Fermion number non-conservation and gravity

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Abstract

It is shown that in the Einstein-Yang-Mills (EYM) theory, as well as in the pure flat space Yang-Mills (YM) theory, there always exists an opportunity to pass over the potential barrier separating homotopically distinct vacuum sectors, because the barrier height may be arbitrarily small. However, at low energies all the overbarrier histories are suppressed by the destructive interference. In the pure YM theory the situation remains the same for any energies. In the EYM theory on the other hand, when the energy is large and exceeds the ground state EYM sphaleron mass, the constructive interference occurs instead. This means that in the extreme high energy limit the exponential suppression of the fermion number violation in pure YM theory is removed due to gravitational effects.

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1 Introduction

The discovery of the vacuum periodicity in a Yang-Mills (YM) theory \cite{1} gave rise to the theoretical description of baryon number non-conservation \cite{2} caused by the anomalous violation of chirality \cite{3}. The standard approach to understanding this phenomenon uses semiclassical barrier penetration between topologically inequivalent vacua, where the tunneling solutions are Euclidean instantons \cite{4}. The amplitude of such a penetration at zero energy is of the order of $\exp(-S_{\text{Eucl}})$, which is incredibly small because of the large value of the instanton action $S_{\text{Eucl}}$. In a pure Yang-Mills theory one can not expect the amplitude to become large with growing energy, because the theory does not contain any dimensional parameters to set the energy scale.

When scalar fields are involved, as in Yang-Mills-Higgs (YMH) theory, a natural energy scale arises, and one may think that the transition amplitude will not be small if the energy is of the order of some threshold value. Indeed, the existence of the sphaleron solution in the electroweak YMH theory \cite{5} implies that the fermion number violating processes may become unsuppressed if the energy (or temperature) is of the order of the sphaleron mass \cite{6, 7}. The idea is that, instead of tunnel penetration under the barrier, the field configuration can pass over it and the value of the sphaleron energy is the minimal barrier height.

If the energy is very high, then the symmetry is restored. The YMH sphalerons then dissociate, and the fermion number violating processes again become suppressed \cite{8}. However, another universal energy scale arises naturally at extremely high energies, namely, the gravitational Planck scale. The existence of this new scale leads one to expect that the fermion number violation in the YM theory will become significant when the energy is comparable with Planck’s energy. This expectation is supported by the discovery of particle-like solutions in the Einstein-Yang-Mills (EYM) theory \cite{9}, which have been interpreted as sphalerons \cite{10}. One might think that the typical mass of these EYM sphalerons, similarly to the situation in the YMH theory, determines the transition threshold.

It turns out however, that, contrary to the situation in the YMH theory, none of the EYM sphalerons relate to the highest energy point on a minimal-energy path connecting inequivalent EYM vacua, for the potential barrier between distinct vacua in the EYM theory can be arbitrarily low \cite{11}. Indeed,
let us consider first the flat space YM theory and define a one-parameter continuous family of static field configurations, \( A(\vec{x}, \lambda) \), such that when the parameter, \( \lambda \), goes from zero to unity, the field \( A(\vec{x}, \lambda) \) interpolates between two vacua with distinct winding numbers. We will call such a family of fields a vacuum-to-vacuum (VTV) path. The simplest choice is \( A(\vec{x}, \lambda) = \lambda A^{(1)}(\vec{x}) \), where \( A^{(1)}(\vec{x}) \) is a pure gauge with unit winding number. The energy for this family, \( U(\lambda) \), has the typical barrier behaviour as it vanishes for the vacuum values, \( \lambda = 0, 1 \), and reaches a maximum in the middle at \( \lambda = 1/2 \). This allows one to say that distinct vacua are divided by a barrier. Next note, that continuous scaling deformation of the family, \( A(\vec{x}, \lambda) \rightarrow A(\beta \vec{x}, \lambda) \), alters the energy as follows: \( U(\lambda) \rightarrow \beta U(\lambda) \), and hence the barrier height, \( \beta U(1/2) \), can be made arbitrarily low by a proper choice of the scaling factor \( \beta \). It is clear that the inclusion of gravity can only reduce the energy and hence does not change the situation (a consistent way to determine of the gravitating energy demands solving the initial value constraints; see below).

Thus, the EYM sphalerons, having finite mass, are not related to the minimal barrier height. One may wonder then what their physical meaning is. Indeed, as the barrier height may be arbitrarily small, an overbarrier passage is possible at any low energy and not only at the (ultra-high) sphaleron energy. In other words, there is no need to climb the mountain in order to reach the neighbouring valley as it is possible to go around it. This certainly obscures the threshold interpretation of the sphalerons.

However, there exists also a counter example to this objection. Indeed, in the flat space limit, where the overbarrier passage is also possible at any non-zero energy, the transition amplitude nevertheless always remains small (in the pure YM theory), as all fermion-violating Green functions contain the small instanton factor, regardless of the value of the energy \[ \text{[7]} \]. Therefore, one may infer that the possibility to pass over the barrier does not yet guarantee in itself an enhancement of the transition rate. To understand why this may happen one has to consider not only one path interpolating between distinct vacua, but all such paths, which corresponds to the inclusion of all the possible transition histories. The contribution of a single path to the transition amplitude may not be small provided that this path lies entirely in the classically allowed region, that is, the energy exceeds the barrier height for the given path – we shall call such a path an overbarrier path. However, when one takes the sum over all histories, it may happen that the contributions of different paths cancel each other. The latter occurs when the set
of all paths does not include a path whose contribution into the transition amplitude is stationary with respect to small variations of the path itself. Such a situation is encountered in the flat space pure YM theory, where all overbarrier histories are strongly suppressed by the destructive interference, regardless of the value of the energy, and a stationary phase path, if any exists, may only be underbarrier, so that the total transition amplitude is always small.

One may infer from this example that for the EYM theory all the fermion number violating transitions should also be suppressed in the low energy limit, despite the existence of the overbarrier paths, for gravitation can not essentially alter the dynamics at low energies, and the theory should resemble the flat space YM theory in this limit. Therefore, the vanishing of the greatest lower bound of the barrier height may be not so fatal for the threshold interpretation of the EYM sphalerons. Indeed, as we will see below, the resemblance between the YM and the EYM theories only holds at low energies. In the high energy limit, the situation changes. Firstly, at high energies the set of the overbarrier paths in the EYM case becomes larger in comparison with that appearing in the flat space theory, because the gravitational binding tends to reduce the barrier height. In addition, and this is the crucial point, when the energy exceeds the ground state sphaleron mass, gravity introduces a stationary phase overbarrier history. This leads us to conclusion that the transition becomes unsuppressed at high energies and restores the initial threshold interpretation of the sphalerons.

The main goal of this paper is to argue, using the quasi-classical arguments, that the suppression of the fermion number violating transitions in the EYM theory is removed when the energy is larger than the ground state sphaleron mass (see also [38]). The theoretical tool relevant for our purposes has been developed by Bitar and Chang in their real time analysis of tunneling transitions in the flat space YM theory [11]. The essence of this method is to introduce firstly a VTV path in the configuration function space of a theory. Then one assumes that the transition between distinct vacuum sectors of the theory proceeds along this single VTV path. This reduces the problem to a one-dimensional quantum-mechanical task and allows one to write down immediately the corresponding WKB amplitude. The last step is to take the sum over all VTV paths.

We apply Bitar and Chang’s technique to represent (in the lowest order WKB approximation) the amplitude of the fermion number violating tran-
tion in the EYM theory at non-zero energy (a similar idea was exploited also within the context of the YMH theory; see Ref. [12]). Following this line, we introduce a set of VTV paths connecting homotopically distinct vacua in the EYM configuration function space (see Eqs. (23),(36) below). Our basic approximation is the assumption that the quantum transition between distinct vacuum sectors occurs only along these paths. We estimate the partial WKB amplitudes for the paths introduced and specify those paths whose amplitudes are not small. Such paths lie entirely in the classically allowed region and they relate to overbarrier passages. Remaining paths pass through the classically forbidden region and we exclude them from our consideration because the corresponding partial amplitudes are small due to the tunneling suppression. The total transition amplitude is the sum over all partial amplitudes which can be estimated by making use of the stationary phase approximation. We show that the overbarrier paths always exist, that is, the overbarrier passages are possible at any non-zero energy. However, when the energy is small, the set of the overbarrier paths does not contain a stationary phase path. This means, that the sum over all overbarrier histories, being evaluated via the stationary phase approximation, will be small, as the stationary point is absent, so the overbarrier passages are suppressed by the destructive interference. On the other hand, when the energy is large and exceeds the ground state sphaleron mass, a stationary phase path appears on the set of the overbarrier paths. This allows us to conclude that, when the energy is large, the total transition amplitude in the EYM theory is not small (because the stationary point is present). Thus, the suppression of the fermion number violation in the EYM theory is removed when the energy is of the order of the EYM sphaleron mass.

The rest of the paper is organized as follows.

In Sec.II we investigate the homotopical classification of the vacua arising in the SU(2) EYM theory and estimate instanton contributions to the fermion number violation rate. In Sec.III we introduce a set of vacuum-to-vacuum paths in the EYM configuration function space such that the time evolution along these paths leads to a change of the topological winding number. In Sec.IV we apply Bitar and Chang’s technique to obtain the WKB transition amplitude. To select the overbarrier paths we study in Sec.V the structure of the potential barrier dividing vacua in the EYM theory. In the vicinity of the $n$-th sphaleron solution, we find that locally, the potential barrier surface is a saddle with one transversal and $n$ longitudinal negative directions. Ex-
ploiting local properties of the potential barrier surface we show in Sec.VI the existence of the (approximative) overbarrier stationary phase path when the energy exceeds the ground state sphaleron mass. Some concluding remarks are made in Sec.VII. There are also three appendices included. Appendix A contains an explanation of our choice of the gravitational action, and also details of the computation of the action for the spherically symmetric case. The Appendices B and C include various formulae relevant for the description of the spherically symmetric EYM fields and an explicit procedure for the first and second variation of the ADM energy functional.

Our sign conventions used are those of Landau & Lifshitz [37].

2 The EYM vacua and the instanton estimates

The starting point of our considerations is the notion of the topological vacua of the EYM fields. The topological vacua of the YM field have been first introduced in Ref.[1] within the context of the flat space theory. It turns out that inclusion of gravity modifies the analysis in [1] only slightly, as long as one does not take into account the topological effects of the gravitational field itself.

Consider the action of the EYM theory with the $SU(2)$ gauge group

$$S_{EYM} = S_G + S_{YM}, \quad (1)$$

with the gravitational part

$$S_G = -\frac{1}{16\pi G} \int R \sqrt{-g} d^4x - \frac{1}{16\pi G} \int_{\Sigma} (g^{\mu\alpha} \Gamma^\beta_{\alpha\beta} - g^{\alpha\beta} \Gamma^\mu_{\alpha\beta}) d\Sigma^\mu, \quad (2)$$

and the Yang-Mills contribution

$$S_{YM} = -\frac{1}{2g^2} \int tr F_{\mu\nu} F^{\mu\nu} \sqrt{-g} d^4x. \quad (3)$$

Here $G$ is Newton’s constant, $g$ is the gauge coupling constant, $F_{\mu\nu} = \partial_{\nu} A_{\mu} - \partial_{\mu} A_{\nu} - i[A_{\mu}, A_{\nu}]$ is the matrix valued gauge field tensor, $A_{\mu} = A^a_{\mu} \tau^a / 2$, and $\tau^a$ ($a = 1, 2, 3$) are the Pauli matrices. We use the gravitational action,
which does not contain second derivatives of the metric; for explanations of our choice see Appendix A.

The action (1) has many stationary points, including solutions with non-trivial space-time topology such as black holes [15]. Consider a sector with spacetime manifolds carrying the trivial $R^4$ topology. Define vacua in this sector as the stationary points with zero ADM energy. As follows from the positive energy theorem [16], such vacua must have a flat metric and hence the YM field is a pure gauge. In the $A_0 = 0$ gauge one can represent the vacuum fields as

$$A_j(\vec{x}) = i U \partial_j U^{-1}, \quad g_{\mu\nu}(\vec{x}) = \eta_{\mu\nu},$$

(4)

where $\eta_{\mu\nu}$ is the flat space metric, $U(\vec{x})$ is a $SU(2)$ valued function, and spatial index $j$ runs over 1,2,3 ($\vec{x} \equiv x^j$).

Such vacua are to be the boundary conditions for the path integral. It seems that not all of them are physically important, but only those satisfying some additional topological conditions, which have the origin in instanton physics. Following the standard line [1], we pass to the Euclidean sector and use the instanton arguments to clarify the structure of the EYM vacua. One should note that restricting ourselves only to spacetime manifolds with the $R^4$ topology, we exclude from consideration all non-trivial (i.e. carrying non-zero topological charges) gravitational instantons and vacua. A complete investigation of quantum gravity effects requires inclusion of instantons and vacua with non-trivial topological indices, both for the gauge and gravitational fields. This lies, however, beyond the scope of our present analysis (description of the non-trivial gravitational vacua has been done in Ref.[17]).

Consider an instanton interpolating between two EYM vacua. The finiteness of the instanton action implies the finiteness of both the matter and the gravitational parts of the action as both of them are positive. Indeed, the matter action is positive. The gravitational Euclidean action may be arbitrary, however the positive action conjecture implies that it should be positive on shell, where the condition $R = 0$ holds [18]. Finiteness of the matter part of the action implies that the YM field tends to a pure gauge, $i U \partial_j U^{-1}$, at Euclidean infinity, $S^2_{Euel}$, which defines a map $S^2_{Euel} \rightarrow S^2_{SU(2)}$, as the $SU(2)$ group is also a three sphere [4]. All maps $S^3 \rightarrow S^3$ fall into homotopy classes labeled by an integer degree of the map. In the case under consideration the
degree of the map is the Pontryagin index

$$\nu = \frac{1}{16\pi^2} \int tr F_{\mu\nu} \bar{F}^{\mu\nu} \sqrt{g} d^4 x, \tag{5}$$

where the dual tensor is $$\bar{F}^{\mu\nu} = \frac{1}{2\sqrt{g}} \varepsilon^{\mu\nu\alpha\beta} F_{\alpha\beta}$$ ($$\varepsilon^{0123} = 1$$). It is worth noting that this equation in fact does not involve the metric, so we infer that instantons interpolating between two vacua in EYM theory may be still described in terms of gauge invariant, topological indices of the YM field.

Let us introduce the Chern-Simons current

$$K^\mu = \frac{1}{8\pi^2} tr \frac{\varepsilon^{\mu\nu\alpha\beta}}{\sqrt{g}} A_\nu (\nabla_\alpha A_\beta - \frac{2i}{3} A_\alpha A_\beta)$$

whose divergence is

$$\nabla_\alpha K^\alpha = \frac{1}{16\pi^2} tr F_{\mu\nu} \bar{F}^{\mu\nu}, \tag{6}$$

where $$\nabla_\alpha$$ is the covariant derivative with respect to the spacetime metric. Pass to the $$A_0 = 0$$ gauge, which violates the equivalence between the Euclidean coordinates. Using Eqs.(5),(6), one obtains

$$\nu = \int \sqrt{g} K^0 d^3 x \bigg|_{\tau=+\infty}^{\tau=-\infty} + \int dt \oint \bar{K} d\Sigma = \int \sqrt{g} K^0 d^3 x \bigg|_{\tau=+\infty}^{\tau=-\infty} \equiv k(\infty) - k(-\infty). \tag{7}$$

The surface integral in this expression can be omitted as the YM field tends to a pure gauge at the spatial infinity. Indeed, for a pure gauge field taken in the $$A_0 = 0$$ gauge, $$K^0$$ is the only non-zero component of the Chern-Simons current. So, the surface term in Eq.(7) vanishes when the boundary moves to infinity. In the limit $$\tau \to \pm \infty$$ the gauge field tends to a pure gauge, $$iU \partial_j U^{-1}$$, which allows one to write

$$k = \int K^0 d^3 x = \frac{1}{24\pi^2} \int \varepsilon^{ijk} tr U \partial_i U^{-1} U \partial_j U^{-1} U \partial_k U^{-1} d^3 x. \tag{8}$$

Thus, one can attach to any EYM vacuum a number, $$k$$ – the Chern-Simons charge – and the difference of two such numbers for vacua connected via an instanton is always an integer and gauge invariant as well. The value of $$k$$ itself is neither gauge invariant nor an integer, however if the function $$U(\bar{x})$$ in Eq.(4) satisfies the following additional condition [1]

$$\lim_{|\bar{x}| \to \infty} U(\bar{x}) = 1, \tag{9}$$
then $k$ will be an integer and may be treated as the topological winding number. Indeed, the condition (9) implies that $U(\vec{x})$ may be viewed as a map $S^3_{\text{spatial}} \to S^3_{\text{SU(2)}}$, where $S^3_{\text{spatial}}$ is the compactified 3-dimensional space $[1,21]$, so an integer degree of the map again appears, which is exactly given by Eq.(8). Also, $k$ will be invariant with respect to small gauge transformations generated by functions obeying (9) and having zero Chern-Simons index (8).

Notice, that the condition (9) arises naturally. Indeed, for the trivial vacuum one has $0 = A_j = iU\partial_j U^{-1}$, which implies that $U(\vec{x}) = 1$ (up to a global gauge transformation), so the condition (9) holds. During the time evolution one has in the $A_0 = 0$ gauge

$$0 = A_0(t, \vec{x}) = \lim_{|\vec{x}| \to \infty} A_0(t, \vec{x}) = \lim_{|\vec{x}| \to \infty} iU\partial_t U^{-1}, \quad (10)$$

which gives $[21]$

$$\lim_{|\vec{x}| \to \infty} U(t, \vec{x}) = 1. \quad (11)$$

Thus, the condition (9) holds for any vacuum which may be connected with the trivial vacuum via an instanton.

So, one can see that the vacuum classification in the EYM theory resembles that arising in the flat space YM theory. The EYM vacua can be classified in terms of the integer winding numbers of the gauge field. Gravitation does not introduce any major peculiarities as long as one considers only manifolds with $R^4$-topology.

It is natural to ask how large the amplitude of transition between two EYM vacua with distinct winding numbers is. As is known, in flat spacetime this amplitude is small because the corresponding instanton action is large $[1]$. It is easy to see that the inclusion of gravity does not improve the situation. Indeed, the inequality

$$tr(F_{\mu\nu} \pm \tilde{F}_{\mu\nu})^2 = 2tr(F_{\mu\nu}F_{\mu\nu} \pm F_{\mu\nu}\tilde{F}_{\mu\nu}) \geq 0$$

implies that the total EYM action satisfies the following relation

$$S_{\text{EYM}}^{\text{Eucl}} = \frac{1}{2g^2} \int trF_{\mu\nu}F_{\mu\nu}\sqrt{g}d^4x + S_{\text{G}}^{\text{Eucl}} \geq \frac{8\pi^2}{g^2} |\nu| + S_{\text{G}}^{\text{Eucl}}. \quad (12)$$

As the Euclidean gravitational action is positive on shell, it is clear that the inclusion of gravity may only lead to an additional suppression of the instanton transitions between sectors with distinct winding numbers.
However, as was first argued within the context of the standard model \cite{7}, instanton estimates for fermion number violating amplitudes may break down in the classical, many-quanta limit, where the sphaleron estimates are relevant. One may reformulate these arguments in the EYM case as follows. Let quarks, $q$, and leptons, $l$, be involved. Then it follows from Eq. (12) that the fermion number violating amplitude, $<qqql>$, is always small. Indeed, such an amplitude may be obtained from the generating functional whose path integral always gives rise to the small factor $\exp(-S_{\text{Eucl}})$. However, the inclusive amplitude, $<qqqlA^n h^m>$, may not be small, where $A$ and $h$ stand for the gauge bosons and gravitons, and $m$ and $n$ are large numbers (see Ref. [7] for details). Notice, that such an inclusive amplitude arises naturally in a process mediated by a sphaleron, as the latter, being a classical object, involves large number of quanta. In other words, one may expect that the collision of two high energy fermions produces the sphaleron as an intermediate state. Then it decays giving a large number of gauge and gravitational quanta and also other fermions as a side effect due to the anomaly. A straightforward evaluation of the amplitude for such a process can be done with the use of perturbation techniques \cite{7}, but we shall proceed along a different line and apply instead non-perturbative approximation.

3 An example of the real time winding number changing history

Now, we return to the Lorenzian sector and introduce a family of paths lying in the EYM configuration function space and connecting homotopically distinct vacua. In the next section this will allow us to apply Bitar and Chang’s technique in order to represent an amplitude of the winding number changing transition. The considerations in this section follow the line sketched in Ref. [10].

We shall consider spherically symmetric fields and so we need the corresponding expressions for the field potentials. Let us use the following parameterization of the spherically symmetric gravitational field

$$ds^2 = R_g^2\{1 - \frac{2m}{r}\sigma^2 dt^2 - \frac{dr^2}{1 - 2m/r} - r^2(\sigma^2 + \sin^2 \vartheta d\varphi^2)\}, \quad (13)$$

where $R_g = \sqrt{4\pi \ell_{pl}/g}$ is the natural length scale arising in the EYM theory.
with $\ell_p$ being Planck’s length, all other quantities in Eq.(13) being dimensionless, and $m$ and $\sigma$ being functions of the radial coordinate, $r$, and time, $t$, as well. Topological triviality of the space time manifold implies that the metric must be regular and asymptotically flat; this means that $m$, $\sigma$ are smooth functions satisfying the following boundary conditions

$$\sigma \to 1, \quad m \to \text{const as } r \to \infty; \quad m = o(r), \quad \sigma = \sigma_0 + o(r) \text{ as } r \to 0.$$  \hspace{1cm} (14)

The spherical YM field is given by Witten’s ansatz [22], which can be represented in the following form

$$A = W_0 L_1 \, dt + W_1 L_1 \, dr + \{(1-p_1) L_2 + p_2 L_3\} \sin \vartheta \, d\varphi,$$  \hspace{1cm} (15)

where

$$L_1 = \sin \vartheta \cos \varphi \left( \frac{T_1}{2} \right) + \sin \vartheta \sin \varphi \left( \frac{T_2}{2} \right) + \cos \vartheta \left( \frac{T_3}{2} \right),$$

$$L_2 = \partial_{\vartheta} L_1, \quad L_3 = \frac{1}{\sin \vartheta} \partial_{\varphi} L_1,$$  \hspace{1cm} (16)

$W_0, W_1, p_1, p_2$ are functions of $r$ and $t$. It is convenient to use also another parameterization of this ansatz introducing functions $\Omega_0, \Omega_1, f, \alpha$ as follows:

$$W_0 = \Omega_0 + \dot{\alpha}, \quad W_1 = \Omega_1 + \alpha', \quad p_1 = f \cos \alpha, \quad p_2 = f \sin \alpha,$$  \hspace{1cm} (17)

where dot and prime denote differentiation with respect to $t$ and $r$ correspondingly. One may see that only three of these functions are essential, the fourth one, $\alpha$, is actually a pure gauge parameter, we shall call it an angle parameter of the field. The gauge transformation

$$A_\mu \to U(A_\mu + i \partial_\mu) U^{-1}$$  \hspace{1cm} (18)

with

$$U = \exp\{i \beta(t, r) L_1\}$$  \hspace{1cm} (19)

preserves the form of the field (15), (17) leaving functions $\Omega_0, \Omega_1, f$ invariant and altering the angle parameter as follows

$$\alpha \to \alpha + \beta.$$  \hspace{1cm} (20)

Also, gauge invariant components of the energy-momentum tensor of the field (13) (see Eq.(B1)) do not include $\alpha$. 

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If one puts in Eq. (15), (17) $\Omega_0 = \Omega_1 \equiv 0$, $f \equiv 1$, allowing for the $\alpha$ to be arbitrary, then the energy of the field vanishes, so the field will correspond to a pure gauge, $iU\partial_\mu U^{-1}$, where $U$ is given by Eq. (19) with $\beta$ replaced by $\alpha$. The topological vacua arise if one imposes the following additional restrictions on the angle parameter: $\dot{\alpha} = 0$, which insures the $A_0 = 0$ gauge condition, and also

$$\alpha(0) = 0, \quad \alpha(\infty) = -2\pi k.$$ (21)

Indeed, the Chern-Simons charge for the field (15) (see Appendix B) is

$$k = \int \sqrt{-g}K^0d^3x = \frac{1}{2\pi} \int_0^\infty dr \{W_1(p_1^2 + p_2^2 - 1) + p_2' p_1 + (1 - p_1) p_2\};$$ (22)

in vacuum this expression takes the form

$$\frac{1}{2\pi}(\sin \alpha - \alpha)(\infty) \bigg|_0,$$

which is the integer, $k$, provided that the condition (21) holds (the condition $\alpha(0) = 0$ in Eq. (21) insures regularity of the field at the origin). It is clear then that the function (19) with time-independent $\beta$ vanishing both at the origin and infinity generates small gauge transformations as the condition (21) remains unchanged.

Summarizing, we conclude that the gauge field (15), (17) with $\Omega_0 = \Omega_1 = f - 1 \equiv 0$ and arbitrary time-independent $\alpha$ obeying the condition (21), represents the topological vacuum, $iU\partial_\mu U^{-1}$ with $U = \exp(i\alpha L_1)$, and the winding number is $k$.

We consider now an explicit example of the winding number changing evolution of the gauge field (10):

$$A_\mu(t, \vec{x}) = \frac{i}{2}(1 - K(r))U\partial_\mu U^{-1}, \quad U = \exp(2i\lambda L_1),$$ (23)

where $\lambda$ is some function of time, $\lambda(t_0) = 0$, $\lambda(t_1) = \pi$, where $t_0 < t_1$, and $K(r)$ satisfies

$$K(0) = 1, \quad K(\infty) = -1.$$ (24)

It is clear that the field (23) is spherical and hence it can be represented in the form (15) giving for the corresponding functions

$$W_0 = (1 - K)\dot{\lambda}, \quad W_1 = 0, \quad p_1 = \cos^2 \lambda + K \sin^2 \lambda, \quad p_2 = (1 - K)\sin \lambda \cos \lambda.$$ (25)
and also
\[ \Omega_0 = -\lambda \frac{K(K^2 - 1)}{K^2 + \cot g^2 \lambda}, \quad \Omega_1 = \frac{K' \cot g \lambda}{K^2 + \cot g^2 \lambda}, \]
\[ f^2 = \sin^2 \lambda K^2 + \cos^2 \lambda, \quad \tan \alpha = \frac{(1 - K) \cot g \lambda}{K + \cot g^2 \lambda}. \]  
(26)

One may see that the field (23) vanishes in the beginning, \( \lambda = 0 \), as well as at the end, \( \lambda = \pi \), of the evolution (provided that \( \dot{\lambda}(t_0) = \dot{\lambda}(t_1) = 0 \)). Thus, at first sight, this field corresponds to a loop in the configuration function space interpolating between the same trivial vacuum. However, the crucial issue here is the right choice of the gauge. The homotopical vacuum classification is based on the \( A_0 = 0 \) gauge condition [1],[21],[24], so one has to pass to this gauge. To do this, perform the gauge transformation (20) and demand
\[ A_0 = (\Omega_0 + \dot{\alpha} + \dot{\beta}) L_1 = 0, \]  
(27)
where \( \Omega_0 \) and \( \alpha \) are given by Eq.(26). In this way we obtain the equation for the gauge parameter \( \beta \):
\[ \Omega_0 + \dot{\alpha} + \dot{\beta} = \dot{\lambda}(1 - K) + \dot{\beta} = 0, \]  
(28)
which yields
\[ \beta = (K - 1) \lambda + B(r), \]  
(29)
where the function \( B(r) \) may be put equal to zero using the residual freedom of the time-independent gauge transformations, which preserve the \( A_0 = 0 \) gauge condition. Thus, in the new gauge, the invariant functions \( \Omega_0, \Omega_1, f \) are still determined by Eq.(26), but the angle parameter \( \alpha \) is replaced by
\[ \alpha_0 = (K - 1) \lambda + \arct g \frac{(1 - K) \cot g \lambda}{K + \cot g^2 \lambda}. \]  
(30)
The corresponding expressions for the functions \( W_0, W_1, p_1, p_2 \) in the new gauge are
\[ W_0 = 0, \quad W_1 = \lambda K', \quad p_1 = \cos \lambda \cos K \lambda + K \sin \lambda \sin K \lambda, \]
\[ p_2 = \cos \lambda \sin K \lambda - K \sin \lambda \cos K \lambda. \]  
(31)
One can see that the field (15) specified by these functions vanishes at \( \lambda = 0 \). On the other hand, at the end of the evolution, at \( \lambda = \pi \), the field (15),(31)
is the non-trivial pure gauge. The angle parameter of the new vacuum can be defined from Eq. (30):

\[ \alpha_0(r) = \pi(K(r) - 1). \]  

One may see also from Eqs. (21), (24) that this indeed corresponds to a vacuum with unit winding number.

Thus the field (23) indeed interpolates between distinct vacua. To show a change of the winding number during the evolution (23) one has to estimate the Chern-Simons number as a function of \( \lambda \) (see also [23] for discussion of the subject in the case when arbitrary gauge groups are involved). Inserting functions (31) into Eq. (22) and using Eq. (24), one obtains

\[ k = \frac{1}{2\pi} \int_{-1}^{1} dK(-\lambda + \sin \lambda \cos \lambda + \lambda \cos \lambda K \lambda - \sin \lambda \cos K \lambda + K \lambda \sin \lambda \sin K \lambda) = \frac{1}{\pi} (\lambda - \sin \lambda \cos \lambda), \]  

which varies from zero to unity as \( \lambda \) goes from zero to \( \pi \). It is worth noting that the middle of the evolution, \( \lambda = \pi/2 \), corresponds to a half-integer Chern-Simons number, \( k = 1/2 \) [10]. It is clear now that the boundary conditions (24) for the function \( K \) in Eq. (23) are important. Imposing instead the conditions

\[ K(0) = K(\infty) = 1, \]  

the value of the integral in the right hand side of Eq. (33) would be zero as would be the Chern-Simons number. So, the boundary conditions (24) insure that the field (23) interpolates between homotopically different vacuum sectors. On the other hand, the conditions (34) would make the field interpolate between vacua inside the same vacuum equivalency class.

Now, being aware that the field (23) indeed relates to the transition between distinct vacua, we adopt a slightly different interpretation of this example. Let us treat the quantity \( \lambda \) in Eq. (23) not as a function of time but rather just as a parameter running from zero to \( \pi \). This means that the field (23) is viewed now as a one-parameter family of static field configurations or a path in the EYM configuration function space.

Next, we want to specify a gravitational field related to this path. The most natural choice is for the matter field (23) and the corresponding gravitational field to form an admissible set of the initial data on an initial time-symmetry hypersurface. So we impose the initial value constraints, from
which the only non-trivial one in our case is Einstein’s equation (B8). For the sake of convenience only, we impose also the equation (B10), which specifies the lapse function $\sigma$. Using Eqs. (B1), (26), (B8), (B10), we get the following equations for the spherical gravitational field related to the path (23)

$$m' = \sin^2 \lambda \left\{ (1 - \frac{2m}{r}) K^2 + \sin^2 \lambda \frac{(K^2 - 1)^2}{2r^2} \right\},$$

$$\sigma' = 2\sin^2 \lambda \frac{K^2}{r} \sigma.$$  

Straightforward integration of these equation with the boundary conditions (14) allows us to express $m$ and $\sigma$ in terms of the quantities, which parameterize the field (23) [10]:

$$\sigma(r) = \exp\{-2\sin^2 \lambda \int_r^\infty K'^2 \frac{dr}{r}\},$$

$$m(r) = \frac{\sin^2 \lambda}{\sigma(r)} \int_0^r \left( K'^2 + \sin^2 \lambda \frac{(K_2 - 1)^2}{2r^2} \right) \sigma dr.$$  

If $K(r)$ is a sufficiently smooth function then these expressions specify a metric, which is asymptotically flat and regular for any $\lambda$ and takes the vacuum value, $\eta_{\mu\nu}$, at the ends of the path, $\lambda = 0, \pi$. The corresponding (dimensionless) ADM mass is

$$M = \lim_{r \to \infty} m(r).$$  

Recall now that $K(r)$ in the above expressions may be arbitrary. This means that Eqs.(23),(36) define not one path but rather a family of vacuum-to-vacuum paths

$$\{ A_j(\vec{x}, \lambda); g_{\mu\nu}(\vec{x}, \lambda) \}|_{K(r)}, \quad \lambda \in [0, \pi]$$  

in the EYM configuration function space, each path of the family being specified by a function $K(r)$ satisfying conditions (24). The energy (37) for each path vanishes for the vacuum values, $\lambda = 0, \pi$, and has a maximum in between at $\lambda = \pi/2$.

Now, one can see that the fields (23),(36) given at the point $\lambda = \pi/2$ coincide with the solutions of the EYM equations found by Bartnik and McKinnon (BK) [9] provided that the function $K(r)$ is identified with the
corresponding BK magnetic amplitude $w_n(r)$ (see Appendix B). If the number $n$ is odd, then $K(r) = w_n(r)$ obeys the boundary conditions (24), so odd-$n$ BK solutions belong to paths connecting distinct vacua (even-$n$ solutions belong to paths interpolating between representatives of the same vacuum class as $w_n$ functions in this case obey the condition (34)). This circumstance has allowed us initially to interpret the odd-$n$ BK solutions in Ref.[10] as sphalerons, because they relate to the top of the potential barrier dividing homotopically distinct EYM vacua, $\lambda = \pi/2$, and also have the Chern-Simons index one-half. However, as is mentioned in the Introduction (see also [10]), the analogy with the standard YMH sphaleron is incomplete because none of the BK solutions relate to an absolute minimum barrier height, and hence an additional investigation is required in order to establish the sphaleron nature of these solutions.

4 The winding number changing transition amplitude.

In this section we represent the amplitude of a quantum transition between homotopically non-equivalent EYM vacuum sectors at non-zero energy a la Bitar and Chang. We assume that the transition occurs only along the paths (23),(36), which is our basic approximation. Notice that Bitar and Chang’s approach gives an adequate description of the transition process provided one is able to take into account all VTV paths. But the main advantage of the method, perhaps, is that it may give a good description of the transition even if one takes only some particular set of the vacuum-to-vacuum paths [11].

To obtain the transition amplitude, let us first choose a path from the family (23),(36). To find the partial amplitude related to this path, allow for the parameter $\lambda$ to depend on time: $\lambda \rightarrow \lambda(t)$, and calculate the action of the fields. Inserting Eqs.(23),(36) with $\lambda = \lambda(t)$ into Eq.(1), represent the action in the form

$$S_{EYM} = \frac{4\pi}{g^2} \int (L_G + L_{YM}) \, dt,$$

where the Lagrangians $(4\pi/g^2)L_G$ and $(4\pi/g^2)L_{YM}$ of gravitational and the YM field are the spatial integrals of the corresponding Lagrangian densities given by Eqs.(2),(3). Consider first the matter part of the action. Using
Eqs. (3), (26), (B2) and taking into account the Einstein equations (35), one obtains

$$L_{YM} = \frac{\mu(\lambda)}{2} \dot{\lambda}^2 - \int_0^\infty m' \sigma dr,$$

(40)

where

$$\mu(\lambda) = \int_0^\infty \frac{r^2}{\sigma} (K'^2 + 2 \sin^2 \lambda \frac{(K^2 - 1)^2}{r^2 - 2mr}) dr,$$

(41)

quantities $m$, $m'$, $\sigma$ being given by Eqs. (35), (36) and $K(r)$ being a function specifying the path under consideration.

The calculation of the gravitational part of the action is performed in the Appendix A. For regular geometry described by Eq. (13) the horizon term in Eq. (A20) disappears, so one gets

$$L_G = - \int_0^\infty m \sigma' dr$$

(42)

(when one passes in Eq. (A20) from dimensional variables $t, r, m$ to dimensionless ones, which are used throughout in the paper, the multiplier $4\pi/g^2$ explicitly depicted in Eq. (39) arises).

One can see that the integral entering Eq. (42) combines with that coming from Eq. (40) to give the total derivative under integration. This allows us to represent the total EYM action in the following form

$$S_{EYM} = \frac{4\pi}{g^2} \int \left( \frac{\mu(\lambda)}{2} \dot{\lambda}^2 - U(\lambda) \right) dt,$$

(43)

where the effective mass term, $\mu(\lambda)$, is given by Eq. (41), and the potential, $U(\lambda)$, reads

$$U(\lambda) = \int_0^\infty (m \sigma)' dr = m(\infty).$$

(44)

This coincides with the ADM mass given by Eq. (37) (that is true provided that the geometry is everywhere regular; see Appendix A). Explicitly one has

$$U(\lambda) \equiv U[K(r), \lambda] =$$

$$= \sin^2 \lambda \int_0^\infty (K'^2 + \sin^2 \lambda \frac{(K^2 - 1)^2}{2r^2}) \exp(-2\sin^2 \lambda \int_r^\infty K'^2 \frac{dr}{r}) dr.$$
external potential \( U(\lambda) \). The potential has the typical barrier shape: for each \( K(r) \) it vanishes for the vacuum values, \( \lambda = 0, \pi \), and reaches a maximum in between at \( \lambda = \pi/2 \) (the latter can be seen if we pass to a new independent variable \( z = r/sin\lambda \) under the integration in Eq.(45)).

Thus, we arrive at the one-dimensional barrier transition problem. The corresponding one-dimensional Schrödinger equation then reads

\[
\mathcal{H} = \frac{p^2}{2\mu(\lambda)} + U(\lambda) = E, \tag{46}
\]

with \( p \) being the momentum conjugated to \( \lambda \), and the quantity \( E \) has the sense of the energy of the asymptotically free quantum states (the operator ordering problem can be avoided in this case \[11\]). This allows us to write down the corresponding partial WKB transition amplitude as follows

\[
A_{K(r)} = B \exp\left\{ i\frac{4\pi}{g^2} \Phi[K(r)] \right\} = B \exp\left\{ i\frac{4\pi}{g^2} \int_0^\pi d\lambda \sqrt{2\mu(\lambda)}[E - U(\lambda)] \right\}, \tag{47}
\]

where \( B \) absorbs all other WKB factors, which are inessential for our present considerations. The subscript \( K(r) \) indicates that this partial amplitude relates to a single path specified by \( K(r) \). If the quantity \( E - U(\lambda) \) is negative, then one should take the value of the square root lying in the upper half of the complex plane.

The last step in finding the transition amplitude is to represent the total amplitude as the sum over all partial amplitudes (47):

\[
A = \sum_{K(r)} A_{K(r)}. \tag{48}
\]

This expression gives the amplitude of the winding number changing transition in the EYM theory at arbitrary energy \( E \). It is clear, however, that taking the sum is virtually impossible. The only way to estimate this sum is to make use of the stationary phase approximation. However, as was shown in Ref.[11], the finding of the exact stationary phase path implies solving a system of the coupled differential equations, which, unfortunately, lies beyond our abilities.

To proceed further we will not calculate the amplitude, but consider instead a more simple problem. Namely, we want to find out under which conditions this amplitude will not be small.
We use the following terminology. Let the energy, $E$, be fixed. Consider a path $(23), (36)$ that lies entirely in the classically allowed region:

$$U[K(r), \lambda] < E, \quad \lambda \in [0, \pi].$$

(49)

Such a path will be called an overbarrier path. If a path passes also through the classically forbidden region then we shall call it an underbarrier path, for the evolution along such a path implies barrier penetration.

For an underbarrier paths the amplitude (47) is small, as it includes the small tunneling factor. This allows us to exclude from the sum (48) the contribution of all underbarrier paths, as this contribution is certainly small. The remaining sum over the overbarrier paths will not be small if only this sum includes a term (or terms), $A_{K(r)}$, whose value is stationary with respect to small variations of $K(r)$. Such a term corresponds to the contribution of a stationary phase path. Thus, the amplitude (48) will not be small provided there exists a stationary phase path on the set of all overbarrier paths. Therefore, to proceed further, we first need to select the overbarrier paths from the set $(23),(36)$, and next to look for a stationary phase path among them.

5 Structure of the energy surface

To select the overbarrier paths in accordance with Eq.(49) one has to know properties of the potential $U[K(r), \lambda]$. It is useful to treat the functional $U[K(r), \lambda]$ in geometrical terms, viewing it as an infinite dimensional surface in the corresponding function space. Let us call this surface an energy surface or a potential barrier surface. It is natural to call the directions in this surface generated by $\partial/\partial \lambda$ and $\partial/\partial K(r)$ a transverse direction and a longitudinal direction respectively. Indeed, changing of $\lambda$ with fixed $K(r)$ corresponds to motion across the barrier towards a neighbouring vacuum. Changing of $K(r)$ implies passing to other paths, i.e. the motion along the barrier. When $K(r)$ is fixed, the potential reaches a maximum at $\lambda = \pi/2$, so we will call the functional

$$\varepsilon[K(r)] = U[K(r), \lambda = \pi/2]$$

(50)

a barrier height functional and say that it defines the profile of the top of the barrier.
Consider the critical points of the energy surface. It is clear that such points have to belong to the top, $\lambda = \pi/2$. Direct variation of the functional (50) yields (see Appendix C)

$$
\delta \varepsilon = 2 \int_0^\infty \left\{ -((1 - \frac{2m}{r})\sigma K')' + \frac{K(K^2 - 1)}{r^2}\sigma \right\} \delta K dr,
$$

(51)

where $\delta K(0) = \delta K(\infty) = 0$, functions $m, \sigma$ are given by Eq.(C3). The vanishing of this variation requires the condition

$$
((1 - \frac{2m}{r})\sigma K')' = \sigma \frac{K(K^2 - 1)}{r^2}
$$

(52)

to hold, which is exactly the YM equation (B11). Noting then, that the functions $m, \sigma$ in this case satisfy the equations

$$
\sigma' = 2\frac{K'^2}{r} \sigma, \quad (m\sigma')' = (K'^2 + \frac{(K^2 - 1)^2}{2r^2})\sigma,
$$

(53)

which are identical to the Einstein equations (B12),(B13), we conclude that the condition $\delta \varepsilon = 0$ insures the on-shell condition $\delta S_{EYM} = 0$. All points of the energy surface relate to the regular field configurations, hence, critical points of the top relate to regular solutions of the EYM equations, i.e. to the EYM sphalerons. One may say that the EYM sphalerons “lie” on the top of the potential barrier. (Of course, these arguments only establish the coincidence of the extrema of the action and those of the truncated mass functional under consideration. When all field degrees of freedom are involved rather than only those which are spherically symmetric, the correspondence between extrema of the energy and the action is not proven, although it is very likely that it indeed holds; see Ref.[25].)

Next, we consider the local properties of the energy surface in the vicinity of a critical point. Firstly, we know that the potential energy decreases when the transversal $\lambda$-coordinate deviates from the $\lambda = \pi/2$ value, so locally, near any critical point, the energy surface is a saddle, possessing at least one negative direction. Therefore, each EYM sphaleron has at least one negative mode which we will refer to as transverse rolling down mode [26]. In the YMH theory this mode is the single negative mode of the sphaleron (provided that the Higgs field self-coupling is not too large [19]), so the energy increases in any longitudinal direction. In the EYM theory there exist additional
sphaleron negative modes, for the potential energy decreases also in some longitudinal directions. Indeed, direct calculation (see Appendix C) shows that in the vicinity of a sphaleron the following expansion holds

$$\varepsilon[w_n(r) + \varphi(r)] = \varepsilon[w_n(r)] + \delta^2 \varepsilon + \ldots, \quad (54)$$

where the first order term vanishes and dots denote higher order terms. The second order term reads

$$\delta^2 \varepsilon = \int_0^\infty \varphi(-\frac{d^2}{dr_*^2} + V) \varphi \, dr_*, \quad (55)$$

with the new radial coordinate, $r_*$, being defined by the following relations $\frac{dr}{dr_*} = \sigma(1 - 2m/r)$, $r_*(r = 0) = 0$, the perturbation $\varphi(r_*)$ obeys $\varphi(0) = \varphi(\infty) = 0$. The effective potential is

$$V = \sigma^2(1 - \frac{2m}{r})\{3w_n^2 - \frac{1}{r^2} + \frac{8}{r^3} w_n^3 w_n(w_n^2 - 1) - \frac{4}{r^2} w_n^2(1 - \frac{(w_n^2 - 1)^2}{r^2})\}, \quad (56)$$

where functions $m$ and $\sigma$ relate to the corresponding sphaleron solutions, the derivatives are calculated with respect to $r$. One may see that the scalar product for different perturbation modes can be naturally defined as $<\varphi_1, \varphi_2> = \int \varphi_1 \varphi_2 \, dr_*$; independent modes being orthogonal. It is obvious that if the differential operator in Eq.(55) has negative eigenvalues,

$$(-\frac{d^2}{dr_*^2} + V) \varphi = \omega^2 \varphi, \quad \omega^2 < 0, \quad (57)$$

corresponding eigenfunctions, $\varphi$, will specify the longitudinal negative directions on the energy surface, that is, the longitudinal negative sphaleron modes. It is worth noting that Eq.(57) coincides (up to a constant multiplier) with that first obtained by Straumann and Zhou \[27\] in their analysis of the linear stability of the BK solutions, provided that one identifies the quantity $\omega$ with the time frequency of perturbations of the background BK solution. (Notice, that Straumann and Zhou derived Eq.(57) linearizing the EYM equation near a background equilibrium solution. It is interesting, that one may obtain the same equation from the energy considerations. A similar situation is also encountered in the theory of relativistic stars \[28\], where the dynamical pulsation equation for a star composed from perfect fluid can be derived by making use of the two different approaches.)
It is known [24], that the equation (57) has exactly \( n \) independent negative eigenvalue solutions. Thus in the vicinity of the \( n \)-th sphaleron the energy surface has, apart from one transversal negative mode, also \( n \) longitudinal negative modes. The total number of the negative modes, \( n + 1 \), is in agreement with the result by Sudarsky and Wald [25].

It is obvious now that when the energy \( E \) exceeds an extremal (non-zero) value of the barrier height, that is, the mass of the \( n \)-th EYM sphaleron,

\[
E > \varepsilon[w_n],
\]

then the condition (58) is satisfied provided that \( K(r) \) is close to \( w_n(r) \), i.e. \( K(r) = w_n(r) + \varphi(r) \) with \( \varphi \) being a small perturbation. This means that the path passing through the \( n \)-th sphaleron (such a path is defined by Eqs.(23),(36) with \( K(r) = w_n(r) \)) as well as neighbouring paths will be overbarrier.

6 The stationary phase path

Let the condition (58) hold. Consider the overbarrier paths defined in the preceding section. Now, we are looking for the stationary phase path on the set of these overbarrier paths. Our aim is to show that the path passing through the sphaleron will be such a stationary phase path. This means that the quantum phase \( \Phi[K(r)] \) defined by Eq.(47) with \( K(r) = w_n(r) \) should be stationary on variations of \( K(r) \). To check the stationarity of the phase let us consider an arbitrary variation of \( K = w_n \):

\[
K(r) = w_n(r) + \alpha \varphi(r),
\]

where \( \alpha \) is the variational parameter, and the perturbation obeys \( \varphi(r) = O(r^2) \) as \( r \to 0 \), \( \varphi(r) = O(1/r) \) as \( r \to \infty \). Inserting this into \( \Phi[K(r)] \), \( U[K(r), \lambda] \) and \( \varepsilon[K(r)] \) defined by Eqs.(17), (15) and (50) respectively, one obtains the phase \( \Phi_{\varphi}(\alpha) \), (index \( \varphi \) refers to the choice of the perturbation), and the one and two-dimensional sections of the energy surface: \( \varepsilon_{\varphi}(\alpha), U_{\varphi}(\alpha, \lambda) \). The phase \( \Phi[K(r)] \) will be stationary if \( \Phi_{\varphi}(\alpha) \) has an extremum for any \( \varphi(r) \) in Eq.(23).

One should note that, strictly speaking, the exact stationary phase path is not contained in the path family (23), (36). Finding such a path implies
solving the general equations of motion [11], which lies beyond the scope of our present analysis. Paths (23),(36) specified by $K = w_n$ may be only approximately stationary. This means that for these approximate paths and any independent perturbations $\varphi(r)$ in (59), the positions of the extremum of the corresponding phase functions $\Phi_\varphi(\alpha)$ do not coincide exactly with the position of the sphaleron, $\alpha = 0$, however they are close together. In other words, functional derivatives $\delta \Phi[K(r)]|_{K=w_n}$ although do not vanish exactly, are nevertheless small.

Consider first such perturbations $\varphi(r)$ which increase the barrier height: $\varepsilon(\alpha \neq 0) > \varepsilon(0)$; there are infinitely many such perturbations. One may see that the phase is stationary with respect to all these perturbations. Indeed, in this case the corresponding two-dimensional section of the energy surface, $U_\varphi(\alpha, \lambda)$, has the typical saddle shape shown in Fig.1. The saddle negative $\lambda$-direction on the picture (shown by the horizontal arrow) specifies the transverse rolling down mode of the sphaleron [26]. It is clear from this picture that the path passing through the sphaleron is the minimal potential energy path interpolating between distinct vacua, so it corresponds to an extremum of the phase. The analogous situation takes place in the YMH theory where the sphaleron is the highest-energy point on a minimal-energy path connecting distinct vacua (at least for small values of the scalar field self-coupling [19]), which implies the stationarity of the phase for such a path [3]. Direct numerical inspection shows that, for several ground state sphaleron energy-increasing perturbation modes $\varphi(r)$ checked, the phase $\Phi_\varphi(\alpha)$ indeed has an extremum (minimum) in the vicinity of zero value of $\alpha$.

However, contrary to the situation in the YMH case, in the EYM theory not all perturbations (59) increase the barrier height, for EYM sphalerons possess also longitudinal negative modes. Inserting such perturbation modes in Eq. (59), one obtains the corresponding two-dimensional sections of the energy surface which have the typical shape shown in Fig.2. One may see from this picture that, for such a section, the path passing through the sphaleron is not the minimal energy path. So, with respect to these negative perturbations, the stationarity of the phase is not obvious (it is usually assumed that a sphaleron has to have one and only one negative mode in order to be significant for the transition processes [19]). Nevertheless, we are able to demonstrate the stationarity of the phase also in this case, at least for the ground state ($n = 1$) sphaleron.

For the ground state sphaleron there exists only one negative eigenmode
to Eq. (57). It turns out that the corresponding perturbation (59) may be related to the rescaling of the sphaleron field (the scaling arguments in the EYM theory have also been considered in Ref. [30]. Namely, instead of Eq. (59) let

$$K(r) = w_1(\beta r),$$  

(60)

where $\beta$ is the scaling parameter. Inserting this in Eq. (45) one obtains the following two dimensional section of the energy surface:

$$U(\beta, \lambda) = \beta \sin^2 \lambda \int_0^\infty \left( w_1'^2 + \sin^2 \lambda \frac{(w_1^2 - 1)^2}{2r^2} \right) \exp(-2\beta^2 \sin^2 \lambda \int_r^\infty \frac{w_1'^2}{r} dr) dr,$$

(61)

(here $w_1 = w_1(r)$), the value $U(1, \pi/2)$ being the sphaleron mass. This function tends to zero when $\beta \to 0, \infty$ as well as when $\lambda \to 0, \pi$, so one may see that the potential barrier indeed can be arbitrarily low. The plot of this function is depicted in Fig. 2. This picture demonstrates explicitly the existence of the two independent sphaleron negative modes. One such mode is the transverse rolling down mode [26] (the $\lambda$-direction on the picture), the other is the longitudinal negative mode [27] (the $\beta$-direction). (Note that the infinitesimal scaling mode, $\varphi = \partial_\beta w_1(\beta r)|_{\beta=1} = rw_1'(r)$, is not the exact eigenmode for the Eq. (57), but rather a superposition of the true negative eigenmode and some other mode).

The scaling behaviour of the quantum phase is defined by the inserting (60) into (47):

$$\Phi_{\text{scale}}(\beta) = \int_0^\pi d\lambda \sqrt{2\mu(\beta, \lambda)[E - U(\beta, \lambda)]},$$

(62)

where $U[\beta, \lambda]$ is given by Eq. (51) and

$$\mu(\beta, \lambda) = \frac{1}{\beta} \int_0^\infty \frac{r^2}{\sigma}(w_1'^2 + 2\sin^2 \lambda \frac{(w_1^2 - 1)^2}{r^2 - 2\beta m r}) dr,$$

(63)

with

$$\sigma(r) = \exp\{-2\beta^2 \sin^2 \lambda \int_r^\infty \frac{w_1'^2}{r} dr\},$$

$$m(r) = \frac{\beta \sin^2 \lambda}{\sigma(r)} \int_r^\infty (w_1'^2 + \sin^2 \lambda \frac{(w_1^2 - 1)^2}{2r^2}) \sigma dr.$$

(64)

Now, one may see that the phase (62) indeed has an extremum. First note that when $\beta$ is small, the phase $\Phi_{\text{scale}}(\beta) \sim 1/\sqrt{\beta}$, i.e. it diverges when
\( \beta \to 0 \) (remember that the energy \( E \) exceeds the sphaleron mass, that is, the maximal value of the potential drawn in Fig.2). Also, and this is the crucial point, \( \Phi_{scale}(\beta) \) diverges when \( \beta \) tends to some value \( \beta_c = 1.465 \), because the integral entering Eq.(63) diverges in this limit. The value \( \beta = \beta_c \) corresponds to such a rescaling of the sphaleron field when the rescaled configuration begins to acquire an event horizon (all of the paths \((23),(36),(60)\) with \( \beta > \beta_c \) pass through virtual black holes and give an infinite contribution to the action). At the horizon the quantity \( r^2 - 2\beta mr \) entering the denominator in Eq.(63) vanishes, so the integral diverges as does the phase. So, somewhere in between, \( 0 < \beta < \beta_c \), the phase must have a minimum (see Fig.3).

Thus we can see that when the energy exceeds the ground state EYM sphaleron mass, the path passing through the sphaleron insures extremality of the quantum phase with respect to any small variations inside the path family \((23),(36)\). The existence of this extremum insures that the sum in Eq.(48), being evaluated via stationary phase approximation, will be not small, as it will include the contribution of a stationary point. Therefore, when the energy is large, the winding number changing transition in the EYM theory is not suppressed.

It turns out that the analysis carried out above for the ground state sphaleron remains valid for the higher (odd \( n > 1 \)) sphalerons. This means that each higher sphaleron introduces a stationary phase path, only if the corresponding phase will be stationary also with respect to the additional \( n - 1 \) longitudinal higher sphaleron negative modes, which is very plausible. Thus, if the energy exceeds the masses of all the EYM sphalerons (i.e. \( E > 1 \) in the dimensionless units used), the sum in Eq.(48) includes the contributions of (infinitely) many stationary points (one point for each higher sphaleron), which may lead to an additional enhancement of the transition rate.

When the energy is less then the ground state sphaleron mass, the situation changes. The overbarrier paths in this case also exist, as can be seen in Fig.2. However, there exists no stationary phase path, so the overbarrier passages are suppressed by the destructive interference. This is always occurs also in the flat space limit, \( G \to 0 \), because in this case the threshold energy value – the sphaleron mass – being proportional to Planck’s mass, goes to infinity exceeding any finite value of energy, so that the transition is always suppressed.
7 Conclusion

The main results of our analysis may be summarized as follows. In the EYM theory, as well as in the pure flat space YM theory, there always exists an opportunity to pass over the potential barrier separating homotopically distinct vacuum sectors. However, at low energies all the overbarrier histories are suppressed by the destructive interference. In the pure YM theory the situation remains the same for any energies. In the EYM theory on the other hand, when the energy is large and exceeds the ground state sphaleron mass, the constructive interference occurs instead. This means that the rate of the inclusive $\nu$ fermion number violating reactions at very high energies is not small. Such reactions lead to the formation of intermediate sphaleron states, the decay of which will produce a large number of gravitons and gauge bosons $\phi$, as well as extra fermions due to the anomaly. It has been argued recently by Gibbons and Steif [32], that the decay of the BK particles should be accompanied by the fermion violation. Our arguments show that, at high energies, the probability of BK particles being born and subsequently decaying is not small. Notice that the energies available are very large – the masses of the EYM sphalerons are of the order of $M_{pl}/g$. Such processes may arise naturally within the context of superstring theory leading to a “primordial” fermion asymmetry. Indeed, lower order terms of the expansion of the superstring action in the string tension, give rise to the coupled Einstein-Yang-Mills-Dilaton (EYMD) equation of motion. These equations possess classical solutions, which resemble in many respects the BK solutions and coincide exactly with them in the vanishing dilatonic coupling limit [33]. It is very likely, that these EYMD particles may also be interpreted as sphalerons [34, 36], and our present analysis may be extended to that case.

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Appendix A  The gravitational action

In this paper we are interested in Lorenzian gravitational action. In view of the importance of this issue, we precisely specify in this Appendix our choice of action and its relation to the other possible forms of action.

Our basic requirement for the action is that it is completely free from second derivatives of the metric. Otherwise, all the analysis carried in the main text becomes cumbersome. As is known \[37\], the second derivatives entering the Einstein-Hilbert gravitational action, \( R\sqrt{-g} \), may be combined to form a total divergence:

\[
R\sqrt{-g} = \Gamma \sqrt{-g} - \partial_\mu (\sqrt{-g} W^\mu), \tag{A1}
\]

where the two-gamma term is

\[
\Gamma = g^{\mu\nu} \left( \Gamma^\beta_{\mu\alpha} \Gamma^\alpha_{\nu\beta} - \Gamma^\alpha_{\mu\nu} \Gamma^\beta_{\alpha\beta} \right), \tag{A2}
\]

and

\[
W^\mu = g^{\mu\alpha} \Gamma^\beta_{\alpha\beta} - g^{\alpha\beta} \Gamma^\mu_{\alpha\beta} = \frac{1}{\sqrt{-g}} \left( g^{\mu\alpha} \partial_\alpha \sqrt{-g} + \partial_\alpha (\sqrt{-g} g^{\alpha\mu}) \right). \tag{A3}
\]

It is natural then to choose for the gravitational action the following expression

\[
- \frac{1}{16\pi G} \int (R\sqrt{-g} + \partial_\mu (\sqrt{-g} W^\mu)) d^4 x = - \frac{1}{16\pi G} \int \Gamma \sqrt{-g} d^4 x, \tag{A4}
\]

because it is manifestly free from all second derivatives. Let the integration in this formula be performed over a 4-volume with a boundary \( \Sigma \) – we are working in an asymptotically flat spacetime and assume the \( \Sigma \) is a distant closed boundary which is shifted to spatial infinity in the end of the calculations. Then (A4) may be represented in the following equivalent form

\[
S_G = - \frac{1}{16\pi G} \int R\sqrt{-g} d^4 x - \frac{1}{16\pi G} \oint_\Sigma n_\mu W^\mu \sqrt{|g|} d^3 x, \tag{A5}
\]

where the integration in the second term is performed over the boundary 3-surface, whose unit outward normal is denoted by \( n^\mu \). This expression is our choice for the action. The boundary term in (A5) is non-covariant, and quasi-Cartesian coordinates are implied for calculation of it \[34\] – we imply...
that such coordinates exist. To represent this term in a more conventional form, assume that there exists an extension of the $n^\mu$ vector field into an open neighbourhood of the boundary; then one may write down

$$ \nabla_\mu n^\mu = \nabla^\mu n_\mu = \partial_\mu n^\mu + \Gamma^\mu_{\mu\nu} n^\nu = \partial^\mu n_\mu - g^{\mu\nu} \Gamma^\sigma_{\mu\nu} n_\sigma, $$  \hspace{1cm} (A6)

which allows us to rewrite the surface term in Eq. (A5) as follows

$$ -\frac{1}{16\pi G} \oint (n^\alpha \Gamma^\beta_{\alpha\beta} - g^{\alpha\beta} \Gamma^\mu_{\alpha\beta} n_\mu) \sqrt{|g|} d^3x = \frac{1}{8\pi G} \oint (K + C) d\Sigma, $$  \hspace{1cm} (A7)

where

$$ K = -\nabla_\mu n^\mu $$  \hspace{1cm} (A8)

is the trace of the second fundamental form of the boundary which does not depend on the choice of the extension of $n^\mu$, and

$$ C = \frac{1}{2} (\partial_\mu n^\mu + \partial^\mu n_\mu). $$  \hspace{1cm} (A9)

is an additional non-covariant quantity. It is easy to see that this latter quantity does not involve the normal derivatives of the metric. Indeed, representing $C$ as $C = \partial_\mu n^\mu + \frac{1}{2} g^{\mu\alpha} n^\rho \partial_\alpha g_{\mu\rho}$, and decomposing the partial derivative operator, $\partial_\mu$, into the normal and tangential components with respect to the boundary:

$$ \partial_\mu = \frac{n_\mu}{n^2} n^\alpha \partial_\alpha + (\delta_\mu^\alpha - \frac{n_\mu n^\alpha}{n^2}) \partial_\alpha, $$  \hspace{1cm} (A10)

one can see that the normal derivative of the metric in (A9) vanishes (the condition $n^2 = \text{const}$ is also to be used). This means that the $C$-term vanishes on variation of the action (A3) with the boundary condition $\delta g_{\mu\nu}|_\Sigma = 0$, so this term does not contribute to the resulting Einstein equations. Therefore, if one is interested in the action which just reproduces the correct Einstein equations on variation, then one may drop the non-covariant $C$-term in Eq. (A7) or replace it by the some other quantity giving no contribution on variation, for example, by the negative trace of the second fundamental form of the boundary imbedded into flat space. In this way one comes to the manifestly covariant Gibbons & Hawking’s (GH) action [20]. It is clear however, that dropping of the surface $C$-term is equivalent to introducing the second (tangential) derivatives into the action, so it is obvious that the GH action is not completely free from second derivatives of the metric.
Other possible choices of the surface term also do not allow all the second derivatives from the action to be excluded; comprehensive discussion of this topic can be found in Ref. [13]. So, we should inevitably keep the surface term (A7) as it stands. The resulting first order action is unique [13]. Thus, Eq. (A5) uniquely defines the gravitational action which is free from the second derivatives and gives correct Einstein equations on variation.

A novel feature arises when an event horizon is present. In this case the volume integration in Eq. (A5) should be performed only over the external with respect to the horizon region, i.e. over the part of the 4-volume enclosed by the external boundary, \(\Sigma\), and the inner boundary – event horizon surface – as well. In obvious notations, the result of such integration may be represented as follows:

\[
S_G = -\frac{1}{16\pi G} \int_{\Sigma_{\text{horizon}}} (\Gamma \sqrt{-g} - \partial_\mu (\sqrt{-g} W^\mu)) d^4x - \frac{1}{16\pi G} \oint_{\Sigma} n_\mu W^\mu \sqrt{|g|} d^3x.
\]

(Integrating the total divergence, one obtains

\[
S_G = -\frac{1}{16\pi G} \int_{(\text{external region})} \Gamma \sqrt{-g} d^4x - \frac{1}{16\pi G} \oint_{\Sigma_{\text{horizon}}} n_\mu W^\mu \sqrt{|g|} d^3x,
\]

(A12)

where the surface integral is to be performed over an inner boundary in the limit when it tends to the event horizon surface, \(n^\mu\) being the unit outward normal to the boundary. One may see that the action (A5) in this case is not equivalent to the manifestly first order expression (A4). Thus, when an event horizon is present, the action (A5) does involve second derivatives. However, these derivatives do not influence the variational procedure, as they only give the horizon surface contribution to the action, and it is this additional term which gives rise to the description of the black hole entropy and other related quantities [34, 35].

Now we perform an explicit computation of the action (A5) in the spherically symmetric case. It is convenient to represent the metric in the form

\[
ds^2 = e^\nu dt^2 - e^\lambda dr^2 - r^2 (d\sigma^2 + \sin^2 \sigma \ d\varphi^2),
\]

(A13)

where \(\nu, \lambda\) are functions of \(t, r\). Pass to the quasi-Cartesian coordinates,

\[
x^1 = r \sin \sigma \cos \varphi, x^2 = r \sin \sigma \sin \varphi, x^3 = r \cos \sigma.
\]

The metric then becomes

\[
ds^2 = e^\nu dt^2 - (\delta_{ij} + \frac{e^\lambda - 1}{r^2} x_i x_j) dx^i dx^j
\]

(A14)
The quantity $W^\mu$ given by Eq.(A3) reads in these coordinates

$$W^\mu = (W^0, W^i) = (\lambda \exp(-\nu), \{\frac{2}{r} - (\nu' + \frac{2}{r})\exp(-\lambda)\} \frac{x^i}{r}).$$  \hspace{1cm} (A15)$$

Let us specify the integration domain in the volume integral in Eq.(A5) as follows. Let the time, $t$, vary from $-t_0$ to $t_0$. We consider the case when an event horizon is present, whose size, $r_H$, may depend on time such that $r_H(t) = r_H(-t), r_H(\pm t_0) = 0$. Such a choice is specified by the situation we encounter in the main text, when the time evolution of the metric, whose action we want to calculate, starts from the flat space metric at $t = -t_0$ and ends again at the flat space value at $t = t_0$ passing, in general case, through the black hole phase in between. So, the integration domain in this case is: $t \in [-t_0, t_0], r \in [r_H(t), R_0]$, the variables $\vartheta, \varphi$ spanning the spatial two-sphere, $S^2$, and the limit $R_0 \to \infty$ being implied. The spacetime boundary is $\Sigma = \Sigma_+ \cup \Sigma_t \cup \Sigma_-$. The spacelike parts of the boundary are $\Sigma_{\pm} = \{t = \pm t_0, r \in [0, R_0], \vartheta \in [0, \pi], \varphi \in [0, 2\pi]\}$; the corresponding unit normals are $n_{(\pm)}^\mu = (\pm \exp(\nu/2), 0)$; the volume elements are $d\Sigma_{\pm} = \sqrt{|g|} d^3x = \exp(\lambda/2) dr d\Omega$, where $d\Omega = \sin\vartheta d\vartheta d\varphi$. The timelike part of the boundary is $\Sigma_t = \{t \in [-t_0, t_0], r = R_0, \vartheta \in [0, \pi], \varphi \in [0, 2\pi]\}$; the corresponding normal is $n_{(t)}^\mu = (0, \exp(\lambda/2) x^i/r)$; the volume element is $d\Sigma_t = \exp(\nu/2) dt d\Omega$.

Let us calculate first the surface integral in Eq.(A3). Using Eq.(A13) and the definitions introduced, one obtains explicitly

$$\frac{1}{4\pi} \int W^\mu n_\mu d\Sigma = \int_{-t_0}^{t_0} dt \int_{r_H(t)}^{R_0} dr \{\partial_t (\lambda r^2 \exp(\frac{-\nu}{2})) + \partial_r ((r^2 \nu' + 2r) \exp(\frac{-\lambda}{2}) - 2r \exp(\frac{\nu - \lambda}{2}) + r (\nu' + \lambda') \exp(\frac{\nu + \lambda}{2}) - \exp(\frac{\nu - \lambda}{2}))\}.$$  \hspace{1cm} (A16)$$

Direct calculation of the curvature scalar gives the Einstein-Hilbert term

$$\frac{1}{4\pi} \int R \sqrt{-g} d^4x = \int_{-t_0}^{t_0} dt \int_{r_H(t)}^{R_0} dr \{-\partial_t (\lambda r^2 \exp(\frac{-\nu}{2})) + \partial_r ((r^2 \nu' + 2r) \exp(\frac{-\lambda}{2}) - 2r \exp(\frac{\nu - \lambda}{2}) + r (\nu' + \lambda') \exp(\frac{\nu + \lambda}{2}) - \exp(\frac{\nu - \lambda}{2}))\}.$$

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Integrating the total derivatives, adding all these terms together and taking the limit \( R_0 \to \infty \) one obtains

\[
S_G = \frac{-1}{16\pi G} \left( \int R \sqrt{-g} dt^4 x + \oint_{\Sigma} n_\mu W^\mu d\Sigma \right) = \int_{-t_0}^{t_0} L_G dt, \quad (A18)
\]

where

\[
L_G = -\frac{1}{4G} \int_0^\infty r (\nu' + \lambda') \left( \exp \left( \frac{\nu + \lambda}{2} \right) - \exp \left( \frac{\nu - \lambda}{2} \right) \right) dr - \\
- \frac{1}{4G} \left\{ (r^2 \nu' + 2r) \exp \left( \frac{\nu - \lambda}{2} \right) - 2r \exp \left( \frac{\nu + \lambda}{2} \right) + \\
+ \dot{r}_H(t) \dot{\lambda} r^2 \exp \left( \frac{\nu - \lambda}{2} \right) \right\} \bigg|_{r=r_H(t)}. \quad (A19)
\]

Finally, introducing functions \( m(t, r), \sigma(t, r) \) such that \( \exp(\nu) = \sigma^2(1 - 2Gm/r), \exp(-\lambda) = (1 - 2Gm/r), 2Gm(t, r_H(t)) = r_H(t) \), one may see that the term involving time derivatives in \( (A19) \) vanishes, and the whole expression reduces to a remarkably simple formula

\[
L_G = - \int_{r_H(t)}^\infty m \sigma' dr - \frac{1}{2} \sigma(mr)' \bigg|_{r=r_H(t)}. \quad (A20)
\]

The two terms in the right hand side of this equation relate to the volume and horizon terms in Eq.\( (A12) \) respectively. One may check that for known spherically symmetric black hole solutions, the expression \( (A20) \) yields a value of the gravitational part of the action, which gives rise to regular thermodynamical parameters of a black hole \cite{34}, \cite{35}.

**Appendix B  Spherically symmetric Einstein-Yang-Mills fields.**

In this Appendix we list the full set of the EYM equations in the spherically symmetric case. It is convenient to represent the gauge-invariant quantities in terms of the variables \( \Omega_0, \Omega_1, f \). The non-zero components of the YM field energy-momentum tensor then are

\[
T_0^0 = \frac{1}{2\sigma^2} (\Omega_0' - \dot{\Omega}_1)^2 + \frac{1}{\Delta \sigma^2} (\dot{f}^2 + f^2 \Omega_0^2) + \frac{\Delta}{r^4} (f'' + f^2 \Omega_1^2) + \frac{(f^2 - 1)^2}{2r^4},
\]
\[ T'_0 = 2 \frac{\Delta}{r^4}(\dot{f} f' + f^2 \Omega_0 \Omega_1), \] (B1)

where \( \Delta = r^2 - 2mr \). The component \( T'_r \) can be obtained from Eq.(B1) by changing the sign before the first and the third terms on the right hand side; other non-zero components are \( T'_\theta = T'_\varphi = -(T'_0 + T'_r) \).

The Lagrangian density is

\[
g^2 \mathcal{L}_{YM} = -\frac{1}{2} \text{tr} F_{\mu\nu} F^{\mu\nu} \sqrt{-g} = \\
= \sin \vartheta \left\{ \frac{r^2}{2\sigma} (\Omega'_0 - \dot{\Omega}_1)^2 + \frac{r^2}{\Delta \sigma} (\dot{f}^2 + f^2 \Omega_0^2) - \frac{\sigma \Delta}{r^2} (f'^2 + f^2 \Omega_1^2) - \sigma \frac{(f^2 - 1)^2}{2r^2} \right\}. \] (B2)

The Pontryagin density reads

\[
\frac{1}{16\pi^2} \text{tr} F_{\mu\nu} \tilde{F}^{\mu\nu} \sqrt{-g} = \frac{\sin \vartheta}{8\pi^2} \{ (\Omega'_0 - \dot{\Omega}_1)(1 - f^2) + 2f(\dot{f} \Omega_1 - f' \Omega_0) \}. \] (B3)

Non-zero components of the Chern-Simons current given by Eq.(6) are

\[
K^0 = \frac{1}{8\pi^2 \sigma r^4} \left\{ W_1 (p_1^2 + p_2^2 - 1) + p_2 \dot{p}_1 + (1 - p_1) \dot{p}_2 \right\},
\]
\[
K^r = -\frac{1}{8\pi^2 \sigma r^4} \left\{ W_0 (p_1^2 + p_2^2 - 1) + p_2 \dot{p}_1 + (1 - p_1) \dot{p}_2 \right\}. \]

(B4)

The current is not, of course, gauge invariant, but its divergence is, and this divergence is just the density (B3) (up to the \( \sqrt{-g} \) factor).

The non-trivial YM equations read

\[
\left( \frac{r^2}{\sigma} (\Omega'_0 - \dot{\Omega}_1) \right)' = 2 \frac{r^2}{\Delta \sigma} f^2 \Omega_0, \] (B5)
\[
\left( \frac{r^2}{\sigma} (\Omega'_0 - \dot{\Omega}_1) \right)' = 2 \frac{\Delta \sigma}{r^2} f^2 \Omega_1, \] (B6)
\[
\left( \frac{\Delta \sigma}{r^2} f' \right)' - \left( \frac{r^2}{\Delta \sigma} \dot{f} \right) = \sigma \frac{f(f^2 - 1)}{r^2} + \left( \frac{\Delta \sigma}{r^2} \Omega_1^2 - \frac{r^2}{\Delta \sigma} \Omega_0^2 \right) f. \] (B7)

The independent Einstein equations are

\[ m' = r^2 T'_0, \] (B8)
\[ \dot{m} = -r^2 T'_r. \] (B9)
\[ \frac{\sigma'}{\sigma} = \frac{r^3}{\Delta} (T^0_0 - T^r_r). \] (B10)

The equations (B5)-(B10) admit non-trivial solutions discovered by Bartnik and McKinnon [9]. These solutions correspond to the static purely magnetic case, \( \Omega_0 = \Omega_1 \equiv 0 \). The non-trivial EYM equations then read

\[ \left( \frac{\Delta \sigma}{r^2} f' \right)' = \sigma f \frac{(f^2 - 1)}{r^2}, \] (B11)

\[ m' = \Delta \frac{f^2}{r^2} + \frac{(f^2 - 1)^2}{2r^2}, \] (B12)

\[ \sigma' = \frac{2}{r} f'^2 \sigma. \] (B13)

The regular asymptotically flat solutions to these equations are parameterized by an integer, \( n \); the corresponding magnetic function, \( f \), is usually denoted by \( w_n \). When \( r \) runs from zero to infinity, the function \( w_n(r) \) starts from unit value at the origin and, after \( n \) oscillations around zero, tends asymptotically to \((-1)^n \). The metric functions, \( m(r) \) and \( \sigma(r) \), monotonically increase from \( m(0) = 0 \) to \( m(\infty) = m_n \) and from \( \sigma(0) = \sigma_n < 1 \) to \( \sigma(\infty) = 1 \) correspondingly. The asymptotic value of the mass function, \( m_n \), grows monotonically with increasing \( n \) from the value \( m_1 = 0 \) to \( m_\infty = 1 \), the physical mass of the solutions is \( M_n = (\sqrt{4\pi/g})M_{pl}m_n \) with \( M_{pl} \) being Planck’s mass.

**Appendix C  Variation of the energy functional**

Consider the barrier height functional defined by Eq.(50):

\[ \varepsilon[K(r)] = \int_0^\infty (K'^2 + \frac{(K^2 - 1)^2}{2r^2}) \exp(-2 \int_r^\infty K'^2 \frac{dr}{r}) dr. \] (C1)

Let \( K(r) \) be a sufficiently smooth function satisfying the following conditions

\[ K'(r) = O(r) \quad \text{as} \; r \to 0; \quad K'(r) = O(1/r^2) \quad \text{as} \; r \to \infty. \] (C2)

Define for convenience two new functions

\[ \sigma(r) = \exp\{ -2 \int_r^\infty K'^2 \frac{dr}{r} \}, \quad m(r) = \frac{1}{\sigma(r)} \int_0^r (K'^2 + \frac{(K^2 - 1)^2}{2r^2}) \sigma dr. \] (C3)
they satisfy the following boundary conditions

\[ \sigma(\infty) = 1, \quad \sigma(0) \neq 0; \quad m(\infty) < \infty, \quad m(r) = O(r^3) \text{ as } r \to 0, \]

(C4)

and also, by definition, the following equations

\[ \sigma'(r) = 2 \frac{K'^2}{r} \sigma, \quad (m\sigma)' = (K'^2 + \frac{(K^2 - 1)^2}{2r^2}) \sigma. \]

(C5)

Consider small variation

\[ K(r) \to K(r) + \varphi(r), \]

(C6)

where

\[ \varphi(0) = \varphi(\infty) = 0. \]

(C7)

To preserve the boundary conditions, the variation must also satisfy

\[ \varphi'(r) = O(r) \text{ as } r \to 0, \quad \varphi'(r) = O(1/r^2) \text{ as } r \to \infty. \]

(C8)

Put (C6) into (C1) and expand the result over \( \varphi \), one then obtains

\[ \varepsilon[K + \varphi] = \varepsilon[K] + \delta \varepsilon + \delta^2 \varepsilon + \ldots, \]

(C9)

where the first variation is

\[ \delta \varepsilon = 2 \int_0^\infty \left\{ K' \varphi' \sigma + \frac{K(K^2 - 1)}{r^2} \sigma \varphi - 2 I(K'^2 + \frac{(K^2 - 1)^2}{2r^2}) \sigma \right\} dr, \]

(C10)

and the second variation is

\[ \delta^2 \varepsilon = \int_0^\infty \left\{ \sigma \varphi^2 + \sigma \frac{3K^2 - 1}{r^2} \varphi^2 - 8 I(K' \varphi' + \frac{K(K^2 - 1)}{r^2} \varphi) \sigma + \right. \]

\[ \left. + (8I^2 - 2J)(K'^2 + \frac{(K^2 - 1)^2}{2r^2}) \sigma \right\} dr, \]

(C11)

dots in (C9) denote higher order terms and the following new functions have been introduced:

\[ I(r) = \int_r^\infty K' \varphi' \frac{dr}{r}, \quad J(r) = \int_r^\infty \varphi'^2 \frac{dr}{r}. \]

(C12)
The boundary conditions (C2),(C8) imply that
\[ I(0) < \infty, \quad J(0) < \infty, \quad I(\infty) = J(\infty) = 0. \] (C13)

Consider the first variation (C10). Using (C5), represent \( \delta \varepsilon \) as follows
\[ \delta \varepsilon = 2 \int_0^\infty \left\{ K' \varphi' \sigma + K(K^2 - 1) \frac{\varphi \sigma}{r^2} - 2I(m \sigma)' \right\} dr. \] (C14)

Integrating by parts one has
\[ \delta \varepsilon = (2K' \sigma \varphi - 4Im \sigma)_0^\infty + 2 \int_0^\infty \left\{ -(K' \sigma)' \varphi + \sigma \frac{K(K^2 - 1)}{r^2} \varphi + 2m \sigma I' \right\} dr. \] (C15)

Note, that the boundary terms in this expression vanish. Finding \( I' \) from (C12) and integrating by parts once more, one finally arrives at
\[ \delta \varepsilon = 2 \int_0^\infty \left\{ -((1 - 2m r) \sigma K')' + \frac{K(K^2 - 1)}{r^2} \sigma \right\} \varphi dr. \] (C16)

This result agrees with Eq.(31). One can see that the vanishing of the first variation implies the following equation
\[ ((1 - 2m \frac{r}{r}) \sigma K')' = \sigma \frac{K(K^2 - 1)}{r^2}. \] (C17)

Assume now that the first variation vanishes and consider next the second variation. Using (C5) one obtains
\[ \delta^2 \varepsilon = \int_0^\infty \left\{ \sigma \varphi'^2 + \sigma \frac{3K^2 - 1}{r^2} \varphi^2 - 8I(K' \varphi' + \frac{K(K^2 - 1)}{r^2} \varphi) \sigma + 8I^2 (m \sigma)' - 2J(m \sigma)' \right\} dr. \] (C18)

Integrating the fourth term in the integrand by parts, and using Eq.(C12), one obtains
\[ 8 \int_0^\infty I^2 (m \sigma)' dr = 8I^2 m \sigma \bigg|_0^\infty - 16 \int_0^\infty I' m \sigma dr = 16 \int_0^\infty I m \sigma K' \varphi' \frac{dr}{r}, \] (C19)
where the boundary terms vanish. Combining this result with the third term in Eq. (C18), one has

\[-8 \int_0^\infty I\{ (1 - \frac{2m}{r})\sigma K' \varphi' + \frac{K(K^2 - 1)}{r^2} \sigma \varphi \} \, dr. \quad (C20)\]

Using (C17), (C12), (C5), represent this expression as follows

\[-8 \int_0^\infty I \{ (1 - \frac{2m}{r})\sigma K' \varphi' \} \, dr = -8 I (1 - \frac{2m}{r})\sigma K' \varphi|_0^\infty +
\quad + 8 \int_0^\infty I' (1 - \frac{2m}{r})\sigma K \varphi \, dr = -8 \int_0^\infty (1 - \frac{2m}{r})\sigma K^2 \varphi \varphi'|_r \, dr =
\quad = -4 \int_0^\infty (1 - \frac{2m}{r})\sigma' \varphi \varphi'|_r \, dr, \quad (C21)\]

where the boundary terms vanish. The fifth term in (C18) yields

\[-2 \int_0^\infty J(m\sigma)' \, dr = -2 Jm\sigma|_0^\infty + 2 \int_0^\infty J'm\sigma \, dr = -2 \int_0^\infty m\sigma' \varphi^2 2 \frac{dr}{r} =
\quad = -2m \sigma' \varphi|_0^\infty + 2 \int_0^\infty (m\sigma' \varphi')' \varphi \, dr = \int_0^\infty (\frac{2m}{r} \sigma \varphi')' \varphi \, dr. \quad (C22)\]

The first term in (C18) is

\[\int_0^\infty \sigma \varphi^2 \, dr = - \int_0^\infty (\sigma \varphi')' \, dr, \quad (C23)\]

where the boundary terms are zero. Consider also the following expression

\[0 = \int_0^\infty ((1 - \frac{2m}{r})\sigma' \varphi^2)' \, dr. \quad (C24)\]

Adding the equations (C21)-(C24) and introducing also the tortoise coordinate, \( r_* \),

\[\frac{dr}{dr_*} = \sigma (1 - \frac{2m}{r}), \quad (C25)\]

one finally arrives at

\[\delta^2 \varepsilon = \int_0^\infty \varphi (-\frac{d^2}{dr_*^2} + V) \varphi \, dr_*, \quad (C26)\]

where

\[V = \sigma (1 - \frac{2m}{r}) \{ 2(\sigma' (1 - \frac{2m}{r}))' + \frac{3K^2 - 1}{r^2} \sigma \}, \quad (C27)\]

which agrees with Eq. (56) provided that Eqs. (C5), (C17) are taken into account.
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Figure captions

**Fig.1** Typical shape of the energy-surface two-dimensional section $U_\varphi(\lambda, \alpha)$ for energy increasing sphaleron perturbation modes $\varphi$. The surface forms a barrier separating distant EYM vacua ($\lambda = 0, \pi$). Position of the sphaleron ($\alpha = 0, \lambda = \pi/2$) is shown by the vertical arrow.

**Fig.2** Plot of the function $U(\beta, \lambda)$ defined by Eq.(61). Vertical arrow shows the sphaleron position ($\beta = 1, \lambda = \pi/2$). Horizontal arrows correspond to the transverse ($\lambda$-direction) and the longitudinal ($\beta$-direction) sphaleron negative modes.

**Fig.3** Behaviour of the quantum phase $\Phi_{scale}(\beta)$ defined by Eq.(62) with $E = 1$. The path passing through the sphaleron is specified by the value $\beta = 1$. 
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