THE VARIANCE OF THE SHOCK IN THE HAD PROCESS

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ABSTRACT. We consider the Hammersley-Aldous-Diaconis (HAD) process with sinks and sources such that there is a microscopic shock at every time \( t \); denote \( Z(t) \) its position. We show that the mean and variance of \( Z(t) \) are linear functions of \( t \) and compute explicitly the respective constants in function of the left and right densities. Furthermore, we describe the dependence of \( Z(t) \) on the initial configuration in the scale \( \sqrt{t} \) and, as a corollary, prove a central limit theorem.

1. Introduction

Let \( S \) and \( W \) be one-dimensional Poisson processes and let \( P \) be a two dimensional Poisson process of rate 1. Assume they are homogeneous and mutually independent. The Hammersley-Aldous-Diaconis process \( \mathcal{H}(S, W, P) = (H_s, s \in [0, t]) \) has been constructed by Groeneboom \( \mathcal{H}(S, W, P) \) in the square \( [0, x] \times [0, t] \) as a deterministic function of \( S, W \) and \( P \) as follows. The point configuration \( H_s \) represents the position of particles. At time zero the particles start at the positions \( H_s \) in \( S \), called the sources. Then (see Figure 1), at the first \( s > 0 \) such that \( (y, s) \) is in \( P \) for some \( y \in [0, x] \) or \( s \) is in \( W \), the closest particle to the right of \( y \) jumps to \( y \) if \( (y, s) \) is in \( P \) or to 0 if \( s \) is in \( W \) (points in \( W \) are called sinks). If there is no particle to the right of \( y \), then a new particle is added at \( y \) at time \( s \). The new configuration does not move until the second \( s \) such that \( (y, s) \) is in \( P \) for some \( y \) or \( s \) is in \( W \), when the second jump occurs, and so on until time \( t \). In other words, we define the process inductively as follows: \( H_0 = S \) and for \( s > 0 \),

\[
H_s = \begin{cases} 
H_{s-} & \text{if } s \notin W \cup \{s', (y, s) \in P \text{ for some } y' \in [0, x]\} \\
H_{s-} \setminus \{R(y, H_{s-})\} \cup \{y\} & \text{if } (y, s) \in P \cup \{(0, s'), s' \in W\}
\end{cases}
\]

where \( R(y, H) = \inf\{y' \in H \cup \{x\} : y' > y\} \) for \( H \subset (0, x) \). Let \( N \) be the positions of particles at time \( t \), and let \( E \) be the set of times a new particle enters the system from the right through the vertical axis \( \{x\} \times [0, t) \).

Let \( \lambda \geq 0 \) and \( \rho \geq 0 \). Assume that \( S \) has intensity \( \lambda \), \( W \) has intensity \( \rho \) and \( P \) has intensity 1. Define the point process \( S' \) by removing the first sink of \( S \) and consider the corresponding HAD process \( \mathcal{H}(S', W, P) \). For time \( t \geq 0 \) the coupled processes \( \mathcal{H}(S, W, P) \) and \( \mathcal{H}(S', W, P) \) will differ at one point denoted \( Z(t) \) and called a second class particle (Figure 1).
Figure 1. The black points represent the Poisson points $S$ (sources), $W$ (sinks) and $P$. The polygonal lines represent the trajectory of HAD particles while the trajectory of the second class particle is represented by the dashed polygonal line.

**Results.** Our main results are the computation of the mean and variance of $Z(t)$ and the dependence of fluctuations on the initial configuration.

**Theorem 1.** Assume that $\lambda \rho > 1$. Then for all $t \geq 0$ we have
\[
\mathbb{E}(Z(t)) = \frac{\rho}{\lambda} t, \tag{1.1}
\]
\[
\text{Var}(Z(t)) = \frac{2(\rho - \frac{1}{\lambda})}{(\lambda - \frac{1}{\rho})^2} t. \tag{1.2}
\]

Let
\[
R(S) = \frac{1}{\lambda} \left( \rho - \frac{1}{\lambda} \right) \quad \text{and} \quad R(W) = \frac{1}{\rho} \left( \rho - \frac{1}{\lambda} \right).
\]

In the next theorem we show that in the scale $\sqrt{t}$ the position of $Z(t)$ is the same as the following random variable, depending only on $S$ and $W$:
\[
N(t) := (\lambda - \frac{1}{\rho})^{-1} \left[ 3 \left( \rho - \frac{1}{\lambda} \right) t - \left( |[0, tR(S)] \cap S| + |[0, tR(W)] \cap W| \right) \right],
\]
where $|A|$ denotes the number of points in the (finite) set $A$.

**Theorem 2.** Assume that $\lambda \rho > 1$. Then
\[
\lim_{t \to \infty} \frac{\mathbb{E} \left[ (Z(t) - N(t))^2 \right]}{t} = 0.
\]
and
\[ \lim_{t \to \infty} \frac{Z(t) - \mathbb{E}(Z(t))}{\sqrt{\text{Var}(Z(t))}} = \mathcal{N}(0, 1) \text{ in distribution}, \]
where \( \mathcal{N}(0, 1) \) is a standard normal random variable.

**Brief historical overview.** The HAD process has a genealogical relation with the famous Ulam’s problem, which studies the limit behavior of the longest increasing subsequence \( L_n \) in a random permutation of the numbers \( 1, \ldots, n \). At the beginning of the seventies Hammersley [6] proposed to solve this problem as follows. Assume \((x_1, t_1), \ldots, (x_n, t_n)\) are \( n \) independent random points distributed uniformly in the rectangle \([0, x] \times [0, t]\). These points specify a uniform random permutation \( \pi \) by setting the point with the \( i^{th} \) smallest \( t \)-coordinate has the \( \pi(i) \) smallest \( x \)-coordinate. The length of the longest increasing subsequence of \( \pi \) equals the maximal number \( M(x, t) \) of points on an up-right path from \((0, 0)\) to \((x, t)\) (last-passage percolation). The variable \( M(x, t) \) has the so called super-additivity property [7], which was the key to show that \( \sqrt{n} L_n \to c \). Hammersley conjectured that \( c = 2 \) by presenting a non-rigorous but quite reasonable argument based on hydrodynamical local equilibrium for a related discrete-time particle system. Later Logan and Shepp [10] and Vershik and Kerov [11] proved \( c = 2 \) with combinatorial methods.

In the middle of the nineties, Aldous and Diaconis [1] constructed a continuous time version of the Hammersley process and use Rost [12] coupling ideas, developed in the exclusion process context, to show that \( c = 2 \). This is the reason we call this process Hammersley-Aldous-Diaconis process. Groeneboom [9] introduced the construction in the quadrant we use here. In this setup \( M(x, t) \) equals the number of particles in \([0, x] \times \{ t \}\) plus the number of particles in \( \{ 0 \} \times [0, t] \) (we are allowing an up-right path to pick Poissonian points in the horizontal or vertical lines). Thus, the trajectory of the \( n^{th} \) particle corresponds to the boundary of the region \( \{ M(x, t) \leq n - 1 \} \) (Figure 1).

The macroscopic hydrodynamical behavior of the HAD process is given by the Burgers equation \( \partial_t u + \partial_x g(u) = 0 \), with \( g(u) = \frac{1}{u} \) [13]. When the initial conditions are \( u(r, 0) = \lambda \) for \( r > 0 \) and \( u(r, 0) = \frac{1}{\rho} \) when \( r \leq 0 \), the solution has a drastic changing when the product \( \lambda \rho \) goes from values smaller than 1 to values greater than 1. For \( \lambda \rho = 1 \), the system is stationary and the solutions are constants in time. For \( \lambda \rho < 1 \) it develops a rarefaction front while for \( \lambda \rho > 1 \) it develops a (macroscopic) shock. Since our results concern the shock regime, from now on we assume \( \lambda \rho > 1 \).

The microscopic structure of a shock is described by the so called second-class particle. Our main result is the explicit expression [12] for the variance of the second class particle at any given time \( t \). The method, inspired in work by one of the authors and Fontes [14, 15] for the exclusion process, relates the mean and the variance of the flux of particles through the origin with the mean and the variance of the position of a single second class particle by a linear equation. In this way, if we calculate one, we get the other. However those works show that for the exclusion process the variance of a second class particle is asymptotically linear in time; the constant can also be computed. In the HAD process context, we show
how to refine this method to get an exact formula for the variance of a second class particle at any given time $t$.

2. Preliminaries, coupling and Burke’s theorem

**Notation.** For a generic point process $M$ we denote

$$\int_0^x f(y, M)M(dy) := \sum_{y \in M \cap [0,x]} f(x, M)$$  \hspace{1cm} (2.3)

and recall the Slivnyak-Mecke Theorem: if $M$ is a Poisson process with intensity $\lambda$, then

$$\mathbb{E}\left( \int_0^x f(y, M)M(dy) \right) = \int_0^x \mathbb{E}f(y, M)\lambda(y)dy$$  \hspace{1cm} (2.4)

**The stationary regime and the Burke’s theorem.** Let $\gamma > 0$ and $S_\chi$ and $W_\chi$ be Poisson processes with intensity $\gamma$ and $\gamma^{-1}$, respectively. Then $\chi = \mathcal{H}(S_\chi, W_\chi, P)$ is a stationary HAD process with parameter $\gamma$. Cator and Groeneboom [2] proved the following Burke’s theorem: $N_\chi$ and $E_\chi$ are independent Poisson processes with the same density as $S_\chi$ and $W_\chi$, respectively.

**Coupling two stationary process.** Let $\lambda \rho > 1$. Let $S_\eta$, $W_\sigma$, $I$ and $J$ be mutually independent one-dimensional Poisson point processes with intensities $\frac{1}{\rho}$, $\frac{1}{\lambda}$, $\lambda - \frac{1}{\rho}$ and $\rho - \frac{1}{\lambda}$ respectively. Set

$$S_\sigma = S_\eta + I \quad \text{and} \quad W_\eta = W_\sigma + J.$$  \hspace{1cm} (2.5)

Thus, $S_\sigma$ and $W_\eta$ are two (independent) one-dimensional Poisson processes with intensity $\lambda$ and $\rho$ respectively. Let $P$ be the rate-1 two-dimensional Poisson process mentioned in the Introduction.

We run three HAD processes simultaneously with different boundary conditions but with the same $P$. This is a realization $(\phi, \sigma, \eta)$ of the so called basic coupling defined by

$$\phi = \mathcal{H}(S_\sigma, W_\eta, P), \quad \sigma = \mathcal{H}(S_\sigma, W_\sigma, P) \quad \text{and} \quad \eta = \mathcal{H}(S_\eta, W_\eta, P).$$

For $\chi \in \{\phi, \sigma, \eta\}$ we denote by $\chi(x, t)$ the number of $\chi$-particles (particles that count for the process $\chi$) in the interval $[0, x]$ at time $t$, with the convention that particles escaping through a sink during the time interval $[0, t]$ are located at 0. The flux of the discrepancies between $\sigma$ and $\eta$ through the space-time line $(0, 0)-(x, t)$ is the process defined by

$$\xi(x, t) = \sigma(x, t) - \eta(x, t).$$  \hspace{1cm} (2.6)

There are two kinds of discrepancies: i) those starting at a point $y$ at time 0 in $I$ (in the $x$-axis); ii) those starting at a added sink in a time $s$ in $J$ (in the $t$-axis). The position at time $t$ of the second class particle starting at time zero at site $y$ is denoted $Z_{(y, 0)}(t)$, or shortly $Z_y(t)$, and the position at time $t$ of the second class particle starting at site 0 at time $s$ is denoted $Z_{(0, s)}(t)$, or shortly $Z_s(t)$. 
The flux of second class particles defined in (2.6) is also given by
\[ \xi(x, t) = \# \{ y \in I : Z_{(y,0)}(t) \leq x \} - \# \{ s \in J : Z_{(0,s)}(t) > x \} \]
:= \xi_+(x, t) - \xi_-(x, t) \tag{2.7}

3. **The Mean and the Variance of the Flux of Second Class Particles**

In this section we show the following.

**Proposition 1.** For all \( x, t \geq 0 \)
\[
\mathbb{E}(\xi(x, t)) = \left( \lambda - \frac{1}{\rho} \right) x - \left( \rho - \frac{1}{\lambda} \right) t. \tag{3.8}
\]
\[
\text{Var}(\xi(x, t)) = \left( \lambda - \frac{1}{\rho} \right) x + \left( \rho - \frac{1}{\lambda} \right) t. \tag{3.9}
\]

Denote by \( \bar{M} := |M| \), the total number of points of a generic finite point process \( M \) and \( \bar{x} = \chi(x, t) \). With this notation,
\[
\bar{x} = \bar{N}_\chi + \bar{W}_\chi = \bar{S}_\chi + \bar{E}_\chi, \tag{3.10}
\]
for \( \chi \in \{ \sigma, \eta \} \) and
\[
\bar{\xi} = \bar{\sigma} - \bar{\eta}. \tag{3.11}
\]

**Proof of Proposition 1.** By definition (2.6) and Burke’s theorem,
\[
\mathbb{E}(\xi(x, t)) = \mathbb{E}(\bar{\xi}) = \mathbb{E}(\bar{\sigma}) - \mathbb{E}(\bar{\eta}) = \mathbb{E}(\bar{N}_\sigma + \bar{W}_\sigma - (\bar{N}_\eta + \bar{W}_\eta)) = \left( \lambda x + \frac{t}{\lambda} \right) - \left( \frac{x}{\rho} + \rho t \right).
\]
This shows (3.8).

To compute the variance combine first (2.5) and Burke’s theorem to get
\[
\text{Cov}(\bar{S}_\sigma, \bar{W}_\eta) = \text{Cov}(\bar{E}_\sigma, \bar{N}_\eta) = 0. \tag{3.12}
\]
Hence, by (3.10),
\[
\text{Cov}(\bar{\sigma}, \bar{\eta}) = \text{Cov}(\bar{S}_\sigma + \bar{E}_\sigma, \bar{N}_\eta + \bar{W}_\eta) = \text{Cov}(\bar{S}_\sigma, \bar{N}_\eta) + \text{Cov}(\bar{E}_\sigma, \bar{W}_\eta) = \text{Cov}(\bar{S}_\eta + \bar{I}, \bar{N}_\eta) + \text{Cov}(\bar{E}_\sigma, \bar{W}_\sigma + \bar{J}) = \text{Cov}(\bar{S}_\eta, \bar{N}_\eta) + \text{Cov}(\bar{E}_\sigma, \bar{W}_\sigma). \tag{3.13}
\]

Analogous (and simpler) reasoning for \( \chi \in \{ \eta, \sigma \} \) gives
\[
\text{Var}(\bar{\chi}) = \text{Cov}(\bar{E}_\chi, \bar{W}_\chi) + \text{Cov}(\bar{N}_\chi, \bar{S}_\chi). \tag{3.14}
\]

On the other hand, since \( \bar{x} = \bar{N}_\chi + \bar{W}_\chi \),
\[
\text{Var}(\bar{x}) = \text{Var}(\bar{N}_\chi) + \text{Var}(\bar{W}_\chi) + 2\text{Cov}(\bar{N}_\chi, \bar{W}_\chi) \tag{3.15}
\]
Using $\bar{W}_x = \bar{S}_x + \bar{E}_x - \bar{N}_x$ and $E_x$ independent of $N_x$,

\[
\text{Cov}(\bar{N}_x, \bar{W}_x) = \text{Cov}(\bar{N}_x, \bar{S}_x + \bar{E}_x - \bar{N}_x) = \text{Cov}(\bar{N}_x, \bar{S}_x) - \text{Var}(\bar{N}_x) .
\]

which implies

\[
\text{Var}(\bar{x}) = \text{Var}(\bar{W}_x) - \text{Var}(\bar{N}_x) + 2\text{Cov}(\bar{N}_x, \bar{S}_x) .
\]

(3.16)

Identities (3.14) and (3.16) imply

\[
\text{Cov}(\bar{E}_x, \bar{W}_x) - \text{Cov}(\bar{N}_x, \bar{S}_x) = \text{Var}(\bar{W}_x) - \text{Var}(\bar{N}_x)
\]

(3.17)

Finally we compute $\text{Var}(\xi)$ using (3.13) and (3.14):

\[
\text{Var}(\xi) = \text{Var}(\sigma) + \text{Var}(\eta) - 2\text{Cov}(\sigma, \eta) = \text{Cov}(\bar{E}_\eta, \bar{W}_\eta) - \text{Cov}(\bar{N}_\sigma, \bar{S}_\sigma) - \text{Cov}(\bar{E}_\sigma, \bar{W}_\sigma)
\]

\[
= \left(\text{Var}(\bar{W}_\eta) - \text{Var}(\bar{N}_\eta)\right) - \left(\text{Var}(\bar{W}_\sigma) - \text{Var}(\bar{N}_\sigma)\right)
\]

\[
= \left(\rho t - \frac{x}{\rho}\right) - \left(\frac{t}{\lambda} - \lambda x\right),
\]

(3.18)

where in the third line we used (3.17).

\[\square\]

4. THE MEAN AND THE VARIANCE OF A SECOND CLASS PARTICLE

Lemma 1. For all $x,t \geq 0$,

\[
\mathbb{E} \xi(x,t)_+ = \left(\lambda - \frac{1}{\rho}\right) \left\{ x - \int_{0}^{x} \mathbb{P}(Z(t) > z)dz \right\} .
\]

(4.19)

\[
\mathbb{E} \xi(x,t)_- = \left(\rho - \frac{1}{\lambda}\right) \left\{ t - \int_{0}^{t} \mathbb{P}(Z(u) \leq x)du \right\}
\]

(4.20)

Proof of Lemma 1. By definition (2.7),

\[
\xi(x,t)_+ = \int_{0}^{x} 1\{Z_y(t) \leq x\} I(dy)
\]

where $Z_y(t)$ denotes the position at time $t$ of a second-class particle that starts from the discrepancy at $y \in [0,x]$. By translation invariance and (2.4),

\[
\mathbb{E} \xi(x,t)_+ = \left(\lambda - \frac{1}{\rho}\right) \int_{0}^{x} \mathbb{P}(Z(t) \leq x - y)dy.
\]

(4.21)

which proves (4.19). Analogously, (4.20) follows from

\[
\mathbb{E} \xi(x,t)_- = \mathbb{E} \left(\int_{0}^{t} \mathbb{P}(Z_s(t) > x)J(ds)\right).
\]

(4.22)
The main step to calculate the mean value of $Z$ is the following.

**Proposition 2.** For all $x, t \geq 0$

\[
\int_{0}^{x} \mathbb{P}(Z(t) > z)dz = \frac{\rho}{\lambda} \int_{0}^{t} \mathbb{P}(Z(u) \leq x)du.
\]  

(4.23)

**Proof of Proposition 2.** Combining (4.19), (4.20) with (3.8),

\[\left(\lambda - \frac{1}{\rho}\right)x - \left(\rho - \frac{1}{\lambda}\right)t = \mathbb{E}(\xi(x, t)) = \mathbb{E}(\xi(x, t)_{+}) - \mathbb{E}(\xi(x, t)_{-})\]

That is,

\[
\left(\lambda - \frac{1}{\rho}\right)\int_{0}^{x} \mathbb{P}(Z(t) > z)dz = \left(\rho - \frac{1}{\lambda}\right)\int_{0}^{t} \mathbb{P}(Z(u) \leq x)du.
\]

(4.24)

which is equivalent to (4.23) for $\lambda \rho > 1$. □

**Proof of (1.1).** It follows directly from Proposition 2:

\[
\mathbb{E}(Z(t)) = \lim_{x \to \infty} \int_{0}^{x} \mathbb{P}(Z(t) > z)dz = \lim_{x \to \infty} \frac{\rho}{\lambda} \int_{0}^{t} \mathbb{P}(Z(u) \leq x)du = \frac{\rho}{\lambda} t.
\]

□

A similar idea works to compute $\text{Var}(Z(t))$:

**Proposition 3.** For all $x, t \geq 0$

\[
\left(\rho - \frac{1}{\lambda}\right)\left(t + \int_{0}^{t} \mathbb{P}(Z(u) \leq x)du\right)
\]

\[= \left(\lambda - \frac{1}{\rho}\right)^{2} \left(2 \int_{0}^{x} z\mathbb{P}(Z(t) > z)dz - \left(\int_{0}^{x} \mathbb{P}(Z(t) > z)dz\right)^{2}\right) + \text{Var}(\xi(x, t)_{-})
\]

(4.25)

**Proof of (1.2).** Taking $x \to \infty$ in both members of (4.25),

\[
\left(\rho - \frac{1}{\lambda}\right)2t = \left(\lambda - \frac{1}{\rho}\right)^{2}\text{Var}(Z(t)),
\]

from where (1.2) follows. (Notice that, for fixed $t \geq 0$, $\xi_{-}(x, t)$ decreases to 0 when $x \to \infty$.) □

**Proof of Proposition 3.** Noticing that if $\bar{y} \leq y$ and $Z_{\bar{y}}(t) \leq x$ then $Z_{y}(t) \leq x$,

\[
\xi(x, t)_{+}^{2} = \xi(x, t)_{+} + 2 \int_{0}^{x} \int_{0}^{y} 1\{Z_{y}(t) \leq x\}1\{y \neq \bar{y}\}I(dy)I(dy)
\]

(4.26)
as we do for the square of a sum of functions with values in \( \{0, 1\} \). The expected value of the double integral in the above equation is given by

\[
2\mathbb{E} \left( \int_0^x \int_0^y 1 \{Z_u(t) \leq x\} 1\{y \neq \bar{y}\} I(dy) I(dy) \right) \\
= 2(\lambda - \frac{1}{\rho})^2 \int_0^x \int_0^y P(Z(t) \leq x - y) dy \\
= 2(\lambda - \frac{1}{\rho})^2 \int_0^x y P(Z(t) \leq x - y) dy ,
\]

Subtracting (4.29) from (4.28):

\[
\mathbb{E}\left( \xi(x,t)^2 \right) = (\lambda - \frac{1}{\rho}) \left( x - \int_0^x P(Z(t) > z) dz \right)
\]

\[
\quad + (\lambda - \frac{1}{\rho})^2 \left( x^2 - 2x \int_0^x P(Z(t) > z) dz + 2 \int_0^x z P(Z(t) > z) dz \right)
\]

On the other hand, taking the square of (4.19),

\[
\left( \mathbb{E}\xi(x,t) \right)^2 = (\lambda - \frac{1}{\rho})^2 \left\{ x^2 - 2x \int_0^x P(Z(t) > z) dz + \left( \int_0^x P(Z(t) > z) dz \right)^2 \right\}
\]

Subtracting (4.29) from (4.28):

\[
\text{Var}(\xi(x,t)_+) = (\lambda - \frac{1}{\rho}) \left( x - \int_0^x P(Z(t) > z) dz \right)
\]

\[
\quad + (\lambda - \frac{1}{\rho})^2 \left( 2 \int_0^x z P(Z(t) > z) dz - \left( \int_0^x P(Z(t) > z) dz \right)^2 \right)
\]

that is, using (4.19) and \( \text{Var}(\xi(x,t)) = \text{Var}(\xi(x,t)_+) + \text{Var}(\xi(x,t)_-) \),

\[
(\rho - \frac{1}{\lambda}) t = - (\lambda - \frac{1}{\rho}) \left( \int_0^x P(Z(t) > z) dz \right)
\]

\[
\quad + (\lambda - \frac{1}{\rho})^2 \left( 2 \int_0^x z P(Z(t) > z) dz - \left( \int_0^x P(Z(t) > z) dz \right)^2 \right)
\]

\[
\quad + \text{Var}(\xi(x,t)_-)
\]

Use (4.23) to get

\[
(\rho - \frac{1}{\lambda}) \left( t + \int_0^t \mathbb{P}(Z(u) \leq x) du \right)
\]

\[
= (\lambda - \frac{1}{\rho})^2 \left( 2 \int_0^x z \mathbb{P}(Z(t) > z) dz - \left( \int_0^x \mathbb{P}(Z(t) > z) dz \right)^2 \right) + \text{Var}(\xi(x,t)_-)
\]
which shows (4.25). □

5. The dependence on the initial condition

Recall that

\[ N(t) = (\lambda - \frac{1}{\rho})^{-1} \left[ c_0 t - \left( \int_0^{t R(S)} S(dz) + \int_0^{t R(W)} W(ds) \right) \right] \]

where

\[ c_0 = 3 \left( \rho - \frac{1}{\lambda} \right), \quad R(S) = \frac{1}{\lambda} \left( \rho - \frac{1}{\lambda} \right) \quad \text{and} \quad R(W) = \frac{1}{\rho} \left( \rho - \frac{1}{\lambda} \right). \]

Thus,

\[ \mathbb{E}(N(t)) = \mathbb{E}(Z(t)) = \frac{\rho}{\lambda} t \quad \text{and} \quad \text{Var}(N(t)) = \text{Var}(Z(t)) = Dt \quad (5.30) \]

where

\[ D = \frac{2(\rho - \frac{1}{\lambda})}{(\lambda - \frac{1}{\rho})^2}. \]

(by calculating \( N(t) \) and using Theorem 1). The next step is to prove:

**Lemma 2.**

\[ \lim_{t \to \infty} \frac{\text{Cov}(Z(t), N(t))}{t} = D \]

**Proof of Lemma 2.** The proof follows in the same lines of an analogous result proved by Ferrari [5] in the TASEP context. To sketch the idea, denote by \( Z^{z|1}(t) \) (respectively, \( Z^{z|0}(t) \)) the position at time \( t \) of a second class particle with respect to an initial configuration conditioned to have a particle at position \( z \) (respectively, conditioned not to have a particle at position \( z \)). Then we can get (by coupling)

\[ \text{Cov}(Z(t), \int_0^{t R(S)} S(dz)) = -\lambda \int_0^{t R(S)} \mathbb{E}(Z^{z|0}(t) - Z^{z|1}(t))dz \quad (5.31) \]

and

\[ \text{Cov}(Z(t), \int_0^{t R(W)} W(ds)) = -\rho \int_0^{t R(W)} \mathbb{E}(Z^{s|0}(t) - Z^{s|1}(t))ds. \quad (5.32) \]

Now, we claim that for all \( \epsilon > 0 \)

\[ \lim_{t \to \infty} \sup_{z \in [0, t(t R(-) - \epsilon)]} |\mathbb{E}(Z^{z|0}(t) - Z^{z|1}(t)) - \frac{1}{(\lambda - \frac{1}{\rho})}| = 0. \quad (5.33) \]

To sketch the idea, call (see Section 3) \( Z_n^{z|1}(t) \) the position at time \( t \) of the second class particles starting at \( \xi \) conditioned to have one particle at \( z \). Let \( Y_t \) be the position at time \( t \) where the processes with initial configuration being different only at \( z \) differ by time \( t \). Let \( \tau_1 \) be the first time the trajectories of \( Y_t \) and \( Z^{z|1}(t) \) meet each other. Note that \( \tau_1 \)
is finite with probability one. Before $\tau_1, Z_{z_{1}}(t) = Z_{z_{0}}(t)$. After $\tau_1, Z_{z_{1}}(t) = Z_{z_{1}}(t)$ and $Z_{t_{0}} = Z_{z_{1}}(t)$. Since

$$\mathbb{E}(Z_{z_{1}}(t) - Z_{z_{1}}(t)) = \frac{1}{(\lambda - \frac{1}{\rho})},$$

we get (5.33). Thus, Lemma 2 follows by combining (5.31), (5.32) together with (5.33). □

Proof of Theorem 2. By (5.30),

$$\mathbb{E}[ (Z(t) - N(t))^2] = 2Dt - 2\text{Cov}(Z(t), N(t)),$$

and hence, Theorem 2 follows from Lemma 2. □

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