On the Gram’s Law in the Theory of Riemann Zeta Function

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Abstract. Some statements concerning the distribution of imaginary parts of zeros of the Riemann zeta-function are established. These assertions are connected with so-called ‘Gram’s law’ or ‘Gram’s rule’. In particular, we give a proof of several Selberg’s formulae stated him without proof in his paper ‘The Zeta Function and the Riemann Hypothesis’ (1946), and some of their equivalents.

Introduction

In the present paper, the author continues his studies begun in [1] and connected with the distribution of ordinates of zeros of the Riemann zeta-function $\zeta(s)$ and with so-called Gram’s law.

Let’s remind some basic notions and definitions.

Definition 1. For positive $t$, we denote by $\vartheta(t)$ the increment of the argument of the function $\pi^{-s/2}\Gamma(s/2)$ along the segment with the end-points $s = \frac{1}{2}$ and $s = \frac{1}{2} + it$.

On can prove (see, for example, [2]) that the asymptotic expansion

$$\vartheta(t) \sim \frac{t}{2} \ln \frac{t}{2\pi} - \frac{t}{2} - \frac{\pi}{8} + \sum_{n=1}^{+\infty} \frac{2^{2n-1} - 1}{2n(2n-1)} \frac{(-1)^{n+1} B_{2n}}{t^{(2n-1)}},$$

holds, as $t$ grows. Here the $B_{2n}$ are Bernoulli numbers: $B_2 = \frac{1}{6}, B_4 = -\frac{1}{30}, B_6 = \frac{1}{42}, B_8 = -\frac{1}{30}, B_{10} = \frac{5}{66}$ and so on.

The function $\vartheta(t)$ is presented in the expression for $N(t)$ – the number of zeros of $\zeta(s)$ in the strip $0 < \text{Im} \, s \leq t$. This expression is called by Riemann - von Mangoldt formula and has the form

$$N(t) = \frac{1}{\pi} \vartheta(t) + 1 + S(t).$$

Here $S(t) = \pi^{-1} \arg \zeta \left( \frac{1}{2} + it \right)$ denotes the argument of the Riemann zeta function on the critical line. For basic properties of $S(t)$, see the survey [3].

Definition 2. For any $n \geq 0$, the Gram point $t_n$ is defined as the unique root of the equation $\vartheta(t_n) = \pi \cdot (n-1)$.

Suppose $0 < \gamma_1 < \gamma_2 < \ldots < \gamma_n \leq \gamma_{n+1} \leq \ldots$ are positive ordinates of zeros of $\zeta(s)$ numbering in ascending order (if several zeros have the same ordinates, we numerate them in arbitrary way). First researchers of zeros of the Riemann zeta function observed that for all the values $n$ not very large, with the exception of small part of cases, the ordinate $\gamma_n$ belongs to the interval $G_n = (t_{n-1}, t_n]$. Later, this phenomenon was called as ‘Gram’s law’ or ‘Gram’s rule’.

If someone wants to formulate the general rule based on the small number of examples, he has some freedom. This is the reason why the words ‘Gram’s rule’ (and ‘Gram’s law’)

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in some papers have the sense different from the above. Thus, one can say that the interval $G_n$ satisfies the Gram’s rule if $G_n$ contains exactly one zero of function $\zeta\left(\frac{1}{2} + it\right)$, and the number of this zero doesn’t matter. This version of Gram’s rule is contained in the papers of J. I. Hutchinson [4] and E. C. Titchmarsh [5], and from this point of view Gram’s rule is studied by T. S. Trudgian [6].

In order to characterize the degree of deviation from the Gram’s rule, Titchmarsh considered the fractions $\frac{c_n - t_n}{t_{n+1} - t_n}$, where $c_n$ denotes the positive zeros of the function $\zeta\left(\frac{1}{2} + it\right)$ numerated in ascending order, and proved their unboundedness. If the interval $G_m$ contains the zero $c_n$ then such fraction is close to the difference $\Delta_n = m - n$. Thus, Titchmarsh’s theorem implies that the number $n$ of zero lying on the critical line can differs from the number of corresponding Gram’s interval by arbitrary large value. In particular, this means that there exist infinitely many exceptions from Gram’s rule.

In the part IV of his report ‘The Zeta-Function and the Riemann Hypothesis’ [7], Selberg formulated Titchmarsh’s theorem in a following way:

$$\lim_{n \to +\infty} \Delta_n = -\infty, \quad \lim_{n \to +\infty} \Delta_n = +\infty.$$  

For given $n \geq 1$, the quantity $\Delta_n$ was defined as $m - n$ if the following inequalities hold: $t_{m-1} < \gamma_n \leq t_m$. In 1940’s, Selberg established that the formulae

$$\sum_{n=1}^{N} \Delta_n^{2k} = \frac{(2k)!}{k!} \frac{N}{(2\pi)^k} \left(\ln \ln N\right)^k + O\left(N(\ln \ln N)^{k-1/2}\right),$$  

$$\sum_{n=1}^{N} \Delta_n^{2k-1} = O\left(N(\ln \ln N)^{k-1}\right)$$

are valid for any fixed integer $k \geq 1$ and stated them (without proof) in [7]. Then he posed the following conjecture: ‘It is probably true, though I have not been able to prove it rigorously, that $\sqrt{\log \log n}$ is the “normal” order of magnitude of $\Delta_n$ in the sense that if $\Phi(n)$ is a positive function of $n$ which tends to infinity with $n$, the inequalities

$$\frac{\sqrt{\log \log n}}{\Phi(n)} < |\Delta_n| < \Phi(n) \sqrt{\log \log n}$$

hold true for almost all $n$. In particular this should imply that $\gamma_\nu$ “almost never” lies in the interval $(t_{\nu-1}, t_\nu)$. It is the first inequality which is difficult point, and which I have not been able to prove completely.’

Thus the cases considered earlier as ‘exceptions’ are normal, and the cases called earlier as a ‘rule’ occur very rarely.

There are some serious reasons to think that in his definition of $\Delta_n$ Selberg denoted by $\gamma_n$ the ordinates of all zeros of $\zeta(s)$ and not only the ordinates of zeros lying on the critical line, and considered the quantities

$$q_n = \frac{\gamma_n - t_n}{t_{n+1} - t_n}$$
instead of Titchmarsh’s fractions. Indeed, if one considers only the zeros on the critical 
line in the definition of $\Delta_n$, then the formulae (2) and (3) imply some very sharp state-
ments concerning the distribution of zeros of $\zeta(s)$. These statements are close to that 
the Riemann hypothesis asserts and, in any case, they are much more powerful that all 
recent results about a part of zeros of zeta-function lying on the critical line. This is the 
reason why we follow the below definitions.

**Definition 3.** We say that the ordinate $\gamma_n$ satisfies to the Gram’s law if the following 
équalities hold: $t_{n-1} < \gamma_n \leq t_n$.

This means that in the case when the interval $G_n$ contains all the ordinates $\gamma_{n-s}, \ldots, 
\gamma_{n-1}, \gamma_n, \gamma_{n+1}, \ldots, \gamma_{n+r}$, all of these ordinates, except $\gamma_n$, do not obey the Gram’s law.

**Definition 4.** Suppose for given $n \geq 1$ the following inequalities hold:

\[ t_{m-1} < \gamma_n \leq t_m. \]  

(5)

Then we set $\Delta_n = m - n$.

Now it’s clear that the ordinate $\gamma_n$ satisfies the Gram’s law if and only if $\Delta_n = 0$.

In [1] the author proved the part of Selberg conjecture that asserts that the ordinate 
$\gamma_n$ ‘almost never’ lies in the interval $G_n = (t_{n-1}, t_n]$. The proof is based on the properties 
of the sequence $\Delta(n)$ defined as follows.

**Definition 5.** Suppose for given Gram point $t_n$ the following inequalities hold:

\[ \gamma_m \leq \gamma_n < \gamma_{m+1}. \]  

(6)

Then we set $\Delta(n) = m - n$.

It appears that for given interval the number of the indexes $n$ such that $\Delta_n$ equals to 
zero, is very close to the number of $\Delta(n)$ with the same property. Next, the study of the 
sequence $\Delta(n)$ is much more simple than the study of $\Delta_n$. The reason is that both [6] 
and Riemann-von Mangoldt formula imply the key equality $\Delta(n) = S(t_n)$. This relation 
reduces the initial problem to some problem concerning the distribution of values of the 
argument of the Riemann zeta function. The behavior of the function $S(t)$ on the 
‘regular’ sequence of Gram points is studied by methods belonging to A. Selberg [8] and 
A. Ghosh [9].

After the paper [1] was published, the author familiarized with the book of collected 
papers of A.Selberg issued in 1989-1991. The text of Selberg’s paper mentioned above 
in the first volume (see [10, с. 341-355]) was provided by the following remark: ‘... it 
follows from these equations (i.e. from the relations [2] and [3] – M.K.) by standard 
theory that the quantity $\Delta_n/\sqrt{\log \log n}$ has a simple Gaussian distribution. In particular 
this answers in the affirmative question raised here (i.e. the Selberg’s conjecture – M.K.). 
In 1946 I did not know that these results about the moments of $\Delta_n$ allow one to determine 
this distribution function.’ This remark needs careful explanations.

Indeed, before the book [10] was printed, the articles of A. G. Postnikov [11] and 
V. F. Gaposchkin [12] were already published. In these papers, the procedure of deriving 
of the distribution function of some random quantity from the formulae for even and odd
moments of this quantity, was introduced. Next, in the later papers of A. Fujii [13] and A. Ghosh [9] such arguments are applied to some problems in the theory of Riemann zeta function and Dirichlet’s $L$-functions (for the Ghosh method, see, for example, the survey [14]).

But there are some uncertainties with the justification of the formulae (2) and (3). As far as the author found out, Selberd had never published his proof. However, the attempt of such proof is presented in the paper of A. Fujii mentioned above, where the similar problem for Dirichlet’s $L$-functions is considered. Let’s consider this attempt more closely.

Suppose $q \geq 5$ is fixed and denote by $\chi_1, \chi_2$ different primitive characters modulo $q$. Let’s enumerate the positive ordinates of complex zeros of the functions $L(s, \chi_1)$ and $L(s, \chi_2)$ in ascending order:

\begin{align*}
0 < \gamma_1(\chi_1) \leq & \gamma_2(\chi_1) \leq \ldots \leq \gamma_n(\chi_1) \leq \gamma_{n+1}(\chi_1) \leq \ldots, \\
0 < \gamma_1(\chi_2) \leq & \gamma_2(\chi_2) \leq \ldots \leq \gamma_n(\chi_2) \leq \gamma_{n+1}(\chi_2) \leq \ldots.
\end{align*}

Next, suppose for given $n$ the following inequalities hold:

\begin{align*}
\gamma_m(\chi_1) < & \gamma_n(\chi_2) \leq \gamma_{m+1}(\chi_1).
\end{align*}

Then we define $\Delta_n(\chi_1, \chi_2) = n - m$. The question is: how often the difference $\Delta_n(\chi_1, \chi_2)$ vanishes as the ordinate $\gamma_n(\chi_2)$ varies in the interval $(T, T + H]$ which is supposed to be long enough?

In order to solve this problem, Fujii proved that the discrete random quantity with the values $\Delta_n(\chi_1, \chi_2)/\sqrt{\ln \ln n}$ has the distribution which tends to Gaussian distribution as $T$ grows. This implies that the values of $\Delta_n(\chi_1, \chi_2)$ differ from zero for ‘almost all’ $n$. However, the proof of these facts is based on the consideration of the integral

\begin{align*}
j = \int_T^{T+H} (S(t, \chi_1) - S(t, \chi_2))^2 dS(t, \chi_2)
\end{align*}

and, in particular, on the estimation

\begin{align*}
j = O((\ln T)^{2k+1}). \quad (7)
\end{align*}

This estimation plays the key role in Fujii’s arguments. Here $S(t, \chi) = \pi^{-1} \arg L(\frac{1}{2} + it, \chi)$ denotes the argument of Dirichlet’s $L$-function $L(s, \chi)$ on the critical line. The proof of (7) is missed.

This proof seems not satisfactory because the integral $j$ as Stiltjes integral does not exist. Indeed, the points of discontinuity of piecewise smooth function $S(t, \chi_2)$ under the differential coincides with the ordinates $\gamma_n(\chi_2)$. The function $S(t, \chi_1) - S(t, \chi_2)$ in the integrand has discontinuities at these points, too. In such case it is easy to construct two sequences of the integral sums for $j$ which tend to different limits (see, for example, [15, no. 584])).

These arguments, applied formally to the problem of calculating of even moments of $\Delta_n$, lead to the integrals

\begin{align*}
\int_T^{T+H} S^{2k}(t) dS(t), \quad k = 1, 2, 3, \ldots,
\end{align*}
which do not exist, too.

Thus the problem of reconstruction of Selberg’s proof of (2) and (3) is still open. In the below, we make such attempt. Now let’s consider the structure of the paper.

This article consist of three parts. The first part (§1) deals with the order of growth of $\Delta_n$ as $n \to +\infty$ (theorem 1). Next, an $\Omega$-estimations, which improve the assertions [1], are proved here (theorem 2). In essence, all these statements follow from the famous $O$- and $\Omega$- theorems for the function $S(t)$.

Lemma 2 in the basic assertion of the second part of the paper (§2). This lemma deals with the number of solutions of the inequality

$$a < \Delta_n \leq b,$$

for given integer $a$ and $b$. It appears that this number is very close to the number of solutions of the inequality

$$-(b + 1) < \Delta(n) \leq -(a + 1)$$

(the domains of $n$ are the same in both inequalities, of course). This fact implies the closure of the distributions of $\Delta_n$ and $\Delta(n)$ (theorem 3) and the closure of their moments of any degree. Thus, both the Selberg’s conjecture and the formulae (2),(3) follow from the corresponding theorems of [1] (see lemmas 3-5 of the present paper).

Finally, in the last, third part some equivalents of Selberg’s conjecture are proved. In particular, the problem of distribution of the differences $\gamma_n - t_n$ is considered (theorems 7-10). If the ordinate $\gamma_n$ obeys the Gram’s law then the order of such difference does not exceed the quantity

$$t_{n+1} - t_n \sim \frac{2\pi}{\ln n}. \tag{8}$$

However, the quantities $|\gamma_n - t_n|$ are much larger then (8) in the mean and their mean value is close to

$$\frac{\sqrt{\ln \ln n}}{\ln n}.$$

The paper ended with some theorems concerning the distribution of upper and lower ‘peaks’ of the graph of $S(t)$, i.e. the distribution of quantities $S(\gamma_n + 0)$, $S(\gamma_n - 0)$ (theorems 11-14). As in the case of differences $\Delta_n$, the major part of these peaks are of order $\sqrt{\ln \ln n}$.

NOTATIONS. Throughout the paper, $\varepsilon$ denotes an arbitrary small fixed positive number, $0 < \varepsilon < 0.001$, $N \geq N_0(\varepsilon) > 1$, $N$ is an integer, $M = \left[ N^{27/82+\varepsilon} \right]$, $L = \ln \ln N$, $\theta, \theta_1, \theta_2, \ldots$ are complex numbers whose absolute value does not exceed one and which are, generally speaking, different in different relations. All other notations are explained in the text.

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§1. The Order of Growth of the Values $\Delta_n$ as $n \to +\infty$

In this part, some statements concerning the order of growth of $\Delta_n$ are established. For the below, we need the following definition.

**Definition 6.** Suppose $\gamma$ is an ordinate for $r$ different zeros of the Riemann zeta-function $\zeta(s)$ with multiplicities $k_1, \ldots, k_r$. Then the sum

$$\kappa = k_1 + \ldots + k_r$$

is called the *multiplicity of the ordinate* $\gamma$. In the case $\gamma = \gamma_l$ we shall use the notation $\kappa_l$ instead of $\kappa$. Thus in the case

$$\gamma_{l-1} < \gamma_l = \gamma_{l+1} = \ldots = \gamma_{l+p-1} < \gamma_{l+p} \quad (9)$$

we obviously have: $\kappa_l = \kappa_{l+1} = \ldots = \kappa_{l+p-1} = p$. By basic properties of $S(t)$, if follows that

$$\kappa_l = S(\gamma_l + 0) - S(\gamma_l - 0). \quad (10)$$

**Theorem 1.** $\Delta_n = O(\ln n)$ as $n \to +\infty$. If the Riemann hypothesis is true then

$$\Delta_n = O\left(\frac{\ln n}{\ln \ln n}\right).$$

**Proof.** Using the definition of $N(t)$, from (5) we get

$$N(t_{m-1} - 0) < N(\gamma_n + 0) \leq N(t_m + 0). \quad (11)$$

Now, $N(\gamma_n + 0) = n + \theta_n(\kappa_n - 1)$, where $0 \leq \theta_n \leq 1$. Indeed, the case $\kappa_n = 1$ is obvious. Suppose the inequalities (9) hold for some $p \geq 2$. Then for $n = l + s$, $0 \leq s \leq p - 1$ we have $\kappa_n = p$. Therefore,

$$N(\gamma_n + 0) = l + p - 1 = n + (p - s - 1) = n + \kappa_n - s - 1 = n + \theta_n(\kappa_n - 1),$$

where

$$0 \leq \theta_n = \frac{\kappa_n - s - 1}{\kappa_n - 1} \leq 1.$$ 

Combining (11) with the Riemann-von Mangoldt formula we conclude

$$\frac{1}{\pi} \vartheta(t_{m-1}) + 1 + S(t_{m-1} - 0) < n + \theta_n(\kappa_n - 1) \leq \frac{1}{\pi} \vartheta(t_m) + 1 + S(t_m - 0).$$

Using the definition of Gram points and replacing $\vartheta(t_{m-1}), \vartheta(t_m)$ by the quantities $\pi(m-2), \pi(m-1)$, we get after some transformations:

$$-S(t_m + 0) - \kappa_n \leq m - n \leq -S(t_{m-1} - 0) + \kappa_n.$$

Now the assertion follows from (10) and from classical upper bounds for $|S(t)|$ (see, for example, [3]; for the values of implied constants in $O$’s, see the remark after lemma 7). This completes the proof.
The below theorem establishes the connection between the fractions (4) and the quantities \( \Delta_n \).

**Lemma 1.** As \( n \to +\infty \), the following relation holds:

\[
\Delta_n = q_n + \theta_n + O\left(\frac{\ln n}{n}\right),
\]

where \( 0 \leq \theta_n \leq 1 \) and the constant in \( O \)-symbol is an absolute.

**Proof.** First, from (5) it follows that

\[
\frac{t_{m-1} - t_n}{t_{n+1} - t_n} < q_n \leq \frac{t_m - t_n}{t_{n+1} - t_n}. \tag{12}
\]

Next, the definition of Gram’s points and the Lagrange’s mean value theorem imply the relation

\[
\pi \cdot (m - n) = \vartheta(t_m) - \vartheta(t_n) = \vartheta'(t_n)(t_m - t_n) + \frac{1}{2} \vartheta''(\xi)(t_m - t_n)^2,
\]

where \( \xi \) lies between \( t_m \) and \( t_n \). Thus,

\[
t_m - t_n = \frac{\pi(m - n)}{\vartheta'(t_n)} \cdot \frac{1}{1 + r}, \quad r = \frac{\vartheta''(\xi)}{2 \vartheta'(t_n)} (t_m - t_n). \tag{13}
\]

Further, theorem 1 implies the following rough estimate

\[
t_m - t_n \ll t_m + t_n \ll \frac{n}{\ln n}.
\]

From the relations

\[
\vartheta''(\xi) = \frac{1}{2\xi} + O\left(\frac{1}{\xi^3}\right) \ll \frac{1}{\xi} \ll \frac{1}{t_n} \ll \frac{\ln n}{n}, \quad \vartheta'(t_n) = \frac{1}{2} \ln \frac{t_n}{2\pi} + O\left(\frac{1}{t_n}\right) \gg \ln n
\]

we obtain the inequality

\[
r \ll \frac{\ln n}{n} \cdot \frac{1}{\ln n} \cdot \frac{n}{\ln n} \ll \frac{1}{\ln n}.
\]

Now the equality (13) implies more precise bounds for the difference \( t_m - t_n \) and for the quantity \( r \), namely

\[
t_m - t_n = \frac{\pi(m - n)}{\vartheta'(t_n)} \left(1 + O\left(\frac{1}{\ln n}\right)\right) \ll \frac{|m - n|}{\ln n} \ll 1, \quad r \ll \frac{\ln n}{n} \cdot \frac{1}{\ln n} \ll \frac{1}{n}.
\]

Thus we get

\[
t_m - t_n = \frac{\pi(m - n)}{\vartheta'(t_n)} \left(1 + O\left(\frac{1}{\ln n}\right)\right).
\]

Further, the equalities

\[
t_{m-1} - t_n = \frac{\pi(m - n - 1)}{\vartheta'(t_n)} \left(1 + O\left(\frac{1}{\ln n}\right)\right),
\]
are proved in the same way. Substituting all the expressions in (12), we obtain

\[
q_n \leq \frac{\pi(m - n) \cdot \vartheta'(t_n) (1 + O\left(\frac{1}{n}\right))}{\vartheta'(t_n) \cdot \pi} = \Delta_n + O\left(\frac{\ln n}{n}\right),
\]

and

\[
q_n > \frac{\pi(m - n - 1) \cdot \vartheta'(t_n) (1 + O\left(\frac{1}{n}\right))}{\vartheta'(t_n) \cdot \pi} = \Delta_n - 1 + O\left(\frac{\ln n}{n}\right).
\]

This completes the proof of the lemma.

**Theorem 2.** The inequalities

\[
\max_{N < n \leq N + M} (\pm \Delta_n) \geq c\left(\frac{\ln N}{\ln \ln N}\right)^{1/3},
\]

hold for some positive constant \( c = c(\varepsilon) \).

**Proof.** Let \( Q = c_1\left(\frac{\ln N}{\ln \ln N}\right)^{1/3} \) where the constant \( c_1 \) will be chosen later. Suppose the inequalities

\[
q_n = \frac{\gamma_n - t_n}{t_{n+1} - t_n} < Q,
\]

hold true for any \( n, N < n \leq N + M \). Taking \( \tau_n = t_n + Q(t_{n+1} - t_n) \), we conclude from (15) that \( \gamma_n < \tau_n \). Then the Riemann-von Mangoldt formula implies that the inequalities

\[
n \leq N(\gamma_n + 0) \leq N(\tau_n + 0) = \frac{1}{\pi} \vartheta(\tau_n) + 1 + S(\tau_n + 0)
\]

hold true for any \( n \) from the interval under considering. Combining the equality (14) with Lagrange’s mean value theorem, we have

\[
\vartheta(\tau_n) = \vartheta(t_n) + \vartheta'(t_n)Q(t_{n+1} - t_n) + \frac{1}{2} \vartheta''(\xi)Q^2(t_{n+1} - t_n)^2 = \\
= \vartheta(t_n) + \pi Q \left(1 + O\left(\frac{1}{N \ln N}\right)\right) + O\left(\frac{Q^2}{N \ln N}\right) = \pi(n - 1 + Q) + O(N^{-1})
\]

for some \( t_n < \xi < t_{n+1} \). Substituting this expression for \( \vartheta(\tau_n) \) in (16), we obtain

\[
S(\tau_n + 0) \geq -Q + O(N^{-1}).
\]

Now let’s show that the inequality (17) can’t hold true for every \( n \) under condition \( N < n \leq N + M \). For arbitrary positive \( x \), by \( t_x = t(x) \) we denote the unique solution of the equation

\[
\vartheta(t_x) = \pi \cdot (x - 1)
\]

and set \( \tau_x = t_x + Q(t_{x+1} - t_x) \). Then

\[
\tau_x' = t_x' + Q(t_{x+1}' - t_x') = t_x' + Q t_x''
\]
for some \( \eta, x < \eta < x + 1 \). By differentiating the equation (18) with respect to \( x \) twice we obtain

\[
\begin{align*}
t'_x &= \frac{\pi}{\vartheta'(t_x)}, \\
t''_x &= -\frac{(t'_x)^2 \vartheta''(t_x)}{\vartheta'(t_x)} = -\pi^2 \frac{\vartheta''(t_x)}{(\vartheta'(t_x))^3} \ll \frac{1}{x \ln^2 x}.
\end{align*}
\]

Therefore, the inequalities

\[
\tau'_x = \frac{\pi}{\vartheta'(t_x)} - O\left(\frac{Q}{x \ln^2 x}\right) = \frac{\pi}{\vartheta'(t_x)} \left(1 - O\left(\frac{Q}{x \ln x}\right)\right) > \frac{\pi}{2 \vartheta'(t_x)} > 0,
\]

holds for \( N < x \leq N + M \). Now it follows that the sequence \( \tau_n \) increases monotonically in the interval under considering. Finally, let’s note that

\[
\tau_{n+1} - \tau_n = \tau'_\zeta < \frac{\pi}{\vartheta'(t_\zeta)} < \frac{\pi}{\vartheta'(t_n)} < \frac{3\pi}{\ln n}
\]

for some \( n < \zeta < n + 1 \). Now let’s show that if the point \( \tau_n \) is close to the minima of \( S(t) \) then the value \( S(\tau_n) \) is very large and negative. This will contradict to (17).

From omega-theorem for \( S(t) \) (see [3]), it follows that there exists a point \( \tau \) in the interval \((\tau_N, \tau_{N+M}]\) such that

\[
S(\tau) < -c_0 \left(\frac{\ln N}{\ln \ln N}\right)^{1/3},
\]

for some constant \( c_0 = c_0(\varepsilon) > 0 \). Choosing \( n \) from the inequalities \( \tau_n < \tau \leq \tau_{n+1} \), we suppose that \( \gamma^{(1)} < \gamma^{(2)} < \ldots < \gamma^{(s)} \) are all different ordinates of zeros of \( \zeta(s) \) lying the interval \((\tau_n, \tau] \). Then, performing the increment of the function \( S(t) \) along this interval as the sum of increments along all subintervals of continuity of \( S(t) \) and the sum of jumps of \( S(t) \) at the points of discontinuity \( \gamma^{(1)}, \gamma^{(2)}, \ldots, \gamma^{(s)} \), we get the following identity:

\[
S(\tau) - S(\tau_n + 0) = \left\{S(\gamma^{(1)} - 0) - S(\tau_n + 0)\right\} + \\
\left\{S(\gamma^{(1)} + 0) - S(\gamma^{(1)} - 0)\right\} + \left\{S(\gamma^{(2)} - 0) - S(\gamma^{(1)} + 0)\right\} + \ldots \\
\quad + \left\{S(\gamma^{(s)} + 0) - S(\gamma^{(s)} - 0)\right\} + \left\{S(\tau) - S(\gamma^{(s)} + 0)\right\}.
\]

Obviously, this relation implies the following inequality:

\[
S(\tau) - S(\tau_n + 0) \geq \left\{S(\gamma^{(1)} - 0) - S(\tau_n + 0)\right\} + \left\{S(\gamma^{(2)} - 0) - S(\gamma^{(1)} + 0)\right\} + \ldots \\
\quad + \left\{S(\tau) - S(\gamma^{(s)} + 0)\right\}. \quad (19)
\]

Let \((a, b)\) be an interval of continuity of \( S(t) \), and suppose that \( 1 < a < b, b - a < 1 \). Then

\[
S(b) - S(a) = (b - a)S'(\xi) = (b - a) \left( -\frac{1}{2} \ln \frac{\xi}{2\pi} + O(\xi^{-2}) \right) = \\
= (b - a) \left( -\frac{1}{2} \ln \frac{\xi}{2\pi} + O(a^{-2}) \right), \quad (20)
\]
for some $a < \xi < b$. Using the inequality
\[
\tau - \tau_n \leq \tau_{n+1} - \tau_n < \frac{3\pi}{\ln n} < 1,
\]
from (19) and (20) we conclude
\[
S(\tau) - S(\tau + 0) \geq \left\{ (\gamma^{(1)} - \tau_n) + (\gamma^{(2)} - \gamma^{(1)}) + \ldots + (\tau - \gamma^{(s)}) \right\} \left( -\frac{1}{2} \ln \frac{\tau}{2\pi} + O(\tau^{-2}) \right) > \\
> - (\tau - \tau_n) \ln \tau > -\frac{3\pi}{\ln N} \ln N = -3\pi.
\]
Hence
\[
S(\tau_n + 0) \leq S(\tau) + 3\pi < -c_0 \left( \frac{\ln N}{\ln \ln N} \right)^{1/3} + 3\pi < -0.9c_0 \left( \frac{\ln N}{\ln \ln N} \right)^{1/3}.
\]
Now it is easy to see that the last inequality contradicts to (17), if we choose $c_1 = 0.8c_0$. Therefore, there exists at least one number $n$ under condition $N < n \leq N + M$, such that the inequality (15) fails. By lemma 1, for such $n$ we get
\[
\Delta_n > q_n - 1 + O \left( \frac{\ln n}{n} \right) \geq Q - 1.1 > c \left( \frac{\ln N}{\ln \ln N} \right)^{1/3},
\]
where $c = 0.7c_0$.

The existence of large negative values of $\Delta_n$ is proved in the same way. If we suppose that $q_n > -Q$ holds for every $n$ under considering then it follows that $\gamma_n > \sigma_n$ where $\sigma_n = t_n - Q(t_{n+1} - t_n)$, and hence
\[
N(\sigma_n + 0) \leq N(\gamma_n - 0) < n.
\]
Transforming the left side as above we get
\[
S(\sigma_n + 0) < Q + O(N^{-1}). \tag{21}
\]

The application of omega-theorem for $S(t)$ yields the existence of a point $\sigma$, $\sigma_N < \sigma \leq \sigma_{N+M}$ such that
\[
S(\sigma) > c_0 \left( \frac{\ln N}{\ln \ln N} \right)^{1/3}.
\]
Since the sequence $\sigma_n$ is monotonically increasing in the interval under considering, it’s possible to point out the number $n$ such that $\sigma_{n-1} < \sigma \leq \sigma_n$. Using the same arguments as above and applying the inequality $\sigma_n - \sigma_{n-1} < 3\pi(\ln n)^{-1}$, we have
\[
S(\sigma_n + 0) - S(\sigma) > (\sigma_n - \sigma) \left( -\frac{1}{2} \ln \frac{\sigma_n}{2\pi} + O(\sigma_n^{-2}) \right) > -(\sigma_n - \sigma) \ln \sigma_n > -3\pi,
\]
\[
S(\sigma_n + 0) > S(\sigma) - 3\pi > 0.9c_0 \left( \frac{\ln N}{\ln \ln N} \right)^{1/3}.
\]
The last relation contradicts to (21), if we choose \( c_1 = 0.8c_0 \). Therefore, there exists at least one number \( n, N < n \leq N + M \), such that \( q_n \leq -Q \). For such \( n \) we have

\[
\Delta_n \leq q_n + 1.1 \leq -Q + 1.1 < -c \left( \frac{\ln N}{\ln \ln N} \right)^{1/3},
\]

where \( c = 0.7c_0 \). The theorem is completely proved.

§2. Selberg Hypothesis and the Moments of \( \Delta_n \)

This section is devoted to the proof of Selberg hypothesis and to the calculation of the moments of \( \Delta_n \), that is, to the calculation of the sums

\[
\sum_{N < n \leq N + M} \Delta_n^a, \quad \sum_{N < n \leq N + M} |\Delta_n|^a
\]

for different \( a \). We need the following definition.

**Definition 7.** Suppose \( a, b \) are an arbitrary real numbers, \( a < b \). Denote by \( e(a, b) \) the number of solutions of the inequalities \( a < \Delta_n \leq b \) with the condition \( N < n \leq N + M \). Similarly, by \( f(a, b) \) we denote the number of solutions of the inequalities \( a < \Delta(n) \leq b \) under the same condition.

The below assertion plays the key role in what follows.

**Lemma 2.** The relation

\[
e(a, b) = f(-b + 1, -(a + 1)) + \theta(|a| + |b| + 2)
\]

holds true for any integers \( a \) and \( b, a < b \).

**Proof.** First we get \( e(a, b) = M - e_1 - e_2 \), where \( e_1 \) and \( e_2 \) denote, respectively, the numbers of solutions of inequalities

\[
\Delta_n \leq a, \quad \Delta_n > b
\]

under the same condition \( N < n \leq N + M \). The first inequality in (22) is equivalent to the following:

\[
\gamma_n \leq t_{n+a}
\]

Indeed, if \( t_{m-1} < \gamma_n \leq t_m \) and \( \Delta_n = m - n \leq a \), then \( m \leq n + a \) and therefore \( \gamma_n \leq t_{n+a} \). Suppose now that (23) holds. Then for the number \( m \) defined above we have: \( \gamma_n \leq t_{n+a} \leq t_{n+a} \). Hence, \( m \leq n + a \) and \( \Delta_n \leq a \).

By setting \( \nu = n + a \) in (23), we obtain that \( e_1 \) equals to the number of solutions of the inequality

\[
\gamma_{\nu-a} \leq t_{\nu}
\]

with the condition \( N + a < \nu \leq N + M + a \). Hence, the difference between \( e_1 \) and the number \( f_1 \) of solutions of (24) under the condition \( N < \nu \leq N + M \) does not exceed \( |a| \). Thus, \( e_1 = f_1 + \theta_1 |a| \).

Using the same arguments we obtain that the inequalities \( \Delta_n > b \) and \( \gamma_n > t_{n+b} \) are equivalent and the quantity \( e_2 \) equals to the number of solutions \( \nu \) such that \( N + b < \nu \leq N + M + b \) and

\[
\gamma_{\nu-b} > t_{\nu}.
\]
Hence, \( e_2 = f_2 + \theta_2|b| \), where \( f_2 \) denotes the number of solutions of (25) with the condition \( N < \nu \leq N + M \).

Suppose the Gram point \( t_\nu \) does not satisfy both (24) and (25). Then

\[
\gamma_{\nu-b} \leq t_\nu < \gamma_{\nu-a}.
\]  

Let’s show that (26) is equivalent to double inequality

\[
-(b+1) < \Delta(\nu) \leq -(a+1).
\]  

Indeed, determining \( m \) from the inequalities \( \gamma_m \leq t_\nu < \gamma_{m+1} \), from (26) we conclude that \( \nu - b \leq m \leq \nu - a - 1 \). Hence, (27) holds true. Next, for \( m \) defined above, from (27) it follows that \( \nu - b \leq m \leq \nu - a - 1 \) and hence

\[
\gamma_{\nu-b} \leq \gamma_m \leq t_\nu < \gamma_{m+1} \leq \gamma_{\nu-a}.
\]

Therefore, the number of solutions of (26) satisfying the condition \( N < \nu \leq N + M \), equals to \( f(-(b+1), -(a+1)) \). Thus,

\[
e(a, b) = M - e_1 - e_2 = M - f_1 - f_2 - \theta_1|a| - \theta_2|b| = \\
= f(-(b+1), -(a+1)) + \theta(|a| + |b|).
\]

Lemma is completely proved.

**Lemma 3.** For a real \( x \) let the quantity \( \nu(x) \) denote the number of integers \( n \), \( N < n \leq N + M \), satisfying the condition

\[
\Delta(n) \leq \frac{x}{\pi} \sqrt{\frac{L}{2}}.
\]

Then

\[
\nu(x) = M \left( \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-u^2/2} du + \theta \delta \right),
\]

where \( \delta = e^{22.4} \varepsilon^{-1.5}(\ln L)^{-0.5} \).

For the proof, see [1].

**Theorem 3.** For any \( a \) and \( b \), \( a < b \), the number of solutions of the inequality \( a < \Delta_n \leq b \) with the condition \( N < n \leq N + M \) satisfies the relation

\[
e(a, b) = M \left( \frac{1}{\sqrt{2\pi}} \int_{\alpha}^{\beta} e^{-u^2/2} du + \theta \Delta \right),
\]

where \( \alpha = \pi a \sqrt{2/L} \), \( \beta = \pi b \sqrt{2/L} \), \( \Delta = 2.2 e^{22.4} \varepsilon^{-1.5}(\ln L)^{-0.5} \).

**Proof.** Let \( a, b \) be an integers and let \( c \) be sufficiently large constant such that the inequalities \( |\Delta_n| \leq l \), \( |\Delta(n)| \leq l \) hold true for any \( N < n \leq N + M \) with \( l = |c \ln N| \). Then
in the case \(-l \leq a < b \leq l\) the assertion follows from lemmas 2 and 3:

\[
e(a, b) = f(-(b + 1), -(a + 1)) + 2\theta_1 l =
\]

\[
= M\left(\frac{1}{\sqrt{2\pi}} \int_{-\pi(a+1)\sqrt{2/L}}^{\pi(b+1)/\sqrt{2/L}} e^{-u^2/2} du + 2\theta_2 \delta\right) + 2\theta_1 l =
\]

\[
= M\left(\frac{1}{\sqrt{2\pi}} \int_{\pi a\sqrt{2/L}}^{\pi b\sqrt{2/L}} e^{-u^2/2} du + \theta_2 \sqrt{\pi/L} + 2\theta_2 \delta\right) + 2\theta_1 l =
\]

\[
= M\left(\frac{1}{\sqrt{2\pi}} \int_{\alpha_1}^{\beta} e^{-u^2/2} du + 2.1\theta \delta\right).
\]

In the case \(-l \leq a < l < b\) the required statement follows from the equality \(e(a, b) = e(a, l)\) and from the estimate

\[
\frac{1}{\sqrt{2\pi}} \int_{\alpha_1}^{\beta} e^{-u^2/2} du < \lambda^{-1} e^{-\lambda^2/2} < 0.01\delta, \quad \lambda = \pi l \sqrt{2/L}.
\]

The cases \(a < -l \leq b \leq l\), \(a < -l < l < b\) are handle as above. If \(a\) or \(b\) is non-integer, then assertion follows from the relation \(e(a, b) = e([a], [b])\) and the above arguments. Theorem is completely proved.

**Corollary.** Selberg hypothesis is true. Moreover, if \(\Phi(x) > 0\) is an arbitrary unbounded monotonically increasing function, then the number of \(n, N < n \leq N + M\), that do not satisfy the condition

\[
\frac{1}{\Phi(n)} \sqrt{\ln \ln n} < |\Delta_n| \leq \Phi(n) \sqrt{\ln \ln n},
\]

is of order not exceeding \(M(\Phi^{-1}(N) + \Delta)\).

Proof of this assertion repeats practically word-for-word the proof of the corollary of theorem 4 in [1].

**Remark.** Theorem 3 asserts that the quantity \(r(a, b)\) in the relation

\[
e(a, b) = M\left(\frac{1}{\sqrt{2\pi}} \int_{\alpha_1}^{\beta} e^{-u^2/2} du + r(a, b)\right)
\]

obeys the estimate

\[
r(a, b) = O(\ln L)^{-0.5} = O((\ln \ln N)^{-0.5}).
\]

for any \(a\) and \(b\). Suppose \(\varepsilon\) be any positive number, \(\varepsilon < \frac{1}{2}\). Then for any \(a\) we have

\[
0 = e(a + \varepsilon, a + 1 - \varepsilon) = M\left(\frac{1}{\sqrt{2\pi}} \int_{\alpha_1}^{\alpha_2} e^{-u^2/2} du + r(a + \varepsilon, a + 1 - \varepsilon)\right),
\]

where \(\alpha_1 = \pi(a + \varepsilon)\sqrt{2/L}, \alpha_2 = \pi(a + 1 - \varepsilon)\sqrt{2/L}\). Hence,

\[
r(a + \varepsilon, a + 1 - \varepsilon) = -\frac{1}{\sqrt{2\pi}} \int_{\alpha_1}^{\alpha_2} e^{-u^2/2} du = O(\alpha_2 - \alpha_1) = O(L^{-0.5}) = O((\ln \ln N)^{-0.5}).
\]

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Probably this estimate holds true for any $a, b$. It’s interesting to compare this assumption with one Selberg’s assertion stated in [16, Theorem 2].

For the below, we need some new notations. For positive $a$, we set

$$\kappa(a) = \frac{2^a}{\sqrt{\pi}} \Gamma\left(\frac{a + 1}{2}\right).$$

Then for integer $k \geq 0$ we have:

$$\kappa(2k) = \frac{(2k)!}{k!}, \quad \kappa(2k + 1) = \frac{2^{2k+1}}{\sqrt{\pi}} k!.$$

In what follows, we shall use the obvious estimations

$$\kappa(2k) < \frac{3}{2} \left(\frac{4k}{e}\right)^k, \quad \frac{1}{\kappa(2k)} < \left(\frac{e}{4k}\right)^k,$$

without special comments, and the same for inequalities

$$k \leq (\sqrt{3})^k, \quad k\sqrt{k} \leq (\sqrt{3})^k, \quad k^2 \leq (\sqrt{9})^k.$$

Finally, we set $A = e^{21} \varepsilon^{-1.5}, B = A^2 e^{-8} = e^{34} \varepsilon^{-3}$. In order to prove Selberg’s formulae, we need some auxiliary assertions from [1].

**Lemma 4.** Let $k$ be an integer with the condition $1 \leq k \leq \sqrt{L}$. Then the following relation holds:

$$\sum_{N < n \leq N + M} \Delta^{2k}(n) = \frac{\kappa(2k)}{(2\pi)^{2k}} M L^{k} \left(1 + \theta A^k L^{-0.5}\right).$$

Moreover, we have the estimate

$$\left|\sum_{N < n \leq N + M} \Delta^{2k-1}(n)\right| \leq \frac{3.5}{\sqrt{B}} (Bk)^k M L^{k-1}.$$

**Lemma 5.** Let $a$ be an arbitrary number with the condition

$$0 < a \leq \frac{e \ln L}{10 A \ln \ln L}.$$

Then the following asymptotic formula holds:

$$\sum_{N < n \leq N + M} |\Delta(n)|^a = \frac{\kappa(a)}{(2\pi)^a} M L^{0.5a} \left(1 + \theta c(A^{-1} \ln L)^{-\gamma}\right).$$

Furthermore, in the case $0 < a \leq 1$ the quantities $c$ and $\gamma$ can be set to be equal to 90 and 0.5a, respectively, and in the case $a > 1$ to $2^{15.4}$ and $\frac{1}{2} + \left\{\frac{a-1}{2}\right\}$.

Similarly to the assertions of lemmas 4 and 5, the below analogues of Selberg’s formulae are uniform to the parameter $k$.

**Theorem 4.** Let $k$ be an integer with the condition $1 \leq k \leq \sqrt{L}$. Then the following equality holds:

$$\sum_{N < n \leq N + M} \Delta^{2k}_n = \frac{\kappa(2k)}{(2\pi)^{2k}} M L^k \left(1 + 1.1 \theta A^k L^{-0.5}\right).$$
In particular, for fixed $k$ we get
\[
\sum_{N < n \leq N + M} \Delta_{n}^{2k} = \frac{(2k)!}{k!} \frac{M}{(2\pi)^{2k}} (\ln \ln N)^k + O(M(\ln \ln N)^{k-1/2}).
\]

**Corollary.** Under the same restrictions on $k$, the following estimate holds:
\[
\sum_{N < n \leq N + M} \Delta_{n}^{2k} \leq \frac{1.1}{A} \frac{x(2k)}{(2\pi)^{2k}} M(AL)^k.
\]

**Theorem 5.** Let $k$ be an integer with the condition $1 \leq k \leq \sqrt{L}$. Then the following estimate holds:
\[
\left| \sum_{N < n \leq N + M} \Delta_{n}^{2k-1} \right| \leq \frac{3.6}{\sqrt{B}} (Bk)^k ML^{k-1}.
\]

In particular, for fixed $k$ we have:
\[
\sum_{N < n \leq N + M} \Delta_{n}^{2k-1} = O(M(\ln \ln N)^{k-1}).
\]

**Theorem 6.** Let $a$ be an arbitrary number with the condition
\[
0 < a \leq \frac{e \ln L}{10A \ln \ln L}.
\]

Then the following asymptotic formula holds:
\[
\sum_{N < n \leq N + M} |\Delta_n|^a = \frac{x(a)}{(2\pi)^a} ML^{0.5a} (1 + \theta_c(A^{-1} \ln L)^{-\gamma}).
\]

Furthermore, in the case $0 < a \leq 1$ the quantities $c$ and $\gamma$ can be set to be equal to 91 and 0.5a, respectively, and in the case $a > 1$ to $2^{15.5}$ and $\frac{1}{2} + \{\frac{a-1}{2}\}$. In particular, in the case $a = 1$ we get the equality:
\[
\sum_{N < n \leq N + M} |\Delta_n| = \frac{M}{\pi \sqrt{\pi}} \sqrt{\ln \ln N} \left(1 + O((\ln \ln \ln N)^{-1/2})\right).
\]

**Proof.** Theorems 4-6 can be proved in a similar way. Using the fact that the numbers $\Delta_n$ are integral, for any $a > 0$ we obtain
\[
\sum_{N < n \leq N + M} |\Delta_n|^a = \sum_{\nu=1}^{l} \nu^a \left( e(\nu - 1, \nu) + e(-\nu + 1) \right).
\]

The number $l = \lceil c \ln N \rceil$ is chosen such that the absolute values of the quantities $\Delta_n$, $\Delta(n)$ do not exceed $l$ for $N < n \leq N + M$. By lemma 2, for $\nu \geq 2$ we get
\[
e(\nu - 1, \nu) = f(-(\nu+1), -\nu) + \theta_1(2\nu-1), \quad e(-\nu+1, -\nu)) = f(\nu - 1, \nu) + \theta_2(2\nu+1).
\]
Hence,

\[
\sum_{N<n \leq N+M} |\Delta_n|^a = \sum_{\nu=1}^{l} \nu^a \left( f(\nu - 1, \nu) + f(-(\nu + 1), -\nu) + 4\theta_3 \nu \right) = \sum_{N<n \leq N+M} |\Delta(n)|^a + 4\theta_4 \sum_{\nu=1}^{l} \nu^{a+1}.
\]

Now theorems 4 and 6 follow from lemmas 4 and 5, respectively, and from obvious bound

\[
4 \sum_{\nu=1}^{l} \nu^{a+1} < \frac{4(l + 1)^{a+2}}{a + 2} < (1.5c \ln N)^{a+2}.
\]

Further, for integer \(k \geq 1\) we have:

\[
\sum_{N<n \leq N+M} \Delta_{2k-1} = \sum_{\nu=1}^{l} \left( \nu^{2k-1}e(\nu - 1, \nu) + (-\nu)^{2k-1}e(-(\nu + 1), -\nu) \right) = \sum_{\nu=1}^{l} \nu^{2k-1} \left( e(\nu - 1, \nu) - e(-(\nu + 1), -\nu) \right) = \sum_{\nu=1}^{l} \nu^{2k-1} \left( f(-(\nu + 1), -\nu) - f(\nu - 1, \nu) + 4\theta_1 \nu \right) = -\sum_{\nu=1}^{l} \left( \nu^{2k-1}f(\nu - 1, \nu) + (-\nu)^{2k-1}f(-(\nu + 1), -\nu) \right) + \theta_2 (1.5c \ln N)^{a+2} = -\sum_{N<n \leq N+M} \Delta_{2k-1}^{a+2} + \theta_3 (1.5c \ln N)^{a+2}.
\]

Thus, the assertion of Theorem 5 is a corollary of lemma 4.

§3. The Distribution of Differences \(\gamma_n - t_n\)

Lemma 1 implies that the difference between quantities \(\Delta_n\) and \(q_n = \frac{\gamma_n - t_n}{t_{n+1} - t_n}\) does not exceed \(O(1)\). This fact together with the assertions of theorems 3-6 allow us to find approximately the distribution function for the differences \(t_n - \gamma_n\) and to calculate the moments of these quantities.

**Lemma 6.** For any \(n\) with the condition \(N < n \leq N + M\) the following equality holds

\[
\gamma_n - t_n = \frac{\pi}{\vartheta'(t_N)} \left( \Delta_n + \varepsilon_n \right),
\]

where \(|\varepsilon_n| \leq 1.01\).

**Proof.** Using (14) and the relation

\[
\frac{1}{\vartheta'(t_n)} = \frac{\pi}{\vartheta'(t_N)} \left( 1 + O \left( \frac{M}{N \ln N} \right) \right).
\]
we get:

$$\gamma_n - t_n = q_n(t_{n+1} - t_n) = \frac{\pi q_n}{\vartheta'(t_n)} \left( 1 + O\left( \frac{1}{N \ln N} \right) \right) = \frac{\pi q_n}{\vartheta'(t_N)} \left( 1 + O\left( \frac{M}{N \ln N} \right) \right).$$

Expressing $q_n$ by $\Delta_n$, from lemma 1 we obtain

$$\gamma_n - t_n = \pi \vartheta'(t_N) \left( \Delta_n - \theta_n + O\left( \frac{\ln N}{N} \right) \right) \left( 1 + O\left( \frac{M}{N \ln N} \right) \right) = \pi \vartheta'(t_N) (\Delta_n + \varepsilon_n),$$

where

$$|\varepsilon_n| = |\theta_n + O(MN^{-1})| \leq 1.01.$$

The lemma is proved.

**Theorem 7.** Let $k$ be an integer with the condition $1 \leq k \leq \sqrt{L}$. Then the following equality holds

$$\sum_{N < n \leq N+M} (\gamma_n - t_n)^{2k} = \frac{\pi(2k)}{(2 \vartheta'(t_N))^{2k}} ML^k \left( 1 + 0.4 \theta(4A^{\sqrt{3}}) L^{-0.5} \right).$$

**Proof.** Using the equality

$$(a + b)^{2k} = a^{2k} + \theta k 2^{2k-1} (|a|^{2k-1}|b| + |b|^{2k})$$

and defining the quantities $\varepsilon_n$ as in lemma 6, we find

$$(\Delta_n + \varepsilon_n)^{2k} = \Delta_n^{2k} + \theta k 2^{2k-1} (|\Delta_n|^{2k-1} |\varepsilon_n| + |\varepsilon_n|^{2k}) =$$

$$= \Delta_n^{2k} + \theta k 2^{2k} (|\Delta_n|^{2k-1} + (1.01)^{2k}) =$$

$$\sum_{N < n \leq N+M} (\Delta_n + \varepsilon_n)^{2k} = \Sigma_1 + \theta k 2^{2k} (\Sigma_2 + (1.01)^{2k} M),$$

where

$$\Sigma_1 = \sum_{N < n \leq N+M} \Delta_n^{2k}, \quad \Sigma_2 = \sum_{N < n \leq N+M} |\Delta_n|^{2k-1}.$$

Further,

$$M = \frac{\pi(2k)}{(2\pi)^{2k}} ML^k \delta_1, \quad \delta_1 = \frac{(2\pi)^{2k}}{\pi(2k)} \frac{1}{L^k} < \left( \frac{4\pi^2}{L} \right)^k \left( \frac{e}{4k} \right)^k = \left( \frac{\pi^2 e}{k L} \right)^k \leq \frac{\pi^2 e}{L},$$

and, furthermore,

$$\delta_1^{1/(2k)} < \sqrt{\frac{\pi^2 e}{k L}}.$$

Applying Hölder’s inequality together with the corollary of theorem 4 we get

$$k 2^{2k} \Sigma_2 \leq k 2^{2k} M^{1/(2k)} \Sigma_1^{1-1/(2k)} = \frac{\pi(2k)}{(2\pi)^{2k}} ML^k \delta_2,$$

$$\delta_2 = k 2^{2k} \sqrt{\frac{\pi^2 e}{k L}} \frac{1.1 A^k}{\sqrt{A}} < 0.1 (4A^{\sqrt{3}})^k \sqrt{L}.$$
Returning to the initial sum, we obtain
\[
\sum_{N<n\leq N+M} (\Delta_n + \varepsilon_n)^{2k} = \frac{\varphi(2k)}{(2\pi)^{2k}} ML^k(1 + \theta\delta),
\]
where
\[
\delta = \frac{1.1A^k}{\sqrt{L}} + 0.1(4A^\sqrt{3})^k + k(2.02)^{2k} \frac{\pi^2e}{L} < \frac{0.4(4A^\sqrt{3})^k}{\sqrt{L}},
\]
\[
\sum_{N<n\leq N+M} (\gamma_n - t_n)^{2k} = \left(\frac{\pi}{\vartheta(t_N)}\right)^{2k} \sum_{N<n\leq N+M} (\Delta_n + \varepsilon_n)^{2k} = \frac{\varphi(2k)}{(2\vartheta(t_N))^{2k}} ML^k(1 + \theta\delta).
\]

The theorem is proved.

**Remark.** If we set \(e_n = (\gamma_n - t_n) \vartheta'(t_N)\sqrt{2/L}\), then the assertion of theorem 7 can be represented in the form:
\[
\sum_{N<n\leq N+M} e_n^{2k} = \frac{(2k)!}{k!} \frac{M}{2^k} (1 + 0.4\theta(4A^\sqrt{3}L^{-0.5})).
\]

**Theorem 8.** Let \(k\) be an integer with the condition \(1 \leq k \leq \sqrt{L}\). Then the following estimation holds:
\[
\left| \sum_{N<n\leq N+M} (\gamma_n - t_n)^{2k-1} \right| < \frac{3.7}{\sqrt{B}} \left(\frac{\pi}{\vartheta(t_N)}\right)^{2k-1} (Bk)^k ML^{k-1}.
\]

**Proof.** By the relation
\[
(a - b)^{2k-1} = a^{2k-1} + \theta k 2^{2k-1} (a^{2k-2}|b| + |b|^{2k-1}),
\]
we obtain:
\[
\sum_{N<n\leq N+M} (\Delta_n + \varepsilon_n)^{2k-1} = \sum_{N<n\leq N+M} \Delta_n^{2k-1} + \theta_1(k 2^{2k} \sum_{N<n\leq N+M} \Delta_n^{2k-2} + k(2.02)^{2k} M).
\]

First, the corollary of theorem 4 implies the estimations
\[
k 2^{2k} \sum_{N<n\leq N+M} \Delta_n^{2k-2} \leq k 2^{2k} \frac{1.1}{A} \frac{\varphi(2k-2)}{(2\pi)^{2k-2}} M(AL)^{k-1} = \frac{3.6}{\sqrt{B}} (Bk)^k ML^{k-1} \delta_1,
\]
where
\[
\delta_1 = \frac{\sqrt{B}}{3.6} \cdot \frac{1.1k 2^{2k}}{(Bk)^k} \frac{(4A)^k}{A^2 (2\pi)^{2k-2}} \varphi(2k-2) = \frac{1.1 \sqrt{B}}{3.6 A^2} \cdot \frac{4\pi^2}{4\pi^2 Bk} \frac{4A}{2k(2k-1)} < \frac{1.1 \sqrt{B}}{3.6 A^2} \cdot \frac{2\pi^2}{\pi^2 Bk} \left(\frac{4A}{2\pi B e}\right)^k < \frac{1.1}{1.2} \frac{\sqrt{B}}{A^2} \frac{4A}{\pi^2 B e} = \frac{4e^{-5}}{B}.
\]

Further,
\[
k(2.02)^{2k} M = \frac{3.6}{\sqrt{B}} (Bk)^k ML^{k-1} \delta_2,
\]
\[
\delta_2 = \frac{\sqrt{B}}{3.6} \cdot \frac{k}{(Bk)^k} \frac{(2.02)^{2k}}{L^{k-1}} < \frac{\sqrt{B}}{3.6} \frac{1}{(kL)^{k-1}} \left(\frac{4.1}{B}\right)^k < \frac{4.1}{3.6} \frac{1}{\sqrt{B}} < \frac{1.2}{\sqrt{B}}.
\]
Finally, combining all these estimations and using the theorem 5, we get
\[
\left| \sum_{N<n \leq N+M} (\Delta_n + \varepsilon_n)^{2k-1} \right| < \frac{3.6}{\sqrt{B}} (Bk)^k ML^{k-1} \left( 1 + \frac{4e^{-5}}{B} + \frac{1.2}{\sqrt{B}} \right) < \\
< \frac{3.7}{\sqrt{B}} (Bk)^k ML^{k-1}.
\]

The last relation implies the required assertion.

**Remark.** The estimate of theorem 8 can be represented in the form:
\[
\left| \sum_{N<n \leq N+M} \varepsilon_n^{2k-1} \right| < \frac{2.4}{\sqrt{B}} (Be\pi^2)^k \cdot \frac{(2k-1)!}{k!} ML^{-0.5}.
\]

**Theorem 9.** Let \( a \) be an arbitrary number with the condition
\[
0 < a < \frac{e \ln L}{10A \ln \ln L}.
\]
Then the following asymptotic formula holds:
\[
\sum_{N<n \leq N+M} |\gamma_n - t_n|^a = \frac{\gamma(a)}{(2\varphi(t_N))^a} ML^{0.5} (1 + \theta c_1 (A^{-1} \ln L)^{-\gamma}).
\]

Furthermore, in the case \( 0 < a \leq 1 \) the quantities \( c_1 \) and \( \gamma \) can be set to be equal \( 97 \) and \( 0.5a \), respectively, and in the case \( a > 1 \) to \( 2^{15.6} \) and \( \frac{1}{2} + \left\{ \frac{a-1}{2} \right\} \). In particular, in the case \( a = 1 \) we get the equality:
\[
\sum_{N<n \leq N+M} |\gamma_n - t_n| = 2M \frac{\ln \ln N}{\ln N} \left( 1 + O((\ln \ln N)^{-0.5}) \right).
\]

**Proof.** First consider the case \( 0 < a \leq 1 \). Let’s show that
\[
|\Delta_n + \varepsilon_n|^a = |\Delta_n|^a + 1.01\theta. \tag{28}
\]
Indeed, in the case \( \Delta_n = 0 \) the equality \( \text{(28)} \) follows from the estimate \( |\varepsilon_n| \leq 1.01 \). If \( \Delta_n = 1 \) then \( |\Delta_n + \varepsilon_n|^a = 2.01\theta \) and hence
\[
|\Delta_n + \varepsilon_n|^a = 1 + 2.01\theta - 1 = 1 + 1.01\theta_1 = |\Delta_n|^a + 1.01\theta_1.
\]
If \( \Delta_n \geq 2 \) then \( \Delta_n + \varepsilon_n > 0 \). By Lagrange’s mean value theorem,
\[
|\Delta_n + \varepsilon_n|^a = (\Delta_n + \varepsilon_n)^a = \Delta_n^a + a \varepsilon_n (\Delta_n + \theta \varepsilon_n)^{a-1}.
\]
Since \( |\Delta_n + \theta \varepsilon_n| \geq 2 - 1.01 = 0.99 \), we get:
\[
|a \varepsilon_n (\Delta_n + \theta \varepsilon_n)^{a-1}| \leq 1.01a \cdot (0.99)^{a-1} \leq 1.01,
|\Delta_n + \varepsilon_n|^a = |\Delta_n|^a + 1.01\theta.
The case of negative $\Delta_n$ is handled as above. Therefore,

$$
\sum_{N<n \leq N+M} |\Delta_n + \varepsilon_n|^a = \sum_{N<n \leq N+M} |\Delta_n|^a + 1.01\theta M.
$$

Now the required assertion follows from theorem 6, from the equality

$$
1.01M = \frac{\pi(a)}{(2\pi)^a} M L^{0.5a} \delta_1
$$

and from the estimations

$$
\delta_1 = \frac{\pi^{a+0.5}}{\Gamma\left(\frac{a+1}{2}\right)} \frac{1.01}{L^{0.5a}} \leq 1.01\pi L^{-\gamma} < 5.7L^{-\gamma} < 6(A^{-1} \ln L)^{-\gamma}.
$$

Suppose now $a > 1$. Using the same arguments as above, we get

$$
|\Delta_n + \varepsilon_n|^a = |\Delta_n|^a + \theta \left(0.6a 2^a |\Delta_n|^{a-1} + a(2.02)^a \right).
$$

Summing the both parts over $n$, we obtain:

$$
\sum_{N<n \leq N+M} |\Delta_n + \varepsilon_n|^a = \sum_{N<n \leq N+M} |\Delta_n|^a + \theta \left(0.6a 2^a \sum_{N<n \leq N+M} |\Delta_n|^{a-1} + a(2.02)^a M \right).
$$

Further, theorem 6 implies that the last sum over $n$ does not exceed

$$
\frac{\Gamma(0.5a)}{\pi^{a-0.5}} M L^{(a-1)/2} d, \quad d = 1 + c_1(A^{-1} \ln L)^{-\gamma},
$$

where $c_1 = 91$, $\gamma = \frac{1}{2}(a-1)$ in the case $1 < a \leq 2$, and $c_1 = 2^{15.5}$, $\gamma = \frac{1}{2} + \left\{\frac{a-2}{2}\right\} = \frac{1}{2} + \left\{\frac{a}{2}\right\}$ in the case $a > 2$. It’s easy to note that the quantity $d$ does not exceed 92 for any $a > 1$. Hence,

$$
0.6a 2^a \sum_{N<n \leq N+M} |\Delta_n|^{a-1} \leq 92 \cdot 0.6a 2^a \frac{\Gamma(0.5a)}{\pi^{a-0.5}} M L^{(a-1)/2} = \frac{\pi(a)}{(2\pi)^a} M L^{0.5a} \delta_1,
$$

where

$$
\delta_1 = \frac{\pi^{a+0.5}}{\Gamma(0.5a) \pi^{a-0.5}} \frac{\Gamma(0.5a)}{\pi^{a-0.5}} 92 \cdot 0.6a 2^a L^{-0.5} < 348 \frac{\Gamma(0.5a+1)}{\Gamma(0.5(a+1))} 2^a L^{-0.5} < \frac{348(a+1) 2^a L^{-0.5}}{L^{-0.49}}.
$$

Finally, we have:

$$
0.5a(2.02)^a M = \frac{\pi(a)}{(2\pi)^a} M L^{0.5a} \delta_2,
$$

$$
\delta_2 = \frac{a(2.02)^a \pi^{a+0.5}}{\Gamma(0.5(a+1)) L^{0.5a}} \leq 2a \sqrt{\pi} \left(\frac{2.02\pi}{\sqrt{L}}\right)^a < \frac{4.04\pi}{\sqrt{L}} < \frac{23}{\sqrt{L}}.
$$

Thus,

$$
\sum_{N<n \leq N+M} |\Delta_n + \varepsilon_n|^a = \frac{\pi(a)}{(2\pi)^a} M L^{0.5a} (1 + \theta \delta),
$$
where
\[ |\delta| \leq 2^{15.5} (A^{-1} \ln L)^{-\gamma} + L^{-0.49} + 23 L^{-0.5} < 2^{15.6} (A^{-1} \ln L)^{-\gamma}, \]
\[ \sum_{N < n \leq N + M} |\gamma_n - t_n|^a = \frac{x(a)}{(2 \vartheta'(t_N))^a} ML^{0.5a} (1 + \theta \delta). \]

The theorem is proved.

The assertions of theorems 7 and 8 allow one to find approximately the distribution function for the differences \( \gamma_n - t_n \).

**Theorem 10.** The quantity \( \nu(x) \) of the numbers \( n, N < n \leq N + M \) that satisfy the condition
\[ e_n = (\gamma_n - t_n) \vartheta'(t_N) \sqrt{2/L} \leq x \]
obeys the following relation:
\[ \nu(x) = M \left( \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-u^2/2} du + \frac{\theta A}{\sqrt{\ln L}} \right). \]

**Proof.** We give only a sketch, since the proof of this assertion repeats practically word-for-word the proof of theorem 4 in \([1]\) (lemma 3 in present paper). That’s why the most part of calculations is missed here.

Suppose \( f(t) \) be a characteristic function of the discrete random quantity with the values \( e_n, N < n \leq N + M \), that is, the sum
\[ f(t) = \frac{1}{M} \sum_{N < n \leq N + M} \exp(ite_n). \]

Taking an integer \( K > 1 \) whose precise value is chosen below, by corollaries of theorems 7 and 8 we get
\[ f(t) = \frac{1}{M} \sum_{N < n \leq N + M} (\cos(ite_n) + i \sin(ite_n)) = \]
\[ = \frac{1}{M} \sum_{N < n \leq N + M} \left( \sum_{k=0}^{K-1} \frac{(-1)^k (te_n)^{2k}}{(2k)!} + i \sum_{k=1}^{K} \frac{(-1)^{k-1} (te_n)^{2k-1}}{(2k-1)!} \right) \]
\[ = \sum_{k=0}^{K-1} \frac{(-1)^k 2^{2k}}{(2k)!} \frac{1}{M} \sum_{N < n \leq N + M} e_n^{2k} + i \sum_{k=1}^{K} \frac{(-1)^{k-1} t^{2k-1}}{(2k-1)!} \sum_{N < n \leq N + M} e_n^{2k-1} + \]
\[ + 2\theta \frac{t^{2K}}{(2K)!} \frac{1}{M} \sum_{N < n \leq N + M} e_n^{2K} = \]
\[ = \sum_{k=0}^{K} \frac{(-1)^k t^{2k} (2k)!}{(2k)!} \left( 1 + \frac{0.2 \theta_1 (4A \sqrt{3})^k}{\sqrt{L}} \right) + \theta_2 \sum_{k=1}^{K} \frac{|t|^{2k-1}}{(2k-1)!} \frac{1.6 \times (Be \pi^2)^k}{\sqrt{B} \sqrt{L}} \]
\[ + 2\theta_3 \frac{t^{2K}}{(2K)!} \frac{1}{2K} \left( 1 + \frac{0.2 (4A \sqrt{3})^K}{\sqrt{L}} \right), \]
where $\theta_1 = 0$ for $k = 0$. After some transformations we have

$$f(t) = g(t) + \theta(r_1 + r_2 + r_3),$$

where $g(t) = e^{-t^2/2}$,

$$r_1 = \frac{3}{K!} \left(\frac{t^2}{2}\right)^K, \quad r_2 = \frac{0.4|t|}{\sqrt{L}} \sum_{k=1}^{K} \frac{(2A^6\sqrt{3})^k}{k!} |t|^{2k-1}, \quad r_3 = \frac{2.4|t|}{\sqrt{BL}} \sum_{k=1}^{K} \frac{(Be\pi^2)^k}{k!} |t|^{2(k-1)}.$$

Now let’s consider the integral $I(\lambda)$,

$$I(\lambda) = \int_{0}^{\lambda} \frac{|f(t) - g(t)|}{|t|} dt,$$

where $\lambda > 1$. The we have the following estimate:

$$I(\lambda) \leq \frac{3}{K} \frac{1}{K!} \left(\frac{\lambda^2}{2}\right)^K + \frac{0.4}{\sqrt{L}} \sum_{k=1}^{K} \frac{(2A^6\sqrt{3})^k}{k!} \frac{\lambda^{2k}}{2k} + \frac{2.4}{\sqrt{BL}} \sum_{k=1}^{K} \frac{(Be\pi^2)^k}{k!} \frac{\lambda^{2k-1}}{2k-1} <$$

$$< \left(\frac{\lambda^2 e}{2K}\right)^K + \frac{0.2}{\sqrt{L}} \exp\left(2A^6\sqrt{3}\lambda^2\right) + \frac{2.4}{\sqrt{BL}} \exp\left(Be\pi^2 \lambda^2\right) <$$

$$< \left(\frac{\lambda^2 e}{2K}\right)^K + \frac{0.5}{\sqrt{L}} \exp\left(Be\pi^2 \lambda^2\right).$$

Setting

$$\lambda = \frac{1}{2\pi} \sqrt{-\frac{\ln L}{B e}}, \quad K = \left[\frac{Be}{2} \lambda^2\right] + 1 = \left[\frac{1}{8\pi^2} \ln L\right] + 1,$$

we get:

$$\frac{0.5}{\sqrt{L}} \exp\left(Be\pi^2 \lambda^2\right) < \frac{0.5}{\sqrt{L}},$$

$$\left(\frac{\lambda^2 e}{2K}\right)^K \leq B^{-K} \leq \exp\left(-\frac{\ln B}{8\pi^2} \ln L\right) \leq \exp\left(-\frac{5}{\pi^2} \ln L\right) < \frac{1}{\sqrt{L}}.$$ 

Thus, $I(\lambda) < 1/\sqrt{L}$. Let $F(x)$ and $G(x)$ be the distribution functions corresponding to the characteristic functions $f(t)$ and $g(t)$, respectively. By the Berry-Esseen inequality (see, for example, [17, p. 20.3]), for any real $x$ we have:

$$|F(x) - G(x)| \leq \frac{2}{\pi} I(\lambda) + \frac{12\sqrt{2}}{\pi \sqrt{n}} \lambda^{-1} < \frac{1}{\sqrt{L}} + \frac{e^{20.5} e^{-1.5}}{\sqrt{\ln L}} < \frac{A}{\sqrt{\ln L}}.$$

Hence,

$$F(x) = G(x) + \frac{\theta A}{\sqrt{\ln L}} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-u^2/2} du + \frac{\theta A}{\sqrt{\ln L}}.$$

The theorem is proved.
Corollary. Let $\Phi(x)$ be an arbitrary positive function that increases unboundedly as $x \to +\infty$. Then the inequalities
\[
\frac{1}{\Phi(n)} \frac{\sqrt{\ln \ln n}}{\ln n} < |\gamma_n - t_n| \leq \Phi(n) \frac{\sqrt{\ln \ln n}}{\ln n}
\]
hold for almost all $n$ in the following sense: the quantity of numbers $n, n \leq N$ that do not satisfy these inequalities, is $o(N)$ as $N \to +\infty$.

§ 4. The Behavior of Quantities $S(\gamma_n + 0), S(\gamma_n - 0)$ “in the Mean”

The function $S(t)$ is a piecewise smooth function with discontinuities at the ordinates of zeros of $\zeta(s)$. It decreases monotonically on every interval of discontinuity of the form $(\gamma_n, \gamma_{n+1})$. Thus, the right and left limits of $S(t)$ at a points of discontinuity have an obvious geometric sense: they are an upper and lower ‘peaks’ of saw-tooth graph of the function $S(t)$ (see fig. 1). The quantities $\Delta(n)$ are connected with the values of $S(t)$ at a Gram’s points ($\Delta(n) = S(t_n)$) and, similarly, the differences $\Delta_n$ are connected with the values of $S(t)$ at the points of discontinuity (lemma 7). This fact allows one to calculate the moments of the quantities $S(\gamma_n \pm 0)$ and to find approximately the distribution function for these quantities.

![Fig. 1. The graph of the function $S(t)$. Vertical segments on the graph correspond to the jumps of $S(t)$ at a points of discontinuity, i.e. at ordinates of zeros of $\zeta(s)$.

Lemma 7. Suppose $n \to +\infty$. Then the following equalities hold
\[
S(\gamma_n + 0) = -\Delta_n + \theta_1 \kappa_n + O\left(\frac{\ln n}{n}\right), \quad S(\gamma_n - 0) = -\Delta_n - \theta_2 \kappa_n + O\left(\frac{\ln n}{n}\right),
\]
where $0 \leq \theta_1, \theta_2 \leq 1$, and the implied constants in $O$’s are absolute.

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Proof. By definition of \( q_n \) we get \( \gamma_n = t_n + r_n \), where \( r_n = q_n(t_{n+1} - t_n) \). Applying Riemann-von Mangoldt formula we obtain

\[
N(\gamma_n + 0) = \frac{1}{\pi} \vartheta(\gamma_n) + 1 + S(\gamma_n + 0) = \frac{1}{\pi} \vartheta(t_n + r_n) + 1 + S(\gamma_n + 0) = \frac{1}{\pi} \vartheta(t_n) + 1 + \frac{r_n}{\pi} \vartheta'(t_n) + \frac{r_n^2}{2\pi} \vartheta''(\xi) + 1 + S(\gamma_n + 0),
\]

where \( \xi \) lies between \( \gamma_n \) and \( t_n \). By (14), we have

\[
\frac{r_n}{\pi} \vartheta'(t_n) = q_n \left( 1 + O\left( \frac{1}{n \ln n} \right) \right) = q_n + O\left( \frac{1}{n} \right),
\]

\[
\frac{r_n^2}{2\pi} \vartheta''(\xi) = O\left( \frac{1}{\xi} \right) = O\left( \frac{\ln n}{n} \right),
\]

and hence

\[
N(\gamma_n + 0) = n + q_n + S(\gamma_n + 0) + O\left( \frac{\ln n}{n} \right).
\]

From the other hand, in the proof of theorem 1 we find that

\[
N(\gamma_n + 0) = n + \theta_n(\kappa_n - 1), \quad 0 \leq \theta_n \leq 1.
\]

Comparing these expressions for \( N(\gamma_n + 0) \) and applying lemma 1 we get

\[
S(\gamma_n + 0) = -q_n + \theta_n(\kappa_n - 1) + O\left( \frac{\ln n}{n} \right) = -\Delta_n + \theta + \theta_n(\kappa_n - 1) + O\left( \frac{\ln n}{n} \right) =
\]

\[
= -\Delta_n + \theta_1 \kappa_n + O\left( \frac{\ln n}{n} \right),
\]

\[
S(\gamma_n - 0) = S(\gamma_n + 0) - \kappa_n = -\Delta_n - \theta_2 \kappa_n + O\left( \frac{\ln n}{n} \right),
\]

where \( 0 \leq \theta_1 \leq 1, \theta_2 = 1 - \theta_1 \). The lemma is proved.

Remark. Lemma 7 allows one to estimate the constant \( c \) in the inequality of Theorem 1: \( |\Delta_n| \leq c\ln n \). Indeed, using an upper bound for \( |S(t)| \), from [3] we obtain

\[
|\Delta_n| = |S(\gamma_n + 0) - \theta_1 \kappa_n| + O\left( \frac{\ln n}{n} \right) =
\]

\[
= |(1 - \theta_1)S(\gamma_n + 0) + \theta_1 S(\gamma_n - 0)| + O\left( \frac{\ln n}{n} \right) \leq
\]

\[
\leq (1 - \theta_1) |S(\gamma_n + 0)| + \theta_1 |S(\gamma_n - 0)| + O\left( \frac{\ln n}{n} \right) \leq
\]

\[
\leq (1 - \theta_1 + \theta_1)8.9 \ln \gamma_n + O\left( \frac{\ln n}{n} \right) < 8.9 \ln n.
\]

More general, suppose one has an estimation of the type \( |S(t)| \leq cf(t) \) where the function \( f(t) \) increases monotonically and \( c \) is a positive constant. Then by the same arguments
we can conclude that \(|\Delta_n| \leq (c + \varepsilon) f(n)\) for any \(\varepsilon > 0\) and \(n \geq n_0(\varepsilon)\). Thus, if the Riemann hypothesis is true then the best known bound for \(|S(t)|\) (see [18]) implies that

\[
|\Delta_n| \leq \left( \frac{1}{2} + \varepsilon \right) \frac{\ln n}{\ln \ln n}.
\]

For the below, we need some assertions concerning the quantities \(\kappa_n\). The main purpose of these assertions is to show that these quantities behave as a constants ‘in the mean’.

**Definition 8.** Let \(j \geq 1\) be an integer. Denote by \(n_j(T)\) the number of ordinates \(\gamma\) of zeros of \(\zeta(s)\), \(0 < \gamma \leq T\), whose multiplicities are equal to \(j\).

First we show that the number \(n_j(T)\) is sufficiently small for large \(j\). In essence, the proof of this fact repeats word-for-word an upper estimate for the number of zeros of \(\zeta(s)\) with given multiplicity. This proof is based on two facts: 1) an absolute value of the difference \(S(t + h) - S(t)\) is very large when \(t\) is close to the ordinate \(\gamma\) with high multiplicity for \(h \asymp (\ln t)^{-1}\) (lemma 8); 2) the measure of the subset of given interval where this difference is large, is small enough (lemma 9).

In what follows, \(T \geq T_0(\varepsilon) > 0\), \(h = \frac{\pi}{3} (\ln \frac{T}{27})^{-1}\). The parameter \(H\) is supposed to be the following. The below lemma is proved in [19] only for \(H = T^{27/82 + \varepsilon}\). This restriction is inconvenient in some cases. The proof of lemma 9 is based on the density theorem for the zeros of the Riemann zeta-function, that is, on the upper bound for the number

\[
N(\sigma, T + H) - N(\sigma, T)
\]

of zeros of \(\zeta(s)\) in the region \(\sigma < \text{Re} s < 1\), \(T < \text{Im} s \leq T + H\) (see [14]). However, the proof of this bound is valid also in the case \(T^{27/82 + \varepsilon_1} \leq H \leq T^{27/82 + \varepsilon_2}\), where \(\varepsilon_1, \varepsilon_2\) are an arbitrary numbers with the condition \(0.9 \varepsilon \leq \varepsilon_1 < \varepsilon_2 \leq \varepsilon\). Therefore we suppose that \(H\) satisfies the above conditions with \(\varepsilon_1 = 0.9 \varepsilon\), \(\varepsilon_2 = \varepsilon\).

**Lemma 8.** Let \(\gamma\) be the imaginary part of a zero of \(\zeta(s)\) such that \(T \leq \gamma - h < \gamma \leq T + H\). Suppose that the interval \((\gamma - h, \gamma]\) contains exactly \(m\) ordinates of zeros of \(\zeta(s)\) whose multiplicities are equal to \(j\). Then we have

\[
S(t + 2h) - S(t) \geq mj - 0.5
\]

for any \(t\) in \((\gamma - 2h, \gamma - h]\).

**Proof.** Suppose that \(\gamma - 2h < t < \gamma - h\) and that \(\gamma^{(1)} < \ldots < \gamma^{(r)}\) are all distinct ordinates of zeros of \(\zeta(s)\) contained in \((t, t + 2h]\). Using the same arguments as above (see theorem 2) we obtain the identity

\[
S(t + 2h) - S(t) = (S(t + 2h) - S(\gamma^{(r)} + 0)) + \sum_{\nu=1}^{r} \left( S(\gamma^{(\nu)} + 0) - S(\gamma^{(\nu)} - 0) \right) + \sum_{\nu=2}^{r} \left( S(\gamma^{(\nu)} - 0) - S(\gamma^{(\nu-1)} + 0) \right) + (S(\gamma^{(1)} - 0) - S(t)).
\]

(30)
All the differences \( S(\gamma^{(\nu)} + 0) - S(\gamma^{(\nu)} - 0) = \kappa(\gamma^{(\nu)}) \) are positive. Moreover, the inequalities \( \gamma - h > t \), \( \gamma < t + 2h \) imply that the interval \( (\gamma - h, \gamma] \) is contained entirely in \( (t, t + 2h] \), and there are at least \( m \) ordinates among \( \gamma^{(1)}, \ldots, \gamma^{(r)} \), whose multiplicities are equal to \( j \). Hence,

\[
\sum_{\nu=1}^{r} (S(\gamma^{(\nu)} + 0) - S(\gamma^{(\nu)} - 0)) \geq mj.
\]

Transforming all other differences in the right-hand side of (30) by Lagrange’s mean value theorem and using the relation

\[
S'(t) = -\frac{1}{2\pi} \ln \frac{t}{2\pi} + O(t^{-2}) = -\left( \frac{1}{6h} + \frac{\theta H}{\pi T} \right),
\]

valid for \( T \leq t \leq T + H \), we obtain

\[
S(t+2h) - S(t) \geq mj - \left( (t+2h-\gamma^{(r)}) + \sum_{\nu=2}^{r} (\gamma^{(\nu)} - \gamma^{(\nu-1)}) + (\gamma^{(1)} - t) \right) \left( \frac{1}{6h} + \frac{H}{\pi T} \right) =
\]

\[
= mj - 2h \left( \frac{1}{6h} + \frac{H}{\pi T} \right) > mj - 0.5.
\]

The lemma is proved.

**Lemma 9.** Let \( D(\lambda) \) be the set of points \( t \) such that \( T \leq t \leq T + H \) and \( |S(t+2h) - S(t)| \geq \lambda \). Then the inequality

\[
\text{mes } D(\lambda) \leq e^4 H e^{-C\lambda},
\]

holds for any \( \lambda \) with \( C = \pi \sqrt{\frac{2}{5}e A^{-1}} \).

For the proof, see [19].

**Lemma 10.** For any \( j \geq 1 \), the following estimation holds:

\[
n_j(T + H) - n_j(T) < e^{7.2} \left( N(T + H) - N(T) \right) e^{-Cj},
\]

where the constant \( C \) is defined in lemma 9.

**Proof.** We follow the proof of theorems B and C from [19]. Let \( j \) be the multiplicity of an ordinate \( \gamma \), \( T < \gamma \leq T + H \). Then the inequality \( |S(t)| \leq 8.9 \ln t \) implies

\[
j = S(\gamma + 0) - S(\gamma - 0) \leq 17.8 \ln \gamma < 18 \ln T.
\]

Hence, the difference \( n_j(T + H) - n_j(T) \) equals to zero for \( j \geq 18 \ln T \). In the case \( 1 \leq j \leq C^{-1} \ln 2 \) we obviously have \( 1 \leq 2e^{-Cj} \) and therefore

\[
n_j(T + H) - n_j(T) \leq N(T + H) - N(T) \leq 2(N(T + H) - N(T)) e^{-Cj}.
\]

Thus, it is sufficient to consider the case

\[
C^{-1} \ln 2 < j < 18 \ln T.
\]
Suppose $\gamma_{(1)}$ is the largest ordinate with multiplicity equal to $j$ in the interval $(T, T + H]$. Then we denote $E_1 = (\gamma_{(1)} - h, \gamma_{(1)}]$ and stand the symbol $\gamma_{(2)}$ for the largest ordinate whose multiplicity equals to $j$ in the interval $(T, \gamma_{(1)} - h]$. Further, we denote $E_2 = (\gamma_{(2)} - h, \gamma_{(2)}]$ and stand the symbol $\gamma_{(3)}$ for the largest ordinate whose multiplicity equals to $j$ in the interval $(T, \gamma_{(2)} - h]$ and so on.

We continue this construction until there are no such ordinates in the interval $(T, \gamma_{(r)} - h]$ or until we find such ordinate $\gamma_{(r)}$ satisfying the condition $\gamma_{(r)} - h < T \leq \gamma_{(r)}$. In both cases we set $E_r = (\gamma_{(r)} - h, \gamma_{(r)}]$.

The intervals $E_1, E_2, \ldots, E_r$ are pairwise disjoint and have the same length $h$. Moreover, their union contains all ordinates of zeros with multiplicity equals to $j$ lying in $(T, T + H]$. Now we partition the intervals constructed into classes $\mathcal{E}_1, \mathcal{E}_2, \ldots$ by putting into class $\mathcal{E}_m$ the intervals containing exactly $m$ of desired ordinates. If $k_m$ is the number of intervals in the class $\mathcal{E}_m$ then

$$n_j(T + H) - n_j(T) \leq 1 \cdot k_1 + 2 \cdot k_2 + \ldots + mk_m + \ldots \quad (31)$$

Let us find an upper bound for each of quantities $k_m$. Suppose the interval $E = (\gamma - h, \gamma]$ belongs to $\mathcal{E}_m$. We then set $E' = E - h = (\gamma - 2h, \gamma - h]$. By lemma 8,

$$S(t + 2h) - S(t) \geq mj - 0.5$$

for any $t \in E'$. Since $E'$ is contained in $(T - 2h, T + H]$ then $E'$ is entirely contained in $D(mj - 0.5)$ where $D(\lambda)$ denotes the set of points $t \in (T - 2h, T + H]$ satisfying the inequality $|S(t + 2h) - S(t)| \geq \lambda$. After carrying out this construction for any interval $E$ in $\mathcal{E}_m$, we see that all the $k_m$ intervals are contained in $D(mj - 0.5)$. Since they are pairwise disjoint and have the same length $h$, their total length does not exceed the measure of $D(mj - 0.5)$, that is

$$k_m h \leq \text{mes } D(mj - 0.5).$$

By lemma 9, we get

$$\text{mes } D(mj - 0.5) \leq e^{A(H + 2h)e^{-C(mj - 0.5)}} < e^{A + CH e^{-Cmj}},$$

$$k_m \leq e^{A + CH h^{-1}} e^{-Cmj}.$$

Returning to (31), we obtain

$$n_j(T + H) - n_j(T) \leq e^{A + CH h^{-1}} \sum_{m=1}^{+\infty} me^{-Cmj} = e^{A + CH h^{-1}} \frac{e^{-Cj}}{(1 - e^{-Cj})^2}.$$ 

Since $e^{-Cj} \leq \frac{1}{2}$, then

$$n_j(T + H) - n_j(T) \leq 4e^{A + CH h^{-1}} e^{-Cj} < e^{7.2} (N(T + H) - N(T)) e^{-Cj}.$$

The lemma is proved.

**Corollary.** Let $\nu(j) = \nu(j; M, N)$ be the number of distinct ordinates $\gamma_n$, $N < n \leq N + M$, whose multiplicities are equal to $j$. Then the following estimation holds: $\nu(j) \leq e^{7.3} Me^{-Cj}$. 

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**Proof.** Setting $T = \gamma_N$, $H = \gamma_{N+M} - \gamma_N$ in lemma and noting that the multiplicities of $\gamma_N$, $\gamma_{N+M}$ are of order $\ln N$, we obtain

$$\nu(j) \leq e^{7.2(M + O(\ln N))e^{-Cj}} < e^{7.3Me^{-Cj}}.$$ 

The bounds of lemma 10 and it’s corollary allows one to study the behavior of multiplicities $\kappa_n$ ‘in the mean’.

**Definition 9.** Suppose $a > 0$. We denote by

$$K_0(a) = K_0(a; M, N) = \sum_{N < n \leq N + M} \kappa_n^a$$

the sum over all ordinates $\gamma_n$, $N < n \leq N + M$. Further, we denote by

$$K_1(a) = K_1(a; M, N) = \sum_{N < n \leq N + M} \kappa_n^a$$

the sum over all distinct ordinates $\gamma_n$, $N < n \leq N + M$.

Thus, if (9) holds then the sum $K_0(a)$ contains, for example, all the terms $\kappa_l^a$, $\kappa_{l+1}^a, \ldots, \kappa_{l+p-1}^a$, though the sum $K_1(a)$ contains only the term $\kappa_l^a$. It’s easy to see that

$$K_0(a) = \sum_{N < n \leq N + M} \kappa_n \cdot \kappa_n^a = K_1(a + 1).$$

**Lemma 11.** Suppose $a \geq 1$. The following estimation holds

$$K_1(a) < e^{8.4 \frac{\Gamma(a + 1)}{C_{a+1}} M},$$

where the constant $C$ is defined in lemma 9.

**Proof.** Using the assertion and the notations of lemma 10 and it’s corollary, we find

$$K_1(a) = \sum_{j \geq 1} j^a \nu(j) \leq e^{7.3M} \sum_{j=1}^{\infty} j^a e^{-Cj}. \quad (32)$$

The function $y(x) = x^a e^{-Cx}$ increases on the segment $1 \leq x \leq x_a$, $x_a = aC^{-1}$, up to it’s maximum

$$\left( \frac{a}{C} \right)^a e^{-a} = \left( \frac{a}{Ce} \right)^a \quad (33)$$

at a point $x_a$ and then decreases monotonically. Let’s define the integer $m$ from the inequalities $m < x_a \leq m + 1$. Then we estimate the terms of the sum (32) corresponding to $j = m, m + 1$ by the quantity (33) and all other terms by the integrals of the function $y(x)$. Thus we get

$$\sum_{j=1}^{\infty} j^a e^{-Cj} \leq \left( \int_0^m + \int_{m+1}^{\infty} \right) x^a e^{-Cx} dx + 2 \left( \frac{a}{Ce} \right)^a <$$

$$< \int_0^{\infty} x^a e^{-Cx} dx + 2 \left( \frac{a}{Ce} \right)^a < \frac{\Gamma(a + 1)}{C_{a+1}} + 2 \left( \frac{a}{Ce} \right)^a < \frac{3\Gamma(a + 1)}{C_{a+1}}. \quad (34)$$

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Therefore, the desired assertion follows from the last inequality.

The estimate of lemma 11 together with the equalities of lemma 7 allow one to calculate the moments of the quantities $S(\gamma_n \pm 0)$.

**Theorem 11.** Let $k$ be an integer with the condition $1 \leq k \leq \sqrt{L}$. Then the following relation holds

$$\sum_{N < n \leq N + M} S^{2k}(\gamma_n + 0) = \frac{\pi(2k)}{(2\pi)^{2k}} ML^k \left(1 + 19\theta_1 A(4A\sqrt{3})^k L^{-0.5}\right),$$

$$\sum_{N < n \leq N + M} S^{2k}(\gamma_n - 0) = \frac{\pi(2k)}{(2\pi)^{2k}} ML^k \left(1 + 19\theta_2 A(4A\sqrt{3})^k L^{-0.5}\right).$$

**Proof.** Using the same arguments as above (see the proof of theorem 7) and the equality of lemma 7 we get

$$S(\gamma_n + 0) = -\Delta_n + 1.1\theta\kappa_n,$$

$$S^{2k}(\gamma_n + 0) = \Delta_n^{2k} + \theta k 2^{2k} (|\Delta_n|^{2k-1}\kappa_n + (1.1)^{2k}\kappa_n^2),$$

and finally

$$\sum_{N < n \leq N + M} S^{2k}(\gamma_n + 0) = \Sigma_1 + \theta k 2^{2k}((1.1)^{2k}\Sigma_2 + \Sigma_3),$$

where

$$\Sigma_1 = \sum_{N < n \leq N + M} \Delta_n^{2k}, \quad \Sigma_2 = \sum_{N < n \leq N + M} \kappa_n^{2k} = \sum_{N < n \leq N + M} \kappa_n^{2k+1},$$

$$\Sigma_3 = \sum_{N < n \leq N + M} |\Delta_n|^{2k-1}\kappa_n.$$

First, by lemma 11 we conclude that

$$\Sigma_2 \leq \frac{e^{8.4}}{C^{2k+2}} (2k+1)! M = \frac{\pi(2k)}{(2\pi)^{2k}} ML^k \delta_2,$$

where

$$\delta_2 = \frac{e^{8.4}}{C^2} \left(\frac{2\pi}{C}\right)^{2k} \frac{k!(2k+1)!}{(2k)!} L^k \leq \frac{e^{8.4}}{C^2} \left(\frac{4\pi^2}{C^2 L}\right)^k (2k+1) 2\sqrt{k} \left(\frac{k}{e}\right)^k \leq \frac{e^{11}}{C^2} k\sqrt{k} \left(\frac{4\pi^2 k}{C^2 L}\right)^k \leq \frac{e^{11}}{C^2} \left(\frac{4\pi^2 \sqrt{3} k}{C^2 L}\right)^k.$$

The right-hand side has its maximum at the point $k = 1$ for $1 \leq k \leq \sqrt{L}$. Hence,

$$\delta_2 \leq \frac{e^{11}}{C^2} \frac{4\pi^2 \sqrt{3}}{C^2 L} < \frac{1}{\sqrt{L}}.$$

Moreover, we have

$$\delta_2^{1/(2k)} = \left(\frac{e^{11}}{C^2}\right)^{1/(2k)} \sqrt{\frac{4\pi^2 \sqrt{3} k}{C^2 L}} \leq \sqrt{\frac{e^{11}}{C^2} \frac{2\pi \sqrt{3}}{C\sqrt{e}} \sqrt{\frac{k}{L}}} < e^{2.8} A^2 \sqrt{\frac{k}{L}}.$$
Further, by the corollary of theorem 4 we obtain
\[ \Sigma_1 \leq \frac{\mathcal{N}(2k)}{(2\pi)^{2k}} ML^k \delta_1, \quad \delta_1 = 1.1A^{k-1}. \]

Thus
\[ k2^{2k}\Sigma_3 \leq k2^{2k}\Sigma_1 \leq \frac{\mathcal{N}(2k)}{(2\pi)^{2k}} ML^k \delta_3, \]
\[ \delta_3 = k2^{2k}\delta_1^{-1/2}\delta_2^{-1/2} \leq k2^{2k}\left(\frac{1.1}{A}\right)^{1-1/2}\delta_1^{-1/2} \leq A^{-1/2}e^{2.8}\sqrt{\frac{k}{L}} \leq \sqrt{1.1e^{2.8}A(4A)^k}. \]

Finally we get
\[ \sum_{N<n \leq N+M} S^{2k}(\gamma_n + 0) = \frac{\mathcal{N}(2k)}{(2\pi)^{2k}} ML^k(1 + \theta\delta), \]
where
\[ \delta = \frac{1.1A^k}{\sqrt{L}} + \frac{18A(4A\sqrt{3})^k}{\sqrt{L}} + \frac{k2^{2k}}{\sqrt{L}} < \frac{19A(4A\sqrt{3})^k}{\sqrt{L}}. \]

The second equality can be proved in a similar way. The theorem is proved.

**Theorem 12.** Let \( k \) be an integer with the condition \( 1 \leq k \leq \sqrt{L} \). Then the following relation holds
\[ \sum_{N<n \leq N+M} \left| S^{2k-1}(\gamma_n + 0) \right| < c_k ML^{k-1}, \quad \sum_{N<n \leq N+M} \left| S^{2k-1}(\gamma_n - 0) \right| < c_k ML^{k-1}, \]
where the quantity \( c_k \) can be set to be equal to \( (eA)^3 \) for \( k = 1 \) and to \( 3.7B^{-0.5}(Bk)^k \), for \( k \geq 2 \).

**Proof.** It’s sufficient to estimate the first sum. Suppose \( k = 1 \). By lemmas 7,11 and theorem 5 we have:
\[ \sum_{N<n \leq N+M} S(\gamma_n + 0) \leq \sum_{N<n \leq N+M} \Delta_n \leq 1.01 \sum_{N<n \leq N+M} \kappa_n \leq 3.6\sqrt{BM} + e^{2.7}A^3M < (eA)^3M. \]

Suppose now \( k \geq 2 \). Summing both parts of the equality
\[ S^{2k-1}(\gamma_n + 0) = -\Delta_n^{2k-1} + 0.5\theta h 2^{2k}((1.01)^{2k-1}\kappa_n^{2k-1} + \Delta_n^{2k-2}\kappa_n), \]
we get
\[ \sum_{N<n \leq N+M} \left| S^{2k-1}(\gamma_n + 0) \right| \leq \sum_{N<n \leq N+M} \Delta_n^{2k-1} + 0.5k 2^{2k}((1.01)^{2k-1}\Sigma_1 + \Sigma_2). \]
where

\[ \Sigma_1 = \sum_{N<n \leq N+M} \kappa_n^{2k-1} = \sum'_{N<n \leq N+M} \kappa_n^{2k}, \quad \Sigma_2 = \sum_{N<n \leq N+M} \Delta_n^{2k-2} \kappa_n. \]

By lemma 11,

\[ \Sigma_1 \leq e^{8,4} C^{2k+1} (2k)! M < e^{7,6} A \sqrt{k} \left( \frac{2k}{Ce} \right)^{2k} M, \]

\[ 0.5k 2^k (1.01)^{2k-1} \Sigma_1 < e^7 A \left( \frac{2k}{C} \right)^{2k} M = k^k ML^{k-1} \delta_1, \]

\[ \delta_1 = e^7 AL \left( \frac{4k}{C^2 L} \right)^k \leq e^7 AL \left( \frac{8}{C^2 L} \right)^2 < \frac{1}{\sqrt{L}} < 0.05 \sqrt{B}. \]

Further, applying Hölder’s inequality to the sum \( \Sigma_2 \) we obtain \( \Sigma_2 \leq \Sigma_3^{1/k} \Sigma_4^{1/k} \), where

\[ \Sigma_3 = \sum_{N<n \leq N+M} \Delta_n^{2k}, \quad \Sigma_4 = \sum_{N<n \leq N+M} \kappa_n^k = \sum'_{N<n \leq N+M} \kappa_n^{k+1}. \]

The corollary of theorem 4 and the estimate of lemma 1 imply together

\[ \Sigma_3 \leq 1.1 \frac{\varkappa(2k)}{A (2\pi)^{2k}} M (AL)^k < \frac{\sqrt{e}}{A} \left( \frac{kAL}{\pi^2 e} \right)^k M, \]

\[ \Sigma_4 \leq e^{8,4} C^{k+2} (k+1)! M < e^6 A^2 k \sqrt{k} \left( \frac{k}{Ce} \right)^k M. \]

Hence

\[ 0.5k 2^k \Sigma_2 \leq 0.5k 2^k \frac{4\sqrt{e}}{\sqrt{A}} \left( \frac{kAL}{\pi^2 e} \right)^{k-1} e^3 A \sqrt{3} k \frac{k}{Ce} M < \]

\[ < 25\sqrt{A} (\frac{k}{3} kA)^k ML^{k-1} < 0.05 \sqrt{B} (Bk)^k ML^{k-1}. \]

Now the desired assertion follows from theorem 5:

\[ \left| \sum_{N<n \leq N+M} S^{2k-1}(\gamma_n + 0) \right| < \frac{(Bk)^k}{\sqrt{B}} ML^{k-1} (3.6 + 0.05 + 0.05) = 3.7 \sqrt{B} (Bk)^k ML^{k-1}. \]

The theorem is proved.

**Theorem 13.** Let \( a \) be an arbitrary real number with the condition

\[ 0 < a < \frac{e \ln L}{10 A \ln \ln L}. \]

Then the following asymptotic formulae hold

\[ \sum_{N<n \leq N+M} |S(\gamma_n + 0)|^a = \frac{\varkappa(a)}{(2\pi)^a} ML^{0.5a} (1 + \theta_1 \delta), \]

\[ \sum_{N<n \leq N+M} |S(\gamma_n - 0)|^a = \frac{\varkappa(a)}{(2\pi)^a} ML^{0.5a} (1 + \theta_2 \delta), \]

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where \( \delta = c(A^{-1} \ln L)^{-7} \). Therefore, the constants \( c \) and \( \gamma \) can be set to be equal to \( e^5 A^3 \) and \( 0.5a \) if \( 0 < a \leq 1 \), and to \( 2^{15.5} \) and \( \frac{1}{2} + \left\{ \frac{a-1}{2} \right\} \) if \( a > 1 \). In particular, in the case \( a = 1 \) we get

\[
\sum_{N < n \leq N + M} |S(\gamma_n + 0)| = \frac{M}{\pi \sqrt{\pi}} \sqrt{\ln \ln N} (1 + O((\ln \ln \ln N)^{-0.5})),
\]

\[
\sum_{N < n \leq N + M} |S(\gamma_n - 0)| = \frac{M}{\pi \sqrt{\pi}} \sqrt{\ln \ln N} (1 + O((\ln \ln \ln N)^{-0.5})).
\]

Proof. Suppose first that \( 0 < a \leq 1 \). Using lemma 7 and considering the cases \( |\Delta_n| \leq 2\kappa_n \) and \( |\Delta_n| > 2\kappa_n \) separately we obtain

\[
|S(\gamma_n + 0)|^a = |\Delta_n|^a + 3.01 \theta \kappa_n.
\]

Summing both parts over \( n \) we get

\[
\sum_{N < n \leq N + M} |S(\gamma_n + 0)|^a = \Sigma_1 + 3.01 \theta \Sigma_2,
\]

where

\[
\Sigma_1 = \sum_{N < n \leq N + M} |\Delta_n|^a, \quad \Sigma_2 = \sum_{N < n \leq N + M} \kappa_n = \sum_{N < n \leq N + M} \kappa_n^2.
\]

Applying lemma 11 to the sum \( \Sigma_2 \) we have

\[
\Sigma_2 \leq \frac{2e^{8.4} M}{C^3} = \frac{\zeta(a)}{(2\pi)^a} M L^{0.5a} \delta_2,
\]

\[
\delta_2 \leq \frac{a^{a+1/2}}{\Gamma \left( \frac{a+1}{2} \right)} \frac{2e^{8.4}}{C^3 L^{0.5a}} \leq 2^{1.5} e^{8.4} C^{-3} L^{-0.5a} < 0.5e^5 A^3 L^{-0.5}.
\]

The application of asymptotic formula for the sum \( \Sigma_1 \) yields

\[
\sum_{N < n \leq N + M} |S(\gamma_n + 0)|^a = \frac{\zeta(a)}{(2\pi)^a} M L^{0.5a} (1 + \theta \delta),
\]

where

\[
\delta \leq 91(A^{-1} \ln L)^{-0.5a} + 0.5e^5 A^3 L^{-0.5a} < e^5 A^3 (A^{-1} \ln L)^{-0.5a}.
\]

Suppose now \( a > 1 \). By easy-to-check relation

\[
|S(\gamma_n + 0)|^a = |\Delta_n|^a + (3.01)^a \theta (2\kappa_n^a + |\Delta_n|^{a-1} \kappa_n^a)
\]

we obtain the following expression for the initial sum:

\[
\sum_{N < n \leq N + M} |S(\gamma_n + 0)|^a = \Sigma_1 + (3.01)^a \theta \left( \Sigma_2 + \Sigma_1^{-1/a} \Sigma_2^{1/a} \right),
\]

where

\[
\Sigma_2 = \sum_{N < n \leq N + M} \kappa_n^a = \sum_{N < n \leq N + M} \kappa_n^{a+1}.
\]
(the symbol \(\Sigma_1\) denotes the same sum as above). Using again the estimation of lemma 11 together with the duplication formula for the gamma-function we obtain

\[
\Sigma_2 \leq \frac{e^{8.4}}{C^{a+2}} \Gamma(a+2)M \leq \frac{\nu(a)}{(2\pi)^a} ML^{0.5a} \delta_1,
\]

\[
\delta_1 = \frac{\pi^{a+1/2}}{\Gamma\left(\frac{a+1}{2}\right)} \frac{\Gamma(a+2)}{C^{a+2}} \frac{e^{8.4}}{L^{0.5a}} \leq e^9 C^{-2} \frac{\Gamma(a+2)}{\Gamma\left(\frac{a+1}{2}\right)} \left(\frac{\pi}{C\sqrt{L}}\right)^a = e^9 C^{-2} \frac{\Gamma(a+2)}{\Gamma\left(\frac{a+1}{2}\right)} \left(\frac{2\pi}{C\sqrt{L}}\right)^a \leq e^9 C^{-2} \left(\frac{4\pi a}{C\sqrt{L}}\right)^a.
\]  

(35)

Since \(a = o(\ln L)\), we obviously have

\[
\delta_1 \leq e^9 C^{-2} \left(\frac{1}{\sqrt[4]{L}}\right)^a < \frac{1}{\sqrt[4]{L}}.
\]

Next, noting that

\[
\Sigma_1 < 1.01 \frac{\nu(a)}{(2\pi)^a} ML^{0.5a}
\]

and applying the bound [35] for \(\delta_1\), we obtain

\[
\Sigma_1^{1-1/a} \Sigma_2^{1/a} < \frac{\nu(a)}{(2\pi)^a} ML^{0.5a} \delta_2, \quad \delta_2 < 1.01 e^9 C^{-2} \frac{4\pi a}{C\sqrt{L}} < \frac{1}{\sqrt[4]{L}}.
\]

Thus,

\[
\sum_{N < n \leq N + M} |S(\gamma_n + 0)|^a = \frac{\nu(a)}{(2\pi)^a} ML^{0.5a} (1 + \theta \delta),
\]

where

\[
\delta = 2^{15.5} (A^{-1} \ln L)^{-\gamma} + 2(3.01)^a L^{-0.25} < 2^{15.6} (A^{-1} \ln L)^{-\gamma}, \quad \gamma = \frac{1}{2} + \left\{\frac{a-1}{2}\right\}.
\]

All the above arguments can be applied without any modifications to the second sum of the theorem. The theorem is proved.

By standard way, we derive the following assertion from the theorems 11 and 12.

**Theorem 14.** For a real \(x\) let quantity \(\nu_1 = \nu_1(x)\) denotes the number of ordinates \(\gamma_n, N < n \leq N + M\) satisfying the condition

\[
S(\gamma_n + 0) \leq \frac{x}{\pi} \sqrt{\frac{L}{2}}.
\]

Then we have the equation

\[
\nu_1(x) = M \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-u^2/2} + \frac{\theta A}{\sqrt{\ln L}}\right).
\]

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The quantity $\nu_2(x)$ of ordinates $\gamma_n$, $N < n \leq N + M$ with the condition

$$S(\gamma_n - 0) \leq \frac{x}{\pi}\sqrt{\frac{L}{2}}$$

obeys the similar relation.

Proof of this theorem repeats word-for-word the proof of theorem 10.

**Corollary.** Let $\Phi(x)$ be an arbitrary positive monotonic function that increases unboundedly as $x \to +\infty$. Then the inequalities

$$\frac{1}{\Phi(n)} \sqrt{\ln \ln n} < |S(\gamma_n + 0)| \leq \Phi(n) \sqrt{\ln \ln n}$$

(36)

holds for ‘almost all’ $n$ (i.e. the quantity of numbers $n \leq N$ that do not satisfy inequalities (36), is $o(N)$ as $N \to +\infty$). This relation holds true if we replace the quantities $S(\gamma_n+0)$ in (36) to $S(\gamma_n - 0)$.

**Remark.** Let $K \geq K_0(\varepsilon) > 0$ be a sufficiently large integer and $N$ runs through all the integers from the interval $(K, 2K]$. Then all the theorems from §§2-4 hold true with $M = \lfloor K^\varepsilon \rfloor$ for each $N$ in this interval, except, may be $K_{1-0.05\varepsilon}$ values.

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