Simple Robust Estimating Method for Generalized Linear Models and its Application to Propensity Score Estimation

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Abstract

A generalized linear model is one of the most well-known model families in statistics. Of course, we should specify the correct model structure to estimate the unbiased result and make a valid inference, however, we would like to consider another rational approach in this paper to make a valid inference. The proposed method is 1) preparing some candidate models, and 2) construct an estimating equation including the candidate models at once (do not have to select just one) to estimate an interested parameter. If the correct model was included in the candidate models, the parameter estimator would have the consistency. By using the idea, we will consider a robust estimating method to estimate a “valid” parameter estimator without any model/variable selection methods. As an application example, an estimation of a propensity score and a generalized propensity score will be considered.

Keywords: Generalized linear model, Propensity score, Robust estimation
1 Introduction

A generalized linear model (GLM, c.f. McCullagh and Nelder, 2019) is one of the most well-known model families in statistics. In fact, a GLM is frequently used in statistical analyses, and the number of citations for McCullagh and Nelder, (2019), a masterpiece of GLM, is 41,304 (2022-02-01, the result of Google Scholar). A GLM can provide more flexible modeling than an ordinary linear model: a noise of an outcome has a distribution in an exponential family; not limited to the normal distribution. The flexible modeling can be achieved by a link function deciding a relationship between an expectation of an outcome and its model structure. Of course, we should specify the correct model structure to estimate the unbiased result and make a valid inference. One of the strategies to achieve them is to use some model/variable selection methods such as an AIC or a BIC (see e.g. Claeskens and Hjort, 2003). These methods can construct a “valid” model in the sense of an expected risk or a selection of covariates, respectively.

Model/variable selection methods are commonly used and appropriate procedures, however, we would like to consider an another rational approach in this paper to make a valid inference. The proposed method is derived from a method called “multiply robust estimation” in causal inference and missing data (see Han, 2014 and Orihara and Hamada, 2019). In brief, we prepare some candidate models, and construct an estimating equation including the candidate models at once (do not have to select just one) to estimate an interested parameter. If the correct model was included in the candidate models, the parameter estimator would have the consistency. By using the idea, we will consider a robust estimating method to estimate a “valid” parameter estimator without any model/variable selection methods. Concretely, many working models are combined into a one integrated model, and the integrated model is used for a subsequent inference. Also, we consider the situation where the true model is not included in the candidate models. Under this situation, we confirm the condition where the estimator becomes a “valid” in the sense of the true Kullback-Leibler (KL) divergence.
As an application example, an estimation of a propensity score (Rosenbaum and Rubin, 1983) and a generalized propensity score (Hirano and Imbens, 2004 and Imai van Dyk, 2004) will be considered (we call them “propensity score” simply). Commonly, the former is assumed to be a logistic regression model and the latter is assumed to be a normal distribution model. A propensity score is defined as the individual treatment allocation probability (e.g. the probability of being a male and elderly smoker) or the individual missing probability (e.g. the probability of being a male and elderly subject who do not visit a hospital), and plays an important role in causal inference and missing data (cf. Rosenbaum and Rubin, 1983 and Little and Rubin, 2019). A point to note that an interested estimator might have some bias if a propensity score model was misspecified (Kang and Schafer, 2007). To overcome the problem, a variety of interesting methods have been proposed (Hainmueller, 2012, Han, 2014, Imai and Ratkovic, 2014, Zubizarreta, 2015, Fong et al., 2018, and Orihara and Hamada, 2019). However, to the best of my knowledge, the properties of the methods are derived for the estimator of the interested parameter, not for the estimator of a propensity score itself. Therefore, the properties of other estimating methods using a propensity score estimator cannot be mentioned. In this sense, our propose method allows us to directly confirm the properties of a propensity score estimator. Also, our proposed method may overcome the problem of a variable selection of a propensity score model (Brookhart et al., 2006 and Austin et al., 2007). For instance, a researcher claims that some covariates have to be included in a propensity score model, whereas, another researcher claims that the covariates do not have to be included. Commonly, in this situation, one of the researchers has to compromise, or the researchers use some model/variable selection methods. Meanwhile, our proposed method provides a simple solution: the both models can be included in a one integrated model.

The remainder of the paper proceeds as follows. In section 2, we propose a new robust GLM estimator and confirm their properties with and without the condition where the true model is included in the candidate models. Regularity conditions are found in the appendix.
2 Constructing a robust estimating method

Let \( n \) be the sample size, and assume that \( i = 1, 2, \ldots, n \) are i.i.d. samples. \( \mathbf{X} \in \mathbb{R}^p \) denotes a vector of covariates. We assume that an outcome has a distribution in an exponential family (McCullagh and Nelder, 2019):

\[
 f(y; \theta, \phi) = \exp \left\{ \frac{y\theta - b(\theta)}{a(\phi)} + c(y, \phi) \right\}, \tag{2.1}
\]

where \((\theta, \phi)\) are parameters; in particular, we are interested in the parameter \( \theta \). Note that a normal distribution and a Bernoulli distribution are included in an exponential family. For a normal distribution, each function and parameter in (2.1) become

\[
 \theta = \mu \ (\text{mean}), \quad a(\phi) = \sigma^2 \ (\text{variance}), \quad b(\theta) = \frac{\theta^2}{2}, \quad c(y, \phi) = -\frac{y^2}{2\sigma^2} - \frac{1}{2} \log(2\pi\sigma^2).
\]

For a Bernoulli distribution, each function and parameter in (2.1) become

\[
 \theta = \log \left( \frac{p}{1-p} \right) \ (p \ is \ a \ binomial \ probability), \quad a(\phi) = 1, \quad b(\theta) = \log(1 + e^\theta), \quad c(y, \phi) = 0.
\]

To construct a GLM, we consider a relationship between an expectation of an outcome and its model structure \( \varphi(\mathbf{x}; \beta) \). Specifically, \( \varphi(\mathbf{x}; \beta) = g(p) \), where a monotonic function \( g \) is called “link function”. For a Bernoulli distribution, when we select a link function as \( g = \log \ (\text{logit link}) \) and assume a linear relationship between \( \mathbf{x} \) and \( \beta \) (\( \varphi(\mathbf{x}; \beta) = \mathbf{x}^\top \beta \)),

\[
 p = \frac{\exp \{\mathbf{x}^\top \beta\}}{1 + \exp \{\mathbf{x}^\top \beta\}} \ (\text{logistic regression model}).
\]

Whereas, when we select a link function as \( g = \Phi^{-1} \ (a \ normal \ distribution \ function; \ probit \ link) \) and assume a linear relationship between \( \mathbf{x} \) and \( \beta \),

\[
 p = \Phi \ (\mathbf{x}^\top \beta) \ (\text{probit model}).
\]
Note that $\theta$ becomes a function of $x$ and $\beta$ clearly from the above example: $\theta = \theta(x; \beta)$. Hereinafter, the model (2.1) will be considered for a while.

From here, we introduce a proposed procedure that many working models are combined into a one integrated model. At first, we consider working models; $k (k = 1, 2, \ldots, K)$ denotes each model:

$$f(y|x; \beta_k) = \exp \left\{ \frac{y\theta_k(x; \beta_k) - b(\theta_k(x; \beta_k))}{a(\phi)} + c(y, \phi) \right\}.$$  

Note that we assume that $\phi$ is known and needs not to be estimated, and the functions $a, b,$ and $c$ are common to each model. For a normal distribution, these assumptions imply that variance parameters are known and common to each model. To estimate parameter $\beta_k$, we consider a KL divergence $D_k(\beta_k)$:

$$D_k(\beta_k) = E[\log g(Y|X)] - E[\log f(Y|X; \beta_k)], \quad (2.2)$$

where $g(y|x)$ is the unknown true probability distribution. Whereas we would like to estimate the parameters so that $D_k(\beta_k)$ is small, we cannot handle (2.2) directly. Therefore, we consider an estimator of the second term of (2.2) to estimate parameter $\beta_k$:

$$\ell(\beta_k) = \frac{1}{n} \sum_{i=1}^{n} \log f(y_i|x_i; \beta_k) \propto \frac{1}{n} \sum_{i=1}^{n} \left[ y_i \theta_k(x_i; \beta_k) - b(\theta_k(x_i; \beta_k)) \right],$$

where $\ell$ is a log-likelihood function. Therefore, a score function $S(\beta_k)$ becomes

$$S(\beta_k) = \frac{\partial}{\partial \beta_k} \ell(\beta_k) = \frac{1}{n} \sum_{i=1}^{n} \left[ y_i - \hat{b}(\theta_k(x_i; \beta_k)) \right] \frac{\partial}{\partial \beta_k} \theta_k(x_i; \beta_k),$$

where $\hat{b}$ is the first differentiable of $b$. The solution of $S(\beta_k) = 0$ becomes a maximum likelihood estimator (MLE) $\hat{\beta}_k$. Next, by using the MLEs $\hat{\beta}_k$, we consider an integrated
model:

\[
f(y|\mathbf{x}; \gamma, \hat{\beta}) = \exp \left\{ \frac{y \sum_{k=1}^{K} \gamma_k \theta_k(\mathbf{x}; \hat{\beta}_k) - b \left( \sum_{k=1}^{K} \gamma_k \theta_k(\mathbf{x}; \hat{\beta}_k) \right)}{a(\phi)} + c(y, \phi) \right\},
\]

where \(\sum_{k=1}^{K} \gamma_k = 1\). The same discussion as above, a KL divergence, a log-likelihood function, and a score function become

\[
D(\gamma, \beta) = E \left[ \log g(Y|\mathbf{X}) \right] - E \left[ \log f(Y|\mathbf{X}; \gamma, \beta) \right], \quad (2.3)
\]

\[
\ell(\gamma, \hat{\beta}) \propto \frac{1}{n} \sum_{i=1}^{n} \left[ y_i \sum_{k=1}^{K} \gamma_k \theta_k(\mathbf{x}_i; \hat{\beta}_k) - b \left( \sum_{k=1}^{K} \gamma_k \theta_k(\mathbf{x}_i; \hat{\beta}_k) \right) \right],
\]

\[
S(\gamma, \hat{\beta}) = \frac{\partial}{\partial \gamma} \ell(\gamma, \hat{\beta}) = \frac{1}{n} \sum_{i=1}^{n} \left[ y_i - b \left( \sum_{k'=1}^{K} \gamma_{k'} \theta_{k'}(\mathbf{x}_i; \hat{\beta}_{k'}) \right) \right] \left( \begin{array}{c}
\theta_1(\mathbf{x}_i; \hat{\beta}_1) \\
\vdots \\
\theta_K(\mathbf{x}_i; \hat{\beta}_K)
\end{array} \right), \quad (2.4)
\]

The solution of \(S(\gamma, \hat{\beta}) = 0\) becomes a MLE \(\hat{\gamma}\), and the integrated model \(f(y|\mathbf{x}; \hat{\gamma}, \hat{\beta})\) is used for a subsequent inference.

In the next step, we confirm properties of our proposed estimating procedure. At first, we consider the situation where the true model is included in the candidate models \((k = 1, 2, \ldots, K)\). Concretely, we assume that \(k = 1\) is the true model. Under this setting, the following theorem is proved:

**Theorem 1.**

Under the regularity conditions from C.1 to C.3,

\[
\hat{\gamma}_1 \xrightarrow{P} 1 \text{ and } \hat{\gamma}_k \xrightarrow{P} 0, \ k = 2, \ldots, K. \quad (2.5)
\]
Therefore, from the continuous mapping theorem,

\[
f(y|x; \hat{\gamma}, \hat{\beta}) = \exp \left\{ y \sum_{k=1}^{K} \hat{\gamma}_k \theta_k(x; \hat{\beta}_k) - b \left( \sum_{k=1}^{K} \hat{\gamma}_k \theta_k(x; \hat{\beta}_k) \right) \right\}
\]

\[
\rightarrow \exp \left\{ y \theta_1(x; \beta_1^0) - \frac{b(\beta_1^0)}{a(\phi)} + c(y, \phi) \right\}
\]

\[
= f(y|x; \beta_1^0),
\]

where the superscript “0” of parameters means the true value of parameters.

Proof. From C.1 and the continuous mapping theorem, (2.4) becomes

\[
S(\gamma, \hat{\beta}) = S(\gamma, \beta^*) + o_p(1).
\]

Also, from (2.4),

\[
\frac{\partial^2}{\partial \gamma \partial \beta^*} \ell(\gamma, \beta^*) = \frac{\partial}{\partial \gamma} S^T(\gamma, \beta^*) = -\frac{1}{n} \sum_{i=1}^{n} \hat{b} \left( \sum_{k' = 1}^{K} \gamma_{k'} \theta_{k'}(x_i; \beta_{k'}^*) \right)^{\otimes 2}
\]

\[
\left( \theta_1(x_i; \beta_1^0) \right)
\]

\[
\left( \theta_2(x_i; \beta_2^*) \right)
\]

\[
\vdots
\]

\[
\theta_K(x_i; \beta_K^*)
\]

Therefore, from C.2, (2.6) becomes the negative definite:

\[
\frac{\partial^2}{\partial \gamma \partial \beta^*} \ell(\gamma, \beta^*) < O.
\]
From (2.7), (2.4) has the unique solution \( \dot{\gamma} \xrightarrow{P} \gamma^0 \) satisfying
\[
E \left[ Y - \hat{b} \left( \sum_{k'=1}^{K} \gamma_{k'}^0 \theta_{k'}(X; \beta_{k'}) \right) \right] \left( \begin{array}{c} \theta_1(X; \beta_1^0) \\ \theta_2(X; \beta_2^0) \\ \vdots \\ \theta_K(X; \beta_K^0) \end{array} \right) = 0 \tag{2.8}
\]

Next, the concrete values of \( \gamma^0 \) is confirmed. The expectation regarding \( Y | x \) of (2.8) is
\[
E \left[ Y - \hat{b} \left( \sum_{k'=1}^{K} \gamma_{k'}^0 \theta_{k'}(x; \beta_{k'}) \right) \right] = 0 \tag{2.9}
\]

From the property of GLM (see McCullagh and Nelder, 2019),
\[
E [Y | x] = \hat{b}(\theta_1(x_i; \beta_1^0)).
\]

Therefore, (2.9) becomes
\[
\hat{b}(\theta_1(x_i; \beta_1^0)) - \hat{b} \left( \sum_{k'=1}^{K} \gamma_{k'}^0 \theta_{k'}(x; \beta_{k'}) \right).
\]

From the above, (2.8) becomes
\[
E \left[ \hat{b}(\theta_1(x_i; \beta_1^0)) - \hat{b} \left( \sum_{k'=1}^{K} \gamma_{k'}^0 \theta_{k'}(x; \beta_{k'}) \right) \right] \left( \begin{array}{c} \theta_1(X; \beta_1^0) \\ \theta_2(X; \beta_2^0) \\ \vdots \\ \theta_K(X; \beta_K^0) \end{array} \right) = 0. \tag{2.10}
\]

From C.3, there exists the unique value of \( \gamma \) such that
\[
\gamma_1^0 = 1 \quad and \quad \gamma_{k}^0 = 0, \; k = 2, \ldots, K.
\]
Therefore, (2.5) is obtained.

From the Theorem 1, the integrated model is consistent with the true model when the true model is included in the candidate models, and \( n \to \infty \). Next, we consider the situation where the true model is not included in the candidate models; the assumption of Theorem 1 is not hold. Even in this situation, the following theorem is hold:

**Theorem 2.**

*Under the regularity conditions from C.2, the following inequality is hold for \( \forall \gamma \):

\[
D(\gamma, \beta^*) \leq \sum_{k=1}^{K} \gamma_k D_k(\beta_k^*)
\]  

(2.11)

*Proof. It is sufficient to show that

\[
-\mathbb{E}[\log f(Y|X; \gamma, \beta^*)] + \sum_{k=1}^{K} \gamma_k \mathbb{E}[\log f(Y|X; \beta_k^*)] \leq 0,
\]

i.e.,

\[
-\int \left( y \sum_{k=1}^{K} \gamma_k \theta_k(x; \beta_k^*) - b \left( \sum_{k=1}^{K} \gamma_k \theta_k(x; \beta_k^*) \right) \right) f(y, x) dy dx \\
\quad + \sum_{k=1}^{K} \gamma_k \left[ \int (y \theta_k(x; \beta_k^*) - b(\theta_k(x; \beta_k^*))) f(y, x) dy dx \right]
\]

\[
= \int b \left( \sum_{k=1}^{K} \gamma_k \theta_k(x; \beta_k^*) \right) f(x) dx - \sum_{k=1}^{K} \gamma_k \int b(\theta_k(x; \beta_k^*)) f(x) dx
\]

\[
= \int \left[ b \left( \sum_{k=1}^{K} \gamma_k \theta_k(x; \beta_k^*) \right) - \sum_{k=1}^{K} \gamma_k \int b(\theta_k(x; \beta_k^*)) \right] f(x) dx \leq 0. 
\]  

(2.12)

From C.2, by using the property of convex functions,

\[
b \left( \sum_{k=1}^{K} \gamma_k \theta_k \right) \leq \sum_{k=1}^{K} \gamma_k b(\theta_k)
\]

for all \( \theta_k \) and \( \gamma \). Therefore, (2.12) is hold; (2.11) is obtained. \( \square \)
The Theorem 2 means that the integrated model becomes better than each candidate model in the sense of the true KL divergence even if the true model is not included in the candidate models. In this sense, the integrated model is better option than any model selection properties when we are not interested in the selection of the “valid” model.
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A Regularity Conditions

C.1 For all $k \in \{1, \ldots, K\}$, $\hat{\beta} \overset{P}{\to} \beta^*$, where $\beta^*$ is the “true” parameter value (see White, 1982).

C.2 For all $\theta \in \mathbb{R}$, $\ddot{b}(\theta) > 0$, where $\ddot{b}$ is the second differentiables of $b$.

C.3 $\exists x \in \mathbb{R}^p \text{ s.t. } \theta_k(x; \beta_k^*) \neq \theta_{k'}(x; \beta_{k'}^*) \text{ for all combinations of } k \neq k'$. 