Extending $p$-divisible groups and Barsotti-Tate deformation ring in the relative case

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Abstract

Let $k$ be a perfect field of characteristic $p > 2$, and let $K$ be a finite totally ramified extension of $W(k)[\frac{1}{p}]$ of ramification degree $e$. We consider an unramified base ring $R_0$ over $W(k)$ satisfying certain conditions, and let $R = R_0 \otimes_{W(k)} O_K$. Examples of such $R$ include $R = O_K[s_1, \ldots, s_d]$ and $R = O_K(t_1^{\pm 1}, \ldots, t_d^{\pm 1})$. We show that the generalization of Raynaud’s theorem on extending $p$-divisible groups holds over the base ring $R$ when $e < p - 1$, whereas it does not hold when $R = O_K[s]$ with $e \geq p$. As an application, we prove that if $R$ has Krull dimension 2 and $e < p - 1$, then the locus of Barsotti-Tate representations of $\text{Gal}(\overline{R}[\frac{1}{p}] / R[\frac{1}{p}])$ cuts out a closed subscheme of the universal deformation scheme. If $R = O_K[s]$ with $e \geq p$, we prove that such a locus is not $p$-adically closed.

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1 Introduction

Let $k$ be a perfect field of characteristic $p > 2$, and $W(k)$ be its ring of Witt vectors. Let $K$ be a finite totally ramified extension of $W(k)[\frac{1}{p}]$ of ramification degree $e$, and let $O_K$ be its ring of integers. We consider an unramified base ring $R_0$ over $W(k)$ satisfying certain conditions (cf. Section 2), and let $R = R_0 \otimes_{W(k)} O_K$. Important examples of such $R$ include
the formal power series ring $R = \mathcal{O}_K[s_1, \ldots, s_d]$, and $R = \mathcal{O}_K[t_1^{\pm 1}, \ldots, t_d^{\pm 1}]$ which is the $p$-adic completion of $\mathcal{O}_K[t_1^{\pm 1}, \ldots, t_d^{\pm 1}]$.

When $R = \mathcal{O}_K$, Raynaud showed the following theorem on extending $p$-divisible groups.

**Theorem 1.1.** ([Ray74, Proposition 2.3.1]) Let $G$ be a $p$-divisible group over $K$. Suppose that for each $n \geq 1$, $G[p^n]$ extends to a finite flat group scheme over $\mathcal{O}_K$. Then $G$ extends to a $p$-divisible group over $\mathcal{O}_K$, and such an extension is unique up to isomorphism.

In this paper, we prove that the generalization of Raynaud’s theorem holds over the relative base $R$ when the ramification is small ($e < p - 1$). On the other hand, using an example from [VZ10] on purity of $p$-divisible groups, we show that such a statement does not hold when the ramification is large.

**Theorem 1.2.** Assume $e < p - 1$. Let $G$ be a $p$-divisible group over $R[p^1]$. Suppose that for each $n \geq 1$, $G[p^n]$ extends to a finite locally free group scheme over $R$. Then $G$ extends to a $p$-divisible group over $R$, and such an extension is unique up to isomorphism.

If $e \geq p$ and $R = \mathcal{O}_K[s]$, there exists a $p$-divisible group $G$ over $R[p^1]$ such that $G[p^n]$ extends to a finite locally free group scheme over $R$ for each $n$ but $G$ does not extend to a $p$-divisible group over $R$.

As an application, we study the geometry of the locus of representations arising from $p$-divisible groups over $R$ when $R$ has Krull dimension 2. Let $G_R$ be the étale fundamental group of $\text{Spec} R[p^1]$. For a fixed absolutely irreducible $\mathbf{F}_p$-representation $V_0$ of $G_R$, there exists a universal deformation ring which parametrizes the deformations of $V_0$ ([SL97]).

We say that a finite continuous $\mathbf{Q}_p$-representation $V$ of $G_R$ is Barsotti-Tate if it arises from a $p$-divisible group over $R$, i.e., if there exists a $p$-divisible group $G_R$ over $R$ such that $V \cong T_p(G_R) \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$ where $T_p(G_R)$ denotes the Tate module of $G_R$. For a torsion $\mathbf{Z}_p$-representation $T$ of $G_R$, we say it is torsion Barsotti-Tate if it is a quotient of a finite free $\mathbf{Z}_p$-representation $T_1$ such that $T_1[p^1]$ is Barsotti-Tate. By using Theorem 1.2, we prove:

**Theorem 1.3.** Suppose $R$ has Krull dimension 2 and $e < p - 1$. Then the locus of Barsotti-Tate representations of $G_R$ cuts out a closed subscheme of the universal deformation scheme.

If $R = \mathcal{O}_K[s]$ and $e \geq p$, then the locus of Barsotti-Tate representations is not $p$-adically closed in the following sense: there exists a finite free $\mathbf{Z}_p$-representation $T$ of $G_R$ such that $T/p^nT$ is torsion Barsotti-Tate for each integer $n \geq 1$ but $T[p^1]$ is not Barsotti-Tate.

We give a more precise statement of Theorem 1.3 in Section 5.

**Acknowledgements**

I would like to express sincere gratitude to Mark Kisin for his guidance while working on this topic. This paper is partly based on author’s Ph.D. thesis under his supervision. I also thank Brian Conrad and Tong Liu for helpful discussions.
2 Relative Breuil-Kisin Classification

We first explain the classification of \( p \)-divisible groups and finite locally free group schemes over \( \text{Spec} R \) via certain Kisin modules, which is proved in [Kis06] when \( R = \mathcal{O}_K \) and generalized in [Kim15] for the relative case.

We will work over the relative base rings as considered in [Bri08] with some additional mild assumptions. Denote by \( W(k) \langle t_1^{\pm 1}, \ldots, t_d^{\pm 1} \rangle \) the \( p \)-adic completion of the polynomial ring \( W(k)[t_1^{\pm 1}, \ldots, t_d^{\pm 1}] \). Let \( R_0 \) be a ring obtained from \( W(k)\langle t_1^{\pm 1}, \ldots, t_d^{\pm 1} \rangle \) by iterations of the following operations:

- \( p \)-adic completion of an étale extension;
- \( p \)-adic completion of a localization;
- completion with respect to an ideal containing \( p \).

We assume that either \( W(k)\langle t_1^{\pm 1}, \ldots, t_d^{\pm 1} \rangle \rightarrow R_0 \) has geometrically regular fibers or \( R_0 \) has Krull dimension less than 2, and that \( k \rightarrow R_0/pR_0 \) is geometrically integral and \( R_0 \) is an integral domain. Furthermore, we suppose that \( R_0 \) is formally smooth formally finite type over some Cohen ring (cf. [Kim15, Section 2.2.2]). In particular, \( R_0 \) is a regular ring.

\( R_0/pR_0 \) has a finite \( p \)-basis given by \( \{t_1, \ldots, t_d\} \) in the sense of [DJ95, Definition 1.1.1]. Let \( \hat{\Omega}_{R_0} = \lim_{\longrightarrow n} \Omega_{(R_0/p^n)/W(k)} \) be the module of \( p \)-adically continuous Kähler differentials. We have \( \hat{\Omega}_{R_0} \cong \bigoplus_{i=1}^d R_0 \cdot d(\log t_i) \) by [Bri08] Proposition 2.0.2. The Witt vector Frobenius on \( W(k) \) extends (not necessarily uniquely) to \( R_0 \). We fix such a Frobenius endomorphism \( \varphi : R_0 \rightarrow R_0 \), and let \( R = R_0 \otimes_{W(k)} \mathcal{O}_K \) be our base ring. Examples of such \( R \) include \( R = \mathcal{O}_K\langle t_1^{\pm 1}, \ldots, t_d^{\pm 1} \rangle \) and \( R = \mathcal{O}_K[s_1, \ldots, s_d] \) (for example, via \( s_i = 1 + t_i \)).

It will be useful later to consider the following natural maps between base rings. Let \( R_{0,g} \) be the \( p \)-adic completion of \( \lim_{\longrightarrow \varphi} (R_0/pR_0) \) with the induced Frobenius, and denote by \( k_g \) the perfect closure \( \varinjlim \text{Frac}(R_0/pR_0) \) of \( \text{Frac}(R_0/pR_0) \). By the universal property of \( p \)-adic Witt vectors, we have a unique continuous (with respect to the \( p \)-adic topology) morphism \( h : W(k_g) \rightarrow R_{0,g} \) commuting with their projections to \( k_g \). By unicity, \( h \) is compatible with Frobenius endomorphisms. Since \( h \) modulo \( p \) is an isomorphism and \( R_{0,g} \) is \( p \)-torsion free and \( p \)-adically complete and separated, \( h \) is an isomorphism. We will make use of this isomorphism later when we apply results from classical \( p \)-adic Hodge theory over \( p \)-adic fields, since such results will hold for the base ring \( R_{0,g} \otimes_{W(k)} \mathcal{O}_K \). Let \( b_g : R_0 \rightarrow R_{0,g} \) be the natural morphism compatible with Frobenius. This induces \( \mathcal{O}_K \)-linearly the base change map \( b_g : R \rightarrow R_{0,g} \otimes_{W(k)} \mathcal{O}_K \).

**Lemma 2.1.** The map \( b_g : R_0 \rightarrow R_{0,g} \) is injective. Furthermore, for each integer \( n \geq 1 \), the map \( R_0/(p^n) \rightarrow R_{0,g}/(p^n) \) induced from \( b_g \) is injective.
Proof. Since $R_0/(p)$ is an integral domain, the map $R_0/(p) \to R_{0,g}/(p) = k_g$ is injective. Thus, $b_g : R_0 \to R_{0,g}$ is injective as $R_0$ is $p$-adically separated and $R_{0,g}$ is $p$-torsion free. It also follows that $R_0/(p^n) \to R_{0,g}/(p^n)$ is injective for each $n \geq 1$.

Let $\mathcal{G} = R_0[u]$ equipped with the Frobenius extending that on $R_0$, given by $\varphi : u \mapsto u^p$. Denote by $E(u)$ the Eisenstein polynomial for the extension $K$ over $W(k)\left[\frac{1}{p}\right]$.

**Definition 2.2.** A quasi-Kisin module of height 1 is a pair $(\mathcal{M}, \varphi_{\mathcal{M}})$ where

- $\mathcal{M}$ is a finitely generated projective $\mathcal{G}$-module;
- $\varphi_{\mathcal{M}} : \mathcal{M} \to \mathcal{M}$ is a $\varphi$-semilinear map such that $\text{coker}(1 \otimes \varphi_{\mathcal{M}} : \mathcal{G} \otimes_{\mathcal{G}, \varphi} \mathcal{M} \to \mathcal{M})$ is annihilated by $E(u)$.

Note that for a quasi-Kisin module $\mathcal{M}$ of height 1, $1 \otimes \varphi_{\mathcal{M}} : \varphi^*\mathcal{M} := \mathcal{G} \otimes_{\mathcal{G}, \varphi} \mathcal{M} \to \mathcal{M}$ is injective since $\mathcal{M}$ is finite projective over $\mathcal{G}$ and $\text{coker}(1 \otimes \varphi_{\mathcal{M}})$ is killed by $E(u)$. Let $\text{Mod}_{\mathcal{G}}(\varphi)$ denote the category of quasi-Kisin modules of height 1 whose morphisms are $\mathcal{G}$-module maps compatible with Frobenius.

Consider the composite $\mathcal{G} \to \mathcal{G}/u\mathcal{G} = R_0 \xrightarrow{\varphi} R_0$. Let $\text{Mod}_{\mathcal{G}}(\varphi, \nabla)$ denote the category whose objects are tuples $(\mathcal{M}, \varphi_{\mathcal{M}}, \nabla_{\mathcal{M}})$ where $(\mathcal{M}, \varphi_{\mathcal{M}})$ is a quasi-Kisin module of height 1, $\mathcal{M} := \mathcal{M} \otimes_{\mathcal{G}, \varphi} R_0$, and $\nabla_{\mathcal{M}} : \mathcal{M} \to \mathcal{M} \otimes_{\mathcal{M}, \mathcal{G}} \mathcal{G}$ is a topologically quasi-nilpotent integrable connection commuting with $\varphi_{\mathcal{M}} := \varphi_{\mathcal{M}} \otimes \varphi_{R_0}$. (Here, $\nabla_{\mathcal{M}}$ being topologically quasi-nilpotent means that the induced connection on $\mathcal{M}/p\mathcal{M}$ is nilpotent). The morphisms in $\text{Mod}_{\mathcal{G}}(\varphi, \nabla)$ are $\mathcal{G}$-module maps compatible with Frobenius and connection. The objects in $\text{Mod}_{\mathcal{G}}(\varphi, \nabla)$ are called Kisin modules of height 1. The following theorem is proved in [?].

**Theorem 2.3.** (cf. [Kim15, Corollary 6.7 and Remark 6.9]) There exists an exact anti-equivalence of categories

$$\mathcal{M}^* : \{p\text{-divisible groups over } R\} \to \text{Mod}_{\mathcal{G}}(\varphi, \nabla).$$

Let $R'_0$ be another unramified ring satisfying the conditions as above equipped with a Frobenius, and let $b : R_0 \to R'_0$ be a $\varphi$-equivariant map. Then the formation of $\mathcal{M}^*$ commutes with the base change $R \to R' := R'_0 \otimes_{W(k)} \mathcal{O}_K$ induced $\mathcal{O}_K$-linearly from $b$.

The classification of $p$-power order finite locally free group schemes over $R$ can be obtained by considering torsion Kisin modules.

**Definition 2.4.** A torsion quasi-Kisin module of height 1 is a pair $(\mathcal{M}, \varphi_{\mathcal{M}})$ where

- $\mathcal{M}$ is a finitely presented $\mathcal{G}$-module killed by a power of $p$, and of $\mathcal{G}$-projective dimension 1;
$\varphi_\mathfrak{M}: \mathcal{M} \to \mathcal{M}$ is a $\varphi$-semilinear endomorphism such that $\text{coker}(1 \otimes \varphi_\mathfrak{M} : \varphi^*\mathcal{M} \to \mathcal{M})$ is killed by $E(u)$.

Let $\text{Mod}^{\text{tor}}(\varphi)$ denote the category of torsion quasi-Kisin modules of height 1 whose morphisms are $\mathcal{S}$-linear maps compatible with $\varphi$. Let $\text{Mod}^{\text{tor}}(\varphi, \nabla)$ denote the category whose objects are tuples $(\mathcal{M}, \varphi_\mathfrak{M}, \nabla_\mathcal{M})$ where $(\mathcal{M}, \varphi_\mathfrak{M})$ is a torsion quasi-Kisin module of height 1, $\mathcal{M} := \mathcal{M} \otimes_{\mathcal{S}, \varphi} R_0$, and $\nabla_\mathcal{M}: \mathcal{M} \to \mathcal{M} \otimes_{R_0} \hat{\Omega}_{R_0}$ is a topologically quasi-nilpotent integrable connection commuting with $\varphi_\mathcal{M} := \varphi_\mathfrak{M} \otimes \varphi_{R_0}$. The morphisms in $\text{Mod}^{\text{tor}}(\varphi, \nabla)$ are $\mathcal{S}$-module maps compatible with $\varphi$ and $\nabla$. The objects are called torsion Kisin modules of height 1.

**Lemma 2.5.** Let $\mathcal{M}$ be a torsion quasi-Kisin module of height 1. Then $1 \otimes \varphi_\mathfrak{M} : \varphi^*\mathcal{M} \to \mathcal{M}$ is injective.

**Proof.** Let $\mathcal{S}_g := R_{0,g}[u]$ equipped with the Frobenius given by $\varphi(u) = u^p$. By the local criterion for flatness, $b_g: R_0 \to R_{0,g}$ is flat since $R_0/(p) \to R_{0,g}/(p) = k_g$ is flat and $R_{0,g}$ is $p$-torsion free, and the map $\mathcal{S} \to \mathcal{S}_g$ is flat. Note that $\mathcal{M}_g := \mathcal{M} \otimes_{\mathcal{S}} \mathcal{S}_g$ equipped with $\varphi_{\mathfrak{M}_g} := \varphi_{\mathfrak{M}} \otimes \varphi_{\mathcal{S}_g}$ is a torsion Kisin module of height 1 over $\mathcal{S}_g$.

We first claim that the natural map $b: \mathcal{M} \to \mathcal{M}_g$ is injective. Since $\mathcal{M}$ has projective dimension $\leq 1$, there exists a short exact sequence $0 \to \mathcal{M}_1 \to \mathcal{M}_2 \to \mathcal{M} \to 0$ where $\mathcal{M}_1$ and $\mathcal{M}_2$ are finite projective $\mathcal{S}$-modules. $\mathcal{M}_1$ and $\mathcal{M}_2$ have the same rank since $\mathcal{M}$ is killed by a power of $p$. We have a commutative diagram

$$
\begin{array}{cccc}
0 & \longrightarrow & \mathcal{M}_1 & \longrightarrow & \mathcal{M}_2 & \longrightarrow & \mathcal{M} & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow b & & \\
0 & \longrightarrow & \mathcal{M}_1 \otimes_{\mathcal{S}} \mathcal{S}_g & \longrightarrow & \mathcal{M}_2 \otimes_{\mathcal{S}} \mathcal{S}_g & \longrightarrow & \mathcal{M}_g & \longrightarrow & 0
\end{array}
$$

whose rows are exact. Since $\mathcal{M}_1$ and $\mathcal{M}_2$ are projective over $\mathcal{S}$, the left and middle vertical maps are injective. Furthermore, for $i = 1, 2$, we have $\text{coker}(\mathcal{M}_i \to \mathcal{M}_i \otimes_{\mathcal{S}} \mathcal{S}_g) \cong \mathcal{M}_i \otimes_{\mathcal{S}} (\mathcal{S}_g/\mathcal{S})$ as $\mathcal{S}$-modules. On the other hand, all elements in the kernel of the induced map $\mathcal{M}_1 \otimes_{\mathcal{S}} (\mathcal{S}_g/\mathcal{S}) \to \mathcal{M}_2 \otimes_{\mathcal{S}} (\mathcal{S}_g/\mathcal{S})$ are killed by some power of $p$ since $\mathcal{M}_1[\frac{1}{p}] \cong \mathcal{M}_2[\frac{1}{p}]$. And $\mathcal{S}_g/\mathcal{S}$ is $p$-torsion free since $R_0/(p) \to R_{0,g}/(p) = k_g$ is injective, so $\mathcal{M}_1 \otimes_{\mathcal{S}} (\mathcal{S}_g/\mathcal{S})$ is $p$-torsion free as $\mathcal{M}_1$ is projective over $\mathcal{S}$. Hence, the map $\mathcal{M}_1 \otimes_{\mathcal{S}} (\mathcal{S}_g/\mathcal{S}) \to \mathcal{M}_2 \otimes_{\mathcal{S}} (\mathcal{S}_g/\mathcal{S})$ is injective. By the snake Lemma, we deduce that $b: \mathcal{M} \to \mathcal{M}_g$ is injective.

Now, consider the following commutative diagram:

$$
\begin{array}{ccc}
\mathcal{S} \otimes_{\varphi, \mathcal{S}} \mathcal{M} & \xrightarrow{1 \otimes \varphi_\mathfrak{M}} & \mathcal{M} \\
\downarrow & & \downarrow b \\
\mathcal{S}_g \otimes_{\varphi, \mathcal{S}_g} \mathcal{M}_g & \xrightarrow{1 \otimes \varphi_{\mathfrak{M}_g}} & \mathcal{M}_g
\end{array}
$$

Since $\varphi: \mathcal{S} \to \mathcal{S}$ is flat by [Bri08, Lemma 7.1.8], $\mathcal{S} \otimes_{\varphi, \mathcal{S}} \mathcal{M}$ has projective dimension 1 as a $\mathcal{S}$-module and is killed by a power of $p$. By the same argument as above, the natural
map $\mathcal{S} \otimes_{\varphi, \mathcal{S}} \mathcal{M} \to \mathcal{S}_g \otimes_{\mathcal{S}} (\mathcal{S} \otimes_{\varphi, \mathcal{S}} \mathcal{M}) \cong \mathcal{S}_g \otimes_{\varphi, \mathcal{S}_g} \mathcal{M}_g$ is injective, which is the left vertical map. The bottom map is injective by [Lim07, Proposition 2.3.2] since $R_{0,g} \cong W(k_g)$. Thus, the top map is injective.

Denote by $(\text{Mod FI})_{\mathcal{S}}(\varphi, \nabla)$ the full subcategory of $\text{Mod}_{\mathcal{S}}(\varphi, \nabla)$ consisting of objects $\mathcal{M}$ such that $\mathcal{M} \cong \bigoplus_i \mathcal{M}_i$ as $\mathcal{S}$-modules where $\mathcal{M}_i$’s are projective over $\mathcal{S}/(p^n_i)$ for some positive integers $n_i$. The following theorem is shown in [Kim15].

**Theorem 2.6.** (cf. [Kim15] Proposition 9.5 and Theorem 9.8] There exists an exact fully faithful functor $\mathcal{M}^*\text{ from the category of } p\text{-power order finite locally free group schemes over } R \text{ to } (\text{Mod FI})_{\mathcal{S}}(\varphi, \nabla)$ with the following properties:

- Let $H$ be a $p$-power order finite locally free group scheme over $R$. If $H = \ker(h : G^0 \to G^1)$ for an isogeny $h$ of $p$-divisible groups over $R$, then there exists a natural isomorphism $\mathcal{M}^*(H) \cong \coker(\mathcal{M}^*(h))$ of torsion Kisin modules of height 1;

- Let $R'_0$ be another unramified ring satisfying the conditions as above equipped with a Frobenius, and let $b : R_0 \to R'_0$ be a $\varphi$-equivariant map. Then the formation of $\mathcal{M}^*$ commutes with the base change $R \to R' := R'_0 \otimes_{W(k)} \mathcal{O}_K$ induced $\mathcal{O}_K$-linearly from $b$.

Moreover, the functor $\mathcal{M}^*$ induces an anti-equivalence from the category of $p$-power order finite locally free group schemes $H$ over $R$ such that $H[p^n]$ is locally free over $R$ for all $n \geq 1$ to $(\text{Mod FI})_{\mathcal{S}}(\varphi, \nabla)$.

We end this section by recalling some necessary results on connections explained in [Kim15, Section 10.2], which is based on [Vas13]. Let $(\mathcal{M}, \varphi_{\mathcal{M}})$ be a quasi-Kisin module of height 1, and let $\mathcal{M} = \mathcal{M} \otimes_{\mathcal{S}, \varphi} R_0$ equipped with the induced Frobenius $\varphi_{\mathcal{M}} \otimes \varphi_{R_0}$. From [Kim15, Eq. (6.1), (6.2) and Remark 3.13], we have the $R_0$-submodule $\text{Fil}^1 \mathcal{M} \subset \mathcal{M}$ associated with $\mathcal{M}$ such that $p\mathcal{M} \subset \text{Fil}^1 \mathcal{M}$, $\mathcal{M}/\text{Fil}^1 \mathcal{M}$ is projective over $R_0/(p)$, and $(1 \otimes \varphi)(\varphi_{\text{Fil}^1 \mathcal{M}}) = p\mathcal{M}$ as $R_0$-modules (cf. [Kim15, Definition 3.4 and 3.6] for the frame $(R_0, pR_0, R_0/(p), \varphi_{R_0}, \varphi_{R_0}/p))$. Fix an $R_0$-direct factor $\mathcal{M}^1 \subset \mathcal{M}$ which lifts $\text{Fil}^1 \mathcal{M}/p\mathcal{M} \subset \mathcal{M}/p\mathcal{M}$, and let $\hat{\mathcal{M}} := (\mathcal{M} + \frac{1}{p^n} \mathcal{M}^1) \otimes_{R_0, \varphi} R_0 \subset \mathcal{M} \otimes_{R_0, \varphi} R_0[\frac{1}{p}]$. For each integer $n \geq 1$, suppose $\nabla_n : R_0/(p^n) \otimes_{R_0} \mathcal{M} \to (R_0/(p^n) \otimes_{R_0} \mathcal{M}) \otimes_{R_0} \hat{\mathcal{M}}$ is a connection such that the following diagram is commutative:

$$
\begin{array}{c}
R_0/(p^n) \otimes_{R_0} \mathcal{M} \\
\downarrow \varphi^*(\nabla_n) \downarrow \varphi^*(\nabla_n) \\
R_0/(p^n) \otimes_{R_0} \hat{\mathcal{M}} & \xrightarrow{\alpha^{(n)}} & R_0/(p^n) \otimes_{R_0} \hat{\mathcal{M}} \otimes_{R_0} \hat{\mathcal{M}} \\
\downarrow \varphi^*(\nabla_n) & & \downarrow \varphi^*(\nabla_n) \\
R_0/(p^n) \otimes_{R_0} \mathcal{M} & \xrightarrow{\nabla_n} & R_0/(p^n) \otimes_{R_0} \mathcal{M} \otimes_{R_0} \hat{\mathcal{M}}
\end{array}
$$

Here, $\varphi^*(\nabla_n)$ is given by choosing an arbitrary lift of $\nabla_n$ on $R_0/(p^{n+1}) \otimes_{R_0} \mathcal{M}$, and $\varphi^*(\nabla_n)$ does not depend on the choice of such a lift (cf. [Vas13, Section 3.1.1 Equation (9)]).
Identify $\hat{\Omega}_{R_0} = \bigoplus_{i=1}^d R_0 \cdot d(\log t_i)$. By passing to a finite Zariski covering of $\text{Spf}(R_0, p)$, we may assume that $\mathcal{M}^1$ and $\mathcal{M}/\mathcal{M}^1$ are free over $R_0$. Fix such a choice of the covering, and fix a $R_0$-basis of $\mathcal{M}$ adapted to the direct factor $\mathcal{M}^1$. By [Vas13, Section 3.2 Basic Theorem] and its proof, the set of connections $\nabla_1$ on $R_0/(p) \otimes_{R_0} \mathcal{M}$ satisfying the commutative diagram (2.11) for $n = 1$ corresponds to the solutions over $R_0/(p)$ of a certain Artin-Schreier system of equations over $R_0/(p)$. In particular, if follows directly that we have finitely many such $\nabla_1$ (cf. [Vas13 Theorem 2.4.1 (b)]). Furthermore, given a connection $\nabla_n$ on $R_0/(p^n) \otimes_{R_0} \mathcal{M}$, the set of connections $\nabla_{n+1}$ on $R_0/(p^{n+1}) \otimes_{R_0} \mathcal{M}$ which lift $\nabla_n$ and satisfy the commutative diagram (2.11) for $n + 1$ corresponds the solutions over $R_0/(p)$ of a certain Artin-Schreier system of equations over $R_0/(p)$ by loc. cit., and we have finitely many such $\nabla_{n+1}$.

3 Étale $\varphi$-modules and Galois Representations

We recall the results in [Kim15, Section 7] on associating Galois representations with étale $\varphi$-modules in the relative setting. The underlying geometry is based on perfectoid spaces (cf. [Sch12]). We will use the results to translate our question on $p$-divisible groups into a question on Kisin modules and étale $\varphi$-modules.

Let $\overline{R}$ denote the union of finite $R$-subalgebras $R'$ of a fixed separable closure of $\text{Frac}(R)$ such that $R'[\frac{1}{p}]$ is étale over $R[\frac{1}{p}]$. Then $\text{Spec}(\overline{R}[\frac{1}{p}])$ is a pro-universal covering of $\text{Spec}(R[\frac{1}{p}])$, and $\overline{R}$ is the integral closure of $R$ in $\overline{R}[\frac{1}{p}]$. Let $G_R := \text{Gal}(\overline{R}[\frac{1}{p}]/R[\frac{1}{p}]) = \pi_1(\text{Spec}(R[\frac{1}{p}]), \eta)$ with a choice of a geometric point $\eta$. Choose a uniformizer $\varpi \in \mathcal{O}_K$. For integers $n \geq 0$, we choose compatibly $\varpi_n \in \overline{R}$ such that $\varpi_0 = \varpi$ and $\varpi_{n+1} = \varpi_n$, and let $L$ be the $p$-adic completion of $\bigcup_{n \geq 0} K(\varpi_n)$. Then $L$ is a perfectoid field and $(\hat{\overline{R}}[\frac{1}{p}], \hat{\overline{R}})$ is a perfectoid affinoid $L$-algebra, where $\hat{\overline{R}}$ denotes the $p$-adic completion of $\overline{R}$.

Let $L^\circ$ denote the tilt of $L$ as defined in [Sch12], and let $\varpi := (\varpi_n) \in L^\circ$. Let $(\overline{R}[\frac{1}{p}], \overline{R})$ be the tilt of $(\overline{R}[\frac{1}{p}], \hat{\overline{R}})$. Let $E_{R_{\infty}}^+ = \mathcal{G}/p\mathcal{G}$, and let $\hat{E}_{R_{\infty}}^+$ be the $u$-adic completion of $\lim_{\leftarrow} E_{R_n}^+$. Let $E_{R_{\infty}} = E_{R_{\infty}}^+[\frac{1}{u}]$ and $\hat{E}_{R_{\infty}} = \hat{E}_{R_{\infty}}^+[\frac{1}{u}]$. By [Sch12, Proposition 5.9], $(\hat{E}_{R_{\infty}}, \hat{E}_{R_{\infty}}^+)$ is a perfectoid affinoid $L^\circ$-algebra, and we have the natural injection $(\hat{E}_{R_{\infty}}, \hat{E}_{R_{\infty}}^+) \hookrightarrow (\overline{R}[\frac{1}{p}], \overline{R})$ given by $u \mapsto \varpi$. Let $(\hat{R}_{\infty}[\frac{1}{p}], \hat{R}_{\infty})$ be a perfectoid affinoid $L$-algebra whose tilt is $(\hat{E}_{R_{\infty}}, \hat{E}_{R_{\infty}}^+)$, and let $\mathcal{G}_{\hat{R}_{\infty}} = \pi_1(\text{Spec}(\hat{R}_{\infty}[\frac{1}{p}], \eta)$. Then we have a continuous map of Galois groups $\mathcal{G}_{\hat{R}_{\infty}} \rightarrow \mathcal{G}_R$, which is a closed embedding by [GR03, Proposition 5.4.54]. By the almost purity theorem in [Sch12], $\overline{R}[\frac{1}{p}]$ can be canonically identified with the $\varpi$-adic completion of $\mathcal{G}[\frac{1}{u}]$. Note that $\varphi$ on $\mathcal{G}$ extends naturally to
Definition 3.1. An étale \((\varphi, \mathcal{O}_E)\)-module is a pair \((M, \varphi_M)\) where \(M\) is a finitely generated \(\mathcal{O}_E\)-module and \(\varphi_M: M \to M\) is a \(\varphi\)-semilinear endomorphism such that \(1 \otimes \varphi_M: \varphi^* M \to M\) is an isomorphism. We say that an étale \((\varphi, \mathcal{O}_E)\)-module is projective (resp. torsion) if the underlying \(\mathcal{O}_E\)-module \(M\) is projective (resp. \(p\)-power torsion).

Let \(\text{Mod}_{\mathcal{O}_E}\) denote the category of étale \((\varphi, \mathcal{O}_E)\)-modules whose morphisms are \(\mathcal{O}_E\)-linear maps compatible with Frobenius. Let \(\text{Mod}^\text{pr}_{\mathcal{O}_E}\) and \(\text{Mod}^\text{tor}_{\mathcal{O}_E}\) respectively denote the full subcategories of projective and torsion objects.

Note that we have a natural notion of a subquotient, direct sum, and tensor product for étale \((\varphi, \mathcal{O}_E)\)-modules, and duality is defined for projective and torsion objects. If \((\mathfrak{M}, \varphi_{\mathfrak{M}})\) is a quasi-Kisin module (resp. torsion quasi-Kisin module) of height 1, then \((\mathfrak{M} \otimes_{\mathcal{O}} \mathcal{O}_E, \varphi_{\mathfrak{M}} \otimes \varphi_E)\) is a projective (resp. torsion) étale \((\varphi, \mathcal{O}_E)\)-module since \(1 \otimes \varphi_{\mathfrak{M}}\) is injective (by Lemma 2.5 for torsion quasi-Kisin modules) and its cokernel is killed by \(E(u)\) which is a unit in \(\mathcal{O}_E\). If we denote by \(\mathcal{O}_{E,g}\) the corresponding ring for \(R_{0,g}\), then for any étale \((\varphi, \mathcal{O}_E)\)-module \(M\), \(M \otimes_{\mathcal{O}_{E,b}} \mathcal{O}_{E,g}\) with the induced Frobenius is an étale \((\varphi, \mathcal{O}_{E,g})\)-module.

We consider \(W(\mathcal{O}_E[\frac{1}{p}])\) as an \(\mathcal{O}_E\)-algebra via mapping \(u\) to the Teichmüller lift \([\omega]\) of \(\varphi\) and let \(\mathcal{O}_E^{\text{ur}}\) be the integral closure of \(\mathcal{O}_E\) in \(W(\mathcal{O}_E[\frac{1}{p}])\). Let \(\mathcal{O}^{\text{ur}}_E\) be its \(p\)-adic completion. Since \(\mathcal{O}_E\) is normal, we have \(\text{Aut}_{\mathcal{O}_E}(\mathcal{O}_E^{\text{ur}}) \cong \mathcal{G}_{E_{R_\infty}} := \pi_1(\text{Spec} E_{R_\infty})\), and by [GR03, Proposition 5.4.54] and the almost purity theorem, we have \(\mathcal{G}_{E_{R_\infty}} \cong \mathcal{G}_{E_{R_\infty}} \cong \mathcal{G}_{R_\infty}\). This induces \(\mathcal{G}_{R_\infty}\)-action on \(\mathcal{O}_E^{\text{ur}}\). The following is shown in [Kim15].

Lemma 3.2. (cf. [Kim15, Lemma 7.5 and 7.6]) We have \((\mathcal{O}^{\text{ur}}_E)^{\mathcal{G}_{R_\infty}} = \mathcal{O}_E\) and the same holds modulo \(p^a\). Furthermore, there exists a unique \(\mathcal{G}_{R_\infty}\)-equivariant ring endomorphism \(\varphi\) on \(\mathcal{O}^{\text{ur}}_E\) lifting the \(p\)-th power map on \(\mathcal{O}^{\text{ur}}_E/p\) and extending \(\varphi\) on \(\mathcal{O}_E\). The inclusion \(\mathcal{O}^{\text{ur}}_E \hookrightarrow W(\mathcal{O}_E[\frac{1}{p}])\) is \(\varphi\)-equivariant where the latter ring is given the Witt vector Frobenius.

Let \(\text{Rep}_{\mathbb{Z}_p}(\mathcal{G}_{R_\infty})\) be the category of finite continuous \(\mathbb{Z}_p\)-representations of \(\mathcal{G}_{R_\infty}\), and let \(\text{Rep}^\text{free}_{\mathbb{Z}_p}(\mathcal{G}_{R_\infty})\) and \(\text{Rep}^\text{tor}_{\mathbb{Z}_p}(\mathcal{G}_{R_\infty})\) respectively denote the full subcategories of free and torsion objects. For \(M \in \text{Mod}_{\mathcal{O}_E}\) and \(T \in \text{Rep}_{\mathbb{Z}_p}(\mathcal{G}_{R_\infty})\), we define \(T(M) := (M \otimes_{\mathcal{O}_E} \mathcal{O}_E^{\text{ur}})^{\varphi=1}\) and \(M(T) := (T \otimes_{\mathbb{Z}_p} \mathcal{O}_E^{\text{ur}})^{\mathcal{G}_{R_\infty}}\). Then we have the following proposition from [Kim15].

Proposition 3.3. ([Kim15, Proposition 7.7]) The constructions \(T(\cdot)\) and \(M(\cdot)\) give exact quasi-inverse equivalences of \(\otimes\)-categories between \(\text{Mod}_{\mathcal{O}_E}\) and \(\text{Rep}_{\mathbb{Z}_p}(\mathcal{G}_{R_\infty})\). Moreover, \(T(\cdot)\) and \(M(\cdot)\) restrict to rank-preserving equivalences of categories between \(\text{Mod}^\text{pr}_{\mathcal{O}_E}\) and \(\text{Rep}^\text{free}_{\mathbb{Z}_p}(\mathcal{G}_{R_\infty})\), and length-preserving equivalences between \(\text{Mod}^\text{tor}_{\mathcal{O}_E}\) and \(\text{Rep}^\text{tor}_{\mathbb{Z}_p}(\mathcal{G}_{R_\infty})\). In both cases, \(T(\cdot)\) and \(M(\cdot)\) commute with taking duals.
For $M$ in $\text{Mod}^{\text{pr}}_{G_{\mathbb{Z}}} \text{ (resp. in } \text{Mod}^{\text{tor}}_{G_{\mathbb{Z}}})$, we define the contravariant functor $T^\vee(\cdot)$ to $\text{Rep}_{\mathbb{Z}}(G_{\mathbb{R}_{\infty}})$ by $T^\vee(M) := \text{Hom}_{\mathbb{O}_E}\phi, (M, \tilde{\mathcal{O}}_{\mathbb{Z}}^{\text{pr}} \otimes \mathbb{Z}/\mathbb{Z}_p))$. Note that if we have a short exact sequence of étale $(\phi, \mathcal{O}_E)$-modules $0 \to M_1 \to M_2 \to M \to 0$ where $M_1, M_2$ are projective over $\mathcal{O}_E$ and $M$ is $p$-power torsion, then it induces a short exact sequence

$$0 \to T^\vee(M_2) \to T^\vee(M_1) \to T^\vee(M) \to 0$$

in $\text{Rep}_{\mathbb{Z}}(G_{\mathbb{R}_{\infty}})$.

Now, if $G_R$ is a $p$-divisible group over $R$, we write $T^\vee(G_R) := \text{Hom}_{\mathbb{R}(\mathbb{Q}_p/\mathbb{Z}_p, G_R \times R \mathbb{R})}$ to be the associated Tate module, which is a finite free $\mathbb{Z}_p$-representation of $G_R$. By [Kim15, Corollary 8.2], we have a natural $G_{\mathbb{R}_{\infty}}$-equivariant isomorphism $T^\vee(M^*G_R) \otimes_{\mathbb{O}_E} \mathcal{O}_E \cong T^\vee(G_R)$. If $H$ is a $p$-power order finite locally free group scheme over $R$, then $H(\mathbb{R})$ is a finite torsion $\mathbb{Z}_p$-representation of $G_R$. By [Kim15, Proposition 9.10], there exists a natural $G_{\mathbb{R}_{\infty}}$-equivariant isomorphism $T^\vee(M^*H) \otimes_{\mathbb{O}_E} \mathcal{O}_E \cong H(\mathbb{R})$, and if $H = \ker(h : G^0 \to G^1)$ for some isogeny $h$ of $p$-divisible groups over $R$, then the isomorphism $T^\vee(M^*H) \otimes_{\mathbb{O}_E} \mathcal{O}_E \cong H(\mathbb{R})$ is compatible with the isomorphisms $T^\vee(M^*(G^i) \otimes_{\mathbb{O}_E} \mathcal{O}_E) \cong T^\vee_{p}(G^i), \ i = 0, 1, 2$.

Note that any $p$-divisible group over $R_{\mathbb{Z}}$ is étale, so the category of $p$-divisible groups over $R_{\mathbb{Z}}$ is equivalent to the category of finite free $\mathbb{Z}_p$-representations of $G_R$. If we are given a $p$-divisible group $G$ over $R_{\mathbb{Z}}$, then the corresponding Galois representation is given by $T^\vee(G) := \text{Hom}_{\mathbb{R}(\mathbb{Q}_p/\mathbb{Z}_p, G \times R_{\mathbb{Z}} \mathbb{R}))}$. By Proposition 3.3, there exists a unique (up to isomorphism) projective étale $(\phi, \mathcal{O}_E)$-module $M$ such that $T^\vee(M) \cong T^\vee(G)$ as $G_{\mathbb{R}_{\infty}}$-representations. We remark that if $G$ extends to a $p$-divisible group $G_R$ over $R$, then $T^\vee(G_R) = T^\vee(G)$ as $G_R$-representations.

4 Extending $p$-divisible Groups

We now prove the generalization of Raynaud’s theorem for the relative base $R$ when $e < p - 1$, and use an example in [VZ10] on purity of $p$-divisible groups to show that when the ramification is large, such a generalization does not hold. We first consider the special case when the base ring $R_0$ as in Section 2 is equal to the formal power series ring over a Cohen ring.

**Proposition 4.1.** Suppose $R_0 = \mathcal{O}[s_1, \ldots, s_r]$ over a Cohen ring $\mathcal{O}$ and $e < p - 1$. Let $G$ be a $p$-divisible group over $R_{\mathbb{Z}}$, and let $n \geq 1$ be an integer. Suppose that $G[p^n]$ extends to a finite flat group scheme $G_{n,R}$ over $R$. Then for each integer $1 \leq m \leq n$, the group scheme $G_{n,R}[p^m]$ is finite flat over $R$.

Furthermore, if $H$ is another finite flat group scheme over $R$ extending $G[p^n]$ and if we identify the associated étale $(\phi, \mathcal{O}_E)$-modules $M_n := \mathfrak{M}^*(G_{n,R}) \otimes_{\mathbb{O}_E} \mathcal{O}_E = \mathfrak{M}^*(H) \otimes_{\mathbb{O}_E} \mathcal{O}_E$, then $\mathfrak{M}^*(G_{R,n}) = \mathfrak{M}^*(H)$ as $\mathfrak{G}$-submodules of $M_n$ with compatible Frobenius.
Proof. Let $M$ be the projective étale $(\varphi, \mathcal{O}_E)$-module such that $T^\vee(M) = T_p(G)$ as $\mathcal{G}_{\mathcal{R}_\infty}$-representations. Denote $\mathcal{M}_n = \mathfrak{M}^*(G_{n,R})$. Since $T_p(G[p^n]) \cong T_p(G)/p^n T_p(G)$, we have $M_n = \mathcal{M}_n \otimes_{\mathcal{O}_E} \mathcal{O}_E \cong M/p^n M$ as étale $(\varphi, \mathcal{O}_E)$-modules.

For proving the first statement, we can make the following choice of Frobenius on $R_0$ without loss of generality. Let $k' = \mathcal{O}/(p)$. Note that since $R_0/pR_0 \cong k'[s_1, \ldots, s_r]$ has a finite $p$-basis, we have $[k': k^p] < \infty$, i.e., $k'$ has a finite $p$-basis. Choose a Frobenius $\varphi : \mathcal{O} \to \mathcal{O}$ lifting the natural Frobenius on $W(k)$, and equip $R_0$ with Frobenius given by $\varphi_\mathcal{O}$ and $\varphi(s_i) = s_i^p$. Let $b_0 : R_0 \to \mathcal{O}$ be the $\mathcal{O}$-linear map given by $s_i \mapsto 0$, which is $\varphi$-equivariant. Let $b_g : R_0 \to R_{0, g} \cong W(k_g)$ be the $\varphi$-equivariant map considered in Section 2. Note that $\mathcal{M}_n \otimes_{\mathcal{O}, b_0} W(k_g)[u]$ and $\mathcal{M}_n \otimes_{\mathcal{O}, b_0} \mathcal{O}[u]$ with the induced diagonal Frobenius are torsion quasi-Kisin modules of height 1 over $W(k_g)[u]$ and $\mathcal{O}[u]$ respectively. Denote by $I_j$ the $j$-th Fitting ideal of $\mathcal{M}_n$ over $\mathcal{S}_n := \mathcal{S}/p^n \mathcal{S}$. Let $I_{j, 0}$ and $I_{j, g}$ be the $j$-th Fitting ideal of $\mathcal{M}_n \otimes_{\mathcal{S}, b_0} W(k_g)[u]$ and $\mathcal{M}_n \otimes_{\mathcal{S}, b_0} \mathcal{O}[u]$ over $W(k_g)[u]/(p^n)$ and $\mathcal{O}[u]/(p^n)$ respectively. Then $I_{j, 0}$ and $I_{j, g}$ are given by the images of $I_j$ under the corresponding maps $b_0$ and $b_g$ respectively.

Let $h$ be the height of $G$. Since $e < p - 1$, we deduce from [Liu07] Lemma 4.3.1 and Corollary 4.2.5 that $\mathcal{M}_n \otimes_{\mathcal{S}, b_g} W(k_g)[u]$ is free of rank $h$ over $W(k_g)[u]/(p^n)$. Furthermore, if we denote by $\mathcal{O}_g$ the $p$-adic completion of $\varprojlim_\varphi \mathcal{O}/(p)$ with the induced Frobenius and $\kappa := \varprojlim_\varphi \mathcal{O}/(p)$, then by the universal property of $p$-adic Witt vectors as in Section 2, $\mathcal{O}_g \cong W(\kappa)$ compatibly with Frobenius endomorphisms. The map $\mathcal{O}[u]/(p^n) \to W(\kappa)[u]/(p^n)$ is faithfully flat, and the induced torsion Kisin module $(\mathcal{M}_n \otimes_{\mathcal{S}, b_0} \mathcal{O}[u]) \otimes_{\mathcal{O}[u]} W(\kappa)[u]$ is free of rank $h$ over $W(\kappa)[u]/(p^n)$ by loc. cit. Hence, $\mathcal{M}_n \otimes_{\mathcal{S}, b_0} \mathcal{O}[u]$ is free of rank $h$ over $\mathcal{O}[u]/(p^n)$. We obtain

$$I_{j, g} = \begin{cases} 0 & \text{if } j < h \\ W(k_g)[u]/(p^n) & \text{if } j \geq h, \end{cases}$$

$$I_{j, 0} = \begin{cases} 0 & \text{if } j < h \\ \mathcal{O}[u]/(p^n) & \text{if } j \geq h. \end{cases}$$

By Lemma 2.1, the map $\mathcal{S}_n \to W(k_g)[u]/(p^n)$ induced from $b_g$ is injective. For $j < h$, the image of $I_j$ under $b_g$ in $W(k_g)[u]/(p^n)$ is equal to $I_{j, g}$ which is 0. Thus, $I_j = 0$ if $j < h$. Suppose $j \geq h$. If $I_j$ is contained in the maximal ideal $(p, s_1, \ldots, s_r, u)$ of $\mathcal{S}_n$, then the image of $I_j$ under $b_0$ would be contained in the maximal ideal of $\mathcal{O}[u]/(p^n)$. Since $I_{j, 0} = \mathcal{O}[u]/(p^n)$, we have $I_j = \mathcal{S}_n$. Hence, $\mathcal{M}_n$ is projective and thus free of rank $h$ over $\mathcal{S}_n$. By Theorem 2.3 $G_{n,R}[p^m]$ is finite flat over $R$ for each $m \geq 1$.

Now we show the second statement, for any choice of Frobenius on $R_0$. Suppose that $G[p^n]$ extends to another finite flat group scheme $H$ over $R$, and let $\mathfrak{N} := \mathfrak{M}^*(H)$ be the associated torsion Kisin module. Identify $\mathfrak{N} \otimes_{\mathcal{O}_E} \mathcal{O}_E = \mathcal{M}_n \otimes_{\mathcal{O}_E} \mathcal{O}_E = M_n$ as étale $(\varphi, \mathcal{O}_E)$-modules, and consider both $\mathfrak{N}$ and $\mathcal{M}_n$ as $\mathcal{S}_n$-submodules of $M_n$. Since $G_{n,R}[p^m]$ is finite flat over $R$ for each $m \geq 1$ and similarly for $H$, and since $M_n$ is projective over $\mathcal{O}_{E,n} := \mathcal{O}_E/(p^n)$,
we have by Theorem 2.6 that \( \mathfrak{M}_n \) and \( \mathfrak{N} \) are projective and thus flat over \( \mathcal{S}_n \). By [Liu07 Corollary 4.2.5], we have \( \mathfrak{M}_n \otimes_{\mathcal{O}_k} W(k)[u] = \mathfrak{N} \otimes_{\mathcal{O}_k} W(k)[u] \) as \( W(k)[u] \)-submodules of \( M_n \otimes \mathcal{O} W(k)[u] \). Note that by Lemma 2.11 the induced map \( \mathcal{O}_{\epsilon,n} \to W_n(k)[u][\frac{1}{u}] \) is injective, and \( \mathcal{O}_{\epsilon,n} \cap W_n(k)[u] = \mathcal{S}_n \) as subrings of \( W_n(k)[u][\frac{1}{u}] \). Since \( \mathfrak{M}_n \) is flat over \( \mathcal{S}_n \), we deduce

\[
(\mathfrak{M}_n \otimes_{\mathcal{S}_n} \mathcal{O}_{\epsilon,n}) \cap (\mathfrak{M}_n \otimes_{\mathcal{S}_n} W_n(k)[u][\frac{1}{u}]) = \mathfrak{M}_n \otimes_{\mathcal{S}_n} \mathcal{S}_n = \mathfrak{M}_n
\]
as \( \mathcal{S}_n \)-submodules of \( \mathfrak{M}_n \otimes_{\mathcal{S}_n} W_n(k)[u][\frac{1}{u}] = M_n \otimes \mathcal{O} W(k)[u] \), and similarly

\[
(\mathfrak{N} \otimes_{\mathcal{S}_n} \mathcal{O}_{\epsilon,n}) \cap (\mathfrak{N} \otimes_{\mathcal{S}_n} W_n(k)[u][\frac{1}{u}]) = \mathfrak{N}
\]
as \( \mathcal{S}_n \)-submodules of \( \mathfrak{N} \otimes_{\mathcal{S}_n} W_n(k)[u][\frac{1}{u}] = M_n \otimes \mathcal{O} W(k)[u] \). Since \( \mathfrak{M}_n \otimes_{\mathcal{S}_n} \mathcal{O}_{\epsilon,n} = M_n = \mathfrak{N} \otimes_{\mathcal{S}_n} \mathcal{O}_{\epsilon,n} \) and \( \mathfrak{M}_n \otimes_{\mathcal{S}_n} W_n(k)[u][\frac{1}{u}] = \mathfrak{N} \otimes_{\mathcal{S}_n} W_n(k)[u][\frac{1}{u}] \) as submodules of \( M_n \otimes \mathcal{O} W(k)[u] \), we obtain \( \mathfrak{M}_n = \mathfrak{N} \) with compatible Frobenius.

We remark that in the second statement of above Proposition 4.1 we do not know whether \( \mathfrak{M}^*(G_{R,n}) \cong \mathfrak{M}^*(H) \) as Kisin modules, i.e., whether the connections on both sides are compatible.

Now we consider the general base ring \( R \) as in Section 2.

**Theorem 4.2.** Assume \( e < p - 1 \). Let \( G \) be a \( p \)-divisible group over \( R[\frac{1}{p}] \). Suppose that for each \( n \), \( G[p^n] \) extends to a finite locally free group scheme \( G_{n,R} \) over \( R \). Then \( G \) extends to a \( p \)-divisible group over \( R \), and such an extension is unique up to isomorphism.

If \( e \geq p \) and \( R = \mathcal{O}_K[s] \), then there exists a \( p \)-divisible group \( G \) over \( R[\frac{1}{p}] \) such that \( G[p^n] \) extends to a finite locally free group scheme \( G_{n,R} \) over \( R \) for each \( n \) but \( G \) does not extend to a \( p \)-divisible group over \( R \).

**Proof.** Suppose \( e < p - 1 \). Let \( M \) be the projective étale \( (\varphi, \mathcal{O}_k) \)-module such that \( T^\vee(M) = T_p(G) \) as \( \mathcal{G}_{R_{\alpha}} \)-representations. For each \( n \geq 1 \), let \( \mathfrak{M}_n := \mathfrak{M}^*(G_{n,R}) \in \text{Mod}_{\varphi}^\text{ét}(\varphi, \nabla) \) be the torsion Kisin module of height 1 corresponding to \( G_{n,R} \). We have \( \mathfrak{M}_n \otimes \mathcal{O}_k \cong M_n := M/p^n M \) as étale \( (\varphi, \mathcal{O}_k) \)-modules. Let \( h \) be the height of \( G \).

For each maximal ideal \( q \) of \( R \), denote \( q_0 := q \cap R_0 \subset R_0 \) the corresponding maximal ideal of \( R_0 \), and let \( b_q : R_0 \to \widehat{R}_{0,q_0} \) be the natural \( \varphi \)-equivariant map where \( \widehat{R}_{0,q_0} \) denotes the \( q_0 \)-adic completion of \( R_{0,q_0} \). By the structure theorem for complete regular local rings, \( \widehat{R}_{0,q_0} \) is isomorphic to a formal power series ring \( \widehat{R}_{0,q_0} \cong \mathcal{O}[s_1, \ldots, s_r] \) over a Cohen ring \( \mathcal{O} \). We have the induced base change \( b_q : R \to \widehat{R}_q \cong \widehat{R}_{0,q_0} \otimes_{\mathcal{O}} \mathcal{O}_K, \) where \( \widehat{R}_q \) is the \( q \)-adic completion of \( R_q \). Denote \( \mathcal{S}_q := \widehat{R}_{0,q_0}[u] \). For the \( p \)-divisible group \( G \times_{R[\frac{1}{p}],b_q} \widehat{R}_q[\frac{1}{p}] \) over \( \widehat{R}_q[\frac{1}{p}] \), note that \( (G \times_{R[\frac{1}{p}],b_q} \widehat{R}_q[\frac{1}{p}])[p^n] \) extends to the finite locally free group scheme \( G_{n,q} := G_{n,R} \times_{R,b_q} \widehat{R}_q \) over \( \widehat{R}_q \) for each \( n \geq 1 \). By Proposition 4.1 \( G_{n,q}[p^n] \) is finite locally free over \( \widehat{R}_q \) for each \( m \geq 1 \), and thus \( \mathfrak{M}^*(G_{n,q}) := \mathfrak{M}_n \otimes_{\mathcal{S}_q} \mathcal{S}_q \) is projective over \( \mathcal{S}_q/(p^n) \).
by Theorem [2.6]. Since this holds for each maximal ideal \( q \) of \( R \), we deduce that \( \mathcal{M}_n \) is projective over \( \mathcal{G}/(p^n) \) of rank \( h \). In particular, \( G_{n,R}[p^m] \) is finite locally free over \( R \) for each \( m \geq 1 \). Note that \( G_{n,R}[p^m] \times_R R[p^{\frac{1}{p}}] \cong (G_{n,R} \times_R R[p^{\frac{1}{p}}])[p^m] \cong G[p^m] \), and \( G_{n,R}[p^m] \) has order \( p^{mh} \) for each \( 1 \leq m \leq n \).

By considering the orders of the groups, we see that the natural sequence of finite locally free group schemes

\[
0 \to G_{n+1,R}[p] \to G_{n+1,R} \to G_{n+1,R}[p^n] \to 0,
\]

where the map \( G_{n+1,R} \to G_{n+1,R}[p^n] \) is induced by multiplication by \( p \), is short exact. Furthermore, it follows easily from the construction of the functor \( \mathcal{M}^*(\cdot) \) in [Kim15] Proof of Proposition 9.5 using isogeny of \( p \)-divisible groups that \( \mathcal{M}^*(G_{n+1,R}[p]) \cong \mathcal{M}_{n+1}/p\mathcal{M}_{n+1} \) as torsion Kisin modules, where \( \mathcal{M}_{n+1}/p\mathcal{M}_{n+1} \) is equipped with Frobenius and connection induced from \( \mathcal{M}_{n+1} \). Since \( \mathcal{M}^*(\cdot) \) is exact, we have \( \mathcal{M}^*(G_{n+1,R}[p^n]) \cong p\mathcal{M}_{n+1} \) where \( p\mathcal{M}_{n+1} \) is equipped with Frobenius and connection induced from \( \mathcal{M}_{n+1} \). We claim that \( \mathcal{M}_n \cong p\mathcal{M}_{n+1} \) as torsion quasi-Kisin modules with compatible Frobenius. Identify \( p\mathcal{M}_{n+1} \otimes_{\mathcal{G}} \mathcal{O}_E = M_n = \mathcal{M}_n \otimes_{\mathcal{G}} \mathcal{O}_E \) as étale \( (\varphi, \mathcal{O}_E) \)-modules, and consider both \( p\mathcal{M}_{n+1} \) and \( \mathcal{M}_n \) as \( \mathcal{G}_n \)-submodules of \( M_n \). For the natural injective map \( \mathcal{M}_n \hookrightarrow \mathcal{M}_n + p\mathcal{M}_{n+1} \) of \( \mathcal{G}_n \)-modules, consider the induced map \( \mathcal{M}_n \otimes_{\mathcal{G}_n, b_q} \mathcal{G}_q \to (\mathcal{M}_n + p\mathcal{M}_{n+1}) \otimes_{\mathcal{G}_n, b_q} \mathcal{G}_q \) for each maximal ideal \( q \) of \( R \). Since \( b_q : \mathcal{G} \to \mathcal{G}_q \) is flat, we have \((\mathcal{M}_n + p\mathcal{M}_{n+1}) \otimes_{\mathcal{G}_n, b_q} \mathcal{G}_q = \mathcal{M}_n \otimes_{\mathcal{G}_n, b_q} \mathcal{G}_q + p\mathcal{M}_{n+1} \otimes_{\mathcal{G}_n, b_q} \mathcal{G}_q \) as submodules of \( \mathcal{M}_n \otimes_{\mathcal{G}_n, b_q} \mathcal{G}_q \). Thus, \( \mathcal{M}_n \otimes_{\mathcal{G}_n} \mathcal{G}_q \cong (\mathcal{M}_n + p\mathcal{M}_{n+1}) \otimes_{\mathcal{G}_n} \mathcal{G}_q \) for each \( q \), which implies that injective map \( \mathcal{M}_n \hookrightarrow \mathcal{M}_n + p\mathcal{M}_{n+1} \) is also surjective. Thus, \( p\mathcal{M}_{n+1} \subset \mathcal{M}_n \), and similarly \( \mathcal{M}_n \subset p\mathcal{M}_{n+1} \). This shows the claim \( \mathcal{M}_n = p\mathcal{M}_{n+1} \) with compatible Frobenius.

Thus, \( \mathcal{M} := \varprojlim_n \mathcal{M}_n \) with the induced Frobenius is a quasi-Kisin module of height 1 over \( \mathcal{G} \). We now equip \( \mathcal{M} := \mathcal{M} \otimes_{\mathcal{G}, \varphi} R_0 \) with a connection. Denote by \( \nabla_{\mathcal{M}_k} : \mathcal{M}_k \otimes_{\mathcal{G}_k, \varphi} R_0 \to (\mathcal{M}_k \otimes_{\mathcal{G}_k, \varphi} R_0) \otimes_{R_0} \widehat{\mathcal{O}}_{R_0} \) the connection for the torsion Kisin module \( \mathcal{M}_n \), and let \( \mathcal{M}_n = \mathcal{M} \otimes_{R_0} R_0/(p^n) \). Consider the multisets

\[
S_n = \{\nabla_{\mathcal{M}_k} \otimes_{R_0} R_0/(p^n) \mid k \geq n + 1\}
\]

of connections on \( \mathcal{M}_n \). Note that for each \( k \geq n + 1 \), the connection \( \nabla_{\mathcal{M}_k} \otimes_{R_0} R_0/(p^n) \) satisfies the commutative diagram (2.1) in Section 2. Using the result discussed at the end of Section 2 we choose a compatible system of connections \( \nabla_n \) on \( \mathcal{M}_n \) inductively as follows. Identify \( \widehat{\mathcal{O}}_{R_0} = \bigoplus_{i=1}^d R_0 \cdot d(\log t_i) \). Let \( \mathcal{M}_1 \subset \mathcal{M} \) be a direct factor lifting \( \text{Fil}^1 \mathcal{M}/p\mathcal{M} \subset \mathcal{M}/p\mathcal{M} \) as in Section 2 and we fix a choice of a finite Zariski covering of \( \text{Spf}(R_0, p) \) over which \( \mathcal{M}_1 \) and \( \mathcal{M}/\mathcal{M}_1 \) are free, and fix a basis of \( \mathcal{M} \) adapted to \( \mathcal{M}_1 \) after passing to the covering. For \( n = 1 \), \( S_1 \) is finite as a set of connections on \( \mathcal{M}_1 \), and we choose a connection \( \nabla_1 \) on \( \mathcal{M}_1 \) which has infinite multiplicity in the multisets \( S_1 \). When we are given a choice of connection \( \nabla_n \) on \( \mathcal{M}_n \), the elements in \( S_{n+1} \) which lift \( \nabla_n \) are contained in a finite set of connections, and we choose a connection \( \nabla_{n+1} \) on \( \mathcal{M}_{n+1} \) which has infinite multiplicity in the multisets \( S_{n+1} \).
multiplicity in $S_{n+1}$. Let $\nabla := \lim_p \nabla_n$ be the induced connection on $\mathcal{M}$. Then $\nabla$ is compatible with Frobenius, integrable, and topologically quasi-nilpotent. Hence, $(\mathfrak{M}, \nabla)$ is a Kisin module of height 1, and the corresponding $p$-divisible group over $R$ extends $G$. The uniqueness of extending $G$ up to isomorphism follows from [Tat67, Theorem 4].

On the other hand, assume $e \geq p$ and $R_0 = W(k)[s]$. Let $U = \text{Spec} R \setminus \{m\}$ be the open subscheme of $\text{Spec} R$, where $m$ is the closed point given by the maximal ideal of $R$. By [VZ10, Theorem 28], there exists a $p$-divisible group $G_U$ over $U$ which does not extend to a $p$-divisible group over $R$. By [FC90, Chapter V, Lemma 6.2], for each $n \geq 1$, the finite locally free group scheme $G_U[p^n]$ extends uniquely to a finite locally free group scheme over $R$ (if $A$ denotes the Hopf algebra for $G_U[p^n] \times_U R[\frac{1}{p}]$ and $B$ denotes the Hopf algebra for $G_U[p^n] \times_U R[\frac{1}{p}]$), then identifying $C := A[\frac{1}{s}] = B[\frac{1}{p}]$ as the Hopf algebra for $G_U[p^n] \times_U R[\frac{1}{p}]$, the unique extension is given by $A \cap B$ with the induced Hopf algebra structure over $R$). Let $G = G_U \times_U R[\frac{1}{p}]$ be the $p$-divisible group over $R_0[\frac{1}{p}]$, and suppose $G$ extends to a $p$-divisible group $G_R$ over $R$. Since $G_U \times_U (R[\frac{1}{s}])[\frac{1}{p}] = G_R \times_R (R[\frac{1}{s}])[\frac{1}{p}]$, we have by [Tat67, Theorem 4] that $G_U \times_U R[\frac{1}{p}] = G_R \times_R R[\frac{1}{p}]$. Thus, $G_R \times_R U = G_U$, which contradicts to that $G_U$ does not extend over $R$. This shows that $G$ cannot be extended to a $p$-divisible group over $R$. \hfill \Box

5 Barsotti-Tate Deformation Ring for Relative Base of Dimension 2

Throughout this section, we assume that the Krull dimension of $R$ is equal to 2. For a finite $\mathbb{Q}_p$-representation $V$ of $G_R$, we say it is Barsotti-Tate if there exists a $p$-divisible group $G_R$ over $R$ such that $V = T_p(G_R) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ as $G_R$-representations.

**Proposition 5.1.** Assume $e < p - 1$. Let $T$ be a finite free $\mathbb{Z}_p$-representation of $G_R$ such that $T[\frac{1}{p}]$ is Barsotti-Tate. Then there exists a $p$-divisible group $G_R$ over $R$ such that $T = T_p(G_R)$.

**Proof.** Since $T[\frac{1}{p}]$ is Barsotti-Tate, there exists a $p$-divisible group $G'_R$ over $R$ such that $T_p(G'_R)[\frac{1}{p}] = T[\frac{1}{p}]$. Denote $T' = T_p(G'_R)$, $G' = G'_R \times_R R[\frac{1}{p}]$, and let $G$ be the $p$-divisible group over $R[\frac{1}{p}]$ corresponding to the representation $T$.

Since $p^n T \subset T'$ and $p^n T' \subset T$ for some positive integer $n$, we have an isogeny $f : G' \rightarrow G$. Let $H := \ker(f)$, which is a finite locally free group scheme over $R[\frac{1}{p}]$. Then we have a closed immersion $h : H \hookrightarrow G'[p^m]$ for some positive integer $m$. Note that $G'[p^m]$ extends to the finite locally free group scheme $G'_R[p^m]$ over $R$.

Let $H_R$ be the scheme theoretic closure of $H$ over $R$ obtained from $h$ and $G'_R[p^m]$, given similarly as in [Ray74, Section 2.1]. By the construction of the scheme theoretic closure, $H_R$ is a finite group scheme. We claim that it is locally free over $R$. For that, let $q$ be a
maximal ideal of $R$ and let $q_0 = q \cap R_0$, and consider the base change map $b_q : R \to \hat{R}_q$ as in the proof of Theorem 12. Since $R$ has Krull dimension 2, we have $\hat{R}_q \cong \mathcal{O}_{q_0}[s] \otimes_{W(k)} \mathcal{O}_K$ for some Cohen ring $\mathcal{O}_{q_0}$ with the maximal ideal $(p)$. Let $U_q \subseteq \text{Spec}\hat{R}_q$ be the closed subscheme obtained by deleting the closed point given by $q$. Since $U_q$ is a Dedekind scheme, $(H_R \times_R \hat{R}_q) \otimes_{\hat{R}_q} U_q$ is locally free over $U_q$ as the corresponding sheaf of Hopf algebras is torsion free. It extends uniquely to a finite locally free group scheme $H_q$ over $\hat{R}_q$ by [FC90, Chapter V, Lemma 6.2]. On the other hand, since $e < p - 1$, note that $p \notin (q \hat{R}_q)^{p-1}$. Since $h$ is a monomorphism, we deduce from [VZ10, Proposition 15] applied for $\hat{R}_q$ that the map $H_q \to G'_R[p^m] \times_R \hat{R}_q$ of finite flat group schemes is a monomorphism and hence a closed immersion. Thus, $H_R \times_R \hat{R}_q = H_q$. Since this holds for every maximal ideal $q$ of $R$, $H_R$ is locally free over $R$.

The map $h$ induces a closed immersion $H_R \hookrightarrow G'_R[p^m]$, and $G_R := G'_R/H_R$ is a $p$-divisible group over $R$. It is clear from the construction that $T_p(G_R) = T$ as $\mathbb{Z}_p[G_R]$-modules.

For a finite free $\mathbb{Z}_p$-representation $T$ of $G_R$, it makes sense by Proposition 5.1 to say that $T$ is Barsotti-Tate if there exists a $p$-divisible group $G_R$ over $R$ such that $T = T_p(G_R)$.

**Lemma 5.2.** Assume $e < p - 1$. Let $H_R$ be a $p$-power order finite locally free group scheme over $R$, and let $T = H_R(\hat{R})$ be the corresponding torsion $\mathbb{Z}_p$-representation of $G_R$. If we have a short exact sequence of $\mathbb{Z}_p[G_R]$-modules

$$0 \to T_1 \to T \to T_2 \to 0,$$

then there exist $p$-power order finite locally free group schemes $H_{1,R}$ and $H_{2,R}$ over $R$ such that $T_i = H_{i,R}(\hat{R})$ for $i = 1, 2$ as $G_R$-representations.

**Proof.** Let $H := H_R \times_R R[[1/p]]$. Let $H_i$ for $i = 1, 2$ be finite locally free group schemes over $R[[1/p]]$ such that $H_i(\hat{R}[[1/p]]) = T_i$ as $G_R$-representations. The given exact sequence of $G_R$-representations induce the short exact sequence

$$0 \to H_1 \to H \to H_2 \to 0$$

of finite locally free group schemes. Let $H_{1,R}$ be the scheme theoretic closure of $H_1$ over $R$ obtained from the closed embedding $H_1 \hookrightarrow H$ and $H_R$. By the same argument as in the proof of Proposition 5.1, $H_{1,R}$ is a finite locally free group scheme over $R$ extending $H_1$. Furthermore, $H_{2,R} := H_R/H_{1,R}$ is a finite locally free group scheme over $R$ extending $H_2$ (cf. [Ray67]). It is clear that $T_i = H_{i,R}(\hat{R})$ for $i = 1, 2$. 

**Corollary 5.3.** Assume $e < p - 1$. Let $A_1 \hookrightarrow A_2$ be an injective map of finite free $\mathbb{Z}_p$-algebras. Let $T_{A_1}$ be a finite free $A_1$-module given the $p$-adic topology and equipped with a continuous $A_1$-linear $G_R$-action. Let $T_{A_2} := T_{A_1} \otimes_{A_1} A_2$ be the induced representation with the $A_2$-linear $G_R$-action. Then $T_{A_1}$ is Barsotti-Tate if and only if $T_{A_2}$ is Barsotti-Tate.
Barsotti-Tate by Theorem 4.2. A given by $T$.

Since by Lemma 5.2, $G$ is injective. Hence, by Lemma 5.2 and Theorem 4.2 similarly as above, $T$ is Barsotti-Tate by Proposition 5.1.

Conversely, suppose $T$ is Barsotti-Tate. Let $B_3$ be the quotient of the induced injection $A_1[\frac{1}{p}] \to A_2[\frac{1}{p}]$ of $\mathbb{Q}_p$-algebras, and let $T \subset T_A$ be the kernel of the induced map of representations $T_A \to T_{A_2} \otimes_{A_1} B_3$. Then for each integer $n \geq 1$, the map $T/p^n \to T_{A_2}/p^n$ is injective. Hence, by Lemma 5.2 and Theorem 4.2 similarly as above, $T$ is Barsotti-Tate. Since $T[\frac{1}{p}] = T_A[\frac{1}{p}]$, $T_A$ is Barsotti-Tate by Proposition 5.1.

We now study the geometry of the locus of Barsotti-Tate representations. Denote by $\mathcal{C}$ the category of topological local $\mathbb{Z}_p$-algebras $A$ satisfying the following conditions:

- the natural map $\mathbb{Z}_p \to A/m_A$ is surjective, where $m_A$ denotes the maximal ideal of $A$;
- the map from $A$ to the projective limit of its discrete artinian quotients is a topological isomorphism.

Note that the first condition implies that the residue field of $A$ is $\mathbb{F}_p$. The second condition is equivalent to the condition that $A$ is complete and its topology can be given by a collection of open ideals $a \subset A$ for which $A/a$ is artinian. Morphisms in $\mathcal{C}$ are continuous $\mathbb{Z}_p$-algebra morphisms. The following proposition is shown in [SL97].

**Proposition 5.4.** (cf. [SL97, Proposition 2.4]) Suppose $A$ is a Noetherian ring in $\mathcal{C}$. Then the topology on $A$ is equal to the $m_A$-adic topology.

For $A \in \mathcal{C}$, we mean by an $A$-representation of $G_R$ a finite free $A$-module equipped with a continuous $A$-linear $G_R$-action. We fix an $\mathbb{F}_p$-representation $V_0$ of $G_R$ which is absolutely irreducible. For $A \in \mathcal{C}$, a deformation of $V_0$ in $A$ is an isomorphism class of $A$-representations of $V$ of $G_R$ satisfying $V \otimes A \mathbb{F}_p \cong V_0$ as $\mathbb{F}_p[G_R]$-modules. We denote by $\text{Def}(V_0, A)$ the set of such deformations. A morphism $f : A \to A'$ in $\mathcal{C}$ induces a map $f_* : \text{Def}(V_0, A) \to \text{Def}(V_0, A')$ sending the class of an $A$-representation $V$ to the class of $V \otimes_{A,f} A'$. The following theorem on universal deformation ring is proved in [SL97].

**Theorem 5.5.** (cf. [SL97, Theorem 2.3]) There exists a universal deformation ring $A_{univ} \in \mathcal{C}$ and a deformation $V_{univ} \in \text{Def}(V_0, A_{univ})$ such that for all $A \in \mathcal{C}$, we have a bijection

$$
\text{Hom}_{\mathcal{C}}(A_{univ}, A) \stackrel{\cong}{\to} \text{Def}(V_0, A)
$$

(5.1)
given by $f \mapsto f_*(V_{univ})$.  

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We remark that $A_{\text{univ}}$ is Noetherian if and only if $\dim_{\mathbb{F}_p} H^1(G_R, \text{End}_{\mathbb{F}_p}(V_0))$ is finite (cf. loc. cit.). Thus, $A_{\text{univ}}$ is not Noetherian in general, even when $R = \mathcal{O}_K$ if $K/\mathbb{Q}_p$ is infinite.

Let $C^0$ be the full subcategory of $C$ consisting of artinian rings. Abusing the notation, we write $V \in \text{Def}(V_0, A)$ for an $A$-representation $V$ to mean that $V \otimes_A \mathbb{F}_p \cong V_0$. For $A \in C^0$ and a representation $V_A \in \text{Def}(V_0, A)$, we say $V_A$ is torsion Barsotti-Tate if there exists a $p$-power order finite locally free group scheme $H_R$ over $R$ such that $V_A \cong H_R(R)$ as $\mathbb{Z}_p[H_R]$-modules. We remark that if $R$ is local, then every $p$-power order finite locally free group scheme over $R$ embeds into a $p$-divisible group over $R$, and thus $V_A$ is torsion Barsotti-Tate if and only if it is a quotient of a finite free $\mathbb{Z}_p$-representation which is Barsotti-Tate. For $A \in C$, denote by $BT(V_0, A)$ the subset of $\text{Def}(V_0, A)$ consisting of the isomorphism classes of representations $V_A$ such that $V_A \otimes_A A/\mathfrak{a}$ is torsion Barsotti-Tate for all open ideals $\mathfrak{a} \subsetneq A$.

**Proposition 5.6.** Assume $e < p - 1$. For any $C$-morphism $f : A \to A'$, we have $f_* (BT(V_0, A)) \subset BT(V_0, A')$. Furthermore, there exists a closed ideal $\mathfrak{a}_{BT}$ of the universal deformation ring $A_{\text{univ}}$ such that the map \((5.7)\) induces a bijection $\text{Hom}_C(A_{\text{univ}}/\mathfrak{a}_{BT}, A) \xrightarrow{\cong} BT(V_0, A)$.

**Proof.** We check the conditions in [SL97, Section 6]. Let $f : A \hookrightarrow A'$ be an injective morphism of artinian rings in $C$, and let $V_A \in \text{Def}(V_0, A)$ be a representation. We first claim that $V_A \in BT(V_0, A)$ if and only if $V_A' := V_A \otimes_A f A' \in BT(V_0, A')$. Suppose that $V_A \in BT(V_0, A)$. Note that $A'$ is a finite $A$-module. Let $x_1, \ldots, x_m$ generate $A'$ over $A$. Then we have a surjective map of $\mathbb{Z}_p[H_R]$-modules $V_A^m \twoheadrightarrow V_A'$ sending the canonical basis elements $e_i$ of $V_A^m$ for $i = 1, \ldots, m$ to $x_i$. Since $V_A^m$ is the direct sum of $m$-copies of $V_A$, it is torsion Barsotti-Tate. Thus, by Lemma 5.2, $V_A' \in BT(V_0, A')$. Conversely, suppose $V_A' \in BT(V_0, A')$. Since we have an injective map of $\mathbb{Z}_p[H_R]$-modules $V_A \hookrightarrow V_A'$, we get $V_A \in BT(V_0, A)$ by Lemma 5.2.

Now, for $A \in C$ and a representation $V_A \in \text{Def}(V_0, A)$, suppose $\mathfrak{a}_1, \mathfrak{a}_2 \subsetneq A$ are open ideals such that $V_A \otimes_A (A/\mathfrak{a}_i) \in BT(V_0, A/\mathfrak{a}_i)$ for $i = 1, 2$. The natural map $A/(\mathfrak{a}_1 \cap \mathfrak{a}_2) \to A/\mathfrak{a}_1 \oplus A/\mathfrak{a}_2$ is injective, and it induces the injective map of $\mathbb{Z}_p[H_R]$-modules $V_A \otimes_A A/(\mathfrak{a}_1 \cap \mathfrak{a}_2) \hookrightarrow (V_A \otimes_A A/\mathfrak{a}_1) \oplus (V_A \otimes_A A/\mathfrak{a}_2)$.

Since the direct sum $(V_A \otimes_A A/\mathfrak{a}_1) \oplus (V_A \otimes_A A/\mathfrak{a}_2)$ is torsion Barsotti-Tate, we see from Lemma 5.2 that $V_A \otimes_A A/(\mathfrak{a}_1 \cap \mathfrak{a}_2) \in BT(V_0, A/(\mathfrak{a}_1 \cap \mathfrak{a}_2))$. The assertion then follows from [SL97, Proposition 6.1].

We now show that when $e < p - 1$, the locus of Barsotti-Tate representations cuts out a closed subscheme of the universal deformation scheme $\text{Spec}(A_{\text{univ}})$:

**Theorem 5.7.** Suppose $e < p - 1$ (and recall that the Krull dimension of $R$ is assumed to be equal to 2). Let $A$ be a finite flat $\mathbb{Z}_p$-algebra equipped with the $p$-adic topology, and let $f : A_{\text{univ}} \to A$ be a continuous $\mathbb{Z}_p$-algebra homomorphism. Then the induced representation $V_{\text{univ}} \otimes_{A_{\text{univ}}, f} A[1/p]$ of $G_R$ is Barsotti-Tate if and only if $f$ factors through the quotient $A_{\text{univ}}/\mathfrak{a}_{BT}$.  

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Proof. Let $A_1 := \text{im}(f) \subset A$, and let $T_{A_1} = V_{\text{univ}} \otimes_{A_{\text{univ}, f}} A_1$. Then $T_{A_1} \otimes_{A_1} A = V_{\text{univ}} \otimes_{A_{\text{univ}, f}} A$, and by Proposition 5.1 and Corollary 5.3 it suffices to show that $T_{A_1}$ is Barsotti-Tate if and only if $f$ factors through $A_{\text{univ}}/a_{\text{BT}}$. Note that $A_1 \in C$, and since $A_1$ is finite flat over $\mathbb{Z}_p$, the topology on $A_1$ is equivalent to the $p$-adic topology and $f : A_{\text{univ}} \to A_1$ is continuous by Proposition 5.4. Suppose first that $T_{A_1}$ is Barsotti-Tate, so that there exists a $p$-divisible group $G_R$ over $R$ such that $T_p(G_R) \cong T_{A_1}$. For each integer $n \geq 1$, we then have $(V_{\text{univ}} \otimes_{A_{\text{univ}, f}} A_1) \otimes_{A_1} A_1/(p^n) = T_{A_1}/p^n \cong (G_R[p^n])(R)$, so $V_{\text{univ}} \otimes_{A_{\text{univ}, f}} A_1/(p^n) \in \text{BT}(V_0, A_1/(p^n))$. Hence, by Proposition 5.6, $f$ factors through $A_{\text{univ}}/a_{\text{BT}}$.

Conversely, suppose $f$ factors through $A_{\text{univ}}/a_{\text{BT}}$. Let $G$ be the $p$-divisible group over $R[[\frac{1}{p}]]$ corresponding to $T_{A_1}$. For each $n \geq 1$, $T_{A_1}/p^n$ is torsion Barsotti-Tate by Proposition 5.6, so $G[p^n]$ extends to a finite locally free group scheme over $R$. Then by Theorem 4.2, $T_{A_1}$ is Barsotti-Tate.

On the other hand, if the ramification is large, we can deduce that the locus of Barsotti-Tate representations is not $p$-adically closed in general:

**Proposition 5.8.** Let $R = \mathcal{O}_K[[s]]$ and suppose $e \geq p$. There exists a $\mathbb{Z}_p$-representation $T$ of $\mathcal{G}_R$ such that $T/p^nT$ is torsion Barsotti-Tate for each $n \geq 1$ but $T$ is not Barsotti-Tate.

**Proof.** By Theorem 4.2 there exists a $p$-divisible group $G$ over $R[[\frac{1}{p}]]$ such that $G[p^n]$ extends to a finite locally free group scheme $G_{n,R}$ over $R$ but $G$ does not extend to a $p$-divisible group over $R$. Let $T$ be the representation corresponding to $G$. Then for each $n$, we have $T/p^nT \cong G_{n,R}(R)$ so it is torsion Barsotti-Tate. However, $T$ is not Barsotti-Tate since $G$ does not extend over $R$.

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