The Baxter Equation for Quantum Discrete Boussinesq Equation

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ABSTRACT

Studied is the Baxter equation for the quantum discrete Boussinesq equation. We explicitly construct the Baxter $Q$ operator from a generating function of the local integrals of motion of the affine Toda lattice field theory, and show that it solves the third order operator-valued difference equation.

1 Introduction

In analysis of the integrable models, realized is the importance of the Baxter equation since it was applied to the eight-vertex model (XYZ model) to compute the spectrum [1]. The Baxter equation is the functional difference relation among the transfer matrix $t(\lambda)$ and the auxiliary function $Q(\lambda)$ which is called the Baxter $Q$ operator, and is powerful enough to determine the spectrum of the transfer matrix only from certain analytic conditions of these functions.

Later Gaudin and Pasquier explicitly constructed the Baxter $Q$ operator in an integral form for the quantum periodic Toda lattice [2]. It was also shown [2,3] that in the classical limit the logarithm of the $Q$ operator is the generating function of the Bäcklund transformation, i.e., the Bäcklund transformation is given by the classical limit of the evolution equation, $O \rightarrow Q O Q^{-1}$ (see, for review, Ref.[4]). This correspondence suggests that the integrable dynamical system on space-time lattice can be constructed by use of the $Q$ operator [4,5]. Recent studies indicate that the $Q$ operator is useful enough to construct the eigenfunction of the Toda lattice [6].

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Since then the explicit form of the Baxter $Q$ operator has been derived for other integrable models such as the XXX model [8, 9], the dimer self-trapping model [10], the quantum KdV model [11], and the Volterra model (discrete KdV equation) [12, 13]. All these models are governed by the quantum algebra associated to $s\ell_2$. In these cases as was pointed out in studies of the integrability of the conformal field theory [11], the Baxter equation may be related with the universal $R$ matrix, $R \in U_q(\hat{s\ell}_2) \otimes U_q(\hat{s\ell}_2)$, with some quantum finite-dimensional representation for the first $U_q(\hat{s\ell}_2)$ algebra and an auxiliary infinite-dimensional $q$-oscillator representation for the second part. In fact proposed was the universal procedure [14] to derive the Baxter equation from the universal $R$ matrix by use of the $q$-oscillator representation. Following this strategy it was shown that the $Q$ operator [12] for the Volterra model can be constructed from the universal $R$ matrix for the quantum algebra $U_q(\hat{s\ell}_2)$ with the $q$-oscillator representation [15].

In this paper we construct the Baxter $Q$ operator for the quantum discrete Boussinesq equation which has $U_q(\hat{s\ell}_3)$ symmetry. In the classical limit the discrete Boussinesq equation is given as

$$\begin{align*}
\frac{dL_n}{dt} &= -W_n + W_{n-1} + L_n (L_{n+1} - L_{n-1}), \\
\frac{dW_n}{dt} &= W_n (L_{n+2} - L_{n-1}).
\end{align*}$$

(1.1)

We stress that $L_n$ and $W_n$ do not mean the Fourier modes of the generators $L(x)$ and $W(x)$ but the discretization thereof. We note that the higher flow of eq. (1.1) reduces in a continuum limit [16] to

$$\begin{align*}
\frac{du}{dt} &= u_{xx} - 2w_x, \\
\frac{dw}{dt} &= -w_{xx} + \frac{2}{3} u_{xxx} + \frac{2}{3} u u_x.
\end{align*}$$

(1.2)

This set of equations gives the Boussinesq equation which is generated from the pseudo-differential operator, $L = \partial^3 + u \partial + w$. In this sense we call eq. (1.1) as the discrete Boussinesq equation. The discrete Boussinesq equation (1.1) was first introduced in Ref. [17] from studies of a discretization of the $W_3$ algebra, motivated from the close connection between the soliton equations and the conformal field theory [18–20]. The integrability of the discrete Boussinesq equation follows from the fundamental commutation relation,

$$\{ T(x) \otimes T(y) \} = [ r(x, y) , T(x) \otimes T(y) ],$$

(1.3)

where we omit the explicit form of the $\mathbb{Z}_3$ symmetric classical $r$ matrix. The monodromy matrix $T(x)$ is given from the Lax matrix as [13]

$$T(x) = \prod_n L_n^W(x),$$

$$L_n^W(x) = \frac{1}{\sqrt{W_n}} \begin{pmatrix} x^2 & -x L_{n+1} & W_n \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$  

(1.4)
As a result we can construct the integrals of motion from the transfer matrix $\text{Tr} T(x)$; some of them are

$$H_0 = \sum_n \log W_n,$$
$$H_1 = \sum_n L_n,$$
$$H_2 = \sum_n \left( \frac{1}{2} (L_n)^2 + L_n L_{n+1} - W_n \right).$$

As is well known, the discrete Boussinesq equation has also an intimate relationship with a lattice analogue of the affine Toda field theory. We have constructed a generating function $Q$ of the local integrals of motion of the affine Toda lattice field theory [21], and have also shown that this also commute with the transfer matrix of the discrete Boussinesq equation. The function $Q$ is constructed in terms of the quantum dilogarithm function as the intertwining operator of the vertex operators. As was conjectured there, we shall show here that this generating function is indeed the Baxter $Q$ operator for the discrete Boussinesq equation.

This paper is organized as follows. In Section 2 we briefly review the integrable structure of the quantum discrete Boussinesq equation. Recalling the relationship with the affine Toda field theory, we construct the transfer matrix and the generating function $Q(\lambda)$ of the local integrals of motion for the affine Toda lattice field theory. In Section 3 we explicitly show that the generating function $Q$ plays a role of the Baxter $Q$ operator for the transfer matrix of the discrete Boussinesq equation. Based on the duality of the affine Toda lattice field theory, we show in Section 4 that we have the dual Baxter equation for the discrete Boussinesq equation. In Section 5 we briefly comment on the continuum limit of the discrete Boussinesq equation, and see a relationship between the previously known results. In Section 6 we consider the generalization for the $N$-reduced discrete KP equation which has the quantum group $\widehat{U}_q(s\ell_N)$ structure. Last section is devoted for the concluding remarks and discussions.

## 2 Quantum Boussinesq Equation

### 2.1 Lattice Vertex Operator and Transfer Matrix

To define the quantization of the discrete Boussinesq hierarchy (1.4), we use two sets of the discretized free fields $\phi_n^{(1)}$ and $\phi_n^{(2)}$, which respectively correspond to the Toda fields associated to the simple roots $\alpha_1$ and $\alpha_2$. We use the free field associated to the maximal root as

$$\phi_n^{(0)} = -\phi_n^{(1)} - \phi_n^{(2)}.$$  \hspace{1cm} (2.1)
The commutation relations are defined as a discrete analogue of the usual relations of the free chiral fields by
\[ [\phi^{(a)}_m, \phi^{(a)}_n] = 2i \gamma, \quad \text{for } m > n, \ a = 1, 2, \]
\[ [\phi^{(1)}_n, \phi^{(2)}_n] = \begin{cases} i \gamma, & \text{for } n \leq m, \\ -i \gamma, & \text{for } n > m. \end{cases} \] (2.2)

Throughout this paper we set the deformation parameter \( q \) as
\[ q = e^{i \gamma}. \] (2.3)

Hereafter for our convention we often use a difference of the lattice free field
\[ \Delta \phi^{(a)}_n = \phi^{(a)}_n - \phi^{(a)}_{n+1}, \] (2.4)
whose commutation relations become local.

With these settings, we define the discrete screening charges as in the case of the conformal field theory (see e.g. Ref. 23);
\[ Q^{(a)} = \sum_n e^{\pi \gamma \phi^{(a)}_n}, \] (2.5)
for \( a = 0, 1, 2 \). We can see that the Serre relation of the algebra \( U_q(\hat{sl}_3) \) is fulfilled;
\[ (Q^{(a)})^2 Q^{(b)} - (q + q^{-1}) \cdot Q^{(a)} Q^{(b)} Q^{(a)} + Q^{(b)} (Q^{(a)})^2 = 0, \] (2.6)
for \( a \neq b \). The generators of the quantum \( W_3 \) algebra are then defined so as to commute with these screening charges. Note that the dual screening charges as a summation of the dual vertex operators,
\[ \tilde{Q}^{(a)} = \sum_n e^{\pi \gamma \phi^{(a)}_n}, \] (2.7)
commute with \( Q^{(a)} \), and that they satisfy the same Serre relation (2.6) replacing \( q \) with
\[ \tilde{q} = e^{i \pi^2 / \gamma}. \] (2.8)

As was shown in Ref. 16, the Lax matrix (1.4) for the discrete Boussinesq equation is gauge-equivalent with
\[ L_n(\lambda) = g(\lambda) \begin{pmatrix} e^{\frac{1}{4} \Delta \phi^{(1)}_n - \frac{1}{4} \Delta \phi^{(2)}_n} & e^{\frac{1}{4} \Delta \phi^{(1)}_n + \frac{1}{4} \Delta \phi^{(2)}_n} \\ e^{-\frac{1}{4} \Delta \phi^{(1)}_n - \frac{1}{4} \Delta \phi^{(2)}_n} & e^{-\frac{1}{4} \Delta \phi^{(1)}_n + \frac{1}{4} \Delta \phi^{(2)}_n} \end{pmatrix} \cdot \begin{pmatrix} e^\lambda & e^{-\lambda} & 1 \\ 1 & e^\lambda & e^{-\lambda} \\ e^{-\lambda} & 1 & e^\lambda \end{pmatrix} \]
\[ \equiv g(\lambda) \cdot C_n \cdot X(\lambda), \] (2.9)

We note that the dynamical variables \( \alpha_n \) and \( \beta_n \) in Ref. 22 are defined by
\[ \alpha_n = e^{\Delta \phi^{(1)}_n - i \gamma}, \quad \beta_n = e^{\Delta \phi^{(1)}_n + \Delta \phi^{(2)}_n - i \gamma}, \]
with \( q = e^{i \gamma} \).
where we have modified the spectral parameter by \( x^{-3} = 1 - e^{-3\lambda} \) from eq. (1.4). The function \( g(\lambda) \) is a normalization function which will be defined below. We introduce diagonal elements of the matrix \( C_n \) as

\[
C_n = \text{diag}\left(c_n^{(1)}, c_n^{(2)}, c_n^{(3)}\right),
\]

which corresponds to the weights in the vector representation \( 3 \) of \( sl_3 \). Following the standard way of studies in the integrable systems, we define the monodromy matrix and the transfer matrix as

\[
T(\lambda) = \prod_n L_n(\lambda), \quad t_1(\lambda) = \text{Tr} \, T(\lambda).
\]

The monodromy matrix satisfies the Yang–Baxter equation \[22\],

\[
R(\lambda - \mu) \cdot T(\lambda) \cdot T(\mu) = T(\mu) \cdot T(\lambda) \cdot R(\lambda - \mu).
\]

Here the \( R \)-matrix is \( \mathbb{Z}_3 \otimes \mathbb{Z}_3 \) symmetric, and reads as

\[
R(\lambda) = \begin{pmatrix}
    a & b & c \\
    \bar{b} & c & \bar{a} \\
    \bar{c} & b & a \\
    c & \bar{b} & \bar{a}
\end{pmatrix},
\]

with

\[
a = \text{sh} \left( \frac{3}{2} \lambda - \frac{1}{2} i \gamma \right),
\]

\[
b = e^{\frac{3i}{2} \gamma} \text{sh} \left( \frac{3}{2} \lambda \right),
\]

\[
\bar{b} = e^{-\frac{3i}{2} \gamma} \text{sh} \left( \frac{3}{2} \lambda \right),
\]

\[
c = e^{-\frac{1}{2} \lambda} \text{sh} \left( - \frac{1}{2} i \gamma \right),
\]

\[
\bar{c} = e^{\frac{1}{2} \lambda} \text{sh} \left( - \frac{1}{2} i \gamma \right).
\]

It follows that the transfer matrix commute with each other,

\[
[t_1(\lambda), t_1(\mu)] = 0,
\]

which supports the quantum integrability of our system.

We note that under the quasi-periodic boundary condition,

\[
\phi_{n+L}^{(a)} = \phi_n^{(a)} + L \cdot i \, P^{(a)}, \quad \text{for } a = 1, 2,
\]

where we have modified the spectral parameter by \( x^{-3} = 1 - e^{-3\lambda} \) from eq. (1.4). The function \( g(\lambda) \) is a normalization function which will be defined below. We introduce diagonal elements of the matrix \( C_n \) as

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\[
R(\lambda) = \begin{pmatrix}
    a & b & c \\
    \bar{b} & c & \bar{a} \\
    \bar{c} & b & a \\
    c & \bar{b} & \bar{a}
\end{pmatrix},
\]

with

\[
a = \text{sh} \left( \frac{3}{2} \lambda - \frac{1}{2} i \gamma \right),
\]

\[
b = e^{\frac{3i}{2} \gamma} \text{sh} \left( \frac{3}{2} \lambda \right),
\]

\[
\bar{b} = e^{-\frac{3i}{2} \gamma} \text{sh} \left( \frac{3}{2} \lambda \right),
\]

\[
c = e^{-\frac{1}{2} \lambda} \text{sh} \left( - \frac{1}{2} i \gamma \right),
\]

\[
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\]

It follows that the transfer matrix commute with each other,

\[
[t_1(\lambda), t_1(\mu)] = 0,
\]

which supports the quantum integrability of our system.

We note that under the quasi-periodic boundary condition,
the transfer matrix (2.11) can be rewritten as

\[ t_1(\lambda) = \text{Tr} \left( e^{-L_1(-P^{(1)} h_3 + P^{(2)} h_2)} \cdot \prod_{n=1}^{\infty} g(\lambda) e^{\frac{i}{2} \gamma} D_n \mathbf{X}(\lambda) D_n^{-1} \right), \]  

(2.16)

where \(3 \times 3\) diagonal matrix \(D_n\) is given by

\[ D_n = \exp\left( -\phi_n^{(1)} h_3 + \phi_n^{(2)} h_2 \right), \]

\[ h_a = E_{aa} - \frac{1}{3} \mathbb{1}. \]

A matrix \(E_{ij}\) is \((E_{ij})_{mn} = \delta_{i,m} \delta_{j,n}\).

For our purpose to construct the Baxter equation for the discrete Boussinesq equation, we use another transfer matrix which is associated with a representation \(3^*\). Generally the monodromy matrix for this type of representation can be constructed from one for the vector representation by the fusion method [24], and in the \(s\ell_3\) case the Lax matrix is simply defined from

\[ L_n(\lambda) \propto \left( (L_n(\lambda))^{-1} \right)^t, \]

where \(t\) denotes a transpose [25];

\[ L_n(\lambda) = \mathbf{g}(\lambda) \left( \begin{array}{ccc} e^{-\frac{1}{2} \Delta \phi_n^{(1)} + \frac{1}{2} \Delta \phi_n^{(2)}} & e^{-\frac{1}{2} \Delta \phi_n^{(1)} - \frac{1}{2} \Delta \phi_n^{(2)}} & 0 \\ e^{\frac{1}{2} \Delta \phi_n^{(1)} + \frac{1}{2} \Delta \phi_n^{(2)}} & e^{\frac{1}{2} \Delta \phi_n^{(1)} - \frac{1}{2} \Delta \phi_n^{(2)}} & 0 \\ 0 & 0 & e^\lambda \end{array} \right) \cdot \left( \begin{array}{ccc} e^\lambda & -1 & 0 \\ 0 & e^\lambda & -1 \\ -1 & 0 & e^\lambda \end{array} \right), \]

(2.17)

where as before \(\mathbf{g}(\lambda)\) is a normalization function. One sees that each element of \(C_n^{-1}\) corresponds to weights of \(3^*\). Correspondingly the monodromy matrix and the transfer matrix for this representation is respectively defined by

\[ \mathbf{T}(\lambda) = \prod_n L_n(\lambda), \quad t_2(\lambda) = \text{Tr} \mathbf{T}(\lambda). \]  

(2.18)

We note that under the boundary condition (2.15) the transfer matrix \(t_2(\lambda)\) has a form,

\[ t_2(\lambda) = \text{Tr} \left( e^{L_1(-P^{(1)} h_3 + P^{(2)} h_2)} \cdot \prod_{n=1}^{\infty} \mathbf{g}(\lambda) e^{\frac{i}{2} \gamma} D_n^{-1} \mathbf{X}(\lambda) D_n \right), \]

(2.19)

The commutativity of this transfer matrix follows from eq. (2.12) as follows. We have the fundamental commutation relation for \(\mathbf{T}(\lambda)\) from eq. (2.12) as

\[ \left( (\mathbf{R}(\lambda - \mu))^{-1} \right)^{t_1 t_2} \mathbf{T}(\lambda) \frac{2}{\mathbf{T}(\mu)} = \frac{2}{\mathbf{T}(\mu)} \mathbf{T}(\lambda) \left( (\mathbf{R}(\lambda - \mu))^{-1} \right)^{t_1 t_2}, \]

(2.20)
we define the fundamental $L$.

Furthermore eq. (2.12) also gives

$$(\mathbf{R}(\lambda - \mu)_{t_2})^{-1} \cdot \mathbf{T}(\lambda) \cdot \mathbf{T}(\mu) = \mathbf{T}(\lambda) \cdot \mathbf{T}(\mu) \cdot \left(\mathbf{R}(\lambda - \mu)_{t_2}\right)^{-1},$$

which proves the commutativity among two transfer matrices,

$$[t_1(\lambda), t_2(\mu)] = 0.$$  

(2.23)

### 2.2 Fundamental $L$ Operator

We shall construct the generating function of the local integrals of motion for the affine Toda lattice field theory.

For the Heisenberg operators $\hat{p}$ and $\hat{q}$ satisfying

$$[\hat{p}, \hat{q}] = -2i\gamma,$

we define the fundamental $\hat{L}$ operator as

$$\hat{L}(\lambda; \hat{q}, \hat{p}) = \frac{1}{\Phi_\gamma(\lambda + \hat{p})} \cdot \frac{1}{\Phi_\gamma(2\lambda + \hat{q})} \cdot \frac{\Theta_\gamma(\hat{q})}{\Phi_\gamma(\lambda - \hat{q})} \cdot \frac{\Theta_\gamma(\hat{p})}{\Phi_\gamma(\lambda - \hat{p})}.$$  

(2.24)

Here $\lambda$ is a spectral parameter, and commute with every quantum operators, and the functions $\Phi_\gamma(\varphi)$ and $\Theta_\gamma(\varphi)$ are respectively defined by

$$\Phi_\gamma(\varphi) = \exp \left( \int_{\mathbb{R} + i0} e^{-i\varphi x} \frac{\mathrm{d}x}{4 \sh(\gamma x) \sh(\pi x)} \right),$$

$$\Theta_\gamma(\varphi) = \Phi_\gamma(\varphi) \cdot \Phi_\gamma(-\varphi).$$

(2.25) (2.26)

The integral $\Phi_\gamma(\varphi)$ was first introduced by Faddeev [23] as a non-compact version of the quantum dilogarithm function. See Appendix A for properties of these functions, and we especially recall that the function $\Theta_\gamma(\varphi)$ is the Gaussian (A.3). By use of the difference equation (A.4), we have

$$\hat{L}(\lambda; \hat{q} - 2i\gamma, \hat{p}) = \hat{L}(\lambda; \hat{q}, \hat{p}) \cdot \left( e^\lambda + e^{-\lambda - i\gamma + \hat{q}} + e^{-i\gamma + \hat{p}} \right) \cdot \frac{1}{1 + e^{-\lambda - i\gamma + \hat{q}} + e^{-\lambda - i\gamma + \hat{p}}} \cdot \hat{L}(\lambda; \hat{q}, \hat{p}),$$

(2.27)

$$\hat{L}(\lambda; \hat{q} + \hat{p} - 2i\gamma) = \hat{L}(\lambda; \hat{q}, \hat{p}) \cdot \left( 1 + e^\lambda + e^{\lambda - i\gamma + \hat{q}} + e^{-\lambda - i\gamma + \hat{p}} \right) \cdot \frac{1}{1 + e^{-\lambda - i\gamma + \hat{q}} + e^{-\lambda - i\gamma + \hat{p}}} \cdot \hat{L}(\lambda; \hat{q}, \hat{p}).$$

(2.28)

\[^2\] Strictly speaking, the operator [2.24] is a non compact analogue of that was introduced in Ref. 23, i.e., we replace the $q$-exponential function [24] with the Faddeev integral integral [22, 23].
As two operators $\Delta \phi_n^{(1)}$ and $\Delta \phi_n^{(2)}$ are the Heisenberg operators, \[ [\Delta \phi_n^{(1)}, \Delta \phi_n^{(2)}] = 2i\gamma, \] we denote for brevity \[ \mathcal{L}_n(\lambda; \phi) = \hat{L}(\lambda; \Delta \phi_n^{(1)}, \Delta \phi_n^{(1)} + \Delta \phi_n^{(2)}). \] (2.29)

We collect useful properties of this operator in Appendix B. This operator acts as the intertwining operator for the vertex operators satisfying \[ L_{\lambda} = e^{\phi_n^{(a)}} + e^{\phi_{n+1}^{(a)}} \] for $a = 0, 1, 2$. The duality (A.2) of the integral (2.25) indicates that the operator $\mathcal{L}_n(\lambda; \phi)$ also intertwines the dual vertex operator; \[ \left( e^{\hat{\phi}_n^{(a)}} + e^{\hat{\phi}_{n+1}^{(a)}} \right) \cdot \mathcal{L}_n(\lambda; \phi) = \mathcal{L}_n(\lambda; \phi) \cdot \left( e^{\hat{\phi}_n^{(a)}} + e^{\hat{\phi}_{n+1}^{(a)}} \right). \] (2.30)

What is important here is that the fundamental $\mathcal{L}$ operator (2.29) itself satisfies the Yang–Baxter equation \[ \mathcal{L}_n(\lambda; \phi) \mathcal{L}_{n+1}(\mu; \phi) \mathcal{L}_n(\mu; \phi) = \mathcal{L}_{n+1}(\mu; \phi) \mathcal{L}_n(\lambda + \mu; \phi) \mathcal{L}_n(\lambda; \phi). \] (2.31)

Thus when we set the $\mathcal{Q}$ operator as \[ \mathcal{Q}(\lambda) = \prod_n \lambda_n f(\lambda) \cdot \mathcal{L}_n(\lambda; \phi), \] (2.33)

with a normalization $f(\lambda)$, we see that they commute between themselves with arbitrary spectral parameters; \[ [\mathcal{Q}(\lambda), \mathcal{Q}(\mu)] = 0. \] (2.34)

Furthermore the fundamental $\mathcal{L}$ operator also satisfies \[ \mathcal{L}_n(\lambda - \mu; \phi) \mathcal{L}_{n+1}(\lambda) \mathcal{L}_{n+1}(\mu) = \mathcal{L}_{n+1}(\mu) \mathcal{L}_n(\lambda) \mathcal{L}_n(\lambda - \mu; \phi), \] (2.35)

which proves that the transfer matrices $t_1(\lambda)$ and $t_2(\lambda)$ commute with the $\mathcal{Q}$ operator (2.33), \[ [\mathcal{Q}(\lambda), t_1(\mu)] = [\mathcal{Q}(\lambda), t_2(\mu)] = 0. \] (2.36)

In conclusion we have a commutative family; the transfer matrices $t_1(\lambda)$ and $t_2(\lambda)$, and the operator $\mathcal{Q}(\lambda)$. We note that intertwining relations (2.30) and (2.31) proves the commutativity between the $\mathcal{Q}$ operator and the screening charges, \[ [\mathcal{Q}(\lambda), \mathcal{Q}^{(a)}] = [\mathcal{Q}(\lambda), \hat{\mathcal{Q}}^{(a)}] = 0. \] (2.37)

As a consequence, the $\mathcal{Q}$ operator is a generating function of the local integrals of motion of the affine $\hat{sL}_3$ Toda field theory \[ [\mathcal{Q}(\lambda), \mathcal{Q}^{(a)}] = [\mathcal{Q}(\lambda), \hat{\mathcal{Q}}^{(a)}] = 0. \] (2.37)

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3 Baxter Equation

We shall show that the $Q$ operator defined in eq. (2.33) plays a role of the Baxter $Q$ operator for the transfer matrices (2.11) and (2.18) of the quantum discrete Boussinesq equation.

**Step 0** We prepare some formulae of the fundamental $L$ operator (2.29); using elements of matrix $C_{n} = \text{diag}(c_{n}^{(1)}, c_{n}^{(2)}, c_{n}^{(3)})$ in eq. (2.9), we have

$$e^{-\lambda \frac{1}{3} i \gamma} \cdot L_{n}(\lambda + \frac{2}{3} \imath \gamma; \phi) = L_{n}^{(1)}(\lambda; \phi) \cdot \left( c_{n}^{(1)} e^\lambda + c_{n}^{(2)} e^{-\lambda} + c_{n}^{(3)} \right)$$

$$= L_{n}^{(2)}(\lambda; \phi) \cdot \left( c_{n}^{(1)} + c_{n}^{(2)} e^\lambda + c_{n}^{(3)} e^{-\lambda} \right)$$

$$= L_{n}^{(3)}(\lambda; \phi) \cdot \left( c_{n}^{(1)} e^{-\lambda} + c_{n}^{(2)} + c_{n}^{(3)} e^\lambda \right),$$

$$e^{-\frac{1}{3} i \gamma (3 \lambda - 1)} \cdot L_{n}(\lambda - \frac{2}{3} \imath \gamma; \phi) = \left( c_{n+1}^{(a)} \right)^{-1} \cdot \left( L_{n}^{(1)}(\lambda; \phi) \cdot e^\lambda - L_{n}^{(2)}(\lambda; \phi) \right)$$

$$= \left( c_{n+1}^{(a)} \right)^{-1} \cdot \left( L_{n}^{(2)}(\lambda; \phi) \cdot e^\lambda - L_{n}^{(3)}(\lambda; \phi) \right)$$

$$= \left( c_{n+1}^{(a)} \right)^{-1} \cdot \left( L_{n}^{(3)}(\lambda; \phi) \cdot e^\lambda - L_{n}^{(1)}(\lambda; \phi) \right).$$

Here we have defined $L_{n}^{(a)}(\lambda; \phi)$ by

$$L_{n}^{(a)}(\lambda; \phi) = \left( c_{n+1}^{(a)} \right)^{-1} \cdot L_{n}(\lambda; \phi) \cdot c_{n+1}^{(a)},$$

which, using the commutation relations (2.3), are explicitly written as

$$L_{n}^{(1)}(\lambda; \phi) = \hat{L}(\lambda; \Delta \phi_{n}^{(1)} - \frac{2}{3} \imath \gamma; \Delta \phi_{n}^{(1)} + \Delta \phi_{n}^{(2)} + \frac{2}{3} \imath \gamma),$$

$$L_{n}^{(2)}(\lambda; \phi) = \hat{L}(\lambda; \Delta \phi_{n}^{(1)} - \frac{2}{3} \imath \gamma; \Delta \phi_{n}^{(1)} + \Delta \phi_{n}^{(2)} - \frac{4}{3} \imath \gamma),$$

$$L_{n}^{(3)}(\lambda; \phi) = \hat{L}(\lambda; \Delta \phi_{n}^{(1)} + \frac{4}{3} \imath \gamma; \Delta \phi_{n}^{(1)} + \Delta \phi_{n}^{(2)} + \frac{2}{3} \imath \gamma).$$

See Appendix C for proof of eqs. (3.1) and (3.2).

**Step 1** We first consider the product of the $Q$ operator (2.33) and the transfer matrix $t_{1}(\lambda)$. As we know from the commutation relations (2.2) of the lattice free field that a difference field $\Delta \phi_{n}^{(a)}$ does not commute only with nearest neighbor $\Delta \phi_{n \pm 1}^{(a)}$, we get

$$Q(\lambda) \cdot t_{1}(\lambda) = \text{Tr} \prod_{n} f(\lambda) g(\lambda) L_{n}(\lambda; \phi) C_{n+1} X(\lambda)$$

$$= \text{Tr} \prod_{n} f(\lambda) g(\lambda) M^{-1} C_{n+1}^{-1} L_{n}(\lambda; \phi) C_{n+1} X(\lambda) C_{n} M. \quad (3.3)$$
Here a matrix $M$ denotes a gauge-transformation. We substitute

$$
M = \begin{pmatrix}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1
\end{pmatrix},
$$

into eq. (3.3), and compute each element directly. We see that both (2,1) and (3,1) components vanishes due to eqs. (3.1). Thus we find that a trace of the product of $3 \times 3$ matrix reduces to a sum of the product of (1,1) component and a trace of the product of remaining $2 \times 2$ matrices;

$$
\mathcal{Q}(\lambda) \cdot t_1(\lambda) = \prod_n f(\lambda) g(\lambda) e^{-\lambda - \frac{2}{3} i \gamma} \cdot \mathcal{L}_n(\lambda + \frac{2}{3} i \gamma; \phi) + \text{Tr} \prod_n f(\lambda) g(\lambda) A_n(\lambda),
$$

where

$$
A_n(\lambda) = \begin{pmatrix}
-\mathcal{L}_n^{(1)}(\lambda; \phi) \cdot e^{-\lambda} + \mathcal{L}_n^{(2)}(\lambda; \phi) \cdot e^{\lambda} & -\mathcal{L}_n^{(1)}(\lambda; \phi) + \mathcal{L}_n^{(2)}(\lambda; \phi) \cdot e^{-\lambda} \\
-\mathcal{L}_n^{(1)}(\lambda; \phi) \cdot e^{-\lambda} + \mathcal{L}_n^{(3)}(\lambda; \phi) & -\mathcal{L}_n^{(1)}(\lambda; \phi) + \mathcal{L}_n^{(3)}(\lambda; \phi) \cdot e^{\lambda}
\end{pmatrix} \cdot \begin{pmatrix}
c_n^{(2)} \\
c_n^{(3)}
\end{pmatrix}.
$$

We now rewrite above $2 \times 2$ matrix $A_n(\lambda)$. Using eqs. (3.2), we have

$$
A_n(\lambda) = e^{-\frac{1}{3} i \gamma}(e^{3\lambda} - 1) \begin{pmatrix}
c_n^{(1)} e^{-\lambda} + c_n^{(3)} & -c_n^{(2)} e^{-\lambda} \\
c_n^{(1)} e^{-\lambda} & c_n^{(1)} e^{-\lambda}
\end{pmatrix} \cdot \mathcal{L}_n(\lambda - \frac{2}{3} i \gamma; \phi) \cdot \begin{pmatrix}
c_n^{(2)} \\
c_n^{(3)}
\end{pmatrix}.
$$

In the last equality, we have used the commutation relations (2.2) and a condition $c_n^{(1)} c_n^{(2)} c_n^{(3)} = e^{\frac{1}{3} i \gamma}$.

As a result we get

$$
\mathcal{Q}(\lambda) \cdot t_1(\lambda) = \prod_n f(\lambda) g(\lambda) e^{-\lambda - \frac{1}{3} i \gamma} \cdot \mathcal{L}_n(\lambda + \frac{2}{3} i \gamma; \phi) = \text{Tr} \prod_n f(\lambda) g(\lambda) e^{-\lambda}(e^{3\lambda} - 1) \begin{pmatrix}
(c_n^{(3)})^{-1} + (c_n^{(1)})^{-1} \cdot e^{\lambda - \frac{2}{3} i \gamma} & - (c_n^{(1)})^{-1} \\
(c_n^{(3)})^{-1} & (c_n^{(2)})^{-1} \cdot e^{\lambda - \frac{2}{3} i \gamma}
\end{pmatrix} \cdot \begin{pmatrix}
L_n^{(3)}(\lambda - \frac{2}{3} i \gamma; \phi) \\
L_n^{(1)}(\lambda - \frac{2}{3} i \gamma; \phi)
\end{pmatrix}.
$$

We keep aside this identity for a while.
**Step 2** In the same manner with Step 1, we study a product of the $Q$ operator and the transfer matrix $t_2(\lambda)$. We have

\[
Q(\lambda) \cdot t_2(\lambda) = \text{Tr} \left( \prod_n f(\lambda) \bar{\Phi}(\lambda) L_n(\lambda; \phi) C_{n+1}^{-1} X(\lambda) \right) \\
= \text{Tr} \left( \prod_n f(\lambda) \bar{\Phi}(\lambda) C_{n+1} L_n(\lambda; \phi) C_{n+1}^{-1} X(\lambda) C_n^{-1} \right) \\
= \text{Tr} \left( \prod_n f(\lambda) \bar{\Phi}(\lambda) M \cdot \left( \begin{array}{ccc}
(c^{(1)}_n)^{-1} e^\lambda \cdot L^{(3)}_n(\lambda; \phi) & -(c^{(2)}_n)^{-1} \cdot L^{(2)}_n(\lambda; \phi) & 0 \\
0 & (c^{(2)}_n)^{-1} e^\lambda \cdot L^{(1)}_n(\lambda; \phi) & -(c^{(3)}_n)^{-1} \cdot L^{(3)}_n(\lambda; \phi) \\
-(c^{(1)}_n)^{-1} \cdot L^{(1)}_n(\lambda; \phi) & 0 & (c^{(3)}_n)^{-1} e^\lambda \cdot L^{(2)}_n(\lambda; \phi) 
\end{array} \right) \right) \cdot M^{-1}.
\]

In the last expression, we have introduced matrix $M$ for gauge-transformation. When we substitute

\[
M = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix},
\]

we see that both $(3,1)$ and $(3,2)$ elements vanish due to eqs. (3.2), and that a trace of the product of $3 \times 3$ matrices can be rewritten as a sum of the product of $(3,3)$-element and a trace of the product of $2 \times 2$ matrices;

\[
Q(\lambda) \cdot t_2(\lambda) = \prod_n f(\lambda) \bar{\Phi}(\lambda) e^{-\frac{4}{3} \gamma} (e^{3\lambda} - 1) \cdot L_n(\lambda - \frac{2}{3} i \gamma, \phi) + \text{Tr} \prod_n f(\lambda) \bar{\Phi}(\lambda) B_n(\lambda),
\]

where $2 \times 2$ matrix $B_n(\lambda)$ is given by

\[
B_n(\lambda) = \left( \begin{array}{cc}
(c^{(1)}_n)^{-1} e^\lambda \cdot L^{(3)}_n(\lambda; \phi) & -(c^{(2)}_n)^{-1} \cdot L^{(2)}_n(\lambda; \phi) \\
(c^{(3)}_n)^{-1} \cdot L^{(3)}_n(\lambda; \phi) & (c^{(2)}_n)^{-1} e^\lambda \cdot L^{(1)}_n(\lambda; \phi) + (c^{(3)}_n)^{-1} \cdot L^{(3)}_n(\lambda; \phi) 
\end{array} \right).
\]

We see that by applying eqs. (3.2) to (2,2)-element of above matrix $B_n(\lambda)$ we have

\[
\text{Tr} \prod_n f(\lambda) \bar{\Phi}(\lambda) B_n(\lambda) = \text{Tr} \prod_n f(\lambda) \bar{\Phi}(\lambda) M \cdot \left( \begin{array}{cc}
(c^{(1)}_n)^{-1} e^\lambda \cdot L^{(3)}_n(\lambda; \phi) & -(c^{(2)}_n)^{-1} \cdot L^{(2)}_n(\lambda; \phi) \\
(c^{(3)}_n)^{-1} \cdot L^{(3)}_n(\lambda; \phi) & (c^{(2)}_n)^{-1} e^\lambda + (c^{(3)}_n)^{-1} \cdot L^{(2)}_n(\lambda; \phi) 
\end{array} \right) \cdot M^{-1},
\]

where the invertible matrix $M$ is included again for gauge-transformation. Substituting

\[
M = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix},
\]

we have
and erasing $\mathcal{L}_n^{(2)}(\lambda; \phi)$ using eqs. (3.3), we get

$$
\text{Tr} \prod_n f(\lambda) \overline{g}(\lambda) B_n(\lambda) = \text{Tr} \prod_n f(\lambda) \overline{g}(\lambda) \begin{pmatrix}
(c_n^{(1)})^{-1} e^{2i\lambda} + (c_n^{(3)})^{-1} - (c_n^{(1)})^{-1} (c_n^{(2)})^{-1} e^{4i\lambda}
\end{pmatrix} \cdot \begin{pmatrix}
\mathcal{L}_n^{(3)}(\lambda; \phi) \\
\mathcal{L}_n^{(1)}(\lambda; \phi)
\end{pmatrix}.
$$

As a result we obtain an identity,

$$
\mathcal{Q}(\lambda) \cdot t_2(\lambda) - \prod_n f(\lambda) \overline{g}(\lambda) e^{-\frac{2}{3} i \gamma} (e^{3\lambda} - 1) \cdot \mathcal{L}_n(\lambda - \frac{2}{3} i \gamma; \phi) = \text{Tr} \prod_n f(\lambda) \overline{g}(\lambda) \begin{pmatrix}
(c_n^{(1)})^{-1} e^{2i\lambda} + (c_n^{(3)})^{-1} - (c_n^{(1)})^{-1} (c_n^{(2)})^{-1} e^{4i\lambda}
\end{pmatrix} \cdot \begin{pmatrix}
\mathcal{L}_n^{(3)}(\lambda; \phi) \\
\mathcal{L}_n^{(1)}(\lambda; \phi)
\end{pmatrix}.
$$

**Step 3** By comparing eq. (3.5) and eq. (3.7), we find that $2 \times 2$ matrices in the right hand sides have same form with each other. By erasing these $2 \times 2$ matrices, we obtain an identity for operators $t_1(\lambda), t_2(\lambda)$, and $\mathcal{Q}(\lambda);

$$
\mathcal{Q}(\lambda) \cdot t_1(\lambda) - \begin{pmatrix}
\prod_n f(\lambda) \\
\prod_n f(\lambda - \frac{2}{3} i \gamma)
\end{pmatrix} g(\lambda) e^{-\lambda - \frac{3}{2} i \gamma} \cdot \mathcal{Q}(\lambda + \frac{2}{3} i \gamma)
= \begin{pmatrix}
\prod_n f(\lambda) \\
\prod_n f(\lambda - \frac{2}{3} i \gamma)
\end{pmatrix} g(\lambda) e^{-\lambda - \frac{3}{2} i \gamma} (e^{3\lambda} - 1) \cdot \mathcal{Q}(\lambda - \frac{2}{3} i \gamma) \cdot t_2(\lambda - \frac{2}{3} i \gamma) - \begin{pmatrix}
\prod_n f(\lambda) \\
\prod_n f(\lambda - \frac{2}{3} i \gamma)
\end{pmatrix} g(\lambda) e^{-\lambda - \frac{3}{2} i \gamma} (e^{3\lambda} - 1) (e^{3\lambda - 2i\gamma} - 1) \cdot \mathcal{Q}(\lambda - \frac{4}{3} i \gamma).
$$

After setting normalization functions as

$$
f(\lambda) = 1,
g(\lambda) = e^{\lambda + \frac{3}{4} i \gamma},
\overline{g}(\lambda) = e^{\frac{1}{2} i \gamma} (e^{3\lambda + 2i\gamma} - 1),
$$

we see that the operator $\mathcal{Q}(\lambda)$ solves the difference equation,

$$
\mathcal{Q}(\lambda + 2 i \gamma) - t_1(\lambda + \frac{4}{3} i \gamma) \cdot \mathcal{Q}(\lambda + \frac{4}{3} i \gamma) + t_2(\lambda + \frac{2}{3} i \gamma) \cdot \mathcal{Q}(\lambda + \frac{2}{3} i \gamma) - \Delta(\lambda + \frac{2}{3} i \gamma) \cdot \mathcal{Q}(\lambda) = 0,
$$

where the function $\Delta(\lambda)$ is defined by

$$
\Delta(\lambda) = ((e^{3\lambda} - 1) (e^{3\lambda + 2i\gamma} - 1))^{\lambda}.
$$

12
As we know from eq. (2.36) that the transfer matrices \( t_{1,2}(\lambda) \) and the \( Q \) operator commute each other, the operator valued difference equation (3.8) can be regarded as the third order difference equation in usual sense, once we apply both hand sides to simultaneous eigenfunction.

We expect in general the third order difference equation (3.8) has three linearly independent solutions, which we write \( Q_a(\lambda) \) for \( a = +, 0, - \). We then introduce the function \( P(\lambda) \) as the quantum Wronskian of solutions;

\[
P_{ab}(\lambda) = \begin{vmatrix} Q_a(\lambda) & Q_a(\lambda - \frac{2}{3}i\gamma) \\ Q_b(\lambda) & Q_b(\lambda - \frac{2}{3}i\gamma) \end{vmatrix},
\]

for \( a, b = 0, \pm \) (we thus have three independent functions \( P_{ab}(\lambda) \)). It is straightforward to see that the function \( P_{ab}(\lambda) \) satisfies the third order difference equation,

\[
P(\lambda + 2i\gamma) - t_2(\lambda + \frac{2}{3}i\gamma) \cdot P(\lambda + \frac{4}{3}i\gamma) \\
+ \Delta(\lambda + \frac{2}{3}i\gamma) t_1(\lambda + \frac{2}{3}i\gamma) \cdot P(\lambda + \frac{2}{3}i\gamma) - \Delta(\lambda + \frac{2}{3}i\gamma) \cdot P(\lambda) = 0.
\]

From two difference equations (3.8) and (3.10), we get the Baxter equation for the discrete Boussinesq equation;

\[
t_1(\lambda) = \Delta(\lambda - \frac{2}{3}i\gamma) \frac{P(\lambda - \frac{2}{3}i\gamma)}{P(\lambda)} + \frac{P(\lambda + \frac{2}{3}i\gamma)}{P(\lambda)} \cdot \frac{Q(\lambda - \frac{2}{3}i\gamma)}{Q(\lambda)} + \frac{Q(\lambda + \frac{2}{3}i\gamma)}{Q(\lambda)},
\]

\[
t_2(\lambda) = \frac{P(\lambda + \frac{2}{3}i\gamma)}{P(\lambda + \frac{2}{3}i\gamma)} + \Delta(\lambda) \frac{P(\lambda)}{P(\lambda + \frac{2}{3}i\gamma)} \cdot \frac{Q(\lambda + \frac{2}{3}i\gamma)}{Q(\lambda)} + \Delta(\lambda) \frac{Q(\lambda - \frac{2}{3}i\gamma)}{Q(\lambda)}.
\]

Once we have a set of functional equations (3.11) we can follow a strategy in Ref. 25 to obtain the spectrum of these transfer matrices; we suppose

\[
Q(\lambda) = \prod_n \text{sh}(\lambda - \lambda_n^{(1)}), \quad P(\lambda) = \prod_n \text{sh}(\lambda - \lambda_n^{(2)}),
\]

and substitute them into eqs. (3.11). From the condition of analyticity of the transfer matrices \( t_a(\lambda) \) we get the nested Bethe ansatz equations. The precise analysis of these equations is for future studies. As a consequence, we have diagonalized the transfer matrices \( t_{1,2}(\lambda) \) by use of the auxiliary functions \( Q(\lambda) \) and \( P(\lambda) \).

4 Duality

We have seen that the \( Q \) operator (2.33) satisfies the third order difference equation (3.8) whose coefficients are given by two transfer matrices \( t_1(\lambda) \) and \( t_2(\lambda) \) for the quantum discrete Boussinesq
equation. As was noticed in Appendix A the \( Q \) operator has a property of duality which follows from eq. (2.31) and eq. (A.2):

\[
\gamma \rightarrow \frac{\pi^2}{\gamma}, \quad \phi_n^{(a)} \rightarrow \frac{\pi}{\gamma} \phi_n^{(a)}, \quad \lambda \rightarrow \frac{\pi}{\gamma} \lambda.
\]  

(4.1)

Above duality implies the existence of the dual Baxter equation; we have the dual third order difference equation,

\[
Q(\lambda + 2i\pi) - u_1(\lambda + \frac{4}{3}i\pi) \cdot Q(\lambda + \frac{4}{3}i\pi) \\
+ u_2(\lambda + \frac{2}{3}i\pi) \cdot Q(\lambda + \frac{2}{3}i\pi) - \tilde{\Delta}(\lambda + \frac{2}{3}i\pi) \cdot \Delta(\lambda + \frac{2}{3}i\pi) = 0,
\]  

(4.2)

where

\[
\tilde{\Delta}(\lambda) = \left((e^{\frac{2\pi}{3}i\lambda} - 1) (e^{\frac{2\pi}{3}(3\lambda + 2i\pi)} - 1)\right)^L.
\]

The operator \( Q(\lambda) \) is as same with eq. (2.33), and the transfer matrix \( u_a(\lambda) \) is defined by

\[
u_1(\lambda) = \text{Tr} \ U(\lambda), \quad U(\lambda) = \prod_n M_n(\lambda),
\]  

(4.3)

\[
u_2(\lambda) = \text{Tr} \ U(\lambda), \quad U(\lambda) = \prod_n M_n(\lambda),
\]  

(4.4)

with the dual Lax matrix \( M(\lambda) \) and \( \overline{M}(\lambda) \),

\[
M_n(\lambda) = h(\lambda) \begin{pmatrix}
e^{\frac{\pi}{\gamma}} \left(\frac{1}{3} \Delta \phi_n^{(1)} - \frac{4}{3} \Delta \phi_n^{(2)}\right) \\
e^{\frac{\pi}{\gamma}} \left(\frac{1}{3} \Delta \phi_n^{(1)} + \frac{2}{3} \Delta \phi_n^{(2)}\right) \\
e^{\frac{\pi}{\gamma}} \left(-\frac{2}{3} \Delta \phi_n^{(1)} - \frac{1}{3} \Delta \phi_n^{(2)}\right)
\end{pmatrix} \cdot \begin{pmatrix} e^{\frac{\pi}{\gamma}} \lambda & e^{-\frac{\pi}{\gamma}} \lambda & 1 \\
1 & e^{\frac{\pi}{\gamma}} \lambda & -e^{-\frac{\pi}{\gamma}} \lambda \\
e^{-\frac{\pi}{\gamma}} \lambda & 1 & e^{\frac{\pi}{\gamma}} \lambda
\end{pmatrix},
\]

\[
\overline{M}_n(\lambda) = \overline{h}(\lambda) \begin{pmatrix} e^{\frac{\pi}{\gamma}} \left(-\frac{1}{3} \Delta \phi_n^{(1)} + \frac{2}{3} \Delta \phi_n^{(2)}\right) \\
e^{\frac{\pi}{\gamma}} \left(-\frac{2}{3} \Delta \phi_n^{(1)} - \frac{1}{3} \Delta \phi_n^{(2)}\right) \\
e^{\frac{\pi}{\gamma}} \left(-\frac{2}{3} \Delta \phi_n^{(1)} + \frac{4}{3} \Delta \phi_n^{(2)}\right)
\end{pmatrix} \cdot \begin{pmatrix} e^{\frac{\pi}{\gamma}} \lambda & -1 & 0 \\
0 & e^{\frac{\pi}{\gamma}} \lambda & -1 \\
-1 & 0 & e^{\frac{\pi}{\gamma}} \lambda
\end{pmatrix}.
\]

To get the dual Baxter equation, another operator \( \mathcal{P} \) should be slightly different from eq. (3.9),

\[
\mathcal{P}_{ab}(\lambda) = \begin{vmatrix} Q_a(\lambda) & Q_a(\lambda - \frac{4}{3}i\pi) \\
Q_b(\lambda) & Q_b(\lambda - \frac{2}{3}i\pi)\end{vmatrix},
\]  

(4.5)

where \( Q_a(\lambda) \) denotes three linearly independent solutions of the difference equation (4.2). This solves the third order difference equation,

\[
\mathcal{P}(\lambda + 2i\pi) - u_2(\lambda + \frac{2}{3}i\pi) \cdot \mathcal{P}(\lambda + \frac{4}{3}i\pi) \\
+ \tilde{\Delta}(\lambda + \frac{2}{3}i\pi) u_1(\lambda + \frac{2}{3}i\pi) \cdot \mathcal{P}(\lambda + \frac{2}{3}i\pi) - \tilde{\Delta}(\lambda + \frac{2}{3}i\pi) \cdot \mathcal{P}(\lambda) = 0.
\]  

(4.6)
As a result, we obtain the dual Baxter equations,

\[ u_1(\lambda) = \tilde{\Delta} (\lambda - \frac{2}{3} i \pi) \frac{\tilde{P}(\lambda - \frac{2}{3} i \pi)}{\tilde{P}(\lambda)} \cdot \frac{Q(\lambda - \frac{2}{3} i \pi)}{Q(\lambda)} + \frac{Q(\lambda + \frac{2}{3} i \pi)}{Q(\lambda)}, \]  

\[ u_2(\lambda) = \tilde{\Delta} (\lambda - \frac{2}{3} i \pi) \frac{\tilde{P}(\lambda + \frac{2}{3} i \pi)}{\tilde{P}(\lambda + \frac{2}{3} i \pi)} \cdot \frac{Q(\lambda + \frac{2}{3} i \pi)}{Q(\lambda)} + \frac{Q(\lambda - \frac{2}{3} i \pi)}{Q(\lambda)}, \]  

from which we get the nested Bethe ansatz equations only from the analytic property of the transfer matrices \( u_a(\lambda) \).

We remark that this duality of the Baxter equation also appeared in studies of the discrete KdV equation [31], where given was the interpretation from the viewpoint of the algebraic geometry.

5 Continuum Limit

We briefly study the continuum limit of the quantum discrete Boussinesq equation. In a continuum limit, the lattice free field \( \phi_n^{(a)} \) is replaced by \( \phi^{(a)}(x) \), and the commutation relations (2.2) then reduce into

\[ [\phi^{(a)}(x), \phi^{(b)}(y)] = C_{ab} \gamma \text{sgn}(x - y), \]  

where \( C \) is the Cartan matrix of \( s\ell_3 \),

\[ C = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}. \]

The quasi-periodic boundary condition (2.15) denotes

\[ \phi^{(a)}(x + L) = \phi^{(a)}(x) + L \cdot i P^{(a)}, \]

which indicates the mode expansion of the free fields as [32]

\[ \phi^{(b)}(x) = i Q^{(b)} + i P^{(b)} x + \sum_{n \neq 0} \frac{a_n^{(b)}}{n} e^{i \frac{2\pi}{L} n x}. \]

We can take the continuum limit of the transfer matrices \( t_{1,2}(\lambda) \) from eq. (2.16) and eq. (2.19).
After a gauge-transformation, we get

\[
(g(\lambda) e^{\lambda + \frac{1}{3} i \gamma})^{-L} \cdot t_1(\lambda) \\
\rightarrow \text{Tr} \left[ e^{iL(P^{(1)} h_1 - P^{(2)} h_3)} \mathcal{P} \exp \left( \int_0^L dx \begin{pmatrix} 0 & e^{-2\lambda - \frac{1}{3} i \gamma} e^{-\phi(1)(x)} & e^{-\lambda + \frac{1}{3} i \gamma} e^{\phi(0)(x)} \\ e^{-2\lambda - \frac{1}{3} i \gamma} e^{\phi(1)(x)} & 0 & e^{-\lambda + \frac{1}{3} i \gamma} e^{\phi(0)(x)} \\ e^{-\lambda + \frac{1}{3} i \gamma} e^{-\phi(2)(x)} & e^{-\lambda + \frac{1}{3} i \gamma} e^{-\phi(2)(x)} & 0 \end{pmatrix} \right) \right],
\]

(5.2)

\[
(g(\lambda) e^{\lambda + \frac{1}{3} i \gamma})^{-L} \cdot t_2(\lambda) \\
\rightarrow \text{Tr} \left[ e^{iL(-P^{(1)} h_1 + P^{(2)} h_3)} \mathcal{P} \exp \left( -e^{-\lambda + \frac{1}{3} i \gamma} \int_0^L dx \begin{pmatrix} 0 & e^{\phi(1)(x)} & 0 \\ e^{\phi(0)(x)} & 0 & e^{\phi(2)(x)} \end{pmatrix} \right) \right],
\]

(5.3)

where \( \mathcal{P} \) denotes the path operator ordering. One finds that the transfer matrix \( t_2(\lambda) \) in eq. (5.3) essentially coincides with that was proposed in [32] as a quantized Drinfeld–Sokolov reduction.

Correspondingly a difference of the free fields (2.4) becomes the differential of the free field,

\[
\Delta \phi_n^{(a)} \rightarrow -\frac{\partial}{\partial x} \phi_n^{(a)}(x),
\]

and the Baxter \( Q \) operator for the quantum Boussinesq equation can be given by the path ordering of the operator \( L_n(\lambda; \phi) \). As a result we recover the third order difference equation equation (3.8). At this stage we are not sure how to relate the \( Q \) operator with the universal \( R \) matrix.

### 6 Generalization: \( N \)-reduced Discrete KP Equation

As the Boussinesq equation (1.2) is the 3-reduced KP equation, we can introduce the quantum \( N \)-reduced discrete KP equation which is related with the affine \( \widehat{\mathfrak{sl}}_N \) Toda field theory. In the classical case, the Lax matrix for the discrete \( N \)-reduced KP equation is given by \([33, 34]\)

\[
L^W_n(x) = \frac{1}{\sqrt{W_n}} \begin{pmatrix} x^{N-1} & -x^{N-2} W_{n+N-2}^{(2)} & x^{N-3} W_{n+N-3}^{(3)} & \ldots & (-1)^{N-2} x W_{n+1}^{(N-1)} & (-1)^{N-1} W_n^{(N)} \\ 1 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 1 & 0 & \ldots & 0 & 0 \\ \vdots & 0 & 1 & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & \ldots & 1 & 0 \end{pmatrix},
\]

(6.1)
where the generators of the lattice $W_N$ algebra are defined as

$$W_{n}^{(s)} = \frac{1}{\prod_{k=0}^{s-1} \left( 1 + \sum_{i=1}^{N-1} \exp(\chi_{n+k}^{(i)}) \right)} \left( \sum_{1 \leq i_1 < i_2 < \cdots < i_{s-1} \leq N-1} \exp \left( \chi_{n+s-1}^{(i_1)} + \chi_{n+s-2}^{(i_2)} + \cdots + \chi_{n+1}^{(i_{s-1})} \right) \right) + \sum_{1 \leq i_1 < i_2 < \cdots < i_s \leq N-1} \exp \left( \chi_{n+s-1}^{(i_1)} + \chi_{n+s-2}^{(i_2)} + \cdots + \chi_{n}^{(i_s)} \right), \quad \text{for } s = 2, 3, \ldots, N-1,$$

$$W_{n}^{(N)} = \frac{\exp \left( \sum_{i=1}^{N-1} \chi_{n+N-i}^{(i)} \right)}{\prod_{k=0}^{N-1} \left( 1 + \sum_{i=1}^{N-1} \exp(\chi_{n+k}^{(i)}) \right)}.$$

(6.2)

Here $\chi_{n}^{(a)}$ is defined in terms of the free fields $\phi_{n}^{(a)}$ (for $a = 1, 2, \ldots, N - 1$) as

$$\chi_{n}^{(a)} = \sum_{b=1}^{a} \Delta \phi_{n}^{(b)},$$

and the quantum algebra for the lattice free fields reads as

$$[\phi_{n}^{(a)}, \phi_{n}^{(a)}] = 2i\gamma, \quad \text{for } m > n,$$

$$[\phi_{n}^{(a)}, \phi_{m}^{(a+1)}] = \begin{cases} i\gamma, & \text{for } n \leq m, \\ -i\gamma, & \text{for } n > m. \end{cases}$$

(6.3)

We note that the transformation (6.2) is the so-called Miura transformation. We set

$$\phi_{n}^{(0)} = -\sum_{a=1}^{N-1} \phi_{n}^{(a)},$$

and the screening charges are defined by (for $a = 0, 1, \ldots, N - 1$)

$$Q^{(a)} = \sum_{n} e^{\phi_{n}^{(a)}}, \quad \tilde{Q}^{(a)} = \sum_{n} e^{\frac{\pi}{N} \phi_{n}^{(a)}}.$$

(6.4)

The sum of the screening charges, $\sum_{a=0}^{N-1} Q^{(a)}$, corresponds to the Hamiltonian of the affine Toda field theory.

In terms of the free fields, the quantum Lax matrix is then defined by

$$L_{n}(\lambda) = \exp \left( \sum_{a=1}^{N-1} h_{a} \chi_{n}^{(a)} \right) \cdot X(\lambda),$$

(6.5)

where $N \times N$ matrices $h_{a}$ and $X(\lambda)$ are defined by

$$h_{a} = E_{a,a} - \frac{1}{N} \mathbb{1}, \quad X(\lambda) = \sum_{k=0}^{N-1} e^{-k\lambda} C^{k}, \quad C = E_{1,N} + \sum_{k=1}^{N-1} E_{k+1,k}.$$
The spectral parameter \( \lambda \) is related with \( x \) in eq. \((6.1)\) by
\[
e^{-N\lambda} = 1 - x^{-N}.
\]

As usual the monodromy and the transfer matrix are respectively given by
\[
T(\lambda) = \prod_n L_n(\lambda), \quad t_1(\lambda) = \text{Tr} T(\lambda). \tag{6.6}
\]

This transfer matrix generates the integrals of motion of the discrete \( N \)-reduced KP equation.

The fundamental \( \mathcal{L} \) operator for \( sl_N \) case is defined by \([22]\):
\[
\mathcal{L}_n(\lambda; \phi) = \frac{1}{\Phi_\gamma(\lambda + (N-2)i\gamma)} \cdots \frac{1}{\Phi_\gamma(\lambda + (N-1)i\gamma)} \times \frac{\Theta_\gamma(\chi_n(1))}{\Phi_\gamma(\lambda - \chi_n(1))} \cdots \frac{\Theta_\gamma(\chi_n(N-1))}{\Phi_\gamma(\lambda - \chi_n(N-1))}, \tag{6.7}
\]

This operator not only satisfies the Yang–Baxter equation \((2.32)\), but fulfills the intertwining relations for the (dual) vertex operators \([35]\):
\[
\left( e^{\phi_n^{(a)}} + e^{\lambda} e^{\phi_{n+1}^{(a)}} \right) \cdot \mathcal{L}_n(\lambda; \phi) = \mathcal{L}_n(\lambda; \phi) \cdot \left( e^{\lambda} e^{\phi_n^{(a)}} + e^{\phi_{n+1}^{(a)}} \right), \tag{6.8}
\]
\[
\left( e^{\phi_n^{(a)}} + e^{\lambda} e^{\phi_{n+1}^{(a)}} \right) \cdot \mathcal{L}_n(\lambda; \phi) = \mathcal{L}_n(\lambda; \phi) \cdot \left( e^{\lambda} e^{\phi_n^{(a)}} + e^{\phi_{n+1}^{(a)}} \right), \tag{6.9}
\]
for \( a = 0, 1, \ldots, N - 1 \). We can easily see that the \( Q \) operator
\[
Q(\lambda) = \prod_n \mathcal{L}_n(\lambda; \phi), \tag{6.10}
\]
commute with the screening charges
\[
[Q(\lambda), Q^{(a)}] = [\hat{Q}(\lambda), \hat{Q}^{(a)}] = 0, \tag{6.11}
\]
which proves that the \( Q \) operator generates the local integrals of motion for the discrete analogue of the affine \( \hat{sl}_N \) Toda field theory.

As we have clarified for \( sl_3 \) case in preceding sections, the generating function of the local integrals of motion for the affine Toda field theory becomes the \( Q \) operator for the quantum Boussinesq equation. Thus it is natural to conjecture that the \( Q \) operator \((6.10)\) satisfies the \( N \)-th order difference equation,
\[
Q(\lambda + 2i\gamma) - \sum_{k=1}^{N-1} t_k(\lambda + 2\frac{N-k}{N}i\gamma) \cdot Q(\lambda + 2\frac{N-k}{N}i\gamma) + \Delta(\lambda + \frac{2}{N}i\gamma) \cdot Q(\lambda) = 0, \tag{6.12}
\]
with some function $\Delta(\lambda)$. Here the transfer matrix $t_1(\lambda)$ is as eq. (6.6) and corresponds to the vector representation. Other transfer matrices $t_{k>1}(\lambda)$ are for $[1^k]$, and can be constructed by the fusion method from the transfer matrix $t_1(\lambda)$ with a suitable normalization. Especially we have

$$
t_{N-1}(\lambda) \propto \text{Tr} \left( \prod_n \exp \left( - \sum_{a=1}^{N-1} h_a \chi_a^{(n)} \right) \cdot \mathbf{X}(\lambda) \right),
$$

with $\mathbf{X}(\lambda) = 1 - e^{-\lambda} \mathbf{G}^{-1}$. The duality of the $Q$ operator also suggests the dual difference equation,

$$
Q(\lambda + 2i\pi) - \sum_{k=1}^{N-1} u_k(\lambda + 2\cdot \frac{N-k}{N} i\pi) \cdot Q(\lambda + 2\cdot \frac{N-k}{N} i\pi) + \Delta(\lambda + 2\cdot \frac{N}{N} i\pi) \cdot Q(\lambda) = 0,
$$

(6.13)

where the transfer matrices $u_k(\lambda)$ follows from $t_k(\lambda)$ under a duality transform [6.1].

## 7 Concluding Remarks

We have explicitly constructed the Baxter $Q$ operator for the quantum discrete Boussinesq equation which has $U_q(\hat{sl}_3)$. As far as the author knows, the explicit form of the $Q$ operator has been known only for the quantum integrable systems associated to $sl_2$, and our $Q$ operator is the first example which solves the third order difference equation. In our construction, the $Q$ operator originates from the generating function of the local integrals of motion for a discrete analogue of the affine $\hat{sl}_3$ Toda field theory, and we hope this can be applied for other Lie algebras, to which some attempts have been made [36]. Important structure is that it has a kind of duality, and that the $Q$ operator satisfies the dual Baxter equation. It was shown [31] in the case of the discrete KdV equation that the dual Baxter equations can be interpreted as the duality between homologies and cohomologies of quantized affine hyper-elliptic Jacobian from the view point of the algebraic geometry. It will be interesting to give such interpretation in our case. In such studies the separation of variables (see Ref. [37] for review, and Ref. [38] for $sl_3$ case) will be useful.

Also in $sl_2$ case, the universal procedure to derive the Baxter equation was proposed [14], where the $Q$ operator was constructed from the universal $\mathcal{R}$ matrix with the infinite-dimensional $q$-oscillator representation. This may be true for $sl_N$ case, but we are not sure how to relate our $Q$ operator (6.10) with the universal $\mathcal{R}$ matrix.

The Baxter equation for $sl_2$ appeared in studies of the second order ordinary differential equations; the spectral determinants for the Schrödinger equation is identified with the $Q$ operator for the CFT [39–41]. Such correspondence is also studied for the higher order differential equation [42, 43]. This “ODE/IM correspondence” seems to come from a similarity between the Stokes multiplier and the Lax matrix, and our results will help further investigations on these topics.
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A Quantum Dilogarithm Function

We review a \( q \)-deformation of the dilogarithm function (2.25);

\[
\Phi_\gamma(\varphi) = \exp \left( \int_{\mathbb{R} + i 0} e^{-i \varphi x} \frac{dx}{4 \sh(\gamma x) \sh(\pi x)} \right). \tag{2.25}
\]

This integral was introduced by Faddeev [27], and it corresponds to a non-compact analogue of the \( q \)-exponential function,

\[
S_q(w) = \prod_{n=0}^{\infty} \left( 1 + q^{2n+1} w \right) = \exp \left( \sum_{k=1}^{\infty} \frac{(-1)^k w^k}{k(q^k - q^{-k})} \right). \tag{A.1}
\]

We list properties of the integral (2.25) in order [44].

• Duality

\[
\Phi_{\frac{\gamma}{\pi}}(\varphi) = \Phi_{\frac{\gamma}{\pi}}(\varphi), \tag{A.2}
\]

• Zero points,

zeros of \( (\Phi_\gamma(\varphi))^\pm 1 = \{ \mp i((2m+1)\gamma + (2n+1)\pi) \} m, n \in \mathbb{Z}_{\geq 0} \} \)

• Inversion relation,

\[
\Theta_\gamma(\varphi) \equiv \Phi_\gamma(\varphi) \cdot \Phi_\gamma(-\varphi)
= \exp \left( -\frac{1}{2i\gamma} \left( \frac{\varphi^2}{2} + \frac{\pi^2 + \gamma^2}{6} \right) \right). \tag{A.3}
\]

A reason why we use notation “\( \Theta \)” is that it is proportional to the Jacobi theta function when we replace the integral \( \Phi_\gamma(\varphi) \) with \( S_q(w), S_q(w) S_q(w^{-1}) \).

• Difference equations,

\[
\frac{\Phi_\gamma(\varphi + i \gamma)}{\Phi_\gamma(\varphi - i \gamma)} = \frac{1}{1 + e^{\varphi}}. \tag{A.4}
\]

The duality (A.2) gives another type of difference equation,

\[
\frac{\Phi_\gamma(\varphi + i \pi)}{\Phi_\gamma(\varphi - i \pi)} = \frac{1}{1 + e^{\pi \varphi}}. \tag{A.5}
\]

• Pentagon relation [44, 45]

\[
\Phi_\gamma(\hat{p}) \Phi_\gamma(\hat{q}) = \Phi_\gamma(\hat{q}) \Phi_\gamma(\hat{p} + \hat{q}) \Phi_\gamma(\hat{p}), \tag{A.6}
\]

where \( \hat{p} \) and \( \hat{q} \) are the Heisenberg operators satisfying a commutation relation,

\[
[\hat{p}, \hat{q}] = -2i \gamma. \tag{A.7}
\]

This identity is a key to apply the integral (2.25) to the quantization of the Teichmüller space [45] and to the construction of the invariant of 3-manifold [46, 47].
B Properties of the Fundamental $\mathcal{L}$ Operator

We study the property of the fundamental $\mathcal{L}$ operator (2.29).

B.1 $\mathbb{Z}_3$ Symmetry

By construction the free fields $\phi^{(1)}_n$, $\phi^{(2)}_n$, and $\phi^{(0)}_n = -\phi^{(1)}_n - \phi^{(2)}_n$ correspond to roots $\alpha_1$, $\alpha_2$, and $\alpha_0$. As the fundamental $\mathcal{L}$ operator $\mathcal{L}_n(\lambda; \phi)$ is defined as the intertwining operators for these $\mathbb{Z}_3$ symmetric vertex operators (2.30), we can suppose the $\mathcal{L}$ operator itself has the $\mathbb{Z}_3$ symmetry. This can be checked by recursive uses of the pentagon identity (A.6) as

$$
\mathcal{L}_n(\lambda; \phi) = \left( \Phi_\gamma(\lambda - \Delta\phi^{(2)}_n) \Phi_\gamma(\lambda - \Delta\phi^{(1)}_n) \Phi_\gamma(2\lambda + \Delta\phi^{(1)}_n + \Delta\phi^{(2)}_n) \right)^{-1} \\
\times \Theta_\gamma(\Delta\phi^{(2)}_n) \Theta_\gamma(\Delta\phi^{(1)}_n) \\
= \left( \Phi_\gamma(\lambda + \Delta\phi^{(1)}_n + \Delta\phi^{(2)}_n) \Phi_\gamma(\lambda - \Delta\phi^{(2)}_n) \Phi_\gamma(2\lambda + \Delta\phi^{(1)}_n + \Delta\phi^{(2)}_n) \Phi_\gamma(\lambda - \Delta\phi^{(2)}_n) \right)^{-1} \\
\times \Theta_\gamma(\Delta\phi^{(1)}_n + \Delta\phi^{(2)}_n) \Theta_\gamma(\Delta\phi^{(2)}_n). 
$$

As a result, we see that

$$
\mathcal{L}_n(\lambda; \phi) \equiv \hat{\mathcal{L}}(\lambda; \Delta\phi^{(1)}_n, -\Delta\phi^{(0)}_n) = \hat{\mathcal{L}}(\lambda; \Delta\phi^{(2)}_n, -\Delta\phi^{(1)}_n) = \hat{\mathcal{L}}(\lambda; \Delta\phi^{(0)}_n, -\Delta\phi^{(2)}_n). \quad (B.1)
$$

B.2 Unitarity

Recalling the fact that the $\Theta$ function (A.3) is proportional to the Gaussian, we can check that the fundamental $\mathcal{L}$ operator satisfies the unitarity condition,

$$
\mathcal{L}_n(\lambda; \phi) \cdot \mathcal{L}_n(-\lambda; \phi) = e^{-\frac{1}{\lambda^2}}. \quad (B.2)
$$

C Proof of Eqs. (5.1) – (5.2)

We sketch the outline of a proof of eqs. (5.1) – (5.2).
To prove eqs. (B.1), we first compute as follows:

$$
\frac{1}{\mathcal{L}_n(\lambda + \frac{2}{3} i \gamma; \phi)} \cdot \mathcal{L}_n^{(1)}(\lambda; \phi) = \frac{\Phi_\gamma(\lambda + \frac{2}{3} i \gamma - \Delta \phi_n^{(1)} - \Delta \phi_n^{(2)})}{\Theta_\gamma(\Delta \phi_n^{(1)} + \Delta \phi_n^{(2)})} \cdot \frac{\Theta_\gamma(\Delta \phi_n^{(1)} - \frac{2}{3} i \gamma)}{\Theta_\gamma(\Delta \phi_n^{(1)})} \\
\times \frac{\Phi_\gamma(2 \lambda + \Delta \phi_n^{(1)} - \frac{2}{3} i \gamma)}{\Phi_\gamma(2 \lambda + \Delta \phi_n^{(1)} - \frac{2}{3} i \gamma)} \cdot \frac{\Theta_\gamma(\Delta \phi_n^{(1)} + \Delta \phi_n^{(2)} + \frac{2}{3} i \gamma)}{\Theta_\gamma(\lambda - \frac{2}{3} i \gamma - \Delta \phi_n^{(1)} - \Delta \phi_n^{(2)})} \\
= \Phi_\gamma(\lambda + \frac{2}{3} i \gamma - \Delta \phi_n^{(1)} - \Delta \phi_n^{(2)}) \cdot \frac{1}{1 + e^{2\lambda + \frac{2}{3} i \gamma + \Delta \phi_n^{(1)}}} \\
\times \frac{\Theta_\gamma(\Delta \phi_n^{(1)} + \Delta \phi_n^{(2)} + \frac{4}{3} i \gamma)}{\Theta_\gamma(\lambda - \frac{4}{3} i \gamma - \Delta \phi_n^{(1)} - \Delta \phi_n^{(2)})} \cdot \frac{1}{e^{\lambda + \frac{2}{3} i \gamma} + e^{2\lambda + \frac{4}{3} i \gamma + \Delta \phi_n^{(1)}} + e^{\Delta \phi_n^{(1)} + \Delta \phi_n^{(2)} + \frac{2}{3} i \gamma}} \\
= e^{\frac{3}{2} \Delta \phi_n^{(1)} + \frac{3}{2} \Delta \phi_n^{(2)}} \cdot \frac{1}{e^{\lambda + \frac{2}{3} i \gamma} + e^{2\lambda + \frac{4}{3} i \gamma + \Delta \phi_n^{(1)}} + e^{\Delta \phi_n^{(1)} + \Delta \phi_n^{(2)} + \frac{2}{3} i \gamma}}.
$$

(C.1)

This proves the first equality in eqs. (B.1). Remaining equalities follow from eqs. (B.27).

To prove eqs. (3.2), we shift parameters in eq. (3.1):

$$
\frac{1}{\mathcal{L}_n(\lambda - \frac{2}{3} i \gamma; \phi)} \cdot \hat{\mathcal{L}}(\lambda; \Delta \phi_n^{(1)} + \frac{2}{3} i \gamma, \Delta \phi_n^{(1)} + \Delta \phi_n^{(2)} - \frac{2}{3} i \gamma) = e^{-\frac{4}{3} i \gamma} \left( c_n^{(1)} e^{2\lambda} + c_n^{(2)} + c_n^{(3)} e^{\lambda} \right).
$$

(C.2)

Using eqs. (3.1) and above identity, we can prove the first equality in eqs. (3.2) as follows;

$$
\mathcal{L}_n^{(1)}(\lambda; \phi) \cdot e^\lambda - \mathcal{L}_n^{(2)}(\lambda; \phi) = \mathcal{L}_n^{(2)}(\lambda; \phi) \cdot \left( (c_n^{(1)} + c_n^{(2)} e^\lambda + c_n^{(3)} e^{-\lambda}) \cdot \frac{e^\lambda}{c_n^{(1)} e^\lambda + c_n^{(2)} e^{-\lambda} + c_n^{(3)}} - 1 \right) \\
= \mathcal{L}_n^{(2)}(\lambda; \phi) \cdot (e^{2\lambda} - e^{-\lambda}) \cdot c_n^{(2)} \cdot \frac{1}{c_n^{(1)} e^\lambda + c_n^{(2)} e^{-\lambda} + c_n^{(3)}} \\
= (e^{3\lambda} - 1) \cdot c_n^{(2)} \cdot \hat{\mathcal{L}}(\lambda; \Delta \phi_n^{(1)} + \frac{2}{3} i \gamma, \Delta \phi_n^{(1)} + \Delta \phi_n^{(2)} - \frac{2}{3} i \gamma) \\
\times \frac{1}{c_n^{(1)} e^\lambda + c_n^{(2)} e^{-\lambda} + c_n^{(3)} e^\lambda} \\
= (e^{3\lambda} - 1) e^{-\frac{4}{3} i \gamma} \cdot c_n^{(2)} \cdot \mathcal{L}_n(\lambda - \frac{2}{3} i \gamma; \phi).
$$

Other equalities directly follow from the $\mathbb{Z}_3$ symmetry (B.1).
References

[1] R. J. Baxter, *Exactly Solved Models in Statistical Mechanics* (Academic Press, London, 1982).

[2] V. Pasquier and M. Gaudin, J. Phys. A: Math. Gen. **25**, 5243 (1992).

[3] V. B. Kuznetsov and E. K. Sklyanin, J. Phys. A: Math. Gen. **31**, 2241 (1998).

[4] E. K. Sklyanin, in *Integrable Systems: From Classical to Quantum*, edited by J. Harnad, G. Sabidussi, and P. Winternitz (Amer. Math. Soc., Providence, 2000), CRM Proceedings & Lecture Notes 26, pp. 227–250.

[5] L. D. Faddeev and A. Yu. Volkov, in *Discrete Integrable Geometry and Physics*, edited by A. I. Bobenko and R. Seiler (Oxford Univ. Press, Oxford, 1998), pp. 301–320.

[6] R. Inoue and K. Hikami, J. Phys. Soc. Jpn. **67**, 1163 (1998).

[7] S. Kharchev and D. Lebedev, Lett. Math. Phys. **50**, 53 (2000).

[8] S. Derkachov, J. Phys. A: Math. Gen. **32**, 5299 (1999).

[9] G. P. Pronko, Commun. Math. Phys. **212**, 687 (2000).

[10] V. B. Kuznetsov, M. Salerno, and E. K. Sklyanin, J. Phys. A: Math. Gen. **33**, 171 (2000).

[11] V. V. Bazhanov, S. L. Lukyanov, and A. B. Zamolodchikov, Commun. Math. Phys. **190**, 247 (1997).

[12] A. Yu. Volkov, Lett. Math. Phys. **39**, 313 (1997).

[13] K. Hikami and R. Inoue, J. Phys. Soc. Jpn. **68**, 376 (1999).

[14] A. Antonov and B. Feigin, Phys. Lett. B **392**, 115 (1997).

[15] A. Antonov, Theor. Math. Phys. **113**, 1520 (1997).

[16] K. Hikami and R. Inoue, J. Phys. A: Math. Gen. **30**, 6911 (1997).

[17] A. A. Belov and K. D. Chaltikian, Phys. Lett. B **309**, 268 (1993).

[18] T. Eguchi and S.-K. Yang, Phys. Lett. B **224**, 373 (1989).

[19] B. A. Kupershmidt and P. Mathieu, Phys. Lett. B **227**, 245 (1989).

[20] R. Sasaki and I. Yamanaka, Commun. Math. Phys. **108**, 691 (1987).

[21] K. Hikami, Phys. Lett. B **443**, 202 (1998).
[22] K. Hikami, Nucl. Phys. B 505, 749 (1997).
[23] P. Di Francesco, P. Mathieu, and D. Sénéchal, Conformal Field Theory (Springer, New York, 1997).
[24] P. P. Kulish, N. Yu. Reshetikhin, and E. K. Sklyanin, Lett. Math. Phys. 5, 393 (1981).
[25] P. P. Kulish and N. Yu. Reshetikhin, J. Sov. Math. 34, 1948 (1986).
[26] K. Hikami, in Physics and Combinatorics 1999 (World Scientific, 2001), to appear.
[27] L. D. Faddeev, Lett. Math. Phys. 34, 249 (1995).
[28] K. Hikami, Chaos, Solitons & Fractals 9, 853 (1998).
[29] B. Enriquez and B. L. Feigin, Theor. Math. Phys. 103, 738 (1995).
[30] S. V. Kryukov, Theor. Math. Phys. 105, 1359 (1995).
[31] F. A. Smirnov, J. Phys. A: Math. Gen. 33, 3385 (2000).
[32] V. A. Fateev and S. L. Lukyanov, Int. J. Mod. Phys. A 7, 1325 (1992).
[33] K. Hikami and R. Inoue, J. Phys. Soc. Jpn. 68, 776 (1999).
[34] K. Hikami, Chaos, Solitons & Fractals 9, 1773 (1998).
[35] K. Hikami, J. Phys. Soc. Jpn. 68, 55 (1999).
[36] R. Inoue and K. Hikami, Nucl. Phys. B 581, 761 (2000).
[37] E. K. Sklyanin, Prog. Theor. Phys. Suppl. 118, 35 (1995).
[38] E. K. Sklyanin, Zap. Nauch. Semin. POMI 205, 166 (1993).
[39] P. Dorey and R. Tateo, J. Phys. A: Math. Gen. 32, L419 (1999).
[40] J. Suzuki, J. Phys. A: Math. Gen. 32, L183 (1999).
[41] V. V. Bazhanov, S. L. Lukyanov, and A. B. Zamolodchikov, hep-th/9812247 (1998).
[42] P. Dorey, C. Dunning, and R. Tateo, J. Phys. A: Math. Gen. 33, 8427 (2000).
[43] J. Suzuki, J. Phys. A: Math. Gen. 33, 3507 (2000).
[44] L. D. Faddeev, R. M. Kashaev, and A. Yu. Volkov, hep-th/0006156 (2000).
[45] L. Chekhov and V. V. Fock, Theor. Math. Phys. 120, 1245 (1999).
[46] R. M. Kashaev, Mod. Phys. Lett. A 10, 1409 (1995).
[47] K. Hikami, RIMS Kôkyûroku 1172, 44 (2000).