Automatic Generation of Efficient Sparse Tensor Format Conversion Routines

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Abstract

This paper shows how to generate code that efficiently converts sparse tensors between disparate storage formats (data layouts) like CSR, DIA, ELL, and many others. We decompose sparse tensor conversion into three logical phases: coordinate remapping, analysis, and assembly. We then develop a language that precisely describes how different formats group together and order a tensor’s nonzeros in memory. This enables a compiler to emit code that performs complex reorderings (remappings) of nonzeros when converting between formats. We additionally develop a query language that can extract complex statistics about sparse tensors, and we show how to emit efficient analysis code that computes such queries. Finally, we define an abstract interface that captures how data structures for storing a tensor can be efficiently assembled given specific statistics about the tensor. Disparate formats can implement this common interface, thus letting a compiler emit optimized sparse tensor conversion code for arbitrary combinations of a wide range of formats without hard-coding for any specific one.

Our evaluation shows that our technique generates sparse tensor conversion routines with performance between 0.99 and 2.2× that of hand-optimized implementations in two widely used sparse linear algebra libraries, SPARSKIT and Intel MKL. By emitting code that avoids materializing temporaries, our technique also outperforms both libraries by between 1.4 and 3.4× for CSC/COO to DIA/ELL conversion.

Keywords sparse tensor conversion, sparse tensor assembly, sparse tensor algebra, sparse tensor formats

1 Introduction

Sparse multidimensional arrays (tensors), which contain mostly zeros, are a convenient representation for data in many domains, including data analytics [2, 6] and machine learning [34, 39] among others. Countless formats for storing sparse tensors in memory have been proposed, many of which are used in real-world applications since no format is universally superior in every circumstance.

It is desirable for a general-purpose sparse tensor algebra library or framework to support efficiently converting tensors between as many combinations of formats as possible. This is because applications typically need to perform different operations on the same tensor, and each operation may require the tensor to be stored in a distinct format for optimal performance. Importing data and transforming it to a sparse tensor, for instance, can be done efficiently if the tensor is constructed in the COO format, since COO supports efficient appends of new nonzeros. If all of the tensor’s nonzeros are clustered along a few dense diagonals though, then storing it in the DIA format reduces its memory footprint, since DIA only has to store each nonempty diagonal’s offset from the main diagonal. Thus, it would be more efficient to compute with the tensor in DIA as that reduces memory traffic. This requires first converting the tensor from COO to DIA.

Existing sparse linear and tensor algebra libraries, such as SPARSKIT [41] and Intel MKL [20], typically provide routines for converting sparse matrices and tensors between specific formats. Many libraries support a large number (N) of formats though, so it is impractical to hand-implement efficient conversion routines for all N² combinations of source and target formats. Instead, these libraries mostly only support direct conversions to and from some arbitrary canonical format (e.g., CSR for SPARSKIT). Thus, to convert a tensor from COO to DIA using SPARSKIT (or Intel MKL), an application must first convert the tensor to CSR and then once more to DIA. This doubles the number of conversions, which is suboptimal given that even efficiently converting a tensor once incurs significant overhead that must be amortized [14, 48]. Worse, this approach is not even an option if an application uses formats that are not supported at all by libraries. The developer must then hand-implement efficient custom conversion routines for each format, which are typically complicated and tedious to write and optimize. This motivates a technique that can instead automatically generate such format conversion routines.

1.1 Challenges

For many reasons, existing sparse tensor and linear algebra compilers such as taco [14, 24, 25] cannot generate efficient sparse tensor conversion routines for many formats.

Disparate orderings of nonzeros Different formats group together and order a tensor’s nonzeros in memory in distinct, potentially complex ways. CSR, for instances, stores nonzeros row by row, while DIA stores them diagonal by diagonal (Figure 1, center) and many other formats store nonzeros...
in even more complex orderings like Morton order [12, 29]. This necessitates a language that can precisely describe all the ways in which different formats order nonzeros, so a compiler knows exactly how generated code needs to reorder a tensor’s nonzeros when converting between formats.

**Reordering nonzeros** Efficiently reordering a sparse tensor’s nonzeros is challenging when the source and target formats use compressed data structures, as these data structures restrict the order in which nonzeros can be efficiently iterated or inserted. CSR, for instance, uses an unpadded segmented vector that stores the segments (representing rows) and the nonzeros within each segment contiguously. This layout makes it inefficient to insert nonzeros in random order since, without knowing how many nonzeros will be in each row, each insertion requires shifting all stored nonzeros in subsequent rows. COO, however, can store nonzeros in any order but can only be efficiently iterated in that order, thus complicating conversions to CSR. One way to resolve such ordering inconsistencies is to simply explicitly sort the nonzeros into the target format’s ordering. However, this incurs significant overhead. So instead, many efficient conversion algorithms first compute statistics about the input tensor (e.g., number of nonzeros per row, bandwidth, etc.) and use those statistics to coordinate the reordering of nonzeros, eliminating the need to sort while minimizing data shuffling. Different target formats require distinct sets of statistics about the input tensor for efficient conversion. And depending on input tensor’s format, efficiently computing those statistics can require very different algorithms. To support conversion between arbitrary formats, a compiler needs to be able to generate efficient code to compute varied statistics about input tensors stored in arbitrary formats.

**Disparate data structures** Different tensor formats use disparate data structures to store nonzeros. Some explicitly store the coordinates of each nonzero, using data structures ranging from segmented vectors to linked lists to hash maps. Others use varying sets of parameters to implicitly encode sequences of coordinates of nonzeros. To support conversion between a wide range of formats, a compiler must be able to reason about how to insert nonzeros into all those different data structures in possibly arbitrary order, even if they do not necessarily support efficient random access.

2 Background

There exist a wide variety of formats for storing sparse tensors in memory. Figure 2 shows four examples of commonly used sparse tensor formats; for an overview of additional formats that have been proposed, we refer readers to Chou et al. [14]. The COO format [7] represents a sparse tensor as a list of its nonzeros, storing the complete coordinates and value of each nonzero. COO supports efficiently appending new nonzeros, though it also wastes significant amounts of memory storing redundant row coordinates. The CSR format [45] compresses out the redundant row coordinates by grouping all nonzeros in the same row together and using a pos array to map nonzeros to each row. However, inserting a nonzero at some arbitrary coordinates into CSR is expensive as all nonzeros in subsequent rows have to be shifted in memory. The DIA [42] format stores nonzeros along the same diagonal, which is particularly efficient for access patterns that follow diagonal traversals. To support conversion algorithms into three logical phases (Section 3). Then, to facilitate generating code for each phase, we develop

**coordinate remapping notation** that precisely describes how different tensor formats group together and order nonzeros in memory (Section 4),

**attribute query language** that specifies what statistics about a tensor are needed in order to efficiently pack the tensor into different formats (Section 5), and a

**tensor assembly abstract interface** that consists of functions to capture how results of attribute queries are used to efficiently assemble many kinds of sparse tensor data structures (Section 6).

As we will show, the conciseness of these abstractions makes it easy to provide specifications that describe how to efficiently construct sparse tensors in many formats. Our technique uses these specifications to emit code that efficiently packs a tensor’s nonzeros in arbitrary order into the desired target format. Combined with recent work that shows how to emit code for iterating tensors in the desired source format, our technique can then generate efficient sparse tensor conversion routines for arbitrary combinations of formats.

We have implemented a prototype of our technique in taco. Our evaluation shows that, for many combinations of source and target formats, our technique generates conversion routines with performance between 0.99 and 2.2× that of hand-optimized implementations in SPARSKIT and Intel MKL, two widely used sparse linear algebra library. Our technique also emits direct conversion routines that are not implemented in either library, which lets us further optimize tensor conversions from CSC and COO to DIA and ELL by between 1.4 and 3.4× (Section 7).
diagonal together in memory, while the ELL [23] format stores the $k$-th nonzero of each row together. Such orderings of nonzeros expose vectorization opportunities for sparse matrix-vector multiplication [16] and can also further reduce memory footprint. However, DIA is only suitable for diagonal and banded matrices, while ELL is only suitable if numbers of nonzeros per row is bounded. As these examples show, there is no universally superior format for storing sparse matrices. The same is true for higher-order sparse tensors, for which even more formats have been proposed [8, 29, 30, 43] that use disparate data structures and ordering schemes to store nonzeros, each with distinct trade-offs.

2.1 Coordinate Hierarchy Abstraction

Chou et al. [14] describe how tensors stored in disparate formats can be represented as coordinate hierarchies that have varying structures but expose the same abstract interface. Figure 3 show examples of coordinate hierarchies that represent a matrix in different formats. Each stored component is represented by a path from the root to a leaf, with labels along the path denoting the component’s coordinates.

We can then decompose sparse tensor formats into level formats that each stores a coordinate hierarchy level, which represents a tensor dimension. CSR (Figure 2b), for instance, can be decomposed into two level formats, dense and compressed, that store the row and column levels respectively. The dense level format implicitly encodes all rows using just a parameter $N$ to store the dimension’s size. By contrast, the compressed level format uses two arrays, pos and crd, to store column coordinates of nonzeros in each row. All level formats, however, expose the same static interface consisting of level functions, which describe how to access a format’s data structures, and properties, which describe characteristics of the data as stored (e.g., if nonzeros are stored in order).

Structured sparse tensor formats like DIA and ELL, which do not group nonzeros lexicographically by their coordinates, can also be decomposed into level formats by casting them as formats for tensors with additional dimensions. For example, a DIA matrix can be cast as a 3rd-order tensor where each slice contains only nonzeros that lie on the same diagonal; Figure 5 demonstrates one way in which this can be done. We can then decompose DIA into three level formats: one that stores the set of nonempty diagonals in a perm array of size $K$, another that encodes the set of rows in each diagonal, and a third that encodes the column coordinates of each nonzero. Such a decomposition lets a sparse tensor algebra compiler better reason about how tensors stored in DIA and similar structured formats can be efficiently iterated, which is crucial for generating fast tensor algebra code.

The coordinate hierarchy abstraction lets a compiler generate efficient code to iterate over sparse tensors in disparate formats by simply emitting code to traverse coordinate hierarchies. This entails recursively generating a set of nested loops that each iterates over a coordinate hierarchy level. The compiler generates each loop by emitting calls to level functions that describe how to efficiently access a level. Then, to obtain code that iterates over a tensor in any desired format, the level function calls are simply replaced with the desired format’s implementations of those level functions. This approach lets a compiler generate efficient code for disparate formats without hard-coding for any specific one.

3 Problem Formulation

Figure 4 shows three examples of sparse tensor conversion routines that efficiently convert tensors between different storage formats. As these examples illustrate, different combinations of source and target formats require vastly dissimilar code. All three routines perform distinct computations to determine how much memory to allocate for the output and where exactly to insert each of the input tensor’s nonzeros. In fact, they do not even iterate over the input nonzeros the same number of times; Figure 4b accesses each nonzero once, while the other two examples iterate over the input twice.

It turns out, however, that efficient algorithms for converting sparse tensors between a wide range of disparate formats can all be decomposed into three logical phases: coordinate remapping, analysis, and assembly. Figure 4 identifies these different phases using distinct background colors.
The coordinate remapping phase iterates over the input tensor and, for each nonzero, computes additional coordinates as functions of the nonzero’s original coordinates. What and how the additional coordinates are computed depend on the target format. For instance, code in Figure 4a, which converts a matrix to DIA, computes a new coordinate as the difference between the column and row coordinates (lines 6 and 24). Coordinate remapping conceptually transforms the input tensor to a higher-order tensor. The lexicographic coordinate ordering of nonzeros in the transformed (remapped) tensor reflects how nonzeros are laid out in the target format, as illustrated in Figure 5.

The analysis phase computes statistics about the input tensor that are later used to ensure sufficient memory is allocated for storing nonzeros in the target format. The exact statistics that are computed also depend on the target format. Figure 4a, for instance, computes the exact set of all nonempty diagonals in the input matrix (lines 1–8), with distinct diagonals identified by offsets (k) computed in the coordinate remapping phase. Figure 4b, by contrast, computes the maximum number of nonzeros in any row of the input matrix (lines 1–5), while Figure 4c computes the number of nonzeros in each row of the input matrix (lines 1–6).

Finally, the assembly phase iterates over the input tensor and inserts each nonzero into the output data structures. Again, where each nonzero is inserted (pB2) depends on the target format. Figure 4a computes pB2 as a function of each nonzero’s row coordinate and its offset k (as computed in the coordinate remapping phase), in such a way that nonzeros with the same offset are grouped together in the output (lines 25–26). By contrast, Figure 4c simply appends each nonzero to its row’s corresponding segment in the crd array (line 19).

In the next three sections, we discuss how our technique generates efficient code to perform coordinate remapping (Section 4), analysis (Section 5), and assembly (Section 6). While each phase is logically distinct, our technique is able to emit code that fuses different phases if it is beneficial to do so. This lets us generate code like Figure 4a, which fuses coordinate remapping with both the analysis and assembly phases to avoid materializing the offsets of the nonzeros.

**Figure 4.** Code (in C++) to convert sparse matrices between different source and target formats. The background colors identify distinct logical phases of format conversion (green for coordinate remapping, yellow for analysis, and blue for assembly).

4 Coordinate Remapping

As explained in Section 3, the first phase of efficient sparse tensor conversion algorithms logically transforms (remaps) the input tensor to a higher-order one, such that the lexicographic coordinate ordering of nonzeros in the remapped tensor reflects how nonzeros are stored in the target format. We propose a new language called coordinate remapping notation, which precisely describes how a tensor can be remapped so as to capture the various ways that different tensor formats group and order nonzeros in memory. We further show how our technique generates code that applies a coordinate remapping to remap the input tensor as part of format conversion. This eliminates the need for end users to hand-implement additional code that separately performs such a remapping, which the technique of Chou et al. [14] requires for conversions to structured tensor formats.
remap_stm := src_indices `->` dst_indices
src_indices := `(ivar `{',ivar `}`)`
dst_indices := `(ivar `(ivar_expr `in`) `ivar_expr`)`
ivar_expr := ivar_xor `(ivar_xor `(ivar_and `ivar_and`)`)` ivar_and := ivar_shift `(ivar_shift `(ivar_add `ivar_add`)`)` ivar_add := ivar_mul `(ivar_mul `ivar_mul`)` ivar_mul := ivar_factor `(ivar_factor `ivar_factor`)` ivar_factor := `(ivar_expr `ivar_counter `ivar `const`)` ivar_counter := `#` {ivar} 

Figure 5. Applying the coordinate remapping $(i,j) \rightarrow (j-i,i,j)$ to a matrix transforms (remaps) it to a 3rd-order tensor where each slice contains all nonzeros that are on the same diagonal in the original matrix. The lexicographic coordinate ordering of nonzeros in the output exactly matches the order in which nonzeros are stored in DIA (Figure 2c).

Figure 6. The syntax of coordinate remapping notation. Expressions in braces may be repeated any number of times.

4.1 Coordinate Remapping Notation

Figure 6 shows the syntax of coordinate remapping notation. Statements in coordinate remapping notation precisely specify how components in the input tensor map to components in a higher-order output tensor. For example, the statement

$$(1,j) \rightarrow (3-i,1,j)$$

maps every component $A_{ij}$ of an input matrix to the corresponding component in the $(j-i)$th slice of the output tensor, which is three-dimensional. Applying this remapping to a matrix stored in any format transforms it to a 3rd-order tensor where each slice contains all nonzeros that lie on the same diagonal in the original matrix. As illustrated in Figure 5, the lexicographic coordinate ordering of nonzeros in the resulting tensor thus accurately reflects the order in which nonzeros are stored in the DIA format.

Similarly, the BCSR format partitions a matrix into fixed-sized $M \times N$ blocks and stores components of each block contiguously in memory [19]. Such grouping of nonzeros can be expressed with the remapping

$$(1,j) \rightarrow (i/M,j/N,1,j)$$

which assigns components that lie within the same block to the same two-dimensional slice (identified by coordinates $(i/M,j/N)$) in the output tensor.

Coordinate remapping notation can express complex tensor reorders. The following remapping, for instance, groups together nonzeros that lie within the same constant-sized $N \times N \times N$ block and also orders the blocks as well as the nonzeros within each block in Morton order [32]:

$$(i,j,k) \rightarrow (r=i/N \text{ in } s=j/N \text{ in } t=k/N \text{ in } (r\&1) | ((s\&1)<<1) | ((t\&1)<<2) \ldots, i/N,j/N,k/N, r=i/N \text{ in } s=j/N \text{ in } t=k/N \text{ in } (r\&1) | ((s\&1)<<1) | ((t\&1)<<2) \ldots, i,j,k).$$

The Morton code of each block and nonzero is computed by interleaving the bits of their coordinates using bitwise operations. This mapping exactly describes how the HiCOO tensor format orders nonzeros in memory [29].

Coordinate remapping notation also provides counters, denoted by the "$#" symbol in Figure 6. Counters map nonzeros that share the same specified coordinates to distinct slices in the output tensor. For instance, the remapping

$$(1,j) \rightarrow (#i,i,j)$$

assigns each nonzero that has the same $i$ coordinate (i.e., that lies on the same row of the input matrix) to a distinct slice between the first and $k$-th slice in the output tensor, where $k$ is the number of nonzeros in row $i$. This remapping effectively groups together (at most) one nonzero from each row in the matrix, which matches how formats like ELL and JAD [40] store nonzeros in memory.

4.2 Code Generation

In order to support generating sparse tensor conversion routines for arbitrary combinations of formats, we independently annotate each supported format with a coordinate remapping that describes how the format groups together and orders nonzeros. As the previous examples demonstrate, it is also simple for end users to add support for additional custom formats by using coordinate remapping notation to specify how the new formats lay out nonzeros in memory. Then, to support converting tensors between two specific formats, our technique emits code that iterates over the input tensor and transforms the coordinates of each nonzero by applying the target format’s coordinate remapping.

To generate a set of nested loops that efficiently iterate over an input tensor in any source format, our technique uses the method Kjolstad et al. [25] proposed and Chou et al. [14] generalized, which we summarize at the end of Section 2.1.
Within the generated loops, our technique then emits code that computes each remapped nonzero’s additional coordinates as functions of the nonzero’s original coordinates, following the target format’s coordinate remapping.

To compute additional coordinates that are defined purely as arithmetic or bitwise expressions of the original coordinates, our technique simply inlines those expressions directly into the emitted code (e.g., lines 6 and 24 in Figure 4a, which compute the first coordinate in the output of the remapping (i,j) \rightarrow (j-1,i,j)). Remappings that contain let expressions are lowered by first emitting code to initialize the local variables and then inlining the expressions that use those local variables. For example, a remapped coordinate \( r = i/N \) in \((r&1) \mid ((r&2)<<2)\) would be lowered to

```c
int r = i/N;
int m = (r&1) | ((r&8)<<2;
```

Coordinate remappings that contain counters are lowered by emitting a set of counter variables for each distinct counter in the remapping. Each counter variable corresponds to a distinct set of coordinates \((i_1, \ldots, i_k)\) that can be used to index into the counter in the remapping, and the counter variable tracks how many input nonzeros with coordinates \((i_1, \ldots, i_k)\) have been iterated so far. Our technique additionally emits code that, for each nonzero with coordinates that correspond to counter variable \(c\), first assigns the nonzero to the output tensor slice indexed by \(c\) and then increments \(c\). So to apply the remapping \((i,j) \rightarrow (#i, i,j)\) to a COO matrix, for instance, our technique emits the following code:

```c
int counter[N] = {0}; // counter variables for #i
for (int p = 0; p < nnz; p++) {
  int i = A1_crd[p];
  int j = A2_crd[p];
  int k = counter[i]++;
  // map A(i,j) to coordinates (k, i,j) ...
```

If the coordinates used to index into a counter are iterated in order though, our technique minimizes the number of counter variables in the generated code by having counter variables be reused across iterations. For instance, if the input matrix is instead stored in CSR, our technique infers from properties of the format (exposed through the abstraction discussed in Section 2.1) that we can efficiently iterate over nonzeros row by row. Thus, to apply the same coordinate remapping as before, our technique emits a single counter variable that is reused to remap each row. The result is optimized code shown on lines 8–13 in Figure 4b.

5 Attribute Queries

As we also saw in Section 3, to avoid having to constantly reallocate and shuffle around stored nonzeros, many efficient tensor conversion algorithms instead allocate memory in one shot based on some statistics about the input tensor. Computing these statistics, however, require very different code depending on how the input tensor is stored. For instance, recall that to convert a matrix to ELL without having to dynamically resize the crd and vals arrays, one must first determine the maximum number of nonzeros \(K\) stored in any row of the input matrix. If the input matrix is stored in COO, then computing \(K\) requires constructing a histogram that records the number of nonzeros in each row, which in turn requires examining all the nonzeros in the matrix. If the input matrix is stored in CSR, however, then the number of nonzeros in each row can instead be directly computed from the pos array. Optimized code for converting CSR matrices to ELL thus does not need to make multiple passes over the input matrix’s nonzeros, reducing memory traffic.

We propose a new language called the **attribute query language** that precisely describes statistics of sparse tensors as aggregations over the coordinates of their nonzeros. The attribute query language is declarative, and attribute queries are specified independently of how the input tensor is actually stored. This lets our technique map attribute queries to equivalent sparse tensor computations and then simply leverage prior work on sparse tensor algebra compilation to generate optimized code for computing tensor statistics.

As we show in Section 6, our technique can thus generate efficient tensor conversion routines while only requiring users to provide simple-to-specify attribute queries for each potential target format, as opposed to complicated loop nests for every combination of source and target formats.

5.1 Attribute Query Language

The attribute query language lets users compute summaries of a tensor’s sparsity structure by performing aggregations over the coordinates of the tensor’s nonzeros. All queries in the attribute query language take the form

```sql
select I1, ..., Im -> <aggr1> as label1, ..., <aggrn> as labeln
```

where each \(I_k\) denotes a dimension of a tensor \(A\) and \(<aggr_k>\) invokes the aggregation function \(\text{max}, \text{min}, \text{or id}\).

For each subtensor \(A'\) of \(A\) with coordinates \((i_1, \ldots, i_m) \in I_1 \times \cdots \times I_m\), \(\text{count}(I_k, \ldots, I_l)\) computes the number of nonempty subtensors with distinct coordinates \((i_1, \ldots, i_m, k, \ldots, l) \in I_1 \times \cdots \times I_m \times I_k \times \cdots \times I_l\) that are contained in \(A'\). For instance, if \(I, J, K\) represent the slice, row, and column dimensions of a 3rd-order tensor \(B\), then the query

```sql
select I -> count(I) as nmr_in_slice
```

computes the number of nonempty rows contained in each slice of \(B\), while the query

```sql
select I -> count(I,K) as nnz_in_slice
```

computes the number of nonzeros in each slice.

\(\max(I_k)\) and \(\min(I_k)\) compute, for each subtensor \(A'\), the largest and smallest \(I_k\) such that the \(I_k\)-th slice of \(A'\) along dimension \(I_k\) is nonempty. For instance, if \(Q\) is the result of
select I ->
min(J) as min_in_row, max(J) as max_in_row

applied to the matrix in Figure 1, then Q[3].min_in_row == 1
and Q[3].max_in_row == 4 since all nonzeros in row 3 (the
last row) lie between columns 1 and 4.

Finally, id simply returns 1 if a subtensor A’ contains
nonzeros and 0 otherwise. So if k is the result of
select J -> id() as ne

again applied to the matrix in Figure 1, then R[4].ne == 1
since column 4 contains a nonzero while R[5].ne == 0 since
the last column is empty.

The attribute query language can be used with coordinate
remapping notation to compute even more complex attributes of structured tensors. For example, let A be a
KxIxJ tensor obtained by applying the remapping (i,j) ->
(j-i,i,j) to a matrix D. Since each slice of A along dimension K
responds to a unique diagonal in D, computing

select K -> id() as ne

on A results in a bit set that encodes the set of all nonempty
diagonals in B. Furthermore, since the coordinate of each slice of A is defined to be the offset of the corresponding
diagonal in B from the main diagonal, applying the query

select -> min(K) as lb, max(K) as ub

to A computes the lower and upper bandwidths of B.

5.2 Code Generation

To generate efficient code that computes an attribute query, our technique reformulates the query as sparse tensor algebra computation. The query is first lowered to a canonical form expressed in a variant of concrete index notation [24], which is a language for specifying tensor operations. The canonical form of the query then subsequently optimized by applying a set of transformations we define that simplify the computation. Finally, the optimized query in concrete index notation is compiled to imperative code by straightforwardly leveraging the techniques of Kjolstad et al. and Chou et al. [14, 24, 25], which take concrete index notation and generate efficient sparse tensor algebra kernels for operands in disparate formats. This approach works as long as query results are stored in a format like dense arrays that can itself be efficiently assembled without requiring attribute queries.

More precisely, let A be an I_1 x ... x I_n tensor obtained by applying some remapping to a J_1 x ... x J_n tensor B. Then, to compute an attribute query of the form

select I_1, ..., I_m -> id() as Q,

which invoke the id function, on A, our technique lowers the query to its canonical form in concrete index notation as

\forall j_1 \cdots \forall j_n Q_{i_1 \cdots i_m} = \text{map}(B_{j_1 \cdots j_n}, 1).

The computation above logically iterates over every component of B, computes the coordinates (i_1, ..., i_m) of each component B_{j_1 \cdots j_n} in the remapped tensor A, and sets the corresponding component in the Boolean result tensor Q to true (1). The map operator returns the second argument if the first argument is nonzero (or true) and zero otherwise, which ensures only the nonzeros in B are aggregated. So if, for instance, C is a K x I x J tensor obtained by applying the remapping (i,j) -> (j-i,i,j) to a matrix D, then to compute select K -> id() as Q on C, our technique lowers the query to the computation \forall i \forall j, Q_{i,j} = \text{map}(D_{j-i,j}, 1). For each and only the nonzeros of D, this computation computes the nonzero’s offset from the main diagonal and sets the corresponding component in Q to true. The query result Q thus strictly encodes the set of diagonals in D that have nonzeros.

In a similar way, our technique lowers count queries

select I_1, ..., I_m -> count(1_k, ..., 1_l) as Q

on A to their canonical form

\forall i, \ldots, \forall i_l Q_{i_1 \cdots i_m} = \text{map}(W_{i_1 \cdots i_l}, 1),

where

\forall j_1 \cdots \forall j_n, W_{i_1 \cdots i_m} = \text{map}(B_{j_1 \cdots j_n}, 1).

The computation above first iterates over the nonzeros of B to compute the intermediate result W, which encodes whether each subtensor of A with coordinates (i_1, ..., i_m) is nonempty. The computation then sums over dimensions I_k to I_l of W in order to compute the number of aforementioned subtensors that are nonempty and contained in each higher-order subtensor with coordinates (i_1, ..., i_m).

Our technique also generates code for max queries

select I_1, ..., I_m -> max(1_k) as Q

by lowering them to their canonical form

\forall j_1 \cdots \forall j_n Q’_{i_1 \cdots i_m} = \text{map}(B_{j_1 \cdots j_n}, i_k - s + 1),

where s denotes the smallest possible coordinate along dimension I_k. The result is assumed to be initialized to the zero matrix, so by mapping each input tensor component to its remapped coordinate i_k plus the constant (1-s), we ensure that only the coordinates of nonzeros are actually aggregated. Q’ can thus be interpreted as the actual result of the original query (i.e., Q) but just shifted by (1-s); in other words, Q’_{i_1 \cdots i_m} \equiv Q_{i_1 \cdots i_m} + s - 1. Similarly, min queries

select I_1, ..., I_m -> min(1_k) as Q

are lowered to their canonical form

\forall j_1 \cdots \forall j_n Q’’_{i_1 \cdots i_m} = \text{map}(B_{j_1 \cdots j_n}, -i_k + t),

where t denotes one plus the largest possible coordinate along dimension I_k and Q’’ is the query result but negated and shifted by t; in other words, Q’’_{i_1 \cdots i_m} \equiv \neg Q’_{i_1 \cdots i_m} + t.

After an attribute query is lowered to its canonical form, our technique eagerly applies a set of predefined transformations on the query computation in order to optimize its performance. Table 1 shows a subset of transformations that our technique uses. In general, these transformations exploit properties of the input tensor and its underlying storage format in order to reduce the number of dimensions that have to be iterated and to eliminate redundant temporaries.
Table 1. Example transformations that our technique applies to optimize attribute queries. We augment level formats with a property that specifies if a dimension stores explicit zeros, which lets our technique determine if a tensor stores only nonzeros.

| Transformation       | Definition                                                                 | Preconditions and Postconditions |
|----------------------|---------------------------------------------------------------------------|----------------------------------|
| reduction-to-assign  | (\(\forall_{j_i} \ldots \forall_{j_n} A_{i_1 \ldots i_m} @= \text{expr}\)) => (\(\forall_{j_i} \ldots \forall_{j_n} A_{i_1 \ldots i_m} = \text{expr}\)) | For each \(j_k\), there exists an \(i_j\) such that \(j_k \equiv i_j\). @ is any reduction operator. |
| inline-temporary     | (\(\forall_{i_j} \ldots \forall_{i_m} A_{i_1 \ldots i_j} @= f(W_{i_1 \ldots i_m})\)) where (\(\forall_{i_j} \ldots \forall_{i_n} W_{i_1 \ldots i_m} = \text{expr}\)) => (\(\forall_{i_j} \ldots \forall_{i_n} A_{i_1 \ldots i_j} @= f(\text{expr})\)) | \(f\) is any function that takes only \(W\) as tensor operand. @ is any reduction operator or a simple assignment. |
| simplify-width-count | (\(\forall_{i_j} \ldots \forall_{i_m} A_{i_1 \ldots i_m} @= \text{map}(B_{i_j \ldots i_n}, c)\)) => (\(\forall_{i_j} \ldots \forall_{i_{n-1}} A_{i_1 \ldots i_{m-1}} @= B'_{i_j \ldots i_{n-1}}, c\)) | \(B\) stores only nonzeros, and \(J_n\) is reduction variable that indexes into the innermost dimension of \(B\) (\(J_n\)). \(c\) is any constant. \(B'\) is a tensor that encodes the number of nonzeros in each slice of \(B\); the values of \(B'\) are not materialized but dynamically computed from level functions that define iteration over dimension \(J_n\) of \(B\). |
| counter-to-histogram | (\(\forall_{i_j} \ldots \forall_{i_n} A_{i_1 \ldots i_m} \max = \text{map}(B_{i_j \ldots i_n} \cap j_k, j_j + 1)\)) => (\(\forall_{i_j} \ldots \forall_{i_n} A_{i_1 \ldots i_m} \max = W_{i_1 \ldots i_n}\)) where (\(\forall_{i_j} \ldots \forall_{i_n} W_{i_1 \ldots i_m} = \text{map}(B_{i_j \ldots i_n}, 1)\)) | None. |

To see how our technique optimizes attribute queries, consider the example query `select 1 -> count(1)` as `Q` applied to an \(I \times J\) matrix \(B\). As described before, our technique first lowers this query to its canonical form

\(\forall_i \forall_j Q_i = \text{map}(W_{ij}, 1)\) where (\(\forall_i \forall_j W_{ij} = \text{map}(B_{ij}, 1)\)).

Our technique then proceeds to iteratively and eagerly apply the transformations shown in Table 1 on the computation above. In particular, each iteration variable bound to a \(\forall\) is used to independently index into a dimension of \(W\), so the sub-statement that defines \(W\) satisfies the preconditions of the reduction-to-assign transformation. Our technique thus applies the aforementioned transformation on the sub-statement that defines \(W\) to get

\(\forall_i \forall_j Q_i = \text{map}(W_{ij}, 1)\) where (\(\forall_i \forall_j W_{ij} = \text{map}(B_{ij}, 1)\)).

Then, since the temporary \(W\) is no longer the result of a reduction operation, our technique eliminates it by applying the inline-temporary transformation to get

\(\forall_i \forall_j Q_i = \text{map}(B_{ij}, 1), 1)\),

which is then trivially rewritten to \(\forall_i \forall_j Q_i = \text{map}(B_{ij}, 1)\) by applying constant folding. If \(B\) is stored in COO without explicit zeros, then our technique additionally applies the simplify-width-count transformation followed by reduction-to-assign again to get the final query

\(\forall_i Q_i = B'_i\),

where each component of \(B'\) is dynamically computed as \(\text{pos}[i+1] - \text{pos}[1]\). The optimized query thus avoids iterating over \(B'\)'s nonzeros, thereby reducing memory traffic.

### 6 Sparse Tensor Assembly

As explained in Section 2.1, a sparse tensor can be modeled as a hierarchical structure of coordinates, where each stored component is represented by a path from the root to a leaf. We can thus view any tensor format simply as a composition of level formats that each stores a level of a coordinate hierarchy. This abstraction lets us reason about sparse tensor assembly as coordinate hierarchy construction.

We extend the coordinate hierarchy abstraction with new primitives (level functions) that describe how each level can be efficiently constructed (assembled). These new level functions, unlike analogous ones that Chou et al. [14] propose, describe how coordinates and edges can be efficiently inserted into a coordinate hierarchy assuming certain statistics about the input tensor have been precomputed.

Figure 7 show how level formats that use disparate data structures to store a coordinate hierarchy level can implement the new assembly level functions. All these implementations expose the same static interface, which lets our code generator reason about and emit efficient code to convert tensors between a wide range of formats. And by using the same level formats to express different tensor formats that share common data structures, our technique can also reuse the same level function implementations to generate conversion routines for many different target formats. For example, the column dimensions of CSR tensors and the row dimensions of COO tensors can both be stored using the same level format (i.e., compressed). Our technique can thus use the same implementations of the new assembly level functions (for the compressed level format) to generate (portions of) code that convert tensors to either CSR or COO. This
limits the one-time effort needed to implement our extended coordinate hierarchy abstraction.

6.1 Assembly Abstraction

Our extended coordinate hierarchy abstraction assumes that coordinate hierarchies can be constructed level by level from top to bottom. The assembly of each level is decomposed into two logical phases: edge insertion and coordinate insertion.

The edge insertion phase, which is optional, logically bulk inserts edges into a coordinate hierarchy, connecting coordinates in one level to coordinates in the preceding parent level. Edge insertion models the assembly of data structures that map nonzeros to their containing subtensors. Depending on whether each position (i.e., node) in the parent level can be iterated in sequence, edge insertion can be done in a sequenced or unsequenced fashion.

Unsequenced edge insertion is defined in terms of three level functions that any level format may implement:

- \texttt{unseq_init_edges(sz_{k-1}, Q_k) \rightarrow void}
- \texttt{unseq_insert_edges(p_{k-1}, i_1, \ldots, i_{k-1}, Q_k) \rightarrow void}
- \texttt{unseq_finalize_edges(sz_{k-1}) \rightarrow void}

\(Q_k\) denotes the (complete) results of attribute queries that a level format specifies must be precomputed, while \(q_k\) denotes only the elements of \(Q_k\) indexed by coordinates \((i_1, \ldots, i_{k-1})\). \(sz_{k-1}\) is the size of the parent level and can be computed as a function of its own parent’s size by calling the level function

\[
\text{get_size(sz_{k-1}) \rightarrow sz_k},
\]

which all level formats must also implement. \texttt{unseq_init_edges} initializes data structures that the level format uses to logically store edges. Then, for each position \(p_{k-1}\) in the parent level, which represents a subtensor with coordinates \((i_1, \ldots, i_{k-1})\), \texttt{unseq_insert_edges} allocates some number of child coordinates to be connected to \(p_{k-1}\). The number of child coordinates allocated can be computed as any function of \(q_k\). Finally, \texttt{unseq_finalize_edges} inserts edges into the coordinate hierarchy such that each coordinate in the parent level is connected to as many children as it was previously allocated. Figure 8 (left) shows how these level functions can logically be invoked to bulk insert edges.

Sequential edge insertion, by contrast, is defined in terms of just two level functions:

- \texttt{seq_init_edges(sz_{k-1}, Q_k) \rightarrow void}
- \texttt{seq_insert_edges(p_{k-1}, i_1, \ldots, i_{k-1}, Q_k) \rightarrow void}

These level functions are analogous to \texttt{unseq_init_edges} and \texttt{unseq_insert_edges} and can be invoked in similar ways. Sequenced edge insertion, however, assumes that all positions in the parent level are iterated in order. Thus, \texttt{seq_insert_edges} is responsible for both allocating the appropriate number of child coordinates to each parent and actually inserting the edges, and a separate \texttt{finalize} function is not necessary.

The coordinate insertion phase logically iterates over the input tensor’s nonzeros and inserts their coordinates into a coordinate hierarchy. Coordinate insertion models the assembly of data structures that store the actual coordinates and values of the nonzeros. This phase is defined in terms of the following level functions:

- \texttt{init_coords(sz_{k-1}, Q_k) \rightarrow void}
- \texttt{init_{get|yield}_pos(sz_{k-1}) \rightarrow void}
- \texttt{(get|yield)_pos(p_{k-1}, i_1, \ldots, i_{k}) \rightarrow p_k}
- \texttt{insert_coord(p_{k-1}, p_k, i_1, \ldots, i_{k}) \rightarrow void}
- \texttt{finalize_{get|yield}_pos(sz_{k-1}) \rightarrow void}

\texttt{init_coords} allocates and initializes data structures for storing coordinates in a coordinate hierarchy level. If a level format implicitly encodes coordinates (e.g., as a fixed range) using some fixed set of parameters, then \texttt{init_coords} also compute those parameters as functions of the attribute query results \(Q_k\). On the other hand, if a level format explicitly stores coordinates in memory, then the coordinates of nonzeros are inserted by invoking \texttt{insert_coord} for each nonzero.

The position \(p_k\) at which each nonzero should be inserted is computed by invoking either \texttt{get_pos} or \texttt{yield_pos}. The former guarantees that nonzeros with the same coordinates are inserted at the same position. The latter allows duplicate coordinates to be inserted at different positions. Both functions, however, may rely on auxiliary data structures to track where to insert coordinates; \texttt{init_{get|yield}_pos} and \texttt{finalize_{get|yield}_pos} initializes and cleans up those data structures. Figure 8 (right) shows how all these level functions can be invoked to perform coordinate insertion.

6.2 Code Generation

To generate code that converts sparse tensors between two formats, our code generator emits loops that iterate over a tensor in the source format and apply the target format’s coordinate remapping to each nonzero. This is done by applying the technique described in Section 4.2. Then, within each loop nest that iterates over the (remapped) input tensor, the code generator emits code that invokes the level functions described in Section 6.1 to store each nonzero into the result. The emitted code is finally specialized to the target format by inlining its implementations of the aforementioned level functions (e.g., as shown in Figure 7). This approach enables the code generator to support disparate target (and source) formats. At the same time, it limits the complexity of the code generator, since the code generator does not need to hard-code for specific data structures but can simply reason about how to invoke a fixed set of level functions.

To minimize memory traffic at runtime, our technique generates code that, wherever possible, fuses the assembly of adjacent levels in the result coordinate hierarchy. Adjacent levels can be assembled together as long as only the parent level requires a separate edge insertion phase (or if none do). As an example, none of the level formats that compose DIA requires edge insertion. Thus, our technique will emit code
To convert any matrix to DIA by iterating over the matrix just once and assembling all output dimensions (levels) together. For each set of levels that can be assembled together, our technique then simply has to emit code like shown in Figure 8 to perform edge insertion (if required) followed by coordinate insertion. If a level format implements both variants of edge insertion, then our technique selects one based on whether the parent level can be iterated in order. The code generator infers this from properties exposed through the coordinate hierarchy abstraction that specify if the parent level stores coordinates in order. If a level format implements yield_pos but does not permit storing duplicates of the same coordinate, then our technique also emits logic to perform deduplication on the fly by keeping track of inserted coordinates.

To see how our technique works, assume we are generating code to convert COO tensors to CSR. To obtain code that assembles the column dimension of the result, the code generator first emits sequenced edge insertion code that has the same structure as Figure 8 (left), except with all level functions replaced by their sequenced counterparts. The emitted code is then specialized to CSR by replacing the level function calls with the compressed level format’s implementations of those functions (Figure 7, top). The result is lines 7–11 in Figure 4c, which iterate over all rows of the output in order and reserve exactly enough space to store each row’s nonzeros. In a similar way, the code generator emits code in Figure 8 (right) to perform coordinate insertion and then specializes it to CSR, yielding lines 12–25 in Figure 4c.

7 Evaluation
We evaluate our technique and find that it generates efficient sparse tensor conversion routines for many combinations...
We implemented a prototype of our technique as extensions widely used Fortran sparse linear algebra library, Auto Generation of Efficient Sparse Tensor Format Conversion Routines Tech Report, 2019, Cambridge, MA.

Table 2. Statistics about matrices used in our experiments.

| #  | Matrix          | Domain                | Dimensions | Nonzeros | Density | Symmetric | Nonempty Diagonals | Maximum Nonzeros/row |
|----|-----------------|-----------------------|------------|----------|---------|-----------|--------------------|---------------------|
| 1  | pdb1HYS         | Protein data base     | 36K × 36K  | 4,344,765 | 3 × 10^{-3} | Yes       | 25,577             | 204                 |
| 2  | jnlibrng1       | Optimization          | 40K × 40K  | 199,200  | 1 × 10^{-4} | Yes       | 5                  | 5                   |
| 3  | obstcla         | Optimization          | 40K × 40K  | 197,608  | 1 × 10^{-4} | Yes       | 5                  | 5                   |
| 4  | chem_master1    | Chemical master equation | 40K × 40K  | 201,201  | 1 × 10^{-4} | No        | 5                  | 5                   |
| 5  | rma10           | 3D CFD                | 46K × 46K  | 2,374,001 | 1 × 10^{-3} | No        | 17,367             | 145                 |
| 6  | dixmaatl        | Optimization          | 60K × 60K  | 299,998  | 8 × 10^{-5} | Yes       | 7                  | 5                   |
| 7  | cant            | FEM/Cantilever        | 62K × 62K  | 4,007,383 | 1 × 10^{-3} | Yes       | 99                 | 78                  |
| 8  | shyy161         | CFD/Navier-Stokes     | 76K × 76K  | 329,762  | 6 × 10^{-5} | No        | 7                  | 6                   |
| 9  | consph          | FEM/Spheres           | 83K × 83K  | 6,010,480 | 9 × 10^{-4} | Yes       | 13,497             | 81                  |
| 10 | denormal        | Counter-example problem | 89K × 89K  | 1,156,224 | 1 × 10^{-4} | Yes       | 13                 | 13                  |
| 11 | Baumann         | Chemical master equation | 112K × 112K | 748,331  | 6 × 10^{-5} | No        | 7                  | 7                   |
| 12 | cop20k_A        | FEM/Accelerator       | 121K × 121K | 2,624,331 | 2 × 10^{-4} | Yes       | 221,205            | 81                  |
| 13 | shipevt         | FEM                   | 141K × 141K | 3,568,176 | 2 × 10^{-4} | Yes       | 10,475             | 102                 |
| 14 | majorbasis      | Optimization          | 160K × 160K | 1,750,416 | 7 × 10^{-5} | No        | 22                 | 11                  |
| 15 | scircuit        | Circuit               | 171K × 171K | 958,936  | 3 × 10^{-5} | No        | 159,419            | 353                 |
| 16 | mac_econo_fwd500 | Economics             | 207K × 207K | 1,273,389 | 9 × 10^{-5} | No        | 511                | 44                  |
| 17 | pwtk            | Wind tunnel           | 218K × 218K | 11,524,432 | 2 × 10^{-4} | Yes       | 19,929             | 180                 |
| 18 | Lin             | Structural problem    | 256K × 256K | 1,766,400 | 3 × 10^{-5} | Yes       | 7                  | 7                   |
| 19 | ecology1        | Animal movement       | 1M × 1M    | 4,996,000 | 5 × 10^{-6} | Yes       | 5                  | 5                   |
| 20 | webbase_1M      | Web connectivity      | 1M × 1M    | 3,105,536 | 3 × 10^{-6} | No        | 564,259            | 4700                |
| 21 | atmosmodd       | Atmospheric model     | 1.3M × 1.3M | 8,814,880 | 5 × 10^{-6} | No        | 7                  | 7                   |

of disparate source and target formats. The generated conversion routines have performance similar to or better than equivalent hand-implemented versions. We also find that, for combinations of source and target formats that are not directly supported by a library, our technique can further optimize conversions between those formats by emitting code that avoids materializing temporaries.

7.1 Experimental Setup

We implemented a prototype of our technique as extensions to the open-source taco tensor algebra compiler [25]. To evaluate it, we compare code that our technique generates against hand-implemented versions in SPARSKIT [41], a widely used [21, 35] Fortran sparse linear algebra library, and Intel MKL [20], a C and Fortran math processing library that is optimized for Intel processors. We also evaluate our technique against taco without our extensions.

All experiments are conducted on a 2.5 GHz Intel Xeon E5-2680 v3 machine with 30 MB of L3 cache and 128 GB of main memory. The machine runs Ubuntu 18.04.3 LTS. We compile code that our technique generates using GCC 7.4.0 and build SPARSKIT from source using GFortran 7.4.0. We run each experiment 100 times under cold cache conditions and report minimum serial execution times.

We run our experiments with real-world matrices of varying sizes and structures as inputs. These matrices, listed in Table 2, come from applications in disparate domains and are obtained from the SuiteSparse Matrix Collection [15].

7.2 Performance Evaluation

We measure the performance of sparse tensor conversion routines that our technique generates for seven distinct combinations of source and target formats:

- COO to CSR (coo_csr)
- COO to DIA (coo_dia)
- CSR to DIA (csr_dia)
- CSR to CSC (csr_csc)
- CSR to ELL (csr_ell)

where inputs and outputs in COO, CSR, or CSC are not assumed to be sorted (though nonzeros are still necessarily grouped by row/column in CSR/CSC). For each combination of formats, we also measure the performance of converting between those formats using SPARSKIT and Intel MKL. Both libraries implement routines that directly convert matrices from COO to CSR, CSR to CSC, and CSR to DIA. Additionally, SPARSKIT supports directly converting matrices from CSR to ELL. However, neither SPARSKIT nor Intel MKL implements routines that directly convert matrices between the remaining combinations of formats. Thus, to perform those conversions for non-symmetric matrices using either library, we first convert the input from its source format to a temporary in CSR and then convert the temporary from CSR to the desired target format. (If the input matrix is symmetric, however, then conversions from CSC to DIA/ELL can simply be cast as conversions from CSR to DIA/ELL.)

Figures 9 to 12 show the results of our experiments. As these results demonstrate, our technique outperforms or is comparable to SPARSKIT and Intel MKL on average for all.
Figure 9. Normalized execution times of COO to CSR conversion (coo_csr), using code that our technique generates and other existing libraries as well as taco without our extensions. Results are normalized to that of our technique for each test matrix.

Figure 10. Normalized execution times of CSR to CSC conversion (csr_csc), using code that our technique generates and other existing libraries. Results are normalized to that of our technique for each test matrix. We only show results for nonsymmetric matrices, since CSR and CSC are equivalent for symmetric matrices.

combinations of source and target formats that we evaluate. On the whole, code that our technique emits to convert matrices from COO to CSR (coo_csr) and from CSR to CSC (csr_csc) exhibit similar performance as hand-implemented routines in SPARSKIT and somewhat better performance than Intel MKL. This is unsurprising since our technique generates code that implement the same algorithms (variations of Gustavson’s HALFPERM algorithm [18]) as SPARSKIT. Our technique also emits code to perform CSR to DIA conversion that is 2.2× faster than SPARSKIT and 1.6× faster than Intel MKL on average. SPARSKIT’s implementation of csr_dia supports extracting a bounded number (K) of nonempty diagonals from an N × N input matrix and storing them in the output. However, SPARSKIT implements this capability inefficiently by performing the first K iterations of selection sort on the set of all 2N − 1 diagonals, in order to identify and extract the densest diagonals. This incurs overhead of (2K − 1)N memory accesses, which is significant if the input matrix only has K nonempty diagonals and ~ KN nonzeros that must be copied to begin with. Furthermore, code that our technique emits to perform CSR to ELL conversion is 1.2× faster than SPARSKIT on average. This is because our technique emits code that invokes calloc to both allocate and zero-initialize the output arrays, whereas SPARSKIT’s csr_ell kernel takes user-allocated output arrays as arguments and separately initializes those arrays.

Our technique also significantly outperforms SPARSKIT by 3.4× on average for COO to DIA conversion, 2.6× for CSC to DIA conversion, and 1.4× for CSC to ELL conversion. Similarly, code that our technique emits outperforms Intel MKL by 3.2× for COO to DIA conversion and 2.2× for CSC to DIA conversion. As mentioned before, neither SPARSKIT nor Intel MKL implements routines to directly convert between any of those combinations of formats. Thus, when the input is nonsymmetric or stored in COO, both libraries must incur additional memory accesses to construct temporary CSR matrices and then iterate over those temporaries again in order to perform the actual conversion to DIA or ELL. By contrast, our technique emits code that directly converts matrices from COO/CSC to DIA/ELL without materializing any temporary. This minimizes memory traffic and lets our technique achieve even greater speedups over SPARSKIT and Intel MKL than we observe for CSR to DIA/ELL conversions, which the libraries directly implement.

Finally, we measure and compare against the performance of taco without our extensions for COO to CSR conversion.
Figure 11. Normalized execution times of conversions to DIA, using code that our technique generates and other existing libraries. Results are normalized to that of our technique for each test matrix. Asterisks denote nonsymmetric matrices. For symmetric matrices, we cast CSC to DIA conversion as CSR to DIA conversion and report the same results shown in Figure 11a. We only show results for matrices that can be stored in DIA with at least 25% of stored values being nonzeros.

By expressing COO to CSR conversion in index notation as tensor assignment, the techniques of Kjolstad et al. and Chou et al. can also generate code that performs the conversion. As Figure 9 also shows though, our technique emits code to perform COO to CSR conversion that is 21.3× faster on average. The techniques of Kjolstad et al. and Chou et al. cannot reason about generating code that inserts nonzeros into CSR indices out of order. Thus, it must also sort the input before performing the actual conversion, incurring significant additional overhead. Furthermore, index notation cannot express conversions between formats that store nonzeros in non-lexicographic coordinate order. So without our extensions, taco cannot emit code to perform conversion end to end with structured formats like DIA or ELL.

8 Related Works

There exists a long line of works [5, 8, 9, 12, 19, 29–31, 40, 43, 48] on developing new sparse tensor formats to accelerate sparse matrix-vector multiplication (SpMV), sparse matrix-dense matrix multiplication (SpDM), matricized tensor times Khatri-Rao products (MTTKRP), and other sparse tensor algebra kernels. These formats organize nonzeros in disparate ways to reduce memory footprint, improve cache utilization, expose parallelization opportunities, and better exploit hardware capabilities such as SIMD vector units for performance. All the aforementioned works rely on hand-implemented routines for converting tensors from more common representations to their proposed formats. These routines are often as complex as the code to actually perform computations.
that the proposed formats are designed to accelerate. Additionally, many techniques [17, 18, 47] have been proposed for accelerating transpositions of CSR matrices, which is equivalent to converting matrices from CSR to CSC.

Existing sparse tensor and linear algebra compilers cannot generate efficient code to convert tensors between arbitrary, disparate formats. taco [14, 24, 25] can emit code to convert tensors between formats that store nonzeros in lexicographic coordinate order, but cannot generate complete conversion routines for structured formats like DIA and ELL. taco without our extensions also cannot emit code that computes and uses statistics about the input tensor to coordinate efficient assembly of the output tensor. LL [3, 4] is a functional language that lets user define sparse matrix formats as nestings of list and pairs that encode nonzeros in a dense matrix. From these specifications, the LL compiler can emit code that convert dense matrices to different sparse matrix formats as nestings of list and pairs that encode nonzeros in a dense matrix. From these specifications, the LL compiler can emit code that converts dense matrices to different sparse matrix formats, but not efficient code that directly convert between sparse matrix formats. SIPR [38], CHiLL [46], and techniques that Bik and Wijshoff [10, 11] proposed can all generate sparse linear algebra kernels that, as sub-operations, convert matrices between different formats. However, conversions in SIPR-generated code are performed by invoking hand-implemented operations that are hard-coded to specific source and target formats. The other techniques, meanwhile, only support a fixed set of standard sparse matrix formats. Bernoulli [26, 27, 44] uses a black-box protocol that provides an abstract interface for describing how sparse matrices stored in different data structures can be efficiently accessed. However, the interface does not support assembly, so Bernoulli cannot generate code that constructs sparse matrix results.

There also exists a separate line of works [13, 22, 28, 33] on generating efficient code for query languages like SQL, which our attribute query language resembles. (Attribute queries are analogous to GROUP BY queries on a table that stores the coordinates of nonzeros.) In particular, HorseIR [13] lowers SQL queries to an array-based intermediate representation that is then optimized and compiled to efficient code. Empty-Headed [1], meanwhile, is a graph processing framework that generates efficient code to compute graph queries expressed in a Datalog-like language. Furthermore, our approach to optimizing attribute queries is reminiscent of query rewriting systems in certain relational database systems like Starburst [36, 37]. All these techniques are designed for queries that may perform complex joins and aggregate data of arbitrary types. By contrast, attribute queries are limited to aggregating (integer) coordinates. This lets our technique lower and optimize attribute queries in ways that would not be valid for aggregations over other types of data.

![Figure 12. Normalized execution times of conversions to ELL, using code that our technique generates and other existing libraries. Results are normalized to that of our technique for each test matrix. Asterisks denote nonsymmetric matrices. For symmetric matrices, we cast CSC to ELL conversion as CSR to ELL conversion and report the same results shown in Figure 12a. We only show results for matrices that can be stored in ELL with at least 25% of stored values being nonzeros.](image-url)
9 Conclusions
We have described a technique for generating sparse tensor conversion routines that efficiently convert tensors between a wide range of disparate formats. Our technique is extensible, and we have shown how even users can easily add support for new source and target formats by simply specifying how to construct and iterate over tensors in those new formats. By making it easy to work with the same data in multiple formats, each suited to a different stage of an application, our technique can greatly reduce the engineering effort needed to optimize sparse tensor algebra applications.

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