General maximum principles for optimal control problems of stochastic Volterra integral equations

Tianxiao Wang

Abstract

Optimal control problems of forward stochastic Volterra integral equations (SVIEs) are formulated and studied. When control region is arbitrary subset of Euclidean space and control enters into the diffusion, necessary conditions of Pontryagin’s type for optimal controls are established via spike variation. Our conclusions naturally cover the analogue of stochastic differential equations (SDEs), and our developed methodology drops the reliance on Itô formula and second-order adjoint equations. Some new features, that are concealed in the SDEs framework, are revealed in our situation. For example, instead of using second-order adjoint equations, it is more appropriate to introduce second-order adjoint processes. Moreover, the conventional way of using one second-order adjoint equation is inadequate here. In other words, two adjoint processes, which just merge into the solution of second-order adjoint equation in SDEs situation, are actually required and proposed in our setting.

Keywords: Stochastic Volterra integral equations, maximum principles, second-order adjoint processes, non-convex control region.

AMS Mathematics subject classification. 93E20, 60H20, 49K45.

1. Introduction

Suppose $(\Omega, \mathcal{F}, \mathbb{F}, P)$ is a complete probability space, $W(\cdot)$ is a one-dimensional Wiener process which generates filtration $\mathbb{F} = \{\mathcal{F}_t\}_{0 \leq t \leq T}$. In this paper, we study optimal control problems for stochastic Volterra integral equations (SVIEs, in short), where the state equation is described as,

\begin{equation}
X(t) = \varphi(t) + \int_0^t b(t, s, X(s), u(s))ds + \int_0^t \sigma(t, s, X(s), u(s))dW(s),
\end{equation}

with $t \in [0, T]$ and the cost functional is

\begin{equation}
J(u(\cdot)) = E[h(X(T)) + \int_0^T l(s, X(s), u(s))ds].
\end{equation}

Here $u(\cdot)$ is a control process taking values in $U \subset \mathbb{R}^m$, and $X(\cdot)$, the (strong) solution of (1.1), is the corresponding state process in $\mathbb{R}^n$. Under proper conditions, (1.1) admits a unique solution $X(\cdot)$. Then the optimal control problem is to find suitable $u(\cdot)$ to minimize (1.2).
One usual way to treat above problem is to establish the Pontryagin’s maximum principle. In 1964, Friedman [10] discussed the case of

\begin{equation}
X(t) = x_0 + \int_0^t h(t-s)b(s, X(s), u(s))ds, \quad t \in [0, T].
\end{equation}

Later in 1969, Vinokurov [23] investigated the case of general nonlinear VIEs with constrained state processes. Some relevant works in the deterministic framework can be found in e.g. Bakke [3], Bonnans et al [5], Dmitruk-Osmolovski [8], Halanay [12] and the references therein. As to the stochastic case, Yong [27], Agram-Øksendal [1] present some investigations with open set above papers can treat the case when both related studies include Bonaccorsi et al [4], Shi et al [22], Wang-Zhang [26] and so on. However, none of this current paper is to spread out detailed discussions with arbitrary $U \subset \mathbb{R}^m$ and control-dependent $\sigma$.

To explain the motivations of studying SVIE (1.1), we start with some comparisons with classical stochastic differential equations (SDEs, in short). If $\varphi$, $b$, $\sigma$ are independent of $t$, then for $t \in [0, T]$, $X(t)$ satisfies the following controlled SDE

\begin{equation}
X(t) = x_0 + \int_0^t b(s, X(s), u(s))ds + \int_0^t \sigma(s, X(s), u(s))dW(s).
\end{equation}

We list out three new features of (1.4), which can not captured by above (1.3).

\begin{itemize}
  \item The diffusion could be $\mathcal{F}_t$-measurable, and thus involves anticipated stochastic integral. Even so, one can still obtain adapted solutions, see e.g. [20].
  \item Due to the reliance on both $t$ and $s$, the drift or diffusion could have singular kernels, which might bring new conclusions and challenges, see e.g. [9], [7], [15].
  \item (1.4) is memoryless in the sense that $X(t + \Delta t) - X(t)$ only depends on the values of $b$, $\sigma$ in $[t, t + \Delta t]$. Nevertheless, in reality, long-term dependence usually exists, and (1.1) is one proper choice to represent the memory effect, see e.g. [1], [22], [24].
\end{itemize}

Based on these facts, we believe that optimal control problems of (1.1) are much richer than that of (1.4). Besides the theoretical parts, we emphasize that stochastic (or deterministic) VIEs can describe many specific models, such as optimal dynamic advertising model (p.53-p.55 in Hartl [13]), optimal capital policy model (p.23-p.25 of Arrow [2] or p.469, p.472 of Kamien-Muller [18]), Ramsey vintage capital model (p.582 of Arrow [2]) or p.469, p.472 of Kamien-Muller [18]), stochastic subdiffusion phenomenon in biophysics experiments (p.506-p.507 of Kou [15]), stochastic inventory-production model (p.2576-p.2578 of [26]), optimal investment model with memory (p.1088-p.1093 of [1]), capital stock model in economics (Example 3.1 of Pardoux-Protter [20]).

Next we introduce one more stochastic epidemic prevention model. Suppose there is a population suffered from one infected disease during $[0, T]$, where $T$ is the time when there is no infective individual, and $0$ is the time when the infective group are separated and received medical treatment. At time $t \in [0, T]$, we denote by $X(t)$ the population of infected people (including the ones who received vaccines before recovering), $u(t)$ the amount of vaccines provided by local government. Suppose any infected individual is likely to get worse and die at future time. Hence we define random variable $\xi_1$ his/her life length, $m_1(r)dr$ his/her dying probability during $[r, r + dr]$ with density function $m_1(\cdot)$. Consequently,

\begin{equation}
F_1(r) := \mathbb{P}\{\xi_1 \leq r\} = \int_0^r m_1(s)ds, \quad r \in [0, T].
\end{equation}

For $t \in [0, T]$, $s \in [0, t]$, at time $t - s$ the infective population is $X(t - s)$. After time $s$ (i.e. at future time
t) the dying population becomes \(X(t-s)m_1(s)ds\). Then, the total number between \([0,t]\) is

\[
\int_0^t X(t-s)m_1(s)ds = \int_0^t X(s)m_1(t-s)ds.
\]

We shall observe that the infected people in different stage (or time period) respond to the vaccine in distinctive ways. Therefore, it is reasonable to introduce the efficiency index \(a(\cdot)\) depending on time. As a result, at time \(t\) there are \(a(t)u(t)\) amount of people whose scenarios become stable. For this group, we define random variables \(\xi(t)\) the duration of recovering completely, \(m_2(r)dr\) the probability of getting normal during \([r,r + dr]\) with density function \(m_2(\cdot)\). In other words,

\[
F_2(r) := \mathbb{P}\{\xi_2 \leq r\} = \int_0^r m_2(s)ds, \quad r \in [0,T].
\]

At time \(t - s\) with \(s \in [0,t]\), the amount of vaccines is \(u(t - s)\), and the population with stable physical condition is \(a(t - s)u(t - s)\). After time \(s\), there are \(a(t - s)u(t - s)m_2(s)ds\) amount of people healing from the disease. Thus the total number between \([0,t]\) is

\[
\int_0^t a(t-s)u(t-s)m_2(s)ds = \int_0^t a(s)u(s)m_2(t-s)ds.
\]

To sum up, the increment of infected individuals at time \(t\) is

\[
\Delta X(t) = \left[-\int_0^t m_1(t-s)X(s)ds - \int_0^t m_2(t-s)a(s)u(s)ds\right] \Delta t.
\]

In other words, for \(t \in [0,T]\),

\[
X(t) = x_0 - \int_0^t \left[ \int_0^s m_1(s-r)X(r)dr \right] ds - \int_0^t \left[ \int_0^s m_2(s-r)a(r)u(r)dr \right] ds
\]

\[
= x_0 - \int_0^t F_1(s-r)X(r)dr - \int_0^t F_2(s-r)a(r)u(r)dr,
\]

where \(x_0\) is the infected individuals at time 0. At this very moment, we make some points on the efficiency coefficient \(a(\cdot)\). Observe that it can be easily influenced by other random factors, like the individual’s physical quality, the epidemic situation in global group, the improvement of the vaccine, etc. Consequently, we may replace \(a(t)\) with \(a(t) + \dot{W}(t)\), where \(\dot{W}(\cdot)\) represents the white noise. Then for \(t \in [0,T]\)

\[
X(t) = x_0 - \int_0^t \left[ F_1(t-r)X(r) + F_2(t-r)a(r)u(r) \right] dr - \int_0^t F_2(t-r)a(r)u(r)dW(r).
\]

Suppose the local government wants to find suitable \(\tilde{u}(\cdot)\) to minimize

\[
J(u(\cdot)) := \mathbb{E} \int_0^T \left( G_1(X(s)) + G_2(u(s)) \right) ds,
\]

where \(G_1(\cdot)\) is the daily cost of living for the infected people, \(G_2(\cdot)\) represents the research and development cost with respect to vaccines. Hence we come up with an optimal control problem associated with (1.6), (1.0).

We return back to our optimal control problem associated with (1.1), (1.2). When (1.1) reduces to (1.4), \(U \subset \mathbb{R}^m\) is arbitrary and \(\sigma\) depends on \(u(\cdot)\), the maximum principles of optimal controls were firstly solved in Peng \(21\) with spike variation. We emphasize that our investigations here are by no means fairly straightforward generalization, and more fresh thoughts have to be injected due to the encountered challenges. We refer to Subsection 3.1 and Section 4 for more details.
The rest of this paper is organized as follows. In Section 2, some notations, spaces are introduced and the optimal control problem is formulated in detail. Section 3 includes four parts. The first part is aimed to illustrate how the obstacles of our study arise, as well as some intuitive introductions of our developed approach. The second part and third part are devoted to treating the encountered difficulties. In the fourth part, two main results of this paper are established and several special cases are discussed. In Section 4, some concluding remarks are present. Finally, a few key lemmas are given in the Appendix.

2. Preliminaries

I. Some notations

First of all, let us introduce some spaces. For $H := \mathbb{R}, \mathbb{R}^n, \mathbb{R}^{n \times m}$, etc., we denote its norm by $\| \cdot \|$. For $0 \leq s < t \leq T$, $p > 1$, we define

$$L^p_{\mathcal{F}}([s, t]; H) := \left\{ X : \Omega \rightarrow H \mid X \text{ is } \mathcal{F}_t\text{-measurable}, \mathbb{E}\|X\|^p < \infty \right\},$$

$$C_{\mathcal{F}}([s, t]; L^p(H)) := \left\{ X : [s, t] \times \Omega \rightarrow H \mid X(\cdot) \text{ is continuous from } [s, t] \text{ to } L^2(\Omega; H), \text{ and measurable, } \mathbb{F}\text{-adapted, sup}_{r \in [s, t]} \mathbb{E}\|X(r)\|^p < \infty \right\},$$

$$L^p_g(s, t; H) := \left\{ X : [s, t] \times \Omega \rightarrow H \mid X(\cdot) \text{ is } \mathbb{F}\text{-adapted, measurable, } \mathbb{E}\int_s^t \|X(r)\|^p dr < \infty \right\},$$

$$L^2(s, t; L^2_g(s, t; H)) := \left\{ Z : [s, t]^2 \times \Omega \rightarrow H \mid Z(u, \cdot) \text{ is measurable, } Z(u, \cdot) \text{ is } \mathbb{F}\text{-adapted} \right\},$$

$$u \in [s, t], \|Z(\cdot, \cdot)\|^2_{L^2(s, t; L^2_g(s, t; H))} \equiv \mathbb{E}\int_s^t \int_s^t |Z(u, v)|^2 dudv < \infty.$$ 

We also need two more spaces for operator-valued processes. To this end, we introduce Banach space $\mathbb{B} := \mathbb{R}^n \times L^4(0, T; \mathbb{R}^n)$. Obviously $\mathbb{B}$ is separable, and there exists a numerable dense subset $\mathbb{B}_0$ and

$$\mathbb{B}_1 := \left\{ \sum_{i=1}^n k_i \alpha^i; \ k_i \in \mathbb{Q}, \ \alpha^i \in \mathbb{B}_0, \ n \in \mathbb{N} \right\} \subset \mathbb{B},$$

where $\mathbb{Q}, \mathbb{N}$ is respectively the set of rational number, integer number in $\mathbb{R}$. We denote $\mathbb{B}' := \mathbb{R}^n \times L^4(0, T; \mathbb{R}^n)$ the dual space of $\mathbb{B}$, $\mathcal{L}(H'; H'')$ the space of linear bounded operators between Banach spaces $H'$ and $H''$,

$$\mathcal{L}_1 := L^2_{1,\mathcal{F}}(0, T; \mathcal{L}(\mathbb{B}; L^4(0, T; \mathbb{R}^n)))$$

$$= \left\{ A : [0, T] \times \Omega \rightarrow \mathcal{L}(\mathbb{B}; L^4(0, T; \mathbb{R}^n)) \mid \text{ for any } b \in \mathbb{B}, \ a(\cdot) \in L^4(0, T; \mathbb{R}^n), \right\},$$

$$\int_0^T a(s)^T A(\cdot, b)(s) ds \in L^2_g(0, T; \mathbb{R}), \ \sup_{t \in [0, T]} \mathbb{E}\|A(t)\|^2_{\mathcal{L}(\mathbb{B}; L^4(0, T; \mathbb{R}^n))} < \infty,$$

(2.1)

$$\mathcal{L}_2 := L^2_{1,\mathcal{F}}(0, T; \mathcal{L}(\mathbb{B}; \mathbb{R}))$$

$$= \left\{ A : [0, T] \times \Omega \rightarrow \mathcal{L}(\mathbb{B}; \mathbb{R}) \mid \sup_{t \in [0, T]} \mathbb{E}\|A(t)\|^2_{\mathcal{L}(\mathbb{B}; \mathbb{R})} < \infty, \ \text{and for any } a(\cdot) \in \mathbb{R}^n, \ b \in \mathbb{B}, \ a^T A(\cdot, b) \in L^2_g(0, T; \mathbb{R}) \right\}.$$ 

In this paper, $K$ is a generic constant which varies in different context.

II. Problem formulation

For FSVIE (1.1), we introduce the following assumptions.

For $F_{SVIE}$ (1.1), we introduce the following assumptions.
(H1) Suppose \( \varphi(\cdot) \in C_p([0, T]; L^p(\Omega; \mathbb{R}^n)) \), given \( p > 2 \), nonempty \( U \subset \mathbb{R}^m, u(\cdot) \in U^p \), where
\[
U^p := \left\{ u(\cdot) : [0, T] \times \Omega \to U \text{ is measurable and } \mathcal{F}-\text{adapted s.t. } \sup_{t \in [0, T]} \mathbb{E}|u(t)|^p < \infty \right\},
\]
b, \( \sigma : [0, T]^2 \times \mathbb{R}^n \times U \times \Omega \to \mathbb{R}^n \) are measurable such that \( s \mapsto (b(t, s, x, u), \sigma(t, s, x, u)) \) is \( \mathcal{F} \)-adapted, \( b, \sigma \) are linear growth of \( x, u \), twice continuously differentiable of \( x \) with bounded first, second order derivatives,
\[
|f(t, s, x, u) - f(t', s, x, u)| \leq \rho(|t - t'|)[1 + |x| + |u|], \quad t, t', s \in [0, T], \quad f := b, \sigma, b_x, \sigma_x,
\]
with \( \rho : [0, \infty) \to [0, \infty) \) a modulus of continuity (continuous, monotone, increasing function with \( \rho(0) = 0 \)). Moreover, \( (s, x, u) \mapsto b_{xx}(t, s, x, u), \sigma_{xx}(t, s, x, u) \) are continuous uniformly in \( t \in [0, T] \).

The next result is concerned with the well-posedness of (1.1), the proof of which is straightforward adaptation of the counterparts in e.g. [22], [23].

Lemma 2.1. Let (H1) hold. Then there exists \( X(\cdot) \in C_p([0, T]; L^p(\Omega; \mathbb{R}^n)) \) satisfying (1.7) such that for absolute constant \( K \),
\[
\sup_{t \in [0, T]} \mathbb{E}|X(t)|^p \leq K\left[ \sup_{t \in [0, T]} \mathbb{E}|\varphi(t)|^p + \sup_{t \in [0, T]} \mathbb{E}\int_0^t |b(t, s, 0, u(s))|^p ds \right.
\]
\[
\left. + \sup_{t \in [0, T]} \mathbb{E}\int_0^t |\sigma(t, s, 0, u(s))|^p ds \right].
\]

Moreover, for \( i = 1, 2 \), if \( X_i(\cdot) \) is the solution associated with \( (\varphi_i, b_i, \sigma_i) \), then
\[
\sup_{t \in [0, T]} \mathbb{E}|X_1(t) - X_2(t)|^p \leq K\left\{ \sup_{t \in [0, T]} \mathbb{E}|\varphi_1(t) - \varphi_2(t)|^p + \sup_{t \in [0, T]} \mathbb{E}\left[ \int_0^T |b_1(t, s, X_2(s), u(s)) - b_2(t, s, X_2(s), u(s))|^p ds \right] \right.
\]
\[
\left. + \sup_{t \in [0, T]} \mathbb{E}\left[ \int_0^T |\sigma_1(t, s, X_2(s), u(s)) - \sigma_2(t, s, X_2(s), u(s))|^2 ds \right]^{\frac{p}{2}} \right\}.
\]

In the following we define \( p := 4 + \kappa \) with constant \( \kappa > 0 \), and \( U_{ad} := U^{4+\kappa} \).

For the involved functions \( h \) and \( l \) in (1.2), we make the following assumption.

(H2) Let \( h : \mathbb{R}^n \times \Omega \to \mathbb{R}, l : [0, T] \times \mathbb{R}^n \times U \times \Omega \to \mathbb{R} \) be measurable such that \( x \mapsto h(x), (x, u) \mapsto l(s, x, u) \) are twice continuously differentiable with
\[
|h_x(x)| + |l_x(t, x, u)| \leq L[1 + |x| + |u|], \quad \left[ |h_{xx}(x)| + |l_{xx}(t, x, u)| \right] \leq L, \quad x \in \mathbb{R}^n, \quad u \in U.
\]

We state the optimal control problem as follows.

Problem (C). Given (1.1), (1.2), we are aiming to find \( \bar{u}(\cdot) \in U_{ad} \) such that \( J(\bar{u}(\cdot)) = \inf_{u(\cdot) \in U_{ad}} J(u(\cdot)) \).

In above, we call \( \bar{u} \) the optimal control, \( \bar{X} \) the optimal state process, \( (\bar{X}, \bar{u}) \) the optimal pair.

3. Maximum Principle for controlled SVIEs

This section is devoted to obtaining optimality necessary conditions of Problem (C) via spike variation. Without further statement, let \( u \in U, \tau \in [0, T], \epsilon > 0 \) be sufficiently small such that \( \tau + \epsilon \leq T \),
\[
E_{\tau, \epsilon} := [\tau, \tau + \epsilon], \quad u^\epsilon(\cdot) := u_{E_{\tau, \epsilon}}(\cdot) + \bar{u}(\cdot)I_{[0, T)/E_{\tau, \epsilon}}(\cdot).
\]
I. First-order variational equations and related quadratic functional

In this part, we derive one quadratic functional of the solutions for first-order variational equations. How to treat this functional appropriately is the crucial step in establishing the maximum principles. These procedures are not necessary with convex control region (224, 277, 283).

Inspired by the SDEs case in e.g. 21, 29, we introduce the first-order variational equations which includes two linear SVIEs as follows,

\[
\begin{aligned}
X_1(t) &= \int_0^t \dot{b}_x(t, s) X_1(s) ds + \int_0^t [\dot{\sigma}_x(t, s) X_1(s) + \delta \sigma(t, s)] dW(s), \\
X_2(t) &= \varphi_2(t) + \int_0^t \dot{b}_x(t, s) X_2(s) ds + \int_0^t \dot{\sigma}_x(t, s) X_2(s) dW(s).
\end{aligned}
\]

Here \( t \in [0, T] \), and for \( f := b, \sigma, \sigma_x \), we make the following conventions,

\[
\begin{aligned}
\varphi_2(t) &= \int_0^t \left[ \frac{1}{2} \dot{b}_{xx}(t, s) X_1^2(s) + \delta b(t, s) \right] ds + \int_0^t \left[ \frac{1}{2} \dot{\sigma}_{xx}(t, s) X_1^2(s) + \delta \sigma_x(t, s) X_1(s) \right] dW(s), \\
\dot{f}_x(t, s) &= f_x(t, s, \bar{X}(s), \bar{u}(s)), \quad \delta f(t, s) := f(t, s, \bar{X}(s), \bar{u}(s)) - f(t, s, \bar{X}(s), \bar{u}(s)), \\
\dot{f}_{xx}(t, s) X_1^2(s) &= \left\{ \text{tr}\{ f_{xx}'(t, s) X_1(s) X_1(s)^\top \} , \cdots , \text{tr}\{ f_{xx}'(t, s) X_1(s) X_1(s)^\top \} \right\}^\top.
\end{aligned}
\]

We give the following standard estimates, the proof of which is similar to the SDEs case.

**Lemma 3.1.** Suppose (H1) is true, \( X_1, X_2 \) satisfy (3.2), \( (\bar{X}, \bar{u}) \) is an optimal pair, \( X^\varepsilon \) is the state process associated with \( u^\varepsilon \), \( u^\varepsilon \) is defined in (3.7). Then

\[
\sup_{t \in [0, T]} \mathbb{E}|X_1(t)|^2 \leq K \varepsilon, \quad \sup_{t \in [0, T]} \mathbb{E}|X^\varepsilon(t) - \bar{X}(t) - X_1(t) - X_2(t)|^2 \leq o(\varepsilon^2).
\]

From Lemma 3.1 one sees that

\[
\begin{aligned}
o(\varepsilon) &\leq \mathbb{E} \int_0^T l_x(s)^\top [X_1(s) + X_2(s)] ds + \mathbb{E} [h_x(T)^\top X_1(T) + X_2(T)] \\
&+ \mathbb{E} \left[ \int_0^T \delta l(s) ds + \frac{1}{2} \int_0^T X_1(s)^\top \dot{l}_{xx}(s) X_1(s) ds + \frac{1}{2} \left[ X_1(T)^\top \dot{h}_{xx}(T) X_1(T) \right] \right],
\end{aligned}
\]

where for example,

\[
\begin{aligned}
\dot{l}_x(s) &= l_x(s, \bar{X}(s), \bar{u}(s)), \quad \dot{l}_{xx}(s) := l_{xx}(s, \bar{X}(s), \bar{u}(s)), \quad \dot{h}_x(T) := h_x(\bar{X}(T)), \\
\dot{h}_{xx}(T) &= h_{xx}(\bar{X}(T)), \quad \delta l(s) := l(s, \bar{X}(s), u^\varepsilon(s)) - l(s, \bar{X}(s), \bar{u}(s)), \quad s \in [0, T].
\end{aligned}
\]

We introduce first-order adjoint equation of the form

\[
\begin{aligned}
\dot{Y}(t) &= \dot{l}_x(t)^\top + \dot{b}_x(T, t)^\top \dot{h}_x(T) + \dot{\sigma}_x(T, t)^\top \dot{\pi}(t) + \int_t^T \dot{b}_x(s, t)^\top \dot{Y}(s) ds \\
&+ \int_t^T \dot{\sigma}_x(s, t)^\top Z(s, t) ds - \int_t^T \dot{Z}(t, s) dW(s),
\end{aligned}
\]

and the Hamiltonian function

\[
H(t, x, \bar{X}(\cdot), \bar{Y}(\cdot), \bar{Z}(\cdot), u) := \mathbb{E}_t \left[ b(T, t, x, u)^\top \bar{h}_x(T) + \sigma(T, t, x, u)^\top \bar{\pi}(t) \right]
\]

\[
+ l(t, x, u) + \mathbb{E}_t \int_t^T b(s, t, x, u)^\top \bar{Y}(s) ds + \int_t^T \sigma(s, t, x, u)^\top \bar{Z}(s, t) ds.
\]
Observe that \( (5.10) \) is a linear backward stochastic Volterra integral equation (BSVIE) which admits a unique pair of solution \( (\bar{Y}(\cdot), \bar{Z}(\cdot, \cdot)) \in L^2_E(0, T; \mathbb{R}^n) \times L^2(0, T; L^2_E(0, T; \mathbb{R}^{n \times n})) \) such that

\[
\bar{Y}(t) = E_x \bar{Y}(t) + \int_t^T \bar{Z}(t, r) dW(r), \quad s \in [0, t], \quad t \in [0, T], \ a.e.
\]

Above \( (\bar{Y}, \bar{Z}) \) is named as M-solutions of BSVIEs, see e.g. \( [22, 25, 28] \). Thanks to Lemma \( [3.1] \) we have

\[
(3.8) \quad E \int_0^T \left[ \delta \sigma_x(T, s) \tilde{\pi}(s) + \int_s^T \delta \sigma_x(t, s) \bar{Z}(t, s) dt \right] X_1(s) ds = o(\varepsilon).
\]

To sum up, by \( (3.8) \) and Theorem 5.1 in \( [28] \), we can transform \( (3.4) \) into

\[
(3.9) \quad o(1) \leq \frac{1}{\varepsilon} E \int_0^T \Delta \bar{H}^\varepsilon(t) dt + \mathcal{E}(\varepsilon),
\]

where \( \Delta \bar{H}^\varepsilon(\cdot) \), \( \mathcal{E}(\varepsilon) \) are defined as

\[
\Delta \bar{H}^\varepsilon(t) := H(t, \bar{X}(t), \bar{Y}(\cdot), \bar{Z}(\cdot, t), u^\varepsilon(t)) - H(t, \bar{X}(t), \bar{X}(\cdot), \bar{Y}(\cdot), \bar{Z}(\cdot, t), \bar{u}(t)),
\]

\[
\mathcal{E}(\varepsilon) := \frac{1}{2\varepsilon} E \left\{ \int_0^T \text{tr} \left[ \bar{H}_{xx}(t) X_1(t) X_1(t)^T \right] dt + \text{tr} \left[ \bar{h}_{xx}(T) X_1(T) X_1(T)^T \right] \right\},
\]

\[
\bar{H}_{xx}(t) := H_{xx}(t, \bar{X}(t), \bar{X}(\cdot), \bar{Y}(\cdot), \bar{Z}(\cdot, t), \bar{u}(t)), \quad t \in [0, T].
\]

In addition, from \( (\bar{Y}(\cdot), \bar{Z}(\cdot, \cdot)) \) in \( (3.6) \), one has \( \bar{h}_{xx}(\cdot) \in L^2_E(0, T; \mathbb{R}^{n \times n}) \). Therefore, to give the maximum principle, we need to deal with the quadratic form \( \mathcal{E}(\varepsilon) \) in \( (3.4) \).

To see the encountered challenges, we recall the corresponding procedures of treating \( \mathcal{E}(\varepsilon) \) in SDEs case (e.g. \( [21, 29] \)): proving some estimates for the solutions of variational equations, deriving the controlled linear SDE satisfied by \( X_1(\cdot) X_1(\cdot)^T \), introducing suitable adjoint equations, and using duality tricks between the forward, backward systems by the well-known Itô formula. Nevertheless, if we follow above techniques in our setting, we immediately meet some fundamental difficulties which actually indicate the distinctions between the two optimal control problems. The first one lies in the inadequate role of Itô formula in deriving the quadratic form \( \mathcal{E}(\varepsilon) \). One may differentiate \( X_1 \) by imposing differentiability conditions on \( b, \sigma \) (see Section 2 of \( [1] \)) and thus makes Itô formula go through. However, this would cause more complicated double integrals with respect to Lebesgue integral and Itô integral. The second one is concerned with the introducing of proper second-order adjoint equations and suitable duality tricks. According to \( [21] \) (see also Remark \( [3.5] \)), even in special linear quadratic framework, it seems impossible to construct one complete second-order adjoint equation which directly covers that in SDEs case. In other words, in SVIEs case we need to introduce other essential notions. As to the duality, it surely can not be realized without overcoming the obstacles aforementioned.

To provide more fundamental ideas, we revisit the particular SDEs case from new viewpoints. We consider the second-order adjoint equation, i.e., a linear BSDE of

\[
(3.11) \quad \begin{cases}
    dP_2(t) = -[\tilde{b}_x(t)^T P_2(t) + P_2(t) \tilde{b}_x(t) + \tilde{\sigma}_x(t)^T \Lambda_2(t) + \Lambda_2(t) \tilde{\sigma}_x(t)] dt + \Lambda_2(t) dW(t), \quad t \in [0, T], \\
    P_2(T) = \tilde{h}_{xx}(T).
\end{cases}
\]

In some existing literature (\( [9, 17] \)), \( P_2 \) is called the \textit{second-order adjoint process}. The maximum principle of SDEs \( (21, 29) \) includes \( \mathcal{H}_\sigma := \delta \tilde{\sigma}(\tau)^T P_2(\tau) \delta \tilde{\sigma}(\tau) \), where

\[
\delta \tilde{\sigma}(\tau) := \left[ \sigma(\tau, \bar{X}(\tau), u) - \sigma(\tau, \bar{X}(\tau), \bar{u}(\tau)) \right].
\]
Notice that $\mathcal{H}_p$ is limit counterpart of $\mathcal{E}(\varepsilon)$, as $\varepsilon \to 0$.

In the following, we introduce another way to obtain $P_2(\cdot)$ without \ref{eq:3.11}. To this end, for any $\tau \in [0, T]$, $\xi \in L^2_{\mathbb{F}}(\Omega; \mathbb{R}^n)$, $i = 1, 2$, we define

\begin{equation}
\begin{aligned}
Y_i(t) &= \xi_i + \int_{\tau}^{t} \tilde{b}_x(s)Y_i(s)ds + \int_{\tau}^{t} \tilde{\sigma}_x(s)Y_i(s)dW(s), \quad i = 1, 2, \\
J(\tau, \xi_1, \xi_2) &= \mathbb{E}_\tau \int_{\tau}^{T} Y_i(s)^T \mathcal{H}_{xx}(s)Y_i(s)ds + \mathbb{E}_\tau \left[ Y_1(T)^T \tilde{h}_{xx}(T)Y_2(T) \right].
\end{aligned}
\end{equation}

By using Itô formula to $Y_1^T P_2 Y_2$ on $[\tau, T]$ and recalling above $\mathcal{H}_p$, we see that

$$
\xi_1^T P_2(\tau) \xi_2 = J(\tau, \xi_1, \xi_2), \quad \mathcal{H}_p = J(\tau, \delta \tilde{\sigma}(\tau), \delta \tilde{\sigma}(\tau)).
$$

We take a closer look at $J(\tau, \xi_1, \xi_2)$. Notice that the conventional functional analysis theories tell us: given a bounded bilinear operator $F$ on Hilbert space $\mathcal{H} \times \mathcal{H}$, there exists a unique operator $G$ on $\mathcal{H}$ such that $F(x, y) = \langle Gx, y \rangle_{\mathcal{H}}$. Inspired by this point, in the next Lemma \ref{lem:3.2} we prove that there exists a unique measurable, continuous, $\mathbb{F}$-adapted, $\mathbb{R}^n$-valued process $\mathcal{B}_3(\cdot)$ such that $J(\tau, \xi_1, \xi_2) = \xi_1^T \mathcal{B}_3(\tau) \xi_2$. The arbitrariness of $\xi_i$ and the continuity of $P_2, \mathcal{B}_3$ lead to

$$
\mathbb{P}\{ \mathcal{B}_3(\tau) = P_2(\tau), \quad \forall \tau \in [0, T] \} = 1.
$$

We observe that the classical maximum conditions only directly relate to $P_2(\cdot)$, but not $\Lambda_2(\cdot)$. Consequently, above proposed procedures indicate another approach to derive maximum principle without second-order adjoint equation \ref{eq:3.11}. Moreover, one can drop the reliance on Itô formula and the system for $X_1(\cdot)X_1(\cdot)^T$. These points provide us the key clues for following-up investigations on SVIEs.

## II. Representations of some quadratic functionals

Given optimal pair $(\bar{X}(\cdot), \bar{u}(\cdot))$, $u \in U$, we define

\begin{equation}
\begin{aligned}
\delta \tilde{\sigma}(t, \tau) &= \sigma(t, \tau, \bar{X}(\tau), u) - \sigma(t, \tau, \bar{X}(\tau), \bar{u}(\tau)), \\
\Delta \tilde{\sigma}(\cdot, \tau) &= (\delta \tilde{\sigma}(T, \tau), \delta \tilde{\sigma}(\cdot, \tau)), \quad t, \tau \in [0, T].
\end{aligned}
\end{equation}

Under (H1) with $p = 4 + \kappa$, one has $\delta \tilde{\sigma}(\cdot, \tau) \in C_p([0, T]; L^{4+\kappa}(\Omega; \mathbb{R}^n))$.

Given $\tilde{b}_x(\cdot), \tilde{\sigma}_x(\cdot), \tilde{h}_{xx}(T), \tilde{h}_{xx}(\cdot)$ in \ref{eq:3.3}, \ref{eq:3.5}, \ref{eq:3.10}, similar as \ref{eq:3.12}, we introduce

\begin{equation}
\begin{aligned}
F^{\delta \tilde{\sigma}, \delta \bar{\sigma}}(\tau) &= \mathbb{E}_\tau \int_{\tau}^{T} \mathcal{X}(s)^T \tilde{h}_{xx}(s)\mathcal{X}(s)ds + \mathbb{E}_\tau \left[ \mathcal{X}(T)^T \tilde{h}_{xx}(T)\mathcal{X}(T) \right], \\
\mathcal{X}(t) &= \delta \tilde{\sigma}(t, \tau) + \int_{\tau}^{t} \tilde{b}_x(s)\mathcal{X}(s)ds + \int_{\tau}^{t} \tilde{\sigma}_x(s, t)\mathcal{X}(s)dW(s), \quad \forall t \in [\tau, T].
\end{aligned}
\end{equation}

Thanks to (H1) and Lemma \ref{lem:3.1}, there exists a unique $\mathcal{X}(\cdot) \in C_p([\tau, T]; L^{4+\kappa}(\Omega; \mathbb{R}^n))$ satisfying \ref{eq:3.14}. To represent $F^{\delta \tilde{\sigma}, \delta \bar{\sigma}}(\tau)$, a quadratic functional with respect to $\mathcal{X}(\cdot)$ for any fixed $\tau$, we state the following result.

**Lemma 3.2.** Suppose (H1), (H2) hold true with $p = 4 + \kappa$, $(\bar{X}(\cdot), \bar{u}(\cdot))$ is optimal. Then

\begin{equation}
F^{\delta \tilde{\sigma}, \delta \bar{\sigma}}(\tau) = \delta \tilde{\sigma}(T, \tau)^T (\mathcal{B}_1(\tau) \Delta \bar{\sigma}(\cdot, \tau)) + \int_{0}^{T} \delta \tilde{\sigma}(s, \tau)^T (\mathcal{B}_2(\tau) \Delta \bar{\sigma}(\cdot, \tau))(s)ds,
\end{equation}
where $\mathcal{B}_1 \in \mathcal{L}_2$ and $\mathcal{B}_2 \in \mathcal{L}_1$ satisfy (see (3.17) for the definitions of $\mathcal{L}_i$)

\begin{equation}
(3.16) \quad \left[ \|B_1(\tau)\|_{\mathcal{L}(\mathbb{R}^n)} + \|B_2(\tau)\|_{\mathcal{L}(\mathbb{L}^4(0,T;\mathbb{R}^n))} \right]
\leq K \left[ \mathbb{E}_\tau \int_\tau^T |\dot{h}_{xx}(s)|^2 ds \right]^{\frac{1}{2}} + K \left[ \mathbb{E}_\tau |\dot{h}_{xx}(T)|^2 \right]^{\frac{1}{2}}, \text{ a.s. } \tau \in [0,T].
\end{equation}

If there exists $(\mathcal{B}_1, \mathcal{B}_2)$ satisfying (3.16), (3.10), then

\begin{equation}
(3.17) \quad \mathbb{P}(\omega \in \Omega; \mathcal{B}_1(\tau,\omega) = \mathcal{B}_1(\tau,\omega)) = \mathbb{P}(\omega \in \Omega; \mathcal{B}_2(\tau,\omega) = \mathcal{B}_2(\tau,\omega)) = 1.
\end{equation}

**Remark 3.1.** In above, we introduce $\mathcal{B}_1, \mathcal{B}_2$ to treat $\delta \bar{\sigma}(\cdot,\tau)$. These two processes are indispensable, and independent with each other (Subsection 3.4.1). If $\delta \bar{\sigma}(\cdot,\tau) \equiv \delta \bar{\sigma}(\tau)$, they will be unified into one $\mathbb{R}^{n \times n}$-valued process (Lemma 3.2).

**Remark 3.2.** For almost $\omega \in \Omega, \tau \in [0,T], \delta \bar{\sigma}(\cdot,\tau,\omega) \in C([0,T];\mathbb{R}^n)$. However, we extend $C([0,T];\mathbb{R}^n)$ into $L^4(0,T;\mathbb{R}^n)$ since the dual space of the later is easier to treat. This illustrates $L^4(0,T;\mathbb{R}^n)$ in $\mathcal{B}_2$.

To prove Lemma 3.2, for $\alpha := (\alpha_1, \alpha_2(\cdot)) \in \mathbb{B}, \tau \in [0,T], t \in [\tau,T]$, consider

\begin{equation}
(3.18) \quad \begin{cases}
X^\alpha(t) = \alpha_2(t) + \int_\tau^t A(t,s)X^\alpha(s)ds + \int_\tau^t B(t,s)X^\alpha(s)dW(s), & \text{a.e.} \\
X^\alpha(T) = \alpha_1 + \int_\tau^T A(T,s)X^\alpha(s)ds + \int_\tau^T B(T,s)X^\alpha(s)dW(s).
\end{cases}
\end{equation}

Moreover, for $\bar{\alpha}, \tilde{\alpha} \in \mathbb{B}, \tau \in [0,T]$, we define

\begin{equation}
(3.19) \quad f^\bar{\alpha},\tilde{\alpha}_1^\tau(x) := \mathbb{E}_\tau \int_\tau^T X^{\tilde{\alpha}}(s)Q(s)X^{\bar{\alpha}}(s)ds + \mathbb{E}_\tau \left[X^{\tilde{\alpha}}(T)^\top GA^{\bar{\alpha}}(T) \right], \text{ a.s.}
\end{equation}

(H3) $Q \in L^q_F(0,T;\mathbb{R}^{n \times n}), G \in L^\infty_T(\Omega;\mathbb{R}^{n \times n}), A, B : [0,T]^2 \times \Omega \mapsto \mathbb{R}^{n \times n}$ are bounded measurable processes such that for $t \in [0,T], s \mapsto A(t,s), B(t,s)$ are $\mathbb{F}$-adapted, and with modulus function $\rho(\cdot)$,

\[ |A(t,s)| + |B(t,s)| \leq K, \quad |A(t,s) - A(t',s)| + |B(t,s) - B(t',s)| \leq \rho(|t - t'|), \quad t, t' \in [0,T]. \]

Under (H3), (3.15) is solvable with

\[ X^\alpha(\cdot) \in L^4_F(\tau,T;\mathbb{R}^n), \quad X^\alpha(T) \in L^1_{\mathcal{F}_T}(\Omega;\mathbb{R}^n). \]

It is meaningless to discuss $X^\alpha(T)$ if $\alpha_2(\cdot) \in L^1(0,T;\mathbb{R}^n)$. Hence we introduce $X^\alpha(T)$ and $\alpha_1 \in \mathbb{R}^n$ in (3.15).

The appearance of both $\alpha_2(\cdot)$ and $\alpha_1$ explains the introducing of $\mathbb{B}$.

To simplify the notations, we define

\begin{equation}
(3.20) \quad M^{Q,G}(\tau) := \left[ \mathbb{E}_\tau \int_\tau^T |Q(s)|^2 ds \right]^{\frac{1}{2}} + \left[ \mathbb{E}_\tau |G|^2 \right]^{\frac{1}{2}}, \text{ a.s. } \forall \tau \in [0,T].
\end{equation}

**Lemma 3.3.** Suppose (H3) holds. Then for $\alpha, \beta \in \mathbb{B}, \tau \in [0,T]$, one has

\begin{equation}
(3.21) \quad f_1^{\alpha,\beta}(\tau) = \alpha_1^\top (B_1(\tau)\beta) + \int_0^T \alpha_2(t)^\top (B_2(\tau)\beta)(t)dt, \text{ a.s.}
\end{equation}
where $B_1 \in \mathcal{L}_2$, $B_2 \in \mathcal{L}_1$, and for $(\tau, \omega) \in [0, T] \times \Omega$

\[(3.22) \quad \left\| B_1(\tau, \omega) \right\|_{L(B; \mathbb{R}^n)} + \left\| B_2(\tau, \omega) \right\|_{L(B; L^4_0(0, T; \mathbb{R}^n))} \leq K M^{Q, G}(\tau, \omega) \]

If there is another pair $(B'_1, B'_2)$, then for any $\tau \in [0, T]$,

\[(3.23) \quad \mathbb{P}(\omega \in \Omega; \quad B'_1(\tau, \omega) = B_1(\tau, \omega)) = \mathbb{P}(\omega \in \Omega; \quad B'_2(\tau, \omega) = B_2(\tau, \omega)) = 1.

**Proof.** For reader’s convenience, we separate the proof into several steps.

**Step 1:** We obtain a pair of operator-valued processes on a subset of $[0, T] \times \Omega$ with full measure.

For any $t \in [\tau, T]$, $\alpha \in \mathbb{B}$, from (5.15) and Gronwall inequality, we see at once that

\[(3.24) \quad \mathbb{E}_t \left[ \int_{\tau}^{T} |X^\alpha(s)|^4 ds + |X^\alpha(T)|^4 \right] \leq K \left[ \int_{\tau}^{T} |\alpha_2(s)|^4 ds + |\alpha_1^4| \right] = K \|\alpha\|_{\mathbb{B}}.

Consequently, given $\bar{\alpha}, \bar{\alpha} \in \mathbb{B}$, $M^{Q, G}(\cdot)$ in (3.21), it follows that

\[(3.25) \quad |f_1^{\bar{\alpha}, \bar{\alpha}}(\tau)| \leq K M^{Q, G}(\tau)\|\bar{\alpha}\|_{\mathbb{B}} \|\bar{\alpha}\|_{\mathbb{B}} \quad \text{a.s.}

For $\alpha^i, \bar{\alpha}^i, \alpha, \bar{\alpha} \in \mathbb{B}_1, k, l \in \mathbb{Q}$, we define $N := N(\alpha^i, \bar{\alpha}^i, \alpha, \bar{\alpha}, k, l, Q, G)$ and $\mathcal{N}$ as,

\[(3.26) \quad \mathcal{N} := \left\{ (\tau, \omega) \in [0, T] \times \Omega; \left| f_1^{\bar{\alpha}, \bar{\alpha}}(\tau, \omega) \right| \leq K M^{Q, G}(\tau)\|\bar{\alpha}\|_{\mathbb{B}} \|\bar{\alpha}\|_{\mathbb{B}}, \right. f_1^{k\bar{\alpha}^1 + l\bar{\alpha}^1}(\tau, \omega) = k f_1^{\bar{\alpha}^1}(\tau, \omega) + l f_1^{\bar{\alpha}^1}(\tau, \omega), \left. f_1^{k\bar{\alpha}^1 + l\bar{\alpha}^1}(\tau, \omega) = k f_1^{\bar{\alpha}^1}(\tau, \omega) + l f_1^{\bar{\alpha}^1}(\tau, \omega) \right\} \bigcap \bigcap \bigcap \bigcap N(\alpha^i, \bar{\alpha}^i, \alpha, \bar{\alpha}, k, l, Q, G).

Notice that $[\lambda \times \mathbb{P}](N) = T$, $[\lambda \times \mathbb{P}](\mathcal{N}) = T$, where $\lambda$ is the Lebesgue measure. In addition, by inequality (3.25), for any $\tau \in [0, T]$, one has $\mathbb{P}(N_\tau) = 1$ with $N_\tau := \{\omega \in \Omega; (\tau, \omega) \in \mathcal{N}\}$.

**Step 2:** We deduce a pair of operator-valued processes on $[0, T] \times \Omega$.

Given $\alpha, \beta \in \mathbb{B}$, there exist $\left\{ \alpha_n \right\}_{n=1}^{\infty}$, $\left\{ \beta_n \right\}_{n=1}^{\infty} \subset \mathbb{B}_1$ such that

\[(3.27) \quad f_1^{\alpha, \beta}(\tau, \omega) = \alpha^\top \left( B_{1,1}(\tau, \omega) \beta \right) + \int_{0}^{T} \alpha_2(t)^\top \left( B_{1,2}(\tau, \omega) \beta \right)(t) dt.

For any $\tau \in [0, T]$, recalling $\mathbb{P}(N_\tau) = 1$, we know that (3.21) holds almost surely.

**Step 2:** We deduce a pair of operator-valued processes on $[0, T] \times \Omega$.

Given $\alpha, \beta \in \mathbb{B}$, there exist $\left\{ \alpha_n \right\}_{n=1}^{\infty}$, $\left\{ \beta_n \right\}_{n=1}^{\infty} \subset \mathbb{B}_1$ such that

\[\left[ \|\alpha_n - \alpha\|_{\mathbb{B}}^4 + \|\beta_n - \beta\|_{\mathbb{B}}^4 \right] \to 0, \quad n \to \infty.

For any $(\tau, \omega) \in \mathcal{N}$, we denote by $\mathcal{B}_n(\tau, \omega), \mathcal{B}(\tau, \omega)$

\[\mathcal{B}_n(\tau, \omega) := \alpha_n^\top \left( B_{1,1}(\tau, \omega) \beta_n \right) + \int_{0}^{T} \alpha_n(t)^\top \left( B_{1,2}(\tau, \omega) \beta_n \right)(t) dt,
\]
\[\mathcal{B}(\tau, \omega) := \alpha^\top \left( B_{1,1}(\tau, \omega) \beta \right) + \int_{0}^{T} \alpha(t)^\top \left( B_{1,2}(\tau, \omega) \beta \right)(t) dt.
\]
One has \( \lim_{n \to \infty} \left| \mathcal{B}(\tau, \omega) - \mathcal{B}_n(\tau, \omega) \right| = 0 \). On the other hand, by (3.27), \( \mathcal{B}_n(\cdot) \) is measurable, adapted process. Hence similar conclusion holds for \( \mathcal{B}(\cdot) \).

At this moment, we define two processes \( \mathcal{B}_i(\tau, \omega) \) on \([0, T] \times \Omega\) as

\[
(3.28) \quad \mathcal{B}_1(\tau, \omega) := B_{1,1}(\tau, \omega)I_N(\tau, \omega), \quad \mathcal{B}_2(\tau, \omega) := B_{1,2}(\tau, \omega)I_N(\tau, \omega).
\]

For any \( \alpha \in \mathbb{R}^n \), by choosing \( \alpha := (\alpha_1, 0) \in \mathcal{B}(\cdot) \), it follows from (3.28) that

\[
\alpha_1^T [\mathcal{B}_1(\tau, \omega)\beta] = \mathcal{B}(\tau, \omega)I_N(\tau, \omega) = \left[ \lim_{n \to \infty} \mathcal{B}_n(\tau, \omega) \right]I_N(\tau, \omega).
\]

It is then evident to deduce the measurability of \((\tau, \omega) \mapsto \alpha_1^T (\mathcal{B}_1(\tau, \omega)\beta)\), as well as the adaptness.

Similarly one can obtain the case of \( \mathcal{B}_2(\cdot, \cdot) \).

**Step 3:** For any \( \tau \in [0, T] \), \( \alpha, \beta \in \mathbb{B} \), we prove (3.21).

From Step 1, for any \( \tau \in [0, T] \) and \( \omega \in \mathcal{N}_\tau \), (3.21) holds true with any \( \alpha_n, \beta_n \in \mathbb{B}_1 \). Similar as (3.24), for any \( \tau \in [0, T] \), the following is true almost surely,

\[
\mathbb{E}_\tau \int_\tau^T |X^{f_n}(t) - X^f(t)|^4 dt + \mathbb{E}_\tau |X^{f_n}(T) - X^f(T)|^4 \leq K \|f_n - f\|_2, \quad f := \alpha, \beta.
\]

As a result, \( \lim_{n \to \infty} |f_1^{\alpha, \beta}(\tau) - f_1^{\alpha_n, \beta_n}(\tau)| = 0 \). Consequently,

\[
f_1^{\alpha, \beta}(\tau) = \mathcal{B}(\tau) := \alpha_1^T (\mathcal{B}_{1,1}(\tau)\beta) + \int_0^T \alpha_2(t)^T (\mathcal{B}_{1,2}(\tau)\beta)(t) dt. \quad \text{a.s.}
\]

Our conclusion then follows from the relation between \( \mathcal{B}_{1,i}(\cdot, \cdot) \) and \( \mathcal{B}_i(\cdot, \cdot) \) in (3.28).

**Step 4:** In this part, we discuss the integrability and uniqueness of \( \mathcal{B}_i \).

For \((t, \omega) \in \mathcal{N}\), by Lemma 5.3 we have

\[
\left[ \|\mathcal{B}_{1,1}(\tau, \omega)\|_{L_2(\mathbb{R}^n)} + \|\mathcal{B}_{1,2}(\tau, \omega)\|_{L_2(\mathbb{B}; L^2(0, T; \mathbb{R}^n))} \right] \leq KM^{Q,G}(\tau, \omega).
\]

Hence (3.22) is lead by (3.28). On the other hand, from (3.25), \( \sup_{\tau \in [0, T]} \mathbb{E}[|f_1^{\alpha, \beta}(\tau)|^2] < \infty \). As a result, for \( \alpha := (\alpha_1, 0) \in \mathbb{B}, \alpha_1 \in \mathbb{R}, \beta \in \mathbb{B} \), it follows from (3.21) that \( \sup_{\tau \in [0, T]} \mathbb{E}[|\alpha_1^T (\mathcal{B}_1(\tau)\beta)|^2] < \infty \).

Similarly, we derive the case of \( \mathcal{B}_2(\cdot) \).

Eventually, we emphasize that the uniqueness of \( \mathcal{B}_i(\cdot) \) is obvious to obtain.

The next lemma yields information about an extension of (3.21). To this end, for \( \tau \in [0, T] \) we denote \( L^4_{\mathcal{F}_\tau}(\Omega; \mathbb{B}) \) the set of \( \mathcal{F}_\tau \)-strongly measurable \( \mathbb{B} \)-valued random variable \( \xi \) such that \( \mathbb{E}[|\xi|^4] < \infty \). Recall that a \( \mathbb{B} \)-valued random variable \( \xi \) is named \( \mathcal{F}_\tau \)-strongly measurable if there exists a sequence of \( \mathbb{B} \)-valued simple random variables \( \xi_k \) converging to \( \xi \).

**Lemma 3.4.** For any \( \tau \in [0, T] \), \( \xi := (\xi_1, \xi_2(\cdot)), \eta := (\eta_1, \eta_2(\cdot)) \in L^4_{\mathcal{F}_\tau}(\Omega; \mathbb{B}) \),

\[
f_1^{\xi, \eta}(\tau) = \xi_1^T [\mathcal{B}_1(\tau)\eta] + \int_0^T \xi_2(s)^T [\mathcal{B}_2(\tau)\eta](s) ds. \quad \text{a.s.}
\]
Proof. We begin with the simple random variable case. By defining
\[\xi(\omega) := \sum_{i=1}^{n} x_i I_{A_i}(\omega), \quad \eta(\omega) := \sum_{j=1}^{m} y_j I_{B_j}(\omega), \quad \omega \in \Omega, \quad A_i, B_j \in \mathcal{F}_\tau, \quad x_i, y_j \in \mathbb{R}\]
it is easy to see that
\[\sum_{i=1}^{n} \sum_{j=1}^{m} [x_i - i - 1] [B_1(\tau) y_j] + \int_0^T x_i, 2(s) T [B_2(\tau) y_j] (s) ds \cdot I_{A_i} I_{B_j}
\]
\[= \xi^T [B_1(\tau) \eta] + \int_0^T \xi_2(s) T [B_2(\tau) \eta] (s) ds.\]

From Lemma 3.3 we conclude that, for any \( x_i := (x_{i,1}, x_{i,2}(\cdot)) \in \mathbb{R}, y_j \in \mathbb{R}, \)
\[f^x, y_j (\tau) = \int_0^T x_i, 2(t) T (B_2(\tau) y_j) (t) dt. a.s.\]

We thus obtain the desirable conclusion by
\[f_1^{\xi, \eta} (\tau) = \sum_{i=1}^{n} \sum_{j=1}^{m} \left( \mathbb{E}_{\tau} \int_0^T X^{x_i}(s) T Q(s) X^{x_i}(s) ds + \mathbb{E}_{\tau} [X^{x_i}(T) T G X^{x_i}(T)] \right) \cdot I_{A_i} I_{B_j}\]

The task now is to treat the general case. For any \( \xi := (\xi_1, \xi_2(\cdot)), \quad \eta := (\eta_1, \eta_2(\cdot)) \in L^4_{\mathcal{F}_\tau} (\Omega; \mathbb{R}), \) there exist \( \{\xi_n\}_{n \geq 1}, \quad \{\eta_n\}_{n \geq 1} \) such that
\[\| k - k_n \|^4_{L^4_{\mathcal{F}_\tau} (\Omega; \mathbb{R})} \to 0, \quad n \to \infty, \quad k := \xi, \eta.\]

Therefore, similar as (3.24), when \( n \to \infty, \) we have
\[\mathbb{E} \int_0^T |X^{k_n}(t) - X^k(t)|^4 dt + \mathbb{E}|X^{k_n}(T) - X^k(T)|^4 \leq K \| k - k_n \|^4_{L^4_{\mathcal{F}_\tau} (\Omega; \mathbb{R})} \to 0,\]
with \( k := \xi, \eta. \) This implies that for any \( \tau \in [0, T], \quad \lim_{n \to \infty} \mathbb{E}|f_1^{\xi, \eta}(\tau) - f_1^{\xi_n, \eta_n}(\tau)| = 0.\]

On the other hand, by the estimates in Lemma 3.3
\[\lim_{n \to \infty} \mathbb{E} \left| \xi_1^T (B_1(\tau) \eta) - \xi_{n,1}^T (B_1(\tau) \eta_n) + \int_0^T [\xi_2(t) T (B_2(\tau) \eta)(t) - \xi_{n,2}(t) T (B_2(\tau) \eta_n)(t)] \right| dt = 0.\]

Since \( \{\xi_n, \eta_n\} \) are all simple random variables, for any \( n \geq 1 \) we have
\[\mathbb{E} \left| f_1^{\xi, \eta}(\tau) - \xi_{n,1}^T (B_1(\tau) \eta_n) - \int_0^T \xi_{n,2}(s) T [B_2(\tau) \eta_n] (s) ds \right| = 0.\]

To sum up, the conclusion is followed by
\[\mathbb{E} \left| f_1^{\xi, \eta}(\tau) - \xi_1^T (B_1(\tau) \eta) - \int_0^T \xi_2(t) T (B_2(\tau) \eta)(t) dt \right| = 0.\]

Now we give the proof of Lemma 3.2
Proof. Given $\delta \bar{\sigma}(-, \tau), \Delta \bar{\sigma}(-, \tau)$ in (3.13), $\mathcal{X}(-)$ in (3.14), like (3.15) we introduce

$$
\mathcal{X}^\omega(T) := \delta \bar{\sigma}(T, \tau) + \int_\tau^T \bar{b}_x(T, s) \mathcal{X}(s)ds + \int_\tau^T \bar{\sigma}_x(T, s) \mathcal{X}(s)dW(s).
$$

We also introduce $f_1^{\Delta \bar{\sigma}, \Delta \bar{\sigma}}(\tau)$

$$
f_1^{\Delta \bar{\sigma}, \Delta \bar{\sigma}}(\tau) := \mathbb{E}_\tau \int_\tau^T \mathcal{X}(s)^T \bar{H}_{xx}(s) \mathcal{X}(s)ds + \mathbb{E}_\tau [\mathcal{X}^\theta(T)^T \bar{H}_{xx}(T) \mathcal{X}^\theta(T)].
$$

By virtue of Lemma 3.3 and Lemma 3.4 there exist $\mathcal{B}_1(-, \cdot), \mathcal{B}_2(-, \cdot)$ satisfying (3.16) and

$$
f_1^{\delta \bar{\sigma}, \delta \bar{\sigma}}(\tau) = \delta \bar{\sigma}(T, \tau)^T (\mathcal{B}_1(\tau) \Delta \bar{\sigma}(\cdot, \tau)) + \int_0^T \delta \bar{\sigma}(s, \tau)^T (\mathcal{B}_2(\tau) \Delta \bar{\sigma}(\cdot, \tau))(s)ds.
$$

The integrability of $\mathcal{X}(-)$ and $\delta \bar{\sigma}(-, \tau)$ yield $\mathbb{E}_\tau |\mathcal{X}^\theta(T) - \mathcal{X}(T)|^4 = 0$, a.s., $\tau \in [0, T]$.

As a result, the conclusion (3.15) follows immediately.

In the rest of this subsection, we discuss the case when $\delta \bar{\sigma}(-, \tau) \equiv \delta \bar{\sigma}(\tau)$.

To this end, for any $a_1, a_2 \in \mathbb{R}^n$, $\tau \in [0, T]$, similar as (3.19) we define $f_{2}^{a_1, a_2}(\tau)$

$$(3.30)\quad f_{2}^{a_1, a_2}(\tau) := \mathbb{E}_\tau \int_\tau^T X^{a_1}(s)^T Q(s)X^{a_2}(s)ds + \mathbb{E}_\tau [X^{a_1}(T)^T GX^{a_2}(T)], \text{ a.s.}$$

associated with $Q(-), G$, where

$$(3.31)\quad X^{a_1}(t) = a_1 + \int_\tau^t A(t, s)X^{a_1}(s)ds + \int_\tau^t B(t, s)X^{a_1}(s)dW(s), \quad \forall t \in [\tau, T].$$

In particular, we have $f_{2}^{e_i, e_i}(\cdot)$ where $e_i \in \mathbb{R}^n$.

**Lemma 3.5.** Suppose (H3) holds true. Then for any $\tau \in [0, T], \xi_i \in L_{F, \tau}^4(\Omega; \mathbb{R}^n)$,

$$(3.32)\quad f_{2}^{\xi_1, \xi_2}(\tau) = \xi_1^T \mathcal{B}_3(\tau)\xi_2, \quad |\mathcal{B}_3(\tau)| \leq K \left\{ \left[ \mathbb{E}_\tau \int_\tau^T |Q(s)|^2 ds \right]^\frac{1}{2} + \left[ \mathbb{E}_\tau |G|^2 \right]^\frac{1}{2} \right\},$$

where $\mathcal{B}_3(\cdot) := \left\{ f_{2}^{e_i, e_i}(\cdot) \right\}_{1 \leq i, j \leq n}$. If there is another continuous process $\mathcal{B}_4(\cdot)$ satisfying (3.20), then

$$(3.33)\quad \mathbb{P}(\omega \in \Omega; \mathcal{B}_4(\tau, \omega) = \mathcal{B}_3(\tau, \omega), \quad \forall \tau \in [0, T]) = 1.$$

**Proof.** The ideas of this proof are essentially the same as Lemma 3.2 and Lemma 3.4. For reader’s convenience, we give a sketch as follows.

Given $\bar{\alpha}, \bar{\alpha} \in \mathbb{Q}^n$ the set of $n$-dimensional vectors with each component being rational number, and $M_{Q, G}^G(\cdot)$ in (3.20), we can deduce that

$$(3.34)\quad |f_{2}^{\bar{\alpha}, \bar{\alpha}}(\tau)| \leq KM_{Q, G}^G(\tau) \cdot \|ar{\alpha}\|_{\mathbb{Q}^n} \cdot \|ar{\alpha}\|_{\mathbb{Q}^n}, \quad \text{a.s.}$$

Moreover, for any $\bar{\alpha}^i, \bar{\alpha}^i, \bar{\alpha}, \bar{\alpha} \in \mathbb{Q}^n$ with $i = 1, 2$, and $k, l \in \mathbb{Q}$, we define

$$(3.35)\quad \tilde{N} := \left\{ (\tau, \omega) \in [0, T] \times \Omega; |f_{2}^{\bar{\alpha}, \bar{\alpha}}(\tau, \omega)| \leq KM_{Q, G}^G(\tau, \omega)\|ar{\alpha}\|_{\mathbb{Q}^n} \cdot \|ar{\alpha}\|_{\mathbb{Q}^n},
$$

$$
\begin{align*}
&f_{2}^{\bar{\alpha}^i + i\bar{\alpha}, \bar{\alpha}^i + i\bar{\alpha}}(\tau, \omega) = k f_{2}^{\bar{\alpha}^i, \bar{\alpha}^i}(\tau, \omega) + l f_{2}^{\bar{\alpha}, \bar{\alpha}}(\tau, \omega), \\
&f_{2}^{k\bar{\alpha}^i + i\bar{\alpha}, \bar{\alpha}^i + i\bar{\alpha}}(\tau, \omega) = k f_{2}^{\bar{\alpha}^i, \bar{\alpha}^i}(\tau, \omega) + l f_{2}^{\bar{\alpha}, \bar{\alpha}}(\tau, \omega),
\end{align*}
$$

$$
\tilde{N} := \bigcap_{\bar{\alpha}, \bar{\alpha} \in \mathbb{Q}^n} \bigcap_{\bar{\alpha}^i, \bar{\alpha}^i \in \mathbb{Q}^n} \bigcap_{k, l \in \mathbb{Q}} \tilde{N}(\bar{\alpha}^i, \bar{\alpha}^i; \bar{\alpha}, \bar{\alpha}, k, l, Q, G).
$$
It is easy to see that \([\lambda \times \mathbb{P}] (\mathcal{N}) = T\) such that for any \((\tau, \omega) \in \mathcal{N}\), \(f_2^\alpha(\tau, \omega) : \mathbb{Q}^n \times \mathbb{Q}^n \to \mathbb{R}\) is a bounded bilinear map. Moreover, for any \(\tau \in [0, T]\), \(\mathbb{P}(\mathcal{N}_\tau) = 1\), where \(\mathcal{N}_\tau := \{\omega \in \Omega, (\tau, \omega) \in \mathcal{N}\}\).

Using Lemma 3.6, there exists a unique \(\mathbb{R}^{n \times n}\)-valued matrix \(B_{1,3}(\tau, \omega)\), \((\tau, \omega) \in \mathcal{N}\), such that the following holds true with \(\alpha, \beta \in \mathbb{Q}^n\),

\[
\begin{align*}
    f_2^{\alpha, \beta}(\tau, \omega) &= \alpha^T B_{1,3}(\tau, \omega) \beta, \\
    |B_{1,3}(\tau, \omega)| &\leq K M^{Q,G}(\tau, \omega).
\end{align*}
\]

Let \(\alpha := e_i, \beta := e_j, i, j = 1, \ldots, n\), we obtain that \(B_{1,3}^{i,j}(\tau, \omega) = f_2^{e_i, e_j}(\tau, \omega)\) which is component in the \(i\)-th line, \(j\)-th column. In other words, given (3.32), \(B_{1,3} = B_3\) in \(\mathcal{N}\). Moreover,

\[\mathbb{P}(\omega \in \Omega; B_{1,3}(\tau, \omega) = B_3(\tau, \omega)) = \mathbb{P}(\mathcal{N}_\tau) = 1, \ \tau \in [0, T].\]

Considering (3.36), we have \(\sup_{\tau \in [0, T]} \mathbb{E} |B_3(\tau)|^2 < \infty\), and the second result in (3.32). The measurability, adaptness and continuity of \(B_3\) are easy to see.

Following the same ideas as in Step 3 of Lemma 3.6 and Lemma 3.7 for any \(\xi \in L_2^{\mathcal{N}}(\Omega; \mathbb{R}^n)\), one has \(f_2^{\xi_i, \xi_j}(\tau) = \xi_i^T B_3(\tau) \xi_j\), a.s.

Eventually, (3.38) follows from the arbitrariness of \(\xi \in L_2^{\mathcal{N}}(\Omega; \mathbb{R}^n)\) and continuity of \(B_3(\cdot)\).

\[\square\]

**Remark 3.3.** Unlike Lemma 3.6, several more clearer pictures are given here. The first one is about the better regularity of \(X^\alpha(\cdot)\), which saves us from introducing new terms such as \(X^\alpha(T)\) of (3.18). The second one is about \(\mathbb{R}^n\)-valued process \(B_3(\cdot)\), which plays the same role as operator-valued processes \(B_1(\cdot), B_2(\cdot)\). Moreover, \(B_3(\cdot)\) has more stronger properties such as measurability, continuity, uniqueness.

Using Lemma 3.5 we give the following result that is comparable with Lemma 3.2.

**Lemma 3.6.** Suppose (H1), (H2) hold true with \(p = 4 + \kappa, \delta(\cdot, \tau) \equiv \delta(\tau), (\bar{X}(\cdot), \bar{u}(\cdot))\) is optimal. Then there exist a unique (in the sense of (3.38)) measurable, adapted, continuous \(\mathbb{R}^{n \times n}\)-valued process \(B_3(\cdot) := \{F^{e_i, e_j}(\cdot)\}_{1 \leq i, j \leq n}\) such that,

\[
F^{\delta, \delta}(\tau) = \delta(\tau)^T B_3(\tau) \delta(\tau), \ \text{a.s.} \ \tau \in [0, T].
\]

**III. Some subtle asymptotic analyses**

In order to obtain maximum principle, we are in a position to explore some essential relations between \(S(\varepsilon)\) in (3.10) and \(F^{\delta, \delta}(\cdot)\) in (3.13). To get more intuitive feelings, we look at the case of \(n = 1\), \(\tilde{\sigma}(\cdot, \cdot) = \bar{\sigma}(\cdot, \cdot) = 0\). In this case, given deterministic \(Q(\cdot)\), using Fubini theorem and

\[X_1(\cdot) = \int_\tau^{\tau + \varepsilon} \tilde{\sigma}(\cdot, s) I_{[\tau, \tau + \varepsilon]}(s) dW(s),\]

we see that

\[
\frac{1}{\varepsilon} \mathbb{E} \int_\tau^{\tau + \varepsilon} Q(t) |X_1(t)|^2 dt = \frac{1}{\varepsilon} \mathbb{E} \int_\tau^{\tau + \varepsilon} \int_\tau^{\tau + \varepsilon} Q(t) |\delta(\tau, s)|^2 dtds.
\]

For \(\tau \in [0, T]\), a.e., by Lebesgue differentiation theorem,

\[
\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \mathbb{E} \int_\tau^{\tau + \varepsilon} Q(t) |X_1(t)|^2 dt = \mathbb{E} \int_\tau^{T} Q(t) |\delta(\tau, t)|^2 dt = \mathbb{E} \int_\tau^{T} Q(t) |\tilde{\sigma}(t, t)|^2 dt.
\]

Above (3.38) indicates certain asymptotic connection between \(X_1(\cdot)\) and \(X(\cdot)\). We use this basic idea in our framework and present the following result.
Lemma 3.7. Suppose (H1-H2) hold true, \((\bar{X}(\cdot), \bar{u}(\cdot))\) is optimal pair. Then there exists \(\{\varepsilon_n\}_{n \geq 1}\) such that \(\varepsilon_n \to 0\) as \(n \to \infty\), and

\[
\lim_{\varepsilon_n \to 0} \frac{1}{\varepsilon_n} \left[ \mathbb{E} \int_0^T X_1(s) \bar{H}_{xz}(s)X_1(s)ds + \mathbb{E}[X_1(T) \bar{h}_{xz}(T)X_1(T)] \right] = \mathbb{E} \int_\tau^T \bar{X}(s) \bar{H}_{xz}(s)\bar{X}(s)ds + \mathbb{E}[\bar{X}(T) \bar{h}_{xz}(T)\bar{X}(T)].
\]

To prove Lemma 3.7, we take a closer look at \(X_1(\cdot)\) in (3.2). Actually, according to its definition, one can rewrite \(X_1(\cdot)\) as,

\[
X_1(t) = \begin{cases} 
0, & t \in [0, \tau], \\
\int_\tau^t \bar{b}_x(t, s)X_1(s)ds + \int_\tau^t [\bar{\sigma}_x(t, s)X_1(s) + \delta \bar{\sigma}(t, s)]dW(s), & t \in [\tau, \tau + \varepsilon], \\
\rho_1(t) + \int_{\tau + \varepsilon}^t \bar{b}_x(t, s)X_1(s)ds + \int_{\tau + \varepsilon}^t \bar{\sigma}_x(t, s)X_1(s)dW(s), & t \geq \tau + \varepsilon.
\end{cases}
\]

where \(\rho_1(\cdot) \in C([\tau + \varepsilon, T]; L^4(\Omega; \mathbb{R}))\) is \(\mathcal{F}_{\tau + \varepsilon}\)-measurable defined as,

\[
\rho_1(t) := \varepsilon^{-\frac{\kappa}{2}} \delta \bar{\sigma}(\cdot, \tau) \left(W(\tau + \varepsilon) - W(\tau)\right).
\]

We claim that \(\rho_1(\cdot) \in C([\tau + \varepsilon, T], L^4(\Omega; \mathbb{R}^n))\). Therefore, thanks to Lemma 2.1 and (H1), (3.40) admits a unique solution \(Y_1(\cdot) \in C([\tau + \varepsilon, T], L^4(\Omega; \mathbb{R}^n))\).

In fact, since \(\bar{u}(\cdot) \in \mathcal{U}_{ad}\), from Lemma 2.1 and (H1), we have \(\sup_{\tau \in [0, T]} \mathbb{E}|\bar{X}(\tau)|^{4+\kappa} < \infty\), and

\[
\sup_{\tau \in [0, T]} \mathbb{E}|\delta \bar{\sigma}(\tau, \tau)|^{4+\kappa} \leq \left[1 + \sup_{\tau \in [0, T]} \mathbb{E}|\bar{X}(\tau)|^{4+\kappa} + \sup_{\tau \in [0, T]} \mathbb{E}|\bar{u}(\tau)|^{4+\kappa}\right] < \infty.
\]

Consequently, by virtue of Hölder inequality and Jensen’s inequality of expectation,

\[
\sup_{t \in [\tau + \varepsilon, T]} \mathbb{E}|\varphi_1(t)|^4 \leq \sup_{t \in [\tau + \varepsilon, T]} \left[\mathbb{E}|\delta \bar{\sigma}(\tau, \tau)|^{4+\kappa}\right]^{\frac{4}{4+\kappa}} \varepsilon^{-2} \left[\mathbb{E}[\left|W(\tau + \varepsilon) - W(\tau)\right|^{6+\kappa}]\right]^{\frac{4}{6+\kappa}} \leq \sup_{t \in [\tau + \varepsilon, T]} \left[\mathbb{E}|\delta \bar{\sigma}(\tau, \tau)|^{4+\kappa}\right]^{\frac{4}{4+\kappa}} \cdot \left[\frac{(4|p| + 4)!}{2[P] + 2, (2|p| + 2)!}\right]^{\frac{p}{p+\kappa}} < \infty.
\]

Here \(p := \frac{4+\kappa}{\kappa}\), \(|p|\) is the integer part of \(p\), and we used one formula on Brownian motions:

\[
\mathbb{E}|W(t)|^{2k} = \frac{(2k)!}{2^k k!} t^k, \quad k \in \mathbb{N}, \quad t \in \mathbb{R}^+.
\]

Moreover,

\[
\lim_{t \to t_0} \mathbb{E}|\varphi_1(t) - \varphi_1(t_0)|^4 = 0, \quad t_0 \in [\tau + \varepsilon, T].
\]
Hence the conclusion of \( \vartheta_1(\cdot) \) is obvious. For natational simplicity, we denote

\[
(3.42) \quad \begin{cases}
\mathbb{H}(\varepsilon, X_1) := \frac{1}{\varepsilon} \mathbb{E} \left[ \int_{\tau + \varepsilon}^{T} X_1(s) \, \mathbb{T} H_{xx}(s) X_1(s) + X_1(T) \, \mathbb{T} \bar{h}_{xx}(T) X_1(T) \right], \\
\mathbb{H}(\varepsilon, Y_1) := \mathbb{E} \left[ \int_{\tau + \varepsilon}^{T} Y_1(s) \, \mathbb{T} \bar{H}_{xx}(s) Y_1(s) + Y_1(T) \, \mathbb{T} \bar{h}_{xx}(T) Y_1(T) \right].
\end{cases}
\]

Recall that \( L^4(0, T; L^4(\Omega; X')) \) is the set of \( \mathcal{B}(0, T) \cap \mathcal{F}_T \)-strongly measurable \( X' \)-valued process \( f(\cdot) \) satisfying \( \mathbb{E} \int_0^T \| f(s) \|_{X'}^4 \, ds < \infty \).

Using similar ideas as in Lemma 2.5 of [17], we present the following result.

**Lemma 3.8.** Given Banach space \( X' \), suppose \( f(\cdot) \in L^4(0, T; L^4(\Omega; X')) \). Then there exists a sequence \( \{\varepsilon_n\} \) such that

\[
\lim_{n \to \infty} \frac{1}{\varepsilon_n} \int_{t}^{t + \varepsilon_n} \mathbb{E} \| f(s) - f(t) \|_{X'}^4 \, ds = 0, \quad t \in [0, T], \text{ a.e.}
\]

**Lemma 3.9.** For \( X_1(\cdot), Y_1(\cdot), \mathbb{H}(\varepsilon, X_1), \mathbb{H}(\varepsilon, Y_1) \) in (3.40), (3.42), (3.43), there exists \( \{\varepsilon_n\}_{n \geq 1} \) such that

\[
\lim_{n \to \infty} \varepsilon_n = 0, \quad \lim_{n \to \infty} \left[ \mathbb{H}(\varepsilon_n, X_1) - \mathbb{H}(\varepsilon_n, Y_1) \right] = 0.
\]

**Proof.** At first, for any \( \varepsilon > 0 \) we make the following observation

\[
(3.43) \quad \mathbb{H}(\varepsilon, X_1) = \mathbb{E} \int_{\tau + \varepsilon}^{T} \left[ \frac{1}{\varepsilon} X_1(s) \, \mathbb{T} H_{xx}(s) X_1(s) - Y_1(s) \, \mathbb{T} \bar{H}_{xx}(s) Y_1(s) \right] ds
\]

We first treat \( \mathbb{H}_1(\varepsilon) \). An easy calculation shows that

\[
\left| \mathbb{H}_1(\varepsilon) \right| \leq \left[ \mathbb{E} \int_{\tau + \varepsilon}^{T} \left[ \frac{1}{\varepsilon} X_1(s) \, \mathbb{T} H_{xx}(s) X_1(s) - Y_1(s) \, \mathbb{T} \bar{H}_{xx}(s) Y_1(s) \right]^2 ds \right]^{1/2} \left[ \mathbb{E} \int_{\tau + \varepsilon}^{T} |H_{xx}(s)|^2 ds \right]^{1/2}
\]

\[
\leq K \left[ \mathbb{E} \int_{\tau + \varepsilon}^{T} |\varepsilon^{-\frac{1}{2}} X_1(s) - Y_1(s)|^4 ds \right]^{1/4} \left[ \mathbb{E} \int_{\tau + \varepsilon}^{T} \left[ |\varepsilon^{-\frac{1}{2}} X_1(s)|^4 + |Y_1(s)|^4 \right] ds \right]^{1/4},
\]

where we use the fact:

\[
|aa^\top - bb^\top| = ||a - b||a^\top + b|a - b| \leq ||a - b||a + b|,
\]

with \( a, b \in \mathbb{R}^n \). From (3.39), (3.40) and (2.3), we immediately have

\[
\sup_{t \in [\tau + \varepsilon, T]} \mathbb{E} |\varepsilon^{\frac{1}{2}} X_1(t) - Y_1(t)|^4 \leq K \sup_{t \in [\tau + \varepsilon, T]} \mathbb{E} |\varepsilon^{\frac{1}{2}} \rho(t) - g_1(t)|^4
\]

\[
\leq K \varepsilon^{-2} \mathbb{E} \left[ \int_{\tau}^{\tau + \varepsilon} |X_1(s)|^2 ds \right]^2 + K \varepsilon^{-2} \sup_{t \in [\tau + \varepsilon, T]} \mathbb{E} \left[ \int_{\tau}^{\tau + \varepsilon} |\delta\bar{\sigma}(t, s) - \delta\bar{\sigma}(t, \tau)|^2 ds \right]^2.
\]

For the first term, denoted by \( G_1(\varepsilon) \), on the right hand, we obtain \( \lim_{\varepsilon \to 0} G_1(\varepsilon) = 0 \) by Lemma 5.1. As to the second term, \( G_2(\varepsilon) \), thanks to Lemma 5.3, there exists \( \{\varepsilon_n\}_{n \geq 1} \) such that

\[
G_2(\varepsilon_n) \leq \varepsilon_n^{-1} \mathbb{E} \int_{\tau}^{\tau + \varepsilon} \| \delta\bar{\sigma}(\cdot, s) - \delta\bar{\sigma}(\cdot, \tau) \|_{C([0, T]; \mathbb{R}^n)}^4 ds \to 0, \quad n \to \infty.
\]
Consequently, for such \( \{ \varepsilon_n \} \), we conclude that
\[
\lim_{n \to \infty} \sup_{t \in [\tau+\varepsilon, T]} \mathbb{E}[\varepsilon_n^{-\frac{1}{2}} X_1(t) - Y_1(t)]^4 = 0.
\]
As a result, \( \lim_{n \to \infty} \mathbb{H}_1(\varepsilon_n) = 0 \).

Similarly we prove that \( \lim_{n \to \infty} \mathbb{H}_3(\varepsilon_n) = 0 \). The conclusion is established via (3.33).

To treat \( Y_1(\cdot) \) in Lemma 3.9 for \( \varphi_2(\cdot) := \delta \bar{\sigma}(\cdot), \tau \), we need \( Y_2(\cdot) \) on \([\tau + \varepsilon, T]\),
\[(3.44) \quad Y_2(t) = \varphi_2(t) + \int_{\tau+\varepsilon}^t \nabla_x(t, s)Y_2(s)ds + \int_{\tau+\varepsilon}^t \sigma_x(t, s)Y_2(s)dW(s).\]

The solvability of \( Y_2(\cdot) \in C_{\mathbb{P}}([\tau + \varepsilon, T]; L^4(\Omega; \mathbb{R}^n)) \) is followed by the similar procedures as that of \( Y_1(\cdot) \) in (3.30). Moreover, by the uniqueness in \( C_{\mathbb{P}}([\tau + \varepsilon, T]; L^4(\Omega; \mathbb{R})) \), we see that
\[
\mathbb{H}(\varepsilon, Y_1) = \mathbb{E}\left[\varepsilon^{-1}|W(\tau + \varepsilon) - W(\tau)|^2 \cdot F_1^{\delta \bar{\sigma}, \delta \bar{\sigma}}(\tau + \varepsilon)\right],
\]
where
\[
F_1^{\delta \bar{\sigma}, \delta \bar{\sigma}}(\tau + \varepsilon) := \mathbb{E}_{\tau + \varepsilon} \left[ \int_{\tau + \varepsilon}^T Y_2(s)^\top \bar{H}_{xx}(s)Y_2(s)ds + Y_2(T)^\top \bar{h}_{xx}(T)Y_2(T) \right].
\]

We establish the following result via \( Y_2(\cdot) \).

**Lemma 3.10.** Given \( \mathcal{X}(\cdot), Y_1(\cdot), \mathbb{H}(\varepsilon, Y_1) \) in (3.14), (3.40), (3.42), we have
\[(3.45) \quad \lim_{\varepsilon \to 0} \mathbb{H}(\varepsilon, Y_1) = \mathbb{E}\int_\tau^T \mathcal{X}(s)^\top \bar{H}_{xx}(s)\mathcal{X}(s)ds + \mathbb{E}[\mathcal{X}(T)^\top \bar{h}_{xx}(T)\mathcal{X}(T)].\]

**Proof.** Recalling \( F_1^{\delta \bar{\sigma}, \delta \bar{\sigma}}(\cdot) \) in (3.31), we shall derive the conclusion if
\[
\lim_{\varepsilon \to 0} \mathbb{E}\left[\varepsilon^{-1}|W(\tau + \varepsilon) - W(\tau)|^2 \cdot F_1^{\delta \bar{\sigma}, \delta \bar{\sigma}}(\tau + \varepsilon) - F_1^{\delta \bar{\sigma}, \delta \bar{\sigma}}(\tau)\right] = 0.
\]
To obtain this result, using similar ideas as in (3.41), we only prove
\[
\lim_{\varepsilon \to 0} \mathbb{E}\left|F_1^{\delta \bar{\sigma}, \delta \bar{\sigma}}(\tau + \varepsilon) - F_1^{\delta \bar{\sigma}, \delta \bar{\sigma}}(\tau)\right|^{p_0} = 0, \quad p_0 = \frac{2(4 + \kappa)}{8 + \kappa} \in (1, 2).
\]
To this end, by defining
\[
\Theta_1(\tau + \varepsilon) := \int_{\tau + \varepsilon}^T Y_2(s)^\top \bar{H}_{xx}(s)Y_2(s)ds + Y_2(T)^\top \bar{h}_{xx}(T)Y_2(T),
\]
\[
\Theta_2(\tau) := \int_\tau^T \mathcal{X}(s)^\top \bar{H}_{xx}(s)\mathcal{X}(s)ds + \mathcal{X}(T)^\top \bar{h}_{xx}(T)\mathcal{X}(T),
\]
we deduce that
\[
\mathbb{E}\left|F_1^{\delta \bar{\sigma}, \delta \bar{\sigma}}(\tau + \varepsilon) - F_1^{\delta \bar{\sigma}, \delta \bar{\sigma}}(\tau)\right|^{p_0} \equiv \mathbb{E}\left|\Theta_1(\tau + \varepsilon) - \Theta_2(\tau)\right|^{p_0}
\]
\[
\leq K\mathbb{E}\left|\Theta_1(\tau + \varepsilon) - \Theta_2(\tau)\right|^{p_0} + K\mathbb{E}\left|\Theta_1(\tau + \varepsilon) - \Theta_2(\tau)\right|^{p_0}.
\]
Hence it is suffice to prove the terms on right hand approach to zero as \( \varepsilon \to 0 \).
Notice that

$$E[\Theta_1(\tau + \varepsilon) - \Theta_2(\tau)]^{p_0}$$

\[\leq K E \left\{ \int_{\tau + \varepsilon}^{T} |Y_2(s)Y_2(s)^\top - \mathcal{X}(s)\mathcal{X}(s)^\top|^2 ds \right\}^{\frac{p_0}{2}} \left[ \int_{\tau + \varepsilon}^{T} |\hat{H}_{xx}(s)|^2 ds \right]^{\frac{p_0}{2}} \]  

(3.46)

\[+ KE \left\{ \int_{\tau}^{\tau + \varepsilon} |\mathcal{X}(s)\mathcal{X}(s)^\top|^2 ds \right\}^{\frac{p_0}{2}} \left[ \int_{\tau}^{T} |\hat{H}_{xx}(s)|^2 ds \right]^{\frac{p_0}{2}} \]

\[+ KE \left\{ |Y_2(T)^2 - \mathcal{X}(T)\mathcal{X}(T)^\top|^{p_0} |\hat{H}_{xx}(T)|^{p_0} \right\}, \]

where $x^\top Ax = Tr(xx^\top A)$ with $x \in \mathbb{R}^n$ and $A \in \mathbb{R}^{n \times n}$.

As to the first term of (3.46), denoted by $M_1(\varepsilon)$,

$$M_1(\varepsilon) \leq K \left[ E \int_{\tau + \varepsilon}^{T} |\tilde{H}_{xx}(s)|^2 ds \right]^{\frac{p_0}{2}} \left[ E \left( \int_{\tau + \varepsilon}^{T} |Y_2(s)Y_2(s)^\top - \mathcal{X}(s)\mathcal{X}(s)^\top|^2 ds \right)^{\frac{p_0}{2}} \right]^{\frac{2 - p_0}{p_0}}$$

(3.47)

\[\leq K \left[ E \left( \int_{\tau + \varepsilon}^{T} |Y_2(s) + |\mathcal{X}(s)||^4 ds \right)^{\frac{p_0}{2}} \left( \int_{\tau + \varepsilon}^{T} |Y_2(s) - \mathcal{X}(s)|^4 ds \right)^{\frac{p_0}{2}} \right]^{\frac{2 - p_0}{p_0}} \]

\[\leq K \left[ E \int_{\tau + \varepsilon}^{T} (|Y_2(s)| + |\mathcal{X}(s)|)^{4p_0} ds \right]^{\frac{2 - p_0}{p_0}} \left[ E \int_{\tau + \varepsilon}^{T} |Y_2(s) - \mathcal{X}(s)|^{4p_0} ds \right]^{\frac{2 - p_0}{p_0}}, \]

where $p_0^* := \frac{4}{1 - 4\kappa} > 1$.

As to the second term of (3.46), denoted by $M_2(\varepsilon)$,

$$M_2(\varepsilon) \leq \left[ \int_{\tau + \varepsilon}^{T} |\tilde{H}_{xx}(s)|^2 ds \right]^{\frac{p_0}{2}} \left[ E \left( \int_{\tau + \varepsilon}^{T} |\mathcal{X}(s)|^4 ds \right)^{\frac{p_0}{2}} \right]^{\frac{2 - p_0}{p_0}} \rightarrow 0, \ \varepsilon \rightarrow 0.$$  

(3.48)

As to the third term of (3.46), denoted by $M_3(\varepsilon)$,

$$M_3(\varepsilon) \leq K \left[ E |\tilde{H}_{xx}(T)|^2 \right]^{\frac{p_0}{2}} \left[ E (|Y_2(T)| + |\mathcal{X}(T)|)^{4p_0} E |Y_2(T) - \mathcal{X}(T)|^{4p_0} \right]^{\frac{2 - p_0}{p_0}}.$$  

(3.49)

Since $4p_0^* = \frac{4p_0}{1 - 4\kappa}$, we first estimate $E |Y_2(\cdot) - \mathcal{X}(\cdot)|^{4+\kappa}$ which is similar as (2.3) or (3.24),

$$\sup_{r \in [\tau + \varepsilon, T]} E |Y_2(r) - \mathcal{X}(r)|^{4+\kappa}$$

\[\leq K \sup_{r \in [\tau + \varepsilon, T]} E \left| \int_{r}^{r + \varepsilon} \tilde{b}_x(t, s)\mathcal{X}(s)ds + \int_{\tau}^{r + \varepsilon} \tilde{\sigma}_x(t, s)\mathcal{X}(s)dW(s) \right|^{4+\kappa} \rightarrow 0, \ \varepsilon \rightarrow 0.$$

Therefore, from (3.47), (3.49), $\lim_{\varepsilon \rightarrow 0} |M_1(\varepsilon) + M_3(\varepsilon)| = 0$. Considering (3.48), one has

$$\lim_{\varepsilon \rightarrow 0} E |\Theta_1(\tau + \varepsilon) - \Theta_2(\tau)|^{p_0} = 0.$$

Our remaining aim is $\lim_{\varepsilon \rightarrow 0} E \left| E_{\tau + \varepsilon} \Theta_2(\tau) - E_{\tau} \Theta_2(\tau) \right|^{p_0} = 0$. By Lemma 2.1

$$E|\Theta_2(\tau)|^{p_0} \leq K E \left[ \int_{\tau}^{T} |\mathcal{X}(s)|^2 |\tilde{H}_{xx}(s)| ds \right]^{\frac{p_0}{2}} + KE \left[ |\mathcal{X}(T)|^{2p_0} |\tilde{H}_{xx}(T)|^{p_0} \right]$$

\[\leq K \left[ E \int_{\tau}^{T} |\tilde{\sigma}(s, \tau)|^{\frac{p_0}{2}} ds \right]^{\frac{2 - p_0}{p_0}} + K \left[ E |\tilde{\sigma}(T, \tau)|^{\frac{p_0}{2}} \right]^{\frac{2 - p_0}{p_0}} < \infty. \]

Because $E_{\tau + \varepsilon} \Theta_2(\tau) \rightarrow E_{\tau} \Theta_2(\tau)$, a.s., $\varepsilon \rightarrow 0$, and for any $r \in [\tau, T]$,

$$E|E_{r} \Theta_2(\tau)|^{p_0} \leq E \sup_{r \in [\tau, T]} E_{r} |\Theta_2(\tau)|^{p_0} \leq \frac{p_0}{p_0 - 1} E|\Theta_2(\tau)|^{p_0} < \infty.$$  

One has the desired conclusion by dominated convergence theorem.
Now it is time for us to show the proof of Lemma 3.7.

**Proof.** By above (3.39), (3.42), one has,
\[
\frac{1}{\varepsilon} \left[ \mathbb{E} \int_0^T X_1(s)^\top \bar{H}_{xx}(s)X_1(s)ds + \mathbb{E} [X_1(T)^\top \bar{h}_{xx}(T)X_1(T)] \right] = \frac{1}{\varepsilon} \mathbb{E} \int_\tau^{\tau+\varepsilon} X_1(s)^\top \bar{H}_{xx}(s)X_1(s)ds + \mathbb{E}(\varepsilon, X_1).
\]
For the first term on right hand, denoted by \(Q(\varepsilon, X_1)\), from Lemma 3.1 we see that, \(\lim_{\varepsilon \to 0} Q(\varepsilon, X_1) = 0\). We thus derive the conclusion by Lemma 3.9 and Lemma 3.10.

**IV. Maximum principles of optimal control problems for SVIEs**

We present the first main result of this paper, the proof of which is based on the arguments from (3.2) to (3.9), as well as Lemma 3.2, Lemma 3.7.

**Theorem 3.1.** Let (H1)-(H2) hold and \((\bar{X}(\cdot), \bar{u}(\cdot))\) be an optimal pair. Then
\[
\min_{u \in U} \mathcal{H}(t, u) = \mathcal{H}(t, \bar{u}(t)) = 0, \quad \mathbb{P}-\text{a.s.}, \quad t \in [0, T], \quad \text{a.e.}
\]
where
\[
(3.50) \quad \mathcal{H}(t, u) := \Delta H^\varepsilon(t) + \frac{1}{2} \delta \bar{\sigma}(T, t)^\top [B_1(t)\Delta \bar{\sigma}(\cdot, t)] + \frac{1}{2} \int_0^T \delta \bar{\sigma}(s, t)^\top [B_2(t)\Delta \bar{\sigma}(\cdot, t)](s)ds,
\]
\(\Delta H^\varepsilon(\cdot), \delta \bar{\sigma}(T, t), \Delta \bar{\sigma}(\cdot, t)\) are in (3.10), (3.13), \(B_1(\cdot), B_2(\cdot)\) satisfy (3.16), (3.17).

Above \(B_1, B_2\) are called the second-order operator-valued adjoint processes of our optimal control problem.

If \(\delta \bar{\sigma}(\cdot, \tau)\) degenerates into \(F_\tau\)-measurable random variable, using again the arguments from (3.2) to (3.9), and Lemma 3.5, Lemma 3.7 we have the second main result in this article,

**Theorem 3.2.** Let (H1)-(H2) hold with \(\delta \bar{\sigma}(\cdot, \tau) \equiv \delta \bar{\sigma}(\tau)\), and \((\bar{X}(\cdot), \bar{u}(\cdot))\) be an optimal pair. Then
\[
\min_{u \in U} \mathcal{H}_0(t, u) = \mathcal{H}_0(t, \bar{u}(t)) = 0, \quad \mathbb{P}-\text{a.s.}, \quad t \in [0, T], \quad \text{a.e.}
\]
where
\[
(3.51) \quad \mathcal{H}_0(t, u) := \Delta H^\varepsilon(t) + \frac{1}{2} \delta \bar{\sigma}(t)^\top B_3(t)\delta \bar{\sigma}(t), \quad u \in U,
\]
\(B_3(\cdot), \Delta H^\varepsilon(\cdot)\) are defined in Lemma 3.5 and 3.7, respectively.

\(B_3(\cdot)\) is referred as the \(\mathbb{R}^{n \times n}\)-valued second-order adjoint process under this framework.

**Remark 3.4.** Suppose \(\sigma_1, \sigma_2\) are two functions that satisfy the same requirements as \(\sigma\) in (H1), and
\[
(3.52) \quad \sigma(t, s, x, u) := \sigma_1(t, s, x) + \sigma_2(s, x, u), \quad t, s \in [0, T], \quad x \in \mathbb{R}^n, \quad u \in U.
\]
For the previous \(\delta \bar{\sigma}(t, \tau)\) in (3.13) with \(t, \tau \in [0, T]\),
\[
(3.53) \quad \delta \bar{\sigma}(t, \tau) := \sigma(t, \tau, \bar{X}(\tau), u) - \sigma(t, \tau, \bar{X}(\tau), \bar{u}(\tau)) = \delta \bar{\sigma}(\tau), \quad u \in U.
\]
The corresponding maximum principle is easy to see in terms of Theorem 3.2.

Next we discuss several special cases.
IV.1. State-independent diffusion and drift terms

If both \( b \) and \( \sigma \) do not depend on \( x \), then

\[
\delta \sigma(t, \tau) := \sigma(t, \tau, u) - \sigma(t, \tau, \bar{u}(\tau)), \quad t, \tau \in [0, T], \quad u \in U,
\]

and \((3.50), (3.54)\) change accordingly. In addition, for \( u \in U, \ H_{xx}(t, x, \bar{X}(\cdot), \bar{Y}(\cdot), \bar{Z}(\cdot, t), u) \) is bounded.

We define two operator-valued processes \( B'_1, B'_2 \), i.e. for \( t \in [0, T], s \in [0, T], \)

\[
B'_1(t) \Delta \bar{\sigma}(\cdot, t) := E_t[\bar{h}_{xx}(T)] \delta \bar{\sigma}(T, t), \quad [B'_2(t) \Delta \bar{\sigma}(\cdot, t)](s) := E_t[\bar{H}_{xx}(s)] \delta \bar{\sigma}(s, t) I_{[t, T]}(s).
\]

It is a direct calculation that \( B'_1 \) and \( B'_2 \) satisfy (3.16) and (3.15). By the uniqueness in (3.17) and Theorem 3.1 for any \( u \in U, \)

\[
(3.54) \quad \Delta H^\varepsilon(t) + \frac{1}{2} \bar{\sigma}(T, t)^\top E_t[\bar{h}_{xx}(T)] \delta \bar{\sigma}(T, t) + \frac{1}{2} \int_t^T \delta \bar{\sigma}(s, t)^\top E_t[\bar{H}_{xx}(s)] \delta \bar{\sigma}(s, t) ds \geq 0. \ a.s.
\]

Corollary 3.1. Suppose \( b \) and \( \sigma \) do not rely on \( x \), and \( (\bar{X}(\cdot), \bar{u}(\cdot)) \) is optimal pair. Then for almost \( t \in [0, T], \)

(3.54) holds true.

When the state equation is linear and the cost functional is quadratic, similar conclusion was obtained in [24]. In other words, our Corollary 3.1 extends theirs into the nonlinear setting.

IV.2. The convex control region

Suppose (H1), (H2) hold with convex \( U \). Moreover, the following maps are continuous differentiable,

\[
(3.55) \quad u \to (l(t, x, u), b(s, t, x, u), \sigma(s, t, x, u)), \quad x \in \mathbb{R}^n, \quad u \in U, \quad s, t \in [0, T].
\]

For \( t \in [0, T], \ a.e. \ \omega \in \Omega, \ a.s., \) we define \( v := \bar{u}(t) + \varepsilon[u - \bar{u}(t)] \) with \( u \in U \). The convexity of \( U \) shows that \( v \in U \). From (3.50) we have,

\[
0 \leq \frac{\mathcal{H}(t, v)}{\varepsilon} \leq \langle H_u(t, \bar{X}(t), \bar{Y}(\cdot), \bar{Z}(\cdot, t), \bar{u}(t) + \theta \varepsilon u - \bar{u}(t)), u - \bar{u}(t) \rangle + K(t) \varepsilon,
\]

where \( K(\cdot) \) is a process, \( 0 < \theta < 1 \) and \( H_u \) is the partial derivative with respect to \( u \). Let \( \varepsilon \to 0 \), one has

\[
(3.56) \quad \langle H_u(t, \bar{X}(t), \bar{Y}(\cdot), \bar{Z}(\cdot, t), \bar{u}(t)), u - \bar{u}(t) \rangle \geq 0. \ a.s.
\]

Corollary 3.2. Let (H1), (H2), (3.55) hold true and \( (\bar{X}(\cdot), \bar{u}(\cdot)) \) be optimal with convex \( U \). Then there exists a pair of \( (\bar{Y}(\cdot), \bar{Z}(\cdot, \cdot)) \) satisfying (3.14) such that for almost \( t \in [0, T], u \in U, \) (3.30) is satisfied.

Above convex region case was studied in e.g. [27], [28]. Therefore, our study (i.e. Theorem 3.1) extends theirs into the non-convex setting.

IV.3. The linear quadratic case.

We discuss the linear quadratic optimal control problem where \( U := \mathbb{R}^m, \)

\[
b = [A_1(t, s)x + B_1(t, s)u], \quad \sigma = [A_2(t, s)x + B_2(t, s)u],
\]

\[
h = \frac{1}{2} x^\top Gx, \quad l = \frac{1}{2} [x^\top Qx + 2u^\top Sx + u^\top Ru].
\]
Here $A_1, B_1, A_2, B_2, Q, S, R, G$ are bounded and random such that for some modulus function $\rho(\cdot)$,

$$|f(t,s) - f(t',s)| \leq \rho(|t - t'|), \quad t, t', s \in [0,T], \quad f := A_1, B_1, A_2, B_2.$$ 

For optimal $(\bar{X}, \bar{u})$, we define

$$\mathcal{H}_1(t) := S(t)\bar{X}(t) + B_1(T, t)^\top \mathbb{E}_t[G\bar{X}(T)] + \mathbb{E}_t \int_t^T B_1(s, t)^\top \bar{Y}(s)ds$$
$$+ B_2(T, t)^\top \pi(t) + \mathbb{E}_t \int_t^T B_2(s, t)^\top \bar{Z}(s)ds,$$

(3.57)

where $(\bar{Y}, \bar{Z}, \pi)$ is in (3.6) accordingly. Next we consider two special cases.

**Case I:**

Suppose $A_1 \equiv A_2 \equiv 0$. For any $u \in \mathbb{R}^m$, $t \in [0, T]$, a.e., (3.57) becomes

$$[u - \bar{u}(t)]^\top \mathcal{H}_1(t) + \frac{1}{2}[u^\top R(t)u - \bar{u}(t)^\top R(t)\bar{u}(t)] + \frac{1}{2}[u - \bar{u}(t)]^\top \mathcal{B}(t)[u - \bar{u}(t)] \geq 0, \text{ a.s.}$$

where $\mathcal{H}_1(t)$ is in (3.57), $\mathcal{B}(t) := \mathcal{B}_1(t) + \mathcal{B}_2(t)$,

$$\mathcal{B}_1(t) := B_2(T, t)^\top [\mathbb{E}_t G] B_2(T, t), \quad \mathcal{B}_2(t) := \mathbb{E}_t \int_t^T B_2(s, t)^\top Q(s)B_2(s, t)ds.$$ 

Here $\mathcal{B}_i$ are $\mathbb{F}$-adapted processes. By the arbitrariness of $u$, one has

$$\mathcal{H}_1(t) + R(t)\bar{u}(t) = 0, \quad R(t) + \mathcal{B}(t) \geq 0, \quad t \in [0, T]. \text{ a.s. a.e.}$$

Notice that these two condition were also obtained in [24] with distinct approach.

**Remark 3.5.** If for $t \in [0, T]$, $B_i(\cdot, t) \equiv B_i(t), \quad i = 1, 2$. Then we have

$$\mathcal{B}(t) = B_2(t)^\top P_2(t)B_2(t), \quad P_2(t) := \mathbb{E}_t \left[G + \int_t^T Q(s)ds\right].$$

Here $P_2$, which is called second-order adjoint process, just satisfies (3.14), see [24]. For this specification, we reveal two interesting facts.

It is our believe that $G$ and $Q(\cdot)$ should originally play its own peculiar role in optimal control problem of SVIEs, such as above introduced processes $\mathcal{B}_i$. In particular SDEs case, such independence disappears and their roles happen to be merged together in suitable manner, like above $P_2$. In other words, compared with the SDEs scenario with only one adjoint process $P_2$, a pair of different forms of adjoint processes are indeed required in SVIEs setting.

For optimal control problems of SDEs, one can directly introduce second-order adjoint equation (3.11). However, as to the case of SVIEs, this adjoint equation idea does not work any more (see [24]). Actually, in our opinion, there is no way to construct the analogue version of (3.11) here. One has to seek more fundamental, appropriate notion to get around this difficulty. In a nutshell, we need to use new stochastic processes to replace classical stochastic equations.

**Case II:**

Suppose $B_2(t, \cdot) \equiv B_2(\cdot)$. Then $\delta\tilde{\sigma}(t) = B_2(t)[u - \bar{u}(t)], \quad t \in [0, T]$. Recall Theorem 3.2 we have

$$[u - \bar{u}(t)]^\top \mathcal{H}_1(t) + \frac{1}{2}[u^\top R(t)u - \bar{u}(t)^\top R(t)\bar{u}(t)]$$
$$+ \frac{1}{2}[u - \bar{u}(t)]^\top B_2(t)^\top \mathcal{B}(t)B_2(t)[u - \bar{u}(t)] \geq 0,$$
where \( u \in \mathbb{R}^m \), \( \mathcal{H}(\cdot) \) is in \((3.51)\), \( \mathcal{P}(\cdot) := \{ \mathcal{P}_{ij}(\cdot) \}_{n \times n} \),

\[
\begin{cases}
\mathcal{P}_{ij}(t) = \frac{1}{2} \mathbb{E}_t \left( \mathcal{X}^{e_i}_i(T)^{\top} G \mathcal{X}^{e_i}_j(T) \right) + \frac{1}{2} \mathbb{E}_t \int_t^T \mathcal{X}^{e_i}_i(s)^{\top} Q(s) \mathcal{X}^{e_i}_j(s) ds, & t \in [0, T], \\
\mathcal{X}^{e_i}(r) = e_i + \int_t^r A_1(t, s) \mathcal{X}^{e_i}(s) ds + \int_t^r A_2(t, s) \mathcal{X}^{e_i}(s) dW(s), & r \in [t, T],
\end{cases}
\]

By the arbitrariness of \( u \in \mathbb{R}^m \), one then obtains the following maximum condition

\[(3.58) \quad \mathcal{H}_1(t) + R(t) \ddot{u}(t) = 0, \quad R(t) + B_2(t)^{\top} \mathcal{P}(t) B_2(t) \geq 0, \quad t \in [0, T]. \quad \text{a.e.}
\]

**Remark 3.6.** If \( A_1, A_2, B_1 \) are independent of \( t \), then according to Lemma 3.3 and the arguments in Subsection 3.1, \( \mathcal{P}(\cdot) \) is just the unique solution of second-order adjoint equation \((3.11)\), and \((3.58)\) naturally reduce into the counterpart in SDEs case.

4. Concluding remarks

This article is devoted to maximum principles of optimal control problems for SVIEs when the control region is arbitrary subset of \( \mathbb{R}^m \) and diffusion relies on control variable. Some novelties are summed up as follows.

- We introduce a class of quadratic functionals associated with linear SVIEs, and represent them in two different ways. To our best, these conclusions are new and may have independent interests. When they are applied in optimal control problems of SDEs, the maximum principles can be established without Itô formula and second-order adjoint equations.

- We establish two maximum principles in terms of operator-valued, and matrix-valued second-order adjoint processes, respectively. For optimal control problems of SVIEs \((1.1)\), there is no existing paper treating the case of closed \( U \) and state-dependent diffusion. Nevertheless, it is one particular case of our study. Moreover, our conclusions can fully cover the SDEs case.

- We obtain some convincing arguments to show that the second-order adjoint equation idea actually fails in our SVIEs setting. Therefore, we propose appropriate second-order adjoint processes instead. In addition, unlike the classical scenario with only one second-order adjoint equation, here we have to rely on two second-order adjoint processes in the maximum conditions, which of course merge into the solution of second-order adjoint equation in particular SDEs setting.

We emphasize that Theorem 3.2 is a refinement of Theorem 3.1. As a trade-off, some requirements are imposed (Remark 3.4). Consequently, it still remains its importance to replace the pair of operator-valued processes in Theorem 3.1 by explicit \( \mathbb{R}^{n \times n} \)-valued counterparts. We hope to discuss this topic in our forthcoming papers.

5. Appendix

To prove the existence of operator-valued processes in Lemma 3.3, we make some preparations in the sequel. The first two results are more or less standard in functional analysis.

**Lemma 5.1.** Suppose \( f : \mathbb{B} \rightarrow \mathbb{R} \) is a bounded linear functional. Then there exists a unique \( y(\cdot) = (y_1, y_2(\cdot)) \in \mathbb{B}' \) such that

\[
\begin{cases}
f(x) = \int_0^T x_2(t)^{\top} y_2(t) dt + x_1^{\top} y_1, & \forall x = (x_1, x_2(\cdot)) \in \mathbb{B}, \\
\|f\|_{L(\mathbb{B}; \mathbb{R})} = \|y\|_{\mathbb{B}'} = \max \left\{ |y_1|, \left[ \int_0^T |y_2(t)|^2 dt \right]^{\frac{1}{2}} \right\}.
\end{cases}
\]
Lemma 5.2. Given Banach space $X$ with numerable dense subset $X_0$, suppose

$$X_1 := \{ \sum_{i=1}^{n} a_i x_i, \ a_i \in \mathbb{Q}, \ x_i \in X_0, \ n \in \mathbb{N} \},$$

with $\mathbb{Q}$ the set of rational number, $f : X_1 \rightarrow Y$ is map to Banach space $Y$ such that for constant $M > 0$,

$$\|f(x)\|_Y \leq M \|x\|_X, \ \forall x \in X_1, \ f(ax + by) = af(x) + bf(y), \ \forall a, b \in \mathbb{Q}.$$ 

Then there exists a unique bounded linear operator $F : X \rightarrow Y$ satisfying

$$F(x) = f(x), \ x \in X_1, \ \|F\|_{\mathcal{L}(X,Y)} \leq M.$$ 

Lemma 5.3. Given constant $M > 0$, suppose $f : \mathbb{B} \times \mathbb{B} \rightarrow \mathbb{R}$ satisfies

$$(5.1) \qquad \|f(x,y)\| \leq M \|x\|_{\mathbb{B}}, \|y\|_{\mathbb{B}}, \ \forall x, y \in \mathbb{B}, \ f(a \bar{x} + b \bar{y}, y) = af(x,y) + bf(\bar{x}, y), \ f(x, a\bar{y} + b\bar{y}) = af(x, \bar{y}) + bf(x, \bar{y}), \ a, b \in \mathbb{Q}, \ x, \bar{x}, y, \bar{y} \in \mathbb{B}.$$ 

Then there exist a unique pair of linear operators $\hat{B}_1 : \mathbb{B} \rightarrow \mathbb{R}^n$, $\hat{B}_2 : \mathbb{B} \rightarrow L^+(0, T ; \mathbb{R}^n)$ such that

$$\left\{ \begin{array}{l} \|\hat{B}_1\|_{\mathcal{L}(\mathbb{B}; \mathbb{R}^n)} \leq M, \ \|\hat{B}_2\|_{\mathcal{L}(\mathbb{B}; L^+(0, T ; \mathbb{R}^n))} \leq M, \\
 f(x,y) = \int_0^T x_2(t) + [\dot{\hat{B}}_2(x)](t) dt + x_1(\hat{B}_1(x)), \ \forall x, y \in \mathbb{B}. \end{array} \right.$$ 

Proof. Step 1 : Fix $y \in \mathbb{B}_1$, we define $\varphi_y : \mathbb{B}_1 \rightarrow \mathbb{R}$, i.e. $\varphi_y(x) = f(x, y), x \in \mathbb{B}_1$. It is easy to see

$$\varphi_y(k \bar{x} + l \bar{y}) = k \varphi_y(\bar{x}) + l \varphi_y(\bar{y}), \ |\varphi_y(x)| \leq M \|x\|_{\mathbb{B}}, \ k, l \in \mathbb{Q}, \ \bar{x}, \bar{y} \in \mathbb{B}_1.$$ 

According to Lemma 5.2 there exists a unique bounded linear functional $\hat{\varphi}_y : \mathbb{B} \rightarrow \mathbb{R}$ such that

$$\hat{\varphi}_y(x) = \varphi_y(x), \ \|\hat{\varphi}_y\|_{\mathcal{L}(\mathbb{B}; \mathbb{R})} \leq M_y := M \|y\|_{\mathbb{B}}, \ \forall x \in \mathbb{B}_1.$$ 

Consequently, on account of Lemma 5.1 there exists a unique $y^*(\cdot) = (y_1^*, y_2^*(\cdot)) \in \mathbb{B}'$ such that

$$\left\{ \begin{array}{l} \hat{\varphi}_y(x) = x_1^* y_1^* + \int_0^T x_2(t)^\top y_2^*(t) dt, \ x = (x_1, x_2(\cdot)) \in \mathbb{B}, \\
 \|\hat{\varphi}_y\|_{\mathcal{L}(\mathbb{B}; \mathbb{R})} = \max\{ |y_1^*|, [\int_0^T |y_2^*(t)|^\top dt]^{\frac{1}{2}} \}. \end{array} \right.$$ 

Step 2 : We introduce operator $\mathcal{B} : \mathbb{B}_1 \rightarrow \mathbb{R}^n \times L^+(0, T ; \mathbb{R}^n)$, i.e.,

$$\mathcal{B}y = ([B_1y], [B_2y](\cdot)), \ [B_1y]_1 = y_1^*, \ [B_2y](\cdot) = y_2^*(\cdot), \ y \in \mathbb{B}_1.$$ 

The well-posedness of $\mathcal{B}$ is obvious. We claim that $\mathcal{B}$ is linear in following sense,

$$\mathcal{B}(kh + lh) = k\mathcal{B}h + l\mathcal{B}h, \ k, l \in \mathbb{Q}, \ h, \bar{h} \in \mathbb{B}_1.$$ 

Notice that $[kh + lh] \in \mathbb{B}_1$. From Step 1 there exists a unique linear bounded $\hat{\varphi}_{kh+l\bar{h}}(\cdot) : \mathbb{B} \rightarrow \mathbb{R}$ such that $\hat{\varphi}_{kh+l\bar{h}} = \varphi_{kh+l\bar{h}}$ in $\mathbb{B}_1$. Since $\varphi_{kh+l\bar{h}} = k\varphi_{h} + l\varphi_{\bar{h}}$ in $\mathbb{B}_1$, it then follows that $k\hat{\varphi}_{h} + l\hat{\varphi}_{\bar{h}}$ is another linear bounded functional on $\mathbb{B}$ such that $[k\hat{\varphi}_{h} + l\hat{\varphi}_{\bar{h}}] = \varphi_{kh+l\bar{h}}$ in $\mathbb{B}_1$. By virtue of the uniqueness in Lemma 5.2

$$\hat{\varphi}_{kh+l\bar{h}} = [k\hat{\varphi}_{h} + l\hat{\varphi}_{\bar{h}}].$$
From Lemma 5.1 we see that there exists
\[ [k\bar{h} + \bar{l}h]^* = \left( [k\bar{h} + \bar{l}h]_1^*, [k\bar{h} + \bar{l}h]_2^*(s) \right) \in \mathbb{R}^n \times L^\frac{2}{3}(0, T; \mathbb{R}^n) \]
such that,
\[ \tilde{\varphi}_{k\bar{h} + \bar{l}h}(x) = \left( [k\bar{h} + \bar{l}h]_1^* \right)^T x_1 + \int_0^T \left( [k\bar{h} + \bar{l}h]_2^*(s) \right)^T x_2(s) ds, \quad \forall x = (x_1, x_2(\cdot)) \in \mathbb{B}. \]

Using again Lemma 5.1,
\[ k\tilde{\varphi}_{\bar{h}}(x) + l\tilde{\varphi}_{\bar{h}}(x) = \left[ k \int_0^T x_2(t)^T \tilde{y}_2^*(t) dt + kx_1^T \tilde{y}_1^* \right] + \left[ l \int_0^T x_1^T \tilde{y}_2^*(t) dt + lx_1^T \tilde{y}_1^* \right] = \int_0^T x_2(t)^T \left[ k\tilde{y}_2^*(t) + l\tilde{y}_2^*(t) \right] dt + x_1^T \left[ k\tilde{y}_1^* + l\tilde{y}_1^* \right]. \]

Therefore, by the arbitrariness of \( x = (x_1, x_2(\cdot)) \),
\[ [k\bar{h} + \bar{l}h]_2^* = k\tilde{y}_2^* + l\tilde{y}_2^*, \quad [k\bar{h} + \bar{l}h]_1^* = k\tilde{y}_1^* + l\tilde{y}_1^*. \]
This directly leads to the desirable linearity of \( \mathcal{B} \).

**Step 3:** We extend the definition of \( \mathcal{B} \) into \( \mathbb{B} \). For any \( y \in \mathbb{B}_1 \), from **Step 1, 2,**
\[ \| \mathcal{B}y \|_{\mathcal{B}} = \| \tilde{\varphi}_y \|_{L(\mathbb{B}, \mathcal{B}')} \leq M \| y \|_{\mathcal{B}}. \]
Hence from the linearity of \( \mathcal{B} \) and Lemma 5.2 there exists a unique linear \( \hat{\mathcal{B}} : \mathbb{B} \mapsto \mathbb{B}' \) such that
\[ \hat{\mathcal{B}}(x) = \mathcal{B}(x), \quad x \in \mathbb{B}_1, \quad \| \hat{\mathcal{B}} \|_{L(\mathbb{B}, L^\frac{2}{3}(0, T; \mathbb{R}^n))} \leq M. \]
Therefore, for any \( x, y \in \mathbb{B}_1, \)
\begin{equation}
(5.2) \quad f(x, y) = \varphi_y(x) = \varphi_{\tilde{y}}(x) = x_1^T (\hat{\mathcal{B}}y)_1 + \int_0^T x_2(t)^T (\hat{\mathcal{B}}y)_2(t) dt.
\end{equation}

**Step 4:** To see the conclusions of \( \hat{\mathcal{B}}_1, \hat{\mathcal{B}}_2, \) for \( y \in \mathbb{B}_1 \), we define \( \mathcal{B}_1 : \mathbb{B} \mapsto \mathbb{R}^n, \mathcal{B}_2 : \mathbb{B} \mapsto L^\frac{2}{3}(0, T; \mathbb{R}^n) \) as
\[ \mathcal{B}_1y = (\mathcal{B}y)_1, \quad (\mathcal{B}_2y)(\cdot) = \left[ (\mathcal{B}y)_2(\cdot) \right]. \]
Of course, both of them are well-defined.

We look at their linearity with \( k, l \in \mathbb{Q}, \bar{h}, \bar{h} \in \mathbb{B}_1 \). By the linearity of \( \mathcal{B} \), we obtain
\[ \mathcal{B}_i(k\bar{h} + \bar{l}h) = [k(\bar{h} + \bar{l}h)]_i, \quad i = 1, 2. \]
On the other hand, for any \( h \in \mathbb{B}_1 \) we know that \( |\mathcal{B}_i h| \leq \| h \|_{\mathcal{B}}, i = 1, 2 \). By using Lemma 5.2 again, there exist a unique linear bounded \( \hat{\mathcal{B}}_1 : \mathbb{B} \mapsto \mathbb{R}^n \) and a unique linear bounded \( \hat{\mathcal{B}}_2 : \mathbb{B} \mapsto L^\frac{2}{3}(0, T; \mathbb{R}^n) \) such that \( \hat{\mathcal{B}}_1 y = \mathcal{B}_1 y, \hat{\mathcal{B}}_2 y = \mathcal{B}_2 y \) with \( y \in \mathbb{B}_1 \). As a result, from 5.2 we obtain the conclusion.

Using almost the same ideas as above, one obtain the following which is useful in proving Lemma 5.6

**Lemma 5.4.** Given positive constant \( M \), suppose map \( f : \mathbb{Q}^n \times \mathbb{Q}^n \mapsto \mathbb{R} \) satisfies
\begin{equation}
(5.3) \quad |f(x, y)| \leq M|x|_{\mathbb{Q}^n} |y|_{\mathbb{Q}^n}, \quad \forall x, y \in \mathbb{Q}^n, \quad f(ax + b\bar{x}, y) = af(x, y) + bf(\bar{x}, y), \quad f(x, a\bar{y} + b\bar{y}) = af(x, \bar{y}) + bf(x, \bar{y}), \quad a, b \in \mathbb{Q}, \ x, \bar{x}, y, \bar{y} \in \mathbb{Q}^n,
\end{equation}
where \( \mathbb{Q}^n \) is the set of \( n \)-dimensional rational vectors. Then there exist a unique \( \mathbb{R}^{n \times n} \)-valued matrix \( \hat{\mathcal{B}}_3 \) such that \( |\hat{\mathcal{B}}_3| \leq M \), and \( f(x, y) = x^T \hat{\mathcal{B}}_3 y, x, y \in \mathbb{Q}^n \).

**Acknowledgements.** The author highly appreciates the anonymous referees’ constructive comments. He also gratefully acknowledges Professor Jiongmin Yong and Professor Xu Zhang for their valuable suggestions and helpful discussions.
References

[1] N. Agram and B. Øksendal, *Malliavin calculus and optimal control of stochastic Volterra equations*, J. Optim. Theory Appl., (2015), DOI 10.1007/s10957-015-0753-5.

[2] K. Arrow, *Optimal capital policy, the cost of capital and myopic decision rules*, Ann. Inst. Stat. Math., 16 (1964), pp. 21–30.

[3] V. Bakke, *A maximum principle for an optimal control problem: with integral constraints*, J. Optim. Theory Appl. 13 (1974), pp. 32–55.

[4] S. Bonaccorsi, F. Confortola and E. Mastrogiacomo, *Stochastic control for stochastic Volterra equations with complete monotone kernels*, SIAM J. Control Optim., 50 (2012), pp. 748–789.

[5] J. Bonnans, X. Dupuis and C. De la Vega, *First and second order optimality conditions for optimal control problems of state constrained integral equations*, J. Optim. Theory Appl. 159 (2013), pp. 1–40.

[6] W. Cochran, J. Lee and J. Potthoff, *Stochastic Volterra equations with singular kernels*, Stochastic Process Appl. 56 (1995), pp. 337–349.

[7] F. De Hoog and R. Weiss, *On the solution of a Volterra integral equation with a weakly singular kernel*, SIAM J. Math. Anal. 4 (1973), pp. 561–573.

[8] A. Dmitruk and N. Osmolovski, *Necessary conditions for a weak minimum in optimal control problems with integral equations subject to state and mixed constraints*, SIAM J. Control Optim. 52 (2014), pp. 3437–3462.

[9] K. Du and Q. Meng, *A maximum principle for optimal control of stochastic evolution equations*, SIAM J. Control Optim. 51 (2013), pp. 4343–4362.

[10] A. Friedman, *Optimal control for hereditary processes*, Arch. Rat. Mech. Anal. 15 (1964), pp. 396–416.

[11] M. Fuhrman, Y. Hu and G. Tessitore, *Stochastic maximum principle for optimal control of SPDEs*, Appl. Math. Optim. 68 (2013), pp. 181–217.

[12] A. Halanay, *Optimal control for systems with time-lag*, SIAM J. Control 6 (1968), pp. 215–234.

[13] R. Hartl, *Optimal dynamic advertising policies for hereditary processes*, J. Optim. Theory Appl. 43 (1984), pp. 51–72.

[14] N. Hritonenko and Y. Yatsenko, *Optimal control of Solow vintage capital model with nonlinear utility*, Optimization, 57 (2008), pp. 581–592.

[15] S. Kou, *Stochastic modeling in nanoscale biophysics: subdiffusion within proteins*, Ann. Appl. Stat. 2 (2008), pp. 501–535.

[16] P. Lin and J. Yong, *Controlled singular Volterra integral equations and Pontryagin maximum principle*, arXiv:1712.05911v1.

[17] Q. Lü and X. Zhang, *General Pontryagin-type stochastic maximum principle and backward stochastic evolution equation in infinite dimensions*, Springer Briefs in Mathematics, 2014.

[18] M. Kamien and E. Muller, *Optimal control with integral state equations*, Rev. Econ. Stud., 43 (1976), pp. 469–473.
[19] B. Øksendal and T. Zhang, *Optimal control with partial information for stochastic Volterra equations*, Int. J. Stoch. Anal., (2010), doi:10.1115/2010/329185

[20] E. Pardoux and P. Protter, *Stochastic Volterra equations with anticipating coefficients*, Ann. Probab., 18 (1990), pp. 1635–1655.

[21] S. Peng, *A general stochastic maximum principle for optimal control problems*, SIAM J. Control Optim., 28 (1990), pp. 966–979.

[22] Y. Shi, T. Wang, and J. Yong, *Optimal control problems of forward-backward stochastic Volterra integral equations*, Math. Control Relat. Fields, 5 (2015), pp. 613–649.

[23] V. Vinokurov, *Optimal control of processes described by integral equations*, I, II, III, Izv. Vyssh. Učebn. Zaved. Matematika 7, 21–33; 8, 16–23; 9, 16–25; (in Russian) English transl. in SIAM J. Control 7 (1967), pp. 324–336, 337–345, 346–355.

[24] T. Wang, *Linear quadratic control problems of stochastic Volterra integral equations*, to appear in ESAIM: Control Optim. Cal. Var. DOI: https://doi.org/10.1051/cocv/2017002.

[25] T. Wang and J. Yong, *Comparison theorems for backward stochastic Volterra integral equations*, Stochastic Process Appl., 125 (2015), pp. 1756–1798.

[26] T. Wang and H. Zhang, *Optimal control problems of forward-backward stochastic Volterra integral equations with closed control regions*, SIAM J. Control Optim. 55 (2017), pp. 2574–2602.

[27] J. Yong, *Backward stochastic Volterra integral equations and some related problems*, Stoch. Process Appl., 116 (2006), pp. 779–795.

[28] J. Yong, *Well-posedness and regularity of backward stochastic Volterra integral equation*, Probab. Theory Relat. Fields, 142 (2008), pp. 21–77.

[29] J. Yong and X. Zhou, *Stochastic Controls: Hamiltonian Systems and HJB Equations*, Springer-Verlag, New York, Berlin, 2000.