Notes on some classes of
3-dimensional contact metric manifolds

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Abstract. A review of the geometry of 3-dimensional contact metric manifolds shows that generalized Sasakian manifolds and $\eta$-Einstein manifolds are deeply interrelated. For example, it is known that a 3-dimensional Sasakian manifold is $\eta$-Einstein. In this paper, we discuss the relationships between several special classes of 3-dimensional contact metric manifolds which are generalizations of 3-dimensional Sasakian manifolds. We also provide examples illustrating our result in this paper.

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1 Introduction

It is well-known that any 3-dimensional compact oriented manifold admits a contact structure [21], and hence, it admits an associated contact metric structure. Therefore, it is natural to investigate 3-dimensional compact oriented manifolds from the contact metric view point. We shall give a brief review of contact metric manifolds focusing on the interrelationships between the generalizations of Sasakian manifolds and $\eta$-Einstein contact metric manifolds. It is well known that a Sasakian manifold is characterized as a contact metric manifold $M = (M, \phi, \xi, \eta, g)$ whose curvature tensor $R$ satisfies

$$R(X, Y)\xi = \eta(Y)X - \eta(X)Y,$$

for any $X, Y \in \mathfrak{X}(M)$, where $\mathfrak{X}(M)$ denotes the Lie algebra of all smooth vector fields on $M$. As a generalization of the Sasakian manifold, Blair, Koufogiorgos and Papantoniou [2] introduced the notion of a contact metric manifold called a $(\kappa, \mu)$-contact metric manifold satisfying the condition

$$R(X, Y)\xi = \kappa(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)hX - \eta(X)hY),$$

for any $X, Y \in \mathfrak{X}(M)$, where $\kappa$ and $\mu$ are constants on $M$ and $h = \frac{1}{2}L_\xi \phi$ (here, $L_\xi$ is the Lie derivative in the direction of $\xi$). $(\kappa, \mu)$-contact metric manifolds have attracted

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by many authors [4, 5, 9, 10, 11, 18, 20]. \((\kappa, \mu)\)-contact metric manifolds include Sasakian manifolds \((\kappa = 1 \text{ and } h = 0)\), and also many examples of non-Sasakian \((\kappa, \mu)\)-contact metric manifolds have been provided. Koufogiorgos and Tsichlias [12] generalized the notion of a \((\kappa, \mu)\)-contact metric manifold by regarding the constants \(\kappa\) and \(\mu\) in (1.2) to be smooth functions on \(M\), called a generalized \((\kappa, \mu)\)-contact metric manifold. Further, the same authors [11] studied 3-dimensional generalized \((\kappa, \mu)\)-contact metric manifolds with \(\xi \mu = 0\) (this condition means the function \(\mu\) is constant along each integral curve of the characteristic vector field \(\xi\)) and showed that it is possible to construct two families of such manifolds in \(\mathbb{R}^3\), for any smooth function \(\kappa < 1\) of one variable. We shall introduce an example belonging to such families in §5, which illustrates Theorem B in the present paper. Koufogiorgos, Markellas and Papantoniou [10] introduced the notion of a \((\kappa, \mu, \nu)\)-contact metric manifold which is a generalization of the generalized \((\kappa, \mu)\)-contact metric manifold, defined as a contact metric manifold \(M = (\phi, \xi, \eta, g)\) satisfying

\[
R(X, Y)\xi = \kappa(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)hX - \eta(X)hY) + \nu(\eta(Y)\phi hX - \eta(X)\phi hY).
\]

(1.3)

for any \(X, Y \in \mathfrak{X}(M)\), where \(\kappa, \mu, \nu\) are smooth functions on \(M\). In the same paper [10], they proved that a \((\kappa, \mu, \nu)\)-contact metric manifold is necessarily a \((\kappa, \mu)\)-contact metric manifold if the dimension of \(M\) is greater than or equal to 5. They also proved that the condition (1.3) is invariant under the \(D\)-homothetic deformations, and further that, if \(\dim M = 3\), then the condition (1.3) is equivalent to the following condition

\[
Q = \left(\frac{r}{2} - \kappa\right) I + \left(-\frac{r}{2} + 3\kappa\right) \eta \otimes \xi + \mu h + \nu \phi h
\]

(1.4)

holding on an open and dense subset of \(M\), where \(Q\) is the Ricci operator and \(r\) is the scalar curvature of \(M\) ([10], Proposition 3.1). We note that \(\kappa \leq 1\) on 3-dimensional \((\kappa, \mu, \nu)\)-contact metric manifold (see (3.13)). A contact metric manifold \(M = (\phi, \xi, \eta, g)\) is called \(\eta\)-Einstein if the Ricci operator \(Q\) takes the following form

\[
Q = \alpha I + \beta \eta \otimes \xi,
\]

(1.5)

where \(\alpha\) and \(\beta\) are some smooth functions on \(M\). From (1.3) and (1.4), taking account of (1.5), we may observe that the geometry of \((\kappa, \mu, \nu)\)-contact metric manifolds and of generalized \((\kappa, \mu)\)-contact metric manifolds is deeply interrelated with the generalization of the \(\eta\)-Einstein contact metric manifold in the 3-dimensional case. On the other hand, a contact metric manifold \(M = (\phi, \xi, \eta, g)\) is said to be \(H\)-contact if the characteristic vector field \(\xi\) is a harmonic vector field. We remark that \((\kappa, \mu, \nu)\)-contact metric manifold is \(H\)-contact. Koufogiorgos, Markellas and Papantoniou [10] proved that a 3-dimensional \(H\)-contact manifold is a \((\kappa, \mu, \nu)\)-contact metric manifold on an open and dense subset of \(M\) ([10], Theorem 1.1). The last two of the present authors worked on the \(H\)-contact unit tangent sphere bundles [6, 7, 14]. Concerning 3-dimensional \((\kappa, \mu, \nu)\)-contact metric manifolds, the present authors previously proved the following theorem.

Theorem A [8] Let \(M = (\phi, \xi, \eta, g)\) be a 3-dimensional \((\kappa, \mu, \nu)\)-contact metric manifold. If the functions \(\mu\) and \(\nu\) are constant on \(M\), then \(M\) is either Sasakian
or a non-Sasakian \((\kappa, \mu)\)-contact metric manifold with constant scalar curvature \(r = 2\kappa - 2\mu\).

In this paper, we shall prove the following theorem.

**Theorem B** Let \(M = (M, \phi, \xi, \eta, g)\) be a 3-dimensional compact \((\kappa, \mu, \nu)\)-contact metric manifold with \(\xi \mu = \xi \nu = 0\) and let \(r\) be the scalar curvature. If either (the inequality) \(r + \frac{\mu^2}{2} \geq 0\) or \(r + \frac{\mu^2}{2} \leq 0\) holds everywhere on \(M\), then \(M\) is a Sasakian manifold or a non-Sasakian \((\kappa, \mu)\)-contact metric manifold with \(\kappa = \mu - \frac{\mu^2}{4}\) and \(r = -\frac{\mu^2}{2}\).

We here remark that the hypothesis “\(M = (M, \phi, \xi, \eta, g)\) is a 3-dimensional \((\kappa, \mu, \nu)\)-contact metric manifold with \(\xi \mu = \xi \nu = 0\)” is preserved under any \(D\)-homothetic transformation \([10]\) of the contact metric structure \((\phi, \xi, \eta, g)\) on \(M\). Unless otherwise specified, the manifolds to be considered in this paper will be assumed to be connected.

## 2 Preliminaries

In this section, we present some basic facts about contact metric manifolds. We refer to [1] for more details. A \((2n + 1)\)-dimensional smooth manifold \(M\) is called a contact manifold if it admits a global 1-form \(\eta\) such that \(\eta \wedge (d\eta)^n \neq 0\) everywhere on \(M\). We call \(\eta\) a contact form of \(M\). It is well-known that given a contact form \(\eta\), there exists a unique vector field \(\xi\), which is called the characteristic vector field, satisfying \(\eta(\xi) = 1\) and \(d\eta(\xi, X) = 0\) for any vector field \(X\) on \(M\). A Riemannian metric \(g\) is said to be an associated metric to a contact form \(\eta\) if there exists a \((1, 1)\)-tensor field \(\phi\) satisfying

\[
\eta(X) = g(X, \xi), \quad d\eta(X, Y) = g(X, \phi Y), \quad \phi^2 X = -X + \eta(X)\xi,
\]

where \(X\) and \(Y\) are vector fields on \(M\). From (2.1), one can easily obtain

\[
\phi \xi = 0, \quad \eta \circ \phi = 0, \quad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y).
\]

The structure \((\phi, \xi, \eta, g)\) is called a contact metric structure, and a manifold \(M\) with a contact metric structure \((\phi, \xi, \eta, g)\) is said to be a contact metric manifold and is denoted by \(M = (M, \phi, \xi, \eta, g)\). Let \(\nabla\) be the Levi-Civita connection and let \(R\) be the corresponding Riemann curvature tensor field given by \(R(X, Y) = [\nabla X, \nabla Y] - \nabla [X, Y]\) for all vector fields \(X, Y\) on \(M\). We denote by \(S\) the Ricci tensor field of type \((0,2)\), by \(Q\) the Ricci operator, and by \(r\) the scalar curvature. We define on \(M\) the operators \(h, l\) by setting

\[
hX = \frac{1}{2}(\mathcal{L}_\xi \phi) X, \quad lX = R(X, \xi)\xi,
\]

where \(\mathcal{L}_\xi\) is the Lie derivative in the direction of \(\xi\). It is easily checked that \(h\) and \(l\) are symmetric operators and satisfy the following equalities

\[
h\xi = 0, \quad l\xi = 0, \quad h\phi = -\phi h.
\]
We also have the following formulas for a contact metric manifold:

\[
\nabla_X \xi = -\phi X - \phi h X, \quad \text{(and hence } \nabla_\xi \xi = 0) \\
\n\nabla_\xi \phi = 0, \quad T r_l = g(Q\xi, \xi) = 2n - tr(h^2), \\
\phi \phi - l = 2(\phi^2 + h^2), \quad \nabla_\xi h = \phi - \phi l - \phi h^2.
\]

On the other hand, a contact metric manifold for which \(\xi\) is a Killing vector field is called a \(K\)-contact manifold. It is well known that a contact metric manifold is \(K\)-contact if and only if \(h = 0\). It is well known that Sasakian manifolds are necessarily \(K\)-contact but the converse is generally not true except in the 3-dimensional case ([1], pp.70 and pp.76). Here, we note that on any \((2n + 1)(n > 1)\)-dimensional \(\eta\)-Einstein \(K\)-contact manifold, the functions \(\alpha\) and \(\beta\) in the defining equation (1.5) are both constant. We may also note that any 3-dimensional Sasakian manifold is \(\eta\)-Einstein ((1.4), [17]) and \(\alpha + \beta\) is constant [3]. Hence, it is natural to ask whether there exists a 3-dimensional Sasakian manifold with non-constant coefficient functions \(\alpha\) and \(\beta\) as a \(\eta\)-Einstein or not. Concerning this question, to our knowledge, it seems that any explicit example of a 3-dimensional Sasakian manifold with non-constant coefficient functions \(\alpha\) and \(\beta\) as an \(\eta\)-Einstein manifold has not yet appeared in any literature. In the last section, we shall provide an explicit example of such a 3-dimensional Sasakian manifold. Based on the above arguments, it seems worthwhile to discuss the coefficient functions in the equation (1.4) for a 3-dimensional \((\kappa, \mu, \nu)\)-contact metric manifold, along with the generalizations of a 3-dimensional Sasakian manifold introduced in the §1.

### 3 Fundamental formulas

In this section, we shall prepare some fundamental formulas which we need in the proof of the Theorem B.

Let \(M = (M, \phi, \xi, \eta, g)\) be a 3-dimensional contact metric manifold, and \(h, l\) be the \((1, 1)\) tensor fields defined by (2.3). First, we recall the following formula by [19]:

\[
(\nabla_X \phi)Y = g(X + hX, Y)\xi - \eta(Y)(X + hX),
\]

for any \(X, Y \in \mathfrak{X}(M)\). Next, we recall that the curvature tensor \(R\) of a 3-dimensional Riemannian manifold satisfies the following identity

\[
R(X, Y)Z = g(Y, Z)QX - g(X, Z)QY - g(QX, Z)Y \\
+ g(QY, Z)X - \frac{r}{2} (g(Y, Z)X - g(X, Z)Y),
\]

for any \(X, Y, Z \in \mathfrak{X}(M)\). Now, let \(U\) be the open subset of \(M\) on which \(h \neq 0\), and \(V\) be the open subset of points \(m \in M\) such that \(h = 0\) on a neighborhood of \(m\). Then, we may easily check that \(U \cup V\) is an open and dense subset of \(M\). If \(U\) is not empty, for any \(m \in U\), we may choose a local orthonormal frame field \(\{\xi, e_1, e_2 = \phi e_1\}\) on a neighborhood of \(m\) in such a way that

\[
h e_1 = \lambda e_1, \quad he_2 = -\lambda e_2,
\]
where $\lambda$ is a smooth positive function on $U$. We may also note that, if $V$ is not empty, then $V$ becomes a Sasakian manifold (see §2).

Now, we assume that $U$ is not empty. Then, by making use of (2.4), (2.5), (3.2) and (3.3), we have the following basic formulas on $U$:

(3.4)
\[
\begin{align*}
\nabla_\xi e_1 &= -ae_2, & \nabla_\xi e_2 &= ae_1, \\
\nabla_{e_1}e_1 &= \frac{1}{2\lambda}(e_2\lambda + A)e_2, & \nabla_{e_1}e_2 &= -\frac{1}{2\lambda}(e_2\lambda + A)e_1 + (\lambda + 1)\xi, \\
\nabla_{e_2}e_1 &= \frac{1}{2\lambda}(e_1\lambda + B)e_1, & \nabla_{e_2}e_2 &= -\frac{1}{2\lambda}(e_1\lambda + B)e_2 + (\lambda - 1)\xi,
\end{align*}
\]

and we have

(3.5)
\[ [e_1, e_2] = -\frac{1}{2\lambda}(e_2\lambda + A)e_1 + \frac{1}{2\lambda}(e_1\lambda + B)e_2 + 2\xi, \]

where $A = S(\xi, e_1)$, $B = S(\xi, e_2)$ and $a$ is a smooth function. Further, the Ricci operator $Q$ [16] on $U$ is given by

(3.6)
\[
\begin{align*}
Q\xi &= 2(1 - \lambda^2)\xi + Ae_1 + Be_2, \\
Qe_1 &= A\xi + \left(\frac{r}{2} - 1 + \lambda^2 + 2a\lambda\right)e_1 + \xi(\lambda)e_2, \\
Qe_2 &= B\xi + \xi(\lambda)e_1 + \left(\frac{r}{2} - 1 + \lambda^2 - 2a\lambda\right)e_2.
\end{align*}
\]

Thus, from (3.2) and (3.6), we get that the components of the curvature tensor are given by

(3.7)
\[
\begin{align*}
R(e_1, e_2)e_1 &= \left(2 - \frac{r}{2} - 2\lambda^2\right)e_2 - B\xi, & R(e_1, e_2)e_2 &= \left(\frac{r}{2} - 2 + 2\lambda^2\right)e_1 + A\xi, \\
R(e_1, e_2)\xi &= Be_1 - Ae_2, & R(e_1, \xi)e_1 &= -Be_2 + (\lambda^2 - 1 - 2a\lambda)\xi, \\
R(e_1, \xi)e_2 &= Be_1 - \xi(\lambda)\xi, & R(e_1, e_2)\xi &= (2a\lambda + 1 - \lambda^2)e_1 + \xi(\lambda)e_2, \\
R(e_2, e_1)\xi &= Ae_2 - \xi(\lambda)\xi, & R(e_2, \xi)e_1 &= Be_2 + (-1 + \lambda^2 + 2a\lambda)\xi, \\
R(e_2, \xi)e_2 &= \xi(\lambda)e_1 + (1 - 2a\lambda - \lambda^2)e_2.
\end{align*}
\]

We have noted that $Trl = 2(1 - \lambda^2)$ by (2.5). In the remaining section, we assume that $M$ (under consideration) is a $(\kappa, \mu, \nu)$-contact metric manifold. Then, from (1.3), we have

(3.8)
\[
\begin{align*}
R(e_1, e_2)\xi &= 0, & R(e_1, \xi)\xi &= (\kappa + \lambda\mu)e_1 + \lambda\nu e_2, & R(e_2, \xi)\xi &= \lambda\nu e_1 + (\kappa - \lambda\mu)e_2.
\end{align*}
\]

Thus, comparing (3.7) and (3.8), we have

(3.9)
\[ A = B = 0, \]

(3.10)
\[ \xi \lambda = \lambda \nu, \]
Thus, from (1.4), (2.5), (3.6), (3.9), and (3.11), we have further

\begin{equation}
\mu = 2a,
\end{equation}

\begin{equation}
\kappa = \frac{1}{2} S(\xi, \xi) = 1 - \frac{1}{2} Tr(h^2) = 1 - \lambda^2.
\end{equation}

On the other hand, from (2.4) and (3.3), taking account of (3.4), (3.9), (3.10) and (3.12), we have

\begin{equation}
(\nabla_{e_i} \eta)(e_2) = -(\lambda + 1), \quad (\nabla_{e_i} \eta)(\xi) = 0,
\end{equation}

\begin{equation}
(\nabla_{e_2} \eta)(\xi) = 0,
\end{equation}

\begin{equation}
(\nabla_{e_2} \eta)(e_1) = -(\lambda - 1),
\end{equation}

\begin{equation}
(\nabla_{e_3} \eta)(e_2) = -((\lambda +1)e_2 + (\lambda - 1)e_1 - \lambda(\lambda + 1)\xi).
\end{equation}

From (1.3), taking account of the second Bianchi identity, we get

\begin{equation}
\mathfrak{G}_{X,Y,Z} R(X,Y)\nabla Z \xi
\end{equation}

\begin{equation}
in (X,Y)X - \eta(X)Y + \kappa ((\nabla \eta)(Y)X - (\nabla \eta)(X)Y)
\end{equation}

\begin{equation}
+ (Z\mu)(\eta(Y)hX - \eta(X)hY) + \mu((\nabla \eta)(Y)hX + \eta(Y)hY)
\end{equation}

\begin{equation}
- (Z\mu)(\eta(Y)hX - \eta(X)hY) + (Z\nu)(\eta(Y)\phi hX - \eta(X)\phi hY)
\end{equation}

\begin{equation}
+ \nu((\nabla \eta)(Y)\phi hX + \eta(Y)\nabla Z \phi hX - (\nabla \eta)(X)\phi hY - \eta(X)(\nabla \phi hY))
\end{equation}

for any \(X, Y, Z \in \mathfrak{X}(M)\), where \(\mathfrak{G}_{X,Y,Z} \) denotes the cycle sum with respect to the vector fields \(X, Y\) and \(Z\). Setting \(X = e_1, Y = e_2\) and \(Z = \xi\) in (3.15), and taking account of (3.4), (3.7) and (3.14), we have

\(-2(\lambda^2 - 1 + \lambda^2 \mu)\xi = 2(\kappa - \lambda^2 \mu)\xi + (\lambda e_1 \nu - \lambda e_2 \mu - e_2 \xi) e_1 + (e_1 \kappa - \lambda e_1 \mu - \lambda e_2 \nu) e_2,\)

and hence, we have

\begin{equation}
e_1 \kappa = \lambda(e_1 \mu + e_2 \nu), \quad e_2 \kappa = \lambda(e_1 \nu - e_2 \mu).
\end{equation}

Thus, from (3.16), taking account of (3.13), we have also

\begin{equation}
e_1 \lambda = -\frac{1}{2}(e_1 \mu + e_2 \nu), \quad e_2 \lambda = \frac{1}{2}(e_2 \mu - e_1 \nu).
\end{equation}
By the second Bianchi identity, we have further

\[ (\nabla_{\xi,e_1,e_2}(\nabla R)(e_1,e_2)e_1 = 0, \]

Taking account of (3.4) and (3.7) with (3.9), (3.10), (3.12) and (3.13), we have

\[ (\nabla_{\xi,e_1,e_2}(\nabla R)(e_1,e_2)e_1 = -\left(\frac{1}{2}\xi r + 4\lambda^2\nu\right)e_2, \]

\[ (\nabla_{e_1}(\nabla R)(e_2,\xi)e_1 = -(e_1(\lambda\nu + \mu e_2)\lambda)\xi + \lambda(\lambda + 1)\nu e_2, \]

\[ (\nabla_{e_2}(\nabla R)(\xi,e_1)e_1 = (e_2(\lambda\mu) - 2\lambda e_2\lambda + \nu e_1\lambda)\lambda + \lambda(\lambda - 1)\nu e_2. \]

Thus, from (3.18) and (3.19), we have

\[ \xi r = -4\lambda^2\nu. \]

From (3.10) and (3.13), we have also

\[ \xi \kappa = -2\lambda^2\nu. \]

Now, from (3.4), (3.9), (3.12) and (3.13), we obtain

\[ R(e_1,e_2)e_1 = \nabla_{e_1}(\nabla_{e_2}e_1) - \nabla_{e_2}(\nabla_{e_1}e_1) - \nabla_{[e_1,e_2]}e_1 = \left\{ -\frac{1}{2}e_1(e_1 \lambda) - \frac{1}{2}e_2(e_2 \lambda) + \frac{1}{4}(e_2 \lambda)^2 + \frac{1}{2}(e_1 \lambda)^2 + \kappa + \mu \right\} e_2. \]

On one hand, taking account of (2.5) and (3.4), we also obtain

\[ -\frac{1}{2} \triangle \log \lambda = -\frac{1}{2} \left\{ e_1(e_1 \lambda) + e_2(e_2 \lambda) + \xi(\xi \lambda) - \frac{1}{2}(e_2 \lambda)^2 - \frac{1}{2}(e_1 \lambda)^2 \right\}. \]

Thus, from the first equality in (3.7), (3.22) and (3.23), we have

\[ r = \triangle \log \lambda + 2\kappa - 2\mu - \xi \nu. \]

4 Proof of Theorem B

Let \( M = (M, \phi, \xi, \eta, g) \) be a 3-dimensional compact \((\kappa, \mu, \nu)\)-contact metric manifold with \( \xi \mu = \xi \nu = 0 \) on \( M \). Now, we assume that the open subset \( U \) of \( M \) on which \( h \neq 0 \), is not empty. We set

\[ F_{\text{min}} = \{ m \in M | \kappa \text{ takes into minimum at } m \}, \]

\[ F_{\text{max}} = \{ m \in M | \kappa \text{ takes into maximum at } m \}. \]
Then, we may easily check that $F_{\min}$ and $F_{\max}$ are both non-empty closed (and hence, compact) subsets of $M$ such that $F_{\min} \subset U$. And, we see that each integral curve of $\xi$ is a geodesic in $M$. We denote by $\gamma(t) = \gamma(t; m)$ the integral curve of $\xi$ though $m \in U$ with the arc-length parameter $t$. Then, from (3.10) and hypothesis $\xi \nu = 0$, we have

\begin{equation}
\lambda(t) \equiv \lambda(\gamma(t)) = \lambda(m)e^{\nu(m)t}.
\end{equation}

for $|t| < \epsilon$, where $\epsilon$ is a certain positive real number. From (3.13), (4.2), we see that $\kappa(t) = \kappa(\gamma(t))$ is given by

\begin{equation}
\kappa(t) = 1 - \lambda(m)^2 e^{2\nu(m)t},
\end{equation}

for $|t| < \epsilon$. Thus from (4.3), we see that, for each point $m \in U$, $\gamma(t) \in U$ for all $t \in \mathbb{R}$. Now, we suppose that there exists a point $m \in U$ with $\nu(m) > 0$. Then, from (4.3), we have

\begin{equation}
\lim_{t \to +\infty} \kappa(t) = -\infty.
\end{equation}

Similarly, if there exists a point $m \in U$ with $\nu(m) < 0$. Then from (4.3), we have also

\begin{equation}
\lim_{t \to -\infty} \kappa(t) = -\infty.
\end{equation}

Since $M$ is compact, we see that $\kappa (\leq 1)$ must bounded on $M$. But, from (4.4) and (4.5), this is a contradiction. Therefore, it follows that $\nu = 0$ on $U$. Since $V$ is Sasakian, it follows immediately $\nu = 0$ on $V$. Since $U \cup V$ is an open and dense subset in $M$, we see that $\nu$ vanishes on $M$ and hence, the $(\kappa, \mu, \nu)$-contact metric manifold $M$ under consideration reduces to a generalized $(\kappa, \mu)$-contact metric manifold with $\xi \mu = 0$. Since $\nu = 0$ on $M$, from (3.17), we have on $U$,

\begin{equation}
A_1 = -\frac{1}{2}B_1, \quad A_2 = \frac{1}{2}B_2,
\end{equation}

where $A_1 = e_1 \lambda, B_1 = e_1 \mu$, $A_2 = e_2 \lambda, B_2 = e_2 \mu$. From (3.4) and (3.5), we have

\begin{equation}
[e_1, \xi] = \left(\frac{\mu}{2} - \lambda - 1\right) e_2, \quad [e_2, \xi] = -\left(\frac{\mu}{2} + \lambda - 1\right) e_1.
\end{equation}

Since $\nu = 0$, from (3.10), we have also

\begin{equation}
\xi \lambda = 0.
\end{equation}

Thus, from (4.7), taking account of (4.6) and (4.8), we obtain

\begin{equation}
\xi A_1 = \left(\lambda + 1 - \frac{\mu}{2}\right) A_2, \quad \xi A_2 = \left(\lambda - 1 + \frac{\mu}{2}\right) A_1.
\end{equation}

Similarly, from (4.7), taking account of (4.6) and $\xi \mu = 0$, we obtain

\begin{equation}
\xi A_1 = -\left(\lambda + 1 - \frac{\mu}{2}\right) A_2, \quad \xi A_2 = -\left(\lambda - 1 + \frac{\mu}{2}\right) A_1.
\end{equation}
Thus, from (4.9) and (4.10), we have

\[(\lambda + 1 - \frac{\mu}{2}) A_2 = 0.\] (4.11)

\[(\lambda - 1 + \frac{\mu}{2}) A_1 = 0.\] (4.12)

**Lemma 4.1.** \(A_1 = 0\) or \(A_2 = 0\) at each point of \(U\).

**Proof.** We assume that \(A_1 \neq 0\) and \(A_2 \neq 0\) at some point \(m \in U\). Then, from (4.11) and (4.12), it follows that \(\lambda + 1 - \frac{\mu}{2} = 0\) and \(\lambda - 1 + \frac{\mu}{2} = 0\) at the point \(m\), and hence, \(\lambda = 0\) at \(m\). But, this is a contradiction. □

Now, we define subsets \(F_1, F_2, G_1, G_2\) and \(F\) of \(U\) by

\[G_1 = \{m \in U | A_1 \neq 0 \text{ (i.e. } A_2 = 0\text{ ) at } m\},\]

\[G_2 = \{m \in U | A_2 \neq 0 \text{ (i.e. } A_1 = 0\text{ ) at } m\},\]

\[F_1 = \{m \in U | \lambda - 1 + \frac{\mu}{2} = 0 \text{ at } m\},\]

\[F_2 = \{m \in U | \lambda + 1 - \frac{\mu}{2} = 0 \text{ at } m\},\]

\[F = \{m \in U | A_1 = A_2 = 0 \text{ (i.e. } B_1 = B_2 = 0\text{ ) at } m\}.\]

Then, taking account of (4.11) and (4.12) and Lemma 4.1, we have the following relations.

\[G_1 \subset F_1, \ G_2 \subset F_2, \ F_1 \cap F_2 = \emptyset, \text{ and} \]

\[U = G_1 \cup G_2 \cup F = F_1 \cup F_2 \text{ (disjoint union).} \] (4.13)

We have denoted by \(F^{(i)}\) the interior of \(F\) in \(U\). Then, taking account of (4.9), we may observe that, if \(F^{(i)} \neq \emptyset\), then \(\lambda\) (and hence, \(\kappa\)) is constant on \(F^{(i)}\). From (4.13), we see that \(G_1 \cup G_2 \cup F^{(i)}\) is an open and dense subset in \(U\). First, we assume that the inequality \(r + \frac{\mu^2}{2} \geq 0\) holds on \(M\). If \(G_1 \neq \emptyset\), then from (3.24), taking account of (4.12), we have

\[\triangle \log \lambda = r - 2(1 - \lambda^2) - 4(\lambda - 1) = r + 2(\lambda - 1)^2 = r + \frac{\mu^2}{2} \geq 0 \] (4.14)

on \(G_1\). Similarly, if \(G_2 \neq \emptyset\), then, from (3.24), taking account of (4.11), we have

\[\triangle \log \lambda = r - 2(1 - \lambda^2) + 4(\lambda + 1) = r + 2(\lambda + 1)^2 = r + \frac{\mu^2}{2} \geq 0 \] (4.15)

on \(G_2\). Therefore, we have the following inequality

\[\triangle \log \lambda \geq 0 \] (4.16)

on \(G_1 \cup G_2\). By direct calculation, we get

\[\triangle \log \lambda = -\frac{1}{\lambda^2} |\text{grad}\lambda|^2 + \frac{1}{\lambda} \triangle \lambda \] (4.17)
on $G_1 \cup G_2$. Further, since $\kappa = 1 - \lambda^2$ on $U$, we get also
\begin{equation}
\Delta \kappa = -2 |g \operatorname{rad}\lambda|^2 - 2\lambda \Delta \lambda
\end{equation}
on $G_1 \cup G_2$. Thus, from (4.17) and (4.18), we have
\begin{equation}
\Delta \kappa = -4 |g \operatorname{rad}\lambda|^2 - 2\lambda^2 \Delta \log \lambda \leq 0
\end{equation}
on $G_1 \cup G_2$. On the other hand, $\kappa = \text{const}$ on $F(i)$. Since $G_1 \cup G_2 \cup F(i)$ is an open and everywhere dense subset of $U$, from (4.19), we have the inequality
\begin{equation}
\Delta \kappa \leq 0 \text{ on } U.
\end{equation}

If $V \neq \emptyset$, $V$ is Sasakian (and has $\kappa = 1$ on $V$), since $\kappa = 1$ on $V$, it is evident that $\Delta \kappa = 0$ holds on $V$. Since $U \cup V$ is open and everywhere dense in $M$, we see finally that
\begin{equation}
\Delta \kappa \leq 0
\end{equation}
on $M$. On the other hand, the function $\kappa$ takes its minimum on the non-empty subset $F_{min}$. Therefore, by Hopf’s theorem, we see that $\kappa$ is constant on $M$, and hence, $\mu$ is also constant on $M$. Next, we assume that the inequality $r + \frac{\mu^2}{\kappa^2} \leq 0$ holds everywhere on $M$. Then, applying the similar arguments as in the previous case where $r + \frac{\mu^2}{\kappa^2} \geq 0$, we have $\Delta \kappa \geq 0$ holds on $M$. Since the function $\kappa$ takes its maximum on the non-empty subset $F_{max}$. Therefore, by Hopf’s theorem, we see also that $\kappa$ and $\mu$ are both constant on $M$.

As the result, we see that $M$ is a non-Sasakian $(\kappa, \mu)$-contact metric manifold with $\kappa = \mu - \frac{\mu^2}{4}$ and hence $r = -\frac{\kappa^2}{2}$ by virtue of (3.24) if $U \neq \emptyset$. On the other hand, it is evident that $M$ is Sasakian ($\kappa = 1$ and $\mu = \nu = 0$) if $U = \emptyset$. This completes the proof of Theorem B.

5 Examples

In this section, we shall provide an example of the 3-dimensional Sasakian manifold $M = (M, \phi, \xi, \eta, g)$ with non-constant coefficient functions $\alpha$ and $\beta$ in the defining equation (1.5) of an $\eta$-Einstein manifold are both non-constant (see Example 1), and also an example of the 3-dimensional generalized $(\kappa, \mu)$-contact metric manifold which illustrates as well as supports Theorem B (see Example 2). Example 1 below is a special case of the example introduced in Blair’s book [1].

Example 1 Let $M = \mathbb{R}^3$ and set
\begin{equation}
\xi = 2 \frac{\partial}{\partial z}, \quad e_1 = 2 \frac{\partial}{\partial y}, \quad e_2 = 2(\frac{\partial}{\partial x} - y^2 \frac{\partial}{\partial y} + y \frac{\partial}{\partial z}).
\end{equation}
Let $\eta$ be the 1-form dual to $\xi$, and define (1,1)-tenser field $\phi$ by $\phi \xi = 0$, $\phi e_1 = e_2$ and $\phi e_2 = -e_1$. Further, let $g$ be the Riemannian metric defined by $g(\xi, \xi) = 1$, $g(\xi, e_i) = 0$ and $g(e_i, e_j) = \delta_{ij}$ for $1 \leq i, j \leq 2$. Then, by direct calculation, we may check that $(M, \phi, \xi, \eta, g)$ is a 3-dimensional Sasakian manifold and the Ricci transformation $Q$ is given by
\begin{equation}
Q = -(2 + 24y^2)I + (4 + 24y^2)\eta \otimes \xi
\end{equation}
on $M$. Therefore, from (5.2), we see that the 3-dimensional Sasakian manifold $M$ provides an explicit example of the $\eta$-Einstein manifold with non-constant coefficient functions $\alpha$ and $\beta$ in (1.5) which is mentioned in §2.

The following example which is constructed by Koufogiorgos and Tsichlias [11], which illustrates Theorem B.

Example 2 Let $M = \{(x, y, z) \in \mathbb{R}^3 | z > 0\}$ and set

$$\xi = \frac{\partial}{\partial x}, \quad e_1 = -2y \frac{\partial}{\partial x} + (2\sqrt{zx} - \frac{1}{4z}y) \frac{\partial}{\partial y} + \frac{\partial}{\partial z}, \quad e_2 = \frac{\partial}{\partial y}.$$ (5.3)

Let $\eta$ be the 1-form dual to $\xi$, and define (1,1)-tenser field $\phi$ by $\phi\xi = 0$, $\phi e_1 = e_2$ and $\phi e_2 = -e_1$. Further, let $g$ be the Riemannian metric defined by $g(\xi, \xi) = 1$, $g(\xi, e_i) = 0$ and $g(e_i, e_j) = \delta_{ij}$ for $1 \leq i, j \leq 2$. Then, by direct calculation, we may check that $(M, \phi, \xi, \eta, g)$ is a 3-dimensional generalized $(\kappa, \mu, \nu)$-contact metric manifold with $\kappa = 1 - z$, $\mu = 2(1 - \sqrt{z})$ (and $\nu = 0$) and $r + \frac{\mu^2}{8\nu^2} = -\frac{5}{8z} < 0$ on $M$.

Thus, Example 2 shows that the compactness assumption in Theorem B plays an essential role.

It is well-known that a 3-dimensional Lie group $G$ admits a discrete subgroup $\Gamma$ such that the space of right cosets $\Gamma \backslash G$ is compact if and only if $G$ is unimodular [13]. Let $G$ be one of the following simply connected unimodular Lie groups: $E(2)$, $E(1,1)$. Then, from the proof of the Theorem B and ([2,§4], [15]), we may check that $M = \Gamma \backslash G$ with a suitable discrete subgroup $\Gamma$ of $G$, provides an example illustrating Theorem B for non-Sasakian case.

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