On the existence of a common quadratic Lyapunov function for a rank one difference

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Abstract

Suppose that $A$ and $B$ are real stable matrices, and that their difference $A - B$ is rank one. Then $A$ and $B$ have a common quadratic Lyapunov function if and only if the product $AB$ has no real negative eigenvalue. This result is due to Shorten and Narendra, who showed that it follows as a consequence of the Kalman-Yacubovich-Popov solution of the Lur’e problem. Here we present a new and independent proof based on results from convex analysis and the theory of moments.

Key words: quadratic Lyapunov function; Hankel matrix; discrete moment problem.

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1 Introduction and statement of results

This paper presents a new proof of the Shorten-Narendra Theorem \cite{8}, which gives a simple spectral condition for the existence of a common quadratic Lyapunov function (CQLF) for two stable matrices whose difference is rank one. Recall that a matrix \( A \) is stable if the spectrum of \( A \) lies wholly in the open left half of the complex plane. An equivalent condition is the existence of a positive definite matrix \( P \) such that \( PA + A^T P \) is negative definite, in which case the function \( x^T P x \) is a quadratic Lyapunov function for the system \( \dot{x} = Ax \). Consideration of the switching system \( \dot{x} = A(t)x \) with \( A(t) \in \{A, B\} \) leads to the notion of a common quadratic Lyapunov function (CQLF), which is determined by a positive definite matrix \( P \) satisfying

\[
PA + A^T P < 0, \quad PB + B^T P < 0
\]  

(1)

The following theorem of Shorten and Narendra provides a simple test for the existence of a CQLF in the case where \( A - B \) is rank one. The theorem is stated in \cite{8} for matrices in companion form, however this is unnecessary \cite{6} and we state the result in its full generality here.

**Theorem 1** [Shorten and Narendra] Let \( A \) and \( B \) be stable matrices and suppose that \( A - B \) is rank 1. Then the necessary and sufficient condition that \( (A, B) \) have a CQLF is that the matrix \( AB \) does not have a real negative eigenvalue.

The proof of Theorem 1 presented in \cite{8} first relates the spectral condition on \( AB \) to the following positivity condition for the resolvent of \( A \) along the imaginary axis:

\[
\text{Re} z = 0 \Rightarrow 1 + \text{Re} v^T (z - A)^{-1} u > 0
\]  

(2)

where \( A - B = uv^T \). The authors then make use of earlier work of Narendra and Goldwyn \cite{7} and Willems \cite{9} which showed that this resolvent condition (known as the circle criterion) is equivalent to the existence of a CQLF. These earlier papers proved the equivalence by transforming the question of the existence of a CQLF for the pair \( (A, B) \) into the existence of a solution for the Lur’e problem. They then used the fundamental results of Kalman \cite{3} who used techniques from analytic function theory to show that the circle criterion gives a necessary and sufficient condition for the existence of a solution of the Lur’e problem.
The result of Theorem 1 is strikingly simple, and it gives an easy way to check for the existence of a CQLF. It also encourages the belief that there should be a direct and independent proof which does not use the equivalence between the CQLF problem and the Lur’e problem. In this paper we provide such a proof, using methods of convex analysis and the theory of moments. The proof has a geometrical flavor which is described in the next paragraph.

There is a dual formulation of the CQLF condition in terms of intersecting cones in the space of symmetric matrices. Given a real matrix $A$, define

$$
C(A) = \{ AX + XA^T | X \geq 0 \}
$$

(3)

That is, $C(A)$ is the cone of symmetric matrices of the form $AX + XA^T$ where $X$ runs over all positive semidefinite matrices. Considering real $n \times n$ matrices as $n^2$-component vectors with the Hilbert-Schmidt inner product, the existence of a quadratic Lyapunov function for $A$ is equivalent to the existence of a positive definite matrix $P$ such that $\langle P, M \rangle = \text{Tr} PM < 0$ for all $M \neq 0$ in $C(A)$. It is convenient to view this in terms of the hyperplane which is the orthogonal complement of $P$, in which case the condition is that the cone $C(A)$ lies on one side of the hyperplane. Correspondingly, the existence of a CQLF for $A$ and $B$ is equivalent to finding such a hyperplane with both cones $C(A)$ and $C(B)$ on the same side, or alternatively with the cones $C(A)$ and $C(-B)$ on opposite sides. Therefore the existence of a CQLF for $A$ and $B$ is equivalent to the non-intersection (except at the origin) of the cones $C(A)$ and $C(-B)$. This observation leads to the following proposition.

**Proposition 2** Let $A$ and $B$ be stable matrices. Then the pair $(A, B)$ does NOT have a CQLF if and only if there are nonzero positive semidefinite matrices $X$ and $Y$ such that

$$
AX + XA^T + BY + YB^T = 0
$$

(4)

The main result of this paper is contained in the following theorem. It describes a special property of the intersection of the cones $C(A)$ and $C(-B)$ in the case of interest here, namely when $A - B$ is rank 1. The extreme points of the cone $C(A)$ have the form $A\mathbf{v}\mathbf{v}^T + \mathbf{v}\mathbf{v}^TA^T$, where $\mathbf{v}$ is a vector. The next theorem shows that whenever the cones $C(A)$ and $C(-B)$ have a nonzero intersection, then the extreme points of the cones must also have a nonzero intersection.
**Theorem 3** Let $A$ and $B$ be stable matrices, with $A - B$ rank one. Suppose that there are nonzero positive semidefinite matrices $X$ and $Y$ such that

$$AX + XA^T + BY + YB^T = 0$$

Then there are nonzero vectors $v$ and $w$ such that

$$Avv^T + vv^TA^T + Bww^T + ww^TB^T = 0$$

Combining Proposition 2 and Theorem 3 shows that the pair $(A, B)$ does not have a CQLF if and only if there are nonzero vectors $v$ and $w$ such that (5) holds. The proof of Theorem 1 is completed by showing that for stable matrices $A$ and $B$, the existence of vectors $v$ and $w$ satisfying (6) is equivalent to the condition that $AB$ has a real negative eigenvalue. This equivalence was first shown in a more general setting by Mason and Shorten [5]. The idea is simple: there are only two possible ways for the equation (6) to hold – either $v = \alpha Bw$ for some $\alpha$, or else $v = \alpha w$ for some $\alpha$. The first possibility leads to $(\alpha AB + \alpha^{-1})w = 0$, which is precisely the condition that $AB$ has eigenvalue $-\alpha^{-2}$. Running the argument in reverse shows that the conditions are equivalent.

The second possibility would imply that $(\alpha^2 A + B)w = 0$, or equivalently that some convex combination $(1 - x)A + xB$ is singular. However writing $A - B = R$ we have

$$det[(1 - x)A + xB] = det[A] det[I - xA^{-1}R] = det[A] \left(1 - xTr(A^{-1}R)\right)$$

The left side of (7) is nonzero and has the same sign at $x = 0$ and $x = 1$, hence the right side cannot vanish for any value of $x$ between 0 and 1. This rules out the second possibility.

Thus we see that Theorem 1 follows directly from Theorem 3 and the rest of the paper is devoted to its proof. In section 2 we show that it is sufficient to assume a special form for the matrices known as companion form. The main work of the paper appears in section 3 where we prove Theorem 3. The proof uses some relations between Hankel matrices and the solution of the discrete moment problem, and some linear algebra arguments. These are derived in Appendices A and B.
2 Reduction to companion form

Let us write

\[ A - B = xy^T \]  \hspace{1cm} (8)

where \( x \) and \( y \) are vectors in \( \mathbb{R}^n \). Let \( V \) be the span of \( x, Ax, A^2x, \ldots \). Suppose first that \( V \) is a proper subspace of \( \mathbb{R}^n \). Then \( \mathbb{R}^n = V \oplus V^\perp \) and with respect to this decomposition \( A \) and \( B \) are block matrices of the form

\[ A = \begin{pmatrix} A_1 & A_2 \\ 0 & A_3 \end{pmatrix}, \quad B = \begin{pmatrix} B_1 & B_2 \\ 0 & A_3 \end{pmatrix} \] \hspace{1cm} (9)

Since the spectrum of \( A \) is the union of the spectra of \( A_1 \) and \( A_3 \), it follows that \( A_1, A_3 \) and \( B_1 \) are also stable. Now suppose that (5) holds, and write \( X \) and \( Y \) in block form

\[ X = \begin{pmatrix} X_1 \\ X_2^T \\ X_3 \end{pmatrix}, \quad Y = \begin{pmatrix} Y_1 \\ Y_2^T \\ Y_3 \end{pmatrix} \] \hspace{1cm} (10)

Then it follows from (5) that

\[ A_3(X_3 + Y_3) + (X_3 + Y_3)A_3^T = 0 \] \hspace{1cm} (11)

which in turn implies that \( X_3 + Y_3 = 0 \) since \( A_3 \) is stable. Then the positivity of \( X \) and \( Y \) imply that \( X_2 = X_3 = Y_2 = Y_3 = 0 \). Therefore (5) reduces to

\[ A_1X_1 + X_1A_1^T + B_1Y_1 + Y_1B_1^T = 0 \] \hspace{1cm} (12)

This means that it is sufficient to prove Theorem 3 for the pair \( (A_1, B_1) \). Since \( A_1 - B_1 = \tilde{y}y^T \) where \( \tilde{y} \) is the projection of \( y \) onto \( V \), the equation (12) is a special case of (5), namely the case where the vectors \( x, Ax, A^2x, \ldots \) span the whole space. Hence without loss of generality we will assume that the vectors \( x, Ax, A^2x, \ldots, A^{n-1}x \) are linearly independent. In this case the pair \( (A, x) \) is called completely controllable. We next show how this allows a change of basis into a special form known as companion form (see [4] for details).

We first introduce the matrix

\[ S = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \ddots & \vdots & & \vdots \\ 0 & \ldots & 0 & 1 \\ 0 & \ldots & 0 & 0 \end{pmatrix} \] \hspace{1cm} (13)
and the vector

\[ g = \begin{pmatrix} 1 \\ \ldots \\ 0 \end{pmatrix} \]  

(14)

Then a matrix \( A \) is said to be in companion form if it can be written

\[ A = S + gh^T \]  

(15)

for some vector \( h \). Suppose that the pair \((A, x)\) is completely controllable, and that the characteristic polynomial for \( A \) is

\[ A^n + a_{n-1}A^{n-1} + \cdots + a_1 I = 0 \]  

(16)

Then we choose the following vectors as a basis:

\[
\begin{align*}
e_n &= x \\
e_{n-1} &= (A + a_n I)x \\
e_{n-2} &= (A^2 + a_n A + a_{n-1} I)x \\
\vdots \\
e_1 &= (A^{n-1} + a_n A^{n-2} + \cdots + a_2 I)x
\end{align*}
\]

The condition that \((A, x)\) is completely controllable guarantees that these vectors form a basis. Furthermore when the matrix \( A \) is written in this basis, it is easily seen to have the form (15). That is, there is a non-singular real matrix \( R \) such that \( Rx = g \) and \( RAR^{-1} \) has the form (15). It follows that \( RBR^{-1} = RAR^{-1} - gy^TR^{-1} \) is also in companion form, and furthermore that the pair \((A, B)\) has a CQLF if and only if the pair \((RAR^{-1}, RBR^{-1})\) has a CQLF. Similarly for the condition that \(AB\) has a negative real eigenvalue. Therefore it is sufficient to prove Theorem 3 for the case that both \( A \) and \( B \) are in companion form.

### 3  Proof of Theorem 3

#### 3.1  Solving \(AX + XA^T + BY + YB^T = 0\)

We assume henceforth that \( A \) and \( B \) are both in companion form, that is

\[ A = S + gh^T, \quad B = S + gk^T \]  

(17)
and that equation (11) holds. Define
\[ Z = X + Y \quad (18) \]
and
\[ w = \langle k - h, Y(k - h) \rangle^{-1/2} Y(k - h) \quad (19) \]
Note that \( \langle k - h, Y(k - h) \rangle \) cannot be zero, as this would imply that \( AZ + ZA^T = 0 \) which is impossible since \( Z \neq 0 \). Then we can rewrite (5) as
\[ SZ + ZS^T + gh^T Z + Zhg^T + \langle k - h, w \rangle \left( gw^T + wg^T \right) = 0 \quad (20) \]
It will be convenient to separate (20) into a pair of equations. Let \( \Pi \) denote the orthogonal projection onto the subspace orthogonal to the vector \( g \), so that
\[ \Pi = I - gg^T \quad (21) \]
Then the equation (20) is equivalent to the following two equations:
\[ \Pi SZ \Pi + \Pi ZS^T \Pi = 0 \quad (22) \]
and
\[ SZg + Zh + \langle w, k - h \rangle w = 0 \quad (23) \]
Furthermore, an application of the Cauchy-Schwarz inequality shows that \( Y \geq ww^T \), and hence
\[ Z \geq ww^T \quad (24) \]
This means that the pair of matrices \( X' = Z - ww^T \) and \( Y' = ww^T \) also satisfy (5). Therefore the existence of any pair \( (X, Y) \) which satisfy (5) implies the existence of a pair \( Z - ww^T \) and \( ww^T \) satisfying (5), where \( Z \) and \( w \) are related by (22) and (23). Conversely, if \( Z \) and \( w \) satisfy (22), (23) and (24), then they provide a solution of (5). Therefore we have the following result which describes the solutions of (5).

**Lemma 4** Suppose that \( Z \geq 0 \) and \( w \) satisfy (22), (23) and (24). Then the pair \( X' = Z - ww^T \) and \( Y' = ww^T \) satisfy (5). Conversely, suppose \( (X, Y) \) satisfy (5). Let \( Z = X + Y \) and define \( w \) by (19). Then \( Z \) and \( w \) satisfy (22), (23), and (24).
We now return to the equation (23) and solve for \( w \). Define
\[
\xi = SZg + Zh
\]  
(25)

If \( Z \) and \( w \) satisfy the equations (22) and (23), then it must be true that
\[
\langle h - k, \xi \rangle > 0
\]  
(26)

This is a condition on the matrix \( Z \). If it is satisfied, then (23) can be solved for \( w \):
\[
w = \langle h - k, \xi \rangle^{-1/2} \xi
\]  
(27)

The condition \( Z \geq ww^T \) is equivalent to \( 1 \geq w^T Z^{-1} w \). Defining
\[
F(Z) = \frac{\langle \xi, Z^{-1} \xi \rangle}{\langle h - k, \xi \rangle}
\]  
(28)

we can combine the two conditions (26) and (24) as
\[
0 < F(Z) \leq 1
\]  
(29)

We can now restate Lemma 4 as follows.

**Lemma 5** Suppose that \( Z \geq 0 \) satisfies (22) and (29). Define \( w \) by (27). Then the pair \( X' = Z - ww^T \) and \( Y' = ww^T \) satisfy (5). Conversely, suppose \( (X, Y) \) satisfy (5), and let \( Z = X + Y \). Then \( Z \) satisfies (22) and (24).

Lemma 5 shows a many-to-one correspondence between the solutions of (5) and the matrices \( Z \) satisfying (22) and (29). Therefore we have reduced the proof of Theorem 3 to the following problem: suppose that there is some matrix satisfying (22) and (29). Then we want to show that there is another such \( Z \) satisfying (22) and (29) for which both \( X' = Z - ww^T \) and \( Y' = ww^T \) are rank 1. Equivalently, we want to show that there is a rank 2 matrix \( Z \) satisfying (22) for which \( F(Z) = 1 \).
### 3.2 Representation using Hankel matrices, and the moment problem

We use the easily verified fact that every symmetric matrix $Z$ which satisfies (22) has the following form:

\[
Z = \begin{pmatrix}
z_0 & 0 & -z_1 & 0 & z_2 & \ldots \\
0 & z_1 & 0 & -z_2 & 0 & \ldots \\
-z_1 & 0 & z_2 & 0 & -z_3 & \ldots \\
0 & -z_2 & 0 & z_3 & 0 & \ldots \\
& \ddots & \ddots & \ddots & \ddots & \ddots \\
\end{pmatrix}
\] (30)

Except for the minus signs, (30) is an example of a Hankel matrix. Positivity of $Z$ requires that $z_i \geq 0$ for all $i = 0, \ldots, n-1$, and also imposes other constraints. To describe the possible values of $\{z_i\}$, we introduce for each real $x$ the following rank 2 matrix of the form (30):

\[
Z(x) = \begin{pmatrix}
1 & 0 & -x & 0 & x^2 & \ldots \\
0 & x & 0 & -x^2 & 0 & \ldots \\
-x & 0 & x^2 & 0 & -x^3 & \ldots \\
0 & -x^2 & 0 & x^3 & 0 & \ldots \\
& \ddots & \ddots & \ddots & \ddots & \ddots \\
\end{pmatrix}
\] (31)

The next result shows that every matrix $Z \geq 0$ satisfying (22) and (23) can be written as a positive linear combination of the matrices (31) for some non-negative values of $x$.

**Theorem 6** Suppose that the $n \times n$ matrix $Z \geq 0$ satisfies equations (22), (23) and (24). Then there is an integer $p \leq (n+1)/2$, non-negative numbers $0 \leq x_0 < x_1 < \ldots < x_{p-1}$, and positive numbers $\mu_0, \ldots, \mu_{p-1}$, such that

\[
Z = \sum_{i=0}^{p-1} \mu_i Z(x_i)
\] (32)

If $p = (n+1)/2$ then $x_0 = 0$.

The proof of Theorem 6 is presented in Appendix A, using standard results from the theory of moments. Indeed, the result is equivalent to the solution of
the discrete moment problem for \( z_0, \ldots, z_{n-1} \), which is the problem of finding distinct non-negative numbers \( x_0, \ldots, x_{p-1} \) and positive \( \mu_0, \ldots, \mu_{p-1} \), so that

\[
  z_j = \sum_{i=0}^{p-1} \mu_i x_i^j, \quad 0 \leq j \leq n - 1
\]  

(33)

If there is a solution of (33), then the special form of the Hankel matrices implies immediately that (32) holds, and vice versa.

The decomposition (32) is the key for solving the problem posed after Lemma 5: we will show that if \( Z \) satisfies the equations (22), (23) and (24) so that the representation (32) holds, then at least one of the matrices \( \{ Z(x_i) \} \) must also satisfy these equations, and hence by Lemma 5 it provides a solution of (5). Since \( Z(x_i) \) has rank 2, this almost completes the proof of Theorem 3. The only remaining obstacle is that the matrix \( X' = Z(x_i) - ww^T \) may have rank 2, or equivalently \( F(Z(x_i)) < 1 \). To complete the proof we will show that in this case there must be another number \( y < x_i \) for which \( F(Z(y)) = 1 \). Then \( Z(y) - ww^T \) will have rank 1, and hence can be written as \( vv^T \), so that (6) holds.

3.3 Completion of the proof

The key to the solution is the following Lemma, which displays a remarkable property of the function \( F(Z) \) when \( Z \) has the form (32).

**Lemma 7** Suppose that \( Z \) satisfies the hypotheses of Theorem 6, so that the representation (32) holds. For each \( i = 0, \ldots, p - 1 \) define

\[
  \xi_i = SZ(x_i)g + Z(x_i)h
\]

(34)

Then

\[
  F(Z) = F \left( \sum_{i=0}^{p-1} \mu_i Z(x_i) \right) = \frac{\sum_{i=0}^{p-1} \mu_i \langle \xi_i, Z(x_i)^{-1} \xi_i \rangle}{\sum_{i=0}^{p-1} \mu_i \langle h - k, \xi_i \rangle}
\]

(35)

Lemma 7 is proved in Appendix B. We now use it to complete the proof of Theorem 3. By assumption there is a matrix \( Z \) satisfying the hypotheses of Theorem 6 and so Lemma 7 can be used to evaluate \( F(Z) \). We now consider
how the right side of (35) varies as the parameters \( \mu_i \) change. Our goal is to show that there is some \( i \) such that \( F(Z) \geq F(Z(x_i)) \).

Notice first that if \( \langle h - k, \xi_i \rangle \leq 0 \) for any \( i \), then we do not increase \( F(Z) \) by setting \( \mu_i = 0 \), so we will assume that \( \langle h - k, \xi_i \rangle > 0 \) for all \( i \). It is straightforward to compute the derivative with respect to the parameter \( \mu_i \). Since the sign of this derivative is independent of the value of \( \mu_i \), it follows that \( F(Z) \) is a monotone function of each \( \mu_i \). Consider first how \( F(Z) \) behaves as \( \mu_0 \) varies. \( F(Z) \) must decrease either as \( \mu_0 \to \infty \) or as \( \mu_0 \to 0 \). In the former case we get

\[
F(Z) \geq F(Z(x_0))
\]

while in the latter case

\[
F(Z) \geq F\left(\sum_{i=1}^{n-1} \mu_i Z(x_i)\right)
\]

By repeating the same argument if necessary with \( \mu_1, \mu_2, \ldots \) we eventually deduce that

\[
1 \geq F(Z) \geq F(Z(x_i)) > 0
\]

for some \( i \). Recall from Lemma 5 that any matrix \( Z \) satisfying (22) and \( F(Z) \leq 1 \) provides a solution of (20), namely \( X' = Z - w w^T \) and \( Y' = w w^T \). Hence (38) implies that \( Z(x_i) \) provides such a solution.

Since \( Z(x_i) \) has rank 2, it follows that \( X' \) has rank 1 or 2. If \( X' \) has rank 1, then we can immediately deduce that (38) holds, and the proof is complete. The condition that \( X' \) has rank 1 is \( F(Z(x_i)) = 1 \), so we are left with the case where \( F(Z(x_i)) < 1 \). We now show that in this case there is another value \( y < x_i \) such that

\[
F(Z(y)) = 1
\]

This fact follows from these observations:

(a) \( F(Z(x)) \) is a rational function of \( x \);

(b) \( F(Z(x)) \neq 0 \) for all \( x \geq 0 \);

(c) either \( F(Z(0)) > 1 \) or \( F(Z(0)) < 0 \).
To see that (b) is true, note that $F(Z(x)) = 0$ would imply $SZ(x)g + Z(x)h = 0$. However this would imply that $AZ + ZA^T = 0$, which is impossible because $A$ is stable. Similarly (c) is the statement that $Z(0)$ cannot arise as a solution of (5). Continuity now implies that there must be some $y < x$, such that (39) holds, and this completes the derivation of (6).

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A Proof of Theorem 6

There are many results known for the discrete moment problem. We have found special cases of our result in the literature (for example when \( n \) is even and \( Z \) is nonsingular [2]) but not the full statement. For this reason we include the proof here; our starting point is the following result from the text of Ahiezer and Krein [1].

**Lemma 8** Given a sequence \( s_0, s_1, \ldots, s_{2m-2} \) let \( K \) denote the \( m \times m \) Hankel matrix with entries

\[
K_{ij} = s_{i+j-2}, \quad i, j = 1, \ldots, m
\]  

If \( K \geq 0 \), then there is an integer \( p \leq m - 1 \), numbers \( x_1, \ldots, x_p \), and positive numbers \( \mu_1, \ldots, \mu_p \) and \( M \) such that

\[
s_k = \sum_{i=1}^{p} \mu_i x_i^k \quad (k = 0, \ldots, 2m - 3) \tag{41}
\]

\[
s_{2m-2} = \sum_{i=1}^{p} \mu_i x_i^{2m-2} + M \tag{42}
\]

We now use Lemma 8 to prove Theorem 6. First suppose that \( n = 2m - 1 \) is odd, and that \( Z \) has the form (30). Then letting \( s_k = z_k \) for \( k = 0, \ldots, 2m - 2 \)
it follows that the matrix $K$ defined by (40) is positive semidefinite (the minus signs of some off-diagonal entries in $Z$ can be removed by conjugation with a diagonal matrix with $\pm 1$ on the diagonal). Hence the representation (41) implies that there is some $p \leq m - 1$ such that

$$z_k = \sum_{i=1}^{p} \mu_i x_i^k \quad (k = 0, \ldots, 2m - 3)$$ \hspace{1cm} (43)

$$z_{2m-2} = \sum_{i=1}^{p} \mu_i x_i^{2m-2} + M$$ \hspace{1cm} (44)

and therefore we get the following representation for $Z$:

$$Z = \sum_{i=1}^{p} \mu_i Z(x_i) + M g g^T$$ \hspace{1cm} (45)

The rank 2 matrix $Z(x)$ was defined in (31). It can be written as

$$Z(x) = u(x) u(x)^T + x v(x) v(x)^T$$ \hspace{1cm} (46)

where $u(x)$ and $v(x)$ are the $n$-vectors

$$u(x) = \begin{pmatrix} 1 \\ 0 \\ -x \\ 0 \\ (-x)^2 \\ \vdots \\ \vdots \end{pmatrix}, \quad v(x) = S^T u(x) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ -x \\ 0 \\ (-x)^2 \end{pmatrix}$$ \hspace{1cm} (47)

Since $x_1, \ldots, x_p$ are distinct, and $2p \leq 2m - 2 < n$, it follows that the $2p + 1$ vectors $\{u(x_i), v(x_i)\}$ and $g$ are linearly independent (this is easily demonstrated using the Hadamard determinant). Hence positivity of $Z$ implies that each term in (45) is separately positive. From (46) it then follows that

$$x_i \geq 0, \quad i = 1, \ldots, p$$ \hspace{1cm} (48)

It remains to show that $M = 0$ in (45). This requires using (23) and (24). Suppose first that $Z$ is singular; then counting dimensions in (45) shows that
either $M = 0$ or else $p \leq m - 2$. In the latter case it follows that the vectors 
$\{u(x_i), v(x_i)\}, g, Sg$ are linearly independent. Furthermore for any $x$,
$$SZ(x)g = (-x)^{(n+1)/2}v(x)$$
(49)
Therefore from (23) it follows that $w$ is a linear combination of the vectors 
$\{u(x_i), v(x_i)\}, g, Sg$. If $M > 0$ then the coefficient of $Sg$ in this combination 
is nonzero. But (45) then implies that $w$ is not in the range of $Z$, which means 
that (24) cannot hold. Therefore we must have $M = 0$.

In the general case where $Z$ is non-singular, we argue as follows. Notice that $Z(0)$ is the matrix whose $(1, 1)$ entry is 1, and all other entries are zero. If $Z > 0$ then there is $c > 0$ such that
$$Z = cZ(0) + W$$
(50)
where $W > 0$ is singular. Since $W$ satisfies (22), Lemma 8 leads to the representation (45) for $W$, that is
$$Z = cZ(0) + \sum_{i=1}^{p}\mu_iZ(x_i) + Mgg^T$$
(51)
where all $x_i > 0$. Since $W$ is singular, either $M = 0$ and $p \leq m - 1$, or else $p \leq m - 2$. In the former case $2p \leq 2m - 2 < n$ so this establishes (32). In the latter case the vectors $\{u(x_i), v(x_i)\}, u(0), g$ and $Sg$ are linearly independent, and $Sg$ is not in the range of $Z$. Hence by the same argument we must have $M = 0$. Again we have $2p < n$.

This completes the argument for the case when $n$ is odd. When $n$ is even, we first create a $(n+1) \times (n+1)$ matrix $\tilde{Z}$ by adding an extra row and column to the matrix $Z$. The new entries are chosen so that $\tilde{Z}$ has the form (30). This determines uniquely all the entries of $\tilde{Z}$ except the bottom right corner. This entry is chosen large enough so that $\tilde{Z} \geq 0$. To see that this is possible, note that the first $n$ entries of the last column of $\tilde{Z}$ are the vector $SZg$. From (28) we have the bound
$$||Z^{-1/2}SZg|| \leq ||Z^{-1/2}(SZg + Zh)|| + ||Z^{1/2}h||$$
$$\leq F(Z)\langle h - k, (SZg + Zh)\rangle^{1/2} + \langle h, Zh\rangle^{1/2}$$
(52)
Let $k$ be the bottom right corner entry of $\tilde{Z}$. Then taking
$$k \geq \left(\langle h - k, (SZg + Zh)\rangle^{1/2} + \langle h, Zh\rangle^{1/2}\right)^2$$
(53)
it follows that $\tilde{Z} \geq 0$. Hence Lemma 8 can be applied and we deduce that

$$\tilde{Z} = \sum_{i=1}^{p} \mu_i \tilde{Z}(x_i) + M\tilde{g}\tilde{g}^T$$

where $\tilde{Z}(x_i)$ and $\tilde{g}$ are the $(n+1)$ dimensional versions, and again $x_i \geq 0$, and \(\mu_i > 0\). Since $2p \leq n$ we immediately deduce (52) by restricting both sides of (54) to the top left $n \times n$ block. This completes the proof of Theorem 6.

**B Proof of Lemma 7**

We assume that the representation (52) holds. Lemma 7 follows from a result in linear algebra which we state and prove in Lemma 9 below. First we verify that the conditions of the lemma are satisfied.

From (46) it follows that the range of $Z$ is spanned by the vectors \{u(\(x_i\)), v(\(x_i\))\} \((i = 0, \ldots, p-1)\). If $2p \leq n$ then these vectors are independent, and hence $rk(Z) = \sum rk(Z(x_i))$. If $2p = n+1$, then $n$ is odd, and Theorem 6 implies that $x_0 = 0$. Since the vectors \(u(0)\) and \{u(\(x_i\)), v(\(x_i\))\} \((i = 1, \ldots, p-1)\) are independent, it is again true that $rk(Z) = \sum rk(Z(x_i))$.

Furthermore,

$$SZ(x_i)g = \left\{ \begin{array}{ll} \quad \n \text{is even} & \quad \left\{ \begin{array}{l} \quad (-x_i)^{n/2}u(x_i) \quad \n \text{is even} \\ \quad (-x_i)^{(n+1)/2}v(x_i) \quad \n \text{is odd} \end{array} \right. \\
\in Ran(Z(x_i)) \end{array} \right.$$  

Since $Z(x_i)h$ is clearly in $Ran(Z(x_i))$, we deduce that

$$\xi_i = Z(x_i)h + S\tilde{Z}(x_i)\tilde{g} \in Ran(Z(x_i))$$

**Lemma 9** Let $Z$ be an $n \times n$ matrix

$$Z = \sum_{i} \mu_i Z_i$$

such that each $Z_i$ is symmetric and $rk(Z) = \sum rk(Z_i)$. Also, let

$$v = \sum_{i} \mu_i v_i$$
where each \( v_i \in \text{Ran}(Z_i) \). Then:

\[
\langle v, Z^{-1}v \rangle = \sum_i \mu_i \langle v_i, Z_i^{-1}v_i \rangle \quad (60)
\]

**Proof:** It is not assumed that \( Z \) is invertible on \( \mathbb{R}^n \); since \( v \in \text{Ran}(Z) \), \( Z^{-1}v \) is always well-defined. We first prove the result in the case that each \( Z_i = \lambda_i u_i u_i^T \) is rank 1. Since each \( v_i \) is in the range of \( Z_i \),

\[
v_i = a_i u_i, \quad v = \sum_i \mu_i a_i u_i \quad (61)
\]

The fact that \( \text{rank}(Z) = \sum \text{rank}(Z_i) \) implies that the \( u_i \)'s form a basis for \( \text{Ran}(Z) \). We write \( Z^{-1}v \) in that basis with arbitrary coefficients:

\[
Z^{-1}v = \sum_j \alpha_j u_j \quad (62)
\]

Now apply \( Z \) to both sides:

\[
v = Z(\sum_j \alpha_j u_j) = \sum_{i,j} \mu_i Z_i \alpha_j u_j = \sum_{i,j} \mu_i \lambda_i \langle u_i, u_j \rangle \alpha_j u_i \quad (63)
\]

Comparing coefficients in (61) and (63), we see that for all \( i \),

\[
a_i = \lambda_i \sum_j \langle u_i, u_j \rangle \alpha_j \quad (64)
\]

Now we can calculate:

\[
\langle v, Z^{-1}v \rangle = \sum_{i,j} \mu_i a_i \langle u_i, u_j \rangle \alpha_j \quad (65)
\]

\[
= \sum_i \mu_i a_i \left( \frac{a_i}{\lambda_i} \right) \quad (66)
\]

\[
= \sum_i \mu_i \left( \frac{a_i^2}{\lambda_i} \right) \quad (67)
\]

\[
= \sum_i \mu_i \langle v_i, Z_i^{-1}v_i \rangle \quad (68)
\]
To get the full result is now straightforward: since each \( Z_i \) is symmetric, it can written in terms of an orthonormal basis:

\[
Z_i = \sum_{k=1}^{r_i} \lambda_{i,k} u_{i,k} u_{i,k}^T, \quad v_i = \sum_{k=1}^{r_i} a_{i,k} u_{i,k}
\]  \hspace{1cm} (69)

Now, we can write \( Z \) as a sum of linearly independent rank 1 projections and apply what was shown above:

\[
Z = \sum_i \mu_i Z_i = \sum_i \mu_i \lambda_{i,k} u_{i,k} u_{i,k}^T \quad \text{(70)}
\]

\[
v = \sum_i \mu_i v_i = \sum_i \mu_i a_{i,k} u_{i,k} \quad \text{(71)}
\]

\[
\langle v, Z^{-1}v \rangle = \sum_{i,k} \mu_i \left( \frac{a_{i,k}^2}{\lambda_{i,k}} \right) \quad \text{(72)}
\]

The final observation is that, for each \( i \), the \( u_{i,k} \) are orthogonal, which means

\[
Z_{i}^{-1} = \sum_{k=1}^{r_i} \left( \frac{1}{\lambda_{i,k}} \right) u_{i,k} u_{i,k}^T \quad \text{(73)}
\]

\[
\langle v, Z_{i}^{-1}v_i \rangle = \sum_{k=1}^{r_i} \left( \frac{a_{i,k}^2}{\lambda_{i,k}} \right) \quad \text{(74)}
\]

Combining (72) and (74), we see

\[
\langle v, Z^{-1}v \rangle = \sum_{i,k} \mu_i \left( \frac{a_{i,k}^2}{\lambda_{i,k}} \right) = \sum_i \mu_i \langle v, Z_{i}^{-1}v_i \rangle \quad \text{(75)}
\]

which was to be shown.