Crossing symmetric potential model of pion-nucleon scattering

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Abstract. A crossing symmetric πN scattering amplitude is constructed through a complete attachment of two external pions to the dressed nucleon propagator of an underlying πN potential model. Our formulation automatically provides expressions also for the crossing symmetric and gauge invariant pion photoproduction and Compton scattering amplitudes. We show that our amplitudes are unitary if they coincide on-shell with the amplitudes obtained by attaching one pion to the dressed πNN vertex of the same potential model.

Keywords: Pion-nucleon scattering, Crossing symmetry, Pion photoproduction, Gauge invariance, Unitarity

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1. INTRODUCTION

The πN scattering amplitude \( t \) is often described using the set of equations [1, 2, 3]

\[
\begin{align*}
  t &= fg\tilde{f} + t_b, \\
  \tilde{f} &= f_0 + f_0 G_0 t_b, \\
  g &= g_0 + g_0 \Sigma g, \\
  \Sigma &= \tilde{f}_0 G_0 f,
\end{align*}
\]

where \( f (f_0) \) is the dressed (bare) \( N \rightarrow \pi N \) vertex, \( \tilde{f} (\tilde{f}_0) \) is the dressed (bare) \( \pi N \rightarrow N \) vertex, \( g (g_0) \) is the dressed (bare) nucleon propagator, \( G_0 \) is the renormalized disconnected \( \pi N \) propagator, and \( t_b (\nu) \) is the "non-pole" \( \pi N \) t-matrix (potential) with the pole term \( fg\tilde{f} (f_0 g_0 \tilde{f}_0) \) removed. Although these equations provide an exact description in full field theory, their main feature is that they allow one to preserve unitarity when making models for the potential \( \nu \) and bare vertex \( f_0 \). However, like all potential models, these equations suffer from a lack of crossing symmetry, a property whose importance has been emphasized for more than 50 years [4, 5, 6].

Similarly, the pion photoproduction amplitude \( t^\gamma \) is often described by a set of equations that essentially result from Eq. (1a) and Eq. (1b) by replacing the initial pion with a photon [7, 8, 9]:

\[
\begin{align*}
  t^\gamma &= fg\tilde{f}^\gamma + t_b^\gamma, \\
  \tilde{f}^\gamma &= f_0^\gamma + f_0^\gamma G_0 t_b^\gamma,
\end{align*}
\]

where \( \tilde{f}^\gamma (f_0^\gamma) \) is the dressed (bare) \( \gamma N \rightarrow N \) vertex, and \( t_b^\gamma (\nu^\gamma) \) is the pion photoproduction amplitude (Born term) with the pole term \( fg\tilde{f}^\gamma (f_0 g_0 \tilde{f}_0^\gamma) \) removed. Once again
the feature of these equations is that they respect unitarity. This time, however, these equations suffer not only from a lack of crossing symmetry, but also from the breaking of manifest gauge invariance (because the photon is not coupled to all places in the underlying field theory). We shall refer to Eqs. (1) and Eqs. (2) as the standard description.

In this paper we present new equations for πN scattering, pion photoproduction, and Compton scattering, that are based on the potential model of Eqs. (1), but that preserve crossing symmetry and manifest gauge invariance. Our approach is based on the idea of coupling external pions and photons to all possible places in the dressed propagator $g$ of Eq. (1c), and is achieved using the gauging of equations method [10, 11]. Like the standard description, our approach is exact in full field theory; however, just opposite to the standard description, when models are made for the potential $v$ and bare vertex $f_0$, our approach preserves crossing symmetry and gauge invariance at the expense of unitarity. The lack of built-in unitarity is not surprising since our approach effectively sums the full perturbation series in a way that is different from the usual method of iterating a kernel. Nevertheless, we show that our amplitudes will satisfy unitarity whenever the crossing symmetric πN amplitude coincides, on-shell, with the one obtained by attaching one pion to the dressed πNN vertex $f$ of Eq. (1b).

2. SINGLE-GAUGED AMPLITUDE

In Refs. [10, 11] we introduced a technique for attaching an external photon to all possible places (vertices, propagators, potentials, etc.) within a strongly interacting system described by dynamical equations. The completeness of the attachment led to the gauge invariance of the resulting electromagnetic currents. Here we use the same technique to also attach first one external pion, and then in the next section, a second external pion, in order to achieve our goal of deriving a crossing symmetric πN scattering amplitude.

We begin by applying our gauging technique to the πN Green function $G$ generated by the non-pole potential $v$:

$$G = G_0 + G_0 v G.$$  \hspace{1cm} (3)

Denoting by $G^\mu$ the 5-point function resulting from a complete attachment of an external pion to $G$, Eq. (3) is "gauged"\(^2\) to obtain

$$G^\mu = G^\mu_0 + G^\mu_0 v G + G_0 v G^\mu + G_0 v G^\mu_0$$  \hspace{1cm} (4)

which is easily solved to get

$$G^\mu = G \Lambda^\mu G, \quad \Lambda^\mu = \Lambda^\mu_0 + v^\mu$$  \hspace{1cm} (5)

where $\Lambda^\mu_0 \equiv G_0^{-1} G^\mu_0 G_0^{-1}$ is a vertex function derived by attaching an external pion to the disconnected πN propagator $G_0$. Note that $G_0 = g_\pi g_N$ where $g_\pi$ is the pion propagator

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1 For simplicity of presentation, we ignore any terms that cannot be obtained by the attachment of external pions or photons. Such contributions, if present, are gauge invariant and crossing symmetric on their own, and can therefore be separately added to our derived amplitudes.

2 We shall use "gauging" to mean the process of attaching any external particle, not just a gauge boson.
FIGURE 1. The dressed $\pi NN$ vertex function $\Gamma^\mu$ resulting from a complete pion attachment to the dressed nucleon propagator $g$ defined by Eq. (1c).

and $g_N$ is the renormalized nucleon propagator (in exact field theory $g_N = g/Z$ where $Z$ is the renormalization constant; however, in model calculations one usually takes $g_N$ to be the bare propagator with physical nucleon mass). As G-parity conservation forbids a three-pion vertex, $\Lambda^\mu \equiv \Gamma^\mu_N g^{-1} - 1 \pi$ where $\Gamma^\mu_N \equiv g^{-1} N g^{-1} N$. Thus

$$\Lambda^\mu = \Gamma^\mu_N g^{-1} + \nu^\mu. \tag{6}$$

The gauged potential $\nu^\mu$ can be constructed phenomenologically, or derived by gauging a specific model for $\nu$. Similarly, the gauging of the dressed nucleon propagator $g$ of Eq. (1c), gives the 3-point function $g_{\mu}$:

$$g^\mu = g \Gamma^\mu g, \quad \Gamma^\mu = \Gamma^\mu_0 + \Sigma^\mu \tag{7}$$

where $\Gamma^\mu$ is the dressed $\pi NN$ vertex function, $\Gamma^\mu_0 \equiv g^{-1}_0 g^\mu_0 g^{-1}_0$ is the bare vertex function, and $\Sigma^\mu$ is the gauged dressing. Gauging $\Sigma = \bar{f}_0 G f_0$ then leads to a simple intuitive expression for the $\pi NN$ dressed vertex function:

$$\Gamma^\mu = \Gamma^\mu_0 + \bar{f}_0 G f_0 + \bar{f}_0 G \nu^\mu + \bar{f}_0 G_0 \nu^\mu G_0 f, \tag{8}$$

which is illustrated in Fig. (1).

The main results of this section come from the gauging of the dressed $\pi NN$ vertex $f$:

$$(G_0 f g)^V = (G f_0 g)^V = G^V f_0 g + G f_0 g^V + G f_0 g^V = GA^V G_0 f g + G f_0 g^V G^V g. \tag{9}$$

Cutting off the external legs immediately gives the amplitude $T^V \equiv G_0^{-1} (G_0 f g)^V g^{-1}$:

$$T^V = (1 + t_0 G_0 ) (\Lambda^\nu_0 G_0 f + \nu^\mu G_0 f + f_0 ^V) + f^V G \nu^V g. \tag{10}$$

If superscript $V$ corresponds to an external photon, then $T^V \sqrt{Z}$ is the properly normalized manifestly gauge invariant pion photoproduction amplitude, originally derived in Ref. [12] using a more involved approach. If superscript $V$ corresponds to an external pion, then $T^V \sqrt{Z}$ is a "hybrid" $\pi N$ scattering amplitude where the initial state pion is due to gauging and the final state pion is due to the original standard description. In a
In exact field theory, relations connecting the bare $\pi NN$ vertex $f_0$ and "non-pole" $\pi N$ potential $v$ of the standard description, Eqs. (1), to the corresponding gauged quantities $f_0^\mu$ and $v^\mu$ of the single-gauged description - see the first two of Eqs. (14).

similar way, vertex $\bar{f}$ can be gauged to obtain the hybrid amplitude $\bar{T}^\mu$ where the final state pion is due to gauging. One can express these amplitudes in a "pole plus non-pole" form analogous to Eq. (1a):

\[
T^\nu = f g \Gamma^\nu + T_b^\nu, \quad T_b^\nu = (1 + t_b G_0) V^\nu, \quad V^\nu = f_0^\nu + \Lambda^\nu G_0 f;
\]

(11a)

\[
\bar{T}^\mu = \Gamma^\mu g \bar{f} + \bar{T}_b^\mu, \quad \bar{T}_b^\mu = \bar{V}^\mu (1 + G_0 t_b), \quad \bar{V}^\mu = \bar{f}_0^\mu + \bar{f} G_0 \Lambda^\mu.
\]

(11b)

It follows that amplitudes $T_b^\nu$ and $\bar{T}_b^\mu$ satisfy non-standard Bethe-Salpeter equations

\[
T_b^\nu = V^\nu + v G_0 T_b^\nu, \quad \bar{T}_b^\mu = \bar{V}^\mu + \bar{f}_0 G_0 V.
\]

(12)

where the kernel and inhomogeneous terms involve $\pi N$ potentials of different origin. The amplitudes are clearly not crossing symmetric.

We can now express Eq. (8) in the following two ways analogous to Eq. (1b):

\[
\Gamma^\nu = \bar{F}_0^\nu + \bar{f} G_0 V, \quad \bar{F}_0^\nu = \bar{\Gamma}_0^\nu + \bar{f}_0 G_0 f, \quad \bar{F}_0^\mu = \Gamma_0^\mu + \bar{f} G_0 f, \quad \bar{F}_0^\nu = \bar{\Gamma}_0^\nu + \bar{f}_0 G_0 f.
\]

(13a)

(13b)

where Eq. (13a) and Eq. (13b) are to be used to describe pion (or photon) absorption and creation vertices, respectively. In exact field theory, $\Gamma^\mu$ can be identified with function $f / \sqrt{Z}$ of the standard description. Eqs. (13) then imply the following identities (in exact field theory) relating the standard description quantities $f_0$ and $v$ (typically the model inputs) to their gauged counterparts:

\[
f_0 / \sqrt{Z} = F_0^\mu, \quad v / \sqrt{Z} = \bar{V}^\mu, \quad \bar{f}_0 / \sqrt{Z} = \bar{F}_0^\nu, \quad \bar{v} / \sqrt{Z} = V^\nu.
\]

(14)

The first two of Eqs. (14) are illustrated in Fig. 2.

2.1. Unitarity

For ease of presentation, we discuss unitarity within the framework of time ordered perturbation theory for which

\[
G_0(E^+) - G_0(E^-) = -2\pi i \delta(E - H_0)
\]

(15)
where $H_0$ is the free Hamiltonian and $E^\pm = E \pm i\epsilon$. We consider only 2-body unitarity and thus restrict the discussion to energies $E$ below the two-pion threshold. In this energy region the potential $v$ and bare vertex $f_0$ are real, and the standard description of Eqs. (1) will therefore satisfy the following unitarity relations:

$$
t - t\dagger = t\dagger \delta t, \quad t_b - t\dagger_b = t\dagger_b \delta t_b, \quad (16a)$$
$$
f - f\dagger = t\dagger \delta f, \quad \bar{f} - f\dagger = f\dagger \delta t, \quad (16b)$$
$$
g - g\dagger = g\dagger (\Sigma - \Sigma\dagger) g, \quad \Sigma - \Sigma\dagger = f\dagger \delta f, \quad (16c)$$
$$
G - G\dagger = (1 + G_t t\dagger_b) \delta (1 + t_b G_0), \quad (16d)$$

where $\delta$ is shorthand for $-2\pi i\delta(E-H_0)$ and where identical quantities with and without a dagger represent the same functions of $E^-$ and $E^+$, respectively (i.e., a dagger does not mean Hermitian conjugate, but rather, $T \equiv T(E^+)$ and $T\dagger \equiv T(E^-)$). Applying these relations to Eqs. (11) and Eqs. (13) one obtains the analogous unitarity relations for the hybrid amplitudes:

$$
T^\nu - T^\nu\dagger = t\dagger \delta T^\nu, \quad T^\nu_b - T^\nu\dagger_b = t\dagger_b \delta T^\nu_b, \quad (17a)$$
$$
\bar{T}^\mu - \bar{T}^\mu\dagger = \bar{T}^\mu\dagger \delta t, \quad \bar{T}^\mu_b - \bar{T}^\mu\dagger_b = \bar{T}^\mu\dagger_b \delta t_b, \quad (17b)$$
$$
\Gamma^\mu - \Gamma^\mu\dagger = \bar{T}^\mu\dagger \delta f, \quad \Gamma^\nu - \Gamma^\nu\dagger = f\dagger \delta T^\nu, \quad (17c)$$

In the case of gauging with photons, Eq. (17a) provides just the usual statement of Watson’s theorem for the gauge invariant pion photoproduction amplitude $T^\nu$. However, in the case of gauging with pions, Eq. (17a) differs from the usual statement of unitarity for the hybrid amplitude in that a $T^\nu\dagger$ has been replaced with a $t\dagger$ from the standard description. Thus the only way for a hybrid $\pi N$ amplitude $T^\nu$ to be unitary is for it to coincide, on-shell, with the standard amplitude $t$.

### 3. Double-Gauged Amplitude

To obtain a crossing symmetric $\pi N$ scattering amplitude, we attach two external pions to the dressed nucleon propagator $g$; that is, we first gauge $g$ to obtain $g^\mu$ as in Eq. (7), and then gauge Eq. (7) to obtain $g^\mu\nu$ with indices $\nu$ and $\mu$ denoting the initial- and final-state pion, respectively. Thus

$$
g^{\mu\nu} = g T^{\mu\nu} g, \quad T^{\mu\nu} = \Gamma^{\mu} g^{\nu} + g^{\nu} \Gamma^{\mu} + \Gamma^0_0 + \Sigma^{\mu\nu} \quad (18)$$

where $Z T^{\mu\nu}$ is the properly normalized crossing symmetric $\pi N$ amplitude. In Eq. (18)

$$
\Gamma^0_0 = (g_0^{-1} g_0^{-1})^\nu = g^{-1}_0 g_0^{-1} - \Gamma^0_0 g_0 \Gamma^0_0 - \Gamma^\nu_0 g_0 \Gamma^\nu_0, \quad (19a)$$
$$
\Sigma^{\mu\nu} = f_0^{\mu\nu} G_0 f + \bar{f} G_0 f_0^{\mu\nu} + \bar{f} G_0 \Lambda^{\mu\nu} G_0 f + A^{\mu\nu} + A^\nu^{\mu} \quad (19b)$$

where

$$
\Lambda^{\mu\nu} = \Gamma_N^{\mu\nu} + v^{\mu\nu}, \quad (20a)$$
$$
A^{\mu\nu} = \bar{V}^{\mu} G^{\nu} = \bar{V}^{\mu} G_0 T^\nu = \bar{T}^\nu b G_0 V^\nu. \quad (20b)$$
One can thus write the crossing symmetric $T^{\mu\nu}$ in terms of pole and non-pole parts as

\[
T^{\mu\nu} = \Gamma^\mu g^\nu + T^{\mu\nu}_b, \quad T^{\mu\nu}_b = V^{\mu\nu} + \bar{T}^{\mu\nu}_b G_0 V^{\nu},
\]

(21a)

\[
V^{\mu\nu} = \Gamma^\nu g^\mu + \Gamma^\mu_0 + \bar{f}^{\mu\nu}_0 G_0 f + f G_0 \bar{f}^{\mu\nu}_0 + f G_0 A^{\mu\nu} G_0 f + A^{\nu\mu}
\]

\[
= (\bar{F}^{\nu}_0)^\mu + \Gamma^\nu g^\mu + \bar{f} G_0 (V^{\nu})^\mu.
\]

(21b)

We note that the second of Eqs. (21a) still has the basic structure of a Bethe-Salpeter equation although neither the two non-pole $\pi N$ potentials $V^{\mu\nu}$ and $V^{\nu}$, nor the two non-pole $t$ matrices $T^{\mu\nu}_b$ and $\bar{T}^{\mu\nu}_b$ are of the same origin. It is also important to note that Eqs. (21) apply also to the cases where either one or both of the superscripts $\mu$ and $\nu$ refer to the gauging by photons. That is, these equations provide a unified, crossing symmetric description of pion-nucleon elastic scattering, pion photoproduction, and Compton scattering. Moreover, the electromagnetic amplitudes are due to a complete attachment of photons and are therefore manifestly gauge invariant.

### 3.1. Unitarity

In the crossing symmetric formulation, the $\pi N$ potential $V^{\mu\nu}$ of Eq. (21b) is real below two-pion threshold, as is the hybrid potential $V^{\nu}$. It is thus straightforward to obtain the 2-body unitarity relations from Eq. (21a) and the unitarity relations for the hybrid amplitudes, Eqs. (17). One obtains

\[
T^{\mu\nu} - T^{\mu\nu\dagger} = \bar{T}^{\mu\nu} \delta T^{\nu}, \quad T^{\mu\nu}_b - T^{\mu\nu\dagger}_b = \bar{T}^{\mu\nu}_b \delta T^{\nu}_b.
\]

(22)

As for the hybrid case, these have the same form as usual unitarity relations, and differ from them only in that they contain $\pi N$ $t$ matrices of different origin. The task of achieving exact 2-body unitarity for the crossing symmetric amplitudes is therefore a numerical one - the standard potential model of Eqs. (1), and its parameters, need to be adjusted so as to ensure that the single- and double-gauged amplitudes coincide on-shell. Finally, it is worth pointing out that 3-body unitarity could be obtained in a similar fashion by gauging a standard Faddeev-like description of the $\pi\pi N$ system [13].

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