Integrability of irrotational silent cosmological models

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Abstract.
We revisit the issue of integrability conditions for the irrotational silent cosmological models. We formulate the problem both in 1+3 covariant and 1+3 orthonormal frame notation, and show there exists a series of constraint equations that need to be satisfied. These conditions hold identically for FLRW–linearised silent models, but not in the general exact non–linear case. Thus there is a linearisation instability, and it is highly unlikely that there is a large class of silent models. We conjecture that there are no spatially inhomogeneous solutions with Weyl curvature of Petrov type I, and indicate further issues that await clarification.

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1. Introduction

The idea behind the silent cosmological models, introduced and discussed by Matarrese et al (see Refs. [24], [4] and [5]), is the following. Starting from any consistent initial configuration for barotropic perfect fluid spacetime geometries ($\mathcal{M}, g, u$), there exist two physically different phenomena that convey information between adjacent worldlines within the preferred timelike reference congruence $u$, which is identified with the average 4-velocity of the fluid matter source. These are sound waves and gravitational waves. Mathematically, in the $1+3$ covariant representation of such models (see Refs. [6], [9], [10] and [11]), these dynamical interactions are represented by the spatial derivative source terms in the evolution equations. A careful look at the perfect fluid time derivative equations, obtained from the Ricci identities for $u$ and the Bianchi identities, reveals that these terms are comprised of either the spatial derivatives of the fluid acceleration, or the spatial rotation terms (often also called the `curls') of both the electric and magnetic parts of the Weyl curvature, $E_{\mu\nu}$ and $H_{\mu\nu}$. Given the barotropic equation of state assumption, the fluid acceleration, and consequently the spatial 3–gradient of the fluid pressure, are linked to the spatial 3–gradient of the fluid’s total energy density. Pressure 3–gradients and their spatial variations generate propagating sound waves within the fluid, while non–zero values of the spatial rotation of either of the Weyl curvature variables generically induce temporal changes in the other, usually interpreted as propagating gravitational waves.

Both from a mathematical and a physical point of view, it is of interest to investigate the case in which these spatial derivative terms vanish, so that the resultant $1+3$ covariant evolution equations reduce to a coupled set of first–order ordinary differential equations, in terms of the comoving time derivative along the fluid flow lines. In this case, provided the constraint equations (involving only orthogonal spatial derivatives) are satisfied on an initial surface and remain satisfied, the subsequent evolution along individual fluid worldlines within $u$ is decoupled, i.e., the covariant time derivatives decouple from the orthogonally projected spatial ones. This technically describes what is called the silent assumption for cosmological models; it expresses the physical idea of an absence of any form of waves or of gravitational induction (hence, any form of causal communication) propagating between the worldlines of neighbouring fluid elements. Thus, there is no exchange of new information (i.e., information not already coded in the initial data) between the fluid elements.

To realise it, Matarrese et al assumed the fluid matter source to be irrotational dust, which generates a spacetime geometry of purely electric Weyl curvature, i.e., the
magnetic part of the Weyl curvature vanishes†:

\[ \omega^\mu = 0, \quad p = 0 \Rightarrow \dot{u}^\mu = 0, \quad H_{\mu\nu} = 0. \] (1)

Well-known exact solutions of the Einstein field equations fall into this category. Spatially inhomogeneous representatives are the dust spacetime geometries given by Szekeres [26] (discussed in a nice geometrical formulation by Goode and Wainwright [18]), as well as Ellis’ dust subclass of LRS class II spacetime geometries, which includes the Lemaitre–Tolman–Bondi model and the orthogonally spatially homogeneous (OSH) Kantowski–Sachs model (see Refs. [8] and [10]). In both examples, the Weyl curvature tensor is of algebraic Petrov type D (see, e.g., Ref. [19]). A Petrov type I example of a silent model is provided by the OSH Bianchi Type–I dust solutions (see, e.g., Ref. [14]).

The question is how many other such silent solutions there are. This depends on the consistency of the constraint equations with the time evolution equations. A previous paper [20] claiming that these equations are generically consistent with each other is, regrettably, wrong, as indicated in [21] and demonstrated by Bonilla et al [3]. However, the latter paper did not determine the set of consistent solutions. We mount a systematic attack on that question here.

In the following, we first give the dynamical equations defining irrotational dust silent spacetimes in 1 + 3 covariant form; this is done in section 2. We derive a covariant integrability condition, and show that it is satisfied at linear level, i.e., for those silent models which are linearised inhomogeneous perturbations of FLRW dust universes. However the condition is non–trivial in the non–linear case. Thus there is a linearisation instability in the silent models. In section 3 the silent configurations are then formulated in 1+3 orthonormal frame (ONF) terminology. By the assumption of vanishing magnetic Weyl curvature a new constraint is generated. The consistency of this constraint with the remaining dynamical equations is investigated in section 4. We show that a sequence of non–trivial differential and algebraic conditions results from repeated propagation of the constraints along the integral curves of \( u \). These conditions are not identically satisfied in general irrotational silent spacetimes, throwing into question the further physical analysis of silent models, and we conjecture, in particular, that there are no consistent spatially inhomogeneous solutions with a Weyl curvature tensor of algebraic Petrov type I. Finally, the results obtained are discussed in section 5.

† Here and in the rest of this paper, we use the standard 1+3 covariant formalism, thoroughly reviewed in Ref. [9] (see also Refs. [10], [11], [12], [24], [4] and [5]). Sign and index conventions are employed according to Ref. [17].
2. 1 + 3 covariant formulation

The silent conditions Eq. (1) describe irrotational and pressure–free fluid matter sources inducing purely electric Weyl curvature. The 1 + 3 covariant dynamical equations then take the form

**Time derivative equations**

\[
\dot{\Theta} = -\frac{1}{3} \Theta^2 - 2 \sigma^2 - \frac{1}{2} \mu \quad (2)
\]

\[
h_{\rho}^{\mu} h_{\sigma}^{\nu} (\dot{\sigma}^{\rho\sigma}) = -\frac{2}{3} \Theta \sigma^{\mu\nu} - \sigma^{\mu}_{\rho} \sigma^{\nu\rho} - E^{\mu\nu} + \frac{2}{3} \sigma^2 h^{\mu\nu} \quad (3)
\]

\[
h_{\rho}^{\mu} h_{\sigma}^{\nu} (\dot{E}^{\rho\sigma}) = -\frac{1}{2} \mu \sigma^{\mu\nu} - \Theta E^{\mu\nu} + 3 \sigma^{(\mu} E^{\nu)\rho} - \sigma_{\rho}^{\sigma} E^{\sigma}_{\rho} h^{\mu\nu} \quad (4)
\]

\[
\dot{\mu} = -\Theta \mu \quad (5)
\]

**Constraint equations**

\[
0 = h_{\rho}^{\mu} h_{\sigma}^{\nu} (\nabla_{\nu} \sigma^{\rho\sigma}) - \frac{2}{3} Z^{\mu} := (C_{1})^{\mu} \quad (6)
\]

\[
0 = h_{\rho}^{\mu} h_{\sigma}^{\nu} (\nabla_{\nu} E^{\rho\sigma}) - \frac{1}{3} X^{\mu} := (C_{2})^{\mu} \quad (7)
\]

\[
0 = \eta^{\mu\nu\rho\sigma} \sigma_{\nu\tau} E^{\tau}_{\rho} u_{\sigma} := (C_{3})^{\mu} \quad (8)
\]

\[
0 = K_{\mu\nu} := h_{\rho}^{\mu} h_{\nu}^{\sigma} \eta_{\rho\tau\kappa\lambda} (\nabla_{\tau} \sigma_{\kappa\lambda}) u^{\lambda} = [\nabla \times \sigma]_{\mu\nu} := (C_{4})_{\mu\nu} \quad (9)
\]

\[
0 = I_{\mu\nu} := h_{\rho}^{\mu} h_{\nu}^{\sigma} \eta_{\rho\tau\kappa\lambda} (\nabla_{\tau} E_{\kappa\lambda}) u^{\lambda} = [\nabla \times E]_{\mu\nu} := (C_{5})_{\mu\nu} \quad (10)
\]

**Gauß equations**

\[
3S_{\mu\nu} = E_{\mu\nu} - \frac{1}{3} \Theta \sigma_{\mu\nu} + \sigma_{\mu\rho} \sigma_{\nu}^{\rho} - \frac{2}{3} \sigma^2 h_{\mu\nu} \quad (11)
\]

\[
3R = 2 \mu - \frac{2}{3} \Theta^2 + 2 \sigma^2 \quad (12)
\]

The constraints in this setting have the following implications. The spatial divergence of the fluid rate of shear \(\sigma_{\mu\nu}\) is related to the spatial 3–gradient of the fluid rate of expansion, \(Z_{\mu} := h_{\nu}^{\rho} \nabla_{\nu} \Theta\), Eq. (6), and, analogously, the spatial divergence of the electric part of the Weyl curvature \(E_{\mu\nu}\) to the spatial 3–gradient of the fluid’s total energy density, \(X_{\mu} := h_{\nu}^{\rho} \nabla_{\nu} \mu\), Eq. (7). Both the fluid rate of shear tensor and the electric part of the Weyl curvature tensor share a common eigenframe, a property expressed by Eq. (8). This result was originally obtained by Barnes and Rowlingson [1]. Additionally, the constraints (9) and (10) express the condition that in silent configurations as defined by Eq. (1), these tensors need to have zero spatial rotation.

Maartens [21] has shown that the constraints for generic irrotational dust spacetime geometries (with non–zero \(H_{\mu\nu}\)) are consistent with each other and preserved along the integral curves of \(u\), if they are satisfied at an initial instant. This result has been extended by van Elst [15] to generic barotropic perfect fluid spacetimes (with non–zero \(H_{\mu\nu}\) and \(E_{\mu\nu}\)). Equations (6) – (9) are just reduced forms of the constraints underlying general barotropic perfect fluid spacetime geometries. The character of Eq.
(10), however, is slightly different. It is a new constraint arising as a consequence of imposing the silent conditions, Eq. (10). Namely, the vanishing of the magnetic part of the Weyl curvature, $H_{\mu\nu}$, results in the reduction of the general evolution equation for $H_{\mu\nu}$ to a constraint, Eq. (10). This is an example of the conversion process where the imposed assumption that a dynamical variable remains zero throughout the time evolution results in a further integrability condition, additional to those generically obtained from the general 1 + 3 covariant dynamical equations. Consistency of Eq. (10) with the remaining set of equations demands that

$$0 = h^\mu_\mu h^\nu_\nu (\dot{\epsilon}^{\rho\sigma}) = h^\mu_\mu h^\nu_\nu (\dot{C}_5)^{\rho\sigma}$$

holds throughout. Using the methods and identities developed in Ref. [21], this is equivalent to

$$0 = - \frac{4}{3} \Theta (C_5)^{\mu\nu} - \frac{3}{2} E^{(\mu}_\rho h^{\nu)\rho\tau} (C_1)_\sigma u_\tau - \frac{3}{2} \sigma^{(\mu}_\rho h^{\nu)\rho\tau} (C_2)_\sigma u_\tau - \frac{1}{2} \mu (C_4)^{\mu\nu} + \frac{3}{2} B^{\mu\nu} ,$$

where

$$B^{\mu\nu} := h^{(\mu}_\rho h^{\nu)}_\sigma \eta^{\rho\tau\kappa\lambda} u_\kappa \sigma^{\xi}_{\tau} (\nabla_\xi E_{\kappa\sigma}) - 3 h^{(\mu}_\rho h^{\nu)}_\sigma \eta^{\rho\tau\kappa\lambda} u_\kappa \nabla_\tau \left[ \sigma^{\xi}_{(\kappa} E_{\sigma)\xi} \right]$$

$$- \left[ \frac{1}{2} \sigma^{(\mu}_\rho X_\sigma + E^{(\mu}_\rho Z_\sigma} \right] \eta^{\nu)\rho\tau\kappa\lambda} u_\lambda .$$

Hence, using Eqs. (6) - (10),

$$0 = B^{\mu\nu} .$$

The 1 + 3 covariant condition (10) may be simplified and clarified by rewriting Eq. (15) in terms of the irreducible parts of the spatial covariant derivatives of $\sigma_{\mu\nu}$ and $E_{\mu\nu}$. This is achieved via the complete covariant decomposition of the spatial derivative of any rank 2 symmetric tracefree tensor field $A_{\mu\nu}$ orthogonal to $u$, presented in Ref. [22]:

$$h^\mu_\mu h^\nu_\nu h^\kappa_\rho (\nabla_\sigma A_{\tau\kappa}) = \hat{A}_{\mu\nu\rho} + \frac{3}{5} h_{\mu<\nu} h^{\rho>_\nu} h^{\tau}_\kappa (\nabla_\tau A^{\kappa}_{\sigma})$$

$$- \frac{2}{3} [\nabla \times A]^{(\mu}_\rho \eta^{\nu)\mu\sigma\tau} u_\tau ,$$

where angle brackets enclosing indices denote the spatially projected symmetric tracefree part, and $\hat{A}_{\mu\nu\rho} = \hat{A}_{<\mu\nu\rho>}$ is the divergence–free and irrotational part of the spatial derivative of $A_{\mu\nu}$, known as the ‘distortion’ of $A_{\mu\nu}$. By Eqs. (9) and (10), the spatial rotation terms in Eq. (17) vanish for $A_{\mu\nu} = \sigma_{\mu\nu}$ and $A_{\mu\nu} = E_{\mu\nu}$ in irrotational silent models. The spatial divergence terms are determined by Eqs. (6) and (7). Then Eq. (15) may be reduced to a form in which all the terms are spatial divergences or distortions of the fluid rate of shear and the electric part of the Weyl curvature:

$$B^{\mu\nu} = \left[ \hat{\sigma}^{(\mu}_\rho E_{\sigma\kappa} + \frac{2}{3} \hat{E}^{(\mu}_\rho \sigma_{\kappa} + \frac{8}{45} \sigma^{(\mu}_\rho X_\sigma + \frac{1}{15} E^{(\mu}_\rho Z_\sigma} \right] \eta^{\nu)\rho\sigma\tau} u_\tau .$$

The form of Eq. (18) makes clear the separate roles of the spatial divergence terms and the totally symmetric tracefree parts of the derivatives.
Lesame et al. [20] recently reported that condition (16) was identically satisfied. However, this is incorrect [3], and a different conclusion is obtained in what follows. On the other hand, considering covariant and gauge–invariant linearised perturbations of FLRW spacetimes in the sense of Ellis and Bruni [12], Eq. (16) is identically satisfied, as all terms occurring in Eq. (15) are of second order. Thus, in FLRW–linearised silent models the constraints evolve consistently. As is seen from our subsequent analysis, Eq. (16) is not identically satisfied in the exact non–linear case. Consequently this leads to a linearisation instability in irrotational silent cosmological models. In other words, there would be consistent FLRW–linearised solutions, which do not correspond to consistent exact solutions. A constraint similar to Eq. (10) arises in the context of the work by Barnes and Rowlingson [1], who, among the conditions (3), allowed the fluid pressure to be non–zero instead (in which case, of course, a configuration no longer can be called silent). However, in their work the integrability of that constraint was never established.

This concludes the 1 + 3 covariant discussion of the irrotational silent models. To pursue a detailed constraint analysis, for which purpose we make use of algebraic computing facilities, it turns out to be of (relative) computational ease to describe the problem within the related 1 + 3 ONF framework instead. After reformulating the set of dynamical equations (2) – (10) in terms of 1 + 3 ONF variables, the consistency of Eq. (10) with the remaining set is further investigated.

3. 1 + 3 ONF formulation

The ONF approach employs a set of linearly independent 1–form fields \( \{ \omega^a \} \) defined at each point of the spacetime manifold \((\mathcal{M}, g)\) such that the line element can be locally expressed as

\[
\mathrm{d}s^2 = \eta_{ab} \omega^a \omega^b ,
\]

where \( \eta_{ab} = \text{diag} \left[ -1, 1, 1, 1 \right], \left( \sqrt{-\eta} = 1 \right) \), i.e., a constant Minkowskian frame metric. The vector fields \( \{ e_a \} \) dual to the 1–forms \( \{ \omega^a \} \) satisfy the relation

\[
\langle \omega^a, e_b \rangle = \delta^a_b .
\]

We choose the timelike frame vector \( e_0 = u \). This setup allows for a 1 + 3 split of the commutator relations as well as of the curvature variables and their field equations (see Ref. [8]), part of which are constituted by the Bianchi identities.

The commutation functions \( \gamma^a_{bc} \) are defined by

\[
[e_a, e_b] = \gamma^c_{ab} e_c ,
\]

where the frame vectors act as differential operators, \( e_a(T) \), on any geometrical objects \( T \). The purely spatial components, \( \gamma^a_{\beta \gamma} \), decompose into an object \( a_\alpha \) and a symmetric
object \( n_{\alpha\beta} \) as follows

\[
\gamma'^{\alpha\beta} := 2a_{[\beta} \delta^{\alpha]} + \epsilon_{\beta\gamma\delta} n^{\delta\alpha},
\]

(22)

where \( \epsilon_{\alpha\beta\gamma} \) is the totally antisymmetric 3-D permutation tensor with \( \epsilon_{123} = 1 = \epsilon^{123} \).

The commutation functions with one or more indices equal to zero can be expressed in terms of the kinematical quantities and the quantity

\[
\Omega^a := \frac{1}{2} \eta^{abcd} e_b \cdot e_c u_d,
\]

(23)

(\text{where} \( \hat{e}_a := \nabla_u e_a \)), which can be interpreted as the local angular velocity of the (to be chosen) spatial frame \( \{ e_\alpha \} \).

Equation (8) expresses the fact that one can choose a common eigenframe for \( \sigma_{\alpha\beta} \) and \( E_{\alpha\beta} \), i.e., it is possible to diagonalise both tensors simultaneously. As was shown by Barnes and Rowlingson \([1]\), it follows from the on-diagonal components of both the \( H \)-constraint, Eq. (9), and the evolution equation for the magnetic part, Eq. (10), that

\[
0 = n_{11} = n_{22} = n_{33}.
\]

(24)

They also showed that the eigenframe \( \{ e_\alpha \} \) of \( \sigma_{\alpha\beta} \) and \( E_{\alpha\beta} \) is Fermi-transported along \( u \),

\[
\Omega^a = 0,
\]

(25)

a condition obtained from the vanishing off-diagonal components of the evolution equations of both the fluid rate of shear, Eq. (3), and the electric part, Eq. (4).

With these conditions and bearing in mind that \( \omega^\mu = 0 \), one has that \( \{ e_\alpha \} \) is spanned by four individually hypersurface orthogonal basis fields, which implies that local coordinates can be found on \(( \mathcal{M}, g, u )\) with respect to which the metric tensor field \( g \) is diagonal. This result holds for Weyl curvature tensors of either algebraic Petrov type I or type D.

In the following analysis, we will employ traceless–adapted irreducible frame components for \( \sigma_{\alpha\beta} \) and \( E_{\alpha\beta} \), defined by (see, e.g., Ref. \([17]\))

\[
A_+ := -\frac{3}{2} A_{11} = \frac{3}{2} (A_{22} + A_{33}), \quad A_- := \frac{\sqrt{3}}{2} (A_{22} - A_{33}).
\]

(26)

This implies

\[
A^2 = \frac{1}{3} \left[ (A_+)^2 + (A_-)^2 \right].
\]

(27)

Given these specialisations the following set describing silent cosmological models according to conditions \([1]\) can be derived from the general \( 1 + 3 \) ONF dynamical equations \([17]\):
Note that this set implies \( \sigma_- = 0 \Leftrightarrow E_- = 0 \).

Remaining decoupled system of ordinary differential evolution equations:

\[
\begin{align*}
e_0(a_1) &= - \frac{1}{3} (\Theta + \sigma_+) a_1 - \frac{1}{\sqrt{3}} \sigma_- n_{23} \\
e_0(a_2) &= \frac{1}{6} (-2 \Theta + \sigma_+ + \sqrt{3} \sigma_-) a_2 - \frac{1}{2} \left( \sigma_+ - \frac{1}{\sqrt{3}} \sigma_- \right) n_{31} \\
e_0(a_3) &= \frac{1}{6} (-2 \Theta + \sigma_+ - \sqrt{3} \sigma_-) a_3 + \frac{1}{2} \left( \sigma_+ + \frac{1}{\sqrt{3}} \sigma_- \right) n_{12} \\
e_0(n_{23}) &= - \frac{1}{3} (\Theta + \sigma_+) n_{23} - \frac{1}{\sqrt{3}} \sigma_- a_1 \\
e_0(n_{31}) &= \frac{1}{6} (-2 \Theta + \sigma_+ + \sqrt{3} \sigma_-) n_{31} - \frac{1}{2} \left( \sigma_+ - \frac{1}{\sqrt{3}} \sigma_- \right) a_2 \\
e_0(n_{12}) &= \frac{1}{6} (-2 \Theta + \sigma_+ - \sqrt{3} \sigma_-) n_{12} + \frac{1}{2} \left( \sigma_+ + \frac{1}{\sqrt{3}} \sigma_- \right) a_3 .
\end{align*}
\]

Tracefree part and trace of 3–Ricci curvature of spacelike 3–surfaces orthogonal to \( u \) (Gauß equation):

\[
\begin{align*}
^*S_+ &= - \frac{1}{2} \left[ 2 e_1(a_1) - e_2(a_2) - e_3(a_3) - 4 (n_{23})^2 + 2 (n_{31})^2 + 2 (n_{12})^2 \\
&\quad - 3 (e_2 - 2 a_2) (n_{31}) + 3 (e_3 - 2 a_3) (n_{12}) \right] \\
&= E_+ - \frac{1}{4} (\Theta + \sigma_+) \sigma_+ + \frac{1}{8} (\sigma_-)^2 \\
^*S_- &= \sqrt{\frac{3}{2}} \left[ e_2(a_2) - e_3(a_3) - 2 (n_{31})^2 + 2 (n_{12})^2 + 2 (e_1 - 2 a_1) (n_{23}) \\
&\quad - (e_2 - 2 a_2) (n_{31}) - (e_3 - 2 a_3) (n_{12}) \right] \\
&= E_- - \frac{1}{3} (\Theta - 2 \sigma_+) \sigma_- .
\end{align*}
\]
respectively. Equations (58) – (60) follow from the Jacobi identities. By use of the covariant form of the constraints: Eqs. (52) – (54) correspond to Eq. (6), Eqs. (55) – (57) to Eq. (7), Eqs. (61) – (63) to Eq. (9), and Eqs. (64) – (66) to Eq. (10), respectively. Equations (58) – (60) follow from the Jacobi identities. By use of the constraint equations:

\[ 0 = (e_1 - 3a_1) (n_{12}) - (e_2 - 3a_2) (n_{23}) + e_1(a_2) + e_2(a_1) + 4n_{23}n_{31} \tag{48} \]

\[ 0 = (e_1 - 3a_1) (n_{23}) - (e_1 - 2a_1) (n_{12}) + e_3(a_1) + e_1(a_3) + 4n_{12}n_{23} \tag{49} \]

\[ 0 = (e_1 - 2a_1) (n_{31}) - (e_2 - 2a_2) (n_{23}) + e_1(a_2) + e_2(a_1) + 4n_{23}n_{31} \tag{50} \]

\[ \ast R = 2 \left[ (2e_1 - 3a_1) (a_1) + (2e_2 - 3a_2) (a_2) + (2e_3 - 3a_3) (a_3) \right. \]

\[ \left. - (n_{23})^2 - (n_{31})^2 - (n_{12})^2 \right] \]

\[ = 2 \mu - \frac{2}{3} \Theta^2 + 2 \sigma^2. \tag{51} \]

Equations (48) – (50) arise from the condition that the 3–Ricci curvature tensor be diagonal.

The constraint equations:

\[ 0 = (e_1 - 3a_1) (\sigma_+) - \sqrt{3} n_{23} \sigma_- + e_1(\Theta) \tag{52} \]

\[ 0 = (e_2 - 3a_2) (\sigma_+ + \sqrt{3} \sigma_-) + 3n_{31} (\sigma_+ - \frac{1}{\sqrt{3}} \sigma_-) - 2e_2(\Theta) \tag{53} \]

\[ 0 = (e_3 - 3a_3) (\sigma_+ - \sqrt{3} \sigma_-) - 3n_{12} (\sigma_+ + \frac{1}{\sqrt{3}} \sigma_-) - 2e_3(\Theta) \tag{54} \]

\[ 0 = (e_1 - 3a_1) (E_+) - \sqrt{3} n_{23} E_- + \frac{2}{3} e_1(\mu) \tag{55} \]

\[ 0 = (e_2 - 3a_2) (E_+ + \sqrt{3} E_-) + 3n_{31} (E_+ - \frac{1}{\sqrt{3}} E_-) - e_2(\mu) \tag{56} \]

\[ 0 = (e_3 - 3a_3) (E_+ - \sqrt{3} E_-) - 3n_{12} (E_+ + \frac{1}{\sqrt{3}} E_-) - e_3(\mu) \tag{57} \]

\[ 0 = (e_2 - 2a_2) (n_{12}) + (e_3 - 2a_3) (n_{31}) + e_2(a_3) - e_3(a_2) \tag{58} \]

\[ 0 = (e_3 - 2a_3) (n_{23}) + (e_1 - 2a_1) (n_{12}) + e_3(a_1) - e_1(a_3) \tag{59} \]

\[ 0 = (e_1 - 2a_1) (n_{31}) + (e_2 - 2a_2) (n_{23}) + e_1(a_2) - e_2(a_1) \tag{60} \]

\[ 0 = (e_1 - a_1) (\sigma_-) - \sqrt{3} n_{23} \sigma_+ \tag{61} \]

\[ 0 = (e_2 - a_2) (\sigma_+ - \frac{1}{\sqrt{3}} \sigma_-) + n_{31} (\sigma_+ + \sqrt{3} \sigma_-) \tag{62} \]

\[ 0 = (e_3 - a_3) (\sigma_+ + \frac{1}{\sqrt{3}} \sigma_-) - n_{12} (\sigma_+ - \sqrt{3} \sigma_-) \tag{63} \]

\[ 0 = (e_1 - a_1) (E_-) - \sqrt{3} n_{23} E_+ \tag{64} \]

\[ 0 = (e_2 - a_2) (E_+ - \frac{1}{\sqrt{3}} E_-) + n_{31} (E_+ + \sqrt{3} E_-) \tag{65} \]

\[ 0 = (e_3 - a_3) (E_+ + \frac{1}{\sqrt{3}} E_-) - n_{12} (E_+ - \sqrt{3} E_-). \tag{66} \]
commutators (28) – (30) and re–substitution from known relations, one can show that
the Jacobi constraints (58) – (60) as well as the Gauß constraints (46) – (51) are
preserved along \( u \).

4. Constraint analysis

In 1+3 ONF variables, the specific silent model condition that an irrotational dust fluid
matter source induces (purely) electric Weyl curvature of zero spatial rotation — Eq.
(10) in 1+3 covariant terms — takes the form of Eqs. (64) – (66). In order for the silent
assumption, as specified by Eq. (1), to lead to a consistent set of dynamical equations,
the zero spatial rotation condition must be preserved along the integral curves of the
preferred timelike reference congruence \( u \). The constraints (64) – (66) are propagated
along \( u \) by application of the commutators (28) – (30), and it is straightforward to show
that they will be preserved, given that the conditions

\[
0 = E_- e_1(\Theta) + \frac{1}{2} \sigma_- e_1(\mu) - 2 (a_1 \sigma_- + \sqrt{3} n_{23} \sigma_+) E_+ - 2 (a_1 \sigma_+ + \frac{1}{\sqrt{3}} n_{23} \sigma_-) E_-
\]

\[
0 = (E_+ - \frac{1}{\sqrt{3}} E_-) e_2(\Theta) + \frac{1}{2} (\sigma_+ - \frac{1}{\sqrt{3}} \sigma_-) e_2(\mu)
+ 2 (a_2 - n_{31}) (\sigma_+ + \frac{1}{\sqrt{3}} \sigma_-) E_+
+ \frac{2}{\sqrt{3}} a_2 (\sigma_+ - \sqrt{3} \sigma_-) E_- - \frac{2}{\sqrt{3}} n_{31} (\sigma_+ + \frac{5}{\sqrt{3}} \sigma_-) E_-
\]

\[
0 = (E_+ + \frac{1}{\sqrt{3}} E_-) e_3(\Theta) + \frac{1}{2} (\sigma_+ + \frac{1}{\sqrt{3}} \sigma_-) e_3(\mu)
+ 2 (a_3 + n_{12}) (\sigma_+ - \frac{1}{\sqrt{3}} \sigma_-) E_+
- \frac{2}{\sqrt{3}} a_3 (\sigma_+ + \sqrt{3} \sigma_-) E_- - \frac{2}{\sqrt{3}} n_{12} (\sigma_+ + \frac{5}{\sqrt{3}} \sigma_-) E_-
\]

hold. The righthand sides of these equations are equivalent to the 1 + 3 covariant
expression (15) derived above. In the following it is assumed that the electric part of
the Weyl curvature is non–zero.

4.1. Petrov type D

If the spacetime geometry is spatially inhomogeneous and its Weyl curvature tensor is
of algebraic Petrov type D \( (E_- = 0 \Rightarrow \sigma_- = 0) \), then, as Barnes and Rowlingson \[4\]
proved, it is identical to the dust models of Szekeres \[23\], or special subcases thereof,
and the equations are consistent. This can be seen as follows. For \( E_- = 0 \iff \sigma_- = 0 \),
one obtains from Eq. (61) that \( n_{23} = 0 \). Then Eq. (67) is identically satisfied, while
Eqs. (68) and (69) yield

\[
0 = E_+ e_2(\Theta) + \frac{1}{2} \sigma_+ e_2(\mu) + 2 (a_2 - n_{31}) \sigma_+ E_+
\]

\[
0 = E_+ e_3(\Theta) + \frac{1}{2} \sigma_+ e_3(\mu) + 2 (a_3 + n_{12}) \sigma_+ E_+.
\]
However, in that case one can derive from the constraints that

\[ 0 = e_2(\Theta) + (a_2 - n_{31}) \sigma_+ \]
\[ 0 = e_3(\Theta) + (a_3 + n_{12}) \sigma_+ \]
\[ 0 = \frac{1}{2} e_2(\mu) + (a_2 - n_{31}) E_+ \]
\[ 0 = \frac{1}{2} e_3(\mu) + (a_3 + n_{12}) E_+ , \]

so that Eqs. (70) and (71) are identically satisfied as well. A special subcase contained
within the Szekeres family are Ellis’ LRS class II dust models (see Refs. [8] and [16]). Here, the further conditions

\[ 0 = e_2(f) = e_3(f), \]
\[ 0 = a_3 = n_{12} \] and \[ 0 = a_2 = n_{31} \]

apply, and
the equations are again consistent.

4.2. Petrov type I

If, on the other hand, the spacetime geometry is spatially inhomogeneous and its Weyl curvature tensor is of algebraic Petrov type I \( E_+ \neq 0 \Rightarrow \sigma_+ \neq 0 \), then, contrary to
what was claimed by Lesame et al [20], Eqs. (67) – (69) do not
vanish identically, but constitute a new set of constraints. Of course, they are trivially satisfied, if a Petrov

type I spacetime geometry is of OSH Bianchi Type–I \( e_\alpha(f) = 0, \) \[
0 = a_\alpha = n_{\alpha \beta} . \]

Equations (67) – (69) can be interpreted as expressions for the spatial 3–gradients
of the fluid rate of expansion, \( e_\alpha(\Theta) \). Propagating them along \( u \) and re–substituting

from known relations, one obtains algebraic expressions for the 3–gradients of the fluid’s
total energy density, \( e_\alpha(\mu) \), in the form

\[ e_1(\mu) = f_1[ a_1, n_{23}, \sigma_+, \sigma_-, E_+, E_-, \mu ] \] (72)
\[ e_2(\mu) = g_1[ a_2, n_{31}, \sigma_+, \sigma_-, E_+, E_-, \mu ] \] (73)
\[ e_3(\mu) = h_1[ a_3, n_{12}, \sigma_+, \sigma_-, E_+, E_-, \mu ] \] (74)

Here, \( f_1, g_1 \) and \( h_1 \) are multivariate rational expressions of the variables indicated. Each
individual term in the numerators therein is linear in either \( a_\alpha \) or \( n_{\alpha \beta} \), and all terms
in the expressions contain a factor of either a power of \( \sigma_+ \) or a power of \( E_- \). As an
example, in an appendix we give the precise form of \( f_1 \) explicitly.

Propagating Eqs. (72) – (74) along \( u \) and re–substituting from known relations, one
obtains algebraic expressions for the spatial commutation functions \( a_\alpha \) in the form

\[ a_1 = n_{23} f_2[ \Theta, \sigma_+, \sigma_-, E_+, E_-, \mu ] \] (75)
\[ a_2 = n_{31} g_2[ \Theta, \sigma_+, \sigma_-, E_+, E_-, \mu ] \] (76)
\[ a_3 = n_{12} h_2[ \Theta, \sigma_+, \sigma_-, E_+, E_-, \mu ] \] (77)

where again \( f_2, g_2 \) and \( h_2 \) are multivariate rational expressions of the variables indicated,
with each individual term containing a factor of either a power of \( \sigma_+ \) or a power of \( E_- \).
Finally, propagating Eqs. (75) – (77) along \( u \) and re-substituting from known relations, one obtains purely algebraic constraints of the form

\[
0 = n_{23} f_3[\Theta, \sigma_+, \sigma_-, E_+, E_-, \mu ] \\
0 = n_{31} g_3[\Theta, \sigma_+, \sigma_-, E_+, E_-, \mu ] \\
0 = n_{12} h_3[\Theta, \sigma_+, \sigma_-, E_+, E_-, \mu ] ,
\]

where \( f_3, g_3 \) and \( h_3 \) are high-order multivariate polynomial expressions of the variables indicated, with each individual term containing a factor of either a power of \( \sigma_- \) or a power of \( E_- \). At this stage, in \( 1 + 3 \) covariant terms one has taken the fourth spatially projected covariant time derivative of the condition (10) that the electric Weyl tensor has vanishing spatial rotation. We emphasise that, although the conditions (78) – (80) are derived via tetrad methods, they reflect the covariant property that the constraints are not identically satisfied and do not become compatible after repeated differentiation. It is clear from Eqs. (78) – (80) that one can attempt to satisfy these conditions in four different ways (modulo a cyclic permutation of the axes of the spatial frame \( \{e_\alpha\} \)), depending on the number of non-zero \( n_{\alpha\beta} \) variables:

(i) \( 0 = n_{23} = n_{31} = n_{12} \); this implies \( a_\alpha = 0 \), and it is straightforward to show that this case simply corresponds to the OSH dust models of Bianchi Type–I,

(ii) \( 0 = n_{31} = n_{12}, \ n_{23} \neq 0 \), which implies \( 0 = a_2 = a_3 \) and \( f_3 = 0 \),

(iii) \( n_{12} = 0, \ n_{23} \neq 0 \neq n_{31} \), which implies \( a_3 = 0 \) and \( 0 = f_3 = g_3 \),

(iv) all of \( n_{23}, n_{31} \) and \( n_{12} \) are non-zero, which implies \( 0 = f_3 = g_3 = h_3 \).

Due to the particular structure of \( f_3, g_3 \) and \( h_3 \), case (iv) could be solved, e.g., by \( \sigma_- = 0 \iff E_- = 0 \), which just reproduces the Petrov type D situation discussed above. The issue is to see whether other non-trivial solutions in the variables \( \Theta, \sigma_+, \sigma_-, E_+, E_- \), and \( \mu \) to the highly complex algebraic conditions \( 0 = f_3 = g_3 = h_3 \) could be found. Additionally this would involve satisfying Eqs. (72) – (77), and then showing that the time derivatives of this solution are consistent. Note that we have not concluded the set of time derivatives needed to prove the consistent result generically, rather we ceased pursuing the consistency conditions beyond Eqs. (78) – (80) because of the number of terms involved. Given that all of these conditions were satisfied, this would establish the existence of spatially inhomogeneous silent models with a Weyl curvature tensor of algebraic Petrov type I. Unfortunately, because of their complexity, we have been unable to determine if there is such a solution. However, we doubt that such a solution exists. Nevertheless, if the contrary was true, solutions of this type would have to arise in a quite different manner compared to the Szekeres and OSH Type–I solutions.

Similarly, we have been unable to find non-trivial solutions in cases (ii) and (iii). So the cases (ii) – (iv) each require further investigation to establish a conclusive result.
5. Conclusion

On the basis of the analysis given in the previous section, we conclude with the following

Conjecture: There are no spatially inhomogeneous irrotational dust silent models, whose Weyl curvature tensor is of algebraic Petrov type I.

In this case, defined according to Eq. (1), the silent assumption for irrotational dust fluid matter sources would only reproduce already known classes of spatially inhomogeneous spacetime geometries \( \Pi \); they would be very restricted. This would support the conjecture of Matarrese et al [24] that realistic gravitational collapse scenarios (where \( \Theta < 0 \)) should involve non–zero magnetic Weyl curvature (with respect to \( u \)), \( H_{\mu
u} \neq 0 \).

This raises interesting issues about how well the Newtonian solutions in a realistic situation can correspond to the (more accurate) general relativistic description, because the magnetic part of the Weyl curvature vanishes in the Newtonian limit (see Ehlers and Buchert [7], confirming the view of Ellis and Dunsby [13], as opposed to claims by Bertschinger and Hamilton [2]). Furthermore, there is a linearisation instability within a purely general relativistic approach to irrotational silent cosmological models, since, as we showed in section 2 the FLRW–linearised silent models are consistent.

We note that analysis of the kind presented here can be conducted either via covariant or tetrad methods. In some cases, covariant methods succeed in the analysis of consistency (e.g. Ref. [21]), in some cases a tetrad analysis seems to be required (e.g. Ref. [23]), and there are cases where either formalism can be used (e.g. Refs. [8] and [27]). We emphasise that whichever formalism is used, this does not alter the underlying covariant nature of the problem.

Two final remarks should be made. First, the silent criterion for barotropic perfect fluids, as introduced by Matarrese et al [24, 4], and as applied in this paper, demanded that in mathematical terms the evolution equations within the set of \( 1 + 3 \) covariant dynamical equations reduce to a coupled set of ordinary differential equations. However, it should be pointed out that a coupled set of ordinary differential equations describing evolutionary behaviour of relativistic cosmological models can also be obtained from the \( 1 + 3 \) ONF dynamical equations. The best-known example of this kind are the OSH perfect fluid models as discussed by, e.g., Ellis and MacCallum \( [14] \) (see also Ref. [28]). Here the reduction of the evolution equations to a set of ordinary differential equations is achieved by choosing a \( 1 + 3 \) ONF \( \{ e_a \} \) that is invariant under the motions induced by the \( G_3 \) isometry group of spacelike translations. Hence, according to the ordinary differential equations criterion, the OSH perfect fluid models can be called \( 1 + 3 \) ONF silent, but, in general, not \( 1 + 3 \) covariantly silent. Most of the OSH perfect fluid models have non–zero electric and magnetic parts of the Weyl curvature, \( E_{\mu
u} \neq 0 \neq H_{\mu
u} \), and, more importantly, non–zero spatial rotation terms thereof, \( I_{\mu\nu} \neq 0 \neq J_{\mu\nu} \). However, they do not represent wave–like solutions that convey information from one worldline to
another. It is not clear at present how to characterise this behaviour in a 1 + 3 covariant manner.

Second, the demand that $H_{\mu\nu} = 0$ and $\omega^\mu = 0$ goes beyond what is needed to establish a silent model in the 1 + 3 covariant approach to perfect fluid spacetime geometries, for all that is required (in the spatially inhomogeneous case) is that the pressure $p$ and the spatial rotations, $I_{\mu\nu}$ of $E_{\mu\nu}$ and $J_{\mu\nu}$ of $H_{\mu\nu}$, should vanish. The case (4) was considered for simplicity (and even here, a full solution is not yet available). At present we do not know, if there are silent models with $E_{\mu\nu} \neq 0 \neq H_{\mu\nu}$, but $0 = I_{\mu\nu} = J_{\mu\nu}$, or if non–zero vorticity will appreciably broaden the class of allowed silent models. Preliminary investigations in the latter case seem to indicate that rotating silent models are tied to severe restrictions and would therefore be quite rare as well. However, both questions need to be clarified.

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Note added in proof: It has been drawn to our attention that the conjecture in Sec. 5 has been independently arrived at by CF Sopuerta [25], who gives an interesting analysis of silent models from a different viewpoint.

Appendix

For illustrative purposes, in this appendix we explicitly give the exact form of Eq. (72). We have that

$$e_1(\mu) = f_1[ a_1, n_{23}, \sigma_+, \sigma_-, E_+, E_-, \mu ]$$

$$= 4 \left[ a_1 A_1 + \frac{1}{2\sqrt{3}} n_{23} B_1 \right] / C_1 ,$$

where

$$A_1 := 6 \sigma_+ \sigma_- E_+ E_- - (\sigma_-)^2 \mu E_+ + (\sigma_-)^2 (E_+)^2$$

$$- (\sigma_-)^2 (E_-)^2 + 2 E_+ (E_-)^2$$

$$B_1 := 3 (\sigma_+)^2 \mu E_- + 24 (\sigma_+)^2 E_+ E_- - 6 \sigma_+ \sigma_- \mu E_+ + 6 \sigma_+ \sigma_- (E_+)^2$$

$$- 6 \sigma_+ \sigma_- (E_-)^2 - (\sigma_-)^2 \mu E_- + 4 (\sigma_-)^2 E_+ E_-$$

$$+ 6 (E_+)^2 E_- + 2 (E_-)^3$$

$$C_1 := 3 \sigma_+ \sigma_- E_- - (\sigma_-)^2 \mu + (\sigma_-)^2 E_+ + 2 (E_-)^2 .$$
The right hand sides of the remaining conditions at this and all subsequent levels of differentiation can be obtained from a transformation rule related to a cyclic permutation of the axes of the spatial frame \( \{ \mathbf{e}_\alpha \} \). This rule is given by making the substitutions

\[
\begin{align*}
1 & \rightarrow 2 \quad 2 \rightarrow 3 \quad 3 \rightarrow 1 \\
A_+ & \rightarrow -\frac{1}{2} (A_+ + \sqrt{3} A_-) \rightarrow -\frac{1}{2} (A_+ - \sqrt{3} A_-) \rightarrow A_+ \\
A_- & \rightarrow \frac{1}{2} (\sqrt{3} A_+ - A_-) \rightarrow -\frac{1}{2} (\sqrt{3} A_+ + A_-) \rightarrow A_-.
\end{align*}
\]

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