The Faber–Krahn inequality for the Hermite operator with Robin boundary conditions

Francesco Chiacchio · Nunzia Gavitone

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Abstract
In this paper we prove a Faber–Krahn type inequality for the first eigenvalue of the Hermite operator with Robin boundary conditions. We prove that the optimal set is a half-space and we also address the case of equality.

Keywords Faber–Krahn inequality · Hermite operator · Robin boundary conditions

Mathematics Subject Classification 35P15 · 35J25

1 Introduction

This paper deals with the eigenvalue problem for the Hermite operator with Robin boundary conditions. Let us denote by

\[ d\gamma_N = \phi_N(x) \, dx \]

the normalized Gaussian measure in \( \mathbb{R}^N \), where its density, \( \phi_N(x) \), is given by

\[ \phi_N(x) = \frac{1}{(2\pi)^{N/2}} \exp \left( -\frac{|x|^2}{2} \right). \]
Let $\Omega$ be a sufficiently smooth domain of $\mathbb{R}^N$ and let $\nu$ be the unit outer normal to $\partial \Omega$. We consider the following eigenvalue problem

$$
\begin{cases}
-\text{div} (\phi_N(x) \nabla u(x)) = \lambda(\Omega) \phi_N(x) u(x) & \text{in } \Omega \\
\frac{\partial u}{\partial \nu} + \beta u = 0 & \text{on } \partial \Omega,
\end{cases}
$$

where $\beta > 0$. Our aim is to prove a Faber–Krahn inequality for the first eigenvalue, $\lambda_1(\Omega)$, of problem (1.1). More precisely, we are interested in finding, in the class of the sets having prescribed Gaussian measure, the one which minimizes $\lambda_1(\Omega)$.

The Faber–Krahn inequality for the first Robin eigenvalue, for the classical Laplace operator, has been established in the planar case in [9]. Subsequently, such a result has been generalized to any dimension in [20] and for the $p$-Laplace operator in [12] (see also [21] and the references therein, for further generalizations). In all these cases the ball turns out to be the optimal set, when the Robin parameter $\beta$ is positive.

For $\beta < 0$, the problem appears more delicate. Indeed many phenomena can occur, depending also on the dimension and the topological properties of the domain (see, e.g., [4, 24, 25] and the references therein).

The difficulties of the problem when $\beta > 0$ are twofold. On the one hand the level sets of the first eigenfunction are not closed, on the other hand, there is no monotonicity of the first eigenvalue with respect to the inclusion of sets, except some special cases (see [12, 27] and the references therein). These facts prevent one from adapting the classical symmetrization methods, which work for Dirichlet boundary conditions. In order to overcome these difficulties, in the papers quoted above, the authors used a different approach. More precisely, in [9], Bossel introduced a sort of “desymmetrization” technique together with a representation formula for the first Robin eigenvalue. Concerning problem (1.1), some further issues arise since, in general, the domain $\Omega$ is unbounded and the first eigenfunction is not in $L^\infty(\Omega)$. These circumstances do not allow us to adapt, in a straightforward way, Bossel’s arguments. Our analysis requires a detailed study of problem (2.5) in one dimension for half-lines (see Section 3) and a suitable modification of the proof of the representation formula (see Section 4).

To motivate the present note, recall that the Hermite operator enters in the description of the harmonic oscillator in quantum mechanics (see, e.g., [13] and the references therein). Moreover the interest in the Hermite operator comes from the fact that Gaussian measure in $\mathbb{R}^N$ can be obtained as a limit, as $k$ goes to infinity, of the normalized surface measures on $S_{k+1}^{k+N}$, the sphere in $\mathbb{R}^{k+N+2}$ of radius $\sqrt{k}$ (a process known in literature as “Poincaré limit”).

Note further that the Robin problem is often regarded as interpolating between the Dirichlet ($\beta = 0$) and Neumann ($\beta = +\infty$) cases.

As far as the Hermite operator is concerned, when $\Omega$ varies in the class of domains with fixed Gaussian measure, it is known (see [5, 22]) that the set which minimizes the first Dirichlet eigenvalue is given by a half-space. Hence it coincides with the isoperimetric set in the Gaussian Isoperimetric inequality (see, e.g., [8, 14, 17, 32] and the references therein). On the other hand, when Neumann boundary conditions are
imposed, the situation is quite different. As is well known, the problem of minimizing the first non-trivial Neumann eigenvalue, \( \mu_1(\Omega) \), is meaningless, and, hence one tries to maximize it instead. In [15] it has been proved that among all smooth domains with prescribed Gaussian measure, symmetric with respect to the origin the ball centered at the origin maximizes \( \mu_1(\Omega) \). Furthermore in [15] it is shown that, even removing such a topological assumption, the half spaces do not maximize \( \mu_1(\Omega) \).

This phenomenon is quite surprising since in the Euclidean case the ball minimizes the first Dirichlet eigenvalue (the classical Faber–Krahn inequality) and, at same time, it maximizes the first nontrivial Neumann eigenvalue (the classical Szegö-Weinberger inequality).

So it is natural to investigate which is the optimal set in the “Gaussian–Faber–Krahn inequality” when Robin boundary conditions are imposed.

We finally point out that in [19] the authors prove an isoperimetric inequality for the Robin torsional rigidity related to the Hermite operator.

Our main result, Theorem 1.1 below, requires the validity of some functional embedding Theorem. In Section 2 (see definition 2.1) we introduce and describe the family \( G \) of those domains of \( \mathbb{R}^N \) for which such results hold true.

**Theorem 1.1** Let \( \Omega \in G \), with \( G \) as in Definition 2.1. Then

\[
\lambda_1(\Omega^\#) \leq \lambda_1(\Omega), \tag{1.2}
\]

where \( \Omega^\# = S_{\sigma^\#} \) is any half-space such that \( \gamma_N(\Omega^\#) = \gamma_N(\Omega) \). Moreover, equality holds in (1.2) if and only if \( \Omega \) is a half-space, modulo a rotation about the origin.

The paper is organized as follows. Section 2 contains the isoperimetric inequality with respect to the Gaussian measure, along with some preliminary results on the eigenvalue problem (1.1). In the third section we analyze the problem in one dimension for half-lines. Among other things, we prove the log-concavity of any positive first eigenfunction. We also deduce that in one dimension the eigenvalue is monotone with respect to the inclusion of sets (half-lines in this case). Note that in \( \mathbb{R}^N \), with \( N \geq 2 \), for the Laplace operator, such a property holds true for concentric and non-concentric balls (see [3,27]).

Section 4 is devoted to the representation formula for the first eigenvalue of the problem (1.1). Finally, in the last section, we prove our main result, Theorem 1.1.

### 2 Preliminary results

#### 2.1 The Gaussian isoperimetric inequality

Let \( \phi_N(x) \) denotes the density of the normalized Gaussian measure in \( \mathbb{R}^N \), i.e.

\[
\phi_N(x) = \frac{1}{(2\pi)^{N/2}} \exp\left(-\frac{|x|^2}{2}\right).
\]
One defines the Gaussian perimeter of any Lebesgue measurable set $A$ of $\mathbb{R}^N$ as follows:

$$
P_{\phi_N}(A) = \begin{cases} 
\int_{\partial A} \phi_N(x) \, d\mathcal{H}^{N-1}(x) & \text{if } \partial A \text{ is } (N-1) - \text{rectifiable} \\
+\infty & \text{otherwise}, 
\end{cases}
$$

where $d\mathcal{H}^{N-1}$ denotes the $(N-1)$—dimensional Hausdorff measure in $\mathbb{R}^N$. The Gaussian measure of $A$ is given by

$$
\gamma_N(A) = \int_A \phi_N(x) \, dx \in [0, 1]. \quad (2.1)
$$

The celebrated Gaussian isoperimetric inequality (see [8, 22, 32]) states that among all Lebesgue measurable sets in $\mathbb{R}^N$, with prescribed Gaussian measure, the half-spaces minimize the Gaussian perimeter. Furthermore the isoperimetric set is unique, clearly, up to a rotation with respect to the origin (see [14, 17]).

Let $a \in \mathbb{R}$, we use the notation

$$
S_a := \{ x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^N : x_1 < a \}. \quad (2.2)
$$

If $\Omega \subset \mathbb{R}^N$ is a Lebesgue measurable set, its Gaussian symmetrization, $\Omega^\#$, is

$$
\Omega^\# = S_{\sigma^\#} \quad (2.3)
$$

where $\sigma^\#$ is such that

$$
\gamma_N(\Omega) = \gamma_N(\Omega^\#) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\sigma^\#} \exp \left( -\frac{t^2}{2} \right) \, dt = \frac{1}{2} + \frac{1}{2} \text{erf} \left( \frac{\sigma^\#}{\sqrt{2}} \right), \quad (2.4)
$$

where erf stands for the standard error function.

The isoperimetric function in Gauss space, $g(s)$, is given by

$$
g : s \in [0, 1] \rightarrow \frac{1}{\sqrt{2\pi}} \exp \left[ -\left( \text{erf}^{-1}(2s - 1) \right)^2 \right].
$$

Note, indeed, that the Gaussian perimeter of any half-space of Gaussian measure $s \in [0, 1]$ is equal to $g(s)$. The isoperimetric property of the half-spaces can finally be stated also in the following analytic form.

**Theorem 2.1** If $\Omega \subset \mathbb{R}^N$ is any Lebesgue measurable set, it holds that

$$
P_{\phi_N}(\Omega) \geq P_{\phi_N}(\Omega^\#) = g(\gamma_N(\Omega)),
$$

where equality holds if and only if $\Omega$ is equivalent to a half-space.
2.2 The Sobolev space $H^1(\Omega, \phi_N)$

Let $\Omega$ be an open connected subset of $\mathbb{R}^N$. We will denote by $L^2(\Omega, \phi_N)$ the set of all real measurable functions defined in $\Omega$, such that

$$\|u\|_{L^2(\Omega, \phi_N)}^2 := \int_{\Omega} u^2(x)\phi_N(x)dx < +\infty.$$ 

For our future purposes we need also to introduce the following weighted Sobolev space

$$H^1(\Omega, \phi_N) := \left\{ u \in W^{1,1}_{\text{loc}}(\Omega) : (u, |\nabla u|) \in L^2(\Omega, \phi_N) \times L^2(\Omega, \phi_N) \right\},$$

endowed with the norm

$$\|u\|_{H^1(\Omega, \phi_N)} = \|u\|_{L^2(\Omega, \phi_N)} + \|\nabla u\|_{L^2(\Omega, \phi_N)}.$$

In the sequel of the paper, we need to introduce the following family of sets.

**Definition 2.1** A Lipschitz domain $\Omega \subset \mathbb{R}^N$ belongs to $\mathcal{G}$ if $\gamma_N(\Omega) \in (0, 1)$ and the following conditions are fulfilled:

(i) $H^1(\Omega, \phi_N)$ is compactly embedded in $L^2(\Omega, \phi_N)$.

(ii) The trace operator $T$

$$T : u \in H^1(\Omega, \phi_N) \rightarrow u|_{\partial\Omega} \in L^2(\partial\Omega, \phi_N),$$

is well defined;

(iii) The operator $T$ is compact from $H^1(\Omega, \phi_N)$ onto $L^2(\partial\Omega, \phi_N)$.

In (ii) and (iii) the functional space $L^2(\partial\Omega, \phi_N)$ is endowed with the norm

$$\|u\|_{L^2(\partial\Omega, \phi_N)}^2 = \int_{\partial\Omega} u^2(x)\phi_N(x)d\mathcal{H}^{N-1}(x).$$

**Remark 2.1** Observe that $\mathcal{G} \neq \emptyset$. Indeed it contains, at least, the following families of sets

(j) All the bounded and Lipschitz domains of $\mathbb{R}^N$.

(jj) All the convex domains of $\mathbb{R}^N$, not necessarily bounded.

(jjj) All the Lipschitz domains of $\mathbb{R}^N$, not necessarily bounded, for which an extension Theorem holds true (see, e.g., [11,23,29,30]).

The study of functional inequalities related to the Gaussian measure, because they are often generalizable to infinite-dimensional spaces, has given rise to a rich line of research, starting from the seminal paper by Gross (see [28]). The related bibliography is very wide, we refer the interested reader to [1,2,6,7,10,16,18,31] and the references therein.
2.3 The eigenvalue problem

Let $\Omega \in \mathcal{G}$, with $\mathcal{G}$ as in Definition 2.1. We consider the eigenvalue Robin boundary value problem

\[
\begin{aligned}
-\nabla \cdot (\phi_N(x) \nabla u(x)) &= \lambda(\Omega)\phi_N(x)u(x) \quad \text{in } \Omega \\
\frac{\partial u}{\partial \nu} + \beta u &= 0 \quad \text{on } \partial \Omega,
\end{aligned}
\]

(2.5)

where $\nu$ is the unit outer normal to $\partial \Omega$ and $\beta > 0$. A real number $\lambda(\Omega)$ and a function $u \in H^1(\Omega, \phi_N)$ are an eigenvalue and a corresponding eigenfunction of problem (2.5) if the following equality holds

\[
\int_{\Omega} \langle \nabla u, \nabla \psi \rangle \phi_N \, dx + \beta \int_{\partial \Omega} u \psi \phi_N \, d\mathcal{H}^{N-1}(x) = \lambda(\Omega) \int_{\Omega} u \psi \phi_N \, dx, \quad \psi \in H^1(\Omega, \phi_N).
\]

(2.6)

By the definition of $\mathcal{G}$, one can apply the standard theory on compact and self-adjoint operators to problem (2.5). Hence we can arrange the eigenvalues of (2.5) in a non-decreasing sequence $\{\lambda_n(\Omega)\}_{n \in \mathbb{N}}$, such that $\lim_{n \to \infty} \lambda_n(\Omega) = +\infty$. Moreover, the smallest eigenvalue of (2.5), $\lambda_1(\Omega)$, has the following variational characterization

\[
\lambda_1(\Omega) = \inf_{v \in H^1(\Omega, \phi_N) \setminus \{0\}} J[\beta, v],
\]

(2.7)

where

\[
J[\beta, v] = \frac{\int_{\Omega} |\nabla v|^2 \phi_N \, dx + \beta \int_{\partial \Omega} v^2 \phi_N \, d\mathcal{H}^{N-1}(x)}{\int_{\Omega} v^2 \phi_N \, dx}.
\]

(2.8)

Furthermore the following result holds.

**Theorem 2.2** Let $\Omega \in \mathcal{G}$, with $\mathcal{G}$ as in Definition 2.1 and let $\lambda_1(\Omega)$ be the first eigenvalue of (2.5). Then $\lambda_1(\Omega)$ is simple, its corresponding eigenfunctions are smooth, and they have one sign in $\Omega$. Finally, $\lambda_1(\Omega)$ is the unique eigenvalue having a positive eigenfunction.

**Proof** By classical regularity results the first eigenfunctions are smooth in $\Omega$. Let $u$ be an eigenfunction corresponding to $\lambda_1(\Omega)$. Then both $u$ and $|u|$ minimize (2.7), since $J(\beta, u) = J(\beta, |u|)$. Therefore $|u|$ is a first eigenfunction too. By Harnack’s inequality we deduce that $|u|$ can not vanishes inside $\Omega$ and therefore $u$ has one sign in $\Omega$. The simplicity of $\lambda_1(\Omega)$ follows repeating the standard arguments for the classical Laplace operator.

Since eigenfunctions corresponding to different eigenvalues are mutually orthogonal in $L^2(\Omega, \phi_N)$, therefore, any positive function $v \in H^1(\Omega, \phi_N)$, which solves problem (2.5) for some $\lambda \in \mathbb{R}$, is a first eigenfunction, that is $\lambda = \lambda_1(\Omega)$. Clearly,
by the considerations above, it follows that $\lambda_1(\Omega)$ is the unique eigenvalue having a corresponding eigenfunction with constant sign in $\Omega$.

□

3 The eigenvalue problem in the case of half-lines

In this Section we consider the following problem

\[
\begin{cases}
-w'' + tw' = \lambda(\sigma)w & t \in I_{\sigma} \\
w'(\sigma) + \beta w(\sigma) = 0,
\end{cases}
\]

where $I_{\sigma} = (-\infty, \sigma)$.

By Theorem 2.2 the first eigenvalue $\lambda_1(\sigma)$ is simple and its corresponding eigenfunctions $w(t)$ are smooth and have one sign in $I_{\sigma}$. In this case, the variational characterization reads as

\[
\lambda_1(\sigma) = \min_{v \in H^1(I_{\sigma}, \phi_N) \setminus \{0\}} \frac{\int_{-\infty}^{\sigma} (v'(t))^2 e^{-\frac{t^2}{4}} dt + \beta (v(\sigma))^2 e^{-\frac{\sigma^2}{2}}}{\int_{-\infty}^{\sigma} (v'(t))^2 e^{-\frac{t^2}{4}} dt}.
\]

(3.2)

The minimizers $w$ of $\lambda_1(\sigma)$ has the following asymptotic behavior (see, for instance, [33], pp. 34–35 and pp. 72–76)

\[
w \propto (-t)^{\lambda_1(\sigma)} \left(1 + O((-t)^{-2})\right) \quad \text{as} \ t \to -\infty.
\]

(3.3)

Problem (3.1) is strictly related to problem (2.5) when $\Omega$ is a half-space. More precisely, let $\sigma \in \mathbb{R}$ and $S_{\sigma}$ as in (2.2), and let us consider the following problem

\[
\begin{cases}
-\text{div} (\phi_N(x) \nabla u) = \lambda_1(S_{\sigma}) \phi_N(x) u(x) & \text{in} \ S_{\sigma}, \\
\frac{\partial u}{\partial x_1} + \beta u = 0 & \text{on} \ \{x_1 = \sigma\},
\end{cases}
\]

(3.4)

where $\lambda_1(S_{\sigma})$ is the first eigenvalue given by

\[
\lambda_1(S_{\sigma}) = \min_{v \in H^1(S_{\sigma}, \phi_N) \setminus \{0\}} \frac{\int_{S_{\sigma}} |\nabla v|^2 \phi_N \, dx + \beta \int_{\{x_1 = \sigma\}} v^2 \phi_N \, dH^{N-1}(x)}{\int_{S_{\sigma}} v^2 \phi_N \, dx}.
\]

(3.5)

Theorem 2.2 ensures that $\lambda_1(S_{\sigma})$ is simple and it admits a positive corresponding eigenfunction $u(x)$. In what follows we observe that the eigenfunctions $u$ are determined by the ones of problem (3.1).

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Theorem 3.1 Let \( u \) be a positive eigenfunction corresponding to \( \lambda_1(\sigma) \). Then there exists a function \( w(t) : I_\sigma \to (0, +\infty) \) such that \( u(x) = w(x_1) \). Moreover \( w \) is strictly monotone decreasing in \( I_\sigma \) and it solves (3.1), with \( \lambda_1(\sigma) = \lambda_1(S_\sigma) \).

**Proof** Let \( w(t) \) be a positive eigenfunction of problem (3.1) corresponding to \( \lambda_1(\sigma) \). Then it solves

\[
\begin{align*}
- \left( w' \phi_1(t) \right)' &= \lambda_1(\sigma) \phi_1(t) w, \quad t \in I_\sigma, \\
w'(\sigma) + \beta w(\sigma) &= 0.
\end{align*}
\]

(3.6)

Since \( w > 0 \) in \( I_\sigma \), we clearly have that

\[
\left( w' \phi_1(t) \right)' < 0, \quad t \in I_\sigma.
\]

(3.7)

We claim that

\[
\lim_{t \to -\infty} w'(t) e^{-\frac{t^2}{2}} = 0.
\]

(3.8)

Note that, by (3.7), such a limit exists. Now assume to the contrary that

\[
\lim_{t \to -\infty} w'(t) e^{-\frac{t^2}{2}} = L \in (-\infty, +\infty) \setminus \{0\}.
\]

This would imply that

\[
\int_{-\infty}^{\sigma} (w'(t))^2 e^{-\frac{t^2}{2}} dt = +\infty,
\]

a contradiction, since \( w \in H^1(I_\sigma, \phi_1) \). The claim (3.8) is therefore proved.

From (3.8) and (3.7) it follows that \( w'(t) < 0 \) in \( I_\sigma \). Defining \( u(x) = w(x_1), x \in S_\sigma \), by Theorem 2.2 and the fact that \( \lambda_1(S_\sigma) \) is the unique eigenvalue having a positive eigenfunction, we get that \( \lambda_1(\sigma) = \lambda_1(S_\sigma) \).

This concludes the proof of the theorem. \( \square \)

Let \( w(t) \) be as in Theorem 3.1 and let us define the following function

\[
\beta(t) = -\frac{w'(t)}{w(t)}, \quad t \in I_\sigma.
\]

(3.9)

It holds that:

**Proposition 3.1** The function \( \beta(t) \) defined in (3.9) is positive, strictly increasing in \( I_\sigma \) and \( \beta(\sigma) = \beta \).

**Proof** As \( w \) is a solution to (3.1), Theorem 3.1 implies that \( \beta > 0 \). Moreover, since \( w' \) verifies in \( I_\sigma \) the following equation

\[
- (w'')' + t(w')' = (\lambda(\sigma) - 1) w',
\]

(3.10)
taking into account the asymptotic behavior given in (3.3), we have that

$$w' \propto (-t)^{\lambda_1(\sigma) - 1} \left(1 + O((-t)^{-2})\right) \text{ for } t \to -\infty.$$  \hfill (3.11)

Formulas (3.3) and (3.11) yield

$$\lim_{t \to -\infty} \beta(t) = 0.$$  \hfill (3.12)

Moreover

$$\beta'(t) = \frac{-w''w + (w')^2}{w^2} \quad t \in I_\sigma.$$

Since $w''$ is an eigenfunction of the one-dimensional Hermite problem corresponding to the eigenvalue $\lambda(\sigma) - 2$, arguing as for (3.12) we get

$$\lim_{t \to -\infty} \beta'(t) = 0.$$  \hfill (3.13)

By (3.1) it holds

$$\beta'(t) = \frac{-tww' + (w')^2 + \lambda(\sigma)w^2}{w^2} = t\beta(t) + \lambda(\sigma) + \beta^2(t), \quad t \in I_\sigma.$$

Denoting $z = \beta'$, we have

$$z' = tz + \beta(t) + 2\beta(t)z > z(2\beta(t) + t),$$

with $\lim_{t \to -\infty} z = 0$. Then $z > 0$, that is, $\beta(t)$ is strictly increasing and this completes the proof. \hfill $\square$

**Remark 3.1** We observe that Theorem 3.1 implies that any positive eigenfunction of problem (3.1) is log-concave and the same clearly is also true for problem (3.4). Indeed if we consider

$$f(t) = \log(w(t)), \quad t \in I_\sigma$$

then

$$f'(t) = -\beta(t) \quad \text{and} \quad f''(t) = -\beta'(t) < 0.$$

Let $\lambda_1(\sigma)$ be as in (3.2). Proposition 3.1 implies the following monotonicity result for the first eigenvalue $\lambda_1(\sigma)$ of the half-spaces.

**Proposition 3.2** Let $r, \sigma \in \mathbb{R}$ such that $r \leq \sigma$. Then $\lambda_1(r) \geq \lambda_1(\sigma)$. 

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**Proof** Let \( \beta(r) \) be the function defined in (3.9) and \( \beta \) the parameter which appears in the Robin boundary condition in (3.1), we have

\[
\lambda_1(\sigma) = \frac{\int_{-\infty}^{r} (w'(t))^2 e^{-\frac{t^2}{2}} dt + \beta(r)(w(r))^2 e^{-\frac{r^2}{2}}}{\int_{-\infty}^{r} (w'(t))^2 e^{-\frac{t^2}{2}} dt}
\]

\[
= \min_{v \in H^1(I_{\sigma}, \phi_1(t)) \setminus \{0\}} \frac{\int_{-\infty}^{r} (v'(t))^2 e^{-\frac{t^2}{2}} dt + \beta(v(r))^2 e^{-\frac{r^2}{2}}}{\int_{-\infty}^{r} (v'(t))^2 e^{-\frac{t^2}{2}} dt}
\]

\[
\leq \int_{-\infty}^{r} (v'(t))^2 e^{-\frac{t^2}{2}} dt + \beta(v(r))^2 e^{-\frac{r^2}{2}}
\]

\[
= \lambda_1(r),
\]

where the last inequality follows by Proposition 3.1. \( \Box \)

**4 A representation formula for \( \lambda_1(\Omega) \)**

Let \( \Omega \in \mathcal{G} \), with \( \mathcal{G} \) as in Definition 2.1 and let \( u \) be the first positive eigenfunction of (2.5) such that \( \|u\|_{L^2(\Omega, \varphi_N)} = 1 \). From now on, for every \( t > 0 \), we will use the following notation

\[
U_t = \{ x \in \Omega : u > t \},
\]

\[
\partial U_t^{\text{int}} = \{ x \in \Omega : u = t \},
\]

\[
\partial U_t^{\text{ext}} = \{ x \in \partial \Omega : u > t \}.
\]

Let \( \psi \in L^2(\Omega, \varphi_N) \) be a non-negative function and let us consider the following functional

\[
\mathcal{F}_\Omega(U_t, \psi) = \frac{1}{\gamma_N(U_t)} \left( - \int_{U_t} \psi^2 \varphi_N \, dx + \int_{\partial U_t^{\text{int}}} \psi \varphi_N \, d\mathcal{H}^{N-1}(x) \right.
\]

\[
+ \beta \int_{\partial U_t^{\text{ext}}} \varphi_N \, d\mathcal{H}^{N-1}(x) \right).
\]

(4.1)

The following level set representation formula for \( \lambda_1(\Omega) \) holds.

**Theorem 4.1** Let \( \Omega \in \mathcal{G} \), with \( \mathcal{G} \) as in Definition 2.1 and let \( u \) be the positive minimizer of (2.5) such that \( \|u\|_{L^2(\Omega, \varphi_N)} = 1 \). Then, for a.e. \( t > 0 \), it holds that

\[
\lambda_1(\Omega) = \mathcal{F}_\Omega(U_t, \bar{\psi}),
\]

\( \diamond \) Springer
where \( \bar{\psi} = \frac{|\nabla u|}{u} \) and \( F_\Omega \) is defined in (4.1).

**Proof** Since \( u \) is smooth and positive in \( \Omega \), we can divide both terms in the equation in (2.5) by \( u \), and integrate on \( U_t \). Since, by Sard’s theorem \( U_t \) is smooth for almost all \( t \), applying the classical Gauss-Green formula, we get

\[
\lambda(\Omega) \gamma_N(U_t) = \int_{U_t} -\text{div} \left( \phi_N(x) \nabla u \right) \frac{u}{u} \, dx
\]

\[
= -\int_{\partial U_t} \frac{\partial u}{\partial u} \phi_N(x) \, d\mathcal{H}^{N-1}(x) - \int_{U_t} \frac{|\nabla u|^2}{u^2} \phi_N(x) \, dx
\]

\[
= \int_{\partial U_t^{\text{int}}} \frac{|\nabla u|}{u} \phi_N(x) \, d\mathcal{H}^{N-1}(x)
\]

\[
+ \beta \int_{\partial U_t^{\text{ext}}} \phi_N(x) \, d\mathcal{H}^{N-1}(x) - \int_{U_t} \frac{|\nabla u|^2}{u^2} \phi_N(x) \, dx
\]

\[
= F_\Omega \left( U_t, \bar{\psi} \right) \gamma_N(U_t),
\]

and this concludes the proof. \( \square \)

**Theorem 4.2** Let \( \Omega \in \mathcal{G} \), with \( \mathcal{G} \) as in Definition 2.1 and let \( \bar{\psi} \) as in Theorem 4.1. Let \( \psi \in L^2(\Omega, \phi_N) \) be a nonnegative function such that \( \psi \neq \bar{\psi} \) and let \( F_\Omega \) be as in (4.1). Set

\[
w(x) := \psi - \bar{\psi}, \quad I(t) := \int_{U_t} w \bar{\psi} \phi_N \, dx.
\]

Then \( I : [0, +\infty[ \to \mathbb{R} \) is locally absolutely continuous and

\[
F_\Omega(U_t, \psi) \leq \lambda_1(\Omega) - \frac{1}{\gamma_N(U_t)} \frac{1}{t} \left( \frac{d}{dt} t^2 I(t) \right),
\]

for almost every \( t > 0 \).

**Proof** In order to prove (4.4), we write the representation formula (4.2) in terms of \( w \). It follows that, for a.e. \( t > 0 \),

\[
F_\Omega(U_t, \psi) = \lambda_1(\Omega) + \frac{1}{\gamma_N(U_t)} \left( \int_{\partial U_t^{\text{int}}} w \phi_N \, d\mathcal{H}^{N-1} - \int_{U_t} \left( \psi^2 - \bar{\psi}^2 \right) \phi_N \, dx \right)
\]

\[
\leq \lambda_1(\Omega) + \frac{1}{\gamma_N(U_t)} \left( \int_{\partial U_t^{\text{int}}} w \phi_N \, d\mathcal{H}^{N-1} - 2 \int_{U_t} w \bar{\psi} \phi_N \, dx \right)
\]

\[
= \lambda_1(\Omega) + \frac{1}{\gamma_N(U_t)} \left( \int_{\partial U_t^{\text{int}}} w \phi_N \, d\mathcal{H}^{N-1} - 2 I(t) \right)
\]

(4.5)
where the inequality in (4.5) follows since \( \psi, \bar{\psi} \geq 0 \). Applying the coarea formula, it is possible to rewrite \( I(t) \) as

\[
I(t) = \int_{U_t} w\bar{\psi} \phi_N \, dx = \int_t^{+\infty} \frac{1}{\tau} d\tau \int_{\partial U_t^{\text{int}}} w \phi_N \, d\mathcal{H}^{N-1}(x).
\]

This assures that \( I(t) \) is locally absolutely continuous in \([0, +\infty[\) and, for almost every \( t > 0 \) we have

\[
-\frac{d}{dt}(t^2 I(t)) = t \left( \int_{\partial U_t^{\text{int}}} w \phi_N \, d\mathcal{H}^{N-1}(x) - 2I(t) \right).
\]

Substituting in (4.5), the inequality (4.4) follows. \( \square \)

**Theorem 4.3** Let \( \Omega \in \mathcal{G} \), with \( \mathcal{G} \) as in Definition 2.1, and let \( \bar{\psi} \) be as in Theorem 4.1. Let \( \psi \in L^2(\Omega, \phi_N) \) be a nonnegative function such that \( \psi \neq \bar{\psi} \) and let \( F_\Omega \) be as in (4.1). Then there exists a set \( T \subset [0, +\infty[ \) with positive Lebesgue measure such that for every \( t \in T \) it holds that

\[
\lambda_1(\Omega) \geq F_\Omega(U_t, \psi). \tag{4.6}
\]

**Proof** Let \( u \) be the first positive eigenfunction of (2.5) such that \( \|u\|_{L^2(\Omega, \phi_N)} = 1 \). We have two cases. If \( u \in L^\infty(\Omega) \) then the claim follows by repeating line by line the arguments in [12,20]. Hence, from now on, we will assume that \( u \) is not bounded. Note that, by the asymptotic behavior given in (3.3), this case occurs surely when \( \Omega \) is any half-space. In order to prove (4.6), we proceed by contradiction by assuming that

\[
\lambda_1(\Omega) < F_\Omega(U_t, \psi), \text{ for a.e. } t > 0. \tag{4.7}
\]

Let \( \psi_n \in C_c^\infty(\Omega) \) be a sequence of nonnegative functions such that \( \psi_n \to \psi \) in \( L^2(\Omega, \phi_N) \). Then by Fatou’s lemma we have

\[
\lambda_1(\Omega) \leq F_\Omega(U_t, \psi) \leq \liminf_{n \to \infty} F_\Omega(U_t, \psi_n).
\]

Let \( \psi_{nk} \) be a subsequence such that

\[
\lim_{k \to \infty} F_\Omega(U_t, \psi_{nk}) = \liminf_{n \to \infty} F_\Omega(U_t, \psi_n).
\]

In order to simplify the notation, we will still denote by \( \psi_n \) such a subsequence. Then by (4.7) and Theorem 4.3 we have

\[
\lambda_1(\Omega) < \lim_{n \to \infty} F_\Omega(U_t, \psi_n) \leq \lambda_1(\Omega) - \lim_{n \to \infty} \frac{1}{\gamma_N(U_t)} \left( \frac{d}{dt} t^2 I_n(t) \right),
\]

where

\[
I_n(t) = \int_{U_t} w_n \bar{\psi} \phi_N \, dx
\]
with \( w_n = \psi_n - \bar{\psi} \). Then for \( n \) large enough it has to hold that

\[
\frac{d}{dt} t^2 I_n(t) < 0,
\]  
(4.8)

almost everywhere in \([0, +\infty[\). Since

\[
\lim_{t \to +\infty} t^2 |I_n(t)| \leq \lim_{t \to +\infty} t^2 \int_{U_t} |\psi_n - \bar{\psi}| \bar{\psi} \phi_N \, dx \leq \lim_{t \to +\infty} \int_{U_t} u^2 |\psi_n - \bar{\psi}| \bar{\psi} \phi_N \, dx
\]

\[
\leq \lim_{t \to +\infty} \int_{U_t \cap \text{supp} \psi_n} \psi_n u |\nabla u| \phi_N \, dx + \int_{U_t} |\nabla u|^2 \phi_N \, dx = 0,
\]  
(4.9)

and then for sufficiently large \( n \) we have

\[
\lim_{t \to +\infty} t^2 I_n(t) = 0.
\]  
(4.10)

On the other hand, denoting \( m = \min_{\bar{\Omega}} u > 0 \), it holds

\[
|I_n(t)| \leq \int_{\Omega} |\psi_n - \bar{\psi}| \bar{\psi} \phi_N \, dx
\]

\[
\leq \frac{\gamma_N(\Omega)^{1/2}}{m} \|\psi_n\|_{L^\infty(\Omega \cap \text{supp} \psi_n)} \|\nabla u\|_{L^2(\Omega, \phi_N)} \left(1 + \frac{1}{m^2} \|\nabla u\|_{L^2(\Omega, \phi_N)}^2\right).
\]

Then for sufficiently large \( n \) we have

\[
\lim_{t \to 0^+} t^2 I_n(t) = 0.
\]  
(4.11)

Since, by (4.8), the function \( t^2 I_n(t) \) is strictly decreasing, equations (4.10) and (4.11) give a contradiction. This concludes the proof. \( \square \)

5 Proof of the main result

In this Section we prove the Faber–Krahn inequality stated in Theorem 1.1.

**Proof** (Proof of Theorem 1.1) We first construct a suitable test function defined in \( \Omega \) for (4.6). Let \( v \) be a positive eigenfunction to the problem (3.4) in \( \Omega^\# \) and \( w \) the solution to (3.1) such that \( v(x) = w(x_1) \). Let

\[
\psi^*(x_1, \ldots, x_n) = \psi^*(x_1) := - \frac{w'(x_1)}{w(x_1)}, \quad x = (x_1, \ldots, x_n) \in \Omega^\#.
\]

By Theorem 4.1, for a.e. \( r > 0 \), we have

\[
\lambda_1(\Omega^\#) = \mathcal{F}_{\Omega^\#}(S_r, \psi^*).
\]
As before, let \( u \) be the first positive eigenfunction of (2.5) in \( \Omega \) such that \( \| u \|_{L^2(\Omega, \phi_N)} = 1 \). Using the same notation of the previous section, for any \( t > 0 \) we consider \( S_t \), the half-space such that \( \gamma_N(U_t) = \gamma_N(S_t) \). Then, if \( x \in \Omega \) and \( u(x) = t \), we define the following test function

\[
\psi(x) := \beta(r(t)) = \psi^*(r(t)),
\]

where \( \beta(r(t)) \) is defined in (3.9).

We claim that the following inequality holds

\[
\mathcal{F}_{\Omega^*}(S_t, \psi^*) \leq \mathcal{F}_\Omega(U_t, \psi) \tag{5.1}
\]

for a.e. \( t > 0 \). Since

\[
\mathcal{F}_{\Omega^*}(S_t, \psi^*) = \frac{1}{\gamma_N(S_t)} \left( - \int_{S_t} (\psi^*)^2 \phi_N dx + \int_{\{ x_1 = r(t) \}} \psi^* \phi_N d\mathcal{H}^{N-1}(x) \right)
\]

and

\[
\gamma_N(U_t) = \gamma_N(S_t),
\]

in order to prove inequality (5.1), we have to show that

\[
- \int_{S_t} (\psi^*)^2 \phi_N dx + \int_{\{ x_1 = r(t) \}} \psi^* \phi_N d\mathcal{H}^{N-1}(x) \leq \int_{U_t} \psi^2 \phi_N dx + \int_{\partial U_t} \psi \phi_N d\mathcal{H}^{N-1}(x) + \beta \int_{\partial U_t} \phi_N d\mathcal{H}^{N-1}(x). \tag{5.2}
\]

We firstly observe that, by construction, the functions \( \psi \) and \( \psi^* \) are equimeasurable, with respect to the Gaussian measure, and therefore

\[
\int_{U_t} \psi^2 \phi_N dx = \int_{S_t} (\psi^*)^2 \phi_N dx. \tag{5.3}
\]

Hence the first integrals which appear in the left- and right-hand sides of (5.2) coincide. Moreover, the isoperimetric inequality (2.1), and Lemma 3.1, ensure that, for all \( t > 0 \), it holds

\[
\int_{\{ x_1 = r(t) \}} \psi^* \phi_N d\mathcal{H}^{N-1}(x) = \beta(r(t)) P_{\phi_N}(S_t) \leq \beta(r(t)) P_{\phi_N}(U_t) \leq \int_{\partial U_t} \psi \phi_N d\mathcal{H}^{N-1}(x) + \beta \int_{\partial U_t} \phi_N d\mathcal{H}^{N-1}(x). \tag{5.4}
\]

Combining (5.3) and (5.4) we get (5.2) and then (5.1). In turn, by Theorems 4.1, 4.3 and equation (5.1) for all \( t \in T \) defined in Theorem 4.3, we have that

\[
\lambda_1(\Omega) = \mathcal{F}_{\Omega^*}(S_t, \psi^*) \leq \mathcal{F}_\Omega(U_t, \psi) \leq \lambda_1(\Omega),
\]

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which gives the claim.

Finally, in order to conclude the proof, we have to consider the equality case. Let us suppose that $\lambda_1(\Omega) = \lambda_1(\Omega^#)$. Then all the inequalities appearing in the first part of the proof become equalities. In particular $P_{\phi_N}(S_{r(t)}) = P_{\phi_N}(U_t)$. By Theorem 2.1 we have that $U_t$ are half-spaces for each $t$. Since the level sets of any function are always nested, we have that $u$ depends on $x_1$ only and it is a decreasing function. Therefore $\Omega = \Omega^#$ and $u$ coincides with $v$ modulo a constant. $\square$

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**Declarations**

**Conflicts of interest** On behalf of all authors, the corresponding author states that there is no conflict of interest.

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