Singular Non-Abelian Toda Theories

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ABSTRACT The algebraic conditions that specific gauged \( G/H \)-WZW model have to satisfy in order to give rise to Non-Abelian Toda models with non singular metric with or without torsion are found. The classical algebras of symmetries corresponding to grade one rank 2 and 3 singular NA-Toda models are derived.

The proliferation of different two dimensional \( G_0 \)-Toda theories (\[2\] - \[10\]) addresses the question about their algebraic classification in terms of a \( G_0 \subset G \) embedding and a \( G \)-invariant WZNW-model. The simplest class of models is the Abelian Toda, which is known to be completely integrable and conformal invariant. These models correspond to a Abelian subgroup \( G_0 \subset G \) and their symmetries generate the \( W_n \)-algebra (\( n=\)rank \( G \)).

The non-Abelian Toda models, in turn are connected to non-abelian embeddings \( G_0 \subset G \) and describe a string propagating on a specific curved background, containing also tachyons, dilatons \( \Phi(\mathbf{X}) \) and, possibly axions. Its general action is of the form \((i,j=1,\ldots,D; \mu, \nu = 0,1)\):

\[
S = \int d^2 z \left\{ (G_{ij}(X)\eta_{\mu\nu} + \epsilon_{\mu\nu} B_{ij}(X)) \partial_\mu X^i \partial_\nu X^j - \alpha' R^{(2)}(\Phi(X) + Tach.\, potential) \right\}.
\]

The properties of the background metric \( G_{ij} \) and of the anti-symmetric term \( B_{ij} \) (torsion) depend upon the embedding \( G_0 \subset G \), where \( G_0 \) is now non-Abelian and classify the models according to singular or non-singular metrics and the presence of axionic or torsionless terms. The symmetries of the singular metric NA-Toda models are described by a non-local algebra, which corresponds to the semi-classical limit of a mixed parafermionic and \( W \)-algebra structure, denoted by \( V \)-algebra (see \[8\], \[3\], \[4\] and \[5\]).

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Consider the $G$-invariant WZNW-model, describing a 2-D conformal invariant field theory on a group manifold, with fields parametrizing the group element

$$g(z, \bar{z}) = \exp(\alpha^a(z, \bar{z})T^a),$$

where $[T^a, T^b] = f^{abc}T^c$ is the Lie bracket satisfied by the generators of the Lie algebra of $G$.

The WZNW-action is given by

$$S_{\text{WZNW}} = \frac{k}{4\pi} \int d^2z \text{Tr}(g^{-1}\partial gg^{-1}\partial g) + \frac{k}{12\pi} \int_B \epsilon_{ijk} \text{Tr}(g^{-1}\partial_i gg^{-1}\partial_j gg^{-1}\partial_k g),$$

where the topological term denotes a surface integral over a ball $B$ identified as space-time.

The equations of motion are

$$\bar{\partial} J = \partial \bar{J} = 0,$$

where the currents $J = g^{-1}\partial g$ and $\bar{J} = -\bar{\partial} gg^{-1}$ satisfy the Kac-Moody algebra

$$\{J^a(x), J^b(y)\} = f^{abc} J^c(x) \delta(x - y) + k\delta^{ab} \partial_x(x - y).$$

We now seek for a field theory in the submanifold $G_0 \subset G$ preserving conformal invariance. In order to eliminate unwanted degrees of freedom, corresponding to the tangent space of $G/G_0$, we may either implement constraints upon specific components of $J$ and $\bar{J}$ or, equivalently, propose a gauge invariant action and eliminate degrees of freedom by choice of gauge. In this note we shall follow the second approach.

In order to construct a gauge invariant action, we first discuss a systematic procedure in defining the $G_0$ subgroup, in terms of a grading operator.

Let $Q$ be a grading operator, decomposing the Lie algebra $\mathcal{G}$ into graded subspaces, i.e.,

$$\mathcal{G} = \bigoplus_i \mathcal{G}_i, \quad [Q, \mathcal{G}_i] = i \mathcal{G}_i, \quad [\mathcal{G}_i, \mathcal{G}_j] \in \mathcal{G}_{i+j}.$$  \hfill (4)

It is clear that $\mathcal{G}_0$ is the subalgebra of $\mathcal{G}$.

Consider now the Gauss decomposition of the group element $g \in G$, i.e.,

$$g = N_- g_0 M_+,$$

where $N_- = \exp \mathcal{G}_{<0}$, $g_0 = \exp \mathcal{G}_0$ and $M_+ = \exp \mathcal{G}_{>0}$. The Toda models correspond to the co-set $N_- \backslash G/M_+$, and we seek for an action invariant under

$$g \rightarrow g' = \alpha_- g \alpha_+,$$

so that $N_-$ and $M_+$ may be eliminated by choice of gauge.

Introduce gauge fields $A = A_\pm \in \mathcal{G}_{<}$ and $\bar{A} = A_+ \in \mathcal{G}_{>}$, transforming as

$$A \rightarrow A' = \alpha_- A \alpha_+^{-1} + \alpha_- \partial \alpha_+^{-1}, \quad \bar{A} \rightarrow \bar{A}' = \alpha_+^{-1} \bar{A} \alpha_+ + \bar{\partial} \alpha_+^{-1} \alpha_+.$$  \hfill (7)

An invariant action under transformations (6) and (7) is

$$S_{G/H}(g, A, \bar{A}) = S_{\text{WZNW}}(g) - \frac{k}{2\pi} \int d^2z \text{Tr} (A(\bar{\partial} gg^{-1} - \epsilon_+) + \bar{A}(g^{-1}\partial g - \epsilon_-) + Ag\bar{A}g^{-1}),$$

where $\epsilon_+ = \partial_+ - \bar{\partial}_+$ and $\epsilon_- = \partial_- + \bar{\partial}_-$. 

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where \( \epsilon_\pm \) are constant generators of grade \( \pm1 \), respectively. Since the action \( S_{G/H} \) is invariant, we may choose \( \alpha_- = N^{-1} \) and \( \alpha_+ = M^{-1}_+ \), so that

\[
S_{G/H}(g, A, \bar{A}) = S_{G/H}(g_0, A', \bar{A}')
\]

\[
= S_{WZNW}(g_0) - \frac{k}{2\pi} \int dz^2 Tr[A'\epsilon_+ + \bar{A}'\epsilon_- + A'g_0\bar{A}'g_0^{-1}],
\]

since from the graded structure \( Tr(A\bar{\partial}_0\partial g_0^{-1}) = Tr(\bar{A}\partial_0\partial g_0) = 0 \).

After integrating the quadratic terms in the functional integral

\[
Z = \int DAD\bar{A} \exp(-F) \sim \exp(-S_{eff}),
\]

where

\[
F = -\frac{k}{2\pi} \int Tr(A'\epsilon_+ + \bar{A}'\epsilon_- + A'g_0\bar{A}'g_0^{-1}),
\]

we find the effective action

\[
S_{eff} = S_{WZNW}(g_0) - \frac{k}{2\pi} \int Tr(\epsilon_+g_0\epsilon_-g_0^{-1}).
\]

1. ABELIAN TODA MODELS. Consider the grading operator

\[
Q = \sum_{i=1}^{r} \frac{2\lambda_i \cdot H}{\alpha_i^2},
\]

where \( r = \text{rank} G \), \( H \) is the Cartan subalgebra of \( G \), \( \alpha_i \) is the \( i \)th simple root of \( G \) and \( \lambda_i \) is the \( i \)th fundamental weight of \( G \), which satisfies

\[
\frac{2\lambda_i \cdot \alpha_j}{\alpha_i^2} = \delta_{ij}.
\]

In this case, the Abelian subalgebra of \( G \), of grade zero, is

\[
G_0 = u(1)^r = \{h_1, h_2, ..., h_r\},
\]

where \( h_i \), defined as \( h_i = \frac{2\alpha_i \cdot H}{\alpha_i^2} \), and satisfy \([h_i, E_{\alpha_j}] = k_{ji}E_{\alpha_j}\); \( k_{ji} \) denote the Cartan matrix of \( G \). Taking all this into account, the group element in \( G_0 \) may be parameterized as \( g_0 = \exp(\phi_i h_i) \).

The most general constant generators of grade \( \pm1 \) are

\[
\epsilon_\pm = \sum_{i=1}^{r} c_{\pm i}E_{\pm \alpha_i}
\]

and the potential term may be evaluated as

\[
Tr(\epsilon_+g_0\epsilon_-g_0^{-1}) = \sum_{i=1}^{r} \frac{2}{\alpha_i^2} c_{\pm i} c_{\pm -i} \exp(k_{ij}\phi_j).
\]
Therefore, the action is given by
\[ S_{G/H} = \int d^2 z \left( \partial \phi_i \bar{\partial} \phi_j Tr(h_i h_j) - \sum_{i=1}^{r} \frac{2}{\alpha_i^2} c_{c-i} \exp(k_{ij} \phi_j) \right). \]  

At this point, let us remark that if \( c_a = c_{-a} = 0 \), then
\[ \epsilon_{\pm} = \sum_{i \neq a} c_{\pm i} E_{\pm \alpha_i}. \]
The structure of this Abelian Toda model is such that, by a change of variables
\[ g_0 = \exp(\sum_{i=1}^{a-1} \varphi_i h_i + \varphi \lambda_a \cdot H + \sum_{i=a+1}^{r} \varphi_i h_i), \]  
the action decomposes into two Toda models, together with a free field, associated to the generator \( g_0^0 = \lambda_a \cdot H \), such that \([g_0^0, \epsilon_{\pm}] = 0\). Hence, the action can be written as
\[ S_{G/H} = S_{G/H}^{(1)}(\varphi_1, \varphi_2, ..., \varphi_{a-1}) + S_{G/H}^{(2)}(\varphi_{a+1}, \varphi_{a+2}, ..., \varphi_r) + \frac{1}{2} \int d^2 z \partial \varphi \bar{\partial} \varphi, \]
where \( S^{(1)} \) and \( S^{(2)} \) denote the abelian Toda models associated to the decomposed Dynkin diagram of \( G \), by deleting the \( a^{th} \) point.

2. NON-ABELIAN TODA MODELS. Consider now the grading operator
\[ Q_a = \sum_{i \neq a} \frac{2 \lambda_i \cdot H}{\alpha_i^2}, \]  
which determines the non-Abelian subalgebra of grade zero
\[ G_0 = u(1)^{r-1} \otimes sl(2). \]
The group element of grade zero may be then, parameterized as
\[ g_0 = \exp(\chi E_{-\alpha_a}) \exp(\sum_{i=1}^{r} \phi_i h_i) \exp(E_{\alpha_a}). \]
Now, take
\[ \epsilon_{\pm} = \sum_{i \neq a} c_{\pm i} E_{\pm \alpha_i}, \quad [\epsilon_{\pm}, \lambda_a \cdot H] = 0 \]
such that
\[ Tr(\epsilon_{+} g_0 \epsilon_{-} g_0^{-1}) = \sum_{i \neq a} \frac{2}{\alpha_i^2} c_{c-i} \exp(\sum_{j \neq a} k_{ij} \phi_j), \]
as before. The pure \( WZNW \)-action for \( g_0 \) acquires a contribution from the non-abelian structure,
\[ S_{WZNW} = \int d^2 z \left( \sum_{i,j=1}^{r} \partial \phi_i \bar{\partial} \phi_j Tr(h_i h_j) + 2 \partial \chi \bar{\partial} \psi \exp(k_{ai} \phi_i) \right). \]
Notice that an attempt to decouple a free field from the action by changing the variables to

\[ g_0 = \exp(\chi E_{-\alpha}) \exp(\sum_{i=1}^{a-1} \phi_i h_i + \varphi \lambda_a \cdot H + \sum_{i=a+1}^{r} \varphi_i h_i) \exp(\psi E_{\alpha_0}) \]  

(23)

decouples only part of the kinetic term, leading to

\[ L_{WZNW}(g_0) = \sum_{i,j \neq a} \partial \phi_i \partial \phi_j Tr(h_i h_j) + \partial \varphi \partial \varphi + 2 \partial \chi \partial \psi \exp(\sum_{i \neq a} k_i \varphi_i + \varphi). \]

It will become clear that the elimination of field \( \varphi \) is responsible for the non-trivial singular metric (of black hole-type) in the string action. In order to gauge away the extra degree of freedom, we need to modify the action to become invariant under transformations within the subspace \( G_0^0 = \lambda_0 \cdot H \) of grade zero (axial gauging), i.e.,

\[ g_0 \longrightarrow g'_0 = \alpha_0 g_0 \alpha_0, \]  

(24)

where \( \alpha_0 = \exp(\varphi \lambda_a \cdot H) \). Following the same line of reasoning as before, we now introduce the new gauge fields \( A = A_0 + A_0, \bar{A} = A_0 + \bar{A}_0 \), where \( A_0, \bar{A}_0 \in G_0^0 \)

transform as

\[ A \longrightarrow A' = \alpha_0^{-1} A_0 + \alpha_0^{-1} \partial \alpha_0, \quad \bar{A} \longrightarrow \bar{A}' = \alpha_0 \bar{A}_0^{-1} + \bar{\partial} \alpha_0 \alpha_0^{-1}, \]

\[ A_0 \longrightarrow A'_0 = A_0 + \alpha_0^{-1} \partial \alpha_0, \quad \bar{A}_0 \longrightarrow \bar{A}'_0 = \bar{A}_0 + \bar{\partial} \alpha_0 \alpha_0^{-1}. \]

The invariant action, under transformations generated by \( H_{(-,0)} \) (in left) and \( H_{(+,0)} \) (in right) is given by

\[ S_{G/H}(g, A, \bar{A}) = S_{WZNW}(g) - \frac{k}{2\pi} \int d^2 z Tr \left( A(\partial gg^{-1} - \epsilon) + \bar{A}(g^{-1} \partial g - \epsilon) + A g \bar{A} g^{-1} + A_0 \bar{A}_0 \right). \]  

(25)

The redundant fields, corresponding to the nilpotent subalgebras \( G_{<0}, G_{>0} \) and to \( G_0^0 \), may be eliminated by subsequent gauge transformations, generated by \( \alpha_-, \alpha_+ \) and \( \alpha_0 \), respectively. Therefore, we arrive at

\[ S_{G/H}(g, A, \bar{A}) = S_{G/H}(g_0^f, A', \bar{A}'), \]  

(26)

where

\[ g_0^f = \exp(\chi E_{-\alpha_0}) \exp(\sum_{i \neq a} \phi_i h_i) \exp(\psi E_{\alpha_0}). \]  

(27)

Due to the trace properties

\[ \text{Tr} \bar{A} \partial g_0^f (g_0^f)^{-1} = \text{Tr} A_0 \partial g_0^f (g_0^f)^{-1}, \quad \text{Tr} \bar{A} (g_0^f)^{-1} \partial g_0^f = \text{Tr} \bar{A}_0 (g_0^f)^{-1} \partial g_0^f \]

\[ \text{Tr} A_0 \bar{A} (g_0^f)^{-1} = \text{Tr} A_0 g_0^f \bar{A}_0 (g_0^f)^{-1} + \text{Tr} A_0 \bar{A} (g_0^f)^{-1}. \]
the action decomposes into three parts, i.e.,
\[ S_{G/H} = S_{WZNW}(g_0^f) + F_0 + F_\pm, \]  
where
\[ F_0 = -\frac{k}{2\pi} \int d^2z Tr \left( A_0 \partial g_0^f(g_0^f)^{-1} + \bar{A}_0(g_0^f)^{-1} \partial g_0^f + A_0 g_0^f \bar{A}_0(g_0^f)^{-1} + A_0 \bar{A}_0 \right), \]
\[ F_\pm = -\frac{k}{2\pi} \int d^2z Tr \left( A_- \epsilon_+ + \bar{A}_+ \epsilon_- + A_- g_0^f \bar{A}_+(g_0^f)^{-1} \right), \]
and the functional integral now factorizes into
\[ Z = \int DA_0 D\bar{A}_0 \exp(-F_0) \int DA_- D\bar{A}_+ \exp(-F_\pm). \]

Explicitly making use of the given parameterization for \( g_0^f \) and writing the gauge fields as \( A_0 = a_0(z, \bar{z}) \lambda_a \cdot H, \bar{A}_0 = \bar{a}_0(z, \bar{z}) \lambda_a \cdot H \), we find
\[ F_0 = -\frac{k}{2\pi} \int d^2z \frac{a_0 \bar{a}_0}{2 \lambda_a^2} \Delta_n - (a_0 \chi \partial \psi + \bar{a}_0 \psi \partial \chi) \exp(k_{ai} \phi_i), \]
where \( \Delta_n = 1 + \frac{\chi \psi \exp(k_{ai} \phi_i)}{2 \lambda_a^2} \).

The total effective action, then, becomes
\[ S_{\text{eff}} = -\frac{k}{2\pi} \int d^2z \left( \frac{1}{2} \partial \phi_i \partial \phi_j \text{Tr}(h_i h_j) + \frac{\partial \chi \partial \psi \Delta_n}{\Delta_n} \exp(k_{ai} \phi_i) \right. \]
\[ - \left. \sum_{i,j \neq a} \frac{2}{\alpha_i^2} \epsilon_{ijk} \exp(-k_{ij} \phi_j) \right). \]

The non-symmetric term may be written as
\[ \partial \chi \partial \psi = g^{\mu \nu} \partial_\mu \chi \partial_\nu \psi - \epsilon_{\mu \nu} \partial_\mu \chi \partial_\nu \psi, \]
and we can rewrite the effective action (discarding the total derivative term) in the form
\[ S_{\text{eff}} = \int d^2z \left( \frac{1}{2} g^{\mu \nu} \partial_\mu \phi_i \partial_\nu \phi_j \text{Tr}(h_i h_j) + g^{\mu \nu} \partial_\mu \chi \partial_\nu \psi \frac{\exp(k_{ai} \phi_i)}{\Delta_n} \right. \]
\[ - \left. \frac{1}{2} \epsilon_{\mu \nu} k_{ai} \partial_\mu \phi_i (\chi \partial_\nu \psi - \psi \partial_\nu \chi) \frac{\exp(k_{ai} \phi_j)}{\Delta_n} - \sum_{i,j \neq a} \frac{2}{\alpha_i^2} \epsilon_{ijk} \exp(-k_{ij} \phi_j) \right). \]

As we shall demonstrate below (see [4], [5]) the symmetries of the non-Abelian Toda model (33), are generated by non-local algebraic structures due to the gauge invariance within the zero grade subspace.

Let us remark that for \( G = sl(2) \) and for \( \phi_i = 0 \) the action coincides with Witten’s black hole action i.e.,
\[ S = \int d^2z g^{\mu \nu} \partial_\mu \chi \partial_\nu \psi \frac{1}{1 + \chi \psi}, \]
3. NO-TORSION THEOREM. We now discuss the \textit{generalized} non-Abelian Toda model, as well as the “no-torsion theorem”. To do so, first we take the very same grading operator

$$Q_a = \sum_{i \neq a}^{r} \frac{2\lambda_i \cdot H}{\alpha_i^2}$$

and consider \textit{the most general constant generators} of grade ±1, i.e.,

$$\epsilon_{\pm} = \sum_{i \neq a}^{r} c_{\pm} E_{\pm \alpha_i} + b_{\pm} E_{\pm(\alpha_a + \alpha_{a+1})} + d_{\pm} E_{\pm(\alpha_a + \alpha_{a-1})},$$

(34)

It is clear that if \(c_{\pm}, b_{\pm}, d_{\pm} \neq 0\), there shall be no \(g_0^j\) \textit{commuting} with \(\epsilon_{\pm}\), since that require an orthogonal direction to all roots appearing in \(\epsilon_{\pm}\). These are the \textit{generalized non-singular} NA-Toda models of ref. \cite{10}. The NA-Toda models of singular metric \(G_{ij}(X)\) correspond to the cases when \(g_0^j = U(1)\) and we impose it as a subsidiary constraint \cite{10}. Depending upon the choice of the constants \(c_{\pm}, b_{\pm}\) and \(d_{\pm}\) we distinguish four families of \textit{singular} NA-Toda models:

(i) \(b_{\pm} = d_{\pm} = 0, \quad g_0^j = \frac{2\lambda_a \cdot H}{\alpha_a^2}\);

(ii) \(c_{\pm(a-1)} = c_{\pm(a+1)} = 0, \quad g_0^0 = \frac{2\lambda_a \cdot H}{\alpha_a^2} - \frac{2\lambda_{a-1} \cdot H}{\alpha_{a-1}^2} - \frac{2\lambda_{a+1} \cdot H}{\alpha_{a+1}^2}\);

(iii) \(c_{\pm(a+1)} = d_{\pm} = 0, \quad g_0^0 = \frac{2\lambda_a \cdot H}{\alpha_a^2} - \frac{2\lambda_{a+1} \cdot H}{\alpha_{a+1}^2}\);

(iv) \(b_{\pm} = c_{\pm(a-1)} = 0, \quad g_0^0 = \frac{2\lambda_a \cdot H}{\alpha_a^2} - \frac{2\lambda_{a-1} \cdot H}{\alpha_{a-1}^2}\).

Of course, if \(c_{\pm j} = 0, \ j \neq a, a \pm 1\), we find \(g_0^j = \lambda_j \cdot H\). However, since \([\lambda_j \cdot H, E_{\pm \alpha_a}] = 0\), there will be no singular metric present and this case shall be neglected. Cases (i) and (ii) are equivalent, since they are related by the Weyl reflection \(\sigma_{\alpha_a}(\alpha_{a \pm 1}) = \alpha_a + \alpha_{a \pm 1}\) and the corresponding fields are related by non-linear change of the variables. This case has already been discussed in refs. \cite{10} and \cite{11}, and shown to present always the \textit{antisymmetric term}, originated by the presence of \(e^{x \cdot \bar{\chi}}\) in \(\Delta_a\) and in the kinetic term as well. Since we are removing all dependence in \(g_0^j\), when parameterizing \(g_0^f\), cases (iii) and (iv) may be studied together with

$$g_0^f = \exp(\chi \cdot E_{- \alpha_a}) \exp(\Phi(H)) \exp(\psi E_{\alpha_a})$$

(35)

where \(\Phi(H) = \sum_{i=1}^{a-2} \varphi_i h_i + \chi \cdot \bar{\varphi} \cdot H + \chi \cdot \bar{\varphi} \cdot H + \sum_{i=a+2}^{r} \varphi_i h_i\),

$$\chi_{\pm i}^j = \alpha_{a-1} + \alpha_a, \quad \chi_{\pm i}^j = \alpha_{a+1}$$

(36)

$$\chi_{\pm i}^j = \alpha_{a-1} + \alpha_a, \quad \chi_{\pm i}^j = \alpha_{a+1}$$

(37)

for cases \(\text{iii}\) and \(\text{iv}\) respectively, and \(g_0^0 = y \cdot H\), such that \(Tr(\chi \cdot H g_0^0) = 0\). It is straightforward to evaluate the integral

$$F_0 = -\frac{k}{2\pi} \int [2y^2 a_0 \bar{a}_0 \Delta_a - 2 \frac{\alpha_a^2}{\alpha_a^2} (a_0 \bar{\chi} \bar{\psi} + \bar{a}_0 \psi \bar{\chi}) \exp(\Phi(\alpha_a))],$$

\(^2\)If we leave \(\partial^0\) unconstrained the resulting model belongs again to the non singular NA-Toda class of models \cite{10}.
where

\[ \Phi(\alpha_a) = \sum_{i=1}^{a-2} k_{ai} \varphi_i + \alpha_a \cdot \chi - \varphi_- + \alpha_a \cdot \chi + \varphi_+ + \sum_{i=a+2}^{r} k_{ai} \varphi_i \]

and \( \Delta_a = 1 + \frac{\chi \psi \exp(\Phi(\alpha_a))}{2\gamma^2} \). The corresponding effective action takes the form

\[
S = \int d^2 z \left( g^\mu_\nu \partial_\mu \varphi_i \partial_\nu \varphi_j Tr(h_i h_j) + \frac{\exp(\Phi(\alpha_a))}{y^2 \Delta_a} g^\mu_\nu \partial_\mu \psi \partial_\nu \chi - \frac{1}{2} \epsilon_{\mu\nu} \partial_\mu \Phi(\alpha_a)(\psi \partial_\nu \chi - \chi \partial_\nu \psi) \frac{\exp(\Phi(\alpha_a))}{\Delta_a} - V \right)
\]  

(38)

where the potential is given in terms of the constant elements \( \epsilon_\pm \) and \( G_f^0 \) as \( V = Tr[\epsilon_+ g_0^f \epsilon_- g_0^{-1}] \). Now, if we consider Lie algebras whose Dynkin diagrams connect only nearest neighbours, i.e.,

\[ \Phi(\alpha_a) = \alpha_a \cdot \chi - \varphi_- + \alpha_a \cdot \chi + \varphi_+ , \]  

(39)

then the “no-torsion condition” implies \( \Phi(\alpha_a) = 0 \).

Considering case (iii), we have

\[ \alpha_a \cdot \chi_- = \alpha_a \cdot (\alpha_{a-1} + \alpha_a) = 0 , \]  

(40)

\[ \alpha_a \cdot \chi_+ = \alpha_a \cdot (\alpha_{a+1}) = 0 . \]  

(41)

In this case, the only solution for both equations above is to take \( a = r \) (in such a way that \( \alpha_{r+1} = 0 \)) and \( G = B_r \) (so that \( \alpha_{a-1} \cdot \alpha_r = -\alpha_r^2 = -1 \)). This is precisely the case proposed by Leznov and Saveliev \[8\] and subsequently discussed by Gervais and Saveliev \[2\] and also by Bilal \[3\], for the particular case of \( B_2 \).

For case (iv), the “no-torsion condition” requires that

\[ \alpha_{a-1} \cdot \alpha_a = 0 , \quad \alpha_a \cdot (\alpha_a + \alpha_{a+1}) = 0 , \]

which are satisfied by \( a = 1 \) and also by \( G = C_2 \), since \( \alpha_{a-1} = 0 \) and also \( \alpha_1 \cdot \alpha_2 = -\alpha_2^2 = -1 \), respectively.

In general, the “no-torsion condition”, i.e., \( \Phi(\alpha_a) = 0 \), may be expressed in terms of the structure of the co-set \( G_0/G_0^0 = u(1)^{r-1} \otimes sl(2) / u(1) \). The crucial ingredient for the appearance of \( \Phi(\alpha_a) \) arises from the conjugation

\[ Tr(A_0 g_0^f A_0 (g_0^f)^{-1} + A_0 A_0) = 2\lambda_a^2 \left( 1 + \frac{2}{\alpha_a^2} \chi \psi \exp(\Phi(\alpha_a)) \right) . \]

Henceforth, if all generators belonging to the Cartan subalgebra parameterizing \( g_0^f \) commute with \( E_{\pm \alpha_a} \), then \( \Phi(\alpha_a) = 0 \), and therefore the structure of the co-set

\[
\frac{G_0}{G_0^0} = \frac{u(1)^{r-1} \otimes sl(2)}{u(1)} = \frac{u(1)^{r-1} \otimes sl(2)}{u(1)}
\]  

(42)

is the general condition for the absence of the antisymmetric term in the action.
4. ALGEBRA OF SYMMETRIES. The symmetries of the action \([8]\) is expressed in terms of its algebra of conserved currents. Those are obtained from the Hamiltonian reduction of the WZW model when particular set of constraints and gauge fixing conditions are implement. The algebra of the remaining currents can be derived using Dirac brackets or following the method employed in refs. \([4]\) and \([5]\), where the explicit form of the algebra of symmetries \(V_n^{(1,1)}\) for \(a = 1\) and \(G = A_n\) (cases \(i\) and \(ii\) ) were derived . For \(B_2\) we consider \(Q_1 = \frac{2M}{m^2}\) and the two Non-Abelian Toda models can be defined by choosing \(\epsilon^{i}\), where the explicit form of the algebra of symmetries \(V\) be derived using Dirac brackets or following the method employed in refs. \([4]\) and \([5]\).

The remaining currents are \(T_i = J_{a_1}^i, V^i_\pm = \sqrt{2}J_{a_1+\alpha_2}\) and \(V^-_i = -\sqrt{2}J_{-\alpha_2}\) and \(T_{iii} = J_{a_1+\alpha_2}^i, V^i_{iii} = \frac{1}{\sqrt{2}}J_{2a_1+2\alpha_2}\) and \(V^-_{iii} = -\frac{1}{\sqrt{2}}J_{a_1}\) respectively. Their algebra is given by

\[
\{T(\sigma), T(\sigma')\} = 2T(\sigma')\delta(\sigma - \sigma') - T'(\sigma')\delta(\sigma - \sigma') - \frac{1}{2}\delta''(\sigma - \sigma') \tag{43}
\]

\[
\{T(\sigma), V^\pm(\sigma')\} = sV^\pm(\sigma')\delta(\sigma - \sigma') - V'^\pm(\sigma')\delta(\sigma - \sigma') \tag{44}
\]

\[
\{V^\pm(\sigma), V^\mp(\sigma')\} = tV^\pm(\sigma)V^\mp(\sigma')\sigma - \frac{1}{2}\delta''\sigma - \sigma') \tag{45}
\]

for \(s = \frac{3}{2}\) and 2, \(t = \frac{1}{4}\) and 1 corresponding to cases \((i)\) and \((iii)\) respectively. We also have

\[
\{V^\pm_1(\sigma), V^{\mp_1}(\sigma')\} = -\frac{1}{4}(V^\pm_1(\sigma)V_1^{\mp}(\sigma') - 2V^\pm_1(\sigma)V_1^{\mp}(\sigma')\sigma - \sigma')
\]

\[
\pm T(\sigma')\delta(\sigma - \sigma') = \sigma - \sigma') \tag{46}
\]

and

\[
\{V^\pm_{iii}(\sigma), V^{\mp_{iii}}(\sigma')\} = -(V^\pm_{iii}(\sigma)V^{\mp_{iii}}(\sigma') - 2V^\pm_{iii}(\sigma)V^{\mp_{iii}}(\sigma')\sigma - \sigma') - T'(\sigma')\delta(\sigma - \sigma') + \frac{1}{2}\delta''\sigma - \sigma') \tag{47}
\]

for cases \((i)\) and \((iii)\) respectively. The nonlocal V-algebra of the \(B_2\) (case \(iii\)) was derived in \([3]\) by explicit construction of the conserved currents. The algebra of symmetries of the \(B_3\) (case \(iii\)) torsionless NA-Toda model is an example of a new type of nonlocal quadratic algebra of VW-type which mixes the features of V- and W-algebras. It is generated by two nonlocal currents \(V^\pm\) of spin 3 and two local, \(T\) and \(W\) of spin 2 and 4 respectively (all \(V^\pm\) and \(W\) are primary fields):

\[
\{V^\pm(\sigma), V^\pm(\sigma')\} = \frac{1}{2}V^\pm(\sigma)V^\pm(\sigma')\delta(\sigma - \sigma') \tag{48}
\]

\[
\{V^\pm(\sigma), V^{\mp}(\sigma')\} = -\frac{1}{2}V^\pm(\sigma)V^{\mp}(\sigma')\delta(\sigma - \sigma')
\]

\[
+ \left(\frac{1}{5}T'''(\sigma) - \frac{4}{25}(T^2(\sigma))' - W'(\sigma))\delta(\sigma - \sigma')
\]

\[
+ \left(\frac{9}{10}T''(\sigma) - \frac{8}{25}T^2(\sigma) - 2W(\sigma))\delta(\sigma - \sigma')
\]

\[
+ \frac{3}{2}T'(\sigma)\delta''(\sigma - \sigma') + T(\sigma)\delta''(\sigma - \sigma') - \frac{1}{2}\delta''(\sigma - \sigma') \tag{49}
\]

\[
\]

\[
\]
\[ \{W(\sigma), V^\pm(\sigma')\} = -\frac{7}{5}V^\pm(\sigma)\delta''(\sigma - \sigma') - \frac{7}{5}(V^\pm(\sigma))^'\delta''(\sigma - \sigma') \\
- \left(\frac{3}{5}(V^\pm(\sigma))^'' - \frac{26}{25}T(\sigma)V^\pm(\sigma)\right)\delta'(\sigma - \sigma') \\
- \left(\frac{1}{10}(V^\pm(\sigma))^''' - \frac{1}{2}T'(\sigma)V^\pm(\sigma) + \frac{9}{25}T(\sigma)(V^\pm(\sigma))^'\right)\delta(\sigma - \sigma') \]

\[ \{W(\sigma), W(\sigma')\} = -\frac{1}{20}\delta^{(vi)}(\sigma - \sigma') + \frac{7}{25}T(\sigma)\delta^{(v)}(\sigma - \sigma') + \frac{7}{25}T'(\sigma)\delta^{(iv)}(\sigma - \sigma') \\
+ \sum_{i=4}^{7} A_i(\sigma)\delta^{(i-i)}(\sigma - \sigma') \]

where

\[
A_4(\sigma) = \frac{21}{25}T''(\sigma) - \frac{49}{125}T^2(\sigma) + \frac{3}{5}W(\sigma) \\
A_5(\sigma) = \frac{14}{25}T''(\sigma) - \frac{147}{250}(T^2(\sigma))^' + \frac{9}{10}W'(\sigma) \\
A_6(\sigma) = \frac{1}{5}T^{(iv)}(\sigma) - \frac{113}{200}(T^2(\sigma))^'' + \frac{213}{500}T(\sigma)T''(\sigma) + \frac{27}{50}(T'(\sigma))^2 + \frac{72}{625}T^3(\sigma) \\
+ \frac{1}{2}W''(\sigma) - \frac{14}{25}T(\sigma)W'(\sigma) - 3V^+(\sigma)V^-(\sigma) \\
A_7(\sigma) = \frac{3}{100}T^{(iv)}(\sigma) - \frac{9}{100}(T^2(\sigma))^''' - \frac{3}{10}(T(\sigma)T''(\sigma))^' + \frac{243}{500}T'(\sigma)T''(\sigma) \\
+ \frac{81}{250}T(\sigma)T'''(\sigma) + \frac{9}{100}(T^3(\sigma))^' - \frac{243}{5000}T(\sigma)(T^2(\sigma))^' + \frac{1}{10}W'''(\sigma) \\
- \frac{7}{25}(T(\sigma)W(\sigma))^' - \frac{3}{2}(V^+(\sigma)V^-(\sigma))^' \]

The axionic $A_3$-NA-Toda model corresponding to $Q = (\lambda_1 + \lambda_3) \cdot H$, $\epsilon_\pm = E_{\alpha_1} + E_{\alpha_3}$, and $g_{0}^{0} = \lambda_2 \cdot H$ provides an example of a new algebraic structure of UV-type. The $V_4^{(2,1)}$-algebra is generated by three nonlocal currents $V^\pm$ and $V^0$ of spin 2 and a local one (stress tensor) $T$ of spin 2:

\[
\{V^\pm(\sigma), V^\pm(\sigma')\} = \frac{1}{8}\epsilon(\sigma - \sigma')V^\pm(\sigma)V^\pm(\sigma'), \\
\{V^0(\sigma), V^\pm(\sigma')\} = \frac{1}{8}\epsilon(\sigma - \sigma')V^0(\sigma)V^\pm(\sigma') \]

\[
\{V^0(\sigma), V^0(\sigma')\} = -\frac{1}{4}\epsilon(\sigma - \sigma')\{V^+(\sigma)V^-(\sigma') + V^-(\sigma)V^+(\sigma')\} \\
+ 2T(\sigma')\partial_{\sigma}\delta(\sigma - \sigma') + \delta(\sigma - \sigma')\partial_{\sigma}T(\sigma') - 4\partial^2_{\sigma}\delta(\sigma - \sigma') \]

\[
\{V^-(\sigma), V^+(\sigma')\} = -\frac{1}{8}\epsilon(\sigma - \sigma')\{V^0(\sigma)V^0(\sigma') + V^-(\sigma)V^+(\sigma')\} \\
+ T(\sigma')\partial_{\sigma}\delta(\sigma - \sigma') + \frac{1}{2}\delta(\sigma - \sigma')\partial_{\sigma}T(\sigma') - 2\partial^2_{\sigma}\delta(\sigma - \sigma') \]

(49)
By another choice of gauge fixing conditions \([7]\) (by nonlocal gauge transformations that relates two different gauge fixings) one can transform this algebra into nonlocal rational algebra \(V_U^{(2,1)}\) generated by the nonlocal \(V^\pm\) currents and now \(V_0\) is local but appears in the denominator of the r.h.s. of \([9]\). The origin of this novelty is that together with the \(\epsilon_\pm\) invariant subalgebra \(g_0^0\) there exists one extra \(\epsilon_\pm\) invariant subalgebra \(K^-\) i.e.
\[
[\epsilon_\pm, E_{-\alpha_1-\alpha_2} - E_{-\alpha_2-\alpha_3}] = 0
\]
(50)

It is important to note that relaxing the \(g_0^0\) constraint, the algebra of symmetries of the corresponding nonsingular NA-Toda is generated by three local currents \(V^\pm, T\) of spin 2, one nonlocal \(V_0\) \((s = 2)\) and one local spin 1 current \(J\): 
\[
\{V^0(\sigma), J(\sigma')\} = \{V^\pm(\sigma), V^\pm(\sigma')\} = \{V^0(\sigma), V^\pm(\sigma')\} = \frac{1}{2} V^\pm(\sigma)V^0(\sigma')\epsilon(\sigma - \sigma'),
\]
\[
\{J(\sigma), V^\pm(\sigma')\} = \frac{1}{4} V^\pm(\sigma)\delta(\sigma - \sigma'), \quad \{J(\sigma), J(\sigma')\} = \frac{1}{8} \partial_\sigma \delta(\sigma - \sigma'),
\]
\[
\{V^\pm(\sigma), V^- (\sigma')\} = \frac{1}{2} \partial^3_\sigma \delta(\sigma - \sigma') + T(\sigma) \partial_\sigma \delta(\sigma - \sigma') + \frac{1}{2} \partial_\sigma T(\sigma) \delta(\sigma - \sigma') - \frac{1}{4} V^0(\sigma)V^0(\sigma')\epsilon(\sigma - \sigma'),
\]
\[
\{V^0(\sigma), V^0 (\sigma')\} = - \partial^3_\sigma \delta(\sigma - \sigma') + 2T(\sigma) \partial_\sigma \delta(\sigma - \sigma') + \partial_\sigma T(\sigma) \delta(\sigma - \sigma') - [V^+(\sigma)V^-(\sigma') + V^-(\sigma')V^+(\sigma)]\epsilon(\sigma - \sigma').
\]
(51)

where \(T(\sigma) = T - 4J^2\). Again by suitable change of the gauge fixing \([7]\), this algebra takes the form of rational \(r^{(1,2)}_4\)-algebra (the current \(V^0\) appears in the denominators in the \(V^+V^-\)-Poisson brackets).

A complete discussion of the symmetries of all grade \(l = 1\) (i.e. \(\epsilon_\pm \in G_1\)), \(Q = \sum_{i \neq a} \lambda_i \cdot H\) NA-Toda models (including the nonsingular cases \(g_0^0 = 0\)) is given in our forthcoming paper \([8]\) (see also ref. \([9]\))

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