Asymptotic expansions of Kummer hypergeometric functions for large values of the parameters

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Abstract
We derive asymptotic expansions of the Kummer functions \(M(a, b, z)\) and \(U(a, b + 1, z)\) for large positive values of \(a\) and \(b\), with \(z\) fixed. For both functions we consider \(b/a \leq 1\) and \(b/a \geq 1\), with special attention for the case \(a \sim b\). We use a uniform method to handle all cases of these parameters.

Keywords Asymptotic expansions; Kummer functions; Confluent hypergeometric functions.

1 Introduction
We derive asymptotic expansions of the Kummer functions \(M(a, b, z)\) and \(U(a, b + 1, z)\) for large values of \(a\) and \(b\), with \(z\) fixed. Special attention is required when \(a \sim b\), in which case we derive expansions that are uniformly valid when the ratio \(a/b\) approaches 1.

For details on the Kummer or confluent hypergeometric functions, we refer to [14]. For \(b \to \infty\) and \(a \ll b\) we can use the defining convergent power series given in (7.1), which has an asymptotic character. An asymptotic expansion in negative powers of \(b\) can be found in Section 13.8(i) of [14]. In [22 Chapter 10] several expansions of the Kummer functions for large \(a\) or \(b\) are considered.

Applications of the confluent hypergeometric functions in mathematics and physics are reviewed in Sections 13.27 and 13.28 of [14]. These functions also arise in several applications in statistics.

The function \(M(a, b, z)\) appears in the characteristic function of the \(F\) ([16]) and beta ([10 p. 218]) distributions, the cumulative distribution function of the noncentral \(F\) ([18]) and noncentral \(t\) ([10 p. 517]) distributions, and the probability density function of the noncentral \(F\) (see, e.g., [8 p. 615]), general hypergeometric ([13]), and compound normal ([10 p. 193]) distributions. It appears in the moments of the doubly noncentral \(t\) ([10 p. 534]), generalized gamma ([9 p. 389]), log-beta ([10 p. 248]), noncentral chi-square ([10 p. 450]), and doubly noncentral \(F\) ([23]). It features in the

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1We note that [10 p. 501] is in error and uses \(_2F_0\) instead of \(_1F_1\).
normalizing constant of the generalized inverse normal (see [20]) and the Kummer beta

generalized (see [15] and references therein) distributions. In a recent paper [5], where
the computation and inversion of the cumulative noncentral beta distribution function
is considered, the $M$-functions play a role in recurrence relations, series expansions and
integral representations. At last, they appear in the theory of estimation and testing of the
multiple correlation coefficient ([4, p. 671], [6]), of the $t$ test ([3]), of the simultaneous
equations model ([19], [17, p. 470]), as well as in other more special cases ([7]).

The function $U(a, b, z)$ appears in the distribution of the signal-to-noise ratio ([11, p. 675],
[12, p. 905]), in the posterior distribution of a parameter of a queueing system ([1, Section 4]),
and in the characteristic functions of the central and noncentral $F$-
distributions ([16]).

Asymptotic forms of the Kummer functions can be found in [14, §13.8] and in [22,
Chapter 10], but usually the asymptotics is in terms of the argument $z$ in combination
with one or both parameters.

Asymptotic expansions for large $a$ and $b$, with fixed argument $z$ and with attention
to the ratio $a/b$ near unity are not considered earlier, as far as we know, and in this
paper we give the results for positive values of these parameters.

We use a special method to derive large-$a$ asymptotic expansions of the Laplace-type integral

\[ F_\lambda(a) = \frac{1}{\Gamma(\lambda)} \int_0^\infty s^{\lambda-1}e^{-as}f(s)ds, \]  

which are uniformly valid with respect to $\lambda \geq 0$. A similar contour integral is also
used. In Appendix A we summarise this method, called the vanishing saddle point. In
Appendix B we cite the most relevant formulas of the Kummer functions used in this
paper.

### 2 $M(a, b, z), \ b \geq a$

In this section we use the notation

\[ \lambda = b - a, \quad \mu = \frac{\lambda}{a} = \frac{b - a}{a}. \]  

We combine the Kummer relation for the $M$-function in ([16] and use (2.2). This gives

\[ M(a, b, z) = \frac{\Gamma(b)e^z}{\Gamma(a)\Gamma(\lambda)} \int_0^1 e^{-zt}e^{-a\phi(t)} \frac{dt}{t(1-t)}. \]  

where

\[ \phi(t) = -\ln(1-t) - \mu \ln t. \]  

The saddle point $t_0$ follows from the zero of $\phi'(t)$. We have

\[ \phi'(t) = \frac{t(1+\mu) - \mu}{t(1-t)} \implies t_0 = \frac{\mu}{1+\mu}. \]  

When the saddle point is properly inside the interval $[0,1]$ we can use the standard
method for obtaining an asymptotic expansion by using the substitution $\phi(t) - \phi(t_0) =
\[ w^2, \quad \text{sign}(w) = \text{sign}(t - t_0). \] However, when \( t_0 \to 0 \), that is, when \( b \downarrow a \), the standard method is no longer applicable, and we use a uniform method in which \( b = a \) can be used.

The uniform method is based on a transformation of the integral in (2.2) into the standard form in (1.1) by writing

\[ \phi(t) - \phi(t_0) = \psi(s) - \psi(s_0), \quad \text{sign}(t - t_0) = \text{sign}(s - s_0), \quad (2.5) \]

where

\[ \psi(s) = s - \mu \ln s, \quad s_0 = \mu; \quad (2.6) \]

\( s_0 \) is the zero of \( \psi'(s) = (s - \mu)/s \).

In Figure 1 we show for \( \mu = \frac{1}{4} \) the curves of the functions \( \phi(t) - \phi(t_0) \) (left) and \( \psi(s) - \psi(s_0) \) (right) that we use in the transformation in (2.5). The convex curves touch the real axes at \( t_0 = \frac{\mu}{1 + \mu} = \frac{1}{4} \) and \( s_0 = \mu = \frac{1}{4} \). The condition \( \text{sign}(t - t_0) = \text{sign}(s - s_0) \) means that the function values at the left of \( t_0 \) and \( s_0 \) correspond to each other, and the same holds true for those at the right of these points. Clearly, in this way, the transformation is one-to-one for \( t \in (0, 1) \) and \( s > 0 \).

The transformation gives

\[ M(a, b, z) = \frac{\Gamma(b)}{\Gamma(a)} e^{-aA} F_{\lambda}(a), \quad F_{\lambda}(a) = \frac{1}{\Gamma(\lambda)} \int_0^\infty e^{-a s^{\lambda-1}} f(s) \, ds, \quad (2.7) \]

where

\[ A = \phi(t_0) - \psi(s_0) = (1 + \mu) \ln(1 + \mu) - \mu, \quad (2.8) \]

and

\[ f(s) = e^{-z s} \frac{s}{(1 - t)} \frac{dt}{ds} \frac{dt}{\phi'(t)} = s - \mu \frac{t(1 - t)}{s \cdot t(1 + s) - \mu}. \quad (2.9) \]

Using the expansion given in (6.7), we obtain

\[ M(a, b, z) \sim e^{-aA} \frac{\Gamma(b)}{\Gamma(a)} a^{-\lambda} \sum_{n=0}^\infty \frac{f_n(\mu)}{a^n}, \quad a \to \infty. \quad (2.10) \]

To find \( f_0(\mu) \) we evaluate

\[ \frac{dt}{ds} \bigg|_{s=s_0} = \frac{\psi''(s_0)}{\phi'(s_0)} = \frac{1}{(1 + \mu)^3} \quad \Rightarrow \quad f_0(\mu) = e^{-z s_0} \sqrt{1 + \mu}. \quad (2.11) \]

We take the coefficient \( f_0(\mu) \) in front of the expansion and write

\[ M(a, b, z) \sim e^{-aA} \frac{\Gamma(b)}{\Gamma(a)} a^{-\lambda} f_0(\mu) \sum_{n=0}^\infty \frac{f_n(\mu)}{a^n}, \quad a \to \infty, \quad \bar{f}_n(\mu) = \frac{f_n(\mu)}{f_0(\mu)}. \quad (2.12) \]

We evaluate the front factors by using the definition of \( A \) in (2.8) and the scaled gamma functions defined in (7.9), and obtain

\[ e^{-aA} \frac{\Gamma(b)}{\Gamma(a)} a^{-\lambda} f_0(\mu) = e^{z / (1 + \mu)} \frac{\Gamma^*(b)}{\Gamma^*(a)}. \quad (2.13) \]
Figure 1: Curves of the functions $\phi(t) - \phi(t_0)$ (left) and $\psi(s) - \psi(s_0)$ (right) that we use in the transformation in (2.5), displayed for $\mu = \frac{1}{3}$.

This gives the final result

$$M(a, b, z) \sim e^{z/(1+\mu)} \frac{\Gamma^*(b)}{\Gamma^*(a)} \sum_{n=0}^{\infty} \frac{\tilde{f}_n(\mu)}{a^n}, \quad a \to \infty. \quad (2.14)$$

If we wish, using $b = a(1 + \mu)$, we can expand the ratio of scaled gamma functions in front of this expansion in powers of $a^{-1}$.

The first few coefficients of this expansion are $\tilde{f}_0(\mu) = 1$,

$$\tilde{f}_1(\mu) = \frac{\mu \left((\mu + 1)^2 + 6z^2\right)}{12(\mu + 1)^3}, \quad (2.15)$$

$$\tilde{f}_2(\mu) = \frac{\mu(\mu(\mu + 1)^4 + 12(\mu - 12)(\mu + 1)^2z^2 + 96(\mu^2 - 1)z^3 + 36\mu z^4))}{288(\mu + 1)^6}. \quad (2.16)$$

These follow from the scheme given in Appendix A.

Remark 2.1. The nonlinear transformation (2.5) and the other ones used in the next sections can be inverted by using the Lambert $W$ function that satisfies the equation

$$W(z)e^{W(z)} = z. \quad (2.17)$$

See [2] for details. For a proper description of $W(z)$ for $z \in \mathbb{R}$ and $z \in \mathbb{C}$, several branches of this function have to be considered. Write $s = -\mu \sigma$. Then for $\mu > 0$ the transformation (2.5) can be written in the form

$$\sigma e^\sigma = -\frac{t}{\mu}(1-t)^{1/\mu}e^{A(\mu)/\mu}, \quad (2.18)$$

where $A(\mu)$ is given in (2.8). We need to solve this equation for $\sigma < 0$, with the condition $\text{sign}(\sigma + 1) = \text{sign}(t_0 - t)$. For $\sigma = -1$ and $t = t_0$ both functions in (2.18) have the value $-1/e$. 

4
3 $M(a, b, z), \ b \leq a$

In this section we use the notation

$$\lambda = a - b, \ \mu = \frac{\lambda}{a} = \frac{a - b}{a}.$$ (3.1)

We use the integral representation given in (7.3) and write it in the form

$$M(a, b, z) = \frac{\Gamma(b) \Gamma(1 + \lambda)}{\Gamma(a)} \frac{1}{2\pi i} \int_0^{(1+)} e^{zt} e^{a \phi(t)} \frac{dt}{t(t - 1)},$$ (3.2)

where

$$\phi(t) = \ln t - \mu \ln(t - 1).$$ (3.3)

The saddle point $t_0$ follows from $\phi'(t) = 0$, where

$$\phi'(t) = \frac{(1 - \mu)t - 1}{t(t - 1)} \implies t_0 = \frac{1}{1 - \mu}.$$ (3.4)

The path of steepest descent $\mathcal{L}$ through $t_0$ follows from the equation $\Re \phi(t) = 0$. Using polar coordinates $t = r \cdot e^{i\theta}$ we find that it is given by

$$r = \frac{\sin(\theta/\mu)}{\sin((1 - \mu)\theta/\mu)}, \ -\mu\pi \leq \theta \leq \mu\pi.$$ (3.5)

In Figure 2 we show this path for $\mu = \frac{3}{4}$.

The standard saddle point method is not valid when $b \uparrow a$ and we use a uniform method transforming the integral in (3.2) into the standard form (6.2). We use the transformation

$$\phi(t) - \phi(t_0) = \psi(s) - \psi(s_0), \ \text{sign}(t - t_0) = \text{sign}(s - s_0),$$ (3.6)
where
\[ \psi(s) = s - \mu \ln s, \quad s_0 = \mu; \]  
(3.7)

\( s_0 \) is the zero of \( \psi'(s) = (s - \mu)/s \).

We obtain
\[ M(a, b, z) = \frac{\Gamma(b)}{\Gamma(a)} e^{aA} G_\lambda(a), \quad G_\lambda(a) = \frac{\Gamma(\lambda + 1)}{2\pi i} \int e^{as^{-\lambda-1}} g(s) \, ds, \]  
(3.8)

where
\[ g(s) = \frac{s e^{zt}}{t(t-1)} \frac{dt}{dw}, \quad \frac{dt}{dw} = \frac{\psi'(s)}{\phi'(t)}, \]  
(3.9)

and
\[ A = \phi(t_0) - \psi(s_0) = -(1 - \mu) \ln(1 - \mu) - \mu. \]  
(3.10)

Using the expansion given in (6.12), we obtain
\[ M(a, b, z) \sim \frac{\Gamma(b)}{\Gamma(a)} e^{aA} a^\lambda \sum_{n=0}^{\infty} (-1)^n \frac{g_n(\mu)}{a^n}, \quad a \to \infty. \]  
(3.11)

To find \( g_0(\mu) \) we evaluate
\[ \frac{dt}{ds} \bigg|_{s=s_0}^2 = \frac{\psi''(s_0)}{\phi''(t_0)} = \frac{1}{(1 - \mu)^3} \quad \implies \quad g_0(\mu) = e^{zt_0} \sqrt{1 - \mu}. \]  
(3.12)

We take the coefficient \( g_0(\mu) \) in front of the expansion and write
\[ M(a, b, z) \sim e^{aA} \frac{\Gamma(b)}{\Gamma(a)} a^\lambda g_0(\mu) \sum_{n=0}^{\infty} (-1)^n \frac{\bar{g}_n(\mu)}{a^n}, \quad a \to \infty, \quad \bar{g}_n(\mu) = \frac{g_n(\mu)}{f g_0(\mu)}. \]  
(3.13)

We evaluate the front factors by using the definition of \( A \) in (3.10) and the scaled gamma functions defined in (7.9), and obtain
\[ e^{aA} \frac{\Gamma(b)}{\Gamma(a)} a^\lambda g_0(\mu) = e^{z/(1-\mu)} \frac{\Gamma^*(b)}{\Gamma^*(a)}. \]  
(3.14)

This gives the final result
\[ M(a, b, z) \sim e^{z/(1-\mu)} \frac{\Gamma^*(b)}{\Gamma^*(a)} \sum_{n=0}^{\infty} (-1)^n \frac{\bar{g}_n(\mu)}{a^n}, \quad a \to \infty. \]  
(3.15)

If we wish, using \( b = a(1 - \mu) \), we can expand the ratio of scaled gamma functions in front of this expansion in powers of \( a^{-1} \).

The first few coefficients of this expansion are \( \bar{g}_0(\mu) = 1 \),
\[ \bar{g}_1(\mu) = \frac{\mu ((1 - \mu)^2 + 6z^2)}{12(1 - \mu)^3}, \]  
(3.16)

\[ \bar{g}_2(\mu) = \frac{\mu (\mu(1 - \mu)^4 + 12(\mu + 12)(1 - \mu)^2 z^2 + 96(1 - \mu^2) z^3 + 36\mu z^4)}{288(1 - \mu)^6}. \]  
(3.17)

These follow from the scheme given in Appendix A.
Remark 3.1. Comparing the expansion in (3.15) with the one in (2.14), we see that the results are very similar. Indeed, the expansions follow from each other by changing the sign of $\mu$. That is, $f_n(-\mu) = (-1)^n g_n(\mu)$. This can be verified by comparing the first given values of the coefficients, but also by comparing in detail the functions used in the transformations to the standard forms and the construction of the asymptotic expansions. In fact, the starting point for the two cases are the integrals in (7.2) and (7.3), and they follow from each other when $\Re(b - a) > 0$ by integrating in (7.3) along the interval $[0, 1]$. The integral in (7.3) is valid for all $b - a$, but the asymptotic expansion for the standard form $G_\lambda(z)$ in (6.12) is developed for $\Re(b - a) \leq 0$.

4 $U(a, b + 1, z), \ b \geq a$

For the $U$-function we consider $U(a, b + 1, z)$ because this yields similar results as for $M(a, b, z)$. We have the special value $U(a, a + 1, z) = z^{-a}$.

In this section we use the notation

$$\lambda = b - a, \quad \mu = \frac{b - a}{a}. \quad (4.1)$$

We use the contour integral in (7.6) and the Kummer relation for the $U$-function. This gives

$$U(a, b + 1, z) = \frac{z^{-b} \Gamma(\lambda + 1)}{2\pi i} \int_C e^{zt} s^{-\lambda - 1}(1 - t)^{-a} \, dt, \quad (4.2)$$

where the contour is as in (7.6). We write this in the form

$$U(a, b + 1, z) = \frac{z^{-b} \Gamma(\lambda + 1)}{2\pi i} \int_C e^{s\phi(t)} e^{zt} \frac{dt}{t}, \quad (4.3)$$

where

$$\phi(t) = -\ln(1 - t) - \mu \ln t. \quad (4.4)$$

The saddle point $t_0$ follows from

$$\phi'(t) = \frac{(1 + \mu)t - \mu}{t(1 - t)} = 0 \quad \iff \quad t_0 = \frac{\mu}{1 + \mu}. \quad (4.5)$$

We use $\psi(s) = s - \mu \ln s$ and the transformation

$$\phi(t) - \phi(t_0) = \psi(s) - \psi(s_0), \quad (4.6)$$

where $s_0 = \mu$ is the zero of $\psi'(s)$. This gives the representation

$$U(a, b + 1, z) = z^{-b} e^{A} G_\lambda(a), \quad G_\lambda(a) = \frac{\Gamma(\lambda + 1)}{2\pi i} \int_C e^{as} s^{-\lambda - 1} p(s) \, ds, \quad (4.7)$$

where

$$p(s) = e^{zt} \frac{s \, dt}{t \, ds}, \quad A = \phi(t_0) - \psi(s_0) = (1 + \mu) \ln(1 + \mu) - \mu. \quad (4.8)$$
We have the expansion
\[ G_\lambda(a) \sim a^\lambda \sum_{n=0}^{\infty} (-1)^n \frac{p_n(\mu)}{a^n}, \quad a \to \infty. \] (4.9)

The first coefficient is
\[ p_0(\mu) = \frac{e^{z\mu/(1+\mu)}}{\sqrt{1 + \mu}}. \] (4.10)

The first-order asymptotic approximation is
\[ U(a, b + 1, z) \sim z^{-b} a^{-a-b} e^{aA} p_0(\mu). \] (4.11)

Using the definition of \( A(\mu) \) given in (4.7) this becomes
\[ U(a, b + 1, z) \sim z^{-b} a^{-a-b} e^{a-b} p_0(\mu). \] (4.12)

When \( a = b \), that is, \( \mu = 0 \), we obtain the value \( z^{-a} \), which is the special value given in (7.8).

The full expansion can be written as
\[ U(a, b + 1, z) \sim z^{-b} a^{-a-b} e^{a-b} \sum_{n=0}^{\infty} (-1)^n \frac{\bar{p}_n(\mu)}{a^n}, \quad a \to \infty, \] (4.13)

where \( \bar{p}_n(\mu) = p_n(\mu)/p_0(\mu) \). We have \( \bar{p}_0(\mu) = 1 \) and
\[ \bar{p}_1(\mu) = \frac{\mu}{12(1 + \mu)^3} \left( (1 + \mu)^2 + 6z(z - 2 + 2\mu) \right), \]
\[ \bar{p}_2(\mu) = \frac{\mu}{288(1 + \mu)^6} \left( \mu(1 + \mu)^4 - 24(\mu - 12)(1 + \mu)^3 z + 12(25\mu - 36)(1 + \mu)^2 z^2 - 48(5\mu - 2)(1 + \mu) z^3 + 36\mu z^4 \right). \] (4.14)

5 \( U(a, b + 1, z), \ b \leq a \)

In this section we use the notation
\[ \lambda = a - b, \quad \mu = \frac{\lambda}{a} = \frac{a - b}{a}. \] (5.1)

We use the Kummer relation in (7.7) and the integral representation in (7.5). This gives
\[ U(a, b + 1, z) = \frac{z^{-b}}{\Gamma(a - b)} \int_0^\infty e^{-zt} s^{a-b-1}(1 + t)^{-a} dt, \quad \Re(a - b) > 0, \quad \Re z > 0 \] (5.2)

which we write in the form
\[ U(a, b + 1, z) = \frac{z^{-b}}{\Gamma(\lambda)} \int_0^\infty e^{-zt} e^{-a\phi(t)} \frac{dt}{t}, \] (5.3)
where
\[ \phi(t) = \ln(1 + t) - \mu \ln t. \] (5.4)

We calculate the saddle point \( t_0 \):
\[ \phi'(t) = \frac{t(1 - \mu) - \mu}{t(1 + t)} = 0 \implies t_0 = \frac{\mu}{1 - \mu}. \] (5.5)

We use the function \( \psi(s) = s - \mu \ln s \) and transform
\[ \phi(t) - \phi(t_0) = \psi(s) - \psi(s_0), \quad s_0 = \mu, \quad \text{sign}(t - t_0) = \text{sign}(s - s_0), \] (5.6)

and write the result in the standard form
\[ U(a, b + 1, z) = z^{-b} e^{-a A} F_\lambda(a), \quad F_\lambda(a) = \frac{1}{\Gamma(\lambda)} \int_0^\infty e^{-as} s^{\lambda-1} q(s) ds, \] (5.7)

where
\[ q(s) = e^{-z_s} \frac{dt}{t}, \quad A = \phi(t_0) - \psi(s_0) = -(1 - \mu) \ln(1 - \mu) - \mu. \] (5.8)

We have the expansion
\[ F_\lambda(a) \sim a^{-\lambda} \sum_{n=0}^\infty \frac{q_n(\mu)}{a^n}, \quad a \to \infty. \] (5.9)

The first-order asymptotic approximation is
\[ U(a, b + 1, z) \sim z^{-b} a^{-a} e^{-a A(\mu)} q_0(\mu), \quad q_0(\mu) = \frac{e^{-z_\mu/(1-\mu)}}{\sqrt{1-\mu}}. \] (5.10)

Using the definition of \( A(\mu) \) given in (5.7), this becomes
\[ U(a, b + 1, z) \sim z^{-b} a^{-a} e^{-a A(\mu)} q_0(\mu). \] (5.11)

When \( a = b \), that is, \( \mu = 0 \), we obtain the value \( z^{-a} \), which is the special value given in (7.8).

The full expansion can be written as
\[ U(a, b + 1, z) \sim z^{-b} a^{-a} e^{-a A(\mu)} q_0(\mu) \sum_{n=0}^\infty \frac{\tilde{q}_n(\mu)}{a^n}, \quad a \to \infty, \] (5.12)

where \( \tilde{q}_n(\mu) = q_n(\mu)/q_0(\mu) \). We have \( \tilde{q}_0(\mu) = 1 \) and
\[ \begin{align*}
\tilde{q}_1(\mu) &= \frac{\mu}{12(1 - \mu)^3} \left( (1 - \mu)^2 + 6z(2 - 2\mu) \right), \\
\tilde{q}_2(\mu) &= \frac{\mu}{288(1 - \mu)^6} \left( \mu(1 - \mu)^4 - 24(\mu + 12)(1 - \mu)^3 z + 12(25\mu + 36)(1 - \mu)^2 z^2 - 48(5\mu + 2)(1 - \mu)z^3 + 36\mu z^4 \right). 
\end{align*} \] (5.13)

Remark 5.1. The expansions in (5.13) and in (4.12) follow from each other when we change the sign of \( \mu \). Compare a similar observation for the expansions of the \( M \)-function in Remark 5.1.
6 Appendix A: The vanishing saddle point

The asymptotic methods that we consider in this paper are for integrals of Laplace-type of the form

$$ F_{\lambda}(z) = \frac{1}{\Gamma(\lambda)} \int_0^\infty s^{\lambda-1} e^{-zt} f(s) \, ds, \quad (6.1) $$

with $z$ as a large parameter. The method is also for loop integrals of the form

$$ G_{\lambda}(z) = \frac{\Gamma(\lambda + 1)}{2\pi i} \int_{-\infty}^{(0+)} s^{-\lambda} f(s) e^{zs} \, ds, \quad (6.2) $$

where the contour runs from $-\infty$ with $\text{ph} \, s = -\pi$, encircles the origin in anti-clockwise direction, and returns to $-\infty$ with $\text{ph} \, s = \pi$. The negative axis serves as a branch cut and we assume that $s^{-\lambda} f(s)$ has real values for $s > 0$ (when $\lambda$ is real). In this paper we assume that $z > 0$, and $\lambda \geq 0$.

In Watson's lemma for the integral in (6.1), with $z$ as the large parameter, the parameter $\lambda$ is assumed to be fixed. When $\lambda$ is not fixed (say, $\lambda = \mathcal{O}(z)$), Watson’s lemma cannot be used. When $z$ and $\lambda$ are large, the dominant part of the integral in (6.1) is

$$ F_{\lambda}(z) = z^{\lambda} f(\mu) - \frac{1}{2\pi i} \int_0^\infty \frac{f(s) - f(\mu)}{s - \mu} \, ds, $$

$$ = z^{\lambda} f(\mu) + \frac{1}{2\pi i} \int_0^\infty s^{\lambda-1} e^{-zs} f_1(s) \, ds, \quad (6.4) $$

where

$$ f_1(s) = \frac{d}{ds} \frac{f(s) - f(\mu)}{s - \mu}. \quad (6.5) $$

Continuing this procedure we obtain for $K = 0, 1, 2, \ldots$

$$ z^{\lambda} F_{\lambda}(z) = \sum_{k=0}^{K-1} \frac{f_k(\mu)}{z^k} + \frac{1}{2\pi} E_K(z, \mu), $$

$$ f_k(s) = \frac{d^k}{ds^k} \frac{f(s) - f(\mu)}{t - \mu}, \quad k = 1, 2, \ldots, \quad f_0(s) = f(s), \quad (6.6) $$

$$ E_K(t, \mu) = \frac{1}{\Gamma(\lambda)} \int_0^\infty s^{\lambda-1} e^{-zs} f_K(s) \, ds. $$
Eventually we obtain the complete asymptotic expansion

\[ F_{\lambda}(z) \sim z^{-\lambda} \sum_{n=0}^{\infty} \frac{f_n(\mu)}{z^n}, \quad z \to \infty. \quad (6.7) \]

The coefficients \( f_n(\mu) \) can be expressed in terms of the coefficients \( a_n(\mu) \). To verify this we write

\[ f_n(s) = \sum_{m=0}^{\infty} c_m^{(n)} (s - \mu)^m. \quad (6.8) \]

Then \( a_m(\mu) = c_m^{(0)} \), \( f_n(\mu) = c_m^{(n)} \), and we have from (6.6)

\[ f_{n+1}(s) = \sum_{m=0}^{\infty} c_m^{(n+1)} (s - \mu)^m = t \sum_{m=1}^{\infty} c_m^{(n)} (-1)(s - \mu)^{m-2}. \quad (6.9) \]

This gives the recursion

\[ c_m^{(n+1)} = mc_m^{(n)} + \mu(m + 1)c_{m+2}^{(n)}, \quad m, n = 0, 1, 2, \ldots, \quad (6.10) \]

and the few first relations are

\[ f_0(\mu) = a_0(\mu), \quad f_1(\mu) = \mu a_2(\mu), \quad f_2(\mu) = \mu(2a_3(\mu) + 3\mu a_4(\mu)), \quad f_3(\mu) = \mu(6a_4(\mu) + 20\mu a_5(\mu) + 15\mu^2 a_6(\mu)) \]

\[ f_4(\mu) = \mu(24a_5(\mu) + 130\mu a_6(\mu) + 210\mu^2 a_7(\mu) + 105\mu^3 a_8(\mu)). \quad (6.11) \]

Under mild conditions on \( a_n(\mu) \), that is, on \( f \), this expansion is uniformly valid with respect to \( \lambda \in [0, \infty) \), and in a larger domain of the complex plane. The main condition on \( f \) is that its singularities are not too close to the point \( t = \mu \). Initially we have assumed for the integral in (6.1) that \( \lambda > 0 \). However, the reciprocal gamma function \( 1/\Gamma(\lambda) \) in front of the integral makes the integral regular when \( \lambda \downarrow 0 \). This can be seen by using integration by parts (writing \( s^{\lambda-1} ds = (1/\lambda) d(s^{\lambda}) \)), and in this way it can be shown that analytic continuation of \( F_{\lambda}(z) \) of (6.1) is possible into the domain \( \Re \lambda \geq 0 \).

We will see that the asymptotic expansion of \( F_{\lambda}(z) \) allows taking \( \lambda = 0 \). In fact the obtained expansion will be valid for \( z \to \infty \), uniformly with respect to \( \lambda \geq 0 \).

A similar integration by parts procedure gives the expansion of the loop integral in (6.2). We have

\[ G_{\lambda}(z) \sim z^{\lambda} \sum_{n=0}^{\infty} (-1)^n \frac{g_n(\mu)}{z^n}, \quad z \to \infty, \quad (6.12) \]

where the coefficients \( g_n(\mu) \) can be obtained by the same recursive procedure as for \( f_n(\mu) \).

### 7 Appendix B

The defining power series is

\[ M(a, b, z) = \sum_{n=0}^{\infty} \frac{(a)_n}{(b)_n} \frac{z^n}{n!}, \quad (a)_n = \frac{\Gamma(a + n)}{\Gamma(a)} \quad (7.1) \]
with the usual condition that $b$ is not a nonpositive integer. The standard integral is

$$M(a, b, z) = \frac{\Gamma(b)}{\Gamma(a)\Gamma(b - a)} \int_0^1 e^{zt} s^{a-1} (1 - t)^{b-a-1} dt, \quad (7.2)$$

where $\Re a > 0$, $\Re(b - a) > 0$. A contour integral is

$$M(a, b, z) = \frac{\Gamma(b)\Gamma(1 + a - b)}{\Gamma(a)2\pi i} \int_{C} e^{zt} s^{a-1} (t - 1)^{b-a-1} dt, \quad \Re a > 0, \quad (7.3)$$

where the contour starts at $t = 0$, encircles the point $t = 1$ in the anti-clockwise direction, and returns to $t = 0$. Also,

$$M(a, b, z) = \frac{\Gamma(b)z^{1-b}}{2\pi i} \int_{C} e^{zu} u^{-b} (1 - 1/u)^{-a} du, \quad (7.4)$$

where the contour $C$ starts at $-\infty$, with $\text{ph } u = -\pi$, encircles the points 0 and 1 in ant-clockwise direction, and returns to $-\infty$, where $\text{ph } u = +\pi$. At the point where the contour crosses the interval $(1, \infty)$ the functions $u^{-b}$ and $(1 - 1/u)^{-a}$ assume their principal values.

The standard integral for $U(a, b, z)$ is

$$U(a, b, z) = \frac{1}{\Gamma(a)} \int_0^\infty e^{-zs} s^{a-1} (1 + t)^{b-a-1} dt, \quad \Re a > 0, \quad \Re z > 0, \quad (7.5)$$

and a loop integral is

$$U(a, b, z) = \frac{\Gamma(1 - a)}{2\pi i} \int_{-\infty}^{(0+)} e^{zt} s^{a-1} (1 - t)^{b-a-1} dt, \quad \Re z > 0, \quad (7.6)$$

where $a \neq 1, 2, 3, \ldots$. The contour cuts the real axis between 0 and 1. At this point the fractional powers are determined by $\text{ph } (1 - t) = 0$ and $\text{ph } t = 0$.

The Kummer relations are

$$M(a, b, z) = e^z M(b - a, b, -z), \quad U(a, b, z) = z^{1-b} U(a - b + 1, 2 - b, z). \quad (7.7)$$

Special values are

$$M(a, a, z) = e^z, \quad U(a, a + 1, z) = z^{-a}. \quad (7.8)$$

We use also the scaled gamma function

$$\Gamma^*(z) = e^z z^{-z} \sqrt{\frac{z}{2\pi}} \Gamma(z) \sim 1 + \frac{1}{12z} + \frac{1}{288z^2} + \ldots, \quad z \to \infty. \quad (7.9)$$

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