CONSTRUCTION OF EXCITED MULTI-SOLITONS FOR THE 5D ENERGY-CRITICAL WAVE EQUATION

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Abstract. For the 5D energy-critical wave equation, we construct excited \(N\)-solitons with collinear speeds, i.e. solutions \(u\) of the equation such that

\[
\lim_{t \to +\infty} \left\| \nabla_{t,x} u(t) - \nabla_{t,x} \left( \sum_{n=1}^{N} Q_n(t) \right) \right\|_{L^2} = 0,
\]

where for \(n = 1, \ldots, N\), \(Q_n(t, x)\) is the Lorentz transform of a non-degenerate and sufficiently decaying excited state, each with different but collinear speeds. The existence proof follows the ideas of Martel-Merle [14] and Côte-Martel [3] developed for the energy-critical wave and nonlinear Klein-Gordon equations. In particular, we rely on an energy method and on a general coercivity property for the linearized operator.

1. Introduction

1.1. Main result. We consider the energy-critical focusing wave equation in dimension 5,

\[
\begin{cases}
\partial_t^2 u - \Delta u - |u|^\frac{4}{3} u = 0, & (t, x) \in [0, \infty) \times \mathbb{R}^5, \\
u|_{t=0} = u_0 \in \dot{H}^1, & \partial_t u|_{t=0} = u_1 \in L^2.
\end{cases}
\]

(1.1)

Recall that the Cauchy problem for equation (1.1) is locally well-posed in the energy space \(\dot{H}^1 \times L^2\). See e.g. [11] and references therein. Let \(f(u) = |u|^\frac{4}{3} u\) and \(F(u) = \frac{3}{10}|u|^\frac{10}{3}\). For any \(\dot{H}^1 \times L^2\) solution \(\bar{u} = (u, \partial_t u)\), the energy \(E\) and the momentum \(P\) are conserved along the flow, where

\[
E(u, \partial_t u) = \frac{1}{2} \int_{\mathbb{R}^5} \left( \nabla u \right)^2 + |\partial_t u|^2 - 2F(u) \, dx, \quad P(u, \partial_t u) = \frac{1}{2} \int_{\mathbb{R}^5} (\partial_t u \nabla u) \, dx.
\]

Looking for stationnary solutions \(u(t, x) = q(x)\) of (1.1) in \(\dot{H}^1\), we reduce to the Yamabe equation

\[
\Delta q + f(q) = 0 \quad \text{in } \mathbb{R}^5.
\]

(1.2)

Denote

\[
\Sigma = \left\{ q \in \dot{H}^1(\mathbb{R}^5)/0 : q \text{ satisfies (1.2)} \right\}.
\]

For \(q \in \Sigma\) and \(\ell \in \mathbb{R}^5\) such that \(|\ell| < 1\), let

\[
qu(x) = q \left( \frac{1}{|\ell|^2} \left( \frac{1}{\sqrt{1 - |\ell|^2}} \ell \cdot x \right) \ell + x \right),
\]

(1.3)

then \(u(t, x) = qu(x - \ell t)\) is a global, bounded solution of (1.1).

It is known (see e.g. [1, 20]) that the unique (up to scaling invariance and sign change) radially symmetric element of \(\Sigma\) is the ground state \(W\) given explicitly by

\[
W(x) = \left( 1 + \frac{|x|^2}{15} \right)^{-\frac{2}{3}}.
\]
The existence of non-radially symmetric, sign changing elements of \( \Sigma \) with arbitrary large energy was first proved by Ding [9], using variational arguments. Functions \( q \in \Sigma \) with \( q \neq W \) (up to invariances) are usually called excited states.

For any \( q \in \Sigma \), we denote the linearized operator around \( q \) by

\[
\mathcal{L} = -\Delta - f'(q).
\]

Set

\[
\mathcal{Z}_q = \left\{ f \in \dot{H}^1 \text{ such that } \mathcal{L}f = 0 \right\},
\]

and

\[
\tilde{\mathcal{Z}}_q = \text{span} \left\{ \frac{3}{2}q + x \cdot \nabla q; (x_i \partial_{x_j} q - x_j \partial_{x_i} q), 1 \leq i < j \leq 5; \right. \\
\left. \partial_{x_i} q; -3x_iq + |x|^2 \partial_{x_i} q - 2x_i(x \cdot \nabla q), 1 \leq i \leq 5 \right\}.
\]

The function space \( \tilde{\mathcal{Z}}_q \subset \mathcal{Z}_q \) is the null space of \( \mathcal{L} \) that is generated by a family of explicit transformations (see [10, Lemma 3.8] and Section 2.1).

We will need the following definitions.

**Definition 1.1.** Let \( q \in \Sigma \).

(i) \( q \) is called a non-degenerate state if \( \mathcal{Z}_q = \tilde{\mathcal{Z}}_q \).

(ii) \( q \) is called a decaying state if \( |q(x)| \lesssim \langle x \rangle^{-\alpha} \) for all \( x \in \mathbb{R}^5 \).

**Remark 1.2.** The existence of non-degenerate sign-changing states was proved in [7, 8, 19]. As noticed in [10, Remark 3.2], the existence of non-degenerate, decaying states then follows by translation and the Kelvin transformation.

The main goal of this article is to construct of \( N \)-excited states.

**Theorem 1.3** (Existence of multi-excited states). Let \( N \geq 2 \) and let \( \mathbf{e} \) be a unit vector of \( \mathbb{R}^N \). For \( n \in \{1, \ldots, N\} \), let \( \ell_n \in \mathbb{R}^5 \) be such that

\[
\ell_n = \ell_{n'} \mathbf{e} \text{ where } -1 < \ell_n < 1 \text{ and for all } n \neq n', \ell_n \neq \ell_{n'}.
\]

Let \( q_1, \ldots, q_N \) be non-degenerate decaying states of (1.2). Then there exist \( T_0 > 0 \) and a solution \( \bar{u} = (u, \partial_t u) \) of (1.1) in the energy space \( \dot{H}^1 \times L^2 \), defined on \( [T_0, +\infty) \) such that

\[
\lim_{t \to +\infty} \left\| \bar{u}(t) - \sum_{n=1}^{N} \bar{Q}_n(t) \right\|_{\dot{H}^1 \times L^2} = 0,
\]

where for \( n = 1, \ldots, N \),

\[
\bar{Q}_n(t, x) = \left( \frac{q_n(x - \ell_n t)}{-\ell_n \partial_{x_i} q_n(x - \ell_n t)} \right).
\]

**Remark 1.4.** By invariance by rotation in \( \mathbb{R}^5 \), we assume without loss of generality that \( \mathbf{e} \) is \( e_1 \), the first vector of the canonical basis of \( \mathbb{R}^5 \).

**Remark 1.5.** The non-degeneracy condition ensures that the null space of \( \mathcal{L} \) is generated by 21-parameter transformations, which is needed in our proof to obtain coercivity properties for the linearized operator around the excited states. See more details in Section 2.1 and Section 2.2.

**Remark 1.6.** The decay condition ensures that the nonlinear interactions between two excited states of different speeds is of order at most \( t^{-4} \). This rate allows us to close the energy estimates (see more details in Section 4.2 and Section 4.3). At this point, we do not known how to prove Theorem 1.3 without this condition. In particular, we do not construct multi-solitons partly based on non-degenerate decaying state and ground state, since the interaction caused by the ground state would be \( t^{-5} \).
Remark 1.7. Using the Lorentz transformation, the existence result extends to the case of 2-solitons for any different, possibly noncollinear speeds $\ell_1, \ell_2$. See [14, Section 5].

The constructions of asymptotic multi-solitons for dispersive and wave equations have been the subject of several previous works, for both stable and unstable solitons. First results in non-integrable contexts were given by Merle [6, 13, 18, 21] for the construction of multi-solitons for supercritical (gKdV) and (NLS), and Combet [2] for a classification result for supercritical (gKdV). We refer to [3, 14] for results on the existence of multi-solitons for the nonlinear Klein-Gordon (NLKG) and 5D energy critical wave equation that inspired the present work. See also [6, 13, 18, 21] for other existence results.

The article is organized as follows. Section 1 describes the spectral theory for any non-degenerate decaying state $q$. Section 2 introduces technical tools involved in a dynamical approach to the $N$-soliton problem for (1.1): estimates of the non-linear interactions between solitons, decomposition by modulation and parameter estimates. Finally, Theorem 1.3 is proved in Section 4 by energy estimates and a suitable compactness argument.

1.2. Notation. We denote

$$(u, v)_{L^2} = \int_{\mathbb{R}^5} u v dx, \quad (u, v)_{H^1} = \int_{\mathbb{R}^5} (\nabla u \cdot \nabla v) dx.$$ 

For

$$\vec{u} = \left( \begin{array}{c} u_1 \\ u_2 \end{array} \right), \quad \vec{v} = \left( \begin{array}{c} v_1 \\ v_2 \end{array} \right),$$

denote

$$(\vec{u}, \vec{v})_{L^2} = \sum_{k=1,2} (u_k, v_k)_{L^2}, \quad \|\vec{u}\|_{L^2} = (\vec{u}, \vec{u})_{L^2},$$

$$(\vec{u}, \vec{v})_{H^1} = (u_1, v_1)_{H^1} + (u_2, v_2)_{L^2}, \quad \|\vec{u}\|_{H^1} = (\vec{u}, \vec{u})_{H^1}.$$  

When $x_1$ is seen as a specific coordinate, denote

$$x = (x_1, x') \quad \text{where} \quad x' = (x_2, x_3, x_4, x_5).$$

Set $\langle x \rangle = (1 + |x|^2)^{\frac{1}{2}}$ and for $\vec{u} = (u, v)$,

$$\|\vec{u}\|_{W^1}^2 = \|u\|_{W^1}^2 + \|v\|_{W^1}^2,$$

where

$$\|u\|_{W^0} = \int_{\mathbb{R}^5} (|u(x)|^2 + |\nabla u(x)|^2) \langle x \rangle dx,$$

$$\|u\|_{W^1} = \int_{\mathbb{R}^5} (|\nabla u(x)|^2 + |\nabla^2 u(x)|^2) \langle x \rangle dx.$$  

Note that, for all $u \in W^1$,

$$\|\nabla u\|_{W^0} \lesssim \|u\|_{H^1}^\frac{1}{2} \|u\|_{W^1}^\frac{1}{2}. \quad (1.5)$$

Thus, it follows from a standard argument that (1.1) is locally well-posed in the related weighted Sobolev space $W^1 \times W^0$ with a time of existence depending only on the size of the initial data in $\| \cdot \|_{W^1 \times W^0}$.

Denote by $O_5$ be the orthogonal group in dimension 5. Let $SO_5$ be the special orthogonal group, i.e. the subgroup of the elements of $O_5$ with determinant 1.

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2. Spectral theory for non-degenerate state

2.1. Transforms of stationary solutions. Following [10], we recall that (1.2) is invariant under the following four transformations:

1. translations: If \( q \in \Sigma \) then \( q(x + a) \in \Sigma \), for all \( a \in \mathbb{R}^5 \);
2. dilation: If \( q \in \Sigma \) then \( \lambda^2 q(\lambda x) \in \Sigma \) for all \( \lambda > 0 \);
3. orthogonal transformation: If \( q \in \Sigma \) then \( q(Px) \in \Sigma \) where \( P \in O_5 \);
4. Kelvin transformation: If \( q \in \Sigma \) then \( |x|^{-3} q(\frac{1}{|x|^3}) \in \Sigma \).

Let \( M \) be the group of one-to-one maps of \( \mathbb{R}^5 \) generated by the above four transformations. Then, from [10, Section 3], \( M \) generates a 21-parameter family of transformations in a neighborhood of the identity. More precisely, we give explicitly the formula for this 21-parameter family of transformations. Set the one-to-one map

\[
\tau : \{(i, j) \in \mathbb{N}^2 : 1 \leq i < j \leq 5\} \to \{1, 2, \ldots, 10\}.
\]

with

\[
\tau(i, j) = 3i + j - \frac{(i-1)(i-2)}{2} - 4.
\]

For \( c = (c_1, c_2, \ldots, c_{10}) \), set

\[
A_c = [\alpha_{i,j}]_{1 \leq i, j \leq 5}, \quad e^{A_c} = \sum_{n=0}^{\infty} \frac{A^c_n}{n!} \in SO_5,
\]

where \( \alpha_{i,j}^c = 0 \), \( \alpha_{i,j} = c_{\tau(i,j)} \) if \( i < j \) and \( \alpha_{i,j} = -c_{\tau(j,i)} \) if \( j < i \). Note that this definition provides a parametrization of \( SO_5 \) by \( \mathbb{R}^{10} \) in a neighborhood of the identity matrix.

For \( A = (\lambda, \xi, a, c) \in (0, \infty) \times \mathbb{R}^5 \times \mathbb{R}^5 \times \mathbb{R}^{10} \), we introduce the following transform \( \theta_A \in M \),

\[
\theta_A(f)(x) = \lambda^2 \frac{x}{|x|} - a|x|^{-3} f \left( \xi + \frac{\lambda e^{A_c}(x - a|x|^2)}{1 - 2(a, x) + |a|^2|x|^2} \right) \quad \text{for all} \ f \in \mathcal{H}^1.
\]

Observe that for all \( q \in \Sigma \), \( \tilde{Z}_q \) is generated by taking partial derivatives of \( \theta_A(q) \) with respect to \( A = (\lambda, \xi, a, c) \) at \( A = (1, 0, 0, 0) \) (see details in [10, Lemma 3.8]).

2.2. Spectral analysis of \( L \). Following [10], we gather some spectral properties of the linearized operator. For \( q \) a non-degenerate decaying state and collinear speeds \( \ell = \ell e_1 \), let

\[
q_\ell = q \left( \frac{x_1}{\sqrt{1 - \ell^2}}, x' \right), \quad - (1 - \ell^2) \partial_{x_1}^2 q_\ell - \Delta q_\ell + |q_\ell|^2 q_\ell = 0.
\]

Define the following operator,

\[
L = -\Delta - f'(q), \quad L_\ell = - (1 - \ell^2) \partial_{x_1}^2 - \Delta - f'(q_\ell),
\]

and

\[
\mathcal{H}_\ell = \begin{pmatrix} -\Delta - f'(q_\ell) & -\ell \partial_{x_1} \\ \ell \partial_{x_1} & 1 \end{pmatrix}.
\]

Lemma 2.1 (Spectral properties of \( L \)). (i) Spectrum. The self-adjoint operator \( L \) has essential spectrum \([0, +\infty)\), a finite number \( J \geq 1 \) of negative eigenvalues (counted with multiplicity), and its kernel is \( \mathbb{Z}_q \). Let \( (Y_j)_{j \in \{1, \ldots, J\}} \) be an \( L^2 \) orthogonal family of eigenvectors of \( L \) corresponding to the eigenvalues \( -\lambda_j^2 \), \( j \in \{1, \ldots, J\} \).

\[
LY_j = -\lambda_j^2 Y_j, \quad \lambda_j > 0.
\]

Let \( \Psi_k \) be an \( \mathcal{H}^1 \)-orthogonal basis of \( \ker L \), i.e. for any \( k, k' \in \{1, \ldots, K\} \)

\[
(\Psi_k, \Psi_{k'})_{\mathcal{H}^1} = \delta_{kk'} \quad \text{and} \quad \text{span}(\Psi_1, \ldots, \Psi_K) = \mathbb{Z}_q.
\]
The functions \( \hat{\mathbf{Y}} \)

First, we prove the following technical identities of decaying condition. The exponential decay of negative functions \( Y \)

It holds, for all \( k = 1, \ldots, K \), \( j = 1, \ldots, J \) and \( \alpha \in \mathbb{N}^5 \) with \( |\alpha| \leq 2 \), on \( \mathbb{R}^5 \),

\[
|\partial_\alpha^c Y_j(x)| \lesssim e^{-\lambda_j|x|} \quad \text{and} \quad |\partial_\alpha^c \Psi_k(x)| \lesssim (x)^{-(3+\alpha)}.
\]

(ii) Nonnegativity under \( (Y_j)_j=1,\ldots,J \) orthogonality. It holds

\[
\langle L v, v \rangle_{L^2} \geq 0 \quad \text{for all } v \in \mathcal{N}^\perp,
\]

where

\[
\mathcal{N}^\perp = \left\{ v \in H^1 : \langle v, Y_j \rangle_{L^2} = 0, \text{ for any } j = 1, \ldots, J \right\}.
\]

(iii) Cancellation. It holds

\[
\int_{\mathbb{R}^5} f'''(q)\Psi_k \Psi_{k'} \Psi_{k''} \, dx = 0 \quad \text{for all } k, k', k'' = 1, \ldots, K.
\]

Proof. Proof of (i). For the spectral properties of \( \mathcal{L} \), see the proof of [10, Claim 3.5]. The algebraic decay of kernel functions \( \Psi_k \) directly follows from the non-degenerate decaying condition. The exponential decay of negative functions \( Y_j \) follows from standard elliptic arguments. See e.g. [17] and [10, Proposition 3.9].

Proof of (ii). See the proof of [10, Proposition 3.6].

Proof of (iii). Without loss of generality, we consider

\[
\Psi_{k''} = -3x_i q + |x|^2 \partial_{x_i} q - 2x_i x \cdot \nabla q.
\]

For all \( k' = 1, \ldots, K \), we have

\[
-\Delta \Psi_{k'} = f'(q) \Psi_{k'} = 0.
\]

We consider the transformation \( T_a = \theta_A \) with \( \theta_A = (1, 0, a, 0) \) for the above identity,

\[
-\Delta T_a \Psi_{k'} = f'(T_a q) T_a \Psi_{k'} = 0.
\]

Taking the derivative of above identity with respect to \( a_i \), and then letting \( a = 0 \), we obtain

\[
f'''(q)\Psi_{k'} \Psi_{k''} = -\Delta \tilde{\Psi}_{k'} - f'(q) \tilde{\Psi}_{k'} = \mathcal{L} \tilde{\Psi}_{k'},
\]

where

\[
\tilde{\Psi}_{k'} = -3x_i \Psi_{k'} + |x|^2 \partial_{x_i} \Psi_{k'} - 2x_i (x \cdot \nabla \Psi_{k'}).\n\]

Therefore, by integration by parts, for any \( k = 1, \ldots, K \),

\[
\int_{\mathbb{R}^5} \Psi_k (f'''(q)\Psi_{k'} \Psi_{k''}) \, dx = \int_{\mathbb{R}^5} \mathcal{L} \tilde{\Psi}_{k'} \Psi_{k''} \, dx = \int_{\mathbb{R}^5} (\mathcal{L} \Psi_k) \tilde{\Psi}_{k'} \Psi_{k''} \, dx = 0.
\]

Using the non-degenerate condition and proceeding similarly for all the parameters in the transformation \( \theta_A \), we complete the proof of (iii). \( \square \)

2.3. Coercivity of \( \mathcal{H}_\ell \). For \(-1 < \ell < 1\), let

\[
\tilde{\mathcal{Y}}_{k,\ell} = \begin{pmatrix} \Psi_{k}(x_{\ell}) \\ -\ell \partial_{x_{\ell}} \Psi_{k}(x_{\ell}) \end{pmatrix} \quad \text{for } k = 1, \ldots, K
\]

and for \( j = 1, \ldots, J \)

\[
Y_{j,\ell} = Y_j(x_{\ell}), \quad \tilde{Y}_{j,\ell}^0 = \begin{pmatrix} Y_{j,\ell} \\ 0 \end{pmatrix}, \quad \tilde{Y}_{j,\ell}^{\pm} = \begin{pmatrix} Y_{j,\ell} e^{\frac{\lambda_j}{\sqrt{1-\ell^2}} x_1} \\ -\ell \partial_{x_1} Y_{j,\ell} e^{\frac{\lambda_j}{\sqrt{1-\ell^2}} x_1} \end{pmatrix}.
\]

Following [4], we define the function \( \tilde{W}_{j,\ell} \) by

\[
\tilde{W}_{j,\ell} = \tilde{Y}_{j,\ell}^{+} + \tilde{Y}_{j,\ell}^{-}, \quad \text{for } j = 1, \ldots, J.
\]

First, we prove the following technical identities of \( \tilde{Y}_{j,\ell}^{\pm} \) and \( \tilde{W}_{j,\ell} \).

Lemma 2.2. The functions \( \tilde{Y}_{j,\ell}^{\pm} \) and \( \tilde{W}_{j,\ell} \) satisfy the following properties.
(i) For $j = 1, \ldots, J$, 
\[
\mathcal{H}_\ell Y^\pm_{j, \ell} = \mp \lambda_j (1 - \ell^2)\frac{\partial}{\partial x} J Y^\pm_{j, \ell}.
\] (2.4)

(ii) For $j = 1, \ldots, J$, 
\[
\left( \mathcal{H}_\ell \bar{W}_{j, \ell}, \bar{W}_{j, \ell} \right)_{L^2} = -4\lambda_j^2 (1 - \ell^2)\frac{\partial}{\partial x}.
\] (2.5)

(iii) For $j, j' = 1, \ldots, J$ with $j \neq j'$, 
\[
\left( \mathcal{H}_\ell \bar{W}_{j, \ell}, \bar{W}_{j', \ell} \right)_{L^2} = 0.
\] (2.6)

**Proof.** Proof of (i). On the one hand, from (i) of Lemma 2.1 and direct computation,
\[
(-\Delta - f'(\eta)) Y_{j, \ell} e^{\frac{\mp \lambda_j}{\sqrt{1 - \ell^2}} x} + \ell \partial_{x_1} \left( \left( \ell \partial_{x_1} Y_{j, \ell} \mp \frac{\lambda_j}{\sqrt{1 - \ell^2}} \right) e^{\frac{\pm \lambda_j}{\sqrt{1 - \ell^2}} x} \right)
\]
\[
= (\mathcal{L}_e Y_{j, \ell} e^{\frac{\pm \lambda_j}{\sqrt{1 - \ell^2}} x} + \ell \partial_{x_1} Y_{j, \ell} \mp \frac{\lambda_j}{\sqrt{1 - \ell^2}} Y_{j, \ell}) e^{\frac{\pm \lambda_j}{\sqrt{1 - \ell^2}} x}
\]
\[
= \pm \lambda_j (1 - \ell^2)\frac{\partial}{\partial x} Y_{j, \ell} e^{\frac{\pm \lambda_j}{\sqrt{1 - \ell^2}} x}.
\]

On the other hand, by direct computation,
\[
\ell \partial_{x_1} Y_{j, \ell} e^{\frac{\pm \lambda_j}{\sqrt{1 - \ell^2}} x} - \left( \ell \partial_{x_1} Y_{j, \ell} \mp \frac{\lambda_j}{\sqrt{1 - \ell^2}} Y_{j, \ell} \right) e^{\frac{\pm \lambda_j}{\sqrt{1 - \ell^2}} x}
\]
\[
= \left( \pm \frac{\lambda_j}{\sqrt{1 - \ell^2}} \pm \frac{\ell^2}{\sqrt{1 - \ell^2}} \right) Y_{j, \ell} e^{\frac{\pm \lambda_j}{\sqrt{1 - \ell^2}} x} = \pm \lambda_j (1 - \ell^2)\frac{\partial}{\partial x} Y_{j, \ell} e^{\frac{\pm \lambda_j}{\sqrt{1 - \ell^2}} x}.
\]

Gathering above identities, we obtain (2.4).

Proof of (ii). To prove (2.5), we first show that for $j, j' = 1, \ldots, J$, 
\[
\left( \mathcal{H}_\ell \bar{Y}^+_{j, \ell}, \bar{Y}^+_{j', \ell} \right)_{L^2} = \left( \mathcal{H}_\ell \bar{Y}^-_{j, \ell}, \bar{Y}^-_{j', \ell} \right)_{L^2} = 0.
\] (2.7)

On the one hand, by (2.4), for $j, j' = 1, \ldots, J$, 
\[
\left( \mathcal{H}_\ell \bar{Y}^+_{j, \ell}, \bar{Y}^+_{j', \ell} \right)_{L^2} = -\lambda_j (1 - \ell^2)\frac{\partial}{\partial x} \left( J \bar{Y}^+_{j, \ell}, \bar{Y}^+_{j', \ell} \right)_{L^2}.
\]

On the other hand, using again (2.4), for $j, j' = 1, \ldots, J$, 
\[
\left( \mathcal{H}_\ell \bar{Y}^+_{j, \ell}, \bar{Y}^+_{j', \ell} \right)_{L^2} = \left( \bar{Y}^+_{j, \ell}, \mathcal{H}_\ell \bar{Y}^+_{j', \ell} \right)_{L^2} = \lambda_j (1 - \ell^2)\frac{\partial}{\partial x} \left( J \bar{Y}^+_{j, \ell}, \bar{Y}^+_{j', \ell} \right)_{L^2}.
\]

Since $\lambda_j, \lambda_j' > 0$, this implies $\left( \mathcal{H}_\ell \bar{Y}^+_{j, \ell}, \bar{Y}^+_{j', \ell} \right)_{L^2} = 0$. The proof of (2.7) for $\left( \mathcal{H}_\ell \bar{Y}^-_{j, \ell}, \bar{Y}^-_{j', \ell} \right)_{L^2} = 0$ follows from similar arguments and it is omitted.

Now, we start to prove (2.5). From (2.7), we have
\[
\left( \mathcal{H}_\ell \bar{W}_{j, \ell}, \bar{W}_{j', \ell} \right)_{L^2} = 2 \left( \mathcal{H}_\ell \bar{Y}^+_{j, \ell}, \bar{Y}^-_{j', \ell} \right)_{L^2}.
\]

Using (2.4) and the explicit expression of $\bar{Y}^\pm_{j, \ell}$, we compute
\[
\left( \mathcal{H}_\ell \bar{Y}^+_{j, \ell}, \bar{Y}^-_{j', \ell} \right)_{L^2} = -\lambda_j (1 - \ell^2)\frac{\partial}{\partial x} \left( J \bar{Y}^+_{j, \ell}, \bar{Y}^-_{j', \ell} \right)_{L^2} = -2\lambda_j^2 (1 - \ell^2)\frac{\partial}{\partial x} \left( Y_j, Y_j' \right)_{L^2}.
\]

Therefore, the identity (2.5) follows from the normalization $(Y_j, Y_j')_{L^2} = 1$.

Proof of (iii). To prove (2.6), we first show that for $j, j' = 1, \ldots, J$ with $j \neq j'$,
\[
\left( \mathcal{H}_\ell \bar{Y}^+_{j, \ell}, \bar{Y}^-_{j', \ell} \right)_{L^2} = 0 \quad \text{when} \quad \lambda_j \neq \lambda_j',
\] (2.8)

and
\[
\left( \mathcal{H}_\ell \bar{Y}^+_{j, \ell}, \bar{Y}^-_{j, \ell} \right)_{L^2} + \left( \mathcal{H}_\ell \bar{Y}^-_{j, \ell}, \bar{Y}^+_{j, \ell} \right)_{L^2} = 0 \quad \text{when} \quad \lambda_j = \lambda_j'.
\] (2.9)
Let $j \neq j'$. In the case where $\lambda_j \neq \lambda_{j'}$. On the one hand, by (2.4), for $j \neq j'$,
\[
\left( \mathcal{H}_j \tilde{Y}_{j,\ell}^+, \tilde{Y}_{j',\ell}^- \right)_{L^2} = -\lambda_j (1 - \ell^2)^{\frac{1}{2}} \left( J \tilde{Y}_{j,\ell}^+, \tilde{Y}_{j',\ell}^- \right)_{L^2}.
\]
On the other hand, using again (2.4),
\[
\left( \mathcal{H}_j \tilde{Y}_{j,\ell}^+, \tilde{Y}_{j',\ell}^- \right)_{L^2} = \left( \tilde{Y}_{j,\ell}^+, \mathcal{H}_j \tilde{Y}_{j',\ell}^- \right)_{L^2} = -\lambda_{j'} (1 - \ell^2)^{\frac{1}{2}} \left( J \tilde{Y}_{j,\ell}^+, \tilde{Y}_{j',\ell}^- \right)_{L^2}.
\]
Since $\lambda_j \neq \lambda_{j'}$, this implies (2.8).
In the case where $\lambda_j = \lambda_{j'}$. From (2.4) and the explicit expression of $\tilde{Y}_{j,\ell}^\pm$, $(Y_j, Y_{j'})_{L^2} = 0$ and integration by parts, we compute
\[
\left( \mathcal{H}_j \tilde{Y}_{j,\ell}^+, \tilde{Y}_{j',\ell}^- \right)_{L^2} = -\lambda_j (1 - \ell^2)^{\frac{1}{2}} \left( J \tilde{Y}_{j,\ell}^+, \tilde{Y}_{j',\ell}^- \right)_{L^2} = -2\ell \lambda_j (1 - \ell^2)^{\frac{1}{2}} (\partial_{x_1} Y_j, Y_{j'})_{L^2}.
\]
Therefore, by (2.7) and integration by parts,
\[
\left( \mathcal{H}_j \tilde{Y}_{j,\ell}^+, \tilde{Y}_{j',\ell}^- \right)_{L^2} + \left( \mathcal{H}_j \tilde{Y}_{j,\ell}^+, \tilde{Y}_{j',\ell}^- \right)_{L^2} = 0,
\]
which proves (2.9).
Now, we start to prove (2.6). Let $j \neq j'$. From (2.7),
\[
\left( \mathcal{H}_j \tilde{W}_{j,\ell}^+, \tilde{W}_{j',\ell}^- \right)_{L^2} = \left( \mathcal{H}_j \tilde{Y}_{j,\ell}^+, \tilde{Y}_{j',\ell}^- \right)_{L^2} + \left( \mathcal{H}_j \tilde{Y}_{j,\ell}^+, \tilde{Y}_{j',\ell}^- \right)_{L^2}.
\]
Thus, the identity (2.6) directly follows from (2.8) and (2.9).

Second, we prove the following coercivity property.

**Proposition 2.3.** There exist $\mu > 0$ such that, for all $\vec{v} \in \dot{H}^1 \times L^2$

\[
(\mathcal{H}_\ell \vec{v}, \vec{v})_{L^2} \geq \mu \| \vec{v} \|_{\dot{H}^1}^2 = \mu^{-1} \left( \sum_{k=1}^K (\vec{v}, \vec{\Psi}_{k,\ell})_{\dot{H}^1}^2 + \sum_{j=1}^J \left( \mathcal{H}_\ell \vec{v}, \vec{Y}_{j,\ell}^\pm \right)_{L^2}^2 \right).
\]

(2.10)

**Proof.** By a standard argument, it suffices to prove that there exists $\mu > 0$ such that for any $\vec{v} \in \dot{H}^1 \times L^2$ with orthogonality conditions $(\mathcal{H}_\ell \vec{v}, \vec{W}_{j,\ell})_{L^2} = (\vec{v}, \vec{\Psi}_{k,\ell})_{\dot{H}^1} = 0$ for all $j = 1, \ldots, J$ and $k = 1, \ldots, K$ there holds

\[
(\mathcal{H}_\ell \vec{v}, \vec{v})_{L^2} \geq \mu \| \vec{v} \|_{\dot{H}^1}^2.
\]

(2.11)

**Step 1.** Negative direction. Note that for $\vec{v} = (v, z)$

\[
\mathcal{H}_\ell \vec{v} = \left( -\Delta v - f'(q(t))v - \ell \partial_{x_1} z, \ell \partial_{x_1} v + z \right), \quad (\mathcal{H}_\ell \vec{v}, \vec{v})_{L^2} = (\mathcal{L}_\ell v, v)_{L^2} + \| \ell \partial_{x_1} v + z \|_{L^2}^2.
\]

(2.12)

Therefore, from (ii) of Lemma 2.1, for any $\vec{v} \in \dot{H}^1 \times L^2$,

\[
(\vec{v}, \vec{Y}_{j,\ell}^0)_{L^2} = 0 \text{ for } j = 1, \ldots, J \implies (\mathcal{H}_\ell \vec{v}, \vec{v})_{L^2} \geq 0.
\]

(2.13)

We claim, for any $\vec{v} \in \dot{H}^1 \times L^2$,

\[
(\mathcal{H}_\ell \vec{v}, \vec{W}_{j,\ell})_{L^2} = 0 \text{ for } j = 1, \ldots, J \implies (\mathcal{H}_\ell \vec{v}, \vec{v})_{L^2} \geq 0.
\]

(2.14)

For the sake of contradiction, assume that there exists a vector $\vec{v} \in \dot{H}^1 \times L^2$ satisfying the orthogonality conditions in (2.14) and $(\mathcal{H}_\ell \vec{v}, \vec{v}) < 0$. For any $(c_j)_{k=0, \ldots, J} \in \mathbb{R}^{J+1} / \{0\}$, set

\[
\vec{\bar{v}} = c_0 \vec{v} + \sum_{j=1}^J c_j \vec{W}_{j,\ell} \in \text{span} \left( \vec{v}, \vec{W}_{1,\ell}, \ldots, \vec{W}_{J,\ell} \right).
\]
By direct computation, (2.5), (2.6) and the orthogonality conditions in (2.14),
\[
\left( \mathcal{H}_t \hat{h}, \hat{h} \right)_{L^2} = c_0^2 (\mathcal{H}_t \hat{v}, \hat{v})_{L^2} + \sum_{j=1}^{J} c_j^2 \left( \mathcal{H}_t \hat{W}_{j,t}, \hat{W}_{j,t} \right)_{L^2} = c_0^2 (\mathcal{H}_t \hat{v}, \hat{v})_{L^2} - 4(1 - \ell^2)\frac{1}{2} \sum_{j=1}^{J} c_j^2 \lambda_j^2 < 0.
\]

It follows that $(\mathcal{H}_t \cdot \cdot)_{L^2} < 0$ on span $(\hat{v}, \hat{W}_{1,t}, \ldots, \hat{W}_{J,t})$ and
\[
\dim \text{span} (\hat{v}, \hat{W}_{1,t}, \ldots, \hat{W}_{J,t}) = J + 1.
\]

This is contradictory with (2.13) which says that $(\mathcal{H}_t \cdot \cdot)_{L^2}$ is nonnegative under only $J$ orthogonality conditions. The proof of (2.14) is complete.

**Step 2.** Null direction. Note that, from (2.12), for any $\vec{v} = (v, z) \in \ker \mathcal{H}_t$,
\[
-\Delta v - f'(q_t)v - \ell \partial_{x_1} z = 0, \quad \ell \partial_{x_1} v + z = 0,
\]
which is equivalent to
\[
-\Delta v + \ell^2 \partial_{x_1}^2 v - f'(q_t)v = 0, \quad z = -\ell \partial_{x_1} v.
\]
Thus,
\[
\ker \mathcal{H}_t = \text{span} \left( \vec{W}^k, k = 1, \ldots, K \right).
\]

We claim, for all $\vec{v} = (v, z) \in H^1 \times L^2$ with orthogonality conditions $(\mathcal{H}_t \vec{v}, \vec{W}_{j,t})_{L^2} = (\vec{v}, \vec{W}_{k,t})_{\mathcal{H}} = 0$ for $j = 1, \ldots, J$, $k = 1, \ldots, K$,
\[
\vec{v} = 0 \quad \text{or} \quad (\mathcal{H}_t \vec{v}, \vec{v}) > 0,
\]
(2.15)

Denote
\[
N_{\ell}^j = \left\{ \vec{v} = (v, z) \in H^1 \times L^2 : (\mathcal{H}_t \vec{v}, \vec{W}_{j,t})_{L^2} = 0 \right\}.
\]

Indeed, it suffices to prove that for any $\vec{v} \in N_{\ell}^j$,
\[
(\mathcal{H}_t \vec{v}, \vec{v})_{L^2} = 0 \implies \vec{v} \in \ker \mathcal{H}_t.
\]

Fix $\vec{v} \in N_{\ell}^j$ with $(\mathcal{H}_t \vec{v}, \vec{v})_{L^2} = 0$. From $(\mathcal{H}_t \vec{v}, \vec{v})_{L^2} \geq 0$ on $N_{\ell}^j$ and Cauchy-Schwarz inequality for $(\mathcal{H}_t \cdot \cdot)_{L^2}$,
\[
(\mathcal{H}_t \vec{v}, \vec{f}) = 0 \quad \text{for all } \vec{f} = (f, g) \in N_{\ell}^j.
\]

For any $\vec{v} = (v, z) \in C^\infty_0 (\mathbb{R}^5) \times C^\infty_0 (\mathbb{R}^5)$, we decompose
\[
\vec{v} = \vec{f} + \sum_{j=1}^{J} \alpha_j \vec{W}_{j,t} \quad \text{where } \vec{f} \in N_{\ell}^j \quad \text{and} \quad \alpha_j = -\frac{1}{4} \lambda_j^{-2} (1 - \ell^2)^{-\frac{1}{2}} \left( \mathcal{H}_t \vec{v}, \vec{W}_{j,t} \right)_{L^2}.
\]

Thus, from (2.16),
\[
(\mathcal{H}_t \vec{v}, \vec{v})_{L^2} = (\mathcal{H}_t \vec{f}, \vec{f})_{L^2} + \sum_{j=1}^{J} \alpha_j \left( \mathcal{H}_t \vec{v}, \vec{W}_{j,t} \right)_{L^2} = 0.
\]

It follows that $\mathcal{H}_t \vec{v} = 0$ in the sense of distribution, i.e. $\vec{v} \in \ker \mathcal{H}_t$. The proof of (2.15) is complete.

**Step 3.** Conclusion. Now, we prove (2.11) by contradiction and using standard compactness argument. For the sake of contradiction, assume that there exists a sequence
\[
\{ \vec{v}_n = (v_n, z_n) \}_{n \in \mathbb{N}} \in H^1 \times L^2
\]
such that
\[
\vec{v}_n \in N_{\ell}^j, \quad (\vec{v}_n, \vec{W}_{k,t})_{\mathcal{H}} = 0 \quad \text{for } k = 1, \ldots, K,
\]
(2.17)
and

$$0 < (\mathcal{H}_t \vec{v}_n, \vec{v}_n)_{L^2} < \frac{1}{n} \|\vec{v}_n\|_{L^2}^2, \quad \int_{\mathbb{R}^3} f'(q_t)v_n^2 = 1 \quad \text{for any } n \in \mathbb{N}.$$  

From the above inequalities and (2.12), the sequence \{\vec{v}_n\}_{n \in \mathbb{N}} is bounded in \(\dot{H}^1 \times L^2\). Upon extracting a subsequence, we can assume

$$\vec{v}_n = (v_n, z_n) \rightharpoonup \vec{v} = (v, z) \in \dot{H}^1 \times L^2 \quad \text{as } n \to \infty. \tag{2.18}$$

On the one hand, by the Rellich theorem, we have

$$\int_{\mathbb{R}^3} f'(q_t)v^2 dx = \lim_{n \to \infty} \int_{\mathbb{R}^3} f'(q_t)v_n^2 dx = 1.$$  

It follows that \(\vec{v} \neq 0\). On the other hand, from \(\vec{v}_n = (v_n, z_n) \to \vec{v} = (v, z) \in \dot{H}^1 \times L^2\),

$$\int_{\mathbb{R}^3} |\nabla v|^2 dx \leq \lim_{n \to \infty} |\nabla v_n|^2 dx, \quad \int_{\mathbb{R}^3} |\ell \partial_{x_1} v + z|^2 dx \leq \lim_{n \to \infty} \int_{\mathbb{R}^3} |\ell \partial_{x_1} v_n + z_n|^2 dx$$

Therefore,

$$(\mathcal{H}_t \vec{v}, \vec{v})_{L^2} \leq \lim_{n \to \infty} (\mathcal{H}_t \vec{v}_n, \vec{v}_n)_{L^2} \leq 0.$$  

Using again (2.18) and taking limit in (2.17), we obtain

$$\vec{v} \in \mathcal{N}^\perp, \quad (\vec{v}, \vec{\Psi}_{k, \ell})_{\mathcal{H}} = 0 \quad \text{for } k = 1, \ldots, K,$$

Therefore, from (2.15), we obtain \(\vec{v} = 0\) which is a contradiction. The proof of (2.11) is complete. \(\square\)

3. Decomposition around the sum of \(N\) solitons

We prove in this section a general decomposition result around the sum of \(N\) solitons. Let \(N \geq 1\) and for any \(n \in \{1, \ldots, N\}\), let \(\ell_n = \ell_n e_1\) where \(-1 < \ell_n < 1\) and \(\ell_n \neq \ell_{n'}\) for \(n \neq n'\). Let \(q_1, \ldots, q_N\) be any non-degenerate decaying excited states. Denote by \(I\) and \(I^0\) the following two sets of indices

$$I^0 = \{(n, k): n = 1, \ldots, N, \ k = 1, \ldots, K_n\}, \quad |I^0| = \text{Card} I^0 = \sum_{n=1}^{N} K_n,$$

$$I = \{(n, j): n = 1, \ldots, N, \ j = 1, \ldots, J_n\}, \quad |I| = \text{Card} I = \sum_{n=1}^{N} J_n.$$  

We denote by \((\lambda_{n,j})_{(n,j) \in I^*}, (Y_{n,j})_{(n,j) \in I^*}, (\Psi_{n,k})_{(n,k) \in I^0}, (\vec{\Psi}_{n,k})_{(n,k) \in I^0}\) the negative eigenvalues, corresponding eigenfunctions and kernel functions for \(q_n\) as defined in Lemma 2.1. For \(n = 1, \ldots, N\), set

$$Q_n(t, x) = q_{n, \ell_n}(x - \ell_n t), \quad \vec{Q}_n(t, x) = \begin{pmatrix} Q_n(t, x) \\ -\ell_n \partial_{x_1} Q_n(t, x) \end{pmatrix}.$$  

Similarly, for \((n, k) \in I^0\),

$$\Psi_{n,k}(t, x) = \Psi_{n,k, \ell_n}(x - \ell_n t), \quad \vec{\Psi}_{n,k}(t, x) = \vec{\Psi}_{n,k, \ell_n}(x - \ell_n t),$$

and for \((n, j) \in I\),

$$\vec{Y}_{n,j}^\pm(t, x) = \vec{Y}_{n,j, \ell_n}(x - \ell_n t), \quad \vec{Z}_{n,j}^\pm = \mathcal{H}_n \vec{Y}_{n,j}^\pm,$$  

where

$$\mathcal{H}_n = \begin{pmatrix} -\Delta - f'(q_n) & -\ell_n \partial_{x_1} \\ \ell_n \partial_{x_1} & 1 \end{pmatrix}.$$  

Consider a time dependent \(C^1\) function \(b\) of the form

$$b = (b_{n,k})_{(n,k) \in I^0} \in \mathbb{R}^{|I^0|} \quad \text{with } |b| \ll 1.$$


We introduce
\[ U = \sum_{n=1}^{N} Q_n, \quad V = \sum_{(n,k) \in I^0} b_{n,k} \Psi_{n,k}, \]
\[ G = f(U + V) - \sum_{n=1}^{N} f(Q_n) - \sum_{(n,k) \in I^0} b_{n,k} f'(Q_n) \Psi_{n,k}. \]

First, we start with a technical lemma.

**Lemma 3.1.** Let \( W_1 \) and \( W_2 \) be continuous functions such that,
\[ |W_1(x)| + |W_2(x)| \lesssim \langle x \rangle^{-4}, \quad \text{for all } x \in \mathbb{R}^3. \]

Define
\[ W_{n_1}(t, x) = W_1(x - \ell_{n_1} t), \quad W_{n_2}(t, x) = W_2(x - \ell_{n_2} t). \]

Let \( 0 < \alpha_1 \leq \alpha_2 \) be such that \( \alpha_1 + \alpha_2 > \frac{5}{4} \). There exist \( T_0 \gg 1 \) such that, for all \( n_1, n_2 \in \{1, \ldots, N\} \) with \( n_1 \neq n_2 \) and \( t \geq T_0 \), the following hold.

(i) If \( \alpha_2 > \frac{5}{4} \),
\[ \int_{\mathbb{R}^3} |W_{n_1}|^{\alpha_1} |W_{n_2}|^{\alpha_2} \, dx \lesssim t^{-4\alpha_1}. \]  \hfill (3.1)

(ii) If \( \alpha_2 \leq \frac{5}{4} \),
\[ \int_{\mathbb{R}^3} |W_{n_1}|^{\alpha_1} |W_{n_2}|^{\alpha_2} \, dx \lesssim t^{5 - 4(\alpha_1 + \alpha_2)}. \]  \hfill (3.2)

**Proof.** For \( k = 1, 2 \), we denote
\[ \rho_k = x - \ell_{n_k} t, \quad \Omega_k(t) = \{ x \in \mathbb{R}^3 : |x| \leq 10^{-1} |\ell_{n_k} - \ell_{n_2}| t \}. \]

Let \( T_0 \gg 1 \) large enough. For \( t \geq T_0 \), from the decay property of \( W \),
\[ |W_{n_2}(t, x)| \lesssim \langle \rho_2 \rangle^{-4} \lesssim \langle \rho_1 + t \rangle^{-4}, \quad \text{for } x \in \Omega_1, \]  \hfill (3.3)
\[ |W_{n_1}(t, x)| \lesssim \langle \rho_1 \rangle^{-4} \lesssim \langle \rho_2 + t \rangle^{-4}, \quad \text{for } x \in \Omega_2, \]  \hfill (3.4)
\[ |W_{n_1}(t, x)| \lesssim \langle \rho_1 \rangle^{-4} \lesssim t^{-4}, \quad \text{for } x \in \Omega_2^c, \]  \hfill (3.5)
\[ |W_{n_2}(t, x)| \lesssim \langle \rho_2 \rangle^{-4} \lesssim t^{-4}, \quad \text{for } x \in \Omega_2^c. \]  \hfill (3.6)

Proof of (i). Case \( \alpha_1 > \frac{5}{4}, \alpha_2 > \frac{5}{4} \). From (3.3) and (3.5), we obtain
\[ \int_{\Omega_1} |W_{n_1}|^{\alpha_1} |W_{n_2}|^{\alpha_2} \, dx \lesssim t^{-4\alpha_2} \int_{\Omega_1} |W_{n_1}|^{\alpha_1} \, dx \lesssim t^{-4\alpha_2}, \]
\[ \int_{\Omega_2^c} |W_{n_1}|^{\alpha_1} |W_{n_2}|^{\alpha_2} \, dx \lesssim t^{-4\alpha_1} \int_{\Omega_2^c} |W_{n_2}|^{\alpha_2} \, dx \lesssim t^{-4\alpha_1}, \]
which implies (3.1).

Case \( 0 < \alpha_1 \leq \frac{5}{4}, \alpha_2 > \frac{5}{4} \). By (3.3) and change of variable,
\[ \int_{\Omega_1} |W_{n_1}|^{\alpha_1} |W_{n_2}|^{\alpha_2} \, dx \lesssim \int_{\mathbb{R}^3} \langle \rho_1 \rangle^{-4\alpha_1} \langle \rho_1 + t \rangle^{-4\alpha_2} \, dx \]
\[ \lesssim \int_{\mathbb{R}^3} \langle x \rangle^{-4\alpha_1} \langle x \rangle^{-4\alpha_2} \, dx \]
\[ \lesssim t^{5 - 4(\alpha_1 + \alpha_2)} \lesssim t^{-4\alpha_1}. \]

Using again (3.5),
\[ \int_{\Omega_2^c} |W_{n_1}|^{\alpha_1} |W_{n_2}|^{\alpha_2} \, dx \lesssim t^{-4\alpha_1} \int_{\Omega_2^c} |W_{n_2}|^{\alpha_2} \, dx \lesssim t^{-4\alpha_1}. \]

These estimates imply (3.1).
Proof of (ii). First, from (3.3), (3.4) and change of variable, as before,
\[ \int_{\Omega_1} |W_{n_1}|^{\alpha_1} |W_{n_2}|^{\alpha_2} dx + \int_{\Omega_2} |W_{n_1}|^{\alpha_1} |W_{n_2}|^{\alpha_2} dx \lesssim \int_{\mathbb{R}^5} (x)^{-4\alpha_1} ((x) + t)^{-4\alpha_2} dx \lesssim t^{5-4(\alpha_1+\alpha_2)}. \]

Second, by (3.5), (3.6), Hölder’s inequality and change of variable,
\[ \int_{(\Omega_1 \cup \Omega_2)^c} |W_{n_1}|^{\alpha_1} |W_{n_2}|^{\alpha_2} dx \lesssim \left( \int_{\Omega_1^c} |W_{n_1}|^{\alpha_1+\alpha_2} \right)^{\frac{\alpha_1}{\alpha_1+\alpha_2}} \left( \int_{\Omega_2^c} |W_{n_2}|^{\alpha_1+\alpha_2} \right)^{\frac{\alpha_2}{\alpha_1+\alpha_2}} \lesssim \int_{\mathbb{R}^5} (x)^{-4(\alpha_1+\alpha_2)} dx \lesssim t^{5-4(\alpha_1+\alpha_2)}. \]

Gathering above estimates, we obtain (3.2). \qed

Second, we introduce some pointwise estimates following from Taylor expansion, and omit its proof.

**Lemma 3.2.** The following estimates hold.
(i) For all \( n = 1, \ldots, N \),
\[ |f'(U) - f'(Q_n)| \lesssim \sum_{n' \neq n} \left( |Q_{n'}| |\Psi_n|^{\frac{1}{2}} + |Q_{n'}|^{\frac{1}{2}} \right). \] (3.7)

(ii) We have
\[ |f'(U + V) - f'(U)| \lesssim \sum_{(n,k) \in I^0} |b_{n,k}| |\Psi_{n,k}| |Q_n|^{\frac{1}{2}} + \sum_{(n,k) \in I^0} |b_{n,k}|^{\frac{1}{2}} |\Psi_{n,k}|^{\frac{1}{2}}. \] (3.8)

(iii) For all \( s \in \mathbb{R} \),
\[ \begin{align*}
|f(U + V + s) - f(U + V) - f'(U + V)s| &\lesssim \left( |U|^{\frac{1}{2}} + |V|^{\frac{1}{2}} \right) |s|^2 + |s|^{\frac{3}{2}}, \quad (3.9) \\
|F(U + V + s) - F(U + V) - f(U + V)s| &\lesssim \left( |U|^{\frac{1}{2}} + |V|^{\frac{1}{2}} \right) |s|^2 + |s|^{\frac{3}{2}}. \quad (3.10)
\end{align*} \]

Third, we state some preliminary estimates related to the nonlinear interaction. We decompose \( G \) as

\[ G = G_1 + G_2 = G_{1,1} + G_{1,2} + G_{1,3} + G_2, \]

where
\[ \begin{align*}
G_{1,1} &= f(U + V) - f(U) - f'(U)V - \frac{1}{2} f''(U)V^2, \\
G_{1,2} &= f'(U)V - \sum_{(n,k) \in I^0} b_{n,k} f''(Q_n) \Psi_{n,k}, \\
G_{1,3} &= f(U) - \sum_{n=1}^N f'(Q_n) \quad \text{and} \quad G_2 = \frac{1}{2} f''(U)V^2.
\end{align*} \]

For \( n = 1, \ldots, N \), set
\[ G_{3,n} = \frac{1}{2} \sum_{k,k'=1}^{K_n} b_{n,k} b_{n,k'} f''(Q_n) \Psi_{n,k} \Psi_{n,k'} \quad \text{and} \quad G_3 = \sum_{n=1}^N G_{3,n}. \]

**Lemma 3.3.** There exists \( T_0 \gg 1 \) such that the following estimates hold.
(i) Estimates on \( G_1 \). For \( t \geq T_0 \),
\[ \|G_1\|_{L^2} \lesssim \|G_{1,1}\|_{L^2} + \|G_{1,2}\|_{L^2} + \|G_{1,3}\|_{L^2} \lesssim \sum_{(n,k) \in I^0} |b_{n,k}|^{\frac{3}{2}} + t^{-4}. \] (3.11)
(ii) Expansion of $G_2$. For $t \geq T_0$,
\[
\|G_2 - G_3\|_{L^2} \lesssim \sum_{(n,k) \in I^0} |b_{n,k}|^2 t^{-4}.
\] (3.12)

(iii) Estimate on $G$. For $t \geq T_0$,
\[
\|G\|_{L^2} \lesssim \sum_{(n,k) \in I^0} |b_{n,k}|^2 + t^{-4}.
\] (3.13)

**Proof.** Proof of (i). From Taylor formula, we have
\[
|G_{1.1}| \lesssim |V|^2 \lesssim \sum_{(n,k) \in I^0} |b_{n,k}|^2 |\Psi_{n,k}|^2, \quad |G_{1.3}| \lesssim \sum_{n' \neq n} |Q_n|^2 |Q_{n'}|,
\]
\[
|G_{1.2}| \lesssim \sum_{(n,k) \in I^0} |b_{n,k}| \left[ \sum_{n' \neq n} |Q_n|^2 |\Psi_{n,k}| + \sum_{n' \neq n} |Q_{n'}|^2 |\Psi_{n,k}| \right].
\]

Therefore, from (i) of Lemma 3.1, Cauchy-Schwarz inequality and Young’s inequality, we obtain (3.11).

Proof of (ii). We observe that
\[
V^2 = \sum_{n=1}^{N} K_n b_{n,k} \Psi_{n,k} \Psi_{n,k'} + \sum_{n \neq n'} b_{n,k} b_{n',k'} \Psi_{n,k} \Psi_{n',k'}.
\]
Thus,
\[
G_2 = G_3 + G_{2.1} + G_{2.2},
\]
where
\[
G_{2.1} = \frac{1}{2} \sum_{n \neq n'} \sum_{k=1}^{K_n} b_{n,k} b_{n',k'} \Psi_{n,k} \Psi_{n',k'},
\]
\[
G_{2.2} = \frac{1}{2} \sum_{n=1}^{N} \sum_{k,k'} b_{n,k} b_{n,k'} (f''(U) - f''(Q_n)) \Psi_{n,k} \Psi_{n',k'}.
\]

By (i) of Lemma 3.1, we have
\[
\|G_{2.1}\|_{L^2} \lesssim \sum_{n \neq n'} \sum_{k=1}^{K_n} |b_{n,k}|^2 \|\Psi_{n,k} \Psi_{n',k'}\|_{L^2} \lesssim \sum_{(n,k) \in I^0} |b_{n,k}|^2 t^{-4}.
\]

From Taylor formula,
\[
|f''(U) - f''(Q_n)| \lesssim \sum_{n' \neq n} |Q_{n'}|^2 \quad \text{for all } n = 1, \ldots, N.
\]

Thus, using again (i) of Lemma 3.1,
\[
\|G_{2.2}\|_{L^2} \lesssim \sum_{(n,k) \in I^0} |b_{n,k}|^2 \left( \sum_{n' \neq n} \|Q_{n'}|^2 \Psi_{n,k}^2\|_{L^2} \right) \lesssim \sum_{(n,k) \in I^0} |b_{n,k}|^2 t^{-4}.
\]

Gathering above estimates, we obtain (3.12).

Proof of (iii). Estimate (3.13) is consequence of (3.11) and (3.12).

Last we prove a standard decomposition result around the sum of $N$ solitons.

**Proposition 3.4** (Properties of the decomposition). There exists $T_0 \gg 1$ and $0 < \delta_0 < 1$ such that if $\tilde{u}(t) = (u(t), \partial_t u(t))$ is a solution of (1.1) on $[T_1, T_2]$, where $T_0 \leq T_1 < T_2 < +\infty$, such that for any $t \in [T_1, T_2]$
\[
\|\tilde{u}(t) - \sum_{n=1}^{N} Q_n(t)\|_{H^s} \leq \delta_0,
\] (3.14)
then there exists $C^1$ functions $b = (b_{n,k})_{(n,k) \in \Gamma^0}$ on $[T_1,T_2]$ such that, $\varepsilon(t)$ being defined by

$$
\varepsilon(t) = \left( \begin{array}{c} \varepsilon \\ \eta \end{array} \right) = \bar{u}(t) - \sum_{n=1}^{N} \bar{Q}_n(t) - \sum_{(n,k) \in \Gamma^0} b_{n,k} \bar{\Psi}_{n,k}
$$

(3.15)

the following hold on $[T_1,T_2]$,

(i) First properties of the decomposition. For all $(n,k) \in \Gamma^0$ and $t \in [T_1,T_2],$

$$
(\varepsilon(t), \bar{\Psi}_{n,k}(t))_{\mathcal{H}} = 0, \quad |b_{n,k}(t)| \lesssim \|\bar{u}(t) - \sum_{n=1}^{N} \bar{Q}_n(t)\|_{\mathcal{H}}.
$$

(3.16)

(ii) Equation of $\bar{\varepsilon}$. It holds

$$
\begin{cases}
\partial_t \bar{\varepsilon} = \eta - \text{Mod}_1, \\
\partial_t \eta = \Delta \varepsilon + f(U + V + \varepsilon) - f(U + V) - \text{Mod}_2 + G
\end{cases}
$$

(3.17)

where

$$
\text{Mod}_1 = \sum_{(n,k) \in \Gamma^0} \bar{b}_{n,k} \bar{\Psi}_{n,k} \quad \text{and} \quad \text{Mod}_2 = - \sum_{(n,k) \in \Gamma^0} \bar{b}_{n,k} \ell_n \partial_{\bar{z}_j} \bar{\Psi}_{n,k}.
$$

(iii) Estimates for $b$. For $t \in [T_1,T_2]$, we have

$$
\sum_{(n,k) \in \Gamma^0} |\bar{b}_{n,k}(t)| \lesssim \|\bar{\varepsilon}(t)\|_{\mathcal{H}} + \sum_{(n,k) \in \Gamma^0} |b_{n,k}|^2 + t^{-4}.
$$

(3.18)

(iv) Unstable directions. Let $a_{n,j}^\pm = \left( \bar{\varepsilon}^\pm \bar{Z}_{n,j} \right)_{L^2}$ for all $(n,j) \in I$. Then

$$
\left| \frac{d}{dt} a_{n,j}^\pm(t) \pm \alpha_{n,j} a_{n,j}^\pm(t) \right| \lesssim \|\bar{\varepsilon}(t)\|_{\mathcal{H}}^2 + \sum_{(n,k) \in \Gamma^0} |b_{n,k}|^2 + t^{-4}.
$$

(3.19)

where $\alpha_{n,j} = \lambda_{n,j} (1 - \ell_n^2)^{\frac{1}{2}} > 0$.

Proof. Proof of (i). Let $T_0 \gg 1$, fix $t \geq T_0$. Note that, the orthogonality conditions in (3.16) is equivalent to the following matrix identity,

$$
\mathcal{M} b = \left( \begin{array}{c} \bar{u} - \sum_{n=1}^{N} \bar{Q}_n, \bar{\Psi}_{n,k} \end{array} \right)_{(n,k) \in \Gamma^0},
$$

where $b = (b_{n,k})_{(n,k) \in \Gamma^0}$ written in one row and

$$
\mathcal{M} = \left( \begin{array}{c} \bar{\Psi}_{n,k}, \bar{\Psi}_{n,k'} \end{array} \right)_{(n,k),(n',k') \in \Gamma^0}.
$$

Moreover, from (ii) of Lemma 3.1,

$$
\mathcal{M} = \text{diag}(\mathcal{M}_1, \ldots, \mathcal{M}_N) + O(t^{-2}) \quad \text{where} \quad \mathcal{M}_n = \left( \begin{array}{c} \bar{\Psi}_{n,k}, \bar{\Psi}_{n,k'} \end{array} \right)_{k,k'=1,\ldots,K_n}.
$$

Note that for fixed $n$, the family $(\bar{\Psi}_{n,k})_{k=1,\ldots,K_n}$ being linearly independent, the Gram matrix $\mathcal{M}_n$ is invertible. Hence, $\mathcal{M}$ is invertible, for $T_0$ large enough, and $\mathcal{M}^{-1}$ has uniform norm in $t \geq T_0$. Therefore, we obtain

$$
b = \mathcal{M}^{-1} \left( \begin{array}{c} \bar{u} - \sum_{n=1}^{N} \bar{Q}_n, \bar{\Psi}_{n,k} \end{array} \right)_{(n,k) \in \Gamma^0},
$$

and

$$
|b| \leq \|\mathcal{M}^{-1}\| \left( \begin{array}{c} \bar{u} - \sum_{n=1}^{N} \bar{Q}_n, \bar{\Psi}_{n,k} \end{array} \right)_{(n,k) \in \Gamma^0} \lesssim \|\bar{u}(t) - \sum_{n=1}^{N} \bar{Q}_n(t)\|_{\mathcal{H}}.
$$
Proof of (ii). First, by the definition of \( \varepsilon \),

\[
\partial_t \varepsilon = \partial_t u - \sum_{n=1}^{N} \partial_t Q_n - \sum_{(n,k) \in I^0} b_{n,k} \partial_k \Psi_{n,k} = \sum_{(n,k) \in I^0} \dot{b}_{n,k} \Psi_{n,k}
\]

\[
= \eta - \sum_{(n,k) \in I^0} \dot{b}_{n,k} \Psi_{n,k}.
\]

Second, by direct computation,

\[
\partial_t \eta = \partial_t^2 u + \sum_{n=1}^{N} \partial_t \left( \ell_n \partial x_1 Q_n \right) - \sum_{(n,k) \in I^0} b_{n,k} \partial_t (\ell_n \partial x_1 \Psi_{n,k}) + \sum_{(n,k) \in I^0} \dot{b}_{n,k} (\ell_n \partial x_1 \Psi_{n,k})
\]

\[
= \Delta u + f(u) - \sum_{n=1}^{N} \ell_n^2 \partial^2_{x_1} Q_n - \sum_{(n,k) \in I^0} b_{n,k} \ell_n^2 \partial^2_{x_1} \Psi_{n,k} + \sum_{(n,k) \in I^0} \dot{b}_{n,k} (\ell_n \partial x_1 \Psi_{n,k}).
\]

From (3.15), \(- (1 - \ell_n^2) \partial^2_{x_1} Q_n - \Delta Q_n - f(Q_n) = 0\), \(- (1 - \ell_n^2) \partial^2_{x_1} \Psi_{n,k} - \Delta \Psi_{n,k} - f'(Q_n) \Psi_{n,k} = 0\) and the definition of \( G \),

\[
\Delta u + f(u) - \sum_{n=1}^{N} \ell_n^2 \partial^2_{x_1} Q_n - \sum_{(n,k) \in I^0} b_{n,k} \ell_n^2 \partial^2_{x_1} \Psi_{n,k} = \Delta \varepsilon + f(U + V + \varepsilon) - f(U + V) + G.
\]

Therefore,

\[
\partial_t \eta = \Delta \varepsilon + f(U + V + \varepsilon) - f(U + V) + \sum_{(n,k) \in I^0} \dot{b}_{n,k} \ell_n \partial x_1 \Psi_{n,k} + G.
\]

Proof of (iii). First, we decompose

\[
f(U + V + \varepsilon) - f(U + V) = \sum_{n=1}^{N} f'(Q_n) \varepsilon + R_1 + R_2 + R_3,
\]

where

\[
R_1 = f(U + V + \varepsilon) - f(U + V) - f'(U + V) \varepsilon,
\]

\[
R_2 = (f'(U + V) - f'(U)) \varepsilon,
\]

\[
R_3 = \left( f'(U) - \sum_{n=1}^{N} f'(Q_n) \right) \varepsilon.
\]

Therefore, from (3.17), we obtain

\[
\partial_t \varepsilon = \bar{\mathcal{L}} \varepsilon - \bar{\text{Mod}} + \bar{G} + \bar{R}_1 + \bar{R}_2 + \bar{R}_3
\]

(3.20)

where

\[
\bar{\mathcal{L}} = \begin{pmatrix}
0 & 0 \\
\Delta + \sum_{n=1}^{N} f'(Q_n) & 1
\end{pmatrix},
\bar{\text{Mod}} = \begin{pmatrix}
\text{Mod}_1 \\
\text{Mod}_2
\end{pmatrix},
\]

and

\[
\bar{G} = \begin{pmatrix}
0 \\
G
\end{pmatrix},
\bar{R}_1 = \begin{pmatrix}
0 \\
R_1
\end{pmatrix},
\bar{R}_2 = \begin{pmatrix}
0 \\
R_2
\end{pmatrix},
\bar{R}_3 = \begin{pmatrix}
0 \\
R_3
\end{pmatrix}.
\]

We differentiate the orthogonality \((\varepsilon, \tilde{\Psi}_{n,k})_\mathcal{H} \equiv 0\) in (3.16) and using (3.20),

\[
0 = \frac{d}{dt} (\varepsilon, \tilde{\Psi}_{n,k})_\mathcal{H} = (\partial_t \varepsilon, \tilde{\Psi}_{n,k})_\mathcal{H} - (\varepsilon, \partial_t \tilde{\Psi}_{n,k})_\mathcal{H}
\]

\[
= (\bar{\mathcal{L}} \varepsilon, \tilde{\Psi}_{n,k})_\mathcal{H} - \sum_{(n',k')} \dot{b}_{n',k'} (\tilde{\Psi}_{n,k}, \tilde{\Psi}_{n',k'})_\mathcal{H} + (\bar{G}, \tilde{\Psi}_{n,k})_\mathcal{H} + (\bar{R}_1, \tilde{\Psi}_{n,k})_\mathcal{H} + (\bar{R}_2 + \bar{R}_3, \tilde{\Psi}_{n,k})_\mathcal{H} - (\varepsilon, \ell_n \partial x_1 \tilde{\Psi}_{n,k})_\mathcal{H}.
\]
By integration by parts and the decay properties of \(\tilde{\Psi}_{n,k}\), the first term is
\[
(\tilde{E}_n, \tilde{\Psi}_{n,k})_{L^2} = \left(\tilde{F}_n, \tilde{\Psi}_{n,k}\right)_{L^2} = O(\|\tilde{\epsilon}\|_{L^2}).
\]
From (3.13),
\[
\left|\left(\tilde{G}, \tilde{\Psi}_{n,k}\right)_{L^2}\right| \lesssim \|G\|_{L^2} \lesssim \sum_{(n,k) \in I^0} |b_{n,k}|^2 + t^{-4}.
\]
Next, from (3.7), (3.8), (3.9), Sobolev embedding theorem and the decay properties of \(\tilde{\Psi}_{n,k}\),
\[
\left|\left(\tilde{R}_1, \tilde{\Psi}_{n,k}\right)_{L^2}\right| \lesssim \left|\left(\|U\|^{\perp} + |V|^{\perp}\right)\tilde{\epsilon}^2 + \|\tilde{\epsilon}\|_{L^2}^2\right|_{L^2} \lesssim \|\tilde{\epsilon}\|_{H^1}^2,
\]
\[
\left|\left(\tilde{R}_2 + \tilde{R}_3, \tilde{\Psi}_{n,k}\right)_{L^2}\right| \lesssim \|R_2\|_{L^2} + \|R_3\|_{L^2} + \|\tilde{\epsilon}\|_{H^1} \lesssim \|\tilde{\epsilon}\|_{H}.\]
Gathering above estimates and proceeding similarly for all \((n,k) \in I^0\),
\[
\mathcal{M} \hat{b} = O\left(\|\tilde{\epsilon}\|_{H} + \sum_{(n,k) \in I^0} |b_{n,k}|^2 + t^{-4}\right),
\]
where \(\hat{b} = (b_{n,k})_{(n,k) \in I^0}\) written in one row. Therefore, from the matrix \(\mathcal{M}^{-1}\) being uniformly bounded, we obtain (3.18).

Proof of (iv). Using (3.20), we compute
\[
\frac{d}{dt} \tilde{z}_{n,j}^{\pm} = \left(\partial_t \tilde{\epsilon}, \tilde{z}_{n,j}^{\pm}\right)_{L^2} + \left(\tilde{\epsilon}, \tilde{\partial}_t \tilde{z}_{n,j}^{\pm}\right)_{L^2}
= \left(\tilde{E}_n \tilde{z}_{n,j}^{\pm}\right)_{L^2} - \ell_n \left(\tilde{\epsilon}, \tilde{\partial}_t \tilde{z}_{n,j}^{\pm}\right)_{L^2} + \left(\tilde{\partial}_t \tilde{\Psi}_{n,j}\right)_{L^2}
+ \left(\tilde{G}, \tilde{z}_{n,j}^{\pm}\right)_{L^2} + \left(\tilde{\partial}_t \tilde{\Psi}_{n,j}\right)_{L^2} + \left(\tilde{G}, \tilde{z}_{n,j}^{\pm}\right)_{L^2}.
\]
From (2.4), integration by parts and \(-J^2\) is identity matrix, we have
\[
\left(\tilde{E}_n \tilde{z}_{n,j}^{\pm}\right)_{L^2} - \ell_n \left(\tilde{\epsilon}, \tilde{\partial}_t \tilde{z}_{n,j}^{\pm}\right)_{L^2} = - \left(\tilde{\epsilon}, \tilde{\partial}_t \tilde{z}_{n,j}^{\pm}\right)_{L^2} + \sum_{n' \neq n} \left(\tilde{\epsilon}, f'(Q_{n'}) \tilde{z}_{n,j}^{\pm}\right)_{L^2}
= \mp \alpha_{n,j} y_{n,j}^{\pm} + \sum_{n' \neq n} \left(\tilde{\epsilon}, f'(Q_{n'}) \tilde{z}_{n,j}^{\pm}\right)_{L^2},
\]
where
\[
\tilde{z}_{n,j}^{\pm} = \mp \alpha_{n,j} \left(Y_{n,j}, \ell_n e^{\frac{\ell_n a_{n,j}}{\sqrt{\nu - \ell_n^2}}}ight) (\cdot - \ell_n t).
\]
By Sobolev embedding theorem and (i) of Lemma 3.1,
\[
\sum_{n' \neq n} \left|\left(\tilde{\epsilon}, f'(Q_{n'}) \tilde{z}_{n,j}^{\pm}\right)_{L^2}\right| \lesssim \sum_{n' \neq n} \|\tilde{\epsilon}\|_{L^4} \|f'(Q_{n'}) \tilde{z}_{n,j}^{\pm}\|_{L^4} \lesssim \|\tilde{\epsilon}\|_{H}^2 + t^{-4}.
\]
Note that, for all \((n,k) \in I^0\) and \((n,j) \in I\), we have
\[
\left(\tilde{\Psi}_{n,k}, \tilde{\partial}_n \tilde{Y}_{n,j}^{\pm}\right)_{L^2} = \left(H_n \tilde{\Psi}_{n,k}, \tilde{Y}_{n,j}^{\pm}\right)_{L^2} = 0.
\]
Therefore, from (i) of Lemma 3.1, (3.18) and concerning the term with \(\tilde{\partial}_n\),
\[
\left(\tilde{\partial}_n, \tilde{z}_{n,j}^{\pm}\right)_{L^2} = - \sum_{n' \neq n} \sum_{k=1}^{K_{n'}} \left(\tilde{\Psi}_{n',k}, \tilde{z}_{n,j}^{\pm}\right)_{L^2} = O\left(\|\tilde{\epsilon}\|_{H}^2 + \sum_{(n,k) \in I^0} |b_{n,k}|^2 + t^{-4}\right).
\]
Next, from (3.13) and Cauchy Schwarz inequality,
\[
\left|\left(\tilde{G}, \tilde{z}_{n,j}^{\pm}\right)_{L^2}\right| \lesssim \|G\|_{L^2} \lesssim \sum_{(n,k) \in I^0} |b_{n,k}|^2 + t^{-4}.
\]
and decay properties of $Z_{n,j}^\pm$, 
$$
\left| \left( \tilde{R}_1, \tilde{Z}_{n,j}^\pm \right) \right|_{L^2} \lesssim \left| \left( |U|^\frac{1}{2} + |V|^\frac{1}{2} \right) |\varepsilon|^2 + |\varepsilon|^2, Z_{n,j}^\pm \right|_{L^2} \lesssim \|\varepsilon\|^2_t,
$$
$$
\left| \left( \tilde{R}_2, \tilde{Z}_{n,j}^\pm \right) \right|_{L^2} \lesssim \|\varepsilon\|_{L^6} \left\| f'(U + V) - f'(U) \right\|_{L^2} \lesssim \|\varepsilon\|^2_t + \sum_{(n,k) \in P^n} \|B_{n,k}\|^2,
$$
$$
\left| \left( \tilde{R}_3, \tilde{Z}_{n,j}^\pm \right) \right|_{L^2} \lesssim \|\varepsilon\|_{L^6} \left\| \sum_{n' \neq n} \left( |Q_{n'}||Q_n|^\frac{1}{2} + |Q_n|^\frac{1}{2} Z_{n,j}^\pm \right) \right\|_{L^\infty} \lesssim \|\varepsilon\|^2_t + t^{-4}.
$$
Gathering above estimates and proceeding similarly for all $(n,j) \in I$, we obtain (3.19).

4. Proof of Theorem 1.3

In this section, we prove the existence of a solution $\tilde{u}(t)$ of (1.1) satisfying (1.4) in Theorem 1.3. We argue by compactness and obtain $\tilde{u}(t)$ as the limit of suitable approximate multi-solitons $\tilde{u}_m(t)$.

We start with a technical lemma which constructs well-prepared initial data at $t = T \gg 1$ with sufficient freedom related to unstable directions.

**Lemma 4.1.** Let $T \gg 1$ and $C \gg 1$. For any $a = (a_{n,j})_{(n,j) \in I} \in \mathbb{R}^{|I|}$, there exists $A = (\tilde{a}_{n,j})_{(n,j) \in I} \in \mathbb{R}^{|I|}$, $B = (\tilde{b}_{n,k})_{(n,k) \in P^0} \in \mathbb{R}^{|I^0|}$, satisfying

$$
\sum_{(n,j) \in I} |\tilde{a}_{n,j}| + \sum_{(n,k) \in I} |\tilde{b}_{n,k}| \leq C \sum_{(n,j) \in I} |a_{n,j}|,
$$

such that the function $\tilde{\varepsilon}(T)$ defined by

$$
\tilde{\varepsilon}(T) = \sum_{(n,j) \in I} \tilde{a}_{n,j}(a) \tilde{Z}_{n,j}^+(T) + \sum_{(n,k) \in P^0} \tilde{b}_{n,k}(a) \tilde{\Psi}_{n,k}(T),
$$

satisfies for all $(n,k) \in I^0$, $(n,j) \in I$,

$$
\left( \tilde{\varepsilon}(T), \tilde{\Psi}_{n,k}(T) \right)_{H^1} = 0, \quad \left( \tilde{\varepsilon}(T), \tilde{Z}_{n,j}^+(T) \right)_{L^2} = a_{n,j}.
$$

Moreover, the initial data defined by $\tilde{u}(T) = \sum_{n=1}^N \tilde{Q}_n(T) + \tilde{\varepsilon}(T)$ is modulated in the sense of Proposition 3.4 with $b_{n,k}(T) = 0$ for all $(n,k) \in I^0$ and $a_{n,j}^+(T) = a_{n,j}$ for all $(n,j) \in I$.

**Proof.** Our goal is to solve for $A = (\tilde{a}_{n,j})_{(n,j) \in I}$ and $B = (\tilde{b}_{n,k})_{(n,k) \in P^0}$ in terms of $a = (a_{n,j})_{(n,j) \in I}$. From the relations, for all $n = 1, \ldots, N$,

$$
\left( \tilde{\Psi}_{n,k}(T), \tilde{\Psi}_{n',k'}(T) \right)_{H^1} = O(T^{-1}), \quad \left( \tilde{\Psi}_{n,k}(T), \tilde{Z}_{n',j}^+(T) \right)_{H^1} = O(T^{-1}),
$$

$$
\left( \tilde{Z}_{n,j}^+(T), \tilde{\Psi}_{n',k}(T) \right)_{L^2} = O(T^{-1}), \quad \left( \tilde{Z}_{n,j}^+(T), \tilde{Z}_{n',j'}^+(T) \right)_{L^2} = O(T^{-1}),
$$

the conditions in (4.1) are equivalent to a linear relation between $A$ and $B$ of the following form, for all $(n,k) \in I^0$ and $(n,j) \in I$,

$$
\sum_{j=1}^{J_{n'}} \left( \tilde{\Psi}_{n,k}(T), \tilde{Z}_{n,j}^+(T) \right)_{H^1} \tilde{a}_{n,j} + \sum_{k'=1}^{K_{n'}} \left( \tilde{\Psi}_{n,k}(T), \tilde{\Psi}_{n,k'}(T) \right)_{H^1} \tilde{b}_{n,k'} = O \left( \sum_{n' \neq n} \sum_{j=1}^{J_{n'}} |\tilde{a}_{n',j'}| + \sum_{k'=1}^{K_{n'}} |\tilde{b}_{n',k'}| \right) T^{-1}.
$$

\[
\sum_{j'=1}^{J_n} \left( \hat{Z}_{n,j'}^{+} \right) \mathbf{\hat{u}}_{n,j'} = a_{n,j} + O \left( \sum_{n' \neq n} \left( \sum_{j'=1}^{J_{n'}} \left| \mathbf{\hat{a}}_{n',j'} \right| + \sum_{k'=1}^{K_{n'}} \left| \mathbf{\hat{b}}_{n',k'} \right| \right) T^{-1} \right).
\]

Since the families \( \left( \Psi_{n,k} \right)_{(n,k) \in I^0} \) and \( \left( \bar{Z}_{n,j}^{+} \right)_{(n,j) \in I} \) are linear independent, the above linear system is invertible for \( T \) large enough. We obtain the existence and desired properties of \( A = \left( \mathbf{\hat{a}}_{n,j} \right)_{(n,j) \in I} \) and \( B = \left( \mathbf{\hat{b}}_{n,k} \right)_{(n,k) \in I^0} \) for \( T \) large enough.

Let \( T_m = m \) for all \( m \in \mathbb{N}^+ \). For \( a_m = \left( a_{n,j}^m \right)_{(n,j) \in I} \in \mathbb{R}^{I^1} \) small to be determined later, we consider the solution \( \mathbf{\bar{u}}_m \) with the initial data \( \mathbf{\bar{u}}_m(T_m) \) given by Lemma 4.1. Note that \( \mathbf{\bar{u}}_m(T_m) \in W^1 \times W^0 \). Therefore the solution \( \mathbf{\bar{u}}_m \) is well-defined in \( W^1 \times W^0 \) at least on a small interval of time around \( T_m \).

The following Proposition is the main part of the proof of Theorem 1.3.

**Proposition 4.2 (Uniform estimates).** Under the assumptions of Theorem 1.3, there exist \( m_0 \in \mathbb{N}^+ \) and \( T_0 \gg 1 \) such that, for any \( m \geq m_0 \), there exist \( a_m = \left( a_{n,j}^m \right)_{(n,j) \in I} \in \mathbb{R}^{I^1} \) such that the solution \( \mathbf{\bar{u}}_m \) of (1.1) with initial data \( \mathbf{\bar{u}}_m(T_m) \) given by Lemma 4.1 is well-defined in \( W^1 \times W^0 \) on the time interval \([T_0, T_m]\) and satisfies

\[
\forall t \in [T_0, T_m], \quad \left\| \mathbf{\bar{u}}_m(t) - \sum_{n=1}^{N} \mathbf{\bar{Q}}_n(t) \right\|_{\mathcal{H}} \leq t^{-\frac{2}{5}}, \quad \left\| \mathbf{\bar{u}}_m(t) - \sum_{n=1}^{N} \mathbf{\bar{Q}}_n(t) \right\|_{\mathcal{W}} \leq t^{-\frac{2}{5}}. \tag{4.2}
\]

**4.1. Proof of Theorem 1.3 from Proposition 4.2.** We follow the strategy by uniform estimates and compactness introduced in [12, 13]. From the uniform estimates obtained in (4.2) at \( T = T_0 \), up to the extraction of a subsequence, there exists \( \mathbf{\bar{u}}_0 = (u_0, u_1) \in \dot{H}^1 \times L^2 \) such that

\[
\mathbf{\bar{u}}_m(T_0) \rightharpoonup \mathbf{\bar{u}}_0 \quad \text{in} \quad \dot{H}^1 \times L^2 \quad \text{weak} \quad \text{as} \quad m \to \infty,
\]

and \( \| \mathbf{\bar{u}}_m(T_0) \|_{\mathcal{W}} \lesssim 1 \) for all \( m \in \mathbb{N} \). Moreover, from \( W^1 \times W^0 \) is compactly embedded in \( \dot{H}^1 \times L^2 \), we have

\[
\| \mathbf{\bar{u}}_m(T_0) - \mathbf{\bar{u}}_0 \|_{\mathcal{H}} \to 0 \quad \text{as} \quad m \to \infty.
\]

Consider the solution \( \mathbf{\bar{u}}(t) \) of (1.1) with the initial data \( \mathbf{\bar{u}}_0 = (u_0, u_1) \) at \( T = T_0 \). Recall that the solution of (1.1) is continuous with respect to its initial data in the energy space \( \dot{H}^1 \times L^2 \) (see e.g. [11] and references therein). Thus, let \( m \to \infty \) in (4.2), we obtain the solution \( \mathbf{\bar{u}} \) is well-defined in \( \dot{H}^1 \times L^2 \) on \([T_0, +\infty)\) and satisfies

\[
\forall t \in [T_0, +\infty), \quad \left\| \mathbf{\bar{u}}(t) - \sum_{n=1}^{N} \mathbf{\bar{Q}}_n(t) \right\|_{\mathcal{H}} \lesssim t^{-\frac{2}{5}}.
\]

The proof of Theorem 1.3 from Proposition 4.2 is complete.

The rest of this section is devoted to the proof of Proposition 4.2.

**4.2. Bootstrap setting.** For \( 0 < t < T_m \), as long as \( \mathbf{\bar{u}}_m(t) \) is well defined in \( \dot{H}^1 \times L^2 \) and satisfies (3.14), we decompose \( \mathbf{\bar{u}}_m(t) \) as in Lemma 3.4. In particular, we denote by \( \varepsilon = (\varepsilon, \eta), \mathbf{b} = (b_{n,k})_{(n,k) \in I^0}, (a_{n,j}^{\pm})_{(n,j) \in I} \) the parameters of the decomposition of \( \mathbf{\bar{u}}_m \).

To prove Proposition 4.2, we introduce the following bootstrap estimates:

\[
\begin{align*}
\| \varepsilon(t) \|_{\mathcal{H}} & \leq t^{-\frac{2}{5}}, \quad \| \varepsilon(t) \|_{\mathcal{W}} \leq t^{-\frac{2}{5}}, \quad \sum_{(n,k) \in I^0} |b_{n,k}(t)| \leq t^{-\frac{2}{5}}, \\
\sum_{(n,j) \in I} |a_{n,j}^+(t)|^2 & \leq t^{-6}, \quad \sum_{(n,j) \in I} |a_{n,j}^-(t)|^2 \leq t^{-\frac{32}{5}}.
\end{align*}
\tag{4.3}
\]
For \( a_m \in \mathbb{R}^{11} \), set
\[
T_\alpha(a_m) = \inf \{ t \in [T_0, T_m]; \bar{u}_m \text{ satisfies (3.14) and (4.3) holds on } [t, T_m] \}. \tag{4.4}
\]
Note that, for the proof of Proposition 4.2, it suffices to prove that there exists \( T_0 \gg 1 \) (independent with \( m \)) large enough and at least one choice of \( a_m \in B_{\mathbb{R}^{11}}(T_m^{-3}) \) such that \( T_\alpha(a_m) = T_0 \).

4.3. Energy functional. Without loss of generality
\[-1 < \ell_1 < \cdots < \ell_N < 1.\]
Denote
\[\ell^* = \max_n(|\ell_n|) < 1.\]
For
\[0 < \delta < \frac{1}{100} \min_n(\ell_{n+1} - \ell_n)\]
small enough to be chosen later, we denote
\[\ell_n = \ell_n + \delta(\ell_{n+1} - \ell_n), \quad \text{for } n = 1, \ldots, N - 1,\]
\[\ell_n = \ell_n - \delta(\ell_{n+1} - \ell_n), \quad \text{for } n = 2, \ldots, N.\]
For \( t > 0 \), denote
\[\Omega(t) = \left( \left( \left( \ell_1, \ell_1 t \right) \cup \ldots \cup \left( \ell_{N-1}, \ell_{N-1} t \right) \right) \times \mathbb{R}^4 \right), \quad \Omega^C(t) = \mathbb{R}^5 \setminus \Omega(t).\]

We define the continuous function \( \chi_N(t, x) = \chi_N(t, x_1) \) as follow (see [14, 21] for a similar choice of cut-off function), for \( t > 0 \),
\[
\begin{align*}
\chi_N(t, x) &= \ell_1 \text{ for } x_1 \in (-\infty, \ell_1 t], \\
\chi_N(t, x) &= \ell_n \text{ for } x_1 \in \left( \ell_n t, \ell_{n+1} t \right], \text{ for } n \in \{2, \ldots, N - 1\}, \\
\chi_N(t, x) &= \ell_N \text{ for } x_1 \in \left[ \ell_N t, +\infty \right), \\
\chi_N(t, x) &= \frac{x_1}{(1 - 2\delta)t} - \frac{\delta}{1 - 2\delta}(\ell_{n+1} + \ell_n) \text{ for } x_1 \in \left[ \ell_n t, \ell_{n+1} t \right], \text{ for } n \in \{1, \ldots, N - 1\}. 
\end{align*}
\]
Note that
\[
\begin{align*}
\partial_{x_1} \chi_N(t, x) &= \frac{1}{(1 - 2\delta)t} \text{ for } x \in \Omega(t), \\
\partial_t \chi_N(t, x) &= - \frac{x_1}{t(1 - 2\delta)t} \text{ for } x \in \Omega(t), \\
\partial_1 \chi_N(t, x) &= 0, \quad \nabla \chi_N(t, x) = 0, \quad \text{for } x \in \Omega^C(t).
\end{align*}
\]

We define (see [14, 15, 21] for similar energy functional)
\[K(t) = F(t) + G(t) = \mathcal{E}(t) + \mathcal{P}(t) + \mathcal{G}(t)\]
where
\[
\mathcal{E}(t) = \int_{\mathbb{R}^5} |\nabla \varepsilon|^2 + |\eta|^2 - 2(F(U + V + \varepsilon) - F(U + V) - f(U + V)\varepsilon)dx,
\]
\[
\mathcal{P}(t) = 2 \int_{\mathbb{R}^5} \left( \chi_N(t, x)\partial_{x_1} \varepsilon(t, x) \right) \eta(t, x)dx \quad \text{and} \quad \mathcal{G}(t) = 2 \int_{\mathbb{R}^5} \varepsilon(t, x) G_3(t, x)dx.
\]

We start with the following technical lemma.

**Lemma 4.3.** Let \( W \) be a continuous function such that
\[|W(x)| \lesssim (x)^{-4+\alpha} \quad \text{for all } x \in \mathbb{R}^5, \tag{4.7}\]
where \( \alpha > 0 \). For all \( n = 1, \ldots, N \), the following estimates hold.
(i) Estimate of \( L^{\infty} \) norm.
\[\| (\ell_n - \chi_N) W(x - \ell_n t) \|_{L^{\infty}} \lesssim t^{-(\frac{1}{2} + \alpha)}. \tag{4.8}\]
(ii) Estimate of $L^2$ norm.
\[
\| (\ell_n - \chi_N) W(x - \ell_n t) \|_{L^2} \lesssim t^{-\frac{1}{4} + \alpha}.
\] (4.9)

(iii) Estimate of $L^{\frac{4}{3}}$ norm.
\[
\| (\ell_n - \chi_N) W(x - \ell_n t) \|_{L^{\frac{4}{3}}} \lesssim t^{-\frac{5}{10} + \alpha}.
\] (4.10)

**Proof.** From (4.7), the definition of $\chi_N$ and change of variable,
\[
\int_{\mathbb{R}^2} |\ell_n - \chi_N|^{\frac{4}{3}} |W(x - \ell_n t)|^{\frac{4}{3}} \, dx \lesssim \int_{|x| \geq \delta t} (x)^{-\frac{4}{3}(4 + \alpha)} \, dx
\]
\[
\lesssim \int_{|x| \geq \delta t} \frac{r^4}{(1 + r^2)^{(4 + \alpha)}} \, dr \lesssim t^{-\frac{4}{3}(4 + \alpha)},
\]
which implies (4.8). The proof of (4.9) and (4.10) follows from similar arguments and it is omitted. \(\square\)

Under the bootstrap setting (4.3), we prove the following estimates.

**Proposition 4.4.** There exists $0 < \nu \ll 1$ such that the following hold.
(i) Coercivity.
\[
\nu \| \tilde{\varepsilon}(t) \|_{H^l}^2 \leq K(t) + \nu^{-1} t^{-\frac{2}{3}}.
\] (4.11)

(ii) Time variation of $K$.
\[
- \frac{d}{dt} (t^2 K)(t) \leq \nu^{-1} t^{-\frac{2}{3}}.
\] (4.12)

**Proof.** Proof of (i). We set
\[
F_{\Omega}(t) = \int_{\Omega} |(\nabla \varepsilon)^2 + \eta^2 + 2(\chi_N \partial_{x_1} \varepsilon) \eta| \, dx,
F_{\Omega C}(t) = \int_{\Omega C} |(\nabla \varepsilon)^2 + \eta^2| \, dx.
\]
First, from $|\chi_N| \leq \ell < 1$,
\[
F_{\Omega} = \ell \int_{\Omega} \left( \frac{\chi_N}{\ell} \partial_{x_1} \varepsilon + \eta \right)^2 \, dx + \int_{\Omega} \left( 1 - \frac{\chi_N}{\ell} \right) \left( \partial_{x_1} \varepsilon \right)^2 + (1 - \ell) \eta^2 + |\nabla \varepsilon|^2 \right) \, dx
\]
\[
\geq \ell \int_{\Omega} \left( \frac{\chi_N}{\ell} \partial_{x_1} \varepsilon + \eta \right)^2 \, dx + (1 - \ell) \int_{\Omega} \left( |\nabla \varepsilon|^2 + \eta^2 \right) \, dx.
\] (4.13)

Second, by (4.3) and Young’s inequality for product,
\[
|G(t)| \lesssim \left( \sum_{(n,k) \in I^0} |b_{n,k}|^2 \right) \| \tilde{\varepsilon} \|_{H^l} \lesssim t^{-\frac{1}{2}}.
\] (4.14)

Therefore the coercivity property for $K$ is a consequence of the following stronger coercivity property,
\[
F(t) \geq F_{\Omega}(t) + \mu F_{\Omega C}(t) - \mu^{-1} \ell^{-\frac{2}{3}} - \mu^{-1} t^{-\frac{1}{3}} \| \varepsilon \|_{H^l}^2 - \mu^{-1} \| \varepsilon \|_{H^l}^3.
\] (4.15)

The coercivity property (4.15) is a standard consequence of the localized coercivity property around one excited solitary wave $q_0$ in Proposition 2.3 with the orthogonality relations (3.16), and an elementary localization argument. We refer to the proof of Proposition 4.2 (ii) of [14] and Lemma 5.4 (ii) of [21] for a similar proof.

Proof of (ii). **Step 1.** Time variation of $E$. We claim
\[
\frac{d}{dt} E = 2 \int_{\mathbb{R}^2} \text{Mod}_1 (\Delta \varepsilon + f'(U) \varepsilon) \, dx - 2 \int_{\mathbb{R}^2} \eta \text{Mod}_2 dx + 2 \int_{\mathbb{R}^2} \eta G_3 dx
\]
\[
+ 2 \sum_{n=1}^N \int_{\mathbb{R}^2} \ell_n \partial_{x_1} q_n (f(U + V + \varepsilon) - f(U + V) - f'(U + V) \varepsilon) \, dx + O(t^{-\frac{2}{3}}).
\] (4.16)
We decompose
\[ \frac{d}{dt} \mathcal{E} = I_1 + I_2 + I_3 + I_4, \]
where
\[ I_1 = -2 \int_{\mathbb{R}^3} \partial_t U (f(U + V + \varepsilon) - f(U + V) - f'(U + V)\varepsilon) dx, \]
\[ I_2 = -2 \int_{\mathbb{R}^3} \partial_t V (f(U + V + \varepsilon) - f(U + V) - f'(U + V)\varepsilon) dx, \]
\[ I_3 = 2 \int_{\mathbb{R}^3} \partial_t \varepsilon (-\Delta \varepsilon - f(U + V + \varepsilon) + f(U + V)) dx, \]
\[ I_4 = 2 \int_{\mathbb{R}^3} \eta \partial_t \eta dx. \]

Estimate on $I_1$. We claim
\[ I_1 = 2 \sum_{n=1}^{N} \int_{\mathbb{R}^3} \ell_n \partial_{x_1} Q_n (f(U + V + \varepsilon) - f(U + V) - f'(U + V)\varepsilon) dx. \] (4.17)
By direct computation, we obtain
\[ \partial_t U = \sum_{n=1}^{N} \partial_t Q_n = - \sum_{n=1}^{N} \ell_n \partial_{x_1} Q_n, \]
which implies (4.17).

Estimate on $I_2$. We claim
\[ |I_2| \lesssim t^{-\frac{35}{44}}. \] (4.18)
By direct computation,
\[ \partial_t V = \sum_{(n,k) \in I^0} b_{n,k} \Psi_{n,k} - \sum_{(n,k) \in I^0} b_{n,k} \ell_n \partial_{x_1} \Psi_{n,k}. \]
Thus, from (3.9), (3.18), (4.3), and the decay properties of $\Psi_{n,k}$,
\[ |I_2| \lesssim \sum_{(n,k) \in I^0} \int_{\mathbb{R}^3} \left( |b_{n,k}| |\Psi_{n,k}| + |b_{n,k}| |\ell_n \partial_{x_1} \Psi_{n,k}| \right) \left( (|U|^4 + |V|^4) |\varepsilon|^2 + |\varepsilon|^3 \right) dx \]
\[ \lesssim \sum_{(n,k) \in I^0} \left( \|\varepsilon\|_{H^4} + |b_{n,k}| + t^{-4} \right) \|\varepsilon\|_{H^4} \lesssim t^{-\frac{35}{44}} + t^{-\frac{35}{44}} + t^{-\frac{35}{44}}, \]
which implies (4.18).

Estimate on $I_3$. We prove the following estimate
\[ I_3 = -2 \int_{\mathbb{R}^3} \eta (\Delta \varepsilon + f(U + V + \varepsilon) - f(U + V)) dx \]
\[ + 2 \int_{\mathbb{R}^3} \text{Mod}_1 (\Delta \varepsilon + f'(U)\varepsilon) dx + O(t^{-7}). \] (4.19)
By direct computation and (3.17),
\[ I_3 = -2 \int_{\mathbb{R}^3} \eta (\Delta \varepsilon + f(U + V + \varepsilon) - f(U + V)) dx \]
\[ + 2 \int_{\mathbb{R}^3} \text{Mod}_1 (\Delta \varepsilon + f'(U)\varepsilon) dx + I_{3,1} + I_{3,2}, \]
where
\[ I_{3,1} = 2 \int_{\mathbb{R}^3} \text{Mod}_1 (f'(U + V) - f'(U)) \varepsilon dx, \]
\[ I_{3,2} = 2 \int_{\mathbb{R}^3} \text{Mod}_1 (f(U + V + \varepsilon) - f(U + V) - f'(U + V)\varepsilon) dx. \]
By the Cauchy-Schwarz inequality, (3.8), (3.9), (3.18) and (4.3),
\[
|\mathcal{I}_{3,1}| \lesssim \|\text{Mod}_1\|_{L^\infty} \|f'(U + V) - f'(U)\|_{L^2} \|\varepsilon\|_{L^4} \\
\lesssim \|\varepsilon\|_H + \sum_{(n,k) \in \mathcal{P}} |b_{n,k}|^2 + t^{-10} \lesssim t^{-\frac{2}{3}} + t^{-9} + t^{-10} \lesssim t^{-7}.
\]
\[
|\mathcal{I}_{3,2}| \lesssim \|\text{Mod}_1\|_{L^\infty} \|f(U + V + \varepsilon) - f(U + V) - f'(U + V)\varepsilon\|_{L^4} \\
\lesssim \left(\|\varepsilon\|_H + \sum_{(n,k) \in \mathcal{P}} |b_{n,k}|^2 + t^{-4}\right) \|\varepsilon\|_H^2 \lesssim t^{-\frac{2}{3}} + t^{-\frac{4}{5}} + t^{-\frac{7}{5}} \lesssim t^{-7}.
\]
We see that (4.19) follows from above estimates.

**Estimate on $\mathcal{I}_4$.** We claim
\[
\mathcal{I}_4 = 2 \int_{\mathbb{R}^5} \eta(\Delta \varepsilon + f(U + V + \varepsilon) - f(U + V))dx \\
- 2 \int_{\mathbb{R}^5} \eta \text{Mod}_2 dx + 2 \int_{\mathbb{R}^5} \eta G_3 dx + O(t^{-\frac{39}{40}}).
\]

We compute, using again (3.17),
\[
\mathcal{I}_4 = 2 \int_{\mathbb{R}^5} \eta(\Delta \varepsilon + f(U + V + \varepsilon) - f(U + V))dx \\
- 2 \int_{\mathbb{R}^5} \eta \text{Mod}_2 dx + 2 \int_{\mathbb{R}^5} \eta G_3 dx + 2 \int_{\mathbb{R}^5} \eta G_1 dx + 2 \int_{\mathbb{R}^5} \eta (G_2 - G_3) dx.
\]

Indeed, from (3.11), (3.12) and (4.3), we obtain
\[
\left| \int_{\mathbb{R}^5} \eta G_1 dx \right| + \left| \int_{\mathbb{R}^5} \eta (G_2 - G_3) dx \right| \lesssim \|\eta\|_{L^3} (\|G_1\|_{L^2} + \|G_2 - G_3\|_{L^2}) \\
\lesssim \|\eta\|_{L^3} \left( \sum_{(n,k) \in \mathcal{P}} |b_{n,k}|^2 + t^{-4} \right) \lesssim t^{-\frac{39}{40}},
\]
which implies (4.20).

In conclusion of estimates (4.17), (4.18), (4.19) and (4.20), we obtain (4.16).

**Step 2.** Time variation of $\mathcal{P}$. We claim
\[
\frac{d}{dt}\mathcal{P} = - \frac{1}{(1 - 2\delta)} \int_\Omega (\eta^2 + (\partial_{x_1} \varepsilon)^2 + 2\frac{x_1}{t}(\partial_{x_1} \varepsilon)\eta - |\nabla \varepsilon|^2) dx \\
- \sum_{n=1}^N \int_{\mathbb{R}^5} \chi_N \partial_{x_1} Q_n (f(U + V + \varepsilon) - f(U + V) - f'(U + V)\varepsilon) dx \\
- \sum_{n=1}^N \chi_N (\eta \partial_{x_1} \text{Mod}_1 + \text{Mod}_2 \partial_{x_1} \varepsilon) dx + 2 \int_{\mathbb{R}^7} \chi_N (\partial_{x_1} \varepsilon) G_3 dx + O(t^{-\frac{19}{20}}).
\]

We decompose
\[
\frac{d}{dt}\mathcal{P} = 2 \int_{\mathbb{R}^5} (\partial_{x_1} \chi_N)(\partial_{x_1} \varepsilon)\eta dx + 2 \int_{\mathbb{R}^5} \chi_N \partial_{x_1} (\partial_{x_1} \varepsilon) \eta dx = \mathcal{I}_5 + \mathcal{I}_6.
\]

**Estimate on $\mathcal{I}_5$.** From (4.6), we obtain
\[
\mathcal{I}_5 = - \frac{2}{(1 - 2\delta)} \int_\Omega \frac{x_1}{t}(\partial_{x_1} \varepsilon)\eta dx.
\]

\[
\text{(4.23)}
\]
Estimate on $I_6$. We claim

$$I_6 = - \frac{1}{(1-t^n) t} \int _{\Omega} (\eta^2 + (\partial_x \eta)^2 - |\nabla \eta|^2) d\Omega$$

$$- 2 \sum_{n=1}^{N} \int _{\mathbb{R}^3} \chi_N \partial_x Q_n(f(U + V + \varepsilon) - f(U + V) - f'(U + V)\varepsilon) d\Omega$$

$$- 2 \int _{\mathbb{R}^3} \chi_N (\eta \partial_x \chi) d\Omega + 2 \int _{\mathbb{R}^3} \chi_N (\eta \partial_x \chi) G d\Omega + O(t^{-\frac{n}{10}}).$$

(4.24)

By direct computation, we decompose

$$I_6 = 2 \int _{\mathbb{R}^3} \chi_N (\partial_x (\partial_t \varepsilon)) \eta d\Omega + 2 \int _{\mathbb{R}^3} \chi_N (\partial_x \varepsilon) \partial_t \eta d\Omega = I_{6,1} + I_{6,2}.$$ From (3.17), (4.6), and integration by parts,

$$I_{6,1} = - \frac{1}{(1-t^n) t} \int _{\Omega} \eta^2 d\Omega - 2 \int _{\mathbb{R}^3} \chi_N (\eta \partial_x \chi) d\Omega. \quad (4.25)$$

Using again (3.17), (4.6), and integration by parts,

$$I_{6,2} = - \frac{1}{(1-t^n) t} \int _{\Omega} ((\partial_x \varepsilon)^2 - |\nabla \varepsilon|^2) d\Omega - 2 \int _{\mathbb{R}^3} \chi_N (\partial_x \varepsilon) \partial_t \eta d\Omega$$

$$+ 2 \int _{\mathbb{R}^3} \chi_N (\partial_x \varepsilon) G d\Omega + 2 \int _{\mathbb{R}^3} \chi_N (\partial_x \varepsilon) (f(U + V + \varepsilon) - f(U + V)) d\Omega$$

$$+ 2 \int _{\mathbb{R}^3} \chi_N (\partial_x \varepsilon) G d\Omega + 2 \int _{\mathbb{R}^3} \chi_N (\partial_x \varepsilon) (G_2 - G_3) d\Omega.$$ (4.26)

Note that, by integration by parts,

$$2 \int _{\mathbb{R}^3} \chi_N \partial_x \varepsilon (f(U + V + \varepsilon) - f(U + V)) d\Omega$$

$$= - 2 \sum_{n=1}^{N} \int _{\mathbb{R}^3} \chi_N \partial_x Q_n(f(U + V + \varepsilon) - f(U + V) + f'(U + V)\varepsilon) d\Omega + I_0^1 + I_0^2,$$

where

$$I_0^1 = - \frac{2}{(1-t^n) t} \int _{\Omega} (F(U + V + \varepsilon) - F(U + V) - f(U + V)\varepsilon) d\Omega,$$

$$I_0^2 = - 2 \sum_{(n,k) \in I^u} b_{n,k} \int _{\mathbb{R}^3} \chi_N \partial_x \Psi_{n,k} (f(U + V + \varepsilon) - f(U + V) - f'(U + V)\varepsilon) d\Omega.$$ From (3.10), (4.3), the decay properties of $Q_n$ and $\Psi_{n,k}$ and Sobolev embedding theorem,

$$|I_0^1| \lesssim t^{-\frac{n}{10}} \int _{\Omega} (|U| + |V|) |\varepsilon|^2 d\Omega + t^{-1} \|\varepsilon\|_{H^1}^{\frac{n}{10}} \lesssim t^{-\frac{n}{10}}.$$ From (3.9), (4.3) and Sobolev embedding theorem,

$$|I_0^2| \lesssim \sum_{(n,k) \in I^u} |b_{n,k}| \int _{\mathbb{R}^3} |\partial_x \Psi_{n,k}| \left( (|U| + |V| |\varepsilon|^2 + |\varepsilon|^2) \right) d\Omega$$

$$\lesssim \sum_{(n,k) \in I^u} |b_{n,k}| \left( \|\varepsilon\|_{H^{\frac{1}{2}}}^2 \|\varepsilon\|_{H^{\frac{3}{2}}} \right) \lesssim t^{-\frac{n}{10}}.$$
Next, by the Cauchy-Schwarz inequality, (3.11), (3.12) and (4.3),
\[
\left| \int _{\mathbb{R}^5} \chi _{N}(\partial _{x_1}\varepsilon )G_{1}dx \right| + \left| \int _{\mathbb{R}^5} \chi _{N}(\partial _{x_1}\varepsilon ) (G_2 - G_3)dx \right|
\]
\[
\lesssim \|\eta\|_{L^2} (\|G_1\|_{L^2} + \|G_2 - G_3\|_{L^2}) \lesssim \|\eta\|_{L^2} \left( \sum_{(n,k)\in I^0} |b_{n,k}|^2 + t^{-4} \right) \lesssim t^{-\frac{31}{40}}.
\]

Gathering above estimates, we obtain (4.24).

In conclusion of estimates (4.23) and (4.24), we obtain (4.21).

**Step 3.** Time variation of \( G \). We claim
\[
\frac{d}{dt} G = 2 \int _{\mathbb{R}^5} \eta G_3 dx + 2 \sum_{n=1}^N \int _{\mathbb{R}^5} \ell_n (\partial _{x_1} \varepsilon ) G_{3,n} dx + O(t^{-7}). \tag{4.27}
\]

By direct computation and integration by parts,
\[
\frac{d}{dt} G = 2 \int _{\mathbb{R}^5} \eta G_3 dx + 2 \sum_{n=1}^N \int _{\mathbb{R}^5} \ell_n (\partial _{x_1} \varepsilon ) G_{3,n} dx + I_7 + I_8 + I_9,
\]
where
\[
I_7 = \sum_{n=1}^N \sum_{k,k'=1}^{K_n} b_{n,k} b_{n,k'} \int _{\mathbb{R}^5} \varepsilon (f''(Q_n) \Psi _{n,k} \Psi _{n,k'}) dx,
\]
\[
I_8 = - \sum_{n=1}^N \sum_{k=1}^{K_n} b_{n,k} \int _{\mathbb{R}^5} \Psi _{n,k} G_{3,n} dx \quad \text{and} \quad I_9 = - \sum_{n \neq n'} \sum_{k=1}^{K_n} b_{n,k} \int _{\mathbb{R}^5} \Psi _{n,k} G_{3,n'} dx.
\]

First, from (3.18), (4.3) and Sobolev embedding,
\[
|I_7| \lesssim \sum_{n=1}^N \sum_{k,k'=1}^{K_n} |b_{n,k} b_{n,k'}| \left| \int _{\mathbb{R}^5} \varepsilon |f''(Q_n) \Psi _{n,k} \Psi _{n,k'}| dx \right|
\]
\[
\lesssim \sum_{(n,k)\in I^0} |b_{n,k}|^2 H (\|\varepsilon\|_H + \sum_{(n,k)\in I^0} |b_{n,k}|^2 + t^{-4}) \lesssim t^{-\frac{31}{40}}.
\]

Next, from (iii) of Lemma 2.1, we have for all \((n,k)\in I^0\),
\[
\int _{\mathbb{R}^5} \Psi _{n,k} G_{3,n} dx = \frac{1}{2} \sum_{k',k''=1}^{K_n} b_{n,k'} b_{n,k''} \int _{\mathbb{R}^5} \Psi _{n,k} (f''(Q_n) \Psi _{n,k'} \Psi _{n,k''}) dx
\]
\[
= \frac{1}{2} \sum_{k',k''=1}^{K_n} b_{n,k'} b_{n,k''} \int _{\mathbb{R}^5} f''(Q) (\Psi _{k'} \Psi _{k''}) dx = 0.
\]

It follows that \( I_8 = 0 \). Last, from Lemma 3.1, (3.18) and (4.3),
\[
|I_9| \lesssim \sum_{n \neq n'} \sum_{k=1}^{K_n} \sum_{k'=1}^{K_{n'}} |b_{n,k} b_{n',k'}| \left| \int _{\mathbb{R}^5} \Psi _{n,k} (f''(Q_{n'}) \Psi _{n',k'} \Psi _{n',k'}) dx \right|
\]
\[
\lesssim t^{-1} \sum_{(n,k)\in I^0} |b_{n,k}|^2 \left( \|\varepsilon\|_H + \sum_{(n,k)\in I^0} |b_{n,k}|^2 + t^{-4} \right) \lesssim t^{-7}.
\]

Gathering above estimates, we obtain (4.27).

**Step 4.** Conclusion. Combining estimates (4.16), (4.21) and (4.27), we obtain
\[
\frac{d}{dt} K = J_1 + J_2 + J_3 + J_4 + J_5 + O(t^{-\frac{31}{40}}),
\]
where
\[
J_1 = - \frac{1}{(1-2\delta)} \int _{\Omega} \left( \eta^2 + (\partial _{x_1} \varepsilon )^2 + 2 \frac{2^a}{t} (\partial _{x_1} \varepsilon )\eta - |\nabla \varepsilon|^2 \right) dx,
\]
\[ J_2 = 2 \int_{\mathbb{R}^5} (\Delta \varepsilon + f'(U)\varepsilon) \text{Mod}_1 dx - 2 \int_{\mathbb{R}^5} \chi_N(\partial_x \varepsilon) \text{Mod}_2 dx, \]

\[ J_3 = 2 \sum_{n=1}^{N} \int_{\mathbb{R}^5} (\ell_n - \chi_N) \partial_{x_1} Q_n (f(U + V + \varepsilon) - f(U + V) - f'(U + V)\varepsilon) dx. \]

\[ J_4 = -2 \int_{\mathbb{R}^5} \eta (\text{Mod}_2 + \chi_N \partial_{x_1} \text{Mod}_1) dx, \]

\[ J_5 = 2 \sum_{n=1}^{N} \int_{\mathbb{R}^5} (\chi_N - \ell_n) (\partial_{x_1} \varepsilon) G_{3,n} dx. \]

**Estimate on \( J_1 \).** For \( 0 < \delta \ll 1 \) small enough, we claim
\[
- J_1 \leq 2t^{-1} \mathcal{K} + O(t^{-\#}).
\] (4.28)

From (4.13) and (4.14), we obtain
\[
-(1 - 2\delta)\ell t J_1 = \ell \int_{\Omega} \left( \frac{\chi_N}{\ell} \partial_{x_1} \varepsilon + \eta \right)^2 dx + \int_{\Omega} \left( \left( 1 - \frac{\chi_N^2}{\ell} \right) (\partial_{x_1} \varepsilon)^2 - |\nabla \varepsilon|^2 \right) dx \\
+ (1 - \ell) \int_{\Omega} \varepsilon^2 dx + 2 \int_{\Omega} \frac{(\partial_{x_1} \ell N - \chi_N) (\partial_{x_1} \varepsilon) \eta dx} \\
\leq \mathcal{F}_\Omega + O\left( \delta \| \varepsilon \|_{H(\Omega)}^2 \right) \leq (1 + O(\delta)) \mathcal{F}_\Omega \leq (1 + O(\delta)) \mathcal{K} + O(t^{-\#}).
\]

Letting \( \delta \) small enough in above estimate, we obtain (4.28).

**Estimate on \( J_2 \).** By integration by parts, (4.6) and \(- (1 - \ell_n^2) \Psi_{n,k} - f'(Q_n) \Psi_{n,k} = 0 \) for all \((n, k) \in I^0\), we obtain
\[
J_2 = J_{2,1} + J_{2,2} + J_{2,3},
\]

where
\[
J_{2,1} = -\frac{2}{(1 - 2\delta)\ell} t \int_{\Omega} \varepsilon \text{Mod}_2 dx,
\]

\[ J_{2,2} = 2 \sum_{(n, k) \in I^0} \partial_{x_1} \varepsilon (\ell_n - \chi_N) \ell_n \partial_{x_1} \Psi_{n,k} dx,
\]

\[ J_{2,3} = 2 \sum_{(n, k) \in I^0} \partial_{x_1} \varepsilon (f'(U) - f'(Q_n)) \Psi_{n,k} dx.
\]

From the Cauchy-Schwarz inequality, (3.18), (4.3), (4.8) and the decay properties of \( \Psi_{n,k} \),
\[
|J_{2,1}| \lesssim t^{-1} \| \varepsilon \|_{L_{x}^{10}(\Omega)} \sum_{(n, k) \in I^0} |\partial_{x_1} \Psi_{n,k}| \| \partial_{x_1} \Psi_{n,k} \|_{L_{x}^{10}(\Omega)} \lesssim t^{-7},
\]
\[
\lesssim t^{-\frac{2}{7}} \| \varepsilon \|_H \left( \| \varepsilon \|_H + \sum_{(n, k) \in I^0} |b_{n,k}|^2 + t^{-4} \right) \lesssim t^{-7},
\]
\[
|J_{2,2}| \lesssim \| \varepsilon \|_{L_{x}^{10}} \sum_{(n, k) \in I^0} |b_{n,k}| \| (\ell_n - \chi_N) \partial_{x_1} \Psi_{n,k} \|_{L_{x}^{10}} \lesssim t^{-7},
\]
\[
|\ell_n^2| |\Psi_{n,k}| \| |Q_n| \|_{H(\Omega)} \| Q_n^\# \|_{L_{x}^{10}} \leq t^{-2}.
\]

From (i) of Lemma 3.1 and (3.9),
\[
\| f'(U) - f'(Q_n) \Psi_{n,k} \|_{L_{x}^{10}} \lesssim \sum_{n' \neq n} \| Q_n^\# | Q_{n'} | \Psi_{n,k} \|_{L_{x}^{10}} + \sum_{n' \neq n} \| Q_{n'}^\# \Psi_{n,k} \|_{L_{x}^{10}} \lesssim t^{-2}.
\]
Thus, by (3.18), (4.3) and Cauchy-Schwarz inequality,

$$|J_{2,3}| \lesssim \|\varepsilon\|_{L^\infty} \sum_{(n,k) \in I^0} |b_{n,k}| \|(f'(U) - f'(Q_n))\Psi_{n,k}\|_{L^\infty}$$

$$\lesssim t^{-2}\|\varepsilon\|_H \left(\|\varepsilon\|_H + \sum_{(n,k) \in I^0} |b_{n,k}|^2 + t^{-4}\right) \lesssim t^{-7}.$$ 

Gathering above estimates, we obtain

$$|J_2| \lesssim |J_{2,1}| + |J_{2,2}| + |J_{2,3}| \lesssim t^{-7}. \quad (4.29)$$

**Estimate on $J_5$.** From (3.9), (4.3), (4.10), Cauchy-Schwarz inequality and the decay properties of $Q_n$,

$$|J_5| \lesssim \sum_{n=1}^N \|\ell_n - \chi N\partial_{x_1}Q_n\|_{L^\infty} \left|\left(\|U|^3 + |V|^3\right) \varepsilon^2 + |\varepsilon|^2\right|_{L^\infty}$$

$$\lesssim \|\varepsilon\|_H^2 \sum_{n=1}^N \|\ell_n - \chi N\partial_{x_1}Q_n\|_{L^\infty} \lesssim t^{-7}. \quad (4.30)$$

**Estimate on $J_4$.** Note that,

$$\text{Mod}_2 + \chi_N\partial_{x_1}\text{Mod}_1 = - \sum_{(n,k) \in I^0} \tilde{b}_{n,k} (\ell_n - \chi N)\partial_{x_1}\Psi_{n,k}.$$ 

Therefore, by Cauchy-Schwarz inequality, (3.18), (4.3) and (4.9),

$$|J_4| \lesssim \|\eta\|_{L^2} \sum_{(n,k) \in I^0} |\tilde{b}_{n,k}| \|\ell_n - \chi N\partial_{x_1}\Psi_{n,k}\|_{L^2}$$

$$\lesssim t^{-\frac{2}{7}}\|\varepsilon\|_H \left(\|\varepsilon\|_H + \sum_{(n,k) \in I^0} |b_{n,k}|^2 + t^{-4}\right) \lesssim t^{-7}. \quad (4.31)$$

**Estimate on $J_6$.** From (4.3), (4.9) and Cauchy-Schwarz inequality,

$$|J_6| \lesssim \|\partial_{x_1}\varepsilon\|_{L^2} \sum_{(n,k) \in I^0} |b_{n,k}|^2 \|\ell_n - \chi N\| f''(Q_n)\Psi_{n,k} - |l_{n,k}|_{L^2} \lesssim t^{-7}. \quad (4.32)$$

In conclusion of estimates (4.28), (4.29), (4.30), (4.31) and (4.32), for $\delta$ small enough, we obtain

$$-\frac{d}{dt}K \leq 2t^{-1}K + O(t^{-\frac{39}{10}}), \quad (4.33)$$

which implies (4.12).

## 4.4. End of the proof of Proposition 4.2.

We start by improving all the estimates in (4.3) except the ones for the unstable directions $(a_{n,j}^+)(n,j) \in I$.

**Lemma 4.5** (Closing estimates except $(a_{n,j}^+)(n,j) \in I$). For all $t \in [T_*, T_m]$,

$$\|\varepsilon(t)\|_W \leq t^{\frac{1}{6}}, \quad \|\varepsilon(t)\|_H \leq t^{-\frac{19}{10}}, \quad (4.34)$$

$$\sum_{(n,k) \in I^0} |b_{n,k}(t)| \leq t^{-\frac{39}{10}}, \quad \sum_{(n,j) \in I} |a_{n,j}^-(t)|^2 \leq t^{-6}. \quad (4.35)$$

**Proof.** **Step 1.** Bound on the $W$ norm. First, from (3.18) and (4.3),

$$\|\text{Mod}_1\|_W \lesssim \sum_{(n,k) \in I^0} |b_{n,k}| \|\Psi_{n,k}\|_W \lesssim t^{\frac{3}{8}},$$

$$\|\text{Mod}_2\|_W \lesssim \sum_{(n,k) \in I^0} |\tilde{b}_{n,k}| \|\partial_{x_1}\Psi_{n,k}\|_W \lesssim t^{\frac{11}{8}}.$$
Second, from Taylor formula,
\[ |R_\epsilon| \lesssim \hat{(U|\hat{\epsilon}| + |V|\hat{\epsilon}|)|\epsilon| + |\hat{\epsilon}|, \]
\[ |\nabla R_\epsilon| \lesssim |\nabla \epsilon|(U|\hat{\epsilon}| + |V|\hat{\epsilon}| + |\epsilon| + |U + V||\hat{\epsilon}| + |\nabla (U + V)|\hat{\epsilon}|, \]
where \( R_\epsilon = f(U + V + \epsilon) - f(U + V) \). Therefore, from (1.5) and (4.3),
\[ \|R_\epsilon\|_{W^0} \lesssim \|\hat{\epsilon}\|_{H^1} \|\epsilon\|_{W^1} + t^\frac{1}{2} \|\epsilon\|_{H^1} \lesssim t^{-\frac{33}{29}}. \]

Third, from (4.3) and the decay properties of \( Q_n \) and \( \Psi_{n,k} \),
\[ \|G\|_{W^0} \lesssim \sum_{i=1}^3 \|G_{1,i}\|_{W^0} + \|G_2\|_{W^0} \lesssim t^{-\frac{7}{2}} + t^{\frac{1}{2}} \sum_{(n,k) \in I^0} |b_{n,k}|^2 \lesssim t^{-3}. \]

Gathering above estimates and Duhamel’s principle,
\[ \|\bar{\epsilon}(t)\|_W \lesssim \|\bar{\epsilon}(T_m)\|_W + \int_t^{T_m} (\|\text{Mod}_1(s)\|_{W^1} + \|\text{Mod}_2(s)\|_{W^0}) \, ds \]
\[ + \int_t^{T_m} (\|R_\epsilon(s)\|_{W^0} + \|G(s)\|_{W^0}) \, ds \lesssim t^{-\frac{33}{29}}, \]

Letting \( T_0 \) large enough, we conclude (4.34) on \( W \) norm.

**Step 2.** Bound on the energy norm. First, from (4.1) and (4.3) at \( t = T_m \), we obtain
\[ |\mathcal{K}(T_m)| \lesssim \|\mathcal{E}(T_m)\| + |\mathcal{P}(T_m)| + |\mathcal{G}(T_m)| \lesssim T_m^{-6}. \]

Thus, integrating (4.12) on \([t, T_m]\) for all \( t \in [T_*, T_m]\), we obtain
\[ \mathcal{K}(t) \lesssim (t/T_m)^{-2} \mathcal{K}(T_m) + \nu^{-1} t^{-\frac{33}{29}} \lesssim t^{-\frac{33}{29}}. \]

Using (4.11), we have
\[ \|\bar{\epsilon}(t)\|_{H^1}^2 \lesssim \mathcal{K}(t) + t^{-\frac{33}{29}} \lesssim t^{-\frac{33}{29}}. \]

Letting \( T_0 \) large enough, we conclude (4.34) on energy norm.

**Step 3.** Parameter estimates. First, from (4.1) and (4.3) at \( t = T_m \), we obtain
\[ \sum_{(n,k) \in I^0} |b_{n,k}(T_m)| + \sum_{(n,j) \in I^0} |a_{n,j}^-(T_m)| \lesssim T_m^{-7/2}. \quad (4.36) \]

Integrating (3.18) on \([t, T_m]\) for all \( t \in [T_*, T_m]\) and using (4.3), (4.36), we obtain
\[ \sum_{(n,k) \in I^0} |b_{n,k}(t)| \lesssim \sum_{(n,k) \in I^0} |b_{n,k}(T_m)| + \int_t^{T_m} s^{-\frac{33}{29}} \, ds \lesssim t^{-\frac{13}{29}}. \]

Letting \( T_0 \) large enough, we conclude (4.35) for \( (b_{n,k})_{(n,k) \in I^0} \).

Now we prove the bound on \( (a_{n,j}^-(t))_{(n,j) \in I} \). By direct computation, (3.19) and (4.3),
\[ \frac{d}{dt} (e^{-2\alpha_{n,j} t} (a_{n,j}(t))^2) = e^{-2\alpha_{n,j} t} \dot{a}_{n,j}(t) \left( \frac{d}{dt} a_{n,j}^- - \alpha_{n,j} a_{n,j}^- \right) \]
\[ = e^{-2\alpha_{n,j} t} O \left( (a_{n,j}^-)(\|\epsilon\|_{H^1}^2 + \sum_{(n,k) \in I^0} |b_{n,k}|^2 + t^{-4}) \right) \]
\[ = e^{-2\alpha_{n,j} t} O \left( t^{-\frac{13}{29}} \right). \]

Integrating on \([t, T_m]\) for all \( t \in [T_*, T_m]\) and using (4.36), we obtain
\[ (a_{n,j}^-(t))^2 \lesssim e^{-2\alpha_{n,j}(T_m-t)} (a_{n,j}^-(T_m))^2 + \int_t^{T_m} e^{2\alpha_{n,j}(t-s)} s^{-\frac{33}{29}} \, ds \lesssim t^{-\frac{33}{29}}, \]
which implies the estimates on \( (a_{n,j}^-)_{(n,j) \in I} \) in (4.35) for taking \( T_0 \) large enough. \( \square \)
Last, we prove the existence of suitable parameters \( a_m = (a_{n,j}^m(t))_{(n,j) \in I} \) by contradiction, using a topological argument.

Lemma 4.6 (Control of unstable directions). There exist \( a_m = (a_{n,j}^m) \in B_{\mathbb{R}^{|I|}}(T_m^{-3}) \) such that, for \( T_0 \) large enough, \( T_\ast(a_m) = T_0 \). In particular, the solution \( \tilde{u}_m \) with the initial data \( \tilde{u}_m(T_m) \) given by Lemma 4.1 satisfies (4.2).

Note that Lemma 4.6 completes the proof of Proposition 4.2.

Proof. Let

\[
a(t) = \sum_{(n,j) \in I} (a_{n,j}^+(t))^2 \quad \text{and} \quad \bar{\alpha} = \min_{(n,j) \in I} \alpha_{n,j} > 0.
\]

We claim the following transversality property, for any \( t \in [T_\ast, T_m] \) where it holds \( a(t) = t^{-6} \), we have

\[
\frac{d}{dt} t^6 a(t) \leq -\bar{\alpha}.
\]

Indeed, from (3.19) and (4.3), for any \( t \in [T_\ast, T_m] \) where it holds \( a(t) = t^{-6} \), we obtain

\[
\frac{d}{dt} a(t) = -2 \sum_{(n,j) \in I} \alpha_{n,j} (a_{n,j}^+(t))^2 + O\left(\|\tilde{\xi}(t)\|_{\mathcal{H}}^3 + \sum_{(n,j) \in I} |a_{n,j}^+(t)|^2 + t^{-\frac{3\alpha}{2}}\right)
\]

\[
+ O\left(\sum_{(n,j) \in I} \sum_{(n,k) \in I^0} |a_{n,j}^+(t)||b_{n,k}(t)|^2\right) \leq -2\bar{\alpha} t^{-6} + O(t^{-\frac{3\alpha}{2}}),
\]

which implies (4.37) for \( T_0 \) large enough. This transversality property is enough to justify the existence of at least a couple \( a_m = (a_{n,j}^m) \in B_{\mathbb{R}^{|I|}}(T_m^{-3}) \) such that \( T_\ast(a_m) = T_0 \).

The proof is by contradiction, we assume that for any \( a_m = (a_{n,j}^m) \in B_{\mathbb{R}^{|I|}}(T_m^{-3}) \), it holds \( T_0 < T_\ast(a_m) \leq T_m \). Then, a construction follows from the following discussion (see for instance more details in [5, 6] and [14, Lemma 4.2]).

Continuity of \( T_\ast \). The above transversality property (4.37) implies that the map

\[
a_m \in B_{\mathbb{R}^{|I|}}(T_m^{-3}) \mapsto T_\ast(a_m) \in (T_0, T_m)
\]

is continuous and

\[
T_\ast(a_m) = T_m \quad \text{for} \quad a_m \in S_{\mathbb{R}^{|I|}}(T_m^{-3}).
\]

Construction of a retraction. We define

\[
\Gamma : B_{\mathbb{R}^{|I|}}(T_m^{-3}) \mapsto S_{\mathbb{R}^{|I|}}(T_m^{-3})
\]

\[
a_m \mapsto (a_{n,j}^+(T_\ast))_{(n,j) \in I}.
\]

From what precedes, \( \Gamma \) is continuous. Moreover, \( \Gamma \) restricted to \( S_{\mathbb{R}^{|I|}}(T_m^{-3}) \) is the identity. The existence of such a map is contradictory with the no retraction theorem for continuous maps from the ball to the sphere. Therefore, the existence of \( a_m \in B_{\mathbb{R}^{|I|}}(T_m^{-3}) \) have proved. Then, the uniform estimates (4.2) is a consequence of Lemma 4.5. The proof of Lemma 4.6 is complete. 

\[\square\]

References

[1] T. Aubin, Problemes isoperimétriques et espaces de Sobolev. J. Differ. Geometry 11 (1976), 573-598.
[2] V. Combet, Multi-soliton solutions for the supercritical gKdV equations. Comm. Partial Differential Equations 36 (2011), 380-419.
[3] R. Côte and Y. Martel, Multi-travelling waves for the nonlinear Klein-Gordon equation. Trans. Amer. Math. Soc., 370: 74617487, 2018.
[4] R. Côte and Y. Martel, Correction: “Multi-travelling waves for the nonlinear Klein-Gordon equation”. Preprint 2020.
[5] R. Côte, Y. Martel and F. Merle, Construction of multi-soliton solutions for the $L^2$-supercritical gKdV and NLS equations. Rev. Mat. Iberoamericana 27 (2011), 273–302.

[6] R. Côte and C. Muñoz, Multi-solitons for nonlinear Klein-Gordon equations. Forum of Mathematics, Sigma 2 (2014).

[7] M. del Pino, M. Musso, F. Pacard and A. Pistoia, Large energy entire solutions for the Yamabe equation, Journal of Differential Equations, 251 (9): 2568-2597, 2011.

[8] M. del Pino, M. Musso, F. Pacard and A. Pistoia, Torus action on $S^n$ and sign changing solutions for conformally invariant equations, Ann. Sc. Norm. Super. Pisa Cl. Sci (5), 12 (1): 209237, 2013.

[9] W. Ding, On a conformally invariant elliptic equation on $\mathbb{R}^n$, Commun. Math. Phys., 107: 331-335, 1986.

[10] T. Duyckaerts, C. Kenig and F. Merle, Solutions of the focusing nonradial critical wave equation with the compactness property. To appear in Ann. Sc. Norm. Super. Pisa Cl. Sci. Preprint arXiv:1402.0365

[11] C. E. Kenig and F. Merle, Global well-posedness, scattering and blow-up for the energy-critical focusing non-linear wave equation. Acta Math. 201 (2008), 147–212.

[12] Y. Martel, Asymptotic $N$-soliton-like solutions of the subcritical and critical generalized Korteweg-de Vries equations. Amer. J. Math. 127 (2005), 1103–1140.

[13] Y. Martel and F. Merle, Multi-solitary waves for nonlinear Schrödinger equations. Annales de l'IHP (C) Non Linear Analysis 23 (2006), 849–864.

[14] Y. Martel and F. Merle, Construction of multi-solitons for the energy-critical wave equation in dimension 5. Arch. Ration. Mech. Anal. 222 (2016), no. 3, 1113–1160.

[15] Y. Martel, F. Merle and T.-P. Tsai, Stability and asymptotic stability in the energy space of the sum of $N$ solitons for subcritical gKdV equations. Commun. Math. Phys. 231 (2002), 347–371.

[16] F. Merle, Construction of solutions with exactly $k$ blow-up points for the Schrödinger equation with critical nonlinearity. Comm. Math. Phys. 129 (1990), no. 2, 223–240.

[17] V. Z. Meshkov, Weighted differential inequalities and their application for estimates of the decrease at infinity of the solutions of second-order elliptic equations. Trudy Mat. Inst. Steklov. 190 (1989), 139158.

[18] Mei Ming, F. Rousset, N. Tzvetkov, Multi-solitons and Related Solutions for the Water-waves System. SIAM J. Math. Anal. 47 (2015), 897-954.

[19] M. Musso and J. Wei, Nondegeneracy of nodal solutions to the critical Yamabe problem, Comm. Math. Phys., 340(3): 10491107, 2015.

[20] G. Talenti, Best constants in Sobolev inequality. Annali di Matematica pura e applicata 10 (1976), 353-372

[21] X. Yuan. On multi-solitons for the energy-critical wave equation in dimension 5. Nonlinearity. 32 (2019), 5017–5048.

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