Cooperative Algorithms for MIMO Interference Channels

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Abstract—Interference alignment is a transmission technique for exploiting all available degrees of freedom in the frequency- or time-selective interference channel with an arbitrary number of users. Most prior work on interference alignment, however, neglects interference from other nodes in the network not participating in the alignment operation. This paper proposes three generalizations of interference alignment for the multiple-antenna interference channel with multiple users that account for colored noise, which models uncoordinated interference. First, a minimum interference-plus-noise leakage (INL) algorithm is presented, and shown to be equivalent to previous subspace methods when noise is spatially white or negligible. This algorithm results in orthornormal precoders that are desirable for practical implementation with limited feedback. A joint minimum mean squared error design is then proposed that jointly optimizes the transmit precoders and receive spatial filters, whereas previous designs neglect the receive spatial filter. Finally, a maximum signal-to-interference-plus-noise ratio (SINR) algorithm is developed that is proven to converge, unlike previous maximum SINR algorithms. The sum throughput of these algorithms is simulated in the context of a network with uncoordinated co-channel interferers not participating in the alignment protocol. It is found that a network with co-channel interference can benefit from employing precoders designed to consider that interference, but in extreme cases, such as when only one receiver has a large amount of interference, ignoring the co-channel interference is advantageous.

I. INTRODUCTION

Interference channels model a network of simultaneously communicating node pairs. In these channels, each transmitter has data to send to only one receiver, which also observes interference from the other transmitters in the network. Analysis of interference channels has shown that interference is not a fundamental limitation. In particular, with any sized interference channel with any number of users, the capacity for any given user will scale at half the rate of its interference-free capacity in the high transmit power regime [1].

The key to achieving a linear capacity scaling is interference alignment (IA) [2], [3]. With IA, interfering transmitters precode their signals to align in the unwanted users’ receive space, allowing these receivers to completely cancel more interferers than they otherwise could. The signals can be aligned in any dimension, including time, frequency, or space. This can be viewed as a cooperative approach because the transmitters neglect the performance of their own link to allow other users to perfectly cancel interference. This is in contrast to a provably suboptimal “selfish” approach where a transmitter ignores the interference it causes and aims simply to maximize its own data rate [4]. Interference alignment has been shown to achieve the maximum capacity scaling, also known as degrees of freedom, of the K-user interference channel, but at finite transmit power it offers suboptimal achievable sum rate. Consequently, there is interest in finding precoders for the interference channel that relax the perfect alignment constraint with the objective of obtaining better nonasymptotic sum rate performance.

Alternative IA precoder designs have been proposed for the single-antenna interference channel with time or frequency selectivity [5], [6]. Closed-form IA precoders and achievable degrees of freedom for the multiple-input multiple-output (MIMO) interference channel with infinitely selective channels have also been found for some asymmetric antenna arrangements [7]. Interference channels where precoding can only be done over one transmission slot are said to have constant or static coefficients. In this case, the degrees of freedom with linear precoding are unknown but have been hypothesized to be less than that with infinite selectivity [8], [9], while nonlinear precoding might achieve KM/2 degrees of freedom with M antennas at each transmitter and receiver [10], [11].

A challenge in constant coefficient MIMO interference channels is that closed form solutions have been found in only a few special cases [1]. Algorithmic techniques, such as alternating minimization [12], have been proposed to find precoders and explore possible degrees of freedom for the general case [13]–[15]. Such algorithms are promising both for their ability to provide precoder solutions in a practical setting and their flexibility in application to arbitrary networks for which closed-form solutions are unknown. The subspace algorithms of [13]–[15], however, still use alignment as the main objective, which is asymptotically optimal for the interference channel but has suboptimal throughput at finite SNR and other regimes. They also neglect colored noise, possibly caused by co-channel interference from outside the coordinating nodes. A maximum-SINR algorithm was proposed in [12], but this algorithm does not optimize a global objective, assumes white Gaussian noise, and is not shown to converge.

In this paper we propose several alternative linear precoding designs for MIMO interference channels. While maximizing the sum rate is the primary objective, we do not directly maximize sum rate due to analytical intractability. Instead we approximate a sum rate maximization via algorithms with varying performance and complexity tradeoffs. First, we derive a generalization of subspace alignment that includes colored...
noise, which biases the preferred alignment subspaces. The resulting objective, which minimizes the interference plus noise leakage (INL), results in orthogonal precoders amenable to quantized CSI. This algorithm is shown to be a special type of minimum mean squared error (MMSE) design, and at high SNR or white noise at all receivers is shown to reduce to the IA subspace methods of [13], [15]. From this, interference alignment is shown to be an MMSE-type solution at infinite SNR, where interference-suppression filters are optimal. As with previous forms of interference alignment, the proposed minimum INL algorithm does not consider the signal power at any given user and is thus suboptimal with finite transmit power. Further, this algorithm and previous designs derive precoders but neglect receiver design, which could be optimized jointly with the precoders.

Inspired by the connection between mutual information and mean-square error [16], we derive an explicit joint MMSE precoder/receiver design for the interference channel. Although it does not directly maximize the sum rate, the joint-MMSE design results in a higher sum rate than subspace methods. It does not lead to orthonormal precoders, making quantized feedback design more difficult. The MMSE design is shown to be a generalization of previous approaches for point-to-point and multiuser settings [17]–[21]. Further, the design is more computationally complex and requires more iterations at high SNR than subspace designs. MMSE-based designs have also recently been developed independently in [22], [23].

To more directly optimize the sum rate we formulate a maximum SINR algorithm, which is proven to converge via alternating minimization of a global performance function. The maximum SINR algorithm derived in [13] is shown to be an approximation to the one derived in this paper. On average, the two are shown to have the same performance, but for any given channel realization may result in different sum rates. This design often has increased throughput relative to MMSE and subspace approaches, but finds nonorthogonal precoders and requires more channel state information if run in a distributed manner.

In summary, this paper proposes three algorithms that span the tradeoff between performance and complexity for the static MIMO interference channel. The minimum INL algorithm has the same complexity as previous work but has improved performance when colored noise exists at any receiver. The joint-MMSE design has further rate enhancements regardless of the noise covariance matrices but has a computationally more complex optimization and non-orthogonal precoders. The maximum SINR design has the best overall performance of all proposed strategies (in most cases, as shown in the simulations), but requires more channel state information than the previous designs and also results in non-orthogonal precoders which are difficult to quantize in a practical setting [24]. The proposed algorithms are then simulated alongside existing methods in regimes previously unconsidered in the literature.

For example, the algorithms are simulated in an environment with an uncoordinated interferer that is not participating in the alignment protocol. This colors the noise at each receiver, and if its power is scaled with the rest of the transmitters, resulting in reduced capacity scaling. Each of the algorithms can outperform the others in different regimes, and each of these regimes is simulated and enumerated.

The rest of this paper is organized as follows: Section II presents the system model under consideration; Section III presents the new MMSE and INL algorithms and derives a maximum SINR algorithm with proven convergence and analyzes each of the methods; Section IV presents simulations under uncoordinated interference and colored noise; and Section V concludes the paper and gives directions for future work.

Before proceeding, we introduce notation. The log refers to \( \log_2 \). Bold uppercase letters, such as \( \mathbf{A} \), denote matrices, bold lowercase letters, such as \( a \), denote column vectors, and normal letters \( a \) denote scalars. The letter \( E \) denotes expectation. \( \mathbb{C} \) is the complex field, \( \mathbb{R}\{a\} \) is the real component of complex scalar \( a \), \( \min\{a, b\} \) denotes the minimum of \( a \) and \( b \), \( \nu^R_{\min}(A) \) is the matrix whose columns are the eigenvectors corresponding to the \( R \) smallest eigenvalues of matrix \( A \). \( \text{tr} \{A\} \) is the trace of matrix \( A \), \( |a| \) is the magnitude of the complex number \( a \), \( \|a\| \) is the Euclidean norm of vector \( a \), and \( |A| \) is the determinant of square matrix \( A \). \( A^* \) is the Hermitian transpose of matrix \( A \) and \( A^{-1} \) is its inverse. The matrices \( I \) and \( 0 \) are the identity matrix and all zero matrix, respectively, of appropriate dimension. Finally, we use \( \{F_i\} \) when referring to the set of precoders and \( F_i \) when referring to the precoder at transmitter \( \ell \), and similarly for receive spatial filters \( \{G_k\} \).

II. SYSTEM MODEL

Consider the \( K \)-user MIMO interference channel illustrated in Figure 1 with \( K \) transmit-receive pairs. A wireless channel links each receiver to each transmitter, but a given transmitter only intends to have its signal decoded by a single receiver. The \( k \)th transmitter possesses \( M_k \) antennas with which to transmit \( S_k \leq M_k \) spatial streams, and the \( k \)th receiver (which is to decode the channel from the \( k \)th transmitter) possesses \( N_k \geq S_k \) antennas. In some analysis and simulations, all users will have the same antenna configurations so that \( M_k = M, N_k = N, S_k = S, \forall k \); we denote this symmetric case as an \( (M, N, K) \) interference channel with \( S \) streams per user.

This paper considers the narrowband MIMO interference channel where each link is static for the duration of a transmission, but may change between successive transmissions. This is the block fading model where all the links in the network are constant for the period of transmission, creating a tractable approximation to more realistic continuous fading models. Linear precoding is done independently over each channel realization, favoring simplicity over the possible degrees of freedom gained by jointly precoding over realizations. This is the same model as previous work on algorithms for the interference channel [13]–[15]. The transmission of all \( K \) users is synchronized such that each begins and ends each transmission simultaneously, and no frequency offsets exist in the network. We therefore take the standard approach [13], [15] and focus on the transmission of a single vector symbol \( s_k \) from transmitter \( k \in \{1, \ldots, K\} \), neglecting any time dependency.
Transmitter $k$ uses linear precoder $\mathbf{F}_k \in \mathbb{C}^{M_k \times S_k}$ to map $S_k$ symbols in $\mathbf{s}_k$ to its $M_k$ transmit antennas,

$$\mathbf{x}_k = \mathbf{F}_k \mathbf{s}_k,$$

where the transmitted symbols are i.i.d. such that $\mathbb{E}\mathbf{s}_k \mathbf{s}_k^* = \mathbf{I}$, the precoder is normalized such that $\|\mathbf{F}_k\|^2 = \rho_k$, and $\rho_k$ is the transmit power at transmitter $k$. Receiver $k$ observes the signal

$$\mathbf{y}_k = \sum_{\ell=1}^{K} \mathbf{H}_{k,\ell} \mathbf{F}_\ell \mathbf{s}_\ell + \mathbf{v}_k,$$

where $\mathbf{H}_{k,\ell}$ is the channel between transmitter $\ell$ and receiver $k$ with spatial covariance matrix $\mathbf{R}_{k} = \mathbb{E}\mathbf{v}_k \mathbf{v}_k^*$. For the analysis in this paper we assume that the channels $\{\mathbf{H}_{k,\ell}\}$ are each full rank and mutually independent, the transmitters send independent data $\mathbb{E}\mathbf{s}_k \mathbf{s}_k^* = \mathbf{0}$ for $k \neq \ell$ and all transmitted signals are statistically independent from the noise at any receiver $\mathbb{E}\mathbf{s}_k \mathbf{v}_k^* = \mathbf{0}$ for all $(k, \ell) \in \{1, \ldots, K\}^2$. No assumptions are made on the noise power or covariance at any receiver. Rewriting (2), receiver $k$ sees

$$\mathbf{y}_k = \mathbf{H}_{k,k} \mathbf{F}_k \mathbf{s}_k + \sum_{\ell=1}^{K} \mathbf{H}_{k,\ell} \mathbf{F}_\ell \mathbf{s}_\ell + \mathbf{v}_k.$$  

The vector $\mathbf{s}_k$ is the signal to be decoded by receiver $k$, and the summation term in (3) is called coordinated interference, since it is caused by transmitters that may coordinate to minimize its effect. Once the precoders are designed, the instantaneous sum rate of the system is

$$R_{\text{sum}} = \sum_{k=1}^{K} \log \left| \mathbf{I} + \hat{\mathbf{R}}_k^{-1} \mathbf{H}_{k,k} \mathbf{F}_k \mathbf{F}_k^* \mathbf{H}_{k,k}^* \right|,$$  

where

$$\hat{\mathbf{R}}_k = \mathbf{R}_k + \sum_{\ell=1}^{K} \mathbf{H}_{k,\ell} \mathbf{F}_\ell \mathbf{F}_\ell^* \mathbf{H}_{k,\ell}^*$$

is the interference plus noise covariance matrix at receiver $k$. The instantaneous sum rate is an important metric for multiuser systems because of its ability to capture the total network throughput in a single scalar. Notice that $R_{\text{sum}}$ assumes ideal non-linear decoding of the signal. Although the proposed algorithms of Section III will design linear processing matrices that can form a part of a linear receiver design, they serve mainly to simplify the optimization and design of the precoders. Design of high performance linear receivers is left to future work, except for the MMSE design presented in Section III-C. Thus, for fair comparison, the sum rate equations assume an ideal decoding for all precoder designs.

Previous authors have shown that $KM/2$ spatial degrees of freedom are achievable in an $(M, M, K)$ interference channel. Degrees of freedom $d$ is

$$d = \lim_{\text{SNR} \to \infty} \frac{C_{\text{sum}}(\text{SNR})}{\log \text{SNR}},$$

where $C_{\text{sum}}$ is the sum capacity of the network, rather than the sum rate for our linear precoding model presented in (4).

The key idea of interference alignment is to make $\sum_{\ell \neq k, \ell \neq \ell} F_{\ell}$ interferers appear as $N_k - S_k$ interferers at receiver $k$ for each $k$ by having them span a subspace of dimension $N_k - S_k$ of the $N_k$-dimensional receive space. Mathematically,

$$\sum_{\ell \neq k} \mathbf{H}_{k,\ell} \mathbf{F}_\ell = \sum_{i=1}^{N_k - S_k} a_i e_k^{(i)}, \forall k,$$

where $\{e_k^{(i)}\}$ are basis vectors for the subspace at receiver $k$ in which all interference must lie. Then receiver $k$ can then resolve its $S_k$ streams with a linear receiver interference-free [1]. For the three user interference channel it is possible to directly find closed-form solutions to $\{F_\ell\}$ for any $S \leq M/2$. Such solutions for obtaining $KM/2$ degrees of freedom, however, in the $(M, M, K)$ interference channel with $K > 3$ users do not appear to be possible. Closed-form solutions, even for a reduced multiplexing gain, are unknown [8] except in special cases [9]. A viable alternative for the general case are alternating minimizations. The next section reviews the existing designs and proposes new algorithms for finding high-rate solutions at finite-SNR in the MIMO interference channel.

III. ITERATIVE ALGORITHMS VIA ALTERNATING MINIMIZATION

This section presents iterative solutions for precoders in the MIMO interference channel using an alternating minimization to solve various optimization objectives. This section proposes three new metrics which aim to approximate a sum rate maximization with better finite-SNR rates than previous work.

The algorithms presented may be implemented in a distributed or centralized manner similar to [13]. These algorithms share a common structure. Each algorithm is designed
to optimize a global objective \( J \) that incorporates the performance of each data link in the network. The objective is a function of the precoders \( \{F_k\} \), the channels \( \{H_{k,\ell}\} \) between all nodes and a processing matrix at each receiver, the structure of which will vary across designs.\(^1\) The free variables are the \( K \) precoders and \( K \) receive processing matrices.

A closed-form solution for a global optimization of any of the objectives in this section is unknown. We therefore turn to an alternating minimization\(^2\) approach for the \( 2K \) variables \( \{F_k\} \). In general, an alternating minimization arbitrarily initializes \( 2K - 1 \) variables and, assuming these variables are fixed, solves for the remaining one. It stores this solution, and moves to another variable, finding a new solution for it assuming the rest of the variables are fixed. Each variable in turn is solved for during each iteration. Note that this procedure is convenient only if there is a simple or even closed-form solution for each of the variables assuming the rest are fixed. Finally, for each of the designs, with the exception of the proposed maximum SINR design, the precoders may be derived in parallel, since their solutions at any step of the algorithm do not depend on each other.

### A. Subspace Optimization

A direct algorithm for the interference channel inspired by interference alignment \( \{1\} \) is to precode the signal at transmitter \( \ell \) such that the coordinated interference caused by transmitter \( \ell \) at receiver \( k \neq \ell \) is nearly orthogonal to a subspace (with orthonormal basis \( \Phi_k \)) of its receive space \( \{13\}, \{15\} \). This subspace is then jointly designed along with the precoders to optimize an appropriate cost function. One way of performing this optimization is to minimize the total “leakage interference” \( \{13\} \) that remains at each receiver after attempting to cancel the coordinated interference by left-multiplication with \( \Phi_k^\ast \) for each \( k \). The global function to optimize is thus

\[
J_{IA} = \sum_{k=1}^K \left\| \Phi_k^\ast \sum_{\ell \neq k} H_{k,\ell} F_{\ell} s_{\ell} \right\|_F^2.
\]  

(8)

The expectation in \( J_{IA} \) and all subsequent analysis is over \( s_k \) (and \( v_k \) where applicable), \( k \in \{1, \ldots, K\} \). Evaluating the expectation and exploiting independence of the signals,

\[
J_{IA} = \sum_{k=1}^K \sum_{\ell \neq k} \left\| \Phi_k^\ast H_{k,\ell} F_{\ell} \right\|_F^2,
\]

(9)

which is termed “interference leakage” in \( \{13\} \). The precoders \( \{F_{\ell}\} \) are constrained to have mutually orthogonal columns with a per-stream power constraint so that \( F_{\ell}^\ast F_{\ell} = \frac{\nu_k}{S_k} I, \forall \ell \).

Although we could enforce a total power constraint on the precoders (and, coincidentally in this case, get the same solution), orthogonality is desired in MIMO precoding designs to aid with feedback of channel state \( \{25\} \). The receive subspace bases \( \{\Phi_k\} \) are orthonormal by definition so that \( \Phi_k^\ast \Phi_k = I \). The objective is thus

\[
\begin{align*}
\text{minimize} & \quad J_{IA}(\{F_{\ell}\}, \{\Phi\}) \\
\text{subject to} & \quad F_{\ell}^\ast F_{\ell} = \frac{\nu_k}{S_k} I, \ell \in \{1, \ldots, K\} \\
& \quad \Phi_k^\ast \Phi_k = I, k \in \{1, \ldots, K\}.
\end{align*}
\]

(10)

The optimization \( \{10\} \) is intuitively pleasing since, with perfect interference alignment, \( J_{IA} = 0 \), and without interference alignment, \( J_{IA} > 0 \). That is, interference alignment, if possible, achieves the global minimum for this function.

Deriving a closed-form solution to \( \{10\} \) for \( K > 3 \) users is difficult due to the inter-dependence of each precoder and receive interference-free subspace. A simple approach, which is guaranteed to converge, is to use an alternating minimization \( \{12\} \). The derivation of this solution is in \( \{13\} \) and our previous work in \( \{15\} \) and is not included here for efficiency. At each step, the solution for each \( F_{\ell} \) is

\[
F_{\ell} = \nu_{\min} \left( \sum_{k=1}^K H_{k,\ell}^\ast \Phi_k^\ast H_{k,\ell} \right),
\]

(11)

and, with all precoders given, the solution for each \( \Phi_k \) is

\[
\Phi_k = \nu_{\min} \left( \sum_{\ell \neq k} H_{k,\ell} F_{\ell}^\ast H_{k,\ell}^\ast \right).
\]

(12)

To run the algorithm, arbitrary receive subspaces for each receiver are used for initialization and an arbitrary orthonormal basis \( \Phi_k \) for each subspace is found. This subspace is ideally reserved for user \( k \)’s signal, thus coordinated interference at receiver \( k \) is ideally orthogonal to this subspace. Then, for each \( \ell \), the algorithm finds the precoder matrix \( F_{\ell} \) such that total coordinated interference caused at each node (other than at node \( \ell \)) has maximum squared Euclidean distance between it and the subspace spanned by the columns of each \( \Phi_k \) using \( \{11\} \). Given these new precoders, the algorithm can update the receive subspaces to be those that span the columns of the matrices with minimum sum squared Euclidean distance to the interference caused by the fixed precoders using \( \{12\} \{15\} \). This can be carried out until \( J_{IA}(t) < \epsilon \) if feasibility conditions are met, or \( J_{IA}(t - 1) - J_{IA}(t) < \epsilon \) otherwise, for an arbitrary convergence threshold \( \epsilon \).

Note that each receiver must still separate the desired spatial streams after the coordinated interference has been canceled with left multiplication of \( \Phi^\ast \). Standard linear designs, such as zero forcing or MMSE, can be employed for this purpose. Thus, the receiver can form a linear receive filter \( G_k \) by multiplying \( \Phi_k \) and the linear spatial filter \( W_k \), which neglects coordinated inter-user interference and equalizes only the desired signal, so that \( G_k = \Phi_k W_k \). Then the vector \( \hat{s}_k = G_k^\ast y_k \) is the interference-free estimate of the original transmitted vector \( s_k \).
B. Minimum Interference Plus Noise Leakage (INL)

The subspace approach of \cite{13, 15}, outlined in Section III-A, aims at aligning interference, which is capacity-optimal as the ratio of signal power to receiver noise power tends to infinity. If colored noise exists in any receiver, however, the IA subspaces might be chosen to align with the noise to cancel it as well as the interference. Such colored noise may be due to an interference source outside of the coordinated portion of the network modeled as an interference channel, as shown in Figure 2. This interference is referred to as *uncoordinated interference*. We therefore focus on algorithms that take noise into account in their optimization. Note that these approaches have a “global” objective function limited to the users cooperating in interference channel, such as inside a single cluster in Figure 2 and thus assume the uncoordinated interferers of other clusters have fixed covariance over the optimization and transmission time.

The objective of the subspace algorithm of Section III-A is to minimize the total post-processing coordinated interference power, also known as *interference leakage or interference power* in \cite{13, 14}. Thus, one intuitive solution is to minimize the total interference plus noise leakage, or INL. Mathematically, this is represented with the global performance function

$$J_{\text{INL}} = \sum_{k=1}^{K} \mathbb{E} \left\| \Phi_k^{*} \left( \sum_{\ell=1}^{K} H_{k,\ell} F_{\ell} s_{\ell} + v_k \right) \right\|_F^2,$$

where $v_k$ is the received noise vector observed at receiver $k$. Expanding the expectation and exploiting the independence of the signal and noise vectors, the objective becomes

$$J_{\text{INL}} = \sum_{k=1}^{K} \sum_{\ell \neq k} \left\| \Phi_k^{*} H_{k,\ell} F_{\ell} \right\|_F^2 + \text{tr} (\Phi_k^{*} R_k \Phi_k),$$

where $R_k = \mathbb{E} v_k v_k^{*}$ is the covariance matrix of the noise at receiver $k$. The objective is then

$$\begin{align*}
\text{minimize} & \quad J_{\text{INL}} (\{F_{\ell}\}, \{\Phi_k\}) \\
\text{subject to} & \quad F_{\ell}^{*} F_{\ell} = \frac{\rho_{\ell}}{N_t} I, \ell \in \{1, \ldots, K\} \\
& \quad \Phi_k^{*} \Phi_k = I, k \in \{1, \ldots, K\}.
\end{align*}$$

The constraints on the precoders and receive subspaces are identical to those in Section III-A. Further, since $J_{\text{INL}}$ is rotation-invariant to each of the variables, the solutions lie on the Grassmann manifold and techniques derived for it can be used. Since $\|A\|_F^2 = \text{tr} (A A^{*})$, $J_{\text{INL}}$ can be rewritten as

$$J_{\text{INL}} = \sum_{k=1}^{K} \sum_{\ell \neq k} \text{tr} (\Phi_k^{*} (H_{k,\ell} F_{\ell} F_{\ell}^{*} H_{k,\ell}^{*} + R_k) \Phi_k),$$

which for fixed $\{F_{\ell}\}$ is minimized by \cite{26}

$$\Phi_k^{\text{opt}} = \rho_{k} S_k \left( R_k + \sum_{\ell=1}^{K} H_{k,\ell} F_{\ell} F_{\ell}^{*} H_{k,\ell}^{*} \right).$$

For the precoders $\{F_{\ell}\}$, it is sufficient to note that, for fixed $\{\Phi_k\}$, minimizing $J_{\text{INL}}$ with respect to $\{F_{\ell}\}$ is equivalent to minimizing $J_{\text{IA}}$ with respect to $\{F_{\ell}\}$, as is seen by comparing \cite{14} and \cite{9}. Thus, the precoder solution is identical to \cite{11}. This solution effectively tries to align the coordinated interference with the dominant directions of the noise (or uncoordinated interference) if the noise has significant energy. In particular, if the noise is highly correlated spatially with a rank-one covariance matrix, then $R_k = \sigma_k^2 a_k a_k^{*}$ and this algorithm will attempt to align the interference to $a_k$ if possible. Such noise, which may correspond to a single-stream uncoordinated interferer not part of the cooperating network, might then be mitigated, although full removal is unlikely. We can also prove the following quantitative conclusions.

**Proposition 1:** If $R_k = \sigma_k^2 I \forall k \in \{1, \ldots, K\}$, then minimizing $J_{\text{INL}}$ is equivalent to minimizing $J_{\text{IA}}$.

**Proof:** By definition,

$$J_{\text{INL}} = \sum_{k=1}^{K} \left\| \Phi_k^{*} \sum_{\ell \neq k} H_{k,\ell} F_{\ell} \right\|_F^2 + \text{tr} (\Phi_k^{*} R_k \Phi_k)$$

$$= \sum_{k=1}^{K} \left\| \Phi_k^{*} \sum_{\ell \neq k} H_{k,\ell} F_{\ell} \right\|_F^2 + \sigma_k^2 \text{tr} (\Phi_k^{*} \Phi_k)$$

$$= J_{\text{IA}} + K \sum_{k=1}^{K} \sigma_k^2 S_k.$$  \hspace{1cm} (18)

Since the summation in (18) is independent of any of the free variables, minimizing $J_{\text{INL}}$ is equivalent to minimizing $J_{\text{IA}}$ when the noise is spatially white at each receiver.

**Proposition 2:** As $\rho_k \to \infty$ or $\|R_k\|_F \to 0$ for all $k$, $J_{\text{INL}}$ converges to $J_{\text{IA}}$. Thus, the subspace algorithm with noise consideration has the same SNR scaling as the pure interference alignment algorithm.

**Proof:** Define $\lambda_k$ as the largest eigenvalue of Hermitian matrix $R_k$. Then,

$$J_{\text{INL}} = J_{\text{IA}} + \sum_{k=1}^{K} \text{tr} (\Phi_k^{*} R_k \Phi_k)$$

$$\leq J_{\text{IA}} + \sum_{k=1}^{K} \lambda_k S_k.$$

(19)
For any arbitrary \( \{ R_k \} \), we define a sequence of functions
\[
J_{\text{INL}}^{(n)} = J_{\text{IA}} + \frac{1}{n} \sum_{k=1}^{K} \lambda_k S_k,
\]
(20)
corresponding to a sequence of noise covariance matrices
\( R_k^{(n)} = R_k / n \), so that \( \| R_k \|_F \to 0 \) as \( n \to \infty \). Then for any \( \epsilon > 0 \),
\[
\left| J_{\text{INL}}^{(n)} - J_{\text{IA}} \right| \leq \epsilon
\]
(21)
for all \( n > \sum_{k=1}^{K} \lambda_k S_k / \epsilon \).

From the proof, we also note that \( J_{\text{INL}} \geq J_{\text{IA}} \) and \( \min J_{\text{INL}} = 0 \) iff \( R_k \) is singular for all \( k \) and the columns of the interference-aligning receiver matrices \( \{ \Phi_k \} \) lie in the null spaces of their respective noise covariance matrices \( \{ R_k \} \). The metric \( J_{\text{INL}} \) is, in fact, likely to have a positive global minimum unless the total number of streams is reduced below the degrees of freedom of the network, even if the noise is correlated, because the noise subspaces at different receivers will not be perfectly alignable almost surely. Adapting the number of streams in the network to improve finite-SNR performance is an interesting problem that is beyond the scope of this paper.

In an idealized system with Gaussian signaling, colored noise may correspond to uncoordinated interference from outside the network of interest. For instance, consider the scenario of a cellular network across a metropolitan area. The strategy for this network may be to coordinate three adjacent sectors to use interference alignment (via subspace optimization) to transmit to one mobile per sector in the downlink. For a regular IA solution, the uncoordinated interference arriving at each receiver from sectors outside the coordination area would be ignored or modeled as spatially white. The min-\( \text{INL} \) algorithm would be able to exploit the knowledge of this uncoordinated interference and account for it as necessary.

The algorithms of Sections III-A and III-B aim to align the coordinated interference, which in turn maximizes capacity in a fully connected high-SNR network. We have seen in Proposition 1 that in finite-SNR environments, white Gaussian noise does not change the solutions of subspace methods. Although this section has presented an approach for networks with colored noise, algorithms with better throughput performance in finite-SNR regimes are desired, especially since most networks are not likely to be fully connected and thus may operate with low interference-to-noise ratio (INR), where subspace methods are not likely optimal even with colored noise considerations. To illustrate the problem of implementing subspace algorithms in a real network, consider the following argument. Suppose all interfering links \( \{ H_{k, \ell} \} \) have a path loss coefficient \( \beta \) whereas direct links have a path loss coefficient of 1. The subspace precoder design will then not depend at all on the value of \( \beta \) since the scalar multiplication does not change the direction of the signal. If the receivers use their interference suppression filters \( \{ U_k \} \) to cancel the interference, then the throughput of the system will be independent of \( \beta \). Thus, subspace algorithms treat weak and strong interferers equally, without exploiting the possible capacity gains available when interference is weakened. If no noise exists in the system, this is perfectly fine, since the receiver will still have an interference-free signal that it could decode perfectly. Realistically, however, a dynamic network would benefit from adapting its behavior to the relative interference energy. As shown numerically in Section IV, the algorithms proposed in Sections III-C and III-D are more suited to such adaptation than the subspace method of Section III-A.

C. Mean Squared Error Minimization

A common metric for accounting for noise in linear receivers in wireless communication systems is the mean squared error. For example, a zero-forcing linear MIMO receiver simply inverts the channel, and results in coloring and amplification of noise. An MMSE receiver balances the effects of noise with that of inverting the channel depending on the relative energy of each. This same concept can be applied to interference alignment, where the transmitter and receiver balance their wish to align the coordinated interference with the need for keeping the signal level well above the noise.

Joint MMSE designs for MIMO channels have been studied for years and have been applied to the point-to-point model [17]-[19] and the broadcast channel [20], [21]. The development for the interference channel is distinguished from previous work in that precoders and receivers need to be designed for multiple transmitters and receivers, rather than just the multiple transmitters or receivers as in the multi-user case, or a single transmitter and receiver in the point-to-point case.

As opposed to objectives discussed in Sections III-A and III-B, the MMSE directly designs the receive spatial filters \( \{ G_k \} \). That is, the output of the product \( G_k^* y_k \) is the estimate \( \hat{s}_k \) of \( s_k \), and the MMSE criterion minimizes the expected sum of the norms between each \( s_k \) and \( s_k \) for all \( k \), yielding the objective
\[
J_{\text{MSE}} = \sum_{k=1}^{K} E \| G_k^* y_k - s_k \|_2^2.
\]
(22)
Substituting (3) for \( y_k \) results in a global performance function of
\[
J_{\text{MSE}} = \sum_{k=1}^{K} E \| G_k^* \left( H_{k, k} F_k s_k + \sum_{\ell=1}^{K} H_{k, \ell} F_{\ell} s_{\ell} + v_k \right) - s_k \|_F^2,
\]
(23)
and an optimization objective of
\[
\text{minimize } J_{\text{MSE}} \left( \{ F_k \}, \{ G_k \} \right) \quad \text{subject to } \| F_{\ell} \|_F^2 \leq \rho_{\ell}, \ell \in \{1, \ldots, K\}.
\]
(24)
Expanding the expectation and simplifying, the optimization is equivalent to
\[
\text{minimize } \sum_{k=1}^{K} \text{tr} \left( G_k^* \left( R_k + H_{k, k} F_k F_k^* H_{k, k} \right) G_k \right) - 2 \text{Re} \left( \text{tr} \left( G_k^* H_{k, k} F_k \right) \right) \quad \text{subject to } \| F_{\ell} \|_F^2 \leq \rho_{\ell}, \ell \in \{1, \ldots, K\}.
\]
(25)
In general, MMSE solutions with an orthogonality constraint are more difficult to derive. Thus, we relax the orthogonality constraint to a total power inequality constraint.
\[ \| F_\ell \|^2_{F^*} \leq \rho_\ell, \forall \ell, \text{ and resort to a solution satisfying the Karush-Kuhn-Tucker (KKT) conditions as in previous joint MMSE solutions for different channel models [27]. As shown in Appendix A at each step the optimal precoders are} \]

\[ F_\ell = \left( \mu_\ell I + \sum_{k=1}^{K} H_{k,\ell}^* G_k G_k^* H_{k,\ell} \right)^{-1} H_{k,\ell}^* G_\ell, \tag{26} \]

where \( \mu_\ell \) is the Lagrangian multiplier chosen to meet the power constraint. This may require a simple optimization (detailed in Appendix A) and has no known closed form. The optimal receivers are

\[ G_k = \left( \sum_{\ell=1}^{K} H_{k,\ell} F_\ell F_\ell^* H_{k,\ell}^* + R_k \right)^{-1} H_{k,k} F_k, \tag{27} \]

where no further optimization needs to be performed because there is no constraint on the receiver. As the following proposition shows, this design can be viewed as a generalization of previous designs for the point-to-point case.

**Proposition 3:** With \( H_{k,\ell} = 0 \) for all \( \ell, k \) such that \( k \neq \ell \), (26) and (27) are equivalent to an MMSE design for a point-to-point scenario. Further, as \( \rho_k \to \infty \), the precoders and receivers diagonalize their respective information links.

**Proof:** This is proven by substituting 0 for each \( H_{k,\ell}, k \neq \ell \), and referring to previous point-to-point results [18], [19].

We also note that at high SNR and no coordinated inter-user interference, the MMSE algorithm will converge with one step, since any initialization precoder \( F_\ell \) is a fixed point of the algorithm and will minimize the MSE.

The MMSE design is unique among those discussed in this paper. As discussed before, the MMSE receiver gives a direct estimate of \( s_k \), while the others require a conventional MIMO receiver after \( G_k \) is applied. The MMSE receiver solution for fixed \( F_\ell, \ell \in \{1, \ldots, K\} \), is simply the conventional MMSE MIMO receiver with colored noise. Further, the solution at each step is not in closed form, as an optimization needs to be done to meet the power constraint for the precoders. Lastly, the precoder solution is not orthogonal (or, conversely, a solution with orthogonal constraints is difficult to find). This algorithm may be difficult to implement because of these properties.

Finally, the min-INL optimization is equivalent to an MMSE problem that compares the post-processing output to \( \Phi_k H_{k,k} F_k s_k \) instead of simply \( s_k \). That is,

\[ J_{\text{INL}} = \sum_{k=1}^{K} \mathbb{E} \| \Phi_k y_k - \Phi_k H_{k,k} F_k s_k \|^2_F. \tag{28} \]

Thus, the receiver \( \Phi_k \) is expected to remove only the effects of coordinated interference and white noise instead of having to correct for distortion created by the channel as well. This output must then be sent to a MIMO equalizer to remove inter-stream interference before symbol-by-symbol demodulation.

**D. Signal-to-Interference-Plus-Noise-Ratio Maximization**

The original subspace algorithm presented in Section III-A minimizes post-processing coordinated interference energy. The min-INL algorithm in Section III-B adds consideration for noise leakage as well, which can improve performance under colored noise. The MMSE solution in Section III-C indirectly accounts for signal power by attempting to force the received signal to look like the intended signal before precoding and transmission. It is clear, however, that a more desirable metric for maximizing the sum throughput would directly account for the post-processing signal-to-interference-plus-noise ratio (SINR).

This section presents an algorithm for maximizing total SINR in the network. The optimization we use is not the only one that could be considered "maximum SINR," however, since total SINR for multiple nodes is not strictly defined in the literature. One may construct any number of global SINR metrics. Previous authors have considered the inter-stream interference for each transmit/receive pair and solved for the precoding and receiver matrices one column at a time [13], resulting in non-orthogonal precoders and receive spatial filters, as in the MMSE case. That approach, however, is not an alternating optimization of a global objective function, and its convergence is unproven. We therefore reformulate the problem into a maximization of the sum signal power across the network divided by the sum interference power, incorporating the inter-stream interference for each user. The performance function becomes

\[ J_{\text{SINR}} = \sum_{k=1}^{K} \sum_{n=1}^{S_k} \mathbb{E} \left| g_k^{(n)} \right|^2 H_{k,k} f_k^{(n)} s_k^{(n)} \right|^2 \]

\[ \sum_{k=1}^{K} \sum_{n=1}^{S_k} \mathbb{E} \left| g_k^{(n)} \right|^2 \left( \left( \sum_{\ell \neq k} R_{\ell} f_k^{(n)} + r_{k,k} + v_k \right) \right)^2, \tag{29} \]

where \( g_k^{(n)} \) is the nth column of matrix \( G_k \),

\[ r_{k,\ell} = \sum_{m=1}^{S_\ell} H_{k,\ell} f_\ell^{(m)} s_\ell^{(m)} \]

is the pre-processing interference at receiver \( k \) from transmitter \( \ell \),

\[ r_{k,n} = \sum_{w=1}^{S_k} H_{k,k} k_w^{(w)} s_w^{(n)} \]

is the pre-processing self-interference from streams \( w \neq n \) at receiver \( k \), and \( s_k^{(n)} \) is the nth entry of vector \( s_k \). Notice that

\[ R_{\text{sum}} = \sum_{i=1}^{K} \sum_{n=1}^{S_i} \log \left( 1 + \frac{P_i^{(n)}}{I_i^{(n)} + N_i^{(n)}} \right) \]

\[ \geq \log \left( 1 + \sum_{i=1}^{K} \frac{S_i}{N_i^{(n)}} \right) \]

\[ \geq \log \left( 1 + \frac{S_i}{N_i^{(n)}} \right) \]

\[ = \log \left( 1 + J_{\text{SINR}} \right), \]

where \( P_i^{(n)} \) is the post-processing signal energy of the nth stream at the ith receiver, \( I_i^{(n)} \) is the post-processing interference energy, and \( N_i^{(n)} \) is the post-processing noise energy seen by the stream. The new objective \( J_{\text{SINR}} \) is the sum of signal power in the network divided by the sum coordinated...
inter-user interference power and inter-stream interference power after processing. By maximizing this ratio the algorithm can design the precoders to either decrease post-processing interference (the denominator) or increase signal power (the numerator) to improve total network performance.

The function $J_{\text{SINR}}$ is a generalized Rayleigh quotient and can be solved using generalized eigen decomposition. The optimization problem is

$$
\text{maximize} \quad J_{\text{SINR}} \left( \{ f^{(n)}_\ell \}, \{ g^{(n)}_k \} \right)
$$

subject to $$||f^{(n)}_\ell||^2 = \frac{\rho}{S}, \forall n, \ell. \quad (32)$$

For tractability we constrain each stream’s precoder to have a norm equality constraint so that $||\mathbf{f}^n_\ell||^2 = \rho$. For a larger objective function, and increased complexity, we could also introduce an inequality constraint on each column and vary the transmit power over the streams. As shown in Appendix [B], the solutions to the columns of the precoders are

$$
f^{(n)}_\ell = \sqrt{\frac{\rho}{S}} \nu_{\max} \left( S \mathbf{q}^{(n)}_\ell \mathbf{I} + \sum_{w=1}^{S} \sum_{w \neq m} \mathbf{H}^*_\ell,\ell \mathbf{g}^{(w)}_k \mathbf{f}^{(w)}_k \mathbf{H}^*_\ell,\ell 
+ \sum_{k=1}^{K} \sum_{m=1}^{S} \mathbf{H}^*_\ell,\ell \mathbf{g}^{(m)}_k \mathbf{f}^{(m)}_k \mathbf{H}^*_\ell,\ell \right)^{-1} 
\left( \mathbf{H}^*_\ell,\ell \mathbf{g}^{(n)}_k \mathbf{f}^{(n)}_k + S \mathbf{q}^{(n)}_\ell \mathbf{I} \right), \quad (33)$$

where $\mathbf{q}^{(n)}_\ell$ is the sum of the terms in the denominator of $\frac{5}{2}$ that do not directly involve $f^{(n)}_\ell$, and $\nu_{\max}$ is the sum of the terms in the numerator of $\frac{5}{2}$ that do not directly involve $f^{(n)}_\ell$. The solutions to the columns of the receivers are

$$
g^{(n)}_k = \nu_{\max} \left( \mathbf{q}^{(n)}_k \mathbf{I} + \sum_{w=1}^{S} \sum_{w \neq m} \mathbf{H}^*_k,k \mathbf{f}^{(w)}_k \mathbf{f}^{(w)}_k \mathbf{H}^*_k,k + \sum_{\ell=1}^{K} \sum_{m=1}^{S} \mathbf{H}^*_\ell,\ell \mathbf{f}^{(m)}_\ell \mathbf{f}^{(m)}_\ell \mathbf{H}^*_k,k \right)^{-1} 
\left( \mathbf{H}^*_k,k \mathbf{f}^{(n)}_k \mathbf{f}^{(n)}_k + \mathbf{q}^{(n)}_k \mathbf{I} \right), \quad (34)$$

where $\nu_{\max}$ is the generalized eigenvector corresponding to the largest generalized eigenvalue of the matrix pair $(\mathbf{A}, \mathbf{B})$, $\mathbf{q}^{(n)}_k$ and $\mathbf{q}^{(n)}_k$ are defined similarly as $(33)$ but with respect to $g^{(n)}_k$. With all other variables fixed, the solutions in $(33)$ and $(34)$ maximize the global SINR function $(29)$, whereas the solutions in $(13)$ give a suboptimal approximation to this solution. As shown in Section [IV], this does not imply that an iterative algorithm using the proposed solutions will converge to a larger objective than that of $(13)$. For any given channel realization and initialization, the two algorithms may give an identical result, or either may outperform the other. The simulation results in Section [V] suggest, however, that the two algorithms perform similarly on average. Since the proposed design requires more network knowledge than that of $(13)$, the latter is more attractive for implementation. If the extra network state knowledge is available, however, an intelligent design would be to run both algorithms and choose the design that works best for each channel realization, resulting in a sum throughput higher than either algorithm could produce individually.

Note that the IA algorithm will minimize the left-hand term of the denominator in $J_{\text{SINR}}$, and the min-IL algorithm will minimize the entire denominator (minus inter-stream interference). Certainly, with no noise (or, more rigorously, as total signal to noise ratio goes to infinity), the two solutions are equivalent since maximizing the SINR will reduce to maximizing the SIR, which, as discussed before, IA does. This fact was proven in [14].

E. Convergence and Initialization

This section analyzes some important details of the algorithms proposed in this paper. In particular, the focus is on variable initialization, algorithm convergence, method of execution, obtainment of channel state, and precoder constraints. We have found heuristically that arriving at a globally optimum point for the minimization algorithms (global optimality cannot be identified with the max SINR algorithm) is highly likely even when initializing the precoders to truncated identity matrices; the throughput, however, is the real objective we wish to optimize, and these algorithms only approximate that optimization. Thus, different initializations of an algorithm may result in drastically different throughputs, even if they result in the same final objective (or cost) function. For example, consider Figure 3. Each of the algorithms discussed in Section III was run on a fixed channel with 10 different random precoder initializations for the $(2,2,3)$ MIMO IC at 10 and 40 dB. Although the MMSE algorithm varies the most for this channel realization, this is not a general trend.
shown that random initializations give as good of rates in these algorithms as any “intelligent” initialization tried. If possible, multiple runs of the algorithm should be made with different initializations for the best performance in terms of throughput, as shown in Figure 5.

Each of the algorithms from Section III are guaranteed to converge because the objectives are bounded and at each step are moving monotonically in the direction of that bound. Convergence to a global optimum is not guaranteed except when the objective has certain convexity-like properties [12] that these algorithms are not proven to possess. Also, convergence of the cost function does not automatically imply convergence of the precoder designs, the analysis of which is beyond our scope.

IV. SIMULATIONS

This section presents simulations of the algorithms presented in Section III to substantiate our claims and show that each of the algorithms can outperform the others in different regimes since none explicitly maximizes throughput. All of the simulations evaluate the expected sum rate with i.i.d. zero-mean unit-variance complex Gaussian coefficients for each channel, with the precoders for each realization calculated with perfect CSI and as if the realization was flat in time and frequency. More realistic channel scenarios are considered in our related work [28]. Transmitter $k$ is assigned a deterministic transmit power $\rho_k$ and the link from transmitter $\ell$ to receiver $k$ has a deterministic path loss coefficient $\alpha_{k,\ell}$. Whereas in preceding analysis $\alpha_{k,\ell}$ was absorbed into $H_{k,\ell}$, in this section we pull it out for exposition. We also define $\gamma_{k,\ell} = \alpha_{k,\ell}\rho_{k,\ell}$ to be the expected SNR at receiver $k$ from transmitter $\ell$. Thus, the sum rate is

$$R_{\text{sum}} = \mathbb{E}_{\{H_{k,\ell}\}} \left\{ \sum_{k=1}^{K} \log \left| I + \hat{R}_k^{-1} H_{k,\ell} F_k F_k^H H_{\ell,k}^* \right| \right\},$$

where

$$\hat{R}_k = R_k + \sum_{\ell \neq k} \alpha_{k,\ell} H_{k,\ell} F_k F_k^H H_{\ell,k}^*$$

is the interference plus noise covariance. Precoders are initialized randomly with orthonormal columns, as discussed in Section III-E and each algorithm is presented with identical initializations. Five random initializations are used for each channel realization, as motivated in Figure 3 and the initialization that maximizes (35) is kept while the others are thrown away. In each plot presented in this section, $R_{\text{sum}}$ is computed via Monte Carlo simulations using 1000 independent channel realizations. Each iterative algorithm is run with 100 iterations each.

Each algorithm from Section III is compared with a random precoding scenario where each precoder $F_\ell$ is chosen as the left singular vectors of a random Gaussian matrix, to enforce an orthogonality constraint. That is, by Random Beamforming, we mean that

$$F_\ell = \sqrt{\frac{P_\ell}{S_\ell}} U_\ell^{(S_\ell)},$$

where $U_\ell^{(S_\ell)}$ are the first $S_\ell$ columns of the left singular matrix of a random matrix with i.i.d. zero-mean unit-variance complex Gaussian coefficients. A greedy approach is also included to show the benefit of cooperation in the MIMO interference channel [4], [29]. In this design, each precoder $F_k$, $k \neq \ell$, is held fixed when designing $F_\ell$. Then

$$U_\ell^{(S_\ell)} \mathbf{v}_\ell^{(S_\ell)} = \left( R_\ell + \sum_{k \neq \ell} \alpha_{\ell,k} H_{\ell,k} F_k F_k^H H_{\ell,k}^* \right)^{-1/2} \sqrt{\alpha_{\ell,k}} H_{\ell,\ell},$$

and

$$F_\ell = \sqrt{\frac{P_\ell}{S_\ell}} \mathbf{v}_\ell^{(S_\ell)}.$$ (39)

The greedy algorithm is not guaranteed to converge since it is not optimizing a global function, but it requires less channel estimation. Finally, when the $K = 3$ user interference channel is considered, the closed-form solution from [11] is also used for a baseline comparison.

We first introduce colored noise into the interference channel via an uncoordinated rank-one interferer in the network, as discussed in Section III-B. Defining $H_{\ell,k,E}$ as the MIMO channel from the uncoordinated rank-one interferer to receiver $k$ in the interference channel, then receiver $k$ observes

$$y_k = \sum_{\ell=1}^{K} \sqrt{\alpha_{k,\ell}} H_{k,\ell} F_E s_\ell + \sqrt{\alpha_{k,E}} H_{k,E} f_E s_E + v_k,$$ (40)

where since the uncoordinated interferer is rank-one, it is sending a single stream $s_E$ precoded with vector $f_E$. Each receiver sees spatially white additive noise on top of the signal and interference (coordinated and uncoordinated). Pure interference alignment will ignore the uncoordinated interference, implicitly assuming it is spatially white. The rest of the algorithms will take the uncoordinated interferer into account but will not be able to fully suppress it without reducing the number of streams in the network since the uncoordinated interferer, which is scaling its power with the transmitters inside the network, is reducing the degrees of freedom of the network, making it interference-limited.

Figure 4 illustrates results for the rank-one interferer scenario for each algorithm discussed in Section III with $K = 3$ users, $M = N = 2$ antennas at each node, and $S = 1$ stream being transmitted between each transmit/receive pair for $\rho_k = \rho_E = \rho$, $\forall k$ and $\alpha = 1$. That is, the transmit power is equal at all transmitters, including the uncoordinated interferer, and the path loss and fading statistics are identical on all links.

The MMSE algorithm has higher degrees of freedom in this case because of its power inequality constraint on the precoders. This allows two transmitters to effectively shut off while the third has a degree of freedom and can cancel the external interferer with its extra receive antenna. This shows the flexibility of the MMSE design. Other than MMSE, the max-SINR algorithm outperforms the others in the power ranges considered. Note that, although on average the max SINR algorithm and the approximate max SINR algorithm have nearly identical performance, for any given channel realization they may have very different sum rates. IA performs the worst of all four iterative algorithms since it is neglecting
the uncoordinated interference. At high $\rho$, considering the colored noise in the algorithm objective results in a roughly 20% increase in sum rate for this scenario. Note that the two best-performing algorithms, MMSE and max-SINR, do not have orthogonal precoders and thus may be more complex to implement in a real system with feedback requirements. With its orthogonal design and improved performance over IA, the min-INL algorithm is a good tradeoff between complexity and performance in this scenario.

Next, we keep the same scenario but with fixed uncoordinated interference power, so that the degrees of freedom are not reduced. Figure 5 gives the results of this experiment. It shows that the uncoordinated interference, which is fixed at $P_E = 0$ dB, has little effect on the system, even at low $\rho_k = \rho$. The algorithms, except random beamforming, all scale at the same rate, and thus all exploit the maximum degrees of freedom in the network. For a fixed number of iterations, however, the MMSE algorithm does not scale, as it appears to require more iterations to converge than the others at high $\rho$. In particular, as shown in Figure 5 when the MMSE design is run with 500 iterations, its performance approaches that of the rest of the designs, while the other algorithms benefit very little from the increase in iterations. This is consistently seen in the rest of the simulations in this section. Analysis of this longer convergence is left to future work. Finally, we note that iterative IA outperforms the closed-form solution because multiple IA solutions exist, and iterative IA is better able to find the best one because of the multiple random initializations. If the closed-form algorithm is modified to explore multiple possible solutions, it would perform equally well in this case.

Now we remove the uncoordinated interferer from all but one receiver in the network, and allow that uncoordinated interference power to scale with internal network transmit power, so that $\rho_k = P_E = \rho$, but $\rho_k,E = 0$ for $k > 1$ and $\alpha_{1,E} = 1$, for each algorithm discussed in Section III. In this case, a rank-one uncoded interferer is sensed at only receiver 1 in the (2,2,3) MIMO interference channel with $S = 1$. The interferer’s transmit power is scaled with the transmitters in the network so the degrees of freedom are reduced. The network is not interference-limited, however, since only one receiver sees the interference.

Fig. 4. Sum rate vs. $\rho_k = \rho$ for each algorithm discussed in Section III for the case where a rank-one uncoordinated interferer is introduced into the (2,2,3) network with $S = 1$ stream per user. The interferer’s transmit power is scaled with the transmitters in the network so the degrees of freedom are reduced, and the network is interference-limited at high values of $\rho$. The MMSE algorithm is an exception because it has a power inequality constraint on its precoders and can thus allow two transmitters to turn off, giving the remaining transmitter one degree of freedom, so the sum capacity scales linearly with $\rho$.

Fig. 5. Sum rate vs. $\rho_k = \rho_E = \rho$ for each algorithm discussed in Section III for the case where a rank-one uncoordinated interferer with fixed transmit power of $P_E = 0$ dB and $P_E = 20$ dB is introduced into the (2,2,3) network with $S = 1$ stream per user. The degrees of freedom in this network are the same as if the interferer did not exist, and each algorithm, with the exception of random beamforming, performs very similarly and exploits all the degrees of freedom in the network.

Fig. 6. Sum rate vs. $\rho_k = \rho$, with $\alpha_{1,E} = 0$ for $k > 1$ and $\alpha_{1,E} = 1$, for each algorithm discussed in Section III. In this case, a rank-one uncoordinated interferer is sensed at only receiver 1 in the (2,2,3) MIMO interference channel with $S = 1$. The interferer’s transmit power is scaled with the transmitters in the network so the degrees of freedom are reduced. The network is not interference-limited, however, since only one receiver sees the interference.
is equally concerned about the average performance but only in terms of coordinated interference, whereas MMSE has flexibility to overcome the interference by reducing transmit power of receiver 1. To maximize sum rate in this case, it appears one should either ignore the external interference or include a power inequality constraint in the precoder design.

We now turn to the case of no uncoordinated interference, considering only the conventional interference channel in isolation. In the first experiment, the transmit power is kept fixed but the path loss coefficient $\alpha_{k,\ell}$ is varied on the interfering links ($k \neq \ell$) only. Figure 7 illustrates the results. The IA algorithm has constant throughput regardless of the interference path loss coefficient, but the other iterative algorithms are able to exploit the decrease in interference, converging to IA when the interference power is high.

V. CONCLUSIONS AND FUTURE WORK

This paper has discussed the application and performance of iterative algorithms in the MIMO $K$-user constant-coefficient interference channel under various operating regimes. The convergence and optimality of the algorithms has been discussed, and similarities between all of them have been derived. If an iterative solution for the interference channel is ever practical in a real system, it is unlikely that a direct interference alignment approach is desirable because of its suboptimality in environments where one or more links have little energy relative to the others. Instead, the max SINR or MMSE metrics are desirable in most environments because they flexibly adapt the solution between interference alignment (high interference power) and SVD precoding (no interference, fixed number of streams), and the MMSE solution in particular has a transmit power inequality constraint. These algorithms, however, have relatively high implementation complexity because of their nonorthogonality and lack of closed-form solutions at each step in general cases. In particular, the MMSE algorithm requires some optimization for meeting the power constraint, and the max-SINR algorithm requires more channel state knowledge at each iteration than the others. The min-INL algorithm is a good tradeoff between the three algorithms, since it has improved performance over IA in scenarios where there is uncoordinated interference or colored noise, but still has relatively low implementation complexity because of its simpler solutions and orthogonal precoders. Future work will focus on analyzing and reducing the overhead associated with solutions such as the ones presented in this paper. Although some studies have been carried out on the application of interference alignment to a cellular network [30]–[32], overhead and feedback analyses need to be performed to find out if the achievable gains are worth the effort.

APPENDIX A

DERIVATION OF MEAN SQUARED ERROR MINIMIZATION

Proof: For completeness, we restate the optimization from (24).

\[
\begin{align*}
\text{minimize} & \quad J_{\text{MSE}} (\{F_\ell\}, \{G_k\}) \\
\text{subject to} & \quad \|F_\ell\|_F^2 \leq \rho_\ell, \ell \in \{1, \ldots, K\}.
\end{align*}
\]

where $J_{\text{MSE}}$ is defined in (25). We use the Karush-Kuhn-Tucker conditions to solve the optimization at each step with all but one variable fixed. The Lagrangian of (24) is

\[
\mathcal{L} = \sum_{k=1}^{K} \text{tr} \left( G_k^* \left( \sum_{\ell=1}^{K} H_{k,\ell} F_\ell F_\ell^* H_{k,\ell}^* + R_k \right) G_k \right) - 2 \Re \left\{ \text{tr} \left( G_k^* H_{k,k} F_\ell \right) \right\} + \sum_{\ell=1}^{K} \mu_\ell \left( \text{tr} (F_\ell^* F_\ell) - \rho_\ell \right),
\]

where $\mu_\ell$ is the Lagrangian multiplier for the power constraint for precoder $\ell$. The KKT conditions are

\[
\begin{align*}
\nabla \mathcal{L} &= 0 \quad (43) \\
\mu_\ell (\text{tr} (F_\ell^* F_\ell) - 1) &= 0, \forall \ell \quad (44) \\
\text{tr} (F_\ell^* F_\ell) &\leq \rho_\ell, \forall \ell \quad (45) \\
\mu_\ell &\geq 0, \forall \ell. \quad (46)
\end{align*}
\]

For fixed $\{F_\ell\}$ and $\{\mu_\ell\}$, $\{G_k\}$ can be found by solving $\nabla_{G_k} \mathcal{L} = \nabla_{G_k} J_{\text{MSE}} = 0$ for $k \in \{1, \ldots, K\}$ and the KKT conditions will be automatically met since there are no constraints on $\{G_k\}$. This yields

\[
G_k = \left( \sum_{\ell=1}^{K} H_{k,\ell} F_\ell F_\ell^* H_{k,\ell}^* + R_k \right)^{-1} H_{k,k} F_\ell \quad (47)
\]

In solving for $\{F_\ell\}$, we must ensure all of (43)–(46) are satisfied. To satisfy $\nabla_{F_\ell} \mathcal{L} = 0$, we must have

\[
F_\ell = \left( \mu_\ell I + \sum_{k=1}^{K} H_{k,\ell}^* G_k G_k^* H_{k,\ell} \right)^{-1} H_{k,\ell}^* G_k \quad (48)
\]

If $\mu_\ell = 0$ satisfies (45), then all the KKT conditions are satisfied and the optimal $F_\ell$ has been found for this step of
the alternating minimization. Otherwise, we must solve for \( \mu_\ell > 0 \) such that \( \| F_\ell \|_F = \rho_\ell \) to satisfy the KKT conditions. Although there is no known closed-form solution for \( \mu_\ell \) in this case [27], \( |F_\ell|_F \) is a monotonically decreasing function of \( \mu_\ell \) for \( \mu_\ell > 0 \), so simple one-dimensional searches such as the bisection method can be done to solve for \( \{ \mu_\ell \} \). ■

APPENDIX B
DERIVATION OF SINR MAXIMIZATION

Proof: For completeness, we restate the optimization from (32),

\[
\begin{align*}
\max & \quad J_{\text{SINR}} \left( \{ f^{(n)}_\ell \}, \{ g^{(n)}_k \} \right) \\
\text{subject to} & \quad \| F^{(n)}_\ell \|_F^2 = \frac{\rho_\ell}{S_\ell}, \forall \ell, \ell.
\end{align*}
\]

(49)

where \( J_{\text{SINR}} \) is defined in (29). The optimization is performed on the columns of each precoder \( F_\ell \) and spatial equalizer \( G_k \).

In solving for the \( n \)th column of precoder \( F_\ell \), we hold fixed every other precoder \( F_k, k \neq \ell \), and other columns \( f^{(w)}_\ell, w \neq n \) of precoder \( F_\ell \), as well as every receive combining matrix \( G_k, \forall k \). The objective of (29) can be rewritten as

\[
J_{\text{SINR}} = \frac{\sum_{\ell=1}^K \sum_{n=1}^{S_\ell} f^{(n)}_\ell (C^{(n)}_\ell + S_\ell y^{(n)}_\ell I) f^{(n)}_\ell}{\sum_{\ell=1}^K \sum_{m=1}^{S_\ell} f^{(m)}_\ell (S_\ell q^{(m)}_\ell I + A^{(m)}_\ell + B^{(m)}_\ell) f^{(m)}_\ell},
\]

(50)

where

\[
\begin{align*}
A^{(n)}_\ell &= \sum_{w=1}^{S_\ell} H^*_\ell g^{(w)}_\ell g^{(w)}_\ell H_{\ell,\ell}, \\
B^{(n)}_\ell &= \sum_{k=1}^K \sum_{m=1}^{S_k} H^*_k g^{(m)}_k g^{(m)}_k H_{k,\ell}, \\
C^{(n)}_\ell &= H^*_\ell g^{(n)}_\ell g^{(n)}_\ell H_{\ell,\ell},
\end{align*}
\]

and \( q^{(n)}_\ell \) is the remaining summation terms in the denominator of (29) that are independent of \( f^{(n)}_\ell \), contracted here for brevity. Similarly, \( r^{(n)}_\ell \) is the remaining summation terms in the numerator of (29) that are independent of \( f^{(n)}_\ell \). The function in (50) is the generalized Rayleigh quotient which is well known to be solved by the generalized eigen-vector of the numerator and denominator matrices,

\[
f^{(n)}_\ell = \sqrt{\frac{\rho_\ell}{S_\ell}} y^{(n)}_\ell \left( S_\ell q^{(n)}_\ell I + A^{(n)}_\ell + B^{(n)}_\ell \right)^{-1} \left( C^{(n)}_\ell + S_\ell r^{(n)}_\ell I \right).
\]

(54)

The derivation of the receive combiner columns follows the same structure as the precoders, but with a unit-norm constraint on the columns for simplicity (this removes the \( S_\ell \) multipliers in front of \( q^{(n)}_\ell \) and \( r^{(n)}_\ell \)). Here we define \( f^{(n)}_k \) to be the terms in the numerator of (29) independent of \( g^{(n)}_k \) and similarly for \( q^{(n)}_k \) in the denominator. The solution for the

\[
g^{(n)}_k = \nu_{\max} \left( \left( q^{(n)}_k I + \sum_{w=1}^{S_k} H_{k,\ell} f^{(w)}_\ell f^{(w)}_k H^*_k,\ell \right)^{\nu_{\max}} \sum_{\ell=1}^K \sum_{m=1}^{S_\ell} H_{k,\ell} f^{(n)}_\ell f^{(n)}_k H^*_k,\ell \right)^{-1} \left( H_{k,\ell} f^{(n)}_\ell f^{(n)}_k H^*_k,\ell + r^{(n)}_k I \right).
\]

(55)

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