deletion set $Y$ of size at most $s(k) + k$, decides whether there is an $tw_H(G)$ witness $X$ of width at most $k - 1$ (respectively $ed_H(G)$ witness $X$ of depth at most $k$) such that $Y \subseteq X$.

Proof. We refer to a witness as either being an $tw_H$ witness of width at most $k - 1$ or an $ed_H$ witness of depth at most $k$. Given a set $X \subseteq V(G)$, we can verify it is a witness by testing whether $tw(T_G(X)) \leq k - 1$ (respectively $td(T_G(X)) \leq k$) in FPT time [328] and verifying that each connected component $C \in cc(G - X)$ is contained in $H$, which can be done in polynomial time by assumption.

We show that we can find a witness if it exists, by doing the above verification for FPT many vertex subsets $D \subseteq V(G)$, as follows.

1. For each $y \in Y$, let $C_y$ be the set of connected vertex sets $S$ with $y \in S$, $|S| \leq s(k)$ and $|N(S)| \leq k$. For each $B \subseteq Y$ with $|B| \leq k$, a choice tuple $t_B$ contains an entry for each $y \in Y \setminus B$, where entry $t_B[y]$ is some set $C_y \in C_y$.
2. For each $B \subseteq Y$ with $|B| \leq k$ and each choice tuple $t_B$, if $G - (Y \cup \cup_{y \in Y \setminus B} N(t_B[y]))$ has one connected component $C$ of size at least $s(k)$ and $|V(G) \setminus N[C]| < s(k)$, apply the witness verification test to $D = Y \cup \cup_{y \in Y \setminus B} N(t_B[y]) \cup Q$ for each $Q \subseteq V(G) \setminus N[C]$.
3. Return the logical or of all witness verification tests.

We argue that the algorithm runs in FPT time. Note that as $|Y| \leq s(k) + k$, there are at most $(s(k) + k)^k$ choices for $B$. Furthermore $C_y$ can be computed in FPT time using Lemma 1, hence the number of choice tuples is also FPT many. For each choice for $B$ and each choice tuple $t_B$, there are at most $2^{s(k)}$ choices for $Q$. Since each vertex set can be verified to be a witness in FPT time, the running time claim follows.

Finally we argue correctness of the algorithm. Since $tw(G) > s(k) + k$ (and also $td(G) > s(k) + k$ as $tw(G) \leq td(G) - 1$), by Proposition 2 any witness $X$ is of size at most $s(k) + k - 1$, the graph $G - X$ has exactly one large connected component $C$ of size at least $s(k)$, and $|V(G) \setminus N[C]| < s(k)$.

Suppose $G$ has a witness that is a superset of $Y$. Fix some witness $X$ of minimal cardinality with $Y \subseteq X$ and let $C$ be the unique component of size at least $s(k)$ of $G - X$. Note that since $C \cap X = \emptyset$, we have $C \cap Y = \emptyset$.

Let $B = N(C) \cap Y$. By Note 1, we have $|N(C)| \leq k$, hence the branching algorithm makes this choice for $B$ at some point. For each $y \in Y \setminus B$, let $C_y$ be the connected component of $G - N[C]$ containing $y$. See Figure 1 for a sketch of the situation. Since $|V(G) \setminus N[C]| < s(k)$ and $|N(C)| \leq k$, we have that $|V(C_y)| < s(k)$ and $|N(C_y)| \leq k$. Note that $N(C_y) \subseteq N(C) \subseteq X$. The branching algorithm at some point tries the choice tuple $t_B$ where $t_B[y] = C_y$ for each $y \in Y \setminus B$. Consider the set $A = Y \cup \cup_{y \in Y \setminus B} N(t_B[y])$. Note that $A \subseteq X$ by construction.

If $N(C) \subseteq A$, then the single large component of $G - A$ of size at least $s(k)$ is exactly $C$. Since $|V(G) \setminus N(C)| < s(k)$, it follows that $X = A \cup Q$ for some $Q \subseteq V(G) \setminus N[C]$. It follows that the algorithm correctly identifies $X$ in this case.

The only remaining case is $N(C) \not\subseteq A$. We argue that this cannot happen when witness $X$ is of minimal cardinality. Suppose $N(C) \not\subseteq A$ and let $v \in N(C) \setminus A$. Let $Z = Y \cup \cup_{y \in Y \setminus B} N[C_y]$ and note that we subtract the open neighborhoods of the components, instead of the closed neighborhoods of the components in the definition of $A$. Let $C_v$ be the connected component of $G - (C \cup Z)$ that contains $v$. We argue that $X \cap C_v$ is a witness. Again consult Figure 1 for an intuition. Note that $C_v \cap Y = \emptyset$ by construction as $Y \subseteq Z$. Because $Y$ is an $H$-deletion set, it follows that for each connected component $C'$ of $G - (X \cap C_v)$ we have $C' \subseteq H$. We argue that $N(C_v) \subseteq N(C)$. First we argue that $N(C_v) \cap X \subseteq N(C)$. Indeed if any vertex in $X \setminus (Z \cup N(C))$ was adjacent to $C_v$, the vertex itself would belong to $C_v$. If any vertex $z \in Z \setminus N(C)$ was adjacent to $w \in C_v$, then either $w \in N(C)$ and hence $w \in N(C_y)$ for some $y \in Y$ and hence $w \in Z$, or $w \in C_y \setminus N(C)$ and $w \in C_y$ for some $y \in Y$ and hence $w \in Z$; in both cases we contradict $w \in C_v$. Similar arguments show that $N(C_v) \setminus N[C] = v$. If $v$ is adjacent to at least one vertex in $C$ as $v \in N(C)$, it follows that $C \cup C_v$ is a connected component of $G - (X \cup C_v)$ with $N(C \cup C_v) \subseteq N(C)$. Therefore $T_G(X \setminus C_v)$ is an induced subgraph of $T_G(X)$. We conclude that $X \cap C_v$ is a witness. Since $X$ was assumed to be of minimal cardinality, we arrive at a contradiction and hence $A \supseteq N(C)$.
Fig. 1. Situation sketch of Lemma 3. The set $X$ in grey denotes a witness and the set $C$ is the single large component of $G - X$.

3.2 Classes $\mathcal{H}$ with finitely many forbidden induced subgraphs

**Theorem 2.** Let $\mathcal{H}$ be a graph class characterized by a finite set of forbidden induced subgraphs. Then $\mathcal{H}$-treewidth and $\mathcal{H}$-elimination distance are non-uniformly fixed-parameter tractable.

**Proof.** By Lemma 2 containment in $\mathcal{H}$ is CMSO expressible, therefore by Note 2 there exists a formula $\varphi(k, H, f)$ for each $f \in \{\text{tw}, \text{td}\}$ such that an instance $(G, k)$ of $\mathcal{H}$-treewidth (respectively $\mathcal{H}$-elimination distance) is a YES-instance if and only if $G \models \varphi(k, H, f)$. Furthermore, containment in $\mathcal{H}$ is polynomial time solvable, as we can verify that a graph does not contain any of the finitely many forbidden induced subgraphs.

We argue that both problems are in FPT when the input graph $G$ is $(s(k), k)$-unbreakable for any $s: \mathbb{N} \rightarrow \mathbb{N}$. If $\text{tw}(G) \leq s(k) + k$, we solve the problems directly using Courcelle’s Theorem 8 using $\varphi(k, H, f)$. Otherwise by Proposition 2 each witness $X$ is of size at most $s(k) + k - 1$. We can enumerate all minimal $\mathcal{H}$-deletion sets $Y$ of size at most $s(k) + k - 1$ in FPT time by finding a forbidden induced subgraph and branching in all finitely many ways of destroying it. Since any witness $X$ is an $\mathcal{H}$-deletion set, for some $Y \in \mathcal{Y}$ we have $Y \subseteq X$. Hence we solve the problem by calling Lemma 3 for each $Y \in \mathcal{Y}$. Applying Theorem 1 concludes the proof.

Using known characterizations by a finite number of forbidden induced subgraphs (cf. 5) we obtain the following corollary to Theorem 2.

**Corollary 1.** Let $\mathcal{H}$ be set of graphs that are either (1) cliques, (2) claw-free, (3) of degree at most $d$ for fixed $d$, (4) cographs, or (5) split graphs. $\mathcal{H}$-treewidth and $\mathcal{H}$-elimination distance are non-uniformly fixed-parameter tractable.

3.3 Bipartite graphs

We use shorthand bip to denote the class of bipartite graphs. The problem of deleting $k$ vertices to obtain a bipartite graph is better known as the Odd Cycle Transversal (OCT) problem. The problem was shown to be FPT for the first time by Reed et al. 27. We use some of their ingredients to show the following.

**Lemma 4.** The bip-treewidth and bip-elimination distance problems are non-uniformly fixed-parameter tractable.

**Proof.** By Lemma 2 containment in the class of bipartite graphs is CMSO expressible, therefore by Note 2 there exists a formula $\varphi(k, \text{bip}, f)$ for each $f \in \{\text{tw}, \text{td}\}$ such that an instance $(G, k)$ of bip-treewidth
(respectively bip-elimination distance) is a yes-instance if and only if $G \models \varphi_{(k, \text{bip}, f)}$. We argue that both problems are FPT in $(s(k), 2k)$-unbreakable graphs for any $s \colon \mathbb{N} \to \mathbb{N}$. Note that the theorem then follows by Theorem 1.

Let $G$ be an $(s(k), 2k)$-unbreakable graph. As before, we use the term witness to either refer to an $tw_H$ witness of width at most $k - 1$ or an $ed_H$ witness of depth at most $k$, depending on the problem being solved. We first test whether $tw(G) \leq s(k) + k$, in FPT time [3]. If so, then we can solve the problems directly using Courcelle’s Theorem [8] using $\varphi_{(k,\text{bip},f)}$. Otherwise by Proposition [9] the size of each witness in $G$ is at most $s(k) + k - 1$, and for each witness $X$ there is a unique connected component of $G - X$ of at least $s(k)$ vertices, henceforth called the large component. We use a two-step process to find an odd cycle transversal that is a subset of some witness (if a witness exists), so that we may invoke Lemma [3] to find a witness.

For a witness $X^*$ in $G$ and an odd cycle transversal $W$ of $G$, we say that a partition $(W_L, W_f)$ of $W$ is weakly consistent with $X^*$ if for the unique large component $C$ of $G - X^*$ we have that $W \cap C = W_L$, $|W_L| \leq k$, and $W \subseteq C \cup X^*$. An odd cycle transversal $W$ is strongly consistent with $X^*$ if $W \subseteq X^*$.

The following claim encapsulates the connection between odd cycle transversals and separators that forms the key of the iterative-compression algorithm for OCT due to Reed, Smith, and Vetta [27].

**Claim 1.** For each partitioned OCT $W = (W_L, W_f)$ of $G$, for each partition of $W_L = W_{L,1} \cup W_{L,2}$ into two independent sets, for each proper 2-coloring $c$ of $G - W$, we have the following equivalence for each $X \subseteq V(G) \setminus W$: the graph $(G - W_f) - X$ has a proper 2-coloring with $W_{L,1}$ color 1 and $W_{L,2}$ color 2 if and only if the set $X$ separates $A$ from $R$ in the graph $G - W$, with:

$$A = (N_{G-W_f}(W_{L,1}) \cap c^{-1}(1)) \cup (N_{G-W_f}(W_{L,2}) \cap c^{-1}(2))$$

$$R = (N_{G-W_f}(W_{L,1}) \cap c^{-1}(2)) \cup (N_{G-W_f}(W_{L,2}) \cap c^{-1}(1)).$$

Observe that $c^{-1}(i) \subseteq V(G - W)$ for each $i \in [2]$, so that $A \cup R \subseteq V(G - W)$, and that the separator $X$ is allowed to intersect $A \cup R$.

**Proof.** ($\Rightarrow$) Suppose that $(G - W_f) - X$ has a proper 2-coloring with $W_{L,1}$ color 1 and $W_{L,2}$ color 2. Suppose for a contradiction that $X$ is not an $(A, R)$-separator in $G - W$, that is, in $(G - W) - X$ there is a connected component $H$ simultaneously containing a vertex $a \in A$ and a vertex $r \in R$. Note that $H$ is also a connected subgraph of $G - W$ and therefore bipartite, which means that if $|V(H)| \geq 2$ there is a unique partition of $H$ into two independent sets, so that $H$ has exactly two proper 2-colorings depending on which independent set is called color 1 and which is called color 2. Note that if $|V(H)| = 1$, the fact that $H$ has exactly two proper 2-colorings is trivial. It follows that any proper 2-coloring of $H$ either coincides with the 2-coloring $c$ of $G - W$, or is such that every vertex gets the opposite of its current color under $c$.

The fact that $a \in A$ means by definition that either we have $c(a) = 1$ and $a$ is adjacent to a vertex of $W_{L,1}$, or $c(a) = 2$ and $a$ is adjacent to a vertex of $W_2$. In either case, it shows that in any proper 2-coloring of $(G - W_f) - X$ in which $W_{L,1}$ gets color 1 and $W_{L,2}$ gets color 2, the color of $a$ must be different from its color under $c$. By an analogous argument, the fact that $r \in R$ means that in any proper 2-coloring of $(G - W_f) - X$ in which $W_{L,1}$ gets color 1 and $W_{L,2}$ gets color 2, the color of $r$ must be identical to its color under $c$.

Since $a$ and $r$ belong to the same connected subgraph $H$ of $(G - W_f) - X$, in any proper 2-coloring they either both change their color compared to $c$, or both keep their color compared to $c$. This is a contradiction to the fact that $a$ changed color and $r$ remained of the same color.

($\Leftarrow$) For the converse, consider a set $X$ that separates $A$ from $R$ in $G - W$. We construct a proper 2-coloring $c'$ of $(G - W_f) - X$ in which $W_{L,1}$ gets color 1 and $W_{L,2}$ gets color 2, as follows. Let $c'(v \in W_{L,1}) = 1$ and $c'(v \in W_{L,2}) = 2$. For each connected component of $(G - W) - X$ that contains a vertex from $R$, let its coloring under $c'$ be identical to its coloring under $c$. For each connected component of $(G - W) - X$ that contains no vertex from $R$, let its coloring under $c'$ be the opposite of its coloring under $c$. Since $c$ was a proper coloring, there are no color conflicts among vertices of $(G - W) - X$. Since both $W_{L,1}$ and $W_{L,2}$ are independent sets, there are no color conflicts among $W_{L,1}$ or among $W_{L,2}$, it remains to verify that each edge connecting $W_L$ to a vertex of $(G - W) - X$ is properly colored. But this follows from our construction: all neighbors of $W_{L,1}$ with color 1 under $c$ belong to $A$ and therefore have their coloring swapped to 2 in $c'$; similarly all neighbors of $W_{L,2}$ with color 2 under $c$ belong to $A$ and have their coloring swapped to 1 in $c'$. Finally, neighbors of $W_{L,1}$ with color 2 in $c$ belong to $R$ and therefore have the same color 2 in $c'$, and
neighbors of $W_{L,2}$ with color 1 in $c$ belong to $R$ and have the same color 1 in $c'$, ensuring these edges are properly colored as well.

The next two claims show that certain types of OCTs can be computed efficiently in the $(s(k), 2k)$-unbreakable input graph $G$.

**Claim 2.** There is an FPT algorithm that outputs a list of partitioned OCTs in $G$ with the guarantee that for each witness $X$, there is a partitioned OCT on the list that is weakly consistent with $X$.

**Proof.** The algorithm proceeds as follows.

1. Initialize an empty list $W$. Compute a minimum cardinality odd cycle transversal $W \subseteq V(G)$ of size at most $s(k) + k - 1$. If no such OCT exists, return the empty list.
2. For each $y \in V(G)$, let $C_y$ be the set of connected vertex sets $S$ with $y \in S$, $|S| \leq s(k)$ and $|N(S)| \leq k$.
   For each partition $P = (W_L, W_I, W_R)$ of $W$, a choice tuple $t_P$ contains an entry for each $y \in W_R$, where entry $t_P[y]$ is some set $C_y \in C_y$.
3. For each partition $P = (W_L, W_I, W_R)$ of $W$ and each choice tuple $t_P$, if $(W \setminus W_R) \cup \bigcup_{y \in W_R} N(t_P[y])$ is an OCT, then add $(W_L, W_I \cup \bigcup_{y \in W_R} N(t_P[y]))$ to $W$.
4. Return the list $W$.

We argue the running time of the steps described above. The first step can be done in time $O^*(3^{s(k)+k})$ [27,10]. For each $y \in V(G)$, computing $C_y$ is in FPT by 4. Since there are $3^{s(k)+k-1}$ possible partitions $P$ and FPT many choice tuples $t_P$, the running time follows. To see the correctness of the algorithm, first note that each partition is an OCT by construction. All that is left is to show that the output is consistent with some witness $X$ by Proposition 2 and for each choice tuple $t_P$, if $(W \setminus W_R) \cup \bigcup_{y \in W_R} N(t_P[y])$ is an OCT, then add $(W_L, W_I \cup \bigcup_{y \in W_R} N(t_P[y]))$ to $W$.

**Claim 3.** There is an FPT algorithm that, given a partitioned OCT that is weakly consistent with some (unknown) witness $X$ in $G$, outputs a list of OCTs in $G$ such that at least one is strongly consistent with $X$.

**Proof.** Let $(W_L, W_I)$ be the given partitioned OCT, where $W_L \cup W_I = W$. If $|W| > s(k) + k - 1$, then no witness is strongly consistent with $W$ by Proposition 2, hence we may assume $|W| \leq s(k) + k - 1$.

1. Initialize an empty list $W$. For each $y \in V(G)$, let $C_y$ be the set of connected vertex sets $S$ with $y \in S$, $|S| \leq s(k)$ and $|N(S)| \leq 2k$.
2. Let $c^*$ be an arbitrary proper 2-coloring of $G - W$ and let $B^*_i = (c^*)^{-1}(i)$ for each $i \in [2]$.
3. For each partition $(W_1, W_2)$ of $W_L$, let $B_1 = N(W_2) \setminus W$ and $B_2 = N(W_1) \setminus W$. Let $A = (B_1 \cap B_2) \cup (B_2 \cap B_1)$ and $R = (B_1 \cap B_1) \cup (B_2 \cap B_2)$.
4. For each choice $Q \in \{A, R\}$ with $|Q| \leq s(k) + k$, for each $D \subseteq Q$ with $|D| \leq k$, choice tuple $t_{Q,D}$ has an entry for each $y \in Q \setminus D$, where entry $t_{Q,D}[y]$ is some vertex set $C_y \in C_y$.
5. For each choice $Q \in \{A, R\}$ with $|Q| \leq s(k) + k$, for each $D \subseteq Q$ with $|D| \leq k$, and for each choice tuple $t_{Q,D}$, add $(W \cup D) \cup \bigcup_{y \in Q \setminus D} N(t_{Q,D}[y]) \setminus W_L$ to $W$ in case it is an OCT.
6. Return the list $W$.

The running time follows from Lemma 1 and the fact that there are FPT many choices for $(W_1, W_2)$, $D$, and tuple $t_{Q,D}$, we argue the correctness of the algorithm. Note that each set in the output list is an OCT by construction. Consider some witness $X$ with $(W_L, W_I)$ weakly consistent with $X$ and let $C$ be the unique large component of $G - X$, which is bipartite by definition of witness. Let $Y \subseteq X$ be an OCT of $G$ with $W_I \subseteq Y$ and $Y \subseteq W_I \cup N(C)$. Note that such an OCT $Y$ exists as $W' = (W \setminus W_I) \cup N(C)$ is such an OCT. Let $c: V(G) \setminus Y \rightarrow [2]$ be a proper 2-coloring of $G - Y$. For some partition $(W_1, W_2)$ of $W_L$ we have $W_i \subseteq c^{-1}(i)$ for each $i \in [2]$. Note that since $W \setminus W_L = W_I$, we have that $|W' \setminus W_I| \leq k$. 

By Claim 3 it follows that $Y \setminus W_I \subseteq N(C)$ separates $A$ and $R$ in $G - W$. Note that $B_i \subseteq N[C]$ for each $i \in [2]$ since $W_i \subseteq C$, therefore $A \subseteq N[C]$ and $R \subseteq N[C]$. Observe that $W_L \cup N(C)$ is an $(A, R)$-separator of size at most $2k$ in $G$. Therefore, since $G$ is $(s(k), 2k)$-unbreakable, it follows that at least one of the two sides has size at most $s(k)$ after deleting $W_L \cup N(C)$. Let $Q \in \{A, R\}$ be the small side, the algorithm tries this choice as $|Q| \leq s(k) + k$ is satisfied. Let $D = N(C) \cap Q$. For each $y \in Q \setminus D$, let $C_y$ be the connected component of $G - (N(C) \cup W_L)$ containing $y$. Note that $|C_y| \leq s(k)$ and $|N(C_y)| \leq 2k$. Let the choice tuple $t_{Q,D}$ be such that $t_{Q,D}[y] = C_y$ for each $y \in Q \setminus D$. Observe that $(D \cup \bigcup_{y \in Q \setminus D} N(t_{Q,D}[y])) \setminus W_L \subseteq N(C)$ is an $(A, R)$-separator in $G - W$. Therefore $(W_I \cup D \cup \bigcup_{y \in Q \setminus D} N(t_{Q,D}[y])) \setminus W_L$ is an OCT by Claim 3 contained in $X$, concluding the proof.

With the two claims above, we can solve the problem as follows. Compute a list of partitions $W$ using Claim 3 and use each $W \in W$ as input to Claim 3. Using the output $U$ of Claim 3 call Lemma 3 for each $U \in U$. By the output guarantee of the claims, for each witness $X$ we call the lemma with $U \subseteq X$ at some point, thus solving the problem.

4 Conclusion

We have shown that $H$-elimination distance and $H$-treewidth are non-uniformly fixed-parameter tractable for $H$ being the class of bipartite graphs, and whenever $H$ is defined by a finite set of forbidden induced subgraphs. An obvious direction for further research is extending this to other graph classes. While the algorithms presented here solve the decision variant of the problem, by self-reduction they can be used to identify a witness if one exists. The main observation driving such a self-reduction is the following: if $\text{tw}_H(G) \leq k$, then for an arbitrary $v \in V(G)$ there exists a $\text{tw}_H(G)$-witness that contains $v$ if and only if the graph $G'$ obtained from $G$ by inserting a minimal forbidden induced subgraph into $H$ and identifying one of its vertices with $v$, still satisfies $\text{tw}_H(G') \leq k$. Hence an iterative process can identify all vertices of a witness in this way.

While we have focused on the established notions of $\text{tw}_H$ and $\text{ed}_H$, the ideas presented here can be generalized using minor-closed graph parameters $f$ other than treewidth and treedepth. As long as $f$ can attain arbitrarily large values, implying its value on a clique grows with the size of the clique, and $H$ is characterized by a finite set of forbidden induced subgraphs, we believe our approach can be generalized to answer questions of the form: does $G$ have an $H$-deletion set $X$ for which $f(T_G(X)) \leq k$?

References

1. Akansha Agrawal, Lawqueen Kanesh, Fahad Panolan, M. S. Ramanujan, and Saket Saurabh. An FPT algorithm for elimination distance to bounded degree graphs. In Proceedings of the 38th International Symposium on Theoretical Aspects of Computer Science, STACS 2021, 2021. doi:10.4230/LIPIcs.STACS.2021.50
2. Akansha Agrawal and M. S. Ramanujan. On the parameterized complexity of clique elimination distance. In Yixin Cao and Marcin Pilipczuk, editors, 15th International Symposium on Parameterized and Exact Computation, IPEC 2020, December 14-18, 2020, Hong Kong, China (Virtual Conference), volume 180 of LIPIcs, pages 1:1–1:13. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2020. doi:10.4230/LIPIcs.IPEC.2020.1
3. Hans L. Bodlaender. A linear-time algorithm for finding tree-decompositions of small treewidth. SIAM J. Comput., 25(6):1305–1317, 1996. doi:10.1137/S0097539793251219
4. Hans L. Bodlaender and Arie M. C. A. Koster. Combinatorial optimization on graphs of bounded treewidth. Comput. J., 51(3):255–269, 2008. doi:10.1016/j.comjnl.200307
5. Andreas Brandstädt, Van Bang Le, and Jeremy P. Spinrad. Graph classes: a survey. Society for Industrial and Applied Mathematics, Philadelphia, PA, USA, 1999.
6. Jannis Bulian and Anuj Dawar. Graph isomorphism parameterized by elimination distance to bounded degree. Algorithmica, 75(2):363–382, 2016. doi:10.1007/s00453-015-0045-3
7. Jannis Bulian and Anuj Dawar. Fixed-parameter tractable distances to sparse graph classes. Algorithmica, 79(1):139–158, 2017. doi:10.1007/s00453-016-0235-7
8. Bruno Courcelle. The monadic second-order logic of graphs. i. recognizable sets of finite graphs. Inf. Comput., 85(1):12–75, 1990. doi:10.1016/0890-5401(90)90043-B
9. Bruno Courcelle and Joost Engelfriet. Graph Structure and Monadic Second-Order Logic: A Language-Theoretic Approach. Encyclopedia of Mathematics and its Applications. Cambridge University Press, 2012. doi:10.1017/CBO9780511977619
10. Marek Cygan, Fedor V. Fomin, Łukasz Kowalik, Daniel Lokshtanov, Dániel Marx, Marcin Pilipczuk, Michal Pilipczuk, and Saket Saurabh. *Parameterized Algorithms*. Springer, 2015. [doi:10.1007/978-3-319-21275-3]

11. Rodney G. Downey and Michael R. Fellows. *Fundamentals of Parameterized Complexity*. Texts in Computer Science. Springer, 2013. [doi:10.1007/978-1-4471-5559-1]

12. Eduard Eiben, Robert Ganian, Théâle Hamm, and O-joung Kwon. Measuring what matters: A hybrid approach to dynamic programming with treewidth. In Peter Rossmannith, Pinar Heggernes, and Joost-Pieter Katoen, editors, *44th International Symposium on Mathematical Foundations of Computer Science, MFCS 2019, August 26–30, 2019, Aachen, Germany*, volume 138 of LIPIcs, pages 42:1–42:15. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2019. [doi:10.4230/LIPIcs.MFCS.2019.42]

13. Michael R. Fellows, Daniel Lokshtanov, Neeldhara Misra, Frances A. Rosamond, and Saket Saurabh. Graph layout problems parameterized by vertex cover. In Seok-Hee Hong, Hiroshi Nagamochi, and Takuro Fukunaga, editors, *Algorithms and Computation, 19th International Symposium, ISAAC 2008, Gold Coast, Australia, December 15-17, 2008. Proceedings*, pages 5369 of *Lecture Notes in Computer Science*, pages 294–305. Springer, 2008. [doi:10.1007/978-3-540-92182-0_28]

14. Till Fluschnik, Rolf Niedermeier, Carsten Schulter, and Philipp Zschoche. Multistage s-t path: Confronting similarity with dissimilarity in temporal graphs. In Yixin Cao, Siu-Wing Cheng, and Minming Li, editors, *31st International Symposium on Algorithms and Computation, ISAAC 2020, December 14-18, 2020, Hong Kong, China (Virtual Conference)*, volume 181 of LIPIcs, pages 43:1–43:16. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2020. [doi:10.4230/LIPIcs.ISAAC.2020.43]

15. Fedor V. Fomin and Yngve Villanger. Treewidth computation and extremal combinatorics. *Comb.*, 32(3):289–308, 2012. [doi:10.1007/s00493-012-2536-z]

16. Robert Ganian, Sebastian Ordyniak, and Stefan Szeider. A join-based hybrid parameter for constraint satisfaction. In Thomas Schiex and Simon de Givry, editors, *Principles and Practice of Constraint Programming - 25th International Conference, CP 2019, Stamford, CT, USA, September 30 - October 4, 2019, Proceedings*, volume 11802 of *Lecture Notes in Computer Science*, pages 195–212. Springer, 2019. [doi:10.1007/978-3-030-30048-7_12]

17. Robert Ganian, M. S. Ramanujan, and Stefan Szeider. Combining treewidth and backdoors for CSP. In Heribert Vollmer and Brigitte Vellée, editors, *34th Symposium on Theoretical Aspects of Computer Science, STACS 2017, March 8-11, 2017, Hannover, Germany*, volume 66 of LIPIcs, pages 36:1–36:17. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2017. [doi:10.4230/LIPIcs.STACS.2017.36]

18. Jiong Guo, Falk Hüffner, and Rolf Niedermeier. A structural view on parameterizing problems: Distance from triviality. In Rodney G. Downey, Michael R. Fellows, and Frank K. H. A. Dehne, editors, *Parameterized and Exact Computation, First International Workshop, IWPEC 2004, Bergen, Norway, September 14-17, 2004*, Proceedings, volume 3162 of *Lecture Notes in Computer Science*, pages 162–173. Springer, 2004. [doi:10.1007/978-3-540-28639-4_15]

19. Eva-Maria C. Hols, Stefan Kratsch, and Astrid Pieterse. Elimination distances, blocking sets, and kernels for vertex cover. In Christophe Paul and Markus Bläser, editors, *37th International Symposium on Theoretical Aspects of Computer Science, STACS 2020, March 10-13, 2020, Montpellier, France*, volume 154 of LIPIcs, pages 36:1–36:14. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2020. [doi:10.4230/LIPIcs.STACS.2020.36]

20. Bart M. P. Jansen and Hans L. Bodlaender. Vertex cover kernelization revisited - upper and lower bounds for a refined parameter. *Theory Comput. Syst.*, 53(2):263–299, 2013. [doi:10.1007/s00224-012-9393-4]

21. Bart M.P. Jansen, Jari J. H. de Kroon, and Michal Wlodarczyk. Vertex deletion parameterized by elimination distance. In Christophe Paul and Markus Bläser, editors, *Algorithm Theory - SWAT 2010, 12th Scandinavian Symposium and Workshops on Algorithm Theory, Bergen, Norway, June 21-23, 2010. Proceedings*, volume 6139 of *Lecture Notes in Computer Science*, pages 81–92. Springer, 2010. [doi:10.1007/978-3-642-13731-0_9]

22. Stefan Kratsch and Pascal Schweitzer. Isomorphism for graphs of bounded feedback vertex set number. In Ioannis Chatzigiannakis, Christos Kaklamanis, Dániel Marx, and Donald Sannella, editors, *45th International Colloquium on Automata, Languages, and Programming, ICALP 2018, July 9-13, 2018, Prague, Czech Republic*, volume 107 of LIPIcs, pages 135:1–135:14. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2018. [doi:10.4230/LIPIcs.ICALP.2018.135]

23. Alexander Lindermayr, Sebastian Siebertz, and Alexandre Vigny. Elimination distance to bounded degree on planar graphs. In Javier Esparza and Daniel Kráľ, editors, *45th International Symposium on Mathematical Foundations of Computer Science, MFCS 2020, August 24-28, 2020, Prague, Czech Republic*, volume 170 of LIPIcs, pages 65:1–65:12. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2020. [doi:10.4230/LIPIcs.MFCS.2020.65]

24. Daniel Lokshtanov, M. S. Ramanujan, Saket Saurabh, and Meirav Zehavi. Reducing CMSO model checking to highly connected graphs. *CoRR*, abs/1802.01453, 2018. URL: [http://arxiv.org/abs/1802.01453](http://arxiv.org/abs/1802.01453) [arXiv:1802.01453]
26. Jaroslav Nesetril and Patrice Ossona de Mendez. *Sparsity - Graphs, Structures, and Algorithms*, volume 28 of *Algorithms and combinatorics*. Springer, 2012. doi:10.1007/978-3-642-27875-4

27. Bruce A. Reed, Kaleigh Smith, and Adrian Vetta. Finding odd cycle transversals. *Oper. Res. Lett.*, 32(4):299–301, 2004. doi:10.1016/j.orl.2003.10.009

28. Felix Reidl, Peter Rossmanith, Fernando Sánchez Villaamil, and Somnath Sikdar. A faster parameterized algorithm for treedepth. In Javier Esparza, Pierre Fraigniaud, Thore Husfeldt, and Elias Koutsoupias, editors, *Automata, Languages, and Programming - 41st International Colloquium, ICALP 2014, Copenhagen, Denmark, July 8-11, 2014, Proceedings, Part I*, volume 8572 of *Lecture Notes in Computer Science*, pages 931–942. Springer, 2014. doi:10.1007/978-3-662-43948-7_77

29. Neil Robertson and Paul D. Seymour. Graph minors .xiii. the disjoint paths problem. *J. Comb. Theory, Ser. B*, 63(1):65–110, 1995. doi:10.1006/jctb.1995.1006
A Proof of Theorem 1

Since we slightly changed the statement of Theorem 1 compared to its original form, we state its proof as given in the full version by Lokshtanov et al. [25] for completeness. We require the following theorem from their paper.

**Theorem 3.** [24, Theorem 22] Let $\varphi$ be a CMSO formula. For all $c \in \mathbb{N}$, there exists $s \in \mathbb{N}$ such that if there exists an algorithm that solves CMSO[$\varphi$] on $(s,c)$-unbreakable structures in time $O(n^d)$ for some $d > 4$, then there exists an algorithm that solves CMSO[$\varphi$] on general structures in time $O(n^d)$.

**Proof of Theorem 1** Let $\hat{c} : \mathbb{N}_0 \to \mathbb{N}_0$ and define $\hat{s} : \mathbb{N}_0 \to \mathbb{N}_0$ as follows. For all $k \in \mathbb{N}_0$, let $\hat{s}(k)$ be the constant $s$ in Theorem 3 and $c = \hat{c}(k)$. Suppose that CMSO[$\varphi$] is FPT on $(\hat{s}(k), \hat{c}(k))$-unbreakable structures. Then for every fixed $k$ we can solve it in $O(n^d)$ time for some fixed $d > 4$. By Theorem 3 it follows that we can solve CMSO[$\varphi$] in $O(n^d)$ time for every fixed $k$ on general structures. Therefore we can solve it in FPT time on general structures. \qed