The cohomology of Bestvina-Brady groups

Ian J. Leary∗ Müge Saadetoğlu†

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Dedicated to Warren Dicks, on the occasion of his 60th birthday.

Abstract

For each subcomplex of the standard CW-structure on any torus, we compute the homology of a certain infinite cyclic regular covering space. In all cases when the homology is finitely generated, we also compute the cohomology ring. For aspherical subcomplexes of the torus, our computation gives the homology of the groups introduced by M. Bestvina and N. Brady in [3]. We compute the cohomological dimension of each of these groups.

1 Introduction

Let \(\mathbb{T}\) be the circle, or 1-dimensional unitary group, given a CW-structure with one 0-cell and one 1-cell. Suppose also that the identity element of the group is chosen to be the 0-cell. For a set \(V\), let \(T(V)\) denote the direct sum \(T(V) = \bigoplus_{v \in V} \mathbb{T}\). There is a natural CW-structure on \(T(V)\) in which the \(i\)-cells are in bijective correspondence with \(i\)-element subsets of \(V\).

For the purposes of this paper, a simplicial complex will be defined abstractly as a non-empty set of finite sets which is closed under inclusion. The one element members of the set of sets are the vertices of the simplicial complex. Every simplicial complex (including the empty simplicial complex) contains a unique \(-1\)-simplex corresponding to the empty set.

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If \( \sigma \) is a finite subset of \( V \), the closure in \( T(V) \) of the cell corresponding to \( \sigma \) is equal to \( T(\sigma) \), and consists of all the cells corresponding to subsets of \( \sigma \). It follows that there is a bijective correspondence between simplicial complexes whose vertex set is contained in \( V \) and non-empty subcomplexes of \( T(V) \) (see [8, 3.23] for this statement in the case when \( V \) is finite). The empty simplicial complex corresponds to the subcomplex \( T_\emptyset \) consisting of just the single 0-cell of \( T(V) \), and a non-empty simplicial complex \( L \) corresponds to the complex \( T_L \) defined by

\[
T_L = \bigcup_{\sigma \in L} T(\sigma).
\]

The fundamental group of \( T_L \) and the cohomology ring of \( T_L \) are easily described in terms of \( L \), and there is a characterisation of those \( L \) for which \( T_L \) is aspherical. (We shall describe all of these results below.)

A point in \( T(V) \) is a vector \((t_v)\) of elements of \( \mathbb{T} \) indexed by \( V \), such that only finitely many \( t_v \) are not the identity element. The group multiplication induces a map \( \mu : T(V) \to \mathbb{T} \) which takes the point \((t_v)\) to the product of all of the non-identity \( t_v \)'s. For each \( L \) this induces a cellular map \( \mu_L : T_L \to \mathbb{T} \), and \( \mu_L \) is surjective when \( L \) is non-empty. Our aim is to study the homology and cohomology of the space \( \tilde{\mathbb{T}}_L \), the infinite cyclic cover of \( T_L \) obtained by pulling back the universal cover of \( \mathbb{T} \) via \( \mu_L \).

\[
\begin{array}{ccc}
\tilde{\mathbb{T}}_L & \longrightarrow & \mathbb{R} \\
\downarrow & & \downarrow \\
T_L & \xrightarrow{\mu_L} & \mathbb{T}
\end{array}
\]

For each \( L \), we describe the homology of \( \tilde{\mathbb{T}}_L \), together with information about the \( \mathbb{Z} \)-action induced by the action of \( \mathbb{Z} \) by deck transformations on \( \tilde{\mathbb{T}}_L \). We deduce that for any non-trivial ring \( R \), the \( R \)-homology of \( \tilde{\mathbb{T}}_L \) is finitely generated as an \( R \)-module if and only if \( L \) is finite and \( R \)-acyclic. In all cases when \( L \) is \( R \)-acyclic, we give a complete description of the cohomology ring \( H^*(\tilde{\mathbb{T}}_L; R) \).

Let \( G_L \) denote the fundamental group of \( T_L \). Since the fundamental group of a CW-complex depends only on its 2-skeleton, the group \( G_L \) depends only on the 1-skeleton of the simplicial complex \( L \). The presentation for \( G_L \) coming from the cell structure on \( T_L \) has one generator for each vertex of \( L \), subject only to the relation that the generators \( v \) and \( w \) commute.
whenever \( \{v, w\} \) is an edge in \( L \). These groups are known as right-angled Artin groups. It can be shown that \( T_L \) is aspherical if and only if \( L \) is a flag complex. Every simplicial complex may be completed to a flag complex with the same 1-skeleton (just add in a simplex for each finite complete subgraph of the 1-skeleton) and so one sees that the spaces of the form \( T_L \) include models for the classifying spaces of all right-angled Artin groups.

When \( L \) is non-empty, \( \tilde{T}_L \) is connected and the fundamental group of \( \tilde{T}_L \) is the kernel of the induced map \( \mu_* : G_L \to \mathbb{Z} \), which sends each of the generators for \( G_L \) to \( 1 \in \mathbb{Z} \). Call this group \( H_L \). The groups \( H_L \) are known as Bestvina-Brady groups. In the case when \( L \) is a finite flag complex, M. Bestvina and N. Brady showed that the homological finiteness properties of \( H_L \) are determined by the homotopy type of \( L \). For example, they show that \( H_L \) is finitely presented if and only if \( L \) is 1-connected \[3\]. For an explicit presentation for \( H_L \) for any \( L \), see \[6\].

Let \( L \) be an \( n \)-dimensional flag complex. It is easy to show that in this case, the cohomological dimension of the group \( G_L \) is equal to \( n + 1 \). The cohomological dimension of \( G_L \) over any non-trivial ring \( R \) is also equal to \( n + 1 \). It also follows easily that the cohomological dimension of \( H_L \) is equal to either \( n \) or \( n + 1 \). Our computations together with some of the results from \[3\] allow us to determine the cohomological dimension of \( H_L \), at least in the case when \( R \) is either a field or a subring of the rationals. If \( n = 0 \) and \( L \) is a single point, then \( H_L \) is the trivial group. Otherwise, if there exists an \( R \)-module \( A \) such that \( H^n(L; A) \neq 0 \), then \( H_L \) has cohomological dimension \( n + 1 \) over \( R \). If there exists no such \( A \), then \( H_L \) has cohomological dimension \( n \) over \( R \). Note that in contrast to the case of \( G_L \), the cohomological dimension of \( H_L \) may vary with the choice of ring \( R \). As a corollary we deduce that the trivial cohomological dimension and cohomological dimension of \( H_L \) are equal.

Some of these results appeared, with slightly different proofs, in the Southampton PhD thesis of the second named author. For some results, we give a brief sketch of a second proof. Some computations of low-dimensional ordinary cohomology (and many other algebraic invariants) for a special class of the Bestvina-Brady groups also appear in a recent preprint of S. Papadima and A. Suciu \[12\].
2 Homology and cohomology of $T_L$

The differential in the cellular chain complex for $T(V)$ is trivial, and hence so is the differential in the cellular chain complex for $T_L$, for any $L$. It follows that for any ring $R$, $H_i(T_L; R)$ is a free $R$-module with basis the $i$-cells of $T_L$, or equivalently the $(i - 1)$-simplices of $L$. The differential in the cellular cochain complex is also trivial. The group $H^i(T_L; R)$ is isomorphic to a direct product of copies of $R$ indexed by the $(i - 1)$-simplices of $L$. To describe the ring structure on the cohomology, we first consider the case of the torus $T(V)$.

The cohomology ring $H^*(T(V); R)$ can be described as the exterior algebra $\Lambda^*(R, V)$. A homogeneous element $f \in \Lambda^i(R, V)$ is an alternating function $f : V^n \to R$, where we say that a function is alternating if the following two conditions are satisfied:

1. $f(v_1, \ldots, v_n) = 0$ whenever there exists $1 \leq i < j \leq n$ with $v_i = v_j$;
2. $f(v_1, \ldots, v_i, v_{i+1}, \ldots, v_n) = -f(v_1, \ldots, v_{i+1}, v_i, \ldots, v_n)$ for any $i$ with $1 \leq i < n$.

If $f \in \Lambda^i$ and $g \in \Lambda^{n-i}$, the product $f.g$ is the so-called ‘shuffle product’. This is defined in terms of the pointwise product by the equation

$$f.g(v_1, \ldots, v_n) = \sum_\pi \epsilon(\pi)f(v_{\pi(1)}, \ldots, v_{\pi(i)})g(v_{\pi(i+1)}, \ldots, v_{\pi(n)}),$$

where $\epsilon(\pi) \in \{\pm 1\}$ denotes the sign of the permutation $\pi$, and the summation ranges over all permutations $\pi$ such that

$$\pi(1) < \pi(2) < \cdots < \pi(i) \quad \text{and} \quad \pi(i + 1) < \pi(i + 2) < \cdots < \pi(n).$$

(The ‘shuffles’ or permutations of the above type are chosen because they are a set of coset representatives in $S_n$ for the subgroup $S_i \times S_{n-i}$, so that each $i$-element subset of $\{1, \ldots, n\}$ is equal to $\{\pi(1), \ldots, \pi(i)\}$ for exactly one such $\pi$. Any other set of coset representatives could be used instead.)

There is a similar description of the ring structure on $H^*(T_L; R)$ for any simplicial complex $L$, as the exterior face ring $\Lambda^*_R(L)$ of $L$. If $V$ is the vertex set of $L$, $\Lambda^*_R(L)$ is the quotient of $\Lambda^*_R(V)$ by the homogeneous ideal $I_L$, with generators the functions that vanish on every $n$-tuple $(v_1, \ldots, v_n)$ which does not span a simplex of $L$. The inclusion of $T_L$ in $T(V)$ induces a homomorphism of cohomology rings

$$\Lambda^*_R(V) \cong H^*(T(V); R) \to H^*(T_L; R),$$
and it is easy to check (from the additive description of $H^*(T_L; R)$ given above) that this homomorphism is surjective and that its kernel is $I_L$. Hence one obtains a theorem which was first stated in [10] in the case when $L$ is finite:

**Theorem 1** For any simplicial complex $L$ and ring $R$, the cohomology ring $H^*(T_L; R)$ is isomorphic to the exterior face ring $\Lambda_R^*(L)$.

For any path-connected space $X$, there is a natural isomorphism between $H^1(X; \mathbb{Z})$ and $\text{Hom}(\pi_1(X), \mathbb{Z})$. The element of $H^1(T_L; \mathbb{Z}) = \Lambda^1_\mathbb{Z}(L)$ that corresponds to the homomorphism $\mu_* : G_L \to \mathbb{Z}$ is the element $\beta_L$, the constant function which takes each vertex of $L$ to $1 \in \mathbb{Z}$. By a slight abuse of notation, write $\beta_L$ also for the element of $\Lambda_\mathbb{R}^1(L)$ that takes each vertex of $L$ to $1 \in R$.

In any anticommutative ring, multiplication by an element of odd degree gives rise to a differential. The cochain complex structure on $\Lambda_\mathbb{R}^*(L)$ given by multiplication by $\beta_L$ is easily described.

**Theorem 2** For any ring $R$, there is a natural isomorphism of cochain complexes

$$\left(\Lambda_\mathbb{R}^*(L), \beta \times \right) \cong C^*_{+1}(L; R)$$

between the exterior face ring of $L$ with differential given by left multiplication by $\beta_L$, and the augmented simplicial cochain complex of $L$ shifted in degree by one.

**Proof.** In degree $i$, each of the two graded $R$-modules is isomorphic to a direct product of copies of $R$ indexed by the $(i-1)$-simplices of $L$, or equivalently the $R$-valued functions on the oriented $(i-1)$-simplices of $L$, where $f(-\sigma) = -f(\sigma)$ if $-\sigma$ is the same simplex as $\sigma$ with the opposite orientation. It remains to show that this isomorphism is compatible with the differentials on the two cochain complexes.

Let $f$ be an $R$-valued function on the $(i-1)$-simplices of $L$, and compare the functions $\beta \cdot f$ and $\delta f$, the image of $f$ under the differential on $C^*_{+1}(L; R)$. 


If \((v_0, \ldots, v_i)\) is the vertex set of an oriented \(i\)-simplex of \(L\), then

\[
\beta.f(v_0, \ldots, v_i) = \sum_{j=0}^{i} (-1)^j \beta(v_j)f(v_0, \ldots, v_{j-1}, v_{j+1}, \ldots, v_i)
\]

\[
= \sum_{j=0}^{i} (-1)^j f(v_0, \ldots, v_{j-1}, v_{j+1}, \ldots, v_i)
\]

\[= \delta f(v_0, \ldots, v_i).\]

This completes the proof.

\[\blacksquare\]

3 Higher homotopy of \(T_L\)

Recall that a full subcomplex \(M\) of a simplicial complex \(L\) is a subcomplex such that if \(\sigma\) is any simplex of \(L\) and each vertex of \(\sigma\) is in \(M\), then \(\sigma\) is in \(M\).

**Proposition 3** If \(M\) is a full subcomplex of \(L\), then \(T_M\) is a retract of \(T_L\).

**Proof.** Let \(W\) be a subset of \(V\). There is an isomorphism of topological groups \(T(V) \cong T(W) \oplus T(V - W)\). The inclusion

\[i : T(W) \cong T(W) \oplus \{1\} \rightarrow T(V)\]

and projection

\[\pi : T(V) \rightarrow T(V)/\{1\} \oplus T(V - W) \cong T(W)\]

satisfy \(\pi \circ i = 1_{T(W)}\), and show that \(T(W)\) is a retract of \(T(V)\).

Now suppose that \(L\) is a simplicial complex with vertex set \(V\) and that \(M\) is the full subcomplex with vertex set \(W \subseteq V\). Then \(T_L\) is a subcomplex of \(T(V)\), and \(T_M\) is a subcomplex of \(T(W)\). The maps \(i\) and \(\pi\), when restricted to \(T_M\) and \(T_L\), show that \(T_M\) is a retract of \(T_L\) as claimed. \[\blacksquare\]

Recall that a simplicial complex \(L\) is said to be flag if every finite complete subgraph of the 1-skeleton of \(L\) is the 1-skeleton of a simplex of \(L\). Any full subcomplex of a flag complex is flag.
Proposition 4 \( T_L \) is aspherical if and only if \( L \) is a flag complex.

Proof. A subset of a CW-complex that meets the interior of infinitely many cells contains an infinite discrete set, so cannot be compact. Hence any map from a sphere to a CW-complex has image inside a finite subcomplex and any homotopy between maps of a sphere into a CW-complex has image contained in a finite subcomplex. Thus it suffices to consider the case when \( L \) is finite.

Suppose that \( L \) is a finite flag complex with vertex set \( V \). If \( L \) is an \( n \)-simplex, then \( T_L \) is an \((n + 1)\)-torus, and so \( T_L \) is aspherical. If \( L \) is not a simplex, then there exist \( v_1, v_2 \in V \) so that there is no edge in \( L \) from \( v_1 \) to \( v_2 \). For \( i = 1, 2 \), let \( L_i \) be the full subcomplex of \( L \) with vertex set \( V - \{v_i\} \), and define \( L_3 \) by \( L_3 = L_1 \cap L_2 \). Then \( L = L_1 \cup L_2 \), and each of \( L_1, L_2 \) and \( L_3 \) is flag. By induction, \( T_{L_i} \) is aspherical for \( i = 1, 2, 3 \). Also \( T_{L_3} = T_{L_1} \cap T_{L_2} \) is a subcomplex of both \( T_{L_1} \) and \( T_{L_2} \). The fundamental group of \( T_{L_3} \) maps injectively to the fundamental group of each of \( T_{L_1} \) and \( T_{L_2} \), since \( T_{L_3} \) is a retract of each of \( T_{L_1} \) and \( T_{L_2} \). A theorem of Whitehead [8, 1.B.11] implies that \( T_L = T_{L_1} \cup T_{L_2} \) is aspherical.

Conversely, suppose that \( L \) is not flag. Then \( L \) contains a full subcomplex \( M \) which is equal to the boundary of an \( n \)-simplex for some \( n > 1 \). Then \( T_M \) is the \( n \)-skeleton of an \((n + 1)\)-torus, and so \( \pi_n(T_M) \) is non-zero. Since \( T_M \) is a retract of \( T_L \) it follows that \( \pi_n(T_L) \) is non-zero too.

Remark 5 There is also a metric proof that \( T_L \) is aspherical whenever \( L \) is a finite flag complex: in this case the geodesic metric induced by the standard product metric on \( T(V) \) is locally CAT(0). A version of the Cartan-Hadamard theorem shows that any locally CAT(0) metric space is aspherical [3]. We give the above proof instead to emphasize that the metric technology is not needed.

4 Homology of \( \tilde{T}_L \)

Let \( Z \) denote the fundamental group of \( \mathbb{T} \), an infinite cyclic group. Since \( \tilde{T}_L \) is defined in terms of \( \mu_{L}: T_L \to \mathbb{T} \) by pulling back the universal covering space of \( \mathbb{T} \), \( Z \) acts via deck transformations on \( \tilde{T}_L \). When \( L \) is non-empty, the map \( \mu_{L} \) induces an isomorphism \( G_L/H_L \cong Z \). In this section we describe the cellular chain complex and homology of \( \tilde{T}_L \) as a \( Z \)-module, for every \( L \). Let \( C^+_\ast(L) \) denote the augmented cellular chain complex of \( L \), and let \( d = d_L \) be its differential.
Proposition 6 The cellular chain complex $C_*(\widetilde{T}_L)$ is isomorphic to $\mathbb{Z}[Z] \otimes C^+_{n-1}(L)$ with differential $(1 - z) \otimes d_L$.

Proof. Each $(n - 1)$-simplex $\sigma$ of $L$ corresponds to a cubical $n$-cell in $T_L$, whose opposite faces are identified. In $\widetilde{T}_L$ this lifts to a free $Z$-orbit of cells. The $i$th opposite pair of faces are no longer identified, but differ by the translation action of $x$. By picking an orbit representative in each orbit of cells, we establish a $Z$-equivariant bijection between the set of $n$-cells of $\widetilde{T}_L$ and the direct product of $Z$ with the set of $(n - 1)$-simplices of $L$. The free abelian group with this basis is naturally isomorphic to $\mathbb{Z}[Z] \otimes C^+_{n-1}(L)$. Let $v$ be any fixed 0-cell of $\widetilde{T}_L$. In each orbit of higher-dimensional cells, pick the orbit representative that has $v$ as a vertex but does not have $z^{-1}v$ as a vertex. With respect to this choice of orbit representatives, the boundary map is as claimed. 

Corollary 7 For any $L$, for any abelian group $A$, and for any $n \geq 0$, there are short exact sequences of $\mathbb{Z}[Z]$-modules:

$$0 \to B^+_{n-1}(L; A) \to H_n(\widetilde{T}_L; A) \to \mathbb{Z}[Z] \otimes \mathcal{H}_{n-1}(L; A) \to 0,$$

$$0 \to \mathbb{Z}[Z] \otimes \mathcal{H}_{n-1}(L; A) \to H_n(\widetilde{T}_L; A) \to \mathbb{Z}^+_{n-1}(L; A) \to 0,$$

where $Z$ acts trivially on $\mathbb{Z}^+_{n-1}(L; A)$ and on $B^+_{n-1}(L; A)$, the cycles and boundaries in $C^+_n(L; A)$. The inclusion of the $Z$-fixed points in $H_n(\widetilde{T}_L; A)$ gives rise to the first sequence, and the map of $H_n(\widetilde{T}_L; A)$ onto its largest $Z$-invariant quotient gives rise to the second sequence.

In the case when $A = R$, a ring, each sequence admits an $R[Z]$-module structure. In this case, the first sequence is split if $\mathcal{H}_{n-1}(L; R)$ is $R$-projective, and the second sequence always admits an $R$-module splitting.

Proof. Take elements $p(z) \in \mathbb{Z}[Z]$, and $c \in C^+_{n-1}(L; A)$. The chain $p(z) \otimes c$ is a cycle for $(1 - z) \otimes d$ if and only if $d(c) = 0$. The boundary of $p(z) \otimes c$ is $(1 - z)p(z) \otimes dc$. Thus the cycles $Z_n$ in $C_*(\widetilde{T}_L; A)$ may be identified with $\mathbb{Z}[Z] \otimes Z^+_{n-1}(L; A)$, and the boundaries $B_n$ may be identified with $(1 - z)\mathbb{Z}[Z] \otimes B^+_{n-1}(L; A)$. Between these lies $B'_n = \mathbb{Z}[Z] \otimes B^+_{n-1}(L; A)$, and $B'_n$ is a $\mathbb{Z}[Z]$-submodule of $Z_n$. This gives a short exact sequence of $\mathbb{Z}[Z]$-modules

$$0 \to B'_n/B_n \to H_n(\widetilde{T}_L; A) \to Z_n/B'_n \to 0,$$
and one sees that $B'_n/B_n \cong B'_{n-1}(L; A)$ with the trivial $\mathbb{Z}$-action and that $Z_n/B'_n \cong \mathbb{Z}[Z] \otimes \mathbb{P}_{n-1}(L; A)$.

Now define $Z'_n$ to be $(1 - z)\mathbb{Z}[Z] \otimes Z_{n-1}^+(L; A)$, a $\mathbb{Z}[Z]$-submodule of $Z_n$. As abelian groups, $Z_n = Z'_n \oplus (1 \otimes Z_{n-1}^+(L; A))$. It follows that the short exact sequence

$$0 \to Z'_n/B_n \to H_n(\tilde{T}_L; A) \to Z_n/Z'_n \to 0$$

is always $\mathbb{Z}$-split. One sees that $Z'_n/B_n \cong \mathbb{Z}[Z] \otimes \mathbb{P}_{n-1}(L; A)$ and that $Z_n/Z'_n \cong Z_{n-1}^+(L; A)$.

To compute the $\mathbb{Z}$-fixed points in $H_n(\tilde{T}; A)$, apply the $\mathbb{Z}$-fixed point functor to the first sequence. Since this functor is left-exact, one obtains an exact sequence:

$$0 \to B^+_{n-1}(L; A) \to H_n(\tilde{T}_L; A)^\mathbb{Z} \to 0.$$ 

To compute the maximal $\mathbb{Z}$-fixed quotient of $H_n(\tilde{T}; A)$, start with the short exact sequence

$$0 \to B_n \to Z_n \to H_n \to 0,$$

and apply the invariant quotient functor $H_0(\mathbb{Z}; -)$. This functor is right-exact, and it is easy to see that $H_0(\mathbb{Z}; Z_n) = Z_n/Z'_n$. Hence one obtains an exact sequence:

$$Z_n/Z'_n \to H_n(\tilde{T}_L; A)^\mathbb{Z} \to 0.$$ 

Since we have already shown that $Z_n/Z'_n$ is a $\mathbb{Z}$-invariant quotient of the homology group $H_n(\tilde{T}_L; A)$, we see that the maximal $\mathbb{Z}$-invariant quotient is isomorphic to $Z_n/Z'_n$ as claimed.

In the case when $A = R$, a ring, the $\mathbb{Z}[Z]$-modules and maps that appear in the short exact sequences also admit an $R$-module structure which commutes with the $\mathbb{Z}$-action. If $\mathbb{P}_{n-1}(\tilde{T}; R)$ is $R$-projective, then $\mathbb{Z}[Z] \otimes \mathbb{P}_{n-1}(\tilde{T}; R)$ is $R[\mathbb{Z}]$-projective, and so the first short exact sequence of $R[\mathbb{Z}]$-modules splits. In any case, $Z_{n-1}^+(L; R)$ is free as an $R$-module, and so the second short exact sequence admits an $R$-splitting.

**Corollary 8** Suppose that $L$ is a finite complex and $R$ is a ring such that $\mathbb{P}_i(L; R) = 0$ for $i < n$. Suppose also that $L$ has $f_i$ $i$-dimensional simplices for $i \geq 0$, and define $f_{-1} = 1$. For each $i$ with $0 \leq i \leq n$, $H_i(\tilde{T}_L; R)$ is a free $R$-module of rank

$$\sum_{j=0}^{i} (-1)^{i+j} f_{j-1}.$$
Proof. For each $j$, $C^+_j(L; R)$ is a free $R$-module of rank $f_j$. For each $i \leq n$, we know that

$$H_i(\tilde{T}_L; R) \cong Z^+_{i-1}(L; R) = B^+_i(L; R),$$

$$C^+_i(L; R) \cong Z^+_{i-1}(L; R) \oplus Z^+_i(L; R),$$

and that $Z^+_i(L; R)$ is a free $R$-module. Solving for the rank of $Z^+_i(L; R)$ gives the claimed result. 

**Corollary 9** If $L$ is finite and $\mathbb{Q}$-acyclic, then the Euler characteristic of $\tilde{T}_L$ is defined and is given by the formula:

$$\chi(\tilde{T}_L) = \sum_{i \geq 0} (-1)^i (i + 1) f_i.$$

Proof. Since $L$ is $\mathbb{Q}$-acyclic, the reduced Euler characteristic $\sum_{i \geq -1} (-1)^i f_i$ of $L$ is equal to zero. Let $L$ be $n$-dimensional. Then the expression for the rank of $H_i(\tilde{T}_L; \mathbb{Q})$ given by the previous corollary gives

$$\chi(\tilde{T}_L) = \sum_{i=0}^{n} (-1)^i \sum_{j=0}^{i} (-1)^{i+j} f_{j-1} = \sum_{i=0}^{n} (-1)^i (n + 1 - i) f_{i-1}.$$

Hence we see that

$$\chi(\tilde{T}_L) = \sum_{i=0}^{n} (-1)^i (n + 1 - i) f_{i-1} - (n + 1) \sum_{i=0}^{n+1} (-1)^i f_{i-1} = \sum_{i=0}^{n+1} (-1)^{i-1} i f_{i-1},$$

as claimed.

**Remark 10** There is a way to deduce Corollary 9 from results of Bestvina and Brady from [3]. This alternative proof was given in the second named author’s PhD thesis [13]. Let $\tilde{\mu}_L$ be the map defined by the pullback square

$$\begin{array}{ccc}
\tilde{T}_L & \xrightarrow{\tilde{\mu}_L} & \mathbb{R} \\
\downarrow & & \downarrow \\
T_L & \xrightarrow{\mu_L} & \mathbb{T}.
\end{array}$$

In the case when $L$ is finite and $\mathbb{Q}$-acyclic, Bestvina and Brady show that the inclusion $\tilde{\mu}_L^{-1}(x)$ in $\tilde{T}_L$ is a rational homology isomorphism for any real number $x$ [3]. In the case when $x$ is not an integer, there is a cellular structure on $\tilde{\mu}_L^{-1}(x)$ with $(i + 1) f_i$ $i$-cells for each $i \geq 0$.

On the other hand, we know of no other proof of Corollary 7 or Corollary 8 than the proofs given above.
5 Cohomology of $\tilde{T}_L$

**Proposition 11** Let $C^*(\tilde{T}; A)$ denote the cellular cochain complex for $\tilde{T}_L$ with coefficients in the abelian group $A$. Each $C^n(\tilde{T}_L; A)$ is isomorphic to a coinduced $\mathbb{Z}$-module:

$$C^n(\tilde{T}_L; A) \cong \operatorname{Hom}(\mathbb{Z}[\mathbb{Z}], C^{n-1}_+(L; A)) \cong \prod_{i \in \mathbb{Z}} C^{n-1}_+(L; A).$$

If $A$ is an $R$-module for some ring $R$, then this is an isomorphism of $R[\mathbb{Z}]$-modules. The coboundary map is given by $\delta((f_i)_{i \in \mathbb{Z}}) = (\delta f_i - \delta f_{i+1})_{i \in \mathbb{Z}}$. The action of $\mathbb{Z}$ is the ‘shift action’: $z(f_i)_{i \in \mathbb{Z}} = (f_{i+1})$. The image of $C^*(T_L)$ in $C^*(\tilde{T}_L)$ is identified with the ‘constant sequences’, i.e., those with $f_i = f_j$ for all $i, j$.

**Proof.** Most of the assertions follow immediately from the description of $C^*(\tilde{T}_L)$ given in Proposition 6, since

$$C^*(\tilde{T}_L; A) \cong \operatorname{Hom}(C_*(\tilde{T}_L), A) \cong \operatorname{Hom}(\mathbb{Z}[\mathbb{Z}] \otimes C^*_+(L; A), A) \cong \operatorname{Hom}(\mathbb{Z}[\mathbb{Z}], C^*_+(L; A)).$$

The claim concerning the image of $C^*(T_L)$ is clear, since cochains that factor through the projection $\tilde{T}_L \to \tilde{T}_L/\mathbb{Z} = T_L$ may be identified with cochains that are fixed by the $\mathbb{Z}$-action. $lacksquare$

**Corollary 12** For any $L$, any abelian group $A$, and any $n \geq 0$ there are short exact sequences of $\mathbb{Z}[\mathbb{Z}]$-modules:

$$0 \to C^{n-1}_+(L; A)/B^{n-1}_+(L; A) \to H^n(\tilde{T}_L; A) \to M \to 0,$$

$$0 \to \prod_Z H^{n-1}_+(L; A) \to H^n(\tilde{T}_L; A) \to C^{n-1}_+(L; A)/Z^{n-1}_+(L; A) \to 0,$$

where $\mathbb{Z}$ acts trivially on $C^*_+(L; A)$, and $Z^*_+(L; A)$ (resp. $B^*_+(L; A)$) denotes the cocycles (resp. coboundaries) in $C^*_+(L; A)$. The module $M$ fits in to a short exact sequence:

$$0 \to \mathbb{H}^{n-1}_+(L; A) \otimes_{\mathbb{Z}} \prod_Z \mathbb{H}^{n-1}_+(L; A) \to M \to 0,$$
where $Z$ acts by the ‘shift action’ on the product and where $\Delta$ is the inclusion of the constant sequences. The first short exact sequence is the inclusion of the $Z$-fixed points in $H^n(\tilde{T}_L; A)$. If $A$ is an $R$-module, then both short exact sequences admit an $R[Z]$-action.

Proof. Using the description of $C^* = C^*(\tilde{T}_L; A)$ given in Proposition 11 we obtain descriptions of the coboundaries $B^n$ and cocycles $Z^n$ in $C^n$:

$$B^n \cong \prod_{i \in \mathbb{Z}} B^n_{i-1}(L; A),$$

$$Z^n \cong \{(f_i)_{i \in \mathbb{Z}} \in \prod_{i \in \mathbb{Z}} C^n_{i-1}(L; A) : \delta f_i = \delta f_{i+1} \forall i \in \mathbb{Z}\}.$$ 

Let $B'^n$ be the submodule of $Z^n$ generated by $B^n$ and the constant sequences $\{(f_i)_{i \in \mathbb{Z}} : f_i = f_j\}$, and let $Z'\! n$ be the submodule of $Z^n$ consisting of sequences of cocycles, i.e., $Z'^n = \prod_{i \in \mathbb{Z}} Z^n_{i-1}(L; A)$. The first short exact sequence in the statement is equal to

$$0 \to B'^n/B^n \to H^n(\tilde{T}_L; A) \to Z'^n/B'^n \to 0,$$

and the second one to

$$0 \to Z'^n/B^n \to H^n(\tilde{T}_L; A) \to Z^n/Z'^n \to 0.$$ 

The computation of the $Z$-fixed points in $H^n(\tilde{T}_L; A)$ follows by applying the $Z$-fixed point functor to the first exact sequence.  

Theorem 13 Let $R$ be a ring. The image of the map

$$H^*(T_L; R) \to H^*(\tilde{T}_L; R)$$

is equal to the $Z$-fixed point subring of $H^*(\tilde{T}_L; R)$ and is isomorphic to the quotient $H^*(T_L; R)/(\beta_L)$. In degree $n$, the cokernel of this map is isomorphic to an infinite product of copies of $H^{n-1}(L; R)$. In particular, the map is a ring isomorphism if and only if $L$ is $R$-acyclic.

Proof. This follows from Corollary 12 and Theorem 2.
Corollary 14 Suppose that $L$ is $R$-acyclic. There is an $R$-algebra isomorphism
\[ H^*(\tilde{T}_L; R) \cong \Lambda^*_R(L)/(\beta_L). \]
For each $n$, $H^n(\tilde{T}_L; R)$ is isomorphic to a direct product of copies of $R$.

Proof. This follows from Theorem 13 and Theorem 1.

Remark 15 The second named author’s PhD thesis contained a different proof of Corollary 14 in the case when $L$ is finite, flag and $R$-acyclic [13].

Recall that we denote by $G_L$ the fundamental group of $T_L$, and by $H_L$ the fundamental group of $\tilde{T}_L$. Also recall from Proposition 4 that when $L$ is flag, each of $T_L$ and $\tilde{T}_L$ is an Eilenberg-Mac Lane space for its fundamental group. In [13] an explicit chain homotopy was used to show that the action of $Z$ on $H^*(\tilde{T}_L; R)$ is trivial when $L$ is finite, flag and $R$-acyclic. The long exact sequence in group cohomology coming from the isomorphism $G_L = H_L : Z$ was then used to establish the isomorphism of Corollary 14.

6 Cohomological dimension

The trivial cohomological dimension of a space $X$, $\text{tcd}(X)$, is defined to be the supremum of those integers $n$ for which there exists an abelian group $A$ for which the singular cohomology group $H^n(X; A)$ is non-zero. For any non-trivial ring $R$, $\text{tcd}_R(X)$ is defined similarly except that only abelian groups $A$ admitting an $R$-module structure are considered.

Now suppose that $X$ is path-connected, and that $G$ is the fundamental group of $X$. For $M$ a $G$-module, we write $H^*(X; M)$ for the singular cohomology of $X$ with twisted coefficients in $M$. If $X$ admits a universal covering space $\tilde{X}$, then this is just the cohomology of the cochain complex $\text{Hom}_G(C_\bullet(\tilde{X}), M)$ of $G$-equivariant singular cochains on $\tilde{X}$. The cohomological dimension $\text{cd}(X)$ of $X$ is the supremum of those integers $n$ such that there is a $G$-module $M$ for which $H^n(X; M) \neq \{0\}$. For a non-trivial ring $R$, $\text{cd}_R(X)$ is defined similarly except that only $RG$-modules $M$ are considered.

Each of the invariants depends only on the homotopy type of $X$. In the case when $X$ is a classifying space or Eilenberg-Mac Lane space for $G$, we write $\text{cd}(G)$, $\text{cd}_R(G)$, $\text{tcd}(G)$ and $\text{tcd}_R(G)$ for the corresponding invariants of $X$.

We summarize the properties of these invariants below.
Proposition 16 In the following statements, $X$ is any path-connected space, and $R$ is any non-trivial ring. Rings are assumed to have units, and ring homomorphisms are assumed to be unital.

1. $\text{tcd}(X) = \text{tcd}_Z(X)$, $\text{cd}(X) = \text{cd}_Z(X)$.

2. $\text{tcd}_R(X) \leq \text{cd}_R(X)$.

3. If there is a ring homomorphism $\phi : R \to S$, then $\text{tcd}_S(X) \leq \text{tcd}_R(X)$ and $\text{cd}_S(X) \leq \text{cd}_R(X)$.

4. If $R$ is a subring of $S$ in such a way that $R$ is a direct summand of $S$ as an $R$-$R$-bimodule, then $\text{tcd}_R(X) = \text{tcd}_S(X)$ and $\text{cd}_R(X) = \text{cd}_S(X)$.

5. If $Y$ is a covering space of $X$, then $\text{cd}_R(Y) \leq \text{cd}_R(X)$.

6. If $H \leq G$, then $\text{cd}_R(H) \leq \text{cd}_R(G)$.

7. For any group $G$, $\text{cd}_R(G) = \text{proj.dim}_{RG}(R)$.

Proof. A $ZG$-module is the same thing as a $G$-module, establishing (1). Similarly, any $R$-module $A$ may be viewed as an $RG$-module by letting each element of $G$ act via the identity, which establishes (2). A ring homomorphism $\phi$ as above allows one to define an $R$-module structure on any $S$-module and an $RG$-module structure on any $SG$-module, which proves (3). Under the hypotheses of (4), any $R$-module $A$ is isomorphic to a direct summand of the $S$-module $S \otimes_R A$ and any $RG$-module $M$ is isomorphic to a direct summand of the $SG$-module $SG \otimes_{RG} M$. Since cohomology commutes with finite direct sums, this shows that $\text{tcd}_R(X) \leq \text{tcd}_S(X)$ and $\text{cd}_R(X) \leq \text{cd}_S(X)$. The opposite inequalities follow from (3).

If the fundamental group of $Y$ is the subgroup $H \leq G$, and $M$ is any $RH$-module, define the coinduced $RG$-module by

$$\text{Coind}(M) = \text{Hom}_{ZH}(ZG, M).$$

Then there is an isomorphism $H^*(X; \text{Coind}(M)) \cong H^*(Y; M)$, which establishes (5). Now (6) is just the special case of (5) in which $X$ is a classifying space for $G$, since then the covering space of $X$ with fundamental group $H$ is a classifying space for $H$.

If $X$ is a classifying space for $G$, then the universal covering space $\tilde{X}$ is contractible, and so $C_*(\tilde{X})$ is a free resolution of $Z$ over $ZG$. Similarly,
\( C_\ast(\tilde{X}; R) \) is a free resolution of \( R \) over \( RG \). Hence \( H^\ast(\tilde{X}; M) \) is isomorphic to \( \text{Ext}_{ZG}(\mathbb{Z}, M) \), and if \( M \) is an \( RG \)-module \( H^\ast(\tilde{X}; M) \) is also isomorphic to \( \text{Ext}_{RG}(R, M) \), which establishes (7). (See for example [4] for more details.)

\[
\text{Remark 17}\quad \text{It is easy to find groups } G \text{ for which } \text{tcd}(G) < \text{cd}(G). \text{ There are many acyclic groups, or groups for which the group homology } H_i(G; \mathbb{Z}) = 0 \text{ for all } i > 0 \text{ (see for example [1]). For any such } G, \text{ tcd}(G) = 0, \text{ while } \text{cd}(G) \neq 0 \text{ unless } G \text{ is the trivial group.}
\]

Before stating the next proposition, we remind the reader that the dimension of a simplicial complex is the supremum of the dimensions of its simplices, so that the empty simplicial complex has dimension \(-1\).

**Proposition 18** For any simplicial complex \( L \) and any non-trivial ring \( R \),

\[
\text{tcd}_R(T_L) = \text{cd}_R(T_L) = \dim(T_L) = \dim(L) + 1.
\]

**Proof.** Immediate from Theorem [1].

**Proposition 19** For any simplicial complex \( L \) with at least two vertices and for any \( R \),

\[
\text{tcd}_R(\tilde{T}_L) = \max\{\dim(L), 1 + \text{tcd}_R(L)\}.
\]

**Proof.** Immediate from Corollary [12].

The following theorem is the first result in this paper for which we rely on techniques from Bestvina-Brady [3].

**Theorem 20** Let \( L \) be a simplicial complex with at least two vertices and let \( R \) be a non-trivial ring.

1. \( \dim(L) \leq \text{tcd}_R(\tilde{T}_L) \leq \text{cd}_R(\tilde{T}_L) \leq \dim(L) + 1. \)

2. If \( L \) is \( R \)-acyclic, then \( \text{tcd}_R(L) = \text{cd}_R(\tilde{T}_L) = \dim(L) \).

**Proof.** The claims for arbitrary \( L \) follow easily from earlier results. Proposition [19] implies that \( \dim(L) \leq \text{tcd}_R(\tilde{T}_L) \). By Proposition [16] one has that \( \text{tcd}_R(\tilde{T}_L) \leq \text{cd}_R(\tilde{T}_L) \leq \text{cd}_R(T_L) \), and of course \( \text{cd}_R(T_L) \leq \dim(T_L) = \dim(L) + 1. \)
It remains to show that when $L$ is $R$-acyclic, $\text{cd}_R(\tilde{T}_L) \leq \dim(L)$. Following Bestvina and Brady \cite{BB}, let $X_L$ be the universal covering space of $T_L$, or equivalently of $\tilde{T}_L$. Now let $f_L : X_L \to \mathbb{R}$ be the composite

$$X_L \to X_L/H_L = \tilde{T}_L \tilde{\mu} \to \mathbb{R},$$

where $\tilde{\mu}$ is the lift of the map $\mu_L : T_L \to \mathbb{T}$. Now define $Y_L = f^{-1}(0) \subseteq X_L$. There is a natural cubical CW-structure on $X_L$, whose cells are the lifts to $X_L$ of the cells of $T_L$. One can also put a CW-structure on $Y_L$, such that each cell of $Y_L$ is the intersection of $Y_L$ with a cell of $X_L$. For this CW-structure, the dimension of $Y_L$ is equal to $\dim(L)$. It can be shown that $X_L$ is homotopy equivalent to $Y_L$ with infinitely many subspaces homotopy equivalent to $L$ coned off (this is from \cite{BB}, but see also \cite{HH} which explicitly checks this in the case when $L$ is infinite).

Since $L$ is $R$-acyclic, it follows that the inclusion of $Y_L$ in $X_L$ induces an isomorphism of $R$-homology. The cellular chain complexes $C_*(Y_L; R)$ and $C_*(X_L; R)$ consist of free $RH_L$-modules, and so it follows that for any $RH_L$-module $M$,

$$H^*(\tilde{T}_L; M) \cong H^*(Y_L/H_L; M).$$

This shows that $\text{cd}_R(\tilde{T}_L) \leq \dim(L)$ as required.

Remark 21 In the case when $L$ is empty, $\tilde{T}_L$ is 0-dimensional, and consists of a single free $\mathbb{Z}$-orbit of points. In the case when $L$ is a single point, $\tilde{T}_L$ is homeomorphic to $\mathbb{R}$, with $\mathbb{Z}$ acting via the translation action of $\mathbb{Z}$. It is clear that the case when $L$ is empty is exceptional for $\tilde{T}_L$. The reason why the case when $L$ is a single point needs to be excluded from Proposition \ref{prop:cd} and from Theorem \ref{thm:cd} is that the formulae for $H^*(\tilde{T}_L; R)$ given in Section 5 involve the reduced cohomology of $L$, whereas the definition of $\text{tcd}(L)$ involves the unreduced cohomology of $L$. This only makes a difference when $L$ is both 0-dimensional and $R$-acyclic, i.e., the case when $L$ is a single point.

7 Bestvina-Brady groups

In this section, we give a complete calculation of the cohomological dimension of Bestvina-Brady groups, or equivalently the cohomological dimension of the spaces $\tilde{T}_L$ for $L$ a flag complex. We impose some conditions on the coefficient ring $R$ that were not previously required. The reason why we work only with
flag complexes in this section is that we need to know that when $L \leq K$ is a full subcomplex of $K$, then $\tilde{T}_L$ is homotopy equivalent to a covering space of $\tilde{T}_K$.

**Theorem 22** Let $R$ be either a field or a subring of $\mathbb{Q}$. Let $L$ be a flag complex with at least two vertices. (This implies that $H_L$ is infinite, and that $\tilde{T}_L$ is a classifying space for $H_L$.) The following equations hold:

$$
\text{cd}_R(H_L) = \text{tcd}_R(H_L) = \max\{\dim(L), 1 + \text{tcd}_R(L)\}.
$$

The proof of the theorem will require two lemmas, the second of which is a strengthening of a lemma from [2].

**Lemma 23** Suppose that $L$ is a flag complex and is a subcomplex of a simplicial complex $K$. The relative barycentric subdivision $(K, L)'$ is a flag complex containing $L$ as a full subcomplex.

Proof. Before recalling the definition of the relative barycentric subdivision $(K, L)'$, we remind the reader that each simplicial complex contains a unique $-1$-simplex corresponding to the empty subset of its vertices. The vertex set of $(K, L)'$ is the disjoint union of the vertex set of $L$ and the set of simplices of $K$ not contained in $L$. An $n$-simplex of $(K, L)'$ has the form $(\sigma_0 < \sigma_1 < \cdots < \sigma_r)$, for some $r$ satisfying $0 \leq r \leq n + 1$. Here $\sigma_0$ denotes an $(n - r)$-simplex of $L$, each $\sigma_i$ is a simplex of $M$, and $\sigma_i$ is not contained in $L$ if $i > 0$.

Suppose that a finite subset $S$ of the vertices of $(K, L)'$ has the property that any two of its members are joined by an edge. Since $L$ is flag, this implies that the set $S \cap L$ is the vertex set of a simplex $\sigma_0$ of $L$. Each element of $S - L$ is a simplex $\sigma_i$ of $M$ not contained in $L$. The existence of an edge between each element of $S \cap L$ and each element of $S - L$ implies that each $\sigma_i$ contains $\sigma_0$, and the existence of an edge between each pair of elements of $S - L$ implies that the $\sigma_i$ are totally ordered by inclusion. This shows that $(K, L)'$ is a flag complex.

From the description of the simplices of $(K, L)'$, it is easy to see that any simplex of $(K, L)'$ whose vertex set lies in $L$ is in fact a simplex in $L$, which verifies that $L$ is a full subcomplex of $(K, L)'$.

**Lemma 24** Let $L$ be a flag complex, let $R$ be either a subring of $\mathbb{Q}$ or a field of prime order, and suppose that $\text{tcd}_R(L) < \dim(L)$. Then there is
an $R$-acyclic flag complex $K$ containing $L$ as a full subcomplex such that $\dim(L) = \dim(K)$.

Proof. We may assume that $\dim(L) = n$ is finite, and we may assume that $n \geq 2$. Let $C'L$ denote the cone on the $(n - 2)$-skeleton of $L$, and let $L_1$ be the union $L_1 = L \cup C'L$. Now $L_1$ is $(n - 2)$-connected, and the inclusion map $L \to L_1$ induces an isomorphism $H^n(L_1; A) \to H^n(L; A)$ for any abelian group $A$.

In each case, $R$ is a principal ideal domain, and so by the universal coefficient theorem for cohomology [9, V.3.3], for any $R$-module $A$, we have that

$$H^n(L_1; A) \cong \Hom_R(H_n(L_1, R), A) \oplus \Ext_R^1(H_{n-1}(L_1, R), A).$$

The hypotheses therefore imply that $H_n(L_1; R) = 0$, and that $H_{n-1}(L_1; R)$ is a projective $R$-module. (Since $L_1$ is $(n - 2)$-connected, one also has that $\overline{H}_i(L_1; R) = 0$ for each $i < n - 1$.) Every projective module for a principal ideal domain is free [9, I.5.1], and so $H_{n-1}(L_1; R)$ is a free $R$-module. Since $\overline{H}_{n-2}(L_1; \mathbb{Z}) = 0$ (because $L_1$ is $(n - 2)$-connected), the universal coefficient theorem for homology [9, V.2.5] tells us that $H_{n-1}(L_1; R) \cong H_{n-1}(L_1; \mathbb{Z}) \otimes R$.

Hence there exist integral cycles $z_i \in C_{n-1}(L_1; \mathbb{Z})$ whose images in the group $C_{n-1}(L_1; R)$ map to an $R$-basis for $H_{n-1}(L_1; R)$. By the Hurewicz theorem, each $z_i$ is realized by some map $f_i : S^{n-1} \to L_1$. Replace $f_i$ by a simplicial approximation $f_i' : S_i \to L_1$, where $S_i$ is some triangulation of the $(n - 1)$-sphere. Using a simplicial mapping cylinder construction as in [8, 2C.5], use each $f_i'$ to attach a triangulated $n$-cell to $L_1$ to produce $L_2$, an $R$-acyclic simplicial complex with $\dim(L_2) = n$ and such that $L \leq L_1 \leq L_2$. By Lemma 23, we may take $K$ to be the relative barycentric subdivision $K = (L_2, L)'$. 

Proof of Theorem 22. If $S$ is any field, and $R$ is the smallest subfield of $S$, then $R$ and $S$ satisfy the conditions of statement (4) of Proposition 16. Hence there are equalities of functions $cd_R = cd_S$ and $\text{tcd}_R = \text{tcd}_S$. Thus it suffices to prove Theorem 22 in the case when $R$ is either the field of $p$ elements or a subring of $\mathbb{Q}$.

We may assume that $\dim(L) = n < \infty$. In the case when $\text{tcd}_R(L) = n$, Proposition 19 and part (1) of Theorem 20 imply that

$$n + 1 = \text{tcd}_R(\tilde{T}_L) \leq cd_R(\tilde{T}_L) \leq \dim(\tilde{T}_L) = n + 1,$$
as claimed. If \( \text{tcd}_R(L) < n \), then by Lemma 24, there is an \( n \)-dimensional \( R \)-acyclic flag complex \( K \) containing \( L \) as a full subcomplex. Part (2) of Theorem 20 tells us that \( \text{tcd}_R(K) = \text{cd}_R(K) = n \). Now \( H_L \) is a retract of \( H_K \), and so by part (6) of Proposition 16

\[
\text{cd}_R(H_L) = \text{cd}_R(T_L) \leq \text{cd}_R(H_K) = \text{cd}_R(T_K) = n.
\]

This gives

\[
n = \text{tcd}_R(T_L) \leq \text{cd}_R(T_L) \leq \text{cd}_R(T_K) = n,
\]
as claimed.

8 Examples

Example 25 Let \( L = L(m) \) be a flag triangulation of the space constructed by attaching a disc to a circle via a map of degree \( m \). This \( L \) has the property that \( \text{tcd}_R(L) = 0 \) if \( m \) is a unit in \( R \) and \( \text{tcd}_R(L) = 2 \) if \( m \) is not a unit in \( R \). Let \( H = H_L \) be the corresponding Bestvina-Brady group. From Theorem 20 and Proposition 19 it follows that for any ring \( R \), \( \text{cd}_R(H) = 2 \) if \( m \) is a unit in \( R \) and \( \text{cd}_R(H) = 3 \) if \( m \) is not a unit in \( R \).

Example 26 Let \( L \) be a flag triangulation of a 2-dimensional Eilenberg-Mac Lane space for the additive group of \( \mathbb{Q} \). Then \( \text{tcd}_F(L) = 1 \) for any field \( F \), while \( \text{tcd}_Z(L) = 2 \). From Theorem 22 it follows that the Bestvina-Brady group \( H = H_L \) has the property that \( \text{cd}_F(H) = 2 \) for any field \( F \), while \( \text{cd}_Z(H) = 3 \).

Groups having similar properties to those given in Example 25 have been constructed previously by many authors, using finite-index subgroups of right-angled Coxeter groups \([2, 5, 7]\). The examples coming from Coxeter groups have the advantage that they are of type \( FP \), whereas they have the disadvantage that the trivial cohomological dimension of these groups appears to be unknown.

Finite-index subgroups of (non-finitely generated) Coxeter groups were used in \([5]\) to construct groups having similar properties to those given in Example 26. Again, the trivial cohomological dimension of these examples appears to be unknown. It can be shown that for any group \( G \) of type \( FP \), there is a field \( F \) such that \( \text{cd}_F(G) = \text{cd}_Z(G) \). (See Proposition 9 of \([5]\).)
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Authors’ addresses:
Ian Leary: Department of Mathematics, The Ohio State University, 231 West 18th Avenue, Columbus, Ohio 43210-1174, United States.

and
School of Mathematics, University of Southampton, Southampton, SO17 1BJ, United Kingdom.

leary@math.ohio-state.edu

Müge Saadetoğlu: School of Mathematics, University of Southampton, Southampton, SO17 1BJ, United Kingdom.

and
AS277, Department of Mathematics, Eastern Mediterranean University, Gazimagusa, Mersin 10, Turkey

muge.saadetoglu@emu.edu.tr