FAST TRACK COMMUNICATION

Mean-field description of ultracold bosons on disordered two-dimensional optical lattices

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Abstract

In the present communication, we describe the properties induced by disorder on an ultracold gas of bosonic atoms loaded into a two-dimensional optical lattice with global confinement ensured by a parabolic potential. Our analysis is centred on the spatial distribution of the various phases, focusing particularly on the superfluid properties of the system as a function of external parameters and disorder amplitude. In particular, it is shown how disorder can suppress superfluidity, while partially preserving the system coherence.

(Some figures in this article are in colour only in the electronic version)

In recent years, the experimental results concerning confined ultracold atoms in optical lattices have attracted much theoretical interest from many different fields, ranging from condensed-matter physics to quantum information theory (see, e.g., [1, 2]). The ability to tune the fundamental physical parameters has provided an unrivalled tool to devise specific physical situations which, beyond their intrinsic physical interest, can be employed as important simulation tools [1]. In particular, the possibility of engineering a defect-free periodic potential appears as one of the most intriguing characteristics of these systems. Such ‘cleanliness’ has allowed the experimental realization of the Bose–Hubbard (BH) model and the observation of the superfluid-Mott quantum phase transition discussed by Fisher in his seminal paper [3], where, in addition, the issue of the effect of disorder on the physical properties of the BH Hamiltonian was first addressed. Furthermore, many experimental techniques such as laser speckle field [4] and the superposition of different optical lattices with incommensurate lattice constants [5–7] have recently substantiated the theoretical investigation of ultracold atoms in disordered optical lattices [8, 9], revealing a rich scenario of new physical situations such as the appearance of new phases (e.g. Bose-glass phase [6, 7] ) and superfluid (SF) percolation in \( d \)-dimensional \( (d > 1) \) lattices [10, 11].
In this communication, we will deal with the properties of a two-dimensional (2D) optical lattice where bosons are confined by an overall parabolic potential and subject to a random potential distribution.

The experimental realization of 2D lattices is discussed in [12–14]. In particular the latter represents a direct experimental realization of the Hamiltonian discussed in the present communication, when disorder is absent. A valuable feature of the setup here considered is the possibility of investigating a wide range of physical situations and geometries according to different external parameter choices. While the parabolic potential confines bosons in a disk-like domain, the interplay of the other external parameters (hopping amplitude and chemical potential) allows one to realize, for example, Mott phases distributed in concentric shells, each with different filling, intercalated with SF shells. The 1D counterpart of this scenario has been thoroughly studied in [15].

Here, we will focus particularly on the superfluid-phase spatial distribution in the presence of a random potential. We analyse first the situation where a (single) central disk-like Mott region is surrounded by a thin SF shell, and then the case when the system is fully SF. We show that, in the first case, the quasi-1D SF domain is strongly influenced by the presence of ‘impurities’, leading to a drop of the SF fraction for a small increase of the disorder amplitude. In the second case, the genuine 2D geometry of the SF domain leads to the formation of circulating streams which do not disappear abruptly when the disorder amplitude is increased. In this respect, our setup therefore appears to be an effective tool to investigate the combined effect of disorder and dimensionality on the SF properties of a bosonic system.

The system considered is modelled by a BH Hamiltonian $H = H_I + H_T$, with

$$H_I = \frac{U}{2} \sum_{i=1}^{M} n_i (n_i - 1) - \sum_{i=1}^{M} \mu_i n_i, \quad H_T = -T \sum_{i,j} A_{ij} a_i^\dagger a_j + \text{h.c.} \quad (1)$$

where $n_i = a_i^\dagger a_i$, $a_i^\dagger$ $(a_i)$ represents the creation (destruction) operator at site $i$ and the total boson number $N = \sum_i n_i$ is a conserved quantity. The random on-site effective chemical potential $\mu_i$ is given by

$$\mu_i = -\Omega_0 (x_i^2 + y_i^2) + \Delta_i + \mu \quad (2)$$

accounting both for the noise distribution $\Delta$ ($\Delta = \Delta \cdot \xi$, with $\xi_i$ uniformly distributed between 0 and 1) and for the parabolic confinement $\Omega_0$. The Hamiltonian parameters $U$ and $T$ represent the two-body interaction and the hopping amplitude between neighbouring sites, respectively, $M$ is the total number of sites (in our simulations we have chosen $M = 31^2 = 961$) and the matrix $A_{ij}$ is the so-called adjacency matrix, which is zero if sites $i$ and $j$ are not connected and one otherwise.

Our analysis is based on the mean-field decoupling scheme already exploited by many authors for the analysis of the disordered BH model [6, 10, 16]. For an application of this scheme to a disordered system see [17]. Despite the well-known shortcoming of not taking into account spatial correlations of quantum origin (see, e.g., the discussion about spatial correlations in [18], and in particular figure 4 therein), this mean-field decoupling scheme has proven the ability to disclose and qualitatively characterize the prominent physical features of bosonic atoms loaded in optical lattices, especially in dimension greater than one (for a detailed discussion on the MF scheme and its application see [19] and references therein). Note however that we take into account the spatial dependence arising from the disordered+harmonic local potential, and hence—at least partially—the spatial correlations thereof. The essence of this approximation is the replacement of two-boson operators $a_i^\dagger a_j^\dagger$ with single-site operators...
weighted by mean-field parameters \( \alpha_i = \langle \psi | a_i | \psi \rangle \) (where \( \alpha_i \in \mathbb{C} \) and \( | \psi \rangle \) represents the mean-field ground state). Namely,

\[
\alpha_i^\dagger a_i = \alpha_i^\dagger \alpha_i + \alpha_i^\dagger \alpha_i \quad \text{with} \quad i \in [1, \ldots, N]
\]

where the \( \alpha_i \) have to be determined by self-consistency standard procedure.

Within this scheme, Hamiltonian \( \mathcal{H} \) can thus be approximated by \( \mathcal{H} \simeq \sum_i \mathcal{H}_i + h \) where \( h = T \sum_{i=1}^{M} \alpha_i^\dagger A_i \alpha_i \) and parameters \( \gamma_i = \sum_{i=1}^{M} A_i \alpha_i \) in the single-site Hamiltonian \( \mathcal{H}_i = U n_i (n_i - 1) / 2 - \mu_i n_i - T \langle \gamma_i a_i^\dagger + \gamma_i^* a_i \rangle \) take into account the original inter-site coupling.

Most of the fundamental physical properties of the system can be obtained from the values of \( \alpha_i \) calculated from Hamiltonian \( \mathcal{H} \) alone where \( T \in \mathbb{R} \). However, in order to properly describe the SF properties of the system it is well known that one has to go through same procedure with the introduction of the Peierls factors [20]. The Hamiltonian with the addition of the Peierls factors reads

\[
\hat{H} = \mathcal{H} + \hat{H}_T, \quad \hat{H}_T = -T \sum_{i,j} e^{-i \theta B_{ij}} A_i a_j + \text{h.c.}
\]

The quantity \( \theta B_{ij} \) accounts for an infinitesimal velocity field \( \mathbf{v} \) which can be considered as a ‘probe’ to test the superfluidity properties of the system through the relation \( \theta B_{ij} = m / h | r_i \rangle \langle r_j | \mathbf{v} \cdot \mathbf{dr} \) [21] where \( r_i \) is the \( i \)th site position. With the Peierls factors \( \hat{H} \) becomes the Hamiltonian in the moving reference frame which is analogous to the Hamiltonian of a charged particle in the presence of a magnetic field (see, e.g., [21]). For the determination of the response of the system, it is necessary to take the limit \( \theta \to 0 \) and hence it is appropriate to consider the presence of the Peierls factors in perturbative terms with respect to the Hamiltonian \( \mathcal{H} \).

Up to first order, it is possible to show that the mean-field parameters for the Hamiltonian \( \hat{H} \), corresponding through the mean-field site-decoupling approach to the Hamiltonian \( \mathcal{H} \), assume the following form \( \alpha_i = \alpha_i^0 \exp[-i \theta \phi_i] \), where \( \alpha_i^0 \) represent the (real) mean-field parameter related to the Hamiltonian \( \mathcal{H} \), and the form of the phase term depends on the first-order approximation in \( \theta \).

In this communication we define the SF fraction (SFF) as the ratio between the current operator, as evaluated in equation (7), and the maximum possible SF current between two sites, i.e.

\[
f_i = \frac{\sum_{ij} \langle \psi | J_{ij} | \psi \rangle}{\sum_{ij} \langle \psi | J_{ij}^{\text{max}} | \psi \rangle} \quad \text{with} \quad \langle \psi | J_{ij}^{\text{max}} | \psi \rangle = -2T \sqrt{m_i} \sqrt{m_j} B_{ij}
\]

where we have employed the current operator defined as

\[
J_{ij} = i T \left( \exp[-i \theta B_{ij}] a_j^\dagger a_i - \exp[i \theta B_{ij}] a_i^\dagger a_j \right)
\]

which describes the supercurrent in the rotating (lattice) frame. Recalling equation (3), the first-order approximation for the expectation value of the current operator can be written in the mean-field scheme as

\[
\mathcal{J}_{ij} = \langle \psi | J_{ij} | \psi \rangle \simeq -2T \alpha_i^0 a_j^\dagger \theta [\phi_i - \phi_j + B_{ij}].
\]

So far, we have not explicitly defined the structure of the site-dependent (through matrix \( B_{ij} \)) Peierls factors. As suggested by the circular symmetry of the parabolic potential, we have considered by definition \( B_{ij} \) an infinitesimal rotation around the paraboloid axis. The generated flow—observed in the rotating frame—must be construed as a response to this infinitesimal rotation. To interpret correctly figures 1, 2 and 4, where the spatial distribution of \( \mathcal{J}_{ij} \) is depicted, it is necessary to recall that supercurrents are depicted in the (rotating) lattice frame and hence, in the laboratory frame, the overall rigid-body rotation effect must be
Figure 1. Upper panels: expectation value for the local number operator (left) and value of $\alpha_i$ (right). Lower panel: expectation value for the current operator $J_{ij}$. Both configurations are considered in the absence of disorder.

subtracted. The SFF at rest appears thus to be surrounded by a rotating rigid body—in this case the non-SF fraction.

In the 1D homogeneous case $f_s$ vanishes iff $\alpha_i = 0$ (the standard condition for Mott regime). In this case, the translational invariance of the system holds, entailing $n_i = n_j = n, \alpha_i^0 = \alpha_j^0 = \alpha_0$ and $\phi_i = \phi_j$ (overall phase coherence), thus $J_{ij} = 0$ iff $\alpha_0 = 0$. In 2D disordered case, conversely, equation (7) shows that the condition $f_s = 0$ is ensured either by $\alpha_i = 0$, as in the 1D homogeneous case or, more interestingly, by means of a phase rearrangement, which is determined by the constraint $\phi_j - \phi_i + B_{ij} = 0$, entailing the emergence of local clusters with $\alpha_i \neq 0$.

It is worth noting that, for a uniform ($\mu_i = \mu$) 1D system, due to translational invariance, equation (5) reproduces the mean-field behaviour of SFF as defined in the literature [22]:

$$f_{s,1D} = \frac{\sum_{ij} 2T \alpha_0^2 B_{ij}}{\sum_{ij} 2T n B_{ij}} = \frac{\alpha_0^2}{n}. \quad (8)$$

The identification of $f_{s,1D}$ with $\alpha_0^2/n$ deserves some comment. In [23] it was shown that the largest eigenvalue of the one-body density matrix $\rho_{ij} = \langle \psi | a_i^\dagger a_j | \psi \rangle / N$ is a measure of the
condensate fraction $f_c$. Within the mean-field scheme, it is possible to prove that the ratio $\langle \alpha_i^2 \rangle / \langle n_i \rangle$—the space average $\langle x_i \rangle$ ($x = \alpha, n$) takes into account the lattice inhomogeneity due to the disorder—can be related to the largest eigenvalue of the mean-field approximation of matrix $\rho_{ij} = [\langle \psi | n_i | \psi \rangle - |\alpha_i|^2 \delta_{ij} + \alpha_i^* \alpha_j] / N$ and thus provides an estimate of the condensate fraction $f_c$. In this spirit, for a uniform system, equation (8) satisfies the condition $f_c \sim f_s$.

From a more general point of view, since the condition $f_c \neq 0$ implies long-range correlations, $\langle \alpha_i^2 \rangle / \langle n_i \rangle$ can be viewed as a good coherence measure even for a disordered system.

In the numerical simulations we have analysed two different situations. In the first case, we have evaluated the effect of noise on a configuration where a single central Mott domain, with $n_i = 1$, is surrounded by a SF shell (figure 1), exhibiting a ring supercurrent. The parameters choice in this case has been performed in order to obtain a quasi-1D ring-shaped domain. In the laboratory frame, the latter corresponds to inert matter decoupled from the
lattice rotation owing to its SF character. This situation was explored in [18], in absence of disorder. In comparison, as a second case, we consider a situation where a central disk-like SF region is present and, consequently, exhibits a 2D behaviour.

Case I. In the first case \( T/U = 1.6 \times 10^{-2}, \Omega_0/U = 6 \times 10^{-3} \) and \( \mu/U = 0.5 \) corresponding to \( N_{\text{tot}} = 199 \) total particles) the increase of the noise amplitude leads to the ‘fracture’ of the ring with \( \alpha_i \neq 0 \) and to the drop of the SFF, leading the rearrangement of the SF flow within the new potential pattern (figure 2).

In the idealized situation of a perfect 1D ring domain, the presence of an impurity cutting the ring is sufficient to determine a sudden drop to zero in the supercurrent induced by the velocity field. In the situation here depicted, due to the finite radial extent of the SF region such an impurity occupies an extended domain. In addition to this effect several vortices emerge which correspond to feeble supercurrents pinned around the sites with large (disorder-induced) local chemical potential \( \mu_i \) (see figure 2, lower panel). It is worth noting that in the laboratory frame the extended impurity rotates together with the pinned vortices dragged by the lattice.

The flow rearrangement in the presence of disorder seems to be driven by the competition between the two terms in equation (7). In a situation where disorder is weak, the term \( B_{ij} \) prevails, leading to a quasi-proportionality between the forcing term and the SF flow. On the other hand, in the presence of large disorder, the phase pattern is adjusted so as to satisfy the condition \( \phi_j - \phi_i \sim -B_{ij} \). In this case the parameter \( \langle \alpha_i^2 \rangle / \langle n_i \rangle \), a good SFF measure for a homogeneous 1D system, while still being a measure of the local coherence of the system, can no longer be interpreted as a measure of the SFF, since the presence of small clusters with \( \alpha_i \neq 0 \) will contribute to the overall \( \langle \alpha_i^2 \rangle / \langle n_i \rangle \) value but not to the SFF. Note that here by ‘small’ we refer to a three-site cluster, in that, in a 2D square lattice four sites are needed to close a loop and thus to support a current.

Figure 3 shows that an increase of the disorder strength induces a sharp drop in the SFF while leaving the quantum coherence of the system almost unchanged. Each data point is the result of the average over ten realizations of the random potential. For the sake of clarity, the

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Figure 3. Comparison between the SFF (solid line) and the value of \( \langle \alpha_i^2 \rangle / \langle n_i \rangle \) (dashed line) as a function of disorder amplitude. Error bars account for the dispersion of the numerical data around their mean value for ten different disorder realizations.

In order to corroborate the general validity of our method, we note that the mean-field results in the absence of disorder described in figure 1 seem to properly reproduce the main findings of the quantum Monte Carlo approach of [18].
error bars are plotted every fifth data point. As we discussed above, the sharp drop in the SFF corresponds to the interruption of the quasi-1D SF domain characterizing the system in the absence of disorder.

A qualitative argument for the determination of the disorder-amplitude range in which the ring starts to break is to consider that, in the absence of disorder, the ring is approximately two-site thick and with a radius of about 12 sites. Hence, the potential energy difference between radially adjacent sites is about $\Omega_0 (12^2 - 11^2)$. In order to have superfluidity, the hopping must be large enough to overcome the potential difference $\Delta_c \simeq \Omega_0 (12^2 - 11^2)$, where $\Delta_c$ can thus be viewed as the disorder amplitude corresponding to the drop in the SFF. As a consequence, if disorder-induced local potentials of up to the same amplitude are present, the SFF is not affected.

**Case II.** The second setup has been conceived in order to analyse the effects of disorder in a fully 2D SF domain. In this case, due to the more markedly SF character of the system, it has
been necessary to consider much larger noise amplitudes to have observable effects on the SF distribution. In figure 4, we have sketched the values for $J_{ij}$ and $\alpha_i$ in a situation exhibiting a strong SF character ($T/U = 0.1$, $\Omega_0/U = 3.6 \times 10^{-2}$, $\mu/U = 0.6$ with $N_{tot} = 313$) for increasing values of the noise amplitude. Low values for $J_{ij}$ on the lattice centre are direct consequence of the velocity pattern imposed on the lattice and must not be interpreted as the absence of superfluidity at the centre of the trap. As previously discussed, in this case we have $J_{ij} \propto -B_{ij}$. For increasing values of the ratio between the noise and the parabola amplitude ($\Delta/\Omega_0$), as in case I, the SFF, due to the rearrangement of the flow, has a slower decrease than in the previous situation, due to the much larger SFF—note the scale difference between figures 3 and 5.

A prominent difference between cases I and II resides in the fact that the first one can be considered a quasi-1D case, and hence the ‘cutting’ of the SF shell is analogous to the interruption of a 1D chain. The second case represents, on the other hand, a genuine 2D situation and, as it is possible to see in figure 4, the presence of noise leads to the formation of percolation patterns with nonzero circulation [10]. The supercurrents in this case will follow closed and possibly merging paths encircling domains characterized by $f_s = 0$ but $\langle \alpha_i^2 \rangle/\langle n_i \rangle \neq 0$. The latter shows then the survival of a local coherence character, connected to $f_s$, even in the absence of supercurrents. The condition $f_s \neq 0$ together with $f_c = 0$, according to the scheme of [17], supplies, in the thermodynamical limit, a possible characterization of a Bose-glass phase.

In conclusion, we have presented a theoretical study of the behaviour of bosonic ultracold atoms in a 2D optical lattice with parabolic radial confinement with disorder. Our investigation has focused on the SF response both in the weakly and in the strongly SF regime. Within our approximation scheme, we have shown that, in the 2D disordered case, the absence of supercurrents is determined by two independent effects: the (usual) vanishing of the parameters $\alpha_i$ and the phase rearrangement implied by $\phi_j - \phi_i + B_{ij} = 0$. While the first effect can be related to the presence of a Mott phase in the thermodynamical limit, the second one, in the same limit, suggests the presence of a Bose-glass phase. In this framework, we have given a possible interpretation of the mechanism through which superfluidity is destroyed by disorder. Our numerical analysis evidences how, as a consequence of the phase rearrangement, the quasi-1D SF domain is destroyed by the appearance of ‘impurities’, with an ensuing drop of the SF fraction for a small increase of the disorder amplitude. Conversely, in case II the
genuine 2D nature of the SF domain leads to a multiple-stream SF flow determined by the increase of the disorder amplitude, with the consequent appearance of percolation patterns. In the laboratory frame, the strong-disorder sources in the rotating lattice have an overall site-dependent dragging effect on lattice bosons indicating the presence of non-superfluid regions locally surrounded by flows of nonzero circulation.

The rich phenomenology here presented exhibits a complex interplay between parameters such as the hopping amplitude, the boson–boson interaction, the disorder amplitude and the parabola coefficient. In view of the intrinsic interest from the experimental point of view, this deserves a more systematic analysis which we will perform elsewhere.

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