Concurrence of two identical atoms in a rectangular waveguide: linear approximation with single excitation

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Abstract
Waveguide electrodynamics has been attracted much attention; however, a waveguide with a cross section has seldom considered. We study two two-level systems (TLSs) interacting with a reservoir of guided modes confined in a rectangular waveguide of a cross section. For the energy separation of the identical TLSs far away from the cutoff frequencies of transverse modes, the delay differential equations are obtained with single-excitation initial in the TLSs. We accurately solved the delay differential equations for the TLSs interacting with either single transverse mode or double transverse modes of the waveguide. The retarded character of multiple reemissions and reabsorptions of photons between the TLSs is revealed, and the effects of the inter-TLS distance on the time evolution of the concurrence of the TLSs is examined.

Keywords
Entanglement · Concurrence · One-dimensional waveguide · Two two-level atoms · Dipolar approximation · Rotating wave approximation

1 Introduction
Quantum entanglement is a nonlocal correlation of multipartite quantum systems, which distinguishes the quantum world from the classical world. Due to its important role in quantum computation and communication, it is a physical resource which quan-
tum technologies are based on. However, the inevitable interaction of quantum systems with their surrounding environments induces decoherence of quantum systems, which degrades the entanglement of quantum systems. Understanding the dynamics of the entanglement is desirable to be able to manipulate entanglement states in a practical way as well as the question of emergent classicality from quantum theory. Entanglement dynamics is studied under local decoherence (two particles in an entangled state are coupled to its own environment individually), and a peculiar dynamical feature of entangled state is that complete disentanglement is achieved in finite time although complete decoherence takes an infinite time, which is termed “entanglement sudden death” [1–3]. It is well known that the radiation field emitted by an atom may influence the dynamics of its closely spaced atoms [4–8]. The entanglement can be generated in a two-atom system after a finite time by their cooperative spontaneous emission, or the destroyed entanglement may reappear suddenly after its death, which is known as sudden birth of entanglement [9,10].

In quantum network, stationary qubits generate, store and process quantum information at quantum nodes, and flying qubit transmits quantum information between the nodes through quantum channels. A distributed quantum network requires coherently transferring quantum information among stationary qubits, flying qubits, and between stationary qubits and flying qubits. With the development of techniques in quantum information, an alternative waveguide-based quantum electrodynamics (QED) system has emerged as a promising candidate for achieving quantum network [11–16]. In this system, atoms are located at quantum nodes and photons propagating along the network are confined in a waveguide. Inside a one-dimensional (1D) waveguide, the electromagnetic field is confined spatially in two dimensions and propagates along the remaining one, which is called guided modes. The spectrum of the guided modes is continuous. The coupling of the electromagnetic field to a TLS can be increased by reducing the transverse size of the guided modes. Therefore, the study of entanglement dynamics in systems embedded in waveguides is of importance. A waveguide with a cross section has many guided modes [17], e.g., transverse magnetic (TM) modes or transverse-electric (TE) ones. However, the most work only considers one guided mode of the waveguide [7,18–35]. In this paper, we consider the dynamic behavior of bipartite entanglement involving two identical TLSs which is implanted into the 1D rectangular hollow metallic waveguide. Since local addressing is difficult, we assume that there is no direct interaction between the TLSs, the TLSs and the field share initially a single excitation. By considering that the energy separation of the TLSs is far away from the cutoff frequencies of the transverse modes, the delay differential equations are obtained for two TLSs’ amplitudes with the field initially in vacuum, where multiple guided modes are included. The spatial separation of the two TLSs introduces the position-dependent phase factor and the time delay (finite time required for light to travel from one TLS to the other) in each transverse mode. The phase factors and the time delays are different in different transverse modes. The effect of the phase factors and the time delays on the entanglement dynamics of the TLSs are studied in details by considering the TLSs interacting with single transverse mode and double transverse modes.

This paper is organized as follows. In Sect. 2, we introduce the model and establish the notation. In Sect. 3, we derive the relevant equations describing the dynamics of
the system for the TLSs being initially excited and the waveguide mode in the vacuum state and investigate the effect of spatial separation on the dynamics of entanglement between two identical TLSs, which is characterized by concurrence. In Sect. 4, we interpret a possible realization for our model. We make a conclusion in Sect. 5, and Appendix is attached in Sect. 6.

2 Two TLSs in a rectangular waveguide

We consider a rectangular hollow metallic waveguide made of perfect conductors with the area $A = ab$ ($a = 2b$) of its cross section, as shown in Fig. 1. The axes of the waveguide are parallel to the $z$ axis, and the waveguide is infinite along the $z$ axis. Since the translation invariance is maintained along the $z$ axis, all the components of the electromagnetic field describing the guided mode depend on the coordinate $z$ as $e^{ikz}$. The guided mode can be characterized by three wave numbers $\{k_x, k_y, k_z\}$. The spatial confinement of the electromagnetic field along the $xy$ plane makes the appearance of the two nonnegative integers $m$ and $n$, which is related to wave numbers along the $x$ and $y$ directions by $k_x = m\pi/a$ and $k_y = n\pi/b$. In this waveguide, there are two types of guiding modes [17,36,38,39,53]: the transverse magnetic modes $\text{TM}_{mn}$ ($H_z = 0$) and the transverse-electric modes $\text{TE}_{mn}$ ($E_z = 0$). Due to the complexity of the calculation, we study the role of the guided modes in this paper and the evanescent modes are not considered. Two identical TLSs, named TLS 1 and TLS 2, are separately located inside the waveguide at positions $r_1 = (a/2, b/2, z_1)$ and $r_2 = (a/2, b/2, z_2)$, and the distance between the TLSs is denoted by $d = |z_2 - z_1|$. The free Hamiltonian of the TLSs reads

$$H_a = \sum_{l=1}^{2} \hbar \omega_A \sigma^+_l \sigma^-_l$$

where $\omega_A$ are the energy difference between the excited state $|e\rangle$ and the ground state $|g\rangle$ and $\sigma^+_l \equiv |e_l\rangle \langle g_l|$ ($\sigma^-_l \equiv |g_l\rangle \langle e_l|$) is the rising (lowing) atomic operator of the
lth TLS. We assume the dipoles of TLSs are along the z axis. In this case, only the TM\(_{mn}\) guided modes are interacted with the TLSs. The free Hamiltonian of the field reads

\[
H_f = \sum_j \int d{\mathbf{k}} \hbar \omega_{jk} \hat{a}_{jk}^\dagger \hat{a}_{jk}
\]

(2)

where \(\hat{a}_{jk}^\dagger (\hat{a}_{jk})\) is the creation (annihilation) operator of the TM\(_{mn}\) modes. Here, we have replaced \((m, n)\) with the sequence number \(j\), i.e., \(j = 1, 2, 3 \ldots\) denoting TM\(_{11}\), TM\(_{31}\), TM\(_{51}\) \cdots, respectively. We note other modes, i.e., TM\(_{12}\), TM\(_{21}\) \ldots, are not permitted due to the location of TLSs. For each guided mode, the dispersion relation is given by \(\omega_{jk} = \sqrt{\Omega^2_j + c^2 k^2}\), where \(\Omega_{mn} = c \sqrt{(m\pi/a)^2 + (n\pi/b)^2}\) is the cutoff frequency. No electromagnetic field can be guided if their frequency is smaller than the cutoff frequency \(\Omega_1\). The atomic size \(r \ll |k|^{-1}\), so the interaction Hamiltonian between the TLSs and the electromagnetic field written in the dipolar approximation is

\[
H_{\text{int}} = i \sum_{l=1}^2 \sum_j \int d{\mathbf{k}} \hbar \frac{g_{jl}}{\sqrt{\omega_{jk}}} S_l^- (\hat{a}_{jk}^\dagger e^{ikzl} + \hat{a}_{jk} e^{-ikzl}) + h.c.
\]

(3)

where \(g_{jl} = \Omega_j \mu_l / \sqrt{A \pi \varepsilon_0}\) and \(\mu_l\) the magnitude of the dipole of the \(l\)th TLS. We assume that \(\mu_1 = \mu_2 = \mu\) is real. Then, the parameter \(g_{jl}\) becomes [38]

\[
g_j = \frac{\Omega_j \mu \sin \left(\frac{m\pi}{2}\right) \sin \left(\frac{n\pi}{2}\right)}{\sqrt{\hbar A \pi \varepsilon_0}},
\]

(4)

where \(\varepsilon_0\) is the permittivity of free space. The TLS’s position is presented in the exponential function in Eq. (3). We assume that the two-atom system is weakly coupled to the field. Under the rotating wave approximation, the interaction Hamiltonian can be reduced to [38]

\[
H_{\text{int}} = i \sum_{l=1}^2 \sum_j \int d{\mathbf{k}} \hbar \frac{g_j}{\sqrt{\omega_{jk}}} e^{ikzl} S_l^- a_{jk}^\dagger + h.c.
\]

(5)

The total system, the two TLSs and the photons in quantum electromagnetic field, is described by the Hamiltonian

\[
H = H_f + H_a + H_{\text{int}}
\]

(6)

The total system is a closed system. However, each subsystem is an open system. When we are only interested in the dynamics of TLSs, the quantum electromagnetic field can be regarded as an environment.
3 Entanglement dynamics

Any state of the two TLSs is linear superposition of the basis of the separable product states \(|1\rangle = |g_1 g_2\rangle, |2\rangle = |e_1 g_2\rangle, |3\rangle = |g_1 e_2\rangle, \text{ and } |4\rangle = |e_1 e_2\rangle\). Since the number of quanta is conserved in this system, the wave function of the total system can be written as:

\[
|\psi(t)\rangle = b_1 |20\rangle + b_2 |30\rangle + \sum_j \int dk b_{jk} a_{jk}^\dagger |10\rangle
\]  

in single-excitation subspace, where \(|0\rangle\) is the vacuum state of the quantum field. The first term in Eq. (7) presents TLS 1 in the excited state with no excitations in the field, \(b_1(t)\) is the corresponding amplitude, the second term in Eq. (7) presents TLS 2 in the excited state with photons in the vacuum, and \(b_2(t)\) is the corresponding amplitude, whereas the third term in Eq. (7) describes all TLSs in the ground state with a photon emitted at a mode \(k\) of the TM\(_j\) guided mode, and \(b_{jk}(t)\) is the corresponding amplitude. The initial state of the system is denoted by the amplitudes \(b_1(0), b_2(0), b_{jk}(0) = 0\). The Schrödinger equation results in the following coupled equation of the amplitudes (for the derivation, see “Appendix”)

\[
\dot{b}_1(t) = -i \omega_A b_1(t) - \sum_j \int dk \frac{b_{jk}(t) g_j}{\sqrt{\omega_{jk}}} e^{-ikz_1}
\]

\[
\dot{b}_2(t) = -i \omega_A b_2(t) - \sum_j \int dk \frac{b_{jk}(t) g_j}{\sqrt{\omega_{jk}}} e^{-ikz_2}
\]

\[
\dot{b}_{jk}(t) = -i \omega_{jk} b_{jk}(t) + \frac{g_j}{\sqrt{\omega_{jk}}} [b_1(t)e^{ikz_1} + b_2(t)e^{ikz_2}]
\]

As the coupling strength is quite weak compared to the atomic frequency \(\omega_A\) and field frequencies \(\omega_{jk}\), we introduce three new variables to remove the high-frequency effect,

\[
b_1(t) = B_1(t) e^{-i\omega_A t},
\]

\[
b_2(t) = B_2(t) e^{-i\omega_A t},
\]

\[
b_{jk}(t) = B_{jk}(t) e^{-i\omega_{jk} t},
\]

we note that Eq. (9) transforms the state in Schrödinger picture represented by coefficients \(b_1(t), b_2(t)\) and \(b_{jk}(t)\) into the state in the interaction picture denoted by coefficients \(B_1(t), B_2(t)\) and \(B_{jk}(t)\). Then, formally integrate equation of \(B_{jk}(t)\), which is later inserted into the equations for \(B_1(t)\) and \(B_2(t)\). The probability amplitude for one TLS being excited is determined by two coupled integro-differential equations. Assuming that the atomic frequency \(\omega_A\) is far away from the cutoff frequencies \(\Omega_j\) (see Fig. 2), we can expand \(\omega_{jk}\) around \(\omega_A\) up to the linear term

\[
\omega_{jk} = \omega_A + \nu_j (k - k_{j0}), k > 0
\]

\[
\omega_{jk} = \omega_A - \nu_j (k + k_{j0}), k < 0
\]
where the wave number of the emitted radiation $k_{0j} = \sqrt{\omega_A^2 - \Omega_j^2}/c$ is determined by $\omega_{jk} = \omega_A$, and the group velocity

$$v_j \equiv \frac{d\omega_{jk}}{dk}\bigg|_{k=k_{0j}} = \frac{c\sqrt{\omega_A^2 - \Omega_j^2}}{\omega_A}$$

is different for different TM$_j$ guided modes. Integrating over all wave vectors $k$ gives rise to a linear combination of $\delta(t - \tau - \tau_j)$ and $\delta(t - \tau_j)$, where $\tau_j = d/v_j$ is the time delay taking by a photon traveling from one TLS to the other TLS in the given transverse mode $j$. The differential equations governing the dynamics of two TLSs read (for the derivation, see “Appendix”)

\begin{align}
(\partial_t + \gamma) B_1(t) &= -\sum_j \gamma_j e^{i\varphi_j} B_2\left(t - \tau_j\right) \Theta\left(t - \tau_j\right), \\
(\partial_t + \gamma) B_2(t) &= -\sum_j \gamma_j e^{i\varphi_j} B_1\left(t - \tau_j\right) \Theta\left(t - \tau_j\right),
\end{align}

where we have defined the phase $\varphi_j = k_{0j}d$ due to the distance between the TLSs, the decay rate $\gamma_j = \pi g_j^2/(v_j \omega_A)$ caused by the interaction between the TLSs and the vacuum field in a given transverse mode $j$. $\Theta(x)$ is the Heaviside unit step function, i.e., $\Theta(x) = 1$ for $x > 0$, and $\Theta(x) = 0$ for $x < 0$. The decay to all TM$_j$ modes is denoted by $\gamma = \sum_j \gamma_j$, and the retard effect [41–49] has been implied by the symbol $\tau_j$. At times less than minimum $\tau_j$, two TLSs decay as if they are isolated in rectangular waveguide. After the time minimum $\tau_j$, the TLS recognizes the other TLS due to its absorption of photons. As time goes on, reemissions and reabsorptions of photons by two TLSs might produce interference, which leads to the change of atomic upper-state population. It is convenient to write Eq. (12) in the Dicke symmetric state $|s\rangle = (|2\rangle + |3\rangle)/\sqrt{2}$ and antisymmetric state $|a\rangle = (|2\rangle - |3\rangle)/\sqrt{2}$

\begin{align}
(\partial_t + \gamma) C_s(t) &= -\sum_j \gamma_j e^{i\varphi_j} C_s\left(t - \tau_j\right) \Theta\left(t - \tau_j\right), \\
(\partial_t + \gamma) C_a(t) &= \sum_j \gamma_j e^{i\varphi_j} C_a\left(t - \tau_j\right) \Theta\left(t - \tau_j\right),
\end{align}

where

\begin{align}
C_s(t) &= \frac{B_1(t) + B_2(t)}{\sqrt{2}}, \\
C_a(t) &= \frac{B_1(t) - B_2(t)}{\sqrt{2}}.
\end{align}

Equation (13) allows either TLS 1 or TLS 2 to be excited with equal probability. They are degenerate eigenstates of Hamiltonian $H_a$. The equations for the amplitudes of the Dicke states are not coupled.
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Fig. 2 (Color online) Dispersion relation of the TM$_j$ guiding modes of the waveguide. $\Omega_1 = \sqrt{5c\pi}/a$, $\Omega_2 = \sqrt{13c\pi}/a$ and $\Omega_3 = \sqrt{29c\pi}/a$. a the atomic frequency $\omega_A \simeq (\Omega_1 + \Omega_2)/2$, the photon propagate in the lowest mode with wave number $k_{10}$ and $-k_{10}$; b the atomic frequency $\omega_A \simeq (\Omega_2 + \Omega_3)/2$, the photon propagate in different modes with wave number $\pm k_{10}$ and $\pm k_{20}$.

To measure the amount of the entanglement, we use concurrence as the quantifier [50]. By taking a partial trace over the waveguide degrees of freedom, the reduced density matrix of the two TLSs is of an X-form in the two-qubit standard basis $\{|1\rangle, |2\rangle, |3\rangle, |4\rangle\}$

$$\rho(t) = \begin{pmatrix}
0 & \sum_j \int dk |B_{jk}(t)|^2 & 0 & 0 \\
0 & |B_1(t)|^2 & B_1(t)B_2^*(t) & 0 \\
0 & B_2(t)B_1^*(t) & |B_2(t)|^2 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}$$

(15)

where $\rho_{14}(t) = \rho_{41}(t) = \rho_{44}(t) = 0$ since the system is constrained in a single-excitation subspace. The concurrence for this type of state can be calculated easily as [47]

$$C(t) = \max(0, 2 |B_1(t)B_2^*(t)|)$$

(16)

which can also be expressed as the function of the amplitudes of the Dicke states by the relation in Eq. (14).

We expect that the position-dependent phase factors $e^{i\varphi_j}$ and the delay times $\tau_j$ will lead to a modification of the entanglement among the TLSs.

3.1 Single transverse mode

In the frequency band between $\Omega_{11}$ and $\Omega_{31}$, the waveguide is said to be single mode. The TLSs with the transition frequency $\omega_A \in (\Omega_{11}, \Omega_{31})$ only emit photons into the TM$_{11}$ ($j = 1$) guided mode. In this case, taking the Laplace transform of Eq. (13) and using the geometric series expansion, the time behavior of Dicke states can be obtained by the inverse Laplace transform,

$$C_s(t) = C_{s0} \sum_{n=0}^{\infty} \frac{(-\gamma_1 e^{i\varphi_1})^n}{n!} t^n e^{-\gamma_1 t}.$$  

(17a)
Fig. 3 (Color online) Concurrence between the TLSs as functions of the dimensionless time $t/\tau_1$ with initial condition $C_s(0) = 1$ for different phases $\varphi_1 = 2n\pi$ (black solid curve), $\varphi_1 = 2n\pi + \pi$ (blue dotted curve), $\varphi_1 = 2n\pi + \pi/2$ (green dot-dashed curve), $\varphi_1 = 2n\pi + \pi/4$ (red dashed curve) in (a) $n = 1.0 \times 10^9$, (b) $n = 8.0 \times 10^9$, (c) $n = 8.0 \times 10^{10}$. We have set the following parameters: $a = 2b$, $\omega_{A} = (\Omega_{11} + \Omega_{31})/2$, $\gamma_1\lambda_1/v_1 = 1.5 \times 10^{-10}$.
where \( t_n = t - n\tau_1 \), \( C_{s0} \) and \( C_{a0} \) are the initial amplitudes. The time axis is divided into intervals of length \( \tau_1 \). A step character is presented in Eq. (17). For \( t \in [0, \tau_1] \), both amplitudes, \( C_s(t) \) and \( C_a(t) \), decay exponentially with decay rate \( \gamma_1 \). The underlying physics is that one TLS requires at least the time \( \tau_1 \) to recognize the other TLS. For \( t \in [\tau_1, 2\tau_1] \), the absorption and reemission of light by each TLS produce the interference, which results in a energy change between two TLSs. From Eq. (17), one can observe that there is a \( \pi \) phase difference between the amplitudes \( C_s(t) \) and \( C_a(t) \). Hence, we assume that the two TLSs are initially prepared in the symmetric state \( C_{s0} = 1 \) to study the effect of the inter-TLS distance on the dynamics of entanglement between the TLSs. In this case, the concurrence takes the maximum value between 0 and \( |C_s(t)|^2 \). In Fig. 3, we have numerically plotted the concurrence as a function of time \( t \) in units of \( \tau_1 \) with \( \gamma_1\lambda_1/\nu_1 = 1.5 \times 10^{-10} \), where the wavelength \( \lambda_1k_{10} = 2\pi \). In the time interval \( t \in [0, \tau_1] \), two TLSs radiate spontaneously, so the concurrence decays exponentially with time. As time goes on, the inter-TLS distance has influenced the entanglement dynamics via phase \( \varphi_1 \) and delay time \( \tau_1 \). When the two TLSs are close together, two TLSs act collectively, and the system dynamics is independent of the finite propagating time of the light, which is shown in Fig. 3a with \( \gamma_1\tau_1 \ll 1 \). There is stationary two-TLS entanglement when the inter-TLS distance equals an odd integer number of \( \lambda_1/2 \), (i.e., \( \varphi_1 = 2n\pi + \pi \)). And a small deviation of the special position leads to the entanglement decaying asymptotically to zero. The entanglement loses fast when the inter-TLS distance equals an integer number of \( \lambda_1 \) corresponding to \( \varphi_1 = 2n\pi \). For \( \gamma_1\tau_1 \ll 1 \), the dependence of the entanglement on phase in Fig. 3a can be understood by letting \( \tau_1 \to 0 \). In this case, the amplitude of state \( |s\rangle \) becomes
\[ C_s(t) = C_{s0} \exp \left[ -i \gamma_1 (1 + \cos \varphi_1) - it \gamma_1 \sin \varphi_1 \right] \]  

(18)

It can be observed from Eq. (18) that \( |C_s(t)| \) exponentially decays with time, it decays fast when \( \varphi_1 = 2n\pi \) and keeps its initial value when \( \varphi_1 = 2n\pi + \pi \). Although one can explain the relation of entanglement with phase by Eq. (18), the probability of finding the TLSs in the initial state is less than unity in Fig. 3a when \( \varphi_1 = 2n\pi + \pi \). Hence, the stationary two-TLS entanglement indicates a superposition of the symmetry state in the absence of photons and the ground TLS state in the presence of a photon. As the inter-TLS separation increases a little bit to meet \( \gamma_1 \tau_1 \sim 1 \) in Fig. 3b, both the decay time and the phase play important roles due to the interference. The interference produced by multiple reemissions and re absorptions of photon results in an oscillatory entanglement. Panel (c) of Fig. 3 illustrates the dynamics of entanglement for a larger inter-TLS distance with \( \gamma_1 \tau_1 \gg 1 \). It can be observed that the phase does not make any sense. At early time, each initially excited TLS emits light to the waveguide, and the entanglement begins to decrease abruptly from one. Then, the concurrence keeps small and approximates to zero until the time \( t = \tau_1 \), at which the excitation gets absorbed by each TLS. As soon as the emitted photon returns to the TLSs, the entanglement is created. Then, the decrease of entanglement begins. The entanglement of the TLSs exhibits peaks due to the iteration of the process where a photon emitted by one of the atoms is reabsorbed by another atom, but its periodic maxima are reduced in magnitude as \( t \) increases because the energy is carried away from TLSs by the forward-going waves emitted by TLS 1 and the backward-going waves emitted by TLS 2. In this case, the amplitudes are approximately described by

\[ C_s(t) \propto \left( -\gamma_1 e^{i\varphi_1} \right)^n \frac{n^n}{n!} e^{-\gamma_1 t_n} \]  

(19)

in each time interval \([n \tau_1, (n + 1) \tau_1]\).

For TLSs initially in the antisymmetry state, the concurrence exhibits the similar behavior to the symmetry state with a \( \pi \) phase difference. However, when the inter-TLS spacing \( d = 0 \), the antisymmetry state is a dark state which does not interact with the electromagnetic field; it means the probability of finding the TLSs in the initial state is unity at any time, so the concurrence is unchanged and remains its initial value one (see the solid black line in Fig. 4). We note that \( d = 0 \) cannot be achieved in reality. We also plot the concurrence as a function of the dimensionless time \( t/\tau_1 \) for phase \( \varphi_1 = 2n\pi \) with \( n = 2.5 \times 10^9 \) (the green dot–dashed line) and \( n = 8.0 \times 10^{10} \) (red dashed line) in Fig. 4, where all the position-dependent phase factors \( e^{i\varphi_1} \) are equal. It can be seen that the cooperative effect become lower and lower as the inter-TLS distance increases, so does the maximum of concurrence.

### 3.2 Two transverse modes

A TLS in its upper state radiates waves into the continua resonant with itself. As the transition frequency of the TLSs increases, there are additional guided modes taking part in the interaction with the TLSs. Due to their different wave numbers, the inter-
The concurrence that only TM_{11} mode is considered is presented by blue dashed lines. The concurrence that both TM_{11} and TM_{31} modes are considered is presented by the red solid lines. We have set the following parameters: $a = 2b$, $\omega_A = (\Omega_{31} + \Omega_{51})/2$, $\gamma_1 \lambda_1/v_1 = 1.1 \times 10^{-11}$.

TLS distance introduces different flight time $\tau_j$ of light between the TLSs and different phases $\phi_j$. From the definition of $\tau_j$ and $\phi_j$, we know that $\tau_j < \tau_{j+1}$ and $\phi_j < \phi_{j+1}$ for the given distance $d$. In this section, we assume that the transition frequency of the TLSs is smaller than the cutoff frequency $\Omega_{51}$ and larger than the cutoff frequency $\Omega_{31}$; this means that the emit photons will propagate in guided modes TM_{11} and TM_{31}. The delay differential equation in Eq. (13) is reduced to

\begin{align}
(\partial_t + \gamma) C_s(t) &= -\alpha_1 C_s(t - \tau_1) \Theta (t - \tau_1) \\
&\quad -\alpha_2 C_s(t - \tau_2) \Theta (t - \tau_2) \\
(\partial_t + \gamma) C_a(t) &= \alpha_1 C_a(t - \tau_1) \Theta (t - \tau_1) \\
&\quad +\alpha_2 C_a(t - \tau_2) \Theta (t - \tau_2)
\end{align}

(20a) (20b)

where $\gamma = \gamma_1 + \gamma_2$ and $\alpha_j = \gamma_j e^{i\phi_j} (j = 1, 2)$. We first discuss the case with $d = 0$. It is clear by inspection of Eq. (20) that the initial entanglement determined by the state $|s\rangle$ decreases more quickly in time; on the contrary, the initial entanglement determined by the state $|a\rangle$ does not change in time as shown by the black lines in Fig. 4.

To analyze the influence of the number of the transverse modes on the entanglement which interact with TLSs, we fix the phase $\varphi_1 = 2n\pi$ with $n = 3.1 \times 10^9$ in Fig. 5a, $n = 7.5 \times 10^9$ in Fig. 5b, $n = 2.3 \times 10^{10}$ in Fig. 5c and $n = 2.3 \times 10^{12}$ in Fig. 5d.
We plotted the time behavior of the concurrence between TLS by only considering the TM$_{11}$ mode in Eq. (20), which is shown in blue dashed lines in Fig. 5. The red solid lines in Fig. 5 present the time behavior of the concurrence by considering both TM$_{11}$ and TM$_{31}$ modes in Eq. (20). It can be found that increasing the number of the transverse modes which interact with TLSs leads to exponentially decay with a rate $\gamma_1 + \gamma_2$ up to time $t = \tau_1$. After this, the phase $\varphi_1$ begins to have an effect until time $t = \tau_2$. After time $\tau_2$, the phase $\phi_2$ and delay time $\tau_2$ in TM$_{31}$ mode play an equal role. The part of excitation emitted into TM$_{11}$ mode is immediately reabsorbed by the other one, but the part of excitation emitted into TM$_{31}$ mode undergoes delay; however, wave interference still produced at each exchange of the excitation between the TLSs in TM$_{31}$ mode. In this case, we can solve Eq. (20) by letting $\tau_1 \to 0$

$$C_s(t) = C_{s0} \exp \left[ -(\gamma_1 + \alpha_1)t - (\gamma_2 + \alpha_2)t \right].$$  

$$C_a(t) = C_{a0} \exp \left[ -(\gamma_1 - \alpha_1)t - (\gamma_2 - \alpha_2)t \right].$$  

The norm of amplitudes is completely determined by phases leading to exponential decay of the entanglement. The red solid line in panel (b) exhibits behavior different from that in panel (a), indicating that the $\varphi_2$ and delay time $\tau_2$ play an equal role. The part of excitation emitted into TM$_{11}$ mode is immediately reabsorbed by the other one, but the part of excitation emitted into TM$_{31}$ mode undergoes delay; however, wave interference still produced at each exchange of the excitation between the TLSs in TM$_{31}$ mode. In this case, we can solve Eq. (20) by letting $\tau_1 \to 0$

$$C_s = C_{s0} \sum_{n=0}^{\infty} \frac{(-\alpha_2)^n}{n!} e^{-(\gamma_1 + \alpha_1)(t-n\tau_2)} (t - n\tau_2)^n$$  

$$C_a = C_{a0} \sum_{n=0}^{\infty} \frac{\alpha_2^n}{n!} e^{-(\gamma_1 - \alpha_1)(t-n\tau_2)} (t - n\tau_2)^n$$  

Panel (c) shows that as $d$ increased, the delay time should be taken into account in TM$_{11}$ mode besides the wave interference. In this case, use the same method of getting Eq. (17), we solve Eq. (20) and obtain the solutions

$$C_s = C_{s0} \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{C_{nk}^k \alpha_1^n \alpha_2^{-k}}{n!} e^{-\gamma(t-n\tau_2)},$$  

$$C_a = C_{a0} \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{C_{nk}^k \alpha_1^n \alpha_2^{-k}}{n!} e^{-\gamma(t-n\tau_2)},$$
are coherent sums over contributions starting at different instants of time \( \tau_{nk} = k \tau_1 + (n - k) \tau_2 \), where \( C_n^k = \frac{k!}{n!(n-k)!} \). Each term of the sum has a well-defined phase and are damped by an exponential function at rate \( \gamma \). Interference is possible if the amplitudes do not decay appreciably over the time \( \tau_{nk} \). As \( d \) is large enough so that \( \min_{p,q} |p \tau_2 - q \tau_1| \gg 1 \) with nonnegative integers \( p \) and \( q \) which are not zero at the same time, the phase factor plays no role as shown in panel (d). The wave packets of the emitted excitation are bouncing back and forth between two TLSs until its intensity is damped to zero. So there are collapses and revivals of the concurrence until the amplitude of the revivals damped to zero.

4 Possible realization

There might be one possible realization of our system in the microwave regime. Replace the superconducting cavity by a cold metallic waveguide (MWG) in Refs. [51–53], then let Rydberg atoms cross one by one the MWG. Circular Rydberg states with principal numbers 51 and 50 are the states \( |e\rangle \) and \( |g\rangle \), with transition frequency \( \omega_A = 2\pi \times 51.1 \text{ GHz} \) and dipole moment \( d \sim 1250e a_B \) where \( e \) is the charge of an electron and \( a_B \) is the Bohr radius [51]. For single transverse mode case, we take the square waveguide of \( a \approx 8.55 \text{ mm}(a = 2b) \) and the atomic transition frequency \( \omega_A \approx (\Omega_{11} + \Omega_{31})/2 \) since the cutoff frequencies of the first two transverse magnetic modes are \( \Omega_{11} = 2\pi \times 39.2 \text{ GHz} \) and \( \Omega_{31} = 2\pi \times 63.3 \text{ GHz} \). If we take \( a \approx 13.19 \text{ mm} \), the cutoff frequencies of the first three transverse magnetic modes are \( \Omega_{11} = 2\pi \times 25.4 \text{ GHz} \), \( \Omega_{31} = 2\pi \times 41.0 \text{ GHz} \) and \( \Omega_{51} = 2\pi \times 61.2 \text{ GHz} \), respectively. In this case, \( \omega_A \approx (\Omega_{51} + \Omega_{31})/2 \) and TLSs interact with the first two transverse magnetic modes TM\(_{11}\) and TM\(_{31}\).

5 Conclusion

We have studied the effects of the inter-TLS distance on the entanglement properties of two identical TLSs located inside a rectangular hollow metallic waveguide of transverse dimensions \( a \) and \( b \). When the energy separation of the TLS is far away from the cutoff frequencies of the transverse modes and there is single excitation in the system, the Schrodinger equation for the wave function with single-excitation initial in the TLSs is reduced to the delay differential equations for the amplitudes of two TLSs, where phase factors and delay times are induced by the finite distance between the TLSs. The delay differential equations are solved exactly for the TLSs interacting with either single transverse mode or double transverse modes of the waveguide, which directly reveals the retarded character of multiple reemissions and reabsorptions of photons between the TLSs. For the inter-TLS distance \( d = 0 \), there exists an antisymmetry state decoupled with the field modes, so the entanglement can be generated if the TLSs are initial in a separate state, later trapped in the antisymmetry state. As the TLSs are close together such that the time delay \( \max \{\tau_j\} \) is much smaller than the TLS decay time \( \gamma^{-1} \), the excitation emitted by one TLS into the field is absorbed immediately by the other. The dynamic of the entanglement is dramatically affected.
by phases, leading to an enhanced and inhibited exponential decay of the concurrence when only one transverse mode are considered, and an exponential decay when more transverse modes are involved. As the inter-TLS distance increases, both phases and delay times affect the concurrence. There is a proper delay of reabsorption after reemission of photons, but interference is possible if the amplitudes of TLSs do not decay appreciably over time $\tau_{nk}$. As $d$ is large enough so that $\min_{p,q} |p \tau_2 - q \tau_1| \gg 1$ with nonnegative integers $p$ and $q$ which are not zero at the same time, the phase factor plays no role. There appear collapses and revivals of the entanglement of the TLSs. We note that our studies focus on the dependence of the concurrence on the inter-TLS distance, but it is easy to study the dependence of the concurrence on the initial state of the system with the exact solution.

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6 Appendix

The detailed derivation of Eqs. (8) and (12).

We consider a system of two two-level atoms weakly coupled to a rectangular hollow metallic waveguide made of perfect conductors. Under the dipolar and rotating wave approximation, the total Hamiltonian in the Schrödinger picture is

$$H = 2 \sum_{l=1}^{2} \hbar \omega \sigma^+_l \sigma^-_l + \sum_j \int d k h \omega_{jk} \hat{a}^+_j \hat{a}_j
$$

$$+ i \sum_{l=1}^{2} \sum_j \int d k h \frac{g_j}{\sqrt{\omega_{jk}}} \left( S^+_l \hat{a}^+_j e^{i k z_l} - S^+_l \hat{a}_j e^{-i k z_l} \right) \quad (24)$$

Any state of the two TLSs is linear superposition of the basis of the separable product states $|1\rangle = |g_1 g_2\rangle$, $|2\rangle = |e_1 g_2\rangle$, $|3\rangle = |g_1 e_2\rangle$ and $|4\rangle = |e_1 e_2\rangle$. While in a single-excitation subspace, the basis $\{ |20\rangle, |30\rangle, \hat{a}^+_j |10\rangle \}$ is complete to describe the system because the number of quanta is conserved. Here, $|0\rangle$ is the vacuum state of the quantum field. So the wave function of the total system can be written as

$$|\psi(t)\rangle = b_1 |20\rangle + b_2 |30\rangle + \sum_j \int d k b_j |10\rangle \quad (25)$$

$b_1(t)$, $b_2(t)$ and $b_{jk}(t)$ are corresponding amplitude. The wave function of the total system obeys the Schrödinger equation

$$i \hbar \frac{\partial |\psi(t)\rangle}{\partial t} = H |\psi(t)\rangle \quad (26)$$
substituting Eqs. (24) and (25) into Eq. (26), we obtain

\[
\dot{b}_1 |20\rangle + \dot{b}_2 |30\rangle + \sum_j \int dk \dot{b}_{jk} \hat{a}_{jk}^\dagger |10\rangle
\]

\[
= -\frac{i}{\hbar} \left[ \sum_{l=1}^{2} \hbar \omega A \sigma_l^+ \sigma_l^- + \sum_j \int dk \hbar \omega_{jk} \hat{a}_{jk}^\dagger \hat{a}_{jk} \right]
\]

\[
+ i \sum_{l=1}^{2} \sum_j \int dk \hbar \frac{g_j}{\sqrt{\omega_{jk}}} (S_l^- \hat{a}_{jk}^\dagger e^{ikz_l} - S_l^+ \hat{a}_{jk} e^{-ikz_l}) \left( b_1 |20\rangle + b_2 |30\rangle + \sum_j \int dk b_{jk} \hat{a}_{jk}^\dagger |10\rangle \right) \]

\[
= -i \omega_A b_1 |20\rangle - i \omega_A b_2 |30\rangle - i \sum_j \int dk \omega_{jk} b_{jk} \hat{a}_{jk}^\dagger |10\rangle
\]

\[
+ \sum_j \int dk \hbar \frac{g_j b_1}{\sqrt{\omega_{jk}}} e^{ikz_1} \hat{a}_{jk}^\dagger |10\rangle
\]

\[
+ \sum_j \int dk \hbar \frac{g_j b_2}{\sqrt{\omega_{jk}}} e^{ikz_2} \hat{a}_{jk}^\dagger |10\rangle
\]

\[- \sum_j \int dk \hbar \frac{g_j b_{jk}}{\sqrt{\omega_{jk}}} e^{-ikz_1} |20\rangle - \sum_j \int dk \hbar \frac{g_j b_{jk}}{\sqrt{\omega_{jk}}} e^{-ikz_2} |30\rangle \]

\[
= \left( -i \omega_A b_1 - \sum_j \int dk \hbar \frac{g_j b_{jk}}{\sqrt{\omega_{jk}}} e^{-ikz_1} \right) |20\rangle
\]

\[
+ \left( -i \omega_A b_2 - \sum_j \int dk \hbar \frac{g_j b_{jk}}{\sqrt{\omega_{jk}}} e^{-ikz_2} \right) |30\rangle
\]

\[
+ \sum_j \int dk \left( -i \omega_{jk} b_{jk} + \frac{g_j b_1}{\sqrt{\omega_{jk}}} e^{ikz_1} + \frac{g_j b_2}{\sqrt{\omega_{jk}}} e^{ikz_2} \right) \hat{a}_{jk}^\dagger |10\rangle \]  

(27)

taking the overlap of the both sides of Eq. (27) with |20⟩, |30⟩ and ⟨10| \hat{a}_{jk}, respectively, gives

\[
\dot{b}_1(t) = -i \omega_A b_1(t) - \sum_j \int dk \frac{g_j b_{jk}(t)}{\sqrt{\omega_{jk}}} e^{-ikz_1} \]  

(28a)

\[
\dot{b}_2(t) = -i \omega_A b_2(t) - \sum_j \int dk \frac{g_j b_{jk}(t)}{\sqrt{\omega_{jk}}} e^{-ikz_2} \]  

(28b)

\[
\dot{b}_{jk}(t) = -i \omega_{jk} b_{jk}(t) + \frac{g_j}{\sqrt{\omega_{jk}}} [b_1(t)e^{ikz_1} + b_2(t)e^{ikz_2}] \]  

(28c)
As the coupling strength is quite weak compared to the atomic frequency $\omega_A$ and field frequencies $\omega_{jk}$, we introduce three new variables to remove the high-frequency effect,

\begin{align}
    b_1(t) &= B_1(t)e^{-i\omega_A t}, \\
    b_2(t) &= B_2(t)e^{-i\omega_A t}, \\
    b_{jk}(t) &= B_{jk}(t)e^{-i\omega_{jk} t},
\end{align}

using Eqs. (29), (28) become

\begin{align}
    \dot{B}_1(t) &= -\int dk \frac{g_j B_{jk}(t)}{\omega_{jk}} e^{-i(\omega_{jk} - \omega_A) t} e^{-i k z_1} \\
    \dot{B}_2(t) &= -\int dk \frac{g_j B_{jk}(t)}{\omega_{jk}} e^{-i(\omega_{jk} - \omega_A) t} e^{-i k z_2} \\
    \dot{B}_{jk}(t) &= \frac{g_j e^{ik z_1}}{\sqrt{\omega_{jk}}} [B_1(t) + B_2(t)e^{ik z_0}] e^{i(\omega_{jk} - \omega_A) t}
\end{align}

where $z_0 = z_2 - z_1$. Integrating equation of $B_{jk}(t)$ and inserting it into the equations for $B_1(t)$ and $B_2(t)$, we have

\begin{align}
    \dot{B}_1(t) &= -\sum_j \int_0^t d\tau \int dk \frac{g_j^2}{\omega_{jk}} [B_1(\tau) + B_2(\tau)e^{ik z_0}] e^{-i(\omega_{jk} - \omega_A)(t-\tau)} \\
    \dot{B}_2(t) &= -\sum_j \int_0^t d\tau \int dk \frac{g_j^2}{\omega_{jk}} [B_1(\tau)e^{ik z_0} + B_2(\tau)] e^{-i(\omega_{jk} - \omega_A)(t-\tau)}
\end{align}

Then, take the linear expansion of dispersion relationship

\begin{align}
    \omega_{jk} &= \omega_A + v_j (k - k j_0), k > 0 \quad (32a) \\
    \omega_{jk} &= \omega_A - v_j (k + k j_0), k < 0 \quad (32b)
\end{align}

into Eq. (31) and we assume the coefficient $\frac{g_j^2}{\omega_{jk}}$ is independent with frequencies, $\frac{g_j^2}{\omega_{jk}} \rightarrow \frac{g_j^2}{\omega_A}$, which is a variant of Weisskopf–Wigner theory [54], Eq. (31) can be expressed as

\begin{align}
    \dot{B}_\mu(t) &= -\sum_j \frac{g_j^2}{\omega_A} \int_0^t d\tau \int_0^\infty dk [B_\mu(\tau) + B_\nu(\tau)e^{i(\omega_{jk} - \omega_A)(t-\tau)}] e^{i(-1)^{\nu} k z_0} e^{i v_j (k-k j_0)(t-\tau)} \\
    &\quad -\sum_j \frac{g_j^2}{\omega_A} \int_0^t d\tau \int_{-\infty}^0 dk [B_\mu(\tau) + B_\nu(\tau)e^{i(\omega_{jk} - \omega_A)(t-\tau)}] e^{i v_j (k+k j_0)(t-\tau)}
\end{align}
\begin{align*}
&= - \sum_j \frac{g_j^2}{\omega_A} \int_0^t d\tau \int_0^\infty dk \left[ B_\mu(\tau) + B_\nu(\tau)e^{i(-1)^\nu k z_0})e^{-i v_j(k-k_j0)(t-\tau)} \right] \\
&- \sum_j \frac{g_j^2}{\omega_A} \int_0^t d\tau \int_0^\infty dk \left[ B_\mu(\tau) + B_\nu(\tau)e^{-i(-1)^\nu k z_0})e^{-i v_j(k-k_j0)(t-\tau)} \right] \\
&= - \sum_j \gamma_j \int_0^t d\tau \left[ 2B_\mu(\tau)\delta(t-\tau) + B_\nu(\tau)\delta \left( t - \frac{(-1)^\nu z_0}{v_j} \right) \right] e^{i v_j k_j0(t-\tau)} \\
&+ B_\nu(\tau)\delta \left( t - \tau + \frac{(-1)^\nu z_0}{v_j} \right) e^{i v_j k_j0(t-\tau)} \\
&\text{(33)}
\end{align*}

where \( \mu, \nu = 1, 2 (\mu \neq \nu) \), \( \gamma_j = \frac{\pi g_j^2}{v_j \omega_A} \). The integral bound are approximated as \( \int_0^{+\infty} dk \simeq \frac{1}{2} \int_{-\infty}^{+\infty} dk \), since we are only interested in photons with a narrow bandwidth in the vicinity of \( \omega_A \) [55]. Then, from Eq. (33) we know that for \( \mu = 1, \nu = 2 \)

\begin{align*}
\dot{B}_1(t) &= - \sum_j \gamma_j \int_0^t d\tau \left[ 2B_1(\tau)\delta(t-\tau) + B_2(\tau)\delta \left( t - \frac{z_0}{v_j} \right) \right] e^{i v_j k_j0(t-\tau)} \\
&+ B_2(\tau)\delta \left( t - \tau + \frac{z_0}{v_j} \right) e^{i v_j k_j0(t-\tau)} \\
&= - \sum_j \gamma_j \left[ B_1(\tau) + e^{i k_j0d} B_2 \left( t - \frac{d}{v_j} \right) \Theta \left( t - \frac{d}{v_j} \right) \right] \\
&\text{(34)}
\end{align*}

and for \( \mu = 2, \nu = 1 \)

\begin{align*}
\dot{B}_2(t) &= - \sum_j \gamma_j \int_0^t d\tau \left[ 2B_2(\tau)\delta(t-\tau) + B_1(\tau)\delta \left( t - \frac{z_0}{v_j} \right) \right] e^{i v_j k_j0(t-\tau)} \\
&+ B_1(\tau)\delta \left( t - \tau - \frac{z_0}{v_j} \right) e^{i v_j k_j0(t-\tau)} \\
&= - \sum_j \gamma_j \left[ B_2(\tau) + e^{i k_j0d} B_1 \left( t - \frac{d}{v_j} \right) \Theta \left( t - \frac{d}{v_j} \right) \right] \\
&\text{(35)}
\end{align*}

where \( d = |z_0| \), \( \Theta(x) \) is the Heaviside unit step function and the formula \( \int_0^t f(\tau)\delta(t-\tau)d\tau = \frac{1}{2} f(t) \) is applied.

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