Group Sum Chromatic Number of Graphs

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Abstract
We investigate the group sum chromatic number ($\chi^\Sigma_g(G)$) of graphs, i.e. the smallest value $s$ such that taking any Abelian group $G$ of order $s$, there exists a function $f : E(G) \rightarrow G$ such that the sums of edge labels properly colour the vertices. We prove that for any graph $G$ with no components of order less than 3, $\chi^\Sigma_g(G) = \chi(G)$ except some infinite family of graphs.

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1 Introduction

A labelling of edges of a graph $G$ is called vertex-colouring if it results in weighted degrees that properly colour the vertices. If we use the elements of $\{1, 2, \ldots, k\}$ to label the edges, such a labelling is called vertex-colouring $k$-edge-labelling.

The concept of colouring the vertices with the sums of edge labels was introduced for the first time by Karoński, Łuczak and Thomason (\cite{[12]}). The authors posed the following question. Given a graph $G$ with no components
of order less than 3, what is the minimum k such that there exists a vertex-colouring k-edge-labelling? As there are some analogies with ordinary proper graph colouring, we will call this minimum value of k the sum chromatic number and denote it with \(\chi^\Sigma(G)\).

Karoński, Łuczak and Thomason in [12] conjectured that \(\chi^\Sigma(G) \leq 3\) for every graph G with no component of order less than 3. First constant bound was proved by Addario-Berry et al. in [1] (\(\chi^\Sigma(G) \leq 30\)) and then improved by Addario-Berry et al. in [2] (\(\chi^\Sigma(G) \leq 16\)), Wan and Yu in [14] (\(\chi^\Sigma(G) \leq 13\)) and finally by Kalkowski, Karoński and Pfender in [10] (\(\chi^\Sigma(G) \leq 5\)).

On the other hand, numerous authors studied various labelling problems when elements of finite Abelian groups were used instead of integers to label either vertices or edges of graph. We give only few examples here. Graham and Sloane in [7] studied harmonious graphs i.e. graphs for which there exists an injection \(f : V(G) \to \mathbb{Z}_q\) that assigns to every edge \((x, y) \in E(G)\) unique sum \(f(x) + f(y) \mod q\). Beals et al. (see [3]) considered the concept of harmoniousness with respect to arbitrary Abelian groups. ˙Zak in [15] generalized the problem and introduced new parameter, harmonious order of \(G\), the smallest number \(t\) such that injection \(f : V(G) \to \mathbb{Z}_t\) (or surjection if \(t < V(G)\)) produces distinct edge sums. Hovey in [8] considers the so-called \(A\)–cordial labellings, where for a given Abelian group \(A\) and a graph \(G\) one wants to obtain such a vertex labelling that the classes of vertices labelled with one label are (almost) equinumerous and so are the classes of edges with the same sums. Cavenagh et al. (see [4]) consider edge-magic total labellings with finite Abelian groups, i.e. the labellings of vertices and edges resulting in equal edge sums. Froncek in [6] defined the notion of group distance magic graphs, i.e. the graphs allowing the bijective labelling of vertices with elements of an Abelian group resulting in constant sums of neighbour labels. Stanley in [13] studied the vertex-magic labellings of edges with the elements of an Abelian group \(A\), i.e. labellings, where the resulting weighted degrees are constant. Kaplan et al. in [11] considered vertex-antimagic edge labellings, i.e. the bijections \(f : E(G) \to A\), where \(A\) is a cyclic group, resulting in distinct weighted degrees of vertices.

The problem considered in this paper is, in a sense, a link between these two subjects. Assume we are given an arbitrary graph \(G\) that is \(\chi(G)\)-colourable and has no components isomorphic to \(K_1\) or \(K_2\). Assume \(G\) is an Abelian group of order \(m \geq \chi(G)\) with the operation denoted by + and neutral element 0. For convenience we will write \(ka\) to denote \(a + a + \cdots + a\) (where element \(a\) appears \(k\) times), \(-a\) to denote the inverse of \(a\) and we will use \(a - b\) instead of \(a + (-b)\). Moreover, the notation \(\sum_{a \in S} a\) will be
used as a short form for \(a_1 + a_2 + a_3 + \ldots\), where \(a_1, a_2, a_3, \ldots\) are all the elements of the set \(S\).

We define edge labelling \(f : E(G) \to \mathcal{G}\) leading us to the weighted degrees defined as the sums:

\[
w(v) = \sum_{e \ni v} f(e)
\]

We call \(f\) vertex-\(G\)-colouring if the weighted degrees of neighbouring vertices are distinct. The group sum chromatic number of \(G\), denoted \(\chi^\Sigma_g(G)\), is the smallest integer \(s\) such that for every Abelian group \(\mathcal{G}\) of order \(s\) there exists vertex-\(G\)-colouring labelling \(f\) of \(G\).

Partial solution to this problem was given by Karoński, Luczak and Thomason in [12]. Namely, they proved that there always exists vertex-\(G\)-colouring labeling of given graph \(G\) if it is \(|\mathcal{G}|\)-colourable and \(|\mathcal{G}|\) is odd. However, for the sake of completeness we present our original proofs covering all the possible cases.

Let us define a special family of graphs.

**Definition 1.1.** Connected graph \(G\) with of order at least 3 is ugly if it belongs to one of the following classes:

- \(\chi(G) = 4k + 2\) for some integer \(k \geq 0\) and in each proper \(\chi(G)\)-colouring of \(G\) all the colour classes of \(G\) are odd,
- \(\chi(G) = 2^q\) for some integer \(q \geq 2\) and in each proper \(\chi(G)\)-colouring of \(G\) either exactly 2 or exactly \(2^q - 2\) colour classes are odd.

The main result of our paper is the following theorem, determining the value of \(\chi^\Sigma_g(G)\) for every graph with no components of order less than 3.

**Theorem 1.2.** Let \(G\) be arbitrary graph with no components of order less than 3. If \(G\) does not contain any ugly component \(C_1\) such that \(\chi(C_1) = \chi(G)\) and in the case when \(\chi(G) = 2^q\) for some integer \(q \geq 2\) no component \(C_2 \cong K_{\chi(G)-2}\), then

\[\chi^\Sigma_g(G) = \chi(G)\]

Otherwise

\[\chi^\Sigma_g(G) = \chi(G) + 1\]

2 Proof of the Theorem [1.2]

In order to properly colour all the vertices of \(G\) we need at least \(\chi(G)\) distinct elements of \(\mathcal{G}\). However it is not always enough, what shows the following lemma.
Lemma 2.1. If graph $G$ is ugly, then

$$\chi_{\Sigma}^g(G) \geq \chi(G) + 1.$$ 

Proof. Assume first that $\chi(G) = 4k + 2$ and all the colour colour classes of $G$ are odd. Assume $\chi_{\Sigma}^g(G) = 4k + 2$. Every Abelian group of order $4k + 2$ is isomorphic to $Z_2 \times G^*$ for some Abelian group $G^*$ of order $2k + 1$. We can express every element of $G$ as $(z, a)$, where $z \in Z_2$ and $a \in G^*$. In such a situation the elements of $2k + 1$ of the colour classes would have weighted degrees equal to $(1, a_j)$ for distinct $a_j \in G^*$ and the remaining $2k + 1$ to $(0, a_j)$. This would imply that the sum of all the weighted degrees equals $(1, a)$ for some $a \in G^*$. But on the other hand, for every edge label $(z, a_j)$ we have $2(z, a_j) = (0, 2a_j)$, so $\sum_{v \in V(G)} w(v) = (0, b)$ for some $b \in G^*$. This contradiction implies that $\chi_{\Sigma}^g(G) \geq \chi(G) + 1$.

Now let us consider the second case. Assume that $G \cong Z_2 \times Z_2 \times \cdots \times Z_2$. It means that there are $2^q - 1 \geq 3$ involutions in $G$, and their sum equals to 0 (see e.g. [5], Lemma 8). Assume $\chi_{\Sigma}^g(G) = 2^q$. The vertices in every colour class must have the weighted degrees equal to distinct element of $G$. If there are 2 odd colour classes, let us assume that their vertices have weighted degrees $a$ and $b$, where $a \neq b$. Sum of the weighted degrees in any even class would be equal to 0 and in the odd classes $a$ and $b$. Thus $\sum_{v \in V(G)} w(v) = a + b$. Similarly, if there are exactly $2^q - 2$ odd colour classes, then assume the weighted degrees of the vertices in two even classes are $a$ and $b$, where $a \neq b$. The sum of all the weighted degrees is equal to the sum of all the elements of $G$ except $a$ and $b$, that is $-a - b = a + b$. In both situations, as in the case of the bipartite graph, $\sum_{v \in V(G)} w(v) = 0$. But the equality $a + b = 0$ means $a = b$. The contradiction implies that $\chi_{\Sigma}^g(G) \geq 2^q + 1 = \chi(G) + 1$.

Given any two vertices $x_1$ and $x_2$ belonging to the same connected component of $G$, there exist walks from $x_1$ to $x_2$. Some of them may consist of even number of vertices (some of them being repetitions). We are going to call them even walks. The walks with odd number of vertices will be called odd walks. We will always choose the shortest even or the shortest odd walk from $x_1$ to $x_2$.

We start with 0 on all the edges of $G$. Then, in every step we will choose $x_1$ and $x_2$ and add some labels to all the edges of chosen walk from $x_1$ to $x_2$. To be more specific, we will add some element $a$ of the group to the labels of all the edges having odd position on the walk (starting from $x_1$) and $-a$ to the labels of all the edges having even position. It is possible
that some labels will be modified more than once, as the walk does not need to be a path. We will denote such situation with $\phi_e(x_1, x_2) = a$ if we label the shortest even walk and $\phi_o(x_1, x_2) = a$ if we label the shortest odd walk. Observe that putting $\phi_e(x_1, x_2) = a$ means adding $a$ to the weighted degrees of both vertices, while $\phi_o(x_1, x_2) = a$ means adding $a$ to the weighted degree of $x_1$ and $-a$ to the weighted degree of $x_2$. In both cases the operation does not change the weighted degree of any other vertex of the walk.

Let us continue with the following lemma, determining the group sum chromatic number of bipartite connected graphs.

**Lemma 2.2.** Let $G$ be a connected bipartite graph of order $n \geq 3$. Then

$$\chi^\Sigma(G) = \begin{cases} 3 & \text{when } G \text{ is ugly} \\ 2 & \text{otherwise.} \end{cases}$$

**Proof.** We already know that the given values are lower bounds. Thus it suffices to present a proper group colouring. Let us denote colour classes with $V_1$ and $V_2$.

In the case when $G$ is not ugly, then for every group $G$ of order 2 we have $G \cong Z_2$, so we can use the last one. At least one of the colour classes, say $V_1$ is even. We choose in any way $|V_1|/2$ disjoint ordered pairs of vertices $(x'_1, x'_2), x'_1, x'_2 \in V_1$ and put $\phi_o(x'_1, x'_2) = 1$ for $j = 1, \ldots, |V_1|/2$. Finally we label all the unlabelled edges with 0. This way we obtain

$$w(x) = \begin{cases} 1 & \text{when } x \in V_1 \\ 0 & \text{otherwise.} \end{cases}$$

If $G$ is ugly (i.e. both colour classes are odd), then we can use $Z_3$, as for every Abelian group $G$ of order 3 we have $G \cong Z_3$. Assume without loss of generality that $|V_1| \geq |V_2|$ and so $|V_1| \geq 3$. We choose three vertices $x_1, x_2, x_3 \in V_1$. We choose in any way $(|V_1| - 3)/2$ disjoint ordered pairs of vertices $(x'_1, x'_2), x'_1, x'_2 \in V_1 \setminus \{x_1, x_2, x_3\}$ and put $\phi_o(x'_1, x'_2) = 2$ for $j = 1, \ldots, (|V_1| - 3)/2$. Moreover we put $\phi_o(x_1, x_2) = \phi_o(x_1, x_3) = 1$. Finally we label all the unlabelled edges with 0. This way we obtain

$$w(x) = \begin{cases} 1 \text{ or } 2 & \text{when } x \in V_1 \\ 0 & \text{otherwise.} \end{cases}$$

Before analysing the case of non-bipartite graphs we need to prove the following technical lemma.
Lemma 2.3. Let $G$ be an Abelian group with involutions set $I^* = \{i_1, \ldots, i_{2k-1}\}$, $k \geq 2$ and let $I = I^* \cup \{0\}$. Then for any given $r$ such that $0 \leq r \leq 2^k$, there exists set $R \subseteq I$, $|R| = r$, such that

$$\sum_{i \in R} i = 0$$

if and only if $r \not\in \{2, 2^k - 2\}$.

Proof. We can consider only the case when $r \leq 2^{k-1}$ because if we can find a set $R$ for given $r$, then the set $I \setminus R$ will also sum up to 0.

If $r = 0$ then $R = \emptyset$ and if $r = 1$, then $R = \{0\}$.

Let $r = 2$. It is impossible to find $a, b \in I$, $a \neq b$ such that $a + b = 0$, because the last condition implies $a = b$, contradiction. Thus we cannot find desired set $R$ neither for $r = 2$, nor for $r = 2^k - 2$.

Now let $3 \leq r \leq 2^{k-1}$. We know that $I \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$, so each of its elements can be expressed as different binary sequence $i_r = (a^r_1, \ldots, a^r_k)$. The addition in $I$ is represented by coordinate-wise addition in $\mathbb{Z}_2$. Assume these sequences are ordered lexicographically in the increasing order. Observe, that for $q < 2^{k-1}$, we have $a^q_1 = 0$.

Let $R^* = (i_1, \ldots, i_{r-1})$ and let

$$i^* = \sum_{i \in R^*} i.$$ 

If $i^* = 0$, then $R = R^* \cup \{0\}$ is the desired set. If $i^* \neq 0$ and $i^* = i_q$ for some $q > r - 1$ (thus $i^* \not\in R^*$), then we put $R = R^* \cup \{i^*\}$. Finally, if $i^* \neq 0$ and $i^* = i_p$ for some $q \leq r - 1$ ($i^* \in R^*$), we choose any $i_p \in R^*$, $p \neq q$ (such $p$ exists as $|R^*| = r - 1 \geq 2$) and define $R$ in the following way:

$$R = R^* \setminus \{i_p\} \cup \{i_p + i_{2k-1}, i_q + i_{2k-1}\}.$$ 

Indeed, we have

$$\sum_{i \in R} i = \sum_{i \in R^*} i - i_p + i_p + i_{2k-1} + i_q + i_{2k-1} = i^* - i_p + i_p + i_{2k-1} + i^* + i_{2k-1} = 0,$$

and $i_p + i_{2k-1} = (1, a^p_2, \ldots, a^p_k) \neq i_q + i_{2k-1} = (1, a^q_2, \ldots, a^q_k)$ are the only two elements of $R$ represented by the sequences with 1 on the first position.

Now we are ready to determine the value of $\chi^G_{\Sigma}(G)$ for arbitrary connected non-bipartite graph.
Lemma 2.4. Let $G$ be a connected non-bipartite graph of order $n \geq 3$. Then

$$\chi^\Sigma_g(G) = \begin{cases} \chi(G) + 1 & \text{when } G \text{ is ugly} \\ \chi(G) & \text{otherwise.} \end{cases}$$

Proof. Just as in the case of bipartite graphs, it suffices to present the construction, proving this way the upper bound. We will denote the colour classes with $V_1, \ldots, V_{\chi(G)}$, assuming that the odd classes precede the even ones in this sequence. We start with labelling all the edges with 0. Now we are going to adjust some of the edge labels.

Observe that choosing any two vertices $x_i$ and $x_j$ of $G$ and any $a \in \mathcal{G}$, both operations $\phi_o(x_1, x_2) = a$ and $\phi_e(x_1, x_2) = a$ are feasible as $G$ is connected and contains at least one odd cycle.

Let us define two special ways of labelling the edges of the subsets of $V(G)$. Assume first that some vertex set $V_j \subseteq V(G)$ is even. For any $a \in \mathcal{G}$, we can increase the weighted degree of every vertex of $V_j$ by $a$ in the following way. If $a = 0$, then we do not change the label of any edge. If $a \neq 0$, then we construct in any way $|V_j|/2$ disjoint pairs of vertices $(x_{i,1}^j, x_{i,2}^j)$, $i = 1, \ldots, |V_j|/2$ and for each such pair we put $\phi_e(x_{i,1}^j, x_{i,2}^j) = a$, increasing $w(x)$ by $a$ for every $x \in V_j$. In order to simplify the notation, we will denote such a labelling of $V_j$ by $\Phi(V_j) = a$. Observe, that the operation $\Phi(V_j) = 0$ is feasible also in the case when $V_j$ is odd.

Assume now, that two vertex sets $V_j, V_k \subseteq V(G)$, are odd. This time, for any couple $(-a, a)$ we can increase by $a$ the weighted degree of every vertex in $V_j$ and increase by $-a$ the weighted degree of every vertex in $V_k$. We do this by choosing arbitrary vertices $x_j \in V_j$ and $x_k \in V_k$ and putting $\phi_o(x_j, x_k) = a$, $\Phi(V_j) = a$ and $\Phi(V_k) = -a$. In order to simplify the notation, we will denote such an operation with $\Phi(V_j, V_k) = (a, -a)$.

In the remainder of the proof the number of odd classes will be denoted by $t$.

If $\chi(G) \equiv 1 \pmod{2}$ or $G$ is ugly, then $|\mathcal{G}| \equiv 1 \pmod{2}$ and we can find $[t/2]$ disjoint couples of distinct non-zero elements the form $\{a_j, -a_j\}$, where $j = 1, \ldots, [t/2]$. We construct $[t/2]$ pairs of odd colour classes $(V_{2j-1}, V_{2j})$, $1 \leq j \leq [t/2]$ and for every such pair we put $\Phi(V_{2j-1}, V_{2j}) = (a_j, -a_j)$. Now, if $t$ is odd, then we put $\Phi(V_t) = 0$. Finally, we assign to each of the even classes $V_j$, $j = t + 1, \ldots, |\mathcal{G}|$ one of the remaining $|\mathcal{G}| - t$ elements of $\mathcal{G}$, $a_j$, and we put $\Phi(V_j) = a_j$.

If $\chi(G)$ is even and $G$ is not ugly, then $|\mathcal{G}| = \chi(G)$ is even. If $t \leq 1$, then we assign distinct $a_j \in \mathcal{G}$, $a_j \neq 0$ to $|\mathcal{G}| - 1$ distinct even classes and put $\Phi(V_j) = a_j$, $j = 2, \ldots, |\mathcal{G}|$. We also put $\Phi(V_1) = 0$, no matter if $V_1$
is even or odd. If $t = 2$, then we choose any $a_1 \in G$, $a_1 \neq 0$, $2a_1 \neq 0$ (such $a_1$ must exist as $G$ is not ugly and so $G \ncong Z_2 \times \cdots \times Z_2$). Let us denote $a_2 = -a_1$ and let $a_3, \ldots, a_{|G|}$ be the remaining elements of $G$ ordered in any way. We put $\Phi(V_1, V_2) = (a_1, -a_1)$ and $\Phi(V_j) = a_j$ for $j = 3, \ldots, |G|$. Finally, if $t \geq 3$, then we have to distinguish few cases depending on the structure of $G$ and the exact value of $t$. Observe that $G \cong Z_{2p_1} \times Z_{2p_2} \times \cdots \times Z_{2p_k} \times G^*$ for some positive integers $p_1, p_2, \ldots, p_k$ and some Abelian group $G^*$ where $|G^*| = 2p_0 + 1$ for some integer $p_0 \geq 0$. There are exactly $2^k - 1$ involutions $i_1, i_2, \ldots, i_{2^k-1}$ in $G$. The remaining non-zero elements of $G$ form $(|G| - 2^k)/2 \geq 1$ disjoint pairs of the form $\{a_j, -a_j\}$.

**Case 1:** $k = 1$. We have exactly one involution $i_1$ in $G$. If $t < |G|$, then we can find $[t/2]$ disjoint couples of distinct non-zero elements the form $\{a_j, -a_j\}$, where $j = 1, \ldots, [t/2]$. We construct $[t/2]$ pairs of odd colour classes $(V_{2j-1}, V_{2j})$, $1 \leq j \leq [t/2]$ and for every such pair we put $\Phi(V_{2j-1}, V_{2j}) = (a_j, -a_j)$. Now, if $t$ is odd, then we put $\Phi(V_t) = 0$. Finally, we assign to each of the even classes $V_j$, $j = t + 1, \ldots, |G|$ one of the remaining $|G| - t$ elements of $G$, $a_j$, and we put $\Phi(V_j) = a_j$.

If $t = |G|$, then $p_1 \geq 2$ as $G$ is not ugly, and so $G$ has a subgroup $G^{**} = \{0, a_1, 2a_1 = i_1, 3a_1\}$. We choose any three vertices $x_1 \in V_1$, $x_2 \in V_2$ and $x_3 \in V_3$ and put $\phi_e(x_1, x_2) = i_1$ and $\phi_e(x_1, x_3) = a_1$. Now, as $V_j \setminus \{x_j\}$ are even for $j = 1, 2, 3$, we put $\Phi(V_1 \setminus \{x_1\}) = 3a_1$, $\Phi(V_2 \setminus \{x_2\}) = i_1$, $\Phi(V_3 \setminus \{x_3\}) = a_1$ and $\Phi(V_4) = 0$. All the remaining $|G| - t$ odd classes we order in pairs $(V_{2j-1}, V_{2j})$, $j = 3, \ldots, |G|/2$, arrange with disjoint pairs of elements $\{a_j, -a_j\}$, $\{a_j, -a_j\} \subset G \setminus G^{**}$ and we put $\Phi((V_{2j-1}, V_{2j})) = (a_j, -a_j)$.

**Case 2:** $k \geq 2$, $t \leq 2^k, t \neq 2^k - 2$. The number of involusions is $2^k - 1 \geq 3$. Using Lemma 2.3, we choose $t$ elements $i_1, \ldots, i_t \in G$ such that $2i_j = 0$, $j = 1, \ldots, t$ and

$$\sum_{j=1}^{t} i_t = 0.$$ 

We choose one vertex $x_j$ from every colour class $V_j$, $j = 1, \ldots, t$, and we put $\phi_e(x_1, x_j) = i_j$ for $j = 2, \ldots, t$. This way we obtain $w(x_j) = i_j$ for $j = 1, \ldots, t$. As in every $V_j$, $j = 1, \ldots, t$, the number of remaining vertices is now even, we put $\Phi(V_j \{x_j\}) = i_j$, $j = 1, \ldots, t$. As the remaining colour classes $V_j$ are even, we can use the remaining $a_j \in G$ by putting $\Phi(V_j) = a_j$, $j = t + 1, \ldots, |G|$.
Case 3: \( t = 2^k - 2 \). We choose any \( a_1 \in G, a_1 \neq 0, 2a_1 \neq 0 \) (such \( a_1 \) must exist as \( G \) is not ugly and so \( G \neq Z_2 \times \cdots \times Z_2 \)). Let us denote \( a_2 = -a_1 \). We put \( \Phi(V_{t-1}, V_t) = (a_1, -a_1) \). Now we are left with \( t - 2 = 2^k - 4 \geq 4 \) odd classes (\( t \geq 3 \) implies \( k \geq 3 \) in this case). Using Lemma 2.3, we choose a set of \( t - 2 \) elements \( R = \{i_1, \ldots, i_t\} \subseteq G \) such that \( 2i_j = 0, j = 1, \ldots, t \) and
\[
\sum_{j=1}^{t-2} i_t = 0.
\]
We choose arbitrary vertex \( x_j \) from every colour class \( V_j, j = 1, \ldots, t - 2 \), and we put \( \phi_o(x_j, x_j) = i_j \) for \( j = 2, \ldots, t \). This way we obtain \( w(x_j) = i_j \) for \( j = 1, \ldots, t - 2 \). As in every \( V_j, j = 1, \ldots, t - 2 \), the number of remaining vertices is now even, we put \( \Phi(V_j \setminus \{x_j\}) = i_j, j = 1, \ldots, t - 2 \). As the remaining colour classes \( V_j, j = t + 1, \ldots, |G| \) are even, we can use the remaining \( |G| - t \) elements \( a_j \in G \setminus (R \cup \{a_1, a_2\}) \) by putting \( \Phi(V_j) = a_j, j = t + 1, \ldots, |G| \).

Case 4: \( t > 2^k \). We colour \( \lceil (t - 2^k) / 2 \rceil \) pairs of odd colour classes with the highest indices using \( \lceil (t - 2^k) / 2 \rceil \) disjoint pairs of non-zero elements of \( G \) of the form \( \{-a_j, a_j\} \) in the similar way as in the case when \( k = 1 \). The remaining \( 2^k - 1 \) or \( 2^k \) odd classes are coloured as in the case 2.

In order to finish the proof of the Theorem we need the following lemma.

Lemma 2.5. Given a connected graph \( G \), it is possible to find a vertex-\( G \)-colouring of \( G \) for every Abelian group \( G \) of order \( h > \chi^G(G) \), except the case when \( h = 2^q \) for some integer \( q \geq 2 \) and \( G = K_{h-2} \).

Proof. As before, let \( t \) denote the number of odd colour classes of \( G \). If \( h \) is odd, then we can choose \( \lfloor t/2 \rfloor \) disjoint pairs of elements of \( G \) of the form \((a_j, -a_j)\) and put \( \Phi(V_{2j-1}, V_{2j}) = (a_j, -a_j) \) for \( j = 1, \ldots, \lfloor t/2 \rfloor \). If \( t \) is odd, then we put \( \Phi(V_t) = 0 \). Finally, we choose any \( \chi(G) - t \) of the remaining elements of \( G \) (let us denote them with \( a_j, j = t + 1, \ldots, \chi(G) \)) and we put \( \Phi(V_j) = a_j, j = t + 1, \ldots, \chi(G) \). If \( h \) is even, then we have to consider three cases.

If \( |V(G)| > h \) then, using \( h - \chi(G) \) additional colours, we can recolour some of the vertices in order to obtain proper \( h \)-colouring of \( G \). Moreover, we are able to control the parity of new colour classes this way in order to obtain the ”good” number of odd colour classes. In such a case we can use the lemma 2.4 in order to obtain the desired \( G \)-colouring.
If $|V(G)| \leq h$ and $G$ is bipartite, then either one of its colour classes is even, or both classes are odd. If there exists even colour class, then we use same method as in the proof of the Lemma 2.2 using one of the involutions of $G$ instead of 1 (such an involution has to exist as $h$ is even). If both colour classes are odd and there is some element $a$ of order greater than 2 in $G$, then we use the method from the proof of lemma 2.2 substituting 1 with $a$ and 2 with $-a$. Finally, if both colour classes are odd and all the non-zero elements of $G$ are involutions, then we recolour the vertices of one of the colour classes of $G$ in order to obtain 4 odd colour classes $V_1, V_2, V_3, V_4$ (assume the classes $V_2, V_3$ and $V_4$ are the new ones, so the bipartition of $G$ is $(V_1, V_2 \cup V_3 \cup V_4)$). Observe that $h \geq 4$ as $\chi^*_G(G) = 3$ in this case.

We apply the lemma 2.3 in order to properly colour the graph. To be more specific, we choose 4 distinct elements $i_j \in G$, $2i_j = 0$, $j = 1, 2, 3, 4$ such that $\sum_{j=1}^{4} i_j = 0$ and then we proceed similarly as in the proof of the Lemma 2.3 (the case when $3 \leq t \leq 2^k$, $t \neq 2^k - 2$): we choose any vertices $x_j \in V_j$, $j = 1, 2, 3, 4$ and we put $\phi_e(x_1, x_j) = i_j$, $j = 2, 3, 4$ and $\Phi(V_j \setminus \{x_j\}) = i_j$, $j = 1, 2, 3, 4$.

Last possibility is that $|V(G)| \leq h$ and $G$ is not bipartite. In such a situation we have to distinguish two cases depending on the relation between $|V(G)|$ and $h$.

If $|V(G)| \neq h - 2$ or $h \neq 2^q$ for any integer $q \geq 2$, then we recolour the vertices in order to obtain $t = |V(G)|$ odd (one-element) colour classes $V_j = \{x_j\}$, $j = 1, \ldots, t$. If there is only one involution in $G$, then we can find $\lfloor t/2 \rfloor$ disjoint pairs $\{a_j, -a_j\}$, $a_j \neq 0$, $2a_j \neq 0$ and we put $\phi_o(x_{2j-1}, x_{2j}) = a_j$, $j = 1, \ldots, \lfloor t/2 \rfloor$. If the set of involutions $I^*$ has $k \geq 2$ (i.e. $k \geq 3$) elements, then we can choose $t$ distinct elements $a_1, a_2, \ldots, a_t \in G$ such that

$$\sum_{j=1}^{t} a_j = 0.$$

To be more specific, if $(h - k - 1)/2 \geq \lfloor t/2 \rfloor$, then we choose $\lfloor t/2 \rfloor$ disjoint pairs $\{a_j, -a_j\}$, $a_j \neq 0$, $2a_j \neq 0$, $j = 1, \ldots, \lfloor t/2 \rfloor$. If $(h - k - 1)/2 < \lfloor t/2 \rfloor$, then we choose all the $(h - k - 1)/2$ disjoint pairs $\{a_j, -a_j\} \subset G$, $a_j \neq 0$, $2a_j \neq 0$, $j = 1, \ldots, (h - k - 1)/2$ and we add to this set $t - (h - k - 1)/2$ elements of $I^* \cup \{0\}$ - their existence is granted by the Lemma 2.3 if only $t - (h - k - 1)/2 \notin \{2, k - 1\}$. If $t - (h - k - 1)/2 \in \{2, k - 1\}$, then we choose $(h - k - 1)/2 - 1$ pairs $\{a_j, -a_j\} \subset G$ and $t - (h - k - 1)/2 + 2$ elements of $I^* \cup \{0\}$. The only situation when we can not do that is when $(h - k - 1)/2 = 0$, i.e. $G \cong Z_2 \times \cdots \times Z_2 = (Z_2)^q$ for some integer $q$ and $t \in \{2, h - 2\}$ (i.e. $t = h - 2$, as $V(G) \geq 3$).
Finally, if $|V(G)| = h - 2$ and $\mathcal{G} \cong Z_2 \times \cdots \times Z_2 = (Z_2)^q$ for some integer $q \geq 2$, then we consider two cases, depending on the structure of $G$. Observe that in both cases $\chi^\Sigma_g(G)$.

If $G$ is not a complete graph, then $\chi(G) < |V(G)|$. That means that either the number of odd colour classes $t < |V(G)|$ satisfies $t \in \{0\} \cup \{3, \ldots, h - 3\}$, or there are exactly 2 odd colour classes and we can recolour one of the even ones in order to obtain $t + 2 = 4 \neq h - 2$ odd colour classes. In both situations we can apply the same methods as in the previous case in order to find vertex-$\mathcal{G}$-colouring labelling of $G$. Observe that we can exclude the case $t = 1$, as it contradicts with the parity of $|V(G)| = h - 2$.

Assume finally that $G \cong K_{2q-2}$. Suppose there is some vertex-$\mathcal{G}$-colouring labelling $f$ of $G$. Then the sum of all weighted degrees would be $-a - b = a + b$ for some distinct elements $a, b \in \mathcal{G}$, as the sum of all the elements of $\mathcal{G}$ is 0. On the other hand we have

$$\sum_{x \in V(G)} w(x) = \sum_{e \in E(G)} 2f(e) = 0,$$

so $a + b = 0$ and $a = b$, a contradiction.

The main result of our paper follows from the above lemmas. If there is no ugly component $C$ such that $\chi(C) = \chi(G)$, then we know that $\chi^\Sigma_g(G) \geq \chi(G)$. On the other hand, using Lemmas 2.2 and 2.4 we are able to find the vertex-$\mathcal{G}$-colouring edge labelling of every component $C_j$ of $G$, for any Abelian group $\mathcal{G}$, $|\mathcal{G}| = \chi^\Sigma_g(C_j) \leq \chi(G)$. Thus by Lemma 2.5 we are able to find the desired $\mathcal{G}$-colouring of $G$ for any Abelian group $\mathcal{G}$, $|\mathcal{G}| = \chi(G)$, except the case when there is some component $C_2 \cong K_{\chi(G)-2}$ and $\chi(G) = 2^q$ for some integer $q \geq 2$. But in such a case Lemma 2.5 guarantees that there exists a vertex-$\mathcal{G}$-colouring edge labelling for every Abelian group $\mathcal{G}$, $|\mathcal{G}| \geq \chi(G) + 1$.

If there is some ugly component $C$ such that $\chi(C) = \chi(G)$, then by Lemma 2.1 we know that $\chi^\Sigma_g(G) \geq \chi(G) + 1$. On the other hand, using Lemmas 2.2 and 2.4 we are able to find the vertex-$\mathcal{G}$-colouring edge labelling of every component $C_j$ of $G$, for any Abelian group $\mathcal{G}$, $|\mathcal{G}| = \chi^\Sigma_g(C_j) \leq \chi(G) + 1$. Thus by Lemma 2.3 we are able to find the desired $\mathcal{G}$-colouring of $G$ for any Abelian group $\mathcal{G}$, $|\mathcal{G}| = \chi(G) + 1$ (note that $\chi(G) + 1$ is odd).

3 Final Remarks

Using the argument similar to the final part of the proof of Theorem 1.2 we can formulate the following corollary.
Corollary 3.1. Let \( G \) be any graph with no components of order less than 3. Then for every \( h \geq \chi^\Sigma(G) \) there exists a vertex-\( G \)-colouring edge labelling of \( G \) for every Abelian group \( \mathcal{G} \), \( |\mathcal{G}| = h \), except the case when \( \mathcal{G} \cong \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2 \cong (\mathbb{Z}_2)^q \) for some integer \( q \geq 1 \) and there exists a component \( C \cong K_{h-2} \) of \( G \).

In the introduction we mentioned that the problem considered here is analogous to the problem of finding \( \chi^\Sigma(G) \), i.e. the smallest \( k \) such that there exists a vertex-colouring \( k \)-edge-labelling. There is also total version of this problem, where vertex labelling is allowed and the weighted degree is calculated as

\[
  w(v) = f(v) + \sum_{e \ni v} f(e)
\]

(see e.g. [9]). In case of the problem presented here, such a modification does not make sense. Trivially, for every graph \( G \), any group of order \( \chi(G) \) would be enough as one can label the vertices of different colour classes with distinct elements of \( \mathcal{G} \) and then put 0 on all the edges.

In the proof of Theorem 1.2 we often use the fact that we are allowed to use 0 on edges. Thus the natural problem is the following.

Problem 3.2. Let \( G \) be a simple graph with no components of order less than 3. For any Abelian group \( \mathcal{G} \), let \( \mathcal{G}^* = \mathcal{G} \setminus \{0\} \). Determine non-zero group sum chromatic number \( (\chi^\Sigma^*(G)) \) of \( G \), i.e. the smallest value \( s \) such that taking any Abelian group \( \mathcal{G} \) of order \( s \), there exists a function \( f : E(G) \to \mathcal{G}^* \) such that the resulting weighted degrees properly colour the vertices.

In this case the variant of the problem with labels on vertices allowed does not have to be trivial.

All the elements of \( \mathcal{G} \) can be obtained as some combination of not necessarily all of its elements, in particular of its generators. The question is, how many elements of \( \mathcal{G} \) we have to use in order to obtain vertex-\( \mathcal{G} \)-colouring edge labelling.

Problem 3.3. Assume that for given simple graph \( G \) with no components of order less than 3 there exists vertex-\( \mathcal{G} \)-colouring edge labelling for every group \( \mathcal{G} \) of order \( s \). What is the minimum number \( k = k(G,s) \) such that for every group \( \mathcal{G} \) of order \( s \) there is a subset \( S \subset \mathcal{G} \), \( |S| \leq k \) such that there exists a vertex-\( \mathcal{G} \)-colouring edge labelling \( f : E(G) \to S \)?

So far we considered only finite Abelian groups. So, next question seems to be natural, as some generalization of the problem of the ordinary vertex-colouring edge-labelling.
Problem 3.4. Let $G$ be a simple graph with no component of order less than 3. Determine the smallest value of $k$ such that for any infinite Abelian group $G$ there exists a subset $S \subseteq G$, $S \leq k$ such that there exists a vertex-$G$-colouring labelling $f : E(G) \to S$.

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