A DYNAMICAL APPROACH TO MAASS CUSP FORMS

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Abstract. For nonuniform cofinite Fuchsian groups $\Gamma$ which satisfy a certain additional geometric condition, we show that the Maass cusp forms for $\Gamma$ are isomorphic to $1$-eigenfunctions of a finite-term transfer operator. The isomorphism is constructive.

1. Introduction

Let $\Gamma$ be a nonuniform cofinite Fuchsian group and consider its action on the hyperbolic plane $\mathbb{H}$ by Möbius transformations. The purpose of this article is to characterize, under a certain additional geometric requirement on $\Gamma$, the Maass cusp forms for $\Gamma$ as $1$-eigenfunctions of a finite-term transfer operator which arises from a discretization of the geodesic flow on $\Gamma \backslash \mathbb{H}$.

Maass cusp forms for $\Gamma$ are specific eigenfunctions of the Laplace-Beltrami operator $\Delta$ acting on $L^2(\Gamma \backslash \mathbb{H})$ which decay rapidly towards any cusp of $\Gamma \backslash \mathbb{H}$. They span the cuspidal spectrum of $\Delta$ in $L^2(\Gamma \backslash \mathbb{H})$, which together with the residual spectrum spans the discrete spectrum.

The discretization of the geodesic flow on $\Gamma \backslash \mathbb{H}$ used in this transfer operator approach to Maass cusp forms was developed in [Poh10] and is specifically adjusted to this purpose. The arising transfer operator families are parametrized by $\mathbb{C}$. The transfer operator with parameter $s$ is given by a finite sum of specific elements of $\Gamma$ acting via the action of principal series representation with spectral parameter $s$ on functions which are defined on certain intervals in the geodesic boundary of $\mathbb{H}$.

Our main result is as follows:

**Theorem A.** Let $s \in \mathbb{C}$, $0 < \text{Re} \, s < 1$. Then the space of Maass cusp forms for $\Gamma$ with eigenvalue $s(1-s)$ is isomorphic to the space of sufficiently regular $1$-eigenfunctions of the transfer operator with parameter $s$.

The regularity required for the eigenfunctions is specified in Theorem 3.1 below. The proof of Theorem A takes advantage of the characterization of Maass cusp forms in parabolic 1-cohomology in [BLZ09]. In this article we show that the parabolic 1-cocycle classes are isomorphic to these highly regular 1-eigenfunctions of the transfer operator. Both of these isomorphisms are constructive, and hence the isomorphism in Theorem A is so.

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Since the transfer operators involve only finitely many terms, their 1-eigenfunctions are the solutions of finite families of functional equations. Moreover, the eigenfunctions of sufficient regularity arise via integral transforms from Maass cusp forms. Therefore these eigenfunctions can be understood as period functions for the Maass cusp forms for \( \Gamma \).

The discretization of the geodesic flow allows for a number of choices, each choice giving rise to a definition of period functions. By Theorem A, all these spaces of period functions are isomorphic. The precise effect of the choices in the discretization is discussed in Section 4 below, where we also provide an explicit formula for the isomorphism between the different spaces of period functions.

In Section 2 below we recall the discretization of the geodesic flow, present the associated transfer operators and the definition of period functions, and provide the necessary background on the parabolic 1-cohomology characterization of Maass cusp forms. Theorem A is then proved in Section 3.

For the sample Fuchsian lattices \( \Gamma_0(p) \), \( p \) prime, and specific choices of the discretization, an adapted version of Theorem A has appeared in [Poh12]. The uniform structure of the Ford fundamental domains for \( \Gamma_0(p) \) allows to present the necessary constructions and definitions in a more depictive way, and the structure of the lattices \( \Gamma_0(p) \) simplifies the proof. Moreover, the period functions for Hecke triangle groups in [MP11] are special instances of this work as well. For several lattices also other approaches are known to define period functions, e.g. by [DH07] and [CM01]. It would be interesting to understand the precise isomorphism between those and the ones provided here.

### 2. Symbolic dynamics, transfer operators, and period functions

This section serves to present the additional geometric condition required of the considered Fuchsian lattices \( \Gamma \) and to briefly recall the discretization of the geodesic flow on \( \Gamma \backslash \mathbb{H} \) from [Poh10] as well as the characterization of Maass cusp forms in parabolic 1-cohomology from [BLZ09]. For proofs we refer to the original articles. Moreover, we provide a definition of period functions.

To simplify the exposition, we use the upper half plane

\[
\mathbb{H} := \{ z \in \mathbb{C} \mid \text{Im} \, z > 0 \}
\]

as model for the hyperbolic plane and identify its geodesic boundary with \( \mathbb{P}^1(\mathbb{R}) \cong \mathbb{R} \cup \{ \infty \} \). In this model, the group of orientation-preserving Riemannian isometries on \( \mathbb{H} \) can be identified with \( \text{PSL}(2, \mathbb{R}) \), whose action on \( \mathbb{H} \) is given by fractional linear transformations and extends continuously to \( \mathbb{P}^1(\mathbb{R}) \). Thus we have

\[
\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot z = \begin{cases} \frac{az + b}{cz + d} & \text{if } cz + d \neq 0 \\ \infty & \text{if } cz + d = 0 \end{cases}
\] and

\[
\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \infty = \begin{cases} \frac{a}{c} & \text{if } c \neq 0 \\ \infty & \text{if } c = 0 \end{cases}
\]

for \( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \text{PSL}(2, \mathbb{R}) \) and \( z \in \mathbb{H} \cup \mathbb{R} \). Throughout let \( \Gamma \) be a nonuniform cofinite Fuchsian lattice and suppose that \( \infty \) is a representative of a cusp of \( \Gamma \backslash \mathbb{H} \). Then the stabilizer group \( \Gamma_\infty = \text{Stab}_\Gamma(\infty) \) of \( \infty \) in \( \Gamma \) is generated by some element

\[
T := \begin{bmatrix} 1 & \lambda \\ 0 & 1 \end{bmatrix} \in \Gamma
\]
with \( \lambda > 0 \). A point in \( \mathbb{P}^1(\mathbb{R}) \) is called \textit{cuspidal} if it is fixed by a parabolic element in \( \Gamma \). We use \( S\mathbb{H} \) to denote the unit tangent bundle of \( \mathbb{H} \). The action of \( \Gamma \) extends to \( S\mathbb{H} \). By \( \Gamma \backslash \mathbb{H} \) resp. \( \Gamma \backslash S\mathbb{H} \) we denote the quotient space of the \( \Gamma \)-action on \( \mathbb{H} \) resp. \( S\mathbb{H} \). We remark that we may identify the unit tangent bundle of \( \Gamma \backslash \mathbb{H} \) with \( \Gamma \backslash S\mathbb{H} \).

If \( U \) is a subset of \( \mathbb{H} \), then we let \( \partial U \) denote its boundary. The complement of a set \( B \) in a set \( A \) is denoted by

\[
A \setminus B = \{ a \in A \mid a \notin B \}.
\]

Finally, a smooth function always refers to a \( C^\infty \) function.

2.1. \textbf{The additional requirement on} \( \Gamma \). The additional condition we require to be satisfied by \( \Gamma \) is of geometric nature and restricts the admissible boundary structure of the subset of \( \mathbb{H} \) which is common to all exteriors of isometric spheres of \( \Gamma \). In short, it says that there is a Ford fundamental domain for \( \Gamma \) constructed with respect to \( \infty \) such that the highest points of all non-vertical bounding complete geodesic segments are contained in the boundary of the fundamental domain but are not intersection points of two non-vertical sides of the fundamental domain.

Let \( g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma \setminus \Gamma_\infty \). Then the isometric sphere of \( g \) is the set

\[
I(g) := \{ z \in \mathbb{H} \mid |cz + d| = 1 \}.
\]

It is identical to the complete geodesic segment connecting \( \frac{-d-1}{c} \) and \( \frac{-d+1}{c} \), or the semi-circle in \( \mathbb{H} \) with center \( -\frac{d}{c} \) and radius \( \frac{1}{|c|} \). The exterior of \( I(g) \) is

\[
\text{ext } I(g) := \{ z \in \mathbb{H} \mid |cz + d| > 1 \}.
\]

The summit of \( I(g) \) is the point

\[
s = -\frac{d}{c} + \frac{i}{|c|} \in \mathbb{H}.
\]

Let

\[
\mathcal{K} := \bigcap_{g \in \Gamma \setminus \Gamma_\infty} \text{ext } I(g)
\]

be the common part of all exteriors of the isometric spheres of \( \Gamma \). This is a convex subset of \( \mathbb{H} \) which contains

\[
\{ z \in \mathbb{H} \mid \text{Im } z > y_0 \}
\]

for a sufficiently large \( y_0 > 0 \) and whose boundary is a locally finite union of geodesic segments which are connected subsets of isometric spheres. An isometric sphere of \( \Gamma \) is called \textit{relevant} if it coincides with the boundary of \( \mathcal{K} \) in more than one point.

From now on we impose the following condition on \( \Gamma \):

\[
(A) \quad \text{If for } g \in \Gamma \setminus \Gamma_\infty \text{ the isometric sphere } I(g) \text{ is relevant, then its summit is contained in } \partial \mathcal{K} \text{ but is not a vertex of } \mathcal{K}.
\]

For \( r \in \mathbb{R} \), let \( \mathcal{F}_\infty(r) := (r, r + \lambda) + i\mathbb{R}_{>0} \). Then

\[
\mathcal{F}(r) := \mathcal{F}_\infty(r) \cap \mathcal{K}
\]

is a Ford fundamental domain for \( \Gamma \). If we choose for \( r \) the center of a relevant isometric sphere, then \( \mathcal{F}(r) \) is a fundamental domain as described above.
2.2. Discretization. The starting point in [Poh10] for the discretization of the geodesic flow on $\Gamma \backslash \mathbb{H}$ is a specific choice of a cross section in the sense of Poincaré for this flow. This cross section is a subset $\hat{C}$ on the unit tangent bundle $\Gamma \backslash S\mathbb{H}$ of $\Gamma \backslash \mathbb{H}$ which is intersected by almost all geodesics infinitely often in the past and the future, and each such intersection is discrete in time. Geodesics are here parametrized by arc length, and “almost all” refers to all geodesics which do not converge to a cusp forward or backward in time. Before we expound the construction of $\hat{C}$ in Section 2.3 below, we briefly explain how it gives rise to a discrete dynamical system on subsets of $\mathbb{R}$.

For $\hat{v} \in \hat{C}$ let $\hat{\gamma}_v$ denote the geodesic on $\Gamma \backslash \mathbb{H}$ determined by

$$\frac{d}{dt}|_{t=0} \hat{\gamma}_v(t) = \hat{v}.$$ 

The choice of $\hat{C}$ yields that if $\hat{\gamma}_v$ does not converge to a cusp, there is a minimal return time of $\hat{\gamma}_v$ to $\hat{C}$, that is a minimal time $t(\hat{v}) > 0$ such that

$$\frac{d}{dt}|_{t=t(\hat{v})} \hat{\gamma}_v(t) \in \hat{C}.$$ 

Therefore $\hat{C}$ induces the (partially defined) first return map

$$\mathcal{R}: \hat{C} \to \hat{C}, \quad \hat{v} \mapsto \frac{d}{dt}|_{t=t(\hat{v})} \hat{\gamma}_v(t).$$

The precise domain of definition for $\mathcal{R}$ is discussed in detail in [Poh10]. The sole property of this domain used hiddenly here is its density in $\hat{C}$.

A major property of the cross section $\hat{C}$ is that it has a set of representatives $C'$ in $S\mathbb{H}$ which decomposes (uniquely) into a finite number $C'_1, \ldots, C'_k$ of subsets each of which is either of the form

$$C'_j = \left\{ X \in S\mathbb{H} \mid X = a \frac{\partial}{\partial x}|_{r_j+iy} + b \frac{\partial}{\partial y}|_{r_j+iy}, \ a > 0, \ b \in \mathbb{R}, \ y > 0 \right\}$$

for some cuspidal point $r_j \in \mathbb{R}$, or

$$C'_j = \left\{ X \in S\mathbb{H} \mid X = a \frac{\partial}{\partial x}|_{r_j+iy} + b \frac{\partial}{\partial y}|_{r_j+iy}, \ a < 0, \ b \in \mathbb{R}, \ y > 0 \right\}$$

for some cuspidal point $r_j \in \mathbb{R}$. In other words, each $C'_j$ consists of the unit tangent vectors in $S\mathbb{H}$ which are based on the complete geodesic segment $(r_j, \infty)$ and point into either of the halfspaces $\{\Re z > r_j\}$ or $\{\Re z < r_j\}$. For given a subset $U$ of $\mathbb{H}$ and a unit tangent vector $v \in S\mathbb{H}$, we say that $v$ points into $U$ if the geodesic $\gamma_v$ determined by

$$\frac{d}{dt}|_{t=0} \gamma_v(t) = v$$

immediately runs into $U$. More precisely, if there exists $\eta > 0$ such that $\gamma_v((0, \eta)) \subseteq U$.

The combination of this partition of $C'$ and the first return map $\mathcal{R}$ allows to define a discrete dynamical system on parts of the geodesic boundary of $\mathbb{H}$ which is conjugate to $\mathcal{R}$. To that end, let

$$\bar{D}_j := \{ (\gamma_v(-\infty), \gamma_v(+\infty), j) \mid v \in C'_j \} \quad \text{for} \ j = 1, \ldots, k,$$
and
\[ \tilde{D} := \prod_{j=1}^{k} \tilde{D}_j. \]

Then the map \( \tau: \hat{C} \to \tilde{D} \) defined by
\[ \tau(\hat{v}) := (\gamma_v(\infty), \gamma_v(+\infty), j) \] if \( \hat{v} \) is represented by \( v \in C'_j \)

is a bijection. Thus, there is a unique (partially defined) self-map \( \tilde{F} \) on \( \tilde{D} \) which is conjugate to \( \mathcal{R} \) via \( \tau \). Its domain of definition corresponds to the domain of definition of \( \mathcal{R} \).

For each \( j \in \{1, \ldots, k\} \), the structure of \( C'_j \) yields that \( \tilde{D}_j \) is either of the form
\[ (-\infty, r_j) \times (r_j, \infty) \times \{j\} \quad \text{or} \quad (r_j, \infty) \times (-\infty, r_j) \times \{j\}. \]

As shown in \( \text{[Poh10]} \), the map \( \tilde{F} \) is locally given by Möbius transformations of specific elements from \( \Gamma \), and thus has an easy structure.

From the discrete dynamical system \( (\tilde{D}, \tilde{F}) \) we will only need its expanding direction, which means its projection to the last two components. We denote this restricted discrete dynamical system with \( (D, F) \), where
\[ D := \prod_{j=1}^{k} D_j, \quad D_j := \{ (\gamma_v(\infty), j) \mid v \in C'_j \} \]

and \( F \) is the self-map of \( D \) which is induced by \( \tilde{F} \). To be precise, \( F \) is a self-map only on
\[ D \setminus \{(r, j) \mid r \text{ cuspidal, } j = 1, \ldots, k\} \]

and can be analytically extended to a map defined on \( D \) up to finitely many points. Here we work with this analytic extension and still write \( D \) for its domain of definition. It will always be clear on which points of \( D \) the map \( F \) is not defined.

2.3. Cross section and choice of set of representatives. For the definition of the cross section \( \hat{C} \) and a choice of its set \( C' \) of representatives we consider \( \mathcal{K} \) as a subset of \( \mathbb{H} \cup \mathbb{P}^1(\mathbb{R}) \).

The vertices of \( \mathcal{K} \) which are contained in \( \mathbb{H} \) are called \textit{inner vertices}, those which are contained in \( \mathbb{R} \) are called \textit{infinite vertices}. We decompose its closure \( \overline{\mathcal{K}} \) as follows into a collection of hyperbolic triangles and rectangles. If \( v \neq \infty \) is a vertex of \( \mathcal{K} \), then \( v \) is either the intersection point or the common endpoint of two (uniquely determined) relevant isometric spheres. Let \( s_1 \) resp. \( s_2 \) be their summits. If \( v \) is inner, then we form the hyperbolic rectangle with vertices \( \infty, s_1, v \) and \( s_2 \). If \( v \) is infinite, then we form the two hyperbolic triangles with vertices \( \infty, v \) and \( s_1 \) resp. \( \infty, v \) and \( s_2 \). In any case, if a side of these triangles and rectangles has \( \infty \) as one endpoint, then we call it \textit{vertical}, otherwise \textit{non-vertical}. Let \( \mathcal{C} \) denote this collection of hyperbolic triangles and rectangles.

Let \( \hat{C} \) be the set of unit tangent vectors \( X \in S\mathbb{H} \) such that \( X \) is based on a vertical side of an element in \( \mathcal{C} \) but not tangent to this side. Further let \( \pi: S\mathbb{H} \to \Gamma \backslash \mathbb{H} \) denote the quotient map. Then we choose
\[ \hat{C} := \pi(\tilde{C}) \]
as cross section for the geodesic flow on $\Gamma \backslash S\mathbb{H}$. To find a set of representatives for $\hat{C}$ with the properties announced in Section 2.2 we proceed as follows.

The elements of $\mathcal{C}$ (now considered as subsets of $\mathbb{H}$) provide a tesselation of $\mathbb{H}$. This means their $\Gamma$-translates cover $\mathbb{H}$ and, whenever two $\Gamma$-translates of elements in $\mathcal{C}$ have a point in common, then it is either a single point which is a common vertex of both translates or they coincide at a common side or they are equal. Out of the family $\mathcal{C}$ we pick a subfamily $\mathcal{A}$ of triangles and rectangles whose union forms a (closed) fundamental region for $\Gamma$ in $\mathbb{H}$. Within the family $\mathcal{A}$, the tesselation property induces a unique and well-defined side-pairing. In analogy with Poincaré’s Fundamental Polyhedron Theorem we use this side-pairing to define cycles as explained in the following. For this we remark that non-vertical sides of rectangles (resp. triangles) in $\mathcal{A}$ can only be paired with non-vertical sides of rectangles (resp. triangles).

Let $\mathcal{A} \in \mathcal{A}$ be a rectangle. Suppose that $v$ is the inner vertex of $\mathcal{K}$ to which $\mathcal{A}$ is associated, and let $b_1, b_2$ denote the two non-vertical sides of $\mathcal{A}$. We denote by $k_1(\mathcal{A}), k_2(\mathcal{A})$ the two elements in $\Gamma \backslash \Gamma_\infty$ such that $b_j \in I(k_j(\mathcal{A}))$ and $k_j(\mathcal{A})b_j$ is a non-vertical side of some rectangle in $\mathcal{A}$. These are the side-pairing elements for the non-vertical sides of $\mathcal{A}$. We define $\mathcal{A}(v) := \mathcal{A}$ and any choice $h \in \{k_1(\mathcal{A}), k_2(\mathcal{A})\}$ we assign a finite sequence in $\mathcal{A} \times \Gamma$ using the following algorithm:

Set $v_1 := v$, $A_1 := \mathcal{A}(v_1)$, $h_1 := h$, $g_1 := id$ and $g_2 := h_1$. Iteratively for $j = 2, 3, \ldots$ set $v_j := g_j(v)$ and $A_j := \mathcal{A}(v_j)$. Let $h_j$ be the element in $\Gamma \backslash \Gamma_\infty$ such that $\{h_j, h_j^{-1}\} = \{k_1(\mathcal{A}_j), k_2(\mathcal{A}_j)\}$. Set $g_{j+1} := h_jg_j$. The algorithm stops if $g_{j+1} = id$. We assign to $(A_1, h_1)$ the sequence (the cycle) $((A_j, h_j))_{j=1,\ldots,k}$ where $k > 2$ is minimal such that $g_{k+1} = id$.

We consider the two sequences determined by $(\mathcal{A}, k_1(\mathcal{A}))$ and $(\mathcal{A}, k_2(\mathcal{A}))$ as equivalent, as well as any sequences determined by any element $(\mathcal{A}', h')$ of these sequences.

Let $\mathcal{A} \in \mathcal{A}$ be a triangle and let $b$ be its non-vertical side. Then there are unique elements $h \in \Gamma \backslash \Gamma_\infty$ and $\mathcal{A}' \in \mathcal{A}$ such that $b \in I(h)$ and $hb$ is the non-vertical side of $\mathcal{A}'$. We assign to $(\mathcal{A}, h)$ the cycle $((\mathcal{A}, h), (\mathcal{A}', h^{-1}))$, which we consider to be equivalent to the cycle $((\mathcal{A}', h^{-1}), (\mathcal{A}, h))$.

For any of these cycles in $\mathcal{A} \times \Gamma$ we call any element $(\mathcal{A}, h)$ contained in it a generator of the sequence or its equivalence class. To define a set of representatives $\hat{\mathcal{C}}$ for $\hat{C}$ we fix a generator for each equivalence class of cycles. Let $\mathcal{S}$ denote the set of chosen generators.

Let $(\mathcal{A}, h)$ be an element of a cycle in $\mathcal{A} \times \Gamma$. Then one of the vertical sides of $\mathcal{A}$ is contained in the geodesic segment $(h^{-1} \cdot \infty, \infty)$. We define

$$
\varepsilon(\mathcal{A}, h) := \begin{cases} +1 & \text{if } \mathcal{A} \subseteq \{ z \in \mathbb{H} \mid \text{Re } z \geq h^{-1} \cdot \infty \}, \\ -1 & \text{if } \mathcal{A} \subseteq \{ z \in \mathbb{H} \mid \text{Re } z \leq h^{-1} \cdot \infty \}. \end{cases}
$$

Let $(\mathcal{A}, h) \in \mathcal{S}$. Suppose first that $\mathcal{A}$ is a rectangle. Let $((\mathcal{A}_j, h_j))_{j=1,\ldots,k}$ be the cycle in $\mathcal{A} \times \Gamma$ determined by $(\mathcal{A}, h)$. Let

$$
\text{cyl}(\mathcal{A}) := \min \{ \{ \ell \in \{1, \ldots, k-1\} \mid A_{\ell+1} = \mathcal{A} \} \cup \{k\} \}.
$$
For $j = 1, \ldots, \text{cyl}(A)$ we set

$$C'_{(A,h),j} := \left\{ \frac{\partial}{\partial x} h_{j}^{-1} + i y h_{j}^{-1} \in S^H \mid \varepsilon_j \cdot a > 0, \ b \in \mathbb{R}, \ y > 0 \right\}$$

where $\varepsilon_j := \varepsilon(A_j, h_j)$. Suppose now that $A$ is a triangle and let $(A, h)$ be the cycle determined by $(A, h)$. Let $v, v' \in \mathbb{R}$ be the infinite vertices of $\mathcal{K}$ to which $A$ resp. $A'$ are associated. Choose an integer $m = m(A, h) \in \mathbb{Z}$ and set $\varepsilon := \varepsilon(A, h)$. We define

$$C_{(A,h),1} := \left\{ \frac{\partial}{\partial x} v + b \frac{\partial}{\partial y} v \in S^H \mid \varepsilon \cdot a > 0, \ b \in \mathbb{R}, \ y > 0 \right\},$$

$$C_{(A,h),2} := \left\{ \frac{\partial}{\partial x} v' + b \frac{\partial}{\partial y} v' \in S^H \mid \varepsilon \cdot a < 0, \ b \in \mathbb{R}, \ y > 0 \right\},$$

$$C_{(A,h),3} := \left\{ \frac{\partial}{\partial x} T^m h_{-1} + i y h_{-1} + b \frac{\partial}{\partial y} T^m h_{-1} + i y \in S^H \mid \varepsilon \cdot a < 0, \ b \in \mathbb{R}, \ y > 0 \right\}.$$ 

The choice of the integer $m$ will affect the subsequent steps. We record it with the map $T : (A, h) \mapsto m(A, h)$ defined on the elements $(A, h) \in \mathbb{S}$ for which $A$ is a triangle. Then

$$C' := \bigcup_{(A,h) \in \mathbb{S}} \text{cyl}(A) \bigcup_{(A,h) \in \mathbb{S}} \text{rectangle} \bigcup_{(A,h) \in \mathbb{S}} \text{triangle} \bigcup_{j=1}^{3} C'_{(A,h),j}$$

is a set of representatives for $\tilde{C}$ with the properties described in Section 2.2.

2.4. The induced discrete dynamical system $(D, F)$. Let

$$\Sigma := \left\{ ((A, h), j) \mid (A, h) \in \mathbb{S}, \ A \text{ rectangle}, j = 1, \ldots, \text{cyl}(A) \right\} \cup \left\{ ((A, h), j) \mid (A, h) \in \mathbb{S}, \ A \text{ triangle}, j = 1, 2, 3 \right\}$$

denote the arising set of symbols. Then

$$C' = \coprod_{\alpha \in \Sigma} C'_\alpha.$$

We call the sets $C'_\alpha, \ \alpha \in \Sigma$, the components of $C'$.

To simplify notations, we use the following conventions. For $r \in \mathbb{R}$ and $\varepsilon \in \{ \pm1 \}$ we let

$$\langle r, \varepsilon \infty \rangle := \begin{cases} (r, \infty) & \text{if } \varepsilon = +1 \\ (-\infty, r) & \text{if } \varepsilon = -1. \end{cases}$$

Let $\alpha \in \Sigma$. Recall that $C'_\alpha$ is based on the complete geodesic segment $(r_\alpha, \infty)$ for a cuspidal point $r_\alpha \in \mathbb{R}$. We define

$$\varepsilon_\alpha := \begin{cases} +1 & \text{if the elements of } C'_\alpha \text{ point into } \{ \Re z > r_\alpha \}, \\ -1 & \text{if the elements of } C'_\alpha \text{ point into } \{ \Re z < r_\alpha \}. \end{cases}$$

Further, we let

$$I_\alpha := \langle r_\alpha, \varepsilon_\alpha \infty \rangle \quad \text{and} \quad D_\alpha := I_\alpha \times \{ \alpha \}.$$ 

Then

$$D = \coprod_{\alpha \in \Sigma} D_\alpha.$$
An explicit expression for the map \( F : D \to D \) can be deduced as follows. Given a point \((r, \alpha) \in D_\alpha\) for some \(\alpha \in \Sigma\) we pick any element \(v \in C'_\alpha\) such that \(\gamma_r(\infty) = r\). Let \(\tilde{\gamma}_v = \Gamma.\gamma_v\) be the corresponding geodesic on \(\Gamma \setminus \mathbb{H}\) and let \(t_0\) be the first return time of \(\tilde{\gamma}_v\) to \(\tilde{C}\). Then \(\gamma'_v(t_0)\) is contained in a (unique) \(\Gamma\)-translate of some component of \(C'_\beta\), say \(\gamma'_v(t_0) \in g.C'_\beta\). Thus,

\[
F(r, \alpha) = (g^{-1}r, \beta).
\]

The exact values for \(g\) and \(\beta\) can be algorithmically calculated from the side-pairing in \(\tilde{A}\). The outcome is that \(F\) restricts to a finite number of local diffeomorphisms of the form

\[
I_\alpha \cap g. I_\beta \to I_\beta, \quad (r, \alpha) \mapsto (g^{-1}r, \beta).
\]

For details we refer to [Poh10].

2.5. \textbf{The associated family of transfer operators.} For each \(s \in \mathbb{C}\), the transfer operator with parameter \(s\) associated to the discrete dynamical system \((D, F)\) is the operator

\[
(\mathcal{L}_{F,s}f)(x) := \sum_{y \in F^{-1}(x)} \frac{f(y)}{|F'(y)|^s}
\]

defined on the space \(\text{Fct}(D; \mathbb{C})\) of complex-valued functions on \(D\). In this section we provide a matrix representation for \(\mathcal{L}_{F,s}\).

For any function \(\varphi : V \to \mathbb{R}\) on some subset \(V\) of \(\mathbb{R}\) and for any \(g \in \Gamma\) we set

\[
(\tau_s(g^{-1})\varphi)(r) := (g'(r))^s \varphi(g.r)
\]

whenever it is well-defined. For appropriate sets \(V\), the map \(\tau_s\) is a left \(\Gamma\)-action. It is essentially (depending on conventions) the left-action variant of the so-called slash action.

For \(\alpha \in \Sigma\) and \(f \in \text{Fct}(D; \mathbb{C})\) we let

\[
f_\alpha := f \cdot 1_{D_\alpha},
\]

where \(1_{D_\alpha}\) denotes the characteristic function of the set \(D_\alpha\). Then \(f = \sum_{\alpha \in \Sigma} f_\alpha\). We may identify \(f\) with the vector \((f_\alpha)_{\alpha \in \Sigma}\) and then \(D_\alpha\) with \(I_\alpha\). Let

\[
(\tilde{f}_\alpha) = \tilde{f} := \mathcal{L}_{F,s}f.
\]

We derive explicit expressions for \(\tilde{f}_\alpha\), \(\alpha \in \Sigma\), in dependence of the component functions \(f_\beta\), \(\beta \in \Sigma\).

Let \(\alpha \in \Sigma\). Let \(v \in C'_\alpha\) and suppose that

\[
\gamma'_\alpha((-\infty, 0)) \cap \Gamma.C' \neq \emptyset.
\]

Then there exists a \textit{previous time of intersection}, namely

\[
t_1 := \max\{t < 0 \mid \gamma'_\alpha(t) \in \Gamma.C'\}.
\]

We call \(\gamma'_\alpha(t_1)\) the \textit{previous intersection} of \(v\). To determine for a given \(x = (r, \alpha) \in D_\alpha\) its preimages under \(F\) is equivalent to determine for the vectors in \(C'_\alpha\) their locations of previous intersection. These can be deduced from the side-pairing in \(\tilde{A}\) as explained in the following. For \(C'_\alpha\), there exists a unique generator \((A, h) \in \mathbb{S}\) and a unique element \((A', h')\) in the cycle determined by \((A, h)\) and a unique \(n \in \mathbb{Z}\) such that one of the vertical sides of \(T^nA'\) is contained in the base set of \(C'_\alpha\) and

\[
T^nA' \subseteq \{z \in \mathbb{H} \mid \varepsilon_\alpha \cdot (\text{Re} z - r_\alpha) < 0\}.
\]
In other words, some subset of $C'_\alpha$ is based on a vertical side of $T^nA'$ and the elements of $C'_\alpha$ do not point into $T^nA'$. In this case, we say that $C'_\alpha$ is neighboring $T^nA'$. We have to distinguish the following three situations:

**Situation (1):** $\mathcal{A}'$ is a rectangle. Then we are in one of the situations shown in Figure 1.

![Figure 1. Situation (1)](image1)

**Situation (2):** $\mathcal{A}'$ is a triangle and $C'_\alpha$ is neighboring $T^n\mathcal{A}'$ on its long side. Then we are in one of the situations shown in Figure 2.

![Figure 2. Situation (2)](image2)

**Situation (3):** $\mathcal{A}'$ is a triangle and $C'_\alpha$ is neighboring $T^n\mathcal{A}'$ on its short side. Then we are in one of the situations shown in Figure 3.

Note that the two sub-situations shown in any of the Figures 1-3 are mirror-inverted and thus are equivalent for all further considerations. We will not distinguish these in the following figures. However, their differences are taken into account in all formulas by $\varepsilon_\alpha$. For the locations of the previous intersections we have to subdivide these situations. All further numbering of situations will refer and extend the one just introduced.

**Situation (1):** Let $((\mathcal{A}_\ell, h_\ell))_{\ell=1,\ldots,k}$ be the cycle determined by $(\mathcal{A}, h)$ and suppose that $\mathcal{A}' = \mathcal{A}_j$ for some $j \in \{1,\ldots,k\}$. We set

$$C'_\ell := C'_{(\mathcal{A}, h), \ell}, \quad f_\ell := f_{(\mathcal{A}, h), \ell} \quad \text{and} \quad \varepsilon_\ell := \varepsilon_{(\mathcal{A}, h), \ell}$$
for \( \ell = 1, \ldots, k \), where the index \( \ell \) is understood modulo \( \text{cyl}(A) \). According to the rotation direction of the cycle related to the orientation of \( C'_{\alpha} \) we are either in Situation (1a) shown in Figure 4 \( (\varepsilon_\alpha = \varepsilon_j) \) or in Situation (1b) shown in Figure 5 \( (\varepsilon_\alpha = -\varepsilon_j) \).

**Figure 3. Situation (3)**

**Figure 4. Situation (1a)**

**Figure 5. Situation (1b)**
In Situation (1a) we have
\[ \tilde{f}_\alpha = \tau_s(T^n)f_j + \tau_s(T^n h_j^{-1})f_{j+1} + \cdots + \tau_s(T^n h_j^{-1} \cdots h_{j+k-3}^{-1})f_{j+k-2} \]
\[ = \sum_{\ell=0}^{k-2} \tau_s(T^n h_j^{-1} \cdots h_{j+\ell-1}^{-1})f_{j+\ell}, \]
whereas in Situation (1b) we have
\[ \tilde{f}_\alpha = \tau_s(T^n h_j^{-1} f_{j+1} + \tau_s(T^n h_j^{-1} h_{j+1}^{-1})f_{j+2} + \cdots + \tau_s(T^n h_j^{-1} \cdots h_{j+k-2}^{-1})f_{j+k-1} \]
\[ = \sum_{\ell=0}^{k-2} \tau_s(T^n h_j^{-1} \cdots h_{j+\ell-1}^{-1})f_{j+\ell+1}. \]

**Situation (2):** Let \(((A, h), (A', h))\) be the cycle determined by \((A, h)\) (note that this \(A'\) is not necessarily the \(A'\) from above) and let \(m := m(A, h)\). We set \(C'_j := C'_j(A, h), f_j := f_j(A, h)\), and \(\varepsilon_j := \varepsilon_j(A, h, j)\) for \(j = 1, 2, 3\). Then we have either
Situation (2a) \((\varepsilon_\alpha = \varepsilon_1)\) or Situation (2b) \((\varepsilon_\alpha = -\varepsilon_1)\) shown in Figure 6.

![Figure 6](image)

**Figure 6.** On the left Situation (2a), on the right Situation (2b)

In Situation (2a) we have
\[ \tilde{f}_\alpha = \tau_s(T^n h) f_1 + \tau_s(T^n h T^{-m}) f_3, \]
and in Situation (2b) we have
\[ \tilde{f}_\alpha = \tau_s(T^n h^{-1}) f_2 + \tau_s(T^{n-m}) f_3. \]

**Situation (3):** We use the notation from Situation (2). Then we have either
Situation (3a) \((\varepsilon_\alpha = \varepsilon_1)\) or Situation (3b) \((\varepsilon_\alpha = -\varepsilon_1)\) shown in Figure 7.

In Situation (3a) we have
\[ \tilde{f}_\alpha = \tau_s(T^n) f_1 + \tau_s(T^n h^{-1}) f_2, \]
whereas in Situation (3b) we have
\[ \tilde{f}_\alpha = \tau_s(T^n h) f_1 + \tau_s(T^n) f_2. \]
2.6. **Period functions.** Let $s \in \mathbb{C}$. We say that a function $\varphi : \mathbb{R} \to \mathbb{C}$ extends smoothly ($C^\infty$) to $P^1(\mathbb{R})$ if for some (and indeed any) element $g \in \Gamma \setminus \Gamma_\infty$, the functions $\varphi$ and $\tau_s(g)\varphi$ are smooth on $\mathbb{R}$. Note that this notion of smooth extension depends on $s$ and $\Gamma$.

For any $\alpha \in \Sigma$, we find a unique $\beta \in \Sigma$ and unique $g \in \Gamma$ such that $C'_\alpha$ and $g.C'_\beta$ are based on the same geodesic segment but are disjoint. We call $(\beta, g)$ the **tuple assigned to $\alpha$**. The precise values for $\beta$ and $g$ can be read off from Situations (1a)–(3b). For $f \in \text{Fct}(D; \mathbb{C})$ we define

$$
\psi_{\alpha,f} := \begin{cases} 
\varepsilon_\alpha f_\alpha & \text{on } \langle r_\alpha, \varepsilon_\alpha \infty \rangle \\
-\varepsilon_\alpha \tau_s(g) f_\beta & \text{on } \langle r_\alpha, -\varepsilon_\alpha \infty \rangle.
\end{cases}
$$

The space of **period functions** $\text{FE}^\omega_{\omega, \text{dec}}(\Gamma; \mathbb{A}, \mathbb{S}, \mathbb{T})$ (relative to the choices of $\mathbb{A}$, $\mathbb{S}$ and $\mathbb{T}$) is defined to be the space of function vectors $f = (f_\alpha)_{\alpha \in \Sigma}$ such that

- (PF1) $f_\alpha \in C^\omega(I_\alpha; \mathbb{C})$ for $\alpha \in \Sigma$,
- (PF2) $f = \mathcal{L}_{F,s} f$,
- (PF3) If for $\alpha \in \Sigma$, the map $\psi_{\alpha,f}$ in (2) arises from Situation (1a) or (3b) (that is, $\beta$ and $g$ are determined by these situations), then $\psi_{\alpha,f}$ extends smoothly to $\mathbb{R}$.
- (PF4) If for $\alpha \in \Sigma$, the map $\psi_{\alpha,f}$ is not determined by Situation (1a) or (3b), then it extends smoothly to $P^1(\mathbb{R})$.

**Remark 2.1.** If (PF3) is satisfied, then the maps considered there actually extend smoothly to $P^1(\mathbb{R})$ by the following argument: If

$$
\psi_\alpha = \begin{cases} 
\varepsilon_\alpha f_\alpha & \text{on } \langle r_\alpha, \varepsilon_\alpha \infty \rangle \\
-\varepsilon_\alpha \tau_s(g) f_\beta & \text{on } \langle r_\alpha, -\varepsilon_\alpha \infty \rangle
\end{cases}
$$
is one of these maps, then

\[ \phi_\beta = \begin{cases} 
\varepsilon_\beta f_\beta & \text{on } (r_\beta, \varepsilon_\beta \infty) \\
-\varepsilon_\beta \tau_s(g^{-1}) f_\alpha & \text{on } (r_\beta, -\varepsilon_\beta \infty) 
\end{cases} \]

is also one and \( \varepsilon_\alpha = \varepsilon_\beta, r_\beta = g^{-1} \infty \) and \( g^{-1} r_\alpha = \infty \). Thus, \( \tau_s(g^{-1}) \psi_\alpha = \psi_\beta \).

### 2.7. Parabolic 1-cohomology

For the proof of Theorem A we take advantage of the characterization in [BLZ09] of Maass cusp forms with eigenvalue \( s(1-s) \) as parabolic 1-cocycle classes with values in the semi-analytic smooth vectors of the principal series representation. In the following we briefly recall this characterization.

Let \( s \in \mathbb{C} \). The space \( \mathcal{V}_s^{\omega, \infty} \) of semi-analytic smooth vectors in the line model of the principal series representation with spectral parameter \( s \) is the space of functions \( \varphi: \mathbb{R} \rightarrow \mathbb{C} \) which are smooth and extend smoothly to \( \mathbb{R} \setminus E \), where \( E \) is a finite subset which may depend on \( \varphi \). The lattice \( \Gamma \) acts on \( \mathcal{V}_s^{\omega, \infty} \) via the action \( \tau_s \) from (1).

Recall that the space of 1-cocycles of group cohomology of \( \Gamma \) with values in \( \mathcal{V}_s^{\omega, \infty} \) is

\[ Z^1(\Gamma; \mathcal{V}_s^{\omega, \infty}) = \{ c: \Gamma \rightarrow \mathcal{V}_s^{\omega, \infty} \mid \forall g, h \in \Gamma: c_{gh} = \tau_s(h^{-1}) c_g + c_h \} \].

We use here the notation of restricted cocycles and write \( c_g \in \mathcal{V}_s^{\omega, \infty} \) for the image \( c(g) \) of \( g \in \Gamma \) under \( c \in Z^1(\Gamma; \mathcal{V}_s^{\omega, \infty}) \). The space of parabolic 1-cocycles is

\[ Z^1_{par}(\Gamma; \mathcal{V}_s^{\omega, \infty}) = \left\{ c \in Z^1(\Gamma; \mathcal{V}_s^{\omega, \infty}) \left| \begin{array}{l}
\forall p \in \Gamma \text{ parabolic } \exists \psi \in \mathcal{V}_s^{\omega, \infty}: \\
c_p = \tau_s(p^{-1}) \psi - \psi \end{array} \right. \right\} \].

The spaces of 1-coboundaries of group cohomology and of parabolic cohomology are equal. They are

\[ B^1_{par}(\Gamma; \mathcal{V}_s^{\omega, \infty}) = B^1(\Gamma; \mathcal{V}_s^{\omega, \infty}) = \{ g \mapsto \tau_s(g^{-1}) \psi - \psi \mid \psi \in \mathcal{V}_s^{\omega, \infty} \} \].

Then the parabolic 1-cohomology space is the quotient space

\[ H^1_{par}(\Gamma; \mathcal{V}_s^{\omega, \infty}) = Z^1_{par}(\Gamma; \mathcal{V}_s^{\omega, \infty}) / B^1_{par}(\Gamma; \mathcal{V}_s^{\omega, \infty}) \].

Let \( \text{MCF}_s(\Gamma) \) denote the space of Maass cusp forms for \( \Gamma \) with eigenvalue \( s(1-s) \).

**Theorem 2.2 ([BLZ09]).** Let \( s \in \mathbb{C} \) with \( \text{Re } s \in (0,1) \). Then the vector spaces \( \text{MCF}_s(\Gamma) \) and \( H^1_{par}(\Gamma; \mathcal{V}_s^{\omega, \infty}) \) are isomorphic.

The isomorphism in Theorem 2.2 is constructive and given by the following integral transform. Let \( u \in \text{MCF}_s(\Gamma) \) and choose any \( z_0 \in \mathbb{H} \) or cuspidal. Then the parabolic 1-cocycle class \( [c] \in H^1_{par}(\Gamma; \mathcal{V}_s^{\omega, \infty}) \) associated to \( u \) is represented by the cocycle \( c \) given by

\[ c_g(r) := \int_{g^{-1} z_0}^{z_0} [u, R(r, \cdot) \ast] \]

for \( g \in \Gamma \). Here we use \( R: \mathbb{R} \times \mathbb{H} \rightarrow \mathbb{H} \),

\[ R(r, z) := \text{Im} \left( \frac{1}{r - z} \right) \]

and

\[ [u, v] := \frac{\partial u}{\partial z} \cdot v \cdot dz + u \cdot \frac{\partial v}{\partial z} \cdot d\overline{z} \quad \text{ (Green form)} \]
for any complex valued smooth function \( v \) on \( \mathbb{H} \). The integration is performed along any differentiable path from \( g^{-1}z_0 \) to \( z_0 \) which is essentially contained in \( \mathbb{H} \). The integral is well-defined since the 1-form \([u, R(r, \cdot)^c]\) is closed. A change of the choice of \( z_0 \) changes \( c \) by a parabolic 1-coboundary. The \( \Gamma \)-action via \( \tau_s \) translates into a change of path of integration. More precisely, we have

\[
\tau_s(g^{-1}) \int_a^b [u, R(r, \cdot)^c] = \int_{g^{-1} \cdot a}^{g^{-1} \cdot b} [u, R(r, \cdot)^c]
\]

for all \( g \in \Gamma \) and cuspidal points \( a, b \).

3. ISOMORPHISM OF PERIOD FUNCTIONS AND MAASS CUSP FORMS

In this section we prove the following statement:

**Theorem 3.1.** For \( s \in \mathbb{C} \) with \( \text{Re } s \in (0, 1) \), the vector spaces \( \text{FE}^{\omega, \text{dec}}_s(\Gamma; \mathbb{A}, \mathbb{S}, \mathbb{T}) \) and \( H^1_{\text{par}}(\Gamma; \mathcal{V}^{\omega_\infty}_s) \) are isomorphic.

This, together with Theorem 2.2, establishes Theorem A. The isomorphism in Theorem 3.1 is provided by the two constructions presented in the following. For these, let

\[
\Sigma' := \{ \alpha \in \Sigma \mid r_\alpha \text{ is } \Gamma\text{-equivalent to } \infty \}.
\]

If \( \alpha \in \Sigma' \), then \( C'_\alpha \) is as in Situation (1) or (3). Hence there exists \( b \in \Gamma \) such that \( C'_\alpha \) is based on the geodesic segment \( (b^{-1} \cdot \infty, \infty) \). The element \( b \) is unique only up to left multiplication with elements in \( \Gamma_\infty \). Let \((\beta, g)\) be the tuple assigned to \( \alpha \) (cf. Section 2.6). For any possible choice of \( b \), we call \((\beta, g, b)\) a triple assigned to \( \alpha \). Let \( \mathcal{S} := \{ b \in \Gamma \mid \exists \alpha \in \Sigma' \exists \beta \in \Sigma \exists g \in \Gamma : (\beta, g, b) \text{ is assigned to } \alpha \} \cup \{ T \} \).

(a) Let \( f \in \text{FE}_s(\Gamma; \mathbb{A}, \mathbb{S}, \mathbb{T}) \). We define a map

\[
c := c(f) : \mathcal{S} \rightarrow \mathcal{V}^{\omega_\infty}_s
\]

by \( c_T := 0 \) and

\[
c_\beta := \psi_{\alpha,f} \quad \text{for } \alpha \in \Sigma',
\]

where \((\beta, g, b) \in \Sigma \times \Gamma \times \Gamma \) is a triple assigned to \( \alpha \). Proposition 3.3 below shows that \( c \) determines a unique parabolic 1-cocycle.

(b) Let \([c] \in H^1_{\text{par}}(\Gamma; \mathcal{V}^{\omega_\infty}_s)\). Pick its unique representative \( c \in Z^1_{\text{par}}(\Gamma; \mathcal{V}^{\omega_\infty}_s) \) for which \( c_T = 0 \). We associate to \([c]\) a function vector \( f([c]) = (f_\alpha)_{\alpha \in \Sigma} \) as follows. Suppose first that \( \alpha \in \Sigma' \). Pick a triple \((\beta, g, b) \in \Sigma \times \Gamma \times \Gamma \) which is assigned to \( \alpha \). We define

\[
f_\alpha := \varepsilon_\alpha c_\beta \cdot 1_{I_\alpha}.
\]

Suppose now that \( \alpha \in \Sigma \setminus \Sigma' \). Recall that \( C'_\alpha \) is then based on a geodesic segment of the form \((v, \infty)\) with \( v \) being a cuspidal point which is not \( \Gamma \)-equivalent to \( \infty \). Let \( p \) be a generator of \( \text{Stab}_\Gamma(v) \). Let \( \psi \in \mathcal{V}^{\omega_\infty}_s \) be the unique element (see [Poh12, Lemma 3.3]) such that

\[
c_p = \tau_s(p^{-1})\psi - \psi.
\]

We define

\[
f_\alpha := -\varepsilon_\alpha \psi \cdot 1_{I_\alpha}.
\]

Instead of Theorem 3.1 we show its following concretization.
**Theorem 3.2.** Let $s \in \mathbb{C}$ with $0 < \Re s < 1$. Then the map

$$\text{FE}_s^{\omega, \text{dec}}(\Gamma; \mathbb{A}, \mathbb{S}, \mathbb{T}) \to H^1_{\text{par}}(\Gamma; V_{s}^{\omega, \infty}), \quad f \mapsto [c(f)]$$

is a linear isomorphism. Its inverse map is given by

$$H^1_{\text{par}}(\Gamma; V_{s}^{\omega, \infty}) \to \text{FE}_s^{\omega, \text{dec}}(\Gamma; \mathbb{A}, \mathbb{S}, \mathbb{T}), \quad [c] \mapsto f([c]).$$

The proof of Theorem 3.2 is split into Propositions 3.3 and 3.5 below.

**Proposition 3.3.** If $f \in \text{FE}_s(\Gamma; \mathbb{A}, \mathbb{S}, \mathbb{T})$, then $c(f)$ determines a unique element in $Z^1_{\text{par}}(\Gamma; V_{s}^{\omega, \infty})$.

**Remark 3.4.** In order to extend $c := c(f)$ to all of $\Gamma$ and to show that this extension is well-defined, unique and a 1-cocycle, we want to apply the Poincaré Fundamental Polyhedron Theorem (see e.g. [Mas71]) to the (closed) fundamental region $\bigcup \mathbb{A}$ for $\Gamma$. The Poincaré Theorem in its usual form however may only be applied if $\bigcup \mathbb{A}$ is connected. For each $\mathbb{A} \in \mathbb{A}$ we can use an appropriate $T$-shift, say $T^n(\mathbb{A})$, such that the union over the family

$$\mathbb{A}' := \{T^n(\mathbb{A}) \mid \mathbb{A} \in \mathbb{A}\}$$

provides a (connected, closed) fundamental domain for $\Gamma$. The side-pairing elements of the non-vertical sides of the elements in $\mathbb{A}'$ are also $T$-shifted compared to the original ones. If we apply the Poincaré Theorem to $\bigcup \mathbb{A}'$ to deduce the relations between the side-pairing elements, the $T$-shifts will cancel and we will find the same relations as if we had applied the Poincaré Theorem to $\bigcup \mathbb{A}$. In short, for our purposes we may apply the Poincaré Theorem to $\bigcup \mathbb{A}$ if we add $T$ to the set of generators even though it need not be a side-pairing element.

**Proof of Proposition 3.3.** Let $c := c(f)$. The Poincaré Theorem shows that $\Gamma$ is generated (as a group) by $\mathcal{S}$. Therefore, using the definition of $c$ on $\mathcal{S}$ we can extend $c$ to all of $\Gamma$ via the cocycle relation. If this extension (which we also call $c$) is well-defined, then it is unique. Moreover, the properties (PF3) and (PF4) and Remark 2.1 yield that $c$ takes values in $V_{s}^{\omega, \infty}$. To show that $c$ is well-defined and indeed defines a 1-cocycle we proceed in the following steps. The Poincaré Theorem yields that these are sufficient.

(i) If $b_1 = T^n b_2$ for some $b_1, b_2 \in \mathcal{S}$, $n \in \mathbb{Z}$, then we have to show that

$$c_{b_1} = \tau_s(b_2^{-1}) c_{T^n} + c_{b_2}.$$  

Now $c_T = 0$ implies $c_{T^n} = 0$, and by definition $c_{b_1} = c_{b_2}$. Thus, this condition is satisfied. Moreover, we may use $c_{T^n g} = c_g$ in any of the following considerations.

(ii) We have $b, b^{-1} \in \mathcal{S} \setminus \{T\}$ (or more general, $b, T^n b^{-1} \in \mathcal{S}$). We need to show $\tau_s(b) c_b = -c_{b^{-1}}$. There exist $\alpha, \beta \in \Sigma'$ and $g \in \Gamma$ such that $C_\alpha', g C_\beta'$ are both based on the geodesic segment $(b^{-1} \cdot \infty, \infty)$ but are disjoint. Moreover, $b C_\alpha', b g C_\beta'$ are the two translates of some components of $C'$ which are both based on the geodesic segment $(b \cdot \infty, \infty)$, see Figure 8.

Thus, $bg = T^n$ for some $n \in \mathbb{Z}$. Using (i) we may suppose $n = 0$. Then

$$c_b = \psi_{\alpha, \beta} \begin{cases} \varepsilon_\alpha f_\alpha & \text{on } (b^{-1} \cdot \infty, \varepsilon_\alpha \infty) \\ -\varepsilon_\alpha \tau_s(b^{-1}) f_\beta & \text{on } (b^{-1} \cdot \infty, -\varepsilon_\alpha \infty) \end{cases}$$
Figure 8. Situation for $b, b^{-1} \in S \setminus \{T\}$

and

$$c_{b^{-1}} = \psi_{\beta,f} = \begin{cases} \varepsilon_{\beta} f_{\beta} & \text{on } (b, \infty, \varepsilon_{\beta}\infty) \\ -\varepsilon_{\beta} \tau_s(b) f_{\alpha} & \text{on } (b, \infty, -\varepsilon_{\beta}\infty) \end{cases}.$$  

Since $\varepsilon_{\alpha} = \varepsilon_{\beta} =: \varepsilon$ it follows

$$\tau_s(b) c_b = \begin{cases} \varepsilon \tau_s(b) f_{\alpha} & \text{on } (b, \infty, -\varepsilon\infty) \\ -\varepsilon f_{\beta} & \text{on } (b, \infty, \varepsilon\infty) \end{cases} = -c_{b^{-1}}.$$  

(iii) Let $(A, h) \in S$ with $A$ being a rectangle and let $((A_j, h_j))_{j=1,\ldots,k}$ be the cycle determined by $(A, h)$. We have to show that

$$c_{h_{j+k-1} \cdots h_{j+1}} = 0$$

for some $j \in \{1, \ldots, k\}$, where we understand the indices of the $h_\ell$ modulo $k$. Let $\ell \in \{1, \ldots, k\}$. Then $h_\ell \in S$. Let $\alpha_\ell := ((A, h), \ell)$ and $(\beta_\ell, g_\ell) \in \Sigma \times \Gamma$ be the tuple assigned to $\alpha_\ell$. Further set $\varepsilon_\ell := \varepsilon_{\alpha_\ell}$ and $f_\ell := f_{\alpha_\ell}$. Then

$$c_{h_\ell} = \begin{cases} \varepsilon_\ell f_\ell & \text{on } (h^{-1}_\ell, \infty, \varepsilon_\ell\infty) \\ -\varepsilon_\ell \tau_s(g_\ell) f_{\beta_\ell} & \text{on } (h^{-1}_\ell, \infty, -\varepsilon_\ell\infty) \end{cases}.$$  

By definition, we have

$$c_{h_{j+k-1} \cdots h_j} = \sum_{\ell=0}^{k-1} \tau_s(h^{-1}_{j+\ell} \cdots h^{-1}_{j+\ell}) c_{h_{j+\ell}}.$$  

Therefore we have to show

$$c_{h_j} = -\sum_{\ell=0}^{k-2} \tau_s(h^{-1}_{j+\ell} \cdots h^{-1}_{j+\ell+1}) c_{h_{j+\ell+1}}.$$  

Preliminary step: Suppose that we are in Situation (1a). We claim that (4) is satisfied on $(h_{j+k-1}, \infty, \varepsilon_j\infty)$. There exist $j \in \{1, \ldots, k\}$ and $n_j \in \mathbb{Z}$ such
that \( g_{j+k-1} = h_{j+k-2} \cdots h_j T^{-n_j} \) (for Figure 4 this means that \( \alpha = \beta_{j+k-1} \) and \( n = n_j \)). Note that
\[
 h_j^{-1} \cdots h_{j+k-2} = h_{j+k-1}
\]
and \( g_{j+k-1} = h_{j+k-1}^{-1} T^{-n_j} \in \mathcal{S} \). Further note that \( \varepsilon \ell = \varepsilon \beta_{j+k-1} =: \varepsilon \) for all \( \ell = 1, \ldots, k \). Then
\[
 c_{h_{j+k-1}^{-1} T^{-n_j}} = \begin{cases} \varepsilon f_{\beta_{j+k-1}} & \text{on } \langle T^{n_j} h_{j+k-1}, \ell \in \mathbb{N}\rangle, \varepsilon \xi \rangle \\ -\varepsilon \tau_s(T^{n_j} h_{j+k-1}) f_{j+k-1} & \text{on } \langle T^{n_j} h_{j+k-1}, \ell \in \mathbb{N}\rangle, -\varepsilon \xi \rangle. \end{cases}
\]
From \( f \) being a 1-eigenfunction of \( \mathcal{L}_{F,s} \) it follows that
\[
 f_{\beta_{j+k-1}} = \sum_{\ell=0}^{k-2} \tau_s(T^{n_j} h_j^{-1} \cdots h_{j+\ell-1}^{-1}) f_{j+\ell}.
\]
On \( \langle T^{n_j} h_{j+k-1}, \ell \in \mathbb{N}\rangle, \varepsilon \xi \rangle \) we have
\[
 c_{h_{j+k-1}^{-1} T^{-n_j}} = \sum_{\ell=0}^{k-2} \tau_s(T^{n_j} h_j^{-1} \cdots h_{j+\ell-1}^{-1}) \varepsilon f_{j+\ell} = \sum_{\ell=0}^{k-2} \tau_s(T^{n_j} h_j^{-1} \cdots h_{j+\ell-1}^{-1}) c_{h_{j+k-1}^{-1}}.
\]
Now
\[
 c_{h_{j+k-1}^{-1} T^{-n_j}} = -\tau_s(T^{n_j}) \tau_s(h_{j+k-1}) c_{h_{j+k-1}^{-1}}
\]
yields
\[
 0 = \sum_{\ell=0}^{k-1} \tau_s(h_j^{-1} \cdots h_{j+\ell-1}^{-1}) c_{h_{j+k-1}^{-1}} & \text{on } \langle h_{j+k-1}, \ell \in \mathbb{N}\rangle, \varepsilon \xi \rangle.
\]

**Main step:** Suppose now that we are in Situation (1b), hence there exist \( j \in \{1, \ldots, k \} \) and \( n_j \in \mathbb{Z} \) such that \( g_j = T^{-n_j} \) (for Figure 5 this means that \( \alpha = \beta_j \) and \( n = n_j \)). Then \( h_j T^{-n_j} \in \mathcal{S} \) and
\[
 c_{h_j T^{-n_j}} = \begin{cases} \varepsilon f_{\beta_j} & \text{on } \langle T^{n_j} h_j^{-1}, \ell \in \mathbb{N}\rangle, \varepsilon \beta_j, \varepsilon \xi \rangle \\ -\varepsilon \beta_j f_j & \text{on } \langle T^{n_j} h_j^{-1}, \ell \in \mathbb{N}\rangle, -\varepsilon \beta_j, \varepsilon \xi \rangle. \end{cases}
\]
From \( f \) being a 1-eigenfunction of \( \mathcal{L}_{F,s} \) it follows that
\[
 f_{\beta_j} = \sum_{\ell=0}^{k-2} \tau_s(T^{n_j} h_j^{-1} \cdots h_{j+\ell}^{-1}) f_{j+\ell+1}.
\]
Note that \( \varepsilon \beta_j = -\varepsilon \ell =: \varepsilon' \) for \( \ell = 1, \ldots, k \). On \( \langle T^{n_j} h_j^{-1}, \ell \in \mathbb{N}\rangle, \varepsilon' \xi \rangle \) we have
\[
 c_{h_j T^{-n_j}} = \varepsilon' f_{\beta_j} = -\sum_{\ell=0}^{k-2} \tau_s(T^{n_j} h_j^{-1} \cdots h_{j+\ell}^{-1}) \varepsilon f_{j+\ell+1} = -\sum_{\ell=0}^{k-2} \tau_s(T^{n_j} h_j^{-1} \cdots h_{j+\ell}^{-1}) c_{h_{j+\ell+1}}.
\]
Since
\[
 c_{h_j T^{-n_j}} = \tau_s(T^{n_j}) c_{h_j},
\]
the equality (4) is verified on \( h_j^{-1}, \infty, \varepsilon' \infty \). We now show that (4) holds on

\[
\begin{align*}
h_j^{-1}, h_{j+1}^{-1}, \infty, \varepsilon' \infty \end{align*}
\]

\[
\begin{cases}
(h_j^{-1}, h_{j+1}^{-1}, \infty, h_j^{-1}, \infty) & \text{if } \varepsilon' = +1 \\
(h_j^{-1}, \infty, h_j^{-1}, h_{j+1}^{-1}, \infty) & \text{if } \varepsilon' = -1.
\end{cases}
\]

If \( g_{j+1} = T^{-n_{j+1}} \) for some \( n_{j+1} \in \mathbb{Z} \), then we process as before to see that

\[
c_{h_{j+1}} = - \sum_{\ell=0}^{k-2} \tau_s(h_{j+1}^{-1} \cdots h_{j+\ell+1}^{-1})c_{h_{j+\ell+2}}
\]

\[
= - \tau_s(h_{j+1}^{-1} \cdots h_{j+k-1}^{-1})c_{h_j} - \sum_{\ell=1}^{k-2} \tau_s(h_{j+1}^{-1} \cdots h_{j+\ell}^{-1})c_{h_{j+\ell+1}}
\]

on \( h_{j+1}^{-1}, \infty, \varepsilon' \infty \). Since

\[
h_j = h_{j+1}^{-1} \cdots h_{j+k-1}^{-1},
\]

an application of \( \tau_s(h_{j+1}^{-1}) \) on both sides shows (4) on \( h_j^{-1}, h_{j+1}^{-1}, \infty, \varepsilon' \infty \).

We now consider the case that \( g_{j+1} \notin \Gamma_\infty \). By Situations (1a) and (3b),

\[
g_{j+1}, \infty = h_{j+1}^{-1}, \infty
\]

and hence \( g_{j+1} = h_{j+1}^{-1} T^m \) for some \( m \in \mathbb{Z} \). This means we are in Situation (1a) with \( j + 2 \) instead of \( j \) and \( -m \) instead of \( n_j \). The preliminary step from above shows that on \( h_{j+1}^{-1}, \infty, \varepsilon' \infty \) we have

\[
0 = \sum_{\ell=0}^{k-1} \tau_s(h_{j+2}^{-1} \cdots h_{j+\ell+1}^{-1})c_{h_{j+\ell+2}}
\]

\[
= \sum_{\ell=0}^{k-2} \tau_s(h_{j+2}^{-1} \cdots h_{j+k}^{-1})c_{h_{j+\ell+2}} + \tau_s(h_{j+2}^{-1} \cdots h_{j+k}^{-1})c_{h_{j+1}}.
\]

Using \( h_{j+2}^{-1} \cdots h_{j+k}^{-1} = h_{j+1} \) and applying \( \tau_s(h_{j+1}^{-1}) \) we find

\[
0 = \sum_{\ell=1}^{k-1} \tau_s(h_{j+1}^{-1} h_{j+2}^{-1} \cdots h_{j+\ell}^{-1})c_{h_{j+\ell+1}} + c_{h_{j+1}}
\]

\[
= \sum_{\ell=0}^{k-2} \tau_s(h_{j+1}^{-1} \cdots h_{j+k-1}^{-1})c_{h_{j+\ell+1}} + \tau_s(h_{j+1}^{-1} \cdots h_{j+k-1}^{-1})c_{h_j}
\]

on \( h_{j+1}^{-1}, \infty, \varepsilon' \infty \) if we start in Situation (1b). The proof for a start in Situation (1a) is analogous.

(iv) We have an order 2 relation of the form \( (T^m h)^2 = \text{id} \) constructed by the Poincaré Theorem from a situation as indicated in Figure 9, where \( \mathcal{A} \in \mathcal{A} \) is a rectangle, \( h \in \Gamma \) a side-pairing element and \( m \in \mathbb{Z} \).

We have to show

\[
c_{(T^m h)^2} = 0.
\]

Suppose first that \( (\mathcal{A}, h) \) is an element of a cycle determined by an element in \( \mathcal{S} \). Without loss of generality, we may assume that \( (\mathcal{A}, h) \in \mathcal{S} \). Then \( h \in \mathcal{S} \).
Let \( \alpha := ( ( A, h ), 1 ) \). Then \( C'_{\alpha} \) and \( h^{-1}T^{-m}C'_{\alpha} \) are both based on the geodesic segment \( (h^{-1}, \infty, \infty) \) and are disjoint. Thus

\[
c_h = \begin{cases} 
\varepsilon_\alpha f_\alpha & \text{on } (h^{-1}, \infty, \varepsilon_\alpha \infty) \\
-\varepsilon_\alpha \tau_s(h^{-1}T^{-m})f_\alpha & \text{on } (h^{-1}, \infty, -\varepsilon_\alpha \infty).
\end{cases}
\]

By definition and since \( c_T = 0 \) we have

\[
c_{(T^{-m}h)^2} = \tau_s(h^{-1}T^{-m})c_h + c_h.
\]

Now an easy calculation establishes (5).

(v) We have an order 2 relation of the form \((T^nh)^2 = \text{id}\) constructed by the Poincaré Theorem from a situation as indicated in Figure 10, where \( A \in \mathbb{A} \) is a triangle, \( h \in \Gamma \) a side-pairing element and \( n \in \mathbb{Z} \). Here one shows along the lines of (iv) that \( c \) vanishes on \((T^nh)^2\).

This shows that \( c \) is a 1-cocycle. It remains to prove that \( c \) is parabolic. Let \((A, h) \in \mathbb{S}\) with \( A \) being a triangle. Let \( v \) be the infinite vertex of \( \mathcal{K} \) to which \( A \) is associated. By the Poincaré Theorem there is a finite sequence in \( \mathbb{A} \times \Gamma \) determining a generator of \( \text{Stab}_1(v) \). This sequence is constructed as follows. Let \( A_1 := A, \; h_1 := h \) and let \(((A_1, h_1), (A'_1, h_1^{-1}))\) be the cycle determined by \((A_1, h_1)\). Iteratively for \( j = 2, 3, \ldots \) we find \( n_{j-1} \in \mathbb{Z} \) and \((A_j, h_j) \in \mathbb{A} \times \Gamma, \; A_j \) a triangle with non-vertical side in \( I(h_j) \) such that \( T^{n_{j-1}}A_j \) coincides with \( A'_{j-1} \) on their long side (and only there). Let \( A'_j \) be the element in \( \mathbb{A} \) such that \(((A_j, h_j), (A'_j, h_j^{-1}))\) is the cycle determined by \((A_j, h_j)\). This construction stops when there exists \( n_j \in \mathbb{Z} \) such that \( T^{n_j}A'_j \) coincides with \( A_1 \) precisely on their long side. Then

\[
p := T^{n_j}h_jT^{-n_{j-1}}h_{j-1}T^{-n_{j-2}} \cdots h_1
\]
is a generator of $\text{Stab}_\Gamma(v)$ and as such a parabolic element. Let $C'_1$ be the component of $C'$ which is based on the long side of $A_1$ and whose elements point into $A_1$, and let $C'_{n_j}$ be the component which is based on the long side of $A_{n_j}$ and whose elements point into $A_{n_j}$. Further let $f_1, f_{n_j}$ denote the corresponding components of $f$. Define

$$\varphi := \begin{cases} -\varepsilon f_1 & \text{on } \langle v, \varepsilon \infty \rangle \\ \varepsilon \tau_s(T^n_{n_j}) f_{n_j} & \text{on } \langle v, -\varepsilon \infty \rangle \end{cases}.$$ 

By (PF4), $\varphi$ extends smoothly to $P^1(\mathbb{R})$. Using $f = \mathcal{L}_{F,s} f$ one proves in analogy to the arguments above that

$$c_p = \tau_s(p^{-1}) \varphi - \varphi.$$

To illustrate these arguments we perform them for the situation shown in Figures 11-12. The Figure 13 is an intermediate step to construct the lower triangle with vertices $v_1, h^{-1}T^n_{n_1}h^{-1}_{2}, \infty, h^{-1}_{1}T^{-n_1}h_{1}$ in Figure 12. We assume here that $(A_1, h_1), (A_2, h_2) \in \mathcal{S}$, let $m_1 := m(A_1, h_1), m_2 := m(A_2, h_2)$ and use obvious abbreviations for the notation. The necessary steps for the proof and their validity actually can be read off these figures. The general situation is proved in exactly the same way by iterating this specific case sufficiently often. In our case, the stabilizer of $v_1$ is generated by

$$T^n_{n_2}h_2T^{-n_1}h_1.$$ 

We set

$$\psi := \begin{cases} -f_{1,1} & \text{on } (v_1, \infty) \\ \tau_s(T^n_{n_2}) f_{2,2} & \text{on } (-\infty, v_1) \end{cases}$$

and claim that

$$c_{T^n_{n_2}h_2T^{-n_1}h_1} = \tau_s(h^{-1}_{1}T^n_{n_1}h^{-1}_{2}T^{-n_2}) \psi - \psi.$$ 

From the cocycle relation and $c_T = 0$ it follows that

$$c_{T^n_{n_2}h_2T^{-n_1}h_1} = \tau_s(h^{-1}_{1}T^n_{n_1})c_h + c_{h_1}.$$
Since
\[
    c_{h_1} = \begin{cases} 
    \tau_s(g_1) f_{\alpha_1} & \text{on } (h_1^{-1}, \infty, \infty) \\
    -\tau_s(T^{-m_1}) f_{3,1} & \text{on } (-\infty, h_1^{-1}, \infty) 
    \end{cases} 
\]
and
\[
    c_{h_2} = \begin{cases} 
    \tau_s(g_2) f_{\alpha_2} & \text{on } (h_2^{-1}, \infty, \infty) \\
    -\tau_s(T^{-m_2}) f_{3,2} & \text{on } (-\infty, h_2^{-1}, \infty) 
    \end{cases} 
\]
we have
\[
    \tau_s(h_1^{-1}T^{n_1}) c_{h_2} = \begin{cases} 
    \tau_s(h_1^{-1}T^{n_1}g_2) f_{\alpha_2} & \text{on } (h_1^{-1}T^{n_1}h_2^{-1}, \infty, h_1^{-1}, \infty) \\
    -\tau_s(h_1^{-1}T^{n_1-m_2}) f_{3,2} & \text{on } (-\infty, h_1^{-1}T^{n_1}h_2^{-1}, \infty) \cup (h_1^{-1}, \infty, \infty) 
    \end{cases} 
\]
and hence
\[ \tau_s(h_1^{-1}T^n) c_{h_2} + c_{h_1} = \begin{cases} 
-\tau_s(h_1^{-1}T^{n_1-m_2}) f_{3,2} - \tau_s(T^{-m_1}) f_{3,1} & \text{on } (-\infty, h_1^{-1}T^{n_1} h_2^{-1} \infty) \\
\tau_s(h_1^{-1}T^{n_1} g_2) f_{2_2} - \tau_s(T^{-m_1}) f_{3,1} & \text{on } (h_1^{-1}T^{n_1} h_2^{-1} \infty, h_1^{-1} \infty) \\
-\tau_s(h_1^{-1}T^{n_1-m_2}) f_{3,2} + \tau_s(g_1) f_{\alpha_1} & \text{on } (h_1^{-1} \infty, \infty).
\end{cases} \]

On the other side we have
\[ \tau_s(h_1^{-1}T^n; h_2^{-1}T^{-n_2}) \psi - \psi = \begin{cases} 
\tau_s(h_1^{-1}T^{n_1} h_2^{-1}) f_{2_2} - \tau_s(T^{n_2}) f_{2_2} & \text{on } (-\infty, v_1) \\
-\tau_s(h_1^{-1}T^{n_1} h_2^{-1} T^{-n_2}) f_{1_1} + f_{1_1} & \text{on } (v_1, h_1^{-1}T^{n_1} h_2^{-1} \infty) \\
\tau_s(h_1^{-1}T^{n_1} h_2^{-1}) f_{2_2} + f_{1_1} & \text{on } (h_1^{-1}T^{n_1} h_2^{-1} \infty, \infty).
\end{cases} \]

Therefore we have to show:

(i) on \((-\infty, v_1)\):
\[ \tau_s(h_1^{-1}T^{n_1} h_2^{-1}) f_{2,2} - \tau_s(T^{n_2}) f_{2,2} = -\tau_s(h_1^{-1}T^{n_1-m_2}) f_{3,2} - \tau_s(T^{-m_1}) f_{3,1}, \]

(ii) on \((v_1, h_1^{-1}T^{n_1} h_2^{-1} \infty)\):
\[ -\tau_s(h_1^{-1}T^{n_1} h_2^{-1} T^{-n_2}) f_{1,1} + f_{1,1} = -\tau_s(h_1^{-1}T^{n_1-m_2}) f_{3,2} - \tau_s(T^{-m_1}) f_{3,1}, \]

(iii) on \((h_1^{-1}T^{n_1} h_2^{-1} \infty, h_1^{-1} \infty)\):
\[ \tau_s(h_1^{-1}T^{n_1} h_2^{-1}) f_{2,2} + f_{1,1} = \tau_s(h_1^{-1}T^{n_1} g_2) f_{2_2} - \tau_s(T^{-m_1}) f_{3,1}, \]

(iv) on \((h_1^{-1} \infty, \infty)\):
\[ \tau_s(h_1^{-1}T^{n_1} h_2^{-1}) f_{2,2} + f_{1,1} = -\tau_s(h_1^{-1}T^{n_1-m_2}) f_{3,2} + \tau_s(g_1) f_{\alpha_1}. \]
From $f = L_{F,s}f$ it follows

(a) $f_{2,2} = \tau_s(T^{-n_2}h_1^{-1})f_{2,1} + \tau_s(T^{n_2-m_1})f_{3,1}$ on $(-\infty, v'_2) = (-\infty, h_2, v_2)$, and

(b) $f_{2,1} = \tau_s(T^{n_1}h_2^{-1})f_{2,2} + \tau_s(T^{n_1-m_2})f_{3,2}$ on $(-\infty, v'_1) = (-\infty, T^{n_1}, v_2)$.

Then (b) is equivalent to

$$\tau_s(T^{-n_2}h_1^{-1})f_{2,1} = \tau_s(T^{-n_2}h_1^{-1}T^{n_1}h_2^{-1})f_{2,2} + \tau_s(T^{-n_2}h_1^{-1}T^{n_1-m_2})f_{3,2}$$
on $(-\infty, T^{-n_2}, v_1) \cup (h_1^{-1}, \infty, \infty)$ since $h_1^{-1}T^{n_1}, v_2 = v_1$. Plugging this in (a) gives

$$f_{2,2} = \tau_s(T^{-n_2}h_1^{-1}T^{n_1}h_2^{-1})f_{2,2} + \tau_s(T^{-n_2}h_1^{-1}T^{n_1-m_2})f_{3,2} + \tau_s(T^{n_2-m_1})f_{3,1}$$
on $(-\infty, T^{-n_2}h_1^{-1}T^{n_1}, v_2)$. Since $h_1^{-1}T^{n_1}, v_2 = v_1$, this proves (i).

For the proof of (ii) we use that $f = L_{F,s}f$ provides

(c) $f_{1,1} = \tau_s(T^{n_2}h_2)f_{1,2} + \tau_s(T^{n_2}h_2T^{-m_2})f_{3,2}$ on $(v_1, \infty)$, and

(d) $f_{1,2} = \tau_s(T^{-n_1}h_1)f_{1,1} + \tau_s(T^{-n_1}h_1T^{-m_1})f_{3,1}$ on $(v_2, \infty)$.

Then (c) is equivalent to

$$\tau_s(h_1^{-1}T^{n_1}h_2^{-1}T^{-n_2})f_{1,1} = \tau_s(h_1^{-1}T^{n_1})f_{1,2} + \tau_s(h_1^{-1}T^{n_1-m_2})f_{3,2}$$
on $h_1^{-1}T^{n_1}h_2^{-1}T^{-n_2}, (v_1, \infty) = (v_1, h_1^{-1}T^{n_1}h_2^{-1}, \infty)$. Equality (d) is equivalent to

(d') $\tau_s(h_1^{-1}T^{n_1})f_{1,2} = f_{1,1} + \tau_s(T^{-m_1})f_{3,1}$
on $(v_1, h_1^{-1}, \infty)$. Combining these two equalities shows (ii).

To prove (iii) we note that $f = L_{F,s}f$ yields

(e) $\tau_s(g_2)f_{\alpha_2} = \tau_s(h_2^{-1})f_{2,2} + f_{1,2}$
on $(h_2^{-1}, \infty, \infty)$. This is equivalent to

$$\tau_s(h_1^{-1}T^{n_1}g_2)f_{\alpha_2} = \tau_s(h_1^{-1}T^{n_1}h_2^{-1})f_{2,2} + \tau_s(h_1^{-1}T^{n_1})f_{1,2}$$
on $(h_1^{-1}T^{n_1}h_2^{-1}, \infty, h_1^{-1}, \infty)$. Combining this with (d') proves (iii).

For the proof of (iv) we remark that $f = L_{F,s}f$ shows

(f) $\tau_s(g_1)f_{\alpha_1} = f_{1,1} + \tau_s(h_1^{-1})f_{2,1}$
on $(h_1^{-1}, \infty, \infty)$. This together with (b) proves (iv). With this the proof is finally complete. \qed

**Proposition 3.5.** If $[c] \in H^1_{\text{par}}(\Gamma; V^{\omega,\infty}_s)$, then $f([c]) \in \text{Ff}^{\omega,\text{dec}}_s(\Gamma; A, S, T)$.

**Proof.** Let $f := f([c]) = (f_n)_{n \in \Sigma}$. Then $f$ is obviously well-defined. We start by establishing the regularity conditions on $f$. Let $c \in Z^1_{\text{par}}(\Gamma; V^{\omega,\infty}_s)$ be the representative of $[c]$ with $\nu_T = 0$. Then there exists a unique Maass cusp form $u$ with eigenvalue $s(1 - s)$ such that

$$c_g(r) = \int_{B^{-1,\infty}} [u, R(r, \cdot)]$$
for all $g \in \Gamma$. If $v$ is a cuspidal point such that $C'_\alpha$ is based on the geodesic segment $(v, \infty)$ and $p$ is a generator of $\text{Stab}_\Gamma(v)$, then [Poh12, Lemma 3.3] shows that
\[
\psi(r) = -\int_v^\infty [u, R(r, \cdot)^s], \quad r \in \mathbb{R},
\]
determines the unique element $\psi$ in $\mathcal{V}_s^{\omega, \infty}$ such that
\[
c_p = \tau_s(p^{-1})\psi - \psi.
\]
By the definition of $f$, for each $\alpha \in \Sigma$ we have
\[
f_\alpha(r) = \pm \varepsilon_\alpha \int_{r_\alpha}^\infty [u, R(r, \cdot)^s], \quad r \in \langle r_\alpha, \varepsilon_\alpha \infty \rangle,
\]
where the sign in front of the integral depends on whether or not $\alpha \in \Sigma'$ and does not matter here. By [Poh12, Lemma 3.2], $f_\alpha$ is real-analytic for each $\alpha \in \Sigma$. Thus, (PF1) holds. Moreover,
\[
\psi_{\alpha, f}(r) = \int_{r_\alpha}^\infty [u, R(r, \cdot)^s] \quad \text{for } r \in \mathbb{R} \setminus \{r_\alpha\}.
\]
These functions clearly satisfy (PF3) and (PF4). To see (PF2), we use that the 1-form $[u, R(r, \cdot)^s]$ is closed (for all $r$) and hence we may change the path of integration. For $\alpha \in \Sigma$ let $\gamma_\alpha$ denote the geodesic from $r_\alpha$ to $\infty$. If we use as path of integration in (6) instead of $\gamma_\alpha$ the sequence $g_1 \gamma_{\beta_1}, \ldots, g_\ell \gamma_{\beta_\ell}$, where $g_j \in \Gamma$, $\beta_j \in \Sigma$ are as indicated in Figures 4-7, then with (3) we have
\[
f_\alpha(r) = \int_{\gamma_\alpha} [u, R(r, \cdot)^s] = \sum_{j=1}^\ell \int_{g_1 \gamma_{\beta_j}} [u, R(r, \cdot)^s] = \sum_{j=1}^\ell \tau_s(g_j) \int_{\gamma_{\beta_j}} [u, R(r, \cdot)^s]
\]
\[
= \sum_{j=1}^\ell \tau_s(g_j) f_{\beta_j}(r).
\]
This completes the proof. \qed

4. The effect of different choices

The definition of period functions in Section 2.6 is subject to the choices of $\mathbb{A}, \mathbb{S}$ and $\mathbb{T}$. Let $\widetilde{\mathbb{A}}, \mathbb{S}, \mathbb{T}$ and $\mathbb{A}, \mathbb{S}, \mathbb{T}$ be two such choices. By Theorem 3.1 (or Theorem A or 3.2) the spaces $FE_{s, \text{dec}}(\Gamma; \mathbb{A}, \mathbb{S}, \mathbb{T})$ and $FE_{s, \text{dec}}(\Gamma; \mathbb{A}, \mathbb{S}, \mathbb{T})$ are isomorphic, and their canonical isomorphism can be determined by composing their isomorphisms with $H^1_{\text{par}}(\Gamma; \mathcal{V}_{s, \text{dec}})$ from Theorem 3.2.

However, the geometric background of the definition of period functions shows that there is a direct approach to this isomorphism. Let $C'$ resp. $\widetilde{C}'$ denote the set of representatives for the cross section $C$ which is associated to $\mathbb{A}, \mathbb{S}, \mathbb{T}$ resp. $\mathbb{A}, \mathbb{S}, \mathbb{T}$, and let $\Sigma$ resp. $\Sigma$ be the arising set of symbols. For each component $C'_\alpha$, $\alpha \in \Sigma$, of $C'$ there is a unique symbol $\alpha \in \Sigma$ and an element $g_\alpha \in \Gamma$ such that $g_\alpha C'_\alpha$ equals the component $\widetilde{C}'_{\alpha}$ of $\widetilde{C}'$. This relation directly translates to the level of period functions as stated in the following proposition. The proof of this proposition is straightforward.
Proposition 4.1. The map
\[ FE_{\alpha}^{dec}(\Gamma; \mathbb{A}, \mathbb{S}, \mathbb{T}) \to FE_{\alpha}^{dec}(\tilde{\Gamma}; \tilde{\mathbb{A}}, \tilde{\mathbb{S}}, \tilde{\mathbb{T}}), \quad f = (f_{\alpha})_{\alpha \in \Sigma} \mapsto \tilde{f} = (\tilde{f}_{\alpha})_{\alpha \in \Sigma}, \]
where
\[ f_{\alpha} := \tau_{s}(g_{\alpha})f_{\alpha}, \]
is an isomorphism of vector spaces. It coincides with the composition of the isomorphisms from Theorem 3.2.

The isomorphism in Proposition 4.1 also shows how the associated transfer operator families transform into each other. In the following we provide an algorithm to calculate the map \( \tilde{\Sigma} : \Sigma \to \tilde{\Sigma} \) and the elements \( g_{\alpha} \) for \( \alpha \in \Sigma \).

For each \( \mathbb{A} \in \mathbb{A} \) there are unique elements \( n(\mathbb{A}) \in \mathbb{Z} \) and \( \tilde{\mathbb{A}} \in \tilde{\mathbb{A}} \) such that
\[ T^{n(\mathbb{A})}\mathbb{A} = \tilde{\mathbb{A}}. \]

If \( h \in \Gamma \) is the side-pairing element (in \( \mathbb{A} \)) which maps the side \( b_{1} \) of \( \mathbb{A}_{1} \) to the side \( b_{2} \) of \( \mathbb{A}_{2} \) for \( \mathbb{A}_{1}, \mathbb{A}_{2} \in \mathbb{A} \), then
\[ \tilde{h} := T^{n(\mathbb{A}_{2})}hT^{-n(\mathbb{A}_{1})} \]
is the side-pairing element (in \( \tilde{\mathbb{A}} \)) which maps the side \( \tilde{b}_{1} = T^{n(\mathbb{A}_{1})}b_{1} \) of \( \tilde{\mathbb{A}}_{1} \) to the side \( \tilde{b}_{2} \) of \( \tilde{\mathbb{A}}_{2} \).

Let \( (\mathbb{A}, h) \in \mathbb{S} \). Suppose first that \( \mathbb{A} \) is a triangle and let \( m := m(\mathbb{A}, h) \). Recall that the cycle in \( \mathbb{A} \times \Gamma \) determined by \( (\mathbb{A}, h) \) is
\[ ((\mathbb{A}, h), (\mathbb{A}', h^{-1})) \]
with a unique element \( \mathbb{A}' \in \mathbb{A} \). Now \( \tilde{\mathbb{S}} \) contains a unique generator of the equivalence class of cycles in \( \tilde{\mathbb{A}} \times \Gamma \) determined by \( (\tilde{\mathbb{A}}, \tilde{h}) \). This generator is either \( (\tilde{\mathbb{A}}, \tilde{h}) \) or \( (\tilde{\mathbb{A}'}, \tilde{h}^{-1}) \).

If \( (\tilde{\mathbb{A}}, \tilde{h}) \in \tilde{\mathbb{S}}, \) then
\[ \tilde{C}_{(\tilde{\mathbb{A}}, \tilde{h})} = T^{m(\mathbb{A})}C_{(\mathbb{A}, h), 1}, \quad \tilde{C}_{(\tilde{\mathbb{A}}, \tilde{h})} = T^{m(\mathbb{A})}C_{(\mathbb{A}, h), 2} \]
and
\[ \tilde{C}_{(\tilde{\mathbb{A}}, \tilde{h})} = T^{m+n(\mathbb{A})}C_{(\mathbb{A}, h), 3}, \]
where \( \tilde{m} := m(\tilde{\mathbb{A}}, \tilde{h}) \) (the contribution from \( \tilde{T} \)).

If \( (\tilde{\mathbb{A}'}, \tilde{h}^{-1}) \in \tilde{\mathbb{S}}, \) then, with \( \tilde{m} := m(\tilde{\mathbb{A}'}, \tilde{h}^{-1}), \)
\[ \tilde{C}_{(\tilde{\mathbb{A}'}, \tilde{h}^{-1})} = T^{m(\mathbb{A})}C_{(\mathbb{A}, h), 1}, \quad \tilde{C}_{(\tilde{\mathbb{A}'}, \tilde{h}^{-1})} = T^{m(\mathbb{A})}C_{(\mathbb{A}, h), 2} \]
and
\[ \tilde{C}_{(\tilde{\mathbb{A}'}, \tilde{h}^{-1})} = T^{m+n(\mathbb{A})}hT^{-m}C_{(\mathbb{A}, h), 3} = T^{\tilde{m}hT^{m(\mathbb{A})}h^{-m}}C_{(\mathbb{A}, h), 3}. \]

Suppose now that \( \mathbb{A} \) is a rectangle and let
\[ ((\mathbb{A}_{j}, a_{j}))_{j=1,...,k} \]
be the cycle in \( \mathbb{A} \times \Gamma \) determined by \( (\mathbb{A}, h) \). The inverse cycle, that is the cycle in \( \mathbb{A} \times \Gamma \) determined by \( (\mathbb{A}, a_{k}^{-1}) \), is
\[ ((\mathbb{A}', b_{j}))_{j=1,...,k} \]
where (see [Poh10])

\[ \mathcal{A}'_j = \mathcal{A}_{k-j+2} \quad \text{and} \quad b_j = a_{k-j+1}^{-1}. \]

Here, the indices are taken modulo \( \text{cyl}(\mathcal{A}) \). Then there is a unique \( j_0 \in \{1, \ldots, k\} \) such that \( \tilde{S} \) contains either \((\tilde{\mathcal{A}}_{j_0}, \tilde{a}_{j_0})\) or \((\tilde{\mathcal{A}}'_{j_0}, \tilde{b}_{j_0})\).

If \((\tilde{\mathcal{A}}_{j_0}, \tilde{a}_{j_0}) \in \tilde{S}\), then

\[ (\tilde{\mathcal{A}}_{j_0+j-1}, \tilde{a}_{j_0+j-1}) \]

is the cycle in \( \tilde{A} \times \Gamma \) determined by \((\tilde{\mathcal{A}}_{j_0}, \tilde{a}_{j_0})\). In this case we have

\[ C'(\tilde{\mathcal{A}}_{j_0}, \tilde{h}_{j_0}), j = T^n(\mathcal{A}_{j_0+j-1}, C'(\mathcal{A}, h), j_0+j-1 \]

for \( j = 1, \ldots, k \).

If \((\tilde{\mathcal{A}}'_{j_0}, \tilde{b}_{j_0}) \in \tilde{S}\), then we have

\[ C'(\tilde{\mathcal{A}}'_{j_0}, \tilde{h}_{j_0}), j = T^n(\mathcal{A}_{k-j_0-j+3}, C'(\mathcal{A}, h), k-j_0-j+2 \]

for \( j = 1, \ldots, k \).

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