Fourth order weighted elliptic problem under exponential nonlinear growth

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Abstract

We deal with nonlinear weighted biharmonic problem in the unit ball of $\mathbb{R}^4$. The weight is of logarithm type. The nonlinearity is critical in view of Adam’s inequalities in the weighted Sobolev space $W^{2,2}_0(B, w)$. We prove the existence of non trivial solutions via the critical point theory. The main difficulty is the loss of compactness due to the critical exponential growth of the nonlinear term $f$. We give a new growth condition and we point out its importance for checking the Palais-Smale compactness condition.

Keywords: Adam’s inequality, Moser-Trudinger’s inequality, Nonlinearity of exponential growth, Mountain pass method, Compactness level.

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1 Introduction and Main results

In this paper, we consider the following elliptic nonlinear problem:

$$\begin{cases}
L(u) := \Delta(w(x))\Delta u = f(x, u) \quad \text{in} \quad B, \\
u = \frac{\partial u}{\partial n} = 0 \quad \text{on} \quad \partial B,
\end{cases}$$

where $B = B(0, 1)$ is the unit open ball in $\mathbb{R}^4$. The weight is given by

$$w(x) = (\log \frac{e}{|x|})^\beta, \beta \in (0, 1),$$

The nonlinearity $f(x, t)$ is continuous in $B \times \mathbb{R}$ and behaves like $\exp\{\alpha t^{\frac{2}{1-\beta}}\}$ as $t \to +\infty$, for some $\alpha > 0$ and where $\frac{\partial u}{\partial n}$ denotes the outer normal derivative of $u$ on $\partial B$.

Problems of critical exponential growth in second order elliptic equations in dimension $N = 2$

$$-\Delta u = f(x, u) \quad \text{in} \quad \Omega \subset \mathbb{R}^2.$$
where $\alpha_N = \omega_{N-1}^{\frac{1}{N}}$ with $\omega_{N-1}$ is the area of the unit sphere $S^{N-1}$ in $\mathbb{R}^N$.

Later, the Trudinger-Moser inequality was improved to weighted inequalities [9, 10]. The influence of the weight in the Sobolev norm was studied as the compact embedding [21].

When the weight is of logarithmic type, Calanchi and Ruf [11] extend the Trudinger-Moser inequality and proved the following results in the weighted Sobolev space for radial functions

$$W^{1,N}_{0,rad}(B, \rho) = \overline{\text{cl}} \{ u \in C^\infty_{0,rad}(B) \mid \int_B |\nabla u|^N \rho(x) dx < \infty \} :$$

**Theorem 1.1.** [10]

(i) Let $\beta \in [0,1)$ and let $\rho$ given by $\rho(x) = (\log \frac{1}{|x|})^\beta$, then

$$\int_B e^{\beta |u|^N} dx < +\infty, \quad \forall \ u \in W^{1,N}_{0,rad}(B, \rho), \text{ if and only if } \gamma \leq \gamma_{N,\beta} = \frac{N}{(N-1)(1-\beta)} = \frac{N'}{1-\beta}$$

and

$$\sup_{u \in W^{1,N}_{0,rad}(B, \rho)} \int_B e^{\alpha |u|^N} dx < +\infty \quad \Leftrightarrow \quad \alpha \leq \alpha_{N,\beta} = N\left[\omega_{N-1}^{\frac{1}{N}}(1-\beta)\right]^\frac{1}{N-1}$$

where $\omega_{N-1}$ is the area of the unit sphere $S^{N-1}$ in $\mathbb{R}^N$ and $N'$ is the Hölder conjugate of $N$.

(ii) Let $\rho$ given by $\rho(x) = (\log \frac{1}{|x|})^{N-1}$, then

$$\int_B e^{\gamma |u|^N} dx < +\infty, \quad \forall \ u \in W^{1,N}_{0,rad}(B, \rho)$$

and

$$\sup_{u \in W^{1,N}_{0,rad}(B, \rho), \|u\| \leq 1} \int_B e^{\beta |u|^N} dx < +\infty \quad \Leftrightarrow \quad \beta \leq N,$$

where $\omega_{N-1}$ is the area of the unit sphere $S^{N-1}$ in $\mathbb{R}^N$ and $N'$ is the Hölder conjugate of $N$.

These results opened the way to study second order weighted elliptic problems in dimension $N \geq 2$. We cite the work of Calanchi et al [12].

\[ \begin{align*}
&\text{with the weight } \nu(x) = \log(\frac{1}{|x|}) \text{ and where the function } f(x, t) \text{ is continuous in } B \times \mathbb{R} \text{ and behaves like } e^{\alpha t^2} \text{ as } t \to +\infty, \text{ for some } \alpha > 0. \\
&\text{Also, recently, Deng et al [15] and Zhang [35] studied the following problem}
\end{align*} \]
where \( N \geq 2 \), the function \( f(x, t) \) is continuous in \( B \times \mathbb{R} \) and behaves like \( \exp\{e^{\alpha t} N - 1\} \) as \( t \to +\infty \), for some \( \alpha > 0 \). The authors proved that there is a non-trivial solution to this problem using Mountain Pass theorem.

Also, we mention that Baraket et al [6] studied the following non-autonomous weighted elliptic equations

\[
\begin{aligned}
L := -\text{div}(\rho(x) |\nabla u|^{N-2} \nabla u) + \xi(x)|u|^{N-2} u &= f(x, u) \quad \text{in} \quad B \\
u > 0 & \quad \text{in} \quad B \\
u = 0 & \quad \text{on} \quad \partial B,
\end{aligned}
\]

where \( B \) is the unit ball of \( \mathbb{R}^N \), \( N > 2 \), \( f(x, t) \) is continuous in \( B \times \mathbb{R} \) and behaves like \( \exp\{e^{\alpha t} N - 1\} \) as \( t \to +\infty \), for some \( \alpha > 0 \). The weight \( \rho(x) \) is given by \( \rho(x) = \left( \log \frac{e}{|x|} \right)^{N-1} \).

The biharmonic equation in dimension \( N > 4 \)

\[
\Delta^2 u = f(x, u) \quad \text{in} \quad \Omega \subset \mathbb{R}^N,
\]

where the nonlinearity \( f \) has subcritical and critical polynomial growth of power less than \( \frac{N+4}{N-4} \), have been extensively studied [7, 17, 20, 31].

For bounded domains \( \Omega \subset \mathbb{R}^4 \), in [1, 29] the authors proved the following Adams’ inequality

\[
\sup_{u \in S} \int_{\Omega} (e^{\alpha u} - 1) dx < +\infty \quad \iff \quad \alpha \leq 32\pi^2
\]

where

\[
S = \{ u \in W_0^{2,2}(\Omega) \mid \left( \int_{\Omega} |\Delta u|^2 w(x) dx \right)^{\frac{1}{2}} \leq 1 \}.
\]

This last result opened the way to study fourth-order problems with subcritical or critical nonlinearity (see [20], [13]).

We study the existence of the non-trivial solutions when the nonlinear terms have the critical exponential growth in the sense of Adams inequalities [34]. Our approach is variational methods such as the Mountain Pass Theorem with Palais-Smale condition combining with a concentration compactness result.

More precisely, let \( \Omega \subset \mathbb{R}^2 \) be a bounded domain and \( w \in L^1(\Omega) \) be a nonnegative function, the weighted sobolev space is defined as \( W_0^{2,2}(\Omega, w) = \text{cl}\{ u \in C_0^\infty(\Omega) \mid \int_{\Omega} |\Delta u|^2 w(x) dx < \infty \} \). We will restrict our attention to radial functions and then consider the subspace

\[
W = W_0^{2,2}(B, w) = \text{cl}\{ u \in C_0^\infty(B) \mid \int_{B} |\Delta u|^2 w(x) dx < \infty \},
\]

equipped with norm

\[
\|u\| = \left( \int_{B} |\Delta u|^2 w(x) dx \right)^{\frac{1}{2}}, \quad w(x) = \left( \log \frac{e}{|x|} \right)^{\beta},
\]

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which comes from the scalar product

\[ <u, v> = \int_B \Delta u \Delta v (\log \frac{e}{|x|})^{\beta} \, dx. \]

The norm \( \|u\| = \left( \int_B |\Delta u|^2 w(x) \, dx \right)^{\frac{1}{2}} \), and

\[ \|u\|_{W^{2,2}}(B, w) = \left( \int_B u^2 \, dx + \int_B |\nabla u|^2 \, dx + \int_B |\Delta u|^2 w(x) \, dx \right)^{\frac{1}{2}}, \]

are equivalent (see Lemma 1). The choice of the weight and the space \( W^{2,2}_{0,\text{rad}}(B, w) \) are motivated by the following inequality of Adam’s type.

**Theorem 1.2.** [34] Let \( \beta \in (0, 1) \) and let \( w \) given by (1.2), then

\[ \sup_{u \in W^{2,2}_{0,\text{rad}}(B, w)} \int_B e^{\alpha |u|^2 \gamma} \, dx < +\infty \iff \alpha \leq \alpha_0 = 4 \frac{8\pi}{2(1-\beta)} \]

(1.4)

Let \( \gamma := \frac{2}{1-\beta} \). In view of inequality (1.4), we say that \( f \) has critical growth at \( +\infty \) if there exists some \( \alpha_0 > 0 \),

\[ \lim_{s \to +\infty} \frac{|f(x, s)|}{e^{\alpha s^\gamma}} = 0, \quad \forall \alpha \text{ such that } \alpha > \alpha_0 \quad \text{and} \quad \lim_{s \to +\infty} \frac{|f(x, s)|}{e^{\alpha s^\gamma}} = +\infty, \quad \forall \alpha < \alpha_0. \]

(1.5)

Let us now state our results. We suppose that \( f(x, t) \) satisfies the following hypothesis:

\( (H_1) \) \( f : B \times \mathbb{R} \to \mathbb{R} \) is continuous, positive, radial in \( x \) and \( f(x, t) = 0 \) for \( t \leq 0 \);

\( (H_2) \) There exists \( t_0 > 0 \) and \( M > 0 \) such that for all \( t > t_0 \) and for all \( x \in B \) we have

\[ 0 < F(x, t) \leq Mt(x, t), \]

where

\[ F(x, t) = \int_0^t f(x, s) \, ds; \]

\( (H_3) \) \( 0 < F(x, t) \leq \frac{1}{2} f(x, t) t, \quad \forall t > 0, \forall x \in B. \)

\( (H_4) \) \( \limsup_{t \to 0} \frac{2F(x, t)}{t^2} < \lambda_1 \) uniformly in \( x \),

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We denote by
\[ \lambda_1 = \inf_{u \in W \setminus \{0\}} \frac{\int_B |\nabla u|^2 w(x) dx}{\int_B |u|^2 dx}, \]
the first eigenvalue of \((L, W)\). It is well known that \(\lambda_1\) is isolated simple positive eigenvalue and has a positive bounded associated eigenfunction, \([16]\).

We say that \(u\) is a solution to the problem (1.1), if \(u\) is a weak solution in the following sense:

**Definition 1.1.** We say that a function \(u \in W\) is a solution of the problem (1.1) if
\[ \int_B \nabla u \cdot \nabla \varphi w(x) dx = \int_B f(x, u) \varphi dx, \quad \forall \varphi \in W. \]

Let \(J : W \to \mathbb{R}\) be the functional given by
\[ J(u) = \frac{1}{2} \int_B |\nabla u|^2 w(x) dx - \int_B F(x, u) dx, \tag{1.6} \]
where
\[ F(x, t) = \int_0^t f(x, s) ds. \]

It is well-known that seeking a weak solution of (1.1) is equivalent to finding a nonzero critical point of \(J\).

Our Euler–Lagrange functional does not satisfy the Palais–Smale condition at all level anymore. To overcome the verification of compactness of Euler–Lagrange functional at some suitable level, we construct Adams type functions, which are extremal to the inequality (1.4). Our result is as follows:

**Theorem 1.3.** Assume that \(f(x, t)\) has a critical growth at \(+\infty\) for some \(\alpha_0\) and satisfies the conditions \((H_1), (H_2), (H_3)\) and \((H_4)\). If in addition \(f(x, t)\) satisfies the asymptotic condition
\[ (H_5) \quad \lim_{t \to +\infty} \frac{f(x, t)}{t^{\alpha_0(1-\beta)}} \geq \gamma_0 \text{ uniformly in } x, \quad \text{with} \quad \gamma_0 > \frac{1024(1-\beta)}{\alpha_0^{1-\beta}}, \]
then the problem (1.1) has a nontrivial solution.

In general the study of fourth order partial differential equations is considered an interesting topic. The interest in studying such equations was stimulated by their applications in micro-electro-mechanical systems, phase field models of multi-phase systems, thin film theory, surface diffusion on solids, interface dynamics, flow in Hele-Shaw cells, see \([14, 18, 27]\). However many applications are generated by the weighted elliptic problems, such as the study of traveling waves in suspension bridges, radar imaging (see, for example \([4, 23]\)).

The main reason for this study is that, to our knowledge, there are few research taking into account both this type of non-linearity for a non-linear fourth order elliptic equation in the framework of Sobolev weighted spaces.
This paper is organized as follows:
In Section 2, we present some necessary preliminary knowledge about working space, and we prove that the energy $J$ satisfied the two geometric properties. Section 3 is devoted for the compactness analysis. More precisely, we prove a concentration compactness result of Lions type and identify the first compactness level of the energy $J$. Finally, we fulfill the proof of the main results in section 4.
In this work, the constant C may change from line to another and sometimes we index the constants in order to show how they change.

2 Functional setting and Variational formulation

2.1 Functional setting

Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$, be a bounded domain in $\mathbb{R}^N$ and let $w \in L^1(\Omega)$ be a nonnegative function. To deal with weighted operator, we need to introduce some functional spaces $L^p(\Omega, w)$, $W^{m,p}(\Omega, w)$, $W_0^{m,p}(\Omega, w)$ and some of their properties that will be used later. Let $S(\Omega)$ be the set of all measurable real-valued functions defined on $\Omega$ and two measurable functions are considered as the same element if they are equal almost everywhere.

Following Drabek et al. and Kufner in [16], the weighted Lebesgue space $L^p(\Omega, w)$ is defined as follows:

$$L^p(\Omega, w) = \{u : \Omega \to \mathbb{R} \text{ measurable; } \int_\Omega w(x)|u|^p \, dx < \infty\}$$

for any real number $1 \leq p < \infty$.

This is a normed vector space equipped with the norm

$$\|u\|_{p,w} = \left( \int_\Omega w(x)|u|^p \, dx \right)^{\frac{1}{p}}.$$

For $m \geq 2$, let $w$ be a given family of weight functions $w_\tau$, $|\tau| \leq m$, $w = \{w_\tau(x) : x \in \Omega, \ |\tau| \leq m\}$.

In [16], the corresponding weighted Sobolev space was defined as

$$W^{m,p}(\Omega, w) = \{u \in L^p(\Omega), D^\tau u \in L^p(\Omega) \ \forall \ 1 \leq |\tau| \leq m - 1, D^\tau u \in L^p(\Omega, w) \ \forall \ |\tau| = m\}$$

endowed with the following norm:

$$\|u\|_{W^{m,p}(\Omega, w)} = \left( \sum_{|\tau| \leq m-1} \int_\Omega |D^\tau u|^p \, dx + \sum_{|\tau| = m} \int_\Omega |D^\tau u|^p \omega(x) \, dx \right)^{\frac{1}{p}},$$

where $w_\tau = 1$ for all $|\tau| < k$, $w_\tau = \omega$ for all $|\tau| = k$. 
If we suppose also that \( w(x) \in L^1_{\text{loc}}(\Omega) \), then \( C_0^\infty(\Omega) \) is a subset of \( W^{m,p}(\Omega, w) \) and we can introduce the space
\[
W_0^{m,p}(\Omega, w)
\]
as the closure of \( C_0^\infty(\Omega) \) in \( W^{m,p}(\Omega, w) \).

\( (L^p(\Omega, w), \| \cdot \|_{p,w}) \) and \( (W^{m,p}(\Omega, w), \| \cdot \|_{W^{m,p}(\Omega, w)}) \) are separable, reflexive Banach spaces provided that \( w(x) \frac{1}{p-1} \in L^1_{\text{loc}}(\Omega) \).

For \( w(x) = 1 \), one finds the standard Sobolev spaces \( W^{m,p}(\Omega) \), \( W_0^{m,p}(\Omega) \) and the Lebesgue spaces \( L^p(\Omega) \).

Our space setting is
\[
\mathbf{W} = \{ u \in W^{2,2}_{0,\text{rad}}(B, w) \mid \int_B |\triangle u|^2 w(x) dx < \infty \}.
\]
\( \mathbf{W} \) is equipped with norm
\[
\| u \| = \left( \int_B |\triangle u|^2 w(x) dx \right)^{\frac{1}{2}},
\]
which comes from the scalar product
\[
< u, v > = \int_B \Delta u \cdot \Delta v \left( \log \frac{e}{|x|} \right)^{\beta} dx.
\]

We have the following result:

**Lemma 1.** \( (\mathbf{W}, \| \cdot \|_{W^{2,2}_{0,\text{rad}}(B, w)}) \) is a Banach space and the norm \( \| \cdot \| \) is equivalent in \( \mathbf{W} \) to the norm \( \| \cdot \|_{W^{2,2}_{0,\text{rad}}(B, w)} \).

**Proof.** The Sobolev weighted space \( (\mathbf{W}, \| \cdot \|_{W^{2,2}_{0,\text{rad}}(B, w)}) \) is a normed linear space. In order to prove that it is a Banach space, let \( \{ u_n \} \) be a Cauchy sequence that
\[
\| u_n - u_m \|_{W^{2,2}_{0,\text{rad}}(B, w)} \to 0 \quad \text{as} \quad n, m \to +\infty.
\]

Therefore \( \{ u_n \} \) is also a Cauchy sequence in \( (W^{2,2}_{0,\text{rad}}(B, w), \| \cdot \|_{W^{2,2}_{0,\text{rad}}(B, w)}) \).

By the completeness of the last space, there exists \( u \in W^{2,2}_{0,\text{rad}}(B, w) \) such that
\[
\| u_n - u \|_{W^{2,2}_{0,\text{rad}}(B, w)} \to 0 \quad \text{as} \quad n \to +\infty. \tag{2.1}
\]

Since \( \| u \|_{W^{1,2}_{0,\text{rad}}(B)}^2 = \int_B |\nabla u|^2 dx \), then
\[
\| u \|_{W^{1,2}_{0,\text{rad}}(B)} \leq \| u \|_{W^{2,2}_{0,\text{rad}}(B, w)},
\]
for all \( u \in \mathbf{W} \), the sequence \( \{ u_n \} \) is also a Cauchy sequence in \( (W^{1,2}_{0,\text{rad}}(B), \| \cdot \|_{W^{1,2}_{0,\text{rad}}(B)}) \). By the completeness of \( (W^{1,2}_{0,\text{rad}}(B), \| \cdot \|_{W^{1,2}_{0,\text{rad}}(B)}) \) there exists \( v \in W^{1,2}_{0,\text{rad}}(B) \) such that
\[
\| u_n - v \|_{W^{1,2}_{0,\text{rad}}(B)} \to 0 \quad \text{as} \quad n \to +\infty. \tag{2.2}
\]
Since \( u \in W^{2,2}_{0, rad}(B, w) \), \( u \in W^{1,2}_{0, rad}(B) \) and by (2.1) we obtain

\[
\|u_n - u\|_{W^{1,2}_{0, rad}(B)} \to 0 \quad \text{as} \quad n \to +\infty, \tag{2.3}
\]

and from (3.2), (2.3), we have

\[
\|u - v\|_{W^{1,2}_{0, rad}(B)} \leq \|u_n - u\|_{W^{1,2}_{0, rad}(B)} + \|u_n - v\|_{W^{1,2}_{0, rad}(B)} \to 0 \quad \text{as} \quad n \to +\infty,
\]

so \( u = v \) a.e in \( B \), \( u \in W \) and satisfies

\[
\|u_n - u\|_{W^{2,2}_{0, rad}(B, w)} \to 0 \quad \text{as} \quad n \to +\infty.
\]

Now we prove that \( \| \cdot \| \) is equivalent to \( \| \cdot \|_{W_{0, rad}^{2,2}(B, w)} \) in \( W \).

\[
\|u\|_{W_0^{2,2}}^2 = \|u\|^2_2 + \|\nabla u\|^2_2 + \int_B |\Delta u|^2 w(x)dx.
\]

For all \( u \in W_{0, rad}^{2,2}(B) \), we have

\[
\|u\|^2 = \int_B |\Delta u|^2 w(x)dx \leq \|u\|^2_2 + \|\nabla u\|^2_2 + \int_B |\Delta u|^2 w(x)dx
\]

On the other hand, for all \( u \in W_{0, rad}^{2,2}(B, w) \), by Poincaré inequality,

\[
\|u\|^2_2 \leq C\|\nabla u\|^2_2,
\]

and using the Green formula we get

\[
\int_B \nabla u \nabla u = -\int_B u \Delta u + \int_{\partial B} u \frac{\partial u}{\partial n} \leq \left| \int_B u \Delta u \right|,
\]

By Young inequality, we get for all \( \varepsilon > 0 \)

\[
\left| \int_B u \Delta u \right| \leq \frac{1}{2\varepsilon} \int_B |\Delta u|^2 + \frac{\varepsilon}{2} \int_B |u|^2,
\]

Again, by the Poincaré inequality and using the fact that \( w(x) \geq 1 \), for all \( x \in B \), we get

\[
\int_B \nabla u \nabla u dx \leq \frac{1}{2\varepsilon} \int_B |\Delta u|^2 dx + \frac{\varepsilon}{2} C^2 \int_B |\nabla u|^2 dx \leq \frac{1}{2\varepsilon} \int_B |\Delta u|^2 w(x)dx + \frac{\varepsilon}{2} C^2 \int_B |\nabla u|^2 dx.
\]

Hence

\[
(1 - \frac{\varepsilon}{2} C^2) \int_B |\nabla u|^2 dx \leq \frac{1}{2\varepsilon} \int_B |\Delta u|^2 w(x)dx,
\]

which implies that

\[
\|\nabla u\|^2_2 \leq C \int_B |\Delta u|^2 w(x)dx \tag{2.4}
\]

and it is easy to conclude.
2.2 The pass mountain geometry

Since the nonlinearity $f(x, t)$ is critical at $+\infty$, there exist $a$, $C > 0$ positive constants and there exists $t_1 > 1$ such for that

$$|f(x, t)| \leq Ce^{-a t}, \quad \forall |t| > t_1. \quad (2.5)$$

So, the functional $J$ given by (1.6) is well defined and of class $C^1$. In order to prove the existence of nontrivial solution to the problem (1.1), we will prove the existence of nonzero critical point of the functional $J$ by using the following theorem which is introduced by Ambrosetti and Rabionowitz in [5] (Mountain Pass Theorem).

**Definition 2.1.** Let $(u_n)$ be a sequence in a Banach space $E$ and $J \in C^1(E, \mathbb{R})$ and let $c \in \mathbb{R}$. We say that the sequence $(u_n)$ is a Palais-Smale sequence at level $c$ (or $(PS)_c$ sequence) for the functional $J$ if

$$J(u_n) \to c \text{ in } \mathbb{R}, \quad \text{as } n \to +\infty$$

and

$$J'(u_n) \to 0 \text{ in } E', \quad \text{as } n \to +\infty.$$ 

We say that the functional $J$ satisfies the Palais-Smale condition $(PS)_c$ at the level $c$ if every $(PS)_c$ sequence $(u_n)$ is relatively compact in $E$.

**Theorem 2.1.** [5] Let $E$ be a Banach space and $J : E \to \mathbb{R}$ a $C^1$ functional satisfying $J(0) = 0$. Suppose that

(i) There exist $\rho$, $\beta > 0$ such that $\forall u \in \partial B(0, \rho), J(u) \geq \beta$;

(ii) There exists $x_1 \in E$ such that $\|x_1\| > \rho$ and $J(x_1) < 0$;

(iii) $J$ satisfies the Palais-Smale condition $(PS)$, that is for all sequence $(u_n)$ in $E$ satisfying

$$J(u_n) \to d \quad \text{as } n \to +\infty \quad (2.6)$$

for some $d \in \mathbb{R}$ and

$$\|J'(u_n)\|_* \to 0 \quad \text{as } n \to +\infty, \quad (2.7)$$

the sequence $(u_n)$ is relatively compact.

Then, $J$ has a critical point $u$ and the critical value $c = J(u)$ verifies

$$c := \inf_{\gamma \in \Gamma} \max_{t \in [0, 1]} J(\gamma(t))$$

where $\Gamma := \{ \gamma \in C([0, 1], X) \text{ such that } \gamma(0) = 0 \text{ and } \gamma(1) = x_1 \} \text{ and } c \geq \beta.$

Before starting the proof of the geometric properties for the function $J$, we recall the following radial Lemma introduced in [34].
Lemma 2. [32] Let $u$ be a radially symmetric function in $C^2_0(B)$. Then, we have

$$|u(x)| \leq \frac{1}{2\sqrt{2\pi}} \left( \frac{\log(\frac{e}{|x|})^{1-\beta} - 1}{1-\beta} \right)^{\frac{1}{2}} \|u\|.$$ 

Since $w(x) \geq 1$, for all $x \in B$, then the following embedding $W^{2,2}_0(B, w) \hookrightarrow W^{2,2}_0(B) \hookrightarrow L^q(B)$ are continuous and also compact for all $q \geq 2$. So there exists a constant $C > 0$ such that $\|u\|_{2q} \leq C\|u\|$, for all $u \in W$.

In the next Lemma, we prove that the $J$ satisfies the first geometric property.

Lemma 3. Suppose that $(H_1)$ and $(H_4)$ hold. Then, there exist $\rho, \beta_0 > 0$ such that $J(u) \geq \beta_0$ for all $u \in W$ with $\|u\| = \rho$.

Proof. It follows from the hypothesis $(H_4)$ that there exists $t_0 > 0$ and there exists $\varepsilon \in (0, 1)$ such that

$$F(x, t) \leq \frac{1}{2} \lambda_1 (1 - \varepsilon_0) |t|^2, \quad \text{for } |t| < t_2.$$ 

Indeed,

$$\limsup_{t \to 0} \sup_{0 \leq t \leq t_2} \frac{2F(x, t)}{t^2} < \lambda_1$$

or

$$\sup_{0 \leq t \leq t_2} \frac{2F(x, t)}{t^2} < \lambda_1 - \varepsilon_0$$

Since this inequality is strict, then there exists $\varepsilon_0 > 0$ such that

$$\sup_{0 \leq t \leq t_2} \frac{2F(x, t)}{t^2} < \lambda_1 - \varepsilon_0,$$

hence, there exists $t_2 > 0$ such that

$$\sup_{0 \leq t \leq t_2} \frac{2F(x, t)}{t^2} < \lambda_1 - \varepsilon_0.$$ 

Hence

$$\forall |t| < t_2 \quad F(x, t) \leq \frac{1}{2} \lambda_1 (1 - \varepsilon_0)t^2.$$ 

From $(H_3)$ and (2.5) and for all $q > 2$, there exist a constant $C > 0$ such that

$$F(x, t) \leq C|t|^q e^{at^\gamma}, \quad \forall |t| > t_1.$$ 

So

$$F(x, t) \leq \frac{1}{2} \lambda_1 (1 - \varepsilon_0)t^2 + C|t|^q e^{at^\gamma}, \quad \text{for all } t \in \mathbb{R}. \quad (2.10)$$

Since

$$J(u) = \frac{1}{2} \|u\|^2 - \int_B F(x, u)dx,$$
we get
\[
J(u) \geq \frac{1}{2} \|u\|^2 - \frac{1}{2} \lambda_1 (1 - \varepsilon_0) \|u\|^2 - C \int_B |u|^q e^{\alpha_0} \gamma \, dx.
\]
But \(\lambda_1 \|u\|_2^2 \leq \|u\|^2\) and from the Hölder inequality, we obtain
\[
J(u) \geq \frac{\varepsilon_0}{2} \|u\|^2 - C \left( \int_B e^{2\alpha_0 \|u\|^{\gamma}} \right)^{\frac{1}{2}} \|u\|_{2q}^q.
\] (2.11)
From the Theorem \cite{1} if we choose \(u \in W\) such that
\[
2\alpha_0 \|u\|^{\gamma} \leq \alpha_0 \beta_0,
\] (2.12)
we get
\[
\int_B e^{2\alpha_0 \|u\|^{\gamma}} \, dx = \int_B e^{2\alpha_0 \|u\|^{\gamma} \left( \frac{\|u\|}{\lambda_1} \right)^{\gamma}} \, dx < +\infty.
\]
On the other hand \(\|u\|_{2q} \leq C \|u\|\), so
\[
J(u) \geq \frac{\varepsilon_0}{2} \|u\|^2 - C \|u\|^q,
\]
for all \(u \in W\) satisfying (2.12). Since \(2 < q\), we can choose \(\rho = \|u\| > 0\) as the maximum point of the function \(g(\sigma) = \frac{\varepsilon_0}{2} \sigma^2 - C \sigma^q\) on the interval \([0, (\frac{\|u\|}{\lambda_1})^{\frac{1}{\gamma}}]\) and \(\beta_0 = g(\rho), J(u) \geq \beta_0 > 0\).

By the following Lemma, we prove the second geometric property for the functional \(J\).

**Lemma 4.** Suppose that \((H_1)\) and \((H_2)\) hold. Let \(\varphi_1\) be a normalized eigenfunction associated to \(\lambda_1\) in \(W\). Then, \(J(t\varphi_1) \to -\infty\), as \(t \to +\infty\).

**Proof.** It follows from the condition \((H_2)\) that
\[
f(x, t) = \frac{\partial}{\partial t} F(x, t) \geq \frac{1}{M} F(x, t),
\]
for all \(t \geq t_0\). So
\[
F(x, t) \geq C \, e^{\frac{t}{M}}, \quad \forall \ t \geq t_0.
\]
It follows that, there exist \(b > \lambda_1\) and \(C > 0\) such that \(F(x, t) \geq \frac{b}{2} t^2 + C\) for all \(t > 0\).
\[
J(t\varphi_1) \leq t^2 \|\varphi_1\|^2 - \frac{b}{2} t^2 \|\varphi_1\|^2 + C|B|,\]
where \(|B| = meas(B) = Vol(B)\). Then, from the definition of \(\lambda_1\), we get
\[
J(t\varphi_1) \leq t^2 \frac{\lambda_1 - b}{2} \|\varphi_1\|^2 < 0 \ \forall t > 0.
\]
So, the Lemma follows.
3 The compactness analysis

3.1 Concentration Compactness Theorem

In order to prove that the functional \( J \) satisfies the \((PS)\) condition, we need a lions type result \[25\] about an improved Adam’s inequality.

**Theorem 3.1.** Let \((u_k)_k\) be a sequence in \(W\). Suppose that, 
\[
\|u_k\| = 1, \quad u_k \rightharpoonup u \text{ weakly in } W, \quad u_k(x) \to u(x) \text{ a.e } x \in B, \quad \text{and } u \neq 0.
\]
Then
\[
\sup_k \int_B e^{p \alpha_\beta |u_k|^\gamma} \, dx < +\infty,
\]
where \(\alpha_\beta = 4[8\pi^2(1 - \beta)]^{\frac{1}{1 - \beta}}\), for all \(1 < p < U(u)\) where \(U(u)\) is given by:
\[
U(u) := \begin{cases} 
\frac{1}{(1 - \|u\|^2)^\frac{\gamma}{2}} & \text{if } \|u\| < 1 \\
+\infty & \text{if } \|u\| = 1
\end{cases}
\]

**Proof.** Since \(\|u\| \leq \liminf_k \|u_k\| = 1\), we will split the evidence into two cases.

**Case 1 : \(\|u\| < 1\).** We assume by contradiction for some \(p_1 < U(u)\), we have
\[
\sup_k \int_B \exp(\alpha_\beta p_1 |u_k|^\gamma) \, dx = +\infty.
\]
Set 
\[
B_L^k = \{x \in B : u_k(x) \geq L\}
\]
where \(L\) is a constant that we will choose later. Let \(v_k = u_k - L\). we have
\[
(1 + a)^q \leq (1 + \epsilon)a^q + (1 - \frac{1}{(1 + \epsilon)^{\frac{1}{q-1}}})^{1-q}, \quad \forall a \geq 0, \quad \forall \epsilon > 0 \quad \forall q > 1.
\]
(3.1)

So, using (3.2), we get
\[
|u_k|^\gamma = |u_k - L + L|^\gamma \\
\leq (|u_k - L| + |L|)^\gamma \\
\leq (1 + \epsilon)|u_k - L|^\gamma + (1 - \frac{1}{(1 + \epsilon)^{\frac{1}{q-1}}})^{1-q}|L|^\gamma \\
\leq (1 + \epsilon)v_k^\gamma + C(\epsilon, \gamma)\mathcal{L}^\gamma.
\]

We have
\[
\int_B \exp(\alpha_\beta p_1 u_k^\gamma) \, dx = \int_{B_k^\delta} \exp(\alpha_\beta p_1 u_k^\gamma) \, dx + \int_{B \setminus B_k^\delta} \exp(\alpha_\beta p_1 u_k^\gamma) \, dx \\
\leq \int_{B_k^\delta} \exp(\alpha_\beta p_1 u_k^\gamma) \, dx + c \exp(\alpha_\beta L^\gamma) \\
\leq \int_{B_k^\delta} \exp(\alpha_\beta p_1 u_k^\gamma) \, dx + c(L, \gamma, |B|),
\]

and then
\[
\sup_k \int_{B_k^\delta} \exp(\alpha_\beta p_1 u_k^\gamma) \, dx = \infty.
\]

By (3.3) we have
\[
\int_{B_k^\delta} \exp(\alpha_\beta p_1 u_k^\gamma) \, dx \leq \exp((1 + \varepsilon)\alpha_\beta p_1 v_k^\gamma) \int_{B_k^\delta} \exp(\alpha_\beta p_1 u_k^\gamma) \, dx.
\]

Since, \( p_1 < U(u) \), there exists \( \varepsilon > 0 \) such that \( \tilde{p}_1 = (1 + \varepsilon)p_1 < U(u) \). Thus
\[
\sup_k \int_{B_k^\delta} \exp(\tilde{p}_1 u_k^\gamma) \, dx = \infty \tag{3.3}
\]

Now, we define
\[
T^\ell(u) = \min\{\ell, u\} \quad \text{and} \quad T^\ell_c(u) = u - T^\ell(u)
\]
and choose \( \ell \) such that
\[
\frac{1 - \|u\|^2}{1 - \|T^\ell u\|^2} > \left( \frac{\tilde{p}_1}{U(u)} \right)^\frac{2}{\gamma} \tag{3.4}
\]

We claim that
\[
\limsup_k \int_{B_k^\delta} \omega(x) |\Delta v_k|^2 \, dx < \left( \frac{1}{\tilde{p}_1} \right)^\frac{2}{\gamma}.
\]

If this is not the case, then up to a subsequence, we get
\[
\int_{B_k^\delta} \omega(x) |\Delta v_k|^2 \, dx = \int_B \omega(x) |\Delta T^\ell C u_k|^2 \, dx \geq \left( \frac{1}{\tilde{p}_1} \right)^\frac{2}{\gamma} + o_k(1).
\]

Thus,
\[
\left( \frac{1}{\tilde{p}_1} \right)^\frac{2}{\gamma} + \int_B \omega(x) |\Delta T^\ell C u_k|^2 \, dx + o_k(1) \leq \int_B \omega(x) |\Delta T^\ell C u_k|^2 \, dx + \int_{B \setminus B_k^\delta} \omega(x) |\Delta u_k|^2 \, dx \\
= \int_{B_k^\delta} \omega(x) |\Delta u_k|^2 \, dx + \int_{B \setminus B_k^\delta} \omega(x) |\Delta u_k|^2 \, dx = 1.
\]

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For $\mathcal{L} > 0$ fixed, $T^\mathcal{L} u_k$ is also bounded in $X$. Therefore, up to a subsequence, $T^\mathcal{L} u_k \rightharpoonup T^\mathcal{L} u$ weakly in $X$ and $T^\mathcal{L} u_k \rightarrow T^\mathcal{L} u$ almost everywhere in $B$. By the lower semicontinuity of the norm in $X$ and the last inequality, we have

$$\tilde{p}_1 \geq \frac{1}{\left(1 - \liminf_{k \rightarrow +\infty} \|T^\mathcal{L} u_k\|^2\right)^\frac{\gamma}{2}} \geq \frac{1}{\left(1 - \|T^\mathcal{L} u\|^2\right)^\frac{\gamma}{2}},$$

combining with (3.4), we obtain

$$\tilde{p}_1 \geq \frac{1}{\left(1 - \|T^\mathcal{L} u\|^2\right)^\frac{\gamma}{2}} > \frac{\tilde{p}_1}{U(u)} \frac{1}{\left(1 - \|T^\mathcal{L} u\|^2\right)^\frac{\gamma}{2}} = \tilde{p}_1,$$

which is a contradiction. Therefore

$$\limsup_k \int_{B^k} \omega(x) |\nabla v_k|^2 \, dx < \left(\frac{1}{\tilde{p}_1}\right)^\frac{\gamma}{2}.$$ 

By the Adam’s inequality (1.4), we deduce that

$$\sup_k \int_{B^k} \exp\left(\tilde{p}_1 \alpha \beta v_k^\gamma \right) \, dx < \infty$$

which is also a contradiction. The proof is finished in this case.

**Case 2**: $\|u\| = 1$. We can then proceed as in case 1 and obtain

$$\sup_k \int_{B^k} \exp\left(\tilde{p}_1 \alpha \beta v_k^\gamma \right) \, dx = \infty$$

where $\tilde{p}_1 = (1 + \varepsilon)p_1$. Then we have

$$\limsup_k \int_{B^k} \omega(x) |\nabla v_k|^2 \, dx = \limsup_k \int_B \omega(x) |\nabla T^\mathcal{L} u_k|^2 \, dx \geq \left(\frac{1}{\tilde{p}_1}\right)^\frac{\gamma}{2}$$

thus,

$$\|T^\mathcal{L} u\|^2 \leq \liminf_k \int_B \omega(x) |\nabla T^\mathcal{L} u_k|^2 \, dx \leq 1 - \limsup_k \int_B |\nabla T^\mathcal{L} u_k|^2 \, dx \leq 1 - \left(1 - \left(\frac{1}{\tilde{p}_1}\right)^\frac{\gamma}{2}\right).$$

On the other hand, since $\|u\| = 1$, we can take $\mathcal{L}$ large enough such that

$$\|T^\mathcal{L} u\|^2 > 1 - \frac{1}{2} \left(\frac{1}{\tilde{p}_1}\right)^\frac{\gamma}{2},$$

which is a contradiction, and the proof is complete in this case.
3.2 The Palais-Smale sequence

The main difficulty in the approach to the critical problem of growth is the lack of compactness. Precisely, the overall condition of Palais-Smale does not hold except for a certain level of energy. In the following proposition, we identify the first level of compactness.

**Proposition 3.1.** Suppose that \((H_1), (H_2)\) and \((H_3)\) hold. If the function \(f(x, t)\) satisfies the condition \((1.5)\) for some \(\alpha_0 > 0\), then the functional \(J\) satisfies the Palais-Smale condition \((PS)_c\) for any

\[ c < \frac{1}{2} \left( \frac{\alpha \beta}{\alpha_0} \right)^\frac{1}{n} , \]

where \(\alpha \beta = 4 \left[ 8 \pi^2 (1 - \beta) \right]^{\frac{1}{n-2}} \).

**Proof.** Consider a \((PS)_c\) sequence in \(W\), for some \(c \in \mathbb{R}\), that is

\[ J(u_n) = \frac{1}{2} \|u_n\|^2 - \int_B F(x, u_n)dx \to c, \ n \to +\infty \]  

(3.5)

and

\[ |\langle J'(u_n), \varphi \rangle| = \left| \int_B w(x) \Delta u_n \Delta \varphi dx - \int_B f(x, u_n) \varphi dx \right| \leq \varepsilon_n \|\varphi\| , \]  

(3.6)

for all \(\varphi \in W\), where \(\varepsilon_n \to 0\), when \(n \to +\infty\).

Also, inspired by [12], it follows from \((H_2)\) that for all \(\varepsilon > 0\) there exists \(t_\varepsilon > 0\) such that

\[ F(x, t) \leq \varepsilon t f(x, t), \quad \text{for all } |t| > t_\varepsilon \text{ and uniformly in } x \in B, \]  

(3.7)

and so, by (3.5), for all \(\varepsilon > 0\) there exists a constant \(C > 0\)

\[ \frac{1}{2} \|u_n\|^2 \leq C + \int_B F(x, u_n)dx , \]

hence

\[ \frac{1}{2} \|u_n\|^2 \leq C + \int_{|u_n| \leq t_\varepsilon} F(x, u_n)dx + \varepsilon \int_B f(x, u_n)u_n dx \]

and so, from (3.6), we get

\[ \frac{1}{2} \|u_n\|^2 \leq C_1 + \varepsilon \varepsilon_n \|u_n\| + \varepsilon \|u_n\|^2 , \]

for some constant \(C_1 > 0\). Since

\[ \left( \frac{1}{2} - \varepsilon \right) \|u_n\|^2 \leq C_1 + \varepsilon \varepsilon_n \|u_n\| , \]  

(3.8)

we deduce that the sequence \((u_n)\) is bounded in \(W\). As consequence, there exists \(u \in W\) such that, up to subsequence, \(u_n \to u\) weakly in \(W\), \(u_n \to u\) strongly in \(L^q(B)\), for all \(1 \leq q < 4\) and \(u_n(x) \to u(x)\)
a.e. in $B$.
Furthermore, we have, from (3.5) and (3.6), that

$$0 < \int_B f(x, u_n) u_n \leq C, \quad (3.9)$$

and

$$0 < \int_B F(x, u_n) \leq C. \quad (3.10)$$

Since by Lemma 2.1 in [19], we have

$$f(x, u_n) \to f(x, u) \text{ in } L^1(B) \text{ as } n \to +\infty, \quad (3.11)$$

then, it follows from $(H_2)$ and the generalized Lebesgue dominated convergence theorem that

$$F(x, u_n) \to F(x, u) \text{ in } L^1(B) \text{ as } n \to +\infty. \quad (3.12)$$

So,

$$\lim_{n \to +\infty} \|u_n\|^2 = 2(c + \int_B F(x, u)dx). \quad (3.13)$$

Using (3.5), we have

$$\lim_{n \to +\infty} \int_B f(x, u_n) u_n dx = 2(c + \int_B F(x, u)dx). \quad (3.14)$$

Then by $(H_3)$ and (3.6), we get

$$\lim_{n \to +\infty} 2 \int_B F(x, u_n)dx \leq \lim_{n \to +\infty} \int_B f(x, u_n) u_n dx = 2(c + \int_B F(x, u)dx). \quad (3.15)$$

As a direct consequence from (3.11) and (3.12), we get $c \geq 0$.
Also, it follows from (3.5), (3.6), (3.11) and (3.12), by passing to the limit, we obtain that $u$ is a weak solution of the problem (1.1) that is

$$\int_B \triangle u \cdot \triangle \varphi w(x)dx = \int_B f(x, u)\varphi dx, \quad \forall \varphi \in W.$$

Taking $\varphi = u$ as a test function, we get

$$\int_B |\triangle u|^2 w(x)dx = \int_B f(x, u)u dx \geq 2 \int_B F(x, u)dx.$$

Hence $J(u) \geq 0$. We also have by the Fatou’s lemma and (3.12)

$$0 \leq J(u) \leq \frac{1}{2} \liminf_{n \to \infty} \|u_n\|^2 - \int_B F(x, u)dx = c.$$
So, we will finish the proof by considering three cases for the level $c$.

**Case 1.** $c = 0$. In this case

$$0 \leq \mathcal{J}(u) \leq \liminf_{n \to +\infty} \mathcal{J}(u_n) = 0.$$  

So,

$$\mathcal{J}(u) = 0$$

and then by (3.12)

$$\lim_{n \to +\infty} \frac{1}{2} \|u_n\|^2 = \int_B F(x, u) \, dx = \frac{1}{2} \|u\|^2.$$  

It follows that $u_n \to u$ in $W$.

**Case 2.** $c > 0$ and $u = 0$. We prove that this case cannot happen.

From (3.5) and (3.6) with $v = u_n$, we have

$$\lim_{n \to +\infty} \|u_n\|^2 = 2c \quad \text{and} \quad \lim_{n \to +\infty} \int_B f(x, u_n) u_n \, dx = 2c.$$  

Again by (3.6) we have

$$\|u_n\|^2 - \int_B f(x, u_n) u_n \, dx \leq C \varepsilon_n.$$  

First we claim that there exists $q > 1$ such that

$$\int_B |f(x, u_n)|^q \, dx \leq C,$$  

so

$$\|u_n\|^2 \leq C \varepsilon_n + \left( \int_B |f(x, u_n)|^q \right)^{\frac{1}{q}} \left( \int_B |u_n|^{q'} \right)^{\frac{1}{q'}}$$

where $q'$ the conjugate of $q$. Since $(u_n)$ converge to $u = 0$ in $L^q(B)$,

$$\lim_{n \to +\infty} \|u_n\|^2 = 0$$

which in contradiction with $c > 0$.

For the proof of the claim, since $f$ has subcritical or critical growth, for every $\varepsilon > 0$ and $q > 1$ there exists $t_\varepsilon > 0$ and $C > 0$ such that for all $|t| \geq t_\varepsilon$, we have

$$|f(x, t)|^q \leq Ce^{\alpha(1+\varepsilon)t}.$$  

Consequently,

$$\int_B |f(x, u_n)|^q \, dx = \int_{\{|u_n| \leq t\}} |f(x, u_n)|^q \, dx + \int_{\{|u_n| > t\}} |f(x, u_n)|^q \, dx \leq 2\pi^2 \max_{B \times [-t, t]} |f(x, t)|^q + C \int_B e^{\alpha(1+\varepsilon)|u_n|^\gamma} \, dx.$$  

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Since \(2c < \left(\frac{\alpha_\beta}{\alpha_0}\right)^{\frac{\gamma}{2}}\), there exists \(\eta \in (0, \frac{1}{2})\) such that \(2c = (1 - 2\eta)\left(\frac{\alpha_\beta}{\alpha_0}\right)^{\frac{\gamma}{2}}\). On the other hand, \(\|u_n\|^{\gamma} \to (2c)^{\frac{\gamma}{2}}\), so there exists \(n_\eta > 0\) such that for all \(n \geq n_\eta\), we get \(\|u_n\|^{\gamma} \leq \left(1 - \eta\right)\left(\frac{\alpha_\beta}{\alpha_0}\right)^{\frac{\gamma}{2}}\). Therefore,

\[
\alpha_0(1 + \varepsilon)(\frac{|u_n|}{\|u_n\|})^{\gamma}\|u_n\|^\gamma \leq (1 + \varepsilon)(1 - \eta)\alpha_\beta.
\]

We choose \(\varepsilon > 0\) small enough to get

\[
\alpha_0(1 + \varepsilon)\|u_n\|^\gamma \leq \alpha_\beta.
\]

Therefore, the second integral is uniformly bounded in view of (1.4) and the claim is proved.

**Case 3.** \(c > 0\) and \(u \neq 0\). In this case, we claim that \(J(u) = c\) and therefore, we get

\[
\lim_{n \to +\infty} \|u_n\|^2 = 2(c + \int_B F(x, u)dx) = 2(J(u) + \int_B F(x, u)dx) = \|u\|^2.
\]

Do not forget that

\[
J(u) \leq \frac{1}{2} \lim_{n \to +\infty} \|u_n\|^2 - \int_B F(x, u)dx = c.
\]

Suppose that \(J(u) < c\). Then,

\[
\|u\|^\gamma < (2\left(c + \int_B F(x, u)dx\right))^{\frac{\gamma}{2}}.
\]  \hspace{1cm} (3.18)

Set

\[
v_n = \frac{u_n}{\|u_n\|},
\]

and

\[
v = \frac{u}{(2(c + \int_B F(x, u)dx))^{\frac{1}{2}}}.
\]

We have \(\|v_n\| = 1\), \(v_n \rightharpoonup v\) in \(W\), \(v \neq 0\) and \(\|v\| < 1\). So, by Theorem 3.1, we get

\[
\sup_n \int_B e^{\alpha_\beta |v_n|^\gamma} dx < \infty,
\]

for \(1 < p < U(v) = (1 - \|v\|^2)^{\frac{-\gamma}{2}}\).

As in the case (2), we are going to estimate \(\int_B |f(x, u_n)|^q dx\).

For \(\varepsilon > 0\), one has

\[
\int_B |f(x, u_n)|^q dx = \int_{|u_n| \leq t_\varepsilon} |f(x, u_n)|^q dx + \int_{|u_n| > t_\varepsilon} |f(x, u_n)|^q dx \leq 2\pi^2 \max_{B \times [-t_\varepsilon, t_\varepsilon]} |f(x, t)|^q + C \int_B e^{\alpha_\beta (1 + \varepsilon)|u_n|^\gamma} dx \leq C_\varepsilon + C \int_B e^{\alpha_\beta (1 + \varepsilon)|u_n|^\gamma |v_n|^\gamma} dx \leq C,
\]

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provided that $\alpha_0(1 + \varepsilon)\|u_n\|^\gamma \leq p \alpha_0$ and $1 < p < U(v) = (1 - \|v\|^2)^{\frac{\gamma}{2}}$.

Since
\[
(1 - \|v\|^2)^{\frac{\gamma}{2}} = \left(\frac{2(c + \int_B F(x, u)dx)}{2(c + \int_B F(x, u)dx - \|u\|^2)}\right)^{\frac{\gamma}{2}} = \left(\frac{c + \int_B F(x, u)dx}{c - \mathcal{J}(u)}\right)^{\frac{\gamma}{2}},
\]
and
\[
\lim_{n \to +\infty} \|u_n\|^\gamma = (2(c + \int_B F(x, u)dx))^{\frac{\gamma}{2}},
\]
then, for large $n$,
\[
\alpha_0(1 + \varepsilon)\|u_n\|^\gamma \leq \alpha_0(1 + 2\varepsilon)(2(c + \int_B F(x, u)dx))^{\frac{\gamma}{2}}.
\]

But $\mathcal{J}(u) \geq 0$ and $c < \frac{1}{2} \left(\frac{\alpha_\beta}{\alpha_0}\right)^{\frac{\gamma}{2}}$, then if we choose $\varepsilon > 0$ small enough such that
\[
\frac{\alpha_0}{\alpha_\beta} (1 + 2\varepsilon) < \left(\frac{1}{2(c - \mathcal{J}(u))}\right)^{\frac{\gamma}{2}},
\]
we get,
\[
(1 + 2\varepsilon)\left((c - \mathcal{J}(u))\right)^{\frac{\gamma}{2}} \leq \frac{\alpha_\beta}{2\alpha_0}.
\]

So, the sequence $(f(x, u_n))$ is bounded in $L^q, q > 1$.

Since $(\mathcal{J}'(u_n), (u_n - u)) = o(1)$, we have from the boundedness of $\{f(x, u_n)\}$ in $L^q(B)$ for $q > 1$, we can prove that $u_n \to u$ strongly in $W$. Indeed, we have
\[
\|u_n - u\|^2 = (u_n, u_n - u) - (u, u_n - u) = (u_n, u_n - u) + o_n(1) \to 0 \text{ as } n \to +\infty.
\]

From (3.6) and using the Hölder inequality, we get
\[
|< u_n, u_n - u>| \leq \varepsilon_n\|u_n - u\| + |\int_B f(x, u_n)(u_n - u)dx|
\leq C\varepsilon_n + \left(\int_B |f(x, u_n)|^q dx\right)^{\frac{1}{q}} \left(\int_B |u_n - u|^q dx\right)^{\frac{1}{q}} \to 0 \text{ as } n \to +\infty.
\]

Hence,
\[
\lim_{n \to +\infty} \|u_n\|^2 = 2(c + \int_B F(x, u)dx) = \|u\|^2
\]
and this contradicts (3.18). So, $\mathcal{J}(u) = c$ and consequently, $u_n \to u$.

\[\square\]

4 Proof of Theorem 1.3

In the sequel, we will estimate the minimax level of the energy $\mathcal{J}$. We will prove that the mountain pass level $c$ in Theorem 2.1 satisfies
\[
c < \frac{1}{2} \left(\frac{\alpha_\beta}{\alpha_0}\right)^{\frac{\gamma}{2}}.
\]
For this purpose, we will prove that there exists \( v_0 \in W \) such

\[
\max_{t \geq 0} J(tv_0) < \frac{1}{2} \left( \frac{\alpha \beta}{\alpha_0} \right)^{\frac{1}{2}}.
\] (4.1)

### 4.1 Adams functions

Now, we will construct particular functions, namely the Adams functions. We consider the sequence defined for all \( n \geq 3 \) by

\[
w_n(x) = \begin{cases} 
\left( \frac{\log(e^{4\sqrt{n}})}{\alpha \beta} \right)^{\frac{1}{\gamma}} & - \frac{|x|^{2(1-\beta)}}{2 \left( \frac{\alpha}{\alpha_0} \right)^{\frac{1}{\gamma}} \log(e^{4\sqrt{n}})} + \frac{1}{2 \left( \frac{\alpha}{\alpha_0} \right)^{\frac{1}{\gamma}} \log(e^{4\sqrt{n}})} & \text{if } 0 \leq |x| \leq \frac{1}{\sqrt{n}} \\
\left( \frac{\log(|x|)}{\alpha \beta} \right)^{1-\beta} & \left( \frac{\alpha}{\alpha_0} \log(e^{4\sqrt{n}}) \right)^{\frac{1}{\gamma}} & \text{if } \frac{1}{\sqrt{n}} \leq |x| \leq \frac{1}{2} \\
\Delta \zeta_n & \zeta_n & \text{if } \frac{1}{2} \leq |x| \leq 1
\end{cases}
\]

where \( \zeta_n \in C_0^\infty(B) \) is such that

\[
\zeta_n(x) = \frac{1}{\left( \frac{\alpha}{\alpha_0} \log(e^{4\sqrt{n}}) \right)^{\frac{1}{\gamma}}} \log(2e)^{1-\beta}. \quad \partial_x \zeta_n(x) = \frac{-2(1-\beta)}{2 \left( \frac{\alpha}{\alpha_0} \right)^{\frac{1}{\gamma}} \log(e^{4\sqrt{n}})} \left( \frac{\alpha}{\alpha_0} \log(e^{4\sqrt{n}}) \right)^{-\beta}
\]

and \( \zeta_n = \partial_x \zeta_n = 0 \) on \( \partial B \) and the functions \( \zeta_n, \nabla \zeta_n, \Delta \zeta_n \) are all \( \text{o} \left( \frac{1}{\log(e^{4\sqrt{n}})} \right) \).

Let \( v_n(x) = \frac{w_n}{\|w_n\|} \). We have, \( v_n \in W, \|v_n\|^2 = 1 \).

We compute \( \Delta w_n(x) \), we get

\[
\Delta w_n(x) = \begin{cases} 
\frac{-(1-\beta)(4-2\beta)|x|^{-2\beta}}{\left( \frac{\alpha}{\alpha_0} \right)^{\frac{1}{\gamma}} \log(e^{4\sqrt{n}})^{\frac{1}{\gamma}}} & \text{if } 0 \leq |x| \leq \frac{1}{\sqrt{n}} \\
\frac{-(1-\beta)(\log(|x|) - \beta \left( \log(e^{4\sqrt{n}}) \right)^{-1} \left( \frac{\alpha}{\alpha_0} \log(e^{4\sqrt{n}}) \right)^{\frac{1}{\gamma}}} & \text{if } \frac{1}{\sqrt{n}} \leq |x| \leq \frac{1}{2} \\
\Delta \zeta_n & \text{if } \frac{1}{2} \leq |x| \leq 1
\end{cases}
\]
we have,

\[ \| w_n \|^2 = 2\pi^2 \int_0^{\frac{1}{2\sqrt{e}}} r^3|\Delta w_n(x)|^2 \left( \log \frac{e}{r} \right)^\beta \, dr + 2\pi^2 \int_{\frac{1}{2\sqrt{e}}}^{\frac{1}{4}} r^3|\Delta w_n(x)|^2 \left( \log \frac{e}{r} \right)^\beta \, dr + 2\pi^2 \int_{\frac{1}{4}}^{1} r^3|\Delta w_n(x)|^2 \left( \log \frac{e}{r} \right)^\beta \, dr \]

\[ I_1 = 2\pi^2 \int_0^{\frac{1}{2\sqrt{e}}} r^3 \left( \frac{1-\beta^2(4-2\beta)^2}{(2\beta-1)^2 \log(e \sqrt{n})} - \beta \right) \left( \log \frac{e}{r} \right)^\beta \, dr \]

\[ = 2\pi^2 \int_0^{\frac{1}{2\sqrt{e}}} r^3 \left( \frac{(1-\beta^2(4-2\beta)^2}{(2\beta-1)^2 \log(e \sqrt{n})} - \beta \right) \left( \log \frac{e}{r} \right)^\beta \, dr \]

\[ I_2 = 2\pi^2 \int_{\frac{1}{2\sqrt{e}}}^{\frac{1}{4}} r^3 \left( \frac{(1-\beta^2(4-2\beta)^2}{(2\beta-1)^2 \log(e \sqrt{n})} - \beta \right) \left( \log \frac{e}{r} \right)^\beta \, dr \]

\[ = 1 + o\left( \frac{1}{(\log(e \sqrt{n})^\beta} \right) \]

and \( I_3 = o\left( \frac{1}{(\log(e \sqrt{n})^\beta} \right) \). Then \( \| w_n \|^2 = 1 + o\left( \frac{1}{(\log(e \sqrt{n})^\beta} \right) \). Also, for \( 0 \leq |x| \leq \frac{1}{\sqrt{e}n} \), \( v_n^\alpha(x) \geq \left( \frac{(\log(e \sqrt{n})}{\alpha \beta} \right) + o(1) \).

### 4.2 Min-Max level estimate

We are going to the desired estimate.

**Lemma 5.** For the sequence \( (v_n) \) identified by (4.1), there exists \( n \geq 1 \) such that

\[ \max_{t \geq 0} J(t v_n) < \frac{1}{2} \left( \frac{(\alpha \beta)}{\alpha_0} \right)^\frac{1}{2}. \]  

**Proof.** By contradiction, suppose that for all \( n \geq 1 \),

\[ \max_{t \geq 0} J(t v_n) \geq \frac{1}{2} \left( \frac{(\alpha \beta)}{\alpha_0} \right)^\frac{1}{2}. \]
Therefore, for any \( n \geq 1 \), there exists \( t_n > 0 \) such that

\[
\max_{t \geq 0} J(tv_n) = J(t_n v_n) \geq \frac{1}{2} \left( \frac{\alpha \beta}{\alpha_0} \right)^{\frac{2}{\gamma}}
\]

and so,

\[
\frac{1}{2} t_n^2 - \int_B F(x, t_n v_n) dx \geq \frac{1}{2} \left( \frac{\alpha \beta}{\alpha_0} \right)^{\frac{2}{\gamma}}.
\]

Then, by using \((H_1)\)

\[
t_n^2 \geq \left( \frac{\alpha \beta}{\alpha_0} \right)^{\frac{2}{\gamma}}, \quad \text{(4.4)}
\]

On the other hand,

\[
\frac{d}{dt} J(tv_n) \bigg|_{t=t_n} = t_n - \int_B f(x, t_n v_n) v_n dx = 0,
\]

that is

\[
t_n^2 = \int_B f(x, t_n v_n) t_n v_n dx. \quad \text{(4.5)}
\]

Now, we claim that the sequence \( (t_n) \) is bounded in \((0, +\infty)\).

Indeed, it follows from \((H_5)\) that for all \( \varepsilon > 0 \), there exists \( t_\varepsilon > 0 \) such that

\[
f(x, t) t \geq (\gamma_0 - \varepsilon) e^{\alpha_0 t^\gamma} \quad \forall |t| \geq t_\varepsilon, \quad \text{uniformly in } x \in B.
\]

Using \((4.4)\) and \((4.5)\), we get

\[
t_n^2 = \int_B f(x, t_n v_n) t_n v_n dx \geq \int_{0 \leq |x| \leq \frac{1}{4} \sqrt{n}} f(x, t_n v_n) t_n v_n dx.
\]

Since

\[
\frac{t_n}{\|w_n\|} \left( \log e^{\sqrt{n} \alpha \beta} \right)^{-\frac{1}{\gamma}} \rightarrow \infty \quad \text{as } n \to +\infty,
\]

then it follows from \((4.6)\) that for all \( \varepsilon > 0 \), there exists \( n_0 \) such that for all \( n \geq n_0 \)

\[
t_n^2 \geq (\gamma_0 - \varepsilon) \int_{0 \leq |x| \leq \frac{1}{4} \sqrt{n}} e^{\alpha_0 t_n^\gamma v_n^\gamma} dx
\]

\[
t_n^2 \geq 2\pi^2 (\gamma_0 - \varepsilon) \int_0^{\frac{1}{4} \sqrt{n}} r^3 e^{\alpha_0 t_n^\gamma v_n^\gamma} dr + o(1) \quad \text{(4.7)}
\]

Hence,

\[
1 \geq 2\pi^2 (\gamma_0 - \varepsilon) e^{\alpha_0 t_n^\gamma v_n^\gamma} + o(1) - 3 \log n - 2 \log t_n
\]
Therefore \((t_n)\) is bounded. Also, we have from the formula (4.5) that
\[
\lim_{n \to +\infty} t_n^2 \geq \left( \frac{\alpha \beta}{\alpha_0} \right)^2.
\]

Now, suppose that
\[
\lim_{n \to +\infty} t_n^2 > \left( \frac{\alpha \beta}{\alpha_0} \right)^2,
\]
then for \(n\) large enough, there exists some \(\delta > 0\) such that \(t_n^2 \geq (\alpha_0 - \epsilon)\). Consequently the right hand side of (4.7) tends to infinity and this contradicts the boundedness of \((t_n)\). Since \((t_n)\) is bounded, we get
\[
\lim_{n \to +\infty} t_n^2 = \left( \frac{\alpha \beta}{\alpha_0} \right)^2.
\]

Let
\[
A_n = \{ x \in B | t_n v_n \geq t \} \quad \text{and} \quad C_n = B \setminus A_n,
\]

\[
t_n^2 = \int_B f(x, t_n v_n) t_n v_n dx = \int_{A_n} f(x, t_n v_n) t_n v_n dx + \int_{C_n} f(x, t_n v_n) t_n v_n dx 
\geq (\gamma_0 - \epsilon) \int_{A_n} e^{\alpha t_n^2 v_n^2} dx + \int_{C_n} f(x, t_n v_n) t_n v_n dx
\]

\[
= (\gamma_0 - \epsilon) \int_B e^{\alpha t_n^2 v_n^2} dx - (\gamma_0 - \epsilon) \int_{C_n} e^{\alpha t_n^2 v_n^2} dx + \int_{C_n} f(x, t_n v_n) t_n v_n dx.
\]

Since \(v_n \to 0\) a.e in \(B\), \(\chi_{C_n} \to 1\) a.e in \(B\), therefore using the dominated convergence theorem, we get

\[
\int_{C_n} f(x, t_n v_n) t_n v_n dx \to 0 \quad \text{and} \quad \int_{C_n} e^{\alpha t_n^2 v_n^2} dx \to \frac{\pi^2}{2}.
\]

Then,

\[
\lim_{n \to +\infty} t_n^2 = \left( \frac{\alpha \beta}{\alpha_0} \right)^2 \geq \lim_{n \to +\infty} (\gamma_0 - \epsilon) \int_B e^{\alpha t_n^2 v_n^2} dx - (\gamma_0 - \epsilon) \frac{\pi^2}{2}.
\]

On the other hand,

\[
\int_B e^{\alpha t_n^2 v_n^2} dx \geq \int_{\|x| \leq \frac{1}{2}} e^{\alpha t_n^2 v_n^2} dx + \int_{C_n} e^{\alpha t_n^2 v_n^2} dx.
\]

Then, using (4.4),

\[
\lim_{n \to +\infty} t_n^2 \geq \lim_{n \to +\infty} (\gamma_0 - \epsilon) \int_B e^{\alpha t_n^2 v_n^2} dx \geq \lim_{n \to +\infty} (\gamma_0 - \epsilon) 2 \pi^2 \int_{\|x\| \leq \frac{1}{2}} \frac{4 (\log 4)^2}{r^3 e^{4 \log(e^4) \|w_n\|^2}} dr
\]

and, making the change of variable

\[
s = \frac{4 \log \frac{r}{4}}{\log(e^4) \|w_n\|^2},
\]

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we get
\[ \lim_{n \to +\infty} t_n^2 \geq \lim_{n \to +\infty} (\gamma_0 - \varepsilon) \int_B e^{\alpha n \gamma} v_n^\gamma dx \]
\[ \geq \lim_{n \to +\infty} 2\pi^2 (\gamma_0 - \varepsilon) \left\| w_n \right\|_{H^{\gamma}}^\gamma e^4 \int_{\left\| w_n \right\|_{H^{\gamma}}} \frac{\left\| w_n \right\|_{H^{\gamma}}^\gamma (e^{\psi} - 4) ds}{4 \log(e^{\psi})} \]
\[ = \lim_{n \to +\infty} (\gamma_0 - \varepsilon) \frac{\pi^2}{2} e^4 (- e^{-4 \log(e^{\psi})} + e^{-4 \log(2e)}) \]
\[ = (\gamma_0 - \varepsilon) \frac{\pi^2}{32} \]

It follows that
\[ \frac{(\alpha \beta)}{\alpha_0} \geq (\gamma_0 - \varepsilon) \frac{\pi^2}{32} \]
for all $\varepsilon > 0$. So,
\[ \gamma_0 \leq \frac{1024 (1 - \beta)}{\alpha_0^{1-\beta}}, \]
which is in contradiction with the condition $(H_3)$.

Now by Proposition 3.1, the functional $\mathcal{J}$ satisfies the $(PS)$ condition at a level $c < \frac{1}{2} \left( \frac{\alpha \beta}{\alpha_0} \right)^{\frac{1}{2}}$. So, by Lemma 3 and Lemma 4, we deduce that the functional $\mathcal{J}$ has a nonzero critical point $u$ in $W$. From maximum principle, the solution $u$ of the problem (1.1) is positive. The Theorem 1.3 is proved.

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