Incomplete Riemann-Liouville fractional derivative operators and incomplete hypergeometric functions

Mehmet Ali Özarslan and Ceren Ustaoğlu
Eastern Mediterranean University
Gazimagusa, TRNC, Mersin 10, Turkey
Email: mehmetali.ozarslan@emu.edu.tr, ceren.ustaoglu@emu.edu.tr

Abstract
In this paper, the incomplete Pochhammer ratios are defined in terms of the incomplete beta function $B_y(x, z)$. With the help of these incomplete Pochhammer ratios we introduce new incomplete Gauss, confluent hypergeometric and Appell’s functions and investigate several properties of them such as integral representations, derivative formulas, transformation formulas and recurrence relation. Furthermore, an incomplete Riemann-Liouville fractional derivative operators are introduced. This definition helps us to obtain linear and bilinear generating relations for the new incomplete Gauss hypergeometric functions.

Key words : incomplete gamma functions, Pochhammer symbols, incomplete Pochhammer ratios, incomplete beta functions, incomplete hypergeometric functions, incomplete Appell’s functions, generating relations.

1 Introduction
In recent years, some extensions of the well known special functions have been considered by several authors (see, for example, [3], [4], [6], [15], [16], [17], [18], [19], [29]). The familiar incomplete gamma functions $\gamma(s, x)$ and $\Gamma(s, x)$ are defined by

$$\gamma(s, x) := \int_0^x t^{s-1} e^{-t} dt \quad (\text{Re}(s) > 0; \; x \geq 0)$$

and

$$\Gamma(s, x) := \int_x^\infty t^{s-1} e^{-t} dt \quad (x \geq 0; \; \text{Re}(s) > 0 \text{ when } x = 0),$$

respectively. They satisfy the following decomposition formula:

$$\gamma(s, x) + \Gamma(s, x) = \Gamma(s) \quad (\text{Re}(s) > 0). \quad (1)$$

The function $\Gamma(s)$ and its incomplete versions $\gamma(s, x)$ and $\Gamma(s, x)$, play important roles in the study of analytical solutions of a variety of problems in diverse areas of science and engineering [14].

The widely used Pochhammer symbol $(\lambda)_\nu$ $(\lambda, \nu \in \mathbb{C})$ is defined, in general, by

$$(\lambda)_\nu := \frac{\Gamma(\lambda + \nu)}{\Gamma(\lambda)} = \begin{cases} \frac{1}{\lambda(\lambda + 1)\ldots(\lambda + \nu - 1)} & (\nu = 0; \; \lambda \in \mathbb{C} \setminus \{0\}) \\ (\nu \in \mathbb{N}; \; \lambda \in \mathbb{C}) & \end{cases} \quad (2)$$

In terms of the incomplete gamma functions $\gamma(s, x)$ and $\Gamma(s, x)$, the incomplete Pochhammer symbols $(\lambda; x)_\nu$ and $[\lambda; x]_\nu$ $(\lambda, \nu \in \mathbb{C}; \; x \geq 0)$ were defined as follows [3]:

$$(\lambda; x)_\nu := \frac{\gamma(\lambda + \nu, x)}{\Gamma(\lambda)} \quad (\lambda, \nu \in \mathbb{C}; \; x \geq 0) \quad (3)$$
The incomplete beta function is defined by
\[ B_y (x, z) := \int_0^y t^{x-1}(1-t)^{z-1}dt, \quad \text{Re}(x) > \text{Re}(z) > 0, \quad 0 \leq y < 1 \] (9)
and can be expressed in terms of the Gauss hypergeometric function

\[ B_y(x, z) := \frac{y^x}{x} (1 - y)^z \, _2F_1(1, x + z; 1 + x; y). \]  

(10)

The incomplete beta function satisfy the following relation:

\[ B_y(b + n, c - b) + B_{1-y}(c - b, b + n) = B(b + n, c - b), \quad n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}. \]  

(11)

In terms of the incomplete beta function \( B_y(x, z) \), the incomplete Pochhammer ratios \([b, c; y]_n\) and \(\{b, c; y\}_n\) are introduced as follows:

\[ [b, c; y]_n := \frac{B_y(b + n, c - b)}{B(b, c - b)} \]  

(12)

and

\[ \{b, c; y\}_n := \frac{B_{1-y}(c - b, b + n)}{B(b, c - b)} \]  

(13)

where \(0 \leq y < 1\). It is clear from (11) that

\[ [b, c; y]_n + \{b, c; y\}_n = \binom{b}{c}_n. \]  

(14)

In view of (10), we have the following relations

\[ [b, c; y]_n := \frac{1}{B(b, c - b) b + n} y^{b+n} (1 - y)^{-b} \, _2F_1(1, c + n; b + n + 1; y) \]  

(15)

and

\[ \{b, c; y\}_n := \frac{1}{B(b, c - b) c - b} y^{b+n} (1 - y)^{-c} \, _2F_1(1, c + n; 1 + c - b; 1 - y). \]  

(16)

In the following theorem, we investigate the \(n\)th derivatives of the incomplete beta function by means of incomplete Pochhammer ratios.

**Theorem 1** The following derivative formulas hold true:

\[ [b, c; y]_n = \frac{(-1)^n \Gamma(c) \Gamma(b + n)}{\Gamma(c - b + n) \Gamma(b)} y^{b+n} \frac{d^n}{dy^n} [y^{-b} B_y(b, c - b + n)], \]  

(17)

and

\[ \{b, c; y\}_n = \frac{\Gamma(b + n)}{\Gamma(b + 2n) B(b, c - b)} (1 - y)^{-c} \frac{d^n}{dy^n} [(1 - y)^{c+b+n} B_{1-y}(c - b - n, b + 2n)]. \]  

(18)

**Proof.** Using (11) and (12), we immediately obtain the following equation:

\[ [b, c; y]_n = \frac{y^{b+n}}{B(b, c - b)} \int_0^1 u^{b+n-1} (1 - uy)^{c-b-1} du. \]

On the other hand, we have

\[ y^{-b} B_y(b, c - b + n) = \int_0^1 u^{b-1} (1 - uy)^{c-b+n-1} du. \]  

(19)

Taking derivatives \(n\) times on both sides of (19) with respect to \(y\), we can obtain a derivative formula for the incomplete beta function \([b, c; y]_n\) asserted by (17). Formula (18) can be proved in a similar way. \(\blacksquare\)
3 The new incomplete Gauss and confluent hypergeometric functions

In this section, we introduce new incomplete Gauss and confluent hypergeometric functions by

\[ 2F_1(a, [b, c; y]; x) := \sum_{n=0}^{\infty} (a)_n [b, c; y]_n \frac{x^n}{n!}, \]  
(20)

\[ 2F_1(a, \{b, c; y\}; x) := \sum_{n=0}^{\infty} (a)_n \{b, c; y\}_n \frac{x^n}{n!}, \]  
(21)

\[ 1F_1([a, b; y]; x) := \sum_{n=0}^{\infty} [a, b; y]_n \frac{x^n}{n!}, \]  
(22)

and

\[ 1F_1(\{a, b; y\}; x) := \sum_{n=0}^{\infty} \{a, b; y\}_n \frac{x^n}{n!}. \]  
(23)

where \(0 \leq y < 1\).

An immediate consequence of (14) and the definitions (20), (21), (22) and (23) are the following decomposition formulas

\[ 2F_1(a, [b, c; y]; x) + 2F_1(a, \{b, c; y\}; x) = 2F_1(a, b; c; x) \]  
(24)

and

\[ 1F_1([a, b; y]; x) + 1F_1(\{a, b; y\}; x) = 1F_1(a; b; x). \]  
(25)

**Theorem 2** The following integral representation holds true:

\[ 2F_1(\{a, b, c; y\}; x) = \frac{y^b}{B(b, c-b)} \int_0^1 u^{b-1}(1-uy)c^{-b-1}(1-xuy)^{-a} du, \]  
(26)

\(\text{Re}(c) > \text{Re}(b) > 0, \ |\text{arg}(1-x)| < \pi). \)

**Proof.** Replacing the incomplete Pochhammer ratio \([b, c; y]\) in the definition (20) by its integral representation given by (9) and interchanging the order of summation and integral which is permissible under the conditions given in the hypothesis of the Theorem, we find

\[ 2F_1(a, [b, c; y]; x) = \frac{1}{B(b, c-b)} \int_0^y u^{b-1}(1-t)^{c-b-1}(1-xt)^{-a} dt, \]  
(27)

which can be written as follows:

\[ 2F_1(a, [b, c; y]; x) = \frac{y^b}{B(b, c-b)} \int_0^1 u^{b-1}(1-uy)^{c-b-1}(1-xuy)^{-a} du. \]  
(28)

In a similar way, we have the following theorem:

**Theorem 3** The following integral representation holds true:

\[ 2F_1(a, \{b, c; y\}; x) = \frac{(1-y)^{c-b}}{B(b, c-b)} \int_0^1 u^{c-b-1}(1-u(1-y))^{b-1}(1-x+ux(1-y))^{-a} du, \]  
(29)

\(\text{Re}(c) > \text{Re}(b) > 0, |\text{arg}(1-x)| < \pi). \)

**Theorem 4** The following result holds true:

\[ 2F_1(a, [b, c; y]; 1) = \frac{\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} - \frac{(1-y)^{c-b-a}y^b}{B(b, c-b)(c-a-b)} 2F_1(c-a, 1; 1+c-b-a; 1-y). \]  
(30)
Proof. Putting \( x = 1 \) in (34), we obtain
\[
2F_1(a, [b, c; y], 1) = 2F_1(a, b; 1) - 2F_1(a, \{b, c; 1 - y\}; 1)
\]  
\[
= \frac{\Gamma(c)\Gamma(c - a - b)}{\Gamma(c - a)\Gamma(c - b)} \frac{(1 - y)^{c - b - a}}{B(b, c - b)(c - b - a)} \int_0^1 u^{c - b - a - 1}(1 - u(1 - y))^{b - 1} du.
\]  
Using the Euler’s integral representation for (31), we have
\[
2F_1(a, [b, c; y], 1) = \frac{\Gamma(c)\Gamma(c - a - b)}{\Gamma(c - a)\Gamma(c - b)} \frac{(1 - y)^{c - b - a}}{B(b, c - b)(c - b - a)} 2F_1(1 - b, c - b - a; 1 + c - b - a; 1 - y).
\]  
Using transformation formula
\[
2F_1(\alpha, \beta; \gamma; z) = (1 - z)^{\gamma - \beta - \alpha} 2F_1(\gamma - \alpha, \gamma - \beta; \gamma; z),
\]  
in (32), we obtain
\[
2F_1(1 - b, c - b - a; 1 + c - b - a; 1 - y) = y^b 2F_1(c - a, 1; 1 + c - b - a; 1 - y).
\]  
Considering (34) in (32), we get
\[
2F_1(a, [b, c; y], 1) = \frac{\Gamma(c)\Gamma(c - a - b)}{\Gamma(c - a)\Gamma(c - b)} \frac{(1 - y)^{c - b - a}y^b}{B(b, c - b)(c - b - a)} 2F_1(c - a, 1; 1 + c - b - a; 1 - y).
\]  

\[\blacksquare\]

Theorem 5 The following result holds true:
\[
2F_1(a, \{b, c; y\}, 1) = \frac{\Gamma(c)\Gamma(c - a - b)}{\Gamma(c - a)\Gamma(c - b)} \frac{(1 - y)^{c - b - a}y^b}{B(b, c - b)} 2F_1(c - a, 1; b + 1; y).
\]

Theorem 6 The following integral representations hold true:
\[
1F_1([a, b; y], x) = \frac{y^a}{B(a, b - a)} \int_0^1 u^{a - 1}(1 - uy)^{b - 1}e^{xu y} du, \quad \text{Re}(b) > \text{Re}(a) > 0
\]  
\[\text{and}\]
\[
1F_1(\{a, b; y\}, x) = \frac{(1 - y)^{-b - a}}{B(a, b - a)} \int_0^1 u^{-b - a - 1}(1 - u(1 - y))^{a - 1}e^{(1-u(1-y))x} du, \quad \text{Re}(b) > \text{Re}(a) > 0.
\]

Proof. Replacing the incomplete Pochhammer ratio \([a, b; y]\) in the definition (22) by its integral representation given by (34), we are led to the desired result (37). Formula (38) can be proved in a similar way.

\[\blacksquare\]

Theorem 7 The following integral representation holds true:
\[
\int_0^1 y^{k - 1} 2F_1(a, [b, c - k; y]; x) dy = \frac{1}{k} \left[ 2F_1(a, b; c - k; x) - \frac{\Gamma(c - k)\Gamma(b + k)}{\Gamma(b)\Gamma(c)} 2F_1(a, b + k; c; x) \right], \quad k \in \mathbb{N}.
\]

Proof. It is known that from the Euler’s formula
\[
2F_1(a, b + k; c; x) = \frac{1}{B(b + k, c - b - k)} \int_0^1 y^{b + k - 1}(1 - y)^{c - b - k - 1}(1 - xy)^{-a} dy, \quad k \in \mathbb{N}.
\]  
Taking \( u = y^k \) and the remaining part as \( du \) and applying the integration by parts, we get
\[
2F_1(a, b + k; c; x) = \frac{\Gamma(b)\Gamma(c)}{\Gamma(c - k)\Gamma(b + k)} \left[ 2F_1(a, b; c - k; x) - k \int_0^1 y^{k - 1} 2F_1(a, [b, c - k; y], x) dy \right].
\]  
By rearranging the terms we get the result. \(\blacksquare\)
Corollary 8 Taking $k = 1$ in Theorem 7, we get the following result:

\[
\int_0^1 2F_1(a, [b, c - 1; y], x)dy = 2F_1(a, b; c - 1; x) - \frac{b}{c - 1} 2F_1(a, b + 1; c; x). \tag{40}
\]

Theorem 9 The following integral representation holds true:

\[
\int_0^1 y^{k-1} 2F_1(a, [b, c; y], x)dy = \frac{1}{k} \frac{\Gamma (c) \Gamma (c-b+k)}{\Gamma (c-b) \Gamma (c+k)} 2F_1(a, b; c+k; x). \tag{41}
\]

Proof. It is known that

\[
2F_1(a, b; c + k; x) = \frac{1}{B(b, c - b + k)} \int_0^1 y^{b-1}(1-y)^{c-b+k-1}(1-xy)^{-a}dy.
\]

Taking $u = (1-y)^k$ and the rest as $dv$ and using integration by parts, we get the result. \(\blacksquare\)

Corollary 10 Taking $k = 1$ in Theorem 9, we get the following result:

\[
2F_1(a, b; c + 1; x) = \frac{c}{c - b} \int_0^1 2F_1(a, [b, c; y], x)dy. \tag{42}
\]

Theorem 11 The following derivative formula holds true:

\[
\frac{d^n}{dx^n} (2F_1(a, [b, c; y]; x)) = \frac{(a)_n (b)_n}{(c)_n} 2F_1(a + n, [b + n, c + n; y]; x). \tag{43}
\]

Proof. Using (27), differentiating on both sides with respect to $x$, we obtain

\[
\frac{d}{dx} (2F_1(a, [b, c; y]; x)) = \frac{a}{B(b, c - b)} \int_0^y t^{b-1}(1-t)^{c-b-1}(1-xt)^{-a-1}dt,
\]

\[
= \frac{a}{B(b, c - b)} \int_0^y t^{(b+1)-1}(1-t)^{(c+1)-(b+1)-1}(1-xt)^{-(a+1)}dt,
\]

\[
= \frac{ab}{c} B(b+1, c-b) \int_0^y t^{(b+1)-1}(1-t)^{(c+1)-(b+1)-1}(1-xt)^{-(a+1)}dt,
\]

which is (43) for $n = 1$. The general result follows by the principle of mathematical induction on $n$. \(\blacksquare\)

Theorem 12 The following derivative formula holds true:

\[
\frac{d^n}{dx^n} (1F_1([a, b; y]; x)) = \frac{(a)_n}{(b)_n} 1F_1([a + n, b + n; y]; x). \tag{44}
\]

Theorem 13 We have the following difference formula for $2F_1(a, [b, b + h; y]; x)$:

\[
\frac{b + h - 1}{B(b, h)} y^{b-1}(1-y)^{h-1}(1-xy)^{-a} = 2F_1(a, [b, b + h - 1; y]; x) + 2F_1(a, [b - 1, b + h - 1; y]; x) - ax(b + h - 1) 2F_1(a + 1, [b, b + h; y]; x). \tag{45}
\]

Proof. Recalling that the Mellin transform operator is defined by

\[
\mathcal{M} \{f(t) : s := \int_0^\infty t^{s-1} f(t)dt, \ Re(s) > 0,
\]
we observe that \( _2F_1(a, [b, b + h; y]; x) \) is the Mellin transform of the function
\[
f(t : x; y, a; h) = H(y - t)(1 - t)^{b - 1}(1 - xt)^{-a},
\]
where
\[
H(t) = \begin{cases} 
1 & \text{if } t > 0 \\
0 & \text{if } t < 0,
\end{cases}
\]
is the Heaviside unit function. Observing the fact that
\[
\begin{align*}
2F_1(a, [b, b + h; y]; x) := \frac{\mathfrak{M}\{f(t : x; y, a; h)\}}{B(b, h)}, \\
\end{align*}
\]
we can write that
\[
\frac{\partial}{\partial t}(f(t : x; y, a; h)) = -[(y - t)(1 - t)^{b - 1}(1 - xt)^{-a} + (h - 1)H(y - t)(1 - t)^{b - 2}(1 - xt)^{-a}]
\]
\[
+ ax(1 - xt)^{-a - 1}H(y - t)(1 - t)^{b - 1},
\]
where \( \frac{\partial}{\partial t}(H(t)) = \delta(t - t_0), \)
\[
\delta(t - t_0) = \begin{cases} 
\infty & \text{if } t = t_0 \\
0 & \text{if } t \neq t_0,
\end{cases}
\]
is the Dirac delta function. Applying Mellin transform on both sides \( \text{(47)} \) and using \( \text{(46)} \) and the fact that
\[
\mathfrak{M}\{f'(t) : x\} = (1 - x)\mathfrak{M}\{f(t) : x - 1\},
\]
we have
\[
\begin{align*}
\frac{b + h - 1}{B(b, h)}y^{b - 1}(1 - y)^{b - 1}(1 - x)^{-a} &= _2F_1(a, [b, b + h - 1; y]; x) \\
+ _2F_1(a, [b - 1, b + h - 1; y]; x) - ax(b + h - 1) & _2F_1(a + 1, [b, b + h; y]; x).
\end{align*}
\]
This completes the proof. \( \blacksquare \)

In the following theorems, we give transformation formulas:

**Theorem 14** The following transformation formula holds true:
\[
_2F_1(a, [\beta, \gamma; y]; z) = (1 - z)^{-a} _2F_1(a, [\gamma - \beta, \gamma; 1 - y]; \frac{z}{z - 1}), \quad |\arg(1 - z)| < \pi. \tag{48}
\]

**Proof.** Using \( \text{(27)} \), we obtain
\[
_2F_1(a, [\beta, \gamma; y]; z) = \frac{(1 - z)^{-a}}{B(\beta, \gamma - \beta)} \int_{1-y}^{1} (1 - s)^{\beta - 1} s^{\gamma - \beta - 1} \left( \frac{1 - z}{s - z} \right)^{-a} ds. \tag{49}
\]
The substitution \( s = 1 - t \) in \( \text{(49)} \) leads to
\[
_2F_1(a, [\beta, \gamma; y]; z) = \frac{(1 - z)^{-a}}{B(\beta, \gamma - \beta)} \int_{0}^{y} (1 - t)^{\gamma - \beta - 1} \left( \frac{z(1 - t)}{z - 1} \right)^{-a} dt
\]

\[
= (1 - z)^{-a} _2F_1(a, [\gamma - \beta, \gamma; 1 - y]; \frac{z}{z - 1}).
\]
\( \blacksquare \)

**Theorem 15** The following transformation formula holds true:
\[
_2F_1(a, [\beta, \gamma; y]; z) = (1 - z)^{-a} _2F_1(a, [\gamma - \beta, \gamma; 1 - y]; \frac{z}{z - 1}), \quad |\arg(1 - z)| < \pi. \tag{50}
\]

**Theorem 16** The following transformation formulas hold true:
\[
_1F_1([\alpha, \beta; 1 - y]; z) = e^z _1F_1([\beta - \alpha, \beta; y]; -z) \tag{51}
\]
and
\[
_1F_1([\alpha, \beta; z]; z) = e^z _1F_1([\beta - \alpha, \beta; 1 - y]; -z). \tag{52}
\]

**Proof.** The proofs of \( \text{(51)} \) and \( \text{(52)} \) are direct consequences of Theorem 6. \( \blacksquare \)
4 The incomplete Appell’s functions

In this section, we introduce the incomplete Appell’s functions $F_1[a, b, c; d; x, z; y]$, $F_1[a, b, c; d; x, z; y]$, $F_2[a, b, c; d; e; x, z; y]$ and $F_2[a, b, c; d; e; x, z; y]$ by

$$F_1[a, b, c; d; x, z; y] := \sum_{m,n=0}^{\infty} [a, d; y]_{m+n} (b)_m (c)_n \frac{x^m z^n}{m! n!}, \quad \max\{|x|, |z|\} < 1$$

and

$$F_1[a, b, c; d; x, z; y] := \sum_{m,n=0}^{\infty} \{a, d; y\}_{m+n} (b)_m (c)_n \frac{x^m z^n}{m! n!}, \quad \max\{|x|, |z|\} < 1$$

and

$$F_2[a, b, c; d; e; x, z; y] := \sum_{m,n=0}^{\infty} (a)_{m+n} [d, y; m][c, y; n] \frac{x^m y^n}{m! n!}, \quad |x| + |z| < 1$$

and

$$F_2[a, b, c; d; e; x, z; y] := \sum_{m,n=0}^{\infty} (a)_{m+n} [d, y; m][c, y; n] \frac{x^m y^n}{m! n!}, \quad |x| + |z| < 1. \quad (56)$$

We proceed by obtaining the integral representations of the functions $F_1[a, b, c; d; x, z; y]$, $F_1[a, b, c; d; x, z; y]$, $F_2[a, b, c; d; e; x, z; y]$ and $F_2[a, b, c; d; e; x, z; y]$.

**Theorem 17** For the incomplete Appell’s functions $F_1[a, b, c; d; x, z; y]$ and $F_1[a, b, c; d; x, z; y]$, we have the following integral representation:

$$F_1[a, b, c; d; x, z; y] = \frac{y^a}{B(a, d - a)} \int_0^1 u^{a-1}(1 - uy)^{d-a-1}(1 - xy)^{b}(1 - zuy)^{-c} du,$$

Re$(d) > 0$, Re$(a) > 0$, Re$(b) > 0$, Re$(c) > 0$, |arg$(1 - x)$| < π, |arg$(1 - z)$| < π.

and

$$F_1[a, b, c; d; x, z; y] = \frac{(1 - y)^{d-a}}{B(a, d - a)} \times \int_0^1 u^{d-a-1}(1 - u(1 - y))^{a-1}(1 - x(1 - u(1 - y)))^{-b}(1 - z(1 - u(1 - y)))^{-c} du,$$

Re$(d) > 0$, Re$(a) > 0$, Re$(b) > 0$, Re$(c) > 0$, |arg$(1 - x)$| < π, |arg$(1 - z)$| < π. \quad (58)

**Proof.** Replacing the integral representation for incomplete beta function which is given by [9], we find that

$$F_1[a, b, c; d; x, z; y] = \frac{1}{B(a, d - a)} \int_0^y t^{a-1}(1 - t)^{d-a-1}(1 - xt)^{-b}(1 - zt)^{-c} dt,$$

which can be written as

$$F_1[a, b, c; d; x, z; y] = \frac{y^a}{B(a, d - a)} \int_0^1 u^{a-1}(1 - uy)^{d-a-1}(1 - xy)^{b}(1 - zuy)^{-c} du.$$

Whence the result. Formula (58) can be proved in a similar way. ■

**Theorem 18** For the incomplete Appell’s functions $F_2[a, b, c; d; e; x, z; y]$ and $F_2[a, b, c; d; e; x, z; y]$, we have the following integral representation:

$$F_2[a, b, c; d; e; x, z; y] = \frac{y^{b+c}}{B(b, d - b)B(c, e - c)} \times \int_0^1 \int_0^1 u^{b-1}(1 - uy)^{d-b-1}v^{c-1}(1 - vy)^{e-c-1}(1 - xuy - zvy)^{-a} dudv,$$

Re$(d) >$ Re$(a) >$ Re$(b) >$ Re$(c) >$ Re$(m) > 0$, |arg$(1 - x - z)$| < π. \quad (59)
Proof. Replacing the integral representation for incomplete beta function which is given by (9), we get
\begin{align*}
F_2\{a, b; c; x; z; y\} &= \frac{(1 - y)^{d-b+c}}{B(b, d - b)B(c, c - d)} \int_0^1 \int_0^1 u^{d-b-1}(1 - u(1 - y))^{b-1}v^{c-1}(1 - v(1 - y))^{c-1} \\
&\quad \times (1 - x(1 - u(1 - y))) - z(1 - v(1 - y)))^{-a}dudv, \\
\text{Re}(d) > 0, &\, \text{Re}(a) > 0, &\, \text{Re}(b) > 0, &\, \text{Re}(c) > 0, &\, \text{Re}(e) > 0, &\, |\arg(1 - x - z)| < \pi.
\end{align*}

(60)

Formula (60) can be proved in a similar way.

5 Incomplete Riemann-Liouville fractional derivative operator

In this section, we introduce and investigate the incomplete Riemann-Liouville fractional derivative operators. The Riemann-Liouville fractional derivative of order \( \mu \) is defined by
\begin{equation}
D_z^\mu \{ f(z) \} := \frac{1}{\Gamma(-\mu)} \int_0^z f(t)(z - t)^{-\mu-1}dt, \quad \text{Re}(\mu) < 0. 
\end{equation}

(61)

Now, we define the incomplete Riemann-Liouville fractional derivative operators \( D_z^\mu[f(z); y] \) and \( D_z^\mu\{ f(z); y \} \) by
\begin{equation}
D_z^\mu[f(z); y] := \frac{z^{-\mu}}{\Gamma(-\mu)} \int_0^y f(uz)(1 - u)^{-\mu-1}du 
\end{equation}

(62)

and its counterpart is by
\begin{equation}
D_z^\mu\{ f(z); y \} := \frac{z^{-\mu}}{\Gamma(-\mu)} \int_0^1 f(uyz)(1 - wy)^{-\mu-1}dw, \quad \text{Re}(\mu) < 0. 
\end{equation}

(63)

We start our investigation by calculating the incomplete fractional derivatives of some elementary functions.
Theorem 19  Let $\text{Re}(\lambda) > -1$, $\text{Re}(\mu) < 0$. Then

$$D_z^\mu[z^\lambda; y] = \frac{B_y(\lambda + 1, -\mu)}{\Gamma(-\mu)} z^{\lambda - \mu}. \quad (64)$$

**Proof.** Using (62) and (9), we get

$$D_z^\mu[z^\lambda; y] = \frac{z^{-\mu}}{\Gamma(-\mu)} \int_0^y (uz)^\lambda (1-u)^{-\mu-1} du = \frac{B_y(\lambda + 1, -\mu)}{\Gamma(-\mu)} z^{\lambda - \mu}.$$ 

Whence the result. ■

Theorem 20  Let $\text{Re}(\lambda) > -1$, $\text{Re}(\mu) < 0$. Then

$$D_z^\mu\{z^\lambda; y\} = \frac{B_{1-y}(-\mu, \lambda + 1)}{\Gamma(-\mu)} z^{\mu-\lambda}. \quad (65)$$

Theorem 21  Let $\text{Re}(\lambda) > 0$, $\text{Re}(\alpha) > 0$, $\text{Re}(\mu) < 0$ and $|z| < 1$. Then

$$D_z^{\mu-\nu}[z^{\lambda-1}(1-z)^{-\alpha}; y] = \frac{\Gamma(\lambda)}{\Gamma(\mu)} z^{\mu-\lambda} 2 F_1(\alpha, [\lambda; \mu]; z), \quad (66)$$

and

$$D_z^{\lambda-\mu}\{z^{\lambda-1}(1-z)^{-\alpha}; y\} = \frac{\Gamma(\lambda)}{\Gamma(\mu)} z^{\mu-\lambda} 2 F_1(\alpha, [\lambda; \mu]; z). \quad (67)$$

**Proof.** Direct calculations yield

$$D_z^{\lambda-\mu}[z^{\lambda-1}(1-z)^{-\alpha}; y] = \frac{z^{\mu-\lambda}}{\Gamma(\mu-\lambda)} \int_0^y (uz)^{\lambda-1}(1-uz)^{-\alpha}(1-u)^{\mu-\lambda-1} du$$

$$= \frac{z^{\mu-\lambda}}{\Gamma(\mu-\lambda)} \int_0^1 (yz)^{\lambda-1}w^{\lambda-1}(1-ywz)^{-\alpha}(1-wy)^{\mu-\lambda-1} dw$$

$$= \frac{z^{\mu-\lambda}}{\Gamma(\mu-\lambda)} \int_0^1 w^{\lambda-1}(1-ywz)^{-\alpha}(1-wy)^{\mu-\lambda-1} dw.$$ 

By (20), we can write

$$D_z^{\lambda-\mu}[z^{\lambda-1}(1-z)^{-\alpha}; y] = \frac{z^{\mu-1}}{\Gamma(\mu-\lambda)} B(\lambda, \mu-\lambda) 2 F_1(\alpha, [\lambda; \mu]; z)$$

$$= \frac{\Gamma(\lambda)}{\Gamma(\mu)} z^{\mu-1} 2 F_1(\alpha, [\lambda; \mu]; z).$$

Hence the proof is completed. Formula (67) can be proved in a similar way. ■

Theorem 22  Let $\text{Re}(\lambda) > \text{Re}(\mu) > 0$, $\text{Re}(\alpha) > 0$, $\text{Re}(\beta) > 0$; $|az| < 1$ and $|bz| < 1$. Then

$$D_z^{\lambda-\mu}[z^{\lambda-1}(1-az)^{-\alpha}(1-bz)^{-\beta}; y] = \frac{\Gamma(\lambda)}{\Gamma(\mu)} z^{\mu-1} 2 F_1(\alpha, [\lambda; \beta; \mu; az, bz]; y), \quad (68)$$

and

$$D_z^{\lambda-\mu}\{z^{\lambda-1}(1-az)^{-\alpha}(1-bz)^{-\beta}; y\} = \frac{\Gamma(\lambda)}{\Gamma(\mu)} z^{\mu-1} 2 F_1(\alpha, [\lambda; \beta; \mu; az, bz]; y). \quad (69)$$

10
Proof. We have
\[ D_z^{\lambda-\mu}[z^{\lambda-1}(1-az)^{-\alpha}(1-bz)^{-\beta}; y] \]
\[ = \frac{z^{\mu-\lambda}}{\Gamma(\mu-\lambda)} \int_0^y (uz)^{\lambda-1}(1-azu)^{-\alpha}(1-buz)^{-\beta}(1-u)^{\mu-\lambda-1} du \]
\[ = \frac{z^{\mu-\lambda}y}{\Gamma(\mu-\lambda)} \int_0^1 (yw)^{\lambda-1}(1-aywz)^{-\alpha}(1-bywz)^{-\beta}(1-wy)^{\mu-\lambda-1} dw \]
\[ = \frac{z^{\mu-1}y^{\lambda}}{\Gamma(\mu-\lambda)} \int_0^1 w^{\lambda-1}(1-aywz)^{-\alpha}(1-bywz)^{-\beta}(1-wy)^{\mu-\lambda-1} dw. \]

By (70), we can write
\[ D_z^{\lambda-\mu}[z^{\lambda-1}(1-az)^{-\alpha}(1-bz)^{-\beta}; y] = \frac{z^{\mu-1}}{\Gamma(\mu-\lambda)} B(\lambda, \mu-\lambda) F_1[\lambda, \alpha, \beta; \mu, az, bz; y] \]
\[ = \frac{\Gamma(\lambda)}{\Gamma(\mu)} z^{\mu-1} F_1[\lambda, \alpha, \beta; \mu, az, bz; y]. \]

Whence the result. Formula (71), can be proved in a similar way. ■

**Theorem 23** Let \( \Re(\lambda) > \Re(\mu) > 0, \Re(\alpha) > 0, \Re(\beta) > 0, \Re(\gamma) > 0; \left| \frac{t}{1-z} \right| < 1 \) and \(|t| + |z| < 1\) we have
\[ D_z^{\lambda-\mu}[z^{\lambda-1}(1-z)^{-\alpha} \sum_{n=0}^{\infty} (\alpha)_n B_n(\beta + n, \gamma - \beta) \frac{(t)^n}{n!} (1-z)^{-\alpha-n}; y] = \frac{\Gamma(\lambda)}{\Gamma(\mu)} z^{\mu-1} F_2[\alpha, \beta, \gamma; \mu, t, z; y], \] (70)
and
\[ D_z^{\lambda-\mu}[z^{\lambda-1}(1-z)^{-\alpha} \sum_{n=0}^{\infty} (\alpha)_n B_n(\beta + n, \gamma - \beta) \frac{(t)^n}{n!} (1-z)^{-\alpha-n}; y] = \frac{\Gamma(\lambda)}{\Gamma(\mu)} z^{\mu-1} F_2[\alpha, \beta, \gamma; \mu, t, z; y]. \] (71)

Proof. Using Theorem 19 and (70), we get
\[ D_z^{\lambda-\mu}[z^{\lambda-1}(1-z)^{-\alpha} \sum_{n=0}^{\infty} (\alpha)_n B_n(\beta + n, \gamma - \beta) \frac{(t)^n}{n!} (1-z)^{-\alpha-n}; y] \]
\[ = \frac{1}{B(\beta, \gamma - \beta)} D_z^{\lambda-\mu} \sum_{n=0}^{\infty} (\alpha)_n B_n(\beta + n, \gamma - \beta) \frac{(t)^n}{n!} (1-z)^{-\alpha-n}; y] \]
\[ = \frac{1}{B(\beta, \gamma - \beta)} \sum_{m,n=0}^{\infty} B_n(\beta + m, \gamma - \beta) \frac{(t)^n}{n!} \frac{(\alpha)_n (\alpha + n)^m}{m!} D_z^{\lambda-\mu}[z^{\lambda-1+m}; y] \]
\[ = \frac{1}{B(\beta, \gamma - \beta)} \sum_{m,n=0}^{\infty} B_n(\beta + m, \gamma - \beta) \frac{(t)^n}{n!} \frac{(\alpha)_n (\alpha + n)^m}{m!} \frac{B_n(\lambda + m, \mu - \lambda)}{\Gamma(\mu-\lambda)} z^{\mu-m-1} \]
\[ = \frac{\Gamma(\lambda)}{\Gamma(\mu)} z^{\mu-1} F_2[\alpha, \beta, \gamma; \mu, t, z; y]. \]

Hence proof is completed. Formula (71), can be proved in a similar way. ■

### 6 Generating Functions

Now, we obtain linear and bilinear generating relations for the incomplete hypergeometric functions \( _2F_1(a, [b, c]; y) \) by following the methods described in [4]. We start with the following theorem:
Theorem 24 For the incomplete hypergeometric functions we have
\[
\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} {}_2F_1(\lambda + n, [\alpha, \beta]; y) t^n = (1 - t)^{-\lambda} {}_2F_1(\lambda, [\alpha, \beta]; y; \frac{z}{1-t})
\] (72)
and
\[
\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} {}_2F_1(\lambda + n, [\alpha, \beta]; z) t^n = (1 - t)^{-\lambda} {}_2F_1(\lambda, [\alpha, \beta]; \frac{z}{1-t})
\] (73)
where \(|z| < \min\{1, |1 - t|\}\) and \(\text{Re}(\lambda) > 0\), \(\text{Re}(\beta) > \text{Re}(\alpha) > 0\).

Proof. Considering the elementary identity
\[
[(1-z) - t]^{-\lambda} = (1-t)^{-\lambda} \left[1 - \frac{z}{1-t}\right]^{-\lambda}
\]
and expanding the left hand side, we have for \(|t| < |1-z|\) that
\[
(1-z)^{-\lambda} \sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} \left(\frac{t}{1-z}\right)^n = (1-t)^{-\lambda} \left[1 - \frac{z}{1-t}\right]^{-\lambda}.
\]
Now, multiplying both sides of the above equality by \(z^{\alpha-1}\) and applying the incomplete fractional derivative operator \(D_z^{\alpha-\beta}[f(z); y]\) on both sides, we can write
\[
D_z^{\alpha-\beta} \left[ \sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} (1-z)^{-\lambda} \left(\frac{t}{1-z}\right)^n z^{\alpha-1}; y \right] = (1-t)^{-\lambda} D_z^{\alpha-\beta} \left[ z^{\alpha-1} \left[1 - \frac{z}{1-t}\right]^{-\lambda}; y \right].
\]
Interchanging the order, which is valid for \(\text{Re}(\alpha) > 0\) and \(|t| < |1-z|\), we get
\[
\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} D_z^{\alpha-\beta} \left[ z^{\alpha-1}(1-z)^{-\lambda-n}; y \right] t^n = (1-t)^{-\lambda} D_z^{\alpha-\beta} \left[ z^{\alpha-1} \left[1 - \frac{z}{1-t}\right]^{-\lambda}; y \right].
\]
Using Theorem 21, we get the desired result. Formula (73), can be proved in a similar way. \(\blacksquare\)

The following theorem gives another linear generating relation for the incomplete hypergeometric functions.

Theorem 25 For the incomplete hypergeometric functions we have
\[
\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} {}_2F_1(\rho - n, [\alpha, \beta]; z) t^n = (1 - t)^{-\lambda} {}_2F_1(\rho, [\alpha, \beta]; \frac{-zt}{1-t}; y)
\] (74)
and
\[
\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} {}_2F_1(\rho - n, [\alpha, \beta]; z) t^n = (1 - t)^{-\lambda} {}_2F_1(\rho, [\alpha, \beta]; \frac{-zt}{1-t}; y)
\] (75)
where \(\text{Re}(\lambda) > 0\), \(\text{Re}(\rho) > 0\), \(\text{Re}(\beta) > \text{Re}(\alpha) > 0\); \(|t| < \frac{1}{1+|z|}\).

Proof. Considering
\[
[1 - (1-z)t]^{-\lambda} = (1-t)^{-\lambda} \left[1 + \frac{zt}{1-t}\right]^{-\lambda}
\]
and expanding the left hand side, we have for \(|t| < |1-z|\) that
\[
\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} (1-z)^n t^n = (1-t)^{-\lambda} \left[1 - \frac{zt}{1-t}\right]^{-\lambda}.
\]
Now, multiplying both sides of the above equality by \( z^{\alpha-1}(1-z)^{-\rho} \) and applying the fractional derivative operator \( D_z^{\alpha-\beta} [f(z); y] \) on both sides, we get

\[
D_z^{\alpha-\beta} \left[ \sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} z^{\alpha-1} (1-z)^{-\rho+n} t^n ; y \right] = (1-t)^{-\lambda} D_z^{\alpha-\beta} \left[ z^{\alpha-1} (1-z)^{-\rho} \left[ 1 - \frac{-zt}{1-t} \right]^{-\lambda} ; y \right].
\]

Interchanging the order, which is valid for \( \text{Re}(\alpha) > 0 \) and \( |zt| < |1-t| \), we get

\[
\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} D_z^{\alpha-\beta} \left[ z^{\alpha-1} (1-z)^{-(\rho-n)} ; y \right] t^n = (1-t)^{-\lambda} D_z^{\alpha-\beta} \left[ z^{\alpha-1} (1-z)^{-\rho} \left[ 1 - \frac{-zt}{1-t} \right]^{-\lambda} ; y \right].
\]

Using Theorem 21 and 22, we get the desired result. Generating relation (75), can be proved in a similar way.

Finally we have the following bilinear generating relation for the incomplete hypergeometric functions.

**Theorem 26** For the incomplete hypergeometric functions we have

\[
\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} \left. 2F1(\gamma, [-n, \delta; y] ; x) \right| 2F1(\gamma, [\alpha + n, \beta; y] ; z) t^n = (1-t)^{-\lambda} F2(\lambda, \alpha, \gamma, \beta, \delta ; \frac{z}{1-t} ; \frac{-xt}{1-t} ; y) \tag{76}
\]

and

\[
\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} \left. 2F1(\gamma, \{ -n, \delta ; y \} ; x) \right| 2F1(\gamma, \{ \alpha + n, \beta ; y \} ; z) t^n = (1-t)^{-\lambda} F2(\lambda, \alpha, \gamma, \beta, \delta ; \frac{z}{1-t} ; \frac{-xt}{1-t} ; y) \tag{77}
\]

where \( \text{Re}(\lambda) > 0, \text{Re}(\gamma) > 0, \text{Re}(\beta) > 0, \text{Re}(\delta) > 0, \text{Re}(\alpha) > 0; |t| < \frac{1-|z|}{1+|z|} \) and \( |z| < 1 \).

**Proof**. Replacing \( t \) by \((1-x)t\) in (72), multiplying the resulting equality by \( x^{\rho-1} \) and then applying the incomplete fractional derivative operator \( D_x^{\rho-\delta} [f(x); y] \), we get

\[
D_x^{\rho-\delta} \left[ \sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} x^{\rho-1} 2F1(\lambda + n, [\alpha, \beta; y] ; z)(1-x)^{n} t^n ; y \right] = D_x^{\rho-\delta} \left[ (1 - (1-x)t)^{-\delta} x^{\rho-1} 2F1(\lambda, [\alpha, \beta; y] ; \frac{z}{1-(1-x)t}; y) \right].
\]

Interchanging the order, which is valid for \( |z| < 1 \), \( \left| \frac{1-z}{1-t} \right| < 1 \) and \( \left| \frac{zt}{1-t} \right| + \left| \frac{xt}{1-t} \right| < 1 \), we can write that

\[
\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} D_x^{\rho-\delta} \left[ x^{\rho-1}(1-x)^{n} ; y \right] 2F1(\lambda + n, [\alpha, \beta; y] ; z) = (1-t)^{-\lambda} D_x^{\rho-\delta} \left[ x^{\rho-1}(1-x)^{n} ; y \right] 2F1(\lambda, [\alpha, \beta; y] ; \frac{z}{1-(1-x)t}; y).
\]

Using Theorems 21 and 23, we get (76). Generating relation (77), can be proved in a similar way.

## 7 Conclusion

Incomplete Pochhammer ratios are defined in (12) and (13) by using the incomplete beta functions. Several properties of these functions are obtained. Incomplete hypergeometric functions are introduced with the help of these incomplete Pochhammer ratios and certain properties such as integral representations, derivative formulas, transformation formulas and recurrence relation are investigated. Furthermore, incomplete Riemann-Liouville fractional derivative operators are defined. The incomplete Riemann-Liouville fractional derivatives for the some elementary functions are given. Linear and bilinear generating relations for incomplete hypergeometric functions are obtained.
References

[1] P. Agarwal, J. Choi, Fractional calculus operators and their image formulas, Journal of the Korean Mathematical Society, 53 (2016), 363-379.
[2] D. Baleanu, P. Agarwal, R. K. Parmar, M. M. Al Qurashi, S. Salahshour, Extension of the fractional derivative operator of the Riemann-Liouville, J. Nonlinear Sci. Appl., 10 (2017), 2914-2924.
[3] M. A. Chaudhry, A. Qadir, M. Raﬁque, S. M. Zubair, Extension of Euler’s beta function, J. Comput. Appl. Math., 78 (1997), 19-32.
[4] M. A. Chaudhry, A. Qadir, H. M. Srivastava, R. B. Paris, Extended hypergeometric and confluent hypergeometric functions, Appl. Math.Comput., 159 (2004), 589-602.
[5] M. A. Chaudhry, S. M. Zubair, On a Class of Incomplete Gamma Functions with Aplications, Dha HRan, Saudi Arabia, (2001).
[6] N. E. Cho, R. Srivastava, Some extended Pochhammer symbols and their applications involving generalized hypergeometric polynomials, Applied Mathematics and Computations, 234 (2014), 277-285.
[7] J. Choi, P. Agarwal, Certain Class of Generating Functions for the Incomplete Hypergeometric Functions, Abstract and Applied Analysis, 2014 (2014).
[8] J. Choi, R. K. Parmar, P. Chopra, The Incomplete Srivastava’s Triple Hypergeometric Functions gamma(H)(B) and Gamma(H)(B)), Filomat, 7 (2016), 1779-1787.
[9] A. Çetinkaya, The incomplete second Appell hypergeometric functions, Applied Mathematics and Computations, 219 (2013), 8332-8337.
[10] A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, Theory and applications of fractional differential equations, North-Holland Mathematical Studies, 204, Elsevier Science B. V., Amsterdam, (2006).
[11] D. Kumar, J. Singh, M. Al Qurashi, D. Baleanu, Analysis of logistic equation pertaining to a new fractional derivative with non-singular kernel, Advances in Mechanical Engineering, 9 (2017), 1-8.
[12] D. Kumar, J. Singh, D. Baleanu, A fractional model of convective radial fins with temperature-dependent thermal conductivity, Romanian Reports in Physics, 69 (2017), 103.
[13] D. Kumar, J. Singh, D. Baleanu, Modified Kawahara equation within a fractional derivative with non-singular kernel, Thermal Science, (2017), DOI: 10.2298/TSCI160826008K.
[14] D. Kumar, R. P. Agarwal, J. Singh, A modified numerical scheme and convergence analysis for fractional model of Lienard’s equation, J. Comput. Appl. Math., (2017).
[15] Shy-Der Lin, H. M. Srivastava, Mu-Ming Wong, Some Applications of Srivatava’s Theorem Involving a Certain Family of Generalized and Extended Hypergeometric Polynomials, Filomat, 29 (2015), 1811-1819.
[16] M. A. Özarslan, E. Özergin, Some generating relations for extended hypergeometric functions via generalized fractional derivative operator, Math. Comput. Modelling, 52 (2010), 1825-1833.
[17] E. Özergin, M. A. Özarslan, A. Altin, Extension of gamma, beta and hypergeometric functions, J. Comput. Appl. Math., 235 (2011), 4601-4610.
[18] E. Özergin, Some properties of hypergeometric functions, Ph.D. Thesis, Eastern Mediterranean University, North Cyprus, Turkey, (2011).
[19] R. K. Parmar, Some Generating Relations For Generalized Extended Hypergeometric Functions Involving Generalized Fractional Derivative Operator, J. Concr. Appl. Math., 12 (2014), 217-228.
[20] R. K. Parmar, R. K. Saxena, The Incomplete Generalized tau-Hypergeometric and Second tau-Appell functions, Journal of the Korean Mathematical Society, 53 (2016), 363-379.
[21] V. Sahai, A. Verma, On an extension of the generalized Pochhammer symbol and its applications to hypergeometric functions, Asian-European Journal of Mathematics, 9 (2016).
[22] H. M. Srivastava, R. Agarwal, S. Jain, Integral transform and fractional derivative formulas involving the extended generalized hypergeometric functions and probability distributions, Mathematical Methods in the Applied Sciences, 40 (2017), 255-273.
[23] H. M. Srivastava, A. Çetinkaya, O. I. Kiywaz, A certain generalized Pochhammer symbol and its applications to hypergeometric functions, Applied Mathematics and Computation, 226 (2014), 484-491.
[24] H. M. Srivastava, M. A. Chaudry, R. P. Agarwal, The incomplete Pochhammer symbols and their applications to hypergeometric and related functions, Integral Transforms and Special Functions, 23 (2012), 659-683.

[25] H. M. Srivastava, H. L. Manocha, A Treatise on Generating Functions, Halsted, Ellis Horwood, Wiley, New York, Chicester, New York, (1984).

[26] R. Srivastava, N. E. Cho, Generating functions for a certain class of incomplete hypergeometric polynomials, Applied Mathematics and Computation, 219 (2012), 3219-3225.

[27] R. Srivastava, Some properties of a family of incomplete hypergeometric functions, Russian Journal of Mathematical Physics, 20 (2013), 121-128.

[28] R. Srivastava, Some generalizations of Pochhammer’s symbol and their associated families of hypergeometric functions and hypergeometric polynomials, Applied Mathematics & Information Sciences, 7 (2013), 2195-2206.

[29] R. Srivastava, Some classes of generating functions associated with a certain family of extended and generalized hypergeometric functions, Applied Mathematics and Computation, 243 (2014), 132-137.

[30] Xiao-Jun Yang, D. Baleanu, H. M. Srivastava, Local fractional integral transforms and their applications, Elsevier/Academic Press, Amsterdam, (2016).