Dynamic Laplace: Efficient Centrality Measure for Weighted or Unweighted Evolving Networks

Mário Cordeiro¹, ², Rui Portocarrero Sarmento¹, ², Pavel Brazdil², and João Gama²

¹Doctoral Program in Informatics Engineering, Faculty of Engineering, University of Porto
²INESC TEC - Institute for Systems and Computer Engineering, Technology and Science

Abstract

With its origin in sociology, Social Network Analysis (SNA), quickly emerged and spread to other areas of research, including anthropology, biology, information science, organizational studies, political science, and computer science. Being it’s objective the investigation of social structures through the use of networks and graph theory, Social Network Analysis is, nowadays, an important research area in several domains. Social Network Analysis cope with different problems namely network metrics, models, visualization and information spreading, each one with several approaches, methods and algorithms. One of the critical areas of Social Network Analysis involves the calculation of different centrality measures (i.e.: the most important vertices within a graph). Today, the challenge is how to do this fast and efficiently, as many increasingly larger datasets are available. Recently, the need to apply such centrality algorithms to non static networks (i.e.: networks that evolve over time) is also a new challenge. Incremental and dynamic versions of centrality measures are starting to emerge (betweenness, closeness, etc). Our contribution is the proposal of two incremental versions of the Laplacian Centrality measure, that can be applied not only to large graphs but also to, weighted or unweighted, dynamically changing networks. The experimental evaluation was performed with several tests in different types of evolving networks, incremental or fully dynamic. Results have shown that our incremental versions of the algorithm can calculate node centralities in large networks, faster and efficiently than the corresponding batch version in both incremental and full dynamic network setups.

1 Introduction

Centrality measures in Social Network Analysis (SNA) have been an important area of research as they help us to identify the relevant nodes. Researchers have invested a lot of effort to develop algorithms that could efficiently calculate the centrality measures of nodes in networks. With the explosion of social networks users, for example, like Twitter or Facebook, the networks have grown to the point where the use of batch algorithms cannot handle this data efficiently. Thus, to perform the analysis of large and changing networks, it is necessary to adopt streaming techniques and use of incremental algorithms. This way, researchers try to speed-up the process and use less memory whenever possible, by avoiding to process the full network in each iteration.

Our contribution, in this paper, is an efficient solution to calculate a particular centrality measure, the Laplacian centrality, in an incremental only or full dynamic setting. By incremental only we mean networks in which only new nodes and edges are added to the network in subsequent time snapshots, by fully dynamic we mean support for addition and removal of nodes and edges in subsequent time snapshots. We present a solution that is accurate and faster than the corresponding batch algorithm for Laplacian centrality on large evolving networks. The proposed incremental algorithm supports both weighted or unweighted networks.

Succinctly, this document starts with an introduction to related work in Section 3. After the introduction to the related work, we briefly explain the nomenclature used throughout the paper in Section 2. We explain our incremental algorithm in Section 4. Then, in Section 5, we write about the results of our experiments with the developed algorithm. Finally, in Section 6, we write about our work and write about possible directions for future action regarding the area covered in this document.
2 Nomenclature

The undirected unweighted graph that represents a network with \( N \) nodes and \( M \) links is given by \( G = (V,E) \). The components of the graph are: the node set \( V \), which is just a list of indices; the edge set \( E \) where each edge consists of two vertices. The undirected graph has \( n = |V| \) nodes, \( V = \{u_1,u_2,\ldots,u_n\} \), and \( e = |E| \) edges, \( E = \{(i_1,j_1),(i_2,j_2),\ldots,(i_e,j_e)\} \). For each node \( u \), \( d_u \) is its respective degree. A more strict type of graph is the so-called directed graph (or directed network). Directed graphs can be defined as graphs whose all edges have an orientation assigned, so the order of the vertices they link matters. Formally, a directed graph \( D \) is an ordered pair \((V(D),A(D))\) consisting of a nonempty set \( V(D) \) of vertices and a set \( A(D) \), disjoint from \( V(D) \), of arcs. If \( e_{ij} \) is an arc and \( u_i \) and \( u_j \) are vertices such that \( e_{ij} = (u_i,u_j) \), then \( e_{ij} \) is said to join \( u_i \) to \( u_j \), being the first vertex \( u_i \) called initial vertex, and the second vertex \( u_j \) called the terminal vertex.

For undirected and unweighted graphs, adjacency matrices are binary (as a consequence of being unweighted) and symmetric (as a consequence of being undirected), with \( a_{ij} = 1 \) representing the presence of an edge between vertices \( u_i \) and \( u_j \), and \( a_{ij} = 0 \) representing the absence of an edge between vertex pair \( (u_i,u_j) \). For undirected and weighted graphs, the entries of such matrices take values from the interval \([0,\max(w)]\) and are symmetric. For directed and weighted graphs, the entries of such matrices take values from the interval \([0,\max(w)]\) and are non-symmetric. In any of these cases, we deal with non-negative matrices.

3 Related Work

3.1 Centrality Measures for Static Networks

In this section, we briefly introduce some of the commonly used centrality measures, in the context of retrieving centrality values for the nodes in a network. Although there is no consensual best centrality measure for graphs, several measures are accepted and give good centrality values that are valuable in different scenarios. Fig. 1 show a comparison on the central node calculated via different methods. All these centrality measures are presented and explained below.

![Figure 1. Different centrality measures in a graph](image)

**Betweenness Centrality:** measures the extent to which a node lies between other nodes in the network. Thus, the nodes with higher betweenness are included in more shortest paths between nodes that are not directly connected. Nodes with high betweenness occupy critical roles in the network structure since they usually have a network position that allows them to work as an interface between different groups of nodes [13]. The flow of information between two nodes that are not directly connected in the network, rely on the central nodes to propagate the information to these non-connected nodes [20]. Thus, the central nodes, i.e., the nodes with higher values of this centrality measure are important nodes that provide better dissemination of information in the network. The Betweenness Centrality of node \( u \) is given by:

\[
b_u = \sum_{s,t\in V(G)\setminus u} \frac{\sigma_{s,t}(u)}{\sigma_{s,t}}
\]
where $\sigma_{s,t}$ denotes the number of shortest paths between vertices $s$ and $t$ (usually $\sigma_{s,t} = 1$) and $\sigma_{s,t}(u)$ expresses the number of shortest paths passing through node $u$.

**Closeness Centrality:** is a measure based on the concept of distance to other nodes. A node has higher closeness centrality when the shortest path to the other nodes are shorter and lower closeness centrality when the shortest paths to all other nodes in the network are longer. Thus, Closeness Centrality is a measure of how fast a given node can reach all other nodes in the network, in average [13]. The Closeness Centrality of node $u$ is given by:

$$C_{lu} = \frac{N - 1}{\sum_{v \in V(G) \setminus u} d(u, v)}$$

(2)

where $d(u, v)$ denotes the shortest paths between vertices $u$ and $v$ and $n$ expresses the number of nodes in the graph $G$. $N - 1$ represents the normalized form where $N$ is the number of nodes in the graph. For large graphs this difference becomes inconsequential so the $-1$ is dropped.

**Eigenvector Centrality:** is given by the first eigenvector of the adjacency matrix. Therefore, eigenvector centrality assumes that the status of a node is recursively defined by the status of his/her first-degree connections, i.e., the nodes that are directly connected to a particular node [13]. Eigenvector Centrality of node $i$ is given by:

$$x_i = \frac{1}{\lambda} \sum_{j=1}^{n} a_{ij} x_j$$

(3)

where $x_i/x_j$ denotes the centrality of node $i/j$, $a_{ij}$ represents an entry of the adjacency matrix $A$ ($a_{ij} = 1$ if nodes $i$ and $j$ are connected by an edge and $a_{ij} = 0$ otherwise) and $\lambda$ denotes the largest eigenvalue of $A$.

Eigenvector centrality is an important measure since it is based not only on the amount of connections but also the quality of the links a node has.

**Laplacian Centrality:** In [15], Xingqin Qi et al. introduced a novel centrality measure. The authors stress that their new measure based on the Laplacian energy of a node would outperform Betweenness and Closeness centrality regarding complexity. Therefore, the authors achieved a more efficient way of retrieving measurements of centrality in a network. In their paper, authors also compare their new measure with Betweenness centrality and Closeness centrality and prove the reliability of their measure by providing similar results per node. The definition of the Laplacian Centrality of a node $u_i$ is based in the Laplacian Energy of the graph $G$ obtained after removing the node $u_i$ from the graph, and by following the expression [15]:

$$C_L(u_i, G) = \frac{(\Delta E)_i}{E_L(G)} = \frac{E_L(G) - E_L(G_{-i})}{E_L(G)}$$

(4)

From equation 4, we can expect a Laplacian Centrality value between 0 and 1. These values increase correspondingly to the increased centrality of the node. Nonetheless, since the definition is in itself a normalization, it is a standard procedure to present the non-normalized values $(\Delta E)_i$ for each node. This way, we can achieve a faster-ranking computation without considering the total graph energy $E_L(G)$. We present adapted versions of the Laplacian Centrality in the scope of this work. Therefore, a detailed analysis of the algorithms is shown here. This is true either for the unweighted or the weighted variants, for further study and discussion within this document. In Algorithm 1, method LapCentUnweighted(), is presented the implementation of the Laplacian Centrality measure as described in [14, 15] for unweighted networks. Presented pseudocode is based on the implementation of the Laplacian Centrality as made available by [6, 21]. The weighted variant of the Laplacian Centrality
is also made available by [6, 21] (see LapCentWeighted()). The main difference to the unweighted variant is the way how the centrality value is calculated for each node. For the weighted version, instead of considering only the degrees of the nodes, it needs to perform the calculation of all the two walks from affected nodes. This approach revealed to be unfeasible. Wheeler [21] developed a different algorithm, the Algorithm 1, LapCentWeighted(), that proven to be quicker, easy to implement and delivers good centrality results. For the calculation of each node Laplace Centrality, both unweighted and weighted methods are shown in LapCentUnweighted() and LapCentWeighted() methods, respectively. Both algorithms were adapted to work in an evolving network setting by processing each one of the temporal snapshots Main(), either incremental or fully dynamic. By observation of the LapCentUnweighted() method, for each loop of the algorithm, it can be concluded that the centrality parameter is a function of the local degree plus the degree of the neighbors (with different weights for each). Therefore, the metric is not a global measure. The local degree and the 1st order neighbors degree are all that is needed to calculate the metric for unweighted networks. The Main() method shows the pseudo-code required for performing batch Laplacian Centrality calculation on an evolving network. The algorithm will require performing a full calculation for each one of the snapshots \{G_0,G_1,\ldots,G_n\} included in the dataset Dataset. Notice that no Laplacian Centrality data nor network data is shared between snapshots. We will explore this inefficiency of the algorithm in Section 4 when proposing the incremental version of the algorithm.

3.2 Centrality Measures for Evolving Networks

Due to requirements for size or dynamics of networks, some centrality measures were already adapted to be incremental or dynamic. The motivation for the use of incremental algorithms vary, but, the main one, is to efficiently process large volumes of data that is subject to many (relatively) small changes over time. In this subsection, we briefly state some of these improvements regarding incremental centrality measure algorithms. The authors of these algorithms argument their solutions are faster than the batch versions. Our objective is to achieve similar improvements, for the batch version of the Laplacian Centrality algorithm.

Betweenness Centrality: The Brandes algorithm [3] (currently widely used), runs in \(O(mn + n^2 \log n)\) time, where \(n = |V|\) and \(m = |E|\). Nasre et al. [12] developed an incremental algorithm to perform Betweenness Centrality (BC) measures in dynamic networks. The BC score of all vertices in \(G\) is updated when a new edge is added to \(G\), or the weight of an existing edge is reduced. Their incremental algorithm runs in \(O(m'n + n^2)\) time, where \(m'\) is bounded by \(m^* = |E^*|\), and \(E^*\) is the set of edges that lie on the shortest path in \(G\). The authors explain that, even for a single edge update, their incremental algorithm is the first algorithm that is faster on sparse graphs when compared with recomputing using the well-known static Brandes algorithm. The authors also stress that their algorithm is also likely to be much faster than Brandes on dense graphs since \(m^*\) is often close to linear in \(n\). The authors explain that with preliminary experimental results for their basic edge update algorithm on random graphs, generated using the Erdős-Rényi model, they achieve 2 to 15 times speed-up over Brandes’ algorithm for graphs with 256 to 2048 nodes, with the larger speed-ups on dense graphs. Previously to this update of Betweenness Centrality measurements, Kas et al. had already tried to adapt the algorithm for evolving graphs in [7]. We consider both approaches relevant for the research that needs to calculate Betweenness Centrality in evolving graphs.

Closeness Centrality: Recently, two significant publications regarding the update of Closeness Centrality in evolving graphs were proposed by Kas et al. and Sariyuce et al. [8, 17]. To provide a conceptual overview to the reader, we will focus on Kas et al. work in this paper. Kas et al. [8] developed an incremental Closeness Centrality algorithm for dynamic networks. To compute the closeness values incrementally, for streaming, dynamically changing social networks, all-pairs shortest-paths algorithm proposed by Ramalingam and Reps [16] was extended, such that closeness values are incrementally updated, in line with the changing shortest path distances in the network. As shown in figure 2, the addition of an edge between \(X\) and \(Y\) nodes would be processed by discovering affected sources, i.e., nodes that are
Algorithm 1 Batch Laplace Centrality Algorithm

1: $V \leftarrow \{u_1, u_2, \ldots, u_N\}$, $E \leftarrow \{(i_1, j_1), (i_2, j_2), \ldots, (i_k, j_k)\}$

2: procedure LapCentUnweighted($G \leftarrow (V, E)$) ▷ for unweighted networks

3: $\mathcal{C}_{centralties} \leftarrow \{\}$
4: $\mathcal{D}_{degrees} \leftarrow G.degrees()$
5: $\mathcal{V}_s \leftarrow G.nodes()$
6: for each $v$ in $\mathcal{V}_s$ do
7: $\mathcal{N}_{neighbors} \leftarrow G.neighbors(v)$
8: $\text{loc} \leftarrow \mathcal{D}_{degrees}[v]$
9: $\mathcal{N}_{neighbors}$
10: $\text{net} \leftarrow 2 \sum_{i=1}^{\mathcal{N}_{neighbors}} \mathcal{D}_{degrees}[i]$
11: $\mathcal{C}_{centralties}[v] \leftarrow (\text{loc}^2 + \text{loc} + \text{net})$
12: end for
13: return $\mathcal{C}_{centralties}, |\mathcal{V}_s|$

14: procedure CW($G \leftarrow (V, E), \mathcal{V}_{vertex}, \mathcal{D}_{degrees}$) ▷ centrality weight for weighted networks

15: $\mathcal{N}_{neighbors} \leftarrow G.neighbors(\mathcal{V}_{vertex})$
16: $\mathcal{N}_{neighbors}$
17: $\text{cw} \leftarrow \sum_{i=1}^{\mathcal{N}_{neighbors}} (\mathcal{N}_{neighbors}[i].weight())^2$
18: $\mathcal{N}_{neighbors}$
19: $\text{sub} \leftarrow \sum_{i=1}^{\mathcal{N}_{neighbors}} [(\mathcal{D}_{degrees}[i] - \mathcal{N}_{neighbors}[i].weight())^2 - (\mathcal{D}_{degrees}[i])^2]$
20: return $\text{cw}, \text{sub}$

21: procedure LapCentWeighted($G \leftarrow (V, E)$) ▷ for weighted networks

22: $\mathcal{C}_{centralties} \leftarrow \{\}$
23: $\mathcal{V}_s \leftarrow G.nodes()$
24: for each $v$ in $\mathcal{V}_s$ do
25: $\text{loc} \leftarrow \mathcal{D}_{degrees}[v]$
26: $\mathcal{C}_{centralties}[v] \leftarrow (\text{loc}^2 - \text{sub} + 2 \times \text{cw})$
27: end for
28: return $\mathcal{C}_{centralties}, |\mathcal{V}_s|$

29: procedure MAIN
30: Dataset $\leftarrow \{G_0, G_1, \ldots, G_n\}$
31: for each snapshot in Dataset do ▷ calculation in the full network
32: if unweighted Dataset then
33: $\mathcal{C}_{centralties}, \mathcal{N}_{numCentralities} \leftarrow \text{LapCentUnweighted}(snapshot)$
34: else
35: $\mathcal{C}_{centralties}, \mathcal{N}_{numCentralities} \leftarrow \text{LapCentWeighted}(snapshot)$
36: end if
37: end for
38: end procedure

on the path of the new edge, and the affected sinks, i.e., the nodes that are beyond the added connection. Finally, the authors update the Closeness Centrality values just for the affected nodes. The author’s argue that, for incremental algorithms, computation times can benefit from early pruning by updating only the affected parts. While the original algorithm for Closeness Centrality can be performed by running an all-pair shortest paths algorithm like Floyd-Warshall [5], which results in $O(n^3)$ time complexity, Kas et al. achieve several improvements to their 2-phase algorithm. They argue that the time complexity of Phase-1 to be limited by $O(|\text{Affected}||2)$ where the subscript 2 denotes the size of two-hop neighborhood of all affected nodes. The complexity of Phase-2 is dominated by the complexity of priority queue, denoted by $O(|\text{Affected}||\log|\text{Affected}|)$. The authors’ effort results in significant speed-ups over the most commonly used method, on various synthetic and real-life Datasets, suggesting that incremental algorithm design is a fruitful research area for social network analysts. The speed-ups they achieve with this algorithm vary regarding the topology of the network, and for synthetic networks, they conclude that the incremental algorithm is, on average, six times faster than Dijkstra’s algorithm.
(a) Batch Algorithm

(b) Incremental Algorithm

Figure 3. Calculated node centralities with edge \{(4, 6)\} added. Dark grey nodes affected by addition of edges. Light grey nodes centralities need to be calculated due to their neighbourhood with affected nodes.

Table 1. Example of Centrality calculation for the network presented in Fig. 3.

| Node | Batch step=1 | Batch step=2 | Incremental step=1 | Incremental step=2 |
|------|--------------|--------------|--------------------|--------------------|
|      | Centrality | Calculated | Centrality | Calculated | Centrality | Calculated | Centrality | Calculated |
| 1    |  6         | yes         |  6         | yes         |  6         | yes         |  6         | no         |
| 2    | 12         | yes         | 12         | yes         | 12         | yes         | 12         | no         |
| 3    | 18         | yes         | 18         | yes         | 18         | yes         | 18         | no         |
| 4    | 18         | yes         | 28         | yes         | 18         | yes         | 28         | yes         |
| 5    | 34         | yes         | 34         | yes         | 34         | yes         | 38         | yes (neighbour) |
| 6    | 10         | yes         | 20         | yes         | 10         | yes         | 20         | yes         |
| 7    | 18         | yes         | 20         | yes         | 18         | yes         | 20         | yes (neighbour) |
| Total| 7 out of 7 nodes | 7 out of 7 nodes | 4 out of 7 nodes | 14 centrality calculations | 11 centrality calculations |

Figure 2. Kas Incremental Closeness Centrality

Other Incremental and Dynamic Centrality Measures: Other existing centrality measures, for example, Eigenvector centrality, have improvements or variants developed to approach evolving graphs. Eigenvector centrality, or Eigencentrality, has Google’s PageRank and the Katz centrality as possible variants of this measure [19]. Regarding these variants, PageRank has been updated to be an incremental algorithm by several researchers, for example, in [1, 4, 9]. These improvements over the original PageRank measure show significantly faster results when compared with the original PageRank that, for being an iterative process, did not scale well to large-scale graphs.

4 Proposal for a Dynamic Laplace Centrality algorithm

4.1 Locality of the Laplacian Centrality

As already stated before, the Laplacian Centrality metric is not a global measure, i.e., is a function of the local degree plus the degree’s of the neighbors (with different weights for each). Xingqin Qi et al. [14, 15], and the pseudo-code presented in Algorithm 1, shown that local degree and the 1st order neighbors degree is all that is needed to calculate the metric for unweighted networks. To demonstrate this property, the toy network example in Fig. 3 is used to show the locality of the Laplacian Centrality.

Due to the determinism of the algorithm, the node centrality results for the incremental Laplace algorithm are equal to the results of the batch version of the same algorithm. This is explained by the fact that we are not dealing with probabilistic phenomena nor any randomness in the initialization. The results of both versions of the algorithm have the same values for each node in the dynamic graph of Fig. 3, as we can see in Table 1. The obtained values for centrality are the same for both algorithms (expected), but for step 2 in the incremental algorithm, we only calculated the centralities of the affected nodes plus their neighbors. In total, the Batch algorithm performed 14 node centrality calculations (7 for step 1 and 7 for step 2) while the incremental achieved the same result using only 11 node centrality calculations (7 for step 1 and 4 for step 2).

In the toy network of Fig. 3 are considered two distinct snapshots: \( G_0 = \{(1, 2), (2, 3), (3, 5), (5, 6), (5, 4), (4, 7), (5, 7)\} \), and \( G_1 \), a second snapshot, where a new edge \( (4, 6) \) is added to the network, so \( G_1 = G_0 \cup \{(4, 6)\} \). In the first snapshot \( (G_0) \), the algorithm needs
4.2 Dynamic Laplace Centrality algorithm

The original Laplace Centrality algorithm proposed by Xingqin Qi et al. [15] was primarily designed for static networks (i.e., networks that do not evolve). Nevertheless, being a static algorithm, it can be used to calculate centralities in changing networks with the penalty of performing a full computation of the centralities in each one of the network snapshots. With our proposal, the same Xingqin Qi et al. [15] principles were adapted for two incremental algorithms. The proposed incremental algorithm in [18] presents better computational efficiency, by performing selective Laplace Centrality calculations only for the nodes affected by the addition and removal of edges in one of the snapshots (i.e., it reuses information of the previous snapshot to perform the Laplace Centrality calculations on the current snapshot). This is true for unweighted networks only. Although the algorithm is called incremental, because it avoids full calculations, in each one of the snapshots, it is also prepared for the addition and removal of edges in each one of the increments (i.e., full dynamic algorithm).

Based on the assumptions devised in the Section 4.1 and having by reference the batch version of the Laplacian centrality shown in Algorithm 1, a new incremental algorithm was presented (Unweighted version Algorithm 2 and Weighted version Algorithm 2) ¹. This incremental algorithm achieves better efficiency than the batch version by performing Laplace centrality calculations only in the nodes affected by addition or removal of edges and their 1st order neighbors (full dynamic incremental algorithm). In both versions, Algorithm 2 and Algorithm 3, the incremental method LapCentAddRemove() receives a full graph (network of the previous snapshot) as parameters. It also receives the lists of edges that will be added and removed from the graph in the current iteration (Add and Remove respectively), and the previously calculated centralities to be updated in the current iteration (Centralities). Previously to the beginning of the for each cycle, the algorithm will calculate the set of nodes affected by the addition and removal of nodes (Vj) and the list of 1st order neighbors of affected nodes (Vs). Laplace centrality will only be calculated for this two sets of nodes using the same centrality calculation function employed in the batch algorithm. Function LapCentAddRemove() returns the updated graph G, the updated centrality list Centralities, and | Vj | the number of nodes for which new centralities were calculated for the current iteration. In the main function, initially, in the first iteration, the centralities are calculated for the full network, and this information is reused in the following iterations. In each of the incremental steps, the function LapCentAddRemove() only receives the list of edges that changed from the previous iteration. By analysis of the proposed algorithm, we can conclude that obtained efficiency can be related to the number of edges that change in each of the snapshots, and also the degree of its nodes. Higher degrees will require performing more computations of 1st order neighbors. Please remark that LapCentWeighted() is the weighted version of the unweighted LapCent() while LapCentWeightedAddRemove() the weighted version of LapCentAddRemove().

The complexity of the original algorithm [15] is O(n ∗ Δ²), where n is the number of vertices and Δ as the maximum degree (Δ = maxv∈V(G)d_v). Thus, the total complexity for computing Laplacian centrality for network G with n vertices, m edges and maximum degree Δ would be no more than O(n ∗ Δ²). Nonetheless, according to [21] it should be something like O(n ∗ a), where a is the average number of neighbors for the entire graph, and n is the number of nodes. In the worst case, a is the maximum number of neighbors any node has in the graph. Our improvement of this algorithm brings the use of locality features of the original algorithm to lower the complexity to O(n′ ∗ a), where n’ are the affected nodes in each snapshot of the evolving changes of the networks. The affected nodes by addition or removal of m’ edges is given by n’ = (2 ∗ m’ + 2 ∗ m’ ∗ a), i.e.: the sum of 2 nodes per modified edge (2 ∗ m’) and their respective 1st order neighbours (2 ∗ m’ ∗ Δ) or (2 ∗ m’ ∗ a) with [21] assumption. Final time complexity is O(2 ∗ m’ ∗ a + 2 ∗ m’ ∗ a²).

¹source code available here: https://github.com/mmfcordeiro/DynamicLaplaceCentrality
Algorithm 2 Dynamic Laplace Centrality Algorithm (unweighted version)

1: \( V \leftarrow \{ v_1, v_2, \ldots, v_n \} \) \(, \ E \leftarrow \{ (i_1, j_1), (i_2, j_2), \ldots, (i_e, j_e) \} \)
2: \( A_{dd} \leftarrow \text{array}[[i_1, j_1], \ldots, (i_n, j_n)] \) \(, \ R_{\text{remove}} \leftarrow \text{array}[[i_1, j_1], \ldots, (i_n, j_n)] \)
3: \text{procedure} LapCentAddRemove\( (G \leftarrow (V, E), A_{dd}, R_{\text{remove}}, C_{\text{centralities}}) \)
4: \( V_0 \leftarrow \{ \}
5: \text{for each} \ edge \ in \ A_{dd} \ do
6: \hspace{1em} V_0 \leftarrow V_0 \cup \text{edge}.source() \cup \text{edge}.destination()
7: \hspace{1em} G.add_edge(edge)
8: \text{end for}
9: \text{for each} \ edge \ in \ R_{\text{remove}} \ do
10: \hspace{1em} V_0 \leftarrow V_0 \cup \text{edge}.source() \cup \text{edge}.destination()
11: \text{end for}
12: \text{for each} \ node \ in \ V_0 \ do
13: \hspace{1em} V_f \leftarrow V_f \cup G.\text{neighbors}(node)
14: \text{end for}
15: \text{for each} \ edge \ in \ R_{\text{remove}} \ do
16: \hspace{1em} G.remove_edge(edge)
17: \text{end for}
18: D_{\text{degrees}} \leftarrow G.\text{degrees}()
19: \text{for each} \ v \ in \ V_f \ do
20: \hspace{1em} N_{\text{neighbors}} \leftarrow G.\text{neighbors}()
21: \hspace{1em} \text{loc} \leftarrow D_{\text{degrees}}[v] \cdot N_{\text{neighbors}}
22: \hspace{1em} \text{nei} \leftarrow 2 \cdot \sum_{i=1}^{\text{loc}} D_{\text{degrees}}[i]
23: \hspace{1em} C_{\text{centralities}}[v] \leftarrow (\text{loc}^2 + \text{loc} + \text{nei})
24: \text{end for}
25: \text{return} C_{\text{centralities}}, |V_f |, G
26: \text{end procedure}
27: \text{procedure} Main
28: \hspace{1em} D_{\text{dataset}} \leftarrow \{ G_0, G_1, \ldots, G_n \}, A_{dd} \leftarrow \{ A_0, A_1, \ldots, A_n \}, R_{\text{remove}} \leftarrow \{ R_0, R_1, \ldots, R_n \}
29: \hspace{1em} C_{\text{centralities}}, N_{\text{numCentralities}} \leftarrow \text{LapCent}(G_0) \quad \triangleright \text{initial step in the full network}
30: \hspace{1em} G \leftarrow G_0, i \leftarrow 1
31: \text{while} (i \leq |D_{\text{dataset}}|) \ do \quad \triangleright \text{calculate centralities for the increments}
32: \hspace{1em} C_{\text{centralities}}, N_{\text{numCentralities}}, G \leftarrow \text{LapCentAddRemove}(G, A_{dd}[i], R_{\text{remove}}[i], C_{\text{centralities}})
33: \text{end while}
34: \text{end procedure}

5 Experimental Results

The experimental results were obtained using different size networks. The High-energy physics theory citation network [11] has 27 770 vertices and 352 807 edges, the Autonomous systems AS-733 dataset [11] has 6 474 vertices and 13 895 edges distributed by 733 daily snapshots, and the AS Caida Relationships Datasets [11] has 27 770 vertices and 352 807 edges, the Autonomous systems AS-Caida network [11] has 27 770 vertices and 352 807 edges in 733 daily snapshots, and the AS Caida relationships Datasets [11] has 27 770 vertices and 352 807 edges.

The performed empirical evaluation consisted mainly in comparing run times of each increment (duration of each increment and cumulative execution time). Additionally, the size of the network (number of nodes and edges), the number of added/removed edges in each snapshot (negative values mean more edges were removed than added), and the total number of calculated centralities for each snapshot was also registered. The reader should note that the batch algorithm will always need to calculate centralities for all nodes in the snapshot while it is expected that the Dynamic algorithm only performs centrality calculations for the affected nodes. In the end, an analysis of the total speed-up ratio obtained in each of the steps is provided. For all the experimentation and development, we used an Intel (R) Core (TM) i7-4702MQ processor computer with 8 GBytes and SSD HDD. Three runs per algorithm/dataset were performed with values presented in graphs as the average values.

5.1 Evaluation Sets

The Dynamic Laplace Centrality algorithm was evaluated in incremental and dynamic network setups, results are presented in Section 5.2 and Section 5.3 respectively. The algorithms were tested in their unweighted and weighted variants in each one of those network scenarios:

Incremental Networks Evaluation: The datasets used for the incremental evaluation were the
Algorithm 3: Dynamic Laplace Centrality Algorithm (weighted version)

1: \( V \leftarrow \{v_1, v_2, \ldots, v_n\} \) \( , \ E \leftarrow \{(i_1, j_1), (i_2, j_2), \ldots, (i_m, j_m)\} \)
2: \( A_{dd} \leftarrow \text{array}\{((i_1, j_1)), \ldots, ((i_m, j_m))\} \) \( , \ Remove \leftarrow \text{array}\{((i_1, j_1)), \ldots, ((i_m, j_m))\} \)
3: \text{procedure} \text{LapCentWeightedAddRemove}(G \leftarrow (V, E), A_{dd}, Remove, Centralties)
4: \( V_f \leftarrow \{\} \)
5: \text{for each} \( edge \) \text{in} \( A_{dd} \) \text{do}
6: \( V_s \leftarrow V_s \cup \text{edge}.source() \cup \text{edge}.destination() \)
7: \( G.add\_edge(edge) \)
8: \text{end for}
9: \text{for each} \( edge \) \text{in} \( Remove \) \text{do}
10: \( V_s \leftarrow V_s \cup \text{edge}.source() \cup \text{edge}.destination() \)
11: \text{end for}
12: \text{for each} \( node \) \text{in} \( V_s \) \text{do}
13: \( V_f \leftarrow V_f \cup G.neighbors(node) \)
14: \text{end for}
15: \text{for each} \( edge \) \text{in} \( Remove \) \text{do}
16: \( G.remove\_edge(edge) \)
17: \text{end for}
18: \( D_{\text{degrees}} \leftarrow G.d\_\text{degrees}() \)
19: \text{for each} \( v \) \text{in} \( V_f \) \text{do}
20: \( \text{loc} \leftarrow D_{\text{degrees}}[v] \)
21: \( \text{cw}, \text{sub} \leftarrow \text{LapCentWeighted}(G, \text{loc}, D_{\text{degrees}}) \) \( \triangleright \) calls centrality weight for weighted networks
22: \( \text{Centralties}[v] \leftarrow (\text{loc}^2 - \text{sub} + 2 \ast \text{cw}) \)
23: \text{end for}
24: \text{return} \( \text{Centralties} \), \( |V_f|, G \)
25: \text{end procedure}
26: \text{procedure} \text{Main}
27: \( \text{Dataset} \leftarrow \{G_0, G_1, \ldots, G_n\} \), \( A_{dd} \leftarrow \{A_0, A_1, \ldots, A_n\} \), \( Remove \leftarrow \{R_0, R_1, \ldots, R_n\} \)
28: \( \text{Centralties}, \text{NumCentralties} \leftarrow \text{LapCentWeighted}(G_0) \) \( \triangleright \) initial step in the full network
29: \( G \leftarrow G_0, i \leftarrow 1 \)
30: \text{while} \( (i \leq |\text{Dataset}|) \) \text{do} \( \triangleright \) calculate centralities for the increments
31: \( \text{Centralties}, \text{NumCentralties}, G \leftarrow \text{LapCentWeightedAddRemove}(G, A_{dd}[i], Remove[i], \text{Centralties}) \)
32: \text{end while}
33: \text{end procedure}

the High-energy physics theory citation network [11] for unweighted networks tests, and the Reuters terror news network (DaysAll) [2] for weighted networks tests. The evaluation of the incremental setup was done using the original Laplace Centrality (now on called Batch) and the proposed Dynamic Laplace Centrality (now on called Dynamic) in an incremental setting configuration. In both cases the original Laplace Centrality served as a baseline. In the incremental setup, we considered all the snapshots of the datasets. In the High-energy physics theory citation network, 136 snapshots were built by aggregating timestamps of citations in a monthly basis. In the Reuters terror news network (DaysAll), 66 snapshots were considered with data but aggregated in a daily basis. Regarding the testing, while in the Batch setup, for every snapshot, the centralities were calculated having the full network as input, for the incremental algorithm, in the first snapshot the full network is passed as input, and in the following snapshots, the algorithm only receives the set of edges added to the network in that snapshot (incremental).

Dynamic Networks Evaluation: The evaluation of the Dynamic Laplace Centrality algorithm was performed in a dynamic network setup (addition and removal of edges between snapshots) for unweighted and weighted networks. The Autonomous systems AS-733 dataset [11] with 733 snapshots from November 8 1997 to January 2 2000 and AS Caida Relationships Datasets [11] with a total of 122 snapshots from January 2004 to November 2007 were used to evaluate unweighted dynamic networks. The evaluation of the Dynamic Laplace Centrality algorithm in a weighted dynamic network setup was performed using the Reuters terror news network (DaysAll) [2] in an 30 days sliding window (addition and removal of edges), and the Bitcoin Alpha trust weighted signed network (Bitcoin-alpha) [10] in two sliding window setups (aggregation by day, 30 day sliding window in the Bitcoin-alpha-day and aggregation by month, 12 month sliding window in the Bitcoin-alpha).
5.2 Incremental Network Results

In this subsection, we introduce the reader to the results obtained by the incremental setup of the algorithm for unweighted and weighted networks.

**Unweighted Networks:** The results presented in Fig. 4 are related to the High-energy physics theory citation network [11] in an incremental unweighted network setting where no edges are removed from previous snapshots. Fig. 4 – Network Size, shows the variation of the network over the 136 snapshots regarding the number of nodes and number of edges. Fig. 4 – # Added Edges, shows the number of added edges in each snapshot. Notice that the total number of edges in this dataset increases over the time. In the first snapshots a few edges are added to the network, but at the final snapshots, more than 6000 edges are added in each snapshot. Fig. 4 – # Centralities, shows that the number of calculated centralities in the incremental version is much lower than in the batch version. The batch version requires to compute centralities for all nodes in the snapshot. Fig. 4 – Elapsed Time, shows the time required to perform the centralities measurements in each increment. This figure also shows that the incremental version is not only more efficient but also deals better with both increases in the size of network and increase in the number of added edges. This is confirmed by the total time required for processing the whole network: 27,806 seconds of the incremental version compared to the 121,345 seconds of the batch. It is clear that, as the number of added edges increases, the processing elapsed time of the batch version of the algorithm grows much faster than the incremental algorithm. Thus, regarding speed-up ratio, we achieve a speed-up of up to 6 times the processing time of the batch version with an incremental network (Fig. 4 – Speedup Ratio).

**Weighted Networks:** The results presented in Fig. 5 are related to the Reuters terror news network (DaysAll) [2] in an incremental weighted network setting, in this setup no edges are removed from previous snapshots. Fig. 5 – Network Size, shows the variation of the network over the 66 snapshots regarding the number of nodes and number of edges. Fig. 5 – # Added Edges, shows the number of added edges in each snapshot, Fig. 5 – # Centralities, the number of calculated centralities in the incremental version (also much lower than in the batch version). The Fig. 5 – Elapsed Time and (Fig. 5 – Speedup Ratio) shown that were achieved a speed-up of up to 4 times the processing time of the batch version.

5.3 Full Dynamic Network Results

In this subsection, we provide results obtained with the full dynamic setup of the algorithm for unweighted and weighted networks.

**Unweighted Networks:** Regarding the evaluation of full dynamic networks, the Autonomous systems AS-733 dataset with 733 snapshots was used. Fig. 6 – Network Size, shows the
evolution of the network concerning number of nodes and edges over the time. Please note that this setting contains removal of edges and therefore, Fig. 6 – # Added Edges, presents negative values (more removed edges that added for some snapshots). Fig. 6 – # Centralities, again shows that the batch version calculated centralities for all nodes in each snapshot, while the incremental version did that only for affected nodes. It can be observed by this figure that the network has many edges and nodes variation causing significant change on the Elapsed Time required to compute centralities in each of the snapshots. Although this variation is also seen in the batch version (Fig. 6 – Elapsed Time), the incremental version keeps elapsed times per snapshot lower, resulting in more efficient final execution times: 14,163 seconds for the incremental version, and the batch took 92,362 seconds. With this setup, we can see that the removal and addition of edges provokes an even more pronounced increase of speed-up ratio between the batch version and the dynamic version. The speed-up ratio achieves values of more than 15 times in some operations. In the second dynamic network setup, the evolution of the AS Caida Relationships Datasets over its 122 snapshots is shown in Fig. 7 – Network Size. This network also has negative values for # added edges, but it might be seen as a more stable network than the AS-733 dataset. Both number of affected centralities (Fig. 7 – # Centralities), and elapsed time per iteration (Fig. 7 – Elapsed Time) are also presented. It can be observed that the removal and addition of edges, in this setup, provokes a pronounced increase of speed-up ratio between the batch version and the dynamic version. The speed-up ratio achieves values of up to 5 times in some operations with this network.

**Weighted Networks:** Regarding the evaluation of full dynamic networks with the weighted version of the algorithm, two datasets were used. Starting this explanation with the DaysAll dataset, Fig. 8 – Network Size, shows the evolution of the network concerning number of nodes and edges over the time. Please note that this setting contains removal of edges and therefore, Fig. 8 – # Added Edges, presents negative values (more removed edges that added for some snapshots). Fig. 8 – # Centralities, again shows that the batch version calculated centralities for all nodes in each snapshot, while the incremental version did that only for affected nodes. It can be observed by this figure that the network has many edges and nodes variation causing significant change on the Elapsed Time required to compute centralities in each of the snapshots. Although this variation is also seen in the batch version (Fig. 8 – Elapsed Time), the incremental version keeps elapsed times per snapshot lower resulting in more efficient final execution times: 23,725 seconds for the incremental version while the batch took 67,357 seconds. The speed-up ratio achieves values of more than 4 times in some operations. In the second dynamic network setup, this time by using the evolution of the Bitcoinalpha Dataset over its 1919 daily snapshots is shown in Fig. 9 – Network Size. This network also has negative values for # added edges. Both number of affected centralities (Fig. 9 – # Centralities), and elapsed time per iteration (Fig. 9 – Elapsed Time) are also presented. It can be observed that the removal and addition of edges, in this setup, provokes
a pronounced increase of speed-up ratio between the batch version and the dynamic version. The speed-up ratio achieves values of up to 11 times in some operations with this network. In the third and last dynamic network setup, this time by using the evolution of the Bitcoin-alpha Dataset with a 12 month snapshot setting, we have the results presented in Fig. 10. Fig. 10 – Network Size again shows the increase and decrease of nodes and edges as the stream evolves and nodes and their connections are added or removed. Thus, this network also has negative values for # added edges. Again, we present both number of affected centralities (Fig. 10 – # Centralities), and elapsed time per iteration (Fig. 10 – Elapsed Time). Once again, it can be observed that the removal and addition of edges, in this setup, provokes an increase of speed-up ratio between the batch version and the dynamic version. The speed-up ratio achieves values above 5 times in some operations with this network.

5.4 Discussion

Table 2 shows that maximum values of speed-up for unweighted networks were obtained for step 122 in the High-energy physics (6,310 times), step 286 for AS 733 (17,909 times) and step 77 for AS-Caida (5,141 times). It can be seen that bigger speedups are obtained for small changes in the network where the number of recalculated centralities decreases significantly between snapshots. Regarding the initialization of the algorithm (step 1), depending on the initial network size, the incremental algorithm achieves the same speed-up for networks above 3k nodes / 5,5k edges, but for smaller networks, it can take more time than the batch algorithm in the first iteration. The degree of the affected nodes by addition or removal of nodes could also change the speed-up ratio, higher average degrees or dense networks reduce the speed-up ratio. Networks with high variability
of addition and removal of nodes can also reduce the speed-up ratio. Finally, large or very large networks with smaller network changes between snapshots achieve the maximum speedup values. For unweighted networks, Table 2 show that maximum values of speed-up were obtained in the Bitcoinalpha (11 times) in the step 1477, step 9 for DaysAll (4.5 times) and step 12 for Bitcoinalpha (5.125 times). It can be seen that bigger speedups are obtained for small changes in the network where the number of calculated centralities decreases significantly between snapshots. Regarding the initialization of the algorithm (step 1), depending on the initial network size, the incremental algorithm achieves reasonable speed-up, even for networks above 7k nodes / 48k edges. Again, the
Table 2. Overview of the speed-up values achieved for the 7 datasets. Table shows the values for the initial step (full network for both algorithms), minimum, maximum and average. Apart from the speedup values, the conditions in terms of network size and number of added/removed edges per increment and the respective number of centralities calculated are also presented.

|                      | step | speedup | network size | centralities | added edges |
|----------------------|------|---------|--------------|--------------|-------------|
|                      |      |         | # nodes      | # edges      | batch       | dynamic     | batch       | dynamic     |
| **High-energy physics** | init: | 1       | 0.507        | 4            | 2           | 2           | 2           | 2           |
| (Unweighted Incremental) | min:  | 2       | 1.000        | 9            | 6           | 9           | 7           | 4           |
|                      | average: | 122     | 6.310        | 24023        | 280041      | 24023       | 15023       | 3751        | 3749        |
|                      |      |         | 3.910        | 11784        | 110842      | 11784       | 7233        | 2594        | 2593        |
| **AS 733** | init: | 1       | 0.989        | 3015         | 5539        | 3015        | 3015        | 5539        | 5539        |
| (Unweighted Full Dynamic) | min:  | 639     | 0.027        | 103          | 248         | 103         | 5629        | -11719      | -11719      |
|                      | max:  | 286     | 17,099       | 4020         | 8030        | 4020        | 2268        | -1          | -1          |
|                      | average: | 7,755  | 4180         | 8533         | 4180        | 2919        | 11          | 11          |
| **AS Calida** | init: | 1       | 1.000        | 16301        | 32955       | 16301       | 16301       | 32955       | 32955       |
| (Unweighted Full Dynamic) | min:  | 114     | 0.269        | 8020         | 18203       | 8020        | 25997       | -34488      | -34488      |
|                      | max:  | 77      | 5,141        | 24013        | 49332       | 24013       | 21057       | 243         | 243         |
|                      | average: | 3,818  | 22518        | 45775        | 22518       | 20973       | 441         | 441         |
| **DaysAll** | init: | 1       | 1.016        | 2420         | 9318        | 2420        | 2420        | 9318        | 9318        |
| (Weighted Incremental) | min:  | 2       | 0.996        | 4169         | 21396       | 4169        | 4115        | 12078       | 12078       |
|                      | max:  | 9       | 4,174        | 6741         | 47283       | 6741        | 4114        | 353         | 353         |
|                      | average: | 2,394  | 10044        | 94778        | 10044       | 8485        | 2242        | 2242        |
| **DaysAll** | init: | 1       | 1.047        | 2420         | 9318        | 2420        | 2420        | 9318        | 9318        |
| (Weighted Full Dynamic) | min:  | 95      | 0.892        | 1153         | 2093       | 1153         | 1584        | -1124       | -1033       |
|                      | max:  | 9       | 4,539        | 6741         | 47283       | 6741        | 4114        | 353         | 353         |
|                      | average: | 2,730  | 7109        | 48295        | 7109        | 6443        | 21         | 21         |
| **Bitcoin-alpha-day** | init: | 1       | 1.000        | 7            | 4           | 7           | 4           | 4           |
| (Weighted Full Dynamic) | min:  | 2       | 0.333        | 9            | 8           | 9           | 8           | 5           | 4           |
|                      | max:  | 1477    | 11           | 100          | 118         | 100         | 8           | 1           | 1           |
|                      | average: | 2,878  | 151           | 250          | 151         | 91          | 1           | 1           |
| **Bitcoin-alpha-Month** | init: | 1       | 1.000        | 41           | 62          | 41          | 41          | 41          |
| (Weighted Full Dynamic) | min:  | 72      | 0.500        | 1518         | 4069        | 50          | 55          | -20         | -20         |
|                      | max:  | 12      | 5,125        | 1518         | 4069        | 1456        | 849         | 296         | 296         |
|                      | average: | 2,227  | 934         | 2602         | 934         | 835         | 1          | 1          |

degree of the affected nodes by addition or removal of nodes could also change the speed-up ratio. We again conclude that higher average degrees or dense networks reduce the speed-up ratio, and networks with high variability of addition and removal of nodes can also reduce the speed-up ratio.

6 Conclusions and Future Work

In this paper, we presented an incremental and dynamic setup of the Laplacian Centrality algorithm. Through empiric experiments, we have shown that, the incremental and the dynamic setup of the algorithm are faster and more efficient than the batch version. Both settings, incremental and full dynamic, have shown improvements over the batch version of the algorithm, with both types of evolving networks (weighted and unweighted). Additionally, the dynamic algorithm can achieve a speed-up of more than 17 times when compared to the batch algorithm. The minimum speedup factor obtained in all tested scenarios was 4 times. This clearly shows the advantage of our Laplacian Centrality approach when dealing with evolving networks.

In the future, we plan to extend our research, by comparing the incremental setup of the algorithm with other incremental centrality measures like, for example, Betweenness Centrality and Closeness Centrality. For this purpose, we expect to improve results by achieving further optimization, both for weighted or unweighted networks.

Acknowledgements

Part of this work is financed by the ERDF – European Regional Development Fund through the Operational Programme for Competitiveness and Internationalisation - COMPETE 2020 Programme within project POCI-01-0145-FEDER-006961, and by National Funds through the FCT – Fundação para a Ciência e a Tecnologia (Portuguese Foundation for Science and Technology) as part of project UID/EEA/50014/2013. Rui Portocarrero Sarmento also gratefully acknowledges funding from FCT (Portuguese Foundation for Science and Technology) through a PhD grant (SFRH/BD/119108/2016)
References

[1] Bahman Bahmani, Abdur Chowdhury, and Ashish Goel. Fast incremental and personalized PageRank. Proc. VLDB Endow., 4(3):173–184, December 2010.

[2] Vladimir Batagelj and Andrej Mrvar. Density based approaches to network analysis - analysis of reuters terror news network, 2003.

[3] Ulrik Brandes. A faster algorithm for betweenness centrality. Journal of Mathematical Sociology, 25:163–177, 2001.

[4] Prasanna Desikan, Nishith Pathak, Jaideep Srivastava, and Vipin Kumar. Incremental page rank computation on evolving graphs. In Special Interest Tracks and Posters of the 14th International Conference on World Wide Web, WWW ’05, pages 1094–1095, New York, NY, USA, 2005. ACM.

[5] Robert W. Floyd. Algorithm 97: Shortest path. Commun. ACM, 5(6):345–, June 1962.

[6] The igraph core team. igraph - python recipes. http://igraph.wikidot.com/python-recipes#toc4, 2014. [Online; accessed July-2017].

[7] M. Kas, M. Wachs, K. M. Carley, and L. R. Carley. Incremental algorithm for updating betweenness centrality in dynamically growing networks. In 2013 IEEE/ACM International Conference on Advances in Social Networks Analysis and Mining (ASONAM 2013), pages 33–40, Aug 2013.

[8] Miray Kas, Kathleen M. Carley, and L. Richard Carley. Incremental closeness centrality for dynamically changing social networks. In Proceedings of the 2013 IEEE/ACM International Conference on Advances in Social Networks Analysis and Mining, ASONAM ’13, pages 1250–1258, New York, NY, USA, 2013. ACM.

[9] Kyung Soo Kim and Yong Suk Choi. Incremental iteration method for fast pagerank computation. In Proceedings of the 9th International Conference on Ubiquitous Information Management and Communication, IMCOM ’15, pages 80:1–80:5, New York, NY, USA, 2015. ACM.

[10] Srijan Kumar, Francesca Spezzano, VS Subrahmanian, and Christos Faloutsos. Edge weight prediction in weighted signed networks. In Data Mining (ICDM), 2016 IEEE 16th International Conference on, pages 221–230. IEEE, 2016.

[11] Jure Leskovec, Jon Kleinberg, and Christos Faloutsos. Graphs over time: densification laws, shrinking diameters and possible explanations. In Proceeding of the eleventh ACM SIGKDD international conference on Knowledge discovery in data mining - KDD ’05, page 177, New York, New York, USA, August 2005. ACM Press.

[12] Meghana Nasre, Matteo Pontecorvi, and Vijaya Ramachandran. Betweenness centrality - incremental and faster. CoRR, abs/1311.2147, 2013.

[13] Márcia D. B. Oliveira and João Gama. An overview of social network analysis. Wiley Interdisc. Rev.: Data Mining and Knowledge Discovery, 2(2):99–115, 2012.

[14] Xingqin Qi, Robert D. Duval, Kyle Christensen, Edgar Fuller, Arian Spahiu, Qin Wu, Yezhou Wu, Wenliang Tang, and Cunquan Zhang. Terrorist Networks, Network Energy and Node Removal: A New Measure of Centrality Based on Laplacian Energy. Social Networking, 02(01):19–31, 2013.

[15] Xingqin Qi, Eddie Fuller, Qin Wu, Yezhou Wu, and Cun-Quan Zhang. Laplacian centrality: A new centrality measure for weighted networks. Inf. Sci., 194:240–253, July 2012.

[16] G. Ramalingam and Thomas Reps. An incremental algorithm for a generalization of the shortest-path problem. J. Algorithms, 21(2):267–305, September 1996.

[17] Ahmet Erdem Sariyuce, Kamer Kaya, Erik Saule, and Umit V. Catalyurek. Incremental algorithms for closeness centrality. In Proceedings - 2013 IEEE International Conference on Big Data, Big Data 2013, pages 487–492, 2013.
[18] Rui Portocarrero Sarmento, Mário Cordeiro, Pavel Brazdil, and João Gama. Efficient incremental laplace centrality algorithm for dynamic networks. In International Workshop on Complex Networks and their Applications, pages 341–352. Springer, Cham, 2017.

[19] American Mathematical Society. How google finds your needle in the web’s haystack. http://www.ams.org/publicoutreach/feature-column/fcarc-pagerank, 2007. [Online; accessed February-2018].

[20] Stanley Wasserman and Katherine Faust. Social network analysis: Methods and applications, volume 8. Cambridge university press, 1994.

[21] Andrew P. Wheeler. Laplacian centrality in networkx (python). https://andrewpwheeler.wordpress.com/2015/07/29/laplacian-centrality-in-networkx-python/, 2015. [Online; accessed April-2017].