Scaling positive random matrices: concentration and asymptotic convergence

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Abstract

It is well known that any positive matrix can be scaled to have prescribed row and column sums by multiplying its rows and columns by certain positive scaling factors (which are unique up to a positive scalar). This procedure is known as matrix scaling, and has found numerous applications in operations research, economics, image processing, and machine learning. In this work, we investigate the behavior of the scaling factors and the resulting scaled matrix when the matrix to be scaled is random. Specifically, letting $\tilde{A} \in \mathbb{R}^{M \times N}$ be a positive and bounded random matrix whose entries assume a certain type of independence, we provide a concentration inequality for the scaling factors of $\tilde{A}$ around those of $A = \mathbb{E}[\tilde{A}]$. This result is employed to bound the convergence rate of the scaling factors of $\tilde{A}$ to those of $A$, as well as the concentration of the scaled version of $\tilde{A}$ around the scaled version of $A$ in operator norm, as $M, N \to \infty$. When the entries of $\tilde{A}$ are independent, $M = N$, and all prescribed row and column sums are 1 (i.e., doubly-stochastic matrix scaling), both of the previously-mentioned bounds are $O(\sqrt{\log N}/N)$ with high probability. We demonstrate our results in several simulations.

1 Introduction

Let $A \in \mathbb{R}^{M \times N}$ be a nonnegative matrix. It was established in a series of classical papers [22, 23, 24, 2, 1, 16] that under certain conditions one can find a positive vector $x = [x_1, \ldots, x_M]$ and a positive vector $y = [y_1, \ldots, y_N]$, such that the matrix $P = D(x)AD(y)$ has prescribed row sums $r = [r_1, \ldots, r_M]$ and column sums $c = [c_1, \ldots, c_N]$, where $D(v)$ is a diagonal matrix with $v$ on its main diagonal. The problem of finding the appropriate $x$ and $y$ that produce $P$ with the prescribed row and column sums is known as matrix scaling or matrix balancing; see [10] for a comprehensive review of the topic and its various extensions. Formally, we use the following definition.

Definition 1 (Matrix scaling). We say that a pair of vectors $(x, y)$ scales $A$ to row sums $r$ and
column sums \( c \), if

\[
r_i = \sum_{j=1}^{N} P_{i,j} = \sum_{j=1}^{N} x_i A_{i,j} y_j, \quad \text{and} \quad c_j = \sum_{i=1}^{M} P_{i,j} = \sum_{i=1}^{M} x_i A_{i,j} y_j,
\]

(1)

for all \( i \in [M] \) and \( j \in [N] \). We refer to \( x \) and \( y \) from (1) (or their entries) as scaling factors of \( A \).

In the special case that \( M = N \) and \( r_i = c_j = 1 \) for all \( i \in [M] \) and \( j \in [N] \), the problem of matrix scaling becomes that of finding a doubly-stochastic normalization of \( A \), originally studied by Sinkhorn [22] with the motivation of estimating doubly-stochastic transition probability matrices.

It is important to mention that (1) is a system of nonlinear equations in \( x \) and \( y \) with no closed-form solution. Nevertheless, if the scaling factors \( x \) and \( y \) exist, they can be found by the Sinkhorn-Knopp algorithm [24] (also known as the RAS algorithm), which is a simple iterative procedure that alternates between computing \( x \) via (1) using \( y \) from the previous iteration, and vice versa (a procedure equivalent to alternating between normalizing the rows of \( A \) and normalizing the columns of \( A \) to have the prescribed row and column sums, respectively).

From a theoretical perspective, given a nonnegative matrix \( A \), existence and uniqueness of the scaling factors and of the scaled matrix \( P \) depend primarily on the particular zero-pattern of \( A \); see [1] and references therein for more details. In this work, we focus on the simpler case that \( A \) is strictly positive, in which case existence and uniqueness of the scaling factors and of the scaled matrix \( P \) are guaranteed by the following theorem (see [23]).

**Theorem 1** (Existence and uniqueness [23]). Suppose that \( A, r, \) and \( c \) are positive, and \( \|r\|_1 = \|c\|_1 \). Then, there exists a pair of positive vectors \( (x, y) \) that scales \( A \) to row sums \( r \) and column sums \( c \). Furthermore, the resulting scaled matrix \( P = D(x)A D(y) \) is unique, and the pair \( (x, y) \) can be replaced only with \( (\alpha x, \alpha^{-1} y) \), for any \( \alpha > 0 \).

Over the years, matrix scaling and the Sinkhorn-Knopp algorithm have found a wide array of applications in science and engineering. In economy and operations research, classical applications of matrix scaling include transportation planning [12], analyzing migration fields [25], and estimating social accounting matrices [21]. In image processing and computer vision, matrix scaling was employed for image denoising [18] and graph matching [5]. Recently, matrix scaling has been attracting a growing interest from the machine learning community, with applications in manifold learning [17, 27], clustering [28, 14], and classification [7]. See also [20, 6] for applications of matrix scaling in data science through the machinery of optimal transport.

In many practical situations, matrix scaling is actually applied to a random matrix that represents a perturbation, or a random observation, of an underlying deterministic population matrix; see for example [13, 27, 18, 4]. Arguably, this is the case in all of the previously-mentioned applications of matrix scaling whenever real data is involved. In particular, applications of matrix scaling in machine learning and data science often involve large data matrices that suffer from corruptions and measurement errors, and hence are more accurately described by random models. Due to such challenges, it is important to understand the influence of random perturbations in \( A \) on the required
scaling factors and on the resulting scaled matrix, particularly in the setting where $A$ is large and the entrywise perturbations are not necessarily small. It is noteworthy that existing literature related to scaling random matrices is mostly concerned with special cases such as the scaling of symmetric kernel matrices [27, 13] and the spectral properties of random doubly-stochastic matrices [19, 3].

Let $\tilde{A} \in \mathbb{R}^{M \times N}$ be a positive random matrix, and define $A = E[\tilde{A}]$. Theorem 1 establishes the existence and uniqueness of a set of scaling factors $\{(\alpha x, \alpha^{-1} y)\}_{\alpha > 0}$ of $A$, together with the existence and uniqueness of the corresponding scaled matrix $P$. Theorem 1 can also be applied analogously to $\tilde{A}$, establishing the existence and uniqueness of a set of random scaling factors $\{(\tilde{\alpha} x, \tilde{\alpha}^{-1} y)\}_{\tilde{\alpha} > 0}$ of $\tilde{A}$, as well as the existence and uniqueness of the corresponding scaled random matrix $\tilde{P} = D(\tilde{x}) \tilde{A} D(\tilde{y})$.

The main purpose of this work is to establish that under suitable conditions on $\tilde{A}$, $r$, and $c$, there is a pair of scaling factors $(\tilde{x}, \tilde{y})$ of $\tilde{A}$ that concentrates around a pair of scaling factors $(x, y)$ of $A$ (in an appropriate sense), and furthermore, the resulting scaled random matrix $\tilde{P}$ concentrates around $P$ in operator norm. Notably, the main technical challenge in deriving such results is the implicit nonlinear representation of $x$ and $y$ in (1), which prohibits the direct application of standard concentration inequalities. Therefore, an important aspect of this work is providing a mechanism for applying standard vector and matrix concentration inequalities in the analysis of random matrix scaling.

The main contributions of this work are as follows. We begin by providing a concentration inequality for the scaling factors of $\tilde{A}$ around those of $A$ assuming the entries of $\tilde{A}$ are bounded from above and from below away from zero, and in addition that they satisfy the property of being independent within each row and each column of $\tilde{A}$ separately; see Theorem 3 in Section 2.1. To that end, we derive a result concerning the stability of the scaling factors of a matrix under perturbations in the prescribed row and column sums, which may be of independent interest; see Lemma 9 in Section 4.2. We then turn to consider an asymptotic setting of $M, N \to \infty$, and employ Theorem 3 to bound the pointwise convergence rate of the scaling factors of $\tilde{A}$ to those of $A$; see Theorem 4 and Corollary 5 in Section 2.2. In addition, under the same asymptotic setting as above but further assuming that all entries of $\tilde{A}$ are independent, we make use of Theorem 4 to bound the asymptotic concentration of $\tilde{P}$ around $P$ in operator norm; see Theorem 6 and Corollary 7 in Section 2.3. We conclude by conducting several numerical experiments that corroborate our theoretical findings and demonstrate that our convergence rates are tight in certain situations; see Section 3.

2 Main results

2.1 Concentration of matrix scaling factors

Let us define

$$ r_i = \frac{r_i}{\sqrt{\|r\|_1}} \quad \text{ and } \quad c_j = \frac{c_j}{\sqrt{\|c\|_1}} \quad \text{ for all } i \in [M] \text{ and } j \in [N]. $$

The following lemma describes a useful normalization of the scaling factors of $A$ and the resulting bounds on their entries.
Lemma 2 (Boundedness of scaling factors). Suppose that $A$, $r$, and $c$ are positive, and $\|r\|_1 = \|c\|_1$. Then, there exists a unique pair of positive vectors $(x, y)$ that satisfies $\|x\|_1 = \|y\|_1$ and scales $A$ to row sums $r$ and column sums $c$. Furthermore, denoting $a = \min_{i,j} A_{i,j}$ and $b = \max_{i,j} A_{i,j}$, we have that for all $i \in [M]$ and $j \in [N]$

\[ \frac{\sqrt{a}}{b} \leq \frac{x_i}{r_i} \leq \frac{\sqrt{b}}{a}, \quad \frac{\sqrt{a}}{b} \leq \frac{y_j}{c_j} \leq \frac{\sqrt{b}}{a}. \]  

The proof can be found in Section 4.1 and is based on a straightforward manipulation of the system of equations in (1). The normalization $\|x\|_1 = \|y\|_1$ is natural and convenient, since it provides a symmetric bound for the entries of $x$ and $y$ in terms of $a$ and $b$ while precisely characterizing their magnitudes according to $r$ and $c$, respectively. Note that the condition $\|r\|_1 = \|c\|_1$ in Theorem 1 and in Lemma 2 is necessary for the existence of the scaling factors, as by Definition 1 each of the quantities $\|r\|_1$ and $\|c\|_1$ should be the sum of all entries in the scaled matrix $P = D(x)AD(y)$. From this point onward we will always assume that $r$ and $c$ are positive and $\|r\|_1 = \|c\|_1$, denoting the sum of all entries in $P$ by

\[ s = \|r\|_1 = \|c\|_1. \]  

We now have the following theorem, which provides a concentration inequality for a certain pair of scaling factors of $\tilde{A}$ around the pair $(x, y)$ from Lemma 2 (taking $A = E[\tilde{A}]$).

Theorem 3 (Concentration of scaling factors). Let $\tilde{A} \in \mathbb{R}^{M \times N}$ be a positive random matrix, $A = E[\tilde{A}]$, and $(x, y)$ be the unique pair of positive vectors that satisfies $\|x\|_1 = \|y\|_1$ and scales $A$ to row sums $r$ and column sums $c$. Suppose that $\tilde{A}_{i,j} \in [a_{i,j}, b_{i,j}]$ a.s. for all $i \in [M]$ and $j \in [N]$, and denote $a = \min_{i,j} a_{i,j}$, $b = \max_{i,j} b_{i,j}$, and $d = \max_{i,j} \{b_{i,j} - a_{i,j}\}$. Suppose further that $\{\tilde{A}_{i,j}\}_{j=1}^{N}$ are independent for each $i \in [M]$, and $\{\tilde{A}_{i,j}\}_{i=1}^{M}$ are independent for each $j \in [N]$. Then, there exists a pair of positive random vectors $(\bar{x}, \bar{y})$ that scales $\tilde{A}$ to row sums $r$ and column $c$, such that for any $\delta \in (0, 1]$, with probability at least

\[ 1 - 2M \exp\left( -\frac{\delta^2 s^2}{C_p^2 \|r\|_2^2} \right) - 2N \exp\left( -\frac{\delta^2 s^2}{C_p^2 \|c\|_2^2} \right), \]  

we have that for all $i \in [M]$ and $j \in [N]$

\[ \frac{|\bar{x}_i - x_i|}{x_i} \leq \frac{C_e \delta s}{M \min_i r_i}, \quad \frac{|\bar{y}_j - y_j|}{y_j} \leq \frac{C_e \delta s}{N \min_j c_j}, \]  

where

\[ C_p = \sqrt{2} \left( \frac{bd}{a^2} \right), \quad C_e = 1 + 2 \left( \frac{b}{a} \right)^{7/2}. \]  

Note that Theorem 3 requires that the entries of $\tilde{A}$ are independent in each of its rows and each of its columns separately. This condition is clearly less restrictive than the requirement that all of the entries of $\tilde{A}$ are independent. For instance, consider the matrix $\tilde{A}_{i,j} = g_{i,j}(u_i v_j)$, where $\{u_i\}_{i=1}^{M}$,
\( \{v_j\}_{j=1}^{N} \) are independent Rademacher variables, and \( g_{i,j} : \{-1, 1\} \rightarrow \{a_{i,j}, b_{i,j}\} \) are deterministic functions with \( 0 < a_{i,j} < b_{i,j} \), for all \( i \in [M], j \in [N] \). Evidently, each row and column of \( \tilde{A} \) contains independent entries, yet the entries of \( \tilde{A} \) are strongly dependant since knowing any single row (column) of \( \tilde{A} \) substantially restricts the distribution of any other row (column).

It is worthwhile to point out that Theorem 3 also implies the following statement, which is perhaps more intuitive than the formulation in Theorem 3. For any pair of scaling factors \((\tilde{x}', \tilde{y}')\) of \( \tilde{A} \), Theorem 3 implies that there exists a pair of scaling factors \((x', y')\) of \( A \) such that with probability at least (5) the bounds in (6) hold if we replace \((\tilde{x}, \tilde{y})\) and \((x, y)\) with \((\tilde{x}', \tilde{y}')\) and \((x', y')\), respectively (under the conditions in Theorem 3). This claim stems simply from the fact that any pair \((x', y')\) of scaling factors of \( \tilde{A} \) can be written as \((\alpha \tilde{x}, \alpha^{-1} \tilde{y})\) for some \( \alpha > 0 \), where \((\tilde{x}, \tilde{y})\) is the specific pair whose existence is guaranteed by Theorem 3. Subsequently, taking \((x', y') = (\alpha x, \alpha^{-1} y)\), where \((x, y)\) is as in Theorem 3, gives that \(|\tilde{x}_i - x_i'|/x_i' = |\tilde{x}_i - x_i|/x_i\) and \(|\tilde{y}_j - y_j'|/y_j' = |\tilde{y}_j - y_j|/y_j\).

The proof of Theorem 3 can be found in Section 4.3 and is based on the following idea, which is a simple two-step procedure. First, we use Lemma 2 in conjunction with Hoeffding’s inequality [8] to prove that the row and column sums of \( D(x)\tilde{A}D(y) \) concentrate around \( r \) and \( c \), respectively; see Lemma 8 in Section 4.3. Second, we prove that if the matrix \( \tilde{A} \) can be approximately scaled by the pair \((x, y)\), it must imply that \((x, y)\) is sufficiently close to a pair of scaling factors of \( A \). This result is based on Sinkhorn’s technique in [23] for proving the uniqueness of the scaling factors, which we substantially extend to describe the stability of the scaling factors under approximate scaling (or equivalently, under perturbations of the prescribed row and column sums); see Lemma 9 in Section 4.3. It is worthwhile to point out that Hoeffding’s inequality in the proof of Theorem 3 can be replaced with any other concentration inequality for sums of random variables, allowing one to relax the assumptions on boundedness and independence.

### 2.2 Asymptotic convergence of scaling factors

We now place ourselves in an asymptotic setting where the dimensions of \( \tilde{A} \) tend to infinity, and apply Theorem 3 to study the asymptotic convergence of the scaling factors of \( \tilde{A} \) to those of \( A \) in relative error. Let \( \{\tilde{A}^{(N)} \in \mathbb{R}^{M_N \times N}_N\}_{N=N_0}^{\infty} \) be a sequence of positive random matrices and define \( A^{(N)} = \mathbb{E}[\tilde{A}^{(N)}] \), where \( N_0 \) is some positive integer, and \( \lim_{N \rightarrow \infty} M_N = \infty \). Suppose that for any positive integer \( N \geq N_0 \), we are given positive row sums \( \mathbf{r}^{(N)} \) and column sums \( \mathbf{c}^{(N)} \) that satisfy \( \|\mathbf{r}^{(N)}\|_1 = \|\mathbf{c}^{(N)}\|_1 \). According to Lemma 2, \( A^{(N)} \) can be scaled to row sums \( \mathbf{r}^{(N)} \) and column sums \( \mathbf{c}^{(N)} \) by a unique pair of positive vectors \((x^{(N)}, y^{(N)})\) that satisfies \( \|x^{(N)}\|_1 = \|y^{(N)}\|_1 \). As a measure of discrepancy between \((x^{(N)}, y^{(N)})\) and another pair of vectors \((\tilde{x}, \tilde{y})\), where \( \tilde{x} \in \mathbb{R}^{M_N} \) and \( \tilde{y} \in \mathbb{R}^N \), we define

\[
\mathcal{E}_N(\tilde{x}, \tilde{y}) = \max \left\{ \frac{\max_{i \in [M_N]} |x_i - x_i^{(N)}|}{x_i^{(N)}}, \frac{\max_{j \in [N]} |y_j - y_j^{(N)}|}{y_j^{(N)}} \right\} \quad (8)
\]

It is important to mention that the scaling factors \( x_i^{(N)} \) and \( y_j^{(N)} \) of \( A^{(N)} \) can potentially converge to 0 or grow unbounded as \( N \rightarrow \infty \), depending on the asymptotic behavior of the prescribed row
sums $r_1^{(N)}$ and column sums $c^{(N)}$. Consequently, the normalizations by $x_i^{(N)}$ and $y_j^{(N)}$ appearing in the error measure (8) are important for making $\mathcal{E}_N(\tilde{x}, \tilde{y})$ meaningful in the asymptotic regime of $N \to \infty$. Let us denote $s^{(N)} = \|r^{(N)}\|_1 = \|c^{(N)}\|_1$, and define the quantities

$$
\rho_1^{(N)} = \max \left\{ \frac{||r^{(N)}||_2}{s^{(N)}}, \frac{||c^{(N)}||_2}{s^{(N)}} \right\}, \quad \rho_2^{(N)} = \max \left\{ \frac{s^{(N)}}{MN \min_i r_i^{(N)}}, \frac{s^{(N)}}{N \min_j c_j^{(N)}} \right\}. \tag{9}
$$

In what follows we use the notation $X_N = O_{w.h.p}(\gamma_N)$, where $\{X_N\}$ is a sequence of random variables and $\{\gamma_N\}$ is a deterministic sequence, to mean order with high probability, namely that there exists a constant $C$ such that $\lim_{N \to \infty} \Pr\{X_N \leq C\gamma_N\} = 1$. Note that $O_{w.h.p}(\cdot)$ is not equivalent to order in probability $O_p(\cdot)$ [15]. In particular, $X_N = O_{w.h.p}(\gamma_N)$ implies that $X_N = O_p(\gamma_N)$ but not the other way around.

We now have the following theorem, which provides a bound on the convergence rate of a certain sequence of scaling factors $\{(\tilde{x}^{(N)}, \tilde{y}^{(N)})\}$ of $\{A^{(N)}\}$ to $\{(x^{(N)}, y^{(N)})\}$.

**Theorem 4** (Convergence rate of scaling factors). Suppose that for all indices $N$ the matrices $\{\tilde{A}^{(N)}\}$ satisfy the conditions in Theorem 3 with universal positive constants $a, b$ (independent of $N$). Then, there exists a sequence of scaling factors $\{(\tilde{x}^{(N)}, \tilde{y}^{(N)})\}$ of $\{A^{(N)}\}$, such that

$$
\mathcal{E}_N(\tilde{x}^{(N)}, \tilde{y}^{(N)}) = O_{w.h.p} \left( \rho_1^{(N)} \rho_2^{(N)} \sqrt{\log(\max\{M_N, N\})} \right). \tag{10}
$$

The proof can be found in Section 4.4 and is largely based on a direct application of Theorem 3 using an appropriate $\delta$.

To exemplify Theorem 4, let us consider the setting of doubly-stochastic matrix scaling, namely $M_N = N$, and $r_i^{(N)} = c_j^{(N)} = 1$ for all $i \in [M], j \in [N]$. According to (9) we have $\rho_1^{(N)} = 1/\sqrt{N}$ and $\rho_2^{(N)} = 1$. Hence, Theorem 4 asserts that there exists a sequence of scaling factors $\{(\tilde{x}^{(N)}, \tilde{y}^{(N)})\}$ of $\{A^{(N)}\}$, such that $\mathcal{E}_N(\tilde{x}^{(N)}, \tilde{y}^{(N)}) = O_{w.h.p}(\sqrt{\log N/N})$. Similarly, it is easy to verify that the same convergence rate of $O_{w.h.p}\left(\frac{\sqrt{\log N/N}}{\max\{M_N, N\}}\right)$ holds whenever $M_N$ grows proportionally to $N$ and $\max_i r_i^{(N)}/\min_i r_i^{(N)} \leq C$, $\max_j c_j^{(N)}/\min_j c_j^{(N)} \leq C$, for some universal constant $C$. If instead $M_N$ grows disproportionately to $N$, the convergence rate of $(\tilde{x}^{(N)}, \tilde{y}^{(N)})$ to $(x^{(N)}, y^{(N)})$ is dominated by the minimum between $M_N$ and $N$, as described in the next corollary of Theorem 4.

**Corollary 5.** Suppose that the conditions in Theorem 4 hold, and in addition $\max_i r_i^{(N)}/\min_i r_i^{(N)} \leq C$, $\max_j c_j^{(N)}/\min_j c_j^{(N)} \leq C$, for all indices $N$ and some universal constant $C$ (independent of $N$). Then,

$$
\mathcal{E}_N(\tilde{x}^{(N)}, \tilde{y}^{(N)}) = O_{w.h.p} \left( \sqrt{\frac{\log(\max\{M_N, N\})}{\min\{M_N, N\}}} \right). \tag{11}
$$

The proof follows immediately from the fact that $\rho_1^{(N)} \leq C \max\{M_N^{-1/2}, N^{-1/2}\}$ and $\rho_2^{(N)} \leq C$ if $\max_i r_i^{(N)}/\min_i r_i^{(N)} \leq C$ and $\max_j c_j^{(N)}/\min_j c_j^{(N)} \leq C$.

Aside from the setting where the $r_i^{(N)}$'s and $c_j^{(N)}$'s have the same orders of magnitude, Theorem 4 provides guarantees on the convergence rate of the scaling factors even if some of the $r_i^{(N)}$'s or $c_j^{(N)}$'s
grow unbounded with $N$ relative to others. For instance, considering again the setting of doubly-stochastic matrix scaling, we can set a fixed number of the $r_i^{(N)}$’s or $c_j^{(N)}$’s to be $\sqrt{N}$ instead of 1 (for all indices $N$), without affecting the behavior of $\|r^{(N)}\|_2$, $\|c^{(N)}\|_2$, and $s^{(N)}$ asymptotically as $N$ grows. Consequently, the convergence rate of $(\tilde{x}^{(N)}, \tilde{y}^{(N)})$ to $(x^{(N)}, y^{(N)})$ in this case would remain $O_{w.h.p}(\sqrt{\log N/N})$.

2.3 Concentration of $\tilde{P}$ around $P$ in operator norm

Let $\tilde{P}^{(N)}$ and $P^{(N)}$ be the matrices obtained from $\tilde{A}^{(N)}$ and $A^{(N)}$, respectively, after scaling them to row sums $r^{(N)}$ and column sums $c^{(N)}$, i.e.,

$$\tilde{P}^{(N)} = D(\tilde{x}^{(N)}) \tilde{A}^{(N)} D(\tilde{y}^{(N)}), \quad P^{(N)} = D(x^{(N)}) A^{(N)} D(y^{(N)}),$$

(12)

where $(\tilde{x}^{(N)}, \tilde{y}^{(N)})$ is any pair of scaling factors of $\tilde{A}$. Note that by Theorem 1 the matrices $\tilde{P}^{(N)}$ and $P^{(N)}$ are uniquely determined ($\tilde{P}^{(N)}$ being a random matrix). In addition, we define the quantity

$$\rho_3^{(N)} = \sqrt{\max_i r_i^{(N)} \cdot \sqrt{N} \max_j c_j^{(N)}} / s^{(N)}.$$  

(13)

We now have the following result, which provides an upper bound on the concentration of $\tilde{P}^{(N)}$ around $P^{(N)}$ in operator norm.

**Theorem 6.** (Asymptotic concentration of $\tilde{P}^{(N)}$ around $P^{(N)}$) Suppose that for each index $N$ the entries of $\tilde{A}^{(N)}$ are independent, and $\tilde{A}_{i,j}^{(N)} \in [a, b]$ a.s. for all $i \in [M_N], j \in [N]$, and some universal positive constants $a, b$ (independent of $N$). Then,

$$\|\tilde{P}^{(N)} - P^{(N)}\|_2 = O_{w.h.p} \left( (p_1^{(N)} \rho_2^{(N)} \rho_3^{(N)} \sqrt{\log(M_N, N)}) \right).$$

(14)

The proof can be found in Section 4.5, and is based on Theorem 4 and the concentration of $\tilde{A}^{(N)}$ around $A^{(N)}$ in operator norm. To exemplify Theorem 6, we consider again the case of doubly-stochastic matrix scaling, where we have $\rho_1^{(N)} = 1/\sqrt{N}$, and $\rho_2^{(N)} = \rho_3^{(N)} = 1$. Therefore, $\|\tilde{P}^{(N)} - P^{(N)}\|_2 = O_{w.h.p}(\sqrt{\log N/N})$. Note that since $P^{(N)}$ is doubly-stochastic, it follows that $\|P^{(N)}\|_2 = 1$ (see [9]). In the more general case where the prescribed row and column sums are not 1, the operator norm of $P^{(N)}$ can converge to zero or grow unbounded with $N$, depending on the asymptotic behavior of the prescribed row and column sums. Consequently, we also consider the normalized error $\|\tilde{P}^{(N)} - P^{(N)}\|_2 / \|P^{(N)}\|_2$, which is the subject of the following Corollary of Theorem 6 for the case that all prescribed row sums and all prescribed column sums admit the same behaviors with $N$.

**Corollary 7.** Suppose that the conditions in Theorem 6 hold, and in addition $\max_i r_i^{(N)} / \min_i r_i^{(N)} \leq C$, $\max_j c_j^{(N)} / \min_j c_j^{(N)} \leq C$, for all indices $N$ and some universal constant $C$ (independent of $N$).
Then,
\[
\frac{\|\bar{P}^{(N)} - P^{(N)}\|_2}{\|P^{(N)}\|_2} = O_{w.h.p.}\left(\sqrt{\frac{\log(\max\{M_N, N\})}{\min\{M_N, N\}}}\right).
\]  

Proof. Observe that
\[
\|P^{(N)}\|_2 \geq \max \left\{ \left\| \frac{P^{(N)} 1_N}{\sqrt{N}} \right\|_2, \left\| \frac{1^T M_N P^{(N)}}{\sqrt{M_N}} \right\|_2 \right\} \geq \max \left\{ \frac{\|r^{(N)}\|_2}{\sqrt{N}}, \frac{\|c^{(N)}\|_2}{\sqrt{M_N}} \right\}
\]
\[
\geq \sqrt{\min_i r_i^{(N)} \min_j c_j^{(N)}},
\]
where $1_N$ is the column vector of ones in $\mathbb{R}^N$. In addition, it is easy to verify that $\rho_1^{(N)} \leq C \max\{M_N^{-1/2}, N^{-1/2}\}$, $\rho_2^{(N)} \leq C$, and
\[
\frac{\rho_3^{(N)}}{\|P^{(N)}\|_2} \leq \frac{\max_i r_i^{(N)} \max_j c_j^{(N)}}{\|P^{(N)}\|_2 \min_i r_i^{(N)} \min_j c_j^{(N)}} \leq C^2,
\]
where we used the fact that $s^{(N)} = \sqrt{s^{(N)} s^{(N)}} \geq \sqrt{M_N \min_i r_i^{(N)} \sqrt{N \min_j c_j^{(N)}}}$. Applying Theorem 6 and using all of the above concludes the proof.

\section{Numerical examples}

We now exemplify our results in several simulations. In all of our experiments, the matrix $A^{(N)}$ was generated by sampling its entries independently and uniformly from $[1.5, 2.5]$, and $\tilde{A}_{i,j}^{(N)}$ were sampled independently and uniformly from $[A_{i,j}^{(N)} - 0.5, A_{i,j}^{(N)} + 0.5]$ for all $i \in [M_N]$ and $j \in [N]$. Then, the Sinkhorn-Knopp algorithm [24, 11] was applied to both $A^{(N)}$ and $\tilde{A}^{(N)}$, where the algorithm’ iterations were terminated once the row and column sums of the scaled matrices reached their targets up to an error of $10^{-12}$. The resulting pairs of scaling factors were normalized so that $\|\bar{x}^{(N)}\|_1 = \|\bar{y}^{(N)}\|_1$ and $\|\tilde{x}^{(N)}\|_1 = \|\tilde{y}^{(N)}\|_1$. This process was repeated 20 times (each time for a different realization of $A^{(N)}$ and $\tilde{A}^{(N)}$) and the error measures appearing in the left-hand sides of (10) and (15) were computed and averaged over the 20 randomized trials.

Figure 1 depicts the behavior of the empirical error $\mathcal{E}_N(\bar{x}^{(N)}, \bar{y}^{(N)})$ (see (8)) as a function of $N$ in several scenarios. Specifically, Figure 1a exemplifies the scenario of doubly-stochastic matrix scaling, i.e., $M_N = N$ and $r_i^{(N)} = c_j^{(N)} = 1$ for all $i \in [M]$ and $j \in [N]$, in which case Corollary 5 guarantees that $\mathcal{E}_N(\bar{x}^{(N)}, \bar{y}^{(N)}) = O_{w.h.p.}(N^{-1/2} \sqrt{\log N})$. Figure 1b illustrates the case of a rectangular matrix with $M_N = 3N$, where the prescribed row and column sums were sampled independently and uniformly from $[0.1, 1]$ and normalized to sum to 1. In this case, since $M_N$ is proportional to $N$, and in addition $\max_i r_i^{(N)} / \min_i r_i^{(N)} \leq 10$, $\max_j c_j^{(N)} / \min_j c_j^{(N)} \leq 10$, Corollary 5 again dictates that $\mathcal{E}_N(\bar{x}^{(N)}, \bar{y}^{(N)}) = O_{w.h.p.}(N^{-1/2} \sqrt{\log N})$ as for the doubly-stochastic case. Figure 1c illustrates
the scenario of a rectangular matrix with $M_N = 10\sqrt{N}$ and \( r_i(N) = N, c_j(N) = M_N \), for all \( i \in [M] \) and \( j \in [N] \). In this case it follows from Corollary 5 that $E_N(\overline{x}(N), \overline{y}(N)) = \mathcal{O}_{w.h.p}(N^{-1/4} \sqrt{\log N})$. It is evident from Figures 1a, 1b, 1c that the asymptotic bound in Theorem 4 agrees very well with the experimental results, suggesting that this bound is tight for the considered scenarios, and in particular that the factor $\sqrt{\log(N)}$ in the corresponding bounds is necessary.

Figure 2 shows the behavior of the empirical error $\|\overline{P}(N) - P(N)\|_2 / \|P(N)\|_2$ as a function of $N$ for the same scenarios as in Figure 1. For these scenarios, the rates that govern the bounds on $\|\overline{P}(N) - P(N)\|_2 / \|P(N)\|_2$ according to Corollary 7 are the same as those for $E_N(\overline{x}(N), \overline{y}(N))$ from Corollary 5 (described previously in the context of Figure 1). In the scenario where $M_N$ grows proportionally to $N$, it is evident from Figures 2a and 2b that the bound in Corollary 7 is tight up to the factor $\sqrt{\log(N)}$, suggesting that the factor $\sqrt{\log(N)}$ is probably not required in the bound on $\|\overline{P}(N) - P(N)\|_2 / \|P(N)\|_2$ (in contrast to the bound on $E_N(\overline{x}(N), \overline{y}(N))$ depicted in Figure 1).

In the scenario where $M_N$ grows disproportionally to $N$, Figure 1c empirically suggests that the bound in Corollary 7 can be improved by a factor of $\log(N)$, which would bring the rate in this case to be $\mathcal{O}_{w.h.p}(N^{-1/4} / \log N)$. Overall, these experiments suggest that the bound in Corollary 7 describes the correct behavior of $\|\overline{P}(N) - P(N)\|_2 / \|P(N)\|_2$ with $N$ up to poly-logarithmic factors.

### 4 Proofs

#### 4.1 Proof of Lemma 2

Theorem 1 guarantees the existence of a pair $(x', y')$ of scaling factors of $A$ and states that all possible pairs of scaling factors of $A$ must be of the form $(\alpha x', \alpha^{-1} y')$, for $\alpha > 0$. Note that setting $\|\alpha x'\|_1 = \|\alpha^{-1} y'\|_1$ determines $\alpha$ uniquely, and consequently, there exists a unique pair $(x, y)$ such
that $\|x\|_1 = \|y\|_1$. According to (1) we have

$$x_i = \frac{r_i}{\sum_{j=1}^{N} A_{i,j} y_j}, \quad y_j = \frac{c_j}{\sum_{i=1}^{M} A_{i,j} x_i},$$

(18)

and since $a \leq A_{i,j} \leq b$ for all $i, j$, it follows that

$$\frac{r_i}{b \sum_{j=1}^{N} y_j} \leq x_i \leq \frac{r_i}{a \sum_{j=1}^{N} y_j}, \quad \frac{c_j}{b \sum_{i=1}^{M} x_i} \leq y_j \leq \frac{c_j}{a \sum_{i=1}^{M} x_i},$$

(19)

for all $i \in [M], j \in [N]$. Summing the inequalities for $x_i$ in (19) over $i = 1, \ldots, M$, and using $\sum_{i=1}^{M} x_i = \sum_{j=1}^{N} y_j$ together with $\sum_{i=1}^{M} r_i = s$, gives

$$\sqrt{\frac{s}{b}} \leq \sum_{j=1}^{N} y_j = \sum_{i=1}^{M} x_i \leq \sqrt{\frac{s}{a}}.$$

(20)

Lastly, plugging the above back into (19) gives the required result.

4.2 Lemmas supporting the proof of Theorem 3

The first step in proving Theorem 3 is to make use of Lemma 2 together with Hoeffding’s inequality [8] to provide a concentration inequality for the sums of the rows and of the columns of $D(x)\tilde{A}D(y)$ around $r$ and $c$, respectively, where $(x, y)$ is any pair of scaling factors of $A$. This is the subject of the following lemma.

Lemma 8 (Concentration of row and column sums). Suppose that $\{\tilde{A}_{i,j}\}_{i=1}^{M}$ are independent for each $j \in [N]$, and $\{\tilde{A}_{i,j}\}_{j=1}^{N}$ are independent for each $i \in [M]$. Furthermore, suppose that $\tilde{A}_{i,j} \in [a_{i,j}, b_{i,j}]$ a.s. for all $i \in [M]$ and $j \in [N]$, and denote $a = \min_{i,j} a_{i,j}, b = \max_{i,j} b_{i,j}$. Then, for any
pair of vectors \((\mathbf{x}, \mathbf{y})\) that scales \(A = \mathbb{E}[\mathbf{A}]\) to row sums \(\mathbf{r}\) can column sums \(\mathbf{c}\), we have

\[
\Pr \left\{ \frac{1}{r_i} \sum_{j=1}^{N} x_i \tilde{A}_{i,j} y_j - 1 > \varepsilon \right\} \leq 2 \exp \left( \frac{-2\varepsilon^2 s^2}{C^2 \sum_{j=1}^{N} c_j^2 (b_{i,j} - a_{i,j})^2} \right),
\]

(21)

\[
\Pr \left\{ \frac{1}{c_j} \sum_{i=1}^{M} x_i \tilde{A}_{i,j} y_j - 1 > \varepsilon \right\} \leq 2 \exp \left( \frac{-2\varepsilon^2 s^2}{C^2 \sum_{i=1}^{M} r_i^2 (b_{i,j} - a_{i,j})^2} \right),
\]

(22)

for any \(\varepsilon > 0\) and all \(i \in [M]\) and \(j \in [N]\), where \(C = b/a^2\).

**Proof.** Observe that if \(\tilde{A}_{i,j} \in [a_{i,j}, b_{i,j}]\) a.s., then \(A_{i,j} = \mathbb{E}[	ilde{A}_{i,j}] \in [a_{i,j}, b_{i,j}]\). Therefore, using the fact that \(\sum_j x_i A_{i,j} y_j = r_i\), Hoeffding’s inequality [8] gives that

\[
\Pr \left\{ \frac{1}{r_i} \sum_{j=1}^{N} x_i \tilde{A}_{i,j} y_j - 1 > \varepsilon \right\} = \Pr \left\{ \sum_{j=1}^{N} x_i \tilde{A}_{i,j} y_j - \sum_{j=1}^{N} x_i A_{i,j} y_j > \varepsilon r_i \right\}
\]

\[
\leq 2 \exp \left( \frac{-2\varepsilon^2 r_i^2}{\sum_{j=1}^{N} x_i^2 y_j^2 (b_{i,j} - a_{i,j})^2} \right),
\]

(23)

for all \(i \in [M]\). Since we can always find a constant \(\alpha > 0\) such that \(\|\alpha \mathbf{x}\|_1 = \|\alpha^{-1} \mathbf{y}\|_1\), Lemma 2 implies that for all \(i \in [M]\) and \(j \in [N]\),

\[
x_i y_j = (\alpha x_i)(\alpha^{-1} y_j) \leq \tau_i \tau_j \frac{b}{a^2} = C \tau_i \tau_j.
\]

(24)

Applying the above inequality to (23), we get

\[
\Pr \left\{ \frac{1}{r_i} \sum_{j=1}^{N} x_i \tilde{A}_{i,j} y_j - 1 > \varepsilon \right\} \leq 2 \exp \left( \frac{-2\varepsilon^2 r_i^2}{C^2 r_i^2 \sum_{j=1}^{N} c_j^2 (b_{i,j} - a_{i,j})^2} \right)
\]

\[
= 2 \exp \left( \frac{-2\varepsilon^2 s^2}{C^2 \sum_{j=1}^{N} c_j^2 (b_{i,j} - a_{i,j})^2} \right),
\]

(25)

for all \(i \in [M]\). Analogously to the derivation of (25), by using \(\sum_i x_i A_{i,j} y_j = c_j\) together with Hoeffding’s inequality, one can verify that

\[
\Pr \left\{ \frac{1}{c_j} \sum_{i=1}^{M} x_i \tilde{A}_{i,j} y_j - 1 > \varepsilon \right\} = \Pr \left\{ \sum_{i=1}^{M} x_i \tilde{A}_{i,j} y_j - \sum_{i=1}^{M} x_i A_{i,j} y_j > \varepsilon c_j \right\}
\]

\[
\leq 2 \exp \left( \frac{-2\varepsilon^2 s^2}{C^2 \sum_{i=1}^{M} r_i^2 (b_{i,j} - a_{i,j})^2} \right),
\]

(26)

for all \(j \in [N]\).

\(\square\)

We next prove that if \(\tilde{A}\) can be approximately scaled by a pair of vectors \((\mathbf{x}, \mathbf{y})\), then there
exists a pair of vectors \((\bar{x}, \bar{y})\) that scales \(\bar{A}\) and is also close to \((x, y)\). The proof relies on extending Sinkhorn’s original proof of uniqueness of the scaling factors in [23] to describe the stability of the scaling factors under approximate scaling. We note that \(\bar{A}\) is not considered as random in the following Lemma.

**Lemma 9** (Stability of scaling factors under approximate scaling). Let \(\bar{A} \in \mathbb{R}^{M \times N}\) be a positive matrix and denote \(a = \min_{i,j} \bar{A}_{i,j}, \ b = \max_{i,j} \bar{A}_{i,j}\). Suppose that there exists \(\varepsilon \in (0, 1)\) and positive vectors \(x = [x_1, \ldots, x_M]\) and \(y = [y_1, \ldots, y_N]\), such that

\[
\left| \frac{1}{c_j} \sum_{i=1}^{M} x_i \bar{A}_{i,j} y_j - 1 \right| \leq \varepsilon, \quad \left| \frac{1}{r_i} \sum_{j=1}^{N} x_i y_j \bar{A}_{i,j} y_j - 1 \right| \leq \varepsilon, \quad (27)
\]

for all \(i \in [M]\) and \(j \in [N]\). Then, \(\bar{A}\) can be scaled to row sums \(r\) and column sums \(c\) by a pair \((\bar{x}, \bar{y})\) that satisfies

\[
\left| \frac{\bar{x}_i - x_i}{x_i} \right| \leq \frac{\varepsilon}{1 - \varepsilon} + \frac{4\varepsilon s \sqrt{b}}{a^2 C_1^{3/2} C_2^{3/2} M \min_i r_i}, \quad (28)
\]

\[
\left| \frac{\bar{y}_j - y_j}{y_j} \right| \leq \frac{\varepsilon}{1 - \varepsilon} + \frac{4\varepsilon s \sqrt{b}}{a^2 C_1^{3/2} C_2^{3/2} N \min_j c_j}, \quad (29)
\]

for all \(i \in [M]\) and \(j \in [N]\), where \(C_1 = \min_{i,j} \{x_i/\tau_i\}\) and \(C_2 = \min_{i,j} \{y_j/\tau_j\}\).

**Proof.** Let \((\bar{x}, \bar{y})\) be the unique pair of scaling factors of \(\bar{A}\) with \(\|\bar{x}\|_1 = \|\bar{y}\|_1\) (see Lemma 2), and define

\[
\bar{P}_{i,j} = x_i \bar{A}_{i,j} y_j, \quad P_{i,j} = \bar{x}_i \bar{A}_{i,j} \bar{y}_j = u_i \bar{P}_{i,j} v_j, \quad u_i = \frac{\bar{x}_i}{x_i}, \quad v_j = \frac{\bar{y}_j}{y_j}, \quad (30)
\]

for all \(i \in [M]\) and \(j \in [N]\). Observe that \(\sum_i \bar{P}_{i,j} = c_j, \sum_j \bar{P}_{i,j} = r_i\), and

\[
\left| \frac{1}{c_j} \sum_{i=1}^{M} \bar{P}_{i,j} - 1 \right| \leq \varepsilon, \quad \left| \frac{1}{r_i} \sum_{j=1}^{N} \bar{P}_{i,j} - 1 \right| \leq \varepsilon, \quad (31)
\]

for all \(i \in [M]\), \(j \in [N]\). The proof of Lemma 9 is based on a manipulation of (31) using \(\bar{P}, \bar{P}, u = [u_1, \ldots, u_M], v = [v_1, \ldots, v_N]\), and their properties. Since this manipulation is somewhat technical, in what follows we break it down into several steps.

### 4.2.1 Deriving bounds on \(\{u_i\}\) and \(\{v_j\}\)

Using the first inequality in (31) for \(j = \arg\min_k v_k\), we have

\[
1 = \frac{1}{c_j} \sum_{i=1}^{M} \bar{P}_{i,j} = \frac{1}{c_j} \sum_{i=1}^{M} u_i \bar{P}_{i,j} v_j \leq \min_j v_j \max_i u_i \frac{1}{c_j} \sum_{i=1}^{M} \bar{P}_{i,j} \leq (1 + \varepsilon) \min_j v_j \max_i u_i. \quad (32)
\]
Similarly, using the second inequality in (31) for \( i = \text{argmax}_i v_i \) gives
\[
1 = \frac{1}{r_i} \sum_{j=1}^{N} \tilde{P}_{i,j} = \frac{1}{r_i} \sum_{j=1}^{N} u_i \tilde{P}_{i,j} v_j \geq \max_i u_i \min_j v_j \frac{1}{r_i} \sum_{j=1}^{N} \tilde{P}_{i,j} \geq (1 - \varepsilon) \max_i u_i \min_j v_j, \tag{33}
\]
and by combining (32) and (33) we obtain
\[
\frac{1}{1 + \varepsilon} \leq \max_i u_i \min_j v_j \leq \frac{1}{1 - \varepsilon}. \tag{34}
\]
Analogously to (32) and (33), it is easy to verify that by using the first inequality in (31) for \( j = \text{argmax}_k v_k \) and using the second inequality in (31) for \( i = \text{argmin}_\ell v_\ell \), one gets
\[
\frac{1}{1 + \varepsilon} \leq \min_i u_i \max_j v_j \leq \frac{1}{1 - \varepsilon}. \tag{35}
\]
Note that (35) can also be obtained directly from (34) by a symmetry argument, that is, by considering (34) in the setting when \( \tilde{A} \) is replaced with its transpose, thereby interchanging the roles of \( u \) and \( v \).

In addition, according to Lemma 2 and using the fact that \( x_i \geq C_1 r_i \) and \( y_j \geq C_2 \bar{c}_j \) (from the conditions in Lemma 9), it follows that for all \( i \in [M] \) and \( j \in [N] \)
\[
u_i \leq \sqrt{b} \frac{a}{C_1}, \quad v_j \leq \frac{\sqrt{b}}{aC_2}. \tag{36}
\]

### 4.2.2 Bounding \( \max_j v_j - \min_j v_j \) and \( \max_i u_i - \min_i u_i \)

Let us denote \( \ell = \text{argmax}_i u_i \). By the second inequality in (31) together with (34), we can write
\[
1 = \frac{1}{r_\ell} \sum_{j=1}^{N} \tilde{P}_{\ell,j} = \frac{1}{r_\ell} \sum_{j=1}^{N} u_\ell \tilde{P}_{\ell,j} v_j \geq \frac{1}{(1 + \varepsilon) r_\ell} \sum_{j=1}^{N} \tilde{P}_{\ell,j} \min_j v_j, \tag{37}
\]
implying that
\[
\frac{1}{r_\ell} \sum_{j=1}^{N} \hat{P}_{\ell,j} \left( \frac{v_j}{\min_j v_j} - 1 \right) \leq 1 + \varepsilon - \frac{1}{r_\ell} \sum_{j=1}^{N} \hat{P}_{\ell,j} \leq 2\varepsilon. \tag{38}
\]
Multiplying (38) by \( \min_j v_j / \min_j \tilde{P}_{\ell,j} \), it follows that
\[
\frac{1}{r_\ell} \sum_{j=1}^{N} (v_j - \min_j v_j) \leq \frac{1}{r_\ell} \sum_{j=1}^{N} \tilde{P}_{\ell,j} (v_j - \min_j v_j) \leq 2\varepsilon \frac{\min_j v_j}{\min_j \tilde{P}_{\ell,j}} \leq \frac{2\varepsilon \min_j v_j}{aC_1 C_2 \bar{c}_j r_\ell \min_j \bar{c}_j}, \tag{39}
\]
where we used the definition of $\hat{P}$ together with the conditions in Lemma 9. Multiplying (39) by $r_t/N$ and employing the definitions of $\bar{\tau}_i$ and $\bar{c}_j$ (see (2)) gives

$$\frac{1}{N} \sum_{j=1}^{N} (v_j - \min_j v_j) \leq \frac{2\varepsilon s \min_j v_j}{aC_1C_2N \min_j c_j} \leq \frac{2\varepsilon s \max_j v_j}{aC_1C_2N \min_j c_j}. \quad (40)$$

We next provide a derivation analogous to (37)–(40) to obtain a bound for $\frac{1}{N} \sum_{j=1}^{N} (\max_j v_j - v_j)$. Let us denote $t = \arg\min_i u_i$. Using the second inequality in (31) together with (35), we have

$$1 = \frac{1}{r_t} \sum_{j=1}^{N} \tilde{P}_{t,j} = \frac{1}{r_t} \sum_{j=1}^{N} u_t \hat{P}_{t,j} v_j \leq \frac{1}{(1 - \varepsilon)r_t} \sum_{j=1}^{N} \hat{P}_{t,j} \frac{v_j}{\max_j v_j}, \quad (41)$$

and therefore

$$\frac{1}{r_t} \sum_{j=1}^{N} \hat{P}_{t,j} \left(1 - \frac{v_j}{\max_j v_j}\right) \leq \frac{1}{r_t} \sum_{j=1}^{N} \hat{P}_{t,j} - (1 - \varepsilon) \leq 2\varepsilon. \quad (42)$$

Multiplying the above by $\max_j v_j / \min_j \hat{P}_{t,j}$, it follows that

$$\frac{1}{r_t} \sum_{j=1}^{N} (\max_j v_j - v_j) \leq \frac{1}{r_t} \sum_{j=1}^{N} \hat{P}_{t,j} \frac{v_j}{\max_j v_j} \leq \frac{2\varepsilon \max_j v_j}{aC_1C_2 \tilde{\tau}_i \min_j \bar{c}_j}. \quad (43)$$

Furthermore, multiplying the above by $r_t/N$ and using the definitions of $\bar{\tau}_i$ and $\bar{c}_j$ (see (2)), we get

$$\frac{1}{N} \sum_{j=1}^{N} (\max_j v_j - v_j) \leq \frac{2\varepsilon \max_j v_j}{aC_1C_2N \min_j \bar{c}_j}. \quad (44)$$

Lastly, summing (40) and (44) gives

$$\max_j v_j - \min_j v_j \leq \frac{4\varepsilon s \max_j v_j}{aC_1C_2 N \min_j c_j}, \quad (45)$$

It is easy to verify that by repeating the derivation of (37) – (45) analogously for $u_i$ instead of $v_j$, we get

$$\max_i u_i - \min_i u_i \leq \frac{4\varepsilon s \max_i u_i}{aC_1C_2 M \min_i r_i}. \quad (46)$$

We omit the full derivation for the sake of brevity. Note that (46) can also be obtained directly from (45) by a symmetry argument, namely by considering (45) in the setting where $\tilde{A}$ is replaced with its transpose, so that $N$ is replaced with $M$, $\textbf{c}$ is replaced with $\textbf{r}$, and $\textbf{v}$ is replaced with $\textbf{u}$.
4.2.3 Bounding $1 - \alpha u_i$ and $1 - \alpha^{-1} v_j$ for some $\alpha > 0$

Observe that $|\tau - v_j| \leq \max_j v_j - \min_j v_j$ for any $\tau \in [\min_j v_j, \max_j v_j]$ and all $j \in [N]$. Taking $\tau$ as the geometric mean of $\max_j v_j$ and $\min_j v_j$, together with (45) gives

$$\left| \sqrt[\max_j v_j \min_j v_j] v_j - v_j \right| \leq \frac{4\varepsilon s \max_j v_j}{aC_1C_2N \min_j c_j}, \tag{47}$$

for all $j \in [N]$. Multiplying both hand sides of (47) by $\alpha^{-1} = \sqrt{\max_i u_i / \max_j v_j}$ we get

$$\left| \sqrt[\max_i u_i \min_i v_j] v_j - \alpha^{-1} v_j \right| \leq \frac{4\varepsilon s \sqrt{aC_1C_2N \max_i u_i \min_j c_j}}{a^2C_1^{3/2}C_2^{3/2}N \min_j c_j}, \tag{48}$$

where we also used (36) in the last inequality. According to (34), we have for all $\varepsilon \in (0, 1)$ that

$$1 - \frac{\varepsilon}{1 - \varepsilon} \leq \sqrt[\max_i u_i \min_i v_j] v_j \leq \frac{\varepsilon}{1 - \varepsilon} \leq 1 + \frac{\varepsilon}{1 - \varepsilon}, \tag{49}$$

which together with (47) implies that

$$|1 - \alpha^{-1} v_j| \leq \frac{\varepsilon}{1 - \varepsilon} + \frac{4\varepsilon s \sqrt{b}}{a^2C_1^{3/2}C_2^{3/2}N \min_j c_j}. \tag{50}$$

Analogously to (47), from (46) we obtain

$$\left| \sqrt[\max_i u_i \min_i u_i] u_i - u_i \right| \leq \frac{4\varepsilon s \max_i u_i}{aC_1C_2M \min_i r_i}, \tag{51}$$

for all $i \in [M]$. Multiplying both hand sides of (51) by $\alpha = \sqrt{\max_j v_j / \max_i u_i}$ we get

$$\left| \sqrt[\max_i u_i \min_i u_i] u_i - \alpha u_i \right| \leq \frac{4\varepsilon s \sqrt{aC_1C_2M \max_i u_i \min_i r_i}}{a^2C_1^{3/2}C_2^{3/2}M \min_i r_i}. \tag{52}$$

Consequently, using (35), and analogously to the derivation of (50), it follows that

$$|1 - \alpha u_i| \leq \frac{\varepsilon}{1 - \varepsilon} + \frac{4\varepsilon s \sqrt{b}}{a^2C_1^{3/2}C_2^{3/2}M \min_i r_i}, \tag{53}$$

which together with the definition of $u$ and $v$ in (30) concludes the proof (since $(\alpha \tilde{x}, \alpha^{-1} \tilde{y})$ is a pair of scaling factors of $\tilde{A}$).

☐
4.3 Proof of Theorem 3

According to Lemma 2 there exists a unique pair of positive vectors \((x, y)\) satisfying \(\|x\|_1 = \|y\|_1\) that scales \(A\) to row sums \(r\) and column sums \(c\), with

\[
\frac{\sqrt{a}}{b} \leq \frac{x_i}{r_i} \leq \frac{\sqrt{b}}{a}, \quad \frac{\sqrt{a}}{b} \leq \frac{y_j}{c_j} \leq \frac{\sqrt{b}}{a},
\]

(54)

for all \(i \in [M]\) and \(j \in [N]\). Additionally, Lemma 8 together with the union bound (over all \(i \in [M]\) and \(j \in [N]\)), asserts that with probability at least

\[
1 - 2 \sum_{i=1}^{M} \exp \left( - \frac{2\varepsilon^2 s^2}{C^2 \sum_{j=1}^{N} c_j^2 (b_{i,j} - a_{i,j})^2} \right) - 2 \sum_{j=1}^{N} \exp \left( - \frac{2\varepsilon^2 s^2}{C^2 \sum_{i=1}^{M} r_i^2 (b_{i,j} - a_{i,j})^2} \right),
\]

(55)

we have

\[
\max_{j \in [N]} \left| \frac{1}{M} \sum_{i=1}^{M} x_i \tilde{A}_{i,j} y_j - 1 \right| \leq \varepsilon, \quad \text{and} \quad \max_{i \in [M]} \left| \frac{1}{N} \sum_{j=1}^{N} x_i \tilde{A}_{i,j} y_j - 1 \right| \leq \varepsilon,
\]

(56)

where \(C = b/a^2\). If (56) holds, we apply Lemma 9 with \(C_1 = \min_i \{x_i/r_i\} \geq \sqrt{a/b}\) and \(C_2 = \min_j \{y_j/c_j\} \geq \sqrt{a/b}\) (using (54)), which guarantees that \(\tilde{A}\) can be scaled to row sums \(r\) and column sums \(c\) by a pair \((\tilde{x}, \tilde{y})\) that satisfies

\[
\frac{\tilde{x}_i - x_i}{x_i} \leq \frac{\varepsilon}{1 - \varepsilon} + \frac{4\varepsilon s \sqrt{b}}{a^2 C_1^{3/2} C_2^{3/2} M \min_i r_i} \leq \varepsilon \left( 2 + \frac{4s}{M \min_i r_i} \left( \frac{b}{a} \right)^{7/2} \right),
\]

(57)

\[
\frac{\tilde{y}_j - y_j}{y_j} \leq \frac{\varepsilon}{1 - \varepsilon} + \frac{4\varepsilon s \sqrt{b}}{a^2 C_1^{3/2} C_2^{3/2} N \min_j c_j} \leq \varepsilon \left( 2 + \frac{4s}{N \min_j c_j} \left( \frac{b}{a} \right)^{7/2} \right),
\]

(58)

for all \(i \in [M], j \in [N]\), and any \(\varepsilon \in (0, 1/2]\). Overall, taking \(\varepsilon = \delta/2\), and using the fact that \(s = \|x\|_1 \geq M \min_i r_i\) and \(s = \|c\|_1 \geq N \min_j c_j\), asserts that that exists a pair \((\tilde{x}, \tilde{y})\) that scales \(\tilde{A}\) to row sums \(r\) and column sums \(c\), such that for any \(\delta \in (0, 1]\)

\[
\frac{\tilde{x}_i - x_i}{x_i} \leq \frac{\delta s}{M \min_i r_i} \left( 1 + 2 \left( \frac{b}{a} \right)^{7/2} \right),
\]

(59)

\[
\frac{\tilde{y}_j - y_j}{y_j} \leq \frac{\delta s}{N \min_j c_j} \left( 1 + 2 \left( \frac{b}{a} \right)^{7/2} \right),
\]

(60)

for all \(i \in [M]\) and \(j \in [N]\), with probability at least

\[
1 - 2 \sum_{i=1}^{M} \exp \left( - \frac{\delta^2 s^2}{2C^2 \sum_{j=1}^{N} c_j^2 (b_{i,j} - a_{i,j})^2} \right) - 2 \sum_{j=1}^{N} \exp \left( - \frac{\delta^2 s^2}{2C^2 \sum_{i=1}^{M} r_i^2 (b_{i,j} - a_{i,j})^2} \right).
\]

(61)
Therefore, using \( b_{i,j} - a_{i,j} \leq \max_{i,j} \{ b_{i,j} - a_{i,j} \} = d \) proves Theorem 3 with

\[
C_p = \sqrt{2} \left( \frac{bd}{aq^2} \right), \quad C_e = 1 + 2 \left( \frac{b}{a} \right)^{7/2}.
\]

4.4 Proof of Theorem 4

We begin by considering indices \( N \) for which \( \rho_1^{(N)} \rho_2^{(N)} \sqrt{\log(\max\{M_N, N\})} \leq 1/\sqrt{2C_p} \), where \( C_p \) is from Theorem 3. In this case, we apply Theorem 3 using \( \delta = \rho_1^{(N)} \sqrt{2C_p \log(\max\{M_N, N\})} \), noting that \( \delta \leq 1 \) as required since \( \rho_2^{(N)} \geq 1 \) for all \( N \). Consequently, for each such index \( N \) there exists a pair of positive vectors \((\tilde{x}^{(N)}, \tilde{y}^{(N)})\) that scales \( \tilde{A}^{(N)} \) to row sums \( r^{(N)} \) and column sums \( c^{(N)} \), such that with probability at least

\[
1 - 2M_N \exp \left( -\frac{\delta^2(s^{(N)})^2}{C_p ||c^{(N)}||^2_2} \right) - 2N \exp \left( -\frac{\delta^2(s^{(N)})^2}{C_p ||r^{(N)}||^2_2} \right)
\geq 1 - 2M \exp (-2 \log(\max\{M_N, N\})) - 2N \exp (-2 \log(\max\{M_N, N\}))
\geq 1 - \frac{4}{\min\{M_N, N\}},
\]

we have for all \( i \in [M_N] \) and \( j \in [N] \),

\[
\frac{|x_i^{(N)} - x_i^{(N)}|}{x_i^{(N)}} \leq \frac{C_e \delta s^{(N)}}{M_N \min_j r_i^{(N)}} \leq C_e \rho_1^{(N)} \rho_2^{(N)} \sqrt{2C_p \log(\max\{M_N, N\})},
\]

\[
\frac{|y_j^{(N)} - y_j^{(N)}|}{y_j^{(N)}} \leq \frac{C_e \delta s^{(N)}}{N \min_j c_j^{(N)}} \leq C_e \rho_1^{(N)} \rho_2^{(N)} \sqrt{2C_p \log(\max\{M_N, N\})}.
\]

Next, we consider indices \( N \) for which \( \rho_1^{(N)} \rho_2^{(N)} \sqrt{\log(\max\{M_N, N\})} > 1/\sqrt{2C_p} \). In this case, we first apply Lemma 2 to \( \tilde{A}^{(N)} \), which states there exists a pair of positive vectors \((\tilde{x}^{(N)}, \tilde{y}^{(N)})\) that scales \( \tilde{A}^{(N)} \) to row sums \( r^{(N)} \) and column sums \( c^{(N)} \), such that

\[
\frac{\sqrt{a}}{b} \leq \frac{x_i^{(N)}}{x_i^{(N)}} \leq \frac{\sqrt{b}}{a}, \quad \frac{\sqrt{a}}{b} \leq \frac{y_j^{(N)}}{c_j^{(N)}} \leq \frac{\sqrt{b}}{a},
\]

for all \( i \in [M_N] \) and \( j \in [N] \). Combining the above inequalities with the analogous inequalities for \( x^{(N)} \) and \( y^{(N)} \) (that scale \( A \) and satisfy \( \|x^{(N)}\|_1 = \|y^{(N)}\|_1 \)) gives

\[
\frac{x_i^{(N)}}{x_i^{(N)}} \leq \left( \frac{b}{a} \right)^{3/2}, \quad \frac{y_j^{(N)}}{c_j^{(N)}} \leq \left( \frac{b}{a} \right)^{3/2},
\]

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for all \( i \in [M_N] \) and \( j \in [N] \). Therefore, we have that

\[
\mathcal{E}_N(\bar{x}^{(N)}, \bar{y}^{(N)}) = \max \left\{ \max_{i \in [M_N]} \left| \frac{\bar{x}_i^{(N)}}{x_i^{(N)}} - 1 \right|, \max_{j \in [N]} \left| \frac{\bar{y}_j^{(N)}}{y_j^{(N)}} - 1 \right| \right\} \leq \left( \frac{b}{a} \right)^{3/2}
\]

\[
< \sqrt{2C_p} \left( \frac{b}{a} \right)^{3/2} \rho_1^{(N)} \rho_2^{(N)} \sqrt{\log(\max\{M_N, N\})}. \tag{67}
\]

Since the lower bound in the right hand-side of (63) converges to 1 as \( N \to \infty \), (64) together with (67) provide the required result for all indices \( N \).

### 4.5 Proof of Theorem 6

Let us define \( \eta^{(N)} = \bar{x}^{(N)} - x^{(N)} \), and \( \xi^{(N)} = \bar{y}^{(N)} - y^{(N)} \). We can write

\[
\left\| D(\bar{x}^{(N)})\bar{A}^{(N)}D(\bar{y}^{(N)}) - D(x^{(N)})A^{(N)}D(y^{(N)}) \right\|_2 \leq \| D(x^{(N)})\bar{A}^{(N)} - A^{(N)}\|D(y^{(N)})\|_2
\]

\[
+ \| D(\eta^{(N)})\bar{A}D(y^{(N)})\|_2 + \| D(x^{(N)})\bar{A}D(\xi^{(N)})\|_2 + \| D(\eta^{(N)})\bar{A}D(\xi^{(N)})\|_2. \tag{68}
\]

We now bound the summands in the right-hand side of (68) one by one. Note that since \( \{A^{(N)}_{i,j}\}_{i,j} \) are independent and are confined to the interval \([a, b]\) for each index \( N \), the conditions in Theorem 3 hold when replacing \( \bar{A} \) with \( A^{(N)} \). For the first summand in (68), applying Lemma 2 to \( A^{(N)} \) we have

\[
\| D(x^{(N)})\bar{A}^{(N)} - A^{(N)}\|D(y^{(N)})\|_2 \leq \| D(x^{(N)})\|_2 \cdot \| \bar{A}^{(N)} - A^{(N)}\|_2 \cdot \| D(y^{(N)})\|_2
\]

\[
\leq \frac{b}{a^2} \max_i \frac{r_i^{(N)}}{\sqrt{s(N)}} \| A^{(N)} - A^{(N)} \|_2 \cdot \max_j \frac{c_j^{(N)}}{\sqrt{s(N)}} \| A^{(N)} - A^{(N)} \|_2 = \frac{b \rho_3^{(N)}}{a^2 \sqrt{M_N}} \| \bar{A}^{(N)} - A^{(N)} \|_2. \tag{69}
\]

Since \( a \leq \bar{A}^{(N)}_{i,j} \leq b \), then also \( a \leq A^{(N)}_{i,j} \leq b \), which implies that \( a - b \leq \bar{A}^{(N)}_{i,j} - A^{(N)}_{i,j} \leq b - a \). Hence, \( \{\bar{A}^{(N)}_{i,j} - A^{(N)}_{i,j}\}_{i,j} \) are independent, have mean zero, and are bounded (and therefore sub-Gaussian). Applying Theorem 4.4.5 in [26] with \( t = \sqrt{\log N} \) gives

\[
\| \bar{A}^{(N)} - A^{(N)} \|_2 = O_{\text{w.h.p}}(\sqrt{N} + \sqrt{M_N} + \sqrt{\log N}). \tag{70}
\]

Combining (70) with (69) asserts that

\[
\| D(x^{(N)})\bar{A}^{(N)} - A^{(N)}\|D(y^{(N)})\|_2 = O_{\text{w.h.p}} \left( \frac{\rho_3^{(N)}}{\sqrt{M_N}} (\sqrt{N} + \sqrt{M_N} + \sqrt{\log N}) \right). \tag{71}
\]
Continuing, for the second summand in (68), we have

\[
\|D(\eta^{(N)}) \overline{A} D(y^{(N)})\|_2 \leq \|D(\eta^{(N)})\|_2 \cdot \|\overline{A}\|_2 \cdot \|D(y^{(N)})\|_2 \leq \|D(\eta^{(N)})\|_2 \cdot \|\overline{A}\|_F \cdot \max_j c_j^{(N)} \frac{\max_i c_i^{(N)}}{\sqrt{s^{(N)}}}
\]

\[
\leq \max_i |\eta_i^{(N)}| \frac{b \sqrt{MN} \max_j c_j^{(N)}}{\sqrt{s^{(N)}}} \leq \max_i \frac{|x_i^{(N)} - x_i^{(N)}|}{x_i^{(N)}} \cdot \max_i x_i^{(N)} \cdot \frac{b \sqrt{MN} \max_j c_j^{(N)}}{\sqrt{s^{(N)}}}
\]

\[
\leq \max_i \frac{|x_i^{(N)} - x_i^{(N)}|}{x_i^{(N)}} \cdot \max_i x_i^{(N)} \cdot \frac{b^{3/2} \sqrt{MN} \max_i r_i^{(N)} \cdot \max_j c_j^{(N)}}{as^{(N)}}
\]

\[
= O_{w.h.p} \left( \rho_1^{(N)} \rho_2^{(N)} \rho_3^{(N)} \sqrt{\log(\max\{M_N, N\})} \right),
\]

where we used Lemma 2 (applied to \(A^{(N)}\)) and Theorem 4. Analogously, it is easy to verify that the third summand in (68) admits the same bound as the second summand, namely

\[
\|D(x^{(N)}) \overline{A} D(\xi^{(N)})\|_2 = O_{w.h.p} \left( \rho_1^{(N)} \rho_2^{(N)} \rho_3^{(N)} \sqrt{\log(\max\{M_N, N\})} \right).
\]

For the fourth summand in (68), we write

\[
\|D(\eta^{(N)}) \overline{A} D(\xi^{(N)})\|_2 \leq \|D(\eta^{(N)})\|_2 \cdot \|\overline{A}\|_2 \cdot \|D(\xi^{(N)})\|_2 \leq \max_i |\eta_i^{(N)}| \cdot \|\overline{A}\|_F \cdot \max_j |\xi_j^{(N)}|
\]

\[
\leq \max_i \frac{|x_i^{(N)} - x_i^{(N)}|}{x_i^{(N)}} \cdot \max_i x_i^{(N)} \cdot \sqrt{MN} \max_j \frac{|y_j^{(N)} - y_j^{(N)}|}{y_j^{(N)}} \cdot \max_j y_j^{(N)}
\]

\[
\leq \max_i \frac{|x_i^{(N)} - x_i^{(N)}|}{x_i^{(N)}} \cdot \max_i x_i^{(N)} \cdot \frac{b \sqrt{MN} \max_i r_i^{(N)} \cdot \sqrt{N} \max_j c_j^{(N)}}{a^2 s^{(N)}}
\]

\[
= O_{w.h.p} \left( \rho_1^{(N)} \rho_2^{(N)} \sqrt{\log(\max\{M_N, N\})} \rho_3^{(N)} \right),
\]

where we again used Lemma 2 (applied to \(A^{(N)}\)) and Theorem 4.

Observe that \(s^{(N)} = \|r^{(N)}\|_1 \leq \sqrt{MN}\|r^{(N)}\|_2\), and \(s^{(N)} = \|c^{(N)}\|_1 \leq \sqrt{N}\|c^{(N)}\|_2\). Therefore,

\[
\rho_1^{(N)} = \max \left\{ \frac{\|r^{(N)}\|_2}{s^{(N)}}, \frac{\|c^{(N)}\|_2}{s^{(N)}} \right\} \geq \max \left\{ \frac{1}{\sqrt{M_N}}, \frac{1}{\sqrt{N}} \right\}.
\]

Using the above together with the fact that \(\rho_2^{(N)} \geq 1\), it follows that

\[
\rho_1^{(N)} \rho_2^{(N)} \rho_3^{(N)} \sqrt{\log(\max\{M_N, N\})} \geq \rho_3^{(N)} \max \left\{ \frac{1}{\sqrt{M_N}}, \frac{1}{\sqrt{N}} \right\} \sqrt{\log(\max\{M_N, N\})}
\]

\[
= \frac{\rho_3^{(N)}}{\sqrt{MN}} \max \left\{ \sqrt{M_N}, \sqrt{N} \right\} \sqrt{\log(\max\{M_N, N\})}
\]

\[
\geq \frac{\rho_3^{(N)}}{2\sqrt{M_N}} (\sqrt{M_N} + \sqrt{N}) \sqrt{\log N} \geq \frac{\rho_3^{(N)}}{4\sqrt{MN}} (\sqrt{M_N} + \sqrt{N} + \sqrt{\log N}),
\]
for all sufficiently large indices $N$. Applying the above inequality to (71) we obtain

$$
\|D(x^{(N)}) (A^{(N)} - A^{(N)}) D(y^{(N)})\|_2 = O_{w.h.p} \left( \rho_1^{(N)} \rho_2^{(N)} \rho_3^{(N)} \sqrt{\log(\max\{M_{N,N}\})} \right).
$$

(77)

We first consider indices $N$ for which $\rho_1^{(N)} \rho_2^{(N)} \sqrt{\log(\max\{M_{N,N}\})} \leq 1$. In this case, by (74) we have

$$
\|D(\eta^{(N)}) \tilde{A} D(\xi^{(N)})\|_2 = O_{w.h.p} \left( \rho_1^{(N)} \rho_2^{(N)} \rho_3^{(N)} \sqrt{\log(\max\{M_{N,N}\})} \right),
$$

(78)

and plugging (77), (72), (73), and (78) into (68) gives that

$$
\|D(\tilde{x}^{(N)}) \tilde{A}^{(N)} D(\tilde{y}^{(N)}) - D(x^{(N)}) A^{(N)} D(y^{(N)})\|_2 = O_{w.h.p} \left( \rho_1^{(N)} \rho_2^{(N)} \rho_3^{(N)} \sqrt{\log(\max\{M_{N,N}\})} \right).
$$

(79)

Next, we consider indices $N$ for which $\rho_1^{(N)} \rho_2^{(N)} \sqrt{\log(\max\{M_{N,N}\})} > 1$. In this case, we write

$$
\|D(\tilde{x}^{(N)}) \tilde{A}^{(N)} D(\tilde{y}^{(N)}) - D(x^{(N)}) A^{(N)} D(y^{(N)})\|_2 \leq \|D(\tilde{x}^{(N)}) \tilde{A}^{(N)} D(\tilde{y}^{(N)})\|_F + \|D(x^{(N)}) A^{(N)} D(y^{(N)})\|_F.
$$

(80)

By applying Lemma 2 to $A^{(N)}$ and $\tilde{A}^{(N)}$, we have that

$$
\tilde{x}_i^{(N)} \tilde{A}^{(N)} \tilde{y}_j^{(N)} \leq \left( \frac{b}{a} \right)^2 r_i^{(N)} c_j^{(N)} \frac{c_j^{(N)}}{s^{(N)}}, \quad \text{and} \quad x_i^{(N)} A^{(N)} y_j^{(N)} \leq \left( \frac{b}{a} \right)^2 r_i^{(N)} c_j^{(N)} \frac{c_j^{(N)}}{s^{(N)}},
$$

(81)

for all $i \in [M_{N}]$ and $j \in [N]$. Therefore, Combining (81) with (80) implies

$$
\|D(\tilde{x}^{(N)}) \tilde{A}^{(N)} D(\tilde{y}^{(N)}) - D(x^{(N)}) A^{(N)} D(y^{(N)})\|_2 \leq 2 \left( \frac{b}{a} \right)^2 \frac{\|r^{(N)}\|_2 \cdot \|c^{(N)}\|_2}{s^{(N)}} \left( \frac{b}{a} \right)^2 \rho_3^{(N)},
$$

$$
\leq 2 \left( \frac{b}{a} \right)^2 \sqrt{M_{N}} \max_i \sqrt{r_i^{(N)} \cdot \sqrt{N}} \max_j c_j^{(N)} \frac{c_j^{(N)}}{s^{(N)}} = 2 \left( \frac{b}{a} \right)^2 \rho_3^{(N)},
$$

$$
< 2 \left( \frac{b}{a} \right)^2 \rho_1^{(N)} \rho_2^{(N)} \rho_3^{(N)} \sqrt{\log(\max\{M_{N,N}\})} = O_{w.h.p} \left( \rho_1^{(N)} \rho_2^{(N)} \rho_3^{(N)} \sqrt{\log(\max\{M_{N,N}\})} \right),
$$

(82)

where we used the fact that $\rho_1^{(N)} \rho_2^{(N)} \sqrt{\log(\max\{M_{N,N}\})} > 1$ for the considered indices $N$.

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