ESTIMATES FOR JACOBI-SOBOLEV TYPE ORTHOGONAL POLYNOMIALS

by

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Abstract. Let the Sobolev-type inner product \( \langle f, g \rangle = \int_{\mathbb{R}} fg d\mu_0 + \int_{\mathbb{R}} f'g' d\mu_1 \) with \( \mu_0 = w + M\delta_c, \mu_1 = N\delta_c \) where \( w \) is the Jacobi weight, \( c \) is either 1 or \(-1\) and \( M, N \geq 0 \). We obtain estimates and asymptotic properties on \([-1,1]\) for the polynomials orthonormal with respect to \( \langle ., . \rangle \) and their kernels. We also compare these polynomials with Jacobi orthonormal polynomials.

AMS Subject Classification (1991): 33C45, 42C05 Key words: Sobolev-type inner products, orthogonal polynomials, kernels, asymptotic properties.

* This research was partially supported by DGICYT PB93-0228-C02-02
** This research was partially supported by DGICYT PB93-0228-C02-01
1. Introduction

Recently the study of polynomials orthogonal with respect to a nonstandard inner product
\[ \langle f, g \rangle = \int_{\mathbb{R}} fg d\mu_0 + \sum_{k=1}^{m} \int_{\mathbb{R}} f^{(k)} g^{(k)} d\mu_k \]
has attracted the interest of many researchers. In particular when \( m = 1 \) and \( \mu_1 \) is an atomic measure supported at a point \( c \in \mathbb{R} \), results concerning algebraic properties of such polynomials and the location of their zeros have been done (see for instance [1]). From an analytic point of view, the relative asymptotic behaviour of such polynomials when \( \mu_0 \) belongs to the class \( M(0, 1) \) has been accomplished in several papers ([2], [6] and [7]). This behaviour is considered in compact sets of \( \mathbb{C} \setminus \text{supp } \mu_0 \).

However, the behaviour of polynomials in \( \text{supp } \mu_0 \) remains an open question. The aim of this paper is to cover this lack in the literature. In fact, a first approach was given by Marcellán and Osilenker [8] when \( m = 1 \), \( d\mu_0 = \chi_{[-1,1]} dx + M(\delta_1 + \delta_{-1}) \) and \( d\mu_1 = N(\delta_1 + \delta_{-1}) \) using some previous work by Bavinck and Meijer ([3], [4]), \( \delta_c \) denotes a Dirac measure supported at the point \( c \).

In our paper, we will consider \( m = 1 \)
\[ d\mu_0(x) = (1 - x)^\alpha (1 + x)^\beta dx + M\delta_1(x) \]
\[ d\mu_1(x) = N\delta_1(x) \]
with \( \alpha > -1 \) and \( \beta > -1 \).

In Section 2 we present the basic tools concerning the polynomials orthogonal with respect to the above inner product with special emphasis in the case of the so-called Jacobi-Sobolev type polynomials and some results about Jacobi polynomials which we will need throughout the paper. Section 3 deals with pointwise analysis and upper bounds for Jacobi-Sobolev type polynomials as well as an upper bound of their uniform norm using the corresponding estimates for standard Jacobi polynomials. Previously, we study the behaviour of the coefficients which appear in their representation in terms of Jacobi polynomials and, as a consequence, an estimate for them at the ends of the interval as well as an estimate for its first derivative are given.

Finally, in Section 4 we obtain some bounds and estimates for the kernels associated with the polynomials considered above. In particular, the analogue of a very well known result by Máté-Nevai-Totik concerning Christoffel functions is deduced.

In such a way we can give a complete answer in order to estimate the behaviour on \([-1, 1]\) of such polynomials. Notice that some of the above results, when \( d\mu_0 = w dx + M\delta_c \) where \( w \) is a generalized Jacobi weight and \( \mu_k = 0 \) \( (k = 1, \ldots, m) \), have been obtained in [5].
2. Representation formulas and basic results

Let \( \mu \) be a positive Borel measure on \( \mathbb{R} \) whose moments are finite and whose support is an infinite set.

We consider the inner product
\[
(f, g) = \int_{\mathbb{R}} f g d\mu + M f(c) g(c) + N f'(c) g'(c) \quad M, N \geq 0 \quad c \in \mathbb{R}
\] (1)

Let \( p_n \) and \( q_n \) be the polynomials orthonormal with respect to the measure \( \mu \) and the inner product (1), respectively. Denote \( q_n(x) = \gamma_n x^n + \ldots \) and \( p_n(x) = k_n x^n + \ldots \). The Fourier expansion of \( q_n \) in terms of \( p_k \) \((k = 0, \ldots, n)\) leads to
\[
q_n(x) = \frac{\gamma_n}{k_n} p_n(x) - M q_n(c) K_{n-1}(x, c) - N q_n'(c) K^{(0,1)}_{n-1}(x, c)
\] (2)

We have used the abbreviation
\[
K_n^{(r,s)}(x, y) = \sum_{k=0}^{n} p_k^{(r)}(x) p_k^{(s)}(y) = \frac{\partial^{r+s}}{\partial x^r \partial y^s} K_n(x, y)
\]
where, as usual, \( K_n(x, y) = \sum_{k=0}^{n} p_k(x) p_k(y) \).

If we take derivatives in (2) with respect to \( x \) and evaluating at \( x = c \), the values of \( q_n(c) \) and \( q_n'(c) \) can be expressed by
\[
q_n(c) = \frac{\gamma_n}{k_n D_n} [p_n(c) \{1 + N K_{n-1}^{(1,1)}(c, c)\} - N p_n'(c) K_{n-1}^{(0,1)}(c, c)]
\]
\[
q_n'(c) = \frac{\gamma_n}{k_n D_n} [-M p_n(c) K_{n-1}^{(0,1)}(c, c) + p_n'(c) \{1 + M K_{n-1}(c, c)\}]
\]
where
\[
D_n = 1 + M K_{n-1}(c, c) + N K_{n-1}^{(1,1)}(c, c) + M N [K_{n-1}(c, c) K_{n-1}^{(1,1)}(c, c) - (K_{n-1}^{(0,1)}(c, c))^2] \quad (3)
\]

( note that \( D_n = D_n(M, N, c) > 0 \) for all \( M \geq 0, N \geq 0 \) and \( c \in \mathbb{R} \).)

Let \( p_n(x; \mu_j) = k_n(\mu_j)x^n + \ldots, \ j = 0, 1, 2, \ldots, \) the orthonormal polynomials with respect to the measure \( d\mu_j = (x - c)^2 d\mu \) (where \( \mu_0 = \mu \) and \( K_n(x, y; \mu_j) \) the corresponding kernels. Expanding \((x - c)p_{n-1}(x; \mu_{j+1})\) in terms of \( p_k(x; \mu_j) \) we obtain (see [1, Lemma 2.1])
\[
(x - c)p_{n-1}(x; \mu_{j+1}) = \frac{k_{n-1}(\mu_{j+1})}{k_n(\mu_j)} [p_n(x; \mu_j) - \frac{p_n(c; \mu_j)}{K_{n-1}(c, c; \mu_j)} K_{n-1}(x, c; \mu_j)]
\]

Using the orthonormality of the polynomials \( p_{n-1}(x; \mu_{j+1}) \) and \( p_n(x; \mu_j) \) and the reproducing property of the kernels \( K_{n-1}(x, c; \mu_j) \) we have
\[
\left( \frac{k_n(\mu_j)}{k_{n-1}(\mu_{j+1})} \right)^2 = 1 + \frac{p_n(c; \mu_j)^2}{K_{n-1}(c, c; \mu_j)}
\]
We want to point out that
\[
\lim_{n} \frac{k_n(\mu_j)}{k_{n-1}(\mu_{j+1})} = 1 \quad \text{whenever} \quad \mu_j \in M(0, 1) \quad c \in [-1, 1] \quad (4)
\]
(see [9, Theorem 3 on p.26]), that we will use later.

Since the polynomials \(p_n(x; \mu_1)\) satisfy
\[
K_n(c, c)p_n(x; \mu_1) = \frac{k_n(\mu_1)}{k_{n+1}}[p'_{n+1}(c)K_n(x, c) - p_{n+1}(c)K_n^{(0,1)}(x, c)]
\]
we can write \(q_n(c)\) and \(q'_n(c)\) as follows
\[
q_n(c) = \frac{\gamma_n}{k_nD_n}[p_n(c) - N\frac{k_n}{k_{n-1}(\mu_1)}p'_{n-1}(c; \mu_1)K_{n-1}(c, c)]
\]
\[
q'_n(c) = \frac{\gamma_n}{k_nD_n}[p'_n(c) + M\frac{k_n}{k_{n-1}(\mu_1)}p_{n-1}(c; \mu_1)K_{n-1}(c, c)] \quad (5)
\]

If we represent the kernels \(K_{n-1}(x, c)\) and \(K_n^{(0,1)}(x, c)\) in terms of the polynomials \(p_n(x)\) and \(p_n(x; \mu_j)\) with \(j = 1, 2\) we can obtain (see [1, Proposition 2.2])

**Proposition 1.** Let \(p_n\) be the orthonormal polynomials for the measure \(\mu\) and \(c \in \mathbb{R}\) such that the condition \(p_n(c)p_{n-1}(c; \mu_1) \neq 0\) is satisfied for every \(n \in \mathbb{N}\). Then, the polynomials \(q_n\) orthonormal with respect to the inner product (1) verify the formula
\[
q_n(x) = A_n p_n(x) + B_n(x - c)p_{n-1}(x; \mu_1) + C_n(x - c)^2p_{n-2}(x; \mu_2) \quad (6)
\]
with
\[
A_n = \frac{\gamma_n}{k_n(1 - \alpha_n)} \quad B_n = \frac{\gamma_n}{k_{n-1}(\mu_1)}(\alpha_n - \beta_n) \quad C_n = \frac{\gamma_n}{k_{n-2}(\mu_2)}\beta_n \quad (6.1)
\]
where
\[
1 - \alpha_n = D_n^{-1}\left[1 - N\frac{k_n}{k_{n-1}(\mu_1)}p'_{n-1}(c; \mu_1)K_{n-1}(c, c)\right] \quad (6.2)
\]
\[
\beta_n = NK_{n-2}(c, c; \mu_1)D_n^{-1}\left[\frac{k_{n-1}(\mu_1)}{k_n}p'_n(c) + MK_{n-1}(c, c)\right] \quad (6.3)
\]

**Remark.** Since all the zeros of the polynomials \(p_n(x)\) and \(p_{n-1}(x; \mu_1)\) are in the interior of the convex hull of \(\text{supp } \mu\), then the formula (6) is true whenever \(c\) is not an interior point of the convex hull of \(\text{supp } \mu\).

From (2), it is obvious that
\[
\frac{\gamma_n}{k_n} = \int_{\mathbb{R}} q_n p_n d\mu = \langle q_n, p_n \rangle - M p_n(c)q_n(c) - Np'_n(c)q'_n(c)
\]

and then by straightforward calculations we find, (see [1] or [2])

$$\frac{\gamma_n}{k_n} = \left(\frac{D_n}{D_{n+1}}\right)^{1/2}$$  \hspace{1cm} (7)

In the sequel we consider the inner product (1) when the measure $\mu$ is the Jacobi weight and $c = 1$, that is

$$\langle f, g \rangle = \int_{[-1,1]} fg w_{\alpha,\beta} dx + M f(1)g(1) + N f'(1)g'(1)$$  \hspace{1cm} (8)

where $w_{\alpha,\beta}(x) = (1-x)^\alpha(1+x)^\beta$ with $\alpha, \beta > -1$ and $M, N \geq 0$. Let $P_{n}^{(\alpha,\beta)}$ be the Jacobi polynomials with the normalization condition

$$P_{n}^{(\alpha,\beta)}(1) = \frac{\Gamma(n + \alpha + 1)}{\Gamma(\alpha + 1)n!}$$

and $p_{n}^{(\alpha,\beta)}$ the Jacobi orthonormal polynomials. We denote by $q_{n}^{(\alpha,\beta)}$ the polynomials orthonormal with respect to the inner product (8).

Some basic properties of Jacobi polynomials, (see [11], Chapter IV), we will need in the following, are given below. Throughout this paper we use the notation $z_n \cong w_n$ when the sequence $z_n/w_n$ converges to 1.

$$P_{n}^{(\alpha,\beta)}(1) \cong \frac{n^\alpha}{\Gamma(\alpha + 1)}$$  \hspace{1cm} (9)

$$\frac{d}{dx} P_{n}^{(\alpha,\beta)}(x) = \frac{n + \alpha + \beta + 1}{2} P_{n-1}^{(\alpha+1,\beta+1)}(x)$$  \hspace{1cm} (10)

$$\|P_{n}^{(\alpha,\beta)}\|^2 = \frac{2^{\alpha+\beta+1}\Gamma(n + \alpha + 1)\Gamma(n + \beta + 1)}{(2n + \alpha + \beta + 1)n!\Gamma(n + \alpha + \beta + 1)} \cong 2^{\alpha+\beta}n^{-1}$$  \hspace{1cm} (11)

$$a_n = \frac{\Gamma(2n + \alpha + \beta + 1)}{2^n n!\Gamma(n + \alpha + \beta + 1)} \cong 2^{n+\alpha+\beta}n^{-1/2}$$  \hspace{1cm} (12)

where $P_{n}^{(\alpha,\beta)}(x) = a_n x^n + ...$

From (9)-(12), we have for Jacobi orthonormal polynomials:

$$p_{n}^{(\alpha,\beta)}(1) \cong \frac{n^{\alpha+(1/2)}}{2^{(\alpha+\beta)/2}\Gamma(\alpha + 1)}$$  \hspace{1cm} (13)

$$(p_{n}^{(\alpha,\beta)})'(1) \cong \frac{n^{\alpha+(5/2)}}{2^{(\alpha+\beta+2)/2}\Gamma(\alpha + 2)}$$  \hspace{1cm} (14)

From these formulas we can deduce
Lemma 1. The following estimates hold:

\[ K_n(1, 1) \cong \frac{n^{2\alpha+2}}{2^{\alpha+1}\Gamma(\alpha+1)\Gamma(\alpha+2)} \]  
(15)

\[ K_n^{(0,1)}(1, 1) \cong \frac{n^{2\alpha+4}}{2^{\alpha+2}\Gamma(\alpha+1)\Gamma(\alpha+3)} \]  
(16)

\[ K_n^{(1,1)}(1, 1) \cong \frac{\alpha + 2}{2^{\alpha+3}\Gamma(\alpha+2)\Gamma(\alpha+4)}n^{2\alpha+6} \]  
(17)

Proof: Because of the reproducing property of the kernels, \( K_n(x, 1) \) is a polynomial of degree \( n \), orthogonal with respect to the weight \( w_{\alpha+1,1} \), that is, for each \( n \) there exists a constant \( c_n \) such that \( K_n(x, 1) = c_n p_n^{(\alpha+1,\beta)}(x) \). Comparing the leading coefficients we get

\[ K_n(x, 1) = \frac{\|P_n^{(\alpha+1,\beta)}\|}{\|P_n^{(\alpha,\beta)}\|} \frac{n + \alpha + \beta + 1}{2n + \alpha + \beta + 1} p_n^{(\alpha,\beta)}(1)p_n^{(\alpha+1,\beta)}(x) \]  
(18)

Now, (15) follows from (11) and (13).

If we derive (18) and evaluating at \( x = 1 \), by using (11), (13) and (14), we deduce (16).

To obtain the estimate for \( K_n^{(1,1)}(1, 1) \) we can consider the formula

\[ K_n(1, 1)K_n^{(1,1)}(1, 1) - (K_n^{(0,1)}(1, 1))^2 = K_{n-1}(1, 1; w_{\alpha+2,1})K_n(1, 1) \]  
(19)

(see [1, Formula (2.9')]). Now (17) follows from (19), taking into account (15) and (16).

The above lemma and (19) allow us to deduce easily the asymptotic behaviour of \( D_n \), (see formula (3)).

From now on \( C \) will denote a positive constant independent of \( n \), but possibly different in each occurrence.

Lemma 2. There exists a positive constant \( C \) such that:

a) if \( MN > 0 \), then

\[ D_n \cong MN[K_{n-1}(1, 1)K_n^{(1,1)}(1, 1) - (K_n^{(0,1)}(1, 1))^2] \cong Cn^{4\alpha+8} \]

b) if \( M = 0 \) and \( N > 0 \) then

\[ D_n \cong NK_n^{(1,1)}(1, 1) \cong Cn^{2\alpha+6} \]

Taking in mind (7), a consequence of the above lemma is the following
Corollary 1. Let \( k_n \) and \( \gamma_n \) be the leading coefficients of the polynomials \( p_n^{(\alpha,\beta)} \) and \( q_n^{(\alpha,\beta)} \) respectively. Then \( \lim_{n \to \infty} \frac{\gamma_n}{k_n} = 1 \).

3. Estimates for Jacobi-Sobolev polynomials \( q_n^{(\alpha,\beta)} \) on \([-1,1]\)

In this section, we analyze the behaviour of the Jacobi Sobolev-type polynomials \( q_n^{(\alpha,\beta)} \) orthonormal with respect to (8) on \([-1,1]\).

In order to do this we will estimate the size of the coefficients which appear in their representation in terms of Jacobi polynomials, see Proposition 1.

Theorem 1. Let \( \mu \) be the Jacobi measure, \( c = 1 \) and \( A_n, B_n \) and \( C_n \) the corresponding coefficients in Proposition 1. There exists a positive constant \( C \) such that:

a) if \( MN > 0 \) then, \( A_n \sim Cn^{-2\alpha-2} \) \( B_n \sim Cn^{-2\alpha-2} \) \( C_n \sim 1 \).

b) if \( M = 0 \) and \( N > 0 \) then, \( A_n \sim \frac{1}{\alpha+2} \) \( B_n \sim 1 \) \( C_n \sim \frac{1}{\alpha+2} \).

Proof: Firstly, note that because of (4) and Corollary 1, \( \frac{\gamma_n}{k_n} \), \( \frac{\gamma_n}{k_n-2(w_{\alpha+4,\beta})} \) and \( \frac{\gamma_n}{k_n-2(w_{\alpha+4,\beta})} \) converge to 1. So, from (6.1), the asymptotic behaviour of \( A_n, B_n \) and \( C_n \) only depends on \( \alpha_n \) and \( \beta_n \).

a) Assume \( MN > 0 \). Using (13)-(15), we can see that, in formula (6.2), the term in brackets tends to \(-\infty\) like \(-n^{2\alpha+6}\). Since, by Lemma 2, \( D_n \sim Cn^{4\alpha+8} \) it follows that \( \alpha_n \to 1 \) and \( A_n \sim -Cn^{-2\alpha-2} \).

Applying formulas (13)-(15) and Lemma 2 in (6.3), we obtain that \( \beta_n \to 1 \); hence \( \alpha_n - \beta_n \to 0 \). Handling as above, it is not difficult to deduce that \( B_n \sim Cn^{-2\alpha-2} \).

The result for \( C_n \) is immediate.

b) Assume \( M = 0 \) and \( N > 0 \). Lemma 2 and formulas (13)-(15) lead to

\[
D_n^{-1} \frac{N_{k_{n-1}}(1,1)(p_n^{(\alpha+2,\beta)})'(1)}{p_n^{(\alpha,\beta)}(1)} \to \frac{1}{\alpha + 2}
\]

which, since \( D_n \sim Cn^{2\alpha+6} \), implies that \( 1 - \alpha_n \to -1/(\alpha + 2) \). As to \( \beta_n \), arguing in a similar way we get that \( \beta_n \to 1/(\alpha + 2) \) and the assertion follows.

Now, we can give the asymptotic behaviour of the polynomials \( q_n^{(\alpha,\beta)} \) and \( (q_n^{(\alpha,\beta)})' \) at the ends of the interval \([-1,1]\) for \( M \geq 0 \) and \( N > 0 \).

Theorem 2. There exists a positive constant \( C \) such that the following estimates

\[
q_n^{(\alpha,\beta)}(-1) \sim p_n^{(\alpha,\beta)}(-1) \sim C(-1)^n n^{\beta+1/2} \\
(q_n^{(\alpha,\beta)})'(-1) \sim (p_n^{(\alpha,\beta)})'(-1) \sim C(-1)^n n^{\beta+5/2}
\]
whenever we summarize for this situation the main results of this section:

Now it suffices to apply (4), (13)-(15), Lemma 2 and Corollary 1.

orthogonal with respect to the inner product (1) with $c(14)$ and Theorem 1, we obtain that $\lim_{q} c(5)$ written for Jacobi polynomials and $\in x$ hold.

To give the asymptotic behaviour at the point 1, we can use similar arguments.

Deriving in the above expression of $q_n^{(\alpha,\beta)}(x)$ and proceeding as before, from (13), (14) and Theorem 1, we obtain that $\lim_{n} \frac{q_n^{(\alpha,\beta)}(-1)}{p_n^{(\alpha,\beta)}(-1)} = 1$, whenever $M \geq 0$ and $N > 0$.

To give the asymptotic behaviour at the point 1, we can use similar arguments. However, we want to point out that to estimate $(q_n^{(\alpha,\beta)})'(1)$ it is easier to apply formula (5) written for Jacobi polynomials and $c = 1$, that is

$$(q_n^{(\alpha,\beta)})'(1) = \frac{\gamma_n}{k_n D_n}[(p_n^{(\alpha,\beta)})'(1) + M \frac{k_n}{k_{n-1}(w_{\alpha+2,\beta})} p_n^{(\alpha+2,\beta)}(1) K_{n-1}(1,1)]$$

Now it suffices to apply (4), (13)-(15), Lemma 2 and Corollary 1.

Note that the polynomials orthogonal with respect to the measure $\mu + M \delta_1$ are orthogonal with respect to the inner product (1) with $c = 1$, $M > 0$ and $N = 0$. Next we summarize for this situation the main results of this section:

**Lemma 3.** Whenever $M > 0$ and $N = 0$, there exists a positive constant $C$ such that,

$D_n \cong M K_{n-1}(1,1) \cong C n^{2\alpha+2}$

$A_n \cong C n^{-2\alpha-2} \quad B_n \cong 1 \quad C_n = 0$

$q_n^{(\alpha,\beta)}(1) \cong C n^{-\alpha-(3/2)} \quad (q_n^{(\alpha,\beta)})'(1) \cong C n^{\alpha+(5/2)}$

$q_n^{(\alpha,\beta)}(-1) \cong p_n^{(\alpha,\beta)}(-1) \cong C(-1)^n n^{\beta+(1/2)}$

$(q_n^{(\alpha,\beta)})'(-1) \cong (p_n^{(\alpha,\beta)})'(-1) \cong C(-1)^n n^{\beta+(5/2)}$
Remark. Compare the asymptotic behaviour of \( q_n^{(\alpha,\beta)}(1) \) and \( q_n^{(\alpha,\beta)}(-1) \) with the one of \( q_n^{(\alpha,\beta)}(x) \) for \( x \in \mathbb{C} \setminus [-1,1] \) which is well known since Lemma 16 on p.132 in [10] and Theorem 4 in [7] lead to \( \lim_{n \to \infty} \frac{q_n^{(\alpha,\beta)}(x)}{p_n^{(\alpha,\beta)}(x)} = 1 \) uniformly for \( x \) on compact sets of \( \mathbb{C} \setminus [-1,1] \), whenever \( M \geq 0 \) and \( N \geq 0 \). Concerning the asymptotic behaviour of \( p_n^{(\alpha,\beta)}(x) \) out of \([-1,1]\), see [10, Theorem 8.21.7].

Next we are going to find bounds for the polynomials \( q_n^{(\alpha,\beta)} \). First, we need to recall a property satisfied by Jacobi polynomials. Theorem 7.32.2 of [11] shows that there is a constant \( C \) independent of \( x \) and \( n \) such that

\[
n^{1/2} |p_n^{(\alpha,\beta)}(x)| \leq C (1 - x + n^{-2})^{-(\alpha/2) - (1/4)} \quad 0 \leq x \leq 1
\]

Using the fact that \( P_n^{(\alpha,\beta)}(x) = (-1)^n P_n^{(\beta,\alpha)}(-x) \), we have that the orthonormal Jacobi polynomials satisfy the estimate

\[
|p_n^{(\alpha,\beta)}(x)| \leq C (1 - x + n^{-2})^{-(\alpha/2) - (1/4)} (1 + x + n^{-2})^{-(\beta/2) - (1/4)} \quad (20)
\]

for all \( x \in [-1,1] \) and \( n \geq 1 \), with \( \alpha, \beta > -1 \). In the sequel \( C \) will denote a positive constant independent of \( n \) and \( x \), but possibly different in each occurrence.

We will find that similar bounds are valid for the polynomials \( q_n^{(\alpha,\beta)} \) with \( M, N \geq 0 \).

**Theorem 3.** There exists a constant \( C \) such that for each \( x \in [-1,1] \), \( n \geq 1 \) and \( \alpha, \beta > -1 \)

\[
|q_n^{(\alpha,\beta)}(x)| \leq C (1 - x + n^{-2})^{-(\alpha/2) - (1/4)} (1 + x + n^{-2})^{-(\beta/2) - (1/4)} \quad (21)
\]

**Proof:** It suffices to prove the result for \( n \) large enough. We know, (see Proposition 1), that the polynomials \( q_n^{(\alpha,\beta)} \) satisfy the representation formula

\[
q_n^{(\alpha,\beta)}(x) = A_n p_n^{(\alpha,\beta)}(x) + B_n (x-1) p_{n-1}^{(\alpha+2,\beta)}(x) + C_n (x-1)^2 p_{n-2}^{(\alpha+4,\beta)}(x)
\]

Since the coefficients \( A_n, B_n \) and \( C_n \) are bounded (Theorem 1 and Lemma 3) and the boundedness (20) for \( p_n^{(\alpha,\beta)}(x) \) is also true for \( (1-x)p_n^{(\alpha+2,\beta)}(x) \) and \( (1-x)^2 p_n^{(\alpha+4,\beta)}(x) \) for all \( x \in [-1,1] \) and \( n \geq 2 \), the statement follows.

As a consequence, whenever \( \alpha, \beta \geq -1/2 \), we get a bound independent of \( n \)

\[
|q_n^{(\alpha,\beta)}(x)| \leq C (1 - x)^{-(\alpha/2) - (1/4)} (1 + x)^{-(\beta/2) - (1/4)}
\]

for all \( x \in (-1,1) \).

In particular, if \( \alpha = \beta = 0 \), we have \( |q_n^{(\alpha,\beta)}(x)| \leq C (1 - x^2)^{-(1/4)} \) for all \( x \in (-1,1) \). A similar result has been obtained in [8] for the polynomials orthonormal with respect to the inner product \( \langle f, g \rangle = \int_{[-1,1]} fg \, d\mu_0 + \int_{[-1,1]} f'g' \, d\mu_1 \) with \( d\mu_0 = \frac{1}{2} dx + M(\delta_1 + \delta_{-1}) \) and \( d\mu_1 = N(\delta_1 + \delta_{-1}) \).

Now, from Theorem 3, we can deduce an upper bound of the maximum of \( q_n^{(\alpha,\beta)}(x) \) on \([-1,1]\).
Corollary 2. There exists a constant $C$ such that for each $n \geq 1$ we have

$$
\max_{-1 \leq x \leq 1} |q_n^{(\alpha, \beta)}(x)| \leq \begin{cases} 
C n^{q+1/2} & \text{if } q \geq -1/2 \\
C & \text{if } q \leq -1/2
\end{cases}
$$

where $q = \max\{\alpha, \beta\}$.

Proof: The inequalities $1 \leq 1 + x + n^{-2} \leq 3$ and $n^{-2} \leq 1 - x + n^{-2} \leq 2$ hold for $x \in [0, 1]$. Therefore, from (21), it follows that

$$
|q_n^{(\alpha, \beta)}(x)| \leq \begin{cases} 
C n^{\alpha+1/2} & \text{if } \alpha \geq -1/2 \\
C & \text{if } \alpha \leq -1/2
\end{cases}
$$

for all $x \in [0, 1]$.

A similar argument leads to

$$
|q_n^{(\alpha, \beta)}(x)| \leq \begin{cases} 
C n^{\beta+1/2} & \text{if } \beta \geq -1/2 \\
C & \text{if } \beta \leq -1/2
\end{cases}
$$

for all $x \in [-1, 0]$. The assertion follows easily.

Concerning the asymptotic behaviour of the $q_n^{(\alpha, \beta)}$ on $[-1, 1]$, by the previous Section we know estimates for these polynomials at the end points of the support of the Jacobi weight. What about the asymptotic behaviour of the $q_n^{(\alpha, \beta)}$ on $(-1, 1)$?

The Jacobi orthonormal polynomials verify

$$
p_n^{(\alpha, \beta)}(x) = r_n^{\alpha, \beta}(1 - x)^{-(\alpha/2)-(1/4)}(1 + x)^{-(\beta/2)-(1/4)} \cos(k\theta + \gamma) + O(n^{-1}) \quad (22)
$$

$k = n + \frac{\alpha + \beta + 1}{2}$, $\gamma = - (\alpha + 1)\pi/2$ and $r_n^{\alpha, \beta} = \frac{2(\alpha + \beta + 1/2)(\pi n)^{-1/2}}{\|P_n^{(\alpha, \beta)}\|} \to \left(\frac{2}{\pi}\right)^{1/2}$ uniformly for $x$ on compact sets of $(-1, 1)$, (see [11], Theorem 8.21.8)).

Now, we will show that the polynomials $q_n^{(\alpha, \beta)}$ have a similar asymptotic behaviour to the one of $p_n^{(\alpha, \beta)}$ on the interval $(-1, 1)$.

Theorem 4. Let $q_n^{(\alpha, \beta)}$ the polynomials orthonormal with respect to (8) and $A_n$, $B_n$ and $C_n$ the corresponding coefficients which appear in Proposition 1. Then

$$
q_n^{(\alpha, \beta)}(x) = s_n^{\alpha, \beta}(1 - x)^{-(\alpha/2)-(1/4)}(1 + x)^{-(\beta/2)-(1/4)} \cos(k\theta + \gamma) + O(n^{-1})
$$

$$
s_n^{\alpha, \beta} = A_n r_n^{\alpha, \beta} + B_n r_n^{\alpha+2, \beta} + C_n r_n^{\alpha+4, \beta} \to \left(\frac{2}{\pi}\right)^{1/2}
$$

uniformly for $x$ on compact sets of $(-1, 1)$. Therefore, $\lim_n [q_n^{(\alpha, \beta)}(x) - p_n^{(\alpha, \beta)}(x)] = 0$ uniformly for $x$ on compact sets of $(-1, 1)$.

Proof: By Proposition 1

$$
q_n^{(\alpha, \beta)}(x) = A_n p_n^{(\alpha, \beta)}(x) + B_n (x - 1) p_{n-1}^{(\alpha+2, \beta)}(x) + C_n (x - 1)^2 p_{n-2}^{(\alpha+4, \beta)}(x)
$$
From (22), we have

\[ q_n^{(\alpha,\beta)}(x) = (1 - x)^{-(\alpha/2)-(1/4)}(1 + x)^{-(\beta/2)-(1/4)} \cos(k\theta + \gamma) \left[A_n r_n^{\alpha,\beta} + B_n r_n^{\alpha+2,\beta} + C_n r_n^{\alpha+4,\beta} \right] \\
+ [A_n + B_n(x - 1) + C_n(x - 1)^2] + O(n^{-1}) \]

uniformly for \( x \) on compact sets of \((-1, 1)\).

Since the asymptotic behaviour of the coefficients \( A_n, B_n \) and \( C_n \) obtained in the previous section, we get

\[ q_n^{(\alpha,\beta)}(x) \sim s_n^{\alpha,\beta}(1 - x)^{-(\alpha/2)-(1/4)}(1 + x)^{-(\beta/2)-(1/4)} \cos(k\theta + \gamma) + O(n^{-1}) \]

and \( \lim_{n \to \infty} s_n^{\alpha,\beta} = \left( \frac{2}{\pi} \right)^{1/2} \).

Therefore \( q_n^{(\alpha,\beta)}(x) = \frac{s_n^{\alpha,\beta}}{r_n^{\alpha,\beta}} p_n^{(\alpha,\beta)}(x) + O(n^{-1}) \) and we can write

\[ q_n^{(\alpha,\beta)}(x) - p_n^{(\alpha,\beta)}(x) = \left( \frac{s_n^{\alpha,\beta}}{r_n^{\alpha,\beta}} - 1 \right) p_n^{(\alpha,\beta)}(x) + O(n^{-1}) \]

uniformly for \( x \) on compact sets of \((-1, 1)\). Thus the result follows.

**Remark.** From (20) we have \( |p_n^{(\alpha,\beta)}(x)| \leq C \) for \( x \) on compact sets of \((-1, 1)\). Then \( \lim_{n \to \infty} [q_n^{(\alpha,\beta)} - p_n^{(\alpha,\beta)}] = 0 \) uniformly on compact sets of \((-1, 1)\) could be also deduced applying Theorem 5 in [7] and formula (10) of Lemma 16 in [10].

4. Estimates for the kernels

It is known, (Nevai [10, Lemma 5 on p. 108]), that the kernels associated with Jacobi polynomials satisfy the estimate

\[ K_n(x, x) \sim n(1 - x + n^{-2})^{-\alpha-(1/2)}(1 + x + n^{-2})^{-\beta-(1/2)} \]

uniformly in \( |x| \leq 1, n \geq 1 \), where by \( f_n(x) \sim g_n(x) \) we mean that there exist some positive constants \( C_1 \) and \( C_2 \) such that \( C_1 f_n(x) \leq g_n(x) \leq C_2 f_n(x) \) for all \( x \in [-1, 1] \) and \( n \in \mathbb{N} \).

We want to find similar estimates for the new kernels.

Let \( L_n(x, y) \) be the kernels relative to the inner product (8). If we consider their expansion in terms of Jacobi orthonormal polynomials, we can deduce, (see [1, p.744]),

\[ L_n(x, y) = K_n(x, y) - ML_n(0, 1)K_n(x, 1) - NL_n^{(0,1)}(y, 1)K_n^{(0,1)}(x, 1) \]
with
\[ L_n(x, 1) = D_{n+1}^{-1} \left( [1 + NK_n^{(1,1)}(1, 1)]K_n(x, 1) - NK_n^{(0,1)}(1, 1)K_n^{(0,1)}(x, 1) \right) \]

\[ L_n^{(0,1)}(x, 1) = D_{n+1}^{-1} \left( [1 + MK_n(1, 1)]K_n^{(0,1)}(x, 1) - MK_n^{(0,1)}(1, 1)K_n(x, 1) \right) \]

Inserting \( L_n(x, 1) \) and \( L_n^{(0,1)}(x, 1) \) in (24) and taking \( y = x \), we get
\[ L_n(x, x) = K_n(x, x) - D_{n+1}^{-1} \left[ M\{1 + NK_n^{(1,1)}(1, 1)\}K_n(x, 1)^2 \right. \]
\[ \left. - 2MK_n^{(0,1)}(1, 1)K_n(x, 1)K_n^{(0,1)}(x, 1) + N\{1 + MK_n(1, 1)\}K_n^{(0,1)}(x, 1)^2 \right] \]

(25)

If, as usual, we define the Christoffel function
\[ \Lambda_n(x) = \min\{\langle p, p \rangle; \deg p \leq n, p(x) = 1 \} \]

it is easy to see that \( \Lambda_n(x) = [L_n(x, x)]^{-1} \).

We will use the representation (25) to obtain some bounds for \( L_n(x, x) \).

**Theorem 5.** Let \((L_n(x, y))\) be the kernels relative to the polynomials \( q_n^{(\alpha, \beta)} \). Then there exists a constant \( C \) such that for each \( x \in [-1, 1] \) and \( n \geq 1 \)
\[ |L_n(x, x)| \leq Cn(1 - x + n^{-2})^{-\alpha-(1/2)}(1 + x + n^{-2})^{-\beta-(1/2)} \]

**Proof:** From (23) we have for each \( x \in [-1, 1] \), \( n \geq 1 \) and \( \alpha, \beta > -1 \)
\[ |K_n(x, x)| \leq Cn(1 - x + n^{-2})^{-\alpha-(1/2)}(1 + x + n^{-2})^{-\beta-(1/2)} \]

(26)

Moreover, from (11), (18) and (20),
\[ |K_n(x, 1)| \leq C|p_n^{(\alpha, \beta)}(1)||p_n^{(\alpha+1, \beta)}(x)| \]
\[ \leq Cn^{\alpha+(1/2)}(1 - x + n^{-2})^{-(\alpha/2)-(3/4)}(1 + x + n^{-2})^{-(\beta/2)-(1/4)} \]

(27)

for all \( x \in [-1, 1] \).

To find a bound for \( K_n^{(0,1)}(x, 1) \), we will use the formula
\[ K_n^{(0,1)}(x, 1) = (x - 1)K_{n-1}(x, 1; w_{\alpha, 2, \beta}) + \frac{K_n^{(0,1)}(1, 1)}{K_n(x, 1)}K_n(x, 1) \]

(28)

(see [1, Formula (2.9)]), from which, using (27) and Lemma 1, it follows that
\[ |K_n^{(0,1)}(x, 1)| \leq Cn^{\alpha+(5/2)}(1 - x + n^{-2})^{-(\alpha/2)-(3/4)}(1 + x + n^{-2})^{-(\beta/2)-(1/4)} \]

(29)

for all \( x \in [-1, 1] \).
Now it suffices to remind that by Lemmas 1 and 2, whenever $MN > 0$
\[
MD_{n+1}^{-1}[1 + NK_n^{(1,1)}(1, 1)] \leq Cn^{-2\alpha - 2}
\]
\[
2MN D_{n+1}^{-1} K_n^{(0,1)}(1, 1) \leq Cn^{-2\alpha - 4}
\]
\[
ND_{n+1}^{-1}[1 + MK_n(1, 1)] \leq Cn^{-2\alpha - 6}
\]
and to observe that for each $x \in [-1, 1]$, the inequality $n^{-1}(1 - x + n^{-2})^{-1} \leq Cn$ holds. For the other values of the parameters $M$ and $N$, we proceed in a similar way. Thus, the result follows.

This result gives us only upper bounds. Now we want to estimate more accurately $L_n(x, x)$. First, we observe the behaviour of $L_n(x, x)$ at the end points of the interval $[-1, 1]$. Evaluating at $x = 1$ the expression of $L_n(x, 1)$ given in (24) and using (19), we get
\[
L_n(1, 1) = D_{n+1}^{-1}[1 + NK_n(1; w_{\alpha+2, \beta})]K_n(1, 1)
\]
Then, the kernels $L_n(1, 1)$ are bounded if $M > 0$, $N \geq 0$ while $L_n(1, 1) \equiv CK_n(1, 1)$ if $M = 0, N \geq 0$. Note that the boundedness of $L_n(1, 1)$ depends on the addition of a mass at 1 and not of the term involving derivatives.

Moreover from the expression of $L_n(1, 1)$ we can recover the mass $M$. Indeed, by using Lemmas 1, 2 and 3 it follows that, when $M > 0$ and $N \geq 0$, $\lim_{n} \Lambda_{n}(1) = M$.

Otherwise, the mass $N$ can be recovered from $L_n^{(1,1)}(1, 1)$ since, when $M \geq 0$ and $N > 0$, $\lim_{n}[L_n^{(1,1)}(1, 1)]^{-1} = N$.

As to $L_n(-1, -1)$, it suffices to take $x = y = -1$ in (24) and we obtain
\[
L_n(-1, -1) \equiv CK_n(-1, -1) \equiv Cn^{2\beta + 2}
\]
Next, we are going to find uniform estimates for the kernels. When $M > 0$, $L_n(1, 1)$ is bounded, so we give uniform estimates on compact sets not containing the mass point 1.

**Theorem 6.** a) Suppose $M > 0$, $N \geq 0$. Let $\varepsilon > 0$, then
\[
L_n(x, x) \sim n(1 - x + n^{-2})^{-\alpha-(1/2)}(1 + x + n^{-2})^{-\beta-(1/2)}
\]
uniformly on $[-1, 1 - \varepsilon]$, $n \geq 1$.

b) Suppose $M = 0$, $N \geq 0$. Then
\[
L_n(x, x) \sim n(1 - x + n^{-2})^{-\alpha-(1/2)}(1 + x + n^{-2})^{-\beta-(1/2)}
\]
uniformly on $|x| \leq 1$, $n \in \mathbb{N}$.

**Proof:** Because of Theorem 5, it suffices to prove that, for $n$ large enough
\[
L_n(x, x) \geq Cn(1 - x + n^{-2})^{-\alpha-(1/2)}(1 + x + n^{-2})^{-\beta-(1/2)}
\]

13
uniformly on $[-1, 1 - \varepsilon]$ when $M > 0$ and on $[-1, 1]$ when $M = 0$.

For the sake of simplicity, we write
\[
d(x, n) = n(1 - x + n^{-2})^{-\alpha - (1/2)}(1 + x + n^{-2})^{-\beta - (1/2)}
\]

a) Let $N > 0$. Using Lemmas 1 and 2 and formulas (27) and (29), we obtain that the three last summands in (25) are bounded by $Cd(x, n)n^{-2}(1 - x + n^{-2})^{-1}$. Thus, taking into account (23), the result follows. For $N = 0$, we handle in a similar way.

b) For $N = 0$ the result is obvious because of $L_n(x, x) = K_n(x, x)$. Suppose $N > 0$, as $D_{n+1} = 1 + NK_n^{(1,1)}(1, 1)$, from (25) we have
\[
L_n(x, x) \geq ND_{n+1}^{-1}[K_n^{(1,1)}(1, 1)K_n(x, x) - K_n^{(0,1)}(x, 1)^2]
\]
and using, again, the estimates for the kernels and (28) we can deduce the result.

Now we consider the analogue of the Szegő extremum problem for the inner product (8).

It is known that the generalized Szegő extremum problem, associated with a finite positive Borel measure on the real line, consists of finding $\lim n\lambda_n(x; \mu)$ with $\lambda_n(x; \mu)$ the Christoffel functions corresponding to $\mu$. A solution of this problem, when $\mu$ belongs to the Szegő class of the interval $[-1, 1]$, has been given in [9, Theorem 5] by proving that $\lim n\lambda_n(x; \mu) = \pi\mu'(x)(1 - x^2)^{1/2}$ for almost every $x \in [-1, 1]$, where $\mu'$ is almost everywhere the Radon-Nikodym derivative of $\mu$, (see [9]).

**Theorem 7.** Let $\Lambda_n$ the Christoffel functions associated with (8). Then
\[
\lim n\Lambda_n(x) = \pi w_{\alpha, \beta}(x)(1 - x^2)^{1/2}
\]
for almost every $x \in [-1, 1]$.

**Proof:** Because of Máté-Nevai-Totik result, above quoted, we only need to prove $\lim n^{-1}L_n(x, x) = \lim n^{-1}K_n(x, x)$, $x \in [-1, 1]$. Thus, by (25), it suffices to deduce
\[
\lim D_{n+1}^{-1}[1 + NK_n^{(1,1)}(1, 1)]K_n(x, 1)^2 = 0
\]
\[
\lim D_{n+1}^{-1}K_n^{(0,1)}(1, 1)K_n(x, 1)K_n^{(0,1)}(x, 1) = 0
\]
\[
\lim D_{n+1}^{-1}[1 + MK_n(1, 1)]K_n^{(0,1)}(x, 1)^2 = 0
\]
for every $x \in (-1, 1)$ and this follows by considering (18), (13), (22) and (28).

From the results of Section 2 and formulas (27) and (29), the following bounds for $L_n(x, 1)$ and $L_n^{(0,1)}(x, 1)$ can also be obtained:
Theorem 8. There exists a constant $C$ such that for each $x \in [-1, 1]$ and $n \geq 1$

\[
|L_n(x, 1)| \leq C(1 + x + n^{-2})^{-(\beta/2) - (1/4)} \quad \text{if} \quad M > 0
\]
\[
|L_n(x, 1)| \leq Cn^{2\alpha+4}(1 + x + n^{-2})^{-(\beta/2) - (1/4)} \quad \text{if} \quad M = 0
\]
\[
|L_n^{(0,1)}(x, 1)| \leq C(1 + x + n^{-2})^{-(\beta/2) - (1/4)} \quad \text{if} \quad N > 0
\]
\[
|L_n^{(0,1)}(x, 1)| \leq Cn^{2\alpha+4}(1 + x + n^{-2})^{-(\beta/2) - (1/4)} \quad \text{if} \quad N = 0
\]

Notice that the above bounds for $L_n(x, 1)$ when $M > 0$ and for $L_n^{(0,1)}(x, 1)$ when $N > 0$ are, respectively, sharper than the ones for $K_n(x, 1)$ and $K_n^{(0,1)}(x, 1)$ (see formulas (27) and (29)).

Remark. Some of the above results about the kernels appear in [5] for $w$ a generalized Jacobi weight and $N = 0$.

Finally, it is worth observing that if in the product (1) $\mu$ is the Jacobi measure and we take $c = -1$, since Jacobi polynomials satisfy $p_n^{(\alpha,\beta)}(-1) = (-1)^n p_n^{(\beta,\alpha)}(1)$, we get the same results as above but exchanging $\alpha$ and $\beta$.

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