Update on pion weak decay constants in nuclear matter

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Abstract

The QCD sum rule calculation of the in-medium pion decay constants using pseudoscalar-axial vector correlation function, \(i \int d^4x \, e^{ip \cdot x} \langle \rho | T[\bar{d}(x)i\gamma_5 u(x) \, \bar{u}(0) \gamma_\mu \gamma_5 d(0)]|\rho \rangle\) is revisited. In particular, we argue that the dimension 5 condensate, \(\langle \bar{q} (iD_0)^2 q \rangle_N + \frac{1}{8} \langle \bar{q} g_s \sigma \cdot G q \rangle_N\), which is crucial for splitting the time \((f_t)\) and space \((f_s)\) components of the decay constant, is not necessarily restricted to be positive. Its positive value is found to yield a tachyonic pion mass. Using the in-medium pion mass as an input, we fix the dimension 5 condensate to be around \(-0.025 \text{ GeV}^2 \sim -0.019 \text{ GeV}^2\). The role of the \(N\) and \(\Delta\) intermediate states in the correlation function is also investigated. The \(N\) intermediate state is found not to contribute to the sum rules. For the \(\Delta\) intermediate state, we either treat it as a part of the continuum or propose a way to subtract explicitly from the sum rules. With (and without) explicit \(\Delta\) subtraction while allowing the in-medium pion mass to vary within \(139 \text{ MeV} \leq m^*_\pi \leq 159 \text{ MeV}\), we obtain \(f_s/f_\pi = 0.37 \sim 0.78\) and \(f_t/f_\pi = 0.63 \sim 0.79\).

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I. INTRODUCTION

The pion decay constant in nuclear matter is one important parameter to be determined in modern nuclear physics. As an order parameter of spontaneous chiral symmetry breaking, its reduction in the matter may indicate a partial restoration of chiral symmetry. In particular, recent measurements of deeply bound pionic atoms [1, 2, 3, 4] give rise to an exciting discussion on pion-nucleus optical potential. The isovector parameter of the s-wave pion nucleus potential is directly related to the pion decay constant, and its observed enhancement in nuclear matter can be interpreted as a partial restoration of chiral symmetry [5, 6, 7, 8], namely by the decrease of the decay constant. This restoration causes the universal softening of $\sigma$ and $\rho$ in the matter [9]. Furthermore, the change in the decay constant is believed to scale the reduction of hadron masses in the medium [10]. Also the decay constant is directly connected to the renormalization of the induced pseudoscalar coupling in the matter, which is believed to control muon captures in nuclei [11, 12, 13, 14].

One interesting feature of the in-medium pion decay constant is its separation into the time ($f_t$) and space ($f_s$) components. For a model-independent prediction, it will be interesting to calculate the decay constants using QCD sum rules [15, 16]. Indeed, one of present authors (H.K) recently performed a QCD sum rule calculation of the decay constants [17] using pseudoscalar-axial vector correlation function in the matter,

$$\Pi^\mu = i \int d^4x \ e^{ipx} \langle \rho | T[\bar{d}(x)i\gamma_5u(x) \ \bar{u}(0)\gamma^\mu\gamma_5d(0)] | \rho \rangle . \ (1)$$

This correlation function is useful because it can reproduce the Gell-Mann–Oakes–Renner (GOR) relation in vacuum as well as its in-medium version. Also one can clearly see the separation of $f_t$ and $f_s$ in QCD calculation. It was found that the splitting between $f_t$ and $f_s$ is mainly driven by the dimension 5 condensate in the nucleon, $\langle \bar{q}(iD_0)^2 q \rangle_N + \frac{1}{8} \langle \bar{q}g_s\sigma \cdot Gq \rangle_N$. But its positive value leads to a somewhat puzzling result of $f_s/f_t \geq 1$, which neither agrees with the result from in-medium chiral perturbation theory [13, 18], $f_s/f_t \sim 0.28$ (or smaller), nor with the causal constraint from Ref. [19], $f_s/f_t \leq 1$.

However, before making a definite claim from QCD sum rules, one may need to re-examine the sum rule calculation in various respects. First, the argument leading to a positive value for the dimension 5 condensate is based on the Hermitian property of the operators involved but, as we will discuss below, it is not sufficient for determining the definite sign of the dimension 5 condensate. One needs an alternative way to constrain the
value of the dimension 5 condensate. One possibility is to constrain it by requiring that the sum rules reproduce a reasonable in-medium pion mass. Another ingredient for this update is to check more carefully the quasi-pion dominance of the correlation function Eq. (1). In particular, we need to calculate the contributions from $N$ and $\Delta$ intermediate states and estimate how large the change is from these intermediate states. In this work, we address these two aspects and improve the previous QCD sum rule calculation for $f_t$ and $f_s$.

This paper is organized as follows. In Sec. II, we briefly re-derive QCD sum rules for $f_t$ and $f_s$. The issue on the sign of the dimension 5 condensate is discussed in Sec. III. The contributions from $N$ and $\Delta$ intermediate states are studied in Sec. IV and Sec. V. For the $\Delta$ contribution, we propose a way to subtract it from our sum rules. In Sec. VI, we constrain the dimension 5 condensate by requiring it to reproduce an acceptable in-medium pion mass within our sum rule and use it to calculate $f_t$ and $f_s$. We then summarize in Sec. VII.

II. THE QCD SUM RULE FOR $f_t$ AND $f_s$.

In this section, we briefly go through the QCD sum rule derivation of the in-medium pion decay constants, $f_t$ and $f_s$, in Ref. [17]. Ref. [17] considered the pseudoscalar-axial vector correlation function in nuclear matter, Eq. (1), and constructed the two sum rules in the following limit

$$
\Pi_t \equiv \lim_{p \to 0} \frac{\Pi^0}{ip_0}; \quad \Pi_s \equiv \lim_{p \to 0} \frac{\Pi^j}{ip^j}.
$$

(2)

Using the in-medium decay constants defined by

$$
\langle \rho | \bar{d} \gamma^\mu \gamma_5 u | \pi^+ (k) \rho \rangle = (if_t k_0, if_s k),
$$

(3)

and its derivative, we obtain the phenomenological sides

$$
\Pi_{phen}^t = -\frac{m_\pi^2}{2m_q p_0^2 - m_\pi^2} f_t^2; \quad \Pi_{phen}^s = -\frac{m_\pi^2}{2m_q p_0^2 - m_\pi^2} f_t f_s,
$$

(4)

when a quasi-pion intermediates the correlation function. On the other hand, the operator product expansion (OPE) allows us to calculate the correlation function using QCD degrees of freedom. Up to dimension 5 in the expansion, the OPE side of the correlation function is given by [17],

$$
\Pi_{ope}^t = -\frac{3}{4\pi^2} \int_0^1 du \, m_q \ln[-u(1-u)p_0^2 + m_q^2] + \frac{2\langle \bar{q}q \rangle}{p_0^2 - m_q^2} + \frac{8m_q \langle q^3 iD_0 q \rangle}{(p_0^2 - m_q^2)^2}.
$$

(5)
\[ \Pi^{\text{ope}}_s = -\frac{3}{4\pi^2} \int_0^1 du \, m_q \ln[-u(1-u)p_0^2 + m_q^2] + \frac{2\langle \bar{q}q \rangle_\rho}{p_0^2 - m_q^2} - \frac{8 m_q \langle q^i D_0 q \rangle_\rho}{3 (p_0^2 - m_q^2)^2} - \frac{2m_q^2 \langle \bar{q}q \rangle_\rho}{(p_0^2 - m_q^2)^2} \]

\[ + \frac{32}{3} \left[ \langle \bar{q}(iD_0)^2 q \rangle_\rho + \frac{1}{8} \langle \bar{q}g_{s\sigma} \cdot Gq \rangle_\rho \right] \frac{1}{(p_0^2 - m_q^2)^2}. \]  

(6)

Here the subscript “\( \rho \)” denotes the nuclear expectation value.

When Eqs. (10), (11), and (13) are put into the Borel weighted sum rules,

\[ \int_0^{S_0} ds \, e^{-s/M^2} \frac{1}{\pi} \Im[\Pi^{\text{phen}}_t(s) - \Pi^{\text{ope}}_t(s)] = 0 \quad (l = t, s), \]  

(7)

we obtain the two sum rules,

\[ \frac{m_n^*}{2m_q} f_t e^{-m_n^*/M^2} = \frac{3m_q}{4\pi^2} \int_{4m_q^2}^{S_0} ds e^{-s/M^2} \sqrt{1 - \frac{4m_q^2}{s}} - \frac{2\langle \bar{q}q \rangle_\rho e^{-m_q^2/M^2}}{s} \]

\[ + \frac{8m_q}{M^2} \langle q^i D_0 q \rangle_\rho e^{-m_q^2/M^2} - \frac{2m_q^2 \langle \bar{q}q \rangle_\rho e^{-m_q^2/M^2}}{M^2}, \]  

(8)

\[ \frac{m_n^*}{2m_q} f_s e^{-m_n^*/M^2} = \frac{3m_q}{4\pi^2} \int_{4m_q^2}^{S_0} ds e^{-s/M^2} \sqrt{1 - \frac{4m_q^2}{s}} - \frac{2\langle \bar{q}q \rangle_\rho e^{-m_q^2/M^2}}{s} \]

\[ - \frac{8m_q}{3M^2} \langle q^i D_0 q \rangle_\rho e^{-m_q^2/M^2} - \frac{2m_q^2 \langle \bar{q}q \rangle_\rho e^{-m_q^2/M^2}}{M^2} \]

\[ + \frac{32}{3M^2} \left[ \langle \bar{q}(iD_0)^2 q \rangle_\rho + \frac{1}{8} \langle \bar{q}g_{s\sigma} \cdot Gq \rangle_\rho \right] e^{-m_q^2/M^2}. \]  

(9)

Here \( S_0 \) is the continuum threshold and \( M \) is the Borel mass. The various nuclear condensates appearing in the right-hand side are evaluated in the linear density approximation, which yields

\[ \langle q^i D_0 q \rangle_\rho = \rho \langle q^i D_0 q \rangle_N + \frac{m_q}{4} \langle \bar{q}q \rangle_\rho, \]  

(10)

\[ \langle \bar{q}(iD_0)^2 q \rangle_\rho + \frac{1}{8} \langle \bar{q}g_{s\sigma} \cdot Gq \rangle_\rho = \rho \left[ \langle \bar{q}(iD_0)^2 q \rangle_N + \frac{1}{8} \langle \bar{q}g_{s\sigma} \cdot Gq \rangle_N \right] + \frac{m_q^2}{4} \langle \bar{q}q \rangle_\rho, \]  

(11)

\[ \langle \bar{q}q \rangle_\rho = \langle \bar{q}q \rangle_0 + \rho \langle \bar{q}q \rangle_N. \]  

(12)

Here the subscript “0” (“\( N \)” ) denotes the vacuum (nucleon) expectation value. It is interesting to note that, when Eqs. (10) and (11) are put into our sum rules, the \( m_q^2 \langle \bar{q}q \rangle_\rho \) terms are canceled away.

To leading order in quark mass, the ratio of the two sum rules gives

\[ \frac{f_s}{f_t} \simeq 1 - \frac{16}{3M^2} \frac{\langle \bar{q}(iD_0)^2 q \rangle_\rho + \frac{1}{8} \langle \bar{q}g_{s\sigma} \cdot Gq \rangle_\rho}{\langle \bar{q}q \rangle_\rho}. \]  

(13)
Thus, the main splitting between the two decay constants is driven by the nuclear dimension 5 condensate,

\[ \langle \bar{q}(iD_0)^2 q \rangle_\rho + \frac{1}{8} \langle \bar{q}g_\sigma \cdot G q \rangle_\rho . \]  

(14)

Note, in vacuum, this dimension 5 condensate is zero and both sum rules reproduce, up to leading order in quark mass, the well-known Gell-Mann–Oakes–Renner (GOR) relation (in vacuum, \( f_t = f_s = f_\pi = 131 \text{ MeV} \)),

\[ m^2_\pi f^2_\pi = -4m_q \langle \bar{q}q \rangle_0 . \]  

(15)

Even including higher orders in quark mass, our sum rules reproduce the vacuum sum rule for the pseudoscalar decay constants \[20\]. Moreover the \( \Pi_t \) sum rule, Eq.(8), satisfies the in-medium GOR relation

\[ m^*_\pi f^2_t = -4m_q \langle \bar{q}q \rangle_\rho , \]  

(16)

to leading order in \( m_q \).

III. THE DIMENSION 5 CONDENSATE

The dimension 5 condensate is crucial for splitting \( f_t \) and \( f_s \). Depending on its sign, we clearly have different prediction on the ratio \( f_s/f_t \). The ratio becomes greater (smaller) than the unity if the dimension 5 condensate is positive (negative). Using \( \langle \bar{q}g_\sigma \cdot G q \rangle = 2\langle \bar{q}D^2 q \rangle \) in the chiral limit, the dimension 5 condensate in the linear density approximation can be rearranged into the form,

\[ \langle \bar{q}(iD_0)^2 q \rangle_\rho + \frac{1}{8} \langle \bar{q}g_\sigma \cdot G q \rangle_\rho = \rho \left[ \frac{3}{4} \langle \bar{q}(iD_0)^2 q \rangle_N + \frac{1}{4} \langle \bar{q}(iD)^2 q \rangle_N \right] . \]  

(17)

Ref. \[17\] argued that, since \( iD_0 \) and \( iD \) are Hermitian operators, their square must be positive definite. Thus, the nucleon dimension 5 condensate, \( \frac{3}{4} \langle \bar{q}(iD_0)^2 q \rangle_N + \frac{1}{4} \langle \bar{q}(iD)^2 q \rangle_N \), is claimed to have the same sign with the positive quantity, \( \langle \bar{q}q \rangle_N \), which seems consistent with its rough estimate from the bag model \[21\]. This Hermitian argument however is not consistent with the vanishing dimension 5 condensate in vacuum \[1\].

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1 If the same Hermitian argument is applied to its vacuum expectation value, we should have \( \frac{3}{4} \langle \bar{q}(iD_0)^2 q \rangle_0 + \frac{1}{4} \langle \bar{q}(iD)^2 q \rangle_0 \leq 0 \) as \( \langle \bar{q}q \rangle_0 < 0 \). Since the vacuum expectation value is zero, the Hermitian argument only leads to a trivial consequence in vacuum, \( \langle \bar{q}(iD_0)^2 q \rangle_0 = \langle \bar{q}(iD)^2 q \rangle_0 = 0 \). This however can not be correct because their difference is well-known to be nonzero, \( \langle \bar{q}D^2 q \rangle_0 = \langle \bar{q}D^2 q \rangle_0 \neq 0 \).
A possible resolution for this inconsistency can be sought for by considering the condensate in the Euclidean space, \( x_0 \rightarrow ix_4 \). In this space, the vacuum expectation value becomes
\[ -\frac{3}{4} \langle \bar{q}(iD_4)^2 q \rangle_0 + \frac{1}{4} \langle \bar{q}(iD)^2 q \rangle_0 \] and, due to the \( O(4) \) symmetry of vacuum, it is clear that the two terms are canceled to make their sum zero, agreeing with the expectation. Its nucleon expectation value, \( -\frac{3}{4} \langle \bar{q}(iD_4)^2 q \rangle_N + \frac{1}{4} \langle \bar{q}(iD)^2 q \rangle_N \), however, is not necessarily zero because the \( O(4) \) symmetry is broken by a preferred direction of the nucleon state. Now, because of the opposite sign, the Hermitian argument does not lead to a definite sign for the nucleon dimension 5 condensate.

We now comment on a rough estimate of the dimension 5 condensate using the bag model [21]. According to this, \( \langle \bar{q}(iD_0)^2 q \rangle_N \) is zero and, depending on how one treats \( \langle \bar{q}g_s \sigma \cdot G q \rangle_N \), the dimension 5 condensate varies from 0.08 GeV\(^2\) to 0.3 GeV\(^2\). Though not precise, the bag model seems to suggest a positive value for the dimension 5 condensate. However, in the bag model estimate, the non-Abelian nature of gluon fields cannot be implemented and the quark field equation is violated on the bag surface [21]. Because of this limitation, in the subsequent analysis of the in-medium nucleon sum rule [22], the dimension 5 condensate is allowed to be negative, varying from \(-0.5\) GeV\(^2\) to \(0.5\) GeV\(^2\). Therefore, the positive value for the dimension 5 condensate as well as its magnitude is not firmly established. One needs an alternative method to constrain this condensate. In the following analysis, we look for this dimension 5 condensate within our sum rules using the in-medium pion mass as an input.

IV. THE NUCLEON INTERMEDIATE STATE

Another motivation for doing this update is to investigate more closely the hadronic content of the correlation function Eq.(1). In particular, the correlation function may pick up some contributions from \( N \) and \( \Delta \) intermediate states and the quasi-pion dominance has to be checked further. The nuclear correlation function, Eq.(1), in the linear density approximation is separated into the vacuum and nucleon parts,
\[ \Pi^\mu = \Pi_0^\mu + \rho \Pi_N^\mu \] (18)
where the nucleon correlation function is given by
\[ \Pi_N^\mu \equiv i \int d^4x \ e^{ip\cdot x} \langle N | T[\bar{d}(x)i\gamma_5 u(x) \ \bar{u}(0)\gamma^\mu \gamma_5 d(0)] | N \rangle . \] (19)
The nucleon intermediate state can make a pole structure at $p_0^2 = 0$, which may affect the sum rules through Eq. (19) though its contribution is down by the nuclear density $\rho$.

To calculate the $N$ intermediate state contribution, we insert

$$\int \frac{d^4q}{(2\pi)^4} \delta(q^2 - m_N^2) \theta(q_0) |N(q)\rangle \langle N(q)|$$

(20)

between the pseudoscalar and axial-vector currents in Eq. (19). The nucleon matrix elements of the axial-vector current and the pseudoscalar current are given respectively by

$$\langle N(q)|\bar{d}\gamma_\mu\gamma_5u|N(k)\rangle = \bar{U}_N(q)\left[g_A\gamma_\mu + (q-k)\gamma_5g_p\right]U_N(k)$$

(21)

$$\langle N(q)|\bar{d}\gamma_5u|N(k)\rangle = -\frac{f_\pi g_\pi NN}{2m_q} \frac{g_\pi NN}{m^2_N - (q-k)^2} \bar{U}_N(q)i\gamma_5U_N(k)$$

(22)

where $U_N$ is the nucleon spinor, $g_A$ the nucleon axial charge and $g_p$ the induced pseudoscalar term. Inserting these into the nucleon correlation function, Eq. (19), we directly evaluate the $N$ intermediate state contribution. We find that this contribution is proportional to $\sqrt{p^2 + m_N^2 - m_N}$, which vanishes in the limit of Eq. (2). Therefore, the nucleon intermediate state does not contribute to the correlation functions, $\Pi_t$ and $\Pi_s$.

V. THE $\Delta$ INTERMEDIATE STATE

We now consider the $\Delta$ contribution in the nucleon correlation function, Eq. (19). A $\Delta$ couples to $\pi N$ channel strongly and the correlation function may pick up a significant contribution from the $\Delta$ intermediate state. The $\Delta$ intermediate state in $\Pi_N^l$ must have a pole at $p_0^2 = (m_\Delta - m_N)^2 \sim 0.09$ GeV$^2$. One can either treat this as a part of the continuum or directly calculate this contribution using an effective model. Unlike the nucleon intermediate case, however, the matrix elements involved, $\langle \Delta|\bar{u}\gamma_\mu\gamma_5d|N\rangle$ and $\langle \Delta|\bar{d}\gamma_5u|N\rangle$, are not well-known. Even if one can establish a form of the $\Delta$ contribution, the parameters involved highly depend on various models: the estimate of this contribution can not be precise.

Roughly, one may eliminate the $\Delta$ contribution by considering the sum rules with an additional weight of $s - (m_\Delta - m_N)^2$,

$$\int_0^{S_0} ds \ e^{-s/M^2}[s - (m_\Delta - m_N)^2] \frac{1}{\pi} \text{Im}[\Pi_t^{\text{phen}}(s) - \Pi_t^{\text{ope}}(s)] = 0 \quad (l = t, s).$$

(23)

The new weight eliminates the pole at $p_0^2 = (m_\Delta - m_N)^2$ and the sum rules are now free from the $\Delta$ contribution. However, the new weight affects the vacuum part of the sum rules.
as well and, in the limit of \( \rho \to 0 \), the right-hand side of the GOR relation, Eq. (15), entails the factor
\[
\frac{m_q^2 - (m_\Delta - m_N)^2}{m_\pi^2 - (m_\Delta - m_N)^2} \sim 1.3
\]
coming from the additional weight. Though this factor is the unity to leading order in the chiral expansion, its numerical value deviated from the unity affects the GOR relation. Thus, this way of eliminating the \( \Delta \) contribution is crude.

A more economical way is to apply the similar prescription only to the nucleon correlation function \( \Pi^\mu_N \) in Eq. (18). That is, we introduce \( \Pi^N_{\text{phen}} \) and \( \Pi^N_{\text{ope}} \) from \( \Pi^\mu_N \) similarly defined as \( \Pi_t \) and \( \Pi_s \) in Eq. (2) and construct
\[
\int_0^{S_0} ds \, e^{-s/M^2} [s - (m_\Delta - m_N)^2] \frac{1}{\pi} \text{Im} \left[ \Pi^\text{phen}_{Nl}(s) - \Pi^\text{ope}_{Nl}(s) \right] = 0 \quad (l = t, s) . \tag{25}
\]
Obviously, this prescription does not suffer from the problem mentioned above: it does not affect the GOR relation in vacuum. To construct \( \Pi^\text{phen}_{Nl} \) and \( \Pi^\text{phen}_{Ns} \), we expand in terms of the density the in-medium parameters appearing in Eq. (4),
\[
f_t = f_\pi + \rho \Delta f_t ; \quad f_s = f_\pi + \rho \Delta f_s ; \quad m^*_\pi = m_\pi + \rho \Delta m_\pi . \tag{26}
\]
The new phenomenological parameters, \( \Delta f_t, \Delta f_s \) and \( \Delta m_\pi \), constitute \( \Pi^\text{phen}_{Nt} \) and \( \Pi^\text{phen}_{Ns} \). In the OPE side, using Eqs. (10), (11) and (12) for various nuclear condensates, we collect the terms proportional to \( \rho \) corresponding to \( \Pi^\text{ope}_{Nt} \) and \( \Pi^\text{ope}_{Ns} \). We put them into Eq. (25) and obtain the \( \Delta \) subtracted sum rules for \( \Delta f_t \) and \( \Delta f_s \),
\[
\Delta f_t = -\frac{\Delta m_\pi}{m_\pi} f_\pi - \frac{2m_q r_\Delta}{m_\pi^2 f_\pi} \langle \bar{q}q \rangle_N e^{(m_\pi^2 - m_q^2)/M^2} \\
+ \frac{8m_q^2}{m_\pi^2 f_\pi} \langle q^i D_0 q \rangle_N \left[ \frac{r_\Delta}{M^2} - \pi_\Delta \right] e^{(m_\pi^2 - m_q^2)/M^2} ,
\]
\[
\Delta f_t + \Delta f_s = -\frac{2\Delta m_\pi}{m_\pi} f_\pi - \frac{4m_q r_\Delta}{m_\pi^2 f_\pi} \langle \bar{q}q \rangle_N e^{(m_\pi^2 - m_q^2)/M^2} \\
- \frac{16m_q^2}{3m_\pi^2 f_\pi} \langle q^i D_0 q \rangle_N \left[ \frac{r_\Delta}{M^2} - \pi_\Delta \right] e^{(m_\pi^2 - m_q^2)/M^2} \\
+ \frac{64m_q}{3m_\pi^2 f_\pi} \left[ \langle \bar{q}q \rangle_N + \frac{1}{8} \langle \bar{q}g \sigma \cdot G q \rangle_N \right] \left[ \frac{r_\Delta}{M^2} - \pi_\Delta \right] e^{(m_\pi^2 - m_q^2)/M^2} ,
\]
where we have defined
\[
r_\Delta = \frac{m_q^2 - (m_\Delta - m_N)^2}{m_\pi^2 - (m_\Delta - m_N)^2} ; \quad \pi_\Delta = \frac{1}{m_\pi^2 - (m_\Delta - m_N)^2} . \tag{29}
\]
Note, the pion mass shift in the exponential, which is an order $O(m^2_\pi)$ or higher, has been neglected in deriving these sum rules. Once $\Delta m_\pi$ is given, these sum rules yield $\Delta f_t$ and $\Delta f_s$, which then, through Eq. (26), lead to $f_t$ and $f_s$ with the $\Delta$ contribution being subtracted. For the justification of this, we have checked that, without $\Delta$ subtraction, this procedure gives $f_t$ and $f_s$ similar in magnitude with those obtained directly from Eqs. (8), (9).

One may worry about the $\Delta$ decay width and argue that the $\Delta$ contribution is not a pole. A $\Delta$ in free space strongly decays to $\pi N$ with its width 115 MeV. However, in nuclear matter, a $\Delta$ at rest can not decay to $\pi N$ by the Pauli blocking. On the other hand, a $\Delta$ in nuclear matter can have a “spreading width” through the mechanism $\Delta + N \rightarrow N + N$. Its width at nuclear saturation density is between 57 to 75 MeV, reasonably small. The pole ansatz for a resonance with this small width is believed to be reasonable.

VI. SUM RULE ANALYSIS

As we have discussed, the dimension 5 condensate is important for splitting $f_s$ and $f_t$ but its value is not well-known. An additional information is necessary to restrict the value of the dimension 5 condensate. Instead of relying on a model calculation, we look for its value directly from our sum rules using in-medium pion mass as an input. The in-medium pion mass has been studied extensively by chiral perturbation theory. Experimentally, it is extracted from the local potential of the deeply bound pionic atom in $^{208}$Pb. A consensus from these studies is that the in-medium pion mass increases slightly up to 20 MeV. We therefore look for an optimal value of the dimension 5 condensate that leads to the in-medium pion mass within 139 ~ 159 MeV from the $\Pi_s$ sum rule, Eq. (9). The $\Pi_t$ sum rule, Eq. (8), can not be used for this purpose as it does not depend on the dimension 5 condensate.

To calculate the pion mass from our sum rules, we take the derivative of Eq. (9) with respect to $1/M^2$ and divide the resulting formula by Eq. (9). Namely, by defining the right-hand side of Eq. (9) by $\Pi_{Borel}(M^2)$ and its derivative with respect to $1/M^2$ by $\Pi'_{Borel}(M^2)$, the in-medium pion mass satisfies

$$-m^*_{\pi} = \frac{\Pi'_{Borel}(M^2)}{\Pi_{Borel}(M^2)}.$$  

Note, the limit of Eq. (2) means that we are considering a $\Delta$ at rest.
Using this formula, we plot $m_{\pi}^2$ versus the Borel mass $M^2$ in fig. 1 for various values of the nucleon dimension 5 condensate,

$$D_5 \equiv \langle \bar{q} (iD_0)^2 q \rangle_N + \frac{1}{8} \langle \bar{q} g_s \sigma \cdot G q \rangle_N .$$  
(31)

The continuum threshold is set to be $s_0 = 0.09$ GeV$^2$ corresponding to the $\Delta$ intermediate state, i.e., $(m_\Delta - m_N)^2$ but the result is not sensitive to this choice. Other parameters used in our analysis are

$$m_q = 0.007 \text{ GeV} ; \quad \langle \bar{q} q \rangle_0 = (-0.225 \text{ GeV})^3$$
$$\langle \bar{q} q \rangle_N = \frac{0.045 \text{ GeV}}{2m_q} ; \quad \langle \bar{q}^\dagger iD_0q \rangle_N = 0.18 \text{ GeV} .$$  
(32)

As shown, we have quite different curves depending on the dimension 5 condensate, $D_5$. In particular, the positive value of $D_5$ leads to a tachyonic pion mass, $m_{\pi}^* < 0$. As the positive value leads to $f_s/f_t \geq 1$ according to Eq.(13), the tachyonic pion mass for $D_5 \geq 0$ is consistent with the claim that $f_s/f_t \geq 1$ breaks the causality [19]. When $D_5$ is fixed to be around $-0.02$ GeV$^2$, the pion effective mass is about 140 MeV and the stability is quite good over a wide range of the Borel masses. For the in-medium pion mass within 139 MeV $\leq m_{\pi}^* \leq 159$ MeV, $D_5$ is negative and its magnitude is within the range, $0.019 \leq |D_5| \leq 0.025$.

We have also checked that this range of $D_5$ is stable under the rough subtraction of the $\Delta$ contribution given in Eq.(23). Note, the $\Delta$ subtraction procedure advocated in Eq.(25) is not applicable for obtaining the in-medium pion mass.

We now move to an analysis for the pion decay constants, $f_t$ and $f_s$. In fig. 2 we plot $f_t/f_\pi$ using the sum rule Eq.(8) at the saturation density $\rho = 0.17$ fm$^{-3}$. When $m_{\pi}^* = 139$ MeV, $f_t/f_\pi = 0.79$ is obtained from the $\Pi_t$ sum rule Eq.(8) at $M^2 = 1$ GeV$^2$ but with $\Delta$ subtracted according to Eqs.(26) and (27) the ratio becomes slightly smaller, 0.77. The Borel curves for these two cases are shown by the two upper curves in fig. 2. The solid curve is from Eq.(8) and the dashed curve is when the $\Delta$ contribution is subtracted. The lower two curves are obtained when we use the larger pion mass, $m_{\pi}^* = 159$ MeV. Increasing the pion mass makes $f_t/f_\pi$ smaller in both cases. Because the $\Pi_t$ sum rule satisfies the in-medium GOR relation, Eq.(16), large $m_{\pi}^*$ is compensated by small $f_t$. Also at $m_{\pi}^* = 159$ MeV, $f_t$ from the $\Delta$ subtracted sum rule is 9% smaller than the one without $\Delta$ subtraction. Large $m_{\pi}^*$ may easily excite a $\Delta$ so that the $\Delta$ contribution becomes larger in the correlation function. It is interesting to note that $f_t/f_\pi$ at $m_{\pi}^* = 139$ MeV, either with or without $\Delta$ subtraction, is not far from the experimental value of 0.8 [1].
TABLE I: The summary table for the decay constants, $f_t/f_\pi$ and $f_s/f_\pi$ (with $f_\pi = 131$ MeV) calculated from our sum rules for given in-medium pion masses. The $D_5$ value is obtained from Eq. (30). The numbers in parenthesis are when the $\Delta$ contribution is subtracted according to Eq. (25).

| $m^*_\pi$ (MeV) | $D_5$ (GeV$^2$) | $f_t/f_\pi$ | $f_s/f_\pi$ |
|-----------------|-----------------|-------------|-------------|
| 139             | -0.019          | 0.79 (0.77) | 0.78 (0.57) |
| 159             | -0.025          | 0.69 (0.63) | 0.68 (0.37) |

Fig. 3 shows the ratio $f_s/f_\pi$. To obtain this, we first calculate $f_s/f_t$ from the ratio of Eqs. (8) and (9), and multiply it by $f_t/f_\pi$ obtained from Fig. 2. The results with $\Delta$ subtraction from Eqs. (27) and (28) are shown by the dashed curves. Unlike to the $f_t/f_\pi$ case, a somewhat sizable suppression of $f_s/f_\pi$ is obtained when the $\Delta$ contribution is subtracted. This shows that the $\Pi_s$ sum rule depends heavily on the $\Delta$ contribution. At $m^*_\pi = 139$ MeV, the $\Delta$ subtracted sum rule shifts $f_s/f_\pi$ from 0.78 to 0.57, 27% change, while at $m^*_\pi = 159$ MeV, from 0.68 to 0.37, 45 % change. Such a large suppression of $f_s$ is similar to the results from the in-medium chiral perturbation theory [13, 18]. These results as well as the results for $f_t$ are summarized in Table I.

Fig. 4 shows the density dependence of $f_t/f_\pi$. The solid curve is obtained from Eq. (8) with $m^*_\pi = 139$ MeV and the dashed curve is when the $\Delta$ contribution is subtracted according to Eqs. (26) and (27). The similar curves for $f_s/f_\pi$ are shown in Fig. 5. Again, we see that $f_s/f_\pi$ is strongly suppressed when the $\Delta$ contribution is subtracted. It should be also noted that $f_s/f_t$ is less than the unity always, agreeing with the causal constraint derived from Ref. [19].

VII. SUMMARY AND OUTLOOK

We have updated the QCD sum rule calculation of the in-medium pion decay constants, $f_t$ and $f_s$, using pseudoscalar and axial vector correlation function. We have argued that the
dimension 5 condensate, which is crucial for splitting between $f_t$ and $f_s$, is not necessarily restricted to be positive. In fact, the pion mass calculated from the $f_t$ sum rule takes a real value when the dimension 5 condensate is restricted to be negative. We have taken into account contributions from the $N$ and $\Delta$ intermediate states in the correlation function. The $N$ intermediate state was found not to contribute to our sum rules. For the $\Delta$ contribution, we have included either in the continuum or explicitly eliminated by putting an additional weight in the sum rules. The $\Delta$ subtraction procedure was found to affect the extraction of $f_t/f_\pi$ slightly by $3-9\%$. For $f_s/f_\pi$, it affects strongly by $27-45\%$. This strong suppression of $f_s$ is similar to the results from the in-medium chiral perturbation theory. In future, it will be interesting to apply our method for kaonic channel and investigate the in-medium $f_K$. As the strange quark mass is not small, the gluonic dimension 5 operator $m_s \langle G^2 \rangle$, which is absent in this work, could be important.

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FIG. 1: The in-medium pion mass squared, obtained from Eq.(30), is plotted with respect to the Borel mass at the saturation density $\rho_0 = 0.17 \text{ fm}^{-3}$. The number indicated in each curve is the value of the dimension 5 condensate, $D_5$, used. The positive value of $D_5$ leads to the tachyonic mass, $m^*_\pi < 0$. 

![Diagram of the in-medium pion mass squared plotted with respect to the Borel mass at the saturation density $\rho_0 = 0.17 \text{ fm}^{-3}$]
FIG. 2: The Borel curves for the ratio $f_t/f_\pi$ are plotted at the nuclear saturation density $\rho_0 = 0.17$ fm$^{-3}$. The solid lines are obtained from the $\Pi_t$ sum rule, Eq.(8), and the dashed lines are when the $\Delta$ contributions are subtracted according to Eqs.(26) and (27). The value of $m_\pi^*$ used to obtain these curves is indicated.
FIG. 3: The Borel curves for the ratio $f_s/f_\pi$ are plotted at the nuclear saturation density $\rho_0 = 0.17$ fm$^{-3}$. The solid lines are obtained from Eq. (9). The dashed lines are when the $\Delta$ contributions are subtracted according to Eqs. (26) and (28).
FIG. 4: The density dependence of the ratios $f_t/f_\pi$ calculated at $m_\pi^* = 139$ MeV and the Borel mass $M^2 = 1$ GeV$^2$. The solid line (the dashed line) is obtained without (with) $\Delta$ subtraction.
FIG. 5: The density dependence of the ratios $f_s/f_\pi$ calculated at $m_\pi^* = 139$ MeV and the Borel mass $M^2 = 1$ GeV$^2$. The solid line (the dashed line) is obtained without (with) $\Delta$ subtraction. This shows that $f_s/f_\pi$ has a strong dependence on the $\Delta$ contribution.