Curves of Restricted Type in Euclidean Spaces

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Abstract. Submanifolds of restricted type were introduced in [7]. In the present study we consider restricted type of curves in $\mathbb{E}^m$. We give some special examples. We also show that spherical curve in $S^2(r) \subset \mathbb{E}^3$ is of restricted type if and only if either $f(s)$ is constant or a linear function of $s$ of the form $f(s) = \pm s + b$ and every closed $W$-curve of rank $k$ and of length $2\pi r$ in $\mathbb{E}^{2k}$ is of restricted type.

1. Introduction

Let $M^n$ be an $n$-dimensional submanifold of a Euclidean space $\mathbb{E}^m$. Let $x, H$ and $\Delta$ respectively be the position vector field, the mean curvature vector field and the Laplace operator of the induced metric on $M^n$. Then, as is well known (see e.g. [2])

$$\Delta x = -nH,$$

which shows, in particular, that $M^n$ is a minimal submanifold in $\mathbb{E}^m$ if and only if its coordinate functions are harmonic (i.e. they are eigenfunctions of $\Delta$ with eigenvalue 0).

As a generalization of T. Takahashi’s condition and following an idea of O. Garay [13], some of the authors together with J. Pas [10] initiated the study of submanifolds $M^n$ in $\mathbb{E}^m$ such that

$$\Delta x = Ax + B$$

for some fixed vector $B \in \mathbb{E}^m$ and a given matrix $A \in \mathbb{R}^{m \times m}$. This study was continued by the first author together with M. Petrovic [5] and independently by T. Hasanis and T. Vlachos [14].

During the study of submanifolds of $\mathbb{R}^m$ satisfying (2), it was observed that all these matrices $A_p$ are equal for all $p \in M$, or equivalently there exists a fixed matrix $A \in \mathbb{E}^{m \times m}$ (determining, of course, a linear endomorphism of $\mathbb{E}^m$) such that for all $p \in M$ and for all $X \in T_pM$,

$$A_H X = (AX)^T.$$
As the relation (3) expresses a strong relationship between differential geometry and linear algebra, we do think it would be worthwhile to study submanifolds satisfying this condition; such submanifolds are said to be of restricted type.

Submanifolds of restricted type were introduced in [7] by the author B.Y. Chen, F. Dillen, L. Verstraelen and L. Vrancken. The class of submanifolds of restricted type is large which includes 1-type submanifolds, pseudo-umbilical submanifolds with constant mean curvature, submanifolds satisfying either Gray’s condition or Dillen Pas Verstraelen’s condition, all \( k \)-type curves lying fully in \( \mathbb{E}^{2k} \), all null \( k \)-type curves lying fully in \( \mathbb{E}^{2k-1} \), the products of submanifolds of restricted type, the diagonal immersions of restricted type submanifolds and equivariant isometric immersions of compact homogeneous spaces. In [7], it is shown that a hypersurface of restricted type is either minimal, or a part of the product of a sphere and a linear subspace, or a cylinder on a plane curve of restricted type, and all planar curves of restricted type are classified.

2. Basic Concepts

In the present section we recall definitions and results of [1]. Let \( x : M \to \mathbb{E}^m \) be an immersion from an \( n \)-dimensional connected Riemannian manifold \( M \) into an \( m \)-dimensional Euclidean space \( \mathbb{E}^m \). We denote by \( g \) the metric tensor of \( \mathbb{E}^m \) as well as the induced metric on \( M \). Let \( \tilde{\nabla} \) be the Levi-Civita connection of \( \mathbb{E}^m \) and \( \nabla \) the induced connection on \( M \). Then the Gaussian and Weingarten formulas are given, respectively, by

\[
\tilde{\nabla}_XY = \nabla_XY + h(X,Y),
\]

\[
\tilde{\nabla}_X\xi = -A_\xi X + D_X\xi,
\]

where \( X, Y \) are vector fields tangent to \( M \) and \( \xi \) normal to \( M \). Moreover, \( h \) is the second fundamental form, \( D \) is the linear connection induced in the normal bundle \( T^\perp M \), called normal connection and \( A_\xi \) the shape operator in the direction of \( \xi \) that is related with \( h \) by

\[
\langle h(X,Y), \xi \rangle = \langle A_\xi X, Y \rangle.
\]

For an \( n \)-dimensional submanifold \( M \) in \( \mathbb{E}^m \). The mean curvature vector \( \vec{H} \) is given by

\[
\vec{H} = \frac{1}{n} \text{trace} h.
\]

A submanifold \( M \) is said to be minimal (respectively, totally geodesic) if \( \vec{H} \equiv 0 \) (respectively, \( h \equiv 0 \)).

Consider an \( n \)-dimensional Riemannian manifold \( M \) and denote by \( (g_{ij}) \) the local components of its metric. Put \( G = \det(g_{ij}) \) and \( (g^{ij}) = (g_{ij})^{-1} \).
Then the Laplacian $\Delta$ of the metric $g$ can be locally defined by

\begin{equation}
\Delta u = -\frac{1}{\sqrt{G}} \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left( \sqrt{G} g^{ij} \frac{\partial u}{\partial x_j} \right),
\end{equation}

for any function $u$ on $M$, where $x_1, x_2, ..., x_n$ are local coordinates [11].

$M$ is said to be of finite type if each component of the position vector $x$ has a finite spectral decomposition [2]

\begin{equation}
x = x_0 + x_1 + x_2 + \cdots + x_k,
\end{equation}

where $x_0$ is a constant vector in $\mathbb{E}^m$ and $x_1, x_2, ..., x_k$ are non-constant maps which satisfy $\Delta x_i = \lambda_i x_i$, $1 \leq i \leq k$, $\lambda_1 < \lambda_2 < \cdots < \lambda_k$.

If all eigenvalues $\lambda_1, \lambda_2, ..., \lambda_k$ are mutually distinct, then the immersion $x$ (or the submanifold $M$) is said to be of $k$–type [2].

3. $W$-CURVES IN $\mathbb{E}^m$

Let $\gamma = \gamma(t) : I \rightarrow \mathbb{E}^m$ be a regular curve in $\mathbb{E}^m$ (i.e. $|\gamma'|$ is nowhere zero), where $I$ is an interval in $\mathbb{R}$. $\gamma$ is called a Frenet curve of rank $r$ ($r \in \mathbb{N}_0, r \leq m$) if $\gamma'(t), \gamma''(t), \ldots, \gamma^{(r)}(t)$ are linearly independent and $\gamma'(t), \gamma''(t), \ldots, \gamma^{(r+1)}(t)$ are no longer linearly independent for all $t$ in $I$. In this case, $\text{Im}(\gamma)$ lies in an $r$-dimensional Euclidean subspace of $\mathbb{E}^m$. To each Frenet curve of rank $r$ there can be associated orthonormal $r$–frame $\{V_1, V_2, \ldots, V_r\}$ along $\gamma$, the Frenet $r$–frame and $r - 1$ functions $\kappa_1, \kappa_2, \ldots, \kappa_{r-1} : I \rightarrow \mathbb{R}$, the Frenet curvatures, such that

\begin{equation}
\begin{bmatrix}
V'_1 \\
V'_2 \\
V'_3 \\
\vdots \\
V'_{r-1} \\
V'_r
\end{bmatrix} = v
\begin{bmatrix}
0 & \kappa_1 & 0 & \cdots & 0 & 0 \\
-\kappa_1 & 0 & \kappa_2 & \cdots & 0 & 0 \\
0 & -\kappa_2 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & \cdots & 0 & \kappa_{r-1} \\
0 & 0 & \cdots & \cdots & -\kappa_{r-1} & 0
\end{bmatrix}
\begin{bmatrix}
V_1 \\
V_2 \\
V_3 \\
\vdots \\
V_{r-1} \\
V_r
\end{bmatrix},
\end{equation}

where $v$ is the speed of the curve.

In fact, to obtain $V_1, V_2, \ldots, V_r$ it is sufficient to apply the Gram-Schmidt orthonormalization process to $\gamma'(t), \gamma''(t), \ldots, \gamma^{(r)}(t)$. Moreover, the functions $\kappa_1, \kappa_2, \ldots, \kappa_{r-1}$ are easily obtained as by-product during this calculation. More precisely, $V_1, V_2, \ldots, V_r$ and $\kappa_1, \kappa_2, \ldots, \kappa_{r-1}$ are determined by
the following formulas:

\[ E_1(t) := \gamma'(t); \quad V_1 := \frac{E_1(t)}{\|E_1(t)\|} \]

\[ E_k(t) := \gamma^{(k)}(t) - \sum_{i=1}^{k-1} \left\langle \gamma^{(k)}(t), E_i(t) \right\rangle \frac{E_i(t)}{\|E_i(t)\|} \]

\[ \kappa_k(t) := \frac{E_k(t)}{\|E_k(t)\|} \]

\[ V_k := \frac{E_k(t)}{\|E_k(t)\|} \]

where \( k \in \{2, 3, \ldots, r\} \). It is natural and convenient to define Frenet curvatures \( \kappa_r = \kappa_{r+1} = \cdots = \kappa_{m-1} = 0 \). It is clear that \( V_1, V_2, \ldots, V_r \) and \( \kappa_1, \kappa_2, \ldots, \kappa_{r-1} \) can be defined for any regular curve (not necessary a Frenet curve) in the neighborhood of a point \( t_0 \) for which \( \gamma'(t_0), \gamma''(t_0), \ldots, \gamma^{(r)}(t_0) \) are linearly independent.

**Definition 1.** Frenet curve of rank \( r \) for which \( \kappa_1, \kappa_2, \ldots, \kappa_{r-1} \) are constant is called (generalized) screw line or helix [6]. Since these curves are trajectories of the 1-parameter group of the Euclidean transformations, so, F. Klein and S. Lie [9] called them \( W^- \) curves.

A unit speed \( W^- \) curve of rank 2 has the parametrization form

\[ \gamma(s) = a_0 + \sum_{i=1}^{k} (a_i \cos \mu_i s + b_i \sin \mu_i s), \]

and a unit speed \( W^- \) curve of rank \( 2k+1 \) has the parametrization form

\[ \gamma(s) = a_0 + b_0 s + \sum_{i=1}^{k} (a_i \cos \mu_i s + b_i \sin \mu_i s), \]

where \( a_0, b_0, a_1, \ldots, a_k, b_1, \ldots, b_k \) are constant vectors in \( \mathbb{E}^m \) and \( \mu_1 < \mu_2 < \cdots < \mu_k \) are positive real numbers.

So, a \( W^- \) curve of rank 1 is a straight line, a \( W^- \) curve of rank 2 is a circle and a \( W^- \) curve of rank 3 is a right circular helix [6].

A \( W^- \) curve is closed if and only if its rank is even and all \( \mu_i \) are rational multiples of a real number. Therefore, up to rigid motions of a Euclidean space, a closed \( W^- \) curve of rank 2k and of length \( 2\pi r \) in \( \mathbb{E}^{2k} \) has an arc length parameterization of the form:

\[ \gamma(s) = \frac{r}{\sqrt{k}} \left( \frac{1}{t_1} \cos \left( \frac{t_1 s}{r} \right), \frac{1}{t_1} \sin \left( \frac{t_1 s}{r} \right), \ldots, \frac{1}{t_k} \cos \left( \frac{t_k s}{r} \right), \frac{1}{t_k} \sin \left( \frac{t_k s}{r} \right) \right) \]

where \( t_1 < \cdots < t_k \) are positive integers [8].
4. Curves of restricted type

**Definition 2.** A submanifold $M^n$ in $\mathbb{E}^m$ is said to be restricted type if the shape operator $A_H$ is the restriction of a fixed endomorphism $A$ of $\mathbb{E}^m$ on the tangent space of $M^n$ at every point of $M^n$, i.e.

$$A_HX = (AX)^T$$

for any vector $X$, tangent to $M^n$, where $(AX)^T$ denotes the tangential component of $AX$ [7].

**Remark 1.** Equation (14) is equivalent to $\langle A_HX, Y \rangle = \langle AX, Y \rangle$ for all tangent vectors $X, Y$ [7].

**Proposition 1.** Every submanifold $M^n$ in $\mathbb{E}^m$ whose position vector field satisfies $\Delta x = \tilde{A}x + B$, where $\Delta$ is the Laplacian of $M^n$, $\tilde{A} \in \mathbb{R}^{m \times m}$ and $B \in \mathbb{E}^m$, is of restricted type. The endomorphism $A$ is given by $\frac{1}{n}\tilde{A}$ in this case [7].

Let $\gamma$ be a regular curve in $\mathbb{E}^m$. The Laplacian of $\gamma$ can be expressed as

$$\Delta \gamma(t) = -\frac{d^2 \gamma(t)}{dt^2} = -\gamma''(t).$$

By the using of (1) and (15),

$$H = -\Delta \gamma(t) = \gamma''(t)$$

where $H$ is the mean curvature of $\gamma$.

**Proposition 2.** Let $\gamma$ be a curve in $\mathbb{E}^m$. If $\gamma$ has the equation

$$-\gamma''(t) = \Delta \gamma(t) = A\gamma(t) + B$$

such that $B$ is a fixed vector in $\mathbb{E}^m$ and $A$ a symmetric matrix in $\mathbb{R}^{m \times m}$, then $\gamma$ is of restricted type.

**Proof.** From Preposition 1 we have the equation

$$\Delta \gamma(t) = A\gamma(t) + B.$$

Thus using (16) and (18), we get (17). \qed

**Corollary 1.** Let $\gamma$ be a curve in $\mathbb{E}^m$. $\gamma$ is of restricted type if and only if

$$-\gamma'''(t) = A\gamma'(t),$$

where $A$ is a symmetric matrix in $\mathbb{R}^{m \times m}$.

**Example 1.** $S^1(a) \subset \mathbb{E}^2$ is of restricted type.

$S^1(a)$ is given by the parametrization $\gamma(t) = (a \cos t, a \sin t)$. From higher order derivatives of $\gamma$ we get

$$\gamma'''(t) = -I_2\gamma'(t).$$

Thus $S^1(a) \subset \mathbb{E}^2$ is of restricted type.
**Example 2.** A helix which is given by the parametrization

\[ \gamma(t) = (r \cos(ct + d), r \sin(ct + d), at + b) \]

is of restricted type.

From higher order derivatives of \( \gamma \) we get \( \gamma''(t) = -A \gamma'(t) \) where

\[ A = \begin{bmatrix} c^2 & 0 & 0 \\ 0 & c^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \]

Thus helix is of restricted type.

**Example 3.** Every \( k \)-type curve which lies fully in \( \mathbb{E}^{2k} \) is of restricted type [7].

**Example 4.** Every \( 2 \)-type curve in \( \mathbb{E}^m \) is of restricted type [7].

**Example 5.** Although every \( 2 \)-type curve in \( \mathbb{E}^m \) and every \( k \)-type curve which lies fully in \( \mathbb{E}^{2k} \) are curves of restricted type, not every curve of finite type (in the sense of [2,4]) is of restricted type. For instance the following \( 6 \)-type curve in \( \mathbb{E}^3 \) is not of restricted type [7]

\[ \gamma(s) = \left( \begin{array}{c} -\frac{2}{3} \cos \frac{12}{17}s + \frac{3}{4} \cos \frac{16}{17}s + \frac{3}{10} \cos \frac{20}{17}s + \frac{1}{8} \cos \frac{24}{17}s + \frac{1}{14} \cos \frac{28}{17}s, \\ -\frac{2}{3} \sin \frac{12}{17}s + \frac{3}{4} \sin \frac{16}{17}s + \frac{3}{10} \sin \frac{20}{17}s + \frac{1}{8} \sin \frac{24}{17}s + \frac{1}{14} \sin \frac{28}{17}s, \\ \sin 8\frac{17}{17}s \end{array} \right). \]

**Proposition 3.** Let \( \gamma \) be a spherical space curve given with

\[ (21) \quad \gamma(s) = (r \cos s \sin(f(s)), r \sin s \sin(f(s)), r \cos(f(s)) ,\]

where \( f(s) \) is polynomial function. Then \( \gamma \) is of restricted type if and only if \( f(s) \) is either constant or a linear function of \( s \) of the form \( f(s) = \pm s + b \).

**Proof.** Suppose that \( \gamma \) is of restricted type, then by the use of (19) the equality

\[ (22) \quad \begin{bmatrix} \gamma_1'''(s) \\ \gamma_2'''(s) \\ \gamma_3'''(s) \end{bmatrix} = \begin{bmatrix} -c_{11} & 0 & 0 \\ 0 & -c_{22} & 0 \\ 0 & 0 & -c_{33} \end{bmatrix} \begin{bmatrix} \gamma_1'(s) \\ \gamma_2'(s) \\ \gamma_3'(s) \end{bmatrix} \]

holds. Here \( \gamma_i', \gamma_i'''(s) \) are the first and the third derivatives of \( i \)-th component of \( \gamma \) and \( c_{ii} \) is the entry of the matrix \( A \).

From higher order derivatives of \( \gamma \) we get

\[ \gamma'(s) = \left( -r \sin s \sin(f(s)) + r \cos s \cos(f(s))f'(s), r \cos s \sin(f(s)) + r \sin s \cos(f(s))f'(s) \right) \]

(\ref{gamma_prime})
\( \gamma'''(s) = (r \cos s \cos(f(s))(f'''(s) - (f'(s))^3 - 3f'(s)) \\
+ r \sin s \sin(f(s))(1 + 3(f'(s))^2) + r \cos s \sin(f(s))(-3f'(s)f''(s)) \\
+ r \sin s \cos(f(s))(-3f''(s)), r \cos s \cos(f(s))(3f''(s)) \\
+ r \sin s \sin(f(s))(-3f'(s)f''(s)) + r \cos s \sin(f(s))(-1 - 3(f'(s))^2) \\
+ r \sin s \cos(f(s))(f'''(s) - (f'(s))^3 - 3f'(s)), \\
r \sin(f(s))(-f''(s) + (f'(s))^3) + r \cos(f(s))(-3f'(s)f''(s))). \)

Using (22), (23) and (24) we have

\[ f'''(s) - (f'(s))^3 - 3f'(s) + c_{11}f'(s) = 0, \]
\[ 1 + 3(f'(s))^2 - c_{11} = 0, \]
\[ -3f'(s)f''(s) = 0, \]
\[ -3f''(s) = 0, \]
\[ -1 - 3(f'(s))^2 + c_{22} = 0, \]
\[ f'''(s) - (f'(s))^3 - 3f'(s) + c_{22}f'(s) = 0, \]
\[ -f''(s) + (f'(s))^3 - c_{33}f'(s) = 0. \]

From (27) and (28) it can be seen that either \( f(s) \) is constant or a linear function of \( s \) of the form \( f(s) = as + b \) where \( a, b \in \mathbb{R} \). If \( f(s) \) is constant, then \( f(s) \) is a circle which is of restricted type. If \( f(s) \) is a linear function of \( s \) of the form \( f(s) = as + b \), then using (25) and (26) we get \( c_{11} = 1 + 3a^2 = a^2 + 3 \). Then \( a = \pm 1 \) and \( c_{11} = 4 \). Similarly, from (29), (30) and (31) we get \( c_{22} = 4 \) and \( c_{33} = 1 \). So we obtain

\[ A = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \]

Conversely, if \( f(s) = \text{const.} \) or \( f(s) = \pm s + b \) then it is easy to show that the curve given with the parametrization (21) is of restricted type. \( \square \)

We also get the following result.

**Proposition 4.** Let \( \gamma \) be closed \( W \)-curve of rank \( k \) and of length \( 2\pi r \) in \( E^{2k} \) given by the parametrization (13). Then \( \gamma \) is of restricted type.
Proof. From higher order derivatives of \( \gamma \) we get
\[
\gamma'(s) = \frac{1}{\sqrt{k}} \left( -\sin \left( \frac{t_1 s}{r} \right), \cos \left( \frac{t_1 s}{r} \right), \ldots, -\sin \left( \frac{t_k s}{r} \right), \cos \left( \frac{t_k s}{r} \right) \right)
\]
\[
\gamma''(s) = -\frac{1}{\sqrt{k}} \left( \frac{t_1}{r} \cos \left( \frac{t_1 s}{r} \right), \frac{t_1}{r} \sin \left( \frac{t_1 s}{r} \right), \ldots, \frac{t_k}{r} \cos \left( \frac{t_k s}{r} \right), \frac{t_k}{r} \sin \left( \frac{t_k s}{r} \right) \right)
\]
\[
\gamma'''(s) = \frac{1}{\sqrt{k}} \left( \frac{t_1^2}{r^2} \sin \left( \frac{t_1 s}{r} \right), -\frac{t_1^2}{r^2} \cos \left( \frac{t_1 s}{r} \right), \ldots, \frac{t_k^2}{r^2} \sin \left( \frac{t_k s}{r} \right), -\frac{t_k^2}{r^2} \cos \left( \frac{t_k s}{r} \right) \right).
\]
So, we have \( \gamma'''(t) = -A \gamma'(t) \) where
\[
A = \begin{bmatrix}
\frac{t_1^2}{r^2} & 0 & \cdots & 0 & 0 \\
0 & \frac{t_1^2}{r^2} & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & \frac{t_k^2}{r^2} & 0 \\
0 & 0 & \cdots & 0 & \frac{t_k^2}{r^2}
\end{bmatrix}.
\]
Thus \( W \)-curve is of restricted type.

Example 6. A closed \( W \)-curve of rank 4 and of length \( 2\pi \) given by the parametrization
\[
\gamma(s) = (\cos ms, \sin ms, \cos ns, \sin ns)
\]
is of restricted type, where \( m, n \) are positive integers. From higher order derivatives of \( \gamma \) we get \( \gamma'''(t) = -A \gamma'(t) \) where
\[
A = \begin{bmatrix}
m^2 & 0 & 0 & 0 \\
0 & m^2 & 0 & 0 \\
0 & 0 & n^2 & 0 \\
0 & 0 & 0 & n^2
\end{bmatrix}.
\]
Thus \( \gamma \) is of restricted type.

Theorem 1 ([7]). Up to rigid motions of \( \mathbb{E}^2 \), a curve in \( \mathbb{E}^2 \) is of restricted type if and only if it is an open portion of one of the following plane curves:

1) a circle,
(2) a line,
(3) a curve with equation: \( \cos(cx) = e^{-cy} \), where \( c \neq 0 \),
(4) a curve with equation: \( a \sin^2(\sqrt{cx}) + b \sinh^2(\sqrt{cx}) = c \), where \( a > b > 0 \), \( c = a - b \),
(5) a curve with equation: \( a \sin^2(\sqrt{cx}) - b \cosh^2(\sqrt{cx}) = c \), where \( a > 0 > b \), \( c = a - b \).

**Proposition 5 ([7]).** Let \( \gamma \) be a planar curve. \( \gamma \) is of restricted type if and only if the curvature \( \kappa \) of \( \gamma \) satisfies the following differential equation

\[(34) \quad \kappa \kappa''' - \kappa' \kappa'' + 4 \kappa^3 \kappa' = 0\]

where the derivatives are taken with respect to the arc length parameter.

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