Equivariant K-Theory of Simply Connected Lie Groups

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Abstract

We compute the equivariant $K$-theory $K^*_G(G)$ for a simply connected Lie group $G$ (acting on itself by conjugation). We prove that $K^*_G(G)$ is isomorphic to the algebra of Grothendieck differentials on the representation ring. We also study a special example of a non-simply connected Lie group $G$, namely $PSU(3)$, and compute the corresponding equivariant $K$-theory.

1 Introduction

For a finite group $G$, the equivariant $K$-theory of the $G$-space $G$ (where the $G$-action is the conjugation action) has a very nice expression: $K^0_G(G) \otimes \mathbb{C} = \{ \text{conjugacy invariant functions } \mathcal{C} \to \mathbb{C} \}$
where \( C \) is the set of \( \{(g, h) \in G \times G \mid gh = hg\} \).

Since the group \( SL(2, \mathbb{Z}) \) acts on the set \( C \), we have an action of \( SL(2, \mathbb{Z}) \) on the group \( K_0^G(G) \). In particular, the element \( \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \) gives the so-called Fourier transformation, which was used by Lusztig [9] in the representation theory of finite reductive groups. The space \( K_0^G(G) \) appears as the space of conformal field theories attached to a finite group [4]. So it is natural to ask what happens when \( G \) is a compact group.

The first author conjectured that for a compact group \( G \), the equivariant K-theory \( K^*_G(G) \) is isomorphic to the algebra \( \Omega^*_R(G)/\mathbb{Z} \) of Grothendieck differentials, where \( R(G) \) is the representation ring of \( G \).

This was motivated by his computation of the equivariant cohomology theory \( H^*_G(G, \mathbb{C}) \) of \( G \)-space \( G \) with adjoint action.

For a compact group \( G \) and a compact \( G \)-space \( X \), the equivariant cohomology \( H^*_G(X) \) is defined as \( H^*_G(X) = H^*(EG \times X) \).

If we view \( G \) as a \( G \)-space where the action is conjugation, we can prove that \( H^*_G(G) \cong \Omega^*_H(BG \times \mathbb{C}) \) (with coefficients \( \mathbb{C} \)). There are several proofs of this result, one of which is based on the fact that \( H^*_G(G) \) is isomorphic to \( H^*(B(LG)) \), where \( LG \) is the smooth loop group of \( G \).

The main result of the paper is the construction of an isomorphism \( \Omega^*_R(G)/\mathbb{Z} \cong K^*_G(G) \) in the case where \( G \) is simply connected (Theorem 3.2). Though we always have an algebra map \( \phi : \Omega^*_R(G)/\mathbb{Z} \to K^*_G(G) \) for any compact group \( G \), we show that in case \( G = PSU(3) \), \( \phi \) is not an isomorphism. In fact in that case \( K^*_L(G) \) has torsion as a \( R(G) \)-module.

For a simply-connected Lie group \( G \), the proof of our result is as follows. By using Hodgkin’s spectral sequence, we describe the \( R(G) \)-module structure of the equivariant K-theory \( K^*_G(G) \), we prove that is a free module over the representation ring \( R(G) \). Then by considering the natural map from \( \Omega^*_R(G)/\mathbb{Z} \) to \( K^*_G(G) \), we obtain the algebra structure.

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2 Background

In this section we recall some basic material we will use later: representation rings, equivariant K-theory and the algebra of Grothendieck differentials.
Here we always assume $G$ is a compact Lie group.

### 2.1 The representation ring

For a compact Lie group $G$, a $G$-module means a finite-dimensional complex vector space $M$ with a continuous linear $G$-action on it. If $M$ and $N$ are $G$-modules, we can form their direct sum $M \oplus N$, and with respect to this operation, the isomorphism classes of $G$-modules form an abelian semigroup; the associated abelian group $R(G)$ is called the representation ring of $G$, the tensor product induces a commutative ring structure in $R(G)$.

A $G$-module $M$ has a character: $\chi_M : G \to \mathbb{C}$, $\chi_M(g) = \text{tr}_M(g)$, and $M$ is determined by $\chi_M$ up to isomorphism. We know $\chi_{M \oplus N} = \chi_M + \chi_N$, $\chi_{M \otimes N} = \chi_M \chi_N$ and $\chi_M(hgh^{-1}) = \chi_M(g)$, thus the map $M \mapsto \chi_M$ identifies $R(G)$ with a subring of the ring of $G$-invariant complex functions on $G$, so $R(G)$ is also called the character ring of $G$.

We already know [12] that the representation ring of the torus $T^n$ is the ring of Laurent polynomials,

$$R(T^n) = \mathbb{Z}[X_1, X_2, \cdots, X_n; (X_1 X_2 \cdots X_n)^{-1}]$$

The following result is critical to us:

**LEMMA 2.1** [2] For a simply connected Lie group $G$, $R(G)$ is a polynomial ring over $\mathbb{Z}$ with $\text{rank}(G)$ generators.

### 2.2 Equivariant $K$-theory

Let $X$ be a locally compact $G$-space, then the equivariant $K$-theory $K^*_G(X)$ is defined and it is a $\mathbb{Z}_2$-graded algebra [3]. In case that $X$ is compact, $K^0_G(X)$ is generated by equivalence classes of complex $G$-vector bundles over $X$.

Equivariant $K$-theory is a common generalization of two important extreme cases. When $G = 1$, $K^*_G(X)$ is reduced to the ring $K^*(X)$ of the ordinary $K$-theory. When $X$ is a single point, $K^0_G(X)$ is nothing but the representation ring $R(G)$ and $K^1_G(X) = 0$.

$K^*_G(X)$ enjoys some very nice properties:

1. $K^*_G$ is functorial with respect to both the proper equivariant maps between spaces and the homomorphisms between groups. So, for any compact
G-space \( X \), the obvious map \( X \to \) point gives rise to a \( R(G) \)-module structure on \( K^*_G(X) \), therefore \( R(G) \) serves as the coefficient ring in equivariant \( K \)-theory.

If \( B \) is a closed subgroup of \( G \), for any \( B \)-space \( X \), we get the map \( i^*: K^*_G(G\times X) \to K^*_B(X) \) induced from the map \( i: B \hookrightarrow G \), which in fact is an isomorphism.

As a special case, we have:

\[
K^q_G(G/B) = K^q_B(\text{point}) = \begin{cases} R(B), & q = 0 \\ 0, & q = 1 \end{cases}
\]

2. Bott periodicity:

The Thom homomorphism \( \phi_*: K^*_G(X) \to K^*_G(E) \) is an isomorphism for any complex \( G \)-vector bundle \( E \) over \( X \).

3. Six term exact sequence: for any pair \((X, A)\), where \( A \) is closed \( G \)-subset in \( X \), we have the exact sequence:

\[
\begin{array}{cccccc}
K^{-1}_G(X, A) & \to & K^{-1}_G(X) & \to & K^{-1}_G(A) \\
\uparrow & & \downarrow & & \\
K^0_G(A) & \leftarrow & K^0_G(X) & \leftarrow & K^0_G(X, A)
\end{array}
\]

Also we have an effective way to compute the equivariant \( K \)-theory in many cases, that is the Segal spectral sequence:

For any covering \( U \) of \( X \) by closed \( G \)-stable subsets, there exists a multiplicative spectral sequence \( E_{n}^{*,*} \Rightarrow K_{G}^{*}(X) \) \([14]\), such that \( E_{2}^{p,q} = \check{H}^{p}(U, \mathcal{K}^{q}_{G}) \), where \( \mathcal{K}^{q}_{G} \) means the coefficient system: \( U \mapsto \mathcal{K}^{q}_{G}(U) \).

### 2.3 Grothendieck Differentials

Let \( A \subset B \) be commutative rings, the algebra of Grothendieck differentials \( \Omega_{B/A}^{*} \) \([3]\) is the differential graded \( A \)-algebra constructed as follows:

Let \( F \) be the free \( B \)-module generated by all elements in \( B \), to be clear, we use \( db \) to denote the generator corresponding to \( b \in B \), so

\[
F = \bigoplus_{b \in B} Bdb.
\]

and let \( I \subset F \) be the \( B \)-submodule generated by

\[
\begin{aligned}
&\{ da, \ \forall a \in A \\
&d(b_1 + b_2) - db_1 - db_2, \ \forall b_1, b_2 \in B \\
&d(b_1 b_2) - b_1 db_2 - b_2 db_1, \ \forall b_1, b_2 \in B \}
\end{aligned}
\]
we then get the quotient $B$-module

$$\Omega_{B/A} = F/I.$$  

Let $\Omega^0_{B/A} = B$, $\Omega^1_{B/A} = \Omega_{B/A}$, and $\Omega^p_{B/A} = \Lambda^p_B \Omega_{B/A}$, there is a differential:

$$d : \Omega^p_{B/A} \to \Omega^{p+1}_{B/A},$$

which maps $b \in B$ to $db$, then

$$\Omega^*_B/A = \bigoplus_{p=0}^{\infty} \Omega^p_{B/A}$$

is the differential graded algebra of Grothendieck differentials of $B$ over $A$. It is the generalization of the algebra of differentials on affine spaces, for example, if $B = A[x_1, \ldots, x_n]$, then $\Omega^p_{A[x_1,\ldots,x_n]/A} = \oplus_{i_1 < i_2 < \cdots < i_p} A[x_1, \cdots, x_n]dx_{i_1} \wedge \cdots \wedge dx_{i_p}$.

### 3 Construction of $\phi$ and main result

In this section, we construct the map $\phi$ from $\Omega^*_R(G)/\mathbb{Z}$ to $K^*_G(G)$ and state our main result.

We can define the map $\phi$ from $\Omega^*_R(G)$ to $K^*_G(G)$ ($G$ acts on itself by conjugation) as follows:

For any representation $\rho : G \to GL(V)$, we let $\phi(\rho)$ be the $G$-vector bundle $G \times V$ over $G$. Then let $\phi(d\rho) \in K^1_G(G)$ be given by the complex of $G$-bundles over $G \times \mathbb{R}$:

$$0 \to G \times \mathbb{R} \times V \to G \times \mathbb{R} \times V \to 0$$

$$(g, t, v) \mapsto (g, t, tp(g)v)$$

This complex is exact out of $G \times \{0\}$, thus it defines an element in $K^1_G(G)$. By the properties of equivariant $K$-theory, we know $\phi(d\rho) \cup \phi(d\rho) = 0$.

**PROPOSITION 3.1** There is a unique algebra homomorphism $\phi : \Omega^*_R(G)/\mathbb{Z} \to K^*_G(G)$ for which $\phi(\rho)$ and $\phi(d\rho)$ are as described above.

**Proof.** By the definition of $\phi$, we only need to prove that for any $\rho_i : G \to GL(V_i)$, $i = 1, 2$, $\phi(d(\rho_1 \rho_2)) = \phi(\rho_1) \phi(d\rho_2) + \phi(\rho_2) \phi(d\rho_1)$ in $K^1_G(G)$. Once we prove this, then we can extend $\phi$ to an algebra map from $\Omega^*_R/G$ to $K^*_G(G)$. 

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\[ \phi(d(\rho_1\rho_2)) \text{ is the complex:} \]
\[ 0 \to G \times \mathbb{R} \times (V_1 \otimes V_2) \to G \times \mathbb{R} \times (V_1 \otimes V_2) \to 0, \]
\[ (g, t, v_1 \otimes v_2) \mapsto (g, t, tp_1(g)(v_1) \otimes \rho_2(g)(v_2)) \]

\[ \phi(\rho_1)\phi(d\rho_2) \text{ is the complex:} \]
\[ 0 \to G \times \mathbb{R} \times (V_1 \otimes V_2) \to G \times \mathbb{R} \times (V_1 \otimes V_2) \to 0, \]
\[ (g, t, v_1 \otimes v_2) \mapsto (g, t, tv_1 \otimes \rho_2(g)(v_2)) \]

And \[ \phi(\rho_2)\phi(d\rho_1) \text{ is the complex:} \]
\[ 0 \to G \times \mathbb{R} \times (V_1 \otimes V_2) \to G \times \mathbb{R} \times (V_1 \otimes V_2) \to 0, \]
\[ (g, t, v_1 \otimes v_2) \mapsto (g, t, t\rho_1(g)(v_1) \otimes v_2) \]

Now we invoke the following fact:
The two maps \( GL(n) \times GL(n) \to GL(2n) \) given by
\[ (A, B) \mapsto \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \]
and
\[ (A, B) \mapsto \begin{pmatrix} AB & 0 \\ 0 & 1 \end{pmatrix} \]
are homotopic. The homotopy is given explicitly by
\[ \rho_s(A, B) = \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos s & \sin s \\ -\sin s & \cos s \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & B \end{pmatrix} \begin{pmatrix} \cos s & -\sin s \\ \sin s & \cos s \end{pmatrix} \]
where \( 0 \leq s \leq \pi/2 \). It is \( GL(n) \)-equivariant in the following sense:

\[ \rho_s(gAg^{-1}, gBg^{-1}) = \begin{pmatrix} g & 0 \\ 0 & g \end{pmatrix} \rho_s(A, B) \begin{pmatrix} g & 0 \\ 0 & g \end{pmatrix}^{-1} \]

Following the same construction of homotopy, we can prove
\[ \phi(d(\rho_1\rho_2)) \oplus E \simeq \phi(\rho_1)\phi(d\rho_2) + \phi(\rho_2)\phi(d\rho_1) \]

where \( E \) is the trivial complex:
\[ 0 \to G \times \mathbb{R} \times (V_1 \otimes V_2) \to G \times \mathbb{R} \times (V_1 \otimes V_2) \to 0 \]
thus
\[
\phi(d(\rho_1 \rho_2)) = \phi(\rho_1)\phi(d\rho_2) + \phi(\rho_2)\phi(d\rho_1)
\]
in \(K^1_G(G)\). Q.E.D.

The following theorem is the main result of this paper and will be proved in sections 4-6.

**THEOREM 3.2** For a compact, simply connected Lie group \(G\), the algebra homomorphism \(\phi : \Omega^*_{R/Z} \to K^*_G(G)\) is an algebra isomorphism.

### 4 Module structure

From now on we assume \(G\) to be a simply connected Lie group. \(T\) is a maximal torus of \(G\), \(W\) is the corresponding Weyl group. In this case, we know by Lemma 2.1 that the representation ring \(R\) of \(G\) is a polynomial ring. As usual, we make \(G\) act on itself by conjugation.

**LEMMA 4.1** \(K^*_G(G) \cong K^*_{G \times G}(G \times G)\) where the action of \(G \times G\) on \(G \times G\) is given by: \((g_1, g_2), (h_1, h_2) = (g_1 h_1 g_2^{-1}, g_1 h_2 g_2^{-1})\)

**Proof.** Consider \(G\) as the diagonal subgroup of \(G \times G\), the functoriality of equivariant \(K\)-theory yields,

\[
K^*_G(G) \cong K^*_{G \times G}((G \times G)^G \times G)
\]

where we view \(G\) as a \(G \times G\)-space by the following action: \((g_1, g_2), (h_1) = g_1 h_1 g_2^{-1}\). Then we have a \(G \times G\)-equivariant homeomorphism:

\[
(G \times G)^G \times G \cong G \times G
\]

where the homeomorphism is:

\[
((g_2, g_3), g_1) \mapsto (g_2 g_1 g_3^{-1}, g_2 g_3^{-1})
\]

Obviously this map is a \(G \times G\)-map. Q.E.D.

Let us recall a theorem of Hodgkin here:
THEOREM 4.2 \[7\] \[8\] For a Lie group $G$, supposing $\pi_1(G)$ is torsion free, then for any $G$-spaces $X, Y$ there is a spectral sequence $E_r \Rightarrow K^*_G(X \times Y)$, with $E_2$-term

$$E_2^{*, *} = \text{Tor}^{*, *}_{R[G]}(K^*_G(X), K^*_G(Y))$$

**Remark on Tor$^{*, *}_{R}$.** Let $R$ be a commutative ring, $M^*$, $N^*$ be graded differential complexes over $R$, then the bi-complex Tor$^{*, *}_{R}(M^*, N^*)$ is defined (for example, see \[10\]). If we take a proper projective resolution $P^* = \{(P^r)^*\}$ of $N^*$, then Tor$^{*, *}_{R}(M^*, N^*)$ is the homology of the total complex of $M^* \otimes \text{Total}(P^*)$. It is bigraded:

$$H^i(M^* \otimes_R \text{Total}(P^*)) = \text{subquotient of } \oplus_{m+n=i} M^m \otimes_R \text{Total}(P^*)^n$$

Those elements in it having degree $m + s = j$ form Tor$^i_j(R^*, N^*)$.

In our case, if we view $G$ as a $G \times G$-space (under the above action), then

$$K^*_G(G) = \begin{cases} R, & i = 0 \\ 0, & i = 1 \end{cases}$$

and we find that there is a spectral sequence $E_r \Rightarrow K^*_G(G)$ such that

$$E_2^{*, *} = \text{Tor}^{*, *}_{R[G]}(R, R)$$

If consider the bidegree, we see that $M^0 = N^0 = R$, and $M^i = N^i = 0$ for $i > 0$, so Tor$^{i,j}_R(M^*, N^*)$ can be non-zero only when $j = 0$.

**LEMMA 4.3** We have Tor$^{*, *}_{R \otimes R}(R, R) \cong \Omega^*_R$. In particular, Tor$^{*, *}_{R \otimes R}(R, R)$ is a free $R$-module of rank $2^{\text{rank}(G)}$.

**Proof.** Because $R$ is a polynomial ring, $R = \mathbb{Z}[x_1, \cdots, x_n]$, we have

$$R \otimes R = \mathbb{Z}[x_1, \cdots, x_n, y_1, \cdots, y_n].$$

By using the Koszul complex we can get a free resolution of $R$ over $R \otimes R$:

$$\cdots \to \bigoplus_{i_1 \cdots < i_r} \mathbb{Z}[x_1, \cdots, x_n, y_1, \cdots, y_n] e_{i_1} \wedge \cdots \wedge e_{i_r} \xrightarrow{e_i \to 2^i y_i}$$

$$\bigoplus_{i_1 < \cdots < i_{r-1}} \mathbb{Z}[x_1, \cdots, x_n, y_1, \cdots, y_n] e_{i_1} \wedge \cdots \wedge e_{i_{r-1}} \to$$

$$\cdots \to \mathbb{Z}[x_1, \cdots, x_n, y_1, \cdots, y_n] \to \mathbb{Z}[x_1, \cdots, x_n]$$

Using this resolution we can prove the statement easily. Q.E.D.
LEMMA 4.4 \( \text{rank}_R(K^*_G(G)) = 2^{\text{rank}(G)} \)

Proof. Firstly by the localization theorem \[13\] we know that

\[ \text{rank}_R(R)K^*_T(G) = \text{rank}_R(R)K^*_T(G^T) = \text{rank}_R(R)K^*_T(T) = 2^{\text{rank}(G)} \]

where \( G^T \) denotes the fixed point set of \( T \) in \( G \).

Secondly, by \[8\] \[7\], we know that

\[ K^*_T(G) = \text{Tor}^*_R(R, R) \text{Tor}^*_R(R, R) \]

So the rank of \( K^*_G(G) \) over \( R \) is the same as the rank of \( K^*_T(G) \) over \( R(T) \).

Q.E.D.

LEMMA 4.5 We have \( K^*_G(G) \cong \Omega^*_R/Z \) as \( R \)-module.

Proof. \( E_2^{*,*} = \text{Tor}^*_R(R, R) \cong \Omega^*_R/Z \) is a free \( R \)-module of rank \( 2^{\text{rank}(G)} \), the spectral sequence converges to \( K^*_G(G) \), which has the same rank as \( E_2 \), hence \( d_r, r \geq 2 = 0 \). Since \( E_2 \) is a free \( R \)-module, \( E_\infty \) is isomorphic to \( E_2 \) as an \( R \)-module. Q.E.D.

5 The Algebra Structure

So far we have discussed the \( R(G) \)-module structure on \( K^*_G(G) \). Let us now consider the algebra structure.

LEMMA 5.1 \( K^*_T(T) \cong \Omega^*_R(T)/Z \)

We will prove the following lemma in the next section.

LEMMA 5.2 The inclusion \( j : R(G) \to R(T) \) induces an isomorphism:

\[ j^* : \Omega^*_R(G)/Z \to (\Omega^*_R(T)/Z)^W. \]

We already constructed the algebra homomorphism \( \phi : \Omega^*_R/Z \to K^*_G(G) \) in section 3. Now we can prove

THEOREM 5.3 \( \phi \) is an isomorphism.
Proof. Let us consider the maps:
\[ \Omega^*_R \to K^*_G(G) \to K^*_T(G) \to K^*_T(T) \cong \Omega^*_R \]
where \( \alpha \) and \( \beta \) are all natural maps.

The composition \( \beta \circ \alpha \circ \phi \) is \( j^* \) above, so it is injective. This implies \( \phi \) is injective.

For the surjectivity, we already know the following:
1. \( K^*_G(G) \) is a free \( R(G) \)-module
2. \( K^*_T(G) = K^*_G(G) \otimes_R R(T) \)

Now as \( \alpha \) is injective, \( \text{Im}(\alpha) \) contains no torsion element. Since \( \beta \) is the localization map, we see that the composition of \( \beta \circ \alpha \) is injective.

Now by considering the Weyl group actions, we have:
\( \text{Im}(\beta \circ \alpha) \subset K^*_T(T)^W \) and \( K^*_T(T)^W = \text{Im} j^* \), so we get the surjectivity.

Q.E.D.

So we finished the proof of our main theorem.

6 The proof of lemma 5.2

First we prove a general result about holomorphic differential forms \( \Omega^i_{hol}(Y) \) on a complex manifold \( Y \).

PROPOSITION 6.1 Let \( X \) and \( Y \) be complex-analytic manifolds and let \( f : X \to Y \) be a holomorphic mapping. Assume there is a finite group which acts on \( X \) holomorphically, in such a way that \( Y \) identifies with the quotient of \( X \) by \( G \). Then we have
\[ \Omega^i_{hol}(Y) = \Omega^i_{hol}(X)^G. \]

Proof. Let \( \omega \) be a differential form on \( X \) which is \( G \)-invariant. Then \( \omega \) is a meromorphic differential form on \( Y \), and we have to show it has no pole along any irreducible divisor \( D \) of \( Y \). Pick a component \( \Delta \) of \( f^{-1}(D) \). Near a general point \( p \) of \( \Delta \), we can find holomorphic coordinates \( (z_1, \ldots, z_n) \) on \( X \) so that \( \Delta \) has equation \( z_1 = 0 \) and that \( (z_1^e, \ldots, z_n) \) are holomorphic coordinates on \( Y \) near \( f(p) \in Y \). We expand \( \omega \) as
\[ \omega = \sum_{I \subseteq \{2, \ldots, n\}} f_I dz_I + dz_1 \wedge \sum_{J \subseteq \{2, \ldots, n\}} g_J dz_J. \]
Then the condition that $\omega$ is invariant under the transformation $T(z_1, \cdots, z_n) = (\zeta z_1, z_2, \cdots, z_n)$ where $\zeta = e^{2\pi i}$ translates into the conditions

1. $f_I$ is $T$-invariant
2. $g_I(Tz) = \zeta^{-1}g_I(z)$.

The first condition implies that each term $f_I dz_I$ is a holomorphic differential form in a neighborhood of $p$. The second condition implies that $g_J$ is divisible by $z_1^{e-1}$, and therefore is of the form $g_J = z_1^{e-1}h_J$, where $h_J$ is $T$-invariant, hence descends to a holomorphic function in a neighborhood of $p$ in $Y$. Then we have $g_J dz_1 \wedge dz_J = \frac{1}{e}h_J d(z_1^e) \wedge dz_J$. This concludes the proof. Q.E.D.

**Proposition 6.2** Let $\mathcal{T}$ be a split torus over $\text{Spec}(\mathbb{Z})$. Let $X(\mathcal{T})$ denote the character group of $\mathcal{T}$, and let $W$ be a finite subgroup of $\text{Aut}(X(\mathcal{T}))$. We can then view as a finite group acting on $\mathcal{T}$. Assume that the quotient scheme $\mathcal{T}/W$ is smooth over $\text{Spec}(\mathbb{Z})$. Then we have

$$\Omega^i(\mathcal{T})^W = \Omega^i(\mathcal{T}/W).$$

**Proof.** Consider the projection map $\pi : \mathcal{T} \to \mathcal{T}/W$. Let $\Omega^i_{\mathcal{T}}$ denote the sheaf of Grothendieck differential forms on $\mathcal{T}$. Then we will show that the $W$-invariant part $[\pi_*\Omega^i_{\mathcal{T}}]^W$ of the direct image sheaf $\pi_*\Omega^i_{\mathcal{T}}$ identifies with $\Omega^i_{\mathcal{T}/W}$. There is an obvious inclusion

$$u : \Omega^i_{\mathcal{T}/W} \hookrightarrow [\pi_*\Omega^i_{\mathcal{T}}]^W$$

As $\mathcal{T}/W$ is smooth, the sheaf $\Omega^i_{\mathcal{T}/W}$ is locally free and $\mathcal{T}/W$ is normal. Hence it is enough to show that $u$ is an isomorphism outside of a subscheme of codimension $\geq 2$.

We first construct a $W$-invariant closed subscheme $D$ of $\mathcal{T}$ such that the restriction of $\pi$ to $U = \mathcal{T} \setminus D$ is an etale map $U \to U/W$. For $g \in W$, denote by $\mathcal{T}^g \subset \mathcal{T}$ the fixed point scheme of $g$, which is a closed group subscheme. Then we set

$$D = \bigcup_{g \in W, g \neq 1} \mathcal{T}^g.$$  

Let $\Delta$ be the image of $D$ in $\mathcal{T}/W$. As $\pi$ is proper, $\Delta$ is a closed subscheme and is the union of the $\Delta_g = \pi(\mathcal{T}^g)$. As the map $U \to U/W$ is finite etale and Galois with Galois group $W$, it follows that the restriction of $u$ to $U/W$ is an isomorphism. It is then enough to verify that $u$ induces an isomorphism near a generic point of a component $C$ of $\Delta$. Then $C$ is a component of $\Delta_g$. 

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for some $g \in W$. We only need to consider $g$ such that $T^g$ is of codimension 1. This is equivalent to the condition that $w$, as an automorphism of $X(T)$, is such that $w - Id$ has rank $n - 1$, where $n = dim(T)$. So if $w \neq 1$, the only eigenvalues of $w$ are 1 with multiplicity $n - 1$ and $-1$ with multiplicity 1. Then there are 2 possibilities:

(a) in a suitable basis $(\chi_1, \cdots, \chi_n)$ of $X(T)$, we have $w\chi_1 = -\chi_1$ and $w\chi_j = \chi_j$ for $j \geq 2$.

(b) in a suitable basis $(\chi_1, \cdots, \chi_n)$ of $X(T)$, we have $w\chi_1 = \chi_2$, $w\chi_2 = \chi_1$ and $w\chi_j = \chi_j$ for $j \geq 3$.

In case (a), $T^g$ is the connected subscheme of $T$ of equation $\chi^2 = 1$. $T^g$ has 2 irreducible components, of equations $\chi_1 = \pm 1$. These two components meet in characteristic 2. Note that each component of $T^g$ maps onto $Spec(\mathbb{Z})$. In case (b), $T^g$ is irreducible and maps onto $Spec(\mathbb{Z})$. Hence the same is true for $\Delta_g = \pi(T^g)$.

Since every component of $\Delta$ which has codimension 1 maps onto $Spec(\mathbb{Z})$, it follows that its generic point $\eta$ maps to the generic point $Spec(\mathbb{Q})$ of $Spec(\mathbb{Z})$. Hence to verify that $u$ is an isomorphism near $\eta$, there is no harm in replacing $T$ with the algebraic torus $T_0 = T \times_{Spec(\mathbb{Z})} Spec(\mathbb{Q})$ over $\mathbb{Q}$. Then as $\mathbb{C}$ is a field extension of $\mathbb{Q}$, we may work instead with $T_\mathbb{C} = T \times_{Spec(\mathbb{Z})} Spec(\mathbb{C})$. Then we are reduced to showing that if $V$ is a $W$-invariant open set (for the transcendental topology) in $T_\mathbb{C}$, then any $W$-invariant Grothendieck differential $\omega$ on $V$ descends to a Grothendieck differential on $V/W$. It is enough to show that $\omega$ defines a holomorphic differential form on $V/W$, which follows from Proposition 6.1. Q.E.D.

**Proof of lemma 5.2** We note that Proposition 6.2 implies lemma 5.2 easily.

For $G, T$, we have their complexification $G_\mathbb{C}$ and $T_\mathbb{C}$, and $R(G_\mathbb{C}) = R(G)$, $R(T_\mathbb{C}) = R(T)$. For the torus $T = Spec(R(T_\mathbb{C}))$ over $Spec(\mathbb{Z})$, $O(T) = R(T)$.

We have

$$\Omega^*_{R(T)/\mathbb{Z}} = \Omega^*_{T/Spec(\mathbb{Z})}$$

hence

$$[\Omega^*_{R(T)/\mathbb{Z}}]^W = \Omega^*_{(T/W)/Spec(\mathbb{Z})}$$

$$= \Omega^*_{O(T/W)/Spec(\mathbb{Z})}$$

$$= \Omega^*_{R(T)^W/Spec(\mathbb{Z})}$$

$$= \Omega^*_{R(G)/\mathbb{Z}}$$

Q.E.D.
7 The Case of $G = PSU(3)$

So far we discussed the case when $G$ is simply connected. What happens when $G$ is non-simply connected? Let us consider $PSU(3)$.

Since this is a long computation, we just list the results here. $PSU(3)$ has a maximal torus $T^2$. There are elements $X_1$ and $X_2$ in $R(T^2)$:

$$X_1(\text{diag}(\lambda_1, \lambda_2, \lambda_3)) = \frac{\lambda_2}{\lambda_1}$$
$$X_2(\text{diag}(\lambda_1, \lambda_2, \lambda_3)) = \lambda_1^2$$

We have

$$R(T^2) = \mathbb{Z}[X_1, X_2, X_1^{-1}, X_2^{-1}]$$

By taking the Weyl group invariants of $R(T^2)$, we can get

**Lemma 7.1** $R(PSU(3)) = \mathbb{Z}[Z_1, Z_2, Z_3] \subset R(T^2) = \mathbb{Z}[X_1, X_2, X_1^{-1}, X_2^{-1}]$, where

$$Z_1 = X_1 + X_1^{-1} + X_1X_2 + X_1^{-1}X_2^{-1} + X_1^2X_2 + X_1^{-2}X_2^{-1}$$
$$Z_2 = X_2 + X_1^3X_2 + X_1^{-3}X_2^{-2}$$
$$Z_3 = X_1^{-1} + X_1^{-3}X_2^{-1} + X_1X_2^2$$

We see in fact that $R(PSU(3))$ is isomorphic to $\mathbb{Z}[X, Y, Z]/(X^3 - YZ)$, so using the result of [4] or [4], we obtain

**Proposition 7.2** : The Hochschild homology of $R = R(PSU(3))$ is given by

$$HH_0(R) = R$$
$$HH_1(R) = \Omega^1_{R/\mathbb{Z}}$$
$$HH_2(R) = \Omega^2_{R/\mathbb{Z}} = T(\Omega^2_{R/\mathbb{Z}})$$
$$HH_k(R) = \Omega^k_{R/\mathbb{Z}}, \ k \geq 4$$

This is very interesting: the fact that $R$ has infinite homological dimension implies the non-smoothness of $R(PSU(3))$. It is easy to see that the algebraic variety $\text{Spec}(R \otimes \mathbb{C})$ is not smooth, as it has a singular point, corresponding to $A = \text{diag}(1, \omega, \omega^2) \in PSU(3)$.

$K^0_{PSU(3)}(PSU(3))$ has a very nice $R(PSU(3))$-torsion part, it comes from following line bundle:
**Lemma 7.3** There exists a $\text{PSU}(3)$-equivariant line bundle $L$ over $\text{PSU}(3)$, such that:

1) $L^\otimes 3 = 1$
2) $L$ is trivial outside the orbit $G.A$
3) $R(\text{PSU}(3))/\text{Ann}(\lfloor L \rfloor - 1) = \mathbb{Z}$
4) $p_*1 = 1 \oplus L \oplus L^2$

Here $\lfloor L \rfloor$ denotes the class of $L$ in $K^*_{\text{PSU}(3)}(\text{PSU}(3))$, $p: \text{SU}(3) \to \text{PSU}(3)$ is the quotient map.

By using the affine Weyl group [6], we get the orbit space as in figure 1, where points in the edges with arrows are identified by $(a, b) \sim (a, 1)$, the orbit passing through $A$ corresponds to $= (-\frac{2}{3}, 1)$.

Now we can construct a very nice covering of $\text{PSU}(3)$, then by the Segal spectral sequence, we find that:

**Proposition 7.4**

$$K^0_{\text{PSU}(3)}(\text{PSU}(3)) = R(\text{PSU}(3))^2 \oplus \mathbb{Z}.(\lfloor L \rfloor - 1) \oplus \mathbb{Z}.(\lfloor L \rfloor^2 - 1)$$

While $K^1_{\text{PSU}(3)}(\text{PSU}(3))$ is torsion free.

So in this case, we have $\Omega^*_{\mathbb{R}/\mathbb{Z}} \not\cong K^*_G(G)$. The general case remains an open problem.

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Figure 1: Orbit Space of PSU(3)