Lorentz violation effects in $2\nu\beta\beta$ decay

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Abstract. Observable effects for the Lorentz invariance violation (LIV) at a low energy scale can also be investigated in double beta decay (DBD). For example, by comparing the theoretical predictions with a precise analysis of the summed energy spectra of electrons in $2\nu\beta\beta$ decay, one can constrain the $\tilde{a}^{(3)}_{\nu f}$ coefficient that governs the time-like component of the Lorentz invariance violating operator that appears in the Standard Model extension theory. In this work, we perform calculations of the phase space factors and summed energy spectra of electrons as well as of their deviations due to LIV necessary in such experimental investigations. The Fermi functions needed in the calculation are built up with exact electron wave functions obtained by numerically solving the Dirac equation in a realistic Coulomb-type potential with the inclusion of the finite nuclear size and screening effects. We compared our results with those used in previous LIV investigations that were obtained with approximate (analytical) Fermi functions and found differences of up to 30% for heavier nuclei. Our work includes eight experimentally interesting nuclei. Next, we estimate and discuss the uncertainties of our calculations associated with uncertainties in Q-values measurements and the differences raised from the inclusion of the kinematic terms in the formalism. Finally, we provide the ratio between the standard phase space factors and their LIV deviations and the energies where the LIV effects are expected to be maximal. We expect our study to be useful in the current LIV investigations in $2\nu\beta\beta$ decay and to lead to improved constraints on the $\tilde{a}^{(3)}_{\nu f}$ coefficient.

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1. Introduction

There is a currently increasing interest in testing possible violations of fundamental symmetries in physical processes, one of them being the Lorentz invariance. The general theory that incorporates LIV is the Standard-Model extension (SME), which was developed by including Lorentz-violating terms in the Lagrange density [1–4]. These terms are constructed as coordinate-invariant products of a Lorentz invariance violating operator and a coefficient that controls the LIV size. The LIV operators can be of arbitrarily large dimension. Still, a particular interest represents the minimal SME, a limiting case of the SME theory that includes only LIV operators of mass dimension four or less [4]. Direct observation of LIV implies the investigation of physics at the Plank scale, which is not possible at the moment. However, it may be possible that LIV effects can manifest at a low energy scale and be potentially observable with current or near-future experimental techniques. For example, the experiments dedicated to the investigation of neutrino properties can be a framework where visible effects of LIV can be searched. The operators that couple to neutrinos in SME affect neutrino oscillations, neutrinos velocity and the electron spectra in beta and double-beta decays [5–9]. Effects of LIV have been searched in many neutrino oscillation experiments as Double-Chooz [10], MiniBooNE [11], IceCube [12], MINOS [13], SuperKamiokande [14] which obtained constraints on the corresponding coefficients. However, LIV in the neutrino sector can also be induced by a Lorentz-violating operator in SME, called countershaded operator, which does not affect the neutrino oscillations and which also breaks the CPT symmetry. The LIV effects related to this operator are controlled by the oscillation-free (of) coefficient with four components, one time-like \((a_{of}^{(3)})_{00}\) and three space-like \((a_{of}^{(3)})_{im}\), with \(m = 0, \pm 1\) [5]. Non-zero values of \((a_{of}^{(3)})_{00}\) coefficient would produce small deviations in the shape of the energy spectra of electrons emitted in beta and double-beta decays [9]. Such investigations were recently performed in the DBD collaborations EXO [15], GERDA [16], SuperNEMO [17], CUORE [18, 19] and CUPID-0 [20], and the non-observation of LIV effects resulted in constrains of the isotropic coefficient \(a_{of}^{(3)} = (a_{of}^{(3)})_{00}/\sqrt{4\pi}\) [7]. However, in most of these analyses, the necessary theoretical ingredients were calculated with approximate (analytical) Fermi functions [21].

In this paper, we perform calculations of the phase space factors (PSF), summed energy spectra of electrons and their deviations necessary in LIV investigations from DBD experiments. The Fermi functions needed in the calculation are built with exact electron wave functions obtained by numerically solving a Dirac equation in a realistic Coulomb-type potential, with the inclusion of finite nuclear size and screening effects using a method described in our previous papers [25, 26]. We compared our results with those used in previous LIV investigations, which were obtained with approximate (analytical) Fermi functions, and found differences up to 30% for heavier nuclei. Our study includes the nuclei \(^{48}\text{Ca}\), \(^{76}\text{Ge}\), \(^{82}\text{Se}\), \(^{100}\text{Mo}\), \(^{110}\text{Pd}\), \(^{116}\text{Cd}\), \(^{130}\text{Te}\) and \(^{136}\text{Xe}\). Next, we estimate and discuss the uncertainties in our calculations associated with uncertainties
in Q-values measurements and the differences raised from the inclusion of the kinematic terms in the formalism. The Q-values and their uncertainties were obtained for each nucleus by averaging values reported in the literature by using a statistical procedure recommended by the Particle Data Group [27]. Finally, we provide for each nucleus the ratio between the standard PSF and their LIV deviations and the energies where the LIV effects are expected to be maximal. We expect our study to be useful in the current LIV investigations in $2\nu\beta\beta$ decays and to lead to improved constraints on the $\hat{\alpha}^{(3)}_{o_f}$ coefficient.

2. Formalism

The interactions of neutrinos with the countershaded operator modify their four-component momentum from the standard expression $q^a = (\omega, q)$ to $q^a = (\omega, q + \hat{a}^{(3)}_{o_f} \hat{q})$ [4, 9, 28]. Considering this modification, the $2\nu\beta\beta$ decay rate can be expressed as a sum of two components:

$$\Gamma = \Gamma_0 + \delta\Gamma,$$

where $\Gamma_0$ is the standard decay rate and $\delta\Gamma$ is the perturbation induced by LIV. As known, the standard $2\nu\beta\beta$ decay rate can be expressed to a good approximation as follows [29]:

$$\Gamma_0 = g_A^4|m_e c^2 M^{2\nu}|^2 G^{2\nu}$$

In this expression $g_A$ is the axial vector constant, $M^{2\nu}$ (in MeV$^{-1}$) is the nuclear matrix element and $G^{2\nu}$ is the PSF of the transition. Lorentz-violating effects in DBD appear as kinematical effects modifying only the PSF. Thus, the decay rate induced by LIV can be expressed in the form:

$$\delta\Gamma = g_A^4|m_e c^2 M^{2\nu}|^2 \delta G^{2\nu}$$

where $\delta G^{2\nu}$ is the PSF perturbation due to LIV. In what follows, we adopt the natural units ($\hbar = c = 1$), and we write the energy variables in units of $m_e$. For a $2\nu\beta\beta$ transition to the ground state (g.s.) of the daughter nucleus, the PSF expression reads:

$$G^{2\nu} = C_1 \int_0^Q d\epsilon_1 \int_0^{Q-\epsilon_1} d\epsilon_2 \int_0^{Q-\epsilon_1-\epsilon_2} d\omega_1 \times F(Z_f, \epsilon_1) F(Z_f, \epsilon_2) \sqrt{\epsilon_1(\epsilon_1+2)(\epsilon_1+1)} \sqrt{\epsilon_2(\epsilon_2+2)(\epsilon_2+1)} \times \omega_1^2(Q - \epsilon_1 - \epsilon_2 - \omega_1)^2 \left(\langle K_N \rangle^2 + \langle L_N \rangle^2 + \langle K_N \rangle \langle L_N \rangle\right)$$

where $C_1 = (\tilde{A}^2 G_F^2 |V_{ud}|^4 m_e^9)/(96\pi^7 \ln 2)$, $G_F$ is the Fermi coupling constant and $V_{ud}$ the first element of the CKM matrix. $\epsilon_{1,2}$ and $\omega_1$ are the energies of the electrons and of one antineutrino emitted in the decay and $F(Z, \epsilon)$ is the Fermi function. $\langle K_N \rangle$, $\langle L_N \rangle$ are kinematic factors that depend on the electrons ($\epsilon_{1,2}$) and antineutrinos ($\omega_{1,2}$) energies,
on the g.s. energy $E_I$ of the parent nucleus and on an averaged energy $\langle E_N \rangle$ of the excited states in the intermediate nucleus (closure approximation) \cite{22}.

$$\langle K_N \rangle = \frac{1}{\epsilon_1 + \omega_1 + \langle E_N \rangle - E_I + 1} + \frac{1}{\epsilon_2 + \omega_2 + \langle E_N \rangle - E_I + 1}$$

$$\langle L_N \rangle = \frac{1}{\epsilon_1 + \omega_2 + \langle E_N \rangle - E_I + 1} + \frac{1}{\epsilon_2 + \omega_1 + \langle E_N \rangle - E_I + 1}$$

Here, the difference in energy in the denominator can be obtained from the antineutrino momentum which, from Eq. (4), becomes $d\omega$ analyizes \cite{15, 17, 20}: DBD works (see for example \cite{24} and references therein) and also in the previous LIV to their use in the computation of the quantities relevant for analysing of LIV effects in $2\nu\beta\beta$ Q-values. In the next sections, we refer to their use in the computation of the quantities

$$\delta G^{2\nu} = 10\delta^{(3)}_{o_f} C_2 \int_0^Q d\epsilon_1 \int_0^{Q-\epsilon_1} d\epsilon_2 \int_0^{Q-\epsilon_1-\epsilon_2} d\omega_1 \times F(Z_f, \epsilon_1)F(Z_f, \epsilon_2)^{\sqrt{\epsilon_1(\epsilon_1 + 2)(\epsilon_1 + 1)}\sqrt{\epsilon_2(\epsilon_2 + 2)(\epsilon_2 + 1)}(Q - \epsilon_1 - \epsilon_2)}$$

where $C_2 = (\bar{A}^2 G^4_F |V_{ud}|^4 m_e^8)/(240 \pi^7 \ln 2)$. The isotropic coefficient $\delta^{(3)}_{o_f}$, which governs the Lorentz violation strength, is the quantity of interest.

We note here that by making the approximation:

$$\langle K_N \rangle \simeq \langle L_N \rangle \simeq \frac{2}{E_I - \langle E_N \rangle - (Q/2 + 1)}$$

where $C_3 = (\bar{A}^2 G^4_F |V_{ud}|^4 m_e^8)/(240 \pi^7 \ln 2)$ and $C_4 = (G^4_F |V_{ud}|^4 m_e^8)/(240 \pi^7 \ln 2)$.

As seen from the above expressions, the main ingredients in the PSF calculation are the Fermi functions, which encodes the distortion of the electron wave functions due to the Coulomb field of the daughter nucleus, the kinematic factors $\langle K_N \rangle$, $\langle L_N \rangle$ and the Q-values. In the next sections, we refer to their use in the computation of the quantities relevant for analysing of LIV effects in $2\nu\beta\beta$ decay.
3. Fermi functions

In this section, different approximation schemes used for the calculation of the Fermi function are briefly presented.

3.1. The approximation scheme A

In an early derivation of the $2\nu\beta\beta$ decay rate Primakoff & Rosen [21] and Konopinski [30], considered a non-relativistic movement of the emitted electrons in a Coulomb potential given by a point-like nucleus. The correction to the electron wave function can be derived by taking the square of the ratio between the Schrödinger scattering solution for a point charge $Z_f$ and a plane wave, evaluated at the origin. In this case,

$$F(Z_f, \epsilon) = F^{NR}(Z_f, \epsilon) = \frac{2\pi\eta}{1 - e^{-2\pi\eta}}$$

(11)

where $\eta = \pm \alpha Z_f \epsilon / p$ ("+" corresponds to electrons and "-" to positrons), $\alpha \simeq 1/137$ is the fine structure constant, $\epsilon$ is the energy of the outgoing particle and $p = |\vec{p}|$ is their momentum.

Although this approximation fails badly for heavy nuclei (for example, a factor of 5 enhancement in the $^{130}\text{Te}$ phase space factor [31]), its use has the advantage that DBD rate can be integrated analytically and the obtained expression of the decay rate is a polynomial in powers of the Q-value.

3.2. The approximation scheme B

Adopting a relativistic treatment, the Fermi function is obtained as solution of a Dirac equation in a Coulomb potential given by a point charge (the finite size of the nucleus is neglected). It can still be expressed in an analytical form as:

$$F(Z_f, \epsilon) = F_0(Z_f, \epsilon) = 4(2pR_A)^{2(\gamma-1)}e^{\pi\eta}\left|\frac{\Gamma(\gamma + i\eta)}{\Gamma(2\gamma + 1)}\right|^2,$$

(12)

with

$$\gamma = \sqrt{1 - (\alpha Z_f)^2}, \quad \eta = \pm \alpha Z_f \epsilon / p$$

(13)

with the same sign convention as above. $R_A$ is the cut-off radius in the evaluation of the Dirac equation which is taken to match the radius of the daughter nucleus (i.e. $R_A = 1.2A^{1/3}$ fm). This approximation scheme was used extensively in the past [22–24] and will also be used here for comparison. We also note that in those works, the Gamma functions were computed using approximate expressions. In our case, the above expression of the Fermi function was computed using an exact form of the Gamma function and the results agree (within a few percentages) with the ones from Refs. [22–24].
3.3. The approximation scheme C

The Fermi function is built up from the radial solutions of the Dirac equation:

\[ F(Z_f, \epsilon) = \frac{f_1^2(\epsilon, R_A) + g_{-1}^2(\epsilon, R_A)}{2p^2}, \]  

(14)

where \( f_1 \) and \( g_{-1} \) are the radial wave functions of an electron in the \( s_{1/2} \) state evaluated at the nuclear radius \( R_A = 1.2A^{1/3} \), which satisfy the radial Dirac equations:

\[
\begin{align*}
\left( \frac{d}{dr} + \frac{\kappa + 1}{r} \right) g_\kappa(\epsilon, r) &= (\epsilon - V(r) + 1)f_\kappa(\epsilon, r) \\
\left( \frac{d}{dr} - \frac{\kappa - 1}{r} \right) f_\kappa(\epsilon, r) &= -(\epsilon - V(r) - 1)g_\kappa(\epsilon, r)
\end{align*}
\]  

(15)

The relativistic quantum number \( \kappa \), takes positive and negative integer values. Total angular momentum of the electron is given by \( j_\kappa = |\kappa| - 1/2 \). The input central potential \( V(r) \) includes the finite nuclear size correction and is built from a realistic proton charge density of the daughter nucleus:

\[ \rho_e(\vec{r}) = \sum_i (2j_i + 1)v_i^2 |\psi_i(\vec{r})|^2, \]  

(16)

where \( j_i \) is the proton spin, \( v_i \) is the occupation amplitude of the proton wave function \( \psi_i \) of the spherical single particle state \( i \), numerically determined as solution of the Schrödinger equation with a Wood-Saxon potential. The daughter nucleus potential is given by

\[ V(Z, r) = \alpha \hbar c \int \frac{\rho_e(\vec{r})}{|\vec{r} - \vec{r}'|} d\vec{r}'. \]  

(17)

This form of the potential includes diffuse nuclear surface corrections. Further, the screening effect of atomic electrons is taken into account following a prescription described in ref. \[33\], by multiplying the expression of \( V(r) \) with a function \( \phi(r) \), which is solution of the Thomas-Fermi equation

\[ \frac{d^2\phi}{dx^2} = \frac{\phi^{3/2}}{\sqrt{x}}, \]  

(18)

with \( x = r/b, b \approx 0.8853a_oZ^{-1/3} \) and \( a_o \) is the Bohr radius. The method used to solve numerically the Thomas-Fermi equation is the Majorana method \[34\]. The solutions of the Dirac equation were obtained using our code that was built up following the numerical procedure presented in the refs. \[35-37\]. This treatment of computing the Fermi function was also used by different authors \[33,38,39\], but the use of a realistic Coulomb potential was introduced only in Refs. \[25,26\].
4. Comparison between Fermi functions

In earlier PSF calculations using approximate (analytical) forms of Fermi functions, the finite nuclear size or screening effects were not considered in computations, which can have a notable impact on the accuracy of the 2νββ decay related predictions, required in precise measurements. In this section, we calculate the PSF values and their deviations, using the Equations (9) and (10) (without the inclusion of kinematic factors, as they were used in previous LIV analyzes). For the LIV perturbation, relevant is the quantity \( \delta G^{2\nu}/(10\alpha_{of}^{(3)}) \), which is independent on the LIV coefficient and is used by the experimentalists to constrain it.

Table 1. Phase space factors computed with Fermi functions in approximation schemes A, B, C. Values are obtained with Equations (9-10). Q-values are also displayed.

| Nucleus | \( Q^a(\text{keV}) \) | \( G^{2\nu} \) in units of \( 10^{-21}\text{yr}^{-1} \) | A | B | C |
|---------|-----------------|-----------------|---|---|---|
| \( ^{48}\text{Ca} \) | 4268.070 ± 0.076 | 12051.21 ± 2.02 | 15941.38 ± 2.66 | 15212.54 ± 2.54 |
| \( ^{76}\text{Ge} \) | 2039.059 ± 0.014 | 25.755 ± 0.002 | 52.834 ± 0.003 | 48.384 ± 0.003 |
| \( ^{82}\text{Se} \) | 754.297 ± 0.3 | 828.96 ± 0.75 | 1779.02 ± 1.58 | 1597.40 ± 1.42 |
| \( ^{100}\text{Mo} \) | 3034.40 ± 0.17 | 1241.83 ± 0.62 | 3822.34 ± 1.89 | 3312.62 ± 1.63 |
| \( ^{110}\text{Pd} \) | 2017.85 ± 0.64 | 40.32 ± 0.11 | 163.07 ± 0.43 | 138.57 ± 0.37 |
| \( ^{116}\text{Cd} \) | 2813.50 ± 0.13 | 776.81 ± 0.32 | 3318.62 ± 1.33 | 2767.61 ± 1.11 |
| \( ^{130}\text{Te} \) | 2527.515 ± 0.260 | 344.24 ± 0.31 | 1891.38 ± 1.66 | 1538.53 ± 1.35 |
| \( ^{136}\text{Xe} \) | 2458.13 ± 0.41 | 287.56 ± 0.42 | 1797.71 ± 2.55 | 1441.47 ± 2.04 |

\[
\delta G^{2\nu}/(10\alpha_{of}^{(3)}) \text{ in units of } 10^{-21}\text{yr}^{-1}\text{MeV}^{-1} \]

| Nucleus | \( Q^a(\text{keV}) \) | \( \delta G^{2\nu}/(10\alpha_{of}^{(3)}) \) in units of \( 10^{-21}\text{yr}^{-1}\text{MeV}^{-1} \) | A | B | C |
|---------|-----------------|-----------------|---|---|---|
| \( ^{48}\text{Ca} \) | 4268.070 ± 0.076 | 5318.08 ± 0.81 | 7002.59 ± 1.06 | 6674.90 ± 1.01 |
| \( ^{76}\text{Ge} \) | 2039.059 ± 0.014 | 21.754 ± 0.001 | 44.195 ± 0.002 | 40.446 ± 0.002 |
| \( ^{82}\text{Se} \) | 2997.9 ± 0.3 | 497.02 ± 0.40 | 1054.26 ± 0.84 | 945.27 ± 0.75 |
| \( ^{100}\text{Mo} \) | 3034.40 ± 0.17 | 733.70 ± 0.33 | 2219.94 ± 0.99 | 1920.26 ± 0.85 |
| \( ^{110}\text{Pd} \) | 2017.85 ± 0.64 | 34.09 ± 0.08 | 135.41 ± 0.32 | 114.94 ± 0.27 |
| \( ^{116}\text{Cd} \) | 2813.50 ± 0.13 | 489.48 ± 0.18 | 2046.83 ± 0.74 | 1703.76 ± 0.61 |
| \( ^{130}\text{Te} \) | 2527.515 ± 0.260 | 238.08 ± 0.19 | 1277.20 ± 1.00 | 1036.90 ± 0.81 |
| \( ^{136}\text{Xe} \) | 2458.13 ± 0.41 | 203.69 ± 0.27 | 1241.58 ± 1.57 | 993.53 ± 1.26 |

\( ^a \) See Appendix for details on Q-values.

The results are shown in Table 1 wherein the upper part are displayed the PSF values, while in the lower part their deviations. Computations are done for the eight nuclei shown in the first column. The second column indicates the \( Q \)-values for each decay (obtained with the recipe described in the Appendix), while in the next three columns are presented the \( G^{2\nu} \) and \( \delta G^{2\nu}/(10\alpha_{of}^{(3)}) \) values with Fermi functions obtained in approximation schemes A, B, and C. As expected, there are significant differences...
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between the non-relativistic approximation A and the relativistic ones B, C. The non-relativistic treatment underestimates much the PSF values. However, there are also relevant differences between the PSF values calculated with Fermi functions obtained with B and C treatments. The approximation scheme B overestimates the PSF values as compared with C, and the differences between the two sets of values increase with the nuclear mass from about 8% for $^{48}$Ca to about 30% for $^{136}$Xe. Thus, the non-relativistic treatment of the Fermi functions is inadequate for LIV analyzes. In contrast, the use of exact radial solutions of the Dirac equation for building Fermi functions brings relevant corrections to the $G^{2\nu}$ and $\delta G^{2\nu}$ values as compared with those obtained with analytical expressions.

5. Electron spectra

The differences between the PSF values obtained with different Fermi functions are reflected in the electron spectra. To obtain the summed energy spectra of electrons, we changed the integration variables from the individual electrons kinetic energies to their sum $K = \epsilon_1 + \epsilon_2$ and their difference $T = \epsilon_1 - \epsilon_2$ with $K \in [0, Q]$ and $T \in [-K, K]$. In Figure 1 and Figure 2 are plotted the energy spectra (upper part) and their deviations due to LIV (lower part) corresponding with Equation (4) and Equation (7), respectively. The legends in Figure 1 and Figure 2 are related to the approximation schemes for the Fermi functions discussed in Section 3. The spectra corresponding to the LIV component are presented independently of the oscillation-free coefficient $a_{of}^{(3)}$.

It can be observed that the use of non-relativistic Fermi functions (A scheme) gives spectra that differ much from those obtained with relativistic Fermi functions (B, C schemes). Also, there are significant differences between the electron spectra obtained with the B and C approximations. Hence, the use of accurate Fermi functions is important for obtaining reliable summed energy spectra of electrons. Further, looking to the lower part of the figures, one observes that these standard electron spectra should be shifted if LIV effects are present. To highlight this shift, a vertical line passing the maximum of the standard spectrum is present in the figures. We note that, at request, we can provide experimentalists with the numerical data needed to build up theoretical electron spectra.

6. Quantities of experimental interest

In LIV investigations, an important parameter is also the value of the summed energy of electrons where the LIV effect is expected to be maximum. This energy, $K_m$, corresponds to the position where the LIV summed energy spectrum of electrons is maximal. This quantity is dependent on all the input ingredients taken into account in spectrum computation (Fermi function, Q-value, the inclusion of kinematic factors). As seen in Figures 1 and 2 the deviation should not be large since all spectra have similar shape. We computed these positions and presented them in Table 2 together with the ones
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Figure 1. Summed energy spectra of electrons in the standard $2\nu\beta\beta$ decay (a) and their deviations due to LIV (b) for the nuclei $^{48}\text{Ca}$, $^{76}\text{Ge}$, $^{82}\text{Se}$ and $^{100}\text{Mo}$.

obtained in [9]. The values in the third column were obtained by employing the full treatment of the $2\nu\beta\beta$ decay rate, namely using numerical Fermi functions obtained with approximation scheme C, taking the Q-values given in Appendix and including the kinematic factors.

The last column of Table 2 contains the ratio of the integrals in Equation (4) and Equation (7) which is independent of the LIV coefficient $\dot{a}_{ij}^{(3)}$ and, as shown in [20], it is
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Figure 2. Summed energy spectra of electrons in the standard $2\nu\beta\beta$ decay (a) and their deviations due to LIV (b) for the nuclei $^{110}$Pd, $^{116}$Cd, $^{130}$Te and $^{136}$Xe.

the needed input in searching for LIV effects. The values presented in this column are computed using the same treatment as for the ones in the third column. We note the difference between our prediction of the ratio $\tilde{a}_{of}^{(3)} G^{2\nu} / \delta G^{2\nu}$ for $^{82}$Se ($16.823 \times 10^{-6}$GeV) and the one given in [20] ($213.3 \times 10^{-6}$GeV) to constrain the $\tilde{a}_{of}^{(3)}$ coefficient.
**Table 2.** $K_m$ - energy where LIV effect is maximal and $\delta (3) G^{2\nu}/G^{2\nu}$ - ratio calculated with formulas (4) and (7).

| Nucleus | $K_m$(keV) | $K_m$(keV) | $\delta (3) G^{2\nu}/G^{2\nu}$ (10$^{-6}$GeV) |
|---------|-------------|-------------|-----------------------------------------------|
| $^{48}$Ca | 1980        | 2002        | 227.199 ± 0.048                               |
| $^{76}$Ge | 810         | 818         | 119.562 ± 0.009                               |
| $^{82}$Se | 1300        | 1297        | 168.823 ± 0.195                               |
| $^{100}$Mo | 1320        | 1294        | 172.365 ± 0.111                               |
| $^{110}$Pd | –           | 777         | 120.518 ± 0.420                               |
| $^{116}$Cd | 1200        | 1165        | 162.339 ± 0.085                               |
| $^{130}$Te | 1050        | 1013        | 148.307 ± 0.170                               |
| $^{136}$Xe | 1020        | 980         | 145.023 ± 0.269                               |

7. Conclusions

Investigations of LIV effects are currently also conducted in DBD, particularly by searching for deviations of the summed energy spectra of electrons in $2\nu/\beta\beta$ decays from their standard form. In the absence of observing such deviations, constraints are placed on the $\delta (3)$ coefficient that controls the strength of the LIV associated with the time-like component of the countershaded operator in SME theory. For the experimental investigations, theoretical calculations of PSF, summed energy spectra of electrons, and their deviations due to LIV are needed. In this work, we provide accurate calculations of these quantities using exact electron wave functions for building the Fermi functions with the inclusion of finite nuclear size and screening effects, Q-values obtained by averaging on values provided by experimental measurements and PSF expressions that include the kinematic factors. Comparing our results with previous ones used in other LIV investigations, we show that the choice of the Fermi functions is the essential ingredient in calculations. We obtained differences up to about 30% even between different relativistic methods of calculation of these functions. We estimate the uncertainties in the computation of these quantities associated with experimental measurements of the Q-values and with the omission of the kinematic factors in the PSF expressions. Uncertainties in the PSF values due to the use of inaccurate Q-values are nucleus-dependent and are estimated at $8 \times \sigma_Q/Q$. Next, we calculated with our method described in Section 3.3 and Refs. [25][26] the quantities of experimental interest in LIV analyses, namely the ratios between the standard PSF and their LIV deviations.

Precise calculations can significantly influence the theoretical data needed in LIV analyses. As an example, we found a relevant difference between our obtained value of this ratio and the one used by the CUPID-0 collaboration. Hence, we hope that our theoretical predictions corroborated with a precise analysis of the summed energy spectra of electrons lead to improved constraints on the $\delta (3)$ coefficient that controls the LIV strength of the time-like component of the countershaded operator in SME.
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Appendix

One of the most important parameters in the computation of the PSFs is the Q-value of the double-beta decay. In this study, we treated the uncertainty associated with the Q-value as the only source of uncertainty in the PSF. There are multiple experimental values reported in the literature for different double-beta decays, and the use of one or another influences the PSF calculated values and predictions of the electron spectra. The choice of the Q-values used in the PSF calculation is made as follows. For each $2\nu\beta\beta$ decay, we collect the $Q_i$-values from the literature together with their statistical errors ($\delta Q_i$). Then, we calculate an average Q-value and the statistical error for each decay following a procedure presented in [27] that we shortly describe here.

Table A1. Experimental Q values and the average Q-values for g.s. to g.s. transitions

| Parent nucleus | Measured Energies(keV) | Average Value | $Q \pm \sigma_Q$(keV) | S |
|----------------|------------------------|---------------|-----------------------|---|
| $^{48}$Ca      | $4262.96 \pm 0.84$      | $4268.121 \pm 0.079$ | $4268.070 \pm 0.076$ | 1.0 |
|                | $4267.98 \pm 0.32$      |               |                       |    |
| $^{76}$Ge      | $2038.56 \pm 0.32$      | $2038.58 \pm 0.31$ |                        |    |
|                | $2039.006 \pm 0.05$     | $2039.04 \pm 0.16$ |                        |    |
|                | $2039.061 \pm 0.007$    |               |                       |    |
| $^{82}$Se      | $2997.9 \pm 0.3$        |               |                       |    |
| $^{100}$Mo     | $3034.0 \pm 6.0$        | $3034.40 \pm 0.17$ |                        |    |
| $^{110}$Pd     | $2017.85 \pm 0.64$      |               |                       |    |
| $^{116}$Cd     | $2813.50 \pm 0.13$      |               |                       |    |
| $^{130}$Te     | $2527.01 \pm 0.32$      | $2527.518 \pm 0.013$ |                        |    |
|                | $2526.97 \pm 0.23$      |               |                       |    |
| $^{136}$Xe     | $2479.0 \pm 79.0$       | $2462.7 \pm 4.3$  |                        |    |
|                | $2457.83 \pm 0.37$      | $2458.73 \pm 0.56$ |                        |    |

A $Q$-value for a particular DBD is measured in $N$ independent experiments. For the set of measurements obtained, $x_i \pm \delta x_i$, a Gaussian distribution is considered. Here $x_i = Q_i$ and $\delta x_i = \delta Q_i$ are the value and the error provided by the $i$-experiment, respectively. The weighted average and the corresponding error are calculated with the following equation:

$$\bar{x} \pm \delta \bar{x} = \frac{\sum w_i x_i}{\sum w_i} \pm \left( \sum w_i \right)^{-1/2}$$

(A.1)
Table A2. Phase space factors (upper part) and their deviations (lower part) computed with numerical Fermi functions and with inclusion of kinematic factors. Relative uncertainties associated with Q-values are displayed in column 3, in terms of $\sigma_Q/Q$. Differences introduced by the omission of kinematic factors: $\xi = (G - G^{\text{No VS}})/G$ and $\delta\xi = (\delta G - \delta G^{\text{No VS}})/\delta G$ are also displayed in the last column ($G^{\text{No VS}}$ and $\delta G^{\text{No VS}}$ are the ones from Table I).

| Nucleus | $G(10^{-21}\text{yr}^{-1})$ | $\sigma_G/G\left(\frac{\sigma_Q}{Q}\right)$ | $\xi(\%)$ |
|---------|-----------------------------|---------------------------------|-----------|
| $^{48}\text{Ca}$ | 15443.23 | 8.83 | 1.494 |
| $^{76}\text{Ge}$ | 48.50 | 8.32 | 0.231 |
| $^{82}\text{Se}$ | 1604.65 | 8.58 | 0.452 |
| $^{100}\text{Mo}$ | 3325.32 | 8.53 | 0.382 |
| $^{110}\text{Pd}$ | 138.79 | 8.20 | 0.158 |
| $^{116}\text{Cd}$ | 2775.55 | 8.43 | 0.286 |
| $^{130}\text{Te}$ | 1541.74 | 8.32 | 0.208 |
| $^{136}\text{Xe}$ | 1444.19 | 8.29 | 0.188 |

| Nucleus | $\delta G/10\delta a_0(10^{-21}\text{yr}^{-1}\text{MeV}^{-1})$ | $\sigma_{SG}/\delta G\left(\frac{\sigma_Q}{Q}\right)$ | $\delta\xi(\%)$ |
|---------|----------------------------------------------------------|---------------------------------|-----------|
| $^{48}\text{Ca}$ | 6797.21 | 7.94 | 1.800 |
| $^{76}\text{Ge}$ | 40.56 | 7.43 | 0.284 |
| $^{82}\text{Se}$ | 950.49 | 7.69 | 0.549 |
| $^{100}\text{Mo}$ | 1929.23 | 7.64 | 0.465 |
| $^{110}\text{Pd}$ | 115.16 | 7.31 | 0.195 |
| $^{116}\text{Cd}$ | 1709.73 | 7.54 | 0.349 |
| $^{130}\text{Te}$ | 1039.56 | 7.43 | 0.255 |
| $^{136}\text{Xe}$ | 995.83 | 7.39 | 0.231 |

where

$$w_i = \frac{1}{(\delta x_i)^2}$$  \hspace{1cm} (A.2)

and the sums run over all $N$ experiments. For each weighted average, we calculated $\chi^2 = \sum w_i(\bar{x} - x_i)^2$ and a scale factor, $S$, defined as

$$S = \left(\frac{\chi^2}{N - 1}\right)^{1/2}$$  \hspace{1cm} (A.3)

to establish if the measurements are indeed from a Gaussian distribution. If the scale factor is less than or equal to 1, the value of $\delta\bar{x}$ is left unchanged, and the result is accepted. If $S$ is larger than 1 and the input $\delta x_i$ are all about the same size, then we increase $\delta\bar{x}$ by the scale factor. In the final case of $S$ larger than 1 and the $\delta x_i$ are of widely varying magnitudes, $S$ is recalculated with only the input for which $\delta x_i \leq 3N^{1/2}\delta\bar{x}$. In all cases, the original value of $\bar{x}$ remains unchanged.

The results of this procedure are displayed in Table A1 where the first column contains the studied nuclei. The second column contains the experimental data available for each nucleus. The average $Q$-values and their uncertainties ($\sigma_Q$) obtained with the procedure described above are presented in the third column, while the resulting scaling factor $S$ is shown in the last column.
Lorentz violation effects in $2\nu\beta\beta$ decay

Since this procedure provides also an averaged uncertainty for each $Q$-value, an estimation of the uncertainties in the PSFs can be easily made as follows. Let $f$ denote both $G$ and $\delta G$ factors which depend on the $Q$-value, which is subject to some uncertainty $\sigma_Q$. Then, since $Q$ is the only source of error accounted for in this study, the uncertainty of $f$ can be computed using the formula

$$\sigma_f = \left( \frac{\partial f}{\partial Q} \right) \sigma_Q \quad \text{(A.4)}$$

As seen in the table, the uncertainty of $Q$ is reflected significantly in the PSF uncertainty.

The use of more accurate expressions for the kinematic factors $\langle K_N \rangle$ and $\langle L_N \rangle$ (Equation (5) and Equation (6), instead of Equation (8)) also gives differences in the calculations of PSF, summed energy electron spectra and their deviations due to LIV, but with smaller effects than in the case of using different $Q$-values. In Table A2 we report the PSF values (upper part) and their deviations (lower part) computed with numerical Fermi functions (approximation C) and inclusion of the kinematic factors.

The third column displays the relative uncertainty of the PSF values in terms of relative uncertainties in the $Q$-values. As a rule of thumb, the relative uncertainty of the PSFS is between 8 and 9 times (Standard) and between 7 and 8 times (LIV) the one of the $Q$-value. The last column of Table A2 contains the differences in percentages between the calculated values with inclusion or not of the kinematic factors. As seen, there are small deviations associated with this approximation, the largest ones being at $^{48}$Ca nucleus.

References

[1] Colladay D and Kostelecký V A 1997 Phys. Rev. D 55 6760
[2] Colladay D and Kostelecký V A 1998 Phys. Rev. D 58 116002
[3] Kostelecký V A 2004 Phys. Rev. D 69 105009
[4] Kostelecký V A and Russell N 2011 Rev. Mod. Phys. 83(1) 11–31
[5] Alan Kostelecký V and Mewes M 2004 Phys. Rev. D 69(1) 016005
[6] Adam T et al. (The OPERA collaboration) 2012 Journal of High Energy Physics 2012 93
[7] Díaz J S, Kostelecký V A and Lelmert R 2013 Phys. Rev. D 88(7) 071902
[8] Díaz J S 2014 Advances in High Energy Physics 2014 962410
[9] Díaz J S 2014 Phys. Rev. D 89(3) 036002
[10] Abe Y et al. (Double Chooz Collaboration) 2012 Phys. Rev. D 86(11) 112009
[11] Aguilar-Arevalo A A et al. (MiniBooNE Collaboration) 2013 Phys. Lett. B 718 1303–1308
[12] Abbasi R et al. (IceCube Collaboration) 2010 Phys. Rev. D 82(11) 112003
[13] Adamson P et al. (The MINOS Collaboration) 2012 Phys. Rev. D 85(3) 031101
[14] Abe K et al. (Super-Kamiokande Collaboration) 2015 Phys. Rev. D 91(5) 052003
[15] Albert J B et al. (EXO-200 Collaboration) 2016 Phys. Rev. D 93(7) 072001
[16] Pertoldi L 2017 Search for Lorentz and CPT symmetries violation in double-beta decay using data from the GERDA experiment Ph.D. thesis Universita degli Studi di Padova
[17] Arnold R et al. 2019 The European Physical Journal C 79 440
[18] Brofferio C, Cremonesi O and Dell’Oro S 2019 Frontiers in Physics 7 86
[19] Nutini I 2019 The CUORE experiment: detector optimization and modelling and CPT conservation limit Ph.D. thesis INFN - Gran Sasso Science Institute
[20] Azzolini O et al. 2019 Phys. Rev. D 100(9) 092002
Lorentz violation effects in $2\nu\beta\beta$ decay

[21] Primakoff H and Rosen S P 1959 Rep. Prog. Phys. 22 121–166
[22] Haxton W and Stephenson G 1984 Progress in Particle and Nuclear Physics 12 409 – 479
[23] Doi M, Kotani T and Takasugi E 1985 Progress of Theoretical Physics Supplement 83 1–175
[24] Suhonen J and Civitarese O 1998 Physics Reports 300 123 – 214
[25] Stoica S and Mirea M 2013 Phys. Rev. C 88(3) 037303
[26] Mirea M, Pahomi T and Stoica S 2015 Romanian Reports in Physics 67(3) 872–889
[27] Olive K A et al. 2014 Chinese Physics C 38 090001
[28] Kostelecký V A and Tasson J D 2009 Phys. Rev. Lett. 102(1) 010402
[29] Vergados J D, Ejiri H and Šimkovic F 2012 Rep. Prog. Phys. 75 106301
[30] Konopinski E J 1966 The theory of beta radioactivity International series of monographs on physics (Oxford: Clarendon Press)
[31] Haxton W C, Stephenson G J and Strottman D 1982 Phys. Rev. D 25(9) 2360–2369
[32] Rose M E 1961 Relativistic electron theory (New York: John Wiley and Sons)
[33] Kotila J and Iachello F 2012 Phys. Rev. C 85(3) 034316
[34] Esposito S 2002 American Journal of Physics 70 852–856
[35] Salvat F and Mayol R 1991 Computer Physics Communications 62 65–79
[36] Salvat F, Fernández-Varea J and Williamson W 1995 Computer Physics Communications 90 151–168
[37] Salvat F, Fernández-Varea J and Williamson W J 2016 RADIAL mannal
[38] Štefánik D c v, Dvornický R, Šimkovic F and Vogel P 2015 Phys. Rev. C 92(5) 055502
[39] Šimkovic F, Dvornický R, Štefánik D c v and Faessler A 2018 Phys. Rev. C 97(3) 034315
[40] Redshaw M, Bollen G, Brodeur M, Bustabad S, Lincoln D L, Novario S J, Ringle R and Schwarz S 2012 Phys. Rev. C 86(4) 041306
[41] Bustabad S, Bollen G, Brodeur M, Lincoln D L, Novario S J, Redshaw M, Ringle R, Schwarz S and Valverde A A 2013 Phys. Rev. C 88(2) 022501
[42] Kwiatkowski A A et al. 2014 Phys. Rev. C 89(4) 045502
[43] Hykawy J G, Nxumalo J N, Unger P P, Lander C A, Barber R C, Sharma K S, Peters R D and Duckworth H E 1991 Phys. Rev. Lett. 67(13) 1708–1711
[44] Hykawy J, Nxumalo J, Unger P, Lander C, Peters R, Barber R, Sharma K and Duckworth H 1993 Nuclear Instruments and Methods in Physics Research Section A 329 423–432
[45] Douysset G, Fritioff T, Carlberg C, Bergström I and Björkhage M 2001 Phys. Rev. Lett. 86(19) 4259–4262
[46] Rahaman S et al. 2008 Physics Letters B 662 111–116
[47] Mount B J, Redshaw M and Myers E G 2010 Phys. Rev. C 81(3) 032501
[48] Lincoln D L, Holt J D, Bollen G, Brodeur M, Bustabad S, Engel J, Novario S J, Redshaw M, Ringle R and Schwarz S 2013 Phys. Rev. Lett. 110(1) 012501
[49] Nelson M A 1995 Ph.D. thesis University of California
[50] Fink D et al. 2012 Phys. Rev. Lett. 108(6) 062502
[51] Rahaman S, Elomaa V V, Eronen T, Hakala J, Jokinen A, Kankainen A, Rissanen J, Suhonen J, Weber C and yst J 2011 Physics Letters B 703 412–416
[52] Scielzo N D et al. 2009 Phys. Rev. C 80(2) 025501
[53] Redshaw M, Mount B J, Myers E G and Avignone F T 2009 Phys. Rev. Lett. 102(21) 212502
[54] Avignone F T, King G S and Zdesenko Y G 2005 New Journal of Physics 7 6–6
[55] Herlert A 2006 International Journal of Mass Spectrometry 251 131 – 137
[56] Redshaw M, Wingfield E, McDaniel J and Myers E G 2007 Phys. Rev. Lett. 98(5) 053003
[57] McCowan P M and Barber R C 2010 Phys. Rev. C 82(2) 024603