The second Betti number of hyperkähler manifolds

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Abstract

Let $M$ be a compact irreducible hyperkähler manifold, from Bogomolov inequality [VII] we obtain forbidden values of the second Betti number $b_2$ in arbitrary dimension.

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1 Introduction

One of the main conjectures in the theory of hyperkähler manifolds is finiteness of number of classes of hyperkähler manifolds up to deformation in each dimension [Bea1]. Huybrechts [H1] has proved finiteness of the number of deformation classes of holomorphic symplectic structures on each smooth manifold. Moreover, in most dimensions as we see below we only know the two examples due to Beauville [Bea1]. Given the properties of a hyperkähler structure, one might be led to suspect that there can not be many more. In two dimensions there are two more examples constructed by O’Grady [O1, O2]. The conjecture is that there are finitely many hyperkähler manifolds up to the deformation in each dimension. It is well-known, e.g. [H2], for compact hyperkähler manifolds the second Betti number $b_2 \geq 3$. Thanks to Guan [G] it is known for 4-dimensional hyperkähler manifolds that $3 \leq b_2 \leq 8$ or $b_2 = 23$ and also there are some bounds on $b_3$. The conjecture is that the second Betti number is bounded in any dimension. In the present paper, we obtain results about the Betti numbers of compact hyperkähler manifolds and state the following

Theorem: Let $M$ be a compact irreducible hyperkähler manifold of complex dimension $n$. Then

$$b_2 \neq \frac{10+n^2-n}{2},$$

where $b_2$ is the second Betti number of $M$. 

1 The author is partially supported by the Simons Foundation grant and AG Laboratory NRU-HSE, RF government grant, ag. 11.G34.31.0023.

2010 Mathematics Subject Classification. 14K22, 53C26.
2 Hyperkähler manifolds

Definition 2.1: (Bes) Let $(M, g)$ be a Riemannian manifold, and $I, J, K$ are endomorphisms of a real tangent bundle satisfying the relation $I \circ J = -J \circ I = K$. If the metric $g$ on $M$ is Kähler with respect to these complex structures than $M$ is called a hyperkähler manifold.

Clearly, complex dimension of hyperkähler manifold $M$ is even. A compact Kähler manifold is hyperkähler (HK) if it is simply connected and the space of its global holomorphic two-forms is spanned by a symplectic form. In algebraic geometry the word "hyperkähler" is synonymous with "holomorphic symplectic".

Definition 2.2: A manifold $M$ is called holomorphically symplectic if it is a complex manifold with a closed holomorphic 2-form $\Omega$ over $M$ such that $\Omega^n = \Omega \wedge \Omega \wedge \ldots \wedge \Omega$ is a nowhere degenerate section of a canonical class of $M$, where $2n = \dim_M(M)$.

Consider the Kähler forms $w_I(\cdot, \cdot) := g(\cdot, I\cdot), w_J(\cdot, \cdot) := g(\cdot, J\cdot), w_K(\cdot, \cdot) := g(\cdot, K\cdot)$ on $M$. A simple algebraic calculation [Bes] shows that the following form

$$\Omega = w_J + \sqrt{-1}w_K$$

is of type $(2, 0)$ on $(M, I)$. It is closed, holomorphic and moreover nowhere degenerate, as another linear algebraic argument shows, hence it is a holomorphic symplectic form. Thus, the underlying complex manifold $(M, I)$ is holomorphically symplectic. The converse assertion is also true:

Theorem 2.3: (Bea2, Bes) Let $M$ be a compact, Kähler, holomorphically symplectic manifold with the holomorphic symplectic form $\Omega$, $w$ its Kähler form, $n = \dim_M M$. Assume that $\int_M w^n = \int_M (Re\Omega)^n$. Then there is a unique hyperkähler structure $(I, J, K, g)$ over $M$ such that the cohomology class of the symplectic form $w_I = g(\cdot, I\cdot)$ is equal to $w$ and $\Omega = w_J + \sqrt{-1}w_K$.

The Bogomolov-Beauville-Fujiki form was defined in [Bo] and [Bea2], but it is easiest to describe it using the Fujiki theorem.

Theorem 2.4: (F) Let $M$ be a simple hyperkähler manifold, $\eta \in H^2(M)$, and $2n = \dim M$. Then $\int_M \eta^{2n} = \lambda q(\eta, \eta)^n$, for some primitive integer quadratic form $q$ on $H^2(M, \mathbb{Z})$, and $\lambda > 0$ an integer number.

Fujiki formula [Theorem 2.4] determines the form $q$ uniquely up to a sign. For odd $n$, the sign is unambiguously determined as well. For even $n$, one singles out one of the two choices by imposing the inequality

$$q(\Omega, \bar{\Omega}) > 0, \quad 0 \neq \Omega \in H^{2,0}(M)$$
The two-dimension examples of irreducible holomorphic symplectic manifolds are called K3 surfaces. In higher dimensions there are only few examples known. Here is the list of known examples, where manifolds of the same deformation type are not distinguished.

(i) If X is a K3 surface then the Hilbert scheme Hilb^n(X) is an irreducible holomorphic symplectic manifold [Bea2]. Its dimension is 2n and for n > 1 its second Betti number is equal to 23. Construction of Hilbert Scheme can be found elsewhere, e.g. [Bea1, Bea2]. Let X be a K3 surface. Take the symmetric product X(r) = Xr/Sr which parametrizes subsets of r points in K3 surface X, counted with multiplicities; it is smooth on the open subset X0 consisting of subsets with r distinct points, but singular otherwise. We obtain a smooth compact manifold, which is called the Hilbert scheme X[r] if we replace “subset” by “subspace”. The natural map X[r] → X(r) is an isomorphism above X0, but it resolves the singularities of X(r).

Let us describe the easiest case Hilb^2(X) explicitly. For any surface X the Hilbert scheme Hilb^2(X) is the blow-up Hilb^2(X) → S^2(X) of the diagonal Δ = \{x, x| x ∈ X\} ⊂ S^2(X) = \{x, y| x, y ∈ X\}. Equivalently, Hilb^2(X) is the Z/2Z-quotient of the blow-up of the diagonal in X × X. Since for a K3 surface there exists only one Z/2Z-invariant two-form on X × X, the holomorphic symplectic structure on Hilb^2(X) is unique.

(ii) If X is a complex torus of dimension two, then the generalized Kummer variety K^n(X) is an irreducible holomorphic symplectic manifold [Bea2]. Its dimension is 2n and for n > 2 its second Betti number is 7. The Hilbert scheme X[n] of two-dimensional torus has the same properties as X[r], but it is not simply connected. This is fixed by considering the composition of maps: X[n] → X(r) → X, where the last map is s(t₁, ..., tₙ) = t₁ + ... + tₙ. The fibre K^n(X) is a hyperkähler manifold of dimension 2n - generalized Kummer manifold.

(iii) O’Grady’s 10-dimensional example [O1]. Let again S be a K3 surface, and M the moduli space of stable rank 2 vector bundles on S, with Chern classes c₁ = 0, c₂ = 4. It admits a natural compactification M, obtained by adding classes of semi-stable torsion free sheaves. It is singular along the boundary, but O’Grady [O1] constructs a desingularization of M which is a new hyperkähler manifold, of dimension 10. Its second Betti number is 24 [R]. Originally, it was proved that it is at least 24 [O1].

(iv) O’Grady’s 6-dimensional example [O2]. The similar construction can be done starting from rank 2 bundles with c₁ = 0, c₂ = 2 on a 2-dimensional complex torus, that gives a new hyperkähler manifold of dimension 6 as in (iii). Its second Betti number is 8.

Thus we have two series, (i) and (ii), and two sporadic examples, (iii) and (iv). All of them have different second Betti numbers. It has been proved [KLS] that the moduli spaces for all other sets of numerical parameters unless Hilb^n(K3) do not admit a smooth symplectic resolution of singularities. Note that in any given dimension and for any given second Betti number b₂ one knows at most one real manifold carrying the structure of an irreducible holomorphic symplectic manifold. So,

Conjecture 2.5: ([Bea1][S]) The number of deformation types of compact irreducible hyperkähler is finite in any dimension (at least for given b₂).
3 On the Betti numbers of hyperkähler manifolds

Let \( M \) be a compact Kähler manifold of complex dimension \( n \). The Hodge number \( h^{p,q} \) denotes the dimension of the corresponding Dolbeault cohomology
\[
h^{p,q} = h^{n-p,n-q} = h^{q,p}
\]
If \( M \) be a compact connected hyperkähler manifold of real dimension \( 4m \). We can find out more equalities on Hodge numbers. By studying the action of \( Sp(m) \) on spaces of harmonic forms, Wakakuwa \( \text{[W]} \) proved that \( b_{2k} \geq \binom{k+2}{2} \) for \( k \leq m \) and that the odd Betti numbers \( b_{2k+1} \) of \( M \) are all divisible by 4. Moreover, Fujiki using Hodge decompositions refined relative to a choice of complex structure. He proved \( \text{[F]} \) that \( h^{p,q} \geq h^{p+1,q} - 1 \) whenever \( p \geq q \).

Note: The first Betti number \( b_1 \) for compact irreducible hyperkähler manifolds is always zero, but in general odd Betti number not, e.g. \( b_3(K_2(T)) = 8 \).

Wedging with the holomorphic symplectic form \( \Omega = w_J + \sqrt{-1} w_K \) induces a mapping \( H^{p,q} \rightarrow H^{p+2,q} \) which is injective for \( p + 1 \leq m \) and its \( (m - p) \)-fold iteration is an isomorphism. In this way, the equalities on Hodge numbers above are supplemented by the equations
\[
h^{p,q} = h^{2m-p,q}, 0 \leq p, q \leq 2m.
\]
Proof of these results has been given by Verbitsky \( \text{[V2]} \) by considering the action of the Lie algebra \( \mathfrak{so} (5) \) on cohomology.

Salamon proved \( \text{[Sa]} \) that if \( X \) is a compact hyperkähler manifold of dimension \( 2n(= 4m) \) then
\[
\sum_{i=0}^{4m} (-1)^i \left( 6i^2 - n \left( 6n + 1 \right) b_i \right) = 0
\]
Using Wakakuwa results this shows in particular that \( ne(X) \) is divisible by 24, where \( e(X) \) is the Euler characteristic. Note that for a K3 surface the Euler number is 24. Indeed, we can rewrite Salamon’s relation above in terms of Euler characteristics:
\[
\sum_{i=0}^{4m} (-1)^i 6i^2 b_i = n \left( 6n + 1 \right) e \left( X \right).
\]
The case of 4-dimensional hyperkähler manifolds has been discussed by Guan \( \text{[G]} \).

Theorem 3.1: If \( M \) is an irreducible compact hyperkähler manifold of complex dimension 4, then

- if \( b_2 = 23 \), then \( b_3 = 0 \). The Hodge diamond of \( M \) is the same as that of the Hilbert scheme of pairs of points on a K3 surface.
• if $b_2 \neq 23$, then $b_2 \leq 8$, and if $b_2 = 8$, then $b_3 = 0$. 
• in the case of $b_2 = 7$, $b_3 = 0$ or 8.
• in the case of $b_2 = 3, 4, 5, 6$, then the following cases are possible

| $b_2$ | 3 | 4 | 5 | 6 |
|-------|---|---|---|---|
| $b_3$ | $4l, l \leq 17$ | $4l, l \leq 15$ | $4l, l \leq 9$ | $4l, l \leq 4$ |

• the second Chern class lies in the algebra $H^{(4)}$ generated by $H^2(M)$ iff

$$(b_2, b_3) = (5, 36), (7, 8), (8, 0), (23, 0).$$

To prove the Theorem 3.2 stated in the Introduction we will use Bogomolov inequality [VII]. Denote by $w$ the Kähler form on $M$ and let $n$ be dim$_C M$, then by Gauss-Bonnet formula, the cohomology class of $Tr(\Theta \wedge \Theta)$ can be expressed via $c_1(F), c_2(F)$:

$$\frac{\sqrt{-1}}{2\pi^2} Tr(\Theta \wedge \Theta) = 2c_2(F) - \frac{n-1}{n} c_1^2(F),$$

where $F$ is torsion-free coherent sheaf and $\theta$ is curvature of hyperholomorphic connection on it. Therefore, the following integral (since $c_1 = 0$)

$$\int_M c_2 \wedge w^{n-2} > 0 \quad (1)$$

is positive that gives Bogomolov inequality [VII].

**Theorem 3.2:** Let $M$ be a compact irreducible hyperkähler manifold of complex dimension $n$.

$$b_2 \neq \frac{10+n^2-n}{2},$$

where $b_2$ is the second Betti number of $M$.

**Proof:**

Let $e_1, ..., e_{b_2}$ be an orthonormal basis in $H^2(M)$ with respect to the Bogomolov-Beauville quadratic form $q$. Denote the dual form of $q$ by $Q \in \text{Sym}^2 H^2(M)$. Then

$$Q = e_1^2 + e_2^2 + e_3^2 - \sum_{i=4}^{b_2} e_i^2.$$

Since $c_2 = \mu Q + p$, where $\mu \in \mathbb{Q}$ and $p \in (\text{Sym}^2 H^2(M))^\perp$ then

$$\int_M Q \wedge e_1^{n-2} \neq 0,$$
otherwise Bogomolov inequality (1) is not fulfilled.

By Fujiki formula we obtain

\[ \int_M e_1^2 \wedge e_1^{n-2} = n!, \]
\[ \int_M e_i^2 \wedge e_1^{n-2} = 2 \cdot (n-2)! \quad (i \neq 1). \]

Then using composition of \( Q \) given above and sum up

\[ Q \wedge e_1^{n-2} = n! + 2 \cdot (n-2)! + 2 \cdot (n-2)! - 2 \cdot (n-2)! \cdot (b_2 - 3). \]

After calculations we obtain

\[ Q \wedge e_1^{n-2} = (n-2)! \cdot (10 + n \cdot (n-1)) - 2 \cdot (n-2)! \cdot b_2. \]

Thus,

\[ 10 + n^2 - n \neq 2b_2. \]

\[ \blacksquare \]

In the case of 4-dimensional hyperkähler manifolds \( b_2 \neq 11 \) that has been also proved by Guan (Theorem 3.1) earlier and in the dimension 6 impossible value for \( b_2 \) is 20. Recall that all known examples of hyperkähler manifolds in dimension 6 have \( b_2 \) equal to 7 (O’Grady example), 8 (generalized Kummer manifold \( K_3(X) \)) or 23 (\( \text{Hilb}_3(X) \)).

Acknowledgements

Author would like to thank M. Verbitsky for valuable discussions and ideas.

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