Numerical modelling of focused relativistic electron-positron beams colliding with crossing angle

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Abstract.
We present a three-dimensional parallel algorithm for numerical modelling of self-consistent colliding beam dynamics in linear supercolliders. The thin ultrarelativistic beams are focused by quadrupole lens field and collide with a crossing angle (2-25 mrad). The small sizes of the beams are necessary for high luminosity achievement. However, their high density may be critical and the beams may significantly change their shape or even disrupt. The crossing angle between the beams and the non-linear focusing complicate the problem of the beam stability and optimization of the collider parameters.

Our algorithm is based on solution of Vlasov-Liouville and Maxwell’s equations in three-dimensional case. We use the particle-in-cell method with PIC form-factor and Langdon-Lasinsky scheme on shifted Yee’s grids. We combine particle and domain decompositions for the purpose of using $10^9$ model particles in the numerical experiments. The beam particle distribution is highly non-linear in space, and the domains with larger numbers of the model particles are computed with higher numbers of the processors. The algorithm and its components may be applied to different problems in relativistic plasma physics.

1. Introduction
The work is devoted to mathematical modelling of interaction effects of charged particle beams in linear colliders. One of important problems in the physical experiments is achieving of high luminosity for a given energy. Luminosity reflects the number of interactions between the colliding beams in a certain time in the cross-section. The luminosity depends on the beam density, which must be maximal in the interaction point (IP), thus the beams become focused by the magnetic fields of quadrupole lenses only in the neighborhood of the IP. The luminosity depends on the beam currents, thus the charge increase may increase the luminosity. The high energetic beams move with relativistic factors ($\gamma \sim 10^3 - 10^6$) [1] and the transversal fields of the particles are higher $\gamma$ times in comparison with non-relativistic case. In the critical regimes the beam deformation or disruption may occur due to the high repulsion forces between the particles of the beam in the transversal direction. The presence of crossing-angle may significantly influence the beam stability. The collider construction and running is expensive and the topical question is the optimization of the beam parameters.

The standard mathematical model is based on the longitudinal division of the beams into slices. The three-dimensional interactions is substituted with the interaction of the opposite beam slices when their longitudinal coordinates coincide. The two-dimensional slice fields are
applied to the particles of the opposite slice, changing their motion, and then the beams switch roles [2, 3]. The approach is inherited from the successfully applied numerical models for the beam interaction in circular colliders [4, 5], where the beams collide thousands of times and the changes on each turn are small. The single collision with crossing angle in critical regimes with the strong particle and field redistribution in the domain require a three-dimensional model.

We present the parallel algorithm for mathematical modelling of the non-stationary three-dimensional beam dynamics in the self-consistent electromagnetic fields of modern colliders. The algorithm takes into account the high particle energies, the beam non-linear focusing (hour-glass effect) and the beam crossing angle. The results of numerical experiments and their comparison with analytical solutions for a non-critical regime are presented.

2. Mathematical model

The electron/positron beam dynamics in the self-consistent electromagnetic fields in vacuum is considered in a paralelepipedal domain $[0 : L_x] \times [0 : L_y] \times [0 : L_z]$ in the laboratory system of coordinates. We use the Vlasov kinetic equation for the distribution function of electrons $f_-(r, p, t)$ and positrons $f_+ = f_+(r, p, t)$ and Maxwell’s equations in three-dimensional case:

$$\begin{align*}
\frac{\partial f_+}{\partial t} + v_+ \cdot \frac{\partial f_+}{\partial r} + F_+ \cdot \frac{\partial f_+}{\partial p} &= 0, \\
rot E &= -\frac{1}{c} \frac{\partial B}{\partial t}, \\
rot B &= \frac{4\pi}{c} j + \frac{1}{c} \frac{\partial E}{\partial t}, \\
\text{div} E &= 4\pi (n_- e^- + n_+ e^+) \\
\text{div} B &= 0,
\end{align*}$$

(1)

where the particle momentum $p_{+,-} = \gamma_{+,-} m_e v_{+,-}$ and the Lorentz force $F_{+,-} = e^+ (-E + \frac{v_{+,-}}{c} \times H)$.

The charge densities and the currents correspond to the distribution function moments. The characteristics of the Vlasov equation represent the motion equations.

We solve the equations in dimensionless variables, assuming the characteristic length $L$ of the beam is 1 cm and the characteristic speed $v$ of the particles is the speed of light $c = 2.9979 \cdot 10^{10}$ cm/s.

We solve the equations with the particle-in-cell method [6] applying PIC form-factor. We apply the Langdon-Lazinsky [7] scheme for Maxwell’s equations and Boris [8] for the coordinates and momenta on the uniform shifted grids [9]. This schemes yield the second order of accuracy in space and time. For current density computation we use method [10].

We suppose the transversal boundaries are in the near wave zone of the beams, they are very close to the beams and the relativistic delay does not play a significant role. At the longitudinal boundaries there are no beam particles and electromagnetic fields are zero due to the high relativistic factor. The boundary conditions and the initial conditions for the fields are defined as the fields from the beams. The electric field of the beam is defined by the fields of the particles, each particle produces the following field:

$$E = \frac{1}{\gamma^2 (1 - \beta^2 \sin^2 \theta)^{3/2}} \frac{q R}{R^2},$$

where $R$ is radius-vector from the charge $q$ to the point of the field calculation [11], $\theta$ is the angle between $v$ and $R$. The formula demonstrates the field is concentrated in a very thin region: the
transversal field component is higher $\gamma$ times and the longitudinal one is smaller $\gamma^2$ times than the corresponding components of a non-relativistic particle. If a beam moves along $v$ and $\theta$ is between $v$ and $z$ axis, only the fine grids with spatial step $h_z \leq L_z/tg\theta \sim 10^4$ provide differences in the field values. As the beams in colliders are thin and long ($\sigma_z >> \sigma_x >> \sigma_y$) and the crossing angles are 2-50 mrad [12, 13, 14], we assume the field in the equatorial plane for $v$ coincides with the corresponding field in equatorial plane for $z$. In the method [15] we use a special shape of the model particles and the computation of beam field $E(x,y,z)$ allows to sum only the field contributions from the density elements in the equatorial plane (with the same $z$ coordinates) [16].

The initial and boundary conditions for the magnetic field of the beams can be computed from $B_{+,-} = v_{+,-} \times E_{+,-}$.

The beam dynamics in critical regimes cannot be described analytically, however for a non-critical regime the beam-beam effects are negligible and the luminosity for beams with equal

$$K$$

where kinematic factor

$$\epsilon$$

rotation by an angle $\theta$ in the interaction cross-section, we consider the integral luminosity:

$$L = KN_+N_- \int \int \rho_+(x)\rho_+(y)\rho_+(s-s_0)\rho_-(x)\rho_-(y)\rho_-(s+s_0)dx dy ds ds_0,$$

where $(x_{IP}, y_{IP}, z_{IP})$ are IP coordinates, $(x_c, y_c, z_c)$ - beam center coordinates. The focusing is defined by the non-constant values of $\sigma_x, \sigma_y, \sigma_{x'}, \sigma_{y'}$: $\sigma_{x,y} = \sqrt{\beta_x \sigma_{x,y} \epsilon_{x,y}}$, $\beta_{x,y} = \beta_{x,y} - (z - z_{IP})^2/\beta_{x,y}^*$, where $\beta_{x,y}^*$ are the beta-functions in the IP and $\epsilon_x, \epsilon_y$ are the beam emittances. The beam envelopes in directions $x, y$ can be described by the corresponding parabolic beta-function, the beam size minimum is achieved at $z = z_{IP}$. We consider the right beam of positrons moves along $z$-axis, and for another beam coordinates and momenta we apply rotation by an angle $\theta$ in plane $(z, x)$.

Luminosity reflects the number of interactions of the colliding beam particles in a certain time in the interaction cross-section, we consider the integral luminosity:

$$L = KN_+N_- \int \int \rho_+(x)\rho_+(y)\rho_+(s-s_0)\rho_-(x)\rho_-(y)\rho_-(s+s_0)dx dy ds ds_0,$$

where kinematic factor $K = \sqrt{(v_1 - v_2)^2 - (v_1 \times v_2)^2}$ and the distance from the IP is $s_0 = ct$ [17], [18].

The beam dynamics in critical regimes cannot be described analytically, however for a non-critical regime the beam-beam effects are negligible and the luminosity for beams with equal emittances and beta-functions can be written in the following way [19]:

$$L = \frac{N_+N_-cos(\theta/2)}{4\pi \sqrt{\pi \sigma_x^* \sigma_y^* \sigma_z}} \int e^{-s^2 A} ds \sqrt{s^2 cos^2(\theta/2)} - \beta_x^2 + 1 \sqrt{s^2 cos^2(\theta/2)} - \beta_y^2 + 1,$$

where

$$A = \frac{cos^2(\theta/2)}{\sigma_x^2} + \frac{sin^2(\theta/2)}{\sigma_y^2} \left(1 + \frac{s^2 cos^2(\theta/2)}{\beta_x^2} \right).$$

This integral can be computed numerically and can be used as a test to compare with the luminosity obtained through our three-dimensional algorithm, where we compute the luminosity
as the following sum:

\[ L = KN_+N_- \sum_{m} \sum_{l,k} \rho_+(x, y, z - ct) \rho_-(x, y, z + ct) h_x h_y h_z \tau \]

for \( x = (i - 1.5)h_x, y = (i - 1.5)h_y, z = (i - 1.5)h_z, t = m\tau \). The integral luminosity behaviour is one of the ways to control the quality of the complicated algorithm for simulation of the non-stationary three-dimensional beam and field dynamics at all time moments in all grid nodes.

3. Numerical experiments

We present the simulation results for two colliding beams with \( \beta_x = 1.1cm, \beta_y = 0.05cm, \epsilon_x = 2.2 \cdot 10^{-8}cm \cdot rad, \epsilon_y = 7.2 \cdot 10^{-9}cm \cdot rad, \sigma_x = 5 \cdot 10^{-5}cm, \sigma_y = 6 \cdot 10^{-6}cm, \sigma_z = 0.03cm \), the particle relativistic factors \( \gamma = 0.990 \cdot 10^6 \) correspond to energy 500GeV. The beams are cut on 3\( \sigma \) in each direction: \( R_x = 3\sigma_x, R_y = 3\sigma_y, R_z = 3\sigma_z \).

On fig. 1, 3 the beam coordinates \((x,z)\) in time moments \( t = 0 \) and \( t = 0.33 \) respectively for the beam charges \( N_- = N_+ = 10^6 \) are shown. The red and the black lines denote the corresponding beam envelopes for 3.5\( \sigma_x \). The value \( \beta_x \) is high enough for the beam envelopes look straight. On fig. 2, 4 the beam coordinates \((y,z)\) in time moments \( t = 0 \) and \( t = 0.33 \) ns are shown, the smaller value of \( \beta_y \) provides the parabolic beam envelope, the black lines correspond to 3.5\( \sigma_y \). As the beam charges are small, the beam interaction is weak and the beam shapes change according to the distribution (2).
On the fig. 5 the luminosity dependence on the angle $\theta$ is shown. The solid line represents the values for the numerical computation of the integral (3) with the Riemann sum of $8 \cdot 10^5$ elements, the red crosses refer to our three-dimensional code results. The relative error maximum is 0.11% and it demonstrates the good quality of the algorithm performance.

![Figure 5. Luminosity dependence on the crossing angle $\theta$](image1)

Figure 6. Luminosity dependence on the disruption parameter $D_y$

On the fig. 6 the luminosity enhancement factor $H_D = (L(N)/N^2)/(L(N_0)/N_0^2)$ as dependence on the disruption parameter $D_y$ with $\theta = 0$ is shown. We normalize the luminosity on the values for $N_0 = 10^6$, the disruption parameter $D_{x,y} = \frac{2r_e N_0 \sigma_z}{\gamma \sigma_{x,y} (\sigma_x + \sigma_y)}$, where $r_e$ is the electron radius. The values $D_{x,y} << 1$ indicate non-critical regimes, the values $D_{x,y} >> 1$ indicate critical regimes [20]. The beam charge increase provides the luminosity enhancement, it reaches 40% due to the beam compression by the fields of the opposite beam for $N = 2 \cdot 10^{11} (D_y = 10)$, the further increase of the charge doesn’t lead to the luminosity enhancement because of the beam disruption. In fig. 7, 8 the beam coordinates for charges $N = 2 \cdot 10^{11}$ and crossing angle $\theta = 0.004$ and time moment $t = 0.33$ ns in both planes are presented, and the beams deformation in (z,y) plane is observed. The crossing angle significantly reduces the luminosity for these parameters from $L = 1.26 \cdot 10^{31} cm^{-2}s^{-1}$ for $\theta = 0$ down to $L = 5.57 \cdot 10^{30} cm^{-2}s^{-1}$ for $\theta = 0$.

![Figure 7. Beam coordinates (z,x) at t=0.33 ns for $N = 2 \cdot 10^{11}$](image2)

![Figure 8. Beam coordinates (z,y) at t=0.33 ns for $N = 2 \cdot 10^{11}$](image3)

For these simulations we used computational domain with sizes $L_x = 0.002$ cm, $L_y = 0.0005$ cm and $L_z = 0.4$ cm, the coordinates of the IP were $(L_x/2, L_y/2, L_z/2)$. The number of grid
nodes was 100 in each direction, the number of the model was particles \(10^8\), the number of the time steps was 64000, the time step was \(\tau = 3.125 \cdot 10^{-6}\). 448 processor cores were distributed among 25 groups, the three central groups had 324 and 49 processor cores. The computation time was 34 hours.

The computations were performed on the supercomputer Lomonosov (MSU, Moscow), Polytechnic (SPBSTU, Saint Petersburg) and SSCC cluster (ICM&MG SB RAS, Novosibirsk).

4. Conclusion

The three-dimensional parallel algorithm for numerical modelling the non-linear beam collision with a crossing angle in supercolliders is presented. The algorithm is based on solution of the Vlasov and Maxwell’s equations in three-dimensional case. The particle and domain decomposition allows performing simulations with strongly focused beams. The results of numerical experiments with the algorithm demonstrate good coincidence of the numerical solution with analytical one in cases of non-critical regimes. The algorithm can be used not only for the beam parameter optimization, but may be applied to different problems in relativistic plasma physics.

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