A NOTE ON MAPPING CLASS GROUP ACTIONS ON DERIVED CATEGORIES.

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Abstract. Let $X_n$ be a cycle of $n$ projective lines, and $\mathbb{T}_n$ a symplectic torus with $n$ punctures. Using the theory of spherical twists introduced by Seidel and Thomas \cite{ST}, I will define an action of the pure mapping class group of $\mathbb{T}_n$ on $\text{D}^b(\text{Coh}(X_n))$. The motivation comes from homological mirror symmetry for degenerate elliptic curves, which was studied by the author with Treumann and Zaslow in \cite{STZ}.

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1. Introduction.

According to Kontsevich’s *Homological Mirror Symmetry* conjecture (from now on HMS, see \cite{K}), given a Calabi-Yau variety $X$ and a symplectic manifold $\tilde{X}$, if $X$ and $\tilde{X}$ are mirror partners, then the derived category of coherent sheaves over $X$, $\text{D}^b(\text{Coh}(X))$, should be equivalent to the Fukaya category of $\tilde{X}$, $\text{Fuk}(\tilde{X})$. Since $\text{Fuk}(\tilde{X})$ is an invariant of the symplectic geometry of $\tilde{X}$, mirror symmetry predicts that the group of symplectic automorphisms of $\tilde{X}$ acts by equivalences on $\text{D}^b(\text{Coh}(X))$. In \cite{ST} Seidel and Thomas investigate this aspect of HMS by introducing the notions of *spherical object* and *twist functor*, which can be defined for general triangulated categories, and axiomatize the formal homological properties enjoyed by equivalences of the Fukaya category induced by *generalized Dehn twists* (these are special symplectic automorphisms introduced by Seidel, see \cite{S}). Using their theory they are able, in many interesting examples, to give a conjectural description of the equivalences of $\text{D}^b(\text{Coh}(X))$ which should be mirror to symplectic automorphisms of $\tilde{X}$. I refer the reader to \cite{ST} for a detailed account of this circle of ideas. A brief overview of the relevant definitions will be given in Section 3 below.

Let $X_n$ be a cycle of $n$ projective lines, i.e. a nodal curve of arithmetic genus 1, with $n$ singular points. Well known mirror symmetry heuristics suggest that the mirror of $X_n$ should be a symplectic torus with $n$ punctures, which I shall denote $\mathbb{T}_n$. In the paper \cite{STZ},

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joint with Treumann and Zaslow, we prove a version of HMS for $X_n$ and $T_n$, by showing that the category of perfect complexes over $X_n$, $\text{Perf}(X_n)$, is quasi-equivalent to a certain conjectural model of $\text{Fuk}(T_n)$ which we develop in the paper. See also the recent work [LP] in which the authors prove, with very different techniques, a HMS statement for the case $n = 1$.

Motivated by [STZ], in this paper I explore the consequences of mirror symmetry for the study of auto-equivalences of $D^b(\text{Coh}(X_n))$. Recall that the mapping class group of an oriented surface $\Sigma$ can be described as the group of symplectic automorphisms of $\Sigma$, modulo isotopy. The existence of an action of the mapping class group of $T_n$ on $D^b(\text{Coh}(X_n))$ does not follow directly from [STZ], as the model of the Fukaya category considered there is not acted upon, in any obvious way, by symplectomorphisms of $T_n$. My main result uses the framework of [ST] to construct an action of the (pure) mapping class group of $T_n$, $\text{PM}(T_n)$, over $D^b(\text{Coh}(X_n))$. In future work, I plan to establish that this action is faithful. It is worth pointing out that the action I will define is, in an appropriate sense, a categorification of the symplectic representation of the mapping class group, which can be recovered by considering the induced action on the numerical Grothendieck group of $\text{Perf}(X_n)$ (see Remark 3.7 below. For a definition of the symplectic representation, the reader can refer to [FM], Chapter 6).

The paper is organized as follows. In Section 2, I give some background on the mapping class group, and then work out a convenient presentation of $\text{PM}(T_n)$. The proof of the main result, Theorem 3.6, is contained in Section 3. Theorem 3.6 generalizes previous results in [ST] and in [BK], where the authors considered, respectively, the case of a smooth elliptic curve, and of the nodal cubic in $\mathbb{P}^2$ (i.e. the case $n = 1$). Equivalences of $D^b(\text{Coh}(X_n))$ were also investigated in [L]. However, as the author in [L] restricts to a subgroup of equivalences satisfying certain homological conditions, which are violated by the spherical twists I shall consider below, there is essentially no overlap between his work and the present project.

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2. The mapping class group of a punctured torus.

In this section I will briefly review some basic facts about the mapping class group, and then give a presentation of the mapping class group of the punctured torus based on [LP]. Also, it will be useful to spell out some relations between mapping classes which were found by Gervais in [G]. For a comprehensive introduction to the mapping class group I refer the reader to [FM].

1Note also that the HMS statement in [STZ] involves $\text{Perf}(X_n)$, rather than the full derived category of $X_n$. However, $D^b(\text{Coh}(X_n))$ has the same group of auto-equivalences of $\text{Perf}(X_n)$. In fact, any equivalence of $D^b(\text{Coh}(X_n))$ gives, by restriction, an equivalence of $\text{Perf}(X_n)$, and it follows from Lemma 3.3 that this assignment is a bijection.
Let \( \Sigma = \Sigma_{g,n,b} \) be a differentiable, oriented surface of genus \( g \), with \( n \) marked points, and \( b \) boundary components. The mapping class group of \( \Sigma \), denoted by \( M(\Sigma) \), is the group of isotopy classes of orientation preserving diffeomorphisms of \( \Sigma \), which send the set of marked points to itself, and restrict to the identity on the boundary components. Note that \( M(\Sigma_{g,n,b}) \) is uniquely determined by the parameters \( g, n, b \). The pure mapping class group of \( \Sigma \) is the subgroup \( PM(\Sigma) \hookrightarrow M(\Sigma) \) of mapping classes fixing pointwise the set of marked points. Alternatively, \( PM(\Sigma) \) can be defined as the subgroup of \( M(\Sigma) \) generated by Dehn twists along simple closed curves (for the definiton of Dehn twist, and a proof of this claim, see Chapter 3 and 4 of [FM]). In making the above definitions, marked points on \( \Sigma \) could be interpreted, equivalently, as punctures, and I shall make use freely of both viewpoints in the following.

A surface \( \Sigma \) with \( n \) punctures and \( b+m \) boundary components can be immersed in a surface with \( n+m \) punctures and \( b \) boundary components (we can trade \( m \) boundary components for \( m \) punctures, by gluing a punctured discs along each boundary component we wish to remove). Further, this immersion induces a map of pure mapping class groups. The details can be found in Section 2 of [LP], together with the following lemma which will be useful later.

**Lemma 2.1.** Let \( (g,r,m) \notin \{(0,0,1), (0,0,2)\} \), then we have the exact sequence

\[
1 \to \mathbb{Z}^m \to PM(\Sigma_{g,n,b+m}) \to PM(\Sigma_{g,n+m,b}) \to 1,
\]

where \( \mathbb{Z}^m \) stands for the free abelian group of rank \( m \) generated by the Dehn twists along the \( m \) boundary components we are removing.

Set \( T_n = \Sigma_{1,n,0} \) and \( T_{n,m} = \Sigma_{1,n,m} \). The pure mapping class group \( PM(T_n) \) is generated by Dehn twists along \( n+1 \) non-separating simple closed curves. In order to fix ideas, it is convenient to choose explicit representatives for this collection of curves. I will mostly follow the notation of [G], to which I refer for further details. Let \( \Lambda = \mathbb{Z}^2 \hookrightarrow \mathbb{R}^2 \) be the standard integral lattice, let \( T = \mathbb{R}^2/\mathbb{Z}^2 \), and fix a fundamental domain for the action of \( \Lambda \), say \( [0,1) \times [0,1) \). Choose as set of marked points \( P = \{p_1 = (\frac{1}{n+1}, \frac{1}{2}), \ldots, p_n = (\frac{n}{n+1}, \frac{1}{2})\} \), and identify the index set \( \{1 \ldots n\} \) with \( \mathbb{Z}/n \) endowed with the natural cyclic order. A cyclic order allows us to speak unambiguously about ordered triples. If \( i, j, k \in \{1 \ldots n\} \) (not necessarily distinct) form an ordered triple, I shall write \( i \preceq j \preceq k \). If I require \( i, j, k \) to be distinct, I will use the symbol \( \prec \).

Let \( \alpha \) and \( \beta_i, i \in \{1 \ldots n\} \) be the following simple closed curves: in the fundamental domain, \( \alpha \) is given by \( [0,1) \times \{\frac{1}{2}\} \), and \( \beta_i \) is given by \( \{\frac{i}{n+1} - \frac{1}{2(n+1)}\} \times [0,1) \). It will be important to consider also separating curves \( \gamma_{i,j} \) indexed by an ordered pair \( i, j \in \{1 \ldots n\} \). The loop \( \gamma_{i,j} \) can be described as the boundary of a tubular neighborhood of a straight segment \( \sigma \) in \( T \), starting at \( p_i \) and ending at \( p_j \), and such that \( p_k \in \sigma \) if and only if \( i \preceq k \preceq j \). A schematic representation of these curves is given in Figure 1.

If \( \mu \) is a simple closed curve in a differentiable surface \( \Sigma \), denote \( T_\mu \) the Dehn twist along it. I will consider \( T_{n-1,1} \) to be the closed subsurface of \( T_n \) obtained by cutting out a small open disc centered in \( p_n \) (small means that its boundary should not intersect any of the loops

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2In order to make sense of the successor operator \( \bullet + 1 \) on the index set, I will also use the additive structure of \( \mathbb{Z}/n \).
Figure 1. The picture above represents the simple closed curves introduced earlier, which are visualized as subsets of the fixed fundamental domain for the action of \( \Lambda \).

described above). It follows from [LP] that both \( \text{PM}(\mathbb{T}_n) \) and \( \text{PM}(\mathbb{T}_{n-1,1}) \) are generated by Dehn twists \( T_\alpha \), and \( T_\beta_i \), \( i \in \{1 \ldots n\} \). I will refer to this collection of Dehn twists as Humphrey generators, in analogy with Humphrey’s set of generators for the mapping class group of a compact surface.

A presentation of \( \text{PM}(\mathbb{T}_{n-1,1}) \) in terms of Humphrey generators can be read off Proposition 3.3 of [LP]. For the reader’s convenience I collect it below.

**Proposition 2.2.** The pure mapping class group \( \text{PM}(\mathbb{T}_{n-1,1}) \) is generated by \( T_\alpha \), and \( T_\beta_i \), \( i \in \{1 \ldots n\} \), subject to the following relations:

- (Braid relations) for every \( i, j \in \{1 \ldots n\} \),
  \[
  T_\beta_i T_\beta_j = T_{\beta_j} T_{\beta_i},
  \]
  \[
  T_\alpha T_\beta_i T_\alpha = T_{\beta_i} T_\alpha T_{\beta_i}.
  \]

- (Commutativity relations) for every \( i, j, k \in \{1 \ldots n\} \), \( i < j < k \),
  \[
  T_\beta_i (T_\alpha^{-1} T_{\beta_{k+1}}^{-1} T_{\beta_j}^{-1} T_{\alpha} T_{\beta_k} T_{\beta_j} T_{\beta_{k+1}} T_\alpha) = (T_\alpha^{-1} T_{\beta_{k+1}}^{-1} T_{\beta_j}^{-1} T_{\alpha} T_{\beta_k} T_{\beta_j} T_{\beta_{k+1}} T_\alpha) T_\beta_i.
  \]

An analogous presentation for \( \text{PM}(\mathbb{T}_n) \) is described by the following Proposition.

**Proposition 2.3.** Let \( i, j \in \{1 \ldots n\} \), and set

\[
A_{i,j} = T_{\beta_{j+1}} T_\alpha T_{\beta_{k+1}}^{-1} T_{\beta_i} T_{\beta_{k+1}}^{-1} T_{\alpha} T_{\beta_j} T_{\beta_{k+1}}^{-1} T_{\alpha}^{-1} T_{\beta_{k+1}}^{-1} T_{\beta_i}.
\]

The pure mapping class group \( \text{PM}(\mathbb{T}_n) \) is generated by \( T_\alpha \), and \( T_\beta_i \), \( i \in \{1 \ldots n\} \), subject to the following relations:

- Braid relations and Commutativity relations (see Proposition 2.2).
- (G-relation) \( (T_\alpha T_\beta_i)^6 = A_{1,n} A_{2,n} \ldots A_{n-1,n} \).

Before giving a proof of Proposition 2.3, it is useful to consider an alternative presentation of \( \text{PM}(\mathbb{T}_2) \), which will play a role in the next Section, and is contained in Corollary 2.4 below. Although Corollary 2.4 follows quite easily from Proposition 2.3, rather than discussing the details of this derivation, I refer the reader to [PS] for a direct proof.
Corollary 2.4. The pure mapping class group PM(\(\mathbb{T}_2\)) is generated by \(T_\alpha\), \(T_{\beta_1}\) and \(T_{\beta_2}\), subject to the following relations:

- Braid relations
- \((G_2\text{-relation})\) \((T_{\beta_1}T_\alpha T_{\beta_2})^4 = 1\).

The proof of Proposition 2.3 depends on Proposition 2.2 and Lemma 2.1. It follows immediately from the definition of Dehn twist that, for all \(i \in \{1 \ldots n\}\), \(T_{\gamma_{i,i}} = 1\) in PM(\(\mathbb{T}_n\)). In \(\mathbb{T}_{n-1,1}\), however, \(\gamma_{n,n}\) is isotopic to the boundary component, and therefore \(T_{\gamma_{n,n}}\) defines a non-trivial mapping class. Lemma 2.1 assures us that the only extra relation needed to obtain PM(\(\mathbb{T}_n\)) from the presentation in Proposition 2.2 is precisely \(T_{\gamma_{n,n}} = 1\). What is left to do is finding an expression for \(T_{\gamma_{n,n}}\) as a product of Humphrey generators. To achieve this, I need to introduce two more ingredients. The first is the following lemma,

Lemma 2.5. Let \(i, j \in \{1 \ldots n\}\), and, as in Proposition 2.3, set

\[
A_{i,j} = T_{\beta_{i+1}} T_\alpha T_{\beta_i}^{-1} T_{\alpha}^{-1} T_{\beta_{i+1}} T_\alpha T_{\beta_i}^{-1} T_{\alpha}^{-1} T_{\beta_{i+1}} T_\beta_i
\]

then \(T_{\gamma_{i,j}} = A_{i,j} T_{\gamma_{i+1,j}}\).

Proof. Let \(\mathbb{T}_{n-1}\) be the torus with \(n-1\) punctures obtained from \(\mathbb{T}_n\) by filling in the puncture \(p_i\). The Birman exact sequence (see [FM], Theorem 4.6), applied to the inclusion \(\mathbb{T}_n \hookrightarrow \mathbb{T}_{n-1}\), yields

\[
1 \rightarrow \pi_1(\mathbb{T}_{n-1}, p_i) \xrightarrow{\text{Push}} \text{PM(}\mathbb{T}_n\text{)} \xrightarrow{\text{Forget}} \text{PM(}\mathbb{T}_{n-1}\text{)} \rightarrow 1,
\]

where \(\pi_1(\mathbb{T}_{n-1}, p_i)\) is the fundamental group of \(\mathbb{T}_{n-1}\) with base-point \(p_i\). The names attached to the maps above follow the conventions of Chapter 4 in [FM], to which I refer the reader for further details on the Birman exact sequence.

The key point is that \(T_{\gamma_{i,j}} T_{\gamma_{i+1,j}}^{-1}\) lies in the image of the morphism \(\text{Push}\). Figure 2 describes the geometry of two classes of simple closed curves in \(\mathbb{T}_n\), called respectively \(\tilde{\alpha}_m\), and \(\tilde{\beta}_{n,r}\), \(m,n,r \in \{1 \ldots n\}\). It immediately follows from the definition of \(\text{Push}\) that, in PM(\(\mathbb{T}_n\)),

\[
T_{\gamma_{i,j}} T_{\gamma_{i+1,j}}^{-1} = T_{\beta_i} T_{\beta_i+1} T_{\beta_j} T_{\beta_j+1}^{-1}.
\]
It is not hard to express $T_{\tilde{\beta}_{i,j}}$ in terms of Humphrey generators. In fact, by simply applying the definition of Dehn twist, one can verify that $\tilde{\beta}_{i,j} = T_{\tilde{\alpha}_i}T_{\alpha_i}^{-1}(\beta_j)$, and therefore Proposition 2.6 yields, for all $i, j$, $T_{\mu}(i,j) = T_{\mu}T_{\mu}T_{\mu}^{-1}$ (this is Fact 3.7 in [FM]). Thus

$$T_{\tilde{\beta}_{i,j}} = T_{\tilde{\alpha}_i}T_{\alpha_i}^{-1}T_{\beta_j}T_{\alpha_i}^{-1}, \quad \text{and} \quad T_{\tilde{\alpha}_i} = T_{\beta_i}^{-1}T_{\beta_{i+1}}T_{\alpha_i}T_{\beta_i}^{-1}T_{\beta_i}.$$  

Using this last identity, we can rewrite first $T_{\tilde{\beta}_{i,j}}$, and then $T_{\tilde{\gamma}_{i,j}}T_{\tilde{\gamma}_{i+1,j}}^{-1}$, as a product of Humphrey generators, and this completes the proof of Lemma 2.5.

The second ingredient is given by a family of relations in the mapping class group, introduced by Gervais in [G] as star relations.

**Proposition 2.6.** Let $i, j, k \in \{1 \ldots n\}$, and $i \neq j \neq k$. Then

$$(T_{\alpha}T_{\beta_i}T_{\beta_j}T_{\beta_k})^3 = T_{\gamma_i,j}T_{\gamma_j,k}T_{\gamma_k,i}.$$  

**Proof.** See Theorem 1 in [G].

Note that, when $i = j = k$, one obtains the following ‘degenerate’ star relations,

$$(T_{\alpha}T_{\beta_i}T_{\beta_j})^3 = T_{\gamma_{i,i-1}}.$$  

Using the braid relations, the product on the LHS of the equality can be rewritten as $(T_{\alpha}T_{\beta_i})^6$, and therefore Proposition 2.6 yields, for all $i \in \{1, \ldots, n\}$, the identity

$$(T_{\alpha}T_{\beta_i})^6 = T_{\gamma_{i,i-1}}.$$  

Let us fix $i \in \{1 \ldots n\}$, say $i = 1$. Then the degenerate star identity for $i = 1$ combined with an iterated application of Lemma 2.5 (from which we import the notation), gives the formula

$$(T_{\alpha}T_{\beta_1})^6 = (A_{1,n}A_{2,n} \ldots A_{n-1,n})T_{\gamma_{1,n}}.$$  

Since the $A_{i,j}$-s are defined as a product of $T_{\alpha}$ and $T_{\beta_i}$-s, this yields the sought after expression of $T_{\gamma_{1,n}}$ in terms of Humphrey generators, and concludes the proof of Proposition 2.3.

Lemma 2.7 below is the last result of this Section, and describes a family of identities in $\text{PM}(T_{n-1,1})$, which will be useful in Section 3.

**Lemma 2.7.** If $i \in \{1, \ldots, n\}$, then

$$(T_{\alpha}T_{\beta_i})^6(A_{1,n}A_{2,n} \ldots A_{n-1,n})^{-1} = (T_{\alpha}T_{\beta_i})^6(A_{i,i+n-1}A_{i+1,i+n-1} \ldots A_{i+n-2,i+n-1})^{-1}$$

as elements of $\text{PM}(T_{n-1,1})$.

Before proceeding with the proof of Lemma 2.7 a few comments are in order. Note that the $G$-relation of Proposition 2.3 depends on the degenerate star identity for $i = 1$. However, because of the evident cyclic symmetry of the problem, in $\text{PM}(T_{n})$ one would have more generally, for any $i \in \{1 \ldots n\}$, the identity

$$(T_{\alpha}T_{\beta_i})^6 = A_{i,j+n-1}A_{i+1,j+n-1} \ldots A_{i+n-2,j+n-1}.$$  

Note that here, as everywhere in the paper, I am considering curves only up to isotopy.
As a consequence, the following chain of equalities holds in $\text{PM}(\mathbb{T}_n)$,
\[(T_aT_{\beta})^6(A_{1,n}A_{2,n} \ldots A_{n-1,n})^{-1} = (T_aT_{\beta})^6(A_{i,i+n-1}A_{i+1,i+n-1} \ldots A_{i+n-2,i+n-1})^{-1} = 1.
\]
Lemma 2.7 asserts that, in fact, the first of these two equalities can be lifted to $\text{PM}(\mathbb{T}_{n-1,1})$.

**Proof of Lemma 2.7.** Consider the element $G' \in \text{PM}(\mathbb{T}_{n-1,1})$ obtained by multiplying the expression on the LHS of the equality, by the inverse of the expression on the RHS, that is
\[G' = (T_aT_{\beta})^6(A_{i,i+n-1}A_{i+1,i+n-1} \ldots A_{i+n-2,i+n-1})^{-1}((T_aT_{\beta})^6(A_{1,n}A_{2,n} \ldots A_{n-1,n})^{-1})^{-1}.
\]
Also, set $G = (T_aT_{\beta})^6(A_{i,i+n-1}A_{i+1,i+n-1} \ldots A_{i+n-2,i+n-1})^{-1}$. As I pointed out above, the image of $G'$ in $\text{PM}(\mathbb{T}_n) = \text{PM}(\mathbb{T}_{n-1,1})/\langle G \rangle$ is equal to 1. Since $G$ is central it follows that $G'$ must be a power of $G$, that is, in $\text{PM}(\mathbb{T}_{n-1,1})$ $G' = G^n$ for some $n \in \mathbb{Z}$. I will show that $n = 0$. This implies that $G' = 1$ in $\text{PM}(\mathbb{T}_{n-1,1})$, and proves Lemma 2.7.

The identity $G' = G^n$ is equivalent to the following,
\[
(1) \quad (T_aT_{\beta})^6(A_{i,i+n-1}A_{i+1,i+n-1} \ldots A_{i+n-2,i+n-1})^{-1} = ((T_aT_{\beta})^6(A_{1,n}A_{2,n} \ldots A_{n-1,n})^{-1})^n + 1.
\]

Recall that there is a homomorphism $\text{PM}(\mathbb{T}_{n-1,1}) \xrightarrow{\text{Forget}} \text{PM}(\mathbb{T}_{0,1})$ which generalizes the map of the same name appearing in Birman exact sequence (see [FM], Section 9.1 for more details). Since the map $\text{Forget}$ is induced by the inclusion $\mathbb{T}_{n-1,1} \hookrightarrow \mathbb{T}_{0,1}$, and all the $\beta_i$-s have identical isotopy class as subsets of $\mathbb{T}_{0,1}$, we have that for all $i, j \in \{1, \ldots, n\}$,
\[
\text{Forget}(T_{\beta_i}) = \text{Forget}(T_{\beta_j}) = : T_{\beta}, \quad \text{and} \quad \text{Forget}(A_{i,j}) = 1.
\]

Applying $\text{Forget}$ to both sides of equation (1), yields therefore the identity
\[
(T_aT_{\beta})^6 = (T_aT_{\beta})^{6(n+1)}
\]
in $\text{PM}(\mathbb{T}_{0,1})$. As explained by Corollary 7.3 in [FM], there are no torsion elements in the mapping class group of a surface $\Sigma$, provided that its boundary set is non-empty. This is indeed the case of $\mathbb{T}_{0,1}$, and thus $(n + 1)$ must equal 1, as desired. \qed

3. **The action of $\overline{\text{PM}}(\mathbb{T}_n)$ on $D^b\text{Coh}(X_n)$.**

Let $X_n$ be a cycle of $n$ projective lines over a field $\kappa$. That is, $X_n$ is a connected reduced curve with $n$ nodal singularities, such that its normalization $\tilde{X}_n \xrightarrow{\pi} X$ is a disjoint union of $n$ projective lines $D_1, \ldots, D_n$, with the property that the pre-image along $\pi$ of the singular set intersects each $D_i$ in exactly two points. Following the discussion in Section 1 of [ST], the group acting on $D^b\text{Coh}(X_n)$ is going to be a suitable central extension of $\text{PM}(\mathbb{T}_n)$, whose elements should be viewed as **graded** symplectic automorphisms of the mirror of $X_n$, i.e. the torus with $n$ punctures.

**Definition 3.1.** Define $\overline{\text{PM}}(\mathbb{T}_n)$ as the $\mathbb{Z}$-central extension of $\text{PM}(\mathbb{T}_n)$,
\[
0 \rightarrow \mathbb{Z} \rightarrow \overline{\text{PM}}(\mathbb{T}_n) \rightarrow \text{PM}(\mathbb{T}_n) \rightarrow 1,
\]
generated by $T_a, T_{\beta_i}, i \in \{1 \ldots n\}$, and a central element $t$, subject to the following relations:
- Braid relations and Commutativity relations, as in Proposition 2.3
- $(G$-relation) $(T_aT_{\beta})^6(A_{1,n}A_{2,n} \ldots A_{n-1,n})^{-1} = t^2$.

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By $\mathbb{T}_{0,1}$ I mean a symplectic torus with no punctures, and one boundary component.
Remark 3.2. By lifting the $G_2$-relation of Corollary 2.4 to the central extension, one can give an alternative presentation of $\tilde{\text{PM}}(\mathbb{T}_2)$ in which the $G$-relation of Definition 3.1 is replaced by the following,

- $(\tilde{G}_2\text{-relation}) (T_{\beta_1}T_nT_{\beta_2})^4 = t^2$.

The theory of spherical objects was introduced by Seidel and Thomas in [ST]. Given a triangulated category $C$, under mild assumptions, to any object $E$ in $C$ such that $\text{Hom}^*(E, E)$ is isomorphic to the cohomology of the $n$-sphere (i.e. a spherical object), one can associate an autoequivalence, called twist, $T_E : C \to C$.

Let $x_1 \ldots x_n \in X_n$ be (closed) smooth points, such that $x_i$ lies on the $i$-th irreducible component of $X_n$. It is easy to see that the sheaves $\mathcal{O} = \mathcal{O}_{X_n}, \kappa(x_i) i \in \{1 \ldots n\}$ in $D^b(\text{Coh}(X_n))$ are 1-spherical, and therefore determine twist functors $T_{\mathcal{O}}, T_{\kappa(x_i)}$. These equivalences, together with the shift functor, will define the action of $\tilde{\text{PM}}(\mathbb{T}_2)$ on $D^b(\text{Coh}(X_n))$. The main reference for the computations below are [ST] and [BK]. In [BK] the reader can find a detailed treatment of the case $n = 1$, while in [ST] Seidel and Thomas discuss the smooth case, i.e. the action of the mapping class group of a torus with no marked points on the derived category of a smooth elliptic curve.

The following lemma will be extremely useful for computations.

Lemma 3.3. Let $F : D^b(\text{Coh}(X_n)) \to D^b(\text{Coh}(X_n))$ be an auto-equivalence of triangulated categories. If

- $F(\mathcal{O}) \cong \mathcal{O}$, and
- for all $i \in \{1 \ldots n\}$, $F(\kappa(x_i)) \cong \kappa(x_i)$,

then there exists an isomorphisms $f : X_n \to X_n$, such that $F$ is naturally equivalent to $F^* : D^b(\text{Coh}(X_n)) \to D^b(\text{Coh}(X_n))$.

Proof. Note that $X_n$ is projective, as $X_1$ is isomorphic to a nodal cubic curve in $\mathbb{P}^2$, $X_2$ can be embedded as the union of a line and a quadric in $\mathbb{P}^2$, and, if $n \geq 3$, $X_n$ can be embedded as a union of $n$ linear subspaces in $\mathbb{P}^{n-1}$. Consider the line bundle $\mathcal{L} = \mathcal{O}(x_1 + \ldots + x_n)$ over $X_n$. $\mathcal{L}$ is ample (and very ample for $n \geq 3$). Since $F$ preserves $\mathcal{O}$ and $\kappa(x_i)$, it is easy to see that $F(\mathcal{L}^{\otimes m}) \cong \mathcal{L}^{\otimes m}$ for all $m \in \mathbb{Z}$. In fact, $\mathcal{L}^{-1}$ is isomorphic to the kernel of any surjective morphism of sheaves $p : \mathcal{O} \to \bigoplus_{i=1}^{i=n} \kappa(x_i)$. $F(\mathcal{L}^{-1})$ is therefore isomorphic to the (co-)cone of the map

$$F(\mathcal{O})(\cong \mathcal{O}) \xrightarrow{F(p)} F\left(\bigoplus_{i=1}^{i=n} \kappa(x_i)\right) \cong \bigoplus_{i=1}^{i=n} \kappa(x_i),$$

where $F(p)$ must be surjective. It follows that $F(\mathcal{L}^{-1}) \cong \mathcal{L}^{-1}$. Similarly $\mathcal{L}$ is isomorphic to the cone of any morphism in $\text{Hom}^1(\bigoplus_{i=1}^{i=n} \kappa(x_i), \mathcal{O})$ corresponding, under Serre duality, to a surjective morphism $p$ as above, and thus $F(\mathcal{L}) \cong \mathcal{L}$. Analogous arguments can be made for all the tensor powers of $\mathcal{L}$.

From here, in order to prove the claim, is sufficient to mimic the proof of Theorem 3.1 of [BO] (see also Proposition 6.18 in [Ba], in which the argument from [BO] is applied, as here, in the context of singular algebraic varieties). A brief summary of the argument goes as follows. Note first that the functor $F$ induces a graded automorphism of the homogeneous coordinate algebra $\bigoplus_{m=0}^{m=\infty} H^0(\mathcal{L}^{\otimes m})$, which, up to rescaling, must be given by the pull-back along an
automorphism \( f : X_n \to X_n \). Call \( C \) the full linear sub-category of \( D^b(Coh(X_n)) \) having as objects \( \{ \mathcal{L}^{\pm m} \}_{m \in \mathbb{Z}} \). As explained in [BO], one can define a natural equivalence between the restrictions to \( C \) of \( F \) and \( f^* \). Further, since \( X_n \) is projective, and \( \mathcal{L} \) is ample, \( \{ \mathcal{L}^{\pm m} \}_{m \in \mathbb{Z}} \) form an ample sequence in the sense of [BO] (for a proof of this, see Proposition 3.18 of [Huy]). The claim then follows from Proposition A.3 of [BO], which implies that the natural equivalence \( F \cong f^* \) over \( C \) can be extended to the full derived category \( D^b(Coh(X_n)) \). □

**Remark 3.4.** Note that if \( f : X_n \to X_n \) is an automorphism such that \( f(x_i) = x_i \), and \( n \geq 3 \), then \( f \) is the identity. If \( n \leq 2 \), \( f \) may be non-trivial and act as a (non-trivial) permutation on the pre-image of the singular locus in the normalization. However it is immediate to see that \( f \) is an involution, i.e. \( f^2 = Id \).

**Lemma 3.5.** Let \( x \in X_n \) be a smooth point, then

- \( T_{\kappa(x)} \cong - \otimes \mathcal{O}(x) \),
- \( T_{\mathcal{O}(\kappa(x))} \cong \mathcal{O}(-x)[1] \),
- \( T_{\mathcal{O}(\mathcal{O}(x))} \cong \kappa(x) \),
- \( T_{\mathcal{O}} \cong \mathcal{O} \).

**Proof.** The first isomorphism is proved in [ST], Section 3.d. For the other isomorphisms, see Lemma 2.13 in [BK]. □

I am now ready to state the main theorem of this paper.

**Theorem 3.6.** The assignment

- for all \( i \in \{1 \ldots n\} \), \( T_{\beta_i} \mapsto T_{\kappa(x_i)} \),
- \( T_{\alpha} \mapsto T_{\mathcal{O}} \), and
- \( t \mapsto [1] \),

defines a weak action of \( \widetilde{PM}(\mathbb{T}_n) \) on \( D^b(Coh(X_n)) \).

Following [ST], by weak action I mean that this assignment defines a homomorphism between \( \widetilde{PM}(\mathbb{T}_n) \), and the group of autoequivalences of \( D^b(Coh(X_n)) \) modulo natural isomorphism of functors. The action defined in Theorem 3.6 depends on the choice of \( x_1, \ldots, x_n \). However, the action is unique up to conjugation. Note that there is a natural \((\mathbb{C}^*)^n\)-action on \( X_n \), with the property that the \( i \)-th copy of \( \mathbb{C}^* \) acts by multiplication on the \( i \)-th component of \( X_n \). Let \( \lambda = (\lambda_1, \ldots, \lambda_n) \in (\mathbb{C}^*)^n \), and let \( m_\lambda : X_n \to X_n \) be the associated automorphism. Then one can show that, for all \( i \in \{1, \ldots, n\} \), \( (m_\lambda^*)T_{\kappa(x_i)}(m_\lambda^*)^{-1} = T_{k(x_i)} \), and \( (m_\lambda^*)T_{\kappa(x_i)}(m_\lambda^*)^{-1} = T_{\mathcal{O}} \).

**Proof of Theorem 3.6.** I will show that Theorem 3.6 gives a well-defined homomorphism, by checking that the relations in Definiton 3.1 hold.

**Braid relations.** For all \( i, j \in \{1, \ldots, n\} \), \( \mathcal{O}, \kappa(x_i) \) and \( \kappa(x_j) \) form an \( A_2 \)-configuration, in the language of [ST]. The fact that such a collection of spherical twists satisfies the braid relations is proved in Proposition 2.13 of [ST].

**Commutativity relations.** By Lemma 2.11 of [ST], if \( E_1, E_2 \) are spherical objects, then \( T_{E_2}T_{E_1}^{-1} \cong T_{E_2}(E_1) \). It follows that, in order to prove the Commutativity relations, is sufficient to show that, for every \( i, j, k \in \{1 \ldots n\} \), \( i < j < k \),

\[
T_{\mathcal{O}}^{-1}T_{\kappa(x_{k+1})}^{-1}(T_{\kappa(x_k)}T_{\mathcal{O}})T_{\kappa(x_j)}T_{\kappa(x_{k+1})}T_{\mathcal{O}}(\kappa(x_i)) \cong \kappa(x_i) \]
Lemma 3.5, it is easy to check that

$$T_{\mathcal{O}}^{-1}T_{\kappa(x_{i+1})}^{-1}T_{\kappa(x_{j})}^{-1}(T_{\kappa(x_{k})}T_{\mathcal{O}}^{-1}T_{\kappa(x_{k})})T_{\kappa(x_{j})}T_{\kappa(x_{k+1})}T_{\mathcal{O}}(\kappa(x_{i})) \cong \kappa(x_{i}) \iff$$

$$T_{\mathcal{O}}T_{\kappa(x_{k})}T_{\kappa(x_{j})}T_{\kappa(x_{k+1})}T_{\mathcal{O}}(\kappa(x_{i})) \cong T_{\kappa(x_{k})}T_{\kappa(x_{j})}T_{\kappa(x_{k+1})}T_{\mathcal{O}}(\kappa(x_{i})).$$

By Lemma 3.5, $T_{\kappa(x_{k})}T_{\kappa(x_{j})}T_{\kappa(x_{k+1})}T_{\mathcal{O}}(\kappa(x_{i})) \cong \mathcal{O}(-x_{i}+x_{j}-x_{k}+x_{k+1})[1]$. Thus, I need to show that $T_{\mathcal{O}}(\mathcal{O}(-x_{i}+x_{j}-x_{k}+x_{k+1})) \cong \mathcal{O}(-x_{i}+x_{j}-x_{k}+x_{k+1})$. Proposition 2.12 of [ST] states that if $\mathcal{E}_{1}$, $\mathcal{E}_{2}$ are spherical objects such that $\text{Hom}^i(\mathcal{E}_{1}, \mathcal{E}_{2}) = 0$ for all $i$, then $T_{\mathcal{E}_{1}}(\mathcal{E}_{2}) \cong \mathcal{E}_{2}$. The Commutativity relations reduce therefore to the claim: for all $i, j, k \in \{1 \ldots n\}$, $i < j < k$,

$$H^0(\mathcal{O}(-x_{i}+x_{j}-x_{k}+x_{k+1})) = H^1(\mathcal{O}(-x_{i}+x_{j}-x_{k}+x_{k+1})) = 0.$$

This follows from Theorem 2.2 of [DGK], which gives a general formula for computing the cohomology groups of indecomposable vector bundles over a cycle of projective lines.

$\mathcal{G}$-relation. Assume first that $n \geq 3$. I will handle separately the case $n = 2$, for which I will use the alternative $\tilde{\mathcal{G}}$-relation of Remark 3.2 (for the case $n = 1$, the reader should refer to [BK]). Let $i, j \in \{1 \ldots n\}$, and define

$$E_{i,j} = T_{\kappa(x_{j+1})}T_{\mathcal{O}}^{-1}T_{\kappa(x_{j+1})}T_{\kappa(x_{i})}T_{\mathcal{O}}^{-1}T_{\kappa(x_{j+1})}T_{\kappa(x_{i})}T_{\mathcal{O}}^{-1}T_{\kappa(x_{j+1})}T_{\kappa(x_{i})}.$$

I need to prove that $(T_{\mathcal{O}}T_{\kappa(x_{i})})^6 \cong (E_{1, n}E_{2, n} \ldots E_{n-1, n})[2]$. After Lemma 3.3 and Remark 3.4, it is sufficient to check the $\tilde{\mathcal{G}}$-relation on $\mathcal{O}$, and $\kappa(x_{i})$ for all $i \in \{1 \ldots n\}$. In fact, in view of Lemma 2.7, it is enough to evaluate the $\tilde{\mathcal{G}}$-relation on $\mathcal{O}$ and $\kappa(x_{i})$, as, for any $k \in \{1 \ldots n\}$,

$$(T_{\mathcal{O}}T_{\kappa(x_{i})})^6(\kappa(x_{k})) \cong (E_{1,n}E_{2,n} \ldots E_{n-1,n})[2](\kappa(x_{k})) \iff$$

$$(T_{\mathcal{O}}T_{\kappa(x_{i})})^6(\kappa(x_{k})) \cong (E_{i,i+n-1}E_{i+1,i+n-1} \ldots E_{i+n-2,i+n-1})[2](\kappa(x_{k})),$$

and, by the cyclic symmetry of the problem, the latter identity is proved in exactly the same way as the claim that the $\tilde{\mathcal{G}}$-relation holds for $\kappa(x_{i})$.

- $\tilde{\mathcal{G}}$-relation on $\mathcal{O}$. Simply by keeping track of the isomorphisms collected in Lemma 3.5, one can see that

$$(T_{\mathcal{O}}T_{\kappa(x_{i})})^6(\mathcal{O}) \cong (T_{\mathcal{O}}T_{\kappa(x_{i})})^4(\mathcal{O}(-x_{1})[1]) \cong (T_{\mathcal{O}}T_{\kappa(x_{i})})^2(\mathcal{O}(x_{1})[1]) \cong \mathcal{O}[2].$$

On the other hand, I will show that, for all $i, j \in \{1, \ldots, n\}$, $E_{i,j}(\mathcal{O}) \cong \mathcal{O}$, and therefore

$$(E_{1,n}E_{2,n} \ldots E_{n-1,n})[2](\mathcal{O}) \cong \mathcal{O}[2],$$

as expected. Note first that if $x, y \in X_{n}$ are smooth points lying on different connected components, then $T_{\mathcal{O}}(\mathcal{O}(x-y)) \cong \mathcal{O}(x-y)$. This again follows from Theorem 2.2 of [DGK], but can also be checked directly using the braid relations. Using this isomorphism, and Lemma 3.5, it is easy to check that

$$E_{i,j}(\mathcal{O}) = T_{\kappa(x_{j+1})}T_{\mathcal{O}}^{-1}T_{\kappa(x_{j+1})}T_{\kappa(x_{i})}T_{\mathcal{O}}^{-1}T_{\kappa(x_{i+1})}T_{\kappa(x_{i})}T_{\mathcal{O}}^{-1}T_{\kappa(x_{i+1})}T_{\kappa(x_{i})}(\mathcal{O}) \cong$$

$$T_{\kappa(x_{j+1})}T_{\mathcal{O}}^{-1}T_{\kappa(x_{j+1})}T_{\kappa(x_{i})}T_{\mathcal{O}}^{-1}T_{\kappa(x_{i+1})}T_{\kappa(x_{i})}(\mathcal{O}(x_{i}-x_{i+1})) \cong$$

$$T_{\kappa(x_{j+1})}T_{\mathcal{O}}^{-1}T_{\kappa(x_{j+1})}T_{\kappa(x_{i})}T_{\mathcal{O}}^{-1}T_{\kappa(x_{j+1})}T_{\kappa(x_{j+1})}[-1] \cong \mathcal{O}.$$

\(^{5}\)The isomorphism $T_{\mathcal{O}}^{-1}T_{\kappa(x_{k})}T_{\mathcal{O}} \cong T_{\kappa(x_{k})}T_{\mathcal{O}}^{-1}T_{\kappa(x_{k})}$ follows immediately from the braid relations.
Let us check this for \( \kappa(1) \). As before, it is enough to apply Lemma 3.5 to see that
\[
(T_o T_{\kappa(1)})^6(\kappa(1)) \cong (T_o T_{\kappa(1)})^4(\mathcal{O}[1]) \cong (T_o T_{\kappa(1)})^2(\mathcal{O}[-x_1][2]) \cong \kappa(1)[2].
\]
Further, for all \( i \in \{1, \ldots, n - 1\} \), \( E_{i,n}(\kappa(1)) \cong \kappa(1) \). In fact,
\[
E_{i,n}(\kappa(1)) = T_{\kappa(x_i)}T_{\kappa(x_{i+1})}T_{\kappa(x_i)}T_{\mathcal{O}}^{-1}T_{\kappa(x_i)}T_{\kappa(x_{i+1})}T_{\kappa(x_i)}T_{\mathcal{O}}^{-1}T_{\kappa(x_i)}(\kappa(1)) \cong T_{\kappa(x_i)}T_{\mathcal{O}}^{-1}T_{\kappa(x_i)}T_{\kappa(x_{i+1})}T_{\kappa(x_i)}(\kappa(1)) \cong T_{\kappa(x_i)}T_{\mathcal{O}}^{-1}T_{\kappa(x_i)}T_{\kappa(x_{i+1})}(\mathcal{O}(x_1)).
\]
Now,
\[
T_{\kappa(x_i)}T_{\mathcal{O}}^{-1}T_{\kappa(x_{i+1})}T_{\kappa(x_i)}T_{\mathcal{O}}^{-1}T_{\kappa(x_i)}T_{\kappa(x_{i+1})}(\mathcal{O}(x_1)) \cong \kappa(1) \Leftrightarrow
T_{\kappa(x_{i+1})}T_{\kappa(x_i)}(T_{\mathcal{O}}^{-1}T_{\kappa(x_i)}T_{\mathcal{O}})^{-1}T_{\kappa(x_i)}(\mathcal{O}(x_1)) \cong \mathcal{O}(x_1) \Leftrightarrow
T_{\kappa(x_{i+1})}T_{\kappa(x_i)}(T_{\mathcal{O}}^{-1}T_{\kappa(x_i)}T_{\mathcal{O}})^{-1}T_{\kappa(x_i)}T_{\kappa(x_{i+1})}(\mathcal{O}(x_1)) \cong \mathcal{O}(x_1) \Leftrightarrow
T_{\mathcal{O}}^{-1}T_{\kappa(x_i)}T_{\kappa(x_{i+1})}(\mathcal{O}(x_1)) \cong \mathcal{O}(x_{i+1} - x_i) \Leftrightarrow
\mathcal{O}(x_{i+1} - x_i) \cong T_{\mathcal{O}}(\mathcal{O}(x_{i+1} - x_i)).
\]
As I pointed out above, this last isomorphism can be proved using the braid relations. Thus
\[
(E_{1,n}E_{2,n} \ldots E_{n-1,n})(\kappa(1)) \cong \kappa(1)[2],
\]
and this concludes the proof of Theorem 3.6 for the case \( n \geq 3 \).

The case \( n = 2 \). Note that there are isomorphisms
\[
\bullet \quad (T_{\kappa(x_1)}T_{\mathcal{O}}T_{\kappa(x_2)})^2(\mathcal{O}) \cong \mathcal{O}[1], \text{ and }
\bullet \quad (T_{\kappa(x_1)}T_{\mathcal{O}}T_{\kappa(x_2)})^2(\kappa(x_1)) \cong \kappa(x_2)[1], \quad (T_{\kappa(x_1)}T_{\mathcal{O}}T_{\kappa(x_2)})^2(\kappa(x_2)) \cong \kappa(x_1)[1].
\]
Let us check this for \( \kappa(x_1) \):
\[
(T_{\kappa(x_1)}T_{\mathcal{O}}T_{\kappa(x_2)})(T_{\kappa(x_1)}T_{\mathcal{O}}T_{\kappa(x_2)})(\kappa(x_1)) \cong (T_{\kappa(x_1)}T_{\mathcal{O}}T_{\kappa(x_2)})(\mathcal{O}[1]) \cong \kappa(x_2)[1].
\]
Consider an involution \( \sigma : X_2 \to X_2 \) such that \( \sigma(x_1) = x_2 \), and \( \sigma(x_2) = x_1 \). It follows from Remark 3.4 that there is an isomorphism \( f : X_2 \to X_2 \), and a natural equivalence \( (T_{\kappa(x_1)}T_{\mathcal{O}}T_{\kappa(x_2)})^2 \cong f^*\sigma^*[1] \). As \( \sigma \) and \( f \) commute, by taking the square of this natural equivalence, one gets
\[
(T_{\kappa(x_1)}T_{\mathcal{O}}T_{\kappa(x_2)})^4 \cong (f^*\sigma^*[1])(f^*\sigma^*[1]) \cong (f^*)^2(\sigma^*)[2] \cong [2].
\]
In view of Remark 3.2, this implies that the action of \( \widetilde{PM}(\mathbb{T}_2) \) on \( D^b(Coh(X_2)) \) is well defined, and proves the case \( n = 2 \) of Theorem 3.6 \( \square \).

Remark 3.7. It follows from results in Appendix D of [3], that the action defined by Theorem 3.6 is, in an appropriate sense, a categorification of the symplectic representation of the mapping class group. Denote \( K^{num}(X_n) \) the quotient of \( K_0(\text{Perf}(X_n)) \) by the radical of the Euler form (see Appendix D of [3] for further details). The Euler form induces a non-degenerate skew-symmetric form on \( K^{num}(X_n) \), and there is an isomorphism of symplectic lattices \( K^{num}(X_n) \cong H_1(\mathbb{T}_n, \mathbb{Z})(\cong \mathbb{Z}^{n+1}) \) (here, \( \mathbb{T}_n \) denotes the torus with the \( n \) marked points removed). Note that the induced action of \( \widetilde{PM}(\mathbb{T}_n) \) on \( K^{num}(X_n) \) factors through \( PM(\mathbb{T}_n) \oplus \mathbb{Z}_2 \). Bodnarchuk’s computations imply that the resulting action of \( \sigma \).
PM($\mathbb{T}_n$) on $K^{num}(X_n)$ is isomorphic to the standard symplectic representation of PM($\mathbb{T}_n$) over $H_1(\mathbb{T}_n, \mathbb{Z})$.

**References**

[B] L. Bodnarchuk, “Vector bundles on degenerations of elliptic curves of type II, III and IV,” available at https://kluedo.ub.uni-kl.de/files/2039/diss.pdf

[Ba] M. Ballard, “Derived categories of sheaves on singular schemes with an application to reconstruction,” Advances in Math. 227, issue 2, 895–919.

[Ba1] M. Ballard, “Equivalences of derived categories of sheaves on quasi-projective schemes,” arXiv:0905.3148

[BO] A. Bondal, D. Orlov, “Reconstruction of a variety from the derived category and groups of autoequivalences,” Compositio Math. 125 (2001), 327–344.

[BK] I. Burban, B. Kreussler, “Fourier-Mukai transforms and semi-stable sheaves on nodal Weierstrass cubics,” J. Reine Angew. Math. 584 (2005) 45–82.

[DGK] Y. A. Drozd, G.-M. Greuel, I. Kashuba, “On Cohen-Macaulay modules over surface singularities,” Moscow Math. J. 3 (2003), 397–418.

[FM] B. Farb, D. Margalit, “A primer on mapping class groups,” PMS 50 Princeton University Press, 2011.

[G] S. Gervais, “A finite presentation of the mapping class group of a punctured surface,” Topology 40 (2001), 703–725.

[Huy] D. Huybrechts, “Fourier-Mukai transforms in algebraic geometry,” Oxford Mathematical Monographs, Clarendon Press, Oxford, 2006.

[K] M. Kontsevich, “Homological algebra of mirror symmetry”, Proceedings of the International Congress of Mathematicians (Zürich, 1994), 1995, 120–139.

[LK] Y. Lekili, T. Perutz, “Fukaya categories of the torus and Dehn surgeries,” arXiv:1102.3160v2.

[LP] C. Labruère, L. Paris, “Presentations for the punctured mapping class groups in terms of Artin groups”, Algebr. Geom. Top. 1 (2001), 73–114.

[L] P. E. Lowrey, “Interactions between autoequivalences, stability conditions, and moduli problems,” arXiv:0905.1731v2.

[PS] J. R. Parker, C. Series “The mapping class group of the twice punctured torus,” in Groups. Topological, Combinatorial and Geometric Aspects, London Mathematical Society Lecture Note Series, no. 311, 2004, 405–486.

[S] P. Seidel, “Graded Lagrangian Submanifolds,” Bull. Soc. Math. France 128 (2000) 103–149.

[ST] P. Seidel, R. Thomas, “Braid group actions on derived categories of coherent sheaves,” Duke Math. J. 108(1) (2001) 37–108.

[STZ] N. Sibilla, D. Treumann, E. Zaslow, “Ribbon Graphs and Mirror Symmetry I,” arXiv:1103.2462

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