Motivic invariants of Artin stacks
and ‘stack functions’

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Abstract

An invariant $\Upsilon$ of quasiprojective $K$-varieties $X$ with values in a commutative ring $\Lambda$ is motivic if $\Upsilon(X) = \Upsilon(Y) + \Upsilon(X \setminus Y)$ for $Y$ closed in $X$, and $\Upsilon(X \times Y) = \Upsilon(X)\Upsilon(Y)$. Examples include Euler characteristics $\chi$ and virtual Poincaré and Hodge polynomials. We first define a unique extension $\Upsilon'$ of $\Upsilon$ to finite type Artin $K$-stacks $F$, which is motivic and satisfies $\Upsilon'(\lbrack X/G \rbrack) = \Upsilon(X)/\Upsilon(G)$ when $X$ is a $K$-variety, $G$ a special $K$-group acting on $X$, and $\lbrack X/G \rbrack$ is the quotient stack. This only works if $\Upsilon(G)$ is invertible in $\Lambda$ for all special $K$-groups $G$, which excludes $\Upsilon = \chi$ as $\chi(\mathbb{G}_m) = 0$. But we can extend the construction to get round this.

Then we develop the theory of stack functions on Artin stacks. These are a universal generalization of constructible functions on Artin stacks. There are several versions of the construction: the basic one $SF(\mathfrak{F})$, and variants $SF(\mathfrak{F}, \Upsilon, \Lambda)$, ‘twisted’ by motivic invariants. We associate a $\mathbb{Q}$-vector space $SF(\mathfrak{F})$ or a $\Lambda$-module $SF(\mathfrak{F}, \Upsilon, \Lambda)$ to each Artin stack $\mathfrak{F}$, with functorial operations of multiplication, pullbacks $\phi^*$ and pushforwards $\phi_*$ under 1-morphisms $\phi : \mathfrak{F} \to \mathfrak{G}$, and so on. They will be important tools in the author’s series on ‘Configurations in abelian categories’.

1 Introduction

Let $K$ be an algebraically closed field. An invariant $\Upsilon$ of isomorphism classes $\lbrack X \rbrack$ of quasiprojective $K$-varieties $X$ taking values in a commutative ring $\Lambda$ is called motivic if whenever $Y \subseteq X$ is a closed subvariety we have $\Upsilon(\lbrack X \rbrack) = \Upsilon(\lbrack X \setminus Y \rbrack) + \Upsilon(\lbrack Y \rbrack)$, and whenever $X, Y$ are varieties we have $\Upsilon(\lbrack X \times Y \rbrack) = \Upsilon(\lbrack X \rbrack)\Upsilon(\lbrack Y \rbrack)$. The name ‘motivic’ refers to motives and motivic integration, where such constructions are common. Well-known examples are the Euler characteristic, virtual Hodge polynomials and virtual Poincaré polynomials.

The first goal of this paper, in §4.1–§4.2, is to extend such invariants to Artin stacks. If $\mathcal{X}$ is a finite type algebraic $K$-stack with affine geometric stabilizers we define $\Upsilon'(\lbrack \mathcal{X} \rbrack) \in \Lambda$ uniquely with the above motivic properties, such that if $\mathcal{X}$ is a quotient $\lbrack X/G \rbrack$ for $X$ a quasiprojective $K$-variety and $G$ a special algebraic $K$-group, then $\Upsilon'(\lbrack \mathcal{X} \rbrack) = \Upsilon(\lbrack X \rbrack)/\Upsilon(\lbrack G \rbrack)$. Naturally, this is only possible if $\Upsilon(\lbrack G \rbrack)$ is invertible in $\Lambda$ for all special algebraic $K$-groups $G$. The most important restriction this imposes is that $\ell - 1$ is invertible, where $\ell = \Upsilon(\lbrack \mathbb{A}^1 \rbrack)$. We can arrange this for virtual Hodge and
Poincaré polynomials, but not for Euler characteristics, since then \( \ell = 1 \). Parts of \( \text{[3]} \) and \( \text{[6]} \) are dedicated to a version of this construction which allows \( \ell = 1 \), and so defines a kind of Euler characteristic of Artin stacks.

Very roughly, the idea when \( \ell = 1 \) is that if \( T^G \) is a maximal torus of \( G \), then \( \Upsilon([G]) = \Upsilon([G/T^G]) \Upsilon([T^G]) \), where \( \Upsilon([G/T^G]) \) is invertible in \( \Lambda \). So we can write \( \Upsilon'([X/G]) = \Upsilon([G/T^G])^{-1} \Upsilon'([X/T^G]) \). Now \( [X/T^G] \) is a finite disjoint union of \( \kappa \)-substacks 1-isomorphic to \( Y_i \times \text{[Spec } \kappa/H_i \text{]} \) for quasiprojective \( \kappa \)-varieties \( Y_i \) and \( \kappa \)-groups \( H_i \) of the form \( \mathbb{G}_m^k \times K \) for \( K \) finite abelian. We then define \( \Upsilon'([X/T^G]) = \sum_i \Upsilon([Y_i])[H_i] \), which takes values in the commutative \( \Lambda \)-algebra \( \Lambda \) with \( \Lambda \)-basis isomorphism classes \( [H] \) of \( \kappa \)-groups \( H \) of the form \( \mathbb{G}_m^k \times K \) for \( K \) finite abelian, and products \( [H_1][H_2] = [H_1 \times H_2] \).

The above is not quite true: to make \( \Upsilon'([X/G]) \) depend only on the stack \( [X/G] \) and not on \( X,G \) we have to introduce in \( \text{[5,2]} \) the idea of virtual rank, which treats a nonabelian \( \kappa \)-group \( G \) as being a kind of finite \( \Lambda \)-linear combination of certain \( \kappa \)-subgroups \( Q \subseteq T^G \), of which \( T^G \) is the largest. Then \( \Upsilon'([X/G]) \) is a \( \Lambda \)-linear combination of \( \Upsilon'([X/Q]) \) over all such \( Q \).

We will apply this in the series \([9–12]\). If \( A \) is a \( \kappa \)-linear abelian category and \( (\tau,T,\leq) \) a stability condition on \( A \), we define invariants of \( A, (\tau,T,\leq) \) by applying \( \Upsilon' \) to the \( \kappa \)-stacks \( \text{Obj}^{\text{ss}}_\tau(T), \text{Obj}^{\text{st}}_\alpha(T) \) of \( \tau \)-stable objects in \( A \) with class \( \alpha \in K(A) \). The motivic properties of \( \Upsilon' \) mean these invariants satisfy attractive identities and transformation laws, and can be computed in examples.

The second goal of the paper, in \( \text{[3]} \) and \( \text{[4,3,6]} \) is to develop the theory of ‘stack functions’. Before discussing this we explain the ideas of \([8]\) on constructible functions on stacks. To each Artin \( \kappa \)-stack \( \mathcal{F} \) we associate a \( \kappa \)-algebra \( \text{CF}(\mathcal{F}) \) of constructible functions on \( \mathcal{F} \), spanned by the characteristic functions of finite type \( \kappa \)-substacks \( \mathcal{G} \subseteq \mathcal{F} \). If \( \phi : \mathcal{G} \to \mathcal{F} \) is a 1-morphism we define the pushforward \( \text{CF}^{\text{st}}(\phi) : \text{CF}(\mathcal{G}) \to \text{CF}(\mathcal{F}) \) (for \( \phi \) representable and char \( \kappa = 0 \)) and the pullback \( \phi^* : \text{CF}(\mathcal{F}) \to \text{CF}(\mathcal{G}) \) (for \( \phi \) of finite type). These have good functorial properties, for instance \( \text{CF}^{\text{st}}(\psi \circ \phi) = \text{CF}^{\text{st}}(\psi) \circ \text{CF}^{\text{st}}(\phi) \), \( (\psi \circ \phi)^* = \phi^* \circ \psi^* \), and pushforwards, pullbacks commute in Cartesian squares.

Stack functions are a universal generalization of constructible functions. The basic version, in \( \text{[3]} \) replaces \( \text{CF}(\mathcal{F}) \) by a \( \kappa \)-vector space \( \text{SF}(\mathcal{F}) \) of \( \kappa \)-constructible functions on \( \mathcal{F} \), spanned by (representable) 1-morphisms \( \rho : \mathcal{R} \to \mathcal{F} \), for \( \mathcal{R} \) of finite type. These have multiplication and pushforwards and pullbacks along 1-morphisms with the same functoriality properties as constructible functions, and maps to and from \( \text{CF}(\mathcal{F}) \) commuting with multiplication and pushforwards and pullbacks in various ways. Thus, stack functions can be used as a substitute for constructible functions in many problems. But as \( \text{SF}(\mathcal{F}), \text{SF}(\mathcal{G}) \) contain much more information than \( \text{CF}(\mathcal{F}) \) they are a more powerful invariant. This will be exploited in \([10–12]\).

For varieties, similar ideas to \( \text{[3]} \) can be found in the subject of motivic integration. In particular, for a \( \kappa \)-variety \( X \), our space \( \text{SF}(X) \) agrees with \( K_0(\text{Var}_X) \otimes \mathbb{Q} \), where \( K_0(\text{Var}_X) \) is the Grothendieck group of \( X \)-varieties defined by Looijenga \([15, \S 2]\) and Bittner \([1, \S 5]\), and the operations we define on such \( \text{SF}(X) \) agree with operations in \([1, \S 6]\). This suggests our spaces \( \text{SF}, \text{SF}(\mathcal{F}) \) may have applications in the extension of motivic integration to Artin stacks (see Yasuda \([17]\) for the extension to Deligne–Mumford stacks).
Sections 4.3, 5 and 6 integrate these ideas with the material of §4.1–§4.2 to produce stack function spaces \( SF(\tilde{\mathcal{X}}, \Upsilon, \Lambda) \) modifying \( SF(F, \Upsilon, \Lambda) \) (or \( \overline{SF}(\tilde{\mathcal{X}}, \Upsilon, \Lambda) \), or \( SF(\tilde{\mathcal{X}}, \Upsilon, \Lambda^0) \), or \( SF(F, \Theta, \Omega) \), or \( \overline{SF}(\tilde{\mathcal{X}}, \chi, Q) \): there are several different versions), with the same operations and functoriality properties. Here is one way to motivate these spaces. The pushforward of constructible functions \( CF_{\text{stk}}(\phi) : CF(F) \to CF(G) \) is defined by ‘integration’ over the fibres of \( \phi \) using the Euler characteristic \( \chi \) as measure.

If \( \Upsilon \) is a \( \Lambda \)-valued motivic invariant as above, we could instead take \( \Lambda \)-valued constructible functions \( CF(F)_{\Lambda} \), and define pushforwards \( CF_{\text{stk}}(\Upsilon) : CF(F)_{\Lambda} \to CF(G)_{\Lambda} \) by ‘integration’ using \( \Upsilon \) as measure. But then \( CF_{\text{stk}}(\psi \circ \phi) = CF_{\text{stk}}(\psi) \circ CF_{\text{stk}}(\phi) \) may no longer hold, as this depends on properties of \( \chi \) on non-Zariski-locally-trivial fibrations which are false for other \( \Upsilon \) such as virtual Poincaré polynomials. This is a pity, as there would be interesting applications such as the Ringel–Hall algebras in [10] if functoriality held.

Our spaces \( SF(\tilde{\mathcal{X}}, \Upsilon, \Lambda) \),..., are designed to overcome this problem. They are a substitute for \( CF(F)_{\Lambda} \), and would reduce to \( CF(F)_{\Lambda} \) if every 1-morphism \( \phi : X \to \tilde{\mathcal{X}} \) for \( X \) an \( \mathbb{K} \)-variety could be broken into finitely many Zariski locally trivial fibrations \( \phi_i : X_i \to \tilde{\mathcal{X}}_i \subseteq \tilde{\mathcal{X}} \), but in general this is impossible. They have important applications in the author’s series [9–12], where we use them to associate algebras and Lie algebras to a \( \mathbb{K} \)-linear abelian category \( \mathcal{A} \), including quantized universal enveloping algebras, and to define invariants in \( \Lambda \) which ‘count’ \( \tau \)-semistable objects in \( \mathcal{A} \).

In a recent paper [16] written independently, Toen defines a Grothendieck ring of Artin \( n \)-stacks which is closely related to ideas below. In particular, [16, Th. 1.1] is similar to our Theorem 4.10, with the same hypotheses Assumption 4.1. Toen’s ring \( K(CH^{\text{sp}}(k)) \otimes_{\mathbb{Z}} \mathbb{Q} \) is also more-or-less the same thing as \( SF(\text{Spec} \mathbb{K}, \Upsilon_{\text{uni}}, \Lambda_{\text{uni}}) \), combining Example 4.5 and Definition 4.11 below.

In [12, §2.4] we will generalize parts of §4.3–§4.4 below. We define spaces of essential stack functions \( ESF(\tilde{\mathcal{X}}) \) with \( SF(\tilde{\mathcal{X}}) \subseteq ESF(\tilde{\mathcal{X}}) \subseteq LSF(\tilde{\mathcal{X}}) \) and a notion of strong convergence of infinite sums in \( ESF(\tilde{\mathcal{X}}) \), and then we extend the motivic invariants \( \Upsilon' \) of §4 to \( ESF(\tilde{\mathcal{X}}) \) in such a way that \( \Upsilon' \) takes strongly convergent sums in \( ESF(\tilde{\mathcal{X}}) \) to convergent sums in \( \Lambda \), and commutes with taking limits.

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2 Background material

We introduce \( \mathbb{K} \)-groups and Artin stacks in §2.1–§2.2 and then review the author’s paper [8] on constructible functions on stacks in §2.3.

2.1 Algebraic \( \mathbb{K} \)-groups

Let \( \mathbb{K} \) be an algebraically closed field. A good reference on algebraic \( \mathbb{K} \)-groups is Borel [2]. Following Borel, we define a \( \mathbb{K} \)-variety to be a \( \mathbb{K} \)-scheme which
is reduced, separated, and of finite type. We do not require our $K$-varieties to be irreducible, as many authors do. This allows algebraic $K$-groups with more than one connected component as $K$-varieties. An algebraic $K$-group is then a $K$-variety $G$ with identity $1 \in G$ (that is, $1 : \text{Spec } K \to G$), multiplication $\mu : G \times G \to G$ and inverse $i : G \to G$ (as morphisms of $K$-varieties) satisfying the usual group axioms. We call $G$ affine if it is an affine $K$-variety.

We will need the following notation and facts about algebraic $K$-groups and tori. Throughout $G$ is an affine algebraic $K$-group.

- Write $G_m$ for $K \setminus \{0\}$ as a $K$-group under multiplication. Write $A^n$ for affine space $K^n$, regarded as a $K$-variety. If $A$ is a finite-dimensional $K$-algebra, write $A^\times$ for the $K$-group of invertible elements of $A$ under multiplication.
- By a torus we mean an algebraic $K$-group isomorphic to $G_m^k$ for some $k \geq 0$. A subtorus of $G$ means a $K$-subgroup of $G$ which is a torus.
- A maximal torus in $G$ is a subtorus $T^G$ contained in no larger subtorus $T$ in $G$. All maximal tori in $G$ are conjugate by Borel [2, Cor. IV.11.3]. The rank $rk G$ is the dimension of any maximal torus. A maximal torus in $\text{GL}(n, K)$ is the subgroup $G_m^m$ of diagonal matrices.
- Let $T$ be a torus and $H$ a closed $K$-subgroup of $T$. Then $H$ is isomorphic to $G_m^k \times K$ for some $k \geq 0$ and finite abelian group $K$.
- If $S$ is a subset of $T^G$, define the centralizer of $S$ in $G$ to be $C_G(S) = \{\gamma \in G : \gamma s = s \gamma \ \forall s \in S\}$, and the normalizer of $S$ in $G$ to be $N_G(S) = \{\gamma \in G : \gamma^{-1} S \gamma = S\}$. They are closed $K$-subgroups of $G$ containing $T^G$, and $C_G(S)$ is normal in $N_G(S)$.
- The quotient group $W(G, T^G) = N_G(T^G)/C_G(T^G)$ is called the Weyl group of $G$. As in [2, IV.11.19] it is a finite group, which acts on $T^G$.
- Define the centre of $G$ to be $C(G) = \{\gamma \in G : \gamma \delta = \delta \gamma \ \forall \delta \in G\}$. It is a closed $K$-subgroup of $G$.
- There is a notion [2, I.4.5] of semisimple elements $\gamma \in G$, which are diagonalizable in any representation of $G$. (It is essential that $G$ is affine here.) Morphisms of affine algebraic $K$-groups take semisimple elements to semisimple elements, [2, Th. I.4.4(4)]. If $G$ is connected then $\gamma \in G$ is semisimple if and only if it lies in a maximal torus of $G$, [2, Th. IV.11.10].

We will also need the notion of special algebraic $K$-group, which is studied in the articles by Serre and Grothendieck in the Chevalley seminar [4, §§I, 5].

**Definition 2.1.** An algebraic $K$-group $G$ is special if every principal $G$-bundle over a $K$-variety is locally trivial in the Zariski topology.

The following facts may be found in [4, §§1.4, 1.5 & 5.5], or easily deduced.

- $G_m$, $G_m^n$ and $\text{GL}(m, K)$ are special. If $A$ is a finite-dimensional $K$-algebra then $A^\times$ is special. Products of special $K$-groups are special.
A $\mathbb{K}$-group $G$ is special if and only if it admits an embedding $G \subseteq \text{GL}(m, \mathbb{K})$ with the $G$-principal bundle $\text{GL}(m, \mathbb{K}) \to \text{GL}(m, \mathbb{K})/G$ Zariski locally trivial. If this holds for some embedding $G \subseteq \text{GL}(m, \mathbb{K})$ it holds for any embedding $G \subseteq \text{GL}(n, \mathbb{K})$.

- Special $\mathbb{K}$-groups are always affine and connected. A semisimple $\mathbb{K}$-group is special if and only if it is isomorphic to a product of $\mathbb{K}$-groups of the form $\text{SL}(m, \mathbb{K})$ and $\text{Sp}(2n, \mathbb{K})$. Connected, soluble $\mathbb{K}$-groups are special. If $H$ is normal in $G$ with $H, G/H$ special then $G$ is special.

## 2.2 Introduction to Artin $\mathbb{K}$-stacks

Fix an algebraically closed field $\mathbb{K}$ throughout. There are four main classes of ‘spaces’ over $\mathbb{K}$ used in algebraic geometry, in increasing order of generality:

$\mathbb{K}$-varieties $\subseteq \mathbb{K}$-schemes $\subseteq$ algebraic $\mathbb{K}$-spaces $\subseteq$ algebraic $\mathbb{K}$-stacks.

**Algebraic stacks** (also known as Artin stacks) were introduced by Artin, generalizing Deligne–Mumford stacks. Our principal reference is Laumon and Moret-Bailly [14], and a good introduction is provided by Gómez [7]. Following [7, 14] we include in the definition of an algebraic stack $\mathcal{F}$ that the diagonal morphism $\Delta_{\mathcal{F}}$ is representable, quasi-compact and separated, but probably the separatedness assumption can be omitted. We make the convention that all algebraic $\mathbb{K}$-stacks in this paper are locally of finite type, and $\mathbb{K}$-substacks are locally closed.

Algebraic $\mathbb{K}$-stacks form a 2-category. That is, we have objects which are $\mathbb{K}$-stacks $\mathcal{F}, \mathcal{G}$, and also two kinds of morphisms, 1-morphisms $\phi, \psi : \mathcal{F} \to \mathcal{G}$ between $\mathbb{K}$-stacks, and 2-morphisms $A : \phi \to \psi$ between 1-morphisms. An analogy to keep in mind is a 2-category of categories, where objects are categories, 1-morphisms are functors between the categories, and 2-morphisms are isomorphisms (natural transformations) between functors.

We define the set of $\mathbb{K}$-points of a stack.

**Definition 2.2.** Let $\mathcal{F}$ be a $\mathbb{K}$-stack. Write $\mathcal{F}(\mathbb{K})$ for groupoid of 1-morphisms $x : \text{Spec} \mathbb{K} \to \mathcal{F}$, and $\mathcal{F}(\mathbb{K})$ for the set of isomorphism classes in $\mathcal{F}(\mathbb{K})$, so that elements of $\mathcal{F}(\mathbb{K})$ are 2-isomorphism classes $[x]$ of 1-morphisms $x : \text{Spec} \mathbb{K} \to \mathcal{F}$. Elements of $\mathcal{F}(\mathbb{K})$ are called $\mathbb{K}$-points, or geometric points, of $\mathcal{F}$. If $\phi : \mathcal{F} \to \mathcal{G}$ is a 1-morphism then composition with $\phi$ induces a map of sets $\phi_* : \mathcal{F}(\mathbb{K}) \to \mathcal{G}(\mathbb{K})$.

Let $\mathcal{F}$ be an algebraic $\mathbb{K}$-stack and $x : \text{Spec} \mathbb{K} \to \mathcal{F}$ a 1-morphism. Then the group of 2-morphisms $x \to x$ has the structure of a group $\mathbb{K}$-scheme, which is not necessarily reduced. Define $\text{Aut}_{\mathbb{K}}(x)$ to be the associated reduced group $\mathbb{K}$-scheme. Then $\text{Aut}_{\mathbb{K}}(x)$ is an algebraic $\mathbb{K}$-group, which we call the stabilizer group of $x$. We say that $\mathcal{F}$ has affine geometric stabilizers if $\text{Aut}_{\mathbb{K}}(x)$ is an affine algebraic $\mathbb{G}$-group for all 1-morphisms $x : \text{Spec} \mathbb{K} \to \mathcal{F}$. As an algebraic $\mathbb{K}$-group up to isomorphism, $\text{Aut}_{\mathbb{K}}(x)$ depends only on the isomorphism class $[x] \in \mathcal{F}(\mathbb{K})$ of $x \in \mathcal{F}(\mathbb{K})$. If $\phi : \mathcal{F} \to \mathcal{G}$ is a 1-morphism, composition induces a morphism of algebraic $\mathbb{K}$-groups $\phi_* : \text{Aut}_{\mathbb{K}}([x]) \to \text{Aut}_{\mathbb{K}}(\phi_*([x]))$, for $[x] \in \mathcal{F}(\mathbb{K})$. 

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One important difference in working with 2-categories rather than ordinary categories is that in diagram-chasing one only requires 1-morphisms to be 2-isomorphic rather than equal. The simplest kind of commutative diagram is:

\[
\begin{array}{ccc}
\mathcal{F} & \xrightarrow{\phi} & \mathcal{G} \\
\downarrow{\psi} & & \downarrow{\chi} \\
\mathcal{H} & \xrightarrow{\phi} & \mathcal{H},
\end{array}
\]

by which we mean that \(\mathcal{F}, \mathcal{G}, \mathcal{H}\) are \(K\)-stacks, \(\phi, \psi, \chi\) are 1-morphisms, and \(F : \psi \circ \phi \rightarrow \chi\) is a 2-isomorphism. Usually we omit \(F\), and mean that \(\psi \circ \phi \cong \chi\).

**Definition 2.3.** Let \(\phi : \mathcal{F} \rightarrow \mathcal{H}\), \(\psi : \mathcal{G} \rightarrow \mathcal{H}\) be 1-morphisms of \(K\)-stacks. Then one can define the *fibre product stack* \(\mathcal{F} \times_{\phi, \mathcal{H}, \psi} \mathcal{G}\), or \(\mathcal{F} \times \mathcal{G}\) for short, with 1-morphisms \(\pi_{\mathcal{F}}, \pi_{\mathcal{G}}\) fitting into a commutative diagram:

\[
\begin{array}{ccc}
\mathcal{F} & \xrightarrow{\pi_{\mathcal{F}}} & \mathcal{F} \times \mathcal{G} \\
\downarrow{\phi} & & \downarrow{\phi} \\
\mathcal{H} & \xrightarrow{\pi_{\mathcal{G}}} & \mathcal{H}.
\end{array}
\]

A commutative diagram

\[
\begin{array}{ccc}
\mathcal{E} & \xrightarrow{\phi} & \mathcal{F} \\
\downarrow{\psi} & & \downarrow{\psi} \\
\mathcal{G} & \xrightarrow{\phi} & \mathcal{H}
\end{array}
\]

is a *Cartesian square* if it is isomorphic to \((1)\), so there is a 1-isomorphism \(\mathcal{E} \cong \mathcal{F} \times \mathcal{G}\). Cartesian squares may also be characterized by a universal property.

**2.3 Constructible functions on stacks**

Finally we discuss *constructible functions* on \(K\)-stacks, following [8]. For this section we need \(K\) to have characteristic zero.

**Definition 2.4.** Let \(\mathcal{F}\) be an algebraic \(K\)-stack. We call \(C \subseteq \mathcal{F}(K)\) *constructible* if \(C = \bigcup_{i \in I} \mathcal{F}_i(K)\), where \(\{\mathcal{F}_i : i \in I\}\) is a finite collection of finite type algebraic \(K\)-substacks \(\mathcal{F}_i\) of \(\mathcal{F}\). We call \(S \subseteq \mathcal{F}(K)\) *locally constructible* if \(S \cap C\) is constructible for all constructible \(C \subseteq \mathcal{F}(K)\).

A function \(f : \mathcal{F}(K) \rightarrow \mathbb{Q}\) is called *constructible* if \(f(\mathcal{F}(K))\) is finite and \(f^{-1}(c)\) is a constructible set in \(\mathcal{F}(K)\) for each \(c \in f(\mathcal{F}(K)) \setminus \{0\}\). A function \(f : \mathcal{F}(K) \rightarrow \mathbb{Q}\) is called *locally constructible* if \(f : \delta_C\) is constructible for all constructible \(C \subseteq \mathcal{F}(K)\), where \(\delta_C\) is the characteristic function of \(C\). Write \(\text{CF}(\mathcal{F})\) and \(\text{LCF}(\mathcal{F})\) for the \(\mathbb{Q}\)-vector spaces of \(\mathbb{Q}\)-valued constructible and locally constructible functions on \(\mathcal{F}\). They are closed under multiplication.

We explain *pushforwards* and *pullbacks* of constructible functions along a 1-morphism \(\phi : \mathcal{F} \rightarrow \mathcal{G}\), following [8, Def.s 4.8, 5.1 & 5.5].

**Definition 2.5.** Let \(\mathcal{F}\) be an algebraic \(K\)-stack with affine geometric stabilizers and \(C \subseteq \mathcal{F}(K)\) be constructible. Then [8, Def. 4.8] defines the *na"ive Euler
characteristic $\chi^{na}(C)$ of $C$. It is called naive as it takes no account of stabilizer groups. For $f \in \text{CF}(\mathcal{F})$, define $\chi^{na}(\mathcal{F}, f)$ in $\mathbb{Q}$ by

$$\chi^{na}(\mathcal{F}, f) = \sum_{c \in f(\mathcal{F}(K)) \setminus \{0\}} c \chi^{na}(f^{-1}(c)).$$

Let $\phi : \mathcal{F} \to \mathcal{G}$ be a 1-morphism between algebraic $K$-stacks with affine geometric stabilizers. For $f \in \text{CF}(\mathcal{F})$, define $\text{CF}^{na}(\phi) f : \mathcal{G}(K) \to \mathbb{Q}$ by

$$\text{CF}^{na}(\phi) f(y) = \chi^{na}(\mathcal{F}, f \cdot \delta_{\phi^{-1}(y)}) \quad \text{for} \quad y \in \mathcal{G}(K),$$

where $\delta_{\phi^{-1}(y)}$ is the characteristic function of $\phi^{-1}(\{y\}) \subseteq \mathcal{G}(K)$ on $\mathcal{G}(K)$. Then $\text{CF}^{na}(\phi) : \text{CF}(\mathcal{F}) \to \text{CF}(\mathcal{G})$ is a $\mathbb{Q}$-linear map called the naive pushforward.

Now suppose $\phi$ is representable. Then for any $x \in \mathcal{F}(K)$ we have an injective morphism $\phi_* : \text{Aut}_K(x) \to \text{Aut}_K(\phi_*(x))$ of affine algebraic $K$-groups. The image $\phi_*(\text{Aut}_K(x))$ is an affine algebraic $K$-group closed in $\text{Aut}_K(\phi_*(x))$, so the quotient $\text{Aut}_K(\phi_*(x))/\phi_*(\text{Aut}_K(x))$ exists as a quasiprojective $K$-variety.

Define a function $m_\phi : \mathcal{F}(K) \to \mathbb{Z}$ by $m_\phi(x) = \chi(\text{Aut}_K(\phi_*(x))/\phi_*(\text{Aut}_K(x)))$ for $x \in \mathcal{F}(K)$. Then for $f \in \text{CF}(\mathcal{F})$, define $\text{CF}^{stk}(\phi) f : \mathcal{G}(K) \to \mathbb{Q}$ by $\text{CF}^{stk}(\phi) f = \text{CF}^{na}(\phi)(m_\phi \cdot f)$. Then $\text{CF}^{stk}(\phi) : \text{CF}(\mathcal{F}) \to \text{CF}(\mathcal{G})$ is a $\mathbb{Q}$-linear map called the stack pushforward.

Let $\phi$ be of finite type, not necessarily representable. If $C \subseteq \mathcal{G}(K)$ is constructible then so is $\phi_*^{-1}(C) \subseteq \mathcal{F}(K)$. Thus if $f \in \text{CF}(\mathcal{G})$ then $f \circ \phi_* \in \text{CF}(\mathcal{F})$. Define the pullback $\phi^* : \text{CF}(\mathcal{G}) \to \text{CF}(\mathcal{F})$ by $\phi^*(f) = f \circ \phi_*$. It is $\mathbb{Q}$-linear.

Here [8, Th.s 4.9, 5.4, 5.6 & Def. 5.5] are some properties of these.

**Theorem 2.6.** Let $\mathcal{E}, \mathcal{F}, \mathcal{G}, \mathcal{H}$ be algebraic $K$-stacks with affine geometric stabilizers, and $\beta : \mathcal{F} \to \mathcal{G}$, $\gamma : \mathcal{G} \to \mathcal{H}$ be 1-morphisms. Then

$$\text{CF}^{na}(\gamma \circ \beta) = \text{CF}^{na}(\gamma) \circ \text{CF}^{na}(\beta) : \text{CF}(\mathcal{F}) \to \text{CF}(\mathcal{H}),$$

$$\text{CF}^{stk}(\gamma \circ \beta) = \text{CF}^{stk}(\gamma) \circ \text{CF}^{stk}(\beta) : \text{CF}(\mathcal{F}) \to \text{CF}(\mathcal{H}),$$

$$(\gamma \circ \beta)^* = \beta^* \circ \gamma^* : \text{CF}(\mathcal{F}) \to \text{CF}(\mathcal{H})$$

supposing $\beta, \gamma$ representable in (3), and of finite type in (2). If

$$\begin{array}{ccc}
\mathcal{E} & \xrightarrow{\eta} & \mathcal{G} \\
\downarrow \phi & \xrightarrow{\eta, \phi \text{ representable and} } & \downarrow \psi \\
\mathcal{F} & \xrightarrow{\theta} & \mathcal{H}
\end{array}$$

is a Cartesian square with $\theta, \psi$ of finite type, then

the following commutes:

$$\begin{array}{ccc}
\text{CF}(\mathcal{E}) & \xrightarrow{\text{CF}^{stk}(\eta)} & \text{CF}(\mathcal{G}) \\
\downarrow \theta^* & \xrightarrow{\text{CF}^{stk}(\phi)} & \downarrow \psi^* \\
\text{CF}(\mathcal{F}) & \xrightarrow{\text{CF}^{stk}(\phi)} & \text{CF}(\mathcal{H})
\end{array}$$

As discussed in [8, §3.3] for the $K$-scheme case, equation (3) is *false* for algebraically closed fields $K$ of characteristic $p > 0$. In [8, §5.3] we extend all these results to *locally constructible functions*. The main differences are in which 1-morphisms must be of finite type.
3 Stack functions, the basic version

We now introduce stack functions, a universal generalization of constructible functions with similar properties under multiplication, pushforwards and pull-backs. Here we study the basic versions \( \text{SF}(\mathfrak{X}) \), \( \text{SF}(\mathfrak{Y}) \), and in \( \text{SF}(\mathfrak{Z}) \) we generalize them to more complicated spaces \( \text{SF}(\mathfrak{Z}, \mathfrak{Y}, \mathfrak{A}), \ldots \). Throughout \( \mathbb{K} \) will be an algebraically closed field of arbitrary characteristic, except when we specify \( \text{char} \mathbb{K} = 0 \) for results comparing stack and constructible functions. The assumption that all \( \mathbb{K} \)-stacks are locally of finite type can be relaxed too. For some related constructions for \( \mathbb{K} \)-varieties rather than \( \mathbb{K} \)-stacks, see Bittner [1, §5–§6].

**Definition 3.1.** Let \( \mathfrak{X} \) be an algebraic \( \mathbb{K} \)-stack with affine geometric stabilizers. Consider pairs \( (\mathfrak{X}, \rho) \), where \( \mathfrak{X} \) is a finite type algebraic \( \mathbb{K} \)-stack with affine geometric stabilizers and \( \rho : \mathfrak{X} \to \mathfrak{X} \) is a 1-morphism. We call two pairs \( (\mathfrak{X}, \rho), (\mathfrak{X}, \rho') \) equivalent if there exists a 1-isomorphism \( \iota : \mathfrak{X} \to \mathfrak{X} \) such that \( \rho' \circ \iota = \rho \) and \( \rho \) and \( \rho' \) are 2-isomorphic 1-morphisms \( \mathfrak{X} \to \mathfrak{X} \). Write \( [(\mathfrak{X}, \rho)] \) for the equivalence class of \( (\mathfrak{X}, \rho) \). If \( (\mathfrak{X}, \rho) \) is such a pair and \( \mathfrak{S} \) is a closed \( \mathbb{K} \)-substack of \( \mathfrak{X} \) then \( (\mathfrak{S}, \rho|_{\mathfrak{S}}), (\mathfrak{X} \setminus \mathfrak{S}, \rho|_{\mathfrak{X} \setminus \mathfrak{S}}) \) are pairs of the same kind. Define

(a) \( \text{SF}(\mathfrak{X}) \) to be the \( \mathbb{Q} \)-vector space generated by equivalence classes \( [(\mathfrak{X}, \rho)] \) as above, with for each closed \( \mathbb{K} \)-substack \( \mathfrak{S} \) of \( \mathfrak{X} \) a relation

\[
[(\mathfrak{X}, \rho)] = [(\mathfrak{S}, \rho|_{\mathfrak{S}})] + [(\mathfrak{X} \setminus \mathfrak{S}, \rho|_{\mathfrak{X} \setminus \mathfrak{S}})].
\]

(b) \( \text{SF}(\mathfrak{X}) \) to be the \( \mathbb{Q} \)-vector space generated by \( [(\mathfrak{X}, \rho)] \) with \( \rho \) representable, with the same relations (a).

Define a multiplication \( \cdot \) on \( \text{SF}(\mathfrak{X}) \) analogous to multiplication of functions by

\[
[(\mathfrak{X}, \rho)] \cdot [(\mathfrak{S}, \sigma)] = [(\mathfrak{X} \times_{\mathfrak{X} \setminus \mathfrak{S}, \sigma} \mathfrak{S}, \rho \circ \pi_{\mathfrak{S}})].
\]

This is compatible with the relations (b), and so extends to a \( \mathbb{Q} \)-bilinear product \( \text{SF}(\mathfrak{X}) \times \text{SF}(\mathfrak{X}) \to \text{SF}(\mathfrak{X}) \). If \( \rho, \sigma \) are representable then so is \( \rho \circ \pi_{\mathfrak{X}} \), so \( \text{SF}(\mathfrak{X}) \) is closed under \( \cdot \). As \( \rho \circ \pi_{\mathfrak{X}} \) is 2-isomorphic to \( \sigma \circ \pi_{\mathfrak{X}} \), \( \cdot \) is commutative, and one can show it is associative using properties of fibre products.

The assumption that \( \mathfrak{X}, \mathfrak{S} \) have affine geometric stabilizers here will be used in this section only in the results below comparing \( \text{SF}(\mathfrak{X}), \text{SF}(\mathfrak{Y}) \) and \( \text{CF}(\mathfrak{Y}) \) — in particular, without it the linear maps \( \pi_{\mathfrak{X}}^\text{na} : \text{SF}(\mathfrak{X}) \to \text{CF}(\mathfrak{Y}) \) and \( \pi_{\mathfrak{X}}^\text{stk} : \text{SF}(\mathfrak{Y}) \to \text{CF}(\mathfrak{X}) \) in Definition 3.2 would not be well-defined. But in \( \text{SF}(\mathfrak{X}), \text{SF}(\mathfrak{Y}), \text{SF}(\mathfrak{Z}) \) we use the assumption in a much more essential way.

We refer to elements of \( \text{SF}(\mathfrak{X}), \text{SF}(\mathfrak{Y}) \) as stack functions. There is an obvious inclusion \( \text{SF}(\mathfrak{X}) \subset \text{SF}(\mathfrak{Y}) \). We could instead work over \( \mathbb{Z} \) rather than \( \mathbb{Q} \), and define \( \text{SF}(\mathfrak{X})_{\mathbb{Z}} \) to be the abelian group generated by equivalence classes \( [(\mathfrak{X}, \rho)] \) of pairs \( (\mathfrak{X}, \rho) \) with relations (b), so that \( \text{SF}(\mathfrak{X}) = \text{SF}(\mathfrak{X})_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q} \), and so on. Or we could work over any ring or abelian group. But for simplicity we consider only \( \mathbb{Q} \). We define maps between \( \text{CF}(\mathfrak{Y}) \) and \( \text{SF}(\mathfrak{X}), \text{SF}(\mathfrak{Y}) \).
Definition 3.2. Let $\mathcal{F}$ be an algebraic $\mathbb{K}$-stack with affine geometric stabilizers and $C \subseteq \overline{\mathcal{F}}(\mathbb{K})$ a constructible subset. Then we may write $C = \coprod_{i=1}^{n} \mathcal{R}_i(\mathbb{K})$, for $\mathcal{R}_1, \ldots, \mathcal{R}_n$ finite type $\mathbb{K}$-substacks of $\mathcal{F}$. Let $\rho_i : \mathcal{R}_i \to \mathcal{F}$ be the inclusion 1-morphism. Then $\rho_i$ is representable, so $[\mathcal{R}_i, \rho_i]$ lies in $SF(\mathcal{F}) \subseteq SF(\mathcal{F})$ by Definition 3.1.

Define

$$\tilde{\delta}_C = \sum_{i=1}^{n} [(\mathcal{R}_i, \rho_i)] \in SF(\mathcal{F}) \subseteq SF(\mathcal{F}).$$

We think of this stack function as the analogue of the characteristic function $\tilde{\delta}_C \in CF(\mathcal{F})$ of $C$. Using $\mathcal{F}$ and the argument of [8, Def. 3.7] we find that $\tilde{\delta}_C$ is independent of the choice of decomposition $C = \coprod_{i=1}^{n} \mathcal{R}_i(\mathbb{K})$, and so is well-defined.

Define a $\mathbb{Q}$-linear map $\iota_\mathcal{F} : CF(\mathcal{F}) \to SF(\mathcal{F}) \subseteq SF(\mathcal{F})$ by

$$\iota_\mathcal{F}(f) = \sum_{0 \neq c \in f(\mathcal{F}(\mathbb{K}))} c \cdot \tilde{\delta}_{f^{-1}(c)}.$$  

This is well-defined as $f(\mathcal{F}(\mathbb{K}))$ is finite and $f^{-1}(c)$ constructible for all $0 \neq c \in f(\mathcal{F}(\mathbb{K}))$. Since $f = \sum_{0 \neq c \in f(\mathcal{F}(\mathbb{K}))} c \cdot \tilde{\delta}_{f^{-1}(c)}$, $\iota_\mathcal{F}$ is the unique $\mathbb{Q}$-linear map which takes $\tilde{\delta}_C$ to $\tilde{\delta}_C$ for all constructible $C \subseteq \mathcal{F}(\mathbb{K})$. When $\mathbb{K}$ has characteristic zero, define $\mathbb{Q}$-linear maps $\pi^{na}_\mathcal{F} : SF(\mathcal{F}) \to CF(\mathcal{F})$ and $\pi^{stk}_\mathcal{F} : SF(\mathcal{F}) \to CF(\mathcal{F})$ by

$$\pi^{na}_\mathcal{F} \left( \sum_{i=1}^{n} c_i ([\mathcal{R}_i, \rho_i]) \right) = \sum_{i=1}^{n} c_i CF^{na}(\rho_i) 1_{\mathcal{R}_i}$$

and

$$\pi^{stk}_\mathcal{F} \left( \sum_{i=1}^{n} c_i ([\mathcal{R}_i, \rho_i]) \right) = \sum_{i=1}^{n} c_i CF^{stk}(\rho_i) 1_{\mathcal{R}_i},$$

where $1_{\mathcal{R}_i}$ is the function 1 in $CF(\mathcal{R}_i)$, which is constructible as $\mathcal{R}_i$ is of finite type. Here in the second line $\rho_i$ is representable by definition of $SF(\mathcal{F})$, so $CF^{stk}(\rho_i) 1_{\mathcal{R}_i}$ makes sense. To see (8) is well-defined, note that if $\mathcal{R}, \rho$ are as in Definition 3.1 and $\mathcal{F}$ is a closed $\mathbb{K}$-substack of $\mathcal{R}$ then

$$CF^{na}(\rho) 1_{\mathcal{R}} = CF^{na}(\rho) \left( \delta_{\mathcal{F}(\mathbb{K})} + \delta_{\mathcal{F}(\mathbb{K})} \cdot \mathcal{F}(\mathbb{K}) \right) = CF^{na}(\rho) 1_{\mathcal{R}} + CF^{na}(\rho) 1_{\mathcal{R}} = CF^{na}(\rho) 1_{\mathcal{R}}.$$

So $\pi^{na}_\mathcal{F}$ is still well-defined after quotienting by relations (7), and for representable $\rho$ the same argument works for $\pi^{stk}_\mathcal{F}$.

Proposition 3.3. For $\mathbb{K}$ of characteristic zero, $\pi^{na}_\mathcal{F} \circ \iota_\mathcal{F}$ and $\pi^{stk}_\mathcal{F} \circ \iota_\mathcal{F}$ are both the identity on $CF(\mathcal{F})$. Hence $\iota_\mathcal{F}$ is injective and $\pi^{na}_\mathcal{F}, \pi^{stk}_\mathcal{F}$ are surjective. Also $\iota_\mathcal{F}, \pi^{stk}_\mathcal{F}$ commute with multiplication in $CF(\mathcal{F}), SF(\mathcal{F})$.

Proof. If $\mathcal{R}$ is a finite type $\mathbb{K}$-substack in $\mathcal{F}$ with inclusion 1-morphism $\rho : \mathcal{R} \to \mathcal{F}$ then $\iota_\mathcal{F}(\delta_{\mathcal{R}(\mathbb{K})}) = \delta_{\mathcal{R}(\mathbb{K})}$, and $\pi^{na}_\mathcal{F}([\mathcal{R}, \rho]) = \pi^{na}_\mathcal{F}([\mathcal{R}, \rho]) = \delta_{\mathcal{R}(\mathbb{K})}$. Thus $\pi^{na}_\mathcal{F} \circ \iota_\mathcal{F}$ and $\pi^{stk}_\mathcal{F} \circ \iota_\mathcal{F}$ take $\delta_{\mathcal{R}(\mathbb{K})}$ to itself. As such $\delta_{\mathcal{R}(\mathbb{K})}$ generate $CF(\mathcal{F})$, we see that $\pi^{na}_\mathcal{F} \circ \iota_\mathcal{F}, \pi^{stk}_\mathcal{F} \circ \iota_\mathcal{F}$ are the identity, so $\iota_\mathcal{F}$ is injective and $\pi^{na}_\mathcal{F}, \pi^{stk}_\mathcal{F}$ surjective.

If $\mathcal{R}$ is a $\mathbb{K}$-substack of $\mathcal{F}$ with inclusion $\rho : \mathcal{R} \to \mathcal{F}$, there is a 1-isomorphism $\mathcal{R} \cong \mathcal{R} \times \mathcal{R}$ which by (7) implies that $[\mathcal{R}, \rho] \cdot [\mathcal{R}, \rho] = [\mathcal{R}, \rho]$. Hence

$$\iota_\mathcal{F}(\delta_{\mathcal{R}(\mathbb{K})}) = \iota_\mathcal{F}(\delta_{\mathcal{R}(\mathbb{K})}) \cdot \iota_\mathcal{F}(\delta_{\mathcal{R}(\mathbb{K})}).$$

(9)
When $\mathcal{R}, \mathcal{S}$ are disjoint $\mathbb{K}$-substacks of $\mathfrak{F}$ it is easy to see that
\[ \iota_\mathfrak{F}(\delta_{\mathcal{R}(\mathbb{K})}\delta_{\mathcal{S}(\mathbb{K})}) = 0 = \iota_\mathfrak{F}(\delta_{\mathcal{R}(\mathbb{K})}) \cdot \iota_\mathfrak{F}(\delta_{\mathcal{S}(\mathbb{K})}). \] (10)

Given any $f, g \in \text{CF}(\mathfrak{F})$ there exist a finite collection of disjoint $\mathbb{K}$-substacks $\mathcal{R}_i$ of $\mathfrak{F}$ such that $f, g$ are $\mathbb{Q}$-linear combinations of the $\delta_{\mathcal{R}_i(\mathbb{K})}$. Therefore $\iota_\mathfrak{F}(f g) = \iota_\mathfrak{F}(f) \cdot \iota_\mathfrak{F}(g)$ follows from (9) and bilinearity.

For $[(\mathcal{R}, \rho)], [(\mathcal{S}, \sigma)] \in \text{SF}(\mathfrak{F})$, apply (8) with $\mathcal{R} \times_\mathfrak{F} \mathcal{S}, \mathcal{S}, \mathcal{R}, \mathfrak{F}$ in place of $\mathfrak{F}, \mathfrak{F}, \mathcal{S}, \mathfrak{F}$ respectively to the function $1_\mathcal{S} \in \text{CF}(\mathfrak{S})$. This gives
\[ \text{CF}^{\text{stk}}(1_{\mathcal{R} \times_\mathfrak{F} \mathcal{S}}) = \text{CF}^{\text{stk}}(\rho_{\mathcal{R}}) \circ \pi_{\mathcal{S}}^*(1_\mathcal{S}) = \rho^* \circ \text{CF}^{\text{stk}}(\sigma) 1_\mathcal{S} = 1_{\mathcal{R}} \cdot \rho^* \circ \text{CF}^{\text{stk}}(\sigma) 1_\mathcal{S}. \]

Applying $\text{CF}^{\text{stk}}(\rho)$ to this and using (3), (7) and (8) gives
\[ \pi^\text{stk}_\mathfrak{F}([(\mathcal{R}, \rho)], [(\mathcal{S}, \sigma)]) = \text{CF}^{\text{stk}}(\rho \circ \pi_{\mathcal{R}}) 1_{\mathcal{R} \times_\mathfrak{F} \mathcal{S}} = \text{CF}^{\text{stk}}(\rho) 1_{\mathcal{R}} \cdot \text{CF}^{\text{stk}}(\sigma) 1_\mathcal{S} = \pi^\text{stk}_\mathfrak{F}([(\mathcal{R}, \rho)]), \]

since multiplication by $\rho^* \circ \text{CF}^{\text{stk}}(\sigma) 1_\mathcal{S}$ and $\text{CF}^{\text{stk}}(\sigma) 1_\mathcal{S}$ commute with $\text{CF}^{\text{stk}}(\rho)$. Thus $\pi^\text{stk}_\mathfrak{F}(f \cdot g) = \pi^\text{stk}_\mathfrak{F}(f)(\pi^\text{stk}_\mathfrak{F}(g))$ for $f, g \in \text{SF}(\mathfrak{F})$ follows by bilinearity. \hfill \Box

In general, $\iota_\mathfrak{F}$ is far from being surjective, and $\text{SF}(\mathfrak{F}), \text{SF}^*(\mathfrak{F})$ are much larger than $\text{CF}(\mathfrak{F})$. For example, $\text{CF}(\text{Spec } \mathbb{K}) \cong \mathbb{Q}$, but one can show $\text{SF}(\text{Spec } \mathbb{K}) \cong K_0(\text{Var}_{\mathbb{K}}) \otimes_{\mathbb{Z}} \mathbb{Q}$ and $\text{SF}^*(\text{Spec } \mathbb{K}) \cong K_0(\text{Sta}_{\mathbb{K}}) \otimes_{\mathbb{Z}} \mathbb{Q}$, where $K_0(\text{Var}_{\mathbb{K}})$, $K_0(\text{Sta}_{\mathbb{K}})$ are the Grothendieck rings of the (2-)categories of $\mathbb{K}$-varieties and algebraic $\mathbb{K}$-stacks respectively. The ring $K_0(\text{Var}_{\mathbb{K}})$ for $\text{char } \mathbb{K} = 0$ is studied by Bittner [1] and is clearly very large, and $K_0(\text{Sta}_{\mathbb{K}})$ is even larger. Also, $\pi^\text{stk}_\mathfrak{F}$ does not usually commute with multiplication. Next we define pushforwards, pullbacks and tensor products on stack functions.

**Definition 3.4.** Let $\phi : \mathfrak{F} \to \mathfrak{G}$ be a 1-morphism of algebraic $\mathbb{K}$-stacks with affine geometric stabilizers. Define the pushforward $\phi_* : \text{SF}(\mathfrak{F}) \to \text{SF}(\mathfrak{G})$ by
\[ \phi_* : \sum_{i=1}^n c_i[(\mathcal{R}_i, \rho_i)] \mapsto \sum_{i=1}^n c_i[(\mathcal{R}_i, \phi \circ \rho_i)]. \]

This intertwines the relations (11) in $\text{SF}(\mathfrak{F}), \text{SF}(\mathfrak{G})$, and so is well-defined. If $\phi$ is representable then the restriction maps $\phi_* : \text{SF}(\mathfrak{F}) \to \text{SF}(\mathfrak{G})$, since the $\phi \circ \rho_i$ are representable as $\phi, \rho_i$ are.

Now let $\phi$ be of finite type. If $\mathcal{R}_i$ is a finite type algebraic $\mathbb{K}$-stack and $\rho_i : \mathcal{R}_i \to \mathfrak{G}$ a 1-morphism then we may form the Cartesian square:
\[ \begin{array}{ccc}
\mathcal{R}_i \times_{\rho_i, \mathfrak{G}, \phi} \mathfrak{F} & \to & \mathfrak{F} \\
\downarrow \pi_{\mathcal{R}_i} & & \downarrow \phi \\
\mathcal{R}_i & \to & \mathfrak{G}.
\end{array} \] (11)

Since $\mathcal{R}_i$ and $\phi$ are of finite type, so are $\pi_{\mathcal{R}_i}$ and $\mathcal{R}_i \times_{\rho_i, \mathfrak{G}, \phi} \mathfrak{F}$ as (11) is Cartesian. Define the pullback $\phi^* : \text{SF}(\mathfrak{G}) \to \text{SF}(\mathfrak{F})$ by
\[ \phi^* : \sum_{i=1}^n c_i[(\mathcal{R}_i, \rho_i)] \mapsto \sum_{i=1}^n c_i[(\mathcal{R}_i \times_{\rho_i, \mathfrak{G}, \phi} \mathfrak{F}, \pi_{\mathcal{R}_i})]. \] (12)
This is well-defined as $R_i \times_{\rho_i, \mathcal{G}, \phi} \mathfrak{F}$ is unique up to 1-isomorphism, and $\phi^*$ intertwines the relations (11) in $SF(\mathcal{G})$, $SF(\mathfrak{F})$. The restriction maps $\phi^* : SF(\mathcal{G}) \to SF(\mathfrak{F})$, since the $\pi_\mathfrak{F}$ are representable as the $\rho_i$ are, and (11) is Cartesian. The tensor product $\otimes : SF(\mathfrak{F}) \times SF(\mathcal{G}) \to SF(\mathfrak{F} \times \mathcal{G})$ and $\otimes : SF(\mathfrak{F} \times \mathcal{G}) \to SF(\mathfrak{F} \times \mathcal{G})$ is

$$\left( \sum_{i \in I} c_i([R_i, \rho_i]) \right) \otimes \left( \sum_{j \in J} d_j([\mathcal{G}_j, \sigma_j]) \right) = \sum_{i \in I, j \in J} c_id_j([R_i \times \mathcal{G}_j, \rho_i \times \sigma_j]), \quad (13)$$

for finite $I, J$. This is compatible with the relations, and so well-defined. It is the analogue of the obvious map $\otimes : CF(\mathfrak{F}) \times CF(\mathcal{G}) \to CF(\mathfrak{F} \times \mathcal{G})$ on constructible functions given by $(f \otimes g)(x, y) = f(x)g(y)$.

We can now justify the name ‘stack function’. Each $[x] \in \mathfrak{F}(K)$ is an isomorphism class of (finite type) 1-morphisms $x : \text{Spec } K \to \mathfrak{F}$. These induce pullbacks $x^* : SF(\mathfrak{F}) \to SF(\text{Spec } K)$ and $x^* : SF(\mathcal{G}) \to SF(\text{Spec } K)$ depending only on $[x]$. Thus, to each $f \in SF(\mathfrak{F})$ or $SF(\mathcal{G})$ we associate a function $\mathfrak{F}(K) \to SF(\text{Spec } K)$ or $SF(\text{Spec } K)$ by $[x] \mapsto x^*(f)$.

By definition $SF(\text{Spec } K)$ and $SF(\text{Spec } K)$ are the $Q$-vector spaces generated by 1-isomorphism classes $[\mathcal{R}]$ of finite type algebraic $K$-stacks with affine geometric stabilizers, and finite type algebraic $K$-spaces $\mathcal{R}$ respectively, with a relation $[\mathcal{R}] = [\mathcal{G}] + [\mathcal{R} \setminus \mathcal{G}]$ whenever $\mathcal{G}$ is a closed $K$-substack of $\mathcal{R}$. Thus, stack functions on $\mathfrak{F}$ are like ‘functions on $\mathfrak{F}(K)$ with values in stacks’.

Here is the analogue of Theorem 2.6.

**Theorem 3.5.** Let $\mathcal{E}, \mathfrak{F}, \mathcal{G}, \mathfrak{H}$ be algebraic $K$-stacks with affine geometric stabilizers and $\beta : \mathfrak{H} \to \mathcal{G}$, $\gamma : \mathcal{G} \to \mathfrak{F}$ be 1-morphisms. Then

$$\begin{align*}
(\gamma \circ \beta)_* &= \gamma_* \circ \beta_* : SF(\mathfrak{F}) \to SF(\mathfrak{H}), \\
(\gamma \circ \beta)^* &= \beta^* \circ \gamma^* : SF(\mathfrak{H}) \to SF(\mathfrak{F}),
\end{align*} \quad (14)$$

for $\beta, \gamma$ representable in the second equation, and of finite type in the third and fourth. If $f, g \in SF(\mathcal{G})$ and $\beta$ is finite type then $\beta^*(f \cdot g) = \beta^*(f) \cdot \beta^*(g)$. If $\mathcal{E} \xrightarrow{\gamma} \mathfrak{F}$ is a Cartesian square with $\mathcal{G}$, $\mathfrak{H}$ of finite type, then

$$\begin{array}{ccc}
\mathcal{E} & \xrightarrow{\eta} & \mathfrak{F} \\
\downarrow \theta & & \downarrow \psi \\
\mathfrak{H} & \xrightarrow{\phi} & \mathfrak{G}
\end{array} \quad \begin{array}{ccc}
SF(\mathcal{E}) & \xrightarrow{\eta_*} & SF(\mathfrak{F}) \\
\downarrow \theta^* & & \downarrow \psi^* \\
SF(\mathfrak{H}) & \xrightarrow{\phi^*} & SF(\mathfrak{G})
\end{array} \quad (15)
$$

The same applies for $SF(\mathcal{E}), \ldots, SF(\mathfrak{H})$ if $\eta, \phi$ are representable.

**Proof.** The first and second equations of (14) follow from

$$\begin{align*}
(\gamma \circ \beta)_* ([R, \rho]) &= [R, \gamma \circ \beta \circ \rho], \\
(\gamma \circ \beta)^* ([R, \rho]) &= \gamma_* ([R, \beta \circ \rho]),
\end{align*} \quad (16)$$

as $(\gamma \circ \beta) \circ \rho = \gamma(\beta \circ \rho)$. For the third and fourth equations, we need to prove that for $\rho : \mathcal{R} \to \mathfrak{H}$ as in Definition 3.1 we have $(\gamma \circ \beta)^* ([R, \rho]) = \beta^* \circ \gamma^* ([R, \rho])$. This follows from the existence of a 1-isomorphism

$$\begin{array}{cc}
t : (\mathcal{R} \times_{\rho, \mathcal{H}, \gamma} \mathfrak{F}) \times_{\pi_\mathcal{R}, \mathcal{G}, \beta} \mathfrak{F} & \to \mathcal{R} \times_{\rho, \mathcal{H}, \gamma \circ \beta} \mathfrak{F}
\end{array} \quad (16)$$
with \( \pi_\mathcal{G} \circ \iota \)' 2-isomorphic to \( \pi_\mathcal{G} \), as 1-morphisms \((\mathcal{R} \times_{\rho, \mathcal{H}} \mathcal{G}) \times_{\pi_\mathcal{G}, \mathcal{H}, \mathcal{G}} \mathcal{F} \rightarrow \mathcal{F} \). We can construct \( \iota \) easily using the explicit definition \([14, 2.2.2]\) of fibre products of \( \mathbb{K} \)-stacks. This proves \([14]\). One can show \( \beta^*(f \cdot g) = \beta^*(f) \cdot \beta^*(g) \) for \( f = [(\mathcal{R}, \rho)] \), \( g = [(\mathcal{G}, \sigma)] \) using properties of fibre products.

For both cases of \([14]\), let \( \rho : \mathcal{R} \rightarrow \mathcal{G} \) be as in Definition \([3.1]\). Then \( \psi^* \circ \phi_* \left([(\mathcal{R}, \rho)]\right) = \eta_\psi \circ \theta^* \left([(\mathcal{R}, \rho)]\right) \) if \( [(\mathcal{R} \times_{\phi, \mathcal{G}, \psi} \mathcal{G}, \pi_\mathcal{G})] = [(\mathcal{R} \times_{\rho, \mathcal{H}} \mathcal{G}, \eta \circ \pi_\mathcal{G})] \). From Definition \([2.3]\) and equivalence in Definition \([3.1]\) we see that we may replace \( \mathcal{E} \) here by \( \mathcal{F} \times_{\phi, \mathcal{G}, \psi} \mathcal{G} \) and \( \theta, \eta \) by \( \pi_\mathcal{G}, \pi_\mathcal{E} \), so this is equivalent to

\[
\left[(\mathcal{R} \times_{\phi, \mathcal{G}, \psi} \mathcal{G}, \pi_\mathcal{G})\right] = \left[(\mathcal{R} \times_{\rho, \mathcal{H}} \mathcal{G}, \pi_\mathcal{G} \circ \pi_{\mathcal{G} \times_{\phi, \mathcal{G}, \psi} \mathcal{G}})\right].
\]

This follows from the existence of a 1-isomorphism

\[
\iota' : \mathcal{R} \times_{\rho, \mathcal{H}, \pi_\mathcal{G}} (\mathcal{F} \times_{\phi, \mathcal{G}, \psi} \mathcal{G}) \rightarrow \mathcal{R} \times_{\phi, \mathcal{G}, \psi} \mathcal{G}
\]

with \( \pi_{\mathcal{G}} \circ \iota' \) 2-isomorphic to \( \pi_\mathcal{G} \circ \iota \circ \phi \) with \( \pi_{\mathcal{G}} \circ \iota' \) by construction using \([14, 2.2.2]\) as for \( \iota \) in \([14]\) above. This completes the proof. \( \square \)

Here are some compatibilities between \( \otimes \) and the other operations. The proofs are all elementary.

**Proposition 3.6.** Let \( \phi : \mathcal{E} \rightarrow \mathcal{G} \) and \( \psi : \mathcal{G} \rightarrow \mathcal{H} \) be 1-morphisms of algebraic \( \mathbb{K} \)-stacks with affine geometric stabilizers and \( e, f, g, h \) lie in \( \text{SF}(\mathcal{E}), \ldots, \text{SF}(\mathcal{H}) \). Then \( \phi_* (e) \otimes \psi_*(g) = (\phi \otimes \psi)_*(e \otimes g) \), and \( \phi^*(f) \otimes \psi^*(h) = (\phi \otimes \psi)^*(f \otimes h) \) when \( \phi, \psi \) are of finite type. Also \( e \circ (f \otimes g) = (e \otimes f) \circ g \in \text{SF}(\mathcal{E} \times \mathcal{G}) \). If \( \mathbb{K} \) has characteristic zero then \( \pi_{\mathcal{E} \times_{\mathcal{G}} \mathcal{H}}(e \otimes f) = \pi_{\mathcal{E}}(e) \otimes \pi_{\mathcal{H}}(f) \) in \( \text{CF}(\mathcal{E} \times \mathcal{G}) \), and \( \pi_{\mathcal{E} \times_{\mathcal{G}} \mathcal{H}}(e \otimes f) = \pi_{\mathcal{H}}(e) \otimes \pi_{\mathcal{E}}(f) \) when \( e, f \) lie in \( \text{SF}(\mathcal{E}), \text{SF}(\mathcal{G}) \).

The next two results consider the relationships between pushforwards and pullbacks of stack and constructible functions, via the maps \( \iota_\mathcal{G}, \pi_{\mathcal{G}, \mathcal{H}}, \pi_{\mathcal{G} \times_{\phi, \mathcal{G}, \psi} \mathcal{G}} \).

**Proposition 3.7.** Let \( \phi : \mathcal{G} \rightarrow \mathcal{H} \) be a finite type 1-morphism of algebraic \( \mathbb{K} \)-stacks with affine geometric stabilizers. Then \( \phi^* \circ \iota_\mathcal{G} = \iota_\mathcal{G} \circ \phi^* : \text{CF}(\mathcal{H}) \rightarrow \text{SF}(\mathcal{G}) \).

**Proof.** Let \( \mathcal{R} \) be a finite type \( \mathbb{K} \)-substack of \( \mathcal{G} \) with inclusion \( \rho : \mathcal{R} \rightarrow \mathcal{G} \). Then \( \mathcal{R} \times_{\rho, \mathcal{G}, \phi} \mathcal{H} \) is a finite type \( \mathbb{K} \)-substack of \( \mathcal{G} \) with inclusion \( \pi_\mathcal{G} \), and \( \phi^* (\mathcal{G}(\mathbb{K})) = (\mathcal{R} \times_{\rho, \mathcal{G}, \phi} \mathcal{H})(\mathbb{K}) \subseteq \mathcal{G}(\mathbb{K}) \). Hence

\[
\phi^* \circ \iota_\mathcal{G}(\delta(\mathcal{G}(\mathbb{K}))) = \phi^* (\delta(\mathcal{G}(\mathbb{K}))) = \phi^* ([\mathcal{R}(\mathbb{K})]) = \left[(\mathcal{R} \times_{\rho, \mathcal{G}, \phi} \mathcal{H}, \pi_\mathcal{G})\right]
\]

As such \( \delta(\mathcal{G}(\mathbb{K})) \) generate \( \text{CF}(\mathcal{H}) \), the proposition follows by linearity. \( \square \)

**Theorem 3.8.** Let \( \mathbb{K} \) have characteristic zero, \( \mathcal{G}, \mathcal{H} \) be algebraic \( \mathbb{K} \)-stacks with affine geometric stabilizers, and \( \phi : \mathcal{G} \rightarrow \mathcal{H} \) a 1-morphism. Then

(a) \( \pi_{\mathcal{E}} \circ \phi_* = \text{CF}_{\mathcal{E}}(\phi) \circ \pi_{\mathcal{E}} : \text{SF}(\mathcal{G}) \rightarrow \text{CF}(\mathcal{E}) \);

(b) \( \pi_{\mathcal{E}} \circ \phi_* = \text{CF}_{\text{stk}}(\phi) \circ \pi_{\mathcal{E}} : \text{SF}(\mathcal{G}) \rightarrow \text{CF}(\mathcal{E}) \) if \( \phi \) is representable; and
(c) \( \pi_{\text{stk}} \circ \phi^* = \phi^* \circ \pi_{\text{na}} \) if \( \phi \) is of finite type.

**Proof.** Let \([([\mathcal{R}, \rho])]) \in SF(\mathfrak{F})\), so that \( \rho : \mathcal{R} \to \mathfrak{F} \). Then

\[
\text{CF}_{\text{na}}(\phi) \circ \pi_{\text{na}}(([([\mathcal{R}, \rho])])) = \text{CF}_{\text{na}}(\phi) \circ \text{CF}_{\text{na}}(\rho)1_{\mathcal{R}} = \text{CF}_{\text{na}}(\phi \circ \rho)1_{\mathcal{R}}
\]

\[
= \pi_{\text{na}}^{\text{na}}([([\mathcal{R}, \phi \circ \rho]))) = \pi_{\text{na}}^{\text{na}} \circ \phi^*([([\mathcal{R}, \rho]))],
\]

using Definitions 3.2 and 3.4 and equation (2). Part (a) follows by linearity. The proof of (b) is the same, using \( \text{CF}_{\text{stk}}, \pi_{\text{stk}}, \pi_{\text{na}} \) and (3). Let \([([\mathcal{R}, \rho])]) \in SF(\mathfrak{G})\). Then by linearity, part (c) follows from

\[
\pi_{\text{stk}} \circ \phi^*([([\mathcal{R}, \rho])]) = \pi_{\text{stk}}([([\mathcal{R} \times_{\rho, \mathcal{S}, \phi} \mathfrak{F}, \mathfrak{F}]))]) = \text{CF}_{\text{stk}}(\pi_{\text{stk}}(\rho))1_{\mathcal{R} \times_{\rho, \mathcal{S}, \phi} \mathfrak{F}} = \phi^* \circ \text{CF}_{\text{stk}}(\rho)1_{\mathcal{R}} = \phi^* \circ \pi_{\text{stk}}^{\text{na}}([([\mathcal{R}, \rho]))]),
\]

using Definitions 3.2 and 3.4 and equation (5) applied to the Cartesian square

\[
\begin{array}{ccc}
\mathcal{R} \times_{\rho, \mathcal{S}, \phi} \mathfrak{F} & \overset{\pi_{\text{stk}}}{\longrightarrow} & \mathfrak{F} \\
\downarrow \rho \downarrow & & \downarrow \phi \\
\mathcal{R} & \overset{\pi_{\text{na}}^{\text{na}}}{\longrightarrow} & \mathfrak{G},
\end{array}
\]

with \( \rho, \pi_{\text{na}} \) representable and \( \phi, \pi_{\text{stk}} \) of finite type.

The other possible commutation relations are in general false. That is, we expect \( \phi_* \circ c \neq c \circ \phi_* \) for \( \phi \in \text{CF}_{\text{na}}(\phi) \) and \( \pi_{\text{na}}^{\text{na}} \circ \phi^* \neq \phi^* \circ \pi_{\text{na}}^{\text{na}} \). This is why we use only the \( \pi_{\text{stk}} \) and not the \( \pi_{\text{na}}^{\text{na}} \) in the applications of [10–12], as the \( \pi_{\text{stk}} \) commute with both pushforwards and pullbacks, but the \( \pi_{\text{na}}^{\text{na}} \) do not.

Suppose \( \mathfrak{F} \) is a \( \mathbb{K} \)-variety, \( \mathbb{K} \)-scheme or algebraic \( \mathbb{K} \)-space, and \([([\mathcal{R}, \rho])]) \in SF(\mathfrak{F})\). Then \( \rho : \mathcal{R} \to \mathfrak{F} \) is representable, so \( \mathcal{R} \) is a finite type algebraic \( \mathbb{K} \)-space. Thus \( \mathcal{R} \) can be written as the disjoint union of finitely many quasiprojective \( \mathbb{K} \)-subvarieties \( X_i \), and \([([\mathcal{R}, \rho])]) = \bigcup_i \([([X_i, \rho_i]))])\). Therefore \( SF(\mathfrak{F}) \) is generated over \( \mathbb{Q} \) by \([([X, \rho])]) \) for \( X \) a quasiprojective \( \mathbb{K} \)-variety, with relations \([([X, \rho])]) = \([([Y, \rho])]) + \([([X \setminus Y, \rho])]) \) for closed subvarieties \( Y \subseteq X \).

This implies that for \( \mathfrak{F} \) a \( \mathbb{K} \)-variety, \( SF(\mathfrak{F}) \) equals \( K_0(\text{Var}_\mathbb{K})_\mathbb{Q} \), where \( K_0(\text{Var}_\mathbb{K}) \) is the Grothendieck group of \( \mathbb{K} \)-varieties studied by Bittner [1, §5]. So this section generalizes the ideas of Bittner to Artin stacks. The operations \( \cdot, \circ, \phi, \phi^* \) of Definitions 3.4 and 3.5 agree with \( \otimes, \phi, \phi^* \) in [1, §6].

This raises two interesting questions. Firstly, Bittner also defines an involution \( D \) and operations \( \phi, \phi^* \) on a modified space \( K_0(\text{Var}_\mathbb{K})[[\mathbb{A}^1]] \). Do these have analogues for Artin stacks? Secondly, modifications of \( K_0(\text{Var}_\mathbb{K}) \) are the natural value groups for motivic integrals, which is the main reason for studying them. Can the theory of motivic integration be extended to Artin stacks, using modifications of our spaces \( SF, SF(\mathfrak{F}) \)?

Finally, we define local stack functions, the analogue of locally constructible functions. Roughly speaking, we want to repeat Definition 3.1 using pairs \((\mathcal{R}, \rho)\) for which \( \mathcal{R} \) is not necessarily of finite type, but \( \rho \) is. However, this must be modified in two ways. Firstly, we allow sums \( \sum_{i \in I} c_i ([([\mathcal{R}_i, \phi_i])]) \) over infinite indexing sets \( I \), because locally constructible functions can take infinitely many
values. Secondly, the relations (6) are no longer sufficient, because for $\mathcal{R}$ not of finite type we should be able to cut $\mathcal{R}$ into infinitely many disjoint pieces, but (6) allows only for finite decompositions.

**Definition 3.9.** Let $\mathfrak{F}$ be an algebraic $\mathbb{K}$-stack with affine geometric stabilizers. Consider pairs $(\mathcal{R}, \rho)$, where $\mathcal{R}$ is an algebraic $\mathbb{K}$-stack with affine geometric stabilizers and $\rho : \mathcal{R} \to \mathfrak{F}$ is a finite type 1-morphism, with equivalence of pairs as in Definition 3.1. Let $V_{\mathfrak{F}}$ be the $\mathbb{Q}$-vector space of formal $\mathbb{Q}$-linear combinations $\sum_{i \in I} c_i ([\mathcal{R}_i, \rho_i])$, where $I$ is a possibly infinite indexing set, $c_i \in \mathbb{Q}$ and $([\mathcal{R}_i, \rho_i])$ is an equivalence class as above, such that for all finite type $\mathbb{K}$-substacks $\mathcal{G}$ in $\mathfrak{F}$ with inclusion 1-morphism $\phi : \mathcal{G} \to \mathfrak{F}$, there are only finitely many $i \in I$ with $c_i \neq 0$ and $\mathcal{R}_i \times_{\mathcal{G}, \phi} \mathcal{G}$ nonempty.

Let $W_{\mathfrak{F}}$ be the vector subspace of $\sum_{i \in I} c_i ([\mathcal{R}_i, \rho_i])$ in $V_{\mathfrak{F}}$ such that for all finite type $\mathbb{K}$-substacks $\mathcal{G}$ in $\mathfrak{F}$ with inclusion 1-morphism $\phi : \mathcal{G} \to \mathfrak{F}$, we have $\sum_{i \in I} c_i ([\mathcal{R}_i \times_{\mathcal{G}, \phi} \mathcal{G}, \pi_\phi]) = 0$ in $\text{SF} (\mathcal{G})$. There are only finitely many nonzero terms in this sum by definition of $V_{\mathfrak{F}}$, so this makes sense. Define $\text{LSF} (\mathfrak{F})$ to be the quotient $V_{\mathfrak{F}} / W_{\mathfrak{F}}$. Define $V_{\mathfrak{F}}, W_{\mathfrak{F}}, \text{LSF} (\mathfrak{F})$ in exactly the same way, but with all 1-morphisms $\rho_i$ representable, and interpreting the relation $\sum_{i \in I} c_i ([\mathcal{R}_i \times_{\mathcal{G}, \phi} \mathcal{G}, \pi_\phi]) = 0$ in $\text{SF} (\mathcal{G})$.

We define commutative, associative multiplications ‘·’ on $\text{LSF} (\mathfrak{F})$, $\text{LSF} (\mathfrak{F})$ by extending (7) bilinearly to sums $\sum_{i \in I} c_i [([\mathcal{R}_i, \rho_i]) \cdot \sum_{j \in J} d_j ([([\mathcal{G}_j, \sigma_j])].

If $\mathfrak{F}$ is of finite type and $\rho : \mathcal{R} \to \mathfrak{F}$ a 1-morphism then $\mathcal{R}$ is of finite type if and only if $\rho$ is, and taking $\mathcal{G} = \mathfrak{F}$ shows sums in $V_{\mathfrak{F}}$ have only finitely many nonzero terms. It follows easily that $\text{LSF} (\mathfrak{F}) = \text{SF} (\mathfrak{F})$ and $\text{LSF} (\mathfrak{F}) = \text{SF} (\mathfrak{F})$ in this case, just as $\text{LCF} (\mathfrak{F}) = \text{CF} (\mathfrak{F})$. All the definitions and results above for $\text{SF} (\mathfrak{F})$, $\text{SF} (\mathfrak{F})$ have straightforward generalizations to $\text{LSF} (\mathfrak{F})$, $\text{LSF} (\mathfrak{F})$, analogous to [8, §5.3]. We just state these, leaving the proofs as an exercise. Note the differences in which 1-morphisms are required to be of finite type.

**Definition 3.10.** Let $\mathfrak{F}$ be an algebraic $\mathbb{K}$-stack with affine geometric stabilizers and $\mathcal{S} \subseteq \mathfrak{F}(\mathbb{K})$ a locally constructible subset. Then we may write $\mathcal{S} = \bigcup_{i \in I} \mathcal{R}_i(\mathbb{K})$, for $\mathbb{K}$-substacks $\mathcal{R}_i$ of $\mathfrak{F}$ with only finitely many intersecting any constructible set $C \subseteq \mathfrak{F}(\mathbb{K})$. Let $\rho_i : \mathcal{R}_i \to \mathfrak{F}$ be the inclusion 1-morphism, which is representable and of finite type. Define a local stack function

$$\delta_S = \sum_{i \in I} ([\mathcal{R}_i, \rho_i]) \in \text{LSF} (\mathfrak{F}) \subseteq \text{LSF} (\mathfrak{F}).$$

This is independent of the choice of $I, \mathcal{R}_i$. Define $\iota_3 : \text{LCF} (\mathfrak{F}) \to \text{LSF} (\mathfrak{F}) \subseteq \text{LSF} (\mathfrak{F})$ by $\iota_3 (f) = \sum_{c \in \mathfrak{F}(\mathbb{K})} c \cdot \delta_{f^{-1}(c)}$. This potentially infinite sum makes sense as only finitely many terms are nonzero over any constructible subset. For $\mathbb{K}$ of characteristic zero, define $\mathbb{Q}$-linear maps $\pi_{na}^{\mathfrak{F}} : \text{LSF} (\mathfrak{F}) \to \text{LCF} (\mathfrak{F})$ and $\pi_{stk}^{\mathfrak{F}} : \text{LSF} (\mathfrak{F}) \to \text{LCF} (\mathfrak{F})$ by

$$\pi_{na}^{\mathfrak{F}} (\sum_{i \in I} c_i ([\mathcal{R}_i, \rho_i])) = \sum_{i \in I} c_i \text{LCF}^{na} (\rho_i) \mathcal{R}_i,$$

and

$$\pi_{stk}^{\mathfrak{F}} (\sum_{i \in I} c_i ([\mathcal{R}_i, \rho_i])) = \sum_{i \in I} c_i \text{LCF}^{stk} (\rho_i) \mathcal{R}_i.$$

Here $\text{LCF}^{na} (\rho_i), \text{LCF}^{stk} (\rho_i)$ make sense as $\rho_i$ is of finite type. On any constructible subset there are only finitely many nonzero terms on the right hand
sides of these equations, so they are well-defined and lie in LCF(\(\mathfrak{F}\)). The analogue of Proposition 3.7 holds for \(\iota_{\mathfrak{F}}, \pi_{\mathfrak{F}}^{na}, \pi_{\mathfrak{F}}^{st}\).

**Definition 3.11.** Let \(\phi : \mathfrak{F} \rightarrow \mathfrak{G}\) be a finite type 1-morphism of algebraic \(\mathbb{K}\)-stacks with affine geometric stabilizers. Define \(\phi_* : \text{LSF}(\mathfrak{F}) \rightarrow \text{LSF}(\mathfrak{G})\) by

\[\phi_* : \sum_{i \in I} c_i([\mathfrak{R}_i, \rho_i]) \mapsto \sum_{i \in I} c_i([\mathfrak{R}_i, \phi \circ \rho_i]).\]

If \(\phi\) is also representable define \(\phi_* : \text{LSF}(\mathfrak{F}) \rightarrow \text{LSF}(\mathfrak{G})\) the same way. For any \(\phi : \mathfrak{F} \rightarrow \mathfrak{G},\) define \(\phi^* : \text{LSF}(\mathfrak{G}) \rightarrow \text{LSF}(\mathfrak{F})\) and \(\phi^* : \text{LSF}(\mathfrak{G}) \rightarrow \text{LSF}(\mathfrak{F})\) by

\[\phi^* : \sum_{i \in I} c_i([\mathfrak{R}_i, \rho_i]) \mapsto \sum_{i \in I} c_i([\mathfrak{R}_i \times \rho, \phi, \mathfrak{F}, \mathfrak{G}]).\]

As in Proposition 3.7 we have \(\phi^* \circ \iota_{\mathfrak{G}} = \iota_{\mathfrak{F}} \circ \phi^* : \text{LSF}(\mathfrak{G}) \rightarrow \text{LSF}(\mathfrak{F}).\) Define \(\otimes : \text{LSF}(\mathfrak{F}) \times \text{LSF}(\mathfrak{G}) \rightarrow \text{LSF}(\mathfrak{F} \times \mathfrak{G})\) and \(\otimes : \text{LSF}(\mathfrak{F}) \times \text{LSF}(\mathfrak{G}) \rightarrow \text{LSF}(\mathfrak{F} \times \mathfrak{G})\) by [13], allowing \(I, J\) infinite.

**Theorem 3.12.** Let \(\mathcal{E}, \mathfrak{F}, \mathfrak{G}, \mathfrak{H}\) be algebraic \(\mathbb{K}\)-stacks with affine geometric stabilizers and \(\beta : \mathfrak{F} \rightarrow \mathfrak{G}, \gamma : \mathfrak{G} \rightarrow \mathfrak{H}\) be 1-morphisms. Then

\[\gamma \circ \beta_* = \gamma_* \circ \beta_* : \text{LSF}(\mathfrak{F}) \rightarrow \text{LSF}(\mathfrak{H}),\]

\[(\gamma \circ \beta)^* = \beta^* \circ \gamma^* : \text{LSF}(\mathfrak{H}) \rightarrow \text{LSF}(\mathfrak{F}),\]

for all \(\gamma, \beta\) of finite type in the first and second equations, and representable in the second. If \(f, g \in \text{LSF}(\mathfrak{G})\) then \(\beta^*(f \cdot g) = \beta^*(f) \cdot \beta^*(g).\)

If

\[
\begin{array}{ccc}
\mathcal{E} & \xrightarrow{\eta} & \mathfrak{G} \\
\downarrow \phi & \downarrow \psi & \downarrow \\
\mathfrak{F} & \xrightarrow{\iota} & \mathfrak{H}
\end{array}
\]

is a Cartesian square with \(\eta, \phi, \psi\) of finite type, then

\[
\begin{array}{ccc}
\text{LSF}(\mathcal{E}) & \xrightarrow{\eta_*} & \text{LSF}(\mathfrak{G}) \\
\downarrow \phi^* & \downarrow & \downarrow \psi^* \\
\text{LSF}(\mathfrak{F}) & \xrightarrow{\iota_*} & \text{LSF}(\mathfrak{H})
\end{array}
\]

the following commutes:

The same applies for \(\text{LSF}(\mathcal{E}), \ldots, \text{LSF}(\mathfrak{H})\) if also \(\eta, \phi\) are representable.

**Theorem 3.13.** Let \(\mathbb{K}\) have characteristic zero, \(\mathfrak{F}, \mathfrak{G}\) be algebraic \(\mathbb{K}\)-stacks with affine geometric stabilizers, and \(\phi : \mathfrak{F} \rightarrow \mathfrak{G}\) a 1-morphism. Then

(a) \(\pi_{\mathfrak{F}}^{na} \circ \phi_* = \text{LCF}^{na}(\phi) \circ \pi_{\mathfrak{F}}^{na} : \text{LSF}(\mathfrak{F}) \rightarrow \text{LCF}(\mathfrak{G})\) if \(\phi\) is of finite type;

(b) \(\pi_{\mathfrak{F}}^{st} \circ \phi_* = \text{LCF}^{st}(\phi) \circ \pi_{\mathfrak{F}}^{st} : \text{LSF}(\mathfrak{F}) \rightarrow \text{LCF}(\mathfrak{G})\) if \(\phi\) is representable and of finite type; and

(c) \(\pi_{\mathfrak{F}}^{st} \circ \phi^* = \phi^* \circ \pi_{\mathfrak{F}}^{st} : \text{LSF}(\mathfrak{G}) \rightarrow \text{LCF}(\mathfrak{F}).\)

## 4 Motivic invariants of stacks

Let \(\mathbb{K}\) be an algebraically closed field, and suppose \(\Upsilon\) is some invariant of quasiprojective \(\mathbb{K}\)-varieties \(X\) up to isomorphism, taking values in a commutative ring or algebra \(\Lambda\). We call \(\Upsilon\) motivic if whenever \(Y \subseteq X\) is a closed subvariety we have \(\Upsilon([X]) = \Upsilon([X \setminus Y]) + \Upsilon([Y])\), and whenever \(X, Y\) are varieties we have \(\Upsilon([X \times Y]) = \Upsilon([X]) \Upsilon([Y])\). The name ‘motivic’ is a reference
to the subject of motives and motivic integration, where such constructions are common. Well-known examples of motivic invariants are the Euler characteristic, and virtual Poincaré polynomials.

This section extends such invariants $\Upsilon$ to Artin stacks, in the special case when $\ell - 1$ and some other elements are invertible in $\Lambda$, where $\ell = \Upsilon(\mathbb{A}^1)$. Roughly speaking, we need this because for a quotient stack $[X/G]$ we want to set $\Upsilon([X/G]) = \Upsilon([X])/\Upsilon([G])$, but $\Upsilon([G])$ is divisible by $\ell - 1$ for any algebraic $K$-group $G$ with $\text{rk} G > 0$, so $(\ell - 1)^{-1}$ must exist if $\Upsilon([G])^{-1}$ does.

For virtual Poincaré polynomials we can make $\ell - 1$ invertible by defining $\Lambda$ appropriately. But if $\Upsilon$ is the Euler characteristic $\chi$ then $\ell = \chi(\mathbb{A}^1) = 1$, so $\ell - 1$ cannot be invertible, and the approach of this section fails. Section 6 defines refined versions of the constructions of this section, which do work when $\ell - 1$ is not invertible, and so for Euler characteristics.

Section 4.1 explains the properties of $\Upsilon$ we need and gives examples, and $4.2$ explains how to extend $\Upsilon$ naturally to $\Upsilon([R])$ for finite type algebraic $K$-stacks $R$ with affine geometric stabilizers. This $\Upsilon'$ is motivic and satisfies $\Upsilon([X/G]) = \Upsilon([X])/\Upsilon([G])$ when $G$ is a special algebraic $K$-group. Section 4.3 combines these ideas with stack functions to define modified spaces $\text{SF}(\mathcal{E}, \Upsilon, \Lambda)$ which will be powerful tools in [10–12].

4.1 Initial assumptions and examples

Here is the data we shall need for our constructions.

Assumption 4.1. Suppose $\Lambda$ is a commutative $\mathbb{Q}$-algebra with identity 1, and

$$\Upsilon : \{\text{isomorphism classes } [X] \text{ of quasiprojective } K\text{-varieties } X \} \rightarrow \Lambda$$

a map, satisfying the following conditions:

(i) If $Y \subseteq X$ is a closed subvariety then $\Upsilon([X]) = \Upsilon([X \setminus Y]) + \Upsilon([Y])$;

(ii) If $X,Y$ are quasiprojective $K$-varieties then $\Upsilon([X \times Y]) = \Upsilon([X])\Upsilon([Y])$;

(iii) Write $\ell = \Upsilon(\mathbb{A}^1)$ in $\Lambda$, where $\mathbb{A}^1$ is the affine line $K$ regarded as a $K$-variety. Then $\ell$ and $\ell^k - 1$ for $k = 1, 2, \ldots$ are invertible in $\Lambda$.

We chose the notation $'\ell'$ as in motivic integration $[\mathbb{A}^1]$ is called the Tate motive and written $L$. We will often use following easy consequence of (i),(ii).

Lemma 4.2. Suppose Assumption 4.1 holds, and $\phi : X \rightarrow Y$ is a Zariski locally trivial fibration of quasiprojective $K$-varieties with fibre $F$, that is, $F$ is a quasiprojective $K$-variety and $Y$ can be covered by Zariski open sets $U$ such that $\phi^{-1}(U) \cong F \times U$. Then $\Upsilon([X]) = \Upsilon([F])\Upsilon([Y])$.

Proof. Let $n \geq 0$ and $k \geq 1$ be given. Suppose by induction that the lemma holds when either $\dim Y < n$ or $\dim Y = n$ and $Y$ has fewer than $k$ irreducible components. Let $\phi : X \rightarrow Y$ be as above, and suppose $\dim Y = n$ and $Y$ has $k$ irreducible connected components. Then $Y \neq \emptyset$, so we can choose a nonempty open set $U \subseteq Y$ with $\phi^{-1}(U) \cong F \times U$. Set $Y' = Y \setminus U$ and $X' = \phi^{-1}(Y')$. Then
$X', Y'$ are quasiprojective $\mathbb{K}$-varieties, and $\phi|_{X'}: X' \to Y'$ is a Zariski locally trivial fibration with fibre $F$, and either $\dim Y' < \dim Y$, or $\dim Y' = \dim Y$ and $Y'$ has fewer irreducible components than $Y$. So by the inductive hypothesis we have $Y([X']) = Y([F])Y([Y'])$. But then

$$Y([X]) = Y([X']) + Y([\phi^{-1}(U)]) = Y([F])Y([Y']) + Y([F \times U])$$

$$= Y([F])(Y([Y']) + Y([U])) = Y([F])Y([Y]),$$

using Assumption 4.1(i),(ii). The lemma follows by induction on $n, k$. □

Here are some examples of suitable $\Lambda, Y$. The first, for $\mathbb{K} = \mathbb{C}$, uses the virtual Hodge polynomials introduced by Danilov and Khovanskii [5, §1.5], and discussed by Cheah [3, §0.1].

**Example 4.3.** Let $\mathbb{K} = \mathbb{C}$. Define $\Lambda_{H_0} = \mathbb{Q}(x, y)$, the $\mathbb{Q}$-algebra of rational functions in $x, y$ with coefficients in $\mathbb{Q}$. Elements of $\Lambda_{H_0}$ are of the form $P(x, y)/Q(x, y)$, for $P, Q$ rational polynomials in $x, y$ with $Q \neq 0$.

Let $X$ be a quasiprojective $\mathbb{C}$-variety of dimension $m$, and $H^k_c(X, \mathbb{C})$ the compactly-supported cohomology of $X$. Deligne defined a mixed Hodge structure on $H^k_c(X, \mathbb{C})$. Let $h^{p,q}(H^k_c(X, \mathbb{C}))$ be the corresponding Hodge–Deligne numbers. Following [5, §1.5], [3, §0.1] define the virtual Hodge polynomial $e(X; x, y)$ of $X$ to be $e(X; x, y) = \sum_{p+q=0}^{m} \sum_{k=0}^{2m}(-1)^k h^{p,q}(H^k_c(X, \mathbb{C})) x^p y^q$. Set $Y_{H_0}([X]) = e(X; x, y)$, and so in $\Lambda_{H_0}$. Assumption 4.1(i),(ii) for $Y_{H_0}$ follow from [5, Props 1.6 & 1.8], and [5, Ex. 1.10] gives $\ell = Y_{H_0}([\mathbb{A}^1]) = xy$, implying (iii). Thus Assumption 4.1 holds.

If $X$ is a smooth projective $\mathbb{C}$-variety then $h^{p,q}(H^k_c(X, \mathbb{C})) = h^{p,q}(X)$ if $p + q = k$ and 0 otherwise, so $e(X; x, y) = \sum_{p,q=0}^{m}(-1)^{p+q} h^{p,q}(X) x^p y^q$ just encodes the usual Hodge numbers of $X$. The point about virtual Hodge polynomials is that they extend ordinary Hodge polynomials to the non-smooth, non-projective case with the additive and multiplicative properties we need.

As Hodge numbers refine Betti numbers, so the virtual Hodge polynomial $e(X; x, y)$ refines the virtual Poincaré polynomial $P(X; z) = e(X; -z, -z)$, as in Cheah [3, §0.1]. However, virtual Poincaré polynomials work for all algebraically closed $\mathbb{K}$, not just $\mathbb{K} = \mathbb{C}$. I have not been able to find a good reference for the general $\mathbb{K}$ case, though some of the ideas can be found in Deligne [6]. I am grateful to Burt Totaro for explaining it to me.

**Example 4.4.** Define $\Lambda_{P_0} = \mathbb{Q}(z)$, the algebra of rational functions in $z$ with coefficients in $\mathbb{Q}$. Let $\mathbb{K} = \mathbb{C}$ and $X$ be a quasiprojective $\mathbb{C}$-variety. Deligne defined a weight filtration on $H^k_c(X, \mathbb{C})$. Write $W_j(H^k_c(X, \mathbb{C}))$ for the $j^{th}$ quotient space of this filtration. Define $P(X; z) = \sum_{j,k=0}^{m}(-1)^{k-j} \dim W_j(H^k_c(X, \mathbb{C})) z^j$ to be the virtual Poincaré polynomial of $X$. Then $P(X; z) = e(X; -z, -z)$ and $P(X; -1) = \chi(X)$, the Euler characteristic of $X$. Set $Y_{P_0}([X]) = P(X; z)$. As in Example 4.3 Assumption 4.1 holds for $\Lambda_{P_0}, Y_{P_0}$, with $\ell = z^2$.

Here is how to extend this to general algebraically closed $\mathbb{K}$. If $\mathbb{K}$ has characteristic zero and $X$ is a quasiprojective $\mathbb{K}$-variety then $X$ is actually defined over a subfield $\mathbb{K}_0$ of $\mathbb{K}$ which is finitely generated over $\mathbb{Q}$. That is,
$X = X_0 \times_{\text{Spec} K_0} \text{Spec} K$, for $X_0$ a quasiprojective $K_0$-variety, and regarding \text{Spec} $K$ as a $K_0$-scheme. We can embed $K_0$ as a subfield of $\mathbb{C}$, and form a quasiprojective $\mathbb{C}$-variety $X_\mathbb{C} = X_0 \times_{\text{Spec} K_0} \text{Spec} \mathbb{C}$. Define $P(X; z) = P(X_\mathbb{C}; z)$, reducing to the $K = \mathbb{C}$ case, and $\Upsilon_{\text{pt}}([X]) = P(X; z)$. This is independent of choices, and Assumption 4.4 holds with $\ell = z^2$.

If $K$ has characteristic $p > 0$ we use some different ideas, sketched in Deligne [6]. Write $\mathbb{F}_p$ for the finite field with $p$ elements, and $\overline{\mathbb{F}}_p$ for its algebraic closure. Let $l$ be a prime different from $p$. First we explain how to define the virtual Poincaré polynomial of a quasiprojective $\mathbb{F}_p$-variety $X$. Then $X_{\overline{\mathbb{F}}_p} = X \times_{\text{Spec} \mathbb{F}_p} \text{Spec} \overline{\mathbb{F}}_p$ is a quasiprojective $\overline{\mathbb{F}}_p$-variety, so we can form the compactly-supported $l$-adic cohomology $H^c_l(X_{\overline{\mathbb{F}}_p}, \mathbb{Q}_l)$, a finite-dimensional vector space over $\mathbb{Q}_l$.

The geometric Frobenius $Fr$ acts on $X_{\overline{\mathbb{F}}_p}$, and so $Fr^*$ acts on $H^c_l(X_{\overline{\mathbb{F}}_p}, \mathbb{Q}_l)$. In his proof of the Weil Conjecture, Deligne showed that the eigenvalues of $Fr^*$ are ‘Weil numbers of weight $j$’ for $j \geq 0$. Thus we may define a weight filtration on $H^c_j(X_{\overline{\mathbb{F}}_p}, \mathbb{Q}_l)$, whose $j$th quotient space $W^j(H^c_l(X_{\overline{\mathbb{F}}_p}, \mathbb{Q}_l))$ is the eigenspaces of $Fr^*$ with eigenvalues of weight $j$. Then we set

$$P(X; z) = \sum_{j,k=0}^{2m} (-1)^{k-j} \dim \mathbb{Q}_l W_j(H^c_l(X_{\overline{\mathbb{F}}_p}, \mathbb{Q}_l)) z^j.$$

Now let $K$ have characteristic $p > 0$, and $X$ be a quasiprojective $K$-variety. Then $X$ is defined over a subfield $K_0$ of $K$ finitely generated over $\mathbb{F}_p$, so $X = X_0 \times_{\text{Spec} K_0} \text{Spec} K$, for $X_0$ a quasiprojective $K_0$-variety. Regard $X_0$ as a $\mathbb{F}_p$-variety with a dominant morphism $X_0 \rightarrow \text{Spec} K_0$, that is, as a family of quasiprojective $\mathbb{F}_p$-varieties. We specialize this to get a quasiprojective $\mathbb{F}_p$-variety $X_{0}^{sp} = \text{the fibre of } X_0 \times_{\text{Spec} K_0} \text{Spec} \overline{\mathbb{F}}_p \rightarrow \text{Spec} K_0 \times_{\text{Spec} \mathbb{F}_p} \overline{\mathbb{F}}_p$, over a general point. Then we set $P(X; z) = P(X_{0}^{sp}; z)$, reducing to the field case, and $\Upsilon_{\text{pt}}([X]) = P(X; z)$. Again, Assumption 4.4 holds with $\ell = z^2$.

Here is the universal example, through which all other examples factor.

**Example 4.5.** Let $K$ be an algebraically closed field. Define $\Lambda_{\text{uni}}$ to be the $\mathbb{Q}$-algebra generated by isomorphism classes $[X]$ of quasiprojective $K$-varieties $X$ and by $\ell^{-1}$ and $(\ell^k - 1)^{-1}$ for $k = 1, 2, \ldots$ for $\ell \in \{i, 1\}$, with relations $[X] = [X \setminus Y] + [Y]$ for $Y$ a closed subvariety of $X$, and $[X \times Y] = [X] \cdot [Y]$ for $X,Y$ quasiprojective $K$-varieties, and identity $[\text{Spec } K]$. Define $\Upsilon_{\text{uni}}([X]) = [X]$. Then Assumption 4.4 holds trivially.

The drawback is that $\Lambda_{\text{uni}}$ is difficult to describe — Examples 4.3 and 4.4 are much more explicit. It is a modification of the **Grothendieck group $K_0(\text{Var}_K)$ of $K$-varieties**, as in Bittner [1]: $\Lambda_{\text{uni}}$ is $(K_0(\text{Var}_K) \otimes \mathbb{Z} \mathbb{Q})((\ell^{-1}, (\ell^k - 1)^{-1}, k = 1, 2, \ldots]$]. Rings and algebras of this kind are often used in **motivic integration**.

Notice that we have not included **Euler characteristics** in our list of examples, though the Euler characteristic $\chi$ is the most well-known and useful motivic invariant. This is because $\Upsilon([X]) = \chi(X)$ does not satisfy Assumption 4.4 since $\ell = \Upsilon([A^1]) = \chi(A^1) = 1$, so $\ell^k - 1 = 0$ is not invertible in $\Lambda = \mathbb{Q}$ for any $k = 1, 2, \ldots$, and Assumption 4.4(iii) fails. Section 6 will modify our approach for the case $\ell = 1$, to include Euler characteristics.
Assumption 4.1 is equivalent to the hypotheses of Toen [16, Cor. 3.18]. In [16, §3.5] he gives Example 4.3 above and two new examples of suitable data $\Upsilon, \Lambda$, motivic Euler characteristics and $l$-adic Euler characteristics.

4.2 Extending $\Upsilon$ to a homomorphism $SF(Spec \, K) \to \Lambda$

We now extend $\Upsilon$ in §4.1 from quasiprojective $K$-varieties to finite type $K$-stacks with affine geometric stabilizers. We express this as an algebra homomorphism $\Upsilon' : SF(Spec \, K) \to \Lambda$. The next lemma is the reason for Assumption 4.1(iii).

**Lemma 4.6.** Let Assumption 4.1 hold. Then for all $m = 1, 2, \ldots$ we have

$$\Upsilon([GL(m, K)]) = \ell^m(m-1)/2 \prod_{k=1}^{m} (\ell^k - 1),$$

which is invertible in $\Lambda$. (17)

**Proof.** Consider the projection morphism $GL(m, K) \to A^m \{0\}$ taking a matrix to its first column. This is a Zariski locally trivial fibration, with fibre $A^{m-1} \times GL(m-1, K)$. So Lemma 4.2 gives

$$\Upsilon([GL(m, K)]) = \Upsilon([A^m \{0\}]) \cdot \Upsilon(GL(m-1, K))$$

$$= (\ell^m - 1)\ell^{m-1} \cdot \Upsilon(GL(m-1, K)).$$

We deduce (17) by induction on $m$, and invertibility by Assumption 4.1(iii).

**Lemma 4.7.** Let Assumption 4.1 hold, and $G$ be a special algebraic $K$-group. Then $\Upsilon([G])$ is invertible in $\Lambda$.

**Proof.** Embed $G \subseteq GL(m, K)$ with $GL(m, K) \to GL(m, K)/G$ a Zariski locally trivial fibration with fibre $G$. Then $\Upsilon([GL(m, K)]) = \Upsilon([G]) \Upsilon([GL(m, K)/G])$ by Lemma 4.2. But $\Upsilon([GL(m, K)])$ is invertible by Lemma 4.6 so $\Upsilon([G])$ is invertible in $\Lambda$.

If a $K$-stack $\mathcal{R}$ is 1-isomorphic to $[X/G]$ for $X$ a quasiprojective $K$-variety and $G$ a special algebraic $K$-group, we intend to define $\Upsilon'([\mathcal{R}]) = \Upsilon([X])/\Upsilon([G])$ in $\Lambda$. This is independent of choices.

**Proposition 4.8.** Let Assumption 4.1 hold, and $\mathcal{R}$ be a finite type algebraic $K$-stack. Suppose $\mathcal{R}$ is 1-isomorphic to a quotient stack $[X/G]$, where $X$ is a quasiprojective $K$-variety, and $G$ is a special algebraic $K$-group acting on $X$. Then $\Upsilon([X])/\Upsilon([G])$ depends only on $\mathcal{R}$, not on the choice of $X,G$.

**Proof.** Suppose $\mathcal{R}$ is 1-isomorphic to $[X/G]$ and $[Y/H]$, for $X,Y$ quasiprojective $K$-varieties and $G,H$ special algebraic $K$-groups acting on $X,Y$. The 1-isomorphisms $[X/G] \cong \mathcal{R} \cong [Y/H]$ give 1-morphisms $\phi : X \to \mathcal{R}$, $\psi : Y \to \mathcal{R}$ which are atlases, invariant under the $G,H$ actions. Form the fibre product $Z = X \times_{\phi, \mathcal{R}, \psi} Y$. This is a finite type algebraic $K$-space with an action of $G \times H$, such that the $G$- and $H$-actions are free, with $Z/H \cong X$ and $Z/G \cong Y$.

The projections $Z \to X$ and $Z \to Y$ are principal $H$- and $G$-bundles respectively. Therefore $Z$ is a quasiprojective $K$-variety, not just an algebraic $K$-space,
as \(H, X\) are quasiprojective. Also \(Z \to X\) and \(Z \to Y\) are Zariski locally trivial fibrations by Definition 2.1 since \(G, H\) are special. So by Lemma 4.2 we have 
\[
\Upsilon((Z)) = \Upsilon((H))\Upsilon((X)) = \Upsilon((G))\Upsilon((Y)).
\]
Dividing by \(\Upsilon((G))\Upsilon((H))\), which is invertible by Lemma 4.7 proves that \(\Upsilon((X))/\Upsilon((G)) = \Upsilon((Y))/\Upsilon((H))\).

We show by example that the condition that \(G\) is special (or something like it) is necessary in Proposition 4.8.

**Example 4.9.** Let \(\mathbb{K} = \mathbb{C}\) and \(A_{\text{ho}}, \Upsilon_{\text{ho}}\) be as in Example 4.3. Take \(X\) to be the quasiprojective variety \(\mathbb{C} \setminus \{0\}\), with affine \(\mathbb{C}\)-groups \(\{1\}\) and \(\{\pm 1\}\) acting freely on \(X\) by \(\epsilon : x \mapsto \epsilon x\). Take \(\mathcal{R}\) to be \(X\), regarded as a \(\mathbb{C}\)-stack. Then \(\mathcal{R}\) is 1-isomorphic to \([X/\{1\}]\), and also to \([X/\{\pm 1\}]\) by \(\{\pm 1\} \to \{x^2\}\). We have \(\Upsilon_{\text{ho}}([X]) = xy - 1, \Upsilon_{\text{ho}}([\{\pm 1\}]) = 2\) and \(\Upsilon_{\text{ho}}([\{1\}]) = 1\).

\[
\Upsilon_{\text{ho}}([X]/\Upsilon_{\text{ho}}([\{1\}])) = xy - 1 \neq 1/2(xy - 1) = \Upsilon_{\text{ho}}([X])/\Upsilon_{\text{ho}}([\{1\}]),
\]
so \(\Upsilon_{\text{ho}}([X])/\Upsilon_{\text{ho}}([G])\) depends on the choice of \(X, G\) with \(\mathcal{R} \cong [X/G]\). This does not contradict Proposition 4.8 since \(\{\pm 1\}\) is not special. Note also that \(x \mapsto x^2\) is a principal \(\{\pm 1\}\)-bundle which is not a Zariski locally trivial fibration.

By definition \(\mathcal{SF}(\text{Spec} \mathbb{K})\) is spanned by \([[\mathcal{R}, \rho]]\) for \(\rho : \mathcal{R} \to \text{Spec} \mathbb{K}\) a 1-morphism. Since for any \(\mathcal{R}\) there is a projection \(\rho : \mathcal{R} \to \text{Spec} \mathbb{K}\) unique up to 2-isomorphism, we omit \(\rho\), and write \([\mathcal{R}]\) instead of \([[\mathcal{R}, \rho]]\). Recall from 4.3 that \(\mathcal{SF}(\text{Spec} \mathbb{K})\) is a commutative \(\mathbb{Q}\)-algebra. We shall construct an algebra morphism \(\Upsilon' : \mathcal{SF}(\text{Spec} \mathbb{K}) \to \Lambda\).

**Theorem 4.10.** Let Assumption 4.7 hold. Then there exists a unique morphism of \(\mathbb{Q}\)-algebras \(\Upsilon' : \mathcal{SF}(\text{Spec} \mathbb{K}) \to \Lambda\) such that if \(G\) is a special algebraic \(\mathbb{K}\)-group acting on a quasiprojective \(\mathbb{K}\)-variety \(X\) then \(\Upsilon'(\left[[X/G]\right]) = \Upsilon([X])/\Upsilon([G])\), where \(\Upsilon([G])\) is invertible by Lemma 4.7.

**Proof.** By linearity it is enough to define \(\Upsilon'(\left[[\mathcal{R}]\right])\) for \([\mathcal{R}] \in \mathcal{SF}(\text{Spec} \mathbb{K})\). Then \(\mathcal{R}\) is a finite type algebraic \(\mathbb{K}\)-stack with affine geometric stabilizers. Thus by Kresch [13, Prop. 3.5.9] \(\mathcal{R}\) can be stratified by global quotient stacks. This means that the associated reduced stack \(\mathcal{R}_{\text{red}}\) is the disjoint union of finitely many locally closed substacks \(\mathcal{U}_i\), for \(i \in I\) with each \(\mathcal{U}_i\) 1-isomorphic to a global quotient \([X_i/G_i]\), with \(X_i\) a quasiprojective \(\mathbb{K}\)-variety and \(G_i\) an affine \(\mathbb{K}\)-group acting on \(X_i\). (Kresch takes the \(X_i\) to be \(\mathbb{K}\)-schemes, but using varieties is equivalent.) As in [13, Lem. 3.5.1] we can take \(G_i = \text{GL}(m_i, \mathbb{K})\), so in particular we can suppose \(G_i\) is special. Since \(\mathcal{R}_{\text{red}}\) is a closed \(\mathbb{K}\)-substack of \(\mathcal{R}\) with \(\mathcal{R} \setminus \mathcal{R}_{\text{red}}\) empty we have

\[
[\mathcal{R}] = [\mathcal{R}_{\text{red}}] = \sum_{i \in I} [\mathcal{U}_i] = \sum_{i \in I} \left[[X_i/G_i]\right]
\]

in \(\mathcal{SF}(\text{Spec} \mathbb{K})\). Thus, if \(\Upsilon'\) exists at all we must have

\[
\Upsilon'(\left[[\mathcal{R}]\right]) = \sum_{i \in I} \Upsilon\left([[X_i]]/\Upsilon([[G_i]])\right).
\]

(18)

This proves uniqueness of \(\Upsilon'\), if it exists. To show it does, suppose \(\mathcal{R}_{\text{red}}\) is also the disjoint union of finitely many locally closed substacks \(\mathcal{U}_j\) for \(j \in J\) with
$\mathcal{Y}_j$ 1-isomorphic to $[Y_j/H_j]$, with $H_j$ a special algebraic $K$-group acting on a quasi-projective $K$-variety $Y_j$.

Since $\mathcal{U}_i$ is the disjoint union of locally closed $K$-substacks $\mathcal{U}_i \cap \mathcal{W}_j$ for $j \in J$, and $\mathcal{U}_i \cong [X_i/G_i]$, we can write $X_i$ as the disjoint union of locally closed $G_i$-invariant quasi-projective $K$-subvarieties $X_{ij}$ for $j \in J$, with $\mathcal{U}_i \cap \mathcal{W}_j \cong [X_{ij}/G_i]$.

Similarly, we write $Y_j$ as the disjoint union of locally closed, $H_j$-invariant $K$-subvarieties $Y_{ij}$ for $i \in I$, with $\mathcal{U}_i \cap \mathcal{W}_j \cong [Y_{ij}/H_j]$. Thus $[X_{ij}/G_i] \cong [Y_{ij}/H_j]$, so $\mathcal{Y}(X_{ij})/\mathcal{Y}(G_i) = \mathcal{Y}(Y_{ij})/\mathcal{Y}(H_j)$ by Proposition 4.3. Therefore

$$
\sum_{i \in I} \mathcal{Y}(X_{ij})/\mathcal{Y}(G_i) = \sum_{j \in J} \sum_{i \in I} \mathcal{Y}(X_{ij})/\mathcal{Y}(G_i) = \sum_{j \in J} \sum_{i \in I} \mathcal{Y}(Y_{ij})/\mathcal{Y}(H_j),
$$

by Assumption 4.1(i). Thus the right hand side of (18) is independent of choices, and we can take (18) as the definition of $\mathcal{T}'([\mathcal{R}])$.

Using Assumption 4.1(i) we find that if $\mathcal{S}$ is a closed $K$-substack in $\mathcal{R}$ then $\mathcal{Y}'([\mathcal{S}]) = \mathcal{Y}'([\mathcal{S}] \setminus \mathcal{G})$. So $\mathcal{Y}'$ is compatible with the relations (1) defining $\mathcal{SF}(\text{Spec } K)$, and extends uniquely to a $Q$-linear map $\mathcal{Y}': \mathcal{SF}(\text{Spec } K) \to \Lambda$. By Assumption 4.1(ii) we see this is a $Q$-algebra morphism. If $X, G$ are as in the theorem then taking $\mathcal{R} = [X/G], I = \{1\}, X_1 = X, G_1 = G$ in the definition (18) gives $\mathcal{Y}'([X/G]) = \mathcal{Y}([X])/\mathcal{Y}([G])$, as we want. 

Theorem 4.10 is similar to Toen [16, Th. 1.1]. Combining it with Examples 4.3 and 4.4 for $\mathcal{R}$ a finite type algebraic $K$-stack with affine geometric stabilizers, we can define the virtual Hodge function $e(\mathcal{R}; x, y) = \mathcal{Y}_{\text{Ho}}([\mathcal{R}])$ when $K = C$, and the virtual Poincaré function $P(\mathcal{R}; z) = \mathcal{Y}_{\text{Poi}}([\mathcal{R}])$ for general $K$. These are $Q$-rational functions in $x, y$ and $z$ respectively, agree with the usual virtual Hodge and Poincaré polynomials when $\mathcal{R}$ is a quasiprojective $K$-variety, and have additive and multiplicative properties.

Let $G$ be an affine algebraic $K$-group, and take $\mathcal{R} = [\text{Spec } K/G]$, which is a single point $r$ with $\text{Aut}_K(r) = G$. Then the theorem gives $\mathcal{Y}'([\mathcal{R}]) = \mathcal{Y}([\text{GL}(m, K)/G])/\mathcal{Y}([\text{GL}(m, K)])$ for any embedding $G \subseteq \text{GL}(m, K)$. If $G$ is special this reduces to $\mathcal{Y}'([\mathcal{R}]) = \mathcal{Y}([G])^{-1}$, but in general this is false — when $K = C$ and $G = \{\pm 1\}$. Example 4.3 gives $\mathcal{Y}_{\text{Ho}}([\mathcal{R}]) = 1$ but $\mathcal{Y}_{\text{Poi}}([G]) = \frac{1}{2}$.

This has surprising implications. In problems involving ‘counting’ points on Deligne–Mumford stacks, say, one would expect a point $r$ with (finite) stabilizer group $G$ to ‘count’ with weight $1/|G|$. But our discussion shows that $\mathcal{Y}'$ ‘counts’ points with stabilizer $G$ with weight $\mathcal{Y}([\text{GL}(m, K)/G])/\mathcal{Y}([\text{GL}(m, K)])$, which is not $1/|G|$ in general. So these ideas, especially the role of special algebraic $K$-groups, may be telling us about the ‘right’ way to approach counting problems on stacks, such as the invariants studied in [12].

### 4.3 The spaces $\mathcal{SF}(\mathfrak{S}, \mathcal{Y}, \Lambda)$ and their operations

We now integrate the ideas of 4.1–4.2 with the stack function material of 3.3. Here is an extension of Definition 3.1.

**Definition 4.11.** Let Assumption 4.1 hold, and $\mathfrak{S}$ be an algebraic $K$-stack with affine geometric stabilizers. Consider pairs $(\mathcal{R}, \rho)$, where $\mathcal{R}$ is a finite...
type algebraic $\mathbb{K}$-stack with affine geometric stabilizers and $\rho : R \to \mathcal{F}$ is a 1-morphism, with equivalence of pairs as in Definition 3.1. Define $SF(\mathcal{F}, \mathcal{Y}, \Lambda)$ to be the $\Lambda$-module generated by equivalence classes $[(R, \rho)]$ as above, with the following relations:

(i) Given $[(R, \rho)]$ as above and $\mathcal{G}$ a closed $\mathbb{K}$-substack of $R$ we have $[(R, \rho)] = [(\mathcal{G}, \rho_{|\mathcal{G}})] + [(R \setminus \mathcal{G}, \rho_{|R \setminus \mathcal{G}})]$, as in (6).

(ii) Let $R$ be a finite type algebraic $\mathbb{K}$-stack with affine geometric stabilizers, $U$ a quasiprojective $\mathbb{K}$-variety, $\pi_R : R \times U \to R$ the natural projection, and $\rho : R \to \mathcal{F}$ a 1-morphism. Then $[(R \times U, \rho \circ \pi_R)] = \pi(R[[U]])[(R, \rho)]$.

(iii) Given $[(R, \rho)]$ as above and a 1-isomorphism $R \cong [X/G]$ for $X$ a quasiprojective $\mathbb{K}$-variety and $G$ a special algebraic $\mathbb{K}$-group acting on $X$, we have $[(R, \rho)] = \pi((G))^{-1}[(X, \rho \circ \pi)]$, where $\pi : X \to R \cong [X/G]$ is the natural projection 1-morphism. Here $\pi((G))^{-1}$ exists in $\Lambda$ by Lemma 4.7.

Similarly, we could define $SF(\mathcal{F}, \mathcal{Y}, \Lambda)$ to be the $\Lambda$-module generated by $[(R, \rho)]$ with $\rho$ representable, and relations (i)–(iii) as above. But using (i),(iii) above we find $SF(\mathcal{F}, \mathcal{Y}, \Lambda)$ is spanned over $\Lambda$ by $[(X, \rho \circ \pi)]$ for $\mathbb{K}$-varieties $X$. Then $\rho \circ \pi$ is automatically representable, so $[(X, \rho \circ \pi)] \in SF(\mathcal{F}, \mathcal{Y}, \Lambda)$, and $SF(\mathcal{F}, \mathcal{Y}, \Lambda) = SF(\mathcal{F}, \mathcal{Y}, \Lambda)$. Thus we shall not bother with the $SF(\mathcal{F}, \mathcal{Y}, \Lambda)$.

Define a $\mathbb{Q}$-linear projection $\Pi^T_{\mathcal{F}, \Lambda} : SF(\mathcal{F}) \to SF(\mathcal{F}, \mathcal{Y}, \Lambda)$ by

$$\Pi^T_{\mathcal{F}, \Lambda} : \sum_{i \in I} c_i[(R_i, \rho_i)] \mapsto \sum_{i \in I} c_i[(R_i, \rho_i)],$$

using the embedding $\mathbb{Q} \subseteq \Lambda$ to regard $c_i \in \mathbb{Q}$ as an element of $\Lambda$. Then $\Pi^T_{\mathcal{F}, \Lambda}$ is well-defined, as the relation (3) in $SF(\mathcal{F})$ maps to relation (i) above.

The important point here is the relations (i)–(iii) above. These are not arbitrary, but lead to interesting spaces, as our results below will show. In defining a space by generators and relations, one should consider two issues. The first is that any operations on the spaces we define by their action on generators must be compatible with all the relations, or they will not be well-defined. We deal with this in Theorem 4.13 below.

The second is that if we impose too many relations, or inconsistent relations, then the space may be much smaller than we expect, even zero. We will show in Proposition 4.16 below that $SF(\mathcal{F}, \mathcal{Y}, \Lambda)$ is at least as large as $CF(\mathcal{F}) \otimes Q \Lambda$. So the spaces $SF(\mathcal{F}, \mathcal{Y}, \Lambda)$ are quite large (though much smaller than $SF(\mathcal{F}) \otimes Q \Lambda$), and (i)–(iii) have some kind of consistency about them. As in Definition 3.1 $SF(\mathcal{F}, \mathcal{Y}, \Lambda)$ has multiplication, pushforwards, pullbacks and tensor products.

**Definition 4.12.** Let Assumption 4.1 hold, $\mathcal{F}, \mathcal{G}$ be algebraic $\mathbb{K}$-stacks with affine geometric stabilizers, and $\phi : \mathcal{F} \to \mathcal{G}$ a 1-morphism. Define a $\Lambda$-bilinear multiplication $\cdot$ on $SF(\mathcal{F}, \mathcal{Y}, \Lambda)$ by (7). This is commutative and associative as in Definition 3.1. Define the pushforward $\phi_* : SF(\mathcal{F}, \mathcal{Y}, \Lambda) \to SF(\mathcal{G}, \mathcal{Y}, \Lambda)$ by (11), taking the $c_i \in \Lambda$ rather than $c_i \in \mathbb{Q}$. For $\phi$ of finite type, define the pullback $\phi^* : SF(\mathcal{G}, \mathcal{Y}, \Lambda) \to SF(\mathcal{F}, \mathcal{Y}, \Lambda)$ by (12). Define the tensor product $\otimes : SF(\mathcal{F}, \mathcal{Y}, \Lambda) \times SF(\mathcal{G}, \mathcal{Y}, \Lambda) \to SF(\mathcal{F} \times \mathcal{G}, \mathcal{Y}, \Lambda)$ by (13).
Notice that we do not define $\pi_{ij}^{sk}$ for the $\text{SF}(\mathfrak{S}, Y, \Lambda)$, as in \(43\). This is because $\pi_{ij}^{sk}$ is defined using the Euler characteristic $\chi$, and to define their analogues for $\text{SF}(\mathfrak{S}, Y, \Lambda)$ we would need an algebra morphism $\Phi: \Lambda \to \mathbb{Q}$ with $\chi(X) \equiv \Phi \circ \Upsilon([X])$. But $\Upsilon([G_{m}])$ is invertible in $\Lambda$ and $\chi([G_{m}]) = 0$, so no such $\Phi$ exists. The analogue of $\iota_{\mathfrak{S}}$ for $\text{SF}(\mathfrak{S}, Y, \Lambda)$ is $\Pi_{\mathfrak{S}}^{Y, \Lambda} \circ \iota_{\mathfrak{S}}$.

**Theorem 4.13.** These operations $\cdot, \phi_{*}, \phi^{*}$ and $\otimes$ are compatible with the relations (i)--(iii) in Definition\(4.11\) and so are well-defined.

*Proof.* Regard $\cdot, \phi_{*}, \phi^{*}$ and $\otimes$ as defined on generators $[(R, \rho)], [(\mathfrak{S}, \sigma)]$ by \(41\), \(42\), \(11\) giving well-defined elements of $\text{SF}(\ast, Y, \Lambda)$. We have to show that applying $\cdot, \phi_{*}, \phi^{*}$ or $\otimes$ to each relation (i)--(iii) above gives a finite $\Lambda$-linear combination of relations (i)--(iii), that is, relations map to relations. All four are compatible with (i), as for the $\text{SF}(\mathfrak{S})$ case in \(43\). For $\phi_{*}$ and $\otimes$ compatibility with (ii)--(iii) is easy. So we must show $\cdot, \phi^{*}$ are compatible with (ii)--(iii).

For $\cdot, \phi^{*}$, compatibility with (ii) follows as the factor $U$ passes through the appropriate fibre products. So, for instance, we have

$$[(R \times U, \rho \circ \pi_{R})] : [(\mathfrak{S}, \sigma)] = \left(\left(\left(\left((R \times U) \times_{\rho \circ \pi_{R}, \mathfrak{S}, \sigma} \mathfrak{S}, \rho \circ \pi_{R \times U}\right)\right) \times_{\mathfrak{S}, \sigma} \mathfrak{S}, \rho \circ \pi_{R \times U} \circ \pi_{R \times \mathfrak{S}}\right)\right).$$

Therefore right multiplication $\cdot : [(\mathfrak{S}, \sigma)]$ maps (ii) to (ii), and left multiplication does too by commutativity, so $\cdot$ is compatible with (ii). A similar argument works for $\phi^{*}$ and (ii).

Let $[(R, \rho)], [(\mathfrak{S}, \sigma)] \in \text{SF}(\mathfrak{S}, Y, \Lambda)$, with $R \cong [X/G]$ for $X$ a quasiprojective $\mathbb{K}$-variety acted on by a special algebraic $\mathbb{K}$-group $G$. Using Kresch [13, Prop. 3.5.9] as in Theorem\(4.11\) we can find finite sets $I, J$ and $\mathbb{K}$-substacks $R_{i}, \mathfrak{S}_{j}, \mathfrak{S}_{ij}$ in $R, \mathfrak{S}, \mathfrak{S}$ for all $i \in I$ and $j \in J$, such that $R = \coprod_{i \in I} R_{i}, \mathfrak{S} = \coprod_{j \in J} \mathfrak{S}_{j}$, and $\rho, \sigma$ map $\rho: R_{i} \to \mathfrak{S}_{ij}, \sigma: \mathfrak{S}_{j} \to \mathfrak{S}_{ij}$, and $R_{i} \cong [X_{i}/G]$ for $X_{i}$ a $G$-invariant subvariety of $X = \coprod_{i \in I} X_{i}$, and $\mathfrak{S}_{j} \cong [Y_{j}/H_{j}], \mathfrak{S}_{ij} \cong [Z_{ij}/K_{ij}]$ for quasiprojective $\mathbb{K}$-varieties $Y_{j}, Z_{ij}$ acted on by special algebraic $\mathbb{K}$-groups $H_{j}, K_{ij}$.

Refining the decompositions if necessary, we can suppose the 1-morphisms $[X_{i}/G] \to [Z_{ij}/K_{ij}]$ and $[Y_{j}/H_{j}] \to [Z_{ij}/K_{ij}]$ corresponding to $\rho: R_{i} \to \mathfrak{S}_{ij}$ and $\sigma: \mathfrak{S}_{j} \to \mathfrak{S}_{ij}$ are induced by $\mathbb{K}$-variety morphisms $\alpha_{ij}: X_{i} \to Z_{ij}, \beta_{ij}: Y_{j} \to Z_{ij}$ equivalent with respect to $\mathbb{K}$-group morphisms $\gamma_{ij}: G \to K_{ij}$ and $\delta_{ij}: H_{j} \to K_{ij}$. By \(41\) and (i) we see that in $\text{SF}(\mathfrak{S}, Y, \Lambda)$ we have

$$[(R, \rho)] : [(\mathfrak{S}, \sigma)] = \sum_{i \in I, j \in J} \left(\left([X_{i}/G] \times_{Z_{ij}/K_{ij}} [Y_{j}/H_{j}], \rho \circ \pi_{R_{i}}\right)\right). \tag{20}$$

The definitions of fibre products and quotients yield a 1-isomorphism

$$[X_{i}/G] \times_{Z_{ij}/K_{ij}} [Y_{j}/H_{j}] \cong \left(\left([X_{i} \times Y_{j}] \times_{\alpha_{ij} \times \beta_{ij}, Z_{ij} \times \pi_{ij}} (Z_{ij} \times K_{ij})\right) / G \times H_{ij}\right), \tag{21}$$

using the fibre product of $\mathbb{K}$-varieties $X_{i} \times Y_{j}$ and $Z_{ij} \times K_{ij}$ over $Z_{ij} \times Z_{ij}$, where $\pi_{ij}: Z_{ij} \times K_{ij} \to Z_{ij} \times Z_{ij}$ is the $\mathbb{K}$-variety morphism $\pi_{ij}: (z, k) \mapsto (z, k \cdot z)$. Here $G \times H_{ij}$ acts on $X_{i} \times Y_{j}$ by $(g, h): (x, y) \mapsto (g \cdot x, h \cdot y)$, on $Z_{ij} \times K_{ij}$ by
The compatibility of $\mathfrak{z}$ in $\text{SF}$ so on. The analogue of Theorem 3.5 holds for the spaces $\mathcal{F}$ necessary and using [13, Prop. 3.5.9] as above, we can assume that $\phi$ again we can write following for the $\text{SF}$ by a special algebraic $c$ for $\mathcal{K}$ for $\phi$ in $\text{SF}$ $K$ $\text{SF}$ and that the 1-morphisms $\mathcal{K}$ for $\mathcal{X}$ and $\mathcal{Y}$ are induced by $\mathcal{K}$-variety and $\mathcal{K}$-group morphisms $\alpha_i : \mathcal{X}_i \to \mathcal{Z}_i$ and $\gamma_i : \mathcal{G} \to \mathcal{K}_i$.

Next we identify $\text{SF}$ on $\mathcal{K}$, $\mathcal{Y}$, and $\Lambda$. The projections $\Pi_{\mathcal{Y},\Lambda}^\mathcal{X}$ commute with the operations $\cdot$, $\ast$, $\phi^\ast$, $\otimes$ on $\text{SF}^\ast(\mathcal{Y}, \Lambda)$, so that $\phi_\ast \circ \Pi_{\mathcal{Y},\Lambda}^\mathcal{X} = \Pi_{\mathcal{Y},\Lambda}^\mathcal{X} \circ \phi_\ast$ for $\phi : \mathcal{Y} \to \mathcal{G}$, and so on. The analogue of Theorem 3.13 holds for the spaces $\text{SF}^\ast(\mathcal{Y}, \Lambda)$.

Next we identify $\text{SF}^\ast(\text{Spec} \mathbb{K}, \mathcal{Y}, \Lambda)$.

**Proposition 4.15.** The map $i_\Lambda : \Lambda \to \text{SF}^\ast(\text{Spec} \mathbb{K}, \mathcal{Y}, \Lambda)$ taking $i_\Lambda : c \to c[\text{Spec} \mathbb{K}]$ is an isomorphism of algebras.
Proof. As in the proof of Theorem 4.10, \( \text{SF}(\text{Spec } K, \Upsilon, \Lambda) \) is generated over \( \Lambda \) by elements \( [X/G] \) for \( X \) a quasiprojective \( K \)-variety acted on by a special algebraic \( K \)-group \( G \). But using Definition 4.11(ii), (iii) and \( X \cong \text{Spec } K \times X \) we see that
\[
[X/G] = \Upsilon([G])^{-1} = \Upsilon([G])^{-1}[\text{Spec } K \times X] = \Upsilon([G])^{-1} \Upsilon([X])|\text{Spec } K,
\]

so \( \text{SF}(\text{Spec } K, \Upsilon, \Lambda) \) is generated over \( \Lambda \) by \( |\text{Spec } K| \), and \( i_\Lambda \) is surjective.

Define \( \pi_\Lambda : \text{SF}(\text{Spec } K, \Upsilon, \Lambda) \to \Lambda \) by \( \pi_\Lambda : \sum_{i \in I} c_i [R_i] \mapsto \sum_{i \in I} c_i \Upsilon([R_i]) \), for \( I \) a finite set, \( c_i \in \Lambda \) and \( \Upsilon' \) as in Theorem 4.11. Using Theorem 4.11 it is easy to check \( \pi_\Lambda \) is compatible with Definition 4.11(i)–(iii) for \( \text{SF}(\text{Spec } K, \Upsilon, \Lambda) \), and so is well-defined. But \( \pi_\Lambda([\text{Spec } K]) = 1 \), so \( \pi_\Lambda \circ i_\Lambda \) is the identity on \( \Lambda \) by \( \Lambda \)-linearity. Thus \( i_\Lambda \) is injective, and so it is an isomorphism. \( \square \)

Using this we show the spaces \( \text{SF}(\tilde{\mathfrak{g}}, \Upsilon, \Lambda) \) are at least as big as \( \text{CF}(\tilde{\mathfrak{g}}) \otimes_\Lambda \Lambda \).

**Proposition 4.16.** The following map is \( \Lambda \)-linear and injective:
\[
(\Pi_{\tilde{\mathfrak{g}}}^{\Upsilon, \Lambda} \circ i_{\tilde{\mathfrak{g}}}) \otimes_\Lambda \text{id}_\Lambda : \text{CF}(\tilde{\mathfrak{g}}) \otimes_\Lambda \Lambda \to \text{SF}(\tilde{\mathfrak{g}}, \Upsilon, \Lambda).
\] (22)

Proof. \( \Lambda \)-linearity is obvious. Let \( f \in \text{CF}(\tilde{\mathfrak{g}}) \otimes_\Lambda \Lambda \) and \( x : \text{Spec } K \to \tilde{\mathfrak{g}} \) be a 1-morphism. It is easy to show from the definitions that \( i_\Lambda^{-1} \circ x_*( ((\Pi_{\tilde{\mathfrak{g}}}^{\Upsilon, \Lambda} \circ i_{\tilde{\mathfrak{g}}}) \otimes_\Lambda \text{id}_\Lambda)(f)) = f([x]) \in \Lambda \). Thus \( ((\Pi_{\tilde{\mathfrak{g}}}^{\Upsilon, \Lambda} \circ i_{\tilde{\mathfrak{g}}}) \otimes_\Lambda \text{id}_\Lambda)(f) = 0 \) only if \( f([x]) = 0 \) for all \( x : \text{Spec } K \to \tilde{\mathfrak{g}} \), that is, only if \( f = 0 \), so (22) is injective. \( \square \)

This prompts the following intuitive explanation of the spaces \( \text{SF}(\tilde{\mathfrak{g}}, \Upsilon, \Lambda) \), which was the author’s motivation for inventing them. In [12] we considered constructible functions \( \text{CF}(\tilde{\mathfrak{g}}) \), with pushforwards \( \text{CF}^{\text{stk}}(\phi) \) defined by ‘integration’ using the Euler characteristic \( \chi \). We can think of \( \text{SF}(\tilde{\mathfrak{g}}, \Upsilon, \Lambda) \) as being like constructible functions \( \text{CF}(\tilde{\mathfrak{g}}) \otimes_\Lambda \Lambda \) with values in \( \Lambda \), with pushforwards \( \phi_* \) defined by ‘integration’ using \( \Upsilon \) instead of \( \chi \).

In fact, pushforwards on \( \text{CF}(\tilde{\mathfrak{g}}) \otimes_\Lambda \Lambda \) using \( \Upsilon \) do not usually satisfy the analogue of (1), because for a non-Zariski-locally-trivial fibration \( \pi : X \to Y \) with fibre \( F \) we have \( \chi(X) = \chi(F) \chi(Y) \) but \( \Upsilon([X]) \neq \Upsilon([F]) \Upsilon([Y]) \) in general, as in Example 4.19. So to get a theory with the properties we want (Theorem 4.10), we must allow \( \text{SF}(\tilde{\mathfrak{g}}, \Upsilon, \Lambda) \) to be larger than \( \text{CF}(\tilde{\mathfrak{g}}) \otimes_\Lambda \Lambda \) to keep track of \( \rho : \mathfrak{R} \to \tilde{\mathfrak{g}} \) which are non-Zariski-locally-trivial fibrations over substacks of \( \tilde{\mathfrak{g}} \). All fibrations over \( \text{Spec } K \) are Zariski locally trivial, so \( \text{SF}(\text{Spec } K, \Upsilon, \Lambda) \) reduces to \( \Lambda = \text{CF}(\text{Spec } K) \otimes_\Lambda \Lambda \), as in Proposition 4.15.

The spaces \( \text{SF}(\tilde{\mathfrak{g}}, \Upsilon, \Lambda) \) will be important tools in the series [9–12]. Given a \( K \)-linear abelian category \( \mathcal{A} \) we shall define the moduli \( K \)-stack \( \text{Obj}_\Lambda \) of objects in \( \mathcal{A} \). Then \( \text{SF}(\text{Obj}_\Lambda, \Upsilon, \Lambda) \) is well-defined, and in [10] using the Ringel–Hall algebra idea we define an associative multiplication \( * \) on it, different from ‘·’, making it into a noncommutative \( \Lambda \)-algebra. Examples of this yield quantized universal enveloping algebras of Kac–Moody algebras.

An advantage of working with spaces \( \text{SF}(\bullet, \Upsilon, \Lambda) \) rather than \( \text{SF}(\bullet) \) is that because of the relations Definition 4.11(i)–(iii), special properties of \( \mathcal{A} \) such as \( \text{Ext}^i(X, Y) = 0 \) for all \( X, Y \in \mathcal{A} \) and \( i > 1 \) are translated in [10] to extra
identities in $\mathcal{SF}(\mathcal{Ob}_\mathcal{A}, Y, \Lambda)$, telling us something special about this algebra. In [12] we use Proposition 4.15 to project elements of $\mathcal{SF}(\mathcal{Ob}_\mathcal{A}, Y, \Lambda)$ to $\Lambda$, and so define interesting invariants in $\Lambda$ which ‘count’ $\tau$-(semi)stable objects in $\mathcal{A}$.

5 Virtual rank and projections $\Pi^v_{ni}$ on $\mathcal{SF}(\mathcal{F})$

Section 4 assumed $\ell - 1$ is invertible in $\Lambda$, and we want to relax this assumption. The basic reason for it is that $[[\text{Spec } K/G]] = \Upsilon([G])^{-1}[[\text{Spec } K]]$ in $\mathcal{SF}(\text{Spec } K, Y, \Lambda)$, and if $G$ has maximal torus $T^G \cong G_n^n$ then $\Upsilon([T^G]) = (\ell - 1)^n$ divides $\Upsilon([G])$. In this section we shall define new spaces $\mathcal{SF}, \mathcal{SF}(\mathcal{F}, Y, \Lambda)$ with finer relations, which keep track of maximal tori. These will satisfy

$$[[\text{Spec } K/G]] = \Upsilon((G/T^G))^{-1}[[\text{Spec } K/T^G]] + \text{‘lower order terms’},$$

and because $\ell - 1$ does not divide $\Upsilon((G/T^G))$ it will no longer be necessary for $\ell - 1$ to be invertible, as we will see in §6.

To do this we need the difficult idea of virtual rank. The rank $\text{rk}(G)$ of an affine algebraic $K$-group $G$ is the dimension of any maximal torus $T^G$. We begin in §5.1 by defining the real rank projections $\Pi^r_n : \mathcal{SF}(\mathcal{F}) \to \mathcal{SF}(\mathcal{F})$ which project $[(\mathcal{R}, \rho)]$ to $[(\mathcal{R}_n, \rho)]$, where $\mathcal{R}_n$ is the $K$-substack of points $r \in \mathcal{R}(K)$ with stabilizer groups $\text{Aut}_K(r)$ of rank $n$. This is primarily for motivation.

Section 5.2 then defines analogous virtual rank projections $\Pi^v_n : \mathcal{SF}(\mathcal{F}) \to \mathcal{SF}(\mathcal{F})$. These coincide with the $\Pi^r_n$ on $[(\mathcal{R}, \rho)]$ when $\mathcal{R}$ has abelian stabilizer groups, but points $r$ with $\text{Aut}_K(r)$ nonabelian of rank $k$ split into components with ‘virtual rank’ $n < k$. Using these ideas, §5.3 defines spaces $\mathcal{SF}, \mathcal{SF}(\mathcal{F}, \mathcal{Y}, \Lambda)$ similar to those of §4.3 on which operations $\phi^*$, $\phi_*$ and $\Pi^v_n$ are well-defined.

5.1 Real rank and projections $\Pi^r_n$

We define a family of commuting projections $\Pi^r_n : \mathcal{SF}(\mathcal{F}) \to \mathcal{SF}(\mathcal{F})$ for $n = 0, 1, \ldots$ which project to the part of $\mathcal{SF}(\mathcal{F})$ spanned by $[(\mathcal{R}, \rho)]$ such that the stabilizer group $\text{Aut}_K(r)$ has rank $n$ for all $r \in \mathcal{R}(K)$. The superscript ‘$r$’ is short for ‘real’, meaning that the $\Pi^r_n$ decompose $\mathcal{SF}(\mathcal{F})$ by the real (actual) rank of stabilizer groups.

**Definition 5.1.** If $\mathcal{R}$ is an algebraic $K$-stack and $r \in \mathcal{R}(K)$ then $\text{Aut}_K(r)$ is an algebraic $K$-group, so the rank $\text{rk} \text{(Aut}_K(r))$ is well-defined. There is a natural topology on $\mathcal{R}(K)$, in which the open sets are $\mathcal{U}(K)$ for open $K$-substacks $\mathcal{U} \subset \mathcal{R}$. In this topology the function $r \mapsto \text{rk} \text{(Aut}_K(r))$ is upper semicontinuous. Thus, there exist locally closed $K$-substacks $\mathcal{R}_n$ in $\mathcal{R}$ for $n = 0, 1, \ldots$, such that $\mathcal{R}(K) = \bigsqcup_{n \geq 0} \mathcal{R}_n(K)$, and $r \in \mathcal{R}(K)$ has $\text{rk} \text{(Aut}_K(r)) = n$ if and only if $r \in \mathcal{R}_n(K)$. If $\mathcal{R}$ is of finite type then $\mathcal{R}_n = \emptyset$ for $n \gg 0$.

Now let $\mathcal{F}$ be an algebraic $K$-stack with affine geometric stabilizers, and $\mathcal{SF}(\mathcal{F})$ be as in §3. Define $K$-linear maps $\Pi^v_n : \mathcal{SF}(\mathcal{F}) \to \mathcal{SF}(\mathcal{F})$ for $n = 0, 1, \ldots$ on the generators $[(\mathcal{R}, \rho)]$ of $\mathcal{SF}(\mathcal{F})$ by $\Pi^v_n : [(\mathcal{R}, \rho)] \mapsto [(\mathcal{R}_n, \rho|_{\mathcal{R}_n})]$, for $\mathcal{R}_n$ defined as above. If $\mathcal{G}$ is a closed substack of $\mathcal{R}$ it is easy to see that $\mathcal{G}_n$ is a closed substack of $\mathcal{R}_n$ and $(\mathcal{R} \setminus \mathcal{G})_n = \mathcal{R}_n \setminus \mathcal{G}_n$. Thus, $\Pi^v_n$ is compatible with
the relations (6) in $\mathcal{SF}(\mathfrak{F})$, and is well-defined. If $\rho : \mathfrak{R} \to \mathfrak{F}$ is representable then so is $\rho|_{\mathfrak{R}_n}$, so the restriction to $\mathcal{SF}(\mathfrak{F})$ maps $\Pi^e_n : \mathcal{SF}(\mathfrak{F}) \to \mathcal{SF}(\mathfrak{F})$.

Here are some easy properties of the $\Pi^e_n$. The proofs are left as an exercise.

**Proposition 5.2.** In the situation above, we have:

(i) $(\Pi^e_n)^2 = \Pi^e_n$, so that $\Pi^e_n$ is a projection, and $\Pi^e_m \circ \Pi^e_n = 0$ for $m \neq n$.

(ii) For all $f \in \mathcal{SF}(\mathfrak{F})$ we have $f = \sum_{n \geq 0} \Pi^e_n(f)$, where the sum makes sense as $\Pi^e_n(f) = 0$ for $n \gg 0$.

(iii) If $\phi : \mathfrak{F} \to \mathfrak{G}$ is a 1-morphism of algebraic $\mathbb{K}$-stacks with affine geometric stabilizers then $\Pi^e_n \circ \phi_* = \phi_* \circ \Pi^e_n : \mathcal{SF}(\mathfrak{F}) \to \mathcal{SF}(\mathfrak{G})$.

(iv) If $f \in \mathcal{SF}(\mathfrak{F})$, $g \in \mathcal{SF}(\mathfrak{G})$ then $\Pi^e_n(f \otimes g) = \sum_{m=0}^n \Pi^e_m(f) \otimes \Pi^e_{n-m}(g)$.

### 5.2 Operators $\Pi^\mu$ and projections $\Pi^\nu_n$

Next we study a family of commuting operators $\Pi^\mu$ on $\mathcal{SF}(\mathfrak{F})$ defined by a weight function $\mu$, which include as special cases projections $\Pi^c_n$ for $n \geq 0$ similar to the $\Pi^c_n$ of Definition 5.1. But the $\Pi^\mu, \Pi^c_n$ are much more subtle and difficult than the $\Pi^c_n$, as applied to $[\mathfrak{R}(\rho)]$ they modify $\mathfrak{R}$ in a very nontrivial way, rather than just restricting to substacks $\mathfrak{R}_n$. Roughly speaking, $\Pi^\mu$ replaces a point in $\mathfrak{R}$ with stabilizer group $G$ by a linear combination of points with stabilizer groups $C_G(T)$, for certain subgroups $T$ of the maximal torus $T^G$ of $G$.

From Definition [5.3] until Lemma [5.9] we take $X$ to be a quasiprojective $\mathbb{K}$-variety acted on by an affine algebraic $\mathbb{K}$-group $G$, with maximal torus $T^G$.

**Definition 5.3.** If $S \subseteq T^G$ define $X^S$ to be the $\mathbb{K}$-subvariety of $X$ fixed by all elements of $S$. Then $X^S$ is closed, but not necessarily irreducible, and $X^S(\mathbb{K}) = \{ x \in X(\mathbb{K}) : t \cdot x = x \text{ for all } t \in S \}$. For such $X, S$ define $P$ to be the $\mathbb{K}$-subgroup of $T^G$ fixing the subvariety $X^S$. Then $P$ is a closed $\mathbb{K}$-subgroup of $T^G$, containing $S$, and $\overline{P}(\mathbb{K}) = \{ t \in T^G(\mathbb{K}) : t \cdot x = x \text{ for all } x \in X^S(\mathbb{K}) \}$. As $S \subseteq P$ we have $X^P \subseteq X^S$. But also $X^S \subseteq X^P$ by definition of $P$, so $X^P = X^S$. Thus, $X^P$ and $P$ determine each other. Define $\mathcal{P}(X, T^G)$ to be the set of closed $\mathbb{K}$-subgroups $P$ of $T^G$ such that $P$ is the $\mathbb{K}$-subgroup of $T^G$ fixing $X^P$.

**Lemma 5.4.** (i) $\mathcal{P}(X, T^G)$ is finite.

(ii) $\mathcal{P}(X, T^G)$ is closed under intersections, with maximal element $T^G$ and minimal element $P_{\text{min}}$ the subgroup of $T^G$ acting trivially on $X$.

(iii) If $S \subseteq T^G$ then $X^S = X^P$, where $P$ is the unique smallest element of $\mathcal{P}(X, T^G)$ containing $S$.

**Proof.** The map $x \mapsto \text{Stab}_{T^G}(x)$ is a constructible map from $X$ to $\mathbb{K}$-subgroups of $T^G$, and so realizes finitely many values $H_1, \ldots, H_n$ say. These stratify $X$ into locally closed subvarieties $X_1, \ldots, X_n$ with $x \in X_i$ if and only if $\text{Stab}_{T^G}(x) = H_i$. For any $S \subseteq T^G$, $X^S$ is the union of those $X_i$ for which $S \subseteq H_i$, and the corresponding $P$ constructed above is the intersection of the corresponding $H_i$. 

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or $T^G$ if there are no $H_i$. Therefore $\mathcal{P}(X, T^G)$ is exactly the set of intersections of nonempty subsets of $\{ T^G, H_1, \ldots, H_n \}$. This proves (i) and the first two parts of (ii). For the last part of (ii), the minimal element of $\mathcal{P}(X, T^G)$ is $T^G \cap H_1 \cap \cdots \cap H_n$, which is $P_{\min}$. Part (iii) follows easily from the discussion in Definition 5.3. \hfill \Box

**Definition 5.5.** If $S \subset T^G$ then $Q = T^G \cap C(C_G(S))$ is a closed $K$-subgroup of $T^G$ containing $S$. As $S \subset Q$ we have $C_G(Q) \subset C_G(S)$. But $Q$ commutes with $C_G(S)$, so $C_G(S) \subset C_G(Q)$. Thus $C_G(S) = C_G(Q)$. So $Q = T^G \cap C(C_G(Q))$, and $Q$ and $C_G(Q)$ determine each other, given $G, T^G$. Define $\mathcal{Q}(G, T^G)$ to be the set of closed $K$-subgroups $Q$ of $T^G$ such that $Q = T^G \cap C(C_G(Q))$.

**Lemma 5.6.** (i) $\mathcal{Q}(G, T^G)$ is finite.

(ii) $\mathcal{Q}(G, T^G)$ is closed under intersections, with maximal element $T^G$ and minimal element $Q_{\min} = T^G \cap C(G)$.

(iii) If $S \subset T^G$ then $C_G(S) = C_G(Q)$, where $Q$ is the unique smallest element of $\mathcal{Q}(G, T^G)$ containing $S$.

**Proof.** The proof is similar to Lemma 5.1. The map $t \mapsto C_G(\{ t \})$ is constructible from $T^G$ to closed $K$-subgroups of $G$, and realizes finitely many values $H_1, \ldots, H_n$. Set $Q_i = T^G \cap C(H_i)$. Then $\mathcal{Q}(G, T^G)$ is the set of intersections of nonempty subsets of $\{ T^G, Q_1, \ldots, Q_n \}$. We leave the details to the reader. \hfill \Box

We calculate $\mathcal{Q}(G, T^G)$ for the case $G = \text{GL}(m, K)$.

**Example 5.7.** Set $G = \text{GL}(m, K)$ with maximal torus $T^G = \mathbb{G}_m^m$, the subgroup of diagonal matrices. Fix $t \in T^G$, which may be written $\text{diag}(t_1, \ldots, t_m)$ for $t_i \in \mathbb{G}_m$. Let $t_1, \ldots, t_m$ realize $n$ distinct values $u_1, \ldots, u_n$. Then there is a unique surjective map $\phi : \{ 1, \ldots, m \} \to \{ 1, \ldots, n \}$ with $t_i = u_{\phi(i)}$. It is easy to show that $C_G(\{ t \})$ is the subgroup of matrices $(A_{ij})_{i,j=1}^m$ in $\text{GL}(m, K)$ with $A_{ij} = 0$ if $\phi(i) \neq \phi(j)$. Hence

$$C_G(\{ t \}) \cong \prod_{k=1}^n \text{GL}(\phi^{-1}(\{ k \}), K). \quad (23)$$

The centre $C(C_G(\{ t \}))$ of $C_G(\{ t \})$, which agrees with $T^G \cap C(C_G(\{ t \}))$, is

$$\{ \text{diag}(q_1, \ldots, q_m) : q_i \in \mathbb{G}_m, q_i = q_j \text{ if } \phi(i) = \phi(j), \text{ all } i, j \} \cong \mathbb{G}_m^n. \quad (24)$$

Since $\mathcal{Q}(G, T^G)$ is the set of $T^G \cap C(C_G(\{ t \}))$ for all $t \in T^G$, we see $\mathcal{Q}(G, T^G)$ is the set of tori $\{ 24 \}$ for all $1 \leq n \leq m$ and surjective $\phi : \{ 1, \ldots, m \} \to \{ 1, \ldots, n \}$.

**Definition 5.8.** For $P' \subset P$ in $\mathcal{P}(X, T^G)$ and $Q' \subset Q$ in $\mathcal{Q}(G, T^G)$, set

$$m_{T^G}(P', P) = \sum_{A \subset \{ \hat{P} \in \mathcal{P}(X, T^G) : \hat{P} \subset P \} : P \in A} (-1)^{|A|-1}, \quad (25)$$

$$n_{T^G}(Q', Q) = \sum_{B \subset \{ \hat{Q} \in \mathcal{Q}(G, T^G) : \hat{Q} \subset Q \} : Q \in B} (-1)^{|B|-1}. \quad (26)$$
Define $\mathcal{R}(X, G, T^G) = \{P \cap Q : P \in \mathcal{P}(X, T^G), Q \in Q(G, T^G)\}$. For $P \in \mathcal{P}(X, T^G)$, $Q \in Q(G, T^G)$ and $R \in \mathcal{R}(X, G, T^G)$ with $R \subseteq P \cap Q$, define

$$M^X_{G, Q}(P, Q, R) = \left| \frac{N_G(T^G)}{C_G(Q) \cap N_G(T^G)} \right|^{-1} \sum_{P' \in \mathcal{P}(X, T^G), Q' \in Q(G, T^G) : P' \subseteq P, Q' \subseteq Q, R = P' \cap Q'} m^X_{T^G}(P', P) n^G_{T^G}(Q', Q).$$

(27)

Here $C_G(T^G) \subseteq C_G(Q)$ as $Q \subseteq T^G$, so $C_G(Q) \cap N_G(T^G)$ is a subgroup of $N_G(T^G)$ containing $C_G(T^G)$. But $C_G(T^G)$ is of finite index in $N_G(T^G)$ as $W(G, T^G) = N_G(T^G)/C_G(T^G)$ is finite. Hence $N_G(T^G)/(C_G(Q) \cap N_G(T^G))$ is finite, and $M^X_{G, Q}(P, Q, R)$ is well-defined.

**Lemma 5.9.** If $M^X_{G, Q}(P, Q, R) \neq 0$ then $P$ is the smallest element of $\mathcal{P}(X, T^G)$ containing $P \cap Q$, and $Q$ the smallest element of $Q(G, T^G)$ containing $P \cap Q$. Therefore $X^P = X^{P \cap Q}$ and $C_G(Q) = C_G(P \cap Q)$.

**Proof.** Using (25)–(26) we may rewrite the sum in (27) as

$$\sum_{A \subseteq \{P \in \mathcal{P}(X, T^G) : P \subseteq P\}, B \subseteq \{Q \in Q(G, T^G) : Q \subseteq Q\}, P \in A, Q \in B, \cap_{P \in A} \cap_{Q \in B} \bar{Q} = R} (-1)^{|A|+|B|}.$$

(28)

Suppose there exists $P' \in \mathcal{P}(X, T^G)$ with $P \cap Q \subseteq P' \subseteq P$ and $P' \neq P$. Then in (28) the intersection $\cap_{P \in A} \cap_{Q \in B} \bar{Q}$ is unchanged by whether $P' \in A$, as it lies in $P \cap Q$. Thus for each pair $A, B$ in (28) with $P' \notin A$ there corresponds another pair $A \cup \{P'\}, B$, and the total contribution of both is $(-1)^{|A|+|B|} + (-1)^{|A|+1+|B|} = 0$. So (28) and $M^X_{G, Q}(P, Q, R)$ are zero. Conversely, if $M^X_{G, Q}(P, Q, R) \neq 0$ there exists no such $P'$, so $P$ is the smallest element of $\mathcal{P}(X, T^G)$ containing $P \cap Q$. The argument for $Q$ is the same. The final part follows from Lemmas 5.4(iii) and 5.5(iii).

Now we define some linear maps $\Pi^\mu : \text{SF}(\mathfrak{g}) \to \text{SF}(\mathfrak{g})$.

**Definition 5.10.** A weight function $\mu$ is a map

$$\mu : \{\text{K-groups } \mathbb{G}_m^k \times K, k \geq 0, K \text{ finite abelian, up to isomorphism}\} \to \mathbb{Q}.$$

For any algebraic K-stack $\mathfrak{g}$ with affine geometric stabilizers, we will define a linear map $\Pi^\mu : \text{SF}(\mathfrak{g}) \to \text{SF}(\mathfrak{g})$. Now $\text{SF}(\mathfrak{g})$ is generated by elements $[[\mathfrak{g}, \rho]]$ with $\mathfrak{g}$ 1-isomorphic to a global quotient $[X/G]$, for $X$ a quasiprojective K-variety and $G$ a special algebraic K-group, with maximal torus $T^G$. For such $\mathfrak{g}, \rho, X, G, T^G$ define

$$\Pi^\mu\left([[\mathfrak{g}, \rho]]\right) = \sum_{P \in \mathcal{P}(X, T^G), Q \in Q(G, T^G) \text{ and } R \in \mathcal{R}(X, G, T^G) : R \subseteq P \cap Q, M^X_{G, Q}(P, Q, R) \mu(R)} [[(X^P/C_G(Q), \rho)_{t^{P \cap Q}}]].$$

(29)

Here $X^P = X^{P \cap Q}$ and $C_G(Q) = C_G(P \cap Q)$ by Lemma 5.9 so $X^P$ is $C_G(Q)$-invariant, and the stack $[X^P/C_G(Q)]$ is well-defined. The inclusions $X^P \subseteq X$, $C_G(Q) \subseteq G$ induce a 1-morphism $t^{P \cap Q} : [X^P/C_G(Q)] \to [X/G]$. As the $t^{P \cap Q}$ are representable, if $[[\mathfrak{g}, \rho]] \in \text{SF}(\mathfrak{g})$ then $\Pi^\mu\left([[\mathfrak{g}, \rho]]\right) \in \text{SF}(\mathfrak{g})$. 29
An informal but helpful way to rewrite (29) is:

\[
\Pi^\mu(\{G, \rho\}) = \int_{t \in T^G} \frac{|\{w \in W(G, T^G) : w \cdot t = t\}|}{|W(G, T^G)|} \left[\left(\frac{X(t)}{C_G(t)}\right), \rho \circ \iota(t)\right] d\mu. \tag{30}
\]

Here \(d\mu\) is a measure on a class of subsets of \(T^G\) described below. Lemmas 5.4(iii), 5.6(iii) give \([([X(t)/C_G(t)], \rho \circ \iota(t)) = ([X_p/C_G(Q)], \rho \circ \iota_p \circ \iota_Q)\), with \(P, Q\) the unique smallest elements of \(P(X, T^G), Q(G, T^G)\) containing \(t\). Also

\[
\frac{|\{w \in W(G, T^G) : w \cdot t = t\}|}{|W(G, T^G)|} = \frac{|(C_G(t)) \cap N_G(T^G)/C_G(T^G)|}{|W(G, T^G)|} = \frac{|(C_G(Q) \cap N_G(T^G))/C_G(T^G)|}{|N_G(T^G)|} = \frac{N_G(T^G)}{C_G(Q) \cap N_G(T^G)}^{-1}. \tag{31}
\]

Thus the integrand in (30) at \(t\) depends only on \(P, Q\).

Therefore the subsets of \(T^G\) the measure \(d\mu\) must be defined upon for (30) to make sense, are those generated from \(P(X, T^G), Q(G, T^G)\) by Boolean operations. This is determined uniquely by setting \(d\mu(R) = \mu(R)\) for \(R \in R(X, G, T^G)\). We find that for \(P \in P(X, T^G)\) and \(Q \in Q(G, T^G)\) we have

\[
d\mu\left[\begin{array}{c}
P \cap Q \setminus \bigcup_{P' \in P, Q' \in Q : P' \not\subseteq P, Q' \not\subseteq Q, (P, Q) \neq (P', Q')} P' \cap Q' \end{array}\right] = \sum_{P' \in P(X, T^G), Q' \in Q(G, T^G): P' \not\subseteq P, Q' \not\subseteq Q} m^X_{\iota_G}(P', P)m_G^G(Q', Q) \mu(P' \cap Q'). \tag{32}
\]

As \(T^G\) is the disjoint union over \(P, Q\) of the sets \([\cdots]\) on the top line of (32), comparing (27) and (29)–(32) we see (31) and (33) are equivalent.

As the integrand in (30) is invariant under the action of \(W(G, T^G)\), we can simplify (30) further by pushing the integration down to \(T^G/W(G, T^G)\):

\[
\Pi^\mu(\{G, \rho\}) = \int_{W(G, T^G)/W(G, T^G)} \left[\left(\frac{X(t)}{C_G(t)}\right), \rho \circ \iota(t)\right] d\mu. \tag{33}
\]

Now \(T^G/W(G, T^G)\) is a natural object in algebraic group theory, as it is isomorphic to \(G_{ss}/\text{Ad}(G)\), where \(G_{ss}\) is the open set of semisimple elements of \(G\). In the quotient stack \([G_{ss}/\text{Ad}(G)]\) the stabilizer group \(\text{Aut}_K(tW(G, T^G))\) is \(C_G(t)\). So (33) is an integral over \([G_{ss}/\text{Ad}(G)]\) of a function of the stabilizer group. Probably there is some extension of this construction to integrate over all of \([G/\text{Ad}(G)]\), replacing \(T^G\) by a Borel subgroup perhaps, but we do not consider it. We show \(\Pi^\mu\) is independent of choices in its definition.

**Theorem 5.11.** In the situation above, \(\Pi^\mu(\{G, \rho\})\) is independent of the choices of \(X, G, T^G\) and 1-isomorphism \(\mathfrak{R} \cong [X/G]\), and \(\Pi^\mu\) extends to unique linear maps \(\Pi^\mu : \text{SF}(\mathfrak{R}) \to \text{SF}(\mathfrak{R})\) and \(\Pi^\mu : \text{SF}(\mathfrak{R}) \to \text{SF}(\mathfrak{R})\).
Proof. Suppose \( R \) is 1-isomorphic to \([X/G]\) and \([Y/H]\), for \( X, Y \) quasiprojective \( \mathbb{K} \)-varieties and \( G, H \) special algebraic \( \mathbb{K} \)-groups acting on \( X, Y \). Define \( Z, I, Z_i, X_i, Y_i \) as in the proof of Proposition 4.8. Since (29) is additive over \( X = \coprod_{i \in I} X_i \), it is enough to prove it gives the same answer for \([X_i/G]\) and \([Y_i/H]\), for all \( i \in I \). Thus for simplicity we replace \( X_i, Y_i, Z_i \) by \( X, Y, Z \). That is, we have a quasiprojective \( \mathbb{K} \)-variety \( Z \) with a \( G \times H \)-action, such that the \( G \)- and \( H \)-actions are free and induce projections \( \pi_Y : Z \to Y \) and \( \pi_X : Z \to X \) which are \( G \)- and \( H \)-principal bundles respectively. Also \([Z/(G \times H)] \cong [X/G] \cong [Y/H]\). Fix maximal tori \( T^G, T^H \) in \( G, H \).

Let \( t \in T^G \). We will relate \( X(t) \) to \( Z(t, \hat{t}) \) for \( \hat{t} \in T^H \). Suppose \( x \in X(t) \), and \( z \in \pi_X^{-1}([x]) \subseteq Z \). Then the projection \( \pi_G : G \times H \to G \) induces an isomorphism \( \text{Stab}_{G \times H}(z) \to \text{Stab}_G(x) \). So as \( t \in \text{Stab}_G(x) \) there is a unique \( h \in H \) with \( (t, h) \in \text{Stab}_{G \times H}(z) \). Now \( G, H \) and \( G \times H \) are connected, as \( G, H \) are special. Thus elements of \( G, H, G \times H \) are semisimple if and only if they lie in a maximal torus.

As \( t \in T^G \) it is semisimple in \( G \). So \( t \) is semisimple in \( \text{Stab}_G(x) \subseteq G \). Thus \( (t, h) \) is semisimple in \( \text{Stab}_{G \times H}(z) \) as \( \text{Stab}_G(x) \cong \text{Stab}_{G \times H}(z) \), and so \( (t, h) \) is semisimple in \( G \times H \). Therefore \( (t, h) \) lies in a maximal torus of \( G \times H \), which we may take to be \( T^G \times T^H \). As all maximal tori in \( H \) are conjugate, \( hT^Hh^{-1} = T^H \) for some \( h \in H \), and so \( \hat{t} = hh^{-1} \) lies in \( T^H \). Hence \( \hat{z} = h \cdot z \) also lies in \( \pi_X^{-1}([x]) \subseteq Z \), and is fixed by \( (t, \hat{t}) \in T^G \times T^H \).

Since \( \pi_X : Z \to X \) is a principal \( H \)-bundle, \( \pi_X^{-1}([x]) \) is a copy of \( H \). It is now easy to see that \( \pi_X^{-1}([x]) \) is fixed by \( (t, t') \) for some \( t' \in T^H \) only if \( t' = w \cdot \hat{t} \) for some \( w \in W(H, T^H) \), and the set of such \( t' \) is a copy of \( C_H([t']) \). Thus \( \pi_H \) induces a morphism of \( \mathbb{K} \)-varieties

\[
\prod_{t' \in T^H, Z([t, t']) \neq \emptyset} Z([t, t'])/C_H([t']) \to X(t),
\]

whose fibre over each \( x \in X(t) \) consists of one point in \( Z([t, t'])/C_H([t']) \) for each \( t' \) in exactly one orbit of \( W(H, T^H) \) in \( T^H \). Dividing by \( C_H([t']) \) and writing as elements of SE(8) shows that

\[
\left[ \left( [X(t)/C_G([t])], \rho \circ \iota(t) \right) \right] = \sum_{t' \in T^H, Z([t, t']) \neq \emptyset} \frac{|\{w \in W(H, T^H) : w \cdot t' = t\}|}{|W(H, T^H)|}
\left[ \left( [Z([t, t'])/C_G([t]) \times C_H([t'])], \rho \circ \iota(t, t') \right) \right],
\]

where \(|\{w \in W(H, T^H) : w \cdot t' = t\}|/|W(H, T^H)|\) is 1/|\(W(H, T^H) \cdot t'|\), and compensates for the multiplicity of \( [H] \).

Multiplying (35) by \(|\{w \in W(G, T^G) : w \cdot t = t\}|/|W(G, T^G)|\), integrating over \( t \in T^G \) as in (30) and using \( C_{G \times H}(\{t, t'\}) = C_G([t]) \times C_H([t']) \) and \( W(G \times H, T^G \times T^H) = W(G, T^G) \times W(G, T^H) \) shows that (30) gives the same answer using \( X, G, T^G \) and \( Z, G \times H, T^G \times T^H \). Here in relating ‘integrals’ over \( T^G \) and \( T^G \times T^H \) we use the fact that the measure \( d\mu \) is defined using \( \mu \), which depends only on isomorphism classes of \( \mathbb{K} \)-groups, and not on anything special to \( T^G \) or \( T^G \times T^H \).
that $\Pi^\mu$ Combining (29), (36) and (37) shows $\Pi^\mu$ in (36) with $\mu$
Substituting this into (29) with $\mu$

Proof. Arguing as in Lemma 5.9 using (29) is compatible with the relations (6) defining $\text{SF}_{H}$ and $Y,H,T$

Theorem 5.12. (a) $\Pi^1$ defined using $\mu \equiv 1$ is the identity on $\text{SF}(\mathfrak{g})$.

(b) If $\phi : \mathfrak{g} \rightarrow \mathfrak{s}$ is a 1-morphism of algebraic $\mathbb{K}$-stacks with affine geometric stabilizers then $\Pi^\mu \circ \phi_* = \phi_* \circ \Pi^\mu : \text{SF}(\mathfrak{g}) \rightarrow \text{SF}(\mathfrak{s})$.

(c) If $\mu_1, \mu_2$ are weight functions as in Definition 5.14 then $\mu_1\mu_2$ is also a weight function and $\Pi^{\mu_2} \circ \Pi^{\mu_1} = \Pi^{\mu_1} \circ \Pi^{\mu_2} = \Pi^{\mu_1\mu_2}$.

Substituting this into (29) with $\mu \equiv 1$ gives

$$
\Pi^1 \left( (\mathfrak{g}, \rho) \right) = \left[ \left( \left[ X^{P_{\text{min}}}/C_G(Q_{\text{min}}) \right], \rho \circ t_{P_{\text{min}}\cap Q_{\text{min}}} \right) \right] = (\mathfrak{g}, \rho),
$$

since $X^{P_{\text{min}}} = X$ and $C_G(Q_{\text{min}}) = G$. This proves (a), and (b) is immediate.

For (c), note that if $P' \in \mathcal{P}(X, T^G)$ then $\mathcal{P}(X^{P'}, T^G) = \{ P \in \mathcal{P}(X, T^G) : P' \subseteq P \}$, and for such $P$ we have $(X^{P'})^P = X^P$. Similarly, if $Q' \in Q(G, T^G)$ then $T^G$ is a maximal torus in $C_G(Q')$, and $Q(C_G(Q'), T^G) = \{ Q \in Q(G, T^G) : Q' \subseteq Q \}$. Therefore $\mathcal{R}(X^{P'}, C_G(Q'), T^G) = \{ R \in \mathcal{R}(X, G, T^G) : P' \cap Q' \subseteq R \}$.

Using these and (29) in the situation of Definition 5.10 gives

$$
\Pi^{\mu_2} \circ \Pi^{\mu_1} \left( (\mathfrak{g}, \rho) \right) = \sum_{P', P'' \in \mathcal{P}(X, T^G), \quad Q' \subseteq Q, \quad R' \subseteq R, \quad P' \subseteq P, \quad Q' \subseteq Q, \quad R' \subseteq R \cap P \cap Q, \quad \text{G}, \rho \in \mathcal{R}(X, G, T^G) :} M^X_G(P', Q', R') \mu_1(R').
$$

Now a combinatorial calculation with (25)–(27) shows for fixed $P, Q, R, R'$ in (36) with $R' \subseteq R \subseteq P \cap Q$ we have

$$
\sum_{P' \in \mathcal{P}(X, T^G), \quad Q' \subseteq Q, \quad R' \subseteq R \cap P \cap Q} M^X_G(P', Q', R') M^X_{C_G(Q')}(P, Q, R)
$$

Combining (29), (36) and (37) shows $\Pi^{\mu_2} \circ \Pi^{\mu_1} ((\mathfrak{g}, \rho)) = \Pi^{\mu_1\mu_2} ((\mathfrak{g}, \rho))$, so that $\Pi^{\mu_2} \circ \Pi^{\mu_1} = \Pi^{\mu_1\mu_2}$. Exchanging $\mu_1, \mu_2$ then gives $\Pi^{\mu_1} \circ \Pi^{\mu_2} = \Pi^{\mu_1\mu_2}$. □
In contrast to (b), the $\Pi^\mu$ do not in general commute with pullbacks $\phi^* : \text{SF}(\mathcal{G}) \to \text{SF}(\mathcal{G})$ for finite type 1-morphisms $\phi : \mathcal{G} \to \mathcal{G}$. We can now define operators $\Pi^\nu_n$, similar to the operators $\Pi^\mu_n$ of §5.1

**Definition 5.13.** For $n \geq 0$, define $\Pi^\nu_n$ to be the operator $\Pi^\mu_n$ defined with weight $\mu_n$ given by $\mu_n([H]) = 1$ if $\dim H = n$ and $\mu_n([H]) = 0$ otherwise, for all $\mathbb{K}$-groups $H \cong \mathbb{G}_m^k \times K$ with $K$ a finite abelian group.

The analogue of Proposition 5.2 holds for the $\Pi^\nu_n$.

**Proposition 5.14.** In the situation above, we have:

(i) $(\Pi^\nu_n)^2 = \Pi^\nu_n$, so that $\Pi^\nu_n$ is a projection, and $\Pi^\nu_n \circ \Pi^\nu_m = 0$ for $m \neq n$.

(ii) For all $f \in \text{SF}(\mathcal{G})$ we have $f = \sum_{n \geq 0} \Pi^\nu_n(f)$, where the sum makes sense as $\Pi^\nu_n(f) = 0$ for $n \gg 0$.

(iii) If $\phi : \mathcal{G} \to \mathcal{G}$ is a 1-morphism of algebraic $\mathbb{K}$-stacks with affine geometric stabilizers then $\Pi^\nu_n \circ \phi_* = \phi_* \circ \Pi^\nu_n : \text{SF}(\mathcal{G}) \to \text{SF}(\mathcal{G})$.

(iv) If $f \in \text{SF}(\mathcal{G})$, $g \in \text{SF}(\mathcal{G})$ then $\Pi^\nu_n(f \otimes g) = \sum_{m=0}^n \Pi^\nu_m(f) \otimes \Pi^\nu_{n-m}(g)$.

**Proof.** Part (i) is immediate from Theorem 5.12(c) and Definition 5.13. For (ii), in Definition 5.10 we have $\Pi^\nu_n([\mathcal{R}, \rho]) = 0$ for $n > \text{rk} G$ as $\mu(R) = 0$ for all $R$ in $\mathcal{G}$, so $\Pi^\nu_n(f) = 0$ for $n \gg 0$. The first part of (ii) then follows from Theorem 5.12(a), as $\Pi^\mu$ is additive in $\mu$ and $1 = \sum_{n \geq 0} \mu_n$. Theorem 5.12(b) gives (iii), and (iv) is not difficult to prove directly from Definition 5.10 and the fact that $\mu_n(R \times R') = \sum_{m=0}^n \mu_m(R)\mu_{n-m}(R')$.

To get a feel for what the operators $\Pi^\mu$ and $\Pi^\nu_n$ do, consider the case $X = \text{Spec} \mathbb{K}$ and $G = \mathbb{G}_m^k$, so that $\mathfrak{X} = [\text{Spec} \mathbb{K}/\mathbb{G}_m^k]$ is a point with torus stabilizer $\mathbb{G}_m^k$. Then $\Pi^\mu([\mathfrak{X}, \rho]) = \mu([\mathbb{G}_m^k])\rho$, so that $\Pi^\nu([\mathfrak{X}, \rho]) = \rho$ if $k = n$ and 0 otherwise. More generally, if $\mathfrak{X} = [\text{Spec} \mathbb{K}/G]$ for $G$ abelian then $\Pi^\nu([\mathfrak{X}, \rho]) = \rho$ if $\text{rk} G = n$ and 0 otherwise. Thus $\Pi^\mu_n$ and $\Pi^\nu_n$ coincide on points with abelian stabilizers.

However, if $G$ is nonabelian then $\Pi^\nu_n([\text{Spec} \mathbb{K}/G], \rho)$ may be nonzero when $\text{rk} C(G) \leq n < \text{rk} G$, and is zero outside this range. We think of $[\text{Spec} \mathbb{K}/G]$ as being like a linear combination of points with virtual range in the range $\text{rk} C(G) \leq n < \text{rk} G$, and $\Pi^\nu_n$ as projecting to the part of $[\text{Spec} \mathbb{K}/G]$ with virtual rank $n$.

We briefly sketch a conjectural alternative approach to the operators $\Pi^\mu$, which may make them seem more natural. Let $G$ be an affine algebraic $\mathbb{K}$-group, and $\mathfrak{X}$ a finite type algebraic $\mathbb{K}$-stack. Then we can form a $\mathbb{K}$-stack $\mathcal{H}\text{om}([\text{Spec} \mathbb{K}/G], \mathfrak{X})$ by defining for each $\mathbb{K}$-scheme $U$ the groupoid

$$\mathcal{H}\text{om}([\text{Spec} \mathbb{K}/G], \mathfrak{X})(U) = \text{Hom}(U \times [\text{Spec} \mathbb{K}/G], \mathfrak{X}),$$

and for each morphism of $\mathbb{K}$-schemes $\phi : U \to V$ the functor

$$\mathcal{H}\text{om}([\text{Spec} \mathbb{K}/G], \mathfrak{X})(\phi) : \mathcal{H}\text{om}([\text{Spec} \mathbb{K}/G], \mathfrak{X})(V) \to \mathcal{H}\text{om}([\text{Spec} \mathbb{K}/G], \mathfrak{X})(U)$$

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induced by composition of 1-morphisms and 2-morphisms with the 1-morphism
\( \phi \times \text{id}_{[\text{Spec} \mathbb{K}/G]} : U \times [\text{Spec} \mathbb{K}/G] \to V \times [\text{Spec} \mathbb{K}/G] \) and its identity 2-morphism.
Taking \( U = \text{Spec} \mathbb{K} \), we see that the \( \mathbb{K} \)-points of \( \text{Hom}([\text{Spec} \mathbb{K}/G], \mathfrak{R}) \) are 1-morphisms \([\text{Spec} \mathbb{K}/G] \to \mathfrak{R}\). There is a projection \( \Pi : \text{Hom}([\text{Spec} \mathbb{K}/G], \mathfrak{R}) \to \mathfrak{R} \) corresponding to composition of 1-morphisms \( \text{Spec} \mathbb{K} \to [\text{Spec} \mathbb{K}/G] \to \mathfrak{R} \).

If we used a general \( \mathbb{K} \)-stack \( \mathfrak{S} \) in place of \([\text{Spec} \mathbb{K}/G] \) here then \( \text{Hom}(\mathfrak{S}, \mathfrak{R}) \) would be not even locally of finite type – essentially, infinite-dimensional. But \( \text{Hom}([\text{Spec} \mathbb{K}/G], \mathfrak{R}) \) is locally of finite type. It may not be of finite type because the fibre of \( \Pi \) over \( r \in \overline{\mathfrak{R}}(\mathbb{K}) \) is \( [\text{Hom}(G, \text{Aut}_\mathbb{K}(r))/\text{Ad}(\text{Aut}_\mathbb{K}(r))] \), where the \( \mathbb{K} \)-group morphisms \( \text{Hom}(G, \text{Aut}_\mathbb{K}(r)) \) may have infinitely many components.

Roughly speaking, one might construct the \( \Pi^\rho \) as follows. For an algebraic \( \mathbb{K} \)-group \( T \) of the form \( \mathbb{G}_m \times K \) for \( K \) finite abelian, we restrict to a \( \mathbb{K} \)-substack \( \text{Hom}([\text{Spec} \mathbb{K}/T], \mathfrak{R})_{\text{reg}} \) of points in \( \text{Hom}([\text{Spec} \mathbb{K}/T], \mathfrak{R}) \) with some extra properties, and \( \text{Hom}([\text{Spec} \mathbb{K}/T], \mathfrak{R})_{\text{reg}}/\text{Aut}(T) \) is of finite type. Then \( \Pi^\rho([\mathfrak{R}, \rho]) \) is a linear combination over \( T \) of terms like \( [(\text{Hom}([\text{Spec} \mathbb{K}/T], \mathfrak{R})_{\text{reg}}/\text{Aut}(T), \rho \circ \Pi)] \).

These ideas might be worth further investigation, if anyone is interested.

Finally we discuss a generalization \( \Pi_\mathfrak{S}^\rho \) of the operators \( \Pi^\rho \) which will be useful in [10, §5]. The idea is that \( \Pi_\mathfrak{S}^\rho([\mathfrak{R}, \rho]) \) depends not just on subtori \( T \) of stabilizer groups \( \text{Aut}_\mathbb{K}(r) \) in \( \mathfrak{R} \), but also on the morphism \( \rho : T \to \text{Aut}_\mathbb{K}(f) \) to the stabilizer group of \( f = \rho_*(r) \) in \( \mathfrak{S} \). Thus the weight function \( \nu \) is a function of all morphisms \( \rho : T \to \text{Aut}_\mathbb{K}(f) \), which makes it unwieldy to define.

**Definition 5.15.** Let \( \mathfrak{S} \) be an algebraic \( \mathbb{K} \)-stack with affine geometric stabilizers. An \( \mathfrak{S} \)-weight function is a map
\[
\nu : \{(T, f, \phi) : T \text{ a } \mathbb{K} \text{-group isomorphic to } \mathbb{G}_m \times K, K \text{ finite abelian, } f \in \mathfrak{S}(\mathbb{K}), \phi : T \to \text{Aut}(f) \text{ a } \mathbb{K} \text{-group morphism} \} \to \mathbb{Q},
\]
which satisfies \( \nu(T, f, \phi) = \nu(T', f, \phi \circ \iota) \) if \( \iota : T' \to T \) is a \( \mathbb{K} \)-group isomorphism, and is locally constructible in \( f, \phi \). That is, \( \nu \) induces a locally constructible function \( \text{Hom}([\text{Spec} \mathbb{K}/T], \mathfrak{S}) \to \mathbb{Q} \) for each fixed \( T \), in the notation above.

Let \( [\mathfrak{R}, \rho] \in \mathsf{SF}(\mathfrak{S}) \) with \( \mathfrak{R} \cong [X/G] \) for \( X \) a quasiprojective \( \mathbb{K} \)-variety and \( G \) a special algebraic \( \mathbb{K} \)-group with maximal torus \( T^G \). For \( P \in \mathcal{P}(X, T^G) \) and \( R \in \mathcal{R}(X, G, T^G) \) with \( R \subseteq P \) and \( c \in \mathbb{Q} \), define
\[
X_{\nu, c}^{P, R} = \{ x \in \overline{X^P}(\mathbb{K}) : \nu(R, (\rho \circ \pi)_*, x, \rho_*|_R) = c \},
\]
writing \( \pi : X \to [X/G] \cong \mathfrak{R} \) for the projection, so that if \( x \in \overline{X^P}(\mathbb{K}) \) then \( r = \pi_*(x) \in \overline{\mathfrak{R}}(\mathbb{K}) \), and \( f = (\rho \circ \pi)_*(x) \in \overline{\mathfrak{S}}(\mathbb{K}) \), and \( \rho_* : \text{Aut}_\mathbb{K}(r) \to \text{Aut}_\mathbb{K}(f) \) is a \( \mathbb{K} \)-group morphism. Identifying \( \text{Aut}_\mathbb{K}(r) \) with \( \text{Stab}_G(x) \) we have \( R \subseteq P \subseteq \text{Stab}_G(x) = \text{Aut}_\mathbb{K}(r) \), so \( \rho_*|_R : R \to \text{Aut}_\mathbb{K}(f) \) is well-defined.

As \( \nu \) is locally constructible \( X_{\nu, c}^{P, R} \) is a constructible set in \( X^P \), and \( \overline{X^P}(\mathbb{K}) = \coprod_{c \in \mathbb{Q}} X_{\nu, c}^{P, R} \) with \( X_{\nu, c}^{P, R} \neq \emptyset \) for only finitely many \( c \in \mathbb{Q} \). So \( X_{\nu, c}^{P, R} \) can be written as the disjoint union of finitely many quasiprojective \( \mathbb{K} \)-varieties. But for simplicity we neglect this, and pretend \( X_{\nu, c}^{P, R} \) is a variety. Define
\[
\hat{\Pi}_\mathfrak{S}^\rho([\mathfrak{R}, \rho]) = \sum_{P \in \mathcal{P}(X, T^G), Q \in \mathcal{Q}(G, T^G), R \in \mathcal{R}(X, G, T^G)} M_\mathfrak{S}^{\nu, \rho, r}(P, Q, R) c \cdot \left[ \frac{[\mathcal{X}^{P, R}(\mathcal{Q}(Q)), \rho \circ \iota]}{\mathcal{C}(\mathcal{Q}(Q))} \right]. \tag{38}
\]
As for (29) we have $X^P = X^{P\cap Q}$ and $C_G(Q) = C_G(P\cap Q)$, so $C_G(Q)$ commutes with $R \subseteq P\cap Q$, which implies $X_{v,c}^{P,R}$ is $C_G(Q)$-invariant, and (38) is well-defined. If $\nu(T,f,\phi) = \mu(T)$ then $X_{v,c}^{P,R}$ is $X^P$ when $c = \mu(R)$ and $\emptyset$ otherwise, and (38) reduces to (29) giving $\hat{\Pi}_\mathfrak{g} = \Pi^\mu$. So $\hat{\Pi}_\mathfrak{g}$ does generalize $\Pi^\mu$.

We can generalize Theorems 5.11 and 5.12(a,c) to $\hat{\Pi}_\mathfrak{g}$ – informally, we can generalize (30) to regard $\hat{\Pi}_\mathfrak{g}$ as a kind of double integral over $t \in T^G$ and $x \in X^{t\{t\}}$ with respect to a measure $d\nu$ derived from $\nu$, and then the proof of Theorem 5.11 needs fewer changes. Thus we have well-defined linear maps $\hat{\Pi}_\mathfrak{g}^\mu : \mathcal{SF}(\mathfrak{g}) \to \mathcal{SF}(\mathfrak{g})$ and $\hat{\Pi}_\mathfrak{g}^\nu : \mathcal{SF}(\mathfrak{g}) \to \mathcal{SF}(\mathfrak{g})$.

### 5.3 The operators $\mathcal{SF}, \mathcal{SF}(\mathfrak{g}, \mathcal{Y}, \Lambda)$ and their operations

The operators $\Pi^\mu, \Pi^\nu, \hat{\Pi}^\nu_\mathfrak{g}$ of §5.2 cannot be defined on the $\mathcal{SF}(\mathfrak{g}, \mathcal{Y}, \Lambda)$ of §4.3 basically since for $[(\mathcal{R}, \rho)]$ the spaces $\mathcal{SF}(\mathfrak{g}, \mathcal{Y}, \Lambda)$ identify $\mathcal{R} = \text{Spec } \mathbb{K}$ and $\mathcal{R} = T \times [\text{Spec } \mathbb{K}/T]$ for a torus $T$, but the $\Pi^\mu, \Pi^\nu, \hat{\Pi}^\nu_\mathfrak{g}$ distinguish them. That is, the relations in $\mathcal{SF}(\mathfrak{g}, \mathcal{Y}, \Lambda)$ are too coarse, and identify things separated by $\Pi^\mu, \Pi^\nu, \hat{\Pi}^\nu_\mathfrak{g}$. We now construct new spaces $\mathcal{SF}, \mathcal{SF}(\mathfrak{g}, \mathcal{Y}, \Lambda)$ with finer relations, on which $\Pi^\mu, \Pi^\nu, \hat{\Pi}^\nu_\mathfrak{g}$ are well-defined.

**Definition 5.16.** An affine algebraic $\mathbb{K}$-group $G$ is called very special if $C_G(Q)$ and $Q$ are special for all $Q \in Q(G,T^G)$, for any maximal torus $T^G$ in $G$. (Since $Q$ is of the form $G_m^T \times K$ for $K$ finite abelian, $Q$ is special if and only if $|K| = 1$, that is, if it is connected.) Then $G$ is special, as $G = C_G(Q)$ for $Q = T^{G_1} \cap C(G)$. Since $\text{GL}(k, \mathbb{K})$ is special and products of special groups are special, Example 5.7 and (23) imply that $\text{GL}(m, \mathbb{K})$ is very special.

When Assumption 4.11 holds, $G$ is very special, $T^G$ is a maximal torus in $G$ and $Q \in Q(G,T^G)$, define $E(G,T^G,Q) \in \Lambda$ by

$$E(G,T^G,Q) = \mathcal{Y}([Q]) \sum_{Q' \in Q(G,T^G)} \left( \frac{N_G(T^G)}{|C_G(Q')\cap N_G(T^G)|} \right)^{-1} \frac{\mathcal{N}_{T^G}(Q',Q)}{\mathcal{Y}([C_G(Q')])}. \quad (39)$$

Here $\mathcal{Y}([C_G(Q')])^{-1}$ exists in $\Lambda$ by Lemma 4.11 as $G$ is very special.

Here is our refinement of Definition 4.11.

**Definition 5.17.** Let Assumption 4.1 hold, and $\mathfrak{g}$ be an algebraic $\mathbb{K}$-stack with affine geometric stabilizers. Consider pairs $[(\mathcal{R}, \rho)]$, where $\mathcal{R}$ is a finite type algebraic $\mathbb{K}$-stack with affine geometric stabilizers and $\rho : \mathcal{R} \to \mathfrak{g}$ is a 1-morphism, with equivalence of pairs as in Definition 3.1. Define $\mathcal{SF}(\mathfrak{g}, \mathcal{Y}, \Lambda)$ to be the $\Lambda$-module generated by equivalence classes $[(\mathcal{R}, \rho)]$ as above, with the following relations:

(i) Given $[(\mathcal{R}, \rho)]$ as above and $\mathfrak{S}$ a closed $\mathbb{K}$-substack of $\mathfrak{g}$ we have $[(\mathcal{R}, \rho)] = [(\mathfrak{S}, \rho|_{\mathfrak{S}})] + [(\mathcal{R} \setminus \mathfrak{S}, \rho|_{\mathcal{R} \setminus \mathfrak{S}})]$, as in (0).
(ii) Let \( \mathcal{R} \) be a finite type algebraic \( k \)-stack with affine geometric stabilizers, \( U \) a quasiprojective \( k \)-variety, \( \pi_\mathcal{R} : \mathcal{R} \times U \rightarrow \mathcal{R} \) the natural projection, and \( \rho : \mathcal{R} \rightarrow \mathfrak{g} \) a 1-morphism. Then \( [(\mathcal{R} \times U, \rho \circ \pi_\mathcal{R})] = \Upsilon(U)[([\mathcal{R}], \rho)] \).

(iii) Given \([([\mathcal{R}], \rho)]\) as above and a 1-isomorphism \( \mathcal{R} \cong [X/G] \) for \( X \) a quasiprojective \( k \)-variety and \( G \) a very special algebraic \( k \)-group acting on \( X \) with maximal torus \( T^G \), we have

\[
[(\mathcal{R}, \rho)] = \sum_{Q \in \mathcal{Q}(G, T^G)} E(G, T^G, Q)[([X/Q], \rho \circ \iota^Q)] ,
\]

where \( \iota^Q : [X/Q] \rightarrow \mathcal{R} \cong [X/G] \) is the natural projection 1-morphism.

Similarly, define \( \mathsf{SF}(\mathfrak{g}, \Upsilon, \Lambda) \) to be the \( \Lambda \)-module generated by \([([\mathcal{R}], \rho)]\) with \( \rho \) representable, and relations (i)--(iii) as above. Since the \( \iota^Q \) are representable, \( \rho \circ \iota^Q \) is representable in (40), so these relations make sense.

Define projections \( \Pi_{\mathfrak{g}}^{\Upsilon, \Lambda} : \mathsf{SF}(\mathfrak{g}) \rightarrow \mathsf{SF}(\mathfrak{g}, \Upsilon, \Lambda) \) and \( \Pi_{\mathfrak{g}}^{\Lambda} : \mathsf{SF}(\mathfrak{g}, \Upsilon, \Lambda) \rightarrow \mathsf{SF}(\mathfrak{g}, \Upsilon, \Lambda) \) by (19). Here \( \Pi_{\mathfrak{g}}^{\Upsilon, \Lambda} \) is well-defined as relation (ii) in \( \mathsf{SF}(\mathfrak{g}) \) maps to (i) above, and restricts to \( \Pi_{\mathfrak{g}}^{\Lambda} : \mathsf{SF}(\mathfrak{g}) \rightarrow \mathsf{SF}(\mathfrak{g}, \Upsilon, \Lambda) \).

To see \( \Pi_{\mathfrak{g}}^{\Upsilon, \Lambda} \) is well-defined we must show (i)--(iii) above map to Definition 4.11(ii)--(iii), which is obvious for (i)--(ii) but nontrivial for (iii). By Definition 4.11(iii), (iii) the l.h.s. of (10) maps under \( \Pi_{\mathfrak{g}}^{\Upsilon, \Lambda} \) to \( \Upsilon([G])^{-1}[([X/Q], \rho \circ \pi)] \), and the term \([([X/Q], \rho \circ \pi)]\) on the r.h.s. maps to \( \Upsilon([Q])^{-1}([X/Q], \rho \circ \pi)] \), since \( Q \) is special. Therefore (10) maps to relations in \( \mathsf{SF}(\mathfrak{g}, \Upsilon, \Lambda) \) provided

\[
\Upsilon([G])^{-1} = \sum_{Q \in \mathcal{Q}(G, T^G)} \Upsilon([Q])^{-1} E(G, T^G, Q).
\]

This follows from (26) and (39) as in the proof of Theorem 5.12(a), since \( \sum_{Q \in \mathcal{Q}(G, T^G) : Q \subset Q'} \nu_{T^G}^{-1}(Q', Q) = 1 \) if \( Q' = Q_{\min} \) and 0 otherwise, and \( C_G(Q_{\min}) = G \). Thus \( \Pi_{\mathfrak{g}}^{\Upsilon, \Lambda} \) is well-defined.

Here is the analogue of Definition 4.12 but also including \( \Pi^\nu_n, \Pi^\nu_{vi}, \hat{\Pi}^\nu_n \).

**Definition 5.18.** Let Assumption 4.1 hold, \( \mathfrak{g}, \mathcal{G} \) be algebraic \( k \)-stacks with affine geometric stabilizers, and \( \phi : \mathfrak{g} \rightarrow \mathcal{G} \) a 1-morphism. Define a \( \Lambda \)-bilinear multiplication \( \cdot \) on \( \mathsf{SF}(\mathfrak{g}, \Upsilon, \Lambda) \) by (7). This is commutative and associative as in Definition 4.11 and \( \mathsf{SF}(\mathfrak{g}, \Upsilon, \Lambda) \) is closed under \( \cdot \). Define the pushforward \( \phi_* : \mathsf{SF}(\mathfrak{g}, \Upsilon, \Lambda) \rightarrow \mathsf{SF}(\mathcal{G}, \Upsilon, \Lambda) \) by (11), taking the \( c_i \in \Lambda \) rather than \( c_i \in \mathbb{Q} \).

If \( \phi \) is representable this restricts to \( \phi_* : \mathsf{SF}(\mathfrak{g}, \Upsilon, \Lambda) \rightarrow \mathsf{SF}(\mathcal{G}, \Upsilon, \Lambda) \). For \( \phi \) of finite type, define the pullback \( \phi^* : \mathsf{SF}(\mathcal{G}, \Upsilon, \Lambda) \rightarrow \mathsf{SF}(\mathfrak{g}, \Upsilon, \Lambda) \) by (11). This restricts to \( \phi^* : \mathsf{SF}(\mathcal{G}, \Upsilon, \Lambda) \rightarrow \mathsf{SF}(\mathfrak{g}, \Upsilon, \Lambda) \).

Define the tensor product \( \otimes : \mathsf{SF}(\mathfrak{g}, \Upsilon, \Lambda) \times \mathsf{SF}(\mathfrak{g}, \Upsilon, \Lambda) \rightarrow \mathsf{SF}(\mathfrak{g} \times \mathfrak{g}, \Upsilon, \Lambda) \) by (11). This restricts to \( \otimes : \mathsf{SF}(\mathfrak{g}, \Upsilon, \Lambda) \times \mathsf{SF}(\mathcal{G}, \Upsilon, \Lambda) \rightarrow \mathsf{SF}(\mathfrak{g} \times \mathcal{G}, \Upsilon, \Lambda) \). Define \( \Pi^\nu_n, \Pi^\nu_{vi}, \hat{\Pi}^\nu_n : \mathsf{SF}(\mathfrak{g}, \Upsilon, \Lambda) \rightarrow \mathsf{SF}(\mathfrak{g}, \Upsilon, \Lambda) \) and \( \mathsf{SF}(\mathfrak{g}, \Upsilon, \Lambda) \rightarrow \mathsf{SF}(\mathfrak{g}, \Upsilon, \Lambda) \) by (26), as in (35).

Here is the analogue of Theorem 4.13.

**Theorem 5.19.** These operations \( \cdot, \phi_* , \phi^* , \otimes, \Pi^\nu_n, \Pi^\nu_{vi} \) and \( \hat{\Pi}^\nu_n \) are compatible with the relations (i)--(iii) in Definition 5.17 and so are well-defined.
Proof. The proof of Theorem 4.13 shows \( \phi \), \( \otimes \) are compatible with (i)–(iii) above and \( \cdot \). \( \phi^* \) is compatible with (i)–(ii). Using all the notation of Theorem 4.13 we find by the same argument that \( \cdot \) is compatible with (iii) provided
\[
\left[ \left[ (X_i^* \times Y_j^*) \times_{\alpha_i \times \beta_j \times z_i \times z_{ij} \times \pi_{ij}} (Z_{ij} \times K_{ij}) \right] / G \times H_{ij} \right], \rho \circ \pi_{ij} \right] = \sum_{Q \in \mathcal{Q}(G,T^G)} E(G,T^G,Q) \left[ \left[ \left[ (X_i \times Y_j) \times_{\alpha_i \times \beta_j \times z_i \times z_{ij} \times \pi_{ij}} (Z_{ij} \times K_{ij}) \right] / Q \times H_{ij} \right], \rho \circ \pi_{ij} \right] \]
\]
in \( \mathcal{S}(\mathcal{S}, \mathcal{T}, \Lambda) \). This holds because by (iii), both sides are equal to
\[
\sum_{Q \in \mathcal{Q}(G,T^G)} E(G,T^G,Q) E(H_{ij},T^{H_{ij}},Q') \left[ \left[ \left[ (X_i \times Y_j) \times_{\alpha_i \times \beta_j \times z_i \times z_{ij} \times \pi_{ij}} (Z_{ij} \times K_{ij}) \right] / Q \times Q' \right], \rho \circ \pi_{ij} \right].
\]
Here we use the facts that \( \mathcal{Q}(G \times H_{ij}, T^G \times T^{H_{ij}}) = \mathcal{Q}(G,T^G) \times \mathcal{Q}(H_{ij},T^{H_{ij}}) \), and \( E(G \times H_{ij}, T^G \times T^{H_{ij}}) \times Q \times Q' = E(G,T^G,Q) E(H_{ij},T^{H_{ij}},Q') \), and each \( Q \in \mathcal{Q}(G,T^G) \) is a torus, so \( \mathcal{Q}(Q,Q) = \{Q\} \) and \( E(Q,Q,Q) = 1 \). Therefore \( \cdot \) is compatible with (iii) and is well-defined. Modifying the argument of Theorem 4.13 in the same way, \( \phi^* \) is well-defined.

Compatibility of \( \Pi^\mu \) with (i)–(ii) above is easy. To show \( \Pi^\mu \) is compatible with (iii) we must show it takes both sides of (40) to the same thing in \( \mathcal{S}(\mathcal{S}, \mathcal{T}, \Lambda) \). That is, we must prove that
\[
\sum_{P \in \mathcal{P}(X,T^G), Q \in \mathcal{Q}(G,T^G)} M^X_G(P,Q,R) \mu(R) \cdot E(G,T^G,Q) \cdot M^X_G(P',Q',R) \mu(R).
\]
\[
\sum_{Q' \in \mathcal{Q}(G,T^G), P' \in \mathcal{P}(X,Q'), R \in \mathcal{R}(X,Q'), Q' \times P' \times Q', R \neq 0} E(G,T^G,Q') \cdot M^X_G(P,R') \mu(R) \cdot \left[ \left[ \left[ (X' / Q') / \rho \circ \rho \circ \rho \circ Q' \right], \rho \circ \rho \circ \rho \circ Q' \right] \right] \]
\]
in \( \mathcal{S}(\mathcal{S}, \mathcal{T}, \Lambda) \), using \( \mathcal{Q}(Q',Q') = \{Q'\} \) and \( C_{Q'}(Q') = Q' \) in the bottom line. We rewrite the top line of (41) using (40). Since \( \mathcal{Q}(C_{Q'}(Q), T^G) = \{Q' \in \mathcal{Q}(G,T^G) : Q \subseteq Q'\} \) for \( Q \subseteq Q' \), this gives
\[
\sum_{P \in \mathcal{P}(X,T^G), Q \in \mathcal{Q}(G,T^G)} M^X_G(P,Q,R) \mu(R) \cdot \sum_{Q \subseteq Q'} E(C_{Q'}(Q), T^G, Q').
\]
\[
\sum_{Q' \in \mathcal{Q}(G,T^G), P' \in \mathcal{P}(X,Q'), R \in \mathcal{R}(X,Q'), Q' \times P' \times Q', R \neq 0} E(C_{Q'}(Q), T^G, Q') \cdot \left[ \left[ \left[ (X' / Q') / \rho \circ \rho \circ \rho \circ Q' \right], \rho \circ \rho \circ \rho \circ Q' \right] \right].
\]

We claim that the term in the bottom line of (41) with fixed \( P', Q', R \) agrees with the sum of terms in (42) with fixed \( P, Q', R \), where \( P' = P \cap Q' \). To explain the relation between \( P \) and \( P' \), note that for \( P, Q, R, Q' \) in (42) we have \( M^X_G(P,Q,R) \neq 0, \) so \( P \) is the smallest element of \( \mathcal{P}(X,T^G) \) containing \( P \cap Q \) by Lemma 5.9 and \( X' \cap P = X \). But \( Q \subseteq Q' \), so \( P \cap Q \subseteq P \cap Q' = P' \subseteq P \), which shows that \( X' = X \). Note too that \( P \) is the smallest element of \( \mathcal{P}(X,T^G) \) containing \( P' \), so \( P \) and \( P' \) determine each other uniquely given \( Q' \), and fixing \( P', Q', R \) in (41) is equivalent to fixing \( P, Q', R \) in (42).

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Thus $\mu(R) : ([X^P/Q'], \rho \circ t^{P\cap Q'})$ are common terms in (11) and (12), and the sums of coefficients of these for fixed $P, P', Q', R$ are equal provided

$$E(G, T^G, Q')M^X_Q(P', Q', R) = E(C_G(Q), T^G, Q') \sum_{Q \in Q(G, T^G): Q \subseteq Q'} M^X_Q(P, Q, R). \quad (43)$$

Now for $Q, Q'' \in Q(G, T^G)$ with $Q \subseteq Q' \subseteq Q''$ we have

$$\left| \frac{N_G(T^G)}{C_G(T^G) \cap N_G(T^G)} \right| = \left| \frac{N_{C_G(Q)}(T^G)}{C_{C_G(Q)}(Q) \cap N_{C_G(Q)}(T^G)} \right| \left| \frac{N_G(T^G)}{C_G(Q) \cap N_G(T^G)} \right|,$$

noting that $C_{C_G(Q)}(Q') = C_G(Q')$ as $Q \subseteq Q''$ and intersecting top and bottom of the l.h.s. with $C_G(Q)$. Thus from (39) we deduce that

$$E(C_G(Q), T^G, Q') = \left| \frac{N_G(T^G)}{C_G(Q) \cap N_G(T^G)} \right| : E(G, T^G, Q'). \quad (44)$$

Combining this with (27) we see that (43) is equivalent to

$$m^X_Q(R, P') = \sum_{Q \in Q(G, T^G): Q \subseteq Q'} \sum_{\hat{P} \in P(X, T^G): \hat{P} \subseteq \hat{Q}, \hat{Q} \in Q(G, T^G): \hat{Q} \subseteq Q'} m^X_{\hat{Q}}(\hat{P}, P) n^G_{\hat{Q}}(\hat{Q}, Q), \quad (45)$$

where the l.h.s. is $M^X_Q(P', Q', R)$. Fixing $\hat{Q} \subseteq Q'$ in the r.h.s. of (45) and summing over $Q$, we find $\sum_{Q \in Q(G, T^G): Q \subseteq Q'} n^G_{\hat{Q}}(\hat{Q}, Q)$ is 1 if $\hat{Q} = Q'$ and 0 otherwise. So the r.h.s. becomes $\sum_{\hat{P} \in P(X, T^G): \hat{P} \subseteq \hat{Q}, \hat{Q} \in Q(G, T^G): \hat{P} \subseteq \hat{Q}, \hat{Q} \subseteq Q'} m^X_{\hat{Q}}(\hat{P}, P)$, which eventually reduces to $m^X_Q(R, P')$ as $P$ is the smallest element of $P(X, T^G)$ containing $P' = P \cap Q'$. This proves (45), and hence (13) and (11), which shows $\Pi''$ is compatible with (iii) and is well-defined. Also $\Pi''_g$ is a special case of $\Pi''$, and the changes to the proof above for $\Pi''_g$ are straightforward. \hfill \Box

The analogue of Corollary 5.14 is immediate.

**Corollary 5.20.** The projections $\Pi^\Lambda \Lambda, \Pi^\Lambda \Lambda$ commute with $\cdot, \phi, \phi^\ast, \ast$ on $\mathbb{SF}(\ast, Y, \Lambda)$, and $\Pi^\Lambda \Lambda$ commutes with $\Pi^\Lambda \Lambda, \Pi^\Lambda \Lambda$. The analogues of Theorems 5.5 and 5.12 and Proposition 5.14 hold for $\mathbb{SF}, \mathbb{SF}(\ast, Y, \Lambda)$.

Here is a useful way of representing elements of $\mathbb{SF}, \mathbb{SF}(\ast, Y, \Lambda)$.

**Proposition 5.21.** $\mathbb{SF}(\ast, Y, \Lambda)$ and $\mathbb{SF}(\ast, Y, \Lambda)$ are generated over $\Lambda$ by elements $[(U \times [\text{Spec} \mathbb{K}/T], \rho)]$, for $U$ a quasiprojective $\mathbb{K}$-variety and $T$ an algebraic $\mathbb{K}$-group isomorphic to $G_k^k \times K$ for $k \geq 0$ and $K$ finite abelian.

**Proof.** As in the proof of Theorem 5.14 $\mathbb{SF}, \mathbb{SF}(\ast, Y, \Lambda)$ are generated over $\Lambda$ by elements $[(X^G, \rho)]$ for $X$ a quasiprojective $\mathbb{K}$-variety and $G$ an affine algebraic $\mathbb{K}$-group which we can take to be $\text{GL}(m, \mathbb{K})$, so in particular for $G$ very special. Definition 7.13(iii) then implies $\mathbb{SF}, \mathbb{SF}(\ast, Y, \Lambda)$ are generated over $\Lambda$ by $[(X/Q, \rho \circ t^Q)]$ for $X$ a quasiprojective $\mathbb{K}$-variety and $Q$ a torus.

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Given such $X, Q$ there is a finite collection of closed $\mathbb{K}$-subgroups $T_i$ in $Q$ for $i \in I$ occurring as $\text{Stab}_Q(x)$ for $x \in X$, and the set of such $x$ is a locally closed $\mathbb{K}$-subvariety $X_i$ of $X$ with $X = \coprod_{i \in I} X_i$. Here $T_i \cong G_m^k \times K_i$ for $K_i$ finite abelian, as $Q$ is a torus. Then $Q/T_i$ acts freely on $X_i$, and $X_i/(Q/T_i)$ is an algebraic $\mathbb{K}$-space which may be written as a disjoint union of finitely many quasiprojective $\mathbb{K}$-subvarieties $U_{ij}$ for $j \in J_i$. Thus $[X/Q]$ is a disjoint union of $\mathbb{K}$-substacks $i$-isomorphic to $U_{ij} \times [\text{Spec } \mathbb{K}/T_i]$. Definition 5.17(1) gives $[[[X/Q], \rho \circ Q]] = \sum_{i,j} [[(U_{ij} \times [\text{Spec } \mathbb{K}/T_i], \rho_{ij})]]$, so $\Sigma\mathcal{SF}(\mathfrak{S}, \mathfrak{T}, \Lambda)$ are generated over $\Lambda$ by such $[(U_{ij} \times [\text{Spec } \mathbb{K}/T_i], \rho_{ij})]$, as we want. □

Such $[(U \times [\text{Spec } \mathbb{K}/T], \rho)]$ are linearly independent for nonisomorphic $T$.

**Proposition 5.22.** Suppose $\sum_{i \in I} c_i [[U_i \times [\text{Spec } \mathbb{K}/T_i], \rho_i]] = 0$ in $\Sigma\mathcal{SF}(\mathfrak{S}, \mathfrak{T}, \Lambda)$, where $I$ is finite set, $c_i \in \Lambda$, $U_i$ a quasiprojective $\mathbb{K}$-variety and $T_i$ an algebraic $\mathbb{K}$-group isomorphic to $G_m^k \times K_i$ for $K_i \geq 0$ and $K_i$ finite abelian, with $T_i \not\cong T_j$ for $i \neq j$. Then $c_j [[U_j \times [\text{Spec } \mathbb{K}/T_j], \rho_j]] = 0$ for all $j \in I$.

**Proof.** Let $j \in I$, and define $\mu$ in Definition 5.10 by $\mu(T) = 1$ if $T \cong T_j$ and $\mu(T) = 0$ otherwise. Then $\Pi^\mu$ is well-defined on $\Sigma\mathcal{SF}(\mathfrak{S}, \mathfrak{T}, \Lambda)$ by Theorem 5.19 and on $[[U, [\text{Spec } \mathbb{K}/T], \rho_j]]$ it is the identity if $i = j$ and 0 otherwise, since $T_i \not\cong T_j$ for $i \neq j$. The result follows by applying $\Pi^\mu$ to $\sum_{i \in I} \cdots = 0$. □

We identify $\Sigma\mathcal{SF}, \Sigma\mathcal{SF}(\text{Spec } \mathbb{K}, \mathfrak{T}, \Lambda)$, as in Proposition 4.15.

**Proposition 5.23.** Define a commutative $\Lambda$-algebra $\hat{\Lambda}$ with $\Lambda$-basis isomorphism classes $[T]$ of $\mathbb{K}$-groups $T$ of the form $G_m^k \times K$, for $k \geq 0$ and $K$ finite abelian, with $\Lambda$-bilinear multiplication given by $[T][T'] = [T \times T']$ on basis elements. Define $\hat{i}_\Lambda : \Lambda \rightarrow \Sigma\mathcal{SF}(\text{Spec } \mathbb{K}, \mathfrak{T}, \Lambda)$ by $\hat{i}_\Lambda : c \mapsto \sum_i c_i [T_i] \mapsto \sum_i c_i [[\text{Spec } \mathbb{K}/T_i]]$. Then $\hat{i}_\Lambda$ is an algebra isomorphism. It restricts to an isomorphism from the sub-algebra $\Lambda[1]$ of $\hat{i}_\Lambda$. □

**Proof.** Since $[[\text{Spec } \mathbb{K}/T]] [[\text{Spec } \mathbb{K}/T']] = [[\text{Spec } \mathbb{K}/T \times T']]$ in $\Sigma\mathcal{SF}(\text{Spec } \mathbb{K}, \mathfrak{T}, \Lambda)$, $\hat{i}_\Lambda$ is an algebra morphism. By Proposition 5.21 $\Sigma\mathcal{SF}(\text{Spec } \mathbb{K}, \mathfrak{T}, \Lambda)$ is generated over $\Lambda$ by $[U \times [\text{Spec } \mathbb{K}/T]]$ with $U$ a quasiprojective $\mathbb{K}$-variety and $T \cong G_m^k \times K$. But $[U \times [\text{Spec } \mathbb{K}/T]] = \mathfrak{T}(U)[[[\text{Spec } \mathbb{K}/T]]$ by Definition 5.17(ii), so $\Sigma\mathcal{SF}(\text{Spec } \mathbb{K}, \mathfrak{T}, \Lambda)$ is generated over $\Lambda$ by such $[[\text{Spec } \mathbb{K}/T]]$, and $\hat{i}_\Lambda$ is surjective.

Elements of $\hat{\Lambda}$ may be written as $\sum_{i \in I} c_i [T_i]$ for $I$ finite, $c_i \in \Lambda$ and $T_i \not\cong T_j$ for $i \neq j$. Suppose $\hat{i}_\Lambda(\sum_{i \in I} c_i [T_i]) = 0$. Then $\sum_{i \in I} c_i [[\text{Spec } \mathbb{K}/T_i]] = 0$, so Proposition 5.22 gives $c_i [[\text{Spec } \mathbb{K}/T_i]] = 0$ for each $i \in I$. Applying $\Pi^\Lambda_{\text{Spec } \mathbb{K}}$ from Definition 5.17 and $\hat{i}_\Lambda^{-1}$ from Proposition 4.16 gives

$$0 = i^{-1}_\Lambda \circ \Pi^\Lambda_{\text{Spec } \mathbb{K}}(c_i [[\text{Spec } \mathbb{K}/T_i]]) = c_i \hat{i}_\Lambda^{-1}([[\text{Spec } \mathbb{K}/T_i]]) = c_i (\ell - 1)^{-\dim T_i}$$

in $\Lambda$. So $c_i = 0$ for all $i \in I$, and $\sum_{i \in I} c_i [T_i] = 0$. Thus $\hat{i}_\Lambda$ is injective, and so an isomorphism. Finally, as $\rho : [\text{Spec } \mathbb{K}/T] \rightarrow \text{Spec } \mathbb{K}$ is representable if and only if $T \cong \{1\}$, $\Sigma\mathcal{SF}(\text{Spec } \mathbb{K}, \mathfrak{T}, \Lambda)$ is the image under $\hat{i}_\Lambda$ of $\Lambda[1]$. □

Since every finite abelian group $K$ is isomorphic to a product of cyclic groups $\mathbb{Z}_{p^k}$ of prime power order, $\hat{\Lambda}$ is the free commutative $\Lambda$-algebra generated by $[G_m]$ and $[\mathbb{Z}_{p^k}]$ for $p$ prime and $k \geq 1$. The proof of Proposition 4.16 gives:
Proposition 5.24. The following maps are $\Lambda$-linear and injective:

$$
(\Pi^T\Lambda \circ \iota_\Lambda) \otimes_{\mathbb{Q}\Lambda} \text{id}_\Lambda : \text{CF}(\tilde{\mathfrak{g}}) \otimes_{\mathbb{Q}} \Lambda \longrightarrow \bar{SF}(\tilde{\mathfrak{g}}, Y, \Lambda),
$$

$$
\mu \circ ((\Pi^T\Lambda \circ \iota_\Lambda) \otimes_{\mathbb{Q}} \text{id}_\Lambda) : \text{CF}(\tilde{\mathfrak{g}}) \otimes_{\mathbb{Q}} \Lambda \longrightarrow \bar{SF}(\tilde{\mathfrak{g}}, Y, \Lambda),
$$

where in the second line $\mu : \bar{SF}(\text{Spec} \, K, Y, \Lambda) \otimes_{\mathbb{Q}} \bar{SF}(\tilde{\mathfrak{g}}, Y, \Lambda) \rightarrow \bar{SF}(\tilde{\mathfrak{g}}, Y, \Lambda)$ is the combination of the tensor product $\otimes : \bar{SF}(\text{Spec} \, K, Y, \Lambda) \times \bar{SF}(\tilde{\mathfrak{g}}, Y, \Lambda) \rightarrow \bar{SF}(\text{Spec} \, K \times \tilde{\mathfrak{g}}, Y, \Lambda)$ of Definition 5.18 with the isomorphism $\text{Spec} \, K \times \tilde{\mathfrak{g}} \cong \tilde{\mathfrak{g}}$.

Again, this shows that the spaces $\bar{SF}$, $SF(\tilde{\mathfrak{g}}, Y, \Lambda)$ are quite large, though not as large as $\bar{SF}$, $SF(\tilde{\mathfrak{g}}) \otimes_{\mathbb{Q}} \Lambda$, and therefore that the relations Definition 5.17(i)–(iii) have some kind of consistency about them.

Given a generator $[([\mathfrak{g}, \rho])$, for each $r \in \tilde{\mathfrak{g}}(K)$ with $\rho_x(r) = x \in \tilde{\mathfrak{g}}(K)$ we have a $K$-group morphism $\rho_x : \text{Aut}_K(r) \rightarrow \text{Aut}_K(x)$. Roughly speaking, the difference between the spaces $\bar{SF}(\tilde{\mathfrak{g}}, Y, \Lambda)$ of and the $\bar{SF}(\tilde{\mathfrak{g}}, Y, \Lambda)$ above is that the $\bar{SF}$, $SF(\tilde{\mathfrak{g}}, Y, \Lambda)$ keep track of the restriction of $\rho_x$ to a maximal torus of $\text{Aut}_K(r)$, but $SF(\tilde{\mathfrak{g}}, Y, \Lambda)$ loses this information.

Proposition 5.24 shows that in $\bar{SF}$, $SF(\tilde{\mathfrak{g}}, Y, \Lambda)$ we can always reduce to $[([\mathfrak{g}, \rho])$ with all stabilizer groups $\text{Aut}_K(r)$ for $r \in \tilde{\mathfrak{g}}(K)$ of the form $G_k \times K$ for $K$ finite abelian. That is, $\bar{SF}$, $SF(\tilde{\mathfrak{g}}, Y, \Lambda)$ abelianize stabilizer groups. We can regard the $\Pi^\mu_i$ of §5.2 as doing the same job: $[\text{Spec} \, K/G]$ is the sum of components $\Pi^\mu_i([\text{Spec} \, K/G])$ of $[\text{Spec} \, K/G]$ which behave like multiples of $[\text{Spec} \, K/G_m]$.

In the applications of [10–12], the concept of virtual rank given by the $\Pi^\mu_i$ is more useful than the real rank given by the $\Pi^\mu_i$ of §5.1. One reason for this is that the author can prove there are no relations analogous to Definition 5.17(i)–(iii) which are compatible with the $\Pi^\mu_i$ in the way that these are compatible with the $\Pi^\mu_i$, and which are also compatible with multiplication of pullbacks $\phi^\mu$. This suggests there is some kind of consistency between the $\Pi^\mu$, $\Pi^\mu_i$, $\Pi^\mu_\tilde{g}$ and $\phi^\mu$. The author does not yet understand.

6 Extension to the case $\ell = 1$

We now extend the constructions of §4 to the case when $\ell - 1$ is not invertible in $\Lambda$, and in particular to the case $\ell = 1$, which includes Euler characteristics $\chi$. We do this in §6.1 by supposing the algebra $\Lambda$ of §4.1 has a subalgebra $\Lambda^\circ$ containing $\Theta([X])$ for varieties $X$ and some rational functions of $\ell$, but not $(\ell - 1)^{-1}$, and that we are given a surjective algebra morphism $\pi : \Lambda^\circ \rightarrow \Omega$ with $\pi(\ell) = 1$. Then $\Theta = \pi \circ \Theta$ is the motivic invariant we are interested in, which takes values in $\Omega$. This can be done in all our examples.

Section 6.2 shows that the coefficients $E(G, T^G, Q)$ of (39) actually lie in $\Lambda^\circ$ (this is not obvious), and computes them when $G = \text{GL}(m, \mathbb{K})$. Therefore the relations Definition 5.17(i)–(iii) for $\bar{SF}$, $SF(\tilde{\mathfrak{g}}, Y, \Lambda)$ make sense with coefficients in $\Lambda^\circ$ rather than in $\Lambda$, and applying $\pi$ they also make sense with coefficients in $\Omega$. So in §6.3 we define new spaces $\bar{SF}$, $SF(\tilde{\mathfrak{g}}, Y, \Lambda^\circ)$ and $\bar{SF}$, $SF(\tilde{\mathfrak{g}}, \Theta, \Omega)$ with these relations, with the usual operations $\cdot$, $\phi_\circ$, $\phi^\circ$, $\Pi^\mu$, $\Pi^\mu_i$, $\Pi^\mu_\tilde{g}$. These will be important in [10–12] for defining invariants counting coherent sheaves on 40
Calabi–Yau 3-folds. When $\Omega = \mathbb{Q}$ and $\Theta = \chi$ we also define smaller spaces $\mathbb{S}^r\mathbb{S}F(\mathcal{J}, \chi, \mathbb{Q})$ exploiting special properties of $\chi$ on fibrations.

### 6.1 Initial assumptions and examples

For our next constructions we need more data than in Assumption 4.1.

**Assumption 6.1.** Suppose Assumption 4.1 holds, and $\Lambda^0$ is a $\mathbb{Q}$-subalgebra of $\Lambda$ containing the image of $\Upsilon$ and the elements $\ell^{-1}$ and $(\ell^k + \ell^{k-1} + \cdots + 1)^{-1}$ for $k = 1, 2, \ldots$, but not containing $(\ell - 1)^{-1}$. Let $\Omega$ be a commutative $\mathbb{Q}$-algebra, and $\pi : \Lambda^0 \to \Omega$ a surjective $\mathbb{Q}$-algebra morphism, such that $\pi(\ell) = 1$. Define

$$\Theta : \{\text{isomorphism classes } [X] \text{ of quasiprojective } \mathbb{K}\text{-varieties } X \} \to \Omega$$

by $\Theta = \pi \circ \Upsilon$. Then $\Theta([k_1]) = 1$.

Note that given $\Lambda^0$ satisfying the above conditions, there is a natural choice for $\Omega$, $\pi$: by assumption $(\ell - 1)\Lambda^0$ is an ideal in $\Lambda^0$, not containing 1, so we may take $\Omega = \Lambda^0/(\ell - 1)\Lambda^0$ to be the quotient algebra, with projection $\pi : \Lambda^0 \to \Omega$. We can satisfy Assumption 6.1 in all the examples of §4.1.

**Example 6.2.** In Example 4.3 let $\Lambda_{Ho}^0$ be the subalgebra of $P(x, y)/Q(x, y)$ in $\Lambda_{Ho}$ for which $xy - 1$ does not divide $Q(x, y)$. Set $\Omega_{Ho} = \mathbb{Q}(x)$, the $\mathbb{Q}$-algebra of rational functions in $x$, and define $\pi_{Ho} : \Lambda_{Ho}^0 \to \Omega_{Ho}$ by $\pi_{Ho} : P(x, y)/Q(x, y) \to P(x, x^{-1})/Q(x, x^{-1})$. Then Assumption 6.1 holds.

**Example 6.3.** In Example 4.3 let $\Lambda_{Po}^0$ be the subalgebra of $P(z)/Q(z)$ in $\Lambda_{Po}$ for which $z \pm 1$ do not divide $Q(z)$. Here are three possibilities for $\Omega_{Po}, \pi_{Po}$:

(a) Set $\Omega_{Po} = \mathbb{Q}$ and $\pi_{Po} : f(z) \mapsto f(-1)$. Then $\Theta_{Po}([X]) = \pi_{Po} \circ \Upsilon_{Po}([X])$ is the Euler characteristic of $X$.

(b) Set $\Omega_{Po} = \mathbb{Q}$ and $\pi_{Po} : f(z) \mapsto f(1)$. Then $\Theta_{Po}([X]) = \pi_{Po} \circ \Upsilon_{Po}([X])$ is the sum of the virtual Betti numbers of $X$.

(c) Set $\Omega_{Po} = \mathbb{Q} \oplus \mathbb{Q}$, a product of algebras, and $\pi_{Po} : f(z) \mapsto (f(-1), f(1))$.

This combines (a) and (b).

Assumption 6.1 holds in each case.

**Example 6.4.** In Example 4.3 let $\Lambda_{uni}^0$ be the subalgebra of $\Lambda_{uni}$ generated by elements $\ell^{-1}, (\ell^k + \ell^{k-1} + \cdots + 1)^{-1}$ for $k = 1, 2, \ldots$, and $[X]$ for quasiprojective $\mathbb{K}$-varieties $X$. Define $\Omega_{uni}$ to be the quotient algebra $\Lambda_{uni}^0/(\ell - 1)\Lambda_{uni}^0$, with projection $\pi_{uni} : \Lambda_{uni}^0 \to \Omega_{uni}$. We have a morphism $\Lambda_{uni} \to \Lambda_{Po}$ taking $\Lambda_{uni}^0 \to \Lambda_{Po}^0$ and $\ell \mapsto \ell$, so $(\ell - 1)^{-1} \notin \Lambda_{uni}^0$ as $(\ell - 1)^{-1} \notin \Lambda_{Po}^0$ in Example 6.3 and Assumption 6.1 holds.

Here are some useful facts about $\Upsilon([G])$.

**Lemma 6.5.** Let Assumptions 4.1 and 6.1 hold, and $G$ be a special algebraic $\mathbb{K}$-group of rank $k$. Then $\Upsilon([G]) \in (\ell - 1)^k \Lambda^0$ and $\Upsilon([G])^{-1} \in (\ell - 1)^{-k} \Lambda^0$.
We shall show \( G \) is a Zariski locally trivial fibration, and \( \Upsilon([G]) = \Upsilon([G/T^G]) \) by Lemma 3.12. Hence \( \Upsilon([G]) \in (\ell - 1)^k \Lambda^\circ \), as \( \Upsilon([T^G]) = (\ell - 1)^k \).

As \( G \) is special we can embed it in \( GL(m, K) \) with \( GL(m, K) \to GL(m, K)/G \) a Zariski locally trivial fibration, so \( \Upsilon([GL(m, K)]) = \Upsilon([G]) \Upsilon([GL(m, K)/G]) \). Applying Lemma 4.6 yields
\[
\Upsilon([G])^{-1} = (\ell - 1)^{-m} \cdot (\ell - m^{-1})^{1/2} \prod_{k=1}^{m} (\ell^{k-1} + 1)^{-1} \cdot \Upsilon([GL(m, K)/G]). \tag{46}
\]
Now the diagonal matrices \( \mathbb{G}_m \) in \( GL(m, K) \) act on \( GL(m, K)/G \), and the stabilizer of each point is isomorphic to \( \mathbb{G}_m \times K \) for \( j \geq 0 \) and finite abelian \( K \). Since this is conjugate to a subgroup of \( G \), we have \( j \leq k \). Hence each orbit of \( \mathbb{G}_m \) is isomorphic \( \mathbb{G}_m^{-j} \) for some \( 0 \leq j \leq k \).

Thus we may write \( GL(m, K)/G \) as a finite disjoint union of \( \mathbb{G}_m^{-j} \)-invariant \( K \)-subvarieties \( X_i \), such that the \( \mathbb{G}_m^{-j} \)-orbits make \( \mathbb{G}_m^{-j} \) into a fibre bundle with fibre \( \mathbb{G}_m^{-j} \) for \( 0 \leq j \leq k \). Now \( \mathbb{G}_m^{-j} \)-bundles are Zariski locally trivial fibrations as \( \mathbb{G}_m \) is special. So refining the decomposition if necessary we can suppose \( X_i = \mathbb{G}_m^{-j_i} \times Y_i \cong \mathbb{G}_m^{-k} \times \mathbb{G}_m^{-j_i} \times Y_i \) for some quasiprojective \( K \)-varieties \( Y_i \).

Thus \( \Upsilon([X_i]) = (\ell - 1)^{-m} \cdot \Upsilon([\mathbb{G}_m^{-j_i} \times Y_i]) \), and \( \Upsilon([GL(m, K)/G]) = \sum_i \Upsilon([X_i]) \in (\ell - 1)^{-m} \Lambda^\circ \). Combining this with (46) and using Assumption 6.1 shows \( \Upsilon([G])^{-1} \in (\ell - 1)^{-k} \Lambda^\circ \).

### 6.2 Properties of the \( E(G, T^G, Q) \)

We shall show \( E(G, T^G, Q) \) in (39) lies in \( \Lambda^\circ \). This is far from obvious, as by Lemma 6.5 each term in (39) lies in \( (\ell - 1)^{\dim Q - \rk G} \Lambda^\circ \). Effectively, in the sum over \( Q \) in (39) the terms in \( (\ell - 1)^{-n} \) for \( 0 < n \leq \rk G - \dim Q \) all cancel.

**Theorem 6.6.** Let Assumptions 4.7 and 6.1 hold and \( G \) be a very special algebraic \( K \)-group with maximal torus \( T^G \). Then \( E(G, T^G, Q) \) in (39) lies in \( \Lambda^\circ \) for all \( Q \in \mathcal{Q}(G, T^G) \).

**Proof.** Let \( X \) be a quasiprojective \( K \)-variety acted on by a very special \( K \)-group \( G \), so that \( [[X/G]] \in \mathcal{SF}(\Spec K) \). For \( \mu \) a weight function, Definition 5.10 defines \( \Pi^\mu : \mathcal{SF}(\Spec K) \to \mathcal{SF}(\Spec K) \). Applying \( \Upsilon' \) of Theorem 4.10 to (29) and noting that \( C_G(Q) \) is special for all \( Q \in \mathcal{Q}(G, T^G) \) as \( G \) is very special yields
\[
\Upsilon' \circ \Pi^\mu([X/G]) = \sum_{P \in \mathcal{P}(X, T^G), Q \in \mathcal{Q}(G, T^G)} \sum_{R \in \mathcal{R}(X, G, T^G) : R \subseteq P \cap Q} M_G^X(P, Q, R) \mu(R) \cdot \Upsilon([X^P]) \Upsilon([C_G(Q)])^{-1}. \tag{47}
\]
Substituting this in (27) gives a sum over \( P, Q, R, P', Q' \) with \( R = P' \cap Q' \). Comparing this with (39) we see that the sum over \( Q \) is proportional to that defining \( E(G, T^G, Q') \), so (47) becomes
\[
\Upsilon' \circ \Pi^\mu([X/G]) = \sum_{P' \subseteq P \in \mathcal{P}(X, T^G), Q' \in \mathcal{Q}(G, T^G)} m_{T^G}^{X}(P', P) \Upsilon([X^P]) \mu(P' \cap Q') \cdot \Upsilon([Q'])^{-1} E(G, T^G, Q'). \tag{48}
\]
If \( P \) is a closed \( \mathbb{K} \)-subgroup of \( T^G \), write \( X_{T,G}^P = \{ x \in X : \text{Stab}_{T,G}(x) = P \} \), a subvariety of \( X \). It is easy to see that if \( X_{T,G}^P \neq \emptyset \) then \( P \in \mathcal{P}(X, T^G) \), and for \( P \in \mathcal{P}(X, T^G) \) we have \( X^P = \bigcap_{P' \in \mathcal{P}(X, T^G), P' \subseteq P} X_{T,G}^{P'} \). Therefore \( \Upsilon([X^P]) = \sum_{P' \in \mathcal{P}(X, T^G), P' \subseteq P} \Upsilon([X_{T,G}^{P'}]). \) Inverting this combinatorially using properties of the \( m_{T,G}(P', P) \) yields \( \Upsilon([X_{T,G}^P]) = \sum_{P \in \mathcal{P}(X, T^G), P' \subseteq P} m_{T,G}(P', P) \Upsilon([X^P]) \).

Comparing this with (48) we see that

\[
\Upsilon' \circ \Pi'([X/G]) = \sum_{P' \in \mathcal{P}(X, T^G), \gamma Q < \gamma Q'} \Upsilon([X_{T,G}^{P'}]) \mu(\gamma Q') \Upsilon([\gamma Q'])^{-1} E(G, T^G, Q').
\] (49)

Now \( E(G, T^G, Q) \in (\ell - 1)^{\dim Q - \rk G} \Lambda^\circ \) as above, proving the theorem for \( Q = T^G \). So suppose \( \dim Q < \dim T^G = \rk G \). Choose a \( \mathbb{K} \)-subgroup \( T \subseteq T^G \) with \( T \cong \mathbb{G}_m^{\rk G - \dim Q} \) such that \( K = T \cap Q \) is finite, and if \( Q \neq Q' \in Q(G, T^G) \) then \( T \cap Q' \neq K \). This is possible if \( \dim Q > 0 \), as there are infinitely many \( T \). But it may not be if \( \dim Q = 0 \), as \( T = T^G \) is the only choice.

Define \( \mu \) in Definition 5.11 by \( \mu([H]) = 1 \) if \( H \cong K \) and \( \mu([H]) = 0 \) otherwise. Set \( X = G/T \). Then we have 1-isomorphisms \([X/G] \cong [\text{Spec} \mathbb{K}[T]/T^G \cong [(T^G/T)/T^G]] \). Since \( T \neq K \) as \( \dim T > 0 \), we find from Definition 5.11 that \( \Pi'([T^G/T]/T^G]) = 0 \), so \( \Pi'([X/G]) = 0 \) by Theorem 5.11 and (48) is zero. Suppose some \( P', Q' \) give a nonzero term on the r.h.s. of (49). Then \( P' \) is conjugate in \( G \) to a subgroup of \( T \) as \( X_{T,G}^{P'} \neq \emptyset \), and \( \gamma Q' \equiv K \) as \( \mu(P' \cap Q') \neq 0 \). Hence \( \dim P' \leq \dim T \), and \( \dim P' + \dim Q' \leq \rk G \) as \( \dim P' \cap Q' = 0 \).

If \( \dim P' = \dim T \) then \( P' \) is conjugate to \( T \) as \( T \) is connected, giving \( P' = \gamma T \) for \( \gamma \in W(G, T^G) \). Then \( \gamma Q' \equiv K \) and the choice of \( T \) imply \( \gamma Q' = \gamma Q \). Rearranging (48) to put terms \( P', Q' = \gamma T, \gamma Q \) on the left gives

\[
\frac{|W(G, T^G)|}{[\gamma \in W(G, T^G) : \gamma T = T, \gamma Q = Q]} \Upsilon([X_{T,G}^T]) \Upsilon([\gamma Q])^{-1} E(G, T^G, Q) =
\]

\[
- \sum_{P', Q' \in \mathcal{Q}(G, T^G), \dim P' + \dim Q' \leq \rk G, P' \cap Q' \equiv K} \Upsilon([X_{T,G}^{P'}]) \Upsilon([\gamma Q'])^{-1} E(G, T^G, Q'). \]
(50)

Since \( X = G/T \) we find that \( X_{T,G}^T = N_G(T)/T \), so that

\[
\Upsilon([X_{T,G}^T]) = \Upsilon([N_G(T)/T]) = (\ell - 1)^{\dim Q - \rk G} |N_G(T)/C_G(T)| \Upsilon([C_G(T)]) \]

As \( G \) is very special \( C_G(T) \) is special, and \( \rk C_G(T) = \rk G \), so \( \Upsilon([C_G(T)])^{-1} \in (\ell - 1)^{-\rk G} \Lambda^\circ \) by Lemma 5.3. The orbits of \( T^G \) on \( X_{T,G}^P \) are all isomorphic to \( T^G/P' \cong G_m^{\rk G - \dim P'} \), so the argument of Lemma 5.3 shows that \( \Upsilon([X_{T,G}^{P'}]) \in (\ell - 1)^{\rk G - \dim P' \Lambda^\circ} \). Combining these with (50) shows that

\[
E(G, T^G, Q) = \sum_{P' \in \mathcal{P}(X, T^G), \gamma Q' \in \mathcal{Q}(G, T^G), \dim P' < \rk G - \dim Q} (\text{term in } (\ell - 1)^{\rk G - \dim P' - \dim Q' \Lambda^\circ}) \cdot \ U(G, T^G, Q'). \] (51)

Let \( k = 1, \ldots, \rk G \) be given, and suppose by induction that
If \( Q \in \mathbb{Q}(G,T^G) \) then \( E(G,T^G,Q) \) lies in \( \Lambda^e \) when \( \dim Q \geq k \), and in 
\((\ell - 1)^{\dim Q - k}\Lambda^e \) otherwise.

When \( k = \text{rk} G \) this is immediate, from above. Supposing \((\star_k)\) holds we can use \((61)\) to prove \((\star_{k-1})\), by applying \((\star_k)\) to \( E(G,T^G,Q') \) in \((61)\) and thinking carefully about the effect of the inequalities \( \dim P' < \text{rk} G - \dim Q, \dim P' + \dim Q' \leq \text{rk} G \). So \((\star_0)\) holds by induction. This completes the proof, except for the problem in choosing \( T \) above when \( \dim Q = 0 \). We can solve this by a similar argument involving \([X/G]\) for \( X = G/Q \) and \( Q \in \mathbb{Q}(G,T^G) \) finite.

So we may define:

**Definition 6.7.** Let Assumptions 4.1 and 6.1 hold and \( G \) be a very special algebraic \( \mathbb{K} \)-group with maximal torus \( T^G \). For all \( Q \in \mathbb{Q}(G,T^G) \) set 
\( F(G, T^G, Q) = \pi(E(G,T^G, Q)) \). This is well-defined by Theorem 6.6.

We now continue Example 5.7, and calculate the \( E,F(G,T^G,Q) \) when \( G = \text{GL}(m,\mathbb{K}) \). Let \( G,T^G,m,n,\phi \) be as in Example 5.7 and define \( Q \in \mathbb{Q}(G,T^G) \) by \((24)\). Write \( m_k = |\phi^{-1}(\{k\})| \) for \( k = 1, \ldots, n \), so that \( m = m_1 + \cdots + m_n \).

From \((44)\) with \( Q' = Q \) we see that
\[
E(G,T^G,Q) = \left\{ \frac{|w \in W(G,T^G) : w|_Q = \text{id}_Q|}{|W(G,T^G)|} \right\} E(C_G(Q),T^G,Q).
\]

Now \( C_G(Q) \cong \prod_{k=1}^n \text{GL}(m_k,\mathbb{K}) \) by \((23)\) with \( Q \cong \prod_{k=1}^n \mathbb{G}_m \cdot \text{id}_{m_k} \), where \text{id}_{m_k} is the identity matrix in \( \text{GL}(m_k,\mathbb{K}) \). As
\[
E(G \times H,T^G \times T^H,Q_G \times Q_H) = E(G,T^G,Q_G) \cdot E(H,T^H,Q_H),
\]
we deduce that
\[
E(\text{GL}(m,\mathbb{K}),\mathbb{G}_m^m, Q) = \frac{1}{m!} \prod_{k=1}^n m_k! E(m_k), \tag{52}
\]
where \( E(m) = E(\text{GL}(m,\mathbb{K}),\mathbb{G}_m^m, \text{id}_m) \). Applying \( \pi \) gives
\[
F(\text{GL}(m,\mathbb{K}),\mathbb{G}_m^m, Q) = \frac{1}{m!} \prod_{k=1}^n m_k! F(m_k), \tag{53}
\]
where \( F(m) = F(\text{GL}(m,\mathbb{K}),\mathbb{G}_m^m, \text{id}_m) \).

So it is enough to compute \( E(m), F(m) \). For small values of \( m \) we can do this directly using \((39)\), Example 5.7 and Lemma 4.6, giving
\[
E(1) = 1, \quad F(1) = 1, \quad E(2) = (\ell + 1)^{-1} (-\ell^{-1} - \frac{1}{2}), \quad F(2) = -\frac{1}{4},
\]
\[
E(3) = (\ell^2 + \ell + 1)^{-1} (\ell^{-3} + \ell^{-2} + \ell^{-1} + \frac{1}{3}), \quad F(3) = \frac{10}{9}, \quad \ldots.
\]

For larger values of \( m \) it is helpful to have an inductive formula for \( E(m), F(m) \).

By writing \([\mathbb{K}^m/\text{GL}(m+1,\mathbb{K})]) = ([\text{Spec} \mathbb{K}/\text{GL}(m,\mathbb{K}) \ltimes \Lambda^m]) \) in two different ways in \( \text{SF}(\text{Spec} \mathbb{K}, \Upsilon, \Lambda) \), after some calculation we find that
\[
\sum_{n=1}^{m+1} \frac{1}{n!} \prod_{m+1=m_1+\cdots+m_n, \ m_k \geq 1 \ k=1} \prod_{\ell - 1}^{m_k} E(m_k) = \sum_{n=1}^{m} \frac{(-1)^n}{n!} \prod_{m+1=m_1+\cdots+m_n, \ m_k \geq 1 \ k=1} \prod_{\ell - 1}^{m_k} E(m_k). 
\tag{54}
\]
Applying $\pi$ then gives

$$
\sum_{n=1}^{m+1} \frac{1}{n!} \sum_{m_1+\cdots+m_n=m, m_k \geq 1} \prod_{k=1}^{n} m_k F(m_k) = \sum_{n=1}^{m} \frac{(-1)^n}{n!} \sum_{m_1+\cdots+m_n=m, m_k \geq 1} \prod_{k=1}^{n} m_k F(m_k).
$$

(55)

Here (54) and (55) contain $(\ell^m + \cdots + 1)E(m+1)$ and $(m+1)F(m+1)$ on the top lines with $n = 1$, and all other terms involve $E(m')$ and $F(m')$ for $m' \leq m$. So we can use (54) to (55) to find $E(m), F(m)$ inductively.

### 6.3 Spaces $\bar{\mathsf{SF}}, \mathsf{SF}(\mathfrak{F}, \mathfrak{Y}, \Lambda^\circ), \bar{\mathsf{SF}}, \mathsf{SF}(\mathfrak{F}, \Theta, \Omega), \bar{\mathsf{SF}}, \mathsf{SF}(\mathfrak{F}, \chi, \mathbb{Q})$

We restrict the spaces $\mathsf{SF}(\ast, \mathfrak{Y}, \Lambda)$ to $\Lambda^\circ$, and then project to $\Omega$.

**Definition 6.8.** Let Assumptions 4.1 and 6.1 hold, and $\mathfrak{F}$ be an algebraic $\mathbb{K}$-stack with affine geometric stabilizers. Consider pairs $(\mathfrak{R}, \rho)$, where $\mathfrak{R}$ is a finite type algebraic $\mathbb{K}$-stack with affine geometric stabilizers and $\rho : \mathfrak{R} \to \mathfrak{F}$ is a 1-morphism, with equivalence of pairs as in Definition 5.1. Define $\bar{\mathsf{SF}}, \mathsf{SF}(\mathfrak{F}, \mathfrak{Y}, \Lambda^\circ)$ to be the $\Lambda^\circ$-modules generated by equivalence classes $[(\mathfrak{R}, \rho)]$ as above, with $\rho$ representable for $\mathsf{SF}(\mathfrak{F}, \mathfrak{Y}, \Lambda^\circ)$, and with relations Definition 5.17(i)--(iii). These make sense with $\Lambda^\circ$ in place of $\Lambda$ since (i)--(iii) only involve multiplying by elements of $\Lambda^\circ$ in $\Lambda$. In particular, $\mathcal{Y}((U)) \in \Lambda^\circ$ in (ii), and $E(G, T^G, Q) \in \Lambda^\circ$ in equation (40) of (iii) by Theorem 6.6.

Define $\bar{\mathsf{SF}}, \mathsf{SF}(\mathfrak{F}, \Theta, \Omega)$ to be the $\Omega$-modules generated by equivalence classes $[(\mathfrak{R}, \rho)]$ as above, with $\rho$ representable for $\mathsf{SF}(\mathfrak{F}, \Theta, \Omega)$, and with relations Definition 5.17(ii)--(iii) projected to $\Omega$ using $\pi$ in the obvious way. That is, in (ii) we have $[(\mathfrak{R} \times U, \rho \circ \pi_{\mathfrak{R}})] = \Theta([U])[(\mathfrak{R}, \rho)]$, and (40) becomes

$$
[(\mathfrak{R}, \rho)] = \sum_{Q \in \mathcal{Q}(G, T^G)} F(G, T^G, Q) [[X/Q, \rho \circ \iota^Q]].
$$

Since $\pi : \Lambda^\circ \to \Omega$ is supposed surjective we have isomorphisms

$$
\mathsf{SF}(\mathfrak{F}, \Theta, \Omega) \cong \mathsf{SF}(\mathfrak{F}, \mathfrak{Y}, \Lambda^\circ)/(\ker \pi : \bar{\mathsf{SF}}(\mathfrak{F}, \mathfrak{Y}, \Lambda^\circ)),
\mathsf{SF}(\mathfrak{F}, \Theta, \Omega) \cong \mathsf{SF}(\mathfrak{F}, \mathfrak{Y}, \Lambda^\circ)/(\ker \pi : \bar{\mathsf{SF}}(\mathfrak{F}, \mathfrak{Y}, \Lambda^\circ)),
$$

(56)

where $\ker \pi$ is an ideal in $\Lambda^\circ$. Define projections

$$
\bar{\Pi}^{Y, \Lambda^\circ}_{\mathfrak{F}} : \bar{\mathsf{SF}}(\mathfrak{F}) \to \bar{\mathsf{SF}}(\mathfrak{F}, \mathfrak{Y}, \Lambda^\circ), \quad \bar{\Pi}^{Y, \Lambda}_{\mathfrak{F}} : \bar{\mathsf{SF}}(\mathfrak{F}, \mathfrak{Y}, \Lambda^\circ) \to \bar{\mathsf{SF}}(\mathfrak{F}, \mathfrak{Y}, \Lambda),
\bar{\Pi}^{\Theta, \Omega}_{\mathfrak{F}} : \bar{\mathsf{SF}}(\mathfrak{F}) \to \bar{\mathsf{SF}}(\mathfrak{F}, \Theta, \Omega), \quad \bar{\Pi}^{\Theta, \Omega}_{\mathfrak{F}} : \bar{\mathsf{SF}}(\mathfrak{F}, \Theta, \Omega) \to \bar{\mathsf{SF}}(\mathfrak{F}, \Theta, \Omega),
$$

by (13), replacing $c_i$ by $\rho(c_i)$ on the r.h.s. for $\bar{\Pi}^{\Theta, \Omega}_{\mathfrak{F}} : \bar{\mathsf{SF}}(\mathfrak{F}, \mathfrak{Y}, \Lambda^\circ) \to \bar{\mathsf{SF}}(\mathfrak{F}, \Theta, \Omega)$. These are well-defined since they map relations in the domain to relations in the target, as in Definitions 4.11 and 5.17. Define multiplication $\cdot$, pushforwards $\phi_\ast$, pullbacks $\phi^\ast$, tensor products $\otimes$ and operators $\Pi^\mu, \Pi^\nu, \Pi^\varepsilon$ on the spaces $\bar{\mathsf{SF}}, \mathsf{SF}(\ast, \mathfrak{Y}, \Lambda^\circ)$ and $\bar{\mathsf{SF}}, \mathsf{SF}(\ast, \Theta, \Omega)$ exactly as in Definition 5.18.
From the proofs of Theorems 4.13 and 5.19 we deduce the analogous result for the $\overline{SF}, SF(\ast, \Upsilon, \Lambda^v)$. This is nearly immediate, as the relations in $\overline{SF}, SF(\ast, \Upsilon, \Lambda^v)$ are the same as in $\overline{SF}, SF(\ast, \Upsilon, \Lambda)$. We know that under the operations $\ldots, \cdot, \cdots$ relations are taken to linear combinations of relations with coefficients in $\Lambda$, and we must check these coefficients may be chosen in $\Lambda^v$, which is fortunately obvious. Projecting coefficients from $\Lambda^v$ to $\Omega$ using $\pi$ proves the same thing for the $\overline{SF}, SF(\ast, \Omega, \Omega)$, giving:

**Theorem 6.9.** These operations $\ldots, \cdot, \cdots, \overline{\Pi}^v$ on $\overline{SF}, SF(\ast, \Upsilon, \Lambda^v)$ and $\overline{SF}, SF(\ast, \Omega, \Omega)$ are compatible with the relations, and so are well-defined.

As for Corollaries 4.14 and 5.20 we deduce:

**Corollary 6.10.** The projections $\overline{\Pi}^v, \overline{\Pi}^\Omega, \overline{\Pi}^\Lambda, \overline{\Pi}^{v, \Omega}$ commute with the operations $\ldots, \cdot, \cdots, \overline{\Pi}^v$ on $\overline{SF}(\ast), SF(\ast, \Upsilon, \Lambda^v), SF(\ast, \Upsilon, \Lambda^v), SF(\ast, \Omega, \Omega)$. The analogues of Theorems 3.12 and 5.22 and Proposition 5.23 hold for the spaces $\overline{SF}, SF(\ast, \Upsilon, \Lambda^v)$ and $\overline{SF}, SF(\ast, \Omega, \Omega)$.

The analogues of Propositions 5.21 and 5.22 apply for the $\overline{SF}, SF(\ast, \Upsilon, \Lambda^v)$ and $\overline{SF}, SF(\ast, \Omega, \Omega)$, replacing $\Lambda$ by $\Lambda^v$. The analogue of Proposition 5.23 is:

**Proposition 6.11.** Define commutative $\Lambda^v$-algebras $\tilde{\Lambda}^v, \tilde{\Omega}$ with $\Lambda^v$- and $\Omega$-bases isomorphism classes $[T]$ of $K$-groups $T$ of the form $\mathbb{G}_m \times K$, for $k \geq 0$ and $K$ finite abelian, with multiplication $[T][T'] = [T \times T']$. Define $\overline{i}_{v, \Lambda^v} : \Lambda^v \to \overline{SF}(\text{Spec } K, \Upsilon, \Lambda^v)$ and $\overline{i}_{\Omega} : \Omega \to \overline{SF}(\text{Spec } K, \Theta, \Omega)$ by $\sum c_i[T_i] \mapsto \sum c_i[[\text{Spec } K/T_i]]$. Then $\overline{i}_{v, \Lambda^v}, \overline{i}_{\Omega}$ are algebra isomorphisms. They restrict to isomorphisms from $\Lambda^v[\{1\}], \Omega[\{1\}]$ to $\overline{SF}(\text{Spec } K, \Upsilon, \Lambda^v), \overline{SF}(\text{Spec } K, \Theta, \Omega)$.

**Proof.** For the $\Lambda^v$ case we follow Proposition 5.23, replacing $\Lambda$ by $\Lambda^v$ throughout, except that to deduce injectivity we apply $i_{v, \Lambda^v}^{-1} \circ \overline{\Pi}^{v, \Lambda, \Lambda} \circ \overline{\Pi}^{v, \Lambda, \Lambda}$ to project to $\Lambda$, not $\Lambda^v$. The $\Omega$ case then follows using 5.20, since $\Omega \cong \Lambda^v/(\text{Ker } \pi \cdot \Lambda^v)$. 

If $\mathfrak{g}$ is a finite type algebraic $K$-stack with affine geometric stabilizers then $[\mathfrak{g}] \in \overline{SF}(\text{Spec } K, \Upsilon, \Lambda^v)$ or $\overline{SF}(\text{Spec } K, \Theta, \Omega)$ and so $\overline{i}_{v, \Lambda^v}^{-1}([\mathfrak{g}]), \overline{i}_{v, \Lambda^v}^{-1}([\mathfrak{g}])$ lie in $\Lambda^v, \tilde{\Omega}$. We can regard these as generalizations of $\Upsilon'$ in Theorem 4.14 which work even when $\ell = 1$. In particular, when $\Omega = \mathbb{Q}$ and $\Theta = \chi$ as in Example 6.3(a), $\overline{i}_{v, \Lambda^v}^{-1}([\mathfrak{g}]) \in \mathbb{Q}$ is a kind of generalized Euler characteristic of $\mathfrak{g}$.

As for Proposition 5.24 we have:

**Proposition 6.12.** The following maps are $\Lambda^v$- or $\Omega$-linear and injective:

\[
\begin{align*}
(\overline{\Pi}^{v, \Lambda^v} \circ \overline{i}_{v, \Lambda^v}) \otimes_{\mathbb{Q}} \text{id}_{\Lambda^v} & : CF(\mathfrak{g}) \otimes_{\mathbb{Q}} \Lambda^v \to \overline{SF}(\mathfrak{g}, \Upsilon, \Lambda^v), \\
(\overline{\Pi}^{\Omega} \circ \overline{i}_{v, \Lambda^v}) \otimes_{\mathbb{Q}} \text{id}_\Omega & : CF(\mathfrak{g}) \otimes_{\mathbb{Q}} \Omega \to \overline{SF}(\mathfrak{g}, \Theta, \Omega), \\
\mu \circ (\overline{\Pi}^{v, \Lambda^v} \circ \overline{i}_{v, \Lambda^v}) & : CF(\mathfrak{g}) \otimes_{\mathbb{Q}} \Lambda^v \to \overline{SF}(\mathfrak{g}, \Upsilon, \Lambda^v), \\
\mu \circ (\overline{\Pi}^{\Omega} \circ \overline{i}_{v, \Lambda^v}) & : CF(\mathfrak{g}) \otimes_{\mathbb{Q}} \Omega \to \overline{SF}(\mathfrak{g}, \Theta, \Omega),
\end{align*}
\]

defining $\mu$ as in Proposition 5.24.

We generalize the $\pi^{\mathfrak{g}k}$ of $\mathfrak{g}$ to $\overline{SF}(\mathfrak{g}, \Upsilon, \Lambda^v)$ and $\overline{SF}(\mathfrak{g}, \Theta, \Omega)$. 

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Definition 6.13. Let Assumptions 4.1 and 6.1 hold with $K$ of characteristic zero. Suppose $X : \Lambda^0 \to \mathbb{Q}$ or $X : \Omega \to \mathbb{Q}$ is an algebra morphism with $X \circ \Theta((U)) = \chi((U))$ for all quasiprojective $K$-varieties $U$, where $\chi$ is the Euler characteristic. Such morphisms $X$ exist in all of Examples 6.2–6.4. Let $\tilde{\mathfrak{g}}$ be an algebraic $K$-stack with affine geometric stabilizers. Define

$$\tilde{\pi}^\text{stk} : SF(\tilde{\mathfrak{g}}, \Theta, \Lambda^0) \to CF(\tilde{\mathfrak{g}})$$

or $\tilde{\pi}^\text{stk} : SF(\tilde{\mathfrak{g}}, \Theta, \Omega) \to CF(\tilde{\mathfrak{g}})$ by

$$\tilde{\pi}^\text{stk} \left( \sum_{i=1}^n c_i([\mathfrak{R}_i, \rho_i]) \right) = \sum_{i=1}^n X(c_i) CF(\rho_i)_{\mathfrak{R}_i},$$

following (57). By a complicated proof similar to Theorems 4.13 and 4.19 we can show that $\tilde{\pi}^\text{stk}$ is compatible with the relations defining $SF(\tilde{\mathfrak{g}}, \Theta, \Lambda^0)$ and $SF(\tilde{\mathfrak{g}}, \Theta, \Omega)$, and so is well-defined. The analogues of Propositions 3.3 and 3.6 can show that $\tilde{SF}((\tilde{\mathfrak{g}}, \Theta, \Omega), \chi)$ is a Zariski locally trivial fibration. This is a special property of $\tilde{\mathfrak{g}}$, like the constructible functions $CF(F(\tilde{\mathfrak{g}}))$ or $\tilde{\pi}^\text{stk} : SF(\tilde{\mathfrak{g}}, \Theta, \Omega) \to CF(\tilde{\mathfrak{g}})$ by

$$\tilde{\pi}^\text{stk} \left( \sum_{i=1}^n c_i([\mathfrak{R}_i, \rho_i]) \right) = \sum_{i=1}^n X(c_i) CF(\rho_i)_{\mathfrak{R}_i}.$$

In the situation of Examples 4.4 and 6.3(a) we have $\Omega = \mathbb{Q}$ and $\Theta = \chi$, the Euler characteristic, so we have defined spaces $SF(\tilde{\mathfrak{g}}, \chi, \mathbb{Q})$ which are ver

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