Linearized gravity in terms of differential forms

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Abstract. A technique to linearize gravitational field equations is developed in which the perturbation metric coefficients are treated as second rank, symmetric, 1-form fields belonging to the Minkowski background spacetime by using the exterior algebra of differential forms.

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1 Introduction

Immediately after the formulation of the field equations for the theory of general relativity a century ago, Einstein studied the field equations bearing his name in the linear approximation. In particular, he derived the amount of deflection of the starlight by a spherically symmetric mass by using the linear approximation and he also introduced gravitational waves and derived the quadrupole formula for the energy emitted by a material source in the form of gravitational waves.

The gravitational waves generated by a merger of two massive black holes has been observed for the first time by gravitational wave detectors of the LIGO collaboration \cite{1} almost exactly a century later. This landmark observation will certainly have an impact on the theoretical front in general relativity. The gravitational waves topic with all its aspects should be expected to gain more popularity in general relativity courses.

The exterior algebra combined with Cartan’s moving frame technique in terms of differential forms \cite{2} is a powerful tool also in diverse topics under the general heading of the general theory of relativity. One of the main aims of the current paper is to show that the linearization of Einstein’s field equations provides another impressive illustrative case for the use of differential forms.

The required mathematical tools are introduced in the following section in a sufficient generality and the linearized quantities are derived starting from scratch in any of the gravitational models studied. The collection of exterior algebra formulas introduced in the preliminary section pays off in the ensuing sections dealing with Einstein’s field equations along with the discussions of the harmonic gauge condition and the quadratic Lagrangian 4-form from which the linearized equations follow. Some further applications of the technique on the Newtonian limit, the definition of total energy for asymptotically flat isolated gravitating systems and the plane gravitational waves in the linearized approximation are briefly discussed in Section 7. As a relatively more involved application, a derivation of the Lagrangian for new massive gravity in three dimensions is presented in some detail. In Section 8, a method to derive the linearized field equations for a given gravitational Lagrangian is developed and then applied to some higher order gravitational models governed by the Lagrangian densities $R^2$ and $g^\mu\nu \partial_\mu R \partial_\nu R$. In sect. 9, the consequences of a nonvanishing torsion on the linearization have been briefly discussed in the context of the Riemann-Cartan geometry. The paper ends with a final section with brief remarks concerning Deser’s construction of the Einstein-Hilbert Lagrangian.

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2 Mathematical preliminary

This section provides a collection of exterior algebra formulas that are used extensively in the linearization technique, which can be found, for example, in [3,4,5] in sufficient detail.

2.1 The basic rules of the exterior algebra

The basic operators and the identities of the exterior algebra required manipulations and calculations are introduced in this section which also serves to fix the notation used. We denote the exterior product of forms by $\wedge$, whereas the exterior derivative by $d$, and the interior product with respect to a vector $X$ by $i_X$. These operators can be defined by their action on the exterior product of a $p$-form $\omega$ and a $q$-form $\sigma$ as

\begin{align*}
  d(\omega \wedge \sigma) &= d\omega \wedge \sigma + (-1)^p \omega \wedge d\sigma, \\
  i_X(\omega \wedge \sigma) &= i_X\omega \wedge \sigma + (-1)^p \omega \wedge i_X\sigma.
\end{align*}

Both operators are linear by definitions above and $i_V i_V$ is linear in its argument $V$ as well: $i_V V^a i_a$ for a vector field $V = V^a X_a$. Acting on a form, both operators are nilpotent $dd = 0$ and $i_V i_V = 0$. Two consecutive exterior derivatives with respect to vector fields $V$ and $W$ anti-commute $i_V i_W = -i_W i_V$.

The set of an orthonormal basis 1-form will be denoted by $\{\theta^a\}$ whereas the metric dual frame fields will be denoted by $\{X_a\}$. The exterior product of basis 1-forms $\theta^a \wedge \theta^b \cdots$ will be abbreviated as $\theta^{ab\cdots}$. In terms of a natural coordinate basis, they can be expressed as $\theta^a = e^a_\mu dx^\mu$ and $X_a = e^a_\mu \partial_\mu$ where the matrices $e^a_\mu$ and $e^a_\mu$ are inverse of each other: $e^a_\mu e^b_\nu = \delta^a_\nu$, $e^a_\mu e^b_\mu = \delta^b_\mu$. The interior product with respect to a basis vector field $X_a$ will be denoted by $i_a \equiv i_{X_a}$ and note that $i_\mu \theta^\mu = e^a_\mu i_a dx^\mu = e^a_\mu e^b_\mu = \delta^a_\mu$. Since tensor components are used along with their components, the index raising/lowering of the tensor components have to be indicated in a slightly different manner. For example, the 1-form associated with a vector $V = V^a \partial_a = V^a X_a$ is denoted by $V = V_\mu dx^\mu = V_\mu \theta^\mu$ with $V_\mu = g_{\mu\nu} V^\nu$. Likewise, the vector field associated with a 1-form $\sigma = \sigma_\mu dx^\mu$ is $\sigma = \sigma^a \partial_a$ with $\sigma^a = g^{\mu\nu} \sigma_\nu$.

The Hodge dual $*$ is a linear operator acting on the forms and it can be defined with a pseudo-Riemannian metric. The $*$ operator maps a $p$-form to a $(4 - p)$-form. With the help of the $*$ operator, the inner product of two forms provided by the metric can be extended to the definition of the inner product of two $p$-forms $\omega$ and $\sigma$: $\omega \wedge * \omega = \sigma \wedge * \sigma$.

The action of the Hodge dual operator on basis $p$-forms can be defined with the help of a permutation symbol $\epsilon_{abcd}$ with $\epsilon_{0123} = +1$ while odd permutations of the indices resulting in $-1$, and vanishing otherwise. The invariant volume 4-form is defined as $*1 = \frac{1}{4!} \epsilon_{abcd} \theta^{abcd}$. The Hodge duals of basis 1-forms and 2-forms, for example, can be written as $*\theta^a = \frac{1}{4!} \epsilon^{abcd} \theta_{bcd}$, $*\theta^{ab} = \frac{1}{4!} \epsilon^{abcd} \theta_{cd}$. Acting on a $p$-form $** = (-1)^{p(4-p)+1} i_d$. Some identities involving $i_X$, $\wedge$ and $*$ which are of considerable use in exterior algebra manipulations are

\begin{align*}
  i_X * \omega &= *(\omega \wedge \tilde{X}), \\
  \sigma \wedge * \omega &= (-1)^{p+1} * i_\delta \omega,
\end{align*}

in terms of a $p$-form $\omega$, a 1-form $\sigma$ and a vector field $X$. In terms of basis coframe and dual vector fields these read $i_a \ast \omega = *(\omega \wedge \theta_a)$ and $\theta^a \wedge * \omega = (-1)^{p+1} * i_\delta \omega$, respectively.

For the flat Minkowski spacetime, the Hodge dual will be denoted by $*$ and the invariant volume element is $*1 = dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3$. The formulas for the orthonormal basis in a curved spacetime in a curved spacetime above hold for the natural basis of the Minkowski spacetime as well. The set of basis frame fields in the Minkowski spacetime is $\{\partial_a\}$ and the interior derivative with respect to the frame fields $\partial_a$ is $i_a = i_{\partial_a}$. $\{\partial_a\}$ is dual to the set of natural basis $\{dx^a\}$, $i_a dx^b = \delta^b_a$. Consequently, an expression of the form $i_a df$ is simply $\partial_a f$ for a 0-form $f$. For convenience of notation, the natural basis $dx^a$ which is the exterior derivative of the Cartesian coordinates $\{x^a\}$ will be denoted by $e^a$.

Finally, a cautionary remark related to the indices in the linearization is in order. The use of the linearized tensorial quantities adopting an orthonormal coframe inevitably blurs the distinction between the indices relative to a coordinate basis labeled by Greek letters with those of an orthonormal basis labeled by Latin letters. Both type of indices are raised and lowered by $\eta^{ib}$ and $\eta_{ab}$, respectively.

2.2 Cartan’s structure equations

The Levi-Civita connection 1-forms $\omega^a_{\ b}$ relative to an orthonormal basis satisfy Cartan’s first structure equations

\[
  d\theta^a + \omega^a_{\ b} \wedge \theta^b = 0,
\]

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satisfying also the metric compatibility condition: \( \omega_{ab} + \omega_{ba} = 0 \). In terms of the Riemann tensor curvature 2-forms can be expressed in the form \( \Omega^a_b = \frac{1}{2} R^b_{\ \ cdef} \), and in terms of the connection 1-forms, Cartan’s second structure equations read

\[
\Omega^a_b = d\omega^a_b + \omega^a_c \wedge \omega^c_b. \tag{6}
\]

The curvature 2-forms satisfy \( \Omega_{ab} + \Omega_{ba} = 0 \) and the first and second Bianchi identities, namely, \( \Omega^a_b \wedge \theta^b = 0 \) and \( D\Omega^a_b = 0 \), respectively.

\( D \) stands for the covariant exterior derivative acting on tensor valued-forms \( \mathbf{5} \). For example, acting on vector components \( V^a \), it yields \( DV^a = dV^a + \omega^a_b V^b \) whereas, the second Bianchi identity explicitly reads \( D\Omega^a_b = d\Omega^a_b + \omega^a_c \wedge \Omega^c_b - \omega^c_b \wedge \Omega^c_b \). The linearization procedure will often turn a covariant derivative into an exterior derivative and the covariant exterior derivative will not play a significant role in the linearized equations.

In a typical curvature calculation, one first solves (5) in terms of a given orthonormal basis and then inserts them into (4), as will be repeated below in a general form at the linearized level. As a typical calculation in exterior algebra, one can verify that the first structure equations can uniquely be solved for the connection 1-forms [4];

\[
\omega^a_b = \frac{1}{2} i^a_b (d\theta^c \wedge \theta^d) - i^a_d db_b + i^b_d db^a \tag{7}
\]

by calculating two successive inner products of (5) with respect to basis vector fields.

Finally, note that the Ricci 1-forms are defined in terms of the Ricci tensor \( R^a = R^a_b \theta^b = R^{ac} b \theta^b \) which can also be given by \( R_a = i^b \Omega_{ba} \) and thus the Ricci scalar is \( R = i_a R^a \). Likewise, Einstein 1-forms can be written as \( G^a = R^a - \frac{1}{2} R \theta^a \).

### 2.3 Metric perturbation 1-forms

Weak gravitational fields, considered as small perturbations to the flat Minkowski space are studied by introducing metric coefficients \( h_{ab}(x^i) \) which are assumed to satisfy \( |h_{ab}| \ll 1 \) for consistency. In a mathematically precise manner, a weak gravitational field has a metric \( g = \eta_{ab} \theta^a \otimes \theta^b \) of the approximate form

\[
g^L = \eta + 2h_{ab} dx^a \otimes dx^b = \eta + h_a \otimes e^a + e^b \otimes h_b. \tag{8}
\]

The flat spacetime metric is of the form \( \eta = \eta_{ab} e^a \otimes e^b \) where \( \eta_{ab} = \text{diagonal}(\pm -++) \). In order to implement the exterior algebra of forms in a flat background, it is convenient to introduce the 1-forms \( h^a = h^a b \) instead of the symmetric scalar components \( h_{ab} \equiv h_{ba} \). Note that \( \mathbf{5} \) implies that the symmetry of \( h_{ab} \) is related with the symmetry \( g^L_{ab} = g^L_{ba} \). To denote the trace of the perturbation coefficients \( \eta_{ab} h_{ab} = h^a a \equiv h \) will be used. To first order in \( h_{ab} \), a total basis 1-forms \( \theta^a \) then decomposes as

\[
\theta^a_L = e^a + h^a \tag{9}
\]

as a 1-form equation. The label \( L \) in the expression, and in what follows, refers to a quantity linear in \( h_{ab} \). For the equations that are valid in the linear approximation, the equality sign will be used as has been done in \( \mathbf{5} \) and \( \mathbf{6} \), the \( O(h^2) \)-terms are assumed to be ignored. The linearized forms of the basis \( p \)-forms and their Hodge duals can simply be found by making use of (9). For example, one can readily obtain the following linear approximations:

\[
(\theta^a \wedge \theta^b)_L = e^a \wedge e^b + e^a \wedge h^b + h^a \wedge e^b, \tag{10}
\]

\[
(\theta^a \wedge \theta^b \wedge \theta^c)_L = e^a \wedge e^b \wedge e^c + e^a \wedge e^b \wedge h^c + e^a \wedge h^b \wedge e^c + h^a \wedge e^b \wedge e^c, \tag{11}
\]

\[
(\theta^a \wedge \theta^b \wedge \theta^c \wedge \theta^d)_L = e^a \wedge e^b \wedge e^c \wedge e^d + e^a \wedge e^b \wedge e^c \wedge h^d + e^a \wedge e^b \wedge h^c \wedge e^d + e^a \wedge h^b \wedge e^c \wedge e^d + h^a \wedge e^b \wedge e^c \wedge e^d, \tag{12}
\]

\[
[\ast \theta^a]_L = \frac{1}{6} \epsilon_{abcd} (\theta^b \wedge \theta^c \wedge \theta^d)_L = *e^a + h_b \wedge *e^b, \tag{13}
\]

\[
[\ast (\theta^a \wedge \theta^b)]_L = \frac{1}{2} \epsilon_{abcd} (\theta^d \wedge \theta^a)_L = e^d \wedge h^c \wedge *e^b, \tag{14}
\]

\[
[\ast (\theta^a \wedge \theta^b \wedge \theta^c)]_L = \epsilon^{ab}_{\ \cd} \theta^d L = *e^{abc} + h_d a^{abc}, \tag{15}
\]

where it is assumed that terms higher then \( O(h) \) are omitted from the expressions on the right hand sides. The linearized forms of the basis \( p \)-forms and their Hodge dual can likewise be found by using expansions akin to (9) for given tensorial expressions. One can also find \( (+1)^{L-1} \ast (+1) = (1 + h) \ast 1 \) using the above formulas.

Finally, note that one can linearize the frame fields as \( X^L_a = \partial_a - h_{ab} \theta^b \) and consequently, \( (i_X \theta^a)_L = i_X L \delta^a = \delta^a \) to the first order in \( h \). Accordingly, the interior product can be linearized as \( (i_X h)_L = i_{X^L} = i_{(\partial_a - h_{ab} \theta^b)} = i_{\partial_a - h_{ab} \theta^b} \). The linearization formulas of this section are sufficient to rewrite the linearized forms of identities \( \mathbf{5} \) and \( \mathbf{6} \) provided in the previous section.
2.4 The linearized connection and curvature forms

The above formulas are sufficient to introduce the first order approximation into the first structure equations \([5]\). One can show that they become

\[
dh_{ia} + \omega^i_{ab} \wedge e^b = 0.
\]

(16)

The linearized Levi-Civita connection is assumed to retain the metric compatibility and therefore one has \(\omega^L_{ia} + \omega^L_{ib} = 0\). As for the structure equations, the linearized structure equations \([16]\) can be inverted to find \(\omega^L_{ab}\), for example, by making use of the linearized version of the formula \([17]\). One finds

\[
\omega^L_{ab} = -i_a dh_b + i_b dh_a.
\]

(17)

Note that the components of the connection 1-forms \(\Omega^L_{ab}\) can be read off from the expression

\[
\omega^L_{ab} = -(\partial_a h_{bc} - \partial_b h_{ac}) e^c.
\]

(18)

Consequently, the linearized curvature 2-forms \(\Omega^L_{ab}\) can be written as

\[
\Omega^L_{ab} = d\omega^L_{ab}
\]

(19)

by dropping \(\omega^2\) terms in \([16]\). The components of the linearized Riemann tensor, \(R^L_{abcd}\), can be found by inserting \([17]\) into the linearized second structure equations and noting that \(\Omega^L_{ab} \equiv \frac{1}{2} R^L_{abcd}(\theta^a \wedge \theta^b)_L\). One finds

\[
R^L_{abcd} = \partial_a \partial_d h_{bc} - \partial_b \partial_d h_{ac} - \partial_a \partial_c h_{bd} + \partial_b \partial_c h_{ad}.
\]

(20)

In contrast to the expressions \([18]\) and \([20]\), the use of the expressions in differential forms, namely \([17]\) and \([19]\), is more convenient as will be justified below.

3 The linearized Einstein tensor

Einstein’s field equations can be expressed in various equivalent forms. The form suitable for the current presentation involves the Einstein 1-forms \(G^a = G^a_b \theta^b\) \([4,5]\). The Hodge dual of the Einstein 1-forms can be conveniently expressed in the form

\[
* G^a = -\frac{1}{2} \Omega_{bc} \wedge * G^{abc}.
\]

(21)

The expression on the right hand side can readily be obtained by a coframe variation of the Einstein-Hilbert Lagrangian expressed in terms of curvature 2-forms \([4,5,6]\). One can also verify that the expression \([21]\) indeed leads to the Einstein tensor by first rewriting it in the form \(\ast G^a = -\frac{1}{4} R^{abc}_{\ bcd} \epsilon^{bmn} \partial^{pmn}\). Taking the Hodge dual, one finds

\[
G^a = \frac{1}{4} \epsilon^{abcd} \epsilon_{kmnd} R^{bmn}_{\ bc} \theta^k.
\]

(22)

Now, by making use of the identity involving the product of permutation symbols \(\epsilon^{abcd} \epsilon_{kmnd} = \delta^{abc}_{kmn}\) where \(\delta^{abc}_{kmn}\) is the generalized Kronecker symbol, \([22]\) can be rewritten in the form

\[
G^a = \frac{1}{4} \delta^{abc}_{kmn} R^{bmn}_{\ bc} \theta^k.
\]

(23)

The familiar expression \(G^a \equiv G^a_b \theta^b = (R^a - \frac{1}{2} \delta^a_b R) \theta^b\) now can be obtained by expanding the generalized Kronecker symbol into a product of Kronecker deltas as

\[
\delta^{aef}_{bce} = \delta^a_b \delta^{def}_{cc} - \delta^d_c \delta^{ef}_{bc} + \delta^{aef}_{bc} \delta^c_d \quad \text{and} \quad \delta^{ab}_{cd} = \delta^a_c \delta^b_d - \delta^b_c \delta^a_d.
\]

(24)

As is now verified, the linearized Einstein tensor can be found by linearizing the expression \([21]\). For convenience, one can define \(\ast G^a_L \equiv (\ast G^a)_L\), where the linearized Einstein 1-forms then have the expansion \(G^a_L \equiv G^a_b e^b\). Explicitly, one has

\[
* G^a_L = -\frac{1}{2} \Omega_{bc} \wedge * e^{abc} = -\frac{1}{2} d\omega^L_{bc} \wedge * e^{abc}.
\]

(25)

The factor \(\Omega^L_{ab}\) survives the exterior multiplication since it has no \(O(h)\) terms and in addition, because \(e^a\) is a natural \((\text{Cartesian})\) basis, \(d \ast e^{abc} = 0\) identically as a consequence of the identity for exterior derivative: \(dd \equiv 0\). Consequently, the linearized Einstein 3-forms take the form

\[
* G^a_L = -\frac{1}{2} d(\omega^L_{bc} \wedge * e^{abc})
\]

(26)
Note that the linearized vacuum equations take the remarkable form $d \ast F^a_L = 0$ having a formal resemblance to Maxwell’s equation $d \ast F = 0$, where $F$ is the Faraday 2-form, in terms of the Thirring 2-forms $F^a$ which are defined as $\ast F^a = -\frac{1}{2}\omega_bc \wedge \ast \theta^{abc}$.

By making use of the expression (17) for the connection 1-forms, (26) reduces to

$$\ast G^L_a = d \left( i_a dh_c \wedge \ast e^{abc} \right).$$

Further simplification of the resulting expression can be achieved by noting that

$$i_a dh_b = i_a (dh_{bc} \wedge e^c) = (i_a dh_{bc}) e^c - dh_{ba}.$$  

By making use of the symmetry property, $h_{ab} = h_{ba}$, one finally ends up with

$$\ast G^L_a = d \left[ e^b \wedge \ast (dh_b \wedge e^a) \right].$$

(29) This concise expression for the Einstein 3-forms can be presented in a variety of alternative forms. For example, it can be rewritten as

$$\ast G^L_a = d \ast dh^a + d \ast \left[ (i_a dh^b) \wedge e^a \right].$$

(30) An immediate observation about the expression (29) is that it implies that the linearized Bianchi identity $(D \ast G^a)_L = 0$ amounts to the identity $dd \equiv 0$. This also directly follows from the linearized Bianchi identity $(DQ^a)_L = dQ^a_L \equiv 0$.

The components of the linearized Einstein tensor can now be derived by noting that $G^L_{ab} \ast 1 = e^a \wedge \ast G^L_b$ and by making use of the identity $e_{abc} \wedge \ast e^{def} = \delta^{abf}_{def} \ast 1$, so that (29) eventually leads to

$$G^L_{ab} = \delta^{abf}_{def} \partial^c \partial_d h^e_f.$$  

(31) The result expressed in (31) can be expanded into the familiar expression

$$G^L_{ab} = -\Box h_{ab} - \partial_a \partial_b \partial_d c^{cd} + \partial_a \partial^c h_{cb} + \partial_b \partial^c h_{ca} - \partial_a \partial_b \partial h - \eta_{ab} \Box h$$

(32) where $\Box \equiv \eta^{ab} \partial_a \partial_b$. The expression on the right hand side of (32) can be obtained by expanding the generalized Kronecker symbol into a product of the Kronecker deltas. The explicit expression on the right hand side shows that the expression in (27) leads to a symmetric linearized Einstein tensor. One can verify that the expression (32) follows from (29) by using $G^L_{ab} = R^L_{ab} - \frac{1}{2}\eta_{ab} R^L$ with $R^L_{ab} = \eta^{cd} R^L_{abcd}$ and $R^L = \eta_{ab} R^L_{ab}$. The particular staggering of partial derivative indices in the expression (32) is reproduced by exterior algebra from the compact expression (29). It is worth to emphasize that the expression (32) is the expression obtained by making use of the linearized Christoffel symbols and the linearized Riemann tensor and that (32) is merely the expression of (29) in component form.

Finally, the linearized scalar curvature $R^L$ can simply be obtained from (32), or else by using the definition $R^L \ast 1 = -e_a \wedge \ast G^a_L$ and (29). One finds

$$R^L \ast 1 = 2d \ast i_a dh^b = 2(\partial_a \partial_b h^{ab} - \Box h) \ast 1.$$  

(33) Using the above results, one can show that the Hodge duals of the Ricci 1-forms are given by

$$\ast R^a_L = d \ast dh^a + L^a \ast i_a dh^b$$

(34) with $L_a \equiv i_a d + di_a$ standing for the Lie derivative with respect to $\partial_a$ on a p-form. As a consistency check, one can calculate the components of the linearized Ricci 1-forms by using $R^L_a = i^b_L D^L_{ba}$ with those of obtained from (34) as well.

4 Gauge fixing: The harmonic gauge

For harmonic coordinates, a gauge condition, also called harmonic gauge condition, can be expressed as

$$d \ast dx^a = 0.$$  

(35) On the other hand, by making use of $e^a \approx \theta^a - h^a$ and the identity $d \ast \theta^a = -\omega^a_b \wedge \ast \theta^b$, one finds

$$d \ast h_a + \omega^L_{ab} \wedge \ast e^b = 0.$$  

(36)
to first order in $h_{ab}$. The expression (36) leads to the following identity on the derivatives of the perturbation 1-forms

$$i_a dh^a + \frac{1}{2} dh = 0.$$  \hspace{1cm} (37)

In the familiar component form, (37) can be rewritten as

$$\partial_a h^a_{\ b} - \frac{1}{2} \partial_b h = 0.$$  \hspace{1cm} (38)

With the gauge condition imposed, the linearized Einstein 3-forms become

$$\star G^L_{ab} = d \star d \bar{h}^a$$  \hspace{1cm} (39)

where new perturbation 1-forms $\bar{h}^a$ are defined in terms of the original as

$$\bar{h}^a = \left( h_{ab} - \frac{1}{2} \eta_{ab} h \right) e^b.$$  \hspace{1cm} (40)

In terms of $\bar{h}^a$, the gauge condition (38) simply reads $\partial_a \bar{h}^a_{\ b} = 0$. Consequently, the linearized Einstein tensor reduces to the form

$$G^L_{ab} = -\Box h_{ab}$$  \hspace{1cm} (41)

in the harmonic gauge. In the component form of (41), the gauge condition $\partial_a \bar{h}^a_{\ b} = 0$ implies the linearized Bianchi identity $\partial^a G^L_{ab} = 0$ as a consequence of the fact that $\partial_a$ commutes with $\Box$.

5 The Pauli-Fierz Lagrangian

Einstein’s vacuum field equations follow from the Einstein-Hilbert Lagrangian 4-form

$$L_{EH} = \frac{1}{2} R * 1 + \frac{1}{2} \Omega_{ab} \wedge \star \theta^{ab}.$$  \hspace{1cm} (42)

Equation (42) is a special Lagrangian in the sense that it leads to second order partial differential equations in the metric components despite the fact that it contains second order derivatives. On the other hand, the parts containing the second order derivatives can be relegated to a boundary term by making use of the identity

$$d \star \theta^{ab} = -\omega^{a}_{\ c \ e} \wedge \star \theta^{eb} - \omega^{b}_{\ c \ e} \wedge \star \theta^{ac}$$  \hspace{1cm} (43)

(that follows from $D \star \theta^{ab} = 0$ satisfied by a Levi-Civita connection) in conjunction with the second structure equations in (42). One finds

$$L_{EH} = -\frac{1}{2} \omega^{ac}_{\ c} \wedge \omega^{eb}_{\ e} \wedge \star \theta^{ab} - \frac{1}{2} d (\omega_{ab} \wedge \star \theta^{ab})$$  \hspace{1cm} (44)

The field equations which are first order in the metric perturbations require a Lagrangian which is second order in the metric perturbations and (41) is evidently in a suitable form to obtain such a Lagrangian:

$$L^{(2)}_{EH} [h] \equiv -\frac{1}{2} \omega^{ab}_{\ c} \wedge \omega^{eb}_{\ e} \wedge (\star \theta^{ab})_L.$$  \hspace{1cm} (45)

Now retaining the terms to order $O(h^2)$ in this expression, one ends up with

$$L^{(2)}_{EH} [h] = -\frac{1}{2} \epsilon_{a} \wedge dh_{b} \wedge \star (e^b \wedge dh^a),$$  \hspace{1cm} (46)

after some exterior algebra calculations using the formulas given in Section 2. The Lagrangian $L^{(2)}_H[h]$ can be regarded as a Lagrangian for the 1-form field $h_{ab}$ in Minkowski spacetime. A Lagrangian density involving the partial derivatives can be explicitly obtained from (46). One finds

$$L^{(2)}_{EH} [h] = \left( -\frac{1}{2} \partial_a h^a_{\ b} \partial^b h + \partial_a h^a_{\ b} \partial_b h - \partial_a h_{ab} \partial_b h + \frac{1}{2} \partial_a h \partial^b h \right) * 1$$  \hspace{1cm} (47)

up to an omitted total derivative. Equation (47) is nothing but the Pauli-Fierz Lagrangian density.
The form of the Lagrangian 4-form suggests still another way to derive the linearized Einstein 3-forms, or equivalently, the linearized Einstein tensor. More precisely, one can work out the variational derivative \( \delta L^{(2)}[h] \) \( \equiv \delta L^{(2)}[h] / \delta h^a \) with respect to the perturbation 1-forms \( h^a \) to obtain

\[
\delta L^{(2)}_{EH}[h] = -\delta h_a \wedge d [e^b \wedge * (dh_b \wedge e^a)],
\]

up to an omitted boundary term and holding the basis 1-forms \( e^a \) of the Minkowski spacetime fixed. Consequently, by a comparison of the result with \( \delta L^{(2)}_{EH}/\delta h_a = -*G^a \), which has the exact variational derivative \( \delta L^{(2)}_{EH}/\delta h_a = -*G^a \) (See, for example, the derivation provided in [13]).

Finally, note that the fixing of the harmonic gauge can also be achieved by supplementing the Pauli-Fierz Lagrangian with a term \( L_{gf} \) of the form

\[
L_{gf} = \frac{1}{2} (i_a dh^a + dh) \wedge * (i_a dh^b + dh)
\]

involving the square of the gauge condition. By simplifying the expression on the right hand side, one can obtain an extended Lagrangian of the form

\[
L^{(2)}_{EH} \equiv L^{(2)}_{EH} + L_{gf} = -\frac{1}{2} dh_a \wedge * dh^a + \frac{1}{2} dh \wedge e^a \wedge * dh_a + \frac{1}{8} dh \wedge * dh.
\]

One can rederive the field equations and the harmonic gauge condition by calculating the variational derivative of the extended Lagrangian with respect to the variables \( h_a \) and \( h \), respectively.

Compared to the coordinate methods making use of the scalar functions \( h_{ab} \), the linearization technique developed above in terms of 1-forms \( h_a = h_{ab}e^b \) provides a practical and powerful calculational technique, often rendering the equations more transparent as well. This claim can be justified by the illustrative examples below.

6 The Lagrangian for a massive spin-2 field in vacuum

The field equations for a massive spin-2 field was introduced almost 80 years ago by Pauli and Fierz [7,8]. The issue of massive spin-2 modes for the gravitational interactions in a cosmological context frequently occurs in the modified gravity models, for example, to explain the cosmic accelerated expansion of the Universe (See, for example, [9,10,11] for a general discussion on the graviton mass). In three dimensions, massive gravity models, such as New massive gravity to be discussed briefly below, are important since they provide toy models in the context of quantum theory of gravity.

The introduction of mass to a spin-2 field has some interesting and subtle features requiring some extra attention even at the linearized level in vacuum. Mathematically admissible terms that one can consider for the massive spin-2 field are of the form \( h^2 \), or \( h^{ab}h_{ab} \) or else some particular combination of these quadratic terms. Let us consider, for example, the Lagrangian 4-form

\[
L^{(2)}_m[h] = L^{(2)}_{EH}[h] + \frac{m^2}{2} e^a \wedge h_b \wedge * (e^b \wedge h_a)
\]

where the mass term can be rewritten in the form

\[
e^a \wedge h_b \wedge * (e^b \wedge h_a) = \left( h_{ab}h^{ab} - h^2 \right) \wedge 1.
\]

The significance of the particular mass term will be apparent as one proceeds with the derivation of the field equations from the Lagrangian 4-form [51].

The linearized coordinate covariance expressed in terms of a vector field \( X = \xi^a \partial_a \) as \( x^a \mapsto x'^a = x^a + \xi^a(x) \) leads to \( h_{ab} \mapsto h_{ab}' \equiv h_{ab} + \partial_a \xi_b + \partial_b \xi_a \). The infinitesimal change in the metric perturbation 1-forms can be expressed in the form language as \( \delta h_a = d(\xi_a X + i_a dX) \equiv L_ah \Omega_a \) with \( X = \xi_a e^a \). One can show that \( \Omega_a h' \) remains invariant under these transformations. Consequently, the “Bianchi identity” for the massive gravity Lagrangian [51] that follows from the linearized coordinate covariance can be obtained by inserting \( \delta h_a = L_a X \) to the variational derivative:

\[
\delta L^{(2)}_m[h] = \delta L^{(2)}_{EH}[h] - m^2 h_{ab} \wedge e^b \wedge * (e^a \wedge h_b) = 0.
\]

Using the fact that \( \delta L^{(2)}_{EH}[h] \) leads to an exact 3-form (linearized Bianchi identity for the massless case), then leads to

\[
d \left[ e^a \wedge * (h_a \wedge e^b) \right] = 0
\]

which may be rewritten conveniently as

\[
d \left[ h^a - he^a \right] = 0.
\]
In the component form, this identity can be rewritten as $\partial_j h^{a} - \partial_a h = 0$, or equivalently, in a concise form as $i_\kappa dh^a = 0$. The latter form of the identity also offers some insight into the massive spin-2 field equations: This condition cancels out the scalar curvature term from the field equations recalling that the linearized scalar curvature can be written in the form $R^i = 2 \star d \star i_\kappa dh^a$.

The identity $[56]$ then can be used to simplify the massive gravity equations. Explicitly, the massive spin-2 equations read

$$d [e^b \star \{dh_b \wedge e^a \}] - m^2 e^b \wedge \star (e^a \wedge h_b) = 0.$$  \[56\]

The trace of the field equations can be calculated by wedging $[56]$ with $e_a$ from the left. The trace of the first term leads to the scalar curvature, which vanishes by the “Bianchi identity” and one can verify that, in vacuum, the trace of the mass term reads $h = 0$. Hence the Pauli-Fierz mass term in $[51]$ is a unique quadratic combination of the perturbation 1-forms that leads to such a simplification. Taking the simplifications into account, and taking the Hodge dual of $[56]$, the field equations for a massive spin-2 field eventually reduce to the form

$$(\star d \star d + m^2) h_a = 0$$  \[57\]

as a 1-form equation. In components, this can be rewritten as

$$(\Box - m^2) h_{ab} = 0$$  \[58\]

and these are compatible with the subsidiary conditions $\partial^b h_{ab} = 0$ and $h = 0$ by construction. Equation $[55]$ is usually referred to as a constraint for the massive spin-2 field $h_{ab}$ because the constraints are not gauge invariant $[13]$. The reason for this is that as a consequence of these constraints $R^i$ vanishes as one can observe from Eq. $[55]$. However, $R^i$ is invariant under the gauge transformations $\delta h_{ab} = \partial_a \xi_b + \partial_b \xi_a$ generated by a vector field $X = \xi^a \partial_a$ and it cannot be set to zero by a gauge choice.

7 Some applications

7.1 Newtonian Limit

In order to recover the Newtonian limit of the field equations, let us consider a static metric of the form

$$g = -F^2 dt \otimes dt + F^{-2} \delta_{ij} dx^i \otimes dx^j$$  \[59\]

with a metric function $F = F(x^a)$ having the approximate form $F^2 \approx 1 + 2h(x^a)$. In this approximation, only the diagonal components of the metric perturbations survive and one has $\theta^0 = e^0 + h e^0$ and $\theta^0 = e^i - h e^i$ for $i = 1, 2, 3$. The linearized Einstein’s equations, $\star G^a_b = \kappa \star T^a_b$, follow from the total Lagrangian density $L_{tot} = L_{int}[h] + L_{int}$, with the matter-interaction term $L_{int} \equiv s h_{ab} \wedge \star T^a_b$ and the coupling constant $\kappa$. An ideal fluid energy-momentum tensor $T \equiv T_{ab} \theta^a \otimes \theta^b$ is of the form $T = (\rho + p) U \otimes \bar{U} + pg$ in terms of the proper energy density $\rho(x^a)$, pressure $p$ and four-velocity $U = U^a \partial_a$. For $p = 0$ and for $a = 0$, the field equations lead to the Poisson equation $\star d \star dh = -4\pi G \rho$ in three space dimensions where one can identify $h$ as the Newtonian gravitational potential and the coupling constant $\kappa$ in the Einstein field equations can be expressed in the form $\kappa^{-1} = 16\pi G / c^4$, in terms of the Newtonian gravitational constant $G$ and the speed of light $c$. In addition, one can show that the geodesics of massive test particles in the metric $[59]$ lead to $\dot{x}^i + \frac{1}{2} \partial_i F^2 \approx 0$ assuming that the slow-motion approximation $|\dot{x}| \ll 1$ is valid in addition to the weak-field approximation $|h_{ab}| \ll 1$. Here a dot over a spatial coordinate refers to the derivative with respect to proper time $\tau$, $\dot{x}^i = dx^i / d\tau$ $[6]$. 

7.2 Total energy of an asymptotically flat system

For an asymptotically flat spacetime of an isolated gravitating system, the total energy defined by Arnowitt-Deser-Misner (ADM) has the explicit coordinate expression of the form

$$M_{ADM} = \frac{1}{16\pi G} \int_{S_\infty} dS^i (\partial_k g_{ki} - \partial_i g_{kk})$$  \[60\]

where $dS^i$ is the area element of a two-dimensional surface $[13]$. It is possible to recover $[60]$ by using the Thirring 2-forms provided that a coordinate basis is adopted in the expression $[61]$ below $[4]$. By taking the decomposition of
Cartesian coordinates as $h$ perturbations constitute their own linearization. The Kerr-Schild form of the metric of the 3-form of the metric perturbation 1-form $h$ in terms of the global null coordinates $x_{\xi}$ where $\zeta$ or equivalently, $G$ metric perturbation 1-form $h$.

In terms of the null coordinates, the Minkowski spacetime metric explicitly reads

$$
P^\mu = -\frac{1}{16\pi G} \int_{S_{\infty}} *F^\mu.
$$

$M_{ADM}$ corresponding to the temporal component of the four vector (61) can be calculated by integrating the exact 3-form part of $G^0$ over a 2-sphere of infinite radius $11$. For instance, one can show that $M_{ADM}$ is the Schwarzschild mass by making use of (61). However, (60) can be recovered at the linearized level as well. Assuming that the metric for a given static spacetime can be brought to the form (8), one can use the linearized form of the Thirring 2-forms,

$$
P^0 = \frac{1}{16\pi G} \int_{S_{\infty}} *F^0 = -\frac{1}{16\pi G} \int_{S_{\infty}} (\partial_0 h^k_i - \partial_i h^k_k) * e^{00}
$$

with $i, k = 1, 2, 3$ and the integration measure is now built into the expression which is to be restricted to the sphere of radius $R$ and a limiting procedure $R \to \infty$, keeping $x^0 = ct$ constant.

The Abbot-Deser-Tekin charges [14] are a generalization of the ADM energy defined on asymptotically curved backgrounds and applied to the models derived from general quadratic curvature gravitational Lagrangians [15]. The energy definition provided by Deser and Tekin [55] is then reformulated in terms of differential forms in [17] using the mathematical formalism as presented above.

### 7.3 Gravitational Waves

A plane gravitational wave solution, propagating along the $x^3$-axis, can be constructed by inserting the plane wave metric ansatz into the vacuum field equations $\Box h_{ab} = 0$. The ansatz involves a null propagation vector of the form $k = e^0 \mp e^3$ and a constant polarization tensor $\epsilon_{ab}$ so that perturbation metric $h_{ab}$ is in the form corresponding to the real part of $\epsilon_{ab} e^{ikx^3}$. By imposing the so-called transverse-traceless gauge conditions ($\partial^a h_{ab} = h = 0$), + and $\times$ transverse polarization modes for the plane gravitational waves are then obtained in terms of the nonvanishing metric perturbations $h_{12} = h_{21}$ and $h_{22} = -h_{11}$ respectively (See, for example, the lucid presentation in [5]).

More generally, it is well known that the plane-fronted gravitational waves with parallel-propagation ($pp$-waves) constitute their own linearization. The Kerr-Schild form of the metric of the $pp$-waves can be written explicitly as

$$
g = \eta - 2H du \otimes du
$$

in terms of the global null coordinates $x^a = \{u, v, \zeta, \bar{\zeta}\}$ where $u, v$ are the real null coordinates related with the Cartesian coordinates as

$$
u = \frac{1}{\sqrt{2}} (x^0 - x^3), \quad v = \frac{1}{\sqrt{2}} (x^0 + x^3),
$$

whereas the complex conjugate coordinate $\zeta$ and $\bar{\zeta}$ are related to the Cartesian as

$$
\zeta = \frac{1}{\sqrt{2}} (x^1 + ix^2), \quad \bar{\zeta} = \frac{1}{\sqrt{2}} (x^1 - ix^2).
$$

In terms of the null coordinates, the Minkowski spacetime metric explicitly reads

$$
\eta = -du \otimes dv - dv \otimes du + d\zeta \otimes d\bar{\zeta} + d\bar{\zeta} \otimes d\zeta
$$

The nonvanishing metric coefficients in this case are $-\eta_{01} = -\eta_{10} = \eta_{12} = \eta_{21} = 1$. The coordinates of the null coframe are related with the Cartesian coordinates by a transformation matrix with constant coefficients and therefore the (linearized) field equations relative to a null coframe retain the form (27). However, the indices relative to the null coframe in the null coframe case are lowered and raised by the $\eta_{ab}$ and $\eta^{ab}$ of the metric (60), respectively.

For the $pp$-wave metric ansatz, the real profile function $H = H(u, \zeta, \bar{\zeta})$ corresponds to the only nonvanishing metric perturbation 1-form $h_0 = -h^1 = -H du$, where the numerical indices refer to the null coframe. An explicit expression for the only nonvanishing Einstein 3-form can be calculated simply by making use of (27) as

$$
*G^1 = e^{00} \wedge d \star (dh_0 \wedge e^1) = -e^{00} \wedge d \star (dH \wedge e^{01})
$$

where $e^0 = du, e^1 = dv$ and $e^2 = d\zeta = e^{\bar{\z}}$. One can show that the expression on the right hand side reduces to

$$
*G^1 = -2\partial_0 \partial_{\bar{\zeta}} H \star du,
$$

or equivalently, $G_{00} = 2\partial_0 \partial_{\bar{\zeta}} H$. Equation (68) is a well-known equation satisfied by the profile function on the plane wave-fronts spanned by $\zeta$ and $\bar{\zeta}$ for a given value of the real null coordinate $u$. 

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7.4 New massive gravity

It is well-known that the Einstein tensor leads to no propagating degrees of freedom at the linearized level for the vacuum field equations in three dimensions. Recently, Bergshoeff, Hohm and Townsend [19–20] constructed a particular 3D gravity theory with a massive spin-2 propagating mode, which is often called New Massive Gravity (NMG) theory (and sometimes BHT gravity). The Lagrangian of the NMG theory composed of the Einstein-Hilbert term is supplemented by a specific combination of curvature-squared terms. NMG theory is closely related to the topologically massive gravity theory [21] that follows from the Einstein-Hilbert Lagrangian supplemented by a Lorentz Chern-Simons term in the sense that the topologically massive gravity field equations can be considered as square root of those of the NMG theory [22] with the field equations written as a form of a tensorial, Klein-Gordon-type equation.

The four dimensional version of NMG, the so-called “Critical Gravity” recently introduced by Liu and Pope [23], involves a Lagrangian with the Einstein-Hilbert term and a cosmological constant extended by quadratic-curvature terms. With the help of fine tuned parameters of the model, the massive scalar fields can be eliminated and the massive spin-2 field becomes massless in curved backgrounds.

Stelle [24] showed that in four spacetime dimensions, the analysis of the linearized field equations for the general quadratic curvature gravity leads to eight degrees of freedom corresponding to massless spin-2, massive spin-2 and a massive scalar mode. The massive modes correspond to the notorious unstable ghost modes, known as Boulware-Deser ghost modes [13]. In the original derivation presented by Bergshoeff, Hohm and Townsend, the massive spin-0 mode is eliminated by construction (which amounts to the relation $F_L = 0$ in the theory, thanks to the property that the trace of the field equations is of second order) and therefore resulted in massive spin-2 modes as the only propagating modes about a flat background.

Recently, de Rham, Gabadadze and Tolley constructed a gravitational theory (dRGT theory) [25–26] of interacting massive spin-2 fields which is not plagued by ghost modes. Later, the dRGT theory is cast in a convenient form in terms of two sets of coframe 1-form fields by Hinterbichler and Rosen [27–28] in a form which is much suited to the spirit of the current work. Similarly, the dRGT model formulated in terms of two sets of basis coframe fields and dual connection 1-forms [29], unifies the massive gravity models in three dimensions and eliminates the Boulware-Deser ghost mode that is a common feature in the nonlinear massive gravity models (See, also [30]).

The construction of the NMG Lagrangian (79) starts with a Pauli-Fierz mass term and a Einstein-Hilbert term together with an auxiliary symmetric, second rank tensor field. One can conveniently apply the technique presented in the previous sections to derive the quadratic curvature form of the NMG Lagrangian. The derivation is particularly transparent in the exterior algebra language. The derivation is valid for any spacetime dimensions $n \geq 3$ which, in effect, eliminates the massive scalar mode arising from the quadratic curvature part eventually.

The NMG Lagrangian involves the gravitational field variables corresponding to the 1-form field $h_{a} = h_{abcdef} \epsilon^{bc}$ and an auxiliary 1-form field $f_{a} = f_{abc} \epsilon^{bc}$ enjoying the same properties as the perturbation 1-forms $h_{a}$. The Lagrangian explicitly reads

$$L^{(2)}_{NMG}[h_{a}, f_{a}] = \frac{1}{2} h_{a} \wedge \ast G^a_L[h_{b}] + f_{a} \wedge \ast G^a_L[h_{b}] + \frac{1}{2} m^2 f_{a} \wedge e_{b} \wedge \ast (f^{b} \wedge e^{a}),$$

where the first term is the Einstein-Hilbert term, quadratic in the perturbation 1-form $h_{a}$, whereas the second term on the right hand side can be regarded as a Lagrange multiplier term. $G^a_L[f_{a}]$ stands for the Einstein form linearized in the metric perturbation 1-forms $f_{a}$ as in (70). The significance of the specific combination of these two terms in (69) will be apparent as one proceeds. The form of quadratic curvature terms given in (69) is also referred to as “natural bimetric form” (see, for example, the recent work [31]) for generic four dimensional quadratic curvature models.

By using the expression (69) for the linearized Einstein form, the second term on the right hand side can be rewritten conveniently as

$$f_{a} \wedge \ast G^a_L[h_{b}] = h_{a} \wedge \ast G^a_L[f_{b}] + \frac{1}{2} d \left[ h_{a} \wedge e_{b} \wedge \ast (d f^{b} \wedge e^{a}) - f_{a} \wedge e_{b} \wedge \ast (d h^{b} \wedge e^{a}) \right],$$

where the total differential on the right hand side can be disregarded when it appears in an action integral. To put it in other words, the second order differential operator implicitly defined by the linearized Einstein form (69) is Hermitian and expressed in terms of the exterior algebra of differential forms, the Hermiticity property readily follows from the identities satisfied by the Hodge dual and the exterior derivative operators without identifying the differential operator in terms of partial derivatives. Consequently, the Lagrangian (69) can be rewritten in the form

$$L^{(2)}_{NMG}[h_{a}, f_{a}] = h_{a} \wedge \ast G^a_L[f_{c}] - \frac{1}{2} h_{c} + \frac{1}{2} m^2 f_{a} \wedge e_{b} \wedge \ast (f^{b} \wedge e^{a}),$$

up to an omitted exact 3-form. The form of the Lagrangian (71) allows one to regard $h_{a}$ as auxiliary variables as well. Thus, $h_{a}$ can be eliminated from (71) simply by using the field equations for $h_{a}$ which explicitly read

$$\frac{\delta L^{(2)}_{NMG}}{\delta h_{a}} = \ast G^a_L[f_{c}] - h_{c} = 0.$$
These are just Einstein’s vacuum equations with the Einstein tensor linearized in the 1-forms $f^c - h^c$, and therefore (72) are satisfied identically for $h_a = f_a$. Eventually, using this result to simplify the Lagrangian (71), one readily recovers the Pauli-Fierz Lagrangian (51) and by construction, the Lagrangian (69) is equivalent to the Pauli-Fierz Lagrangian expressed in terms of an auxiliary field symmetric, second rank tensor field variable and a suitable Lagrange multiplier term.

The elimination of the field variable $h_a$ in the Lagrangian (71) leads to the fact that it has a massive spin-2 particle content. One can now use the form of the Lagrangian (71) to eliminate the auxiliary fields $f_a$ in the same way as it is done for $h_a$ above by simply calculating the corresponding field equations. Explicitly, the field equations that follow from $\delta L^{(2)}_{NMG}/\delta h^a = 0$ can be written in the form

$$\star G^a_L + m^2 \star (f^a - f^e_a) = 0. \quad (73)$$

Assuming that the auxiliary fields $f_a$ stand for their linearized form in the field equations, one can rewrite (73) without the linear approximation as

$$G^a = -m^2 (f^a - f^e_a), \quad (74)$$

where $f \equiv f^a_a$, although no label $L$ has been used for the auxiliary 1-form fields up to Eq. (74) following the original notation. By calculating the trace of the field equations, one can find that the trace of the auxiliary field variable $f$ is proportional to the scalar curvature as

$$f = -\frac{1}{4m^2} R. \quad (75)$$

Using (75) in the field equations (74), one eventually obtains the auxiliary fields in terms of the metric field as

$$f^a = -\frac{1}{m^2} L^a, \quad (76)$$

where $L^a$s are Schouten 1-forms $L_a \equiv L_{ab} \theta^b$ defined in terms of the Ricci tensor and scalar curvature as

$$L_{ab} = R_{ab} - \frac{1}{4} \eta_{ab} R. \quad (77)$$

Consequently, the nonlinear version of the Lagrangian (69), with the auxiliary variable $f_a$ eliminated, explicitly reads

$$L_{NMG} = \frac{1}{2} R \star 1 - \frac{1}{m^2} L_a \wedge \star G^a + \frac{1}{2m^2} L^a \wedge \theta^b \wedge \star (L_b \wedge \theta^a) \quad (78)$$

With the help of the definition (77) and the identities (4), one can show that the NMG Lagrangian (78) can be rewritten in a familiar form as

$$L_{NMG} = \frac{1}{2} R \star 1 - \frac{1}{m^2} \left( R_{ab} R^{ab} - \frac{3}{8} R^2 \right) \star 1. \quad (79)$$

Note that along the derivation of the NMG Lagrangian 3-form, the auxiliary field stands for a tensor as well as its linearization and the use of exterior algebra makes the manipulations in the derivation of the new massive gravity Lagrangian particularly transparent. It is well-known that the general combination of quadratic curvature terms complementing the Einstein-Hilbert action leads to massive spin-0 and massive spin-2 modes upon linearization around a flat background.

In $n$ spacetime dimensions, the above auxiliary fields procedure yields the quadratic curvature part of the form

$$\left( R_{ab} R^{ab} - \frac{n}{4(n-1)} R^2 \right) \star 1 \quad (80)$$

up to an overall constant. For $n = 4$, the Lagrangian 4-form in (80) is equivalent to the conformally invariant theory, namely the Weyl-squared gravity (52).

In a more general geometrical context, Tekin [33] recently used the technique introduced in this section in his study dedicated to the calculation of masses for the spin-0 and spin-2 modes in general quadratic curvature models around a curved background in four dimensions. By making use of the notion of the equivalent quadratic curvature Lagrangian [34,35], he also obtains the particle spectrum for the gravitational theories with the general Lagrangian involving a given function of the Riemann tensor.
8 Some further development

The application of the linearization technique described above to a given gravitational model naturally requires the corresponding field equations to be expressed in terms of the exterior algebra of differential forms. This section is devoted to such an extension of the method to derive linearized field equations in the desired form simply by making use of the variational derivative of a given Lagrangian with respect to the Levi-Civita connection 1-forms.

Let us consider a generic gravitational Lagrangian volume 4-form \( L = L[\omega^a_b, \theta^a] \), depending on the variables \( \omega^a_b \) and \( \theta^a \). For a metric theory these variables are not independent. Furthermore, the local Lorentz invariance of a Lagrangian forbids the explicit appearance of the connection 1-forms in the Lagrangian form because it is not tensorial and therefore, in a metric theory, the connection can enter into the Lagrangian, for example, through a curvature expression. Likewise, the metric theory can be obtained by the first order formalism \([36,37]\) where the \( \omega^a_b \) and \( \theta^a \) are assumed to be independent variables and then constrain the independent connection 1-forms to be metric compatible \( (\omega^a_b + \omega^a_b = 0 \) relative to an orthonormal coframe) and torsion-free. This can be achieved, for example, by introducing the corresponding Lagrange multiplier terms to the original Lagrangian. On the other hand, in a direct manner more suitable to linearization, it is possible to convert the variational derivative with respect to Levi-Civita connection 1-form into a variational derivative with respect to coframe variational derivatives by making use of the variational derivative of the Cartan’s structure equations \([5]\). The method involves the calculation of variational derivative with respect to connection 1-forms and can be described as follows.

For this purpose, note first that the variational derivative of the curvature 2-forms can be expressed in general as

\[
\delta \Omega^b_{ab} = D \delta \omega^a_b
\]

where \( D \) is assumed to be the covariant exterior derivative with respect to a Levi-Civita connection. Consequently, the terms that turn out to be linear in the metric perturbation only arise from the variational derivatives of the terms with respect to the connection 1-forms. Therefore, in order to obtain the linearized field equations, it is sufficient to evaluate the variational derivative with respect to connection 1-forms in a suitable manner.

The metric field equations can be obtained, as mentioned above, by using the first order formalism which treats the connection and the coframe 1-forms independently, and the torsion-free condition \([5]\) satisfied by the connection has to be implemented into the variational derivatives with respect to the connection 1-forms. Because this is a set of dynamical equations among the variables assumed to be independent initially, one can introduce a Lagrange multiplier term of the form \( \lambda_a \wedge (d\theta^a + \omega^a_b \wedge \theta^b) \) to impose the constraint \([5]\) on the independent connection. Eventually, one eliminates the Lagrange multiplier forms \( \lambda_a \) from the coframe variational derivatives to obtain metric field equations in favor of the remaining variable. In this manner, terms that are linear in metric perturbation 1-forms turn out to be those obtained from the Lagrange multiplier term imposing the vanishing torsion constraint on the independent connection. An immediate technical consequence of this observation is that the linearized equation is always a total derivative, for example, through a curvature expression. Likewise, the metric theory can be obtained by the first order formalism \([36,37]\) where

\[
\delta \Omega^b_{ab} = D \delta \omega^a_b
\]

where \( D \) here stands for the covariant exterior derivative with respect to the Levi-Civita connection. These equations relating the variational derivatives \( \delta \theta^a \) and \( \delta \omega^a_b \) can be inverted to have

\[
\delta \omega^a_b = \frac{1}{2} i^a i_b (D \delta \theta^a \wedge \theta_c) - i^a D \delta \theta_b + i_b D \delta \theta^a
\]

in the same way as the connection 1-forms can be inverted to express them in terms of the exterior derivatives of the basis coframe 1-forms. The expression \([83]\) can be used to convert the variational derivatives with respect to the connection 1-forms into the variation of the metric perturbation 1-forms. Consequently, one can use this result and \([29]\) to have

\[
(\delta \omega^a_b)^L = \frac{1}{2} i^a i_b (d \delta h^a \wedge e_c) - i^a d \delta h_b + i_b d \delta h^a
\]

where, the coframe variational derivatives are expressed in terms of the variation of the metric perturbation 1-forms. The convenient expression \([84]\) then can used to identify the coefficient of metric perturbation 1-form \( \delta h_a \) as the linearized equations.

Finally, note that the terms linearized in the metric perturbation survives the variational derivative only if the Lagrangian contains terms that are at most quadratic in curvature components because, schematically, for a typical Lagrangian \( R^n \), a polynomial in curvature \( R \), one has \( \delta R^n \sim (\delta R)R^{(n-1)} \sim \delta q \nabla \nabla R^{(n-1)} \). To obtain the linearized form of field equations following from a Lagrangian involving higher powers of curvature components, Güllü et al.
developed a method to convert a higher curvature theory to an equivalent quadratic curvature theory. In this regard, the linearization of general quadratic curvature models turns out to be of considerable importance since their linearization encompasses all higher curvature theories.

In order to illustrate the details of the above recipe, let us calculate the linearized form of the field equations that follows from the higher order gravitational Lagrangians \( L = R^2 \times 1 \) and \( L = dR \wedge *dR \).

### 8.1 Fourth order gravity with \( L = R^2 \times 1 \)

In general, the field equations that follow from the gravitational Lagrangian of the form \( f(R) \times 1 \) are of fourth order in metric components with \( f \) being a sufficiently smooth algebraic function of scalar curvature \([39]\). The simplest function \( f(R) = R^2 \) displays the main features of generic \( f(R) \) models.

The total variational derivative of the Lagrangian 4-form \( L = R^2 \times 1 \) can be written as

\[
\delta L = 2R(\delta R) \times 1 + R^2 \delta \times 1, \tag{85}
\]

where only the first term is of interest for the linearization. Thus, it is more convenient to rewrite the variational derivative \([34]\) in the form

\[
\delta L = 2R\delta(R \times 1) - R^2 \delta \times 1. \tag{86}
\]

Furthermore, by using \( R \times 1 = \Omega_{ab} \wedge *\theta^{ab} \), \([34]\) can be put into a convenient form as

\[
\delta L = 2R(\delta \Omega_{ab} \wedge *\theta^{ab} + \Omega_{ab} \wedge \delta *\theta^{ab}) - R^2 \delta \times 1. \tag{87}
\]

Evidently, the linear terms arise only from the variational term \( 2R\delta \Omega_{ab} \wedge *\theta^{ab} \) on the right hand side, which contains terms that are of fourth order. With the help of the relation \([34]\), one finds

\[
\delta L = \delta \omega_{ab} \wedge 2D(R * \theta^{ab}) + \delta \theta_c \wedge (2R\Omega_{ab} \wedge *\theta^{abc} - R^2 * \theta^c). \tag{88}
\]

Finally, one can rewrite the first term in terms of the linearized quantities with the help of \([34]\) to obtain

\[
\delta L = (\delta \omega_{ab})^L \wedge 2d(R * \theta^{ab})^L + \ldots = \left( \frac{1}{2} i^a db(d \delta h^c \wedge c^c) - i^a d \delta h^a + i_b d \delta h^a \right) \wedge 2dR^L \wedge *e^{ab} + \ldots = \delta h_a \wedge 4d * (dR^L \wedge e^a) + \ldots \tag{89}
\]

Thus, the result can be conveniently expressed in the desired form, as in the linearized Einstein 3-forms, as

\[
\left( \frac{\delta L}{\delta \theta^a} \right)^L = (*E^a)^L = *E^a_L = 4d* (dR^L \wedge e^a) \tag{90}
\]

in terms of the 1-forms \( E_a = E_{ab} e^b \). One then uses the result \([40]\) to determine the components of \( E^L_{ab} \) which can be expressed in the form

\[
E^L_{ab} = 4 \left( - \partial_a \partial_b R^L + \eta_{ab} \Box R^L \right). \tag{91}
\]

Once more, it is worth emphasizing that as in the case of the linearized Einstein field equations, the equation \([40]\) yields precisely the linearized field equations \([41]\) obtained by the coordinate methods \([40]\) for \( E^L_{ab} \). For completeness, note that the vacuum field equations \( *E^a = 0 \) can be expressed explicitly by using

\[
*E^a = 4D * (dR \wedge \theta^a) - 4R * R^a + R^2 * \theta^a. \tag{92}
\]

Then the linearized form of the field equations \([40]\) can be rearranged as

\[
*E^a_L = 2d [e_b \wedge i_b * (dR^L \wedge e^a)], \tag{93}
\]

where the expression \( *E^a[h_b] \) on the right hand side is proportional to \( *E^L_{ab} [R^L e^b] \), that is, the linearized Einstein form with the perturbation 1-forms \( h^a \) replaced by \( R^L e^a \). The nested structure of the linearized quadratic curvature gravity is readily available by inspection from the field equations using \([41]\).

Yet another interesting observation on the result \([40]\) is that the linearized equations may be expressed as an exact form in flat background. This is not a particular feature of the pure \( R^2 \) gravity. Consequently, a matter energy
momentum \( T^a \) satisfies \( d \ast T^a = 0 \) (which reads \( \partial_a T^b = 0 \) in component form) as in general relativity. However, there is an additional second order identity satisfied by the linearized equations (90) which can be derived as follows.

For simplicity of the argument, let us assume that the field equations with a matter energy-momentum \( \ast T^a \) can be written in the form \( \ast E^a = 4 \ast T^a \). Then, the linearized trace explicitly reads

\[
3d \ast dR^L = - \ast T, \tag{94}
\]

where \( T \) stands for the trace of the matter energy-momentum tensor. The trace (94) can also be written in the form \( 3 \square R^L = -T \). Now, by applying \( \square \) to the linearized equations (90) and taking into account the fact that \( \square \) commutes with the variational derivative with respect to the coframe basis 1-forms. As in the previous example, by using now the relation (84), one finally obtains the corresponding linearized equations in the form

\[
\eta_{ab} \square T - \partial_a \partial_b T = 3 \square T_{ab}. \tag{96}
\]

8.2 Sixth order gravity with \( L = dR \wedge \ast dR \)

Although the sixth order Lagrangian \( dR \wedge \ast dR \) is in a more suitable form for our purposes, note that it can also be rewritten in the form \( -R \square R \ast 1 \) up to an exact form. In this section we have \( \square \equiv - (d^1 d + d^1) \) with \( d^1 \) standing for the exterior coderivative [3]. The coderivative can be defined in terms of the exterior derivative and the Hodge dual as \( d^1 = (-1)^p \ast d^1 \). As for the other operators used above, the operators \( \square \) and \( d^1 \) act on forms defined in curved space as well as those defined in flat background spacetime. In particular, it follows from the definitions that, \( \square \) acting on a 0-form \( \phi \) in flat spacetime yields \( \eta^{ab} \partial_a \partial_b \phi \) defined in the previous sections.

The total variational derivative of the Lagrangian can be written in the form

\[
\delta L = 2d \delta R \wedge \ast dR - \delta \theta^a \wedge (i_a dR \ast dR + dR_i a \wedge \ast dR) \tag{97}
\]

where the second term on the right hand side arises from commuting the variational derivative with the Hodge dual. The expression on the right hand side can be rearranged to a suitable form by using the identity \( d \ast = (-1)^p \ast d^1 \) acting on a p-form and that the variational derivative commutes with the exterior derivative. Because one has to consider only the variational derivative with respect to the connection 1-forms, it is convenient to rewrite the variational derivative in the form

\[
\delta L = 2 \delta (R \ast 1) \square R + \ldots \tag{98}
\]

Now by making use of \( R \ast 1 = \Omega_{ab} \wedge \ast \theta^{ab} \) and subsequently using the variational identity (54), one can obtain

\[
\delta L = 2 \delta \omega_{ab} \wedge D(\square R \ast \theta^{ab}) + \ldots \tag{99}
\]

where the omitted terms are the variational derivative terms with respect to the coframe basis 1-forms. As in the previous example, by using now the relation (54), one finally obtains the corresponding linearized equations in the form

\[
\left( \frac{\delta L}{\delta \theta^a} \right)^L = (\ast E^a)^L = \ast E^a_L = -4d \ast (d(\square R^L) \wedge \ast \theta^a), \tag{100}
\]

where \( \square \) now refers to the flat background spacetime.

Note that in this case \( \ast E^a_L[h_a] \) is proportional to \( \ast G^2_L[\square R^L \ast c^b] \) and compared to the expression for the linearization of the quadratic curvature gravity, one can deduce that the only difference in the corresponding expressions shows up as the argument of the Einstein form.

As in the case of the quadratic curvature gravity, pure sixth order gravity coupled to a matter field also leads to an additional second order differential identity in the linearized approximation for a matter energy-momentum tensor. More precisely, if one assumes that pure sixth order gravitational equations can be written in the form \( \ast E^a = 4 * T^a \) with matter energy-momentum 1-forms \( T^a \), then the linearized trace reads

\[
3d \ast d(\square R^L) = - \ast T \tag{101}
\]
where \( T \equiv T^a_a \) is the trace of the energy momentum tensor. Equivalently, this equation can also be written in the form \( 3 \Box^2 R = -T \). Consequently, by inserting the trace expression given in (101) into the equations (100), one arrives at

\[
d * (dT \wedge e^a) = 3 \Box T^a_{ab} \wedge e^b
\]

which yields the identity (102) written in component form. Thus, the result of Pechlaner and Sexl that is rederived in the previous section is also valid here for pure sixth order gravity. This may be an indication of its validity for a range of general higher order gravitational models, for example, with the Lagrangian of the form \( L = R^2 + dR \wedge *dR \).

Finally, note that the field equations for the sixth order gravity in terms of the exterior algebra differential forms explicitly read

\[
* E^a = -4 D * (d(\Box R) \wedge \theta)^a + 4 \Box R \wedge R^a - 2(i_a dR) \wedge dR + i_a(dR \wedge *dR),
\]

and the general higher order gravitational Lagrangians involving \( R \Box^k R \wedge *1 \) \((k \text{ being a positive integer)}\) were studied previously by Schmidt. Such Lagrangians leading to \((2k + 2)\)th order metric equations are shown to be conformally equivalent to Einstein-multi-scalar gravitating models. More recently, a sixth order gravity where the Einstein-Hilbert term with a cosmological constant term complemented by \( R^2 \) and \( R \Box R \) terms are studied by Bergshoeff et al. in three dimensions. For the fine-tuned parameters of the model, they showed that two massive graviton modes become massless in addition to the already existing massless spin-2 modes after linearizing the field equations around an AdS background.

9 Nonvanishing torsion

Up to this section, the discussion of linearization has been confined to the Riemannian geometry where the torsion and non-metricity vanish identically, and the metric tensor is fundamental to the underlying geometrical structure. For the geometrical setting allowing a metric compatible connection with nonvanishing torsion, namely for the Riemann-Cartan geometry the connection \( \Omega^a_{bc} \) is determined by independent field equations for the connection 1-forms coupling to the spin of matter fields. In general, the connection \( \Omega^a_{bc} \) can be decomposed into its Riemannian and contortion parts

\[
\Omega^a_{bc} = \Theta^a_{bc} + \Theta^a_{bc} \wedge \theta^c
\]

where the contortion 1-forms \( K_{ab} = -K_{ba} = K^c_{bc} \wedge \theta^c \) are related to the torsion 2-forms \( \Theta^a \) by

\[
\Theta^a = \Theta^a_{bc} \wedge \theta^c.
\]

The torsion 2-forms in turn can be expressed in terms of the components of the tensor torsion \( T^a_{bc} \), as \( \Theta^a = \frac{1}{2} T^a_{bc} \wedge \theta^c \). The curvature 2-forms \( \Omega^a_{b}(A) \) corresponding to the Riemann-Cartan connection can be written as

\[
\Omega^a_{b}(A) = dA^a_{b} + A^a_{c} \wedge A^c_{b} - \frac{1}{4} \Box(T^a_{bc} \wedge \theta^c)
\]

Furthermore, with the help of the decomposition given in (101), it is possible to decompose (106) in a similar way as

\[
\Omega^a_{b}(A) = \Omega^a_{b}(\omega) + D(\omega)K^a_{bc} + K^a_{c} \wedge K^c_{b}
\]

where \( \Omega^a_{b}(\omega) \) are the curvature 2-forms corresponding to the Levi-Civita connection where \( D(\omega) \) is the covariant exterior derivative with respect to it. The expressions (107) then allow one to consider various possibilities for the weak field approximations. One can consider weak metric fields, and/or weak torsion fields as well. In general, the nonvanishing torsion introduces a term of the form

\[
[D(\omega)K^a_{b}]^L = d(K^a_{b})^L
\]

to linearized Riemann-Cartan curvature 2-forms to first order in the field variables. Consequently, the corresponding linearized Einstein 3-forms take the form

\[
*[G^a(A)]^L = * [G^a(\omega)]^L - \frac{1}{2} dK^L_{bc} \wedge *e^{abc}
\]

Note that in this general case, one has two independent weak field 1-forms, \( h_a = h_a \theta^b \) and \( K^L_{ab} = K^L_{abc} \theta^c \) and that the contortion 1-forms \( K_{ab} \) are determined by the independent connection equations. It is well-known that if the equations for the Riemannian connection 1-forms are algebraic, then the metric field equations, and its linearized form can be expressed in terms of Riemannian quantities by eliminating torsion terms.

If the pure gravity action at hand is linear in the curvature, (e.g., the Einstein-Hilbert or the Brans-Dicke action) both the zero-torsion constrained and the unconstrained variations lead to the same set of linearized vacuum field equations. If the action is quadratic or higher order in the curvature, such an equivalence does not occur in general.

Finally, it is interesting to note that the linearized form of the field equations for the teleparallel gravity, where one has vanishing curvature but a nonvanishing torsion, has been considered only relatively recently by Obukhov and Pereira.
10 Concluding comments

An interesting remark concerning the linearization presented above is the following. By inserting the connection expression (7) into the reduced Einstein-Hilbert Lagrangian 4-form (44), the Einstein-Hilbert Lagrangian can be expressed solely in terms of basis 1-forms in the form

\[ L_{EH} = -\frac{1}{2} d\theta^a \wedge \theta^b \wedge \ast (d\theta_b \wedge \theta_a) + \frac{1}{4} d\theta^a \wedge \theta_a \wedge \ast (d\theta^b \wedge \theta_b) - d(\theta_a \wedge \ast d\theta^a) \]  

(110)

by eliminating the connection 1-forms in favor of the basis coframe 1-forms [4]. The Pauli-Fierz Lagrangian (46) is obtained by inserting (9) into the expression (110) where the second term on the right hand side drops out as a consequence of the assumption that \( h_{ab} = h_{ba} \Leftrightarrow h_a \wedge \epsilon^a = 0 \). On the other hand, the remaining term turns out to be the germane part of the Einstein-Hilbert Lagrangian in the sense that the full \( L_{EH} \) can be recovered from it by an infinite number of iterations starting from the linearized approximation.

There are several directions that the linearization technique above can be extended. The above formalism can be applied to the linearization of any modified gravitational field equations around a flat background, which will presumably simplify more complicated field equations [13,17] as well. Such a scheme additionally requires that the field equations are expressed in the language of the exterior algebra of differential forms as in the case of the quadratic curvature and the sixth order gravitational models studied above. The calculations for the particular higher order gravitational models studied above imply that the linearization of the gravitational field equations that follows from a Lagrangian involving only the powers (and the derivatives) of the curvature tensor leads to the mathematical structure similar to the linearized Einstein tensor which may be considered as a general feature that is an imprint of the curvature tensor.

In this paper we have considered only the flat background and in a physically important direction of development, one can tackle the technical details of the linearization around an arbitrary curved background [14,15,16,54,55,56]. The extension of the linearization technique above is essential in the linearization of modified gravitational models involving, for example, the general quadratic curvature gravity which admits maximally symmetric vacuum solutions. In this case one has to take both the zeroth and first order terms in the perturbation 1-forms \( h_a \) into account in the linearization calculations as, for example, displayed in the expansions (12) and (15).

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