Hardness and approximation of the Probabilistic $p$-Center problem under Pressure

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Abstract

The Probabilistic $p$-Center problem under Pressure ($\text{Min } PpCP$) is a variant of the usual $\text{Min } p$-Center problem we recently introduced in the context of wildfire management. The problem is to locate $p$ shelters minimizing the maximum distance people will have to cover to reach the closest accessible shelter in case of fire. The landscape is divided in zones and is modeled as an edge-weighted graph with vertices corresponding to zones and edges corresponding to direct connections between two adjacent zones. The uncertainty associated with fire outbreaks is modeled using a finite set of fire scenarios. Each scenario corresponds to a fire outbreak on a single zone (i.e., on a vertex) with the main consequence of modifying evacuation paths in two ways. First, an evacuation path cannot pass through the vertex on fire. Second, the fact that someone close to the fire may not take rational decisions when selecting a direction to escape is modeled using new kinds of evacuation paths. In this paper, for a given instance of $\text{Min } PpCP$ defined by an edge-weighted graph $G = (V,E,L)$ and an integer $p$, we characterize the set of feasible solutions of $\text{Min } PpCP$. We prove that $\text{Min } PpCP$ cannot be approximated with a ratio less than $\frac{56}{55}$ on subgrids (subgraphs of grids) of degree at most 3. Then, we propose some approximation results for $\text{Min } PpCP$. These results require approximation results for two variants of the (deterministic) $\text{Min } p$-Center problem called $\text{Min MAC } p$-Center and $\text{Min Partial } p$-Center.

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1. Introduction

The problem $\text{Min}\ PpCP$ was introduced in Demange et al. (2018) as a variant of the usual $\text{Min}\ p$-Center problem with uncertainty on vertices. In the same paper, we presented our motivation in the context of wildfire management and discussed it further in Demange et al. (2020). In our model, the landscape is represented by an adjacency graph $G = (V, E)$. Each vertex corresponds to a zone and two vertices $i$ and $j$ are linked by an edge if and only if it is possible to go directly from one to the other without passing through another area. We assume this is a symmetric relation, which makes this graph non-directed. Each edge $(i, j)$ is weighted with a positive number $\ell_{ij}$ that can be seen as a distance or a traveling time; we will call it the length of the edge $(i, j)$. For every two vertices $i, j$, $d(i, j)$ will denote the shortest path distance between $i$ and $j$ in $G$ and for any set of vertices $C \subset V$, we denote $d(v, C) = \min_{c \in C} d(v, c)$ the distance from $v$ to $C$. By convention, we will set $d(v, \emptyset) = +\infty$.

For a given integer $p$, the objective is to select a set $C$ of at most $p$ vertices, i.e. zones, where to locate shelters so as to minimize the maximum traveling time from a zone to a shelter. In a deterministic setup, this problem is the classical $\text{Min}\ p$-Center problem that aims to locate facilities on vertices of a network modeled by a graph. Given our motivating context, centers will just be called shelters and, when no ambiguity occurs, we will just use the term shelter to refer to a vertex where to install a shelter. For a set $C$ of shelters and a vertex $j$, $d(j, C)$ will be called distance to shelters of $j$ and the (deterministic) radius of $C$, denoted $r(C)$, corresponds to the longest distance to shelters of vertices: $r(C) = \max_{v \in V} d(v, C)$. $\text{Min}\ p$-Center is to find, for any $p$, a set $C, |C| \leq p$ of minimum radius.

Figure 1: An example where, for $p = 2$, a singleton minimizes the radius.
Since adding a center to $C$ cannot increase its radius, it is straightforward that, if $p \leq |C|$, then there is an optimal solution with exactly $p$ shelters; however, this is not a necessary condition for optimal solutions. Consider indeed the graph of Figure 1 with all edge lengths equal to 1; if $p = 2$, then the minimum possible radius is 2 but $r(\{x\}) = 2$.

Min $p$-Center and numerous versions have been extensively studied both from a graph theory perspective and for various applications (see, for instance Calik et al. (2015)). It is a well known NP-hard problem, even in the class of planar graphs (Kariv and Hakimi (1979)) that is particularly relevant in our motivating context. Min $p$-Center is known to be 2-approximable (Hochbaum and Shmoys (1985)) and is not approximated with a constant ratio strictly smaller than 2, unless P=NP (Hsu and Nemhauser (1979)). Similar results can be obtained for variants of Min $p$-Center. For instance, in Chaudhuri et al. (1998), the generalization of Min $p$-Center where, given a number $k$, we have to place $p$ centers so as to minimize the maximum distance of any non-center node to its $kth$ closest center. They give a 2-approximation algorithm for this problem, and show it is the best possible. In this paper, we will establish new similar results for other variants of Min $p$-Center.

Min $PpCP$ is a version of $p$-center problems where uncertainty is on vertices: with some probability a vertex may become unavailable due to a fire outbreak. We present in detail this problem in the next section after giving useful related definitions. In section 3 we present the main hardness result and in section 4 we propose some approximation results.

2. The Probabilistic $p$-Center problem under Pressure

2.1. Further notations and definitions

Let $G$ be an edge-weighted graph; we will denote it $G = (V,E,L)$ with $L = (\ell_{ij})_{i,j \in V}$ the matrix of lengths. If $\mathbb{Q}$ denotes the set of rational numbers, $L$ has entries in $\mathbb{Q} \cup \{\infty\}$ such that $\ell_{ij} < \infty \iff (i,j) \in E$. We will denote $\ell_m$ and $\ell_M$, respectively the smallest and the largest edge lengths (i.e., $\ell_M$ is the largest finite entry in $L$). We will refer as the uniform case the case where all edge lengths are equal. For all problems we consider in this paper, the objective value is linear with respect to the lengths and feasibility conditions due not depend on the lengths. As a consequence, the uniform case is equivalent to the case where all edge lengths are equal to 1. When dealing with the uniform case we will omit $L$ in the instance. A mixed graph is a graph with both directed and non-directed edges. When no ambiguity occurs we will use similar notations for graphs and mixed graphs. In the mixed case, we will just identify directed edges as denote them with an arrow in the related drawing. All non-directed notions in graphs also apply to mixed graphs by
considering the non-directed version of the mixed graph obtained by replacing directed edges by non-directed ones. Similarly, all directed notions apply to mixed graphs since a mixed graph can be seen as a digraph with non-directed edges replaced by two directed edges in opposite directions. For instance, when speaking about distances in a mixed graph, paths are meant to respect the edge orientations and thus, the matrix of distances is not symmetric anymore. In an edge-weighted graph $G = (V,E,L)$ and two vertices $i,j$, $d(i,j) = +\infty$ if $i$ and $j$ are in different connected components. In a mixed graph, we may have $d(i,j) = +\infty$ with $i$ and $j$ in the same connected component. It just means that there is no path from $i$ to $j$ respecting the orientation of directed edges. For example, in the mixed graph represented in Figure 2, $d(2,6) = 5$ while $d(6,2) = \infty$.

In our motivating application, fire hazards (or any hazard occurring on vertices) is modeled using scenarios. The landscape is represented by an edge-weighted graph $G = (V,E,L)$. A scenario is associated with each specific fire outbreak. We restrict ourselves to single fire outbreak and consequently, each scenario $s$ corresponds to a single vertex $s$ on fire. This restriction is motivated by our primary focus on a relatively short time period after outbreak which assumes an efficient early warning system. In this case everybody can escape to a shelter before the fire spreads to adjacent zones.

The operational graph associated with the scenario $s$, denoted by $G^s$, is a mixed graph $G^s = (V,E^s,L^s)$ obtained from $G = (V,E,L)$ by replacing the edges $(s,v)$ incident to $s$ by directed edges $(s,v)$. All weights are preserved. Consequently, in $G^s$, vertex $s$ is no longer accessible from another vertex.

For every two vertices $i,j$, the distance from $i$ to $j$ in $G^s$ is denoted $d^s(i,j)$. Note that for all $j \in V \setminus \{s\}$, we have $d^s(j,s) = +\infty$.

In this paper, we consider a uniform distribution of probabilities over all scenarios: each scenario $s_i$, $i \in V$ has probability $\frac{1}{|V|}$ and these events are all independent.

In most $p$-Center problems with uncertainty, given a solution $C$ with $p$ vertices or less, and given a scenario $s$, the evacuation distance of a vertex $j$ is usually the shortest distance between $j$ and its nearest shelter, $d(j,C)$. This strategy is not adapted to our context and we consider a different evacuation strategy introduced and explained in our previous paper Demange et al. (2020). This evacuation strategy induces new evacuation distances to shelters. If $s$ is on fire, we have:

1. for people on $s$, two cases have to be considered. If a shelter is located on $s$, then people present on vertex $s$ are considered as safely sheltered in it, otherwise we assume that they first run away from the fire in any direction and after they reach a neighbor $j$, they evacuate to the shortest shelter from $j$ in $G^s$. 
2. for people who are not on \(s\), say on \(j \neq s\), the evacuation distance from \(j\) to shelter \(k\) corresponds to \(d^s(j,k)\) in graph \(G^s\), i.e., avoiding vertex \(s\).

This evaluation of evacuation distances makes our problem specific compared to the literature and induces some additional complexity. The justification of this measure for people escaping from \(s\) is twofold. First, since the area \(s\) may be relatively large, a single scenario may correspond to many possible fire configurations, each prohibiting some paths in the zone. The second motivation is to represent decision under stress, a very important characteristic in emergency management: somebody close to the fire may not take rational decisions when selecting a direction while people in another zone can be assumed to behave more rationally.

For a given set \(C \subset V\) seen as a set shelter’s locations and a given scenario \(s\), the evacuation distance of a zone \(j\) is denoted by \(r^s(C,j)\). If a shelter is located on \(j\), \(r^s(C,j) = 0\) otherwise we have:

\[
r^s(C,j) = \begin{cases} 
d^s(j,C) & \text{if } j \neq s \\
\max_{v \in N^+_{G^s}(s)} \{l_{sv} + d^s(v,C)\} & \text{otherwise}
\end{cases}
\]

where \(N^+_{G^s}(s)\) is the set of all vertices \(v\) such that \((s,v) \in E^s\).

Notice that \(r^s(C,j)\) is equal to \(+\infty\) if \(j\) can’t reach any shelter in \(G^s\).

The evacuation radius associated with scenario \(s\) is defined as \(r^s(C) = \max_{j \in V} r^s(C,j)\). Note that \(r^s(C)\) is not equal to the usual radius computed in \(G^s\): \(r^s(C) \geq \max_{v \in V} d^s(v,C)\).

**Example 1.** This example is adapted from [Demange et al. (2020)](http://example.com) and allows to better understand the evacuation radius \(r^s(C)\) and the operational graph.

![Figure 2: The operational graph associated with scenario 2 with \(C = \{3,10\}\)](http://example.com)

Let us consider \(p = 2\) and the non-directed version \(G = (V,E,L)\) of the graph in Figure 2. We consider the scenario \(s = 2\). The related operational mixed graph is given in
Figure 2. Vertices 3 and 10 represented by pentagons correspond to shelters’ locations (C = \{3, 10\}). In case of fire on vertex 2 (scenario 2), the modification of the graph and the evacuation strategy induce:

• The shortest path length from 1 to 3 is no longer 3 but 23, using the shortest path 1, 6, 7, 8, 3. Consequently, the nearest shelter from vertex 1 is 10 at a distance of 8. Thus the evacuation distance of 1 in scenario 2, is equal to 8 and vertex 1 is evacuated to vertex 10.

• To compute the evacuation distance of vertex 2 in scenario 2, we have to consider three neighbors:
  
  – for neighbor 1, the distance to the nearest shelter 10 is 1 + 8 = 9;
  
  – for neighbor 7, the distance to the nearest shelter 10 is 3 + 9 = 12;
  
  – for neighbor 3 with a shelter, the distance is 2.

Consequently, \( r^2(C, 2) = 12 \).

• The evacuation radius of the scenario 2 is given by \( r^2(C) = \max_{j=1, \ldots, 14} r^2(C, j) = r^2(C, 13) = 15 \).

We are now ready to define the problem \( \text{Min \ PpCP} \). A \( \text{Min \ PpCP} \)-instance will be an edge-weighted graph \( G \) and an integer \( p \) and a solution \( C \) will correspond to a set of at most \( p \) vertices where to locate shelters. A given solution \( C \) corresponds to \( n = |V| \) evacuation radius \( r^1(C), \ldots, r^n(C) \) for \( n \) different scenarios. We associate to \( C \) the expected value \( \mathbb{E}(C) \) of these evacuation radius over all scenarios:

\[
\mathbb{E}(C) = \frac{1}{|V|} \sum_{s \in V} r^s(C) = \frac{1}{|V|} \sum_{s \in V} \max_{j \in V} r^s(C, j)
\]  

\( \mathbb{E}(C) \) is called probabilistic radius. For any set \( C \) of centers, it can be computed in polynomial time: for each scenario it requires to compute the matrix of shortest path values in the related operational graph, which requires \( O(|V|^3) \) operations. So, \( \mathbb{E}(C) \) can be computed in \( O(|V|^3) \). The \( \text{Min \ PpCP} \) problem is then to determine a solution \( C^* \) minimizing \( E \).

We synthesize below the formal definition of the problem:
In a more general setting we could add a probability distribution on vertices but in this work we only consider the uniform probability distribution. In this context, recall that by uniform, we mean $L$ is the matrix $(\ell_{ij})$ with $\ell_{ij} = 1 \iff (i, j) \in E$ and $\ell_{ij} = \infty$ otherwise.

Note that in our definition, $p$ is part of the instance. We can define natural sub-problems by restricting the possible values for $p$. If $p$ is a fixed value, then the related sub-problem is polynomial since all possible $p$-centers can be enumerated in polynomial time and the probabilistic radius (objective value) of each one can be determined in polynomial time.

For a graph $G = (V, E)$ and a set $V' \subset V$, we will denote $G[V']$ the subgraph of $G$ induced by $V'$. $G[V']$ is called a subgraph of $G$. A partial graph of $G$ is a graph $(V, E')$ with $E' \subset E$ obtained from $G$ by deleting 0 or some edges. A partial subgraph of $G$ is a partial graph of a subgraph of $G$. For $U \subset V$, we denote $G \setminus U$ the graph $G[V \setminus U]$. A pending vertex in a graph is a vertex of degree 1. A $n \times m$ grid is the graph $G = (\mathcal{V}, \mathcal{E})$ with vertex set $\mathcal{V} = \{(i, j), i \in \{0, \ldots, n-1\}, j \in \{0, \ldots, m-1\}\}$ and $((i, j), (k, l)) \in \mathcal{E}$ if and only if $|i-k| + |j-l| = 1$. A (partial) subgrid is a (partial) subgraph of a grid. For instance, the graph in Figure 2 is a partial subgrid. Given a subgrid $G = (V, E)$, a grid embedding is a one-to-one function from $V$ to $\mathcal{V}$ for some dimensions $(n, m)$ such that every edge $(u, v) \in E$ maps to an edge of the $n \times m$ grid. If $u \in V$ maps to $(i, j)$ in the grid, $(i, j)$ are called the coordinates of $u$. Unless otherwise stated, each time we will refer to a subgrid, we will assume that a grid embedding is given. As defined in Demange and de Werra (2013), for a partial subgrid $G$ and a positive integer $f$, the $f$-expansion of $G$, denoted $\text{Exp}(G, f)$, is obtained from $G$ by inserting $f-1$ vertices on each edge (each edge becomes a path of $f$ edges). If $f \geq 2$, the $f$-expansion of any partial subgrid is a subgrid. If $G$ is a subgrid embedded in a $n \times m$ grid $G$, then $\text{Exp}(G, f)$ is a subgrid embedded in the $[(n-1)f+1] \times [(m-1)f+1]$ grid $\text{Exp}(G, f)$. The vertex set of $G$ can be seen as a subset of the vertex set of $\text{Exp}(G, f)$ and more precisely, in the related grid embedding of $\text{Exp}(G, f)$, the coordinates of any vertex $u \in V$ are multiplied by $f$ compared to its coordinates in the original grid embedding of $G$ in $G$. Subgrids, and to a lesser extent partial subgrids, constitute a natural class of instances in our motivating application. It corresponds to the case where the landscape is divided into square areas and some areas are not considered since they correspond for instance to natural barriers, like lakes, or to protected private lands that can neither been used for sheltering nor for evacuating.

**Instance:** An edge-weighted graph $G = (V, E, L)$ and an integer $p$; the instance is denoted $(G, p)$;

**Feasible solutions:** Any $p$-center $C \subset V, |C| \leq p$ satisfying $\mathcal{E}(C) < \infty$;

**Objective:** Minimizes $\mathcal{E}(C)$ defined in Relation 2.
2.2. Related work

In [Demange et al. (2018)], we propose an explicit solution for the uniform case (all edge lengths are 1) on paths and cycles. In these cases, a solution is characterized by the list of lengths of segments between two consecutive centers. A \( p \)-center is then called balanced if the maximum difference between two segment lengths is minimized and it is monotone if the sequence of segment lengths is monotone. It is straightforward to show that a balanced solution is optimal for the usual Min \( p \)-Center and, in [Demange et al. (2018)], we have shown that a monotone balanced solution is also optimal for Min \( p \)-\text{CP}. Even though the result is not surprising, the proof was surprisingly non-trivial. In [Demange et al. (2018)], we proposed as well some related hardness results. In particular, we showed that Min \( p \)-\text{CP} is not approximable on planar graphs of degree 2 or 3 within a ratio less than \( \frac{20}{19} \), unless P=NP. Refining this result in restricted classes of subgrids and designing approximation algorithms were proposed open questions.

In [Demange et al. (2020)], we investigated a variant, called robust, where the objective is to minimize the maximum (worst) evacuation radius over all scenarios instead of minimizing their expected value. For this version we proposed NP-hardness results in various classes of graphs that include subgrids. Our application motivates this class. We also proposed exact algorithms based on Integer Linear Programming formulation.

In the next subsection we characterize the set of feasible solutions of Min \( p \)-\text{CP}.

2.3. Feasible solutions

In this subsection we analyze necessary and sufficient conditions for a solution to be feasible for a given Min \( p \)-\text{CP}-instance \((G, p)\). Without loss of generality we will consider that \( G = (V, E, L) \) is a connected graph. A vertex \( a \in G \) is an articulation point if and only if removing \( a \) disconnects the graph \( G \). We denote by \( \mathcal{A}(G) \) the set of articulation points of \( G \).

We call articulation component of \( G \) associated with an articulation point \( a \) a connected component of \( G \setminus \{a\} \). Then every vertex \( a \in \mathcal{A}(G) \) is associated to at least 2 articulation components, and every articulation component is associated to one articulation point. A graph is 2-connected if it has no articulation point; in this case there is no articulation component.

A minimal articulation component, or MAC for short, is an articulation component that does not strictly contain another articulation component. We denote \( \Upsilon(G) \) the set of minimal articulation components. Note that an articulation component that is a singleton \( \{v\} \) is necessarily minimal and this occurs if and only if \( v \) is a vertex of degree 1.
Lemma 1. A is a minimal articulation component of $G$ if and only if $A$ is an articulation component which does not include an articulation point of $G$.

PROOF. ⇒ By contrapositive we prove that if an articulation component $A$ includes an articulation point, then $A$ is not minimal. Let $A$ be an articulation component induced by the articulation point $a \in V$. Suppose $b \in A$ is an articulation point of $G$. Then $b$ induces at least two disjoint connected components in $G \setminus \{b\}$. Since $b \neq a$, $a$ is in one connected component of $G \setminus \{b\}$, consequently $G \setminus A$ is a subset of this connected component. It follows that at least another component of $G \setminus \{b\}$ is contained in $A$, which means that $A$ is not minimal.

⇐ The proof is also by contrapositive. We prove that if $A$ is a non-minimal articulation component, then $A$ includes an articulation point. Let $A$ an articulation component that is not minimal. Then there is an articulation component $B \subset A$ induced by the articulation point $b \in V$, such that $B \neq A$. Consider $x \in A \setminus B$ and $y \in B$. Since $A$ is connected, $x$ and $y$ are connected in $A$ by a path; this path necessarily crosses $b$ and in particular $b \in A$. □

Lemma 2. All minimal articulation components of $G$ are pairwise disjo ints.

PROOF. By contrapositive, we assume $A \in \Upsilon(G)$ and $B$ an articulation component such that $B \neq A$ and $B \cap A \neq \emptyset$. We prove then that $B$ is not minimal.

Let $x \in A \cap B$. Since $A$ is a MAC, $B \not\subseteq A$. Then there is a vertex $y \in B \cap (V \setminus A)$. Every path between $x$ and $y$ in $G$ crosses $a$. As $B$ is a connected component, there is a path from $x$ to $y$ in $B$, thus $a \in B$, and $B$ is not a MAC by Lemma 1. □

Given an edge-weighted graph $G = (V, E, L)$ and $p$, we denote with $\mathcal{C}_p(G)$ the set of feasible solutions of the $\text{Min P}_p\text{CP}$-instance $(G, p)$.

Proposition 1. Let $(G, p)$ be an instance of $\text{Min P}_p\text{CP}$ with $|V| \geq 2$. A solution $C \subset V, |C| \leq p$ is in $\mathcal{C}_p(G)$ if and only if $|C| \geq 2$ and $C$ includes at least one vertex in each minimal articulation component of $G$.

PROOF. Suppose $C$ is a feasible solution for $\text{Min P}_p\text{CP}$ on $G$. We have seen that $C$ is a feasible solution for $\text{Min P}_p\text{CP}$ if and only if $r^i(C, j) \in \mathbb{R}, \forall j, s \in V$, i.e. all the evacuation distances over all vertices and all scenarios are finite.

First suppose there is no articulation point, then $G$ has no articulation components. Let $s \in C$, and $x \in V, x \neq s$. In scenario $s$, $x$ is assigned to a center that is not $s$. Thus $|C| \geq 2$. Conversely, if $|C| \geq 2$, for any scenario $s$, $G \setminus \{s\}$ is connected and contains at least one center.

Second, suppose $G$ has at least one articulation point and consequently at least 2 disjoint
articulation components. In addition, if \( A \) is an articulation component of \( G \) induced by the articulation point \( a \), then \( \forall j \in A, r^a(C, j) \in \mathbb{R} \) if and only if \( C \cap A \neq \emptyset \). Then \( C \) intersects all articulation components. In particular \( |C| \geq 2 \) and \( C \) intersects all minimal ones. Conversely, if \( C \) intersects all MACs then \( |C| \geq 2 \) and it intersects all articulation components since any articulation component contains a MAC.

\[ \square \]

Remark 1. Feasibility is weight-independent.

Corollary 1. If \( G \) has at least 2 vertices, \( \mathcal{C}_1(G) = \emptyset \).

As a consequence, from now we will consider only \( \min PpCP \) instances satisfying \( p \geq 2 \).

Corollary 2. For a given \( p \), we can verify in polynomial time whether \( \mathcal{C}_p(G) \neq \emptyset \).

PROOF. For \( G = (V, E) \), we generate \( \mathcal{A}(G) \) in \( O(|V| + |E|) \) using Tarjan’s Algorithm [Tarjan (1972)]. The minimal connected components of \( G \) are the connected components of \( G \setminus \mathcal{A}(G) \) adjacent to at most one articulation point in \( G \), where a set \( V' \) of vertices is said adjacent to a vertex if this vertex has at least one neighbor in \( V' \).

There is a feasible solution for \( \min PpCP \) on \( G \) if \( p \) is greater or equal to the number of MACs.

Corollary 3. For all \( C \in \mathcal{C}_p(G) \), \( C \) necessarily includes all vertices of degree 1.

PROOF. Every vertex of degree 1 is a MAC of \( G \). Then by Proposition [1] a feasible solution includes all vertices of degree 1.

\[ \square \]

3. Hardness result

In all this section we assume that all edge lengths are 1 (uniform case). We remind that \( \min p\text{-Center} \) is NP-hard for \( p \geq 2 \) on planar graphs of maximum degree at least 3 (Kariv and Hakimi (1979)). This result does not immediately imply the hardness of \( \min PpCP \). Indeed, we defined our model with fixed uniform probabilities, which does not count the classic deterministic \( \min p\text{-Center} \) problem as one of its specific cases.

In Demange et al. (2018), we showed that \( \min PpCP \) cannot be approximated on planar graphs of degree 2 or 3 with a ratio less than \( \frac{20}{19} \). In this section, we prove that it cannot be approximated with a ratio less than \( \frac{56}{55} \) on a restricted subclass of bipartite planar graphs, the class of subgrids with degree at most 3.

The proof uses two classical optimization problems in graphs. A dominating set in a graph \( G = (V, E) \) is a subset \( U \) of \( V \) such that every vertex not in \( U \) is adjacent to at
least one member of $U$. The Min Dominating Set problem is to find a dominating set of minimum size. We will denote by $\gamma(G)$ the minimum size of a dominating set in $G$. The Min Dominating Set problem is shown NP-hard on subgrids in [Clark et al. (1990)]. Note the relation between Min Dominating Set and the deterministic Min $p$-Center problem: for a graph $G = (V, E), U \subseteq V$ is a dominating set, if and only if $U$ is a $|U|$-center of radius 1.

A vertex cover of a graph is a set of vertices such that each edge of the graph is incident to at least one vertex of the set. The Min Vertex Cover problem is to find a vertex cover of minimum size. We will denote by $\tau(G)$ the minimum size of a vertex cover in $G$. In Kariv and Hakimi (1979), Min Vertex Cover is shown NP-hard on planar graphs of maximum degree 3. It is straightforward (see Lemma 8 proved later) that it remains NP-hard on planar graphs with vertices 2 or 3. In Lemma 3, we show that, for a graph $G'$ obtained from $G$ by inserting an even number of vertices on each edge of $G$, we can ensure a direct relationship between $\tau(G')$ and $\tau(G)$.

We first establish a technical lemma about vertex cover in graphs that is independent to our reduction:

**Lemma 3.** Let $G = (V, E)$ be a graph and $G' = (V', E')$ be the graph obtained by inserting $2k_{uv}$ vertices on each edge $(u, v) \in E$, where $k_{uv}$ is a non-negative integer. Then we have

$$\tau(G') = \tau(G) + \sum_{uv \in E} k_{uv}$$

**Proof.** For every edge $(u, v) \in E$ oriented from $u$ to $v$, denote $X_{uv} = \{x_{uv}^1, \ldots, x_{uv}^{2k_{uv}}\}$ the set of vertices inserted on this edge. Note that at least $k_{uv}$ vertices are needed to cover vertices in $X_{uv}$.

Let $U \subseteq V$ a vertex cover of $G$: $\forall (u, v) \in E, \{u, v\} \cap U \neq \emptyset$. We can build $U' \subseteq V'$ in $G'$ as follows. We initialize $U'$ with all vertices of $U$. Then, for every edge $(u, v) \in E$, if $u \in U$, we add vertices $x_{uv}^{2i}, 1 \leq i \leq k_{uv}$ to $U'$. Otherwise, if $v \in U$, we add vertices $x_{uv}^{2i+1}, 0 \leq i \leq k_{uv}-1$ to $U'$. In both cases we have added exactly $k_{uv}$ vertices and all edges of $P_{uv}^{G'}$ are covered by $U'$, with $|U'| = |U| + \sum_{uv \in E} k_{uv}$. Then $\tau(G') \leq \tau(G) + k_{uv}$.

Assume now that $G'$ has a vertex cover $X'$. For every $(u, v) \in E$, $P_{uv}^{G'}$ is covered by at least $k_{uv} + 1$ vertices. If $u, v \notin X'$, we can transform $X'$ into $U'$ such that $u$ or $v$ is in $U'$. Then $|U'| \leq |U| + \sum_{uv \in E} k_{uv}$. Since at least one vertex between $u$ and $v$ is in $U = V \cap U'$, $U$ is a vertex cover for $G$. Then $|U| = |U'| - k_{uv}$, thus $\tau(G) \leq \tau(G') - k_{uv}$.

Hence $\tau(G') = \tau(G) + k_{uv}$ and the proof is complete. $\square$
The remaining of the section is dedicated to prove Theorem 1. In Subsection 3.1 we explain the general scheme of the demonstration before giving all details in Subsections 3.2, 3.3 and 3.4.

3.1. Global blueprint of the proof

In Theorem 1 we will show that a polynomial time approximation algorithm $A$ for $\text{Min } \text{PpCP}$ in subgrid of degrees $\{2, 3\}$ ($p$ being part of the instance) guaranteeing a ratio of at most $\frac{56}{55}$ could be used to compute in polynomial time the size of the minimum vertex cover on a planar graph of degrees $\{2, 3\}$, which is a contradiction.

We start from a planar graph $G = (V, E)$ with degrees $\{2, 3\}$, instance of $\text{Min Vertex Cover}$. We randomly choose an orientation of the edges of $G$ that will be used in our reductions and analysis. We then apply successively two transformations, Transformation 1 denoted $\phi_1$ and Transformation 2 denoted $\phi_2$ that are detailed in Subsection 3.2. Figure 3 gives a simple schematic representation of the whole reduction.

![Figure 3: The different graphs involved in the reduction.](image)

Transformation 1 ($\phi_1$) constructs from $G$ a subgrid $H_q = (V^{H_q}, E^{H_q})$, for some positive integer $q$ specified later, in such a way that:

- $V \subset V^{H_q}$,
- Edges $(u, v)$ of $G$ map to non-crossing paths $P_{uv}^{H_q}$ of even length between $u$ and $v$ in $H_q$.

The subgrid $H$ appearing in Figure 3 is an intermediate stage not directly used in the analysis.
We then apply Transformation 2 ($\phi_2$) to construct a subgrid $F = (V^F, E^F)$ from $H_q$. Roughly speaking, it consists in replacing the first two edges of $P_{uv}^{H_q}$ (where $uv \in E$ is oriented from $u$ to $v$) with a gadget $T^2$, and every other edge of $H_q$ with a gadget $T^1$, both defined in the next subsection.

For the analysis now, we note that there is no direct and easy link between $\tau(G)$ and $\tau(H_q)$ since $\tau(H_q)$ can be obtained in polynomial time ($H_q$ is bipartite) while $G$ is meant to be an instance of an NP-hard restriction of $\text{Min Vertex Cover}$. For this reason, we introduce an auxiliary graph $\tilde{H}_q = (V_{\tilde{H}_q}, E_{\tilde{H}_q})$. It can be seen as a perturbation of $H_q$ with a direct link between $\tau(\tilde{H}_q)$ and $\tau(G)$. It is simply obtained by replacing, for every edge $(u, v) \in E$, the two first edges of the path $P_{uv}^{H_q}$ by a single edge. This way, the path $P_{uv}^{H_q}$ of even length becomes, in $\tilde{H}_q$, a path $P_{uv}^{\tilde{H}_q}$ of odd length and Lemma 3 can be used to write $\tau(\tilde{H}_q)$ as a function of $\tau(G)$.

On the other hand, as outlined in Lemma 4, the properties of the two gadgets allow to establish a direct link between dominating sets in $F$ and vertex covers in $\tilde{H}_q$. In all, it gives a relation between the $\text{Min Dominating Set}$ problem in $F$ and the $\text{Min Vertex Cover}$ problem in $G$.

Then, in Subsection 3.3 we outline different relations between the $\text{Min Dominating Set}$ problem and $\text{Min PP} \text{CP}$ in a triangle-free graphs without pending vertices using three lemmas, Lemma 5, Lemma 6 and Lemma 7. This can be applied to $F$.

Finally, in Subsection 3.4, we use these results to establish Theorem 1. We show that, when applying A on $F$ for $p < \gamma(F)$, the output is a solution of $\text{Min PP} \text{CP}$ of probabilistic radius at least 2, while applying it for $p = \gamma(F)$ gives a solution of probabilistic radius less than 2. Hereby we can use such an algorithm to compute $\gamma(F)$, and consequently $\tau(G)$. Since constructing $H$, $H_q$, $\tilde{H}_q$ and $F$, as well as evaluating the value of a $\text{Min PP} \text{CP}$ solution, can be done in polynomial time, and since algorithm A is applied less than $|V|$ times, the whole process is polynomial.

3.2. Details on the transformations and their properties

**Transformation 1.** From a planar graph $G = (V,E)$ to a subgrid $H_q = (V^{H_q}, E^{H_q})$ with $q > 0$.

Using a result of Yanpei et al. (1991), we can embed $G = (V,E)$ in a grid $H = (V^H, E^H)$ of polynomial size. Vertices of $G$ are mapped to vertices of the grid, and edges $(u,v)$ of $G$ map to non-crossing paths $P_{uv}^H$ between $u$ and $v$ in the grid. Note that we cannot control the length and parity of these paths. The resulting graph is a partial subgrid and not necessarily
a subgrid yet. We then perform a $2q$-expansion for some positive integer $q$ specified later. The resulting graph $H_q = (V_{Hq}, E_{Hq})$ is a subgrid ($q > 0$). In addition, since the expansion multiplies by $2q$ all path lengths from $H$ to $H_q$, edges $(u, v)$ of $G$ map to non-crossing paths $P_{uv}^{H_q}$ of even length between $u$ and $v$ in $H_q$. It means that paths $P_{uv}^{H_q}$ have $2k_{uv} + 1$ internal vertices (excluding $u$ and $v$) for some non-negative integers $k_{uv}$.

**Example 2.** Suppose the planar graph $G = (V, E)$ is a complete graph on four vertices \{a, b, c, d\} as presented in Figure 4 and set $q = 2$. We choose an orientation of $G$ such that the oriented edges of $G$ are \{(a, b), (a, c), (b, c), (c, d)\}. $H = (V^H, E^H)$ corresponds to a possible embedding of $G$ in a grid, where the edge $(a, d) \in E$ maps to the path $\{a, z_{ad}, d\}$ in $H$. Next, we construct the subgrid $H_q$ by applying the $2q$-expansion. The resulting graph $H_q$ can be seen on the right side of Figure 4. Finally, the related graph $\tilde{H}_q$ is represented in Figure 5.

![Planar graph G](image1)

$H_q$ for $q = 2$ embedded in a 9x9 grid.

![Figure 4: Example of Transformation 1](image2)
As already noticed in Subsection 3.1, we cannot establish a direct link between \( \tau(G) \) and \( \tau(H_q) \) but since we now control the parity of paths \( P_{uv}^{H_q} \), it is easy to slightly modify \( H_q \) so as we can apply Lemma 3. This is the role of the graph \( \tilde{H}_q = (V_{\tilde{H}_q}, E_{\tilde{H}_q}) \). Recall that this graph is obtained from \( H_q \) by replacing, for every edge \( (u,v) \in E \), the two first edges of the path \( P_{uv}^{H_q} \) by a single edge, as illustrated in Figure 5. This way, \( \tilde{H}_q \) can directly be obtained from \( G \) by inserting \( 2k_{uv} \) vertices on each edge \( (u,v) \in E \). As a consequence, Lemma 3 allows to establish:

\[
\tau(\tilde{H}_q) = \tau(G) + \sum_{(u,v) \in E} k_{uv}. \tag{3}
\]

In addition, we have:

\[
|V_{\tilde{H}_q}| = |V| + 2 \sum_{e \in E} k_e \\
|E_{\tilde{H}_q}| = |E| + 2 \sum_{e \in E} k_e \tag{4}
\]

By construction, we have \( \forall (u,v) \in E, 2k_{uv} + 1 \geq 2q - 1 \), which gives:

\[
\forall (u,v) \in E, k_{uv} \geq q - 1. \tag{5}
\]

**Transformation 2.** From subgrid \( H_q = (V^{H_q}, E^{H_q}) \) to subgrid \( F = (V^F, E^F) \).

Thanks to the \( 2q \)-expansion, for \( (u,v) \in E \) oriented from \( u \) to \( v \), the first two edges of \( P_{uv}^{H_q} \) in \( H_q \) are both horizontal or vertical. Note as well that the orientation of \( G \) immediately defines an orientation of \( H_q \) and of \( \tilde{H}_q \). We can then construct the subgrid \( F = (V^F, E^F) \) from the subgrid \( H_q \) as follows.

For every edge \( (u,v) \in E \) oriented from \( u \) to \( v \), we replace, in \( H_q \), the first two edges \( (u,i), (i,x) \) of \( P_{uv}^{H_q} \) with \( T_2^{2x} \) defined in Figure 7 and every other edges \( (x,y) \in E^{H_q} \) with \( T_1^{1xy} \) defined in Figure 6.

In the following we use \( T_{xy} \) to refer to \( T_1^{1xy} \) or \( T_2^{2xy} \). Note that two gadgets \( T_{xy} \) never overlap each other in \( F \) and the resulting graph \( F \) is a subgrid. Indeed, if \( G \) is embedded in a grid \( G, H_q \) is embedded in \( Exp(G, 2q) \) and \( F \) is embedded in \( Exp(G, 14q) \).

By construction we have \( |V^F| = |V_{\tilde{H}_q}| + 13|E_{\tilde{H}_q}| + 3|E| \) and \( |E^F| = 15|E_{\tilde{H}_q}| + 3|E| \).
Figure 5: The graph \( \tilde{H}_q \) obtained from \( G \) through \( H_q \).

\[
\begin{align*}
  k_{ab} &= k_{bc} = k_{bd} = 1,
  k_{ad} &= k_{cb} = 3 \quad \text{and} \quad k_{ac} = 7
\end{align*}
\]

Using Relation 4 we deduce:

\[
\begin{align*}
  |V^F| &= |V| + 16|E| + 28 \sum_{e \in E} k_e \\
  |E^F| &= 18|E| + 30 \sum_{e \in E} k_e
\end{align*}
\]

(6)

**Lemma 4.** For any \( t \leq |V| \), \( \tilde{H}_q = (V^{\tilde{H}_q}, E^{\tilde{H}_q}) \) has a vertex cover of size \( t \) if and only if \( F \) has a dominating set \( D \) of size \( t + 4|E^{\tilde{H}_q}| + |E| \) such that, for each edge \( (x, y) \in E^{\tilde{H}_q} \), we have:

- at least one vertex of \( \{a_{xy}, c_{xy}\} \) is in \( D \)
- at least one vertex of \( \{z_{xy}^1, z_{xy}^{13}\} \) if \( (x, y) \) is the first edge of a path \( P_{uv}^{\tilde{H}_q} \) with \( (u, v) \in E \) oriented from \( u \) to \( v \)
- at least one vertex in \( \{z_{xy}^1, z_{xy}^6\} \) in the other cases.
PROOF. For this result, it is convenient to see how \( F \) could be constructed from \( \tilde{H}_q \): for every edge \((u, v) \in E\) oriented from \(u\) to \(v\), the first edge of \(P_{uv}^{\tilde{H}_q}\) we denote \(E_2^{\tilde{H}_q}\) the set of such edges corresponding to two edges of \(P_{uv}^H\) is replaced with \(T_{uv}^2\). All other edges of \(\tilde{H}_q\) we denote \(E_1^{\tilde{H}_q} \subset E^{\tilde{H}_q}\) their set - are replaced with \(T_{uv}^1\). Note that \(|E_2^{\tilde{H}_q}| = |E|\).

⇒ Let \(U \subset V^{\tilde{H}_q}\) be a vertex cover of \(\tilde{H}_q\) of size \(t\). We initialize \(D\) with all vertices of \(U\), seen as a subset of \(V^F\), and complete it in a dominating set of \(F\). Then for every \((x, y) \in E^{\tilde{H}_q}\), oriented from \(x\) to \(y\), we have \(D \cap \{x, y\} \neq \emptyset\). We then apply one of the two following cases:

- if \((x, y) \in E_2^{\tilde{H}_q}\): If \(x \in D\), we add to \(D\) the vertices \(z_3, c_{xy}, z_4, z_10, z_13\) of \(T_{xy}^2\), else if \(y \in D\), we add to \(D\) the vertices \(a_{xy}, z_1, z_8, z_11\) of \(T_{xy}^2\). In both cases, 5 vertices are added to \(D\), and all the vertices of \(T_{xy}^2\) are dominated by \(D\).

- if \((x, y) \in E_1^{\tilde{H}_q}\): If \(x \in D\), we add to \(D\) the vertices \(z_3, c_{xy}, z_4, z_10\) and \(z_6, z_8\) of \(T_{xy}^1\), else if \(y \in D\), we add to \(D\) the vertices \(z_1, a_{xy}, z_4, z_11\) of \(T_{xy}^1\). In both cases, 4 vertices are added to \(D\), and all the vertices of \(T_{xy}^1\) are dominated by \(D\).
The resulting set $D$ is a dominating set of $F$ of size $t + 4|E\overline{H}_q| + |E_2\overline{H}_q| = t + 4|E\overline{H}_q| + |E|$ and for each edge $(x,y) \in E\overline{H}_q$, $D$ has at least one vertex in $\{a_{xy}, c_{xy}\}$ and one vertex in $\{z_{xy}^3, z_{xy}^{10}\}$ (resp. $\{z_{xy}^3, z_{xy}^{13}\}$) if $(x,y) \in E_2\overline{H}_q$ (resp. $E_1\overline{H}_q$).

$\Leftarrow$ Now suppose we have $D$ a dominating set of $F$. Then for every $(x,y) \in E\overline{H}_q$ oriented from $x$ to $y$, we have:

- if $(x,y) \in E_2\overline{H}_q$: $D$ includes at least 6 vertices on $T_{xy}^2$, and 5 vertices on $T_{xy}^2 \setminus \{x,y\}$.
- if $(x,y) \in E_1\overline{H}_q$: $D$ includes at least 5 vertices on $T_{xy}^1$, and 4 vertices on $T_{xy}^1 \setminus \{x,y\}$.

Then $D$ includes at least $t' + 4|E\overline{H}_q| + |E_2\overline{H}_q| = t' + 4|E\overline{H}_q| + |E|$ vertices for some integer $t'$. We then perform the following modifications on $D$:

- for every $(x,y) \in E_2\overline{H}_q$ oriented from $x$ to $y$: if $x \in D$, we can replace at least 5 vertices of $D \cap T_{xy}^2$ by $z_{xy}^3, c_{xy}, z_{xy}^7, z_{xy}^{10}$ and $z_{xy}^{13}$. If $y \in D$, we can replace at least 5 vertices of $D \cap T_{xy}^2$ by $z_{xy}^1, a_{xy}, z_{xy}^6, z_{xy}^8$ and $z_{xy}^{11}$.
- for every $(x,y) \in E_1\overline{H}_q$ oriented from $x$ to $y$: if $x \in D$, we replace at least 4 vertices of $D \cap T_{xy}^1$ by $z_{xy}^3, c_{xy}, z_{xy}^{4} and z_{xy}^{3}$. If $y \in D$, we replace at least 4 vertices of $D \cap T_{xy}^1$ by $z_{xy}^1, a_{xy}, z_{xy}^{4} and z_{xy}^{4}$. If neither $x$ nor $y$ is in $D$, we can induce that $|D \cap T_{xy}^1| \geq 5$.

Thus, we replace at least 5 vertices of $D \cap T_{xy}^1$ by $x, z_{xy}^3, c_{xy}, z_{xy}^{4} and z_{xy}^{3}$.

Note that none of these modifications increases the size of $D$, and $D$ is still a dominating set of $F$. However, we ensured that $|D \setminus \overline{H}_q| \geq 4|E| + |E|$, and $|D \cap \{x,y\}| \geq 1, \forall (x,y) \in E\overline{H}_q$.

Then $U = D \setminus \overline{H}_q$ is a vertex cover for $\overline{H}_q$ of size at least $t$. This completes the proof.

\[\square\]

3.3. Relations between $\text{Min} \: PpCP$ and dominating sets

**Lemma 5.** If $D \subseteq V$ is a dominating set of a triangle-free graph $G = (V,E)$ with degrees $\{2,3\}$, then $E(D) \leq 2$.

**Proof.** For any $v \in D$, we recall that $r_s'(D,v) = 0$ for all scenarios $s$. For any $v \in V \setminus D$, $v$ is at a distance 1 of a vertex of the dominating set $D$. Then any neighbor of $v$ is either in $D$ or at a distance 1 of a vertex of $D$ that is not adjacent to $v$, as $G$ is triangle free. Therefore

18
the evacuation distance of \( v \) in any scenario cannot exceed 2. Thus \( r^s(D, v) \leq 2 \) for all scenarios \( s \) and

\[
\mathbb{E}(D) = \frac{\sum_{s \in V} r^s(D)}{|V|} = \frac{\sum_{s \in V} \max_{v \in V} (r^s(D, v))}{|V|} \leq 2
\]

\( \square \)

**Remark 2.** \( D \) intersects all articulation components.

**Lemma 6.** For \( G = (V, E) \) a graph with degrees \( \{2, 3\} \) and \( p < \gamma(G) \), the minimum expected value of the evacuation radius over all scenarios of any solution of \( \min PpCP \) is greater than 2.

**proof.** Let \( C \) be a solution of \( \min PpCP \) on \( G \) for \( p < \gamma(G) \). As \( C \) cannot be a dominating set, there exists \( v \in V \) such that \( \{v\} \cup N(v) \cap C = \emptyset \), i.e. \( v \) is not adjacent to any vertex of \( C \). For any scenario \( s \), the evacuation distance of \( v \) will be at least 2 as none of its neighboring vertices is in \( C \). Thus \( r^s(C, v) \geq 2, \forall s \in V \), which implies \( r^s(C) \geq 2, \forall s \in V \). In addition, for any vertex \( y \in N(v) \), the evacuation distance of \( y \) in scenario \( y \) is at least 3 since \( y \) has an evacuation path that crosses \( v \). Since \( r^y(C, y) \geq 3 \) and \( r^y(C) \geq 3 \), it follows that \( \mathbb{E}(C) > 2 \).

\( \square \)

**Lemma 7.** Let \( D \) be a minimum dominating set of \( F \) as described in Lemma 4 and of size \( p_t \). \( D \) is a solution of \( \min PpCP \) for \( p = p_t \) of value strictly less than 2.

**Proof.** Note that \( |D| = p_t = \tau(G) + 4|E_{\tilde{H}_q}| + |E_{\tilde{H}_q}^2| \) as shown in Lemma 4. Using Remark 2 and since \( F \) is triangle-free (it is a subgrid), \( D \) can then be seen as a feasible solution for \( \min PpCP \) and \( p = p_t \) in the graph \( F \). We claim the following relation that immediately concludes the proof:

\[
r^s(D) = \begin{cases} 
1 & \text{if } s \in V_{\tilde{H}_q} \subset V_F \text{ and } s \notin D \\
2 & \text{otherwise}
\end{cases}
\]

We recall that every vertex of \( \tilde{H}_q \) maps a vertex in \( F \) by construction, thus we consider \( V_{\tilde{H}_q} \subset V_F \) in the following. Since \( F \) is triangle-free with no pending vertex, and \( D \) is a dominating set, then we have by Lemma 5 \( r^s(D, v) \leq 2, \forall s, v \in V_F \).

Three cases emerge:
1. $s \in V^F \setminus V^\tilde{H}_q$: Denote $(x, y) \in E^\tilde{H}_q$ such that $s \in T_{xy}$. As $D$ is a minimal dominating set of $F$, $D$ is build as the resulting dominating set described in Lemma 4. It follows that there is at least one evacuation distance of length 2 for any scenario $s \in V^F \setminus V^\tilde{H}_q$, i.e. $r^s(D) = 2$.

In the following, $s \in V^\tilde{H}_q$ and we denote by $u_1, \ldots, u_d \in V^\tilde{H}_q$ the neighbors of $s$ in $\tilde{H}_q$.

2. $s \in V^\tilde{H}_q \cap D$: Since $D$ is minimal, $D \cap V^\tilde{H}_q$ is a minimal vertex cover of $\tilde{H}_q$, thus there is at least one neighbor $u \in \{u_1, \ldots, u_d\}$ of $s$ in $\tilde{H}_q$ that is not included in $D$. By construction, $z_{st}^1, z_{st}^2 \notin D$ and $z_{st}^3 \in D$. Then under scenario $s$, the evacuation distance of $z_{st}^1$ is 2, i.e. $r^s(D, z_{st}^1) = 2$. Under scenario $s$, the evacuation distance of any other vertex in $T_{st}$ is less than 2 given that $D$ is a minimal dominating set. For any other neighbor $u' \in \{u_1, \ldots, u_d\}$ of $s$ in $\tilde{H}_q$ ($u' \neq u$), we have $|\{z_{st}^1, z_{st}^2, z_{st}^3\} \cap D| = 1$, and $D$ a minimal dominating set on $T_{st'}$, thus the evacuation distance of any vertex in $T_{st'}$ is at most 2. Therefore $r^s(D) = 2$.

3. $s \in V^\tilde{H}_q \setminus D$: We recall that by definition $D \cap V^\tilde{H}_q$ is a minimal vertex cover of $\tilde{H}_q$, then $\{u_1, \ldots, u_d\} \subset D$. In addition, for any edge $(s, u) \in E^\tilde{H}_q$ oriented from $s$ to $u$, $D$ includes by construction $z_{st}^1$. Then every neighbor of $s$ in $F$ is included in $D$ by construction. Therefore, $r^s(D, s) = 1$. Since $D$ is a dominating set in $F$, it remains a dominating set in $F \setminus \{s\}$, which guarantees $r^s(D, v) = 1, \forall v \in V^F \setminus \{s\}$. Thus $r^s(D) = 1$.

So, in all cases except the last one, $r^s(U) = 2$, and the proof is complete.

\[ \square \]

We now are ready to prove the main result of this section.

### 3.4. The theorem

The next lemma is certainly a known remark but we show it since we did not find any reference for it.

**Lemma 8.** The **Min Vertex Cover** problem is NP-hard in planar graphs with vertices of degree 2 or 3.

**Proof.** The decision version of **Min Vertex Cover** is known to be NP-complete on planar graphs of maximum degree 3 (see Kariv and Hakimi (1979)). Consider a planar graph $G$ of maximum degree 3 and with a pending vertex $v$. Consider the graph $G'$ obtained
from $G$ by adding a triangle and linking one of its vertices with $v$ ($v$ is then of degree 2 in $G'$). $G'$ is planar with maximum degree 3 and one pending vertex less than $G$. Moreover, $G$ has a vertex cover of size $k$ if and only if $G'$ has a minimum vertex cover of size $k + 2$, which concludes the proof.

\[ \square \]

**Theorem 1.** If $P \neq NP$, there is no polynomial time approximation for $\min \ PpCP$ guaranteeing a ratio less than $\frac{56}{55}$ for subgrids with vertex degrees 2 or 3, even in the uniform case (all edge lengths are 1).

**Proof.** The proof is by contradiction. Let us suppose there is a polynomial approximation algorithm $A$ for uniform $\min \ PpCP$ which guarantees the approximation ratio $\rho$ satisfying $1 < \rho < \frac{56}{55}$, on subgrids with vertex degrees 2 or 3 for a parameter $p$. We will show how to use this algorithm to solve the $\min \ Vertex \ Cover$ problem on planar graphs. Lemma 8 gives the contradiction, unless $P=NP$.

Suppose $\varepsilon > 0$ such that $\rho < \frac{56 + 2\varepsilon}{55 + 2\varepsilon} < \frac{56}{55}$. Take an integer $q \geq 2$ such that $\varepsilon \geq \frac{17}{q-1}$.

Consider a planar graph $G = (V, E)$, instance of $\min \ Vertex \ Cover$. Consider the graph $H_q$ obtained by Transformation 1, as well as $\tilde{H}_q = (V, E_{\tilde{H}_q})$ and the vector $\{k_e : e \in E\}$ obtained through $H_q$. In addition, consider the graph $F = (V, E_F)$ obtained from $H_q$ through Transformation 2.

Recall that, from Relations 4 and 6, we have $|V_{\tilde{H}_q}| = |V| + 2 \sum_{e \in E} k_e$, $|E_{\tilde{H}_q}| = |E| + 2 \sum_{e \in E} k_e$, and $|V_F| = |V| + 16|E| + 28 \sum_{e \in E} k_e$.

We also deduce from Lemma 4:

\[
\gamma(F) = \tau(\tilde{H}_q) + 4|E_{\tilde{H}_q}| + |E| = \tau(G) + 5|E| + 9 \sum_{e \in E} k_e \tag{7}
\]

We apply the hypothetical approximation algorithm $A$ on $F$ for different values of $p$, starting with $p = 2$ and augmenting it. Suppose first we use $p < \gamma(F)$ and the algorithm computes a solution $C$. Then $\mathbb{E}(C) \geq 2$ as proven in Lemma 6. Suppose now we set $p = \gamma(F) = \tau(G) + 5|E| + 9 \sum_{e \in E} k_e$. Given Lemma 7 we obtain the following:

\[
\mathbb{E}(C) = \frac{(|V_{\tilde{H}_q}| - \tau(\tilde{H}_q)) + 2(|V_F| - (|V_{\tilde{H}_q}| - \tau(\tilde{H}_q)))}{|V_F|} = 2\frac{|V_F| - (|V_{\tilde{H}_q}| - \tau(\tilde{H}_q))}{|V_F|}
\]

21
We deduce, using Relations 3, 6 and 7:

\[ |V^F|E(C) = |V| + 32|E| + 55 \sum_{e \in E} k_e + \tau(G) \]

\[ < 2|V| + 32|E| + 55 \sum_{e \in E} k_e \]

where the last inequality holds because \( \tau(G) < |V| \). So, we have:

\[ \mathbb{E}(C) < \frac{2|V| + 32|E| + 55 \sum_{e \in E} k_e}{|V| + 16|E| + 28 \sum_{e \in E} k_e} = 2 - \frac{\sum_{e \in E} k_e}{|V| + 16|E| + 28 \sum_{e \in E} k_e} \]

Using Equation 5, we have \( \sum_{e \in E} k_e \geq (q - 1)|E| \). In addition, since \( G \) is of degree 2 or 3, we have \( |V| \leq |E| \). It follows:

\[ \mathbb{E}(C) < 2 - \frac{\sum_{e \in E} k_e}{17|E| + 28 \sum_{e \in E} k_e} \leq 2 - \frac{1}{28 + \frac{17|E|}{\sum_{e \in E} k_e}} \leq 2 - \frac{1}{28 + \frac{17}{q-1}} \]

As \( \varepsilon \geq \frac{17}{q-1} \) we get:

\[ \mathbb{E}(C) \leq 2 - \frac{1}{28 + \varepsilon} \leq \frac{55 + 2\varepsilon}{28 + \varepsilon} \]

As a consequence, and since an optimal probabilistic solution \( C^* \) will satisfy \( \mathbb{E}(C^*) \leq \mathbb{E}(C) \leq \frac{55 + 2\varepsilon}{28 + \varepsilon} \), the approximation algorithm \( A \) will determine an approximated solution \( C \) in \( F \) of value:

\[ \mathbb{E}(C) \leq \rho \times \mathbb{E}(C^*) \leq \rho \times \frac{55 + 2\varepsilon}{28 + \varepsilon} \leq \frac{55 + 2\varepsilon}{28 + 2\varepsilon} \times \frac{55 + 2\varepsilon}{28 + \varepsilon} < 2 \]  (8)

Note that, given a solution \( C \), computing its probabilistic radius can be done in polynomial time. Indeed, for any \( v,s \in V^F \), computing \( r^s(C,v) \) can be performed using any minimum path algorithm. Hence, we can apply successively the approximation algorithm \( A \) on the graph \( F \) for increasing values of \( p \), starting with \( p = 2 \), until the computed solution \( C \)
satisfies $E(C) < 2$. Thanks to Lemma\[6 and Equation\[8 the algorithm stops for $p = \gamma(F) = \tau(G) + 5|E| + 9 \sum_{e \in E} k_e$. Using Equations\[7 we can deduce $\tau(G) = p - 5|E| - 9 \sum_{e \in E} k_e$.

Since constructing $\tilde{H}_q$ and $F$, as well as evaluating $\bar{E}(C)$, can be done in polynomial time, and since algorithm $A$ will be run less than $|V|$ times, the whole process is polynomial. This is a contradiction if $P \neq NP$, and the proof is complete. \[228□\]

4. Approximation results in the uniform case.

We will show that, in graphs of bounded average degree, there is a polynomial approximation algorithm guaranteeing a constant approximation ratio for the uniform $\text{Min } PpCP$ (i.e., with all edge lengths equal to 1). Our result is even valid if edge lengths lie into $[l, 2l]$ for a positive $l$.

Our strategy is to show that, under these assumptions, the ratio $\frac{\bar{E}(C)}{r(C)}$ is bounded for any $p$-center $C$ that is feasible for $\text{Min } PpCP$. In particular, a solution with constant approximation ratio for $\text{Min } p$-Center has a constant ratio for $\text{Min } PpCP$.

Note that in graphs with general lengths we cannot expect the same and thus, another strategy should be taken. Indeed, consider the caterpillar $H$ of Figure\[8 with three internal vertices $x, y, z$ and edges $(x, y)$ and $(y, z)$ of length $Z$ and three pendent vertices $a, b, c$, respectively linked to $x, y, z$ with edges of length 1.

Figure 8: A case where $\frac{\bar{E}(C)}{r(C)} = Z + 1$.

$\{a, b, c\}$ is the unique feasible solution of the $\text{Min } PpCP$-instance $(H, 3)$. We have $r(\{a, b, c\}) = 1$. However, for any scenario $s$, $r'(\{a, b, c\}) = Z + 1$, which implies $E(\{a, b, c\}) = Z + 1$.

Given an edge-weighted graph $G = (V, E, L)$, recall that $\mathcal{C}_p(G)$ denotes the set of feasible solutions of the $\text{Min } PpCP$-instance $(G, p)$. From Proposition\[1 a set $C \subset V$ is in $\mathcal{C}_p(G)$ if and only if $|C| \geq 2$ and $C$ intersects all MACs. For any $p \leq |V|$, we call MAC $p$-center a $p$-center intersecting all MACs. For $p \geq 2$, $\mathcal{C}_p(G)$ is the set of MAC $p$-centers.
For any set $C \subseteq V$ of centers, recall that the radius of $C$ is $r(C) = \max_{v \in V} d(v, C)$. Note that for any scenario $s \in V$, $r^s(C) \geq r(C)$. We consider the $\text{Min MAC}\ p$-Center problem of finding a MAC $p$-center of minimum radius. The $\text{Min MAC} \ p$-Center problem has a feasible solution for a graph $G$ if and only if $p$ is at least the number of MACs in $G$, i.e., $p \geq |\mathcal{Y}(G)|$.

In what follows, we describe an approximation preserving reduction between $\text{Min PP\ CP}$ and $\text{Min MAC} \ p$-Center (Subsection 4.1). A polynomial approximation algorithm for the latter leads to a polynomial approximation algorithm for the former with a ratio that depends on the average degree $\overline{\deg}(G) = \frac{2|E|}{|V|}$ of $G$. More precisely, the reduction is even the identity and we analyze how good for the problem $\text{Min PP\ CP}$ an approximated MAC $p$-center can be. Then, in Subsection 4.2 we show that $\text{Min MAC} \ p$-Center can be approximated within the ratio 2, which leads to a $(4\overline{\deg}(G) + 2)$-approximation for the uniform $\text{Min PP\ CP}$ (all edges are of length 1). Actually, the result still holds if all edge-lengths lie in the interval $[\ell, 2\ell]$ for any positive $\ell$.

4.1. A polynomial approximation preserving reduction

We directly establish the following proposition for general edge lengths. We will denote respectively $\ell_M$ and $\ell_m$ the maximum and minimum edge lengths.

**Proposition 2.** On an edge weighted graph with lengths in $[\ell_m, \ell_M]$, $\forall C \in \mathcal{G}_p(G)$, we have:

$$\mathbb{E}(C) \leq (2\overline{\deg}(G) + 1)r(C) + (\ell_M - 2\ell_m)\overline{\deg}(G)$$

**Proof.** Let us consider any scenario $s \in V$ of degree $\deg(s)$ and number $1, 2, \ldots, \deg(s)$ the edges incident to $s$. We claim that $r^s(C) \leq (2\deg(s) + 1)r(C)$.

Consider indeed $x \in V$ such that $r^s(C, x) = r^s(C) \geq r(C)$. If $r^s(C, x) = r(C)$, then the claim is satisfied. Let us assume $r^s(C, x) > r(C)$. We consider two cases.

**Case 1:** $x \neq s$. $r^s(C, x)$ is the length of a path $\mu = [x_0, x_1, \ldots, x_k]$, where $x_0 = x$, $x_k \in C$ and $\mu$ is a minimum path in $G^s$.

Since $d^s(x, x_k) > r(C)$, we can define $i = \max\{j \in \{0, \ldots, k - 1\}, d^s(x_j, x_k) > r(C)\}$.

Then all vertices $x_j, j \in \{0, \ldots, i\}$ are, in $G$, at distance at most $r(C)$ from $s$. Indeed, the path $x_j, x_k$ is a minimum path of length greater than $r(C)$ in $G^s$. So, in $G$, the evacuation path of vertices $x_j, j \in \{0, \ldots, i\}$ passes through $s$.

Figure 9 illustrates the distance relation between $x, s$ and $x_k$ in the case $x \neq s$. In the figure, no shelter is located on $s$, but the reasoning is the same if there is one.
In $G$, for $j \in \{0, \ldots, i\}$, we consider a minimum path from $x_j$ to $s$, of value at most $r(C)$. We assign to $x_j$ a color in $N_1, \ldots, N_{\deg(s)}$ depending on the last edge of the minimum path we have fixed for $x_j$: $x_j$ is of color $N_i$ if the related minimum path between $x_j$ and $s$ terminates with the $i^{th}$ edge incident to $s$.

Note that the distance in $G$, between two vertices of the same color is at most $2r(C) - 2\ell_m$. Indeed considering, in $G$, two minimum paths from these vertices to $s$ and sharing the last edge, we deduce a walk avoiding $s$ between them of total length at most $2r(C) - 2\ell_m$. This walk includes a path in $G'$ of length at most $2r(C) - 2\ell_m$ between these two vertices.

This allows us to derive an upper bound of $d^s(x, x_j)$. Suppose $x$ is of color $N_{i_1}$ and consider the last vertex $x_{j_i}$ of color $N_{i_1}$ along the path $\mu$; we have $d^s(x, x_{j_i}) \leq 2r(C) - 2\ell_m$. Then, if $j < i$, the vertex $x_{j+1}$ is of color $N_{i_2}$ and $d^s(x_{j}, x_{j+1}) \leq \ell_M$. Using the same reasoning for all non-empty colors gives $d^s(x, x_i) \leq \deg(s)(2r(C) - 2\ell_m) + (\deg(s) - 1)\ell_M$.

Taking into account the edge $x_ix_{i+1}$ and the fact that $d^s(x_{i+1}, x_k) \leq r(C)$ we have:

$$r^s(C) \leq (2\deg(s) + 1)r(C) + \deg(s)(\ell_M - 2\ell_m) \quad (9)$$

**Case 2:** $x = s$ Similarly, $r^s(C, s)$ is the length of a path $\mu = [x_0, x_1, \ldots, x_k]$, where $x_0 = s$, $x_k \in C$ and $[x_1, \ldots, x_k]$ is a minimum path in $G'$. We define $i$ as in the previous case and use the same argument: $x_1$ is color $N_{i_1}$ and we define $x_j$ as previously. The only difference is that for any vertex the fixed minimum path from $x_j$ to $s$ passes through $x_1$ and consequently $d^s(x_1, x_j) \leq r(C) - \ell_m$. For the other colors, the same bound as previously holds. We then get a better bound:

$$r^s(C) \leq r(C) - \ell_m + (\deg(s) - 1)(2r(C) - 2\ell_m) + \deg(s)\ell_M + r(C)$$

$$\leq 2\deg(s)r(C) + \deg(s)(\ell_M - 2\ell_m) + \ell_m \quad (10)$$

Figure 9: Distance relations between vertices $x$, $s$ and $x_k$ used for Proposition 2
This bound is better than in Equation 9 since $\ell_m \leq r(C)$. So, in all cases we have $r^s(C) \leq (2\deg(s) + 1)r(C)$. We deduce, by taking the average value, $E(C) = \frac{1}{|V|} \sum_{s \in V} r^s(C) \leq (2\deg(G) + 1)r(C) + (\ell_M - 2\ell_m)\deg(G)$ which concludes the proof. \hfill \Box

On a tree, the analysis can be improved:

**Proposition 3.** On a tree with edge lengths in $[\ell_m, \ell_M]$, $\forall C \in C_p(G)$, we have:

$$E(C) \leq 3r(C) + \ell_M - 2\ell_m$$

**Proof.** Consider, for a scenario $s$, and a vertex $x, r^s(C, x) = r^s(C)$, the same analysis as in the proof of Proposition 2. Since there is no cycle, all vertices $x, \ldots, x_i$ are of the same color. Equation 9 becomes

$$r^s(C) \leq 3r(C) + \ell_M - 2\ell_m$$

which concludes the proof. \hfill \Box

**Remark 3.** In Demange et al. (2018), we have shown that, on paths with all edge-weight 1, there is an optimal solution $C^*$ of $\text{Min MAC } p$-Center such that $E(C^*) = r(C^*)$.

As noticed in the following example in Figure 10, with general weight system the situation may be totally different. In this example the graph is a path on 8 vertices with only one edge of weight $Z > 1$ and all other edges of weight 1 and $p = 4$. There is a unique optimal MAC 4-center and, for large values of $Z$, its value is very bad compared to an optimal $\text{Min PpCP}$ solution.

![Figure 10: With general weights, an optimal MAC $p$-center can be a very bad $\text{Min PpCP}$ solution.](image)

$\{1, 3, 6, 8\}$ is an optimal MAC 4-center \hspace{1cm} $\{1, 4, 5, 8\}$ is $\text{Min PpCP}$-optimal for $p = 4$;

$r(\{1, 3, 6, 8\}) = 1, E(\{1, 3, 6, 8\}) = \frac{3}{2} + 1$ \hspace{1cm} $r(\{1, 4, 5, 8\}) = 2, E(\{1, 4, 5, 8\}) = 2$

**Proposition 4.** Suppose a class of edge-weighted graphs $G = (V, E, L)$ with $\ell_M \leq 2\ell_m$ for which $\text{Min MAC } p$-Center can be approximated with $\rho(G)$.

Then, $\text{Min PpCP}$ can be approximated with $(2\deg(G) + 1)\rho(G)$ on the same class.
PROOF. Given a graph $G$ in the class, we build a $p$-center $C$ in $\mathcal{C}_p(G)$, if it exists, of value at most $\rho(G)r^*(G)$, where $r^*(G)$ denotes the optimal radius of a MAC $p$-center in $G$. Using Proposition 2 and $\ell_M \leq 2\ell_m$, we have $E(C) \leq (2\deg(G) + 1)r(C) \leq (2\deg(G) + 1)\rho(G)r^*(G)$.

Now if $C^*$ is an optimum solution for $\text{Min } PpCP$, we have $E(C^*) \geq r(C^*) \geq r^*(G)$. This concludes the proof.

4.2. Constant approximation algorithms

Proposition 5. $\text{Min MAC } p$-Center is polynomial on trees with general lengths.

PROOF. Given a tree $T$, for any distance $d$ we consider the tree $T_d$ obtained from $T$ by gluing to each pending vertex $v$ a path of length $d$. Then, $T$ has a MAC $p$-center of radius $d$ if and only if $T_d$ has a $p$-center of radius $d$. The result immediately follows from the fact that $p$-Center is polynomial on trees.

Using Proposition 3 and the analysis of Proposition 4, we get:

Corollary 4. There is a polynomial algorithm for $\text{Min } PpCP$ guaranteeing the ratio 3 on trees with all edge values 1.

Remark 4. Note however that we leave open the problem of whether $\text{Min } PpCP$ is NP-hard or polynomial on trees.

In this subsection we will need another $p$-center problem called $\text{Min Partial } p$-Center problem and introduced in Daskin and Owen (1999). Given a graph $G = (V, E)$ and a set of vertices $U \subset V$, $\text{Min Partial } p$-Center is to minimize the partial radius $r(C, U)$ of a $p$-center $C$, where $r(C, U) = \max_{x \in U} d(x, C)$. The underlying logic is that only vertices in $U$ need to be close to a center. However, centers can be any vertex in $G$ and distances are computed in $G$ (within our terminology, it means that the evacuation paths toward a shelter are not required to stay in $U$). Note that, if $U = V$, then $r(C, V) = r(C)$ and $\text{Min Partial } p$-Center is just the usual $\text{Min } p$-Center problem. So, $\text{Min } p$-Center is a particular case or $\text{Min Partial } p$-Center. In particular, $\text{Min Partial } p$-Center is not approximable within $2 - \varepsilon$ for any $\varepsilon > 0$, unless P=NP by using the same hardness result for $p$-Center proved in Hsu and Nemhauser (1979). Note that this hardness result for $\text{Min } p$-Center, directly obtained from the NP-hardness of $\text{Min Dominating Set}$, holds in the uniform case (all edges have the length 1). Since $\text{Min Dominating Set}$ remains NP-hard in planar bipartite graphs of degree 3, $\text{Min } p$-Center, and by consequence $\text{Min Partial } p$-Center, are not approximable within $2 - \varepsilon$ for any $\varepsilon > 0$ in planar bipartite graphs of degree 3 with all edge lengths 1, unless P=NP.
Note that the argument used for Min Partial $p$-Center cannot be easily adapted to Min MAC $p$-Center since this latter problem is not an immediate generalization of Min $p$-Center. However, for any edge weighted graph $G = (V, E, L)$, instance of $p$-Center, the instance is equivalent to the instance $(K, \tilde{L})$, where $K$ is the complete graph over $V$ and $\tilde{L}$ denotes the minimum path distance, i.e., $\forall i, j \in V, \tilde{l}_{ij} = d(i, j)$, where the distance $d$ is the distance in $G$. Both instances $G$ and $K$ have the same feasible solutions with the same values and thus, the same optimal solutions. To guarantee finite edge lengths in $K$, we just consider $G$ is connected. Since $K$ is 2-connected as soon as $|V| \geq 2$, Min MAC $p$-Center is equivalent to Min $p$-Center on $K$. Since the hardness result for Min $p$-Center still holds in connected graphs, Min MAC $p$-Center is not approximable within $2 - \varepsilon$ for any $\varepsilon > 0$, unless P=NP. We can even easily show that this hardness results already holds for the uniform case where all edge lengths are 1. To this aim, we just need to show that Min Dominating Set is NP-hard in 2-connected graphs. Given a graph $G = (V,E)$ instance of Min Dominating Set, we construct $G'$ from $G$ as follows: for every articulation point $a$ of $G$, create a twin vertex $a'$ linked to $a$ and to all neighbors of $a$. $G'$ is 2-connected and the Min Dominating Set problems in $G$ and $G'$ are equivalent. Now, a set of $p$ vertices in $G'$ is a dominating set if and only if its radius is 1 and else, the minimum radius of a $p$-center is at least 2. It immediately implies that Min MAC $p$-Center in graphs with edge lengths all equal to 1 is not approximable within $2 - \varepsilon$ for any $\varepsilon > 0$, unless P=NP.

In what follows, we propose a polynomial 2-approximation algorithms for Min Partial $p$-Center and Min MAC $p$-Center. These approximation results hold even in the case with general lengths.

Consider an instance $(G, U)$ of Min Partial $p$-Center, where $G = (V, E, L)$ is a graph with positive lengths on edges and $U \subset V$. We denote $n = |V|$. We can compute $K = (V, \tilde{E}, \tilde{L})$ in $O(n^3)$. We denote $SL = \{d(x,y), x,y \in V\}$ the set of edge lengths in $K$ (note that $|SL| \leq n^2$) and for any $d \in SL$, $K_d = (V, E_d)$ is the partial graph of $K$ where $E_d$ is the set of edges of length at most $d$. Note that for any $p$-center, its radius is in $SL$.

Theorem 2 can be obtained using the general method in Hochbaum and Shmoys (1986) or by adapting the 2-approximation algorithm for Min $p$-Center in Hochbaum and Shmoys (1985). However, to make this paper self contained and since it cannot be deduced from existing results, we give a direct proof. We first introduce the main concepts and claims used for this result since they are used as well in Theorem 3.

A $p$-center of partial radius $d$ in $(G, U)$ can be seen as a partial dominating set of $(K_d, U)$, where a partial dominating set $X$ is a set of vertices such that every vertex in $U$ has at least one neighbor in $X$. If $A_d$ is the adjacency matrix of $K_d$ with additional 1s on the diagonal (alternatively $A_d$ is the adjacency matrix of $K_d$ with additional loops on each vertex), we
denote \( A_{d,U} \) the sub-matrix of \( A_d \) corresponding to rows in \( U \) (it has \(|U|\) rows and \(|V|\) columns). The problem of finding a minimum partial dominating set can be formulated by the following mathematical program \( PDS(G,U,d) \), where the 1s on the diagonal represent the fact that a vertex dominates itself:

\[
PDS(G,U,d) : \begin{cases} 
\min \langle 1_{|V|}, x \rangle \\
A_{d,U}x \geq 1_{|U|} \\
x \in \{0,1\}^{|V|}
\end{cases}
\]

We then consider the mathematical program \( SIS(G,U,d) \) that corresponds to finding a maximum strong independent set of \( K_d \) contained in \( U \), where a strong independent set \( S \subset V \) is an independent set (every two vertices in \( S \) are not adjacent) such that every vertex in \( V \setminus S \) has at most one neighbor in \( S \).

\[
SIS(G,U,d) : \begin{cases} 
\max \langle 1_{|U|}, y \rangle \\
A^T_{d,U}y \leq 1_{|V|} \\
y \in \{0,1\}^{|U|}
\end{cases}
\]

**Claim 1.** The cardinality of any strong independent set of \( K_d \) contained in \( U \) is not more than the cardinality of any partial dominating set of \((K_d,U)\).

**Proof.** The relaxations of mathematical programs \( PDS(G,U,d) \) and \( SIS(G,U,d) \), replacing the binary conditions with non negative conditions, are dual linear programming problems. The result is an immediate consequence of the weak duality theorem. \(\square\)

Let \( d_{\max} = \max(SL) \). We denote \( K_{2d,U} \) the graph \( K_{\min(2d,d_{\max})}[U] \).

**Claim 2.** For a given distance \( d \in SL \), let \( S_d \) be a maximal independent set of \( K_{2d,U} \). \( S_d \) is a partial \(|S_d|\)-center in \((G,U)\) of partial radius \( r(S_d,U) \leq 2d \).

**Proof.** Consider any vertex \( u \in U \setminus S_d \). Since \( S_d \) is maximal, \( S_d \cup \{u\} \) is not independent in \( K_{2d,U} \), which means \( d(u,S_d) \leq 2d \) and the claim is proved. \(\square\)

**Claim 3.** Any independent set \( S \) of \( K_{2d,U} \) is a strong independent set of \( K_d \) contained in \( U \).

**Proof.** By definition, \( S \subset U \). Since \( S \) is independent in \( K_{2d,U} \), it is independent in \( K_{d,U} \), a partial graph of \( K_{2d,U} \). So, it is an independent set of \( K_d \). The result then follows by contrapositive: if there is a vertex \( u \in V \setminus S \) adjacent, in \( K_d \), to two vertices of \( S \), then these two vertices would be at distance at most \( 2d \), so would be adjacent in \( K_{2d,U} \). \(\square\)

Claims 1, 2 and 3 immediately allow to derive an approximation algorithm for \text{Min Partial } p\text{-Center}.
Theorem 2. Min Partial $p$-Center is polynomially 2-approximable and this is the best possible constant ratio.

PROOF. We already noted that 2 is a lower bound for any constant approximation ratio of Min Partial $p$-Center. So, we only need to prove that this bound can be guaranteed.

For a given instance $(G, U)$, we can compute $SL$ and all distance $d(i, j), i, j \in V$ in $O(n^3)$. Then, for any $d \in SL$, we can compute a maximal independent set $S_d$ of $K_{2d, U}$ and then select $\tilde{S}_d$, where $\tilde{d} \in \arg\min \{r(S_d)\}$. In other words, $\tilde{S}_d$ is of minimum value among all $d \in SL, |S_d| \leq p$ $S_d$s of cardinality at most $p$. Denote $r_{\tilde{d}U}$ the minimum partial radius of a $p$-center in $(G, U)$. $r_{\tilde{d}U} \in SL$. Using Claim 3 and Claim 1, $|S_{r_{\tilde{d}U}}| \leq p$ and thus, $\tilde{d}$ exists and $r(\tilde{S}_d) \leq r(S_{r_{\tilde{d}U}})$. Using Claim 2, we deduce $r(S_{r_{\tilde{d}U}}) \leq 2r_{\tilde{d}U}$, which completes the proof. □

Note that, using a binary search on the same model as the 2-approximation algorithm for Min $p$-Center proposed in Hochbaum and Shmoys (1985), we can design a 2-approximation algorithm of complexity $O(n^2 \log n)$ as soon as all distances between two vertices in $G$ are computed.

We use similar ideas and the same claims to derive a polynomial 2-approximation algorithm for Min MAC $p$-Center (Algorithm 1).

To simplify the description of Algorithm 1, we introduce some notations used in the description of the algorithm. Given the instance $G = (V, E, L)$, we denote by $k$ the number of MACs of $G$. These MACs are denoted $A_1, \ldots, A_k$ and the related articulation points are called $a_1, \ldots, a_k$ (we may have $a_i = a_j, i \neq j$). As previously $SL = \{d(i, j), i, j \in V, \}$; for any $d \in SL$, we partition $I = \{1, \ldots, k\}$ into $I = I_d^- \sqcup I_d^+$ ($\sqcup$ denotes the disjoint union), where $I_d^- = \{i \in I, \max_{x \in A_i} d(x, a_i) \leq d\}$ and $I_d^+ = \{i \in I, \max_{x \in A_i} d(x, a_i) > d\}$. MACs $A_i, i \in I_d^-$ are seen as small MACs relative to $d$, while MACs $A_i, i \in I_d^+$ are seen as large ones. “No-solution output” is any output we use to indicate that the problem has no feasible solution.

The idea of the Algorithm is as follows:

1. If the number of MAC is more than $p$, then there is obviously no solution.
2. Else, for every distance $d \in SL$, Algorithm 1 tries to compute a MAC $p$-center $C_d$ of radius at most $2d$; only feasible MAC $p$-centers obtained through this process will be kept and $SL$ is the set of distances $d$ for which it will occur;
3. $C_d$ is built as follows:
   (a) The algorithm selects one center per small MAC $A_i, i \in I_d^-$;
   (b) For each $i \in I_d^+$, all vertices at distance at most $d$ from $a_i$ are allocated to the related center (by definition of $I_d^+$, this includes in particular all vertices of $A_i$).
Algorithm 1 2-approximation for $\text{Min MAC } p$-Center.

Require: Edge weighted graph $G = (V, E, L)$ (lengths are non-negative) and $p \geq 2$.
Ensure: Outputs $C$, a MAC $p$-center if it exists.

1: Begin
2: Compute $A_1, \ldots, A_k$, and $a_1, \ldots, a_k$
3: if $k > p$ then
4: No-solution output
5: else
6: Compute $SL$ and all distances $d(i, j), i, j \in V$
7: $\tilde{SL} \leftarrow \emptyset$
8: for $d \in SL$ do
9: Compute $I_d^-$ and $I_d^+$
10: $C_d \leftarrow \emptyset$
11: for $i \in I_d^-$ do
12: Select $x \in A_i$
13: $C_d \leftarrow C_d \cup \{x\}$
14: end for
15: $V_d' \leftarrow \{v \in V, d(v, \{a_i, i \in I_d^\}) > d\}$
16: $S_d \leftarrow \emptyset$
17: for $i \in I_d^+$ do
18: Select $y \in \text{argmax}_{x \in A_i} d(x, a_i)$
19: $S_d \leftarrow S_d \cup \{y\}$
20: end for
21: while $\exists v \in V_d', d(v, S_d) > 2d$ do
22: $S_d \leftarrow S_d \cup \{v\}$
23: end while
24: if $|S_d| \leq p - |I_d^-|$ then
25: $SL \leftarrow SL \cup \{d\}$
26: $C_d \leftarrow C_d \cup S_d$
27: end if
28: end for
29: Let $\tilde{d} \in \text{argmin}_{d \in SL} (r(C_d))$
30: $C \leftarrow C_{\tilde{d}}$
31: return $C$
32: end if
33: End
(c) \( V_d' \) is the set of uncovered vertices. If possible, the algorithm completes \( C_d \) with a partial \((p - |I_d^-|)\)-center of \((G \setminus \bigcup_{i \in I_d^-} A_i, V_d')\) of partial radius at most \(2d\).

To this aim, it uses the same ideas as in Theorem 2: it constructs a maximal independent set \( S_d \) of \( K_{2d, V_d'} \), but to ensure it intersects all \( A_i \)s, \( i \in I_d^+ \), it initializes it by choosing one vertex in each of these components. If \(|S_d| \leq p - |I_d^-|\), then \( d \in \tilde{SL} \).

4. The best solution \( C_{\tilde{d}}, d \in \tilde{SL} \) is selected as an approximated solution for Min MAC \( p \)-Center.

**Theorem 3.** Algorithm \( \tilde{SL} \) is a polynomial 2-approximation algorithm for Min MAC \( p \)-Center and this is the best possible constant ratio.

**Proof.** We already noted that 2 is a lower bound for constant approximation ratios. So, we only need to prove that this bound can be guaranteed.

Assume that \( k \leq p \); then the instance of Min MAC \( p \)-Center has feasible solutions and thus, also an optimal solution.

Fix a distance \( d \in SL \). Note first that, by definition of \( I_d^- \) and \( I_d^+ \), \( V_d' \) computed at line 15 satisfies \( V_d' \subset V \setminus \bigcup_{i \in I_d^-} A_i \) and \( \forall i \in I_d^+, A_i \cap V_d' \neq \emptyset \). Then, the algorithm computes the set \( S_d \) from Lines 16 to Line 23.

**Claim 4.** \( \forall d \in SL, S_d \) is a maximal independent set in \( K_{2d, V_d'} \) that intersects all \( A_i \)s, \( i \in I_d^+ \).

**Proof.** The algorithm initializes \( S_d \) by selecting, in each MAC \( A_i, i \in I_d^+ \), a vertex at maximum distance from \( a_i \). This ensures that, at Line 20, \( S_d \) includes one element per MAC \( A_i, i \in I_d^+ \) and is an independent set (possibly empty) in \( K_{2d, V_d'} \). Indeed, if \( y_i, y_j \) are respectively selected at Line 18 for \( i, j \in I_d^+, i \neq j \), then any path between them passes through \( a_i \) and \( a_j \) (we may have \( a_i = a_j \)) and is of length greater than \( 2d \). As a consequence, \( S_d \) is a maximal independent set in \( K_{2d, V_d'} \).

\( \tilde{SL} \), computed by the algorithm (Lines 25), is the set of distances \( d \) such that \( S_d \) is of size at most \( p - |I_d^-| \). Consider now an optimal MAC \( p \)-center, \( C_{MAC}^* \), of radius \( d^* \).

**Claim 5.** \( d^* \in \tilde{SL} \)

**Proof.** Since \( C_{MAC}^* \) has at least one center per MAC, \( C_{MAC}^* \) has at most \((p - |I_d^-|)\) centers in \( V \setminus \bigcup_{i \in I_d^-} A_i \). In addition, vertices in \( V_d' \) cannot be associated with (i.e., evacuated to)
centers in \( \bigcup_{i \in I^*} A_i \) since these centers are at distance more than \( d^* \). This means that \( C_{MAC}^* \cap (V \setminus \bigcup_{i \in I^*} A_i) \) is a \((p - |I^*|)\)-center of partial radius at most \( d^* \) in \((G \setminus \bigcup_{i \in I^*} A_i, V_{d^*}')\). As a consequence \( C_{MAC}^* \cap (V \setminus \bigcup_{i \in I^*} A_i) \) is a partial dominating set in \((K_{d^*}, V_{d^*}')\).

Using Claims 3 and 1, we get \(|S_{d^*}| \leq |C_{MAC}^* \cap (V \setminus \bigcup_{i \in I^*} A_i)| \leq p - |I^*|\), which means \( d^* \in \tilde{SL} \).

Claim 5 ensures in particular that \( \tilde{SL} \neq \emptyset \) and consequently \( \tilde{d} \) computed at Line 29 is well defined. Since \( d^* \) and \( \tilde{d} \) are both in \( SL \), the algorithm computes both sets \( C_{d^*} \) and \( C_{\tilde{d}} \) by selecting one vertex per \( A_i, i \in I_{d^*} \), and one vertex per \( A_i, i \in I_{\tilde{d}} \), respectively (from Line 10 to Line 14) and completing with \( S_{d^*} \) and \( S_{\tilde{d}} \), respectively. Using Claim 4, this ensures that both \( C_{d^*} \) and \( C_{\tilde{d}} \) are MAC \( p \)-centers.

Finally, \( C_{\tilde{d}} \) is selected as approximated solution and Line 29 ensures

\[
r(C_{\tilde{d}}) \leq r(C_{d^*})
\]  

We complete the proof by showing the following claim.

**Claim 6.** \( r(C_{d^*}) \leq 2d^* \).

**Proof.** Consider first a vertex \( v \in V_{d^*}' \) and use the same argument as in the proof of Theorem 2. We have \( d(v, C_{d^*}) \leq d(v, C_{d^*} \setminus \bigcup_{i \in I_{d^*}} A_i) \leq r(S_{d^*}, V_{d^*}') \). Using Claims 4 and 2, we have \( r(S_{d^*}, V_{d^*}') \leq 2d^* \) and thus:

\[
\forall v \in V_{d^*}', d(v, C_{d^*}) \leq 2d^*.
\]  

Consider now a vertex \( v \in V \setminus V_{d^*}' \). By definition of \( V_{d^*}' \), it means that \( d(v, \{a_i, i \in I_{d^*}\}) \leq d^* \) and by definition of \( I_{d^*} \), it ensures \( \exists i \in I_{d^*}, \forall u \in A_i, d(v, u) \leq 2d^* \). This ensures:

\[
\forall v \in V_{d^*}', d(v, C_{d^*}) \leq 2d^*.
\]  

Equations 12 and 13 ensure \( r(C_{d^*}) \leq 2d^* \). Claim 6 and Equation 11 imply \( r(C_d) \leq 2d^* \), which concludes the proof of Theorem 3.
We immediately deduce from Theorem 2 and Proposition 4.

**Corollary 5.** *For edge weighted graphs with lengths in \([\ell, 2\ell]\), Min \(PPCP\) is approximable within \(4\deg(G) + 2\).*

### 5. Conclusion

In this paper, we strengthen the analysis of Min \(PPCP\) initiated in Demange et al. (2018). In particular, in Section 3, we revisit the reduction we used in this previous paper to get a hardness result on planar graphs of bounded degree. The new reduction allows to prove that Min \(PPCP\) is not approximable with a ratio less than \(\frac{56}{55}\) on subgrids of degree at most 3. Even thought the result does not generalize the one we previously obtained (the class is more restrictive but the new bound is closer to 1), the proof requires a much deeper analysis with techniques that might be useful for other problems. The main originality of our proof is the use of the intermediate graph \(\tilde{H}_q\) (see Figure 3): it can be seen as a perturbation of the subgrid \(H_q\) that leads to a hard class for Min Vertex Cover.

Then, in Section 4, we propose some approximation results for this problem with, in particular, a constant approximation for graphs of bounded degree and with edge lengths in \([\ell, 2\ell]\). To our knowledge, this is the first example of approximation for this problem and in addition it holds for a class of instances on which all our hardness results apply. It provides a first gap between constant approximation ratios and the hardness in approximation results we have obtained. Narrowing this gap for intermediate classes of graphs is a natural open question for further researches. In section 4.2, we even show a stronger approximation result on trees. However, we leave open the problem of whether Min \(PPCP\) is NP-hard or polynomial on trees.

Most of our results apply for the uniform case only. Surprisingly, Proposition 4 and its Corollary 5 are still valid for the case where edge lengths lie in \([\ell, 2\ell]\). Finding polynomial cases and approximation results for Min \(PPCP\) with general length system remains an important open question that would require new methods or tools.

In Demange et al. (2020), we introduce the robust version of the \(p\)-Center problem under uncertainty. In an ongoing work we try to generalize the present results to this case.

Finally, when considering the feasibility conditions for Min \(PPCP\), we have introduced the notion of minimal articulation components (MACs) and the related Min MAC \(p\)-Center Problem. We have shown that this problem is 2-approximable and that this is the best possible constant approximation ratio (Theorem 3). It is also polynomial on trees. Strength-
ening the analysis of this notion and the complexity and approximation of this problem on specific classes of instances is another question raised by the paper.

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