COMPOSITE RAMSEY THEOREMS VIA TREES

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ABSTRACT. We prove a theorem ensuring that the compositions of certain Ramsey families are still Ramsey. As an application, we show that in any finite coloring of $\mathbb{N}$ there is an infinite set $A$ and an as large as desired finite set $B$ with $(A + B) \cup (AB)$ monochromatic, answering a question from a recent paper of Kra, Moreira, Richter, and Robertson. In fact, we prove an iterated version of this result that also generalizes a Ramsey theorem of Bergelson and Moreira that was previously only known to hold for fields. Our main new technique is an extension of the color focusing method that involves trees rather than sequences.

1. INTRODUCTION

Ramsey theory on $\mathbb{N}$ is centered around characterizing the structures of the following form.

Definition 1.1. We say a family $B \subseteq P(\mathbb{N})$ is Ramsey if for any finite coloring $c : \mathbb{N} \rightarrow [r]$ there is some $A \in B$ such that $c$ is constant on $A$.

The linear Rado families, i.e. the Ramsey families $B$ that are generated by the set of solutions to a finite system of linear equations, are completely characterized by Rado’s theorem [12]. Moreover, given any such family we can obtain a geometric Rado family by composing a given coloring with the homomorphism $n \mapsto 2^n$. For example, Schur’s theorem tells us that $\{\{x, y, z\} : x + y = z\}$ is a linear Rado family, which in turn implies that $\{\{x, y, z\} : xy = z\}$ is a geometric Rado family. From now on we will simply write, for example, that $\{x, y, x + y\}$ is Ramsey, a slight abuse of notation.

In contrast to the linear and geometric case, characterizations of other Ramsey families are still quite elusive, and determining if even some of the simplest non-linear families are Ramsey has been the subject of much recent study; see, e.g., [5, 6, 7, 10, 11, 13, 14], for some recent results of this type. Here we further this line of work by showing that certain non-linear compositions of Ramsey families are still Ramsey.

Theorem 1.2. Let $\mathbb{T}$ be the smallest collection of families of subsets of $\mathbb{N}$ such that:

1. If $B$ is a finite linear or geometric Rado family, then $B \in \mathbb{T}$
2. If $B \in \mathbb{T}$, then for any finite set $P$ of integral polynomials and $k \in \mathbb{N}$ the family

$$\{\{x, xa, x + p(a) : a \in A, x \in X, p \in P\} : X \in \binom{\mathbb{N}}{k}, A \in B\} \in \mathbb{T}.$$
Any family $B \in \mathbb{T}$ is Ramsey. Moreover, if the family is as in point (2) then there is an infinite set of $x \in \mathbb{N}$ witnessing this fact.

This theorem is somewhat abstract, so we discuss some concrete applications.

For the first, consider the family $B = \binom{\mathbb{N}}{k}$ consisting of all $k$ element subsets of $\mathbb{N}$. This is a linear Rado family by the pigeonhole principle, and so applying Theorem 1.2 with $B$ and the constant polynomial $n \mapsto n$ we obtain the following:

**Corollary 1.3.** In any finite coloring of $\mathbb{N}$ there is an infinite set $A$ and set $B \in \binom{\mathbb{N}}{k}$ such that $(A + B) \cup (AB)$ is monochromatic.

The $|A| = |B| = 1$ case of this corollary is Moreira’s theorem [10]. Recently, in [9] Question 8.4, Kra, Moreira, Richter, and Robertson asked if the above result is true when $B$ is also required to be infinite and stated that even the $|A| = |B| = 2$ case seemed out of reach.

Applying Theorem 1.2 to the family from Corollary 1.3 again, we find monochromatic sets of the form

$$(A + B + C) \cup (A + (BC)) \cup (A(B + C)) \cup (ABC).$$

Doing this repeatedly, we show:

**Corollary 1.4.** In any finite coloring of $\mathbb{N}$ there are sets $A_1, \ldots, A_n \in \binom{\mathbb{N}}{k}$ such that

$$\{a_1 \circ_1 (a_2 \circ_2 (\ldots \circ_{n-2} (a_{n-1} \circ_{n-1} a_n)\ldots): a_i \in A_i, \circ_i \in \{+, \cdot\}\}$$

is monochromatic.

Previously, the above result for $|A_i| = 1$ was known to hold in finite colorings of $\mathbb{Q}$ by a theorem of Bergelson and Moreira [3]. However, even the $|A_i| = 1$ case of Example 1 was open for colorings of $\mathbb{N}$.

So far we have only considered applications of Theorem 1.2 that begin with the pigeonhole principle as the base family. Instead starting with a geometric family, we obtain the following common extension of the linear and geometric van der Waerden Theorems by applying Theorem 1.2 with the geometric Rado family $B = \{zy^i, y : i \leq k\}$ and the linear polynomials $n \mapsto n$.

**Corollary 1.5.** The family $\{x + iy, xy, z \cdot xy^i : i \leq k\}$ is Ramsey for any $k \in \mathbb{N}$.

Notice that this tells us that any finite coloring of $\mathbb{N}$ contains arbitrarily long arithmetic and geometric progressions of the same color and step size, and moreover with starting points that are a simple multiplicative shift of each other. One hope might be to prove the following refinement of this result.

**Question 1.** Is the family $\{x + iy, xy^i : i \leq k\}$ Ramsey for any $k \in \mathbb{N}$?

Our proof of Theorem 1.2 builds off of the ideas used in Moreira’s proof that the family $\{xy, x + y\}$ is Ramsey [10]. We use the fact that there are many potential choices for $x$ and $y$ in each step of the proof to build a tree of possible choices for these values. From here we deduce new Ramsey theorems by running color focusing arguments on this tree; see especially Figures 1 and 2 for illustrations of this idea.
Acknowledgements. Thanks to Zach Hunter and Marcin Sabok for many helpful comments on an earlier version of this paper.

2. Notation and technical background

In this section we collect the notation and technical facts that will be used throughout the paper. The reader should note that the proofs of the concrete Theorems in Section 3, which contain almost all of the combinatorial ideas needed for the proof of our general theorem, mostly manages to avoid these notations and relies on more well known facts. Namely, the only thing from Subsection 2.2 needed in Section 3 is the more usual \( P_0 = \mathbb{N} \) case of the polynomial van der Waerden theorem, and Subsection 2.3 can be entirely skipped.

The main technical difficulty with the proof of Theorem 1.2 is that each step of our induction will require us to prove Ramsey theoretic results on the space of polynomials with an additional variable (see the discussion after the proof of Theorem 3.3), which leads to notational difficulties if not combinatorial ones.

2.1. Trees. In this paper, by a tree we mean a collection of finite words (including the empty word) where the \( n \)th letter comes from alphabet \( A_n \). More precisely, given finite sets \( A_0, ..., A_R, \) we’ll consider trees of the form \( T = \bigcup_{0 \leq i < R+1} \prod_{j<i} A_j \). In particular, \( T \) is rooted at the empty set, which has \( |A_0| \) children, and so on.

Given a tree \( T \) as above, we define the \( i \)’th level of the tree as \( T_i = \prod_{j<i} A_j \). Further, we will use \( \cdot \) to denote concatenation, i.e., given \( t \in T_i \) and \( p \in A_i \), \( t \cdot p \in T_{i+1} \).

2.2. Polynomial spaces, notions of size, and van der Waerden’s theorem. As mentioned above, the proof of Theorem 1.2 will require us to consider spaces of many variabled polynomials with rational coefficients.

Definition 2.1. Let \( P_0 = \mathbb{N} \), and having defined \( P_n \), let \( P_{n+1} \) be the set of all formal objects of the form \( \frac{a_1}{b_1} x_{n+1} + \frac{a_2}{b_2} x_{n+1}^2 + ... + \frac{a_k}{b_k} x_{n+1}^k \), where \( k \in \mathbb{N} \) and \( a_j, b_j \in \bigcup_{0 \leq i \leq n} P_i \).

In particular, \( P_1 = x\mathbb{Q}[x] \), the space of all formal polynomials of degree at least one with rational coefficients.

Given \( P \subset P_n \) and \( d \in P_m \), by \( P(d) \) we mean the formal object obtained by replacing every instance of \( x_n \) in \( P \) with \( d \). We will only do this in instances where \( m = n \), in which case this is the composition of polynomials, or when \( m = n - 1 \), where we can think of this as evaluation. Note that in the latter case in general this might not be a subset of \( P_{n-1} \), but in all of our uses of this notation \( d \) will have been chosen such that it is.

We will need a piecewise syndetic version of the polynomial van der Waerden theorem for our proofs. To formally state this, we need the following definitions.

Definition 2.2.

- A set \( S \subseteq P_n \) is **syndetic** if there is a finite set \( F \subset P_n \) such that \( P_n = S - F \).
- A set \( T \subseteq P_n \) is **thick** if for any finite set \( F \), there is a \( t \in T \) such that \( tF \subset T \).
- A set \( A \subseteq P_n \) is **piecewise syndetic** if there is a thick set \( T \) and a syndetic set \( S \) with \( S \cap T \subseteq A \).
Unlike for \( \mathbb{N} \), \( x_n^P \) is not piecewise syndetic for \( n > 0 \). This was mentioned by Moreira in [10] Section 7 as a difficulty with extending his proof to rings that are not large ideal domains (such as \( P_n \) for \( n > 0 \)). However, this fact will not be problematic for us, since we will frontload our use of lemmas about piecewise syndetic sets to before we do any multiplications in all of our proofs.

We now state the two facts about piecewise syndetic sets that we will need. The first is well known; see, e.g. [8], Theorem 4.4).

**Lemma 2.3.** If \( A \subseteq P_n \) is piecewise syndetic and \( A = A_0 \cup \ldots \cup A_R \), then some \( A_i \) is piecewise syndetic.

Finally, we will use the following variant of the Polynomial van der Waerden theorem.

**Theorem 2.4** (Polynomial van der Waerden, essentially [2]). Let \( P \subset P_{n+1} \) be a finite set of polynomials and \( A \subseteq P_n \) be piecewise syndetic. Then there is some \( d \in P_n \) such that

\[
A \cap \bigcap_{p \in P} (A - p(d))
\]

is piecewise syndetic.

This follows from any variant of the polynomial van der Waerden theorem for commutative semigroups, such as [1] Theorem 7.8. Note that that the presence of rational rather than the integral coefficients more typically discussed in this context does not lead to any difficulties. For example, the existence of monochromatic progressions of the form \( a, a + d, a + d, a + 2d \) follows from the existence of monochromatic progressions of the form \( a, a + 2d', a + 3d', a + 6d' \) and setting \( d = 6d' \).

2.3. **Ramsey families.** We will use the following generalization of the linear and geometric Rado families discussed in the introduction. This generality is mostly present just to isolate the properties of the Ramsey family’s that are needed for the proof; namely, these are the properties that will be used to glue two Ramsey families together in a way that preserves their structure.

**Definition 2.5.** Let \( B \) be a family of finite subsets of \( P_n \).

- \( B \) is a **\( P_n \)-Ramsey** if for any finite coloring \( c : P_n \to [R] \) there is a \( B \in B \) such that \( c \) is constant on \( B \).
- \( B \) is a **linear Ramsey family** if it is Ramsey and if for every \( B \in B \) and \( c \in P_n \), \( cB \in B \).
- \( B \) is a **geometric Ramsey family** if it is Ramsey and every \( B, C \in B \) can be expressed as \( B = (b_1, \ldots, b_m) \), \( C = (c_1, \ldots, c_m) \), where also \( (b_1c_1, \ldots, b_mc_m) \in B \). Here we insist that this ordering is consistent between elements (for example, the like terms in geometric progressions).

Observe that every linear Rado family is a linear Ramsey family in \( P_n \), which can be seen by considering the polynomials of the form \( \{ix_n : i \leq k \} \) for sufficiently large \( k \), and every geometric Rado family is a geometric Ramsey family in \( P_n \) by considering polynomials of the
form \( \{x_n^i : i \leq k \} \). Throughout this paper we will only be interested in linear and geometric Ramsey families of this form, so we do not specify if they are \( P_n \) Ramsey or \( P_{n+1} \) Ramsey. This will only be relevant in the proofs of Propositions 4.1 and 4.3.

3. Two Concrete Cases

In this section we prove two special cases of our main result that contain essentially all of the ideas needed for the general theorem while requiring much less notation. Our first example is the following.

**Theorem 3.1.** Any 2-coloring of \( \mathbb{N} \) contains a monochromatic set of the form

\[
\{a, ab, ac, abc, a+b, a+c, a+bc, a+b+c\}.
\]

This is the two color case of Theorem 1.2 applied to the geometric Ramsey family \( \{b, c, bc\} \) and the polynomials \( n \mapsto 0 \) and \( n \mapsto n \). The proof here generalizes easily to arbitrary geometric Ramsey families and polynomials, although we save the details for the next section for the sake of clarity. In particular, the methods here can be easily adapted to prove Corollary 1.3.

**Proof.** Suppose that \( \mathbb{N} = C_0 \cup C_1 \). Our goal is to build a tree of infinite sets with edges labeled by polynomials that we will use for a color focusing argument.

**Claim 3.2.** There are finite sets of polynomials \( P_0, P_1 \subset P_1 \), elements \( d_0, d_1 \in \mathbb{N} \), and a tree of infinite monochromatic sets \( A_t \subseteq \mathbb{N} \) for \( t \in T = \bigcup_{0 \leq i \leq 2} \prod_{j<i} P_j \) such that

1. any 2-coloring of \( P_1 \) contains a monochromatic set of the form \( \{b, c, bc\} \) (in this case, \( P_1 = \{x^i : i \leq s\} \), where \( s \) is the 2-color Schur number, would work).
2. any 2\(|P_1|+1\)-coloring of \( P_0 \) contains a monochromatic set of the form \( \{b, c, bc\} \).
3. for each \( t \in T_i \) there is an \( r \in \{0, 1\} \) such that \( A_t \cup \bigcup_{k \geq 1}(A_t + \prod_{j \leq k} P_j(d_j)) \subseteq C_r \).
4. for \( t \in T_i \) and \( p_i \in P_i \) we have \( p_i(d_i)A_t = A_{t \rightarrow p_i} \in T_{i+1} \).

We will call structures similar to the one described above **color focusing trees**. Before proving the claim, let us see how to use it to finish the proof of Theorem 3.1 (see, especially, Figure 1).

By Lemma 2.3, without loss of generality assume that \( A_\emptyset \subseteq C_0 \). Now, consider an auxiliary 2\(|P_1|+1\)-coloring of \( p \in P_0 \) based on the \((|P_1| + 1)\)-tuple consisting of the color of \( A_p \) together with the colors of its \(|P_1| \) descendents in the tree. By Property (2) of the Claim we find \( b_0, c_0 \in P_0 \) such that the set \( P'_0 = \{b_0, c_0, b_0c_0\} \) is monochromatic. By Property (4) of the construction, this means that for any \( a_0 \in A_\emptyset \) the set \( \{a_0p(d_0) : p \in P'_0\} \) is monochromatic.

If this set is monochromatic in \( C_0 \) then we can finish the proof by letting \( a \in A_\emptyset \subseteq C_0 \), \( b = b_0(d_0) \), and \( c = c_0(d_0) \). This is because \( \{ab, ac, abc\} \subset C_0 \) by the above paragraph and \( \{a+b, a+c, a+bc\} \subset A_\emptyset + P_0(d_0) \subseteq C_0 \) by Property (3) of the Claim.

Otherwise, we can assume that \( A_p \subseteq C_1 \) for each \( p \in P'_0 \). By our construction, for \( p_1 \in P_1 \) and \( p'_0 \in P'_0 \), the color of \( p_1(d_1)A_{p'_0} \) depends only on \( p_1 \). Considering the 2-coloring of \( p \in P_1 \) based on the color of \( A_{p' \rightarrow p} \) and using Property (1) of the claim, we find \( b_1, c_1 \in P_1 \) such
Figure 1. The color focusing step of the proof of Theorem 3.1. Each vertex of the tree represents an infinite set and multiplying by the indicated values sends elements of a vertex to its child. The uncolored vertices are all the same color. If these vertices are blue then we can finish the proof by setting $b = b_1(d_1)$ and $c = c_1(d_1)$ and considering the vertices marked with $\times$. Otherwise they are red and we can finish by setting $b = b_0(d_0) \cdot b_1(d_1)$ and $c = c_0(d_0) \cdot c_1(d_1)$ and considering the vertices marked with $\diamond$.

that $P'_1 = \{b_1, c_1, b_1c_1\}$ is monochromatic. This means that the sets $A_{p'_0 \sim p'_1}$ for $p'_0 \in P'_0$ and $p'_1 \in P'_1$ are all monochromatic in the same color.

If $p'_1(d_1)A_{p'_0} \subseteq C_1$ for $p'_1 \in P'_1$ and $p'_0 \in P'_0$, then we can finish as above by taking $a \in A_{p'_0}$, $b = b_1(d_1)$, $c = c_1(d_1)$ and applying Properties (3) and (4) of the Claim as done above.

Otherwise, $p'_1(d_1)A_{p'_0} \subseteq C_0$ for $p'_1 \in P'_1$ and $p'_0 \in P'_0$. In this case we can finish by setting $a \in A_0$, $b = b_0(d_0) \cdot b_1(d_1)$, and $c = c_0(d_0) \cdot c_1(d_1)$, and applying properties (3) and (4) of the Claim as above.

Now, in order to finish the proof of Theorem 3.1 we just need to prove the Claim.

Proof of Claim 3.2. The construction of the sets $A_t$ is routine and follows from a couple of applications of the polynomial van der Waerden theorem. If we where interested in finding a sequence rather than a tree of infinite sets satisfying the conditions of the Claim, then the construction would be essentially the same as that used by Moreira in [10] Corollary 1.5.

First we define $P_1$ and $P_0$. Let $s_1$ be the 2-color Schur number and define $P_1 = \{x^i : i \leq s_1\}$. Let $s_0$ be the $2^{s_1+1}$-color Schur number and $P_0 = \{x^i : i \leq s_0\}$. These satisfy properties (1) and (2).

Now, without loss of generality we may assume that $C_0$ is piecewise syndetic. By Theorem 2.4, we know that there is a $d_0 \in \mathbb{N}$ such that the set $A_0 = C_0 \cap \bigcap_{p \in P_0} (C_0 - p(d_0))$ is piecewise syndetic. Consider the $2^{|P_0|}$-coloring of $a_0 \in A_0$ based on the $|P_0|$-tuple listing the colors of
\(a_0 \cdot p_0(d_0)\) for \(p_0 \in P_0\). By Lemma 2.3, one of these color classes is piecewise syndetic. Let this set be \(A'_0\).

Apply Theorem 2.4 to \(A'_0\) with the set of polynomials
\[
P'_1 = \{p_0(d_0)p_1, \frac{p_1}{p_0(d_0)} : p_i \in P_i\}
\]
to find \(d_1 \in \mathbb{N}\) such that \(A_1 = A'_0 \cap \bigcap_{p_i \in P'_1} (A'_0 - p_1(d_1))\) is piecewise syndetic. Finally, consider the \(2^{|P_0|\cdot|P_1|}\) coloring of the \(a_0 \in A_1\) based on the tuple listing the color of \(a_0 \cdot p_0(d_0) \cdot p_1(d_1)\), and let \(A_0\) be the color class that is piecewise syndetic by Lemma 2.3. Letting \(A_t\) be defined to satisfy Property (4) of the claim gives the desired sets.

Our next special case is the first iterative case of Theorem 1.2, i.e. the \(|A| = |B| = |C| = 1\) case of Example 1.

**Theorem 3.3.** Any 2-coloring of \(\mathbb{N}\) contains a monochromatic set of the form
\[
\{a + b + c, a + bc, a(b + c), abc\}.
\]

As in the proof of Theorem 3.1, our strategy will be to build a color focusing tree. We will need the following version of Moreira’s theorem [10], whose proof requires only a slight variation on Moreira’s arguments.

**Proposition 3.4.** Let \(F \subset \mathbb{N}\) be finite. For any finite coloring of \(\mathbb{P}_1\) there are \(x, y \in \mathbb{P}_1\) such that
\[
\{xy, x + y : f \in F\}
\]
is monochromatic.

We’ll skip the proof of this proposition for now; the only needed change to Moreira’s arguments is that all of the applications of van der Waerden’s theorem are done at the start of the proof before any multiplications. This avoids the problem caused by the set \(x\mathbb{Q}[x]\) not being piecewise syndetic in \(\mathbb{Q}[x]\). Proposition 3.4 will also follow from Proposition 4.1 below applied to the trivial Ramsey family \(\mathcal{B} = (\mathbb{P}_1)^t\).

**Proof of Theorem 3.3.** Fix a 2-coloring \(N = C_0 \cup C_1\). We will use the following color focusing tree.

**Claim 3.5.** There are finite sets \(P_0, P_1 \subset \mathbb{P}_1\) naturals \(d_0, d_1 \in \mathbb{N}\), and a tree of infinite monochromatic sets \(A_t \subseteq \mathbb{N}\) for \(t \in T = \bigcup_{0 \leq i \leq 2} \prod_{j < i} P_j\) satisfying:

1. any 2-coloring of \(P_0\) contains a monochromatic set of the form \(\{b_0c_0, b_0 + c_0\}\).
2. any 2-coloring of \(P_1\) contains a monochromatic set of the form
\[
\{b_1c_1, b_1 + c_1, b_1 + \frac{c_1}{p_0(d_0)} : p_0 \in P_0\}.
\]
3. for each \(t \in T_i\) there is an \(r \in \{0, 1\}\) such that \(A_t \cup \bigcup_{k \geq r} (A_t + \prod_{j \leq k} P_j(d_j)) \subseteq C_r\).
Figure 2. The color focusing tree from the proof of Theorem 3.3. The uncolored vertices form a monochromatic set. If they are blue we can finish the proof by setting \(b = b_1(d_1)\) and \(c = c_1(d_1)\) and considering the vertices marked with \(\times\). Otherwise they are red and we can finish by setting \(b = (b_0 + c_0)(d_0) \cdot b_1(d_1)\) and \(c = c_1(d_1)\) and considering the vertices marked with \(\diamond\).

\((4)\) for \(t \in T_i\) and \(p_i \in P_i\) we have \(p_i(d_i)A_t = A_{t \rightarrow p_i}\).

The proof of Claim 3.5 is nearly identical to that of Claim 3.2, with the only real difference being that instead of Schur’s theorem we use Proposition 3.4 and compactness to ensure that polynomials \(P_i \subseteq \mathbb{P}_1\) satisfying Properties (1) and (2) exist. We omit the details.

We now use the Claim to complete the proof of Theorem 3.3; see especially Figure 2.

Without loss of generality we may assume that \(A_0 \subseteq C_0\). 2-coloring elements \(p_0 \in P_0\) based on the color of \(A_{p_0}\), by item (1) of Claim 3.5 we find a monochromatic set of the form \(\{b_0 c_0, b_0 + c_0\}\).

If this set is monochromatic in \(C_0\) then set \(b = b_0(d_0), c = c_0(d_0), \) and \(a \in A_0\). We know \(a b c\) and \(a(b + c) \in C_0\) by (4) of Claim 3.5. Moreover, \(a + b + c, a + bc \in A_0 + P_0(d_0) \subseteq C_0\) by property (3), so these are as desired.

Otherwise, we know \(A_1(b_0 + c_0) \subseteq C_1\). 2-coloring elements \(p_1 \in P_1\) based on the color of \(A_{(b_0 + c_0) \rightarrow p_1}\), by (2) of Claim 3.5 we find a monochromatic set of the form \(\{b_1 c_1, b_1 + c_1, b_1 + c_1/(b_0 + c_0)\} \subseteq P_1\).

If this set is monochromatic in \(C_1\) then we can finish as above by setting \(b = b_1(d_1), c = c_1(d_1), \) and \(a \in A_{b_0 + c_0}\) and applying properties (2) and (3) of Claim 3.5 as above.
Otherwise, this set is monochromatic in $C_0$. Now set $b = (b_0 + c_0)(d_0) \cdot b_1(d_1)$, $c = c_1(d_1)$, and $a \in A_\emptyset$. Then $abc \in C_0$ and $a(b + c) \in C_0$ by Property (4). Finally, observe that

$$a + b + c = a + (b_0 + c_0)(d_0)(b_1(d_1) + \frac{c_1(d_1)}{(b_0 + c_0)(d_0)}) \in A_\emptyset + P_0(d_0)P_1(d_1) \subseteq C_0$$

by Property (3). Similarly,

$$a + bc = a + (b_0 + c_0)(d_0)b_1c_1(d_1) \in A_\emptyset + P_0(d_0)P_1(d_1) \subseteq C_0,$$

completing the proof. ■

Before concluding this section and moving on to the proof of our general theorem, notice how in the proof of Theorem 3.3 we needed to use a version of Moreira’s theorem 3.4 for colorings of $P_1$. If we wanted to prove the $n = 3$ of Corollary 1.4 we would need to prove a version of Theorem 3.3 for colorings of $P_1$ rather than $\mathbb{N}$ for a similar reason, and this in turn would require us to prove a version of Moreira’s theorem for colorings of $P_2$. This will be the main source of technical difficulties in the next section, although it will lead to mostly notational rather than combinatorial problems.

4. The general case

In this section we prove Theorem 1.2. As mentioned previously, all of the combinatorial ideas needed for the proof are already present in the two concrete cases discussed in Section 3, so we suggest that the reader focus on understanding those proofs. We only include the technical proofs in this section for the purpose of completeness and verification.

The proof will split into three cases depending on the structure of the family $B$. Throughout we will need to work in the space of many variabled rational polynomials for the reasons discussed after the proof of Theorem 3.3.

**Proposition 4.1.** Let $P_n = C_1 \cup ... \cup C_R$ be a finite coloring, $P \subset P_{n+1}$ a finite set of polynomials, and $B$ a geometric Ramsey family. There is an infinite $X \subseteq P_n$ and a $B \in B$ such that

$$\{x, xb, x + p(b) : x \in X, b \in B, p \in P\}$$

is monochromatic.

**Proof.** As in the special cases proved in Section 3, our goal is to build a color focusing tree.

**Claim 4.2.** There are finite sets of polynomials $P_0, ..., P_R \subset P_{n+1}$, elements $d_0, ..., d_R \in P_n$, and a tree of infinite monochromatic sets $A_t$ for $t \in T = \bigcup_{0 \leq i \leq R+1} \prod_{j < i} P_j$ such that

1. any $R$-coloring of $P_R$ contains a monochromatic element of $B$.
2. for $0 \leq r < R$, any $R^{P_{r+1}|...|P_{r+1}}$-coloring of $P_r$ contains a monochromatic element of $B$.
3. for each $t \in T$, there is an $r \in \{1, ..., R\}$ such that

$$A_t \cup \bigcup_{i \leq k \leq R} (A_t + P( \prod_{i \leq j \leq k} P_j(d_j))) \subseteq C_r.$$
(4) for \( t \in T_i \) and \( p_i \in P_i \) we have \( p_i(d_i)A_t = A_{t-p_i} \in T_{i+1} \).

We can use Claim 4.2 to finish the proof in exactly the same manner as we finished the proof of Theorem 3.1 (see especially Figure 1). We include the details here more for the purpose of verification and completeness than understanding.

By coloring vertices \( t \in T \) based on the color of \( A_t \) and the color of all of their descendants in the tree and applying Properties (1) and (2), we find elements \( B_i \subseteq P_i \) of \( \mathcal{B} \) such that for any \( b_i \in B_i \) the color of \( A_{b_0 \cdots b_i} \) depends only on \( j \). By the pigeonhole principle we find \( i < j \) such that these colors agree on color \( r \), i.e. the sets \( A_{b_0 \cdots b_i} \) and \( A_{b_0 \cdots b_j} \) are all subsets of \( C_r \).

Suppose that the sets \( B_i \subseteq \mathcal{B} \) are of the form \( (b_{0,i}, ..., b_{m,i}) \). Let \( t = b_{0,0} \cdots \sim b_{0,i} \). To finish the proof we will let \( X = A_t \subseteq C_r \) and for \( 0 \leq k \leq m \) let \( b_k = b_{k,i+1}(d_{i+1}) \cdots b_{k,j}(d_j) \). Then by Lemma technical, we know \( (b_0, ..., b_m) = B \in \mathcal{B} \). Moreover, by the previous paragraph and Property (4) of the Claim we know \( x b_k \in C_r \) for \( b_k \in B \) and \( x \in X \). Finally, by Property (3) we know that \( x + P(b_k) \in A_t \subseteq C_r \) for each \( x \in X \) and \( b_k \in B \), finishing the proof.

Therefore, in order to complete the proof of Proposition 4.1 it suffices to prove Claim 4.2.

**Proof of Claim 4.2.** All of the combinatorial ideas needed for the proof of the Claim were already in the proof of Claim 3.2; the only new difficulty is the notation. We will use several applications of the polynomial van der Waerden theorem.

First, we construct the polynomials \( P_{R}, ..., P_{0} \subseteq \mathbb{P}_n \). By the definition of geometric Ramsey families and the compactness principle, we know that for any \( k \in \mathbb{N} \) there is a finite set \( F \subseteq \mathbb{P}_{n+1} \) such that any \( k \)-coloring of \( F \) contains a monochromatic element of \( \mathcal{B} \). Let \( P_R \) be such a finite set for \( k = R \), and having defined \( P_{R}, ..., P_{R-i+1} \) let \( P_i \) be such a set for \( k = R^{P_{i+1}} \cdots P_{R-i+1} \). These satisfy Properties (1) and (2) of the claim.

For the other properties, suppose without loss of generality that \( C_1 \) is piecewise syndetic, and let \( A'_{i-1} = C_1 \). We inductively define a sequence of piecewise syndetic sets \( A'_{i-1} \supseteq A_0 \supseteq A_0' \supseteq A_1 \supseteq A_1' \cdots \supseteq A_R' \), elements \( d_0, ..., d_R \in \mathbb{P}_n \), and finite sets of polynomials \( P_0', ..., P_R' \subseteq \mathbb{P}_{n+1} \) such that:

(a) \[
P_i' = \bigcup_{0 \leq a \leq b \leq c < i} \frac{P(P_i \cdot \prod_{b \leq j \leq c} P_j(d_j))}{\prod_{j' < c} P_{j'}(d_{j'})},
\]
where if \( a = 0 \) then the (empty) bottom product is 1.

(b) \( A_i = A'_{i-1} \cap \bigcap_{p' \in P_i'} (A'_{i-1} - p(d_i)) \), where \( d_i \) is chosen such that this set is piecewise syndetic.

(c) Coloring \( a \in A_i \) based on the tuple listing the colors of \( a \cdot p_0(d_0) \cdots p_i(d_i) \) for \( p_j \in P_j \), \( A'_i \) is chosen to be a piecewise syndetic monochromatic subset of \( A_i \).

Finding sets satisfying (2) is possible by the polynomial van der Waerden Theorem 2.4, and finding sets satisfying (3) is possible by Lemma 2.3. Once these have been constructed, set \( A_0 = A_R' \) and define \( A_t \) for \( t \in T \) to satisfy (4) of Claim 4.2. Note that property (2) of the Claim follows from (b) and property (3) from (c).
The proofs of the other two cases use color focusing arguments similar to the one used in Theorem 3.3 and Figure 2.

**Proposition 4.3.** Let $\mathbb{P}_n = C_1 \cup ... \cup C_R$ be a finite coloring, $P \subset \mathbb{P}_{n+1}$ a finite set of polynomials, and $\mathcal{B}$ a linear Ramsey family. There is an infinite $X \subseteq \mathbb{P}_n$ and an $B \in \mathcal{B}$ such that

$$\{x, xb, x + p(b) : x \in X, b \in B, p \in P\}$$

is monochromatic.

**Proof.** As usual, we build a coloring focusing tree.

**Claim 4.4.** There are finite sets of polynomials $P_0, ..., P_R \subset \mathbb{P}_{n+1}$, elements $d_0, ..., d_R \in \mathbb{P}_n$, and a tree of infinite monochromatic sets $A_t \subseteq \mathbb{P}_n$ for $t \in T = \bigcup_{0 \leq i \leq R+1} \prod_{j < i} P_j$ such that

1. any $R$-coloring of $\mathbb{P}_n$ contains a monochromatic element of $\mathcal{B}$.
2. for each $t \in T_i$ there is an $r \in \{0, ..., R\}$ such that
   $$A_t \cup \bigcup_{i \leq k \leq R} (A_t + P(\prod_{i \leq j \leq k} P_j(d_j))) \subseteq C_r.$$
3. for $t \in T_i$ and $p_i \in P_i$ we have $p_i(d_i)A_t = A_{t-p_i} \in T_{i+1}$.

We will omit the proof of Claim 4.4 as it follows exactly the same steps as the constructions above.

To finish the proof, by Property (1) and (3) we find $(b_0, i), ..., (b_m, i) = B_i \subset P_i$ elements of $\mathcal{B}$ such that the color of $A_{b_{0,0}, ..., b_{0,k}, ..., b_{k,j}}$ depends only on $j$. By the pigeonhole principle we find $i < j$ such that the colors of $A_{b_{0,0}, ..., b_{0,i}, ..., b_{k,j}}$ and $A_{b_{0,0}, ..., b_{0,i}, ..., b_{k,j}}$ agree for any choice of $b_{k,j} \in B_j$.

Let $c = \prod_{k=i+1}^{j-1} b_{0,k}(d_k)$. Then $c(b_{0,j}, ..., b_{m,j}) \in \mathcal{B}$ by Lemma tech. We are done by taking $X = A_{b_{0,0}, ..., b_{0,i}, b_{k^*} = cb_{k,j}}$, and applying Property (2) of the Claim.

We are now ready to prove the iterated case.

**Proposition 4.5.** Let $\mathcal{B}$ be a Ramsey family such that for any $n \in \mathbb{N}$ and any finite $P' \subset \mathcal{P}_{n+1}$ the family

$$\mathcal{C} = \{xb, x + p'(b) : b \in B, x \in X, p' \in P'\} : X \in \binom{\mathbb{N}}{k}, B \in \mathcal{B}$$

is $\mathbb{P}_n$-Ramsey.

Then for any $n \in \mathbb{N}$ and any finite $P \subset \mathcal{P}_{n+1}$ the family

$$\{xc, x + p(c) : c \in C, x \in X, p \in P\} : X \in \binom{\mathbb{N}}{\infty}, C \in \mathcal{C}$$

is $\mathbb{P}_n$-Ramsey.
Proof. Fix \( \mathcal{C}, \mathcal{B}, \) and \( P \) as above and consider a finite coloring \( \mathbb{P}_n = C_1 \cup ... \cup C_R. \)

The proof is a more technical version of the proof of Theorem 3.3. We will make use of the following color focusing tree.

**Claim 4.6.** There are finite sets of polynomials \( P_0, ..., P_R \subset \mathbb{P}_{n+1}, \) elements \( d_0, ..., d_R \in \mathbb{P}_n, \) and a tree of infinite monochromatic sets \( A_t \) for \( t \in T = \bigcup_{0 \leq i \leq R} \prod_{j<i} P_j \) such that

1. any \( R \)-coloring of \( P_0 \) contains a monochromatic element of \( \mathcal{C}. \)
2. for \( 0 \leq r < R \) and \( R \)-coloring of \( P_r, \) there is a \( B_r \in \mathcal{B} \) and an \( X_r \in \binom{\mathbb{P}_n}{k} \) such that

\[
B'_r = \{ x_r b_r, x_r + \frac{p'(b_r)}{\prod_{i<j<k<r} P_j(d_j)} : x_r \in X, b_r \in B_r, p' \in P', 0 \leq k < r \}
\]

is monochromatic, where if \( k = 0 \) the (empty) product in the denominator is 1.
3. for each \( t \in T_i \) there is an \( r \in \{1, ..., R\} \) such that

\[
A_t \cup \bigcup_{i \leq k \leq R} (A_t + P(\prod_{i \leq j \leq k} P_j(d_j))) \subseteq C_r.
\]

4. for \( t \in T_i \) and \( p_i \in P_t \) we have \( p_i(d_i) A_t = A_{t-p_i} \in T_{i+1}. \)

Again, we will omit the proof since it follows from only slight modifications of the proofs of Claims 3.2 and 4.2.

The deduction of Proposition 4.5 from Claim 4.6 is just a natural generalisation of the similar deduction in the proof of Theorem 3.3.

Repeatedly using (1), (2), and (4) of the claim, we find a sequence \( B'_0, ..., B'_R \) of sets as in properties (1) and (2) of the claim with distinguished elements \( x_r, b_r \in B'_r \) as in the definition and such that the color of \( A_{(x_0+b_0)-...-(x_t+b_t)} \) for \( b'_t \in B'_t \) depends only on \( r. \)

By the pigeonhole principle, we find \( i < j \) such that these colors agree, i.e. there is some \( r \in \{1, ..., R\} \) \( A_{(x_0+b_0)-...-(x_t+b_t)} \subseteq C_r \) and \( A_{(x_0+b_0)-...-(b'_j)} \subseteq C_r \) for each \( b'_j \in B'_j. \)

To finish the proof, set \( X = A_{(x_0+b_0)-...-(x_t+b_t)} \) and \( C' = (x_0(d_0) + b_0(d_0)) \cdot ... \cdot (x_{j-1}(d_{j-1} + b_{j-1}(d_{j-1}))B'_j(d_j)). \) By the definition of \( B'_j \) in property (2) of the Claim, this contains a subset \( C \subseteq C. \) For \( x \in X \) and \( c \in C, \) we have that \( xc \in C_r \) by the above paragraph, and \( x + P(c) \subseteq C_r \) by property (3) of the Claim, completing the proof.

Finally, Theorem 1.2 follows by combining Propositions 4.2, 4.3, and 4.5.

5. Open problems

In addition to Question 1 and the problem of Kra, Moreira, Richter, and Robertson mentioned in the intro, there are several other natural ways to potentially extend this work. For example, in a recent paper with Sabok [4] we showed that any finite coloring of \( \mathbb{Q} \) contains a monochromatic set of the form \( \{a, b, ab, a + b\}. \) Can we prove a common generalization of this fact and Corollary 1.4? The 3 term case of this would be the following (recall that, given a set of numbers \( A, \) by \( FS(A) \) we mean the set of all finite non-repeating sums of elements of \( A \) and by \( FP(A) \) we mean the set of all non-repeating finite products):
Question 2. Does every finite coloring of $\mathbb{Q}$ contain a monochromatic set of the form

$$FS(a, b, c) \cup FP(a, b, c) \cup (a + FP(b, c)) \cup (a \cdot FS(b, c))?$$

Of course, even the simpler problem of finding a monochromatic set of the form $FS(a, b, c) \cup FP(a, b, c)$ is open, but it seems plausible that finding the correct way to combine the methods from this paper with the methods from [4] is good way to attack both problems. A good place to start would be with showing that Theorem 1.2 can be extended to work for the $\mathbb{Q}$-Ramsey family \{b, c, bc, b + c\}, i.e.

Question 3. Does every finite coloring of $\mathbb{Q}$ contain a monochromatic set of the form

$$(a + FS(b, c)) \cup (a \cdot FS(b, c)) \cup (a + FP(b, c)) \cup (a \cdot FP(b, c))?$$

In another direction, Corollary 1.4 shows that in any finite coloring of the naturals we can find $a_1, \ldots, a_n$ such that no matter how we place $+$ and $\cdot$ between the terms, the resulting expression is the same color so long as the expression is calculated with respect to the increasing bracketing.

Is a similar Ramsey theorem still true if we allow for any bracketing instead of the increasing one? The first open case of this is the following:

Question 4. Suppose the naturals are finitely colored. Are there $a, b, c, d \in \mathbb{N}$ such that the set

$$\{(a \circ_1 b) \circ_2 (c \circ_3 d) : \circ_i \in \{+, \cdot\}\}$$

is monochromatic?

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