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Some examples of ‘second order elliptic integrable systems associated to a 4-symmetric space’

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1 Hamiltonian Stationary Lagrangian (HSL) surfaces

1.1 A variational problem in \( \mathbb{R}^4 \)

\( \mathbb{R}^4 \) has the canonical Euclidean structure \( \langle \cdot, \cdot \rangle \) and the symplectic form \( \omega := dx^1 \wedge dx^2 + dx^3 \wedge dx^4 \). An immersed surface \( \Sigma \subset \mathbb{R}^4 \) is

(i) **Lagrangian** iff \( \omega|_\Sigma = 0 \)

(ii) **Hamiltonian Stationary Lagrangian** (HSL) iff \( \omega|_\Sigma = 0 \) and \( \Sigma \) is a critical point of the area functional \( A \) with respect to all *Hamiltonian vector fields* \( \xi_h \) s.t.:

- \( \exists h \in C^\infty_c(\mathbb{R}^4), \xi_h \cdot \omega + dh = 0 \)
- equivalently, if \( J \) is the complex structure s.t. \( \omega = \langle J \cdot, \cdot \rangle \), \( \xi_h = J \nabla h \).

It means that \( \delta A_\Sigma(\xi_h) = 0, \forall h \in C^\infty_c(\mathbb{R}^4) \).

What is the Euler equation?

The Gauss map is:

\[
\gamma : \Sigma \longrightarrow Gr_{\text{Lagr}}(\mathbb{R}^4) \subset Gr_2(\mathbb{R}^4) \\
S^1 \times S^2 \subset S^2 \times S^2
\]

Denote \( \gamma = (\rho_\Sigma, \sigma_\Sigma) \) the two components of \( \gamma \). For a Lagrangian immersion \( \rho_\Sigma \simeq e^{i\beta} \). Then the *mean curvature vector* is

\( \vec{H} = J \nabla \beta \).
Lemma 1.1 \( \Sigma \) is HSL iff
\[
\begin{align*}
\omega|_{\Sigma} &= 0 \\
\Delta_{\Sigma}\beta &= 0.
\end{align*}
\]

Remark: \( \Sigma \) is special Lagrangian iff
\[
\begin{align*}
\omega|_{\Sigma} &= 0 \\
\beta &= \text{Constant}.
\end{align*}
\] \( \iff \)
\[
\begin{align*}
\omega|_{\Sigma} &= 0 \\
\Sigma \text{ is minimal}.
\end{align*}
\]

An analytic study was done by R. Schoen and J. Wolfson \[6\] (in a 4-dimensional Calabi–Yau manifold).

1.2 It is a completely integrable system (F.H.–P. Romon \[1, 2\])

Let \( \Omega \subset \mathbb{C} \) be an open subset and \( X : \Omega \rightarrow \mathbb{R}^4 \) a (local) conformal parametrization of \( \Sigma \). Set
\[
\rho_X := \rho_{\Sigma} \circ X,
\]
the left Gauss map.

Idea: to lift the pair \((X, \rho_X)\) to a map \( F : \Omega \rightarrow \mathfrak{G} \), where \( \mathfrak{G} \) is a local symmetry group of the problem. The more naive choice is \( \mathfrak{G} = SO(4) \times \mathbb{R}^4 \), the group of isometries of \( \mathbb{R}^4 \). Then
\[
F = \begin{pmatrix} R & X \\ 0 & 1 \end{pmatrix} \simeq (R, X),
\]
where \( R : \Omega \rightarrow SO(4) \) encodes \( \rho_X \simeq e^{i\beta} \). (Alternatively one can choose \( \mathfrak{G} = U(2) \times \mathbb{R}^4 \), with the identification \( \mathbb{C}^2 \simeq (\mathbb{R}^4, J) \) and \( U(2) \): subgroup of \( SO(4) \). Then the way \( R \in U(2) \) encodes \( \beta \) is simply through the relation \( \det e^{i\beta} \).

In all cases there exists an automorphism \( \tau : \mathfrak{G} \rightarrow \mathfrak{G} \) s.t. \( \tau^4 = Id \). This automorphism acts on the Lie algebra \( \mathfrak{g} \) and can be diagonalized with the eigenvalues \( i, 1, -i \) and \(-1\). Hence the vector space decomposition
\[
\mathfrak{g}^\mathbb{C} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0^\mathbb{C} \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2^\mathbb{C}
\]
eigenvalues: \( -i \quad 1 \quad i \quad -1 \)

Then consider the (pull-back of the) Maurer–Cartan form
\[
\alpha := F^{-1}dF
\]
and split \( \alpha = \alpha_{-1} + \alpha_0 + \alpha_1 + \alpha_2 \) according to this decomposition. Then do the further splitting \( \alpha_2 = \alpha_2' + \alpha_2'' \), where \( \alpha_2' = \alpha(\frac{\partial}{\partial z})dz \) and \( \alpha_2'' = \frac{\partial}{\partial z} \). And consider the family of deformations
\[
\alpha_\lambda := \lambda^{-2}\alpha_2' + \lambda^{-1}\alpha_{-1} + \alpha_0 + \lambda_1 + \lambda^2\alpha_2'', \quad \lambda \in \mathbb{C}^*.
\]

Then:
Theorem 1.1  
(i) $X$ is Lagrangian iff $\alpha''_1 = 0$
(ii) $X$ is HSL iff $\alpha''_1 = 0$ and, $\forall \lambda \in \mathbb{C}^*$, $d\alpha_\lambda + \frac{1}{2} [\alpha_\lambda \wedge \alpha_\lambda] = 0$.

Using this characterisation one see easily that HSL surfaces are solutions of a completely integrable system.

Note that analogous formulations work for HSL surfaces in $\mathbb{C}P^2 = SU(3)/S(U(2) \times U(1)) = U(3)/U(2) \times U(1)$, $\mathbb{C}D^2 = SU(2,1)/S(U(2) \times U(1))$, $\mathbb{C}P^1 \times \mathbb{C}P^1$, $\mathbb{C}D^1 \times \mathbb{C}D^1$.

2. Generalizations in $\mathbb{R}^4$ (after I. Khemar [4])

Again $\mathbb{R}^4$ is endowed with its canonical Euclidean structure. We will also use an identification of $\mathbb{R}^4$ with the quaternions $\mathbb{H}$. We recall that this allows to represent rotations $R \in SO(4)$ by a pair $(p,q) \in S^3 \times S^3 \subset \mathbb{H} \times \mathbb{H}$ of unit quaternions such that $\forall z \in \mathbb{H}$, $R(z) = pqz\overline{q}$. In other words, denoting by $L_p : z \mapsto pz$ and $R_\overline{q} : z \mapsto z\overline{q}$, we have $R = L_pR_\overline{q} = R_\overline{q}L_p$. The pair $(p,q)$ is unique up to sign, hence the identification $SO(4) \simeq S^3 \times S^3/\{\pm\}$.

Moreover we can also precise the identification $Gr_2(\mathbb{R}^4) \simeq S^2 \times S^2$. Let $\text{Stiefel}_2(\mathbb{H}) := \{(e_1, e_2) \in \mathbb{H} \times \mathbb{H} | |e_1| = |e_2| = 1, \langle e_1, e_2 \rangle = 0\}$. Observe that $\forall (e_1, e_2) \in \text{Stiefel}_2(\mathbb{H})$, $e_2\overline{e_1}$ (resp. $\overline{e_2}e_1$) is unitary (because $e_1$ and $e_2$ are so) and imaginary (because $\langle e_1, e_2 \rangle = 0$). Hence this defines two maps $\text{Stiefel}_2(\mathbb{H}) \rightarrow S^2$, $(e_1, e_2) \mapsto e_2\overline{e_1}$, $\text{Stiefel}_2(\mathbb{H}) \rightarrow S^2$, $(e_1, e_2) \mapsto \overline{e_1}e_2$.

These maps factor through the natural map $P : (e_1, e_2) \mapsto \text{Span}\{e_1, e_2\}$ from $\text{Stiefel}_2(\mathbb{H})$ to the oriented Grassmannian $Gr_2(\mathbb{H})$: let $\rho : Gr_2(\mathbb{H}) \rightarrow S^2$, s. t. $\rho \circ P(e_1, e_2) = e_2\overline{e_1}$, $\sigma : Gr_2(\mathbb{H}) \rightarrow S^2$, s. t. $\sigma \circ P(e_1, e_2) = \overline{e_1}e_2$.

Then $(\rho, \sigma) : Gr_2(\mathbb{H}) \rightarrow S^2 \times S^2$ is a diffeomorphism.

2.1 Immersions of a surface in $\mathbb{H}$ with a harmonic ‘left Gauss map’

Let $X : \Omega \rightarrow \mathbb{H}$ be a conformal immersion and $\rho_X : \Omega \rightarrow S^2$ its left Gauss map, i.e. $\forall z \in \Omega$, $\rho_X(z)$ is the image of $\text{Span}(\frac{\partial X}{\partial x}(z), \frac{\partial X}{\partial y}(z))$ by $\rho$. It is characterised by $\frac{\partial X}{\partial y} = \rho_X \frac{\partial X}{\partial x} \iff \frac{\partial X}{\partial z} = \rho_X \frac{\partial X}{\partial z}$.
(In the second equation the $i$ on the l.h.s. is the complex structure on $\Omega \subset \mathbb{C}$, whereas the $\rho_X$ on the r.h.s. denotes the left multiplication in $\mathbb{H}$.)

Remark: instead of viewing $\rho_X$ as the left component of the Gauss map in $Gr_2(\mathbb{H}) \simeq S^2 \times S^2$, an alternative interpretation is that $\rho_X$ is a map into the ‘left’ connected component of the manifold of compatible complex structures $\mathcal{J}_\mathbb{H} \simeq S^2 \cup S^2$ on $\mathbb{H}$ (cf. the work of F. Burstall).

Idea: to lift the pair $(X, \rho_X)$ by a framing $F : \Omega \rightarrow \mathcal{G}$, $\mathcal{G}$ is a subgroup of $SO(4) \ltimes \mathbb{R}^4$. How? We fix some constant imaginary unit vector $u \in S^2 \subset \text{Im} \mathbb{H}$.

- First method: we lift $X$ and its full Gauss map $T_X \Sigma \simeq (\rho_X, \sigma_X)$: we let $(e_1, e_2)$ be any moving frame which is an orthonormal basis of $T_X(z) \Sigma$ (e.g. $e_1 = \frac{\partial X}{\partial x}/|\frac{\partial X}{\partial x}|$, $e_2 = \frac{\partial X}{\partial y}/|\frac{\partial X}{\partial y}|$) and we choose $F = (R, X)$ s.t. $R$ satisfies:

$$R(1) = e_1, \quad R(u) = e_2.$$ 

Decompose $R = L_p R_q$, then

$$R(1) = p\overline{q}, R(u) = pu\overline{q},$$

so that $\rho_X = e_2\overline{e_1} = pu\overline{p}$.

Note: In this case we must choose $\mathcal{G} = SO(4) \ltimes \mathbb{R}^4$ (which acts transitively on $Stiefel_2(\mathbb{H})$).

- Second method: we lift only $X$ and $\rho_X$. Then it means that we choose $F = (R, X)$, where $R = L_p R_q$ is s.t.

$$\rho_X = pu\overline{p}.$$ 

Hence the choice of $q$ is not relevant. In other words introducing the (left) Hopf fibration

$$\mathcal{H}^u_L : SO(4) \rightarrow S^2,$$

we choose the lift $F = (R, X)$ in such a way that $\mathcal{H}^u_L \circ R = \rho_X$.

We observe that in this case one may choose $q = 1$ and assume that $R \in \{L_p | p \in S^3\} \simeq Spin3$, i.e. work with $\mathcal{G} = Spin3 \ltimes \mathbb{H}$. The restriction of $\mathcal{H}^u_L$ to $Spin3$ (viewed as a subgroup of $SO(4)$) is just the Hopf fibration $\mathcal{H}^u : S^3 \rightarrow S^2$.

Actually the second point of view is more general and leads to a simpler theory.

Now let $\tau : (R, X) \mapsto (L_u RL_u^{-1}, -L_u X)$, a 4th order automorphism of $\mathcal{G}$ (i.e. $\tau^4 = Id$). It induces a 4th order automorphism on its Lie algebra $\mathfrak{g}$. Let

$$\mathfrak{g}_C = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$$
be its associated eigenspace decomposition. Split the Maurer–Cartan form $\alpha = F^{-1}dF$ according to this decomposition: $\alpha = \alpha_{-1} + \alpha_0 + \alpha_1 + \alpha_2$ and let

$$\beta_{\lambda^2} := \lambda^{-2}\alpha'_2 + \alpha_0 + \lambda^2\alpha''_2,$$

$$\alpha_{\lambda} := \lambda^{-2}\alpha'_2 + \lambda^{-1}\alpha_{-1} + \alpha_0 + \lambda\alpha_1 + \lambda^2\alpha''_2 = \beta_{\lambda^2} + \lambda^{-1}\alpha_{-1} + \lambda\alpha_1.$$

Then:

**Lemma 2.1** If $X : \Omega \to \mathbb{R}^4$ is a conformal immersion and if $R : \Omega \to SO(4)$ is an arbitrary smooth map, then

$$\mathcal{H}_L^\nu \circ R = \rho_X \iff \alpha''_{-1} = 0.$$

In other words $F = (R, X) : \Omega \to SO(4) \times \mathbb{R}^4$ lifts $(X, \rho_X)$ iff $\alpha''_{-1} = 0$.

Remark: $\alpha_1$ is the complex conjugate of $\alpha_{-1}$, so that $\alpha''_{-1} = 0$ iff $\alpha'_1 = 0$.

**Lemma 2.2** We have:

$$d\alpha_{\lambda} + \frac{1}{2}[\alpha_{\lambda} \wedge \alpha_{\lambda}] = d\beta_{\lambda^2} + \frac{1}{2}[\beta_{\lambda^2} \wedge \beta_{\lambda^2}] + (\lambda^{-3} - \lambda)[\alpha'_2 \wedge \alpha''_{-1}] + (\lambda^3 - \lambda^{-1})[\alpha''_2 \wedge \alpha'_1]. \quad (1)$$

Hence in particular, if $F$ lifts $(X, \rho_X)$, then $d\alpha_{\lambda} + \frac{1}{2}[\alpha_{\lambda} \wedge \alpha_{\lambda}] = d\beta_{\lambda^2} + \frac{1}{2}[\beta_{\lambda^2} \wedge \beta_{\lambda^2}]$.

In order to interpret (1) we further observe that

(i) $\mathfrak{g}^\tau$, the fixed subset of $\tau : \mathfrak{g} \to \mathfrak{g}$, is a subgroup of $\mathfrak{g}$ with Lie algebra $\mathfrak{g}_0$

(ii) $\mathfrak{g}^{\tau^2} = \{(R, 0) \in \mathfrak{g}\}$, the fixed subset of $\tau^2 : \mathfrak{g} \to \mathfrak{g}$, is a subgroup of $\mathfrak{g}$ with Lie algebra $\mathfrak{g}_0 \oplus \mathfrak{g}_2$,

with the inclusions

$$\mathfrak{g}^\tau \subset \mathfrak{g}^{\tau^2} \subset \mathfrak{g}.$$

Moreover $\mathfrak{g} / \mathfrak{g}^{\tau^2} \simeq \mathbb{H}$ and $\mathfrak{g}^{\tau^2} / \mathfrak{g}^\tau \simeq S^2$ and the projection map

$$\mathfrak{g}^{\tau^2} \quad \to \quad \mathfrak{g}^{\tau^2} / \mathfrak{g}^\tau \simeq S^2 \quad R \simeq (R, 0) \quad \mapsto \quad R \mod \mathfrak{g}^\tau$$

coincides with the Hopf fibration $\mathcal{H}_L^\nu$. Hence, by applying the standard theory of harmonic maps into symmetric spaces, we deduce that:

$$d\beta_{\lambda^2} + \frac{1}{2}[\beta_{\lambda^2} \wedge \beta_{\lambda^2}] = 0 \iff \mathcal{H}_L^\nu \circ R : \Omega \to S^2$$

is harmonic.

Putting Lemmas 2.1 and 2.2 and these observations together we conclude with the following:
Theorem 2.1  Let $X: \Omega \to \mathbb{H}$ be a conformal immersion and $\rho_X: \Omega \to S^2$ its left Gauss map. Let $F = (R, X): \Omega \to \mathfrak{g}$ be any smooth map. Then

(i) $\mathcal{H}_L^u \circ R = \rho_X$ (i.e. $F$ is a lift of $(X, \rho_X)$) iff $\alpha''_{-1} = 0$

(ii) If so, i.e. if $F$ is a lift of $(X, \rho_X)$, then $\rho_X$ is harmonic iff

$$d\alpha + \frac{1}{2}[\alpha \wedge \alpha] = 0.$$ 

2.2 Examples

2.2.1 HSL surfaces revisited

Let us introduce again the symplectic form $\omega = dx_1 \wedge dx_2 + dx_3 \wedge dx_4$. Note that $\omega = \omega_1 := \langle L_1 \cdot, \cdot \rangle$. Let us introduce also $\omega_2 := \langle L_2 \cdot, \cdot \rangle = dx_1 \wedge dx_3 + dx_4 \wedge dx_2$ and $\omega_3 := \langle L_3 \cdot, \cdot \rangle = dx_1 \wedge dx_4 + dx_2 \wedge dx_4$. Then

$$e_2e_1 = \rho(e_1, e_2) = i\omega_1(e_1, e_2) + j\omega_2(e_1, e_2) + k\omega_3(e_1, e_2).$$

So $X$ is a conformal Lagrangian immersion iff $X^*\omega_1 = 0$, i.e. if $\rho_X$ takes values in

$$S^1 = \{ j \cos \beta + k \sin \beta = e^{i\beta} j \mid \beta \in \mathbb{R} \}.$$ 

Hence a lift of $(X, \rho_X)$ is characterized by

$$pu = \mathcal{H}_L^u \circ R = \rho_X = e^{i\beta} j. $$

A convenient choice for $u$ is to assume that $u \perp i$, e.g. $u = j$. In that case

$$\{ p \in S^3 | pu = e^{i\beta} j \} = \{ e^{i\beta/2}e^{j\theta} \mid \theta \in \mathbb{R} \}$$

and the simplest choices are $p = \pm e^{i\beta/2}$.

With this choice:

- if we start with the group $\mathfrak{g} = SO(4) \ltimes \mathbb{R}^4$, our lift satisfies $R = L_{e^{i\beta/2}R\varphi}$, i.e. we can reduce $SO(4) \ltimes \mathbb{R}^4$ to $U(2) \ltimes \mathbb{C}^2$

- if we start with the group $\mathfrak{g} = Spin3 \ltimes \mathbb{H}$, our lift satisfies $R = L_{e^{i\beta/2}}$, i.e. we can reduce $Spin3 \ltimes \mathbb{R}^4$ to $U(1) \ltimes \mathbb{C}^2$ (cf. spinor lifts, related to the Konopelchenko–Taimanov representation formula).

2.2.2 Constant mean curvature surfaces in $\mathbb{R}^3$

Consider an immersed surface $\Sigma$ in $\mathbb{H}$ with a harmonic left Gauss map. If we assume further that this surface is contained in $\text{Im} \mathbb{H}$, then any orthonormal basis $(e_1, e_2)$ of $T_{X(z)} \Sigma$ is composed of imaginary vectors. Hence

$$\rho_X = e_2e_1 = -\overline{e_1}e_2 = -\sigma_X,$$

so that $\rho_X$ is harmonic iff $\sigma_X$ is so. Actually $\rho_X$ is nothing but the Gauss map of $\Sigma$ in $\text{Im} \mathbb{H} \simeq \mathbb{R}^3$. Hence by Ruh–Vilms theorem we know that $\Sigma$ is a constant mean curvature surface in $\mathbb{R}^3$. Conversely any constant mean curvature surface in $\mathbb{R}^3$ arises that way.
2.3 Other generalizations in dimension 4

This theory can be generalized to surfaces in $S^4$ or $\mathbb{C}P^2$; then $(X, \rho_X)$ is replaced by a lift of the immersion $X$ in the four dimensional manifold into the twistor bundle of complex structures. The condition of $\rho_X$ being harmonic is replaced by the fact this lift is vertically harmonic (the fiber being the set of (left) compatible complex structures, diffeomorphic to $S^2$). This follows from independant works by F. Burstall and I. Hemar.

3 A generalization for surfaces in $\mathbb{R}^8$ (I. Hemar [4])

The following theory is based on the identification of $\mathbb{R}^8$ with octonions $\mathbb{O}$. Again the map $\text{Stiefel}_2(\mathbb{O}) \rightarrow S^6$, $(e_1, e_2) \mapsto \text{Span}\{e_1, e_2\}$ by introducing

$$\rho : Gr_2(\mathbb{O}) \rightarrow S^6, \quad \text{s.t. } \rho \circ P(e_1, e_2) = e_2 e_1.$$ 

Let $\Sigma$ be an immersed surface in $\mathbb{O}$ we say that $\Sigma$ is $\rho$-harmonic iff the composition of the Gauss map $\Sigma \rightarrow Gr_2(\mathbb{O})$ with $\rho$ is harmonic.

This theory is completely similar with the theory of surfaces in quaternions $\mathbb{H}$ which used the group $\mathfrak{G} = Spin3 \ltimes \mathbb{H}$, where $Spin3$ can be seen as the subgroup of $SO(4)$ generated by $L_i, L_j$ and $L_k$ and the induced representation of $Spin3$ was the spinor representation $\mathbb{H}$. Here we will use $\mathfrak{G} = Spin7 \ltimes \mathbb{O}$, where $Spin7$ can be identified with the subgroup of $SO(8)$ generated by $\{L_v | v \in S^6 \subset \text{Im}\mathbb{O}\}$ and the induced representation on $\mathbb{R}^8$ coincides with the spinor representation of $Spin7$ on $\mathbb{O}$. A difference however is that $Spin7$ is ”bigger” than $Spin3$ and in particular acts transitively on $\text{Stiefel}_2(\mathbb{O})$ (with isotropy $SU(3)$) and $Gr_2(\mathbb{O})$ (with isotropy $G_2$), whereas $Spin3$ do not act transitively on $Gr_2(\mathbb{H})$. After fixing an imaginary unit octonion $u \in \mathbb{O}$, a ‘Hopf” fibration

$$\mathcal{H}^u : Spin7 \rightarrow S^6, \quad p \mapsto \mathcal{H}^u(p), \quad \text{s.t. } pL_u p^{-1} = L_{\mathcal{H}^u(p)}$$

can be defined.

Now let $X : \mathbb{C} \supset \Omega \rightarrow \mathbb{O}$ be a conformal immersion and denote $\rho_X := \rho \circ T_X \Sigma$ the composition of the Gauss map $T_X \Sigma$ of $X$ with $\rho$. After having fixed $u \in S^6 \subset \mathbb{O}$ we let

$$F = \begin{pmatrix} R & X \\ 0 & 1 \end{pmatrix} \simeq (R, X) : \Omega \rightarrow Spin7 \ltimes \mathbb{O},$$
be a smooth map. We say that $F$ lifts $(X, \rho_X)$ iff $\mathcal{H} \circ R = \rho_X$. Using the 4th order automorphism $\tau : \mathfrak{g} \rightarrow \mathfrak{g}$ defined by $\tau(R, X) = (L_u R L_u^{-1}, -L_u X)$, we can characterized among all maps $F = (R, X)$ those which lift $\rho_X$ by the condition $\alpha''_1 = 0$ (after a decomposition of the Maurer–Cartan form $\alpha := F^{-1} dF$ along the eigenspaces of the action of $\tau$ on the Lie algebra $\mathfrak{g}$ of $\mathfrak{g}$). Then the $\rho$-harmonic immersions satisfy a zero curvature equation $d\alpha + \frac{1}{2} [\alpha, \alpha] = 0$ similar to the previous case.

Again $\rho_X$ can be interpreted as a map into the manifold $\mathcal{J}_O$ of compatible complex structures on $O$, because of the relation $\rho_X \frac{\partial X}{\partial z} = i \frac{\partial X}{\partial \bar{z}}$. However the embedding $S^6 \subset \mathcal{J}_O$ is much less clear than the inclusion $S^2 \subset \mathcal{J}_H$ that we used previously: we recall indeed that $\mathcal{J}_H \simeq S^2_L \cup S^2_R$ and hence that our $S^2$ was just the (left) connected component of $\mathcal{J}_H$. However $\mathcal{J}_O \simeq SO(8)/U(4)$ is 12 dimensional, so that our $S^6$ is now a particular submanifold of $\mathcal{J}_O$. Hence a twistor interpretation of the theory in $O$ seems less clear.

4 Towards a supersymmetric interpretation

**Observation**: the coefficients of $\alpha_{-1}$ and $\alpha_1$ actually behave like spinors (they turn half less than those of $\alpha_2$ when $\lambda$ run over $S^1$ and they satisfy a kind of Dirac equation). This motivates the following results by I. Khemar [5].

4.1 Superharmonic maps into a symmetric space

For simplicity we restrict ourself to maps into the sphere $S^n \subset \mathbb{R}^{n+1}$. It can be seen as a system of PDE’s on a map $u : \Omega \rightarrow S^n$ (where $\Omega \subset \mathbb{C}$) and odd sections $\psi_1, \psi_2$ of $u^* TS^n$. This system is

\[
\begin{align*}
\nabla_z \frac{\partial u}{\partial z} &= \frac{1}{4} \left( \psi \langle \psi, \frac{\partial u}{\partial z} \rangle - \overline{\psi} \langle \overline{\psi}, \frac{\partial u}{\partial z} \rangle \right) \\
\nabla_{\overline{z}} \psi &= \frac{1}{4} \langle \psi, \psi \rangle \overline{\psi},
\end{align*}
\]

(2)

where $\psi = \psi_1 - i \psi_2$. By “odd” we mean that the components $\psi_1$ and $\psi_2$ are anticommuting (Grassmann) variables. An alternative elegant reformulation of this system can be obtained by adding the extra field $F : \Omega \rightarrow \mathbb{R}^{n+1}$, which satisfies the 0th order PDE’s

\[
F = \frac{1}{2i} \langle \psi, \overline{\psi} \rangle u
\]

(3)

and by setting

\[
\Phi := u + \theta^1 \psi_1 + \theta^2 \psi_2 + \theta^1 \theta^2 F,
\]

where $\theta^1$ and $\theta^2$ are anticommuting coordinates, so that $(x, y, \theta^1, \theta^2)$ forms a complete system of coordinates on the superplane $\mathbb{R}^{2|2}$. Then (2) and (3) are equivalent to

\[
\overline{D} D \Phi + \langle D \Phi, D \Phi \rangle \Phi = 0,
\]

(4)
where $D = \frac{\partial}{\partial \theta} - \theta \frac{\partial}{\partial z}$, $\overline{D} = \frac{\partial}{\partial \theta} - \overline{\theta} \frac{\partial}{\partial \overline{z}}$.

Actually, from (2) and (3) to (4), we have used the fact that $u$, $\psi_1$, $\psi_2$ and $F$ are the components (supermultiplet) of a single map $\Phi$ from $\mathbb{R}^{2|2}$ to $S^n \subset \mathbb{R}^{n+1}$, which satisfies the superharmonic map equation (4).

Now we lift $\Phi$ to a framing supermap $\mathcal{F} : \mathbb{R}^{2|2} \rightarrow SO(n + 1)$ such that the composition of $\mathcal{F}$ with the projection $SO(n + 1) \rightarrow SO(n + 1)/SO(n) \simeq S^n$ is $\Phi$. Set $\alpha := \mathcal{F}^{-1} d\mathcal{F}$ and decompose $\alpha = \alpha_0 + \alpha_1$, according to the splitting of the Lie algebra $so(n + 1)$ by the Cartan involution.

Before giving a characterization of the superharmonic equation, it is useful to present a technical result concerning the exterior calculus of 1-forms on $\mathbb{R}^{2|2}$.

**Lemma 4.1** For a 1-form $\alpha$ on $\mathbb{R}^{2|2}$ with coefficients in a Lie algebra $\mathfrak{g}$, we have the equivalence

$$d\alpha + \frac{1}{2} [\alpha \wedge \alpha] = 0 \iff \overline{D} \alpha(D) + D \alpha(\overline{D}) + [\alpha(\overline{D}), \alpha(D)] = 0.$$  

**Remark:** $\Lambda^1(\mathbb{R}^{2|2})^*$ is spanned by $(d\theta, d\overline{\theta}, dz + (d\theta)\theta, d\overline{z} + (d\overline{\theta})\overline{\theta})$, the dual basis of $(D, \overline{D}, \frac{\partial}{\partial z}, \frac{\partial}{\partial \overline{z}})$. Hence in particular $\Lambda^2(\mathbb{R}^{2|2})^*$ is 6 dimensional. So the expansion of the l.h.s. of $d\alpha + \frac{1}{2} [\alpha \wedge \alpha] = 0$ leads to 6 equations which are a priori independant. The content of this lemma is that these 6 terms vanish as soon as one of these 6 coefficients (namely the coefficient of $d\theta \wedge d\overline{\theta}$) vanishes.

Now the supermap $\mathcal{F}$ is superharmonic iff

$$\overline{D} \alpha_1(D) + [\alpha_0(\overline{D}), \alpha_1(D)] = 0.$$  

We hence deduce:

**Theorem 4.1** $\mathcal{F}$ is superharmonic iff

$$\forall \lambda \in \mathbb{C}^*, \quad \overline{D} \alpha(D)_\lambda + D \alpha(\overline{D})_\lambda + [\alpha(\overline{D})_\lambda, \alpha(D)_\lambda] = 0,$$

where $\alpha(D)_\lambda := \alpha_0(D) + \lambda^{-1} \alpha_1(D)$ and $\alpha(\overline{D})_\lambda := \alpha_0(\overline{D}) + \lambda \alpha_1(\overline{D})$.

It results that this problem has the structure of a completely integrable system (F. O’DEA, I. KHEMAR). In particular the DPW algorithm for harmonic maps works.

The DPW potential is a $\Lambda \mathfrak{g}^\mathbb{C}_\tau$-valued holomorphic 1-form $\mu$ on $\mathbb{R}^{2|2}$ s.t.

$$\mu(D) = \mu_0(D) + \theta \mu_\theta(D) = \lambda^{-1}(\cdot) + \lambda^0(\cdot) + \cdots$$

One integrates the equation

$$Dg = g\mu(D)$$
to get a holomorphic map $g = g_0 + \theta g_0 : \mathbb{R}^{2|2} \longrightarrow \Lambda \mathfrak{g} \mathbb{C}$. This implies in particular that

$$g_0^{-1} \frac{\partial g_0}{\partial z} = -((\mu_0(D))^2 + \mu_0(D)) = \lambda^{-2}(\cdot) + \lambda^{-1}(\cdot) + \lambda^0(\cdot) + \cdots$$

Similarly, if $\mathcal{F} = \mathcal{F}_0 + \theta \mathcal{F}_0 + \bar{\theta} \mathcal{F}_0 + \theta \bar{\theta} \mathcal{F}_0$, it turns out that $\mathcal{F}_0^{-1} d\mathcal{F}_0 = \lambda^{-2}(\cdot) + \lambda^{-1}(\cdot) + \lambda^0(\cdot) + \lambda(\cdot) + \lambda^2(\cdot)$. Hence we recover (for $\mathcal{F}_0$) something similar to a second order elliptic integrable system.

### 4.2 Superprimitive maps \[5\]

More precisely we can recover a second order elliptic integrable system close to the HSL surface theory in $\mathbb{R}^4$ by looking at superprimitive maps from $\mathbb{R}^{2|2}$ to the 4-symmetric space $SU(3)/SU(2)$: if $\Phi : \mathbb{R}^{2|2} \longrightarrow SU(3)/SU(2)$ is a superprimitive map then the first component $u$ in the decomposition $\Phi = u + \theta^1 \psi_1 + \theta^2 \psi_2 + \theta \bar{\theta} F$ is a conformal HSL immersion (with the restriction that the Lagrangian angle $\beta$ is equal to a real constant plus a harmonic non constant nilpotent function).

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