Initialization of Electromagnetic Transient Simulation Using Boundary Value Solution Method

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Abstract. Initialization is an important preliminary work in electromagnetic transient simulation which is typically executed from the periodic steady-state. To this end, researchers have proposed so far a variety of initialization methods, among which the initialization based on fixed-point iteration or the so-called the EMTP-based approach has been widely used. However, this type of method can be very time consuming for a large power system due to its weak convergence. In this paper, the initialization is described as a two-point differential boundary value problem. On this basis, the boundary value solution technique is used to solve this problem, therefore, a new electromagnetic transient initialization method is derived. The proposed method is a strict Newton algorithm, so it has better convergence than the traditional fixed-point iteration method. Case studies conducted on two typical network have confirmed the effectiveness of the proposed method.

Introduction

The electromagnetic transient simulation and related programs (the EMTP-type programs) are widely used in the electromagnetic transient studies of power systems. The transient behavior of a network can be represented mathematically by ordinary differential equations in the continuous time-domain. Using numerical integration methods such as the implicit trapezoidal rule to solve the ordinary differential equations, the transient solution of the network can be obtained in the discrete time-domain.

The transient study of a network normally begins from the network’s steady state. If the system is lightly damped, the time for it to reach its steady state can be significant and the simulation can be prohibitively expensive. The effective initialization for electromagnetic transient simulation or the EMTP-type programs, therefore, becomes unavoidable. Alternatively, the steady state solution for the periodic ordinary differential equations may be of interest in itself for the computer-aided design of nonlinear circuits and harmonic analysis [1].

Some methods to initialize the EMTP-type programs have been proposed [2, 3]. Based on the needs of the EMTP program development, the earliest proposed electromagnetic transient initialization method should be the phasor-solution technique [4]. In this approach all the elements of a network are represented in the phasor-domain. The system solution is obtained at the fundamental system frequency by solving the network matrix equation. The so-obtained system quantities are used to generate the initial state of the network. This solution technique is simple but limited to the linear and lumped-element networks.

Another important initialization method is the so-called EMTP-based approach [5]. This approach performs a network’s initialization within the EMTP solution frame. That is, the initial steady state of a network can be directly established for a given initial state by simply integrating the system equations until the response becomes periodic. From a mathematical point of view, the solution process of this method is equivalent to the classical fixed-point iteration. It is well known that the
fixed-point iteration is a general numerical method, but its convergence is not as good as the Newton method. So, the EMTP-based approach can closely predict the network’s initial state even for a nonlinear and time-variant network, but it is not practical for large systems due to its low efficiency.

Since the phasor-solution approach and the EMTP-based technique may not be appropriate for the initialization of large power systems, the load-flow program-based initialization technique has been sought to initialize an EMTP-type program [2]. This technique is efficient and suitable for large AC power systems, but it is obviously not suitable for DC transmission systems when using the electromagnetic transient model.

Note that the currently widely used EMTP-based approach is just to simply integrate the system differential equations step by step for a sufficient number of periods until the transient response becomes negligible. The main problem with this approach is its expensively slow convergence. To solve this problem, a natural idea is to use Newton iteration. However, a precise expression for the Jacobian matrix may be difficult to obtain in general. In fact, evaluating the Jacobian matrix by finite differences is inefficient and may cause numerical error because we must solve the nonlinear differential equations many times if nonlinearities exist. A more accurate and efficient method involves the solution of related sensitivity systems which describe the linearization of a nonlinear system [1, 3], but this improvement is still not a strict Newton method.

In this paper, the electromagnetic transient initialization is described as a differential boundary value problem, thus, the boundary value solution technique can be used to solve the initialization. In this approach, theoretically we can use different boundary value methods [6], but considering that the EMTP-type programs are mainly using the trapezoidal rule, we also use the trapezoidal integration method to implement initialization. In order to solve the problem that the Jacobian matrix involved in the boundary value solution is too large, a block recursive solution method is proposed in this paper. In theory, the proposed initialization method is a rigorous Newton algorithm and thus has better convergence.

Section 2 introduces the mathematical expression or the formation of electromagnetic transient initialization problem. On this basis, this section also introduces the basic fixed-point iteration method, which is currently widely used for initialization.

Section 3 describes simply the solution technique for differential boundary value problem, meanwhile this section derives the initialization method based on this technique.

To further elaborate the proposed method, in Section 4, the proposed method is applied to a general linear differential system with periodic input, and compared with the fixed-point iteration. This leads to an interesting result.

In section 5, the proposed methods are tested and compared with the fixed-point iteration method using two simple but typical networks.

The Problem Formation and EMTP-Based Methods

From the mathematical point of view, the electromagnetic transient simulation is just the solution of the differential initial value problem. That is

$$\frac{dx}{dt} \triangleq \dot{x} = f(x, t), \quad x(t = 0) = x_0$$

where, \(x\) and \(f(x, t)\) are \(n\) vectors. For a given initial value \(x_0\), we can obtain discrete time-domain numerical solutions of state variables \(x(t)\) by numerical integration. Electromagnetic transient simulation is mainly to study the transient response of the power system under disturbance or fault conditions. Usually, this process must start from a steady state initial condition. This is the so-called initialization problem.

Note that, in steady state of power systems, \(f(x, t)\) is periodic in \(t\) of period \(T\). Henceforth, we can assume that the Eq. 1 has a periodic solution \(x(t)\) of period \(T\). In this case, the goal of the
Initialization is to determine the initial state \( x_0 \triangleq y \) such that integrating Eq. 1 from this initial state over the interval \([0, T]\) we obtain the periodic solution \( x(t) \) of period \( T \). This is essentially a two-point boundary value problem in which the solution to Eq. 1 in the interval \([0, T]\) must satisfy the following boundary condition:

\[
y \triangleq x_0 \quad x(T) = y
\]

Since

\[
x(T) = y + \int_0^T f(x, \tau) d\tau
\]

we can express the above problem in terms of the mapping

\[
y = F(y) \triangleq y + \int_0^T f(x, \tau) d\tau
\]

where, \( F(y) \) is a function of the initial state \( y \).

For the solution of \( y \), the most basic method is the so-called fixed-point iteration method which can be described as

\[
y^{k+1} = F(y^k) \triangleq y^k + \int_0^{(k+1)T} f(x, \tau) d\tau
\]

where, the superscript \( k \) represents the number of iterations. This iteration must be repeated until the following condition is met:

\[
\|A^k\| \triangleq \|y^{k+1} - F(y^k)\| \leq \varepsilon
\]

where, \( \varepsilon \) is the required accuracy. It is easy to understand that numerical integration is required in the iterative process described by Eq. (5). For this, the trapezoidal rule is usually used because most EMTP-Type programs use this integration method. Obviously, we need to integrate the system step-by-step for a full period to complete one iteration, and repeat this process multiple times until the transient response becomes negligible. This is precisely the method currently used with the EMTP-Type programs to arrive at the steady-state response. Therefore, this method is also commonly referred to as the EMTP-based approach. This procedure is satisfactory if the transient decays rapidly. However, for lightly damped networks typical of power systems, the transient usually decays very slowly and this will make the initialization very time-consuming.

Initialization Using Boundary Value Solution Technique

The main purpose of this study is to improve the convergence of initialization, so as to improve the efficiency of electromagnetic transient simulation. To this end, one of the most important ways is to use the Newton method to solve the Eq. 4. That is

\[
(I - J_F^k)A^k = F(y^k) - y^k
\]

where, \( I \) is a unit matrix and \( J_F \) is the Jacobian matrix,

\[
J_F \triangleq \frac{\partial F(y)}{\partial y}
\]

However, it can be seen from Eq. 4 that the precise expression for the Jacobian matrix \( J_F \) is difficult to obtain when \( f(x, t) \) is nonlinear.

In fact, if we use stepwise integration, which is a serial solution to calculate the integral term involved in \( F(y) \), it is really difficult to get the exact expression of \( J_F \). Since the initialization is a
two-point boundary value problem, the efficient method to solve it is naturally to use the boundary value method \[6\], \[7\]. Note that, for the initialization problem, the most important is the solution technique within the boundary value methods (BVM). Different from the traditional step-by-step integration process, the integral rule should be used at multiple discrete time-points when BVM method is applied to boundary value problem, then all the discrete equations can be solved integrally.

Based on the above ideas, we use the classical trapezoidal rule, which can be seen as a special boundary value method, to solve the boundary value problem Eq. 2, then there are

\[
x_1 = y + \frac{1}{2} h (f(y) + f(x_1)) \\
x_2 = x_1 + \frac{1}{2} h (f(x_1) + f(x_2)) \\
\vdots \\
y = x_{N-1} + \frac{1}{2} h (f(x_{N-1}) + f(y))
\]

(9)

where, \(h \triangleq T/N\). This is precisely the solution method used with the BVM. Using the Newton method to solve the Eq. 9, we can get

\[
\begin{bmatrix}
J_0 & -J_1 & \cdots & -J_{N-1} \\
J_1 & \ddots & \cdots & \vdots \\
\vdots & \ddots & \ddots & \vdots \\
-J_N & \cdots & J_{N-1} & 0
\end{bmatrix}
\begin{bmatrix}
\Delta y^{(k)} \\
\Delta x_1^{(k)} \\
\vdots \\
\Delta x_{N-1}^{(k)}
\end{bmatrix}
= \begin{bmatrix}
b_1^{(k)} \\
b_2^{(k)} \\
\vdots \\
b_N^{(k)}
\end{bmatrix}
\]

(10)

where, the superscript \((k)\) represents the number of Newton iterations,

\[
\begin{aligned}
J_j & \triangleq I + \frac{1}{2} h \frac{\partial f(x_j)}{\partial x_j}, \ j \in (0, N-1) \\
x_0 &= y \\
\end{aligned}
\]

(11)

\[
\begin{aligned}
J_j & \triangleq I - \frac{1}{2} h \frac{\partial f(x_j)}{\partial x_j}, \ j \in (1, N) \\
x_N &= y \\
b_j & \triangleq x_j - x_{j-1} - \frac{1}{2} h (f(x_{j-1}) + f(x_j)), \ j \in (1, N)
\end{aligned}
\]

(12)

(13)

Solving the above Newton Eq. 10 until convergence, we can get the required steady state initial solution \(y\).

Obviously, the above initialization method is a strict Newton algorithm, so it should have better convergence than the fixed-point iteration method. However, the Jacobian matrix involved in the procedure Eq. 10 is very large, and direct solution will lead to huge calculations. To this end, we can consider the following improved solution technique since the overall Jacobian matrix has a special structure as shown in Eq. 10.

Define

\[
\Delta X \triangleq \begin{bmatrix} \Delta x_1^T & \cdots & \Delta x_{N-1}^T \end{bmatrix}^T
\]

(14)
\[ B \doteq \left[ b_j^t \right]^T, \quad j \in (2, N) \] (15)

\[ E \doteq \cdots \left[ \bar{J}_1 \mid 0 \cdots 0 \right] \] (16)

\[ \bar{E}^T \doteq \left[ 0 \cdots 0 \mid J_n^t \right] \] (17)

\[ J_u \doteq \begin{bmatrix} J_1 & \bar{J}_2 & \cdots & \bar{J}_{N-1} \\ J_2 & \cdots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \bar{J}_{N-1} \\ \bar{J}_{N-1} & \cdots & \ddots & J_1 \end{bmatrix} \] (18)

then Eq. 10 can be written as

\[
\begin{bmatrix} J_0 \\ E \\ J_u \end{bmatrix} \Delta y^{(k)} + \begin{bmatrix} \Delta y \\ \Delta X \end{bmatrix}^{(k)} = \begin{bmatrix} b_1^{(k)} \\ B \end{bmatrix}
\] (19)

Since matrix \( J_u \) is an block upper triangular matrix, it can be derived from Eq. 19 that,

\[
\Delta y^{(k)} = \phi^{-1}(b_1 - EJ_u^{-1}B)^{(k)}
\] (20)

\[
\phi \doteq J_0 - EJ_u^{-1}E = J_0 - \prod_{j=1}^{N-1}(J_j^{-1}\bar{J}_{j+1})
\] (21)

Note that, with simple recursive calculations, it is easy to get the value of \( EJ_u^{-1}B \). Once the calculation of \( \Delta y \), \( \Delta X \) is easy to complete.

Another way to reduce the amount of calculation involved in the boundary value solution procedure can be described below. In general, for AC transmission systems, we must have

\[
y \doteq x(0), \quad x(T/2) = -y
\] (22)

Therefore, in order to reduce the dimension of the overall Jacobian matrix, we can carry out the boundary value solution procedure in a half cycle \([0, T/2]\). That is

\[
\begin{bmatrix} J_0 \\ -\bar{J}_1 \\ \vdots \\ -\bar{J}_{\nu-1} \\ \bar{J}_\nu \end{bmatrix} \Delta y^{(k)} + \begin{bmatrix} \Delta y \\ \Delta x_1 \\ \vdots \\ \Delta x_{\nu-1} \end{bmatrix} = \begin{bmatrix} b_1^{(k)} \\ \vdots \\ b_{\nu-1}^{(k)} \end{bmatrix}
\] (23)

where, \( \nu = T/(2h) \),

\[
\bar{J}_\nu \doteq I + \frac{1}{2}h \frac{\partial f(-y)}{\partial y}
\] (24)

\[
\bar{b}_\nu \doteq -y - x_{\nu-1} - \frac{1}{2}h(f(x_{\nu-1}) + f(-y))
\] (25)

Since the size of the overall Jacobian matrix is halved \( \nu = N/2 \), the computational complexity of this method is greatly reduced in each step of Newton iteration.
**Initialization of Linear Differential Systems**

In some special cases, the mathematical model of electromagnetic transient simulation can be expressed by the following equation:

\[ \dot{x} = f(x, t) = Ax + g(t) \]  

(26)

where, \( A \) is a constant coefficient matrix, and \( g(t) \) is a periodic input vector. For example, a HVDC transmission system can be expressed by the above equation, where \( A \) will be a variable structure coefficient matrix due to commutations, and \( g(t) \) is usually related to the three-phase AC voltage of the commutated bus bar [8].

For the above general linear differential system with periodic input, we can derive the explicit initialization formula based on the fixed-point iteration method while using the trapezoidal integral rule. That is

\[ y^{(k+1)} = \Phi^N y^{(k)} + \sum_{j=1}^{N} (\Phi^{N-j} \beta^{-1} \bar{g}(\tau_j)) \]

(27)

where,

\[ \Phi \triangleq (I - \frac{1}{2} hA)^{-1}(I + \frac{1}{2} hA) \]  

(28)

\[ \beta \triangleq I - \frac{1}{2} hA \]  

(29)

\[ \{ \bar{g}(\tau_j) \triangleq \frac{1}{2} h(g(t_{j-1}) + g(t_j)) \}, \quad j \in (1, \ N) \]

(30)

According to the fixed-point principle [9], if

\[ \|\Phi\|_2 < 1 \]  

(31)

where, \( \|\bullet\|_2 \) represents the spectral norm of the matrix, then the iteration Eq. 27 will be always convergent. This is equivalent to saying that the initialization will be successful when the eigenvalues of \( A \) have negative real parts or that the system Eq. 26 is asymptotically stable. Note that if the eigenvalues of \( A \) are close to the imaginary axis, but in the left half plane, then the convergence can be quite slow.

If we perform the initialization using the method described in the previous section, the results are completely different. In this case, there are

\[ \begin{bmatrix} \alpha & -\beta \\ \alpha & \ddots & \ddots \\ \vdots & \ddots & \ddots & -\beta \\ -\beta & \cdots & \cdots & \alpha \end{bmatrix} \begin{bmatrix} y \\ x_1 \\ \vdots \\ x_{N-1} \end{bmatrix} = \begin{bmatrix} \bar{g}(\tau_1) \\ \bar{g}(\tau_2) \\ \vdots \\ \bar{g}(\tau_N) \end{bmatrix} \]

(32)

\[ \alpha \triangleq I + \frac{1}{2} hA \]

(33)

So, we can easily get from Eq. 32 that,
\[ y = -[\alpha - \beta \prod_{j=1}^{N-1}(\Phi^j)]^{-1}G(\tau) \]  

(34)

\[ G(\tau) \triangleq \vec{g}(\tau_1) + \vec{p}H_U^{-1}\vec{G}(\tau) \]  

(35)

where, \( H_U \) is an \((N-1) \times (N-1)\) dimension block upper triangular matrix,

\[
H_U \triangleq \begin{bmatrix}
\alpha & -\beta & \\
\alpha & \ddots & -\beta \\
& \ddots & \ddots & -\beta \\
\beta & \ddots & \ddots & \alpha
\end{bmatrix}
\]  

(36)

\[ \vec{p} \triangleq [\beta \; 0 \; \ldots \; 0] \]  

(37)

\[ \vec{G}^T(\tau) \triangleq \begin{bmatrix} \vec{g}^T(\tau_1) \end{bmatrix}, \; j \in \{2, \; N\} \]  

(38)

Obviously, using the initialization method proposed in this paper, we can solve the steady-state initial solution of linear differential system without any iteration, and therefore there is no convergence problem.

If the system satisfies the boundary Eq. 22, then similarly, we can implement the boundary value method procedure in a half cycle. That is

\[
\begin{bmatrix}
\alpha & -\beta & \\
\alpha & \ddots & -\beta \\
& \ddots & \ddots & -\beta \\
\beta & \ddots & \ddots & \alpha
\end{bmatrix}
\begin{bmatrix}
y_1 \\
\vdots \\
x_{v-1}
\end{bmatrix}
= \begin{bmatrix}
\vec{g}(\tau_1) \\
\vec{g}(\tau_2) \\
\vdots \\
\vec{g}(\tau_{v-1})
\end{bmatrix}
\]  

(39)

Again, the initial value \( y \) can be obtained without iteration, and the corresponding amount of calculation is less.

**Numerical Examples**

To test the proposed method, two case test results are shown in this section.

The first example system is an unloaded high-voltage long transmission line powered by an AC voltage source, which is illustrated in Fig. 1.

\[ R = 10 \Omega \quad L = 300 \text{ km} \]

\[ e_c(t) = 220\sin(100\pi t + \frac{\pi}{2}) \text{kV} \]

\[ R = 0.2568 \text{ \Omega/km} \quad L = 2 \times 10^{-3} \text{H/km} \quad C = 8.6 \times 10^{-9} \text{F/km} \]

![Figure 1. No-load single-phase high-voltage transmission line.](image)

It is well known that the electromagnetic transient of high-voltage long transmission line can be described by the famous telegraph equation. By spatially discretizing the telegraph equation where the transmission line is divided into 50 segments, we can derive the differential equation of the system, which has exactly the same formation as Eq. 26.
The steady-state initial solution, for this system, was obtained by two methods, one by classical fixed-point iteration or the so-called EMTP-based approach, and the other by the proposed method but implemented respectively in half cycle and one cycle. The trapezoidal rule with a fixed-step $h = 10^{-4}$ sec. was used for two methods. Fig. 2 is the iterative process convergence curve of the EMTP-based method while the iteration was started with the zero initial state, i.e., $y^{(1)} = \mathbf{0}$. Fig. 3 is the corresponding voltage curve at the end of the line. Obviously, after 12 iterations, the system reached a steady state cycle state ($\varepsilon \leq 10^{-3}$).

![Figure 2. Convergence curve of the EMTP-based method.](image)

![Figure 3. Voltage curve at the end of the line during iterations.](image)

The second initialization method does not require iteration. By solving a simple equation, the initialization result can be obtained in one go. In order to verify the correctness of the results, record the result of the first method as $y_1$, and the results of the second method respectively as $y_{2-1}$ (half cycle) and $y_{2-2}$ (one cycle), so that we can get

$$
\|y_1 - y_{2-1}\| = 3.9436 \times 10^{-4}
$$

$$
\|y_1 - y_{2-2}\| = 3.9437 \times 10^{-4}
$$

Obviously, if $y_1$ is considered to be sufficiently precise, the above results mean that, the proposed initialization method using both half cycle boundary condition and one cycle boundary condition is also precise. Note that here, the latter not only has almost the same accuracy, but also halved the amount of calculation.
The second example is an equivalent power supply circuit given in Fig. 4, where

\[ V_m = 10 \sin(120\pi t). \]

The state equations for this circuit are

\[
\begin{align*}
\dot{x}_1 &= 10^6 \left( (-x_1 - x_2 + 10 \sin(120\pi t))/5 - i_d \right) \\
\dot{x}_2 &= 10^3 \left( (-x_1 - x_2 + 10 \sin(120\pi t))/5 - x_3 \right) \\
\dot{x}_3 &= 10(x_2 - x_4) \\
\dot{x}_4 &= 10^3 x_3 - x_4 \\
i_d &= 10^{-6} (e^{40x_3} - 1)
\end{align*}
\]

Same as the first example, the initialization was performed respectively by two methods both using trapezoidal rule with a fixed-step \( h = 0.1/6 \) ms. This system is not an AC transmission system, so based on generality, we only implement the proposed method using boundary condition Eq. 2. Fig. 5 is the iterative convergence curve of the EMTP-based initialization, while Fig. 6 is the corresponding variation curve of \( x_i(t) \) during iterations. This initialization needs to be iterated 71 times, that is, it needs to perform 71 cycles of numerical integration. The steady state initial value final obtained by the fixed-point iteration method is:

\[
\begin{align*}
x_1(0) &= -9.0787, \quad x_2(0) = 9.0552 \\
x_3(0) &= 0.0090, \quad x_4(0) = 9.1035
\end{align*}
\]

Fig. 7 is the convergence curve of the proposed method, while Fig. 8 is the variation curve of \( x_3(t) \) obtained by the proposed method. The new initialization method requires only 10 iterations, which is significantly faster than the EMTP-based method. The final value obtained by the proposed method is:

\[
\begin{align*}
x_1(0) &= -9.0753, \quad x_2(0) = 9.0564 \\
x_3(0) &= 0.0090, \quad x_4(0) = 9.1025
\end{align*}
\]
By comparison, it has been shown that the proposed method is much superior to the EMTP-based initialization both in convergence and in computational efficiency.

Conclusions

In this paper, a new method has been proposed for the initialization of electromagnetic transient simulation. The key to this is to use the boundary value solution technique. The proposed method is a rigorous Newton algorithm and thus has better convergence than the currently widely used fixed-point iteration method. The algorithm has been shown to be very effective for initialization, therefore, the method can be combined with existing electromagnetic transient simulation program so as to improve the computational efficiency of electromagnetic transient simulation.

By the way, the proposed method can also be applied to other fields, such as the computer-aided design of nonlinear circuits, in which the steady-state analysis of nonlinear circuits with periodic inputs is essential.

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