On the non-symplectic involutions of the Hilbert square of a K3 surface

S. Boissière, A. Cattaneo, D. G. Markushevich, and A. Sarti

Abstract. We investigate the interplay between the moduli spaces of ample \langle 2 \rangle-polarized IHS manifolds of type $K3^{[2]}$ and of IHS manifolds of type $K3^{[2]}$ with a non-symplectic involution with invariant lattice of rank one. In particular, we describe geometrically some new involutions of the Hilbert square of a K3 surface whose existence was proven in a previous paper of Boissière, Cattaneo, Nieper-Wisskirchen, and Sarti.

Keywords: irreducible holomorphic symplectic manifolds, non-symplectic automorphisms, ample cone.

§ 1. Introduction

By a classical result of Saint-Donat [1], every ample \langle 2 \rangle-polarized complex K3 surface is a double cover of the complex projective plane branched along a smooth sextic curve and it admits a non-symplectic involution in a natural way. The cohomological invariant sublattice is generically isometric to the lattice \langle 2 \rangle of rank one generated by the pullback of the class of a line in the plane. The converse is also true: if a K3 surface admits a non-symplectic involution whose invariant lattice has rank one, then this lattice is isometric to the lattice \langle 2 \rangle and it is generated by an ample class, so the K3 surface can be constructed as a double cover of the plane branched along a smooth sextic curve.

The present paper focuses on a generalization of this result to one deformation class of irreducible holomorphic symplectic manifolds (IHS manifolds). These can be seen as a higher-dimensional generalization of K3 surfaces, and they share several properties with them. These are simply connected manifolds with a unique (up to scalar multiplication) holomorphic 2-form which is everywhere non-degenerate. This implies that their dimension is even and their canonical divisor is trivial. Moreover, their second cohomology with integer coefficients is a lattice for the Beauville–Bogomolov–Fujiki quadratic form. One of the most studied families of IHS manifolds is the $2n$-dimensional Hilbert scheme of $n$ points on a smooth complex projective K3 surface, which has a 20-dimensional moduli space.

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AMS 2010 Mathematics Subject Classification. 14C05, 14J50, 14J28.

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surfaces. In this paper, we focus on ample ⟨2⟩-polarized IHS manifolds of type K3[2], that is, those deformation equivalent to the Hilbert square of a projective K3 surface (see §2). The first main result of this paper is Theorem 3.1, which generalizes Saint-Donat’s result for K3 surfaces.

**Theorem.** (1) Let \((X, D)\) be an ample ⟨2⟩-polarized IHS manifold of type K3[2]. Then \(X\) admits a non-symplectic involution \(\sigma\) whose action on \(H^2(X, \mathbb{Z})\) is the orthogonal symmetry in the class of \(D\) in \(H^2(X, \mathbb{Z})\).

(2) Conversely, let \(X\) be an IHS manifold of type K3[2] with a non-symplectic involution \(\sigma\) whose invariant lattice \(T(\sigma)\) has rank 1. Then \(T(\sigma)\) is generated by the class of an ample divisor \(D\) of square 2 and \(\sigma\) acts on \(H^2(X, \mathbb{Z})\) as the orthogonal symmetry in the class of \(D\).

In §4 we interpret this result as an isomorphism between the moduli space of ample ⟨2⟩-polarized IHS manifolds of type K3[2] introduced by Gritsenko, Hulek and Sankaran [4] and recently reconsidered by Debarre and Macrì [5] on one side, and the moduli space of IHS manifolds of type K3[2] having a non-symplectic involution with invariant lattice of rank one on the other side. Then in §5.3 we show that in this setup, two automorphisms can always be deformed to each other and, in particular, to a Beauville involution or to an O’Grady involution. The major part of these results were already known to experts but to our knowledge a complete proof was missing.

In §6 we present geometric constructions of non-symplectic involutions on the Hilbert square of a K3 surface, with invariant lattice isometric to ⟨2⟩, without deforming it to other IHS manifolds of type K3[2]. In general, this is a difficult problem: such involutions were classified in [6] using the Global Torelli theorem of Markman and Verbitsky [7], but the only geometric example known so far was the Beauville involution (see Definition 5.1). We refer to [8], §3, [9], §4.3, for geometric constructions of non-symplectic involutions. In [6] the authors consider a generic projective K3 surface \(S\) having an ample polarization with square \(2t\) where \(t\) is a positive integer, and they use Torelli theorem to show the existence of non-symplectic involutions on \(S^{[2]}\) for certain values of \(t\) (see §6). The question of constructing geometrically these involutions remained open and it is the second goal of this paper. We consider special K3 surfaces admitting two embeddings as quartics in \(\mathbb{P}^3\) and show how one can use Beauville involutions to construct the non-symplectic involution on the Hilbert scheme. Finally, in §6.2 we use nodal K3 surfaces to give other geometric constructions of Beauville involutions.

Part of the work was done during the 2015 Oberwolfach mini-workshop *Singular curves on K3 surfaces and hyperkähler manifolds*. The authors thank this institution for the stimulating working atmosphere, and the anonymous referee for his/her careful reading and helpful comments and references.

**§2. Preliminary notions**

A lattice \(L\) is a free \(\mathbb{Z}\)-module endowed with an integer-valued non-degenerate symmetric bilinear form. A sublattice \(M \subset L\) is said to be primitive if \(L/M\) is a free \(\mathbb{Z}\)-module.

A compact complex Kähler manifold \(X\) is said to be irreducible holomorphic symplectic (IHS) if it is simply connected and it admits a holomorphic 2-form \(\omega_X\).
everywhere non-degenerate and unique up to scalar multiplication. The existence of such a symplectic form implies immediately that the dimension of $X$ is even. The second cohomology group $H^2(X, \mathbb{Z})$ with integer coefficients is a lattice for the Beauville–Bogomolov–Fujiki [3] quadratic form $q_X$. We write $\langle - , - \rangle_X$ for the associated bilinear form.

The group $\text{Aut}(X)$ of biholomorphic automorphisms of $X$ is discrete. An element $\sigma \in \text{Aut}(X)$ is said to be symplectic if $\sigma^* \omega_X = \omega_X$, and non-symplectic otherwise. In particular, a non-symplectic involution $\sigma$ satisfies $\sigma^* \omega_X = -\omega_X$ and may therefore be referred to as anti-symplectic. The invariant lattice of an automorphism $\sigma \in \text{Aut}(X)$ is the primitive sublattice $T(\sigma) \subset H^2(X, \mathbb{Z})$ consisting of the cohomology classes that are invariant under $\sigma^*$.

An IHS manifold $X$ is called an IHS manifold of type $\text{K}_3[2]$ if it is deformation equivalent to the Hilbert scheme (or Douady space if non-algebraic) of 0-dimensional subschemes of length 2 of a K3 surface. In this case, the lattice $(H^2(X, \mathbb{Z}), q_X)$ is isometric to the lattice

$$L := U^{\oplus 3} \oplus E_8(-1)^{\oplus 2} \oplus \langle -2 \rangle$$

(by convention $U$ is the hyperbolic plane and the root lattice $E_8$ is positive definite).

The monodromy group $\text{Mon}^2(X)$ is the subgroup of $O(H^2(X, \mathbb{Z}))$ generated by the images of all monodromy representations of smooth proper holomorphic families with central fibre $X$. By Markman ([10], Theorem 1.2), if $X$ is of type $\text{K}_3[2]$, then $\text{Mon}^2(X)$ is equal to the subgroup $O^+(H^2(X, \mathbb{Z}))$ of isometries whose real extension preserves the orientation of any positive definite 3-dimensional subspace of $H^2(X, \mathbb{R})$. In this context, a natural orientation of $X$ is $\text{Span}(\mathfrak{R}(\omega_X), \mathfrak{S}(\omega_X), \kappa)$, where $\kappa$ is a Kähler class. More generally, when $X$ and $Y$ are IHS manifolds in the same deformation class, an isometry $\varphi : H^2(X, \mathbb{Z}) \to H^2(Y, \mathbb{Z})$ is said to be orientation-preserving if it respects the orientation of any positive definite 3-dimensional subspaces of $H^2(X, \mathbb{Z})$ and $H^2(Y, \mathbb{Z})$.

An ample $\langle 2 \rangle$-polarized IHS manifold is by definition a pair $(X, D)$ consisting of a projective IHS manifold $X$ and an ample divisor $D$ on $X$ with square 2. It has a canonical orientation given by the positive definite 3-plane $\text{Span}(\mathfrak{R}(\omega_X), \mathfrak{S}(\omega_X), [D])$.

For later use, we recall a Hodge-theoretic version of the Torelli theorem in the special case of IHS manifolds of type $\text{K}_3[2]$.

**Theorem 2.1** ([7], Theorems 1.3, 9.1, 9.5, 9.8). Let $X$ and $Y$ be IHS manifolds of type $\text{K}_3[2]$ and let $\varphi : H^2(X, \mathbb{Z}) \to H^2(Y, \mathbb{Z})$ be an orientation-preserving isometry. Assume that $\varphi$ induces an isomorphism of Hodge structures between $H^2(X, \mathbb{C})$ and $H^2(Y, \mathbb{C})$ and sends a Kähler class of $X$ to a Kähler class of $Y$. Then there is a biregular isomorphism $f : X \to Y$ such that $f^* = \varphi$.

**Remark 2.2.** Observe that in this setup, a Hodge isometry preserving a Kähler class is automatically orientation-preserving since $\omega_X$ is mapped to a complex multiple of $\omega_Y$.

### § 3. Ample polarizations and non-symplectic involutions

This section is inspired by the following classical result of Saint-Donat [1] for IHS manifolds of dimension 2. Consider an ample $\langle 2 \rangle$-polarized K3 surface $(S, D)$,
Then orthogonal symmetry in the class of Theorem 3.1.

\[ \langle \text{is true: any double covering of } D \rangle \]

where \( D \) is an ample divisor with square 2. Then the linear system \(|D|\) is base point free and determines a double covering \( S \rightarrow \mathbb{P}^2 \) branched along a smooth sextic curve. The covering involution \( \sigma \) is non-symplectic and acts on \( H^2(S, \mathbb{Z}) \) as the orthogonal symmetry in the class of \( D \). If \((S, D)\) is generic in the moduli space of ample \( \langle 2 \rangle \)-polarized K3 surfaces, then the invariant sublattice for the action of \( \sigma \) on \( H^2(S, \mathbb{Z}) \) is isometric to the lattice \( \langle 2 \rangle \) generated by the divisor \( D \). The converse is true: any double covering of \( \mathbb{P}^2 \) branched along a smooth sextic curve is an ample \( \langle 2 \rangle \)-polarized K3 surface. We extend this result as follows.

**Theorem 3.1.** (1) Let \((X, D)\) be an ample \( \langle 2 \rangle \)-polarized IHS manifold of type \( K3^{[2]} \). Then \( X \) admits a non-symplectic involution \( \sigma \) whose action on \( H^2(X, \mathbb{Z}) \) is the orthogonal symmetry in the class of \( D \) in \( H^2(X, \mathbb{Z}) \).

(2) Conversely, let \( X \) be an IHS manifold of type \( K3^{[2]} \) with a non-symplectic involution \( \sigma \) whose invariant lattice \( T(\sigma) \) has rank 1. Then \( T(\sigma) \) is generated by the class of an ample divisor \( D \) with square 2 and \( \sigma \) acts on \( H^2(X, \mathbb{Z}) \) as the orthogonal symmetry in the class of \( D \).

**Proof.** (1) Consider the orthogonal symmetry

\[ \varphi : H^2(X, \mathbb{Z}) \rightarrow H^2(X, \mathbb{Z}), \quad v \mapsto \langle v, [D] \rangle_X \cdot [D] - v, \]

whose invariant lattice \( Z[D] \) is isometric to \( \langle 2 \rangle \). We know that \( \varphi \in \text{Mon}^2(X) \) by Corollary 1.8 in [11], but this can also be checked as follows using the characterization of \( \text{Mon}^2(X) \) recalled above. Since \( H^{2,0}(X) = C\omega_X \) is orthogonal to the algebraic class \([D]\), we see that \( \varphi \) acts on \( H^{2,0}(X) \) as multiplication by \((-1)\) and, therefore, \( \varphi \) is a Hodge isometry. Since \( \varphi \) leaves the ample class \([D]\) invariant, it preserves the orientation of the positive definite 3-dimensional subspace \( \text{Span}(\Re(\omega_X), \Im(\omega_X), [D]) \) of \( H^2(X, \mathbb{R}) \). Hence \( \varphi \in \text{Mon}^2(X) \). By Theorem 2.1 there is a \( \sigma \in \text{Aut}(X) \) such that \( \sigma^* = \varphi \). Since the map \( \text{Aut}(X) \rightarrow \text{O}(H^2(X, \mathbb{Z})) \) is injective ([12], Lemma 1.2), \( \sigma \) is an involution.

(2) Looking at the lattice-theoretical classification of non-symplectic involutions acting on IHS manifolds of type \( K3^{[2]} \) ([13], Proposition 8.2), we see that the invariant lattice \( T(\sigma) \) is isometric to the lattice \( \langle 2 \rangle \) generated by the class of a divisor \( D \) with square 2. Since \( X \) admits a non-symplectic automorphism, it is projective; see [2], Proposition 6. Consider an ample class \( \ell \) on \( X \). The class \( \ell + \sigma^*(\ell) \) is ample and \( \sigma \)-invariant, so it is a multiple of the generator \([D]\) of the invariant lattice, whence \( D \) is ample. Using the first part of the theorem, we obtain that \( \sigma \) is the orthogonal symmetry in the class \([D]\). \( \square \)

**Remark 3.2.** One can state very similar results for IHS manifolds of type \( K3^{[n]} \), \( n \geq 3 \); see [14]. We refer to [15], Theorem 1.1, for references and similar results for families with a different invariant lattice. In contrast to the 2-dimensional situation, the full understanding of the linear system \(|D|\) of the ample class with square 2 remains open in this 4-dimensional case; see [6].

**§ 4. Modular interpretation**

We interpret the result above as an isomorphism between two moduli spaces. We fix a primitive embedding

\[ j : \langle 2 \rangle \hookrightarrow L = U^{\oplus 2} \oplus E_8(-1)^{\oplus 2} \oplus \langle -2 \rangle. \]
Such an embedding is unique up to isometries of \( L \); see [6], Proposition 8.2. Write \( \rho \in O(L) \) for the orthogonal symmetry in the ray \( j((2)) \).

By §3 of the paper [16] by Gritsenko, Hulek and Sankaran (see also §3.1 of the paper [5] by Debarre and Macr`ı), there is a quasiprojective irreducible 20-dimensional coarse moduli space \( \mathcal{M}_{(2)} \) parametrizing pairs \((X, \iota)\), where \( X \) is an IHS manifold of type \( \text{K3}[2] \) and \( \iota : (2) \hookrightarrow \text{NS}(X) \) is a primitive embedding, with an open subspace \( \mathcal{M}_{\text{ample}}^{(2)} \) parametrizing pairs \((X, \iota)\) such that \( \iota((2)) \) contains the class of an ample divisor.

By Verbitsky’s results (see [5] and references therein), there is an algebraic period map \( P_{(2)} : \mathcal{M}_{(2)} \to D_{(2)} \) sending each pair \((X, \iota)\) to its period \( H_2(X, \mathbb{Z}) \). This map is an open embedding in the irreducible algebraic variety \( D_{(2)} \).

By Theorems 4.5 and 5.6 of the paper [17] by Boissière, Camere and Sarti, there is a quasiprojective irreducible 20-dimensional coarse moduli space \( \mathcal{M}_{\rho}^{(2)} \) parametrizing triples \((X, \sigma, \iota)\), where \( X \) is an IHS manifold of type \( \text{K3}[2] \), \( \iota : (2) \hookrightarrow \text{NS}(X) \) is a primitive embedding and \( \sigma \in \text{Aut}(X) \) is a non-symplectic involution whose action on \( H^2(X, \mathbb{Z}) \) is conjugate to \( \rho \).

**Corollary 4.1.** The moduli spaces \( \mathcal{M}_{\rho}^{(2)} \) and \( \mathcal{M}_{\text{ample}}^{(2)} \) are isomorphic.

**Proof.** A natural map \( \mathcal{M}_{\rho}^{(2)} \to \mathcal{M}_{\text{ample}}^{(2)} \) is given by Theorem 3.1(2). Conversely, given any \((X, \iota) \in \mathcal{M}_{\text{ample}}^{(2)}\), we write \( D \) for an ample divisor generating \( \iota((2)) \). By Theorem 3.1(1), the pair \((X, D)\) admits a non-symplectic involution \( \sigma \) such that \( \sigma^* \) is the orthogonal symmetry in the ray \( \mathbb{Z}[D] \). Then \( \sigma^* \) is conjugate to \( \rho \). Hence \((X, \sigma, \iota) \in \mathcal{M}_{\rho}^{(2)} \). \( \square \)

**§ 5. Deformation of non-symplectic involutions**

We exhibit two famous families of non-symplectic involutions. The first one, the **Beauville family**, gives a subspace of codimension one in \( \mathcal{M}_{\rho}^{(2)} \). The second one, the **O’Grady family**, is dense in \( \mathcal{M}_{(2)}^{\rho} \).

**5.1. The Beauville family.** Let \( S \) be a projective K3 surface. We recall that the Néron–Severi group of the Hilbert scheme \( S^{[2]} \) of two points on \( S \) admits an orthogonal decomposition

\[ \text{NS}(S^{[2]}) \cong \text{NS}(S) \oplus \mathbb{Z} \delta, \]

where \( 2\delta \) is the class of the exceptional divisor of \( S^{[2]} \), with \( q_{S^{[2]}}(\delta) = -2 \).

**Definition 5.1.** A **Beauville involution** on \( S^{[2]} \) is a non-symplectic involution \( \sigma \) with invariant lattice of rank 1 generated by the class of an ample divisor \( D \) with square 2 which decomposes as \([D] = [H] - \delta\), where \( H \) is a very ample divisor with square 4 on \( S \).

Beauville involutions are geometrically realized as follows. Let \( S \) be a smooth quartic K3 surface in \( \mathbb{P}^3 \) containing no line and let \( |H| \) be the linear system of its hyperplane section. The line \( \ell \) through a subscheme \( \xi \in S^{[2]} \) cuts \( S \) in a residual subscheme of length two providing the involution \( \sigma \); see Beauville’s paper [2] for
details. The construction of such non-symplectic involutions depends on 19 parameters corresponding, as above, to the moduli space of ample \( (4) \)-polarized K3 surfaces.

5.2. The O’Grady family. We briefly recall the construction of double EPW-sextics \[18\]. Let \( V \) be a 6-dimensional complex vector space. We choose an isomorphism \( \text{vol}: \bigwedge^6 V \to \mathbb{C} \), thus making \( \bigwedge^3 V \) a symplectic vector space with the form \( (\alpha, \beta) := \text{vol}(\alpha \wedge \beta) \). The 10-dimensional Lagrangian subspaces

\[
\mathcal{F}_v = \left\{ v \wedge \alpha \middle| \alpha \in \bigwedge^2 V \right\} \cong \bigwedge^2 (V/\mathbb{C}v), \quad v \in V,
\]

are the fibres of the sheaf \( \mathcal{F} = \mathcal{O}_{\mathbb{P}(V)}(-1) \otimes \bigwedge^2 Q \), where \( Q \) is the tautological bundle. A computation of Chern classes yields that \( c_1(\mathcal{F}) = -6c_1(\mathcal{O}_{\mathbb{P}(V)}(1)) \).

Let \( A \subseteq \bigwedge^3 V \) be a Lagrangian subspace. By inclusion and projection, one gets a vector bundle map

\[
\lambda_A: \mathcal{F} \to \bigwedge^3 V \otimes \mathcal{O}_{\mathbb{P}(V)} \to \frac{\bigwedge^3 V}{A} \otimes \mathcal{O}_{\mathbb{P}(V)},
\]

whose degeneracy locus \( Y_A \) is the subscheme of zeros of the global section \( \det(\lambda_A) \) of the sheaf \( \det(\mathcal{F}^v) = \mathcal{O}_{\mathbb{P}(V)}(6) \). If \( Y_A \) is a proper subscheme of \( \mathbb{P}(V) \), then it is a sextic hypersurface called an \textit{EPW-sextic} \[18\], \[19\].

Clearly, \( Y_A = \{ [v] \in \mathbb{P}(V) \mid \dim(\mathcal{F}_v \cap A) \geq 1 \} \). For a generic element \( A \) in the Grassmannian variety of Lagrangian subspaces of \( \mathbb{P}(V) \), the fourfold \( Y_A \) has only double points along the smooth surface \( W_A := \{ [v] \in \mathbb{P}(V) \mid \dim(\mathcal{F}_v \cap A) \geq 2 \} \).

By Theorem 1.1 of the paper \[20\] by O’Grady, in this situation \( Y_A \) admits a smooth double cover \( X_A \to Y_A \) branched along \( W_A \). This cover is an IHS manifold of type \( K3^{[2]} \) called a \textit{double EPW-sextic}. By construction, \( X_A \) comes together with its covering involution, which is non-symplectic and leaves invariant an ample class with square 2. We call it an \textit{O’Grady involution}. This family depends on 20 parameters, corresponding to the dimension of the GIT quotient \( LG(\bigwedge^3 V) / \text{PGL}(V) \).

5.3. Deformation equivalences. Recall that two pairs \( (X_1, f_1) \) and \( (X_2, f_2) \), consisting of IHS manifolds of the same deformation type and their non-symplectic automorphisms, are \textit{deformation equivalent} \([13]\), §4, \[21\], Definition 4.5) if one can find a smooth proper family \( \pi: \mathcal{X} \to \Delta \) of IHS manifolds over a connected smooth analytic space \( \Delta \) along with a fibre-preserving holomorphic automorphism \( F \) which is non-symplectic on each fibre \( \mathcal{X}_t \), \( t \in \Delta \), two points \( t_1, t_2 \in \Delta \) and isomorphisms \( \varphi_i: \mathcal{X}_{t_i} \to X_i \) such that \( \varphi_i \circ F_{t_i} = f_i \circ \varphi_i \) for \( i = 1, 2 \) (we write \( F_t \) for the restriction of \( F \) to the fibre \( \mathcal{X}_t \)).

**Theorem 5.2.** Any two points \( (X, i_X), (Y, i_Y) \in \mathcal{M}_p(2) \) are deformation equivalent.

**Proof.** This follows from a general result of Joumaah \[21\], Theorem 9.10. However, in our situation it is more enlightening to write down a direct argument following the same idea. With every triple \( (X, \sigma, i) \in \mathcal{M}_p(2) \) we associate (as in \[13\], §4) the
20-dimensional local deformation space $\mathcal{X} \rightarrow \text{Def}(X, \sigma, \iota)$ and a holomorphic automorphism $F$ of $\mathcal{X}$ extending $\sigma$ to a non-symplectic involution on each fibre. The disjoint union $\bigsqcup_{(X,\sigma,\iota) \in \mathcal{M}_t^\rho} \text{Def}(X, \sigma, \iota)$ is glued by the equivalence relation given by the period map $P: \mathcal{M}(2) \rightarrow D(2)$. Given two such deformations $(F: \mathcal{X} \rightarrow \mathcal{X}'$ over a base $U$ and $F': \mathcal{X}' \rightarrow \mathcal{X}'$ over $U')$, we see that the restrictions of $F$ and $F'$ over the intersection $U \cap U'$ are equal since the period map is injective: the non-symplectic involution is uniquely determined by its period. So, gluing $\mathcal{X}$ and $\mathcal{X}'$ over $U \cap U'$, we can extend $F$ and $F'$ to a single holomorphic automorphism over $U \cup U'$ and, finally, over $\mathcal{M}(2)$, which is connected. □

The invariant lattice of an automorphism is a topological invariant and, therefore, remains constant under deformations. This yields the following result, which first appeared in Proposition 4.1 of the paper [22] by Ferretti, but the point of view developed here provides a more direct proof.

**Corollary 5.3.** Let $X$ be an IHS manifold of type $\text{K3}^{[2]}$ with a non-symplectic involution $\sigma$. Then the following assertions are equivalent:

1. $\text{rank } T(\sigma) = 1$;
2. $(X, \sigma)$ is deformation equivalent to a Beauville involution;
3. $(X, \sigma)$ is deformation equivalent to an O’Grady involution.

**Proof.** By Theorem 3.1(1), Beauville involutions and O’Grady involutions belong to the moduli space $\mathcal{M}_t^\rho(2)$, where as above $\rho \in O(L)$ denotes the orthogonal symmetry in a certain class with square 2. By Theorem 3.1(2), assertion (2) is equivalent to saying that $(X, \sigma) \in \mathcal{M}_t^\rho(2)$. Hence the three assertions are equivalent by Theorem 5.2. □

§ 6. New geometric constructions of non-symplectic involutions

6.1. Double Beauville involutions. Let $S$ be a complex projective K3 surface with Picard number 1, and let $H$ be a very ample polarization of $S$ with square $H^2 = 2t$, $t \geq 2$. Boissière, Cattaneo, Nieper-Wisskirchen and Sarti [6], Theorem 1.1, used the Global Torelli theorem (Theorem 2.1) to prove that the Hilbert square $S^{[2]}$ admits a non-trivial automorphism if and only if the following arithmetical conditions hold:

- $t$ is not a square;
- Pell’s equation $P_t(-1): x^2 - ty^2 = -1$ admits a solution;
- Pell’s equation $P_{4t}(5): x^2 - 4ty^2 = 5$ has no solution.

In this case, $S^{[2]}$ admits a unique non-trivial automorphism, which is a non-symplectic involution. We write $h$ for the class of $H$ in the splitting $\text{NS}(S^{[2]}) \cong \text{NS}(S) \oplus \mathbb{Z}\delta$ (§ 5.1). Let $(a, b)$ be the minimal positive solution of $P_t(-1)$. The involution acts on $H^2(S^{[2]}, \mathbb{Z})$ as the orthogonal symmetry in the class $D = bh - a\delta$, which is ample of square 2 and, therefore, gives a point in the moduli space $\mathcal{M}_t^\rho(2)$. However, it is hard to produce a geometric realization of this involution when $t \neq 2$. This is the goal of this section.

It is easy to check that the arithmetical assumptions above hold when $t = (2\alpha + 1)^2 + 1$ with $\alpha \geq 1$: the minimal solution of $P_t(-1)$ is $(2\alpha + 1, 1)$ and $P_{4t}(5)$ has no solution modulo 8. We write $\Sigma_\alpha$ for any K3 surface with Picard number 1 and polarization of square $(2\alpha + 1)^2 + 1$. Let $\sigma_\alpha$ be the non-symplectic
involution on $\Sigma^{[2]}_{\alpha}$, and let $L_{\alpha} := \mathbb{Z}h_{1} \oplus \mathbb{Z}h_{2}$ be the lattice of rank two with Gram matrix
\[
\begin{pmatrix}
4 & 4 + 2\alpha \\
4 + 2\alpha & 4
\end{pmatrix}.
\]

**Theorem 6.1.** For any $\alpha \geq 1$, the pair $(\Sigma^{[2]}_{\alpha}, \sigma_{\alpha})$ can be deformed to the pair $(S^{[2]}_{\alpha}, \kappa_{\alpha})$, where $S_{\alpha}$ is a K3 surface with Picard lattice $L_{\alpha}$, and $\kappa_{\alpha} := \sigma_{\alpha}^{\dagger} \sigma_{\alpha}^{2} \sigma_{\alpha}^{1}$ is expressed in terms of two Beauville involutions $\sigma_{\alpha}^{1}, \sigma_{\alpha}^{2}$ on $S^{[2]}_{\alpha}$. This deformation follows a deformation path of polarized K3 surfaces from $\Sigma_{\alpha}$ to $S_{\alpha}$.

**Proof.** The lattice $L_{\alpha}$ is even and hyperbolic of rank 2. Hence, by Corollary 2.9 in Morrison’s paper [23], it admits a primitive embedding in the K3 lattice and there is a projective K3 surface $S_{\alpha}$ such that $\text{NS}(S_{\alpha}) \cong L_{\alpha}$. For any $d = xh_{1} + yh_{2}$, $x, y \in \mathbb{Z}$, we have
\[
d^{2} = 4(x^{2} + (2 + \alpha)xy + y^{2}) = (4 + \alpha)(x + y)^{2} - \alpha(x - y)^{2}.
\]
Hence there are no $(-2)$-curves on $S_{\alpha}$. It follows from the Nakai–Moishezon criterion for ampleness ([24], Proposition 1.4) that the ample cone of $S_{\alpha}$ coincides with its positive cone. Recall that the positive cone is the connected component of the cone $\{d \in L_{\alpha} \otimes \mathbb{R} \mid d^{2} > 0\}$ containing the Kähler classes. Since $h_{1}^{2} > 0$ and $h_{2}^{2} > 0$, this cone contains the first quadrant $\mathbb{Z}_{>0}h_{1} + \mathbb{Z}_{>0}h_{2}$ and its image under the symmetry with respect to the origin. We may assume without loss of generality that the Kähler cone intersects the first quadrant. It follows that the positive cone is given by the inequalities
\[
y > \frac{1 + \epsilon \beta}{\epsilon \beta - 1} x, \quad \text{where} \quad \epsilon = \pm 1 \quad \text{and} \quad \beta = \sqrt{\frac{\alpha}{4 + \alpha}}.
\]

**Lemma 6.2.** The classes $h_{1}, h_{2}, (2 + 2\alpha)h_{1} - h_{2}$ and $(2 + 2\alpha)h_{2} - h_{1}$ are very ample.

**Proof.** These four classes are clearly in the ample cone. Their associated linear systems have no base components. Indeed, by Theorem (d) in §3.8 of [25], if any of these linear systems had base components, then it would decompose as $aE + \Gamma$, where $a$ is an integer, $|E|$ is a free pencil and $\Gamma$ is a $(-2)$-curve such that $E \cdot \Gamma = 1$. But $S_{\alpha}$ contains no $(-2)$-curves. These linear systems thus define regular maps
\[
\varphi_{|h_{1}|} : S_{\alpha} \rightarrow \mathbb{P}^{3}, \quad \varphi_{|(2 + 2\alpha)h_{1} - h_{2}|}, \varphi_{|(2 + 2\alpha)h_{2} - h_{1}|} : S_{\alpha} \rightarrow \mathbb{P}^{1+t}.
\]
Let $d$ be any of the four primitive divisors in issue. We show that $d$ is not hyperelliptic by using Saint-Donat’s criterion for determining hyperelliptic divisors ([1], Theorem 5.2). Indeed, it is easy to check by reduction modulo 2 that there is no class $E = xh_{1} + yh_{2}$ such that $E^{2} = 0$ and $E \cdot d = 2$. It follows that the regular map $\varphi_{|d|}$ is birational onto its image, but since $S_{\alpha}$ contains no $(-2)$-curves, it is an embedding (see [1], (4.2) and §6: any contracted curve has square $-2$ by the genus formula). □

The maps $\varphi_{|h_{1}|}$ embed $S_{\alpha}$ in $\mathbb{P}^{3}$ in two different ways as a quartic without lines. Therefore $S^{[2]}_{\alpha}$ has two different Beauville involutions $\sigma_{\alpha}^{i}, i = 1, 2$. The Néron–Severi
orthogonal symmetry in the class $\kappa$ lattice of $S_\kappa$ since the class $\kappa$ and deform to a Beauville involution. Observe that orthogonal symmetry in the class $\phi$ divisor with square $\Sigma$ In this section we give a new geometric construction $6.2. Nodal K3 surfaces.

$6.2.1. K3$ surface with one node. Let $\tilde{S}$ be a general nodal K3 surface in $\mathbb{P}^4$ (thus $\tilde{S}$ is the complete intersection of a quadric and a cubic) and $\beta: S \to \tilde{S}$ its minimal K3 resolution. We write $\tilde{H}$ for the hyperplane section on $\tilde{S}$, put $H := \beta^* \tilde{H}$ and denote the exceptional $(-2)$-curve by $\varepsilon$. Then we have

$$\text{NS}(S) = \mathbb{Z} \tilde{H} \oplus \mathbb{Z} \varepsilon \cong \langle 6 \rangle \oplus \langle -2 \rangle.$$ 

We define a birational involution $\sigma$ on $S^{[2]}$ as follows. Denote the node of $\tilde{S}$ by $p$ and take two general points $q_1, q_2 \in \tilde{S}$ distinct from $p$. The family of hyperplanes through $p$, $q_1$, $q_2$ is a pencil since the general points $q_1$, $q_2$ impose independent conditions on the linear system $|H|$, which is 4-dimensional. Let $H_1, H_2$ be any two generators of this pencil. Then $H_1$ cuts $\tilde{S}$ along a curve of degree six with a singularity at $p$, and $H_2$ cuts this curve at two more points $q_3, q_4$ (generically distinct). So $\{p, q_1, q_2, q_3, q_4\}$ is the base locus of the pencil. We thus define a birational involution $\sigma$ on $S^{[2]}$ by sending $\{q_1, q_2\}$ to $\{q_3, q_4\}$.

**Proposition 6.4.** The involution $\sigma$ is a Beauville involution on $S^{[2]}$.

**Proof.** The linear system $|H - \varepsilon|$ on $S$ has no base components since $H$ is very ample. As in the proof of Lemma 6.2, we see that the divisor $H - \varepsilon$ is not hyperelliptic and, therefore, the regular map $\varphi_{|H - \varepsilon|}: S \to \mathbb{P}^3$ is birational onto its image. This map contracts no $(-2)$-curves, otherwise there would exist a class $\alpha H + \beta \varepsilon$ such that $(\alpha H + \beta \varepsilon)(H - \varepsilon) = \beta + 3\alpha = 0$ and $(\alpha H + \beta \varepsilon)^2 = 6\alpha^2 - 2\beta^2 = -2$, which...
is impossible. Hence $\varphi_{|H - \varepsilon|}$ embeds $S$ in $\mathbb{P}^3$ as a quartic $\Sigma$ containing no lines, otherwise a $(-2)$-class $\alpha H + \beta \varepsilon$ would be sent to a line and we would get

$$1 = (\alpha H + \beta \varepsilon)(H - \varepsilon) = 6\alpha + 2\beta,$$

which has clearly no solution.

The hyperplane sections on $\tilde{S}$ passing through the node $p$ correspond to divisors in the system $|H - \varepsilon|$ on $S$, so the hyperplanes $H_1$, $H_2$ are sent to hyperplane sections $h_1$, $h_2$ of $\Sigma$ which contain the images of the points $q_1$, $q_2$, $q_3$, $q_4$. These four points thus lie on the line $h_1 \cap h_2$. This shows that the birational involution $\sigma$ on $S^{[2]}$ is nothing else than the Beauville involution on $\Sigma^{[2]}$: since the birational map $\sigma$ defines the action on $H^2(S^{[2]}, \mathbb{Z})$ ([27], Lemma 2.6) and since the representation $\text{Aut}(S^{[2]}) \to \text{O}(H^2(S^{[2]}, \mathbb{Z}))$ is faithful, $\sigma$ extends uniquely to a biregular automorphism, which is necessarily the Beauville involution. □

**Remark 6.5.** It follows that the invariant lattice of $\sigma$ on $S^{[2]}$ is generated by the ample divisor $(H - \varepsilon) - \delta$ with square 2. By Bini [28] we have $\text{Aut}(S) = \{\text{id}\}$, but $\text{Aut}(S^{[2]})$ contains at least one involution. We refer to [6] for a similar property on K3 surfaces of Picard rank one.

### 6.3. K3 surfaces with several nodes

Consider the even hyperbolic lattice

$$R_k = \langle 4 + 2k \rangle \oplus \bigoplus_{i=1}^{k} \langle -2 \rangle, \quad k \leq 10.$$

By [23], Corollary 2.9, Remark 2.11, there is a K3 surface whose Néron–Severi lattice is isomorphic to $R_k$. For $k = 1$ we recover the K3 surface with one node described above, and for $k = 2$ we have a K3 surface with two nodes in $\mathbb{P}^5$ (a complete intersection of three quadrics).

The condition on $k$ comes from the observation that if $k > 10$, $R_k$ cannot be the Néron–Severi group of a K3 surface. Indeed, in this situation the Néron–Severi lattice has rank at least 12 and discriminant group of length $k + 1$, so that the rank of the transcendental lattice is at most 10 with a discriminant group again of length $k + 1$, which is impossible. If $k > 10$, the lattice $R_k$ can only be a sublattice of the Néron–Severi group (see [29], Theorem 2.7, for some examples when $k = 16$).

Write $H_k$ for the generator of the summand $\langle 4 + 2k \rangle$, and $\varepsilon_i$ for the generator of the $i$th copy of the lattice $\langle -2 \rangle$, $i = 1, \ldots, k$. The lattice $R_k$ is the Néron–Severi group of the K3 surface $S_k$ obtained as the minimal resolution of a K3 surface $\widetilde{S}_k$ embedded in $\mathbb{P}^{k+3}$ as a surface of degree $4 + 2k$ with $k$ singularities of type $A_1$ at points $p_1, \ldots, p_k$. The curves $\varepsilon_i$, $i = 1, \ldots, k$, correspond to the exceptional divisors obtained by blowing up the singular points. We define a birational involution on $S_k^{[2]}$ as above. Take two general points $q_1, q_2 \in \widetilde{S}_k$ distinct from $p_1, \ldots, p_k$. The family of hyperplanes through $p_1, \ldots, p_k, q_1, q_2$ has dimension $(k + 3) - (k + 2) = 1$. Let $H_1, H_2$ be any two generators of this pencil. Then $H_1$ cuts $\widetilde{S}_k$ along a curve of degree $4 + 2k$ with nodes at the points $p_1, \ldots, p_k$. The divisor $H_2$ cuts this curve twice at the singular points $p_1, \ldots, p_k$ and once at $q_1, q_2$. Hence it cuts the curve at two other points $q_3, q_4$ (generically distinct). We thus define a birational involution $\sigma$ on $S_k^{[2]}$ by sending $\{q_1, q_2\}$ to $\{q_3, q_4\}$. 

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Proposition 6.6. The involution $\sigma$ is a Beauville involution on $S_k^{[2]}$.

Proof. The proof is similar to that of Proposition 6.4. The divisor $H_k - \sum_{i=1}^{k} \varepsilon_i$ has square 4, its associated linear system has no base components, it is not hyperelliptic and it contracts no $(-2)$-curves. So it embeds $S_k$ in $\mathbb{P}^3$ as a quartic $\Sigma_k$ containing no lines: since any two divisors of the lattice $R_k$ have even intersection number, no $(-2)$-curve is sent to a line. □

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**Samuel Boissière**  
Université de Poitiers, Laboratoire de Mathématiques et Applications, France  
*E-mail*: samuel.boissiere@math.univ-poitiers.fr  
Received 8/JUN/2018  
22/OCT/2018

**Andrea Cattaneo**  
Institut Camille Jordan, Université Claude Bernard  
Lyon 1, France  
*E-mail*: cattaneo@math.univ-lyon1.fr

**Dimitri G. Markushevich**  
Université de Lille, Laboratoire Paul Painlevé, France  
*E-mail*: dimitri.markouchevitch@univ-lille.fr

**Alessandra Sarti**  
Université de Poitiers, Laboratoire de Mathématiques et Applications, France  
*E-mail*: sarti@math.univ-poitiers.fr