METASTABLE BEHAVIOR OF WEAKLY MIXING MARKOV CHAINS: THE CASE OF REVERSIBLE, CRITICAL ZERO-RANGE PROCESSES

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Abstract. We present a general method to derive the metastable behavior of weakly mixing Markov chains. This approach is based on properties of the resolvent equations and can be applied to metastable dynamics which do not satisfy the mixing conditions required in [6, 7] or in [27].

As an application, we study the metastable behavior of critical zero-range processes. Let \( r : S \times S \to \mathbb{R}_+ \) be the jump rates of an irreducible random walk on a finite set \( S \), reversible with respect to the uniform measure. For \( \alpha > 0 \), let \( g : \mathbb{N} \to \mathbb{R}_+ \) be given by \( g(0) = 0, g(1) = 1, g(k) = [k/(k-1)]^\alpha \), \( k \geq 2 \). Consider a zero-range process on \( S \) in which a particle jumps from a site \( x \), occupied by \( k \) particles, to a site \( y \) at rate \( g(k)r(x,y) \). For \( \alpha \geq 1 \), in the stationary state, as the total number of particles, represented by \( N \), tends to infinity, all particles but a negligible number accumulate at one single site. This phenomenon is called condensation. Since condensation occurs if and only if \( \alpha \geq 1 \), we call the case \( \alpha = 1 \) critical. By applying the general method established in the first part of the article to the critical case, we show that the site which concentrates almost all particles evolves in the time-scale \( N^2 \log N \) as a random walk on \( S \) whose transition rates are proportional to the capacities of the underlying random walk.

1. Introduction

More than twenty years ago, stochastic dynamics which exhibit condensation [14, 17] have been introduced. These dynamics describe a conservative evolution of particles on a finite or countably infinite set \( S \). Despite the fact that no particles are created nor annihilated, the system condensates in the sense that a macroscopic proportion of particles sit on a single site provided the density exceeds a critical value [20, 19, 1, 3, 37].

We consider here zero-range processes evolving on a finite set \( S \). The dynamics can be described as follows. Let \( r : S \times S \to \mathbb{R}_+ \) be the jump rates of an irreversible random walk on \( S \), reversible with respect to the uniform measure \( m \). For \( \alpha > 0 \), let \( g : \mathbb{N} \to \mathbb{R}_+ \) be given by \( g(0) = 0, g(1) = 1, g(k) = [k/(k-1)]^\alpha \), \( k \geq 2 \). A particle jumps from a site \( x \), occupied by \( k \) particles, to a site \( y \) at rate \( g(k)r(x,y) \).

Phase transitions. This model exhibits phase transitions at \( \alpha = 1 \) and at \( \alpha = 2 \). Indeed, let \( Z_\alpha : \mathbb{R}_+ \to \mathbb{R}_+ \) be the partition function given by

\[
Z_\alpha(\varphi) = 1 + \sum_{k \geq 1} \frac{\varphi^k}{g(1) \cdots g(k)} = 1 + \sum_{k \geq 1} \frac{\varphi^k}{k^\alpha}.
\]

The parameter \( \varphi \) is usually called the fugacity. It is clear that \( \varphi_c = 1 \) is the radius of convergence of the series for all \( \alpha > 0 \). The grand canonical stationary states,
denoted by \{\nu_\varphi^{(\alpha)} : 0 \leq \varphi < \varphi_c\}, of the zero-range processes are given by

\[ \nu_\varphi^{(\alpha)}(\eta) = \prod_{x \in S} \frac{1}{Z_\alpha(\varphi)} \varphi_{\eta_x}^{\eta_x}. \]

In this formula, \( \eta = (\eta_x)_{x \in S} \) represents a configuration of particles and \( \eta_x \) the number of particles at site \( x \) for the configuration \( \eta \).

The density of particles under the stationary state \( \nu_\varphi^{(\alpha)} \), denoted by \( R_\alpha(\varphi) \), is given by \( R_\alpha(\varphi) = \varphi Z'_\alpha(\varphi)/Z_\alpha(\varphi) \), where \( Z'_\alpha \) stands for the derivative of \( Z_\alpha \).

By (1.1), for \( 0 < \alpha \leq 1 \), \( Z_\alpha(\varphi) \) and \( R_\alpha(\varphi) \) diverge as \( \varphi \to \varphi_c \). In particular, for every density \( \rho > 0 \) there exists (a unique) fugacity whose corresponding grand canonical state has density \( \rho \). For \( 1 < \alpha \leq 2 \), \( Z_\alpha(\varphi) \) converges, but \( R_\alpha(\varphi) \) diverges. In this range, it still holds that for every density \( \rho > 0 \) there exists (a unique) fugacity whose corresponding grand canonical state has density \( \rho \). Finally, for \( \alpha > 2 \), \( Z_\alpha(\varphi) \) and \( R_\alpha(\varphi) \) converge as \( \varphi \to \varphi_c \), and there is a critical density \( \rho_c \) above which there is no fugacity whose corresponding grand canonical state has density \( \rho \).

**Condensation.** By the previous considerations, in the thermodynamical limit, condensation appears only for \( \alpha > 2 \). However, in the context of a fixed finite set \( S \) with the total number of particles increasing to infinity, condensation also occurs in the range \( 1 \leq \alpha \leq 2 \). As there is no condensation for \( \alpha < 1 \) when \( S \) is fixed and finite, we call the parameter \( \alpha = 1 \) critical and \( \alpha > 1 \) super-critical.

For each \( N \geq 1 \), representing the total number of particles, denote by \( \mu_N \) the unique stationary state of the zero-range dynamics with \( N \) particles evolving on \( S \). Fix a sequence \( (\ell_N : n \geq 1) \) of integer numbers such that \( \ell_N \to \infty \), \( \ell_N/N \to 0 \). Denote by \( \mathcal{E}_N^x \), \( x \in S \), the set of configurations given by

\[ \mathcal{E}_N^x = \left\{ \eta : \eta_x \geq N - \ell_N \right\}. \]

Hence, \( \mathcal{E}_N^x \) represents the set of configurations with at least \( N - \ell_N \) particles at site \( x \), that is, the configurations in which a condensate has been formed at site \( x \).

By [8] for \( \alpha > 1 \), and by Theorem 3.3 below for \( \alpha = 1 \) (under additional assumptions on the sequence \( \ell_N \), \( \mu_N(\mathcal{E}_N^x) \to 1/|S| \). Therefore, under the stationary state, essentially all particles sit on a single site.

**The evolution of the condensate.** Once condensation has been established, it becomes natural to consider the time evolution of the model. One expects to observe two different regimes. As particles accumulate on a single site in the stationary state, starting from a homogeneous distribution of particles among all sites, coarsening should occur in a certain time-scale, and particles should gradually concentrate on fewer and fewer sites, until the system saturates and almost all of them sit on a single site. This regime is called in physics literature the coarsening phase of the dynamics. It has been established in [5] for \( \alpha > 1 \) and shown to occur in the time-scale \( N^2 \).

Consider a configuration in which all particles sit on the same site. Call condensate the site at which this occurs. On a longer time-scale, one expects to observe an evolution of the condensate. This has been quantitatively analyzed for zero-range processes evolving on a finite set for \( \alpha > 1 \) in [8, 22, 35, 31] and in the thermodynamical limit (when the number of sites increases together with the number of particles) for \( \alpha > 20 \) in [4].
In this article, we examine the evolution of the condensate on a finite set in the critical case \( \alpha = 1 \). In this context, there is an important difference between the case \( \alpha > 1 \), considered previously, and the case \( \alpha = 1 \) studied here. In the former, on a fixed number of sites, starting from a configuration in a set \( \mathcal{E}_N^x \), called from now on “well”, the process visits all configurations of \( \mathcal{E}_N^x \) before hitting a new well \( \mathcal{E}_N^y, y \neq x \). Such dynamics are said to “visit points”. In contrast, in the critical case \( \alpha = 1 \), this property does not hold because the wells are much larger.

In [6, 7], a general theory has been proposed to derive the metastable behavior of dynamics which visit points. This approach has been successfully applied to super-critical zero-range processes in the aforementioned articles and to inclusion processes in [10].

A resolvent approach. Motivated by critical zero-range processes, in the first part of the article, we present a general method to derive the metastable behavior of dynamics which do not satisfy the assumptions of [6, 7].

The approach consists in showing that the Markov chain fulfills two conditions. First, that the solution of some resolvent equations are, in each well \( \mathcal{E}_N^x \), close to a constant in the \( L^1 \) sense. Then, that starting from any point in \( \mathcal{E}_N^x \), the process does not jump immediately to another well \( \mathcal{E}_N^y, y \neq x \).

In Section 5, we show that the first condition follows from a spectral gap estimate for the dynamics obtained by reflecting the process at the boundary of the metastable sets, and from a characterization of the limits of the solution of the resolvent equation over each well.

The fact that the process remains in a well a reasonable amount of time can be derived in two steps. One first show that the process visits a deep region of the well before it reaches another well. This part of the argument relies on the construction of a super-harmonic function. Then, one proves that starting from this deep region, the process does not hit quickly another well.

The method proposed in [27], which also relies on properties of the resolvent equation, is designed for dynamics with good local ergodic properties and requires either an estimate of the mixing time of the reflected process or the property that the process visits a specific point in the well in the metastable time-scale. In contrast, the method proposed here is designed for dynamics where these properties do not hold or cannot be proved. This is the case of diffusions [24], zero-range processes in the thermodynamical limit [34] and of many other dynamics in which the entropy, and not only the energy landscape, plays a role in the metastable behavior [9, 23] and references therein.

Back to critical zero-range processes. In the second part of the article, we apply the method described above to critical zero-range processes. All estimates are delicate in the critical case due to the small difference between time-scales. While coarsening occurs in the diffusive scale \( N^2 \), the evolution of the condensate is observed in the time-scale \( N^2 \log N \), and the equilibration inside the sets \( \mathcal{E}_N^x \) in a time-scale \( (N / \log N)^2 \).

All these time-scales do not depend on the lattice geometry. The asymptotic jump rates of the condensate, however, depend on the geometry.

An interesting problem, left for future investigations, is the description of the coarsening phase of this model.
2. Metastability of weakly mixing Markov chains

In this and the next section, we present the main results of the article. We provide here a set of sufficient conditions for a sequence of continuous-time Markov chains with poor local mixing conditions to exhibit a metastable behavior. All new notation introduced in the text and not in a displayed equation is presented in blue.

We start by introducing the general framework proposed in \[6, 7\] to describe the metastable behavior of a Markovian dynamics as a Markov chain model reduction. Let \( \{ \mathcal{H}_N : N \geq 1 \} \) be a collection of finite sets. Elements of the set \( \mathcal{H}_N \) are designated by the letters \( \eta, \xi, \) and \( \zeta. \)

Consider a sequence \( \{ \xi_N(t) : t \geq 0 \} \) of \( \mathcal{H}_N \)-valued, irreducible, continuous-time Markov chains, whose generator is represented by \( \mathcal{L}_N \). Therefore, for every function \( f : \mathcal{H}_N \to \mathbb{R}, \)

\[
(\mathcal{L}_N f)(\eta) = \sum_{\xi \in \mathcal{H}_N} R_N(\eta, \xi) \left[ f(\xi) - f(\eta) \right],
\]

where \( R_N(\eta, \xi) \) stands for the jump rates. Denote by \( \lambda_N(\eta) \) the holding times of the Markov chain, \( \lambda_N(\eta) = \sum_{\xi \neq \eta} R_N(\eta, \xi), \) and by \( \mu_N \) the unique stationary state.

Denote by \( D(\mathbb{R}_+, \mathcal{H}_N) \) the space of right-continuous functions \( x : \mathbb{R}_+ \to \mathcal{H}_N \) with left-limits, endowed with the Skorohod topology and its associated Borel \( \sigma \)-field. For a probability measure \( \nu \) on \( \mathcal{H}_N \), let \( P_N^\nu \) be the measure on \( D(\mathbb{R}_+, \mathcal{H}_N) \) induced by the process \( \xi_N(\cdot) \) starting from \( \nu \). When \( \nu = \delta_\eta \) is the Dirac measure concentrated on a configuration \( \eta \in \mathcal{H}_N \), we denote \( P_N^\eta \) by \( P_N^\eta \). Expectation with respect to \( P_N^\nu, P_N^\eta \) is represented by \( \mathbb{E}_\nu^N, \mathbb{E}_\eta^N \), respectively.

Fix a finite set \( S \), and denote by \( \mathcal{E}_N^x, x \in S, \) a family of disjoint subsets of \( \mathcal{H}_N \). Let

\[
\mathcal{E}_N = \bigcup_{x \in S} \mathcal{E}_N^x \quad \text{and} \quad \Delta_N = \mathcal{H}_N \setminus \left( \bigcup_{x \in S} \mathcal{E}_N^x \right). \tag{2.1}
\]

The sets \( \mathcal{E}_N^x, x \in S, \) represent the metastable sets of the dynamics \( \xi_N(\cdot), \) in the sense that, as soon as the process \( \xi_N(\cdot) \) enters one of these sets, \( \mathcal{E}_N^x, \) it equilibrates in \( \mathcal{E}_N^x \) before hitting a new set \( \mathcal{E}_N^y, y \neq x. \) These metastable sets are often called wells. The goal of the theory is to describe the evolution between these wells. To this end, we introduce the so-called order process.

For \( \mathcal{A} \subset \mathcal{H}_N \), denote by \( T^\mathcal{A}(t) \) the total time the process \( \xi_N(\cdot) \) spends in \( \mathcal{A} \) in the time-interval \( [0, t] \):

\[
T^\mathcal{A}(t) = \int_0^t \chi_\mathcal{A}(\xi_N(s)) \, ds,
\]

where \( \chi_\mathcal{A} \) represents the characteristic function of the set \( \mathcal{A} \). Denote by \( S^\mathcal{A}(t) \) the generalized inverse of \( T^\mathcal{A}(t) \):

\[
S^\mathcal{A}(t) = \sup\{ s \geq 0 : T^\mathcal{A}(s) \leq t \}. \tag{2.2}
\]

The trace of \( \xi_N(\cdot) \) on \( \mathcal{A} \), denoted by \( \{ \xi_N^\mathcal{A}(t) : t \geq 0 \} \), is defined by

\[
\xi_N^\mathcal{A}(t) = \xi_N(S^\mathcal{A}(t)) ; \quad t \geq 0. \tag{2.3}
\]

It is an \( \mathcal{A} \)-valued, continuous-time Markov chain, obtained by turning off the clock when the process \( \xi_N(\cdot) \) visits the set \( \mathcal{A}^c \), that is, by deleting all excursions to \( \mathcal{A}^c \). For this reason, it is called the trace process of \( \xi_N(\cdot) \) on \( \mathcal{A}. \)
Let $\Psi_N : \mathcal{E}_N \to S$ be the projection given by
$$\Psi_N(\eta) = \sum_{x \in S} x \cdot \chi_{E_N}(\eta) .$$
The order process $(Y_N(t) : t \geq 0)$ is defined as
$$Y_N(t) = \Psi_N(\xi_N^N(t)), \quad t \geq 0 . \quad (2.4)$$
Denote by $Q^N_\nu$ the probability measure on $D(\mathbb{R}_+, S)$ induced by the measure $P^N_\nu$ and the order process $Y_N$: $Q^N_\nu = P^N_\nu \circ Y_N^{-1}$.

Fix a probability measure $\pi$ on $S$ and a generator $L$ of a $S$-valued, continuous-time Markov chain. Denote by $Q^L_\pi$ the measure on $D(\mathbb{R}_+, S)$ induced by the Markov chain whose generator is $L$ and which starts from $\pi$. Denote $Q^L_x, x \in S$, simply by $Q^L_x$.

**Definition 2.1.** Fix a sequence of probability measures $\{\nu_N : N \geq 1\}$ on $\mathcal{H}_N$ such that $\nu_N(\mathcal{E}_N) = 1$ for all $N \geq 1$. The sequence of Markov chains $\{\xi_N(\cdot) : N \geq 1\}$ is said to be $(\nu_N, \{E^N_x : x \in S\}, \pi, L)$-metastable if

(A) As $N \to \infty$, the sequence of laws $(Q^N_\nu)_{N \in \mathbb{N}}$ converges weakly to $Q^L_\pi$.

(B) For all $t > 0$,
$$\lim_{N \to \infty} \mathbb{E}^N_\nu \left[ \int_0^t \chi_{\Delta_N}(\xi_N(s)) ds \right] = 0 . \quad (2.5)$$

Condition (A) asserts that the order process $Y_N(\cdot)$ converges weakly and condition (B) that the process $\xi_N(\cdot)$ spends a negligible amount of time on $\Delta_N$. It ensures, therefore, that the trace process does not differ much from the original one when starting from $\nu_N$. Note that, by condition (A), $\nu_N(\mathcal{E}_N^x) \to \pi(x)$.

**Comments.** The above definition of metastability differs from the one presented in [6, 7] in that the initial state $\nu_N$ is a measure spread over a well and not a Dirac measure concentrated on a configuration. We introduce some notation to explain the reasons of this modification.

Denote by $\tau_A, A \subset \mathcal{H}_N$ the hitting time of the set $A$:
$$\tau_A = \inf\{t \geq 0 : \xi_N(t) \in A\} ,$$
and by $h_{A, B} : \mathcal{H}_N \to \mathbb{R}$ the equilibrium potential between two disjoint, non-empty subsets $A$ and $B$ of $\mathcal{H}_N$:
$$h_{A, B}(\eta) = P^N_\eta [\tau_A < \tau_B] .$$
Recall that $\mu_N$ denotes the unique invariant measure for the Markov chain $\xi_N(t)$, and denote by $\mathcal{D}_N$ the Dirichlet form associated to the generator $L_N$: for each $F : \mathcal{H}_N \to \mathbb{R}$,
$$\mathcal{D}_N(F) = \langle F, (-L_N F) \rangle_{\mu_N} , \quad (2.5)$$
where $\langle \cdot, \cdot \rangle_{\mu_N}$ stands for the scalar product in $L^2(\mu_N)$. The capacity between $A$ and $B$ is given by
$$\text{cap}_N(A, B) = \mathcal{D}_N(h_{A, B}) . \quad (2.6)$$

The method proposed in [6, 7] to derive metastability relies on the following condition.
For each $x \in S$, there exists a sequence of configurations $\{\xi_N^x : N \geq 1\}$ such that $\xi_N^x \in E_N^x$ for all $N \geq 1$ and
\[
\lim_{N \to \infty} \max_{\eta \in E_N^x} \frac{\text{cap}_N(E_N^x, \xi_N^x)}{\text{cap}_N(\xi_N^x, \eta)} = 0 ,
\]
where
\[
\check{E}_N^x = E_N^x \setminus E_N^x = \bigcup_{y : y \neq x} E_y^x .
\]

By [7, Theorem 2.7], condition (H1) implies that the process $\xi_N(\cdot)$ visits all configurations of a well before it hits a new well: for all $x \in S$,
\[
\lim_{N \to \infty} \max_{\eta, \xi \in E_N^x} P_N^\eta \left[ \tau_{\check{E}_N^x} < \tau_\xi \right] = 0 .
\]

Condition (H1) and its aftermath (2.8) have been derived for many dynamics (cf. [23]), but they are clearly not satisfied in many others. For instance, critical zero-range processes, considered in this article, diffusions in potential fields [12, 28] or condensing zero-range dynamics in the thermodynamical limit [4], to mention a few.

In [27], we present a robust method, based on properties of the resolvent equation, to derive the metastable behavior of dynamics satisfying (2.8). In this article, we present a tailor-made approach, also based on properties of the resolvent equation, to handle dynamics which do not.

**Main result.** The main result of this section provides sufficient conditions for a sequence of Markov chains to be metastable in the sense of Definition 2.1. Denote by $\mu_N^x$ the measure $\mu_N$ conditioned on $E_N^x$:
\[
\mu_N^x(\eta) = \frac{\mu_N(\eta)}{\mu_N(E_N^x)} , \quad \eta \in E_N^x .
\]

The first ingredient is the following condition.

**C1** The set $\Delta_N$ is negligible in the sense that for all $x \in S$, $\mu_N(\Delta_N)/\mu_N(E_N^x) \to 0$.

Moreover, the initial state, represented by $\nu_N$, is concentrated on one well: there exists $x_0 \in S$ such that $\nu_N(E_N^{x_0}) = 1$ for all $N \geq 1$. Finally, there exists a finite constant $C_1$ such that
\[
E_{\mu_N^{x_0}} \left[ \left( \frac{d\nu_N}{d\mu_N^{x_0}} \right)^2 \right] = \sum_{\eta \in E_N^{x_0}} \frac{\nu_N(\eta)^2}{\mu_N^{x_0}(\eta)} \leq C_1 \quad \text{for all } N \geq 1 .
\]

The second ingredient reads:

**C2** For all $x \in S$,
\[
\limsup_{a \to 0} \limsup_{N \to \infty} \sup_{\eta \in E_N^x} P_N^\eta \left[ \tau_{\check{E}_N^x} < a \right] = 0 .
\]

The last condition **C3** below requires the solutions of some resolvent equations to be asymptotically constant on each well $E_N^x$, $x \in S$. Recall that $L$ represents the generator of a $S$-valued continuous-time Markov chain. Fix $\lambda > 0$ and a function $f : S \to \mathbb{R}$. Let $G_N : \mathcal{H}_N \to \mathbb{R}$ be given by
\[
G_N(\eta) = \sum_{x \in S} \left( \lambda - L \right) f(x) \chi_{E_N^x}(\eta) .
\]
That is, the function $G_N$ is equal to $[(\lambda - L)f](x)$ on $E_N$, $x \in S$, and it vanishes on $\Delta_N$. Denote by $F_N : H_N \to \mathbb{R}$ the solution of the resolvent equation

$$(\lambda - L_N)F_N = G_N \quad \text{on} \quad H_N.$$  \tag{2.12}

(C3) For all $\lambda > 0$, function $f : S \to \mathbb{R}$ and $t \geq 0$,

$$\lim_{N \to \infty} \mathbb{E}_N^N \left[ \left| F_N(\xi_N(t)) - f(Y_N(t)) \right| \right] = 0.$$ \tag{2.13}

**Theorem 2.2.** Assume that conditions (C1) – (C3) are in force for some $x_0 \in S$. Then, the process $\xi_N(\cdot)$ is $(\nu_N, \{E_N^x : x \in S\}, \delta_{x_0}, L)$-metastable in the sense of Definition 2.1.

**Remark 2.3.** In contrast with the others, condition (C2) requires to estimate an event with respect to a measure on the path space whose initial distribution is a configuration. The initial state in the other two conditions are not too far from the stationary measure conditioned to a well. For this reason, in many dynamics, condition (C2) is the most difficult to prove.

**Remark 2.4.** By [16, Lemma 3.2 in Chapter 4], it is enough to prove condition (C3) for just one $\lambda > 0$. This observation, however, does not simplify the proofs.

In Section 4, we prove Theorem 2.2, and in Section 5 we provide a set of conditions that yield (C2) and (C3). In the second part of the article we show that these conditions are in force for the critical condensing zero-range processes presented below. In [24, 34], we apply the method presented here to diffusions and to condensing zero-range processes in the thermodynamical limit, respectively.

### 3. Condensing zero-range processes

Fix a finite set $S$ and let $\kappa = |S|$, which is assumed to be larger than or equal to 2. Elements of $S$ are denoted by the letters $x, y, z$. Let $\{X(t)\}_{t \geq 0}$ be a continuous-time, irreducible Markov chain on the set $S$. The jump rates are represented by $r$ and the generator by $L_X$ so that

$$(L_Xf)(x) = \sum_{y \in S} r(x, y) \left[ f(y) - f(x) \right]$$

for all $f : S \to \mathbb{R}$. For convenience, we set $r(x, x) = 0$ for all $x \in S$.

Denote by $\mathbb{P}_x$, $x \in S$, the law of the random walk $X$ starting from $x$, and by $m$ its unique invariant probability measure.

Let $\mathbb{N} = \{0, 1, 2, \ldots\}$, fix $\alpha > 0$, and define $a = a_\alpha : \mathbb{N} \to \mathbb{R}^+$ as

$$a(0) = 1 \quad \text{and} \quad a(n) = n^\alpha \quad \text{for} \quad n \geq 1.$$  

Denote by $g = g_\alpha : \mathbb{N} \to \mathbb{R}^+$ the function given by

$$g(0) = 0, \quad g(1) = 1 \quad \text{and} \quad g(n) = \frac{a(n)}{a(n-1)} = \left( \frac{n}{n-1} \right)^\alpha, \quad n \geq 2.$$  

Denote by $a, g : \mathbb{N}^S \to \mathbb{R}$ the functions given by

$$g(\eta) = \prod_{x \in S} g(\eta_x) \quad \text{and} \quad a(\eta) = \prod_{x \in S} a(\eta_x).$$
For each $x \neq y \in S$ and $\eta \in \mathbb{N}^S$, denote by $\sigma^{x,y}\eta$ the configuration obtained from $\eta$ by moving a particle from $x$ to $y$:

$$(\sigma^{x,y}\eta)_z = \begin{cases} 
\eta_x - 1 & \text{if } z = x, \\
\eta_y + 1 & \text{if } z = y, \\
\eta_z & \text{otherwise},
\end{cases}$$

if $\eta_x \geq 1$, and $\sigma^{x,y}\eta = \eta$ if $\eta_x = 0$.

Denote by $\mathcal{H}_N = \mathcal{H}_{S,N}$, $N \geq 1$, the set given by

$$\mathcal{H}_N = \left\{ \eta = (\eta_x)_{x \in S} \in \mathbb{N}^S : \sum_{x \in S} \eta_x = N \right\}.$$ 

The zero-range process with parameters $\alpha$ and $r$ is the $\mathcal{H}_N$-valued, continuous-time Markov chain $\{\eta_N(t)\}_{t \geq 0}$ whose generator, denoted by $A_N$, is given by

$$(A_N F)(\eta) = \sum_{x,y \in S} g(\eta_x) r(x,y) \left[ F(\sigma^{x,y}\eta) - F(\eta) \right], \quad (3.1)$$

for all functions $F : \mathcal{H}_N \to \mathbb{R}$. Clearly, the process $\eta_N(\cdot)$ is ergodic.

### 3.1. Condensation and metastable behavior in the super-critical case

In this subsection, we review the results for the super-critical case $\alpha > 1$ obtained in [8, 22, 35, 31].

**Condensation phenomenon.** To simplify the presentation, we assume that the invariant probability measure $m$ of the underlying random walk $X(\cdot)$ is the uniform measure:

$$m(x) = \frac{1}{\kappa}, \quad x \in S. \quad (3.2)$$

For the general result without this assumption, we refer to [8, 35].

The invariant probability measure for the zero-range process can be written as

$$\mu_N(\eta) = \frac{1}{\hat{Z}_N} \frac{1}{a(\eta)}, \quad (3.3)$$

where $\hat{Z}_N$ is the normalizing constant given by

$$\hat{Z}_N = \sum_{\eta \in \mathcal{H}_N} \frac{1}{a(\eta)}.$$

Let $(\ell_N)_{N \in \mathbb{N}}$ be a sequence of integer numbers satisfying

$$1 \ll \ell_N \ll N.$$ 

Here, for two sequences $(a_N)_{N \in \mathbb{N}}$ and $(b_N)_{N \in \mathbb{N}}$ of positive real numbers, $a_N \ll b_N$ stands for $\lim_{N \to \infty} a_N/b_N = 0$.

Denote by $\mathcal{E}_N^x$, $x \in S$, the set of configurations with at least $N - \ell_N$ particles at site $x$:

$$\mathcal{E}_N^x = \{ \eta : \eta_x \geq N - \ell_N \}, \quad (3.4)$$

and recall from (2.1) the definition of the sets $\mathcal{E}_N$ and $\Delta_N$. The sets $\mathcal{E}_N^x$, $x \in S$, are called the wells. To stress the dependence of $\Delta_N$ on the set $S$, we sometimes write $\Delta_N$ as $\Delta_{S,N}$.

The next result asserts that the dynamics tend to concentrate particles on a single site. This is called the condensation phenomenon.
Theorem 3.1 (Condensation in the super-critical case). For $\alpha > 1$,
\[ \lim_{N \to \infty} \mu_N(E_N^x) = \frac{1}{\kappa} \text{ for all } x \in S. \]

It follows from this result that $\lim_{N \to \infty} \mu_N(\Delta_N) = 0$. Versions of this result have been obtained in [17, 20, 19, 8, 1, 2, 3]. We refer to [8, Section 3] for a proof without the assumption (3.2).

Time-scale and speeded-up process. Since the transition time between two wells is of order $N^{1+\alpha}$, we speed-up the process by this amount: let
\[ \theta_N = N^{1+\alpha}, \quad \xi_N(t) = \eta_N(t \theta_N). \] (3.5)
The process $\xi_N(\cdot)$ is the $N^S$-valued, Markov chain whose generator, denoted by $L_N$, is given by $L_N = \theta_N A_N$, where the generator $A_N$ has been introduced in (3.1).

Warning: We borrow from the previous section all notation introduced there. Besides the measure $\mu_N$, the generator $L_N$ and the process $\xi_N(\cdot)$, which already appeared, this includes the probability measures $P_N^\nu$, $P_N^\eta$ on $D(\mathbb{R}_+, H_N)$, the Dirichlet form $D_N$, the capacity $\text{cap}_N$ and the measures $Q_N^\nu$.

The limiting process $Y(\cdot)$ in the super-critical case. Denote by $D_X(\cdot)$ the Dirichlet form associated to the random walk $X$: for $f : S \to \mathbb{R}$,
\[ D_X(f) = \frac{1}{2} \sum_{x, y \in S} m(x) r(x, y) (f(y) - f(x))^2. \]

Denote by $\tau_C$, $C \subset S$, the hitting time of the set $C$:
\[ \tau_C = \inf\{t \geq 0 : X(t) \in C\}. \]

Fix two non-empty, disjoint subsets $A$, $B$ of $S$. The equilibrium potential $h_{A, B} : S \to \mathbb{R}$ between $A$ and $B$ is defined by
\[ h_{A, B}(x) = \mathbb{P}_x[\tau_A < \tau_B], \quad x \in S. \] (3.6)

It is well-known that $h_{A, B}$ is the unique solution to the Dirichlet problem:
\[
\begin{cases}
(L_X h)(x) = 0 & x \in S \setminus \{A \cup B\}, \\
h(x) = 1 & x \in A, \\
h(x) = 0 & x \in B.
\end{cases}
\]

The capacity $\text{cap}_X(A, B)$ between $A$ and $B$ is given by
\[ \text{cap}_X(A, B) = D_X(h_{A, B}). \] (3.7)

If $A = \{x\}$ is a singleton we write $\text{cap}_X(x, B)$ instead of $\text{cap}_X(\{x\}, B)$.

The limiting process $Y(\cdot)$ is a continuous-time Markov chain on $S$ whose generator, represented by $L_Y$, is given by
\[ (L_Y f)(x) = \frac{\kappa}{\Gamma(1+\alpha)} \sum_{y \in S} \text{cap}_X(x, y) [f(y) - f(x)], \quad f : S \to \mathbb{R}. \]

In this formula,
\[ \Gamma(1+\alpha) = \sum_{j=0}^{\infty} \frac{1}{a(j)} = 1 + \sum_{n \geq 1} \frac{1}{n^\alpha}, \quad I_\alpha = \int_0^1 u^\alpha (1-u)\, du. \] (3.8)

Remark that $\Gamma(1+\alpha)$ is finite because $\alpha > 1$. 
Denote by $Q_Y$ the probability measure on $D(\mathbb{R}_+, S)$ induced by the Markov chain associated to the generator $L_Y$ starting from $x$.

**Metastable behavior.** We may now describe the evolution of the condensate, characterizing the metastable behavior of super-critical zero range processes. Recall from (2.4) the definition of the process $Y_N(t)$ and of the measure $Q_N^\nu$.

**Theorem 3.2.** Suppose that $\alpha > 1$ and that the invariant probability measure $m$ of $X(t)$ is the uniform measure (3.2). Fix $\delta > 0$ small and let the sequence $\ell_N$, introduced in (3.4), be given by $\ell_N = [N^\delta]$, where $[r]$ stands for the integer part of $r > 0$. Fix $x_0 \in S$ and a sequence $\{\eta_N : N \geq 1\}$ such that $\eta_N \in \mathcal{E}_N^0$ for all $N \geq 1$. Then, the sequence of Markov chains $\xi_N(\cdot)$ is $(\delta_{\eta_N}, \{\mathcal{E}_N^x : x \in S\}, \delta_{x_0}, L_Y)$-metastable in the sense of Definition 2.1. Moreover,

$$\lim_{N \to \infty} \sup_{q \in \mathcal{E}_N^\eta} E_q^\eta \left[ \int_0^t \chi_{\Delta_N}(\xi_N(s)) \, ds \right] = 0.$$ 

This result has been proven, without the uniformity assumption (3.2), in [8] for reversible dynamics. It has been extended in [22, 35] to the general case.

### 3.2. Critical zero-range processes.

We turn to the case $\alpha = 1$, under the uniformity condition (3.2). We adopt the same notation as in the previous subsection. The only and important difference lies on the definition of $\ell_N$, which defines the wells, and of $\theta_N$, which describes the time-scale.

**Condensation of particles.** We first describe the condensation. Let $\ell_N$ be the sequence given by

$$\ell_N = \left[ \frac{N}{\log N} \right], \quad (3.9)$$

where, recall, $[r]$ stands for the integer part of $r > 0$.

**Theorem 3.3.** The assertions of Theorem 3.1 hold for $\alpha = 1$ provided $(\ell_N)_{N \in \mathbb{N}}$ is chosen as in (3.9).

The proof of this result, given in Section 7, is similar to the one of the super-critical case, presented in [8, Section 3]. The assumption (3.2) can be removed, at the cost of heavy notation, which we preferred to avoid.

**Remark 3.4.** It follows from the proof of Theorem 3.3 that the sequence $\ell_N$ needs only to satisfy the conditions

$$\lim_{N \to \infty} \frac{\ell_N}{N} = 0 \quad \text{and} \quad \lim_{N \to \infty} \frac{\log \ell_N}{\log N} = 1.$$ 

In particular, we can select $\ell_N = [N/(\log N)^h]$ for any $h > 0$.

Theorem 3.3 fails, however, for $\ell_N = [N^\delta], \delta \in (0, 1)$, see Lemma 7.2. Therefore, in the critical case there are typically $N/\log N$ particles not sitting at the condensate, while in the super-critical case there are less than $k_N$ particles, where $k_N$ is any sequence increasing to $\infty$. In particular, the wells in the critical case are much larger than in the super-critical case. This is a source of problems and explains why the critical case is much more demanding than the super-critical one.
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**Time-scale.** The transition time between two wells can be easily guessed by examining the case with two sites, where the zero-range becomes a birth-and-death process on \(\{0, \ldots, N\}\). In this situation, one can compute explicitly the capacities between two wells and deduce from them the time-scale. In the critical case, it is of order \(N^2 \log N\). In particular, in the critical case, in the definition of the process \(\xi_N(t)\), introduced in (3.5), we take \(\theta_N = N^2 \log N\).

**Metastable behavior.** The statement of the metastable behavior of the condensate requires further notation and hypotheses.

We assume in this article that the underlying random walk is reversible with respect to the invariant (uniform) measure \(m\):

\[
r(x, y) = r(y, x) \quad \forall x, y \in S. \tag{3.10}
\]

The evolution of the condensate is described by the \(S\)-valued Markov chain, denoted by \(\{Z(t) : t \geq 0\}\), whose generator \(L_Z\) is given by

\[
(L_Zf)(x) = \sum_{y \in S} r_Z(x, y) \{f(y) - f(x)\}, \quad f : S \to \mathbb{R}, \tag{3.11}
\]

where

\[
r_Z(x, y) := 6 \kappa \text{cap}_X(x, y), \quad x, y \in S. \tag{3.12}
\]

The factor 6 in this formula represents \(1/I_1\), where \(I_\alpha\) is defined in (3.8). Note that the invariant measure of \(Z(\cdot)\) is the uniform distribution \(m\) on \(S\) since the capacity is symmetric. Denote by \(Q^Z_x\) the probability measure on \(D(\mathbb{R}^+, S)\) induced by the Markov chain associated to the generator \(L_Z\) starting from \(x\).

Recall the definition of the measure \(\mu_N\) introduced in (3.3), and the one of \(\mu^\infty_N\) introduced in (2.9). The second main result of this article reads as follows.

**Theorem 3.5.** Assume that \(\alpha = 1\) and that conditions (3.10) are in force. Fix \(x_0 \in S\) and a sequence of probability measures \(\{\nu_N : N \geq 1\}\) on \(H_N\) satisfying condition (C1). Then, the sequence of Markov chains \(\xi_N(\cdot)\) is \((\nu_N, \{\xi^\infty_N : x \in S\}, \delta_{x_0}, L_Z)\)-metastable in the sense of Definition 2.1.

The proof of this result is outlined in Section 6.

**Remark 3.6.** No assumption on the geometry of the lattice is needed. We only require the stationary measure of the underlying random walk to be uniform, which is not a necessary but simplifying assumption.

**Remark 3.7.** In the cases \(|S| = 2\) or 3, one can prove that the process visits all configurations in a well before hitting a new well in the sense of condition (H1) of [6]. In particular, in these low dimensions, one can repeat the approach presented in [8] to derive the metastable behavior of the critical zero-range process.

**Remark 3.8.** The result should hold without the assumptions that the stationary measure of the random walk \(X(t)\) is uniform and that the process is reversible. The first hypothesis should not be difficult to remove. It is a minor technical point. The second one provides some symmetry in the construction of a super-harmonic function in Section 12. In the general case, another test function has to be created.

**Remark 3.9.** On the diffusive scale \(N^2\), as in the super-critical case [5], we expect the density of particles to converge weakly to a diffusion which is absorbed at the boundary. This is an open problem which deserves to be considered.
Remark 3.10. Another interesting open problem is to derive the evolution of the condensate in the case where $S$ is the one-dimensional torus with $\kappa = \kappa_N \to \infty$ points and $X(t)$ a finite-range, symmetric random walk on $S$.

It has been proven in [4] that the condensate evolves as a Lévy process when $\alpha > 20$ and $N/\kappa_N \to \rho > \rho_c$, where $\rho_c$ is the critical density above which condensation occurs.

It is being examined in [34], with the method presented in the previous section, in the cases where $N/\kappa_N \to \rho > \rho_c$, $\alpha > 2$ and $\kappa_N \to \infty$, $N/\kappa_N \to \infty$ and $\alpha \geq 1$.

Remark 3.11. In [27], we examine the metastable behavior of critical zero-range processes starting from a configuration, instead of starting from a measure.

Remark 3.12. With a little more effort, one can prove that the finite-dimensional distributions of the process $\Psi_N(\xi_N(t))$ converge to the ones of the process $Y(t)$, applying Proposition 2.1 of [26].

4. Proof of Theorem 2.2

The assertions of Theorem 2.2 follow from Lemma 4.1 and Propositions 4.4 and 4.5 below. We first derive condition (B) of Definition 2.1.

Lemma 4.1. Condition (C1) implies condition (B) of Definition 2.1.

Proof. Fix $t > 0$ and define $e_N : \mathcal{H}_N \to \mathbb{R}$ by

$$e_N(\eta) = \mathbb{E}_N^N \left[ \int_0^t \chi_{\Delta_N}(\xi_N(s)) \, ds \right].$$

Clearly, $e_N$ is uniformly bounded by $t$. By definition of $\mu_N^\eta$,

$$\sum_{\eta \in \mathcal{E}_N^\eta} \mu_N^\eta(\eta) e_N(\eta) = \frac{1}{\mu_N(\mathcal{E}_N^\eta)} \sum_{\eta \in \mathcal{E}_N^\eta} \mu_N(\eta) e_N(\eta).$$

By Fubini’s theorem, and since $\mu_N$ is invariant, this expression is bounded by

$$\frac{1}{\mu_N(\mathcal{E}_N^\eta)} \mathbb{E}_N^{\mathcal{E}_N^\eta} \left[ \int_0^t \chi_{\Delta_N}(\xi_N(s)) \, ds \right] = \frac{t \mu_N(\Delta_N)}{\mu_N(\mathcal{E}_N^\eta)}. \tag{4.1}$$

Let $f_N(\eta) = \nu_N(\eta)/\mu_N^\eta(\eta)$ By the Cauchy-Schwarz inequality and since $e_N$ is uniformly bounded by $t$,

$$\mathbb{E}_N^{\mathcal{E}_N^\eta} \left[ \int_0^t \chi_{\Delta_N}(\xi_N(s)) \, ds \right] \leq \sqrt{t} \left( \sum_{\eta \in \mathcal{E}_N^\eta} \mu_N^\eta(\eta) f_N(\eta)^2 \right)^{1/2} \left( \sum_{\eta \in \mathcal{E}_N^\eta} \mu_N^\eta(\eta) e_N(\eta) \right)^{1/2}.$$  

By condition (C1) and (4.1), this expression is bounded from above by

$$C_1^{1/2} t \left( \frac{\mu_N(\Delta_N)}{\mu_N(\mathcal{E}_N^\eta)} \right)^{1/2},$$

where $C_1$ is the constant appearing in (2.10). By condition (C1), this expression vanishes as $N \to \infty$, which proves that (B) of Definition 2.1 holds.

The proof that the sequence of measures $Q_N^{\mathcal{E}_N^\eta}$ converges weakly to $Q_L^{\mathcal{E}_N^\eta}$ is divided in two parts: we show that the sequence is tight and that the limit point is unique.
Tightness. Denote by $\{F_t^0\}_{t \geq 0}$ the natural filtration of $D(\mathbb{R}_+, \mathcal{H}_N)$, $F_t^0 = \sigma(\xi(s) : s \in [0, t])$, and by $\{F_t\}_{t \geq 0}$ its usual augmentation. Recall from [32, Section 1.4, page 43] the definition of the $\sigma$-algebra at a stopping time. The next result is [29, Lemma 7.2 and the paragraph below].

Lemma 4.2. We have that

1. For every $s \geq 0$, the random time $S^\mathcal{E}_N(s)$, introduced in (2.2), is a stopping time with respect to the filtration $\{F_t\}_{t \geq 0}$.
2. Let $G_t^N = F_{S^\mathcal{E}_N(t)}$, $t \geq 0$, and let $\tau$ be a stopping time with respect to the filtration $\{G_t^N\}_{t \geq 0}$. Then, the random time $S^\mathcal{E}_N(\tau)$ is a stopping time with respect to the filtration $\{G_t\}_{t \geq 0}$.
3. The trace process $\{\xi_N(t)\}_{t \geq 0}$ is a $\mathcal{E}_N$-valued, continuous-time Markov chain with respect to the filtration $\{G_t^N\}_{t \geq 0}$.

Denote by $T_M$, $M > 0$, the collection of stopping times, with respect to the filtration $\{G_t^N\}_{t \geq 0}$, bounded by $M$. The proof of the next result is similar to the one of [29, Lemma 7.5]. We present it here for the sake of completeness.

Lemma 4.3. Suppose that the sequence of probability measures $(\nu_N)_{N \in \mathbb{N}}$ satisfies condition (C1). For all $M > 0$, we have

$$\lim_{a_0 \to 0} \limsup_{N \to \infty} \sup_{\tau \in \mathcal{M}} \sup_{a \in (0, a_0)} \mathbf{P}_{\nu_N}^N \left[ S^\mathcal{E}_N(\tau + a) - S^\mathcal{E}_N(\tau) \geq 2a_0 \right] = 0 .$$

Proof. We note first that

$$\{S^\mathcal{E}_N(\tau + a) - S^\mathcal{E}_N(\tau) \geq 2a_0\} \subset \left\{ \int_{S^\mathcal{E}_N(\tau)}^{S^\mathcal{E}_N(\tau) + 2a_0} \chi_{\mathcal{E}_N} (\xi_N(t)) \, dt < a \right\} .$$

Therefore, the probability appearing in the statement of the lemma is bounded by

$$\mathbf{P}_{\nu_N}^N \left[ \int_{S^\mathcal{E}_N(\tau)}^{S^\mathcal{E}_N(\tau) + 2a_0} \chi_{\Delta_N} (\xi_N(t)) \, dt > 2a_0 - a \right] .$$

This expression is less than or equal to

$$\mathbf{P}_{\nu_N}^N \left[ \int_0^{2M + 2a_0} \chi_{\Delta_N} (\xi_N(t)) \, dt > 2a_0 - a \right] + \mathbf{P}_{\nu_N}^N \left[ S^\mathcal{E}_N(\tau) > 2M \right] .$$

By the Markov inequality, the first probability is bounded from above by

$$\frac{1}{2a_0 - a} \mathbf{E}_{\nu_N}^N \left[ \int_0^{2M + 2a_0} \chi_{\Delta_N} (\xi_N(t)) \, dt \right] .$$

Thus, by Lemma 4.1,

$$\lim_{a_0 \to 0} \limsup_{N \to \infty} \sup_{\tau \in \mathcal{M}} \sup_{a \in (0, a_0)} \mathbf{P}_{\nu_N}^N \left[ \int_0^{2M + 2a_0} \chi_{\Delta_N} (\xi_N(t)) \, dt > 2a_0 - a \right] = 0 .$$

We turn to the second term in (4.2). Note that $S^\mathcal{E}_N(\tau) > 2M$ and $\tau \in \mathcal{M}$ implies that

$$\int_0^{2M} \chi_{\Delta_N} (\xi_N(t)) \, dt > M .$$

Thus, the second term in (4.2) can be handled as the previous one, which completes the proof of the lemma.

Now we prove the main result regarding the tightness.
Proposition 4.4. Under conditions (C1) and (C2), the sequence of probability measures \( \{Q^N_t\}_{N \in \mathbb{N}} \) is tight on \( D(\mathbb{R}_+, S) \). Moreover, any limit point \( Q^* \) satisfies
\[
Q^* \{ Y(0) = x_0 \} = 1 \quad \text{and} \quad Q^* \{ Y(t) \neq Y(t-) \} = 0 \quad \text{for all } t > 0 .
\]

Proof. By Aldous’ criterion, it suffices to verify that for all \( a > 0 \),
\[
\lim_{a_0 \to 0} \limsup_{N \to \infty} \sup_{\tau \in \mathcal{T}_M} \sup_{a \in (0, a_0)} P^{N}_\nu [ Y(\tau + a) \neq Y(\tau) ] = 0 .
\]

By Lemma 4.3, it is enough to show that
\[
\limsup_{a_0 \to 0} \lim_{N \to \infty} \sup_{\tau \in \mathcal{T}_M} \sup_{a \in (0, a_0)} P^{N}_\nu [ Y(\tau + a) \neq Y(\tau), S^{E_N}(\tau + a) - S^{E_N}(\tau) < 2a_0 ] = 0 .
\]
The last probability is bounded from above by
\[
P^{N}_\nu \left[ \Psi(\xi(S^{E_N}(\tau) + t)) \neq \Psi(\xi(S^{E_N}(\tau))) \right] \text{ for some } t \in (0, 2a_0) .
\]

By part (2) of Lemma 4.2 and the strong Markov property, this expression is less than or equal to
\[
\sup_{\eta \in E_N} P^{N}_\nu [ \Psi(\xi(t)) \neq \Psi(\eta) \text{ for some } t \in (0, 2a_0) ] .
\]
This expression is bounded from above by
\[
\max_{y \in S} \sup_{\eta \in E_N} P^{N}_\eta \left[ \tau^{E_N}_\nu < 2a_0 \right] .
\]
To complete the proof of the first assertion of the proposition, it remains to recall the content of condition (C2).

For the second assertion, note that \( Q^* \{ Y(0) = x_0 \} = 1 \) follows from the fact that \( \nu_N \) is concentrated on \( E^{\nu_N}_0 \). For the last claim of the proposition, it suffices to prove that
\[
\lim_{a_0 \to 0} \limsup_{N \to \infty} P^{N}_\nu [ Y(t - a) \neq Y(t) \text{ for some } a \in (0, a_0)] = 0 .
\]
The proof of this estimate is identical to the one of the first claim. \( \square \)

Uniqueness. The uniqueness part relies on the uniqueness of solutions of martingale problems. We start with two estimates. The proof of [27, Lemma 4.3] and condition (B) of Definition 2.1 yield that for all \( t > 0 \), the random time \( S^{E_N}(t) \) is close to \( t \) in the sense that, for all \( \lambda > 0 \),
\[
\lim_{N \to \infty} E^{N}_\nu \left[ e^{-\lambda t} - e^{-\lambda S^{E_N}(t)} \right] = 0 ,
\]
and
\[
\lim_{N \to \infty} E^{N}_\nu \left[ \int_0^t \left\{ e^{-\lambda r} - e^{-\lambda S^{E_N}(r)} \right\} dr \right] = 0 .
\]

Proposition 4.5. Assume that conditions (C1) – (C3) are in force. Let \( Q^* \) be a limit point of the sequence \( Q^N_t \) which satisfies (4.3). Then, \( Q^* = Q^L_z \).

Proof. Fix \( \lambda > 0 \), a function \( f : S \to \mathbb{R} \), and let \( g = (\lambda - L)f \). Recall from (2.11) the definition of \( G_N \), and let \( F_N \) be the solution of (2.12).

Under the measure \( P^{N}_\eta \), the process \( M_N(t) \) given by
\[
M_N(t) = e^{-\lambda t} F_N(\xi_N(t)) - F_N(\xi_N(0)) + \int_0^t e^{-\lambda r} \left[ (\lambda - L_N) F_N \right](\xi_N(r)) dr
\]
is a martingale with respect to the filtration \( \{ \mathcal{F}_t \}_{t \geq 0} \) defined in Lemma 4.2 above. By (2.12), we may replace \( (\lambda - \mathcal{L}_N) F_N \) by \( G_N \). Thus, since \( G_N \) vanishes on \( \Delta_N \), we can rewrite \( M_N(t) \) as

\[
M_N(t) = e^{-\lambda t} F_N(\xi_N(t)) - F_N(\xi_N(0)) + \int_0^t e^{-\lambda r} G_N(\xi_N(r)) \chi \mathcal{E}_N(\xi_N(r)) \, dr.
\]

Recall from Lemma 4.2 the definition of the filtration \( \{ \mathcal{G}_t^N \}_{t \geq 0} \). Since \( S^{\mathcal{E}_N}(t) \) is a stopping time with respect to \( \mathcal{F}_t \), the process \( \hat{M}_N(t) = M(S^{\mathcal{E}_N}(t)) \) is a martingale with respect to the filtration \( \{ \mathcal{G}_t^N \}_{t \geq 0} \):

\[
\hat{M}_N(t) = e^{-\lambda S^{\mathcal{E}_N}(t)} F_N(\xi_N^{\mathcal{E}_N}(t)) - F_N(\xi_N^{\mathcal{E}_N}(0)) + \int_0^{S^{\mathcal{E}_N}(t)} e^{-\lambda r} G_N(\xi_N(r)) \chi \mathcal{E}_N(\xi_N(r)) \, dr.
\]

The presence of the indicator of the set \( \mathcal{E}_N \) in the integral permits to perform the change of variables \( r' = T^{\mathcal{E}_N}(r) \). Hence, as \( T^{\mathcal{E}_N}(S^{\mathcal{E}_N}(t)) = t, \)

\[
\hat{M}_N(t) = e^{-\lambda S^{\mathcal{E}_N}(t)} F_N(\xi_N^{\mathcal{E}_N}(t)) - F_N(\xi_N^{\mathcal{E}_N}(0)) + \int_0^t e^{-\lambda T^{\mathcal{E}_N}(r)} G_N(\xi_N^{\mathcal{E}_N}(r')) \, dr'
\]

Recall the definition of \( g : S \to \mathbb{R} \) introduced at the beginning of this proof. By (4.4), condition (C3), the definitions of \( G_N, Y_N(\cdot) \), and since \( F_N, G_N \) are bounded,

\[
\hat{M}_N(t) = e^{-\lambda t} f(Y_N(t)) - f(Y_N(0)) - \int_0^t e^{-\lambda r'} g(Y_N(r')) \, dr' + R_N(t),
\]

where, for all \( t \geq 0, \)

\[
\lim_{N \to \infty} E^N_{\nu_N}[R_N(t)] = 0. \tag{4.5}
\]

Fix \( 0 \leq s < t, p \geq 1, 0 \leq s_1 < s_2 < \cdots < s_p \leq s \) and a bounded measurable functions \( h : S^p \to \mathbb{R} \). Let

\[
\mathfrak{M}_f^{s,t}(Y(\cdot)) := e^{-\lambda t} f(Y(t)) - e^{-\lambda s} f(Y(s)) + \int_s^t e^{-\lambda r} [(\lambda - L)f](Y(r)) \, dr,
\]

\[
h(Y(\cdot)) := h(Y(s_1), \ldots, Y(s_p)),
\]

and let \( Q^* \) be a limit point of the sequence \( Q^N_{\nu_N} \) satisfying the hypothesis of the proposition. As \( \hat{M}_N(t) \) is a martingale, by (4.5),

\[
E_{Q^*} \left[ \mathfrak{M}_f^{s,t}(Y(\cdot)) \, \mathfrak{h}(Y(\cdot)) \right] = \lim_{N \to \infty} E^N_{\nu_N} \left[ \mathfrak{M}_f^{s,t}(Y_N(\cdot)) \, \mathfrak{h}(Y_N(\cdot)) \right] = 0.
\]

To complete the proof, it remains to appeal to the uniqueness of solutions of martingale problems in finite state spaces.

\[\square\]

5. The conditions (C2) and (C3)

In this section, we present mixing properties of the Markov chain \( \xi_N(\cdot) \) which entail conditions (C2) and (C3). We start with the former.

Condition (C2) might be easier to prove if the process starts from the bottom of the well. With this idea in mind, we first prove in Lemma 5.1 that condition (C2) is fulfilled if it holds for a subset \( \mathcal{D}_N^x \) of \( \mathcal{E}_N^x \), to be interpreted as the bottom of the well, and if the set \( \mathcal{D}_N^x \) is attained before the process hits a new well (the set \( \mathcal{E}_N^x \)). Then, in Lemma 5.3, we propose a general strategy, based on the construction of
a super-harmonic function, to show that the bottom of the well is attained before the process hits a new well.

**Lemma 5.1.** Suppose that for all \( x \in S \), there exists a set \( D_N^x \subset \mathcal{E}_N \) such that

\[
\lim_{N \to \infty} \sup_{\eta \in \mathcal{E}_N} P_{\eta}^N [ \tau_{E_N^x} < \tau_{D_N^x} ] = 0, \tag{5.1}
\]

and

\[
\lim_{a \to 0} \sup_{\eta} \lim_{N \to \infty} \sup_{\eta \in \mathcal{E}_N} P_{\eta}^N [ \tau_{E_N^x} < a ] = 0. \tag{5.2}
\]

Then, condition (C2) holds.

**Proof.** Fix \( c > 0 \) and \( \eta \in \mathcal{E}_N \). Clearly,

\[
P_{\eta}^N [ \tau_{E_N^x} < a ] \leq P_{\eta}^N [ \tau_{E_N^x} < a, \tau_{D_N^x} < \tau_{E_N^x} ] + P_{\eta}^N [ \tau_{D_N^x} > \tau_{E_N^x} ].
\]

By the strong Markov property, this expression is bounded by

\[
\sup_{\xi \in \mathcal{D}_N^x} P_{\eta}^N [ \tau_{E_N^x} < a ] + \sup_{\eta' \in \mathcal{E}_N} P_{\eta'}^N [ \tau_{\eta'} > \tau_{E_N^x} ].
\]

The assertion of the lemma follows from this bound and the hypotheses. \( \square \)

**Remark 5.2.** In many models, including super-critical zero-range processes, the condition (5.1) is proved by verifying condition (H1) of [6], reducing the argument to an estimate of capacities. However, as we have seen in (2.8), condition (H1) implies that the process visits all points of a well before it hits a new one, a property which is not observed in many models, including critical zero-range processes, because the wells are too large.

Lemma 5.3 below presents a new method to derive (5.1), based on the construction of a super-harmonic function.

Let \( \mathcal{W}_N^x, x \in S \), be a subset of \( \mathcal{H}_N \) such that \( \mathcal{E}_N^x \subset \mathcal{W}_N^x \subset (\mathcal{E}_N^x)^c \), and recall that \( \mathcal{D}_N^x \subset \mathcal{E}_N^x \). Let \( \partial^- \mathcal{D}_N^x, \partial^+ \mathcal{W}_N^x \) be the inner and outer boundaries of \( \mathcal{D}_N^x, \mathcal{W}_N^x \), respectively:

\[
\partial^- \mathcal{D}_N^x := \{ \xi \in \mathcal{D}_N^x : \exists \eta \notin \mathcal{D}_N^x \cap \mathcal{E}_N^x, R_N(\eta, \xi) > 0 \}, \tag{5.3}
\]

\[
\partial^+ \mathcal{W}_N^x := \{ \xi \in \mathcal{W}_N^x : \exists \eta \in \mathcal{E}_N^x \cap \mathcal{W}_N^x, R_N(\eta, \xi) > 0 \}.
\]

Recall that a function \( F : \mathcal{H}_N \to \mathbb{R} \) is said to be super-harmonic on \( \mathcal{A} \subset \mathcal{H}_N \) if \( \mathcal{L}_N F(\eta) \leq 0 \) for all \( \eta \in \mathcal{A} \).

**Lemma 5.3.** Suppose that for each \( x \in S \) there exists a positive function \( G_N^x : \mathcal{H}_N \to \mathbb{R}_+ \) which is super-harmonic on \( \mathcal{W}_N^x \setminus \mathcal{D}_N^x \) and satisfies

\[
\lim_{N \to \infty} \sup_{\eta \in \mathcal{E}_N^x \setminus \mathcal{D}_N^x} \frac{G_N^x(\eta) - m_N(x)}{M_N(x) - m_N(x)} = 0, \tag{5.4}
\]

where

\[
m_N(x) = \min_{\eta \in \partial^- \mathcal{D}_N^x} G_N(\eta) \quad \text{and} \quad M_N(x) = \min_{\eta \in \partial^+ \mathcal{W}_N^x} G_N(\eta).
\]

Then, (5.1) is in force for all \( x \in S \).

**Proof.** Fix \( x \in S \) and \( \eta \in \mathcal{E}_N^x \setminus \mathcal{D}_N^x \). Let \( \tau = \tau_{(\mathcal{W}_N^x \setminus \mathcal{D}_N^x)^c} \). For every \( t > 0 \),

\[
E_{\eta}^x \left[ G_N^x(\xi_N(\tau \wedge t)) - G_N^x(\eta) - \int_0^{\tau \wedge t} (\mathcal{L}_N G_N^x)(\xi_N(s)) \, ds \right] = 0.
\]
Hence, since $G_N^x$ is super-harmonic on $\mathcal{W}_N \setminus \mathcal{D}_N^x$,
\[
E^N_\eta \left[ G_N^x(\xi_N(\tau \land t)) \right] \leq G_N^x(\eta),
\]
Letting $t \to \infty$ and since the hitting time $\tau$ is finite almost surely, by Fatou’s lemma,
\[
E^N_\eta \left[ G_N^x(\xi_N(\tau)) \right] \leq G_N^x(\eta).
\] (5.5)
To obtain a lower bound for the last expectation, let us write
\[
p_N(\eta) = P^N_\eta \left[ \tau(\mathcal{W}_N)^c < \tau_{\mathcal{D}_N^x} \right], \quad \eta \in \mathcal{E}_N^x \setminus \mathcal{D}_N^x.
\]
Then, by definitions of $m_N(x)$ and $M_N(x)$, we have
\[
E^N_\eta \left[ G_N^x(\xi(\tau)) \right] \geq p_N(\eta) M_N(x) + [1 - p_N(\eta)] m_N(x).
\]
Inserting this bound to (5.5) yields that
\[
p_N(\eta) \leq \frac{G_N^x(\eta) - m_N(x)}{M_N(x) - m_N(x)}, \quad \eta \in \mathcal{E}_N^x \setminus \mathcal{D}_N^x.
\]
Hence, by the hypothesis of lemma,
\[
\lim_{N \to \infty} \sup_{\eta \in \mathcal{E}_N^x \setminus \mathcal{D}_N^x} p_N(\eta) = 0.
\]
Since (5.1) holds trivially for $\eta \in \mathcal{D}_N^x$, and since $\hat{\mathcal{E}}_N^x \subset (\mathcal{W}_N)^c$, the lemma is proved. \qed

We turn to condition (C3). Denote by $\mu_N^{\mathcal{E}_N}$ the measure $\mu_N$ conditioned on $\mathcal{E}_N$:
\[
\mu_N^{\mathcal{E}_N}(\eta) = \frac{\mu_N(\eta)}{\mu_N(\mathcal{E}_N)}, \quad \eta \in \mathcal{E}_N.
\] (5.6)
Let $\{\mu_N : N \geq 1\}$ be a sequence of probability measures on $\mathcal{H}_N$ satisfying condition (C1). Then,
\[
E^N_{\mu_N^{\mathcal{E}_N}} \left[ \left( \frac{d\mu_N}{d\mu_N^{\mathcal{E}_N}} \right)^2 \right] = \frac{\mu_N(\mathcal{E}_N)}{\mu_N(\mathcal{E}_N^{\mathcal{E}_N})} E^N_{\mu_N^{\mathcal{E}_N}} \left[ \left( \frac{d\mu_N}{d\mu_N^{\mathcal{E}_N}} \right)^2 \right] \leq C_1 \frac{\mu_N(\mathcal{E}_N)}{\mu_N(\mathcal{E}_N^{\mathcal{E}_N})},
\] (5.7)
where $C_1$ is the constant appearing in condition (C1).

Fix $\lambda > 0$ and $f : S \to \mathbb{R}$. Denote by $F_N$ the solution of the resolvent equation (2.12), and recall the definition of the Dirichlet form introduced in (2.5). The first result provides elementary estimates of $F_N$.

**Lemma 5.4.** There exists a finite constant $C_0 = C_0(f)$ such that
\[
\max_{\eta \in \mathcal{H}_N} |F_N(\eta)| \leq C_0 \lambda^{-1}, \quad \mathbb{D}_N(F_N) \leq C_0(1 + \lambda^{-1})
\]
for all $N \geq 1$.

**Proof.** The first bound is obtained by recalling the stochastic representation of the solution of the resolvent equation (see equation (4.1) in [27]). We turn to the second. Multiply both sides of (2.12) by $\mu_N(\eta) F_N(\eta)$ and then sum over $\eta \in \mathcal{H}_N$ to get
\[
\lambda E^N_{\mu_N} [F_N^2] + \mathbb{D}_N(F_N) = \langle F_N, G_N \rangle_{\mu_N}.
\]
Since $|G_N|$ is uniformly bounded by $C_0(1 + \lambda)$ for some constant $C_0 = C_0(f)$, by the first estimate of the lemma, the right-hand side is less than or equal to $C_0(1 + \lambda^{-1})$, as claimed. \qed
Let \( f_N : S \to \mathbb{R} \) be the average of \( F_N \) on the well \( \mathcal{E}_N^x \) with respect to the invariant measure \( \mu_N \) conditioned to \( \mathcal{E}_N^x \):

\[
f_N(x) = E_{\mu_N}^N [ F_N ] = \frac{1}{\mu_N(\mathcal{E}_N^x)} \sum_{\eta \in \mathcal{E}_N^x} F_N(\eta) \mu_N(\eta) . \tag{5.8}
\]

Condition (C3) with \( f \) replaced by \( f_N \) asserts that the solution of the resolvent equation is close to its average in the \( L^1 \)-sense. The next result states that this weaker form of condition (C3) follows from a bound on the variance of \( F_N \) in each well.

Define the conditional variance on \( \mathcal{E}_N^x \) of a function \( F : \mathcal{H}_N \to \mathbb{R} \) as

\[\text{Var}_{\mu_N}(F) = E_{\mu_N}^N \Big[ (F - E_{\mu_N}^N(F))^2 \Big]. \tag{5.9}\]

**Proposition 5.5.** Assume that condition (C1) holds, and that

\[
\lim_{N \to \infty} \sum_{x \in S} \frac{\mu_N(\mathcal{E}_N^x)}{\mu_N(\mathcal{E}_N^{N,0})} \text{Var}_{\mu_N}(F_N) = 0 . \tag{5.10}
\]

Then, for all \( t \geq 0 \),

\[
\lim_{N \to \infty} E_{\nu_N}^N \left[ \left| F_N(\xi_N^x(t)) - f_N(Y_N(t)) \right| \right] = 0 . \tag{5.11}
\]

**Proof.** Define \( \overline{F}_N : \mathcal{E}_N \to \mathbb{R} \) as

\[\overline{F}_N(\eta) = \sum_{x \in S} f_N(x) \chi_{\mathcal{E}_N^x}(\eta) . \]

Note that \( f_N(Y_N(t)) = \overline{F}_N(\xi_N^x(t)) \).

Recall the definition of the measure \( \mu_N^{\xi_N^x} \) introduced in (5.6). Let \( \nu_N(t) \), \( t \geq 0 \), be the distribution of \( \xi_N^x(t) \) on \( \mathcal{E}_N \) when the process \( \xi_N(\cdot) \) starts from \( \nu_N \). With these notations, we can write

\[E_{\nu_N}^N \left[ \left| F_N(\xi_N^x(t)) - f_N(Y_N(t)) \right| \right] = E_{\nu_N(t)}^N \left[ \left| F_N(\eta) - \overline{F}_N(\eta) \right| \right] . \]

By the Cauchy-Schwarz inequality, the square of the right-hand side is bounded above by

\[
E_{\nu_N}^N \left[ \left| F_N - \overline{F}_N \right|^2 \right] E_{\nu_N}^N \left[ \left( \frac{d\nu_N(t)}{d\mu_N^{\xi_N^x}} \right)^2 \right] . \tag{5.12}
\]

By [7, Proposition 6.3], \( \mu_N^{\xi_N^x}(\cdot) = \mu_N(\cdot|\mathcal{E}_N) \) is the invariant measure for the trace process \( \eta_N^{\xi_N^x}(\cdot) \). Let \( h_N^x = d\nu_N(t)/d\mu_N^{\xi_N^x} \). Since the process is reversible, by the first displayed equation in [21, Section 5.2], we have \( \partial_t h_N^x = \mathcal{L}_N^{\xi_N^x} h_N^x \), where \( \mathcal{L}_N^{\xi_N^x} \) stands for the generator of the trace process on the set \( \mathcal{E}_N \). In particular,

\[
\frac{d}{dt} E_{\mu_N}^N \left[ (h_N^x)^2 \right] = 2 E_{\mu_N}^N \left[ h_N^x \mathcal{L}_N^{\xi_N^x} h_N^x \right] \leq 0 .
\]

Therefore, by (5.7),

\[
E_{\mu_N}^N \left[ \left( \frac{d\nu_N(t)}{d\mu_N^{\xi_N^x}} \right)^2 \right] \leq E_{\mu_N}^N \left[ \left( \frac{d\nu_N^{\xi_N^x}}{d\mu_N^{\xi_N^x}} \right)^2 \right] \leq C_1 \frac{\mu_N(\mathcal{E}_N)}{\mu_N(\mathcal{E}_N^{N,0})} .
\]
On the other hand, by definition of $\mathcal{F}_N$, and since $f_N(x)$, introduced in (5.8), is the mean of $F_N$ with respect to the measure $\mu_N$, the first expectation in (5.12) can be written as
\[
\sum_{x \in S} \mu_N(\mathcal{E}_N^x) E_{\mu_N}^N \left[ |F_N - \mathcal{F}_N|^2 \right] = \sum_{x \in S} \mu_N(\mathcal{E}_N^x) \text{Var}_{\mu_N^x}(F_N) .
\]
In the last term, $F_N$ is regarded as a function defined only on $\mathcal{E}_N^x$. The assertion of the proposition follows from the previous estimates.  \[\Box\]

The next result is an elementary consequence of the previous proposition. It provides a sufficient condition for (5.10) to hold, stated in terms of a local ergodic property of the dynamics. More precisely, it asserts that (5.10) is fulfilled provided the dynamics restricted to each well has a spectral gap of an order $\epsilon_N$, where $\epsilon_N/\mu_N(\mathcal{E}_N^x) \to 0$.

**Corollary 5.6.** Assume that there exist a finite constant $C_0$ and a sequence $\{\epsilon_N : N \geq 1\}$ such that $\epsilon_N/\mu_N(\mathcal{E}_N^x) \to 0$ and
\[
\mu_N(\mathcal{E}_N^x) \text{Var}_{\mu_N^x}(F) \leq C_0 \epsilon_N \mathbb{D}_N(F) \quad (5.13)
\]
for all $x \in S$ and $F : \mathcal{H}_N \to \mathbb{R}$. Then, (5.10) holds. In particular, if condition (C1) and (5.13) hold, then (5.11) is in force for all $t \geq 0$.

**Proof.** The assertions are simple consequences of Proposition 5.5 and Lemma 5.4.  \[\Box\]

To derive condition (C3), it remains to replace $f_N$ by $f$ in (5.11). We state this observation in the next result.

**Corollary 5.7.** Assume that the hypotheses of Proposition 5.5 are fulfilled and that
\[
\lim_{N \to \infty} \max_{x \in S} |f_N(x) - f(x)| = 0 . \quad (5.14)
\]
Then, condition (C3) is in force.

Here is an approach to derive (5.14) in the reversible case. Consider the bilinear form in $L^2(\mu_N)$ given by
\[
\mathbb{D}_N(F, G) = \langle -\mathcal{L}_N F, G \rangle_{\mu_N} . \quad (5.15)
\]
Fix a function $g : S \to \mathbb{R}$. We construct a test function, denoted by $V^g : \mathcal{H}_N \to \mathbb{R}$, which takes the value $g(x)$ in $\mathcal{E}_N^x$ for each $x \in S$ and such that
\[
\mathbb{D}_N(V^g, F_N) = D(g, f_N) + o_N(1) , \quad (5.16)
\]
where $D$ is the bilinear Dirichlet form associated to the generator $L$. We expect this identity to hold because $V^g, F_N$ are close to $g, f_N$ on each well and the measure $\mu_N$ is concentrated on the union of these wells.

On the other hand, since $F_N$ is the solution of the resolvent equation, $-\mathcal{L}_N F_N = (\lambda - \mathcal{L}_N) F_N = \lambda F_N = G_N - \lambda F_N$, we get
\[
\mathbb{D}_N(V^g, F_N) = \sum_{x \in S} g(x) \langle (\lambda - L)f(x) \rangle_{\mu_N(\mathcal{E}_N^x)} - \lambda \sum_{x \in S} g(x) f_N(x) \mu_N(\mathcal{E}_N^x) + o_N(1)
\]
\[= \lambda \sum_{x \in S} g(x) (f - f_N)(x) \mu_N(\mathcal{E}_N^x) + D(g, f) + o_N(1) . \quad (5.17)
\]
By combining the two previous estimates, we obtain that
\[ D(g, f - f_N) + \lambda \sum_{x \in S} g(x) (f - f_N)(x) \mu_N(\mathcal{E}_N^x) = o_N(1). \]
Choosing \( g = f - f_N \) yields that \( D(f - f_N, f - f_N) \) is asymptotically small, from what we conclude that \( \|f - f_N\|_{\infty} = o_N(1) \).

In Section 10, we apply this strategy to derive (5.14) for critical zero-range processes. This approach first appeared in [33] in the context of metastable diffusions. It has been extended to non-reversible models in [30].

6. Outline of the proof of Theorem 3.5

From this point up to the end of the article, we consider the critical zero-range process \( \xi_N(\cdot) \) whose generator, denoted by \( \mathcal{L}_N \), is defined right after (3.5) with \( \theta_N = N^2 \log N \) and \( \alpha = 1 \).

In this section, we highlight the main steps of the proof of the metastable behavior of \( \xi_N(\cdot) \). Recall the notation introduced in Section 3. In view of Theorem 2.2, to prove Theorem 3.5 we have to show that conditions (C1) – (C3) are fulfilled. In Section 7, we prove Theorem 3.1, which together with the hypothesis of Theorem 3.5 on the initial state, entails Condition (C1).

We turn to condition (C2). Denote by \( \mathcal{D}_N^x \), \( x \in S \), the deep wells given by
\[ \mathcal{D}_N^x = \{ \eta \in \mathcal{H}_N : \eta_x \geq N - N^{\gamma} \}, \]
where \( 0 < \gamma < 2/\kappa \), and by \( \mathcal{W}_N^x \) the shallow wells given by
\[ \mathcal{W}_N^x = \{ \eta \in \mathcal{H}_N : \eta_x \geq N - \frac{N}{(\log N)^{\beta}} \}, \]
where \( 0 < \beta < 1 \). Clearly,
\[ \mathcal{D}_N^x \subset \mathcal{E}_N^x \subset \mathcal{W}_N^x \subset \mathcal{\hat{E}}_N^x. \]

The proof of condition (C2), presented in Section 8, is based on Lemmata 5.1, 5.3 and carried out in two steps. First, in Proposition 8.3, we prove a weaker version of condition (C2), replacing the initial distribution, concentrated on a configuration, by the stationary state conditioned on the deeper well \( \mathcal{D}_N^x \).

Then, in Proposition 8.6, we show that, starting from a configuration in \( \mathcal{E}_N^x \), the process hits any configuration \( \zeta \) in \( \mathcal{D}_N^x \) before reaching another well (that is, the set \( \mathcal{\hat{E}}_N^x \)). This is a stronger version of condition (5.1) and is based on the construction of a super-harmonic function, as indicated in Lemma 5.3. At the end of Section 8, we show that condition (C2) follows from Propositions 8.3 and 8.6. The super-harmonic function is constructed in Section 12.

We turn to condition (C3). By Theorem 3.3 and Corollary 5.6, it is enough to prove that (5.13) and (5.14) are fulfilled for some sequence \( \epsilon_N \) which vanishes in the limit.

In Section 9, we prove Theorem 6.1 below, which provides the estimate (5.13), while (5.14) is the content of Proposition 10.1.

**Theorem 6.1.** There exists a finite constant \( C_0 \) such that
\[ \text{Var}_{\mu_N} (F) \leq \frac{C_0}{(\log N)^3} D_N (F) \]
for all \( x \in S \) and \( F : \mathcal{H}_N \to \mathbb{R} \).
7. Condensation of Critical Zero-range Process

We prove in this section Theorem 3.3. From now on, $\alpha = 1$ and $\ell_N$ is the sequence introduced in (3.9).

Rewrite the invariant measure $\mu_N$, introduced in (3.3), as

$$
\mu_N(\eta) := \frac{1}{Z_{N,\kappa}} \frac{N}{(\log N)^{\kappa-1}} \frac{1}{a(\eta)}, \quad \eta \in H_N,
$$

where the normalizing constant $Z_{N,\kappa}$ is given by

$$
Z_{N,\kappa} := \frac{N}{(\log N)^{\kappa-1}} \sum_{\eta \in H_N} \frac{1}{a(\eta)}.
$$

Sometimes we denote $Z_{N,|S_0|}$ as $Z_{N,S_0}$ to stress on which set the sum is carried out.

**Proposition 7.1.** We have that

$$
\lim_{N \to \infty} Z_{N,\kappa} = \kappa.
$$

**Proof.** The proof is carried out by induction in $\kappa$. For $\kappa = 2$, since $N/|j(N-j)| = \frac{j-1}{j} - \frac{N-j}{N}$,

$$
Z_{N,2} = \frac{N}{(\log N)^2} \left( \frac{2}{N} + \sum_{j=1}^{N-1} \frac{1}{j} \right) = \frac{2}{\log N} + \frac{2}{\log N} \sum_{j=1}^{N-1} \frac{1}{j}.
$$

The assertion of the proposition follows.

Assume that $\lim_{N \to \infty} Z_{N,\kappa} = \kappa$, and write $Z_{N,\kappa+1}$ as

$$
Z_{N,\kappa+1} = \frac{N}{(\log N)^\kappa} \left\{ \frac{1}{N} + \sum_{j=0}^{N-1} \frac{\log(N-j)}{(N-j) a(j)} Z_{N-j,\kappa} \right\}.
$$

The first term inside braces is negligible, as well as the term $j = 0$. We divide the remaining ones in four pieces.

Recall from (3.9) the definition of the sequence $\ell_N$. By the induction hypothesis and the fact that $N-j \simeq N$ for $j \leq \ell_N$,

$$
\lim_{N \to \infty} \frac{N}{(\log N)^\kappa} \sum_{j=1}^{\ell_N} \frac{\log(N-j)}{j (N-j)} Z_{N-j,\kappa} = \kappa \lim_{N \to \infty} \frac{1}{\log N} \sum_{j=1}^{\ell_N} \frac{1}{j} = \kappa.
$$

As $(Z_{N,\kappa})_{N \geq 1}$ is a bounded sequence (because, by the induction hypothesis, it converges), there exists a finite constant $C_0$, independent of $N$ and whose value may change from line to line, such that

$$
\frac{N}{(\log N)^{\kappa}} \sum_{j=\ell_N+1}^{N/2} \frac{\log(N-j)}{j (N-j)} Z_{N-j,\kappa} \leq \frac{C_0}{\log N} \sum_{j=\ell_N+1}^{N/2} \frac{1}{j}.
$$

Since $\log \ell_N/\log N \to 1$, and since

$$
\sum_{j=\ell_N+1}^{N/2} \frac{1}{j} = [1 + o_N(1)] \left( \log \frac{N}{2} - \log \ell_N \right),
$$

...
we have that
\[
\lim_{N \to \infty} \frac{N}{(\log N)^\kappa} \sum_{j=\ell_N+1}^{N/2} \frac{[\log(N-j)]^{\kappa-1}}{j(N-j)} Z_{N-j,\kappa} = 0.
\]

We turn to the third term. By a change of variables,
\[
\frac{N}{(\log N)^\kappa} \sum_{j=N/2+1}^{N-\ell_N} \frac{[\log(N-j)]^{\kappa-1}}{j(N-j)} Z_{N-j,\kappa} = \frac{N}{(\log N)^\kappa} \sum_{j=\ell_N+1}^{N/2-1} \frac{[\log(j)]^{\kappa-1}}{j(N-j)} Z_{j,\kappa}.
\]

This expression is bounded by
\[
\frac{C_0}{\log N} \sum_{j=\ell_N+1}^{N/2-1} \frac{1}{j}.
\]

At this point, we may proceed as for the second term to show that this expression vanishes as \(N \to \infty\).

It remains to consider the sum
\[
\frac{N}{(\log N)^\kappa} \sum_{j=N/2+1}^{N-\ell_N} \frac{[\log(N-j)]^{\kappa-1}}{j(N-j)} Z_{N-j,\kappa} = \frac{N}{(\log N)^\kappa} \sum_{j=1}^{\ell_N} \frac{[\log(j)]^{\kappa-1}}{j(N-j)} Z_{j,\kappa},
\]
where we performed a change of variables.

Let \((m_N)_{N \geq 1}\) be a sequence such that
\[
\lim_{N \to \infty} m_N = \infty \quad \text{and} \quad \lim_{N \to \infty} \frac{\log m_N}{\log N} = 0.
\]

Since the sequence \((Z_{N,\kappa})_{N \geq 1}\) is bounded,
\[
\frac{N}{(\log N)^\kappa} \sum_{j=1}^{m_N} \frac{[\log(j)]^{\kappa-1}}{j(N-j)} Z_{j,\kappa} \leq \frac{C_0}{\log N} \sum_{j=1}^{m_N} \frac{1}{j}.
\]

Thus, by the second property of the sequence \(m_N\), the left-hand side of the previous inequality converges to 0 as \(N \to \infty\).

We turn to the remaining sum. Since \(m_N \to \infty\) and \(Z_{N,\kappa} \to \kappa\), for \(m_N + 1 \leq j \leq \ell_N\), \(Z_{j,\kappa} = \kappa [1 + o_N(1)]\). Hence, as \(\ell_N/N \to 0\),
\[
\frac{N}{(\log N)^\kappa} \sum_{j=m_N+1}^{\ell_N} \frac{[\log(j)]^{\kappa-1}}{j(N-j)} Z_{j,\kappa} = [1 + o_N(1)] \frac{\kappa}{(\log N)^\kappa} \sum_{j=m_N+1}^{\ell_N} \frac{[\log(j)]^{\kappa-1}}{j}.
\]

Estimating sums by integrals yields that
\[
\sum_{j=m_N+1}^{\ell_N} \frac{[\log(j)]^{\kappa-1}}{j} = [1 + o_N(1)] \frac{(\log \ell_N)^\kappa - (\log m_N)^\kappa}{\kappa}.
\]

Thus, since \(\log \ell_N/\log N \to 1\) and \(\log m_N/\log N \to 0\)
\[
\lim_{N \to \infty} \frac{N}{(\log N)^\kappa} \sum_{j=N-\ell_N}^{N-1} \frac{[\log(N-j)]^{\kappa-1}}{j(N-j)} Z_{N-j,\kappa} = 1.
\]

The assertion of the proposition follows. \(\square\)

We turn to the
Proof of Theorem 3.3. Fix $x \in \mathcal{S}$ and write

$$
\mu_N(\mathcal{E}_N^x) = \frac{N}{Z_{N,\kappa} \left( \log N \right)^{\kappa-1}} \sum_{\eta \geq N - \ell_N} \frac{1}{a(\eta)},
$$

where the sum is performed over all configurations $\eta \in \mathcal{H}_N$ such that $\eta_x \geq N - \ell_N$.

We may rewrite this sum as

$$
\frac{N}{Z_{N,\kappa} \left( \log N \right)^{\kappa-1}} \sum_{j = N - \ell_N}^{N} \frac{1}{j} \left[ \frac{\log(N - j)}{N - j} \right]^{\kappa-2} Z_{N-j,\kappa-1}.
$$

By the last part of the proof of the previous proposition,

$$
\lim_{N \to \infty} \frac{N}{Z_{N,\kappa} \left( \log N \right)^{\kappa-1}} \sum_{j = N - \ell_N}^{N} \frac{1}{j} \left[ \frac{\log(N - j)}{N - j} \right]^{\kappa-2} Z_{N-j,\kappa-1} = 1.
$$

Therefore, by Proposition 7.1, $\lim_{N \to \infty} \mu_N(\mathcal{E}_N^x) = 1/\kappa$. □

The condition $\log \ell_N/\log N \to 1$ is crucial in the previous proofs. The next result shows that if it does not hold, the measure of the set $\mathcal{E}_N^x$ is no longer close to $1/\kappa$.

Note that the sequence $p_N = N^d$ fulfills the conditions of the next lemma. In particular, in critical zero-range processes the wells are very large. This is in sharp contrast with super-critical dynamics in which the wells are formed by configurations in which one site contains at least $N - m_N$ particles, where $m_N$ is any sequence such that $m_N \to \infty$, $m_N/N \to 0$.

Lemma 7.2. Let $(p_N)_{N \geq 1}$ be a sequence of positive integers such that

$$
\lim_{N \to \infty} p_N = \infty \quad \text{and} \quad \lim_{N \to \infty} \frac{\log p_N}{\log N} = \delta \in (0,1).
$$

Then, for all $x \in \mathcal{S}$,

$$
\lim_{N \to \infty} \mu_N(\{ \eta_x \geq N - p_N \}) = \frac{1}{\kappa} \delta^{\kappa-1}.
$$

In particular,

$$
\lim_{N \to \infty} \mu_N(\mathcal{D}_N^x) = \frac{1}{\kappa} \gamma^{\kappa-1},
$$

where $\mathcal{D}_N^x$ is the deep well introduced in (6.1).

Proof. The probability $\mu_N(\{ \eta_x \geq N - p_N \})$ can be written as

$$
\frac{1}{Z_{N,\kappa} \left( \log N \right)^{\kappa-1}} \left( \frac{1}{N} + \sum_{j = N - p_N}^{N-1} \left[ \frac{\log(N - j)}{j (N - j)} \right]^{\kappa-2} Z_{N-j,\kappa-1} \right).
$$

At this point, we repeat the steps presented at the end of the proof of Proposition 7.1. Let $m_N$ be the sequence introduced there and note that $m_N \ll p_N$ because $\log m_N/\log p_N \to 0$.

According to the proof of Proposition 7.1, in the previous displayed equation, the sum of the terms $N - m_N \leq j \leq N - 1$ is negligible, while the sum between $N - p_N$ and $N - m_N$ is equal to

$$
[1 + o_N(1)] \frac{1}{Z_{N,\kappa} \left( \log N \right)^{\kappa-1}} \frac{\kappa - 1}{\kappa - 1} \left( \frac{\log p_N}{\log N} - \frac{\log m_N}{\log N} \right)^{\kappa-1}.
$$

The result now follows from the properties of the sequences $m_N$ and $p_N$. □
8. Proof of Condition (C2)

The proof relies on two results. The first one, Proposition 8.3, provides a weaker version of Condition (C2), in which the initial condition, a configuration, is replaced by the invariant measure conditioned to the set $\mathcal{D}_N^x$. The second one, Proposition 8.6, asserts that starting from the well $\mathcal{E}_N^x$, the process visits every configuration of the deep well $\mathcal{D}_N^x$ before it hits a new well $\mathcal{E}_N^y$.

Recall from (2.7) the definition of $\hat{\mathcal{E}}_N^x$. The proof of Proposition 8.3 is based on the enlargement of the zero-range process and requires an estimate of the capacity $\operatorname{cap}_N(\mathcal{E}_N^x, \hat{\mathcal{E}}_N^x)$.

This estimate is provided in Section 8.1. In Section 8.2, we introduce the enlargement process and present a bound, in terms of capacities, for the probability that the hitting time of a set is small. This general result, stated as Proposition 8.4, can be useful in other contexts.

8.1. Upper bound of the capacity. Recall from (2.6) the definition of the capacity. The main result of this subsection reads as follows. Its proof is presented at the end of Section 10.4.

**Proposition 8.1.** There exists a finite constant $C_0$ such that for all $x \in S$

$$\operatorname{cap}_N(\mathcal{E}_N^x, \hat{\mathcal{E}}_N^x) \leq C_0.$$  

**Remark 8.2.** Although we do not provide the detailed proof here, we can compute the sharp asymptotics for the capacity and show that

$$\operatorname{cap}_N(\mathcal{E}_N^x, \hat{\mathcal{E}}_N^x) = \left[1 + o_N(1)\right] \frac{1}{k} \sum_{y \in S \setminus \{x\}} r_Z(x, y).$$

8.2. The enlarged process. Recall the definition of the sets $\mathcal{D}_N^x$, introduced in (6.1). Denote by $\pi_N^x$ the measure $\mu_N$ conditioned on $\mathcal{D}_N^x$:

$$\pi_N^x(\eta) = \frac{\mu_N(\eta)}{\mu_N(\mathcal{D}_N^x)}, \quad \eta \in \mathcal{D}_N^x. \quad (8.1)$$

The main result of this subsection reads as follows.

**Proposition 8.3.** For all $x \in S$,

$$\limsup_{a \to 0} \limsup_{N \to \infty} \frac{1}{\mu_N} \mathbb{P}_{\nu_N}^{\mathcal{E}_N^x} \left[ \tau_{\hat{\mathcal{E}}_N^y} < a \right] = 0.$$  

The proof of this proposition is based on the next result which provides a bound for the transition time in terms of the initial distribution and the capacity. This result is a modification of [9, Corollary 4.2].

**Proposition 8.4.** For every $x \in S$, probability measure $\nu_N$ concentrated on the set $\mathcal{E}_N^x$, $\gamma_N > 0$ and $N \geq 1$,

$$\left( \frac{1}{\gamma_N} \right) \frac{\mathbb{E}_{\nu_N}^{\mathcal{E}_N^x} \left[ \left( \frac{1}{\gamma_N} \right)^2 \right]}{\mathbb{E}_{\nu_N}^{\mathcal{E}_N^x} \left[ \left( \frac{\nu_N}{\mu_N} \right)^2 \right]} \leq \frac{\operatorname{cap}_N(\mathcal{E}_N^x, \hat{\mathcal{E}}_N^x)}{\mu_N(\mathcal{E}_N^x)}.$$  

**Proof of Proposition 8.3.** By the definition (8.1) of $\pi_N^x$,

$$\mathbb{E}_{\nu_N}^{\mathcal{E}_N^x} \left[ \left( \frac{\pi_N^x}{\mu_N} \right)^2 \right] = \sum_{\eta \in \mathcal{E}_N^x} \frac{\pi_N^x(\eta)^2}{\mu_N(\eta)} = \frac{\mu_N(\mathcal{E}_N^x)}{\mu_N(\mathcal{D}_N^x)}. $$
Therefore, by Proposition 8.4 with \( v_N = \pi_N^\eta \) and \( \gamma_N^{-1} = a \),
\[
\left( P_N^{\pi_N^\eta}[\tau_{\tilde{E}_N^x} \leq a] \right)^2 \leq \frac{e^2 a}{\mu_N(D_N^x)} \text{cap}_N(\tilde{E}_N^x, \tilde{E}_N^x).
\]
By Propositions 7.2 and 8.1, there exists a finite constant \( C(\gamma) \), where \( \gamma \) is the parameter appearing in the definition of the set \( D_N^x \), such that
\[
\left( P_N^{\pi_N^\eta}[\tau_{\tilde{E}_N^x} \leq a] \right)^2 \leq C(\gamma) a.
\]
This completes the proof. \( \square \)

Besides Proposition 8.4, the main ingredients of the proof were the strictly positive lower bound for \( \mu_N(D_N^x) \) and the upper bound for the capacity.

We turn to the proof of Proposition 8.4 which relies on an enlargement of the state space, introduced in [9, Section 2]. Denote by \( R^\eta_{\tilde{N}} : \tilde{E}_N \times \tilde{E}_N \to [0, \infty) \) the jump rates of the trace process \( \{\eta_N^{\tilde{E}}(t)\}_{t \geq 0} \).

Let \( \tilde{E}_N \) be a copy of \( \tilde{E}_N \), and denote by \( \eta^* \in \tilde{E}_N \) the copy of \( \eta \in \tilde{E}_N \).

**Definition 8.5** (Enlarged process). Fix \( N \geq 1 \) and \( \gamma_N > 0 \). The \( \gamma_N \)-enlarged process \( \{\eta_N^{\tilde{E}}(t)\}_{t \geq 0} \) is the continuous-time Markov process on \( \tilde{E}_N \cup \tilde{E}_N^* \) whose jump rates \( R^\eta_{\tilde{N}} : \tilde{E}_N \cup \tilde{E}_N^* \to [0, \infty) \) are given by
\[
R^\eta_{\tilde{N}}(\eta, \zeta) = \begin{cases} R^\eta_{\tilde{N}}(\eta, \zeta) & \text{if } \eta, \zeta \in \tilde{E}_N, \\ \gamma_N & \text{if } \zeta = \eta^* \text{ or } \eta = \zeta^*, \\ 0 & \text{otherwise}. \end{cases}
\]

Namely, the process \( \eta_N^{\tilde{E}}(t) \) at \( \eta^* \in \tilde{E}_N^* \) only jumps to \( \eta \) at rate \( \gamma_N \), while at \( \eta \in \tilde{E}_N \) it jumps to other points of \( \tilde{E}_N \) as in the original dynamics of the trace process, and it jumps to \( \eta^* \) at rate \( \gamma_N \).

The invariant measure for the \( \gamma_N \)-enlarged process \( \eta_N^{\tilde{E}}(t) \) is given by
\[
\mu_N^*(\eta) = \mu_N^*(\eta^*) = \frac{1}{2} \mu_N(\eta) \quad \text{for all } \eta \in \tilde{E}_N.
\]
Actually, the process \( \eta_N^{\tilde{E}}(\cdot) \) is reversible with respect to this measure.

Denote by \( \text{cap}_N(\mathcal{A}, \mathcal{B}) \) the capacity between two disjoint, nonempty subsets \( \mathcal{A}, \mathcal{B} \) of \( \tilde{E}_N \cup \tilde{E}_N^* \), defined in a same manner as (2.6).

**Proof of Proposition 8.4.** Denote by \( P^{N, \tilde{E}}_{v_N} \) the law of the trace process \( \tilde{E}_N^{\tilde{E}}(t) \) on \( \tilde{E}_N \) starting from the measure \( v_N \). In view of [9, Corollary 4.2], to prove the proposition, it is enough to show that
\[
P^{N, \tilde{E}}_{v_N}[\tau_{\tilde{E}_N^{\tilde{E}}} \leq \frac{1}{\gamma_N}] \leq P^{N, \tilde{E}}_{v_N}[\tau_{\tilde{E}_N^{\tilde{E}}} \leq \frac{1}{\gamma_N}],
\]
\[
\text{cap}_N(\tilde{E}_N^{\tilde{E}}, \tilde{E}_N^{\tilde{E}}) \leq \frac{1}{2\mu_N(\tilde{E}_N)} \text{cap}_N(\tilde{E}_N^{\tilde{E}}, \tilde{E}_N^{\tilde{E}}),
\]
where \( \tilde{E}_N^{\tilde{E}}, \tilde{E}_N^{\tilde{E}} \) represent the copies of \( \tilde{E}_N^{\tilde{E}}, \tilde{E}_N^{\tilde{E}} \), respectively.

The first estimate holds because the trace process hits the set \( \tilde{E}_N^{\tilde{E}} \) before the original process, as the later one may spend some time on \( \Delta_N \).

We turn to the second estimate. By [18, Lemma 2.2], the capacity is monotone, so that
\[
\text{cap}_N^*(\tilde{E}_N^{\tilde{E}}, \tilde{E}_N^{\tilde{E}}) \leq \text{cap}_N^*(\tilde{E}_N^{\tilde{E}} \cup \tilde{E}_N^{\tilde{E}}),
\]
where \( \tilde{E}_N^{\tilde{E}}, \tilde{E}_N^{\tilde{E}} \) represent the copies of \( \tilde{E}_N^{\tilde{E}}, \tilde{E}_N^{\tilde{E}} \), respectively.
Denote by \( \chi^*_x = \chi^*_{\mathcal{E}^*_N, \gamma} : \mathcal{E}_N \cup \mathcal{E}^*_N \to \mathbb{R} \) the indicator function of the set \( \mathcal{E}^*_N \cup \mathcal{E}^*_{\gamma_N} \). Since \( \chi^*_x \) is the equilibrium potential between the sets \( \mathcal{E}^*_N \cup \mathcal{E}^*_{\gamma} \) and \( \mathcal{E}^*_{\gamma_N} \) for the \( \gamma_N \)-enlarged process, the right-hand side of the previous displayed equation is equal to \( \mathcal{D}_N^\gamma(\chi^*_x) \), where \( \mathcal{D}_N^\gamma \) represents the Dirichlet form associated to the \( \gamma_N \)-enlarged process.

By definition of the enlarged process, in the computation of the Dirichlet form of the indicator function \( \chi^*_x \) the only terms which do not vanish are those which correspond to jumps between \( \mathcal{E}^*_N \) and \( \mathcal{E}^*_N \). Hence,

\[
\mathcal{D}_N(\chi^*_x) = \sum_{\eta \in \mathcal{E}_N, \zeta \in \mathcal{E}^*_N} \mu_N(\eta) \mathcal{R}_N^\gamma(\eta, \zeta) \left[ \chi^*_x(\zeta) - \chi^*_x(\eta) \right]^2.
\]

By definition of \( \mu_N^\gamma \), \( \mathcal{R}_N^\gamma \) and \( \chi^*_x \), this sum is equal to

\[
\mathcal{D}_N(\chi^*_x) = \frac{1}{2} \sum_{\eta \in \mathcal{E}_N, \zeta \in \mathcal{E}^*_N} \mu_N(\eta) \mathcal{R}_N^\gamma(\eta, \zeta) \left[ \chi^*_{\mathcal{E}^*_N}(\zeta) - \chi^*_{\mathcal{E}^*_N}(\eta) \right]^2.
\]

Denote by \( \mathcal{D}_{\mathcal{E}^*_N}^\gamma \) the Dirichlet form associated to the trace process. The previous sum is equal to \((1/2) \mathcal{D}_{\mathcal{E}^*_N}^\gamma(\chi^*_{\mathcal{E}^*_N})\). Since \( \chi^*_{\mathcal{E}^*_N} \) is the equilibrium potential between \( \mathcal{E}^*_N \) and \( \mathcal{E}^*_N \) for the trace process,

\[
\frac{1}{2} \mathcal{D}_{\mathcal{E}^*_N}^\gamma(\chi^*_{\mathcal{E}^*_N}) = \frac{1}{2} \text{cap}_{\mathcal{E}^*_N}^\gamma(\mathcal{E}^*_N, \mathcal{E}^*_N),
\]

where \( \text{cap}_{\mathcal{E}^*_N}^\gamma \) stands for the capacity for the trace process. By [6, Lemma 6.9],

\[
\frac{1}{2} \text{cap}_{\mathcal{E}^*_N}^\gamma(\mathcal{E}^*_N, \mathcal{E}^*_N) = \frac{1}{2 \mu(\mathcal{E}_N^\gamma)} \text{cap}_{\mathcal{E}^*_N}^\gamma(\mathcal{E}^*_N, \mathcal{E}^*_N),
\]

which completes the proof of the proposition. \( \square \)

8.3. Visiting points. The next result asserts that, starting from the well \( \mathcal{E}^*_N \), the process visits every configuration of the deep well \( \mathcal{D}^*_N \) before it hits a new well \( \mathcal{E}^*_N \).

Its proof, presented in Section 11, relies on the construction of a super-harmonic function on \( \mathcal{W}^*_N \setminus \mathcal{D}^*_N \), carried out in Section 12.

**Proposition 8.6.** For each \( x \in S \),

\[
\lim_{N \to \infty} \inf_{\zeta \in \mathcal{D}^*_N} \inf_{\eta \in \mathcal{E}^*_N} \mathcal{P}_N^\eta[\tau_x < \tau_{\mathcal{E}^*_N}] = 1.
\]

**Proof of Condition (C2).** In view of Lemma 5.1 and Proposition 8.6, it is enough to show that condition (5.2) is fulfilled. Fix \( x \in S \), \( a > 0 \) and \( \eta, \zeta \in \mathcal{D}^*_N \). Clearly,

\[
\mathcal{P}_N^\eta[\tau_{\mathcal{E}^*_N} < a] \leq \mathcal{P}_N^\eta[\tau_{\mathcal{E}^*_N} < a, \tau_x < \tau_{\mathcal{E}^*_N}] + \mathcal{P}_N^\eta[\tau_x > \tau_{\mathcal{E}^*_N}].
\]

By the strong Markov property, this expression is bounded by

\[
\mathcal{P}_N^\eta[\tau_{\mathcal{E}^*_N} < a] \leq \mathcal{P}_N^\eta[\tau_{\mathcal{E}^*_N} < a] + \sup_{\eta', \zeta' \in \mathcal{D}^*_N} \mathcal{P}_N^\eta[\tau_{\zeta'} > \tau_{\mathcal{E}^*_N}].
\]

Multiplying both sides by \( \pi_x^\eta(\zeta) \) and summing over \( \zeta \in \mathcal{D}^*_N \) yields that

\[
\mathcal{P}_N^\eta[\tau_{\mathcal{E}^*_N} < a] \leq \mathcal{P}_N^\eta[\tau_{\mathcal{E}^*_N} < a] + \sup_{\eta', \zeta' \in \mathcal{D}^*_N} \mathcal{P}_N^\eta[\tau_{\zeta'} > \tau_{\mathcal{E}^*_N}].
\]

Hence, condition (5.2) follows from Propositions 8.3 and 8.6. \( \square \)
9. Local Spectral Gap

In this section, we prove Theorem 6.1. Fix \( x_0 \in S \), and let \( S_0 = S \setminus \{ x_0 \} \).

9.1. Restricted process. For \( x \in S \), let

\[
\hat{E}_N^x = \{ \eta \in \mathcal{H}_N : \eta_y \leq \ell_N \text{ for all } y \in S \setminus \{ x \} \}.
\]

(9.1)

Thus, \( \mathcal{E}_N^x \subset \hat{E}_N^x \).

The zero-range process (without acceleration) restricted to \( \hat{E}_N^{x_0} \) is the \( \hat{E}_N^{x_0} \)-valued dynamics obtained by removing all jumps from \( \hat{E}_N^{x_0} \) to its complement.

The generator of this process, denoted by \( \mathcal{L}_N^{x_0} \), is given by

\[
(\mathcal{L}_N^{x_0}(F))(\eta) = \sum_{z, w \in S} g(\eta_z) r(z, w) [F(\sigma^{z, w} \eta) - F(\eta)] 1\{\sigma^{z, w} \eta \in \hat{E}_N^{x_0}\},
\]

for \( F : \hat{E}_N^{x_0} \to \mathbb{R} \). Denote by \( \eta_N^{x_0}(t) \) the Markov chain associated to the generator \( \mathcal{L}_N^{x_0} \).

Let

\[
\hat{\mu}_N^{x_0}(\eta) = \frac{\mu_N(\eta)}{\mu_N(\hat{E}_N^{x_0})}, \quad \eta \in \hat{E}_N^{x_0}
\]

be the probability measure obtained by conditioning the invariant measure \( \mu_N \) to the set \( \hat{E}_N^{x_0} \). As \( \mu_N \), this measure fulfills the detailed balance conditions. In particular, it is invariant.

The Dirichlet form associated to the restricted process \( \eta_N^{x_0}(t) \), denoted by \( \mathbb{D}_N^{x_0} \), is given by,

\[
\mathbb{D}_N^{x_0}(F) = \frac{1}{2} \sum_{z, w \in S} \sum_{\eta, \sigma^{z, w} \eta \in \hat{E}_N^{x_0}} \hat{\mu}_N^{x_0}(\eta) g(\eta_z) r(z, w) [F(\sigma^{z, w} \eta) - F(\eta)]^2,
\]

for \( F : \hat{E}_N^{x_0} \to \mathbb{R} \).

Denote by \( \text{Var}_{\hat{\mu}_N^{x_0}}(F) \) the variance of a function \( F : \hat{E}_N^{x_0} \to \mathbb{R} \) with respect to the measure \( \hat{\mu}_N^{x_0}(\cdot) \):

\[
\text{Var}_{\hat{\mu}_N^{x_0}}(F) = \mathbb{E}_{\hat{\mu}_N^{x_0}} \left[ (F - \mathbb{E}_{\hat{\mu}_N^{x_0}}[F])^2 \right].
\]

The next result establishes a lower bound for the spectral gap of the generator \( \mathcal{L}_N^{x_0} \).

**Theorem 9.1.** There exists a finite constant \( C_0 > 0 \) such that, for all \( F : \hat{E}_N^{x_0} \to \mathbb{R} \),

\[
\text{Var}_{\hat{\mu}_N^{x_0}}(F) \leq C_0 \ell_N^2 \mathbb{D}_N^{x_0}(F).
\]

The proof of the local spectral gap is based on an idea presented in [4, Section 4]. It consists in comparing the restricted process with a collection of independent birth-and-death dynamics whose spectral gap is of order \( \ell_N^{-2} \).

9.2. Proof of Theorem 6.1. The argument relies on the next result.

**Lemma 9.2.** We have that \( \mu_N(\hat{E}_N^{x_0}) = [1 + o_N(1)] (1/\kappa) \).

**Proof.** Let \( \hat{F}_N^{x_0} = \{ \eta \in \mathcal{H}_N : \eta_{x_0} \geq N - (\kappa - 1)\ell_N \} \) so that

\[
\hat{E}_N^{x_0} \subset \hat{E}_N^{x_0} \subset \hat{F}_N^{x_0}.
\]

By the proof of Theorem 3.3, \( \mu_N(\hat{F}_N^{x_0}) = [1 + o_N(1)] (1/\kappa) \). The assertion of the lemma follows from this observation and Theorem 3.3. \( \square \)
Proof of Theorem 6.1. Fix $F : \mathcal{H}_N \to \mathbb{R}$. Since $\mathbb{E}_N^\varphi (F) \leq \mathbb{D}_N (F) / \theta_N \mu_N (\hat{E}_N^\varphi)$ and since $\theta_N^{-3} \mathcal{E}_N = (\log N)^{-3}$, it suffices to show that there exists a finite constant $C_0$ such that

$$\text{Var}_\mu^\varphi (G) \leq C_0 \text{Var}_N^\varphi (G)$$

for all functions $G : \mathcal{H}_N \to \mathbb{R}$.

Write $\mathcal{G} = \sum_{\zeta \in \mathcal{E}_N^\varphi} \hat{\nu}_N^\varphi (\zeta) G(\zeta)$. Then,

$$\text{Var}_\mu^\varphi (G) = \min_{c \in \mathbb{R}} \frac{1}{\mu_N (\mathcal{E}_N^\varphi)} \sum_{\eta \in \mathcal{E}_N^\varphi} \mu_N (\eta) [G(\eta) - c]^2 \leq \frac{1}{\mu_N (\mathcal{E}_N^\varphi)} \sum_{\eta \in \mathcal{E}_N^\varphi} \mu_N (\eta) [G(\eta) - \mathcal{G}]^2.$$

Since $\mathcal{E}_N^\varphi \subset \hat{E}_N^\varphi$, this expression is bounded by

$$\frac{\mu_N (\hat{E}_N^\varphi)}{\mu_N (\mathcal{E}_N^\varphi)} \sum_{\eta \in \hat{E}_N^\varphi} \hat{\nu}_N^\varphi (\eta) [G(\eta) - \mathcal{G}]^2 = [1 + o_N (1)] \text{Var}_{\hat{\nu}_N^\varphi} (G),$$

where the last identity follows from Theorem 3.3 and Lemma 9.2. \hfill \Box

9.3. A birth-and-death process. Consider a birth-and-death process $\{w(t)\}_{t \geq 0}$ on $X = X_N = \{0, 1, \ldots, \ell_N\}$ with jump rates given by

$$R(i, j) = \begin{cases} 1 & \text{if } j = i + 1 \text{ and } j \leq \ell_N, \\ g(i) & \text{if } j = i - 1 \text{ and } j \geq 0, \\ 0 & \text{otherwise}. \end{cases}$$

The invariant probability measure, denoted by $\varphi(\cdot) = \varphi_N (\cdot)$, is given by

$$\varphi (k) = \frac{1}{z_N} \frac{1}{a(k)}, \quad k \in X,$$ \hfill (9.2)

where $z_N$ is the normalizing constant satisfying

$$z_N = \sum_{k=0}^{\ell_N} \frac{1}{a(k)} = [1 + o_N (1)] \log N. \quad (9.3)$$

The process is actually reversible with respect to $\varphi(\cdot)$.

Consider independent, birth-and-death processes $\zeta_x (t), x \in S_0$, each one having the same law as $w(\cdot)$. Denote by $\zeta (t)$ the continuous-time Markov chain on $X^{S_0}$ given by $\zeta (t) = (\zeta_x (t))_{x \in S \setminus \{x_0\}}$.

Here and below, elements of $X^{S_0}$ are represented by $\omega = (\omega_x)_{x \in S \setminus \{x_0\}}$. For each $x \in S_0$, let $\mathbf{d}^x \in X^{S_0}$ be the configuration consisting of only one particle at site $x$:

$$\mathbf{d}^x = \{x = y\}, \quad y \in S \setminus \{x_0\}.$$

The next assertions about the process $\zeta (t)$ are elementary. The invariant measure is the product measure $\varphi^{S_0},$ defined by

$$\varphi^{S_0} (\omega) = \prod_{x \in S_0} \varphi (\omega_x), \quad \omega \in X^{S_0}.$$

Actually, $\zeta (\cdot)$ is reversible with respect to $\varphi^{S_0}$. 
The generator of the process \( \zeta(\cdot) \), denoted by \( \mathcal{L}_N^{\text{BDP}} \), is given by
\[
(\mathcal{L}_N^{\text{BDP}} G)(\omega) = \sum_{x \in S_0} \left[ G(\omega + \delta^x) - G(\omega) \right] \mathbf{1}_{\{\omega_x + 1 \in \mathbb{X}\}} + \sum_{x \in S_0} g(\omega_x) \left[ G(\omega - \delta^x) - G(\omega) \right] \mathbf{1}_{\{\omega_x - 1 \in \mathbb{X}\}},
\]
for \( G : \mathbb{X}^{S_0} \to \mathbb{R} \), and the Dirichlet form by
\[
\mathcal{D}_N^{\text{BDP}}(G) = \frac{1}{2} \sum_{x \in S_0} \sum_{\omega \in \mathbb{X}^{S_0}} \varphi^S_0(\omega) \left[ G(\omega + \delta^x) - G(\omega) \right]^2 \mathbf{1}_{\{\omega_x + 1 \in \mathbb{X}\}}.
\]
Denote by \( \text{Var}_N^{\text{BDP}}(G) \) the variance of \( G : \mathbb{X}^{S_0} \to \mathbb{R} \): \[
\text{Var}_N^{\text{BDP}}(G) = E_{\varphi^S_0} \left[ (G - E_{\varphi^S_0}[G])^2 \right].
\]

The next result is [13, Theorem 1.2]. The lower bound is sharp. It can be shown that there exists constants \( 0 < C_1 < C_2 < \infty \) such that \( C_1 \ell_N^{-2} \leq \lambda_N^{\text{BDP}} \leq C_2 \ell_N^{-2} \), where \( \lambda_N^{\text{BDP}} \) represents the spectral gap of the generator \( \mathcal{L}_N^{\text{BDP}} \). We provide a simple proof of Proposition 9.3 based on the Efron-Stein inequality.

**Proposition 9.3.** There exists a finite constant \( C_0 \) such that
\[
\text{Var}_N^{\text{BDP}}(F) \leq C_0 \ell_N^2 \mathcal{D}_N^{\text{BDP}}(F)
\]
for all \( N \geq 1 \), \( F : \mathbb{X}^{S_0} \to \mathbb{R} \).

The next result is [11, Theorem 6, page 214] and follows from the Efron-Stein inequality [15].

**Lemma 9.4.** Let \( X_1, X_2, \ldots, X_n \) be independent random variables, and let \( f : \mathbb{R}^n \to \mathbb{R}, f_1, f_2, \ldots, f_n : \mathbb{R}^{n-1} \to \mathbb{R} \) be measurable, bounded functions. Define the random variables
\[
Z = f(X_1, X_2, \ldots, X_n),
\]
\[
Z_i = f_i(X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_n), \quad 1 \leq i \leq n.
\]
Then,
\[
\text{Var}(Z) \leq \sum_{i=1}^n E[(Z - Z_i)^2].
\]

The proof below is similar to the one of [4, Lemma 4.4].

**Proof of Proposition 9.3.** For \( \omega \in \mathbb{X}^{S_0} \) and \( x \in S_0 \), denote by \( \omega^{x,k} \) the configuration obtained from \( \omega \) by replacing \( \omega_x \) with \( k \):
\[
(\omega^{x,k})_y = \begin{cases} 
\omega_y & \text{if } y \neq x \\
k & \text{if } y = x.
\end{cases}
\]
Observe that \( G(\omega^{x,0}) \), \( x \in S_0 \), is a function of \( \omega_y, y \neq x \). Hence, by Lemma 9.4,
\[
\text{Var}_N^{\text{BDP}}(G) \leq \sum_{x \in S_0} \sum_{\omega \in \mathbb{X}^{S_0}} \varphi^S_0(\omega) \left[ G(\omega) - G(\omega^{x,0}) \right]^2.
\]
By the Cauchy-Schwarz inequality,
\[
\varphi^S_0(\omega) \left[ G(\omega) - G(\omega^{x,0}) \right]^2 \leq \varphi^S_0(\omega) \omega_x \sum_{k=0}^{\omega_x-1} \left[ G(\omega^{x,k+1}) - G(\omega^{x,k}) \right]^2.
\]

Since $k \leq a(k)$ and $\varphi^S_0(\omega) a(\omega_x) = \varphi^S_0(\omega_x, k) a(k)$, the previous expression is less than or equal to
\[
\sum_{k=0}^{\omega-x-1} \varphi^S_0(\omega_x, k) a(k) \left[ G(\omega_x, k + \delta^x) - G(\omega_x, k) \right]^2.
\]

Up to this point we proved that
\[
\text{Var}_{N}^{\text{BDP}}(G) \leq \sum_{x \in S_0} \sum_{\omega \in \mathbb{X}^{x_0}} \sum_{k=0}^{\omega-x-1} \varphi^S_0(\omega_x, k) a(k) \left[ G(\omega_x, k + \delta^x) - G(\omega_x, k) \right]^2.
\]

Changing variables $\zeta = \omega_x, k$, yields that this sum is equal to
\[
\sum_{x \in S_0} \sum_{\zeta \in \mathbb{X}^{x_0}} \varphi^S_0(\zeta) a(\zeta_x) (\ell_N - \zeta_x) \left[ G(\zeta + \delta^x) - G(\zeta) \right]^2 1\{\zeta_x + 1 \in \mathbb{X}\}.
\]
To complete the proof, it remains to observe that $a(\zeta_x) (\ell_N - \zeta_x) \leq \ell^2_N$. \hfill \Box

9.4. Proof of Theorem 9.1. For $N$ sufficiently large, $N \geq (\kappa - 1)\ell_N$, there exists a natural bijection between $\mathbb{X}^{S_0}$ and $\mathbb{E}_{N}^{\infty}$ given by
\[
\omega \in \mathbb{X}^{S_0} \leftrightarrow \tilde{\omega} = (N - |\omega|, \omega) \in \mathbb{E}_{N}^{\infty},
\]
where $(N - |\omega|, \omega) \in \mathcal{H}_N$ represents the configuration with $N - |\omega|$ particles at the site $x_0$, and $\omega_x$ particles at the site $x \in S_0$. Therefore, we can identify a function $G : \mathbb{X}^{S_0} \to \mathbb{R}$ with $\tilde{G} : \mathbb{E}_{N}^{\infty} \to \mathbb{R}$ by
\[
\tilde{G}(\tilde{\omega}) = G(\omega).
\]

The map $G \leftrightarrow \tilde{G}$ is a bijection between the space of real-valued functions on $\mathbb{X}^{S_0}$ and on $\mathbb{E}_{N}^{\infty}$.

**Proposition 9.5.** There exists a finite constant $C_0$ such that,
\[
\text{Var}_{N}^{\infty}(G) \leq C_0 \text{Var}_{N}^{\text{BDP}}(G)
\]
for all $N$ such that $N \geq (\kappa - 1)\ell_N$ and $G : \mathbb{X}^{S_0} \to \mathbb{R}$.

**Proof.** We first claim that there exists a finite constant $C_0$ such that
\[
\tilde{\mu}^{\infty}_N(\tilde{\omega}) \leq C_0 \varphi^S_0(\omega) \quad \text{for all } N \in \mathbb{N} \text{ and } \tilde{\omega} \in \mathbb{E}_{N}^{\infty}.
\]

Indeed, since $|\omega| \leq \ell_N$, by Proposition 7.1 and Lemma 9.2,
\[
\tilde{\mu}^{\infty}_N(\tilde{\omega}) = \frac{1}{\mu_N(\mathbb{E}_{N}^{\infty})} \sum_{x \in S_0} \frac{1}{\ell_N} \frac{1}{a(\omega_x)}.
\]

At this point, (9.6) follows from (9.2) and (9.3).

Fix $G : \mathbb{X}^{S_0} \to \mathbb{R}$. Since the expectation minimizes the square distance,
\[
\text{Var}_{N}^{\infty}(\tilde{G}) \leq \sum_{\tilde{\omega} \in \mathbb{E}_{N}^{\infty}} \left( \tilde{G}(\tilde{\omega}) - E_{\varphi^S_0}(\tilde{G}) \right)^2 \tilde{\mu}^{\infty}_N(\tilde{\omega}).
\]

By (9.4), (9.5), and (9.6), this expression is bounded from above by
\[
C_0 \sum_{\xi \in \mathbb{X}^{S_0}} (G(\xi) - E_{\varphi^S_0}(G))^2 \varphi^S_0(\xi) = C_0 \text{Var}_{\varphi^S_0}(G) = C_0 \text{Var}_{\text{BDP}}(G),
\]

Since $k \leq a(k)$ and $\varphi^S_0(\omega) a(\omega_x) = \varphi^S_0(\omega_x, k) a(k)$, the previous expression is less than or equal to
\[
\sum_{k=0}^{\omega-x-1} \varphi^S_0(\omega_x, k) a(k) \left[ G(\omega_x, k + \delta^x) - G(\omega_x, k) \right]^2.
\]
which completes the proof of the proposition. \hfill \square

**Proposition 9.6.** There exists a finite constant $C_0$ such that

$$D_{\text{RDP}}^N(G) \leq C_0 D_N^\infty(\tilde{G})$$

for all $N$ such that $N \geq (\kappa - 1)\ell_N$ and $G : \mathbb{X}^S \to \mathbb{R}$.

The proof of this result relies on a technical lemma. We say that two configurations $\eta, \eta'$ are neighbors if $\eta' = \sigma^{x,y} \eta$ for some $x, y \in S$ with $r(x, y) > 0$.

**Lemma 9.7.** For all $\eta \in \hat{\mathcal{E}}_N^0$ and $x \in S_0$ such that $\sigma^{x_0, x} \eta \in \hat{\mathcal{E}}_N^0$, there is a path $s(\eta, x) = (\eta^{(0)} = \eta, \eta^{(1)}, \ldots, \eta^{(m)} = \sigma^{x_0, x} \eta)$ in $\hat{\mathcal{E}}_N^0$ from $\eta$ to $\sigma^{x_0, x} \eta$ such that

1. $m \leq \kappa$
2. $\eta^{(i)}$ and $\eta^{(i+1)}$ are neighbors for all $0 \leq i < m$,
3. $\mu_N(\eta) \leq 4 \mu_N(\eta^{(i)})$ for all $0 \leq i \leq m$,
4. Each pair $(\eta^i, \eta^j)$ of neighboring configurations appears as a consecutive pair in no more than $2\kappa^4$ paths $s(\eta, x)$.

**Proof.** Fix $x \in S_0$. As the random walk is irreducible, there exists $m < \kappa$ and a sequence

$$x_0 = v_0, v_1, \ldots, v_m = x$$

such that $r(v_k, v_{k+1}) > 0$ for all $0 \leq k < m$. This sequence depends only on $x$. It is fixed and will be the same for all configurations $\eta \in \hat{\mathcal{E}}_N^0$.

Fix $\eta \in \hat{\mathcal{E}}_N^0$ such that $\sigma^{x_0, x} \eta \in \hat{\mathcal{E}}_N^0$. The natural definition of the path $s(\eta, x)$ is to set $\eta^{(k)} = \sigma^{x_0, v_k} \eta$. However, if $v_{v_k} = \ell_N$ for some $k$, this path leaves the set $\hat{\mathcal{E}}_N^0$, which is not permitted. We modify the natural path to keep it in the set $\hat{\mathcal{E}}_N^0$.

Note that $\eta_{v_m} < \ell_N$ because $v_m = x$ and $\sigma^{x_0, x} \eta \in \hat{\mathcal{E}}_N^0$. If $\eta_{v_k} < \ell_N$ for $1 \leq k < m$, the path $s(\eta, x)$ is the one above.

If this is not the case, let $p$ be the first integer such that $\eta_{v_k} = \ell_N:

$$p = \min \left\{ 1 \leq k \leq m : \eta_{v_k} = \ell_N \right\}.$$ 

Let $q \geq p$ be the last one with the property that all sites in between are occupied by $\ell_N$ particles:

$$q = \max \left\{ p \leq k \leq m : \eta_{v_j} = \ell_N, p \leq j \leq k \right\}.$$ 

Note that $q < m$ because $\eta_{v_m} < \ell_N$ and that $\eta_{v_{q+1}} < \ell_N$.

The path is constructed as follows. We first move a particle from $x_0 = v_0$ to $v_1$, then we move it from $v_1$ to $v_2$, until we reach $v_{p-1}$. At this point, we may not move it to $v_p$. To remove a particle from $v_p$, we move a particle from $v_q$ to $v_{q+1}$, then from $v_{q-1}$ to $v_q$, until we move one from $v_p$ to $v_{p+1}$. At this point we move a particle from $v_{p-1}$ to $v_p$.

Up to this point, a particle has been displaced from $x_0 = v_0$ to $v_{q+1}$. If all sites between $v_{q+2}$ and $v_m$ have less than $\ell_N$ particles, we continue to move the particle up to the end. Otherwise, we repeat the surgery to avoid leaving the set $\hat{\mathcal{E}}_N^0$. This defines the path $s(\eta, x)$.

Note that the path $s(\eta, x)$ does not visit the same configuration twice: $\eta^{(i)} \neq \eta^{(j)}$ for $i \neq j$. 
It is clear that the conditions (1) and (2) are fulfilled. By definition of the path, for each \(1 \leq k < m\), there exists \(w_1, \ldots, w_4\) [which depend on \(k\), naturally], such that \(\eta(k) = \sigma^{w_4,w_3} \eta\) or \(\eta(k) = \sigma^{w_3,w_2} \sigma^{w_2,w_1} \eta\). Since, for every \(x \neq y\),

\[
\frac{\mu_N(\eta)}{\mu_N(\eta+y,x)} = \frac{a(\eta_x+1)}{a(\eta_x)} \frac{a(\eta_y)}{a(\eta_y)} \leq 2,
\]

condition (3) is proved.

We turn to (4). Suppose that a pair \((\eta', \eta'') = \sigma^{w_4,w_3} \eta\) or \(\eta'' = \sigma^{w_3,w_2} \sigma^{w_2,w_1} \eta\) for some \(w_1, \ldots, w_4\). Hence, either \(\eta = \sigma^{w_1,w_2} \eta'\) or \(\eta = \sigma^{w_4,w_3} \sigma^{w_2,w_1} \eta''\). Therefore, there are at most \(\kappa \leq 2\kappa^3\) possible configurations \(\eta\) and \(\kappa\) possible choices for \(x\), making the total number of possible pairs \((\eta, x)\) in which neighbors \((\eta, \eta', \eta'')\) appear to be bounded by \(2\kappa^4\).

Since a pair \((\eta', \eta'')\) of neighbor configurations appears only once in a path \(\sigma(\eta, x, \eta, x, \eta, x)\), there are at most \(2\kappa^4\) different paths in which a fixed pair \((\eta', \eta'')\) may appear. This completes the proof of the lemma.

**Proof of Proposition 9.6.** Note that the bijection \(\omega \leftrightarrow \tilde{\omega}\) given in (9.4) satisfies \(\omega + \tilde{\omega} = x\). Therefore, we can write \(D_{\text{BDP}}^N(G)\) as

\[
D_{\text{BDP}}^N(G) = \frac{1}{2} \sum_{x \in S_0} \sum_{\omega \in \tilde{X}_0} \varphi_{S_0}(\omega) [G(\omega + \tilde{\omega}) - G(\omega)]^2 1 \{\omega + \tilde{\omega} \in \tilde{X}_0\}
\]

By (9.6) and since the map \(\omega \leftrightarrow \tilde{\omega}\) is bijection, it follows from the previous equation that there exists a finite constant \(C_0\), independent of \(N\), such that

\[
D_{\text{BDP}}^N(G) \leq C_0 \sum_{x \in S_0} \sum_{\eta \in \tilde{E}^0_0} \tilde{\mu}^x_0(\eta) [G(\sigma^{x_0,x} \eta) - G(\eta)]^2 1 \{\sigma^{x_0,x} \eta \in \tilde{X}_0\}. \tag{9.7}
\]

Recall from Lemma 9.7 the definition of the path \(\sigma(\eta, x) = (\eta^{(0)}, \ldots, \eta^{(m)})\) for \(\eta \in \tilde{E}^0_0\) and \(x \in S_0\) such that \(\sigma^{x_0,x} \eta \in \tilde{E}^0_0\). By the Cauchy-Schwarz inequality and conditions (1) and (3) of that lemma,

\[
\tilde{\mu}^x_0(\eta) [G(\sigma^{x_0,x} \eta) - G(\eta)]^2 \leq m \sum_{k=0}^{m-1} \tilde{\mu}^x_0(\eta) [G(\eta^{(k+1)}) - G(\eta^{(k)})]^2
\]

\[
\leq 4\kappa \sum_{k=0}^{m-1} \tilde{\mu}^x_0(\eta^{(k)}) [G(\eta^{(k+1)}) - G(\eta^{(k)})]^2.
\]

Inserting this bound in (9.7), changing the order of summations and applying part (4) of Lemma 9.7 yield that \(D_{\text{BDP}}^N(G)\) is bounded above by

\[
C_0(\kappa) \sum_{(x, y)} \sum_{\eta \in \tilde{E}^0_0} \tilde{\mu}^x_0(\eta) [G(\sigma^{x,y} \eta) - G(\eta)]^2 1 \{\sigma^{x,y} \eta \in \tilde{X}_0\},
\]

where the first sum is carried over all pairs \((x, y)\) such that \(r(x, y) > 0\). The last summation is bounded above by \(C_0(\kappa) D_{\text{BDP}}^N(\tilde{G})\) because \(g(k) \geq 1\) for all \(k \geq 1\).

**Proof of Theorem 9.1.** Fix \(F : \tilde{E}^0_0 \rightarrow \mathbb{R}\). By the bijection introduced in Subsection 9.4, there exists \(G : \tilde{S}^0_0 \rightarrow \mathbb{R}\) such that \(F = \tilde{G}\) in the sense of (9.5). By Propositions
9.3, 9.5, and 9.6, there exists a finite constant \(C_0 = C_0(\kappa)\), independent of \(N\), such that
\[
\text{Var}_{\nu_N}(\tilde{G}) \leq C_0 \text{Var}_{\nu_N}^\text{BDP}(G) \leq C_0 \ell^2 N \text{Var}_{\nu_N}^\text{BDP}(G) \leq C_0 \ell^2 \text{Var}_{\nu_N}^\text{BDP}(\tilde{G}) .
\]
This proves Theorem 9.1. \(\square\)

10. **The Resolvent Equation**

In this section, we prove (5.14) for critical zero-range processes. Fix \(\lambda > 0\) and a function \(f : S \to \mathbb{R}\). Let \(F_N : \mathcal{H}_N \to \mathbb{R}\) be the solution of the resolvent equation (2.12), and let \(F_N : S \to \mathbb{R}\) be given by (5.8).

**Proposition 10.1.** For all \(\lambda > 0\), and \(f : S \to \mathbb{R}\),
\[
\lim_{N \to \infty} \max_{x \in S} |f_N(x) - f(x)| = 0 .
\]

10.1. **Sketch of the proof.** The proof follows the strategy presented below Corollary 5.7. In the context of zero-range processes, the bilinear \(\mathbb{D}_N(F, G)\) introduced in (5.15) takes the form in \(L^2(\mu_N)\) given by
\[
\mathbb{D}_N(F, G) = \frac{\theta_N}{2} \sum_{\eta \in \mathcal{H}_N} \sum_{x, y \in S} \mu_N(\eta) g(\eta_x) r(x, y) (T_{x,y}F)(\eta) (T_{x,y}G)(\eta) , \tag{10.1}
\]
where \((T_{x,y}H)(\eta) = H(\sigma^{x,y}\eta) - H(\eta)\).

In Section 10.3, we present a partition of the set \(\mathcal{H}_N\). The idea behind this construction is that in the computation of the form \(\mathbb{D}_N(F, G)\), for Lipschitz functions \(F, G\), in the sense of Lemma 10.8, only a tiny subspace of \(\mathcal{H}_N\), formed by the wells \(\mathcal{E}_N^x\) and tubes connecting them, matters. In Section 10.4, we construct the test function \(V^\eta\) and show that it is Lipschitz. In the last sections, we prove the two estimates (5.16) and (5.17) on the bilinear form \(\mathbb{D}_N(V^\eta, F_N)\) and Proposition 10.1.

10.2. **Energy estimate.** We prove in this section a simple bounded needed in the proof of Proposition 10.1. Recall from (2.5) that we denote by \(\langle \cdot, \cdot \rangle_{\mu_N}\) the scalar product in \(L^2(\mu_N)\). With this notation, we can write the Dirichlet form as
\[
\mathbb{D}_N(F) = \langle F, -\mathcal{L}_N F \rangle_{\mu_N} .
\]

**Lemma 10.2.** There exists a finite constant \(C_0 = C_0(\kappa)\) such that
\[
\sum_{\eta \in \mathcal{H}_N} \mu_N(\eta) \left[ F(\sigma^{x,y}\eta) - F(\eta) \right]^2 \leq C_0 \theta_N^{-1} \mathbb{D}_N(F)
\]
for all \(x, y \in S\) and \(F : \mathcal{H}_N \to \mathbb{R}\).

**Proof.** Suppose first that \(r(x, y) > 0\). Then,
\[
\mu_N(\eta) \left[ F(\sigma^{x,y}\eta) - F(\eta) \right]^2 \leq \frac{1}{r(x, y)} \mu_N(\eta) g(\eta_x) r(x, y) \left[ F(\sigma^{x,y}\eta) - F(\eta) \right]^2
\]
because \(g(\eta_x) \geq 1\) if \(\eta_x \geq 1\), and both sides are 0 when \(\eta_x = 0\). Summing this over \(\eta \in \mathcal{H}_N\) yields the assertion of the lemma.

If \(r(x, y) = 0\), by the irreducibility of the Markov chain \(X\), there exists a sequence \(x = z_0, z_1, \cdots, z_m = y\) such that \(r(z_i, z_{i+1}) > 0\) for \(0 \leq i < m\). Hence, by the Cauchy-Schwarz inequality
\[
\left[ F_N(\sigma^{x,y}\eta) - F_N(\eta) \right]^2 \leq m \sum_{i=0}^{m-1} \left[ F_N(\sigma^{z_0,z_{i+1}}\eta) - F_N(\sigma^{z_0,z_i}) \right]^2 .
\]
Figure 1. An illustration of sets introduced in Section 10.3 when $S = \{x, y, z\}$. We can notice from this figure that the sets $J^{z, x}$, $J^{x, y}$, and $J^{y, z}$ are disjoint.

Applying the previous argument to each term at the right-hand side completes the proof since there exists a finite constant $C_0$ such that

$$
\mu_N(\sigma^{z, w, \eta}) \leq C_0 \mu_N(\eta)
$$

for all $z, w \in S$, $\eta \in H_N$, $N \in \mathbb{N}$.

10.3. Tubes and wells. Fix $\epsilon > 0$ small. The subsets of $H_N$ constructed in this section may depend on $N$ and $\epsilon$, even if these parameters do not appear in the notation. We also refer to Figure 1 for the illustration of the sets described in this subsection.

Define the enlarged wells $V^x$, $\hat{V}^x$, $x \in S$, by

$$
V^x = \{\eta \in H_N : \eta_x \geq N(1 - 2\epsilon)\}, \quad \hat{V}^x = \{\eta \in H_N : \eta_x \geq N(1 - 4\epsilon)\}.
$$

From now on, all the statements may hold only for large enough $N$. More precisely, there exists a constant $N_0 = N_0(\epsilon)$ which is independent of $\eta$ such that the statement holds only for $N > N_0$. With this convention, $E^x_N \subset V^x \subset \hat{V}^x$.

For $x, y \in S$, the tubes $T^{x, y}$ and $\hat{T}^{x, y}$ connecting the wells $E^x_N$ and $E^y_N$ are defined by

$$
T^{x, y} = \{\eta \in H_N : \eta_x + \eta_y \geq N - \ell_N\}, \quad \hat{T}^{x, y} = \{\eta \in H_N : \eta_x + \eta_y \geq N(1 - 3\epsilon)\}.
$$

Let

$$
J^{x, y} = T^{x, y} \setminus [V^x \cup V^y], \quad \hat{J}^{x, y} = \hat{T}^{x, y} \setminus [\hat{V}^x \cup \hat{V}^y],
$$

and note that the definitions are symmetric: $T^{x, y} = T^{y, x}$, $\hat{T}^{x, y} = \hat{T}^{y, x}$, $J^{x, y} = J^{y, x}$ and $\hat{J}^{x, y} = \hat{J}^{y, x}$.

Denote by $G$ and $\hat{G}$ the union of wells and tubes:

$$
G = \bigcup_{x, y \in S} (V^x \cup J^{x, y}), \quad \hat{G} = \bigcup_{x, y \in S} (\hat{V}^x \cup \hat{J}^{x, y}).
$$

(10.2)
Lemma 10.3. The following holds.

1. Suppose that $\eta \in \mathcal{J}^{x, y}$ for some $x, y \in S$. Then, $\eta_x, \eta_y \in (N \epsilon, N(1 - 2\epsilon))$.
2. Suppose that $\eta \in \mathcal{J}^{x, y}$ for some $x, y \in S$. Then, $\eta_x, \eta_y \in (N \epsilon, N(1 - 4\epsilon))$.
3. For $\{x, y\} \neq \{x', y'\}$, $\mathcal{J}^{x, y} \cap \mathcal{J}^{x', y'} = \emptyset$. In particular, the expression (10.2) represents a partition of $\mathcal{G}$.

Proof. For part (1), if $\eta \in \mathcal{J}^{x, y} = \mathcal{I}^{x, y} \setminus [\mathcal{V}^x \cup \mathcal{V}^y]$, the bound $\eta_x < N(1 - 2\epsilon)$ is trivial because $\eta \notin \mathcal{V}^x$. By symmetry this extends to $\eta_y$. By this bound and since $\eta \in \mathcal{J}^{x, y}$,

$$\eta_x + N(1 - 2\epsilon) > \eta_x + \eta_y > N - \epsilon N > N - \epsilon N.$$ 

This proves the lower bound. Proof of part (2) is similar.

For part (3), it suffices to show that

$$\mathcal{J}^{x, y} \cap \mathcal{J}^{x', y'} \subset \mathcal{V}$$(10.3)

to prove this, fix $\eta \in \mathcal{J}^{x, y} \cap \mathcal{J}^{x', y'}$. Since $\eta_x + \eta_y + \eta_z \leq N$,

$$2N - 2\ell N \leq (\eta_x + \eta_y) + (\eta_x + \eta_z) \leq \eta_x + N.$$

Thus, $\eta_x \geq N - 2\ell N$, which implies that $\eta \in \mathcal{V}^x$, proving (10.3). \(\square\)

In the remaining part of this subsection, we provide an estimate of the measures $\mu_N(\hat{\mathcal{G}} \setminus \mathcal{G})$ and $\mu_N(\mathcal{J}^{x, y})$. For $S_0 \subset S$ and $k \in \mathbb{N}$, let

$$\mathcal{H}_{k, S_0} = \{\xi = (\xi_x)_{x \in S_0} \in \mathbb{N}^{S_0} : |\xi| := \sum_{x \in S_0} \xi_x = k\}.$$ 

We adopt the following convention. Fix $c : \mathbb{N} \times (0, 1) \to \mathbb{R}$. We write $c(N, \epsilon) = o_N(1)$ if $\lim_{N \to \infty} c(N, \epsilon) = 0$ for all $\epsilon > 0$, and $c(N, \epsilon) = o_{\epsilon}(1)$ if $\lim_{\epsilon \to 0} \sup_{N \in \mathbb{N}} |c(N, \epsilon)| = 0$.

Mind that we always send $N \to \infty$ before $\epsilon \to 0$.

Hereafter, $C_0$ represents a finite constant independent of $N, \epsilon$ and $\eta$, and $C_{\epsilon}$ a finite one, independent of $N$ and $\eta$, but which may depend on $\epsilon$. The values of $C_0$ and $C_{\epsilon}$ may change from line to line.

Lemma 10.4. We have that

$$\mu_N(\hat{\mathcal{G}} \setminus \mathcal{G}) \leq \frac{1}{\log N} [o_N(1) + o_{\epsilon}(1)].$$

Proof. For $x \in S$, define

$$\mathcal{A}^x = \hat{\mathcal{V}}^x \setminus \mathcal{G} = \hat{\mathcal{V}}^x \setminus \left(\mathcal{V}^x \cup \bigcup_{y \in S \setminus \{x\}} \mathcal{J}^{x, y}\right).$$

With this notation,

$$\hat{\mathcal{G}} \setminus \mathcal{G} \subset \bigcup_{x \in S} \mathcal{A}^x \cup \bigcup_{x, y \in S} (\hat{\mathcal{J}}^{x, y} \setminus \mathcal{J}^{x, y}).$$

Therefore, it is enough to show that

$$\mu_N(\mathcal{A}^x) = \frac{o_N(1)}{\log N} \quad \text{and} \quad \mu_N(\hat{\mathcal{J}}^{x, y} \setminus \mathcal{J}^{x, y}) = \frac{o_{\epsilon}(1)}{\log N}$$

(10.4)

for all $x, y \in S$. 

We first consider $\mathcal{A}^x$. Since $\widehat{\mathcal{V}}^x \cap \mathcal{V}^y = \emptyset$, in the definition of $\mathcal{A}^x$, we may add $\mathcal{V}^y$ to the expression inside parenthesis. At this point, we may replace $\mathcal{J}^{x,y}$ by $\mathcal{J}^{x,y}$, and then remove $\mathcal{V}^y$ to get that

$$\mathcal{A}^x = \widehat{\mathcal{V}}^x \setminus \left( \mathcal{V}^x \cup \bigcup_{y \in S \setminus \{x\}} \mathcal{J}^{x,y} \right).$$

(10.5)

Write

$$\mathcal{A}^x = \bigcup_{m=N(1-2\epsilon)}^{N(1-2\epsilon)-1} \mathcal{A}_m^x,$$

where $\mathcal{A}_m^x = \{ \eta \in \mathcal{A}^x : \eta_x = m \}$. Represent a configuration $\eta$ as $(\eta_x, \xi)$, where $\xi \in \mathbb{N}^S \setminus \{x\}$ stands for the configuration $\eta$ on $S \setminus \{x\}$ such that $\xi_y = \eta_y$ for all $y \in S \setminus \{x\}$. Note that $\xi \in \mathcal{H}_{N-m,S \setminus \{x\}}$ if $\eta \in \mathcal{A}_m^x$. Let $\mathcal{B}_m^x$ the subset of configurations $\xi \in \mathcal{H}_{N-m,S \setminus \{x\}}$ such that $(m, \xi) \in \mathcal{A}_m^x$.

Recall the definition of the set $\Delta_{S,N}$ introduced below (3.4). We claim that $\mathcal{B}_m^x \subset \Delta_{S \setminus \{x\}, N-m}$. Indeed, fix $\xi \in \mathcal{B}_m^x$ and $y \neq x$. Let $\eta$ be the configuration $(m, \xi)$, so that $\eta \in \mathcal{A}_m^x$. By (10.5), $\eta \notin \mathcal{J}^{x,y}$. Hence, as $\eta_x = m$, $\eta_y + m = \eta_y + \eta_x < N - \ell_N$ so that $\eta_y < N - m - \ell_N \leq (N-m) - \ell_{N-m}$ because $\ell_{N-m} \leq \ell_N$.

Therefore, configurations $\xi$ in $\mathcal{B}_m^x$ have a total of $N-m$ particles and each site has strictly less than $(N-m) - \ell_{N-m}$ particles. Thus, by definition of $\Delta_{S,N}$, $\xi$ belongs to $\Delta_{S \setminus \{x\}, N-m}$, which proves the claim.

By definition of $\mu_N$ and $\mathcal{B}_m^x$,

$$\mu_N(\mathcal{A}^x) = \frac{N}{Z_{N,S}(\log N)^{n-1}} \sum_{m=N(1-4\epsilon)}^{N(1-2\epsilon)-1} \frac{1}{\alpha(m)} \sum_{\xi \in \mathcal{B}_m^x} \frac{1}{\alpha(\xi)}.$$

As $\mathcal{B}_m^x$ is contained in $\Delta_{S \setminus \{x\}, N-m}$, this expression is less than or equal to

$$\frac{N}{(\log N)^{n-1}} \sum_{m=N(1-4\epsilon)}^{N(1-2\epsilon)-1} \frac{1}{m} \frac{Z_{N-m,S}}{Z_{N,S}} \frac{\log(N-m)^{n-2}}{N-m} \mu_{N,m}(\Delta_{S \setminus \{x\}, N-m}).$$

By Proposition 7.1, for each $p \geq 1$, $(Z_{N,p})_{N \geq 1}$ is a bounded sequence. Hence, $Z_{N-m,S} \leq C_0$. As $2\epsilon N \leq N - m \leq 4\epsilon N$, $N/(N-m) \leq C_0/\epsilon$, and $\log(N-m)/\log N \leq 1$. The previous expression is thus bounded above by

$$\frac{C_0}{\log N} \frac{1}{1 - \epsilon} \frac{1}{(1 - 4\epsilon)N} \sum_{M=2\epsilon N}^{4\epsilon N} \mu_{N-1,M}(\Delta_{S \setminus \{x\}, M}).$$

By Theorem 3.3, the sequence $\mu_{N-1,M}(\Delta_{S \setminus \{x\}, M})$ vanishes as $M \to \infty$. This shows that the previous sum is bounded by $2\epsilon N_\delta(1)$. This proves the first estimate in (10.4).

We turn to the second bound of (10.4). Write $\mathcal{J}_{N-\ell_N}^{x,y} \setminus \mathcal{J}^{x,y}$ as

$$\mathcal{J}_{N-\ell_N}^{x,y} \setminus \mathcal{J}^{x,y} = \bigcup_{m=N(1-4\epsilon)}^{N-\ell_N-1} \mathcal{I}_m^{x,y},$$

(10.6)

where $\mathcal{I}_m^{x,y} = \{ \eta \in \mathcal{J}_{N-\ell_N}^{x,y} \setminus \mathcal{J}^{x,y} : \eta_x + \eta_y = m \}$. Write $\eta \in \mathcal{I}_m^{x,y}$ as $\eta = (\eta_x, \eta_y, \zeta)$, where $\zeta \in \mathcal{H}_{N-m,S \setminus \{x,y\}}$ represents the configuration of $\eta$ on $S \setminus \{x, y\}$. 

By part (2) of Lemma 10.3, $\eta_x, \eta_y > N\epsilon$ for configurations $\eta$ in $\hat{\mathcal{J}}^{x,y}$. Therefore, by Proposition 7.1, there exists a finite constant $C_0$ such that

$$\mu_N(I_m) \leq \frac{C_0 N}{(\log N)^{N_\epsilon - 1}} \sum_{i=N_\epsilon}^{m-N_\epsilon} \frac{1}{a(i) a(m-i)} \sum_{\zeta \in \mathcal{H}_{N-m, S \setminus \{x,y\}}} \frac{1}{a(\zeta)}.$$ 

An elementary computation yields that there exists a finite constant $C_0$ such that

$$\sum_{i=N_\epsilon}^{m-N_\epsilon} \frac{1}{a(i) a(m-i)} \leq \frac{C_0}{N} \log \frac{1}{\epsilon}$$

for all $(1 - 4\epsilon)N \leq m \leq N$.

On the other hand, by Proposition 7.1 and since $m \leq N - \ell_N$, there exists a constant $C_0$ such that

$$\sum_{\zeta \in \mathcal{H}_{N-m, S \setminus \{x,y\}}} \frac{1}{a(\zeta)} \leq C_0 \frac{[\log(N-m)]^{\kappa-3}}{N-m} \leq C_0 \frac{(\log N)^{\kappa-3}}{\ell_N}.$$ 

Combining the previous estimates yields that

$$\mu_N(I_m) \leq \frac{C_0}{N \log N} \log \frac{1}{\epsilon},$$

and hence by (10.6),

$$\mu_N(J_m) \leq \frac{C_0}{N \log N} \log \frac{1}{\epsilon},$$

as claimed.

\begin{lemma}
There exists a finite constant $C_0$ such that, for all $x, y \in S$,

$$\mu_N(J_{x,y}) \leq \frac{C_0}{N \log N} \log \frac{1}{\epsilon}.$$ 

\end{lemma}

\begin{proof}
The proof is similar to the one of the last part of the previous lemma. Fix $x, y \in S$ and write

$$J_{x,y} = \bigcup_{m=N-\ell_N}^{N} J_{m},$$

where $J_{m} = \{ \eta \in J_{x,y} : \eta_x + \eta_y = m \}$.

Represent a configuration $\eta$ in $J_{x,y}$ as $\eta = (\eta_x, \eta_y, \zeta)$ for $\zeta \in \mathcal{H}_{N-m, S \setminus \{x,y\}}$.

By part (1) of Lemma 10.3, $\eta_x, \eta_y > N\epsilon$ for configurations $\eta$ in $J_{x,y}$. Thus,

$$\mu_N(J_{m}) \leq \frac{C_0 N}{(\log N)^{N_\epsilon - 1}} \sum_{i=N_\epsilon}^{m-N_\epsilon} \frac{1}{a(i) a(m-i)} \sum_{\zeta \in \mathcal{H}_{N-m, S \setminus \{x,y\}}} \frac{1}{a(\zeta)}.$$ 

Clearly, there exists a finite $C_0$ such that

$$\sum_{i=N_\epsilon}^{m-N_\epsilon} \frac{1}{a(i) a(m-i)} \leq \frac{C_0}{N} \log \frac{1}{\epsilon}, \quad (10.7)$$

for all $N - \ell_N \leq m \leq N$.

By Proposition 7.1,

$$\sum_{\zeta \in \mathcal{H}_{N-m, S \setminus \{x,y\}}} \frac{1}{a(\zeta)} \leq C_0 \frac{[\log(N-m)]^{\kappa-3}}{N-m} \leq C_0 \frac{(\log N)^{\kappa-3}}{N-m}.$$
for $N - \ell_N \leq m < N$. For $m = N$, the sum is bounded by 1.

Putting together the previous estimates yields that

$$\mu_N(\mathcal{J}_m^{x,y}) \leq \frac{C_0}{(\log N)^2} \frac{1}{N-m} \log \frac{1}{\epsilon}$$

for $N - \ell_N \leq m < N$ and $\mu_N(\mathcal{J}_N^{x,y}) \leq [C_0/(\log N)^{\kappa-1}] \log(1/\epsilon)$.

Summing over $N - \ell_N \leq m \leq N$ gives that

$$\mu_N(\mathcal{J}_x^{x,y}) \leq \frac{C_0}{(\log N)^2} \log N \sum_{k=1}^{\ell_N} \frac{1}{k} + \frac{C_0}{(\log N)^{\kappa-1}} \log \frac{1}{\epsilon} \leq \frac{C_0}{\log N} \log \frac{1}{\epsilon},$$

as claimed. □

Decompose the tube $\mathcal{J}^{x,y}$ as

$$\mathcal{J}^{x,y} = \mathcal{K}^{x,y} \cup \mathcal{L}^{x,y},$$

where

$$\mathcal{K}^{x,y} = \{ \eta \in \mathcal{J}^{x,y} : \eta_x \text{ or } \eta_y < 6N\epsilon \},$$

$$\mathcal{L}^{x,y} = \{ \eta \in \mathcal{J}^{x,y} : \eta_x, \eta_y \geq 6N\epsilon \}.$$

The next lemma asserts that we can remove the factor $\log(1/\epsilon)$ in the previous lemma replacing $\mathcal{J}^{x,y}$ by $\mathcal{K}^{x,y}$.

**Lemma 10.6.** There exists a finite constant $C_0$ such that, for all $x, y \in S$,

$$\mu_N(\mathcal{K}^{x,y}) \leq \frac{C_0}{\log N}.$$

**Proof.** Assume that $\eta_x \leq 6\epsilon N$, and let $\mathcal{K}^{x,y}_m = \mathcal{K}^{x,y} \cap \mathcal{T}_m^{x,y}$, $N - \ell_N \leq m \leq N$. By (1) of Lemma 10.3, $\eta_x > \epsilon N$. Hence, $\eta_x$ varies from $\epsilon N$ to $6\epsilon N$.

Proceed as in the previous lemma. In the formula for $\mu_N(\mathcal{K}^{x,y}_m)$, let $i$ represent $\eta_x$, so that (10.7) becomes

$$\sum_{i=N \epsilon}^{6N \epsilon} \frac{1}{a(i)} \frac{1}{a(m-i)} \leq \frac{C_0}{N}.$$

The rest of the argument is identical to the one of Lemma 10.5. □

10.4. **Construction of test functions.** In this section, we introduce functions $U_{x,y} : \mathcal{H}_N \to \mathbb{R}$, $x, y \in S$, to examine the Resolvent equation (2.12). These functions are similar to the ones introduced in the super-critical case in [8] to estimate the capacities between wells.

Fix $x, y \in S$ and a small parameter $\epsilon > 0$. Let $\phi_\epsilon : [0, 1] \to [0, 1]$ be a smooth, non-decreasing, bijective function such that

$$\phi_\epsilon(t) + \phi_\epsilon(1-t) = 1, \quad t \in [0, 1],$$

$$\phi_\epsilon(t) = \begin{cases} 0 & t \in [0, 3\epsilon], \\ (t-4\epsilon)/(1-8\epsilon) & t \in [5\epsilon, 1-5\epsilon], \\ 1 & t \in [1-3\epsilon, 1]. \end{cases}$$

$$\phi'_\epsilon(t) \leq 1 + \epsilon^{1/2}, \quad t \in [0, 1].$$

Although, the existence of such a function is straightforward, we refer to [35, Section 7.3] for an explicit construction.
Define $\Phi_\varepsilon : [0, 1] \to [0, 1]$ by
\[
\Phi_\varepsilon(t) := 6 \int_0^{\phi_\varepsilon(t)} u (1 - u) \, du = 3 \phi_\varepsilon(t)^2 - 2 \phi_\varepsilon(t)^3.
\]
Note that
\[
\Phi_\varepsilon(t) = \begin{cases} 
0 & \text{if } t \in [0, 3\varepsilon] \\
1 & \text{if } t \in [1 - 3\varepsilon, 1] 
\end{cases}.
\]

Recall from (3.6) that $h_{x, y} = h_{\{x\}, \{y\}} : S \to \mathbb{R}$ denotes the equilibrium potential between $x$ and $y$ for the random walk $X(\cdot)$. Let
\[
x = z_1, z_2, \ldots, z_\kappa = y
\]
be an enumeration of $S$ satisfying
\[
1 = h_{x, y}(z_1) \geq h_{x, y}(z_2) \geq \cdots \geq h_{x, y}(z_\kappa) = 0.
\]
Define $U_{x, y} : \mathcal{H}_N \to \mathbb{R}$ by
\[
U_{x, y}(\eta) = \sum_{j=1}^{\kappa-1} [h_{x, y}(z_j) - h_{x, y}(z_{j+1})] \Phi_\varepsilon \left( \frac{1}{N} \sum_{i=1}^{j} \eta_{z_i} \right).
\]
The function $U_{x, y}$ approximates the equilibrium potential between $\mathcal{V}^x$ and $\mathcal{V}^y$ in the tube $\mathcal{J}^{x, y}$.

**Remark 10.7.** Fix $x, y \in S$, and denote by $z_1, z_2, \ldots, z_\kappa$, $z'_1, z'_2, \ldots, z'_\kappa$ the sequences (10.9) associated to the functions $U_{x, y}$ and $U_{y, x}$, respectively. We assume that $z_i = z'_{\kappa+1-i}, 1 \leq i \leq \kappa$. Clearly, this conditions holds if $h_{x, y}(z) \neq h_{x, y}(w)$ for all $z \neq w \in S$.

Denote by $\|u\|_\infty$ the sup-norm of a function $u : S \to \mathbb{R}$, $\|u\|_\infty = \max_{x \in S} |u(x)|$. Let $g_N : S \to \mathbb{R}$ be given by $g_N = f - f_N$. The sequence $\{g_N : N \geq 1\}$ is uniformly bounded because, by Lemma 5.4, so is $\{f_N : N \geq 1\}$. Note that the computation below does not depend on the specific form of the function $g_N$ but only on the fact that it is uniformly bounded.

We omit below the dependence of $g$ on $N$. We define a function $V^g : \mathcal{H}_N \to \mathbb{R}$ in few steps. We first construct it on $G$, and then extend it to the whole set. Recall from Lemma 10.3-(3) that the set $G$ can be represented as a disjoint union of the sets $\mathcal{V}^x$ and $\mathcal{J}^{x, y}$. Let
\[
V^g(\eta) = \begin{cases} 
g(x) & \text{if } \eta \in \mathcal{V}^x, \ x \in S, \\
g(y) + [g(x) - g(y)]U_{x, y}(\eta) & \text{if } \eta \in \mathcal{J}^{x, y}, \ x, y \in S.
\end{cases}
\]

By Remark 10.7, we have $U_{y, x} = 1 - U_{x, y}$ on $\mathcal{J}^{x, y}$. Hence,
\[
g(y) + [g(x) - g(y)]U_{x, y}(\eta) = g(x) + [g(y) - g(x)]U_{y, x}(\eta),
\]
and $V^g$ is well-defined on $\mathcal{J}^{x, y}$.

The function $V^g$ is smooth enough on $G$ in the following sense.

**Lemma 10.8.** There exists a finite constant $C_0$ such that,
\[
\max_{\eta \in G} |V^g(\eta)| \leq C_0 \quad \text{and} \quad |V^g(\sigma^z\cdot w\eta) - V^g(\eta)| \leq \frac{C_0}{N}
\]
for all $N \in \mathbb{N}$, $z, w \in S$, and configurations $\eta$ in $G$ such that $\sigma^z\cdot w\eta \in G$. 

Proof. The first bound follows from the definition of \( V^g \) and from the fact that \( U_{x,y} \) is bounded by 1.

We turn to the second. From the definition of the sets \( V^z \), \( J^{z,y} \), for a pair \( (z,w) \) and configurations \( \eta \) and \( \sigma^{z,w} \eta \) in \( \mathcal{G} \), there are three possibilities. Either \( \eta \) and \( \sigma^{z,w} \eta \) belong to some set \( V^z \), or both to some set \( J^{z,y} \) or \( \eta \) belongs to some \( V^z \) and \( \sigma^{z,w} \eta \) to some \( J^{z,y} \) [or the opposite]. We consider separately the three cases.

The inequality is trivial if \( \eta, \sigma^{z,w} \eta \in V^z \) for some \( x \in S \) since in this case \( V^g(\sigma^{z,w} \eta) - V^g(\eta) = 0 \).

By definition of \( \Phi \) and the bound on the derivative of \( \phi \),
\[
|\Phi'(t)| = 6 \left| \phi'(t) \phi(t) [1 - \phi(t)] \right| \leq 6 \left( 1 + \epsilon^{1/2} \right).
\]
Therefore, there exists a finite constant \( C_0 \) such that,
\[
\left| U_{x,y}(\sigma^{z,w} \eta) - U_{x,y}(\eta) \right| \leq \frac{C_0}{N} \quad (10.12)
\]
for all \( N \in \mathbb{N} \), \( \eta \in \mathcal{H}_N \), and \( z, w \in S \). In particular, the inequality stated in Lemma 10.8 holds if \( \eta, \sigma^{z,w} \eta \in J^{z,y} \) for some \( x, y \in S \).

Finally, assume that \( \sigma^{z,w} \eta \in J^{z,y} \) and \( \eta \) belongs to some \( V^z \) for some \( x \leq y \in S \). The same argument applies to the converse situation. In this case, \( U_{x,y}(\eta) = 1 \) because \( \eta_x \geq (1 - 2\epsilon)N \). Thus, by definition of \( V^g \),
\[
V^g(\sigma^{z,w} \eta) - V^g(\eta) = \left[ g(x) - g(y) \right] \left[ U_{x,y}(\sigma^{z,w} \eta) - U_{x,y}(\eta) \right],
\]
and the assertion of the lemma follows from (10.12).

To extend the function \( V^g \) to \( \mathcal{H}_N \setminus \mathcal{G} \), let
\[
V^g(\eta) = 0 \quad \text{for} \ \eta \in \mathcal{H}_N \setminus \hat{\mathcal{G}}. \quad (10.13)
\]
On \( \hat{\mathcal{G}} \setminus \mathcal{G} \), smoothly interpolate the construction (10.11) and (10.13) in such a way that \( \max_{\eta \in \hat{\mathcal{G}} \setminus \mathcal{G}} |V^g(\eta)| \leq C_0 \) (where \( C_0 \) is the constant appeared in Lemma 10.8) and
\[
|V^g(\sigma^{z,w} \eta) - V^g(\eta)| \leq \frac{C_0}{N} \quad \text{for all} \ \eta \in \hat{\mathcal{G}} \setminus \mathcal{G}, \ z, w \in S,
\]
where \( C_0 \) is a constant independent of \( N \). This is possible in view of Lemma 10.8 and since the distance between \( \mathcal{H}_N \setminus \hat{\mathcal{G}} \) and \( \mathcal{G} \) is of order \( N \epsilon \).

The next result summarizes the bounds obtained in the construction. Recall that \( \epsilon > 0 \) is fixed small parameter which appeared in the construction of the function \( \phi \) and that \( V \) depends on \( \epsilon \) though the dependence does not appear in the notation.

Lemma 10.9. For each small \( \epsilon > 0 \), there exist finite constants \( C_0 \) and \( C_\epsilon \) such that,
\[
\max_{\eta \in \mathcal{H}_N} |V^g(\eta)| \leq C_0 \quad \text{and} \quad |V^g(\sigma^{z,w} \eta) - V^g(\eta)| \leq \frac{C_\epsilon}{N}
\]
for all \( N \geq 1 \), \( z, w \in S \), and \( \eta \in \mathcal{H}_N \).

We have now all elements to estimate the capacity between \( \mathcal{E}^x_N \) and \( \hat{\mathcal{E}}^x_N \).

Proof of Proposition 8.1. Fix \( x \in S \). By the Dirichlet principle ([23, equation (B14)]),
\[
\text{cap}_N(\mathcal{E}^x_N, \hat{\mathcal{E}}^x_N) \leq \mathbb{D}_N(F)
\]
for any function \( F : \mathcal{H}_N \to \mathbb{R} \) such that \( F \equiv 1 \) on \( \mathcal{E}^x_N \) and \( F \equiv 0 \) on \( \hat{\mathcal{E}}^x_N \).
Let \( \chi_x = \chi_{\{x\}} : S \to \mathbb{R} \) be the characteristic function on \( x \), i.e.,
\[
\chi_x(y) = \mathbb{1}\{x = y\}, \quad y \in S,
\]
and consider the test function \( F = V^{\chi_x} \), where \( V^{g} \) has been introduced in (10.11).
Let \( C_x \) be the subset of \( \mathcal{H}_N \) given by
\[
C_x = \bigcup_{y \in S \setminus \{x\}} J^{x,y} \cup (\hat{G} \setminus \hat{G}).
\]
Since \( F(\sigma^z,w \eta) = F(\eta) \) unless \( \eta \) or \( \eta^x,y \) belongs to \( C_x \),
\[
\mathbb{D}_N(V^{\chi_x}) \leq C_0 \frac{\theta_N N^2}{\mu_N(C_x)} \leq C_0 \log \frac{1}{\epsilon}.
\]
To completes the proof, it remains to fix some \( 0 < \epsilon < 1 \) and observe that the sets \( \mathcal{E}^\epsilon_N \) do not depend on \( \epsilon \).

### 10.5. Proof of Proposition 10.1.**

The proof is based on Proposition 10.10 stated below. Fix a function \( f : S \to \mathbb{R} \), and recall the definition of \( F_N \), introduced in (2.12), and the one of \( \mathbb{D}_N \) given in (10.1).

Let \( D_Z(u, v) \) be the bilinear form given by
\[
D_Z(u, v) = \frac{1}{\kappa} \sum_{x \in S} u(x) (-L_Z v)(x) = \frac{1}{2N} \sum_{x, y \in S} r_Z(x,y) (u(y)-u(x))(v(y)-v(x)),
\]
for \( u, v : S \to \mathbb{R} \). Here, \( L_Z \) is the generator introduced in (3.11). The next result is proven in Section 10.6.

**Proposition 10.10.** We have that
\[
\mathbb{D}_N(V^{g}, F_N) = D_Z(g, f_N) + o_N(1) + o_\epsilon(1) .
\] (10.14)

**Proposition 10.11.** We have that
\[
\langle V^{g}, F_N \rangle_{\mu_N} = \frac{1}{\kappa} \sum_{x \in S} g(x) f_N(x) + o_N(1) .
\]

**Proof.** Since \( V^{g}(\eta) = g(\eta) \) for \( \eta \in \mathcal{E}^\epsilon_N \), by Theorem 3.3 and the first bound of Lemma 5.4,
\[
\sum_{\eta \in \mathcal{E}^\epsilon_N} V^{g}(\eta) F_N(\eta) \mu_N(\eta) = \frac{1}{\kappa} \sum_{x \in S} g(x) f_N(x) + o_N(1) .
\]

As, by Lemma 10.9, \( |V^{g}(\eta)| \leq C_0 \), it remains to show that
\[
\sum_{\eta \in \Delta_N} |F_N(\eta)| \mu_N(\eta) = o_N(1) .
\]

By Lemma 5.4, the sum is bounded by \( C_0 \mu_N(\Delta_N) \) for some finite constant \( C_0 \). By Theorem 3.3, this expression vanishes as \( N \to \infty \).  \( \square \)
Proof of Proposition 10.1. The main idea of the proof is to compute $\mathbb{D}_N(V^g, F_N)$ in two different ways. The first one is the estimate carried out in Proposition 10.10. The other, and simpler one, is presented below.

Multiply both sides of the equation (2.12) by $V^g(\eta)\mu_N(\eta)$ and sum over $\eta \in \mathcal{H}_N$ to obtain that

$$\lambda \langle V^g, F_N \rangle_{\mu_N} + \mathbb{D}_N(V^g, F_N) = \langle V^g, G_N \rangle_{\mu_N}.$$  

By Theorem 3.3 and Lemma 10.9, since $\mathcal{E}^x \subset \mathcal{V}^x$,

$$\langle V^g, G_N \rangle_{\mu_N} = \frac{1}{\kappa} \sum_{x \in \mathcal{S}} g(x) (\lambda f - L_Z f)(x) + o_N(1)$$

$$= \frac{\lambda}{\kappa} \sum_{x \in \mathcal{S}} g(x) f(x) + D_Z(g, f) + o_N(1).$$  \hspace{2cm} (10.15)

The two previous equations along with Proposition 10.11 yield that

$$\mathbb{D}_N(V^g, F_N) = \frac{\lambda}{\kappa} \sum_{x \in \mathcal{S}} g(x) (f - f_N)(x) + D_Z(g, f) + o_N(1).$$

Thus, by Proposition 10.10,

$$D_Z(g, f - f_N) + \frac{\lambda}{\kappa} \sum_{x \in \mathcal{S}} g(x) (f - f_N)(x) = o_N(1) + o(1).$$

Set $g = f - f_N$ to get that

$$\|f - f_N\|_\infty \leq C_0 \left\{ o_N(1) + o(1) \right\}$$

for some finite constant $C_0 > 0$. Since both $f$ and $f_N$ do not depend on $\epsilon$, this implies that $\|f - f_N\|_\infty = o_N(1)$, which completes the proof. \hfill \Box

10.6. Proof of Proposition 10.10. Let $\partial^x, x \in \mathcal{S}$, be the configuration with one particle at $x$ and no particles at the other sites.

For each set $A \subset \mathcal{H}_N$, denote by $A_-, A_+ \subset \mathcal{H}_{N-1}$ the sets defined by

$$A_- = \{ \xi \in \mathcal{H}_{N-1} : \xi + \partial^x \in A \ \forall \ x \in \mathcal{S} \},$$

$$A_+ = \{ \xi \in \mathcal{H}_{N-1} : \exists x \in \mathcal{S} \ s.t. \ \xi + \partial^x \in A \}.$$  

Recall from (10.2) the definition of the subsets $\mathcal{G}, \mathcal{G}^c$ of $\mathcal{H}_N$. We claim that

$$\mathcal{H}_{N-1} = \mathcal{G}^- \cup (\mathcal{H}_N \setminus \mathcal{G}^-) - \cup (\mathcal{G} \setminus \mathcal{G})_+.$$ \hspace{2cm} (10.17)

It is clear that the right-hand set is contained in $\mathcal{H}_{N-1}$. Fix $\xi \in \mathcal{H}_{N-1}$. Suppose that $\xi + \partial^x$ belongs to $\mathcal{G} \setminus \mathcal{G}$ for some $x \in \mathcal{S}$. In this case, $\xi \in (\mathcal{G} \setminus \mathcal{G})_+$. Suppose, now, that $\xi + \partial^x \notin \mathcal{G} \setminus \mathcal{G}$ for all $x \in \mathcal{S}$. Fix $x_0 \in \mathcal{S}$. Since $\mathcal{H}_N = \mathcal{G} \cup (\mathcal{H}_N \setminus \mathcal{G}^c) \cup (\mathcal{G}^c \setminus \mathcal{G})$, $\xi + \partial^{x_0} \in \mathcal{G} \cup (\mathcal{H}_N \setminus \mathcal{G}^c)$. Suppose that $\xi + \partial^{x_0} \in \mathcal{G}$. The argument applies to the other possibility. Fix $y \in \mathcal{S} \setminus \{x_0\}$. Since $\xi + \partial^{x_0}$ and $\xi + \partial^y$ are neighbors, $\xi + \partial^y$ can not belong to $\mathcal{H}_N \setminus \mathcal{G}$. As it also does not belong to $\mathcal{G} \setminus \mathcal{G}^c$, $\xi + \partial^y$ is in $\mathcal{G}$. Hence, $\xi + \partial^y \in \mathcal{G}$ for all $y \in \mathcal{S}$, so that $\xi \in \mathcal{G}^c$, as claimed in (10.17).

As the sets on the right-hand side of (10.17) are disjoint, this identity provides a partition of the set $\mathcal{H}_{N-1}$.

Lemma 10.12. There exists a finite constant $C_0$ such that

$$\mu_{N-1}(A_+) \leq C_0 \mu_N(A)$$

for all $A \subset \mathcal{H}_N$ and $N \geq 3$. 
Proof. Note that
\[ \mathcal{A}_+ = \bigcup_{x \in S} \{ \eta - d^x : \eta \in \mathcal{A} \text{ with } \eta_x \geq 1 \} . \]

Therefore,
\[ \mu_{N-1}(\mathcal{A}_+) \leq \sum_{x \in S} \sum_{\eta \in \mathcal{A}_{\eta_x \geq 1}} \mu_{N-1}(\eta - d^x) . \]

In particular, it is enough to show that there exists a finite constant \( C_0 \) such that
\[ \frac{\mu_{N-1}(\eta - d^x)}{\mu_N(\eta)} \leq C_0 \text{ for all } \eta \in \mathcal{H}_N \text{ with } \eta_x \geq 1 . \]

By definition of the measure \( \mu_N \) and Proposition 7.1,
\[ \frac{\mu_{N-1}(\eta - d^x)}{\mu_N(\eta)} = \frac{N-1}{Z_{N-1,S}(\log(N-1))^{\kappa-1}} \frac{Z_{N,S}(\log N)^{\kappa-1}}{N} \frac{a(\eta)}{a(\eta - d^x)} \leq C_0 \frac{(\log N)^{\kappa-1}}{\log(N-1)^{\kappa-1}} . \]

This proves the bound and the lemma. \( \square \)

Denote by \( \mathbb{D}_N(F, G; \mathcal{A}) \), \( \mathcal{A} \subset \mathcal{H}_{N-1} \), the bilinear form given by
\[ \frac{\theta_N a_N}{2} \sum_{\xi \in \mathcal{A}, y \in S} \mu_{N-1}(\xi) r(x, y) \left[ F(\xi + \delta^y) - F(\xi + \delta^x) \right] \left[ G(\xi + \delta^y) - G(\xi + \delta^x) \right] , \]
for \( F, G : \mathcal{H}_N \to \mathbb{R} \). In this formula,
\[ a_N = \frac{Z_{N-1,S}(\log(N-1))^{\kappa-1}}{N-1} \frac{N}{Z_{N,S}(\log N)^{\kappa-1}} = 1 + o_N(1) . \quad (10.18) \]

Since
\[ \mu_N(\xi + \delta^x) g(\xi_x + 1) = a_N \mu_{N-1}(\xi) , \]
a change of variables shows that
\[ \mathbb{D}_N(F, G) = \mathbb{D}_N(F, G; \mathcal{H}_{N-1}) . \]

The proof Proposition 10.10 is divided in several lemmata. We start by restricting the computation to the set \( \mathcal{G} \).

Lemma 10.13. We have that
\[ \mathbb{D}_N(V^g, F_N) = \mathbb{D}_N(V^g, F_N; \mathcal{G}_-) + o_N(1) + o_x(1) . \]

Proof. Since (10.17) is a partition of \( \mathcal{H}_{N-1} \), we can write \( \mathbb{D}_N(V^g, F_N) \) as
\[ \mathbb{D}_N(V^g, F_N; \mathcal{G}_-) + \mathbb{D}_N(V^g, F_N; (\mathcal{H}_N \setminus \mathcal{G})_-) + \mathbb{D}_N(V^g, F_N; (\mathcal{G} \setminus \mathcal{G})_+) . \]

On the one hand,
\[ \mathbb{D}_N(V^g, F_N; (\mathcal{H}_N \setminus \mathcal{G})_-) = 0 \]
because \( V^g(\xi + \delta^z) = 0 \) for all \( \xi \in (\mathcal{H}_N \setminus \mathcal{G})_- \), \( z \in S \).

On the other hand, by the Cauchy-Schwarz inequality and Lemma 5.4,
\[ \mathbb{D}_N(V^g, F_N; (\mathcal{G} \setminus \mathcal{G})_+)^2 \leq C_0 \mathbb{D}_N(V^g, V^g; (\mathcal{G} \setminus \mathcal{G})_+) \quad (10.19) \]
for some finite constant $C_0$. By Lemmata 10.9, 10.12 and the bound on $a_N$, the previous Dirichlet form is less than or equal to

$$C_0 \theta_N \sum_{\xi \in (\widehat{G} \setminus G)_+} \sum_{z, w \in S} \mu_{N-1}(\xi) \left[ V^g(\xi + \xi^z) - V^g(\xi + \xi^w) \right]^2$$

$$\leq C_0 \theta_N \mu_{N-1}(\widehat{G} \setminus G)_+ \leq C_0 \theta_N \mu_N(\widehat{G} \setminus G).$$

Hence, by Lemma 10.4,

$$\mathbb{D}_N(V^g, V^g; (\widehat{G} \setminus G)_+) \leq o_N(1) + o_N(1).$$

Inserting this to (10.19) completes the proof of the lemma. □

Recall, from (10.8), the definition of the sets $K^{x,y}$, $L^{x,y}$, and, from (10.9), the definition of the sequence $(z_i)_{i=1}^m$.

**Lemma 10.14.** Fix $x \neq y \in S$. There exists a constant $C_0$ such that for all $\eta \in J^{x,y}$ and $1 \leq m < \kappa$,

$$0 \leq \Phi_\epsilon \left( \frac{1}{N} \sum_{i=1}^m \eta_i + \frac{1}{N} \right) - \Phi_\epsilon \left( \frac{1}{N} \sum_{i=1}^m \eta_i \right) \leq \frac{C_0}{N} \left[ \frac{\eta_x \eta_y}{N^2} + o_N(1) \right].$$

Moreover, for all $\eta \in L^{x,y}$ and $1 \leq m < \kappa$,

$$\Phi_\epsilon \left( \frac{1}{N} \sum_{i=1}^m \eta_i + \frac{1}{N} \right) - \Phi_\epsilon \left( \frac{1}{N} \sum_{i=1}^m \eta_i \right) = \frac{6}{N} \phi_\epsilon(c) \phi_\epsilon(c) \left[ 1 - \phi_\epsilon(c) \right],$$

where $O(\epsilon)$ is a constant which depends on $\epsilon$ and whose absolute value is bounded by $C_0 \epsilon$.

**Proof.** Fix $\eta \in J^{x,y}$. As $\Phi_\epsilon$ is non-decreasing, the first inequality holds. We consider the second one. By definition of $\Phi_\epsilon$ and the mean-value theorem,

$$\Phi_\epsilon \left( \frac{1}{N} \sum_{i=1}^m \eta_i + \frac{1}{N} \right) - \Phi_\epsilon \left( \frac{1}{N} \sum_{i=1}^m \eta_i \right) = \frac{6}{N} \phi_\epsilon(c) \phi_\epsilon(c) \left[ 1 - \phi_\epsilon(c) \right],$$

where $c = N^{-1} \sum_{1 \leq i \leq m} \eta_i + (\delta/N)$ for some $0 \leq \delta \leq 1$. By definition of $\phi_\epsilon$, $1 - \phi_\epsilon(c) = \phi_\epsilon(1 - c)$. Thus, as

$$0 \leq c - \frac{\eta_x}{N} \leq \frac{N - \eta_x - \eta_y + 1}{N} \leq \frac{\ell_N + 1}{N} ,$$

$$0 \leq (1 - c) - \frac{\eta_y}{N} \leq \frac{N - \eta_x - \eta_y}{N} \leq \frac{\ell_N}{N} ,$$

it follows from the uniform bound on $\|\phi'_\epsilon\|_{\infty}$, that

$$\Phi_\epsilon \left( \frac{1}{N} \sum_{i=1}^m \eta_i + \frac{1}{N} \right) - \Phi_\epsilon \left( \frac{1}{N} \sum_{i=1}^m \eta_i \right) \leq \frac{C_0}{N} \left[ \phi_\epsilon \left( \frac{\eta_x}{N} \right) + o_N(1) \right] \left[ \phi_\epsilon \left( \frac{\eta_y}{N} \right) + o_N(1) \right].$$

Since $\phi'_\epsilon(t) \leq 1 + \epsilon^{1/2}$ for all $0 \leq t \leq 1$ and $\phi_\epsilon(0) = 0$, $\phi_\epsilon(t) \leq (1 + \epsilon^{1/2})t \leq 2t$. This completes the proof of the first assertion of the lemma, as $\eta_x/N \leq 1$. 

We turn to the second one. Fix a configuration \( \eta \) in \( \mathcal{L}_{x,y} \). Since
\[
6\epsilon \leq \frac{\eta_x}{N} \leq \frac{1}{N} \sum_{i=1}^{m} \eta_{z_i} \leq \frac{N - \eta_y}{N} \leq 1 - 6\epsilon,
\]
the constant \( c \) belongs to the interval \([6\epsilon, 1 - (11/2)\epsilon]\) [provided \( 1/N \leq \epsilon/2 \)], and \( \phi'_\epsilon(c) = 1/(1 - 8\epsilon) \). On the other hand, since \( \phi_\epsilon \) is linear on the interval \([5\epsilon, 1 - 5\epsilon]\) and \( \eta_x, \eta_y \geq 6\epsilon N \),
\[
\phi_\epsilon(c) = \phi_\epsilon \left( \frac{\eta_x}{N} \right) + o_N(1) = \frac{1}{1 - 8\epsilon} \left( \frac{\eta_x}{N} - 4\epsilon \right) + o_N(1),
\]
\[
\phi_\epsilon(1 - c) = \phi_\epsilon \left( \frac{\eta_y}{N} \right) + o_N(1) = \frac{1}{1 - 8\epsilon} \left( \frac{\eta_y}{N} - 4\epsilon \right) + o_N(1).
\]
To complete the proof of the second assertion, it remains to report these estimates to the right-hand side of (10.20).

By the definitions of \( V^g \), and \( U_{x,y} \), given in (10.11), (10.10), respectively, for \( \eta \in \mathcal{K}_{x,y} \), there exists a finite constant \( C_0 \) such that for all \( z, w \in S \),
\[
| V^g(\sigma^{z,w}\eta) - V^g(\eta) | \leq \frac{C_0}{N} \phi_\epsilon(1). \tag{10.21}
\]
The next result asserts that it is enough to estimate the Dirichlet form on the sets \( \mathcal{L}_{x,y}^- \), \( x, y \in S \).

**Lemma 10.15.** We have that
\[
\mathbb{D}_N(V^g, F_N; \mathcal{G}_-) = \sum_{x,y \in S} \mathbb{D}_N(V^g, F_N; \mathcal{L}_{x,y}^-) + o_N(1).
\]

**Proof.** An argument, similar to the one presented to derive (10.17), yields that the set \( \mathcal{G}_- \) can be decomposed as
\[
\mathcal{G}_- = \bigcup_{x,y \in S} \mathcal{L}_{x,y}^- \cup \bigcup_{x \in S} \mathcal{V}_{x}^- \cup \bigcup_{x,y \in S} (\mathcal{K}_{x,y}^+ \cap \mathcal{G}_-).
\]
On the one hand,
\[
\mathbb{D}_N(V^g, F_N; \mathcal{V}_x^-) = 0
\]
because \( V^g(\xi + d^z) = g(x) \) for all \( \xi \in \mathcal{V}_x^- \) and \( z \in S \).

On the other hand, by Schwarz inequality and the bound on \( o_N \),
\[
\mathbb{D}_N(V^g, F_N; \mathcal{K}_{x,y}^+ \cap \mathcal{G}_-)^2 
\leq C_0 \mathbb{D}_N(F_N) \times \theta_N \sum_{\xi \in \mathcal{K}_{x,y}^+} \sum_{z,w \in S} \mu_{N-1}(\xi) \left[ V^g(\xi + d^z) - V^g(\xi + d^w) \right]^2
\]
for some finite constant \( C_0 \). By Lemma 5.4 and (10.21), this expression is bounded from above by
\[
\frac{C_0}{N^2} \theta_N o_N(1) \mu_{N-1}(\mathcal{K}_{x,y}^+).
\]
By Lemmata 10.6 and 10.12, \( \mu_{N-1}(\mathcal{K}_{x,y}^+) \leq C_0 \mu_N(\mathcal{K}_{x,y}^+) \leq C_0 / \log N \) for some finite constant \( C_0 \).

Putting together the previous estimates yields that
\[
\mathbb{D}_N(V^g, F_N; \mathcal{K}_{x,y}^+ \cap \mathcal{G}_-) \leq C_0 o_N(1).
\]
This completes the proof of the lemma.

\( \square \)
It remains to compute the Dirichlet form on $\mathcal{L}_{x,y}^\times$. The proof of the next lemma is given in Section 10.7.

**Lemma 10.16.** For $x, y \in S$,$$
\mathbb{D}_N(U_{x,y}, F_N; \mathcal{L}_{x,y}^\times) = \frac{r_Z(x,y)}{\kappa} \left[ f_N(x) - f_N(y) \right] + o_N(1) + o_c(1) .
$$

*Proof of Proposition 10.10.* By definition (10.11) of $V^g$ on $J_{x,y}^\times$, we have $$\Delta_N(V^g, F_N; \mathcal{L}_{x,y}^\times) = \left[ g(x) - g(y) \right] \mathbb{D}_N(U_{x,y}, F_N; \mathcal{L}_{x,y}^\times) .$$

By Lemma 10.16, the right-hand side can be rewritten as

$$
\frac{r_Z(x,y)}{\kappa} \left[ g(x) - g(y) \right] \left[ f_N(x) - f_N(y) \right] + o_N(1) + o_c(1) .
$$

It remains to combine this estimate with Lemmata 10.13 and 10.15. □

10.7. **Proof of Lemma 10.16.** We start with a simple lemma which allows to bound a covariance between two functions $F, G : \mathcal{E}_N^\times \to \mathbb{R}$ in terms of the Dirichlet form of one of them and the $L^\infty$-norm of the other.

**Lemma 10.17.** There exists a finite constant $C_0$ such that, for all $x \in \mathbb{R}$ and $F, G : \mathcal{E}_N^\times \to \mathbb{R}$,

$$
\left| \mathbf{E}_{\mu_N}^N [F G] - \mathbf{E}_{\mu_N}^N [F] \mathbf{E}_{\mu_N}^N [G] \right|^2 \leq \frac{C_0}{(\log N)^3} \|G\|^2_{L^\infty} \mathbb{D}_N(F) .
$$

*Proof.* This lemma is a simple consequence of the local spectral gap estimate. By the Cauchy-Schwarz inequality,

$$
\left| \mathbf{E}_{\mu_N}^N [F G] - \mathbf{E}_{\mu_N}^N [F] \mathbf{E}_{\mu_N}^N [G] \right|^2 \leq \|G\|^2_{L^\infty} \text{Var}_{\mu_N}^\times (F) ,
$$

where the variance has been introduced in (5.9). To complete the proof, it remains to recall the local spectral gap, stated in Theorem 6.1. □

For $x, y \in S$, define

$$
\mathcal{B}_{x,y} = \{ \zeta \in \mathbb{N}^{S \setminus \{x,y\}} : |\zeta| \leq \ell_N - 1 \} .
$$

(10.22)

**Lemma 10.18.** For $x, y \in S$,

$$
\sum_{\zeta \in \mathcal{B}_{x,y}} \frac{1}{a(\zeta)} = \left[ 1 + o_N(1) \right] (\log N)^{\kappa - 2} .
$$

*Proof.* Set $\xi = (N - |\zeta|, \zeta) \in \mathcal{H}_{N,S \setminus \{x\}}$. By Theorem 3.3,

$$
\lim_{N \to \infty} \frac{1}{Z_{N, S \setminus \{x\}} (\log N)^{\kappa - 2}} \sum_{\zeta \in \mathcal{B}_{x,y}} \frac{1}{N - |\zeta| a(\zeta)} = \frac{1}{\kappa - 1} .
$$

The assertion of the lemma follows from Proposition 7.1. □

For $\zeta \in \mathbb{N}^{S \setminus \{x,y\}}$ with $|\zeta| \leq N$, define $\eta_{\zeta}^{(i)} \in \mathcal{H}_N$, $0 \leq i \leq N - |\zeta|$, as the configuration on $S$ with $N - i - |\zeta|$ particles at site $x$, $i$ particles at site $y$, and $\zeta$ particles at $z \in S \setminus \{x,y\}$:

$$
\eta_{\zeta}^{(i)} = (N - i - |\zeta|, i, \zeta) .
$$
Lemma 10.19. For all \( x \neq y \in S \),
\[
\frac{1}{(\log N)^{\kappa-2}} \sum_{\zeta \in B_{x,y}} \frac{1}{a(\zeta)} F_N(\eta^l_\zeta) = \left[ 1 + o_N(1) \right] f_N(x) + o_N(1),
\]
\[
\frac{1}{(\log N)^{\kappa-2}} \sum_{\zeta \in B_{x,y}} \frac{1}{a(\zeta)} F_N(\eta^{N-|\zeta|-6\epsilon N}_\zeta) = \left[ 1 + o_N(1) \right] f_N(y) + o_N(1).
\]

Proof. We prove the first assertion, as the second one can be obtained by symmetry. Fix \( x, y \in S \). For \( k \in \mathbb{N} \), let
\[
B_{x,y}^k = \{ \zeta \in B_{x,y}^k : |\zeta| = k \}.
\]

For \( 0 \leq k < \ell_N \), define
\[
c_N(k) = \log N \left( \sum_{i=0}^{\ell_N-k} \frac{a(N-\ell-i)a(i)}{a(\ell)} \right)^{-1},
\]
and set \( c_N(\ell_N) = 0 \). Note that there exists a finite constant \( C_0 \) such that
\[
|c_N(k)| \leq C_0 \log N \text{ for all } 0 \leq k \leq \ell_N.
\]

Define
\[
\tilde{f}_N(x) = \sum_{\eta \in E_N^x} \mu_N(\eta) c_N(N - \eta - \eta_y) F_N(\eta).
\]
We claim that
\[
\tilde{f}_N(x) = \left[ \frac{1}{\kappa} + o_N(1) \right] f_N(x) + o_N(1),
\]
and that
\[
\frac{1}{Z_{N,S} (\log N)^{\kappa-2}} \sum_{\zeta \in B_{x,y}} \frac{1}{a(\zeta)} F_N(\eta^l_\zeta) - \tilde{f}_N(x) = o_N(1).
\]

The assertion of the lemma follows from these two identities and Proposition 7.1.

To prove the first claim, let
\[
d_N = \sum_{\eta \in E_N^x} c_N(N - \eta - \eta_y) \mu_N(\eta).
\]
By definition of \( \mu_N \) and \( c_N \), as \( c_N(\ell_N) = 0 \), we can rewrite \( d_N \) as
\[
d_N = \frac{N}{Z_{N,S} (\log N)^{\kappa-1}} \sum_{k=0}^{\ell_N-1} \sum_{\zeta \in B_{x,y}^k} \sum_{i=0}^{\ell_N-k} \frac{c_N(k)}{a(N-k-i)a(i)} \frac{1}{a(\zeta)}
\]
\[
= \frac{N}{Z_{N,S} (\log N)^{\kappa-1}} \sum_{k=0}^{\ell_N-1} \sum_{\zeta \in B_{x,y}^k} \log N \frac{1}{a(\zeta)} = \frac{1}{Z_{N,S} (\log N)^{\kappa-2}} \sum_{\zeta \in B_{x,y}} \frac{1}{a(\zeta)}.
\]
Hence, by Proposition 7.1 and Lemma 10.18,
\[
d_N = \frac{1}{\kappa} + o_N(1).
\]
To prove (10.26), it remains to show that
\[
\tilde{f}_N(x) - d_N f_N(x) = o_N(1).
\]
Define $U : \mathcal{E}_N^x \to \mathbb{R}$ as $U(\eta) = c_N(N - \eta_x - \eta_y)$, so that

$$\frac{1}{\mu_N(\mathcal{E}_N^x)} \left\{ \tilde{f}_N(x) - d_N f_N(x) \right\} = E_{\mu_N^x}[F_N U] - E_{\mu_N^x}[F_N] E_{\mu_N^x}[U].$$

Thus, by Lemma 10.17 and (10.25),

$$\left[ \tilde{f}_N(x) - d_N f_N(x) \right]^2 \leq \frac{C_0}{\log N} \mathbb{D}_N(F_N)$$

for some finite constant $C_0$. By Lemma 5.4, this expression is bounded by $C_0 / \log N$, which proves (10.28) and (10.26).

We turn to (10.27). By definition, $\tilde{f}_N(x)$ is equal to

$$\frac{N}{Z_{N,S}(\log N)^{\kappa-1}} \sum_{k=0}^{\ell_N - 1} \sum_{\zeta \in \mathcal{B}^{\kappa,y}} \frac{c_N(k)}{a(\zeta)} a(N - k - i)a(i) F_N(\eta^i_\zeta).$$

On the other hand, by definitions of $\mathcal{B}^{\kappa,y}$, $\mathcal{B}^{\kappa,y}_k$ and $c_N(k)$, given in (10.22), (10.23) and (10.24), respectively, we have

$$\frac{1}{Z_{N,S}(\log N)^{\kappa-2}} \sum_{\zeta \in \mathcal{B}^{\kappa,y}} \frac{1}{a(\zeta)} F_N(\eta^i_\zeta) = \frac{N}{Z_{N,S}(\log N)^{\kappa-1}} \sum_{k=0}^{\ell_N - 1} \sum_{\zeta \in \mathcal{B}^{\kappa,y}} \log N \frac{1}{N} \frac{c_N(k)}{a(\zeta)} a(N - k - i)a(i) F_N(\eta^i_\zeta).$$

Therefore, the left-hand side of (10.27) is equal to

$$\sum_{\zeta \in \mathcal{B}^{\kappa,y}} \sum_{i=0}^{\ell_N - |\zeta|} c_N(|\zeta|) \mu_N(\eta^i_\zeta) [F_N(\eta^i_\zeta) - F_N(\eta^i_\zeta)].$$

In view of the previous expression, by the Cauchy-Schwarz inequality and (10.25), the square of the left-hand side of (10.27) is bounded by

$$C_0 (\log N)^2 \sum_{\zeta \in \mathcal{B}^{\kappa,y}} \sum_{i=0}^{\ell_N - |\zeta|} \mu_N(\eta^i_\zeta) \left[ F_N(\eta^i_\zeta) - F_N(\eta^i_\zeta) \right]^2$$

(10.29)

for some finite constant $C_0$. By the Cauchy-Schwarz inequality again, the square inside the previous sum is less than or equal to

$$\leq C_0 N^2 \sum_{j=i}^{6 \epsilon N - 1} \frac{1}{a(j)} a(N - |\zeta| - j) \left[ F_N(\eta^j_\zeta) - F_N(\eta^j_\zeta) \right]^2 \sum_{j=i}^{6 \epsilon N - 1} a(j) a(N - |\zeta| - j).$$


for some finite constant $C_0$. The sum (10.29) is thus bounded above by

$$C_0 (N \log N)^2 \sum_{\xi \in \mathcal{B}^N} \frac{\ell_N - |\xi|}{N^2} \sum_{i=0}^{6N-1} \sum_{j=i}^{6N-1} \mu_{N/2}(\eta_{i,j}^{(j)}) \left[ F_N(\eta_{i,j}^{(j)}) - F_N(\eta_{i,j}^{(j) + 1}) \right]^2 \leq C_0 N (\log N)^2 \sum_{\xi \in \mathcal{B}^N} \sum_{i=0}^{6N-1} \frac{\mu_{N/2}(\eta_{i,j}^{(j)}) \left[ F_N(\eta_{i,j}^{(j)}) - F_N(\eta_{i,j}^{(j) + 1}) \right]^2}{a(i) a(N - |\xi| - i)}$$

Changing the order of summations this expression becomes

$$C_0 N (\log N)^2 \sum_{\xi \in \mathcal{B}^N} \sum_{j=0}^{6N-1} \mu_{N/2}(\eta_{i,j}^{(j)}) \left[ F_N(\eta_{i,j}^{(j)}) - F_N(\eta_{i,j}^{(j) + 1}) \right]^2 \frac{A_N(j, \xi)}{a(i)},$$

where $A_N(j, \xi) = \min\{j, \ell_N - |\xi|\}$. The last summation over $i$ is bounded by $C_0 \log \ell_N \leq C_0 \log N$. Hence, by Lemma 10.2, this expression is less than or equal to $C_0 (\log N)^2/N \mathbb{D}_N(F_N)$, which, by Lemma 5.4, is bounded by $C_0 (\log N)^2/N$, which proves (10.27).

**Proof of Lemma 10.16.** Fix $x, y \in S$, and recall the definition of $U_{x,y}$ given in (10.10) and the one of the sequence $(z_i)_{i=1}^{\infty}$ introduced in (10.9). With this notation, we can write $\mathbb{D}_N(U_{x,y}, F_N; L^x_{-y})$ as

$$\frac{\theta_N a_N}{2} \sum_{i,j=1}^{\kappa} \sum_{\xi \in L^x_{-y}} \mu_{N-1}(\xi) r(z_i, z_j) (T_{i,j} U_{x,y})(\xi) (T_{i,j} F_N)(\xi),$$

where $(T_{i,j} G)(\xi) = G(\xi + \delta z_i) - G(\xi + \delta z_j)$. Assume that $i > j$. By definition, we can write $T_{i,j} U_{x,y}(\xi)$ as

$$\sum_{n=j}^{i-1} [h_{x,y}(z_n) - h_{x,y}(z_{n+1})] \left[ \Phi_\epsilon \left( \frac{1}{N} \sum_{k=1}^{n} \xi_{z_k} + \frac{1}{N} \right) - \Phi_\epsilon \left( \frac{1}{N} \sum_{k=1}^{n} \xi_{z_k} \right) \right].$$

By the second assertion of Lemma 10.14, this sum is equal to

$$\frac{6}{N} \sum_{n=j}^{i-1} [h_{x,y}(z_n) - h_{x,y}(z_{n+1})] \left[ \frac{\xi_{z_n} \xi_{z_j} y}{N^2} + o_N(1) + O(\epsilon) \right]$$

$$= \frac{6}{N} \left[ h_{x,y}(z_j) - h_{x,y}(z_i) \right] \left[ \frac{\xi_{z_n} \xi_{z_j} y}{N^2} + o_N(1) + O(\epsilon) \right].$$

A similar identity holds for $i < j$. Therefore,

$$\mathbb{D}_N(U_{x,y}, F_N; L^x_{-y}) = I_1 + I_2,$$

where

$$I_1 = \frac{3\theta_N a_N}{N^3} \sum_{\xi \in L^x_{-y}} \sum_{i,j=1}^{\kappa} \mu_{N-1}(\xi) r(z_i, z_j) \xi_{z_n} \xi_{z_j} [h_{x,y}(z_j) - h_{x,y}(z_i)] (T_{i,j} F_N)(\xi),$$

$$I_2 = \frac{\theta_N}{N} \sum_{\xi \in L^x_{-y}} \sum_{i,j=1}^{\kappa} \mu_{N-1}(\xi) [h_{x,y}(z_j) - h_{x,y}(z_i)] (T_{i,j} F_N)(\xi).$$
The second term is easy to estimate. By the Cauchy-Schwarz inequality, its square is bounded by
\[
\frac{1}{N^2} \left[ a_N(1) + O(\epsilon) \right] \theta_N \mu_{N-1}(\mathcal{L}^x_{-y}) D_N(F_N) .
\]
By definition of \( \mathcal{L}^x_{-y} \), \( \mathcal{L}^x_{+y} \), and by Lemmata 10.5 and 10.12,
\[
\mu_{N-1}(\mathcal{L}^x_{-y}) \leq \mu_{N-1}(\mathcal{L}^x_{+y}) \leq C_0 \mu_N(\mathcal{L}^x_{-y}) \leq \frac{C_0}{\log N} \log \frac{1}{\epsilon}
\]
for some finite constant \( C_0 \). Hence, by Lemma 5.4,
\[
I_2 = o_N(1) + o_\epsilon(1) .
\]
We turn to \( I_1 \). Write \( \xi = (\xi_x, \xi_y, \zeta) \) for \( \zeta \in \mathbb{N}^{S \setminus \{x, y\}} \). Then, \( I_1 \) is equal to
\[
\frac{3\theta_N}{N^3} \frac{Z_{N-1}}{S} \frac{\log(N - 1)^{\kappa - 1}}{\log(N - 1)} \times \sum_{\xi \in \mathcal{L}^x_{-y}} \sum_{i, j=1}^{\kappa} \frac{1}{a(\xi)} r(z_i, z_j) \left\{ h_{x, y}(z_i) - h_{x, y}(z_j) \right\} (T_{i,j} F_N)(\xi) .
\]
By Proposition 7.1, by definition of \( \theta_N \) and by (10.18), we may rewrite this expression as
\[
\frac{6[1 + o_N(1)]}{\kappa (\log N)^{\kappa - 2}} \sum_{\xi \in \mathcal{L}^x_{-y}} \sum_{i=1}^{\kappa} \frac{F_N(\xi + \delta^z_i)}{a(\xi)} \sum_{j=1}^{\kappa} r(z_i, z_j) \left\{ h_{x, y}(z_i) - h_{x, y}(z_j) \right\} = \frac{6[1 + o_N(1)]}{\kappa (\log N)^{\kappa - 2}} \sum_{\xi \in \mathcal{L}^x_{-y}} \sum_{i=1}^{\kappa} \frac{F_N(\xi + \delta^z_i)}{a(\xi)} (-L_X h_{x, y})(z_i) .
\]
By (12.1),
\[
(L_X h_{x, y})(z) = \begin{cases} \ -\kappa \text{cap}_X(x, y) & \text{if } z = x \\ \ k \text{cap}_X(x, y) & \text{if } z = y \\ 0 & \text{otherwise} \end{cases} .
\]
Thus, by the definition (3.12) of \( r_Z(x, y) \),
\[
I_1 = \frac{r_Z(x, y)[1 + o_N(1)]}{\kappa (\log N)^{\kappa - 2}} \sum_{\xi \in \mathcal{B}^x_{-y}} \frac{1}{a(\xi)} [F_N(\xi + \delta^x) - F_N(\xi + \delta^y)] .
\]
Recall from (10.22) the definition of the set \( \mathcal{B}^x_{-y} \) and that \( \eta_\zeta^{(i)} \) represents the configuration \( (N - |\zeta| - i, i, \zeta) \in \mathcal{H}_N \). With this notation, the set \( \mathcal{L}^x_{-y} \) can be represented as
\[
\mathcal{L}^x_{-y} = \left\{ (N - |\zeta| - i, i, \zeta) : \zeta \in \mathcal{B}^x_{-y}, \ 6N\epsilon \leq i < N - |\zeta| - 6N\epsilon \right\} .
\]
Therefore, we can write \( I_1 \) as
\[
\frac{r_Z(x, y)[1 + o_N(1)]}{\kappa (\log N)^{\kappa - 2}} \sum_{\xi \in \mathcal{B}^x_{-y}} \frac{1}{a(\xi)} \sum_{i=6N\epsilon}^{N-|\zeta|-6N\epsilon-1} \left[ F_N(\eta_\zeta^{(i)}) - F_N(\eta_\zeta^{(i+1)}) \right] = \frac{r_Z(x, y)[1 + o_N(1)]}{\kappa (\log N)^{\kappa - 2}} \sum_{\xi \in \mathcal{B}^x_{-y}} \frac{1}{a(\xi)} \left[ F_N(\eta_\zeta^{(6N\epsilon)}) - F_N(\eta_\zeta^{(N-|\zeta|-6N\epsilon)}) \right] .
\]
Thus, by Lemma 10.19,

$$I_1 = \frac{r_N(x,y)}{\kappa} \left[ f_N(x) - f_N(y) \right] + o_N(1) + o_N(1),$$

which completes the proof of the lemma. □

11. Attractor sets in the wells

The proof of Proposition 8.6 is divided in two steps. We first show that starting from a configuration \( \eta \) in \( E^x_N \), the process hits the set \( D^x_N \) before it leaves the large well \( W^x_N \). The proof of this result requires the construction of a super-harmonic function on \( W^x_N \setminus E^x_N \), a technical and difficult step presented in the next section. Then, we show that starting from \( D^x_N \), the process visits all configurations of this set before hitting a new well \( E^y_N \).

11.1. Deep wells are attractors. The next result asserts that starting from \( E^x_N \) the process hits the deep well \( D^x_N \) before leaving \( W^x_N \).

Proposition 11.1. For all \( x \in S \),

$$\lim_{N \to \infty} \inf_{\eta \in E^x_N} \mathbb{P}^N_\eta \left[ \tau_{D^x_N} < \tau_{(W^x_N)^c} \right] = 1.$$  \hspace{1cm} (11.1)

The proof of this proposition is based on the existence of a super-harmonic function in \( W^x_N \setminus D^x_N \), presented in the next section.

Theorem 11.2. Fix \( x \in S \). There exist positive, finite constants \( c_0, c_1, c_2 \) and a function \( G^x_N : \mathcal{H}_N \to \mathbb{R} \) such that,

$$\mathcal{L}_N G^x_N(\eta) \leq - \frac{c_0 \theta_N}{N - \eta_x} < 0 , \hspace{1cm} (11.1)$$

and \( c_1 (N - \xi_x) \leq G^x_N(\xi) \leq c_2 (N - \xi_x) \) \hspace{1cm} (11.2)

for all \( \eta \in W^x_N \setminus D^x_N, \xi \in W^x_N \setminus D^x_N \) and large enough \( N \).

Proof of Proposition 11.1. In view of Theorem 11.2 and Lemma 5.3, it is enough to check that (5.4) holds. By (11.2) and the definition of the sets \( D^x_N, E^x_N \) and \( W^x_N \), the ratio in (5.4) is bounded by

$$\frac{c_2 \ell_N - c_1 N^\gamma}{c_1 N/(\log N)^\beta - c_1 N^\gamma}, \hspace{1cm} \eta \in E^x_N \setminus D^x_N .$$

Since \( \ell_N = N/\log N, 0 < \beta < 1 \), and \( 0 < \gamma < 1 \), the previous expression vanishes as \( N \to \infty \). This completes the proof of the proposition. □

11.2. Visiting points in deep wells. The main result of this section, Proposition 11.4, asserts that starting from a deep well \( D^x_N \) the process visits all configurations in \( D^x_N \) before hitting a new well \( E^y_N \). This result is a weak version of Proposition 8.6, as it requires the process to start from \( D^x_N \) instead of \( E^x_N \).

The proof of Proposition 11.4 is based on a classical bound of equilibrium potentials in terms of capacities. We first provide a lower bound on the capacities between configurations in \( D^x_N \).

Lemma 11.3. Fix \( x \in S \). There exists a positive constant \( c_0 \) such that for all \( \xi, \eta \in D^x_N \) and \( N \geq 1 \),

$$\text{cap}_N(\xi, \eta) \geq \frac{c_0 \theta_N}{N^{\gamma_x} (\log N)^{\kappa_x - 1}} .$$
Proof. Fix $\xi, \eta \in \mathcal{D}_N^x$. Consider a sequence $\xi = \zeta^{(0)}, \zeta^{(1)}, \ldots, \zeta^{(p)} = \eta$ in $\mathcal{D}_N^x$ such that $\zeta^{(k+1)} = \sigma x_k, y_k \zeta^{(k)}$ for some $x_k, y_k \in S$ satisfying $r(x_k, y_k) > 0$. Since there are at most $N^\gamma$ particles on $S \setminus \{x\}$, there exists such a sequence with length bounded by $C_0 N^\gamma$:

$$p \leq C_0 N^\gamma$$

for some finite constant $C_0$.

Let $F : \mathcal{H}_N \to \mathbb{R}$ be a function such that $F(\xi) = 0$ and $F(\eta) = 1$. By Cauchy-Schwarz inequality, there exists a finite constant $C$ such that

$$\sum_{k=0}^{p-1} \left[ F(\zeta^{(k+1)}) - F(\zeta^{(k)}) \right]^2 \leq C \theta_N^{-1} \mathbb{E}_N(F) \sum_{k=0}^{p-1} \frac{1}{\mu_N(\zeta^{(k)})}.$$ 

Thus, by the Dirichlet principle,

$$\text{cap}_N(\xi, \eta) \geq c_0 \theta_N \left( \sum_{k=0}^{p-1} \frac{1}{\mu_N(\zeta^{(k)})} \right)^{-1}.$$ 

By definition of the set $\mathcal{D}_N^x$, $a(\zeta) \leq N (N^\gamma)^{\kappa - 1} = N^{1+\kappa(1-\gamma)}$ for $\zeta \in \mathcal{D}_N^x$. Hence, by the explicit formula for the invariant measure and Proposition 7.1, there exists a positive constant $c_0$ such that

$$\mu_N(\zeta) \geq c_0 \frac{N}{(\log N)^{\kappa-1}} \frac{1}{N^{1+(\kappa-1)\gamma}} = \frac{1}{N^{\gamma(1-\kappa)}(\log N)^{\kappa-1}}.$$ 

To complete the proof, it remains to put together all previous estimates. 

The bound produced by this argument in the case where $\xi$ belongs to $\mathcal{D}_N^x$ and $\eta$ to $E_N^x$ is too crude to prove Proposition 11.4 below with $\eta \in E_N^x$, instead of $\eta \in \mathcal{D}_N^x$.

Proposition 11.4. For all $x \in S$,

$$\lim_{N \to \infty} \inf_{\eta \in \mathcal{D}_N^x} \inf_{\xi \in \mathcal{D}_N^x} P_N^\eta [\tau_\xi < \tau_\xi'] = 1.$$ 

Proof. By [25, equation (3.3)] and the monotonicity of the capacity,

$$P_N^\eta [\tau_\xi < \tau_\xi'] \leq \frac{\text{cap}_N(\eta, \mathcal{E}_N^x)}{\text{cap}_N(\eta, \zeta)} \leq \frac{\text{cap}_N(E_N^x, \mathcal{E}_N^x)}{\text{cap}_N(\eta, \zeta)}.$$ 

Thus, by Proposition 8.1 and Lemma 11.3,

$$P_N^\eta [\tau_\xi < \tau_\xi'] \leq C \frac{N^\gamma (\log N)^{\kappa-1}}{\theta_N}.$$ 

Since, by hypothesis, $\gamma < 2/\kappa$, this expression vanishes as $N \to \infty$, as claimed. 

Remark 11.5. Proposition 11.4 is enough to derive Condition (C2), the full content of Proposition 8.6 is not needed. See the proof at the end of Section 8. The statement of Proposition 8.6, is however interesting, as it asserts that starting from a well $E_N^x$, the process visits all points of the deep well $\mathcal{D}_N^x$ before hitting a new well.

Proof of Proposition 8.6. Fix $x \in S$, $\eta \in E_N^x$, $\zeta \in \mathcal{D}_N^x$. By the strong Markov property,

$$P_N^\eta [\tau_\zeta < \tau_\xi'] \geq P_N^\eta [\tau_\mathcal{D}_N^x < \tau_\xi', \tau_\zeta < \tau_\xi'] \geq P_N^\eta [\tau_\mathcal{D}_N^x < \tau_\xi'] \inf_{\xi \in \mathcal{D}_N^x} P_N^\zeta [\tau_\zeta < \tau_\xi'].$$
Optimizing over \( \eta \in \mathcal{E}_N^\xi \) yields that
\[
\inf_{\eta \in \mathcal{E}_N^\xi} P_N^\eta[\tau_\xi < \tau_{\mathcal{E}_N^\xi}] \geq \inf_{\eta \in \mathcal{E}_N^\xi} P_N^\eta[\tau_{\mathcal{E}_N^\xi} < \tau_{(\mathcal{W}_N^\xi)^c}] \inf_{\xi \in \mathcal{P}_N^\xi} P_N^\xi[\tau_\xi < \tau_{\mathcal{E}_N^\xi}].
\]

because \( \tau_{(\mathcal{W}_N^\xi)^c} < \tau_{\mathcal{E}_N^\xi} \). To complete the proof, it remains to recall the statements of Propositions 11.1 and 11.4.

12. A super-harmonic function

In this chapter, we prove Theorem 11.2. For the convenience of notation, we will now work with the generator \( A_N \) of the original zero-range process, instead of the speeded-up generator \( \mathcal{L}_N \).

The super-harmonic function \( G_N^\xi \) is introduced in Section 12.4. We explain below the ideas behind its construction. To propose candidates, one interprets the zero-range process as a random walk on the simplex \( \mathcal{H}_N \).

Fix \( x_0 \in S \), and denote by \( \mathcal{H}_N^{x_0} \) the subset of \( \mathcal{H}_N \) of all configurations such that \( N - \alpha_N \leq \eta_{x_0} \leq N - \beta_N \), where \( \beta_N \ll \alpha_N \ll N \) are two sequences. This means that all coordinates \( \eta_x \) are much smaller than \( \eta_{x_0} \) on the set \( \mathcal{H}_N^{x_0} \).

One wishes to show that \( \sum_{x \neq x_0} \eta_x \) decreases with time in this set. This is done by constructing an increasing function \( F : \mathbb{N} \rightarrow \mathbb{R} \) such that \( (A_N F)(\sum_{x \neq x_0} \eta_x) \leq 0 \) on the set \( \mathcal{H}_N^{x_0} \). In fact, it is not difficult to find functions which are super-harmonic in the interior of \( \mathcal{H}_N^{x_0} \) [the points \( \eta \) in this set such that \( \eta_x > 0 \) for all \( x \)]. Indeed, in the interior, it is clear that \( \sum_{x \neq x_0} \eta_x \) decreases in time because the rate of a jump from \( x_0 \) to \( x \) is strictly smaller than the rate of a jump from \( x \) to \( x_0 \). The problem occurs at the boundary. The sum \( \sum_{x \neq x_0} \eta_x \) may increase due to a jump from \( x_0 \) to a site \( x \) such that \( \eta_x = 0 \), and the reverse jumps are forbidden.

In the diffusive scale the random walk should converge weakly to a diffusion on a continuous simplex. Denote by \( \xi^{x_0} \) the corner of this simplex which corresponds to the configuration in which all particles sit at site \( x_0 \). One can write down the drift of this diffusion and define a \(|S| - 2\)-dimensional manifold with the property that at any point of this manifold the scalar product of the drift of the diffusion with the normal vector to the manifold [which point towards the corner] is positive. For \(|S| = 3 \) or \(4\), one can draw pictures of the vector field induced by the drift to create an intuition. We refer to Figure 2 for an illustration of case \(|S| = 3\).

A good choice for this manifold is the one given by
\[
\mathcal{M}_A = \left\{ \eta \in \mathcal{H}_N^{x_0} : \sum_{x,y \in S \setminus \{x_0\}} a_{x,y} \eta_x \eta_y + \sum_{x \neq x_0} b_x \eta_x = A \right\}
\]

for appropriate coefficients. Each value of \( A \) gives a different manifold. The corresponding function should be constant on each manifold and a natural candidate emerges: \( F = F(\sum_{x,y \in S \setminus \{x_0\}} a_{x,y} \eta_x \eta_y + \sum_{x \neq x_0} b_x \eta_x) \).

This is how the function \( \bar{p} \), introduced in Lemma 12.8, emerges. By Proposition 12.7 and the proof of Proposition 12.14,
\[
(A_N P^{1/2})(\eta) \leq \frac{1}{\text{P}(\eta_{1/2})} \left\{ -1 + \sum_{x \in S_0} 1\{\eta_x = 1\} \right\},
\]
where \( S_0 = S \setminus \{x_0\} \). Thus, \( P^{1/2} \) is super-harmonic except when there is more than one coordinate with only one particle.
Figure 2. An illustration of the drift of the diffusion which approximates the zero-range dynamics when $N$ is large in the case where $S = \{1, 2, 3\}$. The red curve represents the manifold $M_A$.

To modify this function at the boundary, we introduce functions $P^A, A \subset S_0$, which, by the second assertion of Proposition 12.7, eliminate the positive part of $(A_N P^{1/2})(\eta)$ if the configuration $\eta$ has two or more particles at the sites in $A^c$. More precisely, in Section 12.4 and below, we prove that there exists a constant $c > 0$ such that

$$(A_N(P - P^A)^{1/2})(\eta) \leq \frac{-c}{P(\eta)^{1/2}}$$

for all $a > 0$, provided that $\eta_x \geq 2$ for all $x \in A^c$.

Therefore, the functions $(P - P^A)^{1/2}$ are super-harmonic in different regions of the space, and the union of these regions contains the annulus $H_{x0}^{x0}$. We use these functions to define one on $H_{y0}^{x0}$. The problem occurs at the boundary of these regions. This obstacle is circumvented by averaging these functions over the free constant $a$.

12.1. Potential theory of underlying random walk. Recall the definition of equilibrium potential (3.6) and the one of capacity (3.7) for the underlying random walk.

**Lemma 12.1.** Let $B$ be a non-empty subset of $S$ and let $x, y \in S \setminus B$. Then,

$$\frac{h_{x,B}(y)}{\text{cap}_X(x, B)} = \frac{h_{y,B}(x)}{\text{cap}_X(y, B)}.$$  

**Proof.** Recall that we denote by $P_x$ the probability on the path space $D(\mathbb{R}_+, S)$ induced by the random walk $X(t)$ starting from $x$, and by $E_x$ the expectation with respect to $P_x$.

By [6, Proposition 6.10],

$$E_x \left[ \int_0^t \chi(y)(X(t)) \, dt \right] = \frac{\langle \chi(y), h_{x,B} \rangle}{\text{cap}_X(x, B)} = \frac{m(y) h_{x,B}(y)}{\text{cap}_X(x, B)},$$

$$E_y \left[ \int_0^t \chi(x)(X(t)) \, dt \right] = \frac{\langle \chi(x), h_{y,B} \rangle}{\text{cap}_X(y, B)} = \frac{m(x) h_{y,B}(x)}{\text{cap}_X(y, B)}.$$
It remains to show that
\[ m(x) \mathbb{E}_x \left[ \int_0^{\tau_B} \chi(y)(X(t)) \, dt \right] = m(y) \mathbb{E}_y \left[ \int_0^{\tau_B} \chi(z)(X(t)) \, dt \right]. \]

Denote by \((Y(n))_{n \in \mathbb{N}}\) the embedded, discrete-time Markov chain. Recall that \(Y(n)\) is a \(S\)-valued chain which jumps from \(x\) to \(y\) with probability \(p(x, y) = r(x, y)/\lambda(x)\), where \(\lambda(x) = \sum_{y \in S} r(x, y)\), and that its invariant measure, denoted by \(\nu\), is given by \(\nu(x) = m(x) \lambda(x)\).

Let \(\epsilon_k, k \geq 0\), be a sequence of independent, mean-one exponential random variables, independent of the chain \(Y(n)\). Denote by \(\mathbb{E}^Y_{x, t}\) the expectation with respect to the chain \(Y(n)\) starting from \(x \in S\) and the sequence \((\epsilon_n)_{n \in \mathbb{N}}\). With this notation,
\[ \mathbb{E}_x \left[ \int_0^{\tau_B} \chi(y)(X(t)) \, dt \right] = \mathbb{E}^Y_{x, \tau_B} \left[ \sum_{n=0}^{\tau_B-1} 1\{Y(n) = y\} \frac{\epsilon_n}{\lambda(Y(n))} \right]. \]

Replacing in the denominator \(Y(n)\) by \(y\), and then integrating over \(\epsilon_k\), yields that the right-hand side is equal to
\[ \frac{1}{\lambda(y)} \mathbb{E}^Y_{x, \tau_B} \left[ \sum_{n=0}^{\tau_B-1} 1\{Y(n) = y, n < \tau_B\} \epsilon_n \right] = \frac{1}{\lambda(y)} \sum_{n=0}^{\infty} \mathbb{E}^Y_{x, \tau_B} \left[ Y(n) = y, n < \tau_B \right]. \]

We are left to show that for all \(n \geq 0\)
\[ M(x) \mathbb{P}^Y_{x, t} \left[ Y(n) = y, n < \tau_B \right] = M(y) \mathbb{P}^Y_{y, t} \left[ Y(n) = x, n < \tau_B \right], \]
which follows from the reversibility of the chain \(Y(n)\) with respect to the stationary measure \(M\).

Note that we did not use in this proof the fact that the stationary measure \(m\) of the random walk \(X\) is the uniform measure. This result holds for general reversible dynamics, and a version for non-reversible ones can be obtained along the same lines.

We conclude this section with an identity used many times in this article. Let \(A, B\) be two non-empty, disjoint subsets of \(S\). Since \(L_X h_{A, B} = -L_X h_{B, A}\), by the last displayed equation in the proof of [23, Lemma B9],
\[ \operatorname{cap}_S(A, B) = - \sum_{x \in A} m(x) (L_X h_{A, B})(x) = \sum_{x \in A} m(x) (L_X h_{B, A})(x). \]  

(12.1)

12.2. Coefficients of a quadratic function. The super-harmonic function is, essentially, the square root of a quadratic function. We introduce in this section the coefficients of this quadratic function.

Fix \(x_0 \in S\), and recall that \(S_0 = S \setminus \{x_0\}\). For each non-empty subset \(A\) of \(S_0\), define the coefficients \((b^A_{x, y})_{x, y \in S}\) by
\[ b^A_{x, y} = \frac{1}{\kappa} \frac{h_{x, A^c}(y)}{\operatorname{cap}_X(x, A^c)}, \quad x, y \in A, \]  

(12.2)

and let \(b^A_{x, y} = 0\) otherwise.

**Lemma 12.2.** For each non-empty subset \(A\) of \(S_0\) and for all \(x, y \in S\), \(b^A_{x, y} = b^A_{y, x}\).

**Proof.** For \(x, y \in A\) this identity follows from Lemma 12.1. If either \(x \notin A\) or \(y \notin A\) (or both), \(b^A_{x, y} = b^A_{y, x} = 0\) by definition. \(\square\)
We present below some properties of this sequence.

**Lemma 12.3.** For two non-empty subsets $A, B$ of $S_0$ satisfying $A \subset B$, $b^A_{x,y} \leq b^B_{x,y}$ for all $x, y \in S$.

**Proof.** As the coefficients are non-negative, it is enough to check the inequality for all $x, y \in A$. In this case, since the measure $m$ is the uniform measure, by [6, Proposition 6.10],

$$b^A_{x,y} = \frac{m(y) h_x, A^c(y)}{\cap_x(x, A^c)} = \mathbb{E}_x \left[ \int_0^{\tau_{A^c}} \chi(y)(X(t)) \, dt \right],$$

$$b^B_{x,y} = \frac{m(y) h_x, B^c(y)}{\cap_x(x, B^c)} = \mathbb{E}_x \left[ \int_0^{\tau_{B^c}} \chi(y)(X(t)) \, dt \right].$$

The first expectation is bounded by the second since $\tau_{A^c} \leq \tau_{B^c}$. □

For a non-empty subset $A$ of $S_0$, let $z^A_x, x \in S$, be given by

$$z^A_x = \frac{1}{2} \sum_{y \in S} r(x, y) \left( b^A_{y,x} + b^A_{y,y} - 2 b^A_{x,y} \right). \quad (12.3)$$

**Lemma 12.4.** For each non-empty $A \subset S_0$, we have that

$$\frac{1}{2} \sum_{y \in S} r(x, y) \left( b^A_{y,y} - b^A_{x,x} \right) = \begin{cases} \frac{z^A_x}{2} - 1 & \text{for } x \in A, \\ \frac{z^A_x}{2} & \text{for } x \in A^c. \end{cases}$$

**Proof.** If $x \in A^c$ the result follows because $b^A_{x,y} = 0$ for all $y \in S$. On the other hand, if $x \in A$, by (12.1)

$$\sum_{y \in S} r(x, y) \left( b^A_{y,x} - b^A_{x,y} \right) = \frac{m(x)}{\cap_x(x, A^c)} \sum_{y \in S} r(x, y) \left( h_x, A^c(x) - h_x, A^c(y) \right)$$

$$= - \frac{m(x) L_x h_x, A^c(x)}{\cap_x(x, A^c)} = 1.$$

Hence,

$$z^A_x - 1 = \frac{1}{2} \sum_{y \in S} r(x, y) \left( b^A_{y,x} + b^A_{y,y} - 2 b^A_{x,y} \right) - \sum_{y \in S} r(x, y) \left( b^A_{y,x} - b^A_{x,y} \right)$$

$$= \frac{1}{2} \sum_{y \in S} r(x, y) \left[ b^A_{y,y} - b^A_{x,x} \right],$$

as claimed. □

**12.3. Linear and quadratic functions.** For a subset $\mathcal{C}$ of $\mathcal{H}_N$, let

$$\text{int} \mathcal{C} = \{ \eta \in \mathcal{C} : \sigma^{x,y} \eta \in \mathcal{C} \text{ for all } x, y \text{ with } r(x, y) > 0 \},$$

$$\partial \mathcal{C} = \mathcal{C} \setminus \text{int} \mathcal{C},$$

$$\overline{\mathcal{C}} = \{ \eta \in \mathcal{H}_N : \eta \in \mathcal{C} \text{ or } \sigma^{x,y} \eta \in \mathcal{C} \text{ for some } x, y \text{ with } r(x, y) > 0 \}.$$

To prove Theorem 11.2, it suffices to construct a function $\Gamma_N^{\infty}$ on $\overline{W_N^{\infty}} \setminus D_N^{\infty}$, satisfying the conditions of the proposition in the set $W_N^{\infty} \setminus D_N^{\infty}$, and to extend it arbitrarily to $\mathcal{H}_N$.

Let

$$\mathcal{U}_N^{\infty} = \overline{W_N^{\infty}} \setminus D_N^{\infty} \text{ so that } \text{int} \mathcal{U}_N^{\infty} = W_N^{\infty} \setminus D_N^{\infty}.$$
Define the quadratic function $Q^A : \mathcal{U}_N^S \to \mathbb{R}$, $A \subset S_0$, and the linear function $U^A : \mathcal{U}_N^S \to \mathbb{R}$ as

$$Q^A(\eta) = \frac{1}{2} \sum_{x,y \in S} b_{x,y} \eta_x \eta_y = \frac{1}{2} \sum_{x \in A} b_{x,x} \eta_x^2 + \sum_{\{x,y\} \subset A} b_{x,y} \eta_x \eta_y,$$

$$U^A(\eta) = \frac{1}{2} \sum_{x \in S} b_{x,x} \eta_x = \frac{1}{2} \sum_{x \in A} b_{x,x} \eta_x.$$

In the last sum of the first line, each pair $\{x,y\}$ appears only once. Let

$$P^A(\eta) = Q^A(\eta) - U^A(\eta) = \frac{1}{2} \sum_{x \in A} b_{x,x} \eta_x (\eta_x - 1) + \sum_{\{x,y\} \subset A} b_{x,y} \eta_x \eta_y.$$  

Note that $P^\emptyset(\eta) = 0$ for all $\eta$.

Fix $A \subset S_0$, $x \in S$ and $\eta \in \text{int} \mathcal{U}_N^S$ such that $\eta_x \geq 1$. An elementary computation yields that

$$U^A(\sigma^x \eta) - U^A(\eta) = \frac{1}{2} \left\{ b_{y,y} 1\{y \in A\} - b_{x,x} 1\{x \in A\} \right\}. \quad (12.4)$$

Lemma 12.5. For $x \in A$, $y \in S \setminus \{x\}$, and $\eta \in \text{int} \mathcal{U}_N^S$,

$$Q^A(\sigma^x \eta) - Q^A(\eta) = \sum_{z \in A} \eta_z \left[ b_{z,y} - b_{z,x} \right] + \frac{1}{2} \left[ b_{x,x}^A + b_{y,y}^A - 2 b_{x,y}^A \right].$$

Proof. First, fix $y \in A$, $y \neq x$. By definition of $Q_A$, $Q^A(\sigma^x \eta) - Q^A(\eta)$ is equal to

$$\frac{1}{2} b_{x,x}^A \left[ (\eta_x - 1)^2 - \eta_x^2 \right] + \frac{1}{2} b_{y,y}^A \left[ (\eta_y + 1)^2 - \eta_y^2 \right]$$

$$+ b_{x,y}^A \left[ (\eta_x - 1)(\eta_y + 1) - \eta_x \eta_y \right] + \sum_{z \in A \setminus \{x,y\}} b_{z,z}^A \left[ (\eta_z - 1) \eta_z - \eta_x \eta_z \right]$$

$$+ \sum_{z \in A \setminus \{x,y\}} b_{z,y}^A \left[ \eta_z (\eta_y + 1) - \eta_x \eta_y \right].$$

We may rewrite this sum as

$$\frac{1}{2} b_{x,x}^A \left[ 1 - 2 \eta_x \right] + \frac{1}{2} b_{y,y}^A \left[ 2 \eta_y + 1 \right] + b_{x,y}^A \left[ \eta_x - \eta_y - 1 \right]$$

$$+ \sum_{z \in A \setminus \{x,y\}} \eta_z \left[ b_{z,y}^A - b_{z,x}^A \right].$$

Since $b_{z,w}^A$ is symmetric by Lemma 12.2, if $y \in A$,

$$Q^A(\sigma^x \eta) - Q^A(\eta) = \sum_{z \in A} \eta_z \left[ b_{z,y}^A - b_{z,x}^A \right] + \frac{1}{2} \left[ b_{x,x}^A + b_{y,y}^A - 2 b_{x,y}^A \right],$$

as claimed.

Assume now that $y$ belongs to $A^c$. In this case, by definition of $Q_A$, $Q^A(\sigma^x \eta) - Q^A(\eta)$ is equal to

$$\frac{1}{2} b_{x,x}^A \left[ (\eta_x - 1)^2 - \eta_x^2 \right] + \sum_{z \in A \setminus \{x\}} b_{z,z}^A \left[ (\eta_z - 1) \eta_z - \eta_x \eta_z \right] = - \sum_{z \in A} b_{z,x}^A \eta_z + \frac{1}{2} b_{z,x}^A.$$

To complete the proof, it remains to recall that $b_{z,y} = 0$ for all $z \in S$. \qed
Fix \( A \subset S_0 \), \( x \notin A \), \( y \neq x \) and \( \eta \in \text{int} \, \mathcal{U}_N^0 \) such that \( \eta_x \geq 1 \). A similar computation yields that

\[
Q^A(\sigma^{x,y} \eta) - Q^A(\eta) = \left\{ \frac{1}{2} (2\eta_y + 1) b^A_{y,y} + \sum_{z \in A, z \neq y} b^A_{y,z} \eta_z \right\} 1_{\{ y \in A \}}.
\]

It follows from (12.4), Lemma 12.5 and the previous estimate that there exists a constant \( C_0 \) such that

\[
| P^A(\sigma^{x,y} \eta) - P^A(\eta) | \leq C_0 \left\{ 1 + \sum_{z \in A} \eta_z \right\}
\]

for all subsets \( A \) of \( S_0 \), \( x, y \in S \), \( y \neq x \) and \( \eta \in \text{int} \, \mathcal{U}_N^0 \) such that \( \eta_x \geq 1 \).

Let \( u^A_{x,y} \), \( x \in A \), \( y \in A^c \), be given by

\[
u^A_{x,y} = \frac{m(x) \left( L_N h_{x,A^c}(y) \right)}{\text{cap}_X(x, A^c)}.
\]

Since \( m(x) = m(y) \) and \( L_N h_{x,A^c} = -L_N h_{A^c,x} \), by (12.1),

\[
\sum_{y \in A^c} u^A_{x,y} = 1, \quad \text{for all } x \in A.
\]

Observe that this identity holds only because \( m \) is the uniform measure, as we replaced \( m(x) \) by \( m(y) \).

**Lemma 12.6.** Fix \( A \subset S_0 \) and \( \eta \in \text{int} \, \mathcal{U}_N^0 \). If \( x \in A \) and \( \eta_x \geq 1 \), then

\[
\sum_{y \in S} r(x, y) \left[ P^A(\sigma^{x,y} \eta) - P^A(\eta) \right] = -\eta_x + 1.
\]

On the other hand, if \( x \in A^c \) and \( \eta_x \geq 1 \), then

\[
\sum_{y \in S} r(x, y) \left[ P^A(\sigma^{x,y} \eta) - P^A(\eta) \right] = \sum_{z \in A} u^A_{z,x} \eta_z.
\]

In particular, for \( A = S_0 \),

\[
\sum_{y \in S} r(x_0, y) \left[ P^{S_0}(\sigma^{x_0,y} \eta) - P^{S_0}(\eta) \right] = \sum_{z \in S_0} \eta_z.
\]

**Proof.** We consider \( Q^A \) and \( U^A \) separately. By Lemma 12.5,

\[
\sum_{y \in S} r(x, y) \left[ Q^A(\sigma^{x,y} \eta) - Q^A(\eta) \right]
\]

\[
eq \sum_{y \in S} r(x, y) \left( \sum_{z \in A} \eta_z \left[ b^A_{z,y} - b^A_{z,x} \right] + \frac{1}{2} \left[ b^A_{x,y} + b^A_{y,x} - 2 b^A_{y,y} \right] \right).
\]

By definition of \( b^A_{x,y} \) and \( z^A_x \) given in (12.2) and (12.3), respectively, and by changing the order of summation yield that the previous expression is equal to

\[
\sum_{z \in A} \eta_z \frac{m(z)}{\text{cap}_X(z, A^c)} \sum_{y \in S} r(x, y) \left[ h_{z,A^c}(y) - h_{z,A^c}(x) \right] + z^A_x.
\]

Thus, by definition of \( u^A_{x,y} \), introduced in (12.6), we have that

\[
\sum_{y \in S} r(x, y) \left[ Q^A(\sigma^{x,y} \eta) - Q^A(\eta) \right] = \sum_{z \in A} \eta_z u^A_{z,x} + z^A_x.
\]
On the other hand, by definition of $U^A$, and since $\eta_x \geq 1$,
\[
\sum_{y \in S} r(x, y) \left[ U^A(\sigma^x y \eta) - U^A(\eta) \right] = \frac{1}{2} \sum_{y \in S} r(x, y) \left[ b_{y, y}^A - b_{x, x}^A \right].
\]
Assume that $x \in A$ and $\eta_x \geq 1$. In this case, by definition of $u^A_{z, x}$ and (12.1),
\[
u^A_{z, x} = 0 \text{ for } z \in A \setminus \{x\} \text{ and } u^A_{x, x} = -1.
\]
Therefore, in this case,
\[
\sum_{y \in S} r(x, y) \left[ Q^A(\sigma^x y \eta) - Q^A(\eta) \right] = -\eta_x + z^A_x.
\]
Moreover, by Lemma 12.4,
\[
\sum_{y \in S} r(x, y) \left[ U^A(\sigma^x y \eta) - U^A(\eta) \right] = z^A_x - 1.
\]
This completes the first part of the proof, in view of the definition of $P^A$.
Assume that $x \in A^c$ and $\eta_x \geq 1$. In this case, by Lemma 12.4,
\[
\sum_{y \in S} r(x, y) \left[ U^A(\sigma^x y \eta) - U^A(\eta) \right] = z^A_x.
\]
This identity together with (12.8) completes the proof of the second assertion of the lemma.

For the last assertion of the lemma, we have to check that $u^{S_0}_{z, x_0} = 1$ for all $z \in S_0$. By (12.6), and since $h_{z, x_0} = 1 - h_{x_0, z}$ and $m(z) = m(x_0)$,
\[
u^{S_0}_{z, x_0} = m(z) \frac{L_X h_{z, x_0}(x_0)}{\text{cap}_X(z, x_0)} - m(x_0) \frac{L_X h_{x_0, z}(x_0)}{\text{cap}_X(z, x_0)}.
\]
By (12.1), this expression is equal to 1, which completes the proof of the lemma. \hfill \Box

The next result is a consequence of Lemma 12.6. Fix a function $J : U^{S_0}_N \to \mathbb{R}$. In the remaining part of the current section, we write $J(\eta) = o_N(1)$ if
\[
\limsup_{N \to \infty} \sup_{\eta \in U^{S_0}_N} |J(\eta)| = 0.
\]

**Proposition 12.7.** Fix a non-empty subset $A$ of $S_0$ and $\eta \in \text{int } U^{S_0}_N$. Then,
\[
(A_N P^A)(\eta) = \sum_{x \in A} g(\eta_x) [1 - \eta_x] + \sum_{x \in A^c} g(\eta_x) \sum_{z \in A} u^A_{z, x} \eta_z.
\]
If $\eta_x \geq 2$ for all $x \in A^c$, then
\[
(A_N P^A)(\eta) \geq \sum_{x \in A} 1 \{ \eta_x = 1 \}.
\]
Finally, if $A = S_0$,
\[
(A_N P^{S_0})(\eta) = \sum_{x \in S_0} 1 \{ \eta_x = 1 \} + o_N(1).
\]

**Proof.** The first assertion is a consequence of Lemma 12.6. For the second one, since $\eta_x \geq 2$ for all $x \in A^c$, we obtain from the first part that
\[
(A_N P^A)(\eta) \geq \sum_{x \in A} g(\eta_x) [1 - \eta_x] + \sum_{x \in A^c} \sum_{z \in A} u^A_{z, x} \eta_z \tag{12.9}
\]
because \( g(\eta_x) \geq 1 \) for \( x \in A^c \). By (12.7), the second term on the right-hand side is equal to \( \sum_{x \in A} \eta_x \), so that

\[
(A_N P^A)(\eta) \geq \sum_{x \in A} \{ \eta_x - g(\eta_x) [\eta_x - 1] \} = \sum_{x \in A} 1 \{ \eta_x = 1 \},
\]

because \( n - g(n)(n-1) = 1 \{ n = 1 \} \). This proves the second assertion of the proposition.

We turn to the last claim. By the first assertion of this proposition and the last one of Lemma 12.6,

\[
(A_N P^{S_0})(\eta) = \sum_{x \in S_0} g(\eta_x) \{ 1 - \eta_x \} + g(\eta_{x_0}) \sum_{z \in S_0} \eta_z.
\]

As \( \eta \) belongs to \( \text{int} U_N \), \( \sum_{z \in S_0} \eta_z/|\eta_{x_0} - 1| = o_N(1) \). Hence, writing \( g(\eta_{x_0}) \) as \( 1 + [\eta_{x_0} - 1] - 1 \), the previous identity becomes

\[
(A_N P^{S_0})(\eta) = \sum_{x \in S_0} \{ \eta_x - g(\eta_x) [\eta_x - 1] \} + o_N(1) .
\]

To complete the argument, it remains to recall that \( n - g(n)(n-1) = 1 \{ n = 1 \} \). □

Let us write \( P = P^{S_0} \).

**Lemma 12.8.** There exist constants \( c_1, c_2 > 0 \) such that

\[
c_1 \left( \sum_{x \in S_0} \eta_x \right)^2 \leq P(\eta) \leq c_2 \left( \sum_{x \in S_0} \eta_x \right)^2
\]

for all \( \eta \in U_N^{x_0} \).

**Proof.** The upper bound follows from the definition of \( P \). To prove the lower bound, note that there exists constants \( 0 < \lambda < A < \infty \) such that \( \lambda < b_{x,y}^{S_0} < A \) for all \( x \in S_0 \). Thus, by definition of \( P \), since \( b_{x,y}^{S_0} \geq 0 \) for all \( x, y \in S, \) and by the Cauchy-Schwarz inequality,

\[
P(\eta) \geq \frac{\lambda}{2} \sum_{x \in S_0} \eta_x^2 - \frac{A}{2} \sum_{x \in S_0} \eta_x \geq \frac{\lambda}{2(\kappa - 1)} \left( \sum_{x \in S_0} \eta_x \right)^2 - \frac{A}{2} \sum_{x \in S_0} \eta_x .
\]

To complete the proof, it remains to recall that \( \sum_{x \in S_0} \eta_x \geq N^\gamma \gg 1 \) for \( \eta \in U_N^{x_0} \). □

**Lemma 12.9.** There exists a positive constant \( c_0 > 0 \) such that

\[
\sum_{x \in S} g(\eta_x) \sum_{y \in S} r(x, y) [P(\sigma^y \eta) - P(\eta)]^2 \geq c_0 P(\eta)
\]

for all \( \eta \in \text{int} U_N^{x_0} \).

**Proof.** Since \( g(\eta_{x_0}) \geq 1 \), it suffices to show that

\[
\sum_{y \in S} r(x_0, y) [P(\sigma^{x_0} y \eta) - P(\eta)]^2 \geq c_0 P(\eta) .
\]
By the Cauchy-Schwarz inequality,
\[
\sum_{y \in S} r(x_0, y) \sum_{y \in S} r(x_0, y) \left[ P(\sigma^{x_0,y} \eta) - P(\eta) \right]^2 \\
\geq \left( \sum_{y \in S} r(x_0, y) \left[ P(\sigma^{x_0,y} \eta) - P(\eta) \right] \right)^2 = \left( \sum_{z \in S_0} \eta_z \right)^2,
\]
where the last identity follows from the third assertion of Lemma 12.6. To complete the proof, recall the upper bound of Lemma 12.8. \qed

12.4. A super-harmonic function. Some notations are required. We first claim that for each non-empty subset \( B \) of \( S_0 \), there exist positive constants \( \alpha_B, \beta_B > 0 \) such that
\[
\frac{1}{2} \sum_{x \in B_0} b_{x,x}^B \eta_x (\eta_x - 1) + \sum_{\{x,y\} \in B_0} b_{x,y}^B \eta_x \eta_y < \alpha_B \sum_{x \in B_0} \eta_x (\eta_x - 1) + \beta_B \tag{12.10}
\]
for all \( \emptyset \neq B_0 \subset B \) and for all \( \eta \in \mathcal{U}_N^{x_0} \).

To prove this claim, let \( a = \max \{ b_{x,x}^A \} \), where the maximum is performed over all nonempty subsets \( C \) of \( S_0 \), and all \( x, w \in C \). Clearly, there exists a finite constant \( C_0 \), depending only on \( \kappa \), such that the left-hand side of (12.10) is bounded by
\[
C_0 a \sum_{x \in B_0} \eta_x^2 \leq 2C_0 a \sum_{x \in B_0} \eta_x (\eta_x - 1) + C_0 \kappa a,
\]
because \( t^2 \leq 2t(t-1) + 1 \). This proves the claim. Clearly, we may assume that
\[
\alpha_B > b_{x,x}^B \quad \text{for all} \ x \in B.
\]

We assign a positive constant \( c_A > 0 \) to each proper, non-empty subset \( A \) of \( S_0 \), i.e., \( \emptyset \subset A \subset S_0 \), as follows.

If \( A \) is a singleton, \( |A| = 1 \), set \( c_A > 0 \) arbitrarily. Fix \( 2 \leq k \leq |S_0| - 1 \), and suppose that \( c_A \) has been assigned to all sets \( A \neq \emptyset \) such that \( |A| < k \). Fix a subset \( B \) of \( S_0 \) such that \( |B| = k \), and let
\[
c_B^0 = \max_A \max_{x \in A} \left\{ \frac{2 \alpha_B c_A}{b_{x,x}^A} + \beta_B \right\},
\]
where the first maximum is performed over all proper, non-empty subsets \( A \) of \( B \). Then, select a constant \( c_B \) larger than \( c_B^0 \) and such that
\[
c_A \neq c_B \quad \text{for all} \ A, B \subset S_0 \text{ with } A \neq B.
\]

Fix a positive integer \( \ell \geq 2 \). For each proper, non-empty subset \( A \) of \( S_0 \), let \( P_\ell^A : \mathcal{U}_N^{x_0} \to \mathbb{R} \) be given by
\[
P_\ell^A(\eta) = P^A(\eta) - c_A \ell^2.
\]

Clearly,
\[
(\Lambda_N P_\ell^A)(\eta) = (\Lambda_N P^A)(\eta) \quad \text{for all} \ \eta \in \mathcal{U}_N^{x_0}.
\]

Let \( P_\ell^\varnothing(\eta) = 0 \) for all \( \eta \in \mathcal{U}_N^{x_0} \), and define the correction function \( W_\ell : \mathcal{U}_N^{x_0} \to \mathbb{R} \) by
\[
W_\ell(\eta) = \min \{ P_\ell^A(\eta) : A \subset S_0, A \neq S_0 \}.
\]

Lemma 12.10. There exists a constant \( C_0 < \infty \) such that, for all \( \eta \in \mathcal{U}_N^{x_0} \),
\[
-C_0 \ell^2 \leq W_\ell(\eta) \leq 0.
\]
Proof. Since \( W_\ell(\eta) \leq P_\ell^{S_0}(\eta) \leq 0 \), the upper bound is clear. We turn to the lower bound. Since \( P^A \), \( A \subsetneq S_0 \), is a non-negative function, \( P^A_\ell(\eta) \geq -c_A \ell^2 \). It remains to set \( C_0 \) as \( \max_{A \subseteq S_0} c_A \).

By Lemmata 12.8 and 12.10, we get \( P(\eta) - W_\ell(\eta) > 0 \) for all \( \eta \in \mathcal{U}_N^{S_0} \). Let \( F_m : \mathcal{U}_N^{S_0} \to \mathbb{R} \), \( m \geq 2 \), be defined by

\[
F_m(\eta) = \sum_{\ell=2}^m \frac{1}{\ell} [P(\eta) - W_\ell(\eta)]^{1/2}.
\]

**Theorem 12.11.** There exists \( m \in \mathbb{N} \), \( c_0 > 0 \) and \( N_0 \geq 1 \), such that for all \( N \geq N_0 \), the function \( F_m \) satisfies

\[
(A_N F_m)(\eta) \leq -\frac{c_0}{N - \eta_{S_0}} \quad \text{for all } \eta \in \text{int } \mathcal{U}_N^{S_0}.
\]

The proof of this theorem is given in Section 12.8. This result, as well as the majority of the next ones, are asymptotic in \( N \). This means that they may fail for small \( N \), but that there exists a constant \( N_0 \), which may depend only on \( \kappa \) and \( \ell \), such that the assertion holds for \( N > N_0 \).

**Proof of Theorem 11.2.** Let

\[
C_{N_0}^{S_0}(\eta) = \begin{cases} F_m(\eta) & \eta \in \mathcal{U}_N^{S_0}, \\ 0 & \text{otherwise}. \end{cases}
\]

The requirement (11.1) follows from Theorem 12.11 and the fact that \( \mathcal{L}_N = \theta_N A_N \), while (11.2) follows from Lemmata 12.8 and 12.10.

12.5. The corrector \( W_\ell \). Let \( \mathcal{D}_\ell(A) \), \( A \subsetneq S_0 \), be the set given by

\[
\mathcal{D}_\ell(A) = \{ \eta \in \mathcal{U}_N^{S_0} : P_\ell^A(\eta) = W_\ell(\eta) \}.
\] (12.15)

Note that some configurations may belong to several \( \mathcal{D}_\ell(A) \)'s.

Let \( A_0 \) be a proper, non-empty subset of \( S_0 \) and let \( \eta \) be a configuration in \( \mathcal{U}_N^{S_0} \) such that \( \eta_x = 0 \) for all \( x \in A_0 \). The next lemma states that \( W_\ell(\eta) = P_\ell^A(\eta) \) for some \( A \) which contains \( A_0 \).

**Lemma 12.12.** Fix a proper, non-empty subset \( A_0 \) of \( S_0 \) and \( \eta \) in \( \mathcal{U}_N^{S_0} \) such that \( \eta_x = 0 \) for all \( x \in A_0 \). Suppose that

\[
P_\ell^A(\eta) = W_\ell(\eta)
\]

for some \( A \subsetneq S_0 \). Then, \( A \supseteq A_0 \) provided \( N \) is large enough.

**Proof.** Fix \( A \subsetneq S_0 \), and assume that

\[
P_\ell^A(\eta) = W_\ell(\eta) = \min\{P_\ell^B(\eta) : B \subsetneq S_0\}.
\]

In particular, since \( \eta_x = 0 \) for all \( x \in A_0 \),

\[
P_\ell^A(\eta) \leq P_\ell^{A_0}(\eta) = -c_{A_0} \ell^2 < 0.
\] (12.16)

We consider separately three cases.

Suppose that \( A \subsetneq A_0 \). By definition, \( c_{A_0} > c_{A_0}^{b_0} \). By (12.12) and Lemma 12.3, this constant is larger than \( 2c_{A_0} c_A b_x^A \). By (12.11), we get \( 2c_{A_0} c_A b_x^{A_0} \geq 2c_A \geq c_A \). This proves that \( c_{A_0} > c_A \). Thus, as \( \eta_x = 0 \) for all \( x \in A_0 \),

\[
P_\ell^A(\eta) = -c_A \ell^2 > -c_{A_0} \ell^2 = P_\ell^{A_0}(\eta).
\]
in contradiction with (12.16).

Assume that \( A_0 \not\subseteq A \), \( A \not\subseteq A_0 \) and \( A_0 \cup A \neq S_0 \). We claim that

\[
P_{A_0 \cup A}(\eta) < P^A_{\ell}(\eta) .
\]  
(12.17)

Since \( \eta_x = 0 \) for all \( x \in A_0 \) and since \( A_0 \cup A \neq S_0 \),

\[
P^A_{\ell}(\eta) = \frac{1}{2} \sum_{x \in A \setminus A_0} b^A_{x,x} \eta_x (\eta_x - 1) + \sum_{x, y \in A \setminus A_0} b^A_{x,y} \eta_x \eta_y - c_A \ell^2 ,
\]

\[
P_{A_0 \cup A}(\eta) = \frac{1}{2} \sum_{x \in A \setminus A_0} b^A_{x,x} \eta_x (\eta_x - 1) + \sum_{x, y \in A \setminus A_0} b^{A_0 \cup A}_{x,y} \eta_x \eta_y - c_{A_0 \cup A} \ell^2 .
\]

These sums are carried over a set which is not empty because we assumed that \( A \not\subseteq A_0 \). Let

\[
M = \max_{x \in A} \frac{\alpha_{A_0 \cup A}}{b^A_{x,x}} .
\]

By Lemma 12.3 and (12.11), \( M > 1 \).

By (12.10) and the explicit formula for \( P^A_{A_0 \cup A}(\eta) \),

\[
P^A_{A_0 \cup A}(\eta) < \alpha_{A_0 \cup A} \sum_{x \in A \setminus A_0} \eta_x (\eta_x - 1) + \beta_{A_0 \cup A} - c_{A_0 \cup A} \ell^2 .
\]

By definition of \( M \), this expression is bounded by

\[
M \sum_{x \in A \setminus A_0} b^A_{x,x} \eta_x (\eta_x - 1) + \beta_{A_0 \cup A} - c_{A_0 \cup A} \ell^2 .
\]

By definition of \( P^A_{\ell} \), this sum is less than or equal to

\[
2 M P^A_{\ell}(\eta) + 2 M c_A \ell^2 + \beta_{A_0 \cup A} - c_{A_0 \cup A} \ell^2 .
\]

By definition, \( c_{A_0 \cup A} > c^0_{A_0 \cup A} \). Since \( A_0 \not\subseteq A \), \( A \not\subseteq A_0 \cup A \). Thus, by (12.12) and by definition of \( M \), \( c^A_{A_0 \cup A} \geq 2 M c_A + \beta_{A_0 \cup A} \). Hence, by the previous estimates, and since \( \ell \geq 1 \),

\[
P^A_{A_0 \cup A}(\eta) < 2 M P^A_{\ell}(\eta) \leq P^A_{\ell}(\eta)
\]

because \( M > 1 \) and \( P^A_{\ell}(\eta) < 0 \) by (12.16). This proves (12.17) and contradicts the fact that \( P^A_{\ell}(\eta) = W_{\ell}(\eta) \).

Assume, finally, that \( A_0 \cup A = S_0 \). Since both are proper subsets of \( S_0 \), \( A_0 \not\subseteq A \) and \( A \not\subseteq A_0 \).

The set \( S_0 \) can be decomposed into \( S_0 = A_0 \cup (A \setminus A_0) \). Since \( \eta \in U_{N0}^S \), and \( \eta_x = 0 \) for all \( x \in A_0 \),

\[
\sum_{x \in A \setminus A_0} \eta_x = \sum_{x \in S_0} \eta_x \geq N^\gamma .
\]

Since \( b^A_{x,x} > 0 \) for all \( x \in A \), a similar computation to the one presented in the proof of Lemma 12.8 yields that

\[
\frac{1}{2} \sum_{x \in A \setminus A_0} b^A_{x,x} \eta_x (\eta_x - 1) \geq c_0 N^{2\gamma}
\]

for some positive constant \( c_0 \). Thus,

\[
P^A_{\ell}(x) > c_0 N^{2\gamma} - c_A \ell^2 \geq 0
\]

for large enough \( N \), which contradicts (12.16).
In conclusion, none of the previous three cases can be in force, so that $A \supset A_0$, as claimed.

**Corollary 12.13.** Fix a proper, non-empty subset $A$ of $S_0$. Then, for all $x \in S_0 \setminus A$, $\eta \in \mathcal{D}_\ell(A)$, we have that $\eta_x \neq 0$.

**Proof.** Fix a proper, non-empty subset $A$ of $S_0$ and $\eta \in \mathcal{D}_\ell(A)$. Let $A_0 = \{ x \in S_0 : \eta_x = 0 \}$.

As $\eta \in \mathcal{D}_\ell(A)$, $P^A_\ell(\eta) = W_\ell(\eta)$. Hence, by Lemma 12.12, $A_0 \subseteq A$, as claimed. \qed

**12.6. The set $\mathcal{D}_\ell(A)$.** The crucial point in the proof of Theorem 12.11 is to estimate $W_\ell$. This is relatively easy in each set $\mathcal{D}_\ell(A)$ because $W_\ell$ is equal to $P^A_\ell$.

In contrast, its behavior at the boundary $\partial \mathcal{D}_\ell(A)$ is problematic.

The next result states that $\sum_{x \in A} \eta_x$ can not be too large for configurations $\eta$ in $\mathcal{D}_\ell(A)$.

**Proposition 12.14.** There exists $\gamma_1 > 0$ such that, for all proper, non-empty subsets $A$ of $S_0$,

$$\mathcal{D}_\ell(A) \subset \mathcal{G}^{\ell_1}(A) := \{ \eta \in \mathcal{U}^{x_0}_N : \eta_x < \gamma_1 \ell \text{ for all } x \in A \}.$$

**Proof.** Fix $\eta \in \mathcal{D}_\ell(A)$. By definition of $\mathcal{D}_\ell(A)$,

$$P^A_\ell(\eta) \leq P^\mathcal{G}(\eta) = 0.$$

On the other hand, by definition of $P^A_\ell$, there exists $\gamma_A > 0$ such that

$$P^A_\ell(\xi) > 0 \text{ if } \xi_x \geq \gamma_A \ell \text{ for some } x \in A.$$

It follows from the two previous remarks that $\mathcal{D}_\ell(A) \subset \mathcal{G}^{\gamma_A}(A)$. To complete the proof, it remains to set $\gamma_1 = \max \gamma_A$. \qed

**Proposition 12.15.** Fix a proper, non-empty subset $A$ of $S_0$ and $\eta \in \text{int } \mathcal{D}_\ell(A)$. Then,

$$(A_N W_\ell)(\eta) = (A_N P^A_\ell)(\eta) \geq \sum_{x \in S_0} 1\{ \eta_x = 1 \}.$$

**Proof.** Fix $\eta \in \text{int } \mathcal{D}_\ell(A)$, so that

$$W_\ell(\eta) = P^A_\ell(\eta) \text{ and } W_\ell(\sigma^{x,y}\eta) = P^A_\ell(\sigma^{x,y}\eta)$$

for all $x, y$ in $S$ with $r(x, y) > 0$. Thus,

$$(A_N W_\ell)(\eta) = (A_N P^A_\ell)(\eta) = (A_N P^A_\ell)(\eta).$$

We turn to the second assertion. By Corollary 12.13, $\eta_x \neq 0$ for all $x \in S_0 \setminus A$. If $\eta_x = 1$ for some $x \in S_0 \setminus A$, by Corollary 12.13, $\sigma^{x,y}\eta \notin \mathcal{D}_\ell(A)$ for any $y \in S$ with $r(x, y) > 0$, so that $\eta \notin \text{int } \mathcal{D}_\ell(A)$ as well. Therefore, $\eta_x \geq 2$ for all $x \in S_0 \setminus A$, and the second claim follows from the second assertion of Proposition 12.7. \qed

**Lemma 12.16.** Fix $x \neq y \in S$, and proper subsets $A, B$ of $S_0$, $A \neq B$. There exists a constant $C_0 > 0$ such that

$$|P^B_\ell(\eta) - P^A_\ell(\eta)| \leq C_0 \ell \text{ and } |P^B_\ell(\sigma^{x,y}\eta) - P^A_\ell(\sigma^{x,y}\eta)| \leq C_0 \ell$$

for all $\eta \in \mathcal{U}^{x_0}_N$ such that $\eta \in \mathcal{D}_\ell(A)$ and $\sigma^{x,y}\eta \in \mathcal{D}_\ell(B)$.
Proof. Since the proof for these two estimates are identical, we only focus on the first one. We regard $P^A_\ell$ and $P^B_\ell$ as quadratic functions on $\mathbb{R}^{n-1}$ whose restriction to $\mathbb{N}^{n-1}$ is given by (12.14).

As $\eta$ belongs to $\mathcal{D}_\ell(A)$ and $\sigma^{x,y}\eta$ to $\mathcal{D}_\ell(B)$,

$$P^A_\ell(\eta) \leq P^B_\ell(\eta) \quad P^A_\ell(\sigma^{x,y}\eta) \geq P^B_\ell(\sigma^{x,y}\eta).$$

Hence, by the intermediate value theorem, there exists $w_0 \in \mathbb{R}^{n-1}$ belonging to the line segment connecting $\eta$ and $\sigma^{x,y}\eta$ such that

$$(P^A_\ell - P^B_\ell)(w_0) = 0.$$

Since

$$|\eta - w_0| \leq |\eta - \sigma^{x,y}\eta| = \sqrt{2},$$

by the Taylor expansion, there exists a finite constant $C_0$ such that

$$|P^A_\ell(\eta) - P^A_\ell(w_0)| \leq C_0 \{ |\nabla P^A_\ell(\eta)| + \|\nabla^2 P^A_\ell\|_{L^{\infty}(\mathbb{R}^{n-1})} \}.$$

As $P^A_\ell$ is a quadratic function, $\|\nabla^2 P^A_\ell\|_{L^{\infty}(\mathbb{R}^{n-1})} \leq C_0$. On the other hand, since $\eta$ belongs to $\mathcal{D}_\ell(A)$, by Proposition 12.14, $\eta_z \leq \gamma_1 \ell$ for all $z \in A$. Hence, there exists a finite constant $C_0$ such that $|\nabla P^A_\ell(\eta)| < C_0 \ell$, and the previous displayed equation becomes

$$|P^A_\ell(\eta) - P^A_\ell(w_0)| \leq C_0 \ell. \quad (12.18)$$

To use the same argument to estimate $P^B_\ell(\eta) - P^B_\ell(w_0)$ we only need to show that $\eta_z \leq C_0 \ell$ for all $z \in B$. Since $\sigma^{x,y}\eta \in \mathcal{D}_\ell(B)$, by Proposition 12.14, $(\sigma^{x,y}\eta)_z \leq \gamma_1 \ell$ for all $z \in B$. Thus, as $|((\sigma^{x,y}\eta)_z - \eta_z| \leq 1$, $\eta_z \leq (\sigma^{x,y}\eta)_z + 1 \leq \gamma_1 \ell + 1$ for all $z \in B$. This proves (12.18) with $A$ replaced by $B$.

Putting together the previous estimates yields that

$$|P^A_\ell(\eta) - P^A_\ell(\eta)| \leq |P^A_\ell(\eta) - P^A_\ell(w_0)| + |P^B_\ell(\eta) - P^B_\ell(w_0)| \leq C_0 \ell,$$

as claimed. \hfill $\Box$

Lemma 12.17. If $\eta$ belongs to $\partial \mathcal{D}_\ell(A)$, there exists a constant $C_0$ such that

$$|\langle A_N W_\ell(\eta) \rangle - \langle A_N P^A_\ell(\eta) \rangle| \leq C_0 \ell.$$

Proof. Assume that $\eta$ belongs to $\partial \mathcal{D}_\ell(A)$. It is enough to show that there exists a constant $C_0$ such that for all $\eta \in \partial \mathcal{D}_\ell(A)$ and $x, y \in S$ with $r(x, y) > 0$,

$$|W_\ell(\sigma^{x,y}\eta) - P^A_\ell(\sigma^{x,y}\eta)| \leq C_0 \ell.$$

This inequality holds clearly when $\sigma^{x,y}\eta \in \mathcal{D}_\ell(A)$. Assume that $\sigma^{x,y}\eta \in \mathcal{D}_\ell(B)$ for some $B \neq A$. Then,

$$|W_\ell(\sigma^{x,y}\eta) - P^A_\ell(\sigma^{x,y}\eta)| = |P^B_\ell(\sigma^{x,y}\eta) - P^A_\ell(\sigma^{x,y}\eta)|.$$

By Lemma 12.16, this quantity is bounded by $C_0 \ell$. \hfill $\Box$

The following proposition is crucial in the proof of Theorem 12.11. It is here that condition (12.13) plays a role. Let

$$\partial \mathcal{D}_\ell = \bigcup_{A \subseteq S_0} \partial \mathcal{D}_\ell(A).$$
Proposition 12.18. There exists a constant $\gamma_2 > 0$ such that, for all $\eta \in \mathcal{U}_N^{x_0}$:

$$\sum_{\ell \geq 2} 1\{\eta \in \partial D_\ell\} \leq \gamma_2.$$ 

In other words, each configuration $\eta \in \mathcal{U}_N^{x_0}$ belongs to a boundary set $\partial D_\ell(A)$ at most $\gamma_2$ times.

Proof. Fix $\eta \in \partial D_\ell(A)$, so that there exists $x, y \in S$ with $r(x, y) > 0$ such that $\sigma^{x, y}\eta \in D_\ell(B)$ for some $B \neq A$. By Lemma 12.16, there exists $C_0 > 0$ such that $|P^A_\ell(\eta) - P^B_\ell(\eta)| \leq C_0 \ell$.

Therefore, it suffices to prove that there exists a finite constant $C_1$ such that

$$\sum_{\ell = 1}^{\infty} 1\{|P^A_\ell(\eta) - P^B_\ell(\eta)| \leq C_0 \ell\} \leq C_1.$$ 

Recall that

$$P^A_\ell(\eta) - P^B_\ell(\eta) = P^A(\eta) - P^B(\eta) - (c_A - c_B)\ell^2.$$ 

Since $c_A \neq c_B$, the left-hand side of the penultimate displayed equation can be written as

$$\sum_{\ell = 1}^{\infty} 1\left\{|\ell^2 - \frac{(P^A - P^B)(\eta)}{c_A - c_B}| \leq \frac{C_0 \ell}{|c_A - c_B|}\right\}.$$ 

By Lemma 12.19 below, this sum is bounded by a constant which only depends on $c_A$, $c_B$ and $C_0$, as claimed. \qed

Lemma 12.19. For $\alpha > 0$ and $t \in \mathbb{R}$, the set

$$A_{\alpha, t} = \{x \in \mathbb{R} : x^2 - 2\alpha x + t \leq 0 \leq x^2 + 2\alpha x + t\}$$

is either an empty set or a closed interval of length at most $2\alpha$.

Proof. If $t > \alpha^2$, the inequality $x^2 + 2\alpha x + t < 0$ cannot hold and the set $A_{\alpha, t}$ is empty. We may, therefore, assume that $t \leq \alpha^2$. In this case, let

$$u^\pm = \alpha \pm \sqrt{\alpha^2 - t}, \quad v^\pm = -\alpha \pm \sqrt{\alpha^2 - t},$$

so that

$$A_{\alpha, t} = [u^-, u^+] \setminus (v^-, v^+).$$

This set is a closed sub-interval of $[v^+, u^+]$ and $u^+ - v^+ = 2\alpha$. This completes the proof. \qed

12.7. The function $h_\ell$. Fix $\ell \geq 2$, and let

$$h_\ell(\eta) := P(\eta) - W_\ell(\eta).$$

The next result is the main step in the construction of a super-harmonic function.

Proposition 12.20. There exist positive constants $c_1$, $c_2$ such that

$$(A_N h_\ell^{1/2})(\eta) \leq \frac{1}{P(\eta)^{1/2}} \left\{-c_1 + c_2 \ell 1\{\eta \in \partial D_\ell\}\right\}$$

for all $\ell \geq 2$, large enough $N$ and $\eta \in \text{int} \mathcal{U}_N^{x_0}$.

To prove this proposition, we first investigate $A_N h_\ell$. 

Lemma 12.21. There exists a finite constant $C_0$ such that

\[(A_N h_\ell)(\eta) \leq C_0 \ell \mathbf{1}\{\eta \in \partial D_\ell\} + o_N(1)\]

for all $\eta \in \text{int} \, U_N^{x_0} \,$.

Proof. Suppose that $\eta \in \text{int} \, D_\ell(A)$ for some proper subset $A$ of $S_0$. By Proposition 12.15 and the third assertion of Proposition 12.7,

\[(A_N h_\ell)(\eta) = (A_N P)(\eta) - (A_N W^\ell)m(\eta) \leq o_N(1).\]

Assume that $\eta \in \partial D_\ell(A)$, for some proper subset $A$ of $S_0$. By Lemma 12.17,

\[(A_N h_\ell)(\eta) \leq (A_N P)(\eta) - (A_N P^A)(\eta) + C_0 \ell\]

for some finite constant $C_0$. By the first assertion of Proposition 12.7,

\[A_N P^A(\eta) \geq -C_0 \sum_{x \in A} \eta_x\]

for some finite constant $C_0$. By Proposition 12.14, this expression is bounded below by $-C_0 \gamma_1 \ell = -C_0 \ell$. On the other hand, by the third assertion of Proposition 12.7, $(A_N P)(\eta) \leq \kappa + o_N(1)$. This completes the proof of the proposition. $\square$

The next result is an extension of Lemma 12.9.

Lemma 12.22. There exists a positive constant $c_0$ such that

\[\sum_{x \in S} g(\eta_x) \sum_{y \in S} r(x, y) \left[ h_\ell(\sigma^{x,y} \eta) - h_\ell(\eta) \right]^2 \geq c_0 P(\eta)\]

for all $\eta \in (\text{int} \, U_N^{x_0}) \setminus \partial D_\ell$.

Proof. Since $g(\eta_{x_0}) > 0$, it suffices to show that

\[\sum_{y \in S} r(x_0, y) \left[ h_\ell(\sigma^{x_0,y} \eta) - h_\ell(\eta) \right]^2 \geq c_0 P(\eta).\]

By the Cauchy-Schwarz inequality, the square of the left-hand side is bounded below by

\[\left\{ \sum_{y \in S} r(x_0, y) \right\}^{-1} \left\{ \sum_{y \in S} r(x_0, y) \left[ h_\ell(\sigma^{x_0,y} \eta) - h_\ell(\eta) \right] \right\}^2.\]

Thus, by Lemma 12.8, it is enough to show that

\[\sum_{y \in S} r(x_0, y) \left[ h_\ell(\sigma^{x_0,y} \eta) - h_\ell(\eta) \right] \geq c_0 \sum_{x \in S_0} \eta_x. \quad (12.19)\]

Since $\eta \notin \partial D_\ell$, $\eta$ belongs to $\text{int} \, D_\ell(A)$ for some $A \subsetneq S_0$. In particular, $\eta$ and $\sigma^{x_0,y} \eta$ belong to $D_\ell(A)$ for all $y$ such that $r(x_0, y) > 0$. The left-hand side of the previous displayed equation is thus equal to

\[\sum_{y \in S} r(x_0, y) \left[ (P - P^A_\ell)(\sigma^{x_0,y} \eta) - (P - P^A_\ell)(\eta) \right].\]

If $A = \emptyset$, then $P^A_\ell(\sigma^{x_0,y} \eta) = P^A_\ell(\eta) = 0$. In this case, (12.19) follows from the third assertion of Lemma 12.6. If $A \neq \emptyset$,

\[P^A_\ell(\sigma^{x_0,y} \eta) - P^A_\ell(\eta) = P^A(\sigma^{x_0,y} \eta) - P^A(\eta).\]
By the second and third statements of Lemma 12.6, the left-hand side of (12.19) is equal to
\[
\sum_{z \in S_0} \eta_z - \sum_{z \in A} u^A_{z,x_0} \eta_z.
\]

On the set \( U^x_N \), \( \sum_{z \in S_0} \eta_z \geq N^\gamma \) and, by Proposition 12.14, \( \eta_z \leq \gamma_1 \ell \) for all \( z \in A \). In particular, the previous expression is greater than \((1/2) \sum_{z \in S_0} \eta_z\) for \( N \) large enough. This completes the proof.

\[\square\]

**Proof of Proposition 12.20.** By definition,
\[
(A_N h^{1/2}_\ell)(\eta) = \sum_{x \in S} g(\eta_x) \sum_{y \in S} r(x, y) \left[ h^{1/2}_\ell(\sigma^x \cdot y \eta) - h^{1/2}_\ell(\eta) \right].
\]

By Lemmata 12.8 and 12.10, there exists a positive constant \( c_0 \) such that
\[
h_\ell(\eta) \geq P(\eta) \geq c_0 \left( \sum_{x \in S_0} \eta_x \right)^2
\]
for all \( \eta \in \text{int} \ U^x_N \). On the other hand, by definition of \( h_\ell \), (12.5) [for \( A = S_0 \) and \( A \) a proper subset of \( S_0 \)] and Lemma 12.16, there exists a finite constant \( C_0 \) such that
\[
| h_\ell(\sigma^x \cdot y \eta) - h_\ell(\eta) | \leq C_0 \left\{ \ell + \sum_{z \in S_0} | \eta_z | \right\}
\]
for all \( x, y \in S \), \( y \neq x \) and \( \eta \in \text{int} \ U^x_N \) such that \( \eta_x \geq 1 \). This expression is bounded by \( C_0 \sum_{z \in S_0} | \eta_z | \) for \( N \) sufficiently large. Since \( \sum_{z \in S_0} | \eta_z | \geq N^\gamma \) on \( \text{int} \ U^x_N \), it follows from the two previous estimates that there exists a finite constant \( C_0 \) such that
\[
| h_\ell(\sigma^x \cdot y \eta) - h_\ell(\eta) | \leq C_0 N^{-\gamma}
\]
(12.20)
for all \( x, y \in S \), \( y \neq x \) and \( \eta \in \text{int} \ U^x_N \) such that \( \eta_x \geq 1 \).

A second order Taylor expansion and the previous bound yield that \( (A_N h^{1/2}_\ell)(\eta) \) is equal to
\[
\frac{(A_N h_\ell)(\eta)}{2 h_\ell(\eta)^{1/2}} - [1 + c_N] \frac{1}{8 h_\ell(\eta)^{3/2}} \sum_{x \in S} g(\eta_x) \sum_{y \in S} r(x, y) \left[ h_\ell(\sigma^x \cdot y \eta) - h_\ell(\eta) \right]^2,
\]
where \( c_N \) is bounded by \( C_0 N^{-\gamma} \). Hence, by Lemmata 12.21 and 12.22, there exist a finite constant \( C_0 \) and a positive constant \( c_0 \) such that
\[
(A_N h^{1/2}_\ell)(\eta) \leq \frac{C_0 \ell}{h_\ell(\eta)^{1/2}} 1\{ \eta \in \partial D_\ell \} + \frac{o_N(1)}{h_\ell(\eta)^{1/2}} - 1\{ \eta \notin \partial D_\ell \} + c_0 \frac{P(\eta)}{h_\ell(\eta)^{1/2}}.
\]

Write \( 1\{ \eta \notin \partial D_\ell \} \) as \( 1 - 1\{ \eta \in \partial D_\ell \} \). Since \( h_\ell(\eta) \geq P(\eta) \) and, by Lemma 12.10, \( P(\eta) \geq (1/2) h_\ell(\eta) \),
\[
(A_N h^{1/2}_\ell)(\eta) \leq \frac{C_0 \ell}{P(\eta)^{1/2}} 1\{ \eta \in \partial D_\ell \} - \frac{c_0}{P(\eta)^{1/2}},
\]
as claimed.

\[\square\]
12.8. Proof of Theorem 12.11.

Proof of Theorem 12.11. The function \( F_m \) can be written as
\[
F_m(\eta) = \sum_{\ell=2}^{m} \frac{1}{\ell} h_\ell(\eta)^{1/2}.
\]

By Proposition 12.20,
\[
P(\eta^{1/2}(A_N F_m)(\eta) \leq -c_0 \sum_{\ell=2}^{m} \frac{1}{\ell} + C_0 \sum_{\ell=2}^{m} \mathbf{1}_{\{\eta \in \partial \mathcal{D}_\ell\}}.
\]

By Proposition 12.18, this expression is bounded by
\[
-c_0 \log m + C_0 \gamma_2.
\]
Thus, by taking \( m \) large enough, there exists \( c'_0 > 0 \) such that \( P(\eta^{1/2}(A_N F_m)(\eta) < -c'_0 < 0 \) for all \( \eta \in \text{int} \mathcal{U}_N^0 \). It remains to recall the statement of Lemma 12.8 to complete the proof.

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