ATTENUATED TENSOR TOMOGRAPHY ON SIMPLE SURFACES

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ABSTRACT. In this paper we consider the geodesic X-ray transform with attenuation coefficient as a combination of smooth complex function and 1-form. We show that attenuated X-ray transform applied to the pair of tensors is injective modulo the natural obstruction.

1. Introduction

The geodesic X-ray transform of a function is defined by integrating over geodesics. It is naturally arises in linearization of the problem of determining a conformal factor of a Riemannian metric in the boundary rigidity problem. The X-ray transform also arises in Computer Tomography, Positron Emission Tomography, geophysical imaging in determining the inner structure of the Earth, ultrasound imaging. Uniqueness result and stability estimates of the geodesic X-ray transform were obtained by R. G. Mukhometov [16] for simple surface. For simple manifolds of any dimension this result was proven in [4, 17], see also V. A. Sharafutdinov’s books [25, 26]. In his paper N. S. Dairbekov generalized this result for nontrapping manifolds without conjugate points [6]. Fredholm type inversion formulas were given in [22] by L. Pestov and G. Uhlmann.

In our paper we consider the uniqueness problem for the attenuated X-ray transform on a surface. Now let us define this transform.

Let $M$ be a surface with smooth boundary $\partial M$ and endowed with Riemannian metric $g$.

Define the sets of inward and outward unit vectors respectively by

$$\partial_{\pm}SM = \{(x, \xi) \in SM : x \in \partial M, \pm \langle \xi, \nu(x) \rangle \geq 0 \},$$

here and further $\nu$ is the unit inner normal to $\partial M$.

We assume that $(M, g)$ is nontrapping, which means that the time $\tau(x, \xi)$ when the geodesic $\gamma_{x, \xi}(t)$ exists is finite for each $(x, \xi) \in \partial_+ SM$.

Let $h \in C^{\infty}(M)$ and $\alpha$ be a smooth 1-form on $M$. Consider an attenuation coefficient $a$ as a combination of $h$ and $\alpha$, i.e. $a(x, \xi) = h(x) + \alpha_x(\xi)$ for $(x, \xi) \in SM$. Let $\psi : SM \to \mathbb{R}$ be a smooth function on $SM$. Define the attenuated X-ray transform of $\psi$ by

$$I^a \psi(x, \xi) := \int_0^{\tau(x, \xi)} \psi(\gamma(t), \dot{\gamma}(t)) \exp \left[ \int_0^t a(\gamma_{x, \xi}(s), \dot{\gamma}_{x, \xi}(s)) \, ds \right] dt,$$

$$:= \int_0^{\tau(x, \xi)} \psi(\gamma(t), \dot{\gamma}(t)) \exp \left[ \int_0^t h(\gamma_{x, \xi}(s)) + \alpha_{\gamma_{x, \xi}(s)}(\dot{\gamma}_{x, \xi}(s)) \, ds \right] dt,$$
where \((x, \xi) \in \partial_+ SM\).

In this paper we will consider the problem of injectivity of attenuated X-ray transform applied to functions on \(M\).

We say that \((M, g)\) is simple if 1) \(\partial M\) is strictly convex and 2) all geodesics have no conjugate points. In this case, \(M\) is diffeomorphic to the unit ball of \(\mathbb{R}^2\). In particular, a simple manifold is nontrapping. The notion of simplicity arises naturally in the context of the boundary rigidity problem [15].

The Euclidean attenuated X-ray transform is the basis of the medical imaging modality SPECT. The attenuated geodesic ray transform arises in inverse transport problems with attenuation [13, 14], when the index of refraction is anisotropic and represented by a Riemannian metric. It also arises in geophysics where there is attenuation of the elastic waves. Rather unexpectedly, this transform also appeared in the recent works [7, 12] in the context of Calderón’s inverse conductivity problem in anisotropic media.

The following result shows that attenuated X-ray transform on 2-dimensional simple Riemannian manifold is injective.

**Theorem A.** Assume \((M, g)\) is a simple Riemannian surface. Let \(h\) be a smooth complex function and let \(\alpha\) be a smooth complex 1-form on \(M\). Consider an attenuation coefficient \(a\) as a combination of \(h\) and \(\alpha\), i.e. \(a = h + \alpha\). Suppose that \(f : M \to \mathbb{C}\) is smooth such that \(I^a f \equiv 0\), then \(f \equiv 0\).

In the case where \(M = \mathbb{R}^2\) with the Euclidean metric, the corresponding injectivity result for the attenuated X-ray transform with \(a = 0\) has been proved by different methods in E. V. Arbuzov, A. L. Bukhgeim and S. G. Kazantsev [3], R. G. Novikov [19], F. Natterer [18], and J. Boman and J.-O. Strömberg [5]. These methods also come with inversion formulas. If \(M\) is the unit disc in \(\mathbb{R}^2\) with Euclidean metric, a direct inversion formula was given by Kazantsev and A. A. Bukhgeim [11] for the case \(\alpha = 0\). See D. V. Finch [8] and P. Kuchment [10] for surveys of these and other developments in Euclidean space. Injectivity of the attenuated X-ray transform for simple surface has been proved in [24] by M. Salo and G. Uhlmann when \(a\) is a function on \(M\) and in [20] by G. P. Paternain, M. Salo, G. Uhlmann when \(a\) is purely imaginary.

Now we are interested in \(I^a\) applied to the functions on \(SM\) of the following type

\[\psi(x, \xi) = f_{i_1 \ldots i_m}(x)\xi^{i_1} \cdots \xi^{i_m} + \beta_{i_1 \ldots i_{m-1}}(x)\xi^{i_1} \cdots \xi^{i_{m-1}},\]

where \(f\) and \(\beta\) are smooth symmetric (covariant) tensors of ranks \(m\) and \(m - 1\) on \(M\). It has nontrivial kernel since

\[I^a(h p + d^* p + \sigma(p \alpha)) = 0\]

for any smooth symmetric \((m - 1)\)-tensor \(p\) with \(p|_{\partial M} = 0\). Here and further \(\sigma\) denotes symmetrization. We investigate if these are the only elements of the kernel.

The case \(m = 1\) corresponds to the geodesic weighted Doppler transform in which one integrates a vector field along geodesics. This transform appears in ultrasound tomography to detect tumors using blood flow measurements and also in non-invasive industrial measurements for reconstructing the velocity of a moving fluid. For \(m \geq 1\),
problems of this kind also appear in inverse problems for hyperbolic equations \[23\], photoelasticity \[1\]. The integration of tensors of order two along geodesics when \(a = 0\) arises as the linearization of the boundary rigidity problem and the linear problem is known as deformation boundary rigidity.

The next theorem describes elements in the kernel of the attenuated X-ray transform applied to combination of smooth symmetric tensors of ranks \(m\) and \((m - 1)\).

**Theorem B.** Let \((M, g)\) be a simple 2-dimensional Riemannian manifold, and consider \(h\) and \(\alpha\) to be a smooth complex function and 1-form (resp.) on \(M\). Denote 
\[ a = h + \alpha. \]
If \(f\) and \(\beta\) are smooth symmetric \(m\) and \((m - 1)\)-tensors (resp.) on \(M\) such that 
\[ I^a(f + \beta) \equiv 0, \]
then \(f = hp\) and \(\beta = d^*p + \sigma(pa)\) for some smooth symmetric \(m - 1\)-tensor \(p\) with \(p|_{\partial M} = 0\).

For \(m = 1\) results of this type were given in the unit disc in \(\mathbb{R}^2\) in \[11\] when \(\alpha = 0\), for simple manifolds with \(\|a\|_{L^\infty}\) small and \(\alpha = 0\) in \[7\], and for simple manifolds with \(g\) and \(a\) close to real analytic in \[9\]. Corresponding result have been proved in \[20\] by G. P. Paternain, M. Salo, G. Uhlmann when \(a\) is purely imaginary.

When attenuation coefficient vanishes this is the case of the ordinary X-ray transform. Some earlier results corresponding to this problem were described in \[21\] and Theorem B was proved for the case when attenuation coefficient is absent.

Proof of Theorem A follows the same scheme as in \[24\] and proof of Theorem B is mimic those in \[21\]. Briefly, the paper is organized as follows. In Section 2 we recall some facts and definitions from \[20, 24\] that will be used in our paper. These are notions of (anti)holomorphic functions, existence of (anti)holomorphic integrating factors for the transport equation. Next we recall regularity of solutions of transport equation. In Section 3.1 we prove injectivity of the attenuated X-ray transform using pseudodifferential arguments. Finally, Section 3.2 is devoted for the proof of Theorem A. Here we use the same method as in the second proof of Theorem 1.1 in \[21\] see Section 5.

2. Preliminaries

For \((x, \xi) \in SM\) we denote by \(\gamma_{x,\xi} : [\tau_-(x, \xi), \tau_+(x, \xi)] \to M\) a maximal geodesic with initial conditions \(\gamma_{x,\xi}(0) = x, \dot{\gamma}_{x,\xi}(0) = \xi\) and \(\gamma_{x,\xi}(\tau_-(x, \xi)), \gamma_{x,\xi}(\tau_+(x, \xi)) \in \partial M.\) Using the implicit function theorem it is easy to show that the functions \(\tau_+\) and \(\tau_-\) are smooth near a point \((x, \xi)\) such that the geodesic \(\gamma_{x,\xi}(t)\) meets \(\partial M\) transversally at \(t = \tau_-(x, \xi)\) and \(t = \tau_+(x, \xi)\) respectively. Since, by strictly convexity this condition holds everywhere on \(SM\setminus S(\partial M)\), the functions \(\tau_-\) and \(\tau_+\) are smooth on \(SM\setminus S(\partial M)\) \[25, Section 4.1\].

The generator \(G\) of the geodesic flow \(\phi_t\) is the vector field on \(SM\) defined as
\[ G = \frac{\partial}{\partial t}\phi_t(x, \xi)\bigg|_{t=0}. \]

Since \(M\) is assumed oriented there is a circle action on the fibers of \(SM\) with infinitesimal generator \(V\) called the vertical vector field. In local coordinates it is
given by

\[ V u(x, \xi) = \langle \xi_\perp, \partial u / \partial \xi \rangle. \]

The space \( L^2(SM) \) decomposes orthogonally as a direct sum

\[ L^2(SM) = \bigoplus_{k \in \mathbb{Z}} H_k \]

where \( H_k \) is the eigenspace of \(-iV\) corresponding to the eigenvalue \( k \). A function \( u \in L^2(SM) \) has a Fourier series expansion

\[ u = \sum_{k=-\infty}^{\infty} u_k, \]

where \( u_k \in H_k \). Also \( \|u\|^2 = \sum \|u_k\|^2 \), where \( \|u\|^2 = (u, u)^{1/2} \). The even and odd parts of \( u \) with respect to velocity are given by

\[ u_+ := \sum_{k \text{ even}} u_k, \quad u_- := \sum_{k \text{ odd}} u_k. \]

Locally we can always choose isothermal coordinates \((x^1, x^2)\) so that the metric can be written as

\[ ds^2 = e^{2\lambda}((dx^1)^2 + (dx^2)^2) \]

where \( \lambda \) is a smooth real-valued function of \( x = (x^1, x^2) \). This gives coordinates \((x^1, x^2, \varphi)\) on \( SM \) where \( \varphi \) is the angle between a unit vector \( v \) and \( \partial / \partial x^1 \). In these coordinates we may write \( V = \partial / \partial \varphi \) and

\[ u_k(x, \varphi) = \left( \frac{1}{2\pi} \int_0^{2\pi} u(x, t)e^{-kt}dt \right)e^{ik\varphi} = \tilde{u}_k(x)e^{ik\varphi}. \]

A function \( u \) on \( SM \) is called \textit{holomorphic} if \( u_k = 0 \) for all \( k < 0 \). Similarly, we say that a function \( u \) on \( SM \) is called \textit{antiholomorphic} if \( u_k = 0 \) for all \( k > 0 \).

The following result that was obtained in [24, Lemma 2.6] will be used in our paper:

\textbf{Lemma 2.1.} The product of two (anti)holomorphic functions is (anti)holomorphic, and \( e^w \) is (anti)holomorphic if \( w \) is (anti)holomorphic.

Now we recall some results about the existence of holomorphic and antiholomorphic integrating factors for the equation

\[ Gu + au = -\psi \text{ in } SM, \]

where we assume that \( a(x, \xi) = h(x) + \alpha_x(\xi) \). All functions in this section will be scalar and complex valued.

By a \textit{(anti)holomorphic integrating factor} we mean a function \( e^{-w} \), where \( w \in C^\infty(SM) \) is \textit{(anti)holomorphic} in the angular variable, such that for all \( r \in C^\infty(SM) \)

\[ e^w G(e^{-w}r) = Gr + ar \text{ in } SM. \]

This is equivalent with the equation

\[ Gw = -a \text{ in } SM. \]

It was proved in [20, Theorem 4.1] that holomorphic and antiholomorphic integrating factors always exist. This result is stated below.
Theorem 2.2. Let \((M, g)\) be a simple Riemannian surface and \(a \in C^\infty(SM)\). The following conditions are equivalent.

(a) There exist a holomorphic \(w \in C^\infty(SM)\) and antiholomorphic \(\bar{w} \in C^\infty(SM)\) such that \(Gw = G\bar{w} = -a\).

(b) \(a(x, \xi) = h(x) + \alpha_x(\xi)\) where \(h\) is a smooth function on \(M\) and \(\alpha\) is a 1-form.

Using this theorem we see that the boundary value problem
\[
Gu + au = -\psi \text{ in } SM, \quad u|_{\partial SM} = 0
\]
is equivalent with the problem
\[
(1) \quad G(e^{-w}u) = -e^{-w}\psi, \quad -e^{-w}u|_{\partial SM} = 0.
\]

Note. If \(\psi = f(x)\) then the right hand side of (1) equals to \(e^{-w}f\) and is holomorphic since \(w\) is holomorphic. The next result also was proved in [24, Proposition 5.1] and tells that in this case \(e^{-w}u\) is holomorphic too.

Lemma 2.3. Let \((M, g)\) be a simple Riemannian surface and let \(\tilde{f}\) be a function on \(SM\) (anti)holomorphic in the angular variable. Suppose that \(v \in C^\infty(SM)\) satisfies
\[
Gv = -\tilde{f} \text{ in } SM, \quad v|_{\partial SM} = 0.
\]
Then \(v\) is (anti)holomorphic in the angular variable, and \(v_0 = 0\).

Before finishing this section we state some regularity result for the transport equation from [20, Proposition 5.2].

Lemma 2.4. Let \(F : SM \to \mathbb{C}\) be smooth and \(I^a(F) = 0\) then
\[
u(x, \xi) = \int_0^{\tau_+(x, \xi)} F(\phi_t(x, \xi)) \exp \left[ \int_0^t a(\phi_s(x, \xi)) \, ds \right] dt, \quad (x, \xi) \in SM
\]
is smooth.

3. Injectivity of the X-ray Transforms

3.1. Proof of Theorem A. Now we proceed with the proof of Theorem A. Let first \(f \in C^\infty_c(M^{\text{int}})\), and assume that \(I^a f \equiv 0\). Then by Lemma 2.3 the function
\[
u(x, \xi) = \int_0^{\tau_+(x, \xi)} f(\gamma(t)) \exp \left[ \int_0^t a(\gamma_s(x, \xi), \dot{\gamma}_s(x, \xi)) \, ds \right] dt,
\]
is smooth in \(SM\) and satisfies
\[
Gu + au = -f \text{ in } SM, \quad u|_{\partial SM} = 0.
\]

By Theorem 2.2 there is a holomorphic \(w \in C^\infty(SM)\) and antiholomorphic \(\bar{w} \in C^\infty(SM)\) such that
\[
G(e^{-w}u) = -e^{-w}f, \quad G(e^{-\bar{w}}u) = -e^{-\bar{w}}f.
\]

Now Lemma 2.3 shows that \(e^{-w}u\) is holomorphic and \(e^{-\bar{w}}u\) is antiholomorphic. Multiplying by \(e^w\) and \(e^{\bar{w}}\), it follows that \(u\) itself is both holomorphic and antiholomorphic.
This is only possible when \( u \) does not depend on \( \xi \), that is, \( u = u_0 \). Now the transport equation reads
\[
du_0(\xi) + au_0 = -f \text{ in } SM, \quad u_0|_{\partial SM} = 0.
\]
Evaluating this at \( \pm \xi \) and subtracting the resulting expressions gives that \( du_0 \equiv -u_0 \alpha \), hence \( u_0 \equiv 0 \) by the boundary condition. Consequently \( f \equiv 0 \).

It remains to prove the result when \( f \in C^\infty(M) \) may have support extending up to the boundary. In fact, this case can be reduced to the result for compactly supported functions by using the general principle that \((I^a)^* I^a\) is an elliptic pseudodifferential operator.

Suppose \( f \in C^\infty(M) \) and \( I^a f \equiv 0 \). We consider more generally the weighted ray transform with weight \( \rho \in C^\infty(SM) \),
\[
I_\rho f(x, \xi) = \int_0^{\tau_+(x, \xi)} \rho(\phi_t(x, \xi)) f(\pi \circ \phi_t(x, \xi)) dt, \quad (x, \xi) \in \partial_+ SM.
\]
With the choice \( \rho = e^{ia} \) we obtain \( I_\rho f \equiv 0 \). Let \( (\tilde{M}, g) \subset (M, g) \) be another simple manifold which is so small that any \( \tilde{M} \)-geodesic starting at a point of \( \partial SM \) never enters \( M \) again. We extend \( a \) to \( SM \) as a smooth function and \( f \) by zero to \( \tilde{M} \), and denote by \( \tilde{I}_\rho \) the corresponding weighted ray transform in \( \tilde{M} \).

It is easy to see that \( \tilde{I}_\rho(\tilde{x}, \tilde{\xi}) \) for all \( (\tilde{x}, \tilde{\xi}) \in \partial_+ S\tilde{M}, \) since either the geodesic starting from \( (\tilde{x}, \tilde{\xi}) \) never touches \( M \) or else \( \tilde{I}_\rho(\tilde{x}, \tilde{y}) = I_\rho f(x, \xi) \) for some \( (x, \xi) \in \partial_+ SM \). By \cite[Proposition 2]{9}, since \( \rho \) is nonvanishing, \( \tilde{I}_\rho I_\rho \) is an elliptic pseudodifferential operator of order \(-1\) in \( \tilde{M}^{\text{int}} \). Now \( \tilde{I}_\rho^* \tilde{I}_\rho f = 0 \), and elliptic regularity shows that \( f \) is smooth. Thus \( f \in C^\infty(\tilde{M}^{\text{int}}) \) and \( \tilde{I}_\rho \equiv 0 \), showing that \( I^a f \equiv 0 \). The result above implies that \( f \equiv 0 \) as required.

### 3.2. Proof of Theorem B

The proof of Theorem B reduces to proving the next result. We say that \( f \in C^\infty(SM) \) has degree \( m \) if \( f_k = 0 \) for \( |k| \geq m + 1 \).

Now, word for word as in \cite{20}, we explain the identification between real-valued symmetric \( m \)-tensor fields and certain smooth functions on \( SM \) with degree \( m \). Given such a tensor \( f = f_{i_1\cdots i_m} dx^{i_1} \otimes \cdots \otimes dx^{i_m} \) we consider the corresponding function on \( SM \) (henceforth referred to as the restriction) defined by
\[
f(x, \xi) = f_{i_1\cdots i_m} \xi^{i_1} \cdots \xi^{i_m}.
\]
Then clearly
\[
f = \sum_{k=-m}^{m} f_k
\]
where \( f_k = f_{-k} \). Moreover if \( m \) is even (resp. odd) all the odd (resp. even) Fourier coefficients vanish.

Conversely suppose that we are given a smooth real-valued function \( f \in C^\infty(SM) \) with degree \( m \). Suppose in addition that if \( m \) is even (resp. odd) then \( f = f_{+} \) (resp. \( f = f_{-} \)), i.e. all \( f_k \) and \( f_{-k} \) parts with \( k < m \) odd (resp. even) will vanish. Since \( f \) is real-valued \( f_k = f_{-k} \). For each \( k \geq 1 \), the function \( f_{-k} + f_k \) gives rise to a unique real-valued symmetric \( k \)-tensor \( F_k \) whose restriction to \( SM \) is precisely \( f_{-k} + f_k \). This can
be seen as follows. Recall that a smooth element $f_k$ can be identified with a section of $T^*M^\otimes k$ hence, its real part defines a symmetric $k$-tensor. (For $k = 0$, $f_0 = f_0$ is obviously a real-valued 0-tensor.)

By tensoring with the metric tensor $g$ and symmetrizing it is possible to raise the degree of a symmetric tensor by two. Hence $\lambda F_k := \sigma(F_k \otimes g)$ will be a symmetric tensor of degree $k + 2$ whose restriction to $SM$ is again $f_k + f_{-k}$ since $g$ restricts as the constant function 1 to $SM$. Now consider the symmetric $m$-tensor

$$F := \sum_{i=0}^{[m/2]} \lambda^i F_{m-2i}.$$ 

It is easy to check that the restriction of $F$ to $SM$ is precisely $f$.

The proof of Theorem B reduces for the proof of the following

**Theorem 3.1.** Let $(M, g)$ be a simple surface, and assume that $u \in C^\infty(SM)$ satisfies $Gu + au = -\psi$ in $SM$ with $u|_{\partial(SM)} = 0$. If $\psi \in C^\infty(SM)$ has degree $m \geq 1$, then $u$ has degree $m - 1$.

**Proof of Theorem B.** Let $f$ and $\beta$ be a symmetric $m$-tensor and $m - 1$-tensor field on $M$ and suppose that $I^a(f + \beta) = 0$. We write

$$u(x, \xi) := \int_0^{r(x,v)} (f + \beta)(\phi_t(x,v)), \quad (x,v) \in SM.$$ 

Then $u|_{\partial(SM)} = 0$, and also $u \in C^\infty(SM)$ by Lemma 2.4. Now $f + \beta$ has degree $m$, and $u$ satisfies $Gu + au = -(f + \beta)$ in $SM$ with $u|_{\partial(SM)} = 0$. Theorem 3.1 implies that $u$ has degree $m - 1$. We let $p := -u$. As we explained above, $p$ gives rise to a symmetric ($m - 1$)-tensor still denoted by $p$. Since $G(p) + ap = f + \beta$, this implies that $d^*p + ap$ and $f + \beta$ agree when restricted to $SM$ and since they are both symmetric tensors of the same degree it follows that $d^*p + ap = f + \beta$. This proves the theorem.

Theorem 3.1 is in turn an immediate consequence of the next two results.

**Lemma 3.2.** Let $(M, g)$ be a simple surface, and assume that $u \in C^\infty(SM)$ satisfies $Gu + au = -\psi$ in $SM$ with $u|_{\partial(SM)} = 0$. If $m \geq 0$ and if $\psi \in C^\infty(SM)$ is such that $\psi_k = 0$ for $k \leq -m - 1$, then $u_k = 0$ for $k \leq -m$.

**Lemma 3.3.** Let $(M, g)$ be a simple surface, and assume that $u \in C^\infty(SM)$ satisfies $Gu + au = -\psi$ in $SM$ with $u|_{\partial(SM)} = 0$. If $m \geq 0$ and if $\psi \in C^\infty(SM)$ is such that $\psi_k = 0$ for $k \geq m + 1$, then $u_k = 0$ for $k \geq m$.

We will only prove Lemma 3.2, the proof of the other result being completely analogous.

**Proof of Lemma 3.2.** Suppose that $u$ is a smooth solution of $Gu + au = -\psi$ in $SM$ where $\psi_k = 0$ for $k \leq -m - 1$ and $u|_{\partial(SM)} = 0$. We choose a nonvanishing function $v \in \Omega_m$ and define the 1-form

$$A := -v^{-1}Gv.$$
Then \( vu \) solves the problem

\[(G + a + A)(vu) = -v\psi \text{ in } SM, \quad vu|_{\partial(SM)} = 0.\]

Note that \( v\psi \) is a holomorphic function. Next we employ a holomorphic integrating factor: by Theorem 2.2 there exists a holomorphic \( w \in C^\infty(SM) \) with \( Gw = a + A \).

The function \( e^w vu \) then satisfies

\[G(e^w vu) = -e^w v\psi \text{ in } SM, \quad e^w vu|_{\partial(SM)} = 0.\]

The right hand side \( e^w v\psi \) is holomorphic. Now Lemma 2.3 implies that the solution \( e^w vu \) is also holomorphic and \((e^w vu)_k = 0\) for \( k \leq 0 \), and therefore \( u_k = 0 \) for \( k \leq -m \) as required.  

References

[1] H. K. Aben, 1979, *Integrated Photoelasticity*, New York: McGraw-Hill
[2] Yu. Anikonov, V. Romanov, *On uniqueness of determination of a form of rst degree by its integrals along geodesics*, J. Inverse Ill-Posed Probl. 5 (1997), 467-480.
[3] E. V. Arbuzov, A. L. Bukhgeim, and S. G. Kazantsev, *Two-dimensional tomography problems and the theory of A-analytic functions*, Siberian Advances in Mathematics 8 (1998), 1-20.
[4] I. N. Bernstein, M. L. Gerver, *Conditions of distinguishability of metrics by godographs*. Methods and Algorithms of Interpretation of Seismological Information. Computerized Seismology. 13. Nauka, Moscow, 50-73 (in Russian) (1980).
[5] J. Boman and J.-O. Stromberg, Novikovs inversion formula for the attenuated Radon transform — a new approach, J. Geom. Anal. 14 (2004), no. 2, 185-198.
[6] N. S. Dairbekov, *Integral geometry problem for nontrapping manifolds*, Inverse Problems, 22, 431-445, 2006
[7] D. Dos Santos Ferreira, C. E. Kenig, M. Salo, G. Uhlmann, Limiting Carleman weights and anisotropic inverse problems, Invent. Math. 178 (2009), 119-171.
[8] D. V. Finch, *The attenuated X-ray transform: recent developments*, in Inside Out: Inverse Problems and Applications (edited by G. Uhlmann), p. 4766, MSRI Publications 47, Cambridge University Press, Cambridge (2003).
[9] B. Frigyik, P. Stefanov, G. Uhlmann, *The X-Ray Transform for a Generic Family of Curves and Weights*, J. Geom. Anal. 18(1), 81-97, 2008
[10] P. Kuchment, *Generalized transforms of Radon type and their applications*, in The Radon Transform, Inverse Problems, and Tomography (edited by G. Olafsson and E. T. Quinto), p. 67-91, Proc. Symp. Appl. Math. 63, AMS, Providence (2006).
[11] S. G. Kazantsev and A. A. Bukhgeim, *Inversion of the scalar and vector attenuated X-ray transforms in a unit disc*, J. Inverse Ill-Posed Probl. 15 (2007), 735-765.
[12] C. E. Kenig, M. Salo, G. Uhlmann, *Inverse problems for the anisotropic Maxwell equations*, preprint, [arXiv:0905.3275](http://arxiv.org/abs/0905.3275)
[13] S. McDowall, *Optical tomography on simple Riemannian surfaces*, Comm. PDE 30 (2005), 1379-1400.
[14] S. McDowall, *Optical tomography for media with variable index of refraction*, Cubo 11 (2009), 71-97.
[15] R. Michel, *Sur la rigidité imposée par la longueur des géodésiques*, Invent. Math. 65 (1981) 71-83.
[16] R. G. Mukhometov, *The reconstruction problem of a two-dimensional Riemannian metric, and integral geometry* (Russian), Dokl. Akad. Nauk SSSR 232 (1977), no. 1, 32-35.
[17] R. G. Mukhometov, *On a problem of reconstructing Riemannian metrics*. Siberian Math. J., 22, no. 3, 420-433 (1982).
[18] F. Natterer, Inversion of the attenuated Radon transform, Inverse Problems 17 (2001), 113-119.
[19] R. G. Novikov, An inversion formula for the attenuated X-ray transformation, Ark. Mat. 40 (2002), 145-167.
[20] G. P. Paternain, M. Salo, G. Uhlmann, The attenuated ray transform for connections and Higgs fields, preprint, arXiv:1108.1118.
[21] G. P. Paternain, M. Salo, G. Uhlmann, Tensor tomography on surfaces, preprint, arXiv:1109.0505.
[22] L. Pestov, G. Uhlmann, On characterization of the range and inversion formulas for the geodesic X-ray transform, Int. Math. Res. Not. (2004), no. 80, 4331-4347.
[23] V. G. Romanov, 1984, Inverse problems of Mathematical Physics, Moscow: Nauka (in Russian)
[24] M. Salo, G. Uhlmann, The attenuated ray transform on simple surfaces, J. Diff. Geom. 88 (2011), no. 1, 161-187.
[25] V. A. Sharafutdinov, Integral Geometry of Tensor Fields, Utrecht: VSP, 1994
[26] V. A. Sharafutdinov, Ray transform on Riemannian Manifolds (Eight Lectures on Integral Geometry), 1999
[27] P. Stefanov, G. Uhlmann, Stability estimates for the X-ray transform of tensor fields and boundary rigidity, Duke Math. J. 123 (2004), 445-467.

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