Binomial tribonacci sums

Kunle Adegoke1, Robert Frontczak2, Taras Goy3

1Department of Physics and Engineering Physics, Obafemi Awolowo University, Ile-Ife, Nigeria
2Landesbank Baden-Württemberg, Stuttgart, Germany
3Faculty of Mathematics and Computer Science, Vasyl Stefanyk Precarpathian National University, Ivano-Frankivske, Ukraine

(Received: 20 August 2021. Received in revised form: 28 September 2021. Accepted: 6 October 2021. Published online: 13 October 2021.)

© 2021 the authors. This is an open access article under the CC BY (International 4.0) license (www.creativecommons.org/licenses/by/4.0/).

Abstract

We derive expressions for several binomials sums involving a generalized tribonacci sequence. We also study double binomial sums involving this sequence. Several explicit examples involving tribonacci and tribonacci–Lucas numbers are stated to highlight the results.

Keywords: generalized tribonacci sequence; tribonacci number; tribonacci–Lucas number; binomial transform.

2020 Mathematics Subject Classification: 11B37, 11B39.

1. Introduction

There is a dearth of tribonacci summation identities including binomial coefficients. Our goal in this paper is to derive several new binomial tribonacci sums such as

\[
\sum_{k=0}^{n} \binom{n}{k} G_{4k+s} = 2^n G_{3n+s},
\]

\[
\sum_{k=1}^{n} \binom{n}{k} G_{4k+s} = \sum_{m=1}^{n} \frac{2^m G_{3m+s} - G_s}{m},
\]

\[
\frac{3n}{2k} G_{2k+s} = 2^{n-1}(G_{4n+s} + (-1)^n G_{s-2n}),
\]

\[
\sum_{k=0}^{n} (-1)^{n-k} \binom{n+k}{2k} G_{4n+2k+s} = \frac{G_{8n+s} + G_s}{2n},
\]

and double binomial tribonacci summation identities such as

\[
\sum_{k=0}^{n} \binom{k}{p} (-1)^{k+p} \binom{n}{k} G_{5k+p+s} = 3^n G_{3n+s},
\]

\[
\sum_{k=0}^{n} \sum_{p=0}^{k} \binom{k}{p} \frac{G_{k+5p+s}}{3^k} = \left(\frac{7}{3}\right)^n G_{3n+s}.
\]

In the above identities, \(n\) denotes a non-negative integer, \(s\) and \(p\) are arbitrary integers and \(G_n\) is a generalized tribonacci number.

The generalized tribonacci sequence \(G_n = G_n(c_0, c_1, c_2), n \geq 0,\) is defined recursively by

\[
G_n = G_{n-1} + G_{n-2} + G_{n-3}, \quad n \geq 3,
\]

with initial values \(G_0 = c_0, G_1 = c_1, G_2 = c_2\) not all being zero. Extension of the definition of \(G_n\) to negative subscripts is provided by writing the recurrence relation as

\[
G_{-n} = G_{-(n-3)} - G_{-(n-2)} - G_{-(n-1)},
\]

so that \(G_n\) is defined for all integers \(n\).

The most prominent representatives of \(G_n\) and widely studied in the literature are \(G_n(0, 1, 1) = T_n\) the sequence of tribonacci numbers and \(G_n(3, 1, 3) = K_n\) the sequence of tribonacci–Lucas numbers (sequences A000073 and A001644 in [19], respectively).

The first few tribonacci numbers and tribonacci–Lucas numbers with positive and negative subscripts are given in Table 1.

*Corresponding author (adegoke00@gmail.com).
Properties of (generalized) tribonacci sequences were investigated in the recent articles [1–4, 7, 8, 10, 12–18, 20, 21], among others. For instance, Janjić [16] found the remarkable combinatorial identity

\[ T_n = 1 + \sum_{k=1}^{n-1} \sum_{i=0}^{k} \sum_{j=1}^{i-1} \binom{k}{i} \binom{j-1}{n-k-2j}. \]

A generalized tribonacci number \( G_n(c_0, c_1, c_2) \) is given by the Binet formula

\[ G_n(c_0, c_1, c_2) = A\alpha^n + B\beta^n + C\gamma^n, \]

where \( \alpha, \beta \) and \( \gamma \) are the distinct roots of the equation \( x^3 - x^2 - x - 1 = 0 \). The coefficients \( A, B \) and \( C \) depend on the initial values and are determined by the system

\[
\begin{align*}
A + B + C &= c_0, \\
A\alpha + B\beta + C\gamma &= c_1, \\
A\alpha^2 + B\beta^2 + C\gamma^2 &= c_2.
\end{align*}
\]

The Binet formulas for \( T_n \) and \( K_n \) are

\[ T_n = \frac{\alpha^{n+1}}{(\alpha - \beta)(\alpha - \gamma)} + \frac{\beta^{n+1}}{(\beta - \alpha)(\beta - \gamma)} + \frac{\gamma^{n+1}}{(\gamma - \alpha)(\gamma - \beta)} \]

and

\[ K_n = \alpha^n + \beta^n + \gamma^n, \]

where

\[ \alpha = 1 + \frac{\sqrt{19} + 3\sqrt{33} + \sqrt{19 - 3\sqrt{33}}}{3}, \quad \beta = 1 + \omega\frac{\sqrt{19} + 3\sqrt{33} + \omega^2\sqrt{19 - 3\sqrt{33}}}{3}, \]

\[ \gamma = 1 + \omega^2\frac{\sqrt{19} + 3\sqrt{33} + \omega\sqrt{19 - 3\sqrt{33}}}{3}, \]

and \( \omega = \frac{-1+i\sqrt{3}}{2} \) is a primitive cube root of unity.

Tribonacci and tribonacci-Lucas numbers with negative indices can be accessed directly, using the following result.

**Lemma 1.1.** For integer \( n \),

\[ T_{-n} = T_{n-1}^2 - T_{n-2}T_n, \]

\[ K_{-n} = \frac{K_n^2 - K_{2n}}{2}. \]

For a proof of (2), see, for example, [8, Theorem 2.2]. The proof of (3) one can find in [6, Formula (9)].

In this article, we study binomial and double binomial sums with terms being a generalized tribonacci sequence. We derive closed forms for several such sums. We also prove a general binomial identity characterizing \( G_{an+b} \) for \( a \geq 1 \) and \( b \) an arbitrary integer.

## 2. Some auxiliary results

In this section we present some results that we will use in the sequel.

**Lemma 2.1.** Let \( \phi \in \{\alpha, \beta, \gamma\} \). Then, for all \( n \geq 0 \), we have

\[ \phi^{n+1} = \phi^2T_n + \phi(T_{n-1} + T_{n-2}) + T_{n-1}. \]

For a proof of (4), see [7, Formula (6)].
Lemma 2.2. We have

\begin{align}
(\alpha - 1)^3 &= 2\alpha^2, \\
(\alpha + 1)^3 &= 2\alpha^4, \\
(\alpha^2 + 1)^3 &= 4\alpha^5, \\
(\alpha^3 - 1)^3 &= 2\alpha^7, \\
\alpha^4 + 1 &= 2\alpha^2, \\
\end{align}

with identical relations for \(\beta\) and \(\gamma\).

Proof. Since

\begin{equation}
1 + \alpha + \alpha^2 = \alpha^3,
\end{equation}

we have

\begin{equation}
\frac{\alpha^2 + 1}{\alpha^2 - 1} = \alpha
\end{equation}

and

\begin{equation}
\frac{\alpha + 1}{\alpha - 1} = \alpha^2.
\end{equation}

Addition of (11) and (12) gives

\begin{equation}
(\alpha + 1)^2(\alpha - 1) = 2\alpha^2,
\end{equation}

while their subtraction produces

\begin{equation}
(\alpha - 1)^2(\alpha + 1) = 2.
\end{equation}

Eliminating \(\alpha + 1\) between (13) and (14) gives identity (5), while the elimination of \(\alpha - 1\) yields (6).

Cubing identity \(\alpha^2 + 1 = 2\alpha\alpha - 1\) and making use of (5) gives (7). Subtracting (10) from \(\alpha + \alpha^2 + \alpha^3 = \alpha^4\) produces identity (8). Identity (9) follows from \(\alpha^4 + 1 = \alpha^4 + \alpha^3 + \alpha^2 + \alpha = (\alpha^2 + 1)(\alpha + 1)\) with the help of (6) and (7). \(\square\)

Lemma 2.3. Let \(a, b, c\) and \(d\) be rational numbers and \(\lambda\) an irrational number. Then

\[ a + \lambda b = c + \lambda d \iff a = c, \ b = d. \]

3. Identities from the binomial theorem and binomial transform

The next lemma will be the key ingredient to derive many results in this paper. For a proof and some applications to Horadam numbers, see [11].

Lemma 3.1. Let \(n\) and \(j\) be integers with \(0 \leq j \leq n\). Then, for each \(x, y \in \mathbb{C}\), we have

\[ \sum_{k=j}^{n} (-1)^{j-k} \binom{n}{k} y^{k} x^{n-k} = \binom{n}{j} y^{j} (x \pm y)^{n-j}. \]

We also mention the standard fact about sequences and their binomial transforms [5]: Let \((a_n)_{n \geq 0}\) be a sequence of numbers and \((b_n)_{n \geq 0}\) its binomial transform. Then we have the following relations:

\[ b_n = \sum_{k=0}^{n} \binom{n}{k} a_k \iff a_n = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} b_k. \]

Furthermore, if \(a_0 = 0\) (so that \(b_0 = 0\) too) the binomial pair exhibits the following properties:

\[ \sum_{k=1}^{n} \binom{n}{k} \frac{a_k}{k} = \frac{1}{m} \sum_{m=1}^{n} b_m, \]

and

\[ \sum_{k=1}^{n} \binom{n}{k} \frac{a_k}{k + 1} = \frac{1}{n + 1} \sum_{m=1}^{n} b_m. \]

Theorem 3.1. Let \(j\) and \(s\) be integers such that \(s\) is arbitrary and \(j \geq 0\). Then

\[ \sum_{k=j}^{n} \binom{k}{j} \binom{n}{k} G_{4k+s} = \binom{n}{j} 2^{n-j} G_{3n+j+s}. \]
Proof. Use identity (9) in Lemma 3.1 with \( x = 1 \) and \( y = \alpha^4 \), taking note of Lemma 2.3.

**Corollary 3.1.** For \( n \) a non-negative integer and \( s \) any integer,

\[
\sum_{k=0}^{n} \binom{n}{k} G_{4k+s} = 2^n G_{3n+s},
\]

(19)

\[
\sum_{k=0}^{n} \binom{n}{k} (-2)^k G_{4k+s} = (-1)^n G_{4n+s},
\]

(20)

\[
\sum_{k=1}^{n} \binom{n}{k} \frac{G_{4k+s}}{k} = \sum_{m=1}^{n} \frac{2^m G_{3m+s} - G_s}{m}
\]

(21)

and

\[
\sum_{k=1}^{n} \binom{n}{k} \frac{G_{4k+s}}{k+1} = \frac{1}{n+1} \left( \sum_{m=1}^{n} 2^m G_{3m+s} - nG_s \right).
\]

(22)

Proof. To obtain (19) set \( j = 0 \) in (18). Identities (20), (21) and (22) follow form (15), (16) and (17), respectively.

From (19) and (20) we immediately obtain the following binomial tribonacci and tribonacci–Lucas relations.

**Corollary 3.2.** For \( n \geq 0 \),

\[
\sum_{k=0}^{n} \binom{n}{k} T_{4k} = 2^n T_{3n}, \quad \sum_{k=0}^{n} \binom{n}{k} K_{4k} = 2^n K_{3n},
\]

\[
\sum_{k=0}^{n} \binom{n}{k} T_{4k-3n+1} = 2^n, \quad \sum_{k=0}^{n} \binom{n}{k} K_{4k-3n+1} = 2^n,
\]

\[
\sum_{k=0}^{n} \binom{n}{k} T_{4k-3n} = 0, \quad \sum_{k=0}^{n} \binom{n}{k} K_{4k-3n} = 3 \cdot 2^n.
\]

**Theorem 3.2.** For non-negative integer \( n \), any integer \( s \), we have

\[
\sum_{k=0}^{3n} \delta^k \binom{3n}{k} G_{pk+s} = \delta^n 2^n G_{rn+s},
\]

where the values of \( \delta, p, q, r \) as given in each column in Table 2.

| \( \delta \) | \( -1 \) | \( 1 \) | \( 1 \) | \( -1 \) |
| \---|---|---|---|---|
| \( p \) | \( 1 \) | \( 1 \) | \( 2 \) | \( 3 \) |
| \( q \) | \( 1 \) | \( 1 \) | \( 2 \) | \( 1 \) |
| \( r \) | \( -2 \) | \( 4 \) | \( 5 \) | \( 7 \) |

Table 2: Values of \( \delta, p, q, r \) from Theorem 3.2.

Proof. Each of the identities (5)–(8) can be written as \( (\alpha^p + \delta)^3 = 2^q \alpha^r \), where the values of \( \delta, p, q, r \) in each case are as given in each column in Table 2. The identity of the theorem then follows from the binomial theorem and Lemma 2.3.

**Lemma 3.2.** For non-negative integer \( n \) and real or complex \( z \),

\[
2 \sum_{k=0}^{[3n/2]} \binom{3n}{2k} z^{2k} = (1 + z)^{3n} + (1 - z)^{3n},
\]

\[
2 \sum_{k=1}^{[3n/2]} \binom{3n}{2k-1} z^{2k-1} = (1 + z)^{3n} - (1 - z)^{3n}.
\]

**Theorem 3.3.** For non-negative integer \( n \) and any integer \( s \),

\[
\sum_{k=0}^{[3n/2]} \binom{3n}{2k} G_{2k+s} = 2^{n-1} (G_{4n+s} + (-1)^n G_{s-2n}),
\]

\[
\sum_{k=1}^{[3n/2]} \binom{3n}{2k-1} G_{2k+s-1} = 2^{n-1} (G_{4n+s} - (-1)^n G_{s-2n}).
\]
Proof. Set $z = \alpha$ in Lemma 3.2, make use of identities (5) and (6), noting Lemma 2.3 with $\lambda = \alpha$.

Setting $s = 0$ in Theorem 3.3, we immediately obtain the following.

Corollary 3.3. For non-negative integer $n$,

\[
\sum_{k=0}^{\lfloor 3n/2 \rfloor} \binom{3n}{2k} G_{2k} = 2^{n-1}(G_{4n} + (-1)^n G_{-2n}),
\]

\[
\sum_{k=1}^{\lfloor 3n/2 \rfloor} \binom{3n}{2k-1} G_{2k-1} = 2^{n-1}(G_{4n} - (-1)^n G_{-2n}).
\]

As special cases of formulas above we have:

\[
\sum_{k=0}^{\lfloor 3n/2 \rfloor} \binom{3n}{2k} T_{2k} = 2^{n-1} (T_{4n} + (-1)^n (T_{2n-1}^2 - T_{2n-2}T_{2n})),
\]

\[
\sum_{k=1}^{\lfloor 3n/2 \rfloor} \binom{3n}{2k-1} T_{2k-1} = 2^{n-1} (T_{4n} - (-1)^n (T_{2n-1}^2 - T_{2n-2}T_{2n}))
\]

and

\[
\sum_{k=0}^{\lfloor 3n/2 \rfloor} \binom{3n}{2k} K_{2k} = 2^{n-2} (2K_{4n} + (-1)^n (K_{2n}^2 - K_{4n})),
\]

\[
\sum_{k=1}^{\lfloor 3n/2 \rfloor} \binom{3n}{2k-1} K_{2k-1} = 2^{n-2} (2K_{4n} - (-1)^n (K_{2n}^2 - K_{4n})).
\]

Theorem 3.4. For non-negative integer $n$ and any integer $s$,

\[
\sum_{k=0}^{\lfloor 3n/2 \rfloor} \binom{3n}{2k} G_{4k+s} = 2^{2n-1}(G_{5n+s} + (-1)^n G_{2n+s}),
\]

\[
\sum_{k=1}^{\lfloor 3n/2 \rfloor} \binom{3n}{2k-1} G_{4k+s-2} = 2^{2n-1}(G_{5n+s} - (-1)^n G_{2n+s}).
\]

Proof. Combining (5) with (6) yields

\[(\alpha^4 - 1)^3 = 4n^2.\] (23)

Now set $z = \alpha^2$ in Lemma 3.2 and make use of identities (7) and (23), noting Lemma 2.3 with $\lambda = \alpha$.

Theorem 3.5. For non-negative integer $n$ and any integer $s$,

\[
\sum_{k=0}^{\lfloor 3n/2 \rfloor} \binom{3n}{2k} G_{6k+s} = 2^{3n-1}(G_{9n+s} + (-2)^n G_{7n+s}),
\]

\[
\sum_{k=1}^{\lfloor 3n/2 \rfloor} \binom{3n}{2k-1} G_{6k+s-4} = 2^{3n-1}(G_{9n+s} - (-2)^n G_{7n+s}).
\]

Proof. Combining (7) and (23) we have

\[(\alpha^4 - 1)^3 = 16\alpha^2.\] (24)

Set $z = \alpha^4$ in Lemma 3.2 and make use of identities (9) and (24), noting Lemma 2.3 with $\lambda = \alpha$. 

\[\square\]
4. Identities from the Waring formulas

Our next result provides two combinatorial identities for generalized tribonacci numbers involving binomial coefficients.

**Lemma 4.1.** The following identities hold for \( n \geq 0 \) and real or complex \( x \) and \( y \):

\[
\sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \binom{n-k}{k} (xy)^k (x+y)^{n-2k} = \frac{x^{n+1} - y^{n+1}}{x-y}
\]

(25)

and

\[
\sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \binom{n-k}{k} \frac{n}{n-k} (xy)^k (x+y)^{n-2k} = x^n + y^n.
\]

(26)

Formulas (25) and (26) are well-known in combinatorics and called Waring (sometimes Girard-Waring) formulas. The proof of these formulas can be found, for example, in [9].

**Theorem 4.1.** Let \( n \) be a non-negative integer and \( s \) any integer. Then

\[
\sum_{k=0}^{\lfloor n/2 \rfloor} \left(-\frac{1}{4}\right)^k \binom{n-k}{k} (G_{3n-2k+s+4} - G_{3n-2k+s}) = \frac{G_{4n+s+4} - G_s}{2^n}
\]

and

\[
\sum_{k=0}^{\lfloor n/2 \rfloor} \left(-\frac{1}{4}\right)^k \binom{n-k}{k} \frac{G_{3n-2k+s}}{n-k} = \frac{G_{4n+s} + G_s}{2^n n}.
\]

**Proof.** Set \((x,y) = (1,\alpha^4)\) in (25) and (26), respectively, Lemma 4.1 and use identity (8) and Lemma 2.3. \(\Box\)

**Corollary 4.1.** For \( n \geq 0 \),

\[
\sum_{k=0}^{\lfloor n/2 \rfloor} \left(-\frac{1}{4}\right)^k \binom{n-k}{k} (G_{n-2k+4} - G_{n-2k}) = \frac{G_{2n+4} - G_{-2n}}{2^n},
\]

and

\[
\sum_{k=0}^{\lfloor n/2 \rfloor} \left(-\frac{1}{4}\right)^k \binom{n-k}{k} \frac{G_{n-2k}}{n-k} = \frac{G_{2n+4} + G_{-2n}}{n2^n}.
\]

In particular,

\[
\sum_{k=0}^{\lfloor n/2 \rfloor} \left(-\frac{1}{4}\right)^k \binom{n-k}{k} (T_{n-2k+4} - T_{n-2k}) = \frac{T_{2n+4} - T_{2n-1} + T_{2n-2} T_{2n}}{2^n},
\]

\[
\sum_{k=0}^{\lfloor n/2 \rfloor} \left(-\frac{1}{4}\right)^k \binom{n-k}{k} (K_{n-2k+4} - K_{n-2k}) = \frac{2K_{2n+4} + K_{2n}^2 + K_{4n}}{n2^n + 1}
\]

and

\[
\sum_{k=0}^{\lfloor n/2 \rfloor} \left(-\frac{1}{4}\right)^k \binom{n-k}{k} \frac{T_{n-2k}}{n-k} = \frac{T_{2n-1} + T_{2n} (1 - T_{2n-2})}{n2^n},
\]

\[
\sum_{k=0}^{\lfloor n/2 \rfloor} \left(-\frac{1}{4}\right)^k \binom{n-k}{k} \frac{K_{n-2k}}{n-k} = \frac{K_{2n}^2 + 2K_{2n} - K_{4n}}{n2^{n+1}}.
\]

5. Double binomial tribonacci sums

**Theorem 5.1.** Let \( n, j \) and \( s \) be integers with \( s \) arbitrary and \( j \geq 0 \). Then,

\[
\sum_{p=0}^{n-j} (-1)^{j-p} \binom{n-j}{p} G_{5k+p+s} = 3^{n-j} \binom{n}{j} \sum_{p=0}^{j} (-1)^{j-p} \binom{j}{p} G_{3n+2j+p+s}.
\]

(27)

**Proof.** The identity can be derived from Lemma 3.1 using \( 3\phi^3 = \phi^6 - \phi^5 + 1 \). \(\Box\)

**Corollary 5.1.** Let \( n \) and \( s \) be integers. Then,

\[
\sum_{k=0}^{n} (-1)^{k-p} \binom{n}{k} p G_{5k+p+s} = 3^{n} G_{3n+s},
\]

\[
\sum_{k=1}^{n} (-1)^{k-p} \binom{n}{k} p k G_{5k+p+s} = n3^{n-1} (G_{3n+s} + G_{3n+s+1}).
\]
Proof. Set \( j = 0 \) and \( j = 1 \) in (27), respectively.

**Theorem 5.2.** Let \( j \) and \( s \) be integers with \( s \) arbitrary and \( j \geq 0 \). Then

\[
\sum_{k=0}^{n} \sum_{p=0}^{k} \binom{n}{k} \binom{k}{p} \frac{G_{k+4p+s}}{2^k} = 2^{2n-j} \sum_{m=0}^{j} \binom{j}{m} G_{3n-2j+4m+s},
\]

(28)

\[
\sum_{k=0}^{n} \sum_{p=0}^{k} \binom{n}{k} \binom{k}{p} \frac{G_{k+5p+s}}{3^k} = \frac{7}{3} \sum_{m=0}^{n} \binom{j}{m} G_{3n-2j+5m+s}.
\]

(29)

**Proof.** Use Lemma 3.1 in conjunction with \( 4\phi^3 = \phi^5 + \phi + 2 \) and \( 7\phi^3 = \phi^6 + \phi + 3 \), respectively.

**Corollary 5.2.** Let \( n \) and \( s \) be integers. Then,

\[
\sum_{k=0}^{n} \sum_{p=0}^{k} \binom{n}{k} \binom{k}{p} \frac{G_{k+4p+s}}{2^k} = 2^n G_{3n+s},
\]

and

\[
\sum_{k=0}^{n} \sum_{p=0}^{k} \binom{n}{k} \binom{k}{p} \frac{G_{k+5p+s}}{3^k} = \left(\frac{7}{3}\right)^n G_{3n+s}.
\]

**Proof.** Set \( j = 0 \) and \( j = 1 \) in (28) and (29), respectively.

### 6. A general binomial sum identity

**Theorem 6.1.** Let \( j, s \) and \( v \) be integers with \( j, v \geq 0 \), \( v \neq 0 \), \( v \neq 1 \). Then,

\[
\sum_{k=0}^{n} \sum_{p=0}^{k} \binom{n}{k} \binom{k}{p} (-1)^{j+m+q} (j-m) \binom{j}{q} \binom{T_v}{\frac{T_v}{T_{v-1}}}^m G_{vn-j(v-1)+q+s} = \frac{T_v^{m-2}}{T_{v-1}^2} \sum_{k=0}^{n} \sum_{p=0}^{k} \binom{k}{p} (-1)^{k+w+p} (j) \binom{j}{p} \binom{k}{p} \binom{k}{p} \binom{T_{v-1}}{\frac{T_{v-1}}{T_{v-2}}}^k \binom{T_v}{\frac{T_v}{T_{v-1}}}^p G_{k+w+s}.
\]

**Proof.** For \( v \geq 1 \) and \( \phi = \alpha \) write (4) in the form

\[
\alpha^v = \alpha \left(T_v + T_{v-1}(\alpha - 1)\right) + T_{v-2}.
\]

Now, identify \( x = \alpha \left(T_v + T_{v-1}(\alpha - 1)\right) \) and \( a = T_{v-2} \) and use Lemma 3.1 and the binomial theorem to get

\[
\sum_{k=0}^{n} \binom{n}{k} \binom{k}{p} (-1)^{k+w+p} (j) \binom{j}{p} \binom{k}{p} \binom{k}{p} \binom{T_{v-1}}{\frac{T_{v-1}}{T_{v-2}}}^k \binom{T_v}{\frac{T_v}{T_{v-1}}}^p G_{k+w+s} = \frac{T_v^{m+s}}{T_{v-2}^m}.
\]

Multiply both sides by \( \alpha^s \) and combine the similar results for \( \beta \) and \( \gamma \) according to the Binet formula (1).

**Corollary 6.1.** We have

\[
\sum_{k=0}^{n} \sum_{p=0}^{k} (-1)^{w+p} \binom{n}{p} \binom{k}{p} \binom{T_{v-1}}{\frac{T_{v-1}}{T_{v-2}}}^k \binom{T_v}{\frac{T_v}{T_{v-1}}}^p G_{n+w+s} = (-1)^w G_{2n+s}
\]

for \( v = 1 \) the left-hand side collapses and we end with \( G_{n+s} \) on both sides of the equality sign. The special values for \( v = 2 \) and \( v = 3 \) are given by

\[
\sum_{k=0}^{n} \sum_{p=0}^{k} (-1)^{w+p} \binom{n}{p} \binom{k}{p} \binom{T_{v-1}}{\frac{T_{v-1}}{T_{v-2}}}^k \binom{T_v}{\frac{T_v}{T_{v-1}}}^p G_{n+w+s} = (-1)^w G_{2n+s}
\]

and

\[
\sum_{k=0}^{n} \sum_{p=0}^{k} (-1)^{w+p+k} \binom{n}{p} \binom{k}{k} \binom{k}{p} \binom{T_{v-1}}{\frac{T_{v-1}}{T_{v-2}}}^k \binom{T_v}{\frac{T_v}{T_{v-1}}}^p G_{k+w+s} = G_{3n+s}.
\]
References

[1] K. Adegoke, Weighted tribonacci sums, Konuralp J. Math. 8 (2020) 355–360.
[2] K. Adegoke, R. Frontczak, T. Goy, Special sums with squared Horadam numbers and generalized Tribonacci numbers, Palest. J. Math. (2021), To appear.
[3] K. Adegoke, A. Olatinwo, W. Oyekanmi, New Tribonacci recurrence relations and addition formulas, Notes Number Theory Discrete Math. 26 (2020) 164–172.
[4] P. Anantakitpaisal, K. Kuhapatanakul, Reciprocal sums of the tribonacci numbers, J. Integer Seq. 19 (2016) #16.2.1.
[5] K. N. Boyadzhiev, Notes on the Binomial Transform: Theory and Table with Appendix on Stirling Transform, World Scientific, Singapore, 2018.
[6] M. Catalani, Identities for Tribonacci–related sequences, arXiv:0209179 [math.CO], (2002).
[7] G. Cerda-Morales, Quadratic approximation of generalized tribonacci sequences, Discuss. Math. Gen. Algebra Appl. 38 (2018) 227–237.
[8] E. Choi, Modular tribonacci numbers by matrix method, J. Korean Soc. Math. Educ. Ser. B Pure Appl. Math. 20 (2013) 207–221.
[9] L. Comtet, Advanced Combinatorics: The Art of Finite and Infinite Expansions, D. Reidel, Dordrecht, 1974.
[10] J. Feng, More identities on the Tribonacci numbers, Ars Combin. 100 (2011) 73–78.
[11] R. Frontczak, A short remark on Horadam identities with binomial coefficients, Ann. Math. Inf. 54 (2021) DOI: 10.33039/ami.2021.03.016, In press.
[12] R. Frontczak, Convolutions for generalized Tribonacci numbers and related results, Int. J. Math. Anal. 12 (2018) 307–324.
[13] R. Frontczak, Relations for generalized Fibonacci and Tribonacci sequences, Notes Number Theory Discrete Math. 25 (2019) 178–192.
[14] R. Frontczak, Sums of Tribonacci and Tribonacci–Lucas numbers, Int. J. Math. Anal. 12 (2018) 19–24.
[15] T. Goy, M. Shattuck, Determinant identities for Toeplitz–Hessenberg matrices with tribonacci number entries, Trans. Comb. 9 (2020) 89–109.
[16] M. Janjić, Words and linear recurrences, J. Integer Seq. 21 (2018) #18.1.4.
[17] T. Komatsu, R. Li, Convolution identities for Tribonacci numbers with symmetric formulae, Math. Rep. 21(71) (2019) 27–47.
[18] K. Kuhapatanakul, L. Sukruan, The generalized tribonacci numbers with negative subscripts, Integers 14 (2014) #A32.
[19] N. J. A. Sloane, The On-Line Encyclopedia of Integer Sequences, https://oeis.org.
[20] Y. Soykan, Tribonacci and Tribonacci–Lucas matrix sequences with negative subscripts, Comm. Math. Appl. 11 (2020) 141–159.
[21] N. Yilmaz, N. Taskara, Tribonacci and Tribonacci–Lucas numbers via the determinants of special matrices, Appl. Math. Sci. 8 (2014) 1947–1955.