1. Introduction

The fingerprint invariant of partitions is used to describe the Kazhdan-Lusztig map for the classical groups. The Kazhdan-Lusztig map is a map from the unipotent conjugacy classes to the set of conjugacy classes of the Weyl group. It can be extended to the case of rigid semisimple conjugacy classes. The rigid semisimple conjugacy classes and the conjugacy classes of the Weyl group are described by pairs of partitions \((\lambda'; \lambda'')\) and pairs of partitions \([\alpha; \beta]\), respectively.

The fingerprint invariant is a map between these two classes of objects.
Gukov and Witten initiated to study the $S$-duality for surface operators in $\mathcal{N} = 4$ super Yang-Mills theories in the ramified case of the Geometric Langlands Program in [6][7][8]. Surface operators are labeled by pairs of certain partitions. The $S$-duality conjecture suggest that surface operators in the theory with gauge group $G$ should have a counterpart in the Langlands dual group $G^L$. The 'rigid' surface operators corresponding to pairs of rigid partitions are expected to be closed under $S$-duality. Fingerprint is an invariant for the dual pairs partitions.

There is another invariant of partitions called symbol related to the Springer correspondence[5]. By using symbol invariant, Wyllard made some explicit proposals for how the $S$-duality maps should act on rigid surface operators in [10]. In [11], we find a new subclass of rigid surface operators related by $S$-duality. In [12], we found a construction of symbol for the rigid partitions in the $B_n$, $C_n$, and $D_n$ theories. Compared to the symbol invariant, the calculation of fingerprint is a complicated and tedious work. In this paper, we attempt to continue the analysis in [6][10][11][12], studying the construction of fingerprint.

2. Fingerprint of rigid partitions

In this section, we introduce some basic results related to the rigid partition and concept of fingerprint. In Section 3, we calculate the fingerprint of the unipotent operators in the $B_n$, $C_n$, and $D_n$ theories. In Section 4, we calculate the fingerprint of rigid semisimple operators ($\lambda^{\prime}; \lambda^{\prime\prime}$). We decompose $\lambda = \lambda^{\prime} + \lambda^{\prime\prime}$ into several blocks. For $I$ type block, we define operators $\mu_{e1}, \mu_{e2}$ and their odd versions $\mu_{o1}, \mu_{o2}$. For $II$ type blocks, we define operator $\mu_{II}$. For $III$ type blocks, we define operators $\mu_{e11}, \mu_{e12}, \mu_{e21}, \mu_{e22}$ and their odd versions $\mu_{o11}, \mu_{o12}, \mu_{o21}, \mu_{o22}$. For $S$ type blocks, the fingerprint should be calculated from case to case.

2.1. Rigid Partitions in the $B_n, C_n$, and $D_n$ theories. A partition $\lambda$ of the positive integer $n$ is a decomposition $\sum_{i=1}^{l} \lambda_i = n$ ($\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_l$), with the length $l$. Partitions are in a one to one correspondence with Young tableaux. For instance the partition $3^2 2^3 1$ corresponds to

```
        + + +
        + + +
        + +
        +
```

Young tableaux occur in a number of branches of mathematics and physics. They are also tools for the construction of the eigenstates of Hamiltonian System [1][2][3] and label the fixed point of the localization of path integral [4].

For the $B_n (D_n)$ theories, unipotent conjugacy classes are in one-to-one correspondence with partitions of $2n + 1(2n)$ where all even integers appear an even number of times. For the $C_n$ theories, unipotent conjugacy classes are in one-to-one correspondence with partitions $2n$ for which all odd integers appear an even number of times. If it has no gaps (i.e. $\lambda_i - \lambda_{i+1} \leq 1$ for all $i$) and no odd (even) integer
appears exactly twice, a partition in the $B_n$ or $D_n$ ($C_n$) theories is called rigid. We will focus on rigid partition in this paper.

The following facts plays an important role in this study.

**Proposition 1.** The longest row in a rigid $B_n$ partition always contains an odd number of boxes. The following two rows of the first row are either both of odd length or both of even length. This pairwise pattern then continues. If the Young tableau has an even number of rows the row of shortest length has to be even.

**Remark:** If the last row of the partition is odd, the number of rows is odd.

**Proposition 2.** For a rigid $D_n$ partition, the longest row always contains an even number of boxes. And the following two rows are either both of even length or both of odd length. This pairwise pattern then continue. If the Young tableau has an even number of rows the row of the shortest length has to be even.

Examples of partitions in the $B_n$ and $D_n$ theories are shown in Fig.1.

**Proposition 3.** For a rigid $C_n$ partition, the longest two rows both contain either an even or an odd number number of boxes. This pairwise pattern then continues. If the Young tableau has an odd number of rows the row of shortest length has contain an even number of boxes.

Examples of partitions in the $C_n$ theory are shown in Fig.2.
2.2. Definition of fingerprint. We introduce the definition of fingerprint. First, add the two partitions $\lambda = \lambda' + \lambda''$, and then calculate the partition $\mu = Sp(\lambda)$

$$\mu_i = \begin{cases} 
\lambda_i + p_\lambda(i) & \text{if } \lambda_i \text{ is odd and } \lambda_i \neq \lambda_i - p_\lambda(i), \\
\lambda_i & \text{otherwise}
\end{cases}$$

Next define the function $\tau$ from an even positive integer $m$ to 1 as follows. For a partition in the $B_n$ and $D_n$ theories, $\tau(m)$ is $-1$ if at least one $\mu_i$ such that $\mu_i = m$ and either of the following three conditions is satisfied.

(i) $\mu_i \neq \lambda_i$

(ii) $\sum_{k=1}^{i} \mu_k \neq \sum_{k=1}^{i} \lambda_k$

(iii)$_{SO}$ $\lambda'_i$ is odd.

Otherwise $\tau$ is 1. For the partitions in the $C_n$ theory, the definition is exactly the same except the condition (iii)$_{SO}$ is replaced by

(iii)$_{Sp}$ $\lambda'_i$ is even.

Finally construct a pair of partitions $[\alpha; \beta]$, which is related to the conjugacy classes of the Weyl group. For each pair of parts of $\mu$ both equal to $a$, satisfying $\tau(a) = 1$, retain one part $a$ as a part of the partition $\alpha$. For each part of $\mu$ of size $2b$, satisfying $\tau(2b) = -1$, retain $b$ as a part of the partition $\beta$.

Note that $\mu_i \neq \lambda_i$ only happen at the end of a row.

2.3. Preliminary. For a partition $\lambda = n^m m^{m-1} (m-2)^{m-2} \cdot \cdot \cdot 1^n$ as shown in Fig. 3, according to Propositions 1, 2, and 3, the number of boxes of the gray part is even. For the partition in the $B_n$ and $D_n$ theories, $j$ is odd. If the $j - \frac{1}{2}$th pairwise rows are even, the number of boxes of the gray parts in Fig. (4)(a) and (b) are even. If the $j - \frac{1}{2}$th pairwise rows are odd, the number of boxes of the gray parts in Fig. (4)(a) and (b) are odd. Since $j$ is even for the partitions in the $C_n$ theory, the numbers of boxes of the gray parts in Fig. (4)(a) and (b) are always even.

Next, we calculate the fingerprint for the first few parts of a $B_n$ or $D_n$ partition as an example. If $n_a$ is the first odd number in the sequence $n_m, n_{m-1}, \cdot \cdot \cdot, n_j$, then the $m, m - 1, \cdot \cdot \cdot, a + 1$th rows are even. The $a$th row is odd, which is the

![Figure 3. Number of boxes of the gray parts is even.](image-url)
which means $L$ second row of a pairwise rows. Let $L(l) = \sum_{i=1}^{l} n_i$. The $L(l)$th column is at the end of the $l$th row as shown in Fig. 4. So we have

$$p_{L(i)} = (-1)^{\sum_{k=1}^{L(i)} \lambda_k} = 1, \quad i \in (m, \ldots, a - 1),$$

which means $\mu_k = \lambda_k, \quad k \in (1, \ldots, L(a) - 1)$.

The height of the $a$th row is $a$, which is odd. So we have

$$p_{L(a)} = (-1)^{\sum_{k=1}^{L(a)} \lambda_k} = -1.$$

According to (2.1), $\mu_{L(a)} = \lambda_{L(a)} - 1$, satisfying the condition (2.2)(i) for the part $\lambda_{L(a)} (= a - 1)$. So we have $\tau(a - 1) = -1$ for the even number $a - 1$. The $a - 1$th row is the first row of a pairwise rows, so $n_{a-1}$ is even. And we have

$$p_{L(a-1)} = (-1)^{\sum_{k=1}^{L(a)} \lambda_k} \cdot (-1)^{\sum_{k=1}^{L(a)-1} \lambda_k} = (-1)^{\sum_{k=1}^{L(a)} \lambda_k} \cdot (-1)^{\sum_{k=L(a)+1}^{L(a)+n_{a-1}} (a-1)} = -1,$$

which means $\mu_{L(a)-1+1} = \lambda_{L(a)-1+1} + 1$. Since $\mu_i \neq \lambda_i$ only happen at the end of a row, we have

$$\mu_k = \lambda_k, \quad i \in (L(a) + 1, \ldots, L(a) + n_{a-1} - 1).$$

Because of $\mu_{L(a)} = \lambda_{L(a)} + 1$, we have

$$\sum_{k=1}^{L(a)-1+1} \mu_k = \sum_{k=1}^{L(a)-1+1} \lambda_k.$$

For a rigid semisimple operator ($\lambda'$, $\lambda''$), following the above procedure, we can construct the partition $\mu$ for the parts $k^{n_k}$ independently. Let $'-'$ and $'+'$ denote $p_{\lambda}(i) = (-1)^{\sum_{k=1}^{L(a)} \lambda_k} = -1$ and $1$, respectively. The results are as shown in Fig. 5 and Fig. 6. The gray box is always appended as the last part of row and the black box is deleted at the end of row. If $k$ is odd, there are four cases according to $p_{\lambda}(L(k+1))$ and the parity of $n_k$ as shown in Fig. 5.

- For Fig. 5(a), we have $p_{\lambda}(L(k+1)) = -1$. Then $p_{\lambda}(L(k+1)+1) = 1$, which means $\mu_{L(k+1)+1} = \lambda_{L(k+1)+1} + 1$. Since $n_k$ is even, we have $p_{\lambda}(L(k)) = -1$ according to Fig. 4, which means $\mu_{L(k)} = \lambda_{L(k)} - 1$.
- For Fig. 5(b), we have $p_{\lambda}(L(k+1)) = -1$. Then $p_{\lambda}(L(k+1)+1) = 1$, which means $\mu_{L(k+1)+1} = \lambda_{L(k+1)+1} + 1$. Since $n_k$ is odd, we have $p_{\lambda}(L(k)) = 1$ according to Fig. 4, which means $\mu_{L(k)} = \lambda_{L(k)}$. 

**Figure 4.** Parity of the number of boxes of the gray parts depends on the parity of the height $j$. 

![Figure 4](image-url)
For Fig.(5)(c), we have \( p_\lambda(L(k+1)) = 1 \). Then \( p_\lambda(L(k+1)+1) = -1 \), which means \( \mu_{L(k+1)+1} = \lambda_{L(k+1)+1} \). Since \( n_k \) is odd, we have \( p_\lambda(L(k)) = -1 \) according to Fig.(4), which means \( \mu_{L(k)} = \lambda_{L(k)} - 1 \).

For Fig.(5)(a), we have \( p_\lambda(L(k+1)) = -1 \), which means \( \mu_{L(k+1)} = \lambda_{L(k+1)} - 1 \). Since \( k \) is even, we have \( p_\lambda(L(k)) = -1 \) according to Fig.(4), which means \( \mu_{L(k)} = \lambda_{L(k)} - 1 \).

For Fig.(6)(b), we have \( p_\lambda(L(k+1)) = 1 \). Then \( p_\lambda(L(k+1)+1) = 1 \), which means \( \mu_{L(k+1)+1} = \lambda_{L(k+1)+1} \). Since \( k \) is even, we have \( p_\lambda(L(k)) = 1 \) according to Fig.(4), which means \( \mu_{L(k)} = \lambda_{L(k)} \).

If \( k \) is even, there are two cases according to \( p_\lambda(L(k+1)) \) and the parity of \( n_k \) as shown in Fig.(4).

- For Fig.(5)(a), we have \( p_\lambda(L(k+1)) = 1 \), which means \( \mu_{L(k+1)} = \lambda_{L(k+1)} + 1 \). Since \( k \) is even, we have \( p_\lambda(L(k)) = 1 \) according to Fig.(4), which means \( \mu_{L(k)} = \lambda_{L(k)} + 1 \).

- For Fig.(6)(b), we have \( p_\lambda(L(k+1)) = -1 \), which means \( \mu_{L(k+1)} = \lambda_{L(k+1)} - 1 \). Since \( k \) is even, we have \( p_\lambda(L(k)) = -1 \) according to Fig.(4), which means \( \mu_{L(k)} = \lambda_{L(k)} - 1 \).

According to the above results, we have the following important lemma. Without confusion, the image of the map \( \mu \) is also called the partition \( \mu \).

**Lemma 1.** Under the map \( \mu \), the change of the last box of a row of a partition depend on the parity of the row and the sign of \( p_\lambda(i) \).

| Parity of row | Sign | Change |
|---------------|------|--------|
| odd           | -    | \( \mu_i = \lambda_i - 1 \) |
| even          | -    | \( \mu_i = \lambda_i + 1 \) |
| even          | +    | \( \mu_i = \lambda_i \) |
| odd           | +    | \( \mu_i = \lambda_i \) |

The partition \( \mu \) of the partition \( \lambda = \lambda' + \lambda'' \) can be constructed by using the building blocks in Fig.(5) and Fig.(6). It consist of several cycles as shown in Fig.(7). Both \( i \) and \( j \) are odd and both \( n_i \) and \( n_j \) are odd. This cycle begin with Fig.(5)(c) corresponding to \( n_j \) and end with Fig.(6)(b) corresponding to \( n_i \). We can decompose this cycle into building blocks in Fig.(5) and Fig.(6) as follows

\[
\text{Fig.}(5)(c) \rightarrow \text{Fig.}(6)(a) \rightarrow \text{Fig.}(5)(a) \rightarrow \text{Fig.}(6)(a) \rightarrow \text{Fig.}(5)(c) \rightarrow \text{Fig.}(6)(a) \rightarrow \text{Fig.}(6)(b).
\]

We introduce several facts to characterise the definition of fingerprint. The following lemma has been illustrated by Eq.(2.3).

**Lemma 2.1.** The condition \( \sum_{k=1}^{i} \mu_i \neq \sum_{k=1}^{i} \lambda_i \) in the definition of fingerprints must follow the condition \( \mu_j \neq \lambda_j \) until another \( \mu_j \neq \lambda_j \).

**Lemma 2.2.** The effect of the operation \( Sp(\lambda) \) is to ensure that the odd parts of the resulting partition never occur an odd number of times.

For the semisimple operator \((\lambda', \lambda'')\) in the \(B_n\) theory, \(\lambda'\) and \(\lambda''\) are partitions in the \(B_n\) and \(D_n\) theories, respectively. Since the first row of the partitions in the \(B_n\) and \(D_n\) theories is not in pairwise rows, \(\lambda'\) and \(\lambda''\) can be seen as in the same theory.

**Lemma 2.3.** When \( \lambda = \lambda' + \lambda'' \) is even, the condition

\[
(iii)_{SO}, \quad \lambda_i \text{ is odd}
\]
Lemma 2.4. If the fingerprint of $\lambda = \lambda' + \lambda''$ is $[\alpha, \beta]$, the fingerprints of the partition with $\lambda_i + 2$ correspond to $\alpha_i + 2$ and $\beta_i + 1$.

Proof. $\lambda_i \rightarrow \lambda_i + 2$ correspond to $\mu_i \rightarrow \mu_i + 2$, which lead to the conclusion. \qed

We have the following fact from Fig.(7).

Proposition 4. The condition (ii) imply (i) for partition $\lambda$ with $\lambda_i - \lambda_{i+1} \leq 1$. 
Figure 7. One cycle of partition \( \mu \) between successive odd number \( n_j \) and \( n_i \).

We would give an example, which the condition \((ii)\) works as shown in Fig. (8). Both the height of the \((j-1)\)th row, the \((j-3)\)th row, and the \((j-5)\)th are even. And the difference of the height of the \((j-1)\)th row and the \((j-3)\)th row violate the rigid condition \( \lambda_i - \lambda_{i+1} \leq 1 \) as well as height of the \((j-3)\)th row and the \((j-5)\)th row. The part \((j-3)\) satisfy the condition \((2.2)(ii)\) but do not satisfy the condition \((i)\).

Figure 8. The difference of the height of the \((j-1)\)th row and the \((j-3)\)th row is two, violating the rigid condition \( \lambda_i - \lambda_{i+1} \leq 1 \).

3. Fingerprint of rigid unipotent operators

For rigid unipotent operators corresponding to unipotent conjugacy classes, the calculations of fingerprint can be simplified using Propositions (1), (2), and (3). The partition \( \mu \) can be described by maps \( X_S \) and \( Y_S \).

3.1. \( B_n \) and \( D_n \) theories. First, we introduce the map \( X_S \) and the map \( Y_S \). The map \( X_S \) map a partition with only odd rows in the \( B_n \) theory to a partition with only even rows in the \( C_n \) theory as shown in Fig. (9).

\[
X_S : m^{2n_m+1}(m-1)^{2n_{m-1}}(m-2)^{2n_{m-2}}\ldots 2^{2n_2}1^{2n_1} \mapsto m^{2n_m}(m-1)^{2n_{m-1}+2}(m-2)^{2n_{m-2}-2}\ldots 2^{n_2+2}1^{n_1-2}.
\]

(3.1)

where \( m \) has to be odd in order for the first object to be a partition in the \( B_n \).

Figure 9. Map \( X_S \).
theory. The black box of the second row of a pairwise rows is deleted and a gray box is appended at the end of the first row of a pairwise rows. The $2k$th and $(2k + 1)$th rows are in a pairwise rows of the partition in the $B_n$ theory. The $(2k + 1)$th and $(2k + 2)$th rows are in a pairwise rows of a partition in the $C_n$ theory under the map $X$. The map $X$ is a bijection so that $X^{-1}$ is well defined. It is essentially the ’$pC$ collapse’ described in [5] and the inver map $X^{-1}$ is essentially the ’$pB$ expansion’.

Second, we propose the following map $Y$ which take a rigid partition with only odd rows in the $C_n$ theory to a rigid partition with only even rows in the $D_n$ theory as shown in Fig.(10).

\[
Y_S : \quad \begin{array}{c}
m^{2n+1} (m-1)^{2n-1} (m-2)^{2n-2} \cdots 2^{2n} 1^{2n} \\
\mapsto \quad m^{2n} (m-1)^{2n+1} (m-2)^{2n-2} \cdots 2^{n-2} 1^{2n+2}
\end{array}
\]

where $m$ has to be even in order for the first element to be a $C_k$ partition. The $2k + 1$th and $(2k + 2)$th rows are in a pairwise rows of a partition in the $C_n$ theory. The $(2k + 2)$th and $(2k + 3)$th rows are in a pairwise rows of a partition in the $D_n$ theory under the map $Y$. This is a bijection. The map is essentially the ’$pD$ collapse’ described in [5] and the inver map $Y^{-1}$ is essentially the ’$pC$ expansion’.

**Partition $\mu$**

Since $\mu_i \neq \lambda_i$ only happen at the end of a row and the number boxes of each pairwise rows is even, we consider the image of each pairwise rows under the map $\mu$ independently. First, consider an odd pairwise rows of the partition in the $B_n$ or $D_n$ theories as shown in Fig.(11). The $i$th row is the second row of the pairwise rows, so $i$ is odd. According to Fig.(11)(a), we have

\[
p_{\lambda}(L(i)) = (-1)^{\sum_{k=1}^{L(i)} \lambda_k} = -1,
\]

which means

\[
\mu_{L(i)} = \lambda_{L(i)} + p_{\lambda}(L(i)) = \lambda_{L(i)} - 1.
\]

Similarly, according to Fig.(11)(b), we have

\[
p_{\lambda}(L(i-1)) = (-1)^{\sum_{k=1}^{L(i-1)} \lambda_k} = -1,
\]

which means

\[
p_{\lambda}(L(i-1) + 1) = (-1)^{\sum_{k=1}^{L(i-1)} \lambda_k + \lambda_{L(i-1)+1}} = 1.
\]

Then we have

\[
\mu_{L(i-1)+1} = \lambda_{L(i-1)+1} + p_{\lambda}(L(i-1) + 1) = \lambda_{L(i-1)+1} + 1.
\]

We can also get this result using Lemma [4] directly. So we reach the right hand side of Fig.(11). In fact, this result is Fig.(11)(a).
Next consider an even pairwise rows of the partition as shown in Fig.(12). According to Fig.(4)(a), we have
\[ p_\lambda(L(i)) = (-1)^{\sum_{k=1}^{L(i)} \lambda_k} = 1, \]
which means
\[ \mu_{L(i)} = \lambda_{L(i)}. \]
Similarly, according to Fig.(4)(b), we have
\[ p_\lambda(L(i - 1)) = (-1)^{\sum_{k=1}^{L(i-1)} \lambda_k} = 1, \]
and then
\[ p_\lambda(L(i - 1) + 1) = (-1)^{\sum_{k=1}^{L(i-1)} \lambda_k + \lambda_{L(i-1)+1}} = -1, \]
which means
\[ \mu_{L(i-1)+1} = \lambda_{L(i-1)+1}. \]
We can also get this result using Lemma 1 directly. So we reach the right hand side of Fig.(12). In fact, this result is Fig.(6)(b).

The first row of a $B_n$ or $D_n$ partition is not in a pairwise rows. Since the number of boxes of a $B_n$ partition is odd, $p_\lambda = -1$ for the last part, which means we should delete the last box of the fist row according to Lemma 1. Since the number of boxes of a $D_n$ partition is even, $p_\lambda = 1$ for the last part, which means nothing change for
Fingerprint

\[
\begin{align*}
\mu &= X_S \rho_{\text{odd}} + \rho_{\text{even}}.
\end{align*}
\]

For a \( D_n \) partition \( \rho \), the \( \mu \) partition is

\[
\mu = Y_S \rho_{\text{odd}} + \rho_{\text{even}}.
\]

Fingerprint

Figure 13. (a) Image of an odd pairwise rows following odd pairwise rows under the map \( \mu \). (b) Image of an odd pairwise rows following even pairwise rows under the map \( \mu \).

Now, we construct fingerprint \([\alpha, \beta]\). The condition '\((iii)_{SO} \lambda_i' is odd' can not be satisfied for even parts. The pairwise rows above an odd pairwise rows can be an even pairwise rows or an odd pairwise rows as shown in Fig. 13. We compute the fingerprint for the parts \( i^n_i (i - 1)^{n_i-1} \). For Fig. 13(a), both \( i \) and \( n_i \) are odd since the pairwise rows following are even. The fingerprint of the parts \( i^n_i - 1 \) of \( \mu \) is

\[
(i^n_i - 1) \frac{n_i}{2}.
\]

Since the part \( i - 1 \) satisfy the condition 2.2, the fingerprint of the parts \( (i-1)^{n_{i-1}+2} \) is

\[
(\frac{i - 1}{2})^{n_{i-1}+2}.
\]

For Fig. 13(a), the fingerprint is denoted as

\[
F_{oe} = [i^{n_i}, (\frac{i - 1}{2})^{n_{i-1}+2}].
\]

For Fig. 13(b), \( n_i \) is even since the pairwise rows following are odd. The fingerprint of the odd parts \( i^{n_i-2} \) is

\[
(i^{n_i-2}) \frac{n_i}{2}.
\]

Since the part \( i - 1 \) satisfy the condition 2.2, the fingerprint of the parts \( (i-1)^{n_{i-1}+2} \) is

\[
(\frac{i - 1}{2})^{n_{i-1}+2}.
\]

For Fig. 13(b), the fingerprint is denoted as

\[
F_{oo} = [i^{n_i-2}, (\frac{i - 1}{2})^{n_{i-1}+2}].
\]
For both cases, (3.4) is equal to (3.6).

\[
\begin{array}{c}
\text{(a)} \\
\begin{array}{c}
\text{\includegraphics[width=0.4\textwidth]{figure14a.png}}
\end{array}
\end{array}
\begin{array}{c}
\text{(b)} \\
\begin{array}{c}
\text{\includegraphics[width=0.4\textwidth]{figure14b.png}}
\end{array}
\end{array}
\]

**Figure 14.** (a), Image of an even pairwise rows following even pairwise rows under the map \( \mu \). (b), Image of an even pairwise rows following even pairwise rows under the map \( \mu \).

The pairwise rows above an even pairwise rows can be an even pairwise rows or an odd pairwise rows as shown in Fig. 14. For Fig. 14(a), \( n_i \) is even since the pairwise rows following are even. The fingerprint of the odd parts \( i^{n_i} \) is

\[
\left( i^{\frac{n_i}{2}} \right) = (3.7)
\]

And the fingerprint of the even parts \( (i - 1)^{n_i - 1} \) is

\[
\left( (i - 1)^{\frac{n_i - 1}{2}} \right) = (3.8)
\]

For Fig. 14(b), \( n_i \) is odd since the pairwise rows following are odd. The fingerprint of the odd parts \( i^{n_i} \) is

\[
\left( i^{\frac{n_i - 1}{2}} \right) = (3.9)
\]

And the fingerprint of the even parts \( (i - 1)^{n_i - 1} \) is

\[
\left( (i - 1)^{\frac{n_i - 1}{2}} \right) = (3.10)
\]

For both cases the formula (3.8) is equal to the formula (3.10). Both the formula (3.7) and the formula (3.10) can be denoted as \( i^{\frac{n_i}{2}} \). Combined the formulas (3.7), (3.8), (3.9) and (3.10), the fingerprint of the parts \( i^{n_i} (i - 1)^{n_i - 1} \) is

\[
F_o = [1^{\frac{n_i - 1}{2}}, 0]
\]

Finally, we discuss the fingerprint of the parts \( 1^{n_1} \) in the \( B_n \) and \( D_n \) theories. Since the number of the boxes of a \( B_n \) partition is odd, the sign of the last part of the first row is \( '+' \), which means a box is deleted under the map \( \mu \). We can refer to the change of the second row of a odd pairwise rows as shown in Fig. 13. If the row following the first row is even, according to Fig. 14(a), the fingerprint is

\[
[1^{\frac{n_i - 1}{2}}, ]
\]

If the row following the first row is odd, according to Fig. 14(b), the fingerprint is

\[
[1^{\frac{n_i - 2}{2}}, ]
\]

Since the number of the boxes of a \( D_n \) partition is even, the sign of the last part of the first row is \( '+' \), which means nothing changes under the map \( \mu \). We can refer to the change of the second row of a odd pairwise rows as shown in Fig. 14. According to Fig. 14(a) and Fig. 14(b), the fingerprint is

\[
[1^{\frac{n_i}{2}}, ]
\]
3.2. $C_n$ theory. Different from the partitions in the $B_n$ and $D_n$ theories, the first two rows of a $C_n$ partition are of a pairwise rows as shown in Fig.(15). If $i$ and $(i - 1)$th rows are of a pairwise rows, according to Fig.(19), $p_\lambda(L(i)) = +1$ and $p_\lambda(L(i - 1)) = +1$ for $\lambda_{L(i)}$ and $\lambda_{L(i-1)}$ at the end of row, respectively. Using Lemma 1, we have

$$\mu = \lambda,$$

as shown in Fig.(16).

The condition '($iii$)$_{SO}$ $\lambda'_i$ is odd' can not be satisfied for even parts. Since $(i - 1)$th row is the first row of a pairwise, $(i - 1)$ is odd and $n_{i-1}$ is even. The parts $(i - 1)^{n_{i-1}}$ contribute to the first factor of the fingerprint as follows

$$(3.13) \quad [(i - 1)^{\frac{n_{i-1}}{2}}, \ ]$$

Since $i$th row is the second row of a pairwise, $i$ is even. The parts $(i)^{n_i}$ contribute to the second factor of the fingerprint as follows

$$(3.14) \quad [i^{\frac{n_i}{2}}, \ ].$$

Combining the formulas $(3.13)$ and $(3.14)$, the fingerprint of $\lambda$ can be written down directly as follows

$$(3.15) \quad [\prod_{i=1}^{\lambda} i^{\frac{n_i}{2}}, \emptyset].$$
4. Fingerprint of rigid semisimple operators

In this section, we propose an algorithm to calculate the fingerprints of rigid semisimple operators \((\lambda', \lambda'')\) in the \(B_n\), \(C_n\), and \(D_n\) theories. We decompose \(\lambda = \lambda' + \lambda''\) into several blocks whose fingerprints can be constructed by using the results of rigid unipotent operators. The fingerprint of the \(C_n\) rigid semisimple operators is a simplified version of that in the \(B_n\) and \(D_n\) theories.

**Figure 17.** Partitions \(\lambda'\) and \(\lambda''\) are in the \(B_n\) and \(C_n\) theories, respectively.

4.1. \(B_n\) theory. \((\lambda', \lambda'')\) is a rigid semisimple operator in the \(B_n\) theory as shown in Fig. (17). The partition \(\lambda\) is constructed inserting rows of \(\lambda'\) into \(\lambda''\) one by one. This procedure would be helpful to understand the construction of the partition \(\mu\).

**Figure 18.** The first row \(a\) of \(\lambda'\) is inserted into the partition \(\lambda''\).

**Figure 19.** Pairwise rows \(b\) and \(c\) of \(\lambda'\) are inserted into the partition \(\lambda''\).

The insertion of the first row \(a\) is shown in Fig. (18). The insertions of \(b\) and \(c\) of a pairwise rows are shown in Figs. (19)(a) and (b). The row \(b\) can be placed...
between or under a pairwise rows, and these three rows form the lower boundary of the block in the box as shown in Fig. (20). Similarly, the row \( c \) can be placed between or above a pairwise rows, and these three rows form the upper boundary of the block in the box. The rows between the boundaries are pairwise rows of \( \lambda'' \).

Since the first row of each pairwise rows is even, they can be regard as the pairwise rows of a \( B_n \) partition. Under these constructions, each block consist of pairwise rows of \( \lambda' \) and \( \lambda'' \), which means the number of total boxes is even. These processes continue until all the rows of \( \lambda' \) are inserted into \( \lambda'' \). The partition \( \lambda = \lambda' + \lambda'' \) is decomposed into several blocks, consisting of types of blocks such as \( I, II, III \) as shown in Fig. (20). We would refine this model in the end.

**Figure 20.** Partition \( \lambda \) and blocks.

**III type block with operators** \( \mu_{e11}, \mu_{e12}, \mu_{e21}, \) and \( \mu_{e22} \)

The fingerprint can be constructed by summing the contribution of each block. First we consider the block of type III as shown in Fig. (20). Let the pairwise rows \( b \) and \( c \) of \( \lambda' \) in the block are even as shown in Fig. (21). The row \( c \) is the second row of the pairwise rows, forming the upper boundary of the block with an odd pairwise rows of \( \lambda'' \). The row \( b \) is the first row of the pairwise rows, forming the lower boundary of the block with an odd pairwise rows of \( \lambda'' \). The number of boxes of rows above the block is even. And the number of boxes under the first row of the block is also even. So the sign of \( p(i) \) corresponding to the last part of a low is determined by the number of boxes in the block before the part. Since the row \( c \) inserted is even, the number of boxes of the upper boundary is even. So the sign of the last part of each row is the same as that of row of pairwise rows of the \( B_n \)
An even pairwise rows $c$, $b$ of $\lambda'$ are inserted into the partition $\lambda''$. 

The first row and the last row will not change under the map $\mu$.

1. The height of $l6$ is even, so the sign of the last part of the row $l6$ is $'+$'.
2. The total number of boxes of each block is even, so the sign of the last part of the block is $'+$' in Fig. (21).

These results are right even though an odd pairwise are inserted into $\lambda''$.

For Fig. (22)(a), row $c$ is above a pairwise rows.

1. The height of $l5$ are odd, so the sign of the last part of row $l5$ is $'-'$.
2. According to Fig. (4), the sign of the last part of row $l4$ is $'-'$.

If the row $c$ is inserted into the middle of the pairwise rows, we can reach to Fig. (21)(b) similarly.

For Fig. (22)(c), the number of boxes above the row $l3$ is even. So the sign of the last parts of $l3$, $l2$, and $l1$ are determined by the number of boxes before the last part.

1. The height of the row $l3$ is odd, so the sign of the last part $l3$ is $'-'$, which means the last box of the row is deleted under the map $\mu$.
2. The height of the row $l2$ is even, so the sign of the last part of $l2$ is $'-'$, which means we should append a box as the last part of the row under the map $\mu$.

If the row $b$ is inserted into the middle of the pairwise rows, we can reach to Fig. (21)(d) similarly.

According to Fig. (21), the first and the last rows of the block do not change under the map $\mu$. The rows $l2$ and $l3$ change as an odd pairwise rows in the $B_n$ theory as well as the rows $l4$ and $l5$. If we replace the rows $l2$ and $l3$ by even rows, they would change as an even pairwise rows in the $B_n$ theory as well as the rows $l4$ and $l5$.

According to the places where the rows $b$ and $c$ are inserted into $\lambda''$, we define four operators $\mu_{e11}$, $\mu_{e12}$, $\mu_{e21}$, and $\mu_{e22}$ as shown in Fig. (23), corresponding to all the combinations of the upper boundaries Fig. (22)(a) Fig. (22)(b) and the
low boundaries Fig. (22)(c), Fig. (22)(d). For example, the upper boundary of the operators $\mu_{c11}$ is Fig. (21)(a) and the low boundary is Fig. (21)(d).

**III type block with operators $\mu_{o11}, \mu_{o12}, \mu_{o21},$ and $\mu_{o22}$**

Next, we consider an odd pairwise rows in $\lambda'$ is inserted into $\lambda''$, comprising a block as shown in Fig. (22). The row $c$ is the second row of the pairwise rows, forming the upper boundary of the block with an even pairwise rows of $\lambda''$. The row $b$ is the first row of a pairwise rows, forming the lower boundary of the block with an even pairwise rows of $\lambda''$. So the numbers of boxes of lower boundary and upper boundary of block are odd. The numbers of boxes above the block are even. *So the sign of the last box of a low in the block is determined by the number of boxes before the part.*

![Figure 22](image-url)

**Figure 22.** An odd pairwise rows $c, b$ of $\lambda'$ are inserted into the partition $\lambda''$.

First, we determine the changes of the upper boundary under the map $\mu$ for Fig. (22)(a).

1. The height of the row $l_6$ is even, so the sign of the last box of row $l_6$ is $'+$'.
2. The number of boxes of $l_6$ is odd. The height of the row $l_5$ is odd and the number of boxes of the row $l_5$ is even. So the sign of the last part of row $l_5$ is $'-'$.
3. According to Fig. (4), the sign of the last box of row $l_4$ is $'-'$.

If the row $c$ is inserted into the middle of the pairwise rows, we can reach to Fig. (22)(b) similarly.

Next we determine changes of pairwise rows of $\lambda''$ between the upper boundary and the lower boundary in the block. Since the number of boxes of upper boundary of block is odd, the even pairwise rows and the odd pairwise rows exchange roles compared with Fig. (22). So we should append a box as the last part of the first row of an even pairwise rows and delete the last box of the second one. While rows of an odd pairwise rows do not change.

Finally we determine the changes of the lower boundary under the map $\mu$ for Fig. (22)(c). The number of boxes above the row $l_3$ is odd since the inserted row $c$ in the upper boundary is odd. The sign of the last boxes of $l_3, l_2,$ and $l_1$ are determined by the number of boxes before the corresponding part.
(1) The height of the row $l3$ is odd and the length is even. So the sign of the last box of $l3$ is $'-'$.

(2) The height of the row $l2$ is even and the length is odd, so the the sign of the last box of $l2$ is $'-'$.

If the row $b$ is inserted into the middle of the pairwise rows, we can reach Fig. (21) (d) similarly. Since the block consist of pairwise rows of $\lambda'$ and $\lambda''$, the total number boxes of the block is even, which means the sign of the last box of the block is $'+'$ in Fig. (22) (a), (b), (c), and (d).

According to Fig. (22), the first row and the last row of the block do not change under the map $\mu$. The rows $l2$ and $l3$ change as an odd pairwise rows in the $B_n$ theory as well as the rows $l4$ and $l5$. If we replace the rows $l2$ and $l3$ by two odd rows, they change as an even pairwise rows in the $B_n$ theory as well as the rows $l4$ and $l5$. Summary, compared with Fig. (21), nothing changes in Fig. (22) excepting the even pairwise and the odd pairwise exchange roles. Similarly, we can define operators $\mu_o11$, $\mu_o12$, $\mu_o21$, and $\mu_o22$.

**Fingerprint of III type block**

![III type block](image)

**Figure 23.** III type block.

Now, we calculate the fingerprints of the blocks as shown in Fig. (23). A comparison of a $B_n$ partition and the III type block is made previously. The III type block have the exactly the same structure with the $B_n$ partition under the map $\mu$, excepting the first and the last row. For example, the condition '$(iii)_{SO} \lambda_i$ is odd' can not be satisfied for even parts of the block since the height of the inserted row $b$ in $\lambda'$ is even. For the pairwise rows of $\lambda''$ in the block, the height of the first row is even. So we can calculate the fingerprint of the III type blocks using the formulas in Section 3. Since the first row change nothing under the map $\mu$, we should use the formula (3.12) to calculate the fingerprint of the last part of the block. The row following the last row of a block do not change under the map $\mu$. But the height satisfy the condition '$(iii)_{SO} \lambda_i$ is odd'. So the fingerprint can be calculated by the formula (3.14).
II type block

Next we calculate the fingerprint of II type blocks as shown in Fig. (25). The height of the first row of the block is odd and all rows are in pairwise pattern, which are exactly the same with a $C_n$ partition. So the fingerprint of this block can be calculated by using the formula (3.15). We define the operator $\mu_{II}$ corresponding to image of the block under the map $\mu$.

\[\mu_{II}\]

Figure 25. II type block of $\lambda$ and its image under the map $\mu$.

I type block

For I type block, the first row $a$ of $\lambda'$ is inserted into $\lambda''$ as shown in Fig. (26) or the first row of $\lambda''$ is inserted into $\lambda'$. The row $a$ is above an even pairwise rows in Fig. (26)(a) and is between an even pairwise rows in Fig. (26)(b). All the rows are in pairwise pattern except the first row of $\lambda'$ which is odd and the first row of $\lambda''$ which is even according to Proposition 1 and 3. So the number of the boxes of the I type block is odd.

Under the map $\mu$, the images of the first row and the last row of I type block are independent of the position of $a$ which are similar to III type block.

- Since the number of the boxes of the I type block is odd, the sign of the last part of the first row is $'-'$, which means we should delete a box at the end of the first row according to Lemma 1.
- Since the number of the boxes above I type block is even and the height of the last row is even, the sign of the last box of the last row is $'+$', which means the last row does not change under the map $\mu$. 

Figure 26. A comparison between a $B_n$ partition under the map $\mu$ and the block $\mu_{e11}$.
For the other rows, the arguments for the partition $\mu$ are exactly the same with the blocks in Fig. (22). Comparing Fig. (26)(a) with Fig. (22)(a), we define the operator $\mu_{o2}$ corresponding to Fig. (26)(a). Comparing Fig. (26)(b) with Fig. (22)(b), we can define the operator $\mu_{o1}$ corresponding to Fig. (26)(b).

**Figure 26.** The first $I$ type block of $\lambda$ and its image under $\mu$.

If the first row of $\lambda''$ is inserted into $\lambda'$ in the $I$ type block, the rows of $\lambda'$ are represented by white rows and the rows of $\lambda''$ are represented by gray rows. Then the $I$ type block is exactly the same as shown in Fig. (26). The row $a$ is even, which is the first row of $\lambda''$. Comparing Fig. (26)(a) with Fig. (23)(a), we define the operator $\mu_{e1}$ corresponding to Fig. (26)(a). Comparing Fig. (26)(b) with Fig. (23)(b), we define the operator $\mu_{e2}$ corresponding to Fig. (26)(b).

With the partition $\mu$ of the $I$ type block, the fingerprint can be calculated using the formulas of the fingerprint of the $III$ type block directly.

**Special blocks**

**Figure 27.** Blocks of $\lambda$.

We ignore one situation in the above discussions. If the first row of a block is in the partition $\lambda'$ and the second one is in the partition $\lambda''$, then the row under the block would be in another block as shown in Fig. (27). However, if the first two rows
of a block are in the partition \( \lambda' \), the situation become complicated as shown in Fig. (28). The fingerprint of the III type block can be still calculated by previous formulas except the first row. We determine the block \( S \) by including one row in \( \lambda' \) and even number of rows in \( \lambda'' \) as little as possible. Unfortunately, we could not use the operators \( \mu_{e11}, \mu_{e12}, \mu_{e21}, \mu_{e22} \) and their odd versions to determine the images of the blocks \( S \) under the map \( \mu \). So we must calculate the partition \( \mu \) of block \( S \) case by case. Since the block \( S \) and the following III type block are consist of pairwise rows of \( \lambda' \) and \( \lambda'' \), the sign of the last box of the first row of the block \( S \) is \( '+' \), which means this first row would not change under the map \( \mu \).

4.2. \( D_n \) theory. For a rigid semisimple operator \((\lambda', \lambda'')\) in the \( D_n \) theory, the fingerprint can be constructed by the same arguments as that of a partition in the \( B_n \) theory. The I type block of \( \lambda = \lambda' + \lambda'' \) need to be paid more attentions since both \( \lambda' \) and \( \lambda'' \) are in the \( D_n \) theory and their first rows are even. We define operators \( \mu_{e1} \) and \( \mu_{e2} \) corresponding to Fig. (26)(a) and Fig. (26)(b) for the I type block, respectively.

Since the total number of the boxes of \( \lambda \) is even, the sign of the last part of \( \lambda \) is \( '+' \), which means nothing change under the map \( \mu \) for the first row of \( \lambda \). The fingerprints of other parts can be calculated by the same exactly formulas as in the \( B_n \) theory.

4.3. \( C_n \) theory. For a rigid semisimple operator \((\lambda', \lambda'')\) in the \( C_n \) theory, both \( \lambda' \) and \( \lambda'' \) are in the \( C_n \) theory. The first two rows of \( C_n \) partitions are in pairwise pattern as shown in Fig. (29). Compared Fig. (30) with Fig. (20), the construction of fingerprint is even simpler than that of the \( B_n \) and \( D_n \) theories because no I type block appears in \( \lambda \). And the fingerprints of other parts can be calculated by the same exactly formulas as in the \( B_n \) and \( D_n \) theories.

**Figure 28.** \( S \) type block.

**Figure 29.** Partitions \( \lambda' \) and \( \lambda'' \) in the \( C_n \) theory
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